Chapter 8

2013

8.1 Section-A

Question-1(a) Find the dimension and a basis of the solution space W of the system $x + 2y + 2z - s + 3t = 0$, $x + 2y + 3z + s + t = 0$, $3x + 6y + 8z + s + 5t = 0$.

[8 Marks]

Solution: The matrix form of the given homogeneous system of linear equations is

$$\begin{bmatrix} 1 & 2 & 2 & -1 & 3 \\ 1 & 2 & 3 & 1 & 1 \\ 3 & 6 & 8 & 1 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ s \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 \to R_2 - R_1, \quad R_3 \to R_3 - 3R_1$$

$$\begin{bmatrix} 1 & 2 & 2 & -1 & 3 \\ 0 & 0 & +1 & 2 & -2 \\ 0 & 0 & 2 & 4 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ s \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_1 \to R_1 - 2R_2, \quad R_3 \to R_3 - 2R_2$$

$$\begin{bmatrix} 1 & 2 & 0 & -5 & 7 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ s \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This is the required row reduced echelon form.

$$x + 2y - 5s + 7t = 0$$
$$z + 2s - 2t = 0$$

$$\therefore \begin{bmatrix} x \\ y \\ z \\ s \\ t \end{bmatrix} = \begin{bmatrix} -2y + 5s - 7t \\ y \\ -2s + 2t \\ s \\ t \end{bmatrix} = y \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 5 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -7 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$\therefore$ Dimension of Solution Space $(W) = 3$.
Basis of Solution Space $= \{(-2,1,0,0,0),(5,0,-2,1,0),(-7,0,2,0,1)\}$.

Question-1(b) Find the characteristic equation of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and hence find the matrix represented by:

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I.$$

[8 Marks]

Solution:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$$

Characterstic Equation of a square matrix is given by :$\mid A - \lambda I \mid = 0$ i.e.

$$\begin{aligned} \lambda^3 - (\text{ trace of A})\lambda^2 + (C_{11} + C_{22} + C_{33})\,\lambda - |A| = 0 \\ \text{trace}(A) = 2 + 1 + 2 = 5 \\ C_{11} + C_{22} + C_{33} = (2-0) + (4-1) + (2-0) \\ = 7 \\ |A| = 2(2-0) + 0 + 1(0-1) = 3 \end{aligned}$$

$\therefore$ Characteristic Equation: $\quad \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$ Cayley-Hamilton theorem states that every square matrix satisfies its characteristic equation.

$$\therefore A^3 - 5A^2 + 7A - 3I = 0 \qquad \ldots(*)$$

We have to find,

$$\begin{aligned} A^8 - &5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I \\ &= A^5\left(A^3 - 5A^2 + 7A - 3I\right) + \left(A^4 - 5A^3 + 7A^2 - 3A\right) \\ &+ A^2 + A + I \\ &= A^5 \cdot 0 + A \cdot 0 + A^2 + A + I \quad (\text{using}(*)) \\ &= A^2 + A + I \\ &= \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix} \end{aligned}$$

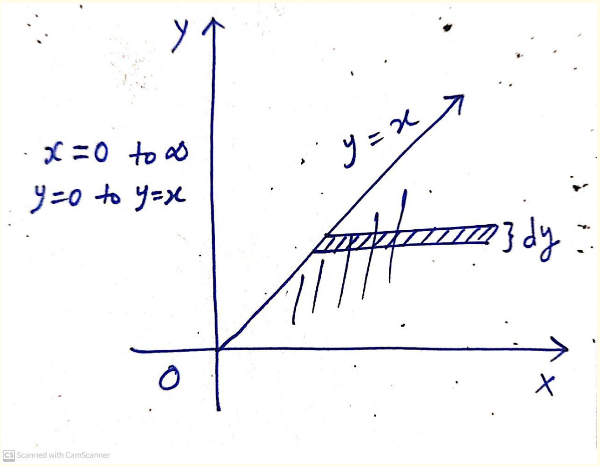
Question-1(c) Evaluate the integral $\int_0^\infty \int_0^x x e^{-x^2/y} dydx$ by changing the order of integration.

[8 Marks]

Solution: Let

$$1 = \int_0^\infty \int_0^x xe^{-x^2/y} dydx$$

Here the limits of integration show that the integration is done first with respect to y from $y = 0$ to $y = x$ and then with respect to x from $x = 0$ and $x = \infty$, i. *e.*, the strip is taken parallel to y -axis in the region bounded by these curves.



On changing the order of integration, we find that the strip parallel to x -axis varies form $x = y$ to $x = \infty$ and then y varies from $y = 0$ to $y = \infty$ to cover the whole region (fig.) Hence on changing the order of integration, we have figure

$$\begin{aligned}
I &= \int_0^\infty \int_{x=y}^\infty xe^{-x^2/y} dxdy \\
&= \int_0^\infty \left[-\frac{y}{2} e^{-x^2/y} \right]_{x=y}^\infty dy \\
&= \frac{1}{2} \int_0^\infty ye^{-y} dy \\
&= \frac{1}{2} \left(\left[y(-e^{-y}) \right]_0^\infty - \int_0^\infty 1 \cdot (-e^{-y}) dy \right) \\
&= \frac{1}{2} \left[\lim_{y \to \infty} \frac{-y}{e^y} - 0 \right] - \frac{1}{2} \left[e^{-y} \right]_0^\infty \qquad \left(\frac{0}{0} \text{form} \right) \\
&= \frac{1}{2} \left[\lim_{y \to \infty} \frac{-1}{e^y} \right] - \frac{1}{2} [0 - 1] \qquad \text{(Using L-Hospital)} \\
&= \frac{1}{2}
\end{aligned}$$

Question-1(d) Find the surface generated by the straight line which intersects the lines $y = z = a$ and $x + 3z = a = y + z$ and is parallel to the plane $x + y = 0$.

[8 Marks]

Solution: The equation of the given lines are

$$y - a = 0, z - a = 0 \qquad \ldots(i)$$

$$x + 3z - a = 0, y + z - a = 0 \qquad \ldots(ii)$$

The equation of any plane through the lines (i) and (ii) are

$$(y - a) - \lambda_1(z - a) = 0$$

$$\Rightarrow y - \lambda_1 z - a + a\lambda_1 = 0 \qquad \ldots(iii)$$

and

$$(x + 3z - a) - \lambda_2(y + z - a) = 0$$

$$(x - \lambda_2 y) + (3 - \lambda_2)\, z - a + a\lambda_2 = 0 \ldots(iv)$$

Any line intersecting the line (i) and (ii) is given by the intersection of the plane (iii) and (iv).
Let λ, μ, v are its dr's, then,

$$0.\lambda + 1.\mu - \lambda_1 \cdot v = 0$$

and

$$1.\lambda - \lambda_2 \cdot \mu + (3 - \lambda_2) \cdot v = 0$$

$$\therefore \frac{\lambda}{3 - \lambda_2 - \lambda_1\lambda_2} = \frac{\mu}{-\lambda_1} = \frac{v}{-1}$$

Now, the line with dr's λ, μ, v is parallel to the plane $x + y = 0$, i.e., this line is perpendicular to the normal to the plane $x+ \ y = 0$, whose dr's are $1, 1, 0$ So, we have

$$1.\,(3 - \lambda_2 - \lambda_1\lambda_2) + 1\,(-\lambda_1) + 0.(-1) = 0$$

$$3 - \lambda_1 - \lambda_2 - \lambda_1\lambda_2 = 0$$

The required locus of the line is obtained by eliminating λ_1 and λ_2 between $(iii), (iv)$ and (v) hence is given by

$$3 - \frac{y - a}{z - a} - \frac{x + 3z - a}{y + z - a} - \frac{y - a}{z - a} \cdot \frac{x + 3z - a}{y + z - a} = 0$$

$$3(y + z - a)(z - a) - (y - a)(y + z - a) - (z - a)(x + 3z - a) - (y - a)(x + 3z - a) = 0$$

$$-yz - y^2 + 2az - xz + 2ax - xy = 0$$

$$yz - y^2 + 2az - xz + 2ax - xy = 0$$

$$yz + y^2 + xz + xy = 2az + 2ax$$

$$(y + z)(x + y) = 2a(x + z)$$

Question-1(e) Find C of the Mean value theorem, if $f(x) = x(x-1)(x-2)$, $a = 0$, $b = \frac{1}{2}$ and C has usual meaning.

[8 Marks]

Solution:

$$f(x) = x(x-1)(x-2) = x^3 - 3x^2 + 2x$$
$$\therefore \quad f(a) = f(0) = 0$$

and

$$\begin{aligned} f(b) &= f\left(\frac{1}{2}\right) \\ &= \frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right) \\ &= \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) \\ &= \frac{3}{8} \end{aligned}$$

$$\therefore \quad \frac{f(b)-f(a)}{b-a} = \frac{\frac{3}{8}-0}{\frac{1}{2}-0} = \frac{3}{4}$$

Also

$$f'(x) = 3x^2 - 6x + 2$$

so that

$$f'(c) = 3c^2 - 6c + 2$$

Substituting these values for Lagrange's mean value theorem,

$$\begin{aligned} \frac{f(b)-f(a)}{b-a} &= f'(c), (a < c < b) \\ \frac{3}{4} &= 3c^2 - 6c + 2 \end{aligned}$$

$$12c^2 - 24c + 5 = 0$$

$$\begin{aligned} c &= \frac{24 \pm \sqrt{(24)^2 - 4 \cdot 12 \cdot 5}}{2 \times 12} \\ &= \frac{24 \pm \sqrt{576 - 240}}{24} \\ &= \frac{24 \pm 4\sqrt{21}}{24} \\ &= 1 \pm \frac{\sqrt{21}}{6} \\ c &= 1 - \frac{\sqrt{21}}{6} \in \left(0, \frac{1}{2}\right) \quad \text{Using Calculator} \end{aligned}$$

Question-2(a) **Let** V **be the vector space of** 2×2 **matrices over** $\mathbb{R}$ **and let** $\mathrm{M} = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$ **Let** $\mathrm{F} : \mathrm{V} \to \mathrm{V}$ **be the linear map defined by** $\mathrm{F(A)} = \mathrm{MA}$**. Find a basis and the dimension of (i) the kernel of W of** F **(ii) the image U of F.**

[10 Marks]

Solution:

$$\begin{aligned}
T\left(\begin{bmatrix} x & y \\ z & w \end{bmatrix}\right) &= \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} \\
&= \begin{bmatrix} x-z & y-w \\ -2x+2z & -2y+2w \end{bmatrix} \\
&= (x-z)\begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} + (y-\omega)\begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \\
&= k_1\begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} + k_2\begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \qquad k_1, k_2 \in \mathbb{R}
\end{aligned}$$

$\therefore$ Range (T)

$$w = \text{span}\left\{\begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}\right\}$$

Dimension $(w) = 2$

($\because$ two vectors $\begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$ are not multiples of each other), hence independent.

For kernel $T(A) = 0$, i.e.

$$\begin{aligned}
T\begin{bmatrix} x & y \\ z & w \end{bmatrix} &= \begin{bmatrix} x-z & y-w \\ -2x+2z & -2y+2w \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \\
x - z &= 0 - 2x + 2z = 0 \\
y - \omega &= 0 \\
-2y + 2w &= 0 \\
\text{i.e. } x &= z \quad \textit{and} \quad y = \omega \\
\therefore \begin{bmatrix} x & y \\ z & w \end{bmatrix} &= \begin{bmatrix} x & y \\ x & y \end{bmatrix} \\
&= x\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + y\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}
\end{aligned}$$

Since vectors $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ are not multiples of each other, hence they are independent therefore they form the basis of kernel (T). Dim (ker T) $= 2$.

Question-2(b) Locate the stationary points of the function $x^4+y^4-2x^2+4xy-2y^2$ and determine their nature.

[10 Marks]

Solution: We have

$$f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$$

$$\frac{\partial f}{\partial x} = 4x^3 - 4x + 4y \qquad \ldots(1)$$

$$\frac{\partial f}{\partial y} = 4y^3 + 4x - 4y \qquad \ldots(2)$$

For extreme values, $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$

$$x^3 - x + y = 0$$

$$\therefore y^3 + x - y = 0$$

Adding (1) and (2), we have

$$x^3 + y^3 = 0$$
$$(x+y)\left(x^2 - xy + y^2\right) = 0$$

$\therefore$ For real $x, x + y = 0$ is the only possibility. Putting $y = -x$ in (1), we get

$$x^3 - x - x = 0$$
$$x^3 - 2x = 0$$
$$x\left(x^2 - 2\right) = 0 \Rightarrow x = 0, \pm\sqrt{2}$$

Hence, the extreme points are $(0,0), (\sqrt{2}, -\sqrt{2})$ and $(-\sqrt{2}, \sqrt{2})$

$$\text{A} = \frac{\partial^2 f}{\partial x^2} = 12x^2 - 4$$

$$\text{B} = \frac{\partial^2 f}{\partial y \partial x} = 4$$

and

$$C = \frac{\partial^2 f}{\partial y^2} = 12y^2 - 4$$

At $(0,0): \quad A = -4, \quad B = 4, \quad C = -4$

$$\therefore \quad AC - B^2 = 16 - 16 = 0$$

$\therefore$ At$(0,0)$, further investigation is required. For small h, k and $h \neq k$, we have

$$f(h,k) - f(0,0) = h^4 + k^4 - 2h^2 + 4hk - 2k^2$$
$$= -2(h-k)^2 < 0 \quad \left[\text{Neglecting } h^4, k^4 ash, k \text{ are small} \right.$$

For $h = k$, we have

$$f(h,k) - f(0,0) = h^4 + h^4 - 2h^2 + 4h^2 - 2h^2$$
$$= 2h^4 > 0$$

As $f(h,k) - f(0,0)$ does not keep the same sign for all small values of h and k, so the point (0,0) is a saddle point.

$$At(\sqrt{2}, -\sqrt{2}) : A = 20, \quad B = 4, \quad C = 20$$

$$\therefore \text{AC} - \text{B}^2 > 0 \text{ and } \text{A} > 0$$

$\Rightarrow f$ has a minimum at $(\sqrt{2}, -\sqrt{2})$

$$\begin{aligned} \text{Minimum value } &= f(\sqrt{2}, -\sqrt{2}) \\ &= (\sqrt{2})^4 + (-\sqrt{2})^4 - 2(\sqrt{2})^2 + 4\sqrt{2}(-\sqrt{2}) - 2(-\sqrt{2})^2 \\ &= 4 + 4 - 4 - 8 - 4 = -8 \end{aligned}$$

$$\text{At } (-\sqrt{2}, \sqrt{2}) : \quad A = 20, B = 4, C = 20$$

$$\therefore C - B^2 = 400 - 16 = 384 > 0 \text{ and } A = 20 > 0$$

$\therefore \quad f(x,y)$ has a minimum at $(-\sqrt{2}, \sqrt{2})$ Minimum value $= f(-\sqrt{2}, \sqrt{2}) = -8$

Question-2(c) Find an orthogonal transformation of co-ordinates which diagonalizes the quadratic form

$$q(x,y) = 2x^2 - 4xy + 5y^2$$

[10 Marks]

Solution:

$$\begin{aligned} q(x,y) &= 2x^2 - 2xy - 2yx + 5y^2 \\ &= \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \\ \therefore \quad A &= \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix} \end{aligned}$$

First we diagonalize this matrix by finding eigenvectors.

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & -2 \\ -2 & 5-\lambda \end{vmatrix} = 0$$

$$(\lambda - 2)(x - 5) - 4 = 0$$

$$\lambda^2 - 7\lambda + 6 = 0$$

$$\Rightarrow \quad \lambda = 1, 6$$

For $\lambda = 1$:

$$\Rightarrow \quad \begin{bmatrix} 2-1 & -2 \\ -2 & 5-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad x - 2y = 0.$$

$$x = 2y$$

$\therefore$ Eigenvector

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2y \\ y \end{bmatrix}$$
$$= y\begin{bmatrix} 2 \\ 1 \end{bmatrix}$$
$$= y\begin{bmatrix} 2/\sqrt{5} \\ 1/5 \end{bmatrix}$$

For $\lambda = 6$:

$$\begin{bmatrix} 2-6 & -2 \\ -2 & 5-6 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$-2x - y = 0 \quad \Rightarrow \quad y = -2x$$
$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -2x \end{bmatrix}$$
$$= x\begin{bmatrix} 1 \\ -2 \end{bmatrix}$$
$$= x\begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}$$

Hence diagonalizing matrix is

$$M = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 \\ 0 & 6 \end{bmatrix}$$

such that

$$M^{-1}AM = D$$

Orthogonal transformation is

$$x = \frac{2}{\sqrt{5}}u + \frac{1}{\sqrt{5}}v$$
$$y = \frac{1}{\sqrt{5}}u - \frac{2}{\sqrt{5}}v$$

Question-2(d) Discuss the consistency and the solutions of the equations

$$x + ay + az = 1, ax + y + 2az = -4, ax - ay + 4z = 2$$

for different values of a.

[10 Marks]

Solution: Matrix eqn. $Ax = B$, therefore,

$$A = \begin{bmatrix} 1 & a & a \\ a & 1 & 2a \\ a & -a & 4 \end{bmatrix}, x = \begin{bmatrix} x \\ y \\ 2 \end{bmatrix}, B = \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix}$$

$$\begin{aligned} \det(A) &= 1\left(4 + 2a^2\right) - a\left(4a - 2a^2\right) + a\left(-a^2 - a\right) \\ &= 4 + 2a^2 - 4a^2 + 2a^3 - a^3 - a^2 \\ &= a^3 - 3a^2 + 4 \\ &= (a+1)(a-2)^2 \end{aligned}$$

Case 1: When $a \neq -1$ and $a \neq 2$

$$\mid A \mid \neq 0 \quad \Rightarrow A^{-1}\text{exist.}$$

Hence, system has unique solution.

Case 2: When $a = -1$

$$[A : B] = \begin{bmatrix} 1 & -1 & -1 & 1 \\ -1 & 1 & -2 & -4 \\ -1 & 1 & 4 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 0 & -3 & -3 \\ 0 & 0 & 3 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & -1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow x - y = 2, \quad z = 1$$

$$\begin{bmatrix} x \\ y \\ 2 \end{bmatrix} = \begin{bmatrix} y+2 \\ y \\ 1 \end{bmatrix} = y \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$$

Hence system has infinitly many solutions.

Case 3: When $a = 2$.

$$[A : B] \sim \begin{bmatrix} 1 & 2 & 2 & 1 \\ 2 & 1 & 4 & -4 \\ 2 & -2 & 4 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & -3 & 0 & -6 \\ 0 & -3 & 0 & 6 \end{bmatrix}$$

$$R_2 \to R_2 - 2R_4 \quad R_3 \to R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & -3 & 0 & -6 \\ 0 & 0 & 0 & 12 \end{bmatrix}$$

The $\text{Rank}(A) = 2$ & $\text{Rank}(A \cdot B) = 3$

Both are not equal, hence system is inconsistent for $a = 2$.

Question-3(a) Prove that if $a_0, a_1, a_2, \ldots\ldots, a_n$ are the real numbers such that

$$\frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \ldots\ldots\ldots + \frac{a_{n-1}}{2} + a_n = 0$$

then there exists at least one real number x between 0 and 1 such that

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \ldots + f_{n-1}x + a_n = 0$$

[10 Marks]

Solution: Consider the function

$$f(x) = a_0\frac{x^{n+1}}{n+1} + a_1\frac{x^n}{n} + a_2\frac{x^{n-1}}{n-1} + \ldots a_{n-1}\frac{x^2}{2} + a_nx$$

over the interval $[0, 1]$.

$$\begin{aligned} f(0) &= 0; \\ f(1) &= \frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \cdots + \frac{a_{n-1}}{2} + a_n \\ &= 0 \quad \text{(given)} \end{aligned}$$

Being a polynomial function, $f(x)$ is continuous and differentiate over interval [0,1].
Hence, Using Rolle's theorem, there exists $C \in (0, 1)$ such that

$$f'(c) = 0$$

$$\text{or} \quad a_0c^n + a_1c^{n-1} + a_2c^{n-2} + \ldots + a_{n-1}c + a_n = 0$$

Hence, Proved

Question-3(b) Reduce the following equation to its canonical form and determine the nature of the conic $4x^2 + 4xy + y^2 - 12x - 6y + 5 = 0$

[10 Marks]

Solution:

$$4x^2 + 4xy + y^2 - 12x - 6y + 5 = 0$$

General equation of second degree:

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

here

$$a = 4, b = 1, c = 5, g = -6, f = -3, h = 2$$

$$\begin{aligned}\Delta &= abc + 2fgh - af^2 - bg^2 - ch^2 \\ &= 20 + 72 - 36 - 36 - 20 \\ &= 0\end{aligned}$$

and

$$ab - h^2$$

$$4 \times 1 - (2)^2 = 0$$

Hence, given equation will represent pair of parallel straight lines.

$$4x^2 + 4xy + y^2 - 12x - 6y + 5 = 0$$

$$(2x + y)^2 - 6(2x + y) + 5 = 0$$
$$(2x + y - 5)(2x + y - 1) = 0$$
$$2x + y - 5 = 0$$

and

$$2x + y - 1 = 0$$

Question-3(c) Let F be a subfield of complex numbers and T a function from $\mathrm{F}^3 \to \mathrm{F}^3$ defined by $\mathrm{T}(\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3) = (\mathrm{x}_1 + \mathrm{x}_2 + 3\mathrm{x}_3, 2\mathrm{x}_1 - \mathrm{x}_2, -3\mathrm{x}_1 + \mathrm{x}_2 - \mathrm{x}_3)$. What are the conditions on (a, b, c) such that (a, b, c) be in the null space of T? Find the nullity of T.

[10 Marks]

Solution: $N_A(T) = \{(x_1, x_2, x_3) \in F \mid T(x_1, x_2, x_1) = (0, 0, 0)\}$ Let $(a, b, c) \in N_A(T)$. Then, $T(a, b, c) = (0, 0, 0)$. ie. $(a + b + 3c, 2a - b, -3a + b - c) = (0, 0, 0)$

$$\begin{array}{lll} \Rightarrow a + b + 3c = 0, & 2a - b = 0, & -3a + b - c = 0 \\ \qquad\downarrow & 2a = b \to & -3a + 2a - c = 0 \\ & & \Rightarrow c = -a. \\ a + b + 3c = 0 & \Rightarrow a + 2a - 3a = 0 & \text{hence it satisfies the values formed} \end{array}$$

$\therefore$ The required conditions are $b = 2a, c = -a$.
ie. $N_A(T) = \{(a, 2a, -a)/a \in \mathbb{F}\}$.
Clearly, the basis of $N_A(T) = \{(1, 2, -1)\}$.
$\therefore$ Nullity $(T) = 1$.

Question-3(d) Find the equations to the tangent planes to the surface $7x^2 - 3y^2 - z^2 + 21 = 0$, which pass through the line $7x - 6y + 9 = 0$, $z = 3$.

[10 Marks]

Solution: Eqn of a plane passing through given line

$$7x - 6y + 9 + \lambda(z - 3) = 0$$

$$7x - 6y + \lambda z + (9 - 3\lambda) = 0$$

Equation of tangent plane to given surface at a point (α, β, γ), lying on surface is

$$7\alpha x - 3\beta y - \gamma z + 21 = 0 \quad - \quad (2)$$

then

$$\frac{7\alpha}{7} = \frac{-3\beta}{-6} = \frac{-\gamma}{\lambda} = \frac{+21}{9 - 3\lambda}$$

(α, β, γ) lies on given surface

$$\therefore 7\left(\frac{1}{3-\lambda}\right)^2 - 3\left(\frac{14}{3-\lambda}\right)^2 - \left(\frac{-7\lambda}{3-\lambda}\right)^2 + 21 = 0$$

$$2\lambda^2 + 9\lambda + 4 = 0$$

$$\Rightarrow \quad \lambda = -4, \frac{-1}{2}$$

Hence, equation of tangent planes are

$$7x - 6y - 4z + 21 = 0$$
$$14x - 12y - z + 21 = 0$$

Question-4(a) Evaluate

$$\int_0^{\pi/2} \frac{x \sin x \cos x dx}{\sin^4 x + \cos^4 x}$$

[10 Marks]

Solution: Using the formula

$$\int_0^a f(x)dx = \int_0^a f(a - x)dx$$

$$I = \int_0^{\pi/2} \frac{\pi/2 \cdot \sin x \cdot \cos x}{\sin^4 x + \cos^4 x} - \int_0^{\pi/2} \frac{x \cdot \sin x \cos x}{\sin^4 x + \cos^4 x} dx (= I)$$

$$\therefore 2I = \frac{\pi}{2} \int_0^{\pi/2} \frac{\sin x \cdot \cos x}{\sin^4 x + \cos^4 x} dx$$

$$I = \frac{\pi}{4} \int_0^{\pi/2} \frac{\tan x \cdot \sec^2 x}{1 + \tan^4 x} dx \text{ (dividing by } \cos^4 x \text{ in numerator and denominator.)}$$

Put $\tan^2 x = t \Rightarrow 2\tan x \sec^2 x dx = dt$

$$\begin{aligned} I &= \frac{\pi}{4} \times \frac{1}{2} \int \frac{dt}{1+t^2} \\ &= \frac{\pi}{8} \tan^{-1} t \Big|_0^{\infty} \\ &= \frac{\pi}{8} \left(\frac{\pi}{2} - 0 \right) \\ &= \boxed{\frac{\pi^2}{16}} \end{aligned}$$

Question-4(b) Let $H = \begin{bmatrix} 1 & \mathbf{i} & 2+i \\ -i & 2 & 1-i \\ 2-i & 1+i & 2 \end{bmatrix}$ be a Hermitian matrix. Find a non-singular matrix P such that $\mathrm{P^t H \overline{P}}$ is diagonal and also find its signature.

[10 Marks]

Solution: Let $H = IHI$

$$\begin{bmatrix} 1 & i & 2+i \\ -i & 2 & 1-i \\ 2-i & 1+i & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} H \begin{bmatrix} 1 & 0 & 0 \\ 0 & 10 & \\ 0 & 0 & 1 \end{bmatrix}$$

Row-operations applied on pre-factor and column operations on post-factor on R H S.

$$R_2 \to R_2 + iR_1, \quad R_3 \to R_3 + (-2+i)R_y$$

$$C_2 \to C_2 - iC_1, \quad C_3 \to C_3 - (2+i)C_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & i \\ 0 & -i & -5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ i & 1 & 0 \\ -2+1 & 0 & 1 \end{bmatrix} H \begin{bmatrix} 1 & -i & -2-1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_3 \to R_3 + iR_2, \quad C_3 \to C_3 - iC_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ i & 1 & 0 \\ -3+i & i & 1 \end{bmatrix} \cdot H \begin{bmatrix} 1 & -i & -3-i \\ 0 & 1 & -i \\ 0 & 0 & 1 \end{bmatrix}$$

$$P^T H \bar{P} = D$$

$$\Rightarrow P = \begin{bmatrix} 1 & i & -3+i \\ 0 & 1 & i \\ 0 & 0 & 1 \end{bmatrix}$$

Rank$(H) = 3$

Index $(H) = 2$ (Positive diagonal entries)

Signature $(H) =$ No. of positive diagonal entries - No. of the negative diagonal entries $= 2 - 1 = 1$.

Question-4(c) Find the magnitude and the equations of the line of shortest distance between the lines

$$\frac{x-8}{3}=\frac{y+9}{-16}=\frac{z-10}{7}$$

and

$$\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}$$

[10 Marks]

Solution: Two lines $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ and $\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$ are coplanar if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0$$

Here

$$\begin{vmatrix} 15-8 & 29-(-9) & 5-10 \\ 3 & -16 & 7 \\ 3 & 8 & -5 \end{vmatrix}=\begin{vmatrix} 7 & 38 & -5 \\ 3 & -16 & 7 \\ 3 & 8 & -5 \end{vmatrix}$$
$$=1176\neq 0$$

Hence given two lines are not coplanar and therefore, not intersecting.

Let $A(3a+8,-16a-9,7a+10)$ and $B(3b+15,8b+29,-5b+5)$ be two general points on the given lines.
Also, let $P(8,-9,10),Q(15,29,5)$ are two given points on the given lines.

$$\therefore \text{D.r of } AB=\langle 3a-3b-7,-16a-8b-38,7a+5b+5\rangle$$

If AB is line of shortest distance, it will be perpendicular to both the lines.

$$\therefore \quad 3(3a-3b-7)-16(-16a-8b-38)+7(7a+5b+5)=0$$
$$157a+77b+311=0$$

&

$$3(3a-3b-7)+8(-16a-8b-38)-5(7a+5b+5)=0$$
$$154a+98b+350=0.$$

Solving, we get $a=-1,b=-2$

$$\therefore A(-3+8,16-9,-7+10) \text{ i.e. } (5,7,3)$$
$$B(-6+15,-16+29,10+5) \text{ i.e. } (9,13,15)$$
$$\begin{aligned}(AB)&=\sqrt{(9-5)^2+(13-7)^2+(15-3)^2}\\&=\sqrt{16+36+144}\\&=\sqrt{196}\\&=14\end{aligned}$$

eqn of AB,

$$\frac{x-5}{4} = \frac{y-7}{6} = \frac{z-3}{12}$$

i.e.

$$\frac{x-5}{2} = \frac{y-7}{3} = \frac{z-3}{6}$$

Question-4(d) Find all the asymptotes of the curve

$$x^4 - y^4 + 3x^2y + 3xy^2 + xy = 0$$

[10 Marks]

Solution: As coefficients of highest power of x and y are constants, hence the given curve has no asymptotes parallel to x-axis or y-axis.

So, we will find only the oblique asymptotes .
Let eqn of asymptote: $y = mx + c$.

$$\phi_4 = x^4 - y^4 \qquad \phi_3 = 3x^2y + 3xy^2$$

Putting $x = 1$, $y = m$

$$\phi_4(m) = 1 - m^4$$

$$\phi_4(m) = 0 \quad \Rightarrow \quad m = 1, -1$$

Also,

$$\begin{aligned} c &= \frac{-\phi_3(m)}{\phi_4'(m)} \\ &= \frac{-3(m)(1+m)}{-4m^3} \\ &= \frac{3(1+m)}{4m^2} \end{aligned}$$

For, $m = 1 \Rightarrow c = \dfrac{3}{2}$

For $m = -1 \Rightarrow c = 0$

Hence, equations of asymptotes are $y = x + 3/2$ & $y = -x$

8.2 Section-B

Question-5(a) Solve:

$$\frac{dy}{dx} + x\sin 2y = x^3\cos^2 y$$

[8 Marks]

Solution:

$$\frac{dy}{dx} + x\cdot\sin 2y = x^3\cdot\cos^2 y$$

Dividing both sides by $\cos^2 y$, we have

$$\sec^2 y\cdot\frac{dy}{dx} + \tan y\cdot(2x) = x^3$$

Let $\tan y = t$ then

$$\sec^2 y\cdot\frac{dy}{dx} = \frac{dt}{dx}$$

$$\therefore\quad \frac{dt}{dx} + 2x\cdot t = x^3$$

$$P = 2x,\quad Q = x^3$$

$$\begin{aligned}\text{I.F. } &\equiv e^{\int p\cdot dx}\\ &= e^{\int 2x\cdot dx}\\ &= \frac{e^{2x}}{2}\end{aligned}$$

$\therefore$ Solution of the differential equation is given as

$$t\ .(\text{I.F.}) \ = \int \text{Q}\ .(\text{I.F.})\ dx + c$$

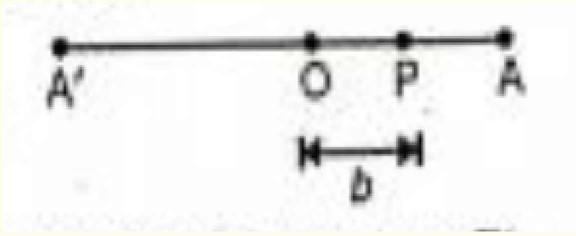
where c is integration constant

$$\begin{aligned}t\cdot\frac{e^{2x}}{2} &= \int x^3\cdot\frac{e^{2x}}{2}\cdot dx + c\\ - &= \frac{e^{2x}}{4}\left(x^3 - \frac{3}{2}x^2 + \frac{3}{2}x - \frac{3}{4}\right) + c\\ 2\tan y e^{2x} &= e^{2x}\left(x^3 - \frac{3}{2}x^2 + \frac{3}{2}x - \frac{3}{4}\right) + c.\end{aligned}$$

Question-5(b) A particle is performing a simple harmonic motion of period T **about centre** O **and it passes through a point** P, **where** $OP = (b$ **with velocity** v **in the direction of** OP. **Find the time which elapses before it returns to** P.

[8 Marks]

Solution: We have to find time taken from P to A and d then A to P.



$$
\begin{aligned}
t &= 2(\text{ time from } A \text{ to } P) \\
&= 2\int_0^{\prime} dt \\
&= 2\int_a^p \frac{dx}{\sqrt{u}\sqrt{a^2-x^2}}
\end{aligned}
$$

$$\text{(Ignoring -ve sign)}\left(\frac{dx}{dt} = \sqrt{u}\sqrt{a^2-x^2}\right)$$

$$
\begin{aligned}
&= \frac{2}{\sqrt{\mu}}\left[\cos^{-1}\frac{x}{a}\right]_a^b \\
&= \frac{2}{\sqrt{\mu}}\left[\cos^{-1}\frac{b}{a} - \cos^{-1}\frac{a}{b}\right] \\
&= \frac{2}{\sqrt{\mu}}\cos^{-1}\frac{b}{a} \\
&\Rightarrow t = \frac{2}{\sqrt{\mu}}\tan^{-1}\left(\frac{\sqrt{a^2-b^2}}{b}\right) \\
&= \frac{2}{\sqrt{\mu}}\tan^{-1}\left(\frac{v}{b\sqrt{\mu}}\right) \\
&= \frac{2}{\frac{2\pi}{\mathrm{T}}}\tan^{-1}\left[\frac{v}{b\left(\frac{2\pi}{\mathrm{T}}\right)}\right] \\
&= \frac{\mathrm{T}}{\pi}\tan^{-1}\left[\frac{vT}{2\pi b}\right]
\end{aligned}
$$

$$
\begin{aligned}
v^2 &= \mu\left(a^2-b^2\right) \\
\Rightarrow \quad v &= \sqrt{a}\sqrt{(a^2-b^2)} \\
\Rightarrow \quad \frac{v}{\sqrt{\mu}} &= \sqrt{a^2-b^2} \\
\mathrm{T} = \frac{2\pi}{\sqrt{\mu}} &\Rightarrow \sqrt{\mu} = \frac{2\pi}{\mathrm{T}}
\end{aligned}
$$

Proved.

Question-5(c) $\overrightarrow{F}$ **being a vector, prove that curl curl** $\overrightarrow{F} = grad\,\text{div}\,\overrightarrow{F} - \nabla^2\overrightarrow{F}$ **where** $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$

[8 Marks]

Solution: Proof

$$\text{Let} \quad \mathbf{A} = A_1\mathbf{i} + A_2\mathbf{j} + A_3\mathbf{k}.$$

$$\text{Then} \quad \nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) + \left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)\mathbf{k}.$$

$$\therefore \nabla \times (\nabla \times \mathbf{A}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} & \frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x} & \partial A_2 - \frac{\partial A_1}{\partial y} \end{vmatrix}$$

$$= \Sigma\left[\left\{\frac{\partial}{\partial y}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial A_1}{\partial z} - \frac{\partial A_3}{\partial x}\right)\right\}\mathbf{i}\right]$$

$$= \Sigma\left[\left\{\left(\frac{\partial^2 A_2}{\partial y \partial x} + \frac{\partial^2 A_3}{\partial z \partial x}\right) - \left(\frac{\partial^2 A_1}{\partial y^2} + \frac{\partial^3 A_1}{\partial z^2}\right)\right\}\mathbf{i}\right]$$

$$= \Sigma\left[\left\{\frac{\partial}{\partial x}\left(\frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) - \left(\frac{\partial^2 A_1}{\partial y^2} + \frac{\partial^2 A_1}{\partial z^2}\right)\right\}\mathbf{i}\right]$$

$$= \Sigma\left[\left\{\frac{\partial}{\partial x}\left(\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right) - \left(\frac{\partial^2 A_1}{\partial x^2} + \frac{\partial^2 A_1^2}{\partial y^2} + \frac{\partial^2 A_1}{\partial z^2}\right)\right\}\mathbf{i}\right]$$

$$= \Sigma\left[\left\{\frac{\partial}{\partial x}(\nabla \cdot \mathbf{A}) - (\nabla^2 A_1)\right\}\mathbf{i}\right]$$

$$= \Sigma\left[\left\{\frac{\partial}{\partial x}(\nabla \cdot \mathbf{A})\right\}\mathbf{i}\right] - \nabla^2 \Sigma A_1 \mathbf{i} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A}.$$

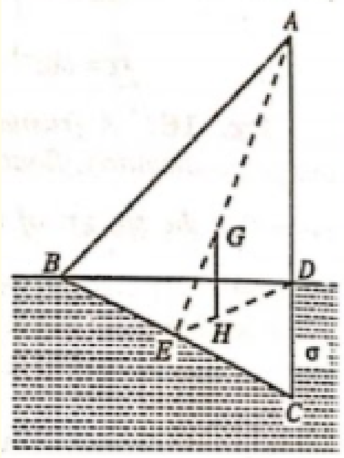
Question-5(d) **A triangular lamina** ABC **of density** ρ **floats in a liquid of density** σ **with its plane vertical, the angle** B **being in the surface of the liquid, and the angle** A **not immersed. Find** p/σ **in terms of the lengths of the sides of the triangle.**

[8 Marks]

Solution: The portion BCD of the ΔABC is immersed in the liquid with BD in contact with the surface Let G and H be the centres of gravity and buoyancy respectively. E is the mid-point of BC The conditions of equilibrium are :

(i) The line GH must be vertical.

(ii) The weight of the lamina must be equal to the weight of the liquid displaced.



Since $EG = \frac{1}{3}EA, EH = \frac{1}{3}ED, GH$ is parallel to AD. But GH is vertical from the first condition so AC must be vertical.

From the second condition of equilibrium, we have

$$\Delta ABC\rho g = \Delta BDC\sigma g$$

$$\begin{aligned} \therefore \quad \frac{\rho}{\sigma} &= \frac{\Delta BDC}{\Delta ABC} \\ &= \frac{\frac{1}{2}BD \cdot DC}{\frac{1}{2}BD \cdot AC} \\ &= \frac{DC}{AC} \\ &= \frac{BC\cos C}{AC} \end{aligned}$$

But

$$\begin{aligned} \frac{AC}{\sin B} &= \frac{BC}{\sin A} \\ BC &= \frac{AC\sin A}{\sin B} \end{aligned}$$

Hence

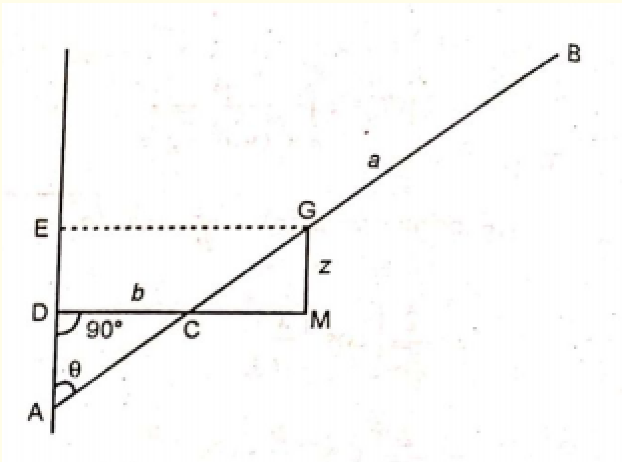
$$\begin{aligned} \frac{\rho}{\sigma} &= \frac{AC\sin A\cos C}{AC\sin B} \\ &= \frac{\sin A\cos C}{\sin B} \\ &= \frac{a}{b} \cdot \frac{a^2+b^2-c^2}{2ab} \\ &= \frac{a^2+b^2-c^2}{2b^2} \end{aligned}$$

Question-5(e) A heavy uniform rod rests with one end against a smooth vertical wall and with a point in its length resting on a smooth peg. Find the position of equilibrium and discuss the nature of equilibrium.

[8 Marks]

Solution: Let AB be a uniform rod of length $2a$. The end A of the rod rests against a smooth vertical wall and the rod rests on a smooth peg C whose distance from the wall is say b i.e.,

$$\text{CD} = b.$$



Suppose the rod makes an angle θ with the wall. The centre of gravity of the rod is at its middle point G. Let z be the height of above the fixed peg C, i.e., GM $= z$. We shall express z in terms of θ. We have,

$$\begin{aligned} z &= \text{GM} = \text{ED} = \text{AE} - \text{AD} \\ &= \text{AG}\cos\theta - \text{CD}\cot\theta \\ &= a\cos\theta - b\cot\theta \\ \therefore \quad dz/d\theta &= -a\sin\theta + b\,\text{cosec}^2\,\theta \end{aligned}$$

and

$$\frac{d^2z}{d\theta^2} = -a\cos\theta - 2b\,\text{cosec}^2\,A$$

For equilibrium of the rod, we have

$$\frac{dz}{d\theta} = 0$$

i.e.,

$$\begin{aligned} -a\sin\theta + b\,\text{cosec}^2\,\theta &= 0 \\ a\sin\theta &= b\,\text{cosec}^2\,\theta \\ \sin^3\theta &= b/a \\ \sin\theta &= (b/a)^{1/3} \\ \theta &= \sin^{-1}\cdot(b/a)^{1/3} \end{aligned}$$

This gives the position of equilibrium of the rod. Again

$$\begin{aligned} \frac{d^2z}{d\theta^2} &= -\left(a\cos\theta + 2b\,\text{cosec}^2\,\theta\cot\theta\right) \\ &= \text{ negative for all acute values of } \theta \end{aligned}$$

Thus $\dfrac{d^2z}{d\theta^2}$ is negative in the position of equilibrium and so z is maximum. Hence the equilibrium is unstable.

Question-6(a) Solve the differential equation

$$\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + \left(4x^2 - 1\right)y = -3e^{x^2}\sin^4 2x$$

by changing the dependent variable.

[13 Marks]

Solution: We have,

$$\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + \left(4x^2 - 1\right)y = -3e^{x^2}\sin 2x$$

Here

$$P = -4x, Q = 4x^2 - 1$$
$$R = -3e^{x^2}\sin 2x$$

In order to remove the first derivative

$$\begin{aligned} v &= e^{-\frac{1}{2}\int pdx} \\ &= e^{-\frac{1}{2}\int -4xdx} \\ &= e^{2\int xdx} \\ &= e^{x^2} \end{aligned}$$

On putting $y = av$, the normal equation is $\frac{d^2u}{dx^2} + \mathrm{Q}_1 u = \mathrm{R}_1$ where

$$\begin{aligned} Q_1 &= Q - \frac{1}{2}\frac{dp}{dx} - \frac{p^2}{4} \\ &= \left(4x^2 - 1\right) - \frac{1}{2}(-4) - \frac{16x^2}{4} \\ &= 4x^2 - 1 + 2 - 4x^2 \\ &= 1 \\ \mathrm{R}_1 &= \frac{\mathrm{R}}{v} \\ &= \frac{-3e^{x^2}\sin 2x}{e^{x^2}} \\ &= -3\sin 2x \end{aligned}$$

Equation (ii) becomes

$$\begin{aligned} &\frac{d^2u}{dx^2} + u = -3\sin 2x \\ \Rightarrow &\left(D^2 + 1\right)u = -3\sin 2x \end{aligned}$$

A.E. is

$$m^2 + 1 = 0$$

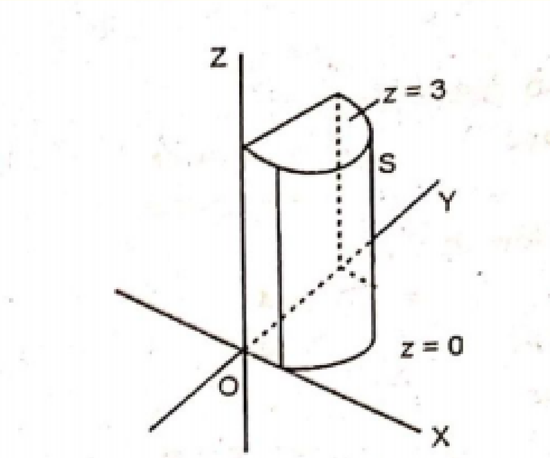
$$\Rightarrow m = \pm i$$

$$\Rightarrow \text{ C.F. } = c_1 \cos x + c_2 \sin x$$

$$\begin{aligned}
\text{P.I. } &= \frac{1}{D_2+1}(-3\sin 2x) \\
&= \frac{-3\sin 2x}{-4+1} \\
&= \sin 2x \\
u &= c_1 \cos x + c_2 \sin x + \sin 2x \\
y &= u.v \\
&= (c_1 \cos x + c_2 \sin x + \sin 2x)\, e^{x^2}
\end{aligned}$$

Question-6(b) Evaluate $\int_S \overrightarrow{F} \cdot d\overrightarrow{s}$, where $\overrightarrow{F} = 4xi - 2y^2j + z^2\overrightarrow{k}$ and s is the surface bounding the region $x^2 + y^2 = 4, z = 0$ and $z = 3$.

[13 Marks]



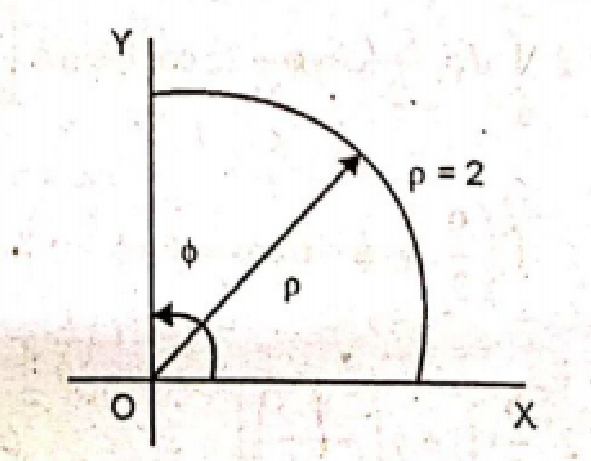
Solution:

Surface S is closed and let us assume that the volume enclosed by it is V.
Then, by Gauss divergence theorem

$$\int_S \vec{F} \cdot d\vec{S} = \iiint\limits_V div(\vec{F})dV, \text{ where } V = \text{Volume enclosed by the surface}$$

$$\begin{aligned}
\text{div}(\vec{F}) &= \frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}\left(-2y^2\right) + \frac{\partial}{\partial z}\left(z^2\right) \\
&= 4 - 4y + 2z = 2(2 - 2y + 2)
\end{aligned}$$

$$\therefore \iiint\limits_V \text{div}\, \vec{F} dV = \iiint\limits_V 2(2 - 2y + z)dV$$



Converting integral to cylindrical co-ordinates.

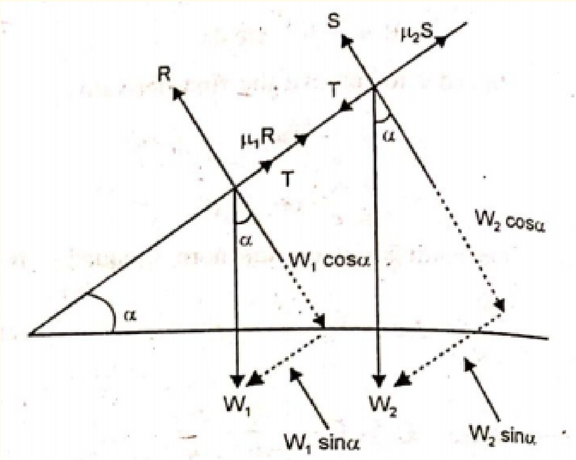
$$z = z, x^2 + y^2 = r^2, \quad x = r\sin\theta, y = r\cos\theta$$
$$r^2 = 4 \Rightarrow 0 \le r \le 2$$
$$\text{and} \quad 0 \le \theta \le 2\pi, \quad \text{also} \quad 0 \le z \le 3$$
$$\text{and} \quad V = rdrd\theta dz$$

$$= \int_{r=0}^{2}\int_{\theta=0}^{2\pi}\int_{z=0}^{3} 2(2 - 2r\sin\theta + z)rdrd\theta dz$$
$$= 2\int_0^2\int_0^{2\pi} \left|2z - 2r\sin\theta z + \frac{z^2}{2}\right|_0^3 rdrd\theta$$
$$= 2\int_0^2\int_0^{2\pi} \left(6 - 6r\sin\theta + \frac{9}{2}\right) rdrd\theta$$
$$= 2\int_0^2 \left|6\theta + 6r\cos\theta + \frac{9}{2}\theta\right|_0^{2\pi} rdr$$
$$= 2\int_0^2 [6(2\pi) + 6r(1-1) + \frac{9}{2}(2\pi)]rdr$$
$$= 2\int_0^2 \frac{21}{2}(2\pi)rdr = 42\pi\int_0^2 rdr = 42\pi\left|\frac{r^2}{2}\right|_0^2$$
$$= 42\pi\left(\frac{4}{2} - 0\right) = 84\pi$$
$$\therefore \int_S \vec{F}d\vec{s} = \iiint_V \int \operatorname{div}\vec{F}dV = 84\pi$$

Question-6(c) Two bodies of weights w_1 and w_2 are placed on an inclined plane and are connected by a light string which coincides with a line of greatest slope of the plane; if the coefficient of friction between the bodies and the plane are respectively μ_1 and μ_2, find the inclination of the plane to the horizontal when both bodies are on the point of motion, it being assumed that smoother body is below the other.

[14 Marks]

Solution: R and S are normal reactions and μ_1R and μ_2 S are forces of friction.



Let T be the tension in the string.

Let α be the inclination of plane to the horizontal. For W_1 : For limiting equilibrium, Horizontally

$$\mu_1 R + T = W_1 \sin\alpha$$

$$\Rightarrow \quad \text{T} = \dot{\text{W}}_1 \sin\alpha - \mu_1\text{R}...(i)$$

Vertically

$$\text{R} = \text{W}_1 \cos\alpha...(ii)$$

From (i) and (ii), we get

$$\text{T} = \text{W}_1 \sin\alpha - \mu_1 \text{ W}_1 \cos\alpha...(iii)$$

For W$_2$: For limiting equilibrium, Horizontally

$$\text{T} + \text{W}_2 \sin\alpha = \mu_2 \text{ S}$$

$$\Rightarrow \quad \text{T} = \mu_2 \text{ S} - \text{W}_2 \sin\alpha...(iv)$$

Vertically,

$$S = W_2 \cos\alpha...(v)$$

From (iv) and (v), we get

$$\text{T}^\circ = \mu_2 \text{ W}_2 \cos\alpha - \text{W}_2 \sin\alpha...(vi)$$

From (iii) and (vi), we get,

$$\begin{aligned}
\mathrm{W}_1 \sin\alpha - \mu_1\ \mathrm{W}_1 \cos\alpha &= \mu_2\ \mathrm{W}_2 \cos\alpha - \mathrm{W}_2 \sin\alpha \\
\Rightarrow \mathrm{W}_1 \sin\alpha + \mathrm{W}_2 \sin\alpha &= \mu_1\ \mathrm{W}_1 \cos\alpha + \mu_2\ \mathrm{W}_2 \cos\alpha \\
\Rightarrow (\mathrm{W}_1 + \mathrm{W}_2) \sin\alpha &= (\mu_1\ \mathrm{W}_1 + \mu_2\ \mathrm{W}_2) \cos\alpha \\
\Rightarrow \quad \tan\alpha &= \frac{\mu_1\ \mathrm{W}_1 + \mu_2\ \mathrm{W}_2}{\mathrm{W}_1 + \mathrm{W}_2} \\
\Rightarrow \quad \alpha &= \tan^{-1}\left(\frac{\mu_1\ \mathrm{W}_1 + \mu_2\ \mathrm{W}_2}{\mathrm{W}_1 + \mathrm{W}_2}\right)
\end{aligned}$$

Question-7(a) Solve

$$\left(D^3 + 1\right) y = e^{x/2} \sin\left(\frac{\sqrt{3}}{2} x\right)$$

where $D = \dfrac{d}{dx}$

[13 Marks]

Solution: Auxiliary Eqn:

$$\begin{aligned}
D^3 + 1 &= 0 \\
D &= -1, \frac{1 \pm \sqrt{3} i}{2} \\
C.F. &= C_1 e^{-x} + e^{x/2}\left(C_2 \cos\frac{\sqrt{3}x}{2} + C_3 \sin\frac{\sqrt{3}x}{2}\right) \\
P.I. &= \frac{1}{D^3 + 1} e^{x/2} \sin\frac{\sqrt{3}x}{2} \\
&= e^{x/2} \frac{1}{\left(D + \frac{1}{2}\right)^3 + 1} \sin\frac{\sqrt{3}x}{2} \\
&\left(\because \frac{1}{f(D)} e^{ax} V = e^x \cdot \frac{1}{f(D+a)} V\right) \\
&= e^{x/2} \frac{1}{D^3 + \frac{1}{8} + \frac{3}{2} D^2 + \frac{3D}{4} + 1} \sin\frac{\sqrt{3}x}{2} \\
f(D) &= D^3 + \frac{1}{8} + \frac{3}{2} D^2 + \frac{3D}{4} + 1 \\
f\left(-\frac{3}{4}\right) &= f\left(-a^2\right) \\
&= D\left(\frac{-3}{4}\right) + \frac{3}{2}\left(\frac{-3}{4}\right) + \frac{3D}{4} + \frac{9}{8} \\
&= 0
\end{aligned}$$

Hence, we take derivative of denominator and multiply by x

$$= xe^{x/2}\frac{1}{3D^2+3D+3/4}\sin\frac{\sqrt{3}x}{2}$$
$$= \frac{xe^{x/2}}{3}\frac{1}{\left(\frac{-3}{4}\right)+D+\frac{1}{4}}\sin\frac{\sqrt{3}x}{2}$$
$$= \frac{xe^{x/2}}{3}\cdot\frac{1}{D-\frac{1}{2}}\cdot\frac{D+1/2}{D+1/2}\sin\frac{\sqrt{3}x}{2}$$
$$= \frac{xe^{x/2}}{3\left(\frac{-3}{24}-1/4\right)}\left(D+\frac{1}{2}\right)\sin\frac{\sqrt{3}x}{2}$$
$$= -1/3xe^{x/2}\left(\frac{\sqrt{3}}{2}\cos\frac{\sqrt{3}x}{2}+\frac{1}{2}\sin\frac{\sqrt{3}x}{2}\right)$$
$$P.I. = \frac{-x}{3}e^{x/2}\cdot\sin\left(\frac{\pi}{3}+\frac{\sqrt{3}x}{2}\right)$$

Question-7(b) A body floating in water has volumes v_1, v_2 and v_3 above the surface, when the densities of the surrounding air are respectively ρ_1, ρ_2, ρ_3. Find the value of:

$$\frac{\rho_2-\rho_3}{v_1}+\frac{\rho_3-\rho_1}{v_2}+\frac{\rho_1-\rho_2}{v_3}$$

[13 Marks]

Solution: Suppose the volume and the density of the body be V and ρ respectively.

Now, weight of the body = weight of air displaced + weight of water displaced Hence,

$$\mathrm{V}'\rho g = \mathrm{V}_1\rho_1 g + (\mathrm{V}-\mathrm{V}_1)\times 1\times g ...(i)$$
$$\mathrm{V}\rho g = \mathrm{V}_2\rho_2 g + (\mathrm{V}-\mathrm{V}_2)\times 1\times g ...(ii)$$
$$\mathrm{V}\rho g = \mathrm{V}_3\rho_3 g + (\mathrm{V}-\mathrm{V}_3)\times 1\times g \quad ......(iii)$$

These relations give,

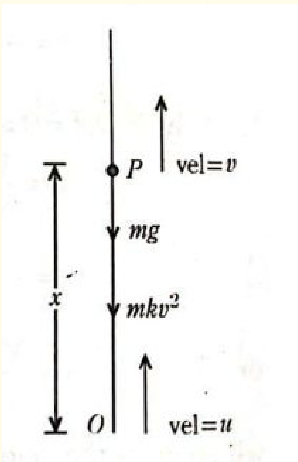
$$V_1 = \frac{\rho-1}{\rho_1-1}\,\mathrm{V}$$
$$\frac{1}{\mathrm{V}_1} = \frac{\rho_1-1}{(p-1)\mathrm{V}}$$
$$V_2 = \frac{\rho-1}{\rho_2-1}V$$
$$\frac{1}{V_2} = \frac{\rho_2-1}{(p-1)V}$$
$$\mathrm{V}_3 = \frac{\rho-1}{\rho_3-1}\,\mathrm{V}$$
$$\frac{1}{\mathrm{V}_3} = \frac{\rho_3-1}{(\rho-1)\mathrm{V}}$$

$$\begin{aligned}\therefore \frac{\rho_2-\rho_3}{V_1}+\frac{\rho_3-\rho_1}{V_2}+\frac{\rho_1-\rho_2}{V_3} &= \frac{(\rho_1-1)}{(p-1)V}(\rho_2-\rho_3)+\frac{(\rho_2-1)}{(\rho-1)V}(\rho_3-\rho_1)\\ &= \frac{1}{(\rho-1)V}\left[(\rho_1-1)(\rho_2-\rho_3)+(\rho_2-1)(\rho_3-\rho_1)\right.\\ &\qquad\left.+(\rho_3-1)(\rho_1-\rho_2)\right]\\ &= 0\end{aligned}$$

Question-7(c) A particle is projected vertically upwards with a velocity u, in a resisting medium which produces a retardation kv^2 when the velocity is v. Find the height when the particle comes to rest above the point of projection.

[14 Marks]

Solution: Let a particle of mass m be projected vertically upwards from the point O with velocity u. Let P be the position of the particle at any time t, where $OP = x$ and let v be the velocity of the particle at P. The forces acting on the particle at P are:
(i) The force mkv^2 due to resistance acting against the direction of motion i.e., acting vertically downwards.
(ii) The weight mg of the particle also acting vertically downwards.



Both these forces act in the direction of x decreasing. Therefore the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2} = -mg - mkv^2$$

$$\text{Or}\quad \frac{d^2x}{dt^2} = -g\left(1+\frac{k}{g}v^2\right)$$

Let V be the terminal velocity of the particle during its downwards motion i.e., the velocity when the resultant acceleration of the particle during its downwards motion is zero. Then

$$0 = mg - mkV^2 \text{ or } k = g/V^2$$

Putting this value of k in the above equation of motion of the particle, we get

$$\frac{d^2x}{dt^2} = -g\left(1+\frac{v^2}{V^2}\right)$$

$$\text{or} \qquad \frac{d^2x}{dt^2} = \frac{-g}{V^2}\left(V^2+v^2\right). \qquad \ldots(1)$$

Relation between v and x : Equation (1) can be written as

$$v\frac{dv}{dx} = \frac{-g}{V^2}\left(V^2+v^2\right) \qquad \left[\because \frac{d^2x}{dt^2} = v\frac{dv}{dx}\right]$$

$$\text{or} \quad \frac{-2g}{V^2}dx = \frac{2vdv}{V^2+v^2}, \quad \text{separating the variables.}$$

Integrating, $\frac{-2gx}{V^2} = \log\left(V^2+v^2\right) + A$, where A is a constant. Initially at $O, x = 0$ and $v = u$

$$\therefore \quad 0 = \log\left(V^2+u^2\right) + A$$

$$\text{or} \quad A = -\log\left(V^2+u^2\right)$$

$$\therefore \quad \frac{-2gx}{V^2} = \log\left(V^2+v^2\right) - \log\left(V^2+u^2\right)$$

$$\text{or} \quad x = \frac{V^2}{2g}\log\frac{V^2+u^2}{V^2+\nu^2} \qquad \ldots(2)$$

which gives the velocity of the particle in any position. If H is the greatest height attained by the particle, then putting $x = H$ and $v = 0$ in (2), we get

$$H = \frac{V^2}{2g}\log\frac{V^2+u^2}{V^2}.$$

Question-8(a) Apply the method of variation of parameters to solve

$$\frac{d^2y}{dx^2} - y = 2\left(1+e^x\right)^{-1}$$

[13 Marks]

Solution: Given DE Eqn:

$$\left(D^2-1\right)y = 2\left(1+e^x\right)^{-1}$$

Auxiliary Eqn:

$$D^2 = 1 = 0 \quad \Rightarrow D = \pm 1$$

$$C \cdot F \cdot \quad = C_1e^x + c_2e^{-x}$$

To find complete solution, we replace constants c_1 and c_2 with functions A and B.

$$y = Ae^x + Be^{-x}$$
$$= Ay_1 + By_1$$

where $y_1 = e^x, \quad y_2 = e^{-x}$

$$\begin{aligned} W &= \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \\ &= \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -1-1 \\ &= -2 \neq 0 \end{aligned}$$

$\Rightarrow y_1 \& y_2$ are independent.

$$\begin{aligned} A &= -\int \frac{y_2 R}{w} dx \\ &= -\int \frac{e^{-x} \cdot 2\left(1+e^x\right)^{-1}}{-2} dx \\ &= \int \frac{dx}{e^x\left(1+e^x\right)} \\ &\begin{pmatrix} \text{put} \quad e^x = t \\ e^x dx = dt \end{pmatrix} \\ &= \int \frac{dt}{t^2(1+t)} \end{aligned}$$

$$\begin{aligned} \frac{1}{t^2(1+t)} &= \frac{A}{t} + \frac{B}{t^2} + \frac{C}{1+t} \\ 1 &= At(t+1) + B(1+t) + ct^2 \\ 1 &= t^2(A+C) + t(A+B) + B \end{aligned}$$

$A+C=0, \quad A+B=0, \quad B=1 \quad \Rightarrow \quad A=-1, B=1, \quad C=1$

$$\begin{aligned} A &= \int \left(\frac{-1}{t} + \frac{1}{t^2} + \frac{1}{t+1}\right) dt \\ &= -\log t - \frac{1}{t} + \log(t+1) + c_1' \\ &= \log\left(\frac{t+1}{t}\right) - \frac{1}{t} + c_1' \\ &= \log\left(1+e^{-x}\right) - e^{-x} + c_1' \end{aligned}$$

$$\begin{aligned} B &= \int \frac{y_1 R}{w} dx \\ &= \int \frac{e^x \cdot 2\left(1+e^x\right)^{-1}}{-2} dx \\ &= -\int \frac{e^x}{1+e^x} dx \\ &= -\log\left(1+e^x\right) + c_2' \end{aligned}$$

Hence, complete general solution is

$$\begin{aligned} y &= Ay_1 + By_2 \\ &= e^x\left[\log\left(1+e^{-x}\right) - e^{-x} + c_1^1\right] + e^{-x}\left[-\log\left(1+e^x\right) + c_2^1\right] \\ y &= e^x \log\left(1+e^{-x}\right) - 1 + e^x \cdot c_1' - e^{-x}\log\left(1+e^x\right) + e^{-x} c_2' \end{aligned}$$

Question-8(b) Verify the divergence theorem for the vector function

$$\overrightarrow{\mathrm{F}} = \left(x^2 - yz\right)\vec{i} + \left(y^2 - xz\right)\vec{j} + \left(z^2 - xy\right)\bar{k}$$

taken over the rectangular parallelopiped

$$0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq c$$

[14 Marks]

Solution: To verify Gauss divergence theorem, we have to show that

$$\iiint\limits_V div\vec{F}dv = \iint\limits_s \vec{F}\cdot\hat{n}\cdot ds$$

Firstly,

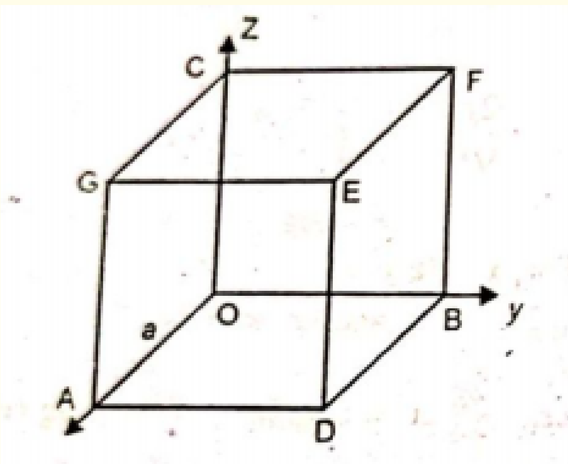
$$\iiint\limits_v \operatorname{div}\overrightarrow{\mathrm{F}}dv = \int_0^c\int_{00}^{ba}\int_0^a\left[\frac{\partial}{\partial x}\left(x^2 - yz\right) + \frac{\partial}{\partial y}\left(y^2 - zx\right) + \frac{\partial}{\partial z}\left(z^2 - xy\right)\right]dxdydz$$

$$= \int_{000}^{1ba}\int_0^b 2(x+y+z)dxdydz$$

$$= a^2bc + ab^2c + abc^2$$

$$= abc(a+b+c)$$

Now to calculate $\iint_s \overrightarrow{\mathrm{F}}.nds$, we divide the surface s of the parallelopiped $0 \leq x \leq a$, $0 \leq y \leq b, 0 \leq z \leq c$ into six parts.



(i) For the face OADB, we have

$$\hat{n} = -\hat{k}, z = 0$$

Therefore,

$$\int_{OADB}\overrightarrow{\mathrm{F}}\cdot\hat{n}\cdot ds = \int_{OADB}\left(x^2\hat{\imath} + y^2\hat{\jmath} - xy\hat{k}\right)\cdot(-\hat{k})ds$$

$$= \int_{00}^{ba} xydxdy$$

$$= \frac{a^2b^2}{4}$$

(ii) For the face $CGEF$, we have $z = c$ -

$$\hat{n} = \hat{k}$$

$$\begin{aligned} z &= \int_{(\text{GEF}} \left[\left(x^2 - cy\right)\hat{i} + \left(y^2 - cx\right)\hat{j} + \left(c^2 - xy\right)\hat{k}\right] \cdot \hat{k} ds \\ &= \int_0^{ba} \int_0^{a} \left(c^2 - xy\right) dxdy \\ &= abc^2 - \frac{a^2b^2}{4} \end{aligned}$$

(iii) For the face $ADEG$, we have $\hat{n} = \hat{i}, x = a$ and $dx = 0$. Therefore,

$$\begin{aligned} \int_{\text{ADEG}} \int_0 \overrightarrow{\mathrm{F}}.n \cdot ds &= \int_0^{c_0 b} \int_0^{2} \left(a^2 - yz\right) dydz \\ &= a^2bc - \frac{b^2c^2}{4} \end{aligned}$$

(iv) For the face OBFC, we have $\hat{n} = -\hat{i}, x = 0\ dx = 0$, Therefore,

$$\begin{aligned} \iint\limits_{\text{OBFC}} \overrightarrow{\mathrm{F}} \cdot \hat{n} \cdot ds &= \int_0^{ab} \int yzdydz \\ &= \frac{b^2c^2}{4} \end{aligned}$$

(v) For the face $OAGC$, we have $\hat{n} = -\hat{j}, y = 0\ dy = 0$, Therefore,

$$\begin{aligned} \iint\limits_{OAGC} \vec{F} \cdot \hat{n}.ds &= \int_0^{ab} \int_0^{b} zxdzdx \\ &= \frac{a^2c^2}{4} \end{aligned}$$

(vi) For the face DBFE, we have $\hat{n} = \hat{j}, y = b\ dy = 0$ Therefore,

$$\begin{aligned} \iint\limits_{\text{DBFE}} \overrightarrow{\mathrm{F}} \cdot \hat{n} \cdot ds &= \int_0^{ab} \int_0^{b} \left(b^2 - zx\right) dzdx \\ &= ab^2c - \frac{a^2c^2}{4} \end{aligned}$$

Hence adding the values of the above integrals, we get

$$\iint\limits_{s} \overrightarrow{\mathrm{F}} \cdot \hat{n} \cdot ds = abc(a + b + c)$$

Hence,

$$\iiint\limits_{V} \int \operatorname{div} \vec{F} dv = \iint\limits_{s} \overrightarrow{\mathrm{F}} \cdot \hat{n}.ds$$

which verifies the Gauss's divergence theorem.

Question-8(c) A particle is projected with a velocity v along a smooth horizontal plane in a medium whose resistance per unit mass is double the cube of the velocity. Find the distance it will describe in time t.

[13 Marks]

Solution: Here since particle is moving in a horizontal plane, the weight mg of the particle will not act. Hence the only force acting on the particle is that due to resistance and is equal to $-m\mu v^3$.
The equation of motion of the particle is

$$m\,(dv/dt) = -m\mu v^3 \quad \text{or} \quad -\left(dv/v^3\right) = \mu dt$$

Integrating, $\frac{1}{2v^2} = \mu t + C$, where C is a constant of integration.
Initially when $t = 0, v = V$,

$$\therefore \quad \frac{1}{2v^2} = \mu t + \frac{1}{2V^2} \text{ or } \frac{1}{v^2} = \frac{2\mu t V^2 + 1}{V^2}$$
$$\text{or} \quad v = V/\sqrt{(1 + 2\mu t V^2)} \quad \ldots (1)$$

If x be the distance descrihed by ine particle in time t, then equation (1) may be written as

$$\frac{dx}{dt} = \frac{V}{\sqrt{1 + 2\mu t V^2}} \quad \text{or} \quad dx = \frac{V}{\sqrt{1 + 2\mu t V^2}} dt$$

Integrating,

$$x = \frac{1}{\mu V}\sqrt{(1 + 2\mu t V^2) + C'} \quad \ldots (2)$$

Initially when $t = 0, x = 0, \quad \Rightarrow \quad C' = -1/\mu V$. Hence equation (2) becomes

$$x = \frac{1}{\mu V}\sqrt{(1 + 2\mu t V^2)} - \frac{1}{\mu}$$
$$\text{or} \quad x = \frac{1}{\mu V}\left[\sqrt{(1 + 2\mu t V^2)} - 1\right] \quad \ldots (3)$$

Equations (1) and (3) give required results.