2018 - IFoS

Q) Air obeying Boyle's law in a uniform tube. ρ the density, v the velocity at a distance x from a fixed pt at time 't'.

Prove: $\dfrac{\partial^2 \rho}{\partial t^2} = \dfrac{\partial^2}{\partial x^2}\{\rho(v^2+k)\}$

Soln: Taking $k = \dfrac{p}{\rho} \Rightarrow p = \rho k$

Eq^n of continuity : $\dfrac{\partial \rho}{\partial t} + \dfrac{\partial}{\partial x}(\rho v) = 0$ —(1)

Eqn of motion : $\dfrac{\partial v}{\partial t} + v\dfrac{\partial v}{\partial x} = -\dfrac{1}{\rho}\dfrac{\partial p}{\partial x}$

$$\Rightarrow \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{k}{\rho}\frac{\partial \rho}{\partial x} \quad -(2)$$

Differentiating (1) w.r.t 't'.

$$\frac{\partial^2 \rho}{\partial t^2} + \frac{\partial}{\partial t}\left(\frac{\partial}{\partial x}(\rho v)\right) = 0$$

$$\Rightarrow \frac{\partial^2 \rho}{\partial t^2} + \frac{\partial}{\partial x}\left(\frac{\partial}{\partial t}(\rho v)\right) = 0$$

$$\Rightarrow \frac{\partial^2 \rho}{\partial t^2} + \frac{\partial}{\partial x}\left(\rho\frac{\partial v}{\partial t} + v\frac{\partial \rho}{\partial t}\right) = 0$$

Using (2) & (1)

$$\Rightarrow \frac{\partial^2 \rho}{\partial t^2} + \frac{\partial}{\partial x}\left(\rho\left(-v\frac{\partial v}{\partial x} - \frac{k}{\rho}\frac{\partial \rho}{\partial x}\right) - v\frac{\partial(\rho v)}{\partial x}\right) = 0$$

$$\Rightarrow \frac{\partial^2 \rho}{\partial t^2} = \frac{\partial}{\partial x}\left(\rho v\frac{\partial v}{\partial x} + k\frac{\partial \rho}{\partial x} + v\frac{\partial}{\partial x}(\rho v)\right)$$

$$\Rightarrow \frac{\partial^2 \rho}{\partial t^2} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}(\rho v^2) + k\frac{\partial \rho}{\partial x}\right)$$

$$\Rightarrow \frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2}{\partial x^2}\left[\rho(v^2+k)\right]$$

Q) In case of 2D motion of a liquid streaming past a fixed circular disc, the velocity at infinity is 'u' in a fixed direcⁿ. Show that max. velocity at any pt of fluid is 2u. Prove that the force necessary to hold the disc is $2mu$.

Soln: Velocity potential $\phi = u\left(r + \frac{a^2}{r}\right)\cos\theta$

where a is the radius of the disc.

$$\frac{\partial \phi}{\partial r} = u\left(1 - \frac{a^2}{r^2}\right)\cos\theta$$

$$\frac{\partial \phi}{\partial \theta} = -u\left(r + \frac{a^2}{r}\right)\sin\theta$$

Then $q^2 = \left(-\frac{\partial \phi}{\partial r}\right)^2 + \left(-\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)^2$

$$\Rightarrow q^2 = u^2\left(1 - \frac{2a^2}{r^2}\cos 2\theta + \frac{a^4}{r^4}\right)$$

For maximum q : $\cos 2\theta = -1$ at $\theta = \pi/2$

$$\Rightarrow q^2 = u^2\left(1 + \frac{2a^2}{r^2} + \frac{a^4}{r^4}\right)$$

$$\Rightarrow q^2 = u^2\left(1 + \frac{a^2}{r^2}\right)^2$$

$$\Rightarrow q = u\left(1 + \frac{a^2}{r^2}\right)$$

For q maximum: $r = a$ (minimum value)

$$\Rightarrow q_{max} = 2u$$

Now, Bernoulli's eqn :

$$\frac{p}{\rho} = \frac{\partial \phi}{\partial t} - \frac{q^2}{2} + F(t)$$

Putting $q^2 = u^2\left(1 - \frac{2a^2}{r^2}\cos 2\theta + \frac{a^4}{r^4}\right)$

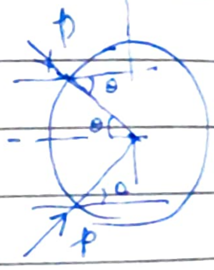
$$\frac{\partial \phi}{\partial t} = \dot{u}\left(r + \frac{a^2}{r}\right)\cos\theta$$

So, $$\frac{p}{\rho} = F(t) - \frac{1}{2}u^2\left(1 - \frac{2a^2}{r^2}\cos 2\theta + \frac{a^4}{r^4}\right) + \dot{u}\left(r + \frac{a^2}{r}\right)\cos\theta$$

Putting $r = a$, the pressure at the boundary will be :

$$p = \rho\left(F(t) - 2u^2\sin^2\theta + 2a\dot{u}\cos\theta\right)$$

The pressure on the disc is



$$= \int_0^{2\pi} (-p\cos\theta)\, a\, d\theta$$

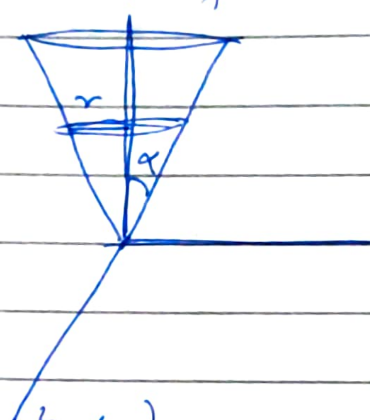
$$= -\rho a \int_0^{2\pi} \left(\underbrace{F(t) - 2u^2\sin^2\theta}_{=0} + 2a\dot{u}\cos\theta\right)\cos\theta\, d\theta$$

$$= -2\rho a^2 \dot{u} \int_0^{2\pi} \cos^2\theta\, d\theta = -2\pi a^2 \rho \dot{u}$$

$$= -2m\dot{u}$$

So, the force required $= 2m\dot{u}$

Q) A particle of mass 'm' is constrained to move on the inner surface of a cone of semi angle α under the action of gravity. Write the eqn of constraints & mention the generalized co-ordinates. Write the eqn of motion.



Soln: $\tan\alpha = r/z$

$\Rightarrow z = r\cot\alpha$

Taking the generalized co-ordinates as r & θ:

$$T = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2 + \dot{r}^2\cot^2\alpha)$$

$$T = \frac{m}{2}(\dot{r}^2\operatorname{cosec}^2\alpha + r^2\dot{\theta}^2)$$

$$U = mgz = mgr\cot\alpha$$

$$L = \frac{m}{2}(\dot{r}^2\operatorname{cosec}^2\alpha + r^2\dot{\theta}^2) - mgr\cot\alpha$$

r constraint: $\partial L/\partial r - d/dt(\partial L/\partial \dot{r}) = 0$

$\Rightarrow mr\dot{\theta}^2 - mg\cot\alpha - m\ddot{r}\operatorname{cosec}^2\alpha = 0$

$\Rightarrow r\dot{\theta}^2 - g\cot\alpha - \ddot{r}\operatorname{cosec}^2\alpha = 0$

θ constraint: $\partial L/\partial\theta - d/dt(\partial L/\partial\dot{\theta}) = 0$

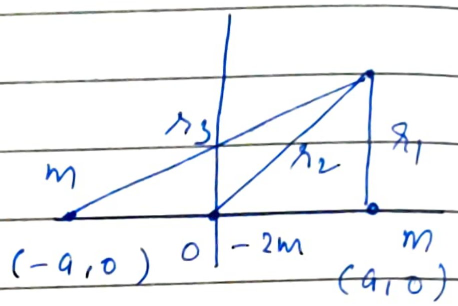
$\Rightarrow -\frac{d}{dt}(mr^2\dot{\theta}) = 0 \Rightarrow mr^2\dot{\theta} = \text{constant}$

The above two equations represent the equation of motion.

Q) Two sources, of strength 'm' are placed at $(-a, 0)$, $(a, 0)$ and a sink of strength '2m' at the origin. Show that streamlines are
$(x^2+y^2)^2 = a^2(x^2-y^2+\lambda xy)$, λ parameter.
Also show that fluid speed at any pt is $2ma^2/(r_1 r_2 r_3)$

Soln. complex potential:

$w = -m\log(z-a)$
$\quad - m\log(z+a)$
$\quad + 2m\log z$



$\Rightarrow w = m[\log z^2 - \log(z^2-a^2)]$
$\Rightarrow \phi + i\psi = m[\log(x^2-y^2+2ixy) - \log(x^2-y^2-a^2+2ixy)]$

using $\log(x+iy) = \frac{1}{2}\log(x^2+y^2) + i\tan^{-1}\frac{y}{x}$

$$\psi = m\left[\tan^{-1}\frac{2xy}{x^2-y^2} - \tan^{-1}\frac{2xy}{x^2-y^2-a^2}\right]$$

$$\Rightarrow \psi = m \,\cancel{\tan}\, \tan^{-1}\left[\frac{-2a^2xy}{(x^2+y^2)^2 - a^2(x^2-y^2)}\right]$$

(using $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)$)

So, ψ = const gives the streamlines.

Putting $\psi = m \tan^{-1}(-2/\lambda)$

$$\boxed{\Rightarrow (x^2+y^2)^2 = a^2(x^2-y^2+\lambda xy)}$$

$$\frac{dw}{dz} = -\frac{m}{z-a} - \frac{m}{z+a} + \frac{2m}{z}$$

$$= -\frac{2a^2 m}{z(z-a)(z+a)}$$

So, $q = \left|\frac{dw}{dz}\right|$

$$= \frac{2a^2 m}{|z|\,|z-a|\,|z+a|}$$

$$= \frac{2a^2 m}{r_1 r_2 r_3}$$