CSE 2019 Calculus

DATE

2019(I).

1) ~~34)~~ $f : [0, \frac{\pi}{2}] \to R$ be cont s.t.

$$f(x) = \frac{\cos^2 x}{4x^2 - \pi^2}, \quad 0 \le x < \frac{\pi}{2}$$

find $f(\frac{\pi}{2})$

Since f is continuous on $[0, \pi/2]$

$$\therefore \quad f(\tfrac{\pi}{2}) = \lim_{x \to \frac{\pi}{2}} f(x).$$

$$\lim_{x \to \frac{\pi}{2}} f(x) = \lim_{x \to \frac{\pi}{2}} \frac{\cos^2 x}{4x^2 - \pi^2} \quad \left(\tfrac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to \frac{\pi}{2}} \frac{-2\cos x \sin x}{8x} \quad (\text{L-Hospital})$$

$$= 0$$

$$\therefore \quad f(\tfrac{\pi}{2}) = 0.$$

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1.(b) Let $f: D(\subseteq \mathbb{R}^2) \to \mathbb{R}$ be a function and $(a,b) \in D$. If $f(x,y)$ is continuous at (a,b), then show that the functions $f(x,b)$ and $f(a,y)$ are continuous at $x=a$ and at $y=b$ respectively.

Solution:

Let $f: D(\subseteq \mathbb{R}^2) \to \mathbb{R}$ be a function and $(a,b) \in D$.

If $f(x,y)$ is continuous at (a,b) then by definition,

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

i.e. $f(x,y) \to f(a,b)$ as $(x,y) \to (a,b)$

i.e. for given $\varepsilon > 0$ (however small)

$\exists\ \delta > 0\ (\delta(\varepsilon))$ such that

$|f(x,y) - f(a,b)| < \varepsilon$ whenever $\|(x,y)-(a,b)\| < \delta$.

A point to be particularly noticed is that if a function of more than

one variable is continuous at a point, it is continuous at that point when considered as a function of single variable.

To be more specific if a function f of two variables x, y is continuous at (a,b) then $f(x,b)$ is continuous at $x=a$ of x and $f(a,y)$ that of f at $y=b$.

Hence, the result.

DATE

33. $f(x) = |\cos x| + |\sin x|$

We check differentiability at $x = \frac{\pi}{2}$.

$$Lf'\left(\frac{\pi}{2}\right) = \lim_{h\to 0} \frac{f\left(\frac{\pi}{2}-h\right) - f\left(\frac{\pi}{2}\right)}{-h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\left|\cos\left(\frac{\pi}{2}-h\right)\right| + \left|\sin\left(\frac{\pi}{2}-h\right)\right| - \left(\left|\cos\frac{\pi}{2}\right| + \left|\sin\frac{\pi}{2}\right|\right)\right]$$

$$= \lim_{h\to 0} \frac{1}{-h}\left[|\sin h| + |\cos h| - (0+1)\right]$$

$$= \lim_{h\to 0} \frac{\sin h + \cos h - 1}{-h} \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{h\to 0} \frac{\cos h - \sin h}{-1} = -1$$

$$Rf'\left(\frac{\pi}{2}\right) = \lim_{h\to 0} \frac{f\left(\frac{\pi}{2}+h\right) - f\left(\frac{\pi}{2}\right)}{h}$$

$$= \lim_{h\to 0} \frac{1}{h}\left[\left|\cos\left(\frac{\pi}{2}+h\right)\right| + \left|\sin\left(\frac{\pi}{2}+h\right)\right| - \left(\left|\cos\frac{\pi}{2}\right| + \left|\sin\frac{\pi}{2}\right|\right)\right]$$

$$= \lim_{h\to 0} \frac{1}{h}\left[|-\sin h| + |\cos h| - (0+1)\right]$$

$$= \lim_{h\to 0} \frac{\sin h + \cos h - 1}{h} \quad \left(\frac{0}{0} \text{ from}\right)$$

$$= \lim_{h\to 0} \frac{\cos h - \sin h}{1} = 1.$$

$Lf'\left(\frac{\pi}{2}\right) \neq Rf'\left(\frac{\pi}{2}\right)$ $\therefore$ Not differentiable at $x = \frac{\pi}{2}$

DATE

4) $f(x) = 2x^3 - 9x^2 + 12x + 6$ in $[2, 3]$

$f'(x) = 6x^2 - 18x + 12$

$= 6(x^2 - 3x + 2)$

$= 6(x-1)(x-2)$

for critical points, $f'(x) = 0$

$6(x-1)(x-2) = 0$

$x = 1, \quad x = 2$

Only $x = 2$ in the given interval

and $f'(x) \geq 0$ in $[2, 3]$

$\therefore$ f is increasing in $[2, 3]$

$f(2) = 2(8) - 9(4) + 12(2) + 6$

$= 10$

$f(3) = 2(27) - 9(9) + 12(3) + 6$

$= 15$

$\therefore$ f has max at $x = 3$

f has min at $x = 2$.

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(c) →

(i) If $u = \sin^{-1}\sqrt{\dfrac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}}$ then

show that $\sin^2 u$ is a homogeneous function of x and y of degree $-1/6$.

Hence show that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{12}\left(\frac{13}{12} + \frac{\tan^2 u}{12}\right)$$

Solution:

Given that $u = \sin^{-1}\sqrt{\dfrac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}}$, we can

write

$$\sin u = \left[\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}\right]^{1/2} = \frac{x^{1/6}}{x^{1/4}}\left[\frac{1+(y/x)^{1/3}}{1+(y/x)^{1/2}}\right]$$

$$= x^{-1/12} f(y/x)$$

$$\Rightarrow \sin^2 u = \left[\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}\right] = \frac{x^{1/3}}{x^{1/2}}\left[\frac{1+(y/x)^{1/3}}{1+(y/x)^{1/2}}\right]$$

$$= x^{-1/6} f(y/x)$$

Thus, $z = \sin^2 u$ is a homogeneous function of x and y of degree $-1/6$. —(i)

Now, by Euler's theorem,

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=-\frac{1}{12}z, \text{ where } z=\sin u$$

$$\Rightarrow x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y}=-\frac{1}{12}\sin u$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{12}\tan u \quad \text{—(1)}$$

Differentiating (1) partially w.r.t. x and y, respectively

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=-\frac{1}{12}\sec^2 u\frac{\partial u}{\partial x} \quad \text{—(2)}$$

$$x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}=-\frac{1}{12}\sec^2 u\frac{\partial u}{\partial y}. \quad \text{—(3)}$$

Multiply (2) by x, (3) by y and add to get

$$\left(x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}\right)+\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$
$$=\frac{-1}{12}\sec^2 u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right).$$

From (1) and (4), we get

$$x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = \frac{1}{12} \tan u \left(1 + \frac{1}{12}(\sec^2 u)\right)$$

$$= \frac{\tan u}{12} \left(\frac{12 + 1 + \sec^2 u}{12} \right)$$

$$= \frac{\tan u}{12} \left(\frac{13}{12} + \frac{\tan^2 u}{12} \right)$$

Hence, Proved.

4.(c)(ii) Using the Jacobian method, show that if $f'(x) = \frac{1}{1+x^2}$ and $f(0) = 0$, then

$$f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right)$$

Solution:

Given that

$$f'(x) = \frac{1}{1+x^2} \quad \text{and} \quad f(0) = 0.$$

Let $u = f(x) + f(y)$ — (1) and $v = \frac{x+y}{1-xy}$ — (2)

We have

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} f'(x) & f'(y) \\ \frac{(1-xy)-(x+y)(-y)}{(1-xy)^2} & \frac{(1-xy)-(x+y)(-x)}{(1-xy)^2} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{1+x^2} & \frac{1}{1+y^2} \\ \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \end{vmatrix}$$

$$= \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2}$$

$$= 0$$

Since, $\frac{\partial(u,v)}{\partial(x,y)} = 0$.

$\therefore$ The given functions are not independent.

i.e. the functions u and v are functionally related.

Let $u = \phi(v)$ then

$$f(x) + f(y) = \phi\left(\frac{x+y}{1-xy}\right)$$

for $y=0$ gives $f(x) = \phi(x)$ [$\because f(0)=0$]

$$\therefore f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right)$$

Hence, the result.