30 (5d)

(d) Prove that the vorticity vector $\vec{\Omega}$ of an incompressible viscous fluid moving in the absence of an external force satisfies the differential equation

$$\frac{D\vec{\Omega}}{Dt} = \left(\vec{\Omega}\cdot\nabla\right)\vec{q} + \nu\ \nabla^2\vec{\Omega},$$

2012 5(a) By Navier Stokes Theorem,

$$\frac{\partial \bar{q}}{\partial t} + \nabla\left(\frac{1}{2}q^2\right) - \bar{q} \times \Omega = \bar{B} - \nabla \int \frac{dp}{\rho} + \nu \nabla^2 \bar{q}$$

In absence of external forces $\bar{B} = 0$

Taking curl on both sides

$$\Rightarrow \nabla \times \left(\frac{\partial \bar{q}}{\partial t}\right) - \nabla \times (\bar{q} \times \Omega) = \nu \nabla \times (\nabla^2 q)$$

$$\Rightarrow \frac{\partial}{\partial t}(\nabla \times \bar{q}) - [(\Omega \cdot \nabla)\bar{q} - (\bar{q} \cdot \nabla)\Omega] = \nu \nabla^2 (\nabla \times q)$$

$$\Rightarrow \frac{\partial \Omega}{\partial t} + (\bar{q} \cdot \nabla)\Omega = (\Omega \cdot \nabla)\bar{q} + \nu \nabla^2(\Omega)$$

$$\Rightarrow \boxed{\frac{D\Omega}{Dt} = (\Omega \cdot \nabla)\bar{q} + \nu \nabla^2 \Omega.}$$

31 (6b)

(b) Derive the differential equation of motion for a spherical pendulum. 13

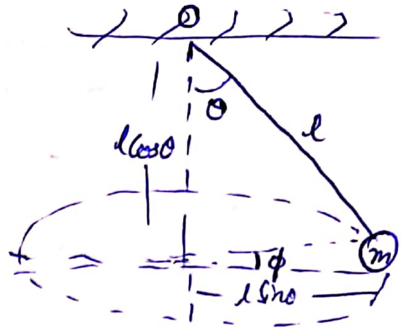
2012 → 5(d) Same IFOS → 6(b)

For mass m,

$$\bar{v} = \dot{r}\,\hat{r} + r\dot{\theta}\,\hat{\theta} + r\sin\theta\,\dot{\phi}\,\hat{\phi}$$

$$\Rightarrow T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right)$$

~~Total~~ Kinetic Energy



Since $r = l$ and $\dot{r} = 0$

$$\Rightarrow T = \frac{1}{2}m\left(l^2\dot{\theta}^2 + l^2\sin^2\theta\,\dot{\phi}^2\right)$$

Potential Energy (V) $= -mgl\cos\theta$

$$\therefore\ L = T - V = \frac{1}{2}m\left[l^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2) + 2gl\cos\theta\right]$$

Now Lagrange's equations are given by –

① $$\frac{d}{dt}\left(\frac{dL}{d\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0$$

$$\Rightarrow ml^2\ddot{\theta} - \frac{1}{2}ml^2\sin 2\theta\,\dot{\phi}^2 + mgl\sin\theta = 0 \quad \text{— ①}$$

② $$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) - \frac{\partial L}{\partial\phi} = 0$$

$$\Rightarrow \quad \frac{d}{dt}\left(ml^2 \sin^2\theta\, \dot{\phi} \right) = 0 \quad - \quad (2)$$

$$\cancel{\Rightarrow ml^2 \left(\sin 2\theta\, \dot{\theta} \dot{\phi} + \sin^2\theta\, \ddot{\phi} \right) = 0 \quad (2)}$$

Eqns (1) and (2) govern motion of spherical pendulum

32 (7a)

.7. (a) Show that

$$u = \frac{A(x^2 - y^2)}{(x^2 + y^2)^2}, \quad v = \frac{2Axy}{(x^2 + y^2)^2}, \quad w = 0$$

are components of a possible velocity vector for inviscid incompressible fluid flow. Determine the pressure associated with this velocity field. 13

YPOS.
2012

7(a) $u = \dfrac{A(x^2-y^2)}{(x^2+y^2)^2}$, $v = \dfrac{2Axy}{(x^2+y^2)^2}$, $w = 0$

For inviscid incompressible fluid, equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\Rightarrow \frac{A\left[2x(x^2+y^2)^2 - 2(x^2+y^2)\,2x\,(x^2-y^2)\right]}{(x^2+y^2)^4} + \frac{2Ax\left[(x^2+y^2)^2 - 2(x^2+y^2)\cdot 2y\cdot y\right]}{(x^2+y^2)^4} = 0$$

$$\Rightarrow \frac{2Ax\left[3y^2-x^2\right]}{(x^2+y^2)^3} + \frac{2Ax\left[x^2-3y^2\right]}{(x^2+y^2)^3} = 0$$

$$\Rightarrow 0 = 0$$

Hence equation of continuity is satisfied.

Now using equations of motion for inviscid incompressible flow,

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + \omega\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + \omega\frac{\partial v}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial y}$$

and $$u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} + \omega\frac{\partial \omega}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z}$$

Using values of $\frac{\partial u}{\partial x}$, $\frac{\partial u}{\partial y}$, $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$, we get

$$\frac{\partial p}{\partial x} = \rho\frac{2A^2x}{(x^2+y^2)^3}, \quad \frac{\partial p}{\partial y} = \rho\frac{2A^2y}{(x^2+y^2)^3}, \quad \frac{\partial p}{\partial z} = 0$$

$$dp = \frac{\partial p}{\partial x}\cdot dx + \frac{\partial p}{\partial y}\cdot dy + \frac{\partial p}{\partial z}\cdot dz$$

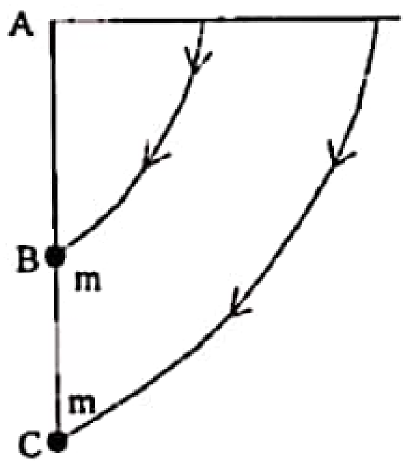
$$= \rho A^2\left[\frac{2x\,dx + 2y\,dy}{(x^2+y^2)^3}\right]$$

$$\Rightarrow p = C - \frac{A^2\rho}{2(x^2+y^2)^2} = \text{Pressure}$$

A, C are constants

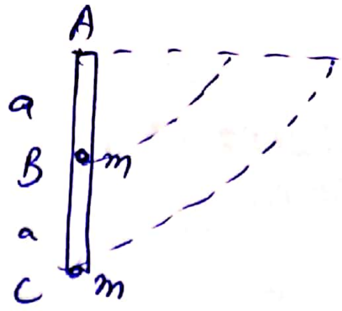
33 (8b)

(b) A weightless rod ABC of length 2a is movable about the end A which is fixed and carries two particles of mass m each one attached to the mid-point B of the rod and the other attached to the end C of the rod. If the rod is held in the horizontal position and released from rest and allowed to move, show that the angular velocity of the rod when it is vertical is $\sqrt{\frac{6g}{5a}}$. 13



2012

8(b) In vertical position, angular velocity of points B and C will be same



$$\Rightarrow \frac{v_B}{a} = \frac{v_C}{2a} \Rightarrow \boxed{v_C = 2v_B} \quad -(1)$$

When Rod comes down from horizontal to vertical position, energy is conserved

$\Rightarrow$ Gain in Kinetic Energy = loss in Potential Energy

$$\Rightarrow \frac{1}{2}mv_B^2 + \frac{1}{2}mv_C^2 = mga + mg(2a)$$

$$\Rightarrow \frac{1}{2}\cancel{m}(v_B^2 + 4v_B^2) = 3\cancel{m}ga \qquad [\text{Using } (1)]$$

$$\Rightarrow v_B = \sqrt{\frac{6ag}{5}}$$

$\therefore$ Angular velocity $= \dfrac{v_B}{a}$

$$= \sqrt{\frac{6g}{5a}}$$

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