Chapter 9

2012

9.1 Section-A

Question-1(a) Let $V = \mathbb{R}^3$ and $\alpha_1 = (1,1,2), \alpha_2 = (0,1,3)$ $\alpha_3 = (2,4,5)$ and $\alpha_4 = (-1,0,-1)\cdot$ be the elements of V. Find a basis for the intersection of the subspace spanned by $\{\alpha_1, \alpha_2\}$ and $\{\alpha_3, \alpha_4\}$.

[8 Marks]

Solution: Let $W_1 = (\alpha_1, \alpha_2) = a(1,1,2) + b(0,1,3) = (a, a+b, 2a+3b)$
Let $W_2 = \sin(\alpha_3, \alpha_4) = c(2,4,5) + d(-1,0,-1) = (2c-d, 4c, 5c-d)$
Let (x,y,z) be an element of intersection of W_1 and W_2 i.e. $(x,y,z) \in W_1 \cap W_2$.

Then,

$$(x,y,z) = (a, a+b, 2a+3b) = (2c-d, 4c, 5c-d)$$
$$\Rightarrow (a, a+b, 2a+3b) - (2c-d, 4c, 5c-d) = (0,0,0)$$
$$\Rightarrow (a-2c+d, a+b-4c, 2a+3b-5c+d) = (0,0,0)$$

Let,

$$A = \begin{bmatrix} 1 & 0 & -2 & 1 \\ 1 & 1 & -4 & 0 \\ 2 & 3 & -5 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 1 \\ 0 & 1 & -2 & 1 \\ 0 & 3 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 1 \\ 1 & 1 & -4 & 0 \\ 0 & 0 & 5 & 2 \end{bmatrix}$$

$$R_1 \to 5R_1 + 2R_3, R_2 \to 5R_2 + 2R_3 \quad R_1 \to R_1/5, R_2 \to R_2/5, R_3 \to R_3/5$$

$$\sim \begin{bmatrix} 5 & 0 & 0 & 9 \\ 0 & 5 & 0 & -1 \\ 0 & 0 & 5 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 9/5 \\ 0 & 1 & 0 & -1/5 \\ 0 & 0 & 1 & 2/5 \end{bmatrix}$$

$$\therefore \quad a + \frac{9}{5}d = 0, b - \frac{1}{5}d = 0, \quad c + \frac{2}{5}d = 0$$

$$a = \frac{-9}{5}d, \quad b = \frac{1}{5}d, \quad c = -\frac{2}{5}d.$$

$$(x,y,z) = (a, a+b, 2a+3b) = \left(\frac{-9}{5}d, -\frac{9}{5}d + \frac{1}{5}d, 2\left(\frac{-9}{5}d\right) + 3\left(\frac{1}{5}d\right)\right)$$
$$= d\left(-\frac{9}{5}, -\frac{8}{5}, -3\right)$$
$$= k(-9,-8,-15)$$
$$= k_1(9,8,15)$$

$\therefore$ Basis of $w_1 \cap \omega_2$ is $\{(9,8,15)\}$.

Question-1(b) Show that the set of all functions which satisfy the differential equation, $\frac{d^2 f}{dx^2} + 3\frac{df}{dx} = 0$ is a vector space.

[8 Marks]

Solution: Let W be the set of all functions which satisfy the differential equation,

$$\frac{d^2 f}{dx^2} + 3\frac{df}{dx} = 0$$

$$\therefore \text{W} = \left\{ f : \frac{d^2 f}{dx^2} + 3\frac{df}{dx} = 0 \right\}$$

Let $y = f(x)$ Obviously $f(x) = 0$ or $y = 0$ satisfy the given differential equation and as such it belongs to W and thus $W \neq \phi$ Now let $y_1, y_2 \in$ W, then

$$\frac{d^2 y_1}{dx^2} + 3\frac{dy_1}{dx} = 0$$

and

$$\frac{d^2 y_2}{dx^2} + 3\frac{dy_2}{dx} = 0$$

Let $a, b \in$ R. If W is to be a subspace then we should show that $ay_1 + by_2$ also belongs to W i.e., it is a solution of the given differential equation. We have

$$\begin{aligned}
\frac{d^2}{dx^2}(ay_1 + by_2) + 3\frac{d}{dx}(ay_1 + by_2) &= a\frac{d^2 y_1}{dx^2} + b\frac{d^2 y_2}{dx^2} + 3a\frac{dy_1}{dx} + 3b\frac{dy_2}{dx} \\
&= a\left(\frac{d^2 y_1}{dx^2} + 3\frac{dy_1}{dx}\right) + b\left(\frac{d^2 y_2}{dx^2} + 3\frac{dy_2}{dx}\right) \\
&= a(0) + b(0) \\
&= 0
\end{aligned}$$

using (1) and (2)

Thus $ay_1 + by_2$ is a solution of the given differential equation and so it belongs to W.

Hence, W is the subspace. Thus, W is a vector space.

Question-1(c) If the three thermodynamic variables P, V, T are connected by a relation $f(P, V, T) = 0$. Show that,

$$\left(\frac{\partial P}{\partial T}\right)_V \cdot \left(\frac{\partial T}{\partial V}\right)_P \left(\frac{\partial V}{\partial P}\right)_T \cong -1$$

[8 Marks]

Solution: Given $f(P, V, T) = 0$ When V is constant;
Taking P as function of T, we have

$$\left(\frac{\partial P}{\partial T}\right)_V = -\frac{\frac{\partial f}{\partial T}}{\frac{\partial f}{\partial P}}$$

Similarly,

$$\left(\frac{\partial T}{\partial V}\right)_p = -\frac{\frac{\partial f}{\partial V}}{\frac{\partial f}{\partial T}}; \left(\frac{\partial V}{\partial P}\right)_T = -\frac{\frac{\partial f}{\partial P}}{\frac{\partial f}{\partial V}}$$

Multiplying the three, we get

$$\left(\frac{\partial P}{\partial T}\right)_V \left(\frac{\partial T}{\partial V}\right)_P \left(\frac{\partial V}{\partial P}\right)_T = -1$$

Question-1(d) If $u = Ae^{-gx}\sin(nt - gx)$, where A, g, n are positive constants, satisfies the heat conduction equation, $\frac{\partial u}{\partial t} = \mu\frac{\partial^2 u}{\partial x^2}$ then show that $g = \sqrt{\left(\frac{n}{2\mu}\right)}$.

[8 Marks]

Solution: $u = \mathrm{A}e^{-gx}\sin(nt - gx)$, where A, g, n positive constants.

This expression satisfies the heat conduction equation.

$$\frac{\partial u}{\partial t} = \mu\frac{\partial^2 u}{\partial x^2}$$

First, finding $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$ from give expression of u, we get

$$\frac{\partial u}{\partial t} = n\ \mathrm{A}e^{-gx}\cos(nt - gx)$$

and

$$\frac{\partial u}{\partial x} = \mathrm{A}\left(\ -ge^{-gx}\cos(nt - gx) - ge^{-tx}\sin(nt - gx)\ \right)$$

$$= -\text{Age}^{-gx}[\cos(nt - gx) + \sin(nt - gx)]$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = -\text{Ag}\begin{pmatrix} e^{-gx}[(g\sin(nt-gx)) \\ -g\cos(nt-gx)] \\ -ge^{-gx}[\cos(nt-gx) \\ +\sin(nt-gx)] \end{pmatrix}$$

$$\frac{\partial^2 u}{\partial x^2} = -\text{Ag}^2 e^{-gx}[\sin(nt - g\dot{x}) - \cos(nt - gx)$$
$$- \sin(nt - gx) - \cos(nt - gx)]$$

$$\frac{\partial^2 u}{\partial x^2} = 2\text{Ag}^2 e^{-gx}\cos(nt - gx)$$

Substituting values of $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$ from (2) and (3) in (1), we get

$$n\ \text{A}e^{-gx}\cos(nt - gx) = 2\text{Ag}^2 e^{-gx}\mu[\cos(nt - gx)$$

$$n = 2\mu g^2$$

$\therefore g = \sqrt{\left(\frac{n}{2\mu}\right)}$

Question-1(e) Find the equations to the lines in which the plane $2x + y - z = 0$ cuts the cone $4x^2 - y^2 + 3z^2 = 0$

[8 Marks]

Solution: Let one of the lines of intersection of the plane

$$2x + y - z = 0 \qquad \ldots(1)$$

and the cone

$$4x^2 - y^2 + 3z^2 = 0 \qquad \ldots(2)$$

be

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \qquad \ldots(3)$$

The line (3) lies in the plane (1) and on the cone (2).

$$\therefore \qquad 2l + m - n = 0 \qquad \ldots(4)$$

and

$$4l^2 - m^2 + 3n^2 = 0 \qquad \ldots(5)$$

Eliminating n between (4) and (5) we get

$$4l^2 - m^2 + 3(2l + m)^2 = 0$$

$$\Rightarrow 16l^2 + 12lm + 2m^2 = 0$$

$$\Rightarrow 8l^2 + 6lm + m^2 = 0$$

$$\Rightarrow (4l + m)(2l + m) = 0$$

$$4l + m = 0, \quad 2l + m = 0$$

$$m = -4l, \quad m = -2l$$

when $m = -4l$, then from $(4), n = -2l$ and when $m = -2l$, then from $(4), n = 0$

Hence, In first case we rearrange as

$$\frac{l}{1} = \frac{m}{-4} = \frac{n}{-2}$$

and in second case, we rearrange as

$$\frac{l}{1} = \frac{m}{-2} = \frac{n}{0}$$

Thus, the equation of the lines in which the given plane cuts the given cone are:

$$\frac{x}{1} = \frac{y}{-4} = \frac{z}{-2}$$

and

$$\frac{x}{1} = \frac{y}{-2} = \frac{z}{0}$$

Question-2(a) Let $f : \mathbb{R} \to \mathbb{R}^3$ be a linear transformation defined by $f(a, b, c) = (a, a+b, 0)$. Find the matrices A and B respectively of the linear transformation f with respect to the standard basis (e_1, e_2, e_3) and the basis (e_1', e_2', e_3') where $e_1' = (1, 1, 0), e_2' = (0, 1, 1)$ $e_3' = (1, 1, 1)$.
Also, show that there exists an invertible matrix P such that

$$B = P^{-1}AP$$

[10 Marks]

Solution: $S_1 = \{e_1, e_2, e_3\}$ where $e_1 = (1, 0, 0), e_2 = (0, 1, 0)$ and $e_3(0, 0, 1)$ is the standard basis of R^3.

$$T(e_1) = (1, 1, 0) = e_1 + e_2 + 0e_3$$

$$T(e_2) = (0, 1, 0) = 0e_1 + e_2 + 0e_3$$

$$T(e_3) = (0, 0, 0) = 0e_1 + 0e_2 + 0e_3$$

$\therefore$ Matrix of T wrt standand basis is $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

Now: $S_2 = \{e_1', e_2', e_3'\}$ where $e_1' = (1, 1, 0), e_2'(0, 1, 1)$ and $e_3' = (1, 1, 1)$.

Let $(x, y, z) = ae_1' + be_2' + ce_3' = (a + c, a + b + c, b + c)$
$a + c = x, b + c \leq z, a + b + c = y$.
On comparing,

$$a + x = c \qquad \ldots (1)$$

$$b + c = z \qquad \ldots (2)$$

$$a + b + c = y \qquad \ldots (3)$$

From (1), (2) and (3), we get:
$a = y - z$,
$b = y - x$,
$c = x - y + z$

$$\begin{aligned}
\therefore (x, y, z) &= (y - z)(1,1,0) + (-x + y)(0,1,1) + (x - y + z)(1,1,1) \\
&= (y - z)e_{\prime} + (-x + y)e_2' + (x - y + z)e_3' \\
T(e_1') &= T(1,1,0) = (1,2,0) = 2e_1' + 1 \cdot e_2' + (-1)e_3' \\
T(e_2') &= T(0,1,1) = (0,1,0) = 1 \cdot e_1' + 1 \cdot e_2' + (-1)e_3' \\
T(e_3') &= T(1,1,1) = (01,2,0) = 2 \cdot e_1' + 1 \cdot e_2' + (-1)e_3'
\end{aligned}$$

$\therefore$ Matrix of T wrt basis S_2 is $B = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix}$

To prove that $B = P^{-1}AP$ for some non-singular matrix P, we need to show that A and B are similar, i.e., the characteristic equation and the roots of A and B are the same.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \quad |A - \lambda I| = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & -\lambda \end{vmatrix} = \begin{array}{c} (1-\lambda)(1-\lambda)(-\lambda) = 0 \\ \Rightarrow \lambda = 1, 1, 0 \end{array}$$

Also,

$$B = \begin{bmatrix} 2 & 1 & 2 \\ 1 & 1 & 1 \\ -1 & -1 & -1 \end{bmatrix} \Rightarrow \quad |B - \lambda I| = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ -1 & -1 & -(1+\lambda) \end{vmatrix} = \begin{array}{c} (2-\lambda)(\lambda^2 - 1) + 1 = 0 \\ \Rightarrow \lambda = 1, 1, 0 \end{array}$$

$\therefore$ A and B are similar.
Hence, $\exists$ a non-singular matrix P such that $B = P^{-1}AP$.

Question-2(b) Verify Cayley-Hamilton theorem for the matrix $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ and find its inverse. Also express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ as a linear polynomial in A.

[10 Marks]

Solution: Cayley-Hamilton theorem states that every square matrix satisfies its characteristic equation. Now, for matrix

$$A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

The characteristic equation is

$$
\begin{aligned}
|A-\lambda I| &= 0\\
\Rightarrow \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} &= 0\\
\Rightarrow (1-\lambda)(3-\lambda)-8 &= 0\\
\Rightarrow \lambda^2-4\lambda-5 &= 0
\end{aligned}
$$

By Cayley-Hamilton theorem the matrix A must satisfy (1).
$\therefore$ We have to verify that

$$A^2-4A-5I=0$$

Now,

$$A^2=\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$$

$$A^2=\begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix}$$

Now

$$
\begin{aligned}
\mathrm{A}^2-4\ \mathrm{A}-5\mathrm{I} &= \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix}-4\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}-5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\\
&= \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix}-\begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix}-\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}\\
&= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}=0
\end{aligned}
$$

Hence, $A^2-4A-5I=0$ Thus, Cayley-Hamilton theorem verified. Now we have to compute A^{-1}. Multiply (2) by A^{-1} we get $A-4I-5A^{-1}=0$

$$
\begin{aligned}
\Rightarrow \quad \mathrm{A}^{-1} &= \frac{1}{5}(\ \mathrm{A}-4\mathrm{I})\\
&= \frac{1}{5}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}-\frac{4}{5}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\\
&= \begin{bmatrix} \frac{1}{5} & \frac{4}{5} \\ \frac{2}{5} & \frac{3}{5} \end{bmatrix}-\begin{bmatrix} \frac{4}{5} & 0 \\ 0 & \frac{4}{5} \end{bmatrix}\\
\therefore \mathrm{A}^{-1} &= \begin{bmatrix} \frac{-3}{5} & \frac{4}{5} \\ \frac{2}{5} & \frac{-1}{5} \end{bmatrix}
\end{aligned}
$$

Now from (2), we get

$$\mathrm{A}^2=4\ \mathrm{A}+5\mathrm{I}\ldots(3)$$

Multiplying both sides of (3) by A, we get

$$A^3=4A^2+5A\ldots(4)$$

$$A^4=4A^3+5A^2\ldots(5)$$

and

$$\mathrm{A}^5=4\ \mathrm{A}^4+5\ \mathrm{A}^3\ldots(6)$$

Now,

$$A^5-4A^4-7A^3+11A^2-A-10I$$

is calculated by substituting for A^5 from (6)

$$
\begin{aligned}
A^5-4A^4-7A^3+11A^2-A-10I &= \left(4\ \mathrm{A}^4+5\ \mathrm{A}^3\right)-4\ \mathrm{A}^4-7\ \mathrm{A}^3+11\ \mathrm{A}^2-\mathrm{A}-10\mathrm{I}\\
&=-2\ \mathrm{A}^3+11\ \mathrm{A}^2-\mathrm{A}-10\mathrm{I}\\
&=-2\left(4\ \mathrm{A}^2+5\ \mathrm{A}\right)+11\ \mathrm{A}^2-\mathrm{A}-10\mathrm{I}[u\,\mathrm{sing}(4)]\\
&=3\ \mathrm{A}^2-11\ \mathrm{A}-10\mathrm{I}\\
&=3(4\ \mathrm{A}+5\mathrm{I})-11\ \mathrm{A}-10\mathrm{I}\quad [using(3)]\\
&=\mathrm{A}+5\mathrm{I}
\end{aligned}
$$

,
which is a linear polynomial in A

Question-2(c) Find the equations of the tangent plane to the ellipsoid

$$2x^2+6y_1^2+3z^2=27$$

which passes through the line

$$x-y-z=0=x-y+2z-9$$

[10 Marks]

Solution: **Method 1:**

Ellipsoid, $2x^2+6y^2+3z^2=27 \qquad \ldots(1).$
Equation of plane passing through the line

$$x-y-z=0=x-y+2z-9$$

is given by:

$$x-y+2z-9+k(x-y-z)=0$$

ie. $\quad (k+1)x-(k+1)y+(-k+2)z=9 \qquad \ldots(2).$
The equation of tangent plane at point (a,b,c) to the ellipsoid (1) is

$$2ax+6by+3cz=27 \qquad \ldots(3)$$

If equations (2) and (3) are idential, then

$$\frac{2a}{k+1}=\frac{6b}{-(k+1)}=\frac{3c}{-k+2}=\frac{27}{9}$$

ie. $a=\frac{3}{2}(k+1), \quad b=-\frac{1}{2}(k+1), \quad c=-k+2.$

Point (a,b,c) lies on ellipsoid (1),
$\therefore \quad 2\cdot\frac{9}{4}(k+1)^2+6\cdot\frac{1}{4}(k+1)^2+3(-k+2)^2=27$

$$x \Rightarrow k=\pm 1$$

When $k=1$, tangent plane: $2x-2y+2=9$
When $k=-1$, tangent plane: $z=3$.

Method 2:
The equation of the plane passing through the line

$$x - y - z = 0 = x - y + 2z - 9$$

is

$$x - y - z + k(x - y + 2z - 9) = 0$$
$$\Rightarrow (1 + k)x - (1 + k)y + (2k - 1)z - 9k = 0$$

Compare it with the general equation of the plane $Lx + my + nz = p$,
we get

$$l = 1 + k, m = -(1 + k)$$
$$n = 2k - 1, p = 9k$$

Now, using the condition of tangency to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

by the plane $Lx + my + nz = p$, is

$$a^2l^2 + b^2m^2 + c^2n^2 = p^2$$

Here, we are given the cquation of the ellipsoid as

$$2x^2 + 6y^2 + 3z^2 = 27$$
$$\Rightarrow \frac{x^2}{\left(\frac{27}{2}\right)} + \frac{y^2}{\left(\frac{27}{6}\right)} + \frac{z^2}{\left(\frac{27}{3}\right)} = 1$$
$$\therefore \quad a^2 = \frac{27}{2}, b^2 = \frac{27}{6}, c^2 = \frac{27}{3}$$

On substituting the values in (2), we get

$$\frac{27}{2}(1 + k)^2 + \frac{27}{6}[-(1 + k)]^2 + \frac{27}{3}(2k - 1)^2 = (9k)^2$$
$$\Rightarrow 18(1 + k)^2 + 9(2k - 1)^2 = 81k^2$$
$$\Rightarrow 2(1 + k)^2 + (2k - 1)^2 = 9k^2$$
$$\Rightarrow 2 + 2k^2 + 4k + 4k^2 + 1 - 4k = 9k^2$$
$$\Rightarrow 3k^2 = 3 \Rightarrow k = \pm 1$$

Putting the values of k in (1), we get two equations of the tangent planes to the given ellipsoid as when $k = 1$

$$\Rightarrow 2x - 2y + z - 9 = 0$$

when

$$k = -1 \Rightarrow -3z + 9 = 0$$
$$\Rightarrow z = 3$$

Question-2(d) Show that there are three real values of λ for which the equations:

$$(a-\lambda)x + by + cz = 0,$$
$$bx + (c-\lambda)y + az = 0,$$
$$cx + ay + (b-\lambda)z = 0$$

are simultaneously true and that the product of these values of λ is $D = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$.

[10 Marks]

Solution: The given equations are:

$$(a-\lambda)x + by + cz = 0$$
$$bx + (c-\lambda)y + az = 0$$
$$cx + ay + (b-\lambda)z = 0$$

The above system of equations are simultaneously true when the determinant of the coefficient matrix is zero i.e.,

$$\begin{vmatrix} a-\lambda & b & c \\ b & c-\lambda & a \\ c & a & b-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda^3 - (a+b+c)\lambda^2 - \left(a^2+b^2+c^2-ab-bc-ca\right)\lambda + (a+b+c)\left(a^2+b^2+c^2-ab-bc-ca\right) = 0$$

This is a cubic equation in λ.
Hence, product of its roots $= \lambda_1\lambda_2\lambda_3$

$$= \frac{(-1)^3(\text{ Constant term })}{(\text{ Coefficient of } \lambda^3)}$$
$$= \frac{-(a+b+c)\left(a^2+b^2+c^2-ab-bc-ca\right)}{(1)}$$

(Using the fact that in $Ax^3 + Bx^2 + Cx + D = 0$, product of roots $= (-1)^3\frac{D}{A}$)

$$\therefore \lambda_1\lambda_2\lambda_3 = -\left(a^3+b^3+c^3-3abc\right)$$
$$= 3abc - a^3 - b^3 - c^3$$

$$Also, \quad D = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = a(bc-a^2) + b(ac-b^2) + c(ab-c^2)$$
$$= -(a^3+b^3+c^3-3abc)$$
$$= \lambda_1\lambda_2\lambda_3$$

Hence, verified.

Question-3(a) Find the matrix representation of linear transformation T on $V_3(IR)$ defined as $T(a,b,c)=(2b+c,a-4b,3a)$ corresponding to the basis $B=\{(1,1,1),(1,1,0),(1,0,0)\}$.

[10 Marks]

Solution: Given basis $B=\{(1,1,1),(1,1,0),(1,0,0)\}$and

$$T(a,b,c)=(2b+c,a-4b,3a)$$

Let $\alpha_1=(1,1,1),\alpha_2=(1,1,0),\alpha_3=(1,0,0)$

By definition of T, we have

$$\mathrm{T}\left(\alpha_1\right)=\mathrm{T}(1,1,1)=(2(1)+1,1-4,3)$$
$$\Rightarrow \mathrm{T}\left(\alpha_1\right)=(3,-3,3)$$

Similarly,

$$\mathrm{T}\left(\alpha_2\right)=\mathrm{T}(1,1,0)=(2,-3,3)$$

and

$$\mathrm{T}\left(\alpha_3\right)=\mathrm{T}(1,0,0)=(0,1,3)$$

Now our aim is to express $\mathrm{T}\left(\alpha_1\right),\mathrm{T}\left(\alpha_2\right)$ and $T\left(\alpha_3\right)$ as linear combination of the vectors in the basis B $\left[\alpha_1,\alpha_2,\alpha_3\right]$

Let

$$(x,y,z)=p\alpha_1+q\alpha_2+r\alpha_3$$
$$(x,y,z)=p(1,1,1)+q(1,1,0)+r(1,0,0)$$
$$(x,y,z)=(p+q+r,p+q,p)$$

$\therefore \quad x=p+q+r,\ y=p+q$ and $z=p$

Solving these equations, we get

$$p=z,q=y-z,r=x-y$$

Putting $x=3,y=-3,z=3$, we get

$$p=3,q=-6,r=6$$
$$\therefore T\left(\alpha_1\right)=3\alpha_1-6\alpha_2+6\alpha_3\ldots(1)$$

Similarly, on putting $x=2,y=-3,z=3$, we get

$$p=3,q=-6,r=5$$
$$\therefore \mathrm{T}\left(\alpha_2\right)=3\alpha_1-6\alpha_2+5\alpha_3$$

Similarly, on putting $x=0,y=1,z=3$, we get

$$p=3,q=-2,r=-1$$
$$\therefore \mathrm{T}\left(\alpha_3\right)=3\alpha_1-2\alpha_2-\alpha_3$$

From $(1),(2)$ and (3), we see that the matrix of T relative to the basis

$$\{\alpha_1,\alpha_2,\alpha_3\}=\begin{bmatrix}3&-6&6\\3&-6&5\\3&-2&-1\end{bmatrix}^T=\begin{bmatrix}3&3&3\\-6&-6&-2\\6&5&-1\end{bmatrix}$$

Question-3(b) Find the dimensions of the rectangular box, open at the top, of maximum capacity whose surface is 432 sq. cm.

[10 Marks]

Solution: Let the dimensions of the rectangular box be x, y and z where these represent length, breadth and height respectively.
Then volume, V $= xyz$ and the surface area of the rectangular box (open at the top) $= xy + 2z(x + y) = 432$ (given)

Define a Lagrangian function

$$\text{F} = xyz + \lambda(xy + 2z(x + y) - 432)$$

Then for extremum value d F $= 0$

$$\Rightarrow d\ \text{F} = [yz + \lambda(y + 2z)]dx + [xz + \lambda(x + 2z)]dy + [xy + \lambda(2(x + y))]d_{z_z}$$

Now equating the coefficients, we

$$yz + \lambda(y + 2z) = 0$$
$$xz + \lambda(x + 2z) = 0$$
$$xy + 2\lambda(x + y) = 0$$

Subtracting (2) from (1) we get 0,

$$\Rightarrow (y - x)z + \lambda(y - x) = 0$$

$$\Rightarrow (y - x)(z + \lambda) = 0$$

$$\Rightarrow y - x = 0,$$

other factors cannot be zero.

$$\therefore y = x$$

Now multiplying equation (2) by 2 and then subtracting the resulting equation from equation (3), we get

$$x(y - 2z) + 2\lambda(x + y - x - 2z) = 0$$

$$\Rightarrow (x + 2\lambda)(y - 2z) = 0$$

$$\Rightarrow y = 2z$$

$\therefore$ The dimensions of the box are of the form

$$x = y = 2z$$
$$xy + 2z(x + y) = 432$$

$$\Rightarrow \quad 12z^2 = 432$$
$$\Rightarrow z^2 = 36$$
$$z = 6$$

Hence, the dimensions of the box are (12,12,6) cm respectively.

Question-3(c) If $2C$ is the shortest distance between the lines

$$\frac{x}{l} - \frac{z}{n} = 1, \quad y = 0$$

and

$$\frac{y}{m} + \frac{z}{n} = 1, \quad x = 0$$

then show that

$$\frac{1}{l^2} + \frac{1}{m^2} + \frac{1}{n^2} = \frac{1}{c^2}$$

[10 Marks]

Solution: The equations of the given lines are:

$$\frac{x}{l} - \frac{z}{n} = 1, y = 0 \qquad \ldots(1)$$

and

$$\frac{y}{m} + \frac{z}{n} = 1, x = 0 \qquad \ldots(2)$$

The equation of the line (1) being put in symmetrical form as

$$\frac{x-l}{l} = \frac{y}{0} = \frac{z}{n} \qquad \ldots(I)$$

The equation of any plane through the line (2) is

$$\left(\frac{y}{m} + \frac{z}{n} - 1\right) + \lambda x = 0$$

$$\Rightarrow \lambda x + \left(\frac{1}{m}\right) y + \left(\frac{1}{n}\right) z - 1 = 0 \qquad \ldots(3)$$

If the plane (3) is parallel to the line (I), then the normal to the plane (3) whose d.c.'s are $\lambda, \frac{1}{m}, \frac{1}{n}$ will be perpendicular to the line (I), and so we have

$$l\lambda + 0\left(\frac{1}{m}\right) + n\left(\frac{1}{n}\right) = 0$$

$$\lambda = \frac{-1}{l}$$

Putting this value of λ in (3), the equation of the plane containing the line (2) and parallel to the line (I) is

$$-\frac{x}{l} + \frac{y}{m} + \frac{z}{n} - 1 = 0$$

$$\frac{x}{l} - \frac{y}{m} - \frac{z}{n} + 1 = 0 \qquad \ldots(4)$$

Clearly, $(l, 0, 0)$ is a point on the line (I) [i.e., (1)]. Hence, the length $2c$ or shortest distance = perpendicular distance of $(l, 0, 0)$ from the plane (4).

$$\therefore \quad 2c = \frac{\left|l\left(\frac{1}{l}\right) - 0 - 0, +1\right|}{\sqrt{\left(\frac{1}{l}\right)^2 + \left(\frac{1}{m}\right)^2 + \left(\frac{1}{n}\right)^2}}$$

$$= \frac{2}{\sqrt{\frac{1}{l^2} + \frac{1}{m^2} + \frac{1}{n^2}}}$$

$$\Rightarrow \quad \sqrt{\frac{1}{l^2} + \frac{1}{m^2} + \frac{1}{n^2}} = \frac{1}{c}$$

$$\text{Hence, } \frac{1}{l^2} + \frac{1}{m^2} + \frac{1}{n^2} = \frac{1}{c^2}$$

Question-3(d) Show that the function defined as

$$f(x) = \begin{cases} \dfrac{\sin 2x}{x} & \textbf{when } x \neq 0 \\ 1 & \textbf{when } x = 0 \end{cases}$$

has removable discontinuity at the origin.

[10 Marks]

Solution:

$$f(x) = \begin{cases} \dfrac{\sin 2x}{x} & \text{when } x \neq 0 \\ 1 & \text{when } x = 0 \end{cases}$$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin 2x}{x}$$

$$= \lim_{x \to 0} \frac{\sin 2x}{2x} \cdot 2 \cdot$$

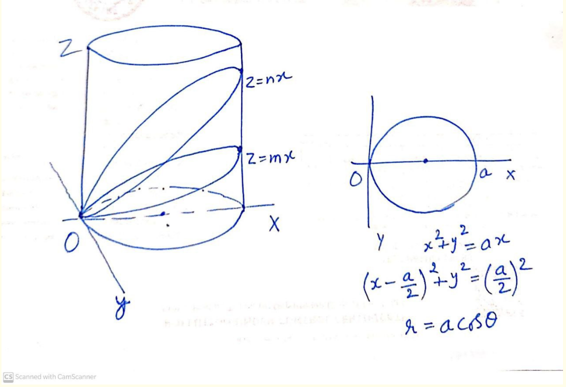
$$= 2$$

$$\text{So that } \lim_{x \to 0} f(x) \neq f(0)$$

Hence, the limit exists but is not equal to the value of the function at the origin. Thus, the function has a removable discontinuity at the origin.

Question-4(a) Find by triple integration the volume cut off from the cylinder $x^2 + y^2 = ax$ by the planes $z = mx$ and $z = nx$.

[10 Marks]



Solution:

Required Volume

$$V = \iint_R (nx - mx) dR$$

Changing to polar co-ordinates

$$V = \int_0^{2\pi} \int_0^{a\cos\theta} (n-m) r\cos\theta (r dr d\theta)$$

$$= (n-m) \int_0^{2\pi} \cos\theta \left[\frac{r^3}{3}\right]_0^{a\cos\theta} d\theta$$

$$= \frac{(n-m)a^3}{3} \int_0^{2\pi} \cos^4\theta d\theta$$

$$= \frac{2\times 2(n-m)a^3}{3} \int_0^{\pi/2} \cos^4\theta d\theta \qquad \left[\because \int_0^{2a} f(x)dx = 2\int_0^a f(x)dx, \text{ if } f(2a-x) = f(x)\right]$$

$$= \frac{4}{3}(n-m)a^3 \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$$

$$= \frac{1}{4}(n-m)\pi a^3$$

Question-4(b) Show that all the spheres that can be drawn through the origin and each set of points where planes parallel to the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$ cut the co-ordinate axes, form a system of spheres which are cut orthogonally by the sphere

$$x^2+y^2+2fx+2gy+2hz=0$$

if $af+bg+ch=0$

[10 Marks]

Solution: The equation of spheres passing through the origin is

$$x^2+y^2+z^2+2ux+2vy+2wz=0$$

Now, the planes parallel to the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$ is given as $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=k$ (where k is any constant) The x -intercept of the above plane is given as

$$\frac{x_{\text{intercept}}}{a}+0+0=k$$

$$x_{\text{intercept}} = ak$$

$\therefore$ Coordinates of the point is $(ak, 0, 0)$ Similarly, y intercept is bk and z intercept is ck Thus, the four points through which the set of spheres passes are

$$(0,0,0), (ak,0,0), (0,bk,0)(0,0,ck)$$

Putting these values one by one in equation (1) we get

$$u = \frac{-ak}{2}, v = \frac{-bk}{2}, w = \frac{-ck}{2}$$

Hence, the equation of a system-spheres passing through the origin and each set of points where planes parallel to the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 0$ cut the coordinate axes is

$$x^2 + y^2 + z^2 - k(ax + by + cz) = 0$$

The equation of other sphere cut orthogonally by the above system of spheres is given as

$$x^2 + y^2 + 2fx + 2gy + 2hz = 0$$

Thus, by the condition of orthogonally, i.e.,

$$2u_1u_2 + 2v_1v_2 + 2w_1w_2 = d_1 + d_2$$

Putting the values, we get

$$2\left(\frac{-ak}{2}\right)(f) + 2\left(\frac{-bk}{2}\right)(g) + 2\left(\frac{-ck}{2}\right)(h) = 0 + 0$$

$$\Rightarrow -afk - bgk - chk = 0$$

$$\Rightarrow k(af + bg + ch) = 0$$

either $k = 0$ or $af + bg + ch = 0$. But $k \neq 0$. (as it will represent the given plane itself, not the plane parallel to the given plane.) Hence,

$$af + bg + ch = 0$$

Question-4(c) A plane makes equal intercepts on the positive parts of the axes and touches the ellipsoid $x^2 + 4y^2 + 9z^2 = 36$. Find its equation.

[10 Marks]

Solution: Let the equation of the plane, making equal intercepts on the positive parts of the axes, be

$$x + y + z = k$$

(where $k > 0$ and indicate the value of the intercept).

Now, it is given that the above plane touch the ellipsoid

$$x^2 + 4y^2 + 9z^2 = 36$$

Therefore, by using the condition of tangency,

$$\left(\begin{array}{c}\text{i.e., when the plane b } x+my+nz= \\ \text{touches the ellipsoid } \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1\end{array}\right)$$

given by

$$a^2l^2+b^2m^2+c^2n^2=p^2$$

we have [from (1)] Here, $l=m=n=1$ and $p=k$ Also, rearranging the given equation of ellipsoid as

$$\frac{x^2}{36}+\frac{4y^2}{36}+\frac{9z^2}{36}=1$$
$$\frac{x^2}{(6)^2}+\frac{y^2}{(3)^2}+\frac{z^2}{(2)^2}=1$$

$\therefore$ We get the values as

$$a=6, b=3, c=2$$

. Now, putting values in equation (2) we get

$$36(1)+9(1)+4(1)=k^2$$
$$\Rightarrow k^2=49$$
$$\Rightarrow \quad k=\pm 7$$

But

$$k\neq -7(\text{ as } k>0)$$
$$\therefore \quad k=7$$

Hence, the equation of the required plane is

$$x+y+z=7$$

Question-4(d) Evaluate the following in terms of Gamma function:

$$\int_0^a \sqrt{\left(\frac{x^3}{a^3-x^3}\right)}\,dx$$

[10 Marks]

Solution: Let

$$\mathrm{I}=\int_0^a \sqrt{\frac{x^3}{a^3-x^3}}dx$$

Let $x^3=a^3\sin^2\theta$ when $x\to 0, \theta\to 0$

$$\Rightarrow \quad x=a\sin^{2/3}\theta \quad \text{when } x\to a, \theta\to \frac{\pi}{2}$$

$$\therefore \quad dx=\frac{2}{3}a\sin^{-1/3}\theta\cos\theta d\theta$$

$$\therefore \quad \mathrm{I} = \int_0^{\pi/2} \sqrt{\frac{a^3 \sin^2\theta}{a^3 - a^3\sin^2\theta}} d\frac{2a}{3}\sin^{-1/3}\theta\cos\theta d\theta$$

$$= \int_0^{\pi/2} \frac{\sin\theta}{\cos\theta}\frac{2}{3}a\sin^{-1/3}\theta\cos\theta d\theta$$

$$= \frac{2}{3}a\int_0^{\pi/2}\sin^{2/3}\theta d\theta$$

Now, using formula

$$\int_0^{\pi/2}\sin^p\theta\cos^q\theta d\theta = \frac{\sqrt{\left(\frac{p+1}{2}\right)}\sqrt{\left(\frac{q+1}{2}\right)}}{2\sqrt{\left(\frac{p+q+2}{2}\right)}}$$

$$\therefore \mathrm{I} = \frac{2}{3}a\frac{\sqrt{\frac{\left(\frac{2}{3}+1\right)}{2}}\sqrt{\left(\frac{0+1}{2}\right)}}{2\sqrt{\left(\frac{\frac{2}{3}+0+2}{2}\right)}}$$

$$\left[i.e., \text{ putting } p = \frac{2}{3} \text{ and } q = 0\right]$$

$$I = \frac{2}{3}a\frac{\sqrt{\frac{5}{6}}\sqrt{\frac{1}{2}}}{2\sqrt{\frac{4}{3}}}$$

$$= \frac{\frac{\sqrt{\pi}a}{3}\sqrt{\frac{5}{6}}}{\sqrt{\left(\frac{1}{3}+1\right)}}$$

$$= \frac{a\sqrt{\pi}}{3}\frac{\sqrt{\frac{5}{6}}}{\frac{1}{3}\sqrt{\frac{1}{3}}}\left(\text{using } \sqrt{n+1} = n\sqrt{n}\right)$$

$$\therefore I = a\sqrt{\pi}\frac{\sqrt{\frac{5}{6}}}{\sqrt{\frac{1}{3}}}$$

9.2 Section-B

Question-5(a) **Solve** $\dfrac{dy}{dx} - \dfrac{\tan y}{1+x} = (1+x)e^x\sec y$

[8 Marks]

Solution:

$$\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x\sec y$$

This is the general form of first degree linear differential equation. It can be rearranged in the form of

$$\frac{dy}{dx} + \mathrm{P}y = \mathrm{Q}$$

where P and Q are function of x or constants.

Dividing by $(\sec y)$ to both sides, we get

$$\frac{1}{\sec y}\frac{dy}{dx} - \frac{\tan y}{\sec y}\left(\frac{1}{1+x}\right) = (1+x)e^x$$

$$\Rightarrow \cos y\frac{dy}{dx} - \sin y\left(\frac{1}{1+x}\right) = e^x(1+x)\ldots(1)$$

Let $\sin y = t$ On differentiation, we get

$$\cos y\frac{dy}{dx} = \frac{dt}{dx}$$

Putting in equation (1) we get

$$\frac{dt}{dx} - \frac{1t}{(1+x)} = e^x(1+x)$$

which is the general form of first order and first degree linear differential equation. Now, solving this linear differential equation

$$\begin{aligned}\text{Integrating factor (I.F.)} &= \int_e \frac{-1}{(1+x)}dx\\ &= e^{-\ln|1+x|}\\ \text{IF.} &= \frac{1}{(1+x)}\end{aligned}$$

$\therefore$ Solution of the differential equation (2) is given as

$$t(\text{L.F.}) = \int \text{Q}(\text{I}\cdot\text{F}\cdot)dx + \text{C}$$

where C is a constant of integration and Q is the right side of equation (2) Putting values of Q and I.F. we get

$$\begin{aligned}\frac{t}{1+x} &= \int e^x(1+x)\cdot\frac{1}{(1+x)}dx + \text{C}\\ &= \int e^x dx + \text{C} = e^x + \text{C}\end{aligned}$$

since, the original differential equation is a function of x and y $\therefore$ Replace t by a function of y (which we let) Hence,

$$\frac{\sin y}{1+x} = e^x + C$$

Thus, the required solution is

$$\frac{\sin y}{1+x} - e^x = \text{C}$$

Question-5(b) Solve and find the singular solution of $x^3p^2 + x^2py + a^3 = 0$.

[8 Marks]

Solution: The given equation is $x^3p^2 + x^2py + a^3 = 0$ solving for y,

$$y = -xp - \frac{a^3}{x^2p}$$

Differentiating (2) with respect to (x) writing p for $\frac{dy}{dx}$, we have

$$\begin{aligned}
& p = -p - x\frac{dp}{dx} - a^3\left(\frac{-2}{x^3p} - \frac{1}{x^2p^2}\frac{dp}{dx}\right) \\
& \Rightarrow 2p + x\frac{dp}{dx} - \frac{2a^3}{x^3p} - \frac{a^3}{x^2p^2}\frac{dp}{dx} = 0 \\
& \Rightarrow 2p\left(1 - \frac{a^3}{x^3p^2}\right) + x\frac{dp}{dx}\left(1 - \frac{a^3}{x^3p^2}\right) = 0 \\
& \left(1 - \frac{a^3}{x^3p^2}\right)\left(2p + x\frac{dp}{dx}\right) = 0
\end{aligned}$$

Omitting the first factor since it does not involve $\frac{dp}{dx}$, we get

$$2p + x\frac{dp}{dx} = 0$$

$$\Rightarrow \quad \frac{1}{p}dp + \frac{2}{x}dx = 0$$

Integrating, we get

$$\log p + 2\log x = \log \mathrm{C}$$

(where $\log \mathrm{C}$ is an integration constant)

$$\Rightarrow \quad \log\left(px^2\right) = \log C$$

$$\Rightarrow \quad px^2 = C$$

$$p = \frac{C}{x^2} \ldots (3)$$

Eliminating p between (1) and (3), the required general solution is

$$x^3\frac{C^2}{x^4} + x^2y\left(\frac{C}{x^2}\right) + a^3 = 0$$

$$\Rightarrow \quad \frac{C^2}{x} + Cy + a^3 = 0$$

$$\Rightarrow \quad C^2 + xyC + a^3x = 0$$

By (4), C-discriminant relation is

$$-(4)(xy)^2 - 4(1)\left(a^3x\right) = 0$$

$$\Rightarrow \quad x\left(xy^2 - 4a^3\right) = 0$$

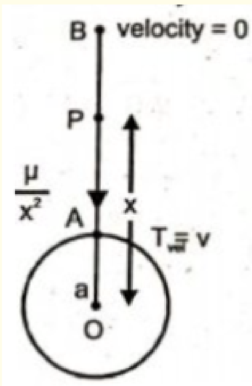
Now, $x = 0$ and $xy^2 - 4a^3 = 0$ both satisfy equation (1) and hence required singular solutions are $x = 0$ and $xy^2 - 4a^3 = 0$

Question-5(c) A particle is projected vertically upwards from the earth's surface with a velocity just sufficient to carry it to infinity. Prove that the time it takes to reach a height h is

$$\frac{1}{3}\sqrt{\left(\frac{2a}{g}\right)}\left[\left(1+\frac{h}{a}\right)^{3/2}-1\right].$$

[8 Marks]

Solution:



Let O be the centre of the earth and A be the point of projection on the earth's surface. If P be the position of the particle at any time t, such that OP $= x$, then the acceleration at

$$\text{P} = \frac{\mu}{x^2}$$

directed towards 0.

$\therefore$ The equation of motion of the particle at P is

$$\frac{d^2x}{dt^2} = \frac{-\mu}{x^2}$$

(Negative sign indicates that acceleration acts in the direction of x decreasing.) But at the point A, on the surface of the earth, $x = a$. and $\frac{d^2x}{dt^2} = -g$

$$\therefore \quad -g = \frac{-\mu}{2} \text{ or } \mu = a^2g$$

$$-g = \frac{-\mu}{a^2}$$

$$\frac{d^2x}{dt^2} = \frac{-a^2g}{x^2}$$

Multiplying by $2\left(\frac{dx}{dt}\right)$ and integrating with respect to (t) we get

$$\left(\frac{dx}{dt}\right)^2 = \frac{2a^2g}{x} + \text{C}$$

where C is a constant. But when

$$x \to \infty, \frac{dx}{dt} \text{ (velocity) } \to 0$$

$$\therefore C = 0$$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2 = \frac{2a^2g}{x}$$

$$\therefore \mathrm{C} = 0$$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2 = \frac{2a^2g}{x}$$

(Here +ve sign is taken because the particle is moving in the direction of x increasing)

$$\Rightarrow \quad \frac{dx}{dt} = a\sqrt{\frac{2g}{x}}$$

Separating the variables, we have

$$dt = \frac{1}{a\sqrt{2g}}\sqrt{x}dx$$

Integrating between the limits $x = a$ to $x = a + h$, the required time t to reach height h is given by

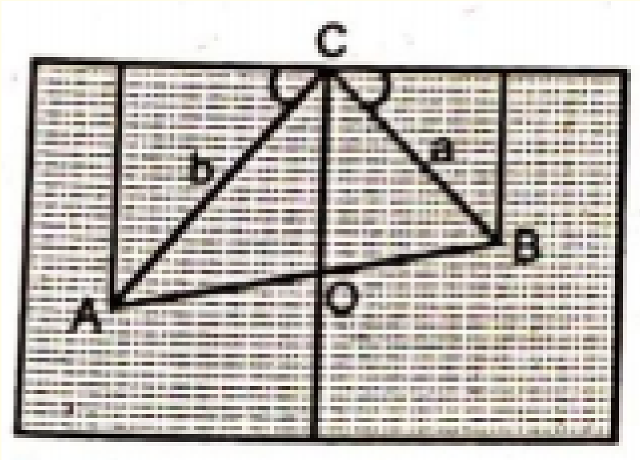
$$\begin{aligned} t &= \frac{1}{a\sqrt{2g}}\int_a^{a+h}\sqrt{x}dx = \frac{1}{a\sqrt{2g}}\left[\frac{2}{3}x^{3/2}\right]_a^{a+h} \\ &= \frac{1}{3a}\sqrt{\frac{2}{g}}\left[(a+h)^{3/2} - a^{3/2}\right] \\ &= \frac{1}{3}\sqrt{\frac{2a}{g}}\left[\left(1+\frac{h}{a}\right)^{3/2} - 1\right] \end{aligned}$$

Question-5(d) A triangle ABC is immersed in a liquid with the vertex C in the surface and the sides AC, BC equally inclined to the surface. Show that the vertical C divides the triangle into two others, the fluid pressures on which are as $b^3 + 3ab^2 : a^3 + 3a^2b$ where a and b are the sides BC & AC respectively.

[8 Marks]

Solution: Let the vertical through C meets AB at O. then

$$\angle \mathrm{ACO} = \angle \mathrm{BCO} = \frac{1}{2}\angle \mathrm{C}$$



Area of ΔAOC $= \frac{1}{2}\text{AC} \cdot OC \sin\angle\text{ACO}$ & Area of ΔBOC $= \frac{1}{2}\text{BC} \cdot \text{OC} \sin\angle\text{BCO}$

The depth of the centre of gravity (C.G.) of ΔAOC below the surface of the liquid

$$= \frac{1}{3}(\text{AC}\cos\angle\text{ACO} + \text{OC})$$

and the depth of the C.G of ΔBOC below the surface of the liquid

$$= \frac{1}{3}(\text{BC}\cos\angle\text{BCO} + \text{OC})$$

$$\begin{aligned}
\therefore \frac{\text{Pressure on }\Delta\text{AOC}}{\text{Pressure on }\Delta\text{BOC}} & \frac{1}{2}AC \cdot OC \sin\angle ACO \cdot \frac{1}{3}(AC\cos\angle ACO \\
&= \frac{1 + \text{OC}) \cdot w}{\frac{1}{2}\text{BC} \cdot \text{OC}\sin\angle\text{BCO} \cdot \frac{1}{3}(\text{BC}\cos\angle\text{BCO}} + \text{OC}) \cdot w \\
&= \frac{\left(\frac{1}{2}bOC\sin\frac{C}{2}\right)\left(\frac{1}{3}\left(b\cos\frac{C}{2} + OC\right)\right)}{\left(\frac{1}{2}aOC\sin\frac{C}{2}\right)\left(\frac{1}{3}\left(a\cos\frac{C}{2} + OC\right)\right)} \\
&= \frac{b\left(b\cos\frac{C}{2} + OC\right)}{a\left(a\cos\frac{C}{2} + OC\right)}
\end{aligned}$$

From Δ 's BCO and ACO, we have

$$\frac{\text{CO}}{\sin\text{B}} = \frac{\text{OB}}{\sin\frac{\text{C}}{2}} \text{ and } \frac{\text{CO}}{\sin\text{A}} = \frac{\text{AO}}{\sin\frac{\text{C}}{2}} \ldots (1)$$

Also

$$\begin{aligned}
\frac{\text{AO}}{b} &= \frac{\text{OB}}{a} \\
&= \frac{\text{AO} + \text{OB}}{b + a} \\
&= \frac{c}{b + a} \ldots (2)
\end{aligned}$$

$$\therefore \text{ The required ratio } = \frac{b\left(b\cos\frac{C}{2} + \frac{OB\sin B}{\sin\frac{C}{2}}\right)}{a\left(a\cos\frac{C}{2} + \frac{AO\sin A}{\sin\frac{C}{2}}\right)}$$

$$= \frac{b(b\sin C + 2OB\sin B)}{a(a\sin C + 2OA\sin A)}$$

$$= \frac{b\left(b\sin C + 2OB\frac{b\sin C}{c}\right)}{a\left(a\sin C + 2OAa\frac{\sin C}{c}\right)}$$

$$= \frac{b^2}{a^2}\cdot\left(\frac{c+2OB}{c+2OA}\right)$$

$$= \frac{b^2}{a^2}\cdot\frac{\left(c+\frac{2ac}{b+a}\right)}{\left(c+\frac{2bc}{b+a}\right)}\left[\text{using } = \frac{b^2}{a^2}\cdot\left[\frac{c(a+b)+2ac}{c(a+b)+2bc}\right]\right]$$

$$= \frac{b^2(3a+b)}{a^2(a+3b)} = \frac{b^3+3ab^2}{a^3+3a^2b}$$

Question-5(e) If $u = x+y+z$, $v = x^2+y^2+z^2$, $w = yz+zx+xy$, prove that grad u, grad v and grad w are coplanar.

[8 Marks]

Solution: Given $u = x+y+z, v = x^2+y^2+z^2$, and $w = yz+zx+xy$

$$\begin{aligned}\operatorname{grad} u &= \nabla u \\ &= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(x+y+z) \\ &= \hat{i}\frac{\partial}{\partial x}(x+y+z) + \hat{j}\frac{\partial}{\partial y}(x+y+z) + \hat{k}\frac{\partial}{\partial z}(x+y+z) \\ \nabla u &= \hat{i}+\hat{j}+\hat{k}\end{aligned}$$

Now,

$$\begin{aligned}\operatorname{grad} v &= \hat{i}\frac{\partial}{\partial x}\left(x^2+y^2+z^2\right) + \hat{j}\frac{\partial}{\partial y}\left(x^2+y^2+z^2\right) + \hat{k}\frac{\partial}{\partial z}\left(x^2+y^2+z^2\right) \\ \nabla v &= 2x\hat{i} + 2y\hat{j} + 2z\hat{k}\end{aligned}$$

Now,

$$\begin{aligned}\operatorname{grad} w &= \hat{i}\frac{\partial}{\partial x}(yz+zx+xy) + \hat{j}\frac{\partial}{\partial y}(yz+zx+xy) + \hat{k}\frac{\partial}{\partial z}(yz+zx+xy) \\ \nabla w &= (y+z)\hat{i} + (z+x)\hat{j} + (x+y)\hat{k}\end{aligned}$$

To prove that $\nabla u, \nabla v$ and ∇w coplanar, we must have the following condition to be true. i.e.,

$$\begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & z+x & x+y \end{vmatrix} = 0$$

On carrying out operations on LHS, we get $C_1 \to C_1 - C_2$ & $C_2 \to C_2 - C_3$, we get

$$\text{LHS} = \begin{vmatrix} 0 & 0 & 1 \\ 2(x-y) & 2(y-z) & 2z \\ y-x & z-y & x+y \end{vmatrix}$$

Solving the determinant we get

$$\begin{aligned} \text{LHS} &= 1[2(x-y)(z-y) - 2(y-z)(y-x)] \\ &= 2[(x-y)(z-y) - (x-y)(z-y)] \\ &= 0 \\ &= \text{RHS} \end{aligned}$$

Hence, we can say that grad u, grad v and grad w are coplanar.

Question-6(a) Solve:

$$x^2 y \frac{d^2y}{dx^2} + \left(x\frac{dy}{dx} - y\right)^2 = 0$$

[10 Marks]

Solution:

$$x^2 y \frac{d^2y}{dx^2} + \left(x\frac{dy}{dx} - y\right)^2 = 0$$

The given equation can be rewritten as

$$x^2\left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] - \left[2xy\frac{dy}{dx} - y^2\right] = 0$$

$$\Rightarrow \left[y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2\right] - \frac{\left[2xy\left(\frac{dy}{dx}\right) - y^2\right]}{x^2} = 0$$

$$\frac{d}{dx}\left(y\frac{dy}{dx}\right) - \frac{d}{dx}\left(\frac{y^2}{x}\right) = 0$$

Integrating, we get

$$y\frac{dy}{dx} - \frac{y^2}{x} = C_1$$

This is Bernoulli form $\therefore$ Putting $y^2 = v$, so that

$$2y\frac{dy}{dx} = \frac{dv}{dx}$$

$\therefore$ (1) becomes

$$\frac{1}{2}\frac{dv}{dx} - \frac{v}{x} = C_1 \Rightarrow \frac{dv}{dx} - \frac{2v}{x} = 2C_1$$

This is the first order linear differential equation. Its I.F.

$$\begin{aligned} \text{I.F.} &= e^{-\int \frac{2}{x}dx} \\ &= e^{-2\ln(x)} \\ &= \frac{1}{x^2} \end{aligned}$$

Hence, solution is

$$\begin{aligned} v\left(\frac{1}{x^2}\right) &= 2C_1 \int \frac{1}{x^2}dx + C_2 \\ \frac{y^2}{x^2} &= \frac{-2C_1}{x} + C_2 \\ \Rightarrow \quad y^2 &= x\left(C_2 x - 2C_1\right) \end{aligned}$$

Question-6(b) Find the value of $\iint_s (\vec{\nabla} \times \vec{F}) \cdot \overrightarrow{ds}$ taken over the upper portion of the surface $x^2 + y^2 - 2ax + az = 0$ and the bounding curve lies in the plane $z = 0$, when
$\vec{F} = (y^2 + z^2 - x)\,\vec{i} + (z^2 + x^2 - y^2)\,\vec{j} + (x^2 + y^2 - z^2)\,\vec{k}$

[10 Marks]

Solution: By Stokes' Theorem

$$I = \iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r}$$

Here,

$$\begin{aligned} \vec{F} &= \left(y^2 + z^2 - x^2\right) i + \left(z^2 + x^2 - y^2\right) j + \left(x^2 + y^2 - z^2\right) k \\ d\vec{r} &= idx + jdy + kdz \end{aligned}$$

Surface $S : x^2 + y^2 - 2ax + az = 0$ with bounding curve lying on $z = 0$.

$$\begin{aligned} \therefore \quad \text{Boundary} \quad C: \quad & x^2 + y^2 - 2ax = 0; z = 0 \\ \text{i.e.} \quad (r\cos\theta)^2 + (r\sin\theta)^2 &- 2ar\cos\theta = 0 \\ r = 2a\cos\theta, \quad & r = 0 \end{aligned}$$

r varies from 0 to $2a\cos\theta$ and θ varies from 0 to 2π.
Hence,

$$\begin{aligned} I &= \int_C \left(y^2 + z^2 - x^2\right) dx + \left(z^2 + x^2 - y^2\right) dy + \left(x^2 + y^2 - z^2\right) \\ &= \int_C \left(y^2 - x^2\right) dx + \left(x^2 - y^2\right) dy \qquad (\because z = 0 \text{ on } C) \\ &= \int_C \left(x^2 - y^2\right)(dy - dx) \end{aligned}$$

$$\begin{aligned} \text{Now, C:} \quad (x - a)^2 + y^2 &= a^2 \\ \therefore \quad x - a = a\cos\theta \quad &; \quad y = a\sin\theta \\ x = a + a\cos\theta \quad &; \quad y = a\sin\theta \end{aligned}$$

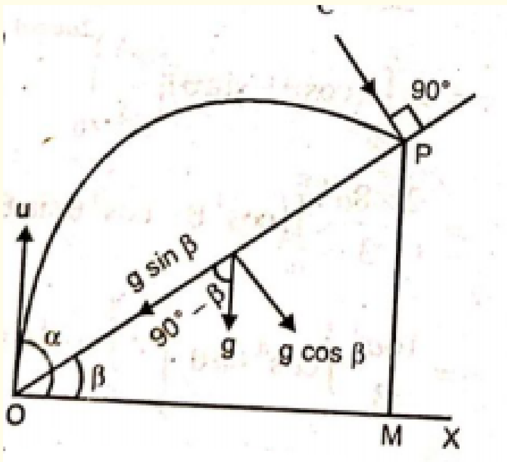
$$\begin{aligned}
\Rightarrow I &= \int_0^{2\pi} \left[a^2(1+\cos\theta)^2 - a^2\sin^2\theta\right]\left[a\cos\theta + a\sin\theta\right]d\theta \\
&= \int_0^{2\pi} a^3\left(1+\cos^2\theta + 2\cos\theta - \sin^2\theta\right)(\cos\theta+\sin\theta)d\theta \\
&= a^3\int_0^{2\pi}\left(2\cos^2\theta + 2\cos\theta\right)(\cos\theta+\sin\theta)d\theta \\
&= 2a^3\int_0^{2\pi}\left[\cos^3\theta + \cos^2\theta + \left(\cos^2\theta+\cos\theta\right)\sin\theta\right]d\theta \\
&= 2a^3\left[2\int_0^{\pi}\left(\cos^3\theta+\cos^2\theta\right)d\theta + \int_0^{2\pi}\left(\cos^3\theta+\sin\theta\right)\sin\theta d\theta\right] \\
&= 2a^3\left[2\times 2\int_0^{2}\cos^2\theta d\theta + 0\right] \\
&= 8a^3\times\frac{1}{2}\times\frac{\pi}{2} = 2\pi a^3
\end{aligned}$$

Question-6(c) A particle is projected with a velocity u and strikes at right angle on a plane through the plane of projection inclined at an angle β to the horizon. Show that the time of flight is $2u\ g\sqrt{(1+3\sin^2\beta)}$ range on the plane is $\frac{2u^2}{g}\cdot\frac{\sin\beta}{1+3\sin^2\beta}$ and the vertical height of the point struck is $\frac{2u^2\sin^2\beta}{g\left(1+3\sin^2\beta\right)}$ above the point of projection.

[10 Marks]

Solution: Let O be the point of projection, u be the velocity of projection, α be the angle of projection and P be the point where the particle strikes the plane at right angles. Let T be the time of flight from O to P. Then by the formula for the time of flight in an inclined plane, we have

$$T = \frac{2u\sin(\alpha-\beta)}{g\cos\beta}$$



Since the particle strikes the inclined plane at right angle at P, therefore the velocity of the particle at P along inclined plane is zero.

Also, the resolved part of the velocity of the particle at O along the inclined plane is $u\cos(\alpha-\beta)$ upwards and the resolved part of the acceleration g along the incline plane is $g\sin\beta$ downwards. So, considering the motion of the particle from O to P along the inclined plane and using the formula $v = u + at$, we have

$$0 = u\cos(\alpha-\beta) - g\sin\beta \mathrm{T}$$

$$\mathrm{T} = \frac{u\cos(\alpha-\beta)}{g\sin\beta}$$

Equating the values of T from (1) and (2) we have

$$\frac{2u\sin(\alpha-\beta)}{g\cos\beta} = \frac{u\cos(\alpha-\beta)}{g\sin\beta}$$

$$\tan(\alpha-\beta) = \frac{1}{2}\cot\beta$$

The condition for striking the plane at right angles.

(i) To prove

$$\mathrm{T} = \frac{2u}{g\sqrt{1+3\sin^2\beta}}$$

Proof: From (2) we have

$$\begin{aligned}
\mathrm{T} &= \frac{u}{g\sin\beta}\cos(\alpha-\beta) \\
&= \frac{u}{g\sin\beta\sec(\alpha-\beta)} \\
&= \frac{u}{g\sin\beta\sqrt{1+\tan^2(\alpha-\beta)}} \\
&= \frac{u}{g\sin\beta\sqrt{1+\frac{1}{4}\cot^2\beta}} [\text{ substituting value from (3)}] \\
&= \frac{2u\sin\beta}{g\sin\beta\sqrt{4\sin^2\beta+\cos^2\beta}} \\
&= \frac{2u}{\sqrt[3]{\sin^2\beta+\cos^2\beta+3\sin^2\beta}}
\end{aligned}$$

$$\therefore \mathrm{T} = \frac{2u}{g\sqrt{1+3\sin^2\beta}}$$

(ii) Range, on the plane

$$R = \frac{2u^2}{8}\frac{\sin\beta}{1+3\sin^2\beta}$$

Proof: Let R be the range on the inclined plane then $R = OP$ considering the motion from O to P along the inclined plane and using the formula $v^2 = u^2 + 2as$, we have

$$\begin{aligned} 0 =& u^2\cos^2(\alpha-\beta) - 2g\sin\beta \mathrm{R} \\ \mathrm{R} =& \frac{u^2\cos^2(\alpha-\beta)}{2g\sin\beta} \\ =& \frac{u^2}{2g\sin\beta\sec^2(\alpha-\beta)} \\ =& \frac{u^2}{2g\sin\beta\left[1+\tan^2(\alpha-\beta)\right]} \\ =& \frac{u^2}{2g\sin\beta\left[1+\frac{1}{4}\cot^2\beta\right]} \quad [\text{ From } (3)] \\ =& \frac{4u^2\sin^2\beta}{2g\sin\beta\left(4\sin^2\beta+\cos^2\beta\right)} \end{aligned}$$

Hence, Range, $\mathrm{R} = \dfrac{2u^2\sin\beta}{g\left(1+3\sin^2\beta\right)}$

(iii) The vertical height of the point struck is

$$\frac{2u^2\sin^2\beta}{g\left(1+3\sin^2\beta\right)}$$

Proof:

$$\begin{aligned} \text{The vertical height of P above O} &= \mathrm{PM} \\ &= \mathrm{OP}\sin\beta \\ &= \mathrm{R}\sin\beta \\ &= \frac{2u^2\sin^2\beta}{g\left(1+3\sin^2\beta\right)} \end{aligned}$$

Question-6(d) Solve $\dfrac{d^4y}{dx^4} + 2\dfrac{d^2y}{dx^2} + y = x^2\mathrm{c}.$

[10 Marks]

Solution: Let $D \equiv \frac{d}{dx}$, then the given differential equation becomes

$$\left(D^4 + 2D^2 + 1\right)y = x^2\cos x$$

This equation is the differential equation of first order with constant coefficients. It is solved by the following method. The auxiliary equation is

$$\begin{aligned} m^4 + 2m^2 + 1 &= 0 \\ \Rightarrow \quad \left(m^2+1\right)^2 &= 0 \\ \Rightarrow \quad m &= \pm i \end{aligned}$$

Thus, the complementary function is given by

$$y = (\mathrm{C}_1 + \mathrm{C}_2x)\cos x + (\mathrm{C}_3 + \mathrm{C}_4x)\sin x$$

where C_1, C_2, C_3 and C_4 are arbitrary constants. Now, the particular integral is given by

$$\begin{aligned} y &= \frac{1}{(1+2D^2+D^4)} x^2 \cos x \\ &= \frac{1}{(D^2+1)^2} x^2 \cos x \\ y &= \text{ Real part of } \left(\frac{1}{(D^2+1)^2} x^2 e^{ix} \right) [\because e^{ix} = \cos x + i \sin x] \end{aligned}$$

Now, solving

$$\frac{1}{(\mathrm{D}^2+1)^2} x^2 e^{ix} = e^{ix} \frac{1}{[(\mathrm{D}+i)^2+1]^2} x^2 \left(\begin{array}{c} \text{Using formula } \frac{1}{f(\mathrm{D})} e^{ax} \mathrm{V} \\ = e^{ax} \cdot \frac{1}{f(\mathrm{D}+a)} \mathrm{V} \end{array} \right)$$

where, V is any function of x

Here $\mathrm{V} = x^2$ $f(\mathrm{D}) = (\mathrm{D}^2+1)^2$ & $a = i$

$$\begin{aligned} &= e^{ix} \frac{1}{[\mathrm{D}^2 + i^2 + 2i\mathrm{D} + 1]^2} x^2 \\ &= e^{ix} \frac{1}{(\mathrm{D}^2 + 2i\mathrm{D})^2} x^2 \quad (\because i^2 = -1) \\ &= e^{ix} \frac{1}{(2iD)^2 \left[1 + \frac{D^2}{2iD}\right]^2} x^2 \\ &= e^{ix} \frac{1}{-4D^2} \left[1 + \frac{D}{2i}\right]^{-2} x^2 \\ &= \frac{-1}{4} e^{ix} \frac{1}{\mathrm{D}^2} \left(1 + (-2)\left(\frac{\mathrm{D}}{2i}\right) + \frac{(-2)(-2-1)}{2!}\left(\frac{\mathrm{D}}{2i}\right)^2 + \dots\right) x^2 \\ &\left[\text{using expansion of } (1+x)^n = 1 + nx + \frac{n(n-1)x^2}{2!} + \dots\right] \\ &= \frac{-1}{4} e^{ix} \frac{1}{D^2} \left(1 - \frac{D}{i} - \frac{3}{4} D^2 + \dots\right) x^2 \\ &= \frac{-e^{ix}}{4} \frac{1}{D^2} \left[x^2 - \frac{1}{i}(2x) - \frac{3}{4}(2) + 0 + 0 + \dots\right] \\ &= \frac{-e^{ix}}{4} \frac{1}{D^2} \left[\left(x^2 - \frac{3}{2}\right) + i(2x)\right] \\ &= \frac{-e^{ix}}{4} \left[\frac{1}{D} \int \left(x^2 - \frac{3}{2}\right) dx + 2i \frac{1}{D} \int x dx\right] \left[\because \frac{1}{D} = \int dx\right] \\ &= \frac{-e^{ix}}{4} \left[\int \left(\frac{x^3}{3} - \frac{3x}{2}\right) dx + 2i \int \frac{x^2}{2} dx\right] \\ &= \frac{-e^{ix}}{4} \left[\frac{x^4}{12} - \frac{3x^2}{4} + 2i\left(\frac{x^3}{6}\right)\right] \\ &= \frac{-e^{ix}}{4} \left[\frac{x^4}{12} - \frac{3x^2}{4} + \frac{ix^3}{3}\right] \end{aligned}$$

Note: While we want the real part of (1), we must open e^{ix} as $(\cos x + i \sin x)$

$\therefore$ (1) equation can be arranged as

$$= \frac{-1}{4}(\cos x + i\sin x)\left[\frac{x^4 - 9x^2}{12} + \frac{i}{3}x^3\right]$$
$$= \left(\frac{9x^2 - x^4}{48} - \frac{i}{12}x^3\right)(\cos x + i\sin x)$$
$$= \left[\left(\frac{9x^2 - x^4}{48}\right)\cos x + \frac{1}{12}x^3 \sin x\right] + i\left[\frac{-1}{12}x^3\cos x + \frac{\sin x}{48}\left(9x^2 - x^4\right)\right]$$

The real part of this is the particular integral
$\therefore$ Particular Integral,
$$y = \frac{x^2}{48}\cos x\left(9 - x^2\right) + \frac{1}{12}x^3 \sin x$$
Thus, the general solution is given by
$$y = \text{C.F} + \text{P.I}$$
$$\therefore y = (\text{C}_1 + \text{C}_2 x)\cos x + (\text{C}_3 + \text{C}_4 x)\sin_2 + \frac{x^2}{48}\left(9 - x^2\right)\cos x + \frac{x^3}{12}\sin x$$
is the required solution.

Question-7(a) A particle is moving with central acceleration $\mu\left[r^5 - c^4 r\right]$ being projected from an apse at a distance c with velocity $\sqrt{\left(\frac{2\mu}{3}\right)}c^3$, show that its path is a curve, $x^4 + y^4 = c^4$.

[14 Marks]

Solution: Here, the central acceleration,
$$p = \mu\left[r^5 - c^4 r\right] = \mu\left[\frac{1}{u^5} - \frac{c^4}{u}\right]\left(\because r = \frac{1}{u}\right)$$
$\therefore$ The differential equation of the path is
$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{p}{u^2} = \frac{\mu}{u^2}\left[\frac{1}{u^5} - \frac{c^4}{u}\right]$$
$$\Rightarrow u^2 = h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{p}{u^2} = \mu\left[\frac{1}{u^7} - \frac{c^4}{u^3}\right]$$
Multiplying both sides by $2\left(\frac{du}{d\theta}\right)$, we get
$$\dot{h}^2\left[2\left(\frac{du}{d\theta}\right)u + 2\left(\frac{du}{d\theta}\right)\frac{d^2u}{d\theta^2}\right] = \frac{2p}{u^2}\left(\frac{du}{d\theta}\right)$$
$$\frac{h^2 d}{d\theta}\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \frac{2p}{u^2}\left(\frac{du}{d\theta}\right)$$

Now, integrating above equation with respect to $'\theta'$, we have

$$h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=2\int\frac{p}{u^2}du+\text{A}$$

where A is a constant

$$v^2=h\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=2\mu\int\left(\frac{1}{u^7}-\frac{c^4}{u^3}\right)+A$$

$$\begin{aligned}v^2&=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]\\&=\mu\left(\frac{-1}{3u^6}+\frac{c^4}{u^2}\right)+\text{A}\end{aligned}$$

But initially when $r=c$ i.e. $u=\frac{1}{c},\frac{du}{d\theta}=0$ (at apse) and $v=c^3\sqrt{\frac{2\mu}{3}}$. $\therefore$ From (1) we have

$$\frac{2\mu c^6}{3}=h^2\cdot\frac{1}{c^2}=\mu\left[\frac{-c^6}{3}+c^6\right]+\text{A}$$

$$\therefore h^2=\frac{2}{3}\mu c^8,\ \text{A}=0$$

Substituting the values of h^2 and A, in (1) we have

$$\frac{2}{3}\mu c^8\cdot\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu\left[\frac{-1}{3u^6}+\frac{c^4}{u^2}\right]$$

$$\begin{aligned}c^8\left(\frac{du}{d\theta}\right)^2&=\frac{-1}{2u^6}+\frac{3c^4}{2u^2}-c^8u^2\\&=\frac{1}{u^6}\left[\frac{-1}{2}+\frac{3}{2}c^4u^4-c^8u^8\right]\\\Rightarrow c^8\left(\frac{du}{d\theta}\right)^2&=\frac{1}{u^6}\left[\frac{-1}{2}-\left(c^8u^8-\frac{3}{2}c^4u^4\right)\right]\\&=\frac{1}{u^6}\left[\frac{-1}{2}-\left(c^4u^4-\frac{3}{4}\right)^2+\frac{9}{16}\right]\\c^8\left(\frac{du}{d\theta}\right)^2&=\frac{1}{u^6}\left[\left(\frac{1}{4}\right)^2-\left(c^4u^4-\frac{3}{4}\right)^2\right]\\\therefore c^4u^3\frac{du}{d\theta}&=\sqrt{\left(\frac{1}{4}\right)^2-\left(c^4u^4-\frac{3}{4}\right)^2}\\d\theta&=\frac{c^4u^3du}{\sqrt{\left(\frac{1}{4}\right)^2-\left(c^4u^4-\frac{3}{4}\right)^2}}\end{aligned}$$

Putting $c^4u^4-\frac{3}{4}=z$, so that $4c^4u^3du=dz$ we have

$$4d\theta=\frac{dz}{\sqrt{\left(\frac{1}{4}\right)^2-z^2}}$$

Integrating,

$$4\theta + \mathrm{B} = \sin^{-1}\left(\frac{z}{1/4}\right)$$

$$\Rightarrow \quad 4\theta + \mathrm{B} = \sin^{-1}(4z)$$

where B is a constant

$$\Rightarrow \quad 4\theta + \mathrm{B} = \sin^{-1}\left(4c^4u^4 - 3\right)$$

But initially when $u = \frac{1}{c}, \theta = 0$

$$\therefore \quad B = \sin^{-1}(1)$$

$$\Rightarrow \quad B = \frac{\pi}{2}$$

$$\therefore \quad 4\theta + \frac{\pi}{2} = \sin^{-1}\left(4c^4u^4 - 3\right)$$

$$\Rightarrow \sin\left(\frac{\pi}{2} + \dot{4}\theta\right) = 4c^4u^4 - 3$$

$$\Rightarrow \cos 4\theta = 4c^4u^4 - 3$$

$$\Rightarrow \quad 4c^4u^4 = 3 + \cos 4\theta$$

$$\Rightarrow \cos 4\theta = 4c^4u^4 - 3$$

$$\Rightarrow 4c^4u^4 = 3 + \cos 4\theta$$

$$\Rightarrow \quad \frac{4c^4}{r^4} = 3 + \cos 4\theta$$

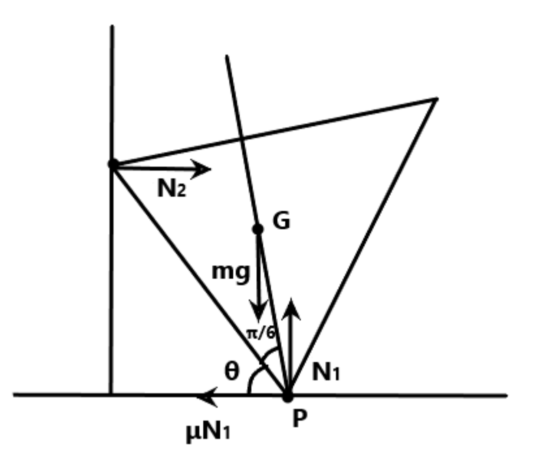
$$\begin{aligned}
\Rightarrow \quad 4c^4 &= r^4\left[3 + 2\cos^2 2\theta - 1\right] \\
&= 2r^4\left[1 + \cos^2 2\theta\right] \\
&= 2r^4\left[\left(\cos^2\theta + \sin^2\theta\right)^2 + \left(\cos^2\theta - \sin^2\theta\right)^2\right] \\
&= 4r^4\left(\cos^4\theta + \sin^4\theta\right) \\
\therefore \quad c^4 &= r^4\left(\cos^4\theta + \sin^4\theta\right) \\
\Rightarrow \quad c^4 &= (r\cos\theta)^4 + (r\sin\theta)^4 \\
\Rightarrow \quad c^4 &= x^4 + y^4 (\because x = r\cos\theta \text{ and } y = r\sin\theta)
\end{aligned}$$

Hence, $x^4 + y^4 = c^4$ is the equation of path.

Question-7(b) A thin equilateral rectangular plate of uniform thickness and density rests with one end of its base on a rough horizontal plane and the other against a small vertical wall. Show that the least angle, its base can make with the horizontal plane is given by $\cot\theta = 2\mu + \frac{1}{\sqrt{3}}\ \mu$, being the coefficient of friction.

[14 Marks]

Solution: Let the side of equilateral triangular plate be $'a'$ and G be its center of gravity. $N_1 =$ Normal reaction by rough horizontal plane.



$$N_1 = \text{mg}, \text{ where m is mass of plate.}$$

N_2 = Normal reaction by small vertical wall

$$N_2 = \mu N_1 = \mu(mg)$$

Taking moments about point P

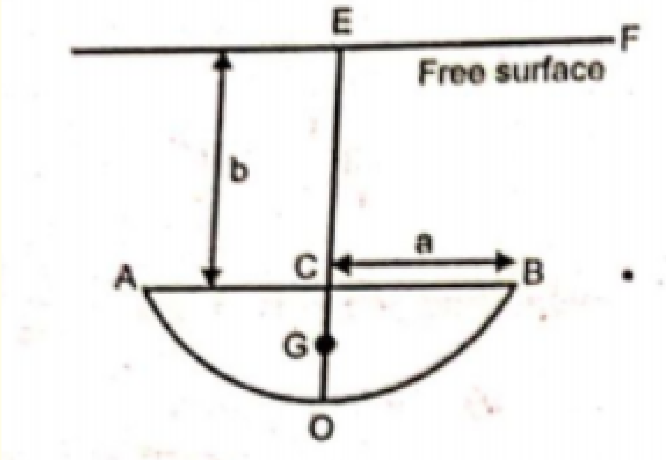
$$mg - CAP\cos\left(\theta + \frac{\pi}{6}\right) = N_2 \times a\sin\theta$$
$$\text{mg} \cdot \frac{\alpha}{\sqrt{3}}\left(\cos\theta \cdot \frac{\sqrt{3}}{2} - \sin\theta\frac{1}{2}\right) = \alpha\sin\theta.\mu mg$$

$$\sqrt{3}\cos\theta - \sin\theta = 2\sqrt{3}\mu\sin\theta$$
$$\sqrt{3}\cos\theta = (1 + 2\sqrt{3}\mu)\sin\theta$$
$$\Rightarrow \cot\theta = 2\mu + \frac{1}{\sqrt{3}}$$

Question-7(c) A semicircular area of radius a is immersed vertically with its diameter horizontal at a depth b. If the circumference be below the centre, prove that the depth of centre of pressure is

$$\frac{1}{4}\frac{3\pi\left(a^2 + 4b^2\right) + 32ab}{4a + 3\pi b}$$

[13 Marks]



Solution:

Depth of the centre of pressure of the semicircular area $= \dfrac{k^2}{h}$, where k is the radius of gyration about the line EF on the free surface and $h =$ depth of CG of the lamina below EF = EG

$$k^2 = "k^2$$

about parallel axis through

$$G + (\mathrm{EG})^2$$

Now,

$$\mathrm{CG} = \frac{4a}{3\pi}$$

and hence

$$\mathrm{EG} = b + \frac{4a}{3\pi}$$

$$\Rightarrow \mathrm{EG} = h = \frac{4a + 3b\pi}{3\pi} \ldots (1)$$

$$\therefore k^2 = "k^{2"}$$

about

$$\mathrm{AB} - (\mathrm{CG})^2 + (\mathrm{EG})^2 = \frac{a^2}{4} - \left(\frac{4a}{3\pi}\right)^2 + \left(\frac{4a + 3b\pi}{3\pi}\right)^2$$

$$= \frac{9\pi^2 a^2 + 36 b^2 \pi^2 + 96 ab\pi}{36\pi^2}$$

$$\therefore k^2 = \frac{3\pi\left(a^2 + 4b^2\right) + 32ab}{12\pi} \ldots (2)$$

From (1) and (2) we get Depth of the centre of pressure $= \dfrac{k^2}{h}$

$$= \left(\frac{3\pi\left(a^2 + 4b^2\right) + 32ab}{12\pi}\right) / \left(\frac{4a + 3b\pi}{3\pi}\right)$$

$$= \frac{1}{4}\left(\frac{3\pi\left(a^2 + 4b^2\right) + 32ab}{4a + 3\pi b}\right)$$

Question-8(a) **Solve** $x = y\dfrac{dy}{dx} - \left(\dfrac{dy}{dx}\right)^2$.

[10 Marks]

Solution: Solving the given differential equation fot x, we get

$$x = py + ap^2 \qquad \ldots(1)$$

Differentiating (1) w.r.t. y and writing $1/p$ for dx/dy, we get

$$\begin{aligned}
\frac{1}{p} &= p + y\frac{dp}{dy} + 2ap\frac{dp}{dy} \\
\text{or} \quad \frac{1-p^2}{p} &= y\frac{dp}{dy} + 2ap\frac{dp}{dy} \\
\text{or} \quad \frac{1-p^2}{p}\frac{dy}{dp} - y &= 2ap, \qquad \text{multiplying both sides by dy/dp} \\
\text{or} \quad \frac{dy}{dp} - \frac{1}{p^2-1}y &= -\frac{2ap^2}{p^2-1} \qquad \ldots(2)
\end{aligned}$$

which is a linear differential equation.
Here the I.F. $= e^{\int (p/(p^2-1))dp} = e^{\frac{1}{2}\log(p^2-1)} = (p^2-1)^{1/2}$ $\therefore$ the solution of (2) is

$$\begin{aligned}
y\left(p^2-1\right)^{1/2} &= \int \frac{-2ap^2}{p^2-1}\left(p^2-1\right)^{1/2} dp + c \\
&= -2a\int \frac{(p^2-1)+1}{\sqrt{(p^2-1)}}dp + c \\
&= -2a\int \left[\sqrt{(p^2-1)} + \frac{1}{\sqrt{(p^2-1)}}\right] dp + c \\
&= -2a\left[\frac{1}{2}p\sqrt{(p^2-1) - \frac{1}{2}\cosh^{-1}p + \cosh^{-1}p}\right] + c \\
&= -ap\sqrt{(p^2-1) - a\cosh^{-1}p + c} \\
\text{or} \quad y &= \frac{c - a\cosh^{-1}p}{\sqrt{(p^2-1)}} - ap\ . \qquad \ldots(3)
\end{aligned}$$

Substituting this value of y in (1), we get,

$$x = \left(\frac{c - a\cosh^{-1}p}{\sqrt{(p^2-1)}} - ap\right) + ap^2$$

$$\Rightarrow x = \frac{p(c - a\cosh^{-1}p)}{\sqrt{(p^2-1)}} \qquad \ldots(4)$$

The equations (3) and (4) constitute the parametric equations of the required solution.

Question-8(b) Find the value of the line integral over a circular path given by
$x^2 + y^2 = a^2, z = 0$ **where the vector field,** $\vec{F} = (\sin y)\vec{i} + x(1 + \cos y)\vec{j}$.

[10 Marks]

Solution: The line integral over a circular path given by C over vector field

$$\overrightarrow{\mathrm{F}} = \int_{\mathrm{C}} \overrightarrow{\mathrm{F}} \cdot dr$$

Here, C is given as $x^2 + y^2 = a^2, z = 0$ and

$$\overrightarrow{\mathrm{F}} = (\sin y)\hat{i} + x(1 + \cos y)\hat{j}$$

As we know that $\vec{r}$ is a position vector and is given as

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$
$$\therefore \quad d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

$$\begin{aligned} \text{Thus, the required integral value } &= \oint_{\mathrm{C}} [\sin y\hat{i} + x(1 + \cos y)\hat{j}] \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k}) \\ &= \oint_{\mathrm{C}} \sin y dx + x(1 + \cos y)dy \\ &= \oint_{\mathrm{C}} \mathrm{M}dx + \mathrm{N}dy \end{aligned}$$

Now, by Green's theorem in plane we have

$$\iint_{\mathbb{R}} \left(\frac{\partial \mathrm{N}}{\partial x} - \frac{\partial \mathrm{M}}{\partial y}\right) dxdy = \oint_{\mathrm{C}} \mathrm{M}dx + \mathrm{N}dy$$

Here $M = \sin y, N = x(1 + \cos y)$

$$\therefore \frac{\partial \mathrm{M}}{\partial y} = \cos y,$$
$$\frac{\partial \mathrm{N}}{\partial x} = 1 + \cos y$$

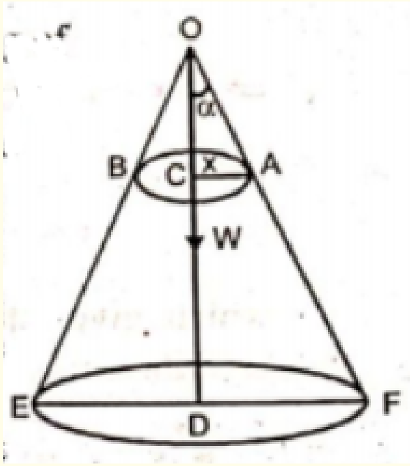
Hence, the given line integral is equal to $= \iint_R (1 + \cos y - \cos y)dxdy = \iint_R dxdy =$ Area of the circle $C = \pi a^2$

Question-8(c) A heavy elastic string, whose natural length is $2\pi a$, is placed round a smooth cone whose axis is vertical and whose semi vertical angle is α. If W be the weight and λ the modulus of elasticity of the string, prove that it will be in equilibrium when in the form of a circle whose radius is

$$a\left(1+\frac{W}{2\pi\lambda}\cot\alpha\right)$$

[10 Marks]

Solution: OEF is a smooth fixed cone of semi-vertical angle α, the axis OD of the cone being vertical.



A heavy elastic string of natural length $2\pi a$ placed round the cone and suppose it rests in the form of a circle whose centre is C and whose radius CA is x.

The weight W of the string acts at its centre of gravity C.

Let T be the tension in this string. Give the string a small displacement in which x changes to $x+\delta x$. The point O remains fixed, the point C is slightly displaced.

$\angle\alpha$ is fixed and the length of the string slight changed. We have the length of the string AB in the form of a circle of radius x is $2\pi x$ and so the work done by the tension T of this string is $-\mathrm{T}\delta(2\pi x)$.
Also, the depth of the point of application C of the weight W below the fixed point O

$$\mathrm{OC}=\mathrm{AC}\cot\alpha=x\cot\alpha$$

the work done by the weight W during this small displacement $=\mathrm{W}\delta(x\cot\alpha)$

Since the reactions at the various points of contact do work, thus by the principle of virtual work,

$$\begin{aligned}-\mathrm{T}\delta(2\pi x)+\mathrm{W}\delta(x\cot\alpha)&=0\\ \Rightarrow -2\pi\mathrm{T}\delta x+\mathrm{W}\cot\alpha\delta x&=0\\ (-2\pi\mathrm{T}+\mathrm{W}\cot\alpha)\delta x&=0\\ \Rightarrow\quad -2\pi\mathrm{T}+\mathrm{W}\cot\alpha&=0(\because\delta x\neq 0)\end{aligned}$$

$$\mathrm{T}=\frac{\mathrm{W}\cot\alpha}{2\pi}$$

Now, by Hooke's law the tension T in the elastic string AB is given by

$$\mathrm{T} = \lambda\frac{(2\pi x - 2\pi a)}{2\pi a}$$
$$\mathrm{T} = \lambda\frac{x-a}{a}$$

Equating the two values of T we get

$$\frac{\mathrm{W}\cot\alpha}{2\pi} = \lambda\frac{(x-a)}{a}$$
$$\Rightarrow \quad x - a = \frac{a}{2\pi\lambda}\mathrm{W}\cot\alpha$$
$$\Rightarrow \quad x = a\left(1 + \frac{w}{2\pi\lambda}\cot\alpha\right)$$

which gives the radius of the string in equilibrium.

Question-8(d) **Solve** $x^2\dfrac{d^2y}{dx^2} + 3x\dfrac{dy}{dx} + y = (1-x)^{-2}$.

[10 Marks]

Solution: Putting $x = e^z$ and denoting d/dz by D', the given differenitial equation becomes

$$\left[D'\left(D'-1\right)+3D'+1\right]y=\frac{1}{\left(1-e^{z}\right)^{2}}$$

$$\text{or}\quad \left(D'+1\right)^{2}y=\frac{1}{\left(1-e^{z}\right)^{2}}$$

$$\text{A.E. is}\quad (m+1)^{2}=0.\quad \Rightarrow\quad m=-1,-1$$

$$\therefore\quad C.F.=\left(c_{1}+c_{2}z\right)e^{-z}=\left(c_{1}+c_{2}\log x\right)\cdot x^{-1}$$

$$\text{P.I.}\ =\frac{1}{\left(D'+1\right)^{2}}\frac{1}{\left(1-e^{z}\right)^{2}}=\frac{1}{\left(D'+1\right)}\cdot\frac{1}{\left(D'+1\right)}\left[\frac{1}{\left(1-e^{z}\right)^{2}}\right]$$

$$\text{Let }\frac{1}{\left(D'+1\right)}\left[\frac{1}{\left(1-e^{z}\right)^{2}}\right]=v\text{ or }\left(D'+1\right)v=\frac{1}{\left(1-e^{z}\right)^{2}}$$

$$\text{or}\quad \frac{dv}{dz}+v=\frac{1}{\left(1-e^{z}\right)^{2}},\qquad \text{which is a linear equation.}$$

$$\text{I.F.}=e^{\int dz}=e^{z}$$

$$\therefore\quad ve^{z}=\int e^{z}\left(1-e^{z}\right)^{-2}dz=\left(1-e^{z}\right)^{-1}$$

$$\text{or}\qquad v=\frac{1}{\left(D'+1\right)}\left[\frac{1}{\left(1-e^{z}\right)^{2}}\right]=e^{-z}\left(1-e^{z}\right)^{-1}$$

$$\begin{aligned}\therefore\quad \text{P.I.}\ &=\frac{1}{\left(D'+1\right)}e^{-z}\left(1-e^{z}\right)^{-1}\\ &=e^{-z}\int e^{z}e^{-z}\left(1-e^{z}\right)^{-1}dz.\\ &=e^{-z}\int\frac{dz}{1-e^{z}}\\ &=e^{-z}\int\frac{1}{x(1-x)}dx,\ \text{putting } x=e^{z},dz=(1/x)dx\\ &=e^{-z}\int\left[\frac{1}{x}+\frac{1}{1-x}\right]dx=e^{-z}[\log x-\log(1-x)]\\ &=\frac{1}{x}\log\frac{x}{1-x}.\end{aligned}$$

Hence the complete solution of the given equation is

$$y=\left(c_{1}+c_{2}\log x\right)\frac{1}{x}+\frac{1}{x}\log\frac{x}{1-x}$$