2. $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

Answer: We use the integral test with $f(x) = x/(x^2+1)$ to determine whether this series converges or diverges. We determine whether the corresponding improper integral $\int_1^{\infty} \frac{x}{x^2+1} dx$ converges or diverges:

$$\int_1^{\infty} \frac{x}{x^2+1} dx = \lim_{b\to\infty} \int_1^{b} \frac{x}{x^2+1} dx = \lim_{b\to\infty} \frac{1}{2} \ln(x^2+1)\Big|_1^b = \lim_{b\to\infty} \left(\frac{1}{2}\ln(b^2+1) - \frac{1}{2}\ln 2\right) = \infty.$$

Since the integral $\int_1^{\infty} \frac{x}{x^2+1} dx$ diverges, we conclude from the integral test that the series $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges.

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Example 9. ***Show that for no value of k, f(x) = $x^3 - 3x + k$ has two distinct zeros in (0, 1).***

(K.U. 1993 S)

Sol. If possible, let α, β be two distinct zeros of $f(x)$ where $0 < \alpha < \beta < 1$.

$f(x)$ being a polynomial is continuous and derivable for all values of x

$\Rightarrow$ $f(x)$ is continuous in $[\alpha, \beta]$ and derivable in (α, β).

Also $f(\alpha) = 0 = f(\beta)$

$\therefore$ By Rolle's Theorem, there must exist a value c of x in (α, β) s.t.

$$f'(c) = 0 \quad \text{or} \quad 3c^2 - 3 = 0 \qquad \Big| \because \ f'(x) = 3x^2 - 3$$

which gives $c = \pm 1$

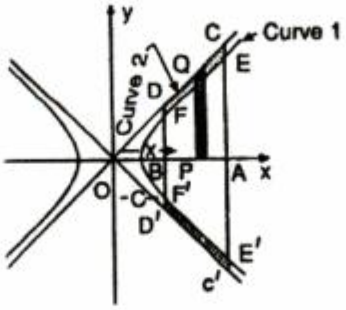
which contradicts the fact that $0 < \alpha < c < \beta < 1$

Hence the result.

Ex. 7. *If the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ revolves about the x-axis, show that the volume included between the surface thus generated, the cone generated by the asymptotes and the two planes perpendicular to the x-axis, at a distance h apart, is equal to that of a cylinder of height h and radius b.*

Sol. The equation of the hyperbola is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \quad ...(1)$$



and the equation of the asymptotes, are $y=\pm(b/a)x$ (ODC & $OD'C'$)

The equation of the asymptote ODC is

$$y=\left(\frac{b}{a}\right)x \quad ...(2)$$

Now let the two planes perpendicular to the x-axis at a distance h apart be $x=c$ and $x=c+h$ respectively.

For curve (1), $y=(b/a)\sqrt{(x^2-a^2)}=f_1(x)$ (say);
and for curve (2), $y=(b/a)x=f_2(x)$ (say).

Then the required volume = Volume generated by revolving area $ACDBA$ about x-axis-Volume generated by revolving area $AEFBA$ about x-axis

$$=\int_c^{c+h}\pi\left[f_2(x)\right]^2dx-\int_c^{c+h}\pi\left[f_1(x)\right]^2dx$$

$$=\pi\int_c^{c+h}\frac{b^2}{a^2}x^2dx-\pi\int_c^{c+h}\frac{b^2}{a^2}(x^2-a^2)\,dx$$

$$=\pi\frac{b^2}{a^2}\int_c^{c+h}(x^2-x^2+a^2)dx=\pi\frac{b^2}{a^2}\left(a^2x\right)_c^{c+h}=\pi b^2h$$

= Volume of the cylinder of height h and radius b.

Example 8. *Show that between any two roots of $e^x \cos x = 1$, there exists at least one root of $e^x \sin x - 1 = 0$.* **[I.A.S. 2009; Delhi Maths (G), 1997]**

Solution. Let α and β be any two distinct roots of $e^x \cos x = 1$

$$\therefore \quad e^{\alpha} \cos \alpha = 1 \quad \text{and} \quad e^{\beta} \cos \beta = 1 \quad \ldots (1)$$

Define a function f as follows $\quad f(x) = e^{-x} - \cos x, \quad [\, x \in [\alpha, \beta] \quad \ldots (2)$

Obviously f is continuous in $[\alpha, \beta]$ and f is derivable in $]\alpha, \beta[$

Indeed, $\quad f'(x) = -e^{-x} + \sin x \quad [\, x \in]\alpha, \beta[\quad \ldots (3)$

From (2), $\quad f(\alpha) = e^{-\alpha} - \cos \alpha = \dfrac{1 - e^{\alpha} \cos \alpha}{e^{\alpha}} = 0$, by (1)

Similarly, $\quad f(\beta) = 0 \quad$ and so $\quad f(\alpha) = f(\beta)$

Thus, f satisfies all the conditions of Rolle's theorem in $[\alpha, \beta]$ and so there exists some $\gamma \in]\alpha, \beta[$ such that $f'(\gamma) = 0$

Then (3) $\Rightarrow \sin \gamma - e^{-\gamma} = 0 \Rightarrow e^{\gamma} \sin \gamma - 1 = 0, \ \alpha < \gamma < \beta$

Hence, γ is a root of $e^x \sin x - 1 = 0, \ \alpha < \gamma < \beta$.

Example 12. *If f' and g' exist for all $x \in [a, b]$ and if $g'(x) \neq 0$ $[\, x \in]a, b[$, then prove that for some $c \in]a, b[$*

$$\frac{f(c) - f(a)}{g(b) - g(c)} = \frac{f'(c)}{g'(c)}.$$ **[I.A.S. 2005; Delhi Maths (G), 1993; Maths (H) 1998, 2002]**

Solution. Define a function ψ on $[a, b]$ as follows :

$$\psi(x) = f(x)\, g(x) - f(a)\, g(x) - g(b)\, f(x), \quad [\, x \in]a, b[\quad \ldots(1)$$

Since f and g are derivable in $[a, b]$, so ψ is continuous in $[a, b]$ and derivable in $]a, b[$.

Also $\quad \psi(a) = \psi(b) = -f(a)\, g(b)$.

Since ψ satisfies all the conditions of Rolle's theorem, therefore, there exists some $c \in]a, b[$ such that $\psi'(c) = 0$. We have, from (1)

$$\psi'(x) = f'(x)\, g(x) + f(x)\, g'(x) - f(a)\, g'(x) - g(b)\, f'(x).$$

$$\therefore \quad \psi'(c) = 0 \quad \Rightarrow \quad f'(c)\, g(c) + f(c)\, g'(c) - f(a)\, g'(c) - g(b)\, f'(c) = 0$$

or $\quad g'(c)\, \{f(c) - f(a)\} = f'(c)\, \{g(b) - g(c)\}$.

Hence, $\quad \dfrac{f(c) - f(a)}{g(b) - g(c)} = \dfrac{f'(c)}{g'(c)}$.

Example 13. *If a function f is such that its derivative f ′ is continuous on [a, b] and derivable on]a, b[, then show that there exists a number c between a and b such that*

$$f(b) = f(a) + (b-a) f'(a) + (1/2) \times (b-a)^2 f''(c).$$ **[Delhi Maths (H), 2005, 08]**

Solution. Define a function ϕ on $[a, b]$ as follows :

$$\phi(x) = f(b) - f(x)\ (b-x) f'(x) - (b-x)^2 A, \qquad \text{...(1)}$$

where A is a constant to be determined by $\phi(a) = \phi(b)$ ...(2)

i.e., $f(b) - f(a) - (b-a) f'(a) - (b-a)^2 A = 0$

i.e., $f(b) = f(a) + (b-a) f'(a) + (b-a)^2 A.$ ...(3)

Since f' is continuous on $[a, b]$, so f is also continuous on $[a, b]$. Also $(b-x)$, $(b-x)^2$ are continuous on $[a, b]$. Then by (1), ϕ is continuous on $[a, b]$. Similarly, ϕ is derivable on $]a, b[$. Further $\phi(a) = \varphi(b)$, by (2).

Thus φ satisfies the conditions of Rolle's theorem and so there exists some point $c \in]a, b[$ such that $\phi'(c) = 0$. ...(4)

From (1), $\phi'(x) = -f'(x) - \{-f'(x) + (b-x) f''(x)\} + 2(b-x) A$

or $\phi'(x) = -(b-x) f''(x) + 2(b-x) A.$ ...(5)

$\therefore$ $\phi'(c) = 0 \Rightarrow -(b-c) f''(c) + 2(b-c) A = 0$

Thus, $A = (1/2) \times f''(c)$, since $a < c < b \Rightarrow b - c \neq 0$.

Substituting this value of A in (3), the result is proved.

Example 6. *Show that* $\dfrac{v-u}{1+v^2} < \tan^{-1} v - \tan^{-1} u < \dfrac{v-v}{1+u^2}$, *if* $0 < u < v$,

and deduce that $\dfrac{\pi}{4} + \dfrac{3}{25} < \tan^{-1} \dfrac{4}{3} < \dfrac{\pi}{4} + \dfrac{1}{6}$. **[Delhi Math (G) 2006; Nagpur 2010]**

Solution. Applying Lagrange's mean value theorem to the function

$f(x) = \tan^{-1} x$ in $[u, v]$, we obtain

$$\frac{f(v) - f(u)}{v-u} = f'(c) \text{ for some } c \in]u, v[$$

or
$$\frac{\tan^{-1} v - \tan^{-1} u}{v-u} = \frac{1}{1+c^2} \text{ for } u < c < v. \qquad \text{...(1)}$$

Now $c > u \Rightarrow 1 + c^2 > 1 + u^2 \Rightarrow 1/(1+c^2) < 1/(1+u^2)$ ...(2)

Again $c < v \Rightarrow 1 + c^2 < 1 + v^2 \Rightarrow 1/(1+c^2) > 1/(1+v^2)$ ...(3)

From (1), (2), (3), we obtain

$$\frac{1}{1+v^2} < \frac{\tan^{-1} v - \tan^{-1} u}{v-u} < \frac{1}{1+u^2}$$

Hence, $\dfrac{v-u}{1+v^2} < \tan^{-1} v - \tan^{-1} u < \dfrac{v-u}{1+u^2}$, as $u < v \Rightarrow v - u > 0$ ...(4)

Taking $u = 1$ and $v = 4/3$ in (4), we obtain

$$\frac{3}{25} < \tan^{-1}\frac{4}{3} - \tan^{-1} 1 < \frac{1}{6} \quad \text{or} \quad \frac{3}{25} < \tan^{-1}\frac{4}{3} - \frac{\pi}{4} < \frac{1}{6}$$

Hence, $\dfrac{\pi}{4} + \dfrac{3}{25} < \tan^{-1}\dfrac{4}{3} < \dfrac{\pi}{4} + \dfrac{1}{6}$.

Example 4. *Prove that* $\dfrac{\tan x}{x} > \dfrac{x}{\sin x}$, *whenever* $0 < x < \dfrac{\pi}{2}$. **[I.A.S. 2008]**

[*Delhi Maths (H), 1999, 2002*; *Delhi Maths (G), 1991*; *Delhi Maths (G), 1992*]

Solution. We have, $$\frac{\tan x}{x} - \frac{x}{\sin x} = \frac{\tan x \sin x - x^2}{x \sin x}.$$

Since $x \sin x > 0$ for all x in $]0, \pi/2[$, therefore, in order to prove the inequality we must prove that $$\tan x \sin x - x^2 > 0, \quad [\, x \in \,]0, \pi/2[.$$

Let c be any real number in $]0, \pi/2[$.

Let $$f(x) = \tan x \sin x - x^2 \quad [\, x \in [0, c].$$

Then f is continuous as well as derivable in $[0, c]$.

Now, $$f'(x) = \sec^2 x \sin x + \tan x \cos x - 2x = \sin x (\sec^2 x + 1) - 2x.$$

Let $$g(x) = f'(x) \text{ for all } x \text{ in } [0, c].$$

Clearly, function g is continuous as well as derivable on $[0, c]$.

Example 7. *Find the volume common to the cylinders* $x^2 + y^2 = a^2$ *and* $x^2 + z^2 = a^2$.

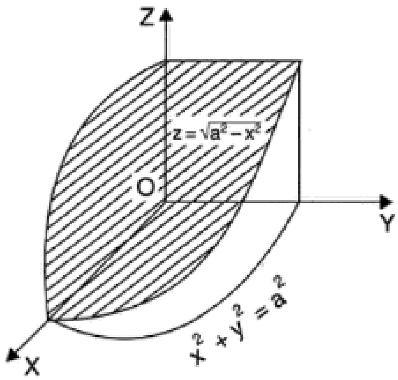
Sol. The section of the cylinder $x^2 + y^2 = a^2$ is the circle $x^2 + y^2 = a^2$ in the xy-plane. In the figure, only one-eighth (in the positive octant) of the required volume is shown.

From the figure, it is evident that $z = \sqrt{a^2 - x^2}$ is to be evaluated over the quadrant of the circle $x^2 + y^2 = a^2$ in the first quadrant for which x varies from 0 to a and y varies from 0 to $\sqrt{a^2 - x^2}$.

$\therefore$ The required volume $= 8 \int_0^a \int_0^{\sqrt{a^2 - x^2}} z \, dy \, dx$

$$= 8 \int_0^a \int_0^{\sqrt{a^2 - x^2}} \sqrt{a^2 - x^2} \, dy \, dx$$

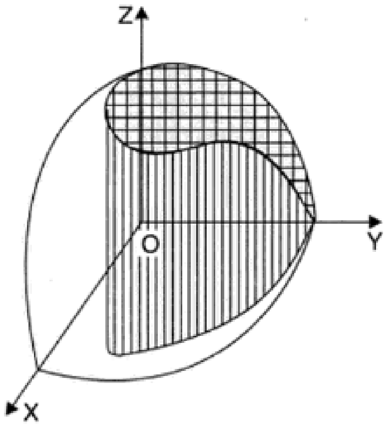
$$= 8 \int_0^a \sqrt{a^2 - x^2} \Big[y \Big]_0^{\sqrt{a^2 - x^2}} dx$$



$$= 8 \int_0^a (a^2 - x^2) \, dx = 8 \left[a^2 x - \frac{x^3}{3} \right]_0^a = 8 \left(a^3 - \frac{a^3}{3} \right) = \frac{16a^3}{3}.$$

Example 8. *Find the volume common to the sphere $x^2 + y^2 + z^2 = a^2$ and the cylinder $x^2 + y^2 = ay$.*

Sol. The required volume is the part of the sphere $x^2 + y^2 + z^2 = a^2$ lying within the cylinder. On account of the symmetry of the sphere, half of it lies above the plane XOY and half below it.



$\therefore$ Required volume $= 2 \iint z \, dy \, dx$

where $z = \sqrt{a^2 - x^2 - y^2}$, and the region of integration is the area inside the circle $x^2 + y^2 = ay$ ...(1)
in the xy-plane.

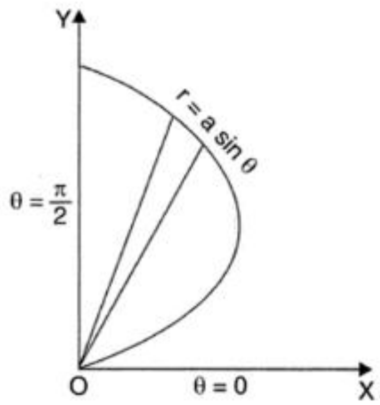
On account of symmetry, the volumes above the two parts of circle (1) in the first and the second quadrants are equal. (The figure shows only the part in the first quadrant).

$\therefore$ The required volume $= 2 \times 2 \iint_R \sqrt{a^2 - x^2 - y^2} \, dy \, dx$

where R is the half of the circle (1) lying in the first quadrant.

Changing to polar coordinates by setting $x = r\cos\theta$, $y = r\sin\theta$ so that $x^2 + y^2 = r^2$, equation (1) becomes

$$r^2 = ar\sin\theta \quad \text{or} \quad r = a\sin\theta.$$



The region of integration is bounded by

$$r = 0, r = a\sin\theta \text{ and } \theta = 0, \theta = \frac{\pi}{2}.$$

$\therefore$ The required volume

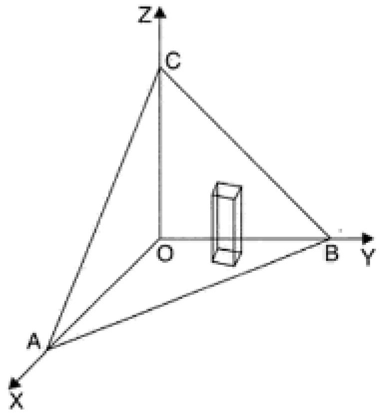
$$= 4\int_0^{\pi/2}\int_0^{a\sin\theta} \sqrt{a^2 - r^2} \, . \, r \, dr \, d\theta$$

$$= 4\int_0^{\pi/2}\int_0^{a\sin\theta} -\frac{1}{2}(a^2 - r^2)^{1/2}(-2r) \, dr \, d\theta$$

$$= 4\int_0^{\pi/2} -\frac{1}{2} \cdot \left[\frac{(a^2 - r^2)^{3/2}}{3/2}\right]_0^{a\sin\theta} d\theta$$

$$= -\frac{4}{3}\int_0^{\pi/2}(a^2\cos^3\theta - a^3) = -\frac{4a^3}{3}\left[\frac{2}{3} - \frac{\pi}{2}\right] = \frac{2}{9}a^3(3\pi - 4).$$

Example 5. *Find the volume of the tetrahedron bounded by the coordinate planes and the plane* $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.



Sol. The required volume OABC lies between the plane $z=c\left(1-\frac{x}{a}-\frac{y}{b}\right)$ and the plane XOY, and is bounded on the sides by the planes $x=0, y=0$.

The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ cuts the plane XOY in the line $\frac{x}{a}+\frac{y}{b}=1, z=0$

$\therefore$ The region OAB above which the volume OABC lies is bounded by $x=0$, $x=a$ and $y=0$,

$$y=b\left(1-\frac{x}{a}\right)$$

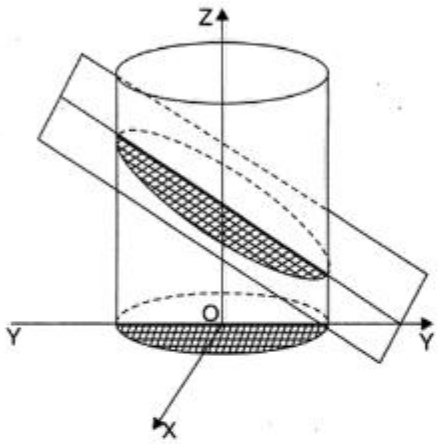
Hence the required volume of the tetrahedron

$$=\int_0^a\int_0^{b(1-x/a)} z\,dy\,dx=\int_0^a\int_0^{b(1-x/a)} c\left(1-\frac{x}{a}-\frac{y}{b}\right)dy\,dx$$

$$=\int_0^a c\left[\left(1-\frac{x}{a}\right)y-\frac{y^2}{2b}\right]_0^{b(1-x/a)}dx=c\int_0^a\left[b\left(1-\frac{x}{a}\right)^2-\frac{b}{2}\left(1-\frac{x}{a}\right)^2\right]dx$$

$$=\frac{bc}{2}\int_0^a\left(1-\frac{x}{a}\right)^2dx=\frac{bc}{2}\cdot\left[\frac{\left(1-\frac{x}{a}\right)^3}{-\frac{3}{a}}\right]_0^a=-\frac{abc}{6}(0-1)=\frac{1}{6}abc.$$

Example 6. *Find the volume bounded by the cylinder* $x^2+y^2=4$ *and the planes* $y+z=4$ *and* $z=0$.



Sol. From the figure, it is clear that $z=4-y$ is to be integrated over the circle $x^2+y^2=4$ in the xy-plane. To cover the shaded half of this circle, x varies from 0 to $\sqrt{4-y^2}$ while y varies from -2 to 2.

$\therefore$ The required volume

$$=2\int_{-2}^{2}\int_0^{\sqrt{4-y^2}} z\,dx\,dy$$

$$=2\int_{-2}^{2}\int_0^{\sqrt{4-y^2}}(4-y)\,dx\,dy$$

$$=2\int_{-2}^{2}(4-y)\Big[x\Big]_0^{\sqrt{4-y^2}}dy$$

$$=2\int_{-2}^{2}(4-y)\sqrt{4-y^2}\,dy=2\int_{-2}^{2}4\sqrt{4-y^2}\,dy-2\int_{-2}^{2}y\sqrt{4-y^2}\,dy$$

$$=8\int_{-2}^{2}\sqrt{4-y^2}\,dy$$ [The second integral is zero since $y\sqrt{4-y^2}$ is an odd function of y]

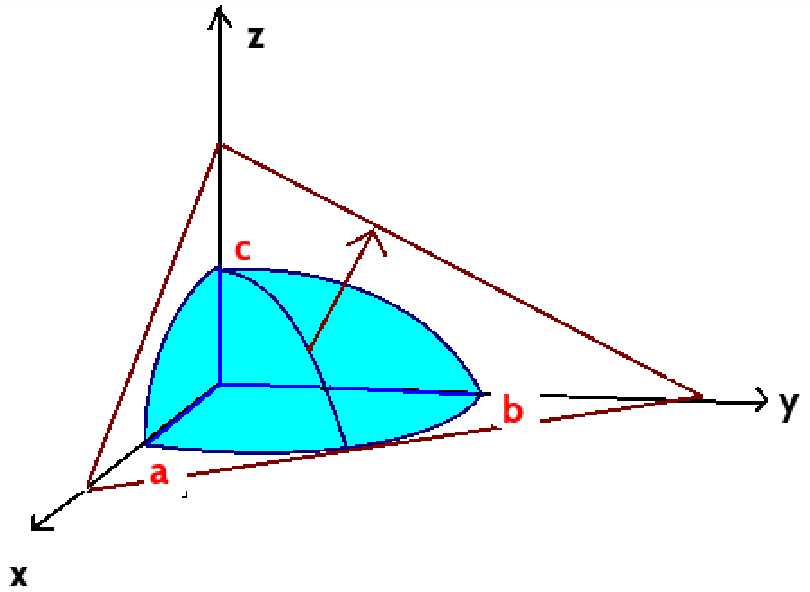
$$=16\int_0^2\sqrt{4-y^2}\,dy$$ [$\because$ $\sqrt{4-y^2}$ is an even function of y]

$$=16\left[\frac{y\sqrt{4-y^2}}{2}+\frac{4}{2}\sin^{-1}\frac{y}{2}\right]_0^2=16\,[2\sin^{-1}1]=32\times\frac{\pi}{2}=16\pi.$$

What is the minimum volume bounded by the planes $x = 0, y = 0, z = 0$ and a plane which is tangent to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

where $x, y, z > 0$



You want to find the the volume of the tetrahedron formed by the $x = y = z = 0$ planes, and the plane tangent to the ellipsoid at some point on its surface, (x, y, z). This volume, of course, will vary with different points on the surface of the ellipse, so the constraint function must be the following equality

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$$

Now the vector normal to the surface of the ellipsoid at some point (x, y, z) is given by the gradient of its equation.

$$\nabla g(x, y, z) = \left\langle \frac{2x}{a^2}, \frac{2y}{b^2}, \frac{2z}{c^2} \right\rangle$$

The equation for the plane is tangent to the ellipsoid at this point is easily worked out to be

$$\frac{x}{a^2}x' + \frac{y}{b^2}b' + \frac{z}{c^2}z' = 1$$

Now, finally, for the volume that this plane bounds within the first quadrant, we make use of the fact that this volume is simply $1/6$ the volume of the rectangular prism whose side lengths are given by the points of where the plane intersects the three axes.

$$V = \frac{(abc)^2}{6xyz}$$

And that is your function f which you are trying to minimize.

$$f = V$$

$$g \to \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2 = 1$$

Example 4.16: Using the transform $x + y = u,\ y = uv$, show that $\int_0^1\int_0^{1-x} e^{\frac{y}{x+y}}dy\,dx = \frac{e-1}{2}$.

Solution: $x + y = u,\ y = uv,$

$\therefore$ $x = u - y = u - uv.$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix} = u - uv + uv = u$$

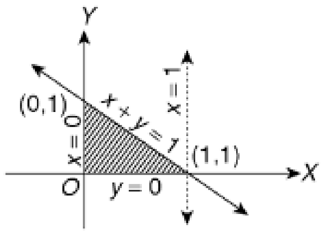


Figure 4.17

Region of integration in x–y plane is bounded by $x = 0,\ x = 1,\ y = 0,\ x + y = 1$.
Line $x = 0$ is transformed to $u = 0,\ v = 1$
Line $y = 0$ is transformed to $u = 0,\ v = 0$
Line $x + y = 1$ is transformed to $u = 1$
$\therefore$ Region of integration in $u - v$ plane is shaded in the figure.

$$\therefore \quad \int_0^1\int_0^{1-x} e^{\frac{y}{x+y}}dy\,dx = \int_0^1\int_0^1 e^v u\,du\,dv = \int_0^1 e^v dv \int_0^1 u\,du$$

$$= \left(e^v\right)_0^1\left(\frac{1}{2}u^2\right)_0^1 = (e-1)\left(\frac{1}{2}\right)$$

$$= \frac{1}{2}(e-1)$$

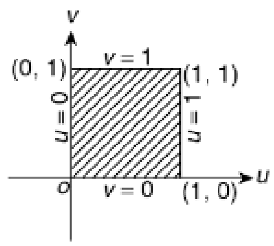


Figure 4.18

Example 4.14: Evaluate $\iint \sqrt{\dfrac{a^2b^2 - b^2x^2 - a^2y^2}{a^2b^2 + b^2x^2 + a^2y^2}}\, dx\, dy$ over the positive quadrant of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$.

Solution: Put $\quad x = ar\cos\theta,\ y = br\sin\theta$

$\therefore \quad dx\,dy = ab\,r\,dr\,d\theta$

and given region of integration R becomes first quadrant of circle $r = 1$. Also, $b^2x^2 + a^2y^2 = a^2b^2r^2$

$$\therefore \quad I = \iint_R \sqrt{\frac{a^2b^2 - b^2x^2 - a^2y^2}{a^2b^2 + b^2x^2 + a^2y^2}}\, dx\, dy = \int_0^{\frac{\pi}{2}}\int_0^1 \sqrt{\frac{1-r^2}{1+r^2}}\, a\,b\,r\,dr\,d\theta$$

$$= ab\int_0^{\frac{\pi}{2}}\int_0^1 \frac{1-r^2}{\sqrt{1-r^4}}\, r\,dr\,d\theta$$

Put $\quad r^2 = \sin\phi$

$\therefore \quad 2r\,dr = \cos\phi\,d\phi$

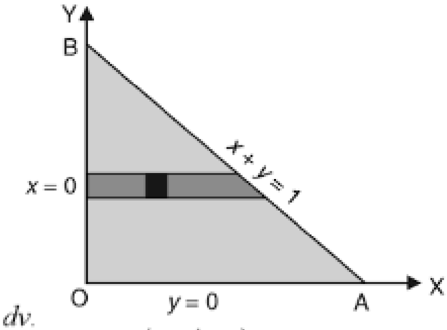
$$\therefore \quad I = ab\int_0^{\pi/2}\int_0^{\frac{\pi}{2}} \frac{1-\sin\phi}{\sqrt{1-\sin^2\phi}}\,\frac{1}{2}\cos\phi\,d\phi\,d\theta$$

$$= \frac{ab}{2}\int_0^{\pi/2}(\phi + \cos\phi)_0^{\pi/2}\,d\theta = \frac{ab}{2}\int_0^{\frac{\pi}{2}}\left(\frac{\pi}{2} - 1\right)d\theta$$

$$= \frac{ab}{2}\left(\frac{\pi}{2} - 1\right)\left(\frac{\pi}{2} - 0\right) = \frac{ab\pi}{8}(\pi - 2)$$

Example 18. *Using the transformation $x + y = u$, $y = uv$, show that*

$$\iint [xy\,(1 - x - y)]^{1/2}\ dx\,dy = \frac{2\pi}{105},$$ *integration being taken over the area of the tringle bounded by the lines $x = 0$, $y = 0$, $x + y = 1$.*



(*xy*-plane)

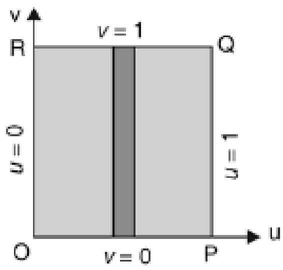
Solution. $\iint [xy\,(1 - x - y)]^{1/2}\ dx\,dy$

$$x + y = u \text{ or } x = u - y = u - uv,$$

$$dx\,dy = \frac{\partial(x, y)}{\partial(u, v)}\,du\,dv = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} du\,dv$$

$$dx\,dy = \begin{vmatrix} 1 - v & -u \\ v & u \end{vmatrix} du\,dv = u\,du\,dv.$$

$$\begin{aligned} x = 0 \quad &\Rightarrow \quad u\,(1 - v) = 0 \\ &\Rightarrow \quad u = 0,\ v = 1 \\ y = 0 \quad &\Rightarrow \quad uv = 0 \\ &\Rightarrow \quad u = 0,\ v = 0 \\ x + y = 1 \quad &\Rightarrow \quad u = 1 \end{aligned}$$



(*uv*-plane)

Hence, the limits of u are from 0 to 1 and the limits of v are from 0 to 1.

The area of integration is a square *OPQR* in *uv*-plane.

On putting $x = u - uv$, $y = uv$, $dx\,dy = u\,du\,dv$ in (1), we get

$$\iint (u - uv)^{1/2}\,(uv)^{1/2}\,(1 - v)^{1/2}\,u\,du\,dv$$

$$= \int_0^1 u^2\,(1 - u)^{1/2}\,du \int_0^1 v^{1/2}\,(1 - v)^{1/2}\,dv = \frac{\Gamma 3}{\Gamma\frac{9}{2}} \times \frac{\Gamma\frac{3}{2}\,\Gamma\frac{3}{2}\,\Gamma\frac{3}{2}}{\Gamma 5}$$

$$= \frac{2.\,\Gamma\frac{3}{2}}{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\Gamma\frac{3}{2}} \times \frac{\frac{1}{2}\Gamma\frac{1}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}}{2} = \frac{1}{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}} \times \frac{\frac{1}{2}\sqrt{\pi}.\ \frac{1}{2}\sqrt{\pi}}{1} = \frac{2\pi}{105}$$ **Ans.**

Q. 3. ***Find the volume bounded above by sphere $x^2 + y^2 + z^2 = 2a^2$ and below the paraboloid $az = x^2 + y^2$.***

Sol. Given limits of z, are

$$z = \frac{x^2 + y^2}{a} \text{ and } z = \sqrt{2a^2 - x^2 - y^2} \qquad \text{...(1)}$$

Also the intersection of

$$x^2 + y^2 + z^2 = 2a^2 \quad \text{and} \quad x^2 + y^2 = az,$$

is given by $x^2 + y^2 + \dfrac{(x^2 + y^2)^2}{a^2} = 2a^2$

$$\Rightarrow \qquad (x^2 + y^2)^2 + a^2(x^2 + y^2) = 2a^4$$

$$\Rightarrow \quad k^2 + ka^2 - 2a^4 = 0, \quad \text{where } k = x^2 + y^2$$

$$\Rightarrow \quad k^2 + 2ka^2 - ka^2 - 2a^4 = 0$$

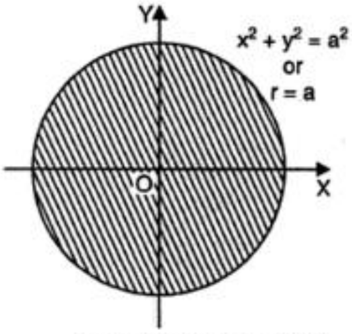
$$k(k + 2a^2) - a^2(k + 2a^2) = 0$$

$$\Rightarrow \quad (k + 2a^2)(k - a^2) = 0$$

$$\Rightarrow \quad (x^2 + y^2 + 2a^2)(x^2 + y^2 - a^2) = 0$$

$$\Rightarrow \quad x^2 + y^2 = a^2 \qquad | \; x^2 + y^2 + 2a^2 \neq 0.$$

Changing to cylinderical coordinates, by putting



$$x = r\cos\phi, \; y = r\sin\phi$$

and $\quad z = z$, so that

$$dx\,dy = r\,dr\,d\phi$$

Also z varies from $\dfrac{r^2}{a}$ to $\sqrt{2a^2 - r^2}$

| From (1)

Therefore the region V is given by

$$V = \{(r, \phi, z) : 0 \le r \le a, \; 0 \le \phi \le 2\pi, \; r^2/a \le z \le \sqrt{2a^2 - r^2}\,\}$$

Hence required volume

$$= \iiint_V dz\,dx\,dy$$

$$= \int_0^{2\pi}\int_0^a \int_{r^2/a}^{\sqrt{2a^2 - r^2}} dz\; r\,dr\,d\phi$$

$$= \int_0^{2\pi}\int_0^a \left| z \right|_{r^2/a}^{\sqrt{2a^2 - r^2}} r\,dr\,d\phi$$

$$= \int_0^{2\pi}\int_0^a \left(\sqrt{2a^2 - r^2} - \frac{r^2}{a}\right) r\,dr\,d\phi$$

$$= \int_0^{2\pi}\int_0^a \left[\frac{-1}{2}(2a^2 - r^2)^{1/2}.(-2r) - \frac{1}{a}r^3\right] dr\,d\phi$$

$$= \int_0^{2\pi} \left| \frac{-1}{2} \cdot \frac{(2a^2 - r^2)^{(1/2)+1}}{\frac{1}{2}+1} - \frac{1}{4a}r^4 \right|_0^a d\phi$$

$$= \int_0^{2\pi} \left(\frac{-1}{3}a^3 - \frac{a^3}{4}\right) - \left(\frac{-1}{3}2\sqrt{2}\,a^3\right) d\phi$$

$$= 2\pi a^3 \left[\left(\frac{-1}{3} - \frac{1}{4}\right) + \frac{2\sqrt{2}}{3}\right]$$

$$= 2\pi a^3 \left[\frac{-4 - 3 + 8\sqrt{2}}{12}\right]$$

$$= \frac{\pi a^3}{6}(8\sqrt{2} - 7) = \pi a^3\left(\frac{4}{3}\sqrt{2} - \frac{7}{6}\right).$$

Example 2: Find the volume bounded by the elliptic paraboloids $z = x^2 + 9y^2$ and $z = 18 - x^2 - 9y^2$.

Solution: The two surfaces intersect on the elliptic cylinder $x^2 + 9y^2 = z = 18 - x^2 - 9y^2$ i.e., $x^2 + 9y^2 = 9$. The projection of this volume onto xy-plane is the plane region D enclosed by ellipse having the same equation $\frac{x^2}{3^2} + \frac{y^2}{1^2} = 1^2$ as shown in (Fig. 4.30).

This volume can be covered as follows:

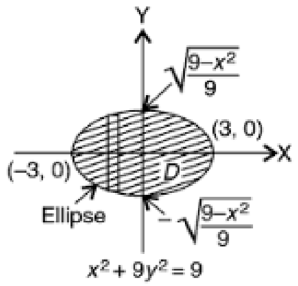
z: from $z_1(x, y) = x^2 + 9y^2$ to
$z_2(x, y) = 18 - x^2 - 9y^2$
y: from $y_1(x) = -\sqrt{\frac{9-x^2}{9}}$ to $y_2(x) = \sqrt{\frac{9-x^2}{9}}$
x: from -3 to 3

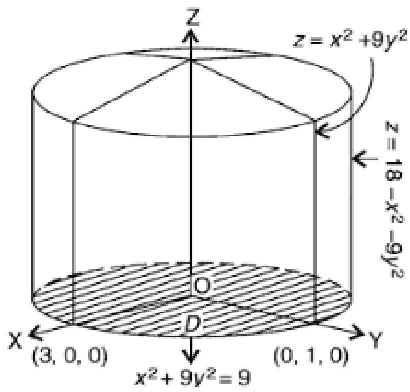
Thus the volume V bounded by the elliptic paraboloids

$$V = \int_{-3}^{3} \int_{y_1(x)}^{y_2(x)} \int_{z_1(x,y)}^{z_2(x,y)} dz\, dy\, dx$$

$$= \int_{-3}^{3} \int_{y_1(x)}^{y_2(x)} \left[(18 - x^2 - 9y^2) - (x^2 + 9y^2)\right] dy\, dx$$

$$= 2 \int_{-3}^{3} \int_{y_1(x)}^{y_2(x)} (9 - x^2 - 9y^2) dy\, dx$$





$$= 2\int_{-3}^{3} (9y - x^2y - 3y^3)\Big|_{-\sqrt{\frac{9-x^2}{9}}}^{\sqrt{\frac{9-x^2}{9}}} dx$$

$$= \frac{8}{9}\int_{-3}^{3} (9 - x^2)^{\frac{3}{2}} dx$$

$$= 72\int_{0}^{\pi} \sin^4\theta \, d\theta \qquad \text{where } x = 3\cos\theta$$

$$= 72\int_{0}^{\pi} \left(\frac{1-\cos 2\theta}{2}\right)^2 d\theta = 27\pi$$

Example 2: $I = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\, dy$.

Solution: Using polar coordinates

$$I = \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} r\, dr\, d\theta$$

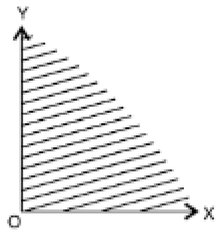


Fig. 4.25

since the first quadrant in xy-plane ($x : 0$ to ∞ and $y : 0$ to ∞) is covered when $r : 0$ to ∞ and $\theta = 0$ to $\pi/2$ (refer Fig. 4.25)

$$I = \frac{\pi}{2}\frac{1}{2}\int_0^\infty e^{-r^2} d(r^2) = \frac{\pi}{4}\frac{e^{-r^2}}{-1}\Big|_0^\infty = \frac{\pi}{4}$$

10. Find the extreme values of $f(x, y) = 2x^2 + 3y^2 - 4x - 5$ on the region

$$D = \{(x, y)|\ x^2 + y^2 \leq 16\}.$$

Solution:

We first need to find the critical points. These occur when

$$f_x = 4x - 4 = 0, \qquad f_y = 6y = 0$$

so the only critical point of f is $(1, 0)$ and it lies in the region $x^2 + y^2 \leq 16$.

On the circle $x^2 + y^2 = 16$, we have $y^2 = 16 - x^2$ and

$$g(x) = f(x, \sqrt{16 - x^2}) = 2x^2 + 3(16 - x^2) - 4x - 5 = -x^2 - 4x + 43\,.$$

$$g'(x) = 0 \Rightarrow -2x - 4 = 0 \Rightarrow x = -2$$

$$y^2 = 16 - x^2 = 16 - 4 = 12 \Rightarrow y = \pm 2\sqrt{3}\,.$$

Now $f(1, 0) = -7$ and $f(-2, \pm 2\sqrt{3}) = 47$. Thus the maximum value of $f(x, y)$ on the disc $x^2 + y^2 \leq 16$ is $f(-2, \pm 2\sqrt{3}) = 47$, and the minimum value is $f(1, 0) = -7$.

9. Find the points on surface $x^2y^2z = 1$ that are closest to the origin.

Solution:

The distance from any point (x, y, z) to the origin is

$$d = \sqrt{x^2 + y^2 + z^2}$$

but if (x, y, z) lies on the surface $x^2y^2z = 1$, then $z = \frac{1}{x^2y^2}$ and so we have

$$d = \sqrt{x^2 + y^2 + x^{-4}y^{-4}}\,.$$

We can minimize d by minimizing the simpler expression

$$d^2 = x^2 + y^2 + x^{-4}y^{-4} = f(x, y)\,.$$

$f_x = 2x - \frac{4}{x^5y^4}$, $f_y = 2y - \frac{4}{x^4y^5}$, so the critical points occur when $2x = \frac{4}{x^5y^4}$ and $2y = \frac{4}{x^4y^5}$ or $x^6y^4 = x^4y^6$ so, $x^2 = y^2$ and $x^{10} = 2 \Rightarrow x = \pm 2^{\frac{1}{10}}$, $y = \pm 2^{\frac{1}{10}}$. The four critical points $(\pm 2^{\frac{1}{10}}, \pm 2^{\frac{1}{10}})$. Thus the points on the surface closes to origin are $(\pm 2^{\frac{1}{10}}, \pm 2^{\frac{1}{10}})$. There is no maximum since the surface is infinite in extent.

8. Find the local maximum and minimum values and saddle point(s) of the function

$$f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2.$$

Solution:

The first order partial derivatives are

$$f_x = 6xy - 6x, \qquad f_y = 3x^2 + 3y^2 - 6y\,.$$

So to find the critical points we need to solve the equations $f_x = 0$ and $f_y = 0$. $f_x = 0$ implies $x = 0$ or $y = 1$ and when $x = 0$, $f_y = 0$ implies $y = 0$ or $y = 2$; when $y = 1$, $f_y = 0$ implies $x^2 = 1$ or $x = \pm 1$. Thus the critical points are $(0, 0)$, $(0, 2)$, $(\pm 1, 1)$.

Now $f_{xx} = 6y - 6$, $f_{yy} = 6y - 6$ and $f_{xy} = 6x$. So $D = f_{xx}f_{yy} - f_{xy}^2 = (6y - 6)^2 - 36x^2$.

Critical point	Value of f	f_{xx}	D	Conclusion
$(0, 0)$	2	-6	36	local maximum
$(0, 2)$	-2	6	36	local minimum
$(1, 1)$	0	0	-36	saddle point
$(-1, 1)$	0	0	-36	saddle point

4. A function f is called **homogeneous of degree n** if it is satisfies the equation $f(tx, ty) = t^n f(x, y)$ for all t, where n is a positive integer. Show that if f is homogeneous of degree n, then

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y)$$

[**Hint:** Use the Chain Rule to differentiate $f(tx, ty)$ with respect t.]

Solution:

Let $u = tx$ and $v = ty$. Then

$$\frac{d}{dt}(f(u, v)) = nt^{n-1}f(x, y)\,.$$

The Chain Rule gives

$$\frac{\partial f}{\partial u}\frac{du}{dt} + \frac{\partial f}{\partial v}\frac{dv}{dt} = nt^{n-1}f(x, y)\,.$$

Therefore

$$x\frac{\partial f}{\partial u} + y\frac{\partial f}{\partial v} = nt^{n-1}f(x, y)\,. \tag{3}$$

Setting $t = 1$ in the Equation (3):

$$x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf(x, y)\,.$$

3. Let f and g be two differentiable real valued functions. Show that any function of the form $z = f(x+at) + g(x-at)$ is a solution of the wave equation $\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2}$.

Solution:

Let $u = x + at$ and $v = x - at$. Then $z = f(u) + g(v)$ and the Chain Rule gives

$$\frac{\partial z}{\partial x} = \frac{df}{du}\frac{\partial u}{\partial x} + \frac{dg}{dv}\frac{\partial u}{\partial x} = \frac{df}{du} + \frac{dg}{dv}.$$

Thus

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{df}{du} + \frac{dg}{dv}\right) = \frac{d^2 f}{du^2} + \frac{d^2 g}{dv^2}. \tag{1}$$

Similarly

$$\frac{\partial z}{\partial t} = \frac{df}{du}\frac{\partial u}{\partial t} + \frac{dg}{dv}\frac{\partial v}{\partial t} = a\frac{df}{du} + a\frac{dg}{dv}.$$

Thus

$$\frac{\partial^2 z}{\partial t^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial t}\right) = \frac{\partial}{\partial x}\left(a\frac{df}{du} + a\frac{dg}{dv}\right) = a^2\frac{d^2 f}{du^2} + a^2\frac{d^2 g}{dv^2} = a^2\left(\frac{d^2 f}{du^2} + \frac{d^2 g}{dv^2}\right). \tag{2}$$

From Equations (1) and (2) we get

$$\frac{\partial^2 z}{\partial t^2} = a^2\frac{\partial^2 z}{\partial x^2}.$$

1. Let $R = \ln(u^2 + v^2 + w^2)$, $u = x + 2y$, $v = 2x - y$, and $w = 2xy$. Use the Chain Rule to find $\frac{\partial R}{\partial x}$ and $\frac{\partial R}{\partial y}$ when $x = y = 1$.

Solution:

The Chain Rule gives

$$\frac{\partial R}{\partial x} = \frac{\partial R}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial R}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial R}{\partial w}\frac{\partial w}{\partial x}$$

$$= \frac{2u}{u^2+v^2+w^2} \times 1 + \frac{2v}{u^2+v^2+w^2} \times 2 + \frac{2w}{u^2+v^2+w^2} \times (2y).$$

When $x = y = 1$, we have $u = 3$, $v = 1$, and $w = 2$, so

$$\frac{\partial R}{\partial x} = \frac{6}{14} \times 1 + \frac{2}{14} \times 2 + \frac{4}{14} \times 2 = \frac{18}{14} = \frac{9}{7}.$$

$$\frac{\partial R}{\partial y} = \frac{\partial R}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial R}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial R}{\partial w}\frac{\partial w}{\partial y}$$

$$= \frac{2u}{u^2+v^2+w^2} \times 2 + \frac{2v}{u^2+v^2+w^2} \times (-1) + \frac{2w}{u^2+v^2+w^2} \times (2x).$$

When $x = y = 1$, we have $u = 3$, $v = 1$, and $w = 2$, so

$$\frac{\partial R}{\partial x} = \frac{6}{14} \times 2 + \frac{2}{14} \times (-1) + \frac{4}{14} \times 2 = \frac{18}{14} = \frac{9}{7}.$$

4. *Find the surface of $x^2+z^2=a^2$ that lies inside the cylinder $x^2+y^2=a^2$.*

Solution. $\frac{\partial z}{\partial x}=-\frac{2x}{2z}, \frac{\partial z}{\partial y}=0.$

$$\therefore \sec \gamma=\sqrt{\left(1+\frac{x^2}{z^2}\right)}=\frac{a}{z}=\frac{a}{\sqrt{(a^2-x^2)}}.$$

Hence $S=8\int_0^a \int_0^{\sqrt{(a^2-x^2)}} \frac{a}{\sqrt{(a^2-x^2)}}\, dx\, dy=8a^2.$

Example 133:

Evaluate the volume of the wedge intercepted between the cylinder $x^2 + y^2 = 2ax$ and the planes $z = x \tan \alpha$ and $z = x \tan \beta$.

Solution:

The required volume is given by

$$V = \iint_E (z_2 - z_1)\, dx\, dy = (\tan \beta - \tan \alpha) \iint_E x\, dx\, dy$$

where E is the region : $x^2 + y^2 = 2ax$. Putting $x = r \cos \theta$, $y = r \sin \theta$;

$$V = (\tan \beta - \tan \alpha) \int_{\theta=0}^{\pi} \int_{r=0}^{2a\cos\theta} r \cos \theta\, r\, dr\, d\theta$$

$$= \frac{1}{3} (\tan \beta - \tan \alpha) \int_0^{\pi} 8a^3 \cos^3 \theta \,.\, \cos \theta\, d\theta$$

$$= \frac{16}{3} a^3 (\tan \beta - \tan \alpha) \int_0^{\pi/2} \cos^4 \theta\, d\theta$$

$$= \frac{16a^3}{3} (\tan \beta - \tan \alpha) \,.\, \frac{3}{4} \,.\, \frac{1}{2} \,.\, \frac{\pi}{2} = a^3 \pi (\tan \beta - \tan\alpha). \textbf{ Ans.}$$

Example 136:

Show that the volume common to the surfaces $y^2 + z^2 = 4ax$ *and* $x^2 + y^2 = 2ax$ *is* $\frac{2}{3}\ (3\pi + 8)a^n$.

Solution:

The paraboloid $y^2 + z^2 = 4ax$ is symmetrical about th plane $z = 0$. Thus the required volumee is given by

$$V = 2 \iiint_{z=0}^{\sqrt{4ax-y^2}} dx\ dy\ dz$$

$$= 4 \int_0^{2a} dx \int_0^{\sqrt{(2ax - x^2)}} \sqrt{4ax - y^2}\ dy$$

$$= 4 \int_0^{2a} \left[\frac{1}{2} y \sqrt{4ax - y^2} + 2ax \sin^{-1} \frac{y}{\sqrt{4ax}}\right]_0^{\sqrt{2ax - x^2}} dx$$

$$= 2 \int_0^{2a} \left[(2ax - x^2)\ (2ax + x^2)^{1/2} + 4ax \sin^{-1} \sqrt{\frac{2ax - x}{4ax}}\right] dx$$

$$= 2 \int_0^{2a} \left[\sqrt{4a^2 - x^2}\ .\ x + 4ax \sin^{-1} \sqrt{\frac{2a - x}{4ax}}\right] dx$$

Put $x = 2a \cos 2\theta$ so that $dx = -4a \sin 2\theta\ d\theta$.

$$\therefore \quad V = 2 \int_0^{\pi/4} (4a^2 \sin 2\theta \cos 2\theta + 8a^2\ \theta \cos 2\theta)\ 4a \sin 2\theta\ d\theta$$

$$= 16a^3 \int_0^{2a} (\sin^2 \phi \cos \phi + \phi \cos \phi \sin \phi)\ d\phi,\ \phi = 2\theta$$

$$= 16a^3 \left[\left|\frac{\sin^3 \phi}{3}\right|_0^{\pi/2} + \left|\phi_n \frac{\sin^2 \phi}{2}\right|_0^{\pi/2} - \frac{1}{2} \int_0^{\pi/2} \sin^2 \phi\ d\phi\right]$$

$$= 16a^3 \left[\frac{1}{3} + \frac{\pi}{4} - \frac{1}{2} . \frac{1}{2} . \frac{\pi}{2}\right] = 16a^3 \left[\frac{1}{3} + \frac{\pi}{8}\right]$$

$$= \frac{2}{3} a^3\ (3\pi + 8).$$

Ans.

Example 128:

Find the volume of the region above the xy-plane bounded by the paraboloid $z = x^2 + y^2$ and the cylinder $x^2 + y^2 = a^2$.

Solution:

The given solid is bounded above by $z = x^2 + y2$ and below by $x^2 + y^2 = a^2$. Moreover the solid being symmetrical, its volume V is four times the volume lying in the first octant.

$$\therefore \quad V = 4 \iint_D z\, dx\, dy = 4 \iint_D (x^2 + y^2)\, dx\, dy$$

Changing to polars, we get

$$V = 4 \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} r^2 (r\, dr\, d\theta) = 4 . \frac{\pi}{2} . \frac{a^4}{4} = \frac{\pi a^4}{2}.$$ **Ans.**

Example 123:

Find the volume of the solid bounded by the surface $z = 1 - 4x^2 - y^2$ and the plane $z = 0$.

Solution:

The paraboloid $z = 1 - 4x^2 - y^2$ cuts the xy-plane ($z = 0$) along the ellipse $D : 4x^2 + y^2 = 1$. Moreover thee solid being symmetrical, its volume V is four times the volume lying in the first octant. Then we have

$$\therefore \quad V = 4 \iint_D z\, dx\, dy$$

$$= 4 \int_0^{1/2} dx \int_0^{\sqrt{(1-4x^2)}} (1 - 4x^2 - y^2)\, dy$$

$$= 4 \int_0^{1/2} \left| (1-4x^2)y - \frac{1}{3} y^3 \right|_0^{\sqrt{1-4x^2}}$$

$$= \frac{8}{3} \int_0^{1/2} (1-4x^2)^{3/2}\, dx$$

$$= \frac{4}{3} \int_0^{\pi/2} \cos^4 t, \text{ where } x = \frac{1}{2} \sin t$$

$$= \frac{4}{3} . \frac{3}{4} . \frac{1}{2} . \frac{\pi}{2} = \frac{\pi}{4}.$$ **Ans.**

Example 127:

Find the volume common to the ellipsoid of revolution $x^2/a^2 + y^2/a^2$ $z^2/b^2 = 1$ and the cylinder $x^2 + y^2 - ay = 0$.

Solution:

The required volume is double the volume that lies above the xy-plane. The given solid is bounded above by $z = \frac{b}{a}\sqrt{a^2 - x^2 - y^2}$ and below by the circular base D : $x^2 + y^2 - ay = 0$ on the xy-plane. Thus the required volume is given by

$$V = 2\iint_D z\,dx\,dy = \iint_D \frac{b}{a}\sqrt{a^2 - x^2 - y^2}\,dx\,dy.$$

Changing to polars, we obtain

$$V = \frac{2b}{a}\int_{\theta=0}^{\pi}\int_{r=0}^{a\sin\theta}\sqrt{a^2 - r^2}\,(r\,dr\,d\theta)$$

$(x^2 + y^2 = ay \;\Rightarrow\; r^2 = ar\sin\theta \;\Rightarrow\; r = a\sin\theta)$

$$= \frac{2b}{a}\int_0^{\pi}\left|-\frac{1}{3}(a^2 - r^2)^{3/2}\right|_0^{a\sin\theta} d\theta$$

$$= \frac{4a^2b}{3}\int_0^{\pi/2}(1 - \cos^2\theta)\,d\theta = \frac{4a^3b}{3}\left[\frac{\pi}{2} - \frac{2}{3}\right]$$

$$= \frac{2a^2b}{9}(3\pi - 4).$$ **Ans.**

1. ***Find the area of the surface of the sphere $x^2+y^2+z^2=a^2$ which lies inside the cylinder $x^2+y^2=ay$.*** **[M.U. 1982 (Type)]**

Solution. $\frac{\partial z}{\partial x} = -\frac{x}{z},\ \frac{\partial z}{\partial y} = -\frac{y}{z}.$

$$\therefore\ \sec\gamma = \sqrt{\left(1+\frac{x^2}{z^2}+\frac{y^2}{z^2}\right)} = \frac{1}{z}\sqrt{(x^2+y^2+z^2)} = \frac{a}{\sqrt{(a^2-x^2-y^2)}}.$$

$$A = 4\iint \frac{a\,dx\,dy}{\sqrt{(a^2-x^2-y^2)}} \text{ over half the circle } x^2+y^2=ay$$

$$= 4a\int_0^{\pi/2}\int_0^{a\sin\theta}\frac{r\,d\theta\,dr}{\sqrt{(a^2-r^2)}} = 4a^2\int_0^{\pi/2}(1-\cos\theta)\,d\theta = 2a^2(\pi-2).$$

Example 124:

Find the volume cut off from the sphere $x^2 + y^2 + z^2 = a^2$ by the cylinder $x^2 + y^2 = ax$.

Solution:

We have

$$V = \iint_D dx\,dy \int_{-\sqrt{(a^2-x^2-y^2)}}^{\sqrt{(a^2-x^2-y^2)}} dz = 2 \iint_D \sqrt{a^2-x^2-y^2}\, dx\,dy$$

or $$V = 2 \int_0^{\pi} d\theta \int_0^{a\cos\theta} \sqrt{a^2 - r^2}\,.\,r\,dr = -\frac{2}{3}\int_0^{\pi} \left|(a^2-r^2)^{3/2}\right|_0^{a\cos\theta}$$

$$= \frac{2}{3}a^3 \int_0^{\pi} (1 - \sin^3\theta) = \frac{2}{3}a^3\left[\pi - 2\,.\,\frac{2}{3}\right]$$

$$= \frac{2}{3}a^3\left(4 - \frac{4}{3}\right).$$

Ans.

1. Use the method of Lagrange multipliers to maximize x^3y^5 subject to the constraint $x + y = 8$.

Answer: Let $f(x,y) = x^3y^5$ and $g(x,y) = x + y - 8$. The points (x,y) which will maximize $f(x,y)$ subject to the constraint $g(x,y) = 0$ will be the set of points (x,y) that satisfy the equations $\nabla f(x,y) = \lambda\nabla g(x,y)$ and $g(x,y) = 0$. Writing out the system of three equations and three unknowns explicitly:

$$\begin{aligned} 3x^2y^5 &= \lambda && (1)\\ 5x^3y^4 &= \lambda && (2)\\ x + y &= 8. && (3) \end{aligned}$$

Suppose $\lambda = 0$. Then eqn.(1) would be $3x^2y^5 = 0$ which would imply that either $x = 0$ or $y = 0$. (They cannot both be equal to 0, because that would contradict eqn.(3).) If $x = 0$, then eqn.(3) says $y = 8$. And if $y = 0$, we would instead have from eqn.(3) that $x = 8$. Either way, we get $f(8,0) = f(0,8) = 0$.

Now suppose that $\lambda \neq 0$. We can set equations (1) and (2) equal to each other and simplify:

$$\begin{aligned} 3x^2y^5 &= 5x^3y^4 \\ 3x^2y^5 - 5x^3y^4 &= 0 \\ x^2y^4(3y - 5x) &= 0 && (4) \end{aligned}$$

Since $\lambda \neq 0$, eqn.(1) implies that $x \neq 0$, $y \neq 0$. Therefore, for eqn.(4) to be true, it must be that $3y - 5x = 0$, and solving this equation for y: $y = \frac{5}{3}x$. Plugging this into eqn.(3) and solving for x yields:

$$x + \frac{5}{3}x = 8 \Longrightarrow \frac{8}{3}x = 8 \Longrightarrow x = 3 \text{ and therefore } y = 5.$$

And so we have $f(3,5) = 3^3 \cdot 5^5 = (27)\cdot(3,125) = 84,375$. Since clearly $f(3,5) > f(8,0)$ and $f(0,8)$, $f(x,y)$ is maximized at the point $(3,5)$.

5. Use the Lagrange multiplier method to find the greatest and least distances from the point $(2, 1, -2)$ to the sphere with the equation $x^2 + y^2 + z^2 = 1$.

Answer: The distance from (x, y, z) to the point $(2, 1, -2)$ is $f(x, y, z) = \sqrt{(x-2)^2 + (y-1)^2 + (z+2)^2}$. This is the function I need to minimize and maximize subject to the constraint that $g(x, y, z) = x^2 + y^2 + z^2 - 1 = 0$. Using the same reasoning given in the previous problem, I will instead minimize and maximize $F(x, y, z) = (x-2)^2 + (y-1)^2 + (z+2)^2$.

Taking the appropriate derivatives, the system of equations I need to solve is

$$\begin{aligned} 2(x-2) = 2x\lambda &\quad \text{which simplifies to:} \quad x(1-\lambda) = 2 &(9)\\ 2(y-1) = 2y\lambda &\quad \text{which simplifies to:} \quad y(1-\lambda) = 1 &(10)\\ 2(z+2) = 2z\lambda &\quad \text{which simplifies to:} \quad z(1-\lambda) = -2 &(11)\\ x^2 + y^2 + z^2 = 1 &\quad \text{which stays the same:} \quad x^2 + y^2 + z^2 = 1. &(12) \end{aligned}$$

Now, eqns.(9-11) all imply that $x, y, z \neq 0$. (Otherwise, we would have $0 = 2$.)

Since I know that $x, y, z \neq 0$, I can legally divide both sides of eqn.(9) by x, both sides of eqn.(10) by y and both sides of eqn.(11) by z. (Before dividing, I need to first make sure that I'm not dividing by 0.) And so

$$\begin{aligned} 1-\lambda &= \frac{2}{x} &(13)\\ 1-\lambda &= \frac{1}{y} &(14)\\ 1-\lambda &= -\frac{2}{z} &(15) \end{aligned}$$

Setting eqns.(13) and (15) equal to each other gets:

$$\frac{2}{x} = -\frac{2}{z} \Longrightarrow z = -x$$

and from eqns.(13) and (14):

$$\frac{2}{x} = \frac{1}{y} \Longrightarrow y = \frac{x}{2}.$$

Let's plug all this into eqn.(12):

$$x^2 + \left(\frac{x}{2}\right)^2 + (-x)^2 = 1 \Longrightarrow \frac{9}{4}x^2 = 1 \Longrightarrow x^2 = \frac{4}{9} \Longrightarrow x = \pm\frac{2}{3}.$$

The two points we need to check are $\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right)$ and $\left(-\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right)$:

$$\begin{aligned} F\left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right) &= 4\\ F\left(-\frac{2}{3}, -\frac{1}{3}, \frac{2}{3}\right) &= 16. \end{aligned}$$

Therefore, the least distance from the point $(2, 1, -2)$ to the unit sphere centered at the origin is $\sqrt{4} = 2$, and the greatest distance is $\sqrt{16} = 4$.

9. Find the maximum and minimum values of $f(x, y, z) = xyz$ on the sphere $x^2 + y^2 + z^2 = 12$.

Answer: The function we want to maximize and minimize is $f(x, y, z) = xyz$, and the constraint we have is $g(x, y, z) = x^2 + y^2 + z^2 - 12 = 0$. The system of equations we need to solve are

$$\begin{aligned} yz &= 2x\lambda && (16)\\ xz &= 2y\lambda && (17)\\ xy &= 2z\lambda && (18)\\ x^2 + y^2 + z^2 &= 12. && (19) \end{aligned}$$

Suppose that $\lambda = 0$. Then exactly *two* variables must be equal to 0. (E.g. Suppose $x = 0$ and $y, z \neq 0$. Then we'd get a contradiction from eqn.(16): $yz = 0$.) And they can't all be 0, because that would contradict eqn.(19). So no matter which variable is not equal to 0, we would have

$$f(\pm\sqrt{12}, 0, 0) = f(0, \pm\sqrt{12}, 0) = f(0, 0, \pm\sqrt{12}) = 0.$$

The $\pm\sqrt{12}$ comes from solving eqn.(19).

Now suppose that $\lambda \neq 0$. Let me multiply both sides of eqn.(16) by x, both sides of eqn.(17) by y, and both sides of eqn.(18) by z, and add those three equations together:

$$\begin{aligned} 3xyz &= 2\lambda(x^2 + y^2 + z^2)\\ 3xyz &= 2\lambda(12)\\ xyz &= 8\lambda. && (20) \end{aligned}$$

That last equation implies that $x, y, z \neq 0$ (since $\lambda \neq 0$) . Let me use eqn.(18) in eqn.(20) and solve for z:

$$\begin{aligned} xyz &= 8\lambda\\ (xy)z &= 8\lambda\\ (2z\lambda)z &= 8\lambda\\ 2z^2 &= 8\\ z^2 &= 4\\ z &= \pm 2. \end{aligned}$$

I can divide by λ in that fourth line because $\lambda \neq 0$.

Similarly, I can use eqn.(17) in eqn.(20) and solve for y: $y = \pm 2$. And with eqn.(16) in eqn.(20), I can get x: $x = \pm 2$. And so there are a few possibilities. Let's make a table of the different combinations:

x	y	z	$f(x, y, z)$
2	2	2	8
2	2	-2	-8
2	-2	2	-8
2	-2	-2	8
-2	2	2	-8
-2	2	-2	8
-2	-2	2	8
-2	-2	-2	-8

Comparing all the numbers (e.g. -8, 0 and 8), we see that the minimum value of $f(x, y, z)$ is -8, and the maximum value is 8.

11. Find the maximum and minimum values of the function $f(x, y, z) = x$ over the curve of intersection of the plane $z = x + y$ and the ellipsoid $x^2 + 2y^2 + 2z^2 = 8$.

Answer: So we have to minimize and maximize the function $f(x, y, z) = x$ subject to the two constraints $g(x, y, z) = x + y - z = 0$ and $h(x, y, z) = x^2 + 2y^2 + 2z^2 - 8 = 0$. Therefore, the equations we have to solve are $\nabla f(x, y, z) = \lambda\nabla g(x, y, z) + \mu\nabla h(x, y, z)$, $g(x, y, z) = 0$, and $h(x, y, z) = 0$. That means, we solve

$$\begin{aligned} 1 &= \lambda + 2x\mu && (21) \\ 0 &= \lambda + 4y\mu && (22) \\ 0 &= -\lambda + 4z\mu && (23) \\ x + y &= z && (24) \\ x^2 + 2y^2 + 2z^2 &= 8. && (25) \end{aligned}$$

Note that it must be the case that $\mu \neq 0$. (If $\mu = 0$, then eqn.(21) would say $\lambda = 1$ and eqn.(22) would say $\lambda = 0$, and we would get a contradiction.)

Since I know I can legally divide by μ, from eqns.(22) and (23) we have

$$4y\mu = -4z\mu \Longrightarrow y = -z$$

and using that in eqn.(24):

$$x + (-z) = z \Longrightarrow x = 2z.$$

Let's plug all this into eqn.(25):

$$\begin{aligned} (2z)^2 + 2(-z)^2 + 2z^2 &= 8 \\ 4z^2 + 2z^2 + 2z^2 &= 8 \\ 8z^2 &= 8 \\ z^2 &= 1 \\ z &= \pm 1. \end{aligned}$$

For $z = 1$, we have $x = 2$ and $y = -1$, and $f(2, -1, 1) = 2$. For $z = -1$, we have $x = -2$ and $y = 1$, and $f(-2, 1, -1) = -2$. Therefore, the minimum value is -2, and the maximum value 2.

EXAMPLE 1 Find the point $P(x, y, z)$ closest to the origin on the plane $2x + y - z - 5 = 0$.

Solution The problem asks us to find the minimum value of the function

$$\begin{aligned}|\overrightarrow{OP}| &= \sqrt{(x-0)^2 + (y-0)^2 + (z-0)^2} \\ &= \sqrt{x^2 + y^2 + z^2}\end{aligned}$$

subject to the constraint that

$$2x + y - z - 5 = 0.$$

Since $|\overrightarrow{OP}|$ has a minimum value wherever the function

$$f(x, y, z) = x^2 + y^2 + z^2$$

has a minimum value, we may solve the problem by finding the minimum value of $f(x, y, z)$ subject to the constraint $2x + y - z - 5 = 0$. If we regard x and y as the independent variables in this equation and write z as

$$z = 2x + y - 5,$$

our problem reduces to one of finding the points (x, y) at which the function

$$h(x, y) = f(x, y, 2x + y - 5) = x^2 + y^2 + (2x + y - 5)^2$$

has its minimum value or values. Since the domain of h is the entire xy-plane, the first derivative test of Section 12.8 tells us that any minima that h might have must occur at points where

$$h_x = 2x + 2(2x + y - 5)(2) = 0, \qquad h_y = 2y + 2(2x + y - 5) = 0.$$

This leads to

$$10x + 4y = 20, \qquad 4x + 4y = 10,$$

and the solution

$$x = \frac{5}{3}, \qquad y = \frac{5}{6}.$$

We may apply a geometric argument together with the second derivative test to show that these values minimize h. The z-coordinate of the corresponding point on the plane $z = 2x + y - 5$ is

$$z = 2\left(\frac{5}{3}\right) + \frac{5}{6} - 5 = -\frac{5}{6}.$$

Therefore, the point we seek is

$$\text{Closest point:} \qquad P\left(\frac{5}{3}, \frac{5}{6}, -\frac{5}{6}\right).$$

The distance from P to the origin is $5/\sqrt{6} \approx 2.04$. ❑

Attempts to solve a constrained maximum or minimum problem by substitution, as we might call the method of Example 1, do not always go smoothly. This is one of the reasons for learning the new method of this section.

EXAMPLE 2 Find the points closest to the origin on the hyperbolic cylinder $x^2 - z^2 - 1 = 0$.

Solution 1 The cylinder is shown in Fig. 12.59. We seek the points on the cylinder closest to the origin. These are the points whose coordinates minimize the value of the function

$$f(x, y, z) = x^2 + y^2 + z^2 \quad \text{Square of the distance}$$

subject to the constraint that $x^2 - z^2 - 1 = 0$. If we regard x and y as independent variables in the constraint equation, then

$$z^2 = x^2 - 1$$

and the values of $f(x, y, z) = x^2 + y^2 + z^2$ on the cylinder are given by the function

$$h(x, y) = x^2 + y^2 + (x^2 - 1) = 2x^2 + y^2 - 1.$$

To find the points on the cylinder whose coordinates minimize f, we look for the points in the xy-plane whose coordinates minimize h. The only extreme value of h occurs where

$$h_x = 4x = 0 \quad \text{and} \quad h_y = 2y = 0,$$

that is, at the point $(0, 0)$. But now we're in trouble—there are no points on the cylinder where both x and y are zero. What went wrong?

What happened was that the first derivative test found (as it should have) the point *in the domain of h* where h has a minimum value. We, on the other hand, want the points *on the cylinder* where h has a minimum value. While the domain of h is the entire xy-plane, the domain from which we can select the first two coordinates of the points (x, y, z) on the cylinder is restricted to the "shadow" of the cylinder on the xy-plane; it does not include the band between the lines $x = -1$ and $x = 1$

We can avoid this problem if we treat y and z as independent variables (instead of x and y) and express x in terms of y and z as

$$x^2 = z^2 + 1.$$

With this substitution, $f(x, y, z) = x^2 + y^2 + z^2$ becomes

$$k(y, z) = (z^2 + 1) + y^2 + z^2 = 1 + y^2 + 2z^2$$

and we look for the points where k takes on its smallest value. The domain of k in the yz-plane now matches the domain from which we select the y- and z-coordinates of the points (x, y, z) on the cylinder. Hence, the points that minimize k in the plane will have corresponding points on the cylinder. The smallest values of k occur where

$$k_y = 2y = 0 \quad \text{and} \quad k_z = 4z = 0,$$

or where $y = z = 0$. This leads to

$$x^2 = z^2 + 1 = 1, \qquad x = \pm 1.$$

The corresponding points on the cylinder are $(\pm 1, 0, 0)$. We can see from the inequality

$$k(y, z) = 1 + y^2 + 2z^2 \geq 1$$

that the points $(\pm 1, 0, 0)$ give a minimum value for k. We can also see that the minimum distance from the origin to a point on the cylinder is 1 unit.

Solution 2 Another way to find the points on the cylinder closest to the origin is to imagine a small sphere centered at the origin expanding like a soap bubble until it just touches the cylinder (Fig. 12.61). At each point of contact, the cylinder and sphere have the same tangent plane and normal line. Therefore, if the sphere and cylinder are represented as the level surfaces obtained by setting

$$f(x, y, z) = x^2 + y^2 + z^2 - a^2 \qquad \text{and} \qquad g(x, y, z) = x^2 - z^2 - 1$$

equal to 0, then the gradients ∇f and ∇g will be parallel where the surfaces touch. At any point of contact we should therefore be able to find a scalar λ ("lambda") such that

$$\nabla f = \lambda \nabla g,$$

or

$$2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k} = \lambda(2x\,\mathbf{i} - 2z\,\mathbf{k}).$$

Thus, the coordinates x, y, and z of any point of tangency will have to satisfy the three scalar equations

$$2x = 2\lambda x, \qquad 2y = 0, \qquad 2z = -2\lambda z. \tag{1}$$

For what values of λ will a point (x, y, z) whose coordinates satisfy the equations in (1) also lie on the surface $x^2 - z^2 - 1 = 0$? To answer this question, we use the fact that no point on the surface has a zero x-coordinate to conclude that $x \neq 0$ in the first equation in (1). This means that $2x = 2\lambda x$ only if

$$2 = 2\lambda, \qquad \text{or} \qquad \lambda = 1.$$

For $\lambda = 1$, the equation $2z = -2\lambda z$ becomes $2z = -2z$. If this equation is to be satisfied as well, z must be zero. Since $y = 0$ also (from the equation $2y = 0$), we conclude that the points we seek all have coordinates of the form

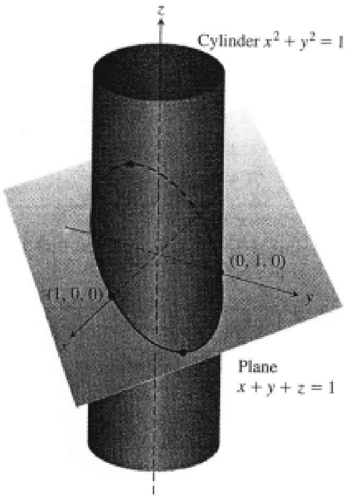
$$(x, 0, 0).$$

What points on the surface $x^2 - z^2 = 1$ have coordinates of this form? The points $(x, 0, 0)$ for which

$$x^2 - (0)^2 = 1, \qquad x^2 = 1, \qquad \text{or} \qquad x = \pm 1.$$

The points on the cylinder closest to the origin are the points $(\pm 1, 0, 0)$. ❑

EXAMPLE 5 The plane $x + y + z = 1$ cuts the cylinder $x^2 + y^2 = 1$ in an ellipse (Fig. 12.66). Find the points on the ellipse that lie closest to and farthest from the origin.



12.66 On the ellipse where the plane and cylinder meet, what are the points closest to and farthest from the origin (Example 5)?

Solution We find the extreme values of

$$f(x, y, z) = x^2 + y^2 + z^2$$

(the square of the distance from (x, y, z) to the origin) subject to the constraints

$$g_1(x, y, z) = x^2 + y^2 - 1 = 0 \tag{3}$$

$$g_2(x, y, z) = x + y + z - 1 = 0. \tag{4}$$

The gradient equation in (2) then gives

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2 \qquad \text{Eq. (2)}$$

$$2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k} = \lambda(2x\,\mathbf{i} + 2y\,\mathbf{j}) + \mu\,(\mathbf{i} + \mathbf{j} + \mathbf{k})$$

$$2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k} = (2\lambda x + \mu)\,\mathbf{i} + (2\lambda y + \mu)\,\mathbf{j} + \mu\,\mathbf{k}$$

or

$$2x = 2\lambda x + \mu, \qquad 2y = 2\lambda y + \mu, \qquad 2z = \mu. \tag{5}$$

The scalar equations in (5) yield

$$\begin{aligned} 2x &= 2\lambda x + 2z \quad &\Rightarrow \quad (1 - \lambda)x = z, \\ 2y &= 2\lambda y + 2z \quad &\Rightarrow \quad (1 - \lambda)y = z. \end{aligned} \tag{6}$$

Equations (6) are satisfied simultaneously if either $\lambda = 1$ and $z = 0$ or $\lambda \neq 1$ and $x = y = z/(1 - \lambda)$.

If $z = 0$, then solving Eqs. (3) and (4) simultaneously to find the corresponding points on the ellipse gives the two points (1, 0, 0) and (0, 1, 0). This makes sense when you look at Fig. 12.66.

If $x = y$, then Eqs. (3) and (4) give

$$\begin{aligned} x^2 + x^2 - 1 &= 0 \qquad & x + x + z - 1 &= 0 \\ 2x^2 &= 1 & z &= 1 - 2x \\ x &= \pm\frac{\sqrt{2}}{2} & z &= 1 \mp \sqrt{2}. \end{aligned}$$

The corresponding points on the ellipse are

$$P_1 = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}, 1 - \sqrt{2}\right) \qquad \text{and} \qquad P_2 = \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}, 1 + \sqrt{2}\right).$$

But here we need to be careful. While P_1 and P_2 both give local maxima of f on the ellipse, P_2 is farther from the origin than P_1.

The points on the ellipse closest to the origin are (1, 0, 0) and (0, 1, 0). The point on the ellipse farthest from the origin is P_2. ❑

Example 7.10. The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1/x & \text{if } x \neq 0, \\ 0 & \text{if } x = 0, \end{cases}$$

is not continuous at 0 since $\lim_{x\to 0} f(x)$ does not exist (see Example 6.9). The left and right limits of f at 0 do not exist either, and we say that f has an essential discontinuity at 0.

Example 7.11. The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

is continuous at $c \neq 0$ (see Example 7.21 below) but discontinuous at 0 because $\lim_{x\to 0} f(x)$ does not exist (see Example 6.10).

Example 7.12. The function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

is continuous at every point of $\mathbb{R}$. (See Figure 1.) The continuity at $c \neq 0$ is proved in Example 7.22 below. To prove continuity at 0, note that for $x \neq 0$,

$$|f(x) - f(0)| = |x\sin(1/x)| \leq |x|,$$

so $f(x) \to f(0)$ as $x \to 0$. If we had defined $f(0)$ to be any value other than 0, then f would not be continuous at 0. In that case, f would have a removable discontinuity at 0.

Example 7.21. The function

$$f(x) = \begin{cases} \sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0 \end{cases}$$

is continuous on $\mathbb{R} \setminus \{0\}$, since it is the composition of $x \mapsto 1/x$, which is continuous on $\mathbb{R} \setminus \{0\}$, and $y \mapsto \sin y$, which is continuous on $\mathbb{R}$.

Example 7.22. The function

$$f(x) = \begin{cases} x\sin(1/x) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

is continuous on $\mathbb{R} \setminus \{0\}$ since it is a product of functions that are continuous on $\mathbb{R} \setminus \{0\}$. As shown in Example 7.12, f is also continuous at 0, so f is continuous on $\mathbb{R}$.

Example 7.26. Define $f : [0, 1] \to \mathbb{R}$ by $f(x) = x^2$. Then f is uniformly continuous on $[0, 1]$. To prove this, note that for all $x, y \in [0, 1]$ we have

$$\left|x^2 - y^2\right| = |x + y|\, |x - y| \leq 2|x - y|,$$

so we can take $\delta = \epsilon/2$ in the definition of uniform continuity. Similarly, $f(x) = x^2$ is uniformly continuous on any bounded set.

Example 7.27. The function $f(x) = x^2$ is continuous but not uniformly continuous on $\mathbb{R}$. We have already proved that f is continuous on $\mathbb{R}$ (it's a polynomial). To prove that f is not uniformly continuous, let

$$x_n = n, \qquad y_n = n + \frac{1}{n}.$$

Then

$$\lim_{n\to\infty} |x_n - y_n| = \lim_{n\to\infty} \frac{1}{n} = 0,$$

but

$$|f(x_n) - f(y_n)| = \left(n + \frac{1}{n}\right)^2 - n^2 = 2 + \frac{1}{n^2} \geq 2 \qquad \text{for every } n \in \mathbb{N}.$$

It follows from Proposition 7.24 that f is not uniformly continuous on $\mathbb{R}$. The problem here is that in order to prove the continuity of f at c, given $\epsilon > 0$ we need to make $\delta(\epsilon, c)$ smaller as c gets larger, and $\delta(\epsilon, c) \to 0$ as $c \to \infty$.

Example 7.28. The function $f : (0, 1] \to \mathbb{R}$ defined by

$$f(x) = \frac{1}{x}$$

is continuous but not uniformly continuous on $(0, 1]$. It is continuous on $(0, 1]$ since it's a rational function whose denominator x is nonzero in $(0, 1]$. To prove that f is not uniformly continuous, we define $x_n, y_n \in (0, 1]$ for $n \in \mathbb{N}$ by

$$x_n = \frac{1}{n}, \qquad y_n = \frac{1}{n+1}.$$

Then $|x_n - y_n| \to 0$ as $n \to \infty$, but

$$|f(x_n) - f(y_n)| = (n + 1) - n = 1 \qquad \text{for every } n \in \mathbb{N}.$$

It follows from Proposition 7.24 that f is not uniformly continuous on $(0, 1]$. The problem here is that given $\epsilon > 0$, we need to make $\delta(\epsilon, c)$ smaller as c gets closer to 0, and $\delta(\epsilon, c) \to 0$ as $c \to 0^+$.

The non-uniformly continuous functions in the last two examples were unbounded. However, even bounded continuous functions can fail to be uniformly continuous if they oscillate arbitrarily quickly.

Example 7.29. Define $f : (0, 1] \to \mathbb{R}$ by

$$f(x) = \sin\left(\frac{1}{x}\right)$$

Then f is continuous on $(0, 1]$ but it isn't uniformly continuous on $(0, 1]$. To prove this, define $x_n, y_n \in (0, 1]$ for $n \in \mathbb{N}$ by

$$x_n = \frac{1}{2n\pi}, \qquad y_n = \frac{1}{2n\pi + \pi/2}.$$

Then $|x_n - y_n| \to 0$ as $n \to \infty$, but

$$|f(x_n) - f(y_n)| = \sin\left(2n\pi + \frac{\pi}{2}\right) - \sin 2n\pi = 1 \qquad \text{for all } n \in \mathbb{N}.$$

It isn't a coincidence that these examples of non-uniformly continuous functions have domains that are either unbounded or not closed. We will prove in Section 7.5 that a continuous function on a compact set is uniformly continuous.

30. Let $f : [1, \infty) \to \mathbb{R}$ be defined by $f(x) = \int_1^x \frac{\ln t}{1+t} dt$. Solve the equation $f(x) + f(\frac{1}{x}) = 2$.

30. Observe that $f(\frac{1}{x}) = \int_1^{1/x} \frac{\ln t}{1+t} dt = \int_1^x \frac{\ln y}{y(1+y)} dy$, by taking $t = \frac{1}{y}$. Therefore $f(x) + f(\frac{1}{x}) = \int_1^x \frac{\ln t}{1+t}(1 + \frac{1}{t})dt = \int_1^x \frac{\ln t}{t} dt = \frac{1}{2}(\ln x)^2$. Now $f(x) + f(\frac{1}{x}) = 2$ implies that $\ln x = \pm 2$ which implies that $x = e^2$ as $x > 1$.

2. Exercise 5.2.5. Let

$$g_a(x) = \begin{cases} x^a \sin(1/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Find a particular (potentially noninteger) value for a so that

(a) g_a is differentiable on $\mathbb{R}$ but such that g_a' is unbounded on $[0, 1]$.

Answer. Let $a = 4/3$ (in fact, a similar proof will work for any $a \in (1, 2)$). Notice that away from zero we can just use the product and chain rules to differentiate g_a:

$$g_{4/3}'(x) = x^{4/3}\cos(1/x) \cdot \frac{-1}{x^2} + \frac{4}{3}x^{1/3}\sin(1/x) = \frac{-\cos(1/x)}{x^{2/3}} + \frac{4}{3}\sqrt[3]{x}\sin(1/x)$$

for $x \neq 0$. If $x_n = \frac{1}{2n\pi}$, then

$$g_{4/3}'(x_n) = \frac{-1}{(1/2n\pi)^{2/3}} = -\left(\sqrt[3]{2n\pi}\right)^2.$$

Hence, the sequence $(g_{4/3}'(x_n))$ is unbounded, so $g_{4/3}'$ is unbounded on $(0, 1)$.
It remains only to show that $g_{4/3}$ is differentiable at zero. To do so, notice that

$$\lim_{x\to 0}\frac{g_{4/3}(x) - g_{4/3}(0)}{x - 0} = \lim_{x\to 0}\frac{g_{4/3}(x)}{x} = \lim_{x\to 0}\frac{x^{4/3}\sin(1/x)}{x} = \lim_{x\to 0}\sqrt[3]{x}\sin(1/x).$$

But now, since $\lim_{x\to 0} -\sqrt[3]{x} = 0$ and $\lim_{x\to 0}\sqrt[3]{x} = 0$, the Squeeze Theorem implies that the above limit is equal to 0, meaning that $g_{4/3}'(0) = 0$.

(b) g_a is differentiable on $\mathbb{R}$ with g_a' continuous but not differentiable at zero.

Answer. Let $a = 7/3$ (again, a similar proof will work for any $a \in (2, 3]$). Then away from zero we have that

$$g_{7/3}'(x) = x^{7/3}\cos(1/x) \cdot \frac{-1}{x^2} + \frac{7}{3}x^{4/3}\sin(1/x) = -\sqrt[3]{x}\cos(1/x) + \frac{7}{3}x^{4/3}\sin(1/x),$$

and a similar proof to the one given above in part (a) implies that $g_{7/3}'(0) = 0$. Notice that the above expression goes to 0 as $x \to 0$, so $g_{7/3}'$ is continuous at zero.

However, if $x_n = \frac{1}{2n\pi}$, then

$$g_{7/3}'(x_n) = -\sqrt[3]{\frac{1}{2n\pi}} \cdot 1 + 0 = \frac{-1}{\sqrt[3]{2n\pi}}.$$

Hence,

$$\frac{g_{7/3}'(x_n) - g_{7/3}'(0)}{x_n - 0} = \frac{g_{7/3}'(x_n)}{x_n} = \frac{-1/\sqrt[3]{2n\pi}}{1/2n\pi} = -\sqrt[3]{2n\pi}^2,$$

which is unbounded, meaning that $\lim_{x\to 0}\frac{g_{7/3}'(x) - g_{7/3}'(0)}{x - 0}$ does not exist, so $g_{7/3}'$ is not differentiable at zero.

(c) g_a is differentiable on $\mathbb{R}$ and g_a' is differentiable on $\mathbb{R}$, but such that g_a'' is not continuous at zero.

Answer. Let $a = 10/3$ (or any number in $(3, 4]$). Then a similar proof to those already given in (a) and (b) will show that $g_{10/3}$ is twice-differentiable, but that the second derivative is not continuous at zero.

Example 16. *Find the area of one of the loops of $x^4 + y^4 = 2a^2xy$, by converting into polar coordinates.* (*M.U. II Semester 2001*)

Solution. Here, we have

$$x^4 + y^4 = 2a^2xy \qquad \text{...(1)}$$

Putting $x = r\cos\theta$, $y = r\sin\theta$ in (1), we get

$$r^4\cos^4\theta + r^4\sin^4\theta = 2a^2r^2\cos\theta\sin\theta$$

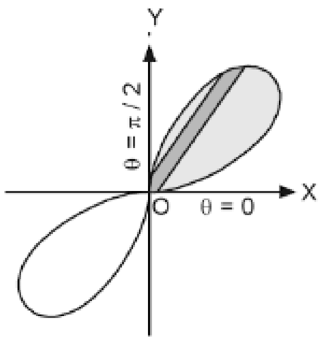
$$r^2(\cos^4\theta + \sin^4\theta) = 2a^2\sin\theta\cos\theta$$

$$r^2 = \frac{2a^2\sin\theta\cos\theta}{\sin^4\theta + \cos^4\theta} \qquad \text{...(2)}$$

Putting $r = 0$, $\quad 2a^2\sin\theta\cos\theta = 0$

$\Rightarrow \qquad \sin\theta = 0 \qquad \Rightarrow \qquad \theta = 0$

$\Rightarrow \qquad \cos\theta = 0 \qquad \Rightarrow \qquad \theta = \frac{\pi}{2}$



$$A = \int_0^{\frac{\pi}{2}} \int_0^{\sqrt{\frac{2a^2\sin\theta\cos\theta}{\sin^4\theta+\cos^4\theta}}} r\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} d\theta \left[\frac{r^2}{2}\right]_0^{\sqrt{\frac{2a^2\sin\theta\cos\theta}{\sin^4\theta+\cos^4\theta}}}$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} \frac{2a^2\sin\theta\cos\theta}{\sin^4\theta+\cos^4\theta}\,d\theta$$

$$= a^2\int_0^{\frac{\pi}{2}} \frac{\tan\theta\sec^2\theta}{\tan^4\theta+1}\,d\theta$$

$$= \frac{a^2}{2}\int_0^{\infty} \frac{dt}{t^2+1} \qquad \begin{pmatrix}\text{Put } \tan^2\theta = t \\ 2\tan\theta\sec^2\theta\,d\theta = dt\end{pmatrix}$$

$$= \frac{a^2}{2}\left[\tan^{-1}t\right]_0^{\infty} = \frac{a^2}{2}(\tan^{-1}\infty) = \frac{a^2}{2}\,\frac{\pi}{2} = \frac{\pi a^2}{4} \qquad \textbf{Ans.}$$

Example 19. *Find the total area of the curve $r = a \sin 2\theta$.* (*M.U. II Semester 2000*)

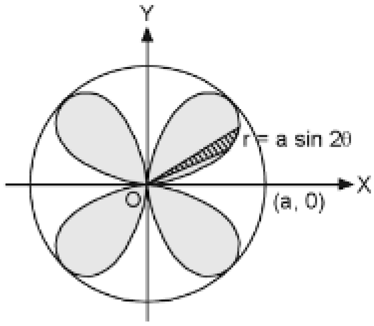
Solution. Here, we have

$$r = a \sin 2\theta$$

The curve has four leaves.

Draw a radial trip from origin to the curve. On this strip the limits of r are 0 and $r = a \sin 2\theta$ and the limits of θ are $\theta = 0$ and $\theta = \frac{\pi}{2}$.

All the leaves are equal in area.



$$\text{The Required Area} = 4\int_0^{\frac{\pi}{2}}\int_0^{a\sin 2\theta} r\,dr\,d\theta$$

$$= 4\int_0^{\frac{\pi}{2}} d\theta\left[\frac{r^2}{2}\right]_0^{a\sin 2\theta}$$

$$= 2\int_0^{\frac{\pi}{2}} a^2\sin^2 2\theta\,d\theta = 2a^2\int_0^{\frac{\pi}{2}}\frac{(1-\cos 4\theta)}{2}\,d\theta$$

$$= a^2\left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\frac{\pi}{2}} = \frac{\pi a^2}{2}$$ **Ans.**

Example 36. *Compute* $\iiint \frac{dx\,dy\,dz}{(x+y+z+1)^3}$ *if the region of integration is bounded by the coordinate planes and the plane $x + y + z = 1$.* (*M.U., II Semester 2007, 2006*)

Solution. Let the given region be R, then R is expressed as

$0 \le z \le 1 - x - y, \quad 0 \le y \le 1 - x, \quad 0 \le x \le 1.$

$$\iiint_R \frac{dx\,dy\,dz}{(x+y+z+1)^3} = \int_0^1 dx\int_0^{1-x} dy\int_0^{1-x-y}\frac{dz}{(x+y+z+1)^3}$$

$$= \int_0^1 dx\int_0^{1-x} dy\left[\frac{1}{-2(x+y+z+1)^2}\right]_0^{1-x-y}$$

$$= -\frac{1}{2}\int_0^1 dx\int_0^{1-x} dy\left[\frac{1}{(x+y+1-x-y+1)^2} - \frac{1}{(x+y+1)^2}\right]$$

$$= -\frac{1}{2}\int_0^1 dx\int_0^{1-x}\left[\frac{1}{4} - \frac{1}{(x+y+1)^2}\right]dy = -\frac{1}{2}\int_0^1 dx\left[\frac{y}{4} + \frac{1}{x+y+1}\right]_0^{1-x}$$

$$= -\frac{1}{2}\int_0^1 dx\left[\frac{1-x}{4} + \frac{1}{x+1+1-x} - \frac{1}{x+1}\right] = -\frac{1}{2}\int_0^1\left[\frac{1-x}{4} + \frac{1}{2} - \frac{1}{x+1}\right]dx$$

$$= -\frac{1}{2}\left[-\frac{(1-x)^2}{8} + \frac{x}{2} - \log(x+1)\right]_0^1 = -\frac{1}{2}\left[\frac{1}{2} - \log 2 + \frac{1}{8}\right] = -\frac{1}{2}\left[\frac{5}{8} - \log 2\right]$$

$$= \frac{1}{2}\log 2 - \frac{5}{16}$$ **Ans.**

Example 21. *Find the area inside the cardioid $r = a(1 + \cos\theta)$ and outside the circle $r = 2a\cos\theta$.*

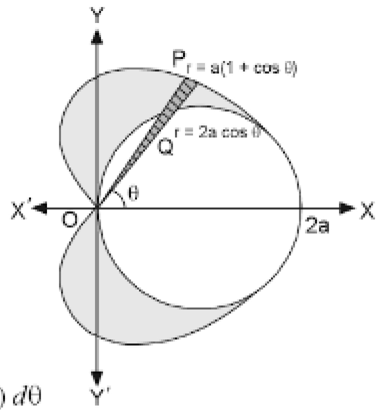
Solution. Here, we have

$$r = a(1 + \cos\theta) \qquad \text{(Cardioid)} \qquad ...(1)$$

$$r = 2a\cos\theta \qquad \text{(Circle)} \qquad ...(2)$$

Draw a radial strip.

On this strip the limits of r are $r = a(1 + \cos\theta)$ and $r = 2a\cos\theta$ and the limits of θ are $\theta = 0$ and $\theta = \pi$ in the first quadrant.



$$\text{Required Area} = 2\iint r\,dr\,d\theta$$

$$= 2\int_0^{\pi} d\theta \int_{2a\cos\theta}^{a(1+\cos\theta)} r\,dr$$

$$= 2\int_0^{\pi} d\theta \left[\frac{r^2}{2}\right]_{2a\cos\theta}^{a(1+\cos\theta)}$$

$$= \int_0^{\pi} d\theta\,[a^2(1+\cos\theta)^2 - 4a^2\cos^2\theta]$$

$$= a^2\int_0^{\pi}(1+\cos^2\theta + 2\cos\theta - 4\cos^2\theta)\,d\theta$$

$$= a^2\int_0^{\pi}(1 - 3\cos^2\theta + 2\cos\theta)\,d\theta$$

$$= a^2\int_0^{\pi}\left(1 - 3\left(\frac{1+\cos 2\theta}{2}\right) + 2\cos\theta\right)d\theta$$

$$= a^2\left[\theta - \frac{3}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) + 2\sin\theta\right]_0^{\pi} = a^2\left[\pi - \frac{3\pi}{2}\right] = -\frac{\pi a^2}{2}$$

$$= \frac{\pi a^2}{2} \text{ (Numerically)}$$ **Ans.**

Example 32. *Evaluate* $\iiint_R (x+y+z)\ dx\,dy\,dz$, *where* $R: 0 \le x \le 1,\ 1 \le y \le 2,\ 2 \le z \le 3.$

Solution. $\int_0^1 dx \int_1^2 dy \int_2^3 (x+y+z)\,dz = \int_0^1 dx \int_1^2 dy \left[\frac{(x+y+z)^2}{2}\right]_2^3$

$$= \frac{1}{2}\int_0^1 dx \int_1^2 dy\,[(x+y+3)^2 - (x+y+2)^2] = \frac{1}{2}\int_0^1 dx \int_1^2 (2x+2y+5)\,.\,1\,.dy$$

$$= \frac{1}{2}\int_0^1 dx \left[\frac{(2x+2y+5)^2}{4}\right]_1^2 = \frac{1}{8}\int_0^1 dx\,[(2x+4+5)^2 - (2x+2+5)^2]$$

$$= \frac{1}{8}\int_0^1 (4x+16)\,.\,2\,dx = \int_0^1 (x+4)\,dx = \left[\frac{x^2}{2}+4x\right]_0^1 = \frac{1}{2}+4 = \frac{9}{2}$$ **Ans.**

Example 33. *Evaluate the integral :* $\int_0^{\log 2} \int_0^x \int_0^{x+\log y} e^{x+y+z}\,dz\,dy\,dx.$

Solution. $\int_0^{\log 2} \int_0^x \int_0^{x+\log y} e^{x+y+z}\,dz\,dy\,dx.$

$$= \int_0^{\log 2} e^x dx \int_0^x e^y dy \int_0^{x+\log y} e^z dz = \int_0^{\log 2} e^x\,dx \int_0^x e^y dy (e^z)_0^{x+\log y}$$

$$= \int_0^{\log 2} e^x dx \int_0^x e^y dy\,(e^{x+\log y} - 1) = \int_0^{\log 2} e^x\,dx \int_0^x e^y dy\,(e^{\log y}\,.\,e^x - 1)$$

$$= \int_0^{\log 2} e^x dx \int_0^x e^y\,(y\,e^x - 1)\,dy = \int_0^{\log 2} e^x dx \left[(ye^x - 1)\,e^y - \int e^x\,.\,e^y\,dy\right]_0^x$$

$$= \int_0^{\log 2} e^x dx \left[(ye^x - 1)\,e^y - e^{x+y}\right]_0^x = \int_0^{\log 2} e^x dx\,[(xe^x - 1)\,e^x - e^{2x} + 1 + e^x]$$

$$= \int_0^{\log 2} e^x dx\,[xe^{2x} - e^x - e^{2x} + 1 + e^x] = \int_0^{\log 2} (xe^{3x} - e^{3x} + e^x)\,dx$$

$$= \left[x\,\frac{e^{3x}}{3} - \int 1\,.\,\frac{e^{3x}}{3}\,dx - \frac{e^{3x}}{3} + e^x\right]_0^{\log 2} = \left[\frac{x}{3}e^{3x} - \frac{e^{3x}}{9} - \frac{e^{3x}}{3} + e^x\right]_0^{\log 2}$$

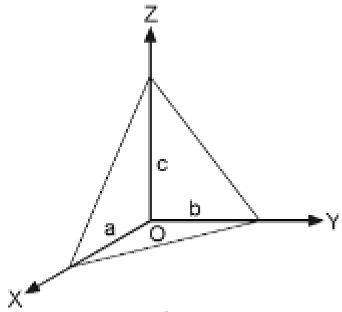
$$= \frac{\log 2}{3}\,e^{3\log 2} - \frac{e^{3\log 2}}{9} - \frac{e^{3\log 2}}{3} + e^{\log 2} + \frac{1}{9} + \frac{1}{3} - 1$$

$$= \frac{\log 2}{3}\,e^{\log 2^3} - \frac{e^{\log 2^3}}{9} - \frac{e^{\log 2^3}}{3} + e^{\log 2} + \frac{1}{9} + \frac{1}{3} - 1$$

$$= \frac{8}{3}\log 2 - \frac{8}{9} - \frac{8}{3} + 2 + \frac{1}{9} + \frac{1}{3} - 1 = \frac{8}{3}\log 2 - \frac{19}{9}$$ **Ans.**

Example 37. *Evaluate* $\iiint x^2yz\ dx\,dy\,dz$ *throughout the volume bounded by the planes* $x = 0$, $y = 0$, $z = 0$, $\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1$. (*M.U. II Semester* 2003, 2002, 2001)

Solution. Here, we have



$$I = \iiint x^2yz\ dx\,dy\,dz \quad ...(1)$$

Putting $x = au$, $y = bv$, $z = cw$

$dx = a\ du$, $dy = b\ dv$, $dz = c\ dw$ in (1), we get

$$I = \iiint a^2bc\,u^2vw\ a\,bc\ du\ dv\ dw$$

Limits are for $u = 0, 1$ for $v = 0, 1 - u$ and for $w = 0, 1 - u - v$

$u + v + w = 1$

$$I = \int_{u=o}^{1}\int_{v=0}^{1-u}\int_{w=0}^{1-u-v} a^3b^2c^2\,u^2vw\ du\ dv\ dw = \int_0^1\int_0^{1-u} a^3b^2c^2u^2v\left[\frac{w^2}{2}\right]_0^{1-u-v} du\ dv$$

$$= \frac{a^3b^2c^2}{2}\int_0^1\int_0^{1-u} u^2v(1-u-v)^2\ du\ dv$$

$$= \frac{a^3b^2c^2}{2}\int_0^1\int_0^{1-u} u^2v\left[(1-u)^2 - 2(1-u)v + v^2\right] du\ dv$$

$$= \frac{a^3b^2c^2}{2}\int_0^1\int_0^{1-u} u^2\,[(1-u)^2v - 2\,(1-u)\,v^2 + v^3]\ du\ dv$$

$$= \frac{a^3b^2c^2}{2}\int_0^1 u^2\left[(1-u)^2\frac{v^2}{2} - 2(1-u)\frac{v^3}{3} + \frac{v^4}{4}\right]_0^{1-u} du$$

$$= \frac{a^3b^2c^2}{2}\int_0^1 u^2\left[\frac{(1-u)^4}{2} - \frac{2(1-u)^4}{3} + \frac{(1-u)^4}{4}\right] du$$

$$= \frac{a^3b^2c^2}{2}\int_0^1 \frac{u^2(1-u)^4}{12}\ du = \frac{a^3b^2c^2}{24}\int_0^1 u^{3-1}\,(1-u)^{5-1}\ du$$

$$= \frac{a^3b^2c^2}{24}\,\beta(3, 5) = \frac{a^3b^2c^2}{24}\cdot\frac{\lceil 3\ \lceil 5}{\lceil 8} = \frac{a^3b^2c^2}{24}\cdot\left(\frac{2!\,4!}{7!}\right) = \frac{a^3b^2c^2}{2520}.$$ **Ans.**

Example 38. *Evaluate the integral* $\iiint (x^2 + y^2 + z^2)\ dx\ dy\ dz$ *taken over the volume enclosed by the sphere* $x^2 + y^2 + z^2 = 1$.

Solution. Let us convert the given integral into spherical polar co-ordinates. By putting

$$x = r \sin\theta \cos\phi\ ; \quad y = r \sin\theta \sin\phi\ ; \quad z = r\cos\theta$$

$$\iiint (x^2 + y^2 + z^2)\ dx\ dy\ dz = \int_0^{2\pi}\int_0^{\pi}\int_0^1 r^2(r^2 \sin\theta\ d\theta\ d\phi\ dr)$$

$$= \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\ d\theta \int_0^1 r^4 dr = \int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta\ d\theta \left(\frac{r^5}{5}\right)_0^1 = \frac{1}{5}\int_0^{2\pi} d\phi\,[-\cos\theta]_0^{\pi} = \frac{2}{5}\int_0^{2\pi} d\phi$$

$$= \frac{2}{5}(\phi)_0^{2\pi} = \frac{4\pi}{5}$$ **Ans.**

Example 39. *Evaluate* $\iiint (x^2 + y^2 + z^2)\ dx\ dy\ dz$ *over the first octant of the sphere* $x^2 + y^2 + z^2 = a^2$. *(M.U. II Semester 2007)*

Solution. Here, we have

$$I = \iiint (x^2 + y^2 + z^2)\,dx\,dy\,dz \qquad ...(1)$$

Putting $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ and $dx\,dy\,dz = r^2 \sin\theta\ dr\ d\theta\ d\phi$ in (1), we get

Limits of r are 0, a for θ are 0, $\frac{\pi}{2}$ for ϕ are 0, $\frac{\pi}{2}$.

$$I = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^a r^2 . r^2 \sin\theta\ dr\ d\theta\ d\phi = \int_0^{\frac{\pi}{2}} d\phi \int_0^{\frac{\pi}{2}} \sin\theta\ d\theta \int_0^a r^4\ dr$$

$$\left(\begin{aligned} x^2 + y^2 + z^2 &= r^2\sin^2\theta\cos^2\phi + r^2\sin^2\theta\sin^2\phi + r^2\cos^2\theta \\ &= r^2\sin^2\theta + r^2\cos^2\theta = r^2 \end{aligned}\right)$$

$$= [\phi]_0^{\pi/2}\,[-\cos\theta]_0^{\pi/2}\left[\frac{r^5}{5}\right]_0^a = \frac{\pi}{2}.(1).\frac{a^5}{5} = \pi.\frac{a^5}{10}.$$ **Ans.**

Example 40. *Evaluate* $\iiint \frac{dx\ dy\ dz}{x^2 + y^2 + z^2}$ *throughout the volume of the sphere* $x^2 + y^2 + z^2 = a^2$.

(M.U. II Semester 2002, 2001)

Solution. Here, we have

$$I = \iiint \frac{dx\ dy\ dz}{x^2 + y^2 + z^2} \qquad ...(1)$$

Putting $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ and $dx\,dy\,dz = r^2\sin\theta\ dr\ d\theta\ d\phi$ in (1), we get

The limits of r are 0 and a, for θ are 0 and $\frac{\pi}{2}$ for ϕ are 0 and $\frac{\pi}{2}$ in first octant.

$$I = 8\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^a \frac{r^2\sin\theta\ dr\ d\theta\ d\phi}{r^2} \qquad \text{[Sphere } x^2 + y^2 + z^2 \text{ lies in 8 quadrants]}$$

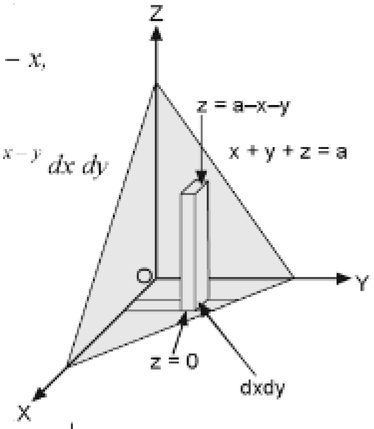
$$I = 8\int_0^{\frac{\pi}{2}} d\phi \int_0^{\frac{\pi}{2}} \sin\theta\ d\theta \int_0^a dr = 8\,[\phi]_0^{\pi/2}\,[-\cos\theta]_0^{\pi/2}\,[r]_0^a = 8\left(\frac{\pi}{2} - 0\right)(0+1)(a+0)$$

$$= 8\frac{\pi}{2}.1.a = 4\pi a$$ **Ans.**

Example 41. *Find the volume of the tetrahedron bounded by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = a$.* *(M.U. II Semester, 2005, 2000)*

Solution. Here, we have a solid which is bounded by $x = 0$, $y = 0$, $z = 0$ and $x + y + z = a$ planes.

The limits of z are 0 and $a - x - y$, the limits of y are 0 and $1 - x$, the limits of x are 0 and a.



$$V = \int_{x=0}^{a} \int_{y=0}^{a-x} \int_{z=0}^{a-x-y} dx\, dy\, dz = \int_{x=0}^{a} \int_{y=0}^{a-x} [z]_0^{a-x-y} dx\, dy$$

$$= \int_{x=0}^{a} \int_{y=0}^{a-x} (a - x - y)\, dx\, dy$$

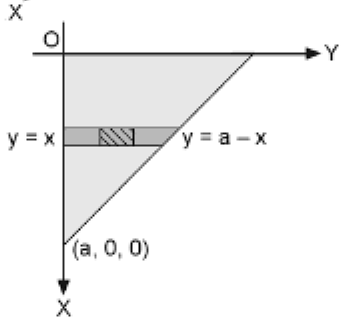
$$= \int_{x=0}^{a} \left[ay - xy - \frac{y^2}{2}\right]_0^{a-x} dx$$

$$= \int_{0}^{a} \left[a(a-x) - x(a-x) - \frac{(a-x)^2}{2}\right] dx$$

$$= \int_{0}^{a} \left[a^2 - ax - ax + x^2 - \frac{a^2}{2} + ax - \frac{x^2}{2}\right] dx$$

$$= \int_{0}^{a} \left(\frac{a^2}{2} - ax + \frac{x^2}{2}\right) dx$$

$$= \left[\frac{a^2}{2} . x - \frac{ax^2}{2} + \frac{x^3}{6}\right]_0^a = a^3\left(\frac{1}{2} - \frac{1}{2} + \frac{1}{6}\right) = \frac{a^3}{6}.$$ **Ans.**

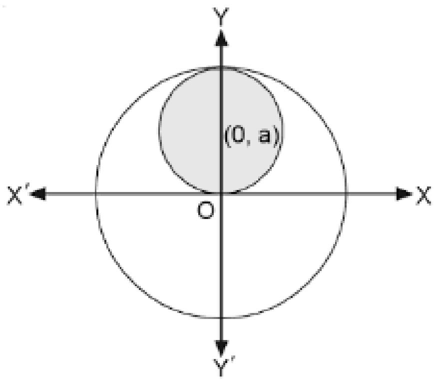


Example 44. *Find the volume of a solid bounded by the spherical surface $x^2 + y^2 + z^2 = 4a^2$ and the cylinder $x^2 + y^2 - 2ay = 0$.*

Solution. $$x^2 + y^2 + z^2 = 4a^2 \qquad ...(1)$$

$$x^2 + y^2 - 2ay = 0 \qquad ...(2)$$

Considering the section in the positive quadrant of the xy-plane and taking z to be positive (that is volume above the xy-plane) and changing to polar co-ordinates, (1) becomes



$$r^2 + z^2 = 4a^2 \quad \Rightarrow \quad z^2 = 4a^2 - r^2$$

$$\therefore \qquad z = \sqrt{4a^2 - r^2}$$

(2) becomes $r^2 - 2ar\sin\theta = 0 \Rightarrow r = 2a\sin\theta$

$$\text{Volume} = \iiint dx\,dy\,dz$$

$$= 4\int_0^{\pi/2} d\theta \int_0^{2a\sin\theta} r\,dr \int_0^{\sqrt{4a^2-r^2}} dz \qquad \text{(Cylindrical coordinates)}$$

$$= 4\int_0^{\pi/2} d\theta \int_0^{2a\sin\theta} r\,dr\,[z]_0^{\sqrt{4a^2-r^2}} = 4\int_0^{\pi/2} d\theta \int_0^{2a\sin\theta} r\,dr\,.\sqrt{4a^2-r^2}$$

$$= 4\int_0^{\pi/2} d\theta \left[-\frac{1}{3}(4a^2-r^2)^{3/2}\right]_0^{2a\sin\theta} = \frac{4}{3}\int_0^{\pi/2}\left[-(4a^2-4a^2\sin^2\theta)^{3/2}+8a^3\right]d\theta$$

$$= \frac{4}{3}\int_0^{\pi/2}(-8a^3\cos^3\theta+8a^3)\,d\theta = \frac{8\times 4a^3}{3}\int_0^{\pi/2}(1-\cos^3\theta)\,d\theta$$

$$= \frac{32a^3}{3}\int_0^{\pi/2}\left(1-\frac{1}{4}\cos 3\theta-\frac{3}{4}\cos\theta\right)d\theta$$

$$= \frac{32a^3}{3}\left[\theta-\frac{1}{12}\sin 3\theta-\frac{3}{4}\sin\theta\right]_0^{\pi/2} = \frac{32a^3}{3}\left(\frac{\pi}{2}+\frac{1}{12}-\frac{3}{4}\right) = \frac{32a^3}{3}\left[\frac{\pi}{2}-\frac{2}{3}\right] \textbf{ Ans.}$$

Example 45. *Find the volume enclosed by the solid*

$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} + \left(\frac{z}{c}\right)^{2/3} = 1$$

Solution. The equation of the solid is

$$\left(\frac{x}{a}\right)^{2/3} + \left(\frac{y}{b}\right)^{2/3} + \left(\frac{z}{c}\right)^{2/3} = 1$$

Putting
$$\left(\frac{x}{a}\right)^{1/3} = u \quad \Rightarrow \quad x = a\,u^3 \quad \Rightarrow \quad d\,x = 3\,au^2\,du$$
$$\left(\frac{y}{b}\right)^{1/3} = v \quad \Rightarrow \quad y = b\,v^3 \quad \Rightarrow \quad d\,y = 3\,bv^2\,dv$$
$$\left(\frac{z}{c}\right)^{1/3} = w \quad \Rightarrow \quad z = c\,w^3 \quad \Rightarrow \quad d\,z = 3\,c\,w^2\,dw$$

The equation of the solid becomes

$$u^2 + v^2 + w^2 = 1 \qquad \ldots(1)$$

$$V = \iiint dx\,dy\,dz \qquad \ldots(2)$$

On putting the values of dx, dy and dz in (2), we get

$$V = \iiint 27abc\,u^2v^2w^2 du\,dv\,dw \qquad \ldots(3)$$

(1) represents a sphere.
Let us use spherical coordinates.

$$u = r\sin\theta\cos\phi, \qquad v = r\sin\theta\sin\phi,$$
$$w = r\cos\theta, \qquad du\,dv\,dw = r^2\sin\theta\,dr\,d\theta\,d\phi$$

On substituting spherical coordinates in (3), we have

$$V = 27abc\,.\,8\int_{r=0}^{1}\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\pi/2} r^2\sin^2\theta\cos^2\phi\,.\,r^2\sin^2\theta\sin^2\phi\,.\,r^2\cos^2\theta\,.\,r^2\sin\theta\,dr\,d\theta\,d\phi$$

$$= 216\,abc\int_{r=0}^{1} r^8\,dr\int_{\phi=0}^{\pi/2}\sin^2\phi\cos^2\phi\,d\phi\int_{\theta=0}^{\pi/2}\sin^5\theta\cos^2\theta\,d\theta$$

$$= 216\,abc\left[\frac{r^9}{9}\right]_0^1 .\left(\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{2\,\Gamma(3)}\right)\left(\frac{\Gamma(3)\,\Gamma\left(\frac{3}{2}\right)}{2\,\Gamma\left(\frac{9}{2}\right)}\right) = 24\,abc\,.\,\frac{1}{2}\,.\,\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma(3)}\,.\,\frac{1}{2}\,.\,\frac{\Gamma(3)\,\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{9}{2}\right)}$$

$$= 6abc\,.\,\frac{\left[\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)\right]^2}{2!}\,.\,\frac{2!\,\Gamma\left(\frac{3}{2}\right)}{\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\frac{3}{2}\,\Gamma\left(\frac{3}{2}\right)} = 6abc\,.\,\frac{1}{4}\,.\,\pi\,\frac{1}{\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)} = \frac{4}{35}\,abc\,\pi$$

Ans.

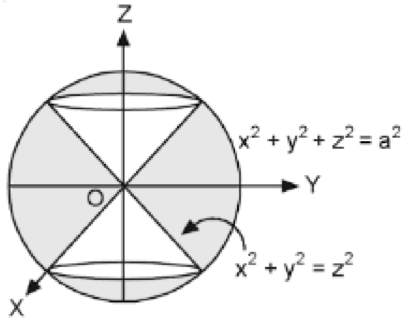
Example 46. *Find the volume bounded above by the sphere $x^2 + y^2 + z^2 = a^2$ and below by the cone $x^2 + y^2 = z^2$.* (*U.P. II Semester 2002*)

Solution. The equation of the sphere is $x^2 + y^2 + z^2 = a^2$ ...(1)

and that of the cone is $x^2 + y^2 = z^2$ ...(2)

In polar coordinates $x = r \sin\theta \cos\phi$, $y = r \sin\theta \sin\phi$, $z = r\cos\theta$

The equation (1) in polar co-ordinates is



$$(r \sin\theta \cos\phi)^2 + (r \sin\theta \sin\phi)^2 + (r\cos\theta)^2 = a^2$$

$$\Rightarrow \quad r^2 \sin^2\theta \cos^2\phi + r^2 \sin^2\theta \sin^2\phi + r^2\cos^2\theta = a^2$$

$$\Rightarrow \quad r^2 \sin^2\theta (\cos^2\phi + \sin^2\phi) + r^2\cos^2\theta = a^2$$

$$\Rightarrow \quad r^2\sin^2\theta + r^2\cos^2\theta = a^2$$

$$\Rightarrow \quad r^2(\sin^2\theta + \cos^2\theta) = a^2$$

$$\Rightarrow \quad r^2 = a^2 \quad \Rightarrow \quad r = a$$

The equation (2) in polar co-ordinates is

$$(r \sin\theta \cos\phi)^2 + (r \sin\theta \sin\phi)^2 = (r\cos\theta)^2$$

$$\Rightarrow \quad r^2\sin^2\theta(\cos^2\phi + \sin^2\phi) = r^2\cos^2\theta \quad \Rightarrow \quad r^2\sin^2\theta = r^2\cos^2\theta$$

$$\Rightarrow \quad \tan^2\theta = 1 \quad \Rightarrow \tan\theta = 1 \Rightarrow \quad \theta = \pm\frac{\pi}{4}$$

The volume in the first octant is one fourth only.

Limits in the first octant : r varies 0 to a, θ from 0 to $\frac{\pi}{4}$ and ϕ from 0 to $\frac{\pi}{2}$.

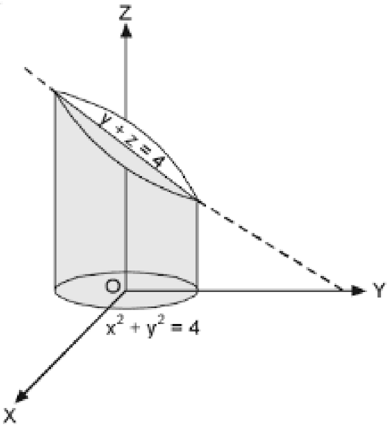
The required volume lies between $x^2 + y^2 + z^2 = a^2$ and $x^2 + y^2 = z^2$.

$$V = 4\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{4}}\int_0^a r^2 \sin\theta \, dr\, d\theta\, d\phi = 4\int_0^{\frac{\pi}{2}} d\phi \int_0^{\frac{\pi}{4}} \sin\theta\, d\theta \left[\frac{r^3}{3}\right]_0^a$$

$$= 4\int_0^{\frac{\pi}{2}} d\phi \int_0^{\frac{\pi}{4}} \sin\theta\, d\theta \,.\, \frac{a^3}{3} = \frac{4a^3}{3}\int_0^{\frac{\pi}{2}} d\phi\,[-\cos\theta]_0^{\frac{\pi}{4}} = \frac{4a^3}{3}(\phi)_0^{\frac{\pi}{2}}\left[-\frac{1}{\sqrt{2}} + 1\right]$$

$$= \frac{2}{3}\pi a^3\left(1 - \frac{1}{\sqrt{2}}\right)$$ **Ans.**

Example 49. *Find the volume bounded by the cylinder $x^2 + y^2 = 4$ and the planes $y + z = 4$ and $z = 0$.*



Solution. $x^2 + y^2 = 4 \Rightarrow y = \pm\sqrt{4 - x^2}$

$y + z = 4 \quad \Rightarrow \quad z = 4 - y$ and $z = 0$

x varies from -2 to $+2$.

$$V = \iiint dx\, dy\, dz = \int_{-2}^{2} dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy \int_0^{4-y} dz$$

$$= \int_{-2}^{2} dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy\, [z]_0^{4-y}$$

$$= \int_{-2}^{2} dx \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dy\, (4 - y) = \int_{-2}^{2} dx \left[4y - \frac{y^2}{2}\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}$$

$$= \int_{-2}^{2} dx \left[4\sqrt{4 - x^2} - \frac{1}{2}(4 - x^2) + 4\sqrt{4 - x^2} + \frac{1}{2}(4 - x^2)\right]$$

$$= 8\int_{-2}^{2} \sqrt{4 - x^2}\, dx = 8\left[\frac{x}{2}\sqrt{4 - x^2} + \frac{4}{2}\sin^{-1}\frac{x}{2}\right]_{-2}^{2} = 16\pi$$

Ans.

Example 28. *Changing the order of integration of* $\int_0^\infty \int_0^\infty e^{-xy} \sin nx\, dx\, dy$ *show that*

$$\int_0^\infty \frac{\sin nx}{x}\, dx = \frac{\pi}{2}.$$

(*AMIETE, Dec. 2010, U.P. I Semester winter 2003, A.M.I.E., Summer 2000*)

Solution. The region of integration is bounded by $x = 0$, $x = \infty$, $y = 0$, $y = \infty$, *i.e.*, first quadrant.

$$\int_0^\infty \int_0^\infty e^{-xy} \sin nx\, dx\, dy = \int_0^\infty dy \int_0^\infty e^{-xy} \sin nx\, dx$$

$$= \int_0^\infty dy \left[\frac{e^{-xy}}{n^2 + y^2}\{-y \sin x - n \cos n x\}\right]_0^\infty$$

$$= \int_0^\infty dy \left[0 + \frac{n}{n^2 + y^2}\right] = \int_0^\infty \frac{n}{n^2 + y^2}\, dy = \left[\tan^{-1} y\right]_0^\infty = \frac{\pi}{2} \qquad \ldots(1)$$

On changing the order of integration

$$\int_0^\infty \int_0^\infty e^{-xy} \sin nx\, dx\, dy = \int_0^\infty \sin nx\, dx \int_0^\infty e^{-xy}\, dy$$

$$= \int_0^\infty \sin nx\, dx \left[\frac{e^{-xy}}{-x}\right]_0^\infty = \int_0^\infty \frac{\sin nx\, dx}{x}\left[-\frac{1}{e^{xy}}\right]_0^\infty$$

$$= \int_0^\infty \frac{\sin nx}{x}\, dx\, [-0 + 1] = \int_0^\infty \frac{\sin nx}{x}\, dx \qquad \ldots(2)$$

From (1) and (2), $\int_0^\infty \frac{\sin nx}{x}\, dx = \frac{\pi}{2}$ **Proved.**

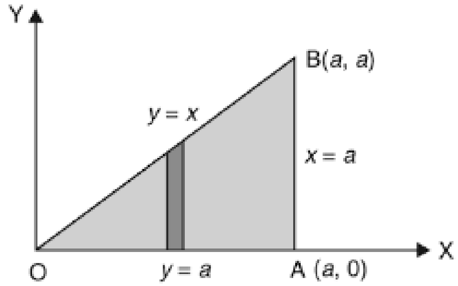
V = a

Example 30. *Evaluate* $\int_0^a \int_0^x \frac{f'(y)\, dy\, dx}{[(a-x)(x-y)]^{1/2}}$

Solution. Let $I = \int_0^a \int_0^x \frac{f'(y)\, dy\, dx}{[(a-x)(x-y)]^{1/2}}$

Here the limits are $x = 0$, $x = a$ and $y = 0$, $y = x$. Evidently the region of integration is $OABO$.

By changing the order of integration, we have



$$I = \int_0^a \int_y^a \frac{f'(y)\, dy\, dx}{[(a-x)(x-y)]^{1/2}}$$

$$= \int_0^a f'(y)\, dy \int_y^a \frac{dx}{\sqrt{(a-x)(x-y)}} \quad ...(1)$$

Let us find the values of $(a-x)$ and $(x-y)$ for (1)

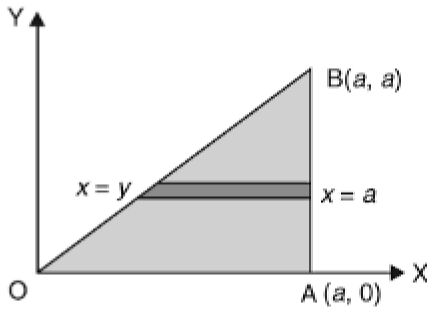
Putting $x = a\cos^2\theta + y\sin^2\theta$

We have $a - x = a - a\cos^2\theta - y\sin^2\theta$

$= a(1-\cos^2\theta) - y\sin^2\theta$

$a - x = a\sin^2\theta - y\sin^2\theta = (a-y)\sin^2\theta \quad ...(2)$

$\Rightarrow \quad -dx = 2(a-y)\sin\theta\cos\theta\, d\theta$, keeping y constant.



Also, $x - y = a\cos^2\theta + y\sin^2\theta - y$

$= a\cos^2\theta - y(1-\sin^2\theta)$

$= a\cos^2\theta - y\cos^2\theta$

$= (a-y)\cos^2\theta \quad ...(3)$

$dx = -2(a-y)\sin\theta\cos\theta\, d\theta \quad ...(4)$

when $x = y$ then $x - y = 0$

Upper limt $x = a$

$x - y = (a-y)\cos^2\theta$

$a - y = (a-y)\cos^2\theta$

$\Rightarrow \cos^2\theta = 1 \Rightarrow \theta = 0$

lower limit $x = y$

$x - y = (a-y)\cos^2\theta$

$x - x = (a-y)\cos^2\theta \Rightarrow 0 = \cos^2\theta \Rightarrow \theta = \frac{\pi}{2}$

Putting the values of $a - x$, $x - y$ and dx from (2), (3) and (4) respectively in (1), we get

$$I = \int_0^a f'(y)\, dy \int_{\frac{\pi}{2}}^0 \frac{-2(a-y)\sin\theta\cos\theta}{\sqrt{(a-y)\sin^2\theta.(a-y)\cos^2\theta}}\, d\theta$$

$$= \int_0^a f'(y)\, dy \int_{\frac{\pi}{2}}^0 \frac{-2(a-y)\sin\theta\cos\theta}{(a-y)\sin\theta\cos\theta}\, d\theta$$

$$= -2\int_0^a f'(y)\, dy \int_{\frac{\pi}{2}}^0 d\theta = 2\int_0^a f'(y)[\theta]_0^{\pi/2} = 2\int_0^a f'(y)\, dy . \frac{\pi}{2} = 2[f(y)]_0^a . \frac{\pi}{2} = [f(a) - f(0)]\,\pi$$

8. (a) Suppose that f is differentiable on $\mathbb{R}$ and has two roots. Show that f' has at least one root.

Noting that differentiability over the reals implies continuity over the reals, we have by Rolle's theorem–using the two roots–that there is a point where $f'(x) = 0$, as requested.

(b) Suppose f is twice differentiable on $\mathbb{R}$ and has three roots. Show that f'' has at least one real root.

Noting that differentiability over the reals implies continuity over the reals, we have by Rolle's theorem–using first the first and second roots, then the second and third roots–that there are two distinct points where $f'(x) = 0$. Thus, reasoning exactly as earlier, we have by Rolle's theorem–using the two roots of $f'(x)$–that there is a point where $f''(x) = 0$, as requested.

(c) Can you generalize parts (a) and (b)?
Yes; if f is n differentiable and has $n+1$ roots, then $f^{(n)}(x)$ has at least one real root.

6. $(\star)$ Using the mean value theorem and Rolle's theorem, show that $x^3 + x - 1 = 0$ has exactly one real root.

Noting that polynomials are continuous over the reals and $f(0) = -1$ while $f(1) = 1$, by the intermediate value theorem we have that $x^3 + x - 1 = 0$ has at least one real root.

We show, then, that $x^3 + x - 1 = 0$ cannot have more than one real root. Assume it does. Then, noting that polynomials are differentiable over the reals, we have by Rolle's theorem–using these two roots–that there is a point where $3x^2 + 1 = 0$ (the derivative of the original function).

$$\begin{aligned} 3x^2 + 1 &= 0 \\ x^2 &= -\frac{1}{3} \\ x &= \sqrt{-\frac{1}{3}} \end{aligned}$$

a contradiction. We have, then, that $x^3 + x - 1 = 0$ cannot have more than one real root which, combined with our earlier result gives that $x^3 + x - 1 = 0$ has exactly one real root as requested.

7. Show that the equation $x^4 + 4x + c = 0$ has at most two real roots.

Assume that the equation $x^4 + 4x + c = 0$ has more than two real roots. Then, noting that polynomials are differentiable and continuous over the reals, we have by Rolle's theorem–using first the first two roots, then the second and third roots–that there are two distinct points where $4x^3 + 4 = 0$ (the derivative of the original function).

$$\begin{aligned} 4x^3 + 4 &= 0 \\ 4x^3 &= -4 \\ x &= \sqrt[3]{-1} \\ &= -1 \end{aligned}$$

a contradiction (we were promised two!). We have, then, that $x^4 + 4x + c = 0$ has at most two real roots, as requested.

4. Let $f(x) = \frac{x^3 - x^2}{x - 1}$ on $[0, 2]$. Show that there is no value of c such that $f'(c) = \frac{f(2) - f(0)}{2 - 0}$. Is this a counterexample to the mean value theorem? Why or why not?

Note first that,

$$\frac{x^3 - x^2}{x - 1} = \frac{x^2(x - 1)}{x - 1}$$
$$= x^2$$

Taking the derivative,

$$f'(x) = 2x$$

Setting it equal to the slope of the secant line through the given points,

$$\begin{aligned} 2x &= \frac{f(0) - f(2)}{0 - 2} \\ &= \frac{0 - 4}{0 - 2} \\ &= 2 \\ x &= 1 \end{aligned}$$

Note, however, that 1 is not in the domain of the given function. This is not a counterexample since there is a point discontinuity at $x = 1$.

Example 128:

Find the volume of the region above the xy-plane bounded by the paraboloid $z = x^2 + y^2$ and the cylinder $x^2 + y^2 = a^2$.

Solution:

The given solid is bounded above by $z = x^2 + y2$ and below by $x^2 + y^2 = a^2$. Moreover the solid being symmetrical, its volume V is four times the volume lying in the first octant.

$$\therefore \quad V = 4 \iint_D z \, dx \, dy = 4 \iint_D (x^2 + y^2) \, dx \, dy$$

Changing to polars, we get

$$V = 4 \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} r^2 \, (r \, dr \, d\theta) = 4 . \frac{\pi}{2} . \frac{a^4}{4} = \frac{\pi a^4}{2}. \qquad \textbf{Ans.}$$

Example 127:

Find the volume common to the ellipsoid of revolution $x^2/a^2 + y^2/a^2$ $z^2/b^2 = 1$ and the cylinder $x^2 + y^2 - ay = 0$.

Solution:

The required volume is double the volume that lies above the xy-plane. The given solid is bounded above by $z = \frac{b}{a}\sqrt{a^2 - x^2 - y^2}$ and below by the circular base D : $x^2 + y^2 - ay = 0$ on the xy-plane. Thus the required volume is given by

$$V = 2\iint_D z\,dx\,dy = \iint_D \frac{b}{a}\sqrt{a^2 - x^2 - y^2}\,dx\,dy.$$

Changing to polars, we obtain

$$V = \frac{2b}{a}\int_{\theta=0}^{\pi}\int_{r=0}^{a\sin\theta}\sqrt{a^2 - r^2}\,(r\,dr\,d\theta)$$

$(x^2 + y^2 = ay \Rightarrow r^2 = ar\sin\theta \Rightarrow r = a\sin\theta)$

$$= \frac{2b}{a}\int_0^{\pi}\left|-\frac{1}{3}(a^2 - r^2)^{3/2}\right|_0^{a\sin\theta} d\theta$$

$$= \frac{4a^2b}{3}\int_0^{\pi/2}(1 - \cos^2\theta)\,d\theta = \frac{4a^3b}{3}\left[\frac{\pi}{2} - \frac{2}{3}\right]$$

$$= \frac{2a^2b}{9}(3\pi - 4).$$

Ans.

Example 124:

Find the volume cut off from the sphere $x^2 + y^2 + z^2 = a^2$ by the cylinder $x^2 + y^2 = ax$.

Solution:

We have

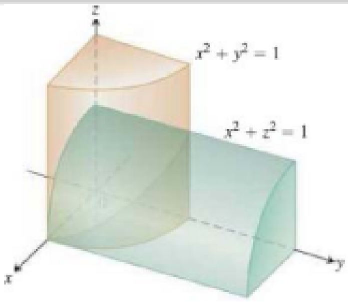
$$V = \iint_D dx\,dy \int_{-\sqrt{(a^2-x^2-y^2)}}^{\sqrt{(a^2-x^2-y^2)}} dz = 2 \iint_D \sqrt{a^2-x^2-y^2}\,dx\,dy$$

or $$V = 2\int_0^{\pi} d\theta \int_0^{a\cos\theta} \sqrt{a^2 - r^2}\,.\,r\,dr = -\frac{2}{3}\int_0^{\pi} \left|(a^2-r^2)^{3/2}\right|_0^{a\cos\theta}$$

$$= \frac{2}{3}a^3 \int_0^{\pi} (1 - \sin^3\theta) = \frac{2}{3}a^3\left[\pi - 2\,.\,\frac{2}{3}\right]$$

$$= \frac{2}{3}a^3\left(4 - \frac{4}{3}\right).$$ **Ans.**

Example. Determine the volume of the region D common to the interiors of the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$, in the first octant.



Solution. One immediately recognizes the solid D has a top given by $x^2 + z^2 = 1$, i.e. $z_{\max}(x) = \sqrt{1-x^2}$, and xy-plane as the bottom. Moreover, the shadow R of the solid D is a circle disk in the 1st quadrant, so R can be described in terms of polar coordinates as $\{\,(r,\theta) \mid 0 \le r \le 1,\ 0 \le \theta \le \pi/2\,\}$. The volume of D is given by $\int_0^{\pi/2}\int_0^1 \sqrt{1 - r^2\cos^2\theta}\,r\,dr\,d\theta \overset{*}{=} \frac{2}{3}$.

(*): the answer $\frac{2}{3}$ is given by integration via rectangular coordinates.

Example 123:

Find the volume of the solid bounded by the surface $z = 1 - 4x^2 - y^2$ and the plane $z = 0$.

Solution:

The paraboloid $z = 1 - 4x^2 - y^2$ cuts the xy-plane ($z = 0$) along the ellipse $D : 4x^2 + y^2 = 1$. Moreover thee solid being symmetrical, its volume V is four times the volume lying in the first octant. Then we have

$$\therefore \quad V = 4 \iint_D z \, dx \, dy$$

$$= 4 \int_0^{1/2} dx \int_0^{\sqrt{(1-4x^2)}} (1-4x^2-y^2) \, dy$$

$$= 4 \int_0^{1/2} \left| (1-4x^2)y - \frac{1}{3} y^3 \right|_0^{\sqrt{1-4x^2}}$$

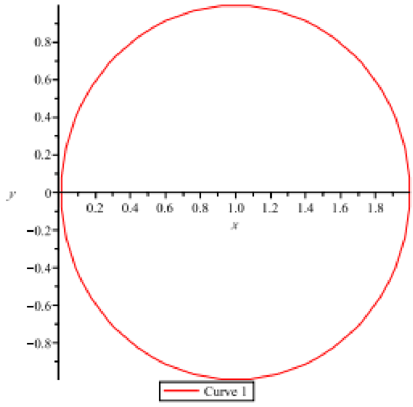
$$= \frac{8}{3} \int_0^{1/2} (1-4x^2)^{3/2} \, dx$$

$$= \frac{4}{3} \int_0^{\pi/2} \cos^4 t, \text{ where } x = \frac{1}{2} \sin t$$

$$= \frac{4}{3} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}. \qquad \textbf{Ans.}$$

Example. Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.

Completing the square, $(x-1)^2 + y^2 = 1$ is the shadow of the cylinder in the xy-plane.



Changing to polar coordinates, the shadow of the cylinder is $r^2 = 2r\cos\theta$ or $r = 2\cos\theta$, so

$$R = \left\{(r,\theta)\Big| -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \text{ and } 0 \le r \le 2\cos\theta\right\}.$$

$$V = \iint_R (x^2+y^2)\,dA = \int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^2\,r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\left(\frac{r^4}{4}\Big|_0^{2\cos\theta}\right)d\theta =$$

$$= \int_{-\pi/2}^{\pi/2} 4\cos^4\theta\,d\theta = 4\int_{-\pi/2}^{\pi/2}\cos^4\theta\,d\theta \underbrace{=}_{\#60} 4\left[\frac{1}{4}\cos^3\theta\sin\theta + \frac{3}{4}\int\cos^2\theta\,d\theta\right]_{-\pi/2}^{\pi/2}$$

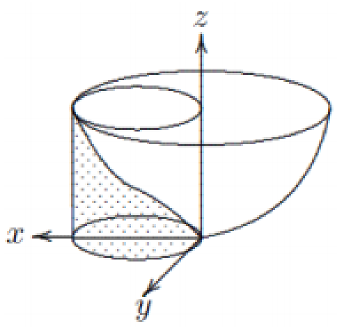
$$= \left[\cos^3\theta\sin\theta + 3\left(\frac{1}{2}\cos\theta\sin\theta + \frac{1}{2}\int d\theta\right)\right]_{-\pi/2}^{\pi/2}$$

$$= \left[\cos^3\theta\sin\theta + \frac{3}{2}\cos\theta\sin\theta + \frac{3}{2}\theta\right]_{-\pi/2}^{\pi/2}$$

$$= \left[\frac{3\pi}{4} - \left(-\frac{3\pi}{4}\right)\right] = \frac{3\pi}{2}.$$

Example. Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.

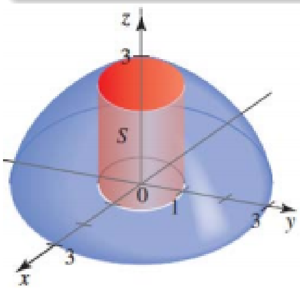
Solution. The cylinder $x^2 + y^2 = 2x$ lies over the circular disk D which can be described as $\{ (r, \theta) \mid -\pi/2 \le \theta \le \pi/2,\ 0 \le r \le 2r\cos\theta \}$ in polar coordinates. The reason is that if we write $(x, y, z) = (r\cos\theta, r\sin\theta, z)$ for any point in the cylinder, then $r^2 = x^2 + y^2 \le 2x = 2r\cos\theta$, i.e. $r \le 2\cos\theta$. As $2\cos\theta \ge r \ge 0$, it follows that



$-\pi/2 \le \theta \le \pi/2$. The height of the solid is the z-value of the paraboloid from the xy-plane. Hence the volume V of the solid is

$$\iint_D (x^2 + y^2)\, dA = \int_{-\pi/2}^{\pi/2} \int_0^{2\cos\theta} r^2 \cdot r\, dr\, d\theta = \frac{3\pi}{2}.$$

Example. Find the volume of the solid that lies below the hemisphere $z = \sqrt{9 - x^2 - y^2}$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 1$.



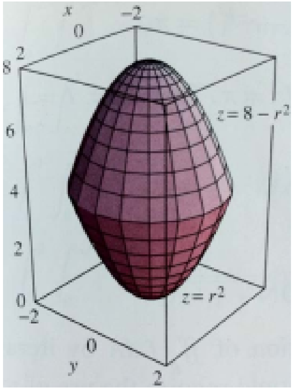
Solution. Let R be the shadow of D after projecting on xy-plane, then R is the circular disk centered at the origin with radius 1, in polar coordinates $\{ (r, \theta) \mid 0 \le r \le 1,\ 0 \le \theta \le 2\pi \}$. Moreover, the top $z_{\text{top}} = \sqrt{9 - x^2 - y^2} = \sqrt{9 - r^2}$, and $z_{\text{bottom}} = 0$.

Hence the volume of the solid D is

$$\int_0^{2\pi} \int_0^1 \sqrt{9 - r^2} \cdot r\, dr\, d\theta = \frac{2\pi}{-2} \int_0^1 \sqrt{9 - r^2}\, d(8 - r^2)$$

$$= -\pi \left[\frac{(9 - r^2)^{3/2}}{3/2} \right]_0^1 = \frac{2\pi}{3} \left[9^{3/2} - 8^{3/2} \right] = \frac{2\pi}{3}(27 - 16\sqrt{2}).$$

Example. Find the volume of the solid bounded above by the paraboloid $z = 8 - x^2 - y^2$, and below by the paraboloid $z = x^2 + y^2$.



Solution. Let $P(x, y, z)$ be the intersection of two paraboloids, then one has $8 - x^2 - y^2 = z = x^2 + y^2$, so $x^2 + y^2 = 4 = 2^2$, which is a circle. The shadow R of the solid D is then the circular disc, in polar coordinates $\{ (r, \theta) \mid 0 \le r \le 2,\ 0 \le \theta \le 2\pi \}$. As $0 \le r \le 2$, we have $r^2 \le 4$, i.e. $r^2 \le 8 - r^2$, so one knows that the top of the solid is given by $z_{\text{top}} = 8 - x^2 - y^2 = 8 - r^2$, and the bottom of the solid is given by $z_{\text{bottom}} = x^2 + y^2 = r^2$.

Hence the volume of the solid D is

$$\int_0^{2\pi} \int_0^2 (8 - r^2 - r^2) \cdot r\, dr\, d\theta = 2\pi \int_0^2 (8r - r^3)\, dr$$

$$= 2\pi \left[4r^2 - \frac{r^4}{4} \right]_0^2 = 24\pi.$$

Determine all p and q for which the following improper integral converges. Justify your answer.

$$\int_0^{\pi/2} \frac{dx}{(\sin^p x)(\cos^q x)}$$

I would consider the behavior at the endpoints. Near $x = 0$, $\sin x \sim x$, so that convergence of the integral requires that $p < 1$. Similarly, near $x = \pi/2$, $\cos x \sim (\pi/2) - x$, so that $q < 1$ for convergence. Therefore, p and q each must be less than 1 for convergence.

This function is known as the Beta function,

$$B(x, y) = 2 \int_0^{\pi/2} \sin(t)^{2x-1} \cos(t)^{2y-1}\, dt.$$

In particular, you are looking at

$$\frac{1}{2} B\left(\frac{1}{2} - \frac{p}{2}, \frac{1}{2} - \frac{q}{2}\right).$$

This converges as long as $p, q < 1$.

For a direct proof, note that by substitution

$$\int_0^{\pi/2} \sin(t)^{-p} \cos(t)^{-q}\, dt = \int_0^{\infty} \frac{t^{-p/2-1/2}}{(1+t)^{1-p/2-q/2}}\, dt.$$

For convergence around 0, we must have that $\frac{p}{2} < \frac{1}{2}$, and for convergence near ∞, we must have $\frac{q}{2} < \frac{1}{2}$. This yields $p, q < 1$.

$\boxed{2}$ For each of the following regions E, write the triple integral $\iiint_E f(x, y, z)\, dV$ as an iterated integral. There may be up to six different ways to do this, depending on whether you write it with $dx\, dy\, dz$ or $dz\, dy\, dx$ or $dx\, dz\, dy$ or...

(a) The tetrahedron bounded by the planes $x + y + z = 1$, $x = 0$, $y = 0$, and $z = 0$.

(b) The (solid) sphere $x^2 + y^2 + z^2 = a^2$.

(c) The region between the paraboloid $x = 1 - y^2 - z^2$ and the yz-plane.

(a) $\iiint_E f(x,y,z)\,dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} f(x,y,z)\,dz\,dy\,dx.$

(b) $\iiint_E f(x,y,z)\,dV = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2-y^2}}^{\sqrt{a^2-x^2-y^2}} f(x,y,z)\,dz\,dy\,dx.$

(c) This region is defined by the inequality $0 \le x \le 1 - y^2 - z^2$. For $1 - y^2 - z^2$ to be nonnegative, we also need (y,z) to lie on the disk D bounded by $y^2 + z^2 = 1$. Thus we get

$$\iiint_E f(x,y,z)\,dV = \iint_D \int_0^{1-y^2-z^2} f(x,y,z)\,dx\,dA$$
$$= \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_0^{1-y^2-z^2} f(x,y,z)\,dx\,dz\,dy.$$

(b) The region bounded by $z = x^2 + 3y^2$ and $z = 4 - y^2$.

Solution. The parabolic cylinder $z = 4 - y^2$ comprises the top of the surface (considered in terms of z) and the paraboloid $z = x^2 + 3y^2$ is the bottom surface in terms of z. To determine the region of the xy-plane which the region bounded by these two surfaces lies over, we intersect the two surfaces, in this case we can set them equal to each other. We see that $x^2 + 3y^2 = 4 - y^2$ if and only if $x^2 + 4y^2 = 4$ if and only if $(x/2)^2 + y^2 = 1$. We will set up and compute the integral of this volume in rectangular coordinates (using a table of integrals to compute the anti-derivative $\int (4 - x^2)^{3/2} dx$).

$$V = \int_{-2}^{2} \int_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2} \int_{x^2+3y^2}^{4-y^2} dzdydx = \int_{-2}^{2} \int_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2} (4 - x^2 - 4y^2)dydx$$
$$= \int_{-2}^{2} \left[(4-x^2)y - (4/3)y^3\right]_{-\sqrt{4-x^2}/2}^{\sqrt{4-x^2}/2} dx = 2\int_{-2}^{2} \left(\frac{(4-x^2)^{3/2}}{2} - \frac{(4-x^2)^{3/2}}{6} \right) dx$$
$$= \frac{2}{3}\int_{-2}^{2} (4-x^2)^{3/2} dx = \frac{2}{3}\left[\frac{x}{8}\left(5 \cdot 2^2 - 2x^2\right)\sqrt{4-x^2} + \frac{3 \cdot 2^4}{8}\sin^{-1}(x/2) \right)\Bigg]_{-2}^{2}$$
$$= 4(\sin^{-1}(1) - \sin^{-1}(-1)) = 4\pi$$

Another way to compute this integral would be to make a substitution $x = 2u$, so $dx = 2du$ and we would be integrate over a circle of radius 1 in (u,y), which we will call $\widetilde{R}$ whereas the ellipse will be called R. This makes everything much simpler. Lets see what happens.

Find the volume of the solid bounded above by the parabolic cylinder $z = 4 - y^2$ and bounded below by the elliptic paraboloid $z = x^2 + 3y^2$.

Solution. The two surfaces intersect in a space curve, whose projection on the xy-plane is the ellipse $x^2 + 4y^2 = 4$ or $\frac{x^2}{4} + y^2 = 1$. Substituting $x = 2r\cos\theta$ and $y = r\sin\theta$, the ellipse becomes $r^2 = 1$.

Further,

$$z = \phi_1(x) = 4 - y^2 \text{ and}$$
$$z = \phi_2(x) = x^2 + 3y^2.$$

Therefore,

$$\phi_1(x) - \phi_2(x) = 4 - y^2 - x^2 - 3y^2$$
$$= 4 - 4y^2 - x^2 = 4(1 - r^2).$$

Also,

$$J = \frac{\partial(x, y)}{\partial(r, \theta)} = 2.$$

Since the solid is symmetrical about x- and y-axis, we have

$$V = 4\int_0^{\frac{\pi}{2}}\int_0^1 4(1 - r^2)2r\,dr\,d\theta$$
$$= 32\int_0^{\frac{\pi}{2}}\int_0^1 (r - r^3)\,dr\,d\theta$$
$$= 32\int_0^{\frac{\pi}{2}}\left[\frac{r^2}{2} - \frac{r^4}{4}\right]_0^1 d\theta = \frac{32\pi}{8} = 4\pi.$$

Example 4. *Consider the sequence* $\{f_n\}$ *of functions defined by*

$$f_n(x) = \frac{\sin(nx+3)}{\sqrt{n+1}} \qquad \text{for all } x \text{ in } \mathbb{R}.$$

Show that $\{f_n\}$ *converges pointwise.*

Solution: For every x in $\mathbb{R}$, we have

$$\frac{-1}{\sqrt{n+1}} \leq \frac{\sin(nx+3)}{\sqrt{n+1}} \leq \frac{1}{\sqrt{n+1}}$$

Moreover,

$$\lim_{n\to\infty} \frac{1}{\sqrt{n+1}} = 0.$$

Applying the squeeze theorem for sequences, we obtain that

$$\lim_{n\to\infty} f_n(x) = 0 \qquad \text{for all } x \text{ in } \mathbb{R}.$$

Therefore, $\{f_n\}$ converges pointwise to the function $f \equiv 0$ on $\mathbb{R}$.

Example 5. *Consider the sequence* $\{f_n\}$ *of functions defined by* $f_n(x) = n^2x^n$ *for* $0 \leq x \leq 1$. *Determine whether* $\{f_n\}$ *is pointwise convergent.*

Solution: First of all, observe that $f_n(0) = 0$ for every n in $\mathbb{N}$. So the sequence $\{f_n(0)\}$ is constant and converges to zero. Now suppose $0 < x < 1$ then $n^2x^n = n^2e^{n\ln(x)}$. But $\ln(x) < 0$ when $0 < x < 1$, it follows that

$$\lim_{n\to\infty} f_n(x) = 0 \quad \text{for } 0 < x < 1$$

Finally, $f_n(1) = n^2$ for all n. So, $\lim\limits_{n\to\infty} f_n(1) = \infty$. Therefore, $\{f_n\}$ is not pointwise convergent on $[0, 1]$.

Example 7. *Consider the sequence of functions defined by*

$$f_n(x) = nx(1-x)^n \quad \text{on } [0,1].$$

Show that $\{f_n\}$ *converges pointwise to the zero function.*

Solution: Note that $f_n(0) = f_n(1) = 0$, for all $n \in \mathbb{N}$. Now suppose $0 < x < 1$, then

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} nxe^{n\ln(1-x)} = x \lim_{n\to\infty} ne^{n\ln(1-x)} = 0$$

because $\ln(1-x) < 0$ when $0 < x < 1$. Therefore, the given sequence converges pointwise to zero.

Example 8. *Let* $\{f_n\}$ *be the sequence of functions on* $\mathbb{R}$ *defined by*

$$f_n(x) = \begin{cases} n^3 & \text{if} \quad 0 < x \leq \frac{1}{n} \\ 1 & \text{otherwise} \end{cases}$$

Show that $\{f_n\}$ *converges pointwise to the constant function* $f = 1$ *on* $\mathbb{R}$.

Solution: For any x in $\mathbb{R}$ there is a natural number N such that x does not belong to the interval $(0,\ 1/N)$. The intervals $(0,\ 1/n)$ get smaller as $n \to \infty$. Therefore, $f_n(x) = 1$ for all $n > N$. Hence,

$$\lim_{n\to\infty} f_n(x) = 1 \quad \text{for all } x.$$

Example 9. *Let* $\{f_n\}$ *be the sequence of functions on* $(0,\ \infty)$ *defined by*

$$f_n(x) = \frac{nx}{1+n^2x^2}.$$

This function converges pointwise to zero. Indeed, $(1+n^2x^2) \sim n^2x^2$ as n gets larger and larger. So,

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{n^2x^2} = \frac{1}{x}\lim_{n\to\infty}\frac{1}{n} = 0.$$

But for any $\varepsilon < 1/2$, we have

$$\left| f_n\left(\frac{1}{n}\right) - f\left(\frac{1}{n}\right) \right| = \frac{1}{2} - 0 > \varepsilon.$$

Hence $\{f_n\}$ is not uniformly convergent.

Ex. 7 (e). *Evaluate $\iiint xyz\,dx\,dy\,dz$ for all positive values of the variables throughout the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.*

Sol. We are to evaluate the integral $\iiint xyz\,dx\,dy\,dz$, for all positive values of the variables x, y, z subject to the condition

$$x^2/a^2 + y^2/b^2 + z^2/c^2 \leq 1.$$

Put $x^2/a^2 = u, y^2/b^2 = v, z^2/c^2 = w$ so that

$$x\,dx = \tfrac{1}{2}a^2\,du, y\,dy = \tfrac{1}{2}b^2\,dv, z\,dz = \tfrac{1}{2}c^2\,dw.$$

Then the required integral

$= \frac{1}{8}a^2b^2c^2 \iiint du\,dv\,dw$, where $u + v + w \leq 1$

$= \frac{1}{8}a^2b^2c^2 \iiint u^{1-1}v^{1-1}w^{1-1}\,du\,dv\,dw$, where $u + v + w \leq 1$

$= \frac{1}{8}a^2b^2c^2 \dfrac{\Gamma(1)\Gamma(1)\Gamma(1)}{\Gamma(1+1+1+1)}$, by Dirichlet's integral

$$= \frac{a^2b^2c^2}{8}\frac{1}{\Gamma(4)} = \frac{a^2b^2c^2}{8} \times \frac{1}{3.2.1} = \frac{a^2b^2c^2}{48}.$$

Example 4.41: Find the volume of the solid bounded above by the parabolic cylinder $z = 4 - y^2$ and bounded below by the elliptic paraboloid $z = x^2 + 3y^2$.

Solution: Curve of intersection of parabolic cylinder $z = 4 - y^2$ and elliptic paraboloid $z = x^2 + 3y^2$ is

$$4 - y^2 = x^2 + 3y^2$$

or $\qquad x^2 + 4y^2 = 4 \qquad$ in x - y plane

$\therefore\ V = \text{volume} = \iiint\limits_E dx\,dy\,dz$, where E is the region $x^2 + (2y)^2 = 4;\ x^2 + 3y^2 \leq z \leq 4 - y^2$.

Let $\qquad x = r\cos\theta, 2y = r\sin\theta, z = z.$

$\therefore\qquad 2dx\,dy\,dz = r\,d\theta\,dr\,dz$

$\therefore\qquad dx\,dy\,dz = \frac{r}{2}d\theta\,dr\,dz,$

and region of integration is

$$r = 2;\ r^2\cos^2\theta + \frac{3}{4}r^2\sin^2\theta \leq z \leq 4 - \frac{r^2}{4}\sin^2\theta$$

$$\therefore\qquad V = \int_0^{2\pi}\int_0^2\int_{\frac{r^2}{4}(4\cos^2\theta+3\sin^2\theta)}^{\frac{1}{4}(16-r^2\sin^2\theta)} \frac{r}{2}\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 \frac{r}{2}\left[\frac{1}{4}\left(16 - r^2\sin^2\theta - 4r^2\cos^2\theta - 3r^2\sin^2\theta\right)\right]dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^2 \frac{r}{8}\left(16 - 4r^2\right)\,dr\,d\theta = \frac{1}{2}\int_0^{2\pi}\int_0^2\left(4r - r^3\right)dr\,d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\left(2r^2 - \frac{r^4}{4}\right)_0^2 d\theta = \frac{1}{2}(2\pi)(8-4) = 4\pi \text{ cubic units.}$$

Example 9.1.1. Prove that between any two real roots of $e^x \sin x = 1$, there exists at least one real root of $e^x \cos x + 1 = 0$. **(C.H., 1991)**

Solution : We consider the function $f(x) = e^{-x} - \sin x$.

Let α and β be two real roots of $e^x \sin x - 1 = 0$.

$\therefore\ e^{\alpha} \sin\alpha - 1 = 0$ and $e^{\beta} \sin\beta - 1 = 0$.

Hence, $f(\alpha) = f(\beta) = 0$.

Both e^{-x} and $\sin x$ are continuous and derivable for all real x.

$\therefore\ f(x)$ is continuous in $[\alpha, \beta]$, derivable in (α, β). Also, $f(\alpha) = f(\beta) = 0$. Therefore, there exists one c in (α, β) such that $f'(c) = 0$, or, $-e^{-c} - \cos c = 0$ or, $e^c \cos c + 1 = 0$ which shows that c is a root of the equation $e^x \cos x + 1 = 0$ and c lies between α and β.

Example 9.1.2. Prove that between two real roots of the equation $e^x \sin x + 1 = 0$, there is at least one real root of $\tan x + 1 = 0$. **(C.H., 1998)**

Solution : We consider the function $f(x) = e^x \sin x + 1$.

Then, $e^x \sin x + 1 = 0$ reduces to $f(x) = 0$.

Now, if α and β are two roots of $f(x) = 0$, then

$$f(\alpha) = f(\beta) = 0.$$

$f(x)$ is continuous for all real x, since e^x and $\sin x$ are continuous for all real x and product of two continuous functions is a continuous function.

$\therefore\ e^x \sin x + 1$ is continuous in $[\alpha, \beta]$.

Further, $e^x \sin x + 1$ is derivable in (α, β).

$\therefore$ By Rolle's theorem there is at least one γ in (α, β) such that $f'(\gamma) = 0$,

i.e., $e^{\gamma} (\sin \gamma + \cos \gamma) = 0$

$\Rightarrow \sin\gamma + \cos\gamma = 0$ (since $e^{\gamma} \neq 0$ for any real γ)

or, $\tan\gamma + 1 = 0$ which implies that γ is a root of $\tan x + 1 = 0$.

Hence the result.

Example 8.4.4. Show that $\dfrac{\tan x}{x} > \dfrac{x}{\sin x}$ in $0 < x < \dfrac{\pi}{2}$.

Solution: $\dfrac{\tan x}{x} - \dfrac{x}{\sin x} = \dfrac{\tan x \ \sin x - x^2}{x \sin x}$

We put $f(x) = \tan x \sin x - x^2$.

Let $0 < c < \dfrac{\pi}{2}$. In $[0, c]$, $f(x)$ is continuous and derivable.

$f'(x) = \sin x\,(1 + \sec^2 x) - 2x$.

We can not determine sign of $f'(x)$ in $[0, c]$.

$\therefore f''(x) = \cos x\,(\sec^2 x + 1) + 2\sec^2 x \tan x.\ \sin x - 2$

$$= \left(\sqrt{\sec x} - \sqrt{\cos x}\right)^2 + 2\sin^2 x.\ \sec^3 x > 0 \quad \text{in } (0, c).$$

$\therefore f'(x) > f'(0)$ in $0 < x < c$.

$$f'(x) > 0 \quad \text{in } 0 < x < c.$$

Therefore $f(x)$ is increasing in $[0, c]$.

$f(x) > f(0) = 0$ implies $f(x) > 0$

or, $\tan x \sin x - x^2 > 0$ or, $\dfrac{\tan x}{x} > \dfrac{x}{\sin x}$ in $0 < x < \dfrac{\pi}{2}$.

Example 8.4.5. If $0 < x < 1$, show that $2x < \log \dfrac{1+x}{1-x} < 2x\left(1 + \dfrac{1}{3}\dfrac{x^2}{1-x^2}\right)$.

Also deduce that $e < \left(1 + \dfrac{1}{n}\right)^{n+\frac{1}{2}} < e^{1+\frac{1}{12n(n+1)}}$.

Solution: Let $f(x) = 2x - \log \dfrac{1+x}{1-x}$ in $[0, 1)$.

If $0 < c < 1$, then f is continuous on $[0, c]$ and derivable there.

Also $f'(x) = \dfrac{-2x^2}{1-x^2} < 0$ in $(0, c)$.

$\therefore$ f is decreasing in $(0, c)$ *i.e.*, $f(c) < f(0) = 0$.

or, $2x - \log \dfrac{1+x}{1-x} < 0$ when $x = c$.

Since $c \in (0, 1)$ it follows that $2x < \log \dfrac{1+x}{1-x}$ in $(0, 1)$.

Let again, $g(x) = \log \dfrac{1+x}{1-x} - 2x\left(1+\dfrac{1}{3}\dfrac{x^2}{1-x^2}\right)$ in $[0, 1)$.

$g'(x) = -\dfrac{4}{3}\dfrac{x^2}{(1-x^2)^2} < 0$. $\quad \therefore g(x) < g(0) = 0$.

or, $\log\dfrac{1+x}{1-x} < 2x\left(1+\dfrac{1}{3}\dfrac{x^2}{1-x^2}\right)$, when $0 < x < 1$.

$\therefore 2x < \log\dfrac{1+x}{1-x} < 2x\left(1+\dfrac{1}{3}\dfrac{x^2}{1-x^2}\right)$.

In the above inequality, let us put $x = \dfrac{1}{2n+1}$.

$\therefore \dfrac{2}{2n+1} < \log\dfrac{n+1}{n} < \dfrac{2}{2n+1}\left(1+\dfrac{1}{12n(n+1)}\right)$.

or, $1 < \left(n+\dfrac{1}{2}\right) \log \dfrac{n+1}{n} < 1 + \dfrac{1}{12n(n+1)}$

or, $e < \left(1+\dfrac{1}{n}\right)^{n+\frac{1}{2}} < e^{1+\frac{1}{12n(n+1)}}$.

Example 8.4.3. Prove that $\frac{2x}{\pi} < \sin x < x$ when $0 < x < \frac{\pi}{2}$. **(C.U. 1996)**

***Solution*:** Let us consider the function $f(x)$ defined as

$$f(x) = \begin{cases} \frac{\sin x}{x} & \text{when } x \neq 0 \\ 1 & \text{when } x = 0 \end{cases}.$$

Here, $\lim\limits_{x \to 0} f(x) = \lim\limits_{x \to 0} \frac{\sin x}{x} = 1 = f(0)$.

$\therefore$ f is continuous at $x = 0$. Also, in $\left(0, \frac{\pi}{2}\right]$, $f(x)$ is continuous

$\therefore$ $f(x)$ is continuous in $\left[0, \frac{\pi}{2}\right]$.

When $x \neq 0$ $\quad f'(x) = \frac{(x - \tan x)\cos x}{x^2}$.

Now, in $0 < x < \frac{\pi}{2}$ $\tan x > x$ and $\cos x > 0$. $\qquad \therefore f'(x) < 0$.

$\therefore$ $f(x)$ is decreasing in $\left[0, \frac{\pi}{2}\right]$.

$\therefore$ $f(0) > f(x) > f\left(\frac{\pi}{2}\right)$ $\quad$ *i.e.*, $1 > \frac{\sin x}{x} > \frac{2}{\pi}$

or, $\frac{2x}{\pi} < \sin x < x$ when $0 < x < \frac{\pi}{2}$.

Example 8.4.1. Show that $x - \frac{x^2}{2} < \log(1 + x) < x - \frac{x^2}{2(1+x)} \quad \forall\, x > 0.$

***Solution*:** Let $F(x) = \log(1 + x) - \left(x - \frac{x^2}{2}\right)$

$$F'(x) = \frac{1}{1+x} - (1 - x) = \frac{x^2}{1+x} > 0 \quad \forall\, x > 0.$$

$\therefore$ $F(x)$ is increasing for $x > 0$ *i.e.*, $F(x) > F(0) = 0.$

$$\therefore\ \log(1 + x) > x - \frac{x^2}{2} \quad \forall\, x > 0.$$

Again, let $\phi(x) = x - \frac{x^2}{2(1+x)} - \log(1 + x)$

$$\therefore\ \phi'(x) = 1 - \frac{x}{1+x} + \frac{x^2}{2(1+x)^2} - \frac{1}{x+1} = \frac{x^2}{2(1+x)^2} > 0.$$

$\therefore$ $\phi(x)$ is also increasing for $x > 0$ *i.e.*, $\phi(x) > \phi(0) = 0.$

$$\therefore\ x - \frac{x^2}{2(1+x)} > \log(1 + x) \quad \forall\, x > 0.$$

$$\therefore\ x - \frac{x^2}{2} < \log(1 + x) < x - \frac{x^2}{2(1+x)} \quad \forall\, x > 0.$$

Example 8.2.14. Let $f(x) = \begin{cases} x\left\{1 + \frac{1}{3}\sin(\log x^2)\right\} & \text{when } x \neq 0 \\ 0 & \text{when } x = 0. \end{cases}$

Show that $f(x)$ is continuous at $x = 0$ but not derivable there.

***Solution*:** We have $f(0+0) = \lim\limits_{h \to 0+} h\left\{1 + \frac{1}{3}\sin(\log h^2)\right\} = \lim\limits_{h \to 0+} \left\{h + \frac{h}{3}\sin(\log h^2)\right\}$

$= 0$ [since $|\sin(\log h^2)| \leq 1$].

Also, $f(0 - 0) = \lim\limits_{h \to 0-} h\left\{1 + \frac{1}{3}\sin(\log h^2)\right\} = 0.$

$f(0) = 0 = \lim\limits_{x \to 0} f(x)$ $\quad\therefore$ $f(x)$ is continuous at $x = 0$.

Now, $\lim_{x \to 0+} \dfrac{f(x)-f(0)}{x-0} = \lim_{x \to 0+} \dfrac{x\left\{1+\frac{1}{3}\sin(\log x^2)\right\}}{x} = \lim_{x \to 0+} \left\{1+\dfrac{1}{3}\sin(\log x^2)\right\}$

Since $\sin(\log x^2)$ oscillates between -1 and 1 as $x \to 0$,

$\lim_{x \to 0} \sin(\log x^2)$ does not exist. $\therefore$ $Rf'(0)$ does not exist.

Similarly, $\lim_{x \to 0-} \dfrac{f(x)-f(0)}{x}$ does not exist.

$\therefore$ $Lf'(0)$ also does not exist. Thus f is not differentiable at $x = 0$.

Example 8.2.12. A function f is defined in $(-1, 1)$ by

$$f(x) = \begin{cases} x^p \sin \frac{1}{x^q} & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \end{cases}$$

Prove that if $0 < q < p - 1$, f' is continuous at $x = 0$ and if $0 < p - 1 \leq q$, f' is discontinuous at $x = 0$.

Solution: When $x \neq 0$ $f'(x) = px^{p-1} \sin \dfrac{1}{x^q} - x^p . \cos\left(\dfrac{1}{x^q}\right)\dfrac{q}{x^{q+1}}$

$$= px^{p-1} \sin \frac{1}{x^q} - x^{p-q-1}.\, q \cos\left(\frac{1}{x^q}\right)$$

Also, $\lim_{x \to 0} \dfrac{f(x)-f(0)}{x-0} = \lim_{x \to 0} x^{p-1} \sin \dfrac{1}{x^q} = \lim_{x \to 0} x^{p-q-1} x^q \sin\left(\dfrac{1}{x^q}\right)$

$= 0$ if $p - q - 1 > 0$ and $q > 0$.

$$\therefore\ f'(x) = \begin{cases} px^{p-1}\sin\frac{1}{x^q} - qx^{p-q-1}\cos\left(\frac{1}{x^q}\right) & \text{when } x \neq 0 \\ 0 \quad \text{if } p-q-1>0, \quad q>0 & \text{when } x=0 \end{cases}$$

$\therefore$ when $p - q - 1 > 0$ and $q > 0$, $\lim_{x \to 0} f'(x) = 0 = f(0)$

i.e., $f'(x)$ is continuous at $x = 0$.

When $p - q - 1 \leq 0$, $\lim_{x \to 0} f'(x)$ does not exits.

$\therefore f'(x)$ is discontinnous at $x = 0$.

Example 8.2.9. Show that, if $f(x) = \begin{cases} x^m \sin\frac{1}{x} & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \end{cases}$

$f(x)$ is derivable at $x = 0$. Also determine m when $f'(x)$ is continuous at $x = 0$.

Solution: $\lim\limits_{x \to 0} \dfrac{f(x) - f(0)}{x - 0} = \lim\limits_{x \to 0} \dfrac{x^m \sin\frac{1}{x}}{x} = \lim\limits_{x \to 0} x^{m-1} \sin \dfrac{1}{x}$

$$= \lim_{x \to 0} x^{m-2} \left(x \sin \frac{1}{x}\right) = \lim_{x \to 0} x^{m-2} . \lim_{x \to 0}\left(x \sin \frac{1}{x}\right) = 0.$$

$\therefore$ $f'(0)$ exists and $f'(0) = 0$

When $x \neq 0$. $f'(x) = m\, x^{m-1} \sin\dfrac{1}{x} - x^{m-2} \cos\dfrac{1}{x}$.

$$\text{Thus } f'(x) = \begin{cases} m\, x^{m-1} . \sin\frac{1}{x} - x^{m-2} \cos\frac{1}{x} & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \end{cases}$$

Now, $\lim\limits_{x \to 0} x^{m-1} \sin \dfrac{1}{x} = 0$ and $\lim\limits_{x \to 0} x^{m-2} \cos\dfrac{1}{x}$ exists if $m > 2$

and the limiting value is then 0.

$\therefore$ $\lim\limits_{x \to 0} f'(x) = f'(0)$ when $m > 2$.

Thus $f'(x)$ is continnous when $m > 2$.

Example 8.2.10. If $f(x) = \begin{cases} \dfrac{x\left(e^{\frac{1}{x}} - e^{-\frac{1}{x}}\right)}{e^{\frac{1}{x}} + e^{-\frac{1}{x}}} & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \end{cases}$

show that $f(x)$ is not derivable at $x = 0$.

Solution: $\lim\limits_{x \to 0+} \dfrac{f(x)-f(0)}{x-0} = \lim\limits_{x \to 0+} \dfrac{e^{\frac{1}{x}} - e^{-\frac{1}{x}}}{e^{\frac{1}{x}} + e^{-\frac{1}{x}}}.$

$$= \lim_{x \to 0+} \frac{1-e^{-\frac{2}{x}}}{1+e^{-\frac{2}{x}}} \quad \begin{bmatrix} \frac{1}{x} \to +\infty \text{ as } x \to 0+ \\ e^{-\frac{1}{x}} \to 0 \text{ as } x \to 0+ \end{bmatrix}$$

$$= 1.$$

$\therefore \; Rf'(0) = 1.$

$$\lim_{x \to 0-} \frac{f(x)-f(0)}{x-0} = \lim_{x \to 0-} \frac{e^{\frac{2}{x}} - 1}{e^{\frac{2}{x}} + 1} = -1 \quad \begin{bmatrix} \frac{1}{x} \to -\infty \quad \text{as } x \to 0- \\ \text{and } e^{\frac{1}{x}} \to 0 \text{ as } x \to 0- \end{bmatrix}$$

$\therefore \; Lf'(0) = -1. \quad \therefore \; f(x)$ is not derivable at $x = 0$.

Example 8.2.6. Show that the function g defined by

$$g(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \end{cases}$$

is differentiable everywhere, but the derived function is not continuous at $x = 0$.
(C.H. 1993, 1998)

Solution: When $x \neq 0$, $g'(x) = 2x \sin \dfrac{1}{x} - \cos \dfrac{1}{x}$.

Again, $\lim\limits_{x \to 0} \dfrac{g(x)-g(0)}{x-0} = \lim\limits_{x \to 0} \dfrac{x^2 \sin \frac{1}{x}}{x} = \lim\limits_{x \to 0} x \sin \dfrac{1}{x} = 0.$

Because, $\left| x \sin \dfrac{1}{x} - 0 \right| = |\,x\,| \left| \sin \dfrac{1}{x} \right| \leq |\,x\,| < \varepsilon$ when $|\,x - 0\,| < \delta \,(= \varepsilon)$

$\therefore \; g(x)$ is derivable at $x = 0$. $\quad \therefore \; g(x)$ is derivable for all $x \in \mathbb{R}$

$$\therefore\ g'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x} & \text{when } x \neq 0 \\ 0 & \text{when } x = 0. \end{cases}$$

Now, $\lim_{x \to 0} \cos\dfrac{1}{x}$ does not exist by Cauchy's condition for existence of limit of a function.

[We choose $\varepsilon = 0.5$ and $x_1 = \dfrac{1}{2n\pi}$; $x_2 = \dfrac{1}{(2n+1)\pi}$ sufficiently close to 0 for large values of n. Then, $|f(x_1) - f(x_2)| = |\cos 2n\pi - \cos(2n+1)\pi| = 2 \not< 0.5$].

$\therefore\ \lim_{x \to 0} g(x)$ does not exist. $\quad\therefore\ g'(x)$ is not continuous at the origin.

Example 8.2.3. Show that the function $f(x) = \begin{cases} x\sin\frac{1}{x} & \text{if } x \neq 0 \\ = 0 & \text{if } x = 0 \end{cases}$ is continuous at $x = 0$, but not derivable at $x = 0$.

***Solution*:** Choose $\varepsilon > 0$.

Here, $|f(x) - f(0)| = |x\sin\dfrac{1}{x} - 0| = |x|\,|\sin\dfrac{1}{x}| \leq |x| < \varepsilon$ if $|x - 0| = |x| < \delta\,(= \varepsilon)$. $\quad\therefore\ f(x)$ is continuous at $x = 0$.

Now, $\lim_{x \to 0} \dfrac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \dfrac{x\sin\frac{1}{x}}{x} = \lim_{x \to 0} \sin\dfrac{1}{x}$

By Cauchy's condition for existence of limit, $\lim_{x \to 0} \sin\dfrac{1}{x}$ does not exist.

[We chose $x_1 = \dfrac{1}{(4n+1)\frac{\pi}{2}}$ and $x_2 = \dfrac{1}{(4n-1)\frac{\pi}{2}}$ sufficiently close to 0 for suitably chosen large n.

$|f(x_1) - f(x_2)| = |\sin(4n+1)\dfrac{\pi}{2} - \sin(4n-1)\dfrac{\pi}{2}| = 2 \not< \varepsilon$

when ε is chosen arbitrarily.]

$\therefore\ \lim_{x \to 0} \dfrac{f(x) - f(0)}{x - 0}$ does not exist. Hence $f(x)$ is not derivable at $x = 0$.

Example 7.5.2. Prove that $f(x) = x^2$ is uniformly continuous in any closed interval $[a, b]$.

***Solution*:** We take two points x_1, x_2 in $[a, b]$ and any $\varepsilon\ (> 0)$.

Now, $| f(x_1) - f(x_2) | = | x_1 - x_2 | | x_1 + x_2 | < 2b | x_1 - x_2 | < \varepsilon$

if $| x_1 - x_2 | < \dfrac{\varepsilon}{2b} = \delta.$

Thus b being known, for a given $\varepsilon > 0$, one single δ will satisfy the definition of uniform continuity of $f(x) = x^2$.

Hence, $f(x) = x^2$ is uniformly continuous on $[a, b]$.

Example 7.5.3. Show that $f(x) = \sin x$ is uniformly continuous on $\mathbb{R}$.

***Solution*:** Let us take $c \in \mathbb{R}$ and for any $x \in \mathbb{R}$

$$| f(x) - f(c) | = | \sin x - \sin c | = 2\left| \sin \frac{x-c}{2} \right| \cdot \left| \cos \frac{x+c}{2} \right|$$

$$\leq 2\left| \sin \frac{x-c}{2} \right| \leq | x - c |.$$

Thus, for any $\varepsilon\ (> 0)$ satisfying $| x - c | < \delta\ (= \varepsilon)$

we find that $| f(x) - f(c) | < \varepsilon$.

Hence, $f(x) = \sin x$ is uniformly continuous on $\mathbb{R}$.

Example 7.5.4. Show that $f(x) = \sin \dfrac{1}{x}\ (x > 0)$ is continuous for all positive x but not uniformly continuous on the set of all positive reals.

***Solution*:** Here, $\sin \dfrac{1}{x}$ is defined for all $x > 0$.

Now, we know that $\sin x$ is continuous at any point c where it is defined.

for, $| \sin x - \sin c | = 2\left| \sin \dfrac{x-c}{2} \right| \left| \sin \dfrac{x+c}{2} \right| \leq 2\left| \sin \dfrac{x-c}{2} \right|$

$\leq | x - c | < \varepsilon$ when $| x - c | < \delta\ (= \varepsilon)$.

To show that $\sin \dfrac{1}{x}$ is not uniformly continuous in $\mathbb{R}^+$ (the set of positive reals) we select $\varepsilon = 1$ and choose

$$x_1 = \frac{1}{(4n-1)\frac{\pi}{2}} \quad \text{and} \quad x_2 = \frac{1}{(4n+1)\frac{\pi}{2}},\ n \in \mathbb{N}$$

Then $|x_1 - x_2| = \left|\frac{2}{(4n-1)\pi} - \frac{2}{(4n+1)\pi}\right| = \frac{4}{(16n^2-1)\pi} < \delta$

[For each positive δ, an integer n can be found].

But $|f(x_1) - f(x_2)| = \left|\sin\left(2n\pi + \frac{\pi}{2}\right) - \sin\left(2n\pi - \frac{\pi}{2}\right)\right| = 2 \nless \varepsilon.$

$\therefore\ f(x) = \sin\frac{1}{x}$ is not uniformly continuous on $\mathbb{R}^+$.

Alternatively : We use the result if for every Cauchy sequence $\{x_n\}_n$ in D. $\{f(x_n)\}_n$ is a not Cauchy sequence in $\mathbb{R}$ then f is not uniformly continuous on D.

We consider the sequence $\{x_n\}_n$ where $x_n = \frac{2}{n\pi}$ $n \in \mathbb{N}$,

which is a Cauchy sequence in (0, 1). But we see that

$\{f(x_n)\}_n$ is $\{1, 0, -1, 0, ...\}$ which is not a Cauchy sequence.

$\therefore$ f is not uniformly continuous on (0, 1).

$\therefore$ f is not uniformly continuous on $\mathbb{R}^+$.

Example 1. A function f is defined as $f(x + y) = f(x) + f(y)$ $\forall\ x, y \in \mathbb{R}$. If $f(1) = k$ and $f : \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$, prove that

$$f(x) = kx \ \forall\ x \in \mathbb{R}.$$

Solution: $f(0 + 0) = f(0) + f(0) \Rightarrow f(0) = 0.$ ... (i)

Again, $0 = f(x - x) = f(x) + f(-x) \Rightarrow f(-x) = -f(x)$ ... (ii)

Let x be a positive integer.

Then $f(x) = f(1 + 1 + \cdots + 1)$

$= f(1) + f(1) + \cdots + f(1)$ (x times) $= kx$.

$\therefore$ $f(x) = kx$ if x is a positive integer ... (iii)

If x is a negative integer $-x$ is a positive integer, then by (iii) $f(-x) = -f(x)$ or, $k(-x) = -f(x) \Rightarrow f(x) = kx$.

$\therefore$ $f(x) = kx$ for any integral value of x.

Let $x = \frac{p}{q}$ where p and q are integers, $q \neq 0$.

$\therefore\ f(p) = f(qx)$ or, $kp = f(qx) = f(x + \cdots + x)$ [q times] $= qf(x)$

$\therefore\ f(x) = k\frac{p}{q} = kx.\quad \therefore\ f(x) = kx$, when x is rational.

Let now, x be an irrational number α.

Since any neighbourhood of α contains both rational and irrational points, we consider a sequence $\{x_n\}_n$ where x_n is rational such that $\{x_n\}_n$ converges to α.

α being a point of continuity, $\lim f(x_n) = f(\alpha)$

i.e., $\lim\ kx_n = f(\alpha)$ or, $k \lim x_n = f(\alpha)$, *i.e.*, $f(\alpha) = k\alpha$.

In other words $f(x) = kx$ when x is irrational.

$\therefore$ Combining all the results we see that $f(x) = kx\ \forall\ x \in \mathbb{R}$.

Example 2. If a continuous function f on $\mathbb{R}$ satisfies the relation $f(x + y) = f(x)\ f(y)\ \forall\ x, y \in \mathbb{R}$, show that either $f(x) = 0\ \forall\ x \in \mathbb{R}$ or there exists $a > 0$ such that $f(x) = a^x\ \forall\ x \in \mathbb{R}$.

Solution: From the given condition $f(x + y) = f(x)\ f(y)$

$$f(1) = f\left(\frac{1}{2}+\frac{1}{2}\right) = f\left(\frac{1}{2}\right).f\left(\frac{1}{2}\right) \geq 0.$$

$f(1) = 0 \Rightarrow f(x) = f(x - 1 + 1) = f(x - 1)\ f(1) = 0$ for all $x \in \mathbb{R}$.

If $f(1) \neq 0$ let $f(1) = a > 0$.

Now for $x = y = 0$. $f(0) = f(0 + 0) = f(0)\ f(0) = \{f(0)\}^2$

$\therefore\ f(0)\ \{f(0) - 1\} = 0\ \ \therefore\ f(0) = 1$, for, $f(0) = 0$ would imply $f(1) = f(0 + 1) = f(0)$

$\Rightarrow f(1) = 0$ which is not the case.

Again $1 = f(0) = f(x - x) = f(x)\ f(- x)\quad \therefore\ f(- x) = \frac{1}{f(x)} = \{f(x)\}^{-1}$

Let x be a positive integer.

$f(x) = f(1 + 1 + \cdots + 1)$ (x times) $= \{f(1)\}^x = a^x$

When x is a negative integer, *say* $x = -y\ \ (y > 0)$

$f(x) = f(-y) = \{f(y)\}^{-1} = a^{-y} = a^x$

Now let $x = \frac{p}{q}$, a rational number.

Then $xq = p$. $f(xq) = f(p) = a^p$ or, $\{f(x)\}^q = a^p$.

or, $f(x) = a^{\frac{p}{q}} = a^x$. $\therefore$ $f(x) = a^x \ \forall \ x \in Q$.

Given that $f(x)$ is continuous at every point.

Let $x = \alpha$ be an irrational number and we consider the sequence $\{x_n\}_n$ of rationals converging to α.

By continuity of f at α, $\{f(x_n)\}_n$ converges to $f(\alpha)$

or, $\lim f(x_n) = f(\alpha)$ or, $\lim a^{x_n} = f(\alpha)$

or, $a^{\alpha} = f(\alpha)$, *i.e.*, $a^x = f(x)$

when x is irrational.

Combining all the results we see that $f(x) = a^x \ \forall \ x \in \mathbb{R}$.

Ex. 8. *Prove that when x and y are positive and $x + y < h$,*

$$\iint f'(x+y)\, x^{l-1} y^{-l}\, dx\, dy = \frac{\pi}{\sin l\pi}\,[f(h) - f(0)].$$

Sol. The given integral

$$I = \iint f'(x+y)\, x^{l-1} y^{(1-l)-1}\, dx\, dy, \text{ where } 0 < x + y < h$$

$$= \frac{\Gamma(l)\,\Gamma(1-l)}{\Gamma(l+1-l)} \int_0^h f'(u)\, u^{l+(1-l)-1}\, du,$$

by Liouville's extension of Dirichlet's theorem

$$= \frac{\Gamma(l)\,\Gamma(1-l)}{\Gamma(1)} \int_0^h f'(u)\, du$$

$$= \frac{\pi}{\sin \pi l}\left[f(u)\right]_0^h = \frac{\pi}{\sin \pi l}\,[f(h) - f(0)].$$

Ex. 9. *Evaluate* $\iiint x^{\alpha} y^{\beta} z^{\gamma} (1-x-y-z)^{\lambda}\, dx\, dy\, dz$ *over the interior of the tetrahedron formed by the coordinate planes and the plane* $x+y+z=1$. **(Meerut 1988)**

Sol. Here the region of integration is bounded by the planes $x=0, y=0, z=0$ and $x+y+z=1$. So the variables x, y, z take all positive values subject to the condition

$$0 < x+y+z < 1.$$

Hence the given integral

$$= \iiint x^{(\alpha+1)-1} y^{(\beta+1)-1} z^{(\gamma+1)-1} [1-(x+y+z)]^{\lambda}\, dx\, dy\, dz$$

$$= \frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)\,\Gamma(\gamma+1)}{\Gamma(\alpha+\beta+\gamma+3)} \cdot \int_0^1 u^{\alpha+1+\beta+1+\gamma+1-1} (1-u)^{\lambda}\, du,$$

by Liouville's extension of Dirichlet's theorem

$$= \frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)\,\Gamma(\gamma+1)}{\Gamma(\alpha+\beta+\gamma+3)} \cdot \int_0^1 u^{(\alpha+\beta+\gamma+3)-1} (1-u)^{(\lambda\quad 1)-1}\, du$$

$$= \frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)\,\Gamma(\gamma+1)}{\Gamma(\alpha+\beta+\gamma+3)} B(\alpha+\beta+\gamma+3, \lambda+1)$$

$$= \frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)\,\Gamma(\gamma+1)}{\Gamma(\alpha+\beta+\gamma+3)} \cdot \frac{\Gamma(\alpha+\beta+\gamma+3)\,\Gamma(\lambda+1)}{\Gamma(\alpha+\beta+\gamma+\lambda+4)}$$

$$= \frac{\Gamma(\alpha+1)\,\Gamma(\beta+1)\,\Gamma(\gamma+1)\,\Gamma(\lambda+1)}{\Gamma(\alpha+\beta+\gamma+\lambda+4)}.$$

EXAMPLE 9.108 **(Application of absolute extrema)**

The flat circular plate has the shape of the region $x^2 + y^2 \le 1$. The plate including the boundary where $x^2 + y^2 = 1$ is heated so that the temperature at the point (x, y) is $T(x, y) = x^2 + 2y^2 - x$. Find the temperatures at the hottest and coldest points on the plate.

Solution:

$$T(x, y) = x^2 + 2y^2 - x$$

$$T_x(x, y) = 2x - 1$$

$$T_y(x, y) = 4y$$

$$T_x(x, y) = 0 \quad \Rightarrow 2x - 1 = 0 \quad \Rightarrow x = \frac{1}{2}$$

$$T_y(x, y) = 0 \quad \Rightarrow 4y = 0 \quad \Rightarrow y = 0$$

$$T\left(\frac{1}{2}, 0\right) = -\frac{1}{4}$$

On boundary $x^2 + y^2 = 1 : T(x, y) = x^2 + 2(1 - x^2) - x = -x^2 - x + 2$

$$T'(x, y) = -2x - 1$$

$$T'(x, y) = 0 \quad \Rightarrow -2x - 1 = 0 \quad \Rightarrow x = -\frac{1}{2} \text{ and } y = \pm\frac{\sqrt{3}}{2}$$

$$T\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = \frac{9}{4}, T\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)a = \frac{9}{4}$$

$$T(-1, 0) = 2, T(1, 0) = 0$$

Therefore, the hottest point is at $\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right), \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ and the hottest temperature is $\frac{9}{4}$ degrees and the coldest point is at $\left(\frac{1}{2}, 0\right)$ and the coldest temperature is $-\frac{1}{4}$ degrees.

Example 7. *Show that the function $f: \mathbf{R} \to \mathbf{R}$ defined by $f(x) = x\,[1 + \frac{1}{3}\sin(\log x^2)]$ when $x \neq 0$ and $f(0) = 0$ is everywhere continuous and monotonic but has no differential coefficient at the origin.*

Solution. For continuity and differentiability of f, see example 4 of § 12. We now show that f is monotonic. We have

$$f'(x) = 1 + \tfrac{1}{3}\sin\log x^2 + \tfrac{2}{3}\cos\log x^2$$
$$\geqslant 0 \text{ for all values of } x \text{ other than zero.}$$

[Note that f' can be zero at the most].

Let x_1, x_2 be two values of x other than zero such that $x_2 > x_1$. Then by the mean value theorem

$$f(x_2) - f(x_1) = (x_2 - x_1)\,f'(c),\ (x_1 < c < x_2)$$
$$\geqslant 0\ [\because\ x_2 - x_1 > 0 \text{ and } f'(c) \geqslant 0].$$

Hence $f(x_2) \geqslant f(x_1)$.

It follows that f is monotonically increasing.

Example 1. A function f is defined as $f(x + y) = f(x) + f(y)\ \forall\ x, y \in \mathbb{R}$. If $f(1) = k$ and $f: \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$, prove that

$$f(x) = kx\ \forall\ x \in \mathbb{R}.$$

Solution: $f(0 + 0) = f(0) + f(0) \Rightarrow f(0) = 0.$ ... (i)

Again, $0 = f(x - x) = f(x) + f(-x) \Rightarrow f(-x) = -f(x)$ ... (ii)

Let x be a positive integer.

Then $f(x) = f(1 + 1 + \cdots + 1)$

$= f(1) + f(1) + \cdots + f(1)$ (x times) $= kx$.

$\therefore$ $f(x) = kx$ if x is a positive integer ... (iii)

If x is a negative integer $-x$ is a positive integer, then by (iii) $f(-x) = -f(x)$ or, $k(-x) = -f(x) \Rightarrow f(x) = kx$.

$\therefore$ $f(x) = kx$ for any integral value of x.

Let $x = \dfrac{p}{q}$ where p and q are integers, $q \neq 0$.

$\therefore$ $f(p) = f(qx)$ or, $kp = f(qx) = f(x + \cdots + x)$ [q times] $= qf(x)$

$\therefore\ f(x) = k\dfrac{p}{q} = kx. \quad \therefore\ f(x) = kx$, when x is rational.

Let now, x be an irrational number α.

Since any neighbourhood of α contains both rational and irrational points, we consider a sequence $\{x_n\}_n$ where x_n is rational such that $\{x_n\}_n$ converges to α.

α being a point of continuity, $\lim f(x_n) = f(\alpha)$

i.e., $\lim\ kx_n = f(\alpha)$ or, $k \lim x_n = f(\alpha)$, *i.e.*, $f(\alpha) = k\alpha$.

In other words $f(x) = kx$ when x is irrational.

$\therefore$ Combining all the results we see that $f(x) = kx\ \forall\ x \in \mathbb{R}$.

Example 2. If a continuous function f on $\mathbb{R}$ satisfies the relation $f(x + y) = f(x)\ f(y)\ \forall\ x, y \in \mathbb{R}$, show that either $f(x) = 0\ \forall\ x \in \mathbb{R}$ or there exists $a > 0$ such that $f(x) = a^x\ \forall\ x \in \mathbb{R}$.

***Solution*:** From the given condition $f(x + y) = f(x)\ f(y)$

$$f(1) = f\left(\frac{1}{2} + \frac{1}{2}\right) = f\left(\frac{1}{2}\right) . f\left(\frac{1}{2}\right) \geq 0.$$

$f(1) = 0 \Rightarrow f(x) = f(x - 1 + 1) = f(x - 1)\ f(1) = 0$ for all $x \in \mathbb{R}$.

If $f(1) \neq 0$ let $f(1) = a > 0$.

Now for $x = y = 0$. $f(0) = f(0 + 0) = f(0)\ f(0) = \{f(0)\}^2$

$\therefore\ f(0)\ \{f(0) - 1\} = 0\ \ \therefore\ f(0) = 1$, for, $f(0) = 0$ would imply $f(1) = f(0 + 1) = f(0)$

$\Rightarrow f(1) = 0$ which is not the case.

Again $1 = f(0) = f(x - x) = f(x)\ f(-x) \quad \therefore\ f(-x) = \dfrac{1}{f(x)} = \{f(x)\}^{-1}$

Let x be a positive integer.

$f(x) = f(1 + 1 + \cdots + 1)$ (x times) $= \{f(1)\}^x = a^x$

When x is a negative integer, *say* $x = -y\ \ (y > 0)$

$f(x) = f(-y) = \{f(y)\}^{-1} = a^{-y} = a^x$

Now let $x = \frac{p}{q}$, a rational number.

Then $xq = p$. $f(xq) = f(p) = a^p$ or, $\{f(x)\}^q = a^p$.

or, $f(x) = a^{\frac{p}{q}} = a^x$. $\therefore$ $f(x) = a^x \ \forall \ x \in Q$.

Given that $f(x)$ is continuous at every point.

Let $x = \alpha$ be an irrational number and we consider the sequence $\{x_n\}_n$ of rationals converging to α.

By continuity of f at α, $\{f(x_n)\}_n$ converges to $f(\alpha)$

or, $\lim f(x_n) = f(\alpha)$ or, $\lim a^{x_n} = f(\alpha)$

or, $a^\alpha = f(\alpha)$, *i.e.*, $a^x = f(x)$

when x is irrational.

Combining all the results we see that $f(x) = a^x \ \forall \ x \in \mathbb{R}$.

Example Evaluate $\iiint_E z\,dV$ where E is the portion of the solid sphere $x^2+y^2+z^2 \le 9$ that is inside the cylinder $x^2+y^2=1$ and above the cone $x^2+y^2=z^2$.

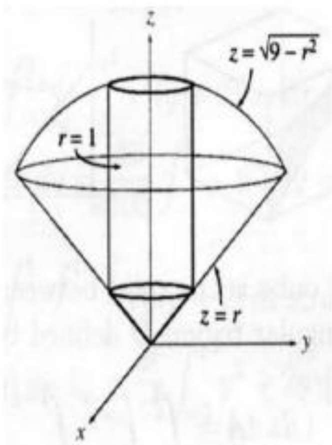


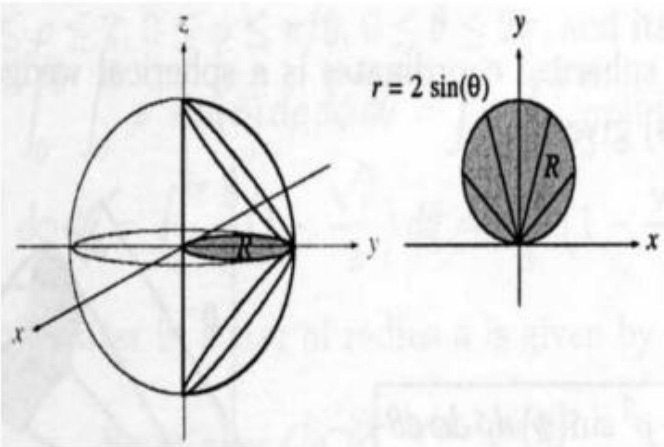
Figure 5:

Soln: The top surface is $z = u_2(x,y) = \sqrt{9-x^2-y^2} = \sqrt{9-r^2}$ and the bottom surface is $z = u_1(x,y) = \sqrt{x^2+y^2} = r$ over the region D defined by the intersection of the top (or bottom) and the cylinder which is a disk $x^2 + y^2 \leq 1$ or $0 \leq r \leq 1$ in the xy-plane.

$$\iiint_E z dV = \iint_D \left[\int_r^{\sqrt{9-r^2}} z dz \right] dA = \int_0^{2\pi} \int_0^1 \int_r^{\sqrt{9-r^2}} z r dz dr d\theta =$$

$$\int_0^{2\pi} \int_0^1 \frac{1}{2}[9 - 2r^2] r dr d\theta = \int_0^{2\pi} \int_0^1 \frac{1}{2}[9r - 2r^3] dr d\theta = \int_0^{2\pi} [9/4 - 1/4] d\theta = 4\pi$$

Example Find the volume of the portion of the sphere $x^2+y^2+z^2 = 4$ inside the cylinder $(y-1)^2 + x^2 = 1$.



Soln: The top surface is $z = \sqrt{4 - x^2 - y^2} = \sqrt{4 - r^2}$ and the bottom is $z = -\sqrt{4 - x^2 - y^2} = -\sqrt{4 - r^2}$ over the region D defined by the cylinder equation in the xy-plane. So rewrite the cylinder equation $x^2 + (y-1)^2 = 1$ as $x^2 + y^2 - 2y + 1 = 1 \Rightarrow r^2 = 2r\sin(\theta) \Rightarrow r = 2\sin(\theta)$.

$$V(E) = \iiint_E 1 dV = \iint_D \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} 1 dz dA = \int_0^{\pi} \int_0^{2\sin(\theta)} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} 1 r dz dr d\theta =$$

$$\int_0^{\pi} \int_0^{2\sin(\theta)} 2r\sqrt{4-r^2} dr d\theta \text{ (by substitution u = 4 − r}^2\text{)} =$$

$$\int_0^{\pi} -\frac{2}{3}[(4 - 4\sin^2(\theta))^{3/2} - (4)^{3/2}] d\theta \text{ (use identity } 1 = \cos^2(\theta) + \sin^2(\theta)\text{)} =$$

$$\int_0^{\pi} \frac{16}{3}[1 - |\cos(\theta)|^3] d\theta = \int_0^{\pi/2} \frac{16}{3}[1 - \cos^3(\theta)] d\theta + \int_{\pi/2}^{\pi} \frac{16}{3}[1 + \cos^3(\theta)] d\theta =$$

$$\int_0^{\pi/2} \frac{16}{3}[1 - (1 - \sin^2\theta)\cos\theta] d\theta + \int_{\pi/2}^{\pi} \frac{16}{3}[1 + (1 - \sin^2\theta)\cos\theta] d\theta =$$

$$16/3[(\theta - \sin\theta + \sin^3\theta/3)|_0^{\pi/2} + (\theta + \sin\theta - \sin^3\theta/3)|_{\pi/2}^{\pi}] = 16\pi/3 - 64/9$$

Example Find the volume of the solid region above the cone $z^2 = 3(x^2 + y^2)$ $(z \geq 0)$ and below the sphere $x^2 + y^2 + z^2 = 4$.

Soln: The sphere $x^2 + y^2 + z^2 = 4$ in spherical coordinates is $\rho = 2$. The cone $z^2 = 3(x^2 + y^2)$ $(z \geq 0)$ in spherical coordinates is $z = \sqrt{3(x^2+y^2)} = \sqrt{3}r \Rightarrow \rho\cos(\phi) = \sqrt{3}\rho\sin(\phi) \Rightarrow \tan(\phi) = 1/\sqrt{3} \Rightarrow \phi = \pi/6$.

Thus E is defined by $0 \leq \rho \leq 2$, $0 \leq \phi \leq \pi/6$, $0 \leq \theta \leq 2\pi$.

$$V(E) = \iiint_E 1dV = \int_0^{2\pi}\int_0^{\pi/6}\int_0^2 \rho^2\sin(\phi)d\rho d\phi d\theta =$$

$$\int_0^{2\pi}\int_0^{\pi/6}\frac{8}{3}\sin(\phi)d\phi d\theta = \frac{16\pi}{3}(1-\frac{\sqrt{3}}{2})$$

17.7.14 Evaluate the surface integral $\iint_S xyz\,dS$, where S is the part of the sphere $x^2 + y^2 + z^2 = 1$ that lies above the cone $z = \sqrt{x^2 + y^2}$

Using the spherical coordinates, let us write the parametric equation as

$$x = \sin\phi\cos\theta, \quad y = \sin\phi\sin\theta, \quad z = \cos\phi,$$

where $0 \leq \phi \leq \pi/4$ and $0 \leq \theta \leq 2\pi$ (we used $\rho = 1$). That is

$$\mathbf{r}(\phi,\theta) = \sin\phi\cos\theta\mathbf{i} + \sin\phi\sin\theta\mathbf{j} + \cos\phi\mathbf{k}.$$

We obtain

$$\begin{aligned}
\mathbf{r}_\phi &= \left\langle \frac{\partial\sin\phi\cos\theta}{\partial\phi}, \frac{\partial\sin\phi\sin\theta}{\partial\phi}, \frac{\partial\cos\phi}{\partial\phi}\right\rangle = \langle\cos\phi\cos\theta, \cos\phi\sin\theta, -\sin\phi\rangle,\\
\mathbf{r}_\theta &= \left\langle \frac{\partial\sin\phi\cos\theta}{\partial\theta}, \frac{\partial\sin\phi\sin\theta}{\partial\theta}, \frac{\partial\cos\phi}{\partial\theta}\right\rangle = \langle-\sin\phi\sin\theta, \sin\phi\cos\theta, 0\rangle.
\end{aligned}$$

Thus,

$$|\mathbf{r}_\phi \times \mathbf{r}_\theta| = \sin\phi.$$

The surface integral is calculated as follows.

$$\begin{aligned}
\iint_S xyz\,dS &= \int_0^{2\pi}\int_0^{\pi/4}(\sin\phi\cos\theta)(\sin\phi\sin\theta)(\cos\phi)\,|\mathbf{r}_\phi\times\mathbf{r}_\theta|\,d\phi d\theta\\
&= \int_0^{2\pi}\int_0^{\pi/4}\cos\theta\sin\theta\sin^3\phi\cos\phi d\phi d\theta\\
&= \int_0^{2\pi}\frac{1}{2}\sin(2\theta)d\theta\left[\frac{\sin^4\phi}{4}\right]_0^{\pi/4}\\
&= 0.
\end{aligned}$$

EXAMPLE 1. Evaluate the integral

$$\iint_R \cos\left(\frac{x-y}{x+y}\right) dxdy,$$

where R is the triangular region with vertices $(0,0)$, $(1,0)$, $(0,1)$.

SOLUTION. Here the region of integration is simple, but the function $f(x,y) = \cos\left(\frac{x-y}{x+y}\right)$ is not. It seems reasonable to set

$$u = x - y, \quad v = x + y.$$

Then, as the point (x,y) varies in R, the point (u,v) varies in the triangular region Q bounded by the lines $u = v$, $v = 1$, $u = -v$.

By the change of variables formula

$$\iint_R \cos\left(\frac{x-y}{x+y}\right) dxdy = \iint_Q \cos(u/v)|J|dudv.$$

Since $x = (u+v)/2$ and $y = (v-u)/2$, we can compute the Jacobian factor:

$$J = x_u y_v - x_v y_u = 1/4 + 1/4 = 1/2.$$

So

$$\begin{aligned}\iint_Q \cos(u/v)|J|dudv &= \frac{1}{2}\int_0^1 \int_{-v}^{v} \cos(u/v)du\, dv \\ &= \frac{1}{2}\int_0^1 v\sin(u/v)\Big|_{u=-v}^{u=v} dv \\ &= \sin 1 \int_0^1 vdv \\ &= \frac{\sin 1}{2}.\end{aligned}$$

Example 6.17. *Evaluate* $\int_0^1 \frac{\sin^{-1} x}{x} dx$.

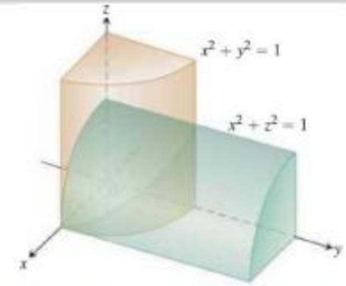
Solution. Put $\sin^{-1} x = \theta$ or $x = \sin\theta$ so that $dx = \cos\theta\, d\theta$
Also when $x = 0, \theta = 0$; when $x = 1, \theta = \pi/2$.

$$\therefore \quad \int_0^1 \frac{\sin^{-1} x}{x} dx = \int_0^{\pi/2} \theta \cdot \frac{\cos\theta}{\sin 0} d\theta \qquad \text{[Integrate by parts]}$$

$$= [\theta \cdot \log\sin\theta]_0^{\pi/2} - \int_0^{\pi/2} 1 \cdot \log\sin\theta\, d\theta$$

$$= -\int_0^{\pi/2} \log\sin\theta\, d\theta = -\left(-\frac{\pi}{2}\log 2\right) = \frac{\pi}{2}\log 2 \qquad \left[\because \operatorname*{Lt}_{x\to 0}(x\log x) = 0\right]$$

Example. Determine the volume of the region D common to the interiors of the cylinders $x^2 + y^2 = 1$ and $x^2 + z^2 = 1$, in the first octant.



Solution. One immediately recognizes the solid D has a top given by $x^2 + z^2 = 1$, i.e. $z_{\max}(x) = \sqrt{1 - x^2}$, and xy-plane as the bottom. Moreover, the shadow R of the solid D is a circle disk in the 1st quadrant, so $R = \{ (x, y) \mid 0 \le x \le 1,\ 0 \le y \le \sqrt{1 - x^2} \}$. The volume of D is given by $\iint_R (\sqrt{1 - x^2} - 0)\, dA$

$$= \int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1 - x^2}\, dy\, dx = \int_0^1 (1 - x^2) dx = \left[x - \frac{x^3}{3} \right]_0^1 = \frac{2}{3}.$$

Example 6.19. *Evaluate* $\underset{n\to\infty}{Lt} \left\{ \left(1 + \frac{1}{n}\right)\left(1 + \frac{2}{n}\right) \dots \left(1 + \frac{n}{n}\right) \right\}^{1/n}$. *(Bhopal, 2008)*

Solution. Let $$P = \underset{n\to\infty}{\text{Lt}} \left\{ \left(1 + \frac{1}{n}\right)\left(1 + \frac{2}{n}\right) \dots \left(1 + \frac{n}{n}\right) \right\}^{1/n}.$$

Taking logs of both sides,

$$\log P = \underset{n\to\infty}{\text{Lt}} \frac{1}{n} \left\{ \log\left(1 + \frac{1}{n}\right) + \log\left(1 + \frac{2}{n}\right) + \dots + \log\left(1 + \frac{n}{n}\right) \right\}$$

Its general term $= \log\left(1 + \frac{r}{n}\right) \cdot \frac{1}{n} = \log(1 + x) \cdot dx$ [Putting $r/n = x$ and $1/n = dx$]

Also for first term $r = 1$ and for the last term $r = n$.

$\therefore$ The lower limit of integration $= \underset{n\to\infty}{\text{Lt}} (1/n) = 0$ and the upper limit $= \underset{n\to\infty}{\text{Lt}} (n/n) = 1$

Hence $\log P = \int_0^1 \log(1 + x)\, dx = \int_0^1 \log(1 + x) \cdot 1\, dx$ [Integrate by parts]

$$= \left| \log(1 + x) \cdot x \right|_0^1 - \int_0^1 \frac{1}{1 + x} \cdot x\, dx$$

$$= \log 2 - \int_0^1 \frac{1 + x - 1}{1 + x} dx = \log 2 - \int_0^1 dx + \int_0^1 \frac{dx}{1 + x}$$

$$= \log 2 - \left| x \right|_0^1 + \left| \log(1 + x) \right|_0^1 = \log 2 - 1 + \log 2$$

$$= \log 2^2 - \log_e e = \log(4/e). \text{ Hence, } P = 4/e.$$

Example 6.23. *Find the area enclosed between one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$; and its base.* (*V.T.U., 2000*)

Solution. To describe its first arch, θ varies from 0 to 2π *i.e.*, x varies from 0 to $2a\pi$ (Fig. 6.8).

$\therefore$ Required area $= \int_{x=0}^{2\pi a} y\,dx$

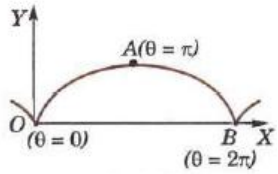


Fig. 6.8

vhere $y = a(1 - \cos\theta)$, $dx = a(1 - \cos\theta)\,d\theta$.

$$= \int_{\theta=0}^{\pi/2} a(1 - \cos\theta)\cdot a(1 - \cos\theta)\,d\theta$$

$$= 2a^2 \int_0^{\pi} (1 - \cos\theta)^2\,d\theta = 8a^2 \int_0^{\pi} \sin^4 \frac{\theta}{2}\,d\theta$$

$$= 16a^2 \int_0^{\pi/2} \sin^4 \phi\,d\phi, \text{ putting } \theta/2 = \phi \text{ so that } d\theta = 2d\phi.$$

$$= 16a^2 \cdot \frac{3\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2} = 3\pi a^2.$$

Example 6.26. *Find the area of the cardioid $r = a(1 - \cos\theta)$.* (*V.T.U., 2004*)

Solution. The curve is as shown in Fig. 6.12. Its upper half is traced from $\theta = 0$ to $\theta = \pi$.

$\therefore$ Area of the curve $= 2\cdot\frac{1}{2}\int_0^{\pi} r^2\,d\theta = a^2 \int_0^{\pi} (1 - \cos\theta)^2\,d\theta$

$$= a^2 \int_0^{\pi} (2\sin^2\theta/2)^2\,d\theta = 4a^2 \int_0^{\pi} \sin^4\theta/2\cdot d\theta$$

$$= 8a^2 \int_0^{\pi/2} \sin^4\phi\,d\phi, \text{ putting } \theta/2 = \phi \text{ and } d\theta = 2d\phi.$$

$$= 8a^2\cdot\frac{3\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2} = \frac{3\pi a^2}{2}.$$

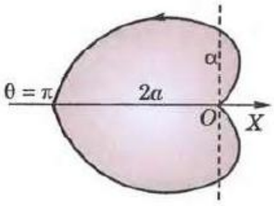


Fig. 6.12

Example 6.27. *Find the area of a loop of the curve $r = a\sin 3\theta$.*

Solution. The curve is as shown in Fig. 4.35. It consists of three loops.

Putting $r = 0$, $\sin 3\theta = 0$ $\therefore$ $3\theta = 0$ or π *i.e.*, $\theta = 0$ or $\pi/3$ which are the limits for the first loop.

$\therefore$ Area of a loop $= \frac{1}{2}\int_0^{\pi/3} r^2\,d\theta = \frac{1}{2}a^2 \int_0^{\pi/3} \sin^2 3\theta\,d\theta = \frac{a^2}{4}\int_0^{\pi/3} (1 - \cos 6\theta)\,d\theta$

$$= \frac{a^2}{4}\left|\theta - \frac{\sin 6\theta}{6}\right|_0^{\pi/3} = \frac{a^2}{4}\left(\frac{\pi}{3} - 0\right) = \frac{\pi a^2}{12}.$$

Example 6.28. *Prove that the area of a loop of the curve $x^3 + y^3 = 3axy$ is $3a^2/2$.*

Solution. Changing to polar form (by putting $x = r\cos\theta$, $y = r\sin\theta$), $r = \dfrac{3a\sin\theta\cos\theta}{\cos^3\theta + \sin^3\theta}$

Putting $r = 0$, $\sin\theta\cos\theta = 0$.

$\therefore$ $\theta = 0, \pi/2$, which are the limits of integration for its loop.

$\therefore$ Area of the loop

$$= \frac{1}{2}\int_0^{\pi/2} r^2\,d\theta = \frac{1}{2}\int_0^{\pi/2} \frac{9a^2\sin^2\theta\cos^2\theta}{(\cos^3\theta + \sin^3\theta)^2}\,d\theta$$

$$= \frac{9a^2}{2}\int_0^{\pi/2} \frac{\tan^2\theta\sec^2\theta}{(1 + \tan^3\theta)^2}\,d\theta \qquad \text{[Dividing num. and denom. by } \cos^6\theta]$$

$$= \frac{3a^2}{2}\int_1^{\infty} \frac{dt}{t^2}, \qquad \text{putting } 1 + \tan^3\theta = t \text{ and } 3\tan^2\theta\sec^2\theta\,d\theta = dt.$$

$$= \frac{3a^2}{2}\left|\frac{t^{-1}}{-1}\right|_1^{\infty} = \frac{3a^2}{2}(-0 + 1) = \frac{3a^2}{2}.$$

Example 7.3. *Evaluate* $\iint_R x^2\, dx\, dy$ *where R is the region in the first quadrant bounded by the lines* $x = y$, $y = 0$, $x = 8$ *and the curve* $xy = 16$.

Solution. The line AL ($x = 8$) intersects the hyperbola $xy = 16$ at A (8, 2) while the line $y = x$ intersects this hyperbola at B (4, 4). Figure 7.5 shows the region R of integration which is the area $OLAB$. To evaluate the given integral, we divide this area into two parts OMB and $MLAB$.

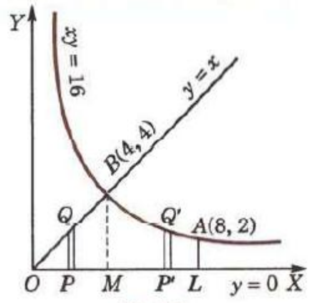


Fig. 7.5

$$\therefore \quad \iint_R x^2\, dxdy = \int_{x \text{ at } 0}^{x \text{ at } M} \int_{y \text{ at } P}^{y \text{ at } Q} x^2\, dxdy + \int_{x \text{ at } M}^{x \text{ at } L} \int_{y \text{ at } P'}^{y \text{ at } Q'} x^2\, dxdy$$

$$= \int_0^4 \int_0^x x^2\, dxdy + \int_4^8 \int_0^{16/x} x^2\, dxdy$$

$$= \int_0^4 x^2\, dx \left| y \right|_0^x + \int_4^8 x^2\, dx \left| y \right|_0^{16/x}$$

$$= \int_0^4 x^3\, dx + \int_4^8 16x\, dx = \left| \frac{x^4}{4} \right|_0^4 + 16 \left| \frac{x^2}{2} \right|_4^8 = 448$$

Example 7.28. *Find the volume of the portion of the sphere* $x^2 + y^2 + z^2 = a^2$ *lying inside the cylinder* $x^2 + y^2 = ay$. (*Rohtak, 2003*)

Solution. The required volume is easily found by changing to cylindrical coordinates (ρ, ϕ, z). We therefore, have

$$x = \rho \cos \phi,\ y = \rho \sin \phi,\ z = z$$

and $$J = \frac{\partial(x, y, z)}{\partial(\rho, \phi, z)} = \rho.$$

Then the equation of the sphere becomes $\rho^2 + z^2 = a^2$ and that of cylinder becomes $\rho = a \sin \phi$.

The volume inside the cylinder bounded by the sphere is twice the volume shown shaded in the Fig. 7.29 for which z varies from 0 to $\sqrt{(a^2 - \rho^2)}$, ρ varies from 0 to $a \sin \phi$ and ϕ varies from 0 to π.

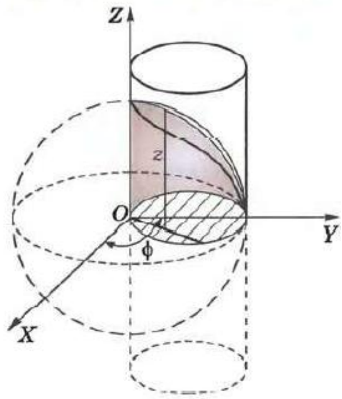


Fig. 7.29

$$\text{Hence the required volume} = 2 \int_0^{\pi} \int_0^{a \sin \phi} \int_0^{\sqrt{(a^2 - \rho^2)}} \rho dz d\rho d\phi$$

$$= 2 \int_0^{\pi} \int_0^{a \sin \phi} \rho \sqrt{(a^2 - \rho^2)}\, d\rho d\phi = 2 \int_0^{\pi} \left| -\frac{1}{3}(a^2 - \rho^2)^{3/2} \right|_0^{a \sin \phi} d\phi$$

$$= \frac{2a^3}{3} \int_0^{\pi} (1 - \cos^3 \phi)\, d\phi = \frac{2a^3}{9}(3\pi - 4).$$

Example 7.31. *Find the area of the portion of the cylinder* $x^2 + z^2 = 4$ *lying inside the cylinder* $x^2 + y^2 = 4$.

Solution. Figure 7.32 shows one-eighth of the required area. Its projection on the xy-plane is a quadrant circle $x^2 + y^2 = 4$.

For the cylinder $\qquad x^2 + z^2 = 4, \qquad \ldots(i)$

we have $$\frac{\partial z}{\partial x} = -\frac{x}{z},\ \frac{\partial z}{\partial y} = 0.$$

so that $$\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1 = \frac{x^2 + z^2}{z^2} = \frac{4}{4 - x^2}.$$

Hence the required surface area = 8 (surface area of the upper portion of (i) lying within the cylinder $x^2 + y^2 = 4$ in the positive octant)

$$= 8 \int_0^2 \int_0^{\sqrt{(4 - x^2)}} \frac{2}{\sqrt{(4 - x^2)}}\, dxdy = 16 \int_0^2 dx = 32 \text{ sq. units.}$$

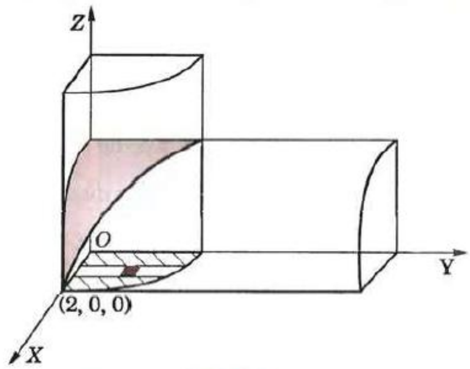


Fig. 7.32

Example 7.32. *Find the area of the portion of the sphere $x^2 + y^2 + z^2 = 9$ lying inside the cylinder $x^2 + y^2 = 3y$.*

Solution. Figure 7.33 shows one-fourth of the required area. Its projection on the xy-plane is the semi-circle $x^2 + y^2 = 3y$ bounded by the Y-axis.

For the sphere

$$x^2 + y^2 + z^2 = 9, \ \frac{\partial z}{\partial x} = -\frac{x}{z} \text{ and } \frac{\partial z}{\partial y} = -\frac{y}{z}$$

$$\therefore \quad \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1 = (x^2 + y^2 + z^2)/z^2$$

$$= \frac{9}{9 - x^2 - y^2} = \frac{9}{9 - r^2} \qquad \text{when} \quad x = r\cos\theta, y = r\sin\theta.$$

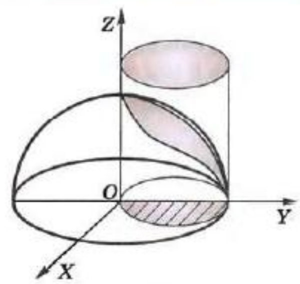


Fig. 7.33

Using polar coordinates, the required area is found by integrating $3/\sqrt{(9 - r^2)}$ over the semi-circle $r = 3\sin\theta$, for which r varies from 0 to $3\sin\theta$ and θ varies from 0 to $\pi/2$.

Hence the required surface area

$$= 4\int_0^{\pi/2}\int_0^{3\sin\theta} \frac{3}{\sqrt{(9 - r^2)}}\, r d\theta\, dr = -6\int_0^{\pi/2} \left| \frac{\sqrt{(9 - r^2)}}{1/2} \right|_0^{3\sin\theta} d\theta$$

$$= 36\int_0^{\pi/2} (1 - \cos\theta)\, d\theta = 36 \left| \theta - \sin\theta \right|_0^{\pi/2} = 18(\pi - 2) \text{ sq. units.}$$

Example 7.15. *Calculate the area included between the curve $r = a(\sec\theta + \cos\theta)$ and its asymptote.*

Solution. The curve is symmetrical about the initial line and has an asymptote $r = a\sec\theta$ (Fig. 7.19).

Draw any line OP cutting the curve at P and its asymptote at P'. Along this line, θ is constant and r varies from $a\sec\theta$ at P' to $a(\sec\theta + \cos\theta)$ at P. Then to get the upper half of the area, θ varies from 0 to $\pi/2$.

$$\therefore \quad \text{required area} \quad = 2\int_0^{\pi/2}\int_{a\sec\theta}^{a(\sec\theta + \cos\theta)} r\, dr d\theta$$

$$= 2\int_0^{\pi/2} \left[\frac{r^2}{2}\right]_{a\sec\theta}^{a(\sec\theta + \cos\theta)} d\theta$$

$$= a^2\int_0^{\pi/2} (2 + \cos^2\theta)\, d\theta = 5\pi a^2/4.$$

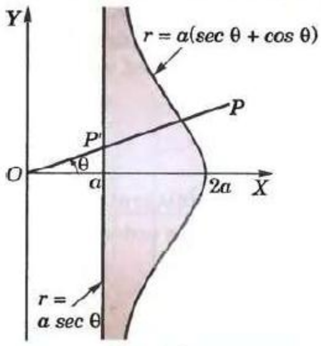


Fig. 7.19

Example 7.49. *The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the axes in A, B and C. Apply Dirichlet's integral to find the volume of the tetrahedron OABC. Also find its mass if the density at any point is kxyz.* (U.P.T.U., 2004)

Solution. Put $x/a = u$, $y/b = v$, $z/c = w$ then the tetrahedron $OABC$ has $u \geq 0$, $v \geq 0$, $w \geq 0$ and $u + v + w \leq 1$.

$\therefore$ volume of this tetrahedron $= \iiint_D dx\, dy\, dz$

$$= \iiint_{D} abc\; du\, dv\, dw \qquad \left[\begin{array}{l} a\ dx = adu,\ dy = bdv,\ dz = cdw \\ \text{for } D' = u \geq 0,\ v \geq 0,\ w \geq 0\ \&\ u+v+w \leq 1. \end{array}\right.$$

$$= abc \iiint_D u^{1-1}\, v^{1-1}\, w^{1-1}\, du\, dv\, dw$$

$$= abc \frac{\Gamma(1)\,\Gamma(1)\,\Gamma(1)}{\Gamma(1+1+1+1)} = \frac{abc}{6} \qquad \text{[By Dirichlet's integral]}$$

$$\text{Mass} = \iiint kxyz dx\, dy\, dz = \iiint k\,(au)\,(bv)\,(cw)\,abc\; du\, dv\, dw$$

$$= ka^2b^2c^2 \iiint u^{2-1}\, v^{2-1}\, w^{2-1}\, du\, dv\, dw$$

$$= ka^2b^2c^2 \;\frac{\Gamma(2)\,\Gamma(2)\,\Gamma(2)}{\Gamma(2+2+2+1)}\, ka^2b^2c^2 \cdot \frac{1}{6!} = \frac{k}{720}\; a^2b^2c^2 .$$