IAS/IFoS MATHEMATICS by K. Venkanna

Set - VII

$\frac{x^2}{A^2}+\frac{y^2}{B^2}=1$

$A^2=\frac{2a^2k}{c}$

$A=a\sqrt{\frac{2k}{c}}$ etc

PARABOLOID

Article 24. (*a*) **To trace elliptic paraboloid**

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2xZ}{c}.$$

The given paraboloid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2z}{c}$. ...(1)

(*i*) **Symmetry.** Since (1) contains even powers of x and y, so it is symmetrical about the YZ and ZX planes.

(*ii*) **Intersection with axes.** The surface (1) meets X-axis ($y=0$, $z=0$)

where $\frac{x^2}{a^2}=0$ or $x^2=0$ $\therefore$ $x=0, 0$.

$\therefore$ The surface (1) touches the X-axis at O(0, 0, 0), Similarly it touches Y-axis at (0, 0, 0). The the surface (1) touches the XY plane. Also (1) meets Z-axis at (0, 0, 0).

(*iii*) **Sections by co-ordinate planes.** (1) meets the YZ plane $(x=0)$

where $$\frac{y^2}{b^2}=\frac{2z}{c} \text{ or } y^2=\frac{2b^2}{c}z$$

which is an upward parabola *in that plane.*

Similarly (1) meets the ZX plane $(y=0)$ in an upward parabola $x^2=\frac{2a^2}{c}z$ *in that plane.*

Again (1) meets the XY plane $(z=0)$ in

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=0$$

which is a point ellipse in that plane.

(*iv*) **Generated by a variable curve.** The surface (1) meets the plane $z=k$, where

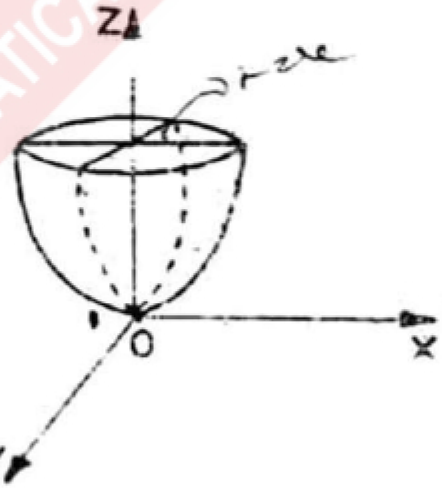
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2k}{c} \quad\text{or}\quad \frac{x^2}{\left(\frac{2a^2k}{c}\right)}+\frac{y^2}{\left(\frac{2b^2k}{c}\right)}=1$$

Thus the surface is generated by a variable ellipse

$$z=k,\ \frac{x^2}{\left(\frac{2a^2k}{c}\right)}+\frac{y^2}{\left(\frac{2b^2k}{c}\right)}=1 \qquad \ldots(2)$$

where k varies.

Its plane is $\parallel$ to XY-plane and the centre $(0, 0, k)$ moves on Z-axis. Now the ellipse (2) is real if k is +ve. Thus the surface lies only above the XY-plane.



Also the semi-axis of the ellipse (2) are $a\sqrt{\frac{2k}{c}}$, $b\sqrt{\frac{2k}{c}}$ which increases as $k>0$ increases and $\to\infty$ as $k\to\infty$. Thus the surface extends to ∞ above the XY-plane.

The shape is as shown in the adjoining figure.

Article 24. (*b*) **To trace the hyperbolic paraboloid**

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{2z}{c}.$$

The equation of the surface is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{2z}{c} \qquad \ldots(1)$$

(*i*) **Symmetry.** Since the equation (1) contains even powers of x and y, so the surface is symmetrical about the YZ and ZX-planes

(*ii*) **Axes intersection.** The surface (1) meets X-axis ($y=0$, $z=0$) where $\frac{x^2}{a^2}=0$ or $x^2=0$ or $x=0, 0$. Thus the surface (1) touches X-axis at the origin. Similarly it touches Y-axis at the origin.

Thus (1) touches XY-plane at O(0, 0, 0).

It meets Z-axis ($x=0$, $y=0$), where $\frac{2z}{c}=0$ or $z=0$ *i.e.*, at the origin.

(*iii*) **Sections by co-ordinate planes.** The surface (1) meets the YZ-planes ($x=0$) where

$$\frac{-y^2}{b^2}=\frac{2z}{c} \quad \text{or} \quad y^2=-2\frac{b^2}{c}z$$

which is a downward parabola *in that plane* (*assuming* c to be +ve).

Similarly (1) meets the ZX-plane in the upward parabola

$$x^2=\frac{2a^2}{c}z \text{ in that plane.}$$

It meets the XY-plane ($z=0$) where

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0 \quad \text{or} \quad y=\pm\frac{b}{a}x$$

which are two straight lines in that plane equally inclined to X-axis.

(*iv*) **Generated by a variable curve.** The surface (1) meets the plane $z=k$ where [putting $z=k$ in (1)].

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{2k}{c} \quad \text{or} \quad \frac{x^2}{\left(\frac{2a^2k}{c}\right)}-\frac{y^2}{\left(\frac{2b^2k}{c}\right)}=1.$$

Thus the surface is generated by a variable hyperbola

$$\frac{x^2}{\left(\frac{2a^2k}{c}\right)}-\frac{y^2}{\left(\frac{2b^2k}{c}\right)}=1,\ z=k \qquad \ldots(2) \text{ [as } k \text{ varies]}$$

whose plane is ∥ to the XY-plane, and centre (0, 0, k) moves on the Z-axis.

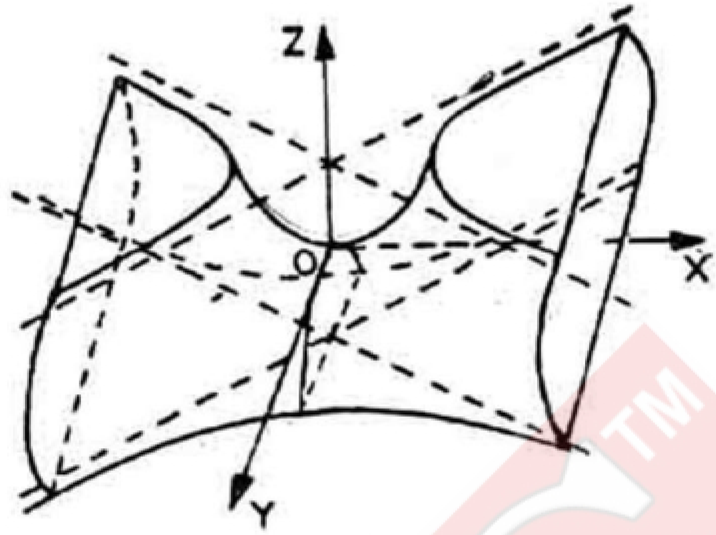
The hyperbola (2) has transverse axis ∥ to X-axis if k is +ve and ∥ to Y-axis if k is −ve. [∵ *c is assumed to* be +ve]

Also the transverse semi-axis is $a\sqrt{\frac{2k}{c}}$ which increases as k(+ve) increases and $\to\infty$ as $k\to\infty$.

Thus surface extends to infinity above the XY-plane.

Similarly the surface extends to infinity below the XY-plane.

∴ The surface extends to infinity both above and below the XY-plane. Hence the shape of the surface is as shown in the figure.



Note. The general equation of the paraboloid is of the form $ax^2+by^2=2z$, which is an elliptic or hyperbolic paraboloid according as a and b are of the same or opposite signs.

Article 25. Intersection of a line with the paraboloid.

Let the line be

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \quad \ldots(1)$$

and the paraboloid be $\qquad ax^2+by^2=2z \qquad \ldots(2)$

Any point on (1) is $P(lr+x_1, mr+y_1, nr+z_1)$. If it lies on (2) then

$$a(lr+x_1)^2+b(mr+y_1)^2=2(nr+z_1)$$

or $\quad r^2(al^2+bm^2)+2r(alx_1+bmy_1-n)+(ax_1^2+by_1^2-2z_1)=0 \quad \ldots(3)$

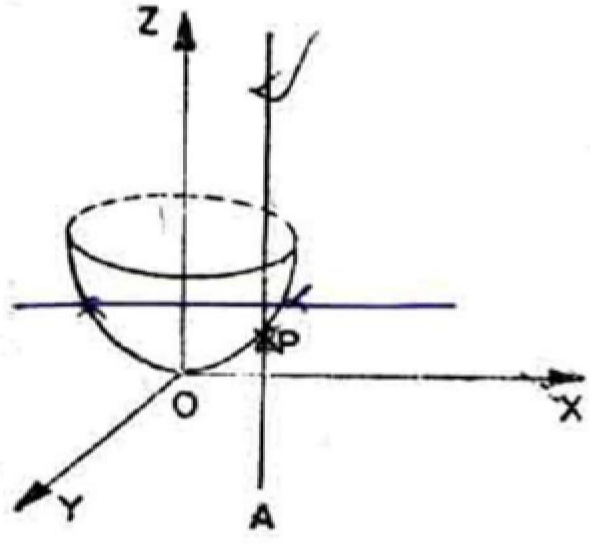
which gives two values of r.

This shows that every line meets the paraboloid in two points, *i.e. every plane section of a paraboloid is a conic.*

Again if $l=m=0$, $n=1$ from (3), one value of r is infinite showing that any line ∥ to the Z-axis meets the paraboloid in one point at an infinite distance from $A(x_1, y_1, z_1)$ and in a finite point. P whose distance from A is given by

$$r=\frac{ax_1^2+by_1^2-2z_1}{2}.$$

Such a line drawn through a point A, which meets the paraboloid in one point at an infinite distance from A and in a point P is called a **diameter** of the paraboloid and P is called the **extremity** of the diameter.



Thus a line ∥ to OZ is a diameter of the paraboloid $ax^2+by^2=2z$

Def. The diameter of a paraboloid which is ⊥ to the tangent plane at its extremity, is called the **axis of the paraboloid**, and its extremity is called the **vertex** of the paraboloid.

Thus OZ is the axis and O the vertex of the paraboloid

$$ax^2+by^2=2z.$$

Cor. A line ∥ to the axis of a paraboloid is diameter.

Article 26. Some standard results [redacted] paraboloid

Following are some of the results about a paraboloid which can be easily proved. The student is advised to prove these results for himself.

Let the paraboloid be

$$ax^2+by^2=2z. \qquad \ldots(1)$$

(*i*) **The tangent plane to (1) at (x_1, y_1, z_1) is**

$\mathbf{axx_1+byy_1=z+z_1}$. (*K.U. 1970*) [*See Art. 1 (g) prove by General Methods*]

(*ii*) **The condition of tangency for a given plane**

$$lx+my+nz=p$$

and the paraboloid (1) is

$$\frac{\mathbf{l^2}}{\mathbf{a}}+\frac{\mathbf{m^2}}{\mathbf{b}}=\mathbf{-2np} \qquad (K.U.\ 1973)$$

[*For proof see page* (*xvii*) *of general methods. Art.* 2(*d*)]

and the *point of contact* is

$$\left(\frac{-l}{an}, \frac{-m}{bn}, \frac{-p}{n}\right)$$

and *any tangent plane* to (1) ∥ to $lx+my+nz=0$ is

$$\mathbf{2n(lx+my+nz)+\frac{l^2}{a}+\frac{m^2}{b}=0.}$$

(*iii*) The plane of contact and the polar plane of (x_1, y_1, z_1) w.r.t. (1) is $\mathbf{axx_1+byy_1=z+z_1}$.

(*iv*) **The enveloping cone of (1) is given by $\mathbf{SS_1=T^2}$**

i.e., $(ax^2+by^2-2z)(ax_1^2+by_1^2-2z_1)=[axx_1+byy_1-(z+z_1)]^2$.

(*v*) **The plane section of (1) with given centre $(\mathbf{x_1, y_1, z_1})$** is given by $\mathbf{T=S_1}$

i.e. $axx_1+byy_1-(z+y_1)=ax_1^2+by_1^2-2z_1$.

Example 1. *Find the condition that the plane $lx+my+nz=l$ may be a tangent plane to the paraboloid $x^2+y^2=2z$.*

Sol. Reproduce Art 2(*d*) page (*xvii*) General Methods replacing *a, b, p* by 1 each).

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Example 2. (*a*) *Show that the plane $8x-6y-z=5$ touches the paraboloid $\frac{x^2}{2}-\frac{y^2}{3}=z$ and find the co-ordinates of the point of contact.*

(Agra 1988)

(*b*) *Show that the plane $2x-4y-z+3=0$ touches the paraboloid $x^2-2y^2=3z$ and find the co-ordinates of the point of contact.*

(Agra 1987 : Madurai 1983)

Sol. (*a*) Let the plane $8x-6x-z=5$ ...(1)

touch the paraboloid $\frac{x^2}{2}-\frac{y^2}{3}=z$ or $3x^2-2y^2=6z$ ...(2)

at the point (x_1, y_1, z_1).

Then the tangent plane to (2) at (x_1, y_1, z_1) is

$$3xx_1-2yy_1=3(z+z_1) \text{ or } 3xx_1-2yy_1-3z=3z_1 \quad ...(3)$$

Now this plane is identical with (1).

$\therefore$ Comparing the co-effs. in (1) and (3), we have

$$\frac{3x_1}{8}=\frac{-2y_1}{-6}=\frac{-3}{-1}=\frac{3z_1}{5}$$

which gives $x_1=8, \quad y_1=9, \quad z_1=5.$

The plane (1) touches the paraboloid (2) if the point of contact (x_1, y_1, z_1), *i.e.* (8, 9, 5) lies on (2) *i.e.* if $3(64)-2(81)=6(5)$

or if $192-162=30$ or $30=30$ which is true.

Hence (1) touches (2) and the point of contact is (8, 9, 5).

(*b*) **Please try yourself.** [**Ans.** (3, 3, −3)]

Example 3. *Prove that the paraboloids*

$$\frac{x^2}{a_1^2}+\frac{y^2}{b_1^2}=\frac{2z}{c_1};\ \frac{x^2}{a_2^2}+\frac{y^2}{b_2^2}=\frac{2z}{c_2};\ \frac{x^2}{a_3^2}+\frac{y^2}{b_3^2}=\frac{2z}{c_3}$$

have a common tangent plane if

$$\begin{vmatrix} a_1^2 & a_2^2 & a_3^2 \\ b_1^2 & b_2^2 & b_3^2 \\ c_1 & c_2 & c_3 \end{vmatrix}=0.$$

Sol. The given paraboloids are $\frac{x^2}{a_1^2}+\frac{y^2}{b_1^2}=\frac{2z}{c_1}$ ...(1)

$\frac{x^2}{a_2^2}+\frac{y^2}{b_2^2}=\frac{2z}{c_2}$ ...(2) and $\frac{x^2}{a_3^2}+\frac{y^2}{b_3^2}=\frac{2z}{c_3}$ ...(3)

Let the common tangent plane be $lx+my+nz=p$ ...(4)

Since it touches the paraboloid (1)

i.e., $$\frac{c_1}{a_1^2}x^2+\frac{c_1}{b_1^2}y^2=2z.$$

$$\therefore \quad \frac{l^2}{\left(\frac{c_1}{a_1^2}\right)}+\frac{m^2}{\left(\frac{c_1}{b_1^2}\right)}=-2np \qquad \Big|\ \textit{Using } \frac{l^2}{a}+\frac{m^2}{b}=-2np$$

or $$l^2a_1^2+m^2b_1^2+2nc_1p=0 \quad ...(5)$$

Similarly (4) touches (2) and (3),

$$\therefore \quad l^2a_2^2+m^2b_2^2+2nc_2p=0 \quad ...(6)$$

and $$l^2a_3^2+m^2b_3^2+2nc_3p=0 \quad ...(7)$$

Eliminating l^2, m^2, $2np$ from (5), (6), (7) by determinants, we have

$$\begin{vmatrix} a_1^2 & b_1^2 & c_1 \\ a_2^2 & b_2^2 & c_2 \\ a_3^2 & b_3^2 & c_3 \end{vmatrix}=0 \text{ or } \begin{vmatrix} a_1^2 & a_2^2 & a_3^2 \\ b_1^2 & b_2^2 & b_3^2 \\ c_1 & c_2 & c_3 \end{vmatrix}=0$$

[*on interchanging rows and columns*]

Example 4. *Show that the equation to two tangent planes to the surface $ax^2+by^2=2z$ which pass through the line*

$$u \equiv lx+my+nz-p=0,\ u' \equiv l'x+m'y+n'z-p'=0$$

is $$u^2\left(\frac{l'^2}{a}+\frac{m'^2}{b}+2n'p'\right)-2uu'\left(\frac{ll'}{a}+\frac{mm'}{b}+np'+n'p\right)+u'^2\left(\frac{l^2}{a}+\frac{m^2}{b}+2np\right)=0.$$ **[Imp.]**

Sol. Any plane through the line

$$u=0,\ u'=0 \text{ is } uk+u'=0 \qquad \ldots(1)$$

i.e., $$lx+my+nz-p+k(l'x+m'y+n'z-p')=0$$

or $$(l+kl')x+(m+km')y+(n+kn')z=p+kp'.$$

If it touches the paraboloid $ax^2+by^2=2z$, then

$$\frac{(l+kl')^2}{a}+\frac{(m+km')^2}{b}=-2(n+kn')\,.\,(p+kp')$$

Using $\frac{l^2}{a}+\frac{m^2}{b}=-2np$

or $$k^2\left(\frac{l'^2}{a}+\frac{m'^2}{b}+2n'p'\right)+2k\left(\frac{ll'}{a}+\frac{mm'}{b}+n'p+np'\right)+\left(\frac{l^2}{a}+\frac{m^2}{b}+2np\right)=0.$$

Putting $k=-\frac{u}{u'}$, from (1) in this, we get the required result.

Example 5. *Find the locus of points from which three mutually perpendicular tangents can be drawn to the paraboloid*

(*i*) $ax+by^2+2z=0$

(*ii*) $ax^2+by^2=2z$.

Sol. (*i*) Let $P(x_1, y_1, z_1)$ be the point. Then enveloping cone of $ax^2+by^2+2z=0$ of the tangents from $P(x_1, y_1, z_1)$ is $SS_1=T^2$

or $$(ax^2+by^2+2z)(ax_1^2+by_1^2+2z_1)=(axx_1+byy_1+z+z_1)^2 \qquad \ldots(1)$$

If the three lines drawn from P to touch the given paraboloid are mutually $\perp$, then the cone must have three mutually $\perp$ generators, *i.e. sum of coeffs.* of x^2, y^2, z^2 *in* (1) *is zero.*

or $$a(by_1^2+2z_1)+b(ax_1^2+2z_1)-1=0$$

or $$ab(x_1^2+y_1^2)+2z_1(a+b)-1=0.$$

$\therefore$ Locus of $P(x_1, y_1, z_1)$ is

$$ab(x^2+y^2)+2(a+b)z-1=0.$$

(*ii*) **Please try yourself.** [Ans. $ab(x^2+y^2)-2(a+b)z-1=0$]

Example 6. (*a*) *Find the equation of the plane which cuts the paraboloid $x^2-2y^2=z$ in a conic with its centre at $(2, \frac{3}{2}, 4)$.*

(*b*) *Find the centre of the conic $ax^2+by^2=2z$; $lx+my+nz=p$.*

Sol. (*a*) The paraboloid is $S \equiv x^2-2y^2-z=0$ and the centre (x_1, y_1, z) is $(2, \frac{3}{2}, 4)$.

Then the plane which has the centre $P(2, \frac{3}{2}, 4)$ is given by $T=S_1$, where T is the expression for tangent plane at P with R.H.S. zero, *i.e.* by

$$x(2)-2y\left(\frac{3}{2}\right)-\frac{1}{2}(z+4)=(2)^2-2\left(\frac{3}{2}\right)^2-4$$

or $$2x-3y-\frac{1}{2}z-2=4-\frac{9}{2}-4$$

or $$4x-6y-z+5=0$$

(*b*) Let (x_1, y_1, z_1) be the centre of the conic given by

$$ax^2+by^2-2z=0 \quad \ldots(1)$$

and $$lx+my+nz-p=0 \quad \ldots(2)$$

Then equation of the plane which cuts (1) in a conic with a centre at (x_1, y_1, z_1) is $T=S_1$ *i.e.*

$$axx_1+byy_1-(z+z_1)=ax_1^2+by_1^2-2z_1$$

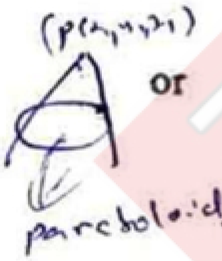
or $$axx_1+byy_1-z-(ax_1^2+by_1^2-z_1)=0 \quad \ldots(3)$$

Now (2) and (3) are identical. $\because$ Comparing coeffs, we have

$$\frac{ax_1}{l}=\frac{by_1}{m}=\frac{-1}{n}=\frac{ax_1^2+by_1^2-z_1}{p}$$

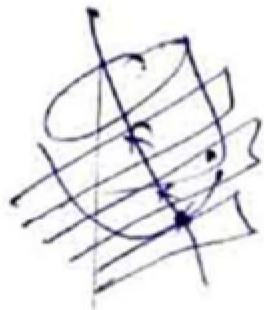
which give $x_1=\dfrac{-l}{an}$, $y_1=\dfrac{-m}{bn}$ and $ax_1^2+by_1^2-z_1=\dfrac{-p}{n}$

Putting the values of x_1, y_1 from first two equations in the third equation, we get



$$a\cdot\frac{l^2}{a^2n^2}+b\cdot\frac{m^2}{b^2n^2}-z_1=-\frac{p}{n}$$

or $$z_1=\frac{l^2}{an^2}+\frac{m^2}{bn^2}+\frac{p}{n}.$$



Thus the centre (x_1, y_1, z_1) of the conic is

$$\left(\frac{-l}{an}, \frac{-m}{bn}, \frac{l^2}{an^2}+\frac{m^2}{bn^2}+\frac{p}{n}\right).$$

Example 7. *Show that the locus of centres of a system of parallel plane sections of a paraboloid is a diameter.*

Prove also that the tangent plane at the extremity of the diameter is parallel to the plane sections. **[Imp.]**

Sol. Let the paraboloid be $ax^2+by^2=2z$ ...(1)
and let (x_1, y_1, z_1) be the centre of one of the plane sections of (1) drawn parallel to a given plane

$$lx+my+nz=p \quad \ldots(2)$$

Then equation of the plane section of (1) whose centre is (x_1, y_1, z_1) is

$$axx_1+byy_1-(z+z_1)=ax_1^2+by_1^2-2z_1 \quad \ldots(3) \quad | \, T=S_1$$

Now (2) is $\parallel$ to plane (3),

$$\therefore \quad \frac{ax_1}{l}=\frac{by_1}{m}=\frac{-1}{n}$$

$\therefore$ Locus of (x_1, y_1, z_1) is $\dfrac{ax}{l}=\dfrac{by}{m}=\dfrac{-1}{n}$ ...(4)

which is the line of intersection of the planes

$nax+l=0$ [from first and third members of (4)]

and $nby+m=0$ [from second and third members of (4)]
which are respectively ∥ to planes $x=0$ and $y=0$.

Thus the line (4) is ∥ to the Z-axis $(x=0, y=0)$ which is the axis of the paraboloid, and consequently (3) is a diameter of (1). Hence the first part.

4

Second part. To find the extremity of diameter (3), we have to solve (4) and (1).

4

From (4), $x=-\dfrac{l}{na}$, $y=-\dfrac{m}{nb}$.

Putting these values of x, y in (1), we get

$$a\left(-\frac{l}{na}\right)^2+b\left(-\frac{m}{nb}\right)^2=2z$$

or $$z=\frac{l^2}{2n^2a}+\frac{m^2}{2n^2b}.$$

Hence the extremity of the diameter is

$$\left(-\frac{l}{na}, -\frac{m}{nb}, \frac{l^2}{2n^2a}+\frac{m^2}{2n^2b}\right).$$

$\therefore$ Equation of the tangent plane to (1) at this extremity is

$$ax\left(-\frac{l}{na}\right)+by\left(-\frac{m}{nb}\right)=z+\left(\frac{l^2}{2n^2a}+\frac{m^2}{2n^2b}\right)$$

| Using $axx_1+byy_1=z+z_1$

or $$lx+my+nz+\frac{l^2}{2na}+\frac{m^2}{2nb}=0$$

which is clearly ∥ to the given plane (2). Hence the result.

Article 27. To find the locus of intersection of three mutually perpendicular tangent planes to the paraboloid

$$\mathbf{ax^2+by^2=2z}$$ **[V. Imp.]**

The given paraboloid is $ax^2+by^2=2z$ ...(1)

Let $l_1x+m_1y+n_1z=p_1$ (l_1, m_1, n_1 being the actual d.c.'s) be one of the three mutually ⊥ tangent planes so that

$$\frac{l_1^2}{a}+\frac{m_1^2}{b}=-2n_1p_1 \quad \textit{Condition of tangency}$$

or $$p_1=-\frac{1}{2n_1}\left(\frac{l_1^2}{a}+\frac{m_1^2}{b}\right).$$

Putting the value of p_1 the equation of one of the three mutually ⊥ tangent planes is

$$l_1x+m_1y+n_1z=-\frac{1}{2n_1}\left(\frac{l_1^2}{a}+\frac{m_1^2}{b}\right)$$

Multiplying both sides by n_1

or $$n_1l_1x+m_1n_1y+n_1^2z+\frac{l_1^2}{2a}+\frac{m_1^2}{2b}=0 \text{ (Note this step)} \quad \ldots(2)$$

Similarly the equations of other two tangent planes is

$$n_2l_2x+m_2n_2y+n_2^2z+\frac{l_2^2}{2a}+\frac{m_2^2}{2b}=0 \quad \ldots(3)$$

$$n_3l_3x+m_3n_3y+n_3^2z+\frac{l_3^2}{2a}+\frac{m_3^2}{2b}=0 \quad \ldots(4)$$

The locus of the point of intersection of (2), (3), (4) is given by eliminating l_1, m_1, n_1 etc. from these equations. Adding (2), (3), (4), we get

$$x\Sigma l_1n_1+y\Sigma m_1n_1+z\Sigma n_1^2+\frac{1}{2a}\Sigma l_1^2+\frac{1}{2b}\Sigma m_1^2=0$$

or $$x(0)+y(0)+z(1)+\frac{1}{2a}(1)+\frac{1}{2b}(1)=0$$

or $$z+\frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)=0$$

$\left[\begin{array}{l}\because\ l_1, m_1, n_1, \text{ etc. are the d.c.'s of three mutually } \perp \text{ lines,} \\ \therefore\ \Sigma l_1^2=\Sigma m_1^2=\Sigma n_1^2=1,\ \Sigma m_1n_1=0 \text{ etc.}\end{array}\right]$

which is the required locus.

It is clearly a plane ‖ to XY plane, *i.e.* ⊥ to the Z-axis, the axis of the paraboloid.

Article 28. Normal to the paraboloid.

To find the equations of the normal at the point (x_1, y_1, z_1) of paraboloid

(*i*) $ax^2+by^2=2z$

(*ii*) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=2z.$

(*i*) The given paraboloid is $ax^2+by^2=2z$ ...(1)

The tangent plane at (x_1, y_1, z_1) to (1) is

$axx_1+byy_1=z+z_1$ | *Using the rule of tanagent plane*

or $axx_1+byy_1-z-z_1=0.$ ...(2)

The d.c.'s of normal to this tangent plane are proportional to ax_1, by_1, -1. | *Coeffs.* of x, y, z

∴ Equations of the normal at (x_1, y_1, z_1), [*i.e.* a line through (x_1, y_1, z_1) and ⊥ to (2)] are

$$\frac{x-x_1}{ax_1}=\frac{y-y_1}{by_1}=\frac{z-z_1}{-1}.$$

(*ii*) **Please try yourself.**

Ans. $$\frac{x-x_1}{\frac{x_1}{a^2}}=\frac{y-y_1}{\frac{y_1}{b^2}}=\frac{z-z_1}{-1}$$

Article 29. Number of normals

To prove that there are five points on an ellipticparabo loid $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2z$, the normals at which pass through a given point (α, β, γ)

[V. Imp] (*Allahabad 1982 ; L.N.M. 1982*)

The given paraboloid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2z$ ...(1)

The normal at (x_1, y_1, z_1) is $\frac{x-x_1}{\frac{x_1}{a^2}}=\frac{y-y_1}{\frac{y_1}{b^2}}=\frac{z-z_1}{-1}$

This passes through (α, β, γ), if

$$\frac{\alpha-x_1}{\frac{x_1}{a^2}}=\frac{\beta-y_1}{\frac{y_1}{b^2}}=\frac{\gamma-z_1}{-1}=\lambda \text{ (say)}$$

From first and last members,

$$\alpha-x_1=\frac{x_1\lambda}{a^2} \text{ or } \alpha=x_1\left(1+\frac{\lambda}{a^2}\right)=\frac{\lambda+a^2}{a^2}x_1$$

$\therefore$ $$x_1=\frac{a^2\alpha}{a^2+\lambda}. \quad \text{...(2)}$$

Similarly $y_1=\frac{b^2\beta}{b^2+\lambda}$ and $z_1=\gamma+\lambda$.

But since (x_1, y_1, z_1) lies on (1), $\therefore$ $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}=2z_1$

or $$\frac{1}{a^2}\cdot\frac{a^4\alpha^2}{(a^2+\lambda)^2}+\frac{1}{b^2}\cdot\frac{b^4\beta^2}{(b^2+\lambda)^2}=2(\gamma+\lambda)$$

or $$a^2\alpha^2(b^2+\lambda)^2+b^2\beta^2(a^2+\lambda)^2=2(\lambda+\gamma)(a^2+\lambda)^2(b^2+\lambda)^2 \quad \text{...(3)}$$

This equation being of fifth degree in λ gives five values of λ. Putting these values of λ in (2), we get five points on the paraboloid the normals at which pass through (α, β, γ). Hence the result.

Cor. From (2), the foot of normal is

$$\left(\frac{a^2\alpha}{a^2+\lambda}, \frac{b^2\beta}{b^2+\lambda}, \gamma+\lambda\right).$$

Article 30. Prove that the feet of normals from a given point (α, β, γ) to an elliptic paraboloid are the five points of intersection of the elliptic paraboloid and a certain cubic curve.

Let as an exercise for the student.

Proceed exactly as in Article 12 in the case of an ellipsoid.

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Example 1. *Prove that the normals from* (α, β, γ) *to the paraboloid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2z$ *lie on the cone*

$$\frac{\alpha}{x-\alpha}-\frac{\beta}{y-\beta}+\frac{a^2-b^2}{z-\gamma}=0.$$ **[V. Imp.]**

(*M.D.U.* 1986, 85)

Sol. The given paraboloid is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=2z \qquad \ldots(1)$$

Let any line through (α, β, γ) be

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad \ldots(2)$$

be the normal at (x_1, y_1, z_1) to (1).

The equation of the tangent plane at (x_1, y_1, z_1) to (1) is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-(z+z_1)=0 \qquad \ldots(3)$$

Since (2) is normal to (3) $\therefore$ it is $\parallel$ to the normal to (3)

$$\therefore \qquad \frac{l}{\frac{x_1}{a^2}}=\frac{m}{\frac{y_1}{b^2}}=\frac{n}{-1}=k \text{ (say)} \qquad \ldots(4)$$

Again if the normal at (x_1, y_1, z_1) to (1) passes through (α, β, γ), then $x_1=\frac{a^2\alpha}{a^2+\lambda}$, $y_1=\frac{b^2\beta}{b^2+\lambda}$, $z_1=\gamma+\lambda$. [From Eqn. (2) of Art. 29] ...(5)

From (4), $l=k\,\frac{x_1}{a^2}=\frac{k}{a^2}\cdot\frac{a^2\alpha}{a^2+\lambda}=\frac{k\alpha}{a^2+\lambda}$ | *Using* (5)

or
$$a^2+\lambda=\frac{k\alpha}{l} \qquad \ldots(6)$$

$$m=k\,\frac{y_1}{b^2}=\frac{k}{b^2}\cdot\frac{b^2\beta}{b^2+\lambda}=\frac{k\beta}{b^2+\lambda} \quad \text{or} \quad b^2+\lambda=\frac{k\beta}{m} \qquad \ldots(7)$$

$$n=-k \qquad \ldots(8)$$

Subtracting (7) from (6), we get

$$a^2-b^2=k\left(\frac{\alpha}{l}-\frac{\beta}{m}\right)$$

$$=-n\left(\frac{\alpha}{l}-\frac{\beta}{m}\right) \qquad \ldots(9) \qquad | \textit{ Using } (8)$$

To find the locus, we have to eliminate l, m, n from (2) and (9). Putting the value of l, m, n from (2) in (9), we have

$$a^2-b^2=-(z-\gamma)\left(\frac{\alpha}{x-\alpha}-\frac{\beta}{y-\beta}\right)$$

or $$\frac{a^2-b^2}{z-\gamma}=-\frac{\alpha}{x-\alpha}+\frac{\beta}{y-\beta}$$

or $$\frac{\alpha}{x-\alpha}-\frac{\beta}{y-\beta}+\frac{a^2-b^2}{z-\gamma}=0$$

which is the required result.

Example 2. *Prove that in general three normals can be drawn from a given point to the paraboloid of revolution $x^2+y^2=2az$.*

Prove also that if the point lies on the surface

$$27a(x^2+y^2)+8(a-z)^3=0,$$

then two of the three normals coincide. **[V. Imp.]**

Sol. The given paraboloid is $x^2+y^2=2az$

or $$\frac{x^2}{a}+\frac{y^2}{a}=2z \quad \text{...(1)}$$

The normal at (x_1, y_1, z_1) to (1) is

$$\frac{x-x_1}{\frac{x_1}{a}}=\frac{y-y_1}{\frac{y_1}{a}}=\frac{z-z_1}{-1} \quad \text{...(2)}$$

If it passes through the given point (α, β, γ), then

$$\frac{\alpha-x_1}{\frac{x_1}{a}}=\frac{\beta-y_1}{\frac{y_1}{a}}=\frac{\gamma-z_1}{-1}=\lambda \text{ (say)} \quad \text{...(3)}$$

which give $$x_1=\frac{a\alpha}{a+\lambda},\ y_1=\frac{a\beta}{a+\lambda},\ z_1=\gamma+\lambda \quad \text{...(4)}$$

or $$\frac{1}{a}\cdot\frac{a^2\alpha^2}{(a+\lambda)^2}+\frac{1}{a}\cdot\frac{a^2\beta^2}{(a+\lambda)^2}=2(\gamma+\lambda)$$

or $$a\alpha^2+a\beta^2=2(\gamma+\lambda)(a+\lambda)^2 \quad \text{...(5)}$$

which being a third degree equation in λ, gives three values of λ. Putting these values of λ in (4), we get three points (x_1, y_1, z_1) on the paraboloid (1) at which the normals pass thro' (α, β, γ). Thus from a given point three normals can be drawn to (1).

Rewriting (5) as

$$f(\lambda)=2(\gamma+\lambda)(a+\lambda)^2-(\alpha^2+\beta^2)=0 \quad \text{...(6)}$$

$$\therefore \quad f'(\lambda)=2(a+\lambda)^2+4(\gamma+\lambda)(a+\lambda)=0 \quad \text{...(7)}$$

If two of the normals coincide, then (6) must have two equal roots showing that $f(\lambda)$ and $f'(\lambda)$ must have a common linear factor.

From (7), $$2(a+\lambda)[a+\lambda+2(\gamma+\lambda)]=0$$

or $$\lambda=-\frac{a+2\gamma}{3} \qquad \Big|\ \because\ a+\lambda\neq 0$$

Putting this value of λ in (6), we get

$$a(\alpha^2+\beta^2)=2\left[\left(\gamma-\frac{a+2\gamma}{3}\right)\left(a-\frac{a+2\gamma}{3}\right)^2\right]$$

$$=2\left(\frac{\gamma-a}{3}\right)\left(\frac{2a-2\gamma}{3}\right)^2=-\frac{8}{27}(a-\gamma)^3$$

or $$a(\alpha^2+\beta^2)+\frac{8}{27}(a-\gamma)^3=0$$

or $$27a(\alpha^2+\beta^2)+8(a-\gamma)^3=0$$

Hence (α, β, γ) lies on

$$27a(x^2+y^2)+8(a-z)^3=0.$$

Example 3. *Show that the feet of the normals from the point (α, β, γ) to the paraboloid $x^2+y^2=2az$ lie on the sphere*

2007 model

$$x^2+y^2+z^2-z(a+\gamma)-\frac{y}{2\beta}(\alpha^2+\beta^2)=0.$$ **[Imp.]**

Sol. If (x_1, y_1, z_1) be the foot of normal through (α, β, γ) to the paraboloid

$$x^2+y^2=2az \text{ or } \frac{x^2}{a}+\frac{y^2}{a}=2z, \text{ then}$$

$$x_1=\frac{a\alpha}{a+\lambda},\ y_1=\frac{a\beta}{a+\lambda},\ z_1=\gamma+\lambda \quad \text{...(1)}$$

[*See Example 2*]

But (x_1, y_1, z_1) lies on the paraboloid, so we have

$$\frac{1}{a}\cdot\frac{a^2\alpha^2}{(a+\lambda)^2}+\frac{1}{a}\cdot\frac{a^2\beta^2}{(a+\lambda)^2}=2(\gamma+\lambda)$$

or $$\frac{a(\alpha^2+\beta^2)}{(a+\lambda)^2}=2(\gamma+\lambda) \quad \text{...(2)}$$

Now (x_1, y_1, z_1) will lie on the given sphere

$$x^2+y^2+z^2-z(a+\gamma)-\frac{y}{2\beta}(\alpha^2+\beta^2)=0$$

if $$\frac{a^2\alpha^2}{(a+\lambda)^2}+\frac{a^2\beta^2}{(a+\lambda)^2}+(\gamma+\lambda)^2-(\gamma+\lambda)(a+\gamma)-\frac{a}{2(a+\lambda)}(\alpha^2+\beta^2)=0$$

or if $$\frac{a^2(\alpha^2+\beta^2)}{(a+\lambda)^2}+(\gamma+\lambda)^2-(\gamma+\lambda)(a+\gamma)-\frac{a(\alpha^2+\beta^2)}{2(a+\lambda)}=0$$

or if $$\frac{a(\alpha^2+\beta^2)}{2(a+\lambda)^2}[2a-(a+\lambda)]+(\gamma+\lambda)[\gamma+\lambda-a-\gamma]=0$$

or if $$\frac{a(\alpha^2+\beta^2)}{2(a+\lambda)^2}(a-\lambda)+(\gamma+\lambda)(\lambda-a)=0$$

or if $$(a-\lambda)\left[\frac{a(\alpha^2+\beta^2)}{2(a+\lambda)^2}-\gamma-\lambda\right]=0$$

or if $$\frac{a(\alpha^2+\beta^2)}{2(a+\lambda)^2}=\gamma+\lambda \qquad \because \quad a-\lambda\neq 0$$

which is true by (2). Hence the result.

Example 4. *Prove that the perpendicular from (α, β, γ) to the polar plane w.r.t. $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2z$ lies on the cone*

$$\frac{\alpha}{x-\alpha}-\frac{\beta}{y-\beta}+\frac{a^2-b^2}{z-\gamma}=0.$$

Sol. The polar plane of (α, β, γ) w.r.t. the paraboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=2z \text{ is } \frac{\alpha x}{b^2}+\frac{\beta y}{b^2}=z+\gamma \qquad ...(1)$$

Any line through (α, β, γ) is $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ ...(2)

If (2) is $\perp$ to the polar plane (1), then it is $\parallel$ to the normal to (1).

$$\therefore \qquad \frac{\frac{\alpha}{a^2}}{l}=\frac{\frac{\beta}{b^2}}{m}=\frac{-1}{n}=k \text{ (say)}$$

$$\therefore \qquad \frac{\alpha}{l}=a^2k,\ \frac{\beta}{m}=b^2k \text{ and } k=\frac{-1}{n}$$

$$\therefore \qquad \frac{\alpha}{l}-\frac{\beta}{m}=(a^2-b^2)k=\frac{-(a^2-b^2)}{n} \qquad ...(3)$$

This shows that line (2) lies on the cone [putting l, m, n from (2) in (3)]

$$\frac{\alpha}{x-\alpha}-\frac{\beta}{y-\beta}=-\frac{(a^2-b^2)}{z-\gamma}$$

or $$\frac{\alpha}{x-\alpha}-\frac{\beta}{y-\beta}+\frac{a^2-b^2}{z-\gamma}=0.$$

Hence the result.

CONJUGATE DIAMETERS

(Vikram 1984)

1. We know that if l, m, n, be proportional to the d.c.s. of a given system of parallel chords of the conicoid $ax^2+by^2+cz^2=1$ and if (α, β, γ) be the mid. pt. of any one of them, then the locus of the mid. pts. (α, β, γ) of the parallel chords is the plane

$$alx+bmy+czn=0 \qquad ...(i)$$

which passes through the centre $(0, 0, 0)$ of the conicoid.

This plane is called the *diametral plane conjugate to the direction* l, m, n. Conversely, any plane $Ax+By+Cz=0$ ...(ii)
through the centre is the diametral plane conjugate to the direction l, m, n given by $\frac{al}{A}=\frac{bm}{B}=\frac{cn}{C}$ [identifying (i) and (ii)]

Thus every central plane is a diametral plane conjugate to some direction.

If P be any point on the conicoid, then the plane bisecting chords parallel to OP is called the *diametral plane of* OP.

Note. In what follows, we shall confine our attention to the ellipsoid only.

2. Definitions

Conjugate Semi-diameters. Any three semi-diameters are called *Conjugate Semi-diameters* if the plane containing any two of them is the diametral plane of the third.

Conjugate Planes. Any three diametral planes are called *Conjugate Planes* if each is the diametral plane of the line of intersection of the other two.

3. Relations between the coordinates of the extremities of a system of conjugate diameters of an ellipsoid.

Let P (x_1, y_1, z_1) be any point on the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad ...(i)$$

Then the diametral plane of OP (*i.e.* the plane bisecting chords parallel to OP) is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=0 \qquad ...(ii)$$

Let Q (x_2, y_2, z_2) be any point on the section of (i) by the plane (ii)

$$\therefore \quad \frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}+\frac{z_1z_2}{c^2}=0 \qquad ...(iii)$$

which shows that Q lies on the diametral plane of OP.

The equation (iii) is also the condition that the diametral plane

$\frac{xx_2}{a^2}+\frac{yy_2}{b^2}+\frac{zz_2}{c^2}=0$ of OQ passes through P.

Thus if the diametral plane of OP passes through Q, then the diametral plane of OQ also passes through P.

Let the line of intersection of the diametral planes of OP and OQ meet the surface (i) in R (x_3, y_3, z_3).

Since R lies on the diametral planes of OP and OQ, the diametral plane of OQ *i.e.* the plane $\frac{xx_3}{a^2}+\frac{yy_3}{b^2}\frac{zz_3}{c^2}=0$ should pass through P and Q.

(The three semi-diameters OP, OQ, OR are such that the plane containing any two of them is the diametral plane of third. Hence they are called conjugate semi-diameters).

Since the points P, Q, R lie on (i)

$$\therefore \quad \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=1, \qquad \frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}+\frac{z_2^2}{c^2}=1,$$

$$\frac{x_3^2}{a^2}+\frac{y_3^2}{b^2}+\frac{z_3^2}{c^2}=1 \qquad \ldots(I)$$

Since the diametral plane of OP passes through Q and R, diametral plane of OQ passes through R and P, diametral plane of OR passes through P and Q

$$\therefore \quad \frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}+\frac{z_1z_2}{c^2}=0$$

$$\frac{x_2x_3}{a^2}+\frac{y_2y_3}{b^2}+\frac{z_2z_3}{c^2}=0, \quad \frac{x_3x_1}{a^2}+\frac{y_3y_1}{b^2}+\frac{z_3z_1}{c^2}=0, \qquad \ldots(II)$$

By virtue of the relations (I), we observe that

$$\frac{x_1}{a}, \frac{y_1}{b}, \frac{z_1}{c}; \frac{x_2}{a}, \frac{y^2}{b}; \frac{z_2}{c} \text{ and } \frac{x_3}{a}, \frac{y_3}{b}, \frac{z_3}{c}$$

can be regarded as the *direction-cosines* of any three lines.

($\because$ if $l^2+m^2+n^2=1$, then l, m, n are *d.cs.* of a line)

Also, by virtue of the relation (II), these lines are mutually perpendicular. Hence, we have $\frac{x_1}{a}, \frac{x_2}{a}, \frac{x_3}{a}, \frac{y_1}{b}, \frac{y_2}{b}, \frac{y_3}{b}$ and $\frac{z_1}{c}, \frac{z_2}{c}, \frac{z_3}{c}$ as the *d.cs.* of another set of three mutually perpendicular lines.

[$\because$ if l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3 are the *d.cs.* of three mutually perpendicular lines, then l_1, l_2, l_3 ; m_1, m_2, m_3 ; n_1, n_2, n_3 are also the *d.cs.* of three mutually perpendicular lines]

Hence we have

$$\left.\begin{aligned}\frac{x_1^2}{a^2}+\frac{x_2^2}{a^2}+\frac{x_3^2}{a^2}=1\\ \frac{y_1^2}{b^2}+\frac{y_2^2}{b^2}+\frac{y_3^2}{b^2}=1\\ \frac{z_1^2}{c^2}+\frac{z_2^2}{c^2}+\frac{z_3^2}{c^2}=1\end{aligned}\right\} \text{ and } \left.\begin{aligned}\frac{x_1y_1}{ab}+\frac{x_2y_2}{ab}+\frac{x_3y_3}{ab}=0\\ \frac{y_1z_1}{bc}+\frac{y_2z_2}{bc}+\frac{y_3z_3}{bc}=0\\ \frac{z_1x_1}{ca}+\frac{z_2x_2}{ca}+\frac{z_3x_3}{ca}=0\end{aligned}\right\}$$

i.e.
$$\left.\begin{aligned}x_1^2+x_2^2+x_3^2=a^2\\ y_1^2+y_2^2+y_3^2=b^2\\ z_1^2+z_2^2+z_3^2=c^2\end{aligned}\right\} \ldots(III) \text{ and } \left.\begin{aligned}x_1y_1+x_2y_2+x_3y_3=0\\ y_1z_1+y_2z_2+y_3z_3=0\\ z_1x_1+z_2x_2+z_3x_3=0\end{aligned}\right\} \quad \ldots(IV)$$

The coordinates of the conjugates semi-diameters are connected by the relations (I), (II), (III) and (IV) above.

4. The sum of the squares of three conjugate semi-diameters of an ellipsoid is constant. (*Agra 1986*)

Let OP, OQ, OR be three conjugate semi-diameters of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

Let the coordinates of P, Q, R be (x_r, y_r, z_r) $r=1, 2, 3.$ Then $x_1^2+x_2^2+x_3^2=a^2,\quad y_1^2+y_2^2+y_3^2=b^2,\quad z_1^2+z_2^2+z_3^2=c^2$

[See relations (III), Art. 3]

Now $OP^2+OQ^2+OR^2$

$$=(x_1^2+y_1^2+z_1^2)+(x_2^2+y_2^2+z_2^2)+(x_3^2+y_3^2+z_3^2)$$
$$=(x_1^2+x_2^2+x_3^2)+(y_1^2+y_2^2+y_3^2)+(z_1^2+z_2^2+z_3^2)$$
$$=a^2+b^2+c^2 \text{ which is constant.}$$

5. The volume of the parallelopiped formed by three conjugate semi-diameters of an ellipsoid as coterminous edges is constant.

Let OP, OQ, OR be any three conjugate semi-diameters with extremities (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3), Then

$$\left.\begin{array}{l} x_1^2+x_2^2+x_3^2=a^2 \\ y_1^2+y_2^2+y_3^2=b^2 \\ z_1^2+z_2^2+z_3^2=c^2 \end{array}\right\} \text{ and } \left.\begin{array}{l} x_1y_1+x_2y_2+x_3y_3=0 \\ y_1z_1+y_2z_2+y_3z_3=0 \\ z_1x_1+z_2x_2+z_3x_3=0 \end{array}\right\} \quad \ldots(A)$$

Volume of the parallelopiped with OP, OQ, OR as coterminous edge$=V=6\times$volume of tetrahedron OPQR.

$$=6\times\frac{1}{6}\begin{vmatrix} 0 & 0 & 0 & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix}=\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \text{ (numerically)}$$

$$\therefore\ V^2=\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}\times\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$$=\begin{vmatrix} \Sigma x_1^2 & \Sigma x_1y_1 & \Sigma x_1z_1 \\ \Sigma x_1y_1 & \Sigma y_1^2 & \Sigma y_1z_1 \\ \Sigma x_1z_1 & \Sigma y_1z_1 & \Sigma z_1^2 \end{vmatrix}=\begin{vmatrix} a^2 & 0 & 0 \\ 0 & b^2 & 0 \\ 0 & 0 & c^2 \end{vmatrix} \text{ Using (A)}$$

$$=a^2b^2c^2$$

$\therefore$ $T=abc$, which is constar.

6. The sum of the squares of the areas of the faces of the parallelopiped with any three conjugate semi-diameters as coterminous edges is constant.

Let OP, OQ, OR be any three conjugate semi-diameters with extremities

$$(x_1, y_1, z_1), (x_2, y_2, z_2) \text{ and } (x_3, y_3, z_3).$$

Since diametral plane of OP passes through Q and R

$$\therefore \quad \begin{cases} \dfrac{x_2x_1}{a^2}+\dfrac{y_2y_1}{b^2}+\dfrac{z_2z_1}{c^2}=0 \\ \dfrac{x_3x_1}{a^2}+\dfrac{y_3y_1}{b^2}+\dfrac{z_3z_1}{c^2}=0 \end{cases}$$

$$\text{or} \quad \begin{cases} \left(\dfrac{x_2}{a}\right)\dfrac{x_1}{a}+\left(\dfrac{y_2}{b}\right)\dfrac{y_1}{b}+\left(\dfrac{z_2}{c}\right)\dfrac{z_1}{c}=0 \\ \left(\dfrac{x_3}{a}\right)\dfrac{x_1}{a}+\left(\dfrac{y_3}{b}\right)\dfrac{y_1}{b}+\left(\dfrac{z_3}{c}\right)\dfrac{z_1}{c}=0 \end{cases}$$

$$\therefore \quad \frac{\dfrac{x_1}{a}}{\dfrac{y_2z_3-y_3z_2}{bc}}=\frac{\dfrac{y_1}{b}}{\dfrac{z_2x_3-z_3x_2}{ca}}=\frac{\dfrac{z_1}{c}}{\dfrac{x_2y_3-x_3y_2}{ab}}$$

$$=\frac{\sqrt{\Sigma \dfrac{x_1^2}{a^2}}}{\sqrt{\Sigma\left(\dfrac{y_2z_3-y_3z_2}{bc}\right)^2}}=\pm\frac{1}{\sin\theta}=\pm\frac{1}{\sin 90^\circ}=\pm 1$$

[Since P lies on $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1 \quad \therefore \quad \Sigma\dfrac{x_1^2}{a^2}=1$

and $\dfrac{x_2}{a}, \dfrac{y_2}{b}, \dfrac{z_2}{c}$; $\dfrac{x_3}{a}, \dfrac{y_3}{b}, \dfrac{z_3}{c}$ are the *d.c.s.* of two perpendicular straight lines.]

$$\therefore \quad \frac{x_1}{a}=\pm\frac{y_2z_3-y_3z_2}{bc}, \quad \frac{y_1}{b}=\pm\frac{z_2x_3-z_3x_2}{ca}, \quad \frac{z_1}{c}=\pm\frac{x_2y_3-x_3y_2}{ab} \qquad \text{...(A)}$$

Let A_1, A_2, A_3 be the areas of the triangles OQR, ORP and OPQ and let l_r, m_r, n_r $(r=1, 2, 3)$ be the *d.c.s.* of normals to these planes.

Projecting the $\triangle$OQR on the plane $x=0$, we get a triangle with vertices $(0, 0, 0)$, $(0, y_2, z_2)$ and $(0, y_3, z_3)$ having area

$=\frac{1}{2}(y_2z_3-y_3z_2)$. The area of the projection is also $A_1 l_1$.

$$\therefore \quad A_1 l_1=\tfrac{1}{2}(y_2z_3-y_3z_2)=\pm\frac{bc\,x_1}{2a} \qquad \text{using (A)}$$

Similarly $A_1m_1=\pm\dfrac{cay_1}{2b}$, $\qquad A_1 n_1=\pm\dfrac{abz_1}{2c}$

Squaring and adding

$$A_1^2=\frac{b^2c^2x_1^2}{4a^2}+\frac{c^2a^2y_1^2}{4b^2}+\frac{a^2b^2z_1^2}{4c^2}$$

$$(\because \quad l_1^2+m_1^2+n_1^2=1)$$

Similarly, projecting the areas ORP and OPQ on the coordinate planes, we get

$$A_2^2 = \frac{b^2c^2x_2^2}{4a^2} + \frac{c^2a^2y_2^2}{4b^2} + \frac{a^2b^2z_2^2}{4c^2}$$

$$A_3^2 = \frac{b^2c^2x_3^2}{4a^2} + \frac{c^2a^2y_3^2}{4b^2} + \frac{a^2b^2z_3^2}{4c^2}$$

Adding, we get

$$A_1^2 + A_2^2 + A_3^2 = \frac{b^2c^2}{4a^2}\ \Sigma x_1^2 + \frac{c^2a^2}{4b^2}\Sigma y_1^2 + \frac{a^2b^2}{4c^2}\Sigma z_1^2$$

$$= \frac{b^2c^2}{4a^2}\ (a^2) + \frac{c^2a^2}{4b^2}\ (b)^2 + \frac{a^2b^2}{4c^2}\ (c^2)$$

$$= \tfrac{1}{4}(b^2c^2 + c^2a^2 + a^2b^2) \text{ which is constant.}$$

7. The sum of the squares of the projection of any three conjugate semi-diameters on any line is constant.

Let l, m, n be the d.cs. of any given line and OP, OQ, OR be the three conjugate semi-diameters.

Projection of OP on this line is $\quad lx_1 + my_1 + nz_1$

" " OQ " " " " $\quad lx_2 + my_2 + nz_2$

" " OR " " " " $\quad lx_3 + my_3 + nz_3$

$\therefore$ the sum of the squares of the projections of OP, OQ and OR on this line

$$= (lx_1 + my_1 + nz_1)^2 + (lx_2 + my_2 + nz_2)^2 + (lx_3 + my_3 + nz_3)^2$$

$$= l^2\Sigma x_1^2 + m^2\Sigma y_1^2 + n^2\Sigma z_1^2 + 2lm\Sigma x_1y_1 + 2mn\ \Sigma y_1z_1 + 2nl\Sigma z_1x_1$$

$$= l^2(a^2) + m^2(b^2) + n^2(c^2) + 2ml(0) + 2mn(0) + 2nl(0)$$

$$= a^2l^2 + b^2m^2 + c^2n^2 \quad \text{which is constant.}$$

8. The sum of the squares of the projections of any three conjugate semi-diameters on any plane is constant.

Let l, m, n be the d.c.s. of the normal to any given plane. Let OP, OQ, OR be the three conjugate semi-diameters.

Sum of the squares of the projections of OP, OQ and OR on this plane

$$= [OP^2 - (lx_1 + my_1 + nz_1)^2] + [OQ^2 - (lx_2 + my_2 + nz_2)^2] + [OR^2 - (lx_3 + my_3 + nz_3)^2]$$

$$= (OP^2 + OQ^2 + OR^2) - (lx_1 + my_1 + nz_1)^2 + (lx_2 + my_2 + nz_2)^2 + (lx_3 + my_3 + nz_3)^2]$$

$$= a^2 + b^2 + c^2 - (a^2l^2 + b^2m^2 + c^2n^2) \quad | \text{ using Arts. 4 and 7}$$

$$= a^2(1 - l^2) + b^2(1 - m^2) + c^2(1 - n^2)$$

$$= a^2(m^2 + n^2) + b^2(n^2 + l^2) + c^2(l^2 + m^2)$$

$$(\because \quad l^2 + m^2 + n^2 = 1)$$

which is constant.

Example 1. *If* (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) *be the extremities of three conjugate semi-dimeters of the ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

find the equation of the plane through these points.

(Agra 1984, 81 ; Allahabad 1981 ; Rohilkhand 1980)

Sol. Let P, Q, R be the extremities (x_r, y_r, z_r), $r=1, 2, 3$ of the three conjugate semi-diameters OP, OQ, OR.

Let the equation of the plane PQR be

$$lx+my+nz=p \qquad \ldots(i)$$

Since P, Q, R all lie on (*i*)

$$\therefore \qquad lx_1+my_1+nz_1=p \qquad \ldots(ii)$$

$$lx_2+my_2+nz_2=p \qquad \ldots(iii)$$

$$lx_3+my_3+nz_3=p \qquad \ldots(iv)$$

Multiplying (*ii*) by x_1, (*iii*) by x_2 and (*iv*) by x_3 and adding, we get

$$l\Sigma x_1^2+m\Sigma x_1y_1+n\Sigma z_1x_1=p(x_1+x_2+x_3)$$

or $$la^2+m(0)+n(0)=p(x_1+x_2+x_3)$$

$$\therefore \qquad l=\frac{p}{a^2}(x_1+x_2+x_3)$$

Similarly multiplying (*ii*), (*iii*), (*iv*) by y_1, y_2, y_3 respectively and adding

$$m=\frac{p}{b^2}(y_1+y_2+y_3)$$

and multiplying (*ii*), (*iii*), (*iv*) by z_1, z_2, z_3 respectively and adding

$$n=\frac{p}{c^2}(z_1+z_2+z_3)$$

Substituting the values of l, m, n in (*i*), equation of plane PQR is

$$\frac{p}{a^2}(x_1+x_2+x_3)x+\frac{p}{b^2}(y_1+y_2+y_3)y+\frac{p}{c^2}(z_1+z_2+z_3)z=p$$

or $$\frac{x}{a^2}(x_1+x_2+x_3)+\frac{y}{b^2}(y_1+y_2+y_3)+\frac{z}{c^2}(z_1+z_2+z_3)=1.$$

Example 2. *Show that the plane PQR, where P, Q, R are the extremities of three conjugate semi-diameters of the ellipsoid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, *touches the ellipsoid*

$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{1}{3}$ *at the centroid of the triangle PQR.*

(Agra 1987 ; Venka 1983 ; Utkal 1981)

Sol. Proceeding as in Ex. 1, the equation of plane PQR is

$$\frac{x}{a^2}(x_1+x_2+x_3)+\frac{y}{b^2}(y_1+y_2+y_3)+\frac{z}{c^2}(z_1+z_2+z_3)=1 \quad ...(i)$$

The centroid of $\triangle$PQR is

$$G\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}, \frac{z_1+z_2+z_3}{3}\right)$$

The tangent plane to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{1}{3} \text{ at G is}$$

$$\frac{x\left(\frac{x_1+x_2+x_3}{3}\right)}{a^2}+\frac{y\left(\frac{y_1+y_2+y_3}{3}\right)}{b^2}+\frac{z\left(\frac{z_1+z_2+z_3}{3}\right)}{c^2}=\frac{1}{3}$$

or $$\frac{x}{a^2}(x_1+x_2+x_3)+\frac{y}{b^2}(y_1+y_2+y_3)+\frac{z}{c^2}(z_1+z_2+z_3)=1$$

which is the same as (i).

Hence the plane PQR touches the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{1}{3} \text{ at G.}$$

Example 3. *Prove that the pole of the plane PQR lies on the ellipsoid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=3$, *where OP, OQ, OR are three conjugate semi-diameters of the ellipsoid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Sol. Equation of the plane PQR is

$$\frac{x}{a^2}(x_1+x_2+x_3)+\frac{y}{b^2}(y_1+y_2+y_3)+\frac{z}{c^2}(z_1+z_2+z_3)=1 \quad ...(i)$$

| See Ex. 1

Let (x', y', z') be the pole of the plane PQR:

Equation to the polar plane of (x', y', z') w.r.t. the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \text{ is}$$

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}+\frac{zz'}{c^2}=1 \quad ...(ii)$$

Comparing (i) and (ii), we have

$$x'=x_1+x_2+x_3, \quad y'=y_1+y_2+y_3, \quad z'=z_1+z_2+z_3$$

$$\therefore \quad \left(\frac{x'}{a}\right)^2+\left(\frac{y'}{b}\right)^2+\left(\frac{z'}{c}\right)^2$$

$$=\frac{1}{a^2}(x_1+x_2+x_3)^2+\frac{1}{b^2}(y_1+y_2+y_3)^2+\frac{1}{c^2}(z_1+z_2+z_3)^2$$

$$=\frac{1}{a^2}\Sigma x_1^2+\frac{1}{b^2}\Sigma y_1^2+\frac{1}{c^2}\Sigma z_1^2+\frac{2}{a^2}\Sigma x_1x_2+\frac{2}{b^2}\Sigma y_1y_2+\frac{2}{c^2}\Sigma z_1z_2$$

$$=\frac{1}{a^2}(a^2)+\frac{1}{b^2}(b^2)+\frac{1}{c^4}(c^2)+\frac{2}{a^2}(0)+\frac{2}{b^2}(0)+\frac{2}{c^2}(0)$$

$$=3$$

Hence (x', y', z') lies on the surface

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=3.$$

Example 4. *Prove that the locus of the foot of the perpendicular from the centre of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ to the plane PQR through the extremities of three conjugate semi-diameters is*

$$a^2x^2+b^2y^2+c^2z^2=3(x^2+y^2+z^2)^2.$$

Sol. Equation to the plane PQR through the extermities of three conjugate semi-diameters OP, OQ, OR is

$$\frac{x}{a^2}(x_1+x_2+x_3)+\frac{y}{b^2}(y_1+y_2+y_3)+\frac{z}{c^2}(z_1+z_2+z_3)=1 \quad \ldots(i)$$

| See Ex. 1

Let $D(\alpha, \beta, \gamma)$ be the foot of the perpendicular from the centre $(0, 0, 0)$ to the plane (i).

Then the d.c.s. of OD, the normal to (i) are proportional to α, β, γ.

$$\therefore \quad \frac{\frac{x_1+x_2+x_3}{a^2}}{\alpha}=\frac{\frac{y_1+y_2+y_3}{b^2}}{\beta}=\frac{\frac{z_1+z_2+z_3}{c^2}}{\gamma}=\frac{1}{\lambda}\text{ (say)} \quad \ldots(ii)$$

Since (α, β, γ) lies on (i)

$$\therefore \quad \Sigma\frac{\alpha(x_1+x_2+x_3)}{a^2}=1 \quad \ldots(iii)$$

From (ii), we have

$$\frac{\alpha^2}{\lambda}+\frac{\beta^2}{\lambda}+\frac{\gamma^2}{\lambda}=\Sigma\frac{\alpha(x_1+x_2+x_3)}{a^2}=1 \quad \text{By } (iii)$$

$$\therefore \quad \lambda=\alpha^2+\beta^2+\gamma^2 \quad \ldots(iv)$$

Also from (*ii*)

$$(x_1+x_2+x_3)^2=\left(\frac{a^2\alpha}{\lambda}\right)^2 \quad \therefore \quad \frac{(x_1+x_2+x_3)^2}{a^2}=\frac{a^2\alpha^2}{\lambda^2}$$

or $$\frac{\Sigma x_1^2+2\Sigma x_1x_2}{a^2}=\frac{a^2\alpha^2}{\lambda^2}$$

or $$\frac{a^2}{a^2}=\frac{a^2\alpha^2}{\lambda^2} \text{ since } \Sigma x_1^2=a^2,\ \Sigma x_1x_2=0$$

or $$\frac{a^2\alpha^2}{\lambda^2}=1$$

Similarly $\dfrac{b^2\beta^2}{\lambda^2}=1,\quad \dfrac{c^2\gamma^2}{\lambda^2}=1$

Adding $\dfrac{a^2\alpha^2}{\lambda^2}+\dfrac{b^2\beta^2}{\lambda^2}+\dfrac{c^2\gamma^2}{\lambda^2}=3$

or $$a^2\alpha^2+b^2\beta^2+c^2\gamma^2=3\lambda^2=3(\alpha^2+\beta^2+\gamma^2)^2 \qquad | \text{ using } (iv)$$

Hence the locus of (α, β, γ) is

$$a^2x^2+b^2y^2+c^2z^2=3(x^2+y^2+z^2).$$

Example 5. *Find the locus of the equal conjugate diameters of the ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1. \quad \text{(Vikram 1984 ; L.N.M. 1982)}$$

Sol. Let OP, OQ, OR be three equal conjugate semi-diameters. Then we have

$$OP^2+OQ^2+OR^2=a^2+b^2+c^2$$

$\because \quad OP^2=OQ^2=OR^2$

$\therefore \quad \text{each}=\frac{1}{3}(a^2+b^2+c^2) \qquad \ldots(i)$

Let P be the point (x_1, y_1, z_1). Equations of OP are

$$\frac{x}{x_1}=\frac{y}{y_1}=\frac{z}{z_1} \qquad \ldots(ii)$$

where $x_1^2+y_1^2+z_1^2=OP^2=\frac{1}{3}(a^2+b^2+c^2) \qquad \ldots(iii)$

| using (*i*)

Also $$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=1 \qquad \ldots(iv)$$

($\because$ P lies on the ellipsoid).

From (*iii*) and (*iv*), we have

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=\frac{3(x_1^2+y_1^2+z_1^2)}{a^2+b^2+c^2} \qquad \ldots(v)$$

| (each=1)

Eliminating x_1, y_1, z_1 from (*ii*) and (*v*), the required locus is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{3(x^2+y^2+z^2)}{a^2+b^2+c^2}.$$

Example 6. *Prove that the plane through a pair of equal conjugate semi-diameters of the ellipsoid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ *touches the cone* $\frac{x^2}{a^2(2a^2-b^2-c^2)}+\frac{y^2}{b^2(2b^2-c^2-a^2)}+\frac{z^2}{c^2(2c^2-a^2-b^2)}=0.$

Sol. Let P, Q, R be the extremities of three equal conjugate semi-diameters OP, OQ, OR.

Let the coordinates of P be (x_1, y_1, z_1). Also let

The plane OQR through the conjugate semi-diameters OQ and OR is the diameter plane of OP.

$\therefore$ The equation of the plane OQR is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=0 \qquad \ldots(i)$$

The plane (*i*) will touch the cone $\Sigma\ \frac{x^2}{a^2(2a^2-b^2-c^2)}=0$

if $\quad \Sigma\ \frac{x_1^2}{a^4}.\ a^2(2a^2-b^2-c^2)=0$

$\left(\because\right.$ The plane $lx+my+nz=p$ touches $ax^2+by^2+cz^2=1$

if $\left.\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p^2\right)$

or $\quad$ if $\Sigma\ \frac{x_1^2}{a^2}[3a^2-(a^2+b^2+c^2)]=0$

or $\quad$ if $\Sigma 3x_1^2-\Sigma\ \frac{x_1^2}{a^2}(a^2+b^2+c^2)=0$

or $\quad$ if $3\Sigma\ x_1^2=(a^2+b^2+c^2)\ \Sigma\ \frac{x_1^2}{a^2}$

or $\quad$ if $3(x_1^2+y_1^2+z_1^2)=(a^2+b^2+c^2)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}\right)$

or $\quad$ if $3\ .\ OP^2=(a^2+b^2+c^2)(1)$ $\qquad | \because$ P lies on the ellipsoid

or $\quad$ if $3r^2=a^2+b^2+c^2$

which is true $\because\ OP^2+OQ^2+OR^2=a^2+b^2+c^2$

and $\qquad OP=OQ=OR=r$

Hence the result.

Example 7. *Prove that the locus of the section of the ellipsoid*

$$\sum\left(\frac{x^2}{a^2}\right)=1$$

by the plane PQR is the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{1}{3}.$$

Sol. The equation of the plane PQR is

$$\left(\frac{x_1+x_2+x_3}{a^2}\right)x+\left(\frac{y_1+y_2+y_3}{b^2}\right)y+\left(\frac{z_1+z_2+z_3}{c^2}\right)z=1 \quad \ldots(i)$$

If (α, β, γ) be the centre of the section of the given ellipsoid by the plane PQR, then the equation of PQR can be written as

"$T=S_1$"

i.e.
$$\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}+\frac{\gamma z}{c^2}=\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2} \quad \ldots(ii)$$

$\therefore$ Equations (*i*) and (*ii*) represent the same plane, therefore, comparing them, we get

$$\frac{x_1+x_2+x_3}{\alpha}=\frac{y_1+y_2+y_3}{\beta}=\frac{z_1+z_2+z_3}{\gamma}=\frac{1}{\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}}$$

where
$$\frac{\alpha}{a}=\left(\frac{x_1+x_2+x_3}{a}\right)\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}\right)$$

Similarly,

$$\frac{\beta}{b}=\left(\frac{y_1+y_2+y_3}{b}\right)\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}\right)$$

and
$$\frac{\gamma}{c}=\left(\frac{z_1+z_2+z_3}{c}\right)\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}\right)$$

Squaring and adding, we get

$$\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}=\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}\right)^2\left[\left(\frac{x_1+x_2+x_3}{a^2}\right)\right.$$
$$\left.+\left(\frac{y_1+y_2+y_3}{b^2}\right)^2+\left(\frac{z_1+z_2+z_3}{c^2}\right)^2\right]$$

$$\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}=3\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}\right)^2$$

or
$$\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}=\frac{1}{3}$$

$\therefore$ The required locus of (α, β, γ) is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{1}{3}.$$

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Example 8. *Prove that the locus of the point of intersection of three tangent planes to* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ *which are parallel to conjugate diameteral planes of* $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=1$ *is*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}.$$

Sol. Let $P(x_1, y_1, z_1)$, $Q(x_2, y_2, z_2)$ and $R(x_3, y_3, z_3)$ be the extremities of the conjugate semi-diameters of the ellipsoid

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=1 \quad \ldots(i)$$

Then the diameteral plane of P w.r.t. to (i) is

$$\frac{xx_1}{\alpha^2}+\frac{yy_1}{\beta^2}+\frac{zz_1}{\gamma^2}=0 \quad \ldots(ii)$$

Any plane parallel to (ii) is

$$\frac{xx_1}{\alpha^2}+\frac{yy_1}{\beta^2}+\frac{zz_1}{\gamma^2}=p_1 \quad \ldots(iii)$$

If (iii) is a tangent plane to $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, then

"$p^2=a^2l^2+b^2m^2+c^2n^2$".

or $\quad p_1^2=a^2\left(\frac{x_1}{\alpha^2}\right)^2+b^2\left(\frac{y_1}{\beta^2}\right)^2+c^2\left(\frac{z_1}{\gamma^2}\right)^2. \quad \ldots(iv)$

Similarly the equation of other planes parallel to OQ and OR are

$$\frac{xx_2}{\alpha_2}+\frac{yy_2}{\beta^2}+\frac{zz_2}{\gamma^2}=p_2 \quad \ldots(v)$$

and $\quad \frac{xx_3}{\alpha^2}+\frac{yy_3}{\beta^2}+\frac{zz_3}{\gamma^2}=p_3 \quad \ldots(vi)$

where $\quad p_2^2=\sum\left[a^2\left(\frac{x_2}{\alpha^2}\right)^2\right]$

and $\quad p_3^2=\sum\left[a^2\left(\frac{x_3}{\alpha^2}\right)^2\right]$

Now for the locus of the point of intersection of (*iii*), (*iv*) and (*vi*), we square and add (*iii*), (*iv*), (*vi*) and get

$$\sum \frac{x^2}{\alpha^4}(x_1^2+x_2^2+x_3^2)+\sum \frac{2xy}{\alpha^2\beta^2}(x_1y_1+x_2y_2+x_3y_3)$$
$$=p_1^2+p_2^2+p_3^2$$

or $$\sum\left[\frac{x^2}{\alpha^4}(\alpha^2)\right]+\sum \frac{2xy}{\alpha^2\beta^2}(0)=\sum \frac{a^2}{\alpha^4}(x_1^2+x_2^2+x_3^2),$$

from (*iv*) and (*vii*) using $\Sigma x_1^2=\alpha^2$ and $x_1y_1=0$ etc.

or $$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=\frac{a^2}{\alpha^4}(\alpha^2)+\frac{b^2}{\beta^4}(\beta^2)+\frac{c^2}{\gamma^4}(\gamma^2),$$

Since $\Sigma x_1^2=\alpha^2$ etc.

or $$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=\frac{a^2}{\alpha^2}+\frac{b^3}{\beta^2}+\frac{c^3}{\gamma^2}$$

Set - VII

Reduction of General Equation of second degree

General equation of the second degree:-

The most general equation of second degree is written as $F(x,y,z) \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$ ——— (1)

(or)

$f(x,y,z) \equiv f(x,y,z)+2ux+2vy+2wz+d=0$ ——— (2)

where $f(x,y,z) \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy.$ ——— (3)

The equation (1) contains ~~ten~~ unknown which can be reduced to nine constants which effective constants by dividing the equation throughout by 'd'.

Thus a surface can be determined with the help of nine conditions which give rise to nine independent relations between the constants.

Note (1):- In all discussions in this chapter, we shall take $F(x,y,z)$ and $f(x,y,z)$ as defined in (1) and (3) above i.e $f(x,y,z)$ will be ~~taken~~ as the homogeneous part of $F(x,y,z)$.

Note:- (2) Here $\frac{\partial f}{\partial x} = 2(ax+hy+gz)$

$\frac{\partial f}{\partial y} = 2(hx+by+fz)$

$\frac{\partial f}{\partial z} = 2(gx+fy+cz)$; and

$\frac{\partial F}{\partial x} = 2(ax+hy+gz+u)$,

$\frac{\partial F}{\partial y} = 2(hx+by+fz+v)$, $\frac{\partial F}{\partial z} = 2(gx+fy+cz+w)$.

* **Determination of the centre of surface $f(x,y,z)=0$:—**

Let (x_1, y_1, z_1) be the centre of the surface $f(x,y,z)=0$.

shifting the origin to the centre (x_1, y_1, z_1), the transformed equation of the surface

$$f(x+x_1, y+y_1, z+z_1)=0$$

i.e
$$a(x+x_1)^2+b(y+y_1)^2+c(z+z_1)^2+2f(y+y_1)(z+z_1)$$
$$+2g(z+z_1)(x+x_1)+2h(x+x_1)(y+y_1)$$
$$+2u(x+x_1)+2v(y+y_1)+2w(z+z_1)$$
$$+d=0.$$

$$\Rightarrow f(x,y,z)+2x(ax_1+hy_1+gz_1+u)+2y(hx_1+by_1+fz_1+v)$$
$$+2z(gx_1+fy_1+cz_1+w)+(ax_1^2+by_1^2+cz_1^2$$
$$+2fy_1z_1+2gz_1x_1+2hx_1y_1+2ux_1+2vy_1+$$
$$2wz_1+d)=0. \qquad \text{—— (I)}$$

where
$$f(x,y,z)=ax^2+by^2+cz^2+2fyz$$
$$+2gzx+2hxy.$$

Now as the centre of (I) is origin, so it should be homogeneous in (x,y,z) (since if (x',y',z') is a point on it, $(-x',-y',-z')$ must also lie on it as $(0,0,0)$, the mid-point of the chord joining (x',y',z') and $(-x',-y',-z')$, is the centre of the surface $f(x,y,z)=0$ and therefore only second degree terms must exist in (I).

$\therefore$ From (I) we have $\quad ax_1 + hy_1 + gz_1 + u = 0 \qquad$ ...(II)

$$hx_1 + by_1 + fz_1 + v = 0 \qquad \text{...(III)}$$

$$gx_1 + fy_1 + cz_1 + w = 0 \qquad \text{...(IV)}$$

Also constant term in (I) can be rewritten as

$x_1(ax_1 + hy_1 + gz_1 + u) + y_1(hx_1 + by_1 + fz_1 + v)$

$$+ z_1(gx_1 + fy_1 + cz_1 + w) + (ux_1 + vy_1 + wz_1 + d)$$

$= ux_1 + vy_1 + wz_1 + d$, with the help of (II), (III), (IV).

$= d'$, (say)

Then (I) reduces to $\quad f(x, y, z) + d' = 0 \qquad$ ...(V)

where $\quad d' = ux_1 + vy_1 + wz_1 + d. \qquad$ ...(VI)

And x_1, y_1, z_1 is obtained from (II), (III) and (IV),

which can be obtained from

$\frac{\partial F}{\partial x} = 0,\ \frac{\partial F}{\partial y} = 0,\ \frac{\partial F}{\partial z} = 0$ replacing x, y, z by x_1, y_1, z_1.

Hence **centre of the surface** $F(x, y, z) = 0$, is given by solving

$$\frac{\partial F}{\partial x} = 0,\ \frac{\partial F}{\partial y} = 0,\ \frac{\partial F}{\partial z} = 0 \text{ for } x, y, z$$

and **the equation of the surface referred to centre as origin** is

$$f(x, y, z) + (ux_1 + vy_1 + wz_1 + d) = 0, \qquad \text{...(VII)}$$

where (x_1, y_1, z_1) is the centre of the surface. **(Remember)**

Note. The equations (II), (III), (IV) may or may not give a unique centre. There may be more than one centre, a line of centres or a plane of centres depending upon the nature of solutions of the above three equations.

From (II), (III), (IV) we find that the centre (x_1, y_1, z_1) lies on the planes

$$\left.\begin{array}{r} ax + hy + gz + u = 0 \\ hx + by + fz + v = 0 \\ \text{and} \quad gx + fy + cz + w = 0 \end{array}\right\} \qquad \text{...(VIII)}$$

These planes are known as **central planes** and any point common to these planes is a **centre.**

§ 12.03. Transformation of f (x, y, z).

To show that by the rotation of axes the expression $f(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$ *transforms to* $\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 = 0$, *where* $\lambda_1, \lambda_2, \lambda_3$ *are the roots of the cubic*

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0$$

or $\quad \lambda^3 - \lambda^2(a + b + c) + \lambda(ab + bc + ca - f^2 - g^2 - h^2)$

$$- (abc + 2fgh - af^2 - bg^2 - ch^2) = 0$$

or $\quad \lambda^3 - \lambda^2(a + b + c) + \lambda(A + B + C) - D = 0,$

where A, B, C are the cofactors of a, b, c respectively in the determinant

$$D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

distance.

We know that the expression $x^2 + y^2 + z^2$ is an invariant when the rectangular axes are rotated through the same origin.

(See chapter on Change of axes)

$\therefore$ If we put $x = l_1x + m_1y + n_1z$, $y = l_2x + m_2y + n_2z$

and $z = l_3x + m_3y + n_3z$ in $x^2 + y^2 + z^2$, then by the relations

$l_1^2 + m_1^2 + n_1^2 = 1$, $l_2^2 + m_2^2 + n_2^2 = 1$, $l_3^2 + m_3^2 + n_3^2 = 1$;

$l_1^2 + l_2^2 + l_3^2 = 1$, $m_1^2 + m_2^2 + m_3^2 = 1$, $n_1^2 + n_2^2 + n_3^2 = 1$,

$l_1l_2 + m_1m_2 + n_1n_2 = 0$, $l_2l_3 + m_2m_3 + n_2n_3 = 0$, $l_3l_1 + m_3m_1 + n_3n_1 = 0$

$l_1m_1 + l_2m_2 + l_3m_3 = 0$, $m_1n_1 + m_2n_2 + m_3n_3 = 0$, $n_1l_1 + n_2l_2 + n_3l_3 = 0$,

it remains unchanged.

Now if the axes are rotated in such a manner that $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$ becomes $\lambda_1x^2 + \lambda_2y^2 + \lambda_3z^2$, then the expression

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy - \lambda(x^2 + y^2 + z^2) \quad \text{...(i)}$$

should reduce to $\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 - \lambda(x^2 + y^2 + z^2)$ ...(ii)

i.e. both the expressions (i) and (ii) will be the product of linear factors for the same value of λ.

Now if (i) *i.e.* if $(a - \lambda)x^2 + (b - \lambda) + (c - \lambda)z^2 + 2fyz + 2gzx + 2hxy$ is the product of two linear factors then we must have

$$\begin{vmatrix} a - \lambda & h & g \\ h & b - \lambda & f \\ g & f & c - \lambda \end{vmatrix} = 0 \quad \text{...(iii)}$$

or $\lambda^3 - \lambda^2(a + b + c) + \lambda(ab + bc + ca - f^2 - g^2 - h^2)$

$$- (abc + 2fgh - af^2 - bg^2 - ch^2) = 0$$

or $\lambda^3 - \lambda^2(a + b + c) + \lambda(A + B + C) - D = 0$,

where A, B, C are the cofactors of corresponding small letters in the determidant

$$D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

And the expression $(\lambda_1 - \lambda)x^2 + (\lambda_2 - \lambda)y^2 + (\lambda_3 - \lambda)z^2$ in (ii) will be the product of two linear factors, if

$$\begin{vmatrix} \lambda_1 - \lambda & 0 & 0 \\ 0 & \lambda_2 - \lambda & 0 \\ 0 & 0 & \lambda_3 - \lambda \end{vmatrix} = 0 \text{ i.e. } (\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda) = 0$$

i.e. when $\lambda = \lambda_1$ or λ_2 or λ_3.

The same values of λ should be obtained from (iii) also.

Hence $\lambda_1, \lambda_2, \lambda_3$ are the roots of the cubic (iii) in λ, which is called **discriminating cubic.**

Let λ be any root (real or imaginary) of the discriminating cubic (iii) and that l, m, n be the principal direction cosines (which may also be real or imaginary) corresponding to this value of λ, then

$$al + hm + gn = \lambda l$$
$$hl + bm + fn = \lambda m$$
$$gl + fm + cn = \lambda n \quad \textbf{[See chapter on Change of axes]}$$

or
$$(a - \lambda) l + hm + gn = 0$$
$$hl + (b - \lambda) m + fn = 0$$
$$gl + fm + (c - \lambda) n = 0$$

where λ is to be replaced by $\lambda_1, \lambda_2, \lambda_3$, to get the corresponding direction-cosines of the axes.

§ 12.04. Various Forms of General Equation of Second Degree.

The general equation of second degree viz $F(x, y, z) = 0$, as given in § 12.01 Page 1 of this chapter, can be reduced to any one of the following forms :—

S. No.	Equation	Name of the surface
1.	$Ax^2 + By^2 + Cz^2 = 1$	Ellipsoid
2.	$Ax^2 + By^2 - Cz^2 = 1$	Hyperboloid of one sheet
3.	$Ax^2 - By^2 - Cz^2 = 1$	Hyperboloid of two sheets
4.	$Ax^2 + By^2 + Cz^2 = 0$	Cone
5.	$Ax^2 + By^2 + 2kz = 0$	Elliptic paraboloid
6.	$Ax^2 - By^2 + 2kz = 0$	Hyperbolic paraboloid
7.	$Ax^2 + By^2 + d = 0$	Elliptic cylinder
8.	$Ax^2 - By^2 + d = 0$	Hyperbolic cylinder
9.	$Ax^2 - By^2 = 0$	Pair of planes
If second degree terms form a perfect square, then		
10.	$Ax^2 + Bx + C = 0$	Pair of parallel planes
11.	$y^2 = Ax$	Parabolic cylinder
12.	$A(x^2 + y^2) + Bz = 0$	Paraboloid of revolution
13.	$A(x^2 + y^2) + Cz^2 = 1$	Ellipsoid of revolution
14.	$A(x^2 - y^2) + Cz^2 = 1$	Hyperboloid of revolution

Note. In discussion to follow we shall take

$$f(l, m, n) \equiv al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm,$$

whence
$$\frac{\partial f}{\partial l} = 2(al + bm + gn).$$

$$\frac{\partial f}{\partial m} = 2\,(hl + bm + fn),\quad \frac{\partial f}{\partial n} = 2\,(gl + fm + cn)$$

§ 12.05. Equation of surface referred to centre as origin.

From § 12.02. result (VIII) on Page 2 of this chapter we know that the centre (x_1, y_1, z_1) of $F(x, y, z) = 0$, lies on the planes given by

$$S_1 \equiv ax + hy + gz + u = 0 \qquad \text{...(i)}$$

$$S_2 \equiv hx + by + fz + v = 0 \qquad \text{...(ii)}$$

$$S_3 \equiv gx + fy + cz + w = 0 \qquad \text{...(iii)}$$

Multiplying (i), (ii) and (iii) by A, H, G respectively and adding we get

$$Dx + (Au + Hv + Gw) = 0, \qquad \text{(iv)}$$

where A, B, C, F, G, H are the cofactors of the corresponding small letters viz. a, b, c, f, g, h in the determinant $D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$.

Similarly multiplying (i), (ii), (iii) by H, B, F and G, F, C respectively and adding separately, we get

$$Dy + (Hu + Bv + Fw) = 0 \qquad \text{...(v)}$$

$$Dz + (Gu + Fv + Cw) = 0 \qquad \text{...(vi)}$$

$\therefore$ From (iv), (v) and (vi) the coordinates of the centre are given by

$$\frac{x}{Au + Hv + Gw} = \frac{y}{Hu + Bv + Fw} = \frac{z}{Gu + Fv + Cw} = -\frac{1}{D} \qquad \text{...(A)}$$

Cor. 1. The equation of a diametral plane of the surface (conicoid) $F(x, y, z) = 0$ is

$$l\frac{\partial F}{\partial x} + m\frac{\partial F}{\partial y} + n\frac{\partial F}{\partial z} = 0 \qquad \text{...(B)}$$

and so any diametral plane passes through the centre or centres.

Cor. 2. From (II), (III), (IV) and (VI) of § 12.02 Page 2 of this chapter we have

$$ax_1 + hy_1 + gz_1 + u = 0$$

$$hx_1 + by_1 + fz_1 + v = 0$$

$$gx_1 + fy_1 + cz_1 + w = 0$$

and

$$ux_1 + vy_1 + wz_1 + (d - d') = 0$$

which on eliminating x_1, y_1, z_1 gives $\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d-d' \end{vmatrix} = 0$

or

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} - d' \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

or

$$P - d'D = 0 \quad \text{or} \quad d' = P/D, \qquad \text{...(C)}$$

where

$$P = \begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} \quad \text{and} \quad D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

Hence referred to centre as origin [See result (VII) of § 12.02. Page 2 of this chapter] the equation of the surface $F(x, y, z) = 0$ is

$$f(x, y, z) + (P/D) = 0 \qquad ...(E)$$

***§ 12.06. Some properties of determinant D.**

We know $D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$ and A, B, C, F, G, H denote the cofactors of the corresponding small letters in this determinant D.

$$\therefore \quad A = bc - f^2,\ B = ca - g^2,\ C = ab - h^2,$$
$$F = gh - af,\ G = hf - bg,\ H = fg - ch.$$

Also $BC - F^2 = (ca - g^2)(ab - h^2) - (gh - af)^2$

$$= a^2bc - abg^2 - ach^2 + g^2h^2 - g^2h^2 - a^2f^2 + 2afgh$$
$$= a(abc + 2fgh - af^2 - bg^2 - ch^2) = aD$$

Similarly $CA - G^2 = bD,\ AB - H^2 = cD,$

$$GH - AF = fD,\ HF - BG = gD,\ FG - CH = hD.$$

And from the properties of determinants **(See Author's Algebra or Matrices)** we know that

$$Aa + Hh + Gg = D,\ Ha + Bh + Fg = 0,\ Ga + Fh + Cg = 0$$

and similar other results.

(i) If $D = 0$, from above we have

$$BC = F^2,\ CA = G^2,\ AB = H^2,\ GH = AF,\ HF = BG,\ FG = CH$$

(ii) If $D = 0$ and $A = 0$, then we have $G = 0,\ H = 0$.

(iii) If $D = 0$ and $A = 0$, $B = 0$, then $F = 0$, $G = 0$, $H = 0$ but C may or may not be zero.

(iv) If $D = 0$ and $H = 0$, then either $A = 0, G = 0$ or $B = 0,\ F = 0$.

(v) If $D = 0$ and $A + B + C = 0$, then

$$A = B = C = F = G = H = 0. \qquad \textbf{(Note)}$$

since, A, B, C have the same sign when $D = 0$ and so $A + B + C = 0$ gives $A = B = C = 0$, whence $F = 0 = G = H$.

§ 12.07. Some facts about planes (to be remembered).

Let there be two equatons

$$\left.\begin{aligned} a_1x + b_1y + c_1z + d_1 = 0 \\ a_2x + b_2y + c_2z + d_2 = 0 \end{aligned}\right\} \qquad ...(I)$$

and each representing a plane,

These two equatons will **represent the same plane,** if

$$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} = \frac{d_1}{d_2} \qquad ...(II)$$

i.e. $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = \begin{vmatrix} c_1 & d_1 \\ c_2 & d_2 \end{vmatrix} = 0$

The planes given by (I) **will be parallel but not the same** provided

$$\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}\neq\frac{d_1}{d_2} \quad ...(III)$$

i.e. $$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} = 0, \begin{vmatrix} c_1 & d_1 \\ c_2 & d_2 \end{vmatrix} \neq 0$$

and will intersect in a line provided

$$\frac{a_1}{a_2}\neq\frac{b_1}{b_2}\neq\frac{c_1}{c_2} \quad ...(IV)$$

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i.e. $$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0, \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \neq 0$$

§ 12.08. Various Cases.

Here we shall consider the various cases which depend on the solution of the equations (i), (ii), (iii) of § 12.05 on Page 5 of this chapter.

Case I. D ≠ 0.

In this case the coordinates of the centre as obtained from (A) of § 12.05 Page 5 of this chapter are finite and unique.

∴ The surface (conicoid) $F(x, y, z)=0$ has a unique centre at a finite distance.

Case II. D = 0 and Au + Hv + Gw ≠ 0.

In this case the coordinates of the centre as obtained from (A) of § 12.05 Page 5 of this chapter are infnite, provided

$Au+Hv+Gw$, $Hu+Bv+Fw$ and $Gu+Fv+Cw$ are not zero.

Thus the surface $F(x, y, z)=0$ has a single centre at infinity.

Case III. D = 0, Au + Hv + Gw = 0.

If we denote the equations (i), (ii) and (iii) of § 12.05 on Page 5 of this chapter by $S_1=0$, $S_2=0$ and $S_3=0$ respectively then we can see that

$$AS_1+HS_2+GS_3=0.$$

∴ The central planes (see definition on § 12.02 Page 2 of this chapter) have a common line of intersection.

Also if $A=bc-f^2\neq 0$, then the planes $S_2=0$ and $S_3=0$ are neither identical nor parallel. So there is a definite line of intersection and the surface $F(x, y, z)=0$ in this case possesses a line of centres at a finite distance.

We can easily see that when $D=0$ and $Au+Hv+Gw=0$ but $A\neq 0$, then $Hu+Bv+Fw=0$ and $Gu+Fv+Cw$ are also zero, since in this case from § 12.06 (i) Page 6 we have $F=\sqrt{(BC)},\ G=\sqrt{(CA)},\ H=\sqrt{(AB)}$ ...(α)

$\therefore Au+Hv+Gw=0 \Rightarrow \sqrt{A}\,(\sqrt{A}u+\sqrt{B}v+\sqrt{C}w)=0$

$$\Rightarrow \sqrt{A}u+\sqrt{B}v+\sqrt{C}w=0,\ \because A\neq 0 \quad ...(\beta)$$

Now $Hu+Bv+Fw=\sqrt{(AB)}\,u+Bv+\sqrt{(BC)}\,w$

$$=\sqrt{B}\,[\sqrt{A}u+\sqrt{B}v+\sqrt{C}w]=0, \text{ from } (\beta)$$

Similarly we can prove that $Gu + Fv + Cw = 0$

[Also we can see from above that if $D = 0$, $A = 0$, then from (α) we get

$$G = 0,\ H = 0.$$

Hence in this case $Au + Hv + Gw = 0$ but $Hu + Bv + Fw$ and $Gu + Fv + Cw$ may or may not be zero].

Case IV. A, B, C, F, G, H are all zero.

As in case III above, the central planes have a common line of intersection. But these planes are parallel as is evident from § 12.07 Page 6 of this chapter.

Also we assume that $fu - gv \neq 0$, because otherwise the two planes given by (i) and (ii) of § 12.05 Page 5 of this chapter would be identical and similarly $fu - hw \neq 0$, as otherwise the planes given by (i) and (iii) of § 12.05 Page 5 of this chapter would be identical.

Hence in this case central planes [given by (i), (ii) and (iii) of § 12.05 Page 5 of this chapter] are parallel but not coincident and so the surface $F(x, y, z) = 0$ has a line of centres at an infinite distance.

Case V. A, B, C, F, G, H are all zero and fu = gv = hw.

In this case if f, g, h are not zero, the central planes (as discussed above in Case IV) are identical and so the surface $F(x, y, z) = 0$ has a plane of centres.

In case all or two of f, g, h are zero, we can deal the case directly.

§ 12.09. Reduction of general equation.

In § 12.04 Page 4 of this chapter, we have seen the various forms of the surfaces represented by the general equation of second degree. Now we shall discuss in articles to follow the reduction to the standard forms depending upon the various cases as given in § 12.08 on Pages 7–8 Ch. XII.

§ 12.10. Case I. D ≠ 0.

In this case there is a unique centre at a finite distance. Also none of the roots of the discriminating cubic (or λ –cubic) vanishes and so $D \neq 0$.

Here the forms to any one of which the given equation can reduce are :—

(i) $\mathbf{Ax^2 + By^2 + Cz^2 = 1}$ **(Ellipsoid)**

(ii) $\mathbf{Ax^2 + By^2 - Cz^2 = 1}$ **(Hyperboloid of one sheet)**

(iii) $\mathbf{Ax^2 - By^2 - Cz^2 = 1}$ **(Hyperboloid of two sheets)**

(iv) $\mathbf{Ax^2 + By^2 + Cz^2 = 0}$ **(Cone)**

Method of Procedure.

(i) Find the coordinates (x_1, y_1, z_1) of the centre of the given surface $F(x, y, z) = 0$ by solving the equations

$$\frac{\partial F}{\partial x} = 0,\ \frac{\partial F}{\partial y} = 0,\ \frac{\partial F}{\partial z} = 0.$$

(ii) Shift the origin to the centre (x_1, y_1, z_1) and then the equation of the surface referred to centre as origin is

$$f(x, y, z) + d' = 0, \text{ where } d' = ux_1 + vy_1 + wz_1 + d.$$

(iii) By rotation of axes, transform the given equation to the form $\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + d' = 0$, where $\lambda_1, \lambda_2, \lambda_3$ are the roots of the discriminating cubic, which can be reduced to any one of the forms given above.

(iv) The **direction ratios of axes** can be obtained by solving any two of the following three equations :—

$$(a-\lambda)\, l + hm + gn = 0$$
$$hl + (b-\lambda)\, m + fn = 0$$
$$gl + fm + (c-\lambda)\, n = 0.$$

Putting the three values of λ, the direction ratios of the three axes can be obtained and so their equations can be obtained *i.e.* we can find the equations of three lines through the centre and having above direction-ratios.

(v) The **principal planes** are given by

$$\lambda (lx + my + nz) + (ul + vm + wn) = 0.$$

(vi) If $d' = 0$, then the surface is a cone. **(Remember)**

Note : For the solution of a cubic equation, students should go through the section on solution of cubic equations from **Author's Theory of Equations.** It is not always possible to solve a cubic equation when all its roots are real, but with the help of Descarte's Rule of signs, we can find the number of its positive and negative roots.

Solved Examples on § 12.10.

***Ex. 1. Reduce the equation $3x^2 + 5y^2 + 3z^2 + 2yz + 2zx + 2xy - 4x - 8z + 5 = 0$ to the standard form. Find the nature of the conicoid, its centre and equations of its axes.**

Sol. Let $F(x, y, z) \equiv 3x^2 + 5y^2 + 3z^2 + 2yz + 2zx + 2xy - 4x - 8z + 5 = 0$

Then the coordinates of the centre are given by

$$\frac{\partial F}{\partial x} = 0,\ \frac{\partial F}{\partial y} = 0,\ \frac{\partial F}{\partial z} = 0,$$

i.e.
$$\left.\begin{array}{lcl} 6x + 2z + 2y - 4 = 0 & \text{or} & 3x + y + z - 2 = 0 \\ 10y + 2z + 2x = 0 & \text{or} & x + 5y + z = 0 \\ 6z + 2y + 2x - 8 = 0 & \text{or} & x + y + 3z - 4 = 0 \end{array}\right\} \quad \text{...(I)}$$

Solving the equations of (I) we get the centre (x_1, y_1, z_1) as

$$(1/3, -1/3, 4/3) \quad i.e. \quad x_1 = 1/3,\ y_1 = -1/3,\ z_1 = 4/3.$$

Shifting the origin to the centre $(1/3, -1/3, 4/3)$, the equation of the surface reduces to $\quad f(x, y, z) + d' = 0, \quad$ ...(II)

where $d' = `ux_1 + vy_1 + wz_1 + d'$

$$= (-2)(1/3) + (0)(-1/3) + (-4)(4/3) + 5 = -1$$

$\therefore$ From (II), the reduced equation of the surface is

$$(3x^2 + 5y^2 + 3z^2 + 2yz + 2zx + 2xy) + (-1) = 0, \quad \text{...(III)}$$

as $f(x, y, z)$ is the homogeneous part of $F(x, y, z)$.

Now the discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 3-\lambda & 1 & 1 \\ 1 & 5-\lambda & 1 \\ 1 & 1 & 3-\lambda \end{vmatrix} = 0$$

or $(3-\lambda)[(5-\lambda)(3-\lambda)-1]-[(3-\lambda)-1]+[1-(5-\lambda)]=0$

or $$\lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0. \qquad \text{...(IV)}$$

By trial we find that $\lambda = 2$ satisfies (IV), so we have $(\lambda - 2)$ as a factor of L.H.S. of (IV) and so we can rewrite (IV) as

$$(\lambda-2)(\lambda^2 - 9\lambda + 18) = 0 \quad \text{or} \quad (\lambda-2)(\lambda-3)(\lambda-6) = 0.$$

$\therefore$ The roots of the discriminating cubic (IV) are 2, 3, 6.

$\therefore$ By rotation of axes, the given equation transforms to

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + d' = 0 \quad i.e. \quad 2x^2 + 3y^2 + 6z^2 - 1 = 0, \qquad \text{...(V)}$$

substituting values of $\lambda_1, \lambda_2, \lambda_3$ and d'.

Then equation (V) can be rewritten as $2x^2 + 3y^2 + 6z^2 = 1$, which represents an ellipsoid.

The direction-ratios of axes can be obtained by solving two of the following three equations

$(a-\lambda)l + hm + gn = 0,\ hl + (b-\lambda)m + fn = 0,\ gl + fm + (c-\lambda)n = 0$

or $(3-\lambda)l + m + n = 0,\ l + (5-\lambda)m + n = 0,\ l + m + (3-\lambda)n = 0 \qquad \text{...(VI)}$

Taking $\lambda = 2$, we have $l + m + n = 0,\ l + 3m + n = 0,\ l + m + n = 0$.

Solving $l + m + n = 0,\ l + 3m + n = 0$, we get

$$\frac{l}{1-3} = \frac{m}{1-1} = \frac{n}{3-1} \quad \text{or} \quad \frac{l}{1} = \frac{m}{0} = \frac{n}{-1}$$

$\therefore$ The equations of the axis, corresponding to $\lambda = 2$, are

$$\frac{x-(1/3)}{1} = \frac{y-(1/3)}{0} = \frac{z-(4/3)}{-1} \qquad \textbf{Ans.}$$

Similarly corresponding to $\lambda = 3$ and $\lambda = 6$ the direction ratios of the axes (*i.e.* the principal directions) are

$$\frac{l}{1} = \frac{m}{-1} = \frac{n}{1} \quad \text{and} \quad \frac{l}{1} = \frac{m}{-2} = \frac{n}{1}$$

As the equation of the corresponding axes are

$$\frac{x-(1/3)}{1} = \frac{y-(1/3)}{-1} = \frac{z-(4/3)}{1};$$

$$\frac{x-(1/3)}{1} = \frac{y-(1/3)}{-2} = \frac{z-(4/3)}{1};$$

Ex. 2. Reduce the equation $3x^2 - y^2 - z^2 + 6yz - 6x + 6y - 2z - 2 = 0$ to the standard form. Also find its centre and the equation referred to centre as origin. (*Avadh* 95)

Solution. Given $F(x, y, z) \equiv 3x^2 - y^2 - z^2 + 6yz - 6x + 6y - 2z - 2 = 0$.

$\therefore$ The coordinates of the centre are given by

$$\frac{\partial F}{\partial x}=0,\ \frac{\partial F}{\partial y}=0,\ \frac{\partial F}{\partial z}=0$$

i.e. $6x-6=0$ or $x-1=0$ or $x=1$ ...(i)

$-2y+6z+6=0$ or $y-3z-3=0$ ...(ii)

and $-2z-2=0$ or $z+1=0$ or $z=-1$ ...(iii)

$\therefore$ Solving (i), (ii) and (iii) we get the centre (x_1, y_1, z_1) as $(1, 0, -1)$. **Ans.**

Shifting the origin to the centre $(1, 0, -1)$ the equation of the surface reduces to $f(x, y, z)+d'=0$,

where $d'=ux_1+vy_1+wz_1+d$

$=(-3)(1)+(3)(0)+(-1)(-1)-2=-4$

$\therefore$ From (iv), the equation of the surface referred to centre as origin is

$(3x^2-y^2-z^2+6yz)+(-4)=0$ or $3x^2-y^2-z^2+6yz-4=0$ **Ans.**

Now the discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix}=0 \text{ or } \begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & -1-\lambda & 3 \\ 0 & 3 & -1-\lambda \end{vmatrix}=0,$$

putting values of a, b, c, f, g, h

or $(3-\lambda)[(1+\lambda)^2-9]=0$ or $(\lambda-3)[\lambda^2+2\lambda-8]=0$

or $(\lambda-3)(\lambda-2)(\lambda+4)=0$, or $\lambda=2, 3, -4$.

$\therefore$ By rotation of axes, the given equation transforms to

"$\lambda_1 x^2+\lambda_2 y^2+\lambda_3 z^2+d'=0$" *i.e.* $2x^2+3y^2-4z^2-4=0$

or $2x^2+3y^2-4z^2=4$, which represents a hyperboloid of one sheet

[$\because$ it is of the form $Ax^2+By^2-Cz^2=1$]. **Ans.**

Ex. 3. Show that the equation $x^2+y^2+z^2-6yz-2zx-2xy-6x-2y-2z+2=0$ represents a hyperboloid of two sheets.

Solution. Comparing the given equation $F(x, y, z)=0$ with the equation $ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$, we have $a=1$, $b=1, c=1, f=-3, g=-1, h=-1, u=-3, v=-1, w=-1, d=2$. ...(i)

Now coordinates of the centre (x_1, y_1, z_1) of the given surface are given by

$$\frac{\partial F}{\partial x}=0,\ \frac{\partial F}{\partial y}=0,\ \frac{\partial F}{\partial z}=0.$$

i.e. $2x_1-2y_1-2z_1-6=0$ or $x_1-y_1-z_1=3$ ...(ii)

$2y_1-6z_1-2x_1-2=0$ or $x_1-y_1+3z_1=-2$ ...(iii)

$2z_1-6y_1-2x_1-2=0$ or $x_1+3y_1-z_1=-2$ ...(iv)

Solving (ii), (iii) and (iv) we get $x_1=1/2$, $y_1=-5/4$, $z_1=-5/4$

$\therefore$ centre of the given surface is $(1/2, -5/4, -5/4)$.

Also $d'=$ '$ux_1+vy_1+wz_1+d$'

$=(-3)(1/2)+(-1)(-5/4)+(-1)(-5/4)+2=3$ ...(v)

Now the discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix}=0 \quad \text{or} \quad \begin{vmatrix} 1-\lambda & -1 & -1 \\ -1 & 1-\lambda & -3 \\ -1 & -3 & 1-\lambda \end{vmatrix}=0$$

or $$\begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 1-\lambda & \lambda-4 \\ -1 & -3 & 4-\lambda \end{vmatrix}=0, \text{ applying, } C_3-C_2.$$

or $\lambda^3-3\lambda^2-8\lambda+16=0$ or $(\lambda-4)(\lambda^2+\lambda-4)=0$

$\therefore$ Either $\lambda=4$ or $\lambda^2+\lambda-4=0$.

Now $\lambda^2+\lambda-4=0$ gives $\lambda=[-1\pm\sqrt{(1+16)}]/2$.

Thus we find that two values of λ are +ve and one −ve.

$\therefore$ By rotation of axes, the given equation transforms to

$\lambda_1x^2+\lambda_2y^2+\lambda_3z^2+d'=0$ *i.e.* $-\frac{\lambda_1}{3}x^2-\frac{\lambda_2}{3}y^2-\frac{\lambda_3}{3}z^2=1, \because d'=3.$

Now two values of λ being positive and one negative, from above the equation of the surface transforms to the form $Ax^2+By^2+Cz^2=1$, where two of A, B, C are negative and third positive, so that the given surface is a hyperboloid of two sheets.

Ex. 4. Reduce the equation $2x^2-7y^2+2z^2-10yz-8zx-10xy+6x+12y-6z+5=0$ to the standard form. What does it represent?

Sol. Comparing the given equation $F(x, y, z)=0$ with the equation

$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0,$

we have $a=2, b=-7, c=2, f=-5, g=-4, h=-5, u=3, v=6, w=-3, d=5.$

Now coordinates of the centre (x_1, y_1, z_1) of the given surface are given by

$$\frac{\partial F}{\partial x}=0, \frac{\partial F}{\partial y}=0, \text{ and } \frac{\partial F}{\partial z}=0.$$

i.e. $4x_1-8z_1-10y_1+6=0$ or $2x_1-5y_1-4z_1+3=0$; ...(ii)

$-14y_1-10z_1-10x_1+12=0$ or $5x_1+7y_1+5z_1-6=0$; ...(iii)

$4z_1-10y_1-8x_1-6=0$ or $4x_1+5y_1-2z_1+3=0$; ...(iv)

Solving (ii), (iii) and (iv) we get $x_1=1/3, y_1=-1/3, z_1=4/3$.

$\therefore$ Centre of the given surface is $(1/3, -1/3, 4/3)$.

Also $d'=ux_1+vy_1+wz_1+d$

$=3(1/3)+6(-1/3)+(-3)(4/3)+5=1-2-4+5=0$ ...(v)

Now the discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix}=0 \quad \text{or} \quad \begin{vmatrix} 2-\lambda & -5 & -4 \\ -5 & -7-\lambda & -5 \\ -4 & -5 & 2-\lambda \end{vmatrix}=0$$

or $(2-\lambda)[-(7+\lambda)(2-\lambda)-25]+5[-5(2-\lambda)-20]-4[25-4(7+\lambda)]=0$

or $\lambda^3+3\lambda^2-90\lambda+216=0$, on simplifying or $(\lambda-3)(\lambda^2+6\lambda-72)=0$ or $(\lambda-3)(\lambda-6)(\lambda+12)=0$

or $\lambda=3, 6, -12$

$\therefore$ Let $\lambda_1=3, \lambda_2=6, \lambda_3=-12$

$\therefore$ By rotation of axes, the given equation transforms to

$$\lambda_1x^2+\lambda_2y^2+\lambda_3z^2+d'=0$$

or $3x^2+6y^2-12z^2+0=0$, substituting values of $\lambda_1, \lambda_2, \lambda_3, d'$

or $x^2+2y^2-4z^2=0$, which is the required standard form and represents a cone. [See § 12.04 (4) Page 4 Ch. XII]

Also the vertex of the cone (and not centre) is $(1/3, -1/3, 4/3)$ as calculated above.

Exercises on § 12.10 (Case I).

Ex. 1. Reduce the equation $11x^2+10y^2+6z^2-8yz+4zx-12xy+72x-72y+36z+150=0$ to the standard form and show that it represents an ellipsoid and find the equations of the axes. (*Avadh* 91; *Garhwal* 94, 92)

Ans. Centre $(-2, 2, -1)$; $3x^2+6y^2+18z^2=12$ (ellipsoid)
d.r's of the axes are 1, 1, 2; 2, 1, −2; −2, 2, −1.

Ex. 2. Reduce $3x^2+6yz-y^2-z^2-6x+6y-2z-2=0$ to the standard form. What surface does it represent?

Ans. $2x^2+3y^2-4z^2=4$; Hyperboloid of one sheet.

Ex. 3. Reduce $2x^2-y^2-10z^2+20yz-8zx-28xy+16x+26y+16z-34=0$ to the standard form. What does it represent ?

Ans. $2x^2-y^2-2z^2=1$; Hyperboloid of two sheets.

Ex. 4. For the conicoid $ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$, show that

(i) all the roots of the discriminating cubic are real.

and (ii) principal directions are perpendicular to each other. (*Rohilkhand* 91)

Ex. 5. Reduce $x^2+3y^2+3z^2-2yz-2x-2y+6z+3=0$ to the standard form and show that the surface represented by it is an ellipsoid.

Ans. $x^2+2y^2+4z^2=1$.

§ 12.11. Case II, $D=0$ and $Au+Hv+Gw\neq 0$.

$D=0 \Rightarrow$ one root of the discriminating cubic is zero. **(Note)**

Here the forms to any one of which the given equation can reduce are

$$Ax^2+By^2+Cz=0 \qquad \text{(Elliptic Paraboloid)}$$

and

$$Ax^2-By^2+Cz=0 \qquad \text{(Hyperbolic Paraboloid)}$$

Method of Procedure.

(i) Find the discriminating cubic viz $\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix}=0$

One root of this cubic will be zero in this case. **(Note)**

(ii) Put $\lambda=0$ in the above determinant and associate each row with l_3, m_3, n_3.

i.e. $al_3+hm_3+gn_3=0$, $hl_3+bm_3+fn_3=0$, $gl_3+fm_3+cn_3=0$.

Solve any two of these, which will give the direction ratios of the axis corresponding to $\lambda=0$.

(iii) Evaluate $k=ul_3+vm_3+wn_3$, **(Remember)**

where l_3, m_3, n_3 are actual direction-cosines.

If $k\neq 0$, then reduced equation is

$\lambda_1 x^2+\lambda_2 y^2+2kz=0$, where λ_1, λ_2 are non-zero roots of the discriminating cubic.

This equation represents an elliptic or hyperbolic paraboloid according as λ_1 and λ_2 have the same or opposite signs.

(iv) **Vertex.** The coordinates of the vertex of the paraboloid in this case are obtained by solving any two of the three equations

$$\frac{\left(\frac{\partial F}{\partial x}\right)}{l_3}=\frac{\left(\frac{\partial F}{\partial y}\right)}{m_3}=\frac{\left(\frac{\partial F}{\partial z}\right)}{n_3}=2k,$$

with the equation $k(l_3x+m_3y+n_3z)+ux+vy+wz+d=0$ **(Remember)**

Solved Examples on § 12.11.

****Ex. 1. Determine completely what is represented by the equation**

$$2x^2+2y^2+z^2+2yz-2zx-4xy+x+y=0.$$

Find the coordinates of its vertex and the equations to its axis

(*Garhwal* 93)

Solution. Here 'a' $=2$, 'b' $=2$, 'c' $=1$, 'f' $=1$, 'g' $=-1$.

'h' $=-2$, 'u' $=1/2$, 'v' $=1/2$, 'w' $=0$ and 'd' $=0$

$\therefore$ The discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix}=0 \quad\text{or}\quad \begin{vmatrix} 2-\lambda & -2 & -1 \\ -2 & 2-\lambda & 1 \\ -1 & 1 & 1-\lambda \end{vmatrix}=0 \qquad \text{...(i)}$$

or $(2-\lambda)[(2-\lambda)(1-\lambda)-1]+2[-2(1-\lambda)+1]-[-2+(2-\lambda)]=0.$

or $\lambda^3-5\lambda^2+2\lambda=0$ or $\lambda(\lambda^2-5\lambda+2)=0$ or $\lambda=0, \frac{1}{2}[5\pm\sqrt{(17)}]$

$\therefore$ Let $\lambda_1=\frac{1}{2}[5+\sqrt{(17)}]$, $\lambda_2=\frac{1}{2}[5-\sqrt{(17)}]$, $\lambda_3=0$

Now putting $\lambda=0$ in the determinant given by (i) and associating each row with l_3, m_3, n_3, we have

$2l_3-2m_3-n_3=0$, $-2l_3-2m_3+n_3=0$, $-l_3+m_3+n_3=0$

Solving last two equations simultaneously for l_3, m_3, n_3, we get

$$\frac{l_3}{2-1}=\frac{m_3}{-1+2}=\frac{n_3}{-2+2}$$

i.e. $$\frac{l_3}{1}=\frac{m_3}{1}=\frac{n_3}{0}=\frac{\sqrt{(l_3^2+m_3^2+n_3^2)}}{\sqrt{(1^2+1^2+0^2)}}=\frac{1}{\sqrt{2}}$$

$\therefore\ l_3 = 1/\sqrt{2},\ m_3 = 1/\sqrt{2},\ n_3 = 0.$

These gives d.c.'s of the axis corresponding to $\lambda = 0$.

Now $k = 'ul_3 + vm_3 + wn_3' = (\frac{1}{2})(1/\sqrt{2}) + (\frac{1}{2})(1/\sqrt{2}) + 0 = 1/\sqrt{2}$

$\therefore$ The required reduced equation is $\lambda_1 x^2 + \lambda_2 y^2 + 2kz = 0$

or $\quad \frac{1}{2}[5+\sqrt{(17)}]\,x^2 + \frac{1}{2}[5-\sqrt{(17)}]\,y^2 + 2(1/\sqrt{2})\,z = 0,$

which represents an elliptic paraboloid as both λ_1, λ_2 are positive. (Note)

Also if $F(x, y, z) = 0$ be the given surface then the coordinates of its vertex are given by solving any two of the three equations

$$\frac{\left(\frac{\partial F}{\partial x}\right)}{l_3} = \frac{\left(\frac{\partial F}{\partial y}\right)}{m_3} = \frac{\left(\frac{\partial F}{\partial z}\right)}{n_3} = 2k.$$

along with $\quad k(l_3x + m_3y + n_3z) + ux + vy + wz + d = 0$

...See § 12.11 (iv) Page 14 Ch. XII

i.e. any two of the equations $4x - 2z - 4y + 1 = 2(1/\sqrt{2})(1/\sqrt{2})$

or $\quad 2x - 2y - z = 0;$

$4y + 2z - 4x + 1 = 2(1/\sqrt{2})(1/\sqrt{2}) \quad$ or $\quad 2x - 2y - z = 0;$

$2z + 2y + 2x = 2(0)(1/\sqrt{2}) \quad$ or $\quad x - y - z = 0$

with $$\frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}x + \frac{1}{\sqrt{2}}y + 0z\right] + \frac{1}{2}x + \frac{1}{2}y + 0 + 0 = 0$$

i.e. $\quad 2x - 2y - z = 0,\ x - y - z = 0,\ x + y = 0$

Solving these we get $x = 0, y = 0, z = 0$ *i.e.* the coordinates of the vertex are $(0, 0, 0)$. **Ans.**

$\therefore$ The equations to its axis are

$$\frac{x-0}{l_3} = \frac{y-0}{m_3} = \frac{z-0}{n_3} \quad i.e. \quad \frac{x-0}{(1/\sqrt{2})} = \frac{y-0}{(1/\sqrt{2})} = \frac{z-0}{0}$$

i.e. $\quad x = y,\ z = 0.$ **Ans.**

***Ex. 2. Reduce the equation $3z^2 - 6yz - 6zx - 7x - 5y + 6z + 3 = 0$ to standard form and find its nature.** *(Avadh 94)*

Sol. Here $'a' = 0$, $'b' = 0$, $'c' = 3$, $'f' = -3$, $'g' = -3$, $'h' = 0$, $'u' = -7/2$, $v' = -5/2$, $'w' = 3$ and $'d' = 3$

$\therefore$ The discriminating cubic is ,

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 0-\lambda & 0 & -3 \\ 0 & 0-\lambda & -3 \\ -3 & -3 & 3-\lambda \end{vmatrix} = 0 \qquad \text{...(i)}$$

or $\quad -\lambda[-\lambda(3-\lambda) - 9] + 0 - 3[-3\lambda] = 0 \quad$ or $\quad \lambda^3 - 3\lambda^2 - 18\lambda = 0$

or $\quad \lambda(\lambda^2 - 3\lambda - 18) = 0 \quad$ or $\quad \lambda(\lambda - 6)(\lambda + 3) = 0 \quad$ or $\quad \lambda = 0, 6, -3$

Let $\lambda_1 = 6, \lambda_2 = -3, \lambda_3 = 0.$

Now putting $\lambda = 0$ in the determinant given by (i) and associating each row with l_3, m_3, n_3, we have

$0 \,.\, l_3 + 0 \,.\, m_3 - 3n_3 = 0,\ 0 \,.\, l_3 + 0 \,.\, m_3 - 3n_3 = 0,\ -3l_3 - 3m_3 + 3n_3 = 0$

These gives $n_3 = 0,\ l_3 + m_3 = 0$

i.e. $\dfrac{l_3}{1} = \dfrac{m_3}{-1} = \dfrac{n_3}{0} = \dfrac{\sqrt{(l_3^2 + m_3^2 + n_3^2)}}{\sqrt{[1^2 + (-1)^2 + 0^2]}} = \dfrac{1}{\sqrt{2}}$

$\therefore\ l_3 = 1/\sqrt{2},\ m_3 = -1/\sqrt{2},\ n_3 = 0$

These gives d.c's of the axis corresponding to $\lambda = 0$.

Now $k = \text{'}ul_3 + vm_3 + wn_3\text{'} = -\dfrac{7}{2}\left(\dfrac{1}{\sqrt{2}}\right) - \dfrac{5}{2}\left(\dfrac{1}{\sqrt{2}}\right) + 3\,(0) = -\dfrac{1}{\sqrt{2}}$

$\therefore$ Required reduced equation is

$$\lambda_1 x^2 + \lambda_2 y^2 + 2kz = 0 \quad \text{or} \quad 6x^2 - 3y^2 - \sqrt{2}z = 0,$$

which represents a hyperbolic paraboloid as λ_1 and λ_2 are of opposite signs.

Ans.

****Ex. 3. Find the coordinates of the vertex and equation to the axis of the hyperbolic paraboloid $4x^2 - y^2 - z^2 + 2yz - 8x - 4y + 8z - 2 = 0$.**

(Rohilkhand 95)

Solution. Here 'a' $= 4$, 'b' $= -1$, 'c' $= -1$, 'f' $= 1$, 'g' $= 0$, 'h' $= 0$, 'u' $= -4$, 'v' $= -2$, 'w' $= 4$ and 'd' $= -2$

$\therefore$ The discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 4-\lambda & 0 & 0 \\ 0 & -1-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{vmatrix} = 0$$

or $\quad (4-\lambda)\,[(1+\lambda)^2 - 1] = 0$ or $\lambda\,(\lambda+2)\,(\lambda-4) = 0$ or $\lambda = 0, -2, 4$.

$\therefore$ Let $\lambda_1 = -2,\ \lambda_2 = 4,\ \lambda_3 = 0$.

Now putting $\lambda = 0$ in the determinant given by (i) and associating each row with l_3, m_3, n_3, we have

$$4l_3 = 0,\ -m_3 + n_3 = 0,\ m_3 - n_3 = 0 \Rightarrow l_3 = 0,\ m_3 = n_3.$$

But $l_3^2 + m_3^2 + n_3^2 = 1$, so $0 + m_3^2 + m_3^2 = 1$ or $m_3 = 1/\sqrt{2} = n_3$

$\therefore$ We have $l_3 = 0, m_3 = 1/\sqrt{2},\ n_3 = 1/\sqrt{2}$

Now $k = ul_3 + vm_3 + wn_3 = -4\,(0) - 2\,(1/\sqrt{2}) + 4\,(1/\sqrt{2}) = \sqrt{2}$

$\therefore$ Required reduced equation is $\lambda_1 x^2 + \lambda_2 y^2 + 2kz = 0$

or $\quad -2x^2 + 4y^2 + 2\sqrt{2}z = 0 \quad$ or $\quad x^2 - 2y^2 - z\sqrt{2} = 0,$

which represents a hyperbolic paraboloid as λ_1 is $-$ ve and λ_2 is $+$ ve.

Also if $F(x, y, z) = 0$ be the given surface then the coordinates of its vertex are given by solving any two of these equations.

$$\frac{(\partial F/\partial x)}{l_3} = \frac{(\partial F/\partial y)}{m_3} = \frac{(\partial F/\partial z)}{n_3} = 2k$$

and $\quad k\,(l_3 x + m_3 y + n_3 z) + ux + vy + wz + d = 0$ ...See § 12.11 (iv) Page 14.

i.e. any two of the equations

$$8x-8=2(\sqrt{2})(0) \text{ i.e. } x=1;$$
$$-2y+2z-4=2(\sqrt{2})(1/\sqrt{2}) \text{ i.e. } y-z+3=0$$
$$-2z+2y+8=2(\sqrt{2})(1/\sqrt{2}) \text{ i.e. } y-z+3=0$$

with $$\sqrt{2}\left[0.x+\frac{1}{\sqrt{2}}y+\frac{1}{\sqrt{2}}z\right]-4x-2y+4z-2=0$$

i.e. $x=1, y-z+3=0, 4x+y-5z+2=0$

Solving these we get $x=1,\ y=-9/4,\ z=3/4$.

$\therefore$ Coordinates of the vertex are $(1,-9/4, 3/4)$ **Ans.**

And the equations of its axis are $\frac{x-1}{l_3}=\frac{y-(-9/4)}{m_3}=\frac{z-(3/4)}{n_3}$

or $$\frac{x-1}{0}=\frac{y+(9/4)}{1/\sqrt{2}}=\frac{z-(3/4)}{1/\sqrt{2}} \quad \text{i.e.} \quad \frac{x-1}{0}=\frac{4y+9}{1}=\frac{4z-3}{1}$$ **Ans.**

***Ex. 4. Show that the following equation represents a paraboloid. Find its vertex and equations to the axis.**

$$4y^2+4z^2+4yz-2x-14y-22z+33=0.$$ (*Rohilkhand* 92, 90)

Solution. Here 'a' $=0$, 'b' $=4$, 'c' $=4$, 'f' $=2$, 'g' $=0$, 'h' $=0$, 'u' $=-1$, 'v' $=-7$, 'w' $=-11$ and 'd' $=33$

$\therefore$ The discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix}=0 \text{ or } \begin{vmatrix} -\lambda & 0 & 0 \\ 0 & 4-\lambda & 2 \\ 0 & 2 & 4-\lambda \end{vmatrix}=0 \qquad ...(i)$$

or $-\lambda[(4-\lambda)^2-4]=0$ or $\lambda[\lambda^2-8\lambda+12]=0$

or $\lambda(\lambda-2)(\lambda-6)=0$ or $\lambda=0, 2, 6$

$\therefore$ Let $\lambda_1=2,\ \lambda_2=6$ and $\lambda_3=0$

Now putting $\lambda=0$ in the determinant given by (i) and associating each row with l_3, m_3, n_3 we have $4m_3+2n_3=0,\ 2m_3+4n_3=0 \Rightarrow m_3=0=n_3$

But $l_3^2+m_3^2+n_3^2=1$, so $l_3^2+0+0=1 \Rightarrow l_3=1$

Now $k=ul_3+vm_3+wn_3=-1(1)-7(0)-11(0)=-1$

$\therefore$ Required reduced equation is $\lambda_1x^2+\lambda_2y^2+2kz=0$

or $$2x^2+6y^2+2(-1)z=0 \quad \text{or} \quad x^2+3y^2-z=0,$$

which represents an elliptic paraboloid as both λ_1 and λ_2 are positive.

Also if $F(x,y,z)=0$ be the given surface then the coordinates of its vertex are given by solving any two of these equations

$$\frac{\partial F/\partial x}{l_3}=\frac{\partial F/\partial y}{m_3}=\frac{\partial F/\partial z}{n_3}=2k$$

and $k(l_3x+m_3y+n_3z)+ux+vy+wz+d=0$ ...See § 12.11 (iv) P. 14.

i.e. any two of the equations

$-2=2(-1)(0)$ which is absurd.

$8y + 4z - 14 = 2(-1).0 \Rightarrow 4y + 2z = 7$;

$8x + 4y - 22 = 2(-1)(0) \Rightarrow 2y + 4z = 11$;

with $(-1)[1.x + 0.y + 0.z] - x - 7y - 11z + 33 = 0$ or $2x + 7y + 11z = 33$

i.e. $4y + 2z = 7,\ 2y + 4z = 11,\ 2x + 7y + 11z = 33$

Solving these we get $x = 1, y = 1/2, z = 5/2$

$\therefore$ Coordinates of the vertex are $(1, 1/2, 5/2)$ **Ans.**

And the equations of its axis are $\dfrac{x-1}{l_3} = \dfrac{y-(1/2)}{m_3} = \dfrac{z-(5/2)}{n_3}$

or $\dfrac{x-1}{1} = \dfrac{y-(1/2)}{0} = \dfrac{z-(5/2)}{0}$ or $\dfrac{x-1}{1} = \dfrac{2y-1}{0} = \dfrac{2z-5}{0}$

or $2y - 1 = 0,\ 2z - 5 = 0$ or $y = 1/2,\ z = 5/2.$

****Ex. 5. Prove that $z(ax + by + cz) + \alpha x + \beta y = 0$ represents a paraboloid and the equations to the axis are**

$$\mathbf{ax + by + 2cz = 0,\ (a^2 + b^2)z + a\alpha + b\beta = 0.}$$ (*Rohilkhand* 93)

Sol. Given equation is $cz^2 + byz + azx + \alpha x + \beta y = 0$

$\therefore$ Here 'a' $= 0$, 'b' $= 0$, 'c' $= c$, 'b' $= b/2$, 'g' $= a/2$, 'h' $= 0$, 'u' $= \alpha/2$, 'v' $= \beta/2$ 'w' $= 0$ and 'd' $= 0$

$\therefore$ The discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 0-\lambda & 0 & a/2 \\ 0 & 0-\lambda & b/2 \\ a/2 & b/2 & c-\lambda \end{vmatrix} = 0 \quad ...(i)$$

or $-\lambda[-\lambda(c-\lambda) - (b^2/4)] + (a/2)[a\lambda/2] = 0$

or $4\lambda^3 - 4a\lambda^2 - (a^2 + b^2)\lambda = 0$ or $\lambda[4\lambda^2 - 4a\lambda - a^2 - b^2] = 0$

or $\lambda = 0$ and $\lambda = \dfrac{4a \pm \sqrt{(16a^2 + 16a^2 + 16b^2)}}{8} = \dfrac{a \pm \sqrt{(2a^2 + b^2)}}{2}$

Let $\lambda_1 = \frac{1}{2}\left[a + \sqrt{(2a^2 + b^2)}\right]$, $\lambda_2 = \frac{1}{2}\left[a - \sqrt{(2a^2 + b^2)}\right]$ $\lambda_3 = 0$

Now putting $\lambda = 0$ in the determinant given by (i) and associating each row with l_3, m_3, n_3, we have

$$0.l_3 + 0.m_3 + (a/2)n_3 = 0,\ 0.l_3 + 0.m_3 + (b/2)n_3 = 0$$

and $$(a/2)l_3 + (b/2)m_3 + cn_3 = 0$$

These gives $n_3 = 0$ and $al_3 + bm_3 = 0$

i.e. $\dfrac{l_3}{b} = \dfrac{m_3}{-a} = \dfrac{n_3}{0} = \dfrac{\sqrt{(l_3^2 + m_3^2 + n_3^2)}}{\sqrt{(b^2 + a^2 + 0)}} = \dfrac{1}{\sqrt{(a^2 + b^2)}}$

$\therefore l_3 = b/\sqrt{(a^2 + b^2)},\ m_3 = -a/\sqrt{(a^2 + b^2)},\ n_3 = 0$

Now $k = $ '$ul_3 + vm_3 + wn_3$'

$$= \frac{\alpha}{2} \cdot \frac{b}{\sqrt{(a^2 + b^2)}} + \frac{\beta}{2}\left[\frac{-a}{\sqrt{(a^2 + b^2)}}\right] + 0 = \frac{b\alpha - a\beta}{2\sqrt{(a^2 + b^2)}} \neq 0$$

$\therefore$ Reduced equation is $\lambda_1 x^2 + \lambda_2 y^2 + 2kz = 0$

or $\quad \frac{1}{2}\left[a+\sqrt{(2a^2+b^2)}\right]x^2+\frac{1}{2}\left[a-\sqrt{(2a^2+b^2)}\right]y^2+\frac{(b\alpha-a\beta)}{\sqrt{(a^2+b^2)}}z=0$ ...(ii)

Now as $a+\sqrt{(2a^2+b^2)}>0$ and $a-\sqrt{(2a^2+b^2)}<0$, so (ii) represents a hypebolic paraboloid.

Also if $F(x,y,z)=0$ be the given surface then the coordinates of its vertex are given by solving any two of the equations

$$\frac{\partial F/\partial x}{l_3}=\frac{\partial F/\partial y}{m_3}=\frac{\partial F/\partial z}{n_3}=2k$$

and $\quad k(l_3x+m_3y+n_3z)+ux+vy+wz+d=0$ ...See § 12.11 (iv) P. 14 Ch. XII

i.e. any two of the equations

$$\frac{\alpha+az}{b/\sqrt{(a^2+b^2)}}=\frac{\beta+bz}{-a/\sqrt{(a^2+b^2)}}=\frac{ax+by+2cz}{0}=\frac{2(b\alpha-a\beta)}{2\sqrt{(a^2+b^2)}}$$

and $\quad k\left[\frac{bx}{\sqrt{(a^2+b^2)}}-\frac{ay}{\sqrt{(a^2+b^2)}}+0\right]+\frac{\alpha}{2}x+\frac{\beta}{2}y=0$, on substituting the values

Thus we have $\frac{\alpha+az}{b}=\frac{\beta+bz}{-a}$, $ax+by+2cz=0$

and $$\frac{b\alpha-a\beta}{2\sqrt{(a^2+b^2)}}\left[\frac{bx-ay}{\sqrt{(a^2+b^2)}}\right]+\frac{1}{2}(\alpha x+\beta y)=0$$

i.e. $\quad (a^2+b^2)z+a\alpha+b\beta=0,\quad ax+by+2cz=0$

and $\quad (b\alpha-a\beta)(bx-ay)+(\alpha x+\beta y)(a^2+b^2)=0$

Now if (x_1,y_1,z_1) be the vertex of the paraboloid then x,y,z satisfies above three equations

i.e. $$(a^2+b^2)z_1+a\alpha+b\beta=0, \quad \text{...(iii)}$$

$$ax_1+by_1+2cz_1=0 \quad \text{...(iv)}$$

and $$(b\alpha-a\beta)(bx-ay)+(\alpha x+\beta y)(a^2+b^2)=0 \quad \text{...(v)}$$

And the equations of the axis are $\frac{x-x_1}{l_3}=\frac{y-y_1}{m_3}=\frac{z-z_1}{n_3}$

or $$\frac{x-x_1}{b/\sqrt{(a^2+b^2)}}=\frac{y-y_1}{-a/\sqrt{(a^2+b^2)}}=\frac{z-z_1}{0}, \quad \text{...(vi)}$$

substituting values of l_3, m_3, n_3

These give $z-z_1=0$ or $z=z_1=-\frac{a\alpha+b\beta}{(a^2+b^2)}$, from (iii)

or $$(a^2+b^2)z+a\alpha+b\beta=0 \quad \text{...(vii)}$$

Again from first two fractions of (vi), we get $a(x-x_1)+b(y-y_1)=0$

or $$ax+by=ax_1+by_1=-2cz_1, \text{ from (iv)}$$

$$=-2c\left[-\frac{a\alpha+b\beta}{a^2+b^2}\right], \text{ from (iii)}$$

$$=-2cz, \text{ from (vii)}$$

or $$ax + by + 2cz = 0 \quad \text{...(viii)}$$

Hence from (vii) and (viii) the equations of the axis of the paraboloid are

$$(a^2 + b^2) z + a\alpha + b\beta = 0, \; ax + by + 2cz = 0$$

Exercises on § 12.11 (Case II)

Ex. 1. Prove that the surface represented by the equation $3x^2 + 4y^2 + 9z^2 + 12yz + 6zx + 4xy + 4x + 6y + 2z + 1 = 0$ is an elliptic paraboloid.

Ans. Reduced form is $[8 + \sqrt{(38)}]x^2 + [8 - \sqrt{(38)}]\, y^2 - [14/\sqrt{(13)}]\, z = 0$

***Ex. 2.** Find the coordinates of the vertex and equation to the axis of the elliptic paraboloid $4x^2 + y^2 + z^2 - 2zx - 2xy + x + y - 4z - 6 = 0$.

Ans. $(-1, 2, -1)$; $x + 1 = -\frac{1}{2}(y - 2) = \frac{1}{2}(z + 1)$.

***Ex. 3.** Find the coordinates of the vertex and equation to the axis of the hyperbolic paraboloid

$$5x^2 - 16y^2 + 5z^2 + 8yz - 14zx + 8xy + 4x + 20y + 4z - 24 = 0.$$

Ans. $(1, 1, 1)$, $\frac{1}{2}(x - 1) = y - 1 = \frac{1}{2}(z - 1)$.

§ 12.12. Case III. $D = 0$, $Au + Hv + Gw = 0$, $A \neq 0$.

In this case the forms to any one of which the given equation can reduce are :—

(i) $Ax^2 + By^2 + C = 0$ **(Elliptic cylinder)**

(ii) $Ax^2 - By^2 + C = 0$ **(Hyperbolic cylinder)**

(iii) $Ax^2 - By^2 = 0$ **(Pair of Planes)**

In this case there is a line centres at a finite distance and the discriminating cubic has one root zero, say λ_3.

The line of centres is given by any two of $\dfrac{\partial F}{\partial x} = 0$, $\dfrac{\partial F}{\partial y} = 0$, $\dfrac{\partial F}{\partial z} = 0$,

where $F(x, y, z) = 0$ is the equation of the given surface.

Let (α, β, γ) be the coordinates of any point lying on this line. Then shifting the origin to (α, β, γ) and rotating the axes in such a manner that these coincide with a set of mutually perpendicular principal directions, the given equation reduces to the form $\lambda_1 x^2 + \lambda_2 y^2 + k = 0$, where $k = u\alpha + v\beta + w\gamma + d$.

Nature : If $k = 0$, this represents a **pair of planes**.

If $k \neq 0$, this represents an **elliptic** or **hyperbolic cylinder** according as the non-zero values of λ (*i.e.* the non-zero roots of the discriminating cubic) are both of the same or opposite signs. **(Remember)**

The line of intersection of the principal planes corresponding to non-zero values of λ is the axis of the cylinder. It is parallel to the principal direction corresponding to λ_3 which is zero and is also the line of the centres.

Solved Examples on § 12.12.

****Ex. 1. Show that the surface $26x^2 + 20y^2 + 10z^2 - 4yz - 16zx - 36xy + 52x - 36y - 16z + 25 = 0$ represents an elliptic cylinder. Also find the equations to its axis.**

Solution. Here the discriminating cubic is given by

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 26-\lambda & -18 & -8 \\ -18 & 20-\lambda & -2 \\ -8 & -2 & 10-\lambda \end{vmatrix} = 0$$

or $\quad (26-\lambda)[(20-\lambda)(10-\lambda)-4]+18[-18(10-\lambda)-16]$
$$-8[36+8(20-\lambda)]=0$$

or $\quad \lambda^3-56\lambda^2+588\lambda=0 \quad$ or $\quad \lambda(\lambda^2-56\lambda+588)=0$

or $\quad \lambda(\lambda-14)(\lambda-42)=0 \quad$ or $\quad \lambda=0, 14, 42.$

Let $\lambda_1=14, \lambda_2=42$ and $\lambda_3=0$. ...(i)

Now putting $\lambda=0$ in the determinant given by (i) and associating each row with l_3, m_3, n_3, we have $26l_3-18m_3-8n_3=0$, $-18l_3+20m_3-2n_3=0$, $-8l_3-2m_3+10n_3=0$.

Solving first and third of these simultaneously, we have

$$\frac{l_3}{1}=\frac{m_3}{1}=\frac{n_3}{1}=\frac{\sqrt{(l_3^2+m_3^2+n_3^2)}}{\sqrt{(1^2+1^2+1^2)}}=\frac{1}{\sqrt{3}}$$

i.e. $$l_3=1/\sqrt{3}=m_3=n_3.$$

The line of centres is given by any two of $\dfrac{\partial F}{\partial x}=0$, $\dfrac{\partial F}{\partial y}=0$, $\dfrac{\partial F}{\partial z}=0$

$\dfrac{\partial F}{\partial x}=0 \Rightarrow 52x-36y-16z+52=0$ *i.e.* $13x-9y-4z+13=0$

$\dfrac{\partial F}{\partial y}=0 \Rightarrow 40y-4z-36x-36=0$ *i.e.* $9x-10y+z+9=0$

$\dfrac{\partial F}{\partial z}=0 \Rightarrow 20z-4y-16x-16=0$ *i.e.* $4x+y-5z+4=0.$

Let (α, β, γ) be any point on the line of centres.

Choosing $\alpha=-1$, $\beta=0$, $\gamma=0$ we find $(-1, 0, 0)$ is a point on the line of centres.

[**Note.** The method of choosing α, β, γ is not unique]

Now $k=u\alpha+v\beta+w\gamma+d=26(-1)+(-18)(0)+(-8)(0)+25=-1\neq 0.$

Hence the given surface reduces to $\lambda_1x^2+\lambda_2y^2+k=0$

i.e. $\quad 14x^2+42y^2-1=0$, which represents an elliptic cylinder as λ_1, λ_2 are both of the same sign. [See § 12.12 Page 20 Ch. XII]

Also the equations of the axis of cylinder are

$$\frac{x-(-1)}{l_3}=\frac{y-0}{m_3}=\frac{z-0}{n_3} \quad \text{or} \quad \frac{x+1}{1}=\frac{y}{1}=\frac{z}{1}, \text{ from (iii)}$$

***Ex. 2. Prove that the surface represented by the equation**

$$\mathbf{5x^2+5y^2+8z^2+8yz+8zx-2xy+12x-12y+6=0}$$

represents a cylinder whose cross-section is an ellipse of eccentricity $1/\sqrt{2}$ and find the equations to its axis. (*Garhwal* 95)

Solution. Here 'a' $=5$, 'b' $=5$, 'c' $=8$, 'f' $=4$, 'g' $=4$, 'h' $=-1$, 'u' $=6$, 'v' $=-6$, 'w' $=0$ and 'd' $=6$.

$\therefore$ The discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix}=0 \quad \text{or} \quad \begin{vmatrix} 5-\lambda & -1 & 4 \\ -1 & 5-\lambda & 4 \\ 4 & 4 & 8-\lambda \end{vmatrix}=0$$

or $(5-\lambda)[(5-\lambda)(8-\lambda)-16]+[-(8-\lambda)-16]+4[-4-4(5-\lambda)]=0$

or $\lambda^3-18\lambda^2+72\lambda=0$ or $\lambda(\lambda^2-18\lambda+72)=0$

or $\lambda(\lambda-6)(\lambda-12)=0$ or $\lambda=0, 6, 12.$

$\therefore$ Let $\lambda_1=6,\ \lambda_2=12,\ \lambda_3=0.$ ...(ii)

Now putting $\lambda=0$ in the determinant given by (i) and associating each row with l_3, m_3, n_3, we have

$5l_3-m_3+4n_3=0,\ -l_3+5m_3+4n_3=0,\ 4l_3+4m_3+8n_3=0.$

Solving last two equations simultaneously, we get

$$\frac{l_3}{4-10}=\frac{m_3}{-2-4}=\frac{n_3}{5+1} \quad \text{or} \quad \frac{l_3}{-1}=\frac{m_3}{-1}=\frac{n_3}{1}=\frac{1}{\sqrt{3}} \qquad \text{...(iii)}$$

Also the line of centres is given by any two of

$$\partial F/\partial x=0,\ \partial F/\partial y=0,\ \partial F/\partial z=0$$

$\dfrac{\partial F}{\partial x}=0 \Rightarrow 10x+8z-2y+12=0$ or $5x-y+4z+6=0$

$\dfrac{\partial F}{\partial y}=0 \Rightarrow 10y+8z-2x-12=0$ or $x-5y-4z+6=0$

$\dfrac{\partial F}{\partial z}=0 \Rightarrow 16z+8y+8x=0$ or $x+y+2z=0$

Let (α, β, γ) be any point on the line of centres. Choosing $\gamma=0,\ \beta=1,\ \alpha=-1$ we find $(-1, 1, 0)$ is a point on the line of centres.

Now $k=u\alpha+v\beta+w\gamma+d=(6)(-1)+(-6)(1)+0+6=-6\neq 0$

Hence the given surface reduces to $\lambda_1x^2+\lambda_2y^2+k=0$

i.e. $6x^2+12y^2-6=0$ *i.e.* $x^2+2y^2-1=0$ ...(iv)

which represents an elliptic cylinder as λ_1, λ_2 are both of the same sign.

Also (iv) can be rewritten as $\dfrac{x^2}{1}+\dfrac{y^2}{(1/2)}=1.$

And so if e be the required eccentricity, then

$$b^2=a^2(1-e^2) \Rightarrow 1/2=(1-e^2) \text{ or } e=1/\sqrt{2}. \qquad \textbf{Ans.}$$

Also the equations of the axis of cylinder are

$$\frac{x-(-1)}{l_3}=\frac{y-1}{m_3}=\frac{z-0}{n_3} \quad \text{or} \quad \frac{x+1}{-1}=\frac{y-1}{-1}=\frac{z}{1} \qquad \textbf{Ans.}$$

****Ex. 3. Determine completely the surface represented by**

$$\mathbf{2y^2-2yz+2zx-2xy-x-2y+3z-2=0.}$$

Sol. Here 'a' $=0$, 'b' $=2$, 'c' $=0$, 'f' $=-1$, 'g' $=1$, 'h' $=-1$, 'u' $=-1/2$, 'v' $=-1$, 'w' $=3/2$ and 'd' $=-2$.

$\therefore$ The discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} -\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & -\lambda \end{vmatrix} = 0 \qquad \text{...(i)}$$

or $\quad -\lambda[-(2-\lambda)\lambda-1]+[\lambda+1]+[1-(2-\lambda)]=0$

or $\quad -\lambda^3+2\lambda^2+\lambda+\lambda+1+1-2+\lambda=0 \quad$ or $\quad \lambda^3-2\lambda^2-3\lambda=0$

or $\quad \lambda(\lambda^2-2\lambda-3)=0 \quad$ or $\quad \lambda(\lambda+1)(\lambda-3)=0 \quad$ or $\quad \lambda=0,-1,3$

or $\quad \lambda_1=3,\ \lambda_2=-1,\ \lambda_3=0.$

Now putting $\lambda=0$ in the determinant given by (i) and associating each row with l_3, m_3, n_3, we have $-m_3+n_3=0,\ -l_3+2m_3-n_3=0,\ l_3-m_3=0$

From these on solving we get $l_3=m_3=n_3=1/\sqrt{3}$ **(Note)**

Further the line of centres is given by any two of

$$\partial F/\partial x=0,\ \partial F/\partial y=0,\ \partial F/\partial z=0$$

Now $\dfrac{\partial F}{\partial x}=0 \Rightarrow 2z-2y-1=0;$

$\dfrac{\partial F}{\partial y}=0 \Rightarrow 4y-2z-2x-2=0$ or $x-2y+z+1=0;$

$\dfrac{\partial F}{\partial z}=0 \Rightarrow -2y+2x+3=0$ or $2x-2y+3=0$

Let (α,β,γ) be any point on the line of centres. Choosing $\gamma=0,\ \beta=-1/2,\ \alpha=-2$ we find that $(-2,-1/2,0)$ is a point on the line of centres.

Now $k=u\alpha+v\beta+w\gamma+d=(-\frac{1}{2})(-2)+(-1)(-\frac{1}{2})+(\frac{3}{2})(0)-2=-\frac{1}{2}\neq 0$

Hence the given surface reduces to $\lambda_1x^2+\lambda_2y^2+k=0$

i.e. $$3x^2-y^2-(1/2)=0$$

which represents a hyperbolic cylinder as λ_1, λ_2 are of different signs.

Also the equations of the axis of the cylinder are

$$\frac{x-(-2)}{l_3}=\frac{y-(-1/2)}{m_3}=\frac{z-0}{n_3} \quad \text{or} \quad \frac{x+2}{1}=\frac{y+(1/2)}{1}=\frac{z}{1} \qquad \textbf{Ans.}$$

***Ex. 4 (a). Prove that the equation $5x^2-4y^2+5z^2+4yz-14zx+4xy+16x+16y+32z+8=0$ represents a pair of planes which pass through the line $x+2=y-1=z$ and are inclined at an angle $2\tan^{-1}(1/\sqrt{2})$.**

Solution. Here 'a' $=5$, 'b' $=-4$, 'c' $=5$, 'f' $=2$, 'g' $=-7$, 'h' $=2$, 'u' $=8$, 'v' $=8$, 'w' $=-16$ and 'd' $=8$.

$\therefore$ The discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 5-\lambda & 2 & -7 \\ 2 & -4-\lambda & 2 \\ -7 & 2 & 5-\lambda \end{vmatrix} = 0 \qquad \text{...(i)}$$

or $\quad (5-\lambda)[-(4+\lambda)(5-\lambda)-4]-2[2(5-\lambda)+14]-7[4-7(4+\lambda)]=0$

or $$\lambda^3 - 6\lambda^2 - 72\lambda = 0$$

or $\lambda(\lambda^2 - 6\lambda - 72) = 0$ or $\lambda(\lambda + 6)(\lambda - 12) = 0$ or $\lambda = 0, -6, 12.$

Let $$\lambda_1 = 12,\ \lambda_2 = -6,\ \lambda_3 = 0.$$

Now putting $\lambda = 0$ in the determinant given by (i) and associating each row with l_3, m_3, n_3, we have

$$5l_3 + 2m_3 - 7n_3 = 0,\ 2l_3 - 4m_3 + 2n_3 = 0,\ -7l_3 + 2m_3 + 5n_3 = 0.$$

Solving first two equations simultaneously, we get

$$\frac{l_3}{4-28} = \frac{m_3}{-14-10} = \frac{n_3}{-20-4} \quad \text{or} \quad \frac{l_3}{1} = \frac{m_3}{1} = \frac{n_3}{1} = \frac{1}{\sqrt{3}} \qquad \text{(Note) ...(ii)}$$

Further the line of centres is given by any two of

$$\frac{\partial F}{\partial x} = 0,\ \frac{\partial F}{\partial y} = 0,\ \frac{\partial F}{\partial z} = 0.$$

Now $\frac{\partial F}{\partial x} = 0 \Rightarrow 10x - 14z + 4y + 16 = 0$ or $5x + 2y - 7z + 8 = 0$

$\frac{\partial F}{\partial y} = 0 \Rightarrow -8y + 4z + 4x + 16 = 0$ or $x - 2y + z + 4 = 0$

$\frac{\partial F}{\partial z} = 0 \Rightarrow 10z + 4y - 14x - 32 = 0$ or $7x - 2y - 5z + 16 = 0$

Let (α, β, γ) be any point on the line of centres.

Choosing $\gamma = 0$, $\beta = 1$, $\alpha = -2$ we find that $(-2, 1, 0)$ is a point on the line of the centres.

Now $k = u\alpha + v\beta + w\gamma + d = 8(-2) + 8(1) - 16(0) + 8 = 0.$

Hence the reduced equation of the given surface is

$$\lambda_1 x^2 + \lambda_2 y^2 + k = 0 \quad \text{or} \quad 12x^2 - 6y^2 + 0 = 0 \quad \text{or} \quad 2x^2 - y^2 = 0 \qquad \text{...(iii)}$$

which represents a pair of planes whose line of section is the line through $(-2, 1, 0)$ and direction ratios from (ii) are 1, 1, 1.

$\therefore$ The equations of this line through which the two planes given by (ii) pass are

$$\frac{x-(-2)}{1} = \frac{y-1}{1} = \frac{z-0}{1} \quad \text{or} \quad x + 2 = y - 1 = z.$$

Again the planes represented by (iii) are $y^2 = 2x^2$.

i.e. $y = x\sqrt{2}$ and $y = -x\sqrt{2}$ *i.e.* $x\sqrt{2} - y = 0$ and $x\sqrt{2} + y = 0$

$\therefore$ The direction ratios of their normals are $\sqrt{2}, -1, 0$ and $\sqrt{2}, 1, 0$.

$\therefore$ If θ be the angle between these planes, then

$$\cos\theta = \frac{\sqrt{2}.\sqrt{2} + (-1).(1) + 0.0}{\sqrt{[(\sqrt{2})^2 + (-1)^2 + 0^2]\,.\,[(\sqrt{2})^2 + (1)^2 + 0^2]}} = \frac{1}{3}$$

$$\therefore \frac{1 - \tan^2(\theta/2)}{1 + \tan^2(\theta/2)} = \frac{1}{3} \quad \text{or} \quad 3 - 3\tan^2\frac{\theta}{2} = 1 + \tan^2\frac{\theta}{2} \quad \text{or} \quad 2\tan^2\frac{\theta}{2} = 1$$

or $$\tan(\theta/2) = 1/\sqrt{2} \quad \text{or} \quad \theta = 2\tan^{-1}(1/\sqrt{2}).$$ Hence proved.

Ex. 4 (b). In Ex. 4 (a) above prove that the two planes pass through the line $x + 3 = y = z + 1$ and the angle between them is $\tan^{-1}(2\sqrt{2})$

Hint. Proceed exactly as in Ex. 4 (a) above.

Here prove that if (α, β, γ) be any point on the line of centres then choosing $\beta = 0, \gamma = -1, \alpha = -3$ we find that $(-3, 0, -1)$ is a point on the line of centres.

$\therefore$ The equations of the line, through which the planes given by (iii) of Ex. 4 (a) above pass, are $\dfrac{x-(-3)}{1} = \dfrac{y-0}{1} = \dfrac{z-(-1)}{1}$ or $x+3 = y = z+1$

Also in Ex. 4 (a) above $\cos\theta = 1/3$ *i.e.* $\sec\theta = 3$

$\Rightarrow \tan^2\theta + 1 = \sec^2\theta = 9 \Rightarrow \tan^2\theta = 8 \Rightarrow \tan\theta = 2\sqrt{2}$

$\Rightarrow \theta = \tan^{-1}(2\sqrt{2})$. Hence proved.

Exercises on § 12.12. (Case III)

Ex. 1. Reduce $2x^2 + 5y^2 + 2z^2 - 2yz + 4zx - 2xy + 14x - 16y + 14z + 26 = 0$ to the standard form. What does is represent ?

Ans. $2x^2 + y^2 = 1$, elliptic cylinder whose axis is $\dfrac{x+3}{-1} = \dfrac{y-1}{0} = \dfrac{z}{1}$

Ex. 2. Reduce $x^2 - y^2 + 4yz + 4zx - 6x - 2y - 8z + 5 = 0$ to the standard form. What does it represent ?

Ans. Hyperbolic cylinder $x^2 - y^2 = 1$, axis is $\dfrac{x-1}{-2} = \dfrac{y-1}{2} = \dfrac{z-1}{1}$

Ex. 3. Find the condition that the homogeneous equation of second degree in x, y, z represent a pair of planes. (*Kanpur* 92)

§ 12.13. Case IV. A, B, C, F, G, H are all zero, f u ≠ g v.

In this case there is a line of centres at infinity and the two roots of discriminating cubic are zero, say λ_2 and λ_3. Also third root $\lambda_1 \neq 0$.

If the axes through the same origin is so rotated that they are parallel to a set of three mutually perpendicular principal directions then the transformed equation is $\lambda_1 x^2 + 2x(ul_1 + vm_1 + wn_1) + 2y(ul_2 + vm_2 + wn_2)$

$$+ 2z(ul_3 + vm_3 + wn_3) + d = 0 \quad \text{...(i)}$$

The direction cosines l_2, m_2, n_2 and l_3, m_3, n_3 corresponding to zero roots λ_2 and λ_3 satisfy the equation $al + hm + gn = 0$ ...(ii)

Choose l_2, m_2, n_2 such that $ul_2 + vm_2 + wn_2 = 0$ ...(iii)

Then from (i), (iii) we have $\lambda_1 x^2 + 2px + 2rz + d = 0,$ ...(iv)

where $p = ul_1 + vm_1 + wn_1, \quad r = ul_3 + vm_3 + wn_3$ ...(v)

From (iv), $\lambda_1\left(x^2 + \dfrac{2p}{\lambda_1}x + \dfrac{p^2}{\lambda_1^2}\right) + 2rz + \left(d - \dfrac{p^2}{\lambda_1}\right) = 0$ **(Note)**

$$\text{or} \quad \lambda_1\left(x + \frac{p}{\lambda_1}\right)^2 + 2r\left[z + \frac{1}{2r}\left(d - \frac{p^2}{\lambda_1}\right)\right] = 0 \quad \text{...(vi)}$$

Shifting the origin to the point $\left[-\frac{p}{\lambda_1}, 0, -\frac{1}{2r}\left(d-\frac{p^2}{\lambda_1}\right)\right]$

the equation (vi) transforms to $\lambda_1 x^2 + 2rz = 0$ or $x^2 + (2r/\lambda_1) z = 0$, ...(vii)

which is the required reduced form and represents a **parabolic cylinder.**

The latus rectum of a normal section is

$$2r/\lambda_1 \text{ i.e. } (2/\lambda_1)(ul_3 + vm_3 + wn_3), \text{ from (v).}$$

Alternative method.

$\because A = 0 = B = C$, so we have $bc - f^2 = 0$, $ca - g^2 = 0$, $ab - h^2 = 0$.

These imply that a, b, c have the same sign, say positive

Also $F = 0$, $G = 0$, $H = 0$ give $gh - af = 0$, $hf - bg = 0$, $fg - ch = 0$

and so either f, g, h are all positive or two negative and one positive. **(Note)**

$$\therefore f(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$
$$= [\sqrt{a}x \pm \sqrt{b}y \pm \sqrt{c}z]^2$$

i.e. the terms of the second degree in the general equation $F(x, y, z) = 0$ form a perfect square.

Now if $fu \neq gv$, then we proceed as follows :—

General equation $F(x, y, z) = 0$ can be rewritten as

$$[\sqrt{a}x + \sqrt{b}y + \sqrt{c}z + \lambda]^2 = 2x[\lambda\sqrt{a} - u] + 2y[\lambda\sqrt{b} - v] + 2z[\lambda\sqrt{c} - w] + (\lambda^2 - d) \quad \text{...(I)}$$

Now choose λ in such a way that the planes $\sqrt{a}x + \sqrt{b}y + \sqrt{c}z + \lambda = 0$ and $2x(\lambda\sqrt{a} - u) + 2y(\lambda\sqrt{b} - v) + 2z(\lambda\sqrt{c} - w) + (\lambda^2 - d) = 0$ are at right angles

so $$\sqrt{a}(\lambda\sqrt{a} - u) + \sqrt{b}(\lambda\sqrt{b} - v) + \sqrt{c}(\lambda\sqrt{c} - w) = 0$$

or $$\lambda(a + b + c) = u\sqrt{a} + v\sqrt{b} + w\sqrt{c}$$

or $$\lambda = (u\sqrt{a} + v\sqrt{b} + w\sqrt{c})/(a + b + c) \quad \text{...(II)}$$

The equation (I) with the help of (II) can be rewritten as

$$\left[\frac{\sqrt{a}x + \sqrt{b}y + \sqrt{c}z + \lambda}{\sqrt{(a+b+c)}}\right]^2 = k\left[\frac{2x(\lambda\sqrt{a} - u) + 2y(\lambda\sqrt{b} - v) + 2z(\lambda\sqrt{c} - w) + (\lambda^2 - d)}{2\sqrt{\{(\lambda\sqrt{a} - u)^2 + (\lambda\sqrt{b} - v)^2 + (\lambda\sqrt{c} - w)^2\}}}\right]$$

where $k = 2[\sqrt{\{(\lambda\sqrt{a} - u)^2 + (\lambda\sqrt{b} - v)^2 + (\lambda\sqrt{c} - w)^2\}}]/(a + b + c)$.

i.e. The above equation takes the from $X^2 = kY$,

where $$X = (\sqrt{a}x + \sqrt{b}y + \sqrt{c}z + \lambda)/\sqrt{(a + b + c)}$$

and $$Y = \frac{2x(\lambda\sqrt{a} - u) + 2y(\lambda\sqrt{b} - v) + 2z(\lambda\sqrt{c} - w) + (\lambda^2 - d)}{2\sqrt{\{(\lambda\sqrt{a} - u)^2 + (\lambda\sqrt{b} - v)^2 + (\lambda\sqrt{c} - w)^2\}}}$$

This represents a parabolic cylinder.

§ 12.14. Case V. A, B, C, F, G, H are all zero and fu = gv = hw.

In this case there is a plane of centres and two roots λ_2, λ_3 (say) of the discriminating cubic are zero.

If l_1, m_1, n_1 be the principal direction cosines corresponding to the non-zero root λ_1 of the discriminating cubic, then

$$\frac{al_1 + hm_1 + gn_1}{l_1} = \frac{hl_1 + bm_1 + fn_1}{m_1} = \frac{gl_1 + fm_1 + cn_1}{n_1} \qquad \text{...(i)}$$

But $f^2 = bc$, $g^2 = ca$ and $h^2 = ab$, so

$al_1 + hm_1 + gn_1 = al_1 + \sqrt{(ab)}\, m_1 + \sqrt{(ca)}\, n_1 = \sqrt{a}\, [\sqrt{a}\, l_1 + \sqrt{b}\, m_1 + \sqrt{c}\, n_1]$

Similarly $hl_1 + bm_1 + fn_1 = \sqrt{b}\, [\sqrt{a}l_1 + \sqrt{b}m_1 + \sqrt{c}n_1]$,

and $\quad gl_1 + fm_1 + cn_1 = \sqrt{c}\, [\sqrt{a}l_1 + \sqrt{b}m_1 + \sqrt{c}n_1]$

$\therefore$ From (i) we have $\quad \dfrac{l_1}{\sqrt{a}} = \dfrac{m_1}{\sqrt{b}} = \dfrac{n_1}{\sqrt{c}} \qquad \text{...(ii)}$

Also here $fu = gv = hw$

$\Rightarrow \sqrt{(bc)}\, u = \sqrt{(ca)}\, v = \sqrt{(ab)}\, w, \quad \because\ f^2 = bc$ etc.

$\Rightarrow u/\sqrt{(a)} = v/\sqrt{(b)} = w/\sqrt{(c)}$

From (ii), $\qquad u/l_1 = v/m_1 = w/n_1 \qquad \text{...(iii)}$

Now if l_2, m_2, n_2 and l_3, m_3, n_3 be the principal direction cosines corresponding to zero roots λ_2 and λ_3, then

$$ul_2 + vm_2 + wn_2 = l_1l_2 + m_1m_2 + n_1n_2 = 0$$
$$ul_3 + vm_3 + wn_3 = l_1l_3 + m_1m_3 + n_1n_3 = 0.$$

Now as in § 12.13 Page 25 Ch. XII rotating the axes we find that the transformed equation is $\quad \lambda_1 x^2 + 2x\,(ul_1 + vm_1 + wn_1) + d = 0 \qquad$ (Note)

or $\quad \lambda_1 x^2 + 2px + d = 0$, where $p = ul_1 + vm_1 + wn_1$

or $\quad \lambda_1 \left(x + \dfrac{p}{\lambda_1}\right)^2 + \left(d - \dfrac{p^2}{\lambda_1}\right) = 0 \quad$ or $\quad \lambda_1 x^2 + k = 0$,

changing the origin to $(-p/\lambda_1, 0, 0)$ and where $k = d - (p^2/\lambda_1)$

This equation represents a **pair of planes** which are identical or parallel according as $k = 0$ or $k \neq 0$

Alternative method.

As in the alternative method given in § 12.13 on Page 26, if A, B, C, F, G and H are zero, we can prove that $\quad f(x, y, z) = [\sqrt{a}\, x \pm \sqrt{b} y \pm \sqrt{c} z]^2$

i.e. the terms of the second degree in the general equation $F(x, y, z) = 0$ form a perfect square.

Now if $fu = gv = hw$, then as above we can get

$$u/\sqrt{a} = v/\sqrt{b} = w/\sqrt{c} = \mu \ \text{(say)} \qquad \text{...(iv)}$$

Also the general equation $F(x, y, z) = 0$ in this case can be written as

$$(\sqrt{a}x + \sqrt{b}y + \sqrt{c}z)^2 + 2\,(ux + vy + wz) + d = 0 \qquad \text{(Note)}$$

or $(\sqrt{ax}+\sqrt{by}+\sqrt{cz})^2+2\mu(\sqrt{ax}+\sqrt{by}+\sqrt{cz})+d=0$, from (iv)

or $\sqrt{ax}+\sqrt{by}+\sqrt{cz}=-\mu\pm\sqrt{(\mu^2-d)}$, solving as a quadratic equation in $\sqrt{ax}+\sqrt{by}+\sqrt{cz}$

This represents a **pair of parallel planes.**

Solved Examples on § 12.13 — § 12.14 (Case IV and V).

***Ex. 1. Reduce the equation $x^2+y^2+z^2-2yz+2zx-2xy+x-4y-z+1=0$ to the standard form and find the latus rectum of the principal section.** (*Avadh* 93)

Solution. As the terms of second degree form a perfect square, so the given equation can be rewritten as $(x-y+z)^2=-x+4y-z-1$

or $(x-y+z+\lambda)^2=(2\lambda-1)x-2(\lambda-2)y+(2\lambda-1)z+(\lambda^2-1)$ ...(i)

adding a constant λ within the brackets on L.H.S. and adding the corresponding terms on the R.H.S. **(Note)**

Now choose λ in such a way that the planes $x-y+z+\lambda=0$ and $(2\lambda-1)x-2(\lambda-2)y+(2\lambda-1)z+(\lambda^2-1)=0$ are at right angles.

Then $1.(2\lambda-1)+(-1)\{-2(\lambda-2)\}+1.(2\lambda-1)=0\Rightarrow\lambda=1$

$\therefore$ From (i), the given equation of the surface can be rewritten as

$$(x-y+z+1)^2=x+2y+z$$

or $$3\left[\frac{x-y+z+1}{\sqrt{3}}\right]^2=\sqrt{6}\left[\frac{x+2y+z}{\sqrt{6}}\right]$$ **(Note)**

or $3X^2=\sqrt{6}Y$ or $X^2=(1/3)\sqrt{6}Y$, which represents a paraboloic cylinder and the latus rectum of the principal paraboloic section is $\sqrt{6}/3$. **Ans.**

****Ex. 2. Show that the equation $x^2+4y^2+9z^2+12yz+6zx+4xy-54x-52y+62z+113=0$ represents a parabolic cylinder, and that the foci of the normal parabolic section lie on the line**

$$x+2y+3z+1=0=x+y-z-5.$$

Solution. As the terms of second degree form a perfect square, so the given equation can be rewritten as $(x+2y+3z)^2=54x+52y-62z-113$

or $(x+2y+3z+\lambda)^2=2(\lambda+27)x+4(\lambda+13)y+2(3\lambda-31)z$
$+(\lambda^2-113)$, ...(i)

adding a constant λ within the brackets on L.H.S. and adding the corresponding terms on R.H.S. **(Note)**

Now choose λ in such a way that the planes $x+2y+3z+\lambda=0$ and $2(\lambda+27)x+4(\lambda+13)y+2(3\lambda-31)z+\lambda^2-113=0$ are at right angles.

Then $1.\{2(\lambda+27)\}+2.\{4(\lambda+13)\}+3.\{2(3\lambda-31)\}=0$

or $2\lambda+54+8\lambda+104+18\lambda-186=0$ or $\lambda=1$.

$\therefore$ From (i), the given equation of the surface reduces to

$$(x+2y+3z+1)^2 = 56x+56y-56z-112$$

or $$(x+2y+3z+1)^2 = 56\,(x+y-z-2)$$

or $$14\left[\frac{x+2y+3z+1}{\sqrt{(1^2+2^2+3^2)}}\right]^2 = 56\sqrt{3}\left[\frac{x+y-z-2}{\sqrt{\{(1^2+1^2+(-1)^2)\}}}\right]$$

or $$Y^2 = 4\sqrt{3}X, \qquad \text{...(ii)}$$

which represents a parabolic cylinder and the latus rectum of the normal parabolic section is $4\sqrt{3}$.

[**Note :** The vertex of the parabolic cylinder lie on the line of intersection of the planes $x+2y+3z+1=0$, $x+y-z-2=0$, the latter being a tangent plane which touches the cylinder along the vertices.]

The foci evidently lie on the line of intersection of the plane $x+2y+3z+1=0$ *i.e.* the plane through the axis and a plane parallel to the tangent plane $x+y-z-2=0$ but at a distance (1/4)th of latus rectum (*i.e.* $\sqrt{3}$) from it. ...(iii)

Now any plane parallel to the tangent plane $x+y-z-2=0$, $x+y-z+k=0$ and it should be at a distance $\sqrt{3}$ from the tangent plane.

Now any point on the tangent plane is (2, 0, 0), putting $y=0$, $z=0$ in $x+y-z-2=0$.

$\therefore$ distance of the plane $x+y-z+k=0$ from (2, 0, 0) must be $\sqrt{3}$.

i.e. $$\frac{2+0-0+k}{\sqrt{[1^2+1^2+(-1)^2]}} = \sqrt{3} \quad \text{or} \quad 2+k=3 \quad \text{or} \quad k=1.$$

$\therefore$ Foci lie on the line of intersection of the planes

$x+2y+3z+1=0$ and $x+y-z+1=0$, from (iii) Hence proved.

****Ex. 4. Show that the equation $4x^2+9y^2+36z^2-36yz+24zx-12xy-10x+15y-30z+6=0$ represents a pair of parallel planes and find the reduced equation.**

Solution. As the second degree terms of the given equation form a perfect square, so it can be rewritten as

$$(2x-3y+6z)^2 = 10x-15y+30z-6 = 5\,(2x-3y+6z)-6 \qquad \text{...(i)}$$

or $(2x-3y+6z)^2 - 5\,(2x-3y+6z)+6=0$

or $X^2-5X+6=0$, where $X=2x-3y+6z$

or $(X-2)(X-3)=0$ or $X=2, X=3$

or $2x-3y+6z=2,\ 2x-3y+6z=3$ ...(ii)

Hence the given equation represents a pair of parallel planes given by (ii).

Also from (i) we have

$$49\left[\frac{2x-3y+6z}{\sqrt{(2^2+3^2+6^2)}}\right]^2 = 7\left[\frac{5\,(2x-3y+6z)}{\sqrt{(2^2+3^2+6^2)}}\right]-6 \qquad \text{...(iii)}$$

Now choose $2x-3y+6z=0$ as $x=0$ *i.e.* if (x, y, z) be the coordinates of any point, then $x = \dfrac{2x-3y+6z}{\sqrt{(2^2+3^2+6^2)}}$

Then (iii) reduces to $49x^2 - 35x + 6 = 0$, which is the required reduced equation. **Ans.**

Exercises on § 12.13 — § 12.14 (Cases IV—V)

****Ex. 1.** Reduce the equation $36x^2 + 4y^2 + z^2 - 4yz - 12zx + 24xy + 4x + 16y - 26z - 3 = 0$ to the standard form. Show that it represents a parabolic cylinder and find the latus rectum of a normal section. Also show that the foci of the normal parabolic sections lie on the line $6x + 2y - z + 1 = 0 = 2x - 3y + 6z + (90/41)$ (*Avadh* 91)

Ans. $41y^2 = 28x$, latus rectum $= 28/41$

Ex. 2. Reduce the equation $9x^2 + 4y^2 + 4z^2 + 8yz + 12zx + 12xy + 4x + y + 10z + 1 = 0$.

Ans. $17y^2 = 7x$, a parabolic cylinder

Ex. 3. What surface is represented by the equation

$$x^2 + 4y^2 + z^2 + 2zx - 4yz - 4xy - 2x + 4y - 2z - 3 = 0\ ?$$

Reduce it to the standard form.

Ans. A pair of parallel planes, $6x^2 - 2\sqrt{6}x - 3 = 0$.

Ex. 4. Show that $(3x - 4y + z)^2 + 9x - 12y + 3z - 10 = 0$ represents a pair of parallel planes. Also reduce it to the standard form $26x^2 - 3\sqrt{(26)}\,x - 10 = 0$.

§ 12.15. Conicoids of revolution.

Here two cases arise viz.

(i) Two roots of the discriminating cubic are equal and third root not equal to zero.

(ii) Two roots of the discriminating cubic are equal and third root equal to zero.

Under (i) the form to which the given surface can reduce are

$$A(x^2 + y^2) + Bz^2 = 1 \qquad \textbf{(Ellipsoid of revolution)}$$

and

$$A(x^2 - y^2) + Bz^2 = 1 \qquad \textbf{(Hyperboloid of revolution)}$$

Under (ii) the form to which the given surface can reduce are

$$A(x^2 + y^2) + Bz = 0 \qquad \textbf{(Paraboloid of revolution)}$$

and

$$A(x^2 + y^2) + D = 0 \qquad \textbf{(Right circular cylinder)}$$

$\therefore$ We conclude that if the two roots of the discriminating cubic are equal, then surface $F(x, y, z) = 0$ represents a surface (or conicoid) of **revolution.**

Here we proceed in the usual way and the direction ratios of the axis of rotation are obtained from the usual equations by taking that value of λ which is different from the equal values.

Solved Examples on § 12.15.

****Ex. 1.** Show that the equation $x^2 + y^2 + z^2 + yz + zx + xy + 3x + y + 4z + 4 = 0$ represents a surface of revolution and determine the equations of its axis of rotation.

Solution. Here the discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 1-\lambda & 1/2 & 1/2 \\ 1/2 & 1-\lambda & 1/2 \\ 1/2 & 1/2 & 1-\lambda \end{vmatrix} = 0 \qquad ...(i)$$

or $(1-\lambda)[(1-\lambda)^2-(1/4)]-(1/2)[(1/2)(1-\lambda)-(1/4)]$
$+(1/2)[(1/4)-(1/2)(1-\lambda)]=0$

or $(1-\lambda)^3-(3/4)(1-\lambda)+(1/4)=0$

or $4(1-\lambda)^3-3(1-\lambda)+1=0$ or $4\lambda^3-12\lambda^2+9\lambda-2=0$

or $(\lambda-2)(2\lambda-1)^2=0$ or $\lambda=2, 1/2, 1/2.$

$\therefore$ We observe that two roots of discriminating cubic are equal and the third is different from zero, so the given equation represents a surface of revolution [either ellipsoid or hyperboloid of revolution]

...See § 12.15 (i) Page 30 Ch. XII.

The central planes are given by $\dfrac{\partial F}{\partial x}=0,\ \dfrac{\partial F}{\partial y}=0,\ \dfrac{\partial F}{\partial z}=0$

i.e. $2x+y+z+3=0,\ x+2y+z+1=0,\ x+y+2z+4=0.$

Solving these we get $x=-1, y=1, z=-2$

$\therefore$ Centre of the given surface is $(-1, 1, -2)$.

$\therefore d'=u\alpha+v\beta+w\gamma+d=(3/2)(-1)+(1/2)(1)+(2)(-2)+4=-1$

$\therefore$ The reduced equation is $\lambda_1x^2+\lambda_2y^2+\lambda_3z^2+d'=0$

or $(1/2)x^2+(1/2)y^2+2z^2-1=0$ or $\dfrac{x^2}{2}+\dfrac{y^2}{2}+\dfrac{z^2}{(1/2)}=1,$

which is an ellipsoid (of revolution), the squares of whose semiaxes are 2, 2, 1/2.

Now putting $\lambda=2$ in the determinant given by (i) and associating each row with l, m, n, the direction cosines of the principal axis (or axis of revolution), we have

$-l+(1/2)m+(1/2)n=0,\ (1/2)l-m+(1/2)n=0,$
$(1/2)l+(1/2)m-n=0$

i.e. $-2l+m+n=0,\ l-2m+n=0,\ l+m-2n=0$

and these gives $l=m=n=1/\sqrt{3}$, $\because l^2+m^2+n^2=1.$

Now the required axis of rotation (or principal axis) is a line through the centre $(-1, 1, -2)$ of the surface of revolution and direction cosines $1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}$ or direction ratios 1, 1, 1.

$\therefore$ The required equations of the axis of rotation are

$$\frac{x-(-1)}{1}=\frac{y-1}{1}=\frac{z-(-2)}{1} \quad \text{or} \quad x+1=y-1=z+2 \qquad \textbf{Ans.}$$

***Ex. 2. Reduce to standard form the equation**

$$\mathbf{7x^2+y^2+z^2+16yz+8zx-8xy+2x+4y-40z-14=0}$$

and find the principal axis.

Solution. Here the discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 7-\lambda & -4 & 4 \\ -4 & 1-\lambda & 8 \\ 4 & 8 & 1-\lambda \end{vmatrix} = 0 \qquad \text{...(i)}$$

or $(7-\lambda)[(1-\lambda)^2-64]+4[-4(1-\lambda)-32]+4[-32-4(1-\lambda)]=0$

or $\lambda^3-9\lambda^2-81\lambda+729=0$ or $(\lambda-9)(\lambda^2-81)=0$

or $(\lambda-9)(\lambda-9)(\lambda+9)=0$ or $\lambda=9, 9, -9$.

i.e. the two roots of discriminating cubic are equal and the third is different from zero, so the given equation represents either an ellipsoid of revolution or a hyperboloid of revolution.

The central planes are given by $\frac{\partial F}{\partial x}=0$, $\frac{\partial F}{\partial y}=0$, $\frac{\partial F}{\partial z}=0$

i.e. $14x-8y+8z+2=0$, $-8x+2y+16z+4=0$, $8x+16y+2z-40=0$

i.e. $7x-4y+4z+1=0$, $-4x+y+8z+2=0$, $4x+8y+z-20=0$

Solving these we get $x=1, y=2, z=0$.

$\therefore$ Centre of the given surface is $(1, 2, 0)$.

$\therefore d' = u\alpha+v\beta+w\gamma+d=(1)(1)+(2)(2)+(-20)(0)-14=-9$

$\therefore$ The reduced equation of the given surface is

$$\lambda_1x^2+\lambda_2y^2+\lambda_3z^2+d'=0 \quad \text{or} \quad 9x^2+9y^2-9z^2-9=0$$

or $x^2+y^2-z^2=1$, which represents a hyperboloid of revolution, the squares of whose semi-axes are 1, 1, 1.

Now putting $\lambda=-9$ in the determinant given by (i) and associating each row with l, m, n, the d.c.'s of the principal axis, we have

$$16l-4m+4n=0, -4l+10m+8n=0, 4l+8m+10n=0$$

and these gives $\frac{l}{1}=\frac{m}{2}=\frac{n}{-2}=\frac{1}{3}$, $\because$ $l^2+m^2+n^2=1$.

$\therefore$ The equations of the principal axis passing through the centre $(1, 2, 0)$ and d.r.'s $1, 2, -2$ are $\frac{x-1}{1}=\frac{y-2}{2}=\frac{z-0}{-2}$. **Ans.**

Ex. 3. Show that the equation $x^2+2yz=1$ represents a surface of revolution and find the axis of revolution.

Solution Given $F(x, y, z) \equiv x^2+2yz-1=0$

$\therefore$ Here 'a' $=1$, $b=c=0$, 'f' $=1$, $g=0=h=u=v=w$, 'd' $=-1$

$\therefore$ The discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0 \qquad \text{...(i)}$$

or $(1-\lambda)[\lambda^2-1]=0$ or $\lambda=1, 1, -1$.

i.e. the two roots of discriminating cubic are equal and the third is different from zero, so the given equation represents either an ellipsoid of revolution or a hyperboloid of revolution.

The central planes are given by $\frac{\partial F}{\partial x}=0,\ \frac{\partial F}{\partial y}=0,\ \frac{\partial F}{\partial z}=0.$

i.e. $2x=0,\ 2z=0,\ 2y=0$ *i.e.* $x=0, y=0, z=0.$

$\therefore$ Centre of the given surface is (0, 0, 0).

$\therefore d'=u\alpha+v\beta+w\gamma+d=0+0+0-1=-1.$

$\therefore$ Reduced equation of the given surface is

$$\lambda_1x^2+\lambda_2y^2+\lambda_3z^2+d'=0 \quad \text{or} \quad x^2+y^2-z^2-1=0$$

or $x^2+y^2-z^2=1$, which represents a hyperboloid of revolution.

Now putting $\lambda=-1$ in the determinant given by (i) and associating each row with l, m, n, the d.c.'s of the axis of revolution (or principal axis) we have

$$2l=0,\ m+n=0,\ m+n=0 \Rightarrow \frac{l}{0}=\frac{m}{1}=\frac{n}{-1}$$

$\therefore$ The equations of required axis of revolution which passes through the centre (0, 0, 0) and whose d.r.'s are 0, 1, – 1 are

$$\frac{x-0}{0}=\frac{y-0}{1}=\frac{z-0}{-1} \quad i.e. \quad x=0, y+z=0. \qquad \textbf{Ans.}$$

****Ex. 4. Show that the surface represented by the equation**

$$\mathbf{x^2+y^2+z^2-yz-zx-xy-3x-6y-9z+21=0}$$

is a paraboloid of revolution the coordinates of the focus being (1, 2, 3) and the equations to axis are x = y – 1 = z – 2. (*Avadh* 95; *Rohilkhand* 97, 96, 94)

Solution. Here the discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix}=0 \quad \text{or} \quad \begin{vmatrix} 1-\lambda & -(1/2) & -(1/2) \\ -(1/2) & 1-\lambda & -(1/2) \\ -(1/2) & -(1/2) & 1-\lambda \end{vmatrix}=0$$

or $(1-\lambda)\left[(1-\lambda)^2-\left(\frac{1}{4}\right)\right]+\left(\frac{1}{2}\right)\left[-\left(\frac{1}{2}\right)(1-\lambda)-\left(\frac{1}{4}\right)\right]-\left(\frac{1}{2}\right)\left[\left(\frac{1}{4}\right)+\frac{1}{2}(1-\lambda)\right]=0$

or $(1-\lambda)[8(1-\lambda)^2-2]-2[2(1-\lambda)+1]=0$ or $4\lambda^3-12\lambda^2+9\lambda=0$

or $\lambda[4\lambda^2-12\lambda+9]=0$ or $\lambda(2\lambda-3)^2=0$ or $\lambda=3/2, 3/2, 0$

As two roots of the discriminating cubic are equal and third root is zero, so it is either a paraboloid of revolution or a right circular cylinder.

[See § 12.15 (ii) Page 30 Ch. XII]

The direction ratios of the axis are given by

$$al+hm+gn=0,\ hl+bm+fn=0,\ gl+fm+cn=0$$

i.e. $l-\frac{1}{2}m-\frac{1}{2}n=0, -\frac{1}{2}l+m-\frac{1}{2}n=0, -\frac{1}{2}l-\frac{1}{2}m+n=0$

i.e. $2l-m-n=0, -l+2m-n=0, -l-m+2n=0.$

These give $l=m=n=1/\sqrt{3}$

Now $k=ul+vm+wn$

or $k=(-3/2)(1/\sqrt{3})+(-3)(1/\sqrt{3})+(-9/2)(1/\sqrt{3})=-3\sqrt{3}\neq 0.$

$\therefore$ The reduced equation is $\lambda_1 x^2 + \lambda_2 y^2 + 2kz = 0$ **(Note)**

...See § 12.11 (iii) Page 14 Ch. XII

or $\quad (3/2)\, x^2 + (3/2)\, y^2 + 2\,(-3\sqrt{3})\, z = 0$

or $\quad x^2 + y^2 = 4\sqrt{3}z$, which reprresents a paraboloid of revolution.

Also the coordinates of the vertex of the paraboloid are obtained by solving any two of the three equations

$$\frac{\left(\dfrac{\partial F}{\partial x}\right)}{l} = \frac{\left(\dfrac{\partial F}{\partial y}\right)}{m} = \frac{\left(\dfrac{\partial F}{\partial z}\right)}{n} = 2k$$ ...See § 12.11 (iv) Page 14 Ch. XII

or $\quad \dfrac{2x - y - z - 3}{1/\sqrt{3}} = \dfrac{2y - z - x - 6}{1/\sqrt{3}} = \dfrac{2z - y - x - 9}{1/\sqrt{3}} = -6\sqrt{3}.$

or $\quad 2x - y - z - 3 = 2y - z - x - 6 = 2z - y - x - 9 = -6$

or $\quad 2x - y - z + 3 = 0,\ x - 2y + z = 0,\ x + y - 2z + 3 = 0$ ...(I)

with the equation $\quad k\,(lx + my + nz) + ux + vy + wz + d = 0$

i.e. $\quad -3\sqrt{3}\left(\dfrac{1}{\sqrt{3}}x + \dfrac{1}{\sqrt{3}}y + \dfrac{1}{\sqrt{3}}z\right) + \left(-\dfrac{3}{2}\right)x + (-3)\,y + \dfrac{-9}{2}z + 21 = 0$

i.e. $$3x + 4y + 5z - 14 = 0 \qquad \text{...(II)}$$

Solving $2x - y - z + 3 = 0,\ x - 2y + z = 0,\ 3x + 4y + 5z - 14 = 0$

we get $x = 0, y = 1, z = 2$. $\quad\therefore\quad$ The required vertex is $(0, 1, 2)$.

$\therefore$ Equations of the axis are $\dfrac{x - 0}{1} = \dfrac{y - 1}{1} = \dfrac{z - 2}{1}$

or $$x = y - 1 = z - 2.$$ **Ans.**

Also the focus will be a point on the axis whose actual direction cosines are $1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3}$ and will be at a distance $(1/4)\, 4\sqrt{3}$ *i.e.* $\sqrt{3}$ from the vertex $(0, 1, 2)$

$\therefore$ Coordinates of the focus are given by

$$\frac{x - 0}{(1/\sqrt{3})} = \frac{y - 1}{(1/\sqrt{3})} = \frac{z - 2}{(1/\sqrt{3})} = \sqrt{3}$$ **(Note)**

or $$x = 1, y = 2, z = 3$$

$\therefore$ The required focus is $(1, 2, 3)$. **Ans.**

****Ex. 5. Show that $13x^2 + 45y^2 + 40z^2 + 12yz + 36zx - 24xy - 49 = 0$ represents a right circular cylinder whose axis is $x/6 = y/2 = z/-3$ and radius 1.**

Solution. Here the discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 13-\lambda & -12 & 18 \\ -12 & 45-\lambda & 6 \\ 18 & 6 & 40-\lambda \end{vmatrix} = 0$$

or $\quad (13 - \lambda)\,[(45 - \lambda)(40 - \lambda) - 36] + 12\,[-12\,(40 - \lambda) - 108]$
$\qquad + 18\,[-72 - 18\,(45 - \lambda)] = 0$

i.e. $\quad \lambda\,(\lambda - 49)^2 = 0 \quad$ or $\quad \lambda = 0, 49, 49.$

As two roots of the discriminating cubic are equal and third root is zero, so it is either a paraboloid of revolution or a right circular cylinder

[See § 12.15 (ii) Page 30 Ch. XII]

The d. ratios of the axis are given by

$$al + hm + gn = 0,\ hl + bm + fn = 0,\ gl + fm + cn = 0$$

i.e. $13l - 12m + 18n = 0, -12l + 45m + 6n = 0,\ 18l + 6m + 40n = 0.$

Solving these we get $\dfrac{l}{6} = \dfrac{m}{2} = \dfrac{n}{-3} = \dfrac{1}{\sqrt{7}}$ (Note)

Now here $k = ul + vm + wn = 0, \because u = 0 = v = w.$

Also the line of centres is given by $\dfrac{\partial F}{\partial x} = 0,\ \dfrac{\partial F}{\partial y} = 0,\ \dfrac{\partial F}{\partial z} = 0$

or $26x - 24y + 36z = 0, -24x + 90y + 12z = 0, 36x + 12y + 80z = 0$

or $13x - 12y + 18z = 0, 8x - 30y - 4z = 0, 9x + 3y + 20z = 0,$

which gives $x = 0, y = 0, z = 0.$

$\therefore$ Any point on the line of centres is $(0, 0, 0)$

Also $d' = u\alpha + v\beta + w\gamma + d = 0 = 0 + 0 - 49 = -49$

$\therefore$ The reduced equation is $\lambda_1 x^2 + \lambda_2 y^2 + d' = 0$ (Note)

or $$49x^2 + 49y^2 - 49 = 0 \quad \text{or} \quad x^2 + y^2 = 1,$$

which is a right circular cylinder of radius 1, as any section of this surface by a plane $z = k$ is a circle $x^2 + y^2 = 1$, whose radius is 1.

And the equations of the axis are

$$\frac{x-0}{l} = \frac{y-0}{m} = \frac{z-0}{n} \quad i.e.\ \frac{x}{6} = \frac{y}{2} = \frac{z}{-3}$$

****Ex. 6. Prove that the equation $2y^2 + 4zx + 2x - 4y + 6z + 5 = 0$ represents a right circular cone. Show also that the semi-veritical angle of this cone is $\pi/4$ and its axis is given by $x + z + 2 = 0,\ y = 1.$** (*Garhwal* 96)

Solution. The discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \quad \text{or} \quad \begin{vmatrix} 0-\lambda & 0 & 2 \\ 0 & 2-\lambda & 0 \\ 2 & 0 & -\lambda \end{vmatrix} = 0 \qquad \text{...(i)}$$

or $$-\lambda[-\lambda(2-\lambda)] + 2[-2(2-\lambda)] = 0.$$

or $$(2-\lambda)(\lambda^2 - 4) = 0 \quad \text{or} \quad \lambda = 2, 2, -2$$

As two roots of this cubic are equal and third is not zero, so the given surface is a surface of revolution. (Note)

Also the line of centres is given by any two of $\dfrac{\partial F}{\partial x} = 0,\ \dfrac{\partial F}{\partial y} = 0,\ \dfrac{\partial F}{\partial z} = 0$

i.e. $$4z + 2 = 0,\ 4y - 4 = 0,\ 4x + 6 = 0.$$

$\therefore$ If (α, β, γ) be any point on the line of centres, then

$4\gamma + 2 = 0,\ 4\beta - 4 = 0,\ 4\alpha + 6 = 0 \Rightarrow \alpha = -3/2,\ \beta = 1, \gamma = -1/2.$

$\therefore$ Any point on the line of centres is $(-3/2, 1, -1/2)$.

$\therefore d' = u\alpha + v\beta + w\gamma + d = 1(-3/2) - 2(1) + 3(-1/2) + 5 = 0.$

$\therefore$ The reduced form of the equation is

$$\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + d' = 0 \quad \text{or} \quad 2x^2 + 2y^2 - 2z^2 + 0 = 0$$

or $$x^2 + y^2 - z^2 = 0 \quad \text{or} \quad x^2 + y^2 = z^2 \tan^2 45°$$

which represents a right ciruclar cone of semi-vertical angle $\pi/4$.

[$\because x^2 + y^2 = z^2 \tan^2 \alpha$ represents a cone whose semi-vertical angle is α].

Now putting the unequal value of λ viz. -2 in the determinaant of (i) and associating each row with l, m, n we have $2l + 2n = 0,\ 4m = 0,\ 2l + 2n = 0$

These gives $$\frac{l}{1} = \frac{m}{0} = \frac{n}{-1} = \frac{1}{\sqrt{2}}$$

$\therefore$ The equations of its axis are

$$\frac{x-(-3/2)}{l} = \frac{y-1}{m} = \frac{z-(-1/2)}{n} \quad \text{or} \quad \frac{x+(3/2)}{1} = \frac{y-1}{0} = \frac{z+(1/2)}{-1}$$

or $$-x-(3/2) = z + (1/2),\ y - 1 = 0 \quad \text{or} \quad x + z + 2 = 0,\ y = 1$$

Hence proved.

Exercises on § 12.15

Ex. 1. Prove that the equation $2x^2 + 5y^2 + z^2 - 4xy - 8x + 14y + 3 = 0$ is a surface of revolution. Also find the equations of its principal axis.

Ans. Reduced equation is $x^2 + y^2 + 6z^2 = 8$, axis $2x + y - 1 = 0 = z$.

Ex. 2. Find the reduced equation of the surface

$$x^2 - y^2 + 2yz - 2zx - x - y + z = 0. \text{ Also find its axis.}$$

Ans. $3(x^2 - y^2) = z\,;\ x - (1/3) = y + (1/3) = z.$

***Ex. 3.** Discuss the nature of the surface $yz + zx + xy = a^2$.

Ans. A hyperboloid of revolution ; reduced equation is

$2x^2 - y^2 - z^2 = 2a^2$, axis is $x = y = z$.

Excercises on Chapter XII

Ex. 1. Reduce the surface $40x^2 + 50y^2 + 9z^2 - 8yz - 16zx + 26xy + 4x + 20y - 28z - 3 = 0$ into the standard form and find the latus rectum of a normal section. *(Avadh 92)*

Ex. 2. Reduce the equation $12x^2 + 10y^2 + 8z^2 - 9yz + zx - 13xy + 75x + 77y - 38z + 100 = 0$ into the standard form and also describe the nature of the surface and find the equations of its axes. *(Avadh 92)*