* Existence and Uniqueness theorem

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = r(x)$$

If a_0, a_1, a_2, r are $C[a,b]$ functions of x and $\underbrace{a_0(x) \neq 0}$ $\forall x$

Then there exists unique solution $y(x)$ satisfying

$$y(x_0) = C_1, \quad y'(x_0) = C_2.$$

$\Downarrow$

If this fails at any x_0, solution fails to be unique

- $\int a^x \, dx = \frac{a^x}{\ln a} + C.$
- $\int \ln x \, dx = x \ln x - x + C.$
- $\int \tan x \, dx = -\ln|\cos x| + C = \ln|\sec x| + C.$
- $\int \sec x \, dx = \ln|\sec x + \tan x| + C.$
- $\int \operatorname{cosec} x \, dx = -\ln|\operatorname{cosec} x + \cot x| + C = \boxed{\ln(\tan x/2) + C}$
- $\int \sinh x \, dx = \cosh x + C$
- $\int \cosh x \, dx = \sinh x + C$
- $\int \tanh x \, dx = \ln(\cosh x) + C$

- $\int \frac{1}{x^2+a^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}$

$\int \frac{1}{x^2-a^2} dx = \frac{1}{2a} \left(\log(x-a) - \log(x+a) \right)$

- $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1} \frac{x}{a}.$
- $\int \frac{dx}{\sqrt{a^2+x^2}} = \sinh^{-1} \left(\frac{x}{a}\right).$
- $\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1} \left(\frac{x}{a}\right)$
- $\int \sqrt{a^2-x^2} \, dx = \frac{x}{2} \sqrt{a^2-x^2} + \frac{a^2}{2} \sin^{-1}\left(\frac{x}{a}\right).$
- $\int \sqrt{x^2-a^2} \, dx = \frac{x}{2} \sqrt{x^2-a^2} - \frac{a^2}{2} \log\left\{x + \sqrt{x^2-a^2}\right\}$
- $\int \sqrt{a^2+x^2} \, dx = \frac{x}{2} \sqrt{a^2+x^2} + \frac{a^2}{2} \log\left\{x + \sqrt{a^2+x^2}\right\}$

$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2} \left\{ a \sin bx - b \cos bx \right\}$

ODE

1. Finding differential equation

- $y = a\cos(mx+b)$ = a, b are arbit constants

$\frac{dy}{dx} = -am\sin(mx+b)$.

$\frac{d^2y}{dx^2} = -am^2\cos(mx+b) = -am^2y \Rightarrow y'' + m^2y = 0$ Ans

- $y = e^x(A\cos x + B\sin x)$

$y' = Ae^x\cos x - Ae^x\sin x + Be^x\sin x + Be^x\cos x = y + (-Ae^x\sin + Be^x\cos x)$

~~$y'' = Ae^x\cos x - Ae^x\sin x - Ae^x\sin x + Be^x\sin x + Be^x\cos x + Be^x\cos x - Be^x\sin$~~

~~$= -2Ae^x\sin x + 2Be^x\cos x$~~

$y'' = -2Ae^x\sin x + 2Be^x\cos x$ — 1

$2y' = 2y + 2(-Ae^x\sin x + Be^x\cos x)$ — 2

1 − 2 $\Rightarrow y'' - 2y' = -2y \Rightarrow \boxed{y'' - 2y' + 2y = 0}$

- $xy = ae^x + be^{-x} + x^2$.

$xy' + y = ae^x + be^{-x} + 2x$

$xy'' + y' + y' = ae^x + be^{-x} + 2 \Rightarrow \boxed{xy'' + 2y' = xy - x^2 + 2}$ Ans

- $y = c(x-c)^2$

$y' = 2c(x-c) \Rightarrow \frac{y}{y'} = \frac{(x-c)}{2} \Rightarrow c = x - \frac{2y}{y'}$

$$\boxed{y' = 2\left(x - \frac{2y}{y'}\right)\left(\cancel{x} - \cancel{x} + \frac{2y}{y'}\right)}$$ Ans

② Show L.I. | Differentiation is allowed to show LI

- $\sin x, \sin 2x, \sin 3x$

$$W(x) = \begin{vmatrix} \sin x & \sin 2x & \sin 3x \\ \cos x & 2\cos 2x & 3\cos 3x \\ -\sin x & -4\sin 2x & -9\sin 3x \end{vmatrix}$$

$$= \sin x(-18\cos 2x \sin 3x - 12\cos 3x \sin 2x) +$$
$$\sin 2x(-3\cos 3x \sin x + 9\sin 3x \cos x) +$$
$$\sin 3x(-4\sin 2x \cos x + 2\cos 2x \sin x)$$

At $x = \pi/2$

$$\boxed{= 1(-18) + 0 + (-1)((-2)) = -20 \neq 0}$$

So, linearly independent.

This is best. $W(x_0) \neq 0$ for any x_0.

JUST Show $W(x) \neq 0$ for any value of x.

- Form DE with $1, x, x^2$ as solutions

$$y = A + Bx + Cx^2$$
$$y' = B + 2Cx$$
$$y'' = 2Cx$$
$$\boxed{y''' = 0}$$ An

(3) 1st Order, 1st degree

- Separable

I $\sqrt{1+x^2+y^2+x^2y^2} + xy\,(dy/dx) = 0$

$$\sqrt{(1+x^2)(1+y^2)} = (-x)\,y\frac{dy}{dx} \Rightarrow \frac{\sqrt{1+x^2}}{x}\,dx = \frac{-y\,dy}{\sqrt{1+y^2}}.$$

$1+y^2 = p^2 \Rightarrow y\,dy = p\,dp \qquad \sqrt{1+x^2} = t \Rightarrow x\,dx = t\,dt$

$$\frac{t.\,t\,dt}{t^2-1} = \frac{-p\,dp}{p}$$

$$1\,dt + \frac{1}{t^2-1}\,dt + dp = 0$$

$$t + p + \frac{1}{2}\log\frac{t-1}{t+1} = 0$$

$$\sqrt{1+x^2} + \sqrt{1+y^2} + \frac{1}{2}\log\frac{\sqrt{1+x^2}-1}{\sqrt{1+x^2}+1} + C = 0 \quad \text{Ans}$$

• *Homogenous*

$$(4y+3x)\,dy + (y-2x)\,dx = 0$$

Put $y = mx \Rightarrow \frac{dy}{dx} = m + x\frac{dm}{dx}$

$$\cancel{(4m+3)x}\left[m + x\frac{dm}{dx}\right] = \cancel{x}\frac{(2-m)}{4m+3} - \frac{-4m^2-3m}{} = \frac{-4m^2-4m+2}{4m+3}$$

$$\frac{dm\,(4m+3)}{(-4m^2-4m+2)} = \frac{dx}{x}.$$

$$-\left[\frac{dm\,(-4m-2)}{(-2m^2-2m+1)} + \frac{(-1)dm}{(\quad)}\right] = \pm 2\frac{dx}{x}$$

$$-\log(1-2m-2m^2) + \int \frac{dm \times (1/2)}{\frac{1}{2} - v - v^2} = 2\log x$$

$$\rightarrow \log(x^2 c\{1-2m-2m^2\}) = \frac{1}{2}\int \frac{dm}{\left(\frac{\sqrt{3}}{2}\right)^2 - \left(v+\frac{1}{2}\right)^2}$$

$$= \frac{1}{2\sqrt{3}} \log\frac{(\sqrt{3}+2v+1)}{(\sqrt{3}-v-1)}$$

Put $m = y/x$ <u>Ans</u>

• $(2x^2+3y^2-7)x\,dx - (3x^2+2y^2-8)y\,dy = 0.$

$x^2 = u \Rightarrow x\,dx = du$, $y^2 = v \Rightarrow y\,dy = dv$

$(2u+3v-7)\,du - (3u+2v-8)\,dv = 0.$ — ①

$u = A+B$ $\qquad$ $v = C+D.$

$2A+2B+3C+3D-7$ $\quad$ / $3A+3B+2C+2D-8$

$2B+3D-7 = 0 = 3B+2D-8 \Rightarrow B = 2,\ D = 1.$

$u = a+2 \qquad v = c+1 \Rightarrow \boxed{a = u-2,\ c = v-1}$

Now ① $\Rightarrow$

$$\frac{dc}{da} = \frac{2a+3c}{3a+2c.}$$

$c = ma$

$$\frac{dc}{da} = m + a\frac{dm}{da}.$$

$$a\frac{dm}{da} = \frac{\cancel{a}(2+\cancel{3m}) - \cancel{3m}-2m^2}{a(3+2m)}$$

$$\Rightarrow \frac{dm\,(3+2m)}{2(1-m^2)} = \frac{da}{a.} \Rightarrow \frac{2da}{a} = \left(\frac{3}{1-m^2} - \frac{(-2m)}{1-m^2}\right)dm$$

$$2\log a = \frac{3}{2}\log\frac{1+m}{1-m} - \log(1-m^2) + \log c$$

Resubstitute all,

I) $\boxed{x\,dx + y\,dy + \frac{x\,dy - y\,dx}{x^2+y^2} = 0.}$

$$dx(x^3 + xy^2 + y) + dy(y^3 + yx^2 + x) = 0$$

$$\frac{\partial N}{\partial x} = 2xy + 1 \neq \frac{\partial M}{\partial y} = 2xy + 1.$$ ← Why Not ??

↓

[As we are discarding a factor of x^2+y^2]

CORRECT WAY :

$$dx\left(x - \frac{y}{x^2+y^2}\right) + dy\left(y + \frac{x}{x^2+y^2}\right) = 0$$

$$\frac{\partial M}{\partial y} = \frac{(-1)(x^2+y^2) + y \cdot 2y}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(y^2+x^2)^2}.$$

$$\frac{\partial N}{\partial x} = \frac{1.(x^2+y^2) - x \cdot 2x}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(y^2+x^2)^2}$$

$$\int x - \frac{y}{x^2+y^2}\,dx + \int y\,dy$$

$$\boxed{\frac{x^2}{2} - \tan^{-1}\frac{x}{y} + \frac{y^2}{2} = C}$$ A

. INSPECTION

$$d\left(\frac{y}{x}\right) = \frac{x\,dy - y\,dx}{x^2}$$

$$d\left(\frac{y^2}{x}\right) = \frac{2xy\,dy - y^2dx}{x^2}$$

$$d(\log xy) = \frac{x\,dy + y\,dx}{xy} \qquad d(xy) = x\,dy + y\,dx.$$

$$d\left(\tan^{-1}\frac{y}{x}\right) = \frac{\frac{x\,dy - y\,dx}{x^2}}{\frac{x^2+y^2}{x^2}} = \frac{x\,dy - y\,dx}{x^2+y^2}$$

$$d\left(\log\left(\frac{y}{x}\right)\right) = \frac{\frac{x\,dy - y\,dx}{x^2}}{\frac{y}{x}} = \frac{x\,dy - y\,dx}{xy}$$

$$d\left(\frac{1}{2}\log(x^2+y^2)\right) = \frac{x\,dx + y\,dy}{x^2+y^2}$$

Q $(1+xy)y\,dx + x(1-xy)dy = 0.$

$$\cancel{x}(y\,dx + x\,dy + xy^2\,dx - x^2y\,dy) = 0$$

$$\cancel{x\,d(xy) + d(x^2y^2)}$$

$$d(xy) + x^2y^2\left(\frac{dx}{x} - \frac{dy}{y}\right) = \frac{d(xy)}{x^2y^2} + d\left(\log\frac{x}{y}\right) = 0$$

$$d\left(\log\frac{x}{y} - \frac{1}{xy}\right) = 0 \Rightarrow \boxed{\log\frac{x}{y} - \frac{1}{xy} = C}$$ Ans

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- Integrating Factor

$$M\,dx + N\,dy = 0.$$

Case I : $M\,dx + N\,dy = 0$ homogenous & $(Mx + Ny) \neq 0$

Then, IF : $\dfrac{1}{(Mx+Ny)}$

$$M\,dx + N\,dy = \frac{1}{2}\left\{(Mx+Ny)\left[\frac{dx}{x} + \frac{dy}{y}\right] + (Mx - Ny)\left[\frac{dx}{x} - \frac{dy}{y}\right]\right\}$$

$$\Rightarrow \frac{M\,dx + N\,dy}{Mx+Ny} = \frac{1}{2}\left\{\left[\frac{dx}{x} + \frac{dy}{y}\right] + \frac{Mx-Ny}{Mx+Ny}\left[\frac{dx}{x} - \frac{dy}{y}\right]\right\} \quad -(1)$$

As $M\,dx + N\,dy$ is homogenous $\Rightarrow$ M, N of same degree in x, y

So, $\dfrac{Mx - Ny}{Mx + Ny}$ = some f^n of $\dfrac{x}{y} = g\left(\dfrac{x}{y}\right)$ — (2)

$$(1) \equiv \frac{M\,dx + N\,dy}{Mx+Ny} = \frac{1}{2}\left\{\left(\frac{dx}{x} + \frac{dy}{y}\right) + g\left(\frac{x}{y}\right)\left(\frac{dx}{x} - \frac{dy}{y}\right)\right\}$$

$$= \frac{1}{2}\left\{d(\log xy) + g\left(e^{\log(x/y)}\right) d\left(\log \frac{x}{y}\right)\right\}$$

Let $\boxed{g\left(e^{\log x/y}\right) = g_1(\log(x/y))}$

$$= \frac{1}{2}\left\{d(\log xy) + g_1\left(\log \frac{x}{y}\right) d\left(\log \frac{x}{y}\right)\right\}$$

$$= d\left[\tfrac{1}{2} \times \log xy + \tfrac{1}{2} \times \int g\{\log(x/y)\}\, d\{\log \tfrac{x}{y}\}\right]$$

So, $\dfrac{1}{Mx+Ny}$ is an I.F for equation $M\,dx + N\,dy = 0$

Case II: $Mdx + Ndy = 0 \equiv f_1(xy)\, y\, dx + f_2(xy)\, x\, dy = 0$

then $\dfrac{1}{Mx - Ny}$ is an integration factor if $Mx - Ny \neq 0$.

$$M\,dx + N\,dy = 0$$

$$f_1(xy)y\,dx + f_2(xy)x\,dy = 0 \qquad \frac{M}{f_1(xy)y} = \frac{N}{f_2(xy)x} = \mu$$

$$M = \mu f_1(xy)y \qquad N = \mu f_2(xy)x$$

$$\frac{M\,dx + N\,dy}{Mx - Ny} = \frac{1}{2}\left\{\frac{Mx+Ny}{Mx-Ny}\left(\frac{dx}{x}+\frac{dy}{y}\right) + \left(\frac{dx}{x}-\frac{dy}{y}\right)\right\}$$

$$= \frac{1}{2}\left\{\frac{f_1(xy)+f_2(xy)}{f_1(xy)-f_2(xy)}\, d(\log xy) + d\log\left(\frac{x}{y}\right)\right\}$$

$$= \frac{1}{2}\left\{h(xy)\, d(\log xy) + d\log\left(\frac{x}{y}\right)\right\}$$

$$= \frac{1}{2}\left\{h(e^{\log xy})\, d(\log xy) + d\left(\log\frac{x}{y}\right)\right\}$$

$$= \frac{1}{2}\left\{F(\log xy)\, d(\log xy) + d\left(\log\frac{x}{y}\right)\right\}$$

$$= \frac{1}{2}\left(\int f(\log xy)\, d(\log xy)\right) + d\left\{\frac{1}{2}\log\frac{x}{y}\right)$$

$\boxed{Q}$ $(xy^2 + 2x^2y^3)dx + (x^2y - x^3y^2)dy = 0$

$$(xy + 2x^2y^2)y\,dx + (xy - x^2y^2)x\,dy = 0 \qquad \boxed{f_1(xy)\,y\,dx + f_2(xy)\,x\,dy}$$

$$IF \Rightarrow \frac{1}{\cancel{x^2y^2} + 2x^3y^3 - \cancel{x^2y^2} + x^3y^3} = \frac{1}{x^3y^3}$$

$$\frac{1}{x^2y + 2x^3y^2 - xy^2 + x^2y^3}\left[\cancel{(xy^2 + 2x^2y^3)dx + (x^2y - x^3y^2)dy} = 0\right]$$

~~y+2xu~~

$$\frac{1}{x^2y+2x^3y^2-xy^2+x^2y^3}\left[(xy^2+2x^2y^3)\,dx+(x^2y-x^3y^2)\,dy\right]$$

Cas

$$\int \frac{y+2xy^2}{x+2x^2y^2-y+xy^2}\,dx+\int\frac{x-x^2y}{x+2x^2y^2-y+xy^2}\,dy$$

$$\frac{1}{x^3y^3}\left[(xy^2+2x^2y^3)\,dx+(x^2y-x^3y^2)\,dy\right]=0$$

$$\left(\frac{1}{x^2y}+\frac{2}{x}\right)dx+\left(\frac{1}{xy^2}-\frac{1}{y}\right)dy=0.$$

$$\frac{y\,dx+x\,dy}{x^2y^2}+\frac{2}{x}\,dx-\frac{1}{y}\,dy=0$$

$$\boxed{d\left(-\frac{1}{xy}\right)+2\log x-\log y=C.}$$ Ans

Case II : $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)$ is $f(x)$, then $e^{\int f(x)dx}$ is IF.

$$f(x)=\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right).$$

$$e^{\int f(x)}(M\,dx+N\,dy)=0 \rightarrow \left[\begin{array}{l} M_1=e^{\int f(x)}M \\ N_1=e^{\int f(x)}N. \end{array}\right.$$

$$M_1\,dx+N_1\,dy=0.$$

TS: $\frac{\partial M_1}{\partial y}=\frac{\partial N_1}{\partial x} \Rightarrow \frac{\partial M_1}{\partial y}=\frac{\partial M}{\partial y}e^{\int f(x)dx}.$ — ①

$$\frac{\partial N_1}{\partial x}=\frac{\partial N}{\partial x}e^{\int f(x)dx}+Ne^{\int f(x)dx}f(x) \quad -②.$$

$$① - ② \Rightarrow \cancel{e^{\int f(x)dx}}[Nf(x)] - N\cancel{e^{\int f(x)dx}}f(x)$$

$$=0 \quad \underline{\underline{\text{An}}}$$

Illy if $\underline{g(y)}$

A. $\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=f(x) \equiv e^{\int f(x)}$

B. $\frac{1}{M}\left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)=g(y) \equiv e^{\int g(y)}$

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· Linear ODE

[Q] $(1+y^2)\,dx = (\tan^{-1}y - x)\,dy$

- Either $\frac{1}{M}(N_x - M_y)$ as IF

OR.

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}.$$

$$IF = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}(y)}.$$

$$x e^{\tan^{-1}y} = \int e^{\tan^{-1}y}\cdot\frac{\tan^{-1}y}{1+y^2}\,dy = \int e^t\, t\, dt$$

$$= e^{\tan^{-1}y}(\tan^{-1}y - 1) + C \quad \text{Ans}$$

· Reducible to Linear. $\boxed{f'(y)\frac{dy}{dx} + P f(y) = Q}$

$\cancel{[Q]\ \frac{1}{x^3}\frac{dy}{dx} + \frac{x\sin 2y}{x^3} = \frac{x^3}{x^3}\cos^2 y}.$

$\cancel{\frac{1}{x^2} = t \Rightarrow -\frac{2}{x^3} = dt}$

[Q] $\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$

$$\sec^2 y\frac{dy}{dx} + 2x\tan y = x^3. \qquad \tan y = t, \quad \sec^2 y\frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dt}{dx} + 2xt = x^3 \Rightarrow e^{\int 2x\,dx} = e^{x^2}$$

$$\tan y\cdot e^{x^2} = \int x^3 e^{x^2}\,dx = \frac{1}{2}e^{x^2}(x^2-1) + C \quad \text{Ans}$$

Q $(x^2 - 2x + 2y^2)\,dx + 2xy\,dy = 0.$

L $\frac{1}{N}(M_y - N_x)$ OR.

$$-2y\frac{dy}{dx} = \frac{1}{x}(x^2 - 2x + 2y^2)$$

$$-2y\frac{dy}{dx} - \frac{2y^2}{x} = x - 2$$

$y^2 = t$, $2y\frac{dy}{dx} = \frac{dt}{dx}.$

$$\frac{dt}{dx} + \frac{2t}{x} = 2 - x$$

$$e^{\int \frac{2}{x}} = x^2$$

$$\boxed{y^2 \cdot x^2 = \frac{2x^3}{3} - \frac{x^4}{4} + c}$$ Ans

Q $(xy^2 + e^{-1/x^3})\,dx - x^2y\,dy = 0$

$$2\left(\frac{y\,dy}{dx} - \frac{y^2}{x} = \frac{e^{-1/x^3}}{x^2}\right)$$

$y^2 = t$

$2y\frac{dy}{dx} = \frac{dt}{dx}.$

$$\frac{dt}{dx} - \frac{2t}{x} = \frac{2e^{-1/x^3}}{x^2}$$

$$e^{\int -\frac{2}{x}dx} = e^{-2\log x} = \frac{1}{x^2}$$

$$\frac{y^2}{x^2} = \int \frac{2e^{-1/x^3}}{x^2 \cdot x^2} = \frac{2}{3}e^{-1/x^3} + c$$

Q $xy - \frac{dy}{dx} = y^3 e^{-x^2}$

$-\frac{1}{y^2} = t$

$$\frac{2}{y^3}\frac{dy}{dx} - \frac{2x}{y^2} = -2e^{-x^2}$$

$\frac{2}{y^3}\frac{dy}{dx} = \frac{dt}{dx}.$

$$\frac{dt}{dx} + 2xt = -2e^{-x^2}$$

$$e^{\int +2x} = e^{+x^2}$$

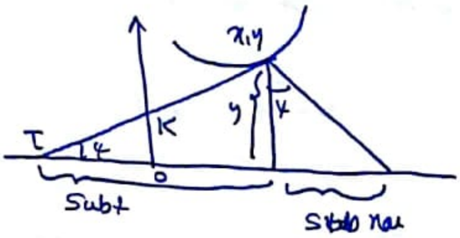
$$\frac{+e^{+x^2}}{y^2} = +\int 2 = 2x + C.$$

③ Geometrical

Subnorm $\equiv y\tan\psi = y\frac{dy}{dx}$

Subtan $\equiv y\cot\psi = y\frac{dx}{dy}$

Normal $\equiv y\sec\psi = y\sqrt{1+(dy/dx)^2}$ — $\frac{Y-y}{X-x} = \frac{-dx}{dy}$



Tangent $\equiv y\,\mathrm{cosec}\,\psi = y\sqrt{1+\left(\frac{dx}{dy}\right)^2}$

$\llcorner\ \frac{Y-y}{X-x} = \frac{dy}{dx}$

Radius of curvature $\equiv \boxed{\dfrac{\left(1+(dy/dx)^2\right)^{3/2}}{(d^2y/dx^2)}}$

- Polar (r,θ)

$\llcorner$ Subtangent $= r^2\,d\theta/dr$ $\left\{ \text{ortho } \frac{dr}{d\theta} \to -r^2\frac{d\theta}{dr} \right\}$

Subnormal $= \frac{dr}{d\theta}$

$\tan\theta = \frac{y}{x}$

Length of tangent $\equiv r\sqrt{1+\left(\frac{r\,d\theta}{dr}\right)^2}$

Length of normal $\equiv \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}$

$r\frac{d\theta}{dr} = \tan\phi \leftarrow$ Angle between tangent, radius

Vectorial Angle $=\theta$.

9. Curve for which portion of y-axis intercepted by tangent varies as cube of abscissa of point of contact.

$$Y - y = \frac{dy}{dx}(X - x) \Rightarrow X = 0 \Rightarrow Y = y - x\frac{dy}{dx}.$$

$$y - x\frac{dy}{dx} = kx^3 \Rightarrow \frac{dy}{dx} - \frac{y}{x} = -kx^2$$

$$e^{-\int \frac{1}{x}dx} = 1/x$$

$$\frac{y}{x} = \int -\frac{kx^2}{x} = -\frac{k}{2}x^2 + C \quad \text{Ans}$$

10. Find family of curves with tangent at $\frac{\pi}{4}$ with $xy = c$.

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} \qquad m_2 = \frac{dy}{dx} = -\frac{c}{x^2}$$

$$m_1 = \frac{dy}{dx}$$

$$1 = \frac{\frac{dy}{dx} + \frac{c}{x^2}}{1 - \frac{dy}{dx}\cdot\frac{c}{x^2}} \Rightarrow 1 - \frac{dy}{dx}\frac{c}{x^2} = \frac{dy}{dx} + \frac{c}{x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{\frac{x^2 - c}{x^2}}{\frac{x^2 + c}{x^2}} = \frac{x^2 + c}{x^2 + c} - \frac{2c}{x^2 + c} = 1 - \frac{2c}{x^2 + c}$$

$$\boxed{y = x - 2c\frac{1}{\sqrt{c}}\tan^{-1}\left(\frac{x}{\sqrt{c}}\right) + c'} \quad \text{Ans.}$$

Q Spherical rain evaporates at rate $\propto$ to surface area. Original $r = 3$ mm, 1 hour later $= 2$ mm. Find expression for r.

$$V = \frac{4}{3}\pi r^3 \qquad S = 4\pi r^2$$

$$\frac{dV}{dt} = -kS \Rightarrow \cancel{4\pi r^2}\frac{dr}{dt} = -k(\cancel{4\pi r^2})$$

$$\Rightarrow r = -kt + C. \Rightarrow C = 3$$

$$r = -kt + 3 \Rightarrow k = 1$$

$$\Rightarrow \boxed{r = 3 - t} \text{ Ans}$$

Q Find family of curves, ratio of y-intercept of tangent to radius vector is constant.

$$Y - y = \frac{dy}{dx}(X - x) \Rightarrow Y = y - x(dy/dx)$$

$$\frac{y - x(dy/dx)}{(x^2+y^2)^{1/2}} = k$$

$$\frac{dy}{dx} = \frac{k\sqrt{x^2+y^2} - y}{(-x)} \Rightarrow y = mx$$

$$\frac{dm}{\sqrt{1+m^2}} + k\frac{dx}{x} = 0$$

$$\sinh^{-1}(m) + k\log x = C \quad \text{Ans}$$

Q Curve for which sum of reciprocal of radius vector and polar subtangent is constant.

$$\frac{1}{r} + \frac{1}{r^2}\frac{dr}{d\theta} = k. \Rightarrow d\theta = \frac{dr}{r(kr-1)} = \left(\frac{k}{kr-1} - \frac{1}{r}\right)dr$$

$$\Rightarrow \boxed{kr - 1 = re^{\theta + c}} \text{ Ans}$$

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$\boxed{Q}$ Curve with radius of curvature $\propto$ Slope of tangent.

LATER

$$\left(1+\left(\frac{dy}{dx}\right)^2\right)^{3/2} = k\,\frac{dy}{dx}\cdot\frac{d^2y}{dx^2}$$

$$\frac{dy}{dx}=p \Rightarrow \quad \frac{(1+p^2)^{3/2}}{p} = k\,\frac{dp}{dx}.$$

$$dx = \frac{k\,p\,dp}{(1+p^2)^{3/2}} \qquad 1+p^2=t^2$$
$$p\,dp = t\,dt.$$

$$dx = \frac{k\,\cancel{t}\,dt}{t^{3}} \Rightarrow x = -\frac{k}{t}+c.$$

$$\Rightarrow x = \frac{-k}{\sqrt{1+p^2}}+C.$$

$$\Rightarrow \frac{(x-c)^2}{k} = \frac{1}{1+\left(\frac{dy}{dx}\right)^2} \Rightarrow \cancel{1+}\left(\frac{dy}{dx}\right)^2 = \frac{k}{(x-c)^2}-1$$

$$\frac{dy}{dx} = \sqrt{\frac{k-(x-c)^2}{(x-c)^2}}$$

$\boxed{8}$ $$\frac{x\,dx + y\,dy}{x\,dy - y\,dx} = \sqrt{\frac{a^2 - x^2 - y^2}{x^2+y^2}}$$

$x^2+y^2 = r^2$

To polar $\equiv$ $x = r\cos\theta$, $y = r\sin\theta$

$\Rightarrow$ $x\,dx + y\,dy = r\,dr$.

$y/x = \tan\theta$.

$$\frac{x\,dy - y\,dx}{x^2} = \sec^2\theta\, d\theta.$$

$$\frac{r\,dr}{r^2\cancel{\cos^2\theta}\,\cancel{\sec^2\theta}\,d\theta} = \sqrt{\frac{a^2-r^2}{r^2}}$$

$$\frac{dr}{\sqrt{a^2-r^2}} = d\theta. \Rightarrow \theta + C = \sin^{-1}\left(\frac{r}{a}\right).$$

$$\Rightarrow \tan^{-1}\left(\frac{y}{x}\right) + C = \sin^{-1}\left(\frac{\sqrt{x^2+y^2}}{a}\right)$$

(4) Orthogonal trajectories

Q. $x^2+y^2+2gx+C=0$ g is parameter - CHECK (Pg 34)

$2x+2y\frac{dy}{dx}+2g=0 \Rightarrow g=-(x+yy')$

$x^2+y^2+2x(-x-yy')+C=0$

$-x^2+y^2-2y\frac{dy}{dx}+C=0. \Rightarrow 2y\frac{dy}{dx}-y^2=x^2-C$

$\Downarrow$ $y^2=t$ $\quad 2y\frac{dy}{dx}=\frac{dt}{dx}$

$\frac{dt}{dx}-t=x^2-c$

$t.e^{-x}=\int e^{-x}(x^2-c)$

$=-e^{-x}x^2+\int e^{-x}2x$
$+ce^{-x}$

$=e^{-x}(c-x^2)-e^{-x}2x+2\int e^{-x}$

$y^2\cancel{e^{-x}}=\cancel{e^{-x}}(2+c-x^2-2x)$

Q. Ortho of $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$

$\frac{x}{a^2+\lambda}+\frac{y}{b^2+\lambda}\frac{dy}{dx}=0 \Rightarrow x(b^2+\lambda)+yy'(a^2+\lambda)=0$

$\Rightarrow \lambda=-\frac{(yy'a^2+xb^2)}{x+yy'} \Rightarrow$ Find $a^2\times T$, $b^2\times T$

$\frac{x^2(x+yy')}{a^2x+\cancel{a^2yy'}-\cancel{a^2yy'}-xb^2}+\frac{y^2(x+yy')}{b^2yy'-a^2yy'}=1$

$\hookrightarrow \frac{(x+yy')(x^2yy'+y^2x)}{xyy'}=a^2-b^2$

For ortho $y'\rightarrow\frac{-1}{y'}$

$\left(x-\frac{y}{y'}\right)\left(\frac{-x^2y}{y'}+y^2x\right)=a^2-b^2$

$\llcorner$ Same as previous obtained

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· Polar ≡

Generally take log to ease process

$$\boxed{\frac{dr}{d\theta} \longrightarrow -r^2 \frac{d\theta}{dr}}$$

(8) Ortho trajectories of $r^n \sin n\theta = a^n$

Take log → $n \log r + \log \sin n\theta = n \log a$

$$\frac{dr}{d\theta} \frac{n}{r} + \frac{n \cos n\theta}{\sin n\theta} = 0 \Rightarrow \cancel{\frac{dr}{\theta} = n \quad -\cot n\theta - \tan n\theta \, d\theta.}$$

$$\frac{dr}{r} = -\cot n\theta \, d\theta.$$

Replace ⇒ $-r^2 \frac{d\theta}{r\, dr} = +\cot n\theta \Rightarrow \tan n\theta \, d\theta = \frac{dr}{-r}$

$$\Rightarrow \log r = \frac{(-1)}{n} \log \cos n\theta + c$$

$p = \frac{dy}{dx}$

(5) Order 1, degree > 1

Q $p^2 - 2p\cosh x + 1 = 0$

$p^2 - p(e^{x}+e^{-x}) + 1 = 0$

$p(p-e^{x}) - e^{-x}(p-e^{x}) = 0 \Rightarrow (p-e^{-x})(p-e^{x}) = 0$

$\frac{dy}{dx} = e^{-x} \Rightarrow y = -e^{-x}+c \;;\; \frac{dy}{dx} = e^{x} \Rightarrow y = e^{x}+c$

$$\boxed{(y+e^{-x}-c)(y-e^{x}-c)=0}$$

For x

Q $(2x-b)p = y - ayp^2 \quad a>0 \quad —(1)$

$2x-b = \frac{y}{p} - ayp$

$\Rightarrow \cancel{\frac{2}{p} = \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - ap - ay\frac{dp}{dy}} \Rightarrow \cancel{\frac{1}{p}+ap - y\frac{dp}{dy}\left(\frac{1}{p^2}+a\right)}$

$\Rightarrow \cancel{\frac{1}{p}+ap^2 - \frac{dp}{dy}\left(-ayp^2-\frac{1}{p^2}\right)} \Rightarrow \cancel{\left(a+\frac{1}{p^2}\right)\left(p-y\frac{dp}{dy}\right)=0}$

$\frac{2}{p} = \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - ap - ay\frac{dp}{dy} = 0$

$\frac{1}{p} + ap + \frac{y}{p}\frac{dp}{dy}\left(ap+\frac{1}{p}\right) = 0$

$\left(1+\frac{y}{p}\frac{dp}{dy}\right)\underbrace{\left(ap+\frac{1}{p}\right)} = 0.$ * This gives singular Solution

No $\frac{dp}{dy}$ So ignore

$\frac{dp}{dy} = \frac{-p}{y} \Rightarrow \log(py) = c \Rightarrow \frac{dy}{dx} = \frac{c}{y} = p.$

Put in (1) $\Rightarrow \boxed{(2x-b)\frac{c}{y} = y - \frac{ac^2}{y}}$ Ans

5 $y - 2xp + yp^2 = 0.$

$$x = \frac{y(1-p^2)}{2p} = \frac{y}{2p} - \frac{yp}{2}$$

$$\frac{\cancel{2}}{p} = \frac{1}{\cancel{2}p} - \frac{y}{\cancel{2}p^2}\frac{dp}{dy} - \frac{p}{\cancel{2}} - \frac{y}{\cancel{2}}\frac{dp}{dy}$$

$$y\frac{dp}{dy}\left(\frac{1}{p^2} - 1\right) + p\left(\frac{\cancel{2}}{p^2} - \cancel{p}1\right) = \left(y\frac{dp}{dy} + p\right)\left(\frac{1}{p^2} - 1\right) = 0.$$

$$\frac{dp}{dy} = \frac{-p}{y} \Rightarrow py = c \Rightarrow p = c/y.$$

$$y - 2\frac{xc}{y} + \frac{\cancel{y}c^2}{y} \Rightarrow \boxed{y^2 - 2cx + c^2 = 0.}$$

. For y

8 $y = 3px + 4p^2.$

$$\cancel{p} = 3x\frac{dp}{dx} + \cancel{3}p + 8p\frac{dp}{dx} \Rightarrow \frac{-2p}{8p+3x} = \frac{dp}{dx} \Rightarrow \frac{dx}{dp} = -4 - \frac{3x}{2p}$$

$$\Rightarrow \frac{dx}{dp} + \frac{3}{2}\frac{x}{p} = -4.$$

$$I.F = e^{\frac{3}{2}\int\frac{1}{p}dp} = p^{3/2}.$$

$$x \cdot p^{3/2} = \int -4p^{3/2}\,dp = \frac{-4.2}{5}p^{5/2} + c.$$

$$x = -\frac{8}{5}p + cp^{-3/2} \rightarrow \text{Put in } (1) \Rightarrow$$

$$y = 3p\left(-\frac{8}{5}p + cp^{-3/2}\right) + 4p^2$$

These give solutions

x & y in terms of p also a solution

No further steps

Clairaut

$y = px + f(p) \rightarrow \boxed{y = cx + f(c)}$

L Standard form = Easy

Q $p^2 x(x-2) + p(2y - 2xy - x + 2) + y^2 + y = 0$

$p^2x^2 - 2pxy + y^2 - 2p^2x + 2py - px + 2p + y = 0$

$(px-y)^2 - 2p(px-y) - 1(px-y) + 2p = 0$

$(px-y)(px-y-2p) - 1(px-y-2p) = 0$

$(px-y-1)(px-y-2p) = 0$

L ~~pc - y = 2c~~

~~(pc - y - 1)~~

$(cx-y-1)(cx-y-2c) = 0$

L Reducible

Q $yp^2 + x^3p - x^2y = 0 \Rightarrow yp^2 + x^3p = yx^2 \Rightarrow \frac{y}{x^2}p^2 + xp = y$

$\Rightarrow \cancel{(y - px) = \frac{yp^2}{x^2}} \Rightarrow \cancel{y^2 = \left(\frac{py}{x}\right)x^2} + \left(\frac{py}{x}\right)^2$

$x^2 = u \quad y^2 = v \Rightarrow \frac{y}{x}\frac{dy}{dx} = \frac{dv}{du} \Rightarrow \frac{yp}{x} = \frac{dv}{du}$

$y^2 = \frac{p^2y^2}{x^2} + \left(\frac{py}{x}\right)x^2$

$v = \left(\frac{dv}{du}\right)^2 + \left(\frac{dv}{du}\right)u$

$\Rightarrow v = cu + c^2 \Rightarrow \boxed{y^2 = cx^2 + c^2}$

Q. $(x^2+y^2)(1+p)^2 - 2(x+y)(1+p)(x+yp) + (x+yp)^2 = 0.$

(2005)

$x^2+y^2 = v$ $\qquad$ $x+y = u.$

Miss IT UP!

$2(x\,dx + y\,dy) = dv$ $\qquad$ $dx+dy = du.$

$$Q = \frac{dv}{du} = \frac{2(x\,dx + y\,dy)}{(dx+dy)} = 2\,\frac{(x+yp)}{1+p}.$$

$$v - \cancel{2}uQ + \frac{Q^2}{4} = 0.$$

$$v = uc - \frac{c^2}{4} \Rightarrow \boxed{x^2+y^2 = c(x+y) - \frac{c^2}{4}}$$ Ans

Substitution for Clairaut's

- $(y-px)(y+px)\ldots \Rightarrow x^2 = u,\ y^2 = v.$
- $e^{ax}, e^{by} \Rightarrow e^{ax} = u,\ e^{by} = v$

$y-px$	$x^2=u,\ y^2=V$
$y+px$	$xy=V$
$x+py$	$x^2+y^2=V$
$p-1$	$x+y=u$

* Singular Solutions

2 approaches

└ W/o general solution.

└ $f(x,y,p)=0$ (+) $\frac{\partial f}{\partial p}=0$.

$\underbrace{\hspace{3cm}}$

Eliminate p

⇓

If $f(x,y,p)=0$ if quadratic in p ⇒

$\boxed{B^2-4ac=0}$

Substitute linear factors obtained with $p=0$.

└ After gen. solution :

└ Eliminate $c = \phi(x,y,c)$ (+) $\frac{\partial \phi}{\partial c}=0$.

└ Direct when quadratic in c.

$B^2-4AC=0$

└ Substitute linear factors.

⇒ If y obtained ⇒ Find $\frac{dy}{dx}$ and put them in given

If x obtained ⇒ Find $\frac{dx}{dy}=\frac{1}{p}$, Rewrite equation in $\frac{1}{p}$

Or Eliminate c between two equations

Q $8ap^3 = 27y$ Find gen, and singular solutions.

$P = \frac{3}{2}\left(\frac{y}{a}\right)^{1/3} \Rightarrow \frac{dy}{y^{1/3}} = \frac{3\,dx}{2a^{1/3}} \Rightarrow \cancel{\frac{y^{2/3}}{2/3} = \frac{3x}{2a^{1/3}} + c}$

$\frac{2 \times 3\, a^{1/3}}{3 \times 2 \times 3} \times y^{2/3} = x + c.$

$ay^2 = (x+c)^3$ —(1)

2nd approach →

$3(x+c)^2 = 0 \Rightarrow \boxed{c = -x}$ from

Put in Sol^n $\Rightarrow ay^2 = 0. \Rightarrow y = 0 \Rightarrow p = 0.$ —(1)

Put in original $\Rightarrow$ Satisfies $\Rightarrow \boxed{y=0}$ is singular sol^n

1st approach →

$8ap^3 = 27y.$

$24ap^2 = 27p \Rightarrow p = 0$

$27y = 0 \Rightarrow \boxed{y=0}$ is sol.^{n} Ans

9. Find gen and sing. Solutions of $y = x - 2ap + ap^2$ and interpret geometrically.

$$p = 1 - 2a\frac{dp}{dx} + 2ap\frac{dp}{dx} \Rightarrow p - 1 = \frac{dp}{dx}a(2p - 2)$$

$$\Rightarrow \frac{p-1}{2(p-1)} = a \cdot \frac{dp}{dx}.$$

$$\Rightarrow \left(2a\frac{dp}{dx} - 1\right)\cancel{(p-1)} = 0$$
as no $\frac{dp}{dx}$ term

$$\frac{dp}{dx} = \frac{1}{2a} \Rightarrow p = \frac{x}{2a} + \frac{c}{2a}.$$

Substitute in given ODE

Do not integrate again or 2 arbit. Constants

Sol^n $\Rightarrow$ $y = x - 2a\left(\frac{x}{2a} + \frac{c}{2a}\right) + a\left(\frac{x}{2a} + \frac{c}{2a}\right)^2$

$$\Rightarrow \boxed{y = \cancel{x} - \cancel{x} - c + \frac{1}{4a}(x+c)^2. \quad -(1)}$$

Quadratic in $c \Rightarrow$

$\cancel{-4ay + c + c^2}$

$$c^2 + 2cx - 4ac + x^2 - 4ay = 0.$$

c-discriminant is $\Rightarrow$

$$B^2 - 4AC \equiv (2x - 4a)^2 - 4.(x^2 - 4ay) = 0$$

$$\Rightarrow \cancel{x^2} + 4a^2 - 4ax - \cancel{x^2} + 4ay = 0$$

$$\Rightarrow 4a^2 - 4a(x - y) = 0$$

$$4a(a - x + y) = 0 \Rightarrow \boxed{x - y = a}$$

Better if solve both p & c-discriminants $\rightarrow$ Check common

Put in original $\Rightarrow$

$y = \cancel{a} + y \cancel{- 2a} + \cancel{a}$ $\Rightarrow$ Satisfies $\Rightarrow$ $x - y - a = 0$ is singular solution

(1) $\equiv (x+c)^2 = 4a(y+c)$ $\rightarrow$ family of parabolas touching $x - y - a = 0$ line

Q Reduce $xyp^2 - p(x^2+y^2-1) + xy = 0$ to Clairaut's $x^2 = u$, $y^2 = v$. Show the equation represents a family of conics touching a square.

$2x\,dx = du \quad 2y\,dy = dv. \quad \Rightarrow \frac{y}{x} p = Q.$

Mult. by y, divide by $x^3 \Rightarrow$

$$p^2 \frac{y^2}{x^2} - p\frac{y}{x} - \frac{py^3}{x^3} + p\frac{y}{x^3} + \frac{y^2}{x^2} = 0.$$

$$Q^2 u - Qu - Q\frac{v}{u} + \frac{Q}{u} + \frac{v}{u} = 0.$$

$$v\,\cancel{(Q-1)} = Qu\frac{(Q-1)}{(Q-1)} + \frac{Q}{(Q-1)}$$

$$v = cu + \frac{c}{c-1} \Rightarrow y^2 = cx^2 + \frac{c}{c-1}.$$

Answer is $y^2 = cx^2 - \frac{c}{c-1}$ ← See How??

$\Downarrow$.

$$c^2x^2 - c(x^2+y^2-1) + y^2 = 0$$

c-discriminant $\Rightarrow (x^2+y^2-1)^2 - 4x^2y^2 = 0$

$$(x^2+y^2-1-2xy)(x^2+y^2-1+2xy) = 0$$

$$((x-y)^2-1)((x+y)^2-1) = 0$$

$(x-y-1)(x-y+1)(x+y+1)(x+y-1) = 0$ ← Check for each

All these singular $\Rightarrow$ form a square

So, General Solution ≡ Conics which touch squares ≡ Sing. Solutions

$\Uparrow$

TRICK

4.31. $4p^2x(x-a)(x-b) = \{3x^2 - 2x(a+b) + ab\}^2$

- $p^2y^2\cos^2\alpha - 2pxy\sin^2\alpha + y^2 - x^2\sin^2\alpha = 0$

$\Downarrow$

Quadratic in py (Take $y^2\cos$ constant)

Linear Differential Equations

- Roots $\equiv \alpha \pm \sqrt{\beta} \Rightarrow e^{\alpha x}\{C_1 \cosh \sqrt{\beta}x + C_2 \sinh x\sqrt{\beta}\}$

→ Finding Particular Integral. $f(D)y = X$

(I) $X = e^{\alpha x}$. —— $\frac{1}{f(\alpha)} e^{ax}$ $\quad f(a) \neq 0$

—— $\frac{x^n}{n!} e^{\alpha x}$ where $f(D) = (D-\alpha)^n$.

(II) $X = \sin ax / \cos ax \longrightarrow \frac{1}{\phi(D^2)} \sin ax/\cos ax = \frac{1}{\phi(-a^2)} \sin ax/\cos ax$

└ Replace all D^2 terms by $(-a^2)$

* Repeated application leads to $\frac{1}{aD+b} \times \frac{aD-b}{aD-b}$ to get D^2 again in D^2.

If $\frac{1}{D^2+a^2} \sin ax/\cos ax = \frac{x}{2}\int \sin ax/\cos ax \, dx$.

(III) $X = x^m \longrightarrow \frac{1}{f(D)} = \frac{1}{D^\alpha[1+\phi(D)]^n} = \frac{[1+\phi(D)]^{-n}}{D^\alpha}$.

$(1+x)^n = 1 + nx + {}^nC_2 x^2 + \dots \dots$

(IV) $X = e^{ax} V \longrightarrow \frac{1}{f(D)} e^{ax} V = e^{ax} \frac{1}{f(D+a)} V$.

(V) $X = xV \rightarrow x\frac{1}{f(D)} V + \cancel{\left[\frac{d}{dD} f(D)\right] V} \left[\frac{d}{dD}\left(\frac{1}{f(D)}\right)\right] V$.

Master formula $\equiv \frac{1}{D-\alpha} X = e^{\alpha x}\int X e^{-\alpha x} dx$

$\boxed{Q}$ $(D^4 - 2D^3 + 5D^2 - 8D + 4)y = e^x$.

Auxiliary Equation $\equiv (D-1)^2(D^2+4) = 0$ $\quad D = 1, 1, \pm 2i$

~~General~~ Homogen Solution $\rightarrow (c_1 + c_2 x)e^x + c_3 \cos 2x + c_4 \sin 2x$

Particular $\rightarrow$ $\frac{1}{(D-1)^2(D^2+4)} e^{1x} = \frac{1}{(D-1)^2(1+4)} e^x = \frac{1}{5} \cdot \frac{e^x}{(D-1)^2}$.

$= \frac{1}{5} \frac{x^2}{2!} e^x$.

$y = (c_1 + c_2 x)e^x + c_3 \cos 2x + c_4 \sin 2x + \frac{x^2}{10} e^x$. Ans

$\boxed{Q}$ If $\frac{\partial^2 x}{\partial t^2} + \frac{g}{b}(x-a) = 0$ $\quad x = a', \frac{dx}{dt} = 0$ when $t = 0$

$\left(D^2 + \frac{g}{b}\right)x = \frac{g}{b}a$

~~General~~ Homoge Solⁿ: $c_1 \cos \sqrt{\frac{g}{b}}\, t + c_2 \sin \sqrt{\frac{g}{b}}\, t$.

$PI \equiv \frac{1}{D^2 + \frac{g}{b}} \frac{g}{b} a e^{0x} \Rightarrow \frac{g}{b} a \cdot \frac{1}{0 + \frac{g}{b}} = a$.

So, Solⁿ $\Rightarrow \boxed{c_1 \cos \sqrt{\frac{g}{b}}\, t + c_2 \sin \sqrt{\frac{g}{b}}\, t + a} = x$.

$$\left.\begin{aligned} a' &= c_1 + a \Rightarrow c_1 = a' - a. \\ 0 &= \sqrt{\frac{g}{b}}\, c_2 \Rightarrow c_2 = 0 \end{aligned}\right\} \text{Ans}$$

Q $(D^4 - m^4)y = \sin mx$

Homog ⇒ $c_1 e^{mx} + c_2 e^{-mx} + c_3 \cos mx + c_4 \sin mx.$

PI ⇒ $\dfrac{1}{(D^2)^2 - m^4} \sin mx = \dfrac{1}{(D^2+m^2)(D^2-m^2)} \sin mx.$

$= -\dfrac{1}{2m^2} \dfrac{1}{D^2+m^2} \sin mx.$

$= -\dfrac{1}{2m^2} \dfrac{x}{2} \int \sin mx = +\dfrac{x}{4m^3} \cos mx$

Q Solve $(D^2 + 2D + 10)y + 37 \sin 3x = 0$ and find y when $x = \pi/2$ if $y = 3$, $\frac{dy}{dx} = 0$ when $x = 0$.

Homog ⇒ $\dfrac{-2 \pm \sqrt{-36}}{2} = (-1 \pm 3i)$

$L e^{-x}(c_1 \cos 3x + c_2 \sin 3x)$

PI ⇒ $\dfrac{1}{D^2 + 2D + 10}(-37 \sin 3x) = \dfrac{-37}{-3^2 + 2D + 10} \sin 3x = \dfrac{-37}{2D+1} \sin 3x$

$= \dfrac{-37(2D-1)}{4D^2 - 1} \sin 3x = (2D-1) \sin 3x = 2.3 \cos 3x - \sin 3x.$

General $= e^{-x}(c_1 \cos 3x + c_2 \sin 3x) + 6 \cos 3x - \sin 3x$

$c_1 + 6 = 3$, $3c_2 = c_1 + 3$ ⇒ $\boxed{c_1 = -3,\ c_2 = 0}$

Q Solve $(D^3 + 3D^2 + 2D)y = x^2.$

Homog → $D = 0, -1, -2$ $\quad c_1 + c_2 e^{-x} + c_3 e^{-2x}.$

$$\frac{1}{D(D^2+3D+2)}(x^2) = \frac{1}{2D}\left[1 + \frac{D^2}{2} + \frac{3D}{2}\right]^{-1}(x^2)$$

$$= \frac{1}{2D}\left[1 - \left(\frac{D^2}{2} + \frac{3D}{2}\right) + \left(\frac{D^2}{2} + \frac{3D}{2}\right)^2 - \cdots\right] x^2$$

$$= \frac{1}{2D}\left(x^2 - 1 - 3x + \frac{9.2}{4}\right)$$

$$= \frac{1}{2.}\int x^2 - 3x + \frac{7}{2}\, dx = \frac{x^3}{6} - \frac{3x^2}{4} + \frac{7x}{4}.$$

Q $(D^4 + D^2 + 1)y = e^{-x/2}\cos\left(\frac{\sqrt{3}x}{2}\right)$ [1993]

Homog →. $D^4 + D^2 + 1 = 0 \Rightarrow (D^2+1)^2 - D^2 = 0$

$$\Rightarrow (D^2+D+1)(D^2-D+1) = 0$$

$$\frac{-1 \pm \sqrt{3}i}{2}, \quad \frac{1 \pm \sqrt{3}i}{2}.$$

$\Rightarrow$ $e^{-x}\left(C_1 \cos\frac{\sqrt{3}x}{2} + C_2 \sin\frac{\sqrt{3}x}{2}\right) + e^{x}\left(C_3 \cos\frac{\sqrt{3}x}{2} + C_4 \sin\frac{\sqrt{3}x}{2}\right)$

PI ⇒ $\frac{1}{D^4 + D^2 + 1} e^{-x/2}\cos\left(\frac{\sqrt{3}x}{2}\right).$

$$e^{-x/2}\frac{1}{\left(D-\frac{1}{2}\right)^4 + \left(D-\frac{1}{2}\right)^2 + 1}\cos\left(\frac{\sqrt{3}x}{2}\right).$$

$$e^{-x/2}\frac{1}{D^4 - \frac{4}{2}D^3 + \frac{6}{4}D^2 - \frac{4}{8}D + \frac{1}{16} + D^2 + \frac{1}{4} - D + 1}\cos\left(\frac{\sqrt{3}x}{2}\right).$$

Put $D^2 = -\frac{3}{4}$.

$$e^{-x/2} \frac{1}{\frac{9}{16} + \frac{4}{2} \cdot \frac{3}{4} D + \frac{6}{4}^3 \times \frac{(-3)}{4_2} - \frac{4}{8} \cancel{D} \cancel{D} + \frac{21}{16} - \frac{3}{4}} = 0.$$

So, $D^2 + 3/4$ is a factor.

$$e^{-x/2} \frac{1}{D^4 - 2D^3 + \frac{5}{2}D^2 - \frac{3}{2}D + \frac{21}{6}} \cos\left(\frac{\sqrt{3}x}{2}\right)$$

$$e^{-x/2} \frac{1}{\left(D^2+\frac{3}{4}\right)\left[D^2 - 2D + 7/4\right]} \cos\left(\frac{\sqrt{3}x}{2}\right) = e^{-x/2} \frac{1}{\left(D^2+\frac{3}{4}\right)(1-2D)} \cos\frac{\sqrt{3}x}{2}.$$

$$= \frac{1}{4} e^{-x/2} \cdot \frac{(1+2D)}{(D^2+3/4)} \cos\left(\frac{\sqrt{3}x}{2}\right)$$

$$= \frac{1}{4} e^{-x/2} \left[\frac{1}{D^2+3/4} \cos\frac{\sqrt{3}x}{2} - \sqrt{3} \cdot \frac{1}{D^2+\frac{3}{4}} \sin\frac{x\sqrt{3}}{2}\right]$$

$$= \frac{1}{4} e^{-x/2} \left[\frac{\cancel{2}x}{2\sqrt{3}} \int \cos\frac{\sqrt{3}x}{2} - \sqrt{3} \cdot \frac{\cancel{2}x}{\cancel{\sqrt{3}}2} \int \sin\frac{x\sqrt{3}}{2}\right]$$

$$= \frac{x}{4\sqrt{3}} \times e^{-x/2} \left\{\left(\sin\frac{\sqrt{3}x}{2}\right) + \sqrt{3}\cos\left(\sqrt{3}x/2\right)\right\}.$$

$\boxed{8}$ $(D^2-5D+6)y = e^{4x}(x^2+9)$.

$\rightarrow C_1e^{3x} + C_2e^{2x}$

$$\rightarrow e^{4x}\frac{1}{(D+4)^2-5(D+4)+6}(x^2+9).$$

$$e^{4x}\frac{1}{D^2+3D+2}(x^2+9).$$

$$\frac{e^{4x}}{2}\left(1+\frac{D^2+3D}{2}\right)^{-1}[x^2+9].$$

$$\frac{e^{4x}}{2}\left[1-\left(\frac{D^2+3D}{2}\right)+\left(\frac{D^2+3D}{2}\right)^2, \dots\right](x^2+9).$$

$$\frac{e^{4x}}{2}\left[x^2+9-\frac{1}{2}(2+6x)+\frac{1}{4}(18)\right].$$

$$\frac{e^{4x}}{2}\left(x^2-3x+\frac{25}{2}\right) = PI.$$

9. $(D^2+2D+1)y = x\sin x$

$C.F \equiv C_1 e^{-x} + C_2 x e^{-x}$

$PI \equiv \frac{1}{(D+1)^2} x\sin x = x\frac{1}{(D+1)^2}\sin x + \cancel{2(D+1)\sin x}$

$\frac{d}{dD}\left(\frac{1}{(D+1)^2}\right)\sin x \quad = x \frac{1}{2D}\sin x + 2\cancel{(\cos x+\sin x)}$

$\Downarrow$. $\quad = \frac{x}{2}(-\cos x) + \cancel{2\cos x+\sin x}$.

$\frac{-2}{(D+1)^3}\sin x = \frac{-2}{D^3+3D^2+3D+1}\sin x = \frac{-2}{-D-3+3D+1}\sin x$

$+$

$\frac{-2}{2(D-1)}\sin x \Rightarrow \frac{-(D+1)}{D^2-1}\sin x = \frac{+(D+1)}{2}\sin x = \left(\frac{\cos x+\sin x}{2}\right)$

Alternative:

9. $(D^2+1)^2 = 24x\cos x$.

$C.F \equiv (C_1+C_2x)\cos x + (C_3+C_4x)\sin x$.

$24.\frac{1}{(D^2+1)^2} x\cos x = 24\frac{1}{(D^2+1)^2} Re\{xe^{ix}\}$

$= 24e^{ix}\frac{1}{((D+i)^2+1)^2} x = 24e^{ix}\frac{1}{(D^2+2Di)^2} x$

$= 24e^{ix}\frac{1}{(2Di)^2}\left[1+\frac{D}{2i}\right]^{-2} x$.

$= -6e^{ix}.\frac{1}{D^2}\left[1+\frac{\cancel{2}iD}{\cancel{2}i}+\dots\right] x$.

$= -6e^{ix}\iint (x+i) = -6e^{ix}\left(\frac{x^3}{6}+\frac{ix^2}{2}\right)$.

$= -6e^{ix}$

$PI = Re\left\{-6e^{ix}\left(\frac{x^3}{6}+\frac{ix^2}{2}\right)\right\} = \boxed{3x^2\sin x - x^3\cos x}$

Q $(D^4 - 4D^2 - 5)y = e^x(x + \cos x)$.

$C.F = \boxed{C_1 e^{5x} + C_2 e^{-x}}$.

$PI \Rightarrow e^x \cdot \dfrac{1}{(D+1)^4 - 4(D+1)^2 - 5}(x + \cos x)$.

$$e^x \cdot \frac{1}{D^4 + 4D^3 + 2D^2 - 4D - 8}(x + \cos x).$$

$$\frac{e^x}{(-8)}\left[1 - \left(+\frac{D}{2} + \frac{D^2}{4} + \dots\right)\right]x.$$

$$\boxed{-\frac{e^x}{8}\left(x + \frac{1}{2}\right)}.$$

$$\frac{e^x}{1 - 4D - 2 - 4D - 8}\cos x$$

$$\frac{e^x(-8D+9)\cos x}{(-8D-9)(-8D+9)}$$

$$\frac{e^x(-8D+9)(\cos x)}{-64-81}.$$

$$\boxed{-\frac{e^x}{145}(8\sin x + 9\cos x)}$$

→ Undetermined Coefficients.

• $x^n \sin ax \longrightarrow (A_0 + A_1 x + .. A_n x^n) \sin ax +$
$(A_0' + A_1' x + .. A_n' x^n) \cos ax.$

• Case ≡ $C.F = (C_1 + C_2 x) e^{2x}$

$RHS = e^{2x}.$

Trial → $(a_1 + a_2 x + a_3 x^2) e^{2x}$

Q $(D^3 + 3D^2 + 2D) y = x^2 + 4x + 8.$

$C.F \Rightarrow C_1 + C_2 e^{-x} + C_3 e^{-2x}.$ | guess = $Ax^2 + Bx + C.$

C_1 is common with $\cancel{x^2+4x+8}$

Multiply x^α to ~~x^2+4x+8~~ guess s.t no common ~~with~~ between C.F and Guess

Here $\alpha = 1 \Rightarrow$

$ax + bx^2 + cx^3 \equiv$ trial. → $a = 11/4.$ $b = 1/4$ $c = 1/6$

Q $(D^2 - 4D + 4) y = x^3 e^{2x} + x e^{2x}.$

$C.F = (C_1 + C_2 x) e^{2x}.$

Trial ⇒ $a_1 x^3 e^{2x} + a_2 x^2 e^{2x} + a_3 x e^{2x} + a_4 e^{2x}.$ (only RHS based)

We need no common between $\boxed{\text{trial \& C.F}}$

New trial ⇒ $x^2 \times$ Old Trial

$= a_1 x^5 e^{2x} + a_2 x^4 e^{2x} + a_3 x^3 e^{2x} + a_4 x^2 e^{2x}$ ✓.

*Cauchy Euler Equations.

$$a_0 x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + \dots = X.$$

Put $x = e^z \rightarrow \log x = z.$

$$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz} \Rightarrow x\frac{dy}{dx} = \frac{dy}{dz} \Rightarrow xD = D_1.$$

Again $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right).$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx}$$

$$\frac{d^2y}{dx^2} = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$

$$x^2D^2 = (D_1^2 - D_1)y. = D_1(D_1 - 1)y.$$

$$x^n D^n = D_1(D_1-1)\dots(D_1 - n+1).$$

$\boxed{x = e^z.}$

Q $(x^3D^3 + 2x^2D^2 + 2)y = 10\left(x + \frac{1}{x}\right).$

$(D_1(D_1-1)(D_1-2) + 2D_1(D_1-1) + 2)y = 10(e^z + e^{-z}).$

C.F $\Rightarrow (D_1^3 - D_1^2 + 2)y = 0 \rightarrow -1, 1\pm i \Rightarrow c_1e^{-z} + e^z(c_2\cos z + c_3 \sin z)$

$10. \frac{1}{(D+1)(D^2-2D+2)}(e^z + e^{-z}) \rightarrow \cancel{10}^2. \frac{1}{(D+1)(1+2+2)}e^{-z} = \frac{2e^{-z}}{D+1}$

$= \frac{2e^{-z}}{(D-1+1)}.1.$

$= 2e^{-z}.z$

$= \boxed{\frac{2\log x}{x}.}$

$\cancel{10}^5. \frac{1}{\cancel{2}(1)}e^z = \boxed{5x}$

Q. Find λ s.t. solutions of $x^2(d^2y/dx^2)+3x(dy/dx)-\lambda y=0$ tends to 0 as $x\to\infty$.

$x=e^z$

$(D_1(D_1-1)+3D_1-\lambda)y=0$

$(D_1^2+2D_1-\lambda)y=0 \Rightarrow D_1=-1\pm\sqrt{1+\lambda}$ here $(\lambda\geq -1)$

Sol^n $\Rightarrow C_1 e^{-[1-\sqrt{1+\lambda}]z}+C_2 e^{-[1+\sqrt{1+\lambda}]z}$

$$=\frac{C_1}{x^{(1-\sqrt{1+\lambda})}}+\frac{C_2}{x^{1+\sqrt{1+\lambda}}}$$

As $x\to\infty$, Sol$^n\to 0$ if $1-\sqrt{1+\lambda}>0 \Rightarrow (1+\lambda)^{1/2}<1$
$\Rightarrow \lambda<0$

$\boxed{-1\leq\lambda<0}$

When RHS is general function

Q. $(x^2D^2+3xD+1)y=1/(1-x)^2$

$(D_1(D_1-1)+3D_1+1)y \equiv (D_1+1)^2y=\frac{1}{(1-x)^2}$ } No change in RHS

$$PI=\frac{1}{(D_1+1)}\cdot\frac{1}{(D_1+1)}\cdot\frac{1}{(1-x)^2}$$

$$=\frac{1}{(D_1+1)}x^{-1}\int \cancel{x^{-(-1)-1}}\cdot\frac{1}{(1-x)^2}=\frac{1}{(D_1+1)}x^{-1}\frac{1}{(1-x)}$$

$$=\boxed{\frac{1}{x}(\log x-\log(1-x))}$$

* General RHS $\Rightarrow$

$$\frac{1}{D_1-a}X=x^a\int x^{-a-1}X\,dx$$ (Remember)

→ Legendre $= \{a_0(a+bx)^n D^n + \dots a_{n-1}(a+bx)D + a_n\} y = X$.

• $e^z = a+bx$

• $(a+bx)^n = b^n D_1(D_1-1)\dots(D_1-n+1)$

(13) Solve $[(1+2x)^2 D^2 - 6(1+2x)D + 16] y = 8(1+2x)^2$

$y(0)=0, y'(0)=2.$

$1+2x = e^z.$ $\quad D_1 \equiv d/dz$

→ $2^2(D_1(D_1-1)) - 6.2.(D_1) + 16)y = (4D_1^2 - 16D_1 + 16)y = 8e^{2z}$.

$(D-2)^2 y = 8e^{2z}$ $\quad$ C.F ⇒ $(C_1 + C_2 z)e^{2z}$.

PI ⇒ $\frac{1}{\cancel{4}(D-2)^2} {}^2\cancel{8}e^{2z} = \cancel{8}\cdot\frac{\cancel{1}}{\cancel{(D-2)}} \frac{z^2}{\cancel{2!}} e^{2z} = \boxed{\cancel{\tfrac14} z^2 e^{2z}}$

$= [\log(1+2x)]^2 \cdot (1+2x)^2.$

C.F ≡ $[C_1 + C_2\log(1+2x)](1+2x)^2$

Total → $(C_1 + C_2\log(1+2x))(1+2x)^2 + (1+2x)^2(\log(1+2x))^2 = y$.

$y(0)=0, y'(0)=2.$ $\qquad C_2\log(\cancel{1+2x})^0 . \cancel{4}(1+2x).$

$\boxed{C_1 = 0}$ $\left(\frac{2C_2}{1+2x}\right)(1+2x)^2 + $ $\quad (1+2x)^2 \cdot \cancel{2}\log(1+2x)^0 \dots = 2$

$+ 4(1+2x)\cancel{(\log}^0(1+2x))^2 +$

$2C_2 = 2 \Rightarrow \boxed{C_2 = 1.}$

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx)$$

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx)$$

* Variation of Parameters ALWAYS in Standard form first

$\boxed{y_2 + Py_1 + Qy = R.}$

. Solve $y_2 + Py_1 + Qy = 0 \rightarrow u, v$ solutions.

. Consider $Au + Bv$ (A, B also f"s of x).

. $\boxed{\text{Assume } uA_1 + vB_1 = 0}$ — ① $A_1 = \frac{\partial A}{\partial x}$ | $B_1 = \frac{\partial B}{\partial x}$

. Put $y = Au + Bv$ and insert in ODE, simplify by ①

$\quad A_1 u_1 + B_1 v_1 = R$ — ②

. Using ① and ② $\rightarrow A_1 = -\frac{vR}{W}$, $B_1 = \frac{uR}{W}$.

$\rightarrow$ Sol" $\equiv$ $y = C.F + P.I.$

$C.F \rightarrow C_1 u + C_2 v$

$P.I \rightarrow u f(x) + v g(x)$

$f(x) = -\int \frac{vR}{W}$ $g(x) = \int \frac{uR}{W}$

Q $y_2 + n^2 y = \sec nx.$

$W = \begin{vmatrix} \cos nx & \sin nx \\ -n\sin nx & n\cos nx \end{vmatrix} = n \neq 0$

C.F $\Rightarrow C_1 \cos nx + C_2 \sin nx$

PI $\Rightarrow \cos nx\, f(x) + \sin nx\, g(x)$

$f(x) = -\int \frac{\sin(nx).\sec(nx)}{n}\,dx = \frac{\log \cos nx}{n^2}$

$g(x) = \int \frac{\cos(nx)\sec nx}{n} = \frac{x}{n}.$

10 $(D^2-2D+1)y = xe^x \sin x$ $y(0)=0$ $\frac{dy}{dx}(0)=0.$

C.F $\Rightarrow$ $(C_1 + C_2 x)e^x$

$$W = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix}.$$

$$= e^{2x} + \cancel{xe^{2x}} - \cancel{xe^{2x}} \neq 0.$$

P.I $\Rightarrow$ $e^x f(x) + xe^x g(x)$

$$f(x) = -\int \frac{x\cancel{e^x} \cdot x\cancel{e^x} \sin x}{\cancel{e^{2x}}}$$

$$= -\int x^2 \sin x$$

$$= +x^2 \cos x - \int 2x\cos x \Rightarrow -\left[2x\sin x - \int 2\sin x\right]$$

$$x^2\cos x - 2x\sin x - 2\cos x.$$

$$g(x) = \int \frac{\cancel{e^x}\, x\cancel{e^x} \sin x}{\cancel{e^{2x}}} = \sin x - x\cos x$$

$$(C_1 + C_2 x)e^x + e^x(x^2\cos x - 2x\sin x - 2\cos x) + xe^x(\sin x - x\cos x)$$

$y(0) = C_1 + (-2) = 0 \Rightarrow C_1 = 2.$

$y'(0) = C_1 + C_2 - 2 = 0 \Rightarrow C_2 = 0$

Ans

[3] Use variation of parameters

$$x^2y'' - 2xy' + 2y = x\log x.$$

~~Put $x = e^z$~~

(1) Find R be rewriting in standard form

$$y'' - \frac{2y'}{x} + \frac{2y}{x^2} = \frac{\log x}{x} \Leftarrow R.$$

$\rightarrow x^2y'' - 2xy' + 2y = 0$ $\qquad x = e^z.$

$$(D_1(D_1 - 1) - 2D_1 + 2)y = 0$$

$$(D_1^2 - 3D_1 + 2)y = 0 \Rightarrow (D_1 - 2)(D_1 - 1)y = 0.$$

$$C_1e^{2z} + C_2e^z = C_1x^2 + C_2x.$$

$\rightarrow PI \equiv x^2f(x) + xg(x).$

$W \equiv \begin{vmatrix} x^2 & x \\ 2x & 1 \end{vmatrix} = x^2 - 2x^2 = -x^2 \neq 0$

Sign retain

$$f(x) = +\int \frac{\cancel{x}}{x^2} \cdot \frac{\log x}{\cancel{x}}\, dx = \int e^{-t}t\, dt.$$

$$= -e^{-t}t - \int -e^{-t} = -te^{-t} - e^{-t} \Rightarrow \boxed{-\left(\frac{\log x}{x} + \frac{1}{x}\right)}.$$

$$g(x) = \int \frac{\cancel{x^2}\log x}{-\cancel{x^2}\,x}\, dx = -\int t\, dt = \boxed{-\frac{(\log x)^2}{2}}.$$

$\boxed{Q}$ $(D^2-2D+2)y = e^x \tan x.$

$$W = \begin{vmatrix} e^x\cos x & e^x\sin x \\ e^x\cos x - e^x\sin & e^x(\sin x+\cos x) \end{vmatrix}$$

$$= e^{2x}(\cancel{\sin\cos} + \cos^2 - \cancel{\sin\cos} + \sin^2)$$

$$= e^{2x} \neq 0$$

$C.F \Rightarrow e^x(c_1\cos x + c_2\sin x)$

$PI \Rightarrow e^x\cos x\, f(x) + e^x\sin x\, g(x)$

$$f(x) = -\int \frac{e^x\sin x\, e^x\tan x}{e^{2x}}\,dx$$

$$= -\int \frac{1-\cos^2 x}{\cos x}\,dx = \int \cos x - \sec x\,dx = \boxed{\sin x - \log(\sec + \tan x)}$$

$$g(x) = \int \sin x\,dx = \boxed{-\cos x}$$

$\boxed{Q}$ $(D^3-6D^2+11D-6)y = e^{2x} \leftarrow S.$

$C.F = c_1e^x + c_2e^{2x} + c_3e^{3x}$ $\quad W = 2e^{6x} \neq 0.$

$uf + vg + wh.$

$$\frac{df}{dx} = \frac{S}{W}\begin{vmatrix} v & w \\ v_1 & w_1 \end{vmatrix} = \frac{e^{2x}}{e^{6x}}\begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = \frac{e^x}{2} \Rightarrow f(x) = \frac{e^x}{2}.$$

$$\frac{dg}{dx} = \frac{-S}{W}\begin{vmatrix} u & w \\ u_1 & w_1 \end{vmatrix} = -1 \Rightarrow g(x) = -x$$

$$\frac{dh}{dx} = \frac{S}{W}\begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \frac{e^{-x}}{2} \Rightarrow h(x) = -\frac{e^{-x}}{2}$$

* Ordinary Simultaneous Diff Equations

$\boxed{Q}$ $\frac{dx}{dt} + \frac{dy}{dt} + 2x - y = 3(t^2 - e^{-t})$,

$2\frac{dx}{dt} - \frac{dy}{dt} - x - y = 3(2t - e^{-t})$

$(D+2)x + (D-1)y = 3(t^2 - e^{-t}) \quad \leftarrow (D+1)$

$(2D-1)x + (D+1)y = -3(2t - e^{-t}) \quad \leftarrow (D-1)$. } Add

$(D^2+3D+2)x + (2D^2-3D+1)x = \cancel{6t} + \cancel{3e^{-t}} + 3t^2 - \cancel{3e^{-t}}$
$6 + 3e^{-t} - \cancel{6t} + 3e^{-t}$

$(3D^2+3)x = 3t^2 + 6 + 6e^{-t} \Rightarrow (D^2+1)x = t^2 + 2 + 2e^{-t}$

C.F $\Rightarrow c_1 \cos t + c_2 \sin t$.

Undet. Coefficients $\Rightarrow (D^2+1)(a_0t^2 + a_1t + a_2 + a_3e^{-t})$

$\Rightarrow 2a_0 + a_3e^{-t} + a_0t^2 + a_1t + a_2 + a_3e^{-t}$.

$a_0 = 1$, $a_1 = 0$, $a_2 = 2$, $a_3 = 1$

$PI \Rightarrow t^2 + 2 + e^{-t}$

Sol^n $\Rightarrow c_1 \cos t + c_2 \sin t + e^{-t} + t^2 = x(t)$. Ans

To find y w/o integration $\Rightarrow$

Add ①,② $\Rightarrow 2y = 3\frac{dx}{dt} + x - 3t^2 - 6t + 6e^{-t}$

$2y = -3c_1 \sin t + 3c_2 \cos t - 3e^{-t} + 6t + c_1 \cos t + c_2 \sin t + e^{-t} + t^2 - 3t^2 - 6t + 6e^{-t}$.

$2y = \cos t\,(c_1 + 3c_2) + \sin t\,(c_2 - 3c_1) + 4e^{-t} - 2t^2$ Ans

$\boxed{9}$ $tDx + 2(x-y) = t$, $tDy + 5y + x = t^2.$

Put $t = e^z.$

$(D_1+2)x - 2y = e^z \Leftrightarrow (D_1+5)y + x = e^{2z}$

$(D_1+5)(D_1+2)x + 2x = 6e^z + 2e^{2z}.$

$[(D_1+4)(D_1+3)]x = 6e^z + 2e^{2z}$

C.F $\Rightarrow$ $\boxed{C_1 e^{-4z} + C_2 e^{-3z}}$

P.I $\Rightarrow$ $6 . \frac{1}{(D_1+4)(D_1+3)} e^z \Rightarrow \boxed{\frac{3}{10} e^z}$

$2 . \frac{1}{\cancel{2}\cdot 6 \cdot 5} e^{2z} = \boxed{\frac{1}{15} e^{2z}}$

$-t$

$-3t^2 6t$
$+6e^{-1}$

* General Linear Equations of 2nd order

$$\boxed{y'' + Py' + Qy = R}$$ ⟸ Put in this form first

① Find first solution - U of Homogenous ≡ ~~y''+PQ'+~~ .

$$e^{ax} \equiv a^2 + aP + Q = 0 \qquad (y'' + Py' + Qy = 0)$$

$$x^m \equiv m(m-1) + Pmx + Qx^2 = 0.$$

② Assume solution $y = uv$

Equation reduces to

$$\frac{d^2v}{dx^2} + \left[P + \frac{2}{u}\frac{du}{dx}\right]\frac{dv}{dx} = \frac{R}{u}$$

③ Put $\frac{dv}{dx} = q$ and Solve

$$\frac{dq}{dx} + \left(P + \frac{2}{u}\frac{du}{dx}\right)q = \frac{R}{u}$$

$\boxed{8}$ $(x+2)y'' - (2x+5)y' + 2y' = (x+1)e^x$

$$\rightarrow \quad y'' - \left(\frac{2x+5}{x+2}\right)y' + \left(\frac{2}{x+2}\right)y' = \left(\frac{x+1}{x+2}\right)e^x \quad -(1)$$

1st Soln $\Rightarrow$ $y'' - \left(\frac{2x+5}{x+2}\right)y' + \frac{2}{x+2}y = 0$

$y = e^{mx}$

$$m=0,2 \qquad \frac{5\pm\sqrt{9}}{4} = 2, \tfrac{1}{2}.$$

$$m^2 - m\left(\frac{2x+5}{x+2}\right) + \frac{2}{x+2} = \frac{(m^2-2m)x + 2m^2 - 5m + 2}{x+2} = 0$$

So, $\boxed{m=2}$

$y = e^{2x}$ is 1st soln.

Let general solution be $\boxed{e^{2x}v = y}$

(1) reduces to

$$\frac{d^2v}{dx^2} + \left[-\frac{2x-5}{x+2} + \frac{2\cdot 2\cancel{e^{2x}}}{\cancel{e^{2x}}}\right]\frac{dv}{dx} = \left(\frac{x+1}{x+2}\right)\frac{e^x}{e^{2x}}.$$

Put $\frac{dv}{dx} = q \Rightarrow$

$$\frac{dq}{dx} + \left[\frac{2x+3}{x+2}\right]q = \left(\frac{x+1}{x+2}\right)e^{-x}.$$

$$I.F \Rightarrow e^{\int\left(2-\frac{1}{x+2}\right)dx} = e^{2x-\log(x+2)} = \frac{e^{2x}}{x+2}$$

$$q \times \frac{e^{2x}}{x+2} = \int \frac{x+1}{x+2}\frac{e^{x}}{(x+2)}\,dx = \int \frac{e^x}{x+2} - \frac{e^x}{(x+2)^2}\,dx.$$

$$= \frac{e^x}{x+2} + \int \cancel{\frac{e^x}{(x+2)^2}} - \int \cancel{\frac{e^x}{(x+2)^2}}$$

$$\rightarrow \frac{dv}{dx} = \left(\frac{e^x}{x+2} + c_1\right)\frac{x+2}{e^{2x}}$$

$$\frac{dv}{dx} = e^{-x} + c_1(x+2)e^{-2x}.$$

$$dv = -e^{-x} + \left(\frac{-2c_1}{2}\right)e^{-2x} + c_1\int xe^{-2x}\,dx$$

$$v = -e^{-x} - c_1e^{-2x} + \frac{c_1xe^{-2x}}{(-2)} + c_1\int \frac{e^{-2x}}{(+2)}.$$

$$\Downarrow$$

$$v = -e^{-x} - c_1e^{-2x} + \frac{c_1xe^{-2x}}{(-2)} + \frac{c_1e^{-2x}}{(-4)}$$

General Solution is $\boxed{y = uv}$

* Solution by reducing to Normal form (No 1st derivative term left)

$y'' + Py' + Qy = R$ —(1)

(1) Choose $u = e^{-\frac{1}{2}\int P\,dx}$.

(2) Assume $y = uv$ as solution

$\frac{d^2v}{dx^2} + Iv = S$.

$I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx}$

$S = \frac{R}{u}$

Derivation = $\boxed{\text{Put } y = uv}$

(1) becomes $\Rightarrow \frac{d^2v}{dx^2} + \underbrace{\left[P + \frac{2}{u}\frac{du}{dx}\right]}\frac{dv}{dx} + \frac{1}{u}\left[\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right]v = \frac{R}{u}$

$P + \frac{2}{u}\frac{du}{dx} = 0 \Rightarrow u = e^{-\frac{1}{2}\int P\,dx}$

(2) Substitute u to get the normal form

(9) Solve by using $y(x) = v(x)\sec x$ in $y'' - 2y'\tan x + 5y = 0$
$y(0) = 0,\ y'(0) = \sqrt{6}$.

$$y'(x) = v'(x)\sec x + v(x)\sec x\tan x$$

$$y''(x) = v''(x)\sec x + 2v'(x)\sec x\tan x + v(x)\sec x\tan^2 x + v(x)\sec^3 x$$

Put in Equation $\Rightarrow$

$$v''(x)\sec x + \cancel{2v'(x)\sec x\tan x} + v(x)(\sec^3 x + \sec x\tan^2 x)$$

$$-\cancel{2v'(x)\sec x\tan x} - 2v(x)\sec x\tan^2 x + 5v(x)\sec x = 0$$

$$v''(x)\sec x + v(x)[\sec^3 x + \cancel{\sec x\tan^2 x} - \cancel{2}\sec x\tan^2 x + 5\sec x] = 0$$

$$v''(x) + v(x)(\sec^2 x - \tan^2 x + 5\cancel{\sec x}] = 0$$

$$v''(x) + 6v(x) = 0 \qquad (D^2+6)v = 0$$

$$v(x) = C_1\cos\sqrt{6}x + C_2\sin\sqrt{6}x$$

Soln $\Rightarrow (C_1\cos\sqrt{6}x + C_2\sin\sqrt{6}x)\sec x = y(x)$.

$C_1 = 0,\ C_2 = 1$ Ans

$$\overset{P}{} \qquad \overset{Q}{} \qquad \overset{R}{}$$

(9) $y'' - 4xy' + (4x^2-1)y = -3e^{x^2}\sin 2x$

Put $u(x) = e^{-\frac{1}{2}\int P\,dx}$

$= e^{-\frac{1}{2}\int 4x\,dx} = e^{+\frac{\cancel{2}x^2}{\cancel{2}}} = e^{+x^2}$

$\boxed{\begin{array}{l} \text{try } e^{mx} \Rightarrow m^2 - 4mx + 4x^2 - 1 \ldots \\ x^m \Rightarrow m(m-1) - 4mx^2 + 4x^2 - 1 \\ m=1 \rightarrow \text{No } x. \\ \text{So, only this way.} \end{array}}$

Let $y = v . e^{+x^2}$

$S = \frac{R}{u} = -3\sin 2x$

$\frac{d^2v}{dx^2} + Iv = S$

$I = Q - \frac{P^2}{4} - \frac{1}{2}\frac{dP}{dx}$

$= (4x^2-1) - 4x^2 + 2 = 1.$

$\frac{d^2v}{dx^2} + v = -3\sin 2x.$

$(D^2+1)v = 0 \Rightarrow C_1\cos x + C_2\sin x \equiv$

$(-3).\frac{1}{D^2+1}\sin(2x) = \frac{(-3)}{-4+1} = 1\sin 2x.$

$v = C_1\cos x + C_2\sin x + \sin 2x.$

$y = uv$ is solution

$\boxed{B}$ Solve by variation of parameters

$$\left[D^2 - \frac{x}{(x-1)}D + \frac{1}{x-1}\right]y = (x-1)$$

$$\boxed{\begin{array}{l} m^2 + Pm + Q = 0 \\ m(m-1) + Pmx + Qx^2 = 0 \end{array}}$$

Consider

$$y_2 - \frac{x}{(x-1)}y_1 + \frac{y}{x-1} = 0.$$

$\cdot\ e^{mx} \Rightarrow\ m^2 - \frac{mx}{(x-1)} + \frac{1}{x-1} = \frac{x(\overset{0,1}{m^2-m}) + 1.(\overset{1,-1}{1-m^2})}{x-1} = 0$

$\boxed{e^x}$

$\cdot\ x^m \Rightarrow\ m(m-1). - \frac{x^2 m}{(x-1)} + \frac{x^2}{(x-1)} = \frac{x^2(1-m). + m(m-1)x + m(m-1)}{(x-1)}$

$m = 1 \Rightarrow \boxed{x}$

We have e^x, x as solutions.

$$W = \begin{vmatrix} e^x & x \\ e^x & 1 \end{vmatrix} = e^x - xe^x$$

$\Rightarrow\ e^x f(x) + x g(x)$ be ~~general~~ particular solution

$f(x) = +\int \frac{x\cancel{(x-1)}}{e^x\cancel{(1-x)}}dx = -xe^{-x} + \int e^{-x}$

$= -e^{-x}(1+x)$

$g(x) = (-1)\int \frac{\cancel{e^x}(x-1)}{\cancel{e^x}(1-x)}dx = -x$

$PI \Rightarrow \boxed{-1-x-x^2}$

Gen Solⁿ $\Rightarrow\ ax + be^x - 1 - x - x^2$ Ans

* Laplace Transforms

① Basic Transforms

$1 \rightarrow 1/S$

$t^n \rightarrow \frac{n!}{S^{n+1}}$

$e^{at} = 1/S-a$

$\cosh at = S/S^2-a^2$ $\quad \cos wt = \frac{S}{S^2+w^2}$

$\sinh at = a/S^2-a^2$ $\quad \sin wt = \frac{w}{S^2+w^2}$

② Basic Theorems

- $\mathcal{L}(e^{at}f(t)) = F(S-a)$.
- $\mathcal{L}(f'(t)) = S\mathcal{L}(f) - f(0). \Rightarrow \mathcal{L}(f^{(n)}) = S^n\mathcal{L}(f) - S^{n-1}f(0) - S^{n-2}f'(0) \ldots$
- $\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{S}F(S)$.
- $\mathcal{L}\{tf(t)\} = -F'(S)$ $\quad \boxed{\mathcal{L}(t^n f(t)) = (-1)^n F^{(n)}(S)}$
- $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_S^\infty F(S)dS$
- $\mathcal{L}\{f(t-t_0)\} = e^{-St_0}X(S)$ or $e^{-St_0}F(S) = \begin{cases} f(t-t_0) & t \geq t_0 \\ 0 & t < t_0 \end{cases}$
- $\mathcal{L}\{f(at)\} = \frac{1}{|a|}F\left(\frac{S}{a}\right)$

(5) Find $\mathcal{L}(\cos \omega t)$, $\mathcal{L}(\sin \omega t)$.

$$\mathcal{L}(\cos \omega t) = \int_0^\infty e^{-st} \cos \omega t = \frac{1}{s} - \frac{\omega}{s}\int_0^\infty e^{-st} \sin \omega t \, dt$$

$$\mathcal{L}(\sin \omega t) = \int_0^\infty e^{-st} \sin \omega t = \frac{\omega}{s}\int_0^\infty e^{-st} \cos \omega t \, dt$$

$$L_c = \frac{1}{s} - \frac{\omega}{s}\left(\frac{\omega}{s} L_c\right) \Rightarrow L_c = \frac{s}{s^2+\omega^2}$$

IIIly $L_s = \dfrac{\omega}{s^2+\omega^2}$

- $$\frac{3s-137}{s^2+2s+401} = \frac{3s-137}{(s+1)^2+400}$$

$$= \frac{3(s+1)}{(s+1)^2+400} - \frac{140}{(s+1)^2+20^2}$$

$$= 3\left(\frac{s+1}{(s+1)^2+400}\right) - 7\left[\frac{20}{(s+1)+20^2}\right]$$

$$= 3e^{-t}\cos 20t - 7e^{-t}\sin 20t.$$

- $$\mathcal{L}(\sinh t \cos t) = \frac{1}{2}\mathcal{L}\left(e^{t}\cos t + e^{-t}\cos t\right)$$

$$= \frac{1}{2}\left(\frac{(s-1)}{(s-1)^2+1^2} - \frac{(s+1)}{(s+1)^2+1^2}\right) \text{ Ans}$$

• $\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+\omega^2)}\right\} = \int_0^t f(\tau)\,d\tau.$

Let $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+\omega^2)}\right\} = f(t)$

$\Downarrow$

$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+\omega^2)}\right\} = \int_0^t \frac{\sin\omega t}{\omega} = \frac{1}{\omega^2}(1-\cos\omega t) = f(t)$

$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+\omega^2)}\right\} = \frac{1}{\omega^2}\int_0^t (1-\cos\omega\tau)\,d\tau = \boxed{\frac{t}{\omega^2} - \frac{\sin\omega t}{\omega^3}}$

• $y''-y = t \qquad y(0)=1,\ y'(0)=1.$

Taking Laplace ⇒

$\left(s^2Y(s) - sy(0) - y'(0)\right) - Y(s) = \frac{1}{s^2}$

$Y(s)(s^2-1) = \frac{1}{s^2} + s + 1 = \frac{s^3+s^2+1}{s^2}$

$Y(s) = \frac{s^3+s^2+1}{s^2(s^2-1)} = \frac{1}{s^2(s^2-1)} + \underbrace{\frac{s}{s^2-1} + \frac{1}{s^2-1}}_{N}$

$Y(s) = \left(\frac{1}{s^2-1} - \frac{1}{s^2}\right) + \frac{1}{s-1}$

$Y(s) = \frac{1}{-2(s+1)} + \frac{1}{2(s-1)} - \frac{1}{s^2} + \frac{1}{s-1}$

$\boxed{y(t) = -\frac{1}{2}e^{-t} + \frac{1}{2}e^{t} - t + e^{t}}$

$y'(0)=0$

(Q) $y'' + y' + 9y = 0 \qquad y(0) = 0.16$

$$\left(s^2 Y(s) - s\times 0.16 - 0\right) + \left(sY(s) - 0.16\right) + 9Y(s) = 0.$$

$$Y\left(s^2+s+9\right) = 0.16\,(s+1)$$

$$Y = \frac{0.16\,(s+1)}{s^2+s+9} = \frac{0.16\,(s+1)}{\left(s+\frac{1}{2}\right)^2+\left(\sqrt{\frac{35}{4}}\right)^2}$$

$$= \frac{0.16\,\left(s+\frac{1}{2}\right) + 0.08}{\left(s+\frac{1}{2}\right)^2+\left(\sqrt{\frac{35}{4}}\right)^2}$$

$$= e^{-t/2}\left(0.16\cos\sqrt{\tfrac{35}{4}}\,t\right) + \frac{0.08}{\sqrt{\frac{35}{4}}}\,e^{-t/2}\sin\left(\sqrt{\tfrac{35}{4}}\,t\right) \text{ Ans}$$

(Q) $\dfrac{e^{-3s}}{(s+2)^2}.$

$e^{-3s} \rightarrow (t \rightarrow t-3)$

$$-\left(\frac{1}{s+2}\right)' = \frac{1}{(s+2)^2} \qquad \left(tf(t) = -F'(s)\right)$$

$$\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) = e^{-2t}.$$

$$\mathcal{L}\left(te^{-2t}\right) = \frac{1}{(s+2)^2}.$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{(s+2)^2}\right\} = \boxed{(t-3)\,e^{-2(t-3)}\,u(t-3)}$$

Also if e^{-as} there

• $\ln\left(\frac{S^2+1}{S(S+1)}\right) = \ln(S^2+1) - \ln(S) - \ln(S+1)$

$\mathcal{L}^{-1}$ — $F'(S) = \frac{2S}{S^2+1} - \frac{1}{S} - \frac{1}{S+1}$

$(-1)\,t\,f(t)$

$\longrightarrow = 2\cos t - 1 - e^{-t}$

$$\boxed{f(t) = \frac{1+e^{-t}-2\cos t}{t}}$$ Ans

• $\tan^{-1}\left(\frac{2}{S^2}\right)$

$F'(s) = \frac{-4s}{S^4+2} = \frac{-4s}{(S^2+2)^2-(2s)^2}$

$(-1)\,t\,f(t) \longrightarrow \frac{1}{S^2-2s+2} - \frac{1}{S^2+2s+2}$

$\frac{1}{(S-1)^2+1^2} - \frac{1}{(S+1)^2+1^2}$

$$f(t) = \frac{(e^t\sin t - e^{-t}\sin t)}{t}$$

$$\mathcal{L}\left(\frac{f(t)}{t}\right) = \int_s^{\infty} F(s).$$

$$\mathcal{L}\left(t\, f(t)\right) = -F'(s)$$

$$\frac{F(s)}{s^2} = \int_0^t \int_0^t f(t)$$

$$\int \int_0^t \sin t$$

$$\int_0^t -\cos t + 1$$

$$\boxed{t - \sin t} \qquad + \cos t - 2\sin t.$$