MAINS TEST SERIES - 2021

TEST-1, PAPER-I

ANSWER KEY

LINEAR ALGEBRA, CALCULUS & THREE DIMENSIONAL GEOMETRY

1(a) Let $V=C^2$, and let $W=\left\{\begin{bmatrix}x_1\\x_2\end{bmatrix} \mid x_1\in R, x_2\in C\right\}$. Is W a subspace of the complex vector space V? Is it a subspace of V when V is considered as a vectorspace over IR?

part-1

Sol Let $V=C^2$ be a given vector space over the field 'C'

let $V=C^2=\left\{\begin{bmatrix}x_1+iy_1\\x_2+iy_2\end{bmatrix} \Big/ \begin{array}{l}x_1+iy_1, x_2+iy_2\in C\\ x_1,y_1,x_2,y_2\in IR\end{array}\right\}$.

let $W=\left\{\begin{bmatrix}x_1\\x_2\end{bmatrix} \Big/ x_1\in IR, x_2\in C\right\}\subseteq V$.

$\because \begin{bmatrix}0\\0\end{bmatrix}\in W \quad \therefore W\neq\phi$.

let $a,b\in C$; $\alpha=\begin{bmatrix}x_1\\x_2\end{bmatrix}$, $\beta=\begin{bmatrix}x_1'\\x_2'\end{bmatrix}\in W$

$x_1,x_1'\in IR$, $x_2,x_2'\in C$.

we have

$$a\alpha+b\beta=a\begin{bmatrix}x_1\\x_2\end{bmatrix}+b\begin{bmatrix}x_1'\\x_2'\end{bmatrix}$$

$$=\begin{bmatrix}ax_1+bx_1'\\ax_2+bx_2'\end{bmatrix}\notin W \text{ because } ax_1+bx_1'\notin IR$$

$\therefore$ W is not a subspace of $V = C^2(\mathbb{C})$.

part-2

Let $V = C^2(\mathbb{R})$

let $W = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} / x_1 \in \mathbb{R}, x_2 \in \mathbb{C} \right\} \subseteq V$.

let $a, b \in \mathbb{R}$; $\alpha = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\beta = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} \in V$

$x_1, x_1' \in \mathbb{R}$;
$x_2, x_2' \in \mathbb{C}$.

We have

$$a\alpha + b\beta = \begin{bmatrix} ax_1 + bx_1' \\ ax_2 + bx_2' \end{bmatrix} \in W$$

as $ax_1 + bx_1' \in \mathbb{R}$ and
$ax_2 + bx_2' \in \mathbb{C}$.

1(b) Let $MSS_n(R)$ denote the vectorspace of $n\times n$ magic squares with real entries. What is the dimension of $MSS_2(R)$? Prove it.

Sol'n: Consider a 2×2 magic square

a	b
c	d

In order for this to indeed be a magic square, we must have $a+b=a+c$, and so $b=c$.

Similarly $a+b=b+d$, and so $a=d$.

Finally $a+b=a+d \Rightarrow b=d$

at which point we can conclude that every 2×2 magic square has the form

$$\begin{bmatrix} \alpha & \alpha \\ \alpha & \alpha \end{bmatrix} = \alpha \cdot \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Thus the set $\left\{ \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right\}$ is a basis for $MSS_2(R)$,

Hence $\dim MSS_2(R) = 1$.

1(C) Evaluate $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^2}$

Sol^n: The given limit is 1^∞ form.

Let $y = \left(\frac{\tan x}{x}\right)^{1/x^2}$ so that $\log y = \frac{1}{x^2}\log\left(\frac{\tan x}{x}\right)$

$$\lim_{x\to 0}\log y = \lim_{x\to 0}\frac{\log(\tan x/x)}{x^2}\left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0}\frac{\frac{x}{\tan x}\left(\frac{x\sec^2 x - \tan x}{x^2}\right)}{2x}$$

$$= \lim_{x\to 0}\frac{x\sec^2 x - \tan x}{2x^2\tan x}\left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0}\frac{\sec^2 x + 2x\sec^2 x\tan x - \sec^2 x}{4x\tan x + 2x^2\sec^2 x}$$

$$= \lim_{x\to 0}\frac{\sec^2 x\tan x}{2\tan x + x\sec^2 x}\left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0}\frac{\sec^4 x + 2\sec x(\sec x\tan x)\tan x}{2\sec^2 x + \sec^2 x + 2x\sec^2 x\tan x}$$

$$= \frac{1}{3}$$

$\therefore \lim_{x\to 0}\log y = \frac{1}{3}$

$\Rightarrow \lim_{x\to 0} y = e^{1/3}$

Hence $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^2} = e^{1/3}$.

1(d) Find the Volume common to cylinders $x^2+y^2=a^2$, $x^2+z^2=a^2$

Sol^n

$$V=\iiint dx\,dy\,dz$$

Limits for

$z \to -\sqrt{a^2-x^2}$ to $+\sqrt{a^2-x^2}$

$y \to -\sqrt{a^2-x^2}$ to $+\sqrt{a^2-x^2}$

$x \to -a$ to a

$$V = 8\int_0^a\int_0^{\sqrt{a^2-x^2}}\int_0^{\sqrt{a^2-x^2}} dx\,dy\,dz$$

$$= 8\int_0^a\int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2}\,dx\,dy$$

$$= 8\int_0^a (a^2-x^2)\,dx = 8\left[a^2x-\frac{x^3}{3}\right]_0^a$$

$$= 8\left(a^3-\frac{a^3}{3}\right) = \frac{16a^3}{3}$$

1(e). Prove that the plane $ax+by+cz=0$ cuts the cone $yz+zx+xy=0$ in perpendicular lines if $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$.

Sol'n: The equation of the plane is $ax+by+cz=0$ —— (1)

and the cone is $yz+zx+xy=0$ —— (2)

Comparing (2) with

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$$

$$\therefore a=0,\ b=0,\ c=0$$

$$\Rightarrow a+b+c=0+0+0=0$$

$\therefore$ The cone (2) has three mutually perpendicular generators.

The plane (1) will cut the cone (2) in $\perp$lar lines if the normal to the plane (1) through the vertex $(0,0,0)$ [where d.c's are proportional to a,b,c] lies on the cone (2).

if $bc+ca+ab=0$ [$\because$ d.c's of the generator satisfy the equation of the cone]

if $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$

(on dividing throughout by abc).

which is the required condition.

2(a) (i) Let V and V' be vectorspaces over K, and let $T: V \to V'$ be an injective linear transformation. If $x_1, \dots, x_n$ are linearly independent elements of V, then $T(x_1), \dots, T(x_n)$ are linearly independent elements of V'. Is converse true? Justify your answers.

(ii) If A is a square matrix, then prove that $\det(A^n) = (\det A)^n$ for all positive integers n.

Sol Let V and V' be vectorspaces over K.

Let $T: V \to V'$ be an injective linear transformation.

Given that $x_1, x_2, x_3 \dots x_n$ are L.I in V

To prove that $T(x_1), T(x_2) \dots T(x_n)$ are L.I.

let $c_1, c_2, \dots c_n \in K$ s.t

$c_1 T(x_1) + c_2 T(x_2) + \dots + c_n T(x_n) = \bar{0}$

$\Rightarrow T(c_1 x_1 + c_2 x_2 + \dots + c_n x_n) = T(\bar{0})$ — (1)

$(\because T(\bar{0}) = \bar{0})$

$\Rightarrow c_1 x_1 + c_2 x_2 + \dots + c_n x_n = \bar{0}$

$\Rightarrow c_1 = c_2 = \dots = c_n = 0$ ($\because T$ is injective i.e 1-1)

($\because x_1, x_2 \dots x_n$ are L.I)

$\therefore T(x_1), T(x_2) \dots T(x_n)$ are L.I in V'.

Converse is also true

i.e if $T(x_1), T(x_2) \dots T(x_n)$ are L.I

then $x_1, x_2 \dots x_n$ are also L.I.

let $a_1, a_2, \dots a_n \in K$ s.t

$a_1x_1 + a_2x_2 + \dots + a_nx_n = \bar{0}$ — (1)

$\Rightarrow T(a_1x_1 + a_2x_2 + \dots + a_nx_n) = T(\bar{0})$

$\Rightarrow a_1T(x_1) + a_2T(x_2) + \dots + a_nT(x_n) = \hat{0}$ ($\because$ T is L.T & $T(\bar{0}) = \hat{0}$)

$\Rightarrow a_1 = a_2 = \dots = a_n = 0$

($\because T(x_1), T(x_2) \dots T(x_n)$ are L.I)

$\therefore x_1, x_2, x_3 \dots x_n$ are L.I in V.

(ii) TOP.T $|A^n| = |A|^n \ \forall n \in \mathbb{Z}^+$

let $S(n) \equiv |A^n| = |A|^n \ \forall n \in \mathbb{Z}^+$ — (1)

if $n=1$ then $|A^1| = |A|^1$

$\therefore S(1)$ is true.

let us assume that $S(n)$ is true for $n=k$.

Then $S(k) \equiv |A^k| = |A|^k$ — (2)

Let us prove that $S(k+1)$ is true.

we have $|A^{k+1}| = |A^k.A| = |A^k||A|$ ($\because |AB| = |A||B|$)

$= |A|^k|A|$ (by (2))

$= |A|^{k+1}$

$\therefore S(k+1)$ is true.

$\therefore S(n)$ is true for all $n \in \mathbb{Z}^+$.

$\therefore |A^n| = |A|^n. n \in \mathbb{Z}$.

2(b) For the function $f(x,y) = \begin{cases} \dfrac{x^2 - x\sqrt{y}}{x^2+y}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$

Sol'n: Let us approach $(0,0)$ along the path $y = x^4$

then

$$\underset{(x,y)\to(0,0)}{Lt} f(x,y) = \underset{x\to 0}{Lt} \frac{x^2 - x(x^2)}{x^2+x^4}$$

$$= \underset{x\to 0}{Lt} \frac{x^2(1-x)}{x^2(1+x^2)}$$

$$= \underset{x\to 0}{Lt} \frac{1-x}{1+x^2} = 1$$

$$\neq f(0,0)$$

$\therefore f(x,y)$ is not continuous at $(0,0)$

$\therefore f$ is not differentiable at $(0,0)$.

(OR)

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(2)(b) For the function

$$f(x,y)=\begin{cases}\dfrac{x^2-x\sqrt{y}}{x^2+y} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0)\end{cases}$$

Examine the continuity and Diff-erentiability.

Ans. Continuity at $\underline{(0,0)}$ –

Taking $f(x,y)$ along $y=m^2x^2$ at $(0,0)$

$$\lim_{\substack{(x,y)\to(0,0)\\ y=m^2x^2}} \frac{x^2-x\sqrt{m^2x^2}}{x^2+m^2x^2}=\frac{1-m}{1+m^2}$$

which is depending on 'm'

Therefore $f(x,y)$ is not continuous at $(0,0)$.

which also implies $f(x,y)$ is not differentiable at $(0,0)$

(2) (xi) Does the point $(4,-6,0)$ lie on the plane which intersects the positive x, y and z-axes at distances 2, 3, 5 units respectively.

Ans Equation of plane, whose x, y, z axes intercept are, 2, 3, 5 is

$$\frac{x}{2}+\frac{y}{3}+\frac{z}{5}=1$$

Point $(4,-6,0)$

$$LHS = \frac{4}{2}-\frac{6}{3}+\frac{0}{5} = 2-2=0$$
$$\neq 1 = RHS$$

Therefore $(4,-6,0)$ does not lies on gi plane.

(2)(c)(ii) Find the eqn of two planes through the points $(0,4,-3)$, $(6,-4,3)$ which cut off from the axes intercepts whose sum is zero.

Soln Let the eqn of plane -

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

Given $a+b+c=0$ — ①

Passes through $(0,4,-3)$

$$\frac{0}{a}+\frac{4}{b}-\frac{3}{c}=1 \Rightarrow \frac{4}{b}-\frac{3}{c}=1 \quad \text{— ②}$$

through $(6,-4,3)$ - $\frac{6}{a}-\frac{4}{b}+\frac{3}{c}=1$

$$\frac{6}{a}-1=1 \quad [\text{By ②}]$$

$$\Rightarrow \boxed{a=3}$$

By ① $\Rightarrow$ $b+c=-3 \Rightarrow b=-3-c$

By ② $\Rightarrow$ $\frac{4}{b}-\frac{3}{c}=1 \Rightarrow 4c-3b=bc$

$\Rightarrow$ $4c-3(-3-c)=(-3-c)c$

$\Rightarrow$ $c^2+10c+9=0$

$\Rightarrow$ $c=-1,-9$

$\Rightarrow$ $b=-3+1, -3+9 = -2, 6$

Eqn of planes $\rightarrow$

$$\boxed{\frac{x}{3}-\frac{y}{2}-\frac{z}{1}=1}, \quad \boxed{\frac{x}{3}+\frac{y}{6}-\frac{z}{9}=1}$$

(3)(a) Let $\alpha = (x_1, x_2)$, $\beta = (y_1, y_2)$ be vectors in R^2 such that $x_1 y_1 + x_2 y_2 = 0$, $x_1^2 + x_2^2 = y_1^2 + y_2^2 = 1$. Prove that $B = \{\alpha, \beta\}$ is a basis for R^2. Find the co-ordinates of the vector (a, b) in the ordered basis $B = \{\alpha, \beta\}$.

Ans. To prove $B = \{\alpha, \beta\}$ is Basis

To P.T B is Linearly independent

Let $a, b \in \mathbb{R}$ such that

$$a(x_1, x_2) + b(y_1, y_2) = (0,0)$$

$$\Rightarrow (ax_1 + by_1, \ ax_2 + by_2) = (0,0)$$

$$\Rightarrow ax_1 + by_1 = 0 \quad -(1), \qquad ax_2 + by_2 = 0 \quad -(2)$$

$(1) \times x_1 + (2) \times x_2$:-

$$ax_1^2 + bx_1y_1 + ax_2^2 + bx_2y_2 = 0$$

$$a(x_1^2 + x_2^2) + b(x_1y_1 + x_2y_2) = 0$$

Given $x_1^2 + x_2^2 = 1$, $x_1y_1 + x_2y_2 = 0$

$$\Rightarrow a \cdot 1 + b \cdot 0 = 0 \Rightarrow \boxed{a = 0}$$

Similarly $\Rightarrow \boxed{b = 0}$ Hence α, β are LI

clearly The number of L.I vectors $= \operatorname{Dim} \mathbb{R}^2 = 2$

Hence B is basis of R^2.

Co-ordinates of (a,b) —

Let $(a,b) = p(x_1, x_2) + q(y_1, y_2)$

$\Rightarrow \quad a = px_1 + qy_1 \quad \text{—} (3)$

$\quad\quad b = px_2 + qy_2 \quad \text{—} (4)$

Multiplying (3) by x_1 and (4) by x_2 and adding →

$\cancel{p} x_1^2 + px_2^2 + qx_1y_1 + qx_2y_2 = ax_1 + bx_2$

$\Rightarrow p(x_1^2 + x_2^2) + q(x_1y_1 + x_2y_2) = ax_1 + bx_2$

$\Rightarrow p + q \cdot 0 = ax_1 + bx_2$ as $x_1^2 + x_2^2 = 1$, $x_1y_1 + x_2y_2 = 0$

$\Rightarrow p = ax_1 + bx_2$

Similarly $q = ay_1 + by_2$

Therefore co-ordinates of (a,b) in the ordered basis B are $(ax_1 + b\cancel{y_2}, ay_1 + by_2)$

3(b) By using Langrange's Multipliers method. find the max and min value of $f(x,y,z) = x+2y$ subject to the constraints

$$x+y+z=1, \; \& \; y^2+z^2=4$$

Sol^n Consider

$$F = (x+2y) + \lambda(x+y+z-1) + \mu(y^2+z^2-4)$$

$$F_x = F_y = F_z = 0$$

$$\Rightarrow F_x = 1+\lambda = 0 \Rightarrow \lambda = -1$$

$$\left.\begin{aligned} F_y &= 2+\lambda+2\mu y = 0 \\ F_z &= 0+\lambda+2\mu z = 0 \end{aligned}\right] \text{using } \lambda = -1$$

we get $y = -\frac{1}{2\mu}$, $z = \frac{1}{2\mu}$

Putting y, z in $y^2+z^2=4$

$$\Rightarrow \left(\frac{-1}{2\mu}\right)^2 + \left(\frac{1}{2\mu}\right)^2 = 4 \Rightarrow \frac{2}{4\mu^2} = 4$$

$$\Rightarrow \mu = \pm\frac{1}{2\sqrt{2}}$$

Therefore $y = \mp\sqrt{2}$, $z = \pm\sqrt{2}$

$x = 1-y-z = 1$ as $y+z=0$

Points $(1,-\sqrt{2},\sqrt{2})$, $(1,\sqrt{2},-\sqrt{2})$

f at $(1,-\sqrt{2},\sqrt{2}) = 1-2\sqrt{2} \to$ Min

f at $(1,\sqrt{2},-\sqrt{2}) = 1+2\sqrt{2} \to$ Max

3(C) A sphere of constant radius r passes through the origin O and cuts the axes in A, B, C. Prove that the locus of the foot of perpendicular from O to the plane ABC is given by $(x^2+y^2+z^2)(x^{-2}+y^{-2}+z^{-2}) = 4r^2$

Sol^n Let A, B and C be $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$

Then the equation of sphere through O, A, B, C is $x^2+y^2+z^2-ax-by-cz=0$

centre $\left(\frac{a}{2}, \frac{b}{2}, \frac{c}{2}\right)$

radius $= r = \sqrt{\frac{a^2}{4}+\frac{b^2}{4}+\frac{c^2}{4}} \Rightarrow 4r^2 = a^2+b^2+c^2$ —(1)

Equation of plane ABC - $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ —(2)

The equation of the line through O perpendicular to the plane (2)

$$\frac{x}{(1/a)} = \frac{y}{(1/b)} = \frac{z}{(1/c)} \quad -(3)$$

Any point on it $\left(\frac{k}{a}, \frac{k}{b}, \frac{k}{c}\right)$

if it is foot of perpendicular let (x_1, y_1, z_1) then

$x_1 = \frac{k}{a}, \; y_1 = k/b, \; z_1 = k/c$

$\Rightarrow a = k/x_1, \; b = k/y_1, \; c = k/z_1$

From eqn ① →

$$4r^2 = k^2\left(\frac{1}{x_1^2} + \frac{1}{y_1^2} + \frac{1}{z_1^2}\right) \quad \text{—④}$$

From eqn ③ →

$$\frac{x}{(1/a)} = \frac{y}{(1/b)} = \frac{z}{(1/c)} = k$$

$$\Rightarrow \cancel{x^2} \cancel{+y^2}. \; \frac{x^2}{(x/a)} = \frac{y^2}{(y/b)} = \frac{z^2}{(z/c)} = k$$

$$\Rightarrow \frac{x^2+y^2+z^2}{\frac{x}{a}+\frac{y}{b}+\frac{z}{c}} = \frac{k}{1}$$

$$\Rightarrow \frac{x^2+y^2+z^2}{1} = k$$ (From equation of plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

Putting value of k in eqn ④ →

$$4r^2 = (x^2+y^2+z^2)^2\,(x_1^{-2}+y_1^{-2}+z_1^{-2})$$

Locus of (x_1, y_1, z_1)

$$\Rightarrow \boxed{4r^2 = (x^2+y^2+z^2)^2\,(x^{-2}+y^{-2}+z^{-2})}$$

(4)(a) Let T be the linear operator on R^3 defined by $T(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3)$

(i) What is the matrix of T in the standard ordered basis.

Soln Standard Basis

$B = \{\alpha, \beta, \gamma\}$ where $\alpha = (1,0,0)$

$\beta = (0,1,0)$, $\gamma = (0,0,1)$

$T(\alpha) = T(1,0,0) = (3,-2,-1)$

$= 3\alpha - 2\beta - \gamma$

$T(0,1,0) = (0,1,2) = 0.\alpha + 1.\beta + 2.\gamma$

$T(0,0,1) = (1,0,4) = 1.\alpha + 0.\beta + 4.\gamma$

Matrix of T w.r.t B

$$= \begin{bmatrix} 3 & 0 & 1 \\ -2 & 1 & 0 \\ -1 & 2 & 4 \end{bmatrix}$$

4(a)(ii) What is the Matrix of T in the ordered basis $\{\alpha_1, \alpha_2, \alpha_3\}$ where $\alpha_1 = (1,0,1)$ $\alpha_2 = (-1,2,1)$ $\alpha_3 = (2,1,1)$

Sol^n

$$T(\alpha_1) = T(1,0,1) = (4,-2,3)$$
$$= \frac{17}{4}(1,0,1) - \frac{3}{4}(-1,2,1) - \frac{1}{2}(2,1,1)$$
$$T(-1,2,1) = (-2,0,-1)$$
$$= -3/4(1,0,1) + \frac{1}{4}(-1,2,1) - \frac{1}{2}(2,1,1)$$
$$T(2,1,1) = (7,-3,4)$$
$$= \frac{11}{2}(1,0,1) - \frac{3}{2}(-1,2,1) + 0(2,1,1)$$

Matrix of T w.r.t. $\{\alpha_1, \alpha_2, \alpha_3\}$

$$= \begin{bmatrix} 17/4 & -3/4 & 11/2 \\ -3/4 & 1/4 & -3/2 \\ -1/2 & -1/2 & 0 \end{bmatrix}$$

4(a)(iii) Prove that T is invertible and give a rule for T^{-1} like the one which defines T.

Sol^n We know T is invertible iff T is nonsingular

Then if $T(x_1, x_2, x_3) = 0$

$\Rightarrow 3x_1 + x_3 = 0$

$-2x_1 + x_2 = 0$

$-x_1 + 2x_2 + 4x_3 = 0$

Solving these equations we get $x_1 = x_2 = x_3 = 0$

Hence proved

Let $T(x_1, x_2, x_3) = (a, b, c)$

$(3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3) = (a, b, c)$

$\Rightarrow a = 3x_1 + x_3$

$b = -2x_1 + x_2$, $c = -x_1 + 2x_2 + 4x_3$

Solving we get $x_1 = \frac{4a + 2b - c}{9}$, $x_2 = \frac{8a + 13b - 2c}{9}$

$x_3 = \frac{-a - 2b + c}{3}$

Hence $T^{-1}(a, b, c) = \left(\frac{4a + 2b - c}{9}, \frac{8a + 13b - 2c}{9}, \frac{-a - 2b + c}{3}\right)$

4(b) (i) Evaluate: $\iint_D x\sin(x+y)\,dx\,dy$, where D is the region bounded by $0\le x\le \pi$ and $0\le y\le \frac{\pi}{2}$.

(ii) If $w=f[xy/(x^2+y^2)]$ is a differentiable function of $u=xy/(x^2+y^2)$, show that $x\left(\frac{\partial w}{\partial x}\right)+y\left(\frac{\partial w}{\partial y}\right)=0$.

Sol^n: (i) Let $u=x$, $v=x+y$

$$J=\begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y\end{vmatrix}=\begin{vmatrix}1 & 0\\ 1 & 1\end{vmatrix}$$

$$=1\neq 0$$

Since $J\neq 0$, this transformation is bijective

$0\le x\le \pi \Rightarrow 0\le u\le \pi$

$0\le y\le \pi/2 \Rightarrow 0\le v-u\le \pi/2$

$$\iint_D x\sin(x,y)\,dx\,dy=\iint_{D'} u\sin v\,du\,dv$$

$$=\int_{u=0}^{\pi}\int_{v=u}^{u+\pi/2} u\sin v\,du\,dv$$

$$=\int_0^{\pi} u(-\cos v)_u^{u+\pi/2}\,du$$

$$=\int_0^{\pi} u[\cos u-\cos(\pi/2+4)]\,du$$

$$=\int_0^{\pi}(u\cos u+u\sin u)\,du=I$$

$$I=\int_0^{\pi}[(\pi-u)\cos(\pi-u)+(\pi-u)\sin(\pi-u)]\,du$$

$$=\int_0^{\pi}[\pi(\sin u-\cos u)+u\cos u-u\sin u]\,du$$

$$2I = \left[\pi(\sin u - \cos u) + 2u\sin u + 2\cos u\right]_0$$

$$= (\pi-2) - (-\pi+2) = 2(\pi-2)$$

$$\therefore I = \pi - 2$$

4(b)(ii). Given $w = f(u)$, where $u = \frac{xy}{x^2+y^2}$

from the chain rule, we have

$$\frac{\partial w}{\partial x} = \frac{dw}{du}\frac{\partial u}{\partial x}$$

$$= f'(u) \cdot y\left[\frac{x^2+y^2-(2x)x}{(x^2+y^2)^2}\right]$$

$$= f'(u)\, y\left[\frac{y^2-x^2}{(x^2+y^2)^2}\right]$$

$$\& \quad \frac{\partial w}{\partial y} = \frac{dw}{du}\frac{\partial u}{\partial y}$$

$$= f'(u) \cdot x\left[\frac{x^2+y^2-(2y)y}{(x^2+y^2)^2}\right]$$

$$= f'(u)\, x\left[\frac{x^2-y^2}{(x^2+y^2)^2}\right]$$

$$\therefore x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = \frac{f'(u)xy}{(x^2+y^2)^2}\left[\cancel{y^2} - \cancel{x^2} + \cancel{x^2} - \cancel{y^2}\right] = 0$$

$$\therefore x\frac{\partial w}{\partial x} + y\frac{\partial w}{\partial y} = 0.$$

(4)(c) Prove that the equation

$2y^2+4zx+2x-4y+6z+5=0$

represents a right circular cone. Show also that the semivertical angle of this cone is $\pi/4$ and its axis is given by $x+z+2=0$, $y=1$

Soln: The discriminating cubic is

$$\begin{vmatrix} a-d & h & g \\ h & b-d & f \\ g & f & c-d \end{vmatrix}=0 \text{ or } \begin{vmatrix} 0-d & 0 & 2 \\ 0 & 2-d & 0 \\ 2 & 0 & -d \end{vmatrix}=0 \quad -(1)$$

$\Rightarrow (-d)[(-d)(2-d)]+2[-2(2-d)]=0$

$\Rightarrow (2-d)(d^2-4)=0 \Rightarrow d=2,2,-2$

As two roots of this cubic are equal and third is not zero, so the given surface is a surface of revolution.

Also the line of centres is given by any two of $\frac{\partial F}{\partial x}=0, \frac{\partial F}{\partial y}=0, \frac{\partial F}{\partial z}=0$

i.e. $4z+2=0,\ 4y-4=0,\ 4x+6=0$

$\therefore$ If (α, β, γ) be any point on the lines of centres, then

$4\gamma + 2 = 0$, $4\beta - 4 = 0$, $4\alpha + 6 = 0$

$\Rightarrow \alpha = -3/2$, $\beta = 1$, $\gamma = -1/2$

$\therefore$ Any point on the line of centres is $(-3/2, 1, -1/2)$

$\therefore d' = u\alpha + v\beta + w\gamma + d = 1(-3/2) + 2(1) + 3(-1/2) + 5$
$= 0$

$\therefore$ The reduced form of the equation is $d_1 x^2 + d_2 y^2 + d_3 z^2 + d' = 0$ or

$2x^2 + 2y^2 - 2z^2 + 0 = 0 \Rightarrow x^2 + y^2 = z^2$

$\Rightarrow x^2 + y^2 = z^2 \tan^2 45^\circ$ which represents the right circular cone of semi vertical angle $\pi/4$.

Now putting the unequal value of d viz -2 in the determinant of (1) and associating each row with l, m, n we have $2l + 2n = 0$, $4m = 0$, $2l + 2n = 0 \Rightarrow \frac{l}{1} = \frac{m}{0} = \frac{n}{-1} = \frac{1}{\sqrt{2}}$

$\therefore$ Eq^n of axes are

$\frac{x - (-3/2)}{l} = \frac{y - 1}{m} = \frac{z - (-1/2)}{n}$ or $\frac{x + 3/2}{1} = \frac{y - 1}{0} = \frac{z + 1/2}{-1}$

$\Rightarrow -x - 3/2 = z + 1/2$, $y - 1 = 0$

$\Rightarrow x + z + 2 = 0$; $y = 1$. Hence proved.

(5)(a) compute A^{25} if $A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 2 \\ 2 & 1 & -1 \end{bmatrix}$

and verify cayley Hamilton theorem.

Ans. Eigen values - $|A - \lambda I| = 0$

$$\Rightarrow \left| \begin{bmatrix} 1-\lambda & 1 & 0 \\ -1 & 1-\lambda & 2 \\ 2 & 1 & -1-\lambda \end{bmatrix} \right| = 0$$

solving - $\lambda^3 - \lambda^2 - 2\lambda = 0$ — (1)

we know by caley Hamilton theorem every square matrix satisfy its characteristic equation.

Therefore $A^3 - A^2 - 2A = 0$

$$A^3 - A^2 - 2A = \begin{bmatrix} 2 & 4 & 2 \\ 0 & 4 & 4 \\ 3 & 4 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 2 & 2 \\ 2 & 2 & 0 \\ -1 & 2 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 2 & 0 \\ -2 & 2 & 4 \\ 4 & 2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = RHS$$

Hence proved

$A^{25} \rightarrow$

We know if A is diagonalizable

then $\exists$ ~~to~~ Invertible matrix P

s.t. $D = P^{-1}AP$

Then $D^n = (P^{-1}AP)^n = (P^{-1}AP)(P^{-1}AP)\cdots(P^{-1}AP)$

$$= P^{-1}A^nP$$

$$\Rightarrow \boxed{A^n = PD^nP^{-1}}$$

Using Equation (1) :-

$$\lambda^3 - \lambda^2 - 2\lambda = 0$$

$$\Rightarrow \lambda(\lambda^2 - \lambda - 2) = 0$$

$$\Rightarrow \lambda(\lambda^2 - 2\lambda + \lambda - 2) = 0$$

$$\Rightarrow \lambda(\lambda - 2)(\lambda + 1) = 0$$

$$\Rightarrow \lambda = 0, 2, -1$$

As all three eigen value is different therefore A is Diagonalizable.

Eigen vector for $\underline{\lambda = 0} \Rightarrow AV_1 = 0$

Solving we get $v_1 = \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$

Similarly for $\lambda = 2 \rightarrow v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and for

$\lambda = -1 \rightarrow v_3 = \begin{bmatrix} -1/2 \\ 1 \\ -5/4 \end{bmatrix}$

Then $P = \begin{bmatrix} -1 & 1 & -1/2 \\ 1 & 1 & 1 \\ 1 & 1 & -5/4 \end{bmatrix} \Rightarrow P^{-1} = \begin{bmatrix} -3/2 & 1/2 & 1 \\ 1/6 & 1/2 & 1/3 \\ 4/3 & 0 & -4/3 \end{bmatrix}$

$A^{25} = P D^{25} P^{-1}$ where $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

$$= \begin{bmatrix} -1 & 1 & -1/2 \\ 1 & 1 & 1 \\ 1 & 1 & -5/4 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2^{25} & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} -3/2 & 1/2 & 1 \\ 1/6 & 1/2 & 1/3 \\ 4/3 & 0 & -4/3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2^{25} & 1/2 \\ 0 & 2^{25} & -1 \\ 0 & 2^{25} & 5/4 \end{bmatrix} \begin{bmatrix} -3/2 & 1/2 & 1 \\ 1/6 & 1/2 & 1/3 \\ 4/3 & 0 & -4/3 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2^{24}+2}{3} & 2^{24} & \frac{2^{24}-2}{3} \\ \frac{2^{24}-4}{3} & 2^{24} & \frac{2^{25}-4}{3} \\ \frac{2^{24}+5}{3} & 2^{24} & \frac{2^{25}-5}{3} \end{bmatrix}$$

(5)(b) Choose the second row of $A = \begin{bmatrix} 0 & 1 \\ * & * \end{bmatrix}$ so that A has Eigenvalue 4 & 7.

Ans. Let $A = \begin{bmatrix} 0 & 1 \\ a & b \end{bmatrix}$

We know from matrix –

Trace A = Sum of Eigen values

Determinant of A = Multiplication of Eigen values

Therefore

$$0 + b = 4 + 7 \Rightarrow \boxed{b = 11}$$

$$|A| = -a = 4 \times 7 \Rightarrow \boxed{a = -28}$$

Matrix $A = \begin{bmatrix} 0 & 1 \\ -28 & 11 \end{bmatrix}$

5(C) If $u = \tan^{-1}\left(\frac{x^3+y^3}{x-y}\right)$, show that

$$x^2\left(\frac{\partial^2 u}{\partial x^2}\right) + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (1-4\sin^2 u)\sin 2u.$$

Sol^n: we have $\tan u = \frac{x^3+y^3}{x-y} = \frac{x^2\left[1+(y/x)^2\right]}{\left[1-y/x\right]}$

Let $z = \tan u$ so that z is a homogeneous function of x and y of degree 2. By Euler's theorem,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$$

$$\Rightarrow x\sec^2 u\frac{\partial u}{\partial x} + y\sec^2 u\frac{\partial u}{\partial y} = 2\tan u$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{2\tan u}{\sec^2 u} = \sin 2u \quad \text{--- (1)}$$

Partially differentiating (1) w.r.t x on bothsides

$$\frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = 2\cos 2u\frac{\partial u}{\partial x}$$

$$\Rightarrow x\frac{\partial u}{\partial x} + x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x \partial y} = 2x\cos 2u\frac{\partial u}{\partial x} \quad \text{--- (2)}$$

Similarly differentiating (1) partially w.r.t y and multiplying by y and multiplying by y on bothsides, we get

$$y\frac{\partial u}{\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + xy\frac{\partial^2 u}{\partial x \partial y} = 2y\cos 2u\frac{\partial u}{\partial y} \quad \text{--- (3)}$$

Adding (2) and (3), we obtain

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}$$

$$= 2\cos 2u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$$

using ①, we obtain

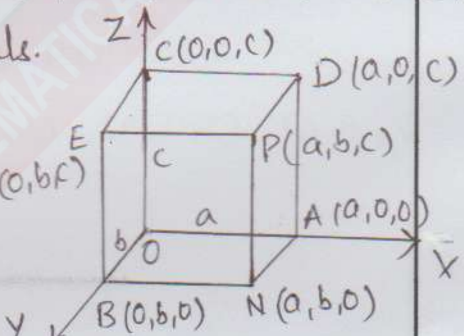
$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+\sin 2u=2\sin 2u\cos 2u \quad \text{—④}$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\sin 2u(2\cos 2u-1)$$
$$=\{2(1-2\sin^2 u)-\}\sin 2u$$
$$=(1-4\sin^2 u)\sin 2u$$

5(d) The edges of a rectangular parallelopiped are a, b, c show that the angles between the four diagonals are given by $\cos^{-1}=\frac{a^2\pm b^2\pm c^2}{a^2+b^2+c^2}$.

Sol^n: Let one corner O of the rectangular parallelopiped be taken as origin and the three edges OA, OB, OC be taken as coordinate axes. Let $OA=a$, $OB=b$ & $OC=c$. Then the coordinates of O, A, B, C, D, P, N and E are respectively $(0,0,0)$, $(a,0,0)$, $(0,b,0)$, $(0,0,c)$, $(a,0,c)$, (a,b,c), $(a,b,0)$ & $(0,b,c)$.

Here OP, CN, AE & BD are the diagonals.



The dr's of OP are

$a-0,\ b-0,\ c-0$ i.e. a, b, c

Again d.r's of CN are

$a-0,\ b-0,\ 0-c$ ie. $a, b, -c$.

Similarly we can find that the direction ratios of AE & BD are $-a, b, c$ & $a, -b, c$ respectively.

If θ be the angle b/w OP & CN, then

$$\cos\theta=\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}=\frac{a.a+b.b+c(-c)}{\sqrt{(a^2+b^2+c^2)}\sqrt{a^2+b^2+(-c)^2}}$$
$$=\frac{a^2+b^2-c^2}{a^2+b^2+c^2}$$

Similarly we can find the angle between other pairs of diagonals & we have six such pairs out of these 4 diagonals & all these angles are given by $\cos^{-1}\left[\frac{\pm a^2\pm b^2\pm c^2}{a^2+b^2+c^2}\right]$. Hence proved.

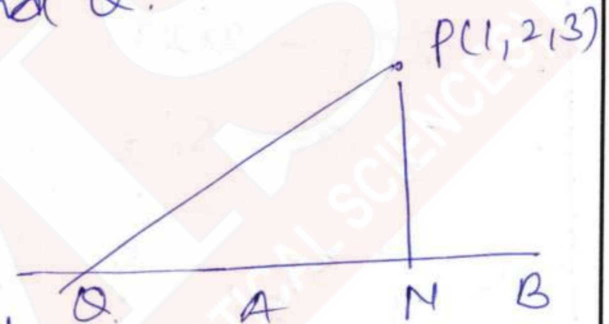
5(e) The equation of AB are $\frac{x}{1}=\frac{y}{-2}=\frac{z}{3}$

Through a point P(1, 2, 3).

PN is drawn perpendicular to AB, and PQ is drawn parallel to the plane $2x+3y+4z=0$ to meet AB in Q. Find the equation of PN and PQ and the co-ordinates of N and Q.

Sol^n

Let point N on AB is $(r,-2r,3r)$



Direction ratios of PN – $(r-1, -2r-2, 3r-3)$

PN is perpendicular to AB so

$1\cdot(r-1)-2(-2r-2)+3(3r-3)=0$

$\Rightarrow 14r=6 \Rightarrow r=3/7$

$N\left(\frac{3}{7}, -\frac{6}{7}, \frac{9}{7}\right)$

Equation of PN $\frac{x-1}{3/7-1}=\frac{y-2}{-6/7-2}=\frac{z-3}{9/7-3}$

$$\Rightarrow \frac{x-1}{-4/7}=\frac{y-2}{-20/7}=\frac{z-3}{-\frac{12}{7}}$$

Equation of plane PQ.

$\Rightarrow \frac{x-1}{1} = \frac{y-2}{5} = \frac{z-3}{3}$ [as PQ is parallel to given plane]

It passes through

$P(1,2,3) \Rightarrow p = 2\cdot1 + 3\cdot2 + 4\cdot3 = 20$

$\Rightarrow 2x+3y+4z = 20$

it meets to line AB at $Q(r_1, -2r_1, 3r_1)$

So Q lies on plane

$2r_1 - 3\times2r_1 + 4\times3r_1 = 20$

$\Rightarrow r_1 = 5/2$

$Q\left(5/2, -5, 15/2\right)$

Equation of PQ $\rightarrow$

$$\frac{x-1}{5/2-1} = \frac{y-2}{-5-2} = \frac{z-3}{15/2-3}$$

$$\Rightarrow \boxed{\frac{x-1}{3} = \frac{y-2}{-14} = \frac{z-3}{9}}$$

6(a). When is a matrix A said to be similar to another matrix B? Prove that

(i) If A is similar to B, then B is similar to A.

(ii) Two similar matrices have the same eigenvalues.

Further, by choosing appropriately the matrices A and B, show that the converse of (ii) above may not be true.

Sol: Let A and B be two square matrices of same order 'n'.

Then A is said to be similar B if $\exists$ $n\times n$ invertible matrix 'C' such that $A = C^{-1}BC$ ie $CA = BC$.

(i) Let A be similar to B.

Then $A = C^{-1}BC$; C is non-singular

$\Rightarrow CA = BC$.

$\Rightarrow CAC^{-1} = B$

$\Rightarrow B = (C^{-1})^{-1}AC^{-1}$

$\Rightarrow B = P^{-1}AP$; $P = C^{-1}$

$\Rightarrow$ B is similar A

(ii) Let us prove that two similar matrices have the same eigen values.

Let A and B be two similar matrices.

Then $\exists$ an invertible matrix P s.t

$$B = P^{-1}AP$$

We have $B - \lambda I = P^{-1}AP - \lambda I$

$$= P^{-1}AP - \lambda P^{-1}P$$

$$= P^{-1}AP - P^{-1}\lambda P$$

$$= P^{-1}(A - \lambda I)P$$

$$\therefore |B - \lambda I| = |P^{-1}(A - \lambda I)P|$$

$$= |P^{-1}||A - \lambda I||P|$$

$$= |A - \lambda I|\,|P^{-1}||P|$$

$$= |A - \lambda I|\,|P^{-1}P|$$

$$\therefore \boxed{|B - \lambda I| = |A - \lambda I|}$$

$\Rightarrow$ A and B have the same characteristic polynomial, and hence ~~Hence same~~ same characteristic roots i.e same eighen values.

But the converse of the above may not be true.

i.e If two matrices (of same order) have same characteristic roots then it is not necessary that they are similar.

for example:

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & -1 \\ -3 & -2 & 3 \end{bmatrix}$$

have the same characteristic roots but are not similar.

(6)(b) Let $S = \{(2,5,-3,-2), (-2,-3,2,-5)$
$(1,3,-2,2)\ (-1,-5,3,5)\} \subseteq \mathbb{R}^4$
Find the bases of the subspace spanned by S.

Soln consider - $S = \begin{bmatrix} 2 & 5 & -3 & -2 \\ -2 & -3 & 2 & -5 \\ 1 & 3 & -2 & 2 \\ -1 & -5 & 3 & 5 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_3}$

$$= \begin{bmatrix} 1 & 3 & -2 & 2 \\ -2 & -3 & 2 & -5 \\ 2 & 5 & -3 & -2 \\ -1 & -5 & 3 & 5 \end{bmatrix} \xrightarrow[\substack{R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 + R_1}]{R_2 \to R_2 + 2R_1} \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 3 & -2 & -1 \\ 0 & -1 & 1 & -6 \\ 0 & -2 & 1 & 7 \end{bmatrix}$$

$$\xrightarrow[R_4 \to R_4 - 2R_3]{R_2 \to R_2 + 3R_3} \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & 0 & 1 & -19 \\ 0 & -1 & 1 & -6 \\ 0 & 0 & -1 & 19 \end{bmatrix}$$

$$\xrightarrow[\substack{R_4 \to R_4 + R_2 \\ R_2 \leftrightarrow R_3}]{} \begin{bmatrix} 1 & 3 & -2 & 2 \\ 0 & -1 & 1 & -6 \\ 0 & 0 & 1 & -19 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$ which is row reduced echelon form.

Therefore Basis of S
$= \{(1,3,-2,2)\ (0,-1,1,-6)\ (0,0,1,-19)\}$

(6)(c) Give an example to show that the eigen values can be changed when a multiple of one row is subtracted from another. Why is zero eigenvalue not changed by the steps of elimination.

Ans Let Matrix $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

Eigenvalues of A $\rightarrow |A - d I| = 0$

$$\begin{vmatrix} 2-d & 1 \\ 1 & 2-d \end{vmatrix} = 0 \Rightarrow (2-d)^2 - 1 = 0$$

$$\Rightarrow d^2 - 4d + 3 = 0 \Rightarrow d_1 = 3, d_2 = 1$$

Now let's make some Row transformation in A $(R_1 \rightarrow R_1 - 2R_2)$ then A will become -

$$A' = \begin{bmatrix} 0 & -3 \\ 1 & 2 \end{bmatrix}$$

$$|A' - d I| = 0 \Rightarrow \begin{vmatrix} 0-d & -3 \\ 1 & 2-d \end{vmatrix} = 0$$

$$(-d)(2-d) + 3 = 0 \Rightarrow d^2 - 2d + 3 = 0$$

$$d = 1 + i\sqrt{2}, 1 - i\sqrt{2}$$

$\Rightarrow$ Hence eigen values changed.

2nd part

zero is eigen value of some matrix iff that matrix have the same values in all rows or all rows is product of one row, which means, if Matrix

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$ and zero is eigen value of matrix A, then one row is product of another

$$a_{11} = c \cdot a_{21}, \quad a_{12} = c \cdot a_{22}$$

where c is constant

$\Rightarrow$ Because of that, if zero eigen value of some matrix, after any transformation eigen value would not change. That matrix is singular, which means, determinant of that matrix is equal zero.

Example: $B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$

Eigenvalues $|B - \lambda I| = 0$

$$\begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^2 - 5\lambda = 0$$

$$\Rightarrow \lambda = 0, 5$$

After transformation

$$\xrightarrow{R_1 \to R_1 - \frac{1}{2}R_2} \begin{bmatrix} 0 & 0 \\ 2 & 4 \end{bmatrix} = B'$$

Eigenvalues $- |B' - \lambda I| = 0$

$$\begin{vmatrix} -\lambda & 0 \\ 2 & 4-\lambda \end{vmatrix} = 0 \Rightarrow \lambda(\lambda - 4) = 0$$

$$\lambda = 0, 4$$

Therefore, after transformation eigenvalue zero is not changed

7(a) (i) Show that $x-\frac{x^2}{2} < \log(1+x) < x-\frac{x^2}{2(1+x)}$, $x>0$.

(ii) Evaluate $\int_0^{\pi} \frac{x\tan x}{\sec x+\tan x}\,dx$.

Sol^n: (i) Let $f(x) = x-\frac{x^2}{2}-\log(1+x)$

$$\therefore f'(x) = 1-x-\frac{1}{1+x} = \frac{1-x^2-1}{1+x}$$

$$= -\frac{x^2}{1+x} < 0 \text{ for } x>0$$

$\Rightarrow f(x)$ is monotonic decreasing for $x>0$,

$\Rightarrow f(x) < f(0)$

But $f(0) = 0-0-\log 1 = 0$

$\therefore f(x)<0 \Rightarrow x-\frac{x^2}{2}-\log(1+x)<0$

$$\Rightarrow x-\frac{x^2}{2} < \log(1+x)$$

Now let $g(x) = \log(1+x)-x+\frac{x^2}{2(1+x)}$

$$\therefore g'(x) = \frac{1}{1+x}-1+\frac{1}{2}\cdot\frac{(1+x).2x-x^2}{(1+x)^2}$$

$$= \frac{1-1-x}{1+x}+\frac{1}{2}\cdot\frac{2x+x^2}{(1+x)^2}$$

$$= -\frac{x}{1+x}+\frac{2x+x^2}{2(1+x)^2}$$

$$= \frac{-2x(1+x)+2x+x^2}{2(1+x)^2}$$

$$= -\frac{x^2}{2(1+x)} < 0 \quad \text{for } x>0$$

$\Rightarrow g(x)$ is monotonic decreasing for $x>0$

$\Rightarrow g(x) < g(0)$

But $g(0)=0$ $\therefore g(x)<0$

$$\Rightarrow \log(1+x)-x+\frac{x^2}{2(1+x)}<0$$

$$\therefore \log(1+x)<x-\frac{x^2}{2(1+x)} \quad \text{———} \; (2)$$

Combining (1) & (2)

$$x-\frac{x^2}{2}<\log(1+x)<x-\frac{x^2}{2(1+x)}$$

(ii) Let $I=\int_0^{\pi}\frac{x\tan x}{\sec x+\tan x}dx$

$$\therefore I=\int_0^{\pi}\frac{(\pi-x)\tan(\pi-x)}{\sec(\pi-x)+\tan(\pi-x)}dx$$

$$=\int_0^{\pi}\frac{(\pi-x)\tan x\,dx}{\sec x+\tan x}$$

Adding the two integrals, we obtain

$$2I=\pi\int_0^{\pi}\frac{\tan x}{\sec x+\tan x}dx=\pi\int_0^{\pi}\frac{(\sec x+\tan x)-\sec x}{\sec x+\tan x}dx$$

$$=\pi\int_0^{\pi}1.dx-\pi\int_0^{\pi}\frac{\sec x}{\sec x+\tan x}dx$$

Put $\sec x+\tan x=t \Rightarrow (\sec x\tan x+\sec^2 x)dx=dt$

$\Rightarrow \sec x(\tan x+\sec x)dx=dt \Rightarrow \sec x\,dx=\frac{dt}{t}$

$$\therefore 2I=\pi^2-\pi\int_{t=1}^{-1}\frac{dt}{t^2}=\pi^2+\pi\int_{-1}^{1}\frac{dt}{t^2}=\pi^2+\pi\left|-\frac{1}{t}\right|_{-1}^{1}$$

$$=\pi^2-2\pi$$

Hence $I=\frac{\pi^2}{2}-\pi$

7(b) By using Lagrange multipliers method. Prove that the rectangular solid of maximum volume which can be inscribed in a sphere is a cube.

Sol^n: Referred to the centre as origin, let the equation of the sphere be $x^2+y^2+z^2=a^2$.

Let (x,y,z) denote the coordinates of that vertex of the rectangular parallelopiped inscribed in the sphere which lies in the positive octant and let V denote the volume of the rectangular parallelopiped. Then, we have to find the maximum value of

$$V = 8xyz \quad \text{——— (1)}$$

Subject to the condition

$$x^2+y^2+z^2=a^2 \quad \text{——— (2)}$$

from (1), $\log V = \log 8 + \log x + \log y + \log z$.

$$\therefore \frac{1}{V}dV = \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz$$

For a maximum or a minimum of V, we must have $dV=0$.

$$\Rightarrow \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0 \quad \text{——— (3)}$$

Also differentiating (2), we have

$$x\,dx + y\,dy + z\,dz = 0 \quad \text{——— (4)}$$

Multiplying (3) by 1 and (4) by λ, and adding and then equating to zero the coefficients of dx, dy, dz, we get

$$\frac{1}{x}+\lambda x=0 \quad , \quad \frac{1}{y}+\lambda y=0 \, , \quad \frac{1}{z}+\lambda z=0$$

$$\Rightarrow -\frac{1}{\lambda}=x^2=y^2=z^2 \quad \text{(or)} \quad x=y=z$$

Thus V is stationary when $x=y=z=a/\sqrt{3}$, from (2)

The lengths of the edges of the rectangular Parallelopiped are $2x, 2y, 2z$. So V is stationary when the rectangular parallelopiped is a cube.

Now regard x and y as independent variables and z as a function of x and y given by (2).

From (1), $\log V=\log 8+\log x+\log y+\log z$

$$\therefore \frac{1}{V}\frac{\partial V}{\partial x}=\frac{1}{x}+\frac{1}{z}\cdot\frac{\partial z}{\partial x}$$

Differentiating (2) partially w.r.t x taking y as constant, we get $2x+2z\left(\partial z/\partial x\right)=0 \Rightarrow \frac{\partial z}{\partial x}=-x/z$.

$$\therefore \frac{1}{V}\frac{\partial V}{\partial x}=\frac{1}{x}+\frac{1}{z}\cdot\frac{-x}{z}=\frac{1}{x}-\frac{x}{z^2}$$

So that
$$\frac{1}{V}\frac{\partial^2 V}{\partial x^2}-\frac{1}{V^2}\left(\frac{\partial V}{\partial x}\right)^2=-\frac{1}{x^2}-\frac{1}{z^2}+\frac{2x}{z^3}\frac{\partial z}{\partial x}$$
$$=-\frac{1}{x^2}-\frac{1}{z^2}-\frac{2x^2}{z^4}.$$

But at the stationary point, we have $\frac{\partial V}{\partial x}=0$.

$\therefore$ at the stationary point found above, we have

$$\frac{\partial^2 V}{\partial x^2}=-V\left[\frac{1}{x^2}+\frac{1}{z^2}+\frac{2x^2}{z^4}\right]=-8xyz\left[\frac{1}{x^2}+\frac{1}{z^2}+\frac{2x^2}{z^4}\right],$$

which is $-$ve when $x=y=z=a/\sqrt{3}$.

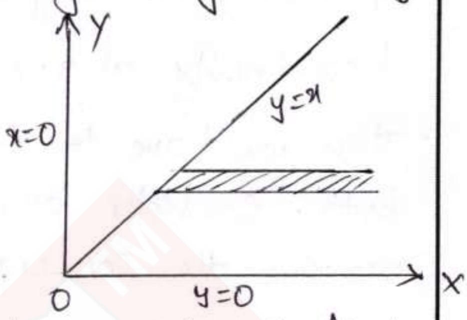
Thus V is maximum when $x=y=z=a/\sqrt{3}$.

Hence the rectangular solid of maximum volume inscribed in a sphere is a cube.

7(c) Evaluate the integral $\int_0^\infty \int_0^x x e^{-x^2/y}\, dx dy$ by changing the order of integration.

Sol'n: The limits of integration are given by the straight line $y=x$, $y=0$, $x=0$ and $x=\infty$.

i.e., the region of integration is bounded $y=0$, $y=x$ and infinite boundary.



Hence taking the strips parallel to x-axis, the limits for y are from 0 to ∞. and the limits for x are from $x=y$ to $x=\infty$.

Hence changing the order of integration, we are

$$\int_0^\infty \int_0^x x e^{-x^2/y}\, dy dx = \int_{y=0}^\infty \int_{x=y}^\infty x e^{-x^2/y}\, dx dy$$

$$= -\int_{y=0}^\infty \int_{-y}^{y} \frac{y}{2} e^t\, dt dy \qquad \frac{-x^2}{y} = t$$

$$\frac{-2x\,dx}{y} = dt$$

$$x\,dx = \frac{-y}{2}\,dt$$

where $x=y$

$$\frac{-y^2}{y} = t \Rightarrow t$$

$$= -\int_0^\infty \frac{y}{2} \left[e^t\right]_{-y}^{\infty} dy$$

$$= -\int_0^\infty \frac{y}{2}\left[0 - e^{-y}\right] dy$$

$$= \int_0^\infty \frac{y}{2} e^{-y}\, dy = \frac{1}{2}\int_0^\infty y e^{-y}\, dy$$

$$= \frac{1}{2}\left[-y(e^{-y}) - \int_0^\infty e^{-y}(-dy)\right]_0^\infty$$

$$= \frac{1}{2}\left[-ye^{-y} + \int_0^\infty e^{-y}\, dy\right]_0^\infty$$

$$= \frac{1}{2}\left[-ye^{-y} - e^{-y}\right]_0^\infty = \frac{1}{2}\left[0 - (0-1)\right]$$

$$= \frac{1}{2}$$

7(d) Show that $\int_0^\infty x^{n-1}e^{-x}\,dx$ converges iff $n>0$.

Sol^n: If $n\geq 1$, the integrated $x^{n-1}e^{-x}$ is continuous at $x=0$.
If $n<1$, the integrated $\frac{e^{-x}}{x^{1-n}}$ has infinite discontinuity at $x=0$.

Thus we have to examine the convergence at 0 to ∞ both. Consider any positive number, say 1, and examine the convergence of

$\int_0^1 x^{n-1}e^{-x}\,dx$ and $\int_1^\infty x^{n-1}e^{-x}\,dx$ at 0 and ∞ respectively.

Convergence at 0, when $n<1$

Let $f(x)=\frac{e^{-x}}{x^{1-n}}$

Take $g(x)=\frac{1}{x^{1-n}}$

Then $\lim_{x\to 0+}\frac{f(x)}{g(x)}=\lim_{x\to 0+}e^{-x}=1$ which is non-zero, finite.

Also $\int_0^1 g(x)\,dx=\int_0^1\frac{dx}{x^{1-n}}$ is convergent iff $1-n<1$ i.e. $n>0$.

$\therefore$ By comparison test

$$\int_0^1 f(x)\,dx=\int_0^1\frac{e^{-x}}{x^{1-n}}\,dx=\int_0^1 x^{n-1}e^{-x}\,dx$$

is convergent at $x=0$ if $n>0$.

Convergence at ∞

We know that $e^x>x^{n+1}$ whatever value n may have

$\therefore e^{-x}<x^{-n-1}$

and $x^{n-1}e^{-x} < x^{n-1}\cdot x^{n-1} = \frac{1}{x^2}$

Since $\int_1^\infty \frac{1}{x^2}dx$ is convergent at ∞

$\therefore \int_1^\infty x^{n-1}e^{-x}\,dx$ is convergent at ∞ for every value of n.

Now $\int_0^\infty x^{n-1}e^{-x}\,dx = \int_0^1 x^{n-1}e^{-x}\,dx + \int_1^\infty x^{n-1}e^{-x}\,dx$

$\therefore \int_0^\infty x^{n-1}e^{-x}\,dx$ converges iff $n>0$.

(8)(a) Obtain the equations of sphere which pass through the circle $y^2+z^2=4, x=0$ and are cut by the plane $2x+2y+z=0$ in a circle of radius 3.

Ans Equation of any sphere through the given circle is

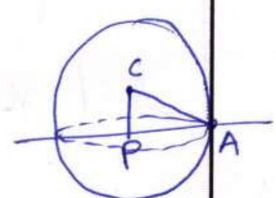
$$(x^2+y^2+z^2-4)+dx=0$$ [Note x^2 term addition]

or $x^2+y^2+z^2+dx-4=0$

Its centre (C) $\left(-d/2, 0, 0\right)$ radius (CA) $=\sqrt{\frac{d^2}{4}+4}$

Given radius of circle

$PA = 3$



CP = Perpendicular distance from C to plane $2x+2y+z=0$

$$=\left|\frac{2(-d/2)+2(0)+0}{\sqrt{4+4+1}}\right| = -d/3$$

Therefore → $CA^2 = PA^2 + CP^2$

$$\left(\frac{d^2}{4}+4\right) = \frac{d^2}{9}+5 \Rightarrow \frac{5}{36}d^2 = 5$$

$\Rightarrow d^2 = 36$ or $d = \pm 6$

Sphere required : $\boxed{x^2+y^2+z^2\pm 6x-4=0}$

(8)(b) Find the equation to the cylinder whose generators are parallel to the line $\frac{x}{1}=\frac{y}{-2}=\frac{z}{3}$ and the guiding curve is the ellipse $x^2+2y^2=1, z=3$

Sol^n Let $P(x_1, y_1, z_1)$ be any point on the cylinder the eq^n of generator through P is
$$\frac{x-x_1}{1}=\frac{y-y_1}{-2}=\frac{z-z_1}{3}$$
And this generator meets the plane $z=2$ in the point given by-
$$\frac{x-x_1}{1}=\frac{y-y_1}{-2}=\frac{3-z_1}{3} \quad (\because z=3)$$
$$\Rightarrow \left(x_1+\left(\frac{3-z_1}{3}\right), -\frac{2(3-z_1)}{3}+y_1, 3\right)$$
Also this generator intersect the conic so
$$\left[x_1+\left(\frac{3-z_1}{3}\right)\right]^2+2\left[y_1-2\left(\frac{3-z_1}{3}\right)\right]^2=1$$
[Guiding curve $x^2+2y^2=1$]
$$\Rightarrow \left(x_1-z_1/3+1\right)^2+2\left(y_1+\frac{2z_1}{3}-2\right)^2=1$$
$$\Rightarrow x_1^2+\frac{z_1^2}{9}+1+2x_1-2z_1/3-\frac{2x_1z_1}{3}+2y_1^2+\frac{8z_1^2}{9}$$
$$+8+\frac{8}{3}y_1z_1-8y_1-\frac{16}{3}z_1=1$$
Solving we get eq^n of cylinder $\rightarrow$ (locus of (x_1, y_1, z_1))
$$x^2+2y^2+z^2-\frac{2}{3}xz+8/_3\, yz+2x-8y-6z+8=0$$

8(c) Find the equations of the tangent planes to $7x^2+5y^2+3z^2=60$ which pass through the line $7x+10y=30$, $5y-3z=0$

Ans Any plane through the given line

$(7x+10y-30)+d(5y-3z)=0$

If it touches the conicoid

$7x^2+5y^2+3z^2=60$

then using condition of tangency

$$\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p^2$$

where plane $\to lx+my+nz=p$

conic $\to ax^2+by^2+cz^2=1$

Therefore $\frac{60(7)^2}{7}+\frac{60(10+5d)^2}{5}+\frac{60(-3d)^2}{3}=30^2$

$\Rightarrow 420+12(100+100d+25d^2)+20(9d^2)=900$

$\Rightarrow 2d^2-5d+3=0 \Rightarrow d=-1,-3/2$

Required tangent planes are

$d=-1 \Rightarrow 7x+5y+3z=30$

$d=-3/2 \Rightarrow 14x+10y+9z=50$

8(d) Find the locus of the point of intersection of perpendicular generators of a hyperboloid of one sheet.

Ans Hyperboloid of one sheet

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

Generator of hyperboloid of λ-system –

$$\left(\frac{x}{a}-\frac{z}{c}\right)=\lambda\left(1-y/b\right) \text{ and } \left(\frac{x}{a}+\frac{z}{c}\right)=\frac{1}{\lambda}\left(1+y/b\right)$$

$$\Rightarrow \frac{x}{a}+\frac{\lambda y}{b}-\frac{z}{c}=\lambda \;\&\; \lambda\frac{x}{a}-y/b+\lambda z/c=1 \quad —(1)$$

If l_1, m_1, n_1 be the d.r's of generator (1) then

$$\frac{l_1}{a}+\frac{\lambda m_1}{b}-\frac{n_1}{c}=0, \quad \frac{\lambda l_1}{a}-\frac{m_1}{b}+\lambda\frac{n_1}{c}=0$$

solving these –

$$\frac{l_1/a}{\lambda^2-1}=\frac{m_1/b}{-\lambda-\lambda}=\frac{n_1/c}{-1-\lambda^2}$$

$$\Rightarrow \frac{l_1}{-a(\lambda^2-1)}=\frac{m_1}{2\lambda b}=\frac{n_1}{c(1+\lambda^2)} \quad —(2)$$

Similarly the direction ratios of l_2, m_2, n_2 of generator belonging to

μ system that is

$$\left(\frac{x}{a}-\frac{z}{c}\right)=\mu\left(1+\frac{y}{b}\right) \;\&\; \left(\frac{x}{a}+\frac{z}{c}\right)=\frac{1}{\mu}\left(1-\frac{y}{b}\right)$$

are given by

$$\frac{l_2}{a(\mu^2-1)}=\frac{m_2}{2b\mu}=\frac{n_2}{-c(\mu^2+1)} \quad -(3)$$

If these two generator are perpendicular then –

$$-a^2(\lambda^2-1)(\mu^2-1)+4b^2\lambda\mu-c^2(1+\lambda^2)(1+\mu^2)=0$$

$$\Rightarrow a^2(\lambda^2\mu^2-\lambda^2-\mu^2+1)+4b^2\lambda\mu+c^2(\lambda^2\mu^2+\lambda^2+\mu^2+1)=0$$

$$\Rightarrow a^2\left[(1+\lambda\mu)^2-(\lambda+\mu)^2\right]+b^2\left[(\lambda-\mu)^2-(\lambda+\mu)^2\right]+c^2\left[(1-\lambda\mu)^2+(\lambda+\mu)^2\right]=0$$

$$\Rightarrow a^2(1+\lambda\mu)^2+b^2(\lambda-\mu)^2+c^2(1-\lambda\mu)^2=(\lambda+\mu)^2(a^2+b^2-c^2) \quad -(4)$$

This relation shows that the point of intersection of the above two generators i.e. $\left[\frac{a(1+\lambda\mu)}{(\lambda+\mu)}, \frac{b(\lambda-\mu)}{(\lambda+\mu)}, \frac{c(1-\lambda\mu)}{(\lambda+\mu)}\right]$

lies on the sphere $x^2+y^2+z^2=a^2+b^2-c^2$, which is known as the director sphere.

Hence the required locus is the curve of intersection of the hyperboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$ and the director sphere

$$x^2+y^2+z^2=a^2+b^2-c^2$$