Krishna's

TEXT BOOK on Mechanics

(*For B.A. and B.Sc. IInd year students of All Colleges affiliated to universities in Uttar Pradesh*)

As per U.P. UNIFIED Syllabus

(*w.e.f.* 2012-2013)

By

A. R. Vasishtha
Retd. Head, Dep't. of Mathematics
Meerut College, Meerut (U.P.)

R.K. Gupta
Retd. Principal & Head, Dep't. of Mathematics
S.S.V. (P.G.) College, Hapur (U.P.)

(Meerut Edition)

KRISHNA Prakashan Media (P) Ltd.
KRISHNA HOUSE, 11, Shivaji Road, Meerut-250 001 (U.P.), India

Jai Shri Radhey Shyam

Dedicated

to

Lord

Krishna

Authors & Publishers

Preface

This book on **MECHANICS** has been specially written according to the latest **Unified Syllabus** to meet the requirements of the **B.A. and B.Sc. Part-II Students** of all Universities in Uttar Pradesh.

The subject matter has been discussed in such a simple way that the students will find no difficulty to understand it. The proofs of various theorems and examples have been given with minute details. Each chapter of this book contains complete theory and a fairly large number of solved examples. Sufficient problems have also been selected from various university examination papers. At the end of each chapter an exercise containing objective questions has been given.

We have tried our best to keep the book free from misprints. The authors shall be grateful to the readers who point out errors and omissions which, inspite of all care, might have been there.

The authors, in general, hope that the present book will be warmly received by the students and teachers. We shall **indeed** be very thankful to our colleagues for their recommending this book to their students.

The authors wish to express their thanks to **Mr. S.K. Rastogi**, *M.D.* and **Mr. Sugam Rastogi**, *Executive Director* and **entire team** of **KRISHNA Prakashan Media (P) Ltd.**, **Meerut** for bringing out this book in the present nice form.

The authors will feel amply rewarded if the book serves the purpose for which it is meant. Suggestions for the improvement of the book are always welcome.

June, 2012 ***—Authors***

Syllabus

Mechanics

U.P. UNIFIED (*w.e.f.* 2012-13)

B.A./B.Sc. IInd Year–Paper-IIIrd M.M. : 34 / 70

Dynamics

Unit-1: Velocity and acceleration along radial and transverse directions, and along tangential and normal directions, Simple harmonic motion, Motion under other laws of forces, Earth attraction, Elastic strings.

Unit-2: Motion in resisting medium, Constrained motion (circular and cycloidal only).

Unit-3: Motion on smooth and rough plane curves, Rocket motion, Central orbits and Kepler's law, Motion of a particle in three dimensions.

Statics

Unit-4: Common catenary, Centre of gravity, Stable and unstable equilibrium, Virtual work.

Unit-5: Forces in three dimensions, Poinsot's central axis, Wrenches, Null line and null plane.

Brief Contents

SECTION

A

STATICS

Chapters

1. Centre of Gravity

2. Strings in Two Dimensions (Common Catenary)

3. Virtual Work

4. Stable and Unstable Equilibrium

5. Equilibrium of Forces in Three Dimensions (Poinsot's Central Axis)

6. Forces in Three Dimensions (Null Lines, Null Planes, Screws and Wrenches)

Chapter

1

Centre of Gravity

1.1 Centre of Gravity

On account of the attraction of the earth (*known as gravity*) every particle on the surface of the earth is attracted towards the centre of the earth by a force proportional to its mass, called the *weight of the particle.*

A rigid body is considered as a collection of particles, rigidly connected with one another, on which are acting the weights of the particles. Such weights are considered to be parallel forces. The resultant of these forces is called the weight of the body and it always passes through a fixed point. This point is called the *centre of gravity* of the body.

Definition: *The centre of gravity of a body is the point, fixed relative to the body, through which the line of action of the weight of the body, always passes, whatever be the position of the body, provided that its size and shape remain unaltered.* **(Agra 2007)**

Centre of gravity is usually written in brief as C.G.

Note 1: *The centre of gravity of a body does not necessarily lie in the body itself.*

Note 2: *The centre of mass (C.M.) of a body practically coincides with its centre of gravity (C.G.). Sometimes the words 'centroid' or 'centre of inertia'* are used in place of the centre of gravity.

1.2 Determination of the C.G. by Integration

If a number of particles of masses $m_1, m_2, m_3, \ldots$ be placed at the points (x_1, y_1), $(x_2, y_2), (x_3, y_3), \ldots$ referred to two rectangular axes, then the co-ordinates $(\bar{x}, \bar{y})$ of the centre of gravity (C.G.) of the body consisting of those particles are given by

$$\bar{x} = \frac{\Sigma\, m_1\, x_1}{\Sigma\, m_1} \quad \text{and} \quad \bar{y} = \frac{\Sigma\, m_1 y_1}{\Sigma\, m_1}$$

These results are a simple consequence of a theorem on the moments of a system of parallel forces and their resultant.

In the case of continuous distribution of matter, the summations can be replaced by definite integrals. Then the C.G. of the body, is given by

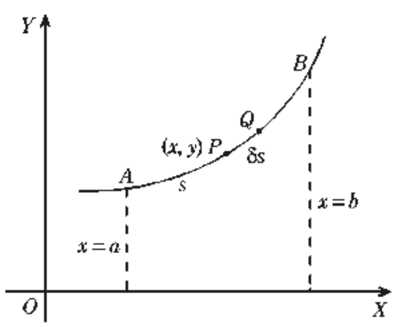
$$\bar{x} = \frac{\int x\, dm}{\int dm} \quad \text{and} \quad \bar{y} = \frac{\int y\, dm}{\int dm},$$

where (x, y) is the C.G. of an elementary mass dm of the given matter.

Important Remark. If a body is symmetrical about a line, the C.G. of the body always lies on the line of symmetry.

1.3 Centre of Gravity of an Arc

Let AB be an arc of the curve $y = f(x)$ extending from $x = a$ to $x = b$. Take $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ any two neighbouring points on the arc. Let A be a fixed point on the curve, arc $AP = s$ and arc $PQ = \delta s$. Let ρ be the density of the arc of the curve at the point P; then ρ may be considered constant from P to Q as arc $PQ = \delta s$ is very small. Hence the mass δm of the elementary arc $PQ = \rho\, \delta s$. Also the centre of gravity of this elementary arc can be approximately taken as the point $P(x, y)$ because the point Q is very close to P and ultimately we have to proceed to the limits as $Q \to P$.



Thus the co-ordinates $(\bar{x}, \bar{y})$ of the centre of gravity of the arc AB are given by the formulae

$$\bar{x} = \frac{\int x\, dm}{\int dm} = \frac{\int x \cdot \rho\, ds}{\int \rho\, ds}, \quad \text{and} \quad \bar{y} = \frac{\int y\, dm}{\int dm} = \frac{\int y \cdot \rho\, ds}{\int \rho\, ds},$$

where in all the above integrals the limits of integration are to be taken from one end of the arc AB to the other end. If the density ρ is constant, the above formulae take the form

$$\bar{x} = \frac{\int x\, ds}{\int ds}, \bar{y} = \frac{\int y\, ds}{\int ds}.$$

The value of the integral in the denominator, *i.e.*, the value of $\int ds$ gives us the length of the arc under consideration.

To perform integration ds will be changed from the following formulae of differential calculus :

$$ds = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\, dx \quad \text{or} \quad ds = \sqrt{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}}\, dy,$$

if the equation of the curve is in Cartesian co-ordinates;

$$ds = \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}\, d\theta \quad \text{or} \quad ds = \sqrt{\left\{r^2 \left(\frac{d\theta}{dr}\right)^2 + 1\right\}}\, dr,$$

if the equation of the curve is in polar co-ordinates ;

or
$$ds = \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}}\, dt,$$

if the equations of the curve are in the parametric form $x = f(t)$, $y = \phi(t)$.

Note: **If the arc under consideration is symmetrical about a straight line, then the centre of gravity will be on the line of symmetry.**

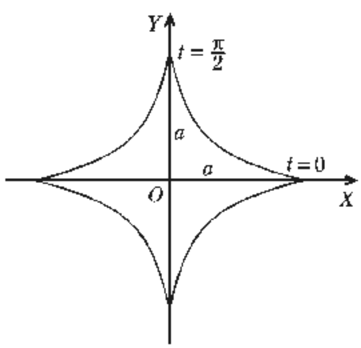
Illustrative Examples

Example 1: *Find the centre of gravity of the arc of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ lying in the first quadrant.* **(Rohilkhand 2006, 10, 11; Garhwal 02)**

Solution: The equation of the curve is

$$x^{2/3} + y^{2/3} = a^{2/3}. \qquad \ldots(1)$$

It meets the x-axis at the points $(\pm a, 0)$ and the y-axis at the points $(0, \pm a)$. The arc lying in the first quadrant is symmetrical about the line $y = x$ and so its C.G. lies on this line. Therefore $\bar{x} = \bar{y}$ where $(\bar{x}, \bar{y})$ are the co-ordinates of the required C.G. of the arc lying in the first quadrant.



Now differentiating (1) w.r.t. x, we get

$$\frac{2}{3} x^{-1/3} + \frac{2}{3} y^{-1/3} \frac{dy}{dx} = 0$$

or
$$\frac{dy}{dx} = -\frac{y^{1/3}}{x^{1/3}}.$$

$\therefore \qquad \dfrac{ds}{dx} = \sqrt{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}}$

$$= \sqrt{\left(1 + \frac{y^{2/3}}{x^{2/3}}\right)} = \sqrt{\left(\frac{x^{2/3} + y^{2/3}}{x^{2/3}}\right)}$$

$$= \sqrt{\left(\frac{a^{2/3}}{x^{2/3}}\right)}, \text{ from (1).}$$

$\therefore \qquad ds = (a/x)^{1/3}\ dx.$

Hence $\qquad \bar{x} = \dfrac{\int x\, ds}{\int ds} = \dfrac{\int_0^a x \cdot (a/x)^{1/3}\, dx}{\int_0^a (a/x)^{1/3}\, dx} = \dfrac{\int_0^a x^{2/3} dx}{\int_0^a x^{-1/3} dx}$

[Note that we cannot cancel any term containing x in the numerator and the denominator because x is a variable]

$$= \frac{\left[\frac{3}{5} x^{5/3}\right]_0^a}{\left[\frac{3}{2} x^{2/3}\right]_0^a} = \frac{2\,a}{5} = \bar{y}, \quad \text{(by symmetry).}$$

$\therefore \qquad \bar{x} = \bar{y} = \dfrac{2\,a}{5} \cdot$

Aliter. The given curve, in parametric form, can be written as

$$x = a \cos^3 t,\ \ y = a \sin^3 t. \qquad \ldots(2)$$

Differentiating (2) w.r.t. t, we get

$$\frac{dx}{dt} = -\,3a \cos^2 t \sin t \ \text{ and } \ \frac{dy}{dt} = 3a \sin^2 t \cos t.$$

$\therefore \qquad \dfrac{ds}{dt} = \sqrt{\left\{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2\right\}}$

$$= \sqrt{\{(-\,3a \cos^2 t \sin t)^2 + (3a \sin^2 t \cos t)^2\}}$$

$$= 3a \cos t \sin t \sqrt{(\cos^2 t + \sin^2 t)} = 3a \cos t \sin t$$

or $\qquad ds = 3a \sin t \cos t\ dt. \qquad \ldots(3)$

Also for the given arc lying in the first quadrant t varies from 0 to $\dfrac{\pi}{2} \cdot$

Hence $\qquad \bar{x} = \dfrac{\int x\, ds}{\int ds} = \dfrac{\int_0^{\pi/2} a \cos^3 t \cdot 3a \cos t \sin t\, dt}{\int_0^{\pi/2} 3a \cos t \sin t\, dt} \qquad$ from (2) and (3)

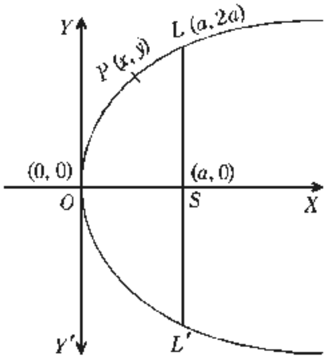
$$= \frac{a \int_0^{\pi/2} \cos^4 t \sin t\, dt}{\int_0^{\pi/2} \cos t \sin t\, dt} = \frac{a\left[-\frac{\cos^5 t}{5}\right]_0^{\pi/2}}{\left[-\frac{\cos^2 t}{2}\right]_0^{\pi/2}} = \frac{2a}{5} \cdot$$

$\therefore$ By symmetry $\bar{y} = \bar{x} = \dfrac{2a}{5}\cdot$

Example 2: *Find the C.G. of the arc of the parabola $y^2 = 4\,ax$ extending from the vertex to an extremity of the latus rectum.* **(Kanpur 2010)**

Solution: The given curve is

$$y^2 = 4\,ax \qquad \ldots(1)$$



The origin $O\,(0,0)$ is the vertex and the point $L\,(a, 2\,a)$ is an extremity of the latus rectum of this parabola. We have to find the C.G. $(\bar{x}, \bar{y})$ of the arc OL.

Differentiating (1) w.r.t. 'x', we get

$$2\,y\left(\frac{dy}{dx}\right) = 4a,$$

so that
$$\frac{dy}{dx} = \frac{2a}{y}$$

and
$$\frac{dx}{dy} = \frac{y}{2a}\cdot$$

Now
$$\frac{ds}{dy} = \sqrt{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}} = \sqrt{\left\{1 + \frac{y^2}{4a^2}\right\}} = \frac{1}{2a}\sqrt{(y^2 + 4a^2)} \qquad \ldots(2)$$

and
$$\frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} = \sqrt{\left(1 + \frac{4a^2}{y^2}\right)}$$

$$= \sqrt{\left(\frac{y^2 + 4a^2}{y^2}\right)} = \sqrt{\left(\frac{4ax + 4a^2}{4ax}\right)}$$

$$= \frac{\sqrt{(x + a)}}{\sqrt{x}}\cdot \qquad \ldots(3)$$

We have $\bar{x} = \dfrac{\int x\,ds}{\int ds}, \bar{y} = \dfrac{\int y\,ds}{ds}$, the limits of integration extending from O to L in each of the integrals.

Now the Nr. of $\bar{x} = \int x\,ds$, between the suitable limits

$$= \int_0^a x\,\frac{ds}{dx}\,dx \qquad \textbf{[Note]}$$

$$= \int_0^a x\,\frac{\sqrt{(x + a)}}{\sqrt{x}}\,dx, \text{ from (3)}$$

$$= \int_0^a \sqrt{x}\,\sqrt{(x + a)}\,dx = \int_0^a \sqrt{(x^2 + ax)}\,dx = \int_0^a \sqrt{\left\{\left(x + \frac{a}{2}\right)^2 - \frac{a^2}{4}\right\}}\,dx$$

$$= \left[\frac{1}{2}\left(x + \frac{a}{2}\right)\sqrt{(x^2 + ax)} - \frac{a^2}{8} \log \left\{\left(x + \frac{a}{2}\right) + (x^2 + ax)\right\}\right]_0^a$$

$$= \left[\frac{1}{2} \cdot (3/2)\, a \,.\, a\sqrt{2} - (1/8)\, a^2 \log \{(3/2)\, a + a\sqrt{2}\} - 0 + (1/8)\, a^2 \log\left(\frac{1}{2}a\right)\right]$$

$$= \left[\frac{3}{4} a^2 \sqrt{2} - \frac{1}{8} a^2 \log \frac{(3/2)\, a + a\sqrt{2}}{\frac{1}{2}a}\right] = \frac{3}{4} a^2 \sqrt{2} - \frac{1}{8} a^2 \log (3 + 2\sqrt{2})$$

$$= (3/4)\, a^2 \sqrt{2} - (1/8)\, a^2 \log (1 + \sqrt{2})^2$$

$$= \left(\frac{3}{4}\right) a^2 \sqrt{2} - \left(\frac{1}{8}\right) \cdot 2\, a^2 \log (1 + \sqrt{2})$$

$$= \frac{1}{4} a^2 [3\sqrt{2} - \log (1 + \sqrt{2})].$$

Also the Dr. of $\bar{x} = \int ds$, between the suitable limits

$$= \int_0^{2a} \frac{ds}{dy}\, dy \qquad \textbf{[Note]}$$

$$= \int_0^{2a} \frac{1}{2a} \sqrt{(y^2 + 4a^2)}\, dy, \text{ from (2)}$$

$$= \frac{1}{2a}\left[\frac{1}{2} y \sqrt{(y^2 + 4a^2)} + \frac{1}{2} \cdot 4a^2 \log \{y + \sqrt{(y^2 + 4a^2)}\}\right]_0^{2a}$$

$$= \frac{1}{2a}\left[a^2 \,.\, 2\sqrt{2} + 2a^2 \log \frac{2a + 2\sqrt{2}\, a}{2a}\right] = a\,[\sqrt{2} + \log (1 + \sqrt{2})]$$

$$= \text{the Dr. of } \bar{y}.$$

Further the Nr. of $\bar{y} = \int y\, ds$, between the suitable limits

$$= \int_0^{2a} y \frac{ds}{dy}\, dy \qquad \textbf{[Note]}$$

$$= \int_0^{2a} y \cdot \frac{1}{2a} \sqrt{(4a^2 + y^2)}\, dy = \frac{1}{2a} \cdot \frac{1}{2} \int_0^{2a} (4a^2 + y^2)^{1/2} (2y)\, dy$$

$$= \frac{1}{4a}\left[(4a^2 + y^2)^{3/2} \cdot \frac{2}{3}\right]_0^{2a} = \frac{1}{6a} [(8a^2)^{3/2} - (4a^2)^{3/2}]$$

$$= \frac{1}{6a} \cdot a^3 \,.\, 4^{3/2} [2^{3/2} - 1] = \frac{4}{3} a^2 (2\sqrt{2} - 1).$$

$$\therefore \qquad \bar{x} = \frac{\text{Nr. of } \bar{x}}{\text{Dr. of } \bar{x}} = \frac{\frac{1}{4} a^2 [3\sqrt{2} - \log (1 + \sqrt{2})]}{a\,[\sqrt{2} + \log (1 + \sqrt{2})]} = \frac{a}{4}\left[\frac{3\sqrt{2} - \log (1 + \sqrt{2})}{\sqrt{2} + \log (1 + \sqrt{2})}\right],$$

$$\text{and} \qquad \bar{y} = \frac{\text{Nr. of } \bar{y}}{\text{Dr. of } \bar{y}} = \frac{(4/3)\, a^2 (2\sqrt{2} - 1)}{a\,[\sqrt{2} + \log (1 + \sqrt{2})]} = \frac{4a\,(2\sqrt{2} - 1)}{3\,[\sqrt{2} + \log (1 + \sqrt{2})]}.$$

Remark 1: If $(\bar{x}, \bar{y})$ be the C.G. of the arc LOL', then $\bar{y} = 0$, by symmetry. Also $\bar{x}$ = the x-co-ordinate of the C.G. of the upper half OL, which we have just found.

Remark 2: If we have to find both $\bar{x}$ and $\bar{y}$ by integration, we should evaluate their numerators and denominators separately because the Dr. of $\bar{x}$ is obviously the same as the Dr. of $\bar{y}$. Thus we avoid the repetition of labour while evaluating the denominator.

Example 3: *Find the position of the C.G. of the arc of the cardioid $r = a\,(1 + \cos\theta)$ lying above the initial line.* **(Agra 2007, 08; Garhwal 01)**

Solution: The given curve is

$$r = a\,(1 + \cos\theta). \qquad \ldots(1)$$

Differentiating (1) w.r.t. θ, we get

$$\frac{dr}{d\theta} = -\,a\sin\theta.$$

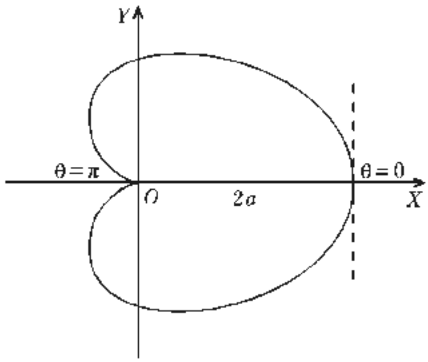
$$\therefore \quad \frac{ds}{d\theta} = \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}$$

$$= \sqrt{\{a^2\,(1 + \cos\theta)^2 + (-a\sin\theta)^2\}}$$

$$= a\sqrt{\{(1 + \cos\theta)^2 + \sin^2\theta\}}$$

$$= a\sqrt{\{2\,(1 + \cos\theta)\}}$$

$$= 2\,a\cos\frac{1}{2}\theta. \qquad \ldots(2)$$



The given curve is symmetrical about the initial line and for the arc of the cardioid lying above the initial line θ varies from 0 to π.

Let $(\bar{x}, \bar{y})$ be the co-ordinates of the C.G. of the arc lying above the initial line. Then

$$\bar{x} = \frac{\int_{\theta=0}^{\pi} x\,ds}{\int_{\theta=0}^{\pi} ds} = \frac{\int_0^{\pi} x\,\frac{ds}{d\theta}\,d\theta}{\int_0^{\pi}\frac{ds}{d\theta}\,d\theta} = \frac{\int_0^{\pi}(r\cos\theta)\cdot 2a\cos\frac{1}{2}\theta\,d\theta}{\int_0^{\pi} 2a\cos\frac{1}{2}\theta\,d\theta} \qquad [\because\ x = r\cos\theta]$$

$$= \frac{\int_0^{\pi} a\,(1 + \cos\theta)\cdot\cos\theta\cos\frac{1}{2}\theta\,d\theta}{\int_0^{\pi}\cos\frac{1}{2}\theta\,d\theta}, \text{ substituting for } r \text{ from (1)}$$

and
$$\bar{y} = \frac{\int_{\theta=0}^{\pi} y\,ds}{\int_{\theta=0}^{\pi} ds} = \frac{\int_0^{\pi} y\,\frac{ds}{d\theta}\,d\theta}{\int_0^{\pi}\frac{ds}{d\theta}\,d\theta} = \frac{\int_0^{\pi} r\sin\theta\cdot 2a\cos\frac{1}{2}\theta\,d\theta}{\int_0^{\pi} 2a\cos\frac{1}{2}\theta\,d\theta} \qquad [\because\ y = r\sin\theta]$$

$$= \frac{\int_0^\pi a\,(1+\cos\theta)\sin\theta\cos\frac{1}{2}\theta\,d\theta}{\int_0^\pi \cos\frac{1}{2}\theta\,d\theta} \qquad [\because\ y = a\,(1+\cos\theta)]$$

Now Nr. of $\bar{x}$ $= \int_0^\pi a\,(1+\cos\theta)\cos\theta\cdot\cos\frac{1}{2}\theta\,d\theta$

$$= a\int_0^\pi 2\cos^2\frac{1}{2}\theta\cdot\left(2\cos^2\frac{1}{2}\theta-1\right)\cdot\cos\frac{1}{2}\theta\,d\theta$$

$$= 2a\int_0^\pi \cos^3\frac{1}{2}\theta\left(2\cos^2\frac{1}{2}\theta-1\right)d\theta$$

$$= 2a\int_0^{\pi/2}\cos^3\phi\,(2\cos^2\phi-1)\cdot 2\,d\phi\text{, putting } \frac{1}{2}\theta=\phi$$

$$= 4a\left[2\int_0^{\pi/2}\cos^5\phi\,d\phi-\int_0^{\pi/2}\cos^3\phi\,d\phi\right]$$

$$= 4a\left[2\cdot\frac{4.2}{5.3.1}-\frac{2}{3.1}\right]=8a\left[\frac{8}{15}-\frac{1}{3}\right]=\frac{8}{5}a.$$

Also Nr. of $\bar{y}$ $= \int_0^\pi a\,(1+\cos\theta)\sin\theta\cos\frac{1}{2}\theta\,d\theta$

$$= a\int_0^\pi 2\cos^2\frac{1}{2}\theta\cdot 2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta\cdot\cos\frac{1}{2}\theta\,d\theta$$

$$= 4a\int_0^\pi \cos^4\frac{1}{2}\theta\sin\frac{1}{2}\theta\,d\theta = -4a\cdot 2\int_0^\pi\cos^4\frac{1}{2}\theta\left(-\frac{1}{2}\sin\frac{1}{2}\theta\right)d\theta$$

$$= -8a\left[\frac{\cos^5\frac{1}{2}\theta}{5}\right]_0^\pi = -\frac{8a}{5}[0-1]=\frac{8a}{5}.$$

Further Dr. of $\bar{x}$ = Dr. of $\bar{y}$

$$= \int_0^\pi \cos\frac{1}{2}\theta\,d\theta = \left[2\sin\frac{1}{2}\theta\right]_0^\pi = 2.$$

$\therefore$ $\bar{x} = \dfrac{(8/5)\,a}{2} = \dfrac{4a}{5}$

and $\bar{y} = \dfrac{(8/5)\,a}{2} = \dfrac{4a}{5}.$

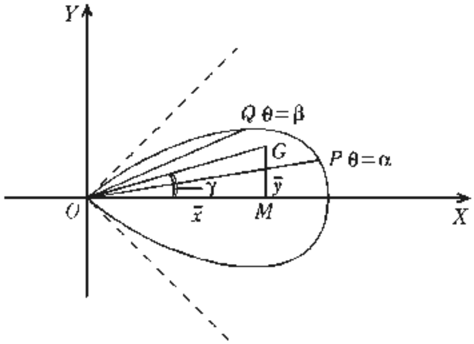
Example 4: *O is the pole of the lemniscate of Bernoulli $r^2 = a^2\cos 2\theta$ and G is the centre of gravity of any arc PQ of the curve ; prove that OG bisects the angle POQ.* **(Kumaun 2002)**

Solution: The given curve is $r^2 = a^2\cos 2\theta.$ ...(1)

Differentiating (1) w.r.t. 'θ', we get

$$2r\left(\frac{dr}{d\theta}\right) = -2a^2\sin 2\theta$$

or $$\frac{dr}{d\theta} = -\frac{(a^2 \sin 2\theta)}{r}.$$



$\therefore$ $$\frac{ds}{d\theta} = \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}$$

$$= \sqrt{\left(r^2 + \frac{a^4 \sin^2 2\theta}{r^2}\right)}$$

$$= \sqrt{\left(\frac{r^4 + a^4 \sin^2 2\theta}{r^2}\right)}$$

$$= \left(\frac{1}{r}\right).\sqrt{(a^4 \cos^2 2\theta + a^4 \sin^2 2\theta)} = \frac{a^2}{r}. \qquad ...(2)$$

Let $(\bar{x}, \bar{y})$ be the cartesian co-ordinates of the centre of gravity (or the centroid) G of an arc PQ of the curve (1). If the vectorial angles of P and Q be α and β respectively, we have $\angle XOP = \alpha$ and $\angle XOQ = \beta$. Let OG make an angle γ with the initial line OX. Then

$$\tan\gamma = \frac{GM}{OM} = \frac{\bar{y}}{\bar{x}} = \frac{\int y\,ds / \int ds}{\int x\,ds / \int ds} = \frac{\int y\,ds}{\int x\,ds}$$

$$= \frac{\int_\alpha^\beta y \frac{ds}{d\theta} d\theta}{\int_\alpha^\beta x \frac{ds}{d\theta} d\theta} = \frac{\int_\alpha^\beta r\sin\theta \cdot \frac{a^2}{r} d\theta}{\int_\alpha^\beta r\cos\theta \cdot \frac{a^2}{r} d\theta}$$

$$\left[\because\ x = r\cos\theta,\ y = r\sin\theta \text{ and } \frac{ds}{d\theta} = \frac{a^2}{r}\right]$$

$$= \frac{\int_\alpha^\beta \sin\theta\, d\theta}{\int_\alpha^\beta \cos\theta\, d\theta} = \frac{[-\cos\theta]_\alpha^\beta}{[\sin\theta]_\alpha^\beta} = \frac{\cos\alpha - \cos\beta}{\sin\beta - \sin\alpha}$$

$$= \frac{2\sin\frac{\alpha+\beta}{2}\sin\frac{\beta-\alpha}{2}}{2\cos\frac{\alpha+\beta}{2}\sin\frac{\beta-\alpha}{2}} = \tan\frac{1}{2}(\alpha+\beta).$$

$\therefore$ $$\gamma = \frac{1}{2}(\alpha+\beta) \text{ i.e., } \angle GOX = \frac{1}{2}(\alpha+\beta).$$

Now $$\angle GOP = \angle GOX - \angle POX = \frac{1}{2}(\alpha+\beta) - \alpha$$

$$= \frac{1}{2}(\beta - \alpha) = \frac{1}{2}\angle QOP.$$

Therefore OG bisects the angle POQ.

Comprehensive Exercise 1

Find the centre of gravity of :

1. The arc of the catenary $y = c \cosh (x / c)$ from the vertex to the point (x, y). **(Kanpur 2011; Agra 01)**
2. The arc of a circle of radius a, which subtends an angle 2α at the centre.
3. A uniform semi-circular wire of radius a.
4. A uniform circular wire of radius a in the form of a quadrant of a circle.
5. The arc of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ which lies in the positive quadrant. **(Agra 2001)**
6. The whole arc of the cardioid $r = a(1 + \cos\theta)$.

Answers 1

1. $\bar{x} = x - \dfrac{c(y - c)}{s}$ and $\bar{y} = \dfrac{1}{2}\left[\dfrac{cx + sy}{s}\right]$
2. C.G. lies on the symmetrical radius at a distance $(a \sin\alpha)/\alpha$ from the centre.
3. C.G. lies on its central radius at a distance $2a/\pi$ from the centre.
4. $\bar{x} = (2\sqrt{2}\,a)/\pi$, $\bar{y} = 0$
5. $\bar{x} = a[\pi - (4/3)]$ and $\bar{y} = 2a/3$
6. $\bar{x} = \dfrac{4a}{5}$, $\bar{y} = 0$

1.4 Centre of Gravity of a Plane Area

Let $(\bar{x}, \bar{y})$ be the co-ordinates of the C.G. of an area A.

Then $$\bar{x} = \frac{\int x\,dm}{\int dm}, \quad \bar{y} = \frac{\int y\,dm}{\int dm}, \qquad \text{[Refer 1.2]}$$

where δm is the mass of an elementary area δA and (x, y) are the co-ordinates of the C.G. of this elementary area.

Now if ρ is the density (*i.e.*, the mass per unit area) of the elementary area δA, then $\delta m = \rho\,\delta A$.

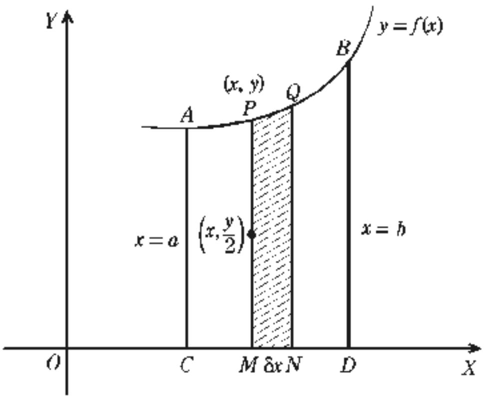
$$\therefore \quad \bar{x} = \frac{\int x\rho\,dA}{\int \rho\,dA} \text{ and } \bar{y} = \frac{\int y\rho\,dA}{\int \rho\,dA}.$$

In case the density ρ **be uniform** (*i.e.*, constant), we have

$$\bar{x} = \frac{\int x\, dA}{\int dA} \quad \text{and} \quad \bar{y} = \frac{\int y\, dA}{\int dA}. \qquad ...(1)$$

The elementary area δA is chosen according to the nature of the problem. The following cases arise :

Case I. *To find the C.G. of a plane area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$ and $x = b$.*



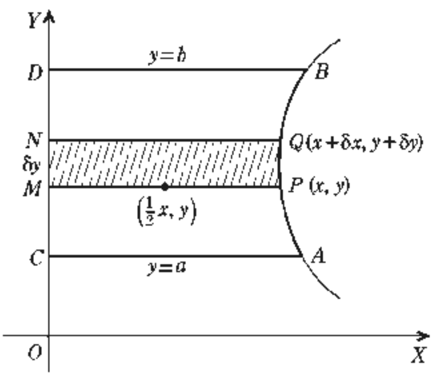
Suppose we have to find the C.G. of the area $ACDB$. Divide this area into elementary strips by drawing lines parallel to the y-axis. Take an elementary strip $PMNQ$ where P and Q are the points (x, y) and $(x + \delta x, y + \delta y)$ respectively. The area δA of this elementary strip is equal to $y\, \delta x$ and the C.G. of this strip can approximately be taken as the middle point $\left(x, \frac{1}{2} y\right)$ of PM because the point Q is in the neighbourhood of P and ultimately we have to take limits when $Q \to P$.

Hence from (1), the co-ordinates $(\bar{x}, \bar{y})$ of the required centre of gravity of the area $ACDB$ are given by

$$\bar{x} = \frac{\int_a^b xy\, dx}{\int_a^b y\, dx}, \quad \bar{y} = \frac{\int_a^b \frac{y}{2} \cdot y\, dx}{\int_a^b y\, dx}.$$

Note: If the given area is symmetrical about the x-axis, then the C.G. of the elementary strip will be the point $(x, 0)$ and therefore we shall have $\bar{y} = 0$.

Case II. *To find the C.G. of a plane area bounded by the curve $x = f(y)$, the y-axis and the abscissae $y = a$ and $y = b$.*



In this case the elementary area δA is the elementary strip $PMNQ$ parallel to the x-axis with area $x\, \delta y$ and the centre of gravity at the point $\left(\frac{1}{2} x, y\right)$.

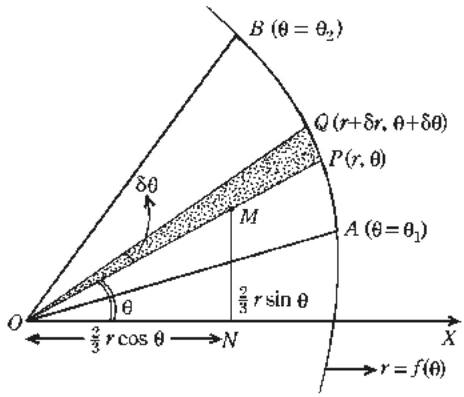
Hence from (1), the co-ordinates of the required C.G. are

$$\bar{x} = \frac{\int_a^b \frac{x}{2} \cdot x\, dy}{\int_a^b x\, dy}, \quad \bar{y} = \frac{\int_a^b y \cdot x\, dy}{\int_a^b x\, dy}.$$

Note: If the given area is symmetrical about the y-axis, then the C.G. of the elementary strip will be $(0, y)$ and therefore we shall have $\bar{x} = 0$.

Case III. *To find the C.G. of a sectorial area bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \theta_1$ and $\theta = \theta_2$.*

Suppose we have to find the C.G. of the sectorial area OAB where OA and OB are the radii vectors $\theta = \theta_1$ and $\theta = \theta_2$ respectively. Divide this area into elementary strips by drawing the radii vectors of the various points of the arc AB of the curve $r = f(\theta)$. Take an elementary strip OPQ where $P(r, \theta)$ and $Q(r + \delta r, \theta + \delta\theta)$ are any two neighbouring points. The area δA of this strip is equal to $\frac{1}{2} r^2 \delta\theta$. Also this strip is triangular in shape and the point Q is in the neighbourhood of P. Therefore the C.G. of this strip OPQ can approximately be taken as the point M on OP such that $OM = \frac{2}{3} OP$. The cartesian co-ordinates of M are



$$\left(\frac{2}{3} r \cos\theta, \frac{2}{3} r \sin\theta\right).$$

Hence from (1), the cartesian co-ordinates $(\bar{x}, \bar{y})$ of the required C.G. are given by

$$\bar{x} = \frac{\int_{\theta_1}^{\theta_2} \frac{2}{3} r \cos\theta \cdot \frac{1}{2} r^2 d\theta}{\int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 d\theta} = \frac{\frac{2}{3} \int_{\theta_1}^{\theta_2} r^3 \cos\theta \, d\theta}{\int_{\theta_1}^{\theta_2} r^2 \, d\theta}$$

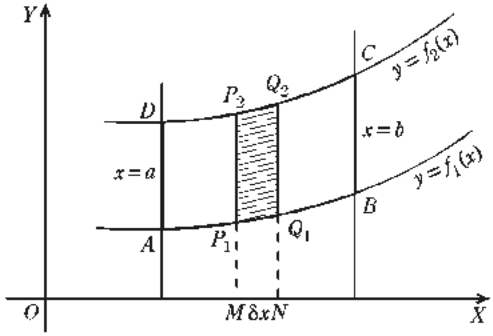
and
$$\bar{y} = \frac{\int_{\theta_1}^{\theta_2} \frac{2}{3} r \sin\theta \cdot \frac{1}{2} r^2 \, d\theta}{\int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 \, d\theta} = \frac{\frac{2}{3} \int_{\theta_1}^{\theta_2} r^3 \sin\theta \, d\theta}{\int_{\theta_1}^{\theta_2} r^2 \, d\theta}.$$

Note: If the given area is symmetrical about the initial line, then the C.G. will lie on the initial line and therefore we shall have $\bar{y} = 0$.

Case IV. *To find the C.G. of an area enclosed between two curves $y = f_1(x)$, $y = f_2(x)$ and the ordinates $x = a$ and $x = b$.*

In this case the elementary area δA is the elementary strip $P_1 Q_1 Q_2 P_2$ drawn parallel to the y-axis and enclosed between the two curves $y = f_1(x)$ and $y = f_2(x)$. Let P_1 and P_2 be the points (x, y_1) and (x, y_2) respectively.

Now δA = area of the elementary strip $P_1Q_1Q_2P_2 = (y_2 - y_1) . \delta x$, where $MN = \delta x$.



The C.G. of the strip $P_1\ Q_1\ Q_2\ P_2$ can approximately be taken as the middle point of $P_1\ P_2$. Thus the co-ordinates of the C.G. of this strip are $\left(x,\ \dfrac{y_1 + y_2}{2}\right)$.

Hence from (1), the co-ordinates of the required C.G. are given by

$$\bar{x} = \frac{\int x\, dA}{\int dA} = \frac{\int_a^b x \cdot (y_2 - y_1)\, dx}{\int_a^b (y_2 - y_1)\, dx}$$

and
$$\bar{y} = \frac{\int y\, dA}{\int dA} = \frac{\int_a^b \frac{1}{2}(y_2 + y_1) \cdot (y_2 - y_1)\, dx}{\int_a^b (y_2 - y_1)\, dx}$$

While evaluating the above integrals we shall have to replace y_1 and y_2 in terms of x from the equations $y = f_1(x)$ and $y = f_2(x)$ respectively.

Important Note: In case the area is enclosed between two curves, the limits of integration will be the values of x for the common points of intersection of the given curves.

Case V. Use of Double Integration. Sometimes it is convenient to consider an elementary area $\delta x\, \delta y$ or $r\, \delta\theta\, \delta r$. Then the C.G. will be given by

$$\bar{x} = \frac{\iint \rho x\, dx\, dy}{\iint \rho\, dx\, dy},\ \bar{y} = \frac{\iint \rho y\, dx\, dy}{\iint \rho\, dx\, dy}$$

or
$$\bar{x} = \frac{\iint r\cos\theta \cdot \rho r\, d\theta\, dr}{\iint \rho\, r\, d\theta\, dr},\quad \bar{y} = \frac{\iint r\sin\theta \cdot \rho r\, d\theta\, dr}{\iint \rho r\, d\theta\, dr}$$

under proper limits of integration. If ρ is constant, it can be cancelled from the numerator and denominator while applying the above formulae.

Working Rule: Take a small element of the area whose centre of gravity is required. Find the C.G. of this elementary area. Then the C.G. of the whole area is given by

$$\bar{x} = \frac{\int (\text{abscissa of the C.G. of the elementary area}) \cdot (\text{elementary area})}{\int \text{elementary area}}$$

$$\bar{y} = \frac{\int (\text{ordinate of the C.G. of the elementary area}) \cdot (\text{elementary area})}{\int \text{elementary area}}$$

under proper limits of integration ; ρ, the density per unit area, being constant.

Remember: If m and n are +ive integers, we have

$$\int_0^{\pi} \sin^n x\,dx = 2\int_0^{\pi/2} \sin^n x\,dx;$$

$$\int_0^{\pi} \cos^n x\,dx = 0 \quad \text{or} \quad 2\int_0^{\pi/2} \cos^n x\,dx,$$

according as n is odd or even;

$$\int_0^{\pi} \sin^m x \cos^n x\,dx = 0 \quad \text{or} \quad 2\int_0^{\pi/2} \sin^m x \cos^n x\,dx,$$

according as n is odd or even;

and $$\int_0^{\pi/2} \sin^m x \cos^n x\,dx = \frac{(m-1)(m-3)\ldots(n-1)(n-3)\ldots}{(m+n)(m+n-2)\ldots}k,$$

where k is $\pi/2$ if m and n are both even otherwise $k=1$.

Also remember that

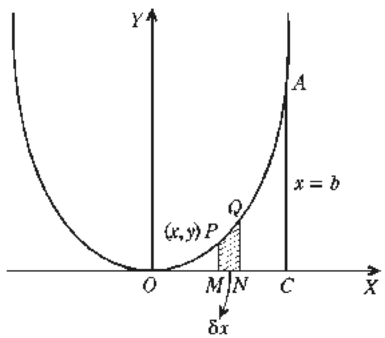
$$\int_0^{\pi/2} \sin^m x \cos^n x\,dx = \frac{\Gamma\{(m+1)/2\}\,\Gamma\{(n+1)/2\}}{2\,\Gamma\{(m+n+2)/2\}},$$

where $m \geq 0$ and $n \geq 0$.

Illustrative Examples

Example 5: *Find the C.G. of the region bounded by the parabola $x^2 = 4ay$, x-axis and the ordinate $x = b$.*

Solution: The given parabola is $x^2 = 4ay$. It is symmetrical about the y-axis. We are required to find the C.G. of the area $OPAC$ for which x varies from 0 to b.



Take $P(x, y)$ and $Q(x+\delta x, y+\delta y)$, the neighbouring points on the arc OA of the parabola $x^2 = 4ay$. Draw PM and QN perpendiculars to the axis of x. We have $PM = y$, $MN = \delta x$.

Then δA = elementary area $PMNQ = y\,\delta x$ and the C.G. of this elementary area can approximately be taken as the point $\left(x, \frac{1}{2}y\right)$ on MP.

$\therefore$ If $(\bar{x}, \bar{y})$ be the required C.G., then

$$\bar{x} = \frac{\int_0^b x \cdot y\,dx}{\int_0^b y\,dx} = \frac{\int_0^b x \cdot \frac{x^2}{4a}\,dx}{\int_0^b \frac{x^2}{4a}\,dx} \qquad \left[\because y = \frac{x^2}{4a}\right]$$

$$= \frac{\int_0^b x^3\, dx}{\int_0^b x^2\, dx} = \frac{\left[\frac{x^4}{4}\right]_0^b}{\left[\frac{x^3}{3}\right]_0^b} = \frac{\frac{b^4}{4}}{\frac{b^3}{3}} = \frac{3b}{4}.$$

And $$\bar{y} = \frac{\int_0^b \frac{1}{2} y \cdot y\, dx}{\int_0^b y\, dx} = \frac{\int_0^b \frac{1}{2}\left(\frac{x^2}{4a}\right)^2 dx}{\int_0^b \left(\frac{x^2}{4a}\right) dx}, \qquad \left[\because\ y = \frac{x^2}{4a}\right]$$

$$= \frac{\frac{1}{8a}\int_0^b x^4\, dx}{\int_0^b x^2\, dx} = \frac{1}{8a}\frac{\left[\frac{x^5}{5}\right]_0^b}{\left[\frac{x^3}{3}\right]_0^b} = \frac{3}{8 \times 5a}\frac{b^5}{b^3} = \frac{3b^2}{40\, a}.$$

$\therefore$ $$\bar{x} = \frac{3b}{4},\ \bar{y} = \frac{3b^2}{40\, a}.$$

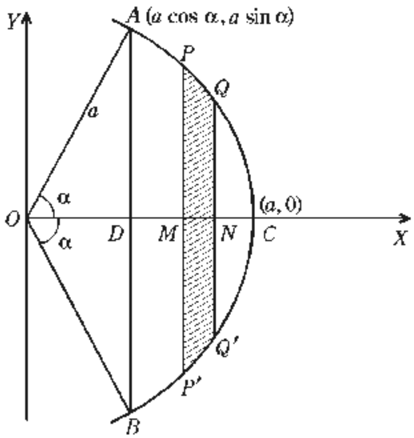
Example 6: *Find the C.G. of a segment of a circle that subtends an angle 2 α at the centre. Hence find the centre of gravity of a uniform disc which is of the shape of a quadrant of a circle.*

Solution: Referred to the centre as origin let the equation of the circle be

$$x^2 + y^2 = a^2, \qquad \ldots(1)$$

a being the radius of the circle.

Let ACB be the segment of this circle subtending an angle 2 α at the centre and let the central radius OC coincide with the x-axis so that $\angle AOC = \alpha$. The segment ACB is symmetrical about the x-axis.



At the point A, $x = a \cos \alpha$ and at the point C, $x = a$.

Take an elementary strip $PP'Q'Q$ of the given segment. Its area is $2\, y\, \delta x$ and its C.G. is on the axis of x at $(x, 0)$.

$\therefore$ If $(\bar{x}, \bar{y})$ be the required C.G. of the segment, then $\bar{y} = 0$ (by symmetry).

Also $$\bar{x} = \frac{\int_{OD}^{OC} x\, 2\, y\, dx}{\int_{OD}^{OC} 2\, y\, dx} = \frac{\int_{a \cos \alpha}^{a} xy\, dx}{\int_{a \cos \alpha}^{a} y\, dx}$$

$$= \frac{\int_{a\cos\alpha}^{a} x\sqrt{(a^2 - x^2)}\, dx}{\int_{a\cos\alpha}^{a} \sqrt{(a^2 - x^2)}\, dx}, \text{ substituting for } y \text{ from (1).}$$

Now the Nr. of $\bar{x}$

$$= \int_{a\cos\alpha}^{a} x\sqrt{(a^2 - x^2)}\, dx = -\frac{1}{2}\int_{a\cos\alpha}^{a} (a^2 - x^2)^{1/2}(-2x)\, dx$$

$$= -\frac{1}{2}\left[\frac{2}{3}(a^2 - x^2)^{3/2}\right]_{a\cos\alpha}^{a}$$

$$= -\frac{1}{3}[0 - (a^2 - a^2\cos^2\alpha)^{3/2}] = -\frac{1}{3}[0 - a^3\sin^3\alpha] = \frac{1}{3}a^3\sin^3\alpha.$$

And the Dr. of $\bar{x} = \int_{a\cos\alpha}^{a} \sqrt{(a^2 - x^2)}\, dx$

$$= \left[\frac{x\sqrt{(a^2 - x^2)}}{2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_{a\cos\alpha}^{a}$$

$$= \left[0 + \frac{1}{2}a^2 \cdot \frac{1}{2}\pi - \frac{1}{2}a^2\sin\alpha\cos\alpha - \frac{1}{2}a^2\sin^{-1}(\cos\alpha)\right]$$

$$= \frac{1}{2}a^2\left[\frac{1}{2}\pi - \sin^{-1}(\cos\alpha) - \sin\alpha\cos\alpha\right]$$

$$= \frac{1}{2}a^2[\cos^{-1}(\cos\alpha) - \sin\alpha\cos\alpha], \qquad \left[\because \frac{1}{2}\pi - \sin^{-1}t = \cos^{-1}t\right]$$

$$= \frac{1}{2}a^2[\alpha - \sin\alpha\cos\alpha].$$

$$\therefore \qquad \bar{x} = \frac{\frac{1}{3}a^3\sin^3\alpha}{\frac{1}{2}a^2(\alpha - \sin\alpha\cos\alpha)} = \frac{2}{3}\cdot\frac{a\sin^3\alpha}{\alpha - \sin\alpha\cos\alpha} \quad \text{and} \quad \bar{y} = 0. \qquad \ldots(2)$$

In particular if the segment is a quadrant of a circle, $\alpha = \frac{\pi}{4}\cdot$

Hence putting $\alpha = \frac{\pi}{4}$ in (2), we get for a uniform disc in the form of a quadrant of a circle,

$$\bar{x} = \frac{2}{3}\cdot\frac{a\sin^3(\pi/4)}{\{(\pi/4) - \sin(\pi/4)\cos(\pi/4)\}} = \frac{2}{3}\cdot\frac{a(1/\sqrt{2})^3}{\{(\pi/4) - (1/\sqrt{2})(1/\sqrt{2})\}}$$

$$= \frac{2}{3}\cdot\frac{4a}{2\sqrt{2}(\pi - 2)} = \frac{2\sqrt{2}\,.\,a}{3(\pi - 2)}, \quad \text{and} \quad \bar{y} = 0.$$

Cor. Putting $\alpha = \frac{\pi}{2}$ in (2), we get the C.G. of a uniform semi-circular disc. Thus for a uniform semi-circular disc,

$$\bar{x} = \frac{2}{3}\cdot\frac{a\sin^3(\pi/2)}{\{(\pi/2) - \sin(\pi/2)\cos(\pi/2)\}} = \frac{2}{3}\frac{a}{(\pi/2)} = \frac{4a}{3\pi},$$

and $\quad \bar{y} = 0.$

Thus the C.G. of a uniform semi-circular disc lies on the symmetrical radius at a distance $\dfrac{4a}{3\pi}$ from the centre.

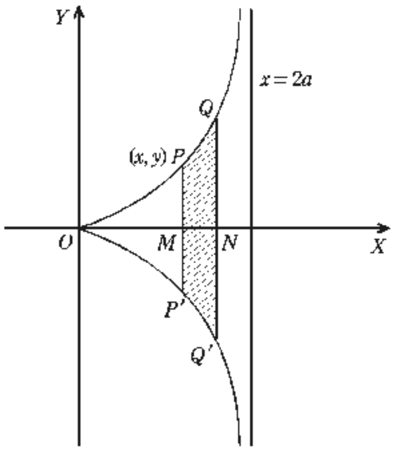
Example 7: *Find the C.G. of the area included between the curve $y^2\,(2a - x) = x^3$ and its asymptote.* **(Rohilkhand 2006; Kanpur 07)**

Solution: The given curve is $y^2\,(2a - x) = x^3$. ...(1)

It is symmetrical about the x-axis. It passes through the origin. Equating the coefficient of the highest power of y to zero, we get

$$2a - x = 0 \quad \text{or} \quad x = 2a$$

as an asymptote parallel to the y-axis.



We are required to find the C.G. of the area included between the curve and its asymptote. Consider an elementary strip $PP'Q'Q$ of this area parallel to y-axis. Its area is $2\,y\,\delta x$ and its C.G. can be taken as the point $M\,(x, 0)$.

If $(\bar{x}, \bar{y})$ be the required C.G., then by symmetry $\bar{y} = 0$.

Also
$$\bar{x} = \frac{\int_0^{2a} x \cdot y\,dx}{\int_0^{2a} y\,dx}$$

$$= \frac{\int_0^{2a} x \cdot x^{3/2}(2a - x)^{-1/2}\,dx}{\int_0^{2a} x^{3/2}(2a - x)^{-1/2}\,dx}, \text{ from (1).}$$

Now put $x = 2a\sin^2\theta$, so that $dx = 4a\sin\theta\cos\theta\,d\theta$.

Also when $x = 0, \theta = 0$ and when $x = 2a, \theta = \dfrac{\pi}{2}\cdot$

$\therefore$
$$\bar{x} = \frac{\int_0^{\pi/2} (2a\sin^2\theta)^{5/2}(2a\cos^2\theta)^{-1/2}\cdot 4a\sin\theta\cos\theta\,d\theta}{\int_0^{\pi/2} (2a\sin^2\theta)^{3/2}(2a\cos^2\theta)^{-1/2}\cdot 4a\sin\theta\cos\theta\,d\theta}$$

$$= \frac{2a\int_0^{\pi/2}\sin^6\theta\,d\theta}{\int_0^{\pi/2}\sin^4\theta\,d\theta} = \frac{2a\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{1}{2}\pi}{\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{1}{2}\pi} = \frac{5a}{3}\cdot$$

Hence $\quad \bar{x} = \dfrac{5}{3}a, \;\; \bar{y} = 0.$

Example 8: *Find the centre of gravity of a plane lamina of uniform density in the form of a quadrant of an ellipse.*

Solution: Let the equation of the ellipse be

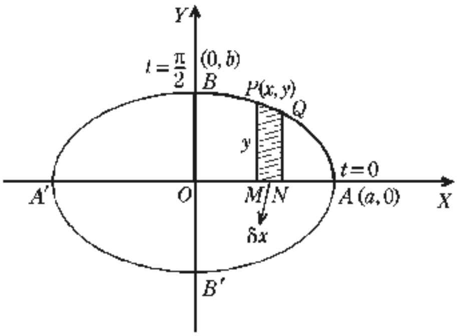
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \qquad \text{...(1)}$$

The parametric equations of (1) are

$$x=a\cos t,\ \ y=b\sin t. \qquad \text{...(2)}$$

At the point $A\,(a,0),\, t=0$

and at the point $B\,(0,b),\, t=\dfrac{\pi}{2}\cdot$



Let $(\bar{x},\bar{y})$ be the C.G. of the area in the form of the quadrant OAB of the ellipse (1). Take an elementary strip $PMNQ$ of this area parallel to the y-axis. The area of this strip is $y\,\delta x$ and its C.G. can be taken as the middle point $\left(x,\dfrac{1}{2}y\right)$ of PM. We have

$$\bar{x}=\frac{\int_{x=0}^{a}xy\,dx}{\int_{x=0}^{a}y\,dx}$$

and

$$\bar{y}=\frac{\int_{x=0}^{a}\frac{1}{2}y\cdot y\,dx}{\int_{x=0}^{a}y\,dx}=\frac{\frac{1}{2}\int_{x=0}^{a}y^2\,dx}{\int_{x=0}^{a}y\,dx}$$

The Nr. of $\bar{x}=\int_{\pi/2}^{0}xy\,\dfrac{dx}{dt}\,dt=\int_{\pi/2}^{0}a\cos t\cdot b\sin t\cdot(-a\sin t)\,dt,$ from (2)

$$=a^2b\int_0^{\pi/2}\cos t\sin^2 t\,dt=a^2b\,\frac{1}{3\cdot1}=\frac{1}{3}a^2b,$$

the Nr. of $\bar{y}=\dfrac{1}{2}\int_{\pi/2}^{0}y^2\,\dfrac{dx}{dt}\,dt=\dfrac{1}{2}\int_{\pi/2}^{0}b^2\sin^2 t\cdot(-a\sin t)\,dt$

$$=\frac{1}{2}ab^2\int_0^{\pi/2}\sin^3 t\,dt=\frac{1}{2}ab^2\,\frac{2}{3\cdot1}=\frac{1}{3}ab^2,$$

and the Dr. of $\bar{x}$ = the Dr. of $\bar{y}$

$$=\int_{\pi/2}^{0}y\,\frac{dx}{dt}\,dt=\int_{\pi/2}^{0}b\sin t\,(-a\sin t)\,dt$$

$$=ab\int_0^{\pi/2}\sin^2 t\,dt=ab\cdot\frac{1}{2}\cdot\frac{\pi}{2}=\frac{\pi}{4}ab.$$

$$\therefore \quad \bar{x} = \frac{\frac{1}{3}a^2 b}{\frac{1}{4}\pi ab} = \frac{4a}{3\pi}, \quad \bar{y} = \frac{\frac{1}{3}ab^2}{\frac{1}{4}\pi\, ab} = \frac{4b}{3\pi}.$$

Example 9: *Find the C.G. of the sector of a circle subtending an angle 2α at the centre of the circle.* **(Agra 2003, 11)**

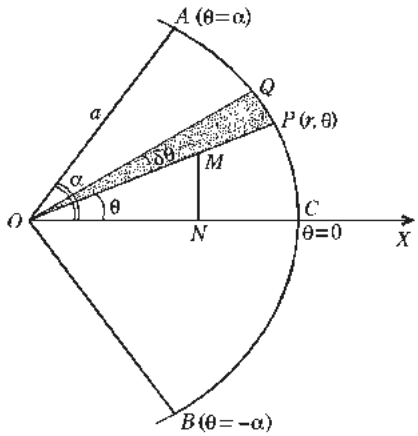
Solution: Referred to the centre as pole, the polar equation of a circle of radius a is

$$r = a. \quad \ldots(1)$$

Let the sector AOB subtend an angle 2α at the centre O of the circle and let the x-axis be along the symmetrical radius OC.

We have $\angle AOC = \alpha$. Consider an elementary strip POQ.

Its area $= \frac{1}{2}r^2\,\delta\theta$ and its C.G. can be taken as the point M on OP such that $OM = \frac{2}{3}OP = \frac{2}{3}r$.



The co-ordinates of M are

$$\left(\frac{2}{3}r\cos\theta,\ \frac{2}{3}r\sin\theta\right).$$

Therefore if $(\bar{x}, \bar{y})$ be the required C.G., then by symmetry $\bar{y} = 0$. Also the x-co-ordinate of the C.G. of the area AOB is the same as the x-co-ordinate of the C.G. of the upper half AOC of this area.

$$\therefore \quad \bar{x} = \frac{\int_0^\alpha \frac{2}{3}r\cos\theta \cdot \frac{1}{2}r^2\,d\theta}{\int_0^\alpha \frac{1}{2}r^2\,d\theta} = \frac{\frac{2}{3}\int_0^\alpha r^3\cos\theta\,d\theta}{\int_0^\alpha r^2\,d\theta}$$

$$= \frac{\frac{2}{3}\int_0^\alpha a^3\cos\theta\,d\theta}{\int_0^\alpha a^2\,d\theta}, \qquad [\because\ r = a, \text{ from } (1)]$$

$$= \frac{2}{3}a\frac{[\sin\theta]_0^\alpha}{[\theta]_0^\alpha} = \frac{2}{3}a\cdot\frac{\sin\alpha}{\alpha}.$$

$$\therefore \quad \bar{x} = \frac{2}{3}\cdot\frac{a\sin\alpha}{\alpha} \quad \text{and} \quad \bar{y} = 0.$$ **(Remember)**

Thus the C.G. of the sector of a circle of radius a lies on the central radius and its distance from the centre of the circle is $\frac{2}{3}\cdot\frac{a\sin\alpha}{\alpha}$.

Example 10: *Find the centre of gravity of the area of a loop of the curve $r = a \cos 2\theta$.*

Solution: The given curve is $r = a \cos 2\theta$. ...(1)

It is symmetrical about the initial line. Putting $r = 0$, we get

$$\cos 2\theta = 0 \;\; i.e., \;\; 2\theta = \pm \frac{1}{2}\pi \;\; i.e., \;\; \theta = \pm \frac{1}{4}\pi.$$

Therefore for one loop of the curve θ varies from $-\frac{1}{4}\pi$ to $\frac{1}{4}\pi$ and this loop is symmetrical about the initial line. Draw the figure of this loop.

If $(\bar{x}, \bar{y})$ be the co-ordinates of the required C.G. of the area of one loop, then

$$\bar{y} = 0 \text{ (by symmetry about } x\text{-axis).}$$

Now consider an elementary strip of area $\frac{1}{2} r^2 \, \delta\theta$ of the given loop of the curve. Clearly the x-co-ordinate of the C.G. of this strip can be taken as $\frac{2}{3} r \cos \theta$.

$$\therefore \quad \bar{x} = \frac{\int_{-\pi/4}^{\pi/4} \frac{2}{3} r \cos\theta \cdot \frac{1}{2} r^2 \, d\theta}{\int_{-\pi/4}^{\pi/4} \frac{1}{2} r^2 \, d\theta} = \frac{\frac{2}{3} \int_{-\pi/4}^{\pi/4} r^3 \cos\theta \, d\theta}{\int_{-\pi/4}^{\pi/4} r^2 \, d\theta} \qquad \text{...(2)}$$

Now the Nr. of $\bar{x}$

$$= \frac{2}{3} \int_{-\pi/4}^{\pi/4} r^3 \cos\theta \, d\theta = \frac{2}{3} \int_{-\pi/4}^{\pi/4} (a \cos 2\theta)^3 \cos\theta \, d\theta, \qquad \text{from (1)}$$

$$= \frac{4a^3}{3} \int_0^{\pi/4} \cos^3 2\theta \cos\theta \, d\theta = \frac{4a^3}{3} \int_0^{\pi/4} (1 - 2\sin^2\theta)^3 \cos\theta \, d\theta.$$

Put $\sqrt{2} \sin\theta = \sin\phi$, so that $\sqrt{2} \cos\theta \, d\theta = \cos\phi \, d\phi$.

Also when $\theta = 0, \phi = 0$ and when $\theta = \frac{\pi}{4}, \phi = \frac{\pi}{2}$.

Therefore the Nr. of $\bar{x}$

$$= \frac{4a^3}{3} \int_0^{\pi/2} (1 - \sin^2\phi)^3 \cdot \frac{\cos\phi}{\sqrt{2}} \, d\phi$$

$$= \frac{4a^3}{3\sqrt{2}} \int_0^{\pi/2} \cos^6\phi \cdot \cos\phi \, d\phi = \frac{4a^3}{3\sqrt{2}} \int_0^{\pi/2} \cos^7\phi \, d\phi$$

$$= \frac{4a^3}{3\sqrt{2}} \cdot \frac{6}{7} \cdot \frac{4}{5} \cdot \frac{2}{3} = \frac{64a^3}{105\sqrt{2}}.$$

Again the Dr. of $\bar{x}$

$$= \int_{-\pi/4}^{\pi/4} r^2 \, d\theta = \int_{-\pi/4}^{\pi/4} a^2 \cos^2 2\theta \, d\theta, \qquad \text{from (1)}$$

$$= 2a^2 \int_0^{\pi/4} \cos^2 2\theta \, d\theta = 2a^2 \int_0^{\pi/2} \cos^2 t \cdot \frac{1}{2} dt, \qquad \text{putting } 2\theta = t$$

$$= a^2 \int_0^{\pi/2} \cos^2 t \, dt = a^2 \cdot \frac{1}{2} \cdot \frac{1}{2} \pi = \frac{a^2 \pi}{4}.$$

Therefore from (2), we get

$$\bar{x} = \frac{64a^3/(105\sqrt{2})}{a^2\,\pi/4} = \frac{64 \times 4 \times a}{105\sqrt{2}\,.\,\pi} = \frac{128\sqrt{2}\,.\,a}{105\,\pi}.$$

Hence $\bar{x} = 128\sqrt{2}\,a/105\,\pi$

and $\bar{y} = 0.$

Remark: To find $\bar{x}$ we can either integrate over the whole loop or we can integrate only over the upper half of this loop. Note that by symmetry about the x-axis, the x-co-ordinate of the C.G. of the whole loop is the same as the x-co-ordinate of the C.G. of the upper half of this loop.

Example 11: *Find the centre of gravity of the area cut off from the parabola $y^2 = 4\,ax$ by the straight line $y = mx$.*

Or

Find the locus of the centroid of the area of the parabola cut off by a variable straight line passing through the vertex.

Solution: The given parabola is

$$y^2 = 4ax \qquad \ldots(1)$$

Its vertex is $(0, 0)$. Equation of the straight line passing through the vertex is

$$y = mx. \qquad \ldots(2)$$

Solving (1) and (2) we get the co-ordinates of the point of intersection A as $\left(\frac{4a}{m^2}, \frac{4a}{m}\right)$.

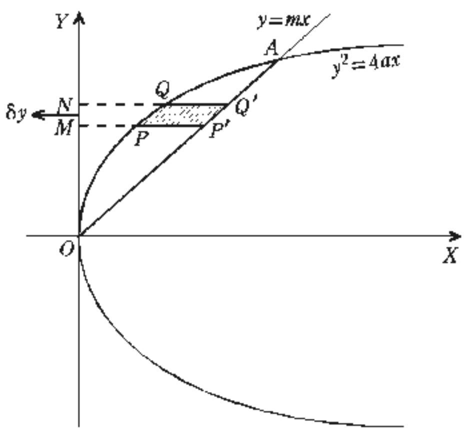
Take a line PP' parallel to the x-axis where $P\,(x_1, y)$ is a point on the arc OA of the parabola $y^2 = 4ax$ and $P'\,(x_2, y)$ a point on the line $y = mx$. Then

$$y^2 = 4ax_1 \quad \text{and} \quad y = mx_2\,. \qquad \ldots(3)$$

We are required to find the C.G. of the area $OPAP'O$.

Consider an elementary strip $PP'Q'Q$ parallel to x-axis, of the enclosed area $OPAP'O$. Then area of this strip $= (x_2 - x_1)\,\delta y$ and the C.G. of this strip can be taken as the middle point

$$\left(\frac{x_1 + x_2}{2}, y\right) \text{ of } PP'.$$



The limits of y for the area enclosed by the parabola and the straight line are 0 to $\frac{4a}{m}$.

If $(\bar{x}, \bar{y})$ be the C.G. of the area $OPAP'O$, then

$$\bar{x} = \frac{\int_0^{4a/m} \frac{x_1 + x_2}{2} \cdot (x_2 - x_1)\, dy}{\int_0^{4a/m} (x_2 - x_1)\, dy} = \frac{\frac{1}{2}\int_0^{4a/m} (x_2^2 - x_1^2)\, dy}{\int_0^{4a/m} (x_2 - x_1)\, dy}$$

and $$\bar{y} = \frac{\int_0^{4a/m} y\,(x_2 - x_1)\, dy}{\int_0^{4a/m} (x_2 - x_1)\, dy}.$$

Now the Nr. of $\bar{x}$

$$= \frac{1}{2}\int_0^{4a/m} (x_2^2 - x_1^2)\, dy = \frac{1}{2}\int_0^{4a/m} \left(\frac{y^2}{m^2} - \frac{y^4}{16a^2}\right) dy, \qquad \text{from (3)}$$

$$= \frac{1}{2}\left[\frac{y^3}{3m^2} - \frac{y^5}{80a^2}\right]_0^{4a/m} = \frac{1}{2}\left[\frac{64a^3}{3m^5} - \frac{64a^3}{5m^5}\right] = \frac{64a^3}{15m^5},$$

the Nr. of $\bar{y}$

$$= \int_0^{4a/m} y\,(x_2 - x_1)\, dy = \int_0^{4a/m} y \cdot \left(\frac{y}{m} - \frac{y^2}{4a}\right) dy, \qquad \text{from (3)}$$

$$= \int_0^{4a/m} \left(\frac{y^2}{m} - \frac{y^3}{4a}\right) dy = \left[\frac{y^3}{3m} - \frac{y^4}{16a}\right]_0^{4a/m}$$

$$= \left[\frac{64a^3}{3m^4} - \frac{16a^3}{m^4}\right] = \frac{16a^3}{3m^4},$$

and the Dr. of both $\bar{x}$ and $\bar{y}$

$$= \int_0^{4a/m} (x_2 - x_1)\, dy = \int_0^{4a/m} \left(\frac{y}{m} - \frac{y^2}{4a}\right) dy, \qquad \text{from (3)}$$

$$= \left[\frac{y^2}{2\,m} - \frac{y^3}{12\,a}\right]_0^{4\,a/m} = \frac{8a^2}{m^3} - \frac{16a^2}{3m^3} = \frac{8a^2}{3m^3}.$$

Thus $$\bar{x} = \frac{64a^3/15m^5}{8a^2/3m^3} = \frac{8a}{5m^2} \quad \text{and} \quad \bar{y} = \frac{16a^3/3m^4}{8a^2/3m^3} = \frac{2a}{m}.$$

Hence the co-ordinates of the required C.G. are $\left(\frac{8a}{5m^2}, \frac{2a}{m}\right)$.

Now as the straight line through the vertex changes, m changes. Therefore to find the locus of the centroid, eliminate the variable m between the values of $\bar{x}$ and $\bar{y}$. Thus

$$\left(\frac{2a}{\bar{y}}\right)^2 = \frac{8a}{5\,\bar{x}} \quad \text{or} \quad \bar{y}^{\,2} = \left(\frac{5}{2}\right) a\,\bar{x}$$

Generalising $(\bar{x}, \bar{y})$, we get the required locus as $y^2 = \left(\frac{5}{2}\right) ax$, which is a parabola.

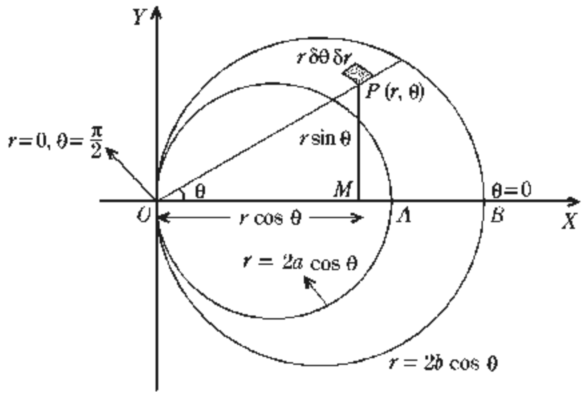
Example 12: *Find the position of the centre of gravity of the area enclosed by the curves* $x^2 + y^2 - 2\,ax = 0$ *and* $x^2 + y^2 - 2\,bx = 0$ *on the positive side of the axis of x.*

Solution: The given curves are

$$x^2 + y^2 - 2\,ax = 0, \quad \ldots(1)$$

and $$x^2 + y^2 - 2\,bx = 0. \quad \ldots(2)$$

Both these curves are circles passing through the origin and having their centres on the axis of x.



Changing to polars, the equation (1) becomes

$$x^2 + y^2 = 2\,ax \quad \text{or} \quad r^2 = 2\,ar\cos\theta$$

$$[\because \; x = r\cos\theta, \; y = r\sin\theta, \; x^2 + y^2 = r^2]$$

or $$r = 2a\cos\theta. \quad \ldots(3)$$

Similarly the equation (2) becomes

$$r = 2b\cos\theta. \quad \ldots(4)$$

The diameter of the circle (3) is $OA = 2\,a$ and that of the circle (4) is $OB = 2b$. We have taken $b > a$.

Let $(\bar{x}, \bar{y})$ be the C.G. of the area enclosed by the circles (3) and (4) and lying above the x-axis. Take a small element $r\,\delta\theta\,\delta r$ of this area at the point $P\,(r, \theta)$ lying within this area. The element $r\,\delta\theta\,\delta r$ being very small, its C.G. can be taken as the point P whose cartesian co-ordinates are $(r\cos\theta, r\sin\theta)$.

$$\therefore \quad \bar{x} = \frac{\int_{\theta=0}^{\pi/2} \int_{r=2a\cos\theta}^{2b\cos\theta} r\cos\theta \cdot r\, d\theta\, dr}{\int_{\theta=0}^{\pi/2} \int_{r=2a\cos\theta}^{2b\cos\theta} r\, d\theta\, dr}$$

and $$\bar{y} = \frac{\int_{\theta=0}^{\pi/2} \int_{r=2a\cos\theta}^{2b\cos\theta} r\sin\theta \cdot r\, d\theta\, dr}{\int_{\theta=0}^{\pi/2} \int_{r=2a\cos\theta}^{2b\cos\theta} r\, d\theta\, dr}$$

[Note that to cover the area under consideration first we regard θ as fixed. For that fixed value of θ, r goes from the circle $r = 2a\cos\theta$ to the circle $r = 2b\cos\theta$ and so these are the limits of r. Now the whole area is covered if θ goes from 0 to $\dfrac{\pi}{2}$ which are therefore the limits of θ. The first integration must be performed with respect to r regarding θ as constant whose limits are in terms of θ and then we integrate with respect to θ.]

Now the Nr. of $\bar{x}$

$$= \int_{\theta=0}^{\pi/2} \cos\theta \left[\frac{r^3}{3}\right]_{2a\cos\theta}^{2b\cos\theta} d\theta = \int_0^{\pi/2} \frac{8}{3}(b^3\cos^3\theta - a^3\cos^3\theta)\cos\theta\, d\theta$$

$$= \frac{8}{3}(b^3 - a^3)\int_{\theta=0}^{\pi/2} \cos^4\theta\, d\theta = \frac{8}{3}(b^3 - a^3)\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi(b^3 - a^3)}{2},$$

the Nr. of $\bar{y}$ $= \int_{\theta=0}^{\pi/2} \sin\theta \left[\frac{r^3}{3}\right]_{2a\cos\theta}^{2b\cos\theta} d\theta = \frac{8}{3}\int_0^{\pi/2} (b^3\cos^3\theta - a^3\cos^3\theta)\sin\theta\, d\theta$

$$= \frac{8}{3}(b^3 - a^3)\int_0^{\pi/2} \cos^3\theta\sin\theta\, d\theta = \frac{8}{3}(b^3 - a^3)\cdot\frac{2}{4\cdot 2} = \frac{2}{3}(b^3 - a^3),$$

the Dr. of $\bar{x}$ = the Dr. of $\bar{y}$

$$= \int_{\theta=0}^{\pi/2} \left[\frac{r^2}{2}\right]_{2a\cos\theta}^{2b\cos\theta} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} 4(b^2 - a^2)\cos^2\theta\, d\theta$$

$$= 2(b^2 - a^2)\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi}{2}(b^2 - a^2).$$

$\therefore$
$$\bar{x} = \frac{\frac{1}{2}\pi(b^3 - a^3)}{\frac{1}{2}\pi(b^2 - a^2)} = \frac{(b-a)(a^2 + ab + b^2)}{(b-a)(b+a)} = \frac{(a^2 + ab + b^2)}{a+b},$$

$$\bar{y} = \frac{\frac{2}{3}(b^3 - a^3)}{\frac{1}{2}\pi(b^2 - a^2)} = \frac{4(a^2 + ab + b^2)}{3\pi(a+b)}.$$

Comprehensive Exercise 2

Find The Centre of Gravity of :

1. The area bounded by the parabola $y^2 = 4ax$, the axis of x and the latus rectum. **(Kumaun 2003)**

2. The area of the curve $ay^2 = x^3$ between the origin and $x = b$.

3. The area enclosed by the parabola $y^2 = 4ax$ and the double ordinate $x = b$.

4. The area bounded by the curve $y = \sin x$, and the lines $x = 0$ and $x = \pi$.

5. The area between the curve $y = \cos x$ from $x = 0$ to $x = \frac{\pi}{2}$, bounded by the line $y = 0$.
6. The area between the curve $y = c \cosh\left(\frac{x}{c}\right)$, the co-ordinate axes and the ordinate $x = a$.
7. The area of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ lying in the positive quadrant.
8. The area of the curve $\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$ lying in the positive quadrant.

(Kanpur 2008; Garhwal 03)

9. The area bounded by the axis of y, the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ and its base.
10. A uniform semi-circular disc.
11. The area of the cardioid $r = a(1 + \cos\theta)$.
12. The area of one loop of the lemniscate $r^2 = a^2 \cos 2\theta$. **(Agra 2009; Kanpur 09)**
13. The area of the loop of the curve $r = a \sin 2\theta$ which lies in the positive quadrant, the density being supposed uniform.
14. Find the centre of gravity of the loop of the curve $r = a \cos 3\theta$, containing the initial line.
15. Find the position of the centroid of the area of the curve $\left(\frac{x}{a}\right)^{1/2} + \left(\frac{y}{b}\right)^{1/2} = 1$ which lies in the positive quadrant.
16. The area included between an ellipse and the line joining the extremities of the major and minor axes.
17. The area enclosed by the curves $y^2 = ax$ and $x^2 = by$.
18. The area bounded by the parabolas $y^2 = 8x$ and $x^2 = 8y$.
19. The area enclosed by the parabolas $y^2 = x$ and $x^2 = 2y$.
20. The area enclosed by the curves $y^2 = ax$ and $y^2 = 2ax - x^2$ on the positive side of the axis of x.
21. The area enclosed by the circle $x^2 + y^2 = a^2$ and the parabola $y^2 = 4ax$ on the positive side of the axis of x.

Answers 2

1. $\bar{x} = \frac{3a}{5}$, $\bar{y} = \frac{3a}{4}$
2. $\bar{x} = \frac{5b}{7}$, $\bar{y} = 0$
3. $\bar{x} = \frac{3b}{5}$, $\bar{y} = 0$
4. $\bar{x} = \frac{\pi}{2}$, $\bar{y} = \frac{\pi}{8}$

5. $\bar{x} = \frac{1}{2}\pi - 1,\ \bar{y} = \frac{\pi}{8}$

6. $\bar{x} = a - c \coth\left(\frac{a}{c}\right) + c \operatorname{cosech}\left(\frac{a}{c}\right),$

$\bar{y} = \frac{1}{4}\left[c \cosh\left(\frac{a}{c}\right) + a \operatorname{cosech}\left(\frac{a}{c}\right)\right]$

7. $\bar{x} = \bar{y} = \frac{256\,a}{315\pi}$

8. $\bar{x} = \frac{256\,a}{315\pi},\ \bar{y} = \frac{256\,a}{315\pi}$

9. $\bar{x} = \frac{9\,(9\pi^2 - 16)}{18\pi},\ \bar{y} = \frac{7a}{6}$

10. $\bar{x} = \frac{4a}{3\pi},\ \bar{y} = 0$

11. $\bar{x} = \frac{5a}{6},\ \bar{y} = 0$

12. $\bar{x} = \left(\frac{1}{8}\right)\pi a\sqrt{2},\ \bar{y} = 0$

13. $\bar{x} = \bar{y} = \frac{128\,a}{105\pi}$

14. $\bar{x} = \frac{81\sqrt{3}a}{80\,\pi},\ \bar{y} = 0$

15. $\bar{x} = \frac{a}{5},\ \bar{y} = \frac{b}{5}$

16. $\bar{x} = \frac{2}{3}\left[\frac{a}{\pi - 2}\right],\ \bar{y} = \frac{2}{3}\left[\frac{b}{\pi - 2}\right]$

17. $\bar{x} = \frac{9}{20}a^{1/3}\,b^{2/3},\ \bar{y} = \frac{9}{20}a^{2/3}\,b^{1/3}$

18. $\bar{x} = \frac{18}{5},\ \bar{y} = \frac{18}{5}$

19. $\bar{x} = \frac{9}{20}2^{2/3},\ \bar{y} = \frac{9}{20}2^{1/3}$

20. $\bar{x} = \frac{a\,(15\pi - 44)}{5\,(3\pi - 8)},\ \bar{y} = \frac{a}{3\pi - 8}$

21. $$\bar{x} = \frac{\frac{1}{2}[(a^2\,b - b^3/3 - b^5/80a^2)]}{\frac{1}{2}b\sqrt{(a^2 - b^2)} + \frac{1}{2}a^2\sin^{-1}(b/a) - (b^3/12a)}$$

$$\bar{y} = \frac{\frac{1}{3}[a^3 - (a^2 - b^2)^{3/2}] - (b^4/16a)}{\frac{1}{2}b\sqrt{(a^2 - b^2)} + \frac{1}{2}a^2\sin^{-1}(b/a) - (b^3/12a)},\ \text{where } b^2 = 4a^2(\sqrt{5} - 2)$$

1.5 Centre of Gravity of a Solid of Revolution

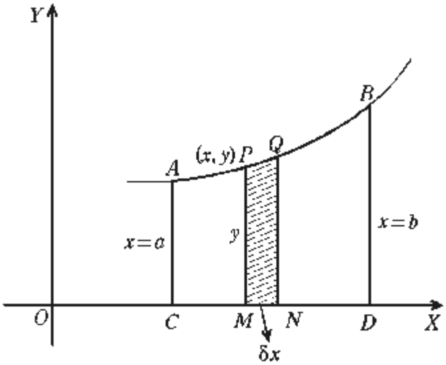
To find the centre of gravity of a solid formed by revolving the curve $y = f(x)$ about the x-axis and cut off between the plane ends $x = a$ and $x = b$.

Let $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ be any two neighbouring points on the arc AB of the curve $y = f(x)$. Draw PM and QN perpendiculars to the x-axis. If we revolve the area between the curve and the x-axis about the x-axis, the elementary strip $PMNQ$ generates a disc of small thickness δx and circular base of area πy^2. Thus volume of the elementary disc $= \pi y^2\,\delta x$ and its mass $= \pi y^2\,\delta x\,\rho$, ρ being the density per unit volume.

The C.G. of this disc may be taken as the point $M(x, 0)$ because $MN = \delta x$ is very small.

$\therefore$ If $(\bar{x}, \bar{y})$ be the required C.G., then $\bar{y} = 0$, (by symmetry).

[Note that the solid of revolution formed by revolving the curve about the x-axis is symmetrical about the x-axis and so its C.G. must lie on the x-axis.]



Also $$\bar{x} = \frac{\int x\,dm}{\int dm} = \frac{\int_a^b x \cdot \rho\pi y^2\,dx}{\int_a^b \rho\,\pi y^2\,dx}$$

$$= \frac{\int_a^b xy^2\,dx}{\int_a^b y^2\,dx},$$ if the density ρ is uniform.

Thus $$\bar{x} = \frac{\int_a^b xy^2\,dx}{\int_a^b y^2\,dx}$$ **and $\bar{y} = 0$, if the density is uniform.**

In case the solid is formed by revolving the curve about the axis of y and cut off between the planes $y = a$ *and* $y = b$, then the volume of the elementary circular disc formed by revolving the elementary strip about the y-axis $= \pi x^2\,\delta y$ and the C.G. of this disc may be supposed to be at $(0, y)$ because its thickness δy is very small.

Then $\bar{x} = 0$, (by symmetry about the y-axis),

and $$\bar{y} = \frac{\int y\,dm}{\int dm} = \frac{\int_{y=a}^b y\rho\,\pi x^2\,dy}{\int_{y=a}^b \rho\pi\,x^2\,dy} = \frac{\int_a^b yx^2\,dy}{\int_a^b x^2\,dy}.$$

if the density ρ is uniform.

Note 1: The centre of gravity of the solid of revolution will always lie on the axis of revolution.

Note 2: In case the solid is formed by revolving the area enclosed by any two given curves **about the x-axis**, then

$$\bar{x} = \frac{\int_a^b x\,(y_2^2 - y_1^2)\,dx}{\int_a^b (y_2^2 - y_1^2)\,dx}$$ and $\bar{y} = 0$,

where the points $P_1\ (x, y_1)$ and $P_2\ (x, y_2)$ are on the respective arcs of the given curves intervened between their common points ; a and b are the abscissae of the common points of intersection.

If the **axis of revolution is y-axis**, then

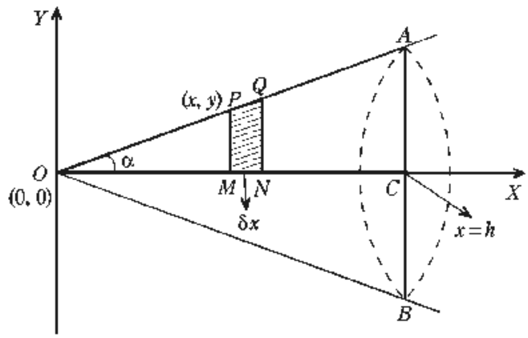
$$\bar{x} = 0 \text{ and } \bar{y} = \frac{\int_a^b y\,(x_2^2 - x_1^2)\,dy}{\int_a^b (x_2^2 - x_1^2)\,dy},$$

where a and b are the ordinates of the common points of intersection ; $P_1\,(x_1, y)$ and $P_2\,(x_2, y)$ being two points on the respective arcs of the two curves intervened between their points of intersection.

Illustrative Examples

Example 13: *Find the centre of gravity of a solid right circular cone of height h.*

Solution: Take a right angled triangle OCA in which $\angle OCA = 90°$. The side OC is of length h and is along the x-axis. If we revolve the triangular area OCA about the x-axis, a solid right circular cone of height h is generated, the axis of the cone being along the x-axis and its vertex being at O.



Let $\angle AOC = \alpha$.

Then the equation of the line OA is

$$y = x \tan \alpha. \qquad \textbf{[Note]} \qquad \ldots(1)$$

Take an elementary strip $PMNQ$ of small width δx parallel to the y-axis, of the area of the triangle OCA. When the area OCA revolves about the line OC (*i.e.*, the x-axis), the elementary strip $PMNQ$ generates a disc of small thickness δx and circular base of area πy^2. The volume of this elementary disc $= \pi y^2\,\delta x$ and the C.G. of this disc may be supposed to be at $M\,(x, 0)$ because the thickness $MN = \delta x$ is very small.

$\therefore$ If $(\bar{x}, \bar{y})$ be the required C.G. of the solid cone, then

$\bar{y} = 0$ (by symmetry about OC *i.e.*, the x-axis).

Also
$$\bar{x} = \frac{\int_0^h x \cdot \pi y^2\,dx}{\int_0^h \pi y^2\,dx} = \frac{\int_0^h x y^2\,dx}{\int_0^h y^2\,dx}$$

$$= \frac{\int_0^h x \cdot (x \tan \alpha)^2\,dx}{\int_0^h x^2 \tan^2 \alpha\,dx} \qquad [\because\ y = x \tan \alpha, \text{ from (1)}]$$

$$= \frac{\int_0^h x^3\,dx}{\int_0^h x^2\,dx} = \frac{\left[\frac{x^4}{4}\right]_0^h}{\left[\frac{x^3}{3}\right]_0^h} = \frac{\frac{1}{4}h^4}{\frac{1}{3}h^3} = \frac{3h}{4}.$$

Hence for the required C.G.,

$$\bar{x} = \frac{3h}{4} \text{ and } \bar{y} = 0.$$

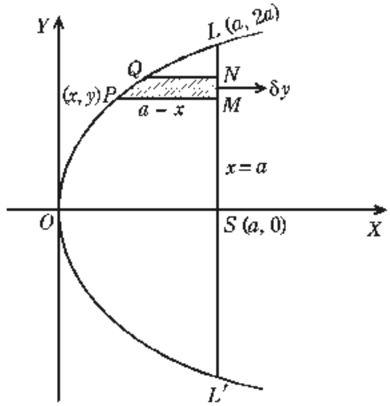
Thus the C.G. of a solid right circular cone lies on its axis at a distance $\left(\frac{3h}{4}\right)$ from the vertex, h being the height of the cone.

Example 14: *Find the centre of gravity of the solid formed by the revolution of the area bounded by the parabola $y^2 = 4ax$, the axis of x and the latus rectum, about the latus rectum.*

Solution: The given parabola is

$$y^2 = 4ax. \quad \text{...(1)}$$

Let O be the vertex and LSL' be the latus rectum. We have to revolve the area LOS about the line LS.



Take an elementary strip $PMNQ$ of small width δy perpendicular to the latus rectum and terminated by the latus rectum. We have $PM = a - x$.

When the area bounded by the parabola $y^2 = 4ax$, the axis of x and the latus rectum is revolved about the latus rectum LL', the elementary strip $PMNQ$ generates a disc of small thickness δy and circular base of area $\pi (a - x)^2$.

Thus volume of this elementary disc $= \pi (a - x)^2\,\delta y$ and the C.G. of this disc may be supposed to be at $M (a, y)$ because the thickness $MN = \delta y$ is very small. Also for the given area y varies from 0 to $2a$.

If $(\bar{x}, \bar{y})$ be the required C.G., then

$\bar{x} = a$, (by symmetry about the axis of revolution LL' *i.e.*, the line $x = a$).

Also
$$\bar{y} = \frac{\int_0^{2a} y \cdot \pi (a - x)^2\,dy}{\int_0^{2a} \pi (a - x)^2\,dy}$$

$$= \frac{\int_0^{2a} y \cdot \left(a - \frac{y^2}{4a}\right)^2 dy}{\int_0^{2a} \left(a - \frac{y^2}{4a}\right)^2 dy} \quad \text{[From (1)]}$$

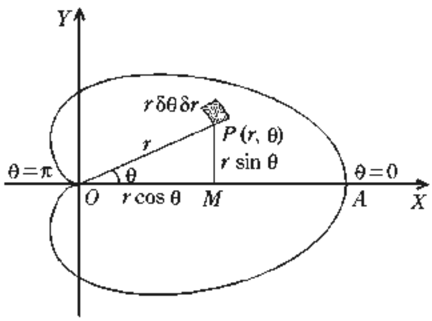
$$= \frac{\int_0^{2a} y\,(4a^2 - y^2)^2\,dy}{\int_0^{2a} (4a^2 - y^2)^2\,dy} = \frac{\int_0^{2a} y\,(16a^4 - 8a^2y^2 + y^4)\,dy}{\int_0^{2a} (16a^4 - 8a^2y^2 + y^4)\,dy}$$

$$= \frac{\left[16a^4 \frac{y^2}{2} - 8a^2 \frac{y^4}{4} + \frac{y^6}{6}\right]_0^{2a}}{\left[16a^4 y - 8\frac{a^2 y^3}{3} + \frac{y^5}{5}\right]_0^{2a}} = \frac{\frac{32a^6}{3}}{\frac{256a^5}{15}} = \frac{5a}{8}.$$

Hence the required C.G. is given by $\bar{x} = a,\ \bar{y} = \frac{5a}{8}$.

Example 15: *Find the centroid of the volume formed by the revolution of the cardioid $r = a(1 + \cos\theta)$ about the x-axis.*

Solution: Obviously the upper half of the cardioid generates the same volume while revolving about the x-axis as is the volume generated by the revolution of the whole of the cardioid about the x-axis.



Consider an elementary area $r\,\delta\theta\,\delta r$ at the point $P(r, \theta)$ lying within the upper half area of the cardioid. When the cardioid is revolved about the x-axis, this elementary area will generate a ring of radius $PM = r\sin\theta$ and of thickness $r\,\delta\theta\,\delta r$.

Volume of this elementary ring $= (2\pi r\sin\theta)\,r\,\delta\theta\,\delta r$ and its C.G. can be taken as the point M on the x-axis whose cartesian co-ordinates are $(r\cos\theta, 0)$. Note that the thickness $r\,\delta\theta\,\delta r$ of the ring is very small.

To cover the whole area of the upper half of the cardioid the limits of r are 0 to $a(1 + \cos\theta)$ and the limits of θ are 0 to π.

If $(\bar{x}, \bar{y})$ be the required C.G., then by symmetry about the axis of revolution, $\bar{y} = 0$. Also

$$\bar{x} = \frac{\int_{\theta=0}^{\pi} \int_{r=0}^{a(1+\cos\theta)} r\cos\theta \cdot (2\pi r\sin\theta)\,r\,d\theta\,dr}{\int_{\theta=0}^{\pi} \int_{r=0}^{a(1+\cos\theta)} (2\pi r\sin\theta)\,r\,d\theta\,dr}$$

$$= \frac{\int_{\theta=0}^{\pi} \int_{r=0}^{a(1+\cos\theta)} r^3 \sin\theta\cos\theta\,d\theta\,dr}{\int_{\theta=0}^{\pi} \int_{r=0}^{a(1+\cos\theta)} r^2 \sin\theta\,d\theta\,dr}$$

$$= \frac{\int_0^{\pi} \left[\frac{r^4}{4} \right]_0^{a(1+\cos\theta)} \sin\theta \cos\theta \, d\theta}{\int_0^{\pi} \left[\frac{r^3}{3} \right]_0^{a(1+\cos\theta)} \sin\theta \, d\theta}$$

$$= \frac{3a}{4} \frac{\int_0^{\pi} (1+\cos\theta)^4 \sin\theta \cos\theta \, d\theta}{\int_0^{\pi} (1+\cos\theta)^3 \sin\theta \, d\theta}.$$

Now put $1+\cos\theta = t$ so that $-\sin\theta \, d\theta = dt$.

Also when $\theta = 0, t = 2$ and when $\theta = \pi, t = 0$.

$\therefore$ $$\bar{x} = \frac{3a}{4} \frac{\int_2^0 t^4 (t-1)(-dt)}{\int_2^0 t^3 (-dt)} = \frac{3a}{4} \frac{\int_0^2 (t^5 - t^4) \, dt}{\int_0^2 t^3 \, dt}$$

$$= \frac{3a}{4} \frac{\left[\frac{t^6}{6} - \frac{t^5}{5} \right]_0^2}{\left[\frac{t^4}{4} \right]_0^2} = \frac{3a}{4} \frac{\left\{ \frac{2^6}{6} - \frac{2^5}{5} \right\}}{\frac{2^4}{4}} = 3a \cdot \frac{2^5}{2^4} \left[\frac{1}{3} - \frac{1}{5} \right]$$

$$= 3a \times 2 \left(\frac{1}{3} - \frac{1}{5} \right) = 6a \cdot \frac{2}{15} = \frac{4a}{5}.$$

$\therefore$ the required C.G. is the point $\left(\frac{4a}{5}, 0 \right)$.

Comprehensive Exercise 3

Find the Centre of Gravity of :

1. A solid uniform hemi-sphere of radius a. **(Kanpur 2009)**
2. The segment of a sphere of radius a cut off by a plane at a distance h from the centre.
3. The volume formed by the revolution of the portion of the parabola $y^2 = 4ax$ cut off by the ordinate $x = h$ about the axis of x. **(Agra 2011)**
4. The volume formed by the revolution of a quadrant of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ about the major axis. **(Agra 2008)**
5. A solid figure formed by revolving a quadrant of an ellipse about its minor axis.
6. The volume formed by the revolution of the cycloid

   $x = a(\theta + \sin\theta), \; y = a(1 - \cos\theta)$

   about the axis of y.

7. The volume formed by revolving the area bounded by the parabolas $y^2 = 4ax$ and $x^2 = 4by$ about the axis of x.

Answers 3

1. $\bar{x} = \frac{3a}{8}, \ \bar{y} = 0$

2. $\bar{x} = \frac{3}{4}\left\{\frac{(a+h)^2}{(2a+h)}\right\}, \ \bar{y} = 0$

3. $\bar{x} = \frac{2h}{3}, \ \bar{y} = 0$

4. $\left(\frac{3a}{8}, 0\right)$

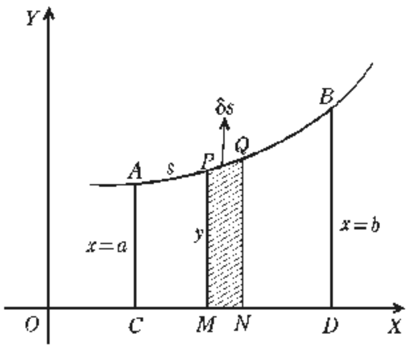
5. $\left(0, \frac{3b}{8}\right)$

6. $\bar{x} = 0, \ \bar{y} = \frac{a(63\pi^2 - 64)}{6(9\pi^2 - 16)}$

7. $\bar{x} = \frac{20}{9} a^{1/3} b^{2/3}, \ \bar{y} = 0$

1.6 Centre of Gravity of Surface of Revolution

To find the centre of gravity of the surface formed by revolving the curve $y = f(x)$ about the x-axis and cut off between the planes $x = a$ and $x = b$.



Let the arc AB of the curve $y = f(x)$ lying between the lines $x = a$ and $x = b$ be revolved about the x-axis. Take an element PQ ($= \delta s$) of the arc AB. When the arc AB revolves about the x-axis, the elementary arc PQ ($= \delta s$) generates an elementary surface area $2\pi y \, \delta s$ of mass $2\pi y \, \delta s \, \rho$, where ρ is the density per unit area of the surface. The C.G. of this elementary surface area may be supposed to be at the point $M(x, 0)$ because δs is very small.

If $(\bar{x}, \bar{y})$ be the required C.G., then $\bar{y} = 0$, (by symmetry about the axis of revolution *i.e.*, the x-axis) and

$$\bar{x} = \frac{\int x \, dm}{\int dm} = \frac{\int x \cdot \rho \, 2\pi y \, ds}{\int \rho \, 2\pi y \, ds}, \text{ between the suitable limits}$$

$$= \frac{\int xy \, ds}{\int y \, ds}, \text{ if the density } \rho \text{ is uniform.}$$

To perform integration, we put $ds = \sqrt{\{1 + (dy/dx)^2\}} \, dx$, for cartesian equation of the curve and we adjust the limits of x suitably.

If the equation of the curve is in the polar form $r = f(\theta)$, then we put $ds = \sqrt{\{r^2 + (dr/d\theta)^2\}}\, d\theta$ and we adjust the limits of θ.

If the equation of the curve is in the parametric form $x = f(t), y = \phi(t)$, then we put $ds = \sqrt{\{(dx/dt)^2 + (dy/dt)^2\}}\, dt$ and we adjust the corresponding limits of t.

If the surface is formed by revolving the curve about the y-axis the corresponding formulae will be

$$\bar{x} = 0, \bar{y} = \frac{\int y \cdot \rho\, 2\pi x\, ds}{\int \rho\, 2\pi x\, ds} = \frac{\int xy\, ds}{\int x\, ds}, \text{ if the density } \rho \text{ is uniform.}$$

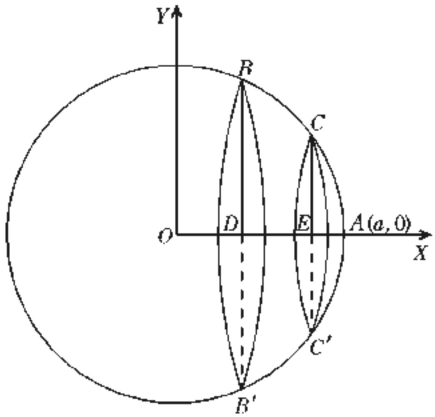
Illustrative Examples

Example 16: *Find the C.G. of a zone of a sphere.*

Solution: Suppose a zone of a sphere is generated by revolving an arc (say, BC) of the circle

$$x^2 + y^2 = a^2, \qquad \ldots(1)$$

about the x-axis.



Then the axis of the zone (*i.e.*, the height of the zone) will be along the x-axis.

Differentiating (1), we get

$$\frac{dy}{dx} = -\left(\frac{x}{y}\right).$$

$$\therefore \qquad \frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}$$

$$= \sqrt{\left(1 + \frac{x^2}{y^2}\right)} = \sqrt{\left(\frac{x^2 + y^2}{y^2}\right)} = \frac{a}{y}. \qquad \ldots(2)$$

Also let BD be the line $x = b$ and CE be the line $x = c$.

If $(\bar{x}, \bar{y})$ be the required C.G. of the zone, then $\bar{y} = 0$, (by symmetry about the axis of revolution *i.e.*, the x-axis).

$$\text{Also} \qquad \bar{x} = \frac{\int x\, 2\pi y\, ds}{\int 2\pi y\, ds} = \frac{\int_{x=b}^{c} xy \cdot \frac{ds}{dx}\, dx}{\int_{x=b}^{c} y \frac{ds}{dx}\, dx} = \frac{\int_b^c x \cdot y \frac{a}{y}\, dx}{\int_b^c y \cdot \frac{a}{y}\, dx} \qquad \text{[from (2)]}$$

$$= \frac{\int_b^c x\,dx}{\int_b^c dx} = \frac{\left[\frac{x^2}{2}\right]_b^c}{[x]_b^c} = \frac{\frac{1}{2}(c^2 - b^2)}{(c-b)} = \frac{1}{2}(c+b).$$

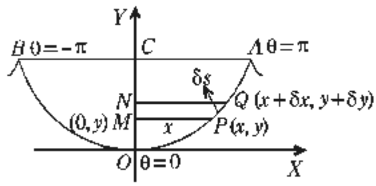
Thus the C.G. of the zone is the middle point of its height *DE*. Hence the required C.G. bisects the height of the zone of the sphere.

Example 17: *Find the centre of gravity of the surface formed by the revolution of the cycloid* $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ *about the axis of* y.

Solution: The given parametric equations of the cycloid are

$$x = a(\theta + \sin\theta), \quad y = a(1 - \cos\theta). \qquad \ldots(1)$$

Differentiating (1) w.r.t. θ, we get



$$\frac{dx}{d\theta} = a(1 + \cos\theta)$$

and $$\frac{dy}{d\theta} = a\sin\theta.$$

$\therefore$ $$\frac{ds}{d\theta} = \sqrt{\left\{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right\}}$$

$$= \sqrt{\{a^2(1+\cos\theta)^2 + a^2\sin^2\theta\}}$$

$$= a\sqrt{\{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta\}} = a\sqrt{[2 + 2\cos\theta]}$$

$$= a\sqrt{[2 \,.\, 2\cos^2\tfrac{1}{2}\theta]} = 2a\cos\tfrac{1}{2}\theta. \qquad \ldots(2)$$

The cycloid is symmetrical about the y-axis and for the portion of the cycloid lying in the positive quadrant θ varies from 0 to π. Obviously the surface formed by revolving the whole cycloid about the y-axis is the same as that formed by revolving the portion of the cycloid lying in the positive quadrant.

Take an elementary arc δs at any point $P(x, y)$ of the portion of the cycloid lying in the positive quadrant. The area of the elementary surface generated by the revolution of the arc δs about the y-axis $= 2\pi x\,\delta s$ and the C.G. of this elementary surface is the point $(0, y)$ on the axis of rotation.

If $(\bar{x}, \bar{y})$ be the required C.G., then

$\bar{x} = 0$, by symmetry about the axis of rotation *i.e.*, the y-axis.

Also $$\bar{y} = \frac{\int y \cdot 2\pi x\,ds}{\int 2\pi x\,ds} = \frac{\int_0^\pi yx\frac{ds}{d\theta}\,d\theta}{\int_0^\pi x\frac{ds}{d\theta}\,d\theta}$$

$$= \frac{\int_0^\pi a(1-\cos\theta)\cdot a(\theta+\sin\theta)\cdot 2a\cos\frac{1}{2}\theta\,d\theta}{\int_0^\pi a(\theta+\sin\theta)\cdot 2a\cos\frac{1}{2}\theta\,d\theta} \qquad \text{[From (1) and (2)]}$$

$$= \frac{a \int_0^{\pi} (1 - \cos\theta)(\theta + \sin\theta) \cos\frac{1}{2}\theta \, d\theta}{\int_0^{\pi} (\theta + \sin\theta) \cdot \cos\frac{1}{2}\theta \, d\theta} \qquad \ldots(3)$$

Now the Nr. of $\bar{y}$

$$= a \int_0^{\pi} 2\sin^2\frac{1}{2}\theta \, . \left(\theta + 2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta\right) . \cos\frac{1}{2}\theta \, d\theta \qquad \textbf{[Note]}$$

$$= 2a \int_0^{\pi/2} \sin^2\phi \, (2\phi + 2\sin\phi\cos\phi) \cos\phi \, 2\, d\phi, \text{ putting } \frac{\theta}{2} = \phi$$

$$= 8a \left[\int_0^{\pi/2} \phi \sin^2\phi \cos\phi \, d\phi + \int_0^{\pi/2} \sin^3\phi \cos^2\phi \, d\phi\right]$$

$$= 8a \left[\left(\phi \cdot \frac{\sin^3\phi}{3}\right)_0^{\pi/2} - \int_0^{\pi/2} 1 \cdot \frac{\sin^3\phi}{3} \, d\phi + \frac{2\,.\,1}{5\,.\,3\,.\,1}\right]$$

$$= 8a \left\{\frac{\pi}{2} \cdot \frac{1}{3} - \frac{1}{3} \cdot \frac{2}{3\,.\,1} + \frac{2}{15}\right\} = 8a \left\{\frac{\pi}{6} - \frac{2}{9} + \frac{2}{15}\right\}$$

$$= \frac{8a}{90}(15\pi - 20 + 12) = \frac{4\,a}{45}(15\pi - 8),$$

and the Dr. of $\bar{y} = \int_0^{\pi} (\theta + \sin\theta) \cos\frac{1}{2}\theta \, d\theta$

$$= \int_0^{\pi} \left(\theta\cos\frac{1}{2}\theta + 2\sin\frac{1}{2}\theta\cos^2\frac{1}{2}\theta\right) d\theta$$

$$= \int_0^{\pi/2} (2\phi\cos\phi + 2\sin\phi\cos^2\phi) \cdot 2d\phi, \text{ putting } \frac{\theta}{2} = \phi$$

$$= 4\left[\int_0^{\pi/2} \phi\cos\phi \, d\phi + \int_0^{\pi/2} \sin\phi\cos^2\phi \, d\phi\right]$$

$$= 4\left\{[\phi \cdot \sin\phi]_0^{\pi/2} - \int_0^{\pi/2} 1 \cdot \sin\phi \, d\phi + \left[-\frac{\cos^3\phi}{3}\right]_0^{\pi/2}\right\}$$

$$= 4\left\{\frac{\pi}{2} + [\cos\phi]_0^{\pi/2} + \frac{1}{3}\right\} = 4\left\{\frac{\pi}{2} - 1 + \frac{1}{3}\right\} = \frac{2}{3}(3\pi - 4).$$

Therefore from (3), we get

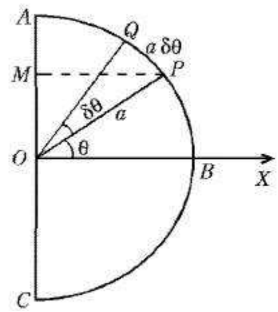
$$\bar{y} = \frac{\{(4a/45)(15\pi - 8)\}}{\frac{2}{3}(3\pi - 4)} = \frac{2\,a\,(15\pi - 8)}{15\,(3\pi - 4)}.$$

Hence the required C.G. is given by

$$\bar{x} = 0, \ \bar{y} = \{2\,a\,(15\pi - 8)/15\,(3\pi - 4)\}.$$

Example 18: *Show that the centre of gravity of a lune of a sphere of angle* $2\,\alpha$ *is at a distance* $\frac{1}{4}\pi\,(a \sin\alpha)/\alpha$ *from its axis.*

Solution: The lune of a sphere of angle $2\,\alpha$ is the surface generated by the revolution of the semi-circular arc ABC about the diameter AC through an angle $2\,\alpha$. The diameter AC is the axis of this lune.



Consider an elementary arc PQ $(= a\,\delta\theta)$ of the arc ABC. If the arc PQ is revolved through an angle $2\,\alpha$ about the diameter AC, it would generate a curved surface in the form of an arc of a circle of radius PM $(= a \cos\theta)$ and having its centre at M.

The mass of this surface is $2\,\alpha \,.\, (a\cos\theta) \,.\, a\,\delta\theta \,.\, \rho$ and the distance of its C.G. from AC

$$= \text{the radius } PM \times \frac{\sin\alpha}{\alpha} = a\cos\theta \,.\, \frac{\sin\alpha}{\alpha} \cdot$$

[Note]

$\therefore$ If $\bar{x}$ be the distance of the C.G. of the lune from its axis AC, then

$$\bar{x} = \frac{\int_{-\pi/2}^{\pi/2} \frac{a\cos\theta\cdot\sin\alpha}{\alpha}\cdot\rho\,2\alpha\cdot(a\cos\theta)\cdot a\,d\theta}{\int_{-\pi/2}^{\pi/2}\rho\,2\alpha\cdot(a\cos\theta)\cdot a\,d\theta}$$

$$= \frac{a\sin\alpha}{\alpha}\cdot\frac{\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta}{\int_{-\pi/2}^{\pi/2}\cos\theta\,d\theta} = \frac{a\sin\alpha}{\alpha}\cdot\frac{2\int_0^{\pi/2}\cos^2\theta\,d\theta}{2\int_0^{\pi/2}\cos\theta\,d\theta}$$

$$= \frac{a\sin\alpha}{\alpha}\cdot\frac{\frac{1}{2}\cdot\frac{1}{2}\pi}{\left[\sin\frac{1}{2}\pi - \sin 0\right]} = \frac{a\sin\alpha}{\alpha}\cdot\frac{\pi}{4}\cdot$$

Hence the C.G. of a lune of a sphere of angle 2α is at a distance $\frac{1}{4}\pi\,\frac{a\sin\alpha}{\alpha}$ from its axis.

Remark: A *lune* of a sphere is a part of its surface bounded by any two planes passing through a fixed diameter. This fixed diameter is called the axis of the lune and the angle between the bounding planes is called '**the angle of the lune**'.

Comprehensive Exercise 4

1. Show that the centre of gravity of the surface of a spherical segment bisects its height.
2. Find the C.G. of a thin uniform hemi-spherical shell.
3. Find the centre of gravity of a thin right conical shell of uniform thickness and density. **(Kumaun 2001)**
4. A parabola revolves round its axis, find the centroid of the portion of the surface between the vertex and a plane perpendicular to the axis at a distance h from the vertex, $4\,a$ being the latus rectum.

5. Find the centre of gravity of the surface formed by the revolution of the cardioid $r = a\,(1 + \cos\theta)$ about its axis. **(Kanpur 2008,10)**

6. Find the centre of gravity of the surface formed by the revolution of one loop of the lemniscate of Bernoulli $r^2 = a^2 \cos 2\theta$ about the initial line.

Answers 4

2. C.G. of a hemi-spherical shell bisects its height

3. C.G. divides the axis of the cone in the ratio 2 : 1, the major portion lying towards the vertex

4. $\bar{x} = \dfrac{h\,(a+h)^{3/2} - (2/5)\,\{(a+h)^{5/2} - a^{5/2}\}}{[(a+h)^{3/2} - a^{3/2}]}, \;\; \bar{y} = 0$

5. $\bar{x} = \dfrac{50}{63}\,a, \;\; \bar{y} = 0$

6. $\bar{x} = a\,(\sqrt{2} + 1)/3\sqrt{2}, \;\; \bar{y} = 0$

1.7 Centre of Gravity when the Density Varies

If we are required to find the C.G. of a body when its density ρ varies from point to point, we cannot cancel ρ from the integrals occurring in the numerator and the denominator of $\bar{x}$ and $\bar{y}$ because, now ρ is not a constant. Further if the body is in the form of a plane lamina and the density is not uniform, we cannot take an element of the body in the form of a strip. Obviously we shall not be able to write the mass of the strip because its density varies from point to point. Here we shall divide the lamina into infinitesimal elements of the second order and thus we shall make use of the double integrals. The procedure to be adopted is as given below :

1. In the Case of Cartesian Curves: Take a small element of area $\delta x\, \delta y$ at any point $P\,(x, y)$ lying in the area whose C.G. is to be found. If ρ be the density at the point P, then the mass of the elementary area $\delta x\, \delta y$ is $\rho\, \delta x\, \delta y$, because the density at every point of this elementary area can be taken the same as that at P. Also the C.G. of this small element $\delta x\, \delta y$ can be taken as the point $P\,(x, y)$. If $(\bar{x}, \bar{y})$ be the required C.G. of the whole area under consideration, we have

$$\bar{x} = \frac{\iint x \cdot \rho\, dx\, dy}{\iint \rho\, dx\, dy} \text{ and } \bar{y} = \frac{\iint y \cdot \rho\, dx\, dy}{\iint \rho \cdot dx\, dy},$$

where the limits of integration are to be so chosen that the whole area under consideration is covered.

2. In the Case of Polar Curves: Take a small element of area $r\, \delta\theta\, \delta r$ at any point $P\,(r, \theta)$ lying within the area whose C.G. is to be found. If ρ be the density at the point P, then the mass of the elementary area $r\, \delta\theta\, \delta r$ is $\rho \,.\, r\, \delta\theta\, \delta r$. Also the C.G. of the small element $r\,\delta\theta\,\delta r$ can be taken as the point P whose cartesian coordinates are $(r\cos\theta, r\sin\theta)$.

If $(\bar{x}, \bar{y})$ be the required C.G. of the whole area under consideration, we have

$$\bar{x} = \frac{\int\int r\cos\theta\cdot\rho r\,d\theta\,dr}{\int\int \rho r\,d\theta\,dr} \text{ and } \bar{y} = \frac{\int\int r\sin\theta\cdot\rho r\,d\theta\,dr}{\int\int \rho r\cdot d\theta\,dr}$$

where the limits of integration are to be so taken as to cover the whole area under consideration.

A similar procedure is adopted in the case of solids and surfaces of revolution when the density varies. The whole procedure will be clear from the worked out examples which follow.

Illustrative Examples

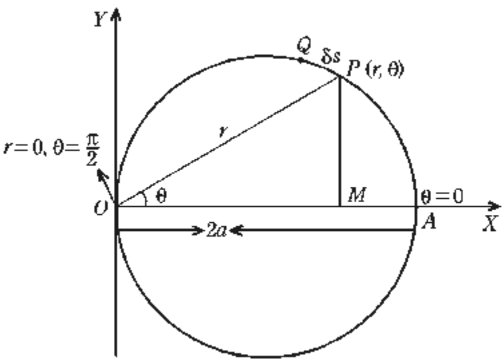
Example 19: *If the density of a circular arc varies as the square of the distance from a point O on the arc, show that the centroid divides the diameter through O in the ratio* $3:1$.

(Agra 2010)

Solution: Take the point O as the pole and the diameter through O as the initial line OX. Then the equation of the circle is

$$r = 2a\cos\theta,$$

where a is the radius of the circle.



Take an elementary arc $PQ = \delta s$ at any point $P(r,\theta)$ on the arc of the circle. If ρ be the density at the point P, then, as given,

$$\rho = k\,.\,OP^2 = kr^2,$$

where k is some constant.

$\therefore$ the mass of the elementary arc $PQ = \rho\,\delta s = kr^2\,\delta s$.

Its C.G. can be taken as the point P whose cartesian co-ordinates are $(r\cos\theta, r\sin\theta)$.

The whole circular arc is symmetrical about OX and its density is also symmetrical about OX as can be easily seen by taking two points on the circular arc on opposite sides of OX and equidistant from OX.

Therefore if $(\bar{x}, \bar{y})$ be the C.G. of the whole circular arc, then $\bar{y} = 0$, by symmetry about OX.

Also $$\bar{x} = \frac{\int r\cos\theta\cdot kr^2 ds}{\int kr^2\,ds} = \frac{\int_{-\pi/2}^{\pi/2} r^3\cos\theta\cdot\frac{ds}{d\theta}\,d\theta}{\int_{-\pi/2}^{\pi/2} r^2\cdot\frac{ds}{d\theta}\,d\theta}.$$

Now $r = 2\,a\cos\theta$, so that $\dfrac{dr}{d\theta} = -2\,a\sin\theta$.

$$\therefore \qquad \frac{ds}{d\theta} = \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}} = \sqrt{(4a^2\cos^2\theta + 4a^2\sin^2\theta)} = 2a.$$

$$\therefore \qquad \bar{x} = \frac{\int_{-\pi/2}^{\pi/2} 8a^3\cos^3\theta\cdot\cos\theta\cdot 2a\,d\theta}{\int_{-\pi/2}^{\pi/2} 4a^2\cos^2\theta\cdot 2a\,d\theta} = \frac{2a\cdot 2\int_0^{\pi/2}\cos^4\theta\,d\theta}{2\cdot\int_0^{\pi/2}\cos^2\theta\,d\theta}$$

$$= 2a\cdot\frac{\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{1}{2}\pi}{\frac{1}{2}\cdot\frac{1}{2}\pi} = 2a\cdot\left(\frac{3}{4}\right) = \left(\frac{3}{4}\right)\cdot 2a = \left(\frac{3}{4}\right)(OA)\cdot$$

Thus if G be the C.G. of the whole circular arc, we have

$$OG = \bar{x} = \left(\frac{3}{4}\right)OA \text{ and } GA = OA - OG = OA - \left(\frac{3}{4}\right)OA = \frac{1}{4}(OA).$$

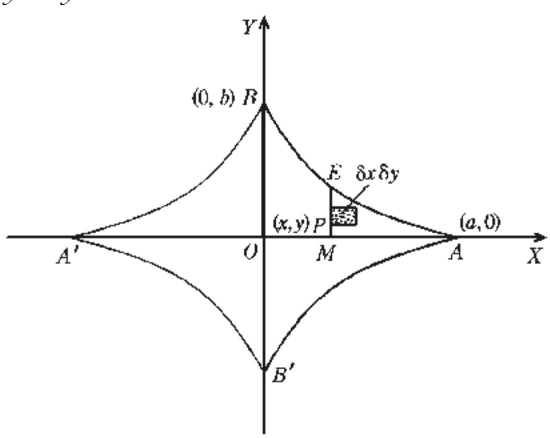
Therefore $OG : GA = \left(\dfrac{3}{4}\right)OA : \dfrac{1}{4}OA = 3:1.$

Example 20: *Find the co-ordinates of the centre of gravity of a lamina in the shape of a quadrant of the curve*

$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} = 1$$

the density being given by $\rho = kxy$.

Solution: The quadrant OAB of the given curve is shown in the figure. Take a small element of area $\delta x\,\delta y$ at any point $P\,(x, y)$ lying within the area OAB. If ρ be the density at P, then according to the question, $\rho = kxy$. Therefore the mass of the element $\delta x\,\delta y = \rho\,\delta x\,\delta y = kxy\,\delta x\,\delta y$.



The element $\delta x\,\delta y$ being very small, its C.G. can be taken as the point P whose co-ordinates are (x, y).

To cover the area OAB the limits of y are 0 to $b\,\{1-(x/a)^{2/3}\}^{3/2}$ and the limits of x are 0 to a.

Therefore if $(\bar{x}, \bar{y})$ be the required C.G. of the area OAB, then

$$\bar{x} = \frac{\int_{x=0}^{a}\int_{y=0}^{b\{1-(x/a)^{2/3}\}^{3/2}} x \cdot kxy\,dx\,dy}{\int_{x=0}^{a}\int_{y=0}^{b\{1-(x/a)^{2/3}\}^{3/2}} kxy\,dx\,dy}$$

$$= \frac{\int_0^a x^2\left[\frac{y^2}{2}\right]_0^{b\{1-(x/a)^{2/3}\}^{3/2}} dx}{\int_0^a x\left[\frac{y^2}{2}\right]_0^{b\{1-(x/a)^{2/3}\}^{3/2}} dx} = \frac{\int_0^a \frac{1}{2}x^2 b^2\left\{1-\left(\frac{x}{a}\right)^{2/3}\right\}^3 dx}{\int_0^a b^2\,\frac{1}{2}\,x\left\{1-\left(\frac{x}{a}\right)^{2/3}\right\}^3 dx}$$

Now put $x = a\sin^3\theta$, so that $dx = 3a\sin^2\theta\cos\theta\,d\theta$.

Also when $x = 0, \theta = 0$ and when $x = a,\ \theta = \frac{\pi}{2}\cdot$

$$\therefore \qquad \bar{x} = \frac{\int_0^{\pi/2} a^2\sin^6\theta\,(1-\sin^2\theta)^3\,.\,3a\sin^2\theta\cos\theta\,d\theta}{\int_0^{\pi/2} a\sin^3\theta\,(1-\sin^2\theta)^3\,.\,3a\sin^2\theta\cos\theta\,d\theta}$$

$$= \frac{a\int_0^{\pi/2}\sin^8\theta\cos^7\theta\,d\theta}{\int_0^{\pi/2}\sin^5\theta\cos^7\theta\,d\theta} = \frac{a\,\dfrac{7.5.3.1.6.4.2}{15.13.11.9.7.5.3.1}}{\dfrac{4.2.6.4.2}{12.10.8.6.4.2}}$$

$$= a\,\frac{6.4.2}{15.13.11.9}\times\frac{12.10.8}{4.2} = \frac{128}{429}a. \quad \therefore \quad \frac{\bar{x}}{a} = \frac{128}{429}\cdot$$

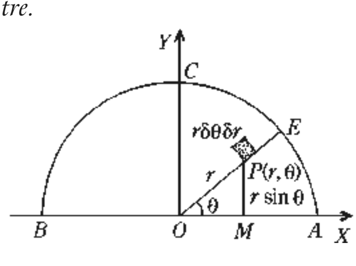
But $\frac{x}{a}$ and $\frac{y}{b}$ are symmetrically placed in the equation of the curve. Therefore by symmetry $\frac{\bar{y}}{b} = \frac{128}{429}\cdot$

Therefore the required C.G. $(\bar{x}, \bar{y})$ is given by

$$\bar{x} = \frac{128}{429}a,\quad \bar{y} = \frac{128}{429}b.$$

Example 21: *Find the centre of gravity of a semi-circular lamina of radius a when the density at any point varies as the square of the distance from the centre.*

Solution: Take the semi-circular area ACB whose central radius OC is along the y-axis and the centre O is the pole. The diameter BA is along the x-axis.



The equation of the bounding circle is $r = a$, a being the radius.

Take a small element of area $r\,\delta\theta\,\delta r$ at any point

$P\ (r, \theta)$ lying within the area ACB whose C.G. is to be found. If ρ be the density at the point P, then according to the question $\rho = \lambda r^2$, where λ is some constant.

Therefore the mass of the element $r\ \delta\theta\ \delta r$

$$= \rho\ .\ r\ \delta\theta\ \delta r = \lambda r^2\ .\ r\ \delta\theta\ \theta r = \lambda r^3\ \delta\theta\ \delta r.$$

The element $r\ \delta\theta\ \delta r$ being very small, its C.G. can be taken as the point P whose cartesian co-ordinates are $(r \cos\theta, r \sin\theta)$.

The semi-circular lamina ACB is symmetrical about OY and its density is also symmetrical about OY as can be easily seen by taking two points of this lamina on opposite sides of OY and at equal distances from it.

Therefore if $(\bar{x}, \bar{y})$ be the required C.G. of the semi-circular lamina ACB, then $\bar{x} = 0$, (by symmetry about the y-axis).

Also $$\bar{y} = \frac{\int_{\theta=0}^{\pi} \int_{r=0}^{a} r \sin\theta \cdot \lambda r^3 d\theta\ dr}{\int_{\theta=0}^{\pi} \int_{r=0}^{a} \lambda r^3\ d\theta\ dr}$$

$$= \frac{\int_{\theta=0}^{\pi} \int_{r=0}^{a} r^4 \sin\theta\ d\theta\ dr}{\int_{\theta=0}^{\pi} \int_{r=0}^{a} r^3\ d\theta\ dr} = \frac{\int_0^{\pi} \left[\frac{r^5}{5}\right]_0^a \sin\theta\ d\theta}{\int_0^{\pi} \left[\frac{r^4}{4}\right]_0^a d\theta}$$

$$= \frac{4a^5 \int_0^{\pi} \sin\theta\ d\theta}{5a^4 \int_0^{\pi} d\theta} = \frac{4a\ [-\cos\theta]_0^{\pi}}{5\ [\theta]_0^{\pi}}$$

$$= \frac{4a\ [-(\cos\pi - \cos 0)]}{5\pi} = \frac{8a}{5\pi}.$$

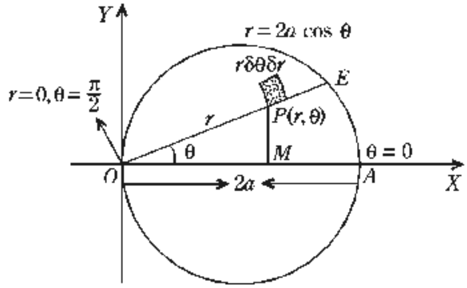
Therefore the required C.G. is given by $\bar{x} = 0,\ \bar{y} = \frac{8a}{5\pi}$.

Hence the C.G. of the lamina lies on the central radius at a distance $\frac{8a}{5\pi}$ from the centre, a being the radius of the lamina.

Example 22: *The density at any point of a circular lamina varies as the n^{th} power of the distance from a point O on the circumference. Show that the centre of gravity of the lamina divides the diameter through O in the ratio $n+2:2$.*

Solution: Let the given point O be taken as the pole and the diameter OA through O as the initial line. Then the equation of the circle is $r = 2\ a \cos\theta$, where the diameter $OA = 2a$.

The circular lamina is symmetrical about the diameter OA. Its density is also symmetrical about OA as can be seen by taking two points of the lamina on the opposite sides of OA and at equal distances from it.



Therefore the C.G. of the lamina lies on the diameter OA.

If $(\bar{x}, \bar{y})$ be the co-ordinates of the C.G. of the whole lamina, then $\bar{y} = 0$, by symmetry about the x-axis.

Also the x-co-ordinate of the C.G. of the whole lamina is the same as the x-co-ordinate of the C.G. of the upper half of this lamina.

Now take a small element of area $r\,\delta\theta\,\delta r$ at any point $P\,(r, \theta)$ lying inside the upper half of the circular lamina. If ρ be the density at the point P, then as given, $\rho = \lambda\,(OP)^n = \lambda r^n$.

Therefore the mass of the element $r\,\delta\theta\,\delta r$ at P

$$= (r\,\delta\theta\,\delta r)\,.\,\lambda r^n = \lambda r^{n+1}\,\delta\theta\,\delta r.$$

The C.G. of this element is the point P whose cartesian co-ordinates are $(r\cos\theta, r\sin\theta)$.

We have $$\bar{x} = \frac{\int_{\theta=0}^{\pi/2}\int_{r=0}^{2a\cos\theta} r\cos\theta\cdot\lambda r^{n+1}\,d\theta\,dr}{\int_{\theta=0}^{\pi/2}\int_{r=0}^{2a\cos\theta} \lambda r^{n+1}\,d\theta\,dr}$$

$$= \frac{\int_0^{\pi/2}\left[\frac{r^{n+3}}{n+3}\right]_0^{2a\cos\theta}\cos\theta\,d\theta}{\int_0^{\pi/2}\left[\frac{r^{n+2}}{n+2}\right]_0^{2a\cos\theta}d\theta} = \frac{(n+2)\cdot 2a\int_0^{\pi/2}\cos^{n+4}\theta\,d\theta}{(n+3)\int_0^{\pi/2}\cos^{n+2}\theta\,d\theta}$$

$$= \frac{2\,a\,(n+2)}{(n+3)}\cdot\frac{\left\{\frac{(n+3)\,(n+1)\,(n-1)\ldots}{(n+4)\,(n+2)\,(n)\ldots}\right\}\times k}{\left\{\frac{(n+1)\,(n-1)\ldots}{(n+2)\,(n)\ldots}\right\}\times k},$$

where k is $\frac{\pi}{2}$ or 1 according as n is even or odd

$$= 2\,a\cdot\frac{(n+2)}{(n+3)}\cdot\frac{(n+3)}{(n+4)}$$

$$= 2a\cdot\frac{n+2}{n+4} = \left(\frac{n+2}{n+4}\right)\cdot 2a = \left(\frac{n+2}{n+4}\right)\cdot OA$$

Hence the centre of gravity G lies on OA and is such that

$$OG = \frac{n+2}{n+4}\,OA\,;$$

$$GA = OA - OG = OA - \frac{n+2}{n+4}\,OA = \frac{2}{(n+4)}\,OA.$$

$\therefore$ $\quad OG : GA = (n+2) : 2$

i.e., G divides OA in the ratio $(n+2) : 2$.

Example 23: *Find the C.G. of a solid hemi-sphere when the density at any point is proportional to the n^{th} power of the distance from the centre. Hence show that the C.G. divides the radius perpendicular to its plane surface in the ratio $(n+3):(n+5)$.*

Solution: Let the hemi-sphere of radius a be generated by revolving the quadrant OAB of the circle $r = a$ about the x-axis. The line OB will generate the plane base of the hemi-sphere and the line OA will be the axis of the hemi-sphere. The centre of the hemi-sphere is at the pole O.

Take a small element of area $r\,\delta\theta\,\delta r$ at any point $P\,(r,\theta)$ lying within the area of the quadrant OAB. When the elementary area $r\,\delta\theta\,\delta r$ is revolved about OA, a circular ring of radius PM $(= r\sin\theta)$ is generated.

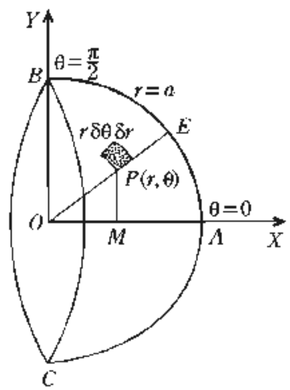
The volume of this elementary ring

$$= (2\pi PM)\, r\,\delta\theta\,\delta r$$
$$= (2\pi r\sin\theta)\, r\,\delta\theta\,\delta r = 2\pi r^2 \sin\theta\,\delta\theta\,\delta r.$$

The distance of each point of this ring from the centre O is r. So according to the question, the density ρ at each point of it is λr^{n}.

$\therefore$ the mass of this ring $= 2\,\pi r^2 \sin\theta\,\delta\theta\,\delta r\,.\rho$

$$= 2\,\pi r^2 \sin\theta\,\delta\theta\,\delta r\,(\lambda r^{n})$$
$$= 2\,\pi\lambda r^{n+2} \sin\theta\,\delta\theta\,\delta r.$$



Its C.G. can be taken at the point M whose cartesian co-ordinates are $(r\cos\theta, 0)$.

If $(\bar{x}, \bar{y})$ be the required C.G. of the hemi-sphere, then $\bar{y} = 0$, (by symmetry about the axis of rotation *i.e.*, the line OX).

Also
$$\bar{x} = \frac{\int_{\theta=0}^{\pi/2}\int_{r=0}^{a} r\cos\theta\cdot 2\pi\lambda r^{n+2}\sin\theta\,d\theta\,dr}{\int_{\theta=0}^{\pi/2}\int_{r=0}^{a} 2\pi\lambda r^{n+2}\sin\theta\,d\theta\,dr}$$

$$= \frac{\int_0^{\pi/2}\left[\frac{r^{n+4}}{n+4}\right]_0^a \cos\theta\sin\theta\,d\theta}{\int_0^{\pi/2}\left[\frac{r^{n+3}}{n+3}\right]_0^a \sin\theta\,d\theta} = \frac{(n+3)\cdot a\int_0^{\pi/2}\sin\theta\cos\theta\,d\theta}{(n+4)\int_0^{\pi/2}\sin\theta\,d\theta}$$

$$= \frac{(n+3)\,a\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2}}{(n+4)\,[-\cos\theta]_0^{\pi/2}} = \frac{(n+3)\,a}{(n+4)}\cdot\frac{1}{2} = \frac{(n+3)\,a}{2\,(n+4)}.$$

$\therefore$ the required C.G. is given by

$$\bar{x} = \{(n+3)\,a/2\,(n+4)\},\ \bar{y} = 0.$$

Also if G be the C.G., then

$$OG = \bar{x} = \{(n+3)\, a \,/\, 2\,(n+4)\}$$

and
$$GA = OA - OG = a - \frac{a}{2} \cdot \frac{n+3}{n+4} = \frac{a}{2} \cdot \frac{n+5}{n+4}.$$

$\therefore$
$$OG : GA = (n+3) : (n+5).$$

Comprehensive Exercise 5

1. Find the centre of gravity of a plate in the form of a quadrant of an ellipse, the density at any point of the plate varying as the product of the distances of the point from the major and minor axes.

2. Find the centre of gravity of a semi-circular lamina of radius a when the density at any point :

(i) varies as the cube of the distance from the centre.

(ii) varies as the distance from the centre **(Garhwal 2004)**

(iii) varies as the square of the distance from the diameter.

(iv) varies as $\sqrt{(a^2 - r^2)}$, where r is the distance of the point from the centre.

3. Find the C.G. of a semi-circular area when the density varies as the distance from one end of the bounding diameter.

4. Find the distance of the centre of gravity of the area of the cardioid $r = a\,(1 + \cos\theta)$ from the cusp, when the density varies as the n^{th} power of the distance from O.

5. Find the centre of gravity of a solid hemi-sphere when the density at any point :

(i) varies directly as the distance from the centre,

(ii) varies inversely as the distance from the centre,

(iii) varies directly as the square of the distance from the centre,

(iv) varies as the square of the distance from the centre of the whole sphere.

6. Show that the C.G. of a sphere, the density at any point of which varies inversely as the square of the distance from a fixed point on the surface of the sphere, bisects the radius through the fixed point.

Answers 5

1. $\bar{x} = \dfrac{8}{15} a,\ \bar{y} = \dfrac{8}{15} b$

2. (i) $\bar{x} = 0,\ \bar{y} = \dfrac{5a}{3\pi}$ (ii) $\bar{x} = 0,\ \bar{y} = \dfrac{3a}{2\pi}$

(iii) $\bar{x} = 0,\ \bar{y} = \dfrac{32\,a}{15\pi}$ (iv) $\bar{x} = 0,\ \bar{y} = 3a$

3. $\bar{x}=\frac{6}{5}a,\ \bar{y}=\frac{9}{20}a$

4. $\bar{x}=a\left(\frac{n+2}{n+3}\right)\left(\frac{2n+5}{n+4}\right),\ \bar{y}=0$

5. (i) $\bar{x}=\frac{2}{5}a,\ \bar{y}=0$ (ii) $\bar{x}=\frac{1}{3}a,\ \bar{y}=0$

(iii) $\bar{x}=\frac{5}{12}a,\ \bar{y}=0$ (iv) same as part (iii)

1.8 Centre of Gravity in three Dimensions. Use of Multiple Integrals to Find the Centre of Gravity of any Volume

Suppose we are to find the centre of gravity of any volume V. Divide the volume V into a large number of small elements. Let $dx\ dy\ dz$ be the volume of an elementary portion of V situated at the point $P(x, y, z)$. If ρ be the density of V at the point P, then the mass of this small element is $\rho\ dx\ dy\ dz$. The C.G. of this small element can be taken as the point $P(x, y, z)$. If $(\bar{x}, \bar{y}, \bar{z})$ be the C.G. of the whole volume V, then

$$\bar{x}=\frac{\iiint \rho x\ dx\ dy\ dz}{\iiint \rho\ dx\ dy\ dz},\ \bar{y}=\frac{\iiint \rho y\ dx\ dy\ dz}{\iiint \rho\ dx\ dy\ dz},\ \bar{z}=\frac{\iiint \rho z\ dx\ dy\ dz}{\iiint \rho\ dx\ dy\ dz}.$$

The limits of integration are to be so taken as to cover the whole volume V.

If the density of the volume V is uniform, ρ can be cancelled from the numerator and denominator in the above formulae.

Illustrative Examples

Example 24: *Find the centre of gravity of the positive octant of the ellipsoid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, *which is of constant density.* **(Agra 2008)**

Solution: Let $(\bar{x}, \bar{y}, \bar{z})$ be the centroid of the given volume.

We have
$$\bar{x}==\frac{\iiint x\ dx\ dy\ dz}{\iiint dx\ dy\ dz},$$

where x, y, z have any positive values subject to the condition

$$\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2\le 1.$$

Put
$$\left(\frac{x}{a}\right)^2=u\ \ i.e.,\ x=au^{1/2}\ \text{so that}\ dx=a\cdot\frac{1}{2}u^{(1/2)-1}\ du,$$

$$\left(\frac{y}{b}\right)^2 = v \quad i.e., \quad y = bv^{1/2} \text{ so that } dy = b \cdot \frac{1}{2} v^{(1/2)-1}\, dv,$$

and
$$\left(\frac{z}{c}\right)^2 = w \quad i.e., \quad z = c\, w^{1/2} \text{ so that } dz = c \cdot \frac{1}{2} w^{\,(1/2)-1}\, dw.$$

Then
$$\bar{x} = \frac{\iiint au^{1/2} \cdot abc \cdot \frac{1}{8} u^{(1/2)-1}\, v^{(1/2)-1}\, w^{(1/2)-1}\, du\, dv\, dw}{\iiint abc \cdot \frac{1}{8} u^{(1/2)-1}\, v^{(1/2)-1}\, w^{(1/2)-1}\, du\, dv\, dw}$$

where u, v, w have any positive values subject to the condition $u + v + w \le 1$

$$= a \frac{\iiint u^{1-1}\, v^{(1/2)-1} w^{(1/2)-1}\, du\, dv\, dw}{\iiint u^{(1/2)-1}\, v^{(1/2)-1}\, w^{(1/2)-1}\, du\, dv\, dw}$$

$$= a \frac{\Gamma(1)\, \Gamma(\frac{1}{2})\, \Gamma(\frac{1}{2})}{\Gamma(1 + \frac{1}{2} + \frac{1}{2} + 1)} \div \frac{\Gamma(\frac{1}{2})\, \Gamma(\frac{1}{2})\, \Gamma(\frac{1}{2})}{\Gamma(\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + 1)}, \text{ by Dirichlet's integrals}$$

$$= a \cdot \frac{\Gamma(1)}{\Gamma(3)} \times \frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}$$

$$= a \cdot \frac{1}{2.1} \cdot \frac{\frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}}{\sqrt{\pi}} = \frac{3}{8} a.$$

Similarly $\bar{y} = \frac{3}{8} b, \bar{z} = \frac{3}{8} c.$

Example 25: *Find the centre of gravity of the volume cut off from the cylinder* $x^2 + y^2 - 2\,ax = 0$ *by the planes* $z = mx$ *and* $z = nx$.

Solution: Let $(\bar{x}, \bar{y}, \bar{z})$ be the centroid of the given volume. Obviously the given volume is symmetrical about the plane XOZ *i.e.*, the plane $y = 0$. [Note that in the equation $x^2 + y^2 - 2\,ax = 0$, the powers of y are all even]. Therefore $\bar{y} = 0$.

We have, $\bar{x} = \frac{\iiint x\, dx\, dy\, dz}{\iiint dx\, dy\, dz}, \quad \bar{z} = \frac{\iiint z\, dx\, dy\, dz}{\iiint dx\, dy\, dz}.$

The limits of integration for z are from mx to nx, for y are from $-\sqrt{(2ax - x^2)}$ to $\sqrt{(2ax - x^2)}$, and for x are from 0 to $2a$.

[Note that from the equation $x^2 + y^2 - 2ax = 0$, we get $y = \pm\sqrt{(2ax - x^2)}$, giving the limits for y, and if we put $y = 0$ in this equation, we get $x^2 - 2ax = 0$, giving $x = 0$ and $2a$ as the limits for x.]

Now $\quad \bar{x} = \dfrac{\int\int x\,[z]_{mx}^{nx}\,dx\,dy}{\int\int [z]_{mx}^{nx}\,dx\,dy}$, integrating w.r.t. z

$$= \frac{\int_{x=0}^{2a}\int_{y=-\sqrt{(2ax-x^2)}}^{\sqrt{(2ax-x^2)}} (n-m)\,x^2\,dx\,dy}{\int_{x=0}^{2a}\int_{y=-\sqrt{(2ax-x^2)}}^{\sqrt{(2ax-x^2)}} (n-m)\,x\,dx\,dy}$$

$$= \frac{\int_0^{2a} x^2 \cdot 2\sqrt{(2ax-x^2)}\,dx}{\int_0^{2a} x \cdot 2\sqrt{(2ax-x^2)}\,dx}, \text{ integrating w.r.t. } y$$

$$= \frac{\int_0^{2a} x^{5/2}\cdot\sqrt{(2a-x)}\,dx}{\int_0^{2a} x^{3/2}\cdot\sqrt{(2a-x)}\,dx}$$

$$= \frac{\int_0^{\pi/2} (2a\sin^2\theta)^{5/2}(2a\cos^2\theta)^{1/2}\cdot 4a\sin\theta\cos\theta\,d\theta}{\int_0^{\pi/2} (2a\sin^2\theta)^{3/2}(2a\cos^2\theta)^{1/2}\cdot 4a\sin\theta\cos\theta\,d\theta}$$

putting $x = 2a\sin^2\theta$, so that $dx = 4a\sin\theta\cos\theta\,d\theta$

$$= 2a\,\frac{\int_0^{\pi/2}\sin^6\theta\cos^2\theta\,d\theta}{\int_0^{\pi/2}\sin^4\theta\cos^2\theta\,d\theta}$$

$$= 2a\,\frac{\frac{5.3.1.1}{8.6.4.2}\cdot\frac{\pi}{2}}{\frac{3.1.1}{6.4.2}\cdot\frac{\pi}{2}} = 2a\cdot\frac{5}{8} = \frac{5a}{4}.$$

Again $\quad \bar{z} = \dfrac{\int\int x\left[\dfrac{z^2}{2}\right]_{mx}^{nx} dx\,dy}{\int\int [z]_{mx}^{nx}\,dx\,dy}$, integrating w.r.t. z

$$= \frac{\int\int x\cdot\frac{1}{2}(n^2x^2 - m^2x^2)\,dx\,dy}{\int\int (nx-mx)\,dx\,dy}$$

$$= \frac{1}{2}(n+m)\frac{\int\int (n-m)\,x^2\,dx\,dy}{\int\int (n-m)\,x\,dx\,dy}$$

$$= \frac{1}{2}(n+m)\,.\,\bar{x}$$

$$= \frac{1}{2}(n+m)\,.\,\frac{5}{4}a$$

$$= \frac{5}{8}(n+m)\,a.$$

Hence $\bar{x} = \frac{5}{4}a,\ \bar{y} = 0,\ \bar{z} = \frac{5}{8}a\,(n+m).$

Example 26: *Find the centroid of the mass which is in the positive octant of the ellipsoid* $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, *where the density* ρ *is given* $\rho = \mu\, x^p y^q z^r$. **(Agra 2007)**

Solution: Let $(\bar{x}, \bar{y}, \bar{z})$ be the centroid of the given mass. We have

$$\bar{x} = \frac{\iiint x\rho\, dx\, dy\, dz}{\iiint \rho\, dx\, dy\, dz}$$

$$= \frac{\iiint x\,\mu\, x^p y^q z^r\, dx\, dy\, dz}{\iiint \mu\, x^p y^q z^r\, dx\, dy\, dz}$$

$$= \frac{\iiint x^{p+1} y^q z^r\, dx\, dy\, dz}{\iiint x^p y^q z^r\, dx\, dy\, dz},$$

where x, y, z have any positive values subject to the condition $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1.$

Put $\frac{x^2}{a^2} = u$ i.e., $x = au^{1/2}$ so that $dx = a \cdot \frac{1}{2} u^{(1/2)-1}\, du,$

$\frac{y^2}{b^2} = v$ i.e., $y = bv^{1/2}$ so that $dy = b \cdot \frac{1}{2} v^{(1/2)-1}\, dv,$

and $\frac{z^2}{c^2} = w$ i.e., $z = c\, w^{1/2}$ so that $dz = c \cdot \frac{1}{2} w^{(1/2)-1}\, dw.$

$$\therefore \quad \bar{x} = \frac{\iiint a^{p+1} u^{(p+1)/2}\, b^q\, v^{q/2}\, c^r\, w^{r/2}\, abc \cdot \frac{1}{8} u^{(1/2)-1} v^{(1/2)-1} w^{(1/2)-1}\, du\, dv\, dw}{\iiint a^p\, u^{p/2} b^q\, v^{q/2} c^r\, w^{r/2} abc \cdot \frac{1}{8} u^{(1/2)-1} v^{(1/2)-1} w^{(1/2)-1} du\, dv\, dw}$$

where u, v, w have any positive values subject to the condition $u + v + w \le 1$

$$= a \frac{\iiint u^{[(p+2)/2]-1}\, v^{[(q+1)/2]-1}\, w^{[(r+1)/2]-1}\, du\, dv\, dw}{\iiint u^{[(p+1)/2]-1}\, v^{[(q+1)/2]-1}\, w^{[(r+1)/2]-1}\, du\, dv\, dw}$$

$$= a \frac{\Gamma\left(\frac{p+2}{2}\right)\Gamma\left(\frac{q+1}{2}\right)\Gamma\left(\frac{r+1}{2}\right)}{\Gamma\left(\frac{p+q+r+4}{2}+1\right)} \div \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)\Gamma\left(\frac{r+1}{2}\right)}{\Gamma\left(\frac{p+q+r+3}{2}+1\right)}$$

by Dirichlet's integrals

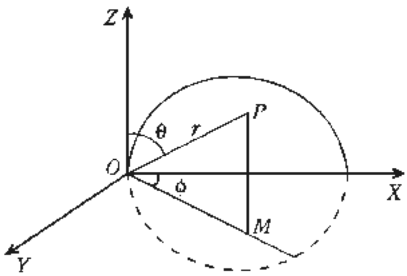
$$= a \frac{\Gamma\left(\frac{p+2}{2}\right)\Gamma\left(\frac{p+q+r+5}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{p+q+r}{2}+3\right)}.$$

Similarly $$\bar{y} = b\,\frac{\Gamma\left(\frac{q+2}{2}\right)\Gamma\left(\frac{p+q+r+5}{2}\right)}{\Gamma\left(\frac{q+1}{2}\right)\Gamma\left(\frac{p+q+r}{2}+3\right)},$$

and $$\bar{z} = c\,\frac{\Gamma\left(\frac{r+2}{2}\right)\Gamma\left(\frac{p+q+r+5}{2}\right)}{\Gamma\left(\frac{r+1}{2}\right)\Gamma\left(\frac{p+q+r}{2}+3\right)}.$$

Example 27: *Find the centre of gravity of a hemi-sphere whose density varies as the distance from a point on its plane edge.*

Solution: Take a point O on the plane edge of a hemi-sphere of radius a as the origin, the diameter through O of the plane base of the hemi-sphere as x-axis, the line OY perpendicular to OX lying in the plane base of the hemi-sphere as y-axis and the line OZ through O perpendicular to the plane base of the hemi-sphere as z-axis.



The cartesian equation of the hemi-sphere is

$$(x-a)^2 + y^2 + z^2 = a^2. \qquad \ldots(1)$$

Let P be any point in the volume of the hemi-sphere such that $OP = r$, $\angle ZOP = \theta$. Draw PM perpendicular to the plane base of the hemi-sphere which is obviously in the xy-plane.

Let $\angle XOM = \phi$.

Then the polar co-ordinates of P are (r, θ, ϕ).

The relations between the cartesian co-ordinates (x, y, z) and the polar co-ordinates (r, θ, ϕ) are

$$z = r\cos\theta,$$
$$x = r\sin\theta\cos\phi,$$
$$y = r\sin\theta\sin\phi.$$

The polar equation of the hemi-sphere (1) is

$$(r\sin\theta\cos\phi - a)^2 + r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta = a^2$$

or $$r^2\sin^2\theta\cos^2\phi - 2\,ar\sin\theta\cos\phi + a^2 + r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta = a^2$$

or $$r^2\sin^2\theta(\cos^2\phi + \sin^2\phi) + r^2\cos^2\theta = 2\,ar\sin\theta\cos\phi$$

or $$r^2 = 2\,ar\sin\theta\cos\phi$$

or $$r = 2\,a\sin\theta\cos\phi.$$

For the volume of the hemi-sphere the limits for r are 0 to $2a \sin\theta \cos\phi$, those for ϕ are $-\dfrac{\pi}{2}$ to $\dfrac{\pi}{2}$ and those for θ are 0 to $\dfrac{\pi}{2}\cdot$

If the density at any point $P(r, \theta, \phi)$ in the volume of the hemi-sphere is λr, of the mass the small element at P is

$$\delta r \,.\, r\,\delta\theta \,.\, r \sin\theta\,\delta\phi \,.\, \lambda r = \lambda r^3 \sin\theta\,\delta r\,\delta\theta\,\delta\phi.$$

Let $(\bar{x}, \bar{y}, \bar{z})$ be the co-ordinates of the C.G. of the hemi-sphere .

We have $$\bar{x} = \frac{\iiint \lambda\, r^3 \sin\theta\, dr\, d\theta\, d\phi \,.\, r\cos\phi \sin\theta}{\iiint \lambda\, r^3 \sin\theta\, dr\, d\theta\, d\phi}$$

$$= \frac{\frac{1}{5}\iint (2\,a\cos\phi\sin\theta)^5 \sin^2\theta\cos\phi\, d\theta\, d\phi}{\frac{1}{4}\iint (2\,a\cos\phi\sin\theta)^4 \sin\theta\, d\theta\, d\phi}$$

[Integrating w.r.t. r between the limits $r = 0$ to $r = 2\,a\cos\phi\sin\theta$]

$$= \frac{8a}{5}\frac{\iint \cos^6\phi\sin^7\theta\, d\theta\, d\phi}{\iint \cos^4\phi\sin^5\theta\, d\theta\, d\phi}$$

$$= \frac{8a}{5}\cdot\frac{\frac{5.3.1}{6.4.2}\cdot\frac{\pi}{2}\times\frac{6.4.2}{7.5.3}}{\frac{3.1}{4.2}\cdot\frac{\pi}{2}\times\frac{4.2}{5.3}}$$

$$= \frac{8a}{7}\cdot$$

Clearly $\bar{y} = 0$, by symmetry.

Also $$\bar{z} = \frac{\iiint \lambda\, r^3 \sin\theta\, dr\, d\theta\, d\phi \cdot r\cos\theta}{\iiint \lambda r^3 \sin\theta\, dr\, d\theta\, d\phi}$$

$$= \frac{\frac{1}{5}\iint (2\,a\cos\phi\sin\theta)^5 \sin\theta\cos\theta\, d\theta\, d\phi}{\frac{1}{4}\iint (2\,a\cos\phi\sin\theta)^4 \sin\theta\, d\theta\, d\phi}$$

$$= \frac{8a}{5}\frac{\iint \cos^5\phi\sin^6\theta\cos\theta\, d\theta\, d\phi}{\iint \cos^4\phi\sin^5\theta\, d\theta\, d\phi}$$

$$= \frac{8a}{5}\cdot\frac{\frac{4.2}{5.3.1}\cdot\frac{1}{7}}{\frac{3.1}{4.2}\cdot\frac{\pi}{2}\cdot\frac{4.2}{5.3.1}}$$

$$= \frac{128\,a}{105\pi}\cdot$$

Hence, $\bar{x} = \dfrac{8a}{7},\ \bar{y} = 0,\ \bar{z} = \dfrac{128\,a}{105\pi}\cdot$

Comprehensive Exercise 6

1. Find the centre of gravity of the volume cut off from the cylinder $2x^2 + y^2 = 2ax$ by the planes $z = mx, z = nx$.

2. Find the centroid of the volume included between the following surfaces :

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, z = 0$$

and $lx + my + nz = 1$.

3. Find the C.G. of the solid bounded by the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

in the positive octant, if the density at any point varies as $xy^2 z^3$.

4. Find the centroid of the volume contained in the positive octant by

$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} + \left(\frac{z}{c}\right)^{2/3} = 1.$$ **(Agra 2009, 11; Kumaun 02; Garhwal 01)**

5. Find the centre of gravity of the volume cut off from the first octant by the plane

$$x + 2y + 4z = 8.$$

6. Find the centroid of the volume included between the following surfaces :

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, x = 0 \text{ and } z = \pm c.$$

(Agra 2000)

7. Find the centroid of the volume included between the following surfaces :

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} - \frac{2x}{a} = 0, x = 2a, y = 0 \text{ and } z = 0.$$

Answers 6

1. $\bar{x} = \frac{5a}{8}, \ \bar{y} = 0, \ \bar{z} = \frac{5a}{16}(m + n)$

2. $\bar{x} = -\frac{a^2 l}{4}, \ \bar{y} = -\frac{b^2 m}{4}, \ \bar{z} = \frac{l^2 a^2 + m^2 b^2 + 4}{8n}$

3. $\bar{x} = \frac{63\,\pi a}{512}, \ \bar{y} = \frac{63b}{128}, \ \bar{z} = \frac{189\pi c}{1024}$

4. $\bar{x} = \frac{21a}{128}, \ \bar{y} = \frac{21b}{128}, \ \bar{z} = \frac{21c}{128}$

5. $\bar{x} = 2, \bar{y} = 1, \bar{z} = \frac{1}{2}$

6. $\bar{x} = \frac{a}{8\pi}\{7\sqrt{2} + 3 \log(1+\sqrt{2})\}, \bar{y} = 0, \bar{z} = 0$

7. $\bar{x} = \frac{4a}{3}, \ \bar{y} = \frac{32b}{15\pi}, \ \bar{z} = \frac{32c}{15\pi}$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. If $(\bar{x}, \bar{y})$ be the centre of gravity of a plane area of uniform density bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$ and $x = b$, then

(a) $\bar{y} = \dfrac{\int_a^b y^2\,dx}{\int_a^b y\,dx}$ (b) $\bar{y} = \dfrac{\frac{1}{2}\int_a^b y^2\,dx}{\int_a^b y\,dx}$

(c) $\bar{y} = \dfrac{\int_a^b xy\,dx}{\int_a^b y\,dx}$ (d) $\bar{y} = \dfrac{\frac{1}{2}\int_a^b xy\,dx}{\int_a^b y\,dx}$

2. The distance of the centre of gravity of a sector of a circle of radius a subtending an angle 2α at the centre of the circle from the centre of the circle is

(a) $\frac{2}{3}\frac{a \sin\alpha}{\alpha}$ (b) $\frac{2}{5}\frac{a \sin\alpha}{\alpha}$

(c) $\frac{1}{3}\frac{a \sin\alpha}{\alpha}$ (d) $\frac{1}{5}\frac{a \sin\alpha}{\alpha}$

3. If $(\bar{x}, \bar{y})$ be the co-ordinates of the C.G. of the area of a loop of the curve $r = a \cos 2\theta$, then

(a) $\bar{x} = \dfrac{\frac{1}{3}\int_{-\pi/4}^{\pi/4} r^3 \cos\theta\,d\theta}{\int_{-\pi/4}^{\pi/4} r^2\,d\theta}$ (b) $\bar{x} = \dfrac{\frac{2}{3}\int_{-\pi/4}^{\pi/4} r^3 \sin\theta\,d\theta}{\int_{-\pi/4}^{\pi/4} r^2\,d\theta}$

(c) $\bar{x} = \dfrac{\frac{2}{3}\int_{-\pi/4}^{\pi/4} r^3 \cos\theta\,d\theta}{\int_{-\pi/4}^{\pi/4} r^2\,d\theta}$ (d) $\bar{x} = 0$.

4. If $(\bar{x}, \bar{y})$ be the centre of gravity of a solid formed by revolving the curve $x = f(y)$ about the y-axis and cut off between the plane ends $y = a$ and $y = b$, then

(a) $\bar{y} = \dfrac{\int_a^b xy^2\,dx}{\int_a^b y^2\,dx}$ (b) $\bar{y} = \dfrac{\int_a^b yx^2\,dy}{\int_a^b y^2\,dy}$

(c) $\bar{y} = 0$

(d) $\bar{y} = \dfrac{\int_a^b yx^2\, dy}{\int_a^b x^2\, dy}$.

5. The distance of the centre of gravity of a uniform semi-circular disc of radius a from the centre of the disc is

(a) $\dfrac{4a}{3\pi}$

(b) $\dfrac{2a}{3\pi}$

(c) $\dfrac{a}{2}$

(d) $\dfrac{a}{4}$

(Agra 2011)

6. The centre of gravity of a semi-circular arc of radius a, lies at the following distance from its centre :

(a) $\dfrac{2a}{\pi}$

(b) $\dfrac{\pi}{2a}$

(c) $\dfrac{\pi}{2}$

(d) $\dfrac{a}{2\pi}$

(Garhwal 2002)

7. The centre of gravity of a semi-circular lamina of radius r from its centre on the central axis, is at the following distance

(a) $\dfrac{3\pi}{4r}$

(b) $\dfrac{4r}{3\pi}$

(c) $\dfrac{3r}{8}$

(d) $\dfrac{r}{2}$

(Garhwal 2003)

8. The C.G. of the area of the first quadrant of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is

(a) $\bar{x} = \bar{y} = 0$

(b) $\bar{x} = 0,\ \bar{y} \neq 0$

(c) $\bar{x} = 0,\ \bar{y} = \dfrac{4b}{3\pi}$

(d) $\bar{x} = \dfrac{4a}{3\pi},\ \bar{y} = \dfrac{4b}{3\pi}$.

(Garhwal 2004)

Fill in the Blank(s)

Fill in the blanks "......" so that the following statements are complete and correct.

1. The centre of gravity of a body is the point, fixed relative to the body, through which the line of action of the ...... of the body, always passes, whatever be the position of the body, provided that its size and shape remain unaltered.
2. If $(\bar{x}, \bar{y})$ be the co-ordinates of the centre of gravity of a given arc AB of some plane curve of uniform density, then

$$\bar{x} = \ldots\ldots \text{ and } \bar{y} = \ldots\ldots.$$

3. If $(\bar{x}, \bar{y})$ be the centre of gravity of the arc of the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ which lies in the first quadrant and $\bar{x} = \dfrac{2a}{5}$, then $\bar{y} = \ldots\ldots.$
4. If $(\bar{x}, \bar{y})$ be the centre of gravity of the whole arc of the cardioid

$$r = a(1 + \cos\theta), \text{ then } \bar{y} = \ldots\ldots.$$

5. If $(\bar{x}, \bar{y})$ be the centre of gravity of a plane lamina which is symmetrical about the axis of y, then $\bar{x} = \ldots\ldots.$

6. If $(\bar{x}, \bar{y})$ be the co-ordinates of the centre of gravity of an area A of uniform density, then

$$\bar{x} = \frac{\ldots\ldots}{\int dA} \quad \text{and} \quad \bar{y} = \frac{\ldots\ldots}{\int dA}$$

7. If $(\bar{x}, \bar{y})$ be the C.G. of a plane area of uniform density bounded by the curve $x = f(y)$, the y-axis and the abscissae $y = a$ and $y = b$, then

$$\bar{x} = \frac{\ldots\ldots}{\int_a^b x\, dy} \quad \text{and} \quad \bar{y} = \frac{\ldots\ldots}{\int_a^b x\, dy}$$

8. If $(\bar{x}, \bar{y})$ be the C.G. of a sectorial area of uniform density bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \theta_1$ and $\theta = \theta_2$, then

$$\bar{x} = \frac{\ldots\ldots}{\int_{\theta_1}^{\theta_2} r^2\, d\theta}$$

9. The distance of the centre of gravity of an arc of a circle of radius a subtending an angle 2α at the centre of the circle from the centre of the circle is ....

10. If $(\bar{x}, \bar{y})$ be the co-ordinates of the centre of gravity of the area of the parabola $y^2 = 4ax$ bounded by the latus rectum, then $\bar{y} = \ldots\ldots$

11. If $(\bar{x}, \bar{y})$ be the centre of gravity of a solid of uniform density formed by revolving the curve $y = f(x)$ about the x-axis and cut off between the plane ends $x = a$ and $x = b$, then

$$\bar{x} = \frac{\ldots\ldots}{\int_a^b y^2\, dx} \quad \text{and} \quad \bar{y} = \ldots\ldots$$

12. If $(\bar{x}, \bar{y})$ be the centroid of the volume formed by the revolution of the cardioid $r = a(1 + \cos\theta)$ about the x-axis, then $\bar{y} = \ldots\ldots$

13. If $(\bar{x}, \bar{y})$ be the centre of gravity of the surface of uniform density formed by revolving the curve $y = f(x)$ about the x-axis and cut off between the planes $x = a$ and $x = b$, then

$$\bar{x} = \frac{\int xy\, ds}{\ldots\ldots} \quad \text{and} \quad \bar{y} = \ldots\ldots$$

14. If $(\bar{x}, \bar{y})$ be the centre of gravity of the surface formed by the revolution of the cycloid

$$x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta)$$

about the axis of y, then $\bar{x} = \ldots\ldots$

15. The distance of the C.G. of a uniform solid hemi-sphere of radius a from its centre is ....

True or False

Write 'T' for true and 'F' for false statement.

1. The centre of gravity of a body does not necessarily lie in the body itself.

2. The distance of the centre of gravity of a uniform hemi-spherical shell of radius a from the centre of the shell is $\frac{a}{4}$.

3. If $(\bar{x}, \bar{y})$ be the C.G. of the whole area of the cardioid $r = a(1 + \cos\theta)$, then $\bar{y} = 0$.

4. If $(\bar{x}, \bar{y})$ be the C.G. of the arch of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ for which $-\pi \le \theta \le \pi$, then $\bar{y} = 0$.

5. If $(\bar{x}, \bar{y})$ be the C.G. of a plane area of uniform density bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$ and $x = b$, then

$$\bar{x} = \frac{\int_a^b xy\, dx}{\int_a^b y\, dx}.$$

6. If $(\bar{x}, \bar{y})$ be the C.G. of a sectorial area bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \alpha$ and $\theta = \beta$, then

$$\bar{x} = \frac{\frac{2}{3}\int_\alpha^\beta r^3 \sin\theta\, d\theta}{\int_\alpha^\beta r^2\, d\theta}$$

7. If $(\bar{x}, \bar{y})$ be the C.G. of the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ of uniform density lying in the first quadrant, then

$$\bar{y} = \frac{\int_{x=0}^{a} y^2\, dx}{\int_{x=0}^{a} y\, dx}.$$

8. If $(\bar{x}, \bar{y})$ be the centroid of the volume formed by the revolution of the cardioid $r = a(1 + \cos\theta)$ about the x-axis, then $\bar{x} = 0$.

9. If $(\bar{x}, \bar{y})$ be the centroid of the surface formed by the revolution of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$ about the axis of y, then $\bar{x} = 0$.

10. The centre of gravity of the area of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ lying in the positive quadrant lies on the line $y = x$.

Answers

Multiple Choice Questions

1. (b)
2. (a)
3. (c)
4. (d)
5. (a)
6. (a)
7. (b)
8. (d)

Fill in the Blank(s)

1. weight
2. $\dfrac{\int x\, ds}{\int ds}, \dfrac{\int y\, ds}{\int ds}$
3. $\dfrac{2a}{5}$
4. 0
5. 0
6. $\int x\, dA, \int y\, dA$
7. $\int_a^b \dfrac{x^2}{2}\, dy, \int_a^b yx\, dy$
8. $\dfrac{2}{3}\int_{\theta_1}^{\theta_2} r^3 \cos\theta\, d\theta$
9. $a\dfrac{\sin\alpha}{\alpha}$
10. 0
11. $\int_a^b xy^2\, dx, 0$
12. 0
13. $\int y\, ds, 0$
14. 0
15. $\dfrac{3a}{8}$

True or False

1. *T*
2. *F*
3. *T*
4. *F*
5. *T*
6. *F*
7. *F*
8. *F*
9. *T*
10. *T*

Chapter

2

Strings in Two Dimensions

(Common Catenary)

2.1 Introduction

In the present chapter we shall consider the equilibrium of perfectly flexible strings. All those strings which offer no resistance on bending at any point are called **flexible strings.** In such cases, the resultant action across any section of the string consists of a single force whose line of action is along the tangent to the curve formed by the string. The normal section of the string is taken to be so small that it may be regarded as a curved line. A chain with short and perfectly smooth links approximates a flexible string.

2.2 The Catenary

When a uniform string or chain hangs freely under gravity between two points not in the same vertical line, the curve in which it hangs, is called a **catenary.**

(Agra 2007)

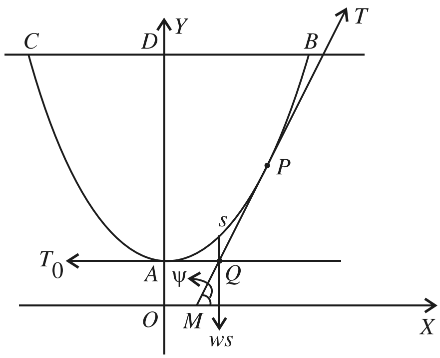
A Uniform or Common Catenary. If the weight per unit length of the suspended flexible string or chain is constant, then the catenary is called the **uniform or common catenary.** **(Kumaun 2000; Lucknow 07, 08)**

Note: Here we shall frequently use the word 'catenary' for the common catenary *i.e.* the word 'catenary' will always mean the common catenary in this chapter.

2.3 Intrinsic Equation of the Common Catenary

(Agra 2006, 08, 09, 11; Rohilkhand 07, 10; Lucknow 08; Bundelkhand 09, 10, 11)

Let the uniform flexible string BAC hang in the form of a uniform catenary with A as its lowest point. Let P be any point on the portion AB of the string and s the distance of P from A measured along the arc length of the string. If w is the weight per unit length of the string, then the weight of the portion AP will be ws and will act vertically downwards through the centre of gravity of AP.



The portion AP of the string is in equilibrium under the action of the following three forces :

1. The weight ws of the string AP acting vertically downwards through its centre of gravity,
2. The tension T_0 at the lowest point A acting along the tangent to the curve at A which is horizontal,
3. The tension T at P acting along the tangent to the curve at P inclined at an angle ψ to the horizontal.

Since the string AP is in equilibrium under the action of the three forces acting in the same vertical plane therefore the line of action of the weight ws must pass through the point Q which is the point of intersection of the lines of action of the tensions T_0 and T.

Resolving the forces acting on AP horizontally and vertically,

we have $\quad T \cos \psi = T_0$, ...(1)

and $\quad T \sin \psi = ws$. ...(2)

Dividing (2) by (1), we have

$$\tan \psi = ws/T_0 . \qquad ...(3)$$

Now let $\quad T_0 = wc$, ...(4)

i.e., let the tension at the lowest point be equal to the weight of the length c of the string, then from (3), we have

$$\tan \psi = s/c, \quad \text{or} \quad s = c \tan \psi, \qquad ...(5)$$

which is the *intrinsic equation of the common catenary.*

Remark 1: From the equation (1) it is clear that the horizontal component of the tension at every point of the catenary is the same and is equal to T_0 , the tension at the lowest point.

Remark 2: From the equation (2) we conclude that the vertical component of the tension at any point of the string is equal to the weight of the string between the vertex and that point.

Remark 3: From the relation (4) it follows that the tension at the lowest point is equal to the weight of the string of length c.

2.4 Cartesian Equation of the Common Catenary

(Avadh 2011; Rohilkhand 06, 11; Lucknow 07; Kanpur 07, 10; Agra 09; Bundelkhand 07; Purvanchal 08, 11)

The intrinsic equation of the common catenary is [see 2.3]

$$s = c \tan \psi \qquad \text{...(1)}$$

where ψ is the angle which the tangent at any point P of the catenary makes with some horizontal line to be taken as the axis of x and s is the arc length of the catenary measured from the vertex A to the point P.

We know that $dy/dx = \tan \psi$.

$\therefore$ from (1), we have $s = c\,(dy/dx)$.

Differentiating both sides with respect to x, we have

$$\frac{ds}{dx} = c\,\frac{d^2y}{dx^2} \quad \text{or} \quad \sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]} = c\,\frac{d^2y}{dx^2}.$$

Putting $\dfrac{dy}{dx} = p$, so that $\dfrac{d^2y}{dx^2} = \dfrac{dp}{dx}$, we get

$$\sqrt{(1 + p^2)} = c\,\frac{dp}{dx} \quad \text{or} \quad \frac{dx}{c} = \frac{dp}{\sqrt{(1 + p^2)}}.$$

Integrating, we have

$$\frac{x}{c} + A = \sinh^{-1}(p) = \sinh^{-1}\left(\frac{dy}{dx}\right), \qquad \text{...(2)}$$

where A is the constant of integration.

Now if we choose the vertical line through the lowest point A of the catenary as the axis of y, then at the point A, we have

$$x = 0 \quad \text{and} \quad dy/dx = 0$$

because the tangent at the point A is horizontal *i.e.*, parallel to the axis of x.

$\therefore$ from (2), we have $A = 0$.

$$\therefore \quad \frac{x}{c} = \sinh^{-1}\left(\frac{dy}{dx}\right) \quad \text{or} \quad \frac{dy}{dx} = \sinh\left(\frac{x}{c}\right).$$

Integrating both sides with respect to x, we have

$$y = c \cosh\left(\frac{x}{c}\right) + B, \qquad \text{...(3)}$$

where B is a constant of integration.

If we take the origin O at a depth c below the lowest point A of the catenary, then at A, we have

$$x = 0 \quad \text{and} \quad y = c.$$

$\therefore$ from (3), we have $B = 0$.

$$\therefore \qquad y = c \cosh\left(\frac{x}{c}\right), \qquad \text{...(4)}$$

which is the *cartesian equation of the common catenary.*

2.5 Definitions

1. **Axis of the catenary.** Since cosh (x / c) is an even function of x, therefore the curve is symmetrical about the axis of y which is along the vertical through the lowest point of the catenary. *This vertical line of symmetry is called the* **axis of the catenary.** **(Garhwal 2001)**
2. **Vertex of the catenary.** *The lowest point A of the common catenary at which the tangent is horizontal is called the* **vertex of the catenary.** **(Garhwal 2001)**
3. **Parameter of the catenary.** *The quantity c occurring in the cartesian equation* $y = c \cosh (x / c)$ *of the catenary is called the* **parameter of the catenary.** **(Garhwal 2001)**
4. **Directrix of the catenary.** *The horizontal line at a depth c below the lowest point i.e., the axis of x, is called the* **directrix of the catenary.** **(Garhwal 2001)**
5. **Span and Sag.** Let the string be suspended from the two points B and C in the same horizontal line. *Then the distance BC is called the* **span of the catenary** *and the depth DA (see fig. of 2.3) of the lowest point below BC is called the* **sag of the catenary.**

2.6 Some Important Relations for the Common Catenary

1. Relation between x and s. **(Rohilkhand 2006; Kanpur 07, 11; Meerut 04, 08; Agra 07; Kumaun 02; Bundelkhand 06, 09)**

For a catenary, we have

$$s = c \tan \psi = c \frac{dy}{dx}. \qquad \left[\because \frac{dy}{dx} = \tan \psi\right]$$

$$\therefore \qquad \frac{dy}{dx} = \frac{s}{c}. \qquad \text{...(1)}$$

Also $\qquad y = c \cosh (x/c).$

Differentiating, we have

$$\frac{dy}{dx} = \sinh\left(\frac{x}{c}\right). \qquad \text{...(2)}$$

From (1) and (2), we have

$$\frac{s}{c} = \sinh\left(\frac{x}{c}\right) \quad \text{or} \quad \boldsymbol{s = c \sinh\left(\frac{x}{c}\right)} \qquad ...(3)$$

which is the relation between x and s.

2. Relation between y and s.

(Lucknow 2006, 09; Meerut 10; Bundelkhand 08; Agra 07, 09; Kanpur 07, 11; Purvanchal 10; Kumaun 01)

For a catenary, we have $y = c \cosh (x/c)$.

Also $\quad s = c \sinh (x/c)$. $\quad$ [see relation (3)]

$\therefore$ squaring and subtracting, we have

$$y^2 - s^2 = c^2 [\cosh^2 (x/c) - \sinh^2 (x/c)] = c^2$$

or $$\boldsymbol{y^2 = c^2 + s^2}, \qquad ...(4)$$

which is the relation between y and s.

3. Relation between y and ψ. **(Avadh 2007)**

For any curve, we have

$$dy / ds = \sin \psi.$$

$$\therefore \quad \frac{dy}{d\psi} = \frac{dy}{ds} \cdot \frac{ds}{d\psi} = \sin \psi \cdot \frac{d}{d\psi} (c \tan \psi) \qquad [\because \ s = c \tan \psi]$$

$$= c \sin \psi \cdot \sec^2 \psi = c \sec \psi \tan \psi.$$

Thus $$\frac{dy}{d\psi} = c \sec \psi \tan \psi.$$

Integrating, we get $y = c \sec \psi + A$, where A is a constant of integration.

But when $\quad y = c, \psi = 0; \ \therefore \ A = 0.$

Hence $$\boldsymbol{y = c \sec \psi}, \qquad ...(5)$$

which is the relation between y and ψ.

Aliter. From relation (4), we have $y^2 = c^2 + s^2$.

Also $\quad s = c \tan \psi$.

$$\therefore \quad y^2 = c^2 + s^2 = c^2 + c^2 \tan^2 \psi = c^2 (1 + \tan^2 \psi) = c^2 \sec^2 \psi$$

or $\quad y = c \sec \psi$.

4. Relation between x and ψ. **(Kumaun 2000; Agra 03 , 07; Meerut 04, 06; Avadh 06, 07; Bundelkhand 08; Lucknow 09; Purvanchal 10)**

For any curve, we have $dx/ds = \cos \psi$.

$$\therefore \quad \frac{dx}{d\psi} = \frac{dx}{ds} \cdot \frac{ds}{d\psi} = \cos \psi \cdot \frac{d}{d\psi} (c \tan \psi) \qquad [\because \ s = c \tan \psi]$$

$$= \cos \psi \cdot c \sec^2 \psi \quad \text{or} \quad \frac{dx}{d\psi} = c \sec \psi.$$

Integrating, we get

$$x = c \log (\sec \psi + \tan \psi) + B, \quad \text{where } B \text{ is a constant of integration.}$$

But when $x = 0, \psi = 0;$ therefore $B = 0$.

Hence $$\boldsymbol{x = c \log (\sec \psi + \tan \psi)}, \qquad ...(6)$$

which is the relation between x and ψ.

Note: The equations (5) and (6) together form the parametric equations of the catenary, ψ being the parameter.

5. Relation between tension and ordinate. **(Agra 2007; Meerut 04, 10)**

From 2.3, we have

$$T \cos \psi = T_0 \text{ and } T_0 = wc.$$

$$T = T_0 \sec \psi = wc \sec \psi.$$

But $\qquad y = c \sec \psi.$

Hence $\qquad \boldsymbol{T = wy,} \qquad \qquad ...(7)$

which is the relation between T and y.

The relation (7) *shows that the tension at any point of a catenary varies as the height of the point above the directrix.*

6. Radius of curvature at any point of a catenary. **(Agra 2007, 08)**

For a catenary, we have

$$s = c \tan \psi. \quad \therefore \quad \rho = \frac{ds}{d\psi} = c \sec^2 \psi. \qquad ...(8)$$

Illustrative Examples

Example 1: *If* $(\bar{x}, \bar{y})$ *be the coordinates of the centre of gravity of the arc measured from the vertex upto the point* $P(x, y)$, *prove that*

$$\bar{x} = x - c \tan (\psi/2), \quad \bar{y} = \frac{1}{2}(c / \cos \psi + x \cot \psi).$$

Solution: The parametric equations of a catenary are

$$x = c \log (\sec \psi + \tan \psi) \text{ and } y = c \sec \psi.$$

Also for a catenary, $s = c \tan \psi. \quad \therefore \quad ds/d\psi = c \sec^2 \psi.$

We have $$\bar{x} = \frac{\int x\, ds}{\int ds} = \frac{\int_0^{\psi} x \frac{ds}{d\psi} d\psi}{\int_0^{\psi} \frac{ds}{d\psi} d\psi} \qquad [\because \text{ At the vertex, } \psi = 0]$$

$$= \frac{\int_0^{\psi} c \log (\sec \psi + \tan \psi)\, c \sec^2 \psi \, d\psi}{\int_0^{\psi} c \sec^2 \psi \, d\psi}$$

$$= \frac{c^2 \left[\{\log (\sec \psi + \tan \psi). \tan \psi\}_0^{\psi} - \int_0^{\psi} \frac{1}{\sec \psi + \tan \psi} (\sec \psi \tan \psi + \sec^2 \psi) \tan \psi \, d\psi\right]}{c\,[\tan \psi]_0^{\psi}}$$

[Integrating the numerator by parts taking $\sec^2 \psi$ as 2nd function]

$$= \frac{c\left[\tan\psi\log(\sec\psi+\tan\psi)-\int_0^{\psi}\sec\psi\tan\psi\,d\psi\right]}{\tan\psi}$$

$$= \frac{c\,[\tan\psi\log(\sec\psi+\tan\psi)-\{\sec\psi\}_0^{\psi}\,]}{\tan\psi} = \frac{x\tan\psi-c\,(\sec\psi-1)}{\tan\psi}$$

$$= x - \frac{c\,(\sec\psi-1)}{\tan\psi} = x - \frac{c\,(1-\cos\psi)}{\sin\psi}$$

$$= x - c\,\frac{2\sin^2\frac{1}{2}\psi}{2\sin\frac{1}{2}\psi\cos\frac{1}{2}\psi} = x - c\tan\frac{1}{2}\psi,$$

and
$$\bar{y} = \frac{\int y\,ds}{\int ds} = \frac{\int_0^{\psi} y\,\frac{ds}{d\psi}\,d\psi}{\int_0^{\psi}\frac{ds}{d\psi}\,d\psi} = \frac{\int_0^{\psi} c\sec\psi\,c\sec^2\psi\,d\psi}{\int_0^{\psi} c\sec^2\psi\,d\psi}$$

$$= \frac{c^2\left[\left\{\frac{1}{2}\sec\psi\tan\psi\right\}_0^{\psi}+\frac{1}{2}\int_0^{\psi}\sec\psi\,d\psi\right]}{c\,[\tan\psi]_0^{\psi}}$$

$$\left[\because \int\sec^n\theta\,d\theta = \frac{\sec^{n-2}\theta\tan\theta}{n-1}+\frac{n-2}{n-1}\int\sec^{n-2}\theta\,d\theta\right]$$

$$= \frac{c\left[\frac{1}{2}\sec\psi\tan\psi+\frac{1}{2}\{\log(\sec\psi+\tan\psi)\}_0^{\psi}\right]}{\tan\psi}$$

$$= \frac{1}{2\tan\psi}\,[c\sec\psi\tan\psi + c\log(\sec\psi+\tan\psi)]$$

$$= \frac{1}{2\tan\psi}\,[c\sec\psi\tan\psi + x] = \frac{1}{2}\left[\frac{c}{\cos\psi}+x\cot\psi\right].$$

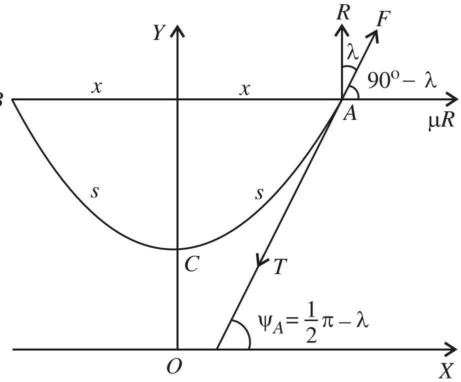
Example 2: *The end links of a uniform chain slide along a fixed rough horizontal rod. Prove that the ratio of the maximum span to the length of the chain is*

$$\mu\ \log\left\{\frac{1+\sqrt{(1+\mu^2)}}{\mu}\right\},$$

where μ *is the coefficient of friction.*

(Agra 2002, 07, 09; Avadh 08)

Solution: Let the end links A and B of a uniform chain slide along a fixed rough horizontal rod. If AB is the maximum span, then A and B are in the state of limiting equilibrium. Let R be the reaction of the rod at A acting perpendicular to the rod. Then the frictional force μR will act at A along the rod in the outward direction BA as shown in the figure. The resultant F of the forces R and μR at A will make an angle λ (where $\tan\lambda=\mu$) with the direction of R. For the equilibrium of A, the resultant F of R and μR at A will be equal and opposite to the tension T at A.



Since the tension at A acts along the tangent to the chain at A, therefore the tangent to the catenary at A makes an angle $\psi_A = \frac{1}{2}\pi - \lambda$ to the horizontal.

Thus for the point A of the catenary, we have

$$\psi = \psi_A = \frac{1}{2}\pi - \lambda.$$

$\therefore$ the length of the chain

$$= 2s = 2c \tan \psi_A$$

$$= 2c \tan\left(\frac{1}{2}\pi - \lambda\right)$$

$$= 2c \cot \lambda = 2c/\mu. \qquad [\because \tan\lambda = \mu]$$

If (x_A, y_A) are the coordinates of the point A, then the maximum span $AB = 2x_A$

$$= 2c \log(\tan\psi_A + \sec\psi_A)$$

$$= 2c \log\{\tan\psi_A + \sqrt{(1 + \tan^2 \psi_A)}\}$$

$$= 2c \log\{\cot\lambda + \sqrt{\{1 + \cot^2 \lambda\}}\} \qquad \left[\because \psi_A = \frac{1}{2}\pi - \lambda\right]$$

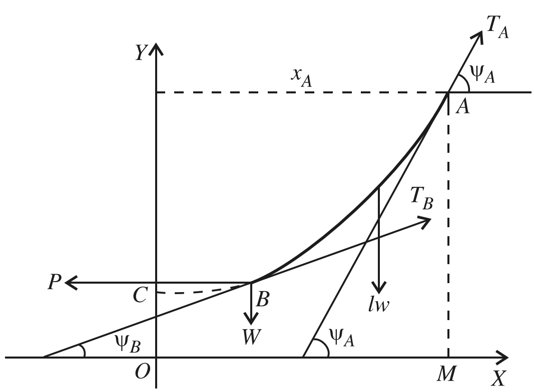
$$= 2c \log\left\{\frac{1}{\mu} + \sqrt{\left(1 + \frac{1}{\mu^2}\right)}\right\} = 2c \log\left\{\frac{1 + \sqrt{(1+\mu^2)}}{\mu}\right\}.$$

Hence the required ratio

$$= \frac{2x}{2s} = \frac{2c \log\left\{\frac{1 + \sqrt{(1+\mu^2)}}{\mu}\right\}}{(2c/\mu)} = \mu \log\left\{\frac{1 + \sqrt{(1+\mu^2)}}{\mu}\right\}.$$

Example 3: *A weight W is suspended from a fixed point by a uniform string of length l and weight w per unit length. It is drawn aside by a horizontal force P. Show that in the position of equilibrium, the distance of W from the vertical through the fixed point is*

$$\frac{P}{w}\left\{\sinh^{-1}\left(\frac{W + lw}{P}\right) - \sinh^{-1}\left(\frac{W}{P}\right)\right\}. \qquad \textbf{(Garhwal 2001)}$$



Solution: The end A of the string AB of length l is attached to the fixed point A and a weight W hanging at the other end B of the string is drawn aside by a horizontal force P.

Let us first consider the equilibrium of the end B of the string. There are three forces acting on it : (i) the horizontal force P applied at

B, (ii) the weight W suspended at B and acting vertically downwards, and (iii) the tension T_B of the string BA acting along the tangent to the string at B which makes an angle ψ_B with OX.

Resolving these forces horizontally and vertically, we have

$$T_B \cos \psi_B = P \qquad ...(1)$$

and $$T_B \sin \psi_B = W. \qquad ...(2)$$

Dividing (2) by (1), we have

$$\tan \psi_B = W/P. \qquad ...(3)$$

Since the horizontal component of the tension at any point of the string is constant and is equal to wc, therefore

$$T_B \cos \psi_B = wc = P,$$

so that $$c = P/w. \qquad ...(4)$$

Now let us consider the equilibrium of the whole string AB. The forces acting on it are :

(i) the horizontal force P applied at B, (ii) the weight W suspended at B, (iii) the weight lw of the string AB acting vertically downwards through the centre of gravity of the string AB, and (iv) the tension T_A at the end A acting along the tangent to the string at A which makes an angle ψ_A with OX.

Resolving these forces horizontally and vertically, we have

$$T_A \cos \psi_A = P \qquad ...(5)$$

and $$T_A \sin \psi_A = W + lw. \qquad ...(6)$$

Dividing (6) by (5), we have

$$\tan \psi_A = (W + lw)/P. \qquad ...(7)$$

Now the distance of the weight W from the vertical AM through the fixed point A

$$= (\text{the } x\text{-coordinate of } A) - (\text{the } x\text{-coordinate of } B) = x_A - x_B .$$

But for a catenary, we have

$$s = c \sinh (x/c).$$

$$\therefore \quad x = c \sinh^{-1}\left(\frac{s}{c}\right) = c \sinh^{-1}\left(\frac{c \tan \psi}{c}\right) \qquad [\because s = c \tan \psi]$$

$$= c \sinh^{-1} (\tan \psi).$$

$$\therefore \quad x_A - x_B = c \sinh^{-1} (\tan \psi_A) - c \sinh^{-1} (\tan \psi_B)$$

$$= \frac{P}{w}\left\{\sinh^{-1}\left(\frac{W + lw}{P}\right) - \sinh^{-1}\left(\frac{W}{P}\right)\right\},$$

substituting for c from (4), for $\tan \psi_A$ from (7) and for $\tan \psi_B$ from (3).

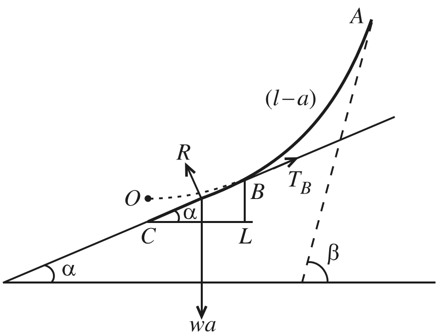
This gives the required distance of W from the vertical through the fixed point A.

Example 4: *One extremity of a uniform string is attached to a fixed point and the string rests partly on a smooth inclined plane; prove that the directrix of the catenary determined by the portion which is not in contact with the plane is the horizontal line drawn through the extremity which rests on the plane.*

If α is the inclination of the plane, β the inclination of the tangent at the fixed extremity, and l the whole length of the string, prove that the length of the portion in contact with the plane is

$$\frac{l \cos \beta}{\cos \alpha \cos (\beta - \alpha)}.$$

Solution: Let ABC be a string of length l of which the portion BC is on the plane inclined at an angle α to the horizontal. Let the length of the portion BC of the string be a. Then the length of the portion AB of the string hanging in the form of an arc of a catenary is $l - a$.



Let us consider the equilibrium of the portion BC of the string lying on the inclined plane. There are three forces acting on it : (i) its weight aw acting vertically downwards through its centre of gravity, (ii) the normal reaction R of the inclined plane acting perpendicular to the plane, and (iii) the tension T_B of the string at B acting along the tangent at B to the string which is along the line CB lying in the inclined plane. Resolving these forces along the inclined plane, we have

$$T_B = wa \sin \alpha. \qquad ...(1)$$

Let BL be the perpendicular from B on the horizontal line through C. Then

$$BL = BC \sin \alpha = a \sin \alpha.$$

$\therefore$ from (1), we have $T_B = w.BL$.

But for a catenary, from $T = wy$, we have $T_B = wy_B$,

where y_B is the vertical distance of the point B from the directrix of the catenary.

Thus we have $wBL = wy_B$, so that $y_B = BL$.

Hence the directrix of the catenary AB is the horizontal line CL through the extremity C of the portion of the string which rests on the inclined plane.

Second Part. Let O be the lowest point *i.e.*, the vertex of the catenary of which AB is a part. The inclinations to the horizontal of the tangents at B and A to the string are α and β respectively.

Then from $s = c \tan \psi$, we have

$$\text{arc } OB = c \tan \alpha \quad \text{and} \quad \text{arc } OA = c \tan \beta.$$

$\therefore$ $\quad \text{arc } AB = \text{arc } OA - \text{arc } OB = c (\tan \beta - \tan \alpha)$

or $\quad l - a = c (\tan \beta - \tan \alpha). \qquad ...(2)$

From (1), we have $wa \sin \alpha = T_B$

$= wy_B = w.c \sec \alpha \qquad [\because$ from $y = c \sec \psi$, we have $y_B = c \sec \alpha]$

$\therefore \quad a = c \sec \alpha \operatorname{cosec} \alpha. \qquad ...(3)$

Dividing (2) by (3), we have

$$\frac{l - a}{a} = \frac{c (\tan \beta - \tan \alpha)}{c \sec \alpha \operatorname{cosec} \alpha}$$

or $$\frac{l-a}{a}=\left(\frac{\sin\beta}{\cos\beta}-\frac{\sin\alpha}{\cos\alpha}\right).\sin\alpha\cos\alpha=\frac{\sin(\beta-\alpha)\sin\alpha}{\cos\beta}$$

or $$(l-a)\cos\beta=a\sin(\beta-\alpha)\sin\alpha$$

or $$\begin{aligned} l\cos\beta &= a\,[\cos\beta+\sin(\beta-\alpha)\sin\alpha] \\ &= a\,[\cos\beta+(\sin\beta\cos\alpha-\cos\beta\sin\alpha)\sin\alpha\,] \\ &= a\,[\sin\beta\cos\alpha\sin\alpha+(1-\sin^2\alpha)\cos\beta] \\ &= a\,[\sin\beta\cos\alpha\sin\alpha+\cos^2\alpha\cos\beta] \\ &= a\cos\alpha\,[\sin\alpha\sin\beta+\cos\alpha\cos\beta] \\ &= a\cos\alpha\cos(\beta-\alpha). \end{aligned}$$

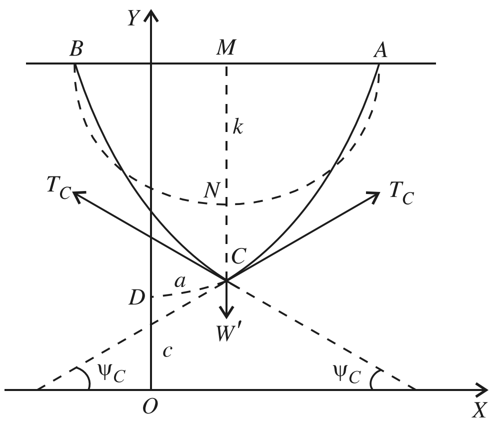
$\therefore$ $$a=\frac{l\cos\beta}{\cos\alpha\cos(\beta-\alpha)}.$$

Example 5: *A uniform chain, of length l and weight W, hangs between two fixed points at the same level, and a weight W′ is attached at the middle point. If k be the sag in the middle, prove that the pull on either point of support is*

$$\frac{k}{2l}W+\frac{l}{4k}W'+\frac{l}{8k}W.$$ **(Agra 2000; Purvanchal 07; Lucknow 11)**

Solution: Let a string of length l and weight W suspended from two points A and B at the same level hang freely under gravity in the form of the catenary ANB.

When a weight W' is attached at the middle point N of the string then it will descend downwards to C, and the two portions AC and BC of the string each of length $\frac{1}{2}l$ will be the parts of two equal catenaries. Let D be the lowest point *i.e.*, the vertex and OX the directrix of the catenary of which AC is an arc.



The weight per unit length of the chain $= w = W/l$.

If T_C, T_C are the tensions at the point C in the strings CA and CB acting along the tangents at C, then resolving vertically the forces acting at C, we have

$$2T_C\sin\psi_C=W', \qquad ...(1)$$

where the tangents at C to the arcs AC and BC are inclined at an angle ψ_C to the horizontal.

But from $T\sin\psi=ws$, we have

$$T_C\sin\psi_C=wa, \text{ where arc } DC=a.$$

$\therefore$ from (1), we have

$$2\,wa=W', \quad\text{or}\quad a=\frac{W'}{2\,w}=\frac{W'l}{2W} \qquad ...(2)$$

Let y_A and y_C be the ordinates of the points A and C respectively.

Now at A, $s = s_A = \text{arc } DA = \text{arc } DC + \text{arc } CA = a + \frac{1}{2} l$ and $y = y_A$;

at C, $s = s_C = \text{arc } DC = a$ and $y = y_C$.

Since sag in the middle $= CM = k$,

$$\therefore \qquad y_C + k = y_A \quad \text{or} \quad y_C = y_A - k.$$

$\therefore$ from $y^2 = c^2 + s^2$, we have

$$y_A^2 = c^2 + (a + \frac{1}{2} l)^2 \quad \text{and} \quad y_C^2 = c^2 + a^2.$$

Subtracting, we have $y_A^2 - y_C^2 = al + \frac{1}{4} l^2$

$$\text{or} \qquad y_A^2 - (y_A - k)^2 = al + \frac{1}{4} l^2 \qquad [\because y_C = y_A - k]$$

$$\text{or} \qquad 2\, k y_A - k^2 = al + \frac{1}{4} l^2 = \frac{W' l^2}{2W} + \frac{l^2}{4} \qquad \left[\because a = \frac{W' l}{2W}, \text{from (2)}\right]$$

$$\text{or} \qquad y_A = \frac{k}{2} + \frac{l^2 W'}{4\, kW} + \frac{l^2}{8k}.$$

Hence the pull (*i.e.*, the tension) at either point of support A or B

$$= T_A = w y_A = \frac{W}{l}\left(\frac{k}{2} + \frac{l^2 W'}{4\, kW} + \frac{l^2}{8k}\right) \qquad \left[\because w = \frac{W}{l}\right]$$

$$= \frac{k}{2l} W + \frac{l}{4k} W' + \frac{l}{8k} W.$$

Example 6: *A uniform chain of length l hangs between two points A and B which are at a horizontal distance a from one another, with B at a vertical distance b above A. Prove that the parameter of the catenary is given by*

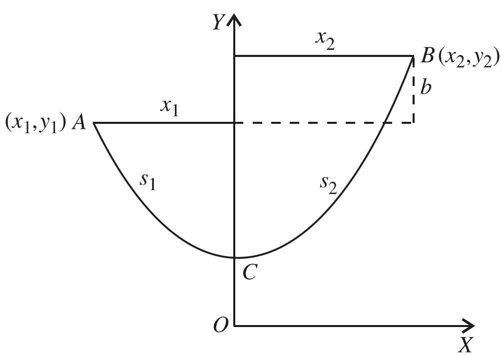
$$2c \sinh (a/2c) = \sqrt{(l^2 - b^2)}.$$

Prove also that, if the tensions at A and B are T_1 and T_2 respectively,

$$T_1 + T_2 = W \sqrt{\left(1 + \frac{4c^2}{l^2 - b^2}\right)}$$

and $\qquad T_2 - T_1 = Wb/l,$

where W is the weight of the chain.



Solution: A uniform chain of length l and weight W hangs between two points A and B. Let C be the vertex, OX the directrix, OY the axis and c the parameter of the catenary in which the chain hangs. Let (x_1, y_1) and (x_2, y_2) be the coordinates of the points A and B respectively and let arc $CA = s_1$ and arc $CB = s_2$.

We have $s_1 + s_2 = l.$

Since the horizontal distance between A and B is a, therefore

$$x_1 + x_2 = a.$$

Again since the vertical distance of B above A is b, therefore

$$y_2 - y_1 = b.$$

Let w be the weight per unit length of the chain. Then

$$W = lw, \quad \text{or} \quad w = W/l.$$

By the formula $s = c \sinh (x/c)$, we have

$$s_1 = c \sinh (x_1/c) \quad \text{and} \quad s_2 = c \sinh (x_2/c).$$

$$\therefore \qquad l = s_1 + s_2 = c\,[\sinh (x_1/c) + \sinh (x_2/c)]. \qquad \text{...(1)}$$

Again by the formula $y = c \cosh (x/c)$, we have

$$y_1 = c \cosh (x_1/c) \text{ and } y_2 = c \cosh (x_2/c).$$

$$\therefore \qquad b = y_2 - y_1 = c\,[\cosh (x_2/c) - \cosh (x_1/c)]. \qquad \text{...(2)}$$

Squaring and subtracting (1) and (2), we have

$$l^2 - b^2 = c^2\,[-\{\cosh^2 (x_1/c) - \sinh^2 (x_1/c)\} - \{\cosh^2 (x_2/c) - \sinh^2 (x_2/c)\}$$
$$+ 2\,\{\cosh (x_1/c) \cosh (x_2/c) + \sinh (x_1/c) \sinh (x_2/c)\}]$$
$$= c^2\,[-1 - 1 + 2 \cosh (x_1/c + x_2/c)]$$
$$= c^2\,[-2 + 2 \cosh \{(x_1 + x_2)/c\}]$$
$$= 2c^2 \left\{\cosh \frac{a}{c} - 1\right\} = 2c^2 \left\{1 + 2 \sinh^2 \frac{a}{2c} - 1\right\}$$
$$= 4c^2 \sinh^2 \frac{a}{2c}. \qquad \text{...(3)}$$

$\therefore$ c is given by

$$2\,c \sinh (a/2c) = \sqrt{(l^2 - b^2)}.$$

[Remember that $[\cosh (\alpha + \beta) = \cosh \alpha \cosh \beta + \sinh \alpha \sinh \beta$, and $\cosh 2\alpha = 1 + 2 \sinh^2 \alpha]$

Now let T_1 and T_2 be the tensions at the points A and B respectively. Then by the formula $T = wy$, we have

$$T_1 = wy_1, T_2 = wy_2\,.$$

$$\therefore \qquad T_2 - T_1 = w\,(y_2 - y_1) = wb = (W/l)\,b = Wb/l.$$

Also $T_1 + T_2 = w\,(y_1 + y_2) = \dfrac{W}{l}\,(y_1 + y_2) = W\,\dfrac{y_1 + y_2}{s_1 + s_2}$

$$= W\,\frac{c \cosh (x_1/c) + c \cosh (x_2/c)}{c \sinh (x_1/c) + c \sinh (x_2/c)}$$

$$= W\,\frac{\cosh (x_1/c) + \cosh (x_2/c)}{\sinh (x_1/c) + \sinh (x_2/c)}$$

$$= W\,\frac{2 \cosh \frac{1}{2}(x_1/c + x_2/c) \cosh \frac{1}{2}(x_1/c - x_2/c)}{2 \sinh \frac{1}{2}(x_1/c + x_2/c) \cosh \frac{1}{2}(x_1/c - x_2/c)}$$

$$= W \coth\left(\frac{x_1 + x_2}{2c}\right) = W \coth \frac{a}{2c}$$

$$= W \sqrt{\left[1 + \text{cosech}^2 \frac{a}{2c}\right]} \qquad [\because \coth^2 \alpha = 1 + \text{cosech}^2 \alpha]$$

$$= W \sqrt{\left[1 + \frac{4c^2}{l^2 - b^2}\right]},$$

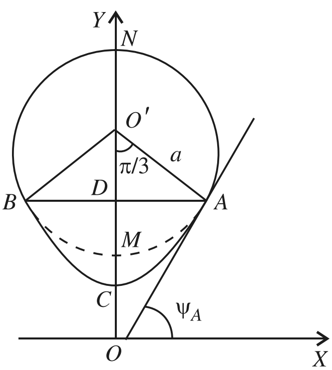
substituting for $\text{cosech}^2 (a/2c)$ from (3).

Example 7: *Show that the length of an endless chain which will hang over a circular pulley of radius a so as to be in contact with two-thirds of the circumference of the pulley is*

$$a \left\{\frac{3}{\log (2 + \sqrt{3})} + \frac{4\pi}{3}\right\}.$$

(Purvanchal 2009; Agra 11; Kanpur 11; Rohilkhand 11; Garhwal 04)

Solution: *Let ANBMA be the circular pulley of radius a and ANBCA the endless chain hanging over it.*



Since the chain is in contact with the two-thirds of the circumference of the pulley, hence the length of this portion ANB of the chain

$$= \frac{2}{3} \text{ (circumference of the pulley)}$$

$$= \frac{2}{3} (2\pi a) = \frac{4}{3} \pi a.$$

Let the remaining portion of the chain hang in the form of the catenary ACB, with AB horizontal . C is the lowest point *i.e.*, the vertex, $CO'N$ the axis and OX the directrix of this catenary.

Let $OC = c =$ the parameter of the catenary.

The tangent at A will be perpendicular to the radius $O'A$.

$\therefore$ If the tangent at A is inclined at an angle ψ_A to the horizontal, then

$$\psi_A = \angle AO'D = \frac{1}{2}(\angle AO'B) = \frac{1}{2}\left(\frac{1}{3} \cdot 2\pi\right) = \frac{1}{3}\pi.$$

From the triangle $AO'D$, we have

$$DA = O'A \sin \frac{1}{3}\pi = a\sqrt{3}/2.$$

$\therefore$ from $x = c \log (\tan \psi + \sec \psi)$, for the point A, we have

$$x = DA = c \log (\tan \psi_A + \sec \psi_A)$$

or
$$\frac{a\sqrt{3}}{2} = c \log\left(\tan\frac{\pi}{3} + \sec\frac{\pi}{3}\right) = c \log (\sqrt{3} + 2).$$

$\therefore$
$$c = \frac{a\sqrt{3}}{2 \log (2 + \sqrt{3})}.$$

From $s = c \tan \psi$ applied for the point A, we have

$$\text{arc } CA = c \tan \psi_A = c \tan \frac{1}{3}\pi = c\sqrt{3} = \frac{3a}{2 \log (2 + \sqrt{3})}.$$

Hence the total length of the chain

$$= \text{arc } ACB + \text{ length of the chain in contact with the pulley}$$

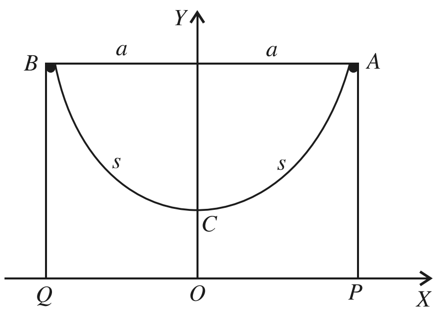
$$= 2 (\text{arc } CA) + \frac{4}{3}\pi a$$

$$= 2 \cdot \frac{3a}{2 \log (2 + \sqrt{3})} + \frac{4}{3}\pi a = a \left\{ \frac{3}{\log (2 + \sqrt{3})} + \frac{4\pi}{3} \right\}.$$

Example 8: *A string hangs over two smooth pegs which are at the same level, its free ends hanging vertically. Prove that when the string is of shortest possible length, the parameter of the catenary is equal to half the distance between the pegs, and find the whole length of the string.*

(Agra 2001)

Solution: Suppose A and B are two smooth pegs at the same level and at a distance $2a$ apart *i.e.*, $AB = 2a$. A string hangs over the pegs A and B. The portions AP and BQ of the string hang vertically and the portion ACB hangs in the form of a catenary whose vertex is C and directrix is OX. Let w be the weight per unit length of the string.



Now the tension of the string remains unaltered while passing over the smooth peg at A. Therefore the tension at the point A due to the string AP is equal to the tension at the point A due to the string ACB hanging in the form of catenary. But the tension at the point A due to the string AP is equal to the weight $w\ AP$ of the string AP and by the formula $T = wy$, the tension at A in the catenary ACB is equal to $w\ y_A$ where y_A is the height of the point A above the directrix OX of the catenary ACB. So we have $w \,.\, y_A = w \,.\, AP$ *i.e.*, $y_A = AP$. Therefore the free end P of the string lies on the directrix OX of the catenary ACB. Similarly the other free end Q of the string also lies on the directrix OX.

Let c be the parameter of the catenary ACB *i.e.*, let $OC = c$. For the point A of the catenary ACB, we have

$$y = y_A = AP \quad \text{and} \quad x = x_A = a.$$

By the formula $s = c \sinh (x/c)$, for the point A, we have

$$s = \text{arc } CA = c \sinh (a/c).$$

Also by the formula $y = c \cosh (x/c)$, for the point A, we have

$$y = y_A = AP = c \cosh (a/c).$$

Hence the total length of the string, say l, is given by

$$l = 2. (\text{arc } CA + y_A) = 2 \{c \sinh (a/c) + c \cosh (a/c)\}$$

$$= 2c \{\tfrac{1}{2} (e^{a/c} - e^{-a/c}) + \tfrac{1}{2} (e^{a/c} + e^{-a/c})\} = 2c\, e^{a/c}. \qquad ...(1)$$

Now l is a function of c. For a maximum or a minimum of l, we must have

$$dl/dc = 0.$$

From (1), we have

$$\frac{dl}{dc} = 2e^{a/c} + 2c\, e^{a/c}.\left(-\frac{a}{c^2}\right) = 2e^{a/c}\left(1-\frac{a}{c}\right).$$

Putting $dl/dc = 0$, we get $2e^{a/c}\{1-(a/c)\} = 0$,

or $\quad 1-(a/c) = 0 \quad$ or $\quad c = a. \qquad [\because\ e^{a/c} \neq 0]$

Now $$\frac{d^2 l}{dc^2} = 2e^{a/c}.\frac{a}{c^2} + \left(1-\frac{a}{c}\right).2e^{a/c}\left(-\frac{a}{c^2}\right)$$

$$= 2e/a, \text{ when } c = a$$

$$= + \text{ ive}$$

$\therefore \quad l$ is minimum when $c = a$.

Thus when the string is of shortest possible length, we have $c = a = \frac{1}{2}(2a)$ = half the distance between the pegs.

Now putting $c = a$ in (1), the length l of the string in this case is given by

$$l = 2a\, e^{a/a} = 2ae.$$

Comprehensive Exercise 1

1. If the normal at any point P of a catenary meets the directrix at Q, show that $PQ = \rho$. **(Bundelkhand 2006)**
2. If T be the tension at any point P of a catenary and T_0 that at the lowest point A, prove that $T^2 - T_0^2 = W^2$, W being the weight of the arc AP of the catenary. **(Lucknow 2007, 11; Bundelkhand 10; Agra 08, 10)**
3. Prove that if a uniform inextensible chain hangs freely under gravity, the difference of the tensions at two points varies as the difference of their heights. **(Agra 2003)**
4. Show that for a common catenary $x = c \log\left(\frac{y+s}{c}\right)$. **(Agra 2000; Kanpur 09; Bundelkhand 09)**
5. A rope of length $2l$ feet is suspended between two points at the same level, and the lowest point of the rope is b feet below the points of suspension. Show that the horizontal component of the tension is $w(l^2 - b^2)/2b$, w being the weight of the rope per foot of its length. **(Garhwal 2002; Meerut 09)**
6. A uniform chain, of length l, is to be suspended from two points A and B, in the same horizontal line, so that either terminal tension is n times that at the lowest point. Show that the span AB must be

$$\frac{l}{\sqrt{(n^2-1)}} \log\{n + \sqrt{(n^2-1)}\}.$$

(Agra 2008; Purvanchal 08; Avadh 09; Bundelkhand 09, 10)

7. A uniform chain of length l, is suspended from two points A, B in the same horizontal line. If the tension at A is twice that at the lowest point show that the span AB is

$$\frac{l}{\sqrt{3}} \log (2 + \sqrt{3}).$$

(Rohilkhand 2006; Kanpur 07, 08, 10; Purvanchal 09; Meerut 10; Agra 11)

8. A uniform chain, of length l, which can just bear a tension of n times its weight, is stretched between two points in the same horizontal line. Show that the least possible sag in the middle is

$$l \{n - \sqrt{(n^2 - \tfrac{1}{4})}\}.$$

9. A given length, $2s$, of a uniform chain has to be hung between two points at the same level and the tension has not to exceed the weight of a length b of the chain. Show that the greatest span is

$$\sqrt{(b^2 - s^2)} \log \left(\frac{b + s}{b - s}\right).$$

(Rohilkhand 2007; Purvanchal 11; Kumaun 03)

10. If α, β be the inclinations to the horizon of the tangents at the extremities of a portion of a common catenary, and l the length of the portion, show that the height of one extremity above the other is

$$\frac{l \sin \frac{1}{2} (\alpha + \beta)}{\cos \frac{1}{2} (\alpha - \beta)},$$

the two extremities being on one side of the vertex of the catenary.

(Avadh 2011)

11. If the ends of a uniform inextensible string of length l hanging freely under gravity slide on a fixed rough horizontal rod whose coefficient of friction is μ, show that at most they can rest at a distance $\mu l \log \left\{\frac{1 + \sqrt{(1 + \mu^2)}}{\mu}\right\}$.

(Meerut 2008, 11; Bundelkhand 08; Kanpur 09)

12. The end links of a uniform chain slide along a fixed rough horizontal rod. If the angle of friction be λ, prove that the ratio of the maximum span to the length of the chain is

$$\left(\log \cot \frac{\lambda}{2}\right) : \cot \lambda.$$

(Agra 2002)

13. The extremities of a heavy string of length $2l$ and weight $2lw$, are attached to two small rings which can slide on a fixed wire. Each of these rings is acted on by a horizontal force equal to lw. Show that the distance apart of the rings is

$$2l \log (1 + \sqrt{2}).$$

14. A uniform chain of length $2l$ is suspended by its ends which are on the same horizontal level. The distance apart $2a$ of the ends is such that the lowest point of the chain is at a distance a vertically below the ends. Prove that if c be the distance of the lowest point from the directrix of the catenary, then

$$\frac{2a^2}{l^2 - a^2} = \log \left(\frac{l + a}{l - a}\right) \quad \text{and} \quad \tanh \frac{a}{c} = \frac{2al}{l^2 + a^2}.$$

15. A heavy uniform string, of length l, is suspended from a fixed point A, and its other end B is pulled horizontally by a force equal to the weight of a length a of the string. Show that the horizontal and vertical distances between A and B are

$a \sinh^{-1} (l/a)$ and $\sqrt{(l^2 + a^2)} - a$, respectively. **(Avadh 2008)**

16. A box kite is flying at a height h with a length l of wire paid out, and with the vertex of the catenary on the ground. Show that at the kite the inclination of the wire to the ground is $2 \tan^{-1} (h/l)$,

and that its tensions there and at the ground are

$w (l^2 + h^2) / 2h$ and $w (l^2 - h^2) / 2h$,

where w is the weight of the wire per unit of length.

(Garhwal 2003; Lucknow 06)

17. A is the lowest point of a uniform thread hanging from two fixed points, B and C. Let a, b be the heights of A and B above the directrix of the catenary formed by the thread. Show that the length of the thread between A and B equals $\sqrt{(b^2 - a^2)}$.

(Agra 2010)

18. The end links of a uniform chain of length l can slide on two smooth rods in the same vertical plane which are inclined in opposite directions at equal angles ϕ to the vertical. Prove that the sag in the middle is $\frac{1}{2} l \tan \frac{1}{2} \phi$.

19. A uniform heavy chain is fastened at its extremities to two rings of equal weight, which slide on smooth rods intersecting in a vertical plane, and inclined at the same angle α to the vertical. Find the condition that the tension at the lowest point may be equal to half the weight of the chain, and in that case, show that the vertical distance of the rings from the point of intersection of the rods is

$l \cot \alpha \log (\sqrt{2} + 1)$,

where $2l$ is the length of the chain.

20. A boat is towed by means of a rope attached to a ship and the lower end of the rope makes an angle of $30°$ to the horizontal. If the length of the rope is 36 feet, and the upper end is 20 feet higher than the lower end, find the resistance of the water to the motion of the boat, the weight of each foot of the rope being ten ounces. [**Ans.** $121 \cdot 2$ lbs. wt.]

21. A length l of a uniform chain has one end fixed at a height h above a rough table, and rests in a vertical plane so that a portion of it lies in a straight line on a table. Prove that if the chain is on the point of slipping, the length on the table is $l + \mu h - \sqrt{\{(\mu^2 + 1) h^2 + 2\mu l h\}}$,

where μ is the coefficient of friction. **(Purvanchal 2011)**

22. A heavy uniform chain AB hangs freely under gravity, with the end A fixed and the other end B attached by a light string BC to a fixed point C at the same level as A. The lengths of the string and chain are such that the ends of the chain at A and B make angles $60°$ and $30°$ respectively with the horizontal. Prove that the ratio of these lengths is $(\sqrt{3} - 1) : 1$. **(Purvanchal 2009)**

23. A heavy chain, of length $2l$, has one end tied at A and the other is attached to a small heavy ring which can slide on a rough horizontal rod which passes through

A. If the weight of the ring be n times the weight of the chain, show that its greatest possible distance from A is

$\frac{2l}{\lambda} \log \{\lambda + \sqrt{(1+\lambda^2)}\}$, where $1/\lambda = \mu\,(2n+1)$ and μ is the coefficient of friction.

24. A uniform inextensible string, of length l and weight wl, carries at one end B, a particle of weight W which is placed on a smooth plane inclined at $30°$ to the horizontal. The other end of the string is attached to a point A, situated at a height h above the horizontal through B and in the vertical plane through the line of greatest slope through B. Prove that the particle will rest in equilibrium with the tangent at B to the catenary lying in the inclined plane, if

$$W/w = (l-h)(l+h)/(h - \tfrac{1}{2}l).$$

25. **(a)** A uniform chain of length l and weight W hangs between two fixed points at the same level and a weight is suspended from its middle point so that the total sag in the middle is h. Show that if P is the pull on either point of support, the weight suspended is

$$\frac{4h}{l} P - \left(\frac{1}{2} + \frac{2h^2}{l^2}\right) W.$$

(b) A uniform chain of length $2l$ and weight W, is suspended from two points A and B, in the same horizontal line. A load P is now suspended from the middle point D of the chain and the depth of this point below AB is found to be h. Show that each terminal tension is

$$\frac{1}{2}\left[P\frac{l}{h} + W\frac{h^2+l^2}{2hl}\right].$$

26. A uniform string of weight W is suspended from two points at the same level and a weight W' is attached to its lowest point. If α and β are now the inclinations to the horizontal of the tangents at the highest and lowest points, prove that

$$\tan\alpha / \tan\beta = 1 + W/W'.$$ **(Avadh 2006; Purvanchal 08, 10)**

27. A uniform chain, of length $2l$ and weight $2w$, is suspended from two points in the same horizontal line. A load w is now suspended from the middle point of the chain and the depth of this point below the horizontal line is h. Show that the terminal tension is

$$\frac{1}{2} w \cdot \frac{h^2 + 2l^2}{hl}.$$

28. A heavy string of uniform density and thickness is suspended from two given points in the same horizontal plane. A weight, an nth that of the string, is attached to its lowest point; show that if θ, ϕ be the inclinations to the vertical of the tangents at the highest and lowest points of the string

$$\tan\phi = (1+n)\tan\theta.$$

29. A and B are two points in the same horizontal line distant $2a$ apart. AO, OB are two equal heavy strings tied together at O and carrying a weight at O. If l is the length of each string, d the depth of O below AB, show that the parameter c of the catenary in which either string hangs is given by

$$l^2 - d^2 = 2c^2\,[\cosh(a/c) - 1].$$

30. An endless uniform chain is hung over two smooth pegs in the same horizontal line. Show that, when it is in a position of equilibrium, the ratio of the distance between the vertices of the two catenaries to half the length of the chain is the tangent of half the angle of inclination of the portions near the pegs.

31. A heavy uniform string, 90 inches long, hangs over two smooth pegs at different heights. The parts which hang vertically are of lengths 30 and 33 inches. Prove that the vertex of the catenary divides the whole string in the ratio 4 : 5, and find the distance between the pegs. **(Meerut 2006; Purvanchal 07)**

32. A heavy string hangs over two fixed small smooth pegs. The two ends of the string are free and the central portion hangs in a catenary. Show that the free ends of the string are on the directrix of the catenary. If the two pegs are on the same level and distant $2a$ apart, show that the equilibrium is impossible unless the string is equal to or greater than $2ae$. **(Agra 2002, 08; Bundelkhand 06)**

2.7 Approximations to the Common Catenary

1. The cartesian equation of the common catenary is

$$y = c \cosh (x / c) = \frac{1}{2} c\,(e^{x/c} + e^{-x/c})$$

$$= \frac{c}{2}\left[\left\{1 + \left(\frac{x}{c}\right) + \frac{1}{2!}\left(\frac{x}{c}\right)^2 + \frac{1}{3!}\left(\frac{x}{c}\right)^3 + \frac{1}{4!}\left(\frac{x}{c}\right)^4 + \dots\right\}\right.$$

$$\left. + \left\{1 - \left(\frac{x}{c}\right) + \frac{1}{2!}\left(\frac{x}{2}\right)^2 - \frac{1}{3!}\left(\frac{x}{c}\right)^3 + \frac{1}{4!}\left(\frac{x}{c}\right)^4 - \dots\right\}\right]$$

$$= c\left[1 + \frac{1}{2!}\left(\frac{x}{c}\right)^2 + \frac{1}{4!}\left(\frac{x}{c}\right)^4 + \dots\right] \qquad \dots(1)$$

Now if x / c is small, then neglecting the powers of x / c higher than two, the equation (1) reduces to

$$y = c\left[1 + \frac{1}{2!}\left(\frac{x}{c}\right)^2\right] \quad \text{or} \quad x^2 = 2c\,(y - c),$$

which is the equation of a parabola of latus rectum $2c$ or $2T_0 / w$.

Thus if x is small compared to c, the common catenary coincides very nearly with a parabola of latus rectum $2c$ or $2T_0 / w$ and vertex at the point $(0, c)$.

Examples of such a case are the electric transmission wires and the telegraphic wires tightly stretched between the poles. Besides such cases of tightly stretched strings, even in the case of a common catenary not tightly stretched if we consider the portion of the curve near the vertex, x is small compared to c.

2. When x is large *i.e.* at points far removed from the lowest point, x / c is large and so $e^{-x/c}$ becomes very small, hence

$$y = c \cosh (x / c) = \frac{1}{2} c\,(e^{x/c} + e^{-x/c}) \text{ behaves as } y = \frac{1}{2} c\, e^{x/c},$$

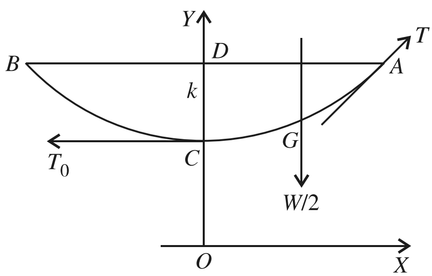
which is an exponential curve.

Hence at points far removed from the lowest point, a common catenary behaves as an exponential curve.

2.8 Sag of Tightly Stretched Wires

Consider a tightly stretched wire which appears nearly a straight line, as for example a telegraphic wire stretched tightly between the poles.

Let A and B be two points in a horizontal line between which a wire is stretched tightly. Let C be the lowest point of the catenary formed by the wire. Let W be the weight and l the length of the wire ACB. Also let T_0 be the horizontal tension at the lowest point C. The portion CA of the wire is in equilibrium under the action of the following forces :



(i) the tension T_0 acting horizontally at the point C,

(ii) the tension T at A acting along the tangent at A,

(iii) the weight $\frac{1}{2}W$ of the wire CA acting vertically downwards through its centre of gravity G.

Since the wire is tightly stretched, the distance of the centre of gravity G of the wire AC from the vertical line through A will be approximately equal to $\frac{1}{2}AC$ *i.e.* $\frac{1}{4}l$. Let k be the sag CD and a the span AB of the catenary.

Taking moments of the forces acting on the portion CA, about A, we have

$$T_0 \,.\, k = \frac{1}{2}W . \frac{1}{4}l$$

or $$T_0 = \frac{lW}{8k}. \qquad ...(1)$$

Now we calculate the increase in the length of the wire on account of the sag.

For a catenary, we have

$$s = c\sinh(x/c) = \frac{1}{2}c\,(e^{x/c} - e^{-x/c})$$

$$= \frac{c}{2}\left[\left\{1 + \left(\frac{x}{c}\right) + \frac{1}{2!}\left(\frac{x}{c}\right)^2 + \frac{1}{3!}\left(\frac{x}{c}\right)^3 + \frac{1}{4!}\left(\frac{x}{c}\right)^4 + \ldots\ldots\right\}\right.$$

$$\left. - \left\{1 - \left(\frac{x}{c}\right) + \frac{1}{2!}\left(\frac{x}{2}\right)^2 - \frac{1}{3!}\left(\frac{x}{c}\right)^3 + \frac{1}{4!}\left(\frac{x}{4}\right)^4 - \ldots\ldots\right\}\right]$$

$$= c\left[\frac{x}{c} + \frac{1}{3!}\left(\frac{x}{c}\right)^3 + \frac{1}{5!}\left(\frac{x}{c}\right)^5 + \ldots\ldots\right]. \qquad ...(2)$$

The radius of curvature ρ of the catenary is given by

$$\rho = c \sec^2 \psi.$$

At the vertex, $\psi = 0$ and so $\rho = c$ at the vertex. This shows that if the curve is flat near its vertex C so that ρ is large at the vertex, then c will be very large and x will be small as compared to c. Thus for a tightly stretched wire x / c is very small.

Thus retaining only the first two terms in (2), we have

$$s = c\left[\frac{x}{c} + \frac{1}{3\,!}\left(\frac{x}{c}\right)^3\right] = x + \frac{x^3}{6c^2}$$

or $$s - x = \frac{x^3}{6c^2}.$$

But $$c = T_0 / w. \qquad [\because T_0 = wc]$$

$\therefore$ $$s - x = (w^2 x^3)/6T_0^{\ 2}.$$

Now putting $x = \frac{1}{2} a$, where a is the span AB, we have

$$s - \frac{a}{2} = \text{arc } CA - DA = \frac{a^3 w^2}{48 T_0^{\ 2}}.$$

$\therefore$ *total increase in the length of the wire due to sagging*

$$= \textit{arc } ACB - \textit{span } AB = 2s - a = a^3\, w^2 / 24 T_0^{\ 2}.$$

Illustrative Examples

Example 9: *Show that the maximum tension in a wire which weighs* $0 \cdot 15$ *lb. per yard and hangs with a sag of* 1 *foot in a horizontal span of* 100 *feet is about* $62\frac{1}{2}$ *lbs. wt.*

(Rohilkhand 2008; Bundelkhand 08)

Solution: Refer figure of 2.8.

From the formula $T = wy$ it is obvious that the maximum tension in the wire will be at the extremities A or B.

Here $w = 0 \cdot 15/3 = 0 \cdot 05$ lb. per foot, and span $AB = 100$ feet.

The sag $CD = k = 1$ foot.

If T is the maximum tension in the wire at A, then

$$T = w y_A, \qquad \text{...(1)}$$

where y_A is the ordinate of the point A.

For a catenary, we have

$$y = c \cosh (x/c) = c\left[1 + \frac{1}{2\,!}\left(\frac{x}{c}\right)^2 + \frac{1}{4\,!}\left(\frac{x}{c}\right)^4 + \ldots\ldots\right] \qquad \text{...(2)}$$

Since sag 1 foot is very small compared to the span 100 feet, hence the wire is tightly stretched and x/c is very small.

$\therefore$ neglecting higher powers of x/c in (2), we have

$$y = c\left[1 + \frac{1}{2\,!}\left(\frac{x}{c}\right)^2\right] = c + \frac{x^2}{2c}. \qquad ...(3)$$

For the point A, $x = DA = \frac{1}{2}AB = 50$ feet. Therefore from (3),

we have $\quad y_A = c + \dfrac{(50)^2}{2c}$.

But $\quad y_A = c + k = c + 1.$

$\therefore \quad c + \dfrac{(50)^2}{2c} = c + 1 \quad$ or $\quad c = 1250.$

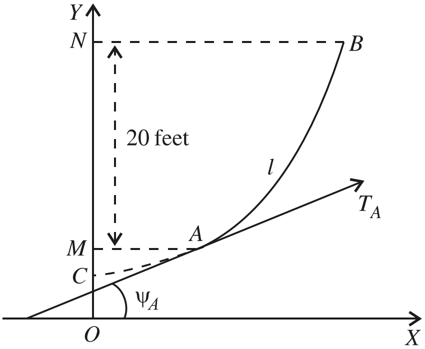
$\therefore \quad y_A = c + 1 = 1250 + 1 = 1251$ feet.

Hence from (3), the required maximum tension

$$T = wy_A = (\cdot 05) \times (1251) = 62 \cdot 55 \text{ lbs. wt.}$$
$$= 62\frac{1}{2} \text{ lbs. wt. nearly.}$$

Example 10: *A uniform chain has its ends fixed at A and B where B is 20 ft. above the level of A, and no part of the chain is below A. At A the chain is inclined at* $\sec^{-1}(5/3)$ *to the horizontal, and the tension there is equal to the weight of* 100 *ft. of chain. Prove that the length of the chain is* 23 *ft.* 11 *inches to the nearest inch.*

Solution: Let AB be the chain of length l feet with the end B at a height 20 ft. above the end A. The tension T_A at the point A acting along the tangent at A is inclined at an angle $\psi_A = \sec^{-1}(5/3)$ to the horizontal.



We have, $\sec\psi_A = 5/3$

and $\quad \tan\psi_A = \sqrt{(\sec^2\psi_A - 1)} = 4/3.$

If w is the weight of one foot length of the chain, then according to the question

$$T_A = 100\,w.$$

Let C be the vertex and OX the directrix of the catenary of which AB is an arc. Let y_A and y_B be the ordinates of the points A and B respectively and let arc $CA = a$ feet.

From $\quad T = wy$, we have $T_A = wy_A$.

But $\quad T_A = 100\,w.$

$\therefore \quad wy_A = 100\,w$

or $\quad y_A = 100$ feet.

Also we have $y_A = c\sec\psi_A$.

$\therefore \quad c = y_A\cos\psi_A = 100\,.\,\dfrac{3}{5} = 60$ feet.

$\therefore$ from $s = c \tan \psi$, for the point A, we have

$$\text{arc } CA = a = c \tan \psi_A = 60 \cdot \frac{4}{3} = 80.$$

Now $y_B = y_A + MN = 100 + 20 = 120$

and $\quad$ arc $CB = a + l = 80 + l$.

$\therefore$ from $y^2 = s^2 + c^2$, we have

$$y_B{}^2 = (l + 80)^2 + 60^2$$

or $\quad 120^2 = (l + 80)^2 + 60^2$

or $\quad (l + 80)^2 = 120^2 - 60^2$

$$= (120 - 60)(120 + 60) = 60 \times 180.$$

$\therefore \quad l + 80 = \pm 60 \sqrt{3}$

or $\quad l = -80 + 60\sqrt{3}$,

neglecting the negative sign otherwise l will be negative

or $\quad l = (60 \times 1 \cdot 732 - 80)$ ft.

$$= 23 \cdot 92 \text{ ft.} = 23 \text{ ft. } 11 \text{ in nearly.}$$

Example 11: *A uniform chain, of length 2l, has its ends attached to two points in the same horizontal line at a distance 2a apart. If l is only a little greater than a, show that the tension of the chain is approximately equal to the weight of a length*

$$\sqrt{\left[\frac{a^3}{6(l-a)}\right]}$$

of the chain, and that the sag or depression of the lowest point of the chain below its ends is nearly

$$\frac{1}{2}\sqrt{\{6a(l-a)\}}.$$

(Rohilkhand 2010)

Solution: [Refer to figure of 2.8]

Since the chain is tightly stretched, the tension at any point shall be the same nearly.

$\therefore$ the tension in the chain $= T = T_0$ (tension at the lowest point)

or $\quad T = wc.$ ...(1)

Here span $= 2a$ and the length of the chain $= 2l$.

$\therefore$ at either of the supports,

$$x = a \text{ and } s = l.$$

$\therefore$ from $s = c \sinh (x/c)$, we have

$$l = c \sinh \frac{a}{c} = c\left[\frac{a}{c} + \frac{1}{3!}\left(\frac{a}{c}\right)^3 + \ldots\ldots\right] \quad \ldots(2)$$

Since the chain is tightly stretched, hence c is large *i.e.*, a/c is very small.

$\therefore$ neglecting higher powers of a/c in (2), we have

$$l = c\left[\frac{a}{c} + \frac{1}{3\,!}\left(\frac{a}{c}\right)^3\right] = a + \frac{a^3}{6c^2}$$

or $$\frac{a^3}{6c^2} = l - a$$

or $$c = \sqrt{\left\{\frac{a^3}{6\,(l-a)}\right\}}.$$

$\therefore$ from (1), we have

$$T = w\sqrt{\left\{\frac{a^3}{6\,(l-a)}\right\}}.$$

Hence the tension of the chain is approximately equal to the weight of a length $\sqrt{\left\{\frac{a^3}{6\,(l-a)}\right\}}$ of the chain.

At the support, $x = a$.

$\therefore$ from $y = c \cosh (x / c)$,

at the support $y = y_A = c \cosh (a/c)$.

$\therefore$ sag in the middle

$$= y_A - c = c \cosh (a/c) - c$$

$$= c\left[1 + \frac{1}{2\,!}\left(\frac{a}{c}\right)^2 + \frac{1}{4\,!}\left(\frac{a}{c}\right)^4 + \ldots - 1\right]$$

$$= c\left[\frac{a^2}{2c^2}\right] \text{ nearly, neglecting higher powers of } a/c$$

$$= \frac{a^2}{2c}$$

$$= \frac{a^2}{2}\sqrt{\left\{\frac{6\,(l-a)}{a^3}\right\}}$$

$$= \frac{1}{2}\sqrt{\{6a\,(l-a)\}}.$$

Example 12: *A telegraph wire is stretched between two poles at a distance 'a' metres apart sags 'n' metres in the middle; prove that the tension at the ends is approximately*

$$w\left(\frac{a^2}{8\,n} + \frac{7}{6}\,n\right).$$

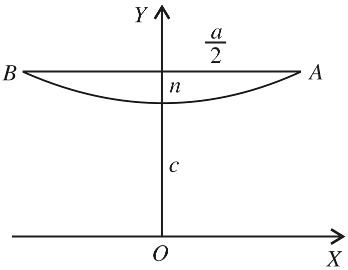
(Avadh 2007, 09; Rohilkhand 09; Bundelkhand 06, 11)

Solution: Applying $y = c \cosh \frac{x}{c}$ for A, we get

$$n + c = c \cosh \frac{a}{2c}$$

$$= c\left[1 + \frac{1}{2\,!}\cdot\frac{a^2}{4c^2} + \frac{1}{4\,!}\cdot\frac{a^4}{16\,c^4} + \ldots\right]$$

$$= c + \frac{a^2}{8c} + \frac{a^4}{24 \times 16\, c^3} + \ldots$$



or $$n = \frac{a^2}{8c} + \frac{a^4}{24 \times 16\, c^3} + \ldots \quad \ldots(1)$$

Since the wire is tightly stretched, c is very large, so that

$$n = \frac{a^2}{8c}, \text{ to a first approximation}$$

i.e., $$c = \frac{a^2}{8\,n}.$$

Substituting this value of c in (1), we get

$$n = \frac{a^2}{8c} + \frac{a^4 \times 8 \times 8 \times 8 \cdot n^3}{24 \times 16 \times a^6}$$

or $$\frac{a^2}{8c} = n - \frac{4n^3}{3a^2} = n\left(1 - \frac{4n^2}{3a^2}\right)$$

or $$\frac{8c}{a^2} = \frac{1}{n}\left(1 - \frac{4n^2}{3a^2}\right)^{-1}$$

or $$c = \frac{a^2}{8\,n}\left[1 + \frac{4n^2}{3a^2} + \ldots\right]$$

[Expanding by binomial theorem and neglecting higher powers of $\frac{n}{a}$ because n is very small]

$$= \frac{a^2}{8\,n} + \frac{1}{6}\,n.$$

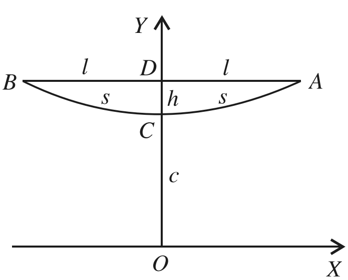
$\therefore$ tension at the ends

$$= w\,(n + c) \qquad [\because\ y = n + c \text{ for } A \text{ and } T = wy]$$

$$= w\left(n + \frac{a^2}{8\,n} + \frac{1}{6}\,n\right) = w\left(\frac{a^2}{8\,n} + \frac{7}{6}\,n\right) \text{ nearly.}$$

Example 13: *ACB is a telegraph wire, the straight line AB being horizontal and of length 2l and C, the middle point of the wire, is at a distance h below AB. Show that the length of the wire is approximately*

$$2\,l + \frac{4h^2}{3\,l} - \frac{28\,h^4}{45\,l^3}. \qquad \textbf{(Agra 2002)}$$



Solution: We have

$$s = c \sinh \frac{x}{c} = c \sinh \frac{l}{c} \quad \text{[For the point } A]$$

$$= c\left[\frac{l}{c} + \frac{l^3}{3\,!\cdot c^3} + \frac{l^5}{5\,!\cdot c^5} + \ldots\right]$$

$$= l + \frac{l^3}{6c^2} + \frac{l^5}{120c^4} + \ldots \qquad \ldots(1)$$

Again $y = c \cosh \frac{x}{c}$ gives

$$h + c = c \cosh \frac{l}{c} \qquad \text{[For the point } A\text{]}$$

$$= c\left[1 + \frac{l^2}{2c^2} + \frac{l^4}{4!.c^4} + \ldots\right] = c + \frac{l^2}{2c} + \frac{l^4}{24c^3} + \ldots$$

or $$h = \frac{l^2}{2c} + \frac{l^4}{24c^3}. \qquad \ldots(2)$$

$\therefore$ $h = \dfrac{l^2}{2c}$, to a first approximation because c is very large

or $$c = \frac{l^2}{2h}.$$

Substituting this value of c in (2), we get

$$h = \frac{l^2}{2c} + \frac{l^4 \times 8h^3}{24 \times l^6}$$

or $$\frac{l^2}{2c} = h - \frac{h^3}{3l^2} = h\left(1 - \frac{h^2}{3l^2}\right)$$

or $$c = \frac{l^2}{2h}\left(1 - \frac{h^2}{3l^2}\right)^{-1}$$

$$= \frac{l^2}{2h}\left(1 + \frac{h^2}{3l^2} + \ldots\right) = \frac{l^2}{2h} + \frac{h}{6}.$$

Substituting this value of c in (1), we get

$$s = l + \frac{l^3}{6}\left(\frac{l^2}{2h} + \frac{h}{6}\right)^{-2} + \frac{l^5}{120}\left(\frac{l^2}{2h} + \frac{h}{6}\right)^{-4}$$

$$= l + \frac{l^3}{6} \cdot \left(\frac{l^2}{2h}\right)^{-2}\left(1 + \frac{h^2}{3l^2}\right)^{-2} + \frac{l^5}{120}\left(\frac{l^2}{2h}\right)^{-4}\left(1 + \frac{h^2}{3l^2}\right)^{-4}$$

$$= l + \frac{2h^2}{3l}\left(1 - \frac{2h^2}{3l^2} + \ldots\right) + \frac{16h^4}{120l^3},$$

neglecting higher powers of h above h^4 because h is very small

or $$2s = 2l + \frac{4h^2}{3l} - \frac{28}{45} \cdot \frac{h^4}{l^3} \text{ nearly.}$$

Comprehensive Exercise 2

1. A telegraph is constructed of No. 8 iron wire which weighs $7 \cdot 3$ lbs. per 100 feet; the distance between the posts is 150 feet and the wire sags 1 foot in the middle. Show that it is screwed upto a tension of about 205 lbs. wt.

(Bundelkhand 2007)

2. The ends of a tightly stretched cable weighing $\frac{1}{2}$ lb. per yard are fixed to two points on a level 80 yards apart and the cable has a sag, at the middle of 1 foot 4 inches. Find the tension in the wire at the lowest point.

[**Ans**. 900 lbs. wt.]

3. A heavy uniform string, 155 ft. long, is suspended from two points A and B, 150 ft. apart on the same horizontal plane. Show that the tension at the lowest point is approximately equal to $1 \cdot 08$ times the weight of the string.

4. A uniform measuring chain of length l is tightly stretched over a river, the middle point just touching the surface of the water, while each of the extremities has an elevation k above the surface. Show that the difference between the length of the measuring chain and the breadth of the river is nearly $\dfrac{8k^2}{3l}$.

5. A chain is suspended in a vertical plane from two fixed supports A and B, which lie in a horizontal line 462 feet apart. If the tangent to the chain at A be inclined at an angle $\tan^{-1}(3/4)$ to the horizon, find the length of the chain.

(Take $\log_e 2 = 0 \cdot 693$).

6. A kite is flown with 600 ft. of string from the hand to the kite, and a spring balance held in hand shows a pull equal to the weight of 100 ft. of the string, inclined at $30°$ to the horizontal. Find the vertical height of the kite above the hand. [**Ans.** $555 \cdot 7$ ft. nearly]

7. A telegraph wire is made of a given material, and such a length l is stretched between two posts, distant d apart and of the same height, as will produce the least possible tension at the posts. Show that $l = (d/\lambda) \sinh \lambda$, where λ is given by the equation $\lambda \tanh \lambda = 1$. **(Kumaun 2002)**

8. If the length of a uniform chain suspended between two posts at the same level is adjusted so that the tension at the posts of support is a minimum for that particular span $2d$, show that the equation to determine c is $\coth \dfrac{d}{c} = \dfrac{d}{c}$.

9. A uniform chain is hung up from two points at the same level and distant $2a$ apart. If z is the sag at the middle, show $z = c\left(\cosh \dfrac{a}{c} - 1\right)$.

If z is small compared with a, show that

$$2cz = a^2 \text{ nearly.}$$

(Kumaun 2001)

10. A telegraph wire is supported by two poles distant 40 yards apart. If the sag be one foot and the weight of the wire half an ounce per foot, show that the horizontal pull on each pole is $\frac{1}{2}$ cwt nearly.

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. For the catenary $y = c \cosh (x/c)$, **(Rohilkhand 2008, 10)**

(a) $s = c \sin (x/c)$ (b) $y = c \tan \psi$

(c) $s = c \sinh (x/c)$ (d) $s = c \sec \psi$.

2. For the catenary $y = c \cosh (x/c)$,

(a) $x = c \log \left(\dfrac{y + s}{c} \right)$ (b) $x = c \tan \psi$

(c) $s = c \cos (x/c)$ (d) $x = c \cos \psi$. **(Bundelkhand 2008)**

3. The point of common catenary, where tangent and hence the tension is horizontal is called:

(a) Axis (b) Directrix

(c) Span (d) Vertex **(Garhwal 2002)**

4. Two catenaries are said to be equal, if:

(a) their parameters are equal

(b) their directrices are the same

(c) their axes are the same

(d) their spans are unique. **(Garhwal 2003)**

5. Tension at any point of the catenary is:

(a) wx (b) wy

(c) wc (d) $w\psi$.

(Rohilkhand 2008; Avadh 06; Garhwal 04)

Fill in the Blank(s)

Fill in the blanks "……" so that the following statements are complete and correct.

1. When a uniform chain whose weight per unit length is constant hangs freely under gravity between two points not in the same vertical line, the curve in which it hangs, is called a …… .

2. The intrinsic equation of a common catenary is …… . **(Agra 2007)**

3. In the equation $s = c \tan \psi$ of a catenary, 'c' is called the …… of the catenary.

4. If a uniform string hangs in the form of a common catenary, then the horizontal component of the tension at every point of the string remains …… .

5. In the case of a common catenary if the vertical line through the vertex is taken as the y-axis and a horizontal line at a depth c below the vertex is taken as the x-axis, then the cartesian equation of the catenary is …… .

6. The horizontal line at a depth c below the vertex of the catenary

$$y = c \cosh (x/c)$$

is called the …… of the catenary.

7. For the catenary $y = c \cosh (x/c)$, $y^2 = c^2 + \dots\dots$ **(Meerut 2004)**

8. The parametric equations of the catenary

$$y = c \cosh (x/c) \text{ are } x = \dots\dots, \; y = c \sec \psi.$$

9. The tension at any point of a string hanging in the form of a common catenary varies as the height of the point above the ....

10. If a uniform string whose weight per unit length is w hangs in the form of a common catenary whose parameter is c, then the tension T_0 at the vertex of the catenary is given by $T_0 = \dots\dots$

True or False

Write 'T' for true and 'F' for false statement.

1. For the catenary $y = c \cosh (x/c)$, $y = c \sec \psi$.
2. If a uniform string suspended from two points B and C in the same horizontal line hangs in the form of a common catenary whose vertex is A, then the depth of A below BC is called the span of the catenary.
3. If a uniform string hangs in the form of a catenary $y = c \cosh (x/c)$ and T is the tension at any point of the string, then $T = wy$ where w is the weight per unit length of the string.
4. If a uniform string hangs in the form of a catenary $y = c \cosh (x/c)$ and T is the tension at any point P of the string, then $T = ws$, where w is the weight per unit length of the string and s is the arc length AP where A is the vertex of the catenary.

Answers

Multiple Choice Question

1. (c)	2. (a)	3. (d)	4. (a)	5. (b)

Fill in the Blank(s)

1. common catenary	2. $s = c \tan \psi$
3. parameter	4. constant
5. $y = c \cosh (x/c)$	6. directrix
7. s^2	8. $c \log (\sec \psi + \tan \psi)$
9. directrix	10. wc

True or False

1. T	2. F
3. T	4. F

□

Chapter

3

Virtual Work

3.1 Displacement

Suppose a particle moves from a position P to any other position Q by whatever path. Then the vector $\vec{PQ}$ is called the displacement of the particle with regard to P. If $\mathbf{r}$ and $\mathbf{r}'$ be the position vectors of the points P and Q referred to some origin O, then the displacement of the particle from P to Q is the vector

$$\vec{PQ} = \mathbf{r}' - \mathbf{r}.$$

3.2 A Rigid Body

A rigid body is a collection of particles such that for any displacement of the body the distance between any two particles of the body remains the same in magnitude. Thus in the case of the rigid body, referred to some origin O, if $\mathbf{r_1}, \mathbf{r_2}$ are respectively the position vectors of the two particles before displacement and $\mathbf{r_1}', \mathbf{r_2}'$ are their respective position vectors after displacement, then the condition of rigidity of the body requires that their mutual distance must remain the same before and after the displacement *i.e.*,

$$|\mathbf{r_2} - \mathbf{r_1}| = |\mathbf{r_2}' - \mathbf{r_1}'|, \quad \text{or} \quad (\mathbf{r_2} - \mathbf{r_1})^2 = (\mathbf{r_2}' - \mathbf{r_1}')^2.$$

3.3 Kinds of Displacement of a Rigid Body

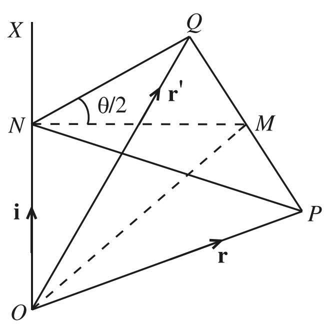
(Translation, Rotation and General).

One way of displacing a particle of a rigid body from one position to any other position is what we call *pure translation*. In this case the displacement is brought about without rotating the body. Thus if **r** be the position vector of a particle P referred to some origin O and if the particle is displaced from P to Q by giving a displacement **u** in the direction of OP only, then this displacement is called *translation* and we say that the displacement $\vec{PQ} = \mathbf{u}$ is a translation.

The other way of displacement of a particle is called *pure rotation*. In this case the displacement of the particle is brought about only by rotating the body about a fixed point, say O, so that the distance of the particle from the fixed point O does not change in the two positions P and Q before and after the displacement. Thus in the case of pure rotation, we have $OP = OQ$ in length but their directions are generally different.

If both the displacements translation and rotation take place simultaneously we call it a *general displacement* of the particle or of the body.

3.4 Rotation of a Rigid Body about a Point



Suppose a rigid body rotates about a fixed point O. On account of this rotation suppose a particle is displaced from P to Q where

$$\vec{OP} = \mathbf{r} \quad \text{and} \quad \vec{OQ} = \mathbf{r}'.$$

Since the displacement of the particle is that of rotation only about O, therefore the length OP = the length OQ.

Let M be the middle point of PQ, so that

$$\vec{OM} = \frac{1}{2}(\mathbf{r} + \mathbf{r}').$$

Draw a line OX, through O, perpendicular to PQ and let **i** be a unit vector in the direction OX. Also let N be the foot of the perpendicular from M to OX.

Since $OP = OQ$ and M is the middle point of PQ, therefore from the $\Delta\ OPQ$ we observe that PQ is perpendicular to OM.

Thus PQ, being perpendicular to OM and ON both, is perpendicular to the plane OMN. Consequently PQ is perpendicular to MN because MN lies in the plane OMN. Thus NM is the perpendicular bisector of PQ and so we have $NP = NQ$.

If $\angle PNQ = \theta$, then $\angle PNM = \angle MNQ = \dfrac{1}{2}\theta$.

The vector $\vec{PQ}$ is perpendicular to the vectors $\mathbf{i}$ and $\vec{OM}$ which implies that $\vec{PQ}$ is parallel to the vector $\mathbf{i} \times \vec{OM}$.

Also $$|\vec{PQ}| = PQ = 2\ PM = 2\ NM \tan \frac{1}{2}\theta$$
$$= 2\ (OM \sin \angle MON)\tan \frac{1}{2}\theta = 2\,|\mathbf{i} \times \vec{OM}|\tan \frac{1}{2}\theta.$$
$$[\because\ |\mathbf{i} \times \vec{OM}| = 1 . OM . \sin \angle MON\ ]$$

Thus $\vec{PQ}$ is parallel to the vector $\mathbf{i} \times \vec{OM}$ and

$$|\vec{PQ}| = \left(2 \tan \frac{1}{2}\theta\right)|\mathbf{i} \times \vec{OM}|.$$

Therefore by the definition of the multiplication of a vector by a scalar, we have

$$\vec{PQ} = \left(2 \tan \frac{1}{2}\theta\right)\mathbf{i} \times \vec{OM} = \left(2 \tan \frac{1}{2}\theta\,\mathbf{i}\right) \times \vec{OM}\ .$$

Thus if $\mathbf{q}$ is the displacement of the particle from P to Q due to this rotation of the rigid body, we have

$$\mathbf{q} = \vec{PQ} = \mathbf{e} \times \mathbf{h}, \text{ where}$$
$$\mathbf{e} = \left(2 \tan \frac{1}{2}\theta\right)\mathbf{i} \quad \text{and} \quad \mathbf{h} = \vec{OM} = \frac{1}{2}(\mathbf{r} + \mathbf{r}').$$

The vector $\mathbf{e}$ is called the finite rotation about O which brings the particle from $\mathbf{r}$ to $\mathbf{r}'$, the direction of the vector $\mathbf{e}$ is called the axis of rotation and θ is called the angle of rotation.

When the rotation is small, Q tends to P *i.e.*, $\mathbf{r}'$ tends to $\mathbf{r}$, and then we have

$$\mathbf{h} = \frac{1}{2}(\mathbf{r} + \mathbf{r}') = \frac{1}{2}(\mathbf{r} + \mathbf{r}) = \mathbf{r}$$

which leads to $\mathbf{q} = \mathbf{e} \times \mathbf{r}.$

Remark. It can be easily seen that the displacement about a point is always a rotation. Also it can be easily shown that any displacement of a rigid body can be reduced to a translation together with a rotation.

3.5 Position Vector of a Point After a General Displacement

Let $\mathbf{r}$ be the position vector of a point P referred to some origin O. If the particle is displaced from P to Q by giving only a displacement $\mathbf{u}$ in the direction of OP (*i.e.*, translation), then

$$\vec{PQ} = \mathbf{u}.$$

Also if P is displaced to Q by giving only a rotation $\mathbf{e}$ about O, then

$$\vec{PQ} = \mathbf{e} \times \frac{1}{2}(\mathbf{r} + \mathbf{r}')$$

where $\mathbf{r}$ and $\mathbf{r}'$ are the position vectors of P and Q respectively.

Now if the particle is displaced from P to Q by giving both the displacements translation $\mathbf{u}$ and rotation $\mathbf{e}$ simultaneously, then it is called a general displacement of the point. In this case, combining the above results, we have

$$\vec{PQ} = \mathbf{u} + \mathbf{e} \times \frac{1}{2}(\mathbf{r} + \mathbf{r}'). \qquad \ldots(1)$$

If this displacement is small, then writing $\mathbf{r}' = \mathbf{r} + d\mathbf{r}$ in the above result (1), we have

$$\vec{PQ} = \mathbf{u} + \mathbf{e} \times \frac{1}{2}(\mathbf{r} + \mathbf{r} + d\mathbf{r})$$

$$= \mathbf{u} + \mathbf{e} \times \mathbf{r} + \frac{1}{2}\mathbf{e} \times d\mathbf{r}$$

$$= \mathbf{u} + \mathbf{e} \times \mathbf{r}, \qquad \text{because both the vectors } \mathbf{e} \text{ and } d\mathbf{r} \text{ are small.}$$

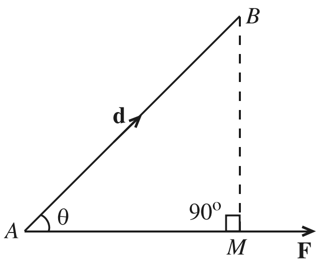
But $\quad \vec{PQ} = \vec{OQ} - \vec{OP} = (\mathbf{r} + d\mathbf{r}) - \mathbf{r} = d\mathbf{r}.$

Therefore if $\mathbf{r}$ is the position vector of a point P and if it is given a small displacement $d\mathbf{r}$ consisting of a small translation $\mathbf{u}$ and a small rotation $\mathbf{e}$, then we have

$$d\mathbf{r} = \mathbf{u} + \mathbf{e} \times \mathbf{r}. \qquad \textbf{[Remember]}$$

3.6 Work Done by a Force

Suppose a force represented by the vector $\mathbf{F}$ acts at the point A. Let the point A be displaced to the point B where $\vec{AB} = \mathbf{d}$.



Then the work W done by the force $\mathbf{F}$ during the displacement $\mathbf{d}$ of its point of application is defined as

$$W = \mathbf{F} \cdot \mathbf{d}, \qquad \ldots(1)$$

where $\mathbf{F} \cdot \mathbf{d}$ is the scalar product of the vectors $\mathbf{F}$ and $\mathbf{d}$.

Let θ be the angle between the vectors $\mathbf{F}$ and $\mathbf{d}$. If $F = |\mathbf{F}|$ and $d = |\mathbf{d}| = AB$, then using the definition of the scalar product of two vectors, the equation (1) defining the work may be written as

$$W = Fd\cos\theta. \qquad \ldots(2)$$

Obviously $d\cos\theta$ is the displacement of the point of application of the force $\mathbf{F}$ in the direction of the force. *Hence the work done by a force is equal to the magnitude of the force multiplied by the displacement of the point of application of the force in the direction of the force.*

From the equation (2) we make the following observations.

(i) If $\theta = \frac{1}{2}\pi$ *i.e.*, if the displacement of the point of application of the force is perpendicular to the direction of the force, then $W = 0$.

(ii) If $0 \le \theta < \frac{1}{2}\pi$ *i.e.*, if the displacement of the point of application of the force parallel to the line of action of the force is in the direction of the force, then W is positive.

(iii) If $\frac{1}{2}\pi < \theta \le \pi$ *i.e.*, if the displacement of the point of application of the force parallel to the line of action of the force is opposite to the direction of the force, then W is negative.

Remark: The work done by a force **F** acting at the point **r** during a small displacement $d\mathbf{r}$ of its point of application is

$$= \mathbf{F} \cdot d\mathbf{r}.$$

3.7 Work Done by a System of Concurrent Forces

Theorem 1: *The work done by the resultant of a number of concurrent forces is equal to the sum of the works done by the separate forces.*

Proof. Let there be n forces represented by the vectors $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ acting at a point P. Then during any displacement of P represented by the vector **d,** the works done by the separate forces are respectively equal to

$$\mathbf{F}_1 \cdot \mathbf{d},\ \mathbf{F}_2 \cdot \mathbf{d}, \ldots, \mathbf{F}_n \cdot \mathbf{d}.$$

The total work done is therefore

$$= \mathbf{F}_1 \cdot \mathbf{d} + \mathbf{F}_2 \cdot \mathbf{d} + \ldots + \mathbf{F}_n \cdot \mathbf{d}$$

$$= (\mathbf{F}_1 + \mathbf{F}_2 + \ldots + \mathbf{F}_n) \cdot \mathbf{d} \qquad [\because \text{ scalar product is distributive}]$$

$= \mathbf{R} \cdot \mathbf{d}$, where $\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \ldots + \mathbf{F}_n$ is the vector representing the resultant of these n concurrent forces.

But $\mathbf{R} \cdot \mathbf{d}$ is the work done by the resultant **R** during the displacement **d** of the point P. Hence we have the result.

Example: *A particle acted on by constant forces* $4\,\mathbf{i} + \mathbf{j} - 3\,\mathbf{k}$ *and* $3\,\mathbf{i} + \mathbf{j} - \mathbf{k}$ *is displaced from the point* $\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$ *to the point* $5\,\mathbf{i} + 4\,\mathbf{j} + \mathbf{k}$*. Find the total work done by the forces.*

Solution: Let **R** be the resultant of the two concurrent forces and **d** be the displacement of their point of application. Then, we have

$$\mathbf{R} = (4\,\mathbf{i} + \mathbf{j} - 3\,\mathbf{k}) + (3\,\mathbf{i} + \mathbf{j} - \mathbf{k}) = 7\,\mathbf{i} + 2\,\mathbf{j} - 4\,\mathbf{k}$$

and $\mathbf{d} = (5\,\mathbf{i} + 4\,\mathbf{j} + \mathbf{k}) - (\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}) = 4\,\mathbf{i} + 2\,\mathbf{j} - 2\,\mathbf{k}.$

$\therefore$ the total work done $= \mathbf{R} \cdot \mathbf{d}$

$$= (7\,\mathbf{i} + 2\,\mathbf{j} - 4\,\mathbf{k}) \cdot (4\,\mathbf{i} + 2\,\mathbf{j} - 2\,\mathbf{k})$$

$$= 28 + 4 + 8 = 40 \quad \text{units of work.}$$

3.8 Work Done by a Couple During a Small Displacement

Let the two forces $\mathbf{F}$ and $-\mathbf{F}$ acting on a rigid body at the points whose position vectors are $\mathbf{r}_1$ and $\mathbf{r}_2$, be equivalent to a couple of moment $\mathbf{G}$, then

$$\mathbf{G} = \mathbf{r}_1 \times \mathbf{F} + \mathbf{r}_2 \times (-\mathbf{F}) = (\mathbf{r}_1 - \mathbf{r}_2) \times \mathbf{F}.$$

Suppose the body undergoes a small displacement consisting of a uniform translation $\mathbf{u}$ and a small rotation $\mathbf{e}$. Then

$$d\mathbf{r}_1 = \mathbf{u} + \mathbf{e} \times \mathbf{r}_1$$

and $$d\mathbf{r}_2 = \mathbf{u} + \mathbf{e} \times \mathbf{r}_2 . \qquad \text{[Refer 3.5]}$$

$\therefore$ the work done by the couple

$$= \mathbf{F} \cdot d\mathbf{r}_1 + (-\mathbf{F}) \cdot d\mathbf{r}_2$$

$$= \mathbf{F} \cdot (\mathbf{u} + \mathbf{e} \times \mathbf{r}_1) + (-\mathbf{F}) \cdot (\mathbf{u} + \mathbf{e} \times \mathbf{r}_2)$$

$$= \mathbf{F} \cdot (\mathbf{e} \times \mathbf{r}_1) - \mathbf{F} \cdot (\mathbf{e} \times \mathbf{r}_2)$$

$$= \mathbf{e} \cdot (\mathbf{r}_1 \times \mathbf{F}) - \mathbf{e} \cdot (\mathbf{r}_2 \times \mathbf{F}), \text{ by a property of scalar triple product}$$

$$= \mathbf{e} \cdot (\mathbf{r}_1 - \mathbf{r}_2) \times \mathbf{F} = \mathbf{e} \cdot \mathbf{G},$$

which is independent of the translation and depends upon rotation only.

3.9 Work Done by a System of Forces During a Small Displacement

Let a system of forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ act at the points of a rigid body whose position vectors with respect to some origin O are $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$ respectively. Suppose this system of forces is equivalent to a single force $\mathbf{R}$ acting at O, together with a couple of moment $\mathbf{G}$. Then

$$\mathbf{R} = \sum_{p=1}^{n} \mathbf{F}_p \quad \text{and} \quad \mathbf{G} = \sum_{p=1}^{n} \mathbf{r}_p \times \mathbf{F}_p . \qquad \ldots(1)$$

If the body undergoes a small displacement consisting of a uniform translation $\mathbf{u}$ and a small rotation $\mathbf{e}$ about O, then for a typical particle displaced from $\mathbf{r}_p$ to $\mathbf{r}_p + d\mathbf{r}_p$, the general displacement $d\mathbf{r}_p$ is given by

$$d\mathbf{r}_p = \mathbf{u} + \mathbf{e} \times \mathbf{r}_p . \qquad \ldots(2)$$

$\therefore$ the work done by the system of forces during this small displacement

$$= \sum_{p=1}^{n} \mathbf{F}_p \cdot d\mathbf{r}_p$$

$$= \sum_{p=1}^{n} \mathbf{F}_p \cdot (\mathbf{u} + \mathbf{e} \times \mathbf{r}_p) \qquad \text{[by (2)]}$$

$$= \mathbf{u} \cdot \sum_{p=1}^{n} \mathbf{F}_p + \mathbf{e} \cdot \sum_{p=1}^{n} \mathbf{r}_p \times \mathbf{F}_p, \text{ as } \mathbf{u} \text{ and } \mathbf{e} \text{ are constant vectors}$$

$$= \mathbf{u} \cdot \mathbf{R} + \mathbf{e} \cdot \mathbf{G}, \text{ by (1).}$$

3.10 Virtual Displacement and Virtual Work

(Meerut 2009)

If a number of forces act on a body and displace it, these forces do some work actually. But if the forces are in equilibrium, then they do not displace their points of application and so there is actually no work done by these forces. However, if we imagine that the forces in equilibrium undergo some small displacement and find out the work done by the forces during that displacement, then such a displacement is called **virtual displacement** and such a work is called **virtual work.**

3.11 The Principle of Virtual Work

The necessary and sufficient condition that a particle or a rigid body acted upon by a system of coplanar forces be in equilibrium is that the algebraic sum of the virtual works done by the forces during any small displacement consistent with the geometrical conditions of the system is zero to the first degree of approximation. **(Lucknow 2006, 08; Meerut 06, 08, 11; Bundelkhand 07,11; Avadh 09, 11; Purvanchal 10, 11)**

Proof. Let a system of forces $\mathbf{F_1}, \ldots\ldots, \mathbf{F}_n$ act at the points of a rigid body whose position vectors with respect to some origin O are $\mathbf{r_1}, \ldots\ldots, \mathbf{r}_n$. Suppose this system of forces is equivalent to a single force $\mathbf{R} = \Sigma\, \mathbf{F_1}$ acting at O, together with a couple of moment $\mathbf{G} = \Sigma\, \mathbf{r_1} \times \mathbf{F_1}$. Then during any small displacement of the body consisting of a uniform translation $\mathbf{u}$ and a small rotation $\mathbf{e}$ about O, the sum of the works done by these forces

$$= \Sigma\, \mathbf{F_1} \cdot d\mathbf{r_1} = \Sigma\, \mathbf{F_1} \cdot (\mathbf{u} + \mathbf{e} \times \mathbf{r_1})$$

$$= \mathbf{u} \cdot \Sigma\, \mathbf{F_1} + \mathbf{e} \cdot \Sigma\, \mathbf{r_1} \times \mathbf{F_1}$$

$$= \mathbf{u} \cdot \mathbf{R} + \mathbf{e} \cdot \mathbf{G}. \qquad \ldots(1)$$

The condition is necessary. Suppose the given system of forces is in equilibrium. Then $\mathbf{R} = \mathbf{0}$ and $\mathbf{G} = \mathbf{0}.$ Therefore, from (1), the sum of the works done by the forces is zero. Hence the condition is necessary.

The condition is sufficient. Suppose the sum of the works done by the forces during **any small displacement** is zero. Then to prove that the forces are in equilibrium. We have, from (1)

$$\mathbf{u} \cdot \mathbf{R} + \mathbf{e} \cdot \mathbf{G} = 0, \qquad \ldots(2)$$

for any small displacement consisting of a uniform translation $\mathbf{u}$ and a small rotation $\mathbf{e}$ about O.

Since the result (2) holds for any small displacement, therefore taking $\mathbf{e} = \mathbf{0}$ and $\mathbf{u} \neq \mathbf{0},$ we get from (2)

$$\mathbf{u} \cdot \mathbf{R} = 0. \qquad \ldots(3)$$

Again taking $\mathbf{u}$ not perpendicular to $\mathbf{R},$ we get from (3)

$$\mathbf{R} = \mathbf{0}.$$

Now taking $\mathbf{e} \neq \mathbf{0}$ and $\mathbf{u} = \mathbf{0},$ we get from (2)

$$\mathbf{e} \cdot \mathbf{G} = 0. \qquad \ldots(4)$$

Again taking **e** not perpendicular to **G,** we get from (4)

$$\mathbf{G} = \mathbf{0}.$$

Thus if the result (2) holds for any small displacement **u** and **e,** we must have **R = 0** and **G = 0.** Hence the forces are in equilibrium and this proves that the condition is sufficient.

Remark 1: The equation (2) formed by equating to zero the sum of the virtual works done by the forces is called **the equation of virtual work.**

Remark 2: The above principle of virtual work and its proof equally holds whether the forces are coplanar or not and whether the forces act upon a particle or upon a rigid body.

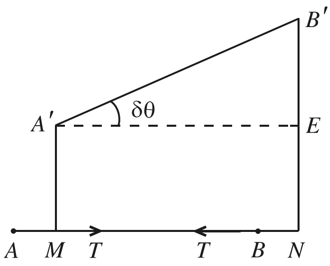
3.12 Forces Which are Omitted in Forming the Equation of Virtual Work

(Meerut 2004, 06; Avadh 06 07; Purvanchal 08; Lucknow 09; Garhwal 02; Kumaun 02)

The principle of virtual work gives us a very powerful method of attacking problems on equilibrium of forces. The mechanical advantage of this principle over other methods is that there are certain forces which are omitted in forming the equation of virtual work and consequently the solution of the problem becomes easy by this method. We now mention with proof the forces which are omitted in forming the equation of virtual work.

(i) The work done by the tension of an inextensible string is zero during a small displacement. **(Lucknow 2007, 10)**

Let AB be an inextensible string of length l joining two points A and B of a rigid body. Let T be the tension in the string AB. After a small displacement let $A'B'$ be the position of the string and $\delta\theta$ be the small angle between AB and $A'B'$. Since the string is inextensible, therefore $A'B' = AB = l$. Draw $A'M$ and $B'N$ perpendiculars to AB. Also draw $A'E$ perpendicular to $B'N$.



On account of the tension in the string AB, there are two forces each equal to T acting on A and B in opposite directions as shown in the figure. After displacement A moves to A' and B moves to B'. The work done by the tension of the string AB during this displacement

$= T \, . \, AM - T \, . \, BN$ [Note that the displacement of B is in a direction opposite to that of the force T]

$= T \, . \, (AB - MB) - T \, . \, (MN - MB)$

$= T \, . \, (AB - MN)$

$= T \, . \, (AB - A'E) = T \, . \, (AB - A'B' \cos \delta\theta)$

$= T \, . \, (l - l \cos \delta\theta)$ [$\because \; AB = A'B' = l$]

$$= T \,.\, l\,(1 - \cos \delta\theta)$$

$$= T \,.\, l\left[1 - \left\{1 - \frac{(\delta\theta)^2}{2\,!} + \ldots\right\}\right], \quad \text{expanding } \cos \delta\theta \text{ in powers of } \delta\theta$$

$$= T \,.\, l \,.\, 0, \quad \text{to the first order of small quantities}$$

$$= 0.$$

Alternative Method

Let T be the tension in an inextensible string connecting two points A and B whose position vectors are $\mathbf{r_1}$ and $\mathbf{r_2}$. Then a force $\mathbf{T}$ acts at $\mathbf{r_1}$ and $-\mathbf{T}$ acts at $\mathbf{r_2}$. Since the string is inextensible, therefore for any displacement of A and B, we have

$$(\mathbf{r_1} - \mathbf{r_2})^2 = \text{constant}.$$

Differentiating,

$$2\,(\mathbf{r_1} - \mathbf{r_2}) \bullet (d\mathbf{r_1} - d\mathbf{r_2}) = 0$$

i.e., $$2\,\mathbf{T} \bullet (d\mathbf{r_1} - d\mathbf{r_2}) = 0 \qquad [\because \ \mathbf{T} \text{ is parallel to } \mathbf{r_1} - \mathbf{r_2}]$$

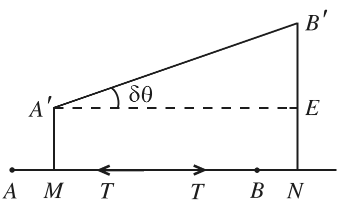
or $$\mathbf{T} \bullet d\mathbf{r_1} + (-\,\mathbf{T}) \bullet d\mathbf{r_2} = 0,$$

showing that the total work done by $\mathbf{T}$ at $\mathbf{r_1}$ and $-\,\mathbf{T}$ at $\mathbf{r_2}$ during a small displacement is zero. Hence the work done by the tension of an inextensible string is zero during a small displacement.

(ii) The work done by the thrust of an inextensible rod is zero during a small displacement. **(Meerut 2004)**

Let T be the thrust in an inextensible rod AB joining two points A and B of a rigid body. Proceed as in part (i). Here the work done by the thrust in the rod AB during a small displacement

$$= -\,T \,.\, AM + T \,.\, BN = 0.$$



Remark: The forces of tension act inward and the forces of thrust act outwards. A common name for tension and thrust is stress. From (i) and (ii) we conclude that *if the distance between two particles of a system is invariable, the work done by the mutual action and reaction between the two particles is zero.*

(iii) *The reaction R of any smooth surface with which the body is in contact does no work.* For, if the surface is smooth, the reaction R on the point of contact A is along the normal to the surface. If A moves to a neighbouring point B, then the displacement AB is right angles to the direction of the force and so the work done by R is zero. If, however, the surface is rough, the work done by the frictional force F *i.e.,* $F\,.(-\,AB)$ will come into the equation of virtual work.

(iv) *If a body rolls without sliding on any fixed surface, the work done in a small displacement by the reaction of the surface on the rolling body is zero.* For, the point of contact of the body is for the moment at rest, and so the normal reaction and the force of friction at the point of contact have zero displacements.

(v) *The work done by the mutual reaction between two bodies of a system is zero in any virtual displacement of the system*. For, action and reaction are equal and opposite and so the work done by the action balances that done by the reaction.

(vi) *If a body is constrained to turn about a fixed point or a fixed axis, the virtual work of the reaction at the point or on the axis is zero.* For in this case the displacement of the point of application of the force is zero.

3.13 Work Done by the Tension and Thrust of an Extensible String During a Small Displacement

(i) *To show that the work done by the tension T of an extensible string of length l during a small displacement is $-T\,.\,\delta l$.* **(Bundelkhand 2007, 08, 10, 11)**

Refer figure of 3.12, part (i).

Let T be the tension in an extensible string AB of length l joining two points A and B of a rigid body. After a small displacement let $A'B'$ be the position of the string and $\delta\theta$ be the small angle between AB and $A'B'$. Since the string is extensible, therefore let $A'B' = l + \delta l$. Draw $A'M$ and $B'N$ perpendiculars to AB. Also draw $A'E$ perpendicular to $B'N$.

On account of the tension in the string AB, there are two forces each equal to T acting on A and B in the opposite directions AB and BA respectively. After displacement A moves to A' and B moves to B'. The work done by the tension of the string AB during the displacement

$$
\begin{aligned}
&= T\,.\,AM - T\,.\,BN \\
&= T\,.\,(AB - MB) - T\,.\,(MN - MB) \\
&= T\,.\,(AB - MN) = T\,.\,(AB - A'E) \\
&= T\,.\,(AB - A'B'\cos\delta\theta) \\
&= T\,.\,[l - (l + \delta l)\cos\delta\theta] \\
&= T\,.\left[l - (l + \delta l)\left\{1 - \frac{(\delta\theta)^2}{2\,!} + \ldots\right\}\right], \text{ expanding } \cos\delta\theta \text{ in powers of } \delta\theta \\
&= T\,.\,[l - l - \delta l], \quad \text{to the first order of small quantities} \\
&= -T\,.\,\delta l.
\end{aligned}
$$

(ii) Similarly it can be shown that *the work done by the thrust T of an extensible rod of length l during a small displacement is $T\,\delta l$.* **(Purvanchal 2007)**

3.14 Application of the Principle of Virtual Work

While applying the principle of virtual work we can give any small displacement to the system provided it is consistent with the geometrical conditions of the system. The displacement should be such as to exclude the forces which are not required and to include those which are required in the final result. After giving the displacement we must note the points and the lengths that change and that do not change during the displacement. If any length or angle etc. is to change during the displacement, we should first find its value in terms of some variable symbol and then after solving the problem we should put its value in the position of equilibrium.

In many cases we are required to find the tension of an inextensible string or the thrust or tension of an inextensible rod. In order to find such a tension or thrust we must give the system a displacement in which the length of the string or the rod changes because otherwise the tension or thrust will not come in the equation of virtual work. But according to the geometrical conditions of the system we cannot give such a displacement to the body. So to get over this difficulty we replace the string or the rod by two equal and opposite forces T which are equivalent to the tension or the thrust in it. By doing so evidently the equation of virtual work is not affected while we become free to give the system a displacement in which the length of the string or the rod changes and consequently T will occur in the equation of virtual work and will thus be determined.

In any problem the virtual work done by the tension T of an extensible string of length l is $-T\,\delta l$ and the virtual work done by the thrust T of an extensible rod of length l is $+T\,\delta l$. In order to find the virtual work done by a force other than a tension or a thrust we first mark a fixed point or a fixed straight line. Then we measure the distance of the point of application of the force from this fixed point or line while moving along the line of action of the force. If this distance is x and the force is P, then the virtual work done by the force P during a small displacement is $P\,\delta x$ in magnitude. If the distance x is measured in the direction of the force P, the virtual work done by P is taken with positive sign and if the distance x is measured in the direction opposite to that of the force P, the virtual work done by P is taken with negative sign.

Equating to zero the total sum of the virtual works done by the forces, we get the equation of the virtual work. Solving this equation we get the value of the required thing to be determined.

If $f\,(x)$ is a function of x, then during a small displacement in which x changes to $x+\delta x$, we have

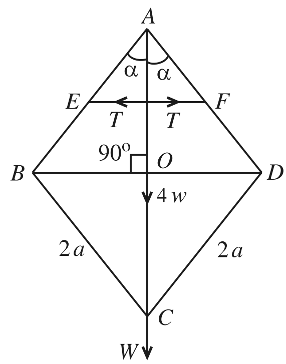
$$\begin{aligned}\delta f\,(x) &= f\,(x+\delta x) - f\,(x)\\ &= f\,(x) + \frac{\delta x}{1!} f\,'(x) + \ldots - f\,(x),\\ &\qquad\qquad \text{expanding } f\,(x+\delta x) \text{ by Taylor's theorem}\\ &= f\,'(x)\,\delta x, \quad \text{to the first order of small quantities.}\end{aligned}$$

In many cases, the only forces that remain in the equation of virtual work are those due to gravity. In such cases if W is the total weight and $\bar{z}$ the height or depth of its point of application (*i.e.*, the centre of gravity of the system) above or below a fixed horizontal level, then by the principle of virtual work for the equilibrium of the body we must have $W\,\delta\,\bar{z} = 0$ *i.e.*, $d\,\bar{z} = 0$, which shows that $\bar{z}$ is a *maximum or minimum* in the position of equilibrium.

Illustrative Examples

Example 1: *Four rods of equal weights w form a rhombus ABCD, with smooth hinges at the joints. The frame is suspended by the point A, and a weight W is attached to C. A stiffening rod of negligible weight joins the middle points of AB and AD, keeping these inclined at α to AC. Show that the thrust in this stiffening rod is* **(Avadh 2008)**

$(2W + 4w)\tan\alpha$.



Solution: $ABCD$ is a framework formed of four equal rods each of weight w and say of length $2a$. It is suspended by the point A and a weight W is attached to C. To keep the system in the form of a rhombus a light rod EF joins the middle points E and F of AB and AD respectively. Obviously the line AC must be vertical and so BD is horizontal.

We have $\angle BAC = \angle DAC = \alpha$.

Let T be the thrust in the rod EF. The total weight $4w$ of all the four rods can be taken acting at the point of intersection O of the diagonals AC and BD.

Replace the rod EF by two equal and opposite forces T as shown in the figure.

Give the system a small symmetrical displacement about the vertical AC in which α changes to $\alpha + \delta\alpha$. The point A remains fixed and so the distances of the points of application of the weights $4w$ and W will be measured from A. The lengths of the rods AB, BC, CD and DA do not change, the length EF changes, the $\angle AOB$ remains $90°$ and the points O and C change.

We have $EF = 2 \,.\, AE \sin\alpha = 2a\sin\alpha$,

AO = depth of O below the fixed point A

$= AB\cos\alpha = 2a\cos\alpha$ and $AC = 2\,AO = 4a\cos\alpha$.

By the principle of virtual work, we have

$$T\,\delta(2a\sin\alpha) + 4w\,\delta(2a\cos\alpha) + W\,\delta(4a\cos\alpha) = 0$$

or $$2aT\cos\alpha\,\delta\alpha - 8aw\sin\alpha\,\delta\alpha - 4aW\sin\alpha\,\delta\alpha = 0$$

or $$2a[T\cos\alpha - 4w\sin\alpha - 2W\sin\alpha]\,\delta\alpha = 0$$

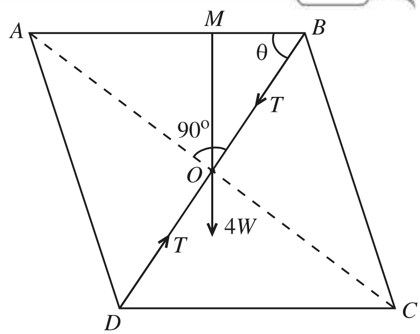
or $$T\cos\alpha - 4w\sin\alpha - 2W\sin\alpha = 0 \qquad [\because \delta\alpha \neq 0]$$

or $$T\cos\alpha = (4w + 2W)\sin\alpha$$

or $$T = (2W + 4w)\tan\alpha.$$

Example 2: *A string, of length a, forms the shorter diagonal of a rhombus formed of four uniform rods, each of length b and weight W, which are hinged together. If one of the rods be supported in a horizontal position, prove that the tension of the string is* $\dfrac{2W(2b^2 - a^2)}{b\sqrt{(4b^2 - a^2)}}$.

(Garhwal 2001; Rohilkhand 11; Kumaun 03)



Solution: $ABCD$ is a framework in the shape of a rhombus formed of four equal uniform rods each of length b and weight W. The rod AB is fixed in a horizontal position and B and D are joined by a string of length a forming the shorter diagonal of the rhombus.

Let T be the tension in the string BD. The total weight $4W$ of the rods AB, BC, CD and DA can be taken as acting at the point of intersection O of the diagonals AC and BD. We have $\angle AOB = 90°$.

Let $\quad \angle ABO = \theta$. Draw OM perpendicular to AB.

Give the system a small symmetrical displacement in which θ changes to $\theta + \delta\theta$. The line AB remains fixed. The points O, C and D change. The lengths of the rods AB, AC, CD and DA do not change while the length BD changes. The $\angle AOB$ will remain 90°.

We have $\quad BD = 2BO = 2\ AB \cos\theta = 2b\cos\theta$.

[Note that in the position of equilibrium $BD = a$. But during the displacement BD changes and so we have found BD in terms of θ.]

The depth of O below the fixed line $AB = MO$

$$= BO\sin\theta = (AB\cos\theta)\sin\theta = b\sin\theta\cos\theta.$$

By the principle of virtual work, we have

$$-T\,\delta\,(2\,b\cos\theta) + 4\,W\,\delta\,(b\sin\theta\cos\theta) = 0$$

or $\quad 2\,bT\sin\theta\,\delta\theta + 4\,bW\,(\cos^2\theta - \sin^2\theta)\,\delta\theta = 0$

or $\quad 2\,b\,[T\sin\theta - 2\,W\,(\sin^2\theta - \cos^2\theta)]\,\delta\theta = 0$

or $\quad T\sin\theta - 2\,W\,(\sin^2\theta - \cos^2\theta) = 0 \qquad [\because\ \delta\theta \neq 0]$

or $\quad T = \dfrac{2\,W\,(\sin^2\theta - \cos^2\theta)}{\sin\theta} = \dfrac{2\,W\,(1 - 2\cos^2\theta)}{\sqrt{(1 - \cos^2\theta)}}.$

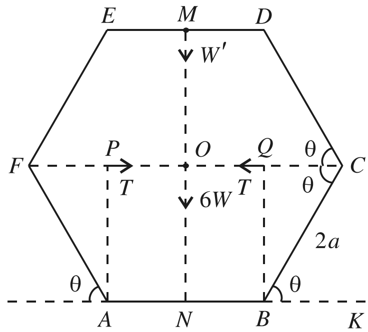
In the position of equilibrium, $BD = a$ or $BO = \frac{1}{2}a$. So in the position of equilibrium,

$$\cos\theta = \frac{BO}{AB} = \frac{\frac{1}{2}a}{b} = \frac{a}{2\,b}.$$

$\therefore \quad T = \dfrac{2\,W\,\{1 - 2\,(a^2/4\,b^2)\}}{\sqrt{\{1 - (a^2/4\,b^2)\}}} = \dfrac{2\,W\,(2\,b^2 - a^2)}{b\sqrt{(4\,b^2 - a^2)}}.$

Example 3: *A regular hexagon ABCDEF consists of six equal rods which are each of weight W and are freely joined together. The two opposite angles C and F are connected by a string, which is horizontal, AB being in contact with a horizontal plane. A weight W' is placed at the middle point of DE. Show that the tension of the string is* $(3W + W')/\sqrt{3}$.

(Avadh 2007; Purvanchal 08; Lucknow 09)



Solution: $ABCDEF$ is a hexagon formed of six equal rods each of weight W and say of length $2\,a$. The hexagon rests in a vertical plane with AB in contact with a horizontal plane. The opposite points C and F are connected by a string and a weight W' is placed at the middle point M of DE. Let T be the tension in the string FC. The total weight $6\,W$ of the six rods AB, BC etc. can be taken acting at O, the middle point of FC.

Suppose BC and AF are inclined at an angle θ to the horizontal. Then in the position of equilibrium $\theta = \pi/3$ because in the position of equilibrium the hexagon is given to be a regular one and so $\angle CBK = 60°$.

Give the system a small symmetrical displacement about the vertical line MON in which θ changes to $\theta + \delta\theta$. The line AB on the horizontal plane remains fixed. The lengths of the rods AB, BC etc. do not change, the length FC changes and the points O and M also change.

We have $\quad FC = FP + PQ + QC = 2a \cos\theta + 2a + 2a \cos\theta$

$$= 2a + 4a \cos\theta,$$

the height of O above AB

$$= NO = BQ = 2\,a \sin\theta,$$

and the height of M above AB

$$= NM = 2\,NO = 4\,a \sin\theta.$$

By the principle of virtual work, we have

$$-T\,\delta(2a + 4a\cos\theta) - 6\,W\,\delta(2a\sin\theta) - W'\delta(4a\sin\theta) = 0$$

or $\quad 4aT\sin\theta\,\delta\theta - 12aW\cos\theta\,\delta\theta - 4aW'\cos\theta\,\delta\theta = 0$

or $\quad 4a\,[T\sin\theta - 3W\cos\theta - W'\cos\theta]\,\delta\theta = 0$

or $\quad T\sin\theta - (3W + W')\cos\theta = 0 \qquad [\because \; \delta\theta \neq 0]$

or $\quad T = (3W + W')\cot\theta.$

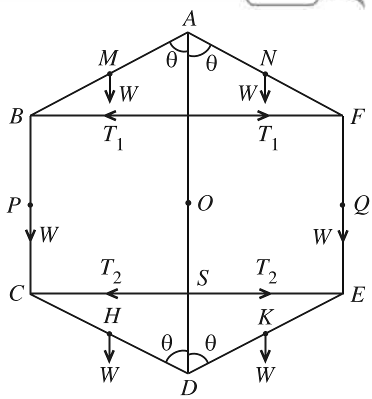
But in the position of equilibrium, $\theta = 60°$. Therefore

$$T = (3W + W')\cot 60° = (3W + W')/\sqrt{3}.$$

Example 4: *Six equal heavy rods, freely hinged at the ends, form a regular hexagon $ABCDEF$, which when hung up by the point A is kept from altering its shape by two light rods BF and CE. Prove that the thrusts of these rods are $(5\sqrt{3}/2)\,W$ and $(\sqrt{3}/2)\,W$, where W is the weight of each rod.* **(Rohilkhand 2008)**

Solution: Let the length of each of the rods AB, BC etc. be $2\,a$ and let θ be the angle which each of the slant rods AB, AF, DC and DE makes with the vertical AD.

Let T_1 and T_2 be the thrusts in the rods BF and CE respectively. Here A is the fixed point. The weights of the rods AB, BC etc. act at their respective middle points as shown in the figure.



Let us first find the thrust T_1.

Replace the rod BF by two equal and opposite forces T_1 as shown in the figure and keep the rod CE intact so that during any displacement the length CE does not change. Now give the system a small symmetrical displacement about the vertical line AD in which θ at the end A changes to $\theta + \delta\theta$ while θ at the end D does not change. The portion $BCDEF$ moves as it is. The length BF changes while the length CE does not change so that during this small displacement the work done by the thrust T_2 of the rod CE is zero. The centres of gravity of all the six rods AB, BC etc. are slightly displaced.

We have $\quad BF = 4\,a \sin\theta.$

In this case we cannot take the total weight of the rods AB, BC etc. act at the middle point O of AD. The depth of each of the points M and N below A is $a\cos\theta$, the depth of each of the points P and Q below A is $2a\cos\theta + a$, and the depth of each of the points H and K below A is

$$2a\cos\theta + 2a + \frac{1}{2}SD$$

where in this case SD is fixed.

By the principle of virtual work, we have

$$T_1\,\delta(4\,a\sin\theta) + 2\,W\,\delta(a\cos\theta) + 2\,W\,\delta(2\,a\cos\theta + a) + 2\,W\,\delta(2\,a\cos\theta + 2\,a + \frac{1}{2}SD) = 0$$

or $\qquad 4\,aT_1\cos\theta\,\delta\theta - 10\,aW\sin\theta\,\delta\theta = 0$

or $\qquad 2\,a\,(2\,T_1\cos\theta - 5W\sin\theta)\,\delta\theta = 0$

or $\qquad 2\,T_1\cos\theta - 5W\sin\theta = 0 \qquad [\because\ \delta\theta \neq 0]$

or $\qquad T_1 = \dfrac{5}{2}W\tan\theta.$

But in the position of equilibrium, the hexagon is a regular one and so $\theta = \pi/3$.

Therefore $\quad T_1 = \dfrac{5}{2}W\tan\dfrac{1}{3}\pi = \dfrac{5}{2}W\sqrt{3}.$

Now let us proceed to find the thrust T_2 .

Replace the rod BF by two equal and opposite forces T_1 as shown in the figure and also replace the rod CE by two equal and opposite forces T_2 as shown in the figure. Give the system a small symmetrical displacement about the vertical line AD in which θ at both the ends A and D changes to $\theta + \delta\theta$ so that both the lengths BF and CE change. In this case the total weight $6W$ of all the six rods AB, BC etc. can be taken acting at the middle point O of AD.

We have $\quad BF = 4a\sin\theta,\ CE = 4a\sin\theta\ $ and $\ AO = 2a\cos\theta + a.$

By the principle of virtual work, we have

$$T_1 \, \delta (4 \, a \sin \theta) + T_2 \, \delta (4 \, a \sin \theta) + 6 \, W \, \delta (2 \, a \cos \theta + a) = 0$$

or $$4 \, a \, T_1 \cos \theta \, \delta\theta + 4 \, a T_2 \cos \theta \, \delta\theta - 12 \, aW \sin \theta \, \delta\theta = 0$$

or $$4 \, a \, [(T_1 + T_2) \cos \theta - 3W \sin \theta] \, \delta\theta = 0$$

or $$(T_1 + T_2) \cos \theta - 3W \sin \theta = 0 \qquad [\because \ \delta\theta \neq 0]$$

or $$T_1 + T_2 = 3W \tan \theta.$$

But in the position of equilibrium $\theta = \pi / 3$.

$\therefore$ $$T_1 + T_2 = 3W \tan \frac{1}{3} \pi = 3W \sqrt{3}.$$

$\therefore$ $$T_2 = 3W \sqrt{3} - T_1$$

$$= 3W \sqrt{3} - \frac{5W \sqrt{3}}{2} = \frac{W \sqrt{3}}{2}.$$

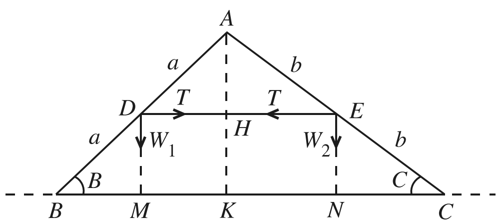
Problems Involving Two Parameters

Example 5: *Two uniform rods AB and AC smoothly jointed at A are in equilibrium in a vertical plane, B and C rest on a smooth horizontal plane and the middle points of AB and AC are connected by a string. Show that the tension of the string is*

$$\frac{W}{\tan B + \tan C},$$

where W is the total weight of the rods AB and AC. **(Meerut 2007; Purvanchal 08, 10)**

Solution: AB and AC are two uniform rods smoothly jointed at A. They rest in a vertical plane with the ends B and C placed on a smooth horizontal plane. Let T be the tension in the string connecting the middle points D and E of AB and AC respectively. Let



$$AB = 2a \text{ and } AC = 2b.$$

The weight W_1 of the rod AB acts at its middle point D and the weight W_2 of the rod AC acts at its middle point E. Therefore the total weight $W = W_1 + W_2$ of the two rods AB and AC acts at some point of the line DE which is parallel to BC.

Give the system a small displacement in which the angle B changes to $B + \delta B$ and C changes to $C + \delta C$. The level of the line BC lying on the horizontal plane remains fixed and the points B and C move on this line. The lengths of the rods AB and AC do not change, the length DE changes and the points D and E move. We have

$$DE = DH + HE = a \cos B + b \cos C,$$

the height of any point of the line DE above BC

$$= DM = a \sin B.$$

The equation of virtual work is

$$-T\,\delta(a\cos B + b\cos C) - W\,\delta(a\sin B) = 0$$

or $$aT\sin B\,\delta B + bT\sin C\,\delta C - aW\cos B\,\delta B = 0$$

or $$a(W\cos B - T\sin B)\,\delta B = bT\sin C\,\delta C. \qquad \ldots(1)$$

From the figure,

$$DM = a\sin B \text{ and } EN = b\sin C.$$

Since $DM = EN$, therefore $a\sin B = b\sin C$.

$\therefore$ $$\delta(a\sin B) = \delta(b\sin C)$$

or $$a\cos B\,\delta B = b\cos C\,\delta C. \qquad \ldots(2)$$

Dividing (1) by (2), we have

$$\frac{W\cos B - T\sin B}{\cos B} = \frac{T\sin C}{\cos C}$$

or $$W - T\tan B = T\tan C$$

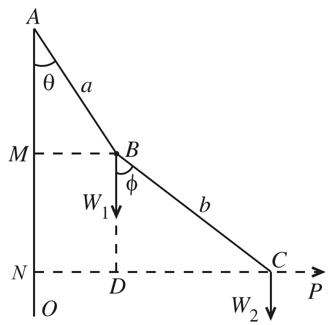
or $$T(\tan B + \tan C) = W$$

or $$T = \frac{W}{\tan B + \tan C}.$$

Example 6: *Weights W_1, W_2 are fastened to a light inextensible string ABC at the points B, C, the end A being fixed. Prove that, if a horizontal force P is applied at C and in equilibrium AB, BC are inclined at angles θ, ϕ to the vertical, then*

$$P = (W_1 + W_2)\tan\theta = W_2\tan\phi. \qquad \textbf{(Lucknow 2007)}$$

Solution: Let the length of the portion AB of the string be a and that of BC be b. The point A is fixed and the vertical line AO through A is a fixed line.



From the fixed point A,

the depth of $B = AM = a\cos\theta$,

and the depth of $C = AN = AM + MN$

$$= AM + BD$$

$$= a\cos\theta + b\cos\phi.$$

Also the horizontal distance of the point C from the fixed line

$$AO = NC = ND + DC = MB + DC$$

$$= a\sin\theta + b\sin\phi.$$

Now give the system a small displacement in which θ changes to $\theta + \delta\theta$, ϕ changes to $\phi + \delta\phi$, the point A remains fixed, the length of the string remains unaltered and the points B and C are slightly displaced. The equation of virtual work is

$$W_1\,\delta(a\cos\theta) + W_2\,\delta(a\cos\theta + b\cos\phi) + P\,\delta(a\sin\theta + b\sin\phi) = 0$$

or $$-aW_1\sin\theta\,\delta\theta - aW_2\sin\theta\,\delta\theta - bW_2\sin\phi\,\delta\phi + aP\cos\theta\,\delta\theta + bP\cos\phi\,\delta\phi = 0$$

or $$a[P\cos\theta - (W_1 + W_2)\sin\theta]\,\delta\theta = b[W_2\sin\phi - P\cos\phi]\,\delta\phi, \qquad \ldots(1)$$

where θ and ϕ are independent of each other.

Now consider a displacement when only θ changes and ϕ does not change so that $\delta\phi = 0$. Then putting $\delta\phi = 0$ in (1), we have

$$a\,[P\cos\theta - (W_1 + W_2)\sin\theta]\,\delta\theta = 0$$

or $$P\cos\theta - (W_1 + W_2)\sin\theta = 0 \qquad [\because\ \delta\theta \neq 0]$$

or $$P = (W_1 + W_2)\tan\theta. \qquad ...(2)$$

Again consider a displacement when only ϕ changes and θ does not change so that $\delta\theta = 0$. Thus putting $\delta\theta = 0$ in (1), we have

$$b\,[W_2\sin\phi - P\cos\phi]\,\delta\phi = 0$$

or $$W_2\sin\phi - P\cos\phi = 0 \qquad [\because\ \delta\phi \neq 0]$$

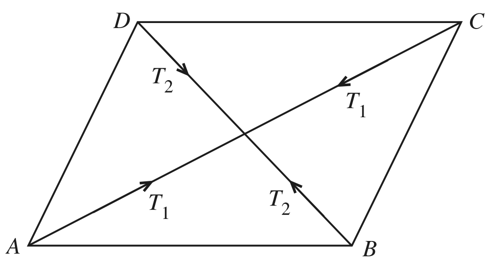
or $$P = W_2\tan\phi. \qquad ...(3)$$

From (2) and (3), we have

$$P = (W_1 + W_2)\tan\theta$$
$$= W_2\tan\phi.$$

Example 7: *Four rods are jointed together to form a parallelogram, the opposite joints are joined by strings forming the diagonals and the whole system is placed on a smooth horizontal table. Show that their tensions are in the same ratio as their lengths.*

Solution: A framework $ABCD$ is in the form of a parallelogram and is placed on a smooth horizontal table. Let T_1 and T_2 be the tensions in the strings AC and BD respectively.



Give the system a small displacement in the plane of the table in which AC changes to $AC + \delta(AC)$ and BD changes to $BD + \delta(BD)$. The lengths of the rods AB, BC, CD, DA do not change. During this displacement the weights of the rods do not work because the displacement of their points of application in the vertical direction is zero. The equation of virtual work is

$$-T_1\,\delta(AC) - T_2\,\delta(BD) = 0$$

or $$\frac{\delta(AC)}{\delta(BD)} = -\frac{T_2}{T_1}. \qquad ...(1)$$

Now let us find a relation between the parameters AC and BD from the figure. Since in a parallelogram the sum of the squares of the diagonals is equal to the sum of the squares of its sides, therefore

$$AC^2 + BD^2 = AB^2 + BC^2 + CD^2 + DA^2 = \text{constant}. \qquad ...(2)$$

Differentiating (2), we get

$$2\,AC\,\delta(AC) + 2\,BD\,\delta(BD) = 0$$

or $$\frac{\delta(AC)}{\delta(BD)} = -\frac{BD}{AC}. \qquad ...(3)$$

From (1) and (3), we get

$$-\frac{T_2}{T_1}=-\frac{BD}{AC} \quad \text{or} \quad \frac{T_1}{AC}=\frac{T_2}{BD},$$

i.e., tensions are in the ratio of the lengths of the strings.

Comprehensive Exercise 1

1. Five weightless rods of equal length are jointed together so as to form a rhombus $ABCD$ with one diagonal BD. If a weight W be attached to C and the system be suspended from A, show that there is a thrust in BD equal to $W/\sqrt{3}$.

(Garhwal 2004; Lucknow 06, 10; Avadh 08; Bundelkhand 09; Rohilkhand 10)

2. Four equal heavy uniform rods are freely jointed so as to form a rhombus which is freely suspended by one angular point, and the middle points of the two upper rods are connected by a light rod so that the rhombus cannot collapse. Prove that the thrust of this light rod is $4\,W\tan\alpha$, where W is the weight of each rod and $2\,\alpha$ is the angle of the rhombus at the point of suspension.

(Rohilkhand 2006; Bundelkhand 10; Meerut 09, 11)

3. Four equal uniform rods, each of weight w, are freely jointed to form a rhombus $ABCD$. The frame work is suspended freely from A and a weight W is attached to each of the joints B, C, D. If two horizontal forces each of magnitude P acting at B and D keep the angle BAD equal to $120°$, prove that

$$P=(W+w)\,2\sqrt{3}.$$ **(Kunpur 2001; Bundelkhand 08, 09)**

4. Four equal uniform rods, each of weight W, are jointed to form a rhombus $ABCD$, which is placed in a vertical plane with AC vertical and A resting on a horizontal plane. The rhombus is kept in the position in which $\angle BAC=\theta$ by a light string joining B and D. Find the tension of the string.

5. A square framework, formed of uniform heavy rods of equal weight W, jointed together, is hung up by one corner. A weight W is suspended from each of the three lower corners and the shape of the square is preserved by a light rod along the horizontal diagonal. Find the thrust of the light rod. **(Purvanchal 2011)**

6. Four uniform rods are freely jointed at their extremities and form a parallelogram $ABCD$, which is suspended by the joint A, and is kept in shape by a string AC. Prove that the tension of the string is equal to half the weight of all the four rods.

7. Four equal uniform rods, each of weight W, are smoothly jointed so as to form a square $ABCD$; the side AB is fixed (clamped) in a vertical position with A uppermost and the figure is kept in shape by a string joining the middle points of AD and DC. Find the tension of the string. **(Lucknow 2007, 11)**

8. Six equal heavy beams are freely jointed at their ends to form a hexagon, and are placed in a vertical plane with one beam resting on a horizontal plane; the middle points of the two upper slant beams, which are inclined at an angle θ to the horizon, are connected by a light cord. Find its tension in terms of W and θ, where W is the weight of each beam. **(Meerut 2006)**

9. A regular hexagon $ABCDEF$ consists of six equal rods which are each of weight W and are freely jointed together. The hexagon rests in a vertical plane and AB is in contact with a horizontal table. If C and F be connected by a light string, prove that its tension is $W\sqrt{3}$. **(Avadh 2011; Purvanchal 09)**

10. Six equal rods AB, BC, CD, DE, EF and FA are each of weight W and are freely jointed at their extremities so as to form a hexagon ; the rod AB is fixed in a horizontal position and the middle points of AB and DE are jointed by a string ; prove that its tension is $3W$.

11. Six equal bars are freely jointed at their extremities forming a regular hexagon $ABCDEF$ which is kept in shape by vertical strings joining the middle points of BC, CD and AF, FE, respectively, the side AB being held horizontal and uppermost. Prove that the tension of each string is three times the weight of a bar.

12. Six equal light rods are joined to form a hexagon $ABCDEF$ which is suspended at A and F so that AF is horizontal. A rod BE, also light, keeps the figure from collapsing and is of such a length that the rods ending in the points B, E are inclined at an angle of 45° to the vertical. Equal weights are suspended from B, C, D, E. Find the stress in BE.

13. Three equal uniform rods AB, BC, CD each of weight w, are freely jointed together at B and C, and rest in a vertical plane, A and D being in contact with a smooth horizontal table. Two equal light strings AC and BD help to support the framework, so that AB and CD are each inclined at an angle α to the horizontal. Show that if a mass of weight W be placed on BC at its middle point, then tension of each string will be of magnitude

$$\left(w + \frac{1}{2}W\right)\cos\alpha \operatorname{cosec}\frac{1}{2}\alpha .$$

14. Two equal beams AC and AB, each of weight W, are connected by a hinge at A and are placed in a vertical plane with their extremities B and C resting on a smooth horizontal plane. They are prevented from falling by strings connecting B and C with the middle points of the opposite beams. Show that the tension of each string is

$$\frac{1}{8}W\sqrt{(1 + 9\cot^2\theta)},$$

where θ is the inclination of each beam to the horizon.

15. A step ladder has a pair of legs which are jointed by a hinge at the top, and are connected by a cord attached at one-third of the distance from the lower end to the top. If the weight of each leg be W_1 and acts at their middle points and if a man of weight W is two-thirds the way up the ladder, show by the principle of virtual work , that the tension in the cord is

$$\frac{1}{2}\left(W + \frac{3}{2}W_1\right)\tan\alpha ,$$

α being the inclination of each leg to the vertical.

16. Four equal uniform bars, each of weight W, are jointed together so as to form a rhombus. This is suspended vertically from one of the joints, and a sphere of weight P is balanced inside the rhombus so as to keep it from collapsing. Show that if 2θ be the angle at the fixed point in the figure of equilibrium, then

$$\frac{\sin^3\theta}{\cos\theta} = \frac{Pr}{4(P + 2W)a},$$

where r is the radius of the sphere and $2a$ is the length of each bar.

17. A quadrilateral $ABCD$, formed of four uniform rods freely jointed to each other at their ends, the rods AB, AD being equal and also the rods BC, CD, is freely suspended from the joint A. A string joins A to C and is such that ABC is a right angle. Apply the principle of virtual work to show that the tension of the string is

$$(W + W') \sin^2 \theta + W',$$

where W is the weight of an upper rod and W' of a lower rod and 2θ is equal to the angle BAD.

18. $ABCD$ is a quadrilateral formed of four uniform freely jointed rods, of which $AB = AD$ and each of weight W, and $BC = CD$ each of weight W'. A string joins A to C. It is freely suspended from A. If $\angle BAD = 2\theta$ and $\angle BCD = 2\phi$, show that the tension in the string is

$$\frac{W \tan\theta + W' (2 \tan\theta + \tan\phi)}{\tan\theta + \tan\phi}.$$

19. Two uniform rods AB, BC of weights W and W' are smoothly jointed at B and their middle points are joined across by a cord. The rods are tightly held in a vertical plane with their ends A, C resting on a smooth horizontal plane. Show by the principle of virtual work that the tension in the cord is

$$(W + W') \cos A \cos C / \sin B.$$

Find the additional tension in the cord caused by suspending a weight W'' from B.

20. Two equal uniform rods AB, AC each of weight W are freely jointed at A and rest with the extremities B and C on the inside of a smooth circular hoop, whose radius is greater than the length of either rod, the whole being in a vertical plane and the middle points of the rods being jointed by a light string. Show that if the string is stretched, its tension is $W(\tan\alpha - 2\tan\beta)$, where 2α is the angle between the rods, and β the angle either rod subtends at the centre.

21. A frame, formed of four light rods, each of length a, freely jointed at A, B, C, D is suspended at A. A mass m is suspended from B and D by two strings of length $l\ (l > a/\sqrt{2})$. The frame is kept in the form of a square by a string AC. Apply the method of virtual work to find the tension T in AC and show that when

$$l = a\sqrt{5},\ T = 2\,mg/3.$$

22. A rod is movable about a point A, and to B is attached a string whose other end is tied to a ring. The ring slides along a smooth horizontal wire passing through A. Prove by the principle of virtual work that the horizontal force necessary to keep the ring at rest is

$$\frac{W \cos\alpha \cos\beta}{2 \sin(\alpha + \beta)},$$

where W is the weight of the rod, and α, β the inclinations of the rod and the string to the horizontal.

23. A solid hemisphere is supported by a string fixed to a point on its rim and to a point on a smooth vertical wall with which the curved surface of the hemisphere is in contact. If θ, ϕ are the inclinations of the string and the plane base of the hemisphere to the vertical, prove that

$$\tan\phi = \frac{3}{8} + \tan\theta.$$

(Garhwal 2001)

24. Five equal uniform rods, freely jointed at their ends, form a regular pentagon *ABCDE* and *BE* is joined by a weightless bar. The system is suspended from *A* in a vertical plane. Prove that the thrust in *BE* is $W \cot \frac{1}{10} \pi$, where W is the weight of the rod. **(Purvanchal 2009)**

25. A regular pentagon *ABCDE* formed of equal uniform rods each of weight W, is suspended from the point *A* and is maintained in shape by a light rod joining the middle points of *BC* and *DE*. Prove that the stress in the light rod is

$$2 W \cot (\pi / 10).$$ **(Meerut 2010)**

26. A freely jointed framework is formed of five equal uniform rods each of weight W. The framework is suspended from one corner which is also joined to the middle point of the opposite side by an inextensible string ; if the two upper and the two lower rods make angles θ and ϕ respectively with the vertical, prove that the tension of the string is to the weight of the rod as

$$(4 \tan \theta + 2 \tan \phi) : (\tan \theta + \tan \phi).$$ **(Purvanchal 2011)**

27. A smoothly jointed framework of light rods forms a quadrilateral *ABCD*. The middle points P, Q of an opposite pair of rods are connected by a string in a state of tension T, and the middle points R, S of the other pair by a light rod in a state of thrust X ; show, by the method of virtual work, that

$$T / PQ = X / RS.$$ **(Lucknow 2007)**

28. The middle points of the opposite sides of a jointed quadrilateral are connected by light rods of lengths l, l'. If T, T' be the tensions in these rods, prove that

$$\frac{T}{l} + \frac{T'}{l'} = 0.$$ **(Bundelkhand 2006)**

Answers 1

4. $2 W \tan \theta$

5. $4 W$

7. $4 W \sqrt{2}$

8. $6 W \cot \theta$

12. $3 W$

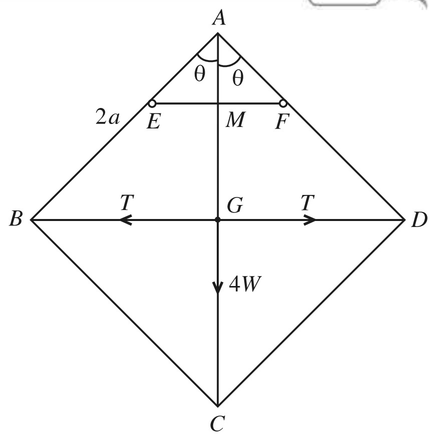
19. $(2 W'' \cos A \cos C) / \sin B$

21. $T = \frac{1}{2} mg \left\{ 1 + \frac{a}{\sqrt{(2 l^2 - a^2)}} \right\}$

Problems Relating to Bodies or Frameworks Resting on Pegs or on Inclined Planes

Example 8: *Four equal rods, each of length $2a$ and weight W, are freely jointed to form a square ABCD which is kept in shape by a light rod BD and is supported in a vertical plane with BD horizontal, A above C and AB, AD in contact with two fixed smooth pegs which are at a distance $2b$ apart on the same level. Find the stress in the rod BD.*

Solution: The rods *AB* and *AD* of the framework rest on two fixed smooth pegs *E* and *F* which are at the same level and $EF = 2b$. Let $2a$ be the length of each of the rods *AB, BC, CD* and *DA*. The total weight $4W$ of all the rods *AB, BC, CD* and *DA* can be taken acting at *G*, the middle point of *AC*.



Let T be the thrust in the rod BD and let

$$\angle BAC = \theta = \angle CAD.$$

Replace the rod BD by two equal and opposite forces T as shown in the figure. Give the system a small symmetrical displacement in which θ changes to $\theta + \delta\theta$. The line EF joining the pegs remains fixed and the distance will be measured from this line. The lengths of the rods AB, BC, CD, DA do not change and the length BD changes. The $\angle AGB$ remains 90°.

The forces contributing to the sum of virtual works are :

(i) the thrust T in the rod BD, and

(ii) the weight $4W$ acting at G. The reactions at the pegs do not work.

We have,

$$BD = 2BG = 2\,.\,2a \sin\theta = 4a \sin\theta.$$

Also the depth of G below the fixed line EF

$$= MG = AG - AM = AB \cos\theta - EM \cot\theta$$

$$= 2a \cos\theta - b \cot\theta.$$

The equation of virtual work is

$$T\,\delta(4\,a \sin\theta) + 4\,W\,\delta(2\,a \cos\theta - b \cot\theta) = 0$$

or $$4\,aT \cos\theta\,\delta\theta - 8\,aW \sin\theta\,\delta\theta + 4\,bW \operatorname{cosec}^2\theta\,\delta\theta = 0$$

or $$4\,(aT \cos\theta - 2aW \sin\theta + bW \operatorname{cosec}^2\theta)\,\delta\theta = 0$$

or $$aT \cos\theta - 2aW \sin\theta + bW \operatorname{cosec}^2\theta = 0 \qquad [\because\ \delta\theta \neq 0]$$

or $$aT \cos\theta = 2aW \sin\theta - bW \operatorname{cosec}^2\theta$$

or $$T = \frac{W}{a\cos\theta}(2a \sin\theta - b \operatorname{cosec}^2\theta) = \frac{W}{a}\tan\theta\,(2\,a - b \operatorname{cosec}^3\theta).$$

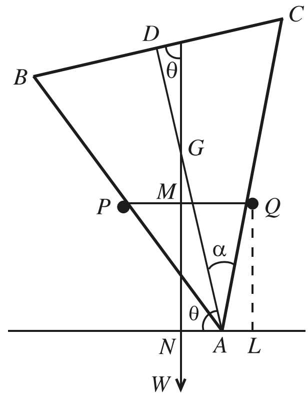
But in the position of equilibrium, $\theta = 45°$.

$$\therefore \quad T = \frac{W}{a}\tan 45°\,(2a - b \operatorname{cosec}^3 45°) = \frac{W}{a}[2a - b\,(\sqrt{2})^3] = \frac{2\,W}{a}(a - b\sqrt{2}).$$

Remark: The pegs E and F may also be taken below the middle points of the rods AB and AD.

Example: 9: *An isosceles triangular lamina, with its plane vertical rests with its vertex downwards, between two smooth pegs in the same horizontal line. Show that there will be equilibrium if the base makes an angle $\sin^{-1}(\cos^2\alpha)$ with the vertical, 2α being the vertical angle of the lamina and the length of the base being three times the distance between the pegs.*

Solution: ABC is an isosceles triangular lamina in which $AB = AC$. The sides AB and AC rest on two smooth pegs P and Q which are in the same horizontal line.



Let $PQ = a$ so that $BC = 3a$.

If D is the middle point of BC, then the centre of gravity G of the lamina lies on the median AD and is such that

$$AG = \frac{2}{3}\, AD.$$

The weight W of the lamina acts vertically downwards at G. We have

$$\angle BAD = \angle CAD = \alpha.$$

Suppose in equilibrium the base BC of the lamina makes an angle θ with the vertical. Since the angle between two lines is equal to the angle between their perpendicular lines, therefore $\angle DAN = \theta$. [Note that DA is perpendicular to BC and AN is perpendicular to the vertical line NMG].

Now $\angle QPA = \angle PAN = \theta - \alpha$, and $\angle QAL = \pi - (\theta + \alpha)$.

Give the lamina a small displacement in which θ changes to $\theta + \delta\theta$. The line PQ joining the pegs remains fixed and the distances will be measured from this line. The angle α remains fixed. The only force contributing to the sum of virtual works is the weight W of the lamina acting at G. We have, the height of G above the fixed line PQ

$$= MG = NG - NM = NG - LQ$$

$$= AG \sin\theta - AQ \sin\{\pi - (\theta + \alpha)\}$$

$$= \frac{2}{3}\, AD \sin\theta - AQ \sin(\theta + \alpha).$$

Now $AD = CD \cot\alpha = \frac{3}{2} a \cot\alpha$. Also from the $\Delta\, AQP$, by the sine theorem of trigonometry, we have

$$\frac{AQ}{\sin APQ} = \frac{PQ}{\sin PAQ} \quad i.e., \quad \frac{AQ}{\sin(\theta - \alpha)} = \frac{a}{\sin 2\alpha}.$$

$\therefore$
$$AQ = \frac{a}{\sin 2\alpha} \sin(\theta - \alpha).$$

$\therefore$
$$MG = \frac{2}{3} \cdot \frac{3}{2}\, a \cot\alpha \sin\theta - \frac{a}{\sin 2\alpha} \sin(\theta - \alpha)\sin(\theta + \alpha)$$

$$= a \cot\alpha \sin\theta - \frac{a}{2\sin 2\alpha}\, 2\sin(\theta - \alpha)\sin(\theta + \alpha)$$

$$= a \cot\alpha \sin\theta - \frac{a}{4\sin\alpha\cos\alpha}(\cos 2\alpha - \cos 2\theta)$$

$$= a \cot\alpha \sin\theta - \frac{a\cos 2\alpha}{4\sin\alpha\cos\alpha} + \frac{a\cos 2\theta}{4\sin\alpha\cos\alpha}.$$

The equation of virtual work is

$$-W\,\delta(MG) = 0, \quad \text{or} \quad \delta(MG) = 0$$

or
$$\delta\left[a\cot\alpha\sin\theta - \frac{a\cos 2\alpha}{4\sin\alpha\cos\alpha} + \frac{a\cos 2\theta}{4\sin\alpha\cos\alpha}\right] = 0$$

or $$\left[a\cot\alpha\cos\theta - \frac{2\,a\sin 2\,\theta}{4\sin\alpha\cos\alpha}\right]\delta\theta = 0$$

or $$a\cot\alpha\cos\theta - \frac{4\,a\sin\theta\cos\theta}{4\sin\alpha\cos\alpha} = 0 \qquad [\because\ \delta\theta \neq 0]$$

or $$a\cos\theta\left(\cot\alpha - \frac{\sin\theta}{\sin\alpha\cos\alpha}\right) = 0.$$

$\therefore$ either $\cos\theta = 0$ *i.e.*, $\theta = \dfrac{\pi}{2}$, giving one position of equilibrium in which the lamina rests symmetrically on the pegs

or $$\cot\alpha - \frac{\sin\theta}{\sin\alpha\cos\alpha} = 0 \ \ i.e.,\ \sin\theta = \cos^2\alpha$$

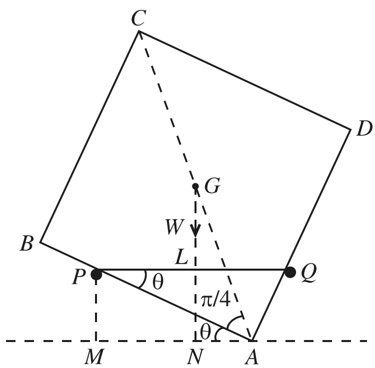
i.e., $\theta = \sin^{-1}(\cos^2\alpha)$, giving the other position of equilibrium.

Example 10: *A square of side 2 a is placed with its plane vertical between two smooth pegs which are in the same horizontal line at a distance c apart ; show that it will be in equilibrium when the inclination of one of its edges to the horizon is either*

$$\frac{\pi}{4} \quad or \quad \frac{1}{2}\sin^{-1}\left(\frac{a^2-c^2}{c^2}\right).$$

(Bundelkhand 2007; Purvanchal 09)

Solution: The sides AB and AD of the square lamina $ABCD$ rest on two smooth pegs P and Q which are in the same horizontal line. It is given that $PQ = c$ and $AB = 2\,a$.



The weight W of the lamina acts at G, the middle point of the diagonal AC. Suppose in the position of equilibrium the side AB of the lamina makes an angle θ with the horizontal so that

$$\angle PAM = \theta = \angle QPA.$$

We have $$\angle BAC = \frac{1}{4}\pi = \text{constant}.$$

Give the lamina a small displacement in which θ changes to $\theta + \delta\theta$. The line PQ joining the pegs remains fixed. The only force contributing to the sum of virtual works is the weight W of the lamina acting at G. We have, the height of G above the fixed line PQ

$$= LG = NG - NL = NG - MP$$

$$= AG\sin\left(\frac{1}{4}\pi + \theta\right) - AP\sin\theta$$

$$= a\sqrt{2}\sin\left(\frac{1}{4}\pi + \theta\right) - PQ\cos\theta\sin\theta$$

$$\left[\because\ AG = \frac{1}{2}AC = \frac{1}{2}2a\sqrt{2} = a\sqrt{2},\ \text{and}\ AP = PQ\cos\theta\right]$$

$$= a\sqrt{2}\left(\sin\frac{1}{4}\pi\cos\theta + \cos\frac{1}{4}\pi\sin\theta\right) - c\cos\theta\sin\theta$$

$$= a(\cos\theta + \sin\theta) - c\cos\theta\sin\theta.$$

The equation of virtual work is

$$-W\,\delta(LG) = 0, \quad \text{or} \quad \delta(LG) = 0$$

or $$\delta\,[a(\cos\theta + \sin\theta) - c\cos\theta\sin\theta] = 0$$

or $$[a(-\sin\theta + \cos\theta) - c(\cos^2\theta - \sin^2\theta)]\,\delta\theta = 0$$

or $$a(\cos\theta - \sin\theta) - c(\cos^2\theta - \sin^2\theta) = 0 \qquad [\because\ \delta\theta \neq 0]$$

or $$(\cos\theta - \sin\theta)\,[a - c(\cos\theta + \sin\theta)] = 0.$$

$\therefore$ either $\cos\theta - \sin\theta = 0$

i.e., $\sin\theta = \cos\theta$ *i.e.,* $\tan\theta = 1$ *i.e.,* $\theta = \frac{1}{4}\pi$,

giving one position of equilibrium in which the lamina rests symmetrically on the pegs

or $$a - c(\cos\theta + \sin\theta) = 0$$

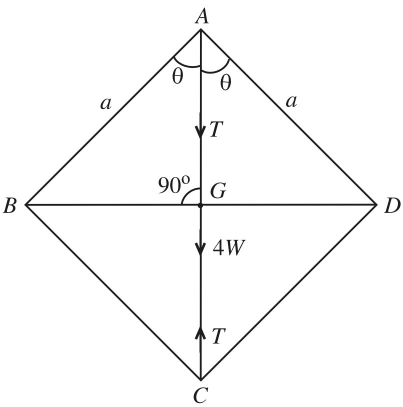
i.e., $$c^2(\cos\theta + \sin\theta)^2 = a^2 \quad i.e., \quad c^2(1 + \sin 2\theta) = a^2$$

i.e., $$\sin 2\theta = \frac{a^2}{c^2} - 1 = \frac{a^2 - c^2}{c^2} \quad i.e., \quad \theta = \frac{1}{2}\sin^{-1}\left(\frac{a^2 - c^2}{c^2}\right),$$

giving the other position of equilibrium.

Problems Involving Elastic Strings

Example 11: *Four equal jointed rods, each of length a are hung from an angular point, which is connected by an elastic string with the opposite point. If the rods hang in the form of a square, and if the modulus of elasticity of the string be equal to the weight of a rod, show that the unstretched length of the string is* $a\sqrt{2}/3$.



Solution: $ABCD$ is a framework formed of four equal rods each of length a and say of weight W. It is suspended from the point A. A and C are connected by an elastic string and in equilibrium $ABCD$ is square. The diagonal AC is vertical and so BD is horizontal. Let T be the tension in the string AC. The total weight $4W$ of all the rods AB, BC, CD and DA can be taken acting at G, the point of intersection of the diagonals AC and BD. Let $\angle BAC = \angle DAC = \theta$.

Give the system a small symmetrical displacement about the vertical line AC in which θ changes to $\theta + \delta\theta$. The point A remains fixed, the length AC changes, the point G is slightly displaced, the lengths of the rods AB, BC, CD, DA do not change, and the $\angle BGA$ remains 90°. We have

$$AC = 2\,AG = 2a\cos\theta.$$

Also the depth of G below A

$$= AG = a\cos\theta.$$

The equation of virtual work is

$$-T\,\delta(2a\cos\theta) + 4W\,\delta(a\cos\theta) = 0$$

or $\quad 2aT\sin\theta\,\delta\theta - 4aW\sin\theta\,\delta\theta = 0$

or $\quad 2a\sin\theta\,(T - 2W)\,\delta\theta = 0$

or $\quad T - 2W = 0 \qquad [\because\ \delta\theta \neq 0 \text{ and } \sin\theta \neq 0]$

or $\quad T = 2W.$

Let l be the natural length of the elastic string AC. In the position of equilibrium, $\angle BAC = 45°$ and so the extended length AC of the elastic string

$$= 2\,AG = 2a\cos 45° = 2a/\sqrt{2} = a\sqrt{2}.$$

By Hooke's law, the tension T in the elastic string AC is given by

$T = \lambda\,\dfrac{AC - l}{l}$, where λ is the modulus of elasticity of the string

$= W\,\dfrac{a\sqrt{2} - l}{l}. \qquad [\because\ \lambda = W]$

Equating the two values of T, we get

$$2W = W\,\frac{a\sqrt{2} - l}{l}$$

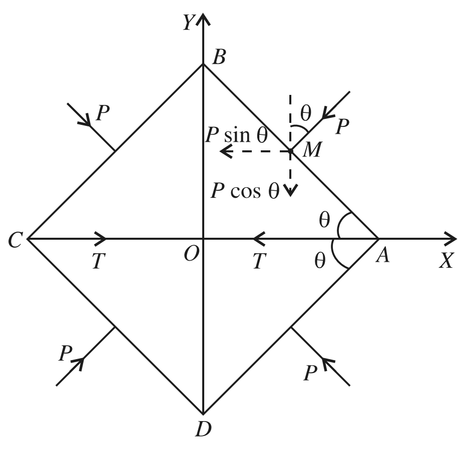
or $\quad 2l = a\sqrt{2} - l \quad$ or $\quad 3l = a\sqrt{2}$

or $\quad l = a\sqrt{2}/3.$

Problems Involving the Nature of Stress

Example 12: *A frame consists of five bars forming the sides of a rhombus $ABCD$ with diagonal AC. If four equal forces P act inwards at the middle points of the sides, and at right angles to the respective sides, prove that the tension in AC is $(P\cos 2\theta/\sin\theta)$ where θ denotes the angle BAC.*

Solution: Let $2a$ be the length of each side of the rhombus $ABCD$ which we shall assume as placed on a smooth horizontal table. The four forces, each equal to P, act inwards at the middle points of AB, BC, CD, DA and at right angles to the respective sides. Let M be the middle point of AB where the force P acts.



Let us assume that the stress in the rod AC is tension and let it be T.

Let $\quad \angle BAC = \theta = \angle CAD.$

Replace the rod AC by two equal and opposite forces T as shown in the figure. Take the centre O of the rhombus as origin, the line OA as the axis of x and the perpendicular line OB as the axis of y. Now give the rhombus a small symmetrical displacement about the centre O in which the centre O and the lines OX and OY remain fixed, θ changes to $\theta + \delta\theta$, the points A and C move on the axis of x and the points B and D move on the axis of y. The length AC changes, the lengths of the rods AB, BC, CD, DA do not change but their middle points are slightly displaced.

We have $AC = 4\,a\cos\theta$ so that the work done by the tension T in the rod AC during this small displacement

$$= -T\,\delta(4\,a\cos\theta) = 4\,aT\sin\theta\,\delta\theta.$$

By symmetry the forces P acting at the middle points of AB, BC, CD, DA contribute equal works, so that the sum of the works done by all of them is four times the work done by P acting at M.

The components (X, Y) of the force P acting at M along the fixed co-ordinate axes OX and OY are given by

$$X = -P\sin\theta,\; Y = -P\cos\theta.$$

Also the co-ordinates of the point M are

$$(a\cos\theta,\, a\sin\theta).$$

$\therefore$ the virtual work of the force P acting at M during this small displacement

$$\begin{aligned} &= X\,\delta(a\cos\theta) + Y\,\delta(a\sin\theta) \\ &= -P\sin\theta\,\delta(a\cos\theta) - P\cos\theta\,\delta(a\sin\theta) \\ &= aP\sin^2\theta\,\delta\theta - aP\cos^2\theta\,\delta\theta \\ &= -aP\cos 2\theta\,\delta\theta. \end{aligned}$$

Hence the total virtual work done by all the four forces P

$$= -4\,aP\cos 2\theta\,\delta\theta.$$

Now the equation of virtual work is

$$4\,aT\sin\theta\,\delta\theta - 4\,aP\cos 2\theta\,\delta\theta = 0$$

or $\quad 4\,a\,(T\sin\theta - P\cos 2\theta)\,\delta\theta = 0$

or $\quad T\sin\theta - P\cos 2\theta = 0 \qquad [\because\ \delta\theta \neq 0]$

or $\quad T = (P\cos 2\theta / \sin\theta).$

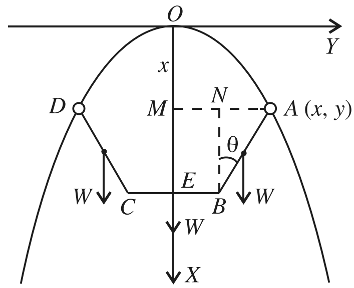
Note: It may be seen that if 2θ is acute, then $\cos 2\theta$ is positive and so the value of T is positive which means that there is tension in the rod AC as we have assumed while solving the problem. But if 2θ is obtuse, then $\cos 2\theta$ is negative and so the value of T is negative which means that there is not tension but thrust in the rod AC.

Problems Involving Curves

Example 13: *Three equal and similar rods AB, BC, CD freely jointed at B and C have small weightless rings attached to them at A and D. The rings slide on a smooth parabolic wire, whose axis is vertical and vertex upwards and whose latus rectum is half the sum of the lengths of the three rods. Prove that in the position of equilibrium, the inclination θ of AB or CD to the vertical is given by*

$$cos\,\theta - sin\,\theta + sin\,2\theta = 0.$$ **(Lucknow 2006, 08, 10)**

Solution: Let $AB = BC = CD = 2a$, so that the sum of their lengths $= 6\,a$.



Then the latus rectum of the parabola $= 3\,a$.

Hence the equation of the parabola is $y^2 = 3ax$.

In the position of equilibrium let θ be the inclination of AB or CD to the vertical. The weights W of the rods AB, BC and CD act at their respective middle points.

Let the coordinates of the point A be (x, y).

Then $x = OM$

and $y = MA = MN + NA = EB + NA = a + 2a\sin\theta.$

Since the point (x, y) lies on the parabola $y^2 = 3ax$, therefore

$$(a + 2a\sin\theta)^2 = 3ax. \qquad ...(1)$$

Differentiating (1), we get

$$2\,(a + 2a\sin\theta)\,2a\cos\theta\,\delta\theta = 3a\,\delta x$$

or $$\delta x = \frac{4}{3}\,a\,(1 + 2\sin\theta)\cos\theta\,\delta\theta. \qquad ...(2)$$

Here OY is the fixed line. The depth of the middle point of AB or CD below $OY = x + a\cos\theta$ and the depth of the middle point of BC below $OY = x + 2a\cos\theta$.

Let the rods be given a small symmetrial displacement about the axis OX in which θ changes to $\theta + \delta\theta$. Then the equation of virtual work is

$$2\,W\,\delta\,(x + a\cos\theta) + W\,\delta\,(x + 2\,a\cos\theta) = 0$$

or $$2\,W\,\delta x - 2\,aW\sin\theta\,\delta\theta + W\,\delta x - 2aW\sin\theta\,\delta\theta = 0$$

or $$3\,W\,\delta x - 4\,aW\sin\theta\,\delta\theta = 0$$

or $$3W\cdot\frac{4}{3}\,a\,(1 + 2\sin\theta)\cos\theta\,\delta\theta - 4\,aW\sin\theta\,\delta\theta = 0,$$

substituting for δx from (2)

or $$4aW\,[\cos\theta + 2\sin\theta\cos\theta - \sin\theta]\,\delta\theta = 0$$

or $$\cos\theta + \sin 2\,\theta - \sin\theta = 0, \qquad [\because\ W \neq 0 \text{ and } \delta\theta \neq 0]$$

which gives the required result.

Comprehensive Exercise 2

1. A rhombus is formed of rods each of weight W and length l with smooth joints. It rests symmetrically with its two upper sides in contact with two smooth pegs at the same level and at a distance $2\,a$ apart. A weight $W\,'$ is hung at the lowest point. If the sides of the rhombus make an angle θ with the vertical, prove that

$$\sin^3\theta = \frac{a\,(4\,W + W')}{l\,(4\,W + 2\,W')}\cdot$$

2. $ABCD$ is a rhombus with four rods each of length l and negligible weight joined by smooth hinges. A weight W is attached to the lowest hinge C, and the frame rests on two smooth pegs in a horizontal line in contact with the rods AB and AD, B and D are in a horizontal line and are joined by a string. If the distance of the pegs apart is $2c$ and the angle at A is 2α, show that the tension in the string is

$$W \tan\alpha \left(\frac{c}{2l}\operatorname{cosec}^3\alpha - 1\right).$$

3. A rhombus $ABCD$ formed of four weightless rods each of length a freely jointed at the extremities, rests in a vertical plane on two smooth pegs, which are in a horizontal line distant $2c$ apart and in contact with AB and AD. Weights each equal to W are hung from the lowest corner C and from the middle points of two lower sides, while B and D are connected by a light inextensible string. If 2α be the angle of the rhombus at A, apply the principle of virtual work to find the tension T of the string.

4. $ABCD$ is a rhombus formed with four rods each of length l and of weight w joined by smooth hinges. A weight W is attached to the lowest hinge C and the frame rests on two smooth pegs in a horizontal line and B and D are joined by a string. If the distance of the pegs apart is $2d$ and the angle at A is 2α, show that the tension in the string is

$$\tan\alpha\left[\frac{d}{2l}(W+4w)\operatorname{cosec}^3\alpha - (W+2w)\right].$$

5. Two bars, each of weight W and length $2a$, are freely jointed at a common extremity and rest symmetrically in contact with two smooth pegs at a distance $2c$ apart in the same horizontal line so as to include an angle 2α, their other extremities being connected by a string. Find the tension T of the string.

6. A frame ABC consists of three light rods, of which AB, AC are each of length a, BC of length $\frac{3}{2}a$, freely jointed together. It rests with BC horizontal, A below BC and the rods AB, AC over two smooth pegs E and F, in the same horizontal line, distant $2b$ apart. A weight W is suspended from A, find the thrust in the rod BC.

7. A rhomboidal framework $ABCD$ is formed of four equal light rods of length a smoothly jointed together. It rests in a vertical plane with the diagonal AC vertical, and the rods BC, CD in contact with two smooth pegs in the same horizontal line at a distance c apart, the joints B, D being kept apart by a light rod of length b. Show that a weight W, being placed on the highest joint A, will produce in BD a thrust of magnitude

$$W(2a^2c - b^3)/b^2(4a^2 - b^2)^{1/2}.$$

8. Three rigid rods AB, BC, CD, each of length $2a$, are smoothly jointed at B and C. The system is placed in a vertical plane so that rods AB, CD are in contact with two smooth pegs distant $2c$ apart in the same horizontal line, the rods AB, CD make equal angle α with the horizon. Prove that the tension of the string AD which will maintain this configuration is

$$\frac{1}{4}W\operatorname{cosec}\alpha\sec^2\alpha\,\{(3c/a) - (3 + 2\cos^3\alpha)\},$$

where W is the weight of either rod.

9. Four light rods are jointed together to form a quadrilateral $OABC$. The lengths are such that

$$OA = OC = a, \quad AB = CB = b.$$

The framework hangs in a vertical plane with OA and OC resting in contact with two smooth pegs distant l apart and on the same horizontal level. A weight hangs at B. If θ, ϕ are the inclinations of OA, AB to the horizontal, prove that these values are given by the equations

$$a \cos \theta = b \cos \phi \text{ and } \frac{1}{2} l \sec^2 \theta \sin \phi = a \sin (\theta + \phi).$$ **(Avadh 2009)**

10. A uniform beam of length $2a$, rests in equilibrium against a smooth vertical wall and upon a smooth peg at a distance b from the wall. Show that in the position of equilibrium the beam is inclined to the wall at an angle $\sin^{-1} (b / a)^{1/3}$.

(Purvanchal 2008; Rohilkhand 06)

11. A heavy uniform rod, of length $2a$, rests with its ends in contact with two smooth inclined planes, of inclinations α and β to the horizon. If θ be the inclination of the rod to the horizon, prove, by the principle of virtual work, that

$$\tan \theta = \frac{1}{2} (\cot \alpha - \cot \beta).$$

(Meerut 2011)

12. A uniform rectangular board rests vertically in equilibrium with its sides a and b on two smooth pegs in the same horizontal line at a distance c apart. Prove by the principle of virtual work that the side of length a makes with the vertical an angle θ given by $2 c \cos 2 \theta = b \cos \theta - a \sin \theta$.

13. Two equal rods, AB and AC, each of length $2b$, are freely jointed at A and rest on a smooth vertical circle of radius a. Show that if 2θ be the angle between them, then $b \sin^3 \theta = a \cos \theta$ or $b = a \cot \theta + b \cos^2 \theta$. **(Lucknow 2008; Kumaun 01, 02)**

14. Two equal rods, each of weight wl and length l, are hinged together and placed astride a smooth horizontal cylindrical peg of radius r. Then the lower ends are tied together by a string and the rods are left at the same inclination ϕ to the horizontal. Find the tension in the string and if the string is slack, show that ϕ satisfies the equation $\tan^3 \phi + \tan \phi = l / 2 r$.

15. Two light rods AOC, BOD are smoothly hinged at O, a point at a distance c from each of the ends A, B which are connected by a string of length $2 c \sin \alpha$. The rods rest in a vertical plane with the ends A and B on a smooth horizontal table. A smooth circular disc of radius a and weight W is placed on the rods above O with its plane vertical so that rods are tangents to the disc. Prove that the tension of the string is $\frac{1}{2} W \{(a/c) \operatorname{cosec}^2 \alpha + \tan \alpha\}$.

16. One end of a uniform rod AB, of length $2 a$ and weight W, is attached by a frictionless joint to a smooth vertical wall, and the other end B is smoothly jointed to an equal rod BC. The middle points of the rods are joined by an elastic string, of natural length a and modulus of elasticity $4 W$. Prove that the system can rest in equilibrium in a vertical plane with C in contact with the wall below A, and the angle between the rods is $2 \sin^{-1} (3 / 4)$.

17. Two equal uniform rods AB, AC each of weight W are freely jointed at A and rest with the extremities B and C on the inside of a smooth circular hoop, whose radius is greater than the length of either rod, the whole being in a vertical plane and the middle points of the rods being jointed by a light string. Show that if the string is stretched, its tension is $W(\tan\alpha - 2\tan\beta)$, where 2α is the angle between the rods, and β the angle either rod subtends at the centre.

18. A heavy elastic string, whose natural length is $2\pi a$, is placed round a smooth cone whose axis is vertical and whose semi-vertical angle is α. If W be the weight and λ the modulus of elasticity of the string, prove that it will be in equilibrium when in the form of a circle whose radius is $a\left(1+\frac{W}{2\lambda\pi}\cot\alpha\right)$. **(Avadh 2009)**

19. An endless chain of weight W rests in the form of a circular band round a smooth vertical cone which has its vertex upwards. Find the tension in the chain due to its weight, assuming the vertical angle of the cone to be 2α. **(Garhwal 2003)**

20. Two small rings of equal weights, slide on a smooth wire in the shape of a parabola whose axis is vertical and vertex upwards, and attract one another with a force which varies as the distance. If they can rest in any symmetrical position on the curve show that they will rest in all symmetrical positions.

21. Two heavy rings slide on a smooth parabolic wire, whose axis is horizontal and plane vertical, and are connected by a string passing round a smooth peg at the focus. Prove that in the position of equilibrium their weights are proportional to their vertical depths below the axis.

22. A smooth parabolic wire is fixed with its axis vertical and vertex downwards, and in it is placed a uniform rod of length $2l$ with its ends resting on the wire. Show that, for equilibrium the rod is either horizontal, or makes with the horizontal an angle θ given by $\cos^2\theta = 2a/l$, $4a$ being the latus rectum of the parabola.

23. A heavy rod, of length $2l$, rests upon a fixed smooth peg at C and with its end B upon a smooth curve. If it rests in all positions, show that the curve is a conchoid whose polar equation, with C as origin, is

$$r = l + \frac{a}{\sin\theta}.$$

24. One end of a beam rests against a smooth vertical wall and the other end on a smooth curve in a vertical plane perpendicular to the wall ; if the beam rests in all positions, show that the curve is an ellipse whose major axis lies along the horizontal line described by the centre of gravity of the beam.

25. A smooth rod passes through a smooth ring at the focus of an ellipse whose major axis is horizontal, and rests with its lower end on the quadrant of the curve which is farthest removed from the focus. Find its position of equilibrium and show that its length must at least be $\frac{3}{4}a + \frac{1}{4}a\sqrt{(1+8e^2)}$ where $2a$ is the major axis and e is the eccentricity.

26. Two small smooth rings of equal weight slide on a fixed elliptical wire, whose major axis is vertical and they are connected by a string which passes over a small smooth peg at the upper focus ; show that the weights will be in equilibrium wherever they are placed. **(Kumaun 2003)**

Answers 2

3. $T = W\,(3c \operatorname{cosec}^2 \alpha - 5a \sin\alpha)\,/\,(2a \cos\alpha)$

5. $T = \dfrac{1}{2} W \left[\dfrac{c}{a} \operatorname{cosec}^2 \alpha \sec\alpha - \tan\alpha \right]$

6. $T = \dfrac{32\, bW}{9\sqrt{7a}}$

14. $w\,(r \sec^2 \phi - \dfrac{1}{2}\, l \cot\phi)$

19. $(W \cot\alpha)/2\pi$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. Five weightless rods of equal length are jointed together so as to form a rhombus $ABCD$ with one diagonal BD. If a weight W be attached to C and the system be suspended from A, then the thrust in the rod BD is equal to

(a) $W/\sqrt{2}$ (b) $W/\sqrt{3}$

(c) $4\,W$ (d) $2\,W$.

2. In a problem on virtual work a body is given a small virtual displacement in which α changes to $\alpha + \delta\alpha$. If the equation of virtual work is

$$T\,\delta(2\,a \sin\alpha) + 4\,w\,\delta(2\,a\cos\alpha) + W\,\delta(4\,a\cos\alpha) = 0,$$

then the value of T is

(a) $(2\,w + 4\,W)\tan\alpha$ (b) $(W + 2\,w)\tan\alpha$

(c) $(2\,W + 4\,w)\tan\alpha$ (d) $(w + 2\,W)\tan\alpha$.

3. Virtual work done by the thrust T in a rod of length l (assumed extensible), is given by :

(a) $-T\,\delta l$ (b) $+T\,\delta l$

(c) $\pm T\,\delta l$ (d) None of these.

(Garhwal 2002; Avadh 06; Bundelkhand 06, 08; Rohilkhand 09, 10)

4. The virtual work done by the reaction of any smooth surface with which the body is in contact, is as follows :

(a) Positive (b) Negative

(c) Zero (d) None of these. **(Garhwal 2003)**

5. The virtual work done by the tension of an extensible string is :

(a) Zero (b) – ve

(c) + ve (d) None of these.

(Garhwal 2004; Rohilkhand 08)

Fill in the Blank(s)

Fill in the blanks "……" so that the following statements are complete and correct.

1. The work done by the tension of an inextensible string during a small displacement is …… .
2. The work done by the tension T of an extensible string of length l during a small displacement is …… .
3. The necessary and sufficient condition that a particle or a rigid body acted upon by a system of coplanar forces be in equilibrium is that the algebraic sum of the …… done by the forces during any small displacement consistent with the geometrical conditions of the system is zero to the first degree of approximation.
4. If a body is constrained to turn about a fixed point or a fixed axis, the virtual work of the reaction at the point or on the axis is …… . **(Bundelkhand 2009)**

True or false

Write 'T' for true and 'F' for false statement.

1. The work done by the thrust T of an inextensible rod of length l during a small displacement is $T\ \delta l$.
2. The work done by the thrust of an inextensible rod is zero during a small displacement.
3. The work done byxz the thrust T of an extensible rod of length l during a small displacement is $T\ \delta l$.
4. The work done by the mutual reaction between two bodies of a system is zero in any virtual displacement of the system.

Answers

Multiple Choice Questions

1. (b)	2. (c)	3. (b)
4. (c)	5. (b)	

Fill in the Blank(s)

1. zero	2. $-T\ \delta l$
3. virtual works	4. zero

True or False

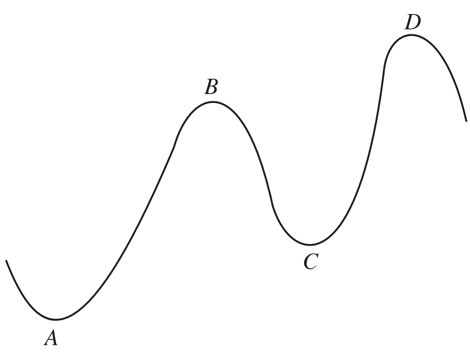
1. *F*	2. *T*	3. *T*
4. *T*		

□

Chapter 4

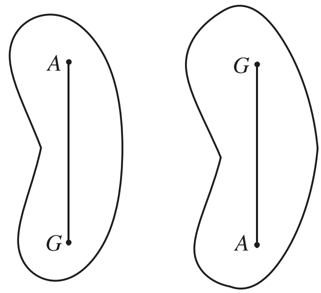
Stable and Unstable Equilibrium

4.1 Introduction



Consider the motion of a body on a smooth curve in a vertical plane as shown in the figure. Obviously the body can rest at points A, B, C and D which are points of maxima or minima of the curve. If the body be slightly displaced from its position of rest at A or C (*i.e.*, the points of minima), it will tend to return to its original position of rest, while if displaced from its position of rest at B or D (*i.e.*, the points of maxima), it will tend to move still further away from its original position of rest. In the first case the equilibrium of the body is said to be *stable* and in the second case it is said to be *unstable.*

Take one more illustration. Consider the equilibrium of a rigid body fixed at one point say A. For the equilibrium of the body the centre of gravity G of the body must lie on the vertical line through the point of support A. There arise three cases.



Case 1. Suppose that the centre of gravity G lies below the point of support A. In this case if the body be slightly displaced from its position of equilibrium its centre of gravity will be raised. If the body be then let free, the force of gravity will bring the body back to its original position of equilibrium. In this case the body is said to be in *stable equilibrium.*

Case 2. Next suppose that the centre of gravity G lies above the point of support A. In this case if the body be slightly displaced from its position of equilibrium, its centre of gravity will be lowered. If the body be then let free, the force of gravity will still further move away the body from its original position of equilibrium. In this case the body is said to be *in unstable equilibrium.*

Case 3. If the centre of gravity G is at the point of support A, the body will still be in equilibrium when displaced. In this case we say that the body is in a state of *neutral equilibrium.*

Remark. It can be seen that among all the possible positions of the body, in the case 1 the height of the centre of gravity of the body above some fixed plane is minimum and in the case 2 it is maximum.

4.2 Definitions

(Meerut 2008)

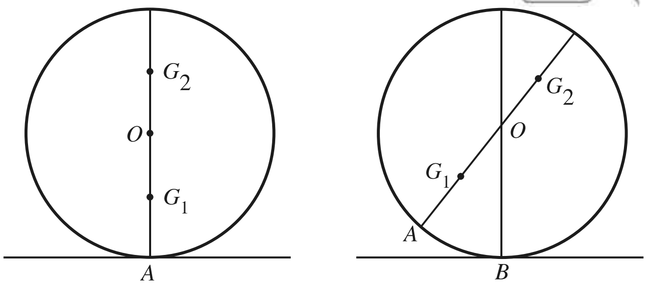
1. Stable equilibrium. *A body is said to be in stable equilibrium if when slightly displaced from its position of equilibrium, the forces acting on the body tend to make it return towards its position of equilibrium.* **(Lucknow 2006, 08, 09; Meerut 11; Rohilkhand 10, 11)**

2. Unstable equilibrium. *The equilibrium of a body is said to be unstable if when slightly displaced from its position of equilibrium, the forces acting on the body tend to move the body further away from its position of equilibrium.* **(Rohilkhand 2010, 11; Meerut 09, 11)**

3. Neutral equilibrium. *A body is said to be in neutral equilibrium if the forces acting on it are such that they keep the body in equilibrium in any slightly displaced position.*

Further example of stable, unstable and neutral equilibrium

Consider the case of a heavy sphere resting on a horizontal plane. Suppose the centre of gravity of the sphere is not at its geometric centre O. It is obvious that for the equilibrium of the sphere its point of contact A with the plane, its geometric centre O and its centre of gravity must be in the same vertical line. One position of equilibrium is in which the centre of gravity G_1 is below the geometric centre O. In this case if the



sphere be slightly displaced it would tend to come back to its original position of equilibrium. This is the position of **stable** equilibrium. The other position of equilibrium is in which the centre of gravity G_2 is above the geometric centre O. In this case if the sphere be slightly displaced it would not come back to its original position of equilibrium but would go further away from that position. This is the position of **unstable equilibrium.**

If, however, the centre of gravity of the sphere is at its geometric centre O, the sphere will still be in equilibrium when displaced. In this case the equilibrium is **neutral.**

If a right circular cone rests on a horizontal plane with its base in contact with the plane and its axis vertical, its equilibrium is stable. But if it rests with its vertex in contact with the plane and its axis vertical, its equilibrium is unstable. Again if it rests along a generator, it is in neutral equilibrium.

The equilibrium of a pendulum is stable when it is displaced from its vertical position of equilibrium, for it returns towards the vertical position again. Any top, heavy thing or a stick placed vertically on a finger is an example of unstable equilibrium.

4.3 The Work Function

Suppose a material system is acted upon by a system of forces X, Y, Z parallel to the axes of coordinates. If during a small displacement whose projections on the coordinate axes are dx, dy, dz the work done by these forces is dW, then

$$dW = Xdx + Ydy + Zdz.$$

The forces X, Y, Z generally depend upon the position of the particle. If we confine ourselves to the class of forces which are single-valued and are functions of x, y, z (and not of time t), then integrating the above equation from some standard position (x_0, y_0, z_0) to any position (x, y, z), we have

$$W = \int_{(x_0, y_0, z_0)}^{(x, y, z)} (Xdx + Ydy + Zdz).$$

Such a function W is called the *work function.* It is work done by the forces in displacing the body from standard position to any position.

If W_A and W_B are the values of the work function at two positions A and B, then $W_B - W_A$ gives the work done by the forces in displacing the body from A to B.

If $Xdx + Ydy + Zdz$ is an exact differential, the forces are called *conservative forces.*

4.4 Work Function Test for the Nature of Stability of Equilibrium

Let A be the position of equilibrium of a rigid body under the action of a given system of forces and let W be the work function of the system in this position A. Suppose the body undergoes a small displacement and takes a position B near to the position of equilibrium A, then the value of the work function in the position B will be $W + dW$. Therefore the work done by the forces in displacing the body from the equilibrium position A to the nearby position B is dW. Since the body is in equilibrium in the position A, therefore by the principle of virtual work, we have $dW = 0$. *Hence the work function W is stationary (maximum or minimum) in the position of equilibrium.*

First suppose that W is maximum at the equilibrium position A. Imagine that the body is slightly displaced to a position B and let W' be the work function there. Since W is maximum at A, therefore, $W' < W$, so that $W' - W$ is negative. It means that in displacing the body from A to B the work done by the forces is negative *i.e.*, the work is done against the forces and hence the forces will have a tendency to bring the body back to the original position of equilibrium A. Hence the equilibrium at A is stable.

Next suppose that W is minimum at the equilibrium position A. If W' is the value of the work function in a slightly displaced position B of the body, then in this case $W' > W$, so that $W' - W$ is positive. It means that in displacing the body from A to B the work done by the forces is positive *i.e.*, the work has been done by the forces and so the forces will have a tendency to move the body further away from the position of equilibrium. Hence in this case the equilibrium at A is unstable.

Thus in the positions of equilibrium of the body the work function W is either maximum or minimum. If it is maximum, the equilibrium is stable and if it is minimum, the equilibrium is unstable.

4.5 Potential Energy Test for the Nature of Stability of Equilibrium

(Lucknow 2006, 08, 09)

Potential energy of a body: *The potential energy of a body, acted upon by a conservative system of forces, is defined as its capacity to do work by virtue of the position it has acquired. It is measured by the amount of work it can do in passing from the present position to some standard position.* If W be the work function of the body in any position referred to some standard position, and V be the potential energy of the body in that position referred to the same standard position, then $V = -W$. If V_A and V_B are the values of the potential energy at the two positions A and B, then $V_A - V_B$ is the work done by the forces in displacing the body from A to B.

Let A be the position of equilibrium of a rigid body under the action of a given system of forces and V be the potential energy of the body in this position A. Suppose the body undergoes a small displacement and takes a position B near to the position of equilibrium A, then the potential energy of the body in the position B will be $V + dV$. Therefore the work done by the forces in displacing the body from the equilibrium position A to the nearby position B is $V - (V + dV)$ *i.e.*, $-dV$. Since the body is in equilibrium in the position A, therefore by the principle of virtual work, we have

$$-dV = 0 \Rightarrow dV = 0.$$

Hence the potential energy V is stationary (maximum or minimum) in the position of equilibrium.

First suppose that V is minimum at the equilibrium position A. Imagine that the body is slightly displaced to a position B and let V' be the potential energy there. Since V is minimum at A, therefore $V' > V$, so that $V - V'$ is negative. It means that in displacing the body from A to B the work done by the forces acting on the body is negative *i.e.*, the work is done against the forces and so the forces will have a tendency to bring the body back to the original position of equilibrium A. Hence the equilibrium at A is stable.

Thus we see that *in the position of stable equilibrium, the potential energy of the body is minimum.*

Next suppose that V is maximum at the equilibrium position A. If V' is the value of the potential energy in a slightly displaced position B of the body, then in this case $V' < V$, so that $V - V'$ is positive. It means that in displacing the body from A to B the work done by the forces is positive *i.e.*, the work is done by the forces and so the forces will tend to move the body further away from the position of equilibrium. Hence the equilibrium at A is unstable.

Thus in the positions of equilibrium of the body the potential energy V is either maximum or minimum. If it is minimum, the equilibrium is stable, and if it is maximum, the equilibrium is unstable.

For example, whenever gravitational energy is the only form of potential energy involved, the height of the centre of gravity of the body above a fixed horizontal plane must be a minimum for stable equilibrium and maximum for unstable equilibrium.

4.6 *z*-Test for the Nature of Stability

(Lucknow 2006)

Suppose a body is in equilibrium under its weight only *i.e.*, the force of gravity is the only external force acting on the body. Let z be the height of the centre of gravity of the body above a fixed horizontal plane. Express z as a function of some variable θ *i.e.*, let $z = f(\theta)$. By the principle of virtual work, for the equilibrium of the body, we must have

$$-W\,\delta z = 0, \text{ where } W \text{ is the weight of the body}$$

$$\Rightarrow \qquad \delta z = 0 \Rightarrow \frac{dz}{d\theta}\,\delta\theta = 0 \Rightarrow \frac{dz}{d\theta} = 0.$$

Thus the equilibrium positions of the body are given by the equation $dz/d\theta = 0$. So in the position of equilibrium the height of the centre of gravity of the body above a fixed level must be either maximum or minimum.

Suppose the equation $dz/d\theta = 0$ on solving gives $\theta = \alpha, \beta, \gamma$ etc. as the positions of equilibrium.

To test the nature of equilibrium at the position $\theta = \alpha$. We find $\dfrac{d^2 z}{d\theta^2}$ for $\theta = \alpha$. If it is positive, then z is minimum for $\theta = \alpha$. So if we give a slight displacement to the body, the height of its centre of gravity will be raised and then on being set free the body will tend to come back to its original position of equilibrium. Therefore in this case the equilibrium is stable.

Again if $d^2 z/d\theta^2$ for $\theta = \alpha$ is negative, then z is maximum for $\theta = \alpha$. So if we give a slight displacement to the body, the height of its centre of gravity will be lowered and then on being set free the force of gravity will still displace the body further away from its original position of equilibrium. Therefore in this case the equilibrium is unstable.

Thus the equilibrium positions of the body are given by the equation $dz/d\theta = 0$. If for a root $\theta = \alpha$ of this equation; $d^2 z/d\theta^2$ is positive, then z is minimum and the equilibrium is stable. But if for $\theta = \alpha$, $d^2 z/d\theta^2$ is negative, then z is maximum and the equilibrium is unstable.

If, however $d^2 z/d\theta^2 = 0$ for $\theta = \alpha$, then we consider $d^3 z/d\theta^3$ and $d^4 z/d\theta^4$. Then for the position of equilibrium $\theta = \alpha$, we must have $d^3 z/d\theta^3 = 0$, and the equilibrium is stable or unstable according as for this position $d^4 z/d\theta^4$ is positive or negative.

Similar tests apply for the other positions of equilibrium $\theta = \beta, \gamma$ etc.

Remark: If $z = f(\theta)$ represents the depth of the centre of gravity of the body below some fixed horizontal plane, then the conditions for the stability and unstability of the equilibrium are reversed. In this case for equilibrium position we must have $dz/d\theta = 0$. If for a root $\theta = \alpha$ of this equation $d^2 z/d\theta^2$ is positive, then z is minimum and the equilibrium is unstable. But if for $\theta = \alpha$, $d^2 z/d\theta^2$ is negative, then z is maximum and the equilibrium is stable. If, however $d^2 z/d\theta^2 = 0$ for $\theta = \alpha$, then we consider higher differential coefficients of z and conclude similarly.

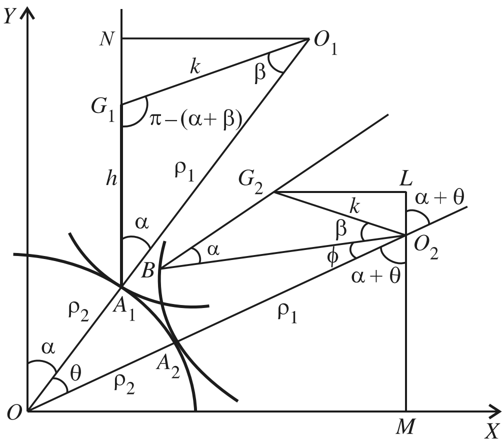
4.7 Stability of a Body Resting on a Fixed Rough Surface

Theorem. *A body rests in equilibrium upon another fixed body, the portions of the two bodies in contact have radii of curvatures ρ_1 and ρ_2 respectively. The centre of gravity of the first body is at a height h above the point of contact and the common normal makes an angle α with the vertical ; it is required to prove that the equilibrium is stable or unstable according as $h <$ or $> \dfrac{\rho_1 \rho_2}{\rho_1 + \rho_2} \cos \alpha$.*

(Meerut 2004; Avadh 11; Purvanchal 10)

Let O and O_1 be the centres of curvature of the lower and upper bodies in the position of rest and A_1 be their point of contact. In this position of equilibrium the common normal OA_1O_1 makes an angle α with the vertical OY. If G_1 is the centre of gravity of the upper body, then for equilibrium the line A_1G_1 must be vertical. It is given that $A_1G_1 = h$.

Let $\quad O_1G_1 = k$ and $\angle OO_1G_1 = \beta$.



Suppose the upper body is slightly displaced by pure rolling over the lower body which is fixed. Let A_2 be the new point of contact. O_2 is the new position of O_1 and the point A_1 of the upper body rolls up to the position B so that $O_2\,B$ is the new position of the original normal $O_1\,A_1$. Also G_2 is the new position of G_1 so that $O_2\,G_2 = O_1\,G_1 = k$.

Suppose the common normal at A_2 makes angles θ and ϕ with the original normals OA_1 and O_2B.

We have $O_1A_1 = \rho_1$

and $\quad OA_1 = \rho_2$. Also $O_2A_2 = \rho_1$

and $\quad OA_2 = \rho_2$.

Since the upper body rolls on the lower body without slipping, therefore

$$\text{arc } A_1A_2 = \text{arc } A_2B \quad i.e., \ \rho_2\theta = \rho_1\,\phi.$$

$$\therefore \quad \frac{d\phi}{d\theta} = \frac{\rho_2}{\rho_1}. \qquad \ldots(1)$$

Let z be the height of G_2 above the fixed horizontal line OX. Then

$$z = LM = LO_2 + O_2M$$
$$= O_2\,G_2 \cos \angle G_2\,O_2\,L + OO_2 \cos(\alpha+\theta)$$
$$= k \cos[\pi - (\alpha+\theta+\phi+\beta)] + (\rho_1+\rho_2)\cos(\alpha+\theta)$$
$$[\because \ O_2G_2 = O_1G_1 = k]$$
$$= (\rho_1+\rho_2)\cos(\alpha+\theta) - k\cos(\alpha+\theta+\phi+\beta).$$

$$\therefore \quad \frac{dz}{d\theta} = -(\rho_1+\rho_2)\sin(\alpha+\theta) + k\sin(\alpha+\theta+\phi+\beta)\left(1+\frac{d\phi}{d\theta}\right)$$

[$\because$ α, β are constants and θ, ϕ are the only variables]

$$= -(\rho_1+\rho_2)\sin(\alpha+\theta) + k\sin(\alpha+\theta+\phi+\beta)\left(1+\frac{\rho_2}{\rho_1}\right) \qquad \text{[From (1)]}$$

$$= \frac{\rho_1+\rho_2}{\rho_1}[-\rho_1\sin(\alpha+\theta) + k\sin(\alpha+\theta+\phi+\beta)]$$

and
$$\frac{d^2z}{d\theta^2} = \frac{\rho_1+\rho_2}{\rho_1}\left[-\rho_1\cos(\alpha+\theta) + k\cos(\alpha+\theta+\phi+\beta)\left(1+\frac{d\phi}{d\theta}\right)\right]$$
$$= \frac{\rho_1+\rho_2}{\rho_1}\left[-\rho_1\cos(\alpha+\theta) + k\cos(\alpha+\theta+\phi+\beta)\left(1+\frac{\rho_2}{\rho_1}\right)\right]$$

$$= \frac{\rho_1 + \rho_2}{\rho_1^2} [-\rho_1^2 \cos(\alpha + \theta) + k(\rho_1 + \rho_2) \cos(\alpha + \theta + \phi + \beta)].$$

In the position of equilibrium $\theta = 0$ and $\phi = 0$.

Thus the equilibrium is stable or unstable according as $d^2 z / d\theta^2$ is positive or negative for $\theta = \phi = 0$, *i.e.*, according as

$$k(\rho_1 + \rho_2) \cos(\alpha + \beta) > \text{ or } < \rho_1^2 \cos\alpha.$$

But from the $\Delta\, A_1 G_1 O_1$, we have

$$h = A_1 G_1 = A_1 N - G_1 N = A_1 O_1 \cos\alpha - O_1 G_1 \cos \angle O_1 G_1 N$$

$$= \rho_1 \cos\alpha - k \cos(\alpha + \beta).$$

$\therefore$ $$k \cos(\alpha + \beta) = \rho_1 \cos\alpha - h.$$

Hence the equilibrium is stable or unstable according as

$$(\rho_1 + \rho_2)(\rho_1 \cos\alpha - h) > \text{ or } < \rho_1^2 \cos^2\alpha$$

i.e., $$(\rho_1 + \rho_2)\rho_1 \cos\alpha - (\rho_1 + \rho_2) h > \text{ or } < \rho_1^2 \cos^2\alpha$$

i.e., $$(\rho_1 + \rho_2) h < \text{ or } > (\rho_1 + \rho_2)\rho_1 \cos\alpha - \rho_1^2 \cos^2\alpha$$

i.e., $$(\rho_1 + \rho_2) h < \text{ or } > \rho_1 \rho_2 \cos\alpha$$

i.e., $$h < \text{ or } > \frac{\rho_1 \rho_2}{\rho_1 + \rho_2} \cos\alpha.$$

Cor. If $\alpha = 0$, the above conditions give that the equilibrium is stable or unstable according as

$$h < \text{ or } > \frac{\rho_1 \rho_2}{\rho_1 + \rho_2} \quad \textit{i.e.,} \quad \frac{1}{h} > \text{ or } < \frac{\rho_1 + \rho_2}{\rho_1 \rho_2}$$

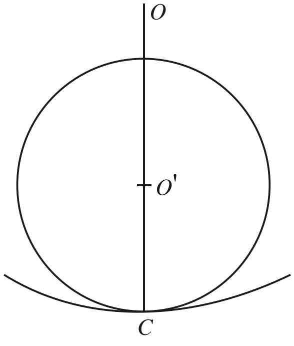
or $$\frac{1}{h} > \text{ or } < \frac{1}{\rho_1} + \frac{1}{\rho_2}.$$

Thus suppose that a body rests in equilibrium upon another body which is fixed and the portions of the two bodies in contact have radii of curvatures ρ_1 and ρ_2 respectively. The C.G. of the first body is at a height h above the point of contact and the common normal coincides with the vertical. Then the equilibrium is stable or unstable according as

$$\frac{1}{h} > \text{ or } < \frac{1}{\rho_1} + \frac{1}{\rho_2}.$$

If the portions of the bodies in contact are spheres of radii r_1 and r_2, then in the above condition we put $\rho_1 = r_1$ and $\rho_2 = r_2$. Thus the equilibrium is stable or unstable according as

$$\frac{1}{h} > \text{ or } < \frac{1}{r_1} + \frac{1}{r_2}.$$



If the surface of the upper body at the point of contact is plane, then $\rho_1 = \infty$ and if the surface of the lower body at the point of contact is plane, then $\rho_2 = \infty$.

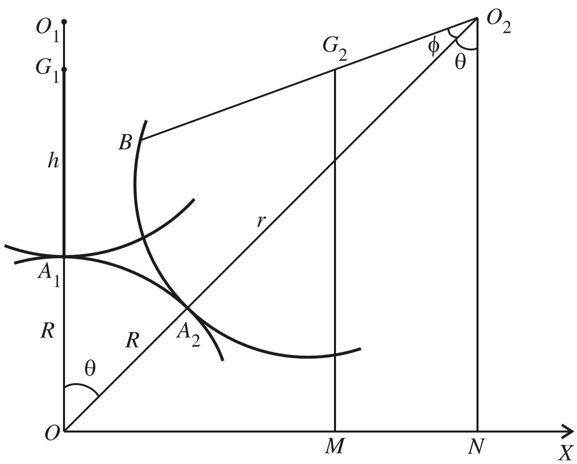
If the surface of the lower body at the point of contact instead of being convex is concave, then ρ_2 is to be taken with negative sign.

On account of its importance we shall now give an independent proof in case the surfaces in contact are spherical.

4.8 Stability of a Body Resting on a Fixed Rough Surface, the Portions of the Two Bodies in Contact being Spheres

A body rests in equilibrium upon another fixed body, the portions of the two bodies in contact being spheres of radii r and R respectively and the straight line joining the centres of the spheres being vertical ; if the first body be slightly displaced, to find whether the equilibrium is stable or unstable, the bodies being rough enough to prevent any sliding.

Let O be the centre of the spherical surface of the lower body which is fixed and O_1 that of the upper body which rests on the lower body, A_1 being their point of contact and the line OO_1 being vertical. If G_1 is the centre of gravity of the upper body, then for the equilibrium of the upper body, the line $A_1\ G_1$ must be vertical ; let $A_1\ G_1$ be h. The figure is a section of the bodies by a vertical plane through G_1 .



Suppose the upper body is slightly displaced by pure rolling over the lower body. Let A_2 be the new point of contact. O_2 is the new position of O_1 and the point A_1 of the upper body rolls up to the position B so that $O_2\ B$ is the new position of $O_1\ A_1$. Also G_2 is the new position of G_1 so that $BG_2 = A_1\ G_1 = h$.

Let $\angle A_1OA_2 = \theta$ and $\angle BO_2A_2 = \phi$; so that $\angle G_2O_2N = \theta + \phi$.

We have $O_1A_1 = r$ and $OA_1 = R$. Also $O_2A_2 = O_2B = r$ and $OA_2 = R$. Since the upper body rolls on the lower body without slipping, therefore

$$\text{arc } A_1A_2 = \text{arc } A_2B \ \ i.e., \ \ R\theta = r\phi \ \ i.e., \ \ \phi = (R/r)\,\theta.$$

Now in order to find the nature of equilibrium, we should find the height z of the centre of gravity G_2 in the new position above the fixed horizontal line OX. We have

$$\begin{aligned} z &= G_2M = O_2N - O_2G_2 \cos(\theta + \phi) \\ &= OO_2 \cos\theta - (O_2B - BG_2)\cos(\theta + \phi) \\ &= (R + r)\cos\theta - (r - h)\cos(\theta + \phi) \end{aligned}$$

$$= (R+r)\cos\theta - (r-h)\cos\{\theta + (R/r)\,\theta\} \qquad [\because \phi = (R/r)\,\theta]$$

$$= (R+r)\cos\theta - (r-h)\cos\left\{\frac{\theta\,(r+R)}{r}\right\}.$$

For equilibrium we have $dz/d\theta = 0$

i.e., $$-(R+r)\sin\theta + (r-h)\sin\left\{\frac{\theta\,(r+R)}{r}\right\}\frac{r+R}{r} = 0.$$

This is satisfied by $\theta = 0$.

Now $$\frac{d^2z}{d\theta^2} = -(R+r)\cos\theta + (r-h)\cos\left\{\frac{\theta\,(r+R)}{r}\right\}\cdot\left(\frac{r+R}{r}\right)^2.$$

$\therefore$ $$\left(\frac{d^2z}{d\theta^2}\right)_{\theta=0} = -(R+r) + (r-h)\left(\frac{r+R}{r}\right)^2$$

$$= \left(\frac{r+R}{r}\right)^2\left\{(r-h) - \frac{r^2}{R+r}\right\} = \left(\frac{r+R}{r}\right)^2\left\{r - \frac{r^2}{R+r} - h\right\}$$

$$= \left(\frac{r+R}{r}\right)^2\left\{\frac{rR}{R+r} - h\right\}.$$

This will be positive if

$$\frac{rR}{R+r} > h \quad i.e., \quad \frac{1}{h} > \frac{R+r}{rR} \quad i.e., \quad \frac{1}{h} > \frac{1}{r} + \frac{1}{R}$$

and negative, if $\dfrac{rR}{R+r} < h$ *i.e.,* $\dfrac{1}{h} < \dfrac{1}{r} + \dfrac{1}{R}$.

Hence the equilibrium is stable or unstable according as

$$\frac{1}{h} > \frac{1}{r} + \frac{1}{R} \quad \text{or} \quad \frac{1}{h} < \frac{1}{r} + \frac{1}{R}.$$

Here R is the radius of the lower body and r that of the upper body and h is the height of the C.G. of the upper body above the point of contact.

Now it remains to discuss the case when $1/h = 1/r + 1/R$ *i.e.,* $h = rR/(R+r)$.

In this case $d^2z/d\theta^2 = 0$. Hence we find $d^3z/d\theta^3$ and $d^4z/d\theta^4$. We have

$$\frac{d^3z}{d\theta^3} = (R+r)\sin\theta - (r-h)\sin\left\{\frac{\theta\,(r+R)}{r}\right)\cdot\left(\frac{r+R}{r}\right)^3$$

and $$\frac{d^4z}{d\theta^4} = (R+r)\cos\theta - (r-h)\cos\left\{\frac{\theta\,(r+R)}{r}\right)\cdot\left(\frac{r+R}{r}\right)^4.$$

Obviously $\left(\dfrac{d^3z}{d\theta^3}\right)_{\theta=0} = 0.$

Also $$\left(\frac{d^4z}{d\theta^4}\right)_{\theta=0} = (R+r) - (r-h)\left(\frac{r+R}{r}\right)^4$$

$$= (R+r)\left\{1 - \frac{r-h}{r}\left(\frac{r+R}{r}\right)^3\right\}$$

$$= (R+r)\left\{1 - \frac{r-h}{r} \cdot \frac{R+r}{r} \cdot \left(\frac{R+r}{r}\right)^2\right\}$$

$$= (R+r)\left\{1 - \left(r - \frac{rR}{R+r}\right) \cdot \frac{R+r}{r^2} \cdot \left(\frac{R+r}{r}\right)^2\right\}$$

$$\left[\because h = \frac{rR}{r+R}\right]$$

$$= (R+r)\left\{1 - \frac{r^2}{R+r} \cdot \frac{R+r}{r^2} \cdot \left(\frac{R+r}{r}\right)^2\right\}$$

$$= (R+r)\left\{1 - \left(\frac{R+r}{r}\right)^2\right\}$$

$$= (R+r)\left\{1 - \left(1 + \frac{R}{r}\right)^2\right\}, \text{ which is negative.}$$

This shows that z is maximum and so in this case the equilibrium is unstable.

Hence if $\frac{1}{h} > \frac{1}{r} + \frac{1}{R}$, the equilibrium is stable

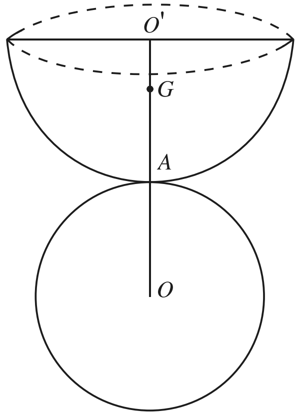
and if $\frac{1}{h} \le \frac{1}{r} + \frac{1}{R}$, the equilibrium is unstable.

Remark: If the upper body has a plane face in contact with the lower body of radius R, then obviously $r = \infty$. And if the lower body be plane, then $R = \infty$.

Illustrative Examples

Example 1: *A hemisphere rests in equilibrium on a sphere of equal radius ; show that the equilibrium is unstable when the curved, and stable when the flat surface of the hemisphere rests on the sphere.* **(Meerut 2007; Rohilkhand 10; Avadh 08, 11; Purvanchal 09; Agra 10; Lucknow 07, 10; 11; Kanpur 08, 11)**

Solution: **(i) When the curved surface of the hemisphere rests on the sphere.** A hemisphere of centre O' rests on a sphere of centre O with its curved surface in contact with the sphere. The point of contact is A and $OA = O'A = a$ (say). Also the line OAO' is vertical. If G is the centre of gravity of the hemisphere, then G lies on $O'A$ and $O'G = \frac{3}{8}a$.



Here ρ_1 = the radius of curvature of the upper body at the point of contact
= the radius of the hemisphere
$= a$,

and ρ_2 = the radius of curvature of the lower body at the point of contact

$= a$.

Also h = the height of the centre of gravity of the upper body above the points of contact A

$$= AG = O'A - O'G = a - \frac{3}{8}a = \frac{5}{8}a.$$

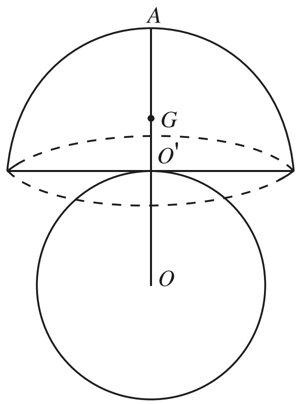
We have $\frac{1}{h} = \frac{1}{5a/8} = \frac{8}{5a}$, and $\frac{1}{\rho_1} + \frac{1}{\rho_2} = \frac{1}{a} + \frac{1}{a} = \frac{2}{a} = \frac{10}{5a}$.

Thus $\frac{1}{h} < \frac{1}{\rho_1} + \frac{1}{\rho_2}$. Hence the equilibrium is unstable in this case.

(ii) When the flat surface of the hemisphere rests on the sphere. In this case a hemisphere of centre O' rests on a sphere of centre O and equal radius a with its flat surface (*i.e.*, the plane base) in contact with the sphere. The point of contact is O' and G is the C.G. of the hemisphere.

Here ρ_1 = the radius of curvature of the upper body at the point of contact = ∞,
[Note that the base of the hemisphere touches the sphere along a straight line]

and ρ_2 = the radius of curvature of the lower body at the point of contact
= the radius of the sphere = a.



Also h = the height of the C.G. of the hemisphere above the point of contact O'

$$= O'G = \frac{3}{8}a.$$

We have $$\frac{1}{h} = \frac{1}{3a/8} = \frac{8}{3a},$$

and $$\frac{1}{\rho_1} + \frac{1}{\rho_2} = \frac{1}{\infty} + \frac{1}{a} = 0 + \frac{1}{a} = \frac{1}{a} = \frac{3}{3a}.$$

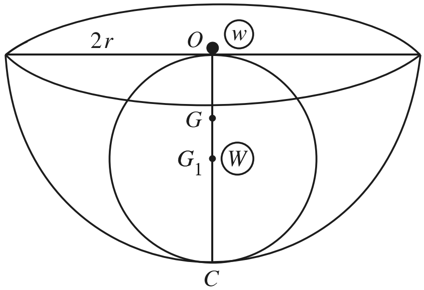
Obviously $$\frac{1}{h} > \frac{1}{\rho_1} + \frac{1}{\rho_2}.$$

Hence in this case the equilibrium is stable.

Remark: Remember that for a straight line the radius of curvature at any point is infinity, and for a circle the radius of curvature at any point is equal to the radius of the circle.

Example 2: *A solid sphere rests inside a fixed rough hemispherical bowl of twice its radius. Show that, however large a weight is attached to the highest point of the sphere, the equilibrium is stable.* **(Meerut 2006, 08; Lucknow 08, 09)**

Solution: Let r be the radius of the solid sphere which rests inside a fixed rough hemispherical bowl of radius $2r$. Their point of contact is C and O is the highest point of the sphere so that $OC = 2r$. Let W and w be weights of the sphere and the weight



attached to the highest point of the sphere. The weight W of the sphere acts at the middle point G_1 of its diameter OC.

If h is the height of the centre of gravity of the combined body consisting of the sphere and the weight w attached to O, then

$$h = \frac{W \,.\, r + w \,.\, 2\,r}{W + w}.$$

Here ρ_1 = the radius of curvature of the upper body at the point of contact C
= the radius of the sphere = r,

and ρ_2 = the radius of curvature of the lower body at the point of contact C
$= -\,2\,r$,

the negative sign is taken because the surface of the lower fixed body *i.e.*, the bowl at C is concave.

The equilibrium will be stable if

$$\frac{1}{h} > \frac{1}{\rho_1} + \frac{1}{\rho_2} \quad i.e., \quad \frac{1}{h} > \frac{1}{r} - \frac{1}{2\,r}$$

i.e., $$\frac{1}{h} > \frac{1}{2\,r} \quad i.e., \quad h < 2\,r$$

i.e., $$\frac{Wr + 2\,wr}{W + w} < 2\,r \quad i.e., \quad Wr + 2\,wr < 2\,Wr + 2\,wr$$

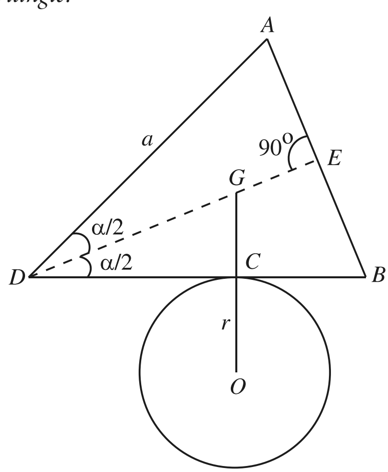
i.e., $Wr < 2\,Wr$, which is so whatever be the value of w.

Hence, however large a weight is attached to the highest point of the sphere, the equilibrium is stable.

Example 3: *A lamina in the form of an isosceles triangle, whose vertical angle is α, is placed on a sphere, of radius r, so that its plane is vertical and one of its equal sides is in contact with the sphere; show that, if the triangle be slightly displaced in its own plane, the equilibrium is stable if $\sin \alpha < 3\,r/a$, where a is one of the equal sides of the triangle.*

Solution: DAB is an isosceles triangular lamina in which

$$DA = DB = a \text{ and } \angle\, ADB = \alpha.$$



The centre of gravity G of the lamina lies on its median DE which is perpendicular to AB and also bisects the angle ADB. We have

$$DG = \frac{2}{3}\,DE = \frac{2}{3}\,a \cos \frac{1}{2}\,\alpha.$$

The lamina rests on a fixed sphere whose centre is O and radius r. Their point of contact is C. For equilibrium the line OCG must be vertical.

If h be the height of the C.G. of the lamina above the point of contact C, then

$$h = GC = DG \sin \frac{1}{2} \alpha$$

$$= \frac{2}{3} a \cos \frac{1}{2} \alpha \sin \frac{1}{2} \alpha = \frac{1}{3} a \sin \alpha.$$

Here ρ_1 = the radius of curvature of the upper body at the point of contact C
$= \infty$

and ρ_2 = the radius of curvature of the lower fixed body at the point $C = r$.

The equilibrium will be stable if

$$\frac{1}{h} > \frac{1}{\rho_1} + \frac{1}{\rho_2} \quad i.e., \quad \frac{1}{h} > \frac{1}{\infty} + \frac{1}{r} \quad i.e., \quad \frac{1}{h} > \frac{1}{r}$$

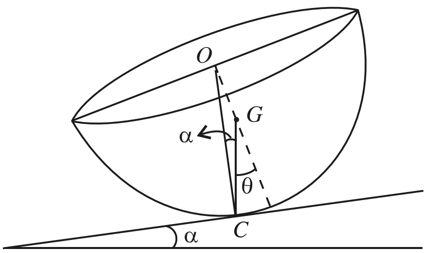
i.e., $\quad h < r \quad i.e., \quad \frac{1}{3} a \sin \alpha < r$

i.e., $\quad \sin \alpha < 3\, r/a.$

Example 4: *A solid hemisphere rests on a plane inclined to the horizon at an angle* $\alpha < \sin^{-1} \frac{3}{8}$, *and the plane is rough enough to prevent any sliding. Find the position of equilibrium and show that it is stable.* **(Avadh 2007; Purvanchal 07)**

Solution: Let O be the centre of the base of the hemisphere and r be its radius. If C is the point of contact of the hemisphere and the inclined plane, then $OC = r$. Let G be the centre of gravity of the hemisphere. Then $OG = 3\, r/8$. In the position of equilibrium the line CG must be vertical.

Since OC is perpendicular to the inclined plane and CG is perpendicular to the horizontal, therefore $\angle\, OCG = \alpha$. Suppose in equilibrium the axis of the hemisphere makes an angle θ with the vertical. From $\Delta\, OGC$, we have



$$\frac{OG}{\sin \alpha} = \frac{OC}{\sin \theta} \quad i.e., \quad \frac{3\, r/8}{\sin \theta} = \frac{r}{\sin \theta}.$$

$$\therefore \quad \sin \theta = \frac{8}{3} \sin \alpha, \quad \text{or} \quad \theta = \sin^{-1} \left(\frac{8}{3} \sin \alpha\right),$$

giving the position of equilibrium of the hemisphere.

Since $\sin \theta < 1$, therefore $\frac{8}{3} \sin \alpha < 1$

i.e., $\quad \sin \alpha < \frac{3}{8} \quad i.e., \quad \alpha < \sin^{-1} \frac{3}{8}.$ ...(1)

Thus for the equilibrium to exist, we must have $\alpha < \sin^{-1} \frac{3}{8}$.

Now let $CG = h$. Then

$$\frac{h}{\sin(\theta-\alpha)}=\frac{3\,r/8}{\sin\alpha}\text{, so that } h=\frac{3\,r\sin(\theta-\alpha)}{8\sin\alpha}.$$

Here $\rho_1 = r$ and $\rho_2 = \infty$.

The equilibrium will be stable if

$$h<\frac{\rho_1\,\rho_2\cos\alpha}{\rho_1+\rho_2} \qquad \text{[See 4.7]}$$

i.e., $\frac{1}{h}>\frac{\rho_1+\rho_2}{\rho_1\,\rho_2}\sec\alpha$ *i.e.,* $\frac{1}{h}>\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)\sec\alpha$

i.e., $\frac{1}{h}>\frac{1}{r}\sec\alpha$ $[\because \rho_1 = r,\ \rho_2=\infty]$

i.e., $h<r\cos\alpha$

i.e., $\frac{3\,r\sin(\theta-\alpha)}{8\sin\alpha}<r\cos\alpha$ [substituting for h]

or $3\sin(\theta-\alpha)<8\sin\alpha\cos\alpha$

or $3\sin\theta\cos\alpha-3\cos\theta\sin\alpha<8\sin\alpha\cos\alpha$

or $8\sin\alpha\cos\alpha-3\sin\alpha\sqrt{\left(1-\frac{64}{9}\sin^2\alpha\right)}<8\sin\alpha\cos\alpha$

$[\because \sin\theta=\frac{8}{3}\sin\alpha]$

or $-\sin\alpha\sqrt{(9-64\sin^2\alpha)}<0$

or $\sin\alpha\sqrt{(9-64\sin^2\alpha)}>0.$ ...(2)

But from (1),

$\sin\alpha<\frac{3}{8}$ *i.e.,* $64\sin^2\alpha<9$

i.e., $\sqrt{(9-64\sin^2\alpha)}$ is a positive real number.

Therefore the relation (2) is true. Hence the equilibrium is stable.

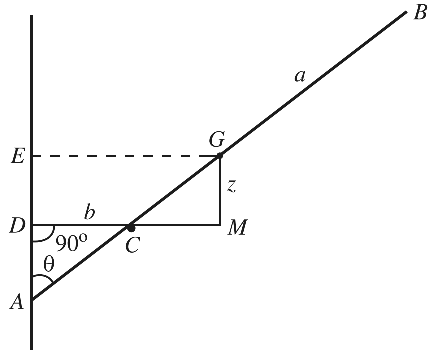
Problems Based Upon z-Test

Example 5: *A heavy uniform rod rests with one end against a smooth vertical wall and with a point in its length resting on a smooth peg ; find the position of equilibrium and show that it is unstable.* **(Avadh 2008; Meerut 10)**

Solution: Let AB be a uniform rod of length $2\,a$. The end A of the rod rests against a smooth vertical wall and the rod rests on a smooth peg C whose distance from the wall is say b *i.e.,* $CD=b$.

Suppose the rod makes an angle θ with the wall. The centre of gravity of the rod is at its middle point G. Let z be the height of G above the fixed peg C *i.e.,* $GM=z$. We shall express z in terms of θ. We have

$$z=GM=ED=AE-AD$$



$$= AG \cos\theta - CD \cot\theta$$
$$= a\cos\theta - b\cot\theta.$$

$\therefore$ $\qquad dz/d\theta = -a\sin\theta + b\,\text{cosec}^2\,\theta$

and $\qquad d^2z/d\theta^2 = -a\cos\theta - 2b\,\text{cosec}^2\,\theta\cot\theta.$

For equilibrium of the rod, we have $dz/d\theta = 0$

i.e., $\qquad -a\sin\theta + b\,\text{cosec}^2\,\theta = 0$

or $\qquad a\sin\theta = b\,\text{cosec}^2\,\theta,$

or $\qquad \sin^3\theta = b/a$

or $\qquad \sin\theta = (b/a)^{1/3}, \quad$ or $\quad \theta = \sin^{-1}(b/a)^{1/3}.$

This gives the position of equilibrium of the rod.

Again $\qquad d^2z/d\theta^2 = -(a\cos\theta + 2b\,\text{cosec}^2\,\theta\cot\theta)$

$$= \text{negative for all acute values of } \theta.$$

Thus $d^2z/d\theta^2$ is negative in the position of equilibrium and so z is maximum. Hence the equilibrium is unstable.

Example 6: *A uniform rod, of length 2l, is attached by smooth rings at both ends of a parabolic wire, fixed with its axis vertical and vertex downwards, and of latus rectum* $4a$. *Show that the angle* θ *which the rod makes with the horizontal in a slanting position of equilibrium is given by* $\cos^2\theta = 2a/l$, *and that, if these positions exist they are stable.*

Show also that the positions in which the rod is horizontal are stable or unstable according as the rod is below or above the focus.

Solution: Let AB be the rod of length $2l$. Take OX and OY as coordinate axes, so that the equation of the parabola be written as
$x^2 = 4ay$.

Let the coordinates of the point A be $(2at, at^2)$ and let the rod AB make an angle θ with the horizontal AC. Then the coordinates of B are

$$(2at + 2l\cos\theta,\ at^2 + 2l\sin\theta).$$

Since B lies on the parabola $x^2 = 4ay$, therefore

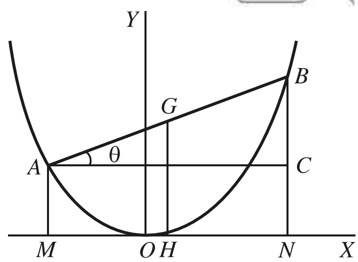
$$(2at + 2l\cos\theta)^2 = 4a(at^2 + 2l\sin\theta)$$

or $\qquad 8atl\cos\theta + 4l^2\cos^2\theta = 8al\sin\theta$

or $\qquad (2al\cos\theta)\,t = 2al\sin\theta - l^2\cos^2\theta$

or $\qquad t = \tan\theta - (l/2a)\cos\theta. \qquad \ldots(1)$

The centre of gravity of the rod AB is at its middle point G. If z be the height of G above the fixed horizontal line OX, then



$$z = GH = \frac{1}{2}(AM + BN)$$

$$= \frac{1}{2}[at^2 + (at^2 + 2\,l\sin\theta)] = at^2 + l\sin\theta$$

$$= a\,[\tan\theta - (l/2\,a)\cos\theta]^2 + l\sin\theta$$

[from (1)]

$$= (l^2/4\,a)\cos^2\theta + a\tan^2\theta = (1/4\,a)\,[l^2\cos^2\theta + 4\,a^2\tan^2\theta].$$

$\therefore$ $$dz/d\theta = (1/4\,a)\,[-2\,l^2\cos\theta\sin\theta + 8\,a^2\tan\theta\sec^2\theta]$$

$$= (1/2\,a)\sin\theta\,[-l^2\cos\theta + 4\,a^2\sec^3\theta].$$

For the equilibrium of the rod, we must have $dz/d\theta = 0$

i.e., $(1/2\,a)\sin\theta\,(-l^2\cos\theta + 4\,a^2\sec^3\theta) = 0.$

$\therefore$ either $\sin\theta = 0$ *i.e.*, $\theta = 0$,

which gives the horizontal position of rest of the rod

or $-l^2\cos\theta + 4\,a^2\sec^3\theta = 0$ *i.e.*, $l^2\cos\theta = 4\,a^2/\cos^3\theta$

i.e., $\cos^4\theta = 4\,a^2/l^2$ *i.e.*, $\cos^2\theta = 2\,a/l$,

which gives the inclined position of rest of the rod.

Now, $$d^2z/d\theta^2 = (1/2\,a)\cos\theta\,[-l^2\cos\theta + 4\,a^2\sec^3\theta] + (1/2\,a)\sin\theta\,[l^2\sin\theta + 12\,a^2\sec^3\theta\tan\theta]. \quad \ldots(2)$$

When $\cos^2\theta = 2\,a/l$ *i.e.*,

when $-l^2\cos\theta + 4\,a^2\sec^3\theta = 0$, we have

$$d^2z/d\theta^2 = (1/2\,a)\sin\theta\,[l^2\sin\theta + 12\,a^2\sec^3\theta\tan\theta]$$

$$= (1/2\,a)\sin^2\theta\,[l^2 + 12\,a^2\sec^4\theta], \text{ which is } > 0.$$

Hence in the inclined position of rest of the rod, z is minimum and so the equilibrium is stable.

Again when the rod is horizontal *i.e.*, $\theta = 0$, we have, from (2)

$$\frac{d^2z}{d\theta^2} = \frac{8\,a^2 - 2\,l^2}{4\,a} = \frac{4\,a^2 - l^2}{2\,a}.$$

The equilibrium in this case is stable or unstable according as $d^2z/d\theta^2$ is positive or negative

i.e., according as $4\,a^2 - l^2 >$ or < 0

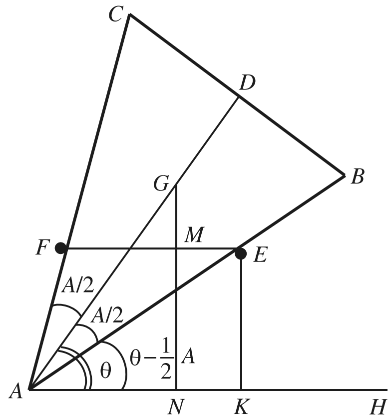
i.e., according as $2\,a >$ or $< l$

i.e., according as $2\,l <$ or $> 4\,a$

i.e., according as the rod is below or above the focus.

Example 7: *A uniform isosceles triangular lamina ABC rests in equilibrium with its equal sides AB and AC in contact with two smooth pegs in the same horizontal line at a distance c apart. If the perpendicular AD upon BC is h, show that there are three positions of equilibrium, of which the one with AD vertical is stable and the other two are unstable, if* $h < 3\,c \operatorname{cosec} A$; *whilst if* $h \geq 3\,c \operatorname{cosec} A$, *there is only one position of equilibrium, which is unstable.*

Solution: ABC is an isosceles triangular lamina resting on two smooth pegs E and F which are in the same horizontal line and $EF = c$. The perpendicular AD from A upon BC is of length h. We have



$$\angle BAD = \angle CAD = \frac{1}{2} A.$$

The weight of the lamina acts at its centre of gravity G, where

$$AD = \frac{2}{3} AD = \frac{2}{3} h.$$

Suppose AD makes an angle θ with the horizontal AH, so that

$$\angle BAH = \theta - \frac{1}{2} A.$$

Let z be the height of G above the fixed horizontal line EF. Then

$$z = GM = GN - MN = GN - EK$$

$$= AG \sin\theta - AE \sin(\theta - \frac{1}{2} A)$$

$$= \frac{2}{3} h \sin\theta - AE \sin(\theta - \frac{1}{2} A). \qquad \ldots(1)$$

Since EF is parallel to AK, therefore

$$\angle FEA = \angle EAK = \theta - \frac{1}{2} A.$$

Now in the Δ AEF, we have

$$\angle EFA = \pi - \{A + (\theta - \frac{1}{2} A)\} = \pi - (\theta + \frac{1}{2} A).$$

Applying the sine theorem of trigonometry for the Δ AEF, we have

$$\frac{AE}{\sin \angle EFA} = \frac{EF}{\sin \angle FAE} \quad i.e., \quad \frac{AE}{\sin\{\pi - (\theta + \frac{1}{2} A)\}} = \frac{c}{\sin A}.$$

$$\therefore \qquad AE = \frac{c}{\sin A} \sin\left(\theta + \frac{1}{2} A\right).$$

Substituting this value of AE in (1), we have

$$z = \frac{2}{3} h \sin\theta - \frac{c}{\sin A} \sin\left(\theta + \frac{1}{2} A\right) \sin\left(\theta - \frac{1}{2} A\right)$$

$$= \frac{2}{3} h \sin\theta - \frac{c}{2 \sin A} [\cos A - \cos 2\theta]$$

$$= \frac{2}{3} h \sin\theta - \frac{c}{2} \cot A + \frac{c}{2 \sin A} \cos 2\theta.$$

$\therefore$ $$\frac{dz}{d\theta} = \frac{2}{3} h \cos\theta - \frac{c}{\sin A} \sin 2\theta. \qquad \ldots(2)$$

For equilibrium, $dz/d\theta = 0$

i.e., $$\frac{2}{3} h \cos\theta - \frac{2c}{\sin A} \sin\theta \cos\theta = 0$$

i.e., $$2 \cos\theta \left[\frac{1}{3} h - \frac{c \sin\theta}{\sin A} \right] = 0.$$

$\therefore$ either $\cos\theta = 0$ *i.e.*, $\theta = \frac{1}{2}\pi$,

or $$\frac{1}{3} h - \frac{c \sin\theta}{\sin A} = 0 \quad i.e., \quad \sin\theta = \frac{h \sin A}{3c} = \frac{h}{3c \operatorname{cosec} A}.$$

Now $\theta = \frac{1}{2}\pi$ gives the position of equilibrium in which AD is vertical and the triangle rests symmetrically on the pegs. The values of θ given by

$$\sin\theta = h/(3c \operatorname{cosec} A)$$

are real and not equal to $\frac{1}{2}\pi$ if $h < 3c \operatorname{cosec} A$. Since

$$\sin(\pi - \theta) = \sin\theta,$$

therefore if $h < 3c \operatorname{cosec} A$, the equation $\sin\theta = h/(3c \operatorname{cosec} A)$ gives two inclined positions of equilibrium, one θ and the other $\pi - \theta$.

Thus if $h < 3c \operatorname{cosec} A$, there are three positions of equilibrium, one symmetrical and the other two inclined.

If $h \geq 3c \operatorname{cosec} A$, then the equation $\sin\theta = h/(3c \operatorname{cosec} A)$ either gives no real value of θ or the value of θ given by it is also equal to $\frac{1}{2}\pi$. Thus in this case the symmetrical position of equilibrium, $\theta = \frac{1}{2}\pi$, is the only position of equilibrium.

Nature of Equilibrium

From (2), $$\frac{d^2 z}{d\theta^2} = -\frac{2}{3} h \sin\theta - \frac{2c}{\sin A} \cos 2\theta. \qquad \ldots(3)$$

For $\theta = \frac{1}{2}\pi$, $$\frac{d^2 z}{d\theta^2} = -\frac{2}{3} h + \frac{2c}{\sin A} = \frac{2}{3}(-h + 3c \operatorname{cosec} A),$$

which is positive or negative according as

$$h < \text{ or } > 3c \operatorname{cosec} A.$$

Thus for $\theta = \frac{1}{2}\pi$, z is minimum or maximum according as

$$h < \text{ or } > 3c \operatorname{cosec} A.$$

Hence for $\theta = \frac{1}{2}\pi$, the equilibrium is stable or unstable according as

$$h < \text{ or } > 3\,c \operatorname{cosec} A.$$

For $\theta = \frac{1}{2}\pi$, $d^2 z/d\theta^2 = 0$ when $h = 3\,c \operatorname{cosec} A$. In this case we can see that

$$d^3 z/d\theta^3 = 0 \quad \text{and} \quad d^4 z/d\theta^4 = -6\,c \operatorname{cosec} A,$$

which is negative. So in this case z is maximum and the equilibrium is unstable. Thus the symmetrical position of equilibrium is stable or unstable according as

$$h < \text{ or } \geq 3\,c \operatorname{cosec} A.$$

Now we consider the inclined positions of equilibrium. From (3), we can write

$$\frac{d^2 z}{d\theta^2} = -\frac{2}{3} h \sin\theta - \frac{2\,c}{\sin A} = \frac{2}{3}(1 - 2\sin^2\theta). \qquad \ldots(4)$$

For the inclined positions of equilibrium, $\sin\theta = (h \sin A)/3\,c$.

Putting $\sin\theta = (h \sin A)/3\,c$ in (4), we get

$$\frac{d^2 z}{d\theta^2} = -\frac{2\,h}{3} \cdot \frac{h \sin A}{3\,c} - \frac{2\,c}{\sin A} + \frac{4\,c}{\sin A} \cdot \frac{h^2 \sin^2 A}{9\,c^2}$$

$$= \frac{2\,h^2}{9c} \sin A - \frac{2\,c}{\sin A} = \frac{2}{9c} \sin A\,(h^2 - 9c^2 \operatorname{cosec}^2 A),$$

which is negative since for inclined positions of equilibrium

$$h < 3\,c \operatorname{cosec} A.$$

Thus for the inclined positions of equilibrium, z is maximum and so they are positions of unstable equilibrium.

Remark: For inclined positions of equilibrium to exist, we must have

$$h < 3\,c \operatorname{cosec} A.$$

For these positions of equilibrium, θ is given by

$$\sin\theta = (h \sin A)/3\,c.$$

Now $\quad \frac{1}{2}A < \theta \;\Rightarrow\; \sin\frac{1}{2}A < \sin\theta$

$\Rightarrow \quad \sin\frac{1}{2}A < (h \sin A)/3c$

$\Rightarrow \quad \sin\frac{1}{2}A < \dfrac{2\,h \sin\frac{1}{2}A \cos\frac{1}{2}A}{3c}$

$\Rightarrow \quad h > \frac{3}{2}c \sec\frac{1}{2}A.$

Thus for inclined positions of equilibrium, we must have

$$\frac{3}{2}c \sec\frac{1}{2}A < h < 3c \operatorname{cosec} A.$$

Comprehensive Exercise 1

1. A uniform cubical box of edge a is placed on the top of a fixed sphere, the centre of the face of the cube being in contact with the highest point of the sphere. What is the least radius of the sphere for which the equilibrium will be stable ?

2. A heavy uniform cube balances on the highest point of a sphere whose radius is r. If the sphere is rough enough to prevent sliding and if the side of the cube be $\pi r/2$, show that the cube can rock through a right angle without falling.

3. A body, consisting of a cone and a hemisphere on the same base, rests on a rough horizontal table the hemisphere being in contact with the table ; show that the greatest height of the cone so that the equilibrium may be stable, is $\sqrt{3}$ times the radius of the hemisphere. **(Lucknow 2006, 11; Avadh 07, 09; Meerut 09)**

4. A solid homogeneous hemisphere of radius r has a solid right circular cone of the same substance constructed on the base ; the hemisphere rests on the convex side of the fixed sphere of radius R. Show that the length of the axis of the cone consistent with stability for a small rolling displacement is
$$\frac{r}{R+r}\left[\sqrt{\{(3R+r)(R-r)\}}-2r\right].$$

5. A uniform beam, of thickness $2b$, rests symmetrically on a perfectly rough horizontal cylinder of radius a ; show that the equilibrium of the beam will be stable or unstable according as b is less or greater than a.
(Rohilkhand 2009; Meerut 10; Avadh 09)

6. A uniform solid hemisphere rests in equilibrium upon a rough horizontal plane with its curved surface in contact with the plane and a particle of mass m is fixed at the centre of the plane face. Show that for any value of m, the equilibrium is stable.

7. A uniform hemisphere rests in equilibrium with its base upwards on the top of a sphere of double its radius. Show that the greatest weight which can be placed at the centre of the plane face without rendering the equilibrium unstable is one-eighth of the weight of the hemisphere. **(Meerut 2011)**

8. A solid sphere rests inside a fixed rough hemispherical bowl of thrice its radius. Find the conditions and nature of equilibrium if a large weight is attached to the highest point of the sphere.

9. A sphere of weight W and radius a lies within a fixed spherical shell of radius b, and a particle of weight w is fixed to the upper end of the vertical diameter. Prove that the equilibrium is stable if **(Meerut 2011)**
$$\frac{W}{w}>\frac{b-2a}{a}.$$

10. A heavy hemispherical shell of radius r has a particle attached to a point on the rim, and rests with the curved surface in contact with a rough sphere of radius R at the highest point. Prove that if $R/r>\sqrt{5}-1$, the equilibrium is stable, whatever be the weight of the particle.

11. A thin hemispherical bowl, of radius b and weight W rests in equilibrium on the highest point of a fixed sphere, of radius a, which is rough enough to prevent any

sliding. Inside the bowl is placed a small smooth sphere of weight w ; show that the equilibrium is not stable unless

$$w < W \frac{a - b}{2b}.$$

12. A uniform beam of length $2a$ rests with its ends on two smooth planes which intersect in a horizontal line. If the inclinations of the planes to the horizontal are α and β $(\alpha > \beta)$, show that the inclination θ of the beam to the horizontal in one of the equilibrium position is given by **(Avadh 2006; Lucknow 07)**

$$\tan\theta = \frac{1}{2}(\cot\beta - \cot\alpha)$$

and show that the beam is unstable in this position.

13. A uniform heavy beam rests between two smooth planes, each inclined at an angle $\frac{1}{4}\pi$ to the horizontal, so that the beam is in a vertical plane perpendicular to the line of action of the planes. Show that the equilibrium is unstable when the beam is horizontal.

14. A heavy uniform rod, length $2a$, rests partly within and partly without a fixed smooth hemispherical bowl of radius r, the rim of the bowl is horizontal, and one point of the rod is in contact with the rim; if θ be the inclination of the rod to the horizon, show that

$$2r\cos 2\theta = a\cos\theta.$$

Show also that the equilibrium of the rod is stable.

15. Two equal uniform rods are firmly jointed at one end so that the angle between them is α, and they rest in a vertical plane on a smooth sphere of radius r. Show that they are in a stable or unstable equilibrium according as the length of the rod is $>$ or $< 4r \operatorname{cosec}\alpha$.

16. A uniform smooth rod passes through a ring at the focus of a fixed parabola whose axis is vertical and vertex below the focus, and rests with one end on the parabola. Prove that the rod will be in equilibrium if it makes with the vertical an angle θ given by the equation

$$\cos^4 \frac{1}{2}\theta = a/2c$$

where $4a$ is the latus rectum and $2c$ the length of the rod. Investigate also the stability of equilibrium in this position.

17. A square lamina rests with its plane perpendicular to a smooth wall one corner being attached to a point in the wall by a fine string of length equal to the side of the square. Find the position of equilibrium and show that it is stable.

18. An isosceles triangular lamina of an angle 2α and height h rests between two smooth pegs at the same level, distant $2c$ apart ; prove that if

$$3c\sec\alpha < h < 6c\operatorname{cosec} 2\alpha,$$

the oblique positions of equilibrium exist, which are unstable. Discuss the stability of the vertical position.

19. A smooth solid right circular cone, of height h and vertical angle 2α, is at rest with its axis vertical in a horizontal circular hole of radius a. Show that the equilibrium is stable; and $16a > 3h\sin 2\alpha$, there are two other positions of unstable equilibrium ; and that if $16\,a < 3h \sin 2\,\alpha$, the equilibrium is unstable, and the position in which the axis is vertical is the only position of equilibrium.

Answers 1

1. $a/2$.
8. Stable or unstable according as the weight attached is less than or greater than the weight of the sphere.
16. Stable.
17. In the position of equilibrium the string makes an angle $\tan^{-1}(1/3)$ with the vertical.
18. The vertical position is stable or unstable according as

$$h < \text{ or } \geq 6\,c \operatorname{cosec} 2\,\alpha.$$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. The centre of gravity of a solid sphere does not coincide with its geometric centre. The sphere rests on a horizontal plane and its centre of gravity is above its geometric centre. The equilibrium of the sphere is
   (a) stable (b) neutral
   (c) unstable (d) none of these.
2. Suppose a body rests in equilibrium upon another body which is fixed and the portions of the two bodies in contact have radii of curvatures ρ_1 and ρ_2 respectively. Let the centre of gravity of the first body be at a height h above the point of contact and the common normal of the two bodies at the point of contact coincide with the vertical. Then the equilibrium is unstable according as
   (a) $\dfrac{1}{h} < \dfrac{1}{\rho_1} + \dfrac{1}{\rho_2}$ (b) $h < \rho_1 + \rho_2$
   (c) $h > \rho_1 + \rho_2$ (d) $\dfrac{1}{h} > \dfrac{1}{\rho_1} + \dfrac{1}{\rho_2}$.

Fill In The Blank(s)

Fill in the blanks "......" so that the following statements are complete and correct.

1. The equilibrium of a body is said to be ...... if when slightly displaced from its position of equilibrium, the forces acting on the body tend to move the body further away from its position of equilibrium.
2. A body is said to be in stable equilibrium if when slightly displaced from its position of equilibrium, the forces acting on the body tend to make it return towards its ....

3. In the position of stable equilibrium, the potential energy of a body is ....
4. Suppose a body rests in equilibrium upon another body which is fixed and the portions of the two bodies in contact have radii of curvatures ρ_1 and ρ_2 respectively. Let the centre of gravity of the first body be at a height h above the point of contact and the common normal of the two bodies at the point of contact coincide with the vertical. Then the equilibrium is stable according as

$$\frac{1}{h} > \ldots\ldots$$

5. Suppose a body is in equilibrium under its weight only. Let z be the height of the centre of gravity of the body above a fixed horizontal plane. Suppose z is expressed as a function of some variable θ *i.e.*, let $z = f(\theta)$. The equilibrium positions of the body are given by the equation $dz/d\theta = 0$. If for a root $\theta = \alpha$ of this equation, $d^2 z/d\theta^2$ is positive, then the equilibrium of the body in this position is ....

True or False

Write 'T' for true and 'F' for false statement.

1. A hemisphere rests in equilibrium on a sphere of equal radius. The equilibrium is stable when the curved, and unstable when the flat surface of the hemisphere rests on the sphere.
2. In the position of unstable equilibrium the potential energy of a body is maximum.
3. The centre of gravity of a solid sphere coincides with its geometric centre. The sphere rests on a horizontal plane. The equilibrium of the sphere is unstable.

Answers

Multiple Choice Questions

1. (c) 2. (a)

Fill in the Blank(s)

1. unstable
2. position of equilibrium.
3. minimum
4. $\dfrac{1}{\rho_1} + \dfrac{1}{\rho_2}$
5. stable

True or False

1. *F*
2. *T*
3. *T*

Chapter

5

Equilibrium of Forces in Three Dimensions

(Poinsot's Central Axis)

5.1 To Find the Resultant of any Given Number of Forces Acting on a Particle

(Purvanchal 2008)

Let $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ be the forces acting on a particle at O and OX, OY, OZ be any three mutually perpendicular axes in space through O.

If $\mathbf{R}$ is the resultant of the above given system of forces, then by a repeated application of the parallelogram law of forces, we have

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \ldots + \mathbf{F}_n. \qquad \ldots(1)$$

Let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be the unit vectors along the three axes OX, OY, OZ respectively, then $\mathbf{R} \cdot \mathbf{i}, \mathbf{R} \cdot \mathbf{j}, \mathbf{R} \cdot \mathbf{k}$ are the resolved parts of the resultant $\mathbf{R}$ along OX, OY, OZ respectively.

From (1), we get

$$\text{and} \quad \left.\begin{aligned} \mathbf{R} \cdot \mathbf{i} &= \mathbf{F}_1 \cdot \mathbf{i} + \mathbf{F}_2 \cdot \mathbf{i} + \ldots \mathbf{F}_\mathbf{n} \cdot \mathbf{i} = X \\ \mathbf{R} \cdot \mathbf{j} &= \mathbf{F}_1 \cdot \mathbf{j} + \mathbf{F}_2 \cdot \mathbf{j} + \ldots \mathbf{F}_n \cdot \mathbf{j} = Y \\ \mathbf{R} \cdot \mathbf{k} &= \mathbf{F}_1 \cdot \mathbf{k} + \mathbf{F}_2 \cdot \mathbf{k} + \ldots \mathbf{F}_n \cdot \mathbf{k} = Z \end{aligned}\right\} \qquad \ldots(2)$$

From the relations (2) we conclude that *"the resolved part of the resultant* **R** *along any axis is equal to the sum of the resolved parts of the forces* $\mathbf{F}_1, \mathbf{F}_2, \dots, \mathbf{F}_n$ *along that axis."*

If X, Y, Z are the algebraic sums of the resolved parts of the given forces along the three axes respectively, then

$$\mathbf{R} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}.$$

The magnitude R of the resultant $\mathbf{R}$ is given by

$$R = \sqrt{(X^2 + Y^2 + Z^2)}.$$

Also $$\hat{\mathbf{R}} = \frac{\mathbf{R}}{|\mathbf{R}|} = \frac{X}{R}\mathbf{i} + \frac{Y}{R}\mathbf{j} + \frac{Z}{R}\mathbf{k}.$$

$\therefore X/R, Y/R, Z/R$ are the direction cosines of the line of action of the resultant force $\mathbf{R}$.

5.2 Necessary and Sufficient Conditions of Equilibrium of a Particle Under the Action of a System of Forces

Theorem: *The necessary and sufficient conditions of equilibrium of a particle under the action of a system of forces, are that the algebraic sums of the resolved parts of the forces along any three mutually perpendicular directions vanish separately.* **(Gorakhpur 2010)**

Proof: Let a particle at O be acted upon by a system of forces $\mathbf{F}_1, \mathbf{F}_2, \dots, \mathbf{F}_n$. Then the resultant $\mathbf{R}$ of this system of forces is given by $\mathbf{R} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}$ where X, Y, Z are the sums of the resolved parts of the given forces along any three mutually perpendicular axes OX, OY, OZ through O and **i, j, k** are unit vectors along these axes respectively. (See 5.1)

The conditions are necessary. If the particle at O is in equilibrium, then the resultant force $\mathbf{R} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k} = \mathbf{0}$ which gives, $X = 0, Y = 0$, and $Z = 0$.

Thus if the particle is in equilibrium, then the sums of the resolved parts of the forces along OX, OY, OZ vanish separately.

The conditions are sufficient. If the sums of the resolved parts of the forces along three mutually perpendicular axes OX, OY, OZ vanish separately *i.e.*, if $X = 0, Y = 0, Z = 0$, then the resultant $\mathbf{R} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k} = \mathbf{0}$.

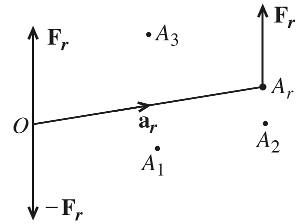
Hence the particle is in equilibrium.

5.3 Reduction of a System of Forces to a Single Force and a Couple

Theorem: *Any given system of forces acting at any given points of a rigid body can be reduced to a single force acting through an arbitrarily chosen point and a couple whose axis passes through that point.* **(Garhwal 2002; Kumaun 03; Avadh 06; 10; Gorakhpur 09; Purvanchal 10)**

Proof: Let there be a system of forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ acting at the points $A_1, A_2, \ldots, A_n$ respectively of a rigid body. Let O be any arbitrarily chosen point and $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ the position vectors of the points $A_1, A_2, \ldots, A_n$ respectively referred to the point O as the origin of vectors.

Consider the force $\mathbf{F}_r$ acting at the point A_r, where $\vec{OA} = \mathbf{a}_r$, and apply two forces $\mathbf{F}_r$ and $-\mathbf{F}_r$ at O parallel to the force $\mathbf{F}_r$ at A_r, in opposite directions as shown in the figure. Since two equal and opposite forces are applied at the same point of the body, therefore they will neutralise each other and will have no extra effect on the body. Thus the force $\mathbf{F}_r$ acting at A_r is equivalent to a force $\mathbf{F}_r$ at A_r and two forces $\mathbf{F}_r$ and $-\mathbf{F}_r$ at O. The forces $\mathbf{F}_r$ at A_r and $-\mathbf{F}_r$ at O will form a couple of moment $\mathbf{a}_r \times \mathbf{F}_r$.



Thus the force $\mathbf{F}_r$ acting at the point A_r of a rigid body is equivalent to an equal force $\mathbf{F}_r$ at O together with a couple of moment $\mathbf{a}_r \times \mathbf{F}_r$.

Similarly all the forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ acting at the points $A_1, A_2, \ldots, A_n$ etc., respectively of the body are equivalent to the forces $\mathbf{F}_1, \mathbf{F}_2$ etc. all acting at O together with the couples of moments $\mathbf{a}_1 \times \mathbf{F}_1, \mathbf{a}_2 \times \mathbf{F}_2$ etc.

Hence the given system of forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ acting at the points $A_1, A_2, \ldots, A_n$ of the body whose position vectors referred to an arbitrary point O are $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ respectively is equivalent to forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ at O together with the couples of moments

$$\mathbf{a}_1 \times \mathbf{F}_1, \mathbf{a}_2 \times \mathbf{F}_2, \ldots, \mathbf{a}_n \times \mathbf{F}_n .$$

If $\mathbf{R}$ is the resultant of the n concurrent forces at O and $\mathbf{G}$ the moment of the resultant couple of the n couples, then we have

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \ldots + \mathbf{F}_n = \sum_{r=1}^{n} \mathbf{F}_r , \qquad \ldots(1)$$

$$\mathbf{G} = \mathbf{a}_1 \times \mathbf{F}_1 + \mathbf{a}_2 \times \mathbf{F}_2 + \ldots + \mathbf{a}_n \times \mathbf{F}_n = \sum_{r=1}^{n} \mathbf{a}_r \times \mathbf{F}_r . \qquad \ldots(2)$$

Hence the given system of forces acting at any given points of a rigid body can be reduced to a single force $\mathbf{R}$, [given by (1)] acting at an arbitrarily chosen point O and a couple of moment $\mathbf{G}$ [given by (2)]. Since the couple is a free vector, therefore the axis of the couple $\mathbf{G}$ can be made to pass through the point O. The point O is called the **base point** for the reduction of the system of forces.

It should be noted that the single force

$$\mathbf{R} = \mathbf{F}_1 + \ldots + \mathbf{F}_n = \sum_{r=1}^{n} \mathbf{F}_r$$

does not depend on the choice of the point O because $\mathbf{R}$ is simply the vector sum of the forces $\mathbf{F}_1, \ldots, \mathbf{F}_n$ which has no connection with the point O. However, the moment $\mathbf{G} = \sum_{r=1}^{n} \mathbf{a}_r \times \mathbf{F}_r$ of the resultant couple depends on the origin of vectors O.

We shall now find the change in **G** with a change in the base point O.

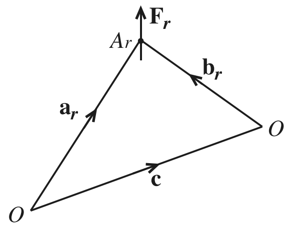
1. Change of the base point

Suppose a system of forces $\mathbf{F}_1, \ldots, \mathbf{F}_n$ acting at different points of a rigid body reduces to a single force **R** and a couple **G** with respect to O as the base point. We have

$$\mathbf{R} = \sum_{r=1}^{n} \mathbf{F}_r \qquad \ldots(1)$$

and
$$\mathbf{G} = \sum_{r=1}^{n} \mathbf{a}_r \times \mathbf{F}_r, \qquad \ldots(2)$$

where $\mathbf{a}_r$ is the position vector of the point of application A_r of the force $\mathbf{F}_r$ with respect to O as the origin of vectors.



Let us change the base point from O to O' where $\vec{OO'} = \mathbf{c}$.

If $\mathbf{b}_r$ be the position vector of the point A_r with respect to O' as the origin of vectors, then

$$\mathbf{b}_r = \vec{O'A_r} = \vec{OA_r} - \vec{OO'} = \mathbf{a}_r - \mathbf{c}.$$

Therefore if the system of forces $\mathbf{F}_1, \ldots, \mathbf{F}_n$ reduces to single force $\mathbf{R}'$ and a couple $\mathbf{G}'$ with respect to O' as the base point, then from (1) and (2), we have

$$\mathbf{R}' = \sum_{r=1}^{n} \mathbf{F}_r = \mathbf{R},$$

and
$$\mathbf{G}' = \sum_{r=1}^{n} \mathbf{b}_r \times \mathbf{F}_r = \sum_{r=1}^{n} (\mathbf{a}_r - \mathbf{c}) \times \mathbf{F}_r$$

$$= \sum_{r=1}^{n} (\mathbf{a}_r \times \mathbf{F}_r - \mathbf{c} \times \mathbf{F}_r) = \sum_{r=1}^{n} \mathbf{a}_r \times \mathbf{F}_r - \sum_{r=1}^{n} \mathbf{c} \times \mathbf{F}_r$$

$$= \mathbf{G} - \mathbf{c} \times \sum_{r=1}^{n} \mathbf{F}_r$$ [$\because$ **c** is a constant vector and can be brought outside the Σ notation]

$$= \mathbf{G} - \mathbf{c} \times \mathbf{R}.$$

Thus remember that $\mathbf{R}' = \mathbf{R}$, and $\mathbf{G}' = \mathbf{G} - \mathbf{c} \times \mathbf{R}$.

5.4 Necessary and Sufficient Conditions of Equilibrium of a Rigid Body Under the Action of a System of Forces Acting at any Points of it

Theorem: *The necessary and sufficient conditions of equilibrium of a rigid body under the action of a system of forces acting at any points of it are that the sums of the resolved parts of the forces along any three mutually perpendicular axes and the sums of the moments of the forces about these axes must vanish separately.*

Proof: Let there be a system of forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ acting at the points $A_1, A_2, \ldots, A_n$ respectively of a rigid body. Let $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ be the position vectors of the points $A_1, A_2, \ldots, A_n$ respectively referred to an arbitrary point O as origin. This system of forces will be equivalent to a single force $\mathbf{R}$ acting through O and a couple of moment $\mathbf{G}$. [See 5.3]. Also we have

$$\mathbf{R} = \sum_{r=1}^{n} \mathbf{F}_r \qquad \ldots(1)$$

and $$\mathbf{G} = \sum_{r=1}^{n} \mathbf{a}_r \times \mathbf{F}_r \,. \qquad \ldots(2)$$

Let **i, j, k** be the unit vectors along three mutually perpendicular axes OX, OY, OZ through O and let (x_r, y_r, z_r) be the coordinates of the point A_r with respect to these axes and X_r, Y_r, Z_r be the components in the directions of these axes of the force $\mathbf{F}_r$ acting at the point A_r. Then we have

$$\mathbf{F}_r = X_r \,\mathbf{i} + Y_r \,\mathbf{j} + Z_r \,\mathbf{k} \quad \text{and} \quad \mathbf{a}_r = x_r \,\mathbf{i} + y_r \,\mathbf{j} + z_r \,\mathbf{k}.$$

$\therefore$ $$\mathbf{R} = \sum_{r=1}^{n} \mathbf{F}_r = \sum_{r=1}^{n} (X_r \,\mathbf{i} + Y_r \,\mathbf{j} + Z_r \,\mathbf{k})$$

or $$\mathbf{R} = \mathbf{i} \sum_{r=1}^{n} X_r + \mathbf{j} \sum_{r=1}^{n} Y_r + \mathbf{k} \sum_{r=1}^{n} Z_r = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k},$$

where $$X = \sum_{r=1}^{n} X_r, Y = \sum_{r=1}^{n} Y_r, Z = \sum_{r=1}^{n} Z_r,$$

are the sums of the components of the given forces along the axes OX, OY, OZ respectively.

Also $$\mathbf{G} = \sum_{r=1}^{n} \mathbf{a}_r \times \mathbf{F}_r = \sum_{r=1}^{n} (x_r \,\mathbf{i} + y_r \,\mathbf{j} + z_r \,\mathbf{k}) \times (X_r \,\mathbf{i} + Y_r \,\mathbf{j} + Z_r \,\mathbf{k})$$

or $$\mathbf{G} = \mathbf{i} \sum_{r=1}^{n} (y_r Z_r - z_r Y_r)$$
$$+ \mathbf{j} \sum_{r=1}^{n} (z_r X_r - x_r Z_r) + \mathbf{k} \sum_{r=1}^{n} (x_r Y_r - y_r X_r)$$

or $$\mathbf{G} = L\,\mathbf{i} + M\,\mathbf{j} + N\,\mathbf{k},$$

where $$L = \sum_{r=1}^{n} (y_r Z_r - z_r Y_r), M = \sum_{r=1}^{n} (z_r X_r - x_r Z_r),$$

$$N = \sum_{r=1}^{n} (x_r Y_r - y_r X_r)$$

are the sums of the moments of the forces about OX, OY, OZ respectively.

The conditions are necessary. If the body is in equilibrium, then there is neither the motion of translation nor the motion of rotation. Hence for the equilibrium of the body it is necessary that

$$\mathbf{R} = \mathbf{0} \quad \text{and} \quad \mathbf{G} = \mathbf{0}.$$

Now $$\mathbf{R} = \mathbf{0} \Rightarrow X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k} = \mathbf{0} \Rightarrow X = 0, Y = 0, Z = 0,$$

and $$\mathbf{G} = \mathbf{0} \Rightarrow L\mathbf{i} + M\mathbf{j} + N\mathbf{k} = \mathbf{0} \Rightarrow L = 0, M = 0, N = 0.$$

Hence for a body acted upon by a number of forces at any points of it to be in equilibrium it is necessary that

$$X = 0 = Y = Z \text{ and } L = 0 = M = N.$$

The conditions are sufficient.

If $\quad X = 0 = Y = Z \quad \text{and} \quad L = 0 = M = N$

then $\quad \mathbf{R} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k} = \mathbf{0}, \quad \text{and} \quad \mathbf{G} = L\mathbf{i} + M\mathbf{j} + N\mathbf{k} = \mathbf{0}.$

Hence the body is in equilibrium.

Remark: In the discussion of 5.4, if

$$R = |\mathbf{R}| = |X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}|,$$

then $\quad R = \sqrt{(X^2 + Y^2 + Z^2)}.$

A unit vector along the force **R** is $(1/R)\,\mathbf{R}$ *i.e.*,

$$(X/R)\,\mathbf{i} + (Y/R)\,\mathbf{j} + (Z/R)\,\mathbf{k}.$$

So the direction cosines of the line of action of the single force **R**

are $\quad X/R, Y/R, Z/R.$

Again if $\quad G = |\mathbf{G}| = |L\mathbf{i} + M\mathbf{j} + N\mathbf{k}|$, then $G = \sqrt{(L^2 + M^2 + N^2)}.$

A unit vector along the axis of the couple **G** is $(1/G)\,\mathbf{G}$ *i.e.*,

$$(L/G)\,\mathbf{i} + (M/G)\,\mathbf{j} + (N/G)\,\mathbf{k}.$$

So the direction cosines of the axis of the couple **G** are $L/G, M/G, N/G$.

5.5 Wrench

Definition: A combination of a single force **R** and a couple **G** is said to constitute a *wrench* if the lines of action of the single force **R** and the axis of the couple **G** are the same *i.e.*, if the vectors **R** and **G** are parallel. **(Kumaun 2002)**

In the case of a wrench $(\mathbf{R}, \mathbf{G})$ the common line of action of the single force **R** and the axis of the couple **G** is said to be the *axis of the wrench.* If $R = |\mathbf{R}|$, then R is called the *intensity* of the wrench. Also if $\mathbf{G} = p\mathbf{R}$, then p is called the *pitch of* the wrench.

In 5.3, we have shown that a system of forces acting at different points of a rigid body can be reduced to a single force **R** acting at an arbitrarily chosen point O and a couple **G**. With respect to an arbitrarily chosen point O the vectors **R** and **G** need not be parallel. However, there exist points with respect to which as base points the system reduces to a wrench $(\mathbf{R}, \mathbf{G})$ *i.e.*, the vectors **R** and **G** are parallel. The locus of all such points is a straight line called the central axis (or Poinsot's central axis) of the system of forces.

Theorem: *Any system of forces acting on a rigid body can be reduced to a single force together with a couple whose axis is along the direction of the single force.*

(Gorakhpur 2006; Kanpur 11)

Or

Every system of forces acting on a rigid body is generally equivalent to a wrench.

Proof: In 5.3, we have proved that a system of forces acting on a rigid body can be reduced to a single force R acting at any arbitrarily chosen point O and a single couple of moment G about an axis through the same point O.

Let the single force R act along the line OA and the couple of moment G about the line OD *i.e.*, OD be the axis of the couple, [as shown in fig. (i)], and let $\angle AOD = \theta$.

Draw a line OB perpendicular to OA such that OA, OB and OD lie in one plane. Draw another line OC perpendicular to this plane containing the lines OA, OB and OD.

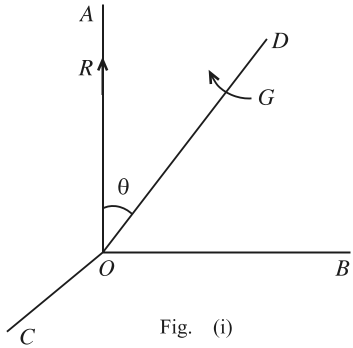


Fig. (i)

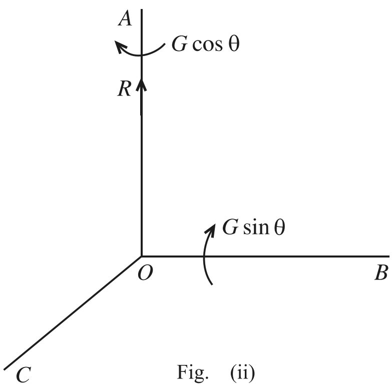


Fig. (ii)

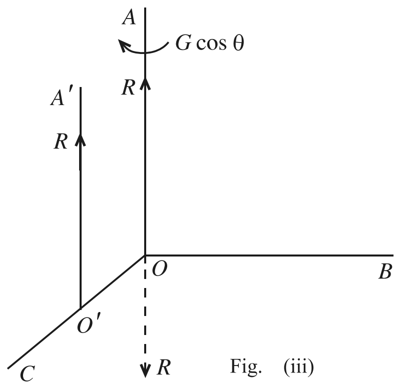


Fig. (iii)

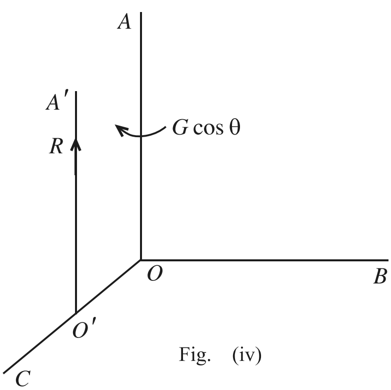


Fig. (iv)

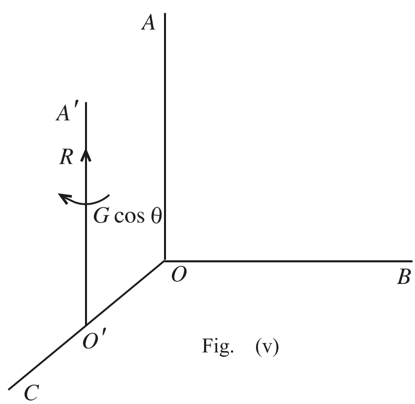


Fig. (v)

The couple of moment G acting about OD is equivalent to a couple of moment $G \cos\theta$ about OA and a couple of moment $G \sin\theta$ about OB [as shown in fig. (ii)].

The line OB is perpendicular to the plane AOC. Therefore the couple of moment $G \sin\theta$ about OB acts in this plane AOC and hence it can be replaced by two equal and unlike parallel forces in the plane AOC whose moment is equal to $G \sin \theta$. Let one of these forces be R at O in the direction opposite to OA, then the other force must be equal to R acting parallel to OA at some point say O' in OC [see fig. (iii)], such that

$$R.OO' = G \sin \theta \ i.e., OO' = \frac{G \sin \theta}{R}.$$

Now two equal forces of magnitude R acting at O in opposite directions balance each other and thus we are left with a force R at O' acting along $O'A'$ and a couple of moment $G \cos \theta$ about the parallel line OA. [see fig. (iv)].

Since the axis of a couple can be transferred to any parallel axis, therefore we transfer the axis of the couple $G \cos \theta$ from OA to $O'A'$. [See fig. (v)].

Hence the system of forces acting on the rigid body is equivalent to a single force R acting along a line $O'A'$ and a single couple of moment $G \cos \theta$ about the same line $O'A'$.

5.6 Central Axis

(Meerut 2006, 07, 08, 10 ; Gorakhpur 07, 08, 10, 11; Lucknow 06, 08, 10; Rohilkhand 09)

Definition: *The line along which the single resultant force of magnitude R acts and which is also the axis of the single couple of moment G* $\cos \theta$ *to which the system of forces acting on a body is reduced, is called the* **Poinsot's Central axis** *of the system of forces.*

5.7 Characteristics of a Central Axis

1. *Central axis for a system of forces acting on a rigid body is unique.*
2. *The moment of the resultant couple with a point on the central axis as the base point is less than the moment of the resultant couple corresponding to any arbitrary point O which is not on the central axis.*

In 5.5 we have shown that if G is the moment of the resultant couple corresponding to any arbitrary point O, then that corresponding to a point on the central axis is $G \cos \theta$. Since the greatest value of $\cos \theta$ is 1 $i.e., \cos \theta \le 1$, hence $G \cos \theta$ is always less than or equal to G $i.e., G \cos \theta \le G$.

5.8 Wrench and Screw

Wrench: Suppose a system of forces acting at different points of a rigid body reduces to a single force R acting at an arbitrarily chosen point O and a couple of moment G. Let θ be the angle between the line of action of the force R and the axis of the couple G. If R is the magnitude of the single force and $K = G \cos \theta$ be the magnitude of the moment of the couple about the central axis, *then R and K taken together are said to constitute the* **wrench** *of the system.* The magnitude of the single resultant force R is known as the **intensity** of the wrench.

(Lucknow 2006, 10; Rohilkhand 09; Gorakhpur 08, 10)

Pitch of the wrench. *The ratio $p = K/R$ i.e. the ratio of the magnitude of the moment of the resultant couple about the central axis and the resultant force is called the pitch.*

(Kumaun 2002; Rohilkhand 10, 11; Gorakhpur 08)

(i) If $p = 0$ *i.e.* $K = 0$, then the given system of forces acting on a rigid body reduces to a single force of magnitude R only; and

(ii) if $p = \infty$ *i.e.* $R = 0$, then the given system of forces acting on a rigid body reduces to a single couple of moment K only.

Screw: *The straight line along which the single force of magnitude R acts, when considered together with the pitch p is called a* **screw.** Thus the screw is a definite straight line associated with a definite pitch. **(Kamaun 2000, 02; Rohilkhand 10, 11)**

5.9 Invariants

1. Whatever origin and the axes are chosen, the quantities

$$X^2 + Y^2 + Z^2 \text{ and } LX + MY + NZ$$

are invariants for any given system of forces acting on a rigid body.

Let O be the origin and OX, OY, OZ be the rectangular coordinate axes; **i, j, k** being the unit vectors along these coordinate axes. Let a system of forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ acting at different points of a rigid body reduce to a single force **R** acting at O and a couple **G** where

$$\mathbf{R} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k} \quad \text{and} \quad \mathbf{G} = L\mathbf{i} + M\mathbf{j} + N\mathbf{k}.$$

Again let $\mathbf{i}', \mathbf{j}', \mathbf{k}'$ be the unit vectors along another system of rectangular coordinate axes $O'X', O'Y', O'Z'$ with O' as the origin. Suppose the same system of forces $\mathbf{F}_1, \ldots, \mathbf{F}_n$ reduces to a single force $\mathbf{R}'$ acting at O' and a couple $\mathbf{G}'$ where

$$\mathbf{R}' = X'\mathbf{i}' + Y'\mathbf{j}' + Z'\mathbf{k}' \quad \text{and} \quad \mathbf{G}' = L'\mathbf{i}' + M'\mathbf{j}' + N'\mathbf{k}'.$$

Then to prove that

$$X^2 + Y^2 + Z^2 \text{ and } LX + MY + NZ \text{ are invariants } i.e.,$$

$$X'^2 + Y'^2 + Z'^2 = X^2 + Y^2 + Z^2,$$

and
$$L'X' + M'Y' + N'Z' = LX + MY + NZ.$$

As shown in 5.3, we have

$$\mathbf{R} = \Sigma\, \mathbf{F}_1 = \mathbf{R}', \text{ and } \mathbf{G}' = \mathbf{G} - \mathbf{c} \times \mathbf{R}, \text{ where } \overrightarrow{OO'} = \mathbf{c}.$$

Now $\mathbf{R} = \mathbf{R}' \Rightarrow |\mathbf{R}| = |\mathbf{R}'|$

$\Rightarrow$ $\quad |X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}| = |X'\mathbf{i}' + Y'\mathbf{j}' + Z'\mathbf{k}'|$

$\Rightarrow$ $\quad \sqrt{(X^2 + Y^2 + Z^2)} = \sqrt{(X'^2 + Y'^2 + Z'^2)}$

$\Rightarrow$ $\quad X^2 + Y^2 + Z^2 = X'^2 + Y'^2 + Z'^2.$

Hence $X^2 + Y^2 + Z^2$ is invariant, or $R = |\mathbf{R}|$ is invariant.

Again $L'X' + M'Y' + N'Z'$

$$= (L'\mathbf{i}' + M'\mathbf{j}' + N'\mathbf{k}') \cdot (X'\mathbf{i}' + Y'\mathbf{j}' + Z'\mathbf{k}')$$

$= \mathbf{G}' \cdot \mathbf{R}' = \mathbf{G}' \cdot \mathbf{R}$ [$\because \mathbf{R}' = \mathbf{R}$]

$= (\mathbf{G} - \mathbf{c} \times \mathbf{R}) \cdot \mathbf{R}$ [$\because \mathbf{G}' = \mathbf{G} - \mathbf{c} \times \mathbf{R}$]

$= \mathbf{G} \cdot \mathbf{R} - (\mathbf{c} \times \mathbf{R}) \cdot \mathbf{R}$

$= \mathbf{G} \cdot \mathbf{R}$ [$\because$ the scalar triple product $(\mathbf{c} \times \mathbf{R}) \cdot \mathbf{R}$ vanishes, the two vectors being equal]

$= (L\,\mathbf{i} + M\,\mathbf{j} + N\,\mathbf{k}) \cdot (X\,\mathbf{i} + Y\,\mathbf{j} + Z\,\mathbf{k}) = LX + MY + NZ.$

Hence $LX + MY + NZ$ *i.e.*, $\mathbf{G} \cdot \mathbf{R}$ is also invariant.

Remark: If $\mathbf{R} = X\,\mathbf{i} + Y\,\mathbf{j} + Z\,\mathbf{k}$, then it is convenient to write $\mathbf{R} = (X, Y, Z)$. Also we have $|\mathbf{R}| = \sqrt{(X^2 + Y^2 + Z^2)}$.

2. *Determination of the pitch and intensity of the wrench with the help of invariants.*

Suppose a system of forces reduces to a single force $\mathbf{R} = (X, Y, Z)$ acting at an arbitrarily chosen point O and a couple $\mathbf{G} = (L, M, N)$. If this system reduces to a wrench $(\mathbf{R}', \mathbf{G}')$, then $\mathbf{R}' = \mathbf{R}$ and $\mathbf{G}'$ is parallel to $\mathbf{R}'$.

The intensity of the wrench

$$= |\mathbf{R}'| = |\mathbf{R}| = \sqrt{(X^2 + Y^2 + Z^2)} = R, \text{ say.}$$

Also $\mathbf{G}' = p\,\mathbf{R}'$, because $\mathbf{G}'$ and $\mathbf{R}'$ are parallel.

The scalar quantity p is called the pitch of the wrench $(\mathbf{R}', \mathbf{G}')$.

Now $\mathbf{G}' = p\,\mathbf{R}' \Rightarrow \mathbf{G}' \cdot \mathbf{R}' = p\,\mathbf{R}' \cdot \mathbf{R}'$

$\Rightarrow \quad \mathbf{G} \cdot \mathbf{R} = p\mathbf{R} \cdot \mathbf{R}$ [$\because \mathbf{G}' \cdot \mathbf{R}' = \mathbf{G} \cdot \mathbf{R}$ and $\mathbf{R} = \mathbf{R}'$]

$$\Rightarrow \quad p = \frac{\mathbf{G} \cdot \mathbf{R}}{\mathbf{R} \cdot \mathbf{R}} = \frac{LX + MY + NZ}{X^2 + Y^2 + Z^2}.$$

Again if $G' = |\mathbf{G}'|$, then from $\mathbf{G}' = p\,\mathbf{R}'$, we have

$$G' = |\mathbf{G}'| = |p\,\mathbf{R}'| = p|\mathbf{R}'| = p|\mathbf{R}|$$
$$= p\sqrt{(X^2 + Y^2 + Z^2)} = pR.$$

Thus for the wrench $(\mathbf{R}', \mathbf{G}')$, we have

$G' = |\mathbf{G}'| = pR$, where p is the pitch of the wrench

and $\quad R = \sqrt{(X^2 + Y^2 + Z^2)}$.

Also
$$G' = pR = \frac{LX + MY + NZ}{X^2 + Y^2 + Z^2}\sqrt{(X^2 + Y^2 + Z^2)}$$
$$= \frac{LX + MY + NZ}{\sqrt{(X^2 + Y^2 + Z^2)}},$$

so that $\quad G'R = LX + MY + NZ.$

5.10 Conditions for a Single Resultant Force

Suppose that a system of forces acting at different points of a rigid body reduces to a single force $\mathbf{R} = (X, Y, Z)$ *acting at an arbitrarily chosen point O and a couple* $\mathbf{G} = (L, M, N)$. *Then the necessary and sufficient conditions for this system of forces to reduce to a single resultant force are*

$$LX + MY + NZ = 0 \quad \textit{and} \quad X^2 + Y^2 + Z^2 \neq 0.$$

The conditions are necessary. Suppose at some point O' as the base point the system $(\mathbf{R}, \mathbf{G})$ reduces to a single force $\mathbf{R}'$ and a couple $\mathbf{G}'$. Then $\mathbf{R}' = \mathbf{R}$.

If at O' the system reduces to a single resultant force, then we must have

$$\mathbf{R}' \neq \mathbf{0} \quad \text{and} \quad \mathbf{G}' = \mathbf{0}$$

$\Rightarrow$ $\qquad \mathbf{R} \neq \mathbf{0} \quad \text{and} \quad \mathbf{R}' \cdot \mathbf{G}' = 0$

$\Rightarrow$ $\qquad \mathbf{R} \neq \mathbf{0} \quad \text{and} \quad \mathbf{R} \cdot \mathbf{G} = 0 \qquad [\because \mathbf{R} \cdot \mathbf{G} \text{ is invariant}]$

$\Rightarrow$ $\qquad X^2 + Y^2 + Z^2 \neq 0 \text{ and } LX + MY + NZ = 0.$

$$[\because \mathbf{R} \cdot \mathbf{G} = LX + MY + NZ\,]$$

Hence the conditions are necessary.

The conditions are sufficient. Suppose

$$X^2 + Y^2 + Z^2 \neq 0 \text{ and } LX + MY + NZ = 0.$$

Then we are to prove that the system must compound to a single resultant force.

Take a point O' on the central axis of the system. Let the system reduce to $(\mathbf{R}', \mathbf{G}')$ at O'. Then $\mathbf{G}'$ is parallel to $\mathbf{R}'$ because $(\mathbf{R}', \mathbf{G}')$ is a wrench.

Now $\qquad LX + MY + NZ = 0 \Rightarrow \mathbf{G} \cdot \mathbf{R} = 0$

$\Rightarrow$ $\qquad \mathbf{G}' \cdot \mathbf{R}' = 0. \qquad \ldots(1)$

But $\mathbf{R}' = \mathbf{R} \neq \mathbf{0}$, because $X^2 + Y^2 + Z^2 \neq 0$. Also $\mathbf{G}'$ is parallel to $\mathbf{R}'$. Therefore the relation (1) holds good only if $\mathbf{G}' = \mathbf{0}$. Consequently at O' the system reduces to a single force only. Hence the conditions are sufficient.

5.11 Equations of the Central Axis

To determine the equations of the central axis of a system of forces acting on a rigid body .

(Lucknow 2006, 08, 09, 10; Meerut 09, 11; Gorakhpur 07, 08, 09; Purvanchal 09; Rohilkhand 07, 11; Avadh 09, 11; Kumaun 01)

Central axis. Definition: *The central axis of a system of forces acting on a rigid body is the straight line which is the locus of the points referred to which as base points the system of forces reduces to a wrench i.e.,reduces to a single force and a single couple such that the line of action of the single force and the axis of the couple are the same.*

Take some conveniently chosen point O as the origin and some mutually perpendicular lines OX, OY, OZ as the coordinate axes. Suppose with respect to O as the base point the system reduces to a single force $\mathbf{R} = (X, Y, Z)$ *i.e.*, $X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}$ and a couple $\mathbf{G} = (L, M, N)$.

Let $P(x, y, z)$ be any point on the central axis of the system. If $\vec{OP} = \mathbf{r}$, then

$$\mathbf{r} = (x, y, z) \text{ i.e., } \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

Since P is on the central axis of the system, therefore the system must reduce to a wrench, say $(\mathbf{R}', \mathbf{G}')$, at P. We have

$$\mathbf{R}' = \mathbf{R}, \text{ and } \mathbf{G}' = \mathbf{G} - \mathbf{r} \times \mathbf{R}. \qquad \text{[Refer 5.3 (ii)]}$$

Now $(\mathbf{R}', \mathbf{G}')$ is a wrench

$\Rightarrow$ the vectors $\mathbf{G}'$ and $\mathbf{R}'$ are collinear

$\Rightarrow$ $\mathbf{G}' = p\,\mathbf{R}'$, where the scalar p is the pitch of the wrench

$\Rightarrow$ $\mathbf{G} - \mathbf{r} \times \mathbf{R} = p\mathbf{R}$

$\Rightarrow$ $(L\mathbf{i} + M\mathbf{j} + N\mathbf{k}) - (x\mathbf{i} + y\mathbf{j} + z\,\mathbf{k}) \times (X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k})$
$$= p\,(X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k})$$

$\Rightarrow$ $(L - yZ + zY)\,\mathbf{i} + (M - zX + xZ)\,\mathbf{j} + (N - xY + yX)\,\mathbf{k}$
$$= pX\mathbf{i} + pY\mathbf{j} + pZ\mathbf{k}.$$

Equating the coefficients of **i, j, k,** we get

$$L - yZ + zY = pX, M - zX + xZ = pY, N - xY + yX = pZ$$

or
$$\frac{L - yZ + zY}{X} = \frac{M - z\,X + xZ}{Y} = \frac{N - xY + yX}{Z} = p. \qquad ...(1)$$

The equations (1) give the central axis of the system. Since any equation of first degree in x, y, z represents a plane, therefore the equations (1) give us three planes. It can be shown that these three planes have a common line of intersection. So the line of intersection of any two of these three planes gives the central axis of the system.

Remark 1: If the pitch $p = 0$ *i.e.*, if $LX + MY + NZ = 0$ *i.e.*, if the system reduces to a single force then the above equations of the central axis can be written as

$$L - yZ + zY = 0, \; M - zX + xZ = 0, \quad \text{and} \quad N - xY + yX = 0.$$

The line of intersection of any two of these three planes, is the central axis of the system.

Remark 2: Suppose a system of forces $\mathbf{R} = (X, Y, Z)$ and $\mathbf{G} = (L, M, N)$ reduces to a wrench consisting of a single force of magnitude R and a couple of moment K. To find the axis of the wrench we write the equations of the central axis of the system. The values of R and K are given by the formulae :

$$R = \sqrt{(X^2 + Y^2 + Z^2)}$$

and
$$K = pR = \frac{LX + MY + NZ}{X^2 + Y^2 + Z^2} \sqrt{(X^2 + Y^2 + Z^2)}$$
$$= \frac{LX + MY + NZ}{R}$$

i.e.,
$$KR = LX + MY + NZ.$$

5.12 Computation of $X, Y, Z; L, M, N$

Suppose a system of forces of magnitudes $F_1, \ldots, F_n$ act at different points of a rigid body. Take some conveniently chosen system of three mutually perpendicular straight lines OX, OY, OZ as the coordinate axes.

Let (x_1, y_1, z_1) be the coordinates of a point on the line of action of the force F_1 and l_1, m_1, n_1 be the direction cosines of this line. Then the equations of this line are

$$\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1}. \qquad \ldots(1)$$

Similarly note the equations of the lines of action of the other forces also. Observe that we need the direction cosines of the line of action of a force and the coordinates of any convenient point on this line of action.

If X_1, Y_1, Z_1 are the components of the force F_1 along the coordinate axes, then $X_1 = F_1 l_1, Y_1 = F_1 m_1, Z_1 = F_1 n_1$. Thus to get X_1, Y_1, Z_1 we multiply F_1 by the direction cosines of the line of action of this force.

Now we make the following scheme :

Coordinates of the point of application of the force	x_1	y_1	z_1	x_2	y_2	z_2 etc.,
Components of the force	X_1	Y_1	Z_1	X_2	Y_2	Z_2

We have $\quad X = \Sigma X_1 = X_1 + X_2 + \ldots, Y = \Sigma Y_1 = Y_1 + Y_2 + \ldots,$

and $\quad Z = \Sigma Z_1 = Z_1 + Z_2 + \ldots.$

Also $\quad L = \Sigma (y_1 Z_1 - z_1 Y_1) = (y_1 Z_1 - z_1 Y_1) + (y_2 Z_2 - z_2 Y_2) + \ldots,$

$\quad M = \Sigma (z_1 X_1 - x_1 Z_1) = (z_1 X_1 - x_1 Z_1) + (z_2 X_2 - x_2 Z_2) + \ldots,$

and $\quad N = \Sigma (x_1 Y_1 - y_1 X_1) = (x_1 Y_1 - y_1 X_1) + (x_2 Y_2 - y_2 X_2) + \ldots.$

Note: In the equation (1), l_1, m_1, n_1 are the direction cosines of the line if $l_1^2 + m_1^2 + n_1^2 = 1$. If l_1, m_1, n_1 are not the direction cosines, but are direction ratios of the line, then the direction cosines will be given by

$$\frac{l_1}{\sqrt{(l_1^2 + m_1^2 + n_1^2)}}, \frac{m_1}{\sqrt{(l_1^2 + m_1^2 + n_1^2)}}, \frac{n_1}{\sqrt{(l_1^2 + m_1^2 + n_1^2)}}.$$

Illustrative Examples

Example 1: *Forces P, Q, R act along three straight lines.*

$$y = b, z = -c; z = c, x = -a \text{ and } x = a, y = -b \text{ respectively.}$$

Show that they will have a single resultant, if

$$\frac{a}{P} + \frac{b}{Q} + \frac{c}{R} = 0;$$

(Garhwal 2002; Lucknow 07, 10; Kanpur 07; Meerut 08)

and that the equations to its line of action are any two of the three

$$\frac{y}{Q}-\frac{z}{R}-\frac{a}{P}=0, \frac{z}{R}-\frac{x}{P}-\frac{b}{Q}=0, \frac{x}{P}-\frac{y}{Q}-\frac{c}{R}=0.$$

(Garhwal 2002)

Solution: The forces P, Q, R act along the lines

$y=b, z=-c\,; z=c, x=-a\,; x=a, y=-b$ respectively.

The equations of these lines can be written as

$$\frac{x-0}{1}=\frac{y-b}{0}=\frac{z+c}{0}; \frac{x+a}{0}=\frac{y-0}{1}=\frac{z-c}{0}; \frac{x-a}{0}=\frac{y+b}{0}=\frac{z-0}{1}.$$

Therefore the forces acting on the body are as follows :

(i) A force P acting at the point $(0, b, -c)$ along the line whose d.c.'s are 1, 0, 0,

(ii) A force Q acting at the point $(-a, 0, c)$ along the line whose d.c.'s are 0, 1, 0,

(iii) A force R acting at the point $(a, -b, 0)$ along the line whose d.c.'s are 0, 0, 1.

The components $(X_1, Y_1, Z_1), (X_2, Y_2, Z_2)$ etc., of these forces parallel to the axes are

$$X_1=P.1=P, \quad X_2=Q.0=0, \quad X_3=R.0=0$$
$$Y_1=P.0=0, \quad Y_2=Q.1=Q, \quad Y_3=R.0=0$$
$$Z_1=P.0=0, \quad Z_2=Q.0=0, \quad Z_3=R.1=R.$$

If the system reduces to a single force $\mathbf{R}=(X, Y, Z)$ acting at O and a couple $\mathbf{G}=(L, M, N)$, then

$$X=\Sigma X_1=X_1+X_2+X_3=P+0+0=P, \; Y=\Sigma Y_1=0+Q+0=Q,$$

and $Z=\Sigma Z_1=0+0+R=R.$

To calculate L, M, N we make the following scheme :

Coordinates of the point of application of the force	x_1	y_1	z_1	x_2	y_2	z_2	x_3	y_3	z_3
	0	b	$-c$	$-a$	0	c	a	$-b$	0
Components of the force	P	0	0	0	Q	0	0	0	R
	X_1	Y_1	Z_1	X_2	Y_2	Z_2	X_3	Y_3	Z_3

Now $L=\Sigma\,(y_1Z_1-z_1Y_1)=(0-0)+(0-cQ)+(-bR-0)=-cQ-bR,$

$M=\Sigma\,(z_1X_1-x_1Z_1)=(-cP-0)+(0-0)+(0-aR)=-cP-aR$

and $N=\Sigma\,(x_1Y_1-y_1X_1)=(0-bP)+(-aQ-0)+(0-0)=-bP-aQ.$

The system of forces will reduce to a single force at some point

if $LX+MY+NZ=0$

or $(-cQ-bR)\,P+(-cP-aR)\,Q+(-bP-aQ)\,R=0$

or $-2a\,QR-2b\,PR-2c\,PQ=0.$

Dividing by $-2PQR$, we have

$$\frac{a}{P}+\frac{b}{Q}+\frac{c}{R}=0, \quad ...(1)$$

i.e., the forces will reduce to a single force if (1) is satisfied.

The equations of the line of action of this single force *i.e.*, of the central axis are

$$\frac{L - yZ + zY}{X} = \frac{M - zX + xZ}{Y}$$

$$= \frac{N - xY + yX}{Z} = \frac{LX + MY + NZ}{X^2 + Y^2 + Z^2} = 0$$

(As the system reduces to a single force)

$\therefore$ the equations of the line of action of the single resultant force are any two of the following three :

$$L - yZ + zY = 0, M - z X + x Z = 0, N - xY + yX = 0$$

or
$$-cQ - bR - yR + zQ = 0, -cP - aR - z P + xR = 0,$$
$$-bP - aQ - xQ + yP = 0.$$

Dividing these equations by QR, PR and PQ respectively, we get

$$-\frac{c}{R} - \frac{b}{Q} - \frac{y}{Q} + \frac{z}{R} = 0,\ -\frac{c}{R} - \frac{a}{P} - \frac{z}{R} + \frac{x}{P} = 0, -\frac{b}{Q} - \frac{a}{P} - \frac{x}{P} + \frac{y}{Q} = 0$$

or
$$\frac{y}{Q} - \frac{z}{R} + \frac{b}{Q} + \frac{c}{R} = 0,\ \frac{z}{R} - \frac{x}{P} + \frac{a}{P} + \frac{c}{R} = 0,\ \frac{x}{P} - \frac{y}{Q} + \frac{a}{P} + \frac{b}{Q} = 0.$$

Subtracting the equation (1) from each of these equations, we get

$$\frac{y}{Q} - \frac{z}{R} - \frac{a}{P} = 0, \frac{z}{R} - \frac{x}{P} - \frac{b}{Q} = 0, \frac{x}{P} - \frac{y}{Q} - \frac{c}{R} = 0.$$

The equations of the line of action of the single force are any two of the above three equations.

Remark: If in the language of the question letters X, Y, Z are given in place of P, Q, R, then in the solution we can take the single force $\mathbf{R} = (X', Y', Z')$ in place of $\mathbf{R} = (X, Y, Z)$.

Example 2: *A force P acts along the axis of x and another force nP along a generator of the cylinder $x^2 + y^2 = a^2$. Show that the central axis lies on the cylinder*

$$n^2 (nx - z)^2 + (1 + n^2)^2 y^2 = n^4 a^2.$$

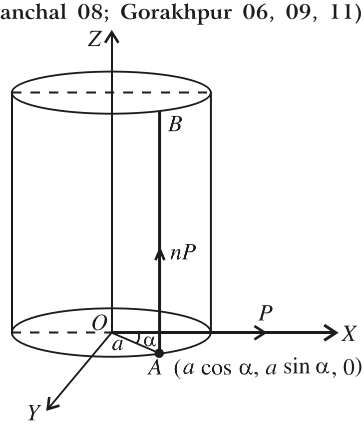
(Meerut 2004, 06, 07, 10; Avadh 07; Lucknow 09; Purvanchal 08; Gorakhpur 06, 09, 11)

Solution: Here one force P acts along the axis of x, whose equations are

$$\frac{x - 0}{1} = \frac{y - 0}{0} = \frac{z - 0}{0},$$

and the other force nP acts along a generator of the cylinder

$$x^2 + y^2 = a^2.$$



Clearly the generators of the cylinder are parallel to the z-axis. Let the force nP act along the generator

AB passing through the point $A\,(a\cos\alpha, a\sin\alpha, 0)$, where α is a parameter.

The equations of the generator AB are

$$\frac{x - a\cos\alpha}{0} = \frac{y - a\sin\alpha}{0} = \frac{z - 0}{1}$$

[$\because$ d.c.'s of AB are 0, 0, 1 as it is parallel to OZ].

Therefore the forces acting on the body are as follows :

(i) The force P acting at the point (0, 0, 0) along x-axis whose d.c.'s are 1, 0, 0; and

(ii) The force nP acting at the point $(a\cos\alpha, a\sin\alpha, 0)$ along the generator AB whose d.c.'s are 0, 0, 1.

The components (X_1, Y_1, Z_1) etc., of these forces parallel to the axes are :

$$X_1 = P.1 = P, \qquad X_2 = nP.0 = 0$$
$$Y_1 = P.0 = 0, \qquad Y_2 = nP.0 = 0$$
$$Z_1 = P.0 = 0, \qquad Z_2 = nP.1 = nP.$$

If the system reduces to a single force $\mathbf{R} = (X, Y, Z)$ acting at O and a couple $\mathbf{G} = (L, M, N)$, then

$$X = X_1 + X_2 = P, Y = Y_1 + Y_2 = 0, Z = Z_1 + Z_2 = nP,$$

$$L = \sum_{r=1}^{2} (y_r Z_r - z_r Y_r) = (y_1 Z_1 - z_1 Y_1) + (y_2 Z_2 - z_2 Y_2)$$

$$= (0 - 0) + (a\sin\alpha.\,nP - 0) = a\,nP\sin\alpha,$$

$$M = \sum_{r=1}^{2} (z_r X_r - x_r Z_r) = (z_1 X_1 - x_1 Z_1) + (z_2 X_2 - x_2 Z_2)$$

$$= (0 - 0) + (0 - a\cos\alpha\,.\,nP) = -\,a\,nP\cos\alpha,$$

and $$N = \sum_{r=1}^{2} (x_r Y_r - y_r X_r) = (x_1 Y_1 - y_1 X_1) + (x_2 Y_2 - y_2 X_2)$$

$$= (0 - 0) + (a\cos\alpha\,.0 - a\sin\alpha\,.0) = 0.$$

$\therefore$ The equations of the central axis are

$$\frac{L - yZ + zY}{X} = \frac{M - zX + xZ}{Y} = \frac{N - xY + yX}{Z}$$

or $$\frac{anP\sin\alpha - y.nP + z.0}{P} = \frac{-\,anP\cos\alpha - z.P + x.nP}{0} = \frac{0 - x.0 + y.P}{nP}$$

or $$\frac{an\sin\alpha - yn}{1} = \frac{-an\cos\alpha + (nx - z)}{0} = \frac{y}{n}.$$

(i) (ii) (iii)

The required surface is obtained by eliminating α from these equations. For this we proceed as follows :

From (i) and (ii), we get

$$-\,a\,n\cos\alpha + (n\,x - z) = 0, \quad \text{or} \quad a\,n\cos\alpha = (n\,x - z) \qquad ...(1)$$

From (i) and (iii), we get

$$an \sin\alpha - yn = y/n, \quad \text{or} \quad an \sin\alpha = (y/n)(n^2+1). \qquad ...(2)$$

Squaring and adding (1) and (2), we have

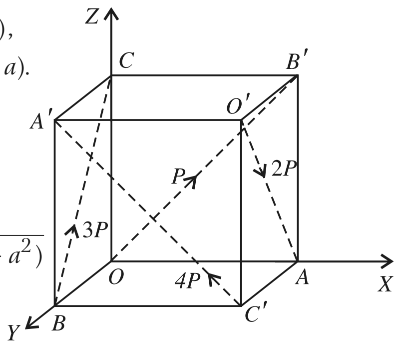
$$a^2 n^2 (\cos^2\alpha + \sin^2\alpha) = (nx - z)^2 + (y^2/n^2)(n^2+1)^2$$

or $$n^2(nx-z)^2 + (1+n^2)^2 y^2 = n^4 a^2,$$ which is the required surface.

Example 3: *OA, OB, OC are the edges of a cube of side a and OO′, AA′, BB′, CC′ are its diagonals; along OB′, O′A, BC and C′A′ act forces equal to P, 2P, 3P and 4P respectively; show that they are equivalent to a force $P\sqrt{(35)}$ at O along a line whose direction cosines are proportional to −3, −5, 6 together with a couple $\frac{1}{2} Pa\sqrt{(114)}$ about a line whose direction cosines are proportional to 7, −2, 2.*

Solution: Let the edges OA, OB, OC of a cube of side a be taken as the three coordinate axes as shown in the figure. The co-ordinates of the vertices of the cube are as follows :

$O(0,0,0)$, $A(a,0,0)$, $B(0,a,0)$, $C(0,0,a)$,
$A'(0,a,a)$, $B'(a,0,a)$, $C'(a,a,0)$, $O'(a,a,a)$.



Direction ratios of the line OB' are

$$a-0, 0-0, a-0, \quad i.e., \quad a, 0, a.$$

$\therefore$ its direction cosines are

$$\frac{a}{\sqrt{(a^2+0+a^2)}}, \frac{0}{\sqrt{(a^2+0+a^2)}}, \frac{a}{\sqrt{(a^2+0+a^2)}}$$

i.e., $$\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}.$$

Similarly the direction cosines of $O'A$, BC and $C'A'$ are

$$0, -1/\sqrt{2}, -1/\sqrt{2}; 0, -1/\sqrt{2}, 1/\sqrt{2} \text{ and } -1/\sqrt{2}, 0, 1/\sqrt{2} \text{ respectively.}$$

Therefore the forces acting on the body are as follows :

(i) The force P acting at the point $O(0,0,0)$ along OB' whose d.c.'s are $1/\sqrt{2}, 0, 1/\sqrt{2}$;

(ii) the force $2P$ acting at the point $O'(a,a,a)$ along $O'A$ whose d.c.'s are $0, -1/\sqrt{2}, -1/\sqrt{2}$;

(iii) the force $3P$ acting at the point $B(0,a,0)$ along BC whose d.c.'s are $0, -1/\sqrt{2}, 1/\sqrt{2}$; and

(iv) the force $4P$ acting at the point $C'(a,a,0)$ along $C'A'$ whose d.c.'s are $-1/\sqrt{2}, 0, 1/\sqrt{2}$.

The components (X_1, Y_1, Z_1) etc., of these forces parallel to the axes are

$$X_1 = P/\sqrt{2}, \quad X_2 = 0, \quad X_3 = 0, \quad X_4 = -4P/\sqrt{2}$$
$$Y_1 = 0, \quad Y_2 = -2P/\sqrt{2}, \quad Y_3 = -3P/\sqrt{2}, \quad Y_4 = 0$$
$$Z_1 = P/\sqrt{2}, \quad Z_2 = -2P/\sqrt{2}, \quad Z_3 = 3P/\sqrt{2}, \quad Z_4 = 4P/\sqrt{2}.$$

If the system of these four forces reduces to a single force $\mathbf{R} = (X, Y, Z)$ acting at O and a couple $\mathbf{G} = (L, M, N)$, then

$$X = \sum_{r=1}^{4} X_r = X_1 + X_2 + X_3 + X_4 = -\frac{3P}{\sqrt{2}}, Y = \sum_{r=1}^{4} Y_r = -\frac{5P}{\sqrt{2}},$$

$$Z = \sum_{r=1}^{4} Z_r = \frac{6P}{\sqrt{2}},$$

$$L = \sum_{r=1}^{4} (y_r Z_r - z_r Y_r)$$

$$= (y_1 Z_1 - z_1 Y_1) + (y_2 Z_2 - z_2 Y_2) + (y_3 Z_3 - z_3 Y_3) + (y_4 Z_4 - z_4 Y_4)$$

$$= (0 - 0) + \left(-\frac{2Pa}{\sqrt 2} + \frac{2Pa}{\sqrt 2}\right) + \left(\frac{3Pa}{\sqrt 2} - 0\right) + \left(\frac{4Pa}{\sqrt 2} - 0\right) = \frac{7Pa}{\sqrt 2},$$

$$M = \sum_{r=1}^{4} (z_r X_r - x_r Z_r)$$

$$= (z_1 X_1 - x_1 Z_1) + (z_2 X_2 - x_2 Z_2) + (z_3 X_3 - x_3 Z_3) + (z_4 X_4 - x_4 Z_4)$$

$$= (0 - 0) + \left(0 + \frac{2Pa}{\sqrt 2}\right) + (0 - 0) + \left(0 - \frac{4Pa}{\sqrt 2}\right) = -\frac{2Pa}{\sqrt 2},$$

and $$N = \sum_{r=1}^{4} (x_r Y_r - y_r X_r)$$

$$= (x_1 Y_1 - y_1 X_1) + (x_2 Y_2 - y_2 X_2) + (x_3 Y_3 - y_3 X_3) + (x_4 Y_4 - y_4 X_4)$$

$$= (0 - 0) + \left(-\frac{2Pa}{\sqrt 2} - 0\right) + (0 - 0) + \left(0 + \frac{4Pa}{\sqrt 2}\right) = \frac{2Pa}{\sqrt 2}.$$

The magnitude of the single force acting at O is

$$R = \sqrt{(X^2 + Y^2 + Z^2)} = \sqrt{\left\{\left(\frac{9}{2} + \frac{25}{2} + \frac{36}{2}\right) P^2\right\}} = P\sqrt{(35)}.$$

Direction cosines of the line of action of this single force R are in the ratio $X : Y : Z$ *i.e.*, $-3P/\sqrt 2 : -5P/\sqrt 2 : 6P/\sqrt 2$ *i.e.*, they are proportional to $-3, -5, 6$.

Also magnitude of the resultant couple is

$$G = \sqrt{(L^2 + M^2 + N^2)} = \sqrt{\left\{\left(\frac{49}{2} + \frac{4}{2} + \frac{4}{2}\right) P^2 a^2\right\}} = \frac{Pa}{2}\sqrt{(114)}$$

and the direction cosines of the line about which this couple of moment G acts are in the ratio

$L : M : N$ *i.e.*, $7Pa/\sqrt 2 : -2Pa/\sqrt 2 : 2Pa/\sqrt 2$ *i.e.*, $7 : -2 : 2$ *i.e.*, they are proportional to $7, -2, 2$.

Comprehensive Exercise 1

1. A single force is equivalent to component forces X, Y, Z about the axes of coordinates and to couples L, M, N about these axes. Prove that the magnitude of the single force is $\sqrt{(X^2 + Y^2 + Z^2)}$

   and the equations of its line of action are

$$\frac{yZ - zY}{L} = \frac{zX - xZ}{M} = \frac{xY - yX}{N} = 1.$$

(Kumaun 2001, 02; Avadh 08)

2. Equal forces act along the coordinate axes and the straight line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}.$$

Find the equations of the central axis of the system.

(Rohilkhand 2007; Purvanchal 09; Kanpur 09)

3. Three forces each equal to P act on a body, one at the point $(a, 0, 0)$ parallel to OY, the second at the point $(0, b, 0)$ parallel to OZ, and the third at the point $(0, 0, c)$ parallel to OX, the axes being rectangular. Find the resultant wrench in magnitude and position.

4. Two forces act, one along the line $y=0, z=0$ and the other along the line $x=0, z=c$. As the forces vary, show that the surface generated by the axis of their equivalent wrench is

$$(x^2+y^2)\,z=cy^2.$$

(Gorakhpur 2007, 10; Avadh 11; Meerut 11)

5. Two equal forces act one along each of the straight lines

$$\frac{x \pm a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{\pm b\cos\theta}=\frac{z}{c}.$$

Show that their central axis must, for all values of θ, lie on the surface

$$y\left(\frac{x}{z}+\frac{z}{x}\right)=b\left(\frac{a}{c}+\frac{c}{a}\right).$$

(Rohilkhand 2008; Purvanchal 11; Meerut 09)

6. Two forces P and Q act along the straight lines whose equations are $y=x\tan\alpha, z=c$ and $y=-x\tan\alpha, z=-c$ respectively. Show that their central axis lies on a straight line

$$\frac{y}{x}=\frac{P-Q}{P+Q}\tan\alpha,\quad \frac{z}{c}=\frac{P^2-Q^2}{P^2+2PQ\cos 2\alpha+Q^2}.$$

For all values of P and Q prove that this line is a generator of the surface

$$(x^2+y^2)\,z\sin 2\alpha=2cxy.$$

7. Forces X, Y, Z act along the three lines given by the equations

$$y=0, z=c;\; z=0, x=a;\; x=0, y=b.$$

Prove that the pitch of the equivalent wrench is

$$(aYZ+bZX+cXY)/(X^2+Y^2+Z^2).$$

(Avadh 2007)

If the wrench reduces to a single force, show that the line of action of the force lies on the surface $(x-a)(y-b)(z-c)=xyz$.

(Garhwal 2003; Kumaun 03; Purvanchal 07; Lucknow 08)

8. **(a)** Three forces act along the straight lines

$$x=0, y-z=a;\; y=0, z-x=a;\; z=0, x-y=a.$$

Show that they cannot reduce to a couple.

Prove also that if the system reduces to a single force, its line of action must lie on the surface $x^2+y^2+z^2-2yz-2zx-2xy=a^2$.

(b) Three forces act along the straight lines

$$x=0, y-z=5;\; y=0, z-x=5;\; z=0, x-y=5.$$

Show that they cannot reduce to a couple.

Prove also that if the system reduces to a single force its line of action must lie on the surface

$$x^2 + y^2 + z^2 - 2yz - 2zx - 2xy = 25.$$

9. A force parallel to z-axis acts at the point $(a, 0, 0)$ and an equal force perpendicular to z-axis acts at the point $(-a, 0, 0)$. Show that the central axis of the system lies on the surface $z^2(x^2 + y^2) = (x^2 + y^2 - ax)^2$. **(Garhwal 2003)**

10. A force F acts along the axis of z, and a force mF along a straight line intersecting the axis of x at a distance c from the origin and parallel to the plane of yz. Show that as this straight line turns round the axis of x, the central axis of the forces generates the surface

$$\{m^2 z^2 + (m^2 - 1) y^2\}(c - x)^2 = x^2 z^2.$$

11. Any number of wrenches of the same pitch act along the generators of the same system of the hyperboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

Show that they will reduce to a single resultant provided their central axis is parallel to a generator of the cone

$$\left(p + \frac{bc}{a}\right)x^2 + \left(p + \frac{ca}{b}\right)y^2 + \left(p - \frac{ab}{c}\right)z^2 = 0.$$

12. OA, OB, OC are three coterminous edges of a cube and AA', BB', CC', OO' are diagonals. Along BC', CA', AB' and OO' act forces equal to X, Y, Z, R respectively. Show that they are equivalent to a single resultant, if

$$(YZ + ZX + XY)\sqrt{3} + R(X + Y + Z) = 0,$$

or $$\sqrt{3}(\Sigma YZ) + R(\Sigma X) = 0.$$

13. Forces P, Q, R act along three non-intersecting edges of a cube, find the central axis. **(Garhwal 2004; Purvanchal 07; Meerut 11)**

14. OA, OB, OC are the edges of a cube of side $2a$ and OO', AA', BB', CC' are its diagonals. Forces $P, 2P, 3P$ and $4P$ act along $OB', O'A, BC$ and $C'A'$ respectively. Find the resultant of the whole system of forces and the direction ratios of its line of action. Also find the couple and the direction ratios of its axis.

15. Equal forces act along two perpendicular diagonals of opposite faces of a cube of side a. Show that they are equivalent to a single force R acting along a line through the centre of the cube and a couple $\frac{1}{2}aR$ with the same line for axis.

16. Six forces each equal to P, act along the edges of a cube taken in order, which do not meet a given diagonal. Show that their resultant is a couple of moment $2\sqrt{3}$. Pa, where a is the edge of the cube.

17. A rigid body is acted upon by three forces $2P \tan A, -P \tan B, 2P \tan C$ along the edges of a cube which do not meet, symmetrically chosen with respect to the axes of co-ordinates drawn parallel to them through the centre of the cube. Prove that the forces are equivalent to a single force acting along the line whose equations are

$$2a \cot B - x \cot A = 2y \cot B + a \cot A = -z \cot C,$$

where $2A, 2B, 2C$ are the angles of a triangle whose sides are in A.P. and $2a$ is the edge of the cube.

Answers 1

2. $$\frac{(\beta n - \gamma m) - y(1+n) + z(1+m)}{(1+l)} = \frac{(\gamma l - \alpha n) - z(1+l) + x(1+n)}{(1+m)}$$
$$= \frac{(\alpha m - \beta l) - x(1+m) + y(1+l)}{(1+n)}$$

3. Wrench (R, K), $R = P\sqrt{3}$, $K = \frac{1}{\sqrt{3}}(a+b+c)P$; central axis is a straight line through the point
$$\left(-\frac{1}{3}(a+2b+3c), -\frac{1}{3}(b+2c+3a), -\frac{1}{3}(c+2a+3b)\right)$$
and having 1, 1, 1 as its direction ratios

13. Referred to three coterminous edges as co-ordinate axes, the equations of central axis are
$$\frac{-aQ - yR + zQ}{P} = \frac{-aR - zP + xR}{Q} = \frac{aP - xQ + yP}{R}$$

14. $R = P\sqrt{(35)}$, d.r.'s $-3, -5, 6$; $G = Pa\sqrt{(114)}$, d.r.'s $7, -2, 2$

5.13 Constrained Bodies

A body, under the influence of some forces, is said to be constrained if it is made to move about one or two fixed points of itself.

For example a bar hinged at any point on it can turn about this point while a door or gate attached to the door post by two hinges can turn about the line joining the hinges.

If three or more non-collinear points of a body are fixed, then the body cannot move. So here, in the next two articles, we shall discuss only two cases of constrained bodies namely (i) equilibrium of a body when one of its points is fixed and (ii) equilibrium of a body when two of its points are fixed.

5.14 Conditions of Equilibrium of a Rigid Body with One Point Fixed

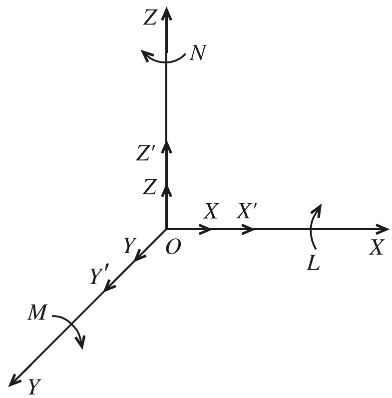
Let O be the fixed point of the body and OX, OY, OZ three mutually perpendicular coordinate axes through the point O. Suppose X', Y', Z' be the components of the force of constraint at O parallel to the axes. Let X, Y, Z be the sums of the components of the external forces parallel to the axes and L, M, N the sums of their moments about these axes.

Thus all the forces acting on the body are equivalent to :

the forces $X + X', Y + Y', Z + Z'$ parallel to the axes

and the couples of moments L, M, N about the axes.

If these forces keep the body in equilibrium, then the conditions of equilibrium to be satisfied are



$$X + X' = 0, Y + Y' = 0, Z + Z' = 0 \quad \text{...(1)}$$

and $$L = 0,\ M = 0, N = 0. \quad \text{...(2)}$$

The equations (1) give the force of constraint at the fixed point O, while the equations (2) give the conditions of equilibrium.

Thus the conditions of equilibrium of a rigid body constrained at one point are that the sums of the moments of the external forces about any three mutually perpendicular axes through the fixed point O must vanish separately.

If the external forces acting on the body are coplanar, then the above conditions of equilibrium of the body reduce to the condition that *"the sum of the moments of the external forces about the fixed point must vanish"*.

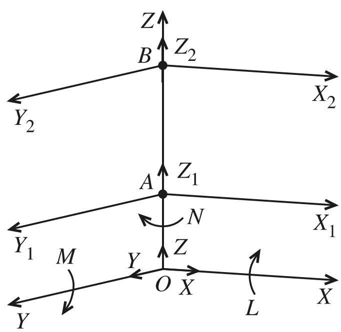
5.15 Conditions of Equilibrium of a Rigid Body with Two Fixed Points

(Rohilkhand 2008)

Let the two points A and B of a body be fixed. Taking this line AB as z-axis, take a point O on it as origin and two mutually perpendicular lines OX and OY, perpendicular to OZ, as the axis of x and axis of y respectively.

Let $OA = z_1$ and $OB = z_2$. Then the coordinates of the two fixed points A and B are $(0, 0, z_1)$ and $(0, 0, z_2)$ respectively.

Suppose X_1, Y_1, Z_1 and X_2, Y_2, Z_2 are the components parallel to the coordinate axes of the forces of constraint at A and B respectively. Let X, Y, Z be the sums of the components of the external forces parallel to the axes and L, M, N the sums of their moments about these axes.



Thus the whole system of forces acting on the body is equivalent to : the forces

$$X + X_1 + X_2, Y + Y_1 + Y_2, Z + Z_1 + Z_2$$

parallel to the axes and the couples of moments

$$L + \sum_{r=1}^{2} (y_r Z_r - z_r Y_r) = L - z_1 Y_1 - z_2 Y_2,$$

$$M + \sum_{r=1}^{2} (z_r X_r - x_r Z_r) = M + z_1 X_1 + z_2 X_2,$$

and $$N + \sum_{r=1}^{2} (x_r Y_r - y_r X_r) = N, \text{ about the axes.}$$

If the above mentioned forces keep the body in equilibrium, then the conditions of equilibrium to be satisfied are

$$X + X_1 + X_2 = 0 \quad ...(1)$$

$$Y + Y_1 + Y_2 = 0 \quad ...(2)$$

$$Z + Z_1 + Z_2 = 0 \quad ...(3)$$

$$L - z_1 Y_1 - z_2 Y_2 = 0 \quad ...(4)$$

$$M + z_1 X_1 + z_2 X_2 = 0 \quad ...(5)$$

and $$N = 0. \quad ...(6)$$

The values of X_1, X_2 are obtained by solving the equations (1) and (5) and the values of Y_1, Y_2 are obtained by solving the equations (2) and (4). The values of Z_1 and Z_2 cannot be determined as there is only one equation (3) connecting them. The last equation (6) gives the condition of equilibrium

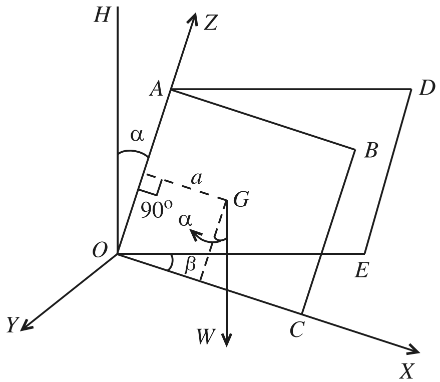
i.e., the condition of equilibrium of a rigid body constrained at two points is that the sum of the moments of the external forces about the fixed axis joining the fixed points must vanish.

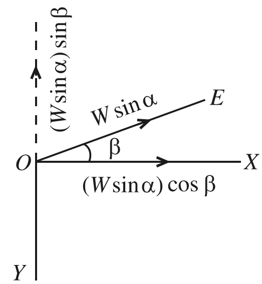
Illustrative Examples

Example 4: *A door of weight W, is free to turn about an axis OA which is inclined at an angle* α *to the vertical. Show that the couple necessary to keep it in a position in which it is inclined at an angle* β *to the vertical plane through OA is* $Wa \sin \alpha \sin \beta$, *where a is the distance of its centre of gravity from OA.* **(Rohilkhand 2007)**

Solution: Let a door $OABC$ of weight W be free to turn about an axis OA which is inclined at an angle α to the vertical OH. Let OE be a line perpendicular to OA such that OA, OE and OH lie in the same vertical plane. Let the door $OABC$ be inclined at an angle β to the vertical plane $OADE$ through OA and OH. Let the line OC be taken as the axis of x, a line OY perpendicular to the gate $OABC$ as axis of y and the fixed line OA as the axis of z. If $OA = 2h$ then the coordinates of the centre of gravity G of the door are $(a, 0, h)$.

The only external force acting on the gate is its weight W acting vertically downwards at its centre of gravity G.





The component of W along OA is $-W \cos \alpha$ and that perpendicular to OA *i.e.* along OE is $W \sin \alpha$. Again the components of $W \sin \alpha$ acting along OE are $(W \sin \alpha) \cos \beta$ along OX and $-W \sin \alpha \sin \beta$ along OY.

Thus the components of W along the coordinate axes are

$$X = W \sin \alpha \cos \beta,$$
$$Y = -W \sin \alpha \sin \beta,$$
$$Z = -W \cos \alpha \text{ respectively.}$$

The couple trying to rotate the gate in the x-y plane is that about OA *i.e.*, about z-axis and its moment N is given by

$$N = xY - yX = a(-W \sin \alpha \sin \beta) - 0$$
$$= -aW \sin \alpha \sin \beta.$$

Hence the couple necessary to keep the gate in this position is of magnitude $W a \sin \alpha \sin \beta$ and acts in the direction opposite to that of N.

Comprehensive Exercise 2

1. A rectangular gate is hung in the ordinary way on two hinges so that the line joining the hinges makes an angle α with the vertical. Show that the work which must be done to move it through an angle θ from its position of equilibrium is

   $Wa \sin \alpha \,.\, (1 - \cos \theta)$, where W is the weight and $2a$ the breadth of the gate.
2. A square table stands on four legs placed respectively at the middle points of its sides, find the greatest weight that can be put at one of the corners without upsetting the table.

Answers 2

2. Weight, equal to the weight of the table.

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. Suppose a system of forces acting at different points of a rigid body reduces to a single force $\mathbf{R} = (X, Y, Z)$ acting at an arbitrarily chosen point O and a couple $\mathbf{G} = (L, M, N)$ whose axis passes through O. The equations of the central axis of this system of forces are

(a) $\dfrac{L - yZ + zY}{X} = \dfrac{M - zX + xZ}{Y} = \dfrac{N - yX + xY}{Z}$

(b) $\dfrac{L - yZ + zY}{L} = \dfrac{M - zX + xZ}{M} = \dfrac{N - xY + yX}{N}$

(c) $\dfrac{L - yZ + zY}{X} = \dfrac{M - zX + xZ}{Y} = \dfrac{N - xY + yX}{Z}$

(d) $\dfrac{L - yZ + zY}{L} = \dfrac{M - zX + xZ}{M} = \dfrac{N - yX + xY}{N}$

2. Forces P, Q, R act along three straight lines

$$y = b,\ z = -c\ ;\ z = c,\ x = -a \ \text{ and } \ x = a,\ y = -b \text{ respectively.}$$

If these forces reduce to a single force $\mathbf{R} = (X, Y, Z)$ acting at O and a couple $\mathbf{G} = (L, M, N)$, then

(a) $X = Q,\ Y = R,\ Z = P$ (b) $X = P,\ Y = Q,\ Z = R$

(c) $X = R,\ Y = P,\ Z = Q$ (d) $X = Q,\ Y = P,\ Z = R.$

3. The straight line along which the single force acts, when considered together with the pitch, is called :

(a) Wrench (b) Axis

(c) Screw (d) Central axis.

(Garhwal 2002; Rohilkhand 11)

4. If at any point of a straight line a system of forces acting on a body can be reduced to a single force and a single couple in which the axis of the couple coincides with the direction of the force, then the line is called as follows :

(a) Wrench (b) Pitch

(c) Central axis (d) Screw. **(Garhwal 2003)**

5. The condition of equilibrium of a system of forces is :

(a) $\mathbf{G} = \mathbf{0}$ (b) $\mathbf{R} = \mathbf{0}, \mathbf{G} \neq \mathbf{0}$

(c) $\mathbf{R} = \mathbf{0}, \mathbf{G} = \mathbf{0}$ (d) $\mathbf{R} = \mathbf{0}$. **(Garhwal 2004)**

Fill In The Blank(s)

Fill in the blanks "……" so that the following statements are complete and correct.

1. Any given system of forces acting at any given points of a rigid body can be reduced to a single force acting through an arbitrarily chosen point and a couple whose …… passes through that point.

2. A wrench consists of a single force and a single couple such that the line of action of the single force and the axis of the couple are the …… .

3. The ...... of a system of forces acting on a rigid body is the straight line which is the locus of the points referred to which as base points the system of forces reduces to a wrench.
4. Suppose any given system of forces acting at any given points of a rigid body is reduced to a single force $\mathbf{R} = (X, Y, Z)$ acting at O and a couple $\mathbf{G} = (L, M, N)$ whose axis passes through O. Then the necessary and sufficient conditions for the forces to be in equilibrium are $X = 0,\ Y = 0,\ Z = 0,$ ....
5. Suppose a system of forces acting at different points of a rigid body reduces to a single force $\mathbf{R} = (X, Y, Z)$ acting at an arbitrarily chosen point O and a couple $\mathbf{G} = (L, M, N)$ whose axis passes through O. Then the necessary and sufficient conditions for this system of forces to reduce to a single resultant force are
$$LX + MY + NZ = \ldots\ldots \text{ and } X^2 + Y^2 + Z^2 \neq 0.$$

True or False

Write 'T' for true and 'F' for false statement.

1. If $(\mathbf{R}, \mathbf{G})$ is a wrench and p is the pitch of this wrench, then $p = |\mathbf{G}|$.
2. Whatever origin and the axes are chosen, the quantities
$$X^2 + Y^2 + Z^2 \text{ and } LX + MY + NZ$$
are invariants for any given system of forces.
3. Suppose a system of forces acting at different points of a rigid body reduces to a single force $\mathbf{R} = (X, Y, Z)$ acting at an arbitrarily chosen point O and a couple $\mathbf{G} = (L, M, N)$ whose axis passes through O. If p is the pitch of the equivalent wrench, then $p = \dfrac{X^2 + Y^2 + Z^2}{LX + MY + NZ}$.
4. Suppose a system of n forces acting at different points of a rigid body reduces to a single force $\mathbf{R}$ and a couple $\mathbf{G}$ with respect to O as the base point. If this system of forces reduces to a single force $\mathbf{R}'$ and a couple $\mathbf{G}'$ with respect to O' as the base point where $\overrightarrow{OO'} = \mathbf{c}$, then $\mathbf{G}' = \mathbf{G} - \mathbf{c} \times \mathbf{R}$.
5. Suppose a system of n forces acting at different points of a rigid body reduces to a single force $\mathbf{R}$ and a couple $\mathbf{G}$ with respect to O as the base point. If this system of forces reduces to a single force $\mathbf{R}'$ and a couple $\mathbf{G}'$ with respect to O' as the base point, then $\mathbf{R}' = \mathbf{R}$.

Answers

Multiple Choice Questions

1. (c) 2. (b) 3. (c) 4. (c) 5. (c)

Fill in the Blank(s)

1. axis 2. same 3. central axis
4. $L = 0,\ M = 0,\ N = 0$ 5. 0

True or False

1. *F* 2. *T* 3. *F* 4. *T* 5. *T*

Chapter

6

Forces in Three Dimensions

(Null Lines, Null Planes, Screws and Wrenches)

6.1 Null Lines, Null Plane and Null Point

(Avadh 2007, 08; Gorakhpur 09)

Null Lines

Null lines of a given system of forces, referred to any origin or base point O', are those lines about which the moment of the system vanishes.

Null Plane **(Avadh 2007, 08)**

The plane in which all these null lines lie is called the null plane of the point O'.

Null point **(Avadh 2007, 08; Gorakhpur 09)**

The point O' itself is called null point.

Example: Suppose a system of forces is represented by a force R and a couple G referred to origin O. Draw a straight line through O and perpendicular to the axis of couple G. Then the moment of the couple about this line is zero. The line of action of R will cut this line and hence the moment of R about this line is zero. Thus we see that the moment of system of forces about this line is zero. Hence this line is called null line at O.

6.2 The Equation to Null Plane

Find the equation to null plane of a given point (a, b, c) referred to any axes Ox, Oy, Oz.

(Avadh 2006)

Proof. Let the system be equivalent to a force $R(X, Y, Z)$ and a couple $G(L, M, N)$ referred to base (origin) O. Let the co-ordinates of a point O' be (a, b, c). Let $O'x', O'y', O'z'$ be the lines through O' and parallel to Ox, Oy, Oz respectively. If L', M', N' are the moments of couples about these lines, then

$$L' = L - (bZ - cY),$$
$$M' = M - (cX - aZ), N' = N - (aY - bX).$$

Obviously, L', M', N' are direction ratios of the axis of the resultant couple at (a, b, c).

Next we suppose that (x, y, z) are the co-ordinates of any point in the null plane of (a, b, c) so that the line joining (x, y, z) to (a, b, c) is a null line and its direction ratios are

$$x - a, y - b, z - c.$$

By definition, null line is a perpendicular to the axis of the couple and hence

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

or $$(x - a)L' + (y - b)M' + (z - c)N' = 0$$

or $$(x - a)[L - bZ + cY] + (y - b)[M - cX + aZ] + (z - c)[N - aY + bX] = 0$$

or $$x(L - bZ + cY) + y(M - cX + aZ) + z(N - aY + bX) = aL + bM + cN \quad \ldots(1)$$

This is the locus of (x, y, z) which is any point in the null plane of (a, b, c). Therefore (1) is the equation of **null plane** of the point (a, b, c). In a determinant form it can be written as

$$\begin{vmatrix} x & y & z \\ a & b & c \\ X & Y & Z \end{vmatrix} = L(x - a) + M(y - b) + N(z - c)$$

6.3 The Null Point of the Plane

Find the null point of the plane

$$lx + my + nz = 1.$$ **(Gorakhpur 2006; Kanpur 08, 10)**

Proof: Let (x', y', z') be null point of the plane

$$lx + my + nz = 1. \quad \ldots(1)$$

Since (x', y', z') lies on this plane, therefore

$$lx' + my' + nz' = 1 \quad \ldots(2)$$

The equation of the null plane through (x', y', z') is

$$x(L - y'Z + z'Y) + y(M - z'X + x'Z) + z(N - x'Y + y'X) = Lx' + My' + Nz' \quad \ldots(3)$$

Since (1) and (3) represent the same plane and hence they must be identical so that on comparing, we get

$$\frac{L - y'Z + z'Y}{l} = \frac{M - z'X + x'Z}{m} = \frac{N - x'Y + y'X}{n}$$
$$= \frac{Lx' + My' + Nz'}{1}$$

From first and second ratios, we get

$$Lm - y'mZ + z'mY - lM + lz'X - x'lZ = 0$$

or $$(Lm - lM) - (my' - lx')Z + mz'Y - lz'X = 0 \qquad ...(4)$$

From first and third ratios, we get

$$Ln - y'nZ + z'nY = lN - lx'Y + ly'X$$

or $$(Ln - lN) - y'nZ + (lx' + nz')Y - ly'X = 0 \qquad ...(5)$$

From second and third ratios, we get

$$nM - nz'X + nx'Z = mN - mx'Y + my'X$$

or $$(nM - mN) - (my' + nz')X + nx'Z + mx'Y = 0 \qquad ...(6)$$

Writing (4), (5), (6) with the help of (2),

$$(Lm - lM) + (nz' - 1)Z + mz'Y + lz'X = 0$$
$$(Ln - lN) - y'nZ + (1 - my')Y - ly'X = 0$$
$$(nM - mN) + (lx' - 1)X + nx'Z + mx'Y = 0$$

or $$-(mL - lM - Z) = (lX + mY + nZ)z'$$
$$-(lN - nL - Y) = (lX + mY + nZ)y'$$
$$-(nM - mN - X) = (lX + mY + nZ)x'.$$

This $\Rightarrow$ $$\frac{x'}{X - nM + mN} = \frac{y'}{Y - lN + nL} = \frac{z'}{Z - mL + lM}$$
$$= \frac{1}{lX + mY + nZ}$$

or $$\frac{x'}{X - \begin{vmatrix} M & N \\ m & n \end{vmatrix}} = \frac{y'}{Y - \begin{vmatrix} N & L \\ n & l \end{vmatrix}} = \frac{z'}{Z - \begin{vmatrix} L & M \\ l & m \end{vmatrix}}$$
$$= \frac{1}{lX + mY + nZ}.$$

This gives the co-ordinates of the required point.

6.4 Condition for a Straight Line

Find the condition that the straight line

$$\frac{x - f}{l} = \frac{y - g}{m} = \frac{z - h}{n}$$

may be a null line for the system of forces $(X, Y, Z; L, M, N)$.

(Avadh 2007; Gorakhpur 07, 08, 10)

Proof: The given system of forces is $(X, Y, Z; L, M, N)$. Let the co-ordinates of O' be (f, g, h).

Let $O'x'$, $O'y'$, O', z' be the lines through O' and parallel to Ox, Oy, Oz respectively. If L', M', N' are the moments of couples about these lines, then

$$L' = L - (gZ - hY), M' = M - (hX - fZ),$$
$$N' = N - (fY - gX).$$

The direction ratios of resultant couple at O' are L', M', N'. The line

$$\frac{x-f}{l} = \frac{y-g}{m} = \frac{z-h}{n}$$

with direction ratios l, m, n will be null line through O if it is perpendicular to the axis of the couple at O'.

Therefore the condition of perpendicularity gives

$$lL' + mM' + nN' = 0$$

or $$l(L - gZ + hY) + m(M - hX + fZ) + n(N - fY + gX) = 0$$

or $$lL + mM + nN = X(-ng + mh) - Y(-nf + hl) + Z(-mf + lg)$$

or $$\begin{vmatrix} X & Y & Z \\ l & m & n \\ f & g & h \end{vmatrix} = lL + mM + nN$$

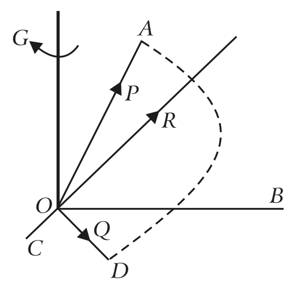
This is the condition for the given line to be a null line through $O'(f, g, h)$.

6.5 Replacement of System of Forces by Two Forces

(Lucknow 2011)

Theorem: *A given system of forces may be replaced by two forces, one of which acts along a given line OA.*

Proof: Let the system reduce to the resultant force R and couple G at the point O. Suppose BOC is the plane of the couple so that the axis of the couple is perpendicular to the plane BOC. Draw a plane through the given line OA and the line of action of R and suppose that this plane cuts the plane BOC in a line OD. Consequently the line of action of R and straight lines OA, OD all are coplanar.



Resolve R along OA and OD. Let the components of R along OA and OD be respectively P and Q. The force Q along OD together with the couple (*i.e.,* two forces of the couple) will give rise to a force Q (in the plane BOC) parallel to OD. The plane BOC is the null plane of O.

Hence the system is reduced to a force P along OA and other force Q in the null plane of O but parallel to OD.

6.6 Conjugate Forces and Lines

Definition: *A given system of forces may be replaced by two forces P and Q, one of which acts along a given line. Such forces P and Q are called* **conjugate forces** *and their lines of action are called* **conjugate lines**. **(Gorakhpur 2011)**

Method for Finding the Conjugate Lines

O is any point on *OA* and as *O* moves on *OA*, the null plane of *O* will always pass through a line conjugate to *OA*. Hence in order find the conjugate line of *OA*, take any two convenient points on *OA* and find their null plane. The intersection of these two null planes will give us the line conjugate to *OA*.

6.7 The Equation of Conjugate Line of the Line

Find the equation of conjugate line of the line

$$\frac{x-f}{l}=\frac{y-g}{m}=\frac{z-h}{n}$$ **(Purvanchal 2007; Gorakhpur 11)**

Proof. Let the system of forces referred to *O* as the origin be $(X, Y, Z; L, M, N)$. In order to find the equations of the conjugate line of the line

$$\frac{x-f}{l}=\frac{y-g}{m}=\frac{z-h}{n} \qquad \ldots(1)$$

we have to find out the equations of null planes of any two chosen points on the line (1). Let one point be (f, g, h) and the other point $(l\rho, m\rho, n\rho)$ at infinity, where $\rho \to \infty$. Evidently these two points lie on (1).

Equation of null plane through (f, g, h) is

$$\begin{vmatrix} x & y & z \\ f & g & h \\ X & Y & Z \end{vmatrix} = L(x-f)+M(y-g)+N(z-h) \qquad \ldots(2)$$

and the null plane through $(l\rho, m\rho, n\rho)$ is

$$\begin{vmatrix} x & y & z \\ l\rho & m\rho & n\rho \\ X & Y & Z \end{vmatrix} = L(x-l\rho)+M(y-m\rho)+N(z-n\rho)$$

Dividing by ρ, we get

$$\begin{vmatrix} x & y & z \\ l & m & n \\ X & Y & Z \end{vmatrix} = L\left(\frac{x}{\rho}-l\right)+M\left(\frac{y}{\rho}-m\right)+N\left(\frac{z}{\rho}-n\right) \qquad \ldots(3)$$

As $\rho \to \infty$, (2) reduces to

$$\begin{vmatrix} x & y & z \\ l & m & n \\ X & Y & Z \end{vmatrix} = -(Ll+Mm+nN) \qquad \ldots(4)$$

The line of intersection of null planes (2) and (3) is a conjugate line to the given line (1).

Illustrative Examples

Example 1: *Find the null point of the plane $x + y + z = 0$ for the force system $(X, Y, Z; L, M, N)$.* **(Kanpur 2007, 11; Avadh 06, 07, 08, 09, 11; Purvanchal 08, 10; Gorakhpur 08, 11)**

Solution: Let (f, g, h) be the null point of the plane

$$x + y + z = 0 \qquad \ldots(1)$$

As (f, g, h) lies on (1), we have

$$f + g + h = 0 \qquad \ldots(2)$$

But the null plane of (f, g, h) is

$$x(L - gZ + hY) + y(M - hX + fZ) + z(N - fY + gX) = Lf + Mg + Nh.$$

Comparing this with the plane (1),

$$\frac{L - gZ + hY}{1} = \frac{M - hX + fZ}{1} = \frac{N - fY + gX}{1} = \frac{Lf + Mg + Nh}{0}$$

Taking first, second; second third; and third, first, we get

$$L - gZ + hY = M - hX + fZ$$
$$M - hX + fZ = N - fY + gX$$
$$N - fY + gX = L - gZ + hY$$

or $\quad L - M = - hY - hX + (f + g) Z = - hY - hX - hZ,$ by (2)

$\quad M - N = (g + h) X - fZ - fY = - fX - fZ - fY,$ by (2)

$\quad N - L = (f + h) Y - gX - gZ = - gY - gX - gZ,$ by (2)

or $\quad h = - (L - M)k, f = - (M - N)k, g = - (N - L)k,$

where $\quad k = 1/(X + Y + Z).$

$$\left(\frac{N - M}{X + Y + Z}, \frac{L - N}{X + Y + Z}, \frac{M - L}{X + Y + Z}\right)$$

are the co-ordinates of the null point.

Example 2: *A system of forces given by $(X, Y, Z; L, M, N)$ is replaced by two forces, one acting along the axis of x and another force.*

Show that the magnitudes of the forces are

$$\frac{LX + MY + NZ}{L} \text{ and } \frac{[(MY + NZ)^2 + L^2 (Y^2 + Z^2)]^{1/2}}{L}$$

and also find the equation of the line of action of the other force. **(Avadh 2008)**

Solution: Suppose the given system of forces reduces to dyname $(X, Y, Z; L, M, N)$. Let a force P act at (0, 0,0) along the axis of x whose equation is

$$\frac{x}{1} = \frac{y}{0} = \frac{z}{0} \qquad \ldots(1)$$

Hence components of this force parallel to axes are $P, 0, 0$. Also the components of the couple to this force at (0, 0, 0) parallel to axes are 0, 0, 0.

[For example $L = yZ - zY = 0.0 - 0 \cdot 0 = 0$]

Therefore the components of other force parallel to axes are $X - P, Y, Z$ and components of couple due to this force parallel to axes are L, M, N. Let this second force act at $(f, g, 0)$. Then

$$\mathbf{i}L + \mathbf{j}M + \mathbf{k}N = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_2 & y_2 & z_2 \\ X_2 & Y_2 & Z_2 \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ f & g & 0 \\ X - P & Y & Z \end{vmatrix}$$

$$= \mathbf{i}(gZ) - \mathbf{j}(fZ) + \mathbf{k}[fY - g(X - P)].$$

This $\Rightarrow$ $L = gZ, M = -fZ, N = fY - g(X - P)$

$\Rightarrow$ $N = \dfrac{-M}{Z}Y - \dfrac{L}{Z}(X - P)$

$\Rightarrow$ $NZ + MY + LX - LP = 0$

$\Rightarrow$ $P = \dfrac{LX + MY + NZ}{L}$

$\Rightarrow$ $X - P = X - \dfrac{LX + MY + NZ}{L} = -\dfrac{(MY + NZ)}{L}$

$\therefore$ magnitude of second force

$$= [(X - P)^2 + Y^2 + Z^2]^{1/2}$$

$$= \left[\left(\frac{MY + NZ}{L}\right)^2 + Y^2 + Z^2\right]^{1/2}$$

$$= (1/L)[(MY + NZ)^2 + L^2(Y^2 + Z^2)]^{1/2}$$

Thus we have shown that :

Magnitude of first force $= P = (LX + MY + NZ)/L$

Magnitude of second force $= (1/L)[(MY + NZ)^2 + L^2(Y^2 + Z^2)]^{1/2}$

This completes the first part of the problem.

To determine the line of action of second force.

Null plane at (f, g, h) is

$$\begin{vmatrix} x & y & z \\ f & g & h \\ X & Y & Z \end{vmatrix} = L(x - f) + M(y - g) + N(z - h)$$

In our case null plane of (0, 0, 0), a point on the line (1), is

$$\begin{vmatrix} x & y & z \\ 0 & 0 & 0 \\ X & Y & Z \end{vmatrix} = L(x - 0) + M(y - 0) + N(z - 0)$$

or $Lx + My + Nz = 0$ ...(2)

Point at infinity lying on the line (1) is $(\rho, 0, 0)$.

Null plane at $(\rho, 0, 0)$ is

$$\begin{vmatrix} x & y & z \\ \rho & 0 & 0 \\ X & Y & Z \end{vmatrix} = L(x - \rho) + M(y - 0) + N(z - 0)$$

Expanding the determinant along the second row,

$$-\rho(yZ - zY) = Lx + My + Nz - L\rho$$

Dividing by ρ and then making $\rho \to \infty$,

$$-(yZ - zY) = -L$$

or $$yZ - zY = L \qquad \ldots(3)$$

Equations (2) and (3) together give the line of action of second force.

Comprehensive Exercise 1

1. Show that among the null lines of any system of forces four are generators of any hyperboloid, two belonging to one system of generators and two to the other system.
2. Show that the wrench $(X, Y, Z; L, M, N)$ is equivalent to two forces, one along the line $x = y = z$, and the other along the line given by

$$Lx + My + Nz = 0,$$
$$x(Y - Z) + y(Z - X) + z(X - Y) = L + M + N$$

   and find the magnitudes of the two forces.
3. A straight line is given by the equations

$$Ax + By + Cz = D, A'x + B'y + C'z = D'$$

   Show that its conjugate is given by equating to zero any two of the determinants

$$\left\| \begin{matrix} L' & M' & N' & Lx + My + Nz \\ A & B & C & D \\ A' & B' & C' & D' \end{matrix} \right\|$$

   where L', M', N' are the component couples at the point (x, y, z) and L, M, N those at the origin.

6.8 Screw, Pitch and Wrench

Screw: If a body rotates about a straight line through some angle and at the same time moves some distance along the line, the body is said to be *screwed* about the line, axis of the screw is often called a *screw*. **(Rolilkhand 2009, 10, 11)**

Pitch: If the body rotates through small angle $d\theta$ about the axis and moves at the same time a distance dx along the axis, then the ratio $\dfrac{dx}{d\theta}$ is called the pitch of the screw. From this it is clear that the pitch is the rate of change of x along the axis as θ increases; so pitch of a screw is a length. **(Rohilkhand 2010, 11; Gorakhpur 08, 10)**

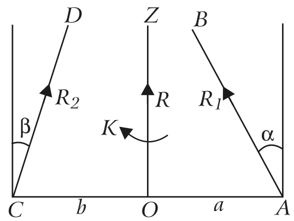
Wrench: *Wrench* is a combination of a force and a couple, the axis of the couple coincides with the direction of the force. If R be the force and K couple, then the ratio K/R is called the *pitch* of the wrench and the single force R is called **intensity** of the wrench. **(Rohilkhand 2009; Avadh 06)**

6.9 Tetrahedron Result for Two Forces

Theorem: *That a given system of forces can be replaced by two forces, equivalent to the given system, in an infinite number of ways and that the tetrahedron formed by the two forces is of constant volume.*

Proof: Let the given system be equivalent to a wrench (R, K) whose central axes is OZ so that R is the resultant force along OZ and K the resultant couple about OZ.

Draw a line CA through O and perpendicular to OZ such that $OA = a, OC = b$.



Suppose that the given system is equivalent to a force R_1 at A in a plane perpendicular to OA and a force R_2 at C in a plane perpendicular to OC such that the lines of action of these make angles α and β with OZ respectively, as shown in Fig. Let α and β be measured positively in opposite directions. Resolving forces along and perpendicular to OZ,

$$R_1 \cos\alpha + R_2 \cos\beta = R \quad \text{...(1)}$$

$$R_1 \sin\alpha - \sin\beta = 0 \quad \text{...(2)}$$

Taking moments of the forces about OZ and a line $\perp$ to OZ, and noting that

moment = force × S.D. × sine of included angle;

we obtain

$$aR_1 \sin\alpha + bR_2 \sin\beta = K \quad \text{...(3)}$$

$$-aR_1 \cos\alpha + bR_2 \cos\beta = 0 \quad \text{...(4)}$$

From (4),

$$\frac{R_1 \cos\alpha}{b} = \frac{R_2 \cos\beta}{a} = \frac{R_1 \cos\alpha + R_2 \cos\beta}{a+b}$$

$$= \frac{R}{a+b}, \text{ by (1).}$$

$$\therefore \quad R_1 \cos\alpha = \frac{bR}{a+b} \quad \text{and} \quad R_2 \cos\beta = \frac{aR}{a+b} \quad \text{...(5)}$$

From (2), $R_1 \sin\alpha = R_2 \sin\beta$

or
$$\frac{a\ R_1 \sin\alpha}{a} = \frac{b\ R_2 \sin\beta}{b} = \frac{aR_1 \sin\alpha + bR_2 \sin\beta}{a+b}$$

$$= \frac{K}{a+b}, \text{ by (3)}$$

This $\Rightarrow R_1 \sin\alpha = R_2 \sin\beta = \dfrac{K}{a+b}$ ...(6)

Squaring (5), (6) and then adding, we get

$$\left.\begin{aligned} R_1^2 &= \frac{b^2R^2 + K^2}{(a+b)^2} \\ R_2^2 &= \frac{a^2R^2 + K^2}{(a+b)^2} \end{aligned}\right] \quad \text{...(7)}$$

Dividing (6) by (5), we get

$$\tan\alpha = \frac{K}{bR}, \ \tan\beta = \frac{K}{aR} \quad \text{...(8)}$$

Relations (7) and (8) give R_1, R_2, α, β in terms of R, K, a, b. But a and b are arbitrary. *Hence there are an infinite number of ways in which the given system can be replaced by two forces.*

Let AB and CD represent the forces R_1 and R_2. Volume V of tetrahedron is given by

$$V = \frac{1}{3} \text{ area of } \Delta ABC \times \text{perpendicular from } D \text{ upon } ABC$$

$$= \frac{1}{3}\cdot\frac{1}{2} AB.\,AC.\,CD \sin(\alpha+\beta) = = \frac{1}{6} R_1(a+b)\cdot R_2 \sin(\alpha+\beta).$$

$\therefore$ $$V = \frac{1}{6}\cdot R_1R_2(a+b)\sin(\alpha+\beta) \quad \text{...(9)}$$

From (5) and (6)

$R_1 \sin\alpha .\, R_2 \cos\beta + R_2 \sin\beta \cdot R_1 \cos\alpha$

$$= \frac{K}{a+b}\cdot\frac{aR}{a+b} + \frac{K}{a+b}\cdot\frac{bR}{a+b} = \frac{RK(a+b)}{(a+b)^2} = \frac{RK}{a+b}$$

or $\quad R_1R_2\sin(\alpha+\beta) = \dfrac{RK}{a+b}$

$\therefore$ from (9), we get

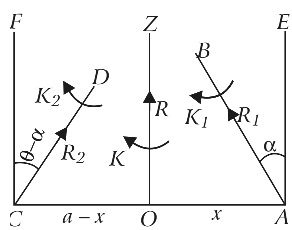
$$V = \frac{1}{6}\cdot\frac{RK(a+b)}{a+b} = \frac{RK}{6} \quad \text{i.e.,} \quad V = \frac{RK}{6}.$$

6.10 The Resultant Wrench of Two Given Wrenches

(Avadh 2006; Gorakhpur 06, 09)

To find the resultant wrench of two given wrenches.

Proof: Suppose the first wrench is (R_1, K_1) whose axis is AB and the other wrench is (R_2, K_2) whose axis is CD. Suppose θ is the angle between AB and CD. Let $AC = a$ be the shortest distance between the axes of the two wrenches. The resultant wrench (R, K) is along OZ perpendicular to AC such that $OA = x$, $CO = a - x$. If the axis AB is inclined at an angle α with OZ then evidently



CD will be inclined at an angle $\theta - \alpha$ to OZ. Resolving forces along and perpendicular to OZ, we get

$$R = R_1 \cos\alpha + R_2 \cos(\theta - \alpha) \quad ...(1)$$

$$0 = R_1 \sin\alpha - R_2 \sin(\theta - \alpha) \quad (2)$$

Taking moments about the above lines,

$$R = K_1 \cos\alpha + K_2 \cos(\theta - \alpha) + xR_1 \sin\alpha + (a - x)R_2 \sin(\theta - a) \quad ...(3)$$

$$0 = K_1 \sin\alpha - K_2 \sin(\theta - \alpha) - xR_1 \cos\alpha + (a - x)R_2 \cos(\theta - \alpha) \quad ...(4)$$

Squaring (1), (2) and then adding, we get

$$R^2 = R_1^2 + R_2^2 + 2R_1R_2 \cos(\alpha + \theta - \alpha)$$

or
$$R^2 = R_1^2 + R_2^2 + 2R_1R_2 \cos\theta \quad ...(5)$$

From (3) and (4), using (1) and (2), we get

$$K = K_1 \cos\alpha + K_2 \cos(\theta - \alpha) + xR_1 \sin\alpha + (\alpha - x)R_2 \sin\alpha$$

$$0 = K_1 \sin\alpha - K_2 \sin(\theta - \alpha) + xR_2 \cos(\theta - \alpha) - Rx + (\alpha - x)R_2 \cos(\theta - \alpha) \quad ...(6)$$

or
$$K = K_1 \cos\alpha + K_2 \cos(\theta - \alpha) + aR_1 \sin\alpha$$

$$0 = K_1 \sin\alpha - K_2 \sin(\theta - \alpha) - XR + aR_2 \cos(\theta - \alpha) \quad ...(7)$$

Our aim is to find out the values of R, K and x.

From (2), $\quad R_1 \sin\alpha - R_2 (\sin\theta\cos\alpha - \cos\theta\sin\alpha) = 0$

or $\quad (R_1 + R_2 \cos\theta)\sin\alpha - R_2 \sin\theta\cos\alpha = 0$

or
$$\frac{\sin\alpha}{R_2 \sin\theta} = \frac{\cos\alpha}{R_1 + R_2\cos\theta}$$

$$= \frac{\sqrt{(\sin^2\alpha + \cos^2\alpha)}}{\sqrt{[R_2^2 \sin^2\theta + (R_1 + R_2\cos\theta)^2]}}$$

$$= \frac{1}{(R_1^2 + R_2^2 + 2R_1R_2\cos\theta)^{1/2}}$$

$$= \frac{1}{R}, \text{ by (5)}$$

$\therefore$
$$\frac{\sin\alpha}{R_2\sin\theta} = \frac{\cos\alpha}{R_1 + R_2\cos\theta} = \frac{1}{R}. \quad ...(8)$$

From (6), we get

$$K = K_1 \cos\alpha + K_2(\cos\theta\cos\alpha + \sin\theta\sin\alpha) + aR_1 \sin\alpha$$

$$= (K_1 + K_2\cos\theta)\cos\alpha + (K_2 \sin\theta + aR_1)\sin\alpha$$

Multiplying by R and then putting the values of $R\sin\alpha, R\cos\alpha$ from (8), we get

$$RK = (K_1 + K_2\cos\theta)(R_1 + R_2 + \cos\theta) + (K_2\sin\theta + aR_1)R_2\sin\theta$$

$$= R_1K_1 + R_1K_2 + K_2R_2\cos\theta + R_1K_2\cos\theta + aR_1R_2\sin\theta$$

or
$$RK = R_1K_1 + R_2K_2 + (R_1K_2 + R_2K_1)\cos\theta + aR_1R_2\sin\theta \quad ...(9)$$

Since θ is given and so R can be determined from (5). Now putting the values of R and θ in (9), we can obtain the value of K.

To determine x.

From (7), $xR = K_1 \sin\alpha - K_2 \sin(\theta - \alpha) + aR_2 \cos(\theta - \alpha)$

or $xR = K_1 \sin\alpha - K_2(\sin\theta\cos\alpha - \cos\theta\sin\alpha) + aR_2(\cos\theta\cos\alpha + \sin\theta\sin\alpha)$

$= (K_1 + K_1 \cos\theta + aR_2 \sin\theta)\sin\alpha + (a R_2 \cos\theta - K_2 \sin\theta)\cos\alpha$

Multiplying by R and then using (8),

$$xR^2 = (K_1 + K_2 \cos\theta + aR_2 \sin\theta)\, R_2 \sin\theta + (aR_2 \cos\theta - K_2 \sin\theta)(R_1 + R_2 \cos\theta)$$

$$= (R_2 K_1 - R_1 K_2)\sin\theta + aR_2^2 \sin^2\theta + aR_2 \cos\theta\,(R_1 + R_2\cos\theta)$$

$$= (R_2 K_1 - R_1 K_2)\sin\theta + aR_2^2 + aR_1 R_2 \cos\theta$$

or $$xR^2 = (R_2 K_1 - R_1 K_2)\sin\theta + aR_2(R_2 + R_1\cos\theta) \qquad \ldots(10)$$

From this equation, we can determine x.

Remark: *The above theorem can also be put as :*

Prove that the invariant $I\,(= RK)$ of two wrenches whose forces are R_1 and R_2 and couples K_1, K_2 is

$$R_1K_1 + R_2K_2 + (R_1K_2 + R_2K_1)\cos\theta + aR_1R_2\sin\theta$$

where θ is the angle between the forces R_1 and R_2 and a the distance between them. Hence deduce the resultant of these two wrenches.

Deduction: *To find the resultant wrench of two given forces R_1 and R_2 inclined at an angle θ.*

Proof: Here $K_1 = 0$, $K_2 = 0$

From (5), $R^2 = R_1^2 + 2R_1R_2\cos\theta \qquad \ldots(11)$

From (9), $RK = R_1 \cdot 0 + R_2 \cdot 0 + (R_1 \cdot 0 + R_2 \cdot 0)\cos\theta + aR_1R_2\sin\theta$

or $RK = aR_1R_2\sin\theta \qquad \ldots(12)$

From (10), $xR^2 = (0 - 0)\sin\theta + aR_2(R_2 + R_1\cos\theta)$

or $xR^2 = aR_2(R_2 + R_1\cos\theta) \qquad \ldots(13)$

The equation (11) gives the intensity R of wrench (R, K). The equation (12) gives the value of K and the equation (13) gives the position of central axis (wrench).

6.11 Locus of the Central Axis

The axes of two given wrenches intersect at right angles. Their intensities are X and Y and their pitches are p_x and p_y. To find the locus of the central axis.

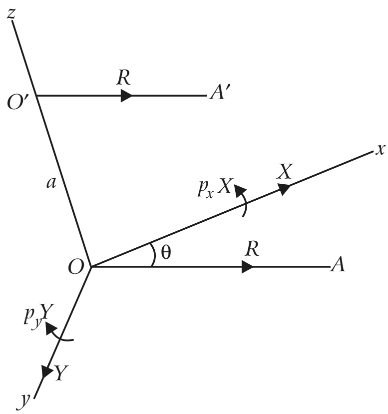
Proof: Suppose the wrenches (X, p_xX) and (Y, p_yY) have their axes along Ox and Oy which are taken as the two co-ordinate axes. Let R be the resultant of the forces X and Y and let OA be the line of action of R such that OA makes an angle θ with Ox. Then

$$X = R\cos\theta, Y = R\sin\theta \qquad \ldots(1)$$

from which, we get

$$R = \sqrt{(X^2 + Y^2)}, \text{ and } \tan\theta = Y/X \qquad \ldots(2)$$

Next we suppose that the resultant couple is G whose axis makes an angle ϕ with OA in the plane xOy.



Then $G \cos\phi$ = couple about OA

$$= p_x X \cos\theta + p_y Y \sin\theta$$
$$= p_x R\cos^2\theta + p_y R\sin^2\theta$$

or $$G\cos\phi = (p_x \cos^2\theta + p_y \sin^2\theta)R$$

and $G\sin\phi$ = couple about a line perpendicular to OA in plane xOy

$$= p_x X\cos(\theta + 90°) + p_y Y \sin(\theta + 90°)$$
$$= -p_x R\cos\theta\sin\theta + p_y R\sin\theta\cos\theta$$
$$= (p_y - p_x) R \sin\theta\cos\theta$$

Thus $$\left.\begin{aligned} G\cos\phi &= (p_x\cos^2\theta + p_y r\sin^2\theta) R \\ G\sin\phi &= (p_y - p_x)R\sin\theta\cos\theta \end{aligned}\right] \quad \ldots(3)$$

Since X and Y are given and hence the resultant R and its line of action, *i.e.,* θ can easily be determined from (2). Also p_x, p_y are given. Hence $G\cos\phi$, $G\sin\phi$ are obtained from (3). From this we can calculate G and the angle ϕ, by the formulae

$$G = \sqrt{(G^2\cos^2\phi + G^2\sin^2\phi)}$$

and $$\tan\phi = \frac{G\sin\phi}{G\cos\phi}.$$

Thus we have a single force R along OA and couple G at O and they are equivalent to a wrench $(R, G\cos\phi)$ whose axis is a line $O'A'$ parallel to OA such that

$$R.OO' = G\sin\phi, \quad i.e., \quad OO' = (G\sin\phi)/R$$

or $$OO' = (p_y - p_x)\sin\theta\cos\theta, \text{ using } (3) \quad \ldots(4)$$

This gives position of central axis.

Pitch p of the wrench is given by

$$p = \frac{G\cos\phi}{R} = p_x\cos^2\theta + p_y\sin^2\theta.$$

Thus p is known.

Equation of central axis is

$$\frac{L-(yZ-zY)}{X}=\frac{M-(zX-xZ)}{Y}=\frac{N-(xY-yX)}{Z}$$

Here $X = X, Y = Y, Z = 0, L = p_x X, M = p_y Y, N = 0.$

$\therefore$ Equation of central axis is

$$\frac{p_x X-(0-zY)}{X}=\frac{p_y Y-(zX-0)}{Y}=\frac{0-(xY-yZ)}{0}$$

From the first two ratios, we get

$$(p_x X + zY)Y = (p_y Y - zX)X$$

or $$z(X^2+Y^2)=(p_y-p_x)XY \qquad ...(5)$$

The last two ratios give $xY - yX = 0$

or $$\frac{X}{x}=\frac{Y}{y} \qquad ...(6)$$

Eliminating X, Y from (5) and (6), we get

$$z(x^2+y^2)=(p_y-p_x)xy \qquad ...(7)$$

This is the surface generated by central axis.

Thus the required locus is given by (7).

Note: *The surface given by (7) is known as the cylindroid and the pitches p_x and p_y are called its* ***principal pitches.***

6.12 Component Wrenches of a Given Wrench

(Purvanchal 2007)

Theorem: *Any wrench may be resolved into two wrenches, whose axes intersect at right angles in an infinite number of ways*

Proof. (See Fig. of 6.11).

Let the given wrenches be $(X, p_x X)$ about OX and $(Y, p_y Y)$ about Oy as axes where p_x and p_y are pitches of the wrenches. Let these two wrenches be equivalent to a single wrench (R, pR) about $O'A'$ as axis where O' is a point on Oz such that $OO' = a$

Draw a line parallel to $O'A'$. Let $\angle xOA = \theta$.

Then $$X = R\cos\theta, Y = R\sin\theta \qquad ...(1)$$

$\therefore$ $$R = \sqrt{(X^2+Y^2)}, \tan\theta = Y/X \qquad ...(2)$$

Also pR = couple about $O'A'$

$$= \text{couple about } OA$$

$$= p_x X\cos\theta + p_y Y\sin\theta$$

$$= p_x R\cos\theta.\cos\theta + p_y R\sin\theta.\sin\theta$$

or $$pR = (p_x\cos^2\theta + p_y\sin^2\theta)R$$

or $$p = (p_x\cos^2\theta + p_y\sin^2\theta) \qquad ...(3)$$

$$OO' = (G\sin\phi)/R \text{ in usual notation.}$$

Here in this case, it gives

$$a = \frac{\text{couple about a line perpendicular to } OA}{R}$$

or $\quad aR = p_x X\cos(\theta + 90^\circ) + p_x Y\sin(\theta + 90^\circ)$

$$= -\,p_x R\cos\theta\sin\theta + p_y R\sin\theta\cos\theta$$

$$= (p_y - p_x)\,R\sin\theta\cos\theta$$

or $\quad a = (p_y - p_x)\sin\theta\cos\theta \qquad \ldots(4)$

Multiplying (3) by $\sin\theta$ and (4) by $\cos\theta$ and then adding, we get

$$p\sin\theta + a\cos\theta = p_y\sin^2\theta + p_y\sin\theta\cos^2\theta$$

or $\quad p\sin\theta + a\cos\theta = p_y\sin\theta$

or $\quad p + a\cot\theta = p_y$

Using this from (3), we get

$$p = p_x\cos^2\theta + (p + a\cot\theta)\sin^2\theta$$

or $\quad p = p_x\cos^2\theta + p\sin^2\theta + a\sin\theta\cos\theta$

or $\quad p\cos^2\theta - a\sin\theta\cos\theta = p_x\cos^2\theta$

or $\quad p - a\tan\theta = p_x.$

Thus we have

$$\left.\begin{aligned} p_x &= p - a\tan\theta \\ p_y &= p + a\cot\theta \end{aligned}\right] \qquad \ldots(5)$$

Now we are given R, θ, a. So we can determine X, Y from (1). Also p_y and p_x can be determined from (5). Since a and θ are arbitrary chosen quantities, consequently X, Y, p_x, p_y may have an infinite number of values according as a and θ change.

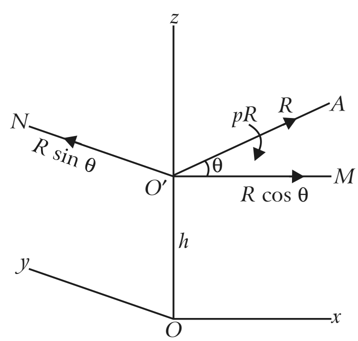
Therefore the component wrenches $(X, p_x X)$ and $(Y, p_y Y)$ of the given wrench (R, pR) can take an infinite values.

6.13 Work Done by a Wrench

Theorem: *To show that the work done by a wrench, of intensity R and pitch p about a given screw, when the body is given small twist $\delta\omega$ about another screw of pitch p_1 is*

$$R\delta\omega\{(p + p_1)\cos\theta - h\sin\theta\}$$

where θ is the angle between the axes of the two screws and h is the shortest distance between them.



Proof: Let $O'A$ be the axis of the wrench (R, pR) and Ox, the axis of the screw whose pitch is p_1 and $OO'(= h)$ is the shortest distance (S.D) between the two axes $O'A$ and Ox. Draw $O'M$ parallel to Ox and $O'N$ parallel to Oy so that $O'N$ is perpendicular to $O'M$ and $O'O$ both.

The force R along $O'A$ when resolved, is equivalent to forces :

$R\cos\theta$ along $O'M$ and $R\sin\theta$ along $O'N$.

Further, the force $R\cos\theta$ along $O'M$ is equivalent to force $R\cos\theta$ along Ox together with a couple $hR\cos\theta$ about Oy; and $R\sin\theta$ along $O'N$ is equivalent to force $R\sin\theta$ along Oy and a couple $-hR\sin\theta$ about Ox.

The couple pR about $O'A$ is equivalent to

couple $pR\cos\theta$ about $O'M$

and couple $pR\sin\theta$ about $O'N$.

Since the axis of any couple can be shifted to any other parallel axis, therefore above two component couples are equivalent to

couple $pR\cos\theta$ about Ox.
couple $pR\sin\theta$ about Oy.

Hence the given wrench (R, pR) is equivalent to:

force $R\cos\theta$ along Ox.
force $R\sin\theta$ along Oy.
couple $R(p\cos\theta - h\sin\theta)$ about Ox
couple $R(p\sin\theta + h\cos\theta)$ about Oy.

Now the body is twisted through a small angle $\delta\omega$ about Ox, the pitch being p_1; the body therefore moves also a distance $p_1\,\delta\omega$ along Ox.

So, on account of any angular displacement $\delta\omega$ about Ox, the work done by the couples is $R\,(p\cos\theta - h\sin\theta)\,\delta\omega$ and the work done by the force is zero; and on account of only linear displacement $p_1\delta\omega$ along Ox, the work done by couples is zero and work done by force is

$$R\cos\theta.\,p_1\,\delta\omega.$$

Work done (due to angular and linear displacement) is

$$R\delta\omega\{(p+p_1)\cos\theta - h\sin\theta\}.$$

Remark. This result is symmetrical in p and p_1, so that if the two screws are interchanged, the work remains unaltered.

6.14 Reciprocal Screws

Definition: *If two screws are such that when a wrench acting on one screw does no work as the body is given a small twist about another screw, the two screws are said to be reciprocal.*
(Rohilkhand 2009)

We know that if wrench (R, pR) acts along a screw of pitch p_1, then the work done by the wrench is

$$R\,\delta\omega[(p+p_1)\cos\theta - h\sin\theta]$$

as the body is given a small angular displacement $\delta\omega$ about the screw of pitch p_1. Hence h is S.D. between the axes of two screws and θ is the angle between their axes.

The work done is zero if

$$(p+p_1)\cos\theta - h\sin\theta = 0 \qquad \ldots(1)$$

This is the necessary condition for the screws of pitches p and p_1 to the reciprocal.

Particular Case. *The axes of two reciprocal screws intersect.*

Then $h = 0$. Now the condition (1) reduces to

$$(p + p_1)\cos\theta - 0\sin\theta = 0 \quad \text{or} \quad (p + p_1)\cos\theta = 0.$$

This $\Rightarrow$ $\quad p + p_1 = 0 \quad$ or $\quad \cos\theta = 0$

$\Rightarrow$ $\quad p = -p_1 \quad$ or $\quad \theta = 90°$

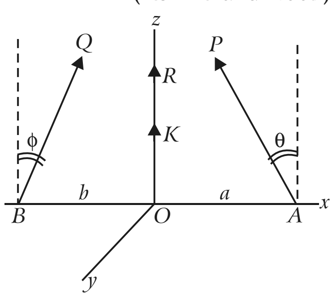
$\Rightarrow$ Either axes of screws are at right angles or the pitches are equal and opposite.

Illustrative Examples

Example 3: *If P and Q be two non-intersecting forces whose directions are perpendicular, show that the ratio of distances of the central axis from their lines of action is Q^2 to P^2.*

(Rohilkhand 2007)

Solution: Let the resultant wrench be (R, K). Also let $OA = a, OB = b$. Let Oz be the central axis.



Let P and Q make angles θ and ϕ with OZ in opposite directions in YOZ plane.

Angle between P and Q is $\theta + \phi = \pi/2$.

Resolving forces along the axes of z and y, we get

$$P\cos\theta + Q\cos\phi = R$$

$$P\sin\theta - Q\sin\phi = 0$$

i.e. $$P\cos\theta + Q\sin\phi = R \qquad \ldots(1)$$

$$P\sin\theta - Q\cos\theta = 0 \qquad \ldots(2)$$

Taking moments about z and y-axes, we get

$$P\sin\theta . a + Q\sin\phi . b = K$$

$$-P\cos\theta . a + Q\cos\phi . b = 0$$

i.e. $$aP\sin\theta + bQ\cos\theta = K \qquad \ldots(3)$$

$$-aP\cos\theta + bQ\sin\theta = 0 \qquad \ldots(4)$$

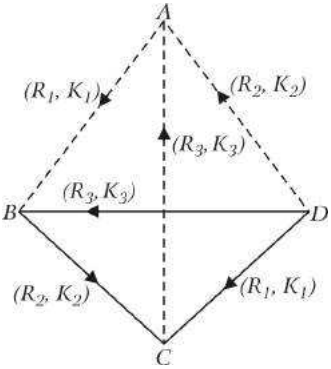
From (4), we have

$$\frac{a}{b} = \frac{Q\sin\theta}{P\cos\theta} = \frac{Q}{P}\tan\theta = \frac{Q}{P}.\frac{Q}{P}, \text{by (2)}$$

$\Rightarrow$ $$\frac{a}{b} = \frac{Q^2}{P^2}.$$

Example 4: *Wrenches of the same pitch p act along the edges of a regular tetrahedron $ABCD$ of side a. If the intensities of the wrenches along AB, DC are the same and also those along BC, DA and DB, CA, show that pitch of the equivalent wrench is* **(Avadh 2007)**

$$p + \frac{a}{2\sqrt{2}}.$$



Solution: Let the wrenches be $(R_1, K_1), (R_2, K_2)$ and (R_3, K_3) acting along the pairs of opposite edges which are at right angles as shown in the fig. such that pitch of each wrench is p so that

$$\frac{K_1}{R_1} = \frac{K_2}{R_2} = \frac{K_3}{R_3} = p. \qquad \ldots(1)$$

We know that if (R, K) be the resultant wrench of the wrenches (R_1, K_1), (R_2, K_2), then $R^2 = R_1^2 + R_2^2 + 2R_1R_2 \cos\alpha$,

α being the angle between the wrenches (R_1, K_1) and (R_2, K_2)

$$RK = R_1K_1 + R_2K_2 + (R_1K_2 + K_1R_{2)} \cos\alpha + \alpha R_1R_2 \sin\alpha$$

a being the shortest distance (S.D.) between the wrenches.

Suppose (R', K') is the resultant of (R_1, K_1) along AB and (R_1, K_1) along DC.

Then $\qquad R'^2 = R_1^2 + R_1^2 + 2R_1^2 \cos 90°$

$$R'K' = R_1K_1 + R_1K_1\,(R_1K_1 + R_1K_1)\cos 90° + \frac{aR_1^2}{\sqrt{2}} \sin 90°$$

or $\qquad R' = R_1\sqrt{2},\ R'K' = 2R_1K_1 + \dfrac{aR_1^2}{\sqrt{2}}$

or $\qquad K' = \dfrac{R'K'}{R'} = \left(2pR_1^2 + \dfrac{aR_1^2}{\sqrt{2}}\right) \cdot \dfrac{1}{R_1\sqrt{2}}$ [by (1)]

$$= \left(p\sqrt{2} + \frac{a}{2}\right)R_1.$$

Thus $\qquad R' = R_1\sqrt{2},\ K' = \left(p\sqrt{2} + \dfrac{a}{2}\right)R_1.$

Similarly if (i), (R'', K'') is the resultant of (R_2, K_2) along BC and (R_2, K_2) along DA, and (ii), (R''', K''') is the resultant of (R_3, K_3) along DB and (R_3, K_3) along CA,

then $\qquad R'' = R_2\sqrt{2}, K'' = \left(p\sqrt{2} + \dfrac{a}{2}\right)R_2$

$$R''' = R_3\sqrt{2}, K''' = \left(p\sqrt{2} + \frac{a}{2}\right)R_3\ .$$

The three resultant wrenches act along the three lines of S.D. of opposite edges of tetrahedron. These lines of S.D. are mutually perpendicular and meet in a point so that the three lines of action of resultant wrenches are also mutually perpendicular and meet in a point.

Take these three lines as co-ordinate axes.

Let (R_0K_0) be the resultant wrench of the three wrenches. Then

$$R_0^2 = R'^2 + R''^2 + R'''^2 = (R_1^2 + R_2^2 + R_3^2) \qquad \ldots(2)$$

$$K_0^2 = K'^2 + K''^2 + K'''^2$$

or $$K_0^2 = \left(p\sqrt{2} + \frac{a}{2}\right)^2 (R_1^2 + R_2^2 + R_3^2) \quad \ldots(3)$$

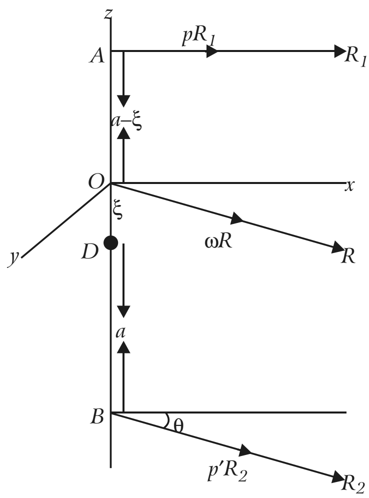
Dividing (3) by (2), $\left(\dfrac{K_0}{R_0}\right)^2 = \left(p\sqrt{2} + \dfrac{a}{2}\right)^2 \cdot \dfrac{1}{2}$

or $$\text{pitch } = \frac{K_0}{R_0} = \frac{1}{\sqrt{2}}\left(p\sqrt{2} + \frac{a}{2}\right) = p + \frac{a}{2\sqrt{2}}.$$

Example 5: *Two wrenches of pitches p, p' whose axes are at a distance $2a$ from each other, have a resultant wrench of pitch ω whose axis intersects the shortest distance between the axes of the given wrenches at a distance ξ form its middle point. Prove that the angle between the given wrenches is given by*

$$\tan\theta = \frac{\xi(p - p') - a(2\omega - p - p')}{\xi^2 - a^2 + (\omega - p)(\omega - p')}.$$

Solution: Let the two wrenches be (R_1, pR_1) and $(R_2, p' R_2)$ and let their resultant wrench be $(R, \omega R)$.



The axis of wrench (R_1, pR_1) is taken along x-axis, the axis of wrench $(R_2, p' R_2)$ makes an angle θ with x-axis such that $AB = 2a$ is S.D. between them. D is the middle point of AB.

Let $\quad OD = \xi$

Hence $\quad AO = a - \xi,$

$\quad OB = a + \xi.$

Let l, m, n be direction cosines of the line of action of $(R, \omega R)$. Resolving forces along x and y-axes,

$$Rl = R_1 + R_2 \cos\theta \quad \ldots(1)$$

$$Rm = R_2 \sin\theta \quad \ldots(2)$$

Taking moments about x- axis,

$$R\omega\, l = pR_1 + p' R_2 \cos\theta + R_2 \sin\theta.(a + \xi) \quad \ldots(3)$$

Taking moments about y-axis,

$$R\omega m = R_1(a - \xi) + p' R_2 \sin\theta - R_2 \cos\theta\,(a + \xi) \quad \ldots(4)$$

Eliminating Rl from (1) and (3),

$$(R_1 + R_2 \cos\theta)\omega - pR_1 = p' R_2 \cos\theta + R_2 \cos\theta.(a + \xi)$$

or $$(\omega - p)R_1 = R_2[(p' - \omega)\cos\theta + (a + \xi)\sin\theta] \quad \ldots(5)$$

Eliminating Rm from (2) and (4),

$$R_2 \sin\theta\omega = R_1(a - \xi) + p' R_2 \sin\theta - R_2 \cos\theta.(a + \xi)$$

or $$(\xi - a)R_1 = R_2[(p' - \omega)\sin\theta - (a + \xi)\cos\theta] \quad \ldots(6)$$

Dividing (5) by (6),

$$\frac{\omega - p}{\xi - p} = \frac{(p' - \omega)\cos\theta + (a + \xi)\sin\theta}{(p' - \omega)\sin\theta - (a + \xi)\cos\theta}$$

or $$(\omega - p)(p' - \omega)\sin\theta - (\omega - p)(a + \xi)\cos\theta$$
$$= (p' - \omega)(\xi - a)\cos\theta + (\xi^2 - a^2)\sin\theta$$

or $$\tan\theta = \frac{(p' - \omega)(\xi - a) + (\omega - p)(a + \xi)}{(\omega - p)(p' - \omega) - \xi^2 + a^2}$$

or $$\tan\theta = \frac{(p - p')\xi - a(p' - \omega + p - \omega)}{-[\xi^2 - a^2 + (\omega - p)(\omega - p')]}$$

or $$\tan\theta = \frac{(p - p')\xi - a(2\omega - p' - p}{\xi^2 - a^2 + (\omega - p)(\omega - p')}.$$

Comprehensive Exercise 2

1. Two forces P and Q are such that their central axis is given in position and the line of action of P is given. Show that the locus of the line of action of Q is a conicoid.
2. Prove that the surface, which is traced out by the axis of principal moment at points lying on a straightline which intersects at right angles the Poinsot's axis of a given system of forces, is a hyperbolic paraboloid.
3. Show that the minimum distance between two forces, which are equivalent to a given system (R, K) and which are inclined at a given angle 2α, is $\frac{2K}{R}\cot\alpha$ and the forces are then each equal to $(R/2)\sec\alpha$.
4. Forces act along the edges of a regular tetrahedron viz. P along BC and DA, Q along CA and DB, and R along AB and DC. Show that the pitch of the equivalent wrench is $\frac{1}{2\sqrt{2}}$ of the edge of tetrahedron.
5. On three given screws, whose axes are mutually perpendicular and concurrent, there act wrenches of pitches p_1, p_2, p_3 whose resultant is on a screw of given pitch p. Show that the locus of this latter screw is the hyperboloid
$$(p - p_1)x^2 + (p - p_2)y^2 + (p - p_3)z^2 + (p - p_1)(p - p_2)(p - p_3) = 0$$
the co-ordinate axes being the axes of the given screws.
6. Two wrenches of pitches p, p' have axes at a distance $2a$ form one another. If the resultant wrench is of pitch ω and its axis is equidistant from the axis of the component wrenches, show that the angle between them is
$$\tan^{-1}\frac{a(2\omega - p - p')}{a^2 - (\omega - p)(\omega - p')}.$$ **(Lucknow 2011)**
7. Show that a wrench , of which the force is R and the pitch is ωc may be replaced by forces inclined at an angle 20 to each other, the shortest distance between them being $2c$, and their magnitudes are
$$(R/2)[\sqrt{(1 + \omega\tan\theta)} \pm \sqrt{(1 - \omega\cot\theta)}]$$
8. At all points on a given straight line are drawn the axes of principal moment corresponding to any given system of forces. Show that these axes lie on a hyperbolic paraboloid and that their ends lie on another given straight line.

□

SECTION

B

DYNAMICS

Chapters

5. Central Orbits

6. Motion of a Particle in Three Dimensions
(Accleration In terms of Different Coordinate Systems)

Chapter

1

Rectilinear Motion With Variable Acceleration

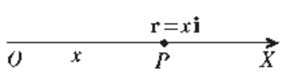
1.1 Introduction

When a point (or particle) moves along a straight line, its motion is said to be a **rectilinear motion.** **(Meerut 2004, 07, 08)**

Here in this chapter we shall discuss the motion of a point (or particle) along a straight line which may be either horizontal or vertical.

1.2 Velocity and Acceleration

Suppose a particle moves along a straight line OX where O is a fixed point on the line. Let P be the position of the particle at time t, where $OP = x$. If $\mathbf{r}$ denotes the position vector of P and $\mathbf{i}$ denotes the unit vector along OX, then $\mathbf{r} = \vec{OP} = x\mathbf{i}$.



Let $\mathbf{v}$ be the velocity vector of the particle at P. Then

$$\mathbf{v}=\frac{d\mathbf{r}}{dt}=\frac{d}{dt}(x\mathbf{i})=\frac{dx}{dt}\mathbf{i}+x\frac{d\mathbf{i}}{dt}=\frac{dx}{dt}\mathbf{i},$$

because **i** is a constant vector. Obviously, the vector **v** is collinear with the vector **i**. Thus for a particle moving along a straight line the direction of velocity is always along the line itself. If at P the particle be moving in the direction of x increasing (*i.e.*, in the direction OX) and if the magnitude of its velocity *i.e.*, its *speed* be v, we have

$$\mathbf{v}=v\,\mathbf{i}=\frac{dx}{dt}\mathbf{i}.\text{ Therefore }\frac{dx}{dt}=v.$$

On the other hand if at P the particle be moving in the direction of x decreasing (*i.e.*, in the direction XO) and if the magnitude of its velocity be v, we have

$$\mathbf{v}=-\,v\,\mathbf{i}=\frac{dx}{dt}\mathbf{i}.\text{ Therefore, }\frac{dx}{dt}=-\,v.$$

Remember: *In the case of a rectilinear motion the velocity of a particle at time t is dx/dt along the line itself and is taken with positive or negative sign according as the particle is moving in the direction of x increasing or x decreasing.*

Now let **a** be the acceleration vector of the particle at P. Then

$$\mathbf{a}=\frac{d\mathbf{v}}{dt}=\frac{d}{dt}\left(\frac{dx}{dt}\mathbf{i}\right)=\frac{d^2x}{dt^2}\mathbf{i}.$$

Thus the vector **a** is collinear with **i** *i.e.*, the direction of acceleration is always along the line itself. If at P the acceleration be acting in the direction of x increasing and if its magnitude be f, we have $\mathbf{a}=f\,\mathbf{i}=\dfrac{d^2x}{dt^2}\mathbf{i}$. Therefore $\dfrac{d^2x}{dt^2}=f$. On the other hand if at P the acceleration be acting in the direction of x decreasing and if its magnitude be f, we have

$$\mathbf{a}=-\,f\,\mathbf{i}=\frac{d^2x}{dt^2}\mathbf{i};\ \text{ therefore }\ \frac{d^2x}{dt^2}=-\,f.$$

Remember: *In the case of a rectilinear motion the acceleration of a particle at time t is d^2x/dt^2 along the line itself and is taken with positive or negative sign according as it acts in the direction of x increasing or x decreasing.*

Since the acceleration is produced by the force, therefore while considering the sign of d^2x/dt^2 we must notice the direction of the acting force and not the direction in which the particle is moving. For example if the direction of the acting force is that of x increasing, then d^2x/dt^2 must be taken with positive sign whether the particle is moving in the direction of x increasing or in the direction of x decreasing.

Other Expressions for acceleration :

Let $\quad v=\dfrac{dx}{dt}\cdot$ We can then write

$$\frac{d^2x}{dt^2}=\frac{d}{dt}\left(\frac{dx}{dt}\right)=\frac{dv}{dt}=\frac{dv}{dx}\cdot\frac{dx}{dt}=v\,\frac{dv}{dx}\cdot$$

Thus $\frac{d^2x}{dt^2}, \frac{dv}{dt}$ and $v\frac{dv}{dx}$ are three expressions for representing the acceleration and any one of them may be used to suit the convenience in working out the problems.

Note: Often we denote $\frac{dx}{dt}$ by $\dot{x}$ and $\frac{d^2x}{dt^2}$ by $\ddot{x}$.

Illustrative Examples

Example 1: *If at time t the displacement x of a particle moving away from the origin is given by $x = a \sin t + b \cos t$, find the velocity and acceleration of the particle.*

Solution: Given that $x = a \sin t + b \cos t$.

Differentiating w.r.t. 't', we get

the velocity $v = \frac{dx}{dt} = a \cos t - b \sin t$.

Differentiating again, we have

the acceleration $= \frac{dv}{dt} = -a \sin t - b \cos t = -x$.

Example 2: *A point moves in a straight line so that its distance s from a fixed point at any time t is proportional to t^n. If v be the velocity and f the acceleration at any time t, show that $v^2 = nfs/(n-1)$.*

Solution: Here, distance $s \propto t^n$.

$\therefore$ let $\quad s = kt^n, \qquad \ldots(1)$

where k is a constant of proportionality.

Differentiating (1), w.r.t. 't', we have

the velocity $v = ds/dt = knt^{n-1}$. $\qquad \ldots(2)$

Again differentiating (2),

the acceleration $f = \frac{dv}{dt} = kn(n-1)t^{n-2}$. $\qquad \ldots(3)$

$$\therefore \quad v^2 = (knt^{n-1})^2 = k^2n^2t^{2n-2}$$

$$= \frac{n \cdot \{kn(n-1)t^{n-2}\} \cdot kt^n}{(n-1)}$$

$$= \frac{nfs}{(n-1)}, \text{ substituting from (1) and (3).}$$

Example 3: *The law of motion in a straight line being given by $s = \frac{1}{2}vt$, prove that the acceleration is constant.*

Solution: We have $s = \frac{1}{2}vt = \frac{1}{2}\frac{ds}{dt}t$. $\qquad \left[\because \; v = \frac{ds}{dt}\right]$

Differentiating w.r.t. 't', we get

$$\frac{ds}{dt} = \frac{1}{2}\frac{d^2s}{dt^2}t + \frac{1}{2}\frac{ds}{dt} \quad \text{or} \quad \frac{1}{2}\frac{ds}{dt} = \frac{1}{2}\frac{d^2s}{dt^2}t \quad \text{or} \quad \frac{ds}{dt} = \frac{d^2s}{dt^2}t.$$

Differentiating again w.r.t. t, we get

$$\frac{d^2s}{dt^2} = \frac{d^2s}{dt^2} + \frac{d^3s}{dt^3}t \quad \text{or} \quad \frac{d^3s}{dt^3}t = 0 \quad \text{or} \quad \frac{d^3s}{dt^3} = 0, \text{ because } t \neq 0.$$

Now $$\frac{d^3s}{dt^3} = 0 \Rightarrow \frac{d}{dt}\left(\frac{d^2s}{dt^2}\right) = 0 \Rightarrow \frac{d^2s}{dt^2} = \text{constant}.$$

Hence the acceleration is constant.

Example 4: *Prove that if a point moves with a velocity varying as any power (not less than unity) of its distance from a fixed point which it is approaching, it will never reach that point.*

Solution: If x is the distance of the particle from the fixed point O at any time t, then its speed v at this time is given by $v = k\,x^n$, where k is a constant and n is not less than 1.

Since the particle is moving towards the fixed point *i.e.*, in the direction x decreasing, therefore

$$dx/dt = -v$$

or $$dx/dt = -kx^n. \quad \ldots(1)$$

Case I. If $n = 1$, then from (1), we have

$$dx/dt = -kx$$

or $$dt = -\frac{1}{k}\frac{dx}{x}.$$

Integrating, $t = -(1/k)\log x + A$, where A is a constant.

Putting $x = 0$, the time t to reach the fixed point O is given by

$$t = -(1/k)\log 0 + A = \infty$$

i.e., the particle will never reach the fixed point O.

Case II. If $n > 1$, then from (1), we have

$$dt = -\frac{1}{k}x^{-n}\,dx.$$

Integrating, $$t = -\frac{1}{k}\cdot\frac{x^{-n+1}}{-n+1} + B, \text{ where } B \text{ is a constant}$$

or $$t = \frac{1}{k(n-1)x^{n-1}} + B.$$

Putting $x = 0$, the time t to reach the fixed point O is given by

$$t = \infty + B = \infty$$

i.e., the particle will never reach the fixed point O.

Hence if $n \geq 1$, the particle will never reach the fixed point, it is approaching.

Example 5: *If t be regarded as a function of velocity v, prove that the rate of decrease of acceleration is given by $f^3\,(d^2t/dv^2)$, f being the acceleration.* **(Kanpur 2007)**

Solution: Let f be the acceleration at time t. Then $f = dv/dt \cdot$ Now the rate of decrease of acceleration $= -\, df/dt$

$$= -\frac{d}{dt}\left(\frac{dv}{dt}\right) = -\frac{d}{dt}\left(\frac{dt}{dv}\right)^{-1}, \text{ regarding } t \text{ as a function of } v$$

$$= -\left[\frac{d}{dv}\left(\frac{dt}{dv}\right)^{-1}\right]\cdot\frac{dv}{dt} = \left(\frac{dt}{dv}\right)^{-2}\frac{d^2t}{dv^2}\cdot\frac{dv}{dt}$$

$$= \left(\frac{dv}{dt}\right)^2\cdot\frac{dv}{dt}\cdot\frac{d^2t}{dv^2} = \left(\frac{dv}{dt}\right)^3\frac{d^2t}{dv^2} = f^3\,\frac{d^2t}{dv^2}\cdot$$

Comprehensive Exercise 1

1. A particle moves along a straight line such that its displacement x, from a point on the line at time t, is given by

   $$x = t^3 - 9t^2 + 24t + 6.$$

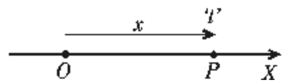
   Determine (i) the instant when the acceleration becomes zero, (ii) the position of the particle at that instant and (iii) the velocity of the particle, then.
2. A particle moves along a straight line and its distance from a fixed point on the line is given by $x = a\cos(\mu t + \varepsilon)$. Show that its acceleration varies as the distance from the origin and is directed towards the origin.
3. A particle moves along a straight line such that its distance x from a fixed point on it and the velocity v there are related by $v^2 = \mu\,(a^2 - x^2)$. Prove that the acceleration varies as the distance of the particle from the origin and is directed towards the origin.
4. The velocity of a particle moving along a straight line, when at a distance x from the origin (centre of force) varies as $\sqrt{\{(a^2 - x^2)/x^2\}}$. Find the law of acceleration.
5. A point moves in a straight line so that its distance from a fixed point in that line is the square root of the quadratic function of the time ; prove that its acceleration varies inversely as the cube of the distance from the fixed point.
6. If a point moves in a straight line in such a manner that its retardation is proportional to its speed, prove that the space described in any time is proportional to the speed destroyed in that time.
7. The velocity of a particle moving along a straight line is given by the relation $v^2 = a\,x^2 + 2\,bx + c$. Prove that the acceleration varies as the distance from a fixed point in the line.

Answers 1

1. (i) 3 seconds (ii) 24 units (iii) 3 units in the direction of x increasing.
4. Accel. varies inversely as the cube of the distance from the origin and is directed towards the origin.

1.3 Motion under Constant Acceleration

A particle moves in a straight line with a constant acceleration f, the initial velocity being u, to discuss the motion.



Suppose a particle moves in a straight line OX starting from O with velocity u. Take O as origin. Let P be the position of the particle at any time t, where $OP = x$. The acceleration of P is constant and is f. Therefore the equation of motion of P is

$$d^2x/dt^2 = f. \quad \text{...(1)}$$

If v is the velocity of the particle at any time t, then $v = dx/dt$. So integrating (1) w.r.t. t, we get

$v = dx/dt = ft + A$, where A is constant of integration.

But initially at O, $v = u$ and $t = 0$; therefore $A = u$. Thus we have

$$v = dx/dt = u + ft. \quad \text{...(2)}$$

The equation (2) gives the velocity v of the particle at any time t.

Now integrating (2) w.r.t. 't', we get

$x = ut + \frac{1}{2} ft^2 + B$, where B is a constant.

But at O, $t = 0$ and $x = 0$; therefore $B = 0$. Thus we have

$$x = ut + \frac{1}{2} ft^2. \quad \text{...(3)}$$

The equation (3) gives the position of the particle at any time t.

The equation of motion (1) can also be written as

$$v\frac{dv}{dx} = f \quad \text{or} \quad 2v\frac{dv}{dx} = 2f.$$

Integrating it w.r.t. x, we get

$v^2 = 2fx + C$. But at O, $x = 0$ and $v = u$; therefore $C = u^2$.

Hence we have

$$v^2 = u^2 + 2fx. \quad \text{...(4)}$$

Thus in equations (2), (3) and (4) we have obtained three well known formulae of rectilinear motion with constant acceleration.

1.4 Newton's Laws of Motion

The Newton's laws of motion are as follows :

Law 1: *Every body continues in its state of rest or of uniform motion in a straight line, unless it is compelled by some external force or forces to change its state.*

Law 2: *The rate of change of momentum of a body is proportional to the impressed force, and takes place in the direction in which the force acts.*

Law 3: *To every action there is an equal and opposite reaction.*

1.5 Equation of Motion

Equation of motion *of a particle moving in a straight line as deduced from the Newton's second law of motion.*

Let v be the velocity at time t of a particle of mass m moving in a straight line under the action of the impressed force P. Since from Newton's second law of motion the rate of change of momentum is proportional to the impressed force, therefore

$$P \propto \frac{d}{dt}(mv), \qquad [\because \text{ By def., momentum} = \text{mass} \times \text{velocity}]$$

or $$P = k\frac{d}{dt}(mv), \text{ where } k \text{ is some constant}$$

or $$P = k\,m\frac{dv}{dt}, \text{ provided } m \text{ is constant}$$

or $$P = kmf. \qquad \ldots(1)$$

$$[\because\ f = \text{acceleration} = dv/dt]$$

Let us suppose that a unit force is that which produces a unit acceleration in a particle of unit mass. Then

$$P = 1, \text{ when } m = 1 \text{ and } f = 1.$$

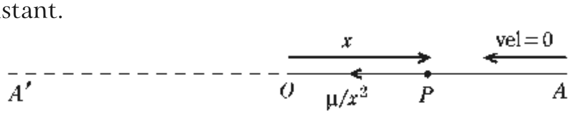
$\therefore$ from (1), we have $k = 1$.

Hence we have $P = mf$, which is the required equation of motion of the particle.

1.6 Motion Under Inverse Square Law

A particle moves in a straight line under an attraction towards a fixed point on the line, which varies inversely as the square of the distance from the fixed point. If the particle was initially at rest, to investigate the motion. **(Kanpur 2010; Garhwal 02)**

Let a particle start from rest from a point A such that $OA = a$, where O is the fixed point (*i.e.*, the centre of force) on the line and is taken as origin. Let P be the position of the particle at any time t, such that $OP = x$. Then the acceleration at $P = \mu / x^2$, towards O, where μ is a constant.



$\therefore$ The equation of motion of the particle at P is

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^2} \qquad \ldots(1)$$

[–ve sign has been taken because d^2x/dt^2 is positive in the direction of x increasing while here μ/x^2 acts in the direction of x decreasing].

Multiplying both sides of (1) by 2 (dx/dt) and then integrating w.r.t. 't', we have

$$\left(\frac{dx}{dt}\right)^2 = \frac{2\mu}{x} + A, \text{ where } A \text{ is constant of integration.}$$

But at A, $x = OA = a$ and $dx/dt = 0$.

$$\therefore \qquad 0 = \frac{2\mu}{a} + A \quad \text{or} \quad A = -\frac{2\mu}{a}.$$

$$\therefore \qquad \left(\frac{dx}{dt}\right)^2 = 2\mu\left(\frac{1}{x} - \frac{1}{a}\right), \qquad \ldots(2)$$

which gives the velocity of the particle at any distance x from the centre of force O.

From (2), we have on taking square root

$$\frac{dx}{dt} = -\sqrt{\left(\frac{2\mu}{a}\right)} \cdot \sqrt{\left(\frac{a-x}{x}\right)}.$$

[Here –ive sign is taken since the particle is moving in the direction of x decreasing].

Separating the variables, we get

$$dt = -\sqrt{\left(\frac{a}{2\mu}\right)} \cdot \sqrt{\left(\frac{x}{a-x}\right)}\, dx.$$

Integrating, $t = -\sqrt{\left(\frac{a}{2\mu}\right)} \cdot \int \sqrt{\left(\frac{x}{a-x}\right)}\, dx + B,$

where B is constant of integration.

Putting $x = a\cos^2\theta$, so that $dx = -2a\cos\theta\sin\theta\, d\theta$, we have

$$t = \sqrt{\left(\frac{a}{2\mu}\right)} \cdot \sqrt{\left(\frac{a\cos^2\theta}{a - a\cos^2\theta}\right)} \cdot 2a\sin\theta\cos\theta\, d\theta + B$$

$$= a\sqrt{\left(\frac{a}{2\mu}\right)} \cdot \int 2\cos^2\theta\, d\theta + B = a\sqrt{\left(\frac{a}{2\mu}\right)} \cdot \int (1 + \cos 2\theta)\, d\theta + B$$

$$= a\sqrt{\left(\frac{a}{2\mu}\right)} \cdot \left(\theta + \frac{\sin 2\theta}{2}\right) + B = a\sqrt{\left(\frac{a}{2\mu}\right)} (\theta + \sin\theta\cos\theta) + B$$

$$= a\sqrt{\left(\frac{a}{2\mu}\right)} [\theta + \sqrt{(1 - \cos^2\theta)} \,.\cos\theta] + B.$$

But $x = a\cos^2\theta$ means $\cos\theta = \sqrt{(x/a)}$ and $\theta = \cos^{-1}\sqrt{(x/a)}$.

$$\therefore \qquad t = a\sqrt{\left(\frac{a}{2\mu}\right)} \cdot \left[\cos^{-1}\sqrt{\left(\frac{x}{a}\right)} + \sqrt{\left(1 - \frac{x}{a}\right)} \cdot \sqrt{\left(\frac{x}{a}\right)}\ \right] + B$$

But initially at $A, t = 0$ and $x = OA = a$.

$$\therefore \qquad 0 = a\sqrt{\left(\frac{a}{2\mu}\right)} \cdot [0+0] + B \quad \text{or} \quad B = 0.$$

$$\therefore \qquad t = a\sqrt{\left(\frac{a}{2\mu}\right)} \cdot \left[\cos^{-1}\sqrt{\left(\frac{x}{a}\right)} + \sqrt{\left(1-\frac{x}{a}\right)} \cdot \sqrt{\left(\frac{x}{a}\right)}\right], \qquad \ldots(3)$$

which gives the time from the initial position A to any point distant x from the centre of force.

Putting $x = 0$ in (3), the time t_1 taken by the particle from A to O is given by

$$t_1 = a\sqrt{\left(\frac{a}{2\mu}\right)} \cdot \left[\frac{\pi}{2} + 0\right] = \frac{\pi}{2}\sqrt{\left(\frac{a^3}{2\mu}\right)}. \qquad \ldots(4)$$

Putting $x = 0$ in (2), we see that the velocity at O is infinite and therefore the particle moves to the left of O. But the acceleration of the particle is towards O, so the particle moves to the left of O under retardation which is inversely proportional to the square of the distance from O. The particle will come to instantaneous rest at A', where $OA' = OA = a$, and then retrace its path. Thus, the particle will oscillate between A and A'.

Time of one complete oscillation $= 4 \times$ (time from A to O)

$$= 4\,t_1 = 2\pi\sqrt{\left(\frac{a^3}{2\mu}\right)}.$$

1.7 Motion of a Particle Under the Attraction of the Earth

Newton's law of gravitation: When a particle moves under the attraction of the earth, the acceleration acting on it towards the centre of the earth will be as follows :

1. When the particle moves (upwards or downwards) outside the surface of the earth, the acceleration varies inversely as the square of the distance of the particle from the centre of the earth.
2. When the particle moves inside the earth through a hole made in the earth, the acceleration varies directly as the distance of the particle from the centre of the earth.
3. The value of the acceleration at the surface of the earth is g. **(Meerut 2004)**

Illustrative Examples

Example 6: *Show that the time of descent to the centre of force, varying inversely as the square of the distance from the centre, through first half of its initial distance is to that through the last half as* $(\pi + 2) : (\pi - 2)$. **(Garhwal 2003; Kanpur 08)**

Solution: Let the particle start from rest from the point A at a distance a from the centre of force O. If x is the distance of the particle from the centre of force at time t, then the equation of motion of the particle at time t is

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^2}.$$

Now proceeding as in 1.6, we find that the time t measured from the initial position $x = a$ to any point distant x from the centre O is given by the equation

$$t = \sqrt{\left(\frac{a^3}{2\mu}\right)} \cdot \left[\cos^{-1}\sqrt{\left(\frac{x}{a}\right)} + \sqrt{\left\{\frac{x}{a}\left(1 - \frac{x}{a}\right)\right\}}\right]. \quad ...(1)$$

[Give the complete proof for deducing this equation here]

Now let B be the middle point of OA. Then at B, $x = a/2$.

Let t_1 be the time from A to B *i.e.*, the time to cover the first half of the initial displacement. Then at B, $x = a/2$ and $t = t_1$. So putting $x = a/2$ and $t = t_1$ in (1), we get

$$t_1 = \sqrt{\left(\frac{a^3}{2\mu}\right)}\left[\cos^{-1}\left(\frac{1}{\sqrt{2}}\right) + \frac{1}{2}\right] = \sqrt{\left(\frac{a^3}{2\mu}\right)}\left[\frac{\pi}{4} + \frac{1}{2}\right].$$

Again let t_2 be the time from A to O. Then at O, $x = 0$ and $t = t_2$. So putting $x = 0$ and $t = t_2$ in (1), we get

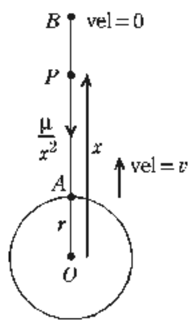
$$t_2 = \sqrt{\left(\frac{a^3}{2\mu}\right)}\,[\cos^{-1} 0 + 0] = \sqrt{\left(\frac{a^3}{2\mu}\right)} \cdot \frac{\pi}{2}.$$

Now if t_3 be the time from B to O (*i.e.*, the time to cover the last half of the initial displacement), then

$$t_3 = t_2 - t_1 = \sqrt{\left(\frac{a^3}{2\mu}\right)}\left[\frac{\pi}{4} - \frac{1}{2}\right].$$

We have $\dfrac{t_1}{t_3} = \dfrac{\frac{1}{4}\pi + \frac{1}{2}}{\frac{1}{4}\pi - \frac{1}{2}} = \dfrac{\pi + 2}{\pi - 2}$, which proves the required result.

Example 7: *If h be the height due to the velocity v at the earth's surface, supposing its attraction constant and H the corresponding height when the variation of gravity is taken into account, prove that $\dfrac{1}{h} - \dfrac{1}{H} = \dfrac{1}{r}$, where r is the radius of the earth.* **(Kanpur 2007)**



Solution: If h is the height of the particle due to the velocity v at the earth's surface, supposing its attraction constant (*i.e.*, taking the acceleration due to gravity as constant and equal to g), then from the formula $v^2 = u^2 + 2fs$, we have

$$0^2 = v^2 - 2gh.$$

$$\therefore \quad v^2 = 2gh. \quad ...(1)$$

When the variation of gravity is taken into account, let P be the position of the particle at any time t measured from the instant the particle is projected vertically upwards from the earth's surface with velocity v, and let $OP = x$.

The acceleration of the particle at P is μ/x^2 directed towards O.

$\therefore$ the equation of motion of the particle at P is

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^2}. \qquad \text{...(2)}$$

[Here the –ive sign is taken since the acceleration acts in the direction of x decreasing.]

But at A *i.e.*, on the surface of the earth,

$$x = OA = r \quad \text{and} \quad \frac{d^2x}{dt^2} = -g.$$

$\therefore$ from (2), we have $-g = -\mu/r^2$ or $\mu = gr^2$.

Substituting in (2), we have

$$\frac{d^2x}{dt^2} = -\frac{gr^2}{x^2}. \qquad \text{...(3)}$$

Multiplying both sides of (3) by 2 (dx/dt) and then integrating w.r.t. 't', we have

$$\left(\frac{dx}{dt}\right)^2 = \frac{2gr^2}{x} + A, \text{ where } A \text{ is a constant of integration.}$$

But at the point A, $x = OA = r$ and $dx/dt = v$, which is the velocity of projection at A.

$$\therefore \quad v^2 = \frac{2gr^2}{r} + A \quad \text{or} \quad A = v^2 - 2gr.$$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2 = \frac{2gr^2}{x} + v^2 - 2gr. \qquad \text{...(4)}$$

Suppose the particle in this case rises upto the point B, where $AB = H$. Then at the point B, $x = OB = OA + AB = r + H$ and $dx / dt = 0$.

$\therefore$ From (4), we have

$$0 = \frac{2gr^2}{r+H} + v^2 - 2gr$$

or

$$v^2 = -\frac{2gr^2}{r+H} + 2gr = \frac{2grH}{r+H}. \qquad \text{...(5)}$$

Equating the values of v^2 from (1) and (5), we have

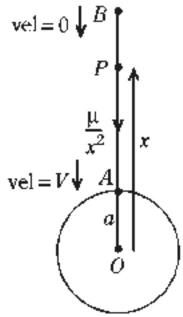
$$2gh = \frac{2grH}{r+H} \quad \text{or} \quad \frac{1}{h} = \frac{r+h}{rH}$$

or

$$\frac{1}{h} = \frac{1}{H} + \frac{1}{r} \quad \text{or} \quad \frac{1}{h} - \frac{1}{H} = \frac{1}{r}.$$

Example 8: *Assuming that a particle falling freely under gravity can penetrate the earth without meeting any resistance, show that a particle falling from rest at a distance b $(b > a)$ from the centre of the earth would on reaching the centre acquire a velocity $\sqrt{[ga(3b-2a)/b]}$ and the* *time to travel from the surface to the centre of the earth is* $\sqrt{\left(\frac{a}{g}\right)}\, sin^{-1} \sqrt{\left[\frac{b}{3b-2a}\right]}$, *where a is the radius of the earth and g is the acceleration due to gravity on the earth's surface.*

Solution: Let the particle fall from rest from the point B such that $OB = b$, where O is the centre of the earth. Let P be the position of the particle at any time t measured from the instant it starts falling from B and let $OP = x$.



Acceleration at $P = \mu / x^2$ towards O. The equation of motion of P is

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^2},$$

which holds good for the motion from B to A *i.e.*, outside the surface of the earth.

But at the point A (on the earth's surface) $x = a$ and $d^2x/dt^2 = -g$.

$$\therefore \qquad -g = -\mu/a^2 \quad \text{or} \quad \mu = a^2 g.$$

$$\therefore \qquad \frac{d^2x}{dt^2} = -\frac{a^2 g}{x^2}. \qquad \ldots(1)$$

Multiplying both sides of (1) by $2\,(dx/dt)$ and then integrating w.r.t. 't', we have

$$\left(\frac{dx}{dt}\right)^2 = \frac{2a^2 g}{x} + A,$$ where A is a constant of integration.

But at B, $x = OB = b$ and $dx/dt = 0$.

$$\therefore \qquad 0 = \frac{2a^2 g}{b} + A \quad \text{or} \quad A = -\frac{2a^2 g}{b}.$$

$$\therefore \qquad \left(\frac{dx}{dt}\right)^2 = 2a^2 g\left(\frac{1}{x} - \frac{1}{b}\right). \qquad \ldots(2)$$

If V is the velocity of the particle at the point A, then at A, $x = OA = a$ and $(dx/dt)^2 = V^2$.

$$\therefore \qquad V^2 = 2a^2 g\left(\frac{1}{a} - \frac{1}{b}\right). \qquad \ldots(3)$$

Now the particle starts moving through a hole from A to O with velocity V at A.

Let x, $(x < a)$, be the distance of the particle from the centre of the earth at any time t measured from the instant the particle starts penetrating the earth at A. The acceleration at this point will be λx towards O, where λ is constant.

The equation of motion (inside the earth) is $d^2x/dt^2 = -\lambda x$, which holds good for the motion from A to O.

At A, $x = a$ and $d^2x/dt^2 = -g$.

Therefore, $\qquad \lambda = g/a$.

$$\therefore \qquad \frac{d^2x}{dt^2} = -\frac{g}{a}x.$$

Multiplying both sides by 2 (dx/dt) and then integrating w.r.t. 't', we have

$$\left(\frac{dx}{dt}\right)^2 = -\frac{g}{a}x^2 + B, \text{where } B \text{ is a constant.} \qquad \ldots(4)$$

But at A, $x = OA = a$

and $$\left(\frac{dx}{dt}\right)^2 = V^2 = 2a^2 g\left(\frac{1}{a} - \frac{1}{b}\right), \text{from (3).}$$

$\therefore$ $$2\,a^2 g\left(\frac{1}{a} - \frac{1}{b}\right) = -\frac{g}{a}a^2 + B$$

or $$B = ag\left(\frac{3b - 2a}{b}\right).$$

Substituting the value of B in (4), we have

$$\left(\frac{dx}{dt}\right)^2 = ag\left(\frac{3b - 2a}{b}\right) - \frac{g}{a}x^2. \qquad \ldots(5)$$

Putting $x = 0$ in (5), we get the velocity on reaching the centre of the earth as

$\sqrt{[\,ga\,(3b - 2\,a)/b]}$.

Again from (5), we have

$$\left(\frac{dx}{dt}\right)^2 = \frac{g}{a}\left[a^2\,\frac{(3b - 2a)}{b} - x^2\right]$$

$$= \frac{g}{a}(c^2 - x^2), \text{ where } c^2 = \frac{a^2}{b}(3b - 2a).$$

$\therefore$ $$\frac{dx}{dt} = -\sqrt{\left(\frac{g}{a}\right)}\cdot\sqrt{(c^2 - x^2)},$$ the –ive sign being taken because the particle is moving in the direction of x decreasing

or $$dt = -\sqrt{\left(\frac{a}{g}\right)}\cdot\frac{dx}{\sqrt{(c^2 - x^2)}},$$ separating the variables.

Integrating from A to O, the required time t is given by

$$t = -\sqrt{\left(\frac{a}{g}\right)}\int_{x=a}^{0}\frac{dx}{\sqrt{(c^2 - x^2)}}$$

$$= \sqrt{\left(\frac{a}{g}\right)}\int_0^a\frac{dx}{\sqrt{(c^2 - x^2)}} = \left[\sqrt{\left(\frac{a}{g}\right)}\sin^{-1}\frac{x}{c}\right]_0^a$$

$$= \sqrt{\left(\frac{a}{g}\right)}\sin^{-1}\frac{a}{c} = \sqrt{\left(\frac{a}{g}\right)}\sin^{-1}\left[\frac{a}{a\sqrt{\{(3b - 2a)/b\}}}\right]$$

[Substituting for c]

$$= \sqrt{\left(\frac{a}{g}\right)}\sin^{-1}\sqrt{\left(\frac{b}{3b - 2a}\right)}.$$

Comprehensive Exercise 2

1. Show that the time occupied by a body, under the acceleration k/x^2 towards the origin, to fall from rest at distance a to distance x from the attracting centre can be put in the form

$$\sqrt{\left(\frac{a^3}{2k}\right)} \cdot \left[\cos^{-1}\sqrt{\left(\frac{x}{a}\right)} + \sqrt{\left\{\frac{x}{a}\left(1-\frac{x}{a}\right)\right\}}\right].$$

Prove also that the time occupied from $x = 3a/4$ to $a/4$ is one-third of the whole time of descent from a to 0.

2. If the earth's attraction vary inversely as the square of the distance from its centre and g be its magnitude at the surface, the time of falling from a height h above the surface to the surface is

$$\sqrt{\left(\frac{a+h}{2g}\right)}\left[\sqrt{\left(\frac{h}{a}\right)} + \frac{a+h}{a}\sin^{-1}\sqrt{\left(\frac{h}{a+h}\right)}\right],$$

where a is the radius of the earth.

3. A particle falls towards the earth from infinity; show that its velocity on reaching the surface of the earth is the same as that which it would have acquired in falling with constant acceleration g through a distance equal to the earth's radius.

4. A particle is shot upwards from the earth's surface with a velocity of one mile per second. Considering variations in gravity find roughly in miles the greatest height attained. Take the radius of the earth as 4000 miles where 1 mile $= 1760 \times 3$ feet.

5. A particle is projected vertically upwards from the surface of earth with a velocity just sufficient to carry it to the infinity. Prove that the time it takes to reach a height h is

$$\frac{1}{3}\sqrt{\left(\frac{2a}{g}\right)}\left[\left(1+\frac{h}{a}\right)^{3/2} - 1\right],$$

where a is the radius of the earth. **(Meerut 2004; Purvanchal 11)**

6. Calculate in miles per second the least velocity which will carry the particle from earth's surface to infinity.

7. Calculate in kilometres per second the least velocity which will carry the particle from earth's surface to infinity. Take the radius of the earth as 6380 km and $g = 9 \cdot 8 \text{ m/sec}^2$.

Answers 2

4. 84 miles approximately
6. 7 miles/sec approximately
7. $11 \cdot 18$ km / sec

1.8 Motion Under Acceleration Varying as the Distance

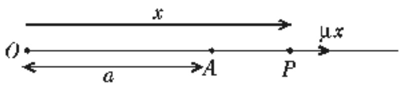
A particle moves under an acceleration varying as the distance and directed away from a fixed point, to investigate the motion.

Let O be the fixed point and x the distance of the particle from O, at any time t. Then the acceleration of the particle at this point is μx in the direction of x increasing.

$\therefore$ the equation of motion of the particle is

$$\frac{d^2x}{dt^2} = \mu x, \quad \ldots(1)$$

where the +ive sign has been taken since the acceleration acts in the direction of x increasing.



Multiplying both sides of (1) by 2 (dx/dt) and then integrating w.r.t. 't', we have

$$\left(\frac{dx}{dt}\right)^2 = \mu x^2 + A, \text{ where } A \text{ is a constant.}$$

Suppose the particle starts from rest at a distance a from O, *i.e.*, $dx/dt = 0$ at $x = a$. Then

$$0 = \mu\, a^2 + A, \quad \text{or} \quad A = -\mu\, a^2.$$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2 = \mu\,(x^2 - a^2), \quad \ldots(2)$$

which gives the velocity at any distance x from O.

From (2), on extracting square root, we have

$$\frac{dx}{dt} = \sqrt{\mu}\,\sqrt{(x^2 - a^2)}$$

[+ive sign being taken because the particle moves in the direction of x increasing]

or $$dt = \frac{1}{\sqrt{\mu}}\,\frac{dx}{\sqrt{(x^2 - a^2)}}.$$

Integrating, $t = \dfrac{1}{\sqrt{\mu}} \cosh^{-1}\dfrac{x}{a} + B.$

But when $t = 0$, $x = a$. Therefore, $B = 0$.

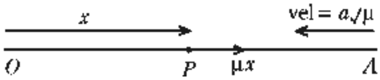
$$\therefore \quad t = \frac{1}{\sqrt{\mu}} \cosh^{-1}\frac{x}{a} \quad \text{or} \quad x = a\cosh(\sqrt{\mu}\, t), \quad \ldots(3)$$

which gives the position of the particle at time t.

Illustrative Examples

Example 9: *If a particle is projected towards the centre of repulsion varying as the distance from the centre, from a distance a from it with a velocity $a\sqrt{\mu}$, prove that the particle will approach the centre but will never reach it.*

Solution: Let the particle be projected from the point A with velocity $a\sqrt{\mu}$ towards the centre of repulsion O and let $OA = a$.



If P is the position of the particle at time t such that $OP = x$, that at P, the acceleration of the particle is μx in the direction PA.

$\therefore$ the equation of motion of the particle is

$$\frac{d^2x}{dt^2} = \mu\, x.$$

[+ive sign is taken because the acceleration is in the direction of x increasing.]

Multiplying by 2 (dx/dt) and integrating w.r.t. 't', we have

$$\left(\frac{dx}{dt}\right)^2 = \mu x^2 + C, \text{ where } C \text{ is a constant.}$$

But at A, $x = a$ and $(dx/dt)^2 = a^2\mu$. Therefore, $C = 0$.

$$\therefore \qquad \left(\frac{dx}{dt}\right)^2 = \mu x^2 \quad \text{or} \quad \frac{dx}{dt} = -\sqrt{\mu}\, x. \qquad \text{...(1)}$$

[–ive sign is taken because the particle is moving in the direction of x decreasing].

The equation (1) shows that the velocity of the particle will be zero when $x = 0$ and not before it and so the particle will approach the centre O.

From (1), we have

$$dt = -\frac{1}{\sqrt{\mu}}\frac{dx}{x}.$$

Integrating between the limits $x = a$ to $x = 0$, the time t_1 from A to O is given by

$$t_1 = -\frac{1}{\sqrt{\mu}}\int_a^0 \frac{dx}{x} = \frac{1}{\sqrt{\mu}}\left[\log x\right]_0^a = \frac{1}{\sqrt{\mu}}(\log a - \log 0) = \infty.$$

[$\because$ $\log 0 = -\infty$].

Hence the particle will take an infinite time to reach the centre O or in other words it will never reach the centre O.

1.9 Motion Under Acceleration Varying as the Cube of the Distance

A particle moves in such a way that its acceleration varies inversely as the cube of the distance from a fixed point and is directed towards the fixed point ; discuss the motion.

Let O be the fixed point and x the distance of the particle from O, at any time t. Then the equation of motion of the particle is

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^3}. \qquad \text{...(1)}$$

[The –ive sign has been taken because the force is given to be attractive.]

Multiplying both sides of (1) by 2 (dx/dt) and then integrating w.r.t. 't' we have

$$\left(\frac{dx}{dt}\right)^2 = \frac{\mu}{x^2} + A.$$

Suppose the particle starts from rest at a distance a from O, *i.e.*, $dx/dt = 0$ at $x = a$.

Then $\quad 0 = \frac{\mu}{a^2} + A \quad$ or $\quad A = -\frac{\mu}{a^2}\cdot$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2 = \mu\left(\frac{1}{x^2} - \frac{1}{a^2}\right), \qquad \ldots(2)$$

which gives the velocity at any distance x from the centre of force O.

From (2), we have

$$\frac{dx}{dt} = -\frac{\sqrt{\mu}}{a}\frac{\sqrt{(a^2 - x^2)}}{x}$$

[the –ive sign has been taken since the particle is moving in the direction of x decreasing]

or $\quad dt = -\frac{a}{\sqrt{\mu}}\cdot\frac{x\,dx}{\sqrt{(a^2 - x^2)}}$, separating the variables

$$= \frac{a}{2\sqrt{\mu}}(a^2 - x^2)^{-1/2}(-2x)\,dx.$$

Integrating, $t = \frac{a}{\sqrt{\mu}}\cdot\sqrt{(a^2 - x^2)} + B.$

But initially when $t = 0, x = a$. Therefore, $B = 0$.

$$\therefore \quad t = \frac{a}{\sqrt{\mu}}\sqrt{(a^2 - x^2)}, \qquad \ldots(3)$$

which gives the position of the particle at any time t.

Illustrative Examples

Example 10: *A particle moves in a straight line towards a centre of force* $\mu / (distance)^3$ *starting from rest at a distance a from the centre of force ; show that the time of reaching a point distant b from the centre of force is a* $\sqrt{\left(\frac{a^2 - b^2}{\mu}\right)}$, *and that its velocity then is* $\sqrt{[\mu(a^2 - b^2)]}/ab$. *Also show that the time to reach the centre is* $a^2/\sqrt{\mu}$. **(Avadh 2011)**

Solution: Let the particle start at rest from A and at time t let it be at P, where $OP = x$; O being the centre force.

Given that the acceleration at P is μ/x^3 towards O, we have

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^3}\cdot \qquad \ldots(1)$$

Multiplying both sides of (1) by 2 (dx/dt) and integrating w.r.t. 't' we have

$$\left(\frac{dx}{dt}\right)^2 = \frac{\mu}{x^2} + C.$$

When $\quad x = a, \dfrac{dx}{dt} = 0$, so that $C = -\dfrac{\mu}{a^2}\cdot$

Hence $\quad \left(\dfrac{dx}{dt}\right)^2 = \mu\left(\dfrac{1}{x^2} - \dfrac{1}{a^2}\right) = \mu\left(\dfrac{a^2 - x^2}{a^2 x^2}\right).$

$$\therefore \qquad \frac{dx}{dt} = -\frac{\sqrt{[\mu(a^2 - x^2)]}}{ax}, \qquad \ldots(2)$$

the negative sign being taken because the particle is moving towards x decreasing. Putting $x = b$ in (2), the velocity at $x = b$ is $\sqrt{[\mu(a^2 - b^2)]}/ab$, in magnitude. This proves the second result.

If t_1 is the time from $x = a$ to $x = b$, then integrating (2) after separating the variables, we get

$$t_1 = -\frac{a}{\sqrt{\mu}}\int_a^b \frac{x}{\sqrt{(a^2 - x^2)}}\,dx = \frac{a}{2\sqrt{\mu}}\int_a^b \frac{-2x}{\sqrt{(a^2 - x^2)}}\,dx$$

$$= \frac{a}{2\sqrt{\mu}}\left[2\sqrt{(a^2 - x^2)}\right]_a^b = \frac{a\sqrt{(a^2 - b^2)}}{\sqrt{\mu}}.$$

This proves the first result.

And if T be the time to reach the centre O, where $x = 0$, then

$$T = \frac{a}{2\sqrt{\mu}}\int_a^0 \frac{-2x}{\sqrt{(a^2 - x^2)}}\,dx = \frac{a}{2\sqrt{\mu}}\left[2\sqrt{(a^2 - x^2)}\right]_a^0 = \frac{a^2}{\sqrt{\mu}}\cdot$$

1.10 Motion Under Miscellaneous Laws of Forces

Now we shall give few examples in which the particle moves under different laws of acceleration.

Illustrative Examples

Example 11: *A particle whose mass is m is acted upon by a force* $m\mu\left[x + \dfrac{a^4}{x^3}\right]$ *towards origin; if it starts from rest at a distance a, show that it will arrive at origin in time* $\pi/(4\sqrt{\mu})\cdot$

(Lucknow 2010; Rohilkhand 11; Purvanchal 10)

Solution: Given $\dfrac{d^2x}{dt^2} = -\mu\left[x + \dfrac{a^4}{x^3}\right]$, $\qquad \ldots(1)$

the –ive sign being taken because the force is attractive.

Integrating it after multiplying throughout by $2\,(dx/dt)$, we get

$$\left(\frac{dx}{dt}\right)^2 = \mu\left[-x^2 + \frac{a^4}{x^2}\right] + C.$$

When $\quad x = a,\ dx/dt = 0$, so that $C = 0$.

$$\therefore \quad \left(\frac{dx}{dt}\right)^2 = \mu\left[\frac{a^4 - x^4}{x^2}\right]$$

or
$$\frac{dx}{dt} = -\frac{\sqrt{\mu}\,\sqrt{(a^4 - x^4)}}{x}, \qquad \ldots(2)$$

the –ive sign is taken because the particle is moving in the direction of x decreasing. If t_1 be the time taken to reach the origin, then integrating (2), we get

$$t_1 = -\frac{1}{\sqrt{\mu}}\int_a^0 \frac{x}{\sqrt{(a^4 - x^4)}}\,dx = \frac{1}{\sqrt{\mu}}\int_0^a \frac{x\,dx}{\sqrt{(a^4 - x^4)}}.$$

Put $x^2 = a^2 \sin\theta$ so that $2x\,dx = a^2\cos\theta\,d\theta$. When $x = 0, \theta = 0$ and when $x = a, \theta = \dfrac{\pi}{2}\cdot$

$$\therefore \quad t_1 = \frac{1}{\sqrt{\mu}}\int_0^{\pi/2}\frac{\frac{1}{2}a^2\cos\theta\,d\theta}{a^2\cos\theta} = \frac{1}{2\sqrt{\mu}}\int_0^{\pi/2}d\theta = \frac{1}{2\sqrt{\mu}}[\theta]_0^{\pi/2}$$

$$= \frac{1}{2\sqrt{\mu}}\cdot\frac{\pi}{2} = \frac{\pi}{4\sqrt{\mu}}.$$

Example 12: *Find the time of descent to the centre of force, when the force varies as (distance)$^{-5/3}$, and show that velocity at the centre is infinite.*

Solution: Let O be the centre of force taken as the origin. Suppose a particle starts at rest from A, where $OA = a$. The particle moves towards O on account of a centre of attraction at O. Let P be the position of the particle at any time t, where $OP = x$. The acceleration of the particle at P is $\mu x^{-5/3}$ directed towards O. Therefore the equation of motion of the particle at P is

$$\frac{d^2x}{dt^2} = -\mu\,x^{-5/3}. \qquad \ldots(1)$$

Multiplying both sides of (1) by $2\,(dx/dt)$ and integrating w.r.t. t, we have

$$\left(\frac{dx}{dt}\right)^2 = -\frac{2\,\mu x^{-2/3}}{-2/3} + k = \frac{3\mu}{x^{2/3}} + k, \text{ where } k \text{ is a constant.}$$

At A, $x = a$ and $dx/dt = 0$, so that $(3\mu / a^{2/3}) + k = 0$

or
$$k = -3\mu / a^{2/3}.$$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2 = \frac{3\mu}{x^{2/3}} - \frac{3\mu}{a^{2/3}} = \frac{3\mu\,(a^{2/3} - x^{2/3})}{a^{2/3}\,x^{2/3}}, \qquad \ldots(2)$$

which gives the velocity of the particle at any distance x from the centre of force O. Putting $x = 0$ in (2), we see that at O, $(dx/dt)^2 = \infty$. Therefore the velocity of the particle at the centre is infinite.

Taking square root of (2), we get

$$\frac{dx}{dt} = -\sqrt{(3\mu)}\sqrt{\left(\frac{a^{2/3} - x^{2/3}}{a^{2/3}\,x^{2/3}}\right)},$$

where the –ive sign has been taken because the particle is moving in the direction of x decreasing.

Separating the variables, we get

$$dt = -\frac{a^{1/3}}{\sqrt{(3\mu)}}\frac{x^{1/3}}{\sqrt{(a^{2/3} - x^{2/3})}}\,dx \qquad \text{...(3)}$$

Let t_1 be the time from A to O. Then at A, $t = 0$ and $x = a$ while at O, $t = t_1$ and $x = 0$. So integrating (3) from A to O, we have

$$\int_0^{t_1} dt = -\frac{a^{1/3}}{\sqrt{(3\mu)}}\int_a^0 \frac{x^{1/3}}{\sqrt{(a^{2/3} - x^{2/3})}}\,dx$$

$$= \frac{a^{1/3}}{\sqrt{(3\mu)}}\int_0^a \frac{x^{1/3}\,dx}{\sqrt{(a^{2/3} - x^{2/3})}}.$$

Put $x = a\sin^3\theta$, so that $dx = 3a\sin^2\theta\cos\theta\,d\theta$. When $x = 0, \theta = 0$ and when $x = a, \theta = \pi/2$.

$$\therefore \qquad t_1 = \frac{a^{1/3}}{\sqrt{(3\mu)}}\int_0^{\pi/2}\frac{a^{1/3}}{a^{1/3}}\frac{\sin\theta}{\cos\theta}\,3a\sin^2\theta\cos\theta\,d\theta$$

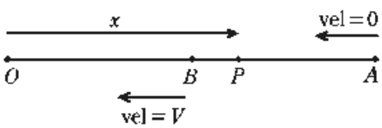
$$= \frac{3a^{4/3}}{\sqrt{(3\mu)}}\int_0^{\pi/2}\sin^3\theta\,d\theta = \frac{3a^{4/3}}{\sqrt{(3\mu)}}\frac{2}{3\cdot 1} = \frac{2a^{4/3}}{\sqrt{(3\mu)}}.$$

Hence the time of descent to the centre of force is $\dfrac{2\,a^{4/3}}{\sqrt{(3\,\mu)}}$.

Example 13: *A particle moves along the axis of x starting from rest at $x = a$. For an interval t_1 from the beginning of the motion the acceleration is $-\mu x$, for a subsequent time t_2 the acceleration is μx, and at the end of this interval the particle is at the origin; prove that*

$$\tan(\sqrt{\mu}t_1)\tanh(\sqrt{\mu}t_2) = 1.$$

Solution: Let the particle moving along the axis of x start from rest at A such that $OA = a$.



Let $-\mu x$ be the acceleration for an interval t_1 from A to B and μx that for an interval t_2 from B to O, where $OB = b$.

For motion from A to B, the equation of motion is

$$d^2x/dt^2 = -\mu x. \qquad \text{...(1)}$$

Multiplying both sides by $2\,(dx/dt)$ and then integrating w.r.t. 't', we have

$$(dx/dt)^2 = -\mu x^2 + A, \text{ where } A \text{ is a constant.}$$

But at $x = a$, $dx/dt = 0$.

$\therefore \qquad 0 = -\mu a^2 + A \quad \text{or} \quad A = \mu a^2.$

$\therefore \qquad (dx/dt)^2 = \mu (a^2 - x^2) \qquad \ldots(2)$

or $\qquad dx/dt = -\sqrt{\mu}\sqrt{(a^2 - x^2)}$ [the –ive sign is taken because the particle is moving in the direction of x decreasing]

or $\qquad dt = -\dfrac{1}{\sqrt{\mu}} \cdot \dfrac{dx}{\sqrt{(a^2 - x^2)}}$, [separating the variables].

Integrating between the limits $x = a$ to $x = b$, the time t_1 from A to B is given by

$$t_1 = -\frac{1}{\sqrt{\mu}} \int_{x=a}^{b} \frac{dx}{\sqrt{(a^2 - x^2)}} = \frac{1}{\sqrt{\mu}} \left[\cos^{-1} \frac{x}{a}\right]_{x=a}^{b}$$

$$= \frac{1}{\sqrt{\mu}} \cos^{-1} \frac{b}{a}.$$

$\therefore \qquad \cos(\sqrt{\mu} t_1) = b/a$

and $\qquad \sin\sqrt{(\mu t_1)} = \sqrt{[1 - \cos^2(\sqrt{\mu} t_1)]} = \sqrt{\left(1 - \dfrac{b^2}{a^2}\right)} = \dfrac{\sqrt{(a^2 - b^2)}}{a}.$

Dividing, $\tan(\sqrt{\mu} t_1) = \dfrac{\sqrt{(a^2 - b^2)}}{b}. \qquad \ldots(3)$

If V is the velocity at B where $x = b$, then from (2),

$$V^2 = \mu (a^2 - b^2). \qquad \ldots(4)$$

For motion from B to O, the velocity at B is V and the particle moves towards O under the acceleration μx.

$\therefore$ the equation of motion is $d^2x/dt^2 = \mu x. \qquad \ldots(5)$

Multiplying both sides by $2(dx/dt)$ and then integrating w.r.t. 't', we have

$(dx/dt)^2 = \mu x^2 + B$, where B is a constant.

But at the point B, $x = b$ and $(dx/dt)^2 = V^2 = \mu (a^2 - b^2)$.

$\therefore \qquad \mu(a^2 - b^2) = \mu b^2 + B \quad \text{or} \quad B = \mu(a^2 - 2b^2).$

$\therefore \qquad \left(\dfrac{dx}{dt}\right)^2 = \mu [x^2 + (a^2 - 2b^2)] \quad \text{or} \quad \dfrac{dx}{dt} = -\sqrt{\mu}\,[x^2 + (a^2 - 2b^2)]$

or $\qquad dt = -\dfrac{1}{\sqrt{\mu}} \dfrac{dx}{\sqrt{[x^2 + (a^2 - 2b^2)]}}.$

Integrating between the limits $x = b$ to $x = 0$, the time t_2 from B to O is given by

$$t_2 = -\frac{1}{\sqrt{\mu}} \int_{x=b}^{0} \frac{dx}{\sqrt{[x^2 + (a^2 - 2b^2)]}}$$

$$= -\frac{1}{\sqrt{\mu}} \left[\sinh^{-1} \frac{x}{\sqrt{(a^2 - 2b^2)}}\right]_b^0 = \frac{1}{\sqrt{\mu}} \sinh^{-1} \frac{b}{\sqrt{(a^2 - 2b^2)}}.$$

$\therefore \qquad \sinh(\sqrt{\mu} t_2) = \dfrac{b}{\sqrt{(a^2 - 2b^2)}}$ so that

$$\cosh(\sqrt{\mu}t_2) = \sqrt{\{1+\sinh^2(\sqrt{\mu}t_2)\}}$$

$$= \sqrt{\left(1+\frac{b^2}{a^2-2b^2}\right)} = \sqrt{\left(\frac{a^2-b^2}{a^2-2b^2}\right)}.$$

Dividing, $\tanh(\sqrt{\mu}t_2) = \dfrac{b}{\sqrt{(a^2-b^2)}}$. ...(6)

Multiplying (3) and (6), we have

$$\tan(\sqrt{\mu}t_1) \cdot \tanh(\sqrt{\mu}t_2) = 1.$$

Comprehensive Exercise 3

1. A particle moves in a straight line with an acceleration towards a fixed point in the straight line, which is equal to $\dfrac{\mu}{x^2} - \dfrac{\lambda}{x^3}$ at a distance x from the given point ; the particle starts from rest at a distance a, show that it oscillates between this distance and the distance $\dfrac{\lambda a}{2\mu a - \lambda}$ and the periodic time is $\dfrac{2\pi\mu a^3}{(2a\mu-\lambda)^{3/2}}$.

2. A particle moves in a straight line under a force to a point in it, varying as $(\text{distance})^{-4/3}$. Show that the velocity in falling from rest at infinity to a distance a is equal to that acquired in falling from rest at a distance a to a distance $a/8$.

3. A particle starts from rest at a distance a from the centre of force which attracts inversely as the distance. Prove that the time of arriving at the centre is $a\sqrt{(\pi/2\mu)}$.

   $\left(\textbf{Remember. } \int_0^\infty e^{-x^2}\,dx = \dfrac{\sqrt{\pi}}{2}\right)$.

4. A particle starts from rest at a distance $2c$ from the centre of force which attracts inversely as the distance. Find the time of arriving at the centre.

5. A particle moves in a straight line, its acceleration directed towards a fixed point O in the line and is always equal to $\mu(a^5/x^2)^{1/3}$ when it is at a distance x from O. If it starts from rest at a distance a from O, show that it will arrive at O with a velocity $a\sqrt{(6\mu)}$ after time $\dfrac{8}{15}\sqrt{\left(\dfrac{6}{\mu}\right)}$. **(Garhwal 2001)**

6. A particle starts with a given velocity v and moves under a retardation equal to k times the space described. Show that the distance traversed before it comes to rest is $v/\sqrt{k}$.

7. Assuming that at a distance x from a centre of force, the speed v of a particle, moving in a straight line is given by the equation $x = ae^{bv^2}$, where a and b are constants, find the law and the nature of the force.

8. A particle of mass m moving in a straight line is acted upon by an attractive force which is expressed by the formula $m\mu a^2/x^2$ for values of $x \geq a$, and by the formula

$m\mu x/a$ for $x \le a$, where x is the distance from a fixed origin in the line. If the particle starts at a distance $2a$ from the origin, prove that it will reach the origin with velocity $(2\mu a)^{1/2}$. Prove further that the time taken to reach the origin is

$$\left(1+\frac{3}{4}\pi\right)\sqrt{(a/\mu)}.$$

9. A particle starts from rest at a distance b from a fixed point, under the action of a force through the fixed point, the law of which at a distance x is $\mu\left(1-\frac{a}{x}\right)$ towards the point when $x > a$ but $\mu\left(\frac{a^2}{x^2}-\frac{a}{x}\right)$ from the same point when $x < a$; prove that the particle will oscillate through a space $(b^2 - a^2)/b$.

Answers 3

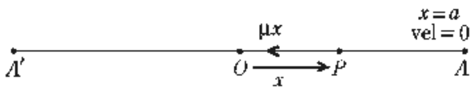
4. $c\sqrt{(2\pi/\mu)}$

7. Inversely proportional to the distance from the centre of force, repulsive or attractive according as $b >$ or < 0

1.11 Simple Harmonic Motion (S.H.M.)

Definition: *The kind of motion, in which a particle moves in a straight line in such a way that its acceleration is always directed towards a fixed point on the line (called the centre of force) and varies as the distance of the particle from the fixed point, is called* ***simple harmonic motion.***

(Avadh 2006; Lucknow 09; Meerut 04, 10; Rohilkhand 10, 11)



Let O be the centre of force taken as origin. Suppose the particle starts from rest from the point A where $OA = a$. It begins to move towards the centre of attraction O. Let P be the position of the particle after time t, where $OP = x$. By the definition of S.H.M. the magnitude of acceleration at P is proportional to x.

Let it be μx, where μ is a constant called the **intensity of force.** Also on account of a centre of attraction at O, the acceleration of P is towards O *i.e.*, in the direction of x decreasing. Therefore the equation of motion of P is

$$\frac{d^2x}{dt^2} = -\mu x, \qquad \ldots(1)$$

where the negative sign has been taken because the **force acting on P is towards** O *i.e.*, in the direction of x decreasing. The equation (1) gives the acceleration of the particle at any position.

Multiplying both sides of (1) by $2\,(dx/dt)$, we get

$$2\frac{dx}{dt}\frac{d^2x}{dt^2} = -2\mu x\frac{dx}{dt}.$$

Integrating with respect to t, we get

$$v^2 = (dx/dt)^2 = -\mu x^2 + C,$$

where C is a constant of integration and v is the velocity at P.
Initially at the point A, $x = a$ and $v = 0$; therefore $C = \mu a^2$.

Thus, we have

$$v^2 = (dx/dt)^2 = -\mu x^2 + \mu a^2$$

or $$\boldsymbol{v^2 = \mu\,(a^2 - x^2)} \qquad \ldots(2)$$

The equation (2) gives the velocity at any point P. From (2) we observe that v^2 is maximum when $x^2 = 0$ or $x = 0$. ***Thus in a S.H.M. the velocity is maximum at the centre of force O***. Let this maximum velocity be v_1. Then at O, $x = 0$, $v = v_1$. So from (2) we get

$$v_1^2 = \mu a^2 \quad \text{or} \quad v_1 = a\sqrt{\mu}.$$

Also from (2) we observe that $v = 0$ when $x^2 = a^2$ *i.e.*, $x = \pm a$. ***Thus in a S.H.M. the velocity is zero at points equidistant from the centre of force.***

Now from (2), on taking square root, we get $dx/dt = -\sqrt{\mu}\sqrt{(a^2 - x^2)}$, where the –ive sign has been taken because at P the particle is **moving** in the direction of x decreasing.
Separating the variables, we get

$$-\frac{1}{\sqrt{\mu}}\frac{dx}{\sqrt{(a^2 - x^2)}} = dt \qquad \ldots(3)$$

Integrating both sides, we get

$$\frac{1}{\sqrt{\mu}}\cos^{-1}\frac{x}{a} = t + D, \quad \text{where } D \text{ is a constant.}$$

But initially at A, $x = a$ and $t = 0$; therefore $D = 0$.
Thus we have

$$\frac{1}{\sqrt{\mu}}\cos^{-1}\frac{x}{a} = t \quad \text{or} \quad \boldsymbol{x = a\cos(\sqrt{\mu}\,t)} \qquad \textbf{(Kanpur 2008)} \ \ldots(4)$$

The equation (4) gives a relation between x and t, where t is the time measured from A. If t_1 be the time from A to O, then at O, we have $t = t_1$ and $x = 0$. So from (4), we get

$$t_1 = \frac{1}{\sqrt{\mu}}\cos^{-1} 0 = \frac{1}{\sqrt{\mu}}\frac{\pi}{2} = \frac{\pi}{2\sqrt{\mu}},$$

which is independent of the initial displacement a of the particle. *Thus in a S.H.M. the time of descent to the centre of force is independent of the initial displacement of the particle.*

Note: The time of descent t_1 from A to O can also be found from (3) with the help of the definite integrals $-\frac{1}{\sqrt{\mu}}\int_a^0 \frac{dx}{\sqrt{(a^2 - x^2)}} = \int_0^{t_1} dt$. For fixing the limits of integration, we observe that at A, $x = a$ and $t = 0$ while at O, $x = 0$ and $t = t_1$.

Nature of Motion: The particle starts from rest at A where its acceleration is maximum and is μa towards O. It begins to move towards the centre of attraction O and as it approaches the centre of force O, its velocity goes on increasing. When the particle reaches O its acceleration is zero and its velocity is maximum and is $a\sqrt{\mu}$ in the direction OA'. Due to this velocity gained at O the particle moves towards the left of O. But on account of the centre of attraction at O a force begins to act upon the particle against its direction of motion. So its velocity goes on decreasing and it comes to instantaneous rest at A' where $OA' = OA$. The rest at A' is only instantaneous. The particle at once begins to move towards the centre of attraction O and retracing its path it again comes to instantaneous rest at A. Thus the motion of the particle is **oscillatory** and it continues to oscillate between A and A'. To start from A and to come back to A is called one *complete oscillation.*

1.12 Few Important Definitions

1. Amplitude: *In a S.H.M. the distance from the centre of force of the position of maximum displacement is called amplitude of the motion.* Thus the amplitude is the distance of a position of instantaneous rest from the centre of force. In the formulae (2) and (4) of the last article the amplitude is a.

2. Time period: *In a S.H.M. the time taken to make one complete oscillation is called time period or periodic time.* Thus if T is the time period of the S.H.M., then

$$T = 4.\text{ (time from } A \text{ to } O) = 4\,.\,[\pi/2\sqrt{\mu}] = 2\pi/\sqrt{\mu},$$

which is **independent of the amplitude** a. **(Kanpur 2007, 10; Rohilkhand 10, 11)**

3. Frequency: *The number of complete oscillations in one second is called the frequency of the motion.* Since the time taken to make one complete oscillation is $2\pi/\sqrt{\mu}$ seconds, therefore if n is the frequency, then $n\,.\,(2\pi/\sqrt{\mu}) = 1$ or $n = \sqrt{\mu}/2\pi$. **(Avadh 2006)**

Thus the frequency is the reciprocal of the periodic time.

Important Remark 1: In a S.H.M. if the centre of force is not at the origin but is at the point $x = b$, then the equation of motion is $d^2x/dt^2 = -\mu\,(x - b)$. Similarly $d^2x/dt^2 = -\mu\,(x + b)$ is the equation of a S.H.M. in which the centre of force is at the point $x = -b$.

Important Remark 2: In the above article when after instantaneous rest at A' the particle begins to move towards A, we have from (2)

$$\frac{dx}{dt} = +\sqrt{\mu}\,\sqrt{(a^2 - x^2)},$$

where the +ive sign has been taken because the particle is moving in the direction of x increasing.

Separating the variables, we have

$$\frac{dx}{\sqrt{(a^2 - x^2)}} = \sqrt{\mu}\,dt.$$

Integrating, we get $-\cos^{-1}(x/a) = \sqrt{\mu}t + B$. Now the time from A to A' is $\pi/\sqrt{\mu}$. Therefore at A', we have $t = \pi/\sqrt{\mu}$ and $x = -a$.

These give $-\cos^{-1}(-a/a) = \sqrt{\mu}.(\pi/\sqrt{\mu}) + B$

or $\qquad -\cos^{-1}(-1) = \pi + B \quad$ or $\quad -\pi = \pi + B \quad$ or $\quad B = -2\pi$.

Thus we have $-\cos^{-1}(x/a) = \sqrt{\mu}t - 2\pi$

or $\qquad \cos^{-1}(x/a) = 2\pi - \sqrt{\mu}t$

or $\qquad x = a\cos(2\pi - \sqrt{\mu}t) \quad$ or $\quad x = a\cos\sqrt{\mu}t$.

Thus in S.H.M. the equation $x = a\cos\sqrt{\mu}t$ is valid throughout the entire motion from A to A' and back from A' to A.

4. Phase and Epoch. From equation (1), we have

$$\frac{d^2x}{dt^2} + \mu x = 0,$$

which is a linear differential equation with constant coefficients and its general solution is given by

$$x = a\cos\sqrt{(\mu t + \varepsilon)}. \qquad \ldots(5)$$

The constant ε is called the **starting phase** or the **epoch** of the motion and the quantity $\sqrt{\mu}t + \varepsilon$ is called the **argument** of the motion. **(Bundelkhand 2006)**

The **phase** *at any time t of a S.H.M. is the time that has elapsed since the particle passed through its extreme position in the positive direction.*

From (5), x is maximum when $\cos(\sqrt{\mu}t + \varepsilon)$ is maximum *i.e.*, when $\cos(\sqrt{\mu}t + \varepsilon) = 1$.

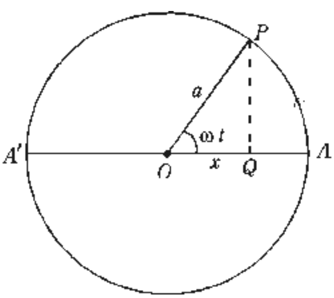
Therefore if t_1 is the time of reaching the extreme position in the positive direction, then

$$\cos(\sqrt{\mu}t_1 + \varepsilon) = 1 \quad \text{or} \quad \sqrt{\mu}t_1 + \varepsilon = 0 \quad \text{or} \quad t_1 = -\varepsilon/\sqrt{\mu}.$$

$\therefore\qquad$ **the phase at time** $t = t - t_1 = t + \dfrac{\varepsilon}{\sqrt{\mu}}$.

5. Periodic Motion. *A point is said to have a* **periodic motion** *when it moves in such a manner that after a certain fixed interval of time called periodic time it acquires the same position and moves with the same velocity in the same direction. Thus S.H.M. is a periodic motion.*

1.13 Geometrical Representation of S.H.M.



Let a particle move with a uniform angular velocity ω round the circumference of a circle of radius a. Suppose AA' is a fixed diameter of the circle. If the particle starts from A and P is its position at time t, then $\angle AOP = \omega t$.

Draw PQ perpendicular to the diameter AA'.

If $\qquad OQ = x$, then $x = a\cos\omega t. \qquad \ldots(1)$

As the particle P moves round the circumference, the foot Q of the perpendicular on the diameter AA' oscillates on AA' from A to A' and from A' to A back. Thus the motion of the point Q is periodic.

From (1), we have

$$\frac{dx}{dt} = -a\omega \sin \omega t \qquad \text{...(2)}$$

and $$\frac{d^2x}{dt^2} = -a\omega^2 \cos \omega t = -\omega^2 x. \qquad \text{...(3)}$$

The equations (2) and (3) give the velocity and acceleration of Q at any time t.

The equation (3) shows that Q executes a simple harmonic motion with centre at the origin O. From equation (1), we see that the amplitude of this S.H.M. is a because the maximum value of x is a.

The periodic time of Q = The time required by P to turn through an angle 2π with a uniform angular velocity ω

$$= 2\pi/\omega.$$

Thus if a particle describes a circle with constant angular velocity, the foot of the perpendicular from it on any diameter executes a S.H.M.

1.14 Important Results about S.H.M.

We summarize the important relations of a S.H.M. as follows : (Remember them).

(i) Referred to the centre as origin the equation of S.H.M. is $\ddot{x} = -\mu x$, or the equation $\ddot{x} = -\mu x$ represents a S.H.M. with centre at the origin.

(ii) The velocity v at a distance x from the centre and the distance x from the centre at time t are respectively given by

$$v^2 = \mu(a^2 - x^2) \quad \text{and} \quad x = a \cos \sqrt{\mu}\, t,$$

where a is the amplitude and the time t has been measured from the extreme position in the positive direction.

(iii) Maximum acceleration $= \mu a$, (at extreme points) **(Bundelkhand 2008)**

(iv) Maximum velocity $= a\sqrt{\mu}$, (at the centre)

(v) Periodic time $T = 2\pi/\sqrt{\mu}$.

(vi) Frequency $n = 1/T = \sqrt{\mu}/2\pi$.

Illustrative Examples

Example 14: *The speed v of the point P which moves in a line is given by the relation*

$$v^2 = a + 2bx - cx^2,$$

where x is the distance of the point P from a fixed point on the path, and a, b, c are constants. Show that the motion is simple harmonic if c is positive ; determine the period and the amplitude of the motion.

Solution: Here given that, $v^2 = a + 2bx - cx^2$. ...(1)

Differentiating both sides of (1) w.r.t. x, we have

$$2v\frac{dv}{dx} = 2b - 2cx.$$

$$\therefore \quad \frac{d^2x}{dt^2} = v\frac{dv}{dx} = -c\left(x - \frac{b}{c}\right). \quad ...(2)$$

Since c is positive, therefore the equation (2) represents a S.H.M. with the centre of force at the point $x = b/c$.

Hence the relation (1) represents a S.H.M. of period

$$T = \frac{2\pi}{\sqrt{\mu}} = \frac{2\pi}{\sqrt{c}},$$ because in the equation (2), $\mu = c$.

To determine the amplitude, putting $v = 0$ in (1), we have

$$a + 2bx - cx^2 = 0 \quad \text{or} \quad cx^2 - 2bx - a = 0.$$

$$\therefore \quad x = \frac{b \pm \sqrt{(b^2 + ac)}}{c}.$$

$\therefore$ the distances of the two positions of instantaneous rest A and A' from the fixed point O are given by

$$OA = \frac{b + \sqrt{(b^2 + ac)}}{c} \text{ and } OA' = \frac{b - \sqrt{(b^2 + ac)}}{c}.$$

The distance of any of these two positions from the centre $x = (b/c)$ is the amplitude of the motion.

$$\therefore \quad \text{the amplitude} = \frac{b + \sqrt{(b^2 + ac)}}{c} - \frac{b}{c} = \frac{\sqrt{(b^2 + ac)}}{c}.$$

Example 15: *In a S.H.M. of period $2\pi/\omega$ if the initial displacement be x_0 and the initial velocity u_0, prove that*

(i) *amplitude* $= \sqrt{\left(x_0^2 + \frac{u_0^2}{\omega^2}\right)}$,

(ii) *position at time* $t = \sqrt{\left(x_0^2 + \frac{u_0^2}{\omega^2}\right)} \cdot \cos\left\{\omega t - \tan^{-1}\left(\frac{u_0}{\omega x_0}\right)\right\}$,

(iii) *time to the position of rest* $= \frac{1}{\omega}\tan^{-1}\left(\frac{u_0}{\omega x_0}\right)$. **(Meerut 2007)**

Solution: We know that in a S.H.M. the time period $= 2\pi/\sqrt{\mu}$.

Since here the time period is $2\pi/\omega$, therefore $2\pi/\sqrt{\mu} = 2\pi/\omega$ *i.e.*, $\mu = \omega^2$.

Now taking the centre of the motion as origin, the equation of the given S.H.M. is

$$\frac{d^2x}{dt^2} = -\omega^2 x. \quad ...(1)$$

Multiplying (1) by $2(dx/dt)$ and integrating w.r.t. 't', we get

$$(dx/dt)^2 = -\omega^2 x^2 + A,$$ where A is a constant.

But initially at $x = x_0$, the velocity $dx/dt = u_0$.

Therefore $\quad u_0^2 = -\omega^2 x_0^2 + A$

or $\quad A = u_0^2 + \omega^2 x_0^2$.

Thus we have

$$\left(\frac{dx}{dt}\right)^2 = -\omega^2 x^2 + u_0^2 + \omega^2 x_0^2 = \omega^2 \left(x_0^2 + \frac{u_0^2}{\omega^2} - x^2\right). \qquad \ldots(2)$$

(i) Now the amplitude is the distance from the centre, of a point where the velocity is zero. Since here the centre is origin, therefore the amplitude is the value of x when velocity is zero.

Putting $\frac{dx}{dt} = 0$ in (2), we get $x = \pm \sqrt{\left(x_0^2 + \frac{u_0^2}{\omega^2}\right)}$.

Hence the required amplitude is $\sqrt{\left(x_0^2 + \frac{u_0^2}{\omega^2}\right)}$.

(ii) Assuming that the particle is moving in the direction of x increasing, we have from (2)

$$\frac{dx}{dt} = \omega \sqrt{\left\{\left(x_0^2 + \frac{u_0^2}{\omega^2}\right) - x^2\right\}}$$

or
$$dt = \frac{1}{\omega} \frac{dx}{\sqrt{\{(x_0^2 + u_0^2/\omega^2) - x^2\}}}.$$

Integrating, we get $t = -\frac{1}{\omega} \cos^{-1}\left\{\frac{x}{\sqrt{(x_0^2 + u_0^2/\omega^2)}}\right\} + B$, where B is a constant.

But initially, when $t = 0, x = x_0$.

$$\therefore \quad B = \frac{1}{\omega} \cos^{-1}\left\{\frac{x_0}{\sqrt{(x_0^2 + u_0^2/\omega^2)}}\right\} = \frac{1}{\omega} \tan^{-1}\left(\frac{u_0}{\omega x_0}\right).$$

$$\therefore \quad t = -\frac{1}{\omega} \cos^{-1}\left\{\frac{x}{\sqrt{(x_0^2 + u_0^2/\omega^2)}}\right\} + \frac{1}{\omega} \tan^{-1}\left(\frac{u_0}{\omega x_0}\right)$$

or
$$\cos^{-1}\left\{\frac{x}{\sqrt{(x_0^2 + u_0^2/\omega^2)}}\right\} = -\left\{\omega t - \tan^{-1}\left(\frac{u_0}{\omega x_0}\right)\right\}$$

or
$$\frac{x}{\sqrt{(x_0^2 + u_0^2/\omega^2)}} = \cos\left[-\left\{\omega t - \tan^{-1}\frac{u_0}{\omega x_0}\right\}\right] = \cos\left(\omega t - \tan^{-1}\frac{u_0}{\omega x_0}\right)$$

or
$$x = \sqrt{\left(x_0^2 + \frac{u_0^2}{\omega^2}\right)} \cos\left(\omega t - \tan^{-1}\frac{u_0}{\omega x_0}\right), \qquad \ldots(3)$$

which gives the position of the particle at time t.

(iii) Substituting the value of x from (3) in (2), we get

$$\left(\frac{dx}{dt}\right)^2 = \omega^2\left(x_0^2 + \frac{u_0^2}{\omega^2}\right) \cdot \sin^2\left\{\omega t - \tan^{-1}\left(\frac{u_0}{\omega x_0}\right)\right\}.$$

Putting $dx/dt = 0$, we get

$$0 = \omega^2\left(x_0^2 + \frac{u_0^2}{\omega^2}\right)\sin^2\left\{\omega t - \tan^{-1}\left(\frac{u_0}{\omega x_0}\right)\right\}$$

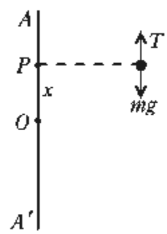
or $$\sin\left\{\omega t - \tan^{-1}\left(\frac{u_0}{\omega x_0}\right)\right\} = 0$$

or $$\omega t - \tan^{-1}\left(\frac{u_0}{\omega x_0}\right) = 0 \quad \text{or} \quad t = \frac{1}{\omega}\tan^{-1}\left(\frac{u_0}{\omega x_0}\right).$$

Hence the time to the position of rest $= \dfrac{1}{\omega}\tan^{-1}\left(\dfrac{u_0}{\omega x_0}\right)$.

Example 16: *A body is attached to one end of an inelastic string, and the other end moves in a vertical line with S.H.M. of amplitude a, making n oscillations per second. Show that the string will not remain tight during the motion unless* $n^2 < g/(4\pi^2 a)$.

Solution: Suppose the string remains tight during the motion so that the body also moves in an identical S.H.M. Let m be the mass of the body.



Let the body move in S.H.M. between A and A' and suppose O is the centre of the motion, where $OA = a$.

Since the body makes n oscillations per second, therefore its time period $\dfrac{2\pi}{\sqrt{\mu}} = \dfrac{1}{n}$.

This gives $\mu = 4\pi^2 n^2$.

At time t, let the body be in position P, where $OP = x$. The impressed force acting on the body is $T - mg$ along OP. Here T is the tension of the string. By Newton's second law of motion, the equation of motion of the body is

$$m\, d^2x/dt^2 = T - mg.$$

$\therefore$ $$T = mg + m\,(d^2x/dt^2).$$

Obviously T is least when d^2x/dt^2 is least. But the least value of d^2x/dt^2 is $-\mu a$. Hence least $T = mg - m\mu a$.

The string will remain tight if this least tension is positive *i.e.*, if $m\mu a < mg$

i.e., if $\quad m 4\pi^2 n^2 a < mg \qquad [\because \mu = 4\pi^2 n^2]$

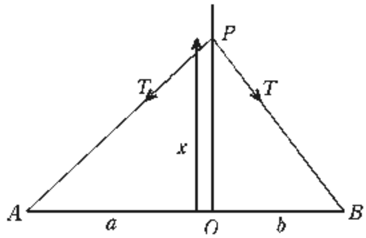
i.e., if $\quad n^2 < g/(4\pi^2 a)$.

Hence the result.

Example 17: *A particle of mass m is attached to a light wire which is stretched tightly between two fixed points with a tension T. If a, b be the distances of the particle from the two ends, prove that the period of small transverse oscillation of mass m is*

$$2\pi \sqrt{\left\{\frac{mab}{T\,(a+b)}\right\}}.$$ **(Avadh 2009)**

Solution: Let a light wire be stretched tightly between the fixed points A and B with a tension T. Let a particle of mass m be attached at the point O of the wire where $AO = a$ and $OB = b$.



Let the particle be displaced slightly perpendicular to AB (*i.e.*, in the transverse direction) and then let go. Let P be the position of the particle at any time t, where $OP = x$. Since the displacement is small, therefore the tension in the string in any displaced position can be taken as T which is the tension in the string in the original position. The equation of motion of the particle is

$$m\frac{d^2x}{dt^2} = -\,(T\cos\angle OPA + T\cos\angle OPB)$$

$$= -T\left\{\frac{OP}{AP} + \frac{OP}{BP}\right\} = -T\left\{\frac{x}{\sqrt{(a^2+x^2)}} + \frac{x}{\sqrt{(b^2+x^2)}}\right\}$$

$$= -T\left\{\frac{x}{a}\left(1+\frac{x^2}{a^2}\right)^{-1/2} + \frac{x}{b}\left(1+\frac{x^2}{b^2}\right)^{-1/2}\right\}$$

$$= -T\left[\frac{x}{a}\left(1-\frac{1}{2}\cdot\frac{x^2}{a^2}+\ldots\right)+\frac{x}{b}\left(1-\frac{1}{2}\cdot\frac{x^2}{b^2}+\ldots\right)\right]$$

$$= -T\left(\frac{x}{a}+\frac{x}{b}\right),$$ neglecting higher powers of $\frac{x}{a}$ and $\frac{x}{b}$ which are very small

$$= -T\left(\frac{a+b}{ab}\right)x.$$

$$\therefore \qquad \frac{d^2x}{dt^2} = -\frac{T\,(a+b)}{mab}\,x = -\mu x, \text{ where } \mu = \frac{T\,(a+b)}{mab}.$$

This is the standard equation of a S.H.M. with centre at the origin. The time period

$$= \frac{2\pi}{\sqrt{\mu}} = 2\pi \Big/ \sqrt{\left\{\frac{T\,(a+b)}{mab}\right\}} = 2\pi\sqrt{\left\{\frac{mab}{T\,(a+b)}\right\}}.$$

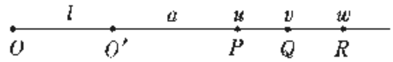
Example 18: *If in a S.H.M. u, v, w be the velocities at distances a, b, c from a fixed point on the straight line which is not the centre of force, show that the period T is given by the equation*

$$\frac{4\pi^2}{T^2}(a-b)(b-c)(c-a) = \begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}.$$

(Rohilkhand 2006; Avadh 08; Lucknow 08, 11; Kanpur 09; Purvanchal 11)

Solution: Let O and O' be the centre of force and the fixed point respectively on the line of motion and let $OO' = l$. Let u, v, w be the velocities of the particle at P, Q, R respectively where

$$O'P = a, O'Q = b, O'R = c.$$



For a S.H.M. of amplitude A, the velocity V at a distance x from the centre of force is given by

$$V^2 = \mu (A^2 - x^2). \qquad \ldots(1)$$

At P, $x = OP = l + a$, $V = u$; at Q, $x = OQ = l + b, V = v$ and

at $R, x = OR = l + c$, $V = w$.

$\therefore$ from (1), we have

$$u^2 = \mu \{A^2 - (l + a)^2\} \quad \text{or} \quad \frac{u^2}{\mu} = A^2 - l^2 - a^2 - 2al$$

or
$$\left(\frac{u^2}{\mu} + a^2\right) + 2l \,.\, a + (l^2 - A^2) = 0. \qquad \ldots(2)$$

Similarly,
$$\left(\frac{v^2}{\mu} + b^2\right) + 2l \,.\, b + (l^2 - A^2) = 0, \qquad \ldots(3)$$

and
$$\left(\frac{w^2}{\mu} + c^2\right) + 2l \,.\, c + (l^2 - A^2) = 0. \qquad \ldots(4)$$

Eliminating $2l$ and $(l^2 - A^2)$ from (2), (3) and (4), we have

$$\begin{vmatrix} \dfrac{u^2}{\mu} + a^2 & a & 1 \\ \dfrac{v^2}{\mu} + b^2 & b & 1 \\ \dfrac{w^2}{\mu} + c^2 & c & 1 \end{vmatrix} = 0$$

or
$$\begin{vmatrix} \dfrac{u^2}{\mu} & a & 1 \\ \dfrac{v^2}{\mu} & b & 1 \\ \dfrac{w^2}{\mu} & c & 1 \end{vmatrix} + \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} = 0$$

or
$$-\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} = \frac{1}{\mu} \begin{vmatrix} u^2 & a & 1 \\ v^2 & b & 1 \\ w^2 & c & 1 \end{vmatrix}$$

or $$\mu \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

or $$\mu\,(a-b)\,(b-c)\,(c-a) = \begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} \qquad \ldots(5)$$

But time period $T = \dfrac{2\pi}{\sqrt{\mu}}$, so that $\mu = \dfrac{4\pi^2}{T^2}$.

Hence from (5), we have

$$\frac{4\pi^2}{T^2}(a-b)\;(b-c)\,(c-a) = \begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}.$$

Comprehensive Exercise 4

1. Define a Simple Harmonic Motion. Show that S.H.M. is periodic and its period is independent of the amplitude. **(Bundelkhand 2006)**
2. The maximum velocity of a body moving with S.H.M. is 2 ft/sec and its period is 1/5 sec. What is its amplitude ? **(Rohilkhand 2006)**
3. At what distance from the centre the velocity in a S.H.M. will be half of the maximum ? **(Bundelkhand 2008)**
4. A particle moves in a straight line and its velocity at a distance x from the origin is $k\sqrt{(a^2 - x^2)}$, where a and k are constants. Prove that the motion is simple harmonic and find the amplitude and the periodic time of the motion.
5. Show that if the displacement of a particle in a straight line is expressed by the equation $x = a\cos nt + b\sin nt$, it describes a simple harmonic motion whose amplitude is $\sqrt{(a^2 + b^2)}$ and period is $2\pi/n$. **(Meerut 2007; Bundelkhand 09)**
6. The speed v of a particle moving along the axis of x is given by the relation $v^2 = n^2(8bx - x^2 - 12b^2)$. Show that the motion is simple harmonic with its centre at $x = 4b$, and amplitude $= 2b$.
7. Show that in a S.H.M. of amplitude a and period 'T', the velocity v at a distance x from the centre is given by the relation $v^2T^2 = 4\pi^2(a^2 - x^2)$.

   Find the new amplitude if the velocity were doubled when the particle is at a distance $\frac{1}{2}a$ from the centre, the period remaining unaltered.
8. Show that the particle executing S.H.M. requires one-sixth of its period to move

from the position of maximum displacement to one in which the displacement is half the amplitude. **(Lucknow 2006; Rohilkhand 06)**

9. A particle is performing a simple harmonic motion of period T about a centre O and it passes through a point P where $OP = b$ with velocity v in the direction OP; prove that the time which elapses before it returns to P is

$$\frac{T}{\pi}\tan^{-1}\left(\frac{vT}{2\pi b}\right).$$

(Kumaun 2001, 02; Meerut 06, 08; Purvanchal 08)

10. A point moving in a straight line with S.H.M. has velocities v_1 and v_2 when its distances from the centre are x_1 and x_2. Show that the period of motion is $2\pi\sqrt{\left(\frac{x_1^2 - x_2^2}{v_2^2 - v_1^2}\right)}$. **(Bundelkhand 2007, 09, 11; Avadh 11)**

11. A particle is moving with S.H.M. and while making an excursion from one position of rest to the other, its distances from the middle point of its path at three consecutive seconds are observed to be x_1, x_2, x_3; prove that the time of a complete oscillation is

$$\frac{2\pi}{\cos^{-1}\{(x_1 + x_3)/2x_2\}}.$$

(Kanpur 2009; Avadh 08; Bundelkhand 06)

12. **(a)** At the ends of three successive seconds the distances of a point moving with S.H.M. from the mean position measured in the same direction are 1, 5 and 5. Show that the period of a complete oscillation is $2\pi/\theta$ where $\cos\theta = 3/5$.

(b) At the ends of three successive seconds, the distances of a point moving with simple harmonic motion from its mean position measured in the same direction are 1, 3 and 4. Show that the period of complete oscillation is $2\pi / \cos^{-1}(5/6)$.

13. A body moving in a straight line OAB with S.H.M. has zero velocity when at the points A and B whose distances from O are a and b respectively, and has velocity v when half way between them. Show that the complete period is $\pi(b - a)/v$.

14. A point executes S.H.M. such that in two of its positions velocities are u, v and the two corresponding accelerations are α, β; show that the distance between the two positions is $(v^2 - u^2)/(\alpha + \beta)$ and the amplitude of the motion is $\dfrac{\{(v^2 - u^2)(\alpha^2 v^2 - \beta^2 u^2)\}^{1/2}}{\alpha^2 - \beta^2}$. **(Meerut 2009, 11)**

15. A particle rests in equilibrium under the attraction of two centres of force which attract directly as the distance, their intensities being μ and μ'; the particle is displaced slightly towards one of them, show that the time of a small oscillation is

$$2\pi/\sqrt{(\mu + \mu')}.$$

(Lucknow 07, 09; Avadh 07)

16. A horizontal shelf is moved up and down with S.H.M. of period $\frac{1}{2}$ sec. What is the amplitude admissible in order that a weight placed on the shelf may not be jerked off ? **(Lucknow 2007)**

Answers 4

2. ·064 ft nearly **3.** $a\sqrt{3}/2$ **4.** $a, 2\pi/k$

7. $\frac{1}{2}a\sqrt{13}$ **16.** $g/(16\pi^2)$

1.15 Hooke's Law

Statement: *The tension of an elastic string is proportional to the extension of the string beyond its natural length.*

If x is the stretched length of a string of natural length l, then by Hooke's law the tension T in the string is given by $T = \lambda \cdot \dfrac{x-l}{l}$, where λ is called the **modulus of elasticity** of the string.

Remember that the direction of the tension is always opposite to the extension.

Theorem: *Prove that the work done against the tension in stretching a light elastic string, is equal to the product of its extension and the mean of its final and initial tensions.*

Proof: Let $OA = a$ be the natural length of a string whose one end is fixed at O. Let the string be stretched beyond its natural length. Let B and C be the two positions of the free end A of the string during its any extension and let $OB = b$ and $OC = c$.

Then by Hooke's law,

$$\text{the tension at } B = T_B = \lambda\,\frac{b-a}{a}, \qquad ...(1)$$

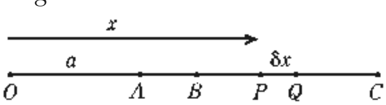
and

$$\text{the tension at } C = T_C = \lambda\,\frac{c-a}{a}, \qquad ...(2)$$

where λ is the modulus of elasticity of the string.

Now we find the work done against the tension in stretching the string from B to C.

Let P be any position of the free end of the string during its extension from B to C and let $OP = x$.



Then the tension at $P = T_P = \lambda \cdot \dfrac{x-a}{a}$.

Now suppose the free end of the string is slightly stretched from P to Q, where $PQ = \delta x$. Then the work done against the tension in stretching the string from P to Q

$$= T_P\,.\delta x = \lambda\,\frac{(x-a)}{a}\cdot \delta x.$$

$\therefore$ The work done against the tension in stretching the string from B to C

$$= \int_b^c \frac{\lambda}{a}(x-a)\,dx = \frac{\lambda}{2a}\left[(x-a)^2\right]_b^c$$

$$= \frac{\lambda}{2a}\left[(c-a)^2 - (b-a)^2\right]$$

$$= \frac{\lambda}{2a}[\{(c-a)-(b-a)\}\{(c-a)+(b-a)\}]$$

$$= (c-b)\cdot\frac{1}{2}\left[\frac{\lambda}{a}(c-a)+\frac{\lambda}{a}(b-a)\right]$$

$$= (c-b)\cdot\frac{1}{2}[T_C+T_B], \qquad \text{[from (1) and (2)]}$$

$$= BC \times (\text{mean of the tensions at } B \text{ and } C).$$

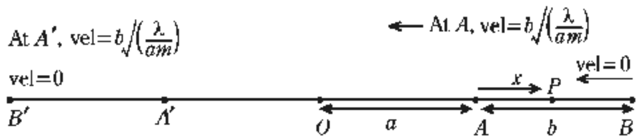
Hence, the work done against the tension in stretching the string is equal to the product of the extension and the mean of the initial and final tensions.

Now we shall discuss few simple and interesting cases of simple harmonic motion.

1.16 Particle Attached to One End of a Horizontal Elastic String

A particle of mass m is attached to one end of a horizontal elastic string whose other end is fixed to a point on a smooth horizontal table. The particle is pulled to any distance in the direction of the string and then let go ; to discuss the motion. **(Avadh 2009, 11)**

Let a string OA of natural length a lie on a smooth horizontal table. The end O of the string is attached to a fixed point of the table and a particle of mass m is attached to the other end A. The mass m is pulled upto B, where $AB = b$, and then let go.



Let P be the position of the particle after time t, where $AP = x$. The table being smooth, the only horizontal force acting on the particle at P is the tension T in the string OP. Since the direction of tension is always opposite to the extension, therefore, the force T acts in the direction PA *i.e.*, in the direction of x decreasing. Also by Hooke's law $T = \lambda\,(x/a)$. Hence the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2} = -\lambda\frac{x}{a} \quad \text{or} \quad \frac{d^2x}{dt^2} = -\frac{\lambda}{am}x. \qquad \ldots(1)$$

The equation (1) shows that the motion of the particle is simple harmonic with centre at the point A. The equation of motion (1) holds good so long as the string is stretched. Since the string becomes slack just as the particle reaches A, therefore the equation (1) holds good for the motion of the particle from B to A.

Multiplying (1) by $2\,(dx/dt)$ and integrating, we get

$$\left(\frac{dx}{dt}\right)^2 = -\frac{\lambda}{am}x^2 + C, \text{ where } C \text{ is a constant.}$$

At the point B, $x = b$ and $dx/dt = 0$; therefore $C = (\lambda/am)\,b^2$.

Thus we have $\left(\dfrac{dx}{dt}\right)^2 = \dfrac{\lambda}{am}(b^2 - x^2)$. ...(2)

This equation gives velocity in any position from B to A. Putting $x = 0$ in (2), we have the velocity at $A = \sqrt{(\lambda/am)}\, b$, in the direction AO.

The time from B to A is $\frac{1}{4}$ of the complete time period of S.H.M. whose equation is (1).

Character of the motion. The motion from B to A is simple harmonic. When the particle reaches A, the string becomes slack and the simple harmonic motion ceases. But due to the velocity gained at A the particle continues to move to the left of A. So long as the string is loose there is no force on the particle to change its velocity because the only force here is that of tension and the tension is zero so long as the string is loose. Thus the particle moves from A to A' with uniform velocity $\sqrt{(\lambda/am)}\, b$ gained by it at A. Here A' is a point on the other side of O such that $OA' = OA$. When the particle passes A' the string again becomes tight and begins to extend. The tension again comes into picture and the particle begins to move in S.H.M. But now the force of tension acts against the direction of motion of the particle. So the velocity of the particle starts decreasing and the particle comes to instantaneous rest at B', where $A'B' = AB$. The time from A' to B' is the same as that from B to A. At B' the particle at once begins to move towards A' because of the tension which attracts it towards A'. Retracing its path the particle again comes to instantaneous rest at B and thus it continues to oscillate between B and B'.

During one complete oscillation the particle covers the distance between A and B and also that between A' and B' twice while moving in S.H.M. Also it covers the distance between A and A' twice with uniform velocity $\sqrt{(\lambda/am)}\, b$. Hence the total time for one complete oscillation

= the complete time period of a S.H.M. whose equation is (1)

+ the time taken to cover the distance $4a$ with uniform velocity $\sqrt{(\lambda/am)}\, b$

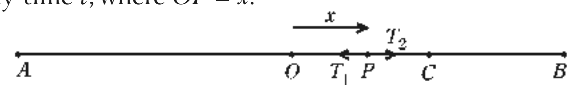
$$= \frac{2\pi}{\sqrt{(\lambda/am)}} + \frac{4a}{\sqrt{(\lambda/am)}\, b}$$

$$= 2\pi\sqrt{\left(\frac{am}{\lambda}\right)} + \frac{4a}{b}\sqrt{\left(\frac{am}{\lambda}\right)} = 2\left(\pi + \frac{2a}{b}\right)\sqrt{\left(\frac{am}{\lambda}\right)}.$$

Illustrative Examples

Example 19: *A light elastic string whose modulus of elasticity is λ is stretched to double its length and is tied to two fixed points distant $2a$ apart. A particle of mass m tied to its middle point is displaced in the line of the string through a distance equal to half its distance from the fixed points and released. Find the time of a complete oscillation and the maximum velocity acquired in the subsequent motion.*

Solution: Let an elastic string of natural length a be stretched between two fixed points A and B distant $2a$ apart, O being the middle point of AB. We have, $OA = OB = a$. Natural length of the portions OA and OB each is $a/2$ (since the string is stretched to double its length). A particle of mass m attached to the middle point O is displaced towards B upto a point C, where $OC = a/2$ and then let go. Let P be the position of the particle after any time t, where $OP = x$.



[**Note** that we have taken O as origin. The direction OP is that of x increasing and the direction PO is that of x decreasing]. At P there are two horizontal forces acting on the particle :

(i) The tension T_1 in the string AP acting in the direction PA *i.e.*, in the direction of x decreasing.

(ii) The tension T_2 in the string BP acting in the direction PB *i.e.*, in the direction of x increasing.

[**Note** that the string AP is extended in the direction AP and so the tension T_1 in it acts in the opposite direction PA].

By Hooke's law, $T_1 = \lambda \dfrac{a + x - \frac{1}{2}a}{a/2}$ and $T_2 = \lambda \dfrac{a - x - \frac{1}{2}a}{a/2}$.

Hence by Newton's second law of motion ($P = mf$), the equation of motion of the particle at P is

$$m \frac{d^2x}{dt^2} = T_2 - T_1 = \lambda \frac{a - x - \frac{1}{2}a}{a/2} - \lambda \frac{a + x - \frac{1}{2}a}{a/2} = -\frac{4\lambda x}{a}.$$

$$\therefore \qquad \frac{d^2x}{dt^2} = -\frac{4\lambda}{am} x. \qquad \ldots(1)$$

Thus the motion is S.H.M. with centre at the origin O. Since we have displaced the particle towards B only upto the point C so that the portion BC of the string is just in its natural length, therefore during the entire motion of the particle both the portions of the string remain taut and so the entire motion of the particle is governed by the above equation. Thus the particle makes oscillations in S.H.M. about O and the time period of one complete oscillation

= the time period of S.H.M. whose equation is (1)

$$= 2\pi \Big/ \sqrt{\left(\frac{4\lambda}{am}\right)} = \pi \sqrt{\left(\frac{am}{\lambda}\right)}.$$

The amplitude (*i.e.*, the maximum displacement from the centre) of this S.H.M. is $a/2$.

$\therefore$ the maximum velocity $= (\sqrt{\mu}) \times$ amplitude $= \sqrt{\left(\dfrac{4\lambda}{am}\right)} \cdot \left(\dfrac{a}{2}\right) = \sqrt{\left(\dfrac{a\lambda}{m}\right)}.$

Example 20: *Two light elastic strings are fastened to a particle of mass m and their other ends to fixed points so that the strings are taut. The modulus of each is λ, the tension T, and lengths a and b. Show that the period of an oscillation along the line of the strings is*

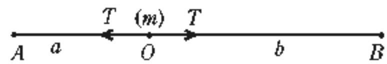
$$2\pi\left[\frac{mab}{(T+\lambda)(a+b)}\right]^{1/2}.$$

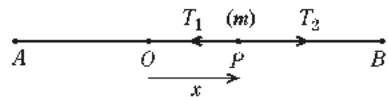
Solution: Let the two light elastic strings be fastened to a particle of mass m at O and their other ends be attached to two fixed points A and B so that the strings are taut and $OA = a, OB = b$. If l and l' are the natural lengths of the strings OA and OB respectively, then in the position of equilibrium of the particle at O,

tension in the string OA = tension in the string $OB = T$, (as given).

Applying Hooke's law, we have

$$T = \lambda\,\frac{a-l}{l} = \lambda\,\frac{b-l'}{l'}. \qquad \ldots(1)$$





From $\quad T = \lambda\,\dfrac{a-l}{l}$, we have $Tl = \lambda a - \lambda l$

i.e., $\quad l(T+\lambda) = \lambda a$, or $\dfrac{\lambda}{l} = \dfrac{T+\lambda}{a}. \quad \ldots(2)$

Similarly $\quad \dfrac{\lambda}{l'} = \dfrac{T+\lambda}{b}. \qquad \ldots(3)$

Now suppose the particle is slightly pulled towards B and then let go. It begins to move towards O. Let P be the position of the particle after any time t, where $OP = x$. The direction OP is that of x increasing and the direction PO is that of x decreasing.

At P there are two horizontal forces acting on the particle.

(i) The tension T_1 in the string AP acting in the direction PA *i.e.*, in the direction of x decreasing.

(ii) The tension T_2 in the string BP acting in the direction PB *i.e.*, in the direction of x increasing.

By Hooke's law, $\quad T_1 = \lambda\,\dfrac{a+x-l}{l},\; T_2 = \lambda\,\dfrac{b-x-l'}{l'}.$

Hence by Newton's second law of motion ($P = mf$), the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2} = T_2 - T_1 = \frac{\lambda(b-x-l')}{l'} - \frac{\lambda(a+x-l)}{l}$$

$$= -\frac{\lambda}{l'}x - \frac{\lambda}{l}x, \qquad \left[\because \text{ From (1), } \frac{\lambda(b-l')}{l'} = \frac{\lambda(a-l)}{l}\right]$$

$$= -\left[\frac{T+\lambda}{b} + \frac{T+\lambda}{a}\right]x, \qquad \text{[from (2) and (3)]}$$

$$= -\frac{(T+\lambda)(a+b)}{ab}x.$$

$$\therefore \quad \frac{d^2x}{dt^2} = -\frac{(T+\lambda)(a+b)}{mab}x, \qquad \ldots(4)$$

showing that the motion of the particle is simple harmonic with centre at the origin O.

Since we have given only a slight displacement to the particle towards B therefore during the entire motion of the particle both the strings remain taut and the entire motion of the particle is governed by the equation (4). Thus the particle makes small oscillations in S.H.M. about O and the time period of one complete oscillation

$$= \frac{2\pi}{\sqrt{\mu}} = \frac{2\pi}{\sqrt{\{(T+\lambda)(a+b)/mab\}}} = 2\pi \left[\frac{mab}{(T+\lambda)(a+b)} \right]^{1/2}.$$

Comprehensive Exercise 5

1. One end of an elastic string (modulus of elasticity λ) whose natural length is a, is fixed to a point on a smooth horizontal table and the other is tied to a particle of mass m, which is lying on the table. The particle is pulled to a distance from the point of attachment of the string equal to twice its natural length and then let go. Show that the time of a complete oscillation is $2(\pi + 2)\sqrt{(am/\lambda)}$.

2. A particle of mass m executes simple harmonic motion in the line joining the points A and B on a smooth table and is connected with these points by elastic strings whose tensions in equilibrium are each T; show that the time of an oscillation is $2\pi \{mll'/T(l+l')\}^{1/2}$, where l, l' are the extensions of the strings beyond their natural lengths. **(Garhwal 2004; Purvanchal 07; Rohilkhand 10)**

3. A particle of unit mass is tied by four elastic strings each of natural length l, and modulus of elasticity λ, to the corners of a square. If the particle is displaced a small distance towards one of the corners and then set free, prove that time of a small oscillation is

   $$\pi \sqrt{[al/\lambda(a-l)]},$$

   where a, the diagonal of the square, is so large that the string remains stretched during motion.

4. An elastic string of natural length $(a+b)$ where $a > b$ and modulus of elasticity λ has a particle of mass m attached to it at a distance a from one end, which is fixed to a point A of a smooth horizontal plane. The other end of the string is fixed to a point B so that the string is just unstretched. If the particle be held at B and then released, show that it will oscillate to and fro through a distance $b(\sqrt{a} + \sqrt{b})/\sqrt{a}$ in a periodic time

   $$\pi(\sqrt{a} + \sqrt{b})\sqrt{(m/\lambda)}.$$

1.17 Particle Suspended By an Elastic String

A particle of mass m is suspended from a fixed point by a light elastic string of natural length a and modulus of elasticity λ. The particle is pulled down a little in the line of the string and released; to discuss the motion.

Let one end of the string OA of natural length a be attached to the fixed point O and a particle of mass m be attached to the other end A. Due to the weight mg of the particle the string OA is stretched and if B is the position of equilibrium of the particle such that $AB = d$, then the tension T_B in the string will balance the weight of the particle *i.e.*,

$$mg = T_B \quad \text{or} \quad mg = \lambda \frac{AB}{OA} = \lambda \frac{d}{a}. \qquad \ldots(1)$$

The particle is pulled down to a point C such that $BC = c$ and then released. At the point C, the tension in the string is greater than the weight of the particle and so the particle starts moving vertically upwards with velocity zero at C. Let P be the position of the particle at any time t, where $BP = x$. The tension in the string when the particle is at P is $T_P = \lambda \dfrac{d+x}{a}$, acting vertically upwards.

The resultant force acting on the particle at P in the vertically upwards direction

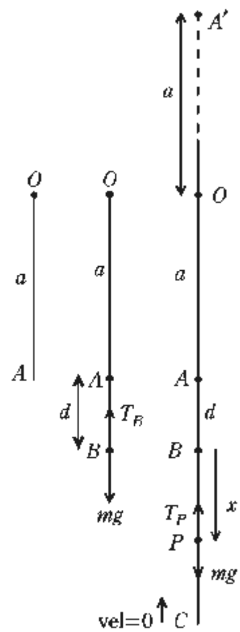
$$= T_P - mg = \lambda\left(\frac{d+x}{a}\right) - mg = \frac{\lambda d}{a} + \frac{\lambda x}{a} - mg$$

$$= \frac{\lambda x}{a}, \qquad \left[\because \frac{\lambda d}{a} = mg, \text{ from (1)}\right]$$

Also the acceleration of the particle at P is d^2x/dt^2 in the direction of x increasing *i.e.*, in the vertically downwards direction.

$\therefore$ By Newton's law, the equation of motion of P is given by

$$m\frac{d^2x}{dt^2} = -\frac{\lambda x}{a} \quad \text{or} \quad \frac{d^2x}{dt^2} = -\frac{\lambda}{am}x. \qquad \ldots(2)$$



This equation holds good so long as the tension operates *i.e.*, when the string is extended beyond its natural length.

Equation (2) is the standard equation of a S.H.M. with centre at the origin B and the amplitude of the motion is $BC = c$.

The periodic time T of the S.H.M. represented by the equation (2) is given by

$$T = 2\pi \Big/ \sqrt{\left(\frac{\lambda}{am}\right)} = 2\pi\sqrt{\left(\frac{am}{\lambda}\right)}. \qquad \ldots(3)$$

The motion of the particle remains simple harmonic as long as there is tension in the string *i.e.*, as long as the particle remains in the region from C to A.

In case the string becomes slack during the motion of the particle, the particle will begin to move freely under gravity.

Now there are two cases.

Case I. If $BC \le AB$ *i.e.*, $c \le d$. In this case the particle will not rise above A and it will come to instantaneous rest before or just reaching A. The whole motion will be S.H.M. with centre at B, amplitude BC and period T given by (3).

Case II. If $BC > AB$ *i.e.*, $c > d$. In this case the particle will rise above A, and the motion will be simple harmonic upto A and above A the particle will move freely under gravity.

Multiplying both sides of (2) by 2 (dx/dt) and then integrating, we have

$$\left(\frac{dx}{dt}\right)^2 = -\frac{\lambda}{am}x^2 + k \text{, where } k \text{ is a constant.}$$

But at C, $x = BC = c$ and $dx/dt = 0$.

$$\therefore \qquad 0 = -\frac{\lambda}{am}c^2 + k \qquad \text{or} \qquad k = \frac{\lambda}{am}c^2.$$

$$\therefore \qquad \left(\frac{dx}{dt}\right)^2 = \frac{\lambda}{am}(c^2 - x^2). \qquad \ldots(4)$$

Now if V is the velocity of the particle at A, where $x = -BA = -d$, then, from (4), we have

$$V^2 = \frac{\lambda}{am}(c^2 - d^2)$$

or

$$V = \sqrt{\left[\frac{\lambda}{am}(c^2 - d^2)\right]}, \qquad \ldots(5)$$

the direction of V being vertically upwards.

If h is the height to which the particle rises above A, then

$$h = \frac{V^2}{2g} = \frac{\lambda(c^2 - d^2)}{2\,amg}, \text{ provided } h \le 2a. \qquad \ldots(6)$$

Also in this case the maximum height attained by the particle during its entire motion

$$= CB + BA + h = c + d + h. \qquad \ldots(7)$$

If $h \le 2a$ *i.e.*, if $h \le AA'$, then the particle, after coming to instantaneous rest, will retrace its path *i.e.*, it will fall freely under gravity upto A and below A it will move in S.H.M. till it comes to instantaneous rest at C.

If $h = 2a = AA'$, then the particle will just come to rest at A' and will then move downwards, retracing its path.

In this case the maximum height attained by the particle

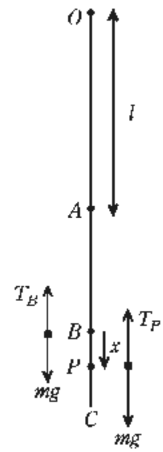
$$= c + d + 2a. \qquad \ldots(8)$$

If $h > 2a$ *i.e.*, if $h > AA'$, then the particle will rise above A' also and so the string will again become stretched and the particle will again begin to move in simple harmonic motion. After coming to instantaneous rest the particle will retrace its path.

Illustrative Examples

Example 21: *An elastic string without weight of which the unstretched length is l and modulus of elasticity is the weight of n gram is suspended by one end and a mass m gram is attached to the other end. Show that the time of a small vertical oscillation is* $2\pi\sqrt{(ml/ng)}$.

(Rohilkhand 2007)



Solution: $OA = l$ is the natural length of a string whose one end is fixed at O. B is the position of equilibrium of a particle of mass m gm. attached to the other end of the string. Considering the equilibrium of the particle at B, we have mg = the tension T_B in the string OB.

$$\therefore \qquad mg = ng\,\frac{AB}{l}, \qquad \ldots(1)$$

because modulus of elasticity of the string is given to be ng.

Now suppose the particle is pulled slightly upto C (so that $BC < AB$), and then let go. It starts moving vertically upwards with velocity zero at C. Let P be its position at any point t, where $BP = x$. The direction BP is that of x increasing and the direction PB is that of x decreasing. At P there are two forces acting on the particle :

(i) The weight mg acting vertically downwards *i.e.*, in the direction of x increasing,

(ii) the tension $T_P = ng\,\dfrac{AB + x}{l}$ in the string OP, acting vertically upwards *i.e.*, in the direction of x decreasing.

Hence by Newton's second law of motion, the equation of motion of the particle at P is

$$m\,\frac{d^2x}{dt^2} = mg - ng\,\frac{AB + x}{l}$$

$$= mg - ng\,\frac{AB}{l} - ng\,\frac{x}{l} = -\,ng\,\frac{x}{l}, \qquad \left[\because \text{ from (1)}, mg = ng\,\frac{AB}{l}\right]$$

$$\therefore \qquad \frac{d^2x}{dt^2} = -\,\frac{ng}{lm}\,x, \qquad \ldots(2)$$

which is the equation of a simple harmonic motion with centre at the origin B and amplitude BC.

Since $BC < AB$, therefore during the entire motion of the particle the string will not become slack.

Thus the entire motion of the particle is governed by the equation (2) and the particle will make oscillations in simple harmonic motion about the centre B.

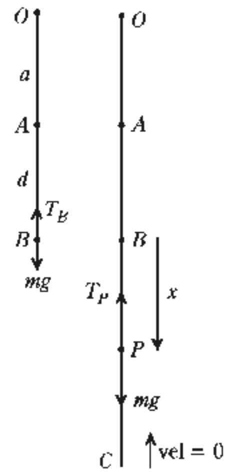
The time of one oscillation $= \dfrac{2\pi}{\sqrt{\mu}} = \dfrac{2\pi}{\sqrt{(ng/lm)}} = 2\pi\sqrt{\left(\dfrac{lm}{ng}\right)}$.

Example 22: *A heavy particle is attached to one end of an elastic string, the other end of which is fixed. The modulus of elasticity of the string is equal to the weight of the particle. The string is* *drawn vertically down till it is four times its natural length and then let go. Show that the particle will return to this point in time* $\sqrt{\left(\frac{a}{g}\right)}\left[\frac{4\pi}{3}+2\sqrt{3}\right]$, *where a is the natural length of the string.*

(Meerut 2004)

Solution: Let $OA = a$ be the natural length of an elastic string whose one end is fixed at O. Let B be the position of equilibrium of a particle of mass m attached to the other end of the string and let $AB = d$. If T_B is the tension in the string OB, then by Hooke's law,

$$T_B = \lambda \frac{OB - OA}{OA} = \lambda \frac{d}{a},$$



where λ is the modulus of elasticity of the string. Considering the equilibrium of the particle at B, we have

$$mg = T_B = \lambda \frac{d}{a} = mg \frac{d}{a} \cdot \quad [\because \lambda = mg, \text{ as given}]$$

$\therefore \qquad d = a.$

Now the particle is pulled down to a point C such that $OC = 4a$ and then let go. It starts moving towards B with velocity zero at C. Let P be the position of the particle at time t, where $BP = x$. [Note that we have taken the position of equilibrium B as origin. The direction BP is that of x increasing and the direction PB is that of x decreasing.]

When the particle is at P, there are two forces acting upon it:

(i) The tension $T_P = \lambda \frac{a+x}{a} = \frac{mg}{a}(a+x)$ in the string OP acting in the direction PO *i.e.*, in the direction of x decreasing.

(ii) The weight mg of the particle acting vertically downwards *i.e.*, in the direction of x increasing.

Hence by Newton's second law of motion ($P = mf$), the equation of motion of the particle at P is

$$m \frac{d^2x}{dt^2} = mg - \frac{mg}{a}(a+x) = -\frac{mgx}{a} \cdot$$

Thus $$\frac{d^2x}{dt^2} = -\frac{g}{a}x, \qquad \ldots(1)$$

Which is the equation of a S.H.M. with centre at the origin B and the amplitude $BC = 2a$ which is greater than $AB = a$.

Multiplying both sides of (1) by $2(dx/dt)$ and integrating w.r.t. t, we have

$$\left(\frac{dx}{dt}\right)^2 = -\frac{g}{a}x^2 + k, \text{ where } k \text{ is a constant.}$$

At the point C, $x = BC = 2a$, and the velocity $dx/dt = 0$.

$\therefore \qquad k = (g/a) \cdot 4a^2.$

$$\therefore \qquad \left(\frac{dx}{dt}\right)^2 = \frac{g}{a}(4a^2 - x^2). \qquad \ldots(2)$$

Taking square root of (2), we have

$$\frac{dx}{dt} = -\sqrt{\left(\frac{g}{a}\right)}\ \sqrt{(4a^2 - x^2)},$$

the –ive sign has been taken because the particle is moving in the direction of x decreasing.

Separating the variables, we have

$$dt = -\sqrt{\left(\frac{a}{g}\right)}\frac{dx}{\sqrt{(4a^2 - x^2)}}. \qquad \ldots(3)$$

If t_1 be the time from C to A, then integrating (3) from C to A, we have

$$\int_0^{t_1} dt = -\sqrt{\left(\frac{a}{g}\right)}\int_{2a}^{-a}\frac{dx}{\sqrt{(4a^2 - x^2)}}$$

or

$$t_1 = \sqrt{\left(\frac{a}{g}\right)}\left[\cos^{-1}\frac{x}{2a}\right]_{2a}^{-a}$$

$$= \sqrt{\left(\frac{a}{g}\right)}\left[\cos^{-1}\left(-\frac{1}{2}\right) - \cos^{-1}(1)\right] = \sqrt{\left(\frac{a}{g}\right)}\cdot\frac{2\pi}{3}.$$

Let v_1 be the velocity of the particle at A. Then at A

$$x = -a \text{ and } (dx/dt)^2 = v_1^2.$$

So from (2), we have

$$v_1^2 = (g/a)(4a^2 - a^2)$$

or $\quad v_1 = \sqrt{(3ag)}$, the direction of v_1 being vertically upwards.

Thus the velocity at A is $\sqrt{(3ag)}$ and is in the upwards direction so that the particle rises above A. Since the tension of the string vanishes at A, therefore at A the simple harmonic motion ceases and the particle when rising above A moves freely under gravity. Thus the particle rising from A with velocity $\sqrt{(3ag)}$ moves upwards till this velocity is destroyed. The time t_2 for this motion is given by

$$0 = \sqrt{(3ag)} - gt_2, \text{ so that } t_2 = \sqrt{\left(\frac{3a}{g}\right)}.$$

Conditions being the same, the equal time t_2 is taken by the particle in falling freely back to A. From A to C the particle will take the same time t_1 as it takes from C to A. Thus the whole time taken by the particle to return to C

$$= 2(t_1 + t_2) = 2\left[\sqrt{\left(\frac{a}{g}\right)}\cdot\frac{2\pi}{3} + \sqrt{\left(\frac{3a}{g}\right)}\right] = \sqrt{\left(\frac{a}{g}\right)}\left[\frac{4\pi}{3} + 2\sqrt{3}\right].$$

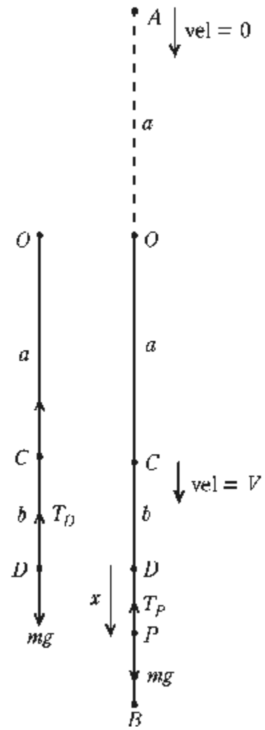
Example 23: *A particle of mass m is attached to one end of an elastic string of natural length a and modulus of elasticity 2 mg whose other end is fixed at O. The particle is let fall from A, where A is vertically above O and OA = a. Show that its velocity will be zero at B, where OB = 3a. Calculate also the time from A to B.*

Solution: Let $OC = a$ be the natural length of an elastic string suspended from the fixed point O. The modulus of elasticity λ of the string is given to be equal to $2\ mg$, where m is the mass of the particle attached to the other end of the string.

If D is the position of equilibrium of the particle such that $CD = b$, then at D the tension T_D in the string OD balances the weight of the particle.

$$\therefore \qquad mg = T_D = \lambda \frac{b}{a} = 2\ mg \frac{b}{a} \quad \text{or} \quad b = \frac{a}{2}.$$

The particle is let fall from A where $OA = a$. Then the motion from A to C will be freely under gravity.



If V is the velocity of the particle gained at the point C, then

$$V^2 = 0 + 2g \cdot 2a \quad \text{or} \quad V = 2\sqrt{(ag)}, \qquad \ldots(1)$$

in the downward direction.

As the particle moves below C, the string begins to extend beyond its natural length and the tension begins to operate. The velocity of the particle continues increasing upto D after which it starts decreasing.

Suppose that the particle comes to instantaneous rest at B. During the motion below C, let P be the position of the particle at any time t, where $DP = x$. If T_P is the tension in the string OP, we have

$$T_P = \lambda \frac{b+x}{a}, \text{ acting vertically upwards.}$$

$\therefore$ The equation of motion of the particle at P is

$$m \frac{d^2x}{dt^2} = mg - T_P = mg - \lambda \cdot \frac{b+x}{a}$$

$$= mg - 2\ mg \cdot \frac{\frac{1}{2}a + x}{a} = -\frac{2mg}{a} x$$

$$\text{or} \qquad \frac{d^2x}{dt^2} = -\frac{2g}{a} x, \qquad \ldots(2)$$

which represents a S.H.M. with centre at D and holds good for the motion from C to B.

Multiplying both sides of (2) by $2\ (dx/dt)$ and then integrating, we have

$$\left(\frac{dx}{dt}\right)^2 = -\frac{2g}{a} x^2 + k, \text{ where } k \text{ is a constant.}$$

But at C, $x = -DC = -b = -a/2$ and $(dx/dt)^2 = V^2 = 4ag$.

$$\therefore \qquad 4ag = -\frac{2g}{a} \cdot \frac{a^2}{4} + k \quad \text{or} \quad k = \frac{9}{2} ag.$$

$$\therefore \qquad \left(\frac{dx}{dt}\right)^2 = -\frac{2g}{a} x^2 + \frac{9}{2} ag$$

or $$\left(\frac{dx}{dt}\right)^2 = \frac{2g}{a}\left(\frac{9}{4}a^2 - x^2\right) \qquad \ldots(3)$$

If the particle comes to instantaneous rest at B where $DB = x_1$, (say), then at B, $x = x_1$ and $dx/dt = 0$. Therefore from (3), we have

$$0 = \frac{2g}{a}\left(\frac{9}{4}a^2 - x_1^2\right), \text{ giving } x_1 = \frac{3}{2}a.$$

Now $$OB = OC + CD + DB = a + \frac{1}{2}a + \frac{3}{2}a = 3a,$$

which proves the first part of the question.

To find the time from A to B:

If t_1 is the time from A to C, then from $s = ut + \frac{1}{2}ft^2$, $2a = 0 + \frac{1}{2}gt_1^2$.

$\therefore$ $$t_1 = 2\sqrt{(a/g)}. \qquad \ldots(4)$$

Now from (3), we have

$$\frac{dx}{dt} = \sqrt{\left(\frac{2g}{a}\right)}\sqrt{\left(\frac{9}{4}a^2 - x^2\right)},$$

the +ive sign has been taken because the particle is moving in the direction of x increasing

or $$dt = \sqrt{\left(\frac{a}{2g}\right)} \cdot \frac{dx}{\sqrt{\left(\frac{9}{4}a^2 - x^2\right)}}.$$

Integrating from C to B, the time t_2 from C to B is given by

$$t_2 = \sqrt{\left(\frac{a}{2g}\right)}\int_{x=-a/2}^{3a/2} \frac{dx}{\sqrt{\left(\frac{9}{4}a^2 - x^2\right)}}$$

$$= \sqrt{\left(\frac{a}{2g}\right)} \cdot \left[\sin^{-1}\left(\frac{x}{3a/2}\right)\right]_{-a/2}^{3a/2}$$

$$= \sqrt{\left(\frac{a}{2g}\right)} \cdot \left[\sin^{-1} 1 - \sin^{-1}\left(-\frac{1}{3}\right)\right]$$

$$= \sqrt{\left(\frac{a}{2g}\right)} \cdot \left[\frac{\pi}{2} + \sin^{-1}\left(\frac{1}{3}\right)\right].$$

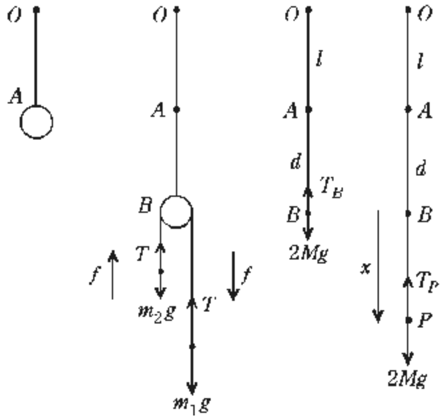
$\therefore$ the time from A to $B = t_1 + t_2$

$$= 2\sqrt{(a/g)} + \sqrt{(a/2g)} \cdot [\pi/2 + \sin^{-1}(1/3)]$$

$$= \frac{1}{2}\sqrt{\left(\frac{a}{2g}\right)}\left[4\sqrt{2} + \pi + 2\sin^{-1}\left(\frac{1}{3}\right)\right].$$

Example 24: *A smooth light pulley is suspended from a fixed point by a spring of natural length l and modulus of elasticity ng. If masses m_1 and m_2 hang at the ends of a light inextensible string* *passing round the pulley, show that the pulley executes simple harmonic motion about a centre whose depth below the point of suspension is* $l\,\{1+(2M/n)\}$, *where* M *is the harmonic mean between* m_1 *and* m_2 .

Solution: Let a smooth **light** pulley be suspended from a fixed point O by a spring OA of natural length l and modulus of elasticity $\lambda = ng$. Let B be the position of equilibrium of the pulley when masses m_1 and m_2 hang at the ends of a light inextensible string passing round the pulley. Let T be the tension in the inextensible string passing round the pulley. Let us first find the value of T.



Let f be the common acceleration of the particles m_1 , m_2 which hang at the ends of a light inextensible string passing round the pulley. If $m_1 > m_2$, then the equations of motion of m_1 , m_2 are

$$m_1 g - T = m_1 f$$

and $$T - m_2 g = m_2 f.$$

Solving, we have

$$T = \frac{2\, m_1\, m_2}{(m_1 + m_2)} \cdot g = Mg,$$

where $$M = \frac{2\, m_1\, m_2}{m_1 + m_2} = \text{the harmonic mean between } m_1 \text{ and } m_2 .$$

Now the pressure on the pulley $= 2\,T = 2\,Mg$ and therefore the pulley, which itself is light, behaves like a particle of mass $2M$.

Now the problem reduces to the vertical motion of a mass $2M$ attached to the end A of the spring OA whose other end is fixed at O. If B is the equilibrium position of the mass $2M$ and $AB = d$, then the tension T_B in the spring OB is $\lambda\,(d/l)$, acting vertically upwards.

For equilibrium of the pulley of mass $2M$ at the point B, we have

$$2Mg = T_B = \lambda\,\frac{d}{l} = ng\,\frac{d}{l}$$

or $$d = \frac{2Ml}{n}. \qquad \ldots(1)$$

Now let the particle of mass $2M$ be slightly pulled down and then let go. If P is the position of this particle at time t such that $BP = x$, then the tension in the spring OP

$$= T_P = \lambda\,\frac{d+x}{l} = ng \cdot \frac{d+x}{l}, \text{ acting vertically upwards.}$$

$\therefore$ The equation of motion of the pulley is given by

$$2M\,\frac{d^2x}{dt^2} = 2Mg - T_P$$

$$= 2Mg - ng\,\frac{d+x}{l}$$

$$= 2\ Mg - ng\,\frac{d}{l} - \frac{ng}{l}\,x = -\,\frac{ng}{l}\,x. \qquad \text{[By (1)]}$$

$$\therefore \qquad \frac{d^2x}{dt^2} = -\,\frac{ng}{2\ Ml}\,x,$$

which represents a simple harmonic motion about the centre B.

Hence the pulley executes simple harmonic motion with centre at the point B whose depth below the point of suspension O is given by

$$OB = OA + AB = l + d = l + \frac{2Ml}{n} = l\left(1 + \frac{2M}{n}\right).$$

Comprehensive Exercise 6

1. A light elastic string of natural length l is hung by one end and to the other end are tied successively particles of masses m_1 and m_2. If t_1 and t_2 be the periods and c_1, c_2 the statical extensions corresponding to these two weights, prove that $g\,(t_1^2 - t_2^2) = 4\,\pi^2\,(c_1 - c_2)$. **(Bundelkhand 2010)**
2. A mass m hangs from a light spring and is given a small vertical displacement. If l is the length of the spring when the system is in equilibrium and n the number of oscillations per second, show that the natural length of the spring is $l - (\,g/4\pi^2 n^2)$.
3. A heavy particle attached to a fixed point by an elastic string hangs freely, stretching the string by a quantity e. It is drawn down by an additional distance f and then let go ; determine the height to which it will arise if $f^2 - e^2 = 4\,ae$, a being the unstretched length of the string.
4. A heavy particle is attached to one point of a uniform elastic string. The ends of the string are attached to two points in a vertical line. Show that the period of a vertical oscillation in which the string remains taut is $2\pi\,\sqrt{(mh/2\,\lambda)}$, where λ is the coefficient of elasticity of the string and h the harmonic mean of the unstretched lengths of the two parts of the string.
5. A light elastic string of natural length l has one extremity fixed at a point O and the other attached to a stone, the weight of which in equilibrium would extend the string to a length l_1. Show that if the stone be dropped from rest at O, it will come to instantaneous rest at a depth $\sqrt{(l_1^2 - l^2)}$ below the equilibrium position. **(Lucknow 2006, 11)**
6. A light elastic string whose natural length is a has one end fixed to a point O, and to the other end is attached a weight which in equilibrium would produce an extention e. Show that if the weight be let fall from rest at O, it will come to stay instantaneously at a point distant $\sqrt{(2ae + e^2)}$ below the position of equilibrium. **(Kumaun 2003)**
7. A light elastic string of natural length a has one extremity fixed at a point O and

the other attached to a body of mass m. The equilibrium length of the string with the body attached is $a \sec \theta$. Show that if the body be dropped from rest at O it will come to instantaneous rest at a depth $a \tan \theta$ below the position of equilibrium.

8. A heavy particle of mass m is attached to one end of an elastic string of natural length l, whose other end is fixed at O. The particle is then let fall from rest at O. Show that, part of the motion is simple harmonic, and that, if the greatest depth of the particle below O is $l \cot^2 \frac{1}{2}\theta$, the modulus of elasticity of the string is $\frac{1}{2} mg \tan^2 \theta$.

9. One end of a light elastic string of natural length a and modulus of elasticity $2mg$ is attached to a fixed point A and the other end to a particle of mass m. The particle initially held at rest at A, is let fall. Show that the greatest extension of the string is $a(1+\sqrt{5})/2$ during the motion and show that the particle will reach back A again after a time $(\pi + 2 - \tan^{-1} 2)\sqrt{(2a/g)}$.

10. A light elastic string AB of length l is fixed at A and is such that if a weight W be attached to B, the string will be stretched to a length $2l$. If a weight $\frac{1}{4}W$ be attached to B and let fall from the level of A, prove that (i) the amplitude of the S.H.M. that ensues is $\frac{3l}{4}$; (ii) the distance through which it falls is $2l$; and (iii) the period of oscillation is

$$\sqrt{(l/4g)}\left(4\sqrt{2} + \pi + 2\sin^{-1}\frac{1}{3}\right).$$

11. A heavy particle of mass m is attached to one end of an elastic string of natural length l ft., whose modulus of elasticity is equal to the weight of the particle and the other end is fixed at O. The particle is let fall from O. Show that a part of the motion is simple harmonic and that the greatest depth of the particle below O is $(2+\sqrt{3})\, l$ ft. Show that this depth is attained in time

$$[\sqrt{2} + \pi - \cos^{-1}(1/\sqrt{3})]\sqrt{(l/g)} \text{ seconds.}$$

12. Two bodies of masses M and M', are attached to the lower end of an elastic string whose upper end is fixed and hang at rest ; M' falls off ; show that the distance of M from the upper end of the string at time t is $a + b + c\cos\{\sqrt{(g/b)}\, t\}$, where a is the unstretched length of the string, b and c the distances by which it would be stretched when supporting M and M' respectively. **(Lucknow 2008; Avadh 07, 09)**

13. A heavy particle is attached to an extensible string to a fixed point from which the particle is allowed to fall freely ; when the particle is in its lowest position the string is of twice the natural length. Prove that the modulus of elasticity is four times the weight of the particle and find the time during which the string is extended beyond its natural length.

14. A heavy particle is supported in equilibrium by two equal elastic strings with their other ends attached to two points in a horizontal line and each inclined at an angle 60° to the vertical. The modulus of elasticity is such that when the particle is suspended from any portion of the string, the extension is equal to its natural length. The particle is displaced vertically a small distance and then released. Prove that the period of its small oscillation is $2\pi\sqrt{(2l/5g)}$ where l is the stretched length of the either string in equilibrium position.

Answers 6

3. $f + e + 2a$

13. $\sqrt{(l/4g)}\left\{\frac{1}{2}\pi + \sin^{-1}\left(\frac{1}{3}\right)\right\}$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. A particle moves in a straight line under an attraction towards a fixed point on the line, which varies inversely as the square of the distance x from the fixed point. Its equation of motion is

(a) $\frac{d^2x}{dt^2} = \frac{\mu}{x^2}$ (b) $\frac{d^2x}{dt^2} = -\mu x$

(c) $\frac{d^2x}{dt^2} = -\frac{\mu}{x^2}$ (d) $\frac{d^2x}{dt^2} = \mu x$

2. If T is the time period of a simple harmonic motion of amplitude a and intensity of force μ, then

(a) $T = \frac{\pi}{2\sqrt{\mu}}$ (b) $T = \frac{2\pi}{\sqrt{\mu}}$

(c) $T = a\sqrt{\mu}$ (d) $T = 2\pi\sqrt{\mu}$

(Bundelkhand 2007, Rohilkhand 10)

3. A particle moving in a straight line OAB with S.H.M. has zero velocity at the points A and B whose distances from O are a and b respectively, and has velocity v when half way between them. If the time period of the motion is T, then

(a) $T = \frac{\pi(b-a)}{v}$ (b) $T = \frac{\pi}{(b-a)} \cdot v$

(c) $T = \frac{\pi b}{v-a}$ (d) $T = \frac{\pi(a-b)}{v}$ **(Garhwal 2002)**

4. If in a simple harmonic motion of amplitude a and intensity of force μ, the displacement of the particle from the centre of force at time t is x, then its equation of motion is

(a) $\frac{dx}{dt} = \mu x$ (b) $\frac{dx}{dt} = -\mu x$

(c) $\frac{d^2x}{dt^2} = \mu x$ (d) $\frac{d^2x}{dt^2} = -\mu x$

(Avadh 2006; Rohilkhand 07)

5. The speed v of a particle moving along the axis of x is given by the relation $v^2 = n^2\ (8bx - x^2 - 12\ b^2)$. The motion is simple harmonic with its centre at

(a) $x = 4b$ (b) $x = 0$
(c) $x = -4b$ (d) $x = 2b$

6. In a simple harmonic motion, at the following distance from the centre will be the velocity half of the maximum :

(a) $\pm \frac{1}{2} a \sqrt{3}$ (b) $\pm \frac{1}{3} a \sqrt{2}$
(c) $+\frac{1}{2} a \sqrt{3}$ (d) $-\frac{1}{2} a \sqrt{3}$

(Garhwal 2003; Rohilkhand 11)

7. In an elastic string Hooke's law is :

(a) $T = \lambda \frac{a}{l}$ (b) $T = \lambda \frac{l}{a}$
(c) $T = \lambda l$ (d) $T = \lambda \frac{dl}{l}$

where l is the natural length of the string and a is the extension in the string beyond its natural length. **(Garhwal 2004)**

Fill in the Blank(s)

Fill in the blanks "……" so that the following statements are complete and correct.

1. If at any time t the displacement of a particle moving in a straight line from any point of its path is x, then the expression for its velocity is …… .
2. If at any time t the displacement of a particle moving in a straight line from any point of its path is x, then the expression for its acceleration is …… .
3. When a particle moves outside the surface of the earth, the acceleration varies inversely as the square of the distance of the particle from the …… of the earth.
4. A particle moves in a straight line in such a way that its acceleration varies inversely as the cube of the distance from a fixed point of its path and is directed towards the fixed point. Its equation of motion is $\frac{d^2x}{dt^2} = \ldots\ldots$
5. If in a simple harmonic motion the intensity of force is μ, then the time taken to make one complete oscillation = …… . **(Meerut 2004)**
6. In a simple harmonic motion the velocity is maximum at the …… . **(Meerut 2004)**
7. If the displacement of a particle moving in a straight line is expressed by the equation $x = a \cos nt + b \sin nt$, then it describes a simple harmonic motion whose time period is … .
8. If in a simple harmonic motion of amplitude a and intensity of force μ, the velocity at a distance x from the centre of force is v, then $v^2 = \ldots\ldots$
9. In a simple harmonic motion the time period is …… of the amplitude.
10. If in a simple harmonic motion of amplitude a and intensity of force μ, the displacement of the particle at time t from the centre of force is x, then $x = \ldots\ldots$

True or False

Write 'T' for true and 'F' for false statement.

1. In a rectilinear motion if v is the velocity at time t at a distance x from a fixed point on the path, then $\dfrac{d^2 x}{dt^2} = v \dfrac{dv}{dx}$.

2. If a particle moves inside the earth through a hole made in the earth, then the acceleration varies inversely as the distance of the particle from the centre of the earth.

3. If a particle moves along a straight line such that its distance x from a fixed point on it and the velocity v there are related by $v^2 = \mu\,(a^2 - x^2)$, then its motion is simple harmonic.

4. In a simple harmonic motion the time of descent to the centre of force is independent of the initial displacement of the particle.

5. In a simple harmonic motion of amplitude a and intensity of force μ, the maximum velocity is $a/\sqrt{\mu}$.

6. A particle moving in a straight line with S.H.M. has velocities v_1 and v_2 when its distances from the centre are x_1 and x_2. The time period of the motion is
$$2\pi \sqrt{\left(\frac{x_1^2 - x_2^2}{v_2^2 - v_1^2}\right)}.$$

7. A particle is moving with S.H.M. and while making an excursion from one position of rest to the other, its distances from the middle point of its path at three consecutive seconds are observed to be x_1, x_2, x_3. The time of one complete oscillation is $2\pi \cos^{-1}\left(\dfrac{x_1 + x_3}{2 x_2}\right)$.

8. A light elastic string of natural length l is hung by one end and to the other end are tied successively particles of masses m_1 and m_2. If t_1 and t_2 be the periods and c_1, c_2 the statical extensions corresponding to these two weights, then $g\,(t_1^2 - t_2^2) = 4\pi\,(c_1 - c_2)$.

9. If the displacement of a particle in a straight line is expressed by the equation
$$x = a \cos nt + b \sin nt,$$
it describes a simple harmonic motion whose amplitude is $\sqrt{(a^2 + b^2)}$.

10. A particle moves in a straight line under an attraction towards a fixed point on the line, which varies inversely as the square of the distance from the fixed point. If its displacement at time t from the fixed point is x, then its equation of motion is
$$\frac{d^2 x}{dt^2} = \frac{\mu}{x^2}.$$

11. A particle is subject to an acceleration $\dfrac{\mu}{(\text{distance})^2}$ towards the origin. If it starts from rest at infinity, its velocity when it is at a distance a from the origin is $\sqrt{2\mu/a}$. **(Meerut 2004)**

Answers

Multiple Choice Questions

1. (c) 2. (b) 3. (a)
4. (d) 5. (a) 6. (a)
7. (a)

Fill in the Blank(s)

1. dx/dt 2. d^2x/dt^2 3. centre
4. $-\mu/x^3$ 5. $2\pi/\sqrt{\mu}$ 6. centre of force
7. $2\pi/n$ 8. $\mu(a^2 - x^2)$ 9. independent
10. $a \cos \sqrt{\mu}\, t$

True or False

1. T 2. F 3. T
4. T 5. F 6. T
7. F 8. F 9. T
10. F 11. T

Chapter 2

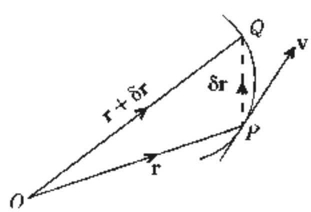
Kinematics in Two Dimensions

2.1 Velocity and Acceleration

1. Velocity (Definition): *The* **velocity** *of a particle moving along a curve is the rate of change of its displacement with respect to time.*

Let P and Q be the positions of a particle moving along a curve at times t and $t + \delta t$ respectively. With respect to O as the origin of vectors, let $\vec{OP} = \mathbf{r}$ and $\vec{OQ} = \mathbf{r} + \delta\mathbf{r}$. Then

$$\vec{PQ} = \vec{OQ} - \vec{OP} = \delta\mathbf{r}$$



represents the displacement of the particle in time δt and $\dfrac{\delta\mathbf{r}}{\delta t}$ represents the average rate of displacement (or the *average velocity*) during the interval δt. The limiting value of the average velocity $\dfrac{\delta\mathbf{r}}{\delta t}$ as $\delta t \to 0$ is the velocity (also known as instantaneous velocity) of the particle at time t. Thus if the vector $\mathbf{v}$ represents the velocity of the particle at time t, then

$$\mathbf{v} = \lim_{\delta t \to 0} \frac{\delta \mathbf{r}}{\delta t} = \frac{d\mathbf{r}}{dt}, \qquad \ldots(1)$$

where $\mathbf{r}$ is the position vector of the particle.

The vector $\frac{d\mathbf{r}}{dt}$ is along the tangent to the path of the particle and consequently the velocity $\mathbf{v}$ is a vector quantity along the tangent at P. The magnitude $v = |\mathbf{v}|$ of the velocity $\mathbf{v}$ is called the *speed* of the particle.

If (x, y, z) are the co-ordinates of the point P and $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ the three unit vectors forming the right handed system, the position vector $\mathbf{r}$ of P is given by

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}. \qquad \ldots(2)$$

$$\therefore \quad \mathbf{v} = \frac{d\,\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} \qquad \ldots(3)$$

Here $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ are called the components or the *resolved* parts of the velocity $\mathbf{v}$ along the axes of x, y and z respectively.

$\therefore$ The speed v of the particle at P is given by

$$v = |\mathbf{v}| = \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right]} \qquad \ldots(4)$$

Remember: The component $\frac{dx}{dt}$, of the velocity $\mathbf{v}$ along the axis of x, is taken with positive or negative sign according as it is in the direction of x increasing or x decreasing. In the case of $\frac{dy}{dt}$ and $\frac{dz}{dt}$ also the positive and negative signs are taken exactly in the same way.

2. Acceleration (Definition): *The* **acceleration vector** *of a particle moving along a curve is defined as the rate of change of its velocity vector.*

If $\mathbf{v}$ and $\mathbf{v} + \delta\mathbf{v}$ are the velocities of a particle moving along a curve at times t and $t + \delta t$ respectively, then $\delta\mathbf{v}$ is the change in velocity in time δt and $\frac{\delta \mathbf{v}}{dt}$ is the average acceleration during the interval δt. If $\mathbf{a}$ is the **acceleration vector** of the particle at time t, then

$$\mathbf{a} = \lim_{\delta t \to 0} \frac{\partial \mathbf{v}}{\delta t} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right) = \frac{d^2\mathbf{r}}{dt^2} \qquad \ldots(5)$$

Substituting for $\mathbf{v}$ from (3), we have

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j} + \frac{d^2z}{dt^2}\mathbf{k}. \qquad \ldots(6)$$

Here $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}$ are called the components of the acceleration $\mathbf{a}$ along the axes of x, y and z respectively.

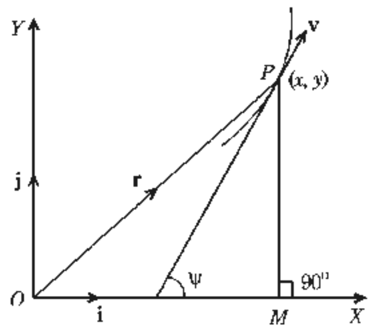
Also magnitude of acceleration

$$= a = |\mathbf{a}| = \sqrt{\left[\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2 + \left(\frac{d^2z}{dt^2}\right)^2\right]}.$$

Remember: The component $\dfrac{d^2x}{dt^2}$ of the acceleration **a** along the axis of x is taken with positive or negative sign according as it is in the direction of x increasing or x decreasing. The positive and negative signs are taken with $\dfrac{d^2y}{dt^2}$ and $\dfrac{d^2z}{dt^2}$ in exactly the same way.

2.2 Components of Velocity and Acceleration Along the Coordinate Axes in Two Dimensions

(Meerut 2004)



Let $P\ (x, y)$ be the position of a particle moving in a plane at any time t. If $\vec{OP} = \mathbf{r}$, we have

$$\mathbf{r} = \vec{OP} = \vec{OM} + \vec{MP} = x\mathbf{i} + y\mathbf{j}.$$

Let **v** be the vector representing the velocity of the particle at P. Then

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d}{dt}(x\mathbf{i} + y\mathbf{j})$$
$$= \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}.$$

Thus the velocity vector **v** has been expressed as a linear combination of the vectors **i** and **j**.

$\therefore$ The x-component of the velocity of $P = \dfrac{dx}{dt} = \dot{x}$, positive in the direction of the vector **i** *i.e.*, positive in the direction of x increasing, and the y-component of the velocity of $P = \dfrac{dy}{dt} = \dot{y}$, positive in the direction of y increasing.

If v is the resultant velocity of P, we have

$$v = \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}} = \frac{ds}{dt}.$$

Also the angle which the direction of v makes with OX

$$= \tan^{-1}\frac{dy/dt}{dx/dt} = \tan^{-1}\frac{dy}{dx} = \tan^{-1}\tan\psi = \psi,$$

showing that the resultant velocity at P is along the tangent at P.

If **a** be the acceleration vector of the particle at P, we have

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left[\frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j}\right] = \frac{d^2x}{dt^2}\mathbf{i} + \frac{d^2y}{dt^2}\mathbf{j}.$$

$\therefore$ the x-component of the acceleration of $P = \dfrac{d^2x}{dt^2} = \ddot{x}$,

positive in the direction of x increasing,

and the y-component of the acceleration of $P = \dfrac{d^2y}{dt^2} = \ddot{y}$, positive in the direction of y increasing.

The resultant acceleration of

$$P = \sqrt{\left[\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2\right]}.$$

Illustrative Examples

Example 1: *A particle moves along the curve $x = t^3 + 1$, $y = t^2$, $z = 2t + 5$ where t is the time. Find the components of the velocity and acceleration at time $t = 1$ in the direction $\mathbf{i} + \mathbf{j} + 3\mathbf{k}$.*

(Agra 2010)

Solution: If $\mathbf{r}$ is the position vector of the particle at time t, then

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = (t^3 + 1)\,\mathbf{i} + t^2\,\mathbf{j} + (2\,t + 5)\,\mathbf{k}.$$

$\therefore$ velocity $\quad \mathbf{v} = \dfrac{d\mathbf{r}}{dt} = 3t^2\,\mathbf{i} + 2t\mathbf{j} + 2\mathbf{k}$

and acceleration $\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = 6t\mathbf{i} + 2\mathbf{j}$.

$\therefore$ at time $t = 1$, $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$ and $\mathbf{a} = 6\,\mathbf{i} + 2\mathbf{j}$.

Now the unit vector in the direction of the vector $\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ is

$$= \frac{\mathbf{i} + \mathbf{j} + 3\mathbf{k}}{\sqrt{(1 + 1 + 9)}} = \frac{\mathbf{i} + \mathbf{j} + 3\mathbf{k}}{\sqrt{(11)}}.$$

$\therefore$ The components of the velocity and acceleration at time $t = 1$ in the direction $\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ are

$$(3\mathbf{i} + 2\mathbf{j} + 2\mathbf{k}) \cdot \frac{\mathbf{i} + \mathbf{j} + 3\mathbf{k}}{\sqrt{(11)}} = \frac{3\,.\,1 + 2\,.\,1 + 2\,.\,3}{\sqrt{(11)}} = \sqrt{(11)}$$

and $\quad (6\mathbf{i} + 2\mathbf{j}) \cdot \dfrac{\mathbf{i} + \mathbf{j} + 3\mathbf{k}}{\sqrt{(11)}} = \dfrac{6\,.\,1 + 2\,.\,1 + 0\,.\,3}{\sqrt{(11)}} = \dfrac{8}{\sqrt{(11)}}$.

Example 2: *The acceleration of a particle at any time $t \geq 0$ is given by*

$$\mathbf{a} = 12\cos 2t\mathbf{i} - 8\sin 2t\mathbf{j} + 16t\mathbf{k}.$$

If the velocity and displacement are zero at $t = 0$, find the velocity and displacement at any time.

Solution: Here, $\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = 12\cos 2t\mathbf{i} - 8\sin 2t\mathbf{j} + 16t\mathbf{k}$.

Integrating w.r.t. 't', we have

$$\mathbf{v} = 6 \sin 2t\mathbf{i} + 4 \cos 2t\mathbf{j} + 8t^2 \mathbf{k} + \mathbf{c}_1 \text{, where } \mathbf{c}_1 \text{ is a constant vector.}$$

But at $t = 0, \mathbf{v} = \mathbf{0}$;

$\therefore$ $\mathbf{0} = 4\mathbf{j} + \mathbf{c}_1$ or $\mathbf{c}_1 = -4\mathbf{j}$.

$\therefore$ Velocity $\mathbf{v} = 6 \sin 2t\mathbf{i} + 4 \cos 2t\mathbf{j} + 8\, t^2\mathbf{k} - 4\mathbf{j}$.

Again $\mathbf{v} = \dfrac{d\mathbf{r}}{dt} = 6 \sin 2t\mathbf{i} + 4 \cos 2\, t\mathbf{j} + 8\, t^2\, \mathbf{k} - 4\mathbf{j}$.

Integrating, w.r.t. 't', we have

$$\mathbf{r} = -3 \cos 2t\mathbf{i} + 2 \sin 2t\mathbf{j} + \frac{8}{3} t^3\, \mathbf{k} - 4t\mathbf{j} + \mathbf{c}_2 \text{,}$$

where $\mathbf{c}_2$ is a constant vector.

But at $t = 0, \mathbf{r} = \mathbf{0}$; $\therefore$ $\mathbf{0} = -3\mathbf{i} + \mathbf{c}_2$ or $\mathbf{c}_2 = 3\mathbf{i}$.

$\therefore$ Displacement from the origin is given by

$$\mathbf{r} = -3 \cos 2t\mathbf{i} + 2 \sin 2t\mathbf{j} + \frac{8}{3} t^3\, \mathbf{k} - 4t\mathbf{j} + 3\mathbf{i}$$

or
$$\mathbf{r} = 3\, (1 - \cos 2t)\, \mathbf{i} + 2\, (\sin 2t - 2t)\, \mathbf{j} + \frac{8}{3} t^3\, \mathbf{k}.$$

Example 3: *A particle is acted on by a force parallel to the axis of y whose acceleration (always towards the axis of x) is $\mu\, y^{-2}$ and when $y = a$, it is projected parallel to the axis of x with velocity $\sqrt{(2\mu/a)}$. Prove that it will describe a cycloid.*

Solution: Here we are given that

$$\frac{d^2 y}{dt^2} = -\mu y^{-2}, \qquad \ldots(1)$$

the negative sign has been taken because the force is in the direction of y decreasing. Also there is no force parallel to the axis of x. Therefore

$$\frac{d^2 x}{dt^2} = 0. \qquad \ldots(2)$$

Multiplying both sides of (1) by $2\,(dy/dt)$ and then integrating w.r.t. t, we have

$$\left(\frac{dy}{dt}\right)^2 = \frac{2\mu}{y} + A, \text{ where } A \text{ is a constant.}$$

Initially, when $y = a, \dfrac{dy}{dt} = 0.$

[Note that initially there is no velocity parallel to the y-axis]

$\therefore$
$$A = -\frac{2\mu}{a}.$$

$\therefore$
$$\left(\frac{dy}{dt}\right)^2 = \frac{2\mu}{y} - \frac{2\mu}{a} = 2\mu\left(\frac{1}{y} - \frac{1}{a}\right) = \frac{2\mu}{a}\left(\frac{a-y}{y}\right)$$

or
$$\frac{dy}{dt} = -\sqrt{\left(\frac{2\mu}{a}\right)} \cdot \sqrt{\left(\frac{a-y}{y}\right)}. \qquad \ldots(3)$$

[–ive sign has been taken because the particle is moving in the direction of y decreasing.]

Now integrating (2), we have

$$\frac{dx}{dt} = B, \text{ where } B \text{ is a constant.}$$

Initially, when $y = a, \dfrac{dx}{dt} = \sqrt{\left(\dfrac{2\mu}{a}\right)}$.

$\therefore$ $$B = \sqrt{\left(\frac{2\mu}{a}\right)}.$$

$\therefore$ $$\frac{dx}{dt} = \sqrt{\left(\frac{2\mu}{a}\right)}. \qquad \ldots(4)$$

Dividing (3) by (4), we have

$$\frac{dy}{dx} = -\sqrt{\left(\frac{a-y}{y}\right)},$$

or $$dx = -\sqrt{\left(\frac{y}{a-y}\right)}\,dy.$$

Integrating, $$x = -\int \sqrt{\left(\frac{y}{a-y}\right)}\,dy + C$$

$$= 2a \int \frac{\cos\theta}{\sin\theta} \cdot \cos\theta \sin\theta \, d\theta + C$$

[putting $y = a\cos^2\theta$, so that $dy = -2\,a\cos\theta\sin\theta\,d\theta$]

$$= a\int (1 + \cos 2\theta)\,d\theta + C$$

$$= a\left(\theta + \frac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{1}{2}a\,(2\,\theta + \sin 2\theta) + C.$$

Let us take $x = 0$, when $y = a$

i.e., when $a\cos^2\theta = a$ *i.e.*, when $\cos\theta = 1$, *i.e.*, when $\theta = 0$.

Then $$0 = \frac{1}{2}a\,(0 + 0) + C \quad \text{or} \quad C = 0.$$

$\therefore$ $$x = \frac{1}{2}(2\,\theta + \sin 2\theta). \qquad \ldots(5)$$

Also $$y = a\cos^2\theta = \frac{1}{2}a\,(1 + \cos 2\theta). \qquad \ldots(6)$$

The equations (5) and (6) give us the path of the particle. But these are the parametric equations of a cycloid.

Hence the path is a cycloid.

Example 4: *A particle moves in the curve* $y = a \log \sec\left(\frac{x}{a}\right)$ *in such a way that the tangent to the curve rotates uniformly ; prove that the resultant acceleration of the particle varies as the square of the radius of curvature.* **(Lucknow 2008; Agra 08; Rohilkhand 08)**

Solution: Since the tangent to the curve rotates uniformly,

$$\therefore \quad \frac{d\psi}{dt} = c \quad \text{(constant)}. \qquad \ldots(1)$$

Equation of the path is

$$y = a \log \sec (x/a).$$

$$\therefore \quad \frac{dy}{dx} = \frac{a}{\sec (x/a)} \cdot \sec \frac{x}{a} \tan \frac{x}{a} \cdot \frac{1}{a}$$

or $$\tan \psi = \frac{dy}{dx} = \tan \frac{x}{a} \cdot$$

$$\therefore \quad \psi = x/a \quad \text{or} \quad x = a\,\psi.$$

$$\therefore \quad \frac{dx}{dt} = a\frac{d\,\psi}{dt} = ac \quad \text{and} \quad \frac{d^2x}{dt^2} = 0.$$

Now $$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = \left(\tan \frac{x}{a}\right). ac\,.$$

$$\therefore \quad \frac{d^2y}{dt^2} = ac \sec^2 \frac{x}{a} \cdot \frac{1}{a} \cdot \frac{dx}{dt}$$

$$= ac \sec^2 \frac{x}{a} \cdot \frac{1}{a} \cdot ac = ac^2 \sec^2 \frac{x}{a} \cdot$$

$$\therefore \quad \text{Resultant acceleration} = \sqrt{\left[\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2\right]}$$

$$= \sqrt{\left[(0)^2 + \left(ac^2 \sec^2 \frac{x}{a}\right)^2\right]} = ac^2 \sec^2 \frac{x}{a} \cdot \qquad \ldots(2)$$

Also the radius of curvature $\rho = \dfrac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2} \cdot$

But $\dfrac{dy}{dx} = \tan \dfrac{x}{a}$ implies that

$$\frac{d^2y}{dx^2} = \frac{1}{a} \sec^2 \frac{x}{a} \cdot$$

$$\therefore \quad \rho = \frac{[1 + \tan^2 (x/a)]^{3/2}}{(1/a) \sec^2 (x/a)} = \frac{a \sec^3 (x/a)}{\sec^2 (x/a)} = a \sec (x/a)$$

or $$\sec (x/a) = \rho/a.$$

$\therefore$ from (2), the resultant acceleration $= ac^2 (\rho/a)^2 = (c^2/a)\,\rho^2$.

Hence the resultant acceleration $\propto \rho^2$.

Comprehensive Exercise 1

1. A particle moves along a curve whose parametric equations are

$x = e^{-t}$, $y = a \cos 3t$, $z = b \sin 3t$, where t is the time. **(Bundelkhand 2009)**

(i) Determine its velocity and acceleration at time t.

(ii) Find the magnitude of the velocity and acceleration at $t = 0$.

(Meerut 2004)

2. The position of a moving point at time t is given by $x = at^2$, $y = 2at$.

Find its velocity and acceleration.

3. The position of a moving point at time t is given by

$x = a \cos t$, $y = a \sin t$.

Find its path, velocity and acceleration.

(Agra 2003; Meerut 06; Bundelkhand 2008)

4. A point moves in a plane, its velocities parallel to the axis of x and y being $u + ey$ and $v + e\, x$ respectively, show that it moves in a conic section.

5. A particle is moving with a constant velocity parallel to the axis of y and a velocity proportional to y parallel to the x-axis ; prove that it will describe a parabola.

6. A particle is acted on by a force parallel to the axis of y whose acceleration is λy, and is initially projected with a velocity $a \sqrt{\lambda}$ parallel to the axis of x at a point where $y = a$, prove that it will describe the catenary $y = a \cosh (x / a)$.

7. A rod moves with its ends sliding on rectangular axes OX and OY. If x, y are the coordinates, at any instant, of a point P on the rod and if the angular velocity ω of the rod is constant, show that the component accelerations of P along the axes are $-x\,\omega^2$ and $-y\,\omega^2$ and the resultant acceleration is $OP \,.\, \omega^2$ towards O.

Answers 1

1. (i) $-e^{-t}\,\mathbf{i} - 3a \sin 3t\mathbf{j} + 3b \cos 3t\mathbf{k}$, $e^{-t}\,\mathbf{i} - 9a \cos 3t\;\mathbf{j} - 9b \sin 3t\mathbf{k}$

(ii) $\sqrt{(1 + 9b^2)}, \sqrt{(1 + 9^2\;a^2)}$

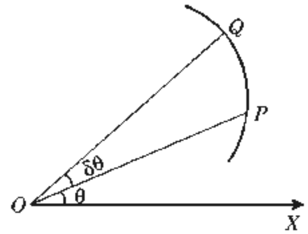
2. $2\,a \sqrt{(t^2 + 1)}, 2a$

3. velocity $= a$, acceleration $= a$, path is $x^2 + y^2 = a^2$

2.3 Angular Velocity and Acceleration **(Bundelkhand 2006)**

1. Angular Velocity (Definition): *Let P be a point moving in a plane. If O is the fixed point and OX a fixed line through O in the plane of motion, then the* **angular velocity** *of the moving point P about O (or the line OP in the plane XOP) is the rate of change of the angle XOP.*

Let P and Q be the positions of a moving particle at times t and $t + \delta t$ respectively such that $\angle\, POX = \theta$ and $\angle\, QOX = \theta + \delta\theta$.



Then the angle turned by the particle in time δt is $\delta\theta$.

$\therefore$ Average rate of change of the angle of P about $O = \dfrac{\delta\theta}{\delta t}$.

$\therefore$ The **angular velocity** of the point P about O

$$= \lim_{\delta t \to 0} \frac{\delta\theta}{\delta t} = \frac{d\theta}{dt} = \dot{\theta},$$

where the dot placed over θ denotes differentiation with respect to the time t.

Since the angular velocity has magnitude as well as direction, it is a vector quantity represented by the vector $\vec{\omega}$. The magnitude of the angular velocity is $\dfrac{d\theta}{dt}$ $(= \dot{\theta} = \omega)$ and its direction is perpendicular to the plane POQ.

Since the angle θ is measured in radians, the unit of angular velocity is radians/sec.

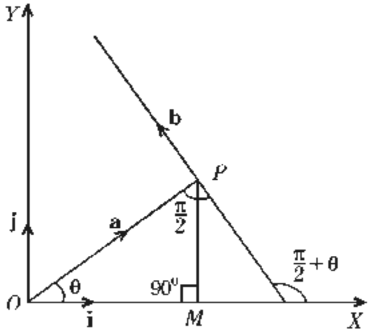
2. Angular Acceleration (Definition): *The rate of change of the angular velocity is called* **angular acceleration.**

$\therefore$ Angular acceleration $= \dfrac{d}{dt}\left(\dfrac{d\theta}{dt}\right) = \dfrac{d^2\theta}{dt^2} = \ddot{\theta}.$

The unit of angular acceleration is radians/sec ^{2}.

2.4 Rate of Change of a Unit Vector in a Plane

Let **i, j** be the unit vectors along two mutually perpendicular fixed lines (say the coordinate axes in the plane).



Let **a** denote a **unit vector** $\vec{OP}$ such that $OP = 1$ and $\angle POX = \theta$.

Then $\mathbf{a} = \vec{OM} + \vec{MP} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$ ...(1)

The vector **a** is a function of θ, where θ is a function of the time t.

Differentiating (1) w.r.t. t, we have

$$\frac{d\mathbf{a}}{dt} = -\sin\theta\frac{d\theta}{dt}\mathbf{i} + \cos\theta\frac{d\theta}{dt}\mathbf{j}$$

[Note that **i** and **j** are constant vectors]

$$= \frac{d\theta}{dt}\left[\cos\left(\frac{1}{2}\pi + \theta\right)\mathbf{i} + \sin\left(\frac{1}{2}\pi + \theta\right)\mathbf{j}\right]$$

or $$\frac{d\mathbf{a}}{dt} = \frac{d\theta}{dt}\mathbf{b},$$

where $\mathbf{b} = \cos\left(\frac{1}{2}\pi + \theta\right)\mathbf{i} + \sin\left(\frac{1}{2}\pi + \theta\right)\mathbf{j}$ is a unit vector inclined at an angle $\frac{1}{2}\pi + \theta$ with OX. Therefore, $\mathbf{b}$ is a unit vector perpendicular to OP in the sense in which θ increases.

Thus remember that if $\mathbf{a}$ is a unit vector which makes a variable angle θ with OX, then

$$\frac{d\mathbf{a}}{dt} = \frac{d\theta}{dt}\mathbf{b}, \qquad \text{...(2)}$$

where $\mathbf{b}$ is a unit vector perpendicular to $\mathbf{a}$ in the direction of θ increasing.

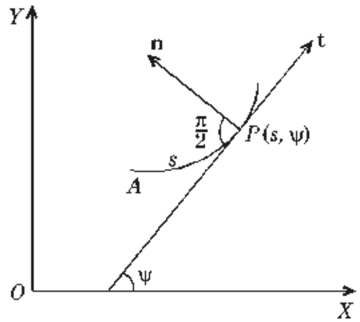
Particular case : If $\mathbf{t}$ and $\mathbf{n}$ are the unit vectors along the tangent and normal respectively at any point P of a plane curve (as shown in the figure), then

$$\frac{d\mathbf{t}}{dt} = \frac{d\psi}{dt}\mathbf{n} = \dot{\psi}\mathbf{n},$$

where ψ is the angle which the tangent at the point P makes with OX.

Also $$\frac{d\mathbf{n}}{dt} = -\frac{d\psi}{dt}\mathbf{t} = -\dot{\psi}\,\mathbf{t}.$$

Here $\mathbf{t}$ is in the direction of s increasing and $\mathbf{n}$ is in the sense in which ψ increases *i.e.*, in the direction of inwards drawn normal.



2.5 Relation between Angular and Linear Velocities

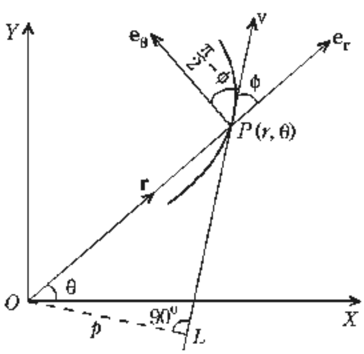
(Agra 2008, 09, 11; Purvanchal 11)

Let OX and OY be two mutually perpendicular fixed lines (say the co-ordinate axes).

If ω is the angular velocity of a moving point P about O, and $\angle POX = \theta$, then

$$\omega = \frac{d\theta}{dt}.$$

Let $\mathbf{r}$ be the position vector of the point P with respect to the origin O and let (r, θ) be the polar co-ordinates of P. If $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are the unit vectors along and perpendicular to OP respectively, then



$$\mathbf{r} = |\mathbf{r}|\,\mathbf{e}_r = r\,\mathbf{e}_r \qquad [\because\ |\mathbf{r}| = |\overrightarrow{OP}| = OP = r]$$

and $$\frac{d\,\mathbf{e}_r}{dt} = \frac{d\theta}{dt}\mathbf{e}_\theta \qquad \text{[Refer 2.4]}$$

Now the linear velocity $\mathbf{v}$ of the point P is along the tangent at P and is given by

$$\mathbf{v} = \frac{d\,\mathbf{r}}{dt} = \frac{d}{dt}(r\,\mathbf{e}_r)$$

$$= \frac{dr}{dt}\mathbf{e}_r + r\frac{d\mathbf{e}_r}{dt} = \frac{dr}{dt}\mathbf{e}_r + r\frac{d\theta}{dt}\mathbf{e}_\theta.$$

Now the component of a vector **a** in the direction of a unit vector **b** is given by $\mathbf{a} \cdot \mathbf{b}$. If v_θ is the component of the velocity **v** in the direction perpendicular to OP, then

$$v_\theta = \mathbf{v} \cdot \mathbf{e}_\theta = \left(\frac{dr}{dt}\,\mathbf{e}_r + r\,\frac{d\theta}{dt}\,\mathbf{e}_\theta\right) \cdot \mathbf{e}_\theta$$

$$= r\,\frac{d\,\theta}{dt} = r\,\omega, \qquad [\because\ \mathbf{e}_r \perp \mathbf{e}_\theta \text{ and } |\mathbf{e}_\theta| = 1]$$

or $$\omega = \frac{v_\theta}{r} = \frac{\text{component of the velocity } v \text{ at } P \text{ perpendicular to } OP}{OP}.$$

(Remember)

Since the angle between **v** and $\mathbf{e}_\theta$ is $\frac{1}{2}\pi - \phi$, therefore

$$v_\theta = \mathbf{v} \cdot \mathbf{e}_\theta = v\,.\,1\,.\cos\left(\frac{1}{2}\pi - \phi\right) = v \sin \phi.$$

$\therefore$ $$\omega = \frac{v \sin \phi}{r} = \frac{vr \sin \phi}{r^2}$$

or $$\omega = \frac{d\theta}{dt} = \frac{vp}{r^2}. \qquad [\because\ p = r \sin \phi]$$

Remark 1: The angular velocity of P about O

$$= \frac{\text{the resolved part of the velocity of } P \perp \text{ to } OP}{OP}.$$

Remark 2: If A and B are both in motion, then the angular velocity of B relative to A

$$= \frac{\text{the resolved part of the velocity of } B \text{ relative to } A \perp \text{ to } AB}{AB}.$$

Alternative method for finding the relation between angular and linear velocities.

Theorem: *If v be the velocity of a point P moving in a plane curve and (r, θ) its coordinates referred to the fixed point O in the plane, then the angular velocity of P about O is equal to vp/r^2, where p is the perpendicular from O drawn to the tangent at P.*

Proof: The angular velocity of P about O

$$= \frac{d\theta}{dt} = \frac{d\theta}{ds} \cdot \frac{ds}{dt} = v\,\frac{d\theta}{ds} \qquad \left[\because\ v = \frac{ds}{dt}\right]$$

$$= \frac{v}{r} \cdot \left(r\,\frac{d\theta}{ds}\right)$$

$$= \frac{v \sin \phi}{r} \qquad \left[\because\ \sin \phi = r\,\frac{d\theta}{ds}, \text{from differential calculus}\right]$$

$$= \frac{v}{r} \cdot \frac{p}{r} \qquad [\because\ p = r \sin \phi]$$

$$= vp/r^2.$$

Illustrative Examples

Example 5: *Prove that the angular acceleration of the direction of motion of a point moving in a plane is* $\frac{v}{\rho}\frac{dv}{ds}-\frac{v^2}{\rho^2}\frac{d\rho}{ds}$. **(Lucknow 2006; Avadh 09; Agra 10)**

Solution: Let P be the position of a moving point at time t. The direction of velocity at P is along the tangent to the path at P. If the tangent at P makes an angle ψ with the axis of x, then

$$\frac{d\psi}{dt}=\frac{d\psi}{ds}\cdot\frac{ds}{dt}=\frac{1}{\rho}\cdot v \qquad \left\{\because \frac{ds}{dt}=v \quad \text{and} \quad \frac{ds}{d\psi}=\rho\right\}$$

Differentiating both sides w.r.t. 't' we have the angular acceleration of the direction of motion

$$=\frac{d^2\psi}{dt^2}=\frac{d}{dt}\left(\frac{1}{\rho}\cdot v\right)=\frac{1}{\rho}\cdot\frac{dv}{dt}-\frac{v}{\rho^2}\cdot\frac{d\rho}{dt}$$

$$=\frac{1}{\rho}\frac{dv}{ds}\cdot\frac{ds}{dt}-\frac{v}{\rho^2}\frac{d\rho}{ds}\cdot\frac{ds}{dt}$$

$$=\frac{1}{\rho}\frac{dv}{ds}\cdot v-\frac{v}{\rho^2}\frac{d\rho}{ds}v=\frac{v}{\rho}\frac{dv}{ds}-\frac{v^2}{\rho^2}\frac{d\rho}{ds}.$$

Example 6: *Prove that the angular velocity of a projectile about the focus of its path varies inversely as its distance from the focus.* **(Agra 2010)**

Solution: We know that the path of a projectile is a parabola whose pedal equation referred to its focus S as the pole is given by

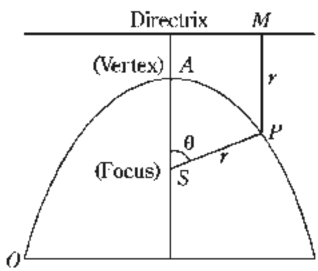
$$p^2=ar. \qquad \ldots(1)$$

Let v be the velocity of the projectile at the point $P\,(r,\theta)$.

Then $MP=PS=r$.

Since the velocity of the projectile at a point of its path is equal to the velocity acquired in falling freely from the directrix to that point, therefore,

$$v=\sqrt{(2\,g\,.\,MP)}=\sqrt{(2gr)}. \qquad \ldots(2)$$



$\therefore$ The angular velocity ω of P about the focus S (*i.e.*, about the pole) is given by

$$\omega=\frac{vp}{r^2}=\frac{\sqrt{(2gr)}\,.\,\sqrt{(ar)}}{r^2}$$

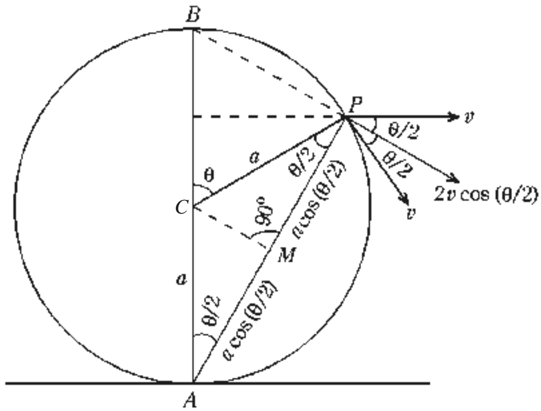
$$=\frac{\sqrt{(2ag)}}{r}=\frac{\sqrt{(2ag)}}{SP}.$$

$$\therefore \qquad \omega\propto\frac{1}{SP}.$$

Hence, the angular velocity of P about the focus S varies inversely as its distance from the focus.

Example 7: *A wheel rolls along a straight road with constant speed v. Show that the actual velocity of P is v.(AP/CP), where A is the point of contact of the wheel with the road and C is the centre of the wheel. Also find its direction. Find also the angular velocity of P relative to A.*

Solution: Let P be a point on the wheel. The point P possesses two velocities (i) v parallel to the road, and (ii) v along the tangent at P to the wheel as shown in the figure.



Let $\angle BCP = \theta$. Then the angle between the horizontal line through P and the tangent to the wheel at P is also θ.

The actual velocity of P = the resultant of the two equal velocities v and v at P

$$= \sqrt{(v^2 + v^2 + 2v^2 \cos\theta)}$$

$$= 2v\cos\frac{1}{2}\theta = v \,.\, \frac{2a\cos\frac{1}{2}\theta}{a}, \qquad \text{where } a \text{ is the radius of the wheel}$$

$$= v\cdot\frac{AP}{CP}\cdot$$

The direction of the actual velocity of P bisects the angle between the two velocities at P and so it makes an angle $\theta/2$ with the horizontal. Now the straight line BP makes an angle $\frac{1}{2}\pi - \frac{1}{2}\theta$ with the vertical BA and so it makes an angle $\frac{1}{2}\theta$ with the horizontal.

Hence the direction of the actual velocity of P is along BP, where B is the highest point of the wheel. The line BP is also perpendicular to AP.

Now the actual velocity of $A = 2v\cos\frac{1}{2}\pi = 0.$ $\qquad [\because \text{ for } A, \theta = \pi]$

The angular velocity of P relative to A

$$= \frac{\text{velocity of } P \text{ relative to } A \text{ in a direction perpendicular to } AP}{AP}\cdot \text{[Refer 2.5]}$$

Since the actual velocity of A is zero, therefore the velocity of P relative to A is the actual velocity of P. But as just shown the actual velocity of P is $v\cdot(AP/CP)$ and its direction is perpendicular to AP.

$\therefore$ the angular velocity of P relative to A

$$= \frac{v \,.\, (AP/CP)}{AP} = \frac{v}{CP} = \frac{v}{a},$$

where a is the radius of the wheel.

Remark: Velocity of P relative to C is v and is along the tangent to the circle at P *i.e.*, in a direction perpendicular to CP.

$\therefore$ angular velocity of P relative to $C = \dfrac{v}{CP} = \dfrac{v}{a}$.

Comprehensive Exercise 2

1. A point P is moving along a fixed straight line AB with uniform velocity v. Show that its angular velocity about a point O is inversely proportional to OP^2.
(Bundelkhand 2007)

2. A particle describes a parabola with uniform speed, show that its angular velocity about the focus S at any point P, varies inversely as $(SP)^{3/2}$.

3. If a point moves along a circle with constant speed, prove that its angular velocity about any point on the circle is half of that about the centre.

4. A body rotates with uniform angular acceleration α. If ω is the angular velocity when the body has turned through an angle θ from rest, show that $\omega^2 = 2\,\alpha\theta$.
(Agra 2003)

5. The line joining two points A, B is of constant length a and the velocities of A, B are in directions which make angles α and β respectively with AB. Prove that the angular velocity of AB is $\dfrac{u \sin(\alpha - \beta)}{a \cos\beta}$, where u is the velocity of A.

6. If a point moves so that its angular velocity about two fixed points is the same, prove that it describes a circle.

7. A point moves with constant velocity v in a circle, find an expression for its angular velocity about any point in the plane of the circle.

8. Two points are moving with uniform velocities u, v in perpendicular lines OX and OY, the motions being towards O. If initially, their distances from the origin are a and b respectively, calculate the angular velocity of the line joining them at the end of t seconds, and show that it is greatest when $t = (au + bv)/(u^2 + v^2)$.

Answers 2

7. vp/r^2, where p is the length of the perpendicular from the point chosen in the plane to the direction of the velocity such that $2\,ap = a^2 + r^2 - b^2$, a being the radius of the circle and b the distance of the point chosen in the plane from the centre of the circle.

8. $(av - bu)/\{(a - ut)^2 + (vt - b)^2\}$

2.6 Radial and Transverse Velocities and Accelerations

(Meerut 2004, 08; Avadh 06; Agra 07, 09; Lucknow 06, 07, 09, 11; Bundelkhand 10; Purvanchal 08, 10; Rohilkhand 11)

Radial and Transverse Velocities (Definition): Let P be the position of a moving particle at time t and $\mathbf{r}$ its position vector w.r.t. the origin O. *Then the resolved parts of the velocity at P along and perpendicular to the radius vector OP are called the* **radial** *and* **transverse velocities** *of the particle at P. Radial and transverse velocities are taken positive in the directions of r and* θ *increasing respectively.*

Let (r, θ) be the polar coordinates of the point P w.r.t. the pole O and the initial line OX. If $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are the unit vectors along and perpendicular to OP, then

$$\frac{d\mathbf{e}_r}{dt} = \frac{d\theta}{dt}\,\mathbf{e}_\theta \quad \text{and} \quad \frac{d\mathbf{e}_\theta}{dt} = -\frac{d\theta}{dt}\,\mathbf{e}_r \qquad \text{[Refer 2.4]} \qquad ...(1)$$

Now $\qquad \vec{OP} = \mathbf{r} = r\,\mathbf{e}_r \qquad [\because\ |\mathbf{r}| = OP = r]$

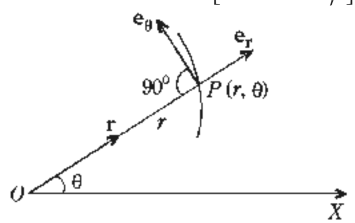
If $\mathbf{v}$ is the velocity vector of the particle at P, then

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d}{dt}(r\,\mathbf{e}_r) \qquad [\because\ \mathbf{r} = r\,\mathbf{e}_r]$$

$$= \frac{dr}{dt}\,\mathbf{e}_r + r\,\frac{d\mathbf{e}_r}{dt}$$

$$= \frac{dr}{dt}\,\mathbf{e}_r + r\,\frac{d\theta}{dt}\,\mathbf{e}_\theta . \qquad ...(2)$$

$$\left[\because \text{ from (1)}, \quad \frac{d\mathbf{e}_r}{dt} = \frac{d\theta}{dt}\,\mathbf{e}_\theta\right]$$



Thus the vector $\mathbf{v}$ has been expressed as a linear combination of the vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$.

$\therefore$ Radial component of velocity at P = the coeff. of the vector $\mathbf{e}_r$ in (2) $= \dfrac{dr}{dt} = \dot{r}$, positive in the direction of the vector $\mathbf{e}_r$ *i.e.*, +ive in the direction of r increasing, and transverse component of velocity at P

= the coeff. of the vector $\mathbf{e}_\theta$ in (2)

$= r\,\dfrac{d\theta}{dt} = r\,\dot{\theta}$, +ive in the direction of θ increasing.

Thus **remember** that at any time t, the **radial velocity** $= \dfrac{dr}{dt}$, +ive in the direction of r increasing and the **transverse velocity** $= r\,\dfrac{d\theta}{dt}$, +ive in the direction of θ increasing.

(Kanpur 2010)

Also the **resultant velocity** $v = \sqrt{\left\{\left(\dfrac{dr}{dt}\right)^2 + \left(r\,\dfrac{d\theta}{dt}\right)^2\right\}}$.

(Kanpur 2008, 11)

Radial and Transverse Accelerations (Definition):

(Meerut 2006, 08, 09, 11; Agra 11)

The resolved parts of the acceleration at P along and perpendicular to the radius vector OP are called the **radial** *and* **transverse accelerations** *of the particle at P.*

If **a** is the acceleration vector of the particle at P, then

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left(\frac{dr}{dt}\,\mathbf{e}_r + r\,\frac{d\theta}{dt}\,\mathbf{e}_\theta\right), \qquad \text{[Substituting for } \mathbf{v} \text{ from (2)]}$$

$$= \left\{\frac{d}{dt}\left(\frac{dr}{dt}\right)\right\}\mathbf{e}_r + \left(\frac{dr}{dt}\right)\frac{d\mathbf{e}_r}{dt} + \left\{\frac{d}{dt}\left(r\,\frac{d\theta}{dt}\right)\right\}\mathbf{e}_\theta + \left(r\,\frac{d\theta}{dt}\right)\frac{d\mathbf{e}_\theta}{dt}$$

$$= \frac{d^2r}{dt^2}\,\mathbf{e}_r + \frac{dr}{dt}\cdot\frac{d\theta}{dt}\,\mathbf{e}_\theta + \left(\frac{dr}{dt}\cdot\frac{d\theta}{dt} + r\,\frac{d^2\theta}{dt^2}\right)\mathbf{e}_\theta - r\,\frac{d\theta}{dt}\cdot\frac{d\theta}{dt}\,\mathbf{e}_r$$

[From (1)]

$$= \left\{\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right\}\mathbf{e}_r + \left\{2\,\frac{dr}{dt}\,\frac{d\theta}{dt} + r\,\frac{d^2\theta}{dt^2}\right\}\mathbf{e}_\theta\,.$$

Thus the vector **a** has been expressed as a linear combination of the vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$. The coefficients of $\mathbf{e}_r$ and $\mathbf{e}_\theta$ in this linear combination will give us respectively the radial and transverse accelerations of the particle at P.

$\therefore$ The radial component of acceleration at $P = \dfrac{d^2r}{dt^2} - r\left(\dfrac{d\theta}{dt}\right)^2$

$$= \ddot{r} - r\,\dot{\theta}^2, \text{ +ive in the direction of } r \text{ increasing,}$$

and the transverse component of acceleration at P

$$= 2\,\frac{dr}{dt}\,\frac{d\theta}{dt} + r\,\frac{d^2\theta}{dt^2} = \frac{1}{r}\,\frac{d}{dt}\left(r^2\,\frac{d\theta}{dt}\right),$$

+ive in the direction of θ increasing.

Thus **remember** that at any time t,

the **radial acceleration** $= \dfrac{d^2r}{dt^2} - r\left(\dfrac{d\theta}{dt}\right)^2$, +ive in the direction of r increasing

and the **transverse acceleration** $= \dfrac{1}{r}\,\dfrac{d}{dt}\left(r^2\,\dfrac{d\theta}{dt}\right)$, +ive in the direction of θ increasing.

(Agra 2007; Kanpur 10)

Also the **resultant acceleration**

$$= \sqrt{\left[\left\{\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right\}^2 + \left\{\frac{1}{r}\,\frac{d}{dt}\left(r^2\,\frac{d\theta}{dt}\right)\right\}^2\right]}.$$

Illustrative Examples

Example 8: *A particle P describes a curve with constant velocity and its angular velocity about a given fixed point O varies inversely as its distance from O, show that the curve is an equiangular spiral.* **(Lucknow 2007; Bundelkhand 11)**

Solution: Let the velocity of the particle be equal to v (constant). Given that the angular velocity $d\theta/dt$ of the particle about the fixed point O varies inversely as its distance r from O, we have

$$d\theta/dt \propto 1/r$$

or $$d\theta/dt = \lambda/r, \qquad \text{...(1)}$$

where λ is a constant.

Now the velocity v of the particle is the resultant of its radial and transverse velocities.

$\therefore$ $$v = \sqrt{[(\text{radial vel.})^2 + (\text{trans. vel.})^2]}$$

$$= \sqrt{\left[\left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2\right]}$$

or $$v^2 = (dr/dt)^2 + \lambda^2, \qquad \text{[From (1)]}$$

or $$dr/dt = \sqrt{(v^2 - \lambda^2)} = \mu, \text{ where } \sqrt{(v^2 - \lambda^2)} = \mu, \text{ a constant}$$

or $$\frac{dr}{d\theta} \cdot \frac{d\theta}{dt} = \mu$$

or $$\frac{dr}{d\theta} \cdot \frac{\lambda}{r} = \mu \qquad \text{[From (1)]}$$

or $$\frac{dr}{r} = \frac{\mu}{\lambda}\, d\theta.$$

Integrating, $\log r = (\mu/\lambda)\,\theta + \log c$, where c is constant

or $$\log (r/c) = (\mu/\lambda)\,\theta.$$

$\therefore$ $$r = ce^{(\mu/\lambda)\,\theta}$$

or $$r = ce^{k\theta}, \quad \text{where } \mu/\lambda = k, \text{ a constant.}$$

This is the equation of an equiangular spiral.

Hence the curve is an equiangular spiral.

Example 9: *The velocities of a particle along and perpendicular to the radius vector are* λr *and* $\mu\,\theta$; *find the path and show that the accelerations along and perpendicular to the radius vector are* $\lambda^2 r - \mu^2 \theta^2/r$ *and* $\mu\,\theta\,(\lambda + \mu/r)$.

(Rohilkhand 2006, 09; Lucknow 10; Purvanchal 11)

Solution: Here, it is given that

radial velocity $= dr/dt = \lambda r \qquad$ ...(1)

and transverse velocity $= r\,(d\theta/dt) = \mu\theta. \qquad$ ...(2)

Dividing (1) by (2), we have

$$\frac{dr}{r\,d\theta} = \frac{\lambda r}{\mu\,\theta} \quad \text{or} \quad \frac{\mu}{\lambda}\frac{dr}{r^2} = \frac{d\theta}{\theta}.$$

Integrating, $-\dfrac{\mu}{\lambda r} = \log\theta + \log c = \log (c\,\theta)$.

$\therefore$ $$c\,\theta = e^{-\mu/\lambda r} \quad \text{or} \quad \theta = a\,e^{b/r},$$

which is the equation of the path, where a and b are constants.

Now radial acceleration $= \dfrac{d^2 r}{dt^2} - r\left(\dfrac{d\theta}{dt}\right)^2$

$$= \frac{d}{dt}\left(\frac{dr}{dt}\right) - \frac{1}{r}\left(r\,\frac{d\theta}{dt}\right)^2$$

$$= \frac{d}{dt}(\lambda\, r) - \frac{1}{r}(\mu\theta)^2 \qquad \text{[From (1) and (2)]}$$

$$= \lambda\,\frac{dr}{dt} - \frac{\mu^2\theta^2}{r} = \lambda^2\, r - \frac{\mu^2\theta^2}{r} \qquad \text{[From (1)]}$$

Again transverse acceleration $= \dfrac{1}{r}\dfrac{d}{dt}\left(r^2\,\dfrac{d\theta}{dt}\right)$

$$= \frac{1}{r}\frac{d}{dt}(r\,.\,\mu\theta) \qquad \text{[From (2)]}$$

$$= \frac{\mu}{r}\left(\frac{dr}{dt}\theta + r\,\frac{d\theta}{dt}\right) = \frac{\mu}{r}(\lambda\, r\,\theta + \mu\theta) \qquad \text{[From (1) and (2)]}$$

$$= \mu\,\theta\,(\lambda + \mu/r).$$

Example 10: *Show that the path of a point P which possesses two constant velocities u and v, the first of which is in a fixed direction and the other is perpendicular to the radius OP drawn from a fixed point O, is a conic whose focus is O and eccentricity is u/v.*

Solution: Take the fixed point O as pole and the fixed direction as the initial line OX. Let $P\,(r,\theta)$ be the position of the particle at any time t. Then according to the question P possesses two constant velocities : (i) u, in the fixed direction OX and (ii) v, perpendicular to OP as shown in the figure.

Resolving the velocities of P along and perpendicular to the radius vector OP, we have

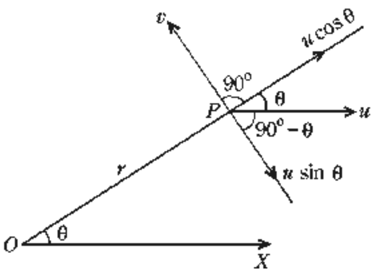
the radial velocity $= dr/dt = u\cos\theta$, ...(1)

and the transverse velocity $= r\,(d\theta/dt) = v - u\sin\theta$. ...(2)

Dividing (1) by (2), we have

$$\frac{dr}{r\,d\theta} = \frac{u\cos\theta}{v - u\sin\theta}$$

or
$$\frac{dr}{r} = \frac{u\cos\theta}{v - u\sin\theta}\,d\theta.$$



Integrating, $\log r = -\log(v - u\sin\theta) + \log C$

or
$$\log\left(\frac{C}{r}\right) = \log(v - u\sin\theta)$$

or
$$\frac{C}{r} = v - u\sin\theta \quad \text{or} \quad \frac{C}{r} = v + u\cos\left(\frac{1}{2}\pi + \theta\right)$$

or
$$\frac{C/v}{r} = 1 + \frac{u}{v}\cos\left(\frac{1}{2}\pi + \theta\right), \qquad ...(3)$$

which is the path of the particle.

The equation (3) is of the form $l/r = 1 + e\cos\theta$, which is a conic whose focus is the pole O and eccentricity e is u/v.

Hence, the path of P is a conic whose focus is O and eccentricity is u/v.

Example 11: *A particle moves along a circle $r = 2\,a\cos\theta$ in such a way that its acceleration towards the origin is always zero. Show that the transverse acceleration varies as the fifth power of cosec θ.* **(Avadh 2007)**

Solution: The equation of the path is $r = 2\,a\cos\theta$. ...(1)

Now according to the question the acceleration of the particle towards the origin is always zero *i.e.*, the radial acceleration is always zero.

$$\therefore \qquad \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = 0. \qquad \text{...(2)}$$

From (1), $\dfrac{dr}{dt} = -2\,a\sin\theta\,\dfrac{d\theta}{dt}$,

$$\text{and} \qquad \frac{d^2r}{dt^2} = -2\,a\sin\theta\,\frac{d^2\theta}{dt^2} - 2\,a\cos\theta\,\frac{d\theta}{dt}\,\frac{d\theta}{dt}$$

$$= -2\,a\sin\theta\,\frac{d^2\theta}{dt^2} - 2\,a\cos\theta\left(\frac{d\theta}{dt}\right)^2.$$

Substituting the values of $\dfrac{d^2r}{dt^2}$ and r in (2), we have

$$-2\,a\sin\theta\,\frac{d^2\theta}{dt^2} - 2\,a\cos\theta\left(\frac{d\theta}{dt}\right)^2 - 2\,a\cos\theta\left(\frac{d\theta}{dt}\right)^2 = 0$$

$$\text{or} \qquad \frac{d^2\theta}{dt^2} = -2\,\frac{\cos\theta}{\sin\theta}\left(\frac{d\theta}{dt}\right)^2$$ **(Bundelkhand 2010; Agra 11)**

$$\text{or} \qquad \frac{d^2\theta/dt^2}{d\theta/dt} = -2\,\frac{\cos\theta}{\sin\theta}\,\frac{d\theta}{dt}.$$

Integrating with respect to t, we get

$$\log(d\theta/dt) = -2\log\sin\theta + \log C, \text{ where } C \text{ is a constant}$$

$$= \log(C/\sin^2\theta) = \log(C\operatorname{cosec}^2\theta).$$

$$\therefore \qquad d\theta/dt = C\operatorname{cosec}^2\theta. \qquad \text{...(3)}$$

Now the transverse acceleration $= \dfrac{1}{r}\dfrac{d}{dt}\left(r^2\dfrac{d\theta}{dt}\right)$

$$= \frac{1}{2\,a\cos\theta}\,\frac{d}{dt}\{4\,a^2\cos^2\theta\,.\,C\operatorname{cosec}^2\theta\}$$

[Substituting for r and $d\theta/dt$ from (1) and (3)]

$$= \frac{4\,Ca^2}{2\,a\cos\theta}\,\frac{d}{dt}(\cot^2\theta) = \frac{2\,Ca}{\cos\theta}\,2\cot\theta\,.\,(-\operatorname{cosec}^2\theta)\cdot\frac{d\theta}{dt}$$

$$= -\frac{4Ca}{\cos\theta}\cot\theta\,.\operatorname{cosec}^2\theta\,.\,C\operatorname{cosec}^2\theta \qquad \left[\because \frac{d\theta}{dt} = C\operatorname{cosec}^2\theta\right]$$

$$= -4C^2\,a\operatorname{cosec}^5\theta.$$

Hence the transverse acceleration $\propto \operatorname{cosec}^5\theta$.

Comprehensive Exercise 3

1. If the angular velocity of a point moving in a plane curve be constant about a fixed origin, show that its transverse acceleration varies as its radial velocity.
(Garhwal 2002; Kanpur 09, 11)

2. A point P describes, with a constant angular velocity about O, the equi-angular spiral $r = ae^{\theta}$, O being the pole of the spiral. Obtain the radial and transverse accelerations of P. **(Lucknow 2006, 07, 09; Rohilkhand 07, 08; Meerut 10)**

3. The velocities of a particle along and perpendicular to a radius vector from a fixed origin are λr^2 and $\mu\theta^2$, where λ and μ are constants. Find the polar equation of the path of the particle and also its radial and transverse accelerations in terms of r and θ only. **(Avadh 2008; Agra 08; Kanpur 10)**

4. The acceleration of a point moving in a plane curve is resolved into two components, one parallel to the initial line and the other along the radius vector; prove that these components are
$$-\frac{1}{r\sin\theta}\cdot\frac{d}{dt}(r^2\dot{\theta}) \quad \text{and} \quad \frac{\cot\theta}{r}\frac{d}{dt}(r^2\dot{\theta}) + \ddot{r} - r\dot{\theta}^2.$$
(Garhwal 2001)

5. An insect crawls at a constant rate u along the spoke of a cart wheel of radius a, the cart is moving with velocity v. Find the acceleration along and perpendicular to the spoke. **(Kumaun 2003; Kanpur 07)**

6. A boat which is rowed with constant velocity u starts from a point A on the bank of a river which flows with a constant velocity v, and it points always towards a point B on the other bank exactly opposite to A; find the equation of the path of the boat.

   If $v = u$, show that the path is a parabola whose focus is B.

7. A straight smooth tube revolves with angular velocity ω in a horizontal plane about one extremity which is fixed. If at zero time a particle inside it be at a distance a from a fixed end and moving with velocity V along the tube, show that its distance at time t is

   $a\cosh\omega t + (V/\omega)\sinh\omega t.$

8. A ring which can slide on a thin long smooth rod rests at a distance d from one end O. The rod is then set revolving uniformly about O in a horizontal plane; show that in space the ring describes the curve $r = d\cosh\theta$.

9. A small ring is at rest on a smooth straight horizontal rod of length a at distance b from one end of the rod. The rod is then suddenly set rotating in a horizontal plane about the end O with constant angular velocity ω. Prove that the ring will leave the rod with velocity $\omega\sqrt{(2a^2 - b^2)}$ after a time $(1/\omega)\cosh^{-1}(a/b)$.

10. A small bead slides with constant speed v on a smooth wire in the shape of the cardioid $r = a(1 + \cos\theta)$. Show that the angular velocity is $(v/2a)\sec\frac{1}{2}\theta$ and that the radial component of the acceleration is constant.

Answers 3

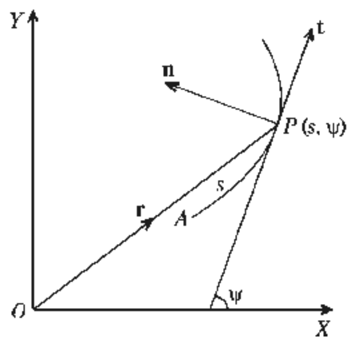
2. $0, 2\,\omega^2 r$
3. $\mu/2r^2 + c = \lambda/\theta, 2\,\lambda^2 r^3 - (1/r)\,\mu^2\,\theta^4, \lambda\,\mu r\,\theta^2 + (1/r)\,2\,\mu^2\,\theta^3$
5. $-\,ut\,v^2/a^2, 2\,uv/a$
6. $c/r = \sin\theta/\left(\tan\frac{1}{2}\theta\right)^{u/v}$

2.7 Tangential and Normal Velocities and Accelerations

(Rohilkhand 2006; Kanpur 07, 08; Avadh 08; Agra 09; Bundelkhand 11)

Tangential and Normal Velocities

Let P be the position of a moving particle at time t and $\mathbf{r}$ its position vector with respect to the origin O. Let arc $AP = s$ and let ψ be the angle which the tangent at P to the path of the particle makes with OX. Then (s, ψ) are the *intrinsic coordinates* of P.



Let $\mathbf{t}$ denote the unit vector along the tangent at P in the direction of s **increasing** and $\mathbf{n}$ the unit vector along the normal at P in the direction of ψ increasing *i.e.*, in the direction of **inwards drawn normal.**

From vector calculus, we have

$$\frac{d\mathbf{r}}{ds} = \mathbf{t}.$$

[Remember that for a curve, $d\mathbf{r}/ds$ denotes the unit tangent vector in the direction of s increasing].

Also $$\frac{d\mathbf{t}}{dt} = \frac{d\psi}{dt}\,\mathbf{n}. \qquad \text{[Refer 2.4]} \quad \ldots(1)$$

If $\mathbf{v}$ is the velocity vector of the particle at P, then

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}}{ds}\,\frac{ds}{dt} = \frac{ds}{dt}\,\mathbf{t} \qquad \left[\because \frac{d\,\mathbf{r}}{ds} = \mathbf{t}\right]$$

$$= \frac{ds}{dt}\,\mathbf{t} + 0\,\mathbf{n}.$$

Thus the vector $\mathbf{v}$ has been expressed as a linear combination of the vectors $\mathbf{t}$ and $\mathbf{n}$.

$\therefore$ **the tangential velocity** at P = the coefficient of $\mathbf{t}$ in (2)

$$= \frac{ds}{dt}, \text{ +ive in the direction of } s \text{ increasing,}$$

and the **normal velocity** at P = the coefficient of $\mathbf{n}$ in (2)

$$= 0.$$

(Kanpur 2010)

If v is the resultant velocity of the particle at P, then $v = \dfrac{ds}{dt}$ and is along the tangent at P.

Thus remember that *the resultant velocity of a particle is always along the tangent to its path.*

Tangential and Normal Accelerations: If $\mathbf{a}$ is the acceleration vector of the particle at P, then

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}(v\mathbf{t}) \qquad \left[\because \ \mathbf{v} = \frac{ds}{dt}\mathbf{t} = v\,\mathbf{t}, \text{ where } v = \frac{ds}{dt}\right]$$

$$= \frac{dv}{dt}\mathbf{t} + v\frac{d\mathbf{t}}{dt}$$

$$= \frac{dv}{dt}\mathbf{t} + v\frac{d\psi}{dt}\mathbf{n} \qquad \text{[From (1)]}$$

$$= \frac{dv}{dt}\mathbf{t} + v\frac{d\psi}{ds}\frac{ds}{dt}\mathbf{n}$$

$$= \frac{dv}{dt}\mathbf{t} + \left(v \cdot \frac{1}{\rho} \cdot v\right)\mathbf{n}, \qquad \left[\because \ \rho = \text{radius of curvature at } P = \frac{ds}{d\psi}\right]$$

$$= \frac{dv}{dt}\mathbf{t} + \frac{v^2}{\rho}\mathbf{n}.$$

Thus the acceleration vector $\mathbf{a}$ has been expressed as a linear combination of the vectors $\mathbf{t}$ and $\mathbf{n}$. The coefficients of $\mathbf{t}$ and $\mathbf{n}$ in this linear combination give us respectively the tangential and normal accelerations of the particle at P.

$\therefore$ the **tangential acceleration** of $P = \dfrac{dv}{dt}$, $+$ive in the direction of s increasing

and the **normal acceleration** of $P = \dfrac{v^2}{\rho}$, $+$ive in the direction of inwards drawn normal. **(Kanpur 2010)**

Other Expressions for the Tangential Acceleration :

(i) Tangential acceleration $= \dfrac{dv}{dt} = \dfrac{d}{dt}\left(\dfrac{ds}{dt}\right) = \dfrac{d^2s}{dt^2}$.

(ii) Tangential acceleration $= \dfrac{dv}{dt} = \dfrac{dv}{ds}\dfrac{ds}{dt} = v\dfrac{dv}{ds}$.

The resultant acceleration of the particle at time t

$$= \sqrt{\left[\left(\frac{d^2s}{dt^2}\right)^2 + \left(\frac{v^2}{\rho}\right)^2\right]}.$$

Illustrative Examples

Example 12: *Prove that the acceleration of a point moving in a curve with uniform speed is* $\rho\,(d\psi/dt)^2$. **(Rohilkhand 2008; Bundelkhand 09; Lucknow 08, 10)**

Solution: Here $v = ds/dt =$ constant.

$\therefore \qquad \dfrac{d^2 s}{dt^2} = 0.$

$$\text{Acceleration} = \sqrt{\left[\left(\frac{d^2 s}{dt^2}\right)^2 + \left(\frac{v^2}{\rho}\right)^2\right]}$$

$$= \frac{v^2}{\rho} = \rho\left(\frac{v}{\rho}\right)^2$$

$$= \rho\left[\frac{ds/dt}{ds/d\psi}\right]^2 \qquad \left[\because\ \rho = \frac{ds}{d\psi}\right]$$

$$= \rho\left(\frac{d\psi}{dt}\right)^2 = \rho\dot{\psi}^2.$$

Example 13: *A particle describes a curve (for which s and ψ vanish simultaneously) with uniform speed v. If the acceleration at any point s be $v^2 c/(s^2 + c^2)$, find the intrinsic equation of the curve.* **(Meerut 2009; Bundelkhand 10; Avadh 11)**

Solution: *Here,* $v = ds/dt = constant.$

$\therefore \qquad \dfrac{d^2 s}{dt^2} = 0.$

$\therefore$ Acceleration at any point 's'

$$= \sqrt{\left[\left(\frac{d^2 s}{dt^2}\right)^2 + \left(\frac{v^2}{\rho}\right)^2\right]} = \frac{v^2}{\rho}.$$

But it is given that the acceleration at any point 's' $= \dfrac{v^2 c}{(s^2 + c^2)}$.

$\therefore \qquad \dfrac{v^2}{\rho} = \dfrac{v^2 c}{(s^2 + c^2)}$

or $\qquad \dfrac{1}{\rho} = \dfrac{d\psi}{ds} = \dfrac{c}{(s^2 + c^2)} \qquad [\because\ v^2 \neq 0]$

or $\qquad d\psi = \dfrac{c}{(s^2 + c^2)}\, ds,$ separating the variables.

Integrating,

$\psi = \tan^{-1}(s/c) + A$, where A is a constant.

Given that $\psi = 0$, when $s = 0$.

$\therefore \qquad 0 = 0 + A \quad \text{or} \quad A = 0.$

$\therefore \qquad \psi = \tan^{-1}(s/c) \quad \text{or} \quad s = c\tan\psi,$

which is the intrinsic equation of the curve and is a catenary.

Example 14: *A point moves in a plane curve so that its tangential acceleration is constant, and the magnitudes of the tangential velocity and normal acceleration are in a constant ratio ; find the intrinsic equation of the curve.*

Solution: Here, it is given that

$$\text{tangential acceleration} = \frac{dv}{dt} = \lambda \text{ (a constant)}, \qquad ...(1)$$

and $$\frac{\text{tangential velocity}}{\text{normal acceleration}} = \frac{v}{v^2/\rho} = \frac{\rho}{v} = \mu \text{ (a constant)} \qquad ...(2)$$

From (2), we have $\dfrac{ds/d\psi}{ds/dt} = \mu$ $\left[\because \rho = \dfrac{ds}{d\psi}, v = \dfrac{ds}{dt}\right]$

or $$\frac{dt}{d\psi} = \mu$$

or $$\frac{d\psi}{dt} = \frac{1}{\mu}. \qquad ...(3)$$

Now from (1), we have

$$\frac{dv}{d\psi} \cdot \frac{d\psi}{dt} = \lambda$$

or $$\frac{dv}{d\psi} \cdot \frac{1}{\mu} = \lambda \qquad \left[\because \frac{d\psi}{dt} = \frac{1}{\mu}\right]$$

or $$dv = \mu\lambda d\psi.$$

Integrating, we have $v = \mu\lambda\psi + k$, ...(4)

where k is a constant.

Now from (2), $\rho = \mu v$.

$\therefore$ $$\frac{ds}{d\psi} = \mu(\mu\lambda\psi + k), \text{ substituting for } v \text{ from (4)}$$

or $$ds = (\mu^2 \lambda\psi + k\mu)\, d\psi.$$

Integrating, $s = \dfrac{1}{2}\mu^2 \lambda\psi^2 + k\mu\psi + C$

or $$s = A\psi^2 + B\psi + C, \text{ when } A = \frac{1}{2}\mu^2 \lambda, B = k\mu.$$

Hence the intrinsic equation of the path is

$$s = A\psi^2 + B\psi + C, \text{ where } A, B, C \text{ are constants.}$$

Example 15: *A particle is moving in a parabola with uniform angular velocity about the focus ; prove that its normal acceleration at any point is proportional to the radius of curvature of its path at that point.* **(Meerut 2006; Rohilkhand 07; Lucknow 09; Agra 10)**

Solution: The pedal equation of a parabola referred to the focus as pole is

$$p^2 = ar. \qquad ...(1)$$

Since the particle moves with uniform angular velocity about the focus (*i.e.*, about the pole), therefore

$$\frac{d\theta}{dt} = \frac{vp}{r^2} = c = \text{a constant}$$

or $$v = r^2 c/p. \qquad ...(2)$$

From (1), we have $2p\dfrac{dp}{dr} = a$, or $\dfrac{dr}{dp} = \dfrac{2p}{a}\cdot$

$\therefore$ $$\rho = r\frac{dr}{dp} = r\cdot\frac{2p}{a}\cdot \qquad \ldots(3)$$

Now $$\frac{\text{Normal acceleration}}{\rho} = \frac{v^2/\rho}{\rho}$$

$$= \frac{v^2}{\rho^2} = \frac{(r^2c/p)^2}{(2\,pr/a)^2}, \qquad \text{substituting for } v \text{ and } \rho \text{ from (2) and (3)}$$

$$= \frac{r^2c^2a^2}{4p^4} = \frac{r^2c^2a^2}{4\,(ar)^2}, \qquad \text{substituting for } p^2 \text{ from (1)}$$

$$= c^2/4.$$

Thus normal acceleration $= (c^2/4)\,\rho$.

Hence normal acceleration $\propto \rho$.

Comprehensive Exercise 4

1. If the velocity of a point moving in a plane curve varies as the radius of curvature, show that the direction of motion revolves with constant angular velocity.
2. A point describes the cycloid $s = 4a\sin\psi$ with uniform speed v. Find its acceleration at any point. **(Rohilkhand 2007, 09)**
3. If the tangential and normal accelerations of a particle describing a plane curve be constant throughout, prove that the radius of curvature at any point t is given by $\rho = (at + b)^2$. **(Avadh 2008; Bundelkhand 08; Meerut 10)**
4. A point moves along the arc of a cycloid in such a manner that the tangent at it rotates with a constant angular velocity. Show that the acceleration of the moving point is constant in magnitude.
5. The rate of change of direction of velocity of a particle moving in a cycloid is constant. Prove that acceleration must be constant in magnitude.
6. A particle moves in a plane in such a manner that its tangential and normal accelerations are always equal and its velocity varies as $\exp\{\tan^{-1}(s/c)\}$, s being the length of the arc of the curve measured from a fixed point on the curve. Find the path. **(Agra 2008; Meerut 11)**
7. A point moves in a plane curve, so that its tangential and normal accelerations are equal and the angular velocity of the tangent is constant. Find the curve. **(Lucknow 2006, 10; Kanpur 09; Avadh 11)**
8. A particle describes a circle of radius r with a uniform speed v, show that its acceleration at any point of the path is v^2/r and is directed towards the centre of the circle. **(Agra 2002)**

9. A particle is describing a plane curve. If the tangential and normal accelerations are each constant throughout the motion, prove that the angle ψ, through which the direction of motion turns in time t is given by $\psi = A \log (1 + Bt)$.

(Avadh 2006; Lucknow 07, 11; Purvanchal 11)

Answers 4

2. $v^2 / \sqrt{(16a^2 - s^2)}$
6. $s = c \tan \psi$
7. $s = (A/C)\, e^{\psi} + B$

Illustrative Examples

Example 16: *A particle describes a cycloid with uniform speed. Prove that the normal acceleration at any point varies inversely as the square root of the distance from the base of the cycloid.*

Solution: The intrinsic equation of a cycloid is $s = 4a \sin \psi$ ...(1)

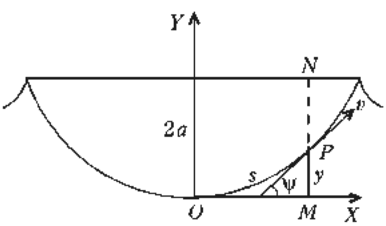
Here $v = ds/dt =$ constant.

Normal acceleration $= \dfrac{v^2}{\rho} = \dfrac{v^2}{ds/d\psi} = \dfrac{v^2}{4\,a \cos \psi} = \dfrac{v^2}{4\,a\,\sqrt{(1 - \sin^2 \psi)}}$

$= \dfrac{v^2}{4\,a\,\sqrt{\{1 - (s^2/16a^2)\}}}$, substituting for $\sin \psi$ from (1)

$= \dfrac{v^2}{\sqrt{(16\,a^2 - s^2)}}$. ...(2)

If y is the distance of any point P of the cycloid from OX (*i.e.*, the tangent at the vertex), then the distance of P from the base of the cycloid



$= NP = MN - PM$

$= 2a - y.$

If arc $OP = s$, then for the cycloid, we have

$s^2 = 8ay.$ **[Remember]**

$\therefore$ from (2), we have normal acceleration

$= \dfrac{v^2}{\sqrt{(16\,a^2 - 8ay)}}$

$= \dfrac{v^2}{\sqrt{[8a\,(2a - y)]}}$

$$= \frac{v^2}{\sqrt{[8a \,.\, NP]}} \propto \frac{1}{\sqrt{(NP)}}, \text{ because } \frac{v^2}{\sqrt{(8a)}} \text{ is constant.}$$

$\therefore$ the normal acceleration varies inversely as the square root of the distance from the base of the cycloid.

Example 17: *A point starts from the origin in the direction of the initial line with velocity f/ω and moves with constant angular velocity ω about the origin and with constant negative radial acceleration f. Show that the rate of growth of the radial velocity is never positive, but tends to the limit zero, and prove that the equation of the path is $\omega^2 r = f(1 - e^{-\theta})$.*

Solution: Here, the angular velocity $= d\theta/dt = \omega =$ constant, ...(1)

and radial acceleration $= \dfrac{d^2r}{dt^2} - r\left(\dfrac{d\theta}{dt}\right)^2 = -f.$ ...(2)

From (1) and (2), we have

$$\frac{d^2r}{dt^2} = r\omega^2 - f. \qquad ...(3)$$

Multiplying both sides of (3) by $2(dr/dt)$ and integrating w.r.t. 't', we have

$$(dr/dt)^2 = r^2\omega^2 - 2fr + A, \qquad ...(4)$$

where A is constant.

But the particle starts from the origin in the direction of the initial line with velocity f/ω.

$\therefore$ initially, when $r = 0$, $\quad dr/dt = f/\omega$.

$\therefore$ from (4), we have $\quad (f/\omega)^2 = A$.

$$\therefore \quad \left(\frac{dr}{dt}\right)^2 = r^2\omega^2 - 2fr + \frac{f^2}{\omega^2} = \left(\frac{f}{\omega} - r\omega\right)^2$$

or $\quad \dfrac{dr}{dt} = \dfrac{f}{\omega} - r\omega \quad$ [the positive sign is taken because the particle moves in the direction of r increasing]

or $\quad \dfrac{dr}{d\theta} \cdot \dfrac{d\theta}{dt} = \dfrac{f}{\omega} - r\omega$

or $\quad \dfrac{dr}{d\theta} \cdot \omega = \dfrac{f}{\omega} - r\omega \quad$ [From (1)]

or $\quad \dfrac{dr}{(f/\omega^2) - r} = d\theta.$

Integrating, we have

$$-\log\left(\frac{f}{\omega^2} - r\right) = \theta + B, \text{ where } B \text{ is a constant.}$$

But initially $\theta = 0$ and $r = 0$; therefore $\quad B = -\log(f/\omega^2)$.

$$\therefore \quad -\log\left(\frac{f}{\omega^2} - r\right) = \theta - \log\left(\frac{f}{\omega^2}\right)$$

or $\qquad \log\left(\dfrac{f}{\omega^2} - r\right) - \log\left(\dfrac{f}{\omega^2}\right) = -\theta$

or $\qquad \log \dfrac{(f/\omega^2 - r)}{f/\omega^2} = -\theta \quad$ or $\quad \dfrac{(f/\omega^2 - r)}{f/\omega^2} = e^{-\theta}$

or $\qquad f/\omega^2 - r = (f/\omega^2)\, e^{-\theta} \quad$ or $\quad r = (f/\omega^2)(1 - e^{-\theta})$

or $\qquad \omega^2 r = f\,(1 - e^{-\theta}), \qquad \ldots(5)$

which is the required equation of the path.

From (3) and (5), we have

$$\frac{d^2 r}{dt^2} = f\,(1 - e^{-\theta}) - f = -\frac{f}{e^\theta}.$$

Thus the rate of growth of the radial velocity

$$= \frac{d}{dt}\left(\frac{dr}{dt}\right) = \frac{d^2 r}{dt^2} = -\frac{f}{e^\theta},$$

which is never positive because f/e^θ is positive for all values of θ.

Also f/e^θ tends to the limit 0 as $\theta \to \infty$.

Example 18: *A particle moves in a catenary $s = c \tan \psi$, the direction of its acceleration at any point makes equal angles with the tangent and normal to the path at the point. If the speed at the vertex (where $\psi = 0$) be u, show that the velocity and acceleration at any other point ψ are ue^ψ and $(\sqrt{2}/c)\, u^2 e^{2\psi} \cos^2 \psi$.*

Solution: It is given that the direction of acceleration at any point makes equal angles with the tangent and normal to the path at the point. Therefore the tangential and normal accelerations will be equal at any time t

i.e., $\qquad v\dfrac{dv}{ds} = \dfrac{v^2}{\rho} \qquad \ldots(1)$

or $\qquad \dfrac{dv}{ds}\cdot \rho = v \quad$ or $\quad \dfrac{dv}{ds}\cdot\dfrac{ds}{d\psi} = v \qquad \left[\because\ \rho = \dfrac{ds}{d\psi}\right]$

or $\qquad \dfrac{dv}{v} = d\psi.$

Integrating, $\log v = \psi + \log A$, where A is a constant.

But at the vertex, where $\psi = 0$, $v = u$.

$\therefore \qquad \log A = \log u \quad$ or $\quad A = u.$

$\therefore \qquad \log v = \psi + \log u \quad$ or $\quad \log v - \log u = \psi$

or $\qquad \log (v/u) = \psi \quad$ or $\quad v = ue^\psi, \qquad \ldots(2)$

which gives the velocity of the particle at any point.

Further it is given that the path of the particle is the catenary

$$s = c \tan \psi.$$

$\therefore \qquad \rho = ds/d\psi = c \sec^2 \psi.$

$\therefore$ the resultant acceleration of the particle

$$= \sqrt{[(\text{tangential accel.})^2 + (\text{normal accel.})^2]}$$

$$= \sqrt{\left\{\left(v\frac{dv}{ds}\right)^2 + \left(\frac{v^2}{\rho}\right)^2\right\}} = \sqrt{\left\{\left(\frac{v^2}{\rho}\right)^2 + \left(\frac{v^2}{\rho}\right)^2\right\}} \quad \text{[From (1)]}$$

$$= \left(\frac{v^2}{\rho}\right)\cdot\sqrt{2} = \sqrt{2}\cdot\frac{(ue^{\psi})^2}{c\sec^2\psi} = \frac{\sqrt{2}}{c}u^2 e^{2\psi}\cos^2\psi.$$

Example 19: *A particle, projected with a velocity u, is acted on by a force which produces a constant acceleration f in the plane of the motion inclined at a constant angle α with the direction of motion. Obtain the intrinsic equation of the curve described, and show that the particle will be moving in the opposite direction to that of projection at time*

$$\frac{u}{f\cos\alpha}(e^{\pi\cot\alpha} - 1).$$

Solution: Let $P(s, \psi)$ be the position of the particle at any time t. It is given that P possesses a constant acceleration f inclined at a constant angle α to the tangent at P. Therefore resolving the acceleration of P along the tangent and normal at P, we have

the tangential acceleration $= v\dfrac{dv}{ds} = f\cos\alpha$, ...(1)

and the normal acceleration $= \dfrac{v^2}{\rho} = f\sin\alpha$. ...(2)

Dividing (1) by (2), we have

$$\frac{1}{v}\frac{dv}{ds}\frac{ds}{d\psi} = \cot\alpha \qquad \left[\because\ \rho = \frac{ds}{d\psi}\right]$$

or $(1/v)\,dv = \cot\alpha\, d\psi$.

Integrating, we have

$\log v = \psi\cot\alpha + A$, where A is a constant.

Let $\psi = 0$ at the point of projection.

Then $\psi = 0$, when $v = u$.

$\therefore$ $\log u = A$.

Thus $\log v = \psi\cot\alpha + \log u$

or $\log(v/u) = \psi\cot\alpha$

or $ds/dt = v = u\,e^{\psi\cot\alpha}$. ...(3)

Now from (2), we have

$$\frac{ds}{d\psi} = \rho = \frac{v^2}{f\sin\alpha} = \frac{u^2 e^{2\psi\cot\alpha}}{f\sin\alpha} \qquad \text{...(4)}$$

[Substituting for v from (3)]

$$\therefore \quad ds = \frac{u^2}{f\sin\alpha}e^{2\psi\cot\alpha}\,d\psi.$$

Integrating, we have

$$s = \frac{u^2}{(f \sin \alpha)(2 \cot \alpha)} e^{2 \psi \cot \alpha} + B \text{, where } B \text{ is a constant}$$

$$= \frac{u^2}{2 f \cos \alpha} e^{2 \psi \cot \alpha} + B.$$

Let $s = 0$, where $\psi = 0$.

Then $B = -\dfrac{u^2}{2 f \cos \alpha}\cdot$

$$\therefore \quad s = \frac{u^2}{2 f \cos \alpha} e^{2 \psi \cot \alpha} - \frac{u^2}{2 f \cos \alpha}$$

$$= \frac{u^2}{2 f \cos \alpha} (e^{2 \psi \cot \alpha} - 1),$$

which is the required intrinsic equation of the path of the particle.

Now dividing (4) by (3), we have

$$\frac{ds}{d\psi} \cdot \frac{dt}{ds} = \frac{u e^{\psi \cot \alpha}}{f \sin \alpha}$$

or $$\frac{dt}{d\psi} = \frac{u}{f \sin \alpha} e^{\psi \cot \alpha}.$$

$$\therefore \quad dt = \frac{u}{f \sin \alpha} e^{\psi \cot \alpha} \, d\psi. \qquad \ldots(5)$$

Now the particle will be moving in the direction opposite to the direction of projection (which is $\psi = 0$), when $\psi = \pi$. Therefore if the particle takes time t_1 to reach the point where $\psi = \pi$, then integrating (5), we have

$$\int_0^{t_1} dt = \int_0^{\pi} \frac{u}{f \sin \alpha} e^{\psi \cot \alpha} d\psi$$

i.e., $$t_1 = \frac{u}{(f \sin \alpha)(\cot \alpha)} \left[e^{\psi \cot \alpha} \right]_0^{\pi}$$

$$= \frac{u}{f \cos \alpha} \left[e^{\pi \cot \alpha} - 1 \right],$$

which gives the required time.

Comprehensive Exercise 5

1. A particle falls down a straight line $x = a$, starting from the axis of x. If the distance from the axis of x be $\frac{1}{2} ft^2$ at time t, find the angular velocity and acceleration of the line joining the particle to the origin. How far has the particle dropped when the angular acceleration becomes zero ?

2. A point P describes a circle of radius r with uniform angular velocity ω about the centre. Show that the angular velocity of P about a point Q distant $r/2$ from the centre O fluctuates between 2ω and $2\omega/3$.

3. A point describes a circle of radius a with a uniform speed v; show that the radial and transverse accelerations are $-(v^2/a)\cos\theta$ and $-(v^2/a)\sin\theta$, if a diameter is taken as initial line and one end of the diameter as pole. **(Lucknow 2009)**

4. A particle moving in a plane, describes the equiangular spiral $r = a\,e^{\theta\cot\alpha}$. If the radius vector to the particle has a constant angular velocity, show that the resultant acceleration of the particle makes an angle 2α with the radius vector and is of magnitude v^2/r, where v is the speed. **(Agra 2011)**

5. If a rod which always passes through the origin, rotates with uniform angular velocity ω, while one end describes the curve $r = a + b\,e^{\theta}$, show that radial acceleration of any point of the rod is the same at every instant, and the radial velocity is the same at every point at a given instant.

6. If the radial and transverse velocities of a particle are always proportional to each other,
   (i) Show that the path is an equiangular spiral.
   (ii) If in addition the radial and transverse accelerations are always proportional to each other, show that the velocity of the particle varies as some power of the radius vector. **(Agra 2002; Purvanchal 09)**

7. A curve is described by a particle having a constant acceleration in a direction inclined at a constant angle to the tangent. Show that the curve is an equiangular spiral. **(Agra 2002; Purvanchal 09)**

8. A particle moves on an equiangular spiral $r = ae^{m\theta}$ with uniform speed v. Find its radial and transverse velocities and accelerations. **(Rohilkhand 2006; Lucknow 08; Kanpur 08, 09)**

9. A particle moves in a plane under a constant acceleration μa parallel to OX and an acceleration $-2\mu y$ parallel to OY, where OX and OY are rectangular axes. If the particle starts from rest at a point $(0, a)$, find the path. **(Meerut 2004)**

Answers 5

1. $4\,aft/(4a^2 + f^2 t^4),\ 4af\,(4a^2 - 3f^2 t^4)/(4a^2 + f^2 t^4)^2,\ a/\sqrt{3}$

8. Radial velocity $= \dfrac{mv}{\sqrt{(1+m^2)}}$; transverse velocity $= \dfrac{v}{\sqrt{(1+m^2)}}$

   Radial acceleration $= -\dfrac{v^2}{r\,(1+m^2)}$; transverse acceleration $= \dfrac{mv^2}{r\,(1+m^2)}$

9. $y = a\cos\left(2\sqrt{x/a}\right)$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. If (x, y) be the position of a particle moving in a plane curve at any time t, then the magnitude of its acceleration at time t is :

(a) $\dfrac{d^2x}{dt^2} + \dfrac{d^2y}{dt^2}$ (b) $\sqrt{\left[\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2\right]}$

(c) $\sqrt{\left[\left(\dfrac{d^2x}{dt^2}\right)^2 + \left(\dfrac{d^2y}{dt^2}\right)^2\right]}$ (d) $\dfrac{dx}{dt} + \dfrac{dy}{dt}$

(Bundelkhand 2009)

2. If (r, θ) be the polar coordinates of the position of a particle moving along a plane curve at time t, then the transverse component of its velocity at time t is :

(a) $\dfrac{d\theta}{dt}$ (b) $\dfrac{dr}{dt}$

(c) $r\dfrac{d\theta}{dt}$ (d) $\dfrac{1}{r}\dfrac{d}{dt}\left(r^2\dfrac{d\theta}{dt}\right)$

3. If (s, ψ) be the intrinsic coordinates of the position of a particle moving along a plane curve at time t and v be its velocity at time t, then the tangential component of its acceleration at time t is :

(a) $\dfrac{ds}{dt}$ (b) $\dfrac{d^2s}{dt^2}$

(c) $v\dfrac{dv}{dt}$ (d) $\dfrac{dv}{ds}$

(Bundelkhand 2007)

4. If (s, ψ) be the intrinsic coordinates of the position of a particle moving along a plane curve at time t and v be its velocity at that instant, then the normal component of its acceleration at time t is :

(a) $v\dfrac{d\psi}{dt}$ (b) $v\dfrac{dv}{ds}$

(c) $\dfrac{d^2\psi}{dt^2}$ (d) $\dfrac{d^2s}{dt^2}\cdot\dfrac{d\psi}{dt}$

(Rohilkhand 2010)

5. The magnitude of the resultant acceleration of a particle moving in a plane curve at time t is :

(a) $\sqrt{\left[\left(\dfrac{d^2s}{dt^2}\right)^2 + \left(\dfrac{v^2}{\rho}\right)^2\right]}$ (b) $\dfrac{d^2s}{dt^2} + \dfrac{v^2}{\rho}$

(c) $\left(\dfrac{d^2s}{dt^2}\right)^2 + \left(\dfrac{v^2}{\rho}\right)^2$ (d) $\sqrt{\left[\left(\dfrac{ds}{dt}\right)^2 + \left(\dfrac{v^2}{\rho}\right)^2\right]}$

(Bundelkhand 2008)

6. The angular velocity of a particle moving in a parabolic path about the focus of its path varies as follows as its distance from the focus :

(a) Inversely proportional
(b) Directly proportional
(c) Inversely proportional to the square
(d) Directly proportional to the square

(Garhwal 2002; Rohilkhand 11)

7. If a particle moves along a circle of radius r with centre at pole, then its radial acceleration is given by :

(a) $r^2 \dot{\theta}$
(b) $\ddot{r} - r\dot{\theta}^2$
(c) $\ddot{r} + r\dot{\theta}^2$
(d) $r\dot{\theta}^2$

(Garhwal 2003; Rohilkhand 06)

8. Radial acceleration at a point is :

(a) $\ddot{r} - r\dot{\theta}^2$
(b) $\ddot{r} - \dot{r}\dot{\theta}$
(c) $\dot{r} - r\dot{\theta}$
(d) $\ddot{r} - r\dot{\theta}$

(Garhwal 2004; Avadh 06; Agra 08, 11; Bundelkhand 08)

Fill in the Blank(s)

Fill in the blanks "......" so that the following statements are complete and correct.

1. If $\mathbf{r}$ is the positon vector of a particle moving along a curve at time t and $\mathbf{v}$ is its velocity at time t, then $\mathbf{v} =$ ......
2. If $\mathbf{r}$ is the position vector of a particle moving along a curve at time t and $\mathbf{a}$ is its acceleration at time t, then $\mathbf{a} =$ ......
3. If a particle moves along a curve, then the direction of its velocity at any point of its path is along the ...... to the path at that point.
4. If (x, y) is the position of a particle moving along a plane curve at time t, then the x-component of its acceleration at time t is ....
5. The position of a moving point at time t is given by $x = at^2$, $y = 2\,at$. The magnitude of its velocity at time t is ....
6. If (r, θ) be the polar coordinates of the position of a particle moving along a plane curve at time t, then its radial velocity at time $t =$ ......
7. If (r, θ) be the polar coordinates of the position of a particle moving along a plane curve at time t, then its transverse acceleration at time $t =$ ......
8. If (r, θ) be the polar coordinates of the position of a particle moving along a plane curve at time t, then its radial acceleration at time $t =$ ......
9. A point P describes, with a constant angular velocity ω about O, the equiangular spiral $r = ae^{\theta}$, O being the pole of the spiral. The transverse acceleration of P is ....
10. If a particle moves along a plane curve and v is the magnitude of its velocity at time t and ρ is the radius of curvature of its path at that point, then its normal acceleration at time $t =$ ......

True or False

Write 'T' for true and 'F' for false statement.

1. If the radial and transverse velocities of a particle are always proportional to each other, then the path of the particle is a circle.

2. If the velocity of a point moving in a plane curve varies as the radius of curvature, then the direction of motion revolves with constant angular velocity.
3. If the rate of change of direction of velocity of a particle moving in a cycloid is constant, then the acceleration of the particle is constant in magnitude.
4. The normal component of the velocity of a particle moving in a plane curve is always zero.
5. The normal component of the accleration of a particle moving in a plane curve is always zero.
6. If the angular velocity of a point moving in a plane curve be constant about a fixed origin, then its transverse acceleration varies as its radial velocity.
7. If (x, y) is the position of a particle moving along a plane curve at time t, then its resultant velocity at time t is $\left(\frac{dx}{dt} + \frac{dy}{dt}\right)$.
8. If (r, θ) be the polar coordinates of the position of a particle moving along a plane curve at time t, then its resultant velocity at time t is
$$\sqrt{\left[\left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2\right]}.$$
9. If a particle moves in a plane curve with a constant velocity parallel to the axis of y and a velocity proportional to y parallel to the x-axis, then its path is a parabola.
10. The radial component of the acceleration of a particle moving in a plane curve is
$$\frac{d^2 r}{dt^2} - r\frac{d^2\theta}{dt^2}.$$

Answers

Multiple Choice Questions

1.	(c)	2.	(c)	3.	(b)	4.	(a)	5.	(a)
6.	(a)	7.	(b)	8.	(a)				

Fill in the Blank(s)

1. $\frac{d\mathbf{r}}{dt}$ 2. $\frac{d^2\mathbf{r}}{dt^2}$ 3. tangent 4. $\frac{d^2x}{dt^2}$ 5. $2a\sqrt{(t^2+1)}$

6. $\frac{dr}{dt}$ 7. $\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$ 8. $\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$

9. $2\omega^2 r$ 10. $\frac{v^2}{\rho}$

True or False

1.	*F*	2.	*T*	3.	*T*	4.	*T*	5.	*F*
6.	*T*	7.	*F*	8.	*T*	9.	*T*	10.	*F*

□

Chapter

3

Constrained Motion on Smooth and Rough Plane Curves
(Vertical Circle and Cycloid)

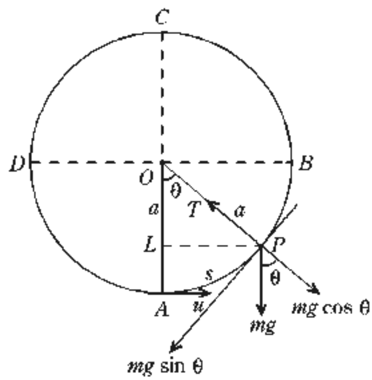
3.1 Introduction

The motion of a particle is called **constrained motion**, *if it is compelled to move along a given curve or surface.* **(Agra 2003, 10; Meerut 04, 06, 07, 11)**

Here in this chapter we shall consider the motion on smooth plane curves, vertical circle and cycloid only.

3.2 Motion in a Smooth Vertical Circle

A heavy particle is tied to one end of a light inextensible string whose other end is attached to a fixed point. It is projected horizontally with a given velocity u from its vertical position of equilibrium; to discuss the subsequent motion.



Let one end of a string of length a be attached to the fixed point O and a particle of mass m be attached at the other end A. Let OA be the vertical position of equilibrium of the string. Let the particle be projected horizontally from A with velocity u. Since the string is inextensible, the particle starts moving in circle whose centre is O and radius a. If P is the position of the particle at time t such that $\angle AOP = \theta$ and arc $AP = s$, the forces acting on the particle at P are :

(i) weight mg of the particle acting vertically downwards,

(ii) tension T in the string acting along PO.

If v be the velocity of the particle at P, the tangential and normal accelerations of P are

$$\frac{d^2s}{dt^2} \text{ (in the direction of } s \text{ increasing)}$$

and $$\frac{v^2}{\rho} \text{ (along inwards drawn normal at } P\text{)}.$$

$\therefore$ the equations of motion of the particle along the tangent and normal are

$$m\frac{d^2s}{dt^2} = -mg\sin\theta \quad \text{...(1)}$$

and $$m\frac{v^2}{\rho} = T - mg\cos\theta. \quad \text{...(2)}$$

Also $$s = \text{arc } AP = a\theta.$$

$\therefore$ $$v = \frac{ds}{dt} = a\frac{d\theta}{dt} \quad \text{and} \quad \frac{d^2s}{dt^2} = a\frac{d^2\theta}{dt^2}. \quad \text{...(3)}$$

$\therefore$ from (1) and (3), we have

$$a\frac{d^2\theta}{dt^2} = -g\sin\theta.$$

Multiplying both sides by $2a\dfrac{d\theta}{dt}$ and integrating w.r.t. 't', we have

$$v^2 = \left(a\frac{d\theta}{dt}\right)^2 = 2ag\cos\theta + A,$$

where A is constant of integration.

But initially at $A, \theta = 0, v = u$.

$\therefore$ $$A = u^2 - 2ag\cos 0 = u^2 - 2ag.$$

$\therefore$ $$v^2 = u^2 - 2ag + 2ag\cos\theta. \quad \text{...(4)}$$

Now for a circle $\rho = a$ (radius).

$\therefore$ from (2), we have

$$T = \frac{m}{a}v^2 + mg\cos\theta = \frac{m}{a}(v^2 + ag\cos\theta).$$

Substituting the value of v^2 from (4), we have

$$T = \frac{m}{a}(u^2 - 2ag + 3ag\cos\theta). \quad \text{...(5)}$$

If the velocity $v = 0$ at $\theta = \theta_1$, then from (4), we have

$$0 = u^2 - 2ag + 2ag\cos\theta_1$$

or $$\cos\theta_1 = \frac{2ag - u^2}{2ag}. \qquad \ldots(6)$$

If h_1 is the height from the lowest point A of the point where the velocity vanishes, then

$$h_1 = OA - a\cos\theta_1 = a - a\cdot\frac{2ag - u^2}{2ag}$$

or $$h_1 = \frac{u^2}{2g}. \qquad \ldots(7)$$

Again if the tension $T = 0$, at $\theta = \theta_2$, then from (5), we have

$$0 = u^2 - 2ag + 3\,ag\cos\theta_2\,.$$

$\therefore$ $$\cos\theta_2 = \frac{2ag - u^2}{3\,ag}. \qquad \ldots(8)$$

If h_2 is the height from the lowest point A of the point where the tension vanishes, then

$$h_2 = OA - a\cos\theta_2 = a - a\,.\frac{2ag - u^2}{3\,ag}$$

or $$h_2 = \frac{u^2 + ag}{3\,g}. \qquad \ldots(9)$$

Now the following cases may arise here :

Case I. The velocity v vanishes before the tension T.

This is possible if and only if $h_1 < h_2$

or $$\frac{u^2}{2\,g} < \frac{u^2 + ag}{3g} \quad \text{or} \quad 3u^2 < 2\,(u^2 + ag)$$

or $$u^2 < 2ag \qquad \text{or} \qquad u < \sqrt{(2ag)}.$$

But when $u < (2\,ag)$, we have from (6), $\cos\theta_1 = +\text{ ive}$ *i.e.*, θ_1 is an acute angle.

Thus if the particle is projected with the velocity $u < \sqrt{(2ag)}$, then it will oscillate about A and will not rise upto the horizontal diameter through O.

Case II. The velocity v and the tension T vanish simultaneously.

This is possible if and only if $h_1 = h_2$

i.e., $$\frac{u^2}{2g} = \frac{u^2 + ag}{3g} \quad i.e., \quad u^2 = 2ag \quad \text{or} \quad u = \sqrt{(2\,ag)}.$$

Also when $u = \sqrt{(2ag)}$, we have from (6) and (8),

$$\theta_1 = \pi/2 = \theta_2\,.$$

Thus if the particle is projected with the velocity $u = \sqrt{(2ag)}$, then it will rise upto the level of the horizontal diameter through O and will oscillate about A in the semi-circular arc BAD.

Case III. Condition for describing the complete circle.

At the highest point C, we have $\theta = \pi$. Therefore from (4) and (5), we have at C,

$$v^2 = u^2 - 4\,ag \quad \text{and} \quad T = \frac{m}{a}(u^2 - 5ag).$$

If $u^2 > 5ag$ *i.e.*, if $u > \sqrt{(5ag)}$, then neither the velocity v nor the tension T is zero at the highest point C, and so the particle will go on describing the complete circle.

And if $u^2 = 5ag$ *i.e.*, if $u = \sqrt{(5ag)}$, then at the highest point C the tension T vanishes whereas the velocity does not vanish.

Hence in this case the string will become momentarily slack at C and the particle will go on describing the complete circle.

Thus the condition for describing the complete circle by the particle is that $u \geq \sqrt{(5ag)}$. **In other words the least velocity of projection for describing the complete circle is $\sqrt{(5ag)}$.**

Case IV. The tension T vanishes before the velocity v

This is possible if and only if $h_1 > h_2$

i.e., $$\frac{u^2}{2g} > \frac{u^2 + ag}{3g} \quad i.e., \quad u^2 > 2ag \quad \text{or} \quad u > \sqrt{(2ag)}.$$

When $u > \sqrt{(2ag)}$, we have from (8), $\cos\theta_2 =$ –ive showing that θ_2 must be $> 90°$.

Now at the point where the tension T is zero, the string becomes slack. Since the velocity v is not zero at that point, therefore the particle will leave the circular path and trace a parabolic path while moving freely under gravity.

Thus if the particle is projected with the velocity u such that $\sqrt{(2ag)} < u < \sqrt{(5ag)}$, then it will leave the circular path at a point somewhere between B and C and trace out a parabolic path.

3.3 Motion of a Particle Projected Along the Inside of a Smooth Fixed Hollow Sphere from its Lowest Point

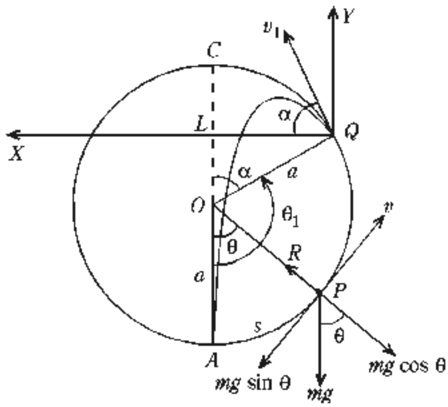
A particle is projected, along the inside of a smooth fixed hollow sphere (or circle) from its lowest point; to discuss the motion.

The discussion is exactly the same as in 3.2 with the difference that in this case the tension T is replaced by the reaction R between the particle and the sphere (or circle).

Illustrative Examples

Example 1: *A particle inside and at the lowest point of a fixed smooth hollow sphere of radius a is projected horizontally with velocity $\sqrt{\left(\frac{7}{2}ag\right)}\cdot$ Show that it will leave the sphere at a height $\frac{3a}{2}$ above the lowest point and its subsequent path meets the sphere again at the point of projection.*

Solution: A particle is projected from the lowest point A of a sphere with velocity $u = \sqrt{\left(\frac{7}{2}ag\right)}$ to move along the inside of the sphere. Let P be the position of the particle at any time t where arc $AP = s$ and $\angle AOP = \theta$. If v be the velocity of the particle at P, the equations of motion along the tangent and normal are



$$m \frac{d^2 s}{dt^2} = -\, mg \sin\theta \qquad \ldots(1)$$

and $$m \frac{v^2}{a} = R - mg\cos\theta \quad \ldots(2)$$

Here R is the reaction of the sphere on the particle at P.

Also $\quad s = a\theta. \qquad \ldots(3)$

From (1) and (3), we have

$$a \frac{d^2\theta}{dt^2} = -\, g\sin\theta.$$

Multiplying both sides by $2a \dfrac{d\theta}{dt}$ and then integrating, we have

$$v^2 = \left(a\frac{d\theta}{dt}\right)^2 = 2\,ag\cos\theta + A.$$

But at the point $A, \theta = 0$ and $v = u = \sqrt{(7ag/2)}$.

$$\therefore \quad A = \frac{7ag}{2} - 2\,ag = \frac{3ag}{2}\cdot$$

$$\therefore \quad v^2 = (3ag/2) + 2\,ag\cos\theta. \qquad \ldots(4)$$

Now from (2) and (4), we have

$$R = \frac{m}{a}\{v^2 + ag\cos\theta\}$$

$$= \frac{m}{a}\left\{\frac{3}{2}ag + 2\,ag\cos\theta + ag\cos\theta\right\} = 3mg\left(\frac{1}{2} + \cos\theta\right)\cdot$$

If the particle leaves the sphere at the point Q, where $\theta = \theta_1$, then

$$0 = 3mg\left(\frac{1}{2} + \cos\theta_1\right) \quad \text{or} \quad \cos\theta_1 = -\frac{1}{2}\cdot$$

If $\quad \angle COQ = \alpha$, then $\alpha = \pi - \theta_1$.

$$\therefore \quad \cos\alpha = \cos(\pi - \theta_1) = -\cos\theta_1 = \frac{1}{2}\cdot \qquad \ldots(5)$$

$$\therefore \quad AL = OA + OL = a + a\cos\alpha = a + \frac{a}{2} = \frac{3a}{2}$$

i.e., the particle leaves the sphere at a height $3a/2$ above the lowest point.

If v_1 is the velocity of the particle at the point Q, then putting $v = v_1$, $R = 0$ and $\theta = \theta_1$ in (2), we get

$$v_1^2 = -\, ag\cos\theta_1 = -\, ag\,.\left(-\frac{1}{2}\right) = \frac{1}{2}\,ag.$$

$\therefore$ the particle leaves the sphere at the point Q with velocity $v_1 = \sqrt{\left(\dfrac{1}{2}ag\right)}$ making an angle α with the horizontal and subsequently describes a parabolic path.

The equation of the parabolic trajectory w.r.t. QX and QY as co-ordinate axes is

$$y = x \tan \alpha - \frac{1}{2} \frac{g x^2}{v_1^2 \cos^2 \alpha}$$

or
$$y = x . \sqrt{3} - \frac{g x^2}{2 . \frac{1}{2} ag . \frac{1}{4}}$$

$[\because \cos \alpha = 1/2$ and so $\sin \alpha = \sqrt{(1 - \cos^2 \alpha)} = \sqrt{3}/2$. Thus $\tan \alpha = \sqrt{3}]$

or
$$y = \sqrt{3} x - \frac{4 x^2}{a}. \qquad \ldots(6)$$

From the figure, for the point A, $x = QL = a \sin \alpha = a \sqrt{3}/2$

and
$$y = - LA = - \frac{3}{2} a.$$

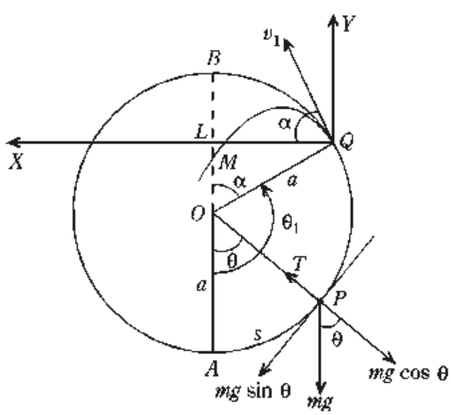
If we put $x = a \sqrt{3}/2$ in the equation (6), we get

$$y = a \frac{\sqrt{3}}{2} \cdot \sqrt{3} - \frac{4}{a} \cdot \frac{3a^2}{4} = \frac{3a}{2} - 3a = - \frac{3}{2} a.$$

Thus the co-ordinates of the point A satisfy the equation (6). Hence the particle, after leaving the sphere at Q, describes a parabolic path which meets the sphere again at the point of projection A.

Example 2: *A heavy particle hangs by an inextensible string of length a from a fixed point and is then projected horizontally with a velocity* $\sqrt{(2 gh)}$. *If* $\frac{5a}{2} > h > a$, *prove that the circular motion ceases when the particle has reached the height* $\frac{1}{3}(a + 2h)$. *Prove also that the greatest height ever reached by the particle above the point of projection is* $(4a - h)(a + 2h)^2 / 27a^2$.

Solution: Let a particle of mass m be attached to one end of a string of length a whose other end is fixed at O. The particle is projected horizontally with a velocity $u = \sqrt{(2 gh)}$ from A. If P is the position of the particle at time t such that $\angle AOP = \theta$ and arc $AP = s$, then the equations of motion of the particle are



$$m \frac{d^2 s}{dt^2} = - mg \sin \theta \qquad \ldots(1)$$

and
$$m \frac{v^2}{a} = T - mg \cos \theta \qquad \ldots(2)$$

Also
$$s = a\theta. \qquad \ldots(3)$$

From (1) and (3), we have

$$a \frac{d^2 \theta}{dt^2} = - g \sin \theta.$$

Multiplying both sides by $2a\dfrac{d\theta}{dt}$ and integrating, we have

$$v^2 = \left(a\frac{d\theta}{dt}\right)^2 = 2ag\cos\theta + A.$$

But at the point $A, \theta = 0$ and $v = u = \sqrt{(2gh)}$.

Therefore $\quad A = 2gh - 2ag.$

$$\therefore \qquad v^2 = 2ag\cos\theta + 2gh - 2ag. \qquad \ldots(4)$$

From (2) and (4), we have

$$T = \frac{m}{a}(v^2 + ag\cos\theta) = \frac{m}{a}(3ag\cos\theta + 2gh - 2ag).$$

If the particle leaves the circular path at Q where $\theta = \theta_1$, then $T = 0$ when $\theta = \theta_1$.

$$\therefore \qquad 0 = \frac{m}{a}(3ag\cos\theta_1 + 2gh - 2ag) \quad \text{or} \quad \cos\theta_1 = -\frac{2h - 2a}{3a}.$$

Since $\dfrac{5}{2}a > h > a$ *i.e.*, $5a > 2h > 2a$, therefore $\cos\theta_1$ is negative and its absolute value is < 1. So θ_1 is real and $\dfrac{1}{2}\pi < \theta_1 < \pi$.

Thus the particle leaves the circular path at Q before arriving at the highest point.

Height of the point Q above A

$$= AL = AO + OL = a + a\cos(\pi - \theta_1) = a - a\cos\theta_1$$

$$= a + a\frac{2h - 2a}{3a} = \frac{1}{3}(a + 2h)$$

i.e., the particle leaves the circular path when it has reached a height $\dfrac{1}{3}(a + 2h)$ above the point of projection.

If v_1 is the velocity of the particle at the point Q, then putting $v = v_1$, $T = 0$ and $\theta = \theta_1$ in (2), we have

$$v_1^2 = -ag\cos\theta_1 = -ag\cdot\left[-\frac{2h - 2a}{3a}\right] = \frac{2}{3}g(h - a).$$

If $\qquad \angle LOQ = \alpha$, then $\alpha = \pi - \theta_1$.

$$\therefore \qquad \cos\alpha = \cos(\pi - \theta_1) = -\cos\theta_1 = \frac{2(h - a)}{3a}.$$

Thus the particle leaves the circular path at the point Q with velocity $v_1 = \sqrt{\left\{\dfrac{2}{3}g(h - a)\right\}}$ at an angle $\alpha = \cos^{-1}\left\{\dfrac{2(h - a)}{3a}\right\}$ to the horizontal and will subsequently describe a parabolic path.

Maximum height of the particle above the point Q

$$= H = \frac{v_1^2\sin^2\alpha}{2g} = \frac{v_1^2}{2g}(1 - \cos^2\alpha)$$

$$= \frac{1}{3}(h-a) . \left[1 - \frac{4}{9a^2}(h-a)^2\right]$$

$$= \frac{1}{27a^2}(h-a)[9a^2 - 4(h^2 - 2ah + a^2)]$$

$$= \frac{(h-a)}{27a^2}[5a^2 + 8ah - 4h^2]$$

$$= \frac{1}{27a^2}(h-a)(a+2h)(5a-2h).$$

$\therefore$ Greatest height ever reached by the particle above the point of projection A

$$= AL + H = \frac{1}{3}(a+2h) + \frac{1}{27a^2}(h-a)(a+2h)(5a-2h)$$

$$= \frac{1}{27a^2}(a+2h)[9a^2 + (h-a)(5a-2h)]$$

$$= \frac{1}{27a^2}(a+2h)[4a^2 + 7ah - 2h^2]$$

$$= \frac{1}{27a^2}(a+2h)(a+2h)(4a-h)$$

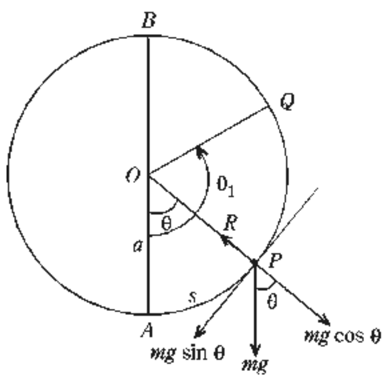
$$= \frac{1}{27a^2}(4a-h)(a+2h)^2.$$

Example 3: *A particle is free to move on a smooth vertical circular wire of radius a. It is projected from the lowest point with velocity just sufficient to carry it to the highest point. Show that the reaction between the particle and the wire is zero after a time*

$$\sqrt{(a/g)}\log(\sqrt{5} + \sqrt{6}).$$

(Kumaun 2001, 03; Garhwal 03, 04; Bundelkhand 06; Rohilkhand 11)

Solution: Let a particle of mass m be projected from the lowest point A of a vertical circle of radius a with velocity u which is just sufficient to carry it to the highest point B.



If P is the position of the particle at any time t such that $\angle AOP = \theta$ and arc $AP = s$, then the equations of motion of the particle along the tangent and normal are

$$m\frac{d^2s}{dt^2} = -mg\sin\theta \qquad \ldots(1)$$

and $$m\frac{v^2}{a} = R - mg\cos\theta. \qquad \ldots(2)$$

Also $$s = a\theta. \qquad \ldots(3)$$

From (1) and (3), we have

$$a\frac{d^2\theta}{dt^2} = -g\sin\theta.$$

Multiplying both sides by $2a(d\theta/dt)$ and integrating, we have

$$v^2 = \left(a\frac{d\theta}{dt}\right)^2 = 2ag\cos\theta + A.$$

But according to the question $v = 0$ at the highest point B, where $\theta = \pi$.

$\therefore \qquad 0 = 2\,ag\cos\pi + A \quad \text{or} \quad A = 2\,ag.$

$$\therefore \qquad v^2 = \left(a\frac{d\theta}{dt}\right)^2 = 2\,ag\cos\theta + 2\,ag. \qquad \text{...(4)}$$

From (2) and (4), we have

$$R = \frac{m}{a}(v^2 + ag\cos\theta) = \frac{m}{a}(2\,ag + 3\,ag\cos\theta). \qquad \text{...(5)}$$

If the reaction $R = 0$ at the point Q where $\theta = \theta_1$, then from (5), we have

$$0 = \frac{m}{a}(2\,ag + 3ag\cos\theta_1) \quad \text{or} \quad \cos\theta_1 = -\frac{2}{3}. \qquad \text{...(6)}$$

From (4), we have

$$\left(a\frac{d\theta}{dt}\right)^2 = 2\,ag(\cos\theta + 1) = 2\,ag\,.\,2\cos^2\frac{1}{2}\theta = 4\,ag\cos^2\frac{1}{2}\theta.$$

$$\therefore \qquad \frac{d\theta}{dt} = 2\sqrt{(g/a)}\cos\frac{1}{2}\theta,$$

the positive sign being taken before the

radical sign because θ increases as t increases

or $\qquad dt = \frac{1}{2}\sqrt{(a/g)}\sec\frac{1}{2}\theta\,d\theta.$

Integrating from $\theta = 0$ to $\theta = \theta_1$, the required time t is given by

$$t = \frac{1}{2}\sqrt{(a/g)}\int_{\theta=0}^{\theta_1}\sec\frac{1}{2}\theta\,d\theta$$

$$= \sqrt{(a/g)}\left[\log\left(\sec\frac{1}{2}\theta + \tan\frac{1}{2}\theta\right)\right]_0^{\theta_1}$$

$$= \sqrt{(a/g)}\log\left(\sec\frac{1}{2}\theta_1 + \tan\frac{1}{2}\theta_1\right). \qquad \text{...(7)}$$

From (6), we have $\quad 2\cos^2\frac{1}{2}\theta_1 - 1 = -\frac{2}{3}$

or $\qquad 2\cos^2\frac{1}{2}\theta_1 = 1 - \frac{2}{3} = \frac{1}{3}$

or $\qquad \cos^2\frac{1}{2}\theta_1 = \frac{1}{6} \quad \text{or} \quad \sec^2\frac{1}{2}\theta_1 = 6.$

$\therefore \qquad \sec\frac{1}{2}\theta_1 = \sqrt{6}$

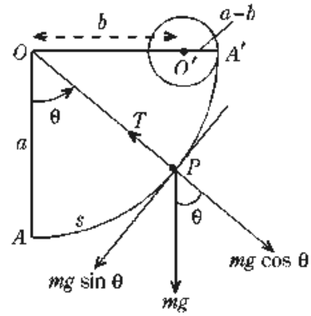
and $\qquad \tan\frac{1}{2}\theta_1 = \sqrt{\left(\sec^2\frac{1}{2}\theta_1 - 1\right)} = \sqrt{(6-1)} = \sqrt{5}.$

Substituting in (7), the required time is given by

$$t = \sqrt{(a/g)}\log(\sqrt{6} + \sqrt{5}).$$

Example 4: *A particle is hanging from a fixed point O by means of a string of length a. There is a small nail at O′ in the same horizontal line with O at a distance b (< a) from O. Find the minimum velocity with which the particle should be projected from its lowest point in order that it may make a complete revolution round the nail without the string becoming slack.*

Solution: Let a particle of mass m hang from a fixed point O by means of a string OA of length a. Let O be a nail in the same horizontal line with O at a distance $OO' = b$ $(< a)$. Let the particle be projected from A with velocity u. It moves in a circle with centre at O and radius as a. If P is the position of the particle at any time t such that $\angle AOP = \theta$ and arc $AP = s$, then the equations of motion of the particle along the tangent and normal are



$$m\frac{d^2s}{dt^2} = -\,mg\sin\theta, \qquad ...(1)$$

and $$m\frac{v^2}{a} = T - mg\cos\theta. \qquad ...(2)$$

Also $$s = a\theta. \qquad ...(3)$$

From (1) and (3), we have

$$a\frac{d^2\theta}{dt^2} = -\,g\sin\theta.$$

Multiplying both sides by $2\,a\,(d\theta/dt)$ and integrating, we have

$$v^2 = \left(a\frac{d\theta}{dt}\right)^2 = 2ag\cos\theta + A.$$

But initially at $A, \theta = 0$ and $v = u$. Therefore,

$$A = u^2 - 2ag.$$

$\therefore$ $$v^2 = u^2 - 2ag + 2ag\cos\theta. \qquad ...(4)$$

At the point $A', \theta = \pi/2$. If v_1 is the velocity at A', then from (4), we have

$$v_1^2 = u^2 - 2ag \quad \text{or} \quad v_1 = \sqrt{(u^2 - 2ag)}.$$

Since there is a nail at O', the particle will describe a circle with centre at O' and radius $O'A' = a - b$.

We know that if a particle is attached to a string of length l, the least velocity of projection from the lowest point in order to make a complete circle is $\sqrt{(5gl)}$. Also in this case, using the result (4), the velocity of the particle when it has described an angle θ from the lowest point is given by

$$v^2 = 5lg - 2lg + 2lg\cos\theta \qquad [\because \text{ here } a = l \text{ and } u^2 = 5gl]$$

$$= 3lg + 2lg\cos\theta.$$

At $\theta = \pi/2$, if $v = v_2$, then $v_2 = \sqrt{(3lg)}$. $\quad [\because \cos\pi/2 = 0]$

Thus in order to describe a complete circle of radius l the minimum velocity of the particle at the end of the horizontal diameter should be $\sqrt{(3gl)}$. Therefore in order to describe a complete circle of radius $l = O'A' = a - b$ round O' the minimum velocity of the particle at A' should be $\sqrt{[3g(a-b)]}$.

But, as already found out, the velocity of the particle at A' is v_1.

$\therefore$ we must have

$$v_1 \geq \sqrt{[3g(a-b)]}$$

or $$\sqrt{(u^2 - 2ag)} \geq \sqrt{[3g(a-b)]}$$

or $$u^2 - 2ag \geq 3g(a-b)$$

or $$u^2 \geq g(5a - 3b)$$

or $$u \geq \sqrt{[g(5a-3b)]}.$$

Hence the required minimum velocity of projection of the particle at the lowest point is

$$\sqrt{[g(5a-3b)]}.$$

Comprehensive Exercise 1

1. A heavy particle of weight W, attached to a fixed point by a light inextensible string describes a circle in a vertical plane. The tension in the string has the values mW and nW respectively when the particle is at the highest and lowest points in the path. Show that $n = m + 6$. **(Meerut 2009)**

2. A heavy particle hanging vertically from a point by a light inextensible string of length l is started so as to make a complete revolution in a vertical plane. Prove that the sum of the tensions at the ends of any diameter is constant.

3. A particle makes complete revolutions in a vertical circle. If ω_1, ω_2 be the greatest and least angular velocities and R_1, R_2 the greatest and least reactions, prove that when the particle projected from the lowest point of the circle makes an angle θ at the centre, its angular velocity is

$$\sqrt{\left[\omega_1^2 \cos^2 \frac{1}{2}\theta + \omega_2^2 \sin^2 \frac{1}{2}\theta\right]}$$

and that the reaction is

$$R_1 \cos^2 \frac{1}{2}\theta + R_2 \sin^2 \frac{1}{2}\theta.$$

4. A heavy particle hangs from a fixed point O, by a string of length a. It is projected horizontally with a velocity $v^2 = (2 + \sqrt{3})\, ag$; show that the string becomes slack when it has described an angle $\cos^{-1}(-1/\sqrt{3})$. **(Agra 2011; Garhwal 01)**

5. Find the velocity with which a particle must be projected along the interior of a smooth vertical hoop of radius a from the lowest point in order that it may leave the hoop at an angular distance of 30° from the vertical. Show that it will strike the hoop again at an extremity of the horizontal diameter.

6. A particle is projected along the inner side of a smooth vertical circle of radius a, the velocity at the lowest point being u. Show that if $2\,ga < u^2 < 5ag$, the particle will leave the circle before arriving at the highest point and will describe a parabola whose latus rectum is $\dfrac{2\,(u^2 - 2\,ag)^3}{27a^2\,g^3}$. **(Agra 2001)**

7. A heavy particle is attached to a fixed point by a fine string of length a; the particle is projected horizontally from the lowest point with velocity $\sqrt{[ag\,(2 + 3\sqrt{3}/2)]}$. Prove that the string would first become slack when inclined to the upward vertical at an angle of 30°, will become tight again when horizontal.

8. A heavy particle hanging vertically from a fixed point by a light inextensible cord of length l is struck by a horizontal blow which imparts it a velocity $2\sqrt{(gl)}$, prove that the cord becomes slack when the particle has risen to a height $\dfrac{2}{3}\,l$ above the fixed point.

   Also find the height of the highest point of parabola subsequently described.

9. A particle is projected, along the inside of a smooth fixed sphere, from its lowest point, with a velocity equal to that due to falling freely down the vertical diameter of the sphere. Show that the particle will leave the sphere and afterwards pass vertically over the point of projection at a distance equal to 25/32 of the diameter.

10. A particle is projected from the lowest point inside a smooth circle of radius a with a velocity due to a height h above the centre. Find the point where it leaves the circle and show that it will afterwards pass through

    (a) the centre if $h = \dfrac{1}{2}\,(a\sqrt{3})$,

    and (b) the lowest point if $h = 3a/4$.

11. A particle is projected along the inside of a smooth vertical circle of radius a from the lowest point. Show that the velocity of projection required in order that after leaving the circle, the particle may pass through the centre is $\sqrt{\left(\dfrac{1}{2}\,ag\right)}\,(\sqrt{3} + 1)$.

12. A particle tied to a string of length a is projected from its lowest point, so that after leaving the circular path it describes a free path passing through the lowest point. Prove that the velocity of projection is $\sqrt{\left(\dfrac{7}{2}\,ag\right)}$.

13. Show that the greatest angle through which a person can oscillate on a swing, the ropes of which can support twice the person's weight at rest is 120°.

    If the ropes are strong enough and he can swing through 180° and if v is his speed at any point, prove that the tension in the rope at that point is $3mv^2/2\,l$ where m is the mass of the person and l the length of the rope.

14. A heavy bead slides on a smooth circular wire of radius a. It is projected from the lowest point with a velocity just sufficient to carry it to the highest point, prove that the radius through the bead in time t will turn through an angle

    $$2\tan^{-1}[\sinh\{t\sqrt{(g/a)}\}]$$

    and that the bead will take an infinite time to reach the highest point.

15. A particle attached to a fixed peg O by a string of length l, is lifted up with the string horizontal and then let go. Prove that when the string makes an angle θ with the horizontal, the resultant acceleration is $g\sqrt{(1+3\sin^2\theta)}$.

16. A particle attached to a fixed peg O by a string of length l, is let fall from a point in the horizontal line through O at a distance $l\cos\theta$ from O; show that its velocity when it is vertically below O is $\sqrt{[2gl(1-\sin^3\theta)]}$.

Answers 1

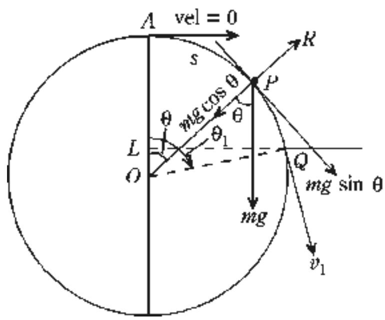
5. $\left[\frac{1}{2}ag(4+3\sqrt{3})\right]^{1/2}$ **8.** $23\,l/27$

10. At a height $2h/3$ above the centre of the circle

3.4 Motion on the Outside of a Smooth Vertical Circle

A particle slides down the outside of a smooth vertical circle starting from rest at the highest point; to discuss the motion. **(Garhwal 2002; Kumaun 02; Agra 09)**

Let a particle of mass m slide down the outside of a smooth vertical circle whose centre is O and radius a, starting from rest at the highest point A. Let P be the position of the particle at any time t such that $\angle AOP=\theta$ and arc $AP=s$. The forces acting on the particle at P are (i) weight mg acting vertically downwards and (ii) the reaction R acting along the outwards drawn normal PO. If v be the velocity of the particle at P the equations of motion of the particle along the tangent and normal are



$$m\frac{d^2s}{dt^2}=mg\sin\theta, \qquad \text{...(1)}$$

(+ive sign is taken on the R.H.S. because $mg\sin\theta$ acts in the direction of s increasing)

and $$m\frac{v^2}{a}=mg\cos\theta-R. \qquad \text{...(2)}$$

[Note that in equation (2) R has been taken with –ive sign because it is in the direction of outwards drawn normal and $mg\cos\theta$ with +ive sign because it is in the direction of inwards drawn normal]

Also $$s=a\theta. \qquad \text{...(3)}$$

From (1) and (3), we have

$$a\frac{d^2\theta}{dt^2}=g\sin\theta.$$

Multiplying both sides by $2a\,(d\theta/dt)$ and integrating, we have

$$v^2 = \left(a\,\frac{d\theta}{dt}\right)^2 = -\,2ag\cos\theta + A.$$

But initially at $A, \theta = 0$ and $v = 0$. Therefore, $A = 2ag$.

$$\therefore \qquad v^2 = 2ag - 2ag\cos\theta = 2ag\,(1 - \cos\theta). \qquad \ldots(4)$$

From (2) and (4), we have

$$R = \frac{m}{a}\,[ag\cos\theta - v^2] = \frac{m}{a}\,[3ag\cos\theta - 2ag]$$

$$= mg\,(3\cos\theta - 2). \qquad \ldots(5)$$

If the particle leaves the circle at Q where $\angle AOQ = \theta_1$, then $R = 0$ when $\theta = \theta_1$. Therefore from (5), we have

$$mg\,(3\cos\theta_1 - 2) = 0 \quad \text{or} \quad \cos\theta_1 = 2/3.$$

Vertical depth of the point Q below A

$$= AL = OA - OL = a - a\cos\theta_1 = a - 2\,a/3 = a/3.$$

Hence if a particle slides down the outside of smooth vertical circle, starting from rest at the highest point, it will leave the circle after descending vertically a distance equal to one-third of the radius of the circle.

If v_1 is the velocity of the particle at Q, then $v = v_1$ when $\theta = \theta_1$.

$\therefore$ from (4), we have

$$v_1^2 = 2ag\,(1 - \cos\theta_1) = 2ag\left(1 - \frac{2}{3}\right) = \frac{2}{3}\,ag.$$

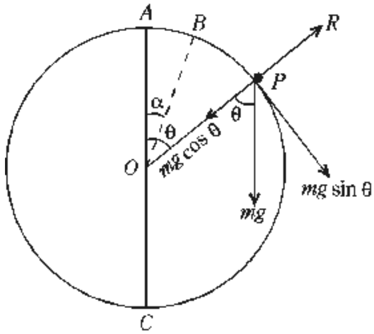
The direction of the velocity v_1 is along the tangent to the circle at Q. Therefore the particle leaves the circle at Q with velocity $v_1 = \sqrt{\left(\frac{2}{3}\,ag\right)}$ making an angle $\theta_1 = \cos^{-1}\left(\frac{2}{3}\right)$ below the horizontal line through Q. After leaving the circle at Q the particle will move freely under gravity and so it will describe a parabolic path.

Illustrative Examples

Example 5: *A particle is placed on the outside of a smooth vertical circle. If the particle starts from a point whose angular distance is α from the highest point of the circle, show that it will fly off the curve when $\cos\theta = \frac{2}{3}\cos\alpha$.* **(Meerut 2008; Agra 10)**

Solution: A particle slides down on the outside of the arc of a smooth vertical circle of radius a, starting from rest at a point B such that $\angle AOB = \alpha$. Let P be the position of the particle at any time t where arc $AP = s$ and $\angle POA = \theta$. The forces acting on the particle at P are : (i) weight mg acting vertically downwards and (ii) the reaction R along the outwards drawn normal OP.

If v be the velocity of the particle at P, the equations of motion of the particle along the tangent and normal are



$$m \frac{d^2 s}{dt^2} = mg \sin \theta, \qquad \ldots(1)$$

and $$m \frac{v^2}{a} = mg \cos \theta - R. \qquad \ldots(2)$$

Also $$s = a\theta. \qquad \ldots(3)$$

From (1) and (3), we have $a \dfrac{d^2\theta}{dt^2} = g \sin \theta$.

Multiplying both sides by $2a\,(d\theta / dt)$ and integrating, we have

$$v^2 = \left(a \frac{d\theta}{dt}\right)^2 = -2ag \cos \theta + A.$$

But initially at B, $\theta = \alpha$ and $v = 0$. Therefore,

$$A = 2ag \cos \alpha \,.$$

$$\therefore \qquad v^2 = 2ag \cos \alpha - 2\, ag \cos \theta. \qquad \ldots(4)$$

From (2) and (4), we have

$$R = \frac{m}{a}(-v^2 + ag \cos \theta)$$

$$= \frac{m}{a}(-2\, ag \cos \alpha + 3ag \cos \theta)$$

$$= mg\,(-2 \cos \alpha + 3 \cos \theta). \qquad \ldots(5)$$

At the point where the particle flies off the circle, we have $R = 0$.

$\therefore$ from (5), we have

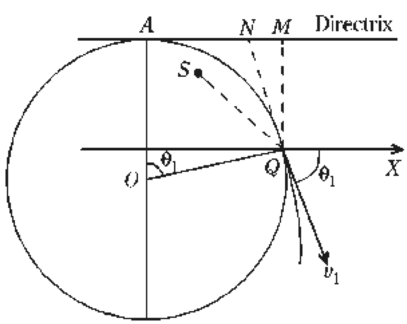
$$0 = mg\,(-2 \cos \alpha + 3 \cos \theta)$$

or $$\cos \theta = \frac{2}{3} \cos \alpha.$$

Example 6: *A particle is placed at the highest point of a smooth vertical circle of radius a and is allowed to slide down starting with a negligible velocity. Prove that it will leave the circle after describing vertically a distance equal to one-third of the radius. Find the position of the directrix and the focus of the parabola subsequently described and show that its latus rectum is* $\dfrac{16}{27}a$.

Solution: For the first part see 3.4.

From 3.4, the particle leaves the circle at the point Q where $\angle AOQ = \theta_1$ and $\cos \theta_1 = \dfrac{2}{3}$.



The velocity v_1 at the point Q is $\sqrt{(2\,ag/3)}$, its direction is along the tangent to the circle at Q. After leaving the circle at the point Q, the particle describes a parabolic path with the velocity of projection $v_1 = \sqrt{(2\,ag/3)}$ making an angle $\theta_1 = \cos^{-1}(2/3)$ below the horizontal line through Q. Latus rectum of the parabola subsequently described

$$= \frac{2 v_1^2 \cos^2 \theta_1}{g}$$

$$= \frac{2}{g} \cdot \frac{2\,ag}{3} \cdot \frac{4}{9} = \frac{16}{27} a.$$

To find the position of the directrix and the focus of the parabola. We know that in a parabolic path of a projectile the velocity at any point of its path is equal to that due to a fall from the directrix to that point.

Therefore if h is the height of the directrix above Q, then the velocity acquired in falling a distance h under gravity $= \sqrt{(2gh)}$.

$$\therefore \qquad v_1 = \sqrt{(2\,ag/3)} = \sqrt{(2gh)} \quad \text{or} \quad h = a/3$$

i.e., the height of the directrix above Q is $a/3$.

Hence the directrix is the horizontal line through the highest point of the circle.

Let QM be the perpendicular from Q on the directrix and QN the tangent at Q. If S is the focus of the parabola subsequently described, we have by the geometrical properties of parabola

$$QS = QM = a/3 \quad \text{and} \quad \angle SQN = \angle NQM.$$

This gives the position of the focus S of the parabola.

Comprehensive Exercise 2

1. A particle is projected horizontally with a velocity $\sqrt{(ag/2)}$] from the highest point of the outside of a fixed smooth sphere of radius a. Show that it will leave the sphere at the point whose vertical distance below the point of projection is $a/6$.

2. A particle moves under gravity in a vertical circle sliding down the convex side of the smooth circular arc. If the initial velocity is that due to a fall to the starting point from a height h, above the centre, show that it will fly off the circle when at a height $\frac{2}{3}h$ above the centre. **(Bundelkhand 2009)**

3. A heavy particle is allowed to slide down a smooth vertical circle of radius $27a$ from rest at the highest point. Show that on leaving the circle it moves in a parabola of latus rectum $16\,a$. **(Agra 2002, 08)**

4. A particle slides down the arc of a smooth vertical circle of radius a, being slightly displaced from rest at the highest point. Find where it will leave the circle and prove that it will strike a horizontal plane through the lowest point of the circle at a distance $\frac{5}{27}(\sqrt{5} + 4\sqrt{2})\,a$ from the vertical diameter.

5. A body is projected, along the arc of a smooth circle of radius a and from the highest point with velocity $\frac{1}{2}\sqrt{(ag)}$; find where it will leave the circle and prove

that it will strike a horizontal plane through the centre of the circle at a distance from the centre

$$\frac{1}{64}[9\sqrt{(39)} + 7\sqrt{7}]\,a.$$

6. A heavy particle slides under gravity down the inside of a smooth vertical tube held in a vertical plane. It starts from the highest point with velocity $\sqrt{(2ag)}$ where a is the radius of the circle. Prove that when in the subsequent motion the vertical component of the acceleration is maximum, the pressure on the curve is equal to twice the weight of the particle.

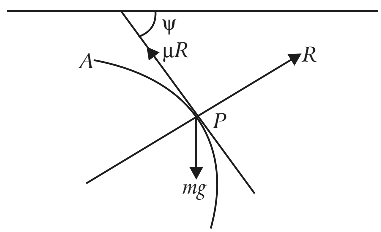
Answers 2

4. At a height $2a/3$ above the centre of the circle.

5. At a height $3a/4$ above the centre.

3.5 Motion on a Rough Curve Under Gravity

A particle slides down a rough curve in a vertical plane under gravity; to discuss the motion.



Let P be the position of the particle at any time t, where the tangent is making an angle ψ with any fixed horizontal line, let the arcual distance of P measured from a fixed point A be s. R is the normal reaction as shown in the figure. Since the particle slides downwards, therefore the force of friction μR acts upwards along the tangent at P.

Resolving forces along the tangent and normal at P, we have equations of motion as

$$mv\,(dv/ds) = mg\sin\psi - \mu R$$

i.e., $$\frac{1}{2}m\,(dv^2/ds) = mg\sin\psi - \mu R \qquad \ldots(1)$$

and $$m(v^2/\rho) = mg\cos\psi - R. \qquad \ldots(2)$$

Eliminating R from (1) and (2) *i.e.* multiplying (2) by μ and subtracting form (1), we get

$$\frac{1}{2}m\frac{dv^2}{ds} - \mu m\frac{v^2}{\rho} = mg\sin\psi - \mu\, mg\cos\psi$$

or $$\frac{dv^2}{ds}.\rho - 2\mu v^2 = 2g\rho(\sin\psi - \mu\cos\psi)$$

or $$\frac{dv^2}{ds}.\frac{ds}{d\psi} - 2\mu v^2 = 2g\rho(\sin\psi - \mu\cos\psi) \qquad \left[\because \rho = \frac{ds}{d\psi}\right]$$

or $$\frac{dv^2}{d\psi} - 2\mu v^2 = 2g\rho(\sin\psi - \mu\cos\psi).$$

This is a linear differential equation in v^2 whose integrating factor is $e^{-2\mu\psi}$. Hence multiplying both sides by $e^{-2\mu\psi}$ and integrating, we get

$$v^2 e^{-2\mu\psi} = 2g \int \rho \cdot e^{-2\mu\psi} (\sin\psi - \mu\cos\psi)\, d\psi + c, \qquad \ldots(3)$$

where c is a constant of integration.

When the equation of the curve is given, ρ can be determined in terms of ψ; therefore, substituting value of ρ in right hand side of (3), it can be integrated and the value of constant c can be determined by initial conditions.

Thus from (3) we can find the value of v^2 in any position and then substituting for v^2 in (2) the value of R is determined.

3.6 Remember

Before solving problems, let us recall some formulae which we shall often use.

(i) The radius of curvature ρ is given by

$$\rho = r\frac{dr}{dp} = \frac{ds}{d\psi} = \frac{\left[\left(\frac{dy}{dx}\right)^2\right]^{3/2}}{\frac{d^2y}{dx^2}} = \frac{(\dot{x}^2 + \dot{y}^2)^{3/2}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}},$$

where $\dot{x} = \dfrac{dx}{dt}$ etc.

(ii) In the ellipse $x = a\cos\phi, y = b\sin\phi$, we have

$$\rho = \frac{(a^2\sin^2\phi - b^2\cos^2\phi)^{3/2}}{ab} = \frac{CD^2}{ab},$$

where CD is the semi-conjugate diameter of the ellipse.

(iii) In case of circle $\rho = a$ = the radius of the circle.

(iv) $\dfrac{ds}{d\theta} = \sqrt{\{r^2 + (\dfrac{dr}{d\theta})^2\}}.$

(v) $\displaystyle\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)$ and

$$\int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx).$$

(vi) In every curve $p = r\sin\phi$.

Illustrative Examples

Example 7: *A particle, under no forces, is projected with velocity V in a rough tube in the form of an equiangular spiral at a distance a from the pole and towards the pole. Show that it will arrive at the pole in time*

$$\frac{a}{V}\,\frac{1}{\cos\alpha - \mu\sin\alpha};$$

α being the angle of the spiral and μ (< $\cot\alpha$) the coefficient of friction.

Solution: Equation of equiangular spiral is

$$r = ae^{\theta \cot \alpha} \quad ...(1)$$

Since particle is moving under no forces, so the only forces acting upon it are the normal reaction R (inward) and frictional force μR (upward). The tangential and normal equations of motion are

$$mv\,(dv/ds) = \mu R \quad ...(2)$$

or $$v^2/\rho = R. \quad ...(3)$$

Multiplying (3) by μ and subtracting from (2), we get

$$v\frac{dv}{ds} - \mu\frac{v^2}{\rho} = 0 \qquad \text{or} \qquad \frac{dv}{ds} - \mu\frac{v}{\rho} = 0$$

or $$\frac{dv}{d\psi}\cdot\frac{d\psi}{ds} - \mu v\frac{d\psi}{ds} = 0 \qquad \text{or} \qquad \frac{dv}{d\psi} - \mu v = 0.$$

$\therefore$ $$\frac{dv}{v} = \mu\, d\psi \qquad i.e. \qquad \log v = \mu\psi + \log A$$

or $$v = Ae^{\mu\psi}. \quad ...(4)$$

From (1), $dr/d\theta = a\cot\alpha\, e^{\theta\cot\alpha}$; $\quad\therefore\ \tan\phi = r(d\theta/dr) = \tan\alpha$ *i.e.*, $\phi = \alpha$.

Hence $\psi = \theta + \phi = \theta + \alpha$.

Substituting this value of ψ in (4), we get

$$v = Ae^{\mu(\theta+\alpha)} = Ae^{\mu\alpha}\cdot e^{\mu\theta} = Ae^{\mu\alpha}\cdot\left[\frac{ae^{\theta\cot\alpha}}{a}\right]^{\mu\tan\alpha}$$

$$= Ae^{\mu\alpha}\cdot\left(\frac{r}{a}\right)^{\mu\tan\alpha} \quad \text{[using (1)]}$$

Initially at $r = a, v = -V$ (being towards pole).

$\therefore$ $$-V = Ae^{\mu\alpha}$$

$\therefore$ $$v = -V.(r/a)^{\mu\tan\alpha}$$

or $$\frac{ds}{dr}\cdot\frac{dr}{dt} = -V\left(\frac{r}{a}\right)^{\mu\tan\alpha} \quad \left[\because\ v = \frac{ds}{dt} = \frac{ds}{dr}\cdot\frac{dr}{dt}\right]$$

or $$\sec\alpha\frac{dr}{dt} = -V\cdot\left(\frac{r}{a}\right)^{\mu\tan\alpha} \quad \left[\because\ \frac{ds}{dr} = \sec\phi = \sec\alpha\right]$$

or $$dt = -\frac{a^{\mu\tan\alpha}}{V\cos\alpha}.r^{-\mu\tan\alpha}dr. \quad ...(5)$$

If t is the time to arrive at the pole, integrating (5) from $r = a$ to $r = 0$, we get

$$t = -\frac{(a)^{\mu\tan\alpha}}{V\cos\alpha}\left[\frac{r^{-\mu\tan\alpha+1}}{-\mu\tan\mu+1}\right]_a^0 = \frac{(a)^{\mu\tan\alpha}}{V\cos\alpha}\cdot\frac{(a)^{-\mu\tan\alpha+1}}{-\mu\tan\alpha+1}$$

$$= \frac{a}{V\cos\alpha}\cdot\frac{1}{1-\mu\tan\mu} = \frac{a}{V(\cos\alpha-\mu\sin\alpha)}$$

Example 8: *A bead slides down a rough circular wire, which is in vertical plane, starting from rest at the end of the horizontal diameter. When it has described an angle* θ *about the centre, show that the square of its angular velocity is*

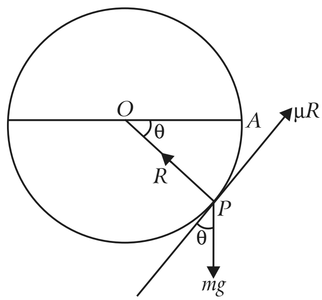
$$\frac{2g}{a(1+4\mu^2)}\left[(1-2\mu^2)\sin\theta+3\mu\left(\cos\theta-e^{-2\mu\theta}\right)\right],$$

where μ *is the coefficient of friction and* a *the radius of the circle.*

Solution: Let bead start from A and P be its position at any time t. The tangential and normal equations of motion are

$$v\frac{dv}{ds}=mg\cos\theta-\mu R \qquad \ldots(1)$$

[since frictional force μR acts upwards]



and $$m\frac{v^2}{a}=R-mg\sin\theta. \qquad \ldots(2)$$

Multiplying (2) by μ and adding to (1), we get

$$mv\frac{dv}{ds}+\mu m\frac{v^2}{a}=mg\,(\cos\theta-\mu\sin\theta)$$

or $$\frac{1}{2}\frac{dv^2}{ds}+\mu\frac{v^2}{a}=g(\cos\theta-\mu\sin\theta)$$

or $$\frac{dv^2}{d\theta}\cdot\frac{d\theta}{ds}+2\mu\frac{v^2}{a}=2g(\cos\theta-\mu\sin\theta) \qquad \ldots(3)$$

If arc $AP=s$ then in a circle $s=a\theta$ $\quad\therefore\quad 1/a=d\theta/ds$

$\therefore$ (3) gives, $$\frac{dv^2}{d\theta}\cdot\frac{1}{a}+2\mu\frac{v^2}{a}=2g(\cos\theta-\mu\sin\theta)$$

or $$\frac{dv^2}{d\theta}+2\mu v^2=2ag\,(\cos\theta-\mu\sin\theta) \qquad \ldots(4)$$

This is a linear differential equation in v^2. Its integrating factor $=e^{\int 2\mu\,d\theta}=e^{2\mu\theta}$. Multiplying both sides of (4) by $e^{2\mu\theta}$ and integrating, we have

$$v^2e^{2\mu\theta}=2ag\int e^{2\mu\theta}(\cos\theta-\mu\sin\theta)\,d\theta+A,$$

where A is constant of integration

or $$v^2e^{2\mu\theta}=2ag\,\frac{e^{2\mu\theta}}{1+4\mu^2}\left[(2\mu\cos\theta+\sin\theta)-\mu(2\mu\sin\theta-\cos\theta)\right]+A,$$

[see 3.5 (v)]

or $$v^2e^{2\mu\theta}=\frac{2gae^{2\mu\theta}}{1+4\mu^2}\left[3\mu\cos\theta+(1-2\mu^2)\sin\theta\right]+A. \qquad \ldots(5)$$

Initially at $A,\theta=0,\,v=0;\ \therefore\quad A=-\dfrac{6ga\mu}{1+4\mu^2}\cdot$

$\therefore$ $$v^2e^{2\mu\theta}=\frac{2gae^{2\mu\theta}}{1+4\mu^2}\left[3\mu\cos\theta+(1-2\mu^2)\sin\theta\right]-\frac{6\mu ag}{1+4\mu^2}$$

or $$\left(a\frac{d\theta}{dt}\right)^2 = \frac{2ga}{1+4\mu^2}[(1-2\mu^2)\sin\theta + 3\mu\cos\theta] - \frac{6\mu ag}{1+4\mu^2}e^{-2\mu\theta}$$

$$\left[\because s = a\theta \;\therefore v = \frac{ds}{dt} = \frac{a\,d\theta}{dt}\right]$$

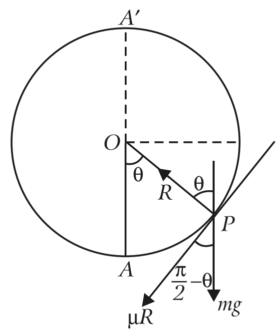
or $$\left(\frac{d\theta}{dt}\right)^2 = \frac{2g}{a(1+4\mu^2)}[(1-2\mu^2)\sin\theta + 3\mu(\cos\theta - e^{-2\mu\theta})].$$

This gives the square of the angular velocity.

Example 9: *A particle is projected horizontally with velocity V along the inside of a rough vertical circle from the lowest point. Prove that if it* ***completes the circuit*** *it will return to the lowest point with a velocity v given by*

$$v^2 = V^2 e^{-4\pi\mu} + \frac{2ga}{1+4\mu^2}(1-2\mu^2)(1-e^{-4\mu\pi})].$$

Solution: Let the particle be projected from lowest point A with velocity V. Let P be its position at any time t, where OP makes an angle θ with the vertical OA. The frictional force μR acts downwards as particle moves in the upward direction.



Hence equations of motion along tangent and normal are

$$mv\,(dv/ds) = -\,mg\sin\theta - \mu R \qquad \ldots(1)$$

$$m(v^2/a) = R - mg\cos\theta. \qquad \ldots(2)$$

Eliminating R between (1) and (2), we get

$$v\frac{dv}{ds} + \mu\frac{v^2}{a} = -\,g(\sin\theta + \mu\cos\theta)$$

or $$\frac{1}{2}\frac{dv^2}{ds} + \mu\frac{v^2}{a} = -\,g(\sin\theta + \mu\cos\theta)$$

or $$\frac{1}{2}\frac{dv^2}{d\theta}\cdot\frac{1}{a} + \mu\frac{v^2}{a} = -\,g(\sin\theta + \mu\cos\theta)$$

$$\left[\because\; s = a\theta \text{ and } \frac{dv^2}{ds} = \frac{dv^2}{d\theta}\cdot\frac{d\theta}{ds} = \frac{dv^2}{d\theta}\cdot\frac{1}{a}\right]$$

or $$\frac{dv^2}{d\theta} + 2\mu v^2 = -\,2ag\,(\sin\theta + \mu\cos\theta).$$

This is a linear differential equation with integrating factor $e^{2\mu\theta}$. Therefore its solution is

$$v^2 e^{2\mu\theta} = -\,2ga\int e^{2\mu\theta}.(\mu\cos\theta + \sin\theta)\,d\theta + C$$

or $$v^2 e^{2\mu\theta} = -\,2ga\frac{e^{2\mu\theta}}{1+4\mu^2}[\mu\,(2\mu\cos\theta + \sin\theta) + 2\mu\sin\theta - \cos\theta] + C$$

[see 3.5 (v)]

or $$v^2 e^{2\mu\theta} = -\frac{2ga\,e^{2\mu\theta}}{1+4\mu^2}[3\mu\sin\theta - (1-2\mu^2)\cos\theta] + C. \qquad \ldots(3)$$

Initially at $A, v = V, \theta = 0$;

$$\therefore \qquad C = V^2 - \frac{2ga}{1+4\mu^2}(1-2\mu^2).$$

$$\therefore \qquad v^2 e^{2\mu\theta} = -\frac{2ga\, e^{2\mu\theta}}{1+4\mu^2}[3\mu \sin\theta - (1-2\mu^2)\cos\theta] + V^2 - \frac{2ga}{1+4\mu^2}(1-2\mu^2) \qquad \ldots(4)$$

Equation (4) gives velocity at any point of the circle when the particle is **moving up.**

If particle completes the circuit it will return back to the point A, when it has described an angle $\theta = 2\pi$. Hence putting $\theta = 2\pi$ in (4), the velocity v on return to the lowest point A is given by

$$v^2 e^{4\mu\pi} = \frac{-2ga\, e^{4\mu\pi}}{1+4\mu^2}[-(1-2\mu^2)] + V^2 - \frac{2ga}{1+4\mu^2}(1-2\mu^2)$$

$$[\because \ \sin 2\pi = 0; \ \cos 2\pi = 1]$$

i.e.

$$v^2 = V^2 e^{-4\mu\pi} + \frac{2ga}{1+4\mu^2}(1-2\mu^2)(1-e^{-4\mu\pi}).$$

Comprehensive Exercise 3

1. A bead moves along a rough curved wire which is such that it changes its direction of motion with constant angular velocity. Show that a possible form of the wire is an equiangular spiral.

2. A ring, which can slide on a rough circular wire in a vertical plane, is projected from the lowest point with such velocity as will take to the **horizontal diameter**; if the ring returns to the lowest point, show that its velocity on arrival is to its velocity of projection as

$$[1-2\mu^2 - 3\mu\, e^{-\mu\pi}]^{1/2} : [1-2\mu^2 + 3\mu e^{\mu\pi}]^{1/2}$$

where μ is the coefficient of friction.

3. A particle is projected horizontally from the lowest point of a rough sphere of radius a. After describing an arc less than a quadrant it returns and comes to rest at the lowest point. Show that the initial velocity must be

$$\sin\alpha \sqrt{\left(2ga\frac{1+\mu^2}{1-2\mu^2}\right)},$$

where μ is the coefficient of friction and $a\alpha$ is the arc through which the particle moves.

4. A particle is projected along the inner surface of a rough sphere and is acted on by no forces; show that it will return to the point of projection at the end of time

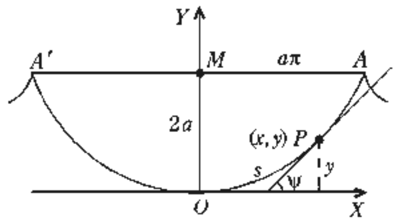
$$\frac{a}{\mu V}(e^{2\mu\pi} - 1).$$

where a is the radius of the sphere, V is the velocity of projection and μ is the coefficient of friction.

3.7 Cycloid

A cycloid is a curve which is traced out by a point on the circumference of a circle as the circle rolls along a fixed straight line.

In the adjoining figure we have shown an inverted cycloid. The point O is called the vertex of the cycloid. The points A and A' are the cusps and the straight line OY is the axis of the cycloid. The line AA' is called the base of the cycloid.



Let $P(x, y)$ be the coordinates of a point on the cycloid w.r.t. OX and OY as coordinate axes and ψ the angle which the tangent at P makes with OX. Then remember the following results :

(i) Parametric equations of the cycloid are given by

$$x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta)$$

where θ is the parameter and we have $\theta = 2\psi$. **(Kanpur 2010)**

(ii) The intrinsic equation of the cycloid is

$$s = 4a \sin\psi, \text{ where arc } OP = s.$$ **(Bundelkhand 2008)**

(iii) Arc $OA = a$ and the height of the cycloid $= OM = 2a$.

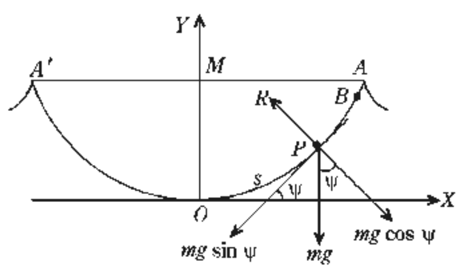
At the point O, $\psi = 0$ and $s = 0$ while at the cusp A, $\psi = \pi/2$ and $s = 4a$.

(iv) For the above cycloid, the relation between s and y is $s^2 = 8ay$.

3.8 Motion on a Cycloid

A particle slides down the arc of a smooth cycloid whose axis is vertical and vertex downwards. To determine the motion.

(Meerut 2004; Rohilkhand 06; Agra 06, 08, 09, 10; Bundelkhand 06, 08; Kanpur 07, 08)



Let O be the vertex of a smooth cycloid and OM its axis. Suppose a particle of mass m slides down the arc of the cycloid starting at rest from a point B where arc $OB = b$. Let P be the position of the particle at any time t where arc $OP = s$ and ψ be the angle which the tangent at P to the cycloid makes with the tangent at the vertex O. The forces acting on the particle at P are : (i) the weight mg acting vertically downwards and (ii) the normal reaction R acting along the inwards drawn normal at P. Resolving these forces along the tangent and normal at P, the tangential and normal equations of motion of P are

$$m\frac{d^2s}{dt^2} = -mg\sin\psi, \quad ...(1)$$

and $$m\frac{v^2}{\rho} = R - mg\cos\psi. \quad ...(2)$$

Here v is the velocity of the particle at P and is along the tangent at P.

[Note that the expression for the tangential acceleration is d^2s/dt^2 and it is positive in the direction of s increasing. In the equation (1) negative sign has been taken because $mg\sin\psi$ acts in the direction of s decreasing. Again the expression for normal acceleration is v^2/ρ and it is positive in the direction of inwards drawn normal. In the equation (2) we have taken R with positive sign because it is in the direction of inwards drawn normal while negative sign has been fixed before $mg\cos\psi$ because it is in the direction of outwards drawn normal].

Now the intrinsic equation of the cycloid is $s = 4a\sin\psi$. ...(3)

From (1) and (3), we have

$$\frac{d^2s}{dt^2} = -\frac{g}{4a}s, \quad ...(4)$$

which is the equation of a simple harmonic motion with centre at the point $s = 0$ *i.e.*, at the point O. Thus the particle will oscillate in S.H.M. about the centre O. The time period T of this S.H.M. is given by

$$T = \frac{2\pi}{\sqrt{(g/4a)}} = \pi\sqrt{(a/g)},$$

which is independent of the amplitude (*i.e.*, the initial displacement b). *Thus from whatever point the particle may be allowed to slide down the arc of a smooth cycloid, the time period remains the same*. Such a motion is called *isochronous* motion.

Multiplying both sides of (4) by 2 (ds/dt) and then integrating w.r.t. 't' we have

$$v^2 = \left(\frac{ds}{dt}\right)^2 = -\frac{g}{4a}s^2 + A.$$

But initially at the point B, $s = b$ and $v = 0$.

Therefore $0 = -(g/4a)b^2 + A$ or $A = (g/4a)b^2$.

$$\therefore \quad v^2 = \left(\frac{ds}{dt}\right)^2 = -\frac{g}{4a}s^2 + \frac{g}{4a}b^2 = \frac{g}{4a}(b^2 - s^2), \quad ...(5)$$

which gives us the velocity of the particle at any position 's'.

Substituting the value of v^2 in (2), we get R which gives us the pressure at any point on the cycloid.

Taking square root of (5), we get

$$\frac{ds}{dt} = -\sqrt{\left(\frac{g}{4a}\right)}\sqrt{(b^2 - s^2)},$$

where the –ive sign has been taken because the particle is moving in the direction of s decreasing.

Separating the variables, we get

$$-\frac{ds}{\sqrt{(b^2-s^2)}}=\sqrt{\left(\frac{g}{4a}\right)}\,t. \qquad \ldots(6)$$

Integrating, we have

$$\cos^{-1}(s/b)=\sqrt{(g/4a)}\,t+C.$$

But initially at $B, s=b$ and $t=0$. Therefore, $\cos^{-1}1=0+C$ or $C=0$.

$$\therefore \qquad \cos^{-1}(s/b)=\sqrt{(g/4a)}\,t \quad \text{or} \quad s=b\cos\sqrt{(g/4a)}\,t,$$

which gives a relation between s and t.

If t_1 be the time from B to O, then integrating (6) from B to O, we have

$$-\int_b^0\frac{ds}{\sqrt{(b^2-s^2)}}=\int_0^{t_1}\sqrt{\left(\frac{g}{4a}\right)}\,dt$$

[Note that at $B, s=b$ and $t=0$ while at $O, s=0$ and $t=t_1$]

or $$\left[\cos^{-1}\frac{s}{b}\right]_b^0=\sqrt{\left(\frac{g}{4a}\right)}\,[t]_0^{t_1}$$

or $$\cos^{-1}0-\cos^{-1}1=\sqrt{\left(\frac{g}{4a}\right)}\,t_1 \quad \text{or} \quad \frac{\pi}{2}=\sqrt{\left(\frac{g}{4a}\right)}\,t_1$$

or $$t_1=\pi\sqrt{(a/g)}.$$

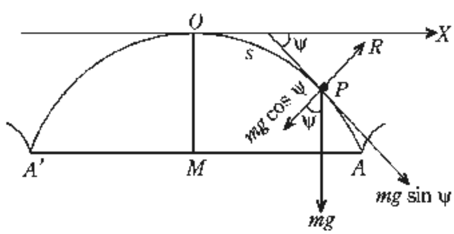
This time t_1 is independent of the initial displacement b of the particle. *Thus on a smooth cycloid the time of descent to the vertex is independent of the initial displacement of the particle.*

If T is time period of the particle *i.e.*, if T is the time for one complete oscillation, we have

$$T=4\times\text{time from } B \text{ to } O=4t_1=4\pi\sqrt{(a/g)}. \qquad \textbf{(Agra 2009)}$$

3.9 Motion on the Outside of a Smooth Cycloid with its Axis Vertical and Vertex Upwards

A particle is placed very close, to the vertex of a smooth cycloid whose axis is vertical and vertex upwards and is allowed to run down the curve, to discuss the motion. **(Kanpur 2010)**



Let a particle of mass m, starting from rest at O, slide down the arc of a smooth cycloid whose axis OM is vertical and vertex O is upwards. Let P be the position of the particle at time t such that arc $OP = s$ and the tangent at P to the cycloid makes an angle ψ with the tangent at the vertex O. The forces acting on the particle at P are : (i) weight mg acting vertically downwards and (ii) the reaction R acting along the outwards drawn normal.

$\therefore$ The equations of motion along the tangent and normal are

$$m \frac{d^2 s}{dt^2} = mg \sin \psi, \quad \text{...(1)}$$

and $$m \frac{v^2}{\rho} = mg \cos \psi - R. \quad \text{...(2)}$$

Also for the cycloid, $s = 4\, a \sin \psi$. ...(3)

From (1) and (3), we have $\dfrac{d^2 s}{dt^2} = \dfrac{g}{4a} s.$

Multiplying both sides by $2\,(ds\,/\,dt)$ and then integrating, we have

$$v^2 = \left(\frac{ds}{dt}\right)^2 = \frac{g}{4\,a} s^2 + A.$$

Initially at O, $s = 0$ and $v = 0$. Therefore, $A = 0$.

$$\therefore \quad v^2 = \frac{g}{4\,a} s^2 = \frac{g}{4\,a} (4\,a \sin \psi)^2 = 4\,ag \sin^2 \psi. \quad \text{...(4)}$$

From (2) and (4), we have

$$R = mg \cos \psi - \frac{mv^2}{\rho}$$

$$= mg \cos \psi - \frac{m \,.\, 4\,ag \sin^2 \psi}{4\,a \cos \psi} \qquad \left[\because \; \rho = \frac{ds}{d\psi} = 4\,a \cos \psi\right]$$

$$= \frac{mg}{\cos \psi} (\cos^2 \psi - \sin^2 \psi). \quad \text{...(5)}$$

The equation (4) gives the velocity of the particle at any position and the equation (5) gives the reaction of the cycloid on the particle at any position. The pressure of the particle on the curve is equal and opposite to the reaction of the curve on the particle.

When the particle leaves the cycloid, we have $R = 0$

i.e., $$\frac{mg}{\cos \psi} (\cos^2 \psi - \sin^2 \psi) = 0$$

i.e., $$\sin^2 \psi = \cos^2 \psi$$

i.e., $$\tan^2 \psi = 1$$

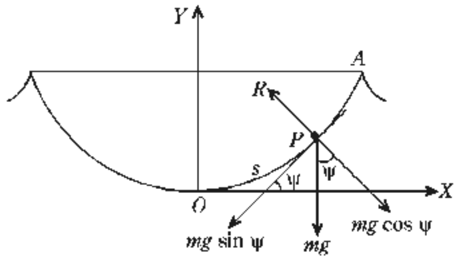
i.e., $$\tan \psi = 1 \quad \textit{i.e.,} \quad \psi = 45^\circ.$$

Hence the particle will leave the curve when it is moving in a direction making an angle 45° downwards with the horizontal.

Illustrative Examples

Example 10: *Prove that for a particle, sliding down the arc and starting from the cusp of a smooth cycloid whose vertex is lowest, the vertical velocity is maximum when it has described half the vertical height.* **(Kumaun 2003)**

Solution: Let a particle of mass m slide down the arc of a cycloid starting at rest from the cusp A. If P is the position of the particle at any time t, then the equations of motion of the particle along the tangent and normal are



$$m\frac{d^2s}{dt^2} = -mg\sin\psi, \qquad \ldots(1)$$

and $$m\frac{v^2}{\rho} = R - mg\cos\psi. \qquad \ldots(2)$$

For the cycloid, $s = 4a\sin\psi$. ...(3)

From (1) and (3), we have

$$\frac{d^2s}{dt^2} = -\frac{g}{4a}s.$$

Multiplying both sides by $2(ds/dt)$ and integrating, we have

$$v^2 = \left(\frac{ds}{dt}\right)^2 = -\frac{g}{4a}s^2 + A.$$

But initially at the cusp A, $s = 4a$ and $v = 0$. Therefore, $A = 4ag$.

$\therefore$ $$v^2 = 4ag - \frac{g}{4a}s^2 = 4ag - \frac{g}{4a}(4a\sin\psi)^2 = 4ag(1 - \sin^2\psi)$$

$$= 4ag\cos^2\psi \quad \text{or} \quad v = 2\sqrt{(ag)}\cos\psi,$$

giving the velocity of the particle at the point P, its direction being along the tangent at P. Let V be the vertical component of the velocity v at the point P. Then

$$V = v\cos(90° - \psi) = v\sin\psi$$

$$= 2\sqrt{(ag)}\cos\psi \, . \sin\psi = \sqrt{(ag)}\sin 2\psi,$$

which is maximum when $\sin 2\psi = 1$ *i.e.*, $2\psi = \pi/2$ *i.e.*, $\psi = \pi/4$.

When $\psi = \pi/4$, $s = 4a\sin(\pi/4) = 2\sqrt{2}\,a$.

Putting $s = 2\sqrt{2}\,a$ in the relation $s^2 = 8ay$, we have

$$(2\sqrt{2}\,a)^2 = 8ay \quad \text{or} \quad y = a.$$

Thus at the point where the vertical velocity is maximum, we have $y = a$.

The vertical depth fallen upto this point

$$= (\text{the } y\text{-coordinate of } A) - a = 2a - a = a = \frac{1}{2}(2a)$$

$$= \text{half the vertical height of the cycloid.}$$

Example 11: *A particle is projected with velocity V from the cusp of a smooth inverted cycloid down the arc, show that the time of reaching the vertex is* $2\sqrt{(a/g)}\tan^{-1}[\sqrt{(4ag)}/V]$.

(Garhwal 2001, 02; Kumaun 02; Purvanchal 11)

Solution: Refer figure of 3.8.

Let a particle be projected with velocity V from the cusp A of a smooth inverted cycloid down the arc. If P is the position of the particle at time t such that the tangent at P is inclined at an angle ψ to the horizontal and arc $OP = s$, then the equations of motion of the particle are

$$m\frac{d^2s}{dt^2} = -mg\sin\psi \quad \text{...(1)}$$

and

$$m\frac{v^2}{\rho} = R - mg\cos\psi. \quad \text{...(2)}$$

For the cycloid, $s = 4a\sin\psi$. ...(3)

From (1) and (3), we have

$$\frac{d^2s}{dt^2} = -\frac{g}{4a}s.$$

Multiplying both sides by $2(ds/dt)$ and integrating, we have

$$v^2 = \left(\frac{ds}{dt}\right)^2 = -\frac{g}{4a}s^2 + A.$$

But initially at the cusp A, $s = 4a$ and $ds/dt = -V$.

$$\therefore \quad V^2 = -(g/4a).16a^2 + A$$

or

$$A = V^2 + 4ag.$$

$$\therefore \quad v^2 = \left(\frac{ds}{dt}\right)^2 = V^2 + 4ag - \frac{g}{4a}s^2 = \left(\frac{g}{4a}\right)\left[\frac{4a}{g}(V^2 + 4ag) - s^2\right]$$

or

$$\frac{ds}{dt} = -\frac{1}{2}\sqrt{\left(\frac{g}{a}\right)}\sqrt{\left[\frac{4a}{g}(V^2 + 4ag) - s^2\right]}$$

(–ive sign is taken because the particle is moving in the direction of s decreasing)

or

$$dt = -2\sqrt{\left(\frac{a}{g}\right)} \cdot \frac{ds}{\sqrt{[(4a/g)(V^2 + 4ag) - s^2]}}.$$

Integrating, the time t_1 from the cusp A to the vertex O is given by

$$t_1 = -2\sqrt{\left(\frac{a}{g}\right)}\int_{s=4a}^{0}\frac{ds}{\sqrt{[(4a/g)(V^2 + 4ag) - s^2]}}$$

$$= 2\sqrt{\left(\frac{a}{g}\right)}\int_{0}^{4a}\frac{ds}{\sqrt{[(4a/g)(V^2 + 4ag) - s^2]}}$$

$$= 2\sqrt{\left(\frac{a}{g}\right)}\left[\sin^{-1}\frac{s}{2\sqrt{(a/g)}\sqrt{(V^2+4ag)}}\right]_0^{4a}$$

$$= 2\sqrt{\left(\frac{a}{g}\right)}\cdot\sin^{-1}\left\{\frac{2\sqrt{(ag)}}{\sqrt{(V^2+4ag)}}\right\} = 2\sqrt{(a/g)}\,.\theta, \qquad \ldots(4)$$

where $\quad \theta = \sin^{-1}\left\{\frac{2\sqrt{(ag)}}{\sqrt{(V^2+4ag)}}\right\}.$

We have $\quad \sin\theta = \left\{\frac{2\sqrt{(ag)}}{\sqrt{(V^2+4ag)}}\right\}.$

$\therefore \quad \cos\theta = \sqrt{(1-\sin^2\theta)} = \sqrt{\left(1-\frac{4ag}{V^2+4ag}\right)} = \frac{V}{\sqrt{(V^2+4ag)}}.$

$\therefore \quad \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{2\sqrt{(ag)}}{V} = \frac{\sqrt{(4ag)}}{V}$

or $\quad \theta = \tan^{-1}[\sqrt{(4ag)}/V].$

$\therefore$ from (4), the time of reaching the vertex is

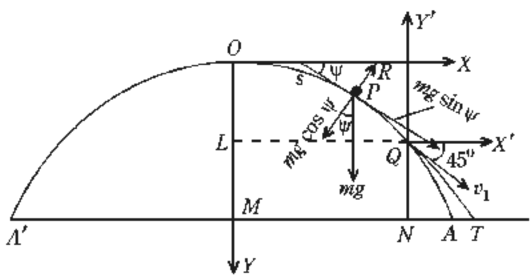
$$= 2\sqrt{(a/g)}\cdot\tan^{-1}[\sqrt{(4ag)}/V].$$

Example 12: *A particle is placed very near the vertex of a smooth cycloid whose axis is vertical and vertex upwards, and is allowed to run down the curve. Prove that it will leave the curve when it has fallen through half the vertical height of the cycloid.* **(Bundelkhand 2010)**

Also prove that the latus rectum of the parabola subsequently described is equal to the height of the cycloid.

Also show that it falls upon the base of the cycloid at a distance $\left(\frac{1}{2}\pi+\sqrt{3}\right)a$ *from the centre of the base, a being the radius of the generating circle.*

Solution: Let a particle of mass m, starting from rest at O, slide down the arc of a smooth cycloid whose axis OM is vertical and vertex O is upwards. Let P be the position of the particle at any time t such that arc $OP = s$. If the tangent at P makes an angle ψ with the horizontal, then the equations of motion of the particle along the tangent and normal at P are



$$m\frac{d^2s}{dt^2} = mg\sin\psi, \qquad \ldots(1)$$

and $\quad m\frac{v^2}{\rho} = mg\cos\psi - R. \qquad \ldots(2)$

Also for the cycloid, $s = 4\,a \sin \psi$. ...(3)

From (1) and (3), we have

$$\frac{d^2 s}{dt^2} = \frac{g}{4\,a}\,s.$$

Multiplying both sides by 2 (ds / dt) and integrating, we have

$$v^2 = \left(\frac{ds}{dt}\right)^2 = \frac{g}{4\,a}\,s^2 + A.$$

Initially at $O, s = 0$ and $v = 0$. Therefore, $A = 0$.

$$\therefore \quad v^2 = \frac{g}{4\,a}\,s^2 = \frac{g}{4\,a}\,(4\,a \sin \psi)^2 = 4\,ag \sin^2 \psi. \quad \text{...(4)}$$

From (2) and (4), we have

$$R = mg \cos \psi - \frac{mv^2}{\rho}$$

$$= mg \cos \psi - m \cdot \frac{4\,ag \sin^2 \psi}{4\,a \cos \psi} \qquad \left[\because \ \rho = \frac{ds}{d\psi} = 4\,a \cos \psi\right]$$

$$= \frac{mg}{\cos \psi}\,(\cos^2 \psi - \sin^2 \psi).$$

If the particle leaves the cycloid at the point Q, then at Q, $R = 0$. When $R = 0$, we have

$$\frac{mg}{\cos \psi}\,(\cos^2 \psi - \sin^2 \psi) = 0 \quad \text{or} \quad \sin^2 \psi = \cos^2 \psi$$

or $\qquad \tan^2 \psi = 1 \quad \text{or} \quad \tan \psi = 1 \quad \text{or} \quad \psi = 45°$.

Thus at Q, we have $\psi = 45°$.

Putting $\psi = \frac{1}{4}\pi$ in $s = 4\,a \sin \psi$, we have at Q,

$$s = 4\,a \sin \frac{1}{4}\pi = 4\,a\,(1/\sqrt{2}) = 2\sqrt{2}a.$$

Again putting $s = 2\sqrt{2}\,a$ in $s^2 = 8ay$, we have at Q,

$$y = \frac{s^2}{8a} = \frac{8a^2}{8a} = a.$$

Thus $OL = a$. Therefore $LM = OM - OL = 2a - a = a$. Hence the particle leaves the cycloid at the point Q, when it has fallen through half the vertical height of the cycloid.

Second part. If v_1 is the velocity of the particle at Q, then from (4), we have

$$v_1^2 = 4\,ag \sin^2 45° = 2ag.$$

Hence the particle leaves the cycloid at Q with velocity $v_1 = \sqrt{(2\,ag)}$ in a direction making an angle 45° downwards with the horizontal. After Q the particle will describe a parabolic path.

Latus rectum of the parabola described after Q

$$= \frac{2\, v_1^2 \cos^2 45°}{g} = \frac{2\,.\,2\, ag \cdot \frac{1}{2}}{g} = 2a$$

i.e., the latus rectum of the parabola subsequently described is equal to the height of the cycloid.

Third part. The equation of the parabolic path described by the particle after leaving the cycloid at Q with respect to the horizontal and vertical lines QX' and QY' as the coordinate axes is

$$y = x \tan(-45°) - \frac{gx^2}{2\, v_1^2 \cos^2(-45°)}$$

[Note that here the angle of projection for the motion of the projectile is $-45°$]

or $$y = -x - \frac{gx^2}{2\,.\,2ag \cdot \frac{1}{2}}$$

or $$y = -x - \frac{x^2}{2a}. \qquad \text{...(5)}$$

Suppose after leaving the cycloid at Q the particle strikes the base of the cycloid at the point T. Let (x_1, y_1) be the coordinates of T with respect to QX' and QY' as the coordinate axes.

Then $x_1 = NT$ and $y_1 = -QN = -a$.

But the point $T(x_1, -a)$ lies on the curve (5).

$$\therefore \quad -a = -x_1 - \frac{x_1^2}{2a}$$

or $$x_1^2 + 2ax_1 - 2a^2 = 0.$$

$$\therefore \quad x_1 = \frac{-2a \pm \sqrt{\{4a^2 - 4\,.\,1\,.\,(-2a^2)\}}}{2\,.\,1}.$$

Neglecting the –ive sign because x_1 cannot be negative, we have

$$x_1 = NT = -a + a\sqrt{3}.$$

The parametric equations of the cycloid w.r.t. OX and OY as the coordinate axes are

$$x = a(\theta + \sin\theta),$$
$$y = a(1 - \cos\theta),$$

where θ is the parameter and $\theta = 2\psi$.

$\therefore$ At the point Q, where $\psi = \frac{1}{4}\pi$, we have

$$x = LQ = a(2\psi + \sin 2\psi)$$
$$= a\left[2\,.\,\frac{1}{4}\pi + \sin\left(2 \cdot \frac{1}{4}\pi\right)\right]$$
$$= a\left(\frac{1}{2}\pi + 1\right).$$

$\therefore$ the horizontal distance of the point T from the centre M of the base of the cycloid

$$= MT = MN + NT = LQ + NT$$

$$= a\left(\frac{1}{2}\pi + 1\right) + (-a + a\sqrt{3})$$
$$= \left(\frac{1}{2}\pi + \sqrt{3}\right)a.$$

Comprehensive Exercise 4

1. A particle slides down a smooth cycloid whose axis is vertical and vertex downwards, starting from rest at the cusp. Find the velocity of the particle and the reaction on it at any point of the cycloid.

2. A particle oscillates from cusp to cusp of a smooth cycloid whose axis is vertical and vertex lowest. Show that the velocity v at any point P is equal to the resolved part of the velocity V at the vertex along the tangent at P *i.e.*, $v = V \cos \psi$.

3. A heavy particle slides down a smooth cycloid starting from rest at the cusp, the axis being vertical and vertex downwards, prove that the magnitude of the acceleration is equal to g at every point of the path and the pressure when the particle arrives at the vertex is equal to twice the weight of the particle.

(Purvanchal 2007; Kumaun 01)

4. A particle oscillates in a cycloid under gravity, the amplitude of the motion being b, and period being T. Show that its velocity at any time t measured from a position of rest is

$$\frac{2\pi b}{T}\sin\left(\frac{2\pi t}{T}\right).$$

5. A particle starts from rest at the cusp of a smooth cycloid whose axis is vertical and vertex downwards. Prove that when it has fallen through half the distance measured along the arc to the vertex, two-thirds of the time of descent will have elapsed. **(Lucknow 2008; Kanpur 09)**

6. A particle slides down the arc of a smooth cycloid whose axis is vertical and vertex lowest, starting at rest from the cusp. Prove that the time occupied in falling down the first half of the vertical height is equal to the time of falling down the second half. **(Meerut 2007)**

7. If a particle starts from rest at a given point of a cycloid with its axis vertical and vertex downwards, prove that it falls $1/n$ of the vertical distance to the lowest point in time

$$2\sqrt{(a/g)}\sin^{-1}(1/\sqrt{n}),$$

where a is the radius of the generating circle.

8. A particle slides down the arc of a smooth cycloid whose axis is vertical and vertex lowest, starting from rest at a given point of the cycloid. Prove that the time occupied in falling down the first half of the vertical height to the lowest point is equal to the time of falling down the second half. **(Bundelkhand 2011)**

9. Two particles are let drop from the cusp of a cycloid down the curve at an interval of time t; prove that they will meet in time $2\pi\sqrt{(a/g)} + (t/2)$.

(Agra 2003, 07, 09; Garhwal 03)

10. A particle starts from rest at any point P in the arc of a smooth cycloid $s = 4\,a \sin\psi$ whose axis is vertical and vertex A downwards ; prove that the time of descent to the vertex is $\pi\sqrt{(a/g)}$. **(Meerut 2006)**

Show that if the particle is projected from P downwards along the curve with velocity equal to that with which it reaches A when starting from rest at P, it will now reach A in half the time taken in the preceding case.

11. If a particle starts from the vertex of a cycloid whose axis is vertical and vertex upwards, prove that its velocity at any point varies as the distance of that point from the vertex measured along the arc.

12. A cycloid is placed with its axis vertical and vertex upwards and a heavy particle is projected from the cusp up the concave side of the curve with velocity $\sqrt{(2\,gh)}$; prove that the latus rectum of the parabola described after leaving the arc is $h^2/2\,a$, where a is the radius of the generating circle.

Answers 4

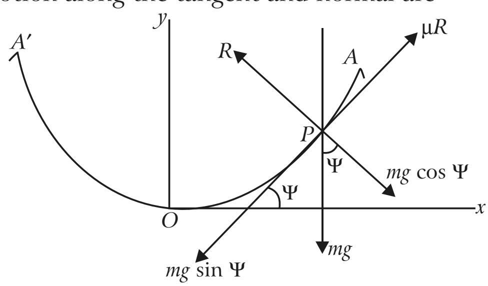
1. $v = 2\sqrt{(ag)}\cos\psi$, $R = 2\,mg\cos\psi$

3.10 Motion on a Rough Cycloid

A particle slides down a rough cycloid of which coefficient of friction is μ to discuss the motion.

Since the particle moves downwards, the force of friction μR acts in the upward direction. Let P be the position of the particle at any time t.

The equations of motion along the tangent and normal are



$$mv\,(dv/ds) = \mu R - mg\sin\psi$$

or $$\frac{1}{2}m(dv^2/ds) = \mu R - mg\sin\psi \qquad \ldots(1)$$

and $$m(v^2/\rho) = R - mg\cos\psi \qquad \ldots(2)$$

Eliminating R between (1) and (2), we get

$$\frac{1}{2}\frac{dv^2}{ds} - \mu\frac{v^2}{\rho} = (\mu\cos\psi - \sin\psi)$$

or $$\frac{dv^2}{d\psi}\cdot\frac{d\psi}{ds}-2\mu\frac{v^2}{\rho}=2g\,(\mu\cos\psi-\sin\psi)$$

or $$\frac{dv^2}{d\psi}-2\mu v^2=2g\rho(\mu\cos\psi-\sin\psi) \qquad \left[\because\ \rho=\frac{ds}{d\psi}\right] \quad ...(3)$$

Now intrinsic equation of cycloid is

$$s=4a\sin\psi.$$

$$\therefore \quad \rho=\left(\frac{ds}{d\psi}\right)=4a\cos\psi.$$ **[Remember]**

Substituting value of ρ in (3), we get

$$\frac{dv^2}{d\psi}-2\mu v^2=8ag\cos\psi(\mu\cos\psi-\sin\psi). \quad ...(4)$$

This is a linear differential equation in v^2.

Integrating factor $= e^{\int -2\mu\, d\psi} = e^{-2\mu\psi}$.

Multiplying both sides of (4) by $e^{-2\mu\psi}$ and integrating, we have

$$v^2e^{-2\mu\psi}=8\,ag\int e^{-2\mu\psi}.\cos\psi\,(\mu\cos\psi-\sin\psi)\,d\psi+c \quad ...(5)$$

where c is a constant of integration to be determined by initial conditions.

$$\therefore \quad v^2e^{-2\mu\psi}=4ag\int e^{-2\mu\psi}\{\mu(1+\cos2\psi)-\sin2\psi\}\,d\psi+c$$

$$=4ag\int[\mu e^{-2\mu\psi}+\mu\cos2\psi\, e^{-2\mu\psi}-\sin2\psi\, e^{-2\mu\psi}]\,d\psi$$

$$=4ag\left[-\frac{e^{-2\mu\psi}}{2}+\frac{e^{-2\mu\psi}}{4+4\mu^2}\{\mu(-2\mu\cos2\psi+2\sin2\psi)\right.$$

$$\left.-(-2\mu\sin2\psi-2\cos2\psi)\}\right]+c$$

[Applying 3.5 (v)]

$$=4ag\left[-\frac{1}{2}e^{-2\mu\psi}+\frac{e^{-2\mu\psi}}{4(1+\mu^2)}\{2(1-\mu^2)\cos2\psi+4\mu\sin2\psi\}\right]+c \quad ...(6)$$

Thus (6) gives velocity of the particle in any position and putting this value of v^2 in (2), R can be found out.

Illustrative Examples

Example 13: *A heavy particle slides from a cusp, down the arc of a rough cycloid whose axis is vertical. Prove that its velocity at the vertex is to the velocity at the same point when the cycloid is smooth as*

$$\sqrt{(e^{-\psi\pi}-\mu^2)}:\sqrt{(1+\mu^2)}$$

where μ is the coefficient of friction. Further show that the particle will certainly come to rest before reaching the vertex if the coefficient of friction be 0.5, having given that

$$\log 2 = 0 \cdot 69315.$$

Solution: Proceeding exactly as in 3.10, we have equation (6) as

$$v^2 e^{-2\mu\psi} = 4ag\left[-\frac{1}{2}e^{-2\mu\psi} + \frac{e^{-2\mu\psi}}{4+4\mu^2}\{2(1-\mu^2)\cos 2\psi + 2\mu\sin 2\psi\} + c\right]. \quad \ldots(6)$$

The particle slides from cusp.

$\therefore$ $v = 0, \psi = \pi/2$; (6) gives

$$c = -4ag\left[-\frac{1}{2}e^{-\mu\pi} + \frac{e^{-\mu\psi}}{4(1+\mu^2)}\{-2(1-\mu^2)\}\right]$$

$$= 2age^{-\mu\pi}\left[1 + \frac{1-\mu^2}{1+\mu^2}\right] = \frac{4age^{-\mu\pi}}{1+\mu^2}$$

Substituting this value of c in (6), we get

$$v^2 e^{-2\mu\psi} = 4ag\left[-\frac{1}{2}e^{-2\psi} + \frac{e^{-2\mu\psi}}{4(1+\mu^2)}\{2(1-\mu^2)\cos 2\psi + 2\mu\sin 2\psi\}\right] + \frac{4age^{-\mu\pi}}{1+\mu^2}. \quad \ldots(7)$$

This gives velocity at any position.

At vertex $\psi = 0$, therefore velocity v at vertex is given by

$$v^2 = 4ag\left[-\frac{1}{2} + \frac{1}{4(1+\mu^2)}\{2(1-\mu^2)\}\right] + \frac{4age^{-\mu\pi}}{1+\mu^2}$$

$$= \frac{4ae}{1+\mu^2}\left[\frac{-2(1+\mu^2)+2(1-\mu^2)}{4} + e^{-\mu\pi}\right]$$

$$= \frac{4ag}{1+\mu^2}[e^{-\mu\pi} - \mu^2]. \quad \ldots(8)$$

Let V be the velocity at the vertex when the cycloid is smooth, hence putting $\mu = 0$ and $v = V$ in (8), we get

$$V^2 = 4ag.$$

$\therefore$ $$(v^2/V^2) = (e^{-\mu\pi} - \mu^2)/(1+\mu^2)$$

or $$v : V = \sqrt{(e^{-\mu\pi} - \mu^2)} : \sqrt{(1+\mu^2)}.$$

This establishes the first result.

The particle will come to rest at the vertex if

$$e^{-\mu\pi} - \mu^2 = 0 \quad \text{[putting } v = 0 \text{ in (8)]}$$

or $$\mu^2 e^{\mu\pi} = 1$$

or $\qquad \mu e^{1/2(\mu\pi)} = 1.$

Hence particle will come to rest before arriving at the vertex if

$$\mu e^{1/2(\mu\pi)} > 1 \qquad ...(9)$$

i.e. if $\qquad \log \mu + \frac{1}{2}\mu\pi \log e > \log 1$

i.e. if $\qquad \log \frac{1}{2} + \frac{1}{2}.\frac{1}{2}\pi > 0 \qquad [\text{as } \mu = 0\cdot 5 = \frac{1}{2}]$

i.e. if $\qquad -\log 2 + \frac{1}{4}\pi > 0$

i.e. if $\qquad -\log 2 + \frac{1}{4}.(22/7) > 0 \qquad [\because \pi = 22/7]$

i.e. if $\qquad -0\cdot 69315 + 0\cdot 78570 > 0$ which is true.

Hence particle will certainly come to rest before reaching the vertex if $\mu = 0\cdot 5$.

Example 14: *A particle slides on a rough wire in the form of a cycloid $s = 4a \sin\psi$, which lies in a vertical plane with its axis vertical and vertex downwards. The particle is projected from the vertex with such a velocity u that it comes to rest at the cusp show that*

$$e^{\mu\pi} = \mu^2 + \frac{u^2}{4ag}(1+\mu^2).$$

Solution: Here particle is projected upwards from the vertex. Let P be its position at any time t. Conceive figure with the difference that μR acts downwards instead of upwards.

$\therefore$ Equations of motion are

$$mv\,(dv/ds) = \mu R - mg \sin\psi \qquad ...(1)$$

$$m\,(v^2/\rho) = R - mg\cos\psi \qquad ...(2)$$

Now proceed as in 3.10 and get equation (6), [with the difference that put $-\mu$ for μ in every step].

$$v^2 e^{2\mu\psi} = 4age^{2\mu\psi}\left[-\frac{1}{2} + \frac{1}{4(1+\mu^2)}\{2(1-\mu^2)\cos 2\psi - 4\mu \sin 2\psi\}\right] + c \qquad ...(6)$$

At the vertex $v = u$, when $\psi = 0$.

$$\therefore \qquad u^2 = 4ga\left[-\frac{1}{2} + \frac{2(1-\mu^2)}{4(1+\mu^2)}\right] + c \qquad ...(7)$$

and at the cusp particle comes to rest *i.e.*, $v = 0$ when $\psi = \frac{1}{2}\pi$, equation (6) becomes

$$c = -4age^{\mu\psi}\left[-\frac{1}{2} + \frac{1}{4(1+\mu^2)}\{-2(1-\mu^2)\}\right] = \frac{4age^{\mu\pi}}{(1+\mu^2)} \qquad ...(8)$$

Eliminating c between (7) and (8), we get

$$u^2 = 4ga\left[-\frac{1}{2} + \frac{2(1-\mu^2)}{4(1+\mu^2)}\right] + \frac{4age^{\mu\pi}}{1+\mu^2}$$

or $$\frac{u^2}{4ag}(1+u^2) = \frac{-2(1+\mu^2)+2(1-\mu^2)}{4} + e^{\mu\pi}$$

or $$\frac{u^2(1+\mu^2)}{4ag} = -\mu^2 + e^{\mu\pi} \quad \text{or} \quad e^{\mu\pi} = \mu^2 + \frac{u^2}{4ag}(1+\mu^2).$$

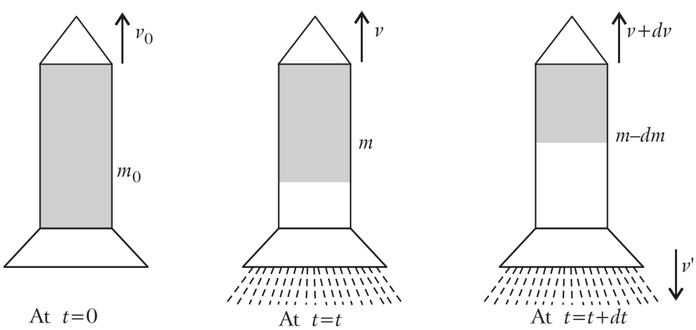
3.11 Rocket Propulsion : Variable Mass in Motion

We have so far considered the motion of bodies with constant mass. The mass of the system is varying in many practical cases. An important example of such motion is the motion of a rocket. Rocket is a system in which fuel is burnt and gases are expelled out of the tail of the rocket and thus the mass of the rocket system varies with time.

Principle of propulsion. Rocket Propulsion is based on the principle of conservation of momentum. Before a rocket is fired, the total momentum of rocket and fuel in it is zero. As the system is essentially an isolated system, the momentum of the system remains unchanged. When a rocket is fired, fuel is burnt and very hot gases are expelled from the tail of the rocket. The momentum acquired by the gases is directed backward. In order to conserve momentum, the rocket must acquire an equal momentum in the forward direction (opposite direction). Thus the rocket is propelled .

Velocity of rocket at any instant. Consider the motion of a rocket moving vertically upwards from the surface of the earth. Let m_0 be the initial mass of the rocket when no gas has escaped yet and v_0 be its velocity with respect to earth at time $t = 0$ [Fig. (a)].

Let m be the mass of the rocket and v be its velocity with respect to earth in upward direction at any time t [Fig. (b)]. Due to combustion of the fuel, the velocity of the rocket increases. Let dm be the mass of the fuel burnt in time dt and dv be the increase in the velocity of the rocket in this time interval. Let the gases formed by the combustion of the mass dm of the fuel leave the rocket in downward direction with velocity v' with respect to earth [Fig. (c)]



(a) (b) (c)

By the principle of conservation of momentum, the initial momentum at the start of time interval dt must be equal to the final momentum at the end of the time interval dt.

Thus linear momentum of mass m of the rocket at time t must be equal to the sum of the momentum of the rocket with mass $m - dm$ and the momentum of the gases formed by the combustion of fuel mass dm.

$$\therefore \quad mv = (m - dm)(v + dv) + dm(-v') \quad \ldots(1)$$

The velocity v' of burnt gases has been taken with negative sign, as the burnt gases move in the direction opposite to that of the rocket. From equation (1), we have

$$mv = mv - v\,dm + m\,dv - dm\,dv - v'\,dm$$

or
$$m\,dv = (v + v')\,dm \quad \ldots(2)$$

The product $dm\,dv$ of two very small quantities has been neglected. If u be the relative velocity of the burnt gases with respect to the rocket, then

$$u = v + v'$$

$\therefore$ from equation (2), we have

$$m\,dv = -u\,dm \quad \ldots(3)$$

The negative sign indicates that the velocity u of the burnt gases with respect to the rocket is in downward direction.

From equation (3), we have

$$dv = -u\frac{dm}{m} \quad \ldots(4)$$

Integrating equation (4) within proper limits, we have

$$\int_{v_0}^{v} dv = -\int_{m_0}^{m} u\frac{dm}{m}$$

$$= -u\int_{m_0}^{m}\frac{dm}{m},$$

assuming the exhaust velocity of the burnt gases to be constant throughout the firing of the rocket.

$$\therefore \quad [v]_{v_0}^{v} = -u[\log m]_{m_0}^{m}$$

or
$$v - v_0 = -u[\log m - \log m_0]$$

or
$$v - v_0 = u\log\frac{m_0}{m}$$

or
$$v = v_0 + u\log\frac{m_0}{m} \quad \ldots(5)$$

Equation (5) gives the velocity of the rocket at any time t, when the mass of the rocket is m. If the initial velocity of the rocket is zero, then equation (5) becomes

$$v = u\log\frac{m_0}{m} \quad \ldots(6)$$

Thus the velocity of the rocket at any instant is directly proportional to

(1) The exhaust speed of the ejecting burnt gases with respect to the rocket.

(2) The natural logarithm of the ratio of initial mass of the rocket to its mass at that instant.

Thrust on the Rocket. Dividing equation (3) by dt, we have

$$m\frac{dv}{dt} = -u\frac{dm}{dt}$$

dv/dt represents the instantaneous acceleration of the rocket, therefore $m(dv/dt)$ is the instantaneous force or thrust on the rocket.

$\therefore$ thrust on the rocket $= F = -u\dfrac{dm}{dt}$ ...(7)

Here $\dfrac{dm}{dt}$ denotes the instantaneous rate of consumption of the fuel in the rocket.

As the velocity u of the burnt gases with respect to the rocket is in downward direction, the thrust on the rocket is in upward direction.

Thus upward thrust on the rocket

$$= F = u\frac{dm}{dt}.$$

Net force on rocket $= F - mg$.

The following points regarding thrust on the rocket may be noted :

(1) The thrust on the rocket at any instant is equal to the product of exhaust speed of the ejecting gases with respect to the rocket and the rate of combustion of fuel at that instant.

(2) The thrust on the rocket can be increased by ejecting the gases at a higher exhaust speed *i.e.*, by ejecting the gases at a greater rate.

Burnt Out Speed of the Rocket

The maximum speed acquired by the rocket when the whole fuel of the rocket gets burnt is called the burnt out speed of the rocket.

If m_r is the mass of the empty container, then from equation (5), the burnt out speed becomes

$$v_b = v_0 + u\log\frac{m_o}{m_r} \quad ...(8)$$

or $$v_b = u\log\frac{m_0}{m_r}, \quad ...(9)$$

when initial velocity is zero.

Illustrative Examples

Example 15: A rocket with lift-off mass 20,000 kg is blasted upwards with an initial acceleration of 5 m s^{-2}. Calculate the initial thrust of the blast.

Solution: Here $m = 20{,}000$ kg.

initial acceleration $= 5$ m s^{-2}

Since the rocket moves up against gravity, therefore the blast has to produce a total acceleration a given by

$$a = 9\cdot 8 + 5 = 14\cdot 8 \text{ m s}^{-2}$$

$\therefore$ the initial thrust of the blast $= F = ma$

$$= 20{,}000 \times 14 \cdot 8$$
$$= 296000 \text{ N}$$

Example 16: A fully fuelled rocket has a mass 21,000 kg of which 15,000 kg is fuel. If the rocket ejects its gases at a speed of 2800 m s^{-1} relative to the rocket and burns fuel at the rate of 190 kg s^{-1}, calculate (i) the thrust of the rocket
(ii) the net force on the rocket at blast off.

Solution: Here $u = 2800 \text{ m s}^{-1}$, $dm/dt = 190 \text{ kg s}^{-1}$

(i) Thrust on rocket $= F = u\dfrac{dm}{dt}$

$$= 2800 \times 190 = 5 \cdot 3 \times 10^5 \text{N}$$

(ii) At blast off, the mass of rocket $= mg = 2100 \times 9 \cdot 8$

$$= 2 \cdot 1 \times 10^5 \text{N}$$

$\therefore$ Net force on rocket at blast off $= F - mg$

$$= 5 \cdot 3 \times 10^5 - 2 \cdot 1 \times 10^5$$
$$= 3 \cdot 2 \times 10^5 \text{N}.$$

Example 17: Fuel is consumed at the rate of 100 kg s^{-1} in a rocket. The exhaust gases are ejected at a speed of $4{\cdot}5 \times 10^4$ m s^{-1}. What is the thrust experienced by the rocket? Also calculate velocity of the rocket at the instant, when its mass is reduced to $\dfrac{1}{10}$th of its initial mass.

Solution: Here $u = 4 \cdot 5 \times 10^4 \text{ m s}^{-1}, \dfrac{dm}{dt} = 100 \text{ kg s}^{-1}$

Thrust on rocket $= F = u\dfrac{dm}{dt}$

$$= 4 \cdot 5 \times 10^4 \times 100 = 4 \cdot 5 \times 10^6 \text{ N}.$$

We have, velocity of the rocket at any instant $= v = u \log \dfrac{m_0}{m}$

Here $\dfrac{m}{m_0} = \dfrac{1}{10}$ or $\dfrac{m_0}{m} = 10$

$\therefore$ $v = 4 \cdot 5 \times 10^4 \log 10 = 4 \cdot 5 \times 10^4 \times 2 \cdot 303 \times \log_{10} 10$

$$= 4 \cdot 5 \times 10^4 \times 2 \cdot 303 \times 1 = 1 \cdot 036 \times 10^5 \text{ m s}^{-1}$$

Comprehensive Exercise 5

1. The base of a rough cycloidal arc is horizontal and its vertex downwards, a bead slides along it starting from rest at the cusp and coming to rest at the vertex. Show that $\mu^2 e^{\mu\pi} = 1$.

2. A particle slides in a vertical plane down a rough cycloidal arc whose axis is vertical and vertex downwards, starting from a point where the tangent makes an angle θ with the horizon and coming to rest at the vertex.
Show that

$$\mu e^{\mu\theta} = \sin\theta - \mu\cos\theta.$$

(Purvanchal 2007)

3. A rough cycloid has its plane vertical and the line joining its cusps horizontal. A heavy particle slides down the curve from rest at a cusp and comes to rest again at the point on the other side of the vertex where the tangent is inclined at 45° in the vertical. Show that the coefficient of friction satisfies the equation

$$3\mu\pi + \log(1+\mu) = 2\log 2.$$

4. A rocket burns fuel at the rate of $50\ \text{g s}^{-1}$ and ejects its gases at a speed of 5×10^5 cm s^{-1}, what force is exerted by the gas on the rocket?
5. A rocket consumed fuel at a rate of 24 kg/s and the burnt gases escaped the rocket at a speed of 64 km/s relative to the rocket. Calculate (i) the upthrust received by the rocket (ii) the velocity acquired by the rocket, when its mass reduced to $\frac{1}{100}$ of its initial mass.
6. A fully fuelled rocket of mass 5000 kg is set to be fired vertically. The burnt fuel is exhausted at a rate of $50\ \text{kg s}^{-1}$ at a speed of $3\times10^3\ \text{m s}^{-1}$ relative to the rocket.

   Including the effect of gravity, calculate the rocket's initial upwards acceleration.
7. A rocket of initial mass 6000 kg ejects gases at the constant rate of $16\ \text{kg s}^{-1}$ with constant relative speed of $11\ \text{m s}^{-1}$. Neglecting the force of gravity, calculate the acceleration of the rocket a minute after the blast.

Answers 5

4. 250 N
5. **(i)** $1\cdot536\times10^5$ N **(ii)** $2\cdot948\times10^4\ \text{m s}^{-1}$
6. $20\ \text{m s}^{-2}$
7. $34\cdot92\ \text{m s}^{-2}$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. A heavy particle of mass m is projected along the inside of a smooth vertical circle of radius a from its lowest point with velocity u. The least value of u for describing the complete circle is

   (a) $\sqrt{ag}$ (b) $\sqrt{4ag}$

   (c) $\sqrt{5ag}$ (d) $\sqrt{2ag}$.

2. A heavy particle of mass m is projected along the inside of a smooth vertical circle of radius a from its lowest point with velocity u. If after rising above the horizontal diameter of the circle the particle is to leave the circle, then we must have

(a) $u < \sqrt{2\,ag}$ (b) $u = \sqrt{2\,ag}$

(c) $u > \sqrt{5ag}$ (d) $\sqrt{2\,ag} < u < \sqrt{5ag}$.

3. A heavy particle of mass m is projected along the inside of a smooth vertical circle of centre O and radius a from its lowest point A with velocity u. If P is the position of the particle at time t such that $\angle AOP = \theta$, arc $AP = s$, the velocity of the particle at P is v and the reaction of the circle on the particle at P is R, then the equation of motion of the particle along the normal to the circle at P is

(a) $m\dfrac{v^2}{a} = mg\cos\theta - R$ (b) $m\dfrac{v^2}{a} = R - mg\cos\theta$

(c) $m\dfrac{v^2}{a} = mg\sin\theta - R$ (d) $m\dfrac{v^2}{a} = R - mg\sin\theta$.

4. A particle slides down the outside of a smooth vertical circle of radius a, starting from rest at the highest point. It will leave the circle after descending vertically a distance equal to

(a) $a/3$ (b) $a/4$

(c) $a/2$ (d) $2\,a/3$. **(Agra 2010)**

5. A heavy particle of mass m is placed very close to the vertex O of a smooth cycloid whose axis is vertical and vertex upwards and is allowed to run down the curve. If P is the position of the particle at any time t such that arc $OP = s$, the tangent at P to the cycloid makes an angle ψ with the tangent at the vertex O, then the equation of motion of the particle along the tangent at P is

(a) $m\dfrac{d^2s}{dt^2} = mg\cos\psi$ (b) $m\dfrac{d^2s}{dt^2} = mg\sin\psi$

(c) $m\dfrac{d^2s}{dt^2} = -\,mg\cos\psi$ (d) $m\dfrac{d^2s}{dt^2} = -\,mg\sin\psi$.

6. A heavy particle is tied to one end of a light inextensible string whose other end is attached to a fixed point. The particle completes the circle if the velocity of projection at the lowest point is least as :

(a) $\sqrt{6ag}$ (b) $\sqrt{3ag}$

(c) $\sqrt{5ag}$ (d) $\sqrt{2ag}$. **(Garhwal 2002)**

7. In order to rise above the horizontal diameter of a vertical circle, the velocity of projection u of a heavy particle tied to a light inextensible string of length a will be related to the length of the string as follows :

(a) $u^2 \geq 2ag$ (b) $u^2 > 2ag$

(c) $u^2 < 2ag$ (d) $u^2 \leq 2ag$. **(Garhwal 2003)**

8. In order to oscillate, the velocity of projection u of a heavy particle tied to a light inextensible string of length a is :

(a) $u^2 \leq 2ag$ (b) $u^2 > 2ag$

(c) $u^2 > 4ag$ (d) $u^2 = 2ag$. **(Garhwal 2004)**

Fill in the Blank(s)

Fill in the blanks "……" so that the following statements are complete and correct.

1. If a particle is compelled to move along a given curve or a surface, then its motion is called a …… . **(Meerut 2004)**

2. A heavy particle of mass m is tied to one end of a light inextensible string of length a whose other end is attached to a fixed point O. It is projected horizontally with a given velocity u from its vertical position of equilibrium A. If P is the position of the particle at time t such that $\angle AOP = \theta$, arc $AP = s$, the velocity of the particle at P is v and the tension in the string PO is T, then the equations of motion of the particle along the tangent and normal at P to its circular path are …… and …… .

3. A heavy particle is tied to one end of a light inextensible string of length a whose other end is attached to a fixed point. It is projected horizontally with a given velocity u from its vertical position of equilibrium. The least value of u for describing the complete circle is …… .

4. A heavy particle of mass m slides down the arc of a smooth cycloid whose axis is vertical and vertex O downwards. If P is the position of the particle at any time t such that arc $OP = s$, the tangent at P to the cycloid makes an angle ψ with the tangent at the vertex O, the velocity of the particle at P is v and the reaction of the cycloid on the particle at P is R, then the tangential and normal equations of motion of P are …… and …… .

5. In the case of the motion of a heavy particle on the arc of a smooth cycloid the time of descent to the vertex is …… of the initial displacement of the particle.

True or False

Write 'T' for true and 'F' for false statement.

1. A heavy particle of mass m is projected along the inside of a smooth vertical circle of radius a from its lowest point with velocity u. The least value of u for describing the complete circle is $\sqrt{4ag}$.

2. A heavy particle of mass m is projected along the inside of a smooth vertical circle of radius a from its lowest point with velocity u. If $u = \sqrt{3ag}$, then after rising above the horizontal diameter of the circle, the particle will leave the circle.

3. A heavy particle of mass m is projected along the inside of a smooth vertical circle of radius a from its lowest point with velocity $\sqrt{6ag}$. The particle will leave the circle at some point of its path.

4. A heavy particle of mass m slides down the arc of a smooth cycloid whose axis is vertical and vertex O downwards. If P is the position of the particle at any time t such that arc $OP = s$ and the tangent at P to the cycloid makes an angle ψ with the tangent at the vertex O, then the tangential equation of motion of P is
$$m \frac{d^2 s}{dt^2} = mg \sin \psi.$$

5. A particle oscillates from cusp to cusp of a smooth cycloid whose axis is vertical and vertex lowest. The velocity v at any point P is equal to the resolved part of the velocity V at the vertex along the tangent at P *i.e.*, $v = V \cos \psi$.

Answers

Multiple Choice Questions

1. (c) 2. (d) 3. (b) 4. (a) 5. (b)
6. (c) 7. (b) 8. (a)

Fill in the Blank(s)

1. constrained motion
2. $m\dfrac{d^2s}{dt^2} = -mg\sin\theta,\ m\dfrac{v^2}{a} = T - mg\cos\theta$
3. $\sqrt{(5ag)}$.
4. $m\dfrac{d^2s}{dt^2} = -mg\sin\psi, m\dfrac{v^2}{\rho} = R - mg\cos\psi$
5. independent

True or False

1. F 2. T 3. F
4. F 5. T

□

Chapter

4

Motion in a Resisting Medium

4.1 Introduction

It is a well known fact that a body moving in a medium (like air) feels a resistance to its motion which increases with the increase in the velocity of the body. Thus the resistance on a body moving in a medium may be assumed to be equal to some function of the velocity of the body. The resistance of the medium always acts opposite to the direction of motion of the body.

Experimentally it has been found out that when a particle is projected in air, the force of resistance varies as the square of the velocity upto a velocity of 800 ft./sec. and as cube of the velocity between 800 ft./sec and 1350 ft./sec. Beyond this velocity the resistance again varies as the square of the velocity.

Therefore in this chapter we shall mostly discuss the motion of a particle (or body) in a resisting medium where the resistance varies as the square of the velocity.

4.2 Terminal Velocity

If a particle falls under gravity in a resisting medium the force of resistance acts vertically upwards on the particle while the force of gravity acts vertically downwards. As the velocity of the particle goes on increasing the force of resistance also goes on increasing. Suppose the force of resistance becomes equal to the force of gravity when the particle has attained the velocity V. Then the resultant downward acceleration of the particle becomes zero and so during its subsequent motion the particle falls with constant velocity V, called the *terminal velocity* or the *limiting velocity*. The terminal velocity is maximum for the downward motion.

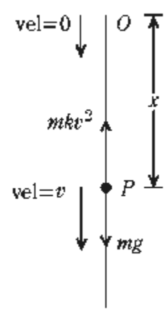
Definition. *If a particle is falling under gravity in a resisting medium, then the velocity V when the downward acceleration is zero is called the* **terminal (or limiting) velocity.**

(Meerut 2004; Rohilkhand 09)

4.3 Motion of a Particle Falling under Gravity

Case I. *A particle is falling from rest under gravity, supposed constant, in a resisting medium whose resistance varies as the square of the velocity; to discuss the motion.*

(Avadh 2007, 11; Purvanchal 08)



Let a particle of mass m fall from rest under gravity from the fixed point O.

Let P be the position of the particle after time t, where $OP = x$. If v is the velocity of the particle at P, then mkv^2 is the resistance of the medium on the particle acting in the upwards direction *i.e.*, in the direction of x decreasing. Here kv^2 is the resistance per unit mass so that the resistance on the particle of mass m is mkv^2.

The weight mg of the particle acts vertically downwards *i.e.*, in the direction of x increasing.

$\therefore$ the equation of motion of the particle at time t is

$$m\frac{d^2x}{dt^2} = mg - mkv^2$$

or
$$\frac{d^2x}{dt^2} = g\left(1 - \frac{k}{g}v^2\right) \qquad \text{...(1)}$$

If V is the terminal velocity, then when $v = V$, $d^2x/dt^2 = 0$.

$\therefore$ from (1), we have $0 = g\left(1 - \dfrac{k}{g}V^2\right)$ or $\dfrac{k}{g} = \dfrac{1}{V^2}$.

$\therefore$
$$\frac{d^2x}{dt^2} = g\left(1 - \frac{v^2}{V^2}\right) \quad \text{or} \quad \frac{d^2x}{dt^2} = \frac{g}{V^2}(V^2 - v^2) \qquad \text{...(2)}$$

To find the relation between v and x:

The equation (2) can be written as

$$v\frac{dv}{dx}=\frac{g}{V^2}(V^2-v^2) \qquad \left[\because \frac{d^2x}{dt^2}=v\frac{dv}{dx}\right]$$

or $$\frac{-2g}{V^2}dx=\frac{-2v\,dv}{V^2-v^2}.$$

Integrating, $\dfrac{-2g}{V^2}x=\log(V^2-v^2)+A$, where A is a constant.

But initially at O, when $x=0, v=0$.

$\therefore$ $$0=\log V^2+A \quad \text{or} \quad A=-\log V^2.$$

$\therefore$ $$\frac{-2gx}{V^2}=\log(V^2-v^2)-\log V^2=\log\left(\frac{V^2-v^2}{V^2}\right)$$

or $$\frac{V^2-v^2}{V^2}=e^{-2gx/V^2}$$

or $$V^2-v^2=V^2e^{-2gx/V^2}$$

or $$v^2=V^2(1-e^{-2gx/V^2}), \qquad ...(3)$$

which gives the velocity of the particle at any position.

Relation between v and t: **(Avadh 2006)**

Again the equation (2) can be written as

$$\frac{dv}{dt}=\frac{g}{V^2}(V^2-v^2) \qquad \left[\because \frac{d^2x}{dt^2}=\frac{dv}{dt}\right]$$

or $$\frac{g}{V^2}dt=\frac{dv}{V^2-v^2}.$$

Integrating, $\dfrac{g}{V^2}t=\dfrac{1}{2V}\log\dfrac{V+v}{V-v}+B$, where B is a constant.

Initially at O, when $t=0, v=0$.

$\therefore$ $$0=\frac{1}{2V}\log 1+B, \quad \text{or} \quad B=0.$$

$\therefore$ $$\frac{gt}{V^2}=\frac{1}{2V}\log\frac{V+v}{V-v}$$

or $$t=\frac{V}{g}\cdot\frac{1}{2}\log\frac{1+(v/V)}{1-(v/V)}$$

$$=\frac{V}{g}\tanh^{-1}\frac{v}{V} \qquad \left[\because \tanh^{-1}z=\frac{1}{2}\log\frac{1+z}{1-z}\right]$$

or $$\frac{gt}{V}=\tanh^{-1}\frac{v}{V} \quad \text{or} \quad v=V\,\mathbf{tanh\,(}\boldsymbol{gt/V}\mathbf{)}, \qquad ...(4)$$

which gives the velocity of the particle at any time.

Relation between x and t: Eliminating v between (3) and (4), we have

$$V^2 \tanh^2 (g t/V) = V^2 (1 - e^{-2 g x/V^2})$$

or $$e^{-2 g x/V^2} = 1 - \tanh^2 (gt/V) = \text{sech}^2 (g t/V)$$

or $$e^{2 g x/V^2} = \cosh^2 (gt/V)$$

or $$\frac{2 g x}{V^2} = 2 \log \cosh (gt/V)$$

or $$\boldsymbol{x = \frac{V^2}{g} \log \cosh (gt/V)} \qquad ...(5)$$

which gives the position of the particle at any time.

Remark. To evaluate $\int \frac{dv}{V^2 - v^2}$, we can directly apply the formula

$$\int \frac{dx}{a^2 - x^2} = \frac{1}{a} \tanh^{-1} \frac{x}{a}.$$ **Remember this formula**.

Case II. *A particle is falling from rest under gravity, supposed constant, in a resisting medium whose resistance varies as the velocity; to discuss the motion.*

(Garhwal 2004; Kanpur 11; Avadh 06; Agra 06; Bundelkhand 09)

Solution: Suppose a particle of mass m starts at rest from a point O and falls vertically downwards in a medium whose resistance on the particle is mk times the velocity of the particle. Let P be the position of the particle at any time t, where $OP = x$ and let v be the velocity of the particle at P.

The forces acting on the particle at P are

(i) The force mkv due to the resistance acting vertically upwards *i.e.*, against the direction of motion of the particle, and

(ii) the weight mg of the particle acting vertically downwards.

By Newton's second law of motion the equation of motion of the particle at time t is

$$m \frac{d^2 x}{dt^2} = mg - mkv$$

or $$\frac{d^2 x}{dt^2} = g - kv. \qquad ...(1)$$

If V is the terminal velocity of the particle during its downward motion, then from (1)

$$0 = g - kV \qquad \text{or} \qquad k = g / V.$$

Putting $k = g / V$ in (1), we get

$$\frac{d^2 x}{dt^2} = g - \frac{g}{V} v = \frac{g}{V} (V - v). \qquad ...(2)$$

Relation between v and x:

The equation (2) can be written as

$$v \frac{dv}{dx} = \frac{g}{V} (V - v)$$

or $$dx = \frac{V}{g} \cdot \frac{v}{V - v} \, dv = -\frac{V}{g} \cdot \frac{-v}{V - v} \, dv$$

$$= -\frac{V}{g} \cdot \frac{(V - v) - V}{V - v} \, dv$$

$$= -\frac{V}{g} \left[1 - \frac{V}{V - v} \right] dv.$$

Integrating, $x = -\frac{V}{g} [v + V \log (V - v)] + A,$ where A is a constant.

But initially at $O, x = 0$ and $v = 0$.

$\therefore$ $$A = \frac{V^2}{g} \log V.$$

$\therefore$ $$x = -\frac{V}{g} v - \frac{V^2}{g} \log (V - v) + \frac{V^2}{g} \log V$$

or $$x = -\frac{V}{g} v + \frac{V^2}{g} \log \frac{V}{V - v}, \qquad \text{...(3)}$$

which gives the velocity of the particle at any position.

Relation between v and t: **(Avadh 2009)**

The equation (2) can also be written as

$$\frac{dv}{dt} = \frac{g}{V} (V - v).$$

$\therefore$ $$dt = \frac{V}{g} \cdot \frac{dv}{V - v}.$$

Integrating, we have

$$t = -\frac{V}{g} \log (V - v) + B, \text{ where } B \text{ is a constant.}$$

Initially at $O, t = 0$ and $v = 0$.

$\therefore$ $$B = \frac{V}{g} \log V.$$

$\therefore$ $$t = -\frac{V}{g} \log (V - v) + \frac{V}{g} \log V$$

or $$t = \frac{V}{g} \log \frac{V}{V - v}, \qquad \text{...(4)}$$

which gives the velocity of the particle at any time t.

Relation between x and t:

From (4), we have

$$\log \frac{V}{V - v} = \frac{gt}{V}$$

or $$\frac{V}{V - v} = e^{gt/V}$$

or $\quad V - v = Ve^{-gt/V}$

or $\quad v = V[1 - e^{-gt/V}]$

or $\quad \dfrac{dx}{dt} = V[1 - e^{-gt/V}]$

or $\quad dx = V[1 - e^{-gt/V}]\,dt$.

Integrating, we get

$$x = Vt + \frac{V^2}{g} e^{-gt/V} + C, \text{where } C \text{ is a constant.}$$

Initially at O, $x = 0$ and $t = 0$.

$\therefore \quad C = -\dfrac{V^2}{g}\cdot$

$\therefore \quad x = Vt + \dfrac{V^2}{g} e^{-gt/V} - \dfrac{V^2}{g}$

or $\quad x = Vt + \dfrac{V^2}{g}(e^{-gt/V} - 1),$ ...(5)

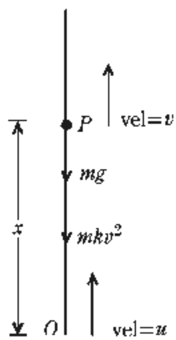
which gives the distance fallen through in time t.

4.4 Motion of a Particle Projected Vertically Upwards

Case I. *A particle is projected vertically upwards under gravity, supposed constant, in a resisting medium whose resistance varies as the square of the velocity; to discuss the motion.*

(Garhwal 2002; Bundelkhand 08; Purvanchal 09, 11)

Let a particle of mass m be projected, vertically upwards from the point O, with velocity u. Let P be the position of the particle at any time t, where $OP = x$ and let v be the velocity of the particle at P. The forces acting on the particle at P are



(i) The force mkv^2 due to resistance acting against the direction of motion *i.e.*, acting vertically downwards.

(ii) The weight mg of the particle also acting vertically downwards.

Both these forces act in the direction of x decreasing. Therefore the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2} = -mg - mkv^2$$

or $$\frac{d^2x}{dt^2} = -g\left(1 + \frac{k}{g}v^2\right).$$

Let V be the terminal velocity of the particle during its downwards motion *i.e.*, the velocity when the resultant acceleration of the particle during its downwards motion is zero. Then

$$0 = mg - mkV^2 \quad \text{or} \quad k = g / V^2.$$

Putting this value of k in the above equation of motion of the particle, we get

$$\frac{d^2 x}{dt^2} = -g\left(1 + \frac{v^2}{V^2}\right)$$

or $$\frac{d^2 x}{dt^2} = \frac{-g}{V^2}(V^2 + v^2). \qquad \text{...(1)}$$

Relation between v and x.

Equation (1) can be written as

$$v\frac{dv}{dx} = \frac{-g}{V^2}(V^2 + v^2) \qquad \left[\because \frac{d^2 x}{dt^2} = v\frac{dv}{dx}\right]$$

or $$\frac{-2g}{V^2}\, dx = \frac{2v\, dv}{V^2 + v^2}, \text{ separating the variables.}$$

Integrating, $\dfrac{-2gx}{V^2} = \log(V^2 + v^2) + A$, where A is a constant.

Initially at O, $x = 0$ and $v = u$.

$\therefore$ $$0 = \log(V^2 + u^2) + A$$

or $$A = -\log(V^2 + u^2).$$

$\therefore$ $$\frac{-2gx}{V^2} = \log(V^2 + v^2) - \log(V^2 + u^2)$$

or $$\boldsymbol{x = \frac{V^2}{2g} \log \frac{V^2 + u^2}{V^2 + v^2}} \qquad \text{...(2)}$$

which gives the velocity of the particle in any position.

If H is the greatest height attained by the particle, then putting $x = H$ and $v = 0$ in (2), we get

$$H = \frac{V^2}{2g} \log \frac{V^2 + u^2}{V^2}. \qquad \textbf{(Kumaun 2001)}$$

Relation between v and t. **(Kumaun 2002; Lucknow 01)**

Equation (1) can be written as

$$\frac{dv}{dt} = -\frac{g}{V^2}(V^2 + v^2) \qquad \left[\because \frac{d^2 x}{dt^2} = \frac{dv}{dt}\right]$$

or $$dt = \frac{-V^2}{g} \cdot \frac{dv}{V^2 + v^2}, \quad \text{separating the variables.}$$

Integrating, $t = \dfrac{-V^2}{g} \cdot \dfrac{1}{V} \tan^{-1} \dfrac{v}{V} + B$, where B is a constant

or $$t = \frac{-V}{g} \tan^{-1} \frac{v}{V} + B.$$

Initially at O, when $t = 0$, $v = u$.

$\therefore \qquad 0 = -\dfrac{V}{g}\tan^{-1}\dfrac{u}{V} + B \quad \text{or} \quad B = \dfrac{V}{g}\tan^{-1}\dfrac{u}{V}.$

$$\therefore \qquad t = \frac{V}{g}\left(\tan^{-1}\frac{u}{V} - \tan^{-1}\frac{v}{V}\right), \qquad \text{...(3)}$$

which gives the velocity of the particle at any time t.

Relation between x and t:

A relation between x and t can be obtained by eliminating v between (2) and (3).

Case II. *A particle is projected vertically upwards under gravity, supposed constant, in a resisting medium whose resistance varies as the velocity; to discuss the motion.*

(Purvanchal 2007; Avadh 08)

Solution: Suppose a particle of mass m is projected vertically upwards from a point O with velocity u in a medium whose resistance on the particle is mk times the velocity of the particle. Let P be the position of the particle at any time t, where $OP = x$ and let v be the velocity of the particle at P. The forces acting on the particle at P are :

(i) The force mkv due to the resistance acting vertically downwards *i.e.*, against the direction of motion of the particle, and

(ii) the weight mg of the particle acting vertically downwards.

Since both these forces act in the direction of x decreasing, therefore the equation of motion of the particle in upwards motion at time t is

$$m\frac{d^2x}{dt^2} = -mg - mkv$$

or
$$\frac{d^2x}{dt^2} = -(g + kv). \qquad \text{...(1)}$$

If the particle moves downwards in the same resisting medium and its velocity is v at time t at distance x from the starting point, then its equation of motion in downwards motion will be

$$m\frac{d^2x}{dt^2} = mg - mkv \quad \text{or} \quad \frac{d^2x}{dt^2} = g - kv.$$

If V is the terminal velocity of the particle during its downward motion, then

$$0 = g - kV \qquad \text{or} \qquad k = g/V.$$

$\therefore$ the equation of motion (1) in upwards motion becomes

$$\frac{d^2x}{dt^2} = -\left(g + \frac{g}{V}v\right) = \frac{-g(V+v)}{V}. \qquad \text{...(2)}$$

Relation between v and x:

The equation (2) can be written as

$$v\frac{dv}{dx} = -\frac{g}{V}(V + v)$$

or
$$dx = -\frac{V}{g}\cdot\frac{v}{v+V}\,dv$$

$$= -\frac{V}{g}\frac{(v+V)-V}{v+V}\,dv$$

$$= -\frac{V}{g}\left[1-\frac{V}{v+V}\right]dv.$$

Integrating, $x = -\dfrac{V}{g}[v - V\log(v+V)] + A$, where A is a constant.

But initially at O, $x = 0$ and $v = u$.

$\therefore$ $$A = \frac{V}{g}[u - V\log(u+V)].$$

$\therefore$ $$x = -\frac{V}{g}[v - V\log(v+V)] + \frac{V}{g}[u - V\log(u+V)]$$

or $$x = \frac{V}{g}\left[(u-v) + V\log\left(\frac{v+V}{u+V}\right)\right], \qquad \text{...(3)}$$

which gives the velocity of the particle at any position.

Relation between v and t:

The equation (2) can also be written as

$$\frac{dv}{dt} = -\frac{g}{V}(v+V).$$

$\therefore$ $$dt = -\frac{V}{g}\frac{dv}{v+V}.$$

Integrating, $t = -\dfrac{V}{g}\log(v+V) + B$, where B is a constant.

But initially at O, $t = 0$ and $v = u$.

$\therefore$ $$B = \frac{V}{g}\log(u+V).$$

$\therefore$ $$t = -\frac{V}{g}\log(v+V) + \frac{V}{g}\log(u+V)$$

or $$t = \frac{V}{g}\log\frac{u+V}{v+V}, \qquad \text{...(4)}$$

which gives the velocity of the particle at any time t.

Relation between x and t:

From (4), we have

$$\log\frac{u+V}{v+V} = \frac{g\,t}{V}$$

or $$\frac{u+V}{v+V} = e^{g\,t/V}$$

or $$v + V = (u+V)\,e^{-g\,t/V}$$

or $$v = \frac{dx}{dt} = -V + (u+V)\,e^{-g\,t/V}$$

or $$dx = [-V + (u + V)\, e^{-g t/V}]\, dt .$$

Integrating, we get

$$x = -Vt - \frac{V}{g}(u + V)\, e^{-g t/V} + C, \text{where } C \text{ is a constant.}$$

Initially at O, $x = 0$ and $t = 0$.

$\therefore$ $$C = \frac{V}{g}(u + V).$$

$\therefore$ $$x = -Vt - \frac{V}{g}(u + V)\, e^{-g t/V} + \frac{V}{g}(u + V)$$

or $$x = -Vt + \frac{V}{g}(u + V)\,[1 - e^{-g t/V}], \qquad \text{...(5)}$$

which gives the distance covered by the particle at any time t.

Remark : The force of resistance is a **non-conservative force.** So if a particle is projected vertically upwards in a resisting medium with velocity u, then its velocity when it returns back to the point of projection is not u. Moreover, the times for the upward motion and for the downward motion are also different.

Illustrative Examples

Example 1: *A particle is projected with velocity V along a smooth horizontal plane in a medium whose resistance per unit mass is* μ *times the cube of the velocity. Show that the distance it has described in time t is* $(1/\mu V)\,[\sqrt{(1 + 2\mu V^2 t)} - 1]$ *and that its velocity then is* $V/\sqrt{(1 + 2\mu V^2 t)}$.

(Purvanchal 2008, 10)

Solution: Take the point of projection O as origin. Let v be the velocity of the particle at time t at a point distant x from the fixed point O. Then the resistance at this point will be $m\mu v^3$, acting in the direction of x decreasing. Here the resistance is the only force acting on the particle during its motion.

$\therefore$ the equation of motion of the particle is

$$m\frac{dv}{dt} = -m\mu v^3$$

or $$\frac{dv}{v^3} = -\mu\, dt.$$

Integrating, $-\dfrac{1}{2v^2} = -\mu t + A$, where A is a constant.

But initially, when $t = 0, v = V$; and so $A = -\dfrac{1}{2V^2}$.

$\therefore$ $$-\frac{1}{2v^2} = -\mu t - \frac{1}{2V^2}$$

or $$1/v^2 = (2\mu V^2 t + 1)/V^2 \quad \text{or} \quad v = V/\sqrt{(1 + 2\mu V^2 t)}, \qquad \text{...(1)}$$

which gives the velocity of the particle at time t.

Since the particle is moving in the direction of x increasing, therefore from the equation (1), we have

$$\frac{dx}{dt} = v = V/\sqrt{(1+2\mu V^2 t)}$$

or $$dx = V(1+2\mu V^2 t)^{-1/2}\, dt.$$

Integrating, $x = \dfrac{1}{\mu V}(1+2\mu V^2 t)^{1/2} + B$, where B is a constant.

But initially when $t = 0, x = 0$; and so $B = -\dfrac{1}{\mu V}$.

$\therefore$ $$x = \frac{1}{\mu V}(1+2\mu V^2 t)^{1/2} - \frac{1}{\mu V}$$

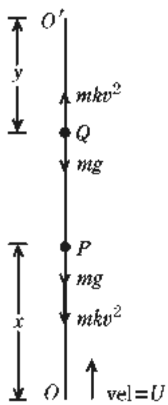
or $$x = \frac{1}{\mu V}[\sqrt{(1+2\mu V^2 t)} - 1],$$

which gives the distance described in time t.

Example 2: *A particle projected upwards with a velocity U, in a medium whose resistance varies as the square of the velocity, will return to the point of projection with velocity $v_1 = \dfrac{UV}{\sqrt{(U^2+V^2)}}$ after a time $\dfrac{V}{g}\left(\tan^{-1}\dfrac{U}{V} + \tanh^{-1}\dfrac{v_1}{V}\right)$, where V is the terminal velocity.*

(Garhwal 2003; Rohilkhand 11)

Solution: **Upward Motion:** Let a particle of mass m be projected vertically upwards from the point O with velocity U. If v is the velocity of the particle at time t at the point P such that $OP = x$, then the resistance at P is mkv^2 acting vertically downwards. Since the weight mg of the particle also acts vertically downwards, therefore the equation of motion of the particle is



$$m\frac{d^2x}{dt^2} = -mg - mkv^2$$

or $$\frac{d^2x}{dt^2} = -(g+kv^2) \qquad \text{...(1)}$$

or $$v\frac{dv}{dx} = -(g+kv^2)$$

or $$\frac{2kv\,dv}{g+kv^2} = -2k\,dx.$$

Integrating, $\log(g+kv^2) = -2kx + A$, where A is a constant.

But initially at O, $x = 0, v = U$;

$\therefore$ $$A = \log(g+kU^2).$$

$\therefore$ $$\log(g+kv^2) = -2kx + \log(g+kU^2)$$

or $$x = \frac{1}{2k} \log \left(\frac{g + kU^2}{g + kv^2} \right).$$

If $OO' = h$ is the maximum height attained by the particle, then at O', $x = h$ and $v = 0$.

$\therefore$ $$h = \frac{1}{2k} \log \left(\frac{g + kU^2}{g} \right). \qquad ...(2)$$

Now to find the time from O to O', we write the equation (1) as

$$\frac{dv}{dt} = -(g + kv^2) = -k \left(\frac{g}{k} + v^2 \right)$$

or $$dt = -\frac{1}{k} \frac{dv}{(g/k) + v^2}.$$

Integrating, the time t_1 from O to O' is given by

$$\int_{t=0}^{t_1} dt = -\frac{1}{k} \int_{v=U}^{0} \frac{dv}{(g/k) + v^2}$$

or $$t_1 = -\frac{1}{k} \cdot \frac{1}{\sqrt{(g/k)}} \left[\tan^{-1} \frac{v}{\sqrt{(g/k)}} \right]_U^0$$

or $$t_1 = \frac{1}{\sqrt{(kg)}} \tan^{-1} \{U \sqrt{(k/g)}\}. \qquad ...(3)$$

(Avadh 2008)

Downward Motion: Now the particle comes to instantaneous rest at the highest point O' and then falls downwards. If v is its velocity at the point Q at the time t (measured from the instant the particle starts falling downwards from O') such that $O'Q = y$, then resistance at Q is mkv^2 acting vertically upwards. Since the weight mg of the particle acts vertically downwards, therefore the equation of motion during this downward motion is

$$m \frac{d^2 y}{dt^2} = mg - mkv^2 \quad \text{or} \quad \frac{d^2 y}{dt^2} = g - kv^2.$$

If V is the terminal velocity of the particle during its downward motion, then $d^2 y / dt^2 = 0$, when $v = V$.

$\therefore$ $$0 = g - kV^2 \quad \text{or} \quad k / g = 1 / V^2 \qquad ...(4)$$

$\therefore$ $$\frac{d^2 y}{dt^2} = g \left(1 - \frac{k}{g} v^2 \right) = g \left(1 - \frac{v^2}{V^2} \right)$$

or $$\frac{d^2 y}{dt^2} = \frac{g}{V^2} (V^2 - v^2) \qquad ...(5)$$

or $$v \frac{dv}{dy} = \frac{g}{V^2} (V^2 - v^2)$$

or $$\frac{-2v\, dv}{V^2 - v^2} = \frac{-2g}{V^2} dy.$$

Integrating, $\log(V^2 - v^2) = \dfrac{-2g}{V^2}\, y + B$, where B is a constant.

But at O', $y = 0$ and $v = 0$; and so $B = \log V^2$.

$$\therefore \qquad \log(V^2 - v^2) = \frac{-2g}{V^2}\, y + \log V^2$$

or
$$y = \frac{V^2}{2g} \log\left(\frac{V^2}{V^2 - v^2}\right).$$

If the particle returns to the point O with velocity v_1, then at $O, v = v_1$ and $y = h$.

$$\therefore \qquad h = \frac{V^2}{2g} \log\left(\frac{V^2}{V^2 - v_1^2}\right). \qquad \text{...(6)}$$

Substituting $\dfrac{k}{g} = \dfrac{1}{V^2}$ in (2), we have

$$h = \frac{V^2}{2g} \log\left(1 + \frac{U^2}{V^2}\right). \qquad \text{...(7)}$$

From (6) and (7), we have

$$\frac{V^2}{2g} \log\left(\frac{V^2 + U^2}{V^2}\right) = \frac{V^2}{2g} \log\left(\frac{V^2}{V^2 - v_1^2}\right)$$

or
$$\frac{V^2 + U^2}{V^2} = \frac{V^2}{V^2 - v_1^2}$$

or
$$V^2 (V^2 + U^2) - (V^2 + U^2)\, v_1^2 = V^4$$

or
$$(V^2 + U^2)\, v_1^2 = U^2 V^2.$$

$$\therefore \qquad v_1 = UV / \sqrt{(U^2 + V^2)},$$

which proves the first part of the question.

To find the time from O' to O, we write the equation (5) as

$$\frac{dv}{dt} = \frac{g}{V^2}(V^2 - v^2)$$

or
$$dt = \frac{V^2}{g} \frac{dv}{(V^2 - v^2)}.$$

Integrating, the time t_2 from O' to O is given by

$$\int_{t=0}^{t_2} dt = \frac{V^2}{g} \int_{v=0}^{v_1} \frac{dv}{V^2 - v^2}$$

or
$$t_2 = \frac{V^2}{g} \cdot \frac{1}{V} \left[\tanh^{-1} \frac{v}{V}\right]_0^{v_1} = \frac{V}{g} \tanh^{-1} \frac{v_1}{V}.$$

$\therefore$ the particle returns to the point of projection O in time

$$= t_1 + t_2 = \frac{1}{\sqrt{(kg)}} \tan^{-1} \{U \sqrt{(k/g)}\} + \frac{V}{g} \tanh^{-1} \frac{v_1}{V}.$$

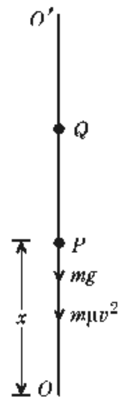
Substituting $k/g = 1/V^2$ from (4), we get

$$t_1 + t_2 = \frac{V}{g} \tan^{-1} \frac{U}{V} + \frac{V}{g} \tanh^{-1} \frac{v_1}{V}$$

$$= \frac{V}{g} \left(\tan^{-1} \frac{U}{V} + \tanh^{-1} \frac{v_1}{V} \right).$$

Example 3: *A heavy particle is projected vertically upwards in a medium the resistance of which varies as the square of velocity. If it has a kinetic energy K in its upwards path at a given point, when it passes the same point on the way down, show that its loss of energy is* $\frac{K^2}{K + K'}$, *where K′ is the limit to which the energy approaches in its downwards course.*

Solution: Let a particle of mass m be projected vertically upwards with a velocity u from the point O. If v is the velocity of the particle at time t at the point P such that $OP = x$, then the resistance at P is $m\mu v^2$ acting vertically downwards. The weight mg of the particle also acts vertically downwards.



∴ The equation of motion of the particle during its upwards motion is

$$m \frac{d^2 x}{dt^2} = -mg - m\mu v^2$$

or $$\frac{d^2 x}{dt^2} = -g \left(1 + \frac{\mu}{g} v^2 \right) \qquad ...(1)$$

If H is the maximum height attained by the particle, then at the highest point O' the particle comes to rest and starts falling vertically downwards. If y is the distance fallen in time t from O' and v is the velocity of the particle at this point, then the resistance is $m\mu v^2$ acting vertically upwards.

∴ the equation of motion of the particle during its downward motion is

$$m \frac{d^2 y}{dt^2} = mg - m\mu v^2$$

or $$\frac{d^2 y}{dt^2} = g - \mu v^2. \qquad ...(2)$$

If V is the terminal velocity of the particle during its downward motion, then $d^2 y/dt^2 = 0$ when $v = V$. Therefore $0 = g - \mu V^2$

or $$\frac{\mu}{g} = \frac{1}{V^2}. \qquad ...(3)$$

∴ From (2), the equation of motion of the particle in downward motion is

$$\frac{d^2 y}{dt^2} = g \left(1 - \frac{1}{V^2} v^2 \right)$$

or $$v \frac{dv}{dy} = \frac{g}{V^2} (V^2 - v^2)$$

or $$\frac{-2v\,dv}{V^2 - v^2} = -\frac{2g}{V^2}\,dy.$$

Integrating, $\log(V^2 - v^2) = -\dfrac{2g}{V^2}\,y + A$, where A is a constant.

But at O', $y = 0$ and $v = 0$; and so $A = \log V^2$.

$\therefore$ $$\log(V^2 - v^2) = -\frac{2g}{V^2}\,y + \log V^2$$

or $$\frac{2gy}{V^2} = \log V^2 - \log(V^2 - v^2)$$

or $$y = \frac{V^2}{2g}\log\left(\frac{V^2}{V^2 - v^2}\right). \qquad \text{...(4)}$$

If v_1 is the velocity of the particle at the point Q at distance h from O', when falling downwards, then from (4),

$$h = \frac{V^2}{2g}\log\left(\frac{V^2}{V^2 - v_1^2}\right). \qquad \text{...(5)}$$

Upward Motion: When the particle is moving upwards from O, then from (1) with the help of (3), the equation of motion of the particle is

$$\frac{d^2x}{dt^2} = -g\left(1 + \frac{v^2}{V^2}\right)$$

$$v\frac{dv}{dx} = -\frac{g}{V^2}(V^2 + v^2)$$

or $$\frac{2v\,dv}{V^2 + v^2} = -\frac{2g}{V^2}\,dx.$$

Integrating, $\log(V^2 + v^2) = -\dfrac{2g}{V^2}\,x + B$, where B is a constant.

But at O, $x = 0$, $v = u$;

and so $$B = \log(V^2 + u^2).$$

$\therefore$ $$\log(V^2 + v^2) = -\frac{2g}{V^2}\,x + \log(V^2 + u^2)$$

or $$x = \frac{V^2}{2g}\log\left(\frac{V^2 + u^2}{V^2 + v^2}\right). \qquad \text{...(6)}$$

If v_2 is the velocity of the particle at the point Q in its upward motion, then at Q, $x = OQ = H - h$, $v = v_2$.

$\therefore$ $$H - h = \left(\frac{V^2}{2g}\right)\log\left(\frac{V^2 + u^2}{V^2 + v_2^2}\right). \qquad \text{...(7)}$$

Since H is the maximum height attained by the particle therefore putting $x = H$ and $v = 0$ in (6), we get

$$H = \frac{V^2}{2g} \log\left(\frac{V^2 + u^2}{V^2}\right). \qquad \text{...(8)}$$

Substituting the values of h and H from (5) and (8) in (7), we get

$$\frac{V^2}{2g} \log \frac{V^2 + u^2}{V^2} - \frac{V^2}{2g} \log \frac{V^2}{V^2 - v_1^2} = \frac{V^2}{2g} \log \frac{V^2 + u^2}{V^2 + v_2^2}$$

or $$\log \frac{(V^2 + u^2)}{V^2} - \log\left(\frac{V^2 + u^2}{V^2 + v_2^2}\right) = \log\left(\frac{V^2}{V^2 - v_1^2}\right)$$

or $$\log\left\{\left(\frac{V^2 + u^2}{V^2}\right) \cdot \left(\frac{V^2 + v_2^2}{V^2 + u^2}\right)\right\} = \log \frac{V^2}{V^2 - v_1^2}$$

or $$\frac{V^2 + v_2^2}{V^2} = \frac{V^2}{V^2 - v_1^2} \quad \text{or} \quad (V^2 + v_2^2)(V^2 - v_1^2) = V^4$$

or $$(V^2 + v_2^2)\, V^2 - (V^2 + v_2^2)\, v_1^2 = V^4$$

or $$v_1^2 = \frac{v_2^2 V^2}{V^2 + v_2^2}. \qquad \text{...(9)}$$

Now the kinetic energy K of the particle at the point Q at depth h below O' during its upward motion $= \frac{1}{2} m v_2^2$ and the K.E. at Q during downward motion $= \frac{1}{2} m v_1^2$.

Also the terminal K.E. $= \frac{1}{2} m V^2$.

The required loss of K.E. $= \frac{1}{2} m v_2^2 - \frac{1}{2} m v_1^2$

$$= \frac{1}{2} m \left[v_2^2 - \frac{v_2^2 V^2}{V^2 + v_2^2}\right], \text{ substituting for } v_1^2 \text{ from (9)}$$

$$= \frac{m}{2} \cdot \frac{v_2^4}{V^2 + v_2^2} = \frac{\left(\frac{1}{2} m v_2^2\right)^2}{\frac{1}{2} m V^2 + \frac{1}{2} m v_2^2} = \frac{K^2}{K' + K},$$

where $K' = \frac{1}{2} m V^2$ = limiting K.E. in the medium.

Example 4: *A particle of mass m falls from rest at a distance a from the centre of the earth, the motion meeting with a small resistance proportional to the square of the velocity v and the retardation being μ for unit velocity; show that the kinetic energy at a distance x from the centre is*

$$mgr^2 \left[\frac{1}{x} - \frac{1}{a} + 2\mu\left(1 - \frac{x}{a}\right) - 2\mu \log\left(\frac{a}{x}\right)\right],$$

the square of μ being neglected and r is the radius of the earth.

Solution: Let a particle of mass m fall from rest at a distance a from the centre O of the earth. If v is the velocity of the particle at time t at the point P whose distance from the centre of the earth is x *i.e.*, $OP = x$, then the two accelerations *i.e.*, the forces acting on the unit mass of the particle at P are :

(i) The attraction of the earth towards its centre $= \lambda / x^2$. But on the surface of the earth, the attraction (acceleration) is g and $x = r =$ the radius of the earth.

$\therefore$ $$\lambda / r^2 = g \quad \text{or} \quad \lambda = r^2 g.$$

$\therefore$ the attraction of the earth towards the centre (*i.e.*, in the direction of x decreasing) is $r^2 g / x^2$.

(ii) The resistance of the medium on the particle $= kv^2$, acting against the direction of motion. But for $v = 1$, the retardation due to the resistance is μ.

$\therefore$ $$\mu = k \,.\, 1^2 \quad \text{or} \quad k = \mu.$$

$\therefore$ the retardation on the particle due to the resistance of the medium is μv^2 acting in the direction of x increasing.

$\therefore$ the equation of motion of the particle is

$$\frac{d^2 x}{dt^2} = -\frac{r^2 g}{x^2} + \mu v^2$$

or $$v \frac{dv}{dx} = -\frac{r^2 g}{x^2} + \mu v^2$$

or $$\frac{1}{2} \frac{d(v^2)}{dx} = -\frac{r^2 g}{x^2} + \mu v^2$$

or $$\frac{d(v^2)}{dx} - (2\mu) v^2 = -\frac{2 r^2 g}{x^2}, \qquad \text{...(1)}$$

which is a linear differential equation in v^2.

$$\text{I.F.} = e^{\int -2\mu \, dx} = e^{-2\mu x}.$$

$\therefore$ the solution of (1) is

$$v^2 e^{-2\mu x} = C - \int \frac{2 r^2 g}{x^2} e^{-2\mu x} \, dx, \text{ where } C \text{ is a constant}$$

or $$v^2 (1 - 2\mu x) = C - 2 r^2 g \,.\int \frac{1}{x^2} (1 - 2\mu x) \, dx,$$

[Expanding $e^{-2\mu x}$ and neglecting the squares and higher powers of μ]

or $$v^2 (1 - 2\mu x) = C - 2 r^2 g \,.\int \left(\frac{1}{x^2} - \frac{2\mu}{x} \right) dx$$

or $$v^2 (1 - 2\mu x) = C + 2 r^2 g \left(\frac{1}{x} + 2\mu \log x \right). \qquad \text{...(2)}$$

But initially at $x = a$, $v = 0$.

$\therefore \qquad 0 = C + 2r^2 g\left(\frac{1}{a} + 2\mu \log a\right).$ ...(3)

Subtracting (3) from (2), we have

$$v^2 (1 - 2\mu x) = 2r^2 g\left(\frac{1}{x} - \frac{1}{a} + 2\mu \log x - 2\mu \log a\right)$$

or
$$v^2 = 2r^2 g\left[\frac{1}{x} - \frac{1}{a} - 2\mu \log\left(\frac{a}{x}\right)\right] . (1 - 2\mu x)^{-1}$$

$$= 2r^2 g\left[\frac{1}{x} - \frac{1}{a} - 2\mu \log\left(\frac{a}{x}\right)\right] . (1 + 2\mu x),$$

[Expanding by binomial theorem and neglecting the squares and higher powers of μ]

$$= 2r^2 g\left[\frac{1}{x} - \frac{1}{a} + 2\mu x\left(\frac{1}{x} - \frac{1}{a}\right) - 2\mu \log\left(\frac{a}{x}\right)\right],$$ [Neglecting μ^2]

$$= 2r^2 g\left[\frac{1}{x} - \frac{1}{a} + 2\mu\left(1 - \frac{x}{a}\right) - 2\mu \log\left(\frac{a}{x}\right)\right].$$

$\therefore$ the kinetic energy of the particle at a distance x from the centre

$$= \frac{1}{2} mv^2$$

$$= m g r^2 \left[\frac{1}{x} - \frac{1}{a} + 2\mu\left(1 - \frac{x}{a}\right) - 2\mu \log\left(\frac{a}{x}\right)\right].$$

Comprehensive Exercise 1

1. A particle is projected with velocity u along a smooth horizontal plane in a medium whose resistance per unit mass is k (velocity), show that the velocity after a time t and the distance s in that time are given by

$$v = ue^{-kt} \quad \text{and} \quad s = u(1 - e^{-kt})/k.$$

2. A particle falls from rest under gravity through a distance x in a medium whose resistance varies as the square of the velocity. If v be the velocity actually acquired by it, v_0 the velocity it would have acquired, had there been no resisting medium and V the terminal velocity, show that

$$\frac{v^2}{{v_0}^2} = 1 - \frac{1}{2}\frac{{v_0}^2}{V^2} + \frac{1}{2 \cdot 3}\frac{{v_0}^4}{V^4} - \frac{1}{2 \cdot 3 \cdot 4}\frac{{v_0}^6}{V^6} + \dots$$

3. A particle of mass m is projected vertically under gravity, the resistance of the air being mk times the velocity. Show that the greatest height attained by the particle is $\frac{V^2}{g}[\lambda - \log(1 + \lambda)]$, where V is the terminal velocity of the particle and λV is the initial velocity. **(Meerut 2004; Lucknow 06, 08, 09; Avadh 06, 07)**

4. A particle of mass m is projected vertically under gravity, the resistance of the air being mk times the velocity. Find the greatest height attained by the particle.

5. A particle of mass m, is falling under the influence of gravity through a medium whose resistance equals μ times the velocity. If the particle were released from rest, show that the distance fallen through in time t is

$$\frac{g\,m^2}{\mu^2}\left[e^{-\mu\,t/m}-1+\frac{\mu t}{m}\right].$$

(Bundelkhand 08; Purvanchal 08)

6. A particle is projected vertically upwards with velocity u, in a medium where resistance is kv^2 per unit mass for velocity v of the particle. Show that the greatest height attained by the particle is

$$\frac{1}{2k}\log\{(g+ku^2)/g\}.$$ **(Kumaun 2001)**

7. A particle is projected vertically upwards with a velocity V and the resistance of the air produces a retardation kv^2, where v is the velocity. Show that the velocity V' with which the particle will return to the point of projection is given by

$$\frac{1}{V'^2}=\frac{1}{V^2}+\frac{k}{g}.$$

(Avadh 2009, 11; Bundelkhand 11)

8. A small stone of mass m is thrown vertically upwards with initial velocity V_0. If the air resistance at speed V is mkV^2, where k is a positive constant show that the stone returns to its starting point with speed $V_0/(1+kV_0^2/g)^{1/2}$.

9. A particle falls from rest in a medium in which the resistance is kv^2 per unit mass. Prove that the distance fallen in time t is $(1/k)\log\cosh\{t\sqrt{(gk)}\}$. **(Kumaun 2003)**

If the particle were ascending, show that at any instant its distance from the highest point of its path is $(1/k)\log\sec\{t\sqrt{(gk)}\}$, where t now denotes the time it will take to reach its highest point.

10. A particle of unit mass is projected vertically upwards with a velocity V in a medium for which the resistance is kv when the speed of the particle is v. Prove that the particle returns to the point of projection with speed V_1 such that

$$V+V_1=\frac{g}{k}\log\left(\frac{g+kV}{g-kV_1}\right).$$

11. A particle of unit mass is projected vertically upwards with velocity v_0 in a medium for which the resistance is kv when the speed of the particle is v, show that the distance covered when the velocity is v is given by

$$x=\frac{v_0-v}{k}+\frac{g}{k^2}\log\left(\frac{kv+g}{kv_0+g}\right).$$

12. A particle is projected upwards with velocity u in a medium, the resistance of which is $g\,u^{-2}\tan^2\alpha$ time the square of the velocity; α being a constant. Show that the particle will return to the point of projection with velocity $u\cos\alpha$ after

a time $$ug^{-1}\cot\alpha\left[\alpha+\log\frac{\cos\alpha}{1-\sin\alpha}\right].$$

13. If the resistance vary as the 4th power of the velocity, the energy of m lbs. at a depth x below the highest point when moving in a vertical line under gravity will be $E \tan (mgx/E)$ when rising, and $E \tanh (mgx/E)$ when falling, where E is the terminal energy in the medium.

14. A particle moving in a straight line is subjected to a resistance kv^3, where v is the velocity. Show that if v is the velocity at time t when the distance is $s, v = u/(1 + kus)$ and $t = (s/u) + \frac{1}{2} ks^2$, where u is the initial velocity.

(Kumaun 2001; Kanpur 07)

15. A heavy particle is projected vertically upwards with velocity U in a medium, the resistance of which varies as the cube of the particle's velocity. Determine the height to which the particle will ascend.

16. A particle moves from rest at a distance a from a fixed point O under the action of a force to O equal to μ times the distance per unit of mass. If the resistance of the medium in which it moves be k times the square of the velocity per unit mass, then show that the square of its velocity when it is at a distance x from O, is

$$\frac{\mu x}{k} - \frac{\mu a}{k} e^{2k(x-a)} + \frac{\mu}{2k^2} [1 - e^{2k(x-a)}].$$

Show also that when it first comes to rest it will be at a distance b given by

$$(1 - 2bk) e^{2bk} = (1 + 2ak) e^{-2ak}.$$

17. What do you understand by 'terminal velocity' ? Give reasons that the terminal velocity obtained from vertically downward motion is also used for the motion vertically upwards. Why is it so ?

Answers 1

4. $\frac{u}{k} - \frac{g}{k^2} \log \left(1 + \frac{ku}{g}\right)$, where u is the velocity of projection

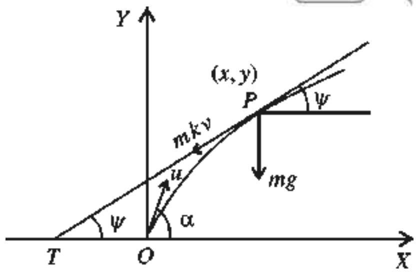
15. $\frac{V^2}{3g} \left[\log \frac{(U^2 - UV + V^2)^{1/2}}{(U + V)} + \sqrt{3} \left(\tan^{-1} \frac{2U - V}{\sqrt{3}V} + \frac{\pi}{6}\right)\right]$

4.5 Motion of a Projectile in a Resisting Medium

A particle is projected under gravity and a resistance equal to mk (velocity), with a velocity u at an angle α to the horizon; to discuss the motion.

(Garhwal 2001; Kanpur 09; Bundelkhand 07)

Let the particle be projected from O (origin) with velocity u at an angle α to the horizon. Let $P(x, y)$ be its position after any time t. The forces acting on the particle at P are :



(i) weight mg of the particle acting vertically downwards, and

(ii) the resisting force mkv along the tangent in the direction PT.

We know that accelerations along x and y axes are $(d^2x/dt^2)\,(=\ddot{x})$ and $(d^2y/dt^2)\,(=\ddot{y})$ respectively. Hence equations of motion along co-ordinate axes are

$$m\ddot{x} = -mkv\cos\psi$$

or $$\ddot{x} = -k\frac{ds}{dt}\cdot\frac{dx}{ds} \qquad \left[\because \cos\psi = \frac{dx}{ds}\right]$$

or $$\ddot{x} = -k\frac{dx}{dt} \quad \text{or} \quad \frac{\ddot{x}}{\dot{x}} = -k. \qquad ...(1)$$

And $$m\ddot{y} = -mkv\sin\psi - mg$$

or $$\ddot{y} = -k\frac{ds}{dt}\cdot\frac{dy}{ds} - g$$

or $$\ddot{y} = -k\frac{dy}{dt} - g. \qquad ...(2)$$

Integrating (1), $\log\dot{x} = -kt + A_1$.

Initially when $t = 0$, $\dot{x} =$ initial horizontal component of velocity $= u\cos\alpha$, therefore $A_1 = \log u\cos\alpha$.

∴ $$\log\dot{x} = -kt + \log u\cos\alpha$$

or $$\log(\dot{x}/u\cos\alpha) = -kt$$

$$\dot{x} = u\cos\alpha\, e^{-kt} \qquad ...(3)$$

Equation (2) may be written as

$$k\ddot{y}/(k\dot{y} + g) = -k.$$

Integrating, $\log(k\dot{y} + g) = -kt + A_2$.

Initially when $t = 0$, $\dot{y} = u\sin\alpha$, therefore

$$A_2 = \log(ku\sin\alpha + g).$$

∴ $$\log\left(\frac{k\dot{y} + g}{ku\sin\alpha + g}\right) = -kt$$

or $$k\dot{y} + g = (ku\sin\alpha + g)\,e^{-kt}. \qquad ...(4)$$

Equations (3) and (4) give the horizontal and vertical components of velocity of the particle at any time *t*.

Equation (3) may be written as

$$dx = u\cos\alpha\, e^{-kt}\,dt \qquad [\because \dot{x} = dx/dt]$$

Integrating, $x = -\dfrac{u}{k}\cos\alpha\, e^{-kt} + A_3$.

Initially at $O, x = 0, t = 0$, therefore $A_3 = \frac{u}{k} \cos \alpha$.

$$\therefore \qquad x = \frac{u}{k} \cos \alpha \, (1 - e^{-k\,t}). \qquad \text{...(5)}$$

Equation (4) may be written as

$$k\,dy + g\,dt = (k\,u \sin \alpha + g)\, e^{-k\,t}\, dt.$$

Integrating, $ky + g\,t = -\frac{1}{k}(k\,u \sin \alpha + g)\, e^{-k\,t} + A_4$.

Initially at $O, y = 0, t = 0$, therefore

$$A_4 = \frac{1}{k}(ku \sin \alpha + g).$$

$$\therefore \qquad ky + gt = \frac{g + k\,u \sin \alpha}{k}(1 - e^{-k\,t}). \qquad \text{...(6)}$$

Thus horizontal and vertical distances described by the particle are given by equations (5) and (6) and these equations are called the **parametric equations of the trajectory.** To find cartesian equation we shall eliminate t between these equations.

From (5), $\quad 1 - e^{-kt} = \frac{k\,x}{u \cos \alpha}$.

$$\therefore \qquad e^{-k\,t} = 1 - \frac{kx}{u \cos \alpha}$$

or

$$t = -\frac{1}{k} \log\left(1 - \frac{kx}{u \cos \alpha}\right).$$

Substituting these values of t and $(1 - e^{-k\,t})$ in (6), we have

$$ky - \frac{g}{k} \log\left(1 - \frac{k\,x}{u \cos \alpha}\right) = \frac{g + k\,u \sin \alpha}{k} \cdot \frac{kx}{u \cos \alpha}$$

or

$$y = \frac{g}{k^2} \log\left(1 - \frac{k\,x}{u \cos \alpha}\right) + \frac{x}{k\,u \cos \alpha} \cdot (g + k\,u \sin \alpha). \qquad \text{...(7)}$$

Equation (7) is the required equation of path (trajectory) of the particle.

Deductions :

(A) Range on the Horizontal Plane Through the Point of Projection:

Let the required range be R. The co-ordinates of the point where the particle strikes the horizontal plane, are $(R, 0)$. This point $(R, 0)$ must satisfy the equation (7) of trajectory. Thus we have

$$0 = \frac{g}{k^2} \log\left(1 - \frac{kR}{u \cos \alpha}\right) + \frac{R}{ku \cos \alpha}(g + ku \sin \alpha) \qquad \text{...(8)}$$

If k is small, to find approximate value of R.

Writing expansion of log in (8), we have

$$\frac{g}{k^2}\left[-\frac{kR}{u\cos\alpha}-\frac{k^2R^2}{2u^2\cos^2\alpha}-\frac{k^3R^3}{3u^3\cos^3\alpha}-\ldots\right]$$
$$+\frac{R}{ku\cos\alpha}(g+ku\sin\alpha)=0\,.$$

Multiplying by $(ku\cos\alpha)/R$ throughout, we have

$$-g-\frac{kRg}{2u\cos\alpha}-\frac{k^2R^2g}{3u^2\cos^2\alpha}+g+ku\sin\alpha=0$$

[Neglecting k^3 ,... etc.]

or $$-\frac{kRg}{2u\cos\alpha}-\frac{k^2R^2g}{3u^2\cos^2\alpha}+ku\sin\alpha=0$$

or $$R=\frac{2u^2\sin\alpha\cos\alpha}{g}-\frac{2kR^2}{3u\cos\alpha} \qquad \ldots(9)$$

$\therefore$ $$R=\frac{2u^2\sin\alpha\cos\alpha}{g} \text{ upto first approximation.}$$

Substituting this value of R in the right hand side of (9), we get value of R upto second approximation as

$$R=\frac{2u^2\sin\alpha\cos\alpha}{g}-\frac{2k}{3u\cos\alpha}\cdot\left\{\frac{2u^2\sin\alpha\cos\alpha}{g}\right\}^2$$
$$=\frac{2u^2\sin\alpha\cos\alpha}{g}-\frac{8ku^3\cos\alpha\sin^2\alpha}{3g^2}\,. \qquad \ldots(10)$$

(B) To Find the Time of Flight:

Let T be the time of flight *i.e.* the time required to strike the horizontal plane through the point of projection. Thus during this time T the particle describes a zero vertical distance, hence putting $y=0, t=T$ in the equation (6), we have

$$gT=\frac{g+ku\sin\alpha}{k}(1-e^{-kT})$$

or $$gT=\frac{1}{k}(g+ku\sin\alpha)\left[1-\left\{1-kT+\frac{k^2T^2}{2\,!}-\frac{k^3T^3}{3\,!}+\ldots\right\}\right]$$

or $$gT=(g+ku\sin\alpha)\left[T-\frac{kT^2}{2}+\frac{k^2T^3}{6}-\ldots\right]$$

or $$gT=gT+k\,Tu\sin\alpha-\frac{1}{2}gkT^2-\frac{1}{2}k^2T^2u\sin\alpha+\frac{1}{6}gk^2T^3+\ldots$$

or $$0=\frac{1}{2}kT\,[2u\sin\alpha-gT-kTu\sin\alpha+\frac{1}{3}gkT^2+\ldots].$$

Cancelling $\frac{1}{2}kT$ and transposing, we get

$$T = \frac{2u \sin \alpha}{g} + \frac{k}{g}\left(\frac{gT^2}{3} - Tu \sin \alpha\right). \qquad ...(11)$$

[Neglecting k^2, k^3, etc.]

$\therefore$ $$T = \frac{2u \sin \alpha}{g}, \text{ upto first approximation.}$$

Substituting this value of T in the right hand side of the equation (11), we have the value of T upto second approximation as

$$T = \frac{2u \sin \alpha}{g} + \frac{k}{g}\left[\frac{g}{3} \cdot \frac{4u^2 \sin^2 \alpha}{g^2} - \frac{2u \sin \alpha}{g} \cdot u \sin \alpha\right]$$

$$= \frac{2u \sin \alpha}{g} - \frac{2}{3} \cdot \frac{ku^2 \sin^2 \alpha}{g^2}. \qquad ...(12)$$

(C) To Find the Greatest Height and the Time to Reach this Height :

(Bundelkhand 2009)

Let h be the greatest height and t_1 the time for the same. At the highest point, the vertical component of velocity (*i.e.*, $\dot{y}$) is zero. Hence putting $\dot{y} = 0$ and $t = t_1$ in the equation (4), we get

$$g = (ku \sin \alpha + g)\, e^{-k t_1}$$

or $$e^{k t_1} = \frac{1}{g}(ku \sin \alpha + g).$$

$\therefore$ $$t_1 = \frac{1}{k} \log\left(1 + \frac{k}{g} u \sin \alpha\right) \qquad ...(13)$$

Now for the greatest height, put $y = h, t = t_1$ in (6), we have

$$kh + gt_1 = \frac{g + ku \sin \alpha}{k}(1 - e^{-k t_1})$$

or $$kh + \frac{g}{k} \log\left(1 + \frac{k}{g} u \sin \alpha\right) = \frac{g + ku \sin \alpha}{k} - \frac{1}{k} g$$

or $$h = \frac{1}{k} u \sin \alpha - \frac{g}{k^2} \log\left(1 + \frac{k}{g} u \sin \alpha\right). \qquad ...(14)$$

(D) To Prove that the Time to the Greatest Height is Less than Half the Time of Flight:

Here we are to prove that $t_1 < \frac{1}{2} T$ *i.e.*, $2t_1 - T$ is negative.

Now $$2t_1 - T = \frac{2}{k} \log\left(1 + \frac{k}{g} u \sin \alpha\right) - \left(\frac{2u \sin \alpha}{g} - \frac{2}{3} \frac{ku^2 \sin^2 \alpha}{g^2}\right)$$

[Using (12) and (13)]

$$= \frac{2}{k}\left[\frac{k}{g} u \sin\alpha - \frac{1}{2}\frac{k^2}{g^2} u^2 \sin^2\alpha\right] - \left(\frac{2u\sin\alpha}{g} - \frac{2}{3}\cdot\frac{ku^2\sin^2\alpha}{g^2}\right)$$

[Expanding log and neglecting higher powers of k]

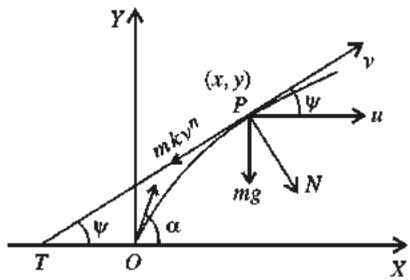
$$= -\frac{ku^2\sin^2\alpha}{g^2} + \frac{2ku^2\sin^2\alpha}{3g^2}$$

$$= -\frac{k\,u^2\sin^2\alpha}{3g^2} = \text{negative}.$$

Hence proved.

4.6 Trajectory in a Resisting Medium when Resistance varies as (Velocity)^n

If a particle describes a trajectory under gravity in a resisting medium whose resistance is equal to mk (velocity)^n, to find the motion.



Let the particle be projected from O. Let P be its position at any time t. Suppose at P, v is its velocity and u is the horizontal component of velocity so that

$$u = v\cos\psi. \qquad \text{...(1)}$$

Equations of motion along the normal and parallel to x-axis are

$$\frac{mv^2}{\rho} = mg\cos\psi$$

or $$\frac{v^2}{\rho} = g\cos\psi \qquad \text{...(2)}$$

and $$m\frac{du}{dt} = -mkv^n\cos\psi$$

or $$\frac{du}{dt} = -kv^n\cos\psi. \qquad \text{...(3)}$$

[Remember]

We shall find the intrinsic equation of the path from (2) and (3).

We have $\rho = -ds/d\psi$ [$\because$ ψ decreases as s increases]

Using this value of ρ in (2), we have

$$v^2\left(-\frac{d\psi}{ds}\right) = g\cos\psi$$

or $$-v\frac{ds}{dt}\cdot\frac{d\psi}{ds} = g\cos\psi \qquad \left[\because\ v = \frac{ds}{dt}\right]$$

or $$v \frac{d\psi}{dt} = - g \cos \psi . \qquad ...(4)$$

From (3), $\frac{du}{d\psi} \cdot \frac{d\psi}{dt} = - kv^n \cos \psi$

or $$\frac{du}{d\psi} \cdot \left(- \frac{g \cos \psi}{v} \right) = - kv^n \cos \psi \qquad \text{[From (4)]}$$

or $$\frac{du}{d\psi} = \frac{k}{g} v^{n+1} = \frac{k}{g} (u \sec \psi)^{n+1} \qquad \text{[From (1)]}$$

or $$- n \frac{du}{u^{n+1}} = - \frac{nk}{g} \sec^{n+1} \psi \, d\psi. \qquad \text{[Multiplying by } - n\text{]}$$

Integrating, we have $\frac{1}{u^n} = - \frac{nk}{g} \int \sec^{n+1} \psi \, d\psi + C$

or $$\frac{1}{(v \cos \psi)^n} = - \frac{nk}{g} \int \sec^{n+1} \psi \, d\psi + C . \qquad ...(5)$$

This gives velocity v at any position, constant C being determined by the initial conditions.

4.7 Trajectory in a Resisting Medium when Resistance varies as (Velocity)^n

It is a special case of 4.6 Proceeding exactly in the same way as in 4.6, we get

u = horizontal component of velocity at $P = v \cos \psi$ ...(1)

The equations of motion along the normal and parallel to x-axis are

$$\frac{v^2}{\rho} = g \cos \psi \qquad ...(2)$$

and $$\frac{du}{dt} = - kv^2 \cos \psi . \qquad ...(3)$$

Also $$\rho = - \frac{ds}{d\psi} \qquad [\because \psi \text{ decreases as } s \text{ increases}]$$

Using the value of ρ in (2), we have

$$v^2 \left(- \frac{d\psi}{ds} \right) = g \cos \psi$$

or $$v \frac{ds}{dt} \cdot \frac{d\psi}{ds} = - g \cos \psi$$

or $$v \frac{d\psi}{dt} = - g \cos \psi . \qquad ...(4)$$

From (3), $$\frac{du}{ds} \cdot \frac{ds}{dt} = - kv^2 \cos \psi$$

or $$\frac{du}{ds} \cdot v = - kv \cdot u \qquad \text{[Using (1)]}$$

or $$\frac{du}{u} = -k\,ds.$$

Integrating, $\log u = k\,s + C$.

Initially at $O, s = 0, u = u_0$ (say); and so $C = \log u_0$.

Hence $$\log \frac{u}{u_0} = -k\,s \quad \text{or} \quad u = u_0\, e^{-ks}. \qquad \ldots(5)$$

Equation (3) may also be written as
$$\frac{du}{d\psi} \cdot \frac{d\psi}{dt} = -kv^2 \cos\psi$$

or $$\frac{du}{d\psi} \cdot \left(\frac{-g\cos\psi}{v} \right) = -kv^2 \cos\psi, \text{ from (4)}$$

or $$\frac{du}{d\psi} = \frac{k}{g} v^3 = \frac{k}{g} (u \sec\psi)^3, \text{ from (1)}$$

or $$\frac{-2\,du}{u^3} = -\frac{2k}{g} \sec^3 \psi.$$

Integrating, $$\frac{1}{u^2} = -\frac{2k}{g} \int \sec^3 \psi\, d\psi + D$$
$$= -\frac{2k}{g} \left[\frac{\sec\psi \tan\psi}{2} + \frac{1}{2} \log(\sec\psi + \tan\psi) \right] + D$$

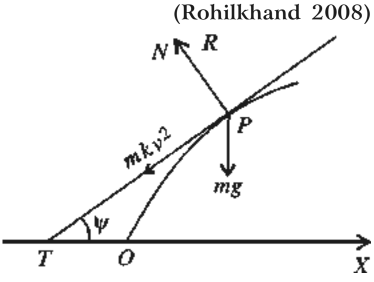
or $$\frac{e^{2ks}}{u_0^2} = -\frac{k}{g} [\sec\psi \tan\psi + \log(\sec\psi + \tan\psi)] + D, \text{ from (5)} \qquad \ldots(6)$$

This is the required intrinsic equation of the path, the constant D being determined by the initial conditions.

Remark: In the present chapter we shall be usually writing equations of motion along the tangent and normal. But it is to be noted that in 4.6 and 4.7 we have written the equations of motion along the normal and parallel to x-axis *i.e.* we have used mixed system.

4.8 Motion on a Smooth Curve under a Resistance which varies as (Velocity)^2

A bead moves on a smooth wire in a vertical plane under a resistance equal to k (velocity)^2; to find the motion. **(Rohilkhand 2008)**



Let P be the position of the bead at any time.

Since bead moves on a wire, therefore, normal reaction will also act. The forces acting on the bead at P are :

(i) Weight mg of the bead acting vertically downwards

(ii) Normal reaction R of the wire along PN.

(iii) The resisting force mkv^2 along the tangent as shown in the figure.

The equations of motion along the tangent and normal at P are

$$mv\frac{dv}{ds} = mg\sin\psi - mkv^2$$

or $$v\frac{dv}{ds} + kv^2 = g\sin\psi \quad \text{or} \quad \frac{1}{2}\frac{dv^2}{ds} + kv^2 = g\sin\psi$$

or $$\frac{dv^2}{d\psi}\frac{d\psi}{ds} + 2kv^2 = 2g\sin\psi$$

or $$\frac{dv^2}{d\psi} + 2k\rho v^2 = 2g\rho\sin\psi \qquad \left[\because \rho\frac{ds}{d\psi}\right] \qquad ...(1)$$

And the other equation (along normal) is :

$$m\frac{v^2}{\rho} = m\,g\cos\psi - R. \qquad ...(2)$$

If the equation to the curve is given, ρ can be determined. Equation (1) is a linear differential equation in v^2 and can be integrated after substituting for ρ.

Particular Case : If the curve is a circle of radius a, then $\rho = a$.

Equation (1) becomes

$$\frac{dv^2}{d\psi} + 2kav^2 = 2ga\sin\psi. \qquad ...(3)$$

This is a linear differential equation. Its integrating factor

$$= e^{\int 2ak\,d\psi} = e^{2ak\psi}.$$

$\therefore$ Solution of (3) is

$$v^2 e^{2ak\psi} = 2ga\int e^{2ak\psi}\sin\psi\,d\psi + C$$

or $$= 2ag[e^{2ak\psi}/(1+4\alpha^2k^2)]\,[2ak\sin\psi - \cos\psi] + C$$

$$v^2 = [2ag/(1+4a^2k^2)]\,[2ak\sin\psi - \cos\psi] + Ce^{-2ak\psi} \qquad ...(4)$$

Equation (4) gives velocity of the bead at any point ψ.

Illustrative Examples

Example 5: *A particle is projected under gravity and a resistance equal to mk (velocity), with a velocity u at an angle α to the horizon. Prove that by a proper choice of axes the equation of the path can be put in the form $y + ax = b\log x$.* **(Garhwal 2004)**

Solution: With usual axes of reference the equation of the path is given by the equation (7) of 4.5 and is

$$y = \frac{g}{k^2}\log\left(1 - \frac{kx}{u\cos\alpha}\right) + \frac{x}{ku\cos\alpha}(g + ku\sin\alpha). \qquad ...(1)$$

Let $$X = 1 - \frac{kx}{u\cos\alpha}$$

or $$\frac{x}{ku\cos\alpha} = \frac{1-X}{k^2}.$$

Equation (1) becomes

$$y = \frac{g}{k^2}\log X + \frac{1-X}{k^2}(g + ku\sin\alpha)$$

or $$y = \frac{g}{k^2}\log X + \left(\frac{g+ku\sin\alpha}{k^2}\right) - \left(\frac{g+ku\sin\alpha}{k^2}\right)X$$

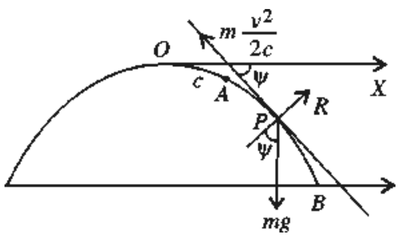
or $$y - \frac{g+ku\sin\alpha}{k^2} = \frac{g}{k^2}\log X - \left(\frac{g+ku\sin\alpha}{k^2}\right)X \cdot$$

Now let $y - \dfrac{g+ku\sin\alpha}{k^2} = Y$, $\dfrac{g}{k^2} = b$ and $\dfrac{g+ku\sin\alpha}{k^2} = a$.

The last equation then becomes

$$Y = b\log X - aX.$$

Example 6: *A heavy bead, of mass m, slides on a smooth wire in the shape of a cycloid, whose axis is vertical and vertex upwards, in a medium whose resistance is* $mv^2/2c$ *and the distance of starting point from the vertex is c; show that the time of descent to the cusp is* $\sqrt{\left[\dfrac{8a(4a-c)}{gc}\right]}$, *where* $2a$ *is the length of the axis of the cycloid.*



Solution: Let O be the vertex and C the cusp of the cycloid. Bead starts from A such that arc $OA = c$.

Also we know that arc $OB = 4a$.

The equation of the cycloid is

$$s = 4a\sin\psi \qquad ...(1)$$

Let P be the position of the bead at any time. The forces acting on the bead at P are :

(i) the resisting force $m(v^2/2c)$ along the tangent,

(ii) the normal reaction R and

(iii) the weight mg of the bead.

The directions of these forces are shown in the figure.

Equation of motion along the tangent (downwards) is

$$mv\frac{dv}{ds} = -m\frac{v^2}{2c} + mg\sin\psi$$

i.e. $$\frac{1}{2}\frac{dv^2}{ds} + \frac{v^2}{2c} = g\sin\psi$$

i.e. $$\frac{dv^2}{ds} + \frac{1}{c}v^2 = \frac{gs}{2a}, \text{ from (1).} \qquad ...(2)$$

This is a linear differential equation in v^2.

Its integrating factor $= e^{\int \frac{1}{c} \cdot ds} = e^{s/c}$.

$\therefore$ Solution of (2) is

$$v^2 . c^{s/c} = \int \frac{gs}{2a} e^{s/c} ds + C = \frac{g}{2a}\left[s \frac{e^{s/c}}{1/c} - \int 1 . ce^{s/c} ds\right] + C$$

$$= \frac{g}{2a}[sce^{s/c} - c^2 e^{s/c}] + C$$

or $$v^2 = \frac{gc}{2a}(s - c) + Ce^{-s/c}.$$

Initially at $A, s = c, v = 0$, $\therefore$ $C = 0$.

$\therefore$ $$v^2 = \left(\frac{gc}{2a}\right)(s - c)$$

or $$v = \frac{ds}{dt} = \sqrt{\left(\frac{gc}{2a}\right)} \sqrt{(s - c)}$$

or $$\frac{ds}{\sqrt{(s - c)}} = \sqrt{\left(\frac{gc}{2a}\right)} dt . \qquad ...(3)$$

Let t be the required time (*i.e.*, the time from A to the cusp B *i.e.*, from $s = c$ to $s = 4a$).

Integrating (3), $\left[2 \sqrt{(s - c)}\right]_c^{4a} = \sqrt{\left(\frac{gc}{2a}\right)} [t]_0^t$

or $$2\sqrt{(4a - c)} = \sqrt{\left(\frac{gc}{2a}\right)} t$$

or $$t = \sqrt{\left\{\frac{8a(4a - c)}{gc}\right\}}.$$

Example 7: *A shot is fired in an atmosphere in which the resistance varies as the cube of the velocity. If f be the retardation when the shot is ascending at an inclination α to the horizon, f_0 when it is moving horizontally, and f' when it is descending at an inclination α to the horizon, prove that*

$$\frac{1}{f} + \frac{1}{f'} = \frac{2\cos^3\alpha}{f_0}$$

and $$\frac{1}{f'} - \frac{1}{f} = \frac{2\sin\alpha}{g}(3 - 2\sin^2\alpha).$$

Solution: Let u be the horizontal component of the velocity v at the point where the tangent to this point makes an angle ψ with the horizontal. Thus

$$u = v\cos\psi . \qquad ...(1)$$

Now when the shot is ascending at an inclination α, the resistance is

$$kv^3 = f.$$

Thus (1) gives $u^3 = (f/k)\cos^3\psi$. ....(2)

The equations of motion along the normal and parallel to x-axis are

$$v^2/\rho = g\cos\psi \qquad \text{...(3)}$$

and $$du/dt = -kv^3\cos\psi. \qquad \text{...(4)}$$

Also $$\rho = ds/d\psi \qquad [\because \psi \text{ decreases as } s \text{ increases}] \qquad \text{...(5)}$$

Using (5) in (3), we get

$$v^2\left(-\frac{d\psi}{ds}\right) = g\cos\psi \quad \text{or} \quad v\cdot\frac{ds}{dt}\frac{d\psi}{ds} = -g\cos\psi$$

or $$v(d\psi/dt) = -g\cos\psi. \qquad \text{...(6)}$$

Equation (4) may be written as

$$\frac{du}{d\psi}\cdot\frac{d\psi}{dt} = -kv^3\cos\psi \quad \text{or} \quad \frac{du}{d\psi} = \frac{k}{g}v^4, \text{ from (6)}$$

or $$\frac{du}{d\psi} = \frac{k}{g}u^4\sec^4\psi, \text{ from (1)}$$

or $$-\frac{3\,du}{u^4} = -\frac{3k}{g}\sec^4\psi.$$

Integrating, $$\frac{1}{u^3} + C = \int -\frac{3k}{g}\sec^2\psi\,(1+\tan^2\psi)\,d\psi$$

$$= -\frac{3k}{g}[\tan\psi + \frac{1}{3}\tan^3\psi].$$

When $\psi = 0$, (*i.e.*, the particle is moving horizontally) let $u = u_0$.

$\therefore$ $$C = -1/u_0^3.$$

$\therefore$ $$\frac{1}{u^3} - \frac{1}{u_0{}^3} = -\frac{3k}{g}[\tan\psi + \frac{1}{3}\tan^3\psi]. \qquad \text{...(7)}$$

Also when the particle is moving horizontally (*i.e.*, $\psi = 0$), $f = f_0$, $u = u_0$.

Thus (2) gives $u_0^3 = f_0/k$ ...(8)

Substituting the values of u^3 and $u_0{}^3$ from (2) and (8) in (7), we get

$$\frac{k}{f\cos^3\psi} - \frac{k}{f_0} = -\frac{3k}{g}[\tan\psi + \frac{1}{3}\tan^3\psi]$$

or $$\frac{1}{f\cos^3\psi} = \frac{1}{f_0} - \frac{3}{g}\cdot\frac{\sin\psi\,(3\cos^2\psi + \sin^2\psi)}{3\cos^3\psi}$$

or $$\frac{1}{f\cos^3\psi} = \frac{1}{f_0} - \frac{1}{g}\cdot\frac{\sin\psi\,(3 - 2\sin^2\psi)}{\cos^3\psi}. \qquad \text{...(9)}$$

Now when $\psi = \alpha$, $f = f$ (given), and when $\psi = -\alpha$, $f = f'$ (given). Thus the equation (9) provides

$$\frac{1}{f\cos^3\alpha} = \frac{1}{f_0} - \frac{1}{g}\,\frac{\sin\alpha\,(3 - 2\sin^2\alpha)}{\cos^3\alpha} \qquad \text{...(10)}$$

and $$\frac{1}{f'\cos^3\alpha} = \frac{1}{f_0} + \frac{1}{g}\cdot\frac{\sin\alpha\,(3 - 2\sin^2\alpha)}{\cos^3\alpha} \qquad \text{...(11)}$$

Adding (10) and (11), $\dfrac{1}{f}+\dfrac{1}{f'}=\dfrac{2\cos^3\alpha}{f_0}$.

Subtracting (10) from (11),

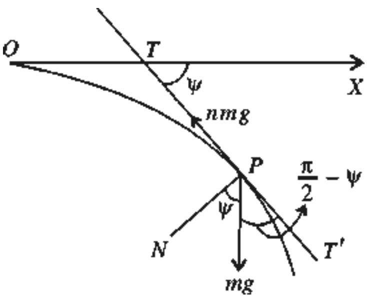
$$\frac{1}{f'}-\frac{1}{f}=\frac{2\sin\alpha\,(3-2\sin^2\alpha)}{g}.$$

Examples in which Equations of Motion are written in Tangential and Normal Form

Example 8: *If the resistance of the air to a particle's motion be n times its weight, and the particle be projected horizontally with velocity V, show that the velocity of the particle, when it is moving at an inclination* ϕ *to the horizontal, is*

$$V(1-\sin\phi)^{(n-1)/2}(1+\sin\phi)^{-(n+1)/2}.$$

Solution: Here resistance $= n\,.\,mg$.



Equations of motion along the tangent and normal are

$$mv\frac{dv}{ds}=mg\cos\left(\frac{\pi}{2}-\psi\right)-nmg$$

i.e., $$v\frac{dv}{ds}=g\sin\psi-ng \qquad \ldots(1)$$

and $$\frac{v^2}{\rho}=g\cos\psi. \qquad \ldots(2)$$

It is required to find a relation beween v and ψ [because $\psi=\phi$ at the point required].

From (2), $v^2(d\psi/ds)=g\cos\psi$ ...(3)

$$\left[\because\ \rho=\frac{ds}{d\psi}\text{ since as } s \text{ increases } \psi \text{ increases}\right]$$

Dividing (1) by (3), we have

$$\frac{1}{v}\cdot\frac{dv}{d\psi}=(\tan\psi-n\sec\psi)$$

or $$dv/v=(\tan\psi-n\sec\psi)\,d\psi$$

Integrating it, $\log v+c=\log\sec\psi-n\log(\sec\psi+\tan\psi)$.

Initially at O, $\psi=0, v=V$ and so $c=-\log V$.

Hence $$\log\frac{v}{V}=\log\sec\psi-n\log(\sec\psi+\tan\psi)$$

$$=\log\sec\psi-\log\left(\frac{1+\sin\psi}{\cos\psi}\right)^n$$

$$=\log\left[\frac{\cos^n\psi}{(1+\sin\psi)^n}\cdot\sec\psi\right]$$

$\therefore$ $$\frac{v}{V} = \frac{\cos^{n-1}\psi}{(1+\sin\psi)^n} = \frac{\{\sqrt{(1-\sin^2\psi)}\}^{n-1}}{(1+\sin\psi)^n}$$

or $$v = V\{(1-\sin\psi)(1+\sin\psi)\}^{(n-1)/2} \cdot (1+\sin\psi)^{-n}$$
$$= V(1-\sin\psi)^{(n-1)/2} . (1+\sin\psi)^{-(n+1)/2}.$$

$\therefore$ Required velocity v when $\psi = \phi$ is
$$v = V(1-\sin\phi)^{(n-1)/2} . (1+\sin\phi)^{-(n+1)/2}.$$

Comprehensive Exercise 2

1. A particle is projected with a velocity whose horizontal and vertical components are U and V from a point in a medium whose resistance per unit of mass is k times the speed. Obtain the equation of the path, and prove that if k is small, the horizontal range is approximately
$$\frac{2UV}{g} - \frac{8UV^2k}{3g^2}.$$

2. A particle acted on by gravity is projected in a medium, the resistance of which varies as the velocity. Show that its acceleration retains a fixed direction and diminishes without limit to zero.

3. A particle of unit mass is projected with velocity u at an inclination α above the horizon in a medium whose resistance is k times the velocity. Show that its direction will again make an angle α with the horizon after a time
$$(1/k)\log\{1 + (2ku/g)\sin\alpha\}.$$

4. If the resistance varies as the velocity and the range on the horizontal plane through the point of projection is maximum, show that the angle α which the direction of projection makes with the vertical is given by
$$\frac{\mu(1+\mu\cos\alpha)}{\mu+\cos\alpha} = \log(1+\mu\sec\alpha),$$
where μ is the ratio of the velocity of projection to the terminal velocity.

5. Show that in the motion of a heavy particle in a medium, the resistance of which varies as the velocity, the greatest height above the level of the point of projection is reached in less than half the total time of the flight above that level.

6. A heavy particle describes a path given by
$$\cos\psi = f(\rho\cos\psi);$$
show that the law of resistance is given by
$$Rv\frac{df}{dv} = -g\sqrt{(1-f^2)}\frac{d}{dv}(vf),$$
where $$f = f\left(\frac{v^2}{g}\right).$$

7. Prove that in the motion of a projectile in a resisting medium the equation $\frac{d^2 y}{dx^2} = -\frac{g}{u^2}$ is satisfied whatever be the law of resistance, u being the horizontal component of the velocity, the axes of x and y being horizontal and vertically upwards.

   If the resistance is constant and equal to kg, show that the velocity v at any point is given by

$$v\,(1 - \sin\psi)^{k} = u_0\,(\cos\psi)^{k-1},$$

   ψ being the slope and u_0 the velocity at the highest point.

8. If the resistance per unit mass is $g(v/V)^2$, prove that

$$du/ds = -(g/V^2)\,u,\; d\psi/ds = (g/u^2)\cos^3\psi,$$

   where u is the horizontal component of velocity.

9. If ρ and ρ' be the radii of curvatures at two points at equal arcual distances from the vertex ; ψ, ψ' inclinations to the horizon of tangents, prove that

$$\rho\rho' \cos^3\psi \cos^3\psi' = \rho_0{}^2,$$

   where ρ_0 is the radius of curvature at the vertex. Supposing resistance varies as the square of the velocity.

10. A particle moving in a resisting medium is acted upon by a central force μ / r^n, if the path be an equiangular spiral of angle α, whose pole is at the centre of force, show that the resistance is

$$\frac{n-3}{2} \cdot \frac{\mu \cos\alpha}{r^n}.$$

11. A heavy particle is projected in a resisting medium. If v be the velocity at any time t and ϕ the inclination to the vertical of the direction of motion and f the retardation, prove that

$$\frac{1}{v}\frac{dv}{d\phi} + \cot\phi + \frac{f}{g\sin\phi} = 0.$$

(Kanpur 2007)

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. The equation of motion of a particle falling vertically downwards in a resisting medium is $m\,\frac{d^2 x}{dt^2} = mg - mk\,v^2$. If V is the terminal velocity, then:

   (a) $k = \frac{V^2}{g}$ (b) $k = \frac{g}{V^2}$

   (c) $k = \frac{V}{g}$ (d) $k = \frac{g}{V}$

(Bundelkhand 2009 Rohilkhand 10)

2. A particle is projected vertically upwards under gravity, supposed constant, in a resisting medium whose resistance varies as the square of the velocity. If v is the velocity of the particle after time t when it has risen through a distance x, then the equation of motion of the particle is

(a) $m\dfrac{d^2x}{dt^2} = mg - mk\,v^2$ (b) $m\dfrac{d^2x}{dt^2} = mg + mk\,v^2$

(c) $m\dfrac{d^2x}{dt^2} = -\,mg - mk\,v^2$ (d) $m\dfrac{d^2x}{dt^2} = -\,mg + mk\,v^2$

3. By increasing the velocity of any body in any medium, the resistance of the medium :

(a) Increases (b) Decreases

(c) Remains the same (d) None of these **(Garhwal 2002)**

4. A particle falls under gravity (constant) from rest in a medium whose resistance varies as the velocity. Its equation of motion is given by :

(a) $m\dfrac{d^2x}{dt^2} = -\,mg - mkv$ (b) $m\dfrac{d^2x}{dt^2} = -\,mg + mkv$

(c) $m\dfrac{d^2x}{dt^2} = mg + mkv$ (d) $m\dfrac{d^2x}{dt^2} = mg - mkv$ **(Garhwal 2003)**

5. The resisting force is :

(a) Conservative (b) Non-conservative

(c) Central (d) None of these

(Garhwal 2004; Rohilkhand 07, 08, 09; Avadh 06)

Fill in the Blank(s)

Fill in the blanks "..." so that the following statements are complete and correct.

1. If a particle is falling under gravity in a resisting medium, then the velocity V when the downward acceleration is zero is called the...

2. A particle is falling from rest under gravity, supposed constant, in a resisting medium whose resistance varies as the square of the velocity. If v is the velocity of the particle after time t when it has fallen through a distance x, then the equation of motion of the particle is

$$m\frac{d^2x}{dt^2} = \ldots$$

3. If a particle falls under gravity in a resisting medium, then the force of resistance acts vertically ... on the particle.

True or False

Write 'T' for true and 'F' for false statement.

1. The direction of the force of resistance is always the same as the direction of motion of the body.

2. The direction of the force of resistance is always opposite to the direction of motion of the body.

3. The force of resistance decreases as the velocity of the particle increases.

4. The equation of motion of a particle falling vertically downwards in a resisting medium is $m\dfrac{d^2x}{dt^2} = mg - mk\,v^3$. If V is the terminal velocity, then $k = \dfrac{g}{V^3}$.

5. A particle is projected with velocity u along a smooth horizontal plane in a medium whose resistance per unit mass is μ times the cube of the velocity. If v is the velocity of the particle at time t at a point distant x from the point of projection, then the equation of motion of the particle is $m\dfrac{dv}{dt} = m\mu\,v^3$.

Answers

Multiple Choice Questions

1. (b) 2. (c) 3. (a)
4. (d) 5. (b)

Fill in the Blank(s)

1. terminal velocity 2. $mg - mk\,v^2$ 3. upwards

True or False

1. F 2. T 3. F
4. T 5. F

□

Chapter 5

Central Orbits

5.1 Definitions

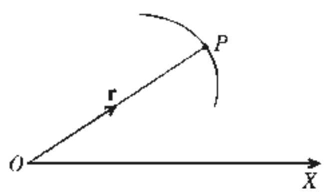
1. Central Force: *A force whose line of action always passes through a fixed point is called a* **central force**. *The fixed point is known as the* **centre of force.**

(Kumaun 2002; Bundelkhand 06; Meerut 07; Agra 07)

2. Central Orbit: ***A* central orbit** *is the path described by a particle moving under the action of a central force.* The motion of a planet about the sun is an important example of a central orbit. **(Garhwal 2001; Kumaun 02; Lucknow 06, 08; Rohilkhand 10)**

Theorem: *A central orbit is always a plane curve.*

(Bundelkhand 2006; Agra 07)



Proof: Take the centre of force O as the origin of vectors. Let P be the position of a particle moving in a central orbit at any time t and let $\vec{OP} = \mathbf{r}$. Then $\dfrac{d^2\mathbf{r}}{dt^2}$ is the expression for the acceleration vector of the particle at the point P. Since the particle moves under the action of a central force with centre at O, therefore the only force acting on the particle at P is along the line OP or PO. So the acceleration vector of P is parallel to the vector $\vec{OP}$.

$\therefore$ $$\frac{d^2\mathbf{r}}{dt^2} \text{ is parallel to } \mathbf{r} \Rightarrow \frac{d^2\mathbf{r}}{dt^2} \times \mathbf{r} = \mathbf{0}$$

$\Rightarrow$ $$\frac{d^2\mathbf{r}}{dt^2} \times \mathbf{r} + \frac{d\mathbf{r}}{dt} \times \frac{d\mathbf{r}}{dt} = \mathbf{0} \qquad \left[\because \frac{d\mathbf{r}}{dt} \times \frac{d\mathbf{r}}{dt} = \mathbf{0}\right]$$

$\Rightarrow$ $$\frac{d}{dt}\left(\frac{d\mathbf{r}}{dt} \times \mathbf{r}\right) = \mathbf{0}$$

$\Rightarrow$ $$\frac{d\mathbf{r}}{dt} \times \mathbf{r} = \text{a constant vector} = \mathbf{h}, \text{say.} \qquad \text{...(1)}$$

Taking dot product of both sides of (1) with the vector $\mathbf{r}$, we get

$$\mathbf{r} \bullet \left(\frac{d\mathbf{r}}{dt} \times \mathbf{r}\right) = \mathbf{r} \bullet \mathbf{h}.$$

But the left hand member is a scalar triple product involving two equal vectors, and so it vanishes.

$\therefore$ $$\mathbf{r} \bullet \mathbf{h} = 0,$$

which shows that $\mathbf{r}$ is always perpendicular to a constant vector $\mathbf{h}$.

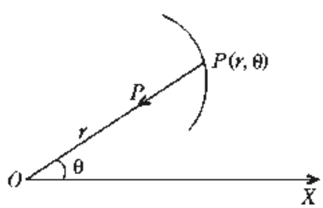
Thus the radius vector OP is always perpendicular to a fixed direction and hence lies in a plane. Therefore the path of P is a plane curve.

5.2 Differential Equation of a Central Orbit

A particle moves in a plane with an acceleration which is always directed to a fixed point O in the plane; to obtain the differential equation of the path.

(Kumaun 2001; Garhwal 04; Agra 07, 08, 11; Avadh 09, 11; Rohilkhand 10)

Let a particle move in a plane with an acceleration P which is always directed to a fixed point O in the plane. Take the centre of force O as the pole. Let OX be the initial line and (r, θ) the polar co-ordinates of the position P of the moving particle at any instant t.



Since the acceleration of the particle is always directed towards the pole O, therefore the particle has only the radial acceleration and the transverse component of the acceleration of the particle is always zero. So the equations of motion of the particle at the point P are

the radial acceleration *i.e.*, $$\frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -P, \qquad \text{...(1)}$$

(the $-$ ive sign has been taken because the radial acceleration P is in the direction of r decreasing)

and the transverse acceleration *i.e.*, $$\frac{1}{r}\frac{d}{dt}\left(r^2 \frac{d\theta}{dt}\right) = 0. \qquad \text{...(2)}$$

From (2), we have $\dfrac{d}{dt}\left(r^2\dfrac{d\theta}{dt}\right)=0.$

Integrating, we get $\boldsymbol{r^2\dfrac{d\theta}{dt}}$ **= constant = *h*, say.** ...(3)

(Lucknow 06, 08)

Let $\qquad r=\dfrac{1}{u}\cdot$

Now from (3), we have

$$d\theta/dt=h/r^2=hu^2$$

Also $$\frac{dr}{dt}=-\frac{1}{u^2}\frac{du}{dt}=-\frac{1}{u^2}\frac{du}{d\theta}\cdot\frac{d\theta}{dt}=-\frac{1}{u^2}\frac{du}{d\theta}\cdot u^2\,h=-h\frac{du}{d\theta},$$

and $$\frac{d^2r}{dt^2}=-h\frac{d^2u}{d\theta^2}\cdot\frac{d\theta}{dt}=-h\frac{d^2u}{d\theta^2}(u^2\,h)=-h^2\,u^2\frac{d^2u}{d\theta^2}\cdot$$

Substituting in (1), we have

$$-h^2u^2\frac{d^2u}{d\theta^2}-\frac{1}{u}\cdot(u^2h)^2=-P \quad \text{or} \quad h^2u^2\frac{d^2u}{d\theta^2}+h^2u^3=P$$

or $$\boldsymbol{\frac{d^2u}{d\theta^2}+u=\frac{P}{h^2u^2}}, \qquad ...(4)$$

which is the differential equation of a central orbit in **polar form** referred to the centre of force as the pole.

Pedal Form: If p is the length of the perpendicular drawn from the origin upon the tangent at the point P, we have

$$\frac{1}{p^2}=\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2.$$

But $u=\dfrac{1}{r}\cdot$ Therefore $\dfrac{du}{d\theta}=-\dfrac{1}{r^2}\dfrac{dr}{d\theta}$ *i.e.*, $\left(\dfrac{du}{d\theta}\right)^2=\dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2.$

So $$\boldsymbol{\frac{1}{p^2}=u^2+\left(\frac{du}{d\theta}\right)^2}. \qquad ...(5)$$

Differentiating both sides of (5) w.r.t. 'θ', we have

$$-\frac{2}{p^3}\frac{dp}{d\theta}=2u\frac{du}{d\theta}+2\frac{du}{d\theta}\cdot\frac{d^2u}{d\theta^2}=2\frac{du}{d\theta}\left(u+\frac{d^2u}{d\theta^2}\right)$$

or $$-\frac{1}{p^3}\frac{dp}{d\theta}=\frac{du}{d\theta}\cdot\frac{P}{h^2\,u^2} \qquad \text{[From (4)]}$$

or $$-\frac{1}{p^3}\cdot\frac{dp}{dr}\cdot\frac{dr}{d\theta}=\left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)\left(\frac{P}{h^2\,u^2}\right), \qquad \left\{\because\ \frac{du}{d\theta}=-\frac{1}{r^2}\frac{dr}{d\theta}\right\}$$

or $$\frac{1}{p^3}\frac{dp}{dr}=\frac{1}{r^2}\cdot\frac{P}{h^2u^2}=u^2\cdot\frac{P}{h^2u^2}=\frac{P}{h^2}$$

or $$\boldsymbol{P=\frac{h^2}{p^3}\frac{dp}{dr}} \qquad ...(6)$$

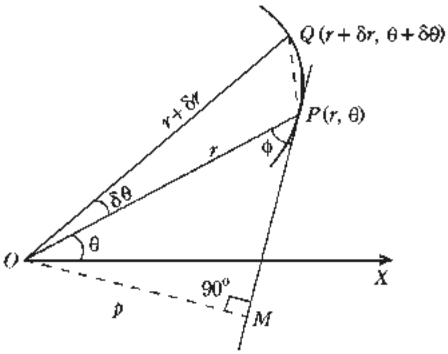
(Garhwal 2001; Kumaun 02; Avadh 06; Agra 07, 08; Purvanchal 09; Bundelkhand 11)

which is the differential equation of a central orbit in **pedal form**.

Angular Momentum or Moment of Momentum: The expression $r^2 (d\theta/dt)$ is called the angular momentum or the moment of momentum about the pole O of a particle of unit mass moving in a plane curve. Since in a central orbit $r^2 (d\theta/dt) =$ constant, therefore *in a central orbit the angular momentum is conserved*.

5.3 Rate of Description of the Sectorial Area

In every central orbit, the sectorial area traced out by the radius vector to the centre of force increases uniformly per unit of time and the linear velocity varies inversely as the perpendicular from the centre upon the tangent to the path.



Take the centre of force O as the pole and OX as the initial line. Let $P(r, \theta)$ and $Q(r + \delta r, \theta + \delta\theta)$ be the positions of a particle moving in a central orbit at times t and $t + \delta t$ respectively.

Sectorial area OPQ described by the particle in time δt

= area of the ΔOPQ

[$\because$ the point Q is very close to P and ultimately we have to take limit as $Q \to P$]

$$= \frac{1}{2} OP.OQ \sin \angle POQ = \frac{1}{2} r(r + \delta r) \sin \delta\theta.$$

$\therefore$ rate of description of the sectorial area

$$= \lim_{\delta t \to 0} \frac{\text{sectorial area } OPQ}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{\frac{1}{2} r(r + \delta r) \sin \delta\theta}{\delta t}$$

$$= \lim_{\delta t \to 0} \frac{1}{2} r(r + \delta r) \cdot \frac{\sin d\theta}{\delta\theta} \cdot \frac{\delta\theta}{\delta t} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} h. \qquad ...(1)$$

[$\because r^2 (d\theta / dt) = h$]

Thus the rate of description of the sectorial area is constant and is equal to $h/2$.

(Purvanchal 2010)

The rate of description of the sectorial area is also called the *areal velocity* of the particle about the fixed point O.

Again for a central orbit, we have $r^2 \dfrac{d\theta}{dt} = h$.

$$\therefore \qquad r^2 \frac{d\theta}{ds} \frac{ds}{dt} = h \quad \text{or} \quad r^2 \frac{d\theta}{ds} \cdot v = h. \qquad ...(2)$$

[$\because ds/dt = v$ (*i.e.*, the linear velocity)]

But from differential calculus, we have

$$r\frac{d\theta}{ds} = \sin\phi,$$

where ϕ is the angle between the radius vector and the tangent.

$\therefore$ $\quad r^2\dfrac{d\theta}{ds} = r\sin\phi = p$, where p is the length of the perpendicular drawn from the pole O on the tangent at P.

Putting $r^2(d\theta/ds) = p$ in (2), we get $vp = h$.

or $$v = \frac{h}{p}\cdot \qquad \text{...(3)}$$

$\therefore$ $\quad v \propto 1/p$ **(Lucknow 2011)**

i.e., the linear velocity at P varies inversely as the perpendicular from the fixed point upon the tangent to the path.

From (3), we have $v^2 = \dfrac{h^2}{p^2}\cdot$

But $$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = u^2 + \left(\frac{du}{d\theta}\right)^2.$$

$\therefore$ $$\boldsymbol{v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right].} \qquad \text{...(4)}$$

The equation (4) gives the linear velocity at any point of the path of a central orbit.

5.4 Elliptic Orbit (Focus as the Centre of Force)

A particle moves in an ellipse under a force which is always directed towards its focus; to find

(i) *the law of force* **(Garhwal 2002; Rohilkhand 11; Agra 10)**

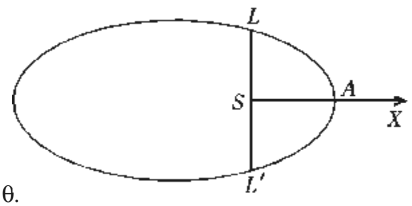
(ii) *the velocity at any point of its path* **(Garhwal 2002; Rohilkhand 11; Bundelkhand 07)**

(iii) *the periodic time.* **(Garhwal 2002; Rohilkhand 11)**

We know that the polar equation of an ellipse referred to its focus S as pole is

$$\frac{l}{r} = 1 + e\cos\theta$$

or $$u = \frac{1}{l} + \frac{e}{l}\cos\theta, \qquad \text{...(1)}$$



where $u = 1/r$. Differentiating, we have

$$\frac{du}{d\theta} = -\frac{e}{l}\sin\theta \text{ and } \frac{d^2u}{d\theta^2} = -\frac{e}{l}\cos\theta.$$

(i) Law of Force: We know that the differential equation of a central orbit referred to the centre of force as pole is

$$\frac{P}{h^2u^2} = u + \frac{d^2u}{d\theta^2}$$

where P is the central acceleration assumed to be attractive.

Now here $P = h^2 u^2 \left[u + \dfrac{d^2 u}{d\theta^2}\right]$

$$= h^2 u^2 \left[\frac{1}{l} + \frac{e}{l}\cos\theta - \frac{e}{l}\cos\theta\right], \text{ substituting for } u \text{ and } d^2u/d\theta^2$$

$$= \frac{h^2 u^2}{l} = \frac{h^2/l}{r^2} = \frac{\mu}{r^2}, \quad \text{...(2)}$$

where $\mu = h^2/l$ or $\boldsymbol{h^2 = \mu l}$. ...(3)

$\therefore$ $P \propto \dfrac{1}{r^2}$.

Hence the acceleration varies inversely as the square of the distance of the particle from the focus. Also the force is attractive because the value of P is positive.

(ii) Velocity: We know that the velocity in a central orbit is given by

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right].$$

$\therefore$ here, $v^2 = h^2\left[\left(\dfrac{1}{l} + \dfrac{e}{l}\cos\theta\right)^2 + \left(\dfrac{-e}{l}\sin\theta\right)^2\right]$

$$= h^2\left[\frac{1}{l^2} + \frac{2e}{l^2}\cos\theta + \frac{e^2}{l^2}\right] = \frac{h^2}{l}\left[\frac{1+e^2}{l} + 2\,\frac{e\cos\theta}{l}\right]$$

$$= \mu\left[\frac{1+e^2}{l} + 2\left(u - \frac{1}{l}\right)\right] \quad \text{[from (1) and (3)]}$$

$$= \mu\left[2\,u - \frac{1-e^2}{l}\right] = \mu\left[\frac{2}{r} - \frac{1-e^2}{l}\right].$$

If $2a$ and $2b$ are the lengths of the major and the minor axes of the ellipse, we have

$$l = \text{the semi-latus rectum} = \frac{b^2}{a} = \frac{a^2(1-e^2)}{a} = a(1-e^2).$$

$\therefore$ $\dfrac{1-e^2}{l} = \dfrac{1}{a}$.

$\therefore$ $\boldsymbol{v^2 = \mu\left(\dfrac{2}{r} - \dfrac{1}{a}\right)}$, ...(4)

which gives the velocity of the particle at any point of its path.

Equation (4) *shows that the magnitude of the velocity at any point of the path depends only on the distance from the focus and that it is independent of the direction of the motion. Also* $v^2 < 2\mu/r$.

(iii) Periodic Time: We know that in a central orbit the rate of description of the sectorial area is constant and is equal to $h/2$. Let T be the time period for one complete revolution *i.e.*, the time taken by the particle in describing the whole of the ellipse. The sectorial area traced in describing the whole arc of the ellipse is equal to the whole area of the ellipse.

$\therefore$ $\quad T(h/2) =$ the whole area of the ellipse $= \pi ab$.

$\therefore$ $$T = \frac{2\pi ab}{h} = \frac{2\pi ab}{\sqrt{(\mu l)}} \qquad [\because \quad h^2 = \mu l]$$

or $$T = \frac{2\pi ab}{\sqrt{\{\mu(b^2/a)\}}} \qquad [\because \quad l = b^2/a]$$

or $$\boldsymbol{T = \frac{2\pi a^{3/2}}{\sqrt{\mu}}}, \qquad \ldots(5)$$

i.e., the time period for one complete revolution is proportional to $a^{3/2}$, *a being semi-major axis*.

5.5 Hyperbolic and Parabolic Orbits (Centre of Force being the Focus)

(Agra 2010)

(i) Hyperbolic Orbit: In the case of hyperbola, we have $e > 1$.

Also $$l = \frac{b^2}{a} = \frac{a^2(e^2 - 1)}{a} = a(e^2 - 1).$$

Proceeding as in 5.4, we have $P = \mu/r^2$, where $h^2 = \mu l$.

[Note that this result does not depend upon the value of e].

Also proceeding as in establishing the result (4) of 5.4, we have here

$$v^2 = \mu\left[\frac{2}{r} + \frac{e^2 - 1}{l}\right] \qquad [\because \ e > 1]$$

or $$\boldsymbol{v^2 = \mu\left[\frac{2}{r} + \frac{1}{a}\right]}.$$ Note that here $v^2 > 2\mu/r$.

(ii) Parabolic Orbit: In this case $e = 1$.

Proceeding as in 5.4, we have here $P = \mu/r^2$ and $v^2 = 2\mu/r$.

5.6 Velocity from Infinity

In connection with the central orbits by the phrase '*velocity from infinity at any point*' we mean the velocity that a particle would acquire if it moved from rest at infinity in a straight line to that point under the action of an attractive force in accordance with the law associated with the orbit.

Suppose a particle falls from rest from infinity in a straight line under the action of a central attractive acceleration P directed towards the centre of force O.

Let Q be the position of the particle at any time t, where $OQ = r$.

Suppose v is the velocity of the particle at Q. The expression for acceleration at the point Q is $v\,(dv/dr)$.

The equation of motion of the particle at the point Q is

$$v\frac{dv}{dr} = -P,$$ [–ive sign has been taken because the acceleration P is in the direction of r decreasing]

or $$v\,dv = -P\,dr. \quad ...(1)$$

Let V be the velocity acquired in falling from rest at infinity to a point distant a from the centre of force O. Then integrating (1) from infinity to the point $r = a$, we get

$$\int_0^V v\,dv = -\int_\infty^a P\,dr$$

or $$\frac{1}{2}V^2 = -\int_\infty^a P\,dr$$

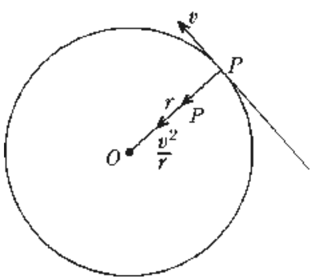
or $$V^2 = -2\int_\infty^a P\,dr,$$ **(Remember)**

which gives the velocity from infinity at a distance a from the centre of force while moving under the central acceleration P associated with the orbit.

5.7 Velocity in a Circle

The phrase '*velocity in a circle*' at any point of a central orbit means the velocity necessary to describe a circle, passing through that point and with centre at the centre of force, while moving under the action of the prescribed force associated with the orbit.

Take the centre of force O as the pole. Let P be the central acceleration, directed towards O, at any point P of the orbit where $OP = r$. Suppose v is the velocity in a circle at P. Then v is the velocity at the point P of a particle which moves, under the same central acceleration P, in a circle with centre at O. But for a circle with centre at the pole O, the radius vector OP is also normal to the circle at P. Therefore



the central radial acceleration P

$$= \text{the inward normal acceleration } v^2/\rho$$

i.e., $$P = v^2/r. \qquad [\because \text{ for the circle}, \rho = r]$$

$\therefore$ $$v^2 = rP.$$ **(Remember)**

Thus while moving under a central attractive acceleration P, the velocity V in a circle at a distance a from the centre of force is given by

$$V^2 = a.\left[P\right]_{r=a}.$$

5.8 Given the Central Orbit, to Find the Law of Force

Case I. *The equation of the orbit being given in the polar form* (r, θ).

We know that referred to the centre of force as pole, the differential equation of a central orbit is

$$\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2}, \qquad ...(1)$$

where P is the central acceleration assumed to be attractive.

From the given equation of the orbit we can easily calculate u and $d^2 u/d\theta^2$ and substituting their values in (1) we can determine P. Thus we find the law of force. If the value of P is positive, the force is attractive and if the value of P is negative, the force is repulsive.

Case II. *The equation of the orbit being given in the pedal form* (p, r). **(Agra 2007)**

The differential equation of a central orbit in (p, r) form is

$$\frac{h^2}{p^3}\frac{dp}{dr} = P. \qquad ...(2)$$

From the given equation of the orbit in (p, r) form, we can find out dp/dr and then substituting its value in (2) we can determine P.

Illustrative Examples

Example 1: *A paritcle describes the curve* $r^n = a^n \cos n\theta$ *under a force to the pole. Find the law of force.* **(Avadh 2006; Purvanchal 08, 10)**

Hence obtain the law of force under which a cardioid can be described.

Solution: The equation of the curve is $r^n = a^n \cos n\theta$.

Putting $r = 1/u$, we have

$$\frac{1}{u^n} = a^n \cos n\theta \quad \text{or} \quad a^n u^n = \sec n\theta. \qquad ...(1)$$

Taking logarithm of both sides of (1), we have

$$n \log a + n \log u = \log \sec n\theta.$$

Differentiating w.r.t. 'θ', we have

$$\frac{n}{u}\frac{du}{d\theta} = \frac{1}{\sec n\theta} n \sec n\theta \tan n\theta \quad \text{or} \quad \frac{du}{d\theta} = u \tan n\theta.$$

Differentiating again w.r.t.. 'θ', we have

$$\frac{d^2u}{d\theta^2} = \frac{du}{d\theta}\tan n\theta + u(\sec^2 n\theta) \,.\, n$$

$$= u \tan n\theta \,.\, \tan n\theta + un \sec^2 n\theta \qquad [\because du/d\theta = u \tan n\theta]$$

$$= u \tan^2 n\theta + un \sec^2 n\theta. \qquad ...(2)$$

The differential equation of the central orbit is

$$\frac{P}{h^2 u^2} = u + \frac{d^2 u}{d\theta^2}.$$

$$\therefore \quad P = h^2 u^2 \left(u + \frac{d^2 u}{d\theta^2} \right) = h^2 u^2 (u + u \tan^2 n\theta + un \sec^2 n\theta)$$

[putting the value of $d^2 u / d\theta^2$ from (2)]

$$= h^2 u^3 (\sec^2 n\theta + n \sec^2 n\theta) = h^2 u^3 (1+n) \sec^2 n\theta$$

[substituting for sec $n\theta$ from (1)]

$$= h^2 (1+n) u^3 . (a^n u^n)^2$$

$$= h^2 a^{2n} (1+n) u^{2n+3} = \frac{h^2 a^{2n} (1+n)}{r^{2n+3}}.$$

$\therefore$ $P \propto 1/r^{2n+3}$ *i.e.*, the force varies inversely as the $(2n+3)$th power of the distance from the pole.

Second part. Putting $n = 1/2$ in the equation of the path, we get

$$r^{1/2} = a^{1/2} \cos \frac{1}{2} \theta$$

or $$r = a \cos^2 \frac{1}{2} \theta$$ [squaring both sides]

or $$r = \frac{1}{2} a . 2 \cos^2 \frac{1}{2} \theta = \frac{1}{2} a (1 + \cos \theta),$$ which is the equation of cardioid.

Now putting $n = \frac{1}{2}$ in the value of P, we get

$$P \propto \frac{1}{r^{1+3}} \quad i.e., \; P \propto \frac{1}{r^4}.$$

Example 2: *A particle describes the curve $r^n = A \cos n\theta + B \sin n\theta$ under a force to the pole. Find the law of force.* **(Kanpur 2009)**

Solution: *Here* $r^n = A \cos n\theta + B \sin n\theta$.

Let $A = k \cos \alpha$ and $B = k \sin \alpha$, where k and α are constants.

Then $r^n = k (\cos \alpha \cos n\theta + \sin \alpha \sin n\theta) = k \cos (n\theta - \alpha)$.

Replacing r by $1/u$, we have

$$r^n = u^{-n} = k \cos (n\theta - \alpha). \quad ...(1)$$

$$\therefore \quad -n \log u = \log k + \log \cos (n\theta - \alpha).$$

Differentiating both sides w.r.t. 'θ', we have

$$\frac{-n}{u} \frac{du}{d\theta} = -n \tan (n\theta - \alpha) \quad \text{or} \quad \frac{du}{d\theta} = u \tan (n\theta - \alpha).$$

$$\therefore \quad \frac{d^2 u}{d\theta^2} = \frac{du}{d\theta} \cdot \tan (n\theta - \alpha) + un \sec^2 (n\theta - \alpha)$$

$$= u \tan^2 (n\theta - \alpha) + un \sec^2 (n\theta - \alpha).$$

The differential equation of the path is

$$\frac{P}{h^2 u^2} = u + \frac{d^2 u}{d\theta^2}.$$

$$\therefore \qquad P = h^2 u^2 [u + u \tan^2 (n\theta - \alpha) + un \sec^2 (n\theta - \alpha)]$$

$$= h^2 u^3 [\sec^2 (n\theta - \alpha) + n \sec^2 (n\theta - \alpha)]$$

$$= (1 + n) h^2 u^3 \sec^2 (n\theta - \alpha)$$

$$= (1 + n) h^2 u^3 (ku^n)^2 \qquad [\because \text{from } (1), \sec (n\theta - \alpha) = ku^n]$$

$$= \frac{(1 + n) h^2 k^2}{r^{2n+3}}.$$

Thus $P \propto \frac{1}{r^{2n+3}}$ *i.e.*, the force is inversely proportional to the $(2n + 3)$th power of the distance from the pole.

Example 3: *Find the law of force towards the pole under which the curve* $\frac{b^2}{p^2} = \frac{2a}{r} - 1$ *is described.* **(Rohilkhand 2008)**

Solution: The equation of the given central orbit is

$$\frac{b^2}{p^2} = \frac{2a}{r} - 1. \qquad ...(1)$$

(1) is the pedal equation of an ellipse referred to the focus as pole.

Differentiating both sides of (1) w.r.t. 'r', we get

$$-\frac{2b^2}{p^3} \frac{dp}{dr} = -\frac{2a}{r^2}, \quad \text{or} \quad \frac{h^2}{p^3} \frac{dp}{dr} = \frac{a}{b^2} \frac{h^2}{r^2}.$$

$$\therefore \qquad P = \frac{h^2}{p^3} \frac{dp}{dr} = \frac{ah^2}{b^2} \frac{1}{r^2}.$$

Thus $P \propto 1 / r^2$ *i.e.*, the acceleration varies inversely as the square of the distance from the focus of the ellipse.

Example 4: *The velocity at any point of a central orbit is* $(1 / n)$*th of what it would be for a circular orbit at the same distance. Show that the central force varies as* $\frac{1}{r^{(2n^2+1)}}$ *and that the equation of the orbit is*

$$r^{n^2 - 1} = a^{n^2 - 1} . \cos (n^2 - 1)\theta.$$

(Agra 2006)

Solution: Under the same central force P, let v and V be the velocities at a distance r from the centre of force in the central orbit and the circular orbit respectively. Then according to the question, we have $v = V / n$

or $\qquad v^2 = V^2 / n^2 \qquad ...(1)$

But $\qquad V^2 / r = P \qquad$ [See 5.7]

or $\qquad V^2 = Pr = P/u. \qquad ...(2)$

$\therefore$ from (1) and (2), we have

$$v^2 = \frac{P}{n^2 u} \quad \text{or} \quad h^2 \left[u^2 + \left(\frac{du}{d\theta} \right)^2 \right] = \frac{P}{n^2 u} \qquad ...(3)$$

[$\because$ for a central orbit, $v^2 = h^2 \{u^2 + (du/d\theta)^2\}$].

Differentiating both sides of (3) w.r.t. 'θ', we have

$$h^2 \left[2u \frac{du}{d\theta} + 2 \frac{du}{d\theta} \frac{d^2 u}{d\theta^2} \right] = \frac{1}{n^2} \left[\frac{1}{u} \frac{dP}{d\theta} - \frac{P}{u^2} \frac{du}{d\theta} \right]$$

$$= \frac{1}{n^2} \left[\frac{1}{u} \frac{dP}{du} \frac{du}{d\theta} - \frac{P}{u^2} \frac{du}{d\theta} \right].$$

$\therefore$
$$2h^2 \frac{du}{d\theta} \cdot \left[u + \frac{d^2 u}{d\theta^2} \right] = \frac{1}{n^2} \frac{du}{d\theta} \left[\frac{1}{u} \frac{dP}{du} - \frac{P}{u^2} \right].$$

Dividing out by $du/d\theta$, we get

$$2h^2 \left[u + \frac{d^2 u}{d\theta^2} \right] = \frac{1}{n^2} \left[\frac{1}{u} \frac{dP}{du} - \frac{P}{u^2} \right]$$

or
$$2 \cdot \frac{P}{u^2} = \frac{1}{n^2} \left[\frac{1}{u} \frac{dP}{du} - \frac{P}{u^2} \right] \qquad \left[\because \frac{P}{h^2 u^2} = u + \frac{d^2 u}{d\theta^2} \right]$$

or
$$2n^2 \cdot \frac{P}{u^2} = \left[\frac{1}{u} \frac{dP}{du} - \frac{P}{u^2} \right] \quad \text{or} \quad (2n^2 + 1) \frac{P}{u^2} = \frac{1}{u} \frac{dP}{du}$$

or
$$\frac{dP}{P} = (2n^2 + 1) \cdot \frac{du}{u}.$$

Integrating, $\log P = (2n^2 + 1) \log u + \log A$.

$\therefore$
$$P = A u^{2n^2 + 1} = \frac{A}{r^{2n^2 + 1}}.$$

$\therefore$ $P \propto \dfrac{1}{r^{2n^2 + 1}}$, which proves the first result.

Substituting $P = A u^{2n^2 + 1}$ in (3), we have

$$h^2 \left[u^2 + \left(\frac{du}{d\theta} \right)^2 \right] = \frac{A u^{2n^2 + 1}}{n^2 u} = \frac{A}{n^2} u^{2n^2}.$$

Putting $u = \dfrac{1}{r}$ so that $\dfrac{du}{d\theta} = -\dfrac{1}{r^2} \dfrac{dr}{d\theta}$, we have

$$\frac{1}{r^2} + \left(-\frac{1}{r^2} \frac{dr}{d\theta} \right)^2 = \frac{A}{n^2 h^2 r^{2n^2}}$$

or
$$r^{2n^2 - 2} + r^{2n^2 - 4} \left(\frac{dr}{d\theta} \right)^2 = \frac{A}{n^2 h^2}$$

or $$r^{2n^2-4}\left(\frac{dr}{d\theta}\right)^2 = \frac{A}{n^2 h^2} - r^{2n^2-2}$$

or $$\left(r^{n^2-2}\right)^2\left(\frac{dr}{d\theta}\right)^2 = a^{2n^2-2} - r^{2n^2-2},$$

setting $A/n^2 h^2 = a^{2n^2-2}$ to get the required form of the answer.

$\therefore$ $$\frac{dr}{d\theta} = \frac{\sqrt{\left\{a^{2n^2-2} - r^{2n^2-2}\right\}}}{r^{n^2-2}}$$

or $$\frac{r^{n^2-2}\,dr}{\sqrt{\left\{\left(a^{n^2-1}\right)^2 - \left(r^{n^2-1}\right)^2\right\}}} = d\theta\,.$$

Putting $r^{n^2-1} = z$ so that $(n^2-1)\,r^{n^2-2}\,dr = dz$, we have

$$\frac{dz}{\sqrt{\left\{\left(a^{n^2-1}\right)^2 - z^2\right\}}} = (n^2-1)\,d\theta.$$

Integrating, $\sin^{-1}\left(\dfrac{z}{a^{n^2-1}}\right) = (n^2-1)\,\theta + B$ or $\sin^{-1}\left(\dfrac{r^{n^2-1}}{a^{n^2-1}}\right) = (n^2-1)\,\theta + B.$

Initially when $\theta = 0$, let $r = a$. Then $B = \sin^{-1} 1 = \pi/2$.

$\therefore$ $$\sin^{-1}\left(\frac{r^{n^2-1}}{a^{n^2-1}}\right) = (n^2-1)\,\theta + \frac{1}{2}\pi$$

or $$\frac{r^{n^2-1}}{a^{n^2-1}} = \sin\left\{(n^2-1)\,\theta + \frac{1}{2}\pi\right\} = \cos\,(n^2-1)\,\theta$$

or $$r^{n^2-1} = a^{(n^2-1)}\cos\,(n^2-1)\,\theta,$$

which is the required equation of the orbit.

Example 5: *A particle moves with a central acceleration $\mu/(distance)^2$; it is projected with velocity V at a distance R. Show that its path is a rectangular hyperbola if the angle of projection is*

$$sin^{-1}\left[\frac{\mu}{VR\left(V^2 - \dfrac{2\mu}{R}\right)^{1/2}}\right].$$

Solution: If the particle describes a hyperbola under the central acceleration $\mu/(\text{distance})^2$, then the velocity v of the particle at a distance r from the centre of force is given by

$$v^2 = \mu \left(\frac{2}{r} + \frac{1}{a} \right), \qquad ...(1)$$

where $2a$ is the transverse axis of the hyperbola.

Since the particle is projected with velocity V at a distance R, therefore from (1), we have

$$V^2 = \mu \left(\frac{2}{R} + \frac{1}{a} \right) \quad \text{or} \quad \frac{\mu}{a} = V^2 - \frac{2\mu}{R}. \qquad ...(2)$$

If α is the required angle of projection to describe a rectangular hyperbola, then at the point of projection from the relation $h = vp$, we have

$$h = Vp = VR \sin \alpha \qquad ...(3)$$

[$\because$ $p = r \sin \phi$ and initially $r = R, \phi = \alpha$]

Also $\qquad h = \sqrt{(\mu l)} = \sqrt{\{(\mu \cdot (b^2/a)\}} + \sqrt{(\mu a)} \qquad ...(4)$

[$\because$ $b = a$ for a rectangular hyperbola]

From (3) and (4), we have

$$VR \sin \alpha = \sqrt{(\mu a)}$$

or $$\sin \alpha = \frac{\sqrt{(\mu a)}}{VR} = \frac{\mu \sqrt{a}}{VR \sqrt{\mu}} = \frac{\mu}{VR \sqrt{(\mu/a)}}$$

Substituting for μ/a from (2), we have

$$\sin \alpha = \mu / \{VR \sqrt{(V^2 - 2\mu/R)}\}$$

or $$\alpha = \sin^{-1} [\mu / \{VR \sqrt{(V^2 - 2\mu/R)}\}],$$

which is the required angle of projection.

Comprehensive Exercise 1

Find the law of force towards the pole under which the following curves are described :

1. (i) $au = e^{n\theta}$ (ii) $r = ae^{\theta \cot \alpha}$

(Bundelkhand 2006 Kanpur 08; Agra 11)

(iii) $r^n \cos n\theta = a^n$. **(Avadh 2008; Kanpur 11)**

2. (i) $a = r \cosh n\theta$

(ii) $a = r \tanh (\theta / \sqrt{2})$. **(Kumaun 2003; Rohilkhand 07)**

3. (i) $r^2 = 2ap$ **(Agra 2007)**

(ii) $p^2 = ar$. **(Garhwal 2004; Kanpur 07, 10)**

4. A particle describes the curve $r^2 = a^2 \cos 2\theta$ under a force to the pole. Find the law of force. **(Kanpur 2010)**

5. A particle describes a circle, pole on its circumference, under a force P to the pole. Find the law of force.

Or

A particle describes the curve $r = 2a \cos \theta$ under a force P to the pole. Find the law of force. **(Bundelkhand 2008)**

6. A particle describes the curve $r = a \sin n\theta$ under a force P to the pole. Find the law of force. **(Rohilkhand 2006; Kanpur 09; Bundelkhand 09, 10)**

7. A particle describes the curve $r^2 = a^2 \cos^2 \theta + b^2 \sin^2 \theta$ under an attraction to the origin, prove that the attraction at the distance r is

$$h^2 [2 (a^2 + b^2) r^2 - 3a^2 b^2] r^{-7}.$$

8. Show that the only law for a central attraction for which the velocity in a circle at any distance is equal to the velocity acquired in falling from infinity to the distance is that of inverse cube.

9. In a central orbit described under a force to a centre, the velocity at any point is inversely proportional to the distance of the point from the centre of force. Show that the path is an equiangular spiral.

10. A particle of unit mass describes an equiangular spiral of angle α, under a force which is always in the direction perpendicular to the straight line joining the particle to the pole of the spiral; show that the force is $\mu r^{2 \sec^2 \alpha - 3}$ and that the rate of description of sectorial area about the pole is $\frac{1}{2} \sqrt{(\mu \sin \alpha \cos \alpha)}\, r^{\sec^2 \alpha}$.

Answers 1

1. (i) $P \propto 1/r^3$ (ii) $P \propto 1/r^3$ (iii) $P \propto r^{2n-3}$
2. (i) $P \propto 1/r^3$ (ii) $P \propto 1/r^5$
3. (i) $P \propto 1/r^5$ (ii) $P \propto 1/r^2$
4. $P \propto 1/r^7$ 5. $P \propto 1/r^5$
6. $P \propto \{2 n^2 a^2 / r^5 - (n^2 - 1)/r^3\}$

5.9 Apse and Apsidal Distance

1. **Apse:** *An apse is a point on the central orbit at which the radius vector from the centre of force to the point has a maximum or minimum value.*

(Lucknow 2007, 10; Rohilkhand 07; Avadh 09)

2. **Apsidal Distance:** *The length of the radius vector at an apse is called an apsidal distance.*

(Locknow 2007, 10; Rohilkhand 07; Avadh 09)

3. **Apsidal Angle:** *The angle between two consecutive apsidal distances is called an apsidal angle.*

Theorem: *At an apse the radius vector is perpendicular to the tangent i.e., at an apse the particle moves at right angles to the radius vector.* **(Bundelkhand 2007, 11)**

From the definition of an apse, r is maximum or minimum at an apse *i.e.*, $u = 1/r$ is minimum or maximum at an apse.

$\therefore$ at an apse, $du/d\theta = 0$.

But we know that $\dfrac{1}{p^2} = u^2 + \left(\dfrac{du}{d\theta}\right)^2$.

$\therefore$ at an apse, $\dfrac{1}{p^2} = u^2 = \dfrac{1}{r^2}$

or $p = r$ or $r \sin \phi = r$

or $\sin \phi = 1$ or $\phi = 90°$.

This proves that at an apse the radius vector is perpendicular to the tangent or in other words at an apse the particle moves at right angles to the radius vector.

Remember: At an apse $dr/d\theta = 0, du/d\theta = 0, \phi = 90°, p = r$ and the direction of motion is at right angles to the radius vector.

5.10 Property of the Apse-Line

Theorem: *If the central acceleration P is a single valued function of the distance, every apse-line divides the orbit into equal and symmetrical portions, and thus there can only be two apsidal distances.*

Proof: Since the central acceleration P is a single valued function of r, therefore the acceleration of the particle is the same at the same distance r.

The differential equation of a central orbit is

$$\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2 u^2} \quad \text{or} \quad h^2 \left[\frac{d^2u}{d\theta^2} + u\right] = \frac{P}{u^2}.$$

Multiplying both sides by 2 $(du / d\theta)$ and integrating w.r.t. 'θ', we have

$$v^2 = h^2 \left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] = 2 \int \frac{P}{u^2}\, du + C,$$

or $$v^2 = C - 2 \int P\, dr. \quad \text{...(1)}$$

$$\left\{\because \frac{1}{u} = r \Rightarrow -\frac{1}{u^2}\, du = dr\right\}$$

The equation (1) shows that if P is a single valued function of the distance r, then the velocity of the particle is the same at the same distance r and is independent of the direction of motion.

Thus we observe that both velocity and acceleration are the same at the same distance from the centre. Therefore if at an apse the direction of velocity is reversed, the particle will describe symmetrical orbit on both sides of the apse-line.

Now when the particle comes to a second apse, the path for the same reasons, is symmetrical about this second apsidal distance also. But this is possible only when the next (third) apsidal distance is equal to the one (first) before it and the angle between the first and the second apsidal distances is the same as the angle between the second and the third apsidal distances. Therefore if the central acceleration is a single valued function of the distance r, there are only two different apsidal distances. Also the angle between any two consecutive apsidal distances always remains the same and is called the *apsidal angle.*

5.11 Condition for Two Apsidal Distances

To prove analytically that when the central acceleration varies as some integral power of the distance, there are at most two apsidal distances.

Let the central acceleration P be given by

$P = \mu r^{n}$, where n is an integer.

Thus $P = \mu u^{-n}$ because $r = 1/u = u^{-1}$.

$\therefore$ the differential equation of the path is

$$h^2 \left\{ u + \frac{d^2 u}{d\theta^2} \right\} = \frac{P}{u^2} = \frac{\mu u^{-n}}{u^2} = \mu u^{-(n+2)}.$$

Multiplying both sides by $2\,(du/d\theta)$ and then integrating, we have

$$h^2 \left\{ u^2 + \left(\frac{du}{d\theta} \right)^2 \right\} = \frac{\mu u^{-(n+1)}}{-(n+1)} + A. \qquad \text{...(1)}$$

But at an apse $du/d\theta = 0$. So putting $du/d\theta = 0$ in (1), we have

$$h^2\, u^2 = -\frac{\mu}{n+1}\, u^{-(n+1)} + A$$

or

$$r^{n+3} - \frac{(n+1)}{\mu} A r^2 + \frac{(n+1)}{\mu} h^2 = 0.$$

Whatever be the values of n or A this equation cannot have more than two changes of sign. Therefore by Descarte's rule of signs it cannot have more than two positive roots. Hence there are at most two positive values of r *i.e.*, at most two apsidal distances.

5.12 Given the Law of Force, to Find the Orbit

This problem is converse to that given in 5.8. For solving such a problem we substitute the given expression for P in the differential equation

$$h^2 \left[\frac{d^2 u}{d\theta^2} + u \right] = \frac{P}{u^2} \qquad \text{...(1)}$$

or $$\frac{h^2}{p^3}\frac{dp}{dr} = P, \qquad ...(2)$$

whichever is convenient. In case the force is repulsive, we take the value of P with negative sign.

Then integrating the resulting differential equation of the central orbit with the help of the given initial conditions, we get the (r, θ) or (p, r) equation of the orbit.

Illustrative Examples

Example 6: *A particle moving with a central acceleration μ / (distance)^3 is projected from an apse at a distance a with a velocity V; show that the path is*

$$r \cosh\left\{\frac{\sqrt{(\mu - a^2V^2)}}{aV}\theta\right\} = a \quad or \quad r \cos\left\{\frac{\sqrt{(a^2V^2 - \mu)}}{aV}\theta\right\} = a$$

according as V is < or > the velocity from infinity. **(Rohilkhand 2008)**

Solution: Here, the central acceleration P

$$= \frac{\mu}{(\text{distance})^3} = \frac{\mu}{r^3} = \mu u^3.$$

The differential equation of the path is

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{\mu u^3}{u^2} = \mu u.$$

Multiplying both sides by $2\,(du/d\theta)$ and integrating, we have

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu u^2 + A, \qquad ...(1)$$

where A is a constant.

But initially when $r = a$ *i.e.*, $u = \frac{1}{a}$, $\frac{du}{d\theta} = 0$ (at an apse) and $v = V$.

$\therefore$ from (1), $V^2 = h^2\left[\frac{1}{a^2}\right] = \frac{\mu}{a^2} + A.$

$$\therefore \qquad h^2 = a^2V^2 \text{ and } A = V^2 - \frac{\mu}{a^2} = \frac{(V^2a^2 - \mu)}{a^2}.$$

Substituting the values of h^2 and A in (1), we have

$$a^2V^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu u^2 + \frac{(V^2a^2 - \mu)}{a^2}$$

or $$a^2V^2\left(\frac{du}{d\theta}\right)^2 = -a^2V^2u^2 + \mu u^2 + \frac{(V^2a^2 - \mu)}{a^2}$$

$$= -(a^2V^2 - \mu)u^2 + (a^2V^2 - \mu)/a^2$$

$$= (a^2 V^2 - \mu)(-u^2 + 1/a^2)$$

or $$a^4 V^2 \left(\frac{du}{d\theta}\right)^2 = (a^2 V^2 - \mu)(1 - a^2 u^2). \qquad ...(2)$$

If V_1 is the velocity acquired by the particle in falling form infinity to the distance a, then

$$V_1^{\ 2} = -2 \int_\infty^a P\, dr = -2 \int_\infty^a \frac{\mu}{r^3}\, dr = -2 \left[-\frac{\mu}{2r^2} \right]_\infty^a = \frac{\mu}{a^2}.$$

Case I. When $V < V_1$ (velocity from infinity), we have

$$V^2 < V_1^2 \quad \text{or} \quad V^2 < \mu/a^2 \quad \text{or} \quad a^2 V^2 < \mu \quad \text{or} \quad \mu - a^2 V^2 > 0.$$

$\therefore$ from (2), we have

$$a^4 V^2 \left(\frac{du}{d\theta}\right)^2 = (\mu - a^2 V^2)(a^2 u^2 - 1)$$

or $$a^2 V \frac{du}{d\theta} = \sqrt{(\mu - a^2 V^2)} . \sqrt{(a^2 u^2 - 1)}$$

or $$\frac{\sqrt{(\mu - a^2 V^2)}}{aV} d\theta = \frac{a\, du}{\sqrt{(a^2 u^2 - 1)}}.$$

Substituting $au = z$, so that $a\, du = dz$, we have

$$\frac{\sqrt{(\mu - a^2 V^2)}}{aV} d\theta = \frac{dz}{\sqrt{(z^2 - 1)}}.$$

Integrating, $$\frac{\sqrt{(\mu - a^2 V^2)}}{aV} \theta + B = \cosh^{-1} z$$

or $$\frac{\sqrt{(\mu - a^2 V^2)}}{aV} \theta + B = \cosh^{-1}(au).$$

But initially when $u = 1/a, \theta = 0$.

$\therefore$ $$0 + B = \cosh^{-1} 1 = 0 \quad \text{or} \quad B = 0.$$

$\therefore$ $$\frac{\sqrt{(\mu - a^2 V^2)}}{aV} \theta = \cosh^{-1}(au)$$

or $$au = \frac{a}{r} = \cosh \left\{ \frac{\sqrt{(\mu - a^2 V^2)}}{aV} \theta \right\}$$

or $$r \cosh \left\{ \frac{\sqrt{(\mu - a^2 V^2)}}{aV} \theta \right\} = a.$$

Case II. When $V > V_1$ (velocity from infinity), we have

$$V^2 > V_1^{\ 2} \quad \text{or} \quad V^2 > \mu/a^2 \quad \text{or} \quad a^2 V^2 - \mu > 0.$$

$\therefore$ from (2), we have

$$a^4 V^2 \left(\frac{du}{d\theta}\right)^2 = (a^2 V^2 - \mu)(1 - a^2 u^2)$$

or $$a^2 V \left(\frac{du}{d\theta}\right) = \sqrt{(a^2 V^2 - \mu)} . \sqrt{(1 - a^2 u^2)}$$

or $$\frac{\sqrt{(a^2 V^2 - \mu)}}{aV} d\theta = \frac{a\, du}{\sqrt{(1 - a^2 u^2)}}.$$

Integrating, $$\frac{\sqrt{(a^2 V^2 - \mu)}}{aV} \theta + C = \sin^{-1}(au).$$

But initially when $u = 1/a, \theta = 0$.

$\therefore$ $$0 + C = \sin^{-1} 1 \quad \text{or} \quad C = \pi/2.$$

$\therefore$ $$\frac{\sqrt{(a^2 V^2 - \mu)}}{aV} \theta + \frac{\pi}{2} = \sin^{-1}(au)$$

or $$au = \frac{a}{r} = \sin\left\{\frac{\sqrt{(a^2 V^2 - \mu)}}{aV} \theta + \frac{\pi}{2}\right\}$$

or $$a = r \cos\left\{\frac{\sqrt{(a^2 V^2 - \mu)}}{aV} \theta\right\}.$$

Example 7: *A particle moves with a central acceleration which varies inversely as the cube of the distance. If it be projected from an apse at a distance a from the origin with a velocity which is $\sqrt{2}$ times the velocity for a circle of radius a, show that the equation to its path is $r \cos(\theta/\sqrt{2}) = a$.*

(Kanpur 2007; Rohilkhand 10; Purvanchal 08)

Solution: Here the central acceleration varies inversely as the cube of the distance *i.e.*, $P = \mu / r^3 = \mu u^3$, where μ is a constant.

If V is the velocity for a circle of radius a, then

$$\frac{V^2}{a} = [P]_{r = a} = \frac{\mu}{a^3}$$

or $$V = \sqrt{(\mu / a^2)}.$$

$\therefore$ the velocity of projection $v_1 = \sqrt{2} V = \sqrt{(2\mu / a^2)}$.

The differential equation of the path is

$$h^2 \left[u + \frac{d^2 u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{\mu u^3}{u^2} = \mu u.$$

Multiplying both sides by $2\,(du/d\theta)$ and integrating, we have

$$v^2 = h^2 \left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu u^2 + A, \qquad \text{...(1)}$$

where A is a constant.

But initially when $r = a$ *i.e.*, $u = 1/a$, $du/d\theta = 0$ (at an apse), and $v = v_1 = \sqrt{(2\mu / a^2)}$.

$\therefore$ from (1), we have

$$\frac{2\mu}{a^2} = h^2 \left[\frac{1}{a^2}\right] = \frac{\mu}{a^2} + A.$$

$\therefore$ $$h^2 = 2\mu \quad \text{and} \quad A = \mu / a^2.$$

Substituting the values of h^2 and A in (1), we have

$$2\mu\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu u^2+\frac{\mu}{a^2}$$

or $$2\left(\frac{du}{d\theta}\right)^2=\frac{1}{a^2}+u^2-2u^2=\frac{1-a^2u^2}{a^2}$$

or $$\sqrt{2}\,a\frac{du}{d\theta}=\sqrt{(1-a^2u^2)} \quad \text{or} \quad \frac{d\theta}{\sqrt{2}}=\frac{a\,du}{\sqrt{(1-a^2u^2)}}.$$

Integrating, $(\theta/\sqrt{2})+B=\sin^{-1}(au)$, where B is a constant.

But initially, when $u=1/a, \theta=0$.

$\therefore$ $$B=\sin^{-1}1=\frac{1}{2}\pi.$$

$\therefore$ $$(\theta/\sqrt{2})+\frac{1}{2}\pi=\frac{1}{2}\sin^{-1}(au)$$

or $$au=a/r=\sin\{\frac{1}{2}\pi+(\theta/\sqrt{2})\}$$

or $a=r\cos(\theta/\sqrt{2})$, which is the required equation of the path.

Example 8: *A particle subject to a force producing an acceleration* $\mu\,(r+2a)/r^5$ *towards the origin is projected from the point* $(a, 0)$ *with a velocity equal to the velocity from infinity at an angle* $\cot^{-1}2$ *with the initial line; show that the equation to the path is*

$$r=a\,(1+2\sin\theta).$$ **(Avadh 2008)**

Solution: Here, the central acceleration

$$P=\frac{\mu\,(r+2a)}{r^5}$$
$$=\mu\left(\frac{1}{r^4}+\frac{2a}{r^5}\right)=\mu\,(u^4+2au^5).$$

Let V be the velocity of the particle acquired in falling from rest from infinity under the same acceleration to the point of projection which is at a distance a from the centre. Then

$$V^2=-2\int_\infty^a P\,dr=-2\int_\infty^a\mu\left(\frac{1}{r^4}+\frac{2a}{r^5}\right)dr$$
$$=-2\mu\left[-\frac{1}{3r^3}-\frac{2a}{4r^4}\right]_\infty^a=2\mu\left[\frac{1}{3a^3}+\frac{1}{2a^3}\right]=\frac{5\mu}{3a^3}$$

or $$V=\sqrt{(5\mu/3a^3)}.$$

According to the question the velocity of projection of the particle is equal to V *i.e.*, $\sqrt{(5\mu/3a^3)}$.

Now the differential equation of the path is

$$h^2\left[u^2+\frac{d^2u}{d\theta^2}\right]=\frac{P}{u^2}=\frac{\mu}{u^2}(u^4+2au^5)=\mu\,(u^2+2au^3).$$

Multiplying both sides by $2\,(du/d\theta)$ and integrating, we have

$$v^2 = h^2 \left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu \left(\frac{2u^2}{3} + au^4\right) + A, \qquad \text{...(1)}$$

where A is a constant.

Initially when $r = a$ *i.e.*, $u = 1/a$, $v = (5\mu/3a^3)$.

Also initially $\phi = \cot^{-1} 2$ or $\cot\phi = 2$ or $\sin\phi = 1/\sqrt{5}$.

But $p = r\sin\phi$. Therefore initially $p = a\,(1/\sqrt{5}) = a/\sqrt{5}$ or $1/p^2 = 5/a^2$.

But $1/p^2 = u^2 + (du/d\theta)^2$. Therefore initially, when $r = a$, we have

$$u^2 + (du/d\theta)^2 = 5/a^2.$$

Applying the above initial conditions in (1), we have

$$\frac{5\mu}{3a^3} = h^2\,\frac{5}{a^2} = \mu\left(\frac{2}{3a^3} + \frac{a}{a^4}\right) + A.$$

$\therefore$ $$h^2 = \mu/3a,\ A = 0.$$

Substituting the values of h^2 and A in (1), we have

$$\frac{\mu}{3a}\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\left(\frac{2}{3}u^3 + au^4\right)$$

or $$\left(\frac{du}{d\theta}\right)^2 = 2au^3 + 3a^2u^4 - u^2.$$

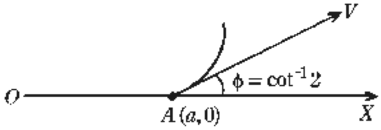
Putting $u = \dfrac{1}{r}$, so that $\dfrac{du}{d\theta} = -\dfrac{1}{r^2}\dfrac{dr}{d\theta}$, we have

$$\left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 = \frac{2a}{r^3} + \frac{3a^2}{r^4} - \frac{1}{r^2}$$

or $$(dr/d\theta)^2 = 2ar + 3a^2 - r^2 = 3a^2 - (r^2 - 2ar)$$
$$= 3a^2 - (r-a)^2 + a^2 = 4a^2 - (r-a)^2$$

or $$dr/d\theta = \sqrt{[(2a)^2 - (r-a)^2]}$$

[Note that as the particle starts moving from A, r increases as θ increases. So we have taken $dr/d\theta$ with +ive sign.]



or $$d\theta = \frac{dr}{\sqrt{[(2a)^2 - (r-a)^2]}}.$$

Integrating, $\theta + B = \sin^{-1}\left(\dfrac{r-a}{2a}\right)$.

But initially when $r = a, \theta = 0$.

$\therefore$ $$B = \sin^{-1} 0 = 0.$$

$\therefore$ $$\theta = \sin^{-1}\left(\frac{r-a}{2a}\right) \quad \text{or} \quad \sin\theta = \frac{r-a}{2a}$$

or $r = a\,(1 + 2\sin\theta)$, which is the required equation of the path.

Example 9: *A particle of mass m moves under a central attractive force* $m\mu\,(5/r^3 + 8c^2/r^5)$, *and is projected from an apse at a distance c with velocity* $3\sqrt{\mu}/c$, *prove that the orbit is* $r = c\cos(2\theta/3)$, *and that it will arrive at the origin after a time* $\pi c^2/(8\sqrt{\mu})$.

Solution: Here, the central acceleration

$$P = \mu\,(5/r^3 + 8c^2/r^5) = \mu\,(5u^3 + 8c^2u^5).$$

The differential equation of the central orbit is

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{\mu}{u^2}(5u^3 + 8c^2u^5) = \mu\,(5u + 8c^2u^3).$$

Multiplying both sides by $2\,(du/d\theta)$ and integrating, we have

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu(5\mu^2 + 4c^2u^4) + A \qquad \text{...(1)}$$

But initially at an apse, $r = c, u = 1/c, du/d\theta = 0, v = 3\sqrt{\mu}/c$.

$\therefore$ from (1), we have

$$\frac{9\mu}{c^2} = h^2 \cdot \frac{1}{c^2} = \mu\left(\frac{5}{c^2} + \frac{4}{c^2}\right) + A.$$

$\therefore$ $h^2 = 9\mu,\ A = 0.$

Substituting the values of h^2 and A in (1), we have

$$9\mu\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu(5\mu^2 + 4c^2u^4)$$

or $\quad 9\,(du/d\theta)^2 = 4\,c^2u^4 - 4u^2.$

Putting $u = \dfrac{1}{r}$, so that $\dfrac{du}{d\theta} = -\dfrac{1}{r^2}\dfrac{dr}{d\theta}$, we have

$$9\,\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = 4\left(\frac{c^2}{r^4} - \frac{1}{r^2}\right)$$

or $\quad 9\,(dr/d\theta)^2 = 4\,(c^2 - r^2)$ or $dr/d\theta = -\dfrac{2}{3}\sqrt{(c^2 - r^2)}$

or $\quad \dfrac{2}{3}\,d\theta = \dfrac{-\,dr}{\sqrt{(c^2 - r^2)}}.$

Integrating, $2\theta/3 + B = \cos^{-1}(r/c)$, where B is a constant.

Initially when $r = c$, let $\theta = 0$. Then $B = \cos^{-1} 1 = 0$.

$\therefore$ $\quad 2\theta/3 = \cos^{-1}(r/c)$ or $r/c = \cos(2\theta/3)$

or $\quad r = c\cos(2\theta/3)$, which is the required equation of the path.

Second Part: We have $h = r^2\,(d\theta/dt)$.

But $h = 3\sqrt{\mu}$ and $r = c\cos(2\theta/3)$, as found above.

$$\therefore \qquad dt = \frac{r^2}{h}\, d\theta = \frac{c^2 \cos^2 (2\theta/3)}{3\sqrt{\mu}}\, d\theta. \qquad ...(2)$$

At the point of projection, we have taken $\theta = 0$. Also at the point $O, r = 0$. Putting $r = 0$ in the equation of the path, we get $0 = c \cos (2\theta/3)$ giving $2\theta/3 = \frac{1}{2}\pi$ or $\theta = 3\pi/4$. So at $O, \theta = 3\pi/4$. Let t_1 be the time from the point of projection to the point O. Then integrating (2), we have

$$t_1 = \int_0^{3\pi/4} \frac{c^2}{3\sqrt{\mu}} \cos^2 (2\theta/3)\, d\theta.$$

Put $2\theta/3 = z$, so that $(2/3)\, d\theta = dz$. When $\theta = 0, z = 0$ and when

$$\theta = \frac{3\pi}{4}, z = \frac{1}{2}\pi.$$

$$\therefore \qquad t_1 = \int_0^{\pi/2} \frac{c^2}{3\sqrt{\mu}} (\cos^2 z) \cdot \left(\frac{3}{2}\right) dz = \frac{c^2}{2\sqrt{\mu}} \int_0^{\pi/2} \cos^2 z\, dz$$

$$= \frac{c^2}{2\sqrt{\mu}} \cdot \frac{1}{2} \cdot \frac{1}{2}\pi = \frac{\pi c^2}{8\sqrt{\mu}}.$$

Example 10: *A particle is attached to a fixed point on a smooth horizontal plane by an elastic string of natural length a. Initially the particle is at rest on the plane with the string just taut and it is projected horizontally in a direction perpendicular to the string with a kinetic energy equal to the potential energy of the string when its extension is $3a/\sqrt{2}$. Prove that the second apsidal distance is equal to 3a.*

Solution: By Hooke's law, the tension in the string when its extension is $3a/\sqrt{2}$

$$= \lambda \cdot \frac{3a/\sqrt{2}}{a} = \frac{3\lambda}{\sqrt{2}},$$

where λ is the modulus of elasticity of the string.

We know that the potential energy of an elastic string in any stretched position

$$= \frac{1}{2}\text{(initial tension + final tension)} \times \text{extension.}$$

$\therefore$ the potential energy of the string when its extension is $3a/\sqrt{2}$

$$= \frac{1}{2}\left[0 + \frac{3\lambda}{\sqrt{2}}\right] \times \frac{3a}{\sqrt{2}} = \frac{9a\lambda}{4}.$$

[Note that the initial tension is zero]

If V is the velocity of projection of the particle, then its kinetic energy at that time

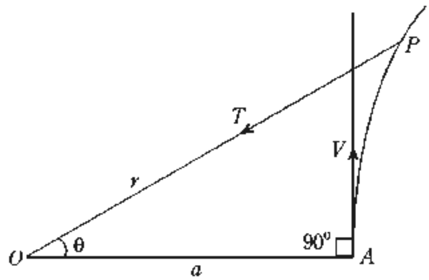
$$= \frac{1}{2} mV^2.$$

According to the question,

$$\frac{1}{2} mV^2 = \frac{9a\lambda}{4} \quad \text{or} \quad V^2 = \frac{9a\lambda}{2m}$$

or
$$V = \sqrt{\left(\frac{9a\lambda}{2m}\right)}.$$

Now suppose the particle is initially at A, where $OA = a$ = natural length of the string. The particle is projected from A perpendicular to OA with velocity $V = \sqrt{(9a\lambda / 2m)}$. Let P be the position of the particle at any time t, where $OP = r$. The only force acting on the particle at P in the plane of motion is the tension T in the string OP and is always directed towards the fixed centre O. By Hooke's law,



$$T = \lambda \frac{OP - a}{a} = \lambda \frac{r - a}{a}.$$

$\therefore$ P = the central acceleration of the particle at the point P

$$= \frac{T}{m} = \frac{\lambda}{am}(r - a) = \frac{\lambda}{am}\left(\frac{1}{u} - a\right).$$

The differential equation of the particle is

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{\lambda}{am}\left(\frac{1}{u^3} - \frac{a}{u^2}\right).$$

Multiplying both sides by $2\,(du/d\theta)$ and integrating, we have

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \frac{\lambda}{am}\left(-\frac{1}{u^2} + \frac{2a}{u}\right) + A. \qquad ...(1)$$

Now the point A is an apse. So initially at A, $r = a, u = 1/a, du/d\theta = 0, v = \sqrt{(9a\lambda/2m)}$.

$\therefore$ from (1), we have

$$\frac{9a\lambda}{2m} = h^2 \cdot \frac{1}{a^2} = \frac{\lambda}{am}(-a^2 + 2a^2) + A.$$

$$\therefore \quad h^2 = \frac{9a^3\lambda}{2\,m},\ A = \frac{7a\lambda}{2\,m}.$$

Substituting the values of h^2 and A in (1), we get

$$\frac{9a^3\lambda}{2m} = \left\{u^2 + \left(\frac{du}{d\theta}\right)^2\right\} = \frac{\lambda}{am}\left(-\frac{1}{u^2} + \frac{2a}{u}\right) + \frac{7a\lambda}{2m}.$$

Putting $du/d\theta = 0$, the apsidal distances are given by

$$\frac{9}{2}a^3u^2 = -\frac{1}{au^2} + \frac{2}{u} + \frac{7a}{2}$$

or $$\frac{9a^3}{2r^2} = -\frac{r^2}{a} + 2r + \frac{7a}{2}$$

or $$9a^4 - 7a^2r^2 - 4ar^3 + 2r^4 = 0$$

or $$2r^4 - 4ar^3 - 7a^2r^2 + 9a^4 = 0$$

or $$(r - a)(r - 3a)(2r^2 + 4ar + 3a^2) = 0.$$

Here $r = a, r = 3a$ are + ive real roots. But $r = a$ is the given apsidal distance. Therefore $r = 3a$ is the other apsidal distance.

Comprehensive Exercise 2

1. **(a)** A particle moves in a plane under a central force which varies inversely as the square of the distance from the fixed point, find the orbit.

   (b) If the central force varies inversely as the cube of the distance from a fixed point, find the orbit.

2. A particle moves with a central acceleration $\mu\left(r + \frac{a^4}{r^3}\right)$ being projected from an apse at a distance a with a velocity $2a\sqrt{\mu}$. Prove that it describes the curve $r^2 (2 + \cos\sqrt{3}\theta) = 3a^2$.

3. A particle subject to the central acceleration $(\mu/r^3) + f$ is projected from an apse at a distance 'a' with the velocity $= \sqrt{\mu}/a$; prove that at any subsequent time t,
$$r = a - \frac{1}{2} f t^2.$$

4. A particle moves under a force
$$m\mu\ \{3au^4 - 2(a^2 - b^2)u^5\}, a > b$$
and is projected from an apse at a distance $(a + b)$ with velocity $\sqrt{\mu}/(a + b)$. Show that the equation of its path is $r = a + b\cos\theta$.
(Garhwal 2002; Lucknow 06, 09, 11; Purvanchal 08)

5. A particle moves under a repulsive force $m\mu$ / (distance)^3 and is projected from an apse at a distance a with a velocity V; show that the equation to the path is $r\cos p\theta = a$, and that the angle θ described in time t is $(1/p)\tan^{-1}(pVt/a)$, where $p^2 = (\mu + a^2V^2)/(a^2V^2)$.

6. A particle moves under a central force $m\lambda\ (3a^3u^4 + 8au^2)$. It is projected from an apse at a distance a from the centre of force with velocity $\sqrt{(10\lambda)}$. Show that the second apsidal distance is half of the first, and that the equation to the path is
$$2r = a[1 + \text{sech}(\theta/\sqrt{5})].$$

7. A particle subject to a central force per unit of mass equal to
$$\mu\ \{2(a^2 + b^2)u^5 - 3a^2b^2u^7\}$$
is projected at the distance a with velocity $\sqrt{\mu}/a$ in a direction at right angles to the initial distance; show that the path is the curve
$$r^2 = a^2\cos^2\theta + b^2\sin^2\theta.$$

8. A particle moves with a central acceleration $\lambda^2\ (8au^2 + a^4u^5)$; it is projected with velocity 9λ from an apse at a distance $a/3$ from the origin; show that the equation to its path is

$$\frac{1}{\sqrt{3}}\sqrt{\left(\frac{au+5}{au-3}\right)} = \cot\left(\frac{\theta}{\sqrt{6}}\right).$$

9. A particle, acted on by a repulsive central force $\mu r/(r^2 - 9c^2)^2$, is projected from an apse at a distance c with velocity $\sqrt{(\mu/8c^2)}$. Find the equation of its path and show that the time to the cusp is $\frac{4}{3}\pi c^2 \sqrt{(2/\mu)}$.

10. A particle is moving with central acceleration $\mu\ (r^5 - c^4 r)$ being projected from an apse at a distance c with velocity $c^3 \sqrt{(2\mu/3)}$, show that its path is the curve $x^4 + y^4 = c^4$. **(Agra 2001; Garhwal 03; Bundelkhand 07; Rohilkhand 09; Purvanchal 10)**

11. If the law of force be $\mu\ (u^4 - \frac{10}{9} au^5)$ and the particle be projected from an apse at a distance $5a$ with a velocity equal to $\sqrt{(5/7)}$ of that in a circle at the same distance, show that the orbit is the limacon $r = a\ (3 + 2\cos\theta)$. **(Garhwal 2001, 03)**

12. **(a)** A particle is projected from an apse at a distance a with the velocity from infinity under the action of a central acceleration μ/r^{2n+3}. Prove that the equation of the path is $r^n = a^n \cos n\theta$. **(Purvanchal 2009; Avadh 11)**

(b) A particle is projected from an apse at a distance a with velocity of projection $\sqrt{\mu}/(a^2\sqrt{2})$ under the action of a central force μu^5. Prove that the path is the circle $r = a\cos\theta$. **(Agra 2011)**

(c) A particle is projected from an apse at a distance a with the velocity from infinity, the acceleration being μu^7; show that the equation to its path is $r^2 = a^2 \cos 2\theta$. **(Lucknow 2008)**

(d) If the central force varies as the cube of the distance from a fixed point then find the orbit.

13. A particle moving under a constant force from a centre is projected at a distance a from the centre in a direction perpendicular to the radius vector with velocity acquired in falling to the point of projection from the centre, show that its path is $(a/r)^3 = \cos^2 (3\theta/2)$.

Also show that the particle will ultimately move in a straight line through the origin in the same way as if its path had always been this line.

If the velocity of projection be double that in the previous case show that the path is

$$\frac{\theta}{2} = \tan^{-1}\sqrt{\left(\frac{r-a}{a}\right)} - \frac{1}{\sqrt{3}}\tan^{-1}\sqrt{\left(\frac{r-a}{3a}\right)}.$$

14. **(a)** A particle moves with a central acceleration $\mu/(\text{distance})^5$ and projected from the apse at a distance a with a velocity equal to n times that which would be

acquired in falling from infinity; show that the other apsidal distance is $a/\sqrt{(n^2-1)}$.

(Purvanchal 2011)

If $n=1$ and particle be projected in any direction, show that the path is a circle passing through the centre of force. **(Rohlikhand 2011)**

(b) If the acceleration at a distance r is μ/r^5 and the particle is projected at a distance a from the centre of force with velocity $\sqrt{(\mu/2a^4)}$, prove that the orbit is a circle through O of diameter a cosec α, where α is the inclination of the direction of projection to the radius vector.

(c) A particle moves under a central attractive force varying inversely as the fifth power of the distance from the centre of force. It is projected from an apse at a distance a with velocity equal to $\sqrt{5}$ times of that which would be acquired in falling from infinity, show that the other apsidal distance is $a/2$.

[**Hint**. Proceed as in part (a), here $n=\sqrt{5}$]

15. A particle describes an orbit with a central acceleration $\mu u^3-\lambda u^5$ being projected from an apse at a distance a with velocity equal to that from infinity. Show that its path is $r=a\cosh(\theta/n)$, where $n^2+1=2\mu a^2/\lambda$.

Prove also that it will be at a distance r at the end of time

$$\sqrt{\left(\frac{a^2}{2\lambda}\right)}\left[a^2\log\left\{\frac{r^2+\sqrt{(r^2-a^2)}}{a}\right\}+r\sqrt{(r^2-a^2)}\right].$$

16. A particle is acted on by a central repulsive force which varies as the nth power of the distance. If the velocity at any point be equal to that which would be acquired in falling from the centre to the point, show that the equation to the path is of the form

$$r^{(n+3)/2}\cos\frac{1}{2}(n+3)\theta=\text{constant}.$$

17. In a central orbit the force is $\mu u^3(3+2a^2u^2)$; if the particle be projected at a distance a with a velocity $\sqrt{(5\mu/a^2)}$ in a direction making an angle $\tan^{-1}(\frac{1}{2})$ with the radius vector, show that the equation to the path is $r=a\tan\theta$.

(Kanpur 2011)

18. A particle moves with a central acceleration $\mu\left(u^5-\frac{1}{8}a^2u^7\right)$; it is projected at a distance a with a velocity $\sqrt{(25/7)}$ times the velocity for a circle at that distance and at an inclination $\tan^{-1}(4/3)$ to the radius vector, show that its path is the curve

$$4r^2-a^2=3a^2/(1-\theta)^2.$$

19. A particle of mass m moves under a central force $m\mu/(\text{distance})^3$ and is projected at a distance a from the centre of force with the velocity which at an angle α to the radius would be acquired by a fall from rest at infinity to the point of projection; prove that the orbit is an equiangular spiral.

20. A particle acted on by a central attractive force μu^3 is projected with a velocity $\sqrt{\mu}/a$ at an angle $\pi/4$ with its initial distance a from the centre of force. Show that its orbit is the equiangular spiral $r=ae^{-\theta}$.

21. A particle moves with a central acceleration $\mu\ (3u^3 + a^2u^5)$ being projected from a distance a at an angle $45°$ with a velocity equal to that in a circle at the same distance. Prove that the time to the centre of force is $a^2\left(2 - \frac{1}{2}\pi\right)/\sqrt{(2\mu)}$.

(Kumaun 2003)

22. A particle moves with central acceleration $\mu\left(r + \frac{2a^3}{r^2}\right)$ being projected from an apse at a distance a with twice the velocity for a circle at that distance, find the other apsidal distance and show that equation to the path is

$$\frac{\theta}{2} = \tan^{-1}(t\sqrt{3}) - \left(\frac{1}{\sqrt{5}}\right)\tan^{-1}[\sqrt{(5/3)}.\, t], \text{ where } t^2 = \frac{(r-a)}{(3a-r)}.$$

23. A particle is projected with velocity $\sqrt{(2\mu/3c^3)}$ from a point P in a field of attractive force μ/r^4 to a point O distant c from P, where r denotes the distance from O.

If the direction of projection makes an angle $45°$ with PO, prove that the orbit is a cardioid and the particle will arrive at O after a time $[(3\pi/4) - 2].\sqrt{(3c^5/\mu)}$.

24. A particle moves in a curve under a central acceleration so that its velocity at any point is equal to that in a circle at the same distance and under the same attraction. Show that the law of force is that of inverse cube, and the path is an equiangular spiral. **(Purvanchal 2009)**

25. A particle moves with central acceleration $(\mu u^2 + \lambda u^3)$ and the velocity of projection at distance R is V; show that the particle will ultimately go off to infinity if $V^2 > \frac{2\mu}{R} + \frac{\lambda}{R^2}$.

26. A particle of mass m is attached to a fixed point by an elastic string of natural length a, the coefficient of elasticity being nmg; it is projected from an apse at a distance a with velocity $\sqrt{(2pgh)}$; show that the other apsidal distance is given by the equation

$$nr^2(r-a) - 2pha(r+a) = 0.$$

Answers 2

1. **(a)** A conic **(b)** An equiangular spiral

9. $8p^2 = 9c^2 - r^2$

12. **(d)** Pedal equation of the orbit is $\frac{h^2}{p^2} = v_0^2 + \frac{\mu r_0^4}{2} - \frac{\mu r^4}{2}$,

where v_0 is the velocity when the particle is at a distance r_0 from the centre of force.

5.13 Planetary Motion

Newton's Law of Gravitation: According to Newton's law of gravitation, *"Every particle of matter attracts every other particle of matter with a force proportional to the product of the masses of the two particles concerned and inversely proportional to the square of the distance between them."* Thus

$$F = G \cdot \frac{m_1 \, m_2}{r^2},$$

where m_1, m_2 are the masses of the particles, r the distance between them, F the gravitational force of attraction and G a constant called the constant of gravitation or the universal constant.

This law holds good in the case of the motion of all planets in the solar system. In particular, the motion of the earth about the sun is governed by this law. Now in this chapter we shall discuss the case of central orbits when the force is an attraction varying inversely as the square of the distance from the centre of force.

5.14 Motion under the Inverse Square Law

To show that the path of a particle which is moving so that its acceleration is always directed to a fixed point and is equal to $\mu / (distance)^2$ *is a conic section and to distinguish between the three cases that arise.*

Here the force is always directed to a fixed point, so it is a case of central orbit.

Also given that the central acceleration $P = \mu / r^2$.

The differential equation of the path (in pedal form) is

$$\frac{h^2}{p^3} \frac{dp}{dr} = P = \frac{\mu}{r^2}.$$

Multiplying both sides by -2, we have

$$\frac{-2h^2}{p^3} dp = \frac{-2\mu}{r^2} dr.$$

Integrating, we have

$$v^2 = \frac{h^2}{p^2} = \frac{2\mu}{r} + B, \qquad \ldots(1)$$

where B is a constant. [Note that in a central orbit, $v = h / p$.]

We know that referred to the focus as pole the pedal equations of ellipse, parabola and hyperbola (that branch which is nearer to the focus taken as pole) are

$$\frac{b^2}{p^2} = \frac{2a}{r} - 1, \quad p^2 = ar \quad \text{and} \quad \frac{b^2}{p^2} = \frac{2a}{r} + 1$$

respectively, where

in the case of ellipse 2 a and 2 b are the lengths of major and minor axes,

in the case of parabola 4 a is the length of latus rectum,

and in the case of hyperbola 2 a and 2 b are the lengths of transverse and conjugate axes.

Now since the equation (1) can be of any of the above three forms, three cases arise here.

Case I. Elliptic Path:

Comparing (1) with $\frac{b^2}{p^2} = \frac{2a}{r} - 1$, the pedal equation of an ellipse, we have

$$\frac{h^2}{b^2} = \frac{\mu}{a} = \frac{B}{-1}.$$

$$\therefore \qquad h^2 = \frac{\mu b^2}{a} \quad \text{and} \quad B = -\frac{\mu}{a}.$$

Substituting in (1), for elliptic path, we have

$$v^2 = \frac{2\mu}{r} - \frac{\mu}{a} = \mu\left(\frac{2}{r} - \frac{1}{a}\right).$$

Obviously here $v^2 < \frac{2\mu}{r}$.

Case II. Parabolic Path. Comparing (1) with $p^2 = ar$, the pedal equation of a parabola, we have

$$\frac{h^2}{1} = \frac{2\mu}{1/a} = \frac{B}{0}.$$

$$\therefore \qquad h^2 = 2\mu a \quad \text{and} \quad B = 0.$$

Substituting in (1), for parabolic path, we have

$$v^2 = \frac{2\mu}{r}.$$

Case III. Hyperbolic Path: Comparing (1) with $\frac{b^2}{p^2} = \frac{2a}{r} + 1$, the pedal equation of a hyperbola, we have

$$\frac{h^2}{b^2} = \frac{\mu}{a} = \frac{B}{1}.$$

$$\therefore \qquad h^2 = \frac{\mu b^2}{a} \quad \text{and} \quad B = \frac{\mu}{a}.$$

Substituting in (1), for hyperbolic path, we have

$$v^2 = \mu\left(\frac{2}{r} + \frac{1}{a}\right).$$

Obviously here $v^2 > \frac{2\mu}{r}$.

Thus from the above three cases, we conclude that the equation (1) always represents a conic section whose focus is at the centre of force. Further the path of the particle is an ellipse, parabola or hyperbola according as B is –ive, zero or +ive. The sign of the value of the constant B depends upon the magnitude of the velocity of the particle at any point. We have found that

if $\quad v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$ or $v^2 < \frac{2\mu}{r}$, then the path is elliptic,

if $\quad v^2 = \frac{2\mu}{r}$, then the path is parabolic,

and if $\quad v^2 = \mu\left(\frac{2}{r} + \frac{1}{a}\right)$ or $v^2 > \frac{2\mu}{r}$, then the path is hyperbolic.

It is to be noted that in each of the three cases the magnitude of the velocity at any point is independent of the direction of the velocity at that point.

Also we have found that

$h^2 = \mu b^2/a = \mu l$ in the case of elliptic path,

$h^2 = 2\,\mu a = \mu l$ in the case of parabolic path,

and $h^2 = \mu b^2/a = \mu l$ in the case of hyperbolic path.

Thus in all the three cases

$h = \sqrt{(\mu l)}$, where l is the length of the semi-latus rectum.

Cor. 1. From the above discussion we see that if a particle is projected from a point at a distance R from the centre of force with velocity V in any direction, then the path is elliptic, parabolic or hyperbolic, according as

$$V^2 < \text{ or } = \text{ or } > \frac{2\mu}{R}.$$

If V_1 is the velocity acquired in falling from infinity to a distance R under the same law of force $P = \frac{\mu}{r^2}$, then as in 5.6, we have

$$V_1^2 = -2\int_\infty^R P\,dr = -2\int_\infty^R \frac{\mu}{r^2}\,dr = 2\mu\left[\frac{1}{r}\right]_\infty^R = \frac{2\mu}{R}.$$

Hence *the path of the particle will be elliptic, parabolic or hyperbolic, according as the velocity at any point < or = or > the velocity from infinity to that point.*

Cor. 2. If V_2 is the velocity for the description of a circle of radius R, then

$$\frac{V_2^2}{R} = \frac{\mu}{R^2} \quad \text{(normal acceleration).}$$

$$V_2 = \sqrt{\left(\frac{\mu}{R}\right)} = \frac{1}{\sqrt{2}}\sqrt{\left(\frac{2\mu}{R}\right)} = \frac{1}{\sqrt{2}}V_1, \qquad \text{[from cor. 1]}$$

where V_1 is the velocity from infinity.

Thus the velocity for the description of a circle of radius R is

$= (1/\sqrt{2})$. *(velocity from infinity to distance R from the centre of force).*

5.15 Kepler's Laws of Planetary Motion

(Agra 2006; Avadh 06; Lucknow 08, 10; Rohilkhand 09, 10, 11)

The laws according to which the planets move with reference to the sun were discovered by the astronomer **John Kepler.** These laws are as follows :

(i) *Each planet describes an ellipse having the sun as one of its foci.*

(ii) *The radius vector drawn from the sun to a planet sweeps out equal areas in equal times.*

(iii) *The squares of the periodic times of the various planets are proportional to the cubes of the semi-major axes of their orbits.*

5.16 Deductions from Kepler's Laws

(i) In this chapter we have already proved in 5.4 that if the orbit is an ellipse under an acceleration towards one of its foci, then the law of acceleration is that of the inverse square of the distance from the focus (centre of force). Hence from the first Kepler's law we conclude that *'the acceleration of each planet towards the sun varies inversely as the square of its distance from the sun'.*

(ii) According to the Kepler's second law the rate of description of sectorial area is constant which is true only in central orbits. Hence from the second Kepler's law we conclude that *'the acceleration of the planet and therefore the force on it is directed towards the sun'.*

(iii) The periodic time T of a closed central orbit under inverse square law is given by

$$T = \frac{\text{area of the ellipse } (i.e., \text{ the curve described })}{\text{rate of description of the sectorial area}}$$

$$= \frac{\pi ab}{\frac{1}{2}h} = \frac{\pi ab}{\frac{1}{2}\sqrt{(\mu l)}} = \frac{2\pi ab}{\sqrt{\{\mu (b^2/a)\}}} = \frac{2\pi a^{3/2}}{\sqrt{\mu}}$$

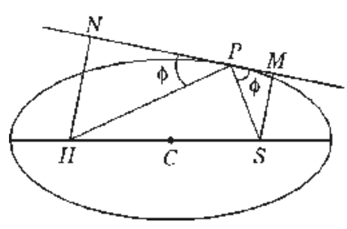
or $$T^2 = \frac{4\pi^2 a^3}{\mu}.$$

But according to the Kepler's third law $T^2 \propto a^3$, from which it follows that μ is a constant.

Hence from the Kepler's third law we conclude that *'the absolute acceleration* μ *is the same for all the planets.'*

5.17 Some Important Geometrical Properties of an Ellipse

In the adjoining figure S and H are the two foci of the ellipse and C is the centre of the ellipse.



1. The product of the perpendiculars drawn from the foci on the tangent at any point of an ellipse is constant and is equal to the square of the semi-minor axis of the ellipse *i.e.*,

$$SM \, . \, HN = b^2.$$

2. The sum of the focal distances of any point on an ellipse is equal to $2a$, where $2a$ is the length of the major axis of the ellipse. Thus

$$SP + HP = 2a.$$

3. The length of the latus rectum of an ellipse is $2(b^2/a)$,

where $b^2 = a^2(1 - e^2)$,

e being the eccentricity of the ellipse.

4. The tangent at any point P of an ellipse is equally inclined to the focal radii SP and HP of that point.

Illustrative Examples

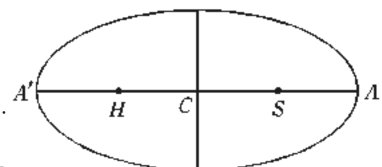
Example 11: *If v_1 and v_2 are the linear velocities of a planet when it is respectively nearest and farthest from the sun, prove that* **(Avadh 2007, 11; Agra 11)**

$$(1 - e)\,v_1 = (1 + e)\,v_2 .$$

Solution: The path of a planet is an ellipse with the sun at its focus. Therefore the velocity v of the planet at a distance r from the focus S (the sun) is given by

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right). \quad \text{...(1)}$$

Let v_1 and v_2 be the velocities of the planet at the points A and A' which are nearest and farthest from the sun at S. Then at A,



$$r = SA = CA - CS = a - ae\,,\, v = v_1$$

and at A', $r = SA' = CS + CA' = ae + a, v = v_2$.

Substituting these values in (1), we have

$$v_1^2 = \mu\left(\frac{2}{a - ae} - \frac{1}{a}\right) = \mu\left\{\frac{2 - (1 - e)}{a(1 - e)}\right\} = \mu\,\frac{(1 + e)}{a(1 - e)}$$

and

$$v_2^2 = \mu\left(\frac{2}{ae + a} - \frac{1}{a}\right) = \mu\left\{\frac{2 - (1 + e)}{a(1 + e)}\right\} = \mu\,\frac{(1 - e)}{a(1 + e)}.$$

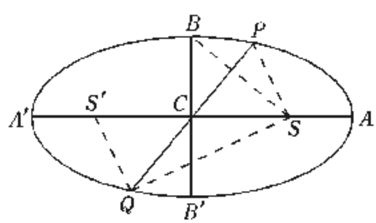
Dividing, we have

$$\frac{v_1^2}{v_2^2} = \frac{(1 + e)^2}{(1 - e)^2} \quad \text{or} \quad \frac{v_1}{v_2} = \frac{1 + e}{1 - e}$$

or

$$(1 - e)\,v_1 = (1 + e)\,v_2 .$$

Example 12: *A particle describes an ellipse as a central orbit about the focus. Prove that the velocity at the end of the minor axis is the geometric mean between the velocities at the ends of any diameter.*



Solution: Let AA' and BB' be the major and minor axes of the ellipse.

Let S, S' be the foci and PQ any diameter of the ellipse.

The velocity v of the particle at any point of the ellipse at a distance r from the focus S is given by

$$v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right), \qquad \ldots(1)$$

where $2\,a$ is the length of the major axis of the ellipse.

We have $SB + S'B = 2\,a$

and $SB = S'B$.

$\therefore$ $SB = a$.

Let V, V_1 and V_2 be the velocities of the particle at the points B, P and Q respectively. Then at $B, r = SB = a, v = V$; at $P, r = SP, v = V_1$; and at $Q, r = SQ, v = V_2$.

$\therefore$ from (1), we have $V^2 = \mu\left(\frac{2}{a} - \frac{1}{a}\right) = \frac{\mu}{a}$, $\qquad \ldots(2)$

$$V_1^2 = \mu\left(\frac{2}{SP} - \frac{1}{a}\right) \text{ and } V_2^2 = \mu\left(\frac{2}{SQ} - \frac{1}{a}\right).$$

Now

$$V_1^2 V_2^2 = \mu^2\left(\frac{2}{SP} - \frac{1}{a}\right)\left(\frac{2}{SQ} - \frac{1}{a}\right)$$

$$= \mu^2\left[\frac{4}{SP.SQ} - \frac{2}{a}\left(\frac{1}{SP} + \frac{1}{SQ}\right) + \frac{1}{a^2}\right]$$

$$= \mu^2\left[\frac{4}{SP.SQ} - \frac{2}{a}\left(\frac{SQ + SP}{SP \cdot SQ}\right) + \frac{1}{a^2}\right] \qquad \ldots(3)$$

We have $SP + SQ = QS' + SQ$ $\qquad [\because \; QS' = SP]$

$= 2\,a$ $\qquad$ [by a property of the ellipse]

Substituting in (3), we have

$$V_1^2\, V_2^2 = \mu^2\left[\frac{4}{SP.SQ} - \frac{2}{a}\left(\frac{2}{SP.SQ}\right) + \frac{1}{a^2}\right] = \frac{\mu^2}{a^2}$$

or $V_1 V_2 = \mu / a$

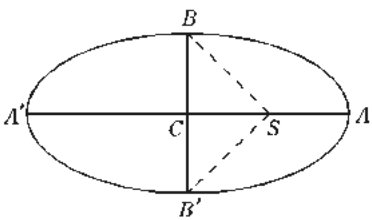
or $V_1 V_2 = V^2$ $\qquad$ [from (2)]

or $V = \sqrt{(V_1 V_2)}$.

Hence V (*i.e.*, the velocity at the end of the minor axis) is equal to the geometric mean between V_1 and V_2 (*i.e.*, the velocities, at the ends of any diameter).

Example 13: *Prove that the time taken by the earth to travel over half its orbit, remote from the sun, separated by the minor axis is two days more than half the year. The eccentricity of the orbit is* 1/60.

Solution: In the figure the path of the earth round the sun is an ellipse, the sun being at the focus S. BB' is the minor axis of the ellipse and C the centre of the ellipse. Remote half from the sun separated by the minor axis is the arc $BA'B'$.



We know that in a central orbit the rate of description of the sectorial area is constant and is equal to $h/2$. The whole area of the ellipse is $\pi\, ab$ and the earth takes one year time to describe the elliptic orbit round the sun.

$$\therefore \qquad \text{one year} = \frac{\pi ab}{h/2}. \qquad \ldots(1)$$

The sectorial area traced out by the earth while describing the arc $BA'B'$

$$= \text{the area } SBA'B'S$$
$$= \frac{1}{2} \text{ area of the ellipse } + \text{ area of the } \Delta\, SBB'$$
$$= \frac{1}{2}\pi ab + \frac{1}{2} BB'.CS$$
$$= \frac{1}{2}\pi ab + \frac{1}{2}\cdot 2\,b\,.\,ae = \frac{1}{2}\pi ab + abe.$$

If t be the time taken by the earth to describe the arc $BA'B'$, then

$$t = \frac{\text{the sectorial area } SBA'B'S}{h/2}$$

i.e.,
$$t = \frac{\frac{1}{2}\pi ab + abe}{h/2}. \qquad \ldots(2)$$

Dividing (2) by (1), we have

$$\frac{t}{\text{one year}} = \frac{\frac{1}{2}\pi ab + abe}{\pi ab} = \frac{1}{2} + \frac{e}{\pi}.$$

$$\therefore \qquad t = \left(\frac{1}{2} + \frac{e}{\pi}\right) \times \text{one year}$$
$$= \frac{1}{2}\text{ year} + \frac{e}{\pi}\text{ year}$$
$$= \frac{1}{2}\text{ year} + \frac{1}{60\,\pi}\text{ year} \qquad \left[\because\ e = \frac{1}{60}\right]$$
$$= \frac{1}{2}\text{ year} + \left[\frac{1}{60} \times \frac{7}{22} \times 365\right]\text{days} \qquad [\because\ \text{one year} = 365 \text{ days}]$$
$$= \frac{1}{2}\text{ year} + 2 \text{ days nearly.}$$

Hence the time of describing half of the orbit remote from the sun is two days more than half the year.

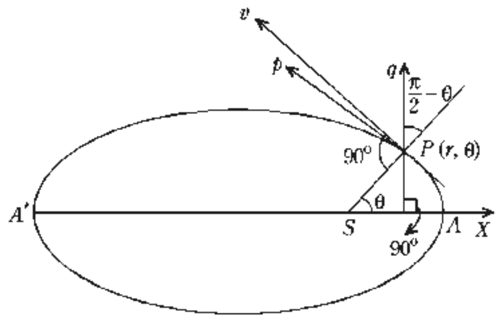
Example 14: *Show that the velocity of a particle moving in an ellipse about a centre of force in the focus is compounded of two constant velocities μ/h perpendicular to the radius and $\mu\, e/h$ perpendicular to the major axis.*

Solution: Referred to the focus S (*i.e.*, the centre of force) as pole, let the equation of the elliptic orbit be

$$\frac{l}{r} = 1 + e\cos\theta, \qquad \ldots(1)$$

where l is the semi-latus rectum of the ellipse.

Let $P(r, \theta)$ be the position of the particle at any time t. The resultant velocity v of the particle at P is along the tangent to the ellipse at P. Suppose the velocity v is the resultant of two velocities p and q where p is perpendicular to the radius vector SP and q is perpendicular to the major axis AA'. Resolving the velocities p and q at P along and perpendicular to the radius vector SP, we have



the radial velocity

$$\frac{dr}{dt} = q \cos\left(\frac{1}{2}\pi - \theta\right) = q \sin\theta, \qquad \ldots(2)$$

and the transverse velocity

$$r\frac{d\theta}{dt} = p + q\sin\left(\frac{1}{2}\pi - \theta\right) = p + q\cos\theta. \qquad \ldots(3)$$

From (2), $q = \dfrac{1}{\sin\theta}\dfrac{dr}{dt}$. $\qquad \ldots(4)$

Differentiating both sides of (1) w.r.t. 't', we have

$$-\frac{l}{r^2}\frac{dr}{dt} = -e\sin\theta\frac{d\theta}{dt}.$$

$$\therefore \qquad \frac{dr}{dt} = \frac{e}{l}\sin\theta\, r^2\frac{d\theta}{dt} = \frac{eh}{l}\sin\theta.$$

$$\left[\because \text{ in a central orbit, } r^2\frac{d\theta}{dt} = h\right]$$

Substituting the value of dr/dt in (4), we get

$$q = \frac{1}{\sin\theta}\cdot\frac{eh}{l}\sin\theta = \frac{eh}{l}$$

$$= \frac{eh}{(h^2/\mu)} \qquad [\because h^2 = \mu l]$$

$$= \frac{e\mu}{h} = \text{constant}.$$

This gives one desired result.

Again from (3), we have

$$p = r\frac{d\theta}{dt} - q\cos\theta$$

$$= \frac{h}{r} - \frac{e\mu}{h}\cos\theta \qquad \left[\because \frac{d\theta}{dt} = h \text{ and } q = \frac{e\mu}{h}\right]$$

$$= \frac{h}{r} - \frac{\mu}{h}\left(\frac{l}{r} - 1\right) \qquad \left[\because \text{ from (1), } e\cos\theta = \frac{l}{r} - 1\right]$$

$$= \frac{h}{r} - \frac{\mu l}{hr} + \frac{\mu}{h}$$

$$= \frac{h}{r} - \frac{h^2}{hr} + \frac{\mu}{h} \qquad [\because \ h^2 = \mu l\,]$$

$$= \frac{\mu}{h} = \text{constant.}$$

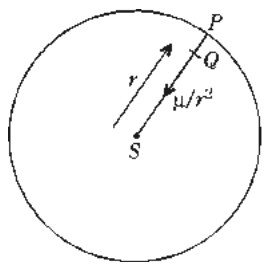
This gives the other desired result.

Example 15: *If a planet were suddenly stopped in its orbit supposed circular, show that it would fall into the sun in a time which is $\sqrt{2}/8$ times the period of the planet's revolution.*

(Rohilkhand 2008)

Solution: Let a planet describing a circular path of radius a and centre S (the sun) be stopped suddenly at the point P of its path. Then it will begin to move towards S along the straight line PS under the acceleration $\mu/(\text{distance})^2$.

If Q is the position of the planet at time t such that $SQ = r$, then the acceleration at Q is μ/r^2 directed towards S.



$\therefore$ the equation of motion of the planet at Q is

$$v\frac{dv}{dr} = -\frac{\mu}{r^2}$$

(–ive sign is taken as the acceleration at Q is in the direction of r decreasing)

or $$v\,dv = -\frac{\mu}{r^2}\,dr.$$

Integrating, $\dfrac{v^2}{2} = \dfrac{\mu}{r} + A$, where A is a constant.

But at $P, r = SP = a$ and $v = 0$.

[Note that the planet begins to move along PS with zero velocity at P]

$\therefore$ $$0 = \frac{\mu}{a} + A \quad \text{or} \quad A = -\frac{\mu}{a}.$$

$\therefore$ $$\frac{v^2}{2} = \frac{\mu}{r} - \frac{\mu}{a} = \frac{\mu\,(a-r)}{ar}$$

or $$v = \frac{dr}{dt} = -\sqrt{(2\mu/a)}.\sqrt{\left(\frac{a-r}{r}\right)}$$

(–ive sign is taken because r decreases as t increases)

or $$dt = -\sqrt{\left(\frac{a}{2\mu}\right)}\cdot\sqrt{\left(\frac{r}{a-r}\right)}\,dr. \qquad \ldots(1)$$

If t_1 is the time taken by the planet from P to S, then integrating (1), we have

$$\int_0^{t_1} dt = -\sqrt{\left(\frac{a}{2\mu}\right)} \int_{r=a}^{0} \sqrt{\left(\frac{r}{a-r}\right)}\, dr$$

or $$t_1 = \sqrt{\left(\frac{a}{2\mu}\right)} \int_0^{\pi/2} \sqrt{\left(\frac{a\cos^2\theta}{a - a\cos^2\theta}\right)} . 2a\cos\theta\sin\theta\, d\theta,$$

putting $r = a\cos^2\theta$, so that $dr = -2a\cos\theta\sin\theta\, d\theta$

$$= a\sqrt{\left(\frac{a}{2\mu}\right)} \int_0^{\pi/2} 2\cos^2\theta\, d\theta$$

$$= a\sqrt{\left(\frac{a}{2\mu}\right)} \int_0^{\pi/2} (1+\cos 2\theta)\, d\theta$$

$$= a\sqrt{\left(\frac{a}{2\mu}\right)} \left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \frac{\pi a^{3/2}}{2\sqrt{(2\mu)}}.$$

But the time period T of the planet's revolution is given by

$$T = \frac{2\pi a^{3/2}}{\sqrt{\mu}}. \qquad \therefore \quad \frac{t_1}{T} = \frac{1}{4\sqrt{2}} = \frac{\sqrt{2}}{8}$$

or $$t_1 = (\sqrt{2}/8)\, T$$

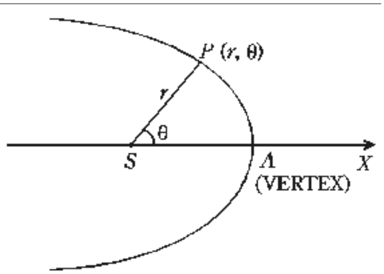
i.e., the time taken by the planet from P to S is $\sqrt{2}/8$ times the period of the planet's revolution.

5.18 Time of Description of an Arc of a Central Orbit

The time of passing from one point of a central orbit to another is usually determined by the equation

$$r^2\left(\frac{d\theta}{dt}\right) = h.$$

5.19 To Find the Time of Description of a Given Arc of a Parabolic Orbit Starting From the Vertex



The polar equation of a parabola of latus rectum $4a$ referred to the focus S as the pole and the axis as the initial line is

$$\frac{2a}{r} = 1 + \cos\theta = 2\cos^2\frac{1}{2}\theta$$

or $$r = a\sec^2\frac{1}{2}\theta.$$

But we have $h = r^2\dfrac{d\theta}{dt}$.

$$\therefore \qquad dt = \frac{r^2}{h}\, d\theta = \frac{a^2 \sec^4 \frac{1}{2}\theta}{\sqrt{(\mu l)}}\, d\theta \qquad [\because\ h^2 = \mu l]$$

$$= \frac{a^2}{\sqrt{(\mu \,.\, 2a)}} \sec^4 \frac{1}{2}\theta \, d\theta. \qquad [\because\ l = 2a]$$

Integrating, the time taken from the vertex A (*i.e.*, $\theta = 0$) to the point $P(r, \theta)$ is given by

$$t = \sqrt{\frac{a^3}{2\mu}} \cdot \int_{\theta=0}^{\theta} \sec^4 \frac{1}{2}\theta \, d\theta$$

$$= \sqrt{\left(\frac{a^3}{2\mu}\right)} \cdot \int_0^{\theta} \left(1 + \tan^2 \frac{1}{2}\theta\right) \sec^2 \frac{1}{2}\theta \, d\theta$$

$$= \sqrt{\left(\frac{a^3}{2\mu}\right)} \cdot 2 \left[\tan \frac{1}{2}\theta + \frac{1}{3} \tan^3 \frac{1}{2}\theta\right]_0^{\theta}$$

or
$$\boldsymbol{t = \sqrt{\left(\frac{2a^3}{\mu}\right)} \left[\tan \frac{\theta}{2} + \frac{1}{3} \tan\theta^3 \frac{\theta}{2}\right]}.$$

5.20 To Find the Time of Description of a Given Arc of an Elliptic Orbit Starting From the Nearer End of The Major Axis

The polar equation of an ellipse referred to its focus S as pole and SA as initial line is

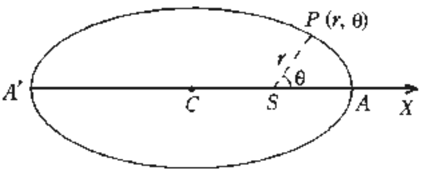
$$\frac{l}{r} = 1 + e \cos \theta, \qquad \ldots(1)$$

where $e < 1$.

Also we have $h = r^2 \dfrac{d\theta}{dt}$.

$$\therefore \qquad dt = \frac{r^2}{h}\, d\theta$$

$$= \frac{l^2}{h} \cdot \frac{d\theta}{(1 + e \cos \theta)^2}.$$



Integrating, the time from A to P is given by

$$t = \frac{l^2}{h} \cdot \int_{\theta=0}^{\theta} \frac{d\theta}{(1 + e \cos \theta)^2}. \qquad \ldots(2)$$

To evaluate the integral we proceed as follows :

We have
$$\frac{d}{d\theta}\left(\frac{\sin\theta}{1 + e\cos\theta}\right) = \frac{\cos\theta\,(1 + e\cos\theta) - \sin\theta.(-e\sin\theta)}{(1 + e\cos\theta)^2}$$

$$= \frac{e + \cos\theta}{(1 + e\cos\theta)^2} = \frac{e^2 + e\cos\theta}{e\,(1 + e\cos\theta)^2}$$

$$= \frac{(1 + e\cos\theta) - (1 - e^2)}{e\,(1 + e\cos\theta)^2}$$

$$= \frac{1}{e\,(1 + e\cos\theta)} - \frac{(1 - e^2)}{e\,(1 + e\cos\theta)^2}.$$

$$\therefore \qquad \frac{1 - e^2}{e\,(1 + e\cos\theta)^2} = \frac{1}{e\,(1 + e\cos\theta)} - \frac{d}{d\theta}\left(\frac{\sin\theta}{1 + e\cos\theta}\right).$$

Integrating both sides, we have

$$\frac{(1 - e^2)}{e}\int \frac{d\theta}{(1 + e\cos\theta)^2} = \frac{1}{e}\int \frac{d\theta}{(1 + e\cos\theta)} - \frac{\sin\theta}{1 + e\cos\theta}. \qquad \ldots(3)$$

Now $\dfrac{d\theta}{1 + e\cos\theta} = \displaystyle\int \frac{d\theta}{1 + e\,\{(1 - \tan^2 \frac{1}{2}\theta) / (1 + \tan^2 \frac{1}{2}\theta)\}}$

$$= \int \frac{(1 + \tan^2 \frac{1}{2}\theta)\, d\theta}{(1 + e) + (1 - e)\tan^2 \frac{1}{2}\theta}$$

$$= \frac{1}{(1 - e)}\int \frac{\sec^2 \frac{1}{2}\theta\, d\theta}{\{(1 + e) / (1 - e)\} + \tan^2 \frac{1}{2}\theta}$$

$$= \frac{2}{1 - e}\int \frac{dz}{\{(1 + e)/(1 - e)\} + z^2}$$

[putting $\tan \frac{1}{2}\theta = z$, so that $\frac{1}{2}\sec^2 \frac{1}{2}\theta\, d\theta = dz$]

$$= \frac{2}{\sqrt{(1 - e)}} \cdot \frac{1}{\sqrt{\{(1 + e) / (1 - e)\}}} \cdot \tan^{-1}\left[\frac{z}{\sqrt{\{(1 + e) / (1 - e)\}}}\right]$$

$$= \frac{2}{\sqrt{(1 - e^2)}} \tan^{-1}\left\{\sqrt{\left(\frac{1 - e}{1 + e}\right)} \tan\frac{\theta}{2}\right\}.$$

$\therefore$ from (3), we have

$$\frac{1 - e^2}{e}\int \frac{d\theta}{(1 + e\cos\theta)^2}$$

$$= \frac{2}{e\sqrt{(1 - e^2)}} \tan^{-1}\left\{\sqrt{\left(\frac{1 - e}{1 + e}\right)} \tan\frac{\theta}{2}\right\} - \frac{\sin\theta}{1 + e\cos\theta}$$

or

$$\int \frac{d\theta}{(1 + e\cos\theta)^2}$$

$$= \frac{2}{e\sqrt{(1 - e^2)^{3/2}}} \tan^{-1}\left\{\sqrt{\left(\frac{1 - e}{1 + e}\right)} \tan\frac{\theta}{2}\right\} - \left(\frac{e}{1 + e^2}\right)\frac{\sin\theta}{1 + e\cos\theta}.$$

$\therefore$ from (2), we have

$$t = \frac{l^2}{h} \cdot \left[\frac{2}{(1-e^2)^{3/2}} \tan^{-1} \left\{ \sqrt{\left(\frac{1-e}{1+e} \right)} \tan \frac{\theta}{2} \right\} - \left(\frac{e}{1-e^2} \right) \frac{\sin \theta}{1 + e \cos \theta} \right]_0^\theta$$

$$= \frac{l^2}{\sqrt{(\mu l)}} \cdot \frac{1}{(1-e^2)^{3/2}}$$

$$\left[2 \tan^{-1} \left\{ \sqrt{\left(\frac{1-e}{1+e} \right)} \tan \frac{\theta}{2} \right\} - e \sqrt{(1-e^2)} . \frac{\sin \theta}{1 + e \cos \theta} \right]$$

$$= \frac{l^{3/2}}{\sqrt{\mu}} \cdot \frac{1}{(1-e^2)^{3/2}}$$

$$\left[2 \tan^{-1} \left\{ \sqrt{\left(\frac{1-e}{1+e} \right)} \tan \frac{\theta}{2} \right\} - e \sqrt{(1-e^2)} . \frac{\sin \theta}{1 + e \cos \theta} \right].$$

But $l = \frac{b^2}{a} = \frac{a^2 (1-e^2)}{a}$

$= a (1 - e^2).$

$$\therefore \quad \boldsymbol{t = \frac{a^{3/2}}{\sqrt{\mu}} \left[2 \tan^{-1} \left\{ \sqrt{\frac{1-e}{1+e}} \tan \frac{\theta}{2} \right\} - e \sqrt{1 - e^2}) . \frac{\sin \theta}{1 + e \cos \theta} \right]}.$$

5.21 To Find the Time of Description of a Given Arc of a Hyperbolic Orbit Starting From the Vertex

The polar equation of a hyperbola referred to its focus S as pole and SA (where A is the vertex) as initial line is

$$\frac{l}{r} = 1 + e \cos \theta, \qquad \ldots(1)$$

where $e > 1$.

Also, we have $h = r^2 \dfrac{d\theta}{dt}$.

$$\therefore \quad dt = \frac{r^2}{h} d\theta = \frac{l^2}{h} \cdot \frac{d\theta}{(1 + e \cos \theta)^2}.$$

Integrating, the time from the vertex A (*i.e.*, $\theta = 0$) to any point $P(r, \theta)$ is given by

$$t = \frac{l^2}{h} \int_0^\theta \frac{d\theta}{(1 + e \cos \theta)^2}. \qquad \ldots(2)$$

Now proceeding as in 5.20, we have

$$\frac{d}{d\theta}\left(\frac{\sin\theta}{1+e\cos\theta}\right)=\frac{1}{e(1+e\cos\theta)}+\frac{e^2-1}{e(1+e\cos\theta)^2}. \qquad [\because \text{ here } e>1]$$

$$\therefore \qquad \frac{e^2-1}{e(1+e\cos\theta)^2}=\frac{d}{d\theta}\left(\frac{\sin\theta}{(1+e\cos\theta)}\right)-\frac{1}{e(1+e\cos\theta)}.$$

Integrating both sides, we have

$$\left(\frac{e^2-1}{e}\right)\int\frac{d\theta}{(1+e\cos\theta)^2}=\frac{\sin\theta}{1+e\cos\theta}-\frac{1}{e}\int\frac{d\theta}{(1+e\cos\theta)}. \qquad \ldots(3)$$

Now $\displaystyle\int\frac{d\theta}{(1+e\cos\theta)}=\int\frac{d\theta}{1+e\,\{(1-\tan^2\frac{1}{2}\theta)/(1+\tan^2\frac{1}{2}\theta)\}}$

$$=\int\frac{(1+\tan^2\frac{1}{2}\theta)\,d\theta}{(e+1)-(e-1)\tan^2\frac{1}{2}\theta}$$

$$=\frac{1}{(e-1)}\int\frac{\sec^2\frac{1}{2}\theta\,d\theta}{\{(e+1)/(e-1)\}-\tan^2\frac{1}{2}\theta}$$

$$=\frac{2}{(e-1)}\int\frac{dz}{\{(e+1)/(e-1)\}-z^2},$$

putting $\tan\frac{1}{2}\theta=z$ so that $\frac{1}{2}\sec^2\frac{1}{2}\theta\,d\theta=dz$

$$=\frac{2}{(e-1)}\cdot\frac{1}{2\sqrt{\{(e+1)/(e-1)\}}}\log\left[\frac{\sqrt{\{(e+1)/(e-1)\}}+z}{\sqrt{\{(e-1)/(e-1)\}}-z}\right]$$

$$=\frac{1}{\sqrt{(e^2-1)}}\log\left[\frac{\sqrt{(e+1)}+\sqrt{(e-1)}\tan\frac{1}{2}\theta}{\sqrt{(e+1)}-\sqrt{(e-1)}\tan\frac{1}{2}\theta}\right].$$

$\therefore$ from (3), we have

or $\displaystyle\left(\frac{e^2-1}{e}\right)\int\frac{d\theta}{(1+e\cos\theta)^2}$

$$=\frac{\sin\theta}{1+e\cos\theta}-\frac{1}{e}\frac{1}{(\sqrt{e^2-1})}\log\left[\frac{\sqrt{(e+1)}+\sqrt{(e-1)}\tan\frac{1}{2}\theta}{\sqrt{(e+1)}-\sqrt{(e-1)}\tan\frac{1}{2}\theta}\right].$$

or $\displaystyle\int\frac{d\theta}{(1+e\cos\theta)^2}=\left(\frac{e}{e^2-1}\right).\frac{\sin\theta}{1+e\cos\theta}-\frac{1}{(e^2-1)^{3/2}}$

$$\log\left[\frac{\sqrt{(e+1)}+\sqrt{(e-1)}\tan\frac{1}{2}\theta}{\sqrt{(e+1)}-\sqrt{(e-1)}\tan\frac{1}{2}\theta}\right].$$

$\therefore$ from (2), we have

$$t = \frac{l^2}{\sqrt{(\mu l)}} \left[\frac{e}{e^2 - 1} \cdot \frac{\sin\theta}{1 + e\cos\theta} - \frac{1}{(e^2 - 1)^{3/2}} \log \left\{ \frac{\sqrt{(e+1)} + \sqrt{(e-1)} \tan\frac{1}{2}\theta}{\sqrt{(e+1)} - \sqrt{(e-1)} \tan\frac{1}{2}\theta} \right\} \right]_0^\theta$$

$$= \frac{l^{3/2}}{\sqrt{\mu}} \cdot \frac{1}{(e^2 - 1)^{3/2}} \left[e\sqrt{(e^2 - 1)} \cdot \frac{\sin\theta}{1 + e\cos\theta} - \log \left\{ \frac{\sqrt{(e+1)} + \sqrt{(e-1)} \tan\frac{1}{2}\theta}{\sqrt{(e+1)} - \sqrt{(e-1)} \tan\frac{1}{2}\theta} \right\} \right]$$

But $\quad l = \dfrac{b^2}{a} = \dfrac{a^2(e^2 - 1)}{a} = a(e^2 - 1).$

$\therefore$

$$\boldsymbol{t = \frac{a^{3/2}}{\sqrt{\mu}} \left[e\sqrt{(e^2 - 1)} \cdot \frac{\sin\theta}{1 + e\cos\theta} - \log \left\{ \frac{\sqrt{(e+1)} + \sqrt{(e-1)} \tan\frac{1}{2}\theta}{\sqrt{(e+1)} - \sqrt{(e-1)} \tan\frac{1}{2}\theta} \right\} \right]}.$$

Illustrative Examples

Example 16: *Prove that in a parabolic orbit the time taken to move from the vertex to a point distant r from the focus is*

$$\frac{1}{3\sqrt{\mu}} (r + l) \sqrt{(2r - l)},$$

where 2 l is the latus rectum.

Solution: For figure refer 5.19.

The polar equation of a parabola of latus rectum $2l$ referred to the focus S as the pole and the axis SA, where A is the vertex, as the initial line is

$$\frac{l}{r} = 1 + \cos\theta = 2\cos^2 \frac{1}{2}\theta$$

or $$r = \frac{1}{2} l \sec^2 \frac{1}{2}\theta. \qquad \ldots(1)$$

But we have $r^2 \left(\dfrac{d\theta}{dt} \right) = h.$

$$\therefore \quad dt = \frac{r^2}{h} d\theta = \frac{\frac{1}{4} l^2 \sec^4 \frac{1}{2}\theta}{\sqrt{(\mu l)}} d\theta \qquad [\because \; h^2 = \mu l]$$

$$= \frac{1}{4}\left(\frac{l^{3/2}}{\sqrt{\mu}}\right) \sec^4 \frac{1}{2}\theta \, d\theta.$$

Integrating, the time taken from the vertex (*i.e.*, $\theta = 0$) to the point $P\ (r, \theta)$ is given by

$$t = \frac{1}{4}\left(\frac{l^{3/2}}{\sqrt{\mu}}\right) \int_0^\theta \sec^4 \frac{1}{2}\theta \, d\theta$$

$$= \frac{1}{4}\left(\frac{l^{3/2}}{\sqrt{\mu}}\right) \int_0^\theta \left(1 + \tan^2 \frac{1}{2}\theta\right) \sec^2 \frac{1}{2}\theta \, d\theta$$

$$= \frac{1}{4}\left(\frac{l^{3/2}}{\sqrt{\mu}}\right) \int_0^\theta \left[\sec^2 \frac{1}{2}\theta + 2\left(\tan^2 \frac{1}{2}\theta\right)\left(\frac{1}{2}\sec^2 \frac{1}{2}\theta\right)\right] d\theta$$

$$= \frac{1}{4}\left(\frac{l^{3/2}}{\sqrt{\mu}}\right) \left[2 \tan \frac{1}{2}\theta + 2 \cdot \frac{1}{3} \tan^3 \frac{1}{2}\theta\right]_0^\theta$$

$$= \frac{1}{2}\left(\frac{l^{3/2}}{\sqrt{\mu}}\right) \left[\tan \frac{1}{2}\theta + \frac{1}{3} \tan^3 \frac{1}{2}\theta\right]$$

$$= \frac{1}{6}\left(\frac{l^{3/2}}{\sqrt{\mu}}\right) \tan \frac{1}{2}\theta \left(3 + \tan^2 \frac{1}{2}\theta\right).$$

But from (1),

$$\sec^2 \frac{1}{2}\theta = \frac{2r}{l} \cdot$$

$\therefore$
$$1 + \tan^2 \frac{1}{2}\theta = \frac{2r}{l}$$

or
$$\tan^2 \frac{1}{2}\theta = \left(\frac{2r}{l}\right) - 1 = \frac{2r - l}{l} \cdot$$

$\therefore$
$$t = \frac{1}{6}\left(\frac{l^{3/2}}{\sqrt{\mu}}\right) \cdot \sqrt{\left[\frac{2r - l}{l}\right]} \left\{3 + \frac{(2r - l)}{l}\right\}$$

$$= \frac{1}{6} \frac{l^{3/2} \sqrt{(2r - l)} . (2l + 2r)}{\sqrt{\mu}\, l^{3/2}}$$

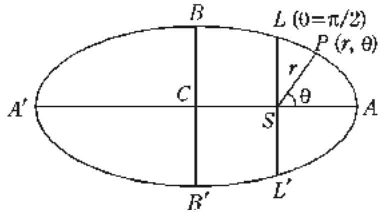
$$= \frac{1}{3\sqrt{\mu}} (r + l) \sqrt{(2r - l)}.$$

Example 17: *Prove that the times taken to describe two portions into which an ellipse is divided by the latus rectum through the centre of force are in a ratio*

$$\{\cos^{-1} e - e\sqrt{(1 - e^2)}\} : \{\pi - \cos^{-1} e + e\sqrt{(1 - e^2)}\}.$$

Solution: In 5.20 we have proved that the time of description t of an arc extending from the vertex A to the point $P\ (r, \theta)$ is given by

$$t = \frac{a^{3/2}}{\sqrt{\mu}} \left[2 \tan^{-1} \left\{\sqrt{\left(\frac{1 - e}{1 + e}\right)} \tan \frac{\theta}{2}\right\} - e\sqrt{(1 - e^2)} \cdot \frac{\sin\theta}{1 + e\cos\theta}\right] \qquad ...(1)$$



Let LL' be the latus rectum through the centre of force S. At the point $L, \theta = \frac{\pi}{2}$.

Substituting $\theta = \frac{\pi}{2}$ in (1), the time t_1 from A to L is given by

$$t_1 = \frac{a^{3/2}}{\sqrt{\mu}}\left[2\tan^{-1}\left\{\sqrt{\left(\frac{1-e}{1+e}\right)}\tan\frac{1}{4}\pi\right\} - e\sqrt{(1-e^2)}.\frac{\sin\frac{1}{2}\pi}{1+e\cos\frac{1}{2}\pi}\right]$$

$$= \frac{a^{3/2}}{\sqrt{\mu}}\left[2\tan^{-1}\left\{\sqrt{\left(\frac{1-e}{1+e}\right)}\right\} - e\sqrt{(1-e^2)}\right].$$

Now $\quad 2\tan^{-1}\sqrt{\left(\frac{1-e}{1+e}\right)} = 2\tan^{-1}\sqrt{\left(\frac{1-\cos\alpha}{1+\cos\alpha}\right)}$, by putting $e = \cos\alpha$

$$= 2\tan^{-1}\tan\frac{1}{2}\alpha$$

$$= 2\times\left(\frac{1}{2}\alpha\right) = \alpha = \cos^{-1} e.$$

$\therefore \quad t_1 = \left(\frac{a^{3/2}}{\sqrt{\mu}}\right)[\cos^{-1} e - e\sqrt{(1-e^2)}].$

$\therefore \quad$ the time T_1 taken to describe the arc $L'AL$

$$= 2t_1 = \left(\frac{2a^{3/2}}{\sqrt{\mu}}\right)[\cos^{-1} e - e\sqrt{(1-e^2)}].$$

Also the time T taken to describe the whole elliptic path $= \left(\frac{2\pi a^{3/2}}{\sqrt{\mu}}\right)$.

$\therefore \quad$ the time T_2 taken to describe the arc $LA'L'$

$$= T - T_1 = \frac{2\pi a^{3/2}}{\sqrt{\mu}} - \frac{2a^{3/2}}{\sqrt{\mu}}[\cos^{-1} e - e\sqrt{(1-e^2)}]$$

$$= \frac{2a^{3/2}}{\sqrt{\mu}}[\pi - \cos^{-1} e + e\sqrt{(1-e^2)}].$$

$\therefore$ the required ratio of the times taken to describe the two portions (arc $L'AL$ and arc $LA'L$) into which an ellipse is divided by the latus rectum through the centre of force S is

$$T_1 : T_2 = \{\cos^{-1} e - e\sqrt{(1-e^2)}\} : \{\pi - \cos^{-1} e + e\sqrt{(1-e^2)}\}.$$

Comprehensive Exercise 3

1. The greatest and least velocities of a certain planet in its orbit round the sun are 30 km / sec and $29 \cdot 2$ km/sec respectively. Find the eccentricity of the orbit.

2. A particle is projected from the earth's surface with velocity v. Show that if the diminution of gravity is taken into account, but the resistance of the air neglected, the path is an ellipse, of major axis $2\ ga^2/(2\ ga - v^2)$, where a is the earth's radius.

3. A particle describes an ellipse under a force μ / (distance)^2 towards the focus. If it was projected with velocity V from a point distant r from the centre of force, show that its periodic time is

$$(2\pi/\sqrt{\mu}) \cdot [2/r - V^2/\mu]^{-3/2}.$$ **(Rohilkhand 2007; Purvanchal 11)**

4. Show that an unresisted particle falling to the earth's surface from a great distance would acquire a velocity $\sqrt{(2\ ga)}$, where a is the radius of the earth.

5. If the velocity of the earth at any point of its orbit, assumed to be circular, were increased by about one-half, prove that it would describe a parabola about the sun as focus.

Show also that, if a body was projected from the earth with a velocity exceeding 7 miles per second, it will not return to the earth and may even leave the solar system.

6. Show that the velocity of a planet at any point of its orbit is the same as it would have been if it had fallen to the point from rest at a distance from the sun equal to the length of the major axis.

7. A particle describes an ellipse under a force μ/r^2 to a focus. Show that the velocity at the end of the minor axis is a geometric mean between the greatest and the least velocities.

8. A particle describes an ellipse under a force to the focus S. When the particle is at one extremity of the minor axis, its kinetic energy is doubled without any change in the direction of motion. Prove that the particle proceeds to describe a parabola.

9. A particle moves with a central acceleration μ / (distance)^2 ; it is projected with velocity V at a distance R. Show that its path is a rectangular hyperbola if the angle of projection is

$$\sin^{-1}[\mu/\{VR\sqrt{(V^2 - 2\mu/R)}\}].$$

10. A body is describing an ellipse of eccentricity e under the action of a force tending to a focus and when at the nearer apse the centre of force is transferred to the other focus. Prove that the eccentricity of the new orbit is $e(3+e)/(1-e)$.
(Avadh 2008)

11. A comet describing a parabola about the sun, when nearest to it suddenly breaks up, without gain or loss of kinetic energy into two equal portions one of which describes a circle ; prove that the other will describe a hyperbola of eccentricity 2.

12. Two particles of masses m_1 and m_2 moving in coplanar parabolas round the sun, collide at right angles and coalesce when their common distance from the sun is

R. Show that the subsequent path of the combined particle is an ellipse of major axis $\dfrac{(m_1 + m_2)^2 \; R}{2 \; m_1 \; m_2}$.

13. Show that if the time of describing an arc bounded by a focal chord of a parabolic orbit under Newtonian law be T, then $T \propto (\text{focal chord})^{3/2}$.

14. If the period of a planet be 365 days and eccentricity be 1/60, then show that the times of describing two halves of the orbit bounded by the latus rectum through the centre of force are $\dfrac{365}{2}\left[1 \pm \dfrac{1}{15\pi}\right]$ days nearly.

Answers 3

1. $\dfrac{1}{74}$

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. In a central orbit if v is the linear velocity at any point of the path, then

(a) $v^2 = h^2 \left[u^2 + \dfrac{d^2 u}{d\theta^2}\right]$ (b) $v^2 = h^2 \left[u^2 + \left(\dfrac{du}{d\theta}\right)^2\right]$

(c) $y^2 = h^2 \left[u^2 + \dfrac{d^2 u}{d\theta^2}\right]$ (d) $v^2 = h^2 \left[u + \dfrac{du}{d\theta}\right]$

(Bundelkhand 2008)

2. If a particle moves in an ellipse of semi-major axis a under a central force directed towards the focus of the ellipse, then the velocity v at any point of the path is given by

(a) $v^2 = \mu \left(\dfrac{2}{r} - \dfrac{1}{a}\right)$ (b) $v^2 = \dfrac{2\mu}{r}$

(c) $v^2 = \mu \left(\dfrac{2}{r} + \dfrac{1}{a}\right)$ (d) $v^2 = \mu \left(\dfrac{2}{r^2} - \dfrac{1}{a}\right)$

3. Referred to the centre of force as pole, the differential equation of a central orbit is

(a) $\dfrac{d^2 u}{d\theta^2} + u^2 = \dfrac{P}{h^2 u^2}$ (b) $\dfrac{du}{d\theta} + u = \dfrac{P}{hu}$

(c) $\dfrac{d^2 u}{d\theta^2} + u = \dfrac{P}{h^2 u^2}$ (d) $\dfrac{d^2 u}{d\theta^2} + u = \dfrac{P}{h^2 u}$

(Bundelkhand 2007, 09; Agra 11)

4. A particle describes the cardioid $r = a\,(1 + \cos\theta)$ under a central force to the pole. If the central acceleration is P, then

(a) $P \propto \frac{1}{r^4}$ (b) $P \propto \frac{1}{r^2}$

(c) $P \propto \frac{1}{r}$ (d) $P \propto \frac{1}{r^3}$

5. If P is the central acceleration towards the pole under which the curve $\frac{b^2}{p^2} = \frac{2a}{r} - 1$ is described, then

(a) $P \propto \frac{1}{r}$ (b) $P \propto \frac{1}{r^3}$

(c) $P \propto \frac{1}{r^4}$ (d) $P \propto \frac{1}{r^2}$

6. A particle describes the curve $r^2 = a^2 \cos 2\theta$ under a force to the pole. If the central acceleration is P, then

(a) $P \propto \frac{1}{r^7}$ (b) $P \propto \frac{1}{r^3}$

(c) $P \propto \frac{1}{r^2}$ (d) $P \propto \frac{1}{r^4}$

7. In a central orbit at an apse

(a) $p = 2r$ (b) $p = r$

(c) $p = \frac{1}{r}$ (d) $p = r^2$

(Bundelkhand 2008; Agra 10)

8. A particle moves with a central acceleration which varies inversely as the cube of the distance. The differential equation of the orbit referred to the centre of force as pole is

(a) $h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \mu u^2$ (b) $h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \mu u$

(c) $h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \mu u^3$ (d) $h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{\mu}{u}$

9. A particle is projected from an apse at a distance a with velocity $\frac{\sqrt{\mu}}{a^2\sqrt{2}}$ under the action of a central force μu^5. The differential equation of the central orbit referred to the centre of force as pole is

(a) $h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \mu u^3$ (b) $h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{\mu}{u^3}$

(c) $h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \mu u^2$ (d) $h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{\mu}{u^2}$

10. A particle moves with a central acceleration $\mu\left(r+\dfrac{a^4}{r^3}\right)$ being projected from an apse at a distance a with a velocity $2\,a\sqrt{\mu}$. At the point $r=a$, we have

(a) $\dfrac{du}{d\theta}=2\,a\sqrt{\mu}$ (b) $\dfrac{du}{d\theta}=a$

(c) $\dfrac{du}{d\theta}=0$ (d) $\dfrac{du}{d\theta}=2a$

11. While describing the central orbit the following part of the acceleration is taken as zero :

(a) Radial (b) Transverse

(c) Tangential (d) Normal

(Garhwal 2002; Rohilkhand 11)

12. If the central force is a single valued function of the distance, then the number of apsidal distances will be :

(a) 2 (b) 1

(c) 3 (d) 0 **(Garhwal 2003)**

13. The pedal equation of a central orbit is :

(a) $\dfrac{h}{p}\dfrac{dp}{dr}=F$ (b) $\dfrac{p^2}{h^3}\dfrac{dp}{dr}=F$

(c) $\dfrac{p}{h}\dfrac{dp}{dr}=F$ (d) $\dfrac{h^2}{p^3}\dfrac{dp}{dr}=F$

(Garhwal 2004; Bundelkhand 06; Rohilkhand 08)

Fill in the Blank(s)

Fill in the blanks "……" so that the following statements are complete and correct.

1. The path described by a particle moving under the action of a central force is called a …… .
2. A central orbit is always a plane … .
3. In a central orbit the transverse acceleration $\dfrac{1}{r}\dfrac{d}{dt}\left(r^2\dfrac{d\theta}{dt}\right)=$ …………
4. The differential equation of a central orbit referred to the centre of force as pole is …… . **(Agra 2008)**
5. Referred to the centre of force as pole, the differential equation of a central orbit in pedal form is …… . **(Agra 2009, 10)**
6. In a central orbit $r^2\dfrac{d\theta}{dt}$ always remains …… .
7. In a central orbit the sectorial area traced out by the radius vector to the centre of force increases …… per unit of time.
8. In a central orbit the linear velocity varies …… as the perpendicular from the centre upon the tangent to the path.

9. If a particle moves in an ellipse under a force which is always directed towards its focus, then the law of force is the ....

10. If the central orbit is an ellipse of semi-major axis a under a central force directed towards the focus of the ellipse, then the time T of one complete revolution is given by $T =$.......

11. If a particle describes the curve $r = 2\,a \cos\theta$ under a force P to the pole, then $P \propto$.......

12. An apse is a point on the central orbit at which the radius vector from the centre of force to the point has a ...... value.

13. In a central orbit at an apse the direction of velocity is ...... to the radius vector drawn from the centre of force.

14. In a central orbit at an apse $\dfrac{du}{d\theta} =$.......

15. Each planet describes an ...... having the sun as one of its foci.

True or False

Write 'T' for true and 'F' for false statement.

1. In a central orbit the angular momentum is conserved.

2. In a central orbit $v = \dfrac{h}{p^2}$ where v is the linear velocity at any point of the orbit and p is the length of the perpendicular drawn from the centre of force to the tangent to the orbit at that point.

3. The radius vector drawn from the sun to a planet sweeps out equal areas in equal times.

4. The squares of the periodic times of the various planets are proportional to the squares of the semi-major axes of their orbits.

5. If P is the central acceleration towards the pole under which the equiangular spiral $r = ae^{\theta \cot \alpha}$ is described, then $P \propto \dfrac{1}{r^3}$.

6. In a central orbit at an apse the radius vector drawn from the centre of force is perpendicular to the tangent at that point.

7. Each planet describes an ellipse having the sun at its centre.

8. If a particle describes the curve $r = a \cos\theta$ under a force P to the pole, then $P \propto \dfrac{1}{r^2}$.

9. If a particle describes the curve $\dfrac{2a}{r} = 1 + \cos\theta$ under a force P to the pole, then $P \propto \dfrac{1}{r^2}$.

10. In a central orbit at an apse $\dfrac{dr}{d\theta} = r$.

Answers

Multiple Choice Questions

1. (b)
2. (a)
3. (c)
4. (a)
5. (d)
6. (a)
7. (b)
8. (b)
9. (a)
10. (c)
11. (b)
12. (a)
13. (d)

Fill in the Blank(s)

1. central orbit
2. curve
3. 0
4. $h^2\left[\frac{d^2u}{d\theta^2}+u\right]=\frac{P}{u^2}$
5. $\frac{h^2}{p^3}\frac{dp}{dr}=P$
6. constant
7. uniformly
8. inversely
9. inverse square law
10. $\frac{2\,\pi a^{3/2}}{\sqrt{\mu}}$
11. $\frac{1}{r^5}$
12. maximum or minimum
13. perpendicular
14. 0
15. ellipse

True or False

1. *T*
2. *F*
3. *T*
4. *F*
5. *T*
6. *T*
7. *F*
8. *F*
9. *T*
10. *F*

□

Chapter 6

Motion of a Particle in Three Dimensions

(Acceleration in Terms of Different Coordinate Systems)

6.1 Acceleration of a Particle in Terms of Cartesian Coordinates

(Rohilkhand 2010)

Let $P(x, y, z)$ be the cartesian coordinates of a point P at time t, w.r.t. the fixed coordinate axes OX, OY and OZ.

If $\mathbf{r} = \vec{OP}$ is the position vector of P w.r.t. the origin O and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the unit vectors along the axes respectively, then

$$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}.$$

If $\mathbf{v}$ is the velocity vector and $\mathbf{a}$ the acceleration vector of P, then

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\,\mathbf{i} + \frac{dy}{dt}\,\mathbf{j} + \frac{dz}{dt}\,\mathbf{k}$$ **(Rohilkhand 2010)**

Thus the velocities of P, parallel to the coordinate axes are

$$\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt} \text{ respectively.}$$

These are positive in the direction of x, y and z increasing respectively.

The resultant velocity of P is given by

$$\sqrt{\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right]}.$$

and $$\mathbf{a}=\frac{d\mathbf{v}}{dt}=\frac{d^2x}{dt^2}\,\mathbf{i}+\frac{d^2y}{dt^2}\,\mathbf{j}+\frac{d^2z}{dt^2}\,\mathbf{k}.$$ **(Rohilkhand 2010)**

Thus acceleration of P, parallel to the coordinate axes are

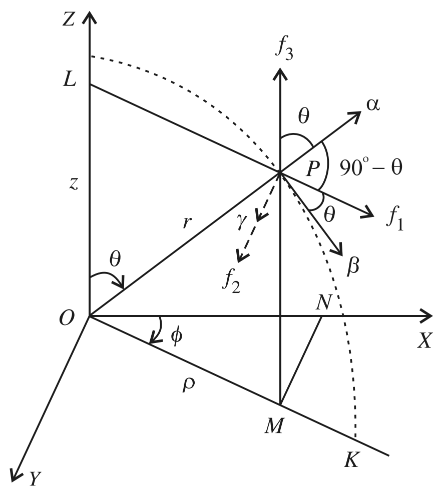
$$\frac{d^2x}{dt^2},\frac{d^2y}{dt^2},\frac{d^2z}{dt^2}\ \textit{respectively.}$$

These are positive in the direction of x, y and z increasing respectively. The resultant acceleration of P is given by

$$\sqrt{\left[\left(\frac{d^2x}{dt^2}\right)^2+\left(\frac{d^2y}{dt^2}\right)^2+\left(\frac{d^2z}{dt^2}\right)^2\right]}.$$

6.2 Acceleration of a Particle in Terms of Polar Coordinates

(Purvanchal 2007, 09, 10; Lucknow 09, 11; Gorakhpur 06, 07, 09, 11)



Let (r,θ,ϕ) be the polar coordinates (spherical polar coordinates) of the point P, at time t, w.r.t. the fixed coordinate axes OX, OY and OZ. Then $OP=r$, θ is the angle that OP makes with the axis OZ and ϕ is the angle that the plane ZPK (the plane through OZ and the point P) also known as plane ZOP makes with the fixed plane ZOX.

The plane ZPK meets the XOY plane in the line OK. Thus $OP=r, \angle ZOP=\theta$ and $\angle XOM=\phi$. The directions of θ and ϕ increasing are shown by arrows $(\rightarrow)$ in the figure.

Let PM be the perpendicular from P on XOY plane. Obviously PM lies in the plane ZPK and M is on the line OK. For all positions of P, the point M, foot of the perpendicular PM will always lie in XOY plane. Therefore in the plane XOY, the polar coordinates of the point M w.r.t. OX as initial line can be taken as (ρ,ϕ), where $\rho=OM=r\sin\theta$.

Thus the accelerations of M in the plane XOY are

$$\frac{d^2\rho}{dt^2}-\rho\left(\frac{d\phi}{dt}\right)^2 \text{ along } OM, \text{(radial acceleration of } M)$$

and $\qquad \dfrac{1}{\rho}\dfrac{d}{dt}\left(\rho^2 \dfrac{d\phi}{dt}\right)$ perpendicular to OM, (transverse acceleration of M)

Also acceleration of P relative to M is $\dfrac{d^2 z}{dt^2}$ along MP where (x, y, z) are in cartesion coordinates of the point P.

Thus the accelerations of the point P are

$$\frac{d^2\rho}{dt^2} - \rho\left(\frac{d\phi}{dt}\right)^2 = f_1 \text{ (say) along } LP$$

$$\frac{1}{\rho}\frac{d}{dt}\left(\rho^2 \frac{d\phi}{dt}\right) = f_2 \text{ (say) perpendicular to the plane } ZPK$$

and $\qquad \dfrac{d^2 z}{dt^2} = f_3$ (say) parallel to

$\therefore$ $\qquad$ if, α = acceleration of P along OP

β = acceleration of P perpendicular to OP in the plane ZPK in the direction of θ increasing

and $\qquad \gamma$ = acceleration of P perpendicular to the plane ZPK, then, we have

$$\left.\begin{aligned}\alpha &= f_1 \sin\theta + f_3 \cos\theta = \left(\frac{d^2\rho}{dt^2}\sin\theta + \frac{d^2 z}{dt^2}\cos\theta\right) - \rho\sin\theta\left(\frac{d\phi}{dt}\right)^2 \\ \beta &= f_1 \cos\theta - f_3 \sin\theta = \left(\frac{d^2\rho}{dt^2}\cos\theta - \frac{d^2 z}{dt^2}\sin\theta\right) - \rho\cos\theta\left(\frac{d\phi}{dt}\right)^2 \\ \text{and} \quad \gamma &= f_2 = \frac{1}{\rho}\frac{d}{dt}\left(\rho^2 \frac{d\phi}{dt}\right)\end{aligned}\right\} \quad \ldots(1)$$

But, we have

$$z = OL = r\cos\theta \quad \text{and} \quad \rho = OM = r\sin\theta.$$

$$\therefore \quad \left.\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial r}\cdot\frac{dr}{dt} + \frac{\partial z}{\partial\theta}\cdot\frac{d\theta}{dt} = \cos\theta\frac{dr}{dt} - r\sin\theta\frac{d\theta}{dt} \\ \frac{d\rho}{dt} &= \frac{\partial\rho}{\partial r}\cdot\frac{dr}{dt} + \frac{\partial\rho}{\partial\theta}\cdot\frac{d\theta}{dt} = \sin\theta\frac{dr}{dt} + r\cos\theta\frac{d\theta}{dt}\end{aligned}\right\} \quad \ldots(2)$$

Also $\qquad \dfrac{d^2 z}{dt^2} = \dfrac{d}{dt}\left(\dfrac{dz}{dt}\right) = \dfrac{d}{dt}\left(\cos\theta\dfrac{dr}{dt} - r\sin\theta\dfrac{d\theta}{dt}\right)$

$$= \cos\theta\frac{d^2 r}{dt^2} - 2\sin\theta\frac{d\theta}{dt}\frac{dr}{dt} - r\cos\theta\left(\frac{d\theta}{dt}\right)^2 - r\sin\theta\frac{d^2\theta}{dt^2}$$

and $\qquad \dfrac{d^2\rho}{dt^2} = \dfrac{d}{dt}\left(\dfrac{d\rho}{dt}\right) = \dfrac{d}{dt}\left(\sin\theta\dfrac{dr}{dt} + r\cos\theta\dfrac{d\theta}{dt}\right)$

$$= \sin\theta\frac{d^2 r}{dt^2} + 2\cos\theta\frac{d\theta}{dt}\frac{dr}{dt} - r\sin\theta\left(\frac{d\theta}{dt}\right)^2 + r\cos\theta\frac{d^2\theta}{dt^2}$$

$\therefore \qquad \dfrac{d^2\rho}{dt^2}\sin\theta + \dfrac{d^2 z}{dt^2}\cos\theta = \dfrac{d^2 r}{dt^2} - r\left(\dfrac{d\theta}{dt}\right)^2$

and $\qquad \dfrac{d^2\rho}{dt^2}\cos\theta - \dfrac{d^2 z}{dt^2}\sin\theta = 2\dfrac{d\theta}{dt}\dfrac{dr}{dt} + r\dfrac{d^2\theta}{dt^2} = \dfrac{1}{r}\dfrac{d}{dt}\left(r^2\dfrac{d\theta}{dt}\right)$

Hence from (1), the acceleration of P in the directions of r, θ and ϕ increasing are given by

$$\left.\begin{aligned}\alpha &= \frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 - r\sin^2\theta\left(\frac{d\phi}{dt}\right)^2\\ \beta &= \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) - r\sin\theta\cos\theta\left(\frac{d\phi}{dt}\right)^2\\ \text{and}\quad \gamma &= \frac{1}{r\sin\theta}\frac{d}{dt}\left(r^2\sin^2\theta\frac{d\phi}{dt}\right)\end{aligned}\right\} \qquad \ldots(3)$$

Note: (i) If (x, y, z) are the cartesian coordinates and (r, θ, ϕ) the polar coordinates of a point P, then

$$x = r\sin\theta\cos\phi,\quad y = r\sin\theta\sin\phi \text{ and } z = r\cos\theta.$$

Obviously $x^2 + y^2 + z^2 = r^2$.

(ii) The velocity v of a particle at (r, θ, ϕ) is given by

$$v^2 = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = r^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2$$

6.3 Equations of Motion of a Particle in Polar Coordinates

If L_1, L_2, L_3 are the algebraic sums of the external forces acting on the particle of mass m at $P\,(r, \theta, \phi)$, along OP (in the direction of r increasing), perpendicular to OP in the plane ZPO (in the direction of θ increasing) and perpendicular to the plane ZPO, respectively, then the equations of motion of the particle are given by

$$m\alpha = m\left[\frac{d^2 r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 - r\sin^2\theta\left(\frac{d\phi}{dt}\right)^2\right] = L_1$$

$$m\beta = m\left[\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) - r\sin\theta\cos\theta\left(\frac{d\phi}{dt}\right)^2\right] = L_2$$

and $$m\gamma = m\cdot\frac{1}{r\sin\theta}\frac{d}{dt}\left(r^2\sin^2\theta\frac{d\phi}{dt}\right) = L_3\,.$$

6.4 Accelerations of a Particle in Terms of Cylindrical Coordinates

(Gorakhpur 2008, 09)

As discussed in the previous article 6.3, the accelerations of $P\,(\rho, \phi, z)$ are

$$\frac{d^2\rho}{dt^2} - \rho\left(\frac{d\phi}{dt}\right)^2, \text{ along } OM$$

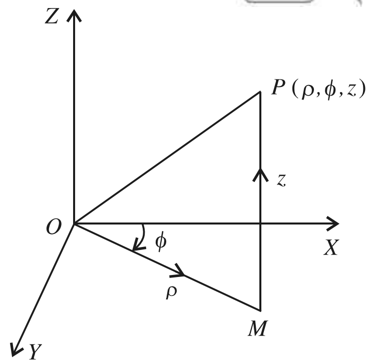
(in the direction of ρ increasing)

$$\frac{1}{\rho}\frac{d}{dt}\left(\rho^2 \frac{d\phi}{dt}\right), \text{perpendicular to } OM \text{ in the plane } ZOP$$

(in the direction of ϕ increasing)

and $$\frac{d^2 z}{dt^2} \text{ parallel to } OZ$$

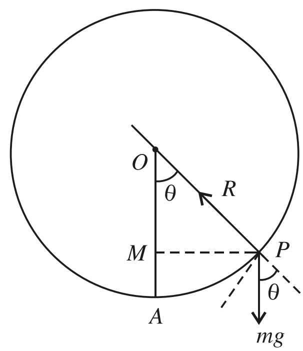
(in the direction of z increasing).



Illustrative Examples

Example 1: *A heavy particle moves in a smooth sphere ; show that, if the velocity be that due to the level of the centre, the reaction of the surface will vary as the depth below the centre.*

Solution: Let m be the mass of the particle and $P(r, \theta, \phi)$ its position at time t, such that OP makes an angle θ with the downward vertical through the centre O of the sphere.



The velocity of the particle at P

$$= \sqrt{(2\ g\,.\,OM)} = \sqrt{(2\ ag\cos\theta)} \qquad ...(1)$$

where a is the radius of the sphere *i.e.*, $OP = a$.

Let R be the reaction of the surface of the sphere at P, which will act along PO.

At the point P, $r = a$ (constant), therefore $\frac{d^2 r}{dt^2} = 0$, and so the equations of motion of the particle are

$$m\left[-a\left(\frac{d\theta}{\delta t}\right)^2 - a\sin^2\theta\left(\frac{d\phi}{dt}\right)^2\right] = mg\cos\theta - R \qquad ...(2)$$

$$m\left[a\frac{d^2\theta}{dt^2} - a\sin\theta\cos\theta\left(\frac{d\phi}{dt}\right)^2\right] = -\,mg\sin\theta \qquad ...(3)$$

$$m\left[\frac{a}{\sin\theta}\frac{d}{dt}\left(\sin^2\theta\cdot\frac{d\phi}{dt}\right)\right] = 0 \qquad ...(4)$$

Integrating (4), we get

$$\sin^2\theta\cdot\frac{d\phi}{dt} = A \ \text{(constant)} \qquad ...(5)$$

Initially let $\theta = \alpha$. When $\theta = \alpha$,

$$\frac{d\theta}{dt} = 0 \ \text{ and } \ \frac{dr}{dt} = 0. \qquad [\because\ r = a]$$

If V is the initial velocity of projection of the particle, then

$$V = \left(a \sin\theta \cdot \frac{d\phi}{dt}\right)_{\theta=\alpha} = a \sin\alpha \left(\frac{d\phi}{dt}\right)_{\theta=\alpha},$$

$\therefore$ $$\left(\frac{d\phi}{dt}\right)_{\theta=\alpha} = \frac{V}{a \sin\alpha}$$

$\therefore$ from (5), we get

$$A = \sin^2\alpha \left(\frac{d\phi}{dt}\right)_{\theta=\alpha} = \sin^2\alpha \cdot \frac{V}{a\sin\alpha} = \frac{V\sin\alpha}{a}$$

$\therefore$ (5) reduces to

$$\sin^2\theta \frac{d\phi}{dt} = \frac{V\sin\alpha}{a} \qquad \text{...(6)}$$

Substituting the value of $\frac{d\phi}{dt}$ from (6) in (3), we get

$$\frac{d^2\theta}{dt^2} - \frac{V^2\sin^2\alpha}{a^2} \cdot \frac{\cos\theta}{\sin^3\theta} = -\frac{g}{a}\sin\theta$$

Multiplying both sides by $2\frac{d\theta}{dt}$ and then integrating, we get

$$\left(\frac{d\theta}{dt}\right)^2 + \frac{V^2\sin^2\alpha}{a^2} \cdot \frac{1}{\sin^2\theta} = \frac{2g}{a}\cos\theta + B \qquad \text{...(7)}$$

Initially when $\theta = \alpha, \frac{d\theta}{dt} = 0,$

$\therefore$ $$B = 0 + \frac{V^2\sin^2\alpha}{a^2} \cdot \frac{1}{\sin^2\alpha} - \frac{2g}{a}\cos\alpha = \frac{V^2}{a^2} - \frac{2g}{a}\cos\alpha$$

$\therefore$ (7) reduces to

$$\left(\frac{d\theta}{dt}\right)^2 + \frac{V^2\sin^2\alpha}{a^2} \cdot \frac{1}{\sin^2\theta} = \frac{2g}{a}\cos\theta + \frac{V^2}{a^2} - \frac{2g}{a}\cos\alpha$$

$\therefore$ $$\left(\frac{d\theta}{dt}\right)^2 + \frac{V^2}{a^2\sin^2\theta}(\sin^2\theta - \sin^2\alpha) - \frac{2g}{a}(\cos\alpha - \cos\theta) \qquad \text{...(8)}$$

Substituting the values of $\left(\frac{d\phi}{dt}\right)^2$ and $\left(\frac{d\theta}{dt}\right)^2$ from (6) and (8) in (2), we get

$$R = mg\cos\theta + ma\left[\frac{V^2}{a^2\sin^2\theta}(\sin^2\theta - \sin^2\alpha) - \frac{2g}{a}(\cos\alpha - \cos\theta)\right] + ma\sin^2\theta \cdot \frac{V^2\sin^2\alpha}{a^2\sin^4\theta}$$

or $$R = 3mg\cos\theta + \frac{mV^2}{a} - 2mg\cos\alpha$$

But when $\theta = \alpha$, velocity $= V$, we have from (1), $V = \sqrt{2ag\cos\alpha}$.

$\therefore$ $$R = 3mg\cos\theta + \frac{m}{a} \cdot 2ag\cos\alpha - 2mg\cos\alpha = 3mg\cos\theta$$

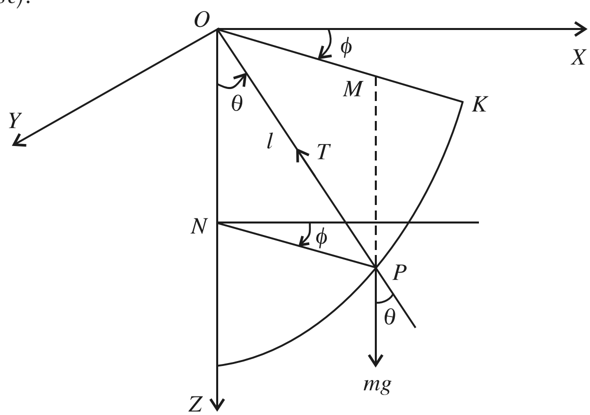
or $$R = \frac{3\,mg}{a} \cdot a\cos\theta = \frac{3\,mg}{a} \cdot OM$$

$\Rightarrow$ $$R \propto OM.$$

Hence the reaction of the surface varies as the depth of the particle below the centre.

Example 2: *A particle is attached to one end of a string, of length l, the other end of which is tied to a fixed point O. When the string is inclined at an acute angle α to the downward-drawn vertical, the particle is projected horizontally and perpendicular to the string with a velocity V. Find the resulting motion of the particle. Also find the tension of the string at any instant.*

Solution: Let P be the position of the particle at time t, such that the polar coordinates of P w.r.t. coordinate axes through O, with Z-axis along downward vertical are (l, θ, ϕ). Here $r = l$ (const).



$\therefore$ If T is the tension in the string, then the equations of motion of the particle are

$$m\left[-l\left(\frac{d\theta}{dt}\right)^2 - l\sin^2\theta\left(\frac{d\phi}{dt}\right)^2\right] = -T + mg\cos\theta \qquad \ldots(1)$$

$$m\left[l\frac{d^2\theta}{dt^2} - l\sin\theta\cos\theta\left(\frac{d\phi}{dt}\right)^2\right] = -mg\sin\theta \qquad \ldots(2)$$

and $$m \cdot \frac{l}{\sin\theta}\frac{d}{dt}\left(\sin^2\theta \cdot \frac{d\phi}{dt}\right) = 0 \qquad \ldots(3)$$

From (3), we get $\dfrac{d}{dt}\left(\sin^2\theta \cdot \dfrac{d\phi}{dt}\right) = 0.$

Integrating, $\sin^2\theta\,\dfrac{d\phi}{dt} = A$ (const). $\qquad \ldots(4)$

But initially $\theta = \alpha$, $V = \left(l\sin\alpha\,\dfrac{d\phi}{dt}\right)_{\theta=\alpha}$ *i.e.*, $\left(\dfrac{d\phi}{dt}\right)_{\theta=\alpha} = \dfrac{V}{(l\sin\alpha)}$

$\therefore$ From (4), $A = \sin^2\alpha \cdot \dfrac{V}{l\sin\alpha} = \dfrac{V\sin\alpha}{l}$

$\therefore$ (4) reduces to, $\sin^2\theta \dfrac{d\phi}{dt} = \dfrac{V \sin\alpha}{l}$ ...(5)

Substituting the value of $\dfrac{d\phi}{dt}$ from (5) in (2), we get

$$\frac{d^2\theta}{dt^2} - \frac{V^2 \sin^2\alpha}{l^2} \cdot \frac{\cos\theta}{\sin^3\theta} = -\frac{g}{l}\sin\theta.$$

Multiplying both sides by $2\dfrac{d\theta}{dt}$ and then integrating, we get

$$\left(\frac{d\theta}{dt}\right)^2 + \frac{V^2 \sin^2\alpha}{l^2} \cdot \frac{1}{\sin^2\theta} = \frac{2g}{l}\cos\theta + B. \qquad \text{...(6)}$$

But initially when $\theta = \alpha$, $\dfrac{d\theta}{dt} = 0$.

$$\therefore \qquad B = \frac{V^2}{l^2} - \frac{2g}{l}\cos\alpha.$$

$\therefore$ From (6), we get

$$\left(\frac{d\theta}{dt}\right)^2 + \frac{V^2 \sin^2\alpha}{l^2} \cdot \frac{1}{\sin^2\theta} = \frac{V^2}{l^2} + \frac{2g}{l}(\cos\theta - \cos\alpha)$$

$$\left(\frac{d\theta}{dt}\right)^2 = \frac{V^2}{l^2}\left(1 - \frac{\sin^2\alpha}{\sin^2\theta}\right) + \frac{2g}{l}(\cos\theta - \cos\alpha) \qquad \text{...(7)}$$

$$= \frac{V^2}{l^2}\,\frac{\sin^2\theta - \sin^2\alpha}{\sin^2\theta} + \frac{2g}{l}(\cos\theta - \cos\alpha)$$

$$= \frac{V^2}{l^2}\,\frac{\cos^2\alpha - \cos^2\theta}{\sin^2\theta} + \frac{2g}{l}(\cos\theta - \cos\alpha)$$

$$= \frac{2g}{l}\,\frac{(\cos\alpha - \cos\theta)}{\sin^2\theta}\left[\frac{V^2}{2gl}(\cos\alpha + \cos\theta) - \sin^2\theta\right]$$

$$= \frac{2g}{l}\,\frac{(\cos\alpha - \cos\theta)}{\sin^2\theta}\,[2n^2 . (\cos\alpha + \cos\theta) - \sin^2\theta]$$

[Taking $V^2 = 4\,lgn^2$]

If $\dfrac{d\theta}{dt} = 0$, then $2n^2(\cos\alpha + \cos\theta) - \sin^2\theta = 0$

[$\because$ $\cos\alpha - \cos\theta \neq 0$ as $\theta = \alpha$ is the initial position]

$\Rightarrow$ $\cos^2\theta + 2n^2\cos\theta - (1 - 2n^2\cos\alpha) = 0$

$\Rightarrow$ $\cos\theta = -n^2 \pm \sqrt{(1 - 2n^2\cos\alpha + n^4)}$

$\therefore$ If $\dfrac{d\theta}{dt} = 0$ for $\theta = \theta_1$, then $\cos\theta_1 = -n^2 + \sqrt{(1 - 2n^2\cos\alpha + n^4)}$ ...(8)

neglecting – sign which is inadmissible as θ is acute angle. Hence the motion of the particle is confined between $\theta = \alpha$ and $\theta = \theta_1$ (given by (8)).

The motion of the particle will remain above or below the starting point $\theta = \alpha$, according as $\theta_1 >$ or $< \alpha$

i.e., according as $\cos\theta_1 <$ or $> \cos\alpha$

i.e., according as $-n^2 \pm \sqrt{(1-2n^2\cos\alpha + n^4)} <$ or $> \cos\alpha$

i.e., according as $1 - 2n^2\cos\alpha + n^4 <$ or $> (n^2 + \cos\alpha)^2$

i.e., according as $n^2 >$ or $< \dfrac{\sin^2\alpha}{4\cos\alpha}$

i.e., according as $V^2 >$ or $< lg\tan\alpha\sin\alpha$.

To find the tension T at any instant.

Substituting the values of $\left(\dfrac{d\phi}{dt}\right)^2$ and $\left(\dfrac{d\theta}{dt}\right)^2$ from (5) and (7) in (1), we get

$$\frac{T}{m} = l\left[\frac{V^2}{l^2}\left(1 - \frac{\sin^2\alpha}{\sin^2\theta}\right) + \frac{2g}{l}(\cos\theta - \cos\alpha)\right] + l\sin^2\theta\cdot\frac{V^2\sin^2\alpha}{l^2\sin^4\theta} + g\cos\theta$$

$$= \frac{V^2}{l} + g(3\cos\theta - 2\cos\alpha)$$

or $\quad T = m[V^2/l + g(3\cos\theta - 2\cos\alpha)]$

which gives the tension in the string at any instant.

Example 3: *A heavy particle is projected with velocity V from the end of a horizontal diameter of a sphere of radius a along the inner surface, the direction of projection making an angle* β *with the equator. If the particle never leaves the surface, prove that* $3\sin^2\beta < 2 + (V^2/3ag)^2$.

Solution: Refer to figure of Ex. 1. Let m be the mass of the particle and P its position at time t such that OP makes an angle θ with the downward vertical through the centre O of the sphere.

Let R be the reaction of the surface of the sphere at P which will act along PO.

The equations of motion of the particle are $\quad [\because\ r = a \text{ (constant)}]$

$$m\left[-a\left(\frac{d\theta}{dt}\right)^2 - a\sin^2\theta\left(\frac{d\phi}{dt}\right)^2\right] = -R + mg\cos\theta \qquad \ldots(1)$$

$$m\left[a\frac{d^2\theta}{dt^2} - a\sin\theta\cos\theta\left(\frac{d\phi}{dt}\right)^2\right] = -mg\sin\theta \qquad \ldots(2)$$

and

$$m\cdot\frac{a}{\sin\theta}\frac{d}{dt}\left(\sin^2\theta\frac{d\phi}{dt}\right) = 0. \qquad \ldots(3)$$

From (3), we have

$$\frac{d}{dt}\left(\sin^2\theta\frac{d\phi}{dt}\right) = 0$$

Integrating, $\sin^2 \theta \dfrac{d\phi}{dt} = A$ (const.) ...(4)

Initially, the particle is projected at an angle β with the equator (*i.e.*, the horizontal diameter), *i.e.*, the particle is projected at an angle $\pi/2 - \beta$ with the vertical.

∴ Initially when $\theta = \dfrac{\pi}{2} - \beta,\ V = \left[a \sin\theta \dfrac{d\phi}{dt}\right]_{\theta = \pi/2 - \beta}$

$$\therefore \quad \left(\frac{d\phi}{dt}\right)_{\theta = \pi/2 - \beta} = \frac{V}{a \cos \beta}.$$

$$\therefore \quad \text{From (4), } A = \sin^2 \left(\frac{\pi}{2} - \beta\right) \cdot \left(\frac{d\phi}{dt}\right)_{\pi/2 - \beta} = \cos^2 \beta \cdot \frac{V}{a \cos \beta} = \frac{V \cos \beta}{a}.$$

$$\therefore \quad (4) \Rightarrow \sin^2 \theta \frac{d\phi}{dt} = \frac{V \cos \beta}{a} \qquad ...(5)$$

Putting the value of $\dfrac{d\phi}{dt}$ from (5) in (2), we get

$$\frac{d^2 \theta}{dt^2} - \frac{V^2 \cos^2 \beta}{a^2} \cdot \frac{\cos \theta}{\sin^3 \theta} = \frac{-g}{a} \sin \theta.$$

Multiplying both sides by $2 \dfrac{d\theta}{dt}$ and then integrating, we get

$$\left(\frac{d\theta}{dt}\right)^2 + \frac{V^2 \cos^2 \beta}{a^2} \cdot \frac{1}{\sin^2 \theta} = \frac{2g}{a} \cos \theta + B$$

Initially when the particle is at rest, $\theta = \dfrac{\pi}{2}, \beta = 0$ and $\dfrac{d\theta}{dt} = 0.$

$$\therefore \quad B = \frac{V^2}{a^2}$$

$$\therefore \quad \left(\frac{d\theta}{dt}\right)^2 + \frac{V^2 \cos^2 \beta}{a^2 \sin^2 \theta} = \frac{2g}{a} \cos \theta + \frac{V^2}{a^2}. \qquad ...(6)$$

Substituting the values of $\dfrac{d\phi}{dt}$ and $\dfrac{d\theta}{dt}$ from (5) and (6) in (1), we get

$$\frac{R}{m} = a \left[\frac{2g}{a} \cos \theta + \frac{V^2}{a^2} - \frac{V^2 \cos^2 \beta}{a^2 \sin^2 \theta}\right] + a \sin^2 \theta \cdot \frac{V^2 \cos^2 \beta}{a^2 \sin^4 \theta} + g \cos \theta$$

or $$\frac{R}{m} = \frac{V^2}{a} + 3g \cos \theta. \qquad ...(7)$$

Particle leaves the surface *i.e.*, $R = 0$, when

$$\frac{V^2}{a} + 3g \cos \theta = 0 \quad i.e., \quad \cos \theta = -\frac{V^2}{(3ag)}$$

From (6), $\dfrac{d\theta}{dt} = 0$, when

$$0 + \frac{V^2 \cos^2 \beta}{a^2 \sin^2 \theta} = \frac{2g}{a} \cos \theta + \frac{V^2}{a^2}$$

i.e., when $$V^2 \cos^2 \beta = a^2 \sin^2 \theta \left(\frac{2g}{a} \cos\theta + \frac{V^2}{a^2} \right)$$

$$= (1 - \cos^2 \theta)(2\, ag \cos\theta + V^2)$$

$$= \left(1 - \frac{V^4}{9a^2 g^2}\right)\left(-\frac{2}{3} V^2 + V^2\right)$$

or $$\cos^2 \beta = \frac{1}{3}\left(1 - \frac{V^4}{9a^2 g^2}\right) \quad \text{or} \quad \sin^2 \beta = 1 - \cos^2 \beta = \frac{1}{3}\left(2 + \frac{V^4}{9a^2 g^2}\right)$$

or $$3 \sin^2 \beta = 2 + \left(\frac{V^2}{3ga}\right)^2 .$$

i.e., Particle just leaves the surface, when

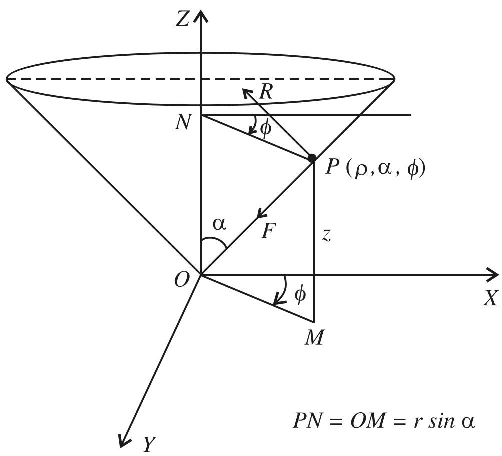
$$3 \sin^2 \beta = 2 + \left(\frac{V^2}{3\, ga}\right)^2 .$$

Hence the particle will never leave the surface, if

$$3 \sin^2 \beta < 2 + \left(\frac{V^2}{3ga}\right)^2 .$$

Example 4: *A particle moves on the inner surface of a smooth cone of vertical angle* $2\,\alpha$, *being acted on by a force towards the vertex of the cone, and its direction of motion always cuts the generators at a constant angle* β. *Find the motion and the law of force.*

Solution: Let $P(r, \alpha, \phi)$ be the position of the particle of mass m on the surface of the cone at any time t and F the force acting on it towards the vertex O. Here $\theta = \alpha \Rightarrow \dot{\theta} = 0$.



Thus the equations of motion of the particle are

$$m\left[\frac{d^2 r}{dt^2} - r \sin^2 \alpha \left(\frac{d\phi}{dt}\right)^2\right] = -F \quad \ldots(1) \; [\because \dot{\theta} = 0, \theta = \alpha]$$

$$m\left[-r \sin\alpha \cos\alpha \left(\frac{d\phi}{dt}\right)^2\right] = -R \quad \ldots(2)$$

and $$m \cdot \frac{1}{r \sin\alpha} \cdot \frac{d}{dt}\left(r^2 \sin^2 \alpha \frac{d\phi}{dt}\right) = 0 \qquad \ldots(3)$$

Since the direction of motion always cuts the generators (*i.e.*, OP) at an angle β,

$\therefore$ $$\tan\beta = \frac{(r \sin\alpha)\, \dot{\phi}}{\dot{r}} \qquad \ldots(4)$$

From (3), we get $\dfrac{d}{dt}\left(r^2 \dfrac{d\phi}{dt}\right) = 0.$

Integrating, $r^2 \dfrac{d\phi}{dt} = A$ (const.) ...(5)

From (4) and (5), we have

$$\frac{dr}{dt} = r \sin\alpha \cot\beta \, . \, \frac{A}{r^2} = \frac{A}{r} \sin\alpha \cot\beta. \qquad \text{...(6)}$$

$$\therefore \qquad \frac{d^2r}{dt^2} = \left(-\frac{A}{r^2}\right)\frac{dr}{dt}\sin\alpha\cot\beta = -\frac{A^2}{r^3}\sin^2\alpha\cot^2\beta.$$

Thus from (1), we get

$$F = -m\left[-\frac{A^2}{r^3}\sin^2\alpha\cot^2\beta - r\sin^2\alpha\left(\frac{A}{r^2}\right)^2\right]$$

$$= \frac{mA^2}{r^3}\sin^2\alpha(\cot^2\beta + 1) = \frac{mA^2\sin^2\alpha}{\sin^2\beta}\cdot\frac{1}{r^3} = \frac{\mu m}{r^3} \qquad \text{...(7)}$$

where $\mu = A^2 \sin^2\alpha / \sin^2\beta$

$$\therefore \qquad F \propto \frac{1}{r^3}\cdot$$

Hence the law of force is inversely proportional to the cube of distance of the particle from the vertex.

From (4), we have

$$\frac{dr}{r} = \sin\alpha\cot\beta \, d\phi$$

Integrating, $\log r = \sin\alpha\cot\beta \, . \, \phi + C$

Initially, when $\phi = 0$, let $r = r_0$, then $C = \log r_0$.

$\therefore \qquad \log r - \log r_0 = \sin\alpha\cot\beta \, . \, \phi$

or $\qquad r = r_0 \, . \, e^{\sin\alpha\cot\beta \, . \, \phi}$ which is the equation of path of the particle.

Substituting the value of $\dfrac{d\phi}{dt}$ from (5) in (2),

$$R = mr\sin\alpha\cos\alpha \, . \, \frac{A^2}{r^4} = mA^2\sin\alpha\cos\alpha \, . \, \frac{1}{r^3}$$

$$= mA^2\sin\alpha\cos\alpha\cdot\frac{F\sin^2\beta}{mA^2\sin^2\alpha}, \text{ using (7)}$$

or $\qquad R = F\cot\alpha \, . \sin^2\beta.$ ...(8)

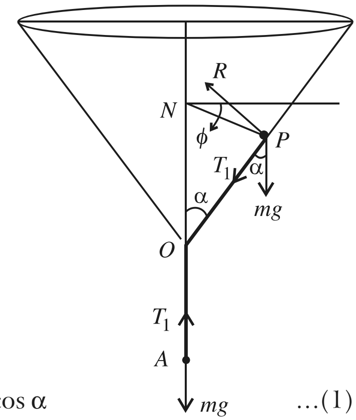
If v is the velocity of the particle at any instant, then

$$v^2 = \left(\frac{dr}{dt}\right)^2 + (r^2\sin^2\theta)\left(\frac{d\phi}{dt}\right)^2 + r^2\left(\frac{d\theta}{dt}\right)^2$$

$$= (A^2/r^2)\sin^2\alpha\cot^2\beta + r^2\sin^2\alpha \, . \, A^2/r^4 + 0 \quad \left[\because \theta = \alpha, \ \frac{d\theta}{dt} = 0\right]$$

$$= (A^2/r^2)\sin^2\alpha(\cot^2\beta + 1) = \frac{A^2\sin^2\alpha}{r^2\sin^2\beta} = \frac{F}{mr^2}\cdot$$

Example 5: *A smooth circular cone, of angle 2 α, has its axis vertical and its vertex, which is pierced with a small hole, downwards. A mass M hangs at rest by a string which passes through the vertex, and a mass m attached to the upper end describe a horizontal circle on the inner surface of the cone. Find the time T of a complete revolution and show that small oscillations about the steady motion take place in the time T cosec* $\alpha \cdot \sqrt{\{(M + m) / 3m\}}$.



Solution: Let l be the total length of the string. Let $P(r, \alpha, \phi)$ be the position of the particle of mass m attached at one end of the string and describing a horizontal circle on the inner surface of the cone.

Here $OP = r$ and $\theta = \alpha$ (const), therefore $\dot{\theta} = 0$.

The length of the hanging portion of the string below $O = OA = l - r$ and let T_1 be the tension in the string.

The equations of motion of the particle of mass m are

$$m\left[\frac{d^2 r}{dt^2} - r\sin^2\alpha\left(\frac{d\phi}{dt}\right)^2\right] = -T_1 - mg\cos\alpha \qquad \ldots(1)$$

$$m\left[-r\sin\alpha\cos\alpha\left(\frac{d\phi}{dt}\right)^2\right] = -R \qquad \ldots(2)$$

and
$$m\cdot\frac{1}{r\sin\alpha}\cdot\frac{d}{dt}\left(r^2\sin^2\alpha\,\frac{d\phi}{dt}\right) = 0 \qquad \ldots(3)$$

And equation of motion of the particle of mass M is

$$M\frac{d^2}{dt^2}(l - r) = Mg - T_1 \qquad \ldots(4)$$

Equation (4) gives, $-M\dfrac{d^2 r}{dt^2} = Mg - T_1$ $\qquad \ldots(5)$

From (3), we get $\dfrac{d}{dt}\left(r^2\dfrac{d\phi}{dt}\right) = 0$

Integrating, $r^2\dfrac{d\phi}{dt} = A$ (const.) $\qquad \ldots(6)$

Subtracting (5) from (1), we get

$$(m + M)\frac{d^2 r}{dt^2} - mr\sin^2\alpha\left(\frac{d\phi}{dt}\right)^2$$

$$= -mg\cos\alpha - Mg \qquad \ldots(7)$$

For steady motion of m, $\dfrac{d^2 r}{dt^2} = 0$ say when $r = d$ (constant), (as the particle describes a horizontal circle *i.e.*, remains at a constant height above O).

$\therefore$ Putting $\dfrac{d\phi}{dt} = \omega$ in (6) and (7), we get

$$A = d^2\,\omega \qquad \text{...(8)}$$

and $$md\,\omega^2 \sin^2 \alpha = g\,(m \cos \alpha + M) \qquad \text{...(9)}$$

and time period $T = 2\pi/\omega$.

For small oscillations about the steady motion at time t, let us take $r = d + \rho$, so that $\dfrac{d^2 r}{dt^2} = \dfrac{d^2 \rho}{dt^2}$.

and from (6) and $\dfrac{d\phi}{dt} = \dfrac{A}{r^2} = \dfrac{d^2 \omega}{(d+\rho)^2}$

$$= \omega\left(1 + \frac{\rho}{d}\right)^{-2} = \omega\left(1 - \frac{2\rho}{d}\right)$$

Neglecting squares and higher powers of ρ, as ρ is very small.

$\therefore$ From (7), we get

$$(M + m)\frac{d^2\rho}{dt^2} = m\,(d+\rho)\sin^2\alpha\,.\,\omega^2\left(1 - \frac{2\rho}{d}\right)^2 - g\,(m\cos\alpha + M)$$

$$= m\omega^2\,(d+\rho)\sin^2\alpha\,.\left(1 - \frac{4\rho}{d}\right) - md\,\omega^2\sin^2\alpha\,, \qquad \text{[By (9)]}$$

$$= m\omega^2\sin^2\alpha\,(d + \rho - 4\rho) - Md\omega^2\sin^2\alpha\,, \qquad [\text{Neglecting } \rho^2]$$

or $$\frac{d^2\rho}{dt^2} = \frac{-3m\omega^2\sin^2\alpha}{(M+m)}\cdot\rho \quad \text{which represents an S.H.M.}$$

and the time period $= 2\pi \Big/ \sqrt{\left(\dfrac{3m\omega^2\sin^2\alpha}{M+m}\right)} = T\cdot\operatorname{cosec}\alpha\,\sqrt{\left(\dfrac{M+m}{3m}\right)}$

Comprehensive Exercise 1

1. A particle moves on a smooth sphere under no forces except the pressure of the surface. Show that its path is given by the equation $\cot\theta = \cot\beta\cos\phi$, where θ and ϕ are its angular coordinates. **(Purvanchal 2007, 09)**
2. A heavy particle is projected horizontally along the inner surface of a smooth spherical shell of radius $a/\sqrt{2}$ with velocity $\sqrt{(7ag/3)}$ at a depth $2a/3$ below the centre. Show that it will rise to a height $a/3$ above the centre, and that the pressure on the sphere just vanishes at the highest point of the path.
3. A particle is projected horizontally along the interior surface of a smooth hemisphere whose axis is vertical and whose vertex is downwards ; the point of projection being at an angular distance β from the lowest point. Show that the

initial velocity so that the particle may just ascend to the rim of the hemisphere is $\sqrt{(2\,ag \sec \beta)}$.

4. A particle constrained to move on a smooth spherical surface is projected horizontally from a point at the level of the centre so that its angular velocity relative to the centre is ω. If $\omega^2 a$ be very great compared with g, show that its depth below the level of the centre at time t is $\frac{2\,g}{\omega^2} \sin^2 \frac{\omega t}{2}$ approximately.

5. A particle is attached to one end of a string, of length l, the other end of which is tied to a fixed point O. When the string is inclined at an acute angle α to the downward-drawn vertical the particle is projected horizontally and perpendicular to the string with a velocity V. Prove that the particle revolves at a constant depth below the centre O as in the ordinary conical pendulum, if $V^2 = gl \frac{\sin^2 \alpha}{\cos \alpha}$.

6. A smooth conical surface is fixed with its axis vertical and vertex downwards. A particle is in steady motion on its concave side in a horizontal circle and is slightly disturbed. Show that the time of a small oscillation about this state of steady motion is $2\pi \sqrt{\{l/(3g \cos \alpha)\}}$, l is the length of the generator to the circle of steady motion.

7. A smooth hollow right circular cone is placed with its vertex downward and axis vertical, and at a point on its interior surface at a height h above the vertex a particle is projected horizontally along the surface with a velocity $\sqrt{\left(\frac{2\,gh}{n^2 + n}\right)}$.
Show that the lowest point of its path will be at a height h/n above the vertex of the cone.

8. At the vertex of a smooth cone of vertical angle 2α, fixed with its axis vertical and vertex downwards, is a centre of repulsive force $\mu / (\text{distance})^4$. A weightless particle is projected horizontally with velocity $\sqrt{\left(\frac{2\mu \sin^3 \alpha}{c^3}\right)}$ from a point, distance c from the axis, along the inside of the surface. Show that it will describe a curve on the cone whose projection on a horizontal plane is

$$1 - c/r = 3 \tanh^2 \left(\frac{1}{2} \theta \sin \alpha\right).$$

9. A particle moves on a smooth right circular cone under a force which is always in a direction perpendicular to the axis of the cone. If the particle describes on the cone a curve which cuts all the generators at a given constant angle, find the law of force and the initial velocity. And show that at any instant the reaction of the cone is proportional to the acting force.

10. A particle moves on the surface of a smooth sphere and is acted on by a force in the direction of the perpendicular from the particle on a diameter and equal to $\mu / (\text{distance})^3$. Show that it can be projected so that its path will cut the meridians at a constant angle.

Objective Type Questions

Multiple Choice Questions

Indicate the correct answer for each question by writing the corresponding letter from (a), (b), (c) *and* (d).

1. Velocity of a particle at the point (r, θ, ϕ) is
   (a) $r\sqrt{(1+\dot{\theta}^2+\sin^2\theta\,\dot{\phi}^2)}$ (b) $r\sqrt{(1+\dot{\theta}^2+\dot{\phi}^2)}$
   (c) $\sqrt{(r^2+\dot{r}^2+r^2\dot{\theta}^2)}$ (d) None of these.

2. Acceleration of a particle at the point $P(a, \theta, \phi)$, perpendicular to OP in the plane ZOP, in the direction of θ increasing is
   (a) $\dfrac{1}{r}\dfrac{d}{dt}\left(r^2\dfrac{d\theta}{dt}\right)-r\left(\dfrac{d\phi}{dt}\right)^2$ (b) $\dfrac{d^2\theta}{dt^2}-\sin\theta\cos\theta\left(\dfrac{d\phi}{dt}\right)^2$
   (c) $\dfrac{d^2z}{dt^2}$ (d) None of these.

3. Acceleration of a particle at the point $P(a, \theta, \phi)$ along OP in the direction of r increasing is
   (a) $\left(\dfrac{d\theta}{dt}\right)^2-\sin^2\theta\left(\dfrac{d\phi}{dt}\right)^2$ (b) $-a\left(\dfrac{d\theta}{dt}\right)^2-a\sin^2\theta\left(\dfrac{d\phi}{dt}\right)^2$
   (c) $\left(\dfrac{d\theta}{dt}\right)^2+\sin^2\theta\left(\dfrac{d\phi}{dt}\right)^2$ (d) None of these.

4. A particle moves on a smooth sphere under no forces except the pressure of the surface, then $\dfrac{d^2\theta}{dt^2}-\sin\theta\cos\theta\left(\dfrac{d\phi}{dt}\right)^2=$
   (a) 1 (b) -1
   (c) 0 (d) None of these.

Fill in the Blank(s)

Fill in the blanks "......" so that the following statements are complete and correct.

1. Acceleration of a particle at the point (x, y, z) is given by ....
2. Velocity of the particle at the point (r, θ, ϕ) in spherial polar coordinates is given by ....
3. Acceleration of a particle at the point $P(r, \theta, \phi)$, along OP, in the direction of r increasing is ....
4. Acceleration of a particle at the point $P(r, \theta, \phi)$, perpendicular to OP in the plane ZOP, in the direction of θ increasing is ....
5. Acceleration of a particle at the point $P(r, \theta, \phi)$, perpendicular to the plane ZOP in the direction of ϕ increasing is ....
6. Acceleration of a particle at the point $P(\rho, \phi, z)$, parallel to XOY plane and in the direction of ρ increasing is ....

7. Acceleration of a particle at the point $P(\rho, \phi, z)$, in the plane ZOP and perpendicular to ρ, in the direction of ϕ increasing is ....

8. Acceleration of a particle at the point $P(\rho, \phi, z)$, parallel to OZ, in the direction of z-increasing is ....

9. A particle moves on a smooth sphere under no forces except the pressure of the surface, then $\dfrac{d^2\theta}{dt^2} = \ldots\ldots$

10. A particle is projected horizontally with velocity V along the interior surface of a smooth hemisphere whose axis is vertical and whose vertex is downwards, the point of projection being at an angular distance β from the lowest point, then $\sin^2\theta \dfrac{d\phi}{dt} = \ldots\ldots$

True or False

Write 'T' for true and 'F' for false statement.

1. Acceleration of a particle at the point $P(\rho, \phi, z)$ parallel to OZ, in the direction of z increasing is $\dfrac{d^2 z}{dt^2}$.

2. Acceleration of a particle at the point $P(a, \theta, \phi)$, perpendicular to the plane ZOP in the direction of ϕ increasing is $\dfrac{1}{\sin\theta}\dfrac{d}{dt}\left(a^2 \sin^2\theta \dfrac{d\phi}{dt}\right)$.

3. Acceleration of a particle at the point $P(r, \theta, \phi)$ along OP in the direction of r increasing is $\dfrac{d^2 r}{dt^2} + r\left(\dfrac{d\theta}{dt}\right)^2 - r\sin^2\theta\left(\dfrac{d\phi}{dt}\right)^2$.

4. A particle moves on a smooth sphere under no forces except the pressure of the surface, then $\dfrac{d^2\theta}{dt^2} = \dfrac{1}{2}\sin 2\theta\left(\dfrac{d\phi}{dt}\right)^2$.

Answers

Multiple Choice Questions

1. $\sqrt{\left[\left(\dfrac{d^2 x}{dt^2}\right)^2 + \left(\dfrac{d^2 y}{dt^2}\right)^2 + \left(\dfrac{d^2 z}{dt^2}\right)^2\right]}$

2. $\sqrt{[r^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2]}$

3. $\dfrac{d^2 r}{dt^2} - r\left(\dfrac{d\theta}{dt}\right)^2 - r\sin^2\theta\left(\dfrac{d\phi}{dt}\right)^2$

4. $\dfrac{1}{r}\dfrac{d}{dt}\left(r^2\dfrac{d\theta}{dt}\right) - r\sin\theta\cos\theta\left(\dfrac{d\phi}{dt}\right)^2$

Fill in the Blank(s)

1. $\frac{1}{r \sin\theta} \frac{d}{dt}\left(r^2 \sin^2\theta \frac{d\phi}{dt}\right)$

2. $\frac{d^2\rho}{dt^2} - \rho\left(\frac{d\phi}{dt}\right)^2$

3. $\frac{1}{\rho} \frac{d}{dt}\left(\rho^2 \frac{d\phi}{dt}\right)$

4. $\frac{d^2 z}{dt^2}$

5. $\sin\theta \cos\theta \left(\frac{d\phi}{dt}\right)^2$

6. $\frac{V}{a} \sin\beta$

7. (a)

8. (b)

9. (b)

10. (c)

True or False

1. *T* 2. *F* 3. *F* 4. *T*