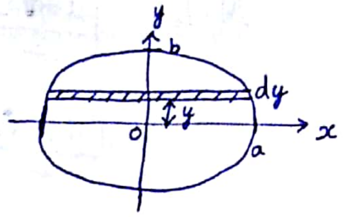
2016 - MECHANICS - IFOS

5.e Ellipse : $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



(i) M.I (rel. x-axis)

$$= \int dm \times y^2$$

$$= \int_{-b}^{b} 2\rho \times \frac{a}{b}\sqrt{b^2-y^2} \times y^2\, dy$$

$$= \frac{2\rho a}{b}\int_{-b}^{b} y^2\sqrt{b^2-y^2}\, dy$$

$$= \frac{4\rho a}{b}\int_{0}^{b} y^2\sqrt{b^2-y^2}\, dy$$

$\rho \rightarrow$ ~~density~~ mass per area

$\rightarrow dm = \rho \times \left\{dy \times 2\sqrt{b^2-y^2}\,\frac{a}{b}\right\}$

$\rightarrow \rho = \frac{M}{\pi a b}$

(Put $y = b\sin\theta$)

$$= \frac{4\rho a}{b}\int_{0}^{\pi/2} b^2\sin^2\theta \times b\cos\theta \times b\cos\theta\, d\theta$$

$$= 4\rho a b^3 \int_{0}^{\pi/2} \sin^2\theta\cos^2\theta\, d\theta$$

$$= 4\rho a b^3 \int_{0}^{\pi/2} \sin^2\theta(1-\sin^2\theta)\, d\theta$$

$$= 4\rho a b^3 \left\{\frac{1}{2}\cdot\frac{\pi}{2} - \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right\}$$

$$= \frac{\pi}{4} a b^3 \rho = \frac{\pi}{4} a b^3 \times \frac{M}{\pi a b}$$

$$= \frac{Mb^2}{4}$$

(ii) M.I (rel. y-axis)

Similar to (i), $= \frac{Ma^2}{4}$

(iii) M.I (rel. origin) = M.I (rel x-axis) + M.I (rel. y-axis)

$$= \frac{M(a^2+b^2)}{4}$$

$\boxed{8.a}$

$$M.I = \int \frac{dm\, r^2}{2}$$

$$= \frac{\rho \pi \tan^4\theta}{2} \int_0^h x^4\, dx$$

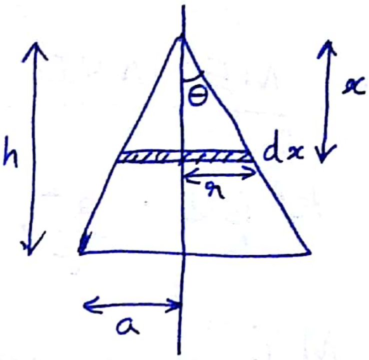
$$= \frac{\rho \pi \tan^4\theta}{2} \times \frac{h^5}{5}$$

$$= \frac{\cancel{\pi}\, \cancel{h}^5}{10} \times \left\{\frac{3M}{\cancel{\pi} a^2 \cancel{h}}\right\}\left\{\frac{a}{\cancel{h}}\right\}^4$$

$$= \underline{\underline{\frac{3}{10} M a^2}}$$

$$\tan\theta = \frac{a}{h}$$

$$\tan\theta = \frac{r}{x}$$



$\rho \rightarrow$ density

$$\rho = \frac{M}{\frac{1}{3}\pi a^2 h}$$

$$dm = \rho\, dV$$

$$= \rho \pi r^2 dx$$

$$dm = \rho \pi x^2 \tan^2\theta\, dx$$

$\boxed{8.b}$ $x = a(\theta - \sin\theta) \Rightarrow \dot{x} = a(1-\cos\theta)\dot{\theta}$

$y = a(1+\cos\theta) \Rightarrow \dot{y} = -a(\sin\theta)\dot{\theta}$

$$T = \frac{1}{2}m\{\dot{x}^2 + \dot{y}^2\} = \frac{1}{2}m\{a^2(1+\cos^2\theta - 2\cos\theta) + a^2\sin^2\theta\}\dot{\theta}^2$$

$$\Rightarrow \underline{T = ma^2\dot{\theta}^2\{1-\cos\theta\}}$$

$$\underline{V = mga(1+\cos\theta)}$$

<u>Lagrangian</u> $(L) = T - V$

$$\boxed{L = ma^2\dot{\theta}^2(1-\cos\theta) - mga(1+\cos\theta)}$$

<u>Lagrange's Eq. of Motion</u>,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\Rightarrow \frac{d}{dt}\left(2\cancel{m}a^{\cancel{2}}\dot{\theta}(1-\cos\theta)\right) - \{\cancel{m}a^{\cancel{2}}\dot{\theta}^2\sin\theta + \cancel{m}g\cancel{a}\sin\theta\} = 0$$

$$\Rightarrow 2a\ddot{\theta}(1-\cos\theta) + \cancel{2}a\dot{\theta}^2\sin\theta - a\cancel{\dot{\theta}^2}\sin\theta + g\sin\theta = 0$$

$$\Rightarrow \boxed{2\ddot{\theta}(1-\cos\theta) + \dot{\theta}^2\sin\theta + \frac{g}{a}\sin\theta = 0} \; //$$

Ex. 18. *A sphere is at rest in an infinite mass of homogeneous liquid of density ρ, the pressure at infinity being Π. If the radius R of the sphere varies in such a way that $R = a + b \cos nt$ where $b < a$, show that pressure at the surface of the sphere at any time is*

$$\Pi + \frac{1}{4} bn^2\rho\,(b - 4a \cos nt - 5b \cos 2nt).$$

Solution. Let v' be the velocity at a distane r' at any time t and p' be the pressure there. Let v be the velocity on the surface of the sphere of radius R where $R = a + b \cos nt$.

The equation of continuity becomes

$$r'^2 v' = f(t) = R^2 v. \qquad \text{...(1)}$$

The equation of motion is given by

$$\frac{\partial v'}{\partial t} + v' \frac{\partial v'}{\partial r'} = -\frac{1}{\rho}\frac{\partial p'}{\partial r'}$$

$$\frac{f'(t)}{r'^2} + v' \frac{\partial v'}{\partial r'} = -\frac{1}{\rho}\frac{\partial p'}{\partial r'}$$

Integrating with regard to r', we have

$$-\frac{f'(t)}{r'} + \frac{1}{2} v'^2 = -\frac{p'}{\rho} + A$$

Initially $\quad r' = \infty, v = 0, p' = F$ then $A = \dfrac{P}{\rho}$

$$\frac{-f'(t)}{r'} + \frac{1}{2} v'^2 = \frac{P - p'}{\rho}$$

Again $\quad r' = R, p' = p$ and $v' = v$

then $\quad -\dfrac{f'(t)}{R} + \dfrac{1}{2}v^2 = \dfrac{1}{\rho}(P - p)$

$$p = P + \rho\left[\frac{f'(t)}{R} - \frac{1}{2}v^2\right] \quad ...(2)$$

R.H.S. $\dfrac{f'(t)}{R} - \dfrac{1}{2}v^2 = 2\left(\dfrac{dR}{dt}\right)^2 + R\dfrac{d^2R}{dt^2} - \dfrac{1}{2}\left(\dfrac{dR}{dt}\right)^2$

$$= \frac{3}{2}\left(\frac{dR}{dt}\right)^2 + R\frac{d^2R}{dt^2}$$

$$= \tfrac{3}{2}(-bn\sin nt)^2 + (a + b\cos nt)(-bn^2\cos nt)$$

$$= \tfrac{1}{2}bn^2(3b\sin^2 nt - 2b\cos^2 nt - 2a\cos nt)$$

$$= \tfrac{1}{4}bn^2[3b(1 - \cos 2nt) - 2b(1 + \cos 2nt) - 4a\cos nt]$$

$$= \tfrac{1}{4}bn^2[b - 4a\cos nt - 5b\cos 2nt]$$

Hence $\quad p = P + \tfrac{1}{4}bn^2\rho\,[b - 4a\cos nt - 5b\cos 2nt]$. **Proved.**