→ Let $|f(z)|$ be a constant in a region where $f(z)$ is analytic. Then $f(z)$ is constant. ①

proof Let $f(z) = u(x,y) + iv(x,y)$

Then $|f(z)| = \sqrt{u^2+v^2}$

Since $|f(z)|$ is constant

$\therefore \sqrt{u^2+v^2} = \underset{(\text{const})}{c}$ (say)

$\Rightarrow \boxed{u^2+v^2 = c^2}$

partially differentiating w.r.t 'x', we get,

$2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{\partial x} = 0$

$\Rightarrow u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} = 0$ —— ①

partially differentiating w.r.t 'y' we get

$u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y} = 0$ —— ②.

Since $f(z)$ is analytic, CR-conditions are satisfied

$\therefore$ On using them in ① & ②, we get

$u\frac{\partial u}{\partial x} - v\frac{\partial u}{\partial y} = 0$, $u\frac{\partial u}{\partial y} + v\frac{\partial u}{\partial x} = 0$

Solving these for $\frac{\partial u}{\partial x}$, we get

$(u^2+v^2)\frac{\partial u}{\partial x} = 0$.

Since $u^2+v^2 \neq 0$, we get $\boxed{\frac{\partial u}{\partial x} = 0}$.

Similarly we can show that $\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$.

Hence u & v are constant

$\therefore f(z)$ is constant.

Miscellaneous Part

The Elementary Functions :-

→ polynomial functions are defined by

$w = a_0 z^n + a_1 z^{n-1} + \cdots\cdots + a_{n-1} z + a_n = P(z)$

where $a_0 \neq 0, a_1, a_2 \cdots a_n$ are complex constants and 'n' is +ve integer called the degree of the polynomial $p(z)$.

→ Rational Algebraic functions are defd by $w = \dfrac{p(z)}{q(z)}$

where $p(z)$ and $q(z)$ are polynomials;

$p(z) = a_0 z^n + a_1 z^{n-1} + \cdots\cdots + a_{n-1} z + a_n$; $a_0 \neq 0$

$q(z) = b_0 z^m + b_1 z^{m-1} + \cdots\cdots + b_{m-1} z + b_m$; $b_0 \neq 0$.

without common factors.

→ Exponential functions are defined by

$w = e^z = e^{x+iy} = e^x (\cos y + i\sin y)$

i.e $w = e^x (\cos y + i\sin y)$ ——— ①

where $e = 2.71828\cdots$ is the natural base of logarithms.

— If 'a' is real and positive, we define

$a^z = e^{z \ln a}$

where $\ln a$ is the natural logarithm of 'a'.

This reduces to ① if $a = e$.

— complex exponential functions have properties similar to those of real exponential functions.

For example: $e^{z_1} \cdot e^{z_2} = e^{z_1 + z_2}$, $\dfrac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2}$

→ Trigonometric functions:

We define the trigonometric functions $\sin z$, $\cos z$ etc., in terms of exponential functions as follows:

$$\sin z = \frac{e^{iz}-e^{-iz}}{2i}, \quad \cos z = \frac{e^{iz}+e^{-iz}}{2}$$

$$\sec z = \frac{1}{\cos z} = \frac{2}{e^{iz}+e^{-iz}}, \quad \csc z = \frac{1}{\sin z} = \frac{2i}{e^{iz}-e^{-iz}}$$

$$\tan z = \frac{\sin z}{\cos z} = \frac{e^{iz}-e^{-iz}}{i(e^{iz}+e^{-iz})}$$

$$\cot z = \frac{i(e^{iz}+e^{-iz})}{e^{iz}-e^{-iz}}.$$

Many of the properties familiar in the case of real trigonometric functions also hold for the complex trigonometric functions.

For example: $\sin^2 z + \cos^2 z = 1$,

$1+\tan^2 z = \sec^2 z$

$1+\cot^2 z = \csc^2 z$.

$\sin(z_1 \pm z_2) = \sin z_1 \cos z_2 \pm \cos z_1 \sin z_2$

$\tan(z_1 \pm z_2) = \dfrac{\tan z_1 \pm \tan z_2}{1 \mp \tan z_1 \tan z_2}$. etc

→ Hyperbolic functions are defined as follows:

$$\sinh z = \frac{e^{z}-e^{-z}}{2}, \quad \cosh z = \frac{e^{z}+e^{-z}}{2}$$

$$\operatorname{sech} z = \frac{1}{\cosh z} = \frac{2}{e^{z}+e^{-z}}, \quad \operatorname{csch} z = \frac{1}{\sinh z} = \frac{2}{e^{z}-e^{-z}}.$$

$$\tanh z = \frac{\sinh z}{\cosh z} = \frac{e^{z}-e^{-z}}{e^{z}+e^{-z}}$$

$$\coth z = \frac{\cosh z}{\sinh z} = \frac{e^{z}+e^{-z}}{e^{z}-e^{-z}} \text{ etc.}$$

→ Logarithmic functions:

If $z = e^w$ then we write $w = \ln z$ called the natural logarithm of z. Thus the natural logarithmic function is the inverse of exponential function and can be defined by
$$w = \ln z \quad \text{—— (1)}$$

Also $e^{w+2in\pi} = e^w \cdot e^{2n\pi i}$
$$= z \quad (\because e^{2in\pi} = 1)$$
$$\Rightarrow e^{w+2in\pi} = z$$
$$\Rightarrow \log_e z = w \pm 2in\pi \quad \text{—— (2)}$$

i.e the logarithm of a complex number has an infinite number of values and is, therefore, a multivalued function.

— The general value of the logarithm of z is written as $\operatorname{Log} z$ (beginning with capital L) so as to distinguish it from its principal value which is written as $\log_e z$. This principal value is obtained by taking $n=0$ in $\operatorname{Log} z$.

∴ from (1) & (2)
$$\operatorname{Log} z = 2in\pi + \log z$$
$$\text{i.e. } \operatorname{Log}(x+iy) = 2in\pi + \log(x+iy)$$

Note: (1) If $y=0$, then $\operatorname{Log} x = 2in\pi + \log x$.

This shows that the logarithm of a real quantity is also multi-valued. Its principal value is real while all other values are

(2) W.K.T the logarithm of a negative quantity has no real value. But we can now evaluate this.

for example:

$$\log_e(-2) = \log_e\{2(-1)\} = \log_e 2 + \log_e(-1)$$

$$= \log_e 2 + i\pi$$

$$(\because -1 = \cos\pi + i\sin\pi = e^{i\pi})$$

$$= 0.6931 + i(3.1416).$$

→ Real and imaginary parts of $\text{Log}(x+iy)$

$$\text{Log}(x+iy) = 2in\pi + \log(x+iy)$$

$$= 2in\pi + \log r(\cos\theta + i\sin\theta)$$

[put $x = r\cos\theta$, $y = r\sin\theta$
$\Rightarrow r = \sqrt{x^2+y^2}$ and $\theta = \tan^{-1}(y/x)$]

$$= 2in\pi + \log re^{i\theta}$$

$$= 2in\pi + \log r + \log e^{i\theta}$$

$$= 2in\pi + \log r + i\theta.$$

$$= 2in\pi + \log(\sqrt{x^2+y^2}) + i\tan^{-1}(y/x)$$

$$= \log\sqrt{x^2+y^2} + i(2n\pi + \tan^{-1} y/x)$$

Cauchy's Inequality and Its Applications.

Cauchy's Inequality

Let $f(z)$ be analytic inside and on a circle 'C' having centre at z_0 and radius r. If $|f(z)| \le M$ on C then $|f^{(n)}(z_0)| \le \frac{M.n!}{r^n}$

proof: By Cauchy's integral formula

$$f^n(z_0) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

$$\therefore |f^n(z_0)| = \left|\frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz\right|$$

$$\le \frac{n!}{2\pi}\int_C \frac{|f(z)|}{|z-z_0|^{n+1}}\,|dz| \quad (\because |i|=1)$$

$$\le \frac{n!}{2\pi}\frac{M}{r^{n+1}}\int_C |dz| \quad \left(\because |f(z)|\le M \;\&\; |z-z_0|=r\right)$$

$$= \frac{n!}{2\pi}\frac{M}{r^{n+1}}(2\pi r)$$

$$= \frac{M.n!}{r^n}$$

$$\text{i.e., } |f^n(z_0)| \le \frac{M.n!}{r^n}$$

Liouville's theorem:

A bounded entire function is constant.

(or)

If for all z in the entire complex plane,
i) $f(z)$ is analytic and (ii) $f(z)$ is bounded.

proof: Since $f(z)$ is an entire function with $|f(z)| \le M \;\forall z$,

by Cauchy's inequality

$$|f^n(z_0)| \le \frac{\lfloor n \, M}{r^n} \quad \text{for } n \ge 1$$

$\therefore |f'(z_0)| \le \frac{\lfloor 1 \cdot M}{r}$ for every real r.

for finite complex plane $r \to \infty$.

As $r \to \infty$, $|f'(z_0)| \longrightarrow 0$.

Since z_0 is an arbitrary point

$\therefore$ we see that the derivative of $f(z)$ vanishes everywhere

$\therefore f(z)$ must be a constant.

→ Is a non constant entire function not bounded ?

Soln: Yes, To prove this let us suppose that a non-constant entire function is bounded. But by Liouville's theorem a bounded entire function is constant.

This is a contradiction.

Hence a non-constant entire function is not bounded.

→ **Note:** Liouville's theorem says that for a non constant entire function there exists a sequence of points $\{z_n\}$ such that $f(z_n) \longrightarrow \infty$.

Theorem

→ A non-constant entire function comes arbitrarily close to every complex number.

We start with the supposition that there exists a complex number 'a' such that the entire function $f(z)$ come close to it.

Therefore there exists an $\epsilon > 0$ such that

$|f(z)-a| \geq \epsilon \ \forall z.$

Now consider a function $g(z)$ defined as

$$g(z) = \frac{1}{[f(z)-a]}$$

Since $f(z)$ is entire, so $f(z)-a$ is also entire.

Hence $g(z)$ is also an entire function.

$$\therefore |g(z)| = \frac{1}{|f(z)-a|} \leq \frac{1}{\epsilon}.$$

Hence $g(z)$ is bounded entire function.

Hence by Liouville's theorem $g(z)$ is constant.

from this it follows that $\frac{1}{f(z)-a}$ is also constant.

$\therefore f(z) = \frac{1}{g(z)} + a$ is also constant.

This is a contradiction to the hypothesis.

This is due to the supposition that $f(z)$ doesnot come close to the complex number a.

Hence $f(z)$ comes close to a.

$\therefore$ A non constant entire function comes close to every <u>complex</u> number.

→ Suppose $f(z)$ is a non constant entire function. Given any complex number a, there exists a sequence $\{z_n\}$ such that $f(z_n) \longrightarrow a$.

Proof: Since $f(z)$ is a non-constant entire function, by previous theorem, it must come arbitrarily close to the given complex number 'a'.

Since $f(z)$ is an entire function, it is continuous throughout the complex-plane.

Hence there must exist a sequence of points $\{z_n\}$ such that $f(z_n)$ approaches a, as $n \to \infty$, or for each n we can find a point z_n such that $f(z_n) \in N(a, \frac{1}{n})$. Since $\frac{1}{n} \to 0$ as $n \to \infty$,

then $f(z_n) \to a$

Note: Though a non constant entire function comes arbitrarily close to every complex value, it doesnot necessarily assume every complex value.

for example: $f(z) = e^z$ is never equal to zero.

However $f(-n) = e^{-n} \to 0$ as $n \to \infty$.

But e^z assumes every other complex value.

Theorem: Suppose $f(z)$ is an entire function and that $|f(z)| \le Mr^{\lambda}$, $(|z| = r > r_0)$. for some non negative real number λ. Then $f(z)$ is a polynomial of degree atmost λ.

proof: Since $f(z)$ is an entire function, by Taylor's theorem, we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \text{ where } a_n = \frac{f^n(z_0)}{n!}$$

which is convergent in certain radius of convergence.

Now let $z_0 = 0$, then we get

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad (1) \qquad a_n = \frac{f^n(0)}{n!}$$

Again since $|f(z)| \le M r^{\lambda}$ on the circle $|z| = r$

by Cauchy's inequality we get

$$|f^n(z_0)| \le \frac{M r^{\lambda} \lfloor n}{r^n} = M r^{\lambda - n} \lfloor n$$

$$\therefore |a_n| = \left| \frac{f^n(0)}{\lfloor n} \right|$$

$$\le \frac{M r^{\lambda - n} \lfloor n}{\cancel{\lfloor n}} = M r^{\lambda - n}$$

Let $r \to \infty$, then from the above relation we can observe that $a_n \to 0$ when $n > \lambda$.

i.e, the terms in the infinite series given in (1), will be zero whenever $n > \lambda$.

$\therefore f(z)$ can be atmost a polynomial of degree λ.

Note: when $\lambda = 0$, the above theorem reduces to Liouville's theorem.

Now we are going to consider a theorem which is partially converse of the above theorem. This helps us in proving the fundamental theorem of Algebra.

Theorem: Suppose $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots\cdots + a_n z^n$, $a_n \neq 0$. Then if $|z| = r$ sufficiently large,

$$\frac{|a_n|}{2} r^n \le |P(z)| \le \frac{3|a_n|}{2} r^n$$

proof: Consider

$$p(z) = a_0 + a_1 z + a_2 z^2 + \cdots\cdots + a_{n-1}z^{n-1} + a_n z^n$$

$$= z^n\Big[\underbrace{a_n}_{z_1} + \underbrace{\left(\frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \cdots\cdots + \frac{a_0}{z^n}\right)}_{z_2 \text{ say}}\Big] \quad \text{—— (1)}$$

By triangle inequality,

we know that

$$|z_1| - |z_2| \le |z_1 - z_2| \le |z_1 + z_2| \le |z_1| + |z_2|$$

keeping the above relation in view and taking $|z| = r$, from eqn (1) we get

$$r^n\left[|a_n| - \left|\frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \cdots + \frac{a_0}{z^n}\right|\right] \le r^n\left|a_n + \left(\frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \cdots + \frac{a_0}{z^n}\right)\right| = |z|^n\left|a_n + \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \cdots + \frac{a_0}{z^n}\right|$$

$$= \left|z^n\left[a_n + \left(\frac{a_{n-1}}{z} + \cdots \frac{a_0}{z^n}\right)\right]\right|.$$

$$\Rightarrow r^n\left[|a_n| - \left|\frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \cdots\cdots \frac{a_0}{z^n}\right|\right] \le |p(z)| \quad \text{(from (1))}. \; (\because |a||b| = |ab|)$$

$$\le r^n\left[|a_n| + \left|\frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \cdots\cdots + \frac{a_0}{z^n}\right|\right] \quad \text{—— (2)}$$

Let us suppose that $r > 1$, then

$$\left|\frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \cdots\cdots + \frac{a_0}{z^n}\right| \le \frac{|a_{n-1}|}{|z|} + \frac{|a_{n-2}|}{|z|^2} + \cdots + \cdots + \frac{|a_0|}{|z|^n}$$

$$\le \frac{|a_{n-1}|}{r} + \frac{|a_{n-2}|}{r} + \cdots\cdots + \frac{|a_0|}{r} \qquad \left(\because |z| = r > 1,\ \frac{1}{|z|} = \frac{1}{r},\ \frac{1}{|z|^2} = \frac{1}{r^2} < \frac{1}{r},\ \frac{1}{|z|^3} = \frac{1}{r^3} < \frac{1}{r} \cdots\right)$$

$$= \frac{|a_{n-1}| + |a_{n-2}| + \cdots\cdots + |a_0|}{r}$$

Let $|a_{n-1}| + |a_{n-2}| + \cdots\cdots + |a_0| = K$ (say),

then $\left|\frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \cdots\cdots + \frac{a_0}{z^n}\right| \le \frac{K}{r}$

On using the above relation in ②, we get

$$r^n\left[|a_n| - \frac{k}{r}\right] \leq |P(z)| \leq r^n\left[|a_n| + \frac{k}{r}\right]$$

Now further let us suppose that $\frac{k}{r} < \frac{|a_n|}{2}$.

Then we get

$$r^n \frac{|a_n|}{2} \leq |P(z)| \leq r^n \frac{3|a_n|}{2}$$

where $r > \frac{2k}{|a_n|}$, i.e, for sufficiently large values of r.

Hence the result.

Fundamental theorem of Algebra:-

Every non-constant polynomial has atleast one zero (root)

(OR) Every polynomial equation $P(z) = a_0 + a_1 z + a_2 z^2 + \cdots \cdots + a_n z^n = 0$, $n \geq 1$ and $a_n \neq 0$, has atleast one root.

proof: Let us start with a contradiction to prove the theorem.

i.e, let $P(z) = a_0 + a_1 z + \cdots\cdots + a_n z^n$, $a_n \neq 0$

has no zeros.

$\therefore P(z) \neq 0 \;\; \forall z$.

Then $\frac{1}{P(z)}$ is defined throughout the complex-plane.

Hence it is an entire function.

Now by the above theorem

$$\frac{1}{r^n}\left|\frac{2}{\ldots}\right| \geq \left|\frac{1}{P(z)}\right| \geq \frac{1}{r^n}\left|\frac{2}{3|a_n|}\right|$$

$\therefore$ As $r \to \infty$, i.e, $|z| \to \infty$, $\frac{1}{P(z)} \longrightarrow 0$

Thus $\left|\frac{1}{P(z)}\right| < 1$ for $|z| \geq R$.

But $\frac{1}{P(z)}$ is continuous in the bounded closed domain $|z| \leq R$.

Hence $\frac{1}{P(z)}$ is bounded in the whole plane and by Liouville's theorem, it must be constant. This means $P(z)$ is also constant.

This is a contradiction to the hypothesis.

This is due to the supposition that $P(z) \neq 0$ for any z.

Hence $P(z) = 0$ has atleast one zero

→ Show that every polynomial of degree n has exactly n zeros.

Sol[n] By the fundamental theorem of Algebra every non-constant polynomial $P(z)$ will have atleast one zero.

$\therefore$ let us suppose $z = r_1$ is a zero. (root)

Then $P(z) = (z - r_1)\, Q(z)$, where $Q(z)$ is a polynomial of $(n-1)^{\text{th}}$ degree.

Again $Q(z)$ also must have atleast one zero r_2.

proceeding in this way we can prove that $P(z)$ has exactly n zeros.

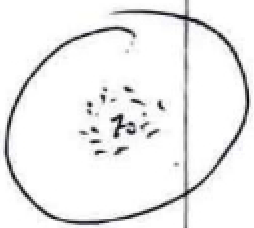
Theorem (I)

Suppose $f(z)$ is analytic in the disk $|z-z_0|<R$, and that $\{z_n\}$ is a sequence of distinct points converging to z_0. If $f(z_n)=0$ for each n, then $f(z)\equiv 0$ everywhere in $|z-z_0|<R$.

proof:

Since $f(z)$ is analytic in the disk $|z-z_0|<R$,

by Taylor's theorem we have



$$f(z)=\sum_{k=0}^{\infty} a_n (z-z_0)^k$$

$$= a_0+a_1(z-z_0)+a_2(z-z_0)^2+\cdots\cdots$$

$$= a_0+\sum_{k=1}^{\infty} a_k (z-z_0)^k \longrightarrow (1)$$

Let us take $z=z_n$ then we get

$$f(z_n)=a_0+\sum_{k=1}^{\infty} a_k (z_n-z_0)^k$$

$$\Rightarrow a_0+\sum_{k=1}^{\infty} a_k (z_n-z_0)^k = 0 \quad \left(\because f(z_n)=0 \ \forall n\right)$$

Taking limit as $n\to\infty$.

$$\lim_{n\to\infty}\left[a_0+\sum_{k=1}^{\infty} a_k (z_n-z_0)^k\right]=0$$

$$\Rightarrow a_0+0+0+\cdots+0=0 \quad \left(\because z_n\to z_0 \text{ as } n\to\infty \Rightarrow z_n-z_0\to 0 \text{ as } n\to\infty\right)$$

$$\Rightarrow a_0=0$$

So now (1) reduces to

$$f(z)=a_1(z-z_0)+a_2(z-z_0)^2+\cdots\cdots$$

$$=(z-z_0)\left[a_1+a_2(z-z_0)+\cdots\cdots\right]$$

$$=(z-z_0)\left[a_1+\sum_{k=2}^{\infty} a_k (z-z_0)^{k-1}\right]$$

$$\Rightarrow \frac{f(z)}{z-z_0}=a_1+\sum_{k=2}^{\infty} a_k (z-z_0)^{k-1}$$

Now let $z=z_n$ then we get

$$\frac{f(z_n)}{z_n-z_0} = a_1 + \sum_{k=2}^{\infty} a_k (z_n - z_0)^{k-1}$$

$$\Rightarrow 0 = a_1 + \sum_{k=2}^{\infty} a_k (z_n - z_0)^{k-1} \quad \left(\because f(z_n)=0\ \forall n\right)$$

As Lt $n\to\infty$ we get

$$a_1 + \underset{k\to\infty}{Lt} \sum_{k=2}^{\infty} a_k (z_n - z_0)^{k-1} = 0$$

$$\Rightarrow a_1 + 0 + 0 + \cdots + 0 = 0$$

$$\Rightarrow a_1 = 0$$

Similarly, we can show that $a_k = 0$, $k = 2, 3, 4, \cdots$

Substituting these values in ①, we get

$$f(z) \equiv 0.$$

Hence the result.

Corollary:

→ Suppose $f(z)$ is analytic at a point $z = z_0$. Then either $f(z) \equiv 0$ in some neighbourhood of z_0 or there exists a real number r such that $f(z) \neq 0$ in the punctured disk $0 < |z - z_0| \leq r$

Proof: To prove the result we start with a contradiction to the alternative and then we prove that the original result holds good.

So we first suppose that there doesnot exist any real number r such that $f(z) \neq 0$ in the punctured disk $0 < |z - z_0| \leq r$.

z_0 $\quad 0<|z-z_0|<r$

Then in each punctured disk $0 < |z - z_0| \leq \frac{1}{n}$ there exist atleast one point z_n such that $f(z_n) = 0$.

$\therefore \exists$ a sequence of points $\{z_n\}$ in the disk converging to z_0 and $f(z_n) = 0$ for each n

Then by above theorem $f(z) \equiv 0$ in some nbd of z_0.

Note → We now generalize the theorem [5] to arbitrary domains.

Suppose $f(z)$ is analytic in a domain D, and that $\{z_n\}$ is a sequence of distinct points converging to a point z_0 in D. If $f(z_n) = 0$ for each 'n' then $f(z) \equiv 0$ throughout D.

Identity Theorem:

Suppose $\{z_n\}$ is a sequence of points having a limit point in a domain D. If $f(z)$ and $g(z)$ are analytic in D, with $f(z_n) = g(z_n)$ for each 'n' then $f(z) \equiv g(z)$ throughout D.

proof: Since $\{z_n\}$ is a sequence of points having a limit point in D

Let z_0 be the limit point of $\{z_n\}$

$\therefore \exists$ a subsequence $\{z_{n_k}\}$ converging to z_0.

Let $h(z) = f(z) - g(z)$ —— (1)

Since $f(z)$ and $g(z)$ are analytic in D

with $f(z_n) = g(z_n)$ for each 'n'

i.e $f(z)$ & $g(z)$ are analytic in D

with $f(z_n) - g(z_n) = 0$ for each 'n'.

Since $f(z)$ & $g(z)$ are analytic in D

$\therefore$ $f(z) - g(z)$ is also analytic in D

$\therefore$ $h(z)$ is analytic in D.

Now we see that $h(z_{n_k}) = 0$ for all points of the subsequence $\{z_{n_k}\}$.

$\therefore$ $h(z) = 0$ in D (By previous note condition)

$\therefore$ $f(z) - g(z) = 0$ in D

$\Rightarrow f(z) = g(z)$ in D.

<u>Note</u>:- The requirement that the limit point z_0 be in the domain of analyticity is essential.

— The nonconstant function $f(z) = e^{\frac{1}{1-z}}$ is analytic in $|z| < 1$

— for $z_n = 1 - \frac{1}{2n\pi i}$, $\forall n \in \mathbb{N}$

we have $f(z_n) = e^{\frac{1}{1-\left(1-\frac{1}{2n\pi i}\right)}}$ $\forall n$

$= e^{2n\pi i}$

$= 1$ $\forall n \in \mathbb{N}$.

Note that $\{z_n\} \longrightarrow 1$ (limit point), as $n \to \infty$

a point at which $f(z)$ is not analytic.

→ Does there exist a function $f(z)$ analytic in $|z|<1$ and satisfying

$$f\left(\frac{1}{2n}\right) = f\left(\frac{1}{2n+1}\right) = \frac{1}{2n}; \quad (n=1,2,3\ldots) \;?$$

$\longleftarrow$ (1)

Sol Let $f(z) = z$ is an analytic function and it satisfies the condition

$$f\left(\frac{1}{2n}\right) = \frac{1}{2n}; \quad n=1,2,\ldots$$

By the identity theorem, this is the only such analytic function.

Since $f\left(\frac{1}{(2n+1)}\right) = \frac{1}{2n+1}$

$$\neq \frac{1}{2n}; \quad n=1,2,\ldots$$

— there does not exist an analytic function that satisfies (1).

Note: However, that we can construct a function $f(z)$ analytic in $|z|<1$ that satisfies $f\left(\frac{1}{2n+1}\right) = \frac{1}{2n}$ for every 'n'.

setting $z = \frac{1}{2n+1}$

we have $2n+1 = \frac{1}{z}$

$$\Rightarrow 2n = \frac{1}{z} - 1 = \frac{1-z}{z}$$

$$\Rightarrow \boxed{\frac{1}{2n} = \frac{z}{1-z}},$$

so that $f(z) = \frac{z}{1-z}$

satisfies the condition.

Note:- — Cauchy's integral formula says that the behaviour of analytic function on a simple closed contour determines its behaviour inside.

— The identity theorem tells us even more. It says the behaviour at any sequence of points, inside or on the simple closed contour determines the behaviour of the analytic function at all points of the domain.

2002 12M → Suppose that f & g are two analytic functions on the set C of all complex numbers with $f(\frac{1}{n}) = g(\frac{1}{n})$ for $n=1,2,3,\ldots$. Then show that $f(z)=g(z)$ for each z in C.

Sol) Let $\{z_n\}$ be the sequence of complex numbers.

Let $z_n = \frac{1}{n}$, $n=1,2,\ldots$

then '0' is the limit point of $\{z_n\}$ in C

$\therefore \exists$ a subsequence $\{z_{n_k}\}$ converging to zero.

Let $\{z_{n_k}\} = \{\frac{1}{2n}\}$ be the subsequence of $\{z_n\}$

that cgs to zero.

Let $h(z) = f(z) - g(z)$ ——— ①.

Since f & g are analytic functions on the set 'C' of complex numbers.

$\therefore f-g$ is also analytic on the set 'C' of all complex numbers.

$\therefore h(z)$ is analytic for each z in $\mathbb{C}$.

put $z = z_{n_k}$ in (1)

$\therefore h(z_{n_k}) = f(z_{n_k}) - g(z_{n_k})$ —— (2).

Given that

$f(1/n) = g(1/n)$ for $n=1,2,3\ldots$

i.e $f(1/n) - g(1/n) = 0$ for $n=1,2,\ldots$

i.e $f(z_n) - g(z_n) = 0$ for $n=1,2,\ldots$

$\therefore h(z_{n_k}) = 0$ for all points of $\{z_{n_k}\}$.

$\therefore h(z) = 0$

$\Rightarrow f(z) - g(z) = 0$

$\Rightarrow f(z) = g(z)$ for each z in $\mathbb{C}$.

Example

Let $\{z_n\} = \{(-1)^n\}$ be a seq. of points having limit point -1 in a domain D.

Let $f(z)$ & $g(z)$ be two analytic functions with $f(z_n) = g(z_n)$ for each 'n'

Let $f(z_n) = (-1)^{2n}$, $g(z_n) = (-1)^{4n}$

then $f(z_n) = g(z_n)$ for each 'n'

Let $h(z) = f(z) - g(z)$ —— (1)

then $h(z)$ is an analytic in D.

Let $\{z_{n_k}\} = \{z_{2n+1}\} = \{(-1)^{2n+1}\}$

put $z = z_{n_k}$ in ①

$h(z_{n_k}) = f(z_{n_k}) - g(z_{n_k})$

$= (-1)^{4n+2} - (-1)^{8n+4}$

$= 0$ · for all points of $\{z_{n_k}\}$.

$\therefore$ $h(z) = 0$ for each z in D.

$\therefore$ $f(z) = g(z)$ for each z in D.

→ Let $f(z)$ be an entire function satisfying $|f(z)| \le k|z|^2$ for some +ve constant k and all z. Show that $f(z) = az^2$ for some const. 'a'.

Now we are going to discuss about the extreme values of an analytic function in a domain D. we prove that $|f(z)|$ will attain its maximum value on the domain only and minimum also on the domain if $f(z) \neq 0$ inside D. This means that the value of $f(z)$ inside D will vary between these two values only.

we also prove that the function $M(r)$, which indicates the maximum value of $|f(z)|$ on the disk $|z|=r$ will be a continuous function. In this process we discuss the following theorems.

(i) Gauss-Mean value theorem

(ii) Maximum modulus theorem

(iii) Minimum modulus theorem

(iv) Schwarz's theorem.

Gauss-Mean value theorem:

This theorem states that for a function to be analytic inside and on a circle, the average of the values on the circumference is equal to the value of the function at the centre of the circle.

Statement: Let $f(z)$ be analytic inside and on a disk $|z-z_0| \leq r$ Then $f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta$

proof: Since $f(z)$ is analytic inside and on a disk $|z-z_0| \leq r$.

$C: |z-z_0| = r$ and z_0 is a point inside C, by Cauchy's Integral formula we have

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)}{z-z_0}\,dz$$

Now let $|z-z_0| = r$

$\Rightarrow z - z_0 = re^{i\theta}$

$\Rightarrow z = z_0 + re^{i\theta}$.

$dz = ire^{i\theta}d\theta \quad , \quad 0 \le \theta \le 2\pi$

$$\therefore f(z_0) = \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(z_0+re^{i\theta})}{re^{i\theta}}\, ire^{i\theta}d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(z_0+re^{i\theta})\,d\theta$$

Hence the result

→ Maximum-Modulus theorem (first form):

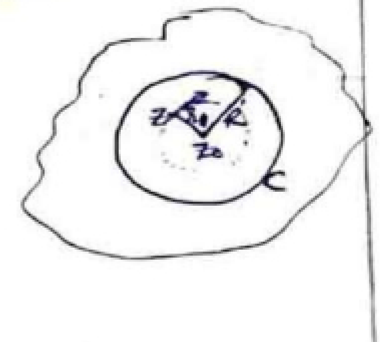
If $f(z)$ is analytic in a domain D, then $|f(z)|$ cannot attain maximum in D, unless $f(z)$ is constant.

Proof: The proof of the theorem is divided into two cases.

Case(i): Let $|f(z)|$ be not constant inside D.

Then for proving the theorem we start with a contradiction. i.e, we start with the supposition that $|f(z)|$ attains its maximum inside D at a point z_0.

Then $|f(z_0)|$ is the maximum value that $|f(z)|$ can take.



Now let us construct a circle C with z_0 as centre and Radius R, where $0 \le r \le R$.

Then by Gauss-Mean value theorem.

$\frac{1}{2\pi}\int_0^{2\pi} f(z_0 + re^{i\theta})\,d\theta$.

But by assumption

$$|f(z_0+re^{i\theta})| < |f(z_0)|.$$

Then for some point z, $z = z_0 + r_1 e^{i\theta_1}$, $r_1 \le R$,

we have $|f(z_0+r_1 e^{i\theta_1})| < |f(z_0)|$

Since $f(z)$ is analytic in D, $f(z)$ and $|f(z_0)|$ are continuous in D.

Then by continuity of $|f(z)|$, the strict inequality must also hold on some arc of the circle $|z-z_0| = r_1$

$$\therefore f(z_0) = \frac{1}{2\pi}\int_0^{2\pi} f(z_0+r_1 e^{i\theta})\, d\theta.$$

$$\Rightarrow |f(z_0)| \le \frac{1}{2\pi}\int_0^{2\pi} |f(z_0+r_1 e^{i\theta})|\, d\theta$$

$$< \frac{1}{2\pi}\int_0^{2\pi} |f(z_0)|$$

$$= \frac{1}{2\pi}|f(z_0)|\,[\theta]_0^{2\pi}$$

$$= |f(z_0)|$$

i.e., $|f(z_0)| < |f(z_0)|$, which is a contradiction. we got this contradiction because of the supposition that $|f(z)|$ attains its maximum value inside D.

Hence $|f(z)|$ cannot attain its maximum inside D when $|f(z)|$ is not constant.

Case ii: Let $|f(z)|$ be constant in D.

[W.K.T if $|f(z)|$ is constant in a region where $f(z)$ is analytic. Then $f(z)$ is constant.]

Then $f(z)$ must also be constant in D.

point inside D unless $f(z)$ is constant.

Hence the theorem.

Maximum modulus theorem (second form):

If $f(z)$ is analytic in a bounded domain D and continuous on its closure $\bar{D}$, then $|f(z)|$ attains a maximum on the boundary. Furthermore $|f(z)|$ doesnot attain a maximum at an interior point unless $f(z)$ is constant.

→ Minimum Modulus theorem:

Suppose $f(z)$ is analytic in a domain D. and $f(z) \neq 0$. in D. Then $|f(z)|$ cannot attain a minimum in D unless $f(z)$ is constant. If $f(z)$ is also continuous on $\bar{D}$, then $|f(z)|$ attains minimum on the boundary.

proof: Since $f(z) \neq 0$ in D, $\frac{1}{f(z)}$ is analytic in D. By Maximum modulus theorem $\frac{1}{|f(z)|}$ attains maximum on $\bar{D}$.

So we conclude that $|f(z)|$ attains minimum on $\bar{D}$.

→ Notation:

Let us denote the maximum value that $|f(z)|$ attains on $|z|=r$ by $M(r)$.

i.e, $M(r) = M(r,f) = \max\limits_{|z|=r} |f(z)|$

→ Suppose $f(z)$ is analytic in the disk $|z| \leq R$. Then $M(r) = M(r,f) = \max |f(z)|$ is a continuous function for $0 < r < R$.

SCHWARZ'S LEMMA:

Suppose $f(z)$ is analytic for $|z|<R$ with $f(0)=0$ and $|f(z)| \leq M$ in $|z|<R$ then $|f(z)| \leq \left(\frac{r}{R}\right) M$ $(|z|=r<R)$ and the equality only for $f(z)=\frac{M}{R}e^{i\alpha}z$,

where α is real.

proof: Since $f(0)=0$, we can write

$$f(z) = a_1 z + a_2 z^2 + a_3 z^3 + \cdots$$
$$= z(a_1 + a_2 z + a_3 z^2 + \cdots)$$
$$= z\, g(z)$$

where $g(z) = a_1 + a_2 z + a_3 z^2 + \cdots$

Since $f(z)$ is analytic in $|z|<R$, $z\,g(z)$ is also analytic.

$\therefore$ $g(z)$ is analytic in $|z|<R$.

Therefore by applying Maximum Modulus theorem to $g(z)$ we get

$$\max_{|z|=r} |g(z)| \leq \max_{|z|=R'} |g(z)| \quad \text{—— (1)}$$

where $r<R'<R$.

$$\therefore \max_{|z|=r} \left|\frac{f(z)}{z}\right| \leq \max_{|z|=R'} \left|\frac{f(z)}{z}\right|$$

$$\Rightarrow \max_{|z|=r} \frac{|f(z)|}{|z|} \leq \max_{|z|=R'} \frac{|f(z)|}{|z|}$$

$$\Rightarrow \max_{|z|=r} \frac{|f(z)|}{r} \leq \max_{|z|=R'} \frac{|f(z)|}{R'}$$

$$\Rightarrow \max_{|z|=r} |f(z)| \leq \max_{|z|=R'} \frac{r}{R'} |f(z)|$$

Since $|f(z)| \leq M$ in $|z|<R$

Then $\max_{|z|=r} |f(z)| \leq \frac{r}{R'}.M$

Since R' can come close to R, we have

$$\max_{|z|=r} |f(z)| \le \frac{r}{R} M.$$

If the equality in relation ① is to hold good, by Maximum modulus theorem, we conclude that $|g(z)|$ is constant.

Then $|f(z)| = |z|\,|g(z)|$

$$\therefore |g(z)| = \frac{|f(z)|}{|z|}$$

$$< \frac{r}{R} M \cdot \frac{1}{r} = \frac{M}{R}.$$

Since $g(z)$ is analytic and $|g(z)|$ is constant in $|z|<R$.

$\therefore$ we conclude that $g(z)$ is constant.

$$\therefore f(z) = z\, g(z) = z \frac{M}{R} e^{i\alpha},$$

where α is real.

Hence the result.

→ Suppose $f(z)$ is analytic for $|z|<R$. If $|f(z)| \le M$ in $|z|<R$, then $|f(z)-f(0)| \le \frac{2r}{R} M$; $(|z|=r<R)$.

<u>proof</u>: Let $g(z) = f(z) - f(0)$

$\therefore g(0) = 0$ and

$$|g(z)| = |f(z) - f(0)| \le |f(z)| + |f(0)|$$
$$\le M + M = 2M.$$

$\therefore g(z)$ is a function such that it is analytic in $|z|<R$ and $|g(z)| \le 2M$.

Then by applying Schwarz's lemma, we get

$$|g(z)| \le \frac{r}{R} 2M \quad ; \quad |z| = r < R$$

$$\therefore |f(z) - f(0)| \le \frac{2rM}{R} \quad ; \quad |z| = r < R.$$

Hence the result

ARGUMENT PRINCIPLE

Introduction:

In this lesson we consider an analytic function $f(z)$ which will become zero at finite number of points with in a closed contour C. Then we construct an expression $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)} dz$ which accounts for the number of zeros of $f(z)$ with in C.

We also prove a theorem called Rouche's theorem which compares the number of zeros of two analytic functions in the domain of analyticity.

Logarithmic derivative:

Suppose that $f(z)$ is analytic at z_0 and $f(z_0)=0$. Then by the corollary to theorem-I there exists a neighbourhood of z_0 that contains no zeros of $f(z)$.

Therefore we can write

$$f(z) = (z-z_0)^K F(z), \text{ where } K \text{ is a +ve integer.}$$

and $F(z)$ is analytic at z_0 with no zeros in the neighbourhood or on its boundary C.

Then $f'(z) = K(z-z_0)^{K-1} F(z) + (z-z_0)^K F'(z)$

$$\therefore \frac{f'(z)}{f(z)} = \frac{K}{z-z_0} + \frac{F'(z)}{F(z)}$$

$$\therefore \int \frac{f'(z)}{f(z)} dz = \int \frac{K}{z-z_0} dz + \int_C \frac{F'(z)}{F(z)} dz$$

Since $\frac{F'(z)}{F(z)}$ is analytic inside and on C,

by Cauchy's theorem, we get

$$\int_C \frac{F'(z)}{F(z)} dz = 0.$$

Let $z - z_0 = \epsilon e^{i\theta}$

$\Rightarrow z = z_0 + \epsilon e^{i\theta}$

$\Rightarrow dz = i\epsilon e^{i\theta} d\theta$

$$\therefore \int_C \frac{f'(z)}{f(z)} dz = \int_0^{2\pi} \frac{k}{\epsilon e^{i\theta}} i\epsilon e^{i\theta} d\theta$$

$$= ik \int_0^{2\pi} d\theta$$

$$= ik\,2\pi = 2\pi i k.$$

$$\therefore k = \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz$$

From this relation it can be said that the order of zero of a function $f(z)$ at a point z_0 is given by $\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)} dz$.

→ The expression $\frac{f'(z)}{f(z)}$ is known as logarithmic derivative of $f(z)$. Because it is the derivative of $\log f(z)$ at points where $f(z)$ is analytic and non-zero.

→ Now let us consider the analyticity of $f(z)$ in a domain instead of at a point.

Counting function:- Let us suppose that $f(z)$ is analytic inside and on a simple closed C with no zeros on C. Then $f(z)$

Type(4)
MISCELLANEOUS

→ Show that $\int_0^\infty \frac{\sin x}{x} = \pi/2$
IAS-1986

Sol^n: for finding the value of integral we approximate the given function by $\frac{e^{iz}}{z}$, whose imaginary part on real axis $\frac{\sin x}{x}$. The contour 'C' along which the function is to be integrated will consists of real axis from ϵ to R, a semi-circle in the upper half plane from R to $-R$, then the real axis from $-R$ to $-\epsilon$ and and finally a semi-circle in the upper half plane from $-\epsilon$ to $+\epsilon$ as shown in the figure.

The origin $z=0$ is avoided because it is a pole for $\frac{e^{iz}}{z}$.
A function cannot be integrated along a path which consists of a singularity.
Since the function $\frac{e^{iz}}{z}$ is analytic in and on C. ($\because z=0$ is outside C)
we have $\int_C \frac{e^{iz}}{z} = 0$

$$\therefore \int_C \frac{e^{iz}}{z} = \int_\epsilon^R \frac{e^{ix}}{x}dx + \int_0^\pi \frac{e^{iz}}{z}dz + \int_{-R}^{-\epsilon} \frac{e^{ix}}{x}dx + \int_\pi^0 \frac{e^{iz}}{z}dz = 0$$

Let $z = Re^{i\theta}$ in the 2nd integral, $z = \epsilon e^{i\theta}$ in the 4th integral
$dz = Rie^{i\theta}$ $\Rightarrow dz = \epsilon i e^{i\theta}$
Let x be replaced by $-x$ in the 3rd integral and combining with the first integral, then we get

$$\int^R \frac{e^{ix}-e^{-ix}}{} dx + \int^\pi \frac{e^{iRe^{i\theta}}}{} iRe^{i\theta}d\theta + \int^0 \frac{e^{i\epsilon e^{i\theta}}}{} i\epsilon e^{i\theta}d\theta = 0$$

But $e^{ix} - e^{-ix} = 2i\sin x$

$$\therefore \int_{\epsilon}^{R} \frac{2i\sin x}{x}dx + i\int_0^{\pi} e^{iRe^{i\theta}}d\theta - i\int_0^{\pi} e^{i\epsilon e^{i\theta}}d\theta = 0$$

$$\Rightarrow 2\int_{\epsilon}^{R} \frac{\sin x}{x}dx + \int_0^{\pi} e^{iRe^{i\theta}}d\theta = \int_0^{\pi} e^{i\epsilon e^{i\theta}}d\theta \quad \text{—— (1)}$$

Now consider the 2nd integral of L.H.S in eqn (1)

we get

$$\left|\int_0^{\pi} e^{iRe^{i\theta}}d\theta\right| \leq \int_0^{\pi} \left|e^{iRe^{i\theta}}\right| d\theta = \int_0^{\pi} \left|e^{iR\cos\theta}\right|\left|e^{-R\sin\theta}\right| d\theta$$

$$= \int_0^{\pi} e^{-R\sin\theta}d\theta$$

$$= 2\int_0^{\pi/2} e^{-R\sin\theta}d\theta$$

$$\leq 2\int_0^{\pi/2} e^{(-2R/\pi)\theta}d\theta$$

$\left(\because \sin\theta \geq \frac{2\theta}{\pi} \text{ for } 0 \leq \theta \leq \pi/2\right)$

$$= \frac{\pi}{R}(1 - e^{-R})$$

This tends to zero as $R \to \infty$.

$\therefore$ As $R \to \infty$, (1) reduces to

$$2\int_{\epsilon}^{\infty} \frac{\sin x}{x}dx = \int_0^{\pi} e^{i\epsilon e^{i\theta}}d\theta.$$

Now taking as $\epsilon \to 0$ we get

$$\lim_{\epsilon \to 0} 2\int_{\epsilon}^{\infty} \frac{\sin x}{x}dx = \lim_{\epsilon \to 0} \int_0^{\pi} e^{i\epsilon e^{i\theta}}d\theta.$$

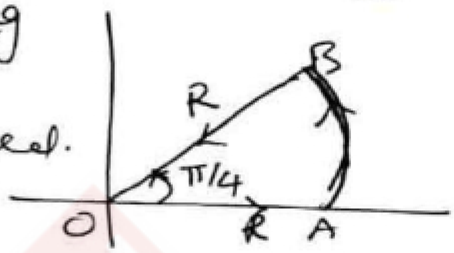
$$\therefore 2\int_0^{\infty} \frac{\sin x}{x}dx = \int_0^{\pi} d\theta = \pi$$

$$\therefore \int_0^{\infty} \frac{\sin x}{x}dx = \pi/2.$$

→ S.T $\int_0^{\infty} \sin x^2 \, dx = \int_0^{\infty} \cos x^2 \, dx = \frac{1}{2}\sqrt{\pi/2}$

Soln: Let C be the contour consisting of the line

line segment $\underline{BO}$ as shown in the figure.
we integrate the function e^{iz^2} along the contour 'C' which ultimately leads to the evaluation of the integral.



Since e^{iz^2} is analytic in and on C.

By Cauchy's theorem

$$\int_C e^{iz^2}dz = 0$$

But $\int_C e^{iz^2}dz = \int_{OA} e^{iz^2}dz + \int_{AB} e^{iz^2}dz + \int_{BO} e^{iz^2}dz = 0$ ——— (1)

- In the first integral on R.H.S, OA is the line segment along the real axis.
  $\therefore$ $y=0$ and x varies from 0 to R.

- In the second integral the path integration is the arc AB. we convert the variable of integration into polar co-ordinates by taking $z=Re^{i\theta}$.
  The variable R is constant and the angle θ varies from 0 to $\pi/4$.

- In the third integral the path of integration is the line BO. Here we take $z=te^{i\pi/4}$, $R \geq t \geq 0$.
  (i.e, $t: R \to 0$)

Now the relation (1) becomes

$$\int_0^R e^{ix^2}dx + \int_0^{\pi/4} e^{iR^2 e^{i2\theta}} iRe^{i\theta}d\theta + \int_R^0 e^{it^2 \cdot e^{i\pi/2}} e^{i\pi/4}dt = 0$$ ——— (2)

Let us consider each integral on L.H.S again and consider what happens when $R \to \infty$.

- from the first integral

— from the second integral

$$\left|\int_0^{\pi/4} e^{iR^2 e^{i2\theta}}\, iRe^{i\theta}d\theta\right| \le \int_0^{\pi/4} \left|e^{iR^2e^{i2\theta}}\right|\left|iRe^{i\theta}\right|d\theta$$

$$= R\int_0^{\pi/4}\left|e^{iR^2\cos 2\theta}\right|\left|e^{-R^2\sin 2\theta}\right|d\theta \qquad \left(\because |ie^{i\theta}|=1\right)$$

$$\le R\int_0^{\pi/4} e^{-R^2\sin 2\theta}\,d\theta \qquad \left(\because \left|e^{iR^2\cos 2\theta}\right|=1\right)$$

$$\le R\int_0^{\pi/4} e^{-R^2 4\theta/\pi} \qquad \left(\because \sin\theta \ge 2\theta/\pi \Rightarrow \sin 2\theta \ge 4\theta/\pi\right)$$

$$= \frac{\pi}{4R}\left(1-e^{-R^2}\right)$$

$e^{iR^2e^{i2\theta}} = e^{iR^2(\cos 2\theta + i\sin 2\theta)} = e^{iR^2\cos 2\theta}e^{-R^2\sin 2\theta}$
$\Rightarrow \left|e^{iR^2e^{i2\theta}}\right| = \left|e^{iR^2\cos 2\theta}\right|\left|e^{-R^2\sin 2\theta}\right| = 1\cdot e^{-R^2\sin 2\theta}$

As $R\to\infty$ the value of the integral tends to zero..

— from the third integral

$$e^{i\pi/4}\int_R^{0} e^{-t^2}dt = -e^{i\pi/4}\int_0^R e^{-t^2}dt \qquad \left[\because \int_a^b f(x)\,dx = -\int_b^a f(x)\,dx\right]$$

As $R\to\infty$, we get

$$-e^{i\pi/4}\int_0^{\infty} e^{-t^2}dt = -\left(\frac{1+i}{\sqrt2}\right)\int_0^\infty e^{-t^2}dt$$

$$= -\left(\frac{1+i}{\sqrt2}\right)\cdot\frac{\sqrt\pi}{2} \qquad \left[\because \int_0^\infty e^{-t^2}dt = \frac{\sqrt\pi}{2}\right]$$

Substituting these values in (2) we get

$$\int_0^\infty (\cos x^2 + i\sin x^2)\,dx + 0 - \left(\frac{1+i}{\sqrt2}\right)\frac{\sqrt\pi}{2} = 0$$

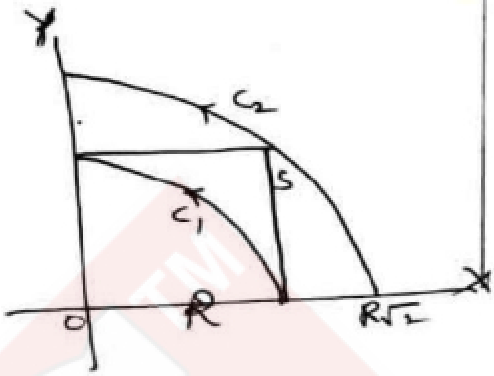
$$\Rightarrow \int_0^\infty (\cos x^2 + i\sin x^2)\,dx = \left(\frac{1+i}{\sqrt2}\right)\frac{\sqrt\pi}{2}$$

comparing the real and imaginary parts

we get
$$\int_0^\infty \sin x^2\,dx = \frac12\sqrt{\pi/2}$$
$$\int_0^\infty \cos x^2\,dx = \frac12\sqrt{\pi/2}$$

→ Show that $\int_0^\infty e^{-x^2}dx = \sqrt{\pi}/2$

Sol^n: Let us denote $I = \int_0^R e^{-x^2}dx$. —(i)



Then

$$I^2 = I \cdot I$$
$$= \int_0^R e^{-x^2}dx \int_0^R e^{-y^2}dy$$
$$= \int_0^R\int_0^R e^{-(x^2+y^2)}dx\,dy$$

Here we are integrating along the sides of a square whose side is R and which is in first quadrant as shown in the figure. Let C_1 and C_2 be two quarter circles in the first quadrant centered at the origin having radius R and $R\sqrt{2}$ respectively.

Evaluating along the circles in polar coordinates and the rectangle using rectangular coordinates and observing the figure.

$$\iint_{C_1} e^{-(x^2+y^2)}dx\,dy \le \iint_S e^{-(x^2+y^2)}dx\,dy \le \iint_{C_2} e^{-(x^2+y^2)}dx\,dy$$

$$\Rightarrow \int_0^{\pi/2}\int_0^R e^{-r^2}r\,dr\,d\theta \le \int_0^R\int_0^R e^{-(x^2+y^2)}dx\,dy \le \int_0^{\pi/2}\int_0^{R\sqrt{2}} e^{-r^2}r\,dr\,d\theta \quad \text{—(1)}$$

consider

$$\int_0^{\pi/2}\int_0^R e^{-r^2}r\,dr\,d\theta$$

Let $r^2 = t \Rightarrow 2r\,dr = dt$

when $r = 0 \Rightarrow t = 0$

$r = R \Rightarrow t = R^2$

Now we get

$$\int_0^{\pi/2}\int_0^R e^{-r^2}r\,dr\,d\theta = \int_0^{\pi/2}\int_0^{R^2} e^{-t}\frac{dt}{2}\,d\theta = \frac{1}{2}\int_0^{\pi/2}(1-e^{-R^2})\,d\theta$$

$$\frac{\pi}{}(1 - e^{-R^2})$$

Similarly $\int_0^{\pi/2}\int_0^{R\sqrt{2}} e^{-r^2}\, r\, dr\, d\theta = \frac{\pi}{4}(1-e^{-2R^2})$ ——(iii)

Using (i) (ii) & (iii) eqn (1) reduces to

$$\frac{\pi}{4}(1-e^{-R^2}) \le \left(\int_0^R e^{-x^2}dx\right)^2 \le \frac{\pi}{4}(1-e^{-2R^2})$$

Now as $R\to\infty$

we get

$$\frac{\pi}{4} \le \left(\int_0^R e^{-x^2}dx\right)^2 \le \frac{\pi}{4}$$

$$\Rightarrow \left(\int_0^R e^{-x^2}dx\right)^2 = \frac{\pi}{4}$$

$$\Rightarrow \int_0^R e^{-x^2}dx = \sqrt{\pi}/2$$

→ Show that $\sin\left\{c\left(z+\frac{1}{z}\right)\right\}$ can be expanded in a series of the type

$\sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} b_n z^{-n}$ in which the coefficients of both z^n and z^{-n} are $\frac{1}{2\pi}\int_0^{2\pi} \sin(2c\cos\theta)\cos n\theta \, d\theta$.

Sol^n:

The function $f(z) = \sin\left\{c\left(z+\frac{1}{z}\right)\right\}$ is analytic except at $z=0$.

i.e., $\sin\left\{c\left(z+\frac{1}{z}\right)\right\}$ is analytic in the annulus $r<z<R$.

where r is small and R is large. Therefore $f(z)$ can be expanded in Laurent's series.

∴ by Laurent's theorem.

$$\sin\left\{c\left(z+\frac{1}{z}\right)\right\} = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n}.$$

where $a_n = \frac{1}{2\pi i}\int_C \sin\left\{c\left(z+\frac{1}{z}\right)\right\} \frac{dz}{z^{n+1}}$ ——— ①

and $b_n = \frac{1}{2\pi i}\int_C \sin\left\{c\left(z+\frac{1}{z}\right)\right\} \frac{dz}{z^{-n+1}}$

Where 'C' is any circle with origin as centre.

Let us in particular choose 'C' to be a circle of radius 1 having centre at the origin.

i.e, the eqn of C is $|z|=1$ or $z=e^{i\theta}$

Since $z = e^{i\theta}$

$\Rightarrow dz = ie^{i\theta}d\theta$.

Then ① becomes

$$a_n = \frac{1}{2\pi i}\int_0^{2\pi} \sin\left\{c\left(e^{i\theta}+e^{-i\theta}\right)\right\} \frac{ie^{i\theta}d\theta}{e^{(n+1)i\theta}}.$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \sin(2c\cos\theta) \cdot ie^{-in\theta} \, d\theta.$$

$\left(\because \frac{e^{i\theta}+e^{-i\theta}}{2} = \cos\theta\right)$

$$= \frac{1}{2\pi}\int_0^{2\pi} \sin(2c\cos\theta)\{\cos n\theta - i\sin n\theta\}\, d\theta.$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \sin(2c\cos\theta)\cos n\theta\, d\theta - i\frac{1}{2\pi}\int_0^{2\pi} \sin(2c\cos\theta)\sin n\theta\, d\theta.$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \sin(2c\cos\theta)\cos n\theta\, d\theta$$

$\boxed{\text{W.K.T if } f(2a-x) = -f(x) \text{ then } \int_0^{2a} f(x)\,dx = 0}$

$\boxed{\because f(\theta) = \sin(2c\cos\theta)\sin n\theta \quad \text{Here } f(2\pi-\theta) = -f(\theta) \quad \therefore \int_0^{2\pi} f(\theta)\,d\theta = 0}$

Since the integral

$$\int_0^{2\pi} \sin(2c\cos\theta)\sin n\theta\, d\theta = 0$$

by a property of definite integral

Now the function $\sin\{c(z+\frac{1}{z})\}$ remains unaltered if $\frac{1}{z}$ is written for z.

Hence $b_n = a_{-n} = \frac{1}{2\pi}\int_0^{2\pi} \sin(2c\cos\theta)\cos(-n\theta)\, d\theta$

$$= \frac{1}{2\pi}\int_0^{2\pi} \sin(2c\cos\theta)\cos n\theta\, d\theta$$

$$= a_n.$$

Hence $\sin\{c(z+\frac{1}{z})\} = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n}$ —— (2)

where $a_n = b_n = \frac{1}{2\pi}\int_0^{2\pi} \sin(2c\cos\theta)\cos n\theta\, d\theta.$

$\therefore$ eqn (2) may be expressed as

$$\sin(c(z+\tfrac{1}{z})) = a_0 + \sum_{n=1}^{\infty}\left(z^n + \frac{1}{z^n}\right)a_n.$$

where $a_n = \frac{1}{2\pi}\int_0^{2\pi} \sin(2c\cos\theta)\cos n\theta\, d\theta.$

H.W. → P.T $\cosh(z+\frac{1}{z}) = a_0 + \sum_{n=0}^{\infty} a_n\left(z^n + \frac{1}{z^n}\right)$

where $a_n = \frac{1}{2\pi}\int_0^{2\pi} \cosh n\theta \cosh(2\cos\theta)\, d\theta.$

IAS 2010 → Show that

$$e^{\frac{1}{2}c(z-\frac{1}{z})} = \sum_{n=-\infty}^{\infty} a_n z^n.$$

where $a_n = \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - c\sin\theta)\, d\theta.$

Sol: The function $e^{\frac{1}{2}c\left(z-\frac{1}{z}\right)}$ is analytic at every point except at $z=0$ and $z=\infty$. i.e, it is analytic in the ring shaped region (annulus) $r \le z \le R$ where r is small and R is large. Hence it can be expanded as a Laurent's series in the form. (20)

$$e^{\frac{1}{2}c\left(z-\frac{1}{z}\right)} = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n}.$$

Where $a_n = \frac{1}{2\pi i}\int_C e^{\frac{1}{2}c\left(z-\frac{1}{z}\right)} \frac{dz}{z^{n+1}}$

$b_n = \frac{1}{2\pi i}\int_C e^{\frac{1}{2}c\left(z-\frac{1}{z}\right)} \frac{dz}{z^{n+1}}$

where C is any circle with origin as centre.

Let us in particular choose 'C' to be a circle of radius 1 having centre at the origin

$\therefore |z|=1$ or $z=e^{i\theta}$.

$\Rightarrow dz = ie^{i\theta}d\theta$

$$\therefore a_n = \frac{1}{2\pi i}\int_0^{2\pi} e^{\frac{1}{2}c\left(z-\frac{1}{z}\right)} \frac{dz}{z^{n+1}}$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} e^{\frac{1}{2}c\left(e^{i\theta}-e^{-i\theta}\right)} \frac{ie^{i\theta}d\theta}{e^{(n+1)i\theta}}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{c\sin\theta} e^{-in\theta} d\theta \qquad \left(\because \sin\theta = \frac{e^{i\theta}-e^{-i\theta}}{2i}\right)$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{-i(n\theta - c\sin\theta)} d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - c\sin\theta) - i\sin(n\theta - c\sin\theta)\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - c\sin\theta)\, d\theta$$

Since the integral $\int_0^{2\pi} \sin(n\theta - c\sin\theta)\, d\theta = 0$

by the property of definite integrals.

2009

Let $f(z) = \dfrac{a_0 + a_1 z + \cdots + a_{n-1} z^{n-1}}{b_0 + b_1 z + \cdots + b_n z^n}$, $b_n \neq 0$.

Assume that the zeros of the denominator are simple. Show that the sum of the residues of $f(z)$ at its poles is equal to $\dfrac{a_{n-1}}{b_n}$.

Sol^n: Let $f(z) = \dfrac{a_0}{b_0 + b_1 z}$, where $b_1 \neq 0$

$$= \frac{a_0}{b_1\left[\frac{b_0}{b_1} + z\right]}$$

$z = -\dfrac{b_0}{b_1}$ is a pole of order 1. i.e., simple pole

The residue at $z = -\dfrac{b_0}{b_1}$ is

$$= \lim_{z \to -\frac{b_0}{b_1}} \left(\frac{b_0}{b_1} + z\right) \frac{a_0}{b_1\left(\frac{b_0}{b_1} + z\right)}$$

$$= \frac{a_0}{b_1}.$$

Now let us assume ~~that~~ $f(z) = \dfrac{a_0 + a_1 z}{b_0 + b_1 z + b_2 z^2}$.

$$= \frac{a_0 + a_1 z}{b_2\left(\frac{b_0}{b_2} + \frac{b_1}{b_2} z + z^2\right)}.$$

Let α, β be the simple poles of

$$\left(\frac{b_0}{b_2} + \frac{b_1}{b_2} z + z^2\right).$$

Since α, β are the roots of $z^2 + \frac{b_1}{b_2} z + \frac{b_0}{b_2}$

Now

$$z^2 + \frac{b_1}{b_2} z + \frac{b_0}{b_2} = (z-\alpha)(z-\beta)$$

Residue at $z=\alpha$ is $= \lim_{z\to\alpha} (\cancel{z-\alpha}) \dfrac{a_0+a_1 z}{b_2 \cancel{(z-\alpha)}(z-\beta)}$

$$= \frac{a_0+a_1\alpha}{b_2(\alpha-\beta)}.$$

Residue at $z=\beta$ is $= \lim_{z\to\beta} (\cancel{z-\beta}) \dfrac{a_0+a_1 z}{b_2(z-\alpha)\cancel{(z-\beta)}}$

$$= \frac{a_0+a_1\beta}{b_2(\beta-\alpha)}$$

$$= -\frac{a_0+a_1\beta}{b_2(\alpha-\beta)}$$

$\therefore$ Sum of the residues of $f(z) = \dfrac{a_0+a_1 z}{b_2\left(z^2+\frac{b_1}{b_2}z+\frac{b_0}{b_2}\right)}$

$$= \frac{1}{b_2}\left[\frac{a_0+a_1\alpha}{\alpha-\beta} - \frac{a_0+a_1\beta}{\alpha-\beta}\right]$$

$$= \frac{1}{b_2}\left[a_1\frac{(\cancel{\alpha-\beta})}{(\cancel{\alpha-\beta})}\right] = \frac{a_1}{b_2}.$$

Let $f(z) = \dfrac{a_0+a_1 z+a_2 z^2}{b_0+b_1 z+b_2 z^2+b_3 z^3}$.

$$= \frac{a_0+a_1 z+a_2 z^2}{b_3[(z-\alpha)(z-\beta)(z-\gamma)}$$

where α, β, γ are the simple poles.

Residue at $z=\alpha$ is $\lim_{z\to\alpha} (\cancel{z-\alpha}) \dfrac{a_0+a_1 z+a_2 z^2}{b_3\cancel{(z-\alpha)}(z-\beta)(z-\gamma)}$

$$= \frac{a_0+a_1\alpha+a_2\alpha^2}{b_3(\alpha-\beta)(\alpha-\gamma)}$$

Residue at $z=\beta$ is $\lim_{z\to\beta} (\cancel{z-\beta}) \dfrac{a_0+a_1 z+a_2 z^2}{b_3(z-\alpha)\cancel{(z-\beta)}(z-\gamma)}$

$$= \frac{a_0+a_1\beta+a_2\beta^2}{b_3(\beta-\alpha)(\beta-\gamma)}$$

Residue at $z=\gamma$ is $= \lim_{z\to\gamma} \cancel{(z-\gamma)} \frac{a_0+a_1z+a_2z^2}{b_3(z-\alpha)(z-\beta)\cancel{(z-\gamma)}}$

$$= \frac{a_0+a_1\gamma+a_2\gamma^2}{b_3(\gamma-\alpha)(\gamma-\beta)}$$

Sum of the residues of $f(z)$ at its poles α, β, γ

$$= \frac{1}{b_3}\left[\frac{a_0+a_1\alpha+a_2\alpha^2}{(\alpha-\beta)(\alpha-\gamma)} + \frac{a_0+a_1\beta+a_2\beta^2}{(\beta-\alpha)(\beta-\gamma)} + \frac{a_0+a_1\gamma+a_2\gamma^2}{(\gamma-\alpha)(\gamma-\beta)}\right]$$

$$= \frac{1}{b_3}\left[\frac{a_0+a_1\alpha+a_2\alpha^2}{(\alpha-\beta)(\alpha-\gamma)} - \frac{a_0+a_1\beta+a_2\beta^2}{(\alpha-\beta)(\beta-\gamma)} + \frac{a_0+a_1\gamma+a_2\gamma^2}{(\alpha-\gamma)(\beta-\gamma)}\right]$$

$$= \frac{1}{b_3}\left[\frac{(a_0+a_1\alpha+a_2\alpha^2)(\beta-\gamma)-(a_0+a_1\beta+a_2\beta^2)(\alpha-\gamma)+(a_0+a_1\gamma+a_2\gamma^2)(\alpha-\beta)}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}\right]$$

$$= \frac{1}{b_3}\left[\frac{a_0(0)+a_1(0)+a_2\left(\alpha^2(\beta-\gamma)-\beta^2(\alpha-\gamma)+\gamma^2(\alpha-\beta)\right)}{(\alpha-\beta)(\beta-\gamma)(\alpha-\gamma)}\right]$$

$$= \frac{1}{b_3}\left[\frac{a_2\left(\alpha^2\beta-\gamma\alpha^2-\beta^2\alpha+\gamma\beta^2+\gamma^2\alpha-\gamma^2\beta\right)}{\alpha^2\beta-\cancel{\alpha\gamma\beta}-\alpha^2\gamma+\alpha\gamma^2-\beta^2\alpha+\beta^2\gamma+\cancel{\alpha\beta\gamma}-\gamma^2\beta}\right]$$

$$= \frac{a_2}{b_3}, \quad (b_3\neq 0)$$

$\therefore$ we conclude that

$$f(z) = \frac{a_0+a_1z+a_2z^2+\dots+a_{n-1}z^{n-1}}{b_0+b_1z+b_2z^2+\dots+b_nz^n}$$

then the sum of the residues of $f(z)$ at its poles is $= \frac{a_{n-1}}{b_n}$, where $b_n\neq 0$

2011 **Example 17.** *If the function $f(z)$ is analytic and one valued in $|z-a|<R$, prove that when $0<r<R$,*

$$f'(a)=\frac{1}{\pi r}\int_0^{2\pi} P(\theta)\, e^{-i\theta}\, d\theta$$

where $P(\theta)$ is the real part of $f(a+re^{i\theta})$. (Agra 1957, 70)

Solution. Since $f(z)$ is analytic in $|z-a|<R$ and $r<R$, it follows that $f(z)$ is also analytic inside the circle C defined by

$$|z-a|=r.$$

Hence by Cauchy's formula for the derivative, we have

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\, dz=f'(a). \qquad \ldots(1)$$

Also $f(z)$ can be expanded as a Taylor's series about $z=a$ in the form

$$f(z)=\sum_{m=0}^{\infty} a_m (z-a)^m.$$

Putting $z-a=re^{i\theta}$, we have

$$f(z)=f(a+re^{i\theta}) \sum_{m=0}^{\infty} a_m\, r^m\, e^{mi\theta}.$$

Then $\overline{f(z)}=\sum_{m=0}^{\infty} \bar{a}_m\, r^m\, e^{-mi\theta}$

$$\therefore \quad \frac{1}{2\pi i}\int \frac{\overline{f(z)}}{(z-a)^2}\, dz=\frac{1}{2\pi i}\int_0^{2\pi} \frac{\sum_{m=0}^{\infty} \bar{a}_m\, r^m\, e^{-mi\theta}}{r^2\, e^{2i\theta}}\,.\, r\, ie^{i\theta}\, d\theta$$

$$=\frac{1}{2\pi}\sum_{m=0}^{\infty} \bar{a}_m\, r^{m-1}\int_0^{2\pi} e^{-(m+1)i\theta}\, d\theta$$

$$=0\left[\because \quad \int_0^{2\pi} e^{-(m+1)i\theta}\, d\theta=0\right]. \qquad \ldots(2)$$

Adding (1) and (2), we have

$$f'(a)=\frac{1}{2\pi i}\int_C \frac{f(z)+\overline{f(z)}}{(z-a)^2}\, dz$$

$$=\frac{1}{2\pi i}\int_C \frac{2 \text{ real part of } f(z)}{(z-a)^2}\, dz$$

$$=\frac{1}{2\pi i}\int_0^{2\pi} \frac{\text{real part of } (a+re^{i\theta})}{r^2\, e^{2i\theta}}\,.\, rie^{i\theta}\, d\theta$$

$[\because \quad z=a+re^{i\theta}]$

$$=\frac{1}{\pi r}\int_0^{2\pi} P(\theta)\, e^{-i\theta}\, d\theta$$

where $P(\theta)$ is the real part of $f(a+re^{i\theta})$

Example 17. *If the function $f(z)$ is analytic and one valued in $|z-a| < R$, prove that when $0 < r < R$,*

$$f'(a)=\frac{1}{\pi r}\int_0^{2\pi} P(\theta)\, e^{-i\theta}\, d\theta$$

where $P(\theta)$ is the real part of $f(a+re^{i\theta})$. (Agra 1957, 70)

Solution. Since $f(z)$ is analytic in $|z-a| < R$ and $r < R$, it follows that $f(z)$ is also analytic inside the circle C defined by

$$|z-a|=r.$$

Hence by Cauchy's formula for the derivative, we have

$$\frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}\, dz=f'(a). \qquad \ldots(1)$$

Also $f(z)$ can be expanded as a Taylor's series about $z=a$ in the form

$$f(z)=\sum_{m=0}^{\infty} a_m (z-a)^m.$$

Putting $z-a=re^{i\theta}$, we have

$$f(z)=f(a+re^{i\theta}) \sum_{m=0}^{\infty} a_m\, r^m\, e^{mi\theta}.$$

Then $\overline{f(z)}=\sum_{m=0}^{\infty} \bar{a}_m\, r^m\, e^{-mi\theta}$

$$\therefore \quad \frac{1}{2\pi i}\int \frac{\overline{f(z)}}{(z-a)^2}\, dz=\frac{1}{2\pi i}\int_0^{2\pi} \frac{\sum_{m=0}^{\infty} \bar{a}_m\, r^m\, e^{-mi\theta}}{r^2\, e^{2i\theta}} . r\, ie^{i\theta}\, d\theta$$

$$=\frac{1}{2\pi}\sum_{m=0}^{\infty} \bar{a}_m\, r^{m-1}\int_0^{2\pi} e^{-(m+1)i\theta}\, d\theta$$

$$=0\left[\because \int_0^{2\pi} e^{-(m+1)i\theta}\, d\theta=0\right]. \qquad \ldots(2)$$

Adding (1) and (2), we have

$$f'(a)=\frac{1}{2\pi i}\int_C \frac{f(z)+\overline{f(z)}}{(z-a)^2}\, dz$$

$$=\frac{1}{2\pi i}\int_C \frac{2 \text{ real part of } f(z)}{(z-a)^2}\, dz$$

$$=\frac{1}{2\pi i}\int_0^{2\pi} \frac{\text{real part of } (a+re^{i\theta})}{r^2\, e^{2i\theta}} . rie^{i\theta}\, d\theta$$

$$[\because \quad z=a+re^{i\theta}]$$

$$=\frac{1}{\pi r}\int_0^{2\pi} P(\theta)\, e^{-i\theta}\, d\theta$$

where $P(\theta)$ is the real part of $f(a+re^{i\theta})$.

$2z^4 + 5z^2 + 2$

$2z^4 + 4z^2 + z^2 + 2$

$2z^2(z^2 + \cancel{2z}) + 1(z^2 + 2)$

$(2z^2 + 1)(z^2 + 2) = 0$

$z^2 = -\frac{1}{2}$ $\quad z^2 = -2$

$z = \sqrt{\ }$ $\quad z = \pm \frac{1}{\sqrt{2}} i$ $\quad z = \pm \sqrt{2}\, i$

$z = \sqrt{-\frac{1}{2}}$

$= \pm \frac{1}{\sqrt{2}} i$