CSE - 2015

Q1 Test for convergence of $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{n}{n^2+1}\right)$

sol. Let $\sum_{n=1}^{\infty} (-1)^{n+1} \left(\frac{n}{n^2+1}\right) = \sum_{n=1}^{\infty} (-1)^{n+1} u_n$

where $u_n = \frac{n}{n^2+1}$

∴ According Leibnitz test for alternating series, the series will be convergent if

i) $u_n > u_{n+1}$ (u_n is monotonically decreasing)

ii) $\lim_{n\to\infty} u_n = 0$

Here $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{1}{n} \frac{1}{1+1/n^2} = 0$

Also $u_n - u_{n+1} = \frac{n}{n^2+1} - \frac{(n+1)}{(n+1)^2+1}$

$= \frac{n(n+1)^2 + n - (n+1)(n^2+1)}{(n^2+1)((n+1)^2+1)}$

$= \frac{n^3 + 2n^2 + \cancel{n} + n - \cancel{n^3} - \cancel{n} - n^2 - 1}{(n^2+1)(n^2+2n+2)}$

$= \frac{n^2+n-1}{(n^2+1)(n^2+2n+2)}$

$$= \frac{(n+\frac{1}{2})^2 - \frac{5}{4}}{(n^2+1)((n+1)^2+1)}$$

As $n \geq 1$ $\therefore (n+\frac{1}{2})^2 \geq \frac{9}{4}$

$\Rightarrow (n+\frac{1}{2})^2 - \frac{5}{4} \geq 1$

As n^2+1 and $(n+1)^2+1$ are +ve ($\because$ sum of squares is positive)

$\therefore u_n - u_{n+1} > 0 \Rightarrow u_n > u_{n+1}$

Hence, By Leibnitz test, Series is convergent

Q2 Is the function $f(x) = \begin{cases} 1/n, & \frac{1}{n+1} < x \le \frac{1}{n} \\ 0, & x = 0 \end{cases}$ Reimann

Integrable? If yes, Find $\int_0^1 f(x)\,dx$

$$f(x) = \begin{cases} 1 & \frac{1}{2} < x \le 1 \\ 1/2 & 1/3 < x \le 1/2 \\ 1/3 & 1/4 < x \le 1/3 \\ \vdots & \\ 1/n & \frac{1}{n+1} < x \le \frac{1}{n} \\ 0 & , x = 0 \end{cases}$$

f(x) is discontinuos at $1/2, 1/3, \frac{1}{4}, \frac{1}{5} \cdots$

As the set of discontinuos point has limit point 0. Therefore f(x) ~~\$~~ is reimann integrable as set of discontinuities has finite limit points.

Now, $\int_0^1 f(x)\,dx = \int_{1/2}^{1} f(x)dx + \int_{1/3}^{1/2} f(x)dx + \cdots\cdots$

$$= \lim_{n\to\infty}\left[\int_{1/2}^{1} f(x)dx + \int_{1/3}^{1/2} f(x)dx + \cdots + \int_{\frac{1}{n+1}}^{1/n} f(x)dx\right]$$

$$= \lim_{n\to\infty}\left[\int_{1/2}^{1} 1\,dx + \int_{1/3}^{1/2} 1/2\,dx + \cdots + \int_{1/n+1}^{1/n} \frac{1}{n}\,dx\right]$$

$$= \lim_{n\to\infty}\left[1\left(1-\frac{1}{2}\right)+\frac{1}{2}\left(\frac{1}{2}-\frac{1}{3}\right)+\dots+\frac{1}{n}\left(\frac{1}{n}-\frac{1}{n+1}\right)\right]$$

$$= \lim_{n\to\infty}\left[\left(1^2+\frac{1}{2^2}+\frac{1}{3^2}+\dots+\frac{1}{n^2}\right)-\left(\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\dots+\frac{1}{n(n+1)}\right)\right]$$

$$= \lim_{n\to\infty}\left[\left(1^2+\frac{1}{2^2}+\frac{1}{3^2}+\dots+\frac{1}{n^2}\right)-\left(1-\cancel{\frac{1}{2}}+\cancel{\frac{1}{2}}-\cancel{\frac{1}{3}}+\dots+\cancel{\frac{1}{n-1}}-\cancel{\frac{1}{n}}+\cancel{\frac{1}{n}}-\frac{1}{n+1}\right)\right]$$

$$= \lim_{n\to\infty}\left[\sum_{m=1}^{n}\frac{1}{m^2}+\left(1-\frac{1}{n+1}\right)\right]$$

$$= \lim_{n\to\infty}\sum_{m=1}^{n}\frac{1}{m^2}-\lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)$$

$$= \sum_{m=1}^{\infty}\frac{1}{m^2}-1$$

$$= \frac{\pi^2}{6}-1 \qquad \left(\because \sum_{m=1}^{\infty}\frac{1}{m^2}=\frac{\pi^2}{6}\right)$$

Q3 Test the series function for uniform convergence. $\sum_{n=1}^{\infty} \frac{nx}{1+n^2x^2}$

Sol: Let $S_n(x) = \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \frac{nx}{1+n^2x^2}$

Let $S_n(x)$ be uniformly convergent. in $(-\infty, \infty)$
Then for every $\varepsilon > 0$, there exist m such that $\forall x \in (-\infty, \infty)$ and $p \in N$ (By Cauchy's criterion)

$$|S_{n+p-1} - S_{n-1}| < \varepsilon \;, \; \forall n \geq m$$

Let $\varepsilon = 1/4$, take $x = 1/m$ and $p = 1$

$$|f_n(x)| < \varepsilon \;, \quad \forall n \geq m$$

$$\left| \frac{nx}{1+n^2x^2} \right| \leq \varepsilon, \ \forall \ n \geq m$$

$$\left| \frac{n(1/n)}{1+n^2(1/n)^2} \right| \leq \varepsilon$$

$\frac{1}{2} \leq \varepsilon$ which is a contradiction

$\because \varepsilon = 1/4$

$\therefore S_n(x) = \sum_{n=1}^{\infty} \frac{nx}{1+n^2x^2}$ is not uniformly convergent

Q Find the absolute maximum and minimum values of function

$f(x,y) = x^2+3y^2-y$ over the region $x^2+2y^2 \leq 1$

sol. Given $f(x,y) = x^2+3y^2-y$

Let $g(x,y) = x^2+2y^2-1$

The constraint is $g(x,y) \le 0$ i.e. $x^2+2y^2-1 \le 0$

The lagrangian is

$F = f(x,y) + \lambda g(x,y)$ where λ = multiplier

$F = x^2+3y^2-y+\lambda(x^2+2y^2-1)$

Case I: Constraint is non binding i.e. $g(x,y) < 0$ then $\lambda = 0$.

$F = f(x,y) = x^2+3y^2-y$

$\frac{\partial F}{\partial x} = 2x$

$\frac{\partial F}{\partial y} = 6y-1$

Critical points are given by $\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = 0$

$\therefore$ $2x = 0$ and $6y-1 = 0$

$\Rightarrow$ $x = 0$ and $y = 1/6$

$g(0, 1/6) = 0^2+2\times\frac{1}{36}-1 = -\frac{17}{18} \le 0$

$\therefore$ $x=0$, $y=1/6$ satisfy the constraint.

Now, $\frac{\partial^2 f}{\partial x^2} = 2$, $\frac{\partial^2 f}{\partial y^2} = 6$

and $\frac{\partial f}{\partial x \partial y} = 0.$

Second-derivative test

$$f_{xx} f_{yy} - f_{xy}^2 = (2)(6) - 0^2 = 12 > 0$$

As $f_{xx} > 0$ and $f_{xx} f_{yy} - f_{xy}^2 > 0$

$\therefore$ $f(x,y)$ attains ~~a~~ minima at $(0, 1/6)$

with $f_{min}(0, 1/6) = 0^2 + 3\left(\frac{1}{6}\right)^2 - \frac{1}{6}$

$f_{min}(0, 1/6) = -\frac{1}{12}$

Case II: Constraint is binding i.e. $g(x) = 0$

$\lambda \neq 0$

$$F = f(x,y) + \lambda g(x,y)$$

$$F = x^2 + 3y^2 - y + \lambda(x^2 + 2y^2 - 1)$$

First order conditions:-

$$\frac{\partial F}{\partial x} = 2x + 2\lambda x = 0 \quad -(1)$$

$$\frac{\partial F}{\partial y} = 6y - 1 + 4\lambda y = 0 \quad -(2)$$

$\frac{\partial f}{\partial \lambda} = x^2 + 2y^2 - 1 = 0 \quad -(3)$

Using (1) $\quad 2x = -2x\lambda$

$\Rightarrow$ either $x = 0$ or $\lambda = -1$

From (2), if $\lambda = -1$, $6y - 1 - 4y \equiv 0$

$\Rightarrow 2y - 1 = 0 \Rightarrow y = 1/2$

Using (3)

$x = 0,$	$\lambda = -1,\ y = 1/2$
$0^2 + 2y^2 - 1 = 0$	$x^2 + 2\left(\frac{1}{2}\right)^2 - 1 = 0$
$y = \pm \frac{1}{\sqrt{2}}$	$x^2 = 1/2$
	$x = \pm 1/\sqrt{2}$

$\therefore$ we get, $\left(0, \pm\frac{1}{\sqrt{2}}\right)$ and $\left(\pm\frac{1}{\sqrt{2}}, \frac{1}{2}\right)$

$$f\left(0, \frac{+1}{\sqrt{2}}\right) = 0^2 + 3\left(\frac{1}{\sqrt{2}}\right)^2 - \frac{1}{\sqrt{2}}$$

$$= \frac{3}{2} - \frac{1}{\sqrt{2}}$$

$$f\left(0, -\frac{1}{\sqrt{2}}\right) = 0^2 + 3\left(-\frac{1}{\sqrt{2}}\right)^2 - \left(-\frac{1}{\sqrt{2}}\right)$$

$$= \frac{3}{2} + \frac{1}{\sqrt{2}}$$

$$f\left(\pm\frac{1}{\sqrt{2}}, \frac{1}{2}\right) = \left(\frac{\pm 1}{\sqrt{2}}\right)^2 + 3\left(\frac{1}{2}\right)^2 - \frac{1}{2}$$

$$= \frac{1}{2} + \frac{3}{4} - \frac{1}{2} = 3/4$$

From case I and II we have

$$f_{min} = -\frac{1}{12} \quad \text{at} \quad (0, 1/6)$$

$$f_{max} = \frac{3}{2} + \frac{1}{\sqrt{2}} \quad \text{at} \quad \left(0, -\frac{1}{\sqrt{2}}\right)$$