1(c) Determine the values of p and q for which $\lim_{x\to 0} \frac{x(1+p\cos x)-q\sin x}{x^3}$ exists and equals 1.

Sol^n: It is a $\frac{0}{0}$ form and so by L Hospital's rule

$$\lim_{x\to 0} \frac{x(1+p\cos x)-q\sin x}{x^3}$$

$$= \lim_{x\to 0} \frac{(1+p\cos x)-px\sin x-q\cos x}{3x^2}$$

$$= \lim_{x\to 0} \frac{1+(p-q)\cos x-px\sin x}{3x^2} \quad \left(\frac{1+p-q}{0}\right)$$

To get the required limit, we take $1+p-q=0$

Thus $\lim_{x\to 0} \frac{1+(p-q)\cos x-px\sin x}{3x^2} \quad \left(\frac{0}{0}\right)$

$$= \lim_{x\to 0} \frac{-(p-q)\sin x-p\sin x-px\cos x}{6x} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x\to 0} \frac{-(p-q)\cos x-2p\cos x+px\sin x}{6}$$

$$= \frac{-p+q-2p}{6} = \frac{-3p+q}{6}$$

we are given $\frac{-3p+q}{6}=1 \Rightarrow -3p+q=6$

Also $1+p-q=0$ and $-6-3p+q=0$

On solving these equations for p and q, we obtain $p=-5/2$, $q=-3/2$

1(d) Find and classify all the critical points of the following function $f(x,y) = 7x - 8y + 2xy - x^2 + y^3$.

Soln: $f(x,y) = 7x - 8y + 2xy - x^2 + y^3$

For critical points, we get

$f_x(x,y) = 7 + 2y - 2x = 0$ — (1)

& $f_y(x,y) = -8 + 2x + 3y^2 = 0$ — (2)

(1)+(2) ≡, $3y^2 + 2y - 1 = 0$

$\Rightarrow y = -1, \frac{1}{3}$.

∴ From (1), we get $x = \frac{5}{2}$, if $x = -1$

$x = 23/6$. if $x = \frac{1}{3}$

∴ critical points are

$\left(\frac{5}{2}, -1\right)$ and $\left(\frac{23}{6}, \frac{1}{3}\right)$.

Now $f_{xy} = 2$

$f_{xx} = -2$ and $f_{yy} = 6y$.

At $\left(\frac{5}{2}, -1\right)$: $f_{xx} = -2 < 0$.

$D = f_{xx} f_{yy} - f_{xy}^2 = (-2)(-6) - 4 = 12 - 4 = 8 > 0$.

∴ $\left(\frac{5}{2}, -1\right)$ is a point of maxima of the function

At $\left(\frac{23}{6}, \frac{1}{3}\right)$:

$D = f_{xx} f_{yy} - f_{xy}^2 = (-2)(2) - 4 = -8 < 0$

∴ We have

$f_{xx} < 0$ and $D < 0$.

∴ the function has neither a maxima nor a minima at $\left(\frac{23}{6}, \frac{1}{3}\right)$.

which is known as a saddle point

1(e) Find the equation of the sphere that passes through the points $(4,1,0)$, $(2,-3,4)$, $(1,0,0)$ and touches the plane $2x+2y-z=11$.

Sol^n. Let the equation of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad \text{---} (1).$$

Its Centre is $(-u,-v,-w)$ & radius $=\sqrt{(u^2+v^2+w^2-d)}$

Since the Sphere (1) passes through the point $(4,1,0)$

so
$$4^2+1^2+0^2+2u(4)+2v(0)+2w(0)+d=0$$
$$8u+2v+d+17=0 \quad \text{---} (2).$$

Similarly if the Sphere (1) passes through $(2,-3,4)$ and $(1,0,0)$ then we have

$$4u-6v+8w+d+29=0 \quad \text{---} (3)$$

and
$$2u+d+1=0 \quad \text{---} (4)$$

Also as the sphere (1) touches the given plane, so the length of $\perp$lar from its Centre $(-u,-v,-w)$ to the given plane $2x+2y-z-11=0$ must be equal to the radius $\sqrt{(u^2+v^2+w^2-d)}$ of the sphere (1)

i.e.
$$\frac{2(-u)+2(-v)-(-w)-11}{\sqrt{[2^2+2^2+(-1)^2]}}=\sqrt{(u^2+v^2+w^2-d)}$$

$$\Rightarrow (-2u-2v+w-11)^2=9(u^2+v^2+w^2-d)$$

$$\Rightarrow 5u^2+5v^2+8w^2-8uv+4vw+4uw-44u-44v+22w-9d-121=0 \quad \text{---} (5)$$

from (4) $\quad u=\frac{1}{2}(-d-1)=-\frac{1}{2}(d+1) \quad \text{---} (6)$

from ②, $\therefore 2v = -8u - d - 17 = 4d + 4 - d - 17 = 3d - 13$

$v = \frac{1}{2}(3d - 13)$ ——— ⑦

from ③, $8w = -4u + 6v - d - 29$

$= (2d+2) + (9d-39) - d - 29$ from ⑥, ⑦

$\Rightarrow 8w = 10d - 66$

$\Rightarrow w = \frac{1}{4}(5d - 33)$ ——— ⑧

Substituting values of u, v, w from ⑥, ⑦, ⑧ in ⑤ and simplifying we get $72d^2 - 747d + 1935 = 0$

which gives $d = 5$

$\therefore$ from ⑥, ⑦ & ⑧ we get $u = -3$, $v = 1$, $w = -2$

$\therefore$ from ① the required equation is

$x^2 + y^2 + z^2 - 6x + 2y - 4z + 5 = 0.$

2(b) Evaluate $\iiint_E z\,dv$ where E is the region between the two planes $x+y+z=2$ and $x=0$ and inside the cylinder $y^2+z^2=1$

Sol^n:

$$\iiint_E z\,dv = \iiint z\,dx\,dy\,dz.$$

By using the cylindrical co-ordinates

$x=x,\ y=r\sin\theta,\ z=r\cos\theta$

where $0\le\theta\le 2\pi$

$0\le r\le 1$

$0\le x\le 2-r\sin\theta-r\cos\theta$

$$= \int_0^{2\pi}\int_0^1\int_{x=0}^{2-r\sin\theta-r\cos\theta} (r\cos\theta)\, r\,dx\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 (r^2\cos\theta)\left[x\right]_0^{2-r\sin\theta-r\cos\theta}$$

$$= \int_0^{2\pi}\int_0^1 r^2\cos\theta\,(2-r\sin\theta-r\cos\theta)\,dr\,d\theta.$$

$$= \int_0^{2\pi}\left(\frac{2}{3}r^3\cos\theta-\frac{1}{4}r^3\sin\theta\cos\theta-\frac{1}{4}r^3\cos^2\theta\right)_0^1 d\theta.$$

$$= \int_0^{2\pi} \frac{2}{3}\cos\theta-\frac{1}{4}\sin\theta\cos\theta-\frac{1}{4}\cos^2\theta\,d\theta.$$

$$= \left[\frac{2}{3}\sin\theta-\frac{1}{8}\cos 2\theta-\frac{1}{4}\left(\frac{\theta+\sin 2\theta}{2}\right)\right]_0^{2\pi}$$

$$= \left[\frac{2}{3}\sin\theta+\frac{1}{16}\cos 2\theta-\frac{1}{8}\left(\theta+\frac{1}{2}\sin 2\theta\right)\right]_0^{2\pi}$$

$$= -\pi/4.$$

2(c)(i) Find the equation of the plane which passes through the points $(0,1,1)$ and $(2,0,-1)$ and is parallel to the line joining the points $(-1,1,-2)$, $(3,-2,4)$. Find also the distance between the line and the plane.

Sol^n: Equation of plane through $(0,1,1)$ is

$$a(x-0)+b(y-1)+c(z-1)=0$$

$$\Rightarrow ax+by+cz-b-c=0 \quad \text{——} \ (1)$$

this also passes through $(2,0,-1)$ then

$$a(2)+b(0)+c(-1)-b-c=0$$

$$\Rightarrow 2a-b-2c=0 \quad \text{——} \ (2)$$

Given plane is parallel to line joining $(-1,1,-2)$ and $(3,-2,4)$

Now direction ratios of line joining $(-1,1,-2)$ and $(3,-2,4)$ is $(4,-3,6)$.

as plane is parallel to this line its normal will be $\perp$lar to it, Then $4a-3b+6c=0$ —— (3)

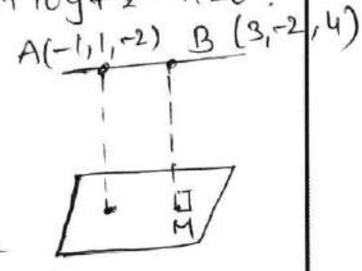
from (2) & (3)

$$\frac{a}{-6-6}=\frac{b}{-8-12}=\frac{c}{-6+4}$$

$$\Rightarrow \frac{a}{-12}=\frac{b}{-20}=\frac{c}{-2}$$

$$\Rightarrow \frac{a}{6}=\frac{b}{10}=\frac{c}{1}$$

Thus equation of plane is $P \equiv 6x+10y+z-11=0$.



and the equation of line is

$$\frac{x+1}{4}=\frac{y-1}{-3}=\frac{z+2}{6}$$

from figure, let M be the foot of the $\perp$lar from point B(3,-2,4) on the plane P.

Equation of BM $\equiv \dfrac{x-3}{6/\sqrt{137}}=\dfrac{y}{10/\sqrt{137}}=\dfrac{z}{1/\sqrt{137}}=r$

where $r = |BM|$

then $\left(3+\frac{6r}{\sqrt{137}}\right), -2+\frac{10r}{\sqrt{137}}, 4+\frac{r}{\sqrt{137}}$ lies on the plane P

thus $6\left(3+\frac{6r}{\sqrt{137}}\right)+10\left(-2+\frac{10r}{\sqrt{137}}\right)+\left(4+\frac{r}{\sqrt{137}}\right)-11=0$

$$\Rightarrow \frac{137r}{\sqrt{137}}+18-20+4-11=0$$

$$\Rightarrow \sqrt{137}\, r = 9$$

$$\Rightarrow r = \frac{9}{\sqrt{137}}$$

2(C)(iii) Find the equation of the tangent plane at point $(1,1,1)$ to the conicoid $3x^2-y^2=2z$.

Sol'n: The equation of the tangent plane to the given conicoid at (α,β,γ) is

$$3x\alpha - y\beta - 2(z+\gamma)=0.$$

$\therefore$ The required tangent plane at $(1,1,1)$ is

$$3x(1)-y(1)-2(z+1)=0, \text{ Putting } \alpha=1, \beta=1, \gamma=1$$

$$3x-y-2z-1=0$$

3(a) Let V and W be two finite-dimensional vector spaces such that $\dim V = \dim W$ and $T: V \to W$ a linear transformation. Then the following conditions are equivalent.

(i) T is invertible (ii) T is non-singular (iii) T is onto

(iv) If $\{v_1, v_2 \ldots v_n\}$ is a basis of V, then $\{T(v_1), T(v_2), \ldots T(v_n)\}$ is a basis of W.

Sol'n: To prove (i) $\Rightarrow$ (ii)

Let T be invertible. Then T is one-to-one.

Let $v \in \text{Ker}\,T$ be arbitrary, so that $T(v) = 0$

$\Rightarrow T(v) = T(0)$, since T is a L.T.

$\Rightarrow v = 0$, since T is one-to-one

$\therefore \text{Ker}\,T = \{0\}$. Hence T is non-singular.

To prove (ii) $\Rightarrow$ (iii)

Let T be non-singular, so that $\text{Ker}\,T = \{0\}$.

It means $\dim \text{Ker}\,T = 0$. By Sylvester's law, we have

$$\dim \text{Range}\,T + \dim \text{Ker}\,T = \dim V$$

$$\Rightarrow \dim \text{Range}\,T = \dim V \quad (\because \dim \text{Ker}\,T = 0)$$

$$\Rightarrow \dim \text{Range}\,T = \dim W \quad (\because \dim V = \dim W),$$

where Range T is a subspace of W. Consequently,

Range $T = W$ i.e. $T(V) = W$. Hence T is onto.

To prove (iii) $\Rightarrow$ (i)

Let T be onto. Then $T(V) = W$ i.e, Range $T = W$

$\therefore \dim \text{Range}\,T = \dim W = \dim V$ —— (1)

By Sylvester's law, $\dim \text{Range}\,T + \dim \text{Ker}\,T = \dim V$ —— (2)

from (1) and (2), $\dim \text{Ker}\,T = 0$ ie, $\text{Ker}\,T = \{0\}$ —— (3)

Now we show that T is 1-1. Let $x, y \in V$ be such that

$T(x) = T(y) \Rightarrow T(x-y) = 0$, as T is a L.T

$\Rightarrow x - y \in \ker T = \{0\}$

$\Rightarrow x - y = 0$

$\Rightarrow x = y \Rightarrow T$ is one to one.

Since T is one-to-one and onto, T is invertible (by definition).

Hence (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (i).

Now we prove (i) $\Rightarrow$ (iv).

Let T be invertible. By definition, T is one-to-one and onto.

Thus T is an isomorphism of V onto W and $\dim V = \dim W$.

By theorem If T is an isomorphism of V onto W, then T maps a basis of V onto a basis of W. Condition (iv) is satisfied.

finally, we prove (iv) $\Rightarrow$ (i).

Let $\{v_1, v_2, ---, v_n\}$ be a basis of V. Then

$\{T(v_1), T(v_2), ---, T(v_n)\}$ is a basis of W, by Condition (iv).

Let $w \in W$ be arbitrary. we can write

$w = \alpha_1 T(v_1) + \alpha_2 T(v_2) + \cdots + \alpha_n T(v_n)$, $\alpha_i \in F$

$w = T(\alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n)$, as T is a L.T.

$w = T(v)$, where $v = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_n v_n \in V$

This means that T is onto and hence T is invertible. $[\because$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i)$]$

Remark: It is useful to remember that if $T: V \to W$ is a L.T. such that $\dim V = \dim W$. Then

T is invertible $\Leftrightarrow$ T is 1-1 $\Leftrightarrow$ T is non-singular $\Leftrightarrow$ T is onto.

Rule: T is invertible iff $\ker T = \{0\}$.

3(b)(i) If $u = \tan^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, Prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\sin 2u}{8}(2\cos^2 u - 3).$$

Sol'n: $u = \tan^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right) \Rightarrow \tan u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$

Let $z = \tan u$ ——— (1)

$$\therefore z = \frac{x+y}{\sqrt{x}+\sqrt{y}} = \frac{x(1+y/x)}{\sqrt{x}(1+\sqrt{y/x})} = x^{1/2} f(y/x)$$

So z is a homogeneous function in x & y of degree $\frac{1}{2}$.

By Euler's theorem, $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{1}{2}z$

$$\Rightarrow x\left(\sec^2 u\frac{\partial u}{\partial x}\right) + y\left(\sec^2 u\frac{\partial u}{\partial y}\right) = \frac{1}{2}\tan u, \text{ by (1)}.$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\frac{\tan u}{\sec^2 u} = \frac{1}{2}\sin u\cos u = \frac{1}{4}\sin 2u.$$

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{4}\sin 2u \text{ ——— (2)}$$

Partially differentiating (2) w.r.t x, on both sides, we get

$$\frac{\partial u}{\partial x} + x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x\partial y} = \frac{1}{2}\cos 2u\frac{\partial u}{\partial x}.$$

$$\Rightarrow x\frac{\partial u}{\partial x} + x^2\frac{\partial^2 u}{\partial x^2} + xy\frac{\partial^2 u}{\partial x\partial y} = \frac{x}{2}\cos 2u.\frac{\partial u}{\partial x}. \text{ ——— (3)}$$

partially diff (3) w.r.t y, on both sides, we get

$$x\frac{\partial^2 u}{\partial x\partial y} + \frac{\partial u}{\partial y} + y\frac{\partial^2 u}{\partial y^2} = \frac{1}{2}\cos 2u\frac{\partial u}{\partial y}.$$

$$\Rightarrow xy\frac{\partial^2 u}{\partial x\partial y} + y\frac{\partial u}{\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{y}{2}\cos 2u\frac{\partial u}{\partial y} \text{ ——— (4)}$$

Adding (3) & (4), we obtain

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}.$$
$$=\frac{\cos 2u}{2}\left[x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right]$$

Using (1), we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}+\frac{1}{4}\sin 2u$$
$$=\frac{\cos 2u}{2}\cdot\frac{1}{4}\sin 2u.$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{1}{4}\sin 2u\left[\frac{\cos 2u}{2}-1\right]$$
$$=\frac{1}{4}\sin 2u\left[\frac{2\cos^2 u-1}{2}-1\right]$$
$$=\frac{\sin 2u}{4}\left[\frac{2\cos^2 u-3}{2}\right]$$
$$=\frac{\sin 2u}{8}\left[2\cos^2 u-3\right]$$

3(b)(ii) For the function

$$f(x,y) = \begin{cases} \dfrac{x^2 - x\sqrt{y}}{x^2+y} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$$

Examine the continuity and differentiability

Sol) let us approach (0,0) along the path $y = x^4$

then $\underset{(x,y)\to(0,0)}{Lt} f(x,y) = \underset{x\to 0}{Lt} \dfrac{x^2 - x(x^2)}{x^2 + x^4}$

$= \underset{x\to 0}{Lt} \dfrac{x^2(1-x)}{x^2(1+x^2)}$

$= \underset{x\to 0}{Lt} \dfrac{1-x}{1+x^2} = 1.$

$\neq f(0,0).$

$\therefore$ $f(x,y)$ is not continuous at $(0,0)$

$\therefore$ f is not differentiable at $(0,0)$.

3(c) Show that the enveloping cylinders of the ellipsoid $ax^2+by^2+cz^2=1$ with generators perpendicular to z-axis meet the plane $z=0$ in parabolas.

Sol^n: The d.c's of the z-axis are 0,0,1

$\therefore$ The d.r's of the line perpendicular to z-axis are $l, m, 0$.

Let $P(\alpha,\beta,\gamma)$ be a point on the enveloping cylinder. Then the equations of the generator through $P(\alpha,\beta,\gamma)$ are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{0}=r \text{ (say)}$$

Any point on it is $(\alpha+lr, \beta+mr, \gamma)$

If this point lies on the given conicoid, we get

$$a(\alpha+lr)^2+b(\beta+mr)^2+c(\gamma)^2=1$$

$$\Rightarrow r^2(al^2+bm^2)+2r(a\alpha l+b\beta m)+(a\alpha^2+b\beta^2+c\gamma^2-1)=0 \quad \text{—(1)}$$

Since this generator is tangent to the given conicoid so the two values of r obtained from (1) must be equal and the condition for the same is

$$(a\alpha l+b\beta m)^2=(al^2+bm^2)(a\alpha^2+b\beta^2+c\gamma^2-1)$$

$\therefore$ The equation of the enveloping cylinder of the given conicoid i.e. the locus of $P(\alpha,\beta,\gamma)$ is

$$(alx+bmy)^2=(al^2+bm^2)(ax^2+by^2+cz^2-1)$$

Its section by the plane $z=0$ is

$$(alx+bmy)^2=(al^2+bm^2)(ax^2+by^2-1),\ z=0$$

$$\Rightarrow a^2l^2x^2+b^2m^2y^2+2ablmxy=a^2l^2x^2+abl^2y^2-al^2+abm^2x^2+b^2m^2y^2-bm^2;\ z=0$$

$$\Rightarrow ab(m^2x^2+l^2y^2-2lmxy)=al^2+bm^2,\ z=0.$$

$$\Rightarrow ab(mx-ly)^2=al^2+bm^2,\ z=0$$

which represents a parabola as the second degree terms form a perfect square.

4(a) Let T be the linear operator on R^3 which is represented in the standard ordered basis by the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix}.$$ Find the minimal polynomial for T.

Sol^n: The characteristic equation of T is $|A-\lambda I|=0$

$$\begin{vmatrix} 2-\lambda & 1 & 0 \\ 0 & 1-\lambda & -1 \\ 0 & 2 & 4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)\left[(1-\lambda)(4-\lambda)+2\right]=0$$

$$\Rightarrow (2-\lambda)(\lambda-2)(\lambda-3)=0$$

$$\Rightarrow \lambda = 2,2,3$$

Hence the characteristic values of T are 2, 2, 3.

The characteristic vector corresponding to $\lambda=2$ is given by $(A-2I)X=0$.

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad R_3 \to R_3 + 2R_2$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \quad R_2 \to R_2 + R_1$$

$\Rightarrow x_2 = 0,\ x_3 = 0$ and x_1 can be given any value

we take $x_1 = 1,\ x_2 = 0,\ x_3 = 0$.

Clearly, there is only one L.I vector corresponding to the characteristic value 2.

$$x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Thus the geometric multiplicity of the eigen value 2 is one while its algebraic multiplicity is 2. Since the geometric multiplicity of this eigen value is not equal to its algebraic multiplicity therefore A is not similar to a diagonal matrix.

i.e. T is not diagonalizable.

we know that the minimal polynomial for T divides its characteristic polynomial.

Thus the possible minimal polynomials for T can be either

$$P(\lambda) = (3-\lambda)(\lambda-2) \text{ (or) } (\lambda-2)^2(3-\lambda)$$

Let us take $P(\lambda) = (3-\lambda)(\lambda-2)$

we have

$$P(A) = (3I-A)(A-2I)$$

$$= \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & 1 \\ 0 & -2 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & -1 \\ 0 & 2 & 2 \end{bmatrix}$$

$$P(A) = \begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \neq 0$$

this shows that $P(\lambda) = (3-\lambda)(\lambda-2)$ is not the minimal polynomial for T.

Hence the minimal polynomial for T is

$$P(\lambda) = (3-\lambda)(\lambda-2)^2$$

which is same as the characteristic polynomial of T.

4(c) By using Lagrange's Multipliers method find the maximum and minimum values of $f(x,y,z) = y^2 - 10z$ subject to the constraint $x^2+y^2+z^2 = 36$.

Sol^n Given that $f(x,y,z) = y^2 - 10z$

subject to

$$x^2+y^2+z^2 = 36. \quad \text{---} (1)$$

Let us consider a function f of independent variables x, y, z.

where $F = y^2 - 10z + \lambda(x^2+y^2+z^2-36)$.

$dF = 2\lambda x\,dx + 2y(1+\lambda)dy + (-10+2z\lambda)dz$

for stationary points.

$f_x = 0 \Rightarrow 2\lambda x = 0$ —(i)

$f_y = 0 \Rightarrow 2y(1+\lambda) = 0$ —(ii)

$f_z = 0 \Rightarrow -10+2z\lambda = 0$ —(iii) $\Rightarrow z \neq 0$ and $\lambda \neq 0$. otherwise $-10 = 0$ which is not true.

If $\lambda \neq 0$, from (i) $x = 0$.

and from (ii); $y = 0$ (or) $\lambda = -1$.

from (1), by putting $\underline{x=0, y=0}$, we get

$z^2 = 36 \Rightarrow \underline{z = \pm 6}$.

$\therefore$ The possible two extrema points are $(0,0,-6), (0,0,6)$

Let $\lambda = -1$ from (iii), $-10-2z = 0 \Rightarrow \boxed{z = -5}$.

and from (i) $x = 0$

$\therefore$ from (1), $0 + y^2 + 25 = 36 \Rightarrow y = \pm\sqrt{11}$

∴ The two more potential absolute extrema

$(0, -\sqrt{11}, -5), (0, \sqrt{11}, -5)$

$f(0,0,-6) = 60$; $f(0,0,6) = -60$

$f(0,-\sqrt{11},-5) = 61$; $f(0,\sqrt{11},-5) = 61$.

The absolute maximum is 61 which occurs $(0, \pm\sqrt{11}, -5)$

and the absolute minimum is -60 which occurs at $(0, 0, 6)$

4(d) Show that the generators through any one of the ends of an equiconjugate diameter of the principal elliptic section of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ are inclined to each other at an angle of 60° if $a^2+b^2=6c^2$. Find also the condition for the generators to be ⊥ar to each other.

Sol'n: Let the point on the diameter as $(a\cos\theta, b\sin\theta, 0)$.

So the two generators through this point are

$$\frac{x-a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{-b\cos\theta}=\frac{z}{\pm c}$$

So the direction ratio of two generators are $(a\sin\theta, -b\cos\theta, c)$ and $(a\sin\theta, -b\cos\theta, -c)$ respectively.

$\alpha=60^\circ$ is the angle between two generators. θ the parameter of end points of conjugal diameters.

$$\cos\alpha=\frac{l_1l_2+m_1m_2+n_1n_2}{\sqrt{l_1^2+m_1^2+n_1^2}\sqrt{l_2^2+m_2^2+n_2^2}}$$

$$\Rightarrow \cos\alpha=\frac{a^2\sin^2\theta+b^2\cos^2\theta-c^2}{a^2\sin^2\theta+b^2\cos^2\theta+c^2}$$

Putting $\alpha=60^\circ$ and $\theta=45^\circ$

($\because$ Equi conjugal diameters means equal length of conjugal diameters.

i.e. $\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}=\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}$

(which is possible for $\theta=45^\circ$)

Putting values of θ and α

$$\Rightarrow a^2+b^2=6c^2$$

5(a) Solve the differential equation

$$x^2\frac{d^3y}{dx^3} + 2x\frac{d^2y}{dx^2} + 2\frac{y}{x} = 10\left(1+\frac{1}{x^2}\right)$$

Sol'n: The given equation can be written as

$$x^3\frac{d^3y}{dx^3} + 2x^2\frac{d^2y}{dx^2} + 2y = 10\left(x+\frac{1}{x}\right)$$

$$\Rightarrow (x^3D^3 + 2x^2D^2 + 2)y = 10\left(x+\frac{1}{x}\right), \text{ where } D = \frac{d}{dx} \quad \text{---} (1)$$

Let $x = e^z$ so that $z = \log x$ and $D_1 = \frac{d}{dz}$ — (2)

Then $xD = D_1$, $x^2D^2 = D_1(D_1-1)$, $x^3D^3 = D_1(D_1-1)(D_1-2)$ — (3)

using (2) and (3), (1) reduces to

$$[D_1(D_1-1)(D_1-2) + 2D_1(D_1-1) + 2]y = 10(e^z + e^{-z})$$

$$\Rightarrow (D_1^3 - D_1^2 + 2)y = 10e^z + 10e^{-z} \quad \text{---} (4)$$

A.E of (4) is $D_1^3 - D_1^2 + 2 = 0$

$$\Rightarrow (D_1+1)(D_1^2 - 2D_1 + 2) = 0$$

$$\Rightarrow D_1 = -1,\ 1 \pm i$$

$$\therefore C.F = c_1e^{-x} + e^z(c_1\cos z + c_2\sin z)$$

$$= c_1x^{-1} + x\,(c_1\cos\log x + c_2\sin\log x)$$

P.I corresponding to $10e^z$

$$= 10\frac{1}{(D_1+1)(D_1^2-2D_1+2)}e^z$$

$$= 10\frac{1}{2(1-2+2)}e^z = 5x \text{ by } (2).$$

and P.I corresponding to $10e^{-z}$

$$= 10 \frac{1}{(D_1+1)(D_1^2-2D_1+2)} e^{-z}$$

$$= 10 \frac{1}{(D_1+1)} \frac{1}{1+2+2} e^{-z}$$

$$= 2 \frac{1}{D_1+1} e^{-z}.1$$

$$= 2e^{-z} \frac{1}{D_1-1+1}.1$$

$$= 2e^{-z} \frac{1}{D_1}.1$$

$$= 2e^{-z}. z$$

$$= 2x^{-1} \log x \text{ , by } (2)$$

$$\therefore Y = c_1 x^{-1} + x(c_1 \cos \log x + c_2 \sin \log x) + 5x + 2x^{-1} \log x.$$

5(b) Use the method of variation of parameters to find the general solution of $x^2y''-4xy'+6y=-x^4\sin x$.

Sol'n: Given that $x^2y''-4xy'+6y=-x^4\sin x$

The given equation in standard form $y_2+Py_1+Qy=R$

is $y''-\frac{4y'}{x}+\frac{6}{x^2}y=-x^2\sin x$

Consider $y''-\frac{4}{x}y'+\frac{6}{x^2}y=0$

$\Rightarrow x^2y''-4xy'+6y=0$

$\Rightarrow (x^2D^2-4xD+6)y=0$, $D\equiv\frac{d}{dx}$ ——— (2)

which is a homogeneous equation.

Putting $x=e^z$ and $D_1=\frac{d}{dz}$

$\Rightarrow \log x=z$

Then from (2), we have

$[D_1(D_1-1)-4D_1+6]y=0$

$(D_1^2-D_1-4D_1+6)y=0$

$(D_1^2-5D_1+6)y=0$ ——— (3)

Auxiliary equation of (3) is $D_1^2-5D_1+6=0$

$\Rightarrow (D_1-2)(D_1-3)=0$

$\Rightarrow D_1=2,3$

$\therefore$ The complementary function of (3)

is $y_c=c_1e^{2z}+c_2e^{3z}$

$\Rightarrow y_c=c_1x^2+c_2x^3$ $(\because x=e^z)$

Now let $u=x^2$, $v=x^3$ & $-x^2\sin x$

Let $\boxed{y_p=Au+Bv}$ be a particular integral of (1) where A and B are functions of x.

and $u = x^2$, $v = x^3$.

Now $\begin{vmatrix} u & u' \\ v & v' \end{vmatrix} = uv' - u'v$
$= x^2(3x^2) - 2x(x^3) = 3x^4 - 2x^4 = x^4 \neq 0.$

Now $A = \int \frac{-vR}{uv' - u'v} dx = \int \frac{-x^3(-x^2 \sin x)}{x^4} dx$

$= \int x \sin x \, dx$

$= \left[x(-\cos x) - \int 1(-\cos x) dx\right]$

$= -x\cos x + \sin x$

and $B = \int \frac{uR}{uv' - u'v} dx$

$= \int \frac{x^2(-x^2 \sin x)}{x^4} dx$

$= -\int \sin x \, dx = \cos x$

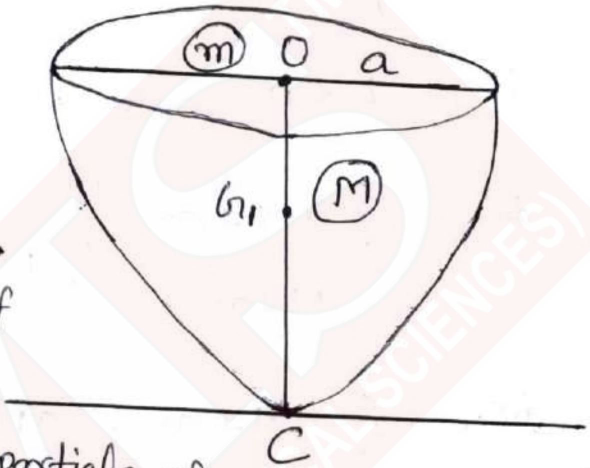
$\therefore y_p = x^2[-x\cos x + \sin x] + x^3 \cos x = x^2 \sin x$

$\therefore$ The general solution of (1) is

$\boxed{y = c_1 x^2 + c_2 x^3 + x^2 \sin x}$

5(C) → A uniform solid hemisphere rests in equilibrium upon a rough horizontal plane with its curved surface in contact with the plane and a particle of mass m is fixed at the centre of the plane face. Show that for any value of m, the equilibrium is stable.

Sol: C is the point of contact of the hemisphere and the plane and O is the centre of the base of the hemisphere. Let M be the mass of the hemisphere and



a be the radius. A particle of mass m is placed at O. The mass M of the hemisphere acts at G_1 where $OG_1 = 3a/8$.

If h be the height of the centre of gravity of the combined body consisting of the hemisphere and the mass m above the point of contact C, then

$$h = \frac{M\cdot\frac{5}{8}a + m\cdot a}{M+m}$$

Here ρ_1 = the radius of curvature of the upper body at the point of contact $C = a$;

and ρ_2 = the radius of curvature of the lower body at the point of contact $C = \infty$.

The equilibrium will be stable if

$$\frac{1}{h} > \frac{1}{\rho_1} + \frac{1}{\rho_2}$$

i.e., $\frac{1}{h} > \frac{1}{a} + \frac{1}{\infty}$

i.e., $\frac{1}{h} > \frac{1}{a}$

i.e., $h < a$

i.e., $\dfrac{\frac{5}{8}aM + am}{M+m} < a$

i.e., $\frac{5}{8}aM + am < aM + am$

i.e., $\frac{5}{8}aM < aM$

i.e., $\frac{5}{8}a < a$, which is so whatever may be the value of m. Hence for any value of m, the equilibrium is stable.

5(d) Prove that curl $(A\times B) = (B\cdot\nabla)A - B\,\mathrm{div}\,A - (A\cdot\nabla)B + A\,\mathrm{div}\,B$.

Sol^n: we have curl $(A\times B) = \nabla\times(A\times B)$

$$= \Sigma\left\{i\times\frac{\partial}{\partial x}(A\times B)\right\} = \Sigma\left\{i\times\left(A\times\frac{\partial B}{\partial x} + \frac{\partial A}{\partial x}\times B\right)\right\}$$

$$= \Sigma\left\{i\times\left(A\times\frac{\partial B}{\partial x}\right)\right\} + \Sigma\left\{i\times\left(\frac{\partial A}{\partial x}\times B\right)\right\}$$

$$= \Sigma\left\{\left(i\cdot\frac{\partial B}{\partial x}\right)A - (i\cdot A)\frac{\partial B}{\partial x}\right\} + \Sigma\left\{(i\cdot B)\frac{\partial A}{\partial x} - \left(i\cdot\frac{\partial A}{\partial x}\right)B\right\}$$

$$= \Sigma\left\{\left(i\cdot\frac{\partial B}{\partial x}\right)A\right\} - \Sigma\left\{(A\cdot i)\frac{\partial B}{\partial x}\right\} + \Sigma\left\{(B\cdot i)\frac{\partial A}{\partial x}\right\} - \Sigma\left\{\left(i\cdot\frac{\partial A}{\partial x}\right)B\right\}$$

$$= \left\{\Sigma\left(i\cdot\frac{\partial B}{\partial x}\right)\right\}A - \left\{A\cdot\Sigma i\frac{\partial}{\partial x}\right\}B + \left\{B\cdot\Sigma i\frac{\partial}{\partial x}\right\}A - \left\{\Sigma\left(i\cdot\frac{\partial A}{\partial x}\right)\right\}B$$

$$= (\mathrm{div}\,B)A - (A\cdot\nabla)B + (B\cdot\nabla)A - (\mathrm{div}\,A)B.$$

5(e) Determine $\int_C (y\,dx + z\,dy + x\,dz)$ by using stoke's theorem, where 'C' is the curve defined by $(x-a)^2 + (y-a)^2 + z^2 = 2a^2$, $x+y=2a$ that starts from the point $(2a,0,0)$ and goes at first below the z-plane.

Sol^n: The Centre of the Sphere

$x^2+y^2+z^2-2ax-2ay=0$ is the point $(a,a,0)$

Since the plane $x+y=2a$ passes through the point $(a,a,0)$.

$\therefore$ The Circle C is great Circle of this sphere.

$\therefore$ Radius of the Circle = Radius of the Sphere

$$= \sqrt{a^2+a^2}$$
$$= \sqrt{2a^2}$$
$$= \sqrt{2}a$$

Now $\int_C y\,dx + z\,dy + x\,dz = \int (yi + zj + xk)\cdot dr$

$$= \iint_S [\text{Curl}\,(yi + zj + xk)]\cdot \hat{n}\,ds$$

$\left(\because \int_C F\cdot dr = \iint_S \text{Curl}\,F\cdot\hat{n}\,ds \text{ by stoke's theorem.}\right)$

where S is any surface of which Circle C is boundary.

Now Curl $(xi+yj+zk) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix}$

$$= -i-j-k = -(i+j+k)$$

Let us take S as the surface of the plane $x+y=2a$ bounded by the circle 'C'.

Then a vector normal to S is

$$\text{grad}\,(x+y) = i+j$$

$\therefore\ \hat{n}$ = unit normal to $S = \frac{1}{\sqrt{2}}(i+j)$

$$\therefore \iint_C y\,dx + z\,dy + x\,dz = \iint_S -(i+j+k)\cdot\left(\frac{i}{\sqrt{2}}+\frac{j}{\sqrt{2}}\right)ds$$

$$= \frac{-2}{\sqrt{2}}\iint ds$$

$= -\sqrt{2}$ (area of the circle of radius $a\sqrt{2}$)

$$= -\sqrt{2}\left[\pi(a\sqrt{2})^2\right]$$

$$= -\sqrt{2}\,\pi a^2(2)$$

$$= -2\sqrt{2}\,\pi a^2$$

6(a) verify that $\frac{1}{2}(Mx+Ny)d(\log_e(xy))+\frac{1}{2}(Mx-Ny)d\left(\log_e\left(\frac{x}{y}\right)\right)=Mdx+Ndy$

Hence show that

(i) If the differential equation $Mdx+Ndy=0$ is homogeneous, then $(Mx+Ny)$ is an integrating factor. unless $Mx+Ny\equiv 0$.

(ii) If the differential equation $Mdx+Ndy=0$ is not exact but is of the form.

$$f_1(xy)y\,dx+f_2(xy)x\,dy=0.$$

then $(Mx-Ny)^{-1}$ is an integrating factor unless $Mx+Ny=0$.

sol'n: The given equation is $Mdx+Ndy=0$ —— ①

where M and N are homogeneous functions of the same degree in x and y.

we have

$$Mdx+Ndy=\frac{1}{2}\left[(Mx+Ny)\left(\frac{dx}{x}+\frac{dy}{y}\right)+(Mx-Ny)\left(\frac{dx}{x}-\frac{dy}{y}\right)\right]$$

$$\Rightarrow Mdx+Ndy=\frac{1}{2}\left[(Mx+Ny)\,d\,(\log xy)+(Mx-Ny)\,d\log\left(\frac{x}{y}\right)\right]$$

Now dividing by $Mx+Ny$ (which is $\neq 0$)

$$\frac{Mdx+Ndy}{Mx+Ny}=\frac{1}{2}\left[d\,(\log(xy))+\frac{Mx-Ny}{Mx+Ny}\,d\left(\log\frac{x}{y}\right)\right]$$

Since M and N are homogeneous functions of the same degree in x and y.

The expression $\frac{Mx-Ny}{Mx+Ny}$ is homogeneous and equal to a function of x/y say $f(x/y)$

$$\therefore \frac{Mdx+Ndy}{Mx+Ny}=\frac{1}{2}d(\log xy)+\frac{1}{2}f\left(\frac{x}{y}\right)d\left(\log\left(\frac{x}{y}\right)\right)$$

(i) $\Rightarrow \frac{x}{y} = e^{\log\left(\frac{x}{y}\right)}$

$\Rightarrow f\left(\frac{x}{y}\right) = f\left(e^{\log\left(\frac{x}{y}\right)}\right) = F\left(\log\left(\frac{x}{y}\right)\right)$

$$\Rightarrow \frac{Mdx+Ndy}{Mx+Ny} = \frac{1}{2}d(\log xy) + \frac{1}{2}F\left(\log\frac{x}{y}\right)d\left(\log\left(\frac{x}{y}\right)\right)$$

which is an exact differential equation.

$$\Rightarrow \frac{Mdx+Ndy}{Mx+Ny} = 0$$

$\Rightarrow \frac{M}{Mx+Ny}dx + \frac{N}{Mx+Ny}dy = 0$ is an exact differential equation.

(ii) The given equation is $Mdx + Ndy = 0$ —— (1)

where $M = yf_1(xy)$; $N = xf_2(xy)$

Now we have

$$Mdx + Ndy = \frac{1}{2}\left[(Mx+Ny)\left(\frac{dx}{x}+\frac{dy}{y}\right) + (Mx-Ny)\left(\frac{dx}{x}-\frac{dy}{y}\right)\right]$$

$$\Rightarrow Mdx+Ndy = \frac{1}{2}\left[(Mx+Ny)d(\log xy) + (Mx-Ny)d\left(\log\left(\frac{x}{y}\right)\right)\right]$$

Dividing by $Mx-Ny$, $(Mx-Ny \neq 0)$ we get,

$$\frac{Mdx+Ndy}{Mx-Ny} = \frac{1}{2}\left[\frac{Mx+Ny}{Mx-Ny}d(\log(xy)) + d\left(\log\left(\frac{x}{y}\right)\right)\right]$$

$$= \frac{1}{2}\left[\frac{xyf_1(xy)+xyf_2(xy)}{xyf_1(xy)-xyf_2(xy)}.d(\log(xy)) + d\left(\log\left(\frac{x}{y}\right)\right)\right]$$

Since $xy = e^{\log xy}$

$$\Rightarrow f(xy) = f(e^{\log xy}) = F(\log xy)$$

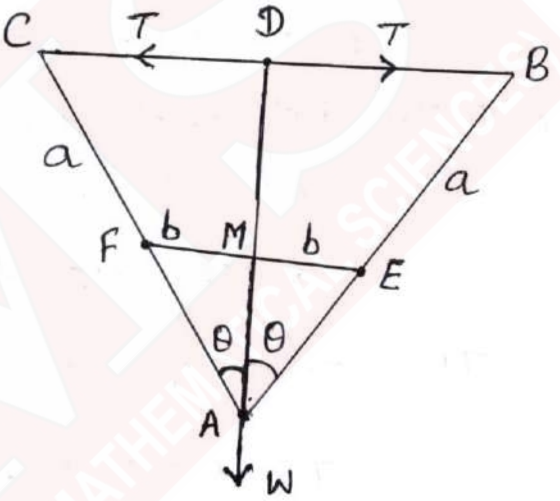
$$\therefore \frac{M\,dx + N\,dy}{Mx - Ny} = 0$$

$$\Rightarrow \frac{M}{Mx-Ny}\,dx + \frac{N}{Mx-Ny}\,dy = 0$$ is an exact differential equation.

$\therefore \frac{1}{Mx-Ny}$ is an integrating factor.

6(b) A frame ABC consists of three light rods, of which AB, AC are each of length a, BC of length $\frac{3}{2}a$, freely jointed together. It rests with BC horizontal, A below BC and the rods AB, AC over two smooth pegs E and F, in the same horizontal line, distant $2b$ apart.
A weight W is suspended from A, find the thrust in the rod BC.

Sol: ABC is a framework consisting of three light rods AB, AC and BC. The rods AB and AC rest on two smooth pegs E and F which are in the same horizontal line and $EF = 2b$.



Each of the rods AB and AC is of length a. Let T be the thrust in the rod BC which is given to be of length $\frac{3}{2}a$. A weight W is suspended from A. The line AD joining A to the middle point D of BC is vertical. Let

$$\angle BAD = \theta = \angle CAD.$$

Replace the rod BC by two equal and opposite forces T as shown in the above figure. Now give the system a small symmetrical displacement in which θ changes to $\theta+\delta\theta$. The line EF joining the pegs remains fixed, the lengths of the rods AB and AC do not change and the length BC changes.

The forces contributing to the sum of virtual works are: (i) the thrust T in the rod BC, and
(ii) the weight W acting at A.

We have,

$$BC = 2BD = 2AB\sin\theta = 2a\sin\theta.$$

Also the depth of the point of application A of the weight W below the fixed line EF

$$= MA = ME\cot\theta = b\cot\theta.$$

The equation of virtual work is

$$T\delta(2a\sin\theta) + W\delta(b\cot\theta) = 0$$

or, $$2aT\cos\theta\,\delta\theta - bW\operatorname{cosec}^2\theta\,\delta\theta = 0$$

or, $$(2aT\cos\theta - bW\operatorname{cosec}^2\theta)\,\delta\theta = 0$$

or, $$2aT\cos\theta - bW\operatorname{cosec}^2\theta = 0 \quad [\because \delta\theta \neq 0]$$

or, $$2aT\cos\theta = bW\operatorname{cosec}^2\theta$$

or, $$T = \frac{Wb}{2a}\operatorname{cosec}^2\theta\sec\theta$$

But in the position of equilibrium,

$$BC = \frac{3}{2}a \quad \text{and so} \quad BD = \frac{3}{4}a.$$

Therefore $\sin\theta = \dfrac{BD}{AB} = \dfrac{\frac{3}{4}a}{a} = \dfrac{3}{4}$

and
$$\cos\theta = \sqrt{(1-\sin^2\theta)}$$
$$= \sqrt{\left(1-\frac{9}{16}\right)}$$
$$= \frac{1}{4}\sqrt{7}.$$

$$\therefore\ T = \frac{Wb}{2a}\cdot\frac{16}{9}\cdot\frac{4}{\sqrt{7}}$$
$$= \frac{32}{9\sqrt{7}}\cdot\frac{b}{a}W$$

6(c)(i). If the acceleration of an object is given by $\vec{a} = \vec{i} + 2\vec{j} + 6t\vec{k}$, find the object's velocity and position functions given that the initial velocity is $\vec{v}(0) = \vec{j} - \vec{k}$ and the initial position is $\vec{r}(0) = \vec{i} - 2\vec{j} + 3\vec{k}$.

Sol'n: Integrating the equation

$$\vec{a} = i + 2j + 6tk,$$

we get, $\vec{v}(t) = \int \vec{a}(t)\,dt = \int(\hat{i} + 2\hat{j} + 6t\hat{k})\,dt$

$$\vec{v}(t) = t\hat{i} + 2t\hat{j} + 3t^2\hat{k} + \vec{b}$$

where $\vec{b}$ is any arbitrary constant vector.

But it is given that when $t=0$, $v(0) = j - k = \vec{b}$

$$\Rightarrow \vec{b} = j - k.$$

$$\therefore v(t) = t\hat{i} + (2t+1)\hat{j} + (3t^2-1)\hat{k}$$

Integrating again w.r.t. t, i.e., $\vec{r} = \int v(t)\,dt$ we get

$$\vec{r}(t) = \frac{t^2}{2}\hat{i} + (t^2+t)\hat{j} + (t^3-t)\hat{k} + \vec{c}$$

where $\vec{c}$ is any arbitrary constant.

Given that $\vec{r}(0) = \hat{i} - 2\hat{j} + 3\hat{k} = \vec{c}$

$$\Rightarrow \vec{c} = \hat{i} - 2\hat{j} + 3\hat{k}$$

$$\therefore \vec{r}(t) = \left(\frac{t^2}{2}+1\right)\hat{i} + (t^2+t-2)\hat{j} + (t^3-t+3)\hat{k}.$$

6(c)(ii) If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, find the value(s) of n in order that $r^n \vec{r}$ may be (i) Solenoidal (ii) irrotational

Sol'n: Let $F = r^n \vec{r}$

The vector F is irrotational if $\nabla \times F = 0$

i.e. $\nabla \times (r^n \vec{r}) = 0$

we know that $\text{Curl}(\phi A) = (\nabla\phi) \times A + \phi \,\text{Curl}\, A$ — (1)

putting $\phi = r^n$ and $A = \vec{r}$ in (1) we get

$$\text{Curl}(r^n \vec{r}) = (\nabla r^n) \times \vec{r} + r^n (\text{Curl}\, \vec{r}) \quad \text{—— (2)}$$

Now $\nabla r^n = \sum i \frac{\partial}{\partial x} r^n$

$$= \sum i \, n r^{n-1} \frac{\partial r}{\partial x} = n r^{n-1} \sum i \frac{\partial r}{\partial x} \qquad \left(\because r = \sqrt{x^2+y^2+z^2} \right.$$

$$= n r^{n-1} \sum i \frac{x}{r} = n r^{n-1} \frac{1}{r} \sum i x \qquad \frac{\partial r}{\partial x} = \frac{1}{\cancel{2}\sqrt{x^2+y^2+z^2}} \cdot \cancel{2}x$$

$$= n r^{n-1} \frac{1}{r} \vec{r} \qquad \left. = \frac{x}{r} \right)$$

$$= n r^{n-2} \vec{r} \quad \text{(3)}$$

Also $\text{Curl}\, \vec{r} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & y & z \end{vmatrix} = i(0) + j(0) + k(0) = 0$

$\therefore$ from (2), we have

$$\text{Curl}(r^n \vec{r}) = (n r^{n-2} \vec{r}) \times \vec{r} + r^n (0)$$

$$= n r^{n-2} (\vec{r} \times \vec{r}) = 0$$

$\therefore \text{Curl}(r^n \vec{r}) = 0$ for any value of n.

$\therefore r^n \vec{r}$ is an irrotational vector for any value of n.

The vector F is solenoidal if $\text{div}\, F = 0$

i.e., $\text{div}\, r^n \vec{r} = 0$.

we know that $\text{div}(\phi A) = \phi \,\text{div} A + A \cdot \text{grad}\, \phi$ — (1).

putting $A = \vec{r}$ and $\phi = r^n$ in (1)

we get

$$\text{div}(r^n\vec{r}) = r^n \text{div}\,\vec{r} + \vec{r}\cdot \text{grad}\, r^n$$

$$= r^n(3) + \vec{r}\cdot n r^{n-2}\vec{r} \quad (\text{from } 3 \quad (\text{div}\,\vec{r} = 3))$$

$$= 3r^n + n(r^{n-2})(\vec{r}\cdot\vec{r})$$

$$= 3r^n + nr^n$$

$$= (n+3)r^n$$

$\therefore$ The vector $r^n\vec{r}$ is solenoidal if

$$(n+3)r^n = 0$$

i.e. only if $n+3=0 \Rightarrow \boxed{n=-3}$

$\therefore$ The vector $r^n\vec{r}$ is solenoidal if $\boxed{n=-3}$ and irrotational for any value of n.

7(a)(i) Solve $(px^2+y^2)(px+y)=(p+1)^2$ by reducing it to Clairaut's form and find its singular solution.

Sol^n: Given differential equation is

$$(px^2+y^2)(px+y)=(p+1)^2.$$

put $u=xy$, $v=x+y$

$$\Rightarrow \frac{du}{dx}=y+x\frac{dy}{dx} \;;\; \frac{dv}{dx}=1+\frac{dy}{dx}=1+p$$

$$=y+xp$$

$$\Rightarrow \frac{dv}{du}=\frac{1+p}{y+xp}$$

$$\Rightarrow P=\frac{1+p}{y+xp} \quad \text{where } P=\frac{dv}{du}\;;\; p=\frac{dy}{dx}$$

$$\Rightarrow P(y+xp)=1+p$$

$$\Rightarrow p(xP-1)=1-Py \Rightarrow p=\frac{1-Py}{xP-1} \quad \text{--- (1)}$$

using (1), the given equation becomes

$$\left[\frac{1-Py}{(xP-1)}x^2+y^2\right]\left[\frac{(1-Py)x}{xP-1}+y\right]=\left[\frac{1-Py}{xP-1}+1\right]^2$$

$$\Rightarrow \left[x^2(1-Py)+y^2(xP-1)\right]\left[(1-Py)x+y(xP-1)\right]=(1-Py+xP-1)^2$$

$$\Rightarrow (x^2-Px^2y+y^2xP-y^2)(x-\cancel{Pxy}+\cancel{Pxy}-y)=P^2(x-y)^2$$

$$\Rightarrow \left[(x^2-y^2)-Pxy(x-y)\right](x-y)=P^2(x-y)^2$$

$$\Rightarrow (x-y)^2\left[(x+y)-Pxy\right]=P^2(x-y)^2$$

$$\Rightarrow x+y-Pxy=P^2$$

$$\Rightarrow v-Pu=P^2$$

$$\Rightarrow v=uP+P^2$$ which is a clairaut's equation.

$\therefore$ The general solution is

$$v=cu+c^2$$

i.e., $x+y=cxy+c^2$

7(a)(ii) Prove that the orthogonal trajectories $r^n \cos n\theta = a^n$ is $r^n \sin n\theta = c^n$.

Sol^n: Given $r^n \cos n\theta = a^n$, where a is a parameter. —— (1)

from (1), $n \log r + \log \cos n\theta = \log a^n$ —— (2)

Differentiating (2) $\frac{n}{r}\frac{dr}{d\theta} - n \tan n\theta = 0$

$\Rightarrow \frac{1}{r}\frac{dr}{d\theta} - \tan n\theta = 0$ —— (3)

which is the differential equation of family (1).

Replacing $dr/d\theta$ by $-r^2 (d\theta/dr)$ in (3), the differential equation of the required orthogonal trajectories is

$\frac{1}{r}(-r^2)(d\theta/dr) - \tan n\theta = 0$

$\Rightarrow \frac{1}{r} dr + \cot n\theta \, d\theta = 0$.

Integrating, $\log r + \frac{1}{n} \log \sin n\theta = \log c$, c being arbitrary constant.

$\Rightarrow n \log r + \log \sin n\theta = n \log c$

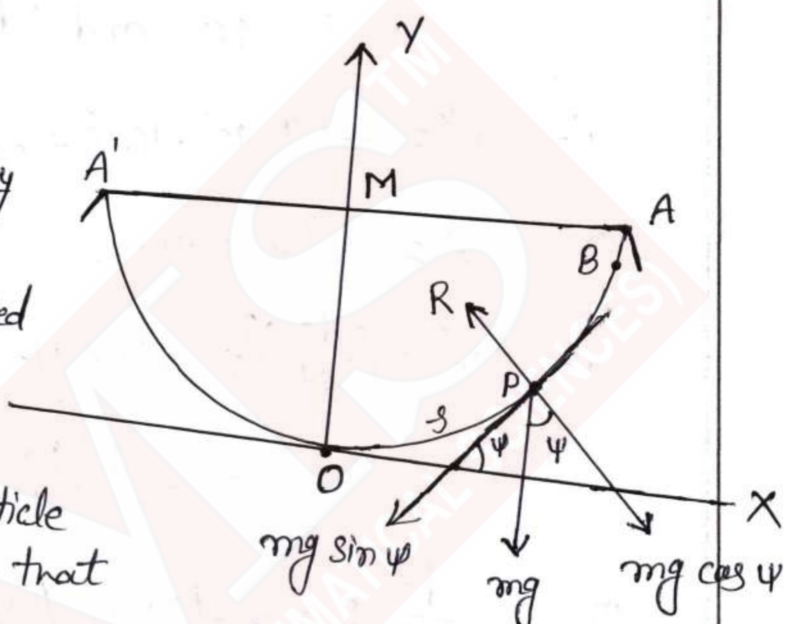
$\Rightarrow r^n \sin n\theta = c^n$

which is the required equation of orthogonal trajectories.

7(b) A particle is projected with velocity V from the cusp of a smooth inverted cycloid down the arc, show that the time of reaching the vertex is $2\sqrt{(a/g)}\tan^{-1}\left[\sqrt{(4ag)}/V\right]$.

Sol:

Let a particle be projected with velocity V from the cusp A of a smooth inverted cycloid down the arc. If P is the position of the particle at time t such that



the tangent at P is inclined at an angle ψ to the horizontal and arc OP = s, then the equations of motion of the particle are

$$m\frac{d^2s}{dt^2} = -mg\sin\psi \quad \text{——— (1)}$$

and $$m\frac{v^2}{\rho} = R - mg\cos\psi \quad \text{——— (2)}$$

for the cycloid, $s = 4a\sin\psi$ ——— (3)

From (1) and (3), we have

$$\frac{d^2s}{dt^2} = -\frac{g}{4a}s.$$

Multiplying both sides by $2(ds/dt)$ and Integrating, we have

$$v^2 = \left(\frac{ds}{dt}\right)^2 = -\frac{g}{4a}s^2 + A$$

But initially at the cusp A,

$$s = 4a \text{ and } (ds/dt)^2 = v^2$$

$$\therefore \quad v^2 = -(g/4a)\cdot 16a^2 + A$$

or $\quad A = v^2 + 4ag.$

$$\therefore \quad v^2 = \left(\frac{ds}{dt}\right)^2 = v^2 + 4ag - \frac{g}{4a}s^2$$

$$= \left(\frac{g}{4a}\right)\left[\frac{4a}{g}(v^2+4ag) - s^2\right]$$

or $\quad \dfrac{ds}{dt} = -\dfrac{1}{2}\sqrt{(g/a)}\sqrt{\left[\dfrac{4a}{g}(v^2+4ag) - s^2\right]}$

(-ive sign is taken because the particle is moving in the direction of s decreasing).

or $\quad dt = -2\sqrt{(a/g)}\dfrac{ds}{\sqrt{[(4a/g)(v^2+4ag) - s^2]}}$

Integrating, the time t_1 from the cusp A to the vertex O is given by

$$t_1 = -2\sqrt{(a/g)}\int_{s=4a}^{0}\frac{ds}{\sqrt{[(4a/g)(v^2+4ag) - s^2]}}$$

$$= 2\sqrt{(a/g)}\int_{0}^{4a}\frac{ds}{\sqrt{[(4a/g)(v^2+4ag) - s^2]}}$$

$$= 2\sqrt{(a/g)}\left[\sin^{-1}\frac{s}{2\sqrt{(a/g)}\sqrt{(v^2+4ag)}}\right]_0^{4a}$$

$$= 2\sqrt{(a/g)}\cdot\sin^{-1}\left\{\frac{2\sqrt{(ag)}}{\sqrt{(v^2+4ag)}}\right\}$$

$$= 2\sqrt{(a/g)}\cdot\theta, \quad \text{——— } (4)$$

where $\theta = \sin^{-1}\left\{\dfrac{2\sqrt{(ag)}}{\sqrt{(v^2+4ag)}}\right\}$

We have $\sin\theta = \dfrac{2\sqrt{(ag)}}{\sqrt{(v^2+4ag)}}$.

$$\therefore \cos\theta = \sqrt{(1-\sin^2\theta)}$$

$$= \sqrt{\left[1-\frac{4ag}{v^2+4ag}\right]}$$

$$= \frac{v}{\sqrt{(v^2+4ag)}}.$$

$$\therefore \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{2\sqrt{(ag)}}{v} = \frac{\sqrt{(4ag)}}{v}$$

or $\quad \theta = \tan^{-1}\left[\sqrt{(4ag)}/v\right]$.

$\therefore$ from (4), the time of reaching the vertex is

$$= 2\sqrt{(a/g)}\tan^{-1}\left[\sqrt{(4ag)}/v\right].$$

7(c)(i) Find the normal and binormal vectors for

$$\vec{r}(t) = \langle t, 3\sin t, 3\cos t \rangle$$

Sol'n: $\vec{r} = t\hat{i} + 3\sin t\hat{j} + 3\cos t\hat{k}$

$$\frac{d\vec{r}}{dt} = \hat{i} + 3\cos t\hat{j} - 3\sin t\hat{k}$$

$$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{1+9(1)} = \sqrt{10}$$

$$\therefore \hat{T} = \frac{d\vec{r}}{ds} = \frac{1}{\sqrt{10}}(\hat{i} + 3\cos t\hat{j} - 3\sin t\hat{k})$$

$$\frac{d\hat{T}}{dt} = \frac{-3\sin t\hat{j} - 3\cos t\hat{k}}{\sqrt{10}} \; ; \; \frac{d\hat{T}}{ds} = \frac{-3}{10}(\sin t\hat{j} + \cos t\hat{k})$$

$$\frac{d\hat{T}}{dt} = \frac{3}{10}$$

$$\therefore \hat{N} = \frac{1}{K}\frac{d\hat{T}}{ds} = \frac{1}{(3/10)}\left(\frac{-3}{10}\right)(\sin t\hat{j} + \cos t\hat{k})$$

$$\Rightarrow \hat{N} = -\sin t\hat{j} - \cos t\hat{k}$$

$$\hat{B} = \hat{T} \times \hat{N}$$

$$\therefore \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{1}{\sqrt{10}} & \frac{3\cos t}{\sqrt{10}} & \frac{-3\sin t}{\sqrt{10}} \\ 0 & -\sin t & -\cos t \end{vmatrix}$$

$$= \hat{i}\left(\frac{-3}{\sqrt{10}}\right) - \hat{j}\left(\frac{-\cos t}{\sqrt{10}}\right) + \hat{k}\left(-\frac{\sin t}{\sqrt{10}}\right)$$

$$\Rightarrow \hat{B} = \frac{1}{\sqrt{10}}(-3\hat{i} + \cos t\hat{j} - \sin t\hat{k})$$

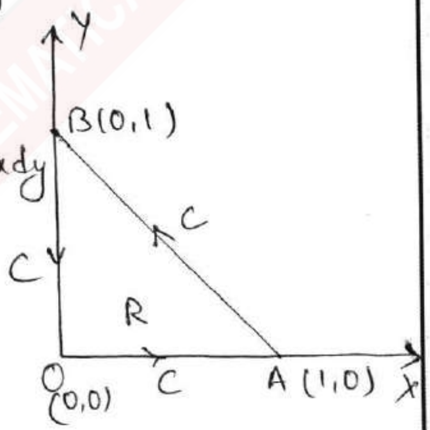
7(c)(ii) Verify Green's theorem in plane for $\oint_C [(3x^2-8y^2)dx+(4y-6xy)dy]$, where C is the boundary of the region defined by $x=0, y=0$ and $x+y=1$.

Sol'n: By Green's theorem in plane, we have.

$$\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy = \oint_C (M dx + N dy)$$

Here $M = 3x^2-8y^2$, $N = 4y-6xy$.

The closed curve C consists of the straight line OA, the st. line AB and straight line BO. The positive direction is traversing C is as shown in the figure and R is the region bounded by C.



we have $\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy$

$$= \iint_R \left[\frac{\partial}{\partial x}(4y-6xy)-\frac{\partial}{\partial y}(3x^2-8y^2)\right]dx\,dy$$

$$= \iint_R [-6y+16y]dx\,dy = 10\iint_R y\,dx\,dy$$

$$= 10\int_{x=0}^{1}\int_{y=0}^{1-x} y\,dx\,dy$$ [for the region R, x varies from 0 to 1 and y varies from 0 to $1-x$]

$$= 10\int_0^1 \left[\frac{y^2}{2}\right]_{y=0}^{1-x} dx$$, integrating with respect to y regarding x as constant.

$$= 5\int_0^1 (1-x)^2 dx = 5\int_0^1 (x-1)^2 dx = \frac{5}{3}\left[(x-1)^3\right]_0^1$$

$$= \frac{5}{3}\left[0-(-1)^3\right] = \frac{5}{3}$$

Now let us evaluate the line integral along the curve C.

Along the st. line OA, we have $y=0$, $dy=0$ and x varies from 0 to 1.

$\therefore$ line integral along $OA = \int_0^1 3x^2 dx = [x^3]_0^1 = 1$

Along the st. line AB, we have $x=1-y$, $dx=-dy$ and y varies from 0 to 1.

$\therefore$ line integral along AB

$$= \int_0^1 \left[\{3(1-y)^2 - 8y^2\}(-dy) + \{4y - 6y(1-y)\}dy\right]$$

$$= \int_0^1 \left[-3(1-2y+y^2) + 8y^2 + 4y - 6y + 6y^2\right] dy$$

$$= \int_0^1 (11y^2 + 4y - 3)dy = \left[\frac{11}{3}y^3 + 2y^2 - 3y\right]_0^1$$

$$= \frac{11}{3} + 2 - 3 = 8/3$$

Along the st. line BO, we have $x=0$, $dx=0$ and y varies from 1 to 0.

$\therefore$ line integral along $BO = \int_1^0 4y\,dy = 2[y^2]_1^0 = -2$

$\therefore$ Total line integral along the closed curve C.

$= 1 + 8/3 - 2$

$= 5/3$ ——— (2).

from (1) & (2), we see that Green's theorem is verified.

8(a). Solve the initial value problem

$$\frac{d^2y}{dt^2} + y = 8e^{-2t}\sin t, \; y(0)=0, \; y'(0)=0.$$

by using Laplace-transform.

Sol^n: Given equation is $\frac{d^2y}{dt^2} + y = 8e^{-2t}\sin t$

$$\Rightarrow y'' + y = 8e^{-2t}\sin t \quad \text{——— (1)}$$

Taking Laplace transform of bothsides of (1)

we get $L(y'') + L(y) = 8L(e^{-2t}\sin t)$

$$\Rightarrow p^2 L\{y(t)\} - p\,y(0) - y'(0) + L\{y(t)\} = \frac{8}{(p+2)^2+1}$$

$$\Rightarrow p^2 L\{y(t)\} + L\{y(t)\} = \frac{8}{p^2+4p+5}$$

$$\Rightarrow L\{y(t)\}(p^2+1) = \frac{8}{p^2+4p+5}$$

$$\Rightarrow L\{y(t)\} = \frac{8}{(p^2+1)(p^2+4p+5)}$$

$$\Rightarrow y(t) = L^{-1}\left\{\frac{8}{(p^2+1)(p^2+4p+5)}\right\}$$

$$y(t) = L^{-1}\left[\frac{-p+1}{p^2+1} + \frac{p+3}{p^2+4p+5}\right]$$

$$= L^{-1}\left(\frac{-p}{p^2+1}\right) + L^{-1}\left(\frac{1}{p^2+1}\right) + L^{-1}\left(\frac{(p+2)+1}{(p+2)^2+1}\right)$$

$$= -\cos t + \sin t + e^{-2t} L^{-1}\left(\frac{p+1}{p^2+1}\right)$$

$$= -\cos t + \sin t + e^{-2t}\left\{L^{-1}\left(\frac{p}{p^2+1}\right) + L^{-1}\left(\frac{1}{p^2+1}\right)\right\}$$

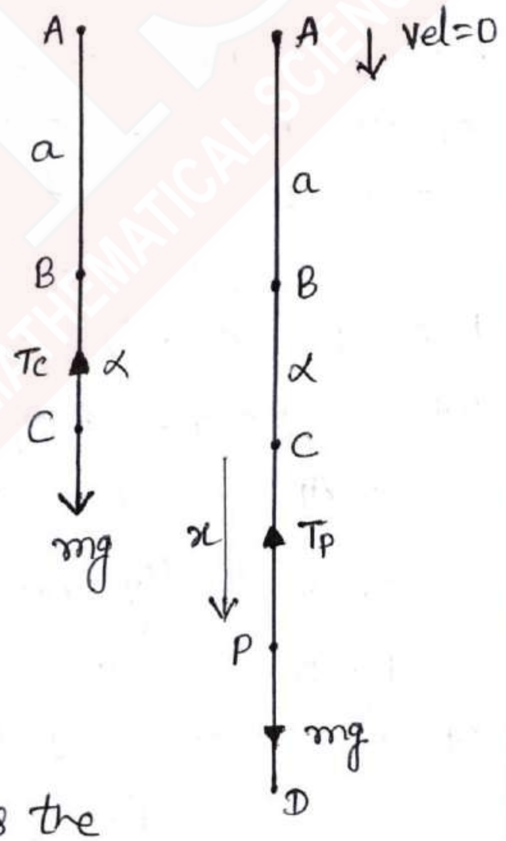
$$= -\cos t + \sin t + e^{-2t}\cos t + e^{-2t}\sin t$$

$$= (e^{-2t}-1)\cos t + (e^{-2t}+1)\sin t$$

which is the required solution.

8(b) One end of a light elastic string of natural length a and modulus of elasticity $2mg$ is attached to a fixed point A and the other end to a particle of mass m. The particle initially held at rest at A, is let fall. Show that the greatest extension of the string is $\frac{1}{2}a(1+\sqrt{5})$ during the motion and show that the particle will reach back A again after a time $(\pi+2-\tan^{-1}2)\sqrt{(2a/g)}$.

Sol: $AB = a$ is the natural length of an elastic string whose one end is fixed at A. Let C be the position of equilibrium of a particle of mass m attached to the other end of the string and let $BC = d$.



In the position of equilibrium of the particle at C, the tension $T_C = \lambda \frac{d}{a} = 2mg\frac{d}{a}$ in the string AC balances the weight mg of the particle.

$$\therefore mg = 2mg(d/a) \text{ or } d = a/2 \quad \text{——} (1)$$

Now the particle is dropped at rest from A.

It falls the distance AB freely under gravity. If v_1 be the velocity gained at B, we have $v_1 = \sqrt{(2ga)}$ in the downward direction. When the particle falls below B, the string begins to extend beyond its natural length and the tension begins to operate. During the fall from B to C the velocity of the particle goes on increasing as the tension remains less than the weight of the particle and when the particle begins to fall below C, its velocity goes on decreasing because now the force of tension exceeds the weight of the particle. Let the particle come to instantaneous rest at D.

During the motion of the particle below B, let P be its position after any time t, where $CP = x$. If T_P be the tension in the string AP, we have $T_P = \lambda \frac{d+x}{a} = 2mg \frac{\frac{1}{2}a+x}{a}$, acting vertically upwards.

By Newton's second law of motion, the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2} = mg - T_P = mg - 2mg\frac{\frac{1}{2}a+x}{a}$$

$$= -\frac{2mg}{a}x$$

$$\therefore \frac{d^2x}{dt^2} = -\frac{2g}{a}x, \quad \text{——— (2)}$$

which is the equation of a S.H.M. with centre at the point C and amplitude CD.

Multiplying (2) by $2(dx/dt)$ and Integrating w.r.t. 't', we get

$$\left(\frac{dx}{dt}\right)^2 = -\frac{2g}{a}\cdot x^2 + K,$$ where K is a constant.

At the point B, the velocity

$$= dx/dt = \sqrt{(2ga)} \text{ and } x = -d = -\frac{a}{2}.$$

$$\therefore K = 2ga + \frac{2g}{a}\cdot\frac{a^2}{4} = 2ga + \frac{2ga}{4} = \frac{5ag}{2}.$$

$$\therefore \text{ we have } \left(\frac{dx}{dt}\right)^2 = -\frac{2g}{a}x^2 + \frac{5ag}{2} \quad \text{——— (3)}$$

The equation (3) gives the velocity of the particle at any point between B and D. At D, $x = CD$ and $dx/dt = 0$. so putting $dx/dt = 0$ in (3), we have

$$0 = -\frac{2g}{a}x^2 + \frac{5ag}{2} \text{ or } x^2 = \frac{5a^2}{4}$$

or $\quad x = \frac{a}{2}\sqrt{5} = CD.$

$\therefore$ the greatest extension of the string

$$= BC + CD$$

$$= \frac{1}{2}a + \frac{1}{2}a\sqrt{5} = \frac{1}{2}a(1+\sqrt{5}).$$

Now from (3), we have

$$\left(\frac{dx}{dt}\right)^2 = \frac{2g}{a}\left[\frac{5}{4}a^2 - x^2\right]$$

$\therefore \quad \frac{dx}{dt} = \sqrt{\left(\frac{2g}{a}\right)}\sqrt{\left[\frac{5}{4}a^2 - x^2\right]}$, the +ive sign has been taken because the particle is moving in the direction of x increasing.

Seperating the variables, we have $dt =$

$$\left(\frac{a}{2g}\right)\frac{dx}{\sqrt{\left[\frac{5}{4}a^2 - x^2\right]}}$$

If t_1 is the time from B to D, then

$$\int_0^{t_1} dt = \sqrt{\left(\frac{a}{2g}\right)}\int_{-a/2}^{(a\sqrt{5})/2}\frac{dx}{\sqrt{\left[\frac{5}{4}a^2 - x^2\right]}}$$

or $\quad t_1 = \sqrt{\left(\frac{a}{2g}\right)}\left[\sin^{-1}\left\{\frac{x}{(a\sqrt{5})/2}\right\}\right]_{-a/2}^{(a\sqrt{5})/2}$

$$= \sqrt{\left(\frac{a}{2g}\right)}\left[\sin^{-1}1 + \sin^{-1}\frac{1}{\sqrt{5}}\right]$$

$$= \sqrt{\left(\frac{a}{2g}\right)}\left(\frac{\pi}{2} + \tan^{-1}\frac{1}{2}\right)$$

$$= \sqrt{\left(\frac{a}{2g}\right)}\left(\frac{\pi}{2} + \cot^{-1}2\right)$$

$$= \sqrt{\left(\frac{a}{2g}\right)}\left(\frac{\pi}{2} + \frac{\pi}{2} - \tan^{-1}2\right) = \sqrt{\left(\frac{a}{2g}\right)}\left(\pi - \tan^{-1}2\right)$$

And if t_2 is the time from A to B, (while falling freely under gravity), then

$$a = 0 \cdot t_2 + \frac{1}{2} g t_2^2$$

or $t_2 = \sqrt{\left(\frac{2a}{g}\right)}$

$\therefore$ the total time to return back to $A = 2$ (time from A to D)

$$= 2(t_2 + t_1)$$

$$= 2\left[\sqrt{\left(\frac{a}{2g}\right)}\left(\pi - \tan^{-1} 2\right) + \sqrt{\left(\frac{2a}{g}\right)}\right]$$

$$= \sqrt{\left(\frac{2a}{g}\right)}\left(\pi - \tan^{-1} 2 + 2\right)$$

Hence this proves the required result.

8(c) Verify divergence theorem for $F = (x^2-yz)\hat{i} + (y^2-zx)\hat{j} + (z^2-xy)\hat{k}$. Taken over the rectangular parallelopiped $0 \le x \le a$, $0 \le y \le b$, $0 \le z \le c$.

Sol^n: we have $\operatorname{div} F = \nabla \cdot F$.

$$= \frac{\partial}{\partial x}(x^2-yz) + \frac{\partial}{\partial y}(y^2-zx) + \frac{\partial}{\partial z}(z^2-xy)$$

$$= 2x+2y+2z.$$

$\therefore$ volume integral $= \iiint_V \nabla \cdot F \, dV$

$$= \iiint_V 2(x+y+z)\, dV$$

$$= 2\int_{z=0}^{c}\int_{y=0}^{b}\int_{x=0}^{a}(x+y+z)\, dV$$

$$= 2\iiint (x+y+z)\,dx\,dy\,dz$$

$$= 2\int_{z=0}^{c}\int_{y=0}^{b}\left(\frac{x^2}{2}+yx+zx\right)_0^a dy\,dz$$

$$= 2\int_{z=0}^{c}\int_{y=0}^{b}\left(\frac{a^2}{2}+ay+az\right)dy\,dz.$$

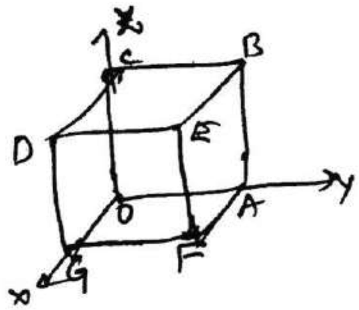
$$= 2\int_{z=0}^{c}\left(\frac{a^2y}{2}+\frac{ay^2}{2}+azy\right)_0^b dz$$

$$= 2\int_0^c\left(\frac{a^2b}{2}+\frac{ab^2}{2}+abz\right)dz = 2\left[\frac{a^2bz}{2}+\frac{ab^2z}{2}+ab\frac{z^2}{2}\right]_0^c$$

$$= abc(a+b+c)$$

Surface integral:

Now calculate $\iint_S F\cdot n\,ds$



over the face DEFG

$n = \hat{i}$, $x = a$

$\therefore \iint_{DEFG} F\cdot n\,ds = \int_{z=0}^{c}\int_{y=0}^{b}\left[(a^2-yz)\hat{i}+(y^2-za)+(z^2-ay)\right]\cdot \hat{i}\,dy\,dz$

$= \int_{z=0}^{c}\int_{y=0}^{b}(a^2-yz)\,dy\,dz$

$= \int_{z=0}^{c}\left(a^2y-\frac{y^2z}{2}\right)_0^b dz$

$= \int_{z=0}^{c}\left(a^2b-\frac{zb^2}{2}\right)dz = \left(a^2bz-\frac{z^2}{4}b^2\right)_0^c$

$= a^2bc-\frac{b^2c^2}{4}.$

over the face ABCO, $n=-\hat{i}$, $x=0$.

$\therefore \iint_{ABCO} F\,n\,ds = \iint\left[(0-yz)\hat{i}+(y^2)\hat{j}+z^2k\right]\cdot(-\hat{i})\,dy\,dz$

$= \int_{z=0}^{c}\int_{y=0}^{b} yz\,dy\,dz = \int_0^c\left(\frac{y^2z}{2}\right)_0^b dz$

$= \int_{z=0}^{c}\frac{b^2}{2}z\,dz = \frac{b^2c^2}{4}.$

over the face ABEF, $n=\hat{j}$, $y=b$.

$\therefore \iint_{ABEF} F\,\hat{n}\,ds = \int_{z=0}^{c}\int_{x=0}^{a}\left[(x^2-bz)\hat{i}+(b^2-zx)\hat{j}-(z^2-bx)k\right]\cdot\hat{j}\,dx\,dz$

$= \int_{z=0}^{c}\int_{x=0}^{a}(b^2-zx)\,dx\,dz = b^2ca-\frac{a^2c^2}{4}$

over the face OGDC, $n=-\hat{j}$, $y=0$.

$\therefore \iint_{OGDC} F\cdot\hat{n}\,ds = \int_{z=0}^{c}\int_{x=0}^{a} zx\,dx\,dz = \frac{c^2a^2}{4}.$

over the face BCDE, $n=k$, $z=c$.

$\therefore \iint_{BCDE} F\,n\,ds = \int_{y=0}^{b}\int_{x=0}^{a}(c^2-xy)\,dx\,dy = c^2ab-\frac{a^2b^2}{4}$

over the face AFGO, $n=-k$, $z=0$.

$\therefore \iint_{AFGO} F\cdot n\,ds = \int_{y=0}^{b}\int_{x=0}^{a} xy\,dx\,dy = \frac{a^2b^2}{4}.$

Adding the six surface integrals, we get

$\iint F\,n\,ds = \left(a^2bc-\frac{c^2b^2}{4}+\frac{c^2b^2}{4}\right)+\left(b^2ca-\frac{a^2c^2}{4}+\frac{a^2c^2}{4}\right)+\left(c^2ab-\frac{a^2b^2}{4}+\frac{a^2b^2}{4}\right)$

$= (abc)(a+b+c)$ Hence the theorem is verified