Forest 2017

1(c) Using the Mean Value Theorem, show that

(i) $f(x)$ is constant in $[a,b]$, if $f'(x)=0$ in $[a,b]$

(ii) $f(x)$ is a decreasing function in (a,b), if $f'(x)$ exists and is <0 everywhere in (a,b).

(i) Let x_1 & x_2 be any two distinct points of $[a,b]$ such that $[x_1, x_2] \subset [a,b]$.

Then f is continuous and derivable on $[x_1, x_2]$

$\therefore \exists\ c \in [x_1, x_2)$ such that $\dfrac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c)$ —① (using Mean Value theorem)

But $f'(x)=0\ \forall\ x \in (a,b)$ and

$x_1 < c < x_2 \quad \therefore f'(c)=0$

from ①, $\dfrac{f(x_2)-f(x_1)}{x_2-x_1} = 0$

$\Rightarrow f(x_2)-f(x_1)=0$

$\Rightarrow f(x_2)=f(x_2)$

Since x_1 & x_2 are any two distinct points of $[a,b]$.

it follows that $f(x)$ is constant on $[a,b]\ \forall\ x \in [a,b]$.

(ii) Suppose $f'(x)<0 \quad \forall\ x \in (a,b)$

Let x_1 and x_2 be two distinct points of $[a,b]$ such that $[x_1, x_2] \subset [a,b]$.

$\Rightarrow$ ~~Sin~~ f is differentiable on $[x_1, x_2] \subset [a,b]$ and therefore continuous also.

(using Mean value theorem)

$\therefore \exists\ c \in (x_1, x_2)$ such that

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c)$$

$\Rightarrow f(x_2) - f(x_1) = (x_2 - x_1) f'(c)$ — ①

Since $x_1 < x_2 \Rightarrow x_2 - x_1 > 0$

$f'(x) \leq 0 \quad \forall x \in [a,b]$ and $x_1 < c < x_2$

$\Rightarrow f'(c) \leq 0$

$\therefore$ from ① : $f(x_2) - f(x_1) \leq 0$

$\Rightarrow f(x_1) \geq f(x_2)$

Since $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$

~~$\therefore$ f is an increasing functn~~

$\therefore$ f is decreasing in (a,b).

1(d) Let $u(x,y) = ax^2 + 2hxy + by^2$ and $v(x,y) = Ax^2 + 2Hxy + By^2$ find the Jacobian $J = \frac{\partial(u,v)}{\partial(x,y)}$, and hence show that u, v are independent unless $\frac{a}{A} = \frac{b}{B} = \frac{h}{H}$.

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 2ax + 2hy & 2hx + 2by \\ 2Ax + 2Hy & 2Hx + 2By \end{vmatrix}$$

$$= (2ax + 2hy)(2Hx + 2By) - (2hx + 2by)(2Ax + 2Hy)$$

$=$

2(b) Show that $\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2} \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}$, $p, q > -1$

Hence evaluate (i) $\int_0^{\pi/2} \sin^4 x \cos^5 x \, dx$

(ii) $\int_0^1 x^2 (1-x^2)^{5/2} dx$

(iii) $\int_0^1 x^4 (1-x)^3 dx$

Put $\sin^2 x = z$ $\therefore$ $2 \sin x \cos x \, dx = dz$

when $x = 0$, $z = 0$ & when $x = \pi/2$, $z = 1$

Also $\cos^2 x = 1 - \sin^2 x = 1 - z$

$$\therefore \int_0^{\pi/2} \sin^p x \cos^q x \, dx$$

$$= \int_0^{\pi/2} (\sin^2 x)^{\frac{p-1}{2}} (\cos^2 x)^{\frac{q-1}{2}} \cdot \sin x \cos x \, dx$$

$$= \int_0^1 z^{\frac{p-1}{2}} (1-z)^{\frac{q-1}{2}} \cdot 1/2 \, dz$$

$$= \frac{1}{2} \int_0^1 z^{\frac{p-1}{2}} (1-z)^{\frac{q-1}{2}} dz$$

$$= \frac{1}{2} \cdot B\left(\frac{p-1}{2} + 1, \frac{q-1}{2} + 1\right)$$

$$= \frac{1}{2} \cdot B\left(\frac{p+1}{2}, \frac{q+1}{2}\right) \qquad \left[B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \right]$$

$$= \frac{1}{2} \cdot \frac{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+1}{2} + \frac{q+1}{2}\right)}$$

$$= \frac{1}{2} \cdot \frac{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}$$

(i) $\int_0^{\pi/2} \sin^4 x \cos^5 x \, dx = \frac{1}{2} \frac{\Gamma\left(\frac{4+1}{2}\right) \Gamma\left(\frac{5+1}{2}\right)}{\Gamma\left(\frac{4+5+2}{2}\right)}$

$$= \frac{\frac{1}{2} \Gamma\left(\frac{5}{2}\right) \Gamma(3)}{\Gamma\left(\frac{11}{2}\right)}$$

$\Gamma(n) = (n-1)!$

$\Gamma(3) = 2$

$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$$= \frac{\frac{1}{2} \frac{3}{2} \frac{1}{2} \Gamma\left(\frac{1}{2}\right) \times 2}{\frac{9}{2} \times \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \Gamma(1/2)}$$

$$= \frac{\cancel{\frac{1}{2}} \times \cancel{\frac{3}{2}} \times \cancel{1/2} \times \cancel{\sqrt{\pi}} \times \cancel{2}}{\frac{9}{2} \times \frac{7}{2} \times \frac{5}{2} \times \cancel{\frac{3}{2}} \times \cancel{\frac{1}{2}} \times \cancel{\sqrt{\pi}}}$$

$63 \times 5 = 315$

$$= \frac{8}{315}$$

(ii) $\int_0^1 x^p (1-x^q)^n \, dx$ where $p, q, n > 0$

Put $x^q = z$ or $x = z^{1/q}$

so that $dx = \frac{1}{q} z^{1/q - 1} dz = \frac{1}{q} z^{\frac{1-q}{q}} dz$

when $x=0$, $z=0$ and $x=1$, $z=1$

$$\therefore \int_0^1 x^p (1-x^q)^n \, dx = \int_0^1 z^{p/q} (1-z)^n \cdot \frac{1}{q} z^{\frac{1-q}{q}} dz$$

$$= \frac{1}{q} \int_0^1 z^{\frac{p+1-q}{q}} (1-z)^n \, dz$$

$$= \frac{1}{q} B\left(\frac{p+1-q}{q} + 1, n+1\right)$$

$$= \frac{1}{q} B\left(\frac{p+1}{q}, n+1\right)$$

$$= \frac{1}{q} \frac{\Gamma\left(\frac{p+1}{q}\right) \Gamma(n+1)}{\Gamma\left(\frac{p+1}{q}+n+1\right)}$$

$$\therefore \text{(ii)} \int_0^1 x^3 (1-x^2)^{5/2}\, dx = \frac{1}{2} \frac{\Gamma\left(\frac{3+1}{2}\right)\Gamma\left(\frac{5}{2}+1\right)}{\Gamma\left(\frac{3+1}{2}+\frac{5}{2}+1\right)} = \Gamma\left(\frac{10}{2}+1\right) = \Gamma(6)$$

$$= \frac{\frac{1}{2} \cdot 1 \cdot \frac{\cancel{5}}{2} \times \frac{\cancel{3}}{2} \times \frac{1}{2} \times \sqrt{\pi}}{5! = \cancel{5} \times 4 \times \cancel{3} \times 2 \times 1}$$

$$= \frac{\sqrt{\pi}}{2^7}$$

$$\text{(iii)} \int_0^1 x^4 (1-x)^3\, dx = \frac{1 .\, \Gamma(4+1)\ \Gamma(3+1)}{\Gamma(4+1+3+1)}$$

$$= \frac{4! \times 3!}{8!} = \frac{\cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{3} \times \cancel{2}}{8 \times 7 \times \cancel{6} \times 5 \times \cancel{4} \times \cancel{3} \times \cancel{2}} = \frac{1}{280}$$

2(c) Find the maxima and minima for the function $f(x,y) = x^3 + y^3 - 3x - 12y + 20$. Also find the Saddle points (if any) for the function.

$f(x,y) = x^3 + y^3 - 3x - 12y + 20$

$f_x(x,y) = 3x^2 - 3$

$f_y(x,y) = 3y^2 - 12$

Equating to zero, these f_x & f_y, we get

$3x^2 - 3 = 0 \Rightarrow x = \pm 1$

& $3y^2 - 12 = 0 \Rightarrow y = \pm 2$

$\therefore$ The func f has 4 Saddle stationary points
$(1,2)$, $(-1,2)$, $(1,-2)$, $(-1,-2)$.

Now, $f_{xx}(x,y) = 6x$

$f_{yy}(x,y) = 6y$

$f_{xy}(x,y) = 0$

At $(1,2)$

$f_{xx} = 6 > 0$, $f_{yy} = 12 > 0$ & $f_{xy} = 0$

& $f_{xx} f_{yy} - f_{xy}^2 = 6 \times 12 - 0$
$= 72 > 0$

Hence $(1,2)$ is the minimum point of $f(x,y)$

At $(-1,2)$

$f_{xx} = -6$, $f_{yy} = 12$, $f_{xy} = 0$

and $f_{xx} f_{yy} - f^2_{xy} = -6 \times 12 = -72 < 0$

$\therefore$ func is neither max. nor min. at $(-1,2)$

At $(1,-2)$

$f_{xx} = 6$, $f_{xy} = 0$, $f_{yy} = -12$

& $f_{xx} f_{yy} - f_{xy}^2 = -72 < 0$

$\therefore$ func is neither max nor min. at $(1,-2)$

At $(-1,-2)$

$f_{xx} = -6 < 0$, $f_{yy} = -12 < 0$ & $f_{xy} = 0$

and $f_{xx} f_{yy} - f_{xy}^2 = 72 > 0$

$\therefore (x,y) = (-1,2)$ is the max. point of $f(x,y)$.

$\therefore$ $(1,2)$ and $(-1,-2)$ are Min. and Max. points of $f(x,y)$ resp.

and stationary points like $(-1,2)$ and $(1,-2)$ which are not extreme points are saddle points.

3(c) Evaluate the integral $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$, by changing to polar coordinates. Hence show that $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}$.

The region of integration being the first quadrant of the xy-plane, r varies from 0 to ∞ and θ varies from 0 to $\pi/2$.

$$I = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy = \int_{\theta=0}^{\pi/2} \int_{r=0}^{\infty} e^{-r^2}\,r\,dr\,d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2} \left\{ \int_0^\infty e^{-r^2}(-2r)\,dr \right\} d\theta$$

$$= -\frac{1}{2}\int_0^{\pi/2} \left| e^{-r^2} \right|_0^\infty d\theta = \frac{1}{2}\int_0^{\pi/2} d\theta = \pi/4$$

Also, $I = \int_0^\infty e^{-x^2}dx \times \int_0^\infty e^{-y^2}dy = \left\{ \int_0^\infty e^{-x^2}dx \right\}^2$

$$\Rightarrow \int_0^\infty e^{-x^2} = \sqrt{\pi/4} = \frac{\sqrt{\pi}}{2}.$$

4 (b) A fun^c $f(x,y)$ is defined as $f(x,y) = \begin{cases} \frac{x^2y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

Show that $f_{xy}(0,0) = f_{yx}(0,0)$

$$f_{xy}(0,0) = \underset{h\to 0}{Lt} \frac{f_y(h,0) - f_y(0,0)}{h}$$

$$f_y(h,0) = \underset{t\to 0}{Lt} \frac{f(h,t) - f(h,0)}{t}$$

$$= \underset{t\to 0}{Lt} \frac{\frac{h^2t^2}{h^2+t^2} - 0}{t}$$

$$= \underset{t\to 0}{Lt} \frac{h^2t}{h^2+t^2} = 0$$

$$f_y(0,0) = \underset{t\to 0}{Lt} \frac{f(0,t) - f(0,0)}{t}$$

$$= \underset{t\to 0}{Lt} \frac{0-0}{t} = 0$$

$$\therefore f_{xy}(0,0) = \underset{h\to 0}{Lt} \frac{0-0}{h} = \underline{\underline{0}}.$$

Now Since $f_{yx}(0,0) = \underset{k\to 0}{Lt} \frac{f_x(0,k) - f_x(0,0)}{k}$

$$f_x(0,k) = \underset{t\to 0}{Lt} \frac{f(t,k) - f(0,k)}{t}$$

$$= \underset{t\to 0}{Lt} \frac{\frac{t^2k^2}{t^2+k^2} - 0}{t} = \underset{t\to 0}{Lt} \frac{tk^2}{t^2+k^2} = 0$$

and $f_x(0,0) = \underset{t\to 0}{Lt} \frac{f(t,0) - f(0,0)}{t} = \underset{t\to 0}{Lt} \frac{0-0}{t} = 0$

$$f_{yx}(0,0) = 0$$

$$\therefore f_{xy}(0,0) = f_{yx}(0,0)$$