PDE

2012 IFoS

1. Solve $(D^3D'^2 + D^2D'^3)z = 0$, where D stands for $\frac{\partial}{\partial x}$ and D' stands for $\frac{\partial}{\partial y}$.

We have,

$$(D^3D'^2 + D^2D'^3)z = 0$$

The given equation is homogeneous but neither D^5 nor D'^5 is present. The given equation can be written as

$$D^2D'^2(D+D')z = 0$$

Taking $D = m$, $D' = 1$, the auxiliary equation is $m^2(m+1) = 0$

$$\Rightarrow m = 0, 0, -1$$

$\therefore$ The required solution is

$$z = \phi_1(y) + x\phi_2(y) + \phi_3(y-x),$$

where ϕ_1, ϕ_2 and ϕ_3 are arbitrary functions.

2. Using Method of Separation of Variables, solve Laplace Equation in three dimensions:

The Laplace equation in three dimension is given by

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} = 0 \text{ or } \nabla^2 \psi = 0$$

Let the solution be

$$\psi(x, y, z) = X(x) Y(y) Z(z)$$

The given equation becomes

$$YZX'' + ZXY'' + XYZ'' = 0$$

On dividing by $XYZ \neq 0$, we get

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0$$

Since each term in the above equation is a function of a single and different independent variable, each ratio must be a constant i.e.

$$\frac{X''}{X} = -\alpha^2, \quad \frac{Y''}{Y} = -\beta^2, \quad \frac{Z''}{Z} = \gamma^2, \text{ where}$$

$$\gamma^2 = \alpha^2 + \beta^2$$

The solutions of the differential equations:

$$X'' + \alpha^2 X = 0, \quad Y'' + \beta^2 Y = 0, \quad Z'' - \gamma^2 Z = 0$$

$$X(x) = c_1 \cos \alpha x + c_2 \sin \alpha x$$

$$Y(y) = c_3 \cos \beta y + c_4 \sin \beta y$$

$$Z(z) = c_5 e^{\gamma z} + c_6 e^{-\gamma z}$$

$\therefore\ \psi(x,y,z) = (c_1\cos\alpha x + c_2\sin\alpha x)(c_3\cos\beta y + c_4\sin\beta y)(c_5 e^{\gamma z} + c_6 e^{-\gamma z})$

OR

$X(x) = c_1 e^{\pm i\alpha x}$, $Y(y) = c_2 e^{\pm i\beta y}$

$Z(z) = c_3 e^{\pm \gamma z}$

Thus $\psi(x,y,z) = C e^{\pm i\alpha x \pm i\beta y \pm \gamma z}$.

3 Solve $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$ using Lagrange's method. (MD-RAIS.)

We have,

$(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$

Lagrange's subsidiary equations are

$$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy}$$

Using multiplier's $1,-1,0$ and $0,1,-1$;

each fraction $= \dfrac{dx - dy}{(x-y)(x+y+z)} = \dfrac{dy - dz}{(y-z)(x+y+z)}$

$$\Rightarrow \frac{d(x-y)}{x-y} = \frac{d(y-z)}{y-z}$$

On integrating, $\log(x-y) = \log(y-z) + \log c_1$

$$\Rightarrow \frac{x-y}{y-z} = c_1 \quad \text{--- (1)}$$

Again each fraction $= \dfrac{dx + dy + dz}{x^2 + y^2 + z^2 - xy - yz + zx}$ — (A)

And choosing x, y, z as multiplier,

each fraction $= \dfrac{x\,dx + y\,dy + z\,dz}{x^3+y^3+z^3-3xyz}$

$$= \frac{x\,dx+y\,dy+z\,dz}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)} \quad -(B)$$

$\therefore$ Equaling (A) and (B), we get

$$dx+dy+dz = \frac{x\,dx+y\,dy+z\,dz}{x+y+z}$$

$\Rightarrow (x+y+z)(dx+dy+dz) = x\,dx+y\,dy+z\,dz$

$\Rightarrow y\,dx + z\,dx + x\,dy + z\,dy + x\,dz + y\,dz = 0$

$\Rightarrow x\,dy + y\,dx + y\,dz + z\,dy + z\,dx + x\,dz = 0$

$\Rightarrow d(xy) + d(yz) + d(zx) = 0$

On integration, we get

$xy + yz + zx = c_2$ — (2)

where c_2 = arbitrary constant

$\therefore$ The required general solution is

$$\frac{x-y}{y-z} = \phi(xy+yz+zx).$$