ANALYTIC GEOMETRY

: IFOS - 2018 :

① (d) Find the equations of the tangent planes to the ellipsoid $2x^2+6y^2+3z^2=27$ which pass through the line $x-y-z=0=x-y+2z-9$

→ Any plane through the given line is

$x-y-z+\lambda(x-y+2z-9)=0 \Rightarrow (1+\lambda)x-(1+\lambda)y-(1-2\lambda)z-9\lambda=0$ — ①

Cond^n for tangency for a plane $lx+my+nz=p$ to a conicoid $ax^2+by^2+cz^2=1$ is $\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p^2$

Here, $a=\frac{2}{27}$, $b=\frac{6}{27}$, $c=\frac{3}{27}$ & $l=1+\lambda$, $m=-(1+\lambda)$, $n=2\lambda-1$, $p=9\lambda$

$\therefore \frac{27(1+\lambda)^2}{2}+\frac{27(1+\lambda)^2}{6}+\frac{27(2\lambda-1)^2}{3}=81\lambda^2$

$\rightarrow (1+\lambda)^2\left[\frac{1}{2}+\frac{1}{6}\right]+\frac{(2\lambda-1)^2}{3}=3\lambda^2$

$\rightarrow 2[1+\lambda^2+2\lambda]+[4\lambda^2+1-4\lambda]=9\lambda^2$

$\Rightarrow 2+2\lambda^2+\cancel{4\lambda}+4\lambda^2+1-\cancel{4\lambda}=9\lambda^2$

$\Rightarrow 3\lambda^2=3 \Rightarrow \lambda=\pm 1$

∴ Tangent planes are:

$x-y-z\pm 1(x-y+2z-9)=0$

$\Rightarrow x-y-z+(x-y+2z-9)=0$ & $x-y-z-(x-y+2z-9)=0$

$\Rightarrow 2x-2y+z-9=0$ & $z=3$

② (a) Find the equation of cylinder whose generators are parallel to the line $\frac{x}{1}=\frac{y}{-2}=\frac{z}{3}$ & whose guiding curve is $x^2+y^2=4,\ z=2$.

→ Any point on the cylinder be (α,β,γ). ~~Any~~ The line passing through (α,β,γ) & parallel to the line $\frac{x}{1}=\frac{y}{-2}=\frac{z}{3}$ is a generator of the cylinder.

$\frac{x-\alpha}{1}=\frac{y-\beta}{-2}=\frac{z-\gamma}{3}$ — ①

It passes through $z=2$:

$$\frac{x-\alpha}{1} = \frac{y-\beta}{-2} = \frac{2-\gamma}{3} \Rightarrow x = \alpha + \frac{2-\gamma}{3},\ y = \beta - 2\left[\frac{2-\gamma}{3}\right],\ z=2$$

This point lies on the conic $x^2+y^2=4$.

$$\rightarrow \left(\alpha + \frac{2-\gamma}{3}\right)^2 + \left(\beta - 2\left[\frac{2-\gamma}{3}\right]\right)^2 = 4$$

$$\rightarrow (3\alpha-\gamma+2)^2 + (3\beta+2\gamma-4)^2 = 36$$

$$\rightarrow 9\alpha^2+\gamma^2+4-6\alpha\gamma+12\alpha-4\gamma+9\beta^2+4\gamma^2-16\gamma-24\beta+12\beta\gamma = 36+16$$

$$\rightarrow 9\alpha^2+9\beta^2+5\gamma^2-6\alpha\gamma+12\beta\gamma+12\alpha-24\beta-20\gamma-16=0$$

Locus of (α,β,γ) gives the eq^n of cylinder:

$$9x^2+9y^2+5z^2-6xz+12yz+12x-24y-20z=16$$

3(a) Find the equation of the tangent plane that can be drawn to the sphere $x^2+y^2+z^2-2x+6y+2z+8=0$ through the straight line $3x-4y-8=0=y-3z+2$.

→ Given sphere: $S \equiv x^2+y^2+z^2-2x+6y+2z+8=0$ —— (1)

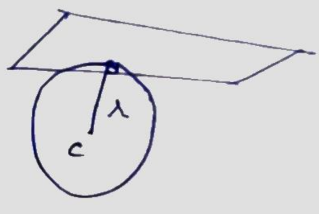
Centre of the sphere is $C(1,-3,-1)$ —— (2)

Radius of the sphere is $r=\sqrt{1+9+1-8} \Rightarrow r=\sqrt{3}$ —— (3)

Any plane through the given line is

$3x-4y-8+\lambda(y-3z+2)=0$ —— (4) $\Rightarrow 3x+(-4+\lambda)y-3\lambda z+(-8+2\lambda)=0$ —— (5)

If the plane (5) is tangent plane to the sphere (1), then the distance of this plane from centre C of the sphere is equal to radius of the sphere.



$$\therefore \frac{3\cdot1+(-4+\lambda)(-3)+(-3\lambda)(-1)+(-8+2\lambda)}{\sqrt{9+(-4+\lambda)^2+(-3\lambda)^2}} = \sqrt{3}$$

$$\rightarrow \frac{(7+2\lambda)^2}{9+(\lambda-4)^2+9\lambda^2} = 3 \Rightarrow 49+4\lambda^2+28\lambda = 27+27\lambda^2+3\lambda^2+48-24\lambda$$

$$\Rightarrow 26\lambda^2-52\lambda+26=0 \Rightarrow \lambda^2-2\lambda+1=0 \Rightarrow (\lambda-1)^2=0 \Rightarrow \lambda=1$$

(4) : $3x-4y-8+1(y-3z+2)=0$

$\Rightarrow 3x+3y-3z-6=0 \Rightarrow \boxed{x-y-z=2}$ which is the reqd tangent plane.

4(a): Find the equations of straight lines in which the plane $2x+y-z=0$ cuts the cone $4x^2-y^2+3z^2=0$. Find the angle between the two straight lines.

$\longrightarrow$ The cone and the ~~straig~~ plane pass through the origin. Then, the straight line of intersection of cone & the plane passes through the origin.

Any line through the origin is $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ — (1)

If the line (1) is line of intersection of cone & the plane, then line (1) is a generator of the cone & is $\perp$ar to the normal to the given plane.

Therefore, we have

$4l^2-m^2+3n^2=0$ & $2l+m-n=0 \Rightarrow n=2l+m$. — (2)

$\Rightarrow 4l^2-m^2+3(2l+m)^2=0$

$\rightarrow 4l^2-m^2+3[4l^2+m^2+4lm]=0$

$\rightarrow 16l^2+12lm+2m^2=0 \Rightarrow 8\frac{l^2}{m^2}+6\frac{l}{m}+1=0$

$\rightarrow 8\frac{l^2}{m^2}+4\frac{l}{m}+2\frac{l}{m}+1=0 \Rightarrow (4\frac{l}{m}+1)(2\frac{l}{m}+1)=0$

$\frac{l}{m}=\frac{-1}{4}$, $\frac{l}{m}=\frac{-1}{2}$ $\Rightarrow \frac{l}{-1}=\frac{m}{4}$ & $\frac{l}{-1}=\frac{m}{2}$.

(2) : $n=2l+m$.

: (a) $m=-4l$ $\therefore$ $n=2l+(-4l)= n=-2l$
$\Rightarrow \frac{n}{2}=\frac{l}{-1}$

(b) $m=-2l$ $\therefore$ $n=2l+m=2l-2l=0 \Rightarrow \frac{n}{0}$

$\therefore \frac{l}{-1}=\frac{m}{4}=\frac{n}{2}$ & $\frac{l}{-1}=\frac{m}{2}=\frac{n}{0}$.

$\therefore$ Reqd lines of intersection are $\frac{x}{-1}=\frac{y}{4}=\frac{z}{2}$ and

$\frac{x}{-1}=\frac{y}{2}=\frac{z}{0}$.

If θ be the angle between the two lines, then.

$$\cos\theta = \frac{(-1)(-1) + (4)(2) + (2)(0)}{\sqrt{1+16+4}\ \sqrt{1+4}} = \frac{9}{\sqrt{21}\sqrt{5}}$$

$$\theta = \cos^{-1}\left(\frac{9}{\sqrt{21}\sqrt{5}}\right)$$

4 (c) : Find the locus of point of intersection of the perpendicular generator of hyperbolic paraboloid $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2z$

→ The generator of hyperbolic paraboloid of λ & μ system are:

$$\frac{x}{a} - \frac{y}{b} = \lambda z, \quad \frac{x}{a} + \frac{y}{b} = \frac{2}{\lambda} \quad \text{——— (1)}$$

and
$$\frac{x}{a} - \frac{y}{b} = \frac{2}{\mu}, \quad \frac{x}{a} + \frac{y}{b} = \mu z \quad \text{——— (2)}$$

Eqn (1) : $\frac{x}{a} - \frac{y}{b} - \lambda z = 0, \quad \frac{x}{a} + \frac{y}{b} + 0z - \frac{2}{\lambda} = 0$

If l_1, m_1, n_1 be the dcs of this line, then,

$$\frac{l_1}{a} - \frac{m_1}{b} - \lambda n_1 = 0 \quad \text{and} \quad \frac{l_1}{a} + \frac{m_1}{b} + 0n_1 = 0$$

$$\therefore \frac{l_1}{\lambda/b} = \frac{m_1}{-\lambda/a} = \frac{n_1}{2/ab} \Rightarrow \frac{l_1}{a\lambda} = \frac{m_1}{-b\lambda} = \frac{n_1}{2} \quad \text{——— (3)}$$

Eqn (2) : $\frac{x}{a} - \frac{y}{b} - \frac{2}{\mu} = 0$, and $\frac{x}{a} + \frac{y}{b} - \mu z = 0$

If l_2, m_2, n_2 are its dcs, then

$$\frac{l_2}{a} - \frac{m_2}{b} + 0n_2 = 0 \quad \& \quad \frac{l_2}{a} + \frac{m_2}{b} - \mu n_2 = 0$$

$$\Rightarrow \frac{l_2}{\mu/b} = \frac{m_2}{\mu/a} = \frac{n_2}{2/ab} \Rightarrow \frac{l_2}{a\mu} = \frac{m_2}{b\mu} = \frac{n_2}{2} \quad \text{——— (4)}$$

Since the two generators are $\perp$ar, we have

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

$$\Rightarrow a\lambda \cdot a\mu - b\lambda \cdot b\mu + 2 \cdot 2 = 0$$

$$\Rightarrow (a^2 - b^2)\lambda\mu + 4 = 0 \quad \text{——— (5)}$$

At the point of intersection, the points are given by

$\alpha = \dfrac{a(\lambda+\mu)}{\lambda\mu}$, $\beta = \dfrac{b(\mu-\lambda)}{\lambda\mu}$, $\gamma = \dfrac{2}{\lambda\mu}$

$\therefore$ (5) $\equiv$ $(a^2-b^2)\mu\lambda + 4 = 0$ $\Rightarrow (a^2-b^2)\dfrac{2}{\gamma} + 4 = 0$

$\therefore$ Reqd locus of points of intersection of generators is

$(a^2-b^2)\dfrac{2}{z} + 4 = 0$ $\Rightarrow$ $\boxed{a^2-b^2+2z=0}$