Mains Test Series - 2019

Test no. 15 (Paper-I Full Syllabus)

Answer key

Section - A

Ques: 1·a) Prove that the set V of vectors (x_1, x_2, x_3, x_4) in $\mathbb{R}^4$ which satisfy the equations
$x_1 + x_2 + 2x_3 + x_4 = 0$ and $2x_1 + 3x_2 - x_3 + x_4 = 0$, is a subspace of $\mathbb{R}^4$. What is the dimension of this subspace? Find one of its bases?

Solution:-

Let $\mathbb{R}^4 = \{(x_1, x_2, x_3, x_4) / x_1, x_2, x_3, x_4 \in \mathbb{R}\}$ be the given vector space.

Let $V = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \middle/ \begin{array}{l} x_1 + x_2 + 2x_3 + x_4 = 0 \\ 2x_1 + 3x_2 - x_3 + x_4 = 0 \\ x_1, x_2, x_3, x_4 \in \mathbb{R} \end{array} \right\} \subseteq \mathbb{R}^4$

Since, $(0,0,0,0) \in \mathbb{R}^4$;

$$0 + 0 + 2(0) + 0 = 0 \text{ and.}$$
$$2(0) + 3(0) - 0 + 0 = 0$$
$$\therefore (0,0,0,0) \in V$$

$\therefore$ V is non-empty subset of $\mathbb{R}^4$

let $\alpha = (x_1, x_2, x_3, x_4)$
$\beta = (y_1, y_2, y_3, y_4) \in V$

then
$$x_1 + x_2 + 2x_3 + x_4 = 0$$
$$2x_1 + 3x_2 - x_3 + x_4 = 0$$
$$y_1 + y_2 + 2y_3 + y_4 = 0$$
$$2y_1 + 3y_2 - y_3 + y_4 = 0.$$

Let $a, b \in \mathbb{R}$, then we have.

$$a\alpha + b\beta = (ax_1+by_1, ax_2+by_2, ax_3+by_3, ax_4+by_4) \quad \text{— (1)}$$

Since;

$$(ax_1+by_1) + (ax_2+by_2) + 2(ax_3+by_3) + (ax_4+bx_4)$$
$$= a(x_1+x_2+2x_3+x_4) + b(y_1+y_2+2y_3+y_4)$$
$$= a(0) + b(0)$$
$$= 0$$

and $2(ax_1+by_1) + 3(ax_2+by_2) - (ax_3+by_3) + (ax_4+by_y)$
$$= a(2x_1+3x_2-x_3+x_4) + b(2y_1+3y_2-y_3+y_4)$$
$$= a(0) + b(0) = 0$$

Since, the number of elements in a basis 'S' is 2

$$\therefore \boxed{\dim V = 2}$$

Q.1 (b) let $A = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ and C be a non-singular matrix of order 3×3.

Find the eigen values of matrix B^3, where $B = C^{-1}AC$.

Sol^n $A = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{vmatrix} = 2((1+3)) + 2(-1-1) + 2(3-1)$$

$$= 2\times -4 + 2\times -2 \cancel{+ 2\times 2}$$

$$|A| = -8 \neq 0$$

$\rightarrow$ A is linearly independent

$\rightarrow$ A is diagonalizable ——— (1)

let λ be an eigen value of A,

characteristic polynomial is

$$|A-\lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & -2 & 2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} = 0$$

$$0 = (2-\lambda)[(1-\lambda)^2 - 3] + 2[-1-\lambda-1] + 2[3-1+\lambda]$$

$$\Rightarrow (2-\lambda)(\lambda+2)(\lambda-2) = 0$$

$$\boxed{\lambda = 2, 2, -2} \text{ ——— (2)}$$

$\because$ A is diagonalizable (from eq^n (1)),

$\exists$ P invertible matrix, such that

$A = P^{-1}DP$ where $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ ———

$$\Rightarrow A = P^{-1}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix} P$$

$$A^2 = (P^{-1}DP)(P^{-1}DP)$$

$$A^2 = P^{-1}D(PP^{-1})DP$$

$$A^2 = P^{-1}DIDP$$

$$A^2 = P^{-1}D^2P$$

Similarly, $A^3 = P^{-1}D^3P = P^{-1}\begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -8 \end{bmatrix} P$ —— (3)

Now, since,

$$B = C^{-1}AC$$

$$B = C^{-1}P^{-1}DPC$$

$$B^2 = (C^{-1}P^{-1}DPC)(C^{-1}P^{-1}DPC)$$

$$B^2 = C^{-1}P^{-1}DP(CC^{-1})P^{-1}DPC$$

$$B^2 = C^{-1}P^{-1}DPIP^{-1}DPC = C^{-1}P^{-1}DPP^{-1}DPC$$

$$B^2 = C^{-1}P^{-1}DIDPC = C^{-1}P^{-1}D^2PC$$

Similarly,

$$B^3 = C^{-1}P^{-1}D^3PC$$

$$\therefore B^3 = C^{-1}P^{-1}\begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -8 \end{bmatrix} PC$$

where, PC is invertible [$\because$ P & C both are non-singular $\Rightarrow$ invertible)

$\Rightarrow B^3$ is diagonalizable and diagonal elements are eigen values of B^3.

$\Rightarrow$ Eigen values of $B^3 = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 = 8, 8, -8$

Ques: 1(c)) Prove that if $a_0, a_1, a_2, \ldots, a_n$, are the real numbers such that

$$\frac{a_0}{n+1}+\frac{a_1}{n}+\frac{a_2}{n-1}+\frac{a_3}{n-2}+\cdots+\frac{a_{n-1}}{2}+a_n=0$$

Then there exists at least one real number 'x' between 0 and 1 such that

$$a_0x^n+a_1x^{n-1}+a_2x^{n-2}+\cdots+a_{n-1}x+a_n=0.$$

Solution:- Given, $\frac{a_0}{n+1}+\frac{a_1}{n}+\frac{a_2}{n-1}+\cdots+\frac{a_{n-1}}{2}+a_n=0$ —— (1)

Let us consider,

$$f(y)=\frac{a_0y^{n+1}}{n+1}+\frac{a_1y^n}{n}+\frac{a_2y^{n-1}}{n-1}+\cdots+\frac{a_{n-1}y^2}{2}+a_ny. \quad \text{—— (2)}$$

is a polynomial function in y.

$\therefore$ (i) $f(y)$ is continuous in $[0,1]$

(ii) $f(y)$ being a polynomial is differentiable in $(0,1)$.

(iii) $f(0)=0$ and

$$f(1)=\frac{a_0}{n+1}+\frac{a_1}{n}+\frac{a_2}{n-1}+\cdots+\frac{a_{n-1}}{2}+a_n$$

$$=0 \quad \text{—— from (1)}$$

$\therefore f(1)=0=f(0)$.

Thus all the conditions of Rolle's theorem are satisfied in the interval $[0,1]$.

$\therefore$ By Rolle's theorem, $\exists$ at least one real number $x\in(0,1)$ such that

$$\boxed{f'(x)=0} \quad \text{—— (3)}$$

$\therefore$ from (2); we get

$$f'(y) = a_0 y^n + a_1 y^{n-1} + a_2 y^{n-2} + \cdots + a_{n-1} y + a_n$$

put $y = x$

$$f'(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n \quad \text{--- (4)}$$

comparing (3) and (4).

$$\boxed{a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n = 0}$$

which is the required result.

Ques:1(d) Find all the asymptotes of the curve
$$x^4-y^4+3x^2y+3xy^2+xy=0.$$

Solution:-

Given the curve is
$$x^4-y^4+3x^2y+3xy^2+xy=0 \quad \text{——— (1)}$$

Since, the coefficient of the highest powers of x and y are constant.

$\therefore$ There are no asymptotes parallel to x-axis and y-axis.

To find out oblique asymptotes

Let $y=mx+c$, is an oblique asymptote.
then substituting it in (1), we get

$$x^4-(mx+c)^4+3x^2(mx+c)+3x(mx+c)^2+x(mx+c)=0$$

$$\Rightarrow x^4-(m^4x^4+c^4+4m^2c^2x^2+2m^2x^2c^2+4mc^3x+4m^3cx^3)+3mx^3+3x^2c+3x(m^2x^2+2mxc+c^2)+mx^2+cx=0$$

$$\Rightarrow x^4-(m^4x^4+c^4+6m^2c^2x^2+4mxc^3+4m^3x^3c)+3mx^3+3x^2c+3m^2x^3+6mx^2c+3xc^2+mx^2+cx=0$$

$$\Rightarrow x^4(1-m^4)+(-4m^3c+3m+3m^2)x^3+(-6m^2c^2+3c+6mc+m)x^2+(-4mc^3+3c^2+c)x-c^4=0$$

Now to find out m & c we will equate two highest degree terms to zero.

ie $1-m^4=0$

$\Rightarrow (1-m^2)(1+m^2)=0$

$(1+m)(1-m)(1+m^2)=0$

$m=\pm 1$, $m=\pm i$ (neglected as only taking real values of m).

and $-4m^3c+3m+3m^2=0$

put $m=1$.

$-4c+3+3=0$

$-4c=-6 \Rightarrow \boxed{c=\frac{3}{2}}$

at $m=-1$.

$+4c-\cancel{3}+\cancel{3}=0$

$\boxed{c=0}$

$\therefore$ putting the values of m & c in $y=mx+c$

The two asymtotes are

at $m=-1$, $c=0$

$\boxed{y=-x}$ or $\boxed{y+x=0}$

at $m=1$, $c=3/2$

$\boxed{y=x+3/2}$

Q.1 (e) find the equation of the sphere which passes through the points (1,0,0), (0,1,0) and (0,0,1) and has its radius as small as possible.

Sol: let the Equation of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+c=0 \longrightarrow \text{(i)}$$

If it passes through (1,0,0), (0,1,0), (0,0,1) then

$$1+2u+c=0,\ 1+2v+c=0,\ 1+2w+c=0$$

$$u=v=w=-\tfrac{1}{2}(1+c) \longrightarrow \text{(ii)}$$

∴ If r be the radius of the sphere (i), then

$$r^2=u^2+v^2+w^2-c=R \text{ (say)}$$

$$R=\frac{3}{4}(1+c)^2-c \quad \text{from(ii)}$$

If 'r' is least then R least

Now $\frac{dR}{dc}=\frac{3}{2}(1+c)-1$ and $\frac{d^2R}{dc^2}=\frac{3}{2}$ = positive

Equating $\frac{dR}{dc}$ to zero we get $\frac{3}{2}c+\frac{1}{2}=0$ (or) $c=-\frac{1}{3}$

and $\frac{d^2R}{dc^2}$ being positive R is least when $c=-\frac{1}{3}$

∴ from(ii) when R i.e r^2 is least we have

$$u=v=w=-\tfrac{1}{2}\left(1-\tfrac{1}{3}\right)=-\tfrac{1}{3}$$

∴ from(i), the required equation is $x^2+y^2+z^2-\frac{2}{3}(x+y+z)-\frac{1}{3}=0$

(or) $3(x^2+y^2+z^2)-2(x+y+z)-1=0.$ ⟵ Ans.

Q.2 (a) (i) If λ is a characteristic root of a non-singular matrix A, then prove that $\frac{|A|}{\lambda}$ is a characteristic root of Adj A.

Sol'n: Since 'λ' is a characteristic root of a non-singular matrix, therefore $\lambda \neq 0$. Also λ is a characteristic root of A implies there exists a non-zero vector X such that $AX = \lambda X$

$$\Rightarrow (\text{Adj } A)(Ax) = (\text{Adj } A)(\lambda x)$$

$$\Rightarrow [(\text{Adj } A) A]x = \lambda (\text{Adj } A)x$$

$$\Rightarrow |A| I x = \lambda (\text{Adj } A)x \quad (\because (\text{Adj} A) A = |A| I)$$

$$\Rightarrow \frac{|A|}{\lambda} x = (\text{Adj } A)x$$

$$\Rightarrow (\text{Adj } A)x = \frac{|A|}{\lambda} x$$

$\Rightarrow \frac{|A|}{\lambda}$ is a characteristic root of the matrix Adj A.

Q.2 (a) (ii) If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, show that for every integer $n \geq 3$, $A^n = A^{n-2} + A^2 - I$. Hence, determine A^{50}.

Sol'n: If $n = 3$ then

$$A^3 = A + A^2 - I \quad \text{——— (1)}$$

since $A^2 = A \cdot A$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Now $A^3 = A^2 \cdot A$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Now $A + A^2 - I$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$\therefore A^n = A^{n-2} + A^2 - I$ is true for $n = 3$.

Suppose $A^n = A^{n-2} + A^2 - I$ is true for $n=k$

$\therefore A^k = A^{k-2} + A^2 - I$

Now for $n = k+1$;

$$A^{k+1} = A \cdot A^k$$
$$= A \cdot \left[A^{k-2} + A^2 - I\right]$$
$$= A^{k-1} + A^3 - A$$
$$= A^{k-1} + A + A^2 - I - A \quad (\text{from } 1)$$
$$= A^{k-1} + A^2 - I$$

$\therefore A^n = A^{n-2} + A^2 - I$ is true for $n = k+1$.

$\therefore$ By mathematical induction, it is true for every integer $n \geq 3$.

$$A^n = A^{n-2} + A^2 - I \quad \text{——} \quad (2)$$

Now $A^3 = A + A^2 - I$

$$\boxed{A^4 = 2A^4 - I}$$

$$A^6 = A^4 + A^2 - I$$

$$\boxed{A^6 = 3A^2 - 2I}$$

$$A^8 = A^6 + A^2 - I$$

$$\boxed{A^8 = 4A^2 - 3I}$$

$$A^{10} = A^8 + A^2 - I$$

$$\boxed{A^{10} = 5A^2 - 4I}$$

Similarly $A^{12} = 6A^2 - 5I$

(i.e. $A^{12} = \frac{12}{2}A^2 - \left(\frac{12}{2} - 1\right)I$)

$A^{50} = 25A^2 - 24I$ (i.e. $A^{50} = \frac{50}{2}A^2 - \left(\frac{50}{2} - 1\right)I$)

$$\therefore A^{50} = 25\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - 24\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}$$

Ques: 2(b)) Show that

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2} \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left[\frac{q+1}{2}\right]}{\Gamma\left[\frac{p+q+2}{2}\right)}, \; p, q > -1.$$

Hence evaluate the following integrals :-

(i) $\int_0^{\pi/2} \sin^4 x \cos^5 x \, dx$

(ii) $\int_0^1 x^3(1-x^2)^{5/2} dx$

(iii) $\int_0^1 x^4(1-x)^3 dx$.

Solution:- we define,

$$\Gamma(m) = \int_0^\infty x^{m-1} \cdot e^{-x} dx \quad , \; m > 0$$

$$\beta(m,n) = \int_0^1 x^{m-1} \cdot (1-x)^{n-1} dx \quad ; \; m, n > 0$$

we put, $\sin^2\theta = x \qquad \therefore 2\sin\theta\cos\theta \, d\theta = dx$

$$\int_0^{\pi/2} \sin^p\theta \cdot \cos^q\theta \, d\theta$$

$$= \int_0^{\pi/2} (\sin^2\theta)^{\frac{p-1}{2}} \cdot (\cos^2\theta)^{\frac{q-1}{2}} \cdot \sin\theta \cdot \cos\theta \, d\theta$$

$$= \int_0^1 x^{\frac{p-1}{2}} \cdot (1-x)^{\frac{q-1}{2}} \frac{dx}{2}.$$

$$= \frac{1}{2} \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$= \frac{1}{2} \frac{\Gamma\left[\frac{p+1}{2}\right]\Gamma\left[\frac{q+1}{2}\right]}{\Gamma\left[\frac{p+q+2}{2}\right]} \quad \left[\because \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma[m+n]}\right]$$

$$\boxed{\therefore \int_0^{\pi/2} \sin^p\theta \cdot \cos^q\theta \, d\theta = \frac{1}{2} \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}}$$

Hence proved

Now to evaluate:

(i) $$\int_0^{\pi/2} \sin^4 x \cdot \cos^5 x \, dx = \frac{1}{2} \frac{\Gamma\left[\frac{4+1}{2}\right]\Gamma\left[\frac{5+1}{2}\right]}{\Gamma\left[\frac{4+5+2}{2}\right]}$$

$$= \frac{1}{2} \frac{\Gamma\left(\frac{5}{2}\right)\Gamma[3]}{\Gamma\left[11/2\right]}$$

$$\boxed{\begin{gathered}\Gamma(m) = (m-1)! \\ \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi} \\ \Gamma(m) = m-1\,\Gamma[m-1]\end{gathered}}$$

$$= \frac{\cancel{\frac{1}{2}} \times \frac{3}{\cancel{2}} \times \frac{1}{\cancel{2}} \times \cancel{\Gamma\left[\frac{1}{2}\right]} \cdot 2!}{\frac{9}{\cancel{2}} \times \frac{7}{\cancel{2}} \times \frac{5}{\cancel{2}} \times \frac{3}{2} \times \frac{1}{2} \times \cancel{\Gamma[1/2]}}$$

$$= \frac{2 \times 2 \times 2 \times \cancel{3}}{9 \times 7 \times 5 \times \cancel{3}} = \frac{8}{315}$$

(ii) $$\int_0^1 x^3 (1-x^2)^{5/2} dx$$

put $x = \sin\theta$ $dx = \cos\theta \, d\theta$.
$0 < \theta < \pi/2$

$$\int_0^{\pi/2} \sin^3\theta \cdot (\cos^2\theta)^{5/2} \cos\theta \, d\theta$$

$$= \int_0^{\pi/2} \sin^3\theta \cdot \cos^5\theta \cdot \cos\theta \, d\theta = \int_0^{\pi/2} \sin^3\theta \cdot \cos^6\theta \, d\theta$$

$$= \frac{1}{2} \frac{\Gamma[2]\Gamma[7/2]}{\Gamma\left[\frac{11}{2}\right]}$$

$$= \frac{1}{2} \; \frac{(1!) \times \cancel{\tfrac{7}{2}} \times \cancel{\tfrac{5}{2}} \times \cancel{\tfrac{3}{2}} \times \cancel{\tfrac{1}{2}}\, \cancel{\Gamma[1/2]}}{\tfrac{9}{2} \times \tfrac{7}{2} \times \cancel{\tfrac{5}{2}} \times \cancel{\tfrac{3}{2}} \times \cancel{\tfrac{1}{2}}\, \cancel{\Gamma[1/2]}}$$

$$= \frac{1}{\cancel{2}} \times \frac{\cancel{4}\,2}{9 \times 7} = \frac{2}{63}.$$

(iii) $\int_0^1 x^4 (1-x)^3 dx.$

$$= \beta(4+1, 3+1) = \beta(5,4)$$

$$= \frac{\Gamma(5)\Gamma[4]}{\Gamma[5+4]} = \frac{\cancel{\Gamma[5]} \cdot 3!}{8 \times 7 \times 6 \times 5 \times \cancel{\Gamma[5]}}$$

$$= \frac{\cancel{3} \times \cancel{2} \times 1}{8 \times 7 \times \cancel{6} \times 5} = \frac{1}{280}$$

$$\boxed{\therefore \int_0^1 x^4 (1-x)^3 dx = \frac{1}{280}}$$ required solution.

Ques: 2(c)) Find the point of intersection P, Q of the generators of opposite system drawn through the points $A(a\cos\alpha, b\sin\alpha, 0)$ and $B(a\cos\beta, b\sin\beta, 0)$ of the principal elliptic section of the hyperboloid $\left(\frac{x^2}{a^2}\right)+\left(\frac{y^2}{b^2}\right)-\left(\frac{z^2}{c^2}\right)=1$.

Hence, show that if A and B are extremities of semi-conjugate diameters, the loci of the points P and Q are the ellipses

$$(x^2/a^2)+(y^2/b^2)=2, \quad z=\pm c.$$

Solution:-

Let the coordinates of the point P be (x_1, y_1, z_1)

The equation of the tangent plane to the given hyperboloid at P is—

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-\frac{zz_1}{c^2}=1$$ and it meets the plane $z=0$ in the line —

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1, \quad z=0 \quad \text{——— (1)}$$

which is the same as the line joining the points A and B.

i.e. $\frac{x}{a}\cos\frac{(\alpha+\beta)}{2}+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$; $z=0$ ——— (2)

Comparing (1) and (2), we get

$$\frac{x_1/a^2}{\frac{1}{a}\cos\left(\frac{\alpha+\beta}{2}\right)} = \frac{y_1/b^2}{\frac{1}{b}\sin\left(\frac{\alpha+\beta}{2}\right)} = \frac{1}{\cos\left(\frac{\alpha-\beta}{2}\right)}$$

$$\Rightarrow \frac{x_1}{a} = \frac{\cos((\alpha+\beta)/2)}{\cos((\alpha-\beta)/2)} \; ; \; \frac{y_1}{b} = \frac{\sin\{(\alpha+\beta)/2\}}{\cos\{(\alpha-\beta)/2\}} \quad \text{--- (3)}$$

Again $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - \frac{z_1^2}{c^2} = 1.$

$$\Rightarrow \left[\frac{1}{\cos^2\left(\frac{\alpha-\beta}{2}\right)}\right] - \frac{z_1^2}{c^2} = 1 \quad [\because \text{from (3)}]$$

$$\Rightarrow \frac{z_1^2}{c^2} = \sec^2\left(\frac{\alpha-\beta}{2}\right) - 1 = \tan^2\left(\frac{\alpha-\beta}{2}\right)$$

$$\Rightarrow \boxed{\frac{z_1}{c} = \pm \frac{\sin\{(\alpha-\beta)/2\}}{\cos\{(\alpha-\beta)/2\}}} \quad \text{--- (4)}$$

from (3) and (4), we get the coordinates of $P(x_1, y_1, z_1)$ as

$$\left[\frac{a\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}, \frac{b\sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}, \pm\frac{c\sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}\right]$$

Again, as A and B are extremities of two semi-conjugate diameters, we have

$$\alpha - \beta = \pi/2 \quad \text{--- (5)}$$

$\therefore$ from (3), we get

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = \frac{1}{\cos^2\left(\frac{\alpha-\beta}{2}\right)} = \frac{1}{\cos^2(\pi/4)} \quad (\text{from (5)})$$

$$\therefore \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = \frac{1}{(1/\sqrt{2})^2} = 2$$

$$\Rightarrow \boxed{\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 2}$$

And from (4),

$$z_1 = \pm c \tan\left(\frac{\alpha-\beta}{2}\right)$$

$$= \pm c \tan\left(\frac{\pi/2}{2}\right) = \pm c \tan(\pi/4) \quad [\text{from } (5)]$$

$$\Rightarrow \boxed{z_1 = \pm c} \qquad [\because \tan \pi/4 = 1]$$

$\therefore$ The locus of P and Q are

$$\boxed{\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2 \; ; \; z = \pm c}$$

Hence, the result

Ques: 3(a)(i) Let T be the linear operator in $\mathbb{R}^3$ defined by $T(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3)$ what is the matrix of T in the standard ordered basis for $\mathbb{R}^3$? What is the basis of range space of T and a basis of null space of T?

Solution:- Let $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be given, such that

$$T(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3)$$

The standard ordered basis for $\mathbb{R}^3$ is

$$S = \{(1,0,0), (0,1,0), (0,0,1)\}$$

Now;

$$T(1,0,0) = (3,-2,-1) = 3(1,0,0) + (-2)(0,1,0) + (-1)(0,0,1)$$
$$T(0,1,0) = (0,1,2) = 0(1,0,0) + 1(0,1,0) + (2)(0,0,1).$$
$$T(0,0,1) = (1,0,4) = 1(1,0,0) + 0(0,1,0) + 4(0,0,1)$$

$\therefore$ The matrix of L.T. with respect to standard basis for $\mathbb{R}^3$ is given by-

$$[T] = \begin{bmatrix} 3 & 0 & 1 \\ -2 & 1 & 0 \\ -1 & 2 & 4 \end{bmatrix}$$

(ii) Range space of T:

$$T = \{\beta \in \mathbb{R}^3 / T(\alpha) = \beta \text{ for } \alpha \in \mathbb{R}^3\}$$

$\therefore$ The range space consists of all vectors of the type $(3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3)$ for all $(x_1, x_2, x_3) \in \mathbb{R}^3$

Let $\beta = (a, b, c) \in R(T)$ be arbitrary.

$T(x_1,x_2,x_3) = (a,b,c)$ for some $(x_1,x_2,x_3) \in \mathbb{R}^3$

$\Rightarrow (3x_1+x_3 , -2x_1+x_2 , -x_1+2x_2+4x_3) = (a,b,c)$

$\Rightarrow x_1(3,-2,-1) + x_2(0,1,2) + x_3(1,0,4).$

Here; $S = \{(1,0,0),(0,1,0),(0,0,1)\}$ is linearly Independent.

Let $S_1 = \{(3,-2,-1),(0,1,2),(1,0,4)\}$

$\therefore S_1 \subseteq R(T)$

Now; verify whether S_1 is L.I or not.

Hence, transform the matrix into Echelon form.

$$\begin{bmatrix} 3 & -2 & -1 \\ 0 & 1 & 2 \\ 1 & 0 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 3 & -2 & -1 \end{bmatrix} [R_3 \leftrightarrow R_1].$$

$$\sim \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & -2 & -13 \end{bmatrix} R_3 \to R_3 - 3R_1 \quad \sim \begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_3 \to R_3 + 2R_2 \\ R_3 \to -\frac{1}{9}R_3 \end{matrix}$$

Hence, $S_1 \subseteq R(T)$ is linearly dependent

$\therefore S_1$ is the required basis for Range space of T.

i.e Basis of Range space $= S_1 = \{(3,-2,-1),(0,1,2),(1,0,4)\}$

(ii) <u>Null space of T:-</u>

$N(T) = \{ \alpha \in \mathbb{R}^3 \ / \ T(\alpha) = \hat{0} \}$.

Let $\alpha \in N(T) \Rightarrow T(\alpha) = \hat{0}$

$\therefore T(a,b,c) = \hat{0}$; where $\hat{0} = (0,0,0) \in V_3(\mathbb{R}^3)$.

$\Rightarrow (3a+c , -2a+b , -a+2b+4c) = (0,0,0)$

$\Rightarrow 3a+c=0 \quad -2a+b=0$

$-a+2b+4c=0.$

from; this $c = -3a$
$b = 2a$

put this value in 3rd equation and it should satisfy the system of equations

$$-a + 4a - 12a = 0$$
$$-9a = 0$$
$$\boxed{a = 0} \text{ hence } \boxed{b = 0 = c}$$

Hence; $S_2 = \{(0,0,0)\}$ is the required basis of null space of T.

Q.3 (a) (ii) Let T be a linear transformation from a vector space V over reals into V such that $T-T^2=I$. Show that T is invertible.

Solution: $\text{Ker } T = \{\alpha \in V \mid T(\alpha)=\hat{0}\}$

$0 \in \text{Ker } T$ ($\because$ by the property of linear transformation $T(0)=\hat{0}$)

$\Rightarrow \text{Ker } T \neq \phi$, Let $\alpha \in \text{ker } T$

now given $T-T^2=I$

$\Rightarrow (T-T^2)(\alpha) = I(\alpha)$ $(\alpha \in \text{ker } T)$

$\Rightarrow T(\alpha)-T^2(\alpha)=I(\alpha)$

$\Rightarrow \hat{0}-T(T(\alpha))=\alpha$ $(\because \alpha \in \text{Ker } T)$

$\Rightarrow -T(0)=\alpha$

$\Rightarrow \hat{0}=\alpha$

$\Rightarrow \text{Ker } T=\{0\}$.

$\Rightarrow T$ is non-singular.

$\Rightarrow T$ is invertible.

Ques: 3(b)(i)) For $x>0$, let $f(x)=\int_1^x \frac{\ln t}{1+t}\,dt$

Evaluate $f(e)+f\left(\frac{1}{e}\right)$?

Solution:- for $x>0$, $f(x)=\int_1^x \frac{\ln t}{1+t}\,dt$

put $t=\frac{1}{u} \Rightarrow dt=-\frac{1}{u^2}du$.

at $t=1 \quad u=1$.

$t=x \quad u=1/x$.

$$\therefore f(x)=\int_1^{1/x} \frac{\log(1/u)}{1+\frac{1}{u}} \times \frac{-1}{u^2}\,du$$

$$=\int_1^{1/x} \frac{\log u}{(u+1)} \times \frac{u}{u^2}\,du.$$

$$f(x)=\int_1^{1/x} \frac{\log u}{u(u+1)}\,du$$

Putting x as $1/x$

$$f(1/x)=\int_1^x \frac{\log u}{u(u+1)}\,du=\int_1^x \left(\frac{\log u}{u}-\frac{\log u}{u+1}\right)du.$$

$$=\left.\frac{(\log u)^2}{2}\right|_1^x - f(x)$$

$$\therefore f(x)+f\left(\frac{1}{x}\right)=\left.\frac{(\log u)^2}{2}\right|_1^x=\frac{1}{2}(\log x)^2.$$

Now put $x = e$.

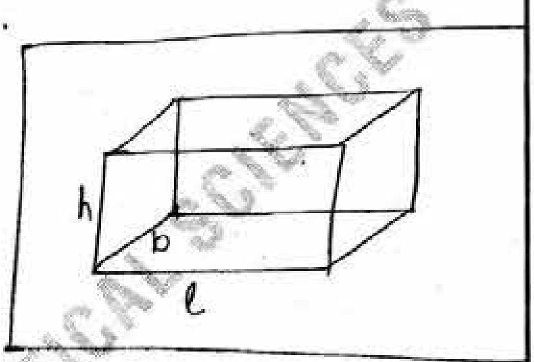
$\therefore f(e) + f\left(\frac{1}{e}\right) = \frac{1}{2}(\log e)^2 = \frac{1}{2}$ $\left[\because \log e = 1\right]$

$\therefore \boxed{f(e) + f(1/e) = \frac{1}{2}}$ is required solution

Ques: 3(b)(ii)) A rectangular box, open at the top, is said to have a volume of 32 cubic metres. Find the dimensions of the box so that the total surface is a minimum.

Solution:- Given; volume of rectangular box = 32 m^3

Volume = lbh.



let l = length of Box
b = Breadth of Box
h = Height of Box.

$V = lbh = 32$

$V \Rightarrow \; h = \frac{32}{lb}$ ——— (1)

S = total surface area of rectangular box.

$S = lb + 2(bh + hl)$ as [Box is open at the top]. ——— (2)

Using (1) in (2)

$$S = lb + 2\left(\cancel{b}\cdot\frac{32}{l\cancel{b}} + \cancel{l}\cdot\frac{32}{\cancel{l}b}\right)$$

$$S = lb + \frac{64}{l} + \frac{64}{b}$$

$$\frac{\partial S}{\partial l} = b - \frac{64}{l^2} = 0 \Rightarrow b = \frac{64}{l^2} \Rightarrow l^2 b = 64 \quad \text{——— (A)}$$

$$\frac{\partial S}{\partial b} = l - \frac{64}{b^2} = 0 \Rightarrow l = \frac{64}{b^2} \Rightarrow b^2 l = 64. \quad \text{——— (B)}$$

from (A) and (B)

we get

$$l = b = \sqrt[3]{64}$$

$$\boxed{l = b = 4}$$

put the value of l, b in (1)

$$h = \frac{32}{4\times4} = \frac{32}{16} = 2.$$

Hence, the dimension of the rectangular box is $l\times b\times h = 4\times4\times2$

i.e $\boxed{\begin{array}{l}\text{length} = 4m \\ \text{Breadth} = 4m \\ \text{height} = 2m\end{array}}$ required result

Q.3 (c) Show that the locus of points from which three mutually perpendicular tangents can be drawn to the paraboloid $ax^2+by^2=2z$ is given by

$$ab(x^2+y^2)-2(a+b)z-1=0$$

Soln: Enveloping cone of the paraboloid $ax^2+by^2=2cz$ with vertex at the point (α,β,γ).

The Equations of a line through (α,β,γ) are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \longrightarrow (i)$$

Any point on this line is $(\alpha+lr,\beta+mr,\gamma+nr) \longrightarrow (ii)$

If the line (i) meets the given paraboloid at a distance r from the point (α,β,γ) ~~then~~ the point given by (ii) must lie on the given paraboloid and so we have

$$a(\alpha+lr)^2+b(\beta+mr)^2=2c(\gamma+nr)$$

(or) $r^2(al^2+bm^2)+2r(al\alpha+bm\beta-cn)+(a\alpha^2+b\beta^2-2c\gamma)=0 \longrightarrow (iii)$

If the line (i) is a tangent of the given paraboloid ~~then~~ the line (i) should meet the paraboloid in two coincident points, the condition for the same is that the roots of (iii) are equal i.e "$B^2=4AC$"

(or) $4(al\alpha+bm\beta-cn)^2=4(al^2+bm^2)(a\alpha^2+b\beta^2-2c\gamma) \longrightarrow (iv)$

The locus of line (i) which is tangent to the given paraboloid is obtained by eliminating l,m,n between (i) and (iv) and is

$$(a\alpha(x-\alpha)+b\beta(y-\beta)-c(z-\gamma))]^2$$

$$=\left[a(x-\alpha)^2+b(y-\beta)^2\right][(a\alpha^2+b\beta^2-2c\gamma)] \longrightarrow (V)$$

If $S\equiv ax^2+by^2-2cz$, $S_1=a\alpha^2+b\beta^2-2c\gamma$ and

$P\equiv a\alpha x+b\beta y-c(z+\gamma)$ then eqn (V) can be written as

$(T-S_1)^2 = (S+S_1-2T)S_1$

$T^2+S_1^2-2TS_1 = SS_1+S_1^2-2TS_1$ (or) $SS_1 = T^2$

$$(ax^2+by^2-2cz)(a\alpha^2+b\beta^2-2c\gamma) = [a(\alpha x)+b\beta y + c(z+\gamma)]^2 \longrightarrow \text{(vi)}$$

the required equation of the enveloping cone of the given paraboloid.

Cor. To find the locus of the points from which three mutually perpendicular tangents can be drawn to the paraboloid $ax^2+by^2=2cz$.

Here we are to apply the condition that the enveloping cone, of the given paraboloid with vertex at (α,β,γ) may have three mutually perpendicular generators and we know that the condition for the same is that the sum of the co-efficients of x^2, y^2 and z^2 in the equation of the cone is zero.

$\therefore$ from (vi) above we get

$$[a(a\alpha^2+b\beta^2-2c\gamma)-a^2\alpha^2]+[b(a\alpha^2+b\beta^2-2c\gamma)-b^2\beta^2]-c^2=0$$

$$ab\beta^2-2ca\gamma+ba\alpha^2-2cb\gamma-c^2=0$$

$$ab(\alpha^2+\beta^2)-2c(a+b)\gamma-c^2=0$$

Hence the required locus of the point (α,β,γ) is

$$ab(x^2+y^2)-2c(a+b)z-c^2=0 \rightarrow \text{(vii)}$$

putting '$c=1$' in the above ~~equation~~ we get the required answer

$$ab(x^2+y^2)-2(a+b)z-1=0$$

Ans.

Q.4 (a) Consider the vectorspace $X := \{P(x) \mid P(x)$ is a polynomial of degree less than or equal to 3 with real coefficients$\}$, over the real field $\mathbb{R}$. Define the map $D : X \to X$ by

$$D(p(x)) := p_1 + 2p_2 x + 3p_3 x^2$$

where $p(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3$

Is D a linear transformation on X? If it is, then construct the matrix representation for D with respect to the basis $\{1, x, x^2, x^3\}$ for X.

Sol'n: Let $p(x), q(x) \in X$, $a, b \in \mathbb{R}$

Given map $D : X \to X$ defined by

$$D(p(x)) = p_1 + 2p_2 x + 3p_3 x^2$$

where $p(x) = p_0 + p_1 x + p_2 x^2 + p_3 x^3$

i.e. $D(p_0 + p_1 x + p_2 x^2 + p_3 x^3) = p_1 + 2p_2 x + 3p_3 x^2$ ——— (1)

Now $D[ap(x) + bq(x)] = D[a(p_0 + p_1 x + p_2 x^2 + p_3 x^3) + b(q_0 + q_1 x + q_2 x^2 + q_3 x^3)]$

$$= D[(ap_0 + bq_0) + (ap_1 + bq_1)x + (ap_2 + bq_2)x^2 + (ap_3 + bq_3)x^3]$$

$$= (ap_1 + bq_1) + 2(ap_2 + bq_2)x + 3(ap_3 + bq_3)x^2$$

$$= a(p_1 + 2p_2 x + 3p_3 x^2) + b(q_1 + 2q_2 x + 3q_3 x^2)$$

$$= aD(p(x)) + bD(q(x))$$

$\therefore D : X \to X$ is a linear transformation.

Now from ①,

$D(1) = 0 = 0 + 0x + 0x^2 + 0x^3$

$D(x) = 1 = 1 + 0x + 0x^2 + 0x^3$

$D(x^2) = 2x = 0 + 2.x + 0x^2 + 0x^3$

$D(x^3) = 3x^2 = 0 + 0.x + 0.x^2 + 3x^2$

Hence the matrix representation of D w.r.to the ordered basis $\{1, x, x^2, x^3\}$

is $$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Ques: 4(b)) By means of the Substitution

$$x+y+z=u\ ,\ y+z=uv\ ,\ z=uvw$$

evaluate $\iiint (x+y+z)^n\, xyz\, dx\, dy\, dz$, taken over the volume bounded by $x=0, y=0, z=0, x+y+z=1$.

Solution:-

Given; $x+y+z=u\ ,\ y+z=uv\ ,\ z=uvw$

as $z=uvw \Rightarrow y+z=uv$

$$y+uvw=uv$$

$$\boxed{y=uv(1-w)}$$

put values of y & z in $x+y+z=u$.

$$x+uv(1-w)+uvw=u$$

$$x+uv-\cancel{uvw}+\cancel{uvw}=u.$$

$$\boxed{x=u(1-v)}$$

Limits are $x=0, y=0, z=0, x+y+z=1$

$$x+y+z=u-\cancel{uv}+\cancel{uv}-\cancel{uvw}+\cancel{uvw} \Rightarrow \boxed{u=1}$$

$$J\left(\frac{x,y,z}{u,v,w}\right)=\begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix}$$

$$=\begin{vmatrix} 1-v & -u & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix}$$

$$R_1 \to R_1+R_2+R_3.$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ v(1-w) & u(1-w) & -uv \\ vw & uw & uv \end{vmatrix}.$$

$$\Rightarrow 1\left(uv(u(1-w) + u^2vw\right)$$

$$\Rightarrow 1\left[uv(u-uw) + u^2vw)\right]$$

$$\Rightarrow u^2v - \cancel{u^2vw} + \cancel{u^2vw} = u^2v$$

$$\therefore \boxed{J\left(\frac{x,y,z}{u,v,w}\right) = u^2v.}$$

$$\therefore \iiint (x+y+z)^n \cdot xyz\, dx\,dy\,dz$$

$$= \iiint u^n \cdot u(1-v)\,uv(1-w)\,uvw\; du\,dv\,dw \cdot u^2v$$

$$= \int_0^1\int_0^1\int_0^1 u^{n+5}\, v^3(1-v)\,w(1-w)\,du\,dv\,dw$$

$$= \int_0^1\int_0^1 v^3(1-v)w(1-w)\,\frac{1}{n+6}\cdot\left(u^{n+6}\right)_0^1\,dv\,dw$$

$$= \int_0^1 w(1-w)\cdot\frac{1}{n+6}\left[\frac{v^4}{4} - \frac{v^5}{5}\right]_0^1 dw \quad \left[\because u^{n+1}\Big|_0^1 = 1\right]$$

$$= \frac{1}{n+6}\left[\frac{1}{4} - \frac{1}{5}\right]\int_0^1 w - w^2\,dw$$

$$= \frac{1}{n+6}\left[\frac{1}{4} - \frac{1}{5}\right]\left[\frac{w^2}{2} - \frac{w^3}{3}\right]_0^1$$

$$= \frac{1}{n+6}\left[\frac{1}{20}\right]\left[\frac{1}{2} - \frac{1}{3}\right] = \frac{1}{n+6}\left[\frac{1}{20}\right]\left[\frac{1}{6}\right].$$

$$\boxed{\therefore \iiint (x+y+z)^n xyz\,dx\,dy\,dz = \frac{1}{120}\left[\frac{1}{n+6}\right]}$$ Required Solution.

Q.4 (c) (i) find the equation of the plane through the point $(2,1,1)$, $(1,-2,3)$ and parallel to the x-axis.

(ii) If $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ represent one of a set of three mutually perpendicular generators of the cone $5yz-8zx-3xy=0$, find the equations of the other two.

Sol^n: (i) The Equation of any plane through $(2,2,1)$ is

$$A(x-2)+B(y-2)+C(z-1)=0 \longrightarrow (i)$$

If it passes through $(1,-2,3)$ then

$$A(1-2)+B(-2-2)+C(3-1)=0$$

$$-A-4B+2C=0 \text{ (or) } A+4B-2C=0 \longrightarrow (ii)$$

The Equation of x-axis is $\frac{x}{1}=\frac{y}{0}=\frac{z}{0} \longrightarrow (iii)$

If the line given by (iii) is parallel to the plane given by (i), then this line is perpendicular to the normal to the plane given by (i) and so we have

$$A\cdot 1+B\cdot 0+C\cdot 0=0 \text{ (or) } A=0 \longrightarrow (iv)$$

Solving (ii) and (iv) we have $4B-2C=0$ (or) $C=2B$.

Substituting these values in (i) we get the required equation as

$$B(y-2)+2B(z-1)=0 \text{ (or) } y+2z-4=0$$

Ans.

(ii) The equation of the given cone as in last example suggests that it has three mutually perpendicular generators one of them is normal to a plane which cuts the cone in two mutually perpendicular generators.

Thus if $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ is one the three mutually perpendicular generators, then it is normal to the plane through the vertex cutting the cone in two perpendicular generators and therefore the equation of the plane is

$$x+2y+3z=0$$

Now we are to find the lines of intersection of this plane and the given cone and one of the lines of intersection be

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$$

Then we have $l+2m+3n=0$ and $5mn-8nl+3lm=0$

Eliminating 'l' between these we get

$5mn-(8n+3m)\left[-(2m+3n)\right]=0$ (or) $24n^2+30mn+6m^2=0$

$m^2+5mn+4n^2=0$ (or) $(m+n)(m+4n)=0$

when $m=-n$ from (ii) we get $l+n=0$ (or) $l=-n$

$$\frac{l}{1}=\frac{m}{1}=\frac{n}{-1} \longrightarrow \text{(iii)}$$

when $m=-4n$ from (ii) we get $l-5n=0$ (or) $l=5n$

$$\frac{l}{5}=\frac{m}{-4}=\frac{n}{1} \rightarrow \text{(iv)}$$

Hence from (iii) and (iv) the other two generators are $\frac{x}{1}=\frac{y}{1}=\frac{z}{-1}$ and $\frac{x}{5}=y/-4=z/1$ and evidently these are perpendicular as $1.5+1.(-4)+(-1)1=0$ and also each of them is perpendicular to the given generation.

Q.5 (a) Find the orthogonal trajectories of the family of curve $\frac{x^2}{a^2}+\frac{y^2}{b^2+\lambda}=1$, λ being the parameter.

Sol'n: The given family of curves is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2+\lambda}=1, \text{ with } \lambda \text{ as parameter.} \quad \text{——— (1)}$$

Differentiating (1) w.r.t x, we get

$$\frac{2x}{a^2}+\frac{2y}{a^2+\lambda}\frac{dy}{dx}=0$$

$$\Rightarrow \frac{1}{a^2+\lambda}=\frac{-x}{a^2y}\frac{dx}{dy} \quad \text{—— (2)}$$

Eliminating λ from (1) & (2), we get

$$\frac{x^2}{a^2}+y^2\left(\frac{-x}{a^2y}\frac{dx}{dy}\right)=1$$

$$\Rightarrow \frac{x^2}{a^2}-\frac{xy}{a^2}\frac{1}{dy/dx}=1 \quad \text{——— (3)}$$

which is the differential equation of the given family of curves (1).

Replacing dy/dx by $-\frac{dx}{dy}$ in (3), the differential equation of the required orthogonal trajectories is

$$\frac{x^2}{a^2}-\frac{xy}{a^2}\left(-\frac{1}{dx/dy}\right)=1$$

$$\Rightarrow \frac{x^2}{a^2}+\frac{xy}{a^2}\frac{dy}{dx}=k$$

$$\Rightarrow \frac{xy}{a^2}\frac{dy}{dx}=\frac{a^2-x^2}{a^2}$$

$$\Rightarrow y\,dy=\left(\frac{a^2}{x}-x\right)dx$$

Integrating, we get

$$\frac{y^2}{2}=a^2\log x-\frac{1}{2}x^2+\frac{1}{2}c$$

$$x^2+y^2=2a^2\log x+c$$

$$\Rightarrow x^2+y^2-2a^2\log x=c$$

which is the required equation of the orthogonal trajectories.

Ques: 5(b)) Solve $y = 2xp - yp^2$ and examine for singular solutions?

Solution :-

Given; $y = 2xp - yp^2$ —— ①

Solving ① for x.

$$① \equiv x = \frac{y}{2p} + \frac{yp}{2} \quad \text{—— ②}$$

Differentiate ②, w.r.t y, and put $\frac{dx}{dy} = \frac{1}{p}$, we get —

$$\frac{1}{p} = \frac{1}{2p} - \frac{y}{2p^2}\frac{dp}{dy} + \frac{p}{2} + \frac{y}{2}\cdot\frac{dp}{dy}$$

$$\Rightarrow \frac{y}{2}\frac{dp}{dy}\left[1 - \frac{1}{p^2}\right] + \frac{p}{2}\left[1 - \frac{1}{p^2}\right] = 0$$

$$\Rightarrow \frac{1}{2}\left(1 - \frac{1}{p^2}\right)\left(y\frac{dp}{dy} + p\right) = 0$$

Omitting the first factor for general solution we have —

$$y\frac{dp}{dy} + p = 0 \Rightarrow \frac{dp}{p} + \frac{dy}{y} = 0$$

$$\Rightarrow \log p + \log y = \log c$$

$$\Rightarrow \log py = \log c.$$

$$\Rightarrow py = c$$

$$\Rightarrow \boxed{p = c/y} \quad \text{—— ③}$$

Eliminating p from ① and ③, the general solution is

$$y = \frac{2xc}{y} - \frac{yc^2}{y^2}$$

$$\Rightarrow y \cdot y = 2xc - c^2$$

$$\Rightarrow \boxed{y^2 = 2xc - c^2} \text{ ——— } ④$$

The p-disc. relation from ① is

$$4x^2 - 4y^2 = 0$$

$$\Rightarrow x^2 - y^2 = 0 \Rightarrow (x-y)(x+y) = 0$$

The c-disc relation from ④ is

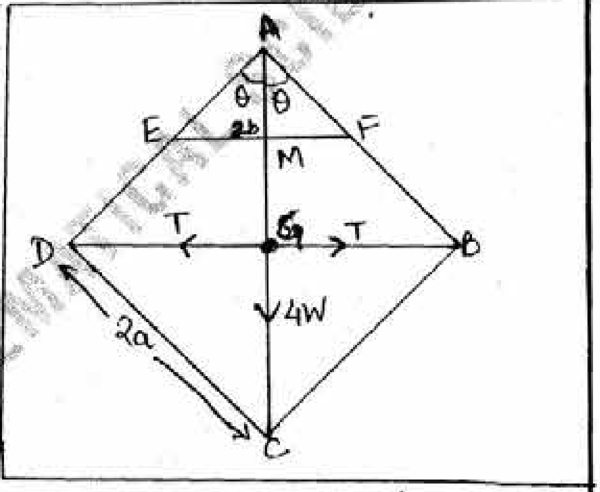
$$4x^2 - 4y^2 = 0 \Rightarrow x^2 - y^2 = 0$$

$$\Rightarrow (x+y)(x-y) = 0$$

Hence; $\boxed{x - y = 0 \text{ and } x + y = 0}$ are singular solutions because these appear once in both the discriminant and clearly, they satisfies the given differential equation.

Ques:- 5(c)) Four equal rods, each of length 2a and weight W, are freely jointed to form a square ABCD which is kept in shape by a light rod BD and is supported in a vertical plane with BD horizontal, A above C and AB, AD in contact with two fixed smooth pegs which are at a distance 2b apart on the same level. Find the stress in the rod BD.

Solution:- The rods AB and AD of the framework rest on two fixed smooth pegs E&F which are at the same level and EF = 2b.



Let, 2a be the length of each of the rods AB, BC, CD & DA.

The total weight 4W of all the rods can be taken acting at G (the mid point of AC.

Let 'T' be the thrust in the rod BD and

let $\boxed{\angle BAC = \theta = \angle CAD}$

Replace the rod BD by two equal and opposite force 'T' as shown in figure.

Give the system a small symmetrical displacement in which θ changes to $\theta + \delta\theta$. The line EF joining the pegs remains fixed and the distance will be measured from this line. The lengths of the rods do not change and BD (length) changes.

The angle $\angle AGB = 90^\circ$ (remains)

The force contributing to the sum of virtual works are:

(i) the thrust T in the rod AB, and

(ii) the weight 4W acting at G.

The reactions at the Pegs do no work.

We have;

$$BD = 2BG = 2 \cdot 2a \sin\theta = 4a\sin\theta.$$

Also the depth of G below the fixed line EF

$$= MG = AG - AM$$

$$= AB\cos\theta - EM\cot\theta$$

$$= 2a\cos\theta - b\cot\theta$$

The equation of virtual work is

$$T\delta(4a\sin\theta) + 4W\delta(2a\cos\theta - b\cot\theta) = 0$$

or $\quad 4aT\cos\theta\,\delta\theta + (-8Wa\sin\theta\,\delta\theta) + 4bW\operatorname{cosec}^2\theta\,\delta\theta = 0$

$$\Rightarrow 4(aT\cos\theta - 2Wa\sin\theta + bW\operatorname{cosec}^2\theta)\delta\theta = 0$$

$$\Rightarrow aT\cos\theta - 2Wa\sin\theta + bW\operatorname{cosec}^2\theta = 0 \quad [\because \delta\theta \neq 0$$

$$aT\cos\theta = 2aW\sin\theta - bW\operatorname{cosec}^2\theta$$

$$T = \frac{W}{a\cos\theta}(2a\sin\theta - b\operatorname{cosec}^2\theta)$$

$$\boxed{T = \frac{W}{a}\tan\theta\,(2a - b\operatorname{cosec}^3\theta)}$$

But in the position of equilibrium, $\theta = 45°$

$$\therefore\ T = \frac{W}{a}\tan 45° \,(2a - b\,\mathrm{cosec}^3\theta)$$

$$T = \frac{W}{a}(2a - b\,\mathrm{cosec}^3\theta)$$

$$T = \frac{W}{a}\left[2a - b(\sqrt{2})^3\right]$$

$$\boxed{T = \frac{2W}{a}(a - b\sqrt{2})}$$

which is required solution

Ques: 5(d) Find the curvature and torsion of the curve $x = a\cos t,\ y = a\sin t,\ z = bt$.

Solution:- From the given,

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = a\cos t\,\hat{i} + a\sin t\,\hat{j} + bt\,\hat{k}$

$$\frac{d\vec{r}}{dt} = -a\sin t\,\hat{i} + a\cos t\,\hat{j} + b\hat{k}$$

$$\frac{d^2\vec{r}}{dt^2} = -a\cos t\,\hat{i} + (-a)\sin t\,\hat{j} + 0\hat{k}$$

$$\frac{d^3\vec{r}}{dt^3} = +a\sin t\,\hat{i} - a\cos t\,\hat{j} + 0\hat{k}$$

$$\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2} = \begin{vmatrix} i & j & k \\ -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \end{vmatrix}$$

$$= \hat{i}\,(0 + ab\sin t) + \hat{j}\,(-ab\cos t - 0) + k\,(a^2)$$

$$\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2} = ab\sin t\,\hat{i} - ab\cos t\,\hat{j} + a^2\hat{k}$$

$$\left[\frac{d\vec{r}}{dt} \cdot \frac{d^2\vec{r}}{dt^2} \cdot \frac{d^3\vec{r}}{dt^3}\right] = a^2 b\sin^2 t + a^2 b\cos^2 t = a^2 b$$

$$\text{Curvature } (K) = \frac{\left|\frac{dr}{dt} \times \frac{d^2r}{dt^2}\right|}{\left|\frac{dr}{dt}\right|^3} = \frac{\sqrt{a^2b^2 + a^4}}{\left(\sqrt{a^2+b^2}\right)^3}$$

$$\text{Curvature } (K) = \frac{a\sqrt{a^2+b^2}}{(a^2+b^2)^{3/2}} = \frac{a}{a^2+b^2}.$$

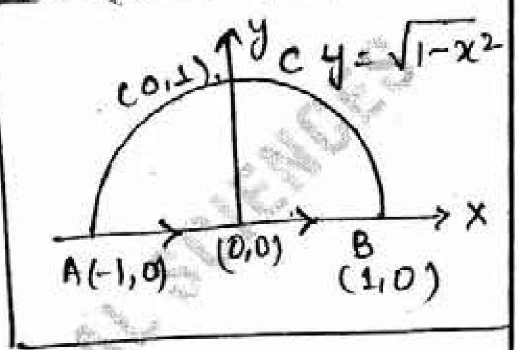
$$\text{Torsion } (\tau) = \frac{\left[\frac{d\vec{r}}{dt} \cdot \frac{d^2\vec{r}}{dt^2} \cdot \frac{d^3\vec{r}}{dt^3}\right]}{\left|\frac{dr}{dt} \times \frac{d^2r}{dt^2}\right|^2} = \frac{a^2 b}{\left[\sqrt{a^2b^2 + a^4}\right]^2} = \frac{b}{a^2+b^2}$$

$$\therefore \boxed{\text{Curvature } (K) = \frac{a}{a^2+b^2}\ ;\ \text{Torsion } (\tau) = \frac{b}{a^2+b^2}}$$

required solution.

Ques: 5(e)) Apply Green's Theorem in the plane to evaluate $\int_C [(2x^2-y^2)dx + (x^2+y^2)dy]$ where C is the boundary of the surface enclosed by the x-axis and the semi-circle $y=(1-x^2)^{1/2}$.

Solution:-



Here, C is the closed curve traversed in positive direction by the straight line AOB and the semi-circle BDA. Also R is the region bounded by this curve 'C'.

we have $\int_C (2x^2-y^2)\,dx + (x^2+y^2)\,dy$.

$$= \int_C M\,dx + N\,dy$$

where $M = 2x^2-y^2$ & $N = x^2+y^2$

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \quad \left[\text{by Green's theorem}\right]$$

$$= \iint_R \left[\frac{\partial}{\partial x}(x^2+y^2) - \frac{\partial}{\partial y}(2x^2-y^2)\right] dx\,dy.$$

$$= \iint [2x + 2y]\,dx\,dy.$$

$$= \int_{x=-1}^{1} \int_{y=0}^{\sqrt{1-x^2}} 2(x+y)\,dx\,dy. \quad \left[\text{Since, for the region } R, y \text{ varies from } 0 \text{ to } \sqrt{1-x^2}\right]$$

$$= 2\int_{x=-1}^{1}\left[xy + \frac{y^2}{2}\right]_{y=0}^{\sqrt{1-x^2}} dx.$$ and x varies from -1 to 1.

$$= 2\int_{-1}^{1}\left[x\sqrt{1-x^2} + \frac{1}{2}(1-x^2)\right] dx.$$

$$= \frac{2}{2}\int_{-1}^{1}(1-x^2)dx \qquad \left[\because \int_{-1}^{1} x\sqrt{1-x^2}\,dx = 0\right]$$

$$= 2\int_{0}^{1}(1-x^2)\,dx.$$

$$= 2\left[x - \frac{x^3}{3}\right]_0^1$$

$$= 2\left[1 - \frac{1}{3} + 0 + 0\right] = 2\left[\frac{2}{3}\right]$$

$$= \frac{4}{3}.$$

$$\therefore \boxed{\int_C (2x^2-y^2)dx + (x^2+y^2)dy = \frac{4}{3}}$$

Hence, the result.

Q.6 (a) Solve $(D^4+D^2+1)y = e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right)$

Sol^n: Given that $(D^4+D^2+1)y = e^{-x/2}\cos\frac{\sqrt{3}}{2}x$

The auxiliary equation is $D^4+D^2+1=0$

$$\Rightarrow (D^2+1)^2-D^2=0$$

$$\Rightarrow (D^2+D+1)(D^2-D+1)=0$$

$$\Rightarrow D^2+D+1=0, \text{ (or) } D^2-D-1=0$$

$$\Rightarrow D = \frac{-1\pm i\sqrt{3}}{2}, \frac{1\pm i\sqrt{3}}{2}$$

$$\therefore C.F = e^{-x/2}\left[c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right] + e^{x/2}\left[c_3\cos\left(\frac{\sqrt{3}}{2}x\right) + c_4\sin\left(\frac{\sqrt{3}}{2}x\right)\right]$$

$$P.I = \frac{1}{D^4+D^2+1}e^{-x/2}\cos\frac{\sqrt{3}}{2}x$$

$$= e^{-x/2}\frac{1}{(D-\frac{1}{2})^4+(D-\frac{1}{2})^2+1}\cos\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{-x/2}\frac{1}{D^4-2D^3+\frac{5}{2}D^2-\frac{3}{2}D+\frac{21}{16}}\cos\frac{\sqrt{3}}{2}x$$

$$= e^{-x/2}\frac{1}{(D^2+\frac{3}{4})(D^2-2D+\frac{7}{4})}\cos\frac{\sqrt{3}}{2}x$$

$$= e^{-x/2}\frac{1}{(D^2+\frac{3}{4})}\frac{1}{(-\frac{3}{4}-2D+\frac{7}{4})}\cos\frac{\sqrt{3}}{2}x$$

$$= e^{-x/2}\frac{1}{(D^2+3/4)}\frac{1}{(1-2D)}\cos\frac{\sqrt{3}}{2}x$$

$$= e^{-x/2}\frac{1}{(D^2+\frac{3}{4})}\frac{1+2D}{(1-4D^2)}\cos\sqrt{3}/2\,x$$

$$= e^{-x/2}\frac{1}{(D^2+3/4)}\frac{1}{4}(1+2D)\cos\frac{\sqrt{3}}{2}x$$

$$= \frac{e^{-x/2}}{4}\frac{1}{(D^2+3/4)}\left(\cos\frac{\sqrt{3}}{2}x - \sqrt{3}\sin\frac{\sqrt{3}}{2}x\right)$$

$$= \frac{1}{4} e^{-x/2} \left[\frac{1}{D^2+\left(\frac{\sqrt{3}}{2}\right)^2} \cos\frac{\sqrt{3}}{2}x - \frac{\sqrt{3}}{D^2+\left(\frac{\sqrt{3}}{2}\right)^2} \sin\frac{\sqrt{3}}{2}x \right.$$

$$= \frac{1}{4} e^{-x/2} \left[\frac{x}{2(\sqrt{3}/2)} \sin\frac{\sqrt{3}}{2}x - \frac{\sqrt{3}\,x}{2(\sqrt{3}/2)} \left(-\cos\frac{\sqrt{3}}{2}x\right) \right]$$

$$= \frac{x}{4\sqrt{3}} e^{-x/2} \left[\sin\frac{\sqrt{3}}{2}x + \sqrt{3}\cos\frac{\sqrt{3}}{2}x \right]$$

$$\therefore y = C.F + P.I$$

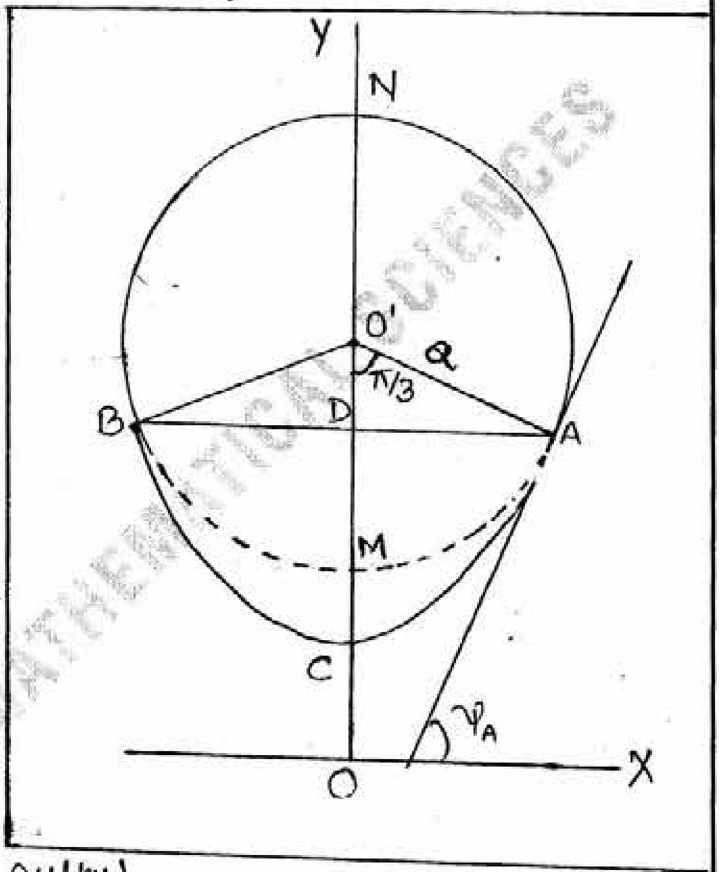
$$= e^{-x/2} \left[c_1 \cos\frac{\sqrt{3}}{2}x + c_2 \sin\frac{\sqrt{3}}{2}x \right]$$

$$+ e^{x/2} \left[c_3 \cos\frac{\sqrt{3}}{2}x + c_4 \sin\frac{\sqrt{3}}{2}x \right]$$

$$+ \frac{x}{4\sqrt{3}} e^{-x/2} \left[\sin\frac{\sqrt{3}}{2}x + \sqrt{3}\cos\frac{\sqrt{3}}{2}x \right]$$

Ques: 6(b)> Find the length of an endless chain which will hang over a circular pulley of radius 'a' so as to be in contact with the two thirds of the circumference of the pulley.

Solution:-



Let ANBMA be the circular pulley of radius 'a' and ANBCA the endless chain hanging over it.

Since, the chain is in contact with the $2/3$rd of the circumference of the pulley, hence the length of this portion ANB of the chain.

$= \frac{2}{3}$(circumference of the pulley).

$= \frac{2}{3}(2\pi a) = \frac{4}{3}\pi a.$

Let the remaining portion of the chain hang in the form of the catenary ACB, with AB horizontal. C is the lowest point i.e., the vertex, CO'N the axis and OX the directrix of this catenary.

Let OC = c = the parameter of the catenary.

The tangent at A will be perpendicular of the radius O'A.

$\therefore$ If the tangent at A is inclined at an angle ψ_A to the horizontal, then

$$\psi_A = \angle AO'D = \frac{1}{2}(\angle AO'B) = \frac{1}{2}\left[\frac{1}{3}\times 2\pi\right) = \frac{1}{3}\pi = \frac{\pi}{3}$$

From the triangle $AO'D$, we have

$$DA = O'A \sin\frac{\pi}{3} = a\cdot\frac{\sqrt{3}}{2} = \frac{\sqrt{3}a}{2}.$$

$\therefore$ from $x = c\log(\tan\psi + \sec\psi)$; for the point A.

we have, $\quad x = DA = c\log(\tan\psi_A + \sec\psi_A)$

or $\quad a\cdot\frac{\sqrt{3}}{2} = c\log\left[\tan\frac{\pi}{3} + \sec\frac{\pi}{3}\right]$

$$a\frac{\sqrt{3}}{2} = c\cdot\log(\sqrt{3}+2)$$

$$\therefore \boxed{c = \frac{a\sqrt{3}}{2\log(\sqrt{3}+2)}} \quad \text{——} (A)$$

From $s = c\tan\psi$ applied for the point A, we have

$$\text{arc } CA = c\tan\psi_A = c\tan\frac{\pi}{3} = c\cdot\sqrt{3}$$

$$\boxed{\text{arc } CA = \frac{3a}{2\log(\sqrt{3}+2)}} \quad \text{——} \left[\text{from } (A)\right] \quad \text{——} (B)$$

Hence, the total length of the chain = arc ABC + length of chain in the contact with pulley.

$$= 2(\text{arc. } CA) + \frac{4}{3}\pi a.$$

$$= \cancel{2}\cdot\frac{3a}{\cancel{2}\log(\sqrt{3}+2)} + \frac{4}{3}\pi a.$$

$$\boxed{\therefore \text{Total length of Chain} = a\left\{\frac{3}{\log(2+\sqrt{3})} + \frac{4}{3}\pi\right\}}$$

Required Result.

Q.6 (c) (i) Find the directional derivative of the function $f = x^2 - y^2 + 2z^2$ at the point $P(1,2,3)$ in the direction of the line PQ where Q is the point $(5,0,4)$.

Sol'n: Here $\text{grad } f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$

$$= 2xi - 2yj + 4zk = 2i - 4j + 12k \text{ at the point } (1,2,3)$$

Also $\overrightarrow{PQ}$ = position vector of Q − position vector of P.

$$= (5i + 0j + 4k) - (i + 2j + 3k) = 4i - 2j + k$$

If $\hat{a}$ be the unit vector in the direction of the vector $\overrightarrow{PQ}$, then

$$\hat{a} = \frac{4i - 2j + k}{\sqrt{(16+4+1)}} = \frac{4i - 2j + k}{\sqrt{21}}$$

∴ the required directional derivative

$$= (\text{grad} f) \cdot \hat{a} = (2i - 4j + 12k) \cdot \left\{ \frac{4i - 2j + k}{\sqrt{21}} \right\}$$

$$= \frac{28}{\sqrt{21}} = \frac{28}{21}\sqrt{21}$$

$$= \frac{4}{3}\sqrt{21}.$$

Q.6 (c) (ii) Show that $F = (2xy + z^3)i + x^2 j + 3xz^2 k$ is a conservative force field. Find the scalar potential. Find also the workdone in moving an object in this field from $(1,-2,1)$ to $(3,1,4)$.

Sol'n: The field F will be conservative if $\nabla \times F = 0$

we have.

$$\nabla\times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy+z^3 & x^2 & 3xz^2 \end{vmatrix} = 0$$

Therefore F is a conservative force field.

Let $F = \nabla\phi$

$\Rightarrow (2xy+z^3)i + x^2 j + 3xz^2 k = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y} j + \frac{\partial\phi}{\partial z} k.$

$\frac{\partial\phi}{\partial x} = 2xy + z^3 \Rightarrow \phi = x^2 y + z^3 x + f_1(y,z)$ —— (1)

$\frac{\partial\phi}{\partial y} = x^2 \Rightarrow \phi = x^2 y + f_2(x,z)$ —— (2)

$\frac{\partial\phi}{\partial z} = 3xz^2 \Rightarrow \phi = xz^3 + f_3(x,y)$ —— (3)

(1), (2), (3) each represents ϕ. these agree if we choose

$f_1(y,z) = 0,\ f_2(x,z) = z^3 x,\ f_3(x,y) = x^2 y$

$\therefore\ \phi = x^2 y + xz^3$ to which may be added any constant.

$\therefore\ \phi = x^2 y + xz^3 + C$

workdone $= \int_{(1,-2,1)}^{(3,1,4)} F\cdot dr$

$= \int_{(1,-2,1)}^{(3,1,4)} d\phi = [\phi]_{(1,-2,1)}^{(3,1,4)}$

$= [x^2 y + xz^3]_{(1,-2,1)}^{(3,1,4)} = 202.$

Q.7 (a) (i) Solve $(2x^2+3y^2-7)x\,dx-(3x^2+2y^2-8)y\,dy=0$ —(1)

(ii) Solve $(2xy^4e^y+2xy^3+y)dx+(x^2y^4e^y-x^2y^2-3x)dy=0$

Sol. (i) Let $x^2=X,\ y^2=Y$ so that $2x\,dx=dX$, $2y\,dy=dY$

$\therefore$ from (1) $\dfrac{dY}{dX}=\dfrac{2X+3Y-7}{3X+2Y-8}$

Now putting $X=u+h$, $Y=v+k$
$dX=du$, $dY=dv$

so that $\dfrac{dv}{du}=\dfrac{2u+3v+(2h+3k-7)}{3u+2v+(3h+2k-8)}$ —(2)

Choose h,k so that $2h+3k-7=0$
& $3h+2k-8=0$ —(3)

Solving (3), we get $h=2,\ k=1$
$X=u+2,\quad Y=v+1$
$\Rightarrow u=X-2,\quad v=Y-1$

Equation (2) becomes
$$\frac{dv}{du}=\frac{2u+3v}{3u+2v}=\frac{2+3(v/u)}{3+2(v/u)} \quad —(4)$$

Taking $\dfrac{v}{u}=w \Rightarrow v=uw$
$\Rightarrow \dfrac{dv}{du}=w+u\dfrac{dw}{du}$ —(5)

from (4) & (5)
$$w+u\frac{dw}{du}=\frac{2+3w}{3+2w}$$
$$\Rightarrow u\frac{dw}{du}=\frac{2+3w}{3+2w}-w$$
$$\Rightarrow u\frac{dw}{du}=\frac{2-2w^2}{3+2w}$$
$$\Rightarrow \frac{3+2w}{2(1-w^2)}dw=\frac{du}{u}$$
$$\Rightarrow \frac{du}{u}=\frac{1}{2}\left[\frac{1}{2}\frac{dw}{1+w}+\frac{5}{2}\frac{dw}{1-w}\right]$$
$$\Rightarrow \log u=\frac{1}{4}\log(1+w)-\frac{5}{4}\log(1-w)$$
$$\Rightarrow 4\log u+\log c=\log(1+w)(1-w)^{-5}$$
$$\Rightarrow cu^4=(1+w)(1-w)^{-5}$$

$$\Rightarrow c = \frac{1+\omega}{(1-\omega)^5 u^4}$$

$$\Rightarrow c = \frac{1+v/u}{(1-v/u)^5 u^4} = \frac{u+v}{(u-v)^5}$$

$$\Rightarrow c = \frac{x-2+y-1}{(x-2-y+1)^5} = \frac{x^2+y^2-3}{(x^2-y^2-1)^5}$$

$$\Rightarrow (x^2+y^2-3) = c(x^2-y^2-1)^5$$

which is the required solution

(ii) Here $M = 2xy^4e^y + 2xy^3 + y$; $N = x^2y^4e^y - x^2y^2 - 3x$

$$\frac{\partial M}{\partial y} = 8xy^3e^y + 2xy^4e^y + 1 \; ; \; \frac{\partial N}{\partial x} = 2xy^4e^y - 2xy^2 - 3$$

$\therefore$ the given eqn is not exact.

Now $\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x} = 8xy^3e^y + 8xy^2 + 4$

and $\dfrac{\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}}{M} = -\dfrac{4}{y}$

which is a function of y alone.

$$I.F = e^{-\int \frac{4}{y} dy} = e^{-4\log y} = \frac{1}{y^4}$$

Multiplying (1) by $\frac{1}{y^4}$ we have

$$\left(2xe^y + \frac{2x}{y} + \frac{1}{y^3}\right)dx + \left[x^2e^y - \frac{x^2}{y^2} - 3\left(\frac{x}{y^4}\right)\right]dy = 0$$

which must be exact and its solution is given by

$$\boxed{x^2e^y + \frac{x^2}{y} + \frac{x}{y^3} = c}$$

Ques:- 7(a)(iii) If $L^{-1}\left\{\frac{p}{(p^2+1)^2}\right\} = \frac{1}{2}t\sin t$, find

$$L^{-1}\left\{\frac{32p}{(16p^2+1)^2}\right\}.$$

Solution:- Given that; $L^{-1}\left\{\frac{p}{(p^2+1)^2}\right\} = \frac{1}{2}t\sin t = F(t)$ (say)

Since; $f(ap) = \frac{1}{a}F(t/a)$.

$$\therefore L^{-1}\left\{\frac{ap}{[(ap)^2+1]^2}\right\} = \frac{1}{2}\cdot\left(\frac{1}{a}\right)\frac{t}{a}\sin\frac{t}{a}$$

putting $a = 4$

$$\Rightarrow L^{-1}\left\{\frac{32p}{((ap)^2+1)^2}\right\} = L^{-1}\left\{\frac{8.4p}{[(4p)^2+1]^2}\right\}.$$

$$= 8\,L^{-1}\left\{\frac{4p}{((4p)^2+1)^2}\right\}.$$

$$= 8\cdot\frac{1}{\cancel{2}}\cdot\frac{1}{\cancel{4}}\cdot\frac{t}{4}\cdot\sin\frac{t}{4}$$

$$= \frac{t}{4}\cdot\sin\frac{t}{4}.$$

$$\therefore \boxed{L^{-1}\left\{\frac{32p}{(16p^2+1)^2}\right\} = \frac{t}{4}\sin\frac{t}{4}}$$

required solution

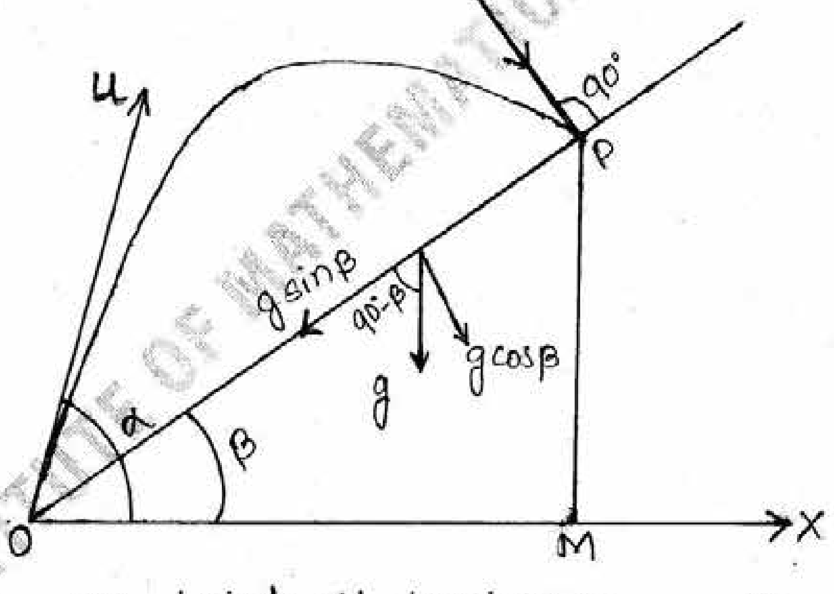
Ques: 7(b)): A particle is projected with a velocity u from a point on an inclined plane whose inclination to the horizontal is β, and strikes it at right angles. Show that –

(i) the time of flight is $\dfrac{2u}{g\sqrt{(1+3\sin^2\beta)}}$,

(ii) the range on the inclined plane is $\dfrac{2u^2}{g} \cdot \dfrac{\sin\beta}{1+3\sin^2\beta}$ and

(iii) the vertical height of the point struck, above the point of projection is $\dfrac{2u^2\sin^2\beta}{g(1+3\sin^2\beta)}$.

Solution:-



Let O be the point of projection, u the velocity of projection, α the angle of projection and 'P' the point where the particle strikes the plane at right angles.

Let T be the time of flight from O to P. Then by formula for the time of flight on an inclined plane, we have,

$$\boxed{T = \frac{2u\sin(\alpha-\beta)}{g\cos\beta}} \quad \text{——— (1)}$$

Since, the particle strikes the inclined plane at right angles at P, therefore the velocity of the particle at 'P' along the inclined plane is zero. Also, the resolved part of the velocity of the particle at 'O' along the inclined plane is $u\cos(\alpha-\beta)$ upwards and the resolved part of the accelaration 'g' along the inclined plane is $g\sin\beta$ downwards. So considering the motion of the particle from O to P along the inclined plane and using the formula

$v = u + ft$, we have

$$0 = u\cos(\alpha-\beta) - g\sin\beta T$$

or $\rightarrow$ $\boxed{T = \dfrac{u\cos(\alpha-\beta)}{g\sin\beta}}$ ——— (2)

Equating T values from (1) and (2), we get -

$$\frac{2\cancel{u}\sin(\alpha-\beta)}{\cancel{g}\cos\beta} = \frac{\cancel{u}\cos(\alpha-\beta)}{\cancel{g}\sin\beta}$$

$$\Rightarrow \frac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)} = \frac{1}{2}\frac{\cos\beta}{\sin\beta}$$

$$\Rightarrow \boxed{\tan(\alpha-\beta) = \frac{1}{2}\cot\beta} \text{ ——— (3)}$$

as the condition for striking the plane at right angles.

(i) from(2)

$$T = \frac{u}{g\sin\beta\sec(\alpha-\beta)} = \frac{u}{g\sin\beta\sqrt{1+\tan^2(\alpha-\beta)}}$$

$$= \frac{u}{g\sin\beta\sqrt{1+\frac{1}{4}\cot^2\beta}} \quad [\text{from } ③].$$

$$= \frac{2u\cancel{\sin\beta}}{g\cdot\cancel{\sin\beta}(\sqrt{4\sin^2\beta+\cos^2\beta})} = \frac{2u}{g\sqrt{\sin^2\beta+\cos^2\beta+3\sin^2\beta}}$$

$$\boxed{T = \frac{2u}{g\sqrt{1+3\sin^2\beta}}}$$

(ii) Let R be the range on the inclined plane; then $R = OP$.

Considering the motion from O to P along the inclined plane using the formula

$v^2 = u^2 + 2fs$, we have

$$0 = u^2\cos^2(\alpha-\beta) - 2g\sin\beta R.$$

or $$R = \frac{u^2\cos^2(\alpha-\beta)}{2g\sin\beta} = \frac{u^2}{2g\cdot\sin\beta\cdot\sec^2(\alpha-\beta)}$$

$$R = \frac{u^2}{2g\sin\beta[1+\tan^2(\alpha-\beta)]}$$

$$R = \frac{u^2}{2g\sin\beta[1+\frac{1}{4}\cot^2\beta]} \quad [\text{from } ③]$$

$$R = \frac{\cancel{4}^{2} u^2 \sin^{\cancel{2}}\beta}{\cancel{2} g \cancel{\sin}\beta\,(4\sin^2\beta + \cos^2\beta)}.$$

$$\boxed{R = \frac{2u^2 \sin\beta}{g\,[4\sin^2\beta + \cos^2\beta)}}$$

(iii) The vertical height of P above O = PM

$PM = OP\sin\beta = R\sin\beta.$

$$\boxed{\therefore \text{ The vertical height P above O} = \frac{2u^2\sin^2\beta}{g(4\sin^2\beta+\cos^2\beta)}}$$

Required solution.

Ques: 7.(C)(i) Prove that $\text{curl}\, \frac{\vec{a}\times\vec{r}}{r^3} = \frac{-\vec{a}}{r^3} + \frac{3\vec{r}}{r^5}(\vec{a}\cdot\vec{r})$, where, $\vec{a}$ is a constant vector.

Solution:- We have,

$$\text{Curl}\, \frac{\vec{a}\times\vec{r}}{r^3} = \nabla\times\left(\frac{\vec{a}\times\vec{r}}{r^3}\right) = \sum i\times\frac{\partial}{\partial x}\left[\frac{\vec{a}\times\vec{r}}{r^3}\right]$$

$$= \left[\hat{i}\times\frac{\partial}{\partial x} + \hat{j}\times\frac{\partial}{\partial y} + \hat{k}\times\frac{\partial}{\partial z}\right]\left[\frac{\vec{a}\times\vec{r}}{r^3}\right] \quad \text{--- (1)}$$

Now, $$\frac{\partial}{\partial x}\left(\frac{\vec{a}\times\vec{r}}{a^3}\right) = \frac{-3}{r^4}\cdot\frac{\partial r}{\partial x}(\vec{a}\times\vec{r}) + \frac{1}{r^3}\left(\vec{a}\times\frac{\partial\vec{r}}{\partial x}\right) + \frac{1}{r^3}\left(\frac{\partial\vec{a}}{\partial x}\times\vec{r}\right).$$

Now, $\frac{\partial\vec{a}}{\partial x} = 0$; because $\vec{a}$ is a constant vector.

Also $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ $\quad\therefore \frac{\partial\vec{r}}{\partial x} = \hat{i}$

Further; $\frac{\partial r}{\partial x} = \frac{x}{r}$

$\therefore$ (1) becomes.

$$\frac{\partial}{\partial x}\left(\frac{\vec{a}\times\vec{r}}{r^3}\right) = \frac{-3}{r^4}\cdot\frac{x}{r}(\vec{a}\times\vec{r}) + \frac{1}{r^3}(a\times\hat{i})$$

$$\therefore \frac{\partial}{\partial x}\left(\frac{\vec{a}\times\vec{r}}{r^3}\right) = -\frac{3x}{r^5}(\vec{a}\times\vec{r}) + \frac{1}{r^3}(a\times\hat{i}).$$

$$\therefore \hat{i}\times\frac{\partial}{\partial x}\left(\frac{\vec{a}\times\vec{r}}{r^3}\right) = -\frac{3x}{r^5}\, i\times(\vec{a}\times\vec{r}) + \frac{1}{r^3}\, i\times[\vec{a}\times\hat{i}].$$

$$= -\frac{3x}{r^5}\left[(i\cdot\vec{r})\vec{a} - (\hat{i}\cdot\vec{a})\cdot\vec{r}\right] + \frac{1}{r^3}\left[(\hat{i}\cdot\hat{i})\vec{a} - (\hat{i}\cdot\vec{a})\cdot\hat{i}\right]$$

$$\therefore \hat{i}\times\frac{\partial}{\partial x}\left(\frac{\vec{a}\times\vec{r}}{r^3}\right) = -\frac{3x}{r^5}x\vec{a} + \frac{3x}{r^5}a_1\vec{r} + \frac{1}{r^3}\vec{a} - \frac{1}{r^3}a_1\hat{i}$$

$\left[\because \hat{i}\cdot\vec{r} = x \text{ and } \hat{i}\cdot\vec{a} = a_1 \text{ ; if } a = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}\right]$

$$= -\frac{3x^2}{r^5}\cdot\vec{a} + \frac{3}{r^5}\cdot a_1 x\vec{r} + \frac{1}{r^3}\vec{a} - \frac{1}{r^3}a_1\hat{i} \quad \text{—} (2).$$

Similarly,

$$\left(\hat{j}\times\frac{\partial}{\partial y}\right)\left(\frac{\vec{a}\times\vec{r}}{r^3}\right) = -\frac{3y^2}{r^5}\cdot\vec{a} + \frac{3}{r^5}\cdot a_2 y\vec{r} + \frac{1}{r^3}\vec{a} - \frac{1}{r^3}a_2\hat{j} \quad \text{—} (3)$$

and $$\left(\hat{k}\times\frac{\partial}{\partial z}\right)\left(\frac{\vec{a}\times\vec{r}}{r^3}\right) = -\frac{3}{r^5}\cdot z^2\vec{a} + \frac{3}{r^5}\cdot a_3 z\vec{r} + \frac{1}{r^3}\vec{a} - \frac{1}{r^3}a_3\hat{k} \quad \text{—} (4)$$

Substituting (3), (4) and (5) in (1), we get.

$$\text{Curl}\left(\frac{\vec{a}\times\vec{r}}{r^3}\right) = -\frac{3x^2\vec{a}}{r^5} + \frac{3}{r^5}a_1 x\vec{r} + \frac{1}{r^3}\left[\vec{a} - a_1\hat{i}\right]$$
$$-\frac{3}{r^5}y^2\vec{a} + \frac{3}{r^5}a_2 y\vec{r} + \frac{1}{r^3}\left[\vec{a} - a_2\hat{j}\right]$$
$$-\frac{3}{r^5}z^2\vec{a} + \frac{3}{r^5}a_3 z\vec{r} + \frac{1}{r^3}\left[\vec{a} - a_3\hat{k}\right].$$

$$\therefore \text{Curl}\left[\frac{\vec{a}\times\vec{r}}{r^3}\right] = -\frac{3}{r^5}(x^2+y^2+z^2)\vec{a} + \frac{3}{r^5}(a_1x + a_2y + a_3z)\vec{r}$$
$$+ \frac{3}{r^3}\vec{a} - \frac{1}{r^3}(a_1\hat{i} + a_2\hat{j} + a_3\hat{k})$$

$$\therefore \text{Curl}\left[\frac{\vec{a}\times\vec{r}}{r^3}\right] = -\frac{3}{r^5}\cdot r^2\vec{a} + \frac{3}{r^5}(\vec{r}\cdot\vec{a})\cdot\vec{r} + \frac{3}{r^3}\vec{a} - \frac{\vec{a}}{r^3}$$
$$= \cancel{-\frac{3}{r^3}\vec{a}} + \frac{3}{r^5}(\vec{r}\cdot\vec{a})\cdot\vec{r} + \cancel{\frac{3}{r^3}\vec{a}} - \frac{\vec{a}}{r^3}$$

$$\boxed{\text{Curl}\left[\frac{\vec{a}\times\vec{r}}{r^3}\right] = -\frac{\vec{a}}{r^3} + \frac{3}{r^5}(\vec{r}\cdot\vec{a})\cdot\vec{r}}$$

Hence proved

Ques: 7(c) ii) If $F = (3x+y)\vec{i} + (y-2z)\vec{j} + (y+x+az)\vec{K}$ determine a so that F is solenoidal.

Solution:-

Given; $F = (3x+y)\vec{i} + (y-2z)\hat{j} + (y+x+az)\hat{k}$

For being solenoidal.

$$\nabla \cdot F = 0$$

$$\left[\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right]\vec{F} = 0$$

$$\left[\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right]\left[(3x+y)\hat{i} + (y-2z)\hat{j} + (y+x+az)\hat{k}\right] = 0$$

$$\left[\frac{\partial}{\partial x}(3x+y)\,\hat{i}\cdot\hat{i} + \frac{\partial}{\partial y}(y-2z)\,\hat{j}\cdot\hat{j} + \frac{\partial}{\partial z}(y+x+az)\,\hat{k}\cdot\hat{k}\right] = 0$$

$$[\because \hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1]$$

$$\frac{\partial}{\partial x}(3x+y) + \frac{\partial}{\partial y}(y-2z) + \frac{\partial}{\partial z}(y+x+az) = 0$$

$$3 + 1 + a = 0$$

$$4 + a = 0$$

$$\boxed{a = -4}$$

Hence; $F = (3x+y)\hat{i} + (y-2z)\hat{j} + (y+x-4z)\hat{k}$, i.e $a = -4$ so that F is solenoidal.

Ques: 8(a)) By using Laplace transformation, solve

$$(D^3 - D^2 - D + 1)y = 8te^{-t} \text{ if } y = Dy = D^2y = 0$$

when $t = 0$.

Solution:- Taking the Laplace transform on both sides of the given equation, we have

$$L(y''') - L\{y''\} - L\{y'\} + L\{y\} = 8L\{te^{-t}\}$$

$$\Rightarrow [p^3 L\{y\} - p^2\{y(0)\} - py'(0) - y''(0)] - [p^2 L\{y\} - py(0) - y'(0)]$$

$$- [pL\{y\} - y(0)] + L\{y\} = -8\frac{d}{dp}[L\{e^{-t}\}]$$

$$\Rightarrow [p^3 - p^2 - p + 1]L\{y\} = -8\frac{d}{dp}\left(\frac{1}{p+1}\right)$$

$$\Rightarrow [p^3 - p^2 - p + 1]L\{y\} = 8\frac{1}{(p+1)^2}$$

$$\Rightarrow \quad L\{y\} = \frac{8}{(p-1)^2(p+1)^3}$$

$$\Rightarrow L\{y\} = \frac{-3}{2(p-1)} + \frac{1}{(p-1)^2} + \frac{3}{2(p+1)} + \frac{2}{(p+1)^2} + \frac{2}{(p+1)^3}$$

$$\Rightarrow y = -\frac{3}{2}L^{-1}\left\{\frac{1}{p-1}\right\} + L^{-1}\left\{\frac{1}{(p-1)^2}\right\} + \frac{3}{2}L^{-1}\left\{\frac{1}{p+1}\right\} + 2L^{-1}\left\{\frac{1}{(p+1)^2}\right\} + 2L^{-1}\left\{\frac{1}{(p+1)^3}\right\}$$

$$\Rightarrow y = -\frac{3}{2}e^t + e^t L^{-1}\left(\frac{1}{p^2}\right) + \frac{3}{2}e^{-t} + 2e^{-t}L^{-1}\left\{\frac{1}{p^2}\right\} + 2e^{-t}L^{-1}\left\{\frac{1}{p^3}\right\}$$

$$\Rightarrow y = -\frac{3}{2}e^t + e^t \cdot t + \frac{3}{2}e^{-t} + 2e^{-t}\cdot t + t^2 e^{-t}$$

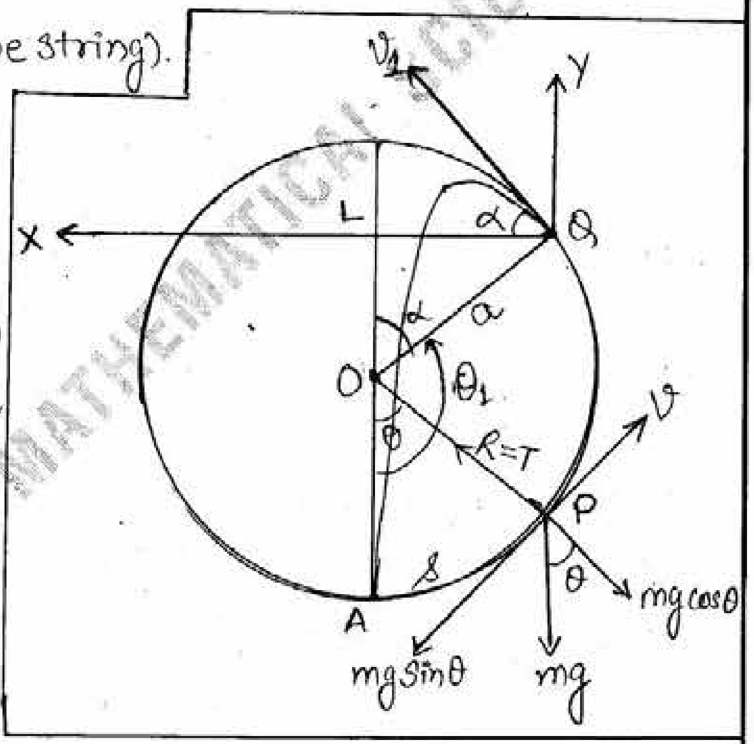
$$\Rightarrow \boxed{y = \left[t^2 + 2t + \frac{3}{2}\right]e^{-t} + \left[t - \frac{3}{2}\right]e^t}$$ Required solution

Ques: 8(b)} A particle of mass m, hanging vertically from a fixed point by a light inextensible cord of length l, is struck by a horizontal blow which imparts to its a velocity $2\sqrt{gl}$. Find the velocity and height of the particle from the level of its initial position when the cord becomes slack.

Solution:-

Take $R = T$

(i.e, the tension in the string).



Let a particle tied to a cord OA of length l be struck by a horizontal blow which imparts it a velocity $2\sqrt{gl}$.

If P is the position of the particle at time t such that $\angle AOP = \theta$, then the equations of motion are —

$$m\frac{d^2s}{dt^2} = -mg\sin\theta \quad \text{——} \ (1)$$

and $$m\frac{v^2}{l} = T - mg\cos\theta \quad \text{——} \ (2)$$

$$s = l\theta \quad \text{——} \ (3)$$

From (1) and (3), we have $l\frac{d^2\theta}{dt^2} = -g\sin\theta$

Multiplying both sides by $2l\frac{d\theta}{dt}$ and integrating, we have –

$$v^2 = \left(l\frac{d\theta}{dt}\right)^2 = 2lg\cos\theta + A$$

But at point A, $\theta = 0$ and $v = 2\sqrt{(gl)}$

$\therefore\ 4gl = 2gl + A$ so that $A = 2gl$

$\therefore\ v^2 = 2gl(\cos\theta + 1)$ ——— (4)

From (2) and (4), we have

$$T = \frac{m}{l}(v^2 + gl\cos\theta) = mg(3\cos\theta + 2) \quad \text{——— (5)}$$

If the cord becomes slack at the point Q, where $\theta = \theta_1$, then from (5), we have –

$$\boxed{T = 0 = mg(3\cos\theta_1 + 2)}$$

giving $\cos\theta_1 = -2/3$

If $\angle COQ = \alpha$, then $\alpha = \pi - \theta_1$.

$\cos\alpha = 2/3$

If v_1 is the velocity of the particle at Q, then $v = v_1$, where $\theta = \theta_1$.

Therefore from (4), we have

$$v_1^2 = 2lg(1 + \cos\theta_1) = 2lg(1 - 2/3)$$

$$v_1^2 = \frac{2lg}{3}$$

Now $OL = l\cos\alpha = \frac{2}{3}l$.

Thus, the particle leaves the circular path at the point Q at a height $2l/3$ above the fixed point O with velocity $v_1 = \sqrt{(2lg/3)}$ at an angle α to the horizontal and subsequently it describes a parabolic path.

Max. height H of the particle above Q.

$$= \frac{v_1^2 \sin^2\alpha}{2g} = \frac{v_1^2}{2g}(1-\cos^2\alpha) = \frac{\frac{2}{3}lg}{2g}\left(1-\frac{4}{9}\right)$$

Max. height H of the particle above Q $= \frac{5l}{27}$

$\therefore$ Height of the highest point of the parabolic path above the fixed point O $= OL + H$

$$= \frac{2}{3}l + \frac{5}{27}l = \frac{23}{27}l.$$

Required solution.

Q.8 (c) verify the divergence theorem for $A=4xi-2y^2j+z^2k$ taken over the region bounded by $x^2+y^2=4$, $z=0$ and $z=3$.

Sol^n: Let S denote the closed surface bounded by the cylinder $x^2+y^2=4$ and the planes $z=0$, $z=3$.

Also let V be the volume bounded by the surface S.

By Gauss divergence theorem, we have

$$\iint_S F.\hat{n}\, ds = \iiint_V \operatorname{div} F\, dv$$

Volume integral $= \iiint_V \operatorname{div} F\, dv$

$$= \iiint_V \nabla . F\, dv$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2)\right] dv$$

$$= \iiint_V (4-4y+2z)\, dv$$

$$= 2\int_{z=0}^{3}\int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (2-2y+z)\, dz\,dx\,dy$$

$$= 2\int_{z=0}^{3}\int_{x=-2}^{2}\left[2y-y^2+zy\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dz\,dx.$$

$$= 2\int_{z=0}^{3}\int_{x=-2}^{2}\left[(2+z)\,2\sqrt{4-x^2}-0\right] dz\,dx$$

$$= 4\int_{z=0}^{3}\int_{x=-2}^{2}\left[2\sqrt{4-x^2}+z\sqrt{4-x^2}\right] dz\,dx.$$

$$= 4\int_{x=-2}^{2}\left[2z\sqrt{4-x^2}+\frac{z^2}{2}\sqrt{4-x^2}\right]_{z=0}^{3} dx$$

$$= 4\int_{x=-2}^{2}\left[6\sqrt{4-x^2}+\frac{9}{2}\sqrt{4-x^2}\right]dx$$

$$= 4\times\frac{21}{2}\int_{x=-2}^{2}\sqrt{4-x^2}\,dx$$

$$= \frac{84}{2}\times 2\int_{0}^{2}\sqrt{4-x^2}\,dx$$

$$= 84\left[\frac{x}{2}\sqrt{4-x^2}+2\sin^{-1}\frac{x}{2}\right]_0^2$$

$$= 84\left[0+2\times\frac{\pi}{2}\right] = 84\pi$$

Now we shall evaluate the surface integral

$$\iint_S F\cdot\hat{n}\,ds$$

Here the surface S consists of three surfaces

(i) the surface S_1 of the base i.e, the plane face $z=0$ of the cylinder.

(ii) the surface S_2 of the top i.e, the plane $z=3$ of the cylinder and

(iii) the surface S_3 of the convex portion of the cylinder.

For the surface S_1 i.e., $z=0$, $F = 4xi - 2y^2 j$, putting $z=0$ in F.

A unit vector $\hat{n}$ along the outward drawn normal to S_1 is obviously $-\hat{K}$.

$$\therefore \iint_{S_1} F\cdot n\,ds = \iint_{S_1}(4xi - 2y^2 j)\cdot(-K)\,ds$$

$$= \iint_{S_1} 0\,ds = 0.$$

For the surface S_2 i.e, $z=3$, $F=4xi-2y^2j+9k$

putting $z=3$ in F.

A unit vector n along the outward drawn normal to S_2 is obviously $\bar{k}$.

$$\therefore \iint_{S_2} F.n\,ds = \iint_{S_2} (4xi-2y^2j+9k).k\,ds$$

$$= \iint_{S_2} 9\,ds$$

$$= 9\,2\pi(2)$$

$$= 36\pi \quad (\because \text{area of } S_2 = 2\pi r \text{ here } r=2)$$

For the convex portion S_3 i.e. $x^2+y^2=4$, a vector normal to S_3 is given by

$$\nabla(x^2+y^2) = 2xi+2yj$$

$\therefore$ n = a unit vector along outward drawn normal at any point of S_3.

$$= \frac{2xi+2yj}{\sqrt{4x^2+4y^2}} = \frac{xi+yj}{2} \quad (\because x^2+y^2=4 \text{ on } S_3)$$

$$\therefore \text{ on } S_3 \; F.\hat{n}\,ds = (4xi-2y^2j+z^2k).\frac{1}{2}(xi+yj)$$

$$= 2x^2-y^3$$

Also ds = elementry area on the surface S_3

$= 2\,d\theta\,dz$, using Cylindrical coordinates r,θ,z

$$\therefore \iint_{S_3} F.\hat{n}\,ds = \iint_{S_3} (2x^2-y^3)\,2\,d\theta\,dz \quad \text{where } x=2\cos\theta,\; y=2\sin\theta$$

$$= \int_{z=0}^{3}\int_{\theta=0}^{2\pi} (8\cos^2\theta - 8\sin^3\theta)\, 2\, d\theta\, dz$$

$$= 16\int_{\theta=0}^{2\pi} \left[\cos^2\theta - \sin^3\theta\right] (z)_{z=0}^{3}\, d\theta$$

$$= 48\int_{0}^{2\pi} (\cos^2\theta - \sin^3\theta)\, d\theta$$

$$= 48\left[4\int_{0}^{\pi/2} \cos^2\theta\, d\theta - \int_{0}^{2\pi} \sin^3\theta\, d\theta\right]$$

$$= 48\left[4\,\frac{1}{2}\,\frac{\pi}{2} - 0\right] \quad \left(\because \sin^3(2\pi-\theta) = -\sin^3\theta \text{ odd function } \int_0^{2\pi} \sin^3\theta\, d\theta = 0\right)$$

$$= 48\pi$$

Hence the required surface integral

$$= \iint_S F\cdot\hat{n}\, ds = \iint_{S_1} F\cdot\hat{n}\, ds + \iint_{S_2} F\cdot\hat{n}\, ds + \iint_{S_3} F\cdot\hat{n}\, ds$$

$$= 0 + 36\pi + 48\pi$$

$$= 84\pi$$

$$\therefore \iint_S F\cdot\hat{n}\, ds = 84\pi$$

$$\therefore \iint_S F\cdot\hat{n}\, ds = \iiint_V \operatorname{div} F\, dV.$$