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DYNAMICS

Volume-I

(For Honours & Post-Graduate Students of Various Universities & also for I.A.S. & P.C.S. Competitive Examinations)

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PREFACE TO THE LATEST EDITION

This book has been thoroughly revised and enlarged and whereever found necessary, new articles and questions have been added. All misprints crept in the previous edition, as far as possible, have also been removed. Suggestions for further improvement from all corners will be highly appreciated.

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PREFACE TO THE FIRST EDITION

The authors feel great pleasure in bringing out this book Rigid Dynamics (Dynamics of Rigid Bodies). This book has been written keeping in mind all points at which students feel difficulty. Efforts have been made to explain such points in depth, so that students can follow the subject easily. Examples are well graded and their solution are given in a systematic way.

We do not claim to express anything in original, but after going through many standard works on the subject, we have the satisfaction of making the book alround complete. This main feature of the book is that it deals with the course on 'Rigid Dynamics' commonly prescribed by all Indian Universities at graduate or post graduate level.

Suggestions for improvement form all corners will be thankfully acknowledged.

—*Authors*

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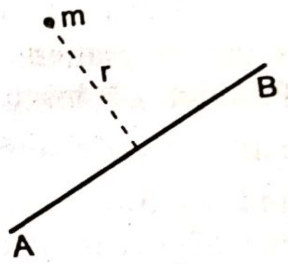
Moment Of Inertia

0· 01 Definitions.

(a) Rigid Body : A rigid body is the system of particles such that the mutual distance of every pair of specified particles in it is invariable and the body does not expand or contract or change its shape in any way. *i.e. the rigid body has invariable size and shape and the distance between any two particles remains always same.*

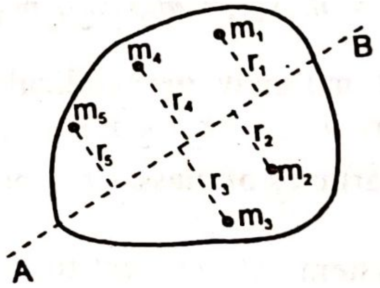
(b) Moment of inertia of a particle : Consider a particle of mass m and a line a line AB, then the moment of inertia of the particle of mass m about the line AB is defined as $I = mr^2$, where r is the perp. distance of the particle from the line.



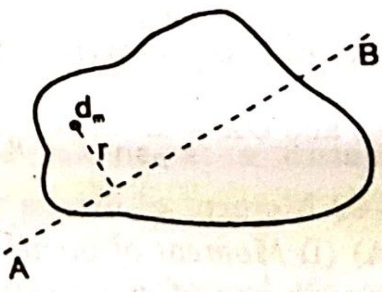
(c) Moment of inertia of a system of particles: Let there be a number of particles $m_1, m_2, m_3, ..., m_p$, and let $r_1, r_2, r_3, ..., r_p$ be the perp. distances of these masses from the given line AB, then the moment of inertia of the system is defined as

$$I = m_1 r_1^2 + m_2 r_2^2 + m_3 r_3^2 + ... + m_p r_p^2$$

$$= \sum_{p=1}^{n} m_p r_p^2$$



(d) Moment of inertia of a continuous distribution of mass : Consider a rigid body and let dm be mass of the elementary portion of the body which is at a perpendicular distance r from the given line AB, then the moment of inertia of the whole body is defined as $I = \int r^2\, dm$, where the integration is taken over the whole body.



(e) Radius of Gyration : The moment of inertia of a system of particles about the line AB is

$$I = \sum_{p=1}^{n} m_p r_p^2.$$

Let the total mass of the system of particles be M, then

$M = \sum_{p=1}^{n} m_p$ and further define a quantity K such that

$$I = MK^2 \Rightarrow K^2 = \left(\frac{1}{M}\right) = \frac{\sum_{p=1}^{n} m_p r_p^2}{\sum_{p=1}^{n} m_p},$$

Then K is called the **radius of gyration** of the system about AB. In the case of *continuous mass distribution*, we similarly have

$$K^2 = \left(\frac{I}{M}\right) = \left[\frac{\int r^2\, dm}{\int dm}\right]$$

where the integration is taken over the whole body.

(f) Product of inertia : If $(x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots (x_p, y_p)$ be the respective coordinates of the particles of masses $m_1, m_2, m_3, \ldots m_p$, referred to two mutually perpendicular lines OX and OY, then the product of inertia of the system of particles with respect to the lines OX and OY, is defined as,

$$P = m_1 x_1 y_1 + m_2 x_2 y_2 + m_3 x_3 y_3 \ldots + m_p x_p y_p = \sum_{p=1}^{n} m_p x_p y_p.$$

If mutually perpendicular axes OX, OY, OZ be taken in space and $(x_1, y_1, z_1), (x_2, y_2, z_2), \ldots (x_p, y_p, z_p)$ be the respective co-ordintes of the particles of masses $m_1, m_2, \ldots, m_p$, then we have, product of inertia of the system with respect to the axes OX and $OY = \sum_{p=1}^{n} m_p x_p y_p$

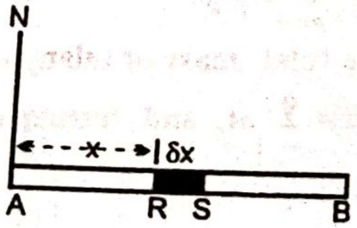
Product of inertia of the system with respect to the axes OY and $OZ = \sum_{p=1}^{n} m_p y_p z_p$.

Product of inertia of the system with respect to the axes OZ and $OX = \sum_{p=1}^{n} m_p x_p z_p$.

0·02 Moment of inertia in some simple cases.

(A) (i) *Moment of inertia of a rod of length 2a and mass M about a line through one of its extremities perp. to its length.*

Consider an element RS of breadth δx of the rod AB at distance x from the line AN, where AN is perp. to AB, M.I. of the element RS about $AN = \frac{M}{2a}\delta x\, x^2$ where $(M/2a)\,\delta x =$ mass of the element.



$\therefore$ M.I. of the whole rod

$$= \int_0^{2a} \frac{M}{2a} x^2\, dx = \frac{M}{2a}\left[\frac{x^3}{3}\right]_0^{2a} = M.\frac{4a^2}{3}.$$

$I = \frac{1}{3} M l^3$

(ii) *Moment of inertia of a rod of length 2a and of mass M about a line through its centre perpendicular to its length.*

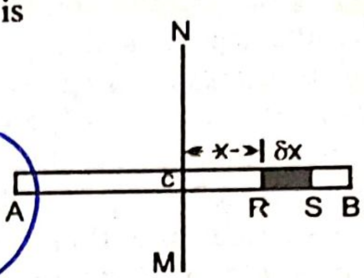
Consider an element RS of breadth δx at distance x from the centre C.

$\therefore$ M.I. of the element RS about NCM is

$$= \frac{M}{2a}\,\delta x . x^2$$

$\Rightarrow$M.I. of the whole rod about

$$MN = \int_{-a}^{a} \frac{M}{2a} x^2\, dx = \frac{M}{2a}\left[\frac{x^3}{3}\right]_{-a}^{a} = M . \frac{a^2}{3}.$$



(B) Rectangular Lamina.

(i) *Moment of inertia of a rectangular lamina about a line through its centre and parallel to one of its edges.*

Consider the strip $RSPQ$ of breadth δx of the rectangular lamina $ABCD$ such that $AB = 2a$ and $AD = 2b$. Let M be the mass of the rectangular lamina. Then mass per unit area $= \dfrac{M}{4ab} = \rho$ (*say*).

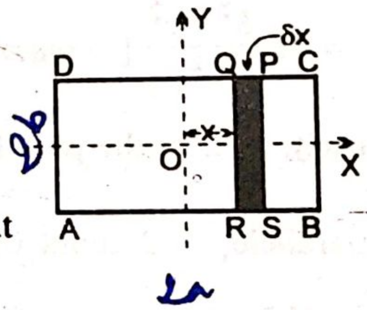
$\therefore$ Mass of the strip $RSPQ = 2b\,\delta x\,\rho = \dfrac{M}{4ab}(2b\,.\delta x)$

Now using [A case (ii)], we get M.I. of the strip about

$$OX = \frac{M}{4ab} 2b\,\delta x \left(\frac{b^2}{3}\right) = \frac{M}{2a}.\frac{b^2}{3}\,\delta x$$

$\therefore$ M.I. of the rectangular lamina about OX

$$= \int_{-a}^{a} \frac{M}{2a}.\frac{b^2}{3}\,dx = \frac{1}{3} M b^2.$$

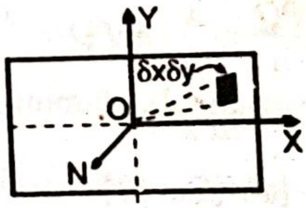


Similarly M.I. of the rectangular lamina about OY is $\frac{1}{3} M a^2$.

(ii) *Moment of inertia of a rectangular lamina about a line through its centre and perp. to its plane.*

Consider an elementary area $\delta x\,\delta y$ of the lamina at a distance $\sqrt{(x^2+y^2)}$ from O. Mass of the elementary area$= \dfrac{M}{4ab}\,\delta x.\,\delta y.$

M.I. of this elementary area about the line ON through O and perpendicular to the plane of the rectangular lamina$= \dfrac{M}{4ab}\,\delta x.\,\delta y\,(x^2+y^2)$



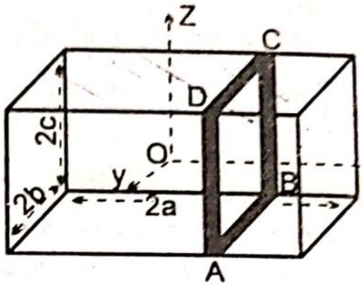
$\therefore$ M.I. of the rectangular lamina about ON is

$$= \frac{M}{4ab}\int_{x=-a}^{a}\int_{y=-b}^{b}(x^2+y^2)\,dx\,dy = \frac{M}{4ab}4\int_{x=0}^{a}\int_{y=0}^{b}(x^2+y^2)\,dx\,dy$$

$$= \frac{M}{ab}\left[\tfrac{1}{3}b\,x^3 + \tfrac{1}{3}b^3x\right]_0^a = \frac{M}{3}(a^2+b^2).$$

(iii) Rectangular Parallelopiped :

Let O be the centre and $2a, 2b, 2c$ the lengths of the edges of the parallelopiped and further let OX, OY, OZ, be the axes of reference, parallel to the edges of lengths $2a$, $2b$ and $2c$ respectively. Divide the parallelopiped into thin rectangular slices perp. to OX, $ABCD$ being one such slice at a distance x. Let the width of the slice be δx.



$\therefore$ M.I. of the rectangular slice about OX

$$= \text{mass} \times \frac{b^2+c^2}{3} = 2b\,.2c.\,\rho.\,\delta x.\frac{b^2+c^2}{3} = 4\,b\,c\,\rho\,\frac{b^2+c^2}{3}\,\delta x$$

[mass of the slice $ABCD = 2b\,2c\,\delta x\,\rho$]

$\Rightarrow$M.I. of the parallelopiped about OX

$$= 4bc\,\rho\,\frac{b^2+c^2}{3}\int_{-a}^{a} dx = 8abc\,\rho\,\frac{b^2+c^2}{3}$$

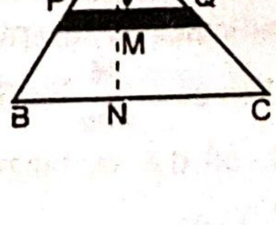
$$= M\,\frac{b^2+c^2}{3} \quad [\because \text{ mass of the parallelopiped } = 2a\,2b.2c\rho = 8abc\,\rho]$$

Similarly, M.I. of the parallelopiped about $OY = M\,\dfrac{c^2+a^2}{3}$ and M.I. of the parallelopiped about $OZ = M\,\dfrac{a^2+b^2}{3}$.

Note : M.I. of the cube of side $2a$ about any of its axis is $\frac{2}{3}Ma^2$.

(c) *Moment of inertia of a uniform triangular lamina about one side.*

Let us divide the lamina ABC by strips parallel to BC. Let PQ be one of such strips of breadth δx at distance x from A and let p be the length of perpendicular AN.



Now $\dfrac{PQ}{a} = \dfrac{x}{p}$ *i.e.* $PQ = \dfrac{xa}{p}$. If M is the mass of the triangular lamina, then mass per unit area$= [M/(\frac{1}{2}a.p)] = \rho\ (say)$.

$$\therefore \text{Mass of the strip} = \frac{M}{\frac{1}{2}ap}\,PQ\,\delta x = \frac{2M}{p^2}x^2\,\delta x$$

M.I. of the strip about $BC = \frac{2M}{p^2} x\,\delta x\,(p-x)^2$

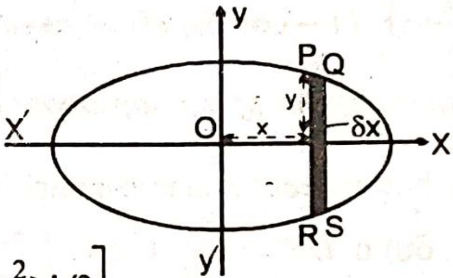
$\therefore$ M.I. of the triangle about $BC = \frac{2M}{p^2}\int_0^p (p-x)^2 x\,dx = \frac{1}{6}Mp^2$

(D) Elliptic disc : *Moment of inertia of an elliptic disc about its major axis.*

Let $PRSQ$ be an elementary strip of breadth δx at a distance x from O, where O is the centre of the disc. M.I. of the strip about

$OX = 2y.\,\delta x.\,\rho.\,\frac{y^2}{3}$, where ρ is the mass per unit area.

$\therefore$ M.I. of the elliptic lamina about OX



$$= \int_{-a}^{a} 2y\rho.\frac{y^2}{3}dx = \frac{2\rho}{3}\int_{-a}^{a} y^3\,dx$$

$$= \frac{2\rho}{3}b^3\int_{-a}^{a}\left(1-\frac{x^2}{a^2}\right)^{3/2}dx$$

$$\left[\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow y = b\left\{1-\frac{x^2}{a^2}\right\}^{1/2}\right]$$

Put $x = a\sin\phi$, so that $dx = a\cos\phi\,d\phi$

$\therefore$ M.I. of the elliptic lamina about OX

$$= \frac{2\rho b^3}{3}\int_{-\pi/2}^{\pi/2}\cos^3\phi\,.a\cos\phi\,d\phi = \frac{4\rho b^3 a}{3}\int_0^{\pi/2}\cos^4\phi\,d\phi$$

$$= \frac{4\rho b^3 a}{3}\cdot\frac{3\pi}{16} = \frac{\pi a b^3\rho}{4}.$$

Again mass of the elliptic lamina

$$M = \rho\int_{-a}^{a} 2y\,dx = 2\rho b\int_{-a}^{a}\left\{1-\frac{x^2}{a^2}\right\}^{1/2}dx$$

$$= 2\rho b\int_{-\pi/2}^{\pi/2}\cos\phi\,.\,a\cos\phi\,d\phi = \pi ab\rho \Rightarrow \rho = \frac{M}{\pi ab}.$$

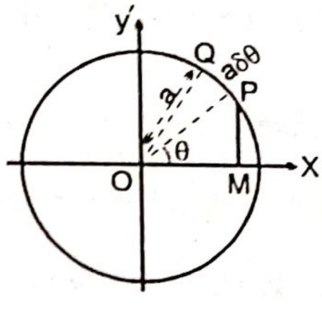
Hence from (1), M.I. of the elliptic lamina about OX *i.e.* about major axes $= \frac{\pi ab^3}{4}\cdot\frac{M}{\pi ab} = \frac{1}{4}Mb^2$.

Similarly M.I. of the elliptic lamina about OY *i.e.* about minor axes $= \frac{1}{4}Ma^2$.

(E) Hoop or Circumference of a circle.

(i) *Moment of inertia of a hoop about a diameter.*

Consider an element PQ of the hoop and let it subtend an angle $\delta\theta$ at its centre O i.e. $P\hat{O}Q = \delta\theta$ where $P\hat{O}X = \theta$.



By the figure it is obvious that arc $PQ = a\,\delta\theta$, where a is the radius of the hoop. Now M.I. of the element PQ about $OX = (a\,\delta\theta)\,\rho.a^2\sin^2\theta$,

where ρ is the mass per unit length of the hoop.

M.I. of the hoop about $OX = \int_{\theta=0}^{2\pi} (a\,d\theta)\,\rho.a^2\sin^2\theta$

$$= \frac{Ma^2}{4\pi}\int_0^{2\pi}(1-\cos 2\theta)\,d\theta = \tfrac{1}{2}Ma^2$$

(ii) *Moment of inertia of a hoop about a line through its centre and perp. to its plane.*

M.I. of the hoop about a line through O and perp. to its plane

$$= (a.\,\delta\theta)\,\rho.\,OP^2 = \frac{M}{2\pi}a^2\,\delta\theta \qquad (\because\ OP = a,\ M = 2\pi a\rho)$$

$\therefore$ M.I. of the hoop about a line through O and perp. to its plane

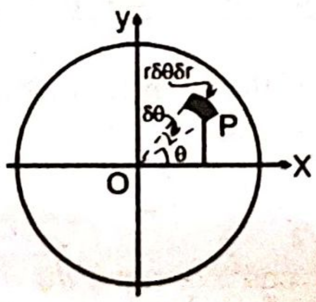
$$= \frac{Ma^2}{2\pi}\int_0^{2\pi} d\theta = \frac{Ma^2}{2\pi}[\theta]_0^{2\pi} = Ma^2.$$

(F) Circular Disc.

(ii) *Moment of inertia of a circular disc of radius a about its diameter.*

Consider an element $r\,\delta\theta\,\delta r$ of the disc at P such that OP makes an angle θ with the axis OX. The perp. distance of P from OX is $r\sin\theta$.

Let ρ be the mass per unit area of the disc. M.I. of this element about $OX = r\,\delta\theta\,\delta r.\,\rho.(r\sin\theta)^2$.



$\therefore$ M.I. of this element about the diameter OX

$$= \int_{r=0}^{a}\int_{\theta=0}^{2\pi} r^3\rho\sin^2\theta\,d\theta\,dr = \frac{M}{\pi a^2}\int_{r=0}^{a}\int_{\theta=0}^{2\pi} r^3\sin^2\theta\,d\theta\,dr \qquad (\because\ M = \pi a^2\rho)$$

$$= \frac{M}{2\pi a^2}\int_0^a r^3\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{2\pi} dr = \frac{Ma^2}{4}.$$

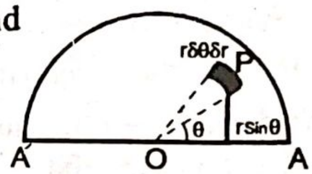
(ii) *Moment of inertia of a circular disc of radius a about a line through its centre perp. to its plane.*

M.I. of the element $r\delta\theta\,\delta r$ about a line through O and perp. to the plane of the disc.$= (r\delta\theta\,\delta r).\,\rho.OP^2 = \dfrac{M r^3}{\pi a^2} d\theta\, dr$ $\quad (\because \; \pi a^2 \rho = M)$

M.I. of the circular disc about a line through O and perp. to the plane of the disc $= \displaystyle\int_{\theta=0}^{2\pi}\int_{r=0}^{a} \frac{M r^3}{\pi a^2} d\theta\, dr = \frac{M a^2}{2}$

(G) Solid Sphere.

If a semi-circular area is revolved about its bounding diameter then the solid so generated is called sphere. Now consider an element of area $r\delta\theta\,\delta r$ at P such that $OP = r$ and makes an angle θ with the diameter. When this area is revolved about the diameter $A'A$, it will generate a ring of cross-section $r\delta\theta\,\delta r$ and radius $r\sin\theta$.



$\therefore$ Mass of this elementary ring

$= 2\pi r\sin\theta.\, r\delta\theta\,\delta r.\,\rho$

M.I. of the elementary ring about $A'A$

$= (2\pi r\sin\theta. r\delta\theta\,\delta r.\,\rho)\,(r\sin\theta)^2$ **[refer E, (ii)]**

$= 2\pi\, \rho r^4 (\sin^3\theta)\, \delta\theta\, \delta r.$

M.I. of the solid sphere about the diameter $A'A$

$$= 2\pi\rho \int_0^{\pi}\int_0^{a} r^4 \sin^3\theta\, d\theta\, dr = 4\pi\rho \int_0^{\pi/2}\int_0^{a} r^4 \sin^3\theta\, d\theta\, dr$$

$$= 4\pi\rho\left[\frac{r^5}{5}\right]_0^a . \frac{2}{3} = 4\pi\rho.\frac{a^5}{5}.\frac{2}{3} = \frac{8\pi a^5 \rho}{15} = I \text{ say}$$

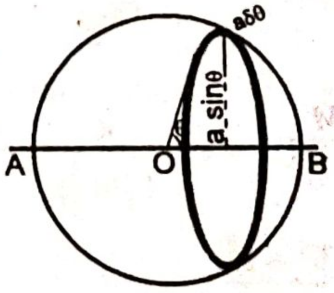
But mass of the sphere, $M = \dfrac{4}{3}\pi a^3 \rho \Rightarrow \rho = \dfrac{3M}{4\pi a^3}$ ...(1)

$$\Rightarrow I = \frac{8}{15}\pi a^5 . \frac{3M}{4\pi a^3} = \frac{2}{5}(M a^2)$$

(H) Hollow sphere.

If a semicircular arc is revolved about its diameter, then the surface so formed is known as hollow sphere.

Consider an elementary arc $a\,\delta\theta$. This arc $a\,\delta\theta$ will generate a circular ring of radius $a\sin\theta$ when revolved about the diameter AB. Now mass of the elementary ring $= 2\pi\, a\sin\theta.\, a\,\delta\theta.\,\rho$.



M.I. of the elementary ring about AB

$= (2\pi a\sin\theta.\, a\,\delta\theta.\,\rho).\, a^2\sin^2\theta$

$= 2\pi a^4 \rho \sin^3\theta\, \delta\theta.$ [refer E, (iii)]

$\Rightarrow$M.I. of the hollow sphere about the diameter AB

$$= 2\pi a^4 \rho \int_0^{\pi} \sin^3\theta\, d\theta = 2\pi a^4 . \frac{M}{4\pi a^2} \int_0^{\pi} \sin^3\theta\, d\theta \quad (\because\ M = 4\pi a^2 \rho)$$

$$= \frac{M a^2}{2} . 2\int_0^{\pi/2} \sin^3\theta\, d\theta = M a^2 \int_0^{\pi/2} \sin^3\theta\, d\theta = M a^2 . \frac{2}{3} = \frac{2}{3} M a^2.$$

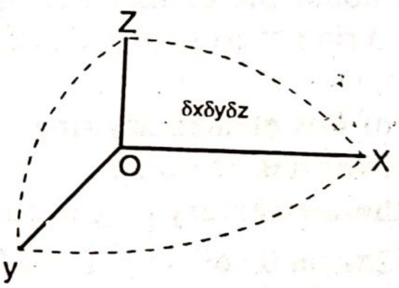
(I) Ellipsoid. Consider an elementary volume $\delta x\, \delta y\, \delta z$ in the positive octant of the ellipsoid $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1.$ **[Kanpur 92]**

Let ρ be mass per unit volume then mass of the elementary volume

$= \rho.(\delta x.\ \delta y.\ \delta z).$

Distance of this element from $OX = \sqrt{(y^2 + z^2)}.$

$\therefore$ M.I. of the ellipsoid about OX $= 8\iiint \rho\, dx\, dy\, dz\, (y^2 + z^2)$, the integration being taken over the positive octant of the ellipsoid and



$(x^2/a^2) + (y^2/b^2) + (z^2/c^2) \leq 1.$

Putting $(x^2/a^2) = u$, $(y^2/b^2) = v$ and $(z^2/c^2) = w$, we get,

$x = au^{1/2}$, $dx = \frac{1}{2} au^{-1/2}\, du$; $y = bv^{1/2}$, $dy = \frac{1}{2} bv^{-1/2}\, dv$;

$z = cw^{1/2}$, $dz = \frac{1}{2} cw^{-1/2}\, dw.$

Now, M.I. of the ellipsoid about OX

$$= 8\iiint \frac{\rho}{8} abc\, (b^2 v + c^2 w) u^{-1/2} v^{-1/2} w^{-1/2}\, du\, dv\, dw$$

where $u + v + w \leq 1$

$$= abc\,\rho \iiint (b^2 u^{-1/2} v^{-1/2} w^{-1/2} + c^2 u^{-1/2} v^{-1/2} w^{-1/2})\, du\, dv\, dw$$

$$= abc\rho \iiint (b^2 u^{1/2 - 1} v^{3/2 - 1} w^{1/2 - 1} + c^2 u^{1/2 - 1} v^{1/2 - 1} w^{3/2 - 1})\, du\, dv\, dw$$

$$= abc\,\rho \left[b^2 . \frac{\Gamma(\frac{1}{2})\, \Gamma(\frac{1}{2})\, \Gamma(\frac{3}{2})}{\Gamma(\frac{7}{2})} + c^2 \frac{\Gamma(\frac{1}{2})\, \Gamma(\frac{1}{2})\, \Gamma(\frac{3}{2})}{\Gamma(\frac{7}{2})} \right]$$

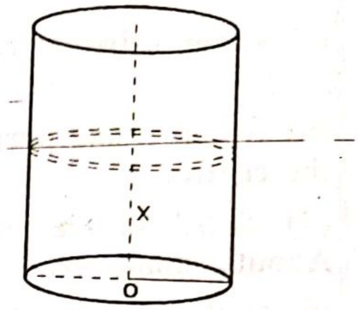
(using Dirichlet's theorem)

$$= abc\,\rho\, (b^2 + c^2) \frac{\pi}{(\frac{5}{2})(\frac{3}{2})} = \frac{4abc\,\rho\,\pi}{3} . \frac{b^2 + c^2}{5}$$

$$= M . \frac{b^2 + c^2}{5} \qquad \text{where } M = \frac{4}{3}\pi abc\,\rho$$

(J) Right circular cylinder.

Let there be a right circular cylinder of radius a and height h. Consider a circular disc of thickness δx at a distance x from O the centre of base. Mass of the disc $= \pi a^2 \delta x . \rho$, where ρ is mass per unit volume.



$\therefore$M.I. of the disc about the axes perp. to the plane of the disc

$$= \pi a^2 \delta x \rho . \frac{1}{2} a^2 \qquad \text{[refer F (ii)]}$$

$$\Rightarrow \text{M.I. of the cylinder} = \frac{\pi a^4 \rho}{2} \int_0^h dx = \frac{\pi a^4 h \rho}{2}$$

$$= \frac{1}{2} M a^2 \qquad [\because M = \pi a^2 h \rho]$$

Reference table : The following table shows the moments of inertia of various rigid bodies considered above. In all cases it is assumed that the body has uniform density.

Rigid Body	Moments of inertia
(1) *Uniform rod of length 2a and mass M.*	
(i) About an axis perp. to the rod through the centre of mass.	$\frac{1}{3} M a^2$
(ii) About a line perp. to the rod through an end.	$\frac{4}{3} M a^2$
(2) *Rectangular plate of sides 2a, 2b and mass M*	
(i) About an axis perp. to the plate through the centre of mass.	$\frac{M}{3}(a^2 + b^2)$
(ii) About a line through centre parallel to the side 2*a*.	$\frac{1}{3} M b^2$
(3) *Rectangular parallelopiped of edges 2a, 2b, 2c.* About a line parallel to the edge 2a, through the centre	$\frac{M}{3}(b^2 + c^2)$
(4) *Circular plate of radius a and mass M.*	
(i) About its diameter.	$\frac{1}{4} M a^2$
(ii) About a line perp. to the plate through the centre.	$\frac{1}{2} M a^2$
(5) *Elliptic disc of axes 2a and 2b and mass M.*	
(i) About the axis 2a.	$\frac{1}{4} M b^2$

(ii) About a line perp. to the disc through its centre.	$\frac{M}{4}(a^2+b^2)$
(6) *Circular ring of radius a and mass M .*	
(i) About a diameter.	$\frac{1}{2}Ma^2$
(ii) About a line perp. to the plate and through the centre.	Ma^2
(7) *Solid sphere of radius a and mass M.* About a diameter.	$\frac{2}{5}Ma^2$
(8) *Hollow sphere of radius a and mass M.* About a diameter (thickness negligible)	$\frac{2}{3}Ma^2$
(9) *Ellipsoid of axes 2a, 2b and 2c.* about the axis 2a.	$\frac{M}{5}(b^2+c^2)$

Routh's Rule : For remembering the moment of inertia of symmetric rigid bodies. M.I. about an axis of symmetry

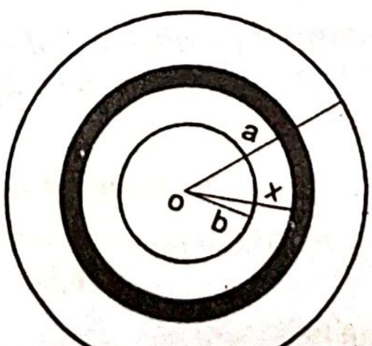
$$= \text{Mass} \times \frac{\text{Sum of the squares of perp. semi axes}}{3, 4 \text{ or } 5}$$

The denominator is 3, 4 or 5 according as the body is rectangular (including rod) elliptical (including circular) or ellipsoid (including sphere).

ILLUSTRATIVE EXAMPLES

Ex.1. *Find the moment of inertia of a hollow sphere about a diameter, its external and internal radii being a and b.*

Sol. Consider a hollow sphere, the external radius of which is a and the internal radius is b. Take a spherical shell in it of radius x and of width δx.



$\therefore$ Moment of inertia of this shell about diameter $= \frac{2}{3}$(Mass of the shell)

$\times$ (radius of the shell)^2 [refer H]

$$= \frac{2}{3}.\, 4\pi x^2 \rho\delta x.\, x^2 = \frac{8\pi\rho}{3}x^4\,\delta x.$$

Hence M.I. of the given hollow sphere

$$= \int_0^a \frac{8\pi\rho}{3}x^4\,dx = \frac{8\pi\rho}{3}\left[\frac{x^5}{5}\right]_0^a = \frac{8}{15}\pi\rho(a^5-b^5)$$

$$= \frac{8}{15}\pi.\frac{3M}{4\pi(a^3-b^3)}.(a^5-b^5) \quad [\because\ M = \tfrac{4}{3}\pi(a^3-b^3)\rho]$$

$$= \frac{2M}{5}\frac{a^5 - b^5}{a^3 - b^3}.$$

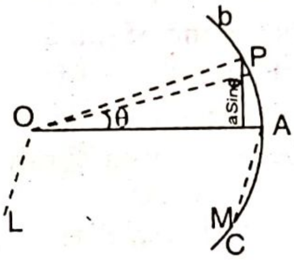
Ex.2. *Find the moment of inertia of the arc of a circle about (i) the diameter bisecting the arc,*
(ii) an axes through the centre perp. to its plane. **[Kanpur 91]**
(iii) an axes through its middle point perp. to its plane. **[Kanpur 91]**

Sol. Consider the arc BC, such that $\angle BOC = 2\alpha$, where O is the centre of the circular arc.

Let OA be the semi diameter bisecting the arc. Take an elementary arc $a\,\delta\theta$ at P.

Mass of this element $= \rho\, a\, \delta\theta$,

(i) Distance of the element from $OA = a \sin\theta$



$\therefore$ M.I. of the element about $OA = \rho a\, \delta\theta\, (a \sin\theta)^2 = \rho a^3 \sin^2\theta\, \delta\theta$

$\Rightarrow$M.I. of the whole arc about

$$OA = \rho a^3 \int_{-\alpha}^{\alpha} \sin^2\theta\, d\theta$$

$$= 2a^3\rho \int_0^{\alpha} \sin^2\theta\, d\theta = a^3\rho \int_0^{\alpha} (1 - \cos 2\theta)\, d\theta = a^3\rho(\alpha - \sin\alpha.\cos\alpha)$$

$$= a^3 . \frac{M}{2\alpha a}(\alpha - \sin\alpha\cos\alpha) = \frac{M a^2}{2\alpha}(\alpha - \sin\alpha\cos\alpha) \quad [\because \; M = 2\alpha a \rho]$$

(*ii*) Let OL be a line through centre O and perp. to the plane of the arc. Distance of the element from $OL = a$

$\therefore$ M.I. of elementary arc about $OL = (\rho a\, \delta\theta) a^2 = \rho\, a^3\, \delta\theta.$

Now M.I. of the whole arc about OL is given by

$$I = \int_{-\alpha}^{\alpha} \rho a^3\, d\theta = 2a^3 \alpha \rho = 2a\,\alpha\,\rho a^2 = M\, a^2 \qquad [\because \; M = 2\alpha\, a\, \rho]$$

(*iii*) Let AM be the line through the middle point A of the arc and perp. to the plane of the arc, then distance of the element from

$$AM = AP = \{(AN)^2 + (NP)^2\}^{1/2}$$

$$= \{(OA - ON)^2 + (NP)^2\}^{1/2} = \{(a - a\cos\theta)^2 + (a\sin\theta)^2\}^{1/2}$$

$$= \{a^2(1 - \cos\theta)^2 + a^2 \sin^2\theta\}^{1/2} = 2a \sin\frac{\theta}{2}.$$

M.I. of the elementary arc about AM is given by

$$I_1 = \rho a\, \delta\theta (2a \sin\tfrac{1}{2}\theta)^2 = 4a^3\, \rho \sin^2\frac{\theta}{2}\, \delta\theta.$$

M.I. of the whole arc about AM

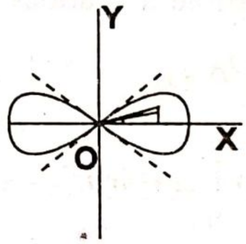
$$= \int_{-\alpha}^{\alpha} 4a^3 \rho \sin^2 \tfrac{1}{2}\theta\, d\theta = 4a^3\rho . 2\int_0^{\alpha} \sin^2\frac{\theta}{2}\, d\theta = 4a^3\rho \int_0^{\alpha} 2\sin^2\frac{\theta}{2}\, d\theta$$

$$= \frac{2Ma^2}{\alpha}(\alpha - \sin\alpha) \qquad \left[\because \; \rho = \frac{M}{2a\alpha}\right]$$

Ex.3. *Show that the moment of inertia of the area bounded by* $r^2 = a^2\cos 2\theta$. **(i)** *about its axis is* $\frac{Ma^2}{16}(\pi - \frac{8}{3})$. **(ii)** *about a line through the origin, in its plane and perp. to its axis is* $\frac{Ma^2}{16}(\pi + \frac{8}{3})$. **(iii)** *about a line through the origin and perp. to its plane is* $\frac{Ma^2\pi}{8}$.

Sol. The curve consists of two loops and the variation of θ is from $-(\pi/4)$ to $(\pi/4)$. Let OX be its axis and OY a line in the plane of the curve perp. to OX. Consider an elementary area $r\delta\theta\,\delta r$, then its mass $= r\delta\theta\,\mathcal{E}\,\rho$.

$\therefore$ Mass of the whole area (both loops)



$$M = 2\int_{-\pi/4}^{\pi/4}\int_{0}^{a\sqrt{(\cos 2\theta)}} \rho r\, d\theta\, dr$$

$$= 4\int_{0}^{\pi/4}\int_{0}^{a\sqrt{(\cos 2\theta)}} \rho r\, d\theta\, dr$$

$$= 4\rho\int_{0}^{\pi/4}\left(\frac{r^2}{2}\right)_0^{a\sqrt{(\cos 2\theta)}} d\theta = 4\rho\int_{0}^{\pi/4}\frac{a^2\cos 2\theta}{2}\, d\theta$$

$$= 2\rho a^2\int_{0}^{\pi/4}\cos 2\theta\, d\theta = 2\rho a^2\int_{0}^{\pi/2}\frac{1}{2}\cos\phi\, d\phi, \text{ Putting } 2\theta = \phi$$

$$= \rho a^2\int_{0}^{\pi/2}\cos\phi\, d\phi = \rho a^2, \quad \rho = \frac{M}{a^2} \qquad \text{...(1)}$$

(i) Distance of the elementary area $r\delta\theta\,\delta r$ from $OX = r\sin\theta$

$\therefore$ M.I. of the elementary area about $OX = \rho\, r\delta\theta\,\delta r.(r\sin\theta)^2$

$\Rightarrow$ M.I. of whole area $= 2\int_{-\pi/4}^{\pi/4}\int_{0}^{a\sqrt{(\cos 2\theta)}} \rho r\, d\theta\, dr\,(r\sin\theta)^2$

$$= 2\rho\int_{-\pi/4}^{\pi/4}\int_{0}^{a\sqrt{(\cos 2\theta)}} r^3\sin^2\theta\, d\theta\, dr = 4\rho\int_{0}^{\pi/4}\left[\frac{r^4}{4}\right]_0^{a\sqrt{(\cos 2\theta)}}\sin^2\theta\, d\theta$$

$$= a^4\rho\int_{0}^{\pi/4}\cos^2 2\theta\sin^2\theta\, d\theta = \frac{a^4\rho}{2}\int_{0}^{\pi/4}(\cos^2\theta)(1-\cos 2\theta)\, d\theta$$

$$= \frac{a^4\rho}{4}\int_{0}^{\pi/2}\cos^2\phi(1-\cos\phi)\, d\phi, \text{ putting } 2\theta = \phi \text{ i.e. } d\theta = \tfrac{1}{2}d\phi$$

$$= \frac{1}{4} a^4 \rho \int_0^{\pi/2} (\cos^2\phi - \cos^3\phi)\, d\phi = \frac{M a^2}{16} [\pi - \tfrac{8}{3}] \qquad \text{[using (1)]}$$

(ii) Distance of the elementary area $r\,\delta\theta\,\delta r$ from $OX = r\cos\theta$

$\therefore$ Its M.I. about $OY = \rho . r\,\delta\theta\,\delta r (r\cos\theta)^2 = \rho r^3 \cos^2\theta\,\delta\theta\,\delta r$

Hence M.I. of the whole area about $OY = 2\int_{-\pi/4}^{\pi/4}\int_0^{a\sqrt{(\cos 2\theta)}} \rho r^3 \cos^2\theta\, d\theta\, dr$

$$= 4\int_0^{\pi/4}\int_0^{a\sqrt{(\cos 2\theta)}} \rho r^3 \cos^2 d\theta\, dr = 4\rho\int_0^{\pi/4}\left(\frac{r^4}{4}\right)_0^{a\sqrt{(\cos 2\theta)}} \cos^2\theta\, d\theta$$

$$= 4\rho\int_0^{\pi/4} \frac{a^4\cos^2 2\theta}{4}\cos^2\theta\, d\theta = \rho a^4 \int_0^{\pi/4} \cos^2 2\theta . \tfrac{1}{2}(1+\cos 2\theta)\, d\theta,$$

putting $2\theta = \phi$ and $d\theta = \frac{1}{2} d\phi$

$$= \frac{\rho a^4}{4}\int_0^{\pi/2} \cos^2\phi\,(1+\cos\phi)\, d\phi = \int_0^{\pi/2} (\cos^2\phi + \cos^3\phi)\, d\phi$$

$$= \frac{\rho a^4}{4}\left[\frac{1}{2}.\frac{\pi}{2} + \frac{2}{3}\right] = \frac{M a^2}{16}(\pi + \tfrac{8}{3}), \qquad \left[\because \rho = \frac{M}{a^2}\right]$$

(iii) Distance of the element $r\,\delta\theta\,\delta r$ from the line which passes through the origin and is perp. to the plane of the curve is r.

$\therefore$ M.I. of the element $= \rho r\,\delta\theta\,\delta r . r^2 = \rho r^3\,\delta\theta\,\delta r$.

Hence required M.I. about the given line

$$= 2\int_{-\pi/4}^{\pi/4}\int_0^{a\sqrt{(\cos 2\theta)}} \rho r^3\, d\theta\, dr = 4\int_0^{\pi/4}\int_0^{a\sqrt{(\cos 2\theta)}} \rho r^3\, d\theta\, dr$$

$$= 4\rho\int_0^{\pi/4}\left[\frac{r^4}{4}\right]_0^{a\sqrt{(\cos 2\theta)}} d\theta = \rho a^4 \int_0^{\pi/4} \cos^2 2\theta\, d\theta$$

$$= \frac{\rho a^4}{2}\int_0^{\pi/2} \cos^2\phi\, d\phi, \text{ where } 2\theta = \phi \text{ etc.} = \frac{M a^2 \pi}{8}.$$

Ex. 4. *From a uniform sphere of radius a, a spherical sector of vertical angle 2α is removed. Show that the moment of inertia of the remainder of mass M about the axis of symmetry is*

$$\frac{1}{5} M a^2 (1+\cos\alpha)(2-\cos\alpha).$$

Sol. Let the spherical sector that has been removed be $OADB$. Let M be the mass of the sphere after removing the portion $OADB$.

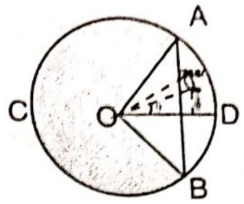
$\therefore M =$ mass of the sphere $-$ mass of the sector

$$= \frac{4}{3}\pi a^3 \rho - \int_0^{a}\int_0^{\alpha} \rho(2\pi r\sin\theta) r\, d\theta\, dr$$

$$= \frac{4}{3}\pi a^3\rho - \frac{2}{3}\pi a^3\rho(1-\cos\alpha)$$

$$= \frac{2\pi a^3\rho}{3}(1+\cos\alpha)$$

Thus we have $\rho = \dfrac{3M}{(1+\cos\alpha)\, 2\pi a^3}$ ...(1)



$\therefore$ Required M.I. of the portion $OACB$ about the axis of symmetry, COD = M.I. of the sphere $-$ M.I. of the sector $(OADB)$

$$= \frac{2}{5}(\textit{mass of sphere}) \times (\textit{radius})^2 - \int_0^a\int_0^\alpha \rho(2\pi\, r\sin\theta) \times r^2\sin^2\theta\, d\theta\, dr$$

$$= \frac{2}{5}\left(\frac{4}{3}\pi a^3\rho\right)a^2 - 2\pi\rho\int_0^\alpha \left(\frac{r^5}{5}\right)_0^a \sin^3\theta\, d\theta$$

$$= \frac{8\pi a^5\rho}{15} - \frac{2\pi a^5\rho}{5}\int_0^\alpha\left(\frac{3\sin\theta - \sin 3\theta}{4}\right)d\theta$$

$$= \frac{8\pi a^5\rho}{15} - \frac{\pi a^5\rho}{10}\left[-3\cos\alpha + \frac{1}{3}\cos 3\alpha + 3 - \frac{1}{3}\right]$$

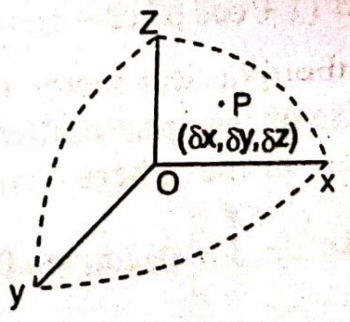
$$= \frac{2\pi a^5\rho}{15}\left[2 + 3\cos\alpha - \cos^3\alpha\right]$$

$$= \frac{2\pi a^5\rho}{15}\cdot\frac{3M}{2\pi a^3(1+\cos\alpha)}(1+\cos\alpha)^2(2-\cos\alpha) \qquad \text{[from (1)]}$$

$$= \frac{Ma^2}{5}(1+\cos\alpha)(2-\cos\alpha).$$

Ex.5. *Find the moment of inertia about the x-axis of the portion of the ellipsoid* $(x^2/a^2)+(y^2/b^2)+(z^2/c^2) = 1$ *which lies in the positive octant, supposing the law of volume density to be* $\rho = \mu xyz$.

Sol. Take an elementary volume $\delta x\, \delta y\, \delta z$ around the point $P(x, y, z)$, inside the octant $OXYZ$. Perpendicular distance of this element from x-axis $= \sqrt{(y^2+z^2)}$.



$\therefore$ M.I. of the element about x-axis

$= \rho(y^2+z^2)\,\delta x\,\delta y\,\delta z$

$= \mu\, xyz\,(y^2+z^2)\,\delta x\,\delta y\,\delta z.$

Hence M.I. of the positive octant of the ellipsoid $= \iiint \mu\, xyz\,(y^2+z^2)\,\delta x\,\delta y\,\delta z$

where $(x^2/a^2)+(y^2/b^2)+(z^2/c^2) \le 1.$

Substituting $(x^2/a^2) = u$, $(y^2/b^2) = v$, $(z^2/c^2) = w$

$\Rightarrow x\,dx = \frac{1}{2}a^2\,du$, $y\,dy = \frac{1}{2}b^2\,dv$, and $z\,dz = \frac{1}{2}c^2\,dw$, we get

$$\text{M.I.} = \mu \iiint \tfrac{1}{8}a^2b^2c^2(b^2v + c^2w)\,du\,dv\,dw, \qquad \text{where } u+v+w \le 1$$

$$= \tfrac{1}{8}\mu a^2b^2c^2 \iiint (b^2u^0v^1w^0 + c^2u^0v^0w^1)\,du\,dv\,dw$$

$$= \tfrac{1}{8}\mu a^2b^2c^2 \iiint (b^2u^{1-1}v^{2-1}w^{1-1} + c^2u^{1-1}v^{1-1}w^{2-1})\,du\,dv\,dw$$

$$= \tfrac{4}{8}\mu a^2b^2c^2\left[\frac{b^2\,\Gamma(1)\,\text{F}(2)\,\Gamma(1)}{\Gamma(1+2+1+1)} + \frac{c^2\,\Gamma(1)\,\Gamma(1)\,\Gamma(2)}{\Gamma(1+1+2+1)}\right]$$

(using Dirichlet's theorem)

$$= \tfrac{1}{8}\mu a^2b^2c^2\left[b^2\frac{\Gamma(2)}{\Gamma(5)} + c^2\frac{\Gamma(2)}{\Gamma(5)}\right] = \frac{1}{192}\mu a^2b^2c^2(b^2+c^2)$$

Now, M = mass of the positive octant of the ellipsoid

$OXYZ = \iiint \mu xyz\,dx\,dy\,dz$

under the condition $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) \le 1.$

$$= \tfrac{1}{8}\mu\, a^2b^2c^2 \iiint du\,dv\,dw$$

[under the condition $u+v+w \le 1$, and $u = (x^2/a^2)$ *etc.*]

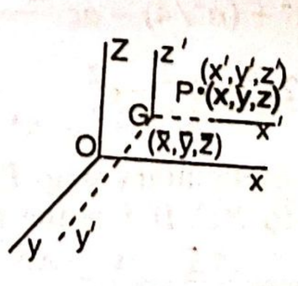
$$= \tfrac{1}{8}\mu a^2b^2c^2\,\frac{\Gamma(1)\,\Gamma(1)\,\Gamma(1)}{\Gamma(4)} = \tfrac{1}{48}\mu a^2b^2c^2; \qquad \therefore \mu = \frac{48M}{a^2b^2c^2}$$

Substituting this value of μ, we get the required moment of inertia $= \frac{1}{4}M(b^2+c^2)$.

0·03. Theorem of Parallel Axes

The Moment Of Inertia And The Products Of inertia about axes through the centre of gravity are given, to find the moments and products of inertia about parallel axes.

Let OX, OY, OZ be a set of co-ordinate axes through any point O, parallel to a set of co-ordinate axes GX', GY', GZ' through G, the centre of gravity. Let $(\bar{x}, \bar{y}, \bar{z})$ be the co-ordinates of G with regard to co-ordinate axes OX, OY, OZ. Let the co-ordinates of any element of mass m situated at the point P with regard to axes OX, OY, OZ be (x, y, z) and with regard to parallel axes through G be (x', y', z')



$\therefore x = \bar{x} + x'$, $y = \bar{y} + y'$, $z = \bar{z} + z'$

M.I. of the body about $OX = \Sigma m\,(y^2 + z^2)$

$$= \Sigma m[(\overline{y} + y')^2 + (\overline{z} + z')^2]$$
$$= \Sigma m[(\overline{y}^2 + \overline{z}^2 + 2y'\,\overline{y} + 2z'\,\overline{z} + y'^2 + z'^2]$$
$$= \Sigma m(\overline{y}^2 + \overline{z}^2) + \Sigma m[y'^2 + z'^2) + 2\overline{y}\,\Sigma my' + 2\overline{z}\,\Sigma mz'$$

Now referred to G as origin, the co-ordinates of the centre of the gravity of the body

$$\frac{\Sigma mx'}{\Sigma m} = 0, \frac{\Sigma my'}{\Sigma m} = 0, \frac{\Sigma mz'}{\Sigma m} = 0.$$

$\therefore \Sigma mx' = 0, \Sigma my' = 0, \Sigma mz' = 0.$

Hence M.I. of the body about $OX = \Sigma m\,(\overline{y}^2 + \overline{z}^2) + \Sigma m(y'^2 + z'^2)$

$= (\overline{y}^2 + \overline{z}^2)\,\Sigma m +$ M.I. of the body about GX'

$= M\,(\overline{y}^2 + \overline{z}^2) +$ M.I. about GX'

$=$ M.I. of mass M placed at G about $OX +$ M.I. about GX'.

Again product of inertia of the body about OX and OY.

$$= \Sigma mxy = \Sigma m(x' + \overline{x})\,(y' + \overline{y})$$
$$= \Sigma mx'y' + \Sigma mx'\overline{y} + \Sigma m\overline{x}y' + \Sigma m\overline{x}\overline{y} + \Sigma mx'y' + \overline{y}\,\Sigma mx' + \overline{x}\,\Sigma my' + \overline{x}\overline{y}\Sigma m$$

$= \Sigma mx'y' + M\,\overline{x}\overline{y} =$ The product of inertia about $(GX' + GY')$.

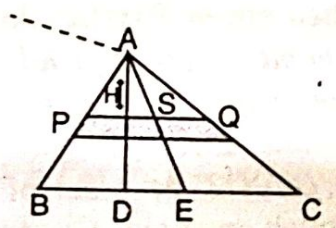
$+$ Product of inertia of mass M placed at G about the axes OX and OY.

ILLUSTRATIVE EXAMPLES

Ex.6. *Find the moment of inertia of the triangle ABC abot a perp. to its plane through A.*

Sol. Let AE be the median and AD the perp. through A on BC.

Let $AD = h$ and $AE = r$, then we have



$$AE^2 = AD^2 + DE^2 = AD^2 + (BE - BD)^2$$
$$= (AD^2 + BD^2) + BE^2 - 2BE.\,BD$$
$$= AB^2 + (\tfrac{1}{2}BC)^2 - 2.\tfrac{1}{2}BC\,AB\cos B$$
$$= c^2 + (a^2/4) - ac.\frac{a^2 + c^2 - b^2}{2ac}$$
$$= \frac{2b^2 + 2c^2 - a^2}{4}, \therefore r^2 = [(2b^2 + 2c^2 - a^2)/4] \qquad ...(1)$$

Take an elementary strip PQ of thickness δx at a distance x from A and parallel to BC. If S is the mid point of the strip PQ, then we get

$$\frac{x}{AD} = \frac{AS}{AE} \Rightarrow \frac{x}{h} = \frac{AS}{r} \Rightarrow AS = \frac{xr}{h}.$$

Also $\dfrac{x}{h} = \dfrac{PQ}{BC} \Rightarrow PQ = \dfrac{xa}{h}$

Let AL be the line through A perp. to the plane of the triangle, then M.I. of the strip about AL.

= M.I. of the strip about a line through the mid points of the strip parallel to AL + (mass of this strip) × AS^2

$$= \frac{1}{3} \cdot \frac{ax}{h} \delta x \rho \cdot \left(\frac{ax}{2h}\right)^2 + \frac{ax}{h} \delta x \, \rho \left(\frac{xr}{h}\right)^2 = \frac{1}{12} \frac{\rho x^3 a}{h^3} \delta x \, [a^2 + 12r^2]$$

∴Required M.I. of the whole triangle about AL

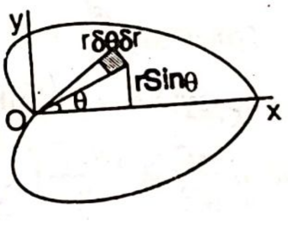
$$= \frac{1}{12}(a^2 + 12r^2) \frac{\rho a}{h^3} \int_0^h x^3 \, dx = \frac{1}{12}(a^2 + 12r^2) \frac{\rho a}{h^3} \cdot \frac{1}{4} h^4$$

$$= \frac{1}{48}[a^2 + 3(2b^2 + 2c^2 - a^2)] \rho a h \qquad \text{[using (1)]}$$

$$= \frac{M}{12}(3b^2 + 3c^2 - a^2), \text{ since } M = \frac{1}{2} a h \rho.$$

Ex.7. *A solid body, of density* ρ, *is in the shape of the solid formed by the revolution of the cardiod* $r = a(1 + \cos\theta)$ *about the initial line; show that its moment of inertia about a straight line through the pole perp. to the initial line is* $\left(\frac{352}{105}\right) \pi \rho a^5$. **[Meerut 1989]**

Sol. Let OX be the initial line and OY a line through the pole perp. to the initial line. Consider an elementary area $r\delta\theta\,\delta r$. This area when revolved about OX generates a circular ring of radius $r\sin\theta$. Its mass $= 2\pi r \sin\theta \, r\delta\theta \, \delta r \rho$. M.I. of the ring about a diameter parallel to



$$OY = (2\pi r \sin\theta \, r\delta\theta \, \delta r \rho) \frac{r^2 \sin^2\theta}{2}$$

[∴ M.I. of the ring about a diameter $= \frac{Ma^2}{2}$]

∴ M.I. of the ring about OY = M.I. of the ring about a diameter parallel to OY + mass of ring × $(r\cos\theta)^2$

$$= (2\pi r \sin\theta \, r\delta\theta \, \delta r \rho) \left(\frac{r^2 \sin^2\theta}{2} + r^2\cos^2\theta\right)$$

Hence moment of inertia of the whole solid of revolution about

$$OY = 2\pi\rho \int_0^\pi \int_0^{a(1+\cos\theta)} r^4 \sin\theta \left(\frac{\sin^2\theta}{2} + \cos^2\theta\right) d\theta \, dr$$

$$= \pi\rho \int_0^\pi \int_0^{a(1+\cos\theta)} r^4 \sin\theta \, (1 + \cos^2\theta) \, d\theta \, dr$$

$$= \pi\rho \int_0^\pi \left(\frac{r^5}{5}\right)_0^{a(1+\cos\theta)} \sin\theta \, (1 + \cos^2\theta) \, d\theta$$

$$= \frac{\pi\rho a^5}{5}\int_0^{\pi} (1+\cos\theta)^5 \sin\theta\,(1+\cos^2\theta)\,d\theta$$

$$= \frac{\pi\rho a^5}{5}\int_0^{\pi} \left(2\cos^2\frac{\theta}{2}\right)^5 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\times\left[1+\left(2\cos^2\frac{\theta}{2}-1\right)^2\right]d\theta$$

Putting $2t=\theta$ so that $2\,dt=d\theta$, we get

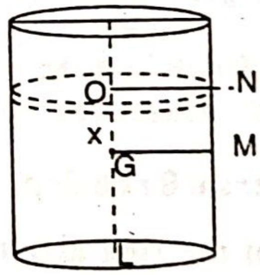
M.I. of the whole solid about OY

$$= \frac{256\,\pi\rho a^5}{5}\int_0^{\pi/2}[\cos^{11}t\sin t+2\cos^{15}t\sin t-2\cos^{13}t\sin t]\,dt$$

$$= \frac{256\,\pi\rho a^5}{5}\left[-\frac{\cos^{12}t}{12}-2\frac{\cos^{16}t}{16}+2\frac{\cos^{14}t}{14}\right]_0^{\pi/2} = (352\pi\rho a^5/105).$$

Ex.8. *Find the moment of inertia of a right circular cylinder about* (i) *its axis* (ii) *a straight line through its centre of gravity perp. to its axis.*

Sol. (i). Let a be the radius of the base, h the height of the cylinder. Take any elementary disc of breadth δx at a distance x from the centre of gravity G. M.I. of this disc about OL, where O is the centre of the disc and L the centre of the base of the



cylinder $= (\rho\pi a^2\delta x)\,\dfrac{a^2}{2}$

$\therefore$ M.I. of the cylinder about the axis OGL

$$= \int_{-h/2}^{h/2}\rho\pi a^2\,dx\,\frac{a^2}{2} = \frac{\rho\pi a^4}{2}\Big[x\Big]_{-h/2}^{h/2}$$

$$= \frac{\rho\pi a^4}{2}h = \frac{M a^2}{2} \quad (\because\ M=\pi a^2 h\rho)$$

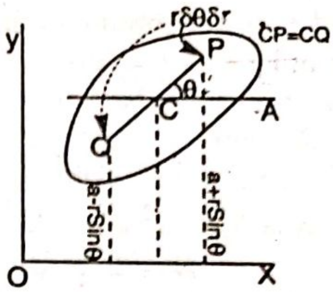
(ii) M.I. of the elementary disc about GM (line through CG and perp. to the axis)= M.I. of the disc about ON + M.I. of the mass of the disc placed at O, about $GM = (\rho\pi a^2\delta x)\,\dfrac{a^2}{4}+(\rho\pi a^2\delta x)x^2 = \rho\pi a^2\left[\dfrac{a^2}{4}+x^2\right]\delta x$

Hence M.I. of the whole cylinder about GM

$$= \pi a^2\rho\int_{-h/2}^{h/2}\left(\frac{a^2}{4}+x^2\right)dx = \frac{M}{4}\left(a^2+\frac{h^2}{3}\right).$$

0·04., Theorem : *A closed curve revolves round any line OX in its own plane which does not intersect it. Show that the moment of inertia of the solid of revolution so formed about OX is equal to* $M\,(a^2+3K^2)$, *where M is the mass of the solid generated, a is the distance from OX of the centre C of the curve, and K is the radius of gyration of the curve about a line through C parallel to OX.*

Proof : Let CA be a line parallel to OX at a distance a from OX. If M is the mass of the solid of revolution formed about OX, then $M = 2\pi a\,\rho S$ where S is the area of the closed curve (using *Pappus theorem*). Take an element $r\delta\theta\,\delta r$ at a distance r from C, which makes an angle θ with CA. Now corresponding to the element $r\delta\theta\,\delta r$ there will be an equal element for the same value θ in the opposite direction at the point Q. The distance of these elements from OX are $a+r\sin\theta$ and $a-r\sin\theta$ respectively. Now $S = \iint 2r\,d\theta\,dr$ ...(i)



where integration is taken to cover the upper half of the area. But moment of inertia of the area S about $CA = S\rho.\,K^2$

$$\therefore S\,\rho K^2 = \iint \rho.\,2r\,d\theta\,dr.\,r^2\sin^2\theta \qquad ...(ii)$$

Further M.I. of the solid of revolution about OX

$$= \iint r\,d\theta\,dr\,\rho\{2\pi(a+r\sin\theta)\,(a+r\sin\theta)^2 + 2\pi(a-r\sin\theta)(a-r\sin\theta)^2\}$$

$$= 4\pi\rho\iint r(a^3+3ar^2\sin^2\theta)\,d\theta\,dr$$

$$= 2\pi\rho a^3\iint 2r\,d\theta\,dr + 2\pi\rho a^3\iint 2\,d\theta\,dr\,r^2\sin^2\theta$$

$$= M(a^2+3K^2) \qquad \text{[using (i) and (ii)]}$$

0·05. Theorem : *Prove a theorem similar to the one proved previously for the moment of inertia of the surface generated by the arc of the curve.*

Proof : Let l be the length of the whole curve, so that we have

$$l = 2\int ds \qquad ...(1)$$

Also mass of the surface of revolution $= 2\pi a\rho l = M$ (say) ...(2)

Let K be the radius of gyration of the arc of the curve about CA, then it is evident that $lK^2 =$ M.I. of the arc about CA $\left\{\begin{array}{l}\text{Ref. figure of the}\\ \text{Previous theorem}\end{array}\right.$

$$= 2\int \rho r^2\sin^2\theta\,ds \qquad ...(3)$$

Hence M.I. of thesurface of revolution about OX

$$= \int \rho[2\pi(a+r\sin\theta)^3 + 2\pi(a-r\sin\theta)^3]\,ds$$

$$= 4\pi\rho\int (a^3+3ar^2\sin^2\theta)\,ds$$

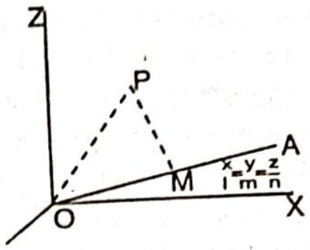
$$= 2\pi\rho a^3\int 2\,ds + 6\pi\rho a\int 2r^2\sin^2\theta\,ds = Ma^2 + 3MK^2$$

[using (1), (2) and (3)]

$$= M(a^2+3K^2).$$

0·06. Moment of Inertia about a line : *To find the moment of inertia about any axis through the meeting point of three perp. axes, the moments and products of inertia about these three axes being known.*

Proof : Let OX, OY, OZ be a set of three mutually perp. axes.



Let $A =$ M.I. about OX, $B =$ M.I. about OY, $C =$ M.I. about OZ, $D =$ Product of inertia w.r.t. axes of y and z. $E =$ Product of inertia w.r.t. axes of z and x and $F =$ Product of inertia w.r.t. axes of x and y. Now if m' is the mass of the element at P whose co-ordinates are (x, y, z), then we easily have

$$\left.\begin{array}{l} A = \Sigma m'(y^2+z^2),\ B = \Sigma m(x^2+z^2)\ ;\ C = \Sigma m'(x^2+y^2) \\ D = \Sigma m'\, yz\ ,\ E = \Sigma m'\, zx,\ F = \Sigma m'\, xy \end{array}\right\}$$

Let OA be a line with direction cosines (l, m, n). From P draw $PM \perp$ to OA, then $PM^2 = OP^2 - OM^2 = (x^2+y^2+z^2) - (lx+my+nz)$

$$[\because\ OP = (x^2+y^2+z^2),\ ON = (lx+my+nz)]$$

$$= x^2(1-l^2)+y^2(1-m^2)+z^2(1-n^2) - 2mnyz - 2lnzx - 2lmxy$$

$$= x^2(m^2+n^2)+y^2(l^2+n^2)+z^2(l^2+m^2) - 2mnyz - 2lnzx - 2lmxy$$

$$[\text{using } l^2+m^2+n^2 = 1]$$

$$= l^2(y^2+z^2)+m^2(x^2+z^2)+n^2(x^2+y^2) - 2mnyz - 2lnzx - 2lmxy$$

$\therefore$ Moment of inertia of the body abou OA,

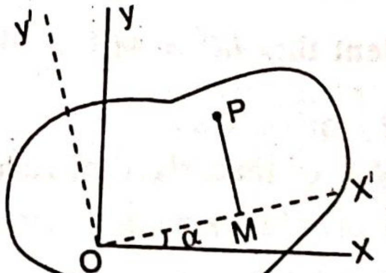
$$= \Sigma m'\, PM^2 = l^2\,\Sigma m'(y^2+z^2) + m^2\,\Sigma m'(x^2+z^2) + n^2\,\Sigma m'(x^2+y^2)$$
$$- 2mn\,\Sigma m'yz - 2ln\,\Sigma m'zx - 2lm\,\Sigma m'xy$$

$$= Al^2 + Bm^2 + Cn^2 - 2Dmn - 2Eln - 2Flm \qquad [\text{using (1)}]$$

0·07. Theorem : *If the moments and products of inertia of a plane lamina about two perp. axes in the plane are known, to find the moment of inertia about any other axis through their point of intersection.*

Proof : Consider an elementary mass m at the point P whose co-ordinates with reference to the axes OX, OY be (x, y). If A and B are the moments of inertia and F the product of inertia of the plane lamina about these axes in the plane, then we have,

$$A = \Sigma my^2,\ B = \Sigma mx^2,\ F = \Sigma mxy \qquad ...(1)$$



Now let OX' be a line inclined at an angle α to OX about which the moment of inertia is required. Let OX', OY' be the new system of co-ordinates axes and let (x', y') be the co-ordinates of the point P with respect to the new axes. Then we will have the relation

$$x' = x\cos\alpha + y\sin\alpha\ ,\quad y' = y\cos\alpha - x\sin\alpha$$

So M.I. of the lamina about $OX' = \Sigma my'^2 = \Sigma m(y\cos\alpha - x\sin\alpha)^2$

$$= \cos^2\alpha\,\Sigma my^2 - 2\sin\alpha\cos\alpha\,\Sigma mxy + \sin^2\alpha\,\Sigma mx^2$$

$= A\cos^2\alpha - 2F\sin\alpha\cos\alpha + B\sin^2\alpha = A\cos^2\alpha + B\sin^2\alpha - F\sin 2\alpha$

Further product of inertia of the lamina about OX', OY'

$= \Sigma mx'y' = \Sigma m(x\cos\alpha + y\sin\alpha)(y\cos\alpha - x\sin\alpha)$

$= \cos\alpha\sin\alpha(\Sigma my^2 - \Sigma mx^2) + (\cos^2\alpha - \sin^2\alpha)\,\Sigma mxy$

$= (A - B)\sin\alpha\cos\alpha + F\cos 2\alpha \equiv \frac{1}{2}(A - B)\sin 2\alpha + F\cos 2\alpha$

0·08. Some Elementary Theorems.

Theorem I. *If A, B, C are the moments and D, E, F; the product of inertia about the axes, then the sum of any two of them is greater than the third.*

Proof : $A + B - C = \Sigma m(y^2 + z^2) + \Sigma m(x^2 + z^2) - \Sigma m(x^2 + y^2)$

$= 2\Sigma mz^2 = +ive$ and hence $A + B > C$.

Theorem II. *The sum of moments of inertia about any three axes (rectangular) meeting at a given point is always constant and is equal to twice the moment of inertia about that point.*

Proof : $A + B + C = 2\Sigma m(x^2 + y^2 + z^2) = 2\Sigma mr^2$

$\Rightarrow A + B + C$ is independent of the direction of the axes.

Theorem III. *The sum of the moments of inertia of a body with reference to any plane through a given point and its normal at that point is constant and is equal to the moment of inertia of the body with reference to that point.*

Proof : Choose the given point as origin and plane as the plane of xy. Now let R' be the moment of inertia w.r.t. xy plane and R the moment of inertia about the normal at origin (z-axis) then we (on applying Theorem II) get $R' + R = \Sigma mr^2 = \frac{1}{2}(A + B + R)$

which is independent of the direction of axis

$\Rightarrow R' = \frac{1}{2}(A + B - R)$

Hence if P', Q', R' are the moments of inertia with reference to the planes of (y, z), (z, x) and (x, y) then we easily obtain

$$P' = \frac{1}{2}(B + R - A)\,,\ Q' = \frac{1}{2}(A + R - B),\ R' = \frac{1}{2}(A + B - R)$$

Theorem IV. *To prove that $A > 2D$, $B > 2E$ and $C > 2F$ where the meanings have their usual significance.*

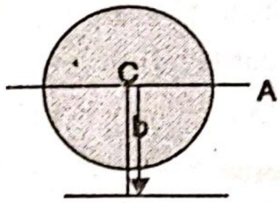
Proof : $(y^2 + z^2) > 2yz$ *etc.* $(\because\ AM > G.M.)$

$\Rightarrow A > 2D$, similarly other results follow.

Ex.9. *The moment of inertia about its axis of a solid rubber tyre of mass M and circular cross section of radius a is $(M/4)(4b^2 + 3a^2)$ where b is the radius of the curve. If the tyre be hollow and of small uniform thickness, show that the corresponding result is $(M/2)(2b^2 + 3a^2)$*

Sol. The tyre solid/hollow is formed by the revolution of a circular area/circular ring about an axis.

Case I. For Solid Tyre . Moment of inertia of circular area about $CA = mass \times (a^2/4) = m\,K^2 \Rightarrow K^2 = (a^2/4)$.
By theorem 0·04 P.21, we have required
M.I.= $M\,(b^2 + 3a^2/4) = (M/4)\,(4b^2 + 3a^2)$



Case II. Hollow Tyre. We know that the moment of inertia of circular arc about $CA = mass.\,(a^2/2) = m\,K^2 \Rightarrow K^2 = (a^2/2)$
By theorem 0·05, P.22, we have M.I. of the hollow tyre

$$= M\,(b^2 + 3a^2/2) = (M/2)\,(2b^2 + 3a^2).$$

Ex.10. *Show that the moment of inertia of an elliptic area of mass M and semi axes a and b about a diameter of length r is* $\frac{1}{4} M \frac{a^2b^2}{r^2}$.

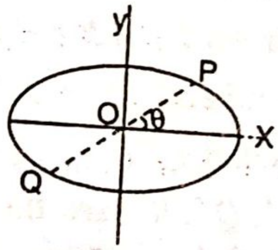
Sol. Consider an elliptic area with its centre at the origin (i) its moment of inertia about major axis $OX = (Mb^2)/4$ (ii) Its moment of inertia about $OY = (Ma^2)/4$ (iii) Its product of inertia about $OX, OY = 0$. Consider a diameter PQ making an angle θ with the axis OX, so that moment of inertia of the ellipse about the diameter

$$PQ = (Mb^2/4)\cos^2\theta + (Ma^2/4)\sin^2\theta + 0 = (M/4)\,(b^2\cos^2\theta + a^2\sin^2\theta) \quad ....(1)$$

If $OP = r$, then co-ordinates of P are $(r\cos\theta, r\sin\theta)$. Since P lies on the ellipse $(x^2/a^2) + (y^2/b^2) = 1$

$\therefore (r^2\cos^2\theta/a^2) + (r^2\sin^2\theta/b^2) = 1$

$\Rightarrow b^2\cos^2\theta + a^2\sin^2\theta = (a^2b^2/r^2)$



Hence moment of inertia of the elliptic area about $OP = (Ma^2b^2/4r^2)$

Ex.11 *Show that the moment of intertia of a right solid cone whose height is h and radius of whose base is a, is* $\frac{3Ma^2}{20}\left\{\frac{6h^2 + a^2}{h^2 + a^2}\right\}$ *about a slant side and* $(3M/80)\,(h^2 + 4a^2)$ *about a line through the centre of gravity of the cone perpendicular to its axis.* **[Agra 1988]**

Sol. Consider an elementary disc of thickness δx at a distance x from A, then we have radius of the disc $= x\tan\alpha$, mass of the disc

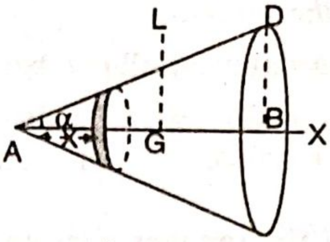
$$= \pi x^2 \tan^2\alpha\,\delta x\,\rho.$$

From the figure , $\tan\alpha = \dfrac{a}{h}$

Now M.I. of the disc about

$$ab = \pi x^2 \tan^2 \alpha \rho \frac{x^2 \tan^2 \alpha}{2} \delta x.$$

$$= \frac{1}{2} \rho \pi \tan^4 \alpha x^4 \delta x$$



Hence M.I. of the cone about

$$AB = \frac{\rho\pi \tan^4 \alpha}{2} \int_0^h x^4 \, dx$$

$$= \frac{\pi \tan^4 \alpha \, \rho h^5}{10} \quad \text{...(1)}$$

Now M.I. of the cone about a line through the vertex A and $\perp$ to AB.

$$= \int_0^h \rho\pi x^2 \tan^2 \alpha \, dx \left[\frac{x^2 \tan^2 \alpha}{4} + x^2 \right] = \rho\pi \tan^2 \alpha \int_0^h \left(\frac{\tan^2 \alpha}{4} + 1 \right) x^4 dx$$

$$= \rho\pi \tan^2 \alpha \left(\frac{\tan^2 \alpha}{4} + 1 \right) \int_0^h x^4 dx = \frac{\rho\pi \tan^2 \alpha h^5}{5} \left(\frac{\tan^2 \alpha}{4} + 1 \right) \quad \text{...(2)}$$

Now product of inertia of the cone about AB and AK.

= P.I. of the cone about GX and GL + P.I. of the mass of the cone (being concentrated at G) about AG and $AK = 0 + M \cdot 0 (AG) = 0$.

Again M.I. of the cone about slant side AD

$$= \frac{\pi \tan^4 \alpha \, \rho h^5}{10} \cos^2 \alpha + \frac{\pi \tan^2 \alpha \, \rho h^5}{5} \left(\frac{\tan^2 \alpha}{4} + 1 \right) \sin^2 \alpha + 0. \text{ (using 0· 06)}$$

$$= \frac{\pi \tan^4 \alpha \, \rho h^5}{20} \left[2 \cos^2 \alpha + \sin^2 \alpha \right] + \frac{\pi \tan^2 \alpha \, \rho h^5}{5} \sin^2 \alpha$$

$$= \frac{\pi a^4 \rho h}{20} \left[1 + \frac{h^2}{a^2 + h^2} \right] + \frac{\pi a^2 \rho h^3}{5} \cdot \frac{a^2}{a^2 + h^2} \qquad \text{since } h \tan \alpha = a$$

$$= \frac{\pi a^4 \rho h}{20} \left[\frac{a^2 + 2h^2 + 4h^2}{a^2 + h^2} \right] = \frac{3Ma^2}{20} \left[\frac{6h^2 + a^2}{a^2 + h^2} \right], \text{ since } M = \tfrac{1}{3} \pi a^2 h \rho.$$

(ii) To find the moment of inertia about a line GL (through centre of gravity and perp. to the axis).

M.I. about the line through A and $\perp$ AB = M.I. about GL + M.I. of mass placed at G, about the line through A and $\perp$ AB.

$\therefore$ M.I. about GL = M.I. about the line through $A \perp AB$ – M.I. of mass M placed at G about the line through A and $\perp$ AB

$$= \frac{\pi a^2 \rho h^3}{5} \left[\frac{a^2}{4h^2} + 1 \right] - M \frac{9h^2}{16} = \frac{3M}{5} \left(\frac{a^2}{4} + h^2 \right) - M \frac{9h^2}{16}$$

$$= \frac{3M}{80} [4a^2 + 16h^2 - 15h^2] = \frac{3M}{80} [h^2 + 4a^2]$$

Ex.12. *Show that the moment of inertia of an ellipse of mass M and semi-axes a and b, about a tangent is* $(5M/4)p^2$, *where p is the perp. from the centre on the tangent.*

Sol. Let the equation of ellipse be $(x^2/a^2)+(y^2/b^2)=1$. Then equation of the tangent is $y=mx+\sqrt{[(a^2m^2+b^2)]}$ where m is the slope of the tangent. If the tangent is inclined at an angle θ to the *x-axis* then $m=\tan\theta$.

$\therefore$ equation of the tangent is given by

$$y=x\tan\theta+\sqrt{[a^2\tan^2\theta+b^2]}\Rightarrow x\tan\theta-y+\sqrt{[a^2\tan^2\theta+b^2]}=0$$

$p=$ distance of the perp. from the centre (0, 0) to the tangent

$$=[\sqrt{\{(a^2\tan^2\theta+b^2)\}}/\sqrt{(1+\tan^2\theta)}]=\sqrt{(a^2\sin^2\theta+b^2\cos^2\theta)}$$

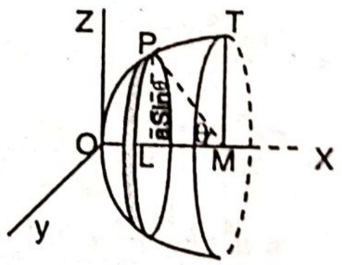
$\therefore$ Moment of inertia about a diameter parallel to the given tangent will be $=(M/4)b^2\cos^2\theta+(M/4)a^2\sin^2\theta+0=(M/4)(b^2\cos^2\theta+a^2\sin^2\theta)$

$$=(M/4)p^2.$$

Hence moment of inertia about the tangent $=(M/4)p^2+Mp^2=\frac{5}{4}Mp^2$.

Ex.13. *Show that for a thin hemi-spherical shell of mass M and radius a, the moment of inertia about any line through the vertex is* $\frac{2}{3}Ma^2$.

Sol. Let O be the vertex and OX the axis of symmetry of the semi-spherical shell. Take the two perp. lines OY and OZ each perp. to OX, so that OX, OY, OZ form the axes of reference. The quadrant of the circle when revolved about OX will generate the hemi-spherical shell. $\therefore$ M.I. about



OX *i.e.* $A=\int_0^{\pi/2}(\rho\, 2\pi a\sin\theta)\,a\,d\theta\, a^2\sin^2\theta$

$$=2\pi\rho a^4\int_0^{\pi/2}\sin^3\theta\,d\theta=2\pi\rho a^4\,\frac{2}{3}$$

$$=\frac{4\pi\rho a^4}{3}=\frac{2}{3}Ma^2\;(\because\;M=2\pi a^2\rho)$$

and $B=$ M.I. about $OY=\Sigma($ M.I. of circular disc about a line through L and parallel to $OY+$ mass of the circular disc $\times\ OL^2)$

$$=\int_0^{\pi/2}2\pi a\rho\sin\theta\, a\,d\theta[a^2\sin^2\theta/2+(a-a\cos\theta)^2]$$

$$=\pi\rho a^4\int_0^{\pi/2}\sin\theta\,(3-4\cos\theta+\cos^2\theta)\,d\theta$$

$$=\pi\rho a^4\,[3.2+\tfrac{1}{3}]=4\pi a^4\rho/3=(\tfrac{2}{3})\,Ma^2$$

Also M.I. about OZ i.e $C = B = (\frac{2}{3})\, M a^2$

Clearly the products of inertia D, E, F will vanish about these axes. Since the co-ordinates of $C.G.$ are $(a/2, 0, 0)$

Now if $[l, m, n]$ are Direction cosines of a line through O, then M.I. about this line $= Al^2 + Bm^2 + Cn^2 - 2Dmn - 2Eln - 2Flm$

$$= (\tfrac{2}{3})\, M a^2 (l^2 + m^2 + n^2) = (\tfrac{2}{3})\, M a^2$$

Ex.14. *If K_1, K_2 be the radii of gyration of a elliptic lamina about two conjugate diameters, then $(1/K_1^2) + (1/K_2^2) = 4\,[(1/a^2) + (1/b^2)]$*

Sol. Let OP and OQ be the conjugate semi diameters of the ellipse.

Let $OP = r_1, OQ = r_2$, then $M K_1^2 = \dfrac{M a^2 b^2}{4 r_1^2}$ and $M K_2^2 = \dfrac{M a^2 b^2}{4 r_2^2}$

$$\therefore (1/K_1^2) + (1/K_2^2) = (4/a^2b^2)\,(r_1^2 + r_2^2)$$

$$= (4/a^2b^2)\,[(a^2\cos^2\theta + b^2\sin^2\theta) + \{a^2\cos^2(\pi/2+\theta) + b^2\sin^2(\pi/2+\theta)\}]$$

$$= (4/a^2b^2)(a^2 + b^2) = 4[(1/a^2) + (1/b^2)].$$

Ex.15. *Show that the sum of moments of inertia of an elliptic area about any two tangents at right angles is always the same.*

Sol. Proceeding similarly as in Ex.12, we have

Moment of inertia about a tangent inclined at an angle $\theta = (\frac{3}{4}) M p^2 = (\frac{5}{4})\, M\,(a^2\sin^2\theta + b^2\cos^2\theta)$

Also moment of inertia about a tangent perp. to the first tangent [by changing θ to $\{(\pi/2) + \theta\}$] $= (\frac{5}{4})\ m(a^2\cos^2\theta + b^2\sin^2\theta)$

Hence sum of the moments of inertia about two perp. tangents

$= (\frac{5}{4})\, M\,(a^2 + b^2)$, this being independent of θ, remains constant.

Ex.16. *Show that the moment of inertia of an elliptic area of mass M and equation $ax^2 + 2hxy + by^2 + 2gx + 2yf + c = 0$ about a diameter parallel to the axis of x is $-\ \{aM\Delta/4(ab - h^2)^2\}$, where*

$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$.

Sol. Equation of the ellipse is given to be

$$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0.$$

Shift the origin to the centre of the ellipse then the equation of the ellipse takes the form $ax^2 + 2hxy + by^2 + \{\Delta/(ab - h^2)\} = 0$ ...(1)

where $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

Putting $y = 0$ in (1), we get $ax^2 = -\ \{\Delta/(ab - h^2)\}$

Now if the length of the semi diameter parallel to the axis of x is r then, we have $r^2 = \{(\Delta/(ab - h^2)\}$

Now Putting $\{\Delta/(ab-h^2)\} = c'$, the equation of the ellipse becomes as $ax^2 + 2hxy + c' = 0$ or $-(ax^2/c') - (2hxy/c') - (by^2/c') = 1$, which is of the standard form $Ax^2 + 2Hxy + By^2 = 1$. If α, β are the semi axes of ellipse, then α, β are the values of R in the equation $\{A - (1/R^2)\}/\{B - (1/R^2)\} = H^2$

or $(-a/c' - 1/R^2)(-b/c' - 1/R^2) = (-h/c')^2$

$\therefore (1/R^4) + (1/R^2)[(a/c') + (b/c')] + [(ab-h^2)/c^2] = 0$

$$\therefore \frac{1}{\alpha^2\beta^2} = \frac{ab-h^2}{c'^2} = \frac{(ab-h^2)^3}{\Delta^2}$$

M.I. about the diameter is $= (M/4)(\alpha^2\beta^2/r^2)$

$= (-M/4)\{\Delta^2/(ab-h^2)^3\}.\{a(ab-h^2)/\Delta\} = -\{aM\Delta/4(ab-h^2)^2\}$

0·09. Method of Differentiation : We know that if y is a function of x and if δx and δy are small changes in the value of x and y, then

$$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx} \Rightarrow \delta y = \frac{dy}{dx}\delta x$$

For example (i) if area of the circle is $A = \pi r^2$, then

$$\delta A = \frac{d}{dr}(\pi r^2)\,\delta r = 2\pi r\,\delta r = \text{circumference of the circle} \times \delta r$$

(ii) If V is the volume of sphere of radius r, then

$$V = (\tfrac{4}{3})\pi r^3 \Rightarrow \delta V = \frac{d}{dr}(\tfrac{4}{3}\pi r^3).\delta r = 4\pi r^2\delta r$$

= surface of a sphereical shell of radius r and thickness δr.

Ex. 17. *Show that the moment of inertia of a thin homogeneous ellipsoidal shell (bounded by similar and similarly situated concentric ellipsoids) about an axis is $M\{(b^2+c^2)/3\}$, where M is the mass of the shell.*

[Agra 1986]

Sol. Let a, b, c be the length of the semi axes and ρ the density of the uniform solid ellipsoid, then moment of inertia of the solid ellipsoid about x- axis $= \{(\tfrac{4}{3})\pi abc\,\rho\}.\{(b^2+c^3)/5\}$

Now let the ellipsoid increase indefinitely small in size.

$$\therefore \text{M.I. of the enclosed ellipsoidal shell} = d\left\{\frac{4}{3}\pi abc\,\rho.\frac{b^2+c^2}{5}\right\} \quad ...(1)$$

Since the concentric bounded ellipsoids are similar ellipsoid then we have $(a/a') = (b/b') = (c/c')$

$\therefore b = (b'/a')\,a$ and $c = (c'/a')\,a \Rightarrow b \equiv pa$ and $c = qa$

$\therefore$ M.I. of the shell $d\,[(\tfrac{4}{3})\,\pi\,\rho\,p\,q\,(p^2+q^2)\,a^2/5]$ [Using (1)]

$= (\tfrac{4}{3})\,\pi\,\rho\,p\,q\,(p^2+q^2)\,a^4 da.$

Now mass of the solid ellipsoid $= (\frac{4}{3})\ \pi abc\ \rho = (\frac{4}{3})\ \pi\ \rho pq\ a^3$.

$\therefore$ Mass of the ellipsoidal shell $M = d\ [\frac{4}{3}\pi\rho pqa^3] = 4\pi\ \rho pqa^2 da$.

HenceM.I. of the ellipsoidal shall$= \frac{4}{3}\pi\ \rho pq\ (p^2 + q^2)a^4 . \dfrac{M}{4\pi\ \rho pqa^2}$

$$= \frac{1}{3}M\ (p^2 + q^2)a^2 = \frac{1}{3}M(p^2a^2 + q^2a^2) = \frac{1}{3}M(b^2 + c^2).$$

0· 10. Moment of Inertia of Heterogeneous Bodies :

In the case of a heterogeneous body whose boundary is a surface of uniform density, the method of differentiation can be successfully used in finding the moment of inertia of the body, the method is as follows:

(i) Suppose the M.I. of a homogeneous solid body of density ρ is known

(ii) Let this M.I. be expressed as a function of single parameter α (say) i.e.

$$\text{M.I.} = \rho\phi\ (\alpha).$$

Then the M.I. of a shell which is considered to be made of a layer of a uniform density $\rho = \rho\phi'\ (\alpha)\ d\ \alpha$. ...(1)

In case the density is not uniform and the variable density is given to be σ then we have, $M.I. = \int \sigma\phi'\ (\alpha)\ d\ \alpha$ ...(2)

Ex. 18. *The moment of inertia of a heterogeneous ellipse about minor axis, the strata of uniform density being confocal ellipse and density along minor axis varying as the distance from the centre is* $\dfrac{3M}{20} \cdot \dfrac{4a^5 + c^5 - 5a^3c^2}{2a^3 + c^3 - 3ac^2}$

[Agra 1981]

Sol. Equation of the confocal ellipse may be written as $\dfrac{x^2}{b^2 + c^2} + \dfrac{y^2}{b^2} = 1$

with the condition $a^2 = b^2 + c^2$. Now mass of ellipse of uniform density $\rho = \pi\ ab\ \rho = \pi b\ \sqrt{(b^2 + c^2)}\ \rho =$ function of (b).

Mass of stratum of density $\rho = \dfrac{d}{db}\{\pi b\ \sqrt{[(b^2 + c^2)]}\ \rho\}\ db$

$$= \pi\rho\ \sqrt{[(b^2 + c^2)} + b^2/\sqrt{(b^2 + c^2)}]\ db$$

$$= \pi\rho\ \{(b^2 + c^2 + b^2)/\sqrt{(b^2 + c^2)}\}\ db = \pi\rho\ \{2b^2 + c^2/\sqrt{(b^2 + c^2)}\}.db$$

Now the density $= \lambda b$ (given)

$\therefore$ Mass of the heterogeneous ellipse, i.e.

$$M = \int_0^b \pi\lambda b \frac{2b^2 + c^2}{\sqrt{(b^2 + c^2)}}\ db = \pi\lambda \int_0^b \left[2\sqrt{(b^2 + c^2)} - \frac{c^2}{\sqrt{(b^2 + c^2)}}\right] db$$

$$= \pi\lambda\left[\int_0^b (b^2 + c^2)^{1/2} 2b\ db - c^2\int_0^b \frac{b\ db}{\sqrt{(b^2 + c^2)}}\right]$$

$$= \pi\lambda\left\{\frac{2}{3}(b^2+c^2)^{3/2} - c^2(b^2+c^2)^{1/2}\right\}_0^b$$

$$= \pi\lambda\left[\tfrac{2}{3}\pi\lambda(a^3-c^3) - c^2(a-c)\right] \qquad \text{Using } a^2 = b^2+c^2$$

$$\therefore M = \tfrac{1}{3}\pi\lambda(2a^3+c^3-3ac^2) \qquad \text{...(1)}$$

Now M.I. of the elliptic disc of uniform density ρ about its minor axis

$$\doteq \pi ab\rho\frac{a^2}{4} = \frac{\pi a^3}{4}b\rho = \rho\pi\frac{b}{4}(b^2+c^2)^{3/2} = \tfrac{1}{4}\pi\rho b(b^2+c^2)^{3/2}$$

which is a function of b as c is constant.

M.I. of an elliptic stratum of density ρ is

$$\frac{d}{db}\left[\tfrac{1}{4}\pi\rho b(b^2+c^2)^{3/2}\right]db = \tfrac{1}{4}\pi\rho\left[\sqrt{(b^2+c^2)}(4b^2+c^2)\,db\right]$$

$$= \tfrac{1}{4}\pi\lambda\left[b\sqrt{(b^2+c^2)}\,(4b^2+c^2)\,db\right] \quad \text{putting } \rho = \lambda b.$$

Now moment of inertia of heterogeneous elliptic disc about minor axis

$$= \tfrac{1}{4}\pi\lambda\int_0^b b\sqrt{(b^2+c^2)}\,(4b^2+c^2)\,db$$

$$= \tfrac{1}{4}\pi\lambda\int_0^b b\sqrt{\{(b^2+c^2).[(4b^2+c^2)-3c^2]\}}\,db$$

$$= \tfrac{1}{4}\pi\lambda\int_0^b \left[4b(b^2+c^2)^{3/2} - 3bc^2(b^2+c^2)^{1/2}\right]db$$

$$= \tfrac{1}{4}\pi\lambda\left[\frac{4}{5}(b^2+c^2)^{5/2} - c^2(b^2+c^2)^{3/2}\right]_0^b$$

$$= \tfrac{1}{4}\pi\lambda\left[\tfrac{4}{5}(b^2+c^2)^{5/2} - c^2(b^2+c^2)^{3/2} - (\tfrac{4}{5}c^5 - c^2c^3)\right]$$

$$= \frac{1}{20}\pi\lambda[4a^5+c^5-5c^2a^3] = \frac{3}{20}M\frac{4a^5+c^5-5c^2a^3}{2a^3+c^3-3ac^2}$$

$$\text{Since } M = \tfrac{1}{3}\pi\lambda(2a^3+c^3-3ac^2)$$

Ex.19. *Show that the M.I. of a heterogeneous ellipsoid about the major axis is $\frac{2}{9}M(b^2+c^2)$, the starta of uniform density being similarly concentric ellipsoids and density along the major axis varying as the distance from the centre.*

Sol. Since the bounding surfaces are similar ellipsoids, so

$(a'/a) = (b'/b) = (c'/c) \Rightarrow (b'/a') = (b/a) = p$ and $(c'/a') = (c/a) = q$

$\therefore b = ap$ and $c = aq$.

Mass of the ellipsoid $= (4/3)\pi\rho abc = (4/3)\pi\rho pq a^3$.

Also M.I. of the homogeneous ellipsoid about *x- axis* = (mass of the ellipsoid) $\times \frac{b^2+c^2}{5} = \frac{4}{3}\pi\rho pqa^3 \frac{p^2+q^2}{5} a^2 = \frac{4}{15}\pi\rho pqa^5 (p^2+q^2)$,

which is a function of single parameter a .

Now the variable density , $\rho = \lambda a$ (given)

$\therefore$ Moment of inertia of the ellipsoidal shell

$$= \frac{d}{da}\left[\frac{4}{15}\pi\rho pq\,(p^2+q^2)\,a^5\right] da = \frac{4}{3}\pi\rho pq\,(p^2+q^2)a^4 da.$$

$\therefore$ M.I. of ellipsoid $= \int_0^a \frac{4}{3}\lambda\pi pqa^5 (p^2+q^2)da.$ (putting $\rho = \lambda a$).

$$= \frac{4}{3}\lambda\pi pq\,(p^2+q^2)\frac{a^6}{6} = \frac{2}{9}\lambda pq\,(p^2+q^2)a^6 \qquad \text{...(1)}$$

Again mass of the shell $\frac{d}{da}[\frac{4}{3}\pi pq\,\rho a^3 da] = 4\pi pq\,\lambda a^3 da$

$\Rightarrow M =$ Mass of given ellipsoid $= 4\pi\lambda pq \int_0^a a^3 da = \pi\lambda pqa^4.$

Required M.I. $= \frac{2}{9}M\,(p^2+q^2)a^2 = \frac{2}{9}M\,(b^2+c^2).$

0·11. Momental Ellipsoid : *If A, B, C, D, E, F, are the moments and products of inertia about the axes then the moment of inertia about as line OQ whose direction cosines are [l, m, n] is given by*

$Al^2 + Bm^2 + Cn^2 - Dmn - 2Enl - 2Flm.$

Let a length $OP = (r)$ along the line OQ be such that the moment of inertia of the body about OQ may be inversely proportional to OP^2, so that $(Al^2 + Bm^2 + Cn^2 - 2Dmn - 2Enl - 2Fml)\alpha \frac{1}{OP^2} = \frac{Constant}{OP^2}$

$\Rightarrow Al^2 + Bm^2 + Cn^2 - 2Dmn - 2Enl - 2Flm = (MK^4/r^2)$

$\Rightarrow Al^2r^2 + Bm^2r^2 + Cn^2r^2 - 2Dmnr^2 - 2Enlr^2 - 2Flmr^2 = MK^4$

where M is mass of the body and K is suitable constant .

$$\Rightarrow Ax^2 + By^2 + Cz^2 - 2Dyz - 2Ezx - 2Fxy = MK^4 \qquad \text{...(1)}$$

Since A, B, C are essentially positive, so this equation represents an ellipsoid . This is called the momental ellipsoidal of the body at the point O, We know from Solid Geometry with suitable change of axis, the equation of the ellipsoid is transformed into the form

$$A_1x^2 + B_1y^2 + C_1z^2 = MK^4 \qquad \text{...(2)}$$

Obviously the product of inertia with respect to the new axes vanish . These three new axes will be called the principal axes of the body at the point O.

Momental Sphere. *This forms a particular case of the above result. If the three principal moments of inertia at the point O are equal to each other then the ellipsoid becomes a sphere. In this case every diameter is a principal diameter and all radii vectors are equal.*

0·12. Momental Ellipse : *We know that in the case of a plane lamina, where A, B are moments of inertia about the axes and F the product of inertia about them, the moment of inertia if lamina about a line OQ which makes an angle θ with OX is given by,*

$$A\cos^2\theta - 2F\sin\theta\cos\theta + B\sin^2\theta.$$ [Meerut 1981. Agra 1981]

Take a length OP along OQ such that this moment of inertia is inversely proportional to the square of OP. If $OP = r$, then we have

$$A\cos^2\theta - 2F\sin\theta\cos\theta + B\sin^2\theta = (Mk^4/r^2)$$

$$\Rightarrow A r^2\cos^2\theta - 2F\sin\theta\cos\theta\, r^2 + B r^2\sin^2\theta = Mk^4$$

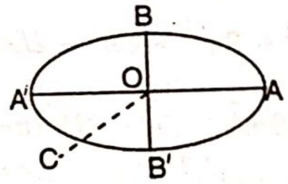
$$\Rightarrow Ax^2 - 2Fxy + By^2 = Mk^4$$

This equation represents an ellipse, because A and B are always + ive being sum of a number of squares. This ellipse is called the momental ellipse at the point O.

Note. The momental ellipse is the section of the momental ellipsoid at O by the plane of the lamina.

Ex. 20. *Show that the momental ellipsoid at the centre of an elliptic plate is* $(x^2/a^2) + (y^2/b^2) + z^2[(1/a^2) + (1/b^2)] =$ constant.

Sol. Let $ABA'B'$ be the elliptic plate. Take the major axis OA, minor axis OB and a perpendicular line OC as the axes of x, y and z respectively. Then we will have $A =$ moment of ineretia about $OA = \frac{1}{4}Mb^2$



$B =$ Moment of inertia about $OB = \frac{1}{4}Ma^2$

$C =$ moment of inertia about $OC = \frac{1}{4}M(a^2 + b^2)$

Clearly $D = E = F = 0$ where D, E and F are products of inertia about the axes y and z, z and x, x and y respectively.

Hence the equation of the momental ellipsoid at O is

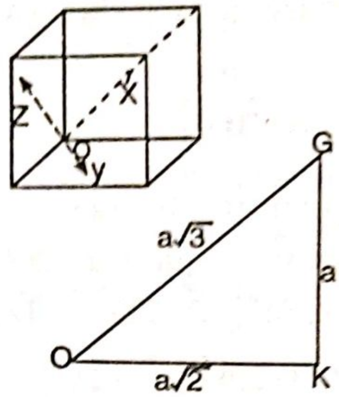
$$Ax^2 + By^2 + Cz^2 - 2Dyz - 2Ezx - 2Fxy = \text{constant}$$

$$\Rightarrow (1/4)\,Mb^2x^2 + (1/4)\,Ma^2y^2 + (1/4)\,M(a^2 + b^2)\,z^2 = \text{constant}$$

$$\Rightarrow (x^2/a^2) + (y^2/b^2) + z^2[(1/a^2) + (1/b^2)] = \text{constant.}$$

Ex. 21. *Show that the equation of the momental ellipsoid at the corner of a cube of side 2a referred to its principal axes is given by* $2x^2 + 11(y^2 + z^2) = c$, *where c is constant.*

Sol. Let O be the corner of the cube and G its centre of gravity . Take the line OX passing through G, the centre of gravity of the cube as x- axis and the two mutually perpendicular lines OY and OZ as the axes of y and z. We know that moment of inertia of the cube of the side $2a$ about any axis through $G = (2/3)\,Ma^2$.



Now the products of inertia of the cube about any two mutually perpendicular lines through G are zero. So the products of inertia about the axes OX, OY, OZ taken in pairs will be zero. It implies OX, OY, OZ are the principal axes of the momental ellipsoid at O. $\therefore A = M.I.$ about $OX = \frac{2}{3} Ma^2$

$B =$ M.I. about $OY =$ M.I. about a line through $G \parallel$ to $OY +$ M.I. of mass M placed at G about $OY = (2/3)M\,a^2 + M(3a^2) = (11/3)\,Ma^2$,

since $OG = a\sqrt{3}$

By symmetry, $C =$ M.I. about $OZ = (11/3)\,Ma^2$

But $D = E = F = 0$. Hence equation of the momental ellipsoid is

$$Ax^2 + By^2 + Cz^2 - 2Dyz - 2Ezx - 2Fxy = \text{constant}$$

$$\Rightarrow (2/3)Ma^2x^2 + (11/3)\,Ma^2y^2 + (11/3)\,Ma^2z^2 = c'(\text{constant})$$

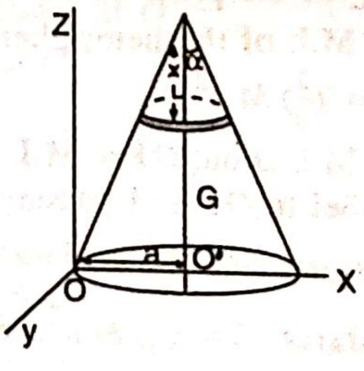
$$\Rightarrow 2x^2 + 11(y^2 + z^2) = c\,(\text{constant}).$$

Ex.22 *Prove that the equation of the momental ellipsoid at a point on the circular edge of a solid cone is*

$(3a^2 + 2h^2)x^2 + (23a^2 + 2h^2)\,y^2 + 26a^2z^2 - 10ahxz =$ *constant. where h is the height of the cone and a is the radius of the base.*

[Vikram 63,Nagpur 64,Agra 66]

Sol. Consider a cone with vertex at V, semi vertical angle α and height h. Let O be the point on the circular edge of the cone where we want to determine the momental ellipsoid. Now take a disc of breadth δx at a depth x from the vertex V of the cone.



$\therefore A =$ Moment of inertia of the cone about $OX = \Sigma$ mass of the circular disc $[(\text{radius})^2/4) + (O'K)^2]$

$$= \int_0^h \pi x^2 \tan^2\alpha\,\rho\,[(x^4\tan^2\alpha)/4 + (h-x)^2]\,dx$$

$$= \pi \tan^2\alpha \, \rho \int_0^h [(x^4 \tan^2\alpha)/4 + h^2x^2 - 2hx^3 + x^4]\, dx$$

$$= \pi \tan^2\alpha \, \rho \Big[(x^5 \tan^2\alpha/20) + (h^2x^3/3) - (hx^4/2) + (x^5/5) \Big]_0^h$$

$$= \pi \tan^2\alpha \, \rho \; [(h^5 \tan^2\alpha/20) + (h^5/3) - (h^5/2) + (h^5/5)]$$

$$= (\pi a^2/h^2) \, \rho h^5 \, [(a^2/20h^2) + (1/30)]$$

$$= 3\, Mh^2 \Big[a^2/20h^2 + (1/30) \Big] \qquad [\because \; M = (1/3)\, \pi a^2 h \rho]$$

$$= (M/20)\, [3a^2 + 2h^2]$$

$B =$ M.I. about $OY =$ M.I. about a line parallel to axis of y through $O' + M a^2 = (M/20)\,(3a^2 + 2h^2) + M a^2 = (M/20)(23a^2 + 2h^2)$

$C =$ M.I. about a line parallel to OZ through O'

$M a^2 = (3/10) M a^2 + M a^2 = (13/10)\, M a^2.$

Since the co-ordinates of the centre of gravity G are $(a, 0, h/4)$

$\therefore D = F = 0$ and $E = M a (1/4) h = (\frac{1}{4})\; Mah$

Hence the equation of the momental ellipsoid at O is

$$Ax^2 + By^2 + Cz^2 - 2Exz = \text{constant}$$

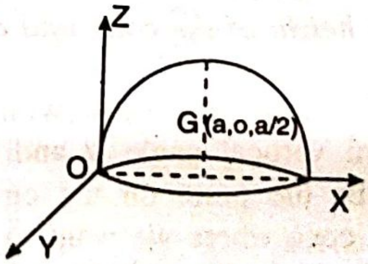
$$\Rightarrow (M/20)(3a^2 + 2h^2)x^2 + (M/20)(23a^2 + 2h^2)y^2$$
$$+ (13/10) M a^2 z^2 - 2(1/4)\, Mahxz = \text{constant}.$$

$$= (3a^2 + 2h^2)x^2 + (23a^2 + 2h^2)y^2 + 26a^2z^2 - 10ahxz = \text{constant}$$

Ex.23. *Show that the momental ellipsoid at a point on the edge of the circular base of a thin hemi-spherical shell is*

$$2x^2 + 5\,(y^2 + z^2) - 3zx = \text{constant}.$$

Sol. Let O be the point on the circular edge of the hemispherical shell. Take OX the diameter as *x-axis*, the line OY in the plane of the base perp. OX as *y-axis*, the line $OZ \perp$ to the plane of the base as *z-axis*.



$A =$ M.I. of the hemispherical shell about $OX = (\frac{2}{3})\, M a^2$

$B =$ M.I. about $OY =$ M.I. about a line parallel to OY and passing through C, the centre of the circular base $+ M a^2 = (\frac{2}{3}) M a^2 + M a^2 = (\frac{5}{3})\, Ma^2$

Similarly $C = (\frac{5}{3})\, M a^2$. Since co-ordinates of centre of gravity G are $(a, 0, a/2)$; $D = F = 0, E = M a (a/2) = (1/2)\, M a^2.$

Hence equation of momental ellipsoid at O is

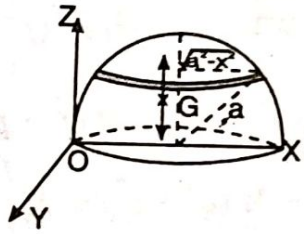
$$Ax^2 + By^2 + Cz^2 - 2Dyz - 2Ezx - 2Fxy = \text{constant}$$

$$\Rightarrow (\tfrac{2}{3})\, M a^2 x^2 + (\tfrac{5}{3})\, M a^2 y^2 + (\tfrac{5}{3})\, M a^2 z^2 - 2(\tfrac{1}{2})\, M a^2 zx = \text{constant}$$

$$\Rightarrow 2x^2 + 5(y^2 + z^2) - 3zx = \text{constant.}$$

Ex.24. *Show that the momental ellipsoid at a point on the rim of a hemisphere is* $2x^2 + 7(y^2 + z^2) - (\frac{15}{4})xz =$ constant. **[Nagpur 1990]**

Sol. Let O be the point on the rim of hemisphere and the diameter OX through O, the axis of x. Take OY a line in the plane of the base and perp. to OX as y– *axis* and a line OZ perp. to the plane of base as z– *axis*.



If G is centre of gravity of the hemisphere then the co-ordinates of G are $[a, 0, 3a/8]$. Let A, B, C be the moments and D, E, F the products of inertia of the hemisphere about these axis. Now take an elementary disc of width δx at a distance x from OX.

The radius of the disc $= \sqrt{(a^2 - x^2)}$

M.I. of the disc about $OX = \pi(a^2 - x^2)\, \rho\, dx\, [(\frac{1}{4})(a^2 - x^2) + x^2]$

$$\therefore A = \frac{1}{4}\pi\rho \int_0^a (a^2 - x^2)(a^2 + 3x^2)\, dx$$

$$= \frac{1}{4}\pi\rho \int_0^a (a^4 + 2a^2x^2 - 3x^4)\, dx$$

$$= \frac{1}{4}\pi\rho\, [a^5 + \frac{2}{3}a^5 - \frac{3}{5}a^5] = \frac{4\pi\rho a^5}{15} = \frac{2}{5} M a^2 \quad (\because \; M = \frac{2}{3}\pi a^3 \rho)$$

Similarly $B = C = (\frac{2}{5})\, M a^2 + M a^2 = (\frac{7}{5})\, M a^2$

Also $D = F = 0$ and $E = M\, [a.\, (\frac{3}{8}a)] = (3M a^2/8)$.

Hence the equation of the required momental ellipsoid at O is

$$Ax^2 + By^2 + Cz^2 - 2Dyz - 2Ezx - 2Fxy = \text{constant}$$

$$\Rightarrow M a^2 [(\tfrac{2}{5})x^2 + (\tfrac{7}{5})y^2 + (\tfrac{7}{5})z^2 - (\tfrac{3}{4})xz] = \text{constant.}$$

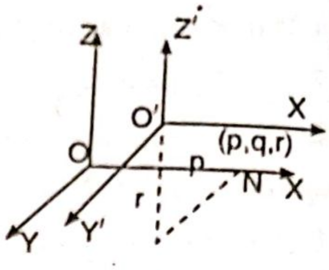
$$\Rightarrow 2x^2 + 7(y^2 + z^2) - (\tfrac{15}{4})xz = \text{constant..}$$

Ex.25. *The principal axes at the centre of gravity being the axes of reference, prove that the momental ellipsoid at the point* (p, q, r) *is*

$$\left(\frac{A}{M} + q^2 + r^2\right)x^2 + \left(\frac{B}{M} + p^2 + r^2\right)y^2 + \left(\frac{C}{M} + q^2 + p^2\right)z^2$$
$$- 2qryz - 2rpzx - 2pqxy = \text{constant.}$$

when referred to its centre of gravity as origin.

Sol. Let GX, GY, GZ be the principal axes through G, the C.G. of the body. Further let A, B, C be the moments of inertia of the body about the principal axes GX, GY, GZ and D, E, F the products of inertia about these axes taken two at a time. Then we will have $D = E = F = 0$



Now if we take A', B', C' as the moment of inertia and D', E', F' the products of inertia about parallel axes through $O' \equiv (p, q, r)$ then we will have

$$A' = A + M(q^2 + r^2) = M\left(\frac{A}{M} + q^2 + r^2\right)$$

Similarly $B' = B + M(r^2 + p^2) = M\left(\frac{B}{M} + r^2 + p^2\right)$

$$C' = C + M(q^2 + p^2) = M\left(\frac{C}{M} + q^2 + p^2\right)$$

Also $D' = D + M\,qr, E' = E + M\,rp, F' = F + M\,pq = Mpq$

Hence the equation of momental ellipsoid at O' is

$$A'x^2 + B'y^2 + C'z^2 - 2D'yz - 2E'zx - 2F'xy = \text{constant}$$

$$\Rightarrow \left(\frac{A}{M} + q^2 + r^2\right)x^2 + \left(\frac{B}{M} + p^2 + r^2\right)y^2 + \left(\frac{C}{M} + q^2 + p^2\right)z^2$$
$$- 2qryz - 2rpzx - 2pqxy = \text{constant.}$$

Ex.26 (a) *If $S \equiv A\,x^2 + B\,y^2 + C\,z^2 - 2D\,yz - 2E\,zx - 2F\,xy = \text{constant}$ be the equation of momental ellipsoid at the centre of gravity O of a body referred to any rectangular axes through O, then prove that the momental ellipsoid at the point (p, q, r) is*

$$S + M\,[(qz - ry)^2 + (rx - pz)^2 + (py - qx)^2] = \text{constant}$$

where M is the mass of the body.

Sol. If A', B', C' are the moments of inertia and D', E', F' are the products of inertia of the body with reference to a set of parallel axes through (p, q, r), then we have

$$A' = A + M(q^2 + r^2), B' = B + M(r^2 + p^2), C' = C + M(q^2 + p^2).$$

Also $D' = D + M\,qr, E' = E + M\,rp, F' = D + M\,pq.$

Hence the momental ellipsoid at (p, q, r) is given by

$$A'x^2 + B'y^2 + C'z^2 - 2D'yz - 2E'zx - 2F'xy = \text{constant}$$

Substituting the values of A', B', C' etc. and simplifying, we get

$$S + M\,[\Sigma\,(qz - ry)^2] = \text{constant.}$$

0·13. Equimomental Bodies : *Two systems or bodies are said to be equimomental or kinetically equivalent when moments and products of inertia of one system or body about the axes are each correspondingly equal* *to moments and products of inertia of the other system or body about the same axes.*

Necessary and sufficient conditions. *The two systems will be equimomental, iff the following conditions are satisfied.*

(i) The centres of gravity of the two system should coincide.

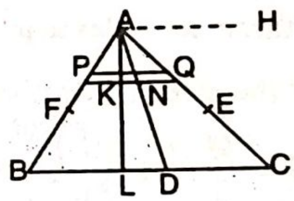
(ii) Both the systems should have the same mass.

(iii) The two systems should have the same principal axes and same principal moments about centre of gravity. **[Rajasthan 1991]**

0·14 *To show that moments and products of inertia of a uniform triangle about any lines are the same as the moments and products of inertia about the same lines, of three particles placed at the mid points of the sides, each equal to one third of the mass of the triangle.* **[Agra 84, Rajasthan 83]**

Let PQ be an elementary strip of breadth δx at a distance x from the vertex A. If the height of triangle $AL = h$, then from similar triangles ABC and APQ, we have

$$\frac{BC}{PQ} = \frac{h}{x} \Rightarrow PQ = \frac{ax}{h} \quad (\because \; BC = a)$$



Now draw the line AH in the plane of the lamina and parallel to BC. M.I. of the lamina about

$$AH = \int_0^h \left[\frac{ax}{h}\rho\,dx\right]x^2 = \frac{a\rho}{h}\int_0^h x^2\,dx = \frac{a\rho h^4}{4h} = \frac{a\rho h^3}{4} = \frac{1}{2}Mh^2$$

Since $M = \dfrac{a\rho h}{2}$

M.I. of the lamina about AL

$$= \int_0^h \left[\frac{ax}{h}\rho\,dx\right]\left[NK^3 + \frac{1}{3}\left(\frac{ax}{2h}\right)^2\right] = \frac{a\rho}{h}\int_0^h \left[x\left(\frac{x}{h}LD\right)^2 + \frac{1}{3}\left(\frac{ax}{2h}\right)^2\right]dx$$

$$= \frac{a\rho}{h}\int_0^h \left[\frac{LD^2}{h^2} + \frac{a^2}{12h^2}\right]x^3\,dx = \frac{a\rho}{h}\left[\frac{LD^2}{h^2} + \frac{a^2}{12h^2}\right]\frac{h^4}{4}$$

$$= \frac{1}{4}a\rho h\left[LD^2 + \frac{a^2}{12}\right] = \frac{1}{4}a\rho h\left[(BD - BL)^2 + \frac{a^2}{12}\right]$$

$$= \frac{1}{4}a\rho h\left[\left(\frac{BC}{2} - BL\right)^2 + \frac{a^2}{12}\right] = \frac{1}{4}a\rho h\left[\left(\frac{a}{2} - C\cos b\right)^2 + \frac{a^2}{12}\right]$$

$$= \frac{1}{4}a\rho h\left[\left(\frac{b\cos C + c\cos B}{2} - c\cos B\right)^2 + \left(\frac{b\cos C + c\cos B}{2\sqrt{3}}\right)^2\right]$$

$$= \frac{1}{4}a\rho h\left[\left(\frac{b\cos C - c\cos B}{2}\right)^2 + \left(\frac{b\cos C + c\cos B}{2\sqrt{3}}\right)^2\right]$$

$$= \frac{1}{48} a\rho h\left[3(b \cos C - c \cos B)^2 + (b \cos C + c \cos B)^2 \right]$$

$$= \frac{M}{6}\left[b^2\cos^2 C + c^2\cos^2 B - bc \cos B . \cos C \right] \quad ...(2)$$

The products of inertia about AL and AH

$$= \int_0^h \left(\frac{ax}{h} \rho x \right) x . NK = \int_0^h \frac{a\rho}{h} x^2 \frac{x}{h} . LD = \frac{a\rho}{h^2} \int_0^h x^3 (BD - BL)\, dx$$

$$= \frac{a\rho}{h^2} \int_0^h x^3 (\tfrac{1}{2}BC - BL)\, dx = \frac{a\rho}{h^2}\left[\tfrac{1}{2}(b \cos C + c \cos B) - c \cos B \right] \frac{h^4}{4}$$

$$= \frac{a\rho}{h^2} \frac{1}{2}(b \cos C - c \cos B)\frac{h^4}{4} = \frac{a\rho h^2}{8}(b\cos C - c \cos B)$$

$$= \tfrac{1}{4} Mh(b \cos C - c \cos B). \quad ...(3)$$

Now let three particles each of mass $\frac{M}{3}$ be placed at the mid points D, E, F of the sides respectively. Then we have M.I. of the three particles about $AH = \frac{M}{3}h^2 + \frac{M}{3}\left(\frac{h}{2}\right)^2 + \frac{M}{3}\left(\frac{h}{2}\right)^2 = \frac{1}{2} M h^2$ ...(4)

M.I. of three particles about AL

$$= \frac{M}{3} LD^2 + \frac{M}{3}\left(\frac{b}{2} \cos C \right)^2 + \frac{M}{3}\left(\frac{1}{2} c \cos B \right)^2$$

$$= \frac{M}{3} [(a/2 - c \cos B)^2 + (\tfrac{1}{2} b \cos C)^2 + (\tfrac{1}{2} c \cos B)^2]$$

$$= \frac{M}{6} [b^2\cos^2 C + c^2\cos^2 B - bc \cos B \cos C] \quad ...(5)$$

Product of inertia of three particles about AL and AH

$$= \frac{M}{3} AL . LD - \frac{M}{3}\frac{1}{2} AL . (\tfrac{1}{2} BL) + \frac{M}{3}\frac{1}{2} AL . (\tfrac{1}{2} CL)$$

$$= \frac{Mh}{3} [LD - \tfrac{1}{4} BL + \tfrac{1}{4} CL]$$

$$= \frac{Mh}{3} [(a/2 - c \cos B) - \tfrac{1}{4} c \cos B + \tfrac{1}{4} b \cos C]$$

$$= \frac{Mh}{3}\left[\frac{b \cos C + c \cos B}{2} - c \cos B - \tfrac{1}{4} c \cos B + \tfrac{1}{4} b \cos C \right]$$

$$= \frac{Mh}{3} [\tfrac{3}{4} b \cos C - \tfrac{3}{4} c \cos B] = \frac{Mh}{4} [b \cos C - c \cos B] \quad ...(6)$$

Comparing the above results, we conclude that moments and products of inertia of the triangle about AH and AL are the same as those of three particles each of mass $\frac{1}{3}M$ placed at the mid points of the sides. Therefore moments and poducts of inertia about any two perp. lines will be the same.

Hence a triangle of mass M is kinetically equivalent to three particles each of mass $\frac{1}{3}M$ placed at the mid points of the side.

Ex.26.(b) *Show that a uniform triangular lamina of mass m is equimomental with three particles, each of mass (1/12)m placed at the angular points and a particle of mass (3/4)m placed at the centre of inertia of the triangle.*

Sol. Clearly C.G. of the four particles is the same as the C.G. of the triangular lamina. Also mass of the four particles

$= (3/12)m + (3/4)m = m =$ mass of the triangle.

Now if the distances of the vertices of the triangle from a line are α, β, γ respectively and the distance of its C.G. from this line is h, then M.I. of the four particles about the line

$$= (1/12)m\alpha^2 + (1/12)m\beta^2 + (1/12)\gamma^2 + (3/4)mh^2$$

$$= (1/12)m[\alpha^2 + \beta^2 + \gamma^2 + 9h^2] \qquad \because [h = (\alpha+\beta+\gamma)/3]$$

$$= (1/12)\, m[\alpha^2 + \beta^2 + \gamma^2 + 9\{(\alpha+\beta+\gamma)/3\}^2]$$

$$= (1/6)\, m[\alpha^2 + \beta^2 + \gamma^2 + \beta\gamma + \gamma\alpha + \alpha\beta]$$

= M.I. of the triangle about the same line.

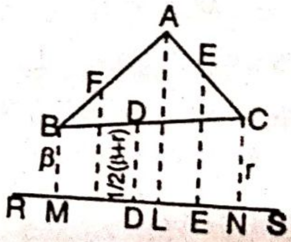
Hence the systems are equi-momental.

Ex.27. *If α, β, γ be the distances of the vertices of a triangle of mass m from any straight line in its plane, show that the moment of inertia of the triangle about this line is $(1/6)\, m(\alpha^2 + \beta^2 + \gamma^2 + \beta\gamma + \gamma\alpha + \alpha\beta)$.*

Hence deduce that if h be the distance of the centre of inertia of the triangle from the line, then M.I. about this line

$$= (1/12)\, m(\alpha^2 + \beta^2 + \gamma^2 + 9h^2).$$

Sol. The triangle of mass m is equimomental to three particles each of mass $\frac{1}{3}m$ placed at the middle points of the sides. Let ABC be the triangle and D, E, F the middle points of its sides. Also let RS be the line in the plane of the triangle. Draw perpendiculars from A, B, C to the line RS.



If $AL = \alpha$, $BM = \beta$, $CN = \gamma$ then the lengths of the perpendiculars from D, E, and F are as given below.

$$DD' = (\tfrac{1}{2})(\beta+\gamma),\ EE' = (\tfrac{1}{2})(\alpha+\gamma),\ FF' = (\tfrac{1}{2})(\alpha+\beta)$$

$$\therefore \text{M.I.} = (\tfrac{1}{3})\, m\{(\beta+\gamma)/2\}^2 + (\tfrac{1}{3})\, m\{(\beta+\alpha)/2\}^2 + (\tfrac{1}{3})\, m\{(\alpha+\beta)/2\}^2$$

$$= (1/12)\, m(2\alpha^2 + 2\beta^2 + 2\gamma^2 + 2\beta\gamma + 2\gamma\alpha + 2\alpha\beta)$$

$$= (1/6)\, m(\alpha^2 + \beta^2 + \gamma^2 + \beta\gamma + \gamma\alpha + \alpha\beta)$$

If h is the distance of C.G. from the line RS, then we have

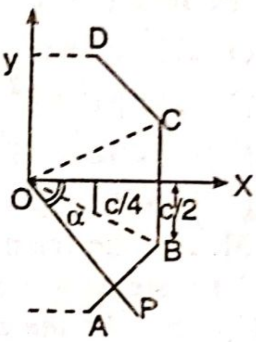
$$h = \left(\tfrac{1}{3}\right)(\alpha+\beta+\gamma)$$

$$\therefore \text{M.I.} = \left(\tfrac{1}{12}\right) m[\alpha^2+\beta^2+\gamma^2+(\alpha+\beta+\gamma)^2]$$

$$= (m/12)(\alpha^2+\beta^2+\gamma^2+9h^2).$$

Ex.28. *Show that the moment of inertia of a regular polygon of n sides about any straight line through its centre is* $\dfrac{Mc^2}{24}\cdot\dfrac{2+\cos(2\pi/n)}{1-\cos(2\pi/n)}$, *where n is the number of sides and C is the length of each side.*

Sol. Consider a polygon $ABCDEF$ of n sides, each of length c. Let O be the centre of the polygon. The polygon can be divided into n equal isosceles triangles. If M is the mass of polygon then mass of each of the n isosceles triangles $= (M/n)$. Consider one of the isosceles triangle, say $\Delta\, OBC$.



Let the right bisector of BC be the *x-axis* and a line through O perp. to OX in the plane of the polygon be the *y– axis*. Clearly $\angle BOC = (2\pi/n)$ and $\angle COX = (\pi/n)$. Now the triangle BOC of mass (M/n) placed at the middle point of its sides.

$\therefore$ M.I. of the triangle about

$$OX = (M/3n)\left[\left(\tfrac{1}{4}c\right)^2+\left(\tfrac{1}{4}c\right)^2\right] = (M/24n)\,c^2.$$

[The length of the perp. from the mid point of OB on $OX = (c/4)$].

M.I. of the triangle about OY

$$= (M/3n)\left[\left(\tfrac{1}{2}c\cot\pi/n\right)^2+2\left(\tfrac{1}{4}c\cot\pi/n\right)^2\right] = (M\,c^2/8n)\cot^2\pi/n$$

If OP is line making an angle α with OX about which we want to determine moment of inertia then M.I. of the triangle OBC about OP

$$= \left(\frac{M}{24n}c^2\right)\cos^2\alpha + \left(\frac{Mc^2}{8n}\cot^2\pi/n\right)\sin^2\alpha$$

Taking the other triangles one by one, we get the M.I. of the polygon

$$\text{about } OP = \frac{Mc^2}{24n}\left\{\cos^2\alpha+\cos^2\left(\alpha+\frac{2\pi}{n}\right)+\cos^2\left(\alpha+\frac{4\pi}{n}\right)+\ldots n \text{ terms}\right\}$$

$$+\frac{Mc^2}{8n}\cot^2\frac{\pi}{n}\left\{\sin^2\alpha+\sin^2\left(\alpha+\frac{2\pi}{n}\right)+\sin^2\left(\alpha+\frac{4\pi}{n}\right)+\ldots\ldots n \text{ terms}\right\}$$

$$= \frac{Mc^2}{24n}\,\frac{1}{2}\,[1+\cos 2\alpha+1+\cos(2\alpha+4\pi/n)+\ldots]$$

$$+\frac{Mc^2}{8n}\cot^2(\pi/n)\cdot\frac{1}{2}\cdot[1-\cos 2\alpha+1-\cos(2\alpha+4\pi/n)\ldots]$$

$$= \frac{Mc^2}{24n}\frac{1}{2}n + \frac{Mc^2}{8n}\cot^2\left(\frac{\pi}{n}\right)\frac{1}{2}n + \frac{Mc^2}{48n}\left[\cos 2\alpha + \cos\left(2\alpha + \frac{4\pi}{n}\right) + \text{....}n\ terms\right]$$

$$- \frac{Mc^2}{16n}\cot^2\left(\frac{\pi}{n}\right)\left[\left[\cos 2\alpha + \cos\left(2\alpha + \frac{4\pi}{n}\right) + \text{....}n\ terms\right]\right.$$

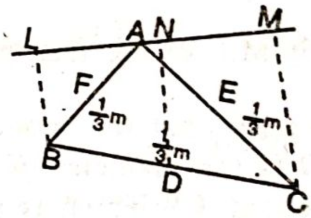
$$= \frac{Mc^2}{48}\left[1 + 3\cot^2\left(\frac{\pi}{n}\right)\right] + 0 - 0 = \frac{Mc^2}{48}\left\{\frac{\sin^2(\pi/n) + 3\cos^2(\pi/n)}{2\sin^2\pi/n}\right\}$$

$$= \frac{Mc^2}{24}\left[\frac{\sin^2(\pi/n) + \cos^2(\pi/n) + 2\cos^2(\pi/n)}{2\sin^2\pi/n}\right]$$

$$= \frac{Mc^2}{24}\left[\frac{1 + \{1 + \cos(2\pi/n)\}}{1 - \cos(2\pi/n)}\right] = \frac{Mc^2}{24}\left[\frac{2 + \cos(2\pi/n)}{1 - \cos(2\pi/n)}\right]$$

Ex.29. *Obtain the moment of inertia of a triangular lamina ABC about a straight line through A (or any vertex) in the plane of the triangle.*

Sol. Let LAM be the line about which moment of inertia is to be determined, and let β, γ be the respective distance of two vertices B and C of the triangle from the straight line LAM. If m is the mass of the triangle, then the triangle is equimomental to three particles, each of mass $\frac{1}{3}m$, placed at the mid points D, E, F of the sides BC, CA, AB. Length of the perp. from F on $LAM = (\beta/2)$



Length of perp. from D on $LAM = (\gamma + \beta)/2$

Length of perp. from E on $LAM = (\gamma/2)$

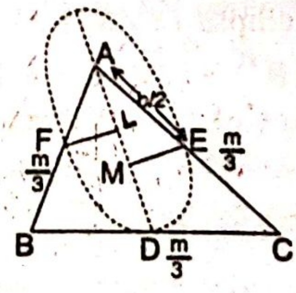
$\therefore$ M.I. of the triangle about LAM

$$= (1/3)\, m[(\beta/2)^2 + \{(\beta+\gamma)/2\}^2 + (\gamma/2)^2] = (1/6)\, m[\beta^2 + \beta\gamma + \gamma^2].$$

Ex.30. *Show that there is a momental ellipse at an angular point of a triangular area which touches the opposite side at its middle point and bisects at the adjacent sides.* **[Delhi Hons. 64]**

Sol. We know that the momental ellipsoid at A passes through D, if moment of inertia about $AD = (mK^4/AD^2)$ where K is some constant.

Similarly the momental ellipsoid will pass through E and F if moment of inertia about $AE = (mK^4/AE^2)$ etc. Now replace the triangle of mass m by three particles, each of mass $(1/3)\, m$ at D, E and F.



M.I. of the triangle about

$A = m/3\, [FL^2 + EM^2]$

$$= m/3\,[\{(c/2)\sin BAD\}^2 + \{(b/2)\sin CAD\}^2]$$
$$= m/12\,[c^2\sin^2 BAD + b^2\sin^2 CAD]$$

Using Lami's theorem in $\Delta s\, ABD$ and ACD, we get

$$\frac{\sin BAD}{(a/2)} = \frac{\sin B}{AD} \text{ and } \frac{\sin CAD}{(a/2)} = \frac{\sin C}{AD}$$

$$\therefore \sin BAD = (a/2)\frac{\sin B}{AD} \text{ and } \sin CAD = (a/2)\frac{\sin C}{AD}$$

$$\therefore \text{M.I. about } AD \text{ becomes } = \frac{m}{12}\left(\tfrac{1}{4}a^2c^2\sin^2 B + \tfrac{1}{4}a^2b^2\sin^2 C\right)\frac{1}{AD^2}$$

$$= \frac{m}{12}\left[\Delta^2 + \Delta^2\right]\frac{1}{AD^2} = \left(\frac{m\Delta^2}{6}\right)\frac{1}{AD^2}$$

M.I. of the triangle about AF

$$= \frac{m}{3}\left[\left(\frac{b}{2}\sin A\right)^2 + \left(\frac{b}{2}\sin A\right)^2\right] = \frac{mb^2}{6}\sin^2 A = \frac{m}{6}\frac{1}{c^2}b^2c^2\sin^2 A$$

$$= \frac{m}{6c^2}4\Delta^2 = \frac{m}{6}\frac{4\Delta^2}{(2AF)^2} = \left(\frac{M\Delta^4}{6}\right)\frac{1}{AF^2}$$

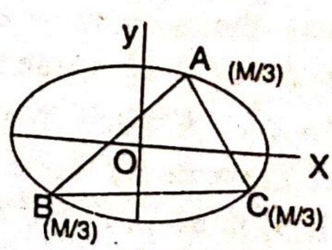
Similarly M.I. of the triangle about $AE = \left(\dfrac{m\Delta^2}{6}\right)\dfrac{1}{AE^2}$.

Thus we see that the momental ellipse at A passes through D, E and F and AD is the diameter of the ellipse. The tangent at D will be parallel to the chord EF which is bisected by the diameter and obviously BC is parallel to EF. Hence BC is tangent to the ellipse at D.

Ex. 31. *Show that any lamina is dynamically equivalent to the three particles each one third of the mass of the lamina, placed at the centre of a maximum triangle inscribed in the ellipse, whose equation referred to the principal axes at the centre of inertia is* $(x^2/B') + (y^2/A') = 2$ *where* mA' *and* mB' *are the principal moments of inertria about* OX *and* OY *and* m *is the mass.*

Sol. If a maximum triangle is inscribed in an ellipse, then the eccentric angles of its angular points will be of the form $\phi, \phi + (2\pi/3), \phi + (4\pi/3)$.

Let ABC be the maximum triangle inscribed in the ellipse $(x^2/B') + (y^2/A') = 2$
$\Rightarrow (x^2/2B') + (y^2/2A') = 1$ then the co-oridinates of A, B, C, will be



$[\sqrt{(2B')}\cos\phi, \sqrt{(2A')}\sin\phi]$,

$\left[\sqrt{(2B')}\cos\left(\phi + \dfrac{2\pi}{3}\right),\right.$

$$\left[\sqrt{(2A')}\sin\left(\phi+\frac{2\pi}{3}\right)\right];\left[\sqrt{(2B')}\cos\left(\phi+\frac{4\pi}{3}\right),\sqrt{(2A')}\sin\left(\phi+\frac{4\pi}{3}\right)\right]$$

Now if $(\bar{x},\bar{y})$ be co- ordinates of C.G. of the three particles placed at A, B, C and each of mass m/3 then

$$\bar{x}=\frac{\frac{m}{3}\sqrt{(2B')}}{m}\left\{\cos\phi+\cos\left(\phi+\frac{2\pi}{3}\right)+\cos\left(\phi+\frac{4\pi}{3}\right)\right\}$$

$$=\frac{\sqrt{(2B')}}{3}\left\{\cos\phi+2\cos(\phi+\pi).\cos\frac{\pi}{3}\right\}$$

$$=\frac{\sqrt{(2B')}}{3}\left\{\cos\phi+2\cos\phi.\frac{1}{2}\right\}=0,\text{ similarly } \bar{y}=0.$$

Thus the centres of inertia of the two system coincide. Also masses of the two systems are equal. M.I. of the particles each of mass m/3 placed at A, B, C *about* OX .

$$=\frac{m}{3}2A'\left[\sin^2\phi+\sin^2\left(\phi+\frac{2\pi}{3}\right)+\sin^2\left(\phi+\frac{4\pi}{3}\right)\right\}$$

$$=\frac{1}{3}mA'\left\{3-\cos 2\phi-\left\{\cos\left(2\phi+\frac{4\pi}{3}\right)+\sin\left(2\phi+\frac{8\pi}{3}\right)\right\}\right]$$

$$=\frac{1}{3}mA'\left[3-\cos 2\phi-2\cos(2\phi+2\pi)\cos\frac{2\pi}{3}\right]$$

$$=\frac{1}{3}mA'\left[3-\cos 2\phi+\cos 2\phi\right]=mA'=\text{M.I. of the lamina about } OX.$$

Similarly M.I. about $OY= m\,B'$ = M.I. of the lamina about OY. Thus M.I. of the three particles about OX and OY are the same as that of lamina. Product of inertia of the three particles about OX, OY.

$$=\frac{m}{3}\sqrt{(2B')}.\sqrt{(2A')}\left[\cos\phi\sin\phi+\cos\left(\phi+\frac{2\pi}{3}\right)\sin\left(\phi+\frac{2\pi}{3}\right)\right.$$
$$\left.+\cos\left(\phi+\frac{4\pi}{3}\right)\sin\left(\phi+\frac{4\pi}{3}\right)\right]$$

$$=\frac{m}{3}\sqrt{(A'B')}\left[\sin 2\phi+\sin\left(2\phi+\frac{4}{3}\pi\right)+\sin\left(2\phi+\frac{8\pi}{3}\right)\right]$$

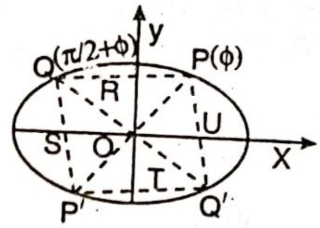
$$=\frac{m}{3}\sqrt{(A'B')}\left[\sin 2\phi+2\sin(2\phi+2\pi).\cos\frac{2\pi}{3}\right]$$

$$=\frac{1}{3}m\sqrt{(A'B')}\left[\sin 2\phi+2\sin 2\phi\left(-\frac{1}{2}\right)\right]=0.$$

Thus the two systems are dynamically equivalent as the two systems have same mass, same C.G., same M.I. and same product of inertia at their C.G.

Ex.32. *Particles each equal to one quarter of the mass of an elliptic area are placed at the middle points of the chords joining the extremities of a pair of conjugate diameters. Prove that these four particles are equimomental to the elliptic area.*

Sol. Let POP' and QOQ' be the conjugate diameters of the elliptic area of mass m. If ϕ is the eccentric angle of P then eccentric angle of Q is $\{(\pi/2)+\phi\}$. The co-ordinates of P, Q respectively are $(a\cos\phi, b\sin\phi)$, $(-a\sin\phi, b\cos\phi)$ and that of P', Q' are $(-a\cos\phi, -b\sin\phi)$ respectively.



If (x_1, y_1) be the co-ordinates of R, the mid point of PQ then

$$x_1 = (\tfrac{1}{2})\, a(\cos\phi - \sin\phi)\ ;\ y_1 = (\tfrac{1}{2})\, b(\sin\phi + \cos\phi).$$

Similarly if (x_2, y_2) are co-ordinates of S then

$$x_2 = -(a/2)(\sin\phi + \cos\phi),\ y_2 = (b/2)(\cos\phi - \sin\phi)$$

Also if (x_3, y_3) and (x_4, y_4) are the co-ordinates of the mid points T and U then $x_3 = -(a/2)(\cos\phi - \sin\phi)$, $y_3 = -(b/2)(\sin\phi + \cos\phi)$, and $x_4 = (a/2)(\sin\phi + \cos\phi)$, $y_4 = (b/2)(\sin\phi - \cos\phi)$.

Now if the co-ordinates of the C.G. of the four particles, each of mass (m/4) placed at the mid points R, S, T and U be $(\bar{x}, \bar{y})$ then

$$\bar{x} = \frac{(\frac{1}{4})m\,(x_1 + x_2 + x_3 + x_4)}{(\frac{1}{4})m + (\frac{1}{4})m + (\frac{1}{4})m + (\frac{1}{4})m}$$

$$= \frac{(m/4)(a/2)[(\cos\phi - \sin\phi) - (\sin\phi + \cos\phi) - (\cos\phi - \sin\phi) + (\sin\phi + \cos\phi)]}{m} = 0.$$

Similarly $\bar{y} = 0$. Thus the C.G. of the particles is the same as the C.G. of the elliptic area. M.I. of the four particles about the major axis OX

$$= \frac{m}{4}[y_1^2 + y_2^2 + y_3^2 + y_4^2] = \frac{m}{4}\left[\frac{b^2}{4}\{(\sin\phi + \cos\phi)^2 + (\cos\phi - \sin\phi)^2 + (\sin\phi + \cos\phi)^2 + (\sin\phi - \cos\phi)^2\}\right]$$

$$= \frac{mb^2}{4} = \text{M.I. of the elliptic area about } OX.$$

Similarly M.I. of the particles about $OY = \dfrac{ma^2}{4}$ = M.I. of the elliptic plate about OY. Now product of inertia about OX, OY

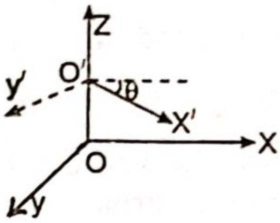
$$= \frac{m}{4}[x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4] = 0$$

= Product of inertia of the elliptic area about OX, OY. Hence the four particles are equimomental to the elliptic area.

0·15. Principal Axes : *To find whether a given straight line is, at any point of its length, a principal axis of a material system. And if the line is principal axis, then to determine the other two principal axis.*

Take the given straight line OZ as y-axis and two perpendicular lines OX, OY as x axis and y- axis respectively , through any point O on the given line.

Now suppose that OZ is a principal axis of the system at a point O' where $OO' = h$.
Assume $O'X', O'Y'$ to be the other two principal axes , such that $O'Y'$ is inclined at an angle θ to a line parallel to OX. Now consider a particle of mass m in the material system . If (x, y, z) be the co-ordinates of that particle with reference to axes OX, OY, OZ and (x', y', z') its co-ordinates with reference to $O'X', O'Y', O'Z'$ as axes , then we will have



$x' = x\cos\theta + \sin\theta,\ y' = -x\sin\theta + y\cos\theta,\ z' = z - h.$

The necessary and sufficient condition , that the new axes $O'X', O'Y', O'Z'$ become the principal axes of the system, is, that the product of inertia with reference to these axes must vanish,

i.e., $\Sigma my'z' = 0, \Sigma mz'x' = 0, \Sigma mx'y' = 0.$

Now $\Sigma my'z' = \Sigma m(-x\sin\theta + y\cos\theta)(z - h)$

$$= \Sigma m\,(yz\cos\theta - xz\sin\theta + hx\sin\theta - hy\cos\theta)$$
$$= \cos\theta\,\Sigma myz - \sin\theta\,\Sigma mzx + h\sin\theta\,\Sigma mx - h\cos\theta\,\Sigma my$$
$$= D\cos\theta - E\sin\theta + h\sin\theta\,M\bar{x} - h\cos\theta\,M\bar{y}$$
$$= D\cos\theta - E\sin\theta + Mh\,(\bar{x}\sin\theta - \bar{y}\cos\theta)$$

Similarly $\Sigma mz'x' = \Sigma m\,[yz\sin\theta + xz\cos\theta - hx\cos\theta - hy\sin\theta]$

$$= D\sin\theta + E\cos\theta - Mh\,(\bar{x}\cos\theta + \bar{y}\sin\theta)$$

and $\Sigma mx'y' = \Sigma m\,[-x^2\sin\theta\cos\theta + xy(\cos^2\theta - \sin^2\theta) + y^2\sin\theta.\cos\theta]$

$$= \frac{1}{2}(A - B)\sin 2\theta + F\cos 2\theta.$$

Now taking $\Sigma mx'y' = 0$, we get

$$\tan 2\theta = \frac{2F}{B - A} \Rightarrow \theta = \frac{1}{2}\tan^{-1}\frac{2F}{B - A} \qquad ...(1)$$

taking $\Sigma my'z' = 0$, we get $Mh = \dfrac{E\sin\theta - D\cos\theta}{\bar{x}\sin\theta - \bar{y}\cos\theta}$

and taking $\Sigma mz'x' = 0$, we have $Mh = \dfrac{D\sin\theta + E\cos\theta}{\bar{x}\cos\theta + \bar{y}\sin\theta}$

$$\therefore\ Mh = \frac{E\sin\theta - D\cos\theta}{\bar{x}\sin\theta - \bar{y}\cos\theta} = \frac{D\sin\theta + E\cos\theta}{\bar{x}\cos\theta + \bar{y}\sin\theta}.$$

Now $\dfrac{E\sin\theta - D\cos\theta}{\bar{x}\sin\theta - \bar{y}\cos\theta} = \dfrac{D\sin\theta + E\cos\theta}{\bar{x}\cos\theta + \bar{y}\sin\theta}$

$$= \frac{(E\sin\theta - D\cos\theta)\sin\theta + (D\sin\theta + E\cos\theta)\cos\theta}{(\bar{x}\sin\theta - \bar{y}\cos\theta)\sin\theta + (\bar{x}\cos\theta + \bar{y}\sin\theta)\cos\theta} = \frac{E}{\bar{x}} \qquad ...(2)$$

and $\dfrac{E\sin\theta - D\cos\theta}{\bar{x}\sin\theta - \bar{y}\cos\theta} = \dfrac{D\sin\theta + E\cos\theta}{\bar{x}\cos\theta + \bar{y}\sin\theta}$

$$= \frac{(E\sin\theta - D\cos\theta)(-\cos\theta) + (D\sin\theta + E\cos\theta)\sin\theta}{(\bar{x}\sin\theta - \bar{y}\cos\theta)(-\cos\theta) + (\bar{x}\cos\theta + \bar{y}\sin\theta)\sin\theta} = \frac{D}{y} \quad ...(3)$$

so, we have $Mh = \dfrac{E}{\bar{x}} = \dfrac{D}{\bar{y}}$. ...(4)

Thus the condition that the line OZ may be principal axis of the system at some point of its length is $\dfrac{E}{\bar{x}} = \dfrac{D}{\bar{y}}$ and then from (4) we get that point at which the line OZ is a principal axis and (1) gives the co-ordinates of the other principal axes at O'.

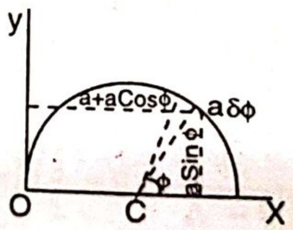
Cor. I. *If an axis passes through the centre of gravity of a body and it is a principal axis at any point of its length, then it is a principal axis at all points of its length.* **[Meerut 1978,Raj.62]**

If z-axis is a principal axis at O, then $D = E = 0$ and further $h = D/M\bar{y} = E/M\bar{x}$ implies that $h = 0$ which means that there is no such other point as O'. But if $\bar{x} = 0$, $\bar{y} = 0$ and $D = E = 0$, then in this case the value of h becomes indeterminate.

So in a case where the axis passes through C.G. of the body and is a principal axis at any point of its length, it is a principal axis at all the points of its length.

Ex. 33. *A wire is in the form of a semi-circle of radius a, show that at an end of its diameter the principal axes in its plane are inclined to the diameter at angles* $(\frac{1}{2})\tan^{-1}(4/\pi)$ *and* $(\pi/2) + (\frac{1}{2})\tan^{-1}(4/\pi)$.

Sol. Let O be an end of the diameter of the semicircular wire. Now choose OX and OY as the axes of reference.



Consider an elementary arc $a\,\delta\phi$, where a is the radius of the circle, then we easily have

A = M.I. of the wire about OX.

$$= \int_0^{\pi} (\rho a\, d\phi)(a\sin\phi)^2$$

$$= \int_0^{\pi/2} 2a^3\rho\sin^2\phi\, d\phi = 2a^3\rho \int_0^{\pi/2} \sin^2\phi\, d\phi$$

$$= 2a^3\rho.\frac{1}{2}.\frac{\pi}{2} = (\pi a^3\rho/2) = (Ma^2/2), \quad [\because\ M = \tfrac{1}{2}(2\pi a\rho) = \pi a\rho]$$

$$B = M.I. \text{ about } OY = \int_0^{\pi} \rho a(a + a\cos\phi)^2\, d\phi$$

$$= \rho a^3 \int_0^{\pi} (1 + 2\cos\phi + \cos^2\phi)\, d\phi$$

$$= \rho a^3 (\pi + 2 . \tfrac{1}{2} . \pi/2) = \frac{3\rho a^3 \pi}{2} = \frac{3Ma^2}{2}$$

$F =$ Product of inertia about OX , OY

$$= \int_0^{\pi} \rho a (a \sin\phi)(a + a\cos\phi)\, d\phi = \rho a^3 \int_0^{\pi} (\sin\phi + \sin\phi\cos\phi)\, d\phi$$

$$= \rho a^3 \left[-\cos\phi + \frac{1}{2}\sin^2\phi \right]_0^{\pi} = 2\rho a^3 = 2Ma^2/\pi .$$

Now let θ be the inclination of one of the principal axes to the diameter OX then, we have

$$\tan 2\theta = \frac{2F}{B-A} \Rightarrow \theta = \tfrac{1}{2}\tan^{-1}\frac{2F}{B-A} = \tfrac{1}{2}\tan^{-1}\left\{ \frac{(4Ma^2/\pi)}{\tfrac{3}{2}Ma^2 - \tfrac{1}{2}Ma^2} \right\}$$

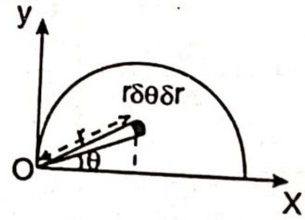
$$= \tan^{-1}(4/\pi).$$

The other principal axis being perpendicular to the above axis will make an angle $[(\pi/2) + (\tfrac{1}{2})\tan^{-1}(4/\pi)]$ with OX .

Ex. 34. *Show that at extremity of the boundary diameter of a semi-circular lamina, the principal axis makes an angle $\tfrac{1}{2}\tan^{-1}(8/3\pi)$ to the diameter.*

Sol. We know that the equation of the circle reffered to O (the extremity of the diameter OX) as pole and OX as the initial line is $r = 2a\cos\theta$ Consider an element $r\delta\theta\delta r$ at P , then M.I. of the lamina O OX is given

by $A = \rho \int_0^{\pi/2} \int_0^{2a\cos\theta} r\, d\theta\, dr (r^2 \sin^2\theta)$



$$= \rho \int_0^{\pi/2} \left(\frac{r^4}{4} \right)_0^{2a\cos\theta} \sin^2\theta\, d\theta$$

$$= \frac{1}{4}\rho (2a)^4 \int_0^{\pi/2} \cos^4\theta \sin^2\theta\, d\theta = \tfrac{1}{3}\pi \rho a^4$$

$B = M.I.$ about $OY = \rho \int_0^{\pi/2} \int^{2a\cos\theta} r\, d\theta\, dr\, (r^2\cos^2\theta)$

$$= \rho \int_0^{\pi/2} \left(\frac{r^4}{4} \right)_0^{2a\cos\theta} \cos^2\theta\, d\theta = \frac{1}{4}\rho\, (2a)^4 \int_0^{\pi/2} \cos^6\theta\, d\theta$$

$$= 4\rho a^4 \frac{5.3.1}{6.4.2} \cdot \frac{\pi}{2} = \frac{5}{3}\pi \rho a^4$$

and $F =$ Product of inertia about OX , OY

$$= \rho \int_0^{\pi/2} \int_0^{2a\cos\theta} r\, d\theta\, dr (r\sin\theta)(r\cos\theta)$$

$$= \frac{1}{4}\rho (2a)^4 \int_0^{\pi/2} \cos^5\theta \sin\theta\, d\theta = 4\rho a^4 \frac{1}{6} = \frac{2}{3}\rho a^4$$

$\Rightarrow \phi = \frac{1}{2}\tan^{-1}\frac{2F}{B-A} = \frac{1}{2}\tan^{-1}(8/3\pi)$ where ϕ is the angle that a principal axis at O makes with OX.

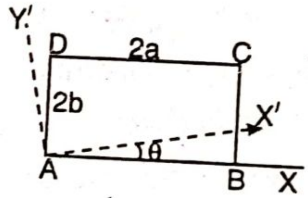
Ex.35. *The lengths AB and AD of the sides of a rectangle $ABCD$ are $2a$ and $2b$. Show that the inclination to AB of one of the principal axes at A is* $\frac{1}{2}\tan^{-1}[3ab/2(a^2-b^2)]$. **[Agra 1989,Delhi Hons.66]**

Sol. Let AB be the axis of x and AD the axis of y and a line through A perpendicular to the plane of the rectangle, the axis of z.

Then $B =$ M.I. of the rectangle about

$AD = \frac{1}{3}Ma^2 + Ma^2$

$= \frac{4}{3}Ma^2$. Similarly $A = \frac{4}{3}Mb^2$



$F =$ product of inertia about $AB, AD = Mab$

Now if θ is the inclination of a principal axis to AB, then we have

$$\tan 2\theta = \frac{2F}{B-A} = \frac{2Mab}{\frac{4}{3}Ma^2 - \frac{4}{3}Mb^2} = \frac{3ab}{2(a^2-b^2)}$$

$$\Rightarrow \theta = \frac{1}{2}\tan^{-1}\left[\frac{3ab}{2(a^2-b^2)}\right].$$

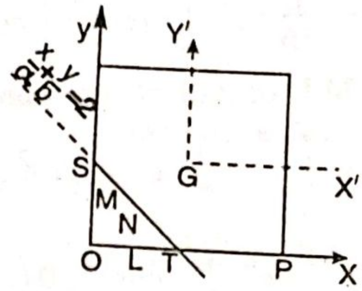
Ex. 36. *A uniform square lamina is bounded by the axes of x and y and the lines $x = 2c$, $y = 2c$ and a corner is out off by the line $x/a + y/b = 2$. Show that the principal axes at the centre of the square are inclined to the axes of x at angles given by*

$$\tan 2\theta = \frac{ab - 2(a+b)c + 3c^2}{(a-b)(a+b-2c)}.$$ **[Lucknow 1990]**

Sol. Let OST be the triangular lamina cut off from the square and let L, M, N be the mid points of its sides. This triangular lamina of mass m can be replaced by three particles each of mass $m/3$ placed at L, M and N. Let ST be the line whose equation is

$(x/a) + (y/b) = 2 \Rightarrow (x/2a) + (y/2b) = 1$

Then $OT = 2a$ and $OS = 2b$, also $OP = 2c$ (given). Let $m_1 =$ mass of the square and $m =$ mass of the triangle OST. Let G be the centre of the square and GX', GY' be the new axes of reference.



With reference to the new axes the co-ordinates of L are $[-(c-a), -c]$, coordinates of M are $[-c, -(c-b)]$ and co-ordinates of N are $[-(c-a), -(c-b)]$

Then $A =$ M.I. of the remaining area about GX'

$=$ M.I. of the whole square - M.I. of the $\Delta\, OST$.

$=$ M.I. of the square $-$ M.I. of the three particles each of mass $\frac{1}{3}m$ placed at the points L, M, N

$$= \frac{1}{3}m_1c^2 - \frac{1}{3}m[c^2 + (c-b)^2 + (c-b)^2]$$

$B =$ M.I. about $GY =$ M.I. of whole square - M.I. of $\Delta\, OST$

$$= \frac{1}{3}mc^2 - \frac{m}{3}[c^2 + (c-a)^2 + (c-a)^2]$$

and $F =$ Product of inertia about GX', GY' of the remaining area

$=$ P.I. of the whole square $-$ P.I. of $\Delta\, OST$

$$= 0 - \frac{1}{3}m\,[(c-a)\,c + c(c-b) + (c-a)(c-b)]$$

$$= -\frac{1}{3}\{3c^2 - 2c\,(a+b) + ab\}$$

$$\therefore \tan 2\theta = \frac{2F}{B-A} = \frac{-\frac{2}{3}\{3c^2 - 2c\,(a+b) + ab\}}{\frac{1}{3}[2\,(c-b)^2 - 2\,(c-a)^2]}$$

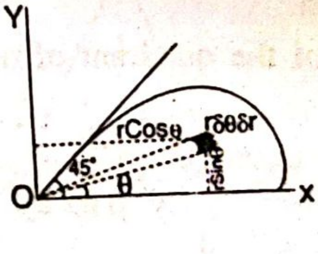
$$= \frac{-\{3c^2 - 2c\,(a+b) + ab\}}{(a-b)(2c-a-c)} = \frac{ab - 2c\,(a+b) + 3c^2}{(a-b)\,(a+b-2c)}.$$

Ex. 37. *Show that the principal axes at the node of a half loop of the leminiscate $r^2 = a^2\cos 2\theta$ are inclined to the initial line at angles* $\frac{1}{2}\tan^{-1}\frac{1}{2}$ and $(\pi/2) + \frac{1}{2}\tan^{-1}\frac{1}{2}$. **[Agra 1986,Delhi Hons.86]**

Sol. The equation of the lemniscate is $r^2 = a^2\cos 2\theta$. Consider an element of area $r\,\delta\theta\;\delta r$ at P.

Then $A =$ M.I. of half of the loop about OX.

$$= \int_0^{\pi/4}\int_0^{a\sqrt{(\cos 2\theta)}} \rho r\,d\theta\,dr(r^2\sin^2\theta)$$



$$= \left(\tfrac{1}{4}\right)\rho a^4 \int_0^{\pi/4} \cos^2 2\theta \sin^2\theta\; d\theta.$$

$$= \left(\tfrac{1}{8}\right)\rho a^4 \int_0^{\pi/4} \cos^2 2\theta\,(1-\cos 2\theta)\; d\theta$$

[Put $2\theta = t$ so that $2d\theta = dt$]

$$\therefore A = \frac{1}{16}\rho a^4 \int_0^{\pi/2} (\cos^2 t - \cos^3 t)\, dt = \frac{\rho a^4}{192}(3\pi - 8)$$

$B =$ M.I. of half of the loop about OY.

$$= \int_0^{\pi/4} \int_0^{a\sqrt{(\cos 2\theta)}} \rho r\, d\theta\, dr . r^2 \cos^2\theta = \frac{1}{4} a^4 \int_0^{\pi/4} \cos^2 2\theta \cos^2\theta\, d\theta$$

$$= \frac{1}{8}\rho a^4 \int_0^{\pi/4} \cos^2 2\theta (1 + \cos 2\theta)\, d\theta \qquad (\text{Putting } 2\theta = t)$$

$$= \frac{1}{16}\rho a^4 \int_0^{\pi/2} (\cos^2 t + \cos^3 t) dt = \frac{\rho a^4}{192}(3\pi + 8)$$

and $F =$ Product of inertia of the half loop about OX, OY

$$= \int_0^{\pi/4} \int_0^{a\sqrt{(\cos 2\theta)}} \rho r\, d\theta\, dr (r \cos\theta)(r \sin\theta)$$

$$= \frac{1}{4}\rho a^4 \int_0^{\pi/4} \cos^2 2\theta \sin\theta . \cos\theta\, d\theta$$

$$= \frac{1}{8}\rho a^4 \int_0^{\pi/4} \cos^2 2\theta . \sin 2\theta\, d\theta = \frac{1}{16}\rho a^4 \int_0^{\pi/2} \cos^2 t \sin t\, dt = \left(\frac{1}{48}\right)\rho a^4$$

If θ is the angle which the principle axis at O makes with OX,

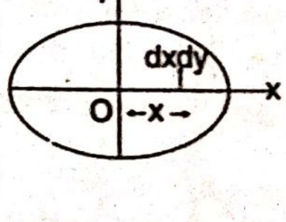
then $\theta = \frac{1}{2}\tan^{-1}\left(\frac{2F}{B - A}\right) = \frac{1}{2}\tan^{-1}\left\{\frac{8}{(3\pi + 8) - (3\pi - 8)}\right\}$

$$= \frac{1}{2}\tan^{-1}\left(\frac{8}{16}\right) = \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right).$$

The other principal axis being at right angles to this principal axis will make an angle $(\pi/2) + \frac{1}{2}\tan^{-1}\left(\frac{1}{2}\right)$ with OX.

Ex.38. *Show that at the centre of a quadrant of an ellipse, the principal axes in its plane are inclined at angle* $\left(\frac{1}{2}\right)\tan^{-1}\left(\frac{4ab}{\pi(a^2 - b^2)}\right)$ *to the axis.*

Sol. Let the equation of the ellipse be $(x^2/a^2) + (y^2/b^2) = 1$. Consider an element $\delta x\, \delta y$ at the point (x, y), then



$A =$ M.I. of the quadrant of an ellipse about OX.

$$= \int_0^a \int_0^{(b/a)\sqrt{(a^2 - x^2)}} y^2 \rho\, dx\, dy$$

$$= \rho \int_0^a \left| \frac{y^3}{3} \right|_0^{(b/a)\sqrt{(a^2 - x^2)}} dx$$

$$= \frac{1}{3}\rho \int_0^a \frac{b^3}{a^3} (a^2 - x^2)^{3/2}\, dx = \frac{1}{3}\frac{\rho b^3}{a^3} \int_0^{\pi/2} a^3 \cos^3\theta \; a\cos\theta \; d\theta$$

[Putting $x = a \sin\theta \Rightarrow dx = a\cos\theta \, d\theta$]

$$= \frac{1}{3}\rho\, ab^3 \, \frac{3.1}{4.2}\cdot\frac{\pi}{2} = \frac{1}{16}\rho a \pi b^3$$

$= \frac{1}{4} M b^2$ where $M =$ mass of the quadrant $= \dfrac{\pi ab \rho}{4}$

Similarly $B =$ M.I. of the quadrant of an ellipse about $OY = \frac{1}{4} Ma^2$

and $F =$ P.I. about OX, OY

$$= \int_0^a \int_0^{(b/a)\sqrt{(a^2 - x^2)}} xy \rho \, dx\, dy = \int_0^a \left| y^2 \right|_0^{(b/a)\sqrt{(a^2 - x^2)}} dx$$

$$= \frac{\rho}{2}\int_0^a x \cdot \frac{b^2}{a^2}(a^2 - x^2)\, dx = \frac{\rho}{2}\cdot\frac{b^2}{a^2}\left[a^2 \cdot \tfrac{1}{2}x^2 - \tfrac{1}{4}x^4 \right]_0^a$$

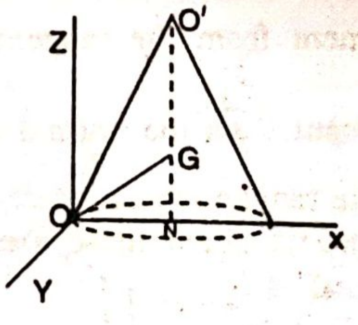
$$= \frac{\rho b^2}{2a^2}\cdot\frac{1}{4}a^4 = \frac{\rho}{8}a^2 b^2 = \frac{1}{2}\left(\frac{1}{4}\pi ab\rho\right)\frac{ab}{\pi} = \frac{Mab}{2\pi}$$

Now if θ be the inclination of principal axis with OX, then

$$\tan 2\theta = \frac{2F}{B - A)} = \frac{4ab}{(a^2 - b^2)\pi} \Rightarrow \theta = \frac{1}{2}\tan^{-1}\left(\frac{4ab}{\pi(a^2 - b^2)}\right)$$

Ex.39. *Find the principal axes of a right circular cone at a point on the circumference of the base, and show that one of them will pass through its C G. if the vertical angle of the cone is* $2\tan^{-1}\left(\frac{1}{2}\right)$.

Sol. We know that in the case of a cone of height h and semi vertical angle α. $A =$ M.I. about $OX = \dfrac{M}{20}(3a^2 + 2h^2)$, where a is the radius of the base,



$B =$ M.I. about $OY = \dfrac{M}{20}(3a^2 + 2h^2)$,

$C =$ *M.I.* about $OZ = \dfrac{13}{10}Ma^2$.

Also

$D = F = 0, E = Mah/4$, since $D = F = 0$; the axis OY as shown in figure is the principal axis at O. Let one of the other two principal axes make an angle θ with OX. Then in that case we have

$$\tan 2\theta = \frac{2E}{C-A} = \frac{\frac{1}{2}Mah}{\frac{13}{10}Ma^2 - \frac{1}{20}M(3a^2+2h^2)} = \frac{10ah}{23a^2-2h^2} \qquad ...(1)$$

Now if the axis OG is to pass through G the C.G. of the cone, then

$$\tan\theta = \frac{GN}{ON} = \frac{(h/4)}{a} = h/(4a)$$

$$\text{Now } \tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = \frac{h/2a}{1-(h^2/16a^2)} = \frac{8ah}{16a^2-h^2} \qquad ...(2)$$

From (1) and (2), we get $\dfrac{10ah}{23a^2-2h^2} = \dfrac{8ah}{16a^2-h^2} \Rightarrow h = 2a$

But $\tan\alpha = a/h = a/2a = \frac{1}{2} \Rightarrow \alpha = \tan^{-1}(\frac{1}{2})$

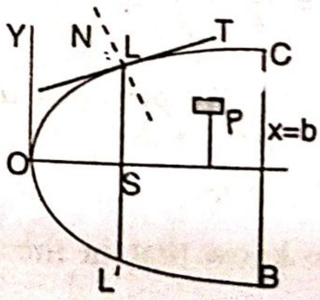
$2\alpha = 2\tan^{-1}(\frac{1}{2})$ [2α being the vertical angle of cone].

Ex.40. *A uniform lamina is bounded by a parabolic are of latus rectum $4a$, and a double ordinates at a distance b from the vertex. If $b = (a/3)(7+4\sqrt{7})$. Show that two of the principal axes at the end of a latus rectum are the tangent and normal there.*

Sol. Let $y^2 = 4ax$ be the parabola and LSL' the latus rectum. The co-ordinates of L are $(a, 2a)$. Now slope of the tangent at

$L = \left(\dfrac{dy}{dx}\right)_{(a,2a)}$. But $\dfrac{dy}{dx} = \sqrt{\left(\dfrac{a}{x}\right)}$

$\therefore$ Slope $= \left(\dfrac{dy}{dx}\right)_{(a,2a)} = \sqrt{\left(\dfrac{a}{a}\right)} = 1$



$\therefore$ Equation of the tangent at L is given by

$y - 2a = 1(x-a) \Rightarrow y - x - a = 0$

Again slope of the normal at $L = -1$

$\therefore$ Equation of the normal is

$y - 2a = -1(x-a) \Rightarrow y + x - 3a = 0$

Consider an element $\delta x\, \delta y$ at the point $P \equiv (x, y)$, then distance of the element from the tangent at $L = \dfrac{y-x-a}{\sqrt{2}} = PK$ and distance of the element from the normal at $L = \dfrac{y+x-3a}{\sqrt{2}} = PH$.

If the tangent and normal at L are the principal axes, then the product of inertia about these axes vanish. Now P. I. about the tangent and normal at L.

$$= \int_0^b \int_{-2\sqrt{(ax)}}^{2\sqrt{(ax)}} \left(\frac{y+x-a}{\sqrt{2}}\right) \cdot \left(\frac{y-x-3a}{\sqrt{2}}\right) dx\, dy$$

$$= \frac{1}{2}\int_0^b \int_{-2\sqrt{(ax)}}^{2\sqrt{(ax)}} (y^2 - x^2 + 2ax - 4ay + 3a^2)\, dx\, dy$$

$$= \int_0^b \int_0^{2\sqrt{(ax)}} (y^2 - x^2 + 2ax + 3a^2)\, dx\, dy - \frac{1}{2}\int_0^b \int_{-2\sqrt{(ax)}}^{2\sqrt{(ax)}} 4ay\, dx\, dy$$

$$= \int_0^b \left\{ \left| (2ax - x^2 + 3a^2) y \right|_0^{2\sqrt{(ax)}} + \left| \frac{y^3}{3} \right|_0^{2\sqrt{(ax)}} \right\} dx - 0$$

$$= \int_0^b \left[\frac{1}{3} \{2\sqrt{(ax)}\}^3 + (3a^2 + 2ax - x^2)\, 2\sqrt{(ax)}\, \right] dx$$

$$= \int_0^b \left[\frac{8}{3} a^{3/2} x^{3/2} + 4a^{3/2} x^{3/2} + 6a^{5/2} x^{1/2} - 2a^{1/2} x^{5/2}\right] dx$$

$$= \frac{4}{21} a^{1/2} b^{3/2} (14ab + 21a^2 - 3b^2)$$

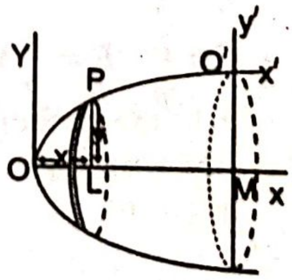
If this P.I. is zero then, we have $21a^2 + 14ab - 3b^2 = 0$

$$\Rightarrow 3b^2 - 14ab - 21a^2 = 0 \Rightarrow b = \frac{14a \pm \sqrt{[196a^2 + 12 \times 21a^2]}}{6}$$

$= \frac{1}{3} a\, (7 + 4\sqrt{7})$, which is equal to the given value of b.

Ex. 41. *The length of the axis of a solid parabola of revolution is equal to the latus rectum of the generating parabola. Prove that one principal axis at a point on the circular rim meets the axis of revolution at an angle* $\frac{1}{2}\tan^{-1}\left(\frac{2}{3}\right)$.

Sol. Let the equation of the generating parabola be $y^2 = 4ax$.
Length of the axis $OM = 4a$ given.
Let O' be a point on the circular rim. Also let $O'X'$ and $O'Y'$ the new axes. Now Consider a circular disc of breath δx at a distance x from the vertex O' then



$A =$ M.I. about $O'X'$

$$= \int_0^{4a} (\pi y^2 \rho dx)\, (y^2/2 + O'M^2)$$

Evidently

$O'M^2 = 4a \,.\, 4a$ (giving) $O'M = 4a$

$$\therefore A = \pi\rho \int_0^{4a} 4ax\,[\,2ax + (4a^2)\,]dx$$

$$= 8a^2\pi\rho\left[\,x^3/3 + 8a\,x^2/2\,\right]_0^{4a} \quad = \tfrac{1}{3}.\,8.64.4a^5\pi\rho$$

$B =$ M.I. about $O'Y'$

$$= \int_0^{4a} \pi y^2\rho\, dx\left[\frac{y^2}{4} + LM^2\right] = \pi\rho\int_0^{4a} 4ax\,[ax + (4a - x)^2\,]dx$$

$$= 4a\pi\rho \int_0^{4a} (16a^2x^2 - 7ax^2 + x^3)dx$$

$$= 4a\pi\rho\left[\,8a^2x^2 - (7/3)\,ax^3 + \tfrac{1}{4}x^4\right]_0^{4a} = (2/3)\times 4\times 64a^5\pi\,\rho$$

and $F =$ product of inertia about $O'X'$ and $O'Y'$

$$= \int_0^{4a} \pi y^2\rho d\,x\,LM.O'M = \pi\rho\int_0^{4a} 4ax\,(4a - x)\,4a dx$$

$$= 16a^2\pi\rho\int_0^{4a} (4ax - x^2)\,dx = \tfrac{1}{3}\times 16\times 32a^5\pi\rho$$

Let the principal axis at O' make an angle ϕ with $O'X'$, then we have

$$\tan 2\phi = \frac{2F}{B - A} = \frac{2\times\frac{1}{3}\times 16\times 32a^5\pi\rho}{\frac{2}{3}\times 4\times 64\times a^6\pi\rho - \frac{1}{3}\times 8\times 64\times 4a^5\pi\rho} = -\frac{2}{3}$$

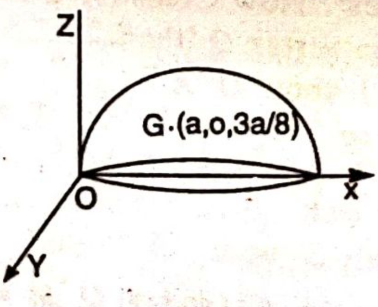
$\Rightarrow \phi = \frac{1}{2}\tan^{-1}\left(\frac{2}{3}\right)$ (numerically).

Ex. 42. *Show that one of the principal axes at a point on the circular rim of the solid hemisphere is inclined at an angle* $\tan^{-1}\left(\frac{1}{3}\right)$ *to the radius through the point.*

Sol. Let O be the point on the base of the solid hemisphere. Let OX, OY, OZ be the axes of reference, then we know in this case

$A = \frac{2}{5}Ma^2,\ B = \frac{7}{5}Ma^2,$

$C = \frac{7}{5}Ma^2, D = F = 0,\ E = \frac{3}{8}Ma^2$



If θ is angle which the principal axis makes with OX; then we have

$$\tan 2\theta = \frac{2E}{C - A}$$

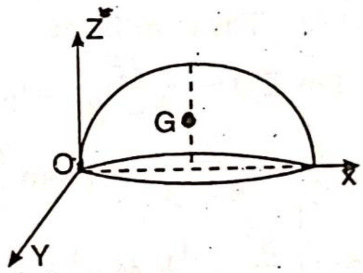
$$= \frac{(\frac{3}{4})\,Ma^2}{(\frac{7}{5})\,Ma^2 - (\frac{2}{5})\,Ma^2} = \frac{3}{4}$$

$$\Rightarrow \frac{2\tan\theta}{1-\tan^2\theta} = \frac{3}{4} \Rightarrow (\tan\theta + 3)(3\tan\theta - 1) = 0$$

$\Rightarrow \tan\theta = (\frac{1}{3})$, the other value of θ being inadamissible.

Ex. 43. *Show that one of the principal axes at any point on the edge of the circular base of a thin hemispherical shell is inclined at an angle, $\pi/8$ to the radius through the point .*

Sol. Let O be the point on the circular rim of the hemispherical shell. Let OX, OY, OZ as shown in the fig. be the axes of reference , then we have



$A =$ M.I. about $OX = \frac{2}{3} ma^2$

$B =$ M.I. about $OY =$ M.I. about a line through the C.G. parallel to OY

$$= \frac{2}{3} Ma^2 + Ma^2 = \frac{5}{3} Ma^2 .$$

Similarly $C = \frac{5}{3} Ma^2$. If G is the C.G. of the shell then co-ordinates of G are $(a, 0, a/2)$. Then we have .

$D = F = 0$ and $E = Ma\ \frac{1}{2} a = \frac{1}{2} Ma^2$

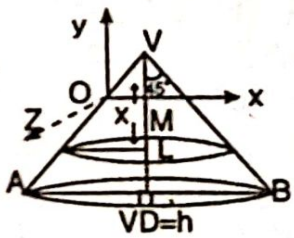
Now if θ is the angle which the principal axis makes with OX then we have

$$\tan 2\theta = \frac{2F}{C-A} = \frac{Ma^2}{(\frac{5}{3})\,Ma^2 - (\frac{2}{3})\,Ma^2} = 1 = \tan(\pi/4)$$

$\therefore 2\theta = (\pi/4) \Rightarrow \theta = (\pi/8)$.

Ex. 44. *If the vertical angle of the cone is 90°. The point at which a generator is a principal axis divides the generator in the ratio 3 : 7 .*

Sol. Consider $\Delta\ VAB$ as the section of the cone through the generator VA and *axis* VD of the cone. In this plane section let OX and OY be the axes of x and y and a line OZ perpendicular to this section as *z-axis*. Obviously the *z*-coordinates of C.G. is zero. $D = E = 0 \Rightarrow$ *z-axis* is a principal axis at O .



Now M.I. of the cone about OZ

$$= \int_0^h \pi x^2 \rho\, [\tfrac{1}{4}x^2 + ML^2]\, dx$$

$$= \int_0^h \pi x^2 \rho\, [\tfrac{1}{4}x^2 + (x - VM)^2\, dx$$

$$= \pi\rho \int_0^h \left[\tfrac{5}{4}x^4 - 2VMx^3 + VM^2x^2\right] dx$$

$$= \pi\rho\left[\frac{h^5}{4} - \tfrac{1}{2}VMh^4 + \tfrac{1}{3}VM^2h^3\right] = A \text{ (say)}$$

and M.I. about $OY = \int_0^h \pi x^2 \rho \left[\frac{x^2}{2} + OM^2\right] dx$

$$= \pi\rho \int_0^h \left[\frac{x^4}{2} + VM^2x^2\right] dx \qquad \left[\because \frac{OM}{VM} = \tan \pi/4 = 1\right]$$

$$= \pi\rho\left[\frac{1}{10}h^5 + \frac{1}{3}VM^2h^3\right] = B \text{ (say)}$$

But VA is principal axis and it makes an angle $\frac{1}{4}\pi$ with *x-axis* at the point O . Thus we have

$$\tan 2\theta = \frac{2F}{B-A} \Rightarrow \tan 2.\frac{\pi}{4} = \frac{2F}{B-A}$$

$$\Rightarrow \tan\frac{\pi}{2} = \frac{2F}{B-A} \text{ but } \tan\frac{\pi}{2} = \infty \therefore B - A = 0 \Rightarrow A = B$$

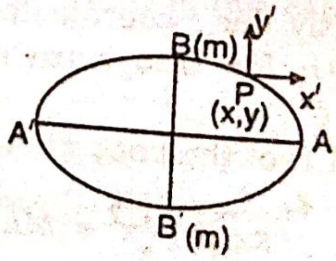
$$\therefore \pi\rho\left[\frac{h^5}{4} - \tfrac{1}{2}VMh^4 + \tfrac{1}{3}VM^2h^3\right] = \pi\rho\left[\tfrac{1}{10}h^5 + \tfrac{1}{2}VM^2h^3\right]$$

$\Rightarrow \frac{VM}{h} = \frac{3}{10}$. By similarity of triangles , we have

$$\frac{VO}{VA} = \frac{3}{10} \text{ or } \frac{VO}{VA - VO} = \frac{3}{7}, \therefore \frac{VO}{OA} = \frac{3}{7}.$$

Ex.45. *Two particles each of mass m are placed at the extremities of the minor axis of an elliptic area of mass M. Prove that principal axes at any point of the circumference of the ellipse will be the tangent and normal to the ellipse if* $\frac{m}{M} = \frac{5}{8}\cdot\frac{e^2}{1-2e^2}$

Sol. Let P be a point on the circumference of the ellipse $x^2/a^2 + y^2/b^2 = 1$ whose co-ordinates are (x, y) . Let PX' and PY' be a set of parallel axes through P. Let particles, each of mass m be placed at the extremities B and B' . of the minor axis.



$A =$ moment of the inertia of the elliptic lamina and the two particles about PX'

$$= M\left(\tfrac{1}{4}b^2 + y^2\right) + m\{(b-y)^2 + (b+y)^2\}$$

$$= \tfrac{1}{4}M\left(b^2 + 4y^2\right) + 2m\left(b^2 + y^2\right)$$

B = M.I. of the elliptic lamina and the two particles about PY'

$$= M\left(\tfrac{1}{4}a^2 + x^2\right) + m\left(x^2 + x^2\right) = \tfrac{1}{4}M\left(a^2 + 4x^2\right) + 2mx^2$$

F = Product of inertia about PX' and PY'

$$= M(-x)(-y) + m\{(-x)(b-y) + (-x)(-b-y)\}$$

$$= (M + 2m)xy.$$

$$\therefore \tan 2\theta = \frac{2F}{B - A}$$

$$= \frac{2(M+2m)xy}{\{(M/4)(a^2 + 4x^2) + 2mx^2\} - \{(M/4)b^2 + 4y^2) + 2m(b^2 + y^2)\}}$$

$$= \frac{8(M+2m)xy}{M(4x^2 - 4y^2 + a^2 - b^2) + 8m(x^2 - y^2 - b^2)} \qquad \text{...(1)}$$

Now equation of the tangent at P is $\dfrac{Xx}{a^2} + \dfrac{Yy}{b^2} = 1$

$\therefore$ Slope of the tangent $= -\dfrac{b^2x}{a^2y}$.

If the tangent at P is the principal axis, then

$\tan\theta = -(b^2x/a^2y)$

$$\Rightarrow \tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta} = \frac{-(2b^2x/a^2y)}{1 - (b^4x^2/a^4y^2)} = \frac{-2xya^2b^2}{a^4y^2 - b^4x^2} \qquad \text{...(2)}$$

Hence (1) and (2) $\Rightarrow \dfrac{8(M + 2m)xy}{M(4x^2 - 4y^2 + a^2 - b^2) + 8m(x^2 - y^2 - b^2)}$

$$= \frac{-2xya^2b^2}{a^4y^2 - b^4x^2} \Rightarrow 5(a^2 - b^2)M = 8m(2b^2 - a^2)$$

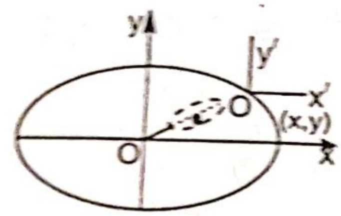
$$\Rightarrow \frac{m}{M} = \frac{5}{8}\cdot\frac{a^2 - b^2}{2b^2 - a^2} = \frac{5}{8}\cdot\frac{e^2}{1 - 2e^2}$$

Ex. 46. *A uniform lamina bounded by the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ has an elliptic hole (semi-axes c, d) in it whose major axis lies in the line $x = y$, the centre being at a distance r from the origin, prove that if one of the principal axis at the point (x,y) makes an angle θ with x-axis then*

$$\tan 2\theta = \frac{8abxy - cd\,\{4(x\sqrt{2} - r)(y\sqrt{2} - r) - (c^2 - d^2)\}}{ab\,\{4(x^2 - y^2) + a^2 - b^2\} - 2cd\,\{(x\sqrt{2} - r)^2 - (y\sqrt{2} - r)^2\}}$$

Sol. Consider C to be the centre of the elliptic hole whose major axis lies along the line $y = x$ i.e. $y = (\tan 45)\,x$. Let O' be any point (x, y) and let $O'X'$, $O'y'$ be the set of parallel axes through o' then

$oc = r$ and $\angle COX = \pi/4$



Mass of the elliptic plate $= \pi ab\,\rho$.

Its M.I. about $O'X'$

$$= \pi ab\,\rho\,(b^2/4 + y^2).$$

Mass of the elliptic hole $= \pi c\,d\,\rho$

($\because$ its semi axes are c and d)

Its M.I. about $O'X'$

$$= \pi cd\rho \int \left\{ (d^2/4)\cos^2 45 + (c^2/4)\sin^2 45 + \left(y - \frac{r}{\sqrt{2}}\right)^2 \right\}$$

$\therefore A =$ M.I. of the remainder about $O'X'$

$$= \pi\rho ab\left(\frac{b^2}{4} + y^2\right) - \frac{1}{8}\pi\,\rho cd\,\{d^2 + c^2 + 4(\sqrt{2}y - r)^2\} \qquad ...(1)$$

Now M.I. of elliptic plate about $O'Y' = \pi ab\,\rho\left(\frac{a^2}{4} + x^2\right)$

M.I. of the remainder hole about $O'Y'$

$$= \pi\rho ab\left[\frac{d^2}{4}\sin^2 45 + \frac{c^2}{4}\cos^2 45 + \left(x - \frac{r}{\sqrt{2}}\right)^2\right]$$

M.I. of the remainder about $O'Y'$

$$= \pi\rho ab\left[\frac{a^2}{4} + x^2\right] - \frac{1}{8}\pi\rho cd\,\{d^2 + c^2 + 4(\sqrt{2}x - r)^2\} = B \text{ (say)}$$

Further product of inertia of the elliptic plate about $(O'X', O'Y') = \pi ab\,\rho xy$

and product of inertia of the elliptic hole about $O'X', O'Y'$

$$= \pi cd\rho\left\{\frac{1}{2}\,\frac{d^2 - c^2}{4}\sin 90 + \left(x - \frac{r}{\sqrt{2}}\right)\left(x - \frac{r}{\sqrt{2}}\right)\right\}$$

$$= \frac{1}{8}\pi cd\,\rho\,\{d^2 - c^2 + 4(x\sqrt{2} - r)(y\sqrt{2} - r)\}$$

$\therefore$ P.I. of the remainder about $O'X', O'Y'$

$$= \pi ab\,\rho\,xy - \frac{1}{8}\pi cd\,\rho\,\{d^2 - c^2 + 4(x\sqrt{2} - r)(y\sqrt{2} - r)\} = F \text{ (say).}$$

Let θ be the angle, which the principal axis makes with *x-axis* then we have $\tan 2\theta = \dfrac{2F}{B - A}$

$$= \frac{8ab\,xy - cd\,\{4(x\sqrt{2} - r)(y\sqrt{2} - r) - (c^2 - d^2)\}}{ab\,\{4(x^2 - y^2) + a^2 - b^2\} - cd\,\{2(x\sqrt{2} - r)^2 - 2(y\sqrt{2} - r)^2\}}$$

0·16. Principal Moments : *Moments of inertia of any body about its principal axes at any point are called its principal moments at that point.* The equation of ellipsoid at any point is given by

$Ax^2 + By^2 + Cz^2 - 2Dyz - 2Ezx - 2Fxy = MK^4$

This equation when referred to principal axes as co-ordinate axes takes the from as $A'x^2 + B'y^2 + C'z^2 = MK^4$, where A' B', C' are principal moments and are the roots of the cubic-equation given below :

$$\begin{vmatrix} A-\lambda & H & G \\ H & B-\lambda & F \\ G & F & C-\lambda \end{vmatrix} = 0. \qquad \textbf{Reduction Cubic.}$$

Ex.47. *Three rods AB, BC, CD each of mass m and length $2a$ are such that each is perpendicular to the other two. Show that the principal moments of inertia at the centre of mass are ma^2, $\frac{11}{3}ma^2$ and $4ma^2$.*

Sol. Draw a line BY, parallel to CD and let BA, BY and BC be the axes of x, y and z respectively.

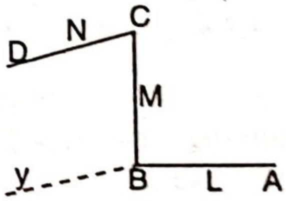
Let L, M, N be the mid points of the rod. Then their co-ordinates are given by $(a, 0, 0)$, $(0, 0, a)$, $(0, a, 2a)$.

Further let $(\bar{x}, \bar{y}, \bar{z})$ be the co-ordinates of $C.G.$ 'G' of the three rods then we easily obtain

$$\bar{x} = \frac{m.a + m.0 + m.0}{m + m + m} = \frac{a}{3}, \bar{y} = \frac{m.0 + m.0 + m.a}{m + m + m} = \frac{a}{3},$$

$$\bar{z} = \frac{m.0 + m.a + m.2a}{m + m + m} = a.$$

i.e. $(\bar{x}, \bar{y}, \bar{z}) = (a/3, a/3, a)$.



Now GX, GY, GZ be the set oi parallel axes through G, then the new co-ordinates of L, M, N referred to G as origin are $(2a/3, -a/3, -a)$, $(-a/3, -a/3, 0)$, $(-a/3, 2a/3, a)$

$\therefore$ M.I. of AB about GX

$$= m\left[\left(\frac{-a}{3}\right)^2 + (-a)^2\right] = \frac{10}{9}ma^2$$

M.I. of BC about $GX = \frac{1}{3}ma^2 + m\left[\left(\frac{-a}{3}\right)^2 + 0^2\right] = \frac{4}{9}ma^2$,

M.I. of CD about $GX = \frac{1}{3}ma^2 + m\,[(\frac{2}{3}a)^2 + a^2] = \frac{16ma^2}{9}$

$\Rightarrow$ M.I. of three rods about $GX = ma^2\left[\frac{16}{9} + \frac{4}{9} + \frac{10}{9}\right] = \frac{10}{3}ma^2$

$= A_1$, say .

M.I. of the three rods about GY

$$= \frac{1}{3}ma^2 + m[(\tfrac{2}{3}a)^2 + (-a)^2] + \frac{1}{3}ma^2 + m[(-a/3)^2 + 0^2]$$
$$+ m[(-a/3)^2 + a^2]$$

$= \frac{10}{3} ma^2 = B_1$, say .

Similarly M.I. of the three rods about GZ

$$= \frac{1}{3} ma^2 + [(2a/3)^2 + (a/3)^2] + m[(-a/3)^2 + (-a/3)^2] + \frac{1}{3} ma^2$$

$$+ m[(-a/3)^2 + (2a/3)^2]$$

$= 2ma^2 = G$, say. Now if D_1, E_1, F_1 are products of inertia , about the parallel axes through G, then we have

$$D_1 = \Sigma m y_1 z_1 = m(-a)(-a/3) + m(-a/3)\,0 + m(\tfrac{2}{3}a)\,a = -ma^2$$

$$E_1 = \Sigma m z_1 x_1 = m(-a)(\tfrac{2}{3}a) + m(0)(-a/3) + m(a)(-a/3) = -ma^2$$

$$F = m(\tfrac{2}{3}a)(-a/3) + m(-a/3)(-a/3) + m(-a/3)(2a/3) = -\frac{1}{3}ma^2$$

$$(\because\ F_1 = \Sigma m x_1 y_1)$$

Hence the momental ellipsoid at G is given by

$$\frac{10}{3} ma^2x^2 + \frac{10}{3} ma^2y^2 + 2ma^2z^2 - 2ma^2\,yz + 2ma^2\,zx + 2.\frac{1}{3}\,ma^2xy$$

$$= 3mK^4$$

$$\Rightarrow \frac{1}{3} ma^2 (10x^2 + 10y^2 + 6z^2 - 6yz + 6zx + 2xy) = 3mK^4$$

Whence the discrimininiating cubic is

$$\lambda^3 - (a + b + c)\lambda^2 + (ab + bc + ca - f^2 - g^2 - h^2)\lambda$$

$$-(abc + 2fgh - af^2 - bg^2 - ch^2) = 0.$$

Reducing $10x^2 + 10y^2 + 6z^2 - 6yz + 6zx + 2xy$ by means of reducing cubic , we get

$$\lambda^3 - (10 + 10 + 6)\lambda^2 + \lambda(100 + 60 + 60 - 9 - 9 - 1) - 396 = 0$$

$$\Rightarrow \lambda^3 - 26\lambda^2 + 201\lambda - 396 = 0$$

$$\Rightarrow (\lambda - 3)\lambda - 11(\lambda - 12) = 0$$

$\Rightarrow \lambda = 3, 11, 12$. Hence the equation of the momental ellipsoid referred to principal axes through G is $\frac{1}{3} ma^2 (3x^2 + 11y^2 + 12z^2) = 3mK^4$

$$\Rightarrow ma^2x^2 + \frac{11}{3} ma^2y^2 + 4ma^2z^2 = 2mK^4 = \text{Constant.}$$

Or in other words , the principal moments are the coefficients of x^2, y^2, z^2 *i.e.* $ma^2, \frac{11}{3}ma^2, 4ma^2$

Ex.48. *Show that for a thin hemispherical shell of radius a and mass M, the principal moments of inertia at the centre of gravity are*

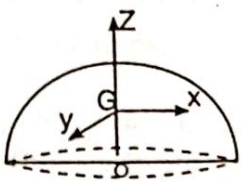
$$\frac{5}{12}Ma^2,\ \frac{5}{12}Ma^2,\ \frac{2}{3}Ma^2.$$

Sol. Consider GX, GY and GZ as co-ordinate axes where G is the C.G.of the body. Now M.I. of the hemispherical shell

about $GX = \frac{2}{3}Ma^2 - M(GO)^2$

$$= \frac{2}{3}Ma^2 - \frac{Ma^2}{4} = \frac{5}{12}Ma^2 = A \text{ (say)}$$

($\because$ G bisects the central radius OZ)



Similarly M.I. about $GY = \frac{5}{12}Ma^2 = B$ (say)

and M.I. about $GZ = \frac{2}{3}Ma^2 = C$ (say)

Again $D = E = F = O \Rightarrow$ that GX, GY, GZ are principal axes at G. So the principal moments at the C.G. are

$$\frac{5}{12}Ma^2, \frac{5}{12}Ma^2, \frac{2}{3}Ma^2.$$

Ex.49. *The principal axes at the C.G. being the axes of reference, obtain the equation of the ellipsoid at the point (p, q, r) and show that the principal M.I., at this point are the roots of*

$$\begin{vmatrix} \{(I-A)/M\} - q^2 - r^2 & pq & rp \\ pq & \{(I-B)/M\} - r^2 - p^2 & qr \\ rp & qr & \{(I-C)/M\} - p^2 - q^2 \end{vmatrix} = 0$$

where I, M, A, B, C have their usual meanings.

Sol. Let A', B', C' be the moments of inertia about parallel axes through (p, q, r) and D', E', F', the products of inertia about them, then we easily obtain

$A' = A + M(q^2 + r^2)$, $B' = B + M(r^2 + p^2)$, $C' = C + M(p^2 + q^2)$,

$D' = D + Mqr = Mqr$, $E' = Mrp$, $F' = Mpq$

$\therefore$ Equation of the momental ellipsoid at (p, q, r) will be

$$A'x^2 + B'y^2 + C'z^2 - 2D'yz - 2E'zx - 2F'xy = \text{Constant}.$$

$$\Rightarrow \Sigma[\{(A/M) + q^2 + r^2\}x^2 - 2\Sigma\, qryz = K^4 \text{ (say)} \qquad \text{....(1)}$$

Now if (l, m, n) are the direction cosines of one of the principal axes of the momental ellipsoid (1) and R the length of principal axis then the equation of the tangent plane at the end of the principal axis will be $lx + my + nz = R$. ...(2)

If I is the M.I. about that principal axis then $I = (MK^4/R)$. ...(3)

Co-ordinates of the end of the principal axis will be given by (lR, mR, nR)

Equation of tangent plane to (1) at the end of the principal axis is also given by $x\dfrac{\partial F}{\partial \alpha}+ y \dfrac{\partial F}{\partial \beta}+ z \dfrac{\partial F}{\partial \gamma}+ t \dfrac{\partial F}{\partial t}= 0$

where $\alpha= lR,\ \beta= mR,\ \lambda= nR$

$\Rightarrow x\,[\{(A/M)+q^2+r^2\}\,lR-pqmR-pmR\,]+y[\,\{(B/M)+r^2+p^2\}mR$
$-qmR-pq\,lR\,]+z\,[\{(C/M)+p^2+q^2\}nR-rplR-qrmR\,]=K^4$ ...(4)

Equation (4) and (2) represent the same tangent plane, therefore comparing the coeffcients of x, y, z and constant term, we get

$$\frac{\{(A/M)+q^2+r^2\}\,lR-pqmR-pmR}{l}$$
$$=\frac{\{(B/M)+r^2+p^2\}\,mR-pq\,lR-qmR}{m}$$
$$=\frac{\{(C/M)+p^2+q^2\}\,nR-rplR-qrmR}{n}=\frac{K^4}{R}=\frac{RI}{M}$$

$$\therefore\ \left.\begin{aligned} lR\,[\{(I-A)/M\}-q^2-r^2]+mR\,.\,pq+nR\,.\,pr&=0\\ lR\,.\,pq+mR\,[\{(I-B)/M\}-r^2-p^2]-nR\,.\,qr&=0\\ lR\,.\,rp+mR\,.\,qr+nR\,[\{(I-C)/M\}-p^2-q^2]&=0\end{aligned}\right\}$$

Eliminating lR, mR, nR from the equation (5) we get

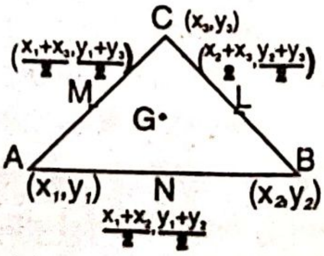
$$\begin{vmatrix}\{(I-A)/M\}-q^2-r^2 & pq & rp\\ pq & \{(I-B)/M\}-r^2-p^2 & qr\\ rp & qr & \{I-C)/M\}-p^2-q^2\end{vmatrix}=0.$$

Ex.50. *Prove that the principal radii of gyration at the C.G. of a triangle are the roots of the equation* $x^4-\dfrac{a^2+b^2+c^2}{30}x^2+\dfrac{\Delta^2}{108}=0,$ *where* Δ *is area of the triangle.*

Sol. Let G be the $C.G.$ of the $\Delta\ ABC$ and let the principal axes through G be taken as the co-ordinate axes.

Let the co-ordinates of A, B, C be (x_1, y_1); (x_2, y_2); (x_3, y_3); respectively then the co-ordinates of the mid point L, M, N, are

$$\left(\frac{x_2+x_3}{2},\frac{y_2+y_3}{2}\right),\left(\frac{x_3+x_1}{2},\frac{y_3+y_1}{2}\right),$$
$$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right).$$



Now co-ordinates of G are also given as $\left(\dfrac{x_1+x_2+x_3}{3},\dfrac{y_1+y_2+y_3}{3}\right)$ and they are evidently $(0, 0)$.

$\therefore x_1 + x_2 + x_3 = 0,\ y_1 + y_2 + y_3 = 0$ ...(1)

$\Rightarrow (x_1 + x_2 + x_3)^2 = 0 \Rightarrow x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_2x_3 + 2x_3x_1 = 0$

$\Rightarrow x_1^2 + x_2^2 + x_3^2 = -2(x_1x_2 + x_2x_3 + x_3x_1)$

Similarly we get $y_1^2 + y_2^2 + y_3^2 = -2(y_1y_2 + y_2y_3 + y_3y_1)$

The length of the side $AB = c = \{(x_1 - x_2)^2 + (y_1 - y_2)^2\}$

or $c^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2$

$$\text{Similarly} \quad \left.\begin{aligned} a^2 &= (x_2 - x_3)^2 + (y_2 - y_3)^2 \\ b^2 &= (x_3 - x_1)^2 + (y_3 - y_1)^2 \end{aligned}\right\} \qquad ...(2)$$

$$\therefore a^2 + b^2 + c^2 = 2(x_1^2 + x_2^2 + x_3^2) + 2(y_1^2 + y_2^2 + y_3^2) - 2(x_1x_2 + x_2x_3 + x_3x_1) - 2(y_1y_2 + y_2y_3 + y_3y_1)$$

$$= 3(x_1^2 + x_2^2 + x_3^2 + y_1^2 + y_2^2 + y_3^2)$$

Again a triangular lamina of mass m is equimomental to three particles of mass $(m/3)$ each placed at the mid points L, M, N.

$\therefore A =$ M.I. of three particles about *x-axis*

$$= \frac{m}{3}\left[\left(\frac{y_1 + y_2}{2}\right)^2 + \left(\frac{y_2 + y_3}{2}\right)^2 + \left(\frac{y_3 + y_1}{2}\right)^2\right]$$

$$= \frac{m}{12}[2(y_1^2 + y_2^2 + y_3^2) + 2(y_1y_2 + y_2y_3 + y_3y_1)] = \frac{m}{12}(y_1^2 + y_2^2 + y_3^2)$$

$B =$ M.I. about y-axis

$$= \frac{m}{3}\left[\left(\frac{x_1 + x_2}{2}\right)^2 + \left(\frac{x_2 + x_3}{2}\right)^2 + \left(\frac{x_3 + x_1}{2}\right)^2\right]$$

$$= \frac{m}{12}[2(x_1^2 + x_2^2 + x_3^2) + 2(x_1x_2 + x_2x_3 + x_3x_1)] = \frac{m}{12}(x_1^2 + x_2^2 + x_3^2).$$

Further we have taken the co-ordinate axes as the principal axes, therefore the P.I. about these axes is zero *i.e.*

$$\frac{m}{3}\left[\frac{x_1 + x_2}{2} \cdot \frac{y_1 + y_2}{2} + \frac{x_2 + x_3}{2} \cdot \frac{y_2 + y_3}{2} + \frac{x_3 + x_1}{2} \cdot \frac{y_3 + y_1}{2}\right] = 0$$

$\Rightarrow (x_1 + x_2)(y_1 + y_2) + (x_2 + x_3)(y_2 + y_3) + (x_3 + x_1)(y_3 + y_1) = 0$

$\Rightarrow 2(x_1y_1 + x_2y_2 + x_3y_3) + x_1(y_2 + y_3) + x_2(y_3 + y_1) + x_3(y_1 + y_2) = 0$

$\Rightarrow x_1y_1 + x_2y_2 + x_3y_3 + (y_1 + y_2 + y_3)(x_1 + x_2 + x_3) = 0$

$\Rightarrow x_1y_1 + x_2y_2 + x_3y_3 = 0$ ...(3)

Now suppose $A = mK_1^2, B = mK_2^2$, then we have

$$m(K_1^2 + K_2^2) = A + B$$

$$A + B = \frac{m}{12}[(x_1^2 + x_2^2 + x_3^2) + (y_1^2 + y_2^2 + y_3^2)]$$

$$= \frac{m}{12} \cdot \frac{a^2+b^2+c^2}{3} = \frac{m(a^2+b^2+c^2)}{36}$$

Again $m^2 K_1^2 K_2^2 = A \times B = \frac{m^2}{144} (x_1^2 + x_2^2 + x_3^2)(y_1^2 + y_2^2 + y_3^2)$

$$= \frac{m^2}{144} [x_1^2 y_1^2 + x_2^2 y_2^2 + x_3^2 y_3^2 + x_1^2 (y_2^2 + y_3^2) + x_2^2 (y_3^2 + y_1^2) + x_3^2 (y_1^2 + y_2^2)]$$

$$= \frac{m}{144} [(x_1 y_1 + x_2 y_2 + x_3 y_3)^2 + (x_1 y_2 - x_2 y_1)^2 + (x_1 y_3 - x_3 y_1)^2 + (x_2 y_3 - x_3 y_2)^2]$$

$$= \frac{m^2}{144} [(x_1 y_2 - x_2 y_1)^2 + (x_2 y_3 - x_3 y_2)^2 + (x_3 y_1 - x_1 y_3)^2] \quad \text{[Using (3)]}$$

$$= \frac{m^2}{144} \left[\frac{4\Delta^2}{9} + \frac{4\Delta^2}{9} + \frac{4\Delta^2}{9} \right] = \frac{m^2 \Delta^2}{108}$$

$[\Delta = \frac{1}{2} \{x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2)\} = \frac{3}{2}(x_1 y_2 - x_2 y_2)$, when $x_1 + x_2 + x_3 = 0$ and $y_1 + y_2 + y_3 = 0]$

Let K_1^2, K_2^2 are the roots of the equation $x^4 - (K_1^2 + K_2^2) x^2 + K_1^2 \times K_2^2 = 0$

$$\Rightarrow x^4 - \frac{a^2+b^2+c^2}{36} x^2 + \frac{\Delta^2}{108} = 0$$

Supplementary Problems

1. Show that the moment of inertia of a semi-circular lamina about a tangent parallel to the bounding diameter is $Ma^2\left(\frac{5}{3} - \frac{8}{3}\pi\right)$, where a is the radius and M is the mass of the lamina.

2. Show that the M.I. of the paraboloid of revolution about its axis is $\frac{M}{3} \times$ the square of the radius of its base.

3. Find the product of inertia of a semi-circular wire about diameter and tangent at its extremity.

4. Find the moment of inertia of a truncoted cone about its axis, the radii of its ends being a and b.

5. Show that the moment of inertia of a parabolic area (of latus rectum 4a) cut off by an ordinate of distance h from the vertex is $\frac{3}{7}Mh^2$ about the tangent at the vertex and $\frac{4}{5}Mah$ about the axis.

6. Show that M.I. of a rectangle of mass M and sides $2a, 2b$ about a diagonal is

$$\frac{2M}{3} \cdot \frac{a^2 b^2}{a^2 + b^2}.$$

7. A closed shell of total mass M, made of thin uniform sheet metal is in the form of a right circular cone of slant height l and base radius **r**. Prove that M.I. of the shell about its axis of symmetry is $\frac{1}{2}Mr^2$ and that about a line through the vertex perpendicular to the axis is $\frac{M}{4}(2l^2 + 2rl - 3r^2)$

8. Find the momenental ellipsoid at any point O of a material straight rod PQ *of mass* M and *length* $2a$.

9. $ABCD$ is a uniform parallelogram of mass **m**. At the middle points of the four sides are placed particles, each equal to $\frac{m}{6}$ and at the intersection of the diagonals a particle of mass $\frac{m}{3}$. Show that these five particles and the parallelogram are equimomental system.

10. Show that a uniform rod of mass m, is kinetically equivalent to three particles rigidly connected and situated one at each end of the rod and at its middle point the masses of the particles being $\frac{m}{6}, \frac{m}{6}$ and $\frac{2}{3}m$.

11. Show that there is a momental ellipse at the centre of inertia of a uniform triangle which touches the side of the triangle at the middle points.

12. At the vertex C of a triangle ABC, which is right angle at C, the principal axes are, a perpendicular to the plane and two others inclined to the sides at an angle

$$\frac{1}{2}\tan^{-1}\frac{ab}{a^2 - b^2}.$$

13. A uniform rectangular plate whose sides are of lengths $2a, 2b$ has a portion cut out in the form of a square whose centre is the centre of rectangle and whose mass is half the mass of the plate. Show that the axes of greastes and least moment of inertia at a corner of the rectangle makes angle $\theta, (\pi/2) + \theta$ with a sidewhere $2\theta = \frac{6ab}{5(a^2 - b^2)}$.

14. ABC is a triangular area and AD is perpendicular to BC and AE is a medim, O is the middle point of DE, show that B is a principal axis of the triangle at O.

15. A uniform circular solid cone of semi-vertical angle α and height h is cut in half by a plane through its axis: Show that the principal moments of inertia at the vertex for one of the halves are $\frac{3}{5}Mh^2(1 + \frac{1}{4}\tan^2\alpha)$ and $\frac{3}{10}Mh^2(1 + \frac{3}{4}\tan^2\alpha)$

$$\pm \frac{3}{10}Mh^2 \sqrt{\{(1 - \frac{1}{4}\tan^2\alpha)^2 + (64/9\pi^2)\tan^2\alpha\}}$$