Chapter 11

Riemann Integral

Motivation: problems and outstanding issues. Another basic concept of mathematical analysis is the concept of the Riemann integral. Historically, a number of problems led to the introduction of this concept. Next, we shall mention some of them.

Problem 1. One of the issues that worried mathematicians from the earliest times was the determination of the surface area of a planar object. Before we focus on the solution to this problem let us recall few basic facts about surface areas of simple planar objects such as a triangle, rectangle, or polygon. (Note that a polygon can be constructed from triangles.) In elementary geometry we learn how to calculate surface areas of these simple planar objects. Their surface areas have some simple natural properties. For example, the surface area of a planar object that is obtained by putting together several planar objects is equal to the sum of the surface areas of the individual objects. Furthermore, the surface area of any part of a planar object is less than the area of the whole object. We shall keep these properties in mind as we shall start thinking about the surface areas of more complicated planar objects.

Let there be an orthogonal coordinate system defined in a plane. Let $I \subset \mathbb{R}$ be an interval and $a,\, b\, (a < b)$ are given points in I. Let function $f : I \to \mathbb{R}$ be bounded and non-negative on $[a, b]$.

The set of points in the plane

$$M(f;\, a,\, b) = \{(x, y) \in \mathbb{R}^2;\ x \in [a, b],\ 0 \le y = f(x)\}$$

will be called a *curved trapezoid* associated with function f and interval $[a, b]$ (Figure 11.1). Let us further assume that function f is continuous at every point of interval $[a, b]$ and calculate the surface area of the curved trapezoid $M(f;\, a,\, b)$.
(Note that we have shown in section 10.2 that this task leads to the determination of the primitive function. However, its existence has not been guaranteed and will be proven only in section 11.5 based on the properties of the Riemann integral.)

Let n be an integer. Let $x_0,\, x_1,\, \ldots\, x_n$ be numbers for which

$$a = x_0 < x_1 < \ldots < x_{n-1} < x_n = b$$

and let

$$m_i = \min_{x\in[x_{i-1}-x_i]} f(x)\, M_i = \max_{x\in[x_{i-1}-x_i]} f(x) \quad i = 1,\, 2,\, \ldots,\, n.$$

Then numbers

$$\sum_{i=1}^{n} m_i(x_{i-1} - x_i), \quad \sum_{i=1}^{n} M_i(x_{i-1} - x_i)$$

represent the surface areas of the polygons consisting of the rectangles with sides of length $x_{i-1} - x_i$, m_i, or $x_{i-1} - x_i$, M_i; $i = 1, 2, \ldots, n$ (Figures 11.2, and 11.3, respectively, where $n = 7$).

THEOREM 11.0.1. *Proof.* ☐

Note 11.0.1.

Example 11.0.1. *Calculate the following indefinite integrals on the maximal sets*
a) $\int \frac{2}{x^4-1}\, dx$; *b)* $\int P_m(x)\, dx$, *where* P_m *is a polynome of the m-th degree* $x \mapsto a_m x^m + \ldots + a_1 x + a_0$, $x \in \mathbb{R}$ *and* $a_i \in \mathbb{R}$ *for* $i = 0, 1, \ldots, m$; *c)* $\int \frac{1}{\sin x}\, dx$.
`Solution`*: a) Using the definition of the Riemann integral the answer is 0.*
b) The integral diverges.

The definition of the derivative of a function depending on a single variable (chapter 8) was motivated by two basic problems — a geometric one (construction of a tangential to the graph of the function) and a physical one (need to find the instantaneous velocity of a motion with known trajectory). Another two problems: the evaluation of a surface area and the calculation of a trajectory of a moving point whose velocity is known, despite being at first glance very different, lead to mathematical problems closely related to the concept of a function differentiability.

The goal of the next two chapters is to develop the constructs that are needed to solve problems of this type. Hence in this chapter we shall deal with the concept and basic propreties of a primitive function, Newton and indefinite integral of a real-valued function of a single variable, and we shall introduce some methods to calculate them.

In the next chapters, on the Riemann and generalised integral, further application and extension of these results will be presented.

11.1 Primitive Function

The task to find a primitive function is in some sense the inverse problem to the problem to find a proper derivative of a given function. (A function f is primitive to its derivative f'.) However, contrary to the calculation of derivatives there is no simple set of rules for the calculation of the primitive function in terms of the elementary functions.

Strongly Primitive Function

One of the basic types of the primitive function is introduced in the following definition:

Definition 11.1.1. Let $f : (J \subset \mathbb{R}) \to \mathbb{R}$ and $I \subset J$ is an interval (of any type). We say that *function* $F : I \to \mathbb{R}$ *is strongly primitive* (or is *a strong potential*) *to function* f *on interval* I if

$$\forall x \in I : \; F'(x) = f(x).$$

Note 11.1.1. 1. From the definition 10.9.1 it follows directly that function F is differentiable on interval I which, according to theorem **??**, means that F is continuous on I, $F \in C(I)$.
2. The *sine* function is strongly primitive to the *cosine* function on interval $(-\infty, +\infty)$. It can be seen from example 6.8.4 that the strongly primitive function on $\mathbb{R}$ to a discontinuous function

$$f : x \mapsto \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x}; & x \in \mathbb{R} \setminus \{0\} \\ 0, & x = 0 \end{cases}$$

is a function F

$$F : x \mapsto \begin{cases} x^2 \sin \frac{1}{x}, & x \in \mathbb{R} \setminus \{0\} \\ 0, & x = 0. \end{cases}$$

Similarly, there exists a strongly primitive function on $(0, 1]$ to a function $f : x \mapsto x^{-2/3}$, $x \in (0, 1]$, namely $F : x \mapsto 3x^{1/3}$, $x \in (0, 1]$.

Example 11.1.1. Prove that there exists no strongly primitive function on $\mathbb{R}$ to a function

$$sgn : x \mapsto \begin{cases} 1, & x \in \mathbb{R}^+ \\ 0, & x = 0 \\ -1, & x \in \mathbb{R}^-. \end{cases}$$

`Solution`: If the strongly primitive function F on $\mathbb{R}$ to function sgn existed then according to the corollary to theorem **??** the derivative $F'(x)$ would not have points of discontinuity of the first kind on $\mathbb{R}$. This is, however, in contradiction with the requirement that $F'(x) = \operatorname{sgn} x$, $x \in \mathbb{R}$. (Point 0 is a point of discontinuity of the first kind of the sgn function.)

The sufficient conditions for the existence of the strongly primitive function are listed in chapter 12.

Now let us focus on the problem of the correctness (uniqueness) of the strongly primitive function.

THEOREM 11.1.1. (i) Let F be the strongly primitive function to function f on interval I. Then for any $C \in \mathbb{R}$ the function $x \mapsto F(x) + C$, $x \in I$, is also strongly primitive to function f on I.
(ii) Let F and G be strongly primitive functions to function f on interval I. Then there exists a unique constant $C \in \mathbb{R}$ such that

$$\forall x \in I : \; F(x) = G(x) + C.$$

Proof. (i) The statement follows directly from definition 10.9.1.
(ii) If both F and G are strongly primitive functions to f on I then $\forall x \in I : (F-G)'(x) = 0$. According to corollary 1 (part e′) to theorem 2.8.4 function $F - G$ is a constant function on I. The uniqueness of this constant can be shown indirectly. □

Note 11.1.2. 1. The strongly primitive function on an interval is not defined uniquely. If it exists then there are infinitely many of them, differing from each other by a constant.

2. It is, however, easy to see that if function f has a strongly primitive function on interval I then to every point $(x_0, y_0) \in I \times \mathbb{R}$ there exists one and only one strongly primitive function F to f on I such that $(x_0, y_0) \in F$. (The graph of F passes through the point (x_0, y_0).)

Primitive Function

In example 1 we saw that even such a simple function as the sgn function does not have a strongly primitive function on $\mathbb{R}$. Hence it is sensible to extend the concept to a wider class of functions.

Definition 11.1.2. Let $f\colon (J \subset \mathbb{R}) \to \mathbb{R}$ and $I \subset J$ is again an interval. We say that *function* $F : I \to \mathbb{R}$ *is primitive* (or *is a potential*) to function f on interval I if

(i) $F \in C(I)$,

(ii) a finite set $M \subset I$ exists such that

$$\forall x \in I \setminus M : F'(x) = f(x).$$

Note 11.1.3. 1. The concept of the primitive function is the generalisation of the concept of the strongly primitive function. In fact, definition 10.9.1 is obtained from definition 10.9.2 for $M = \emptyset$.

2. Function $f\colon x \mapsto |x|$, $x \in \mathbb{R}$ is a primitive function to the sgn function on $\mathbb{R}$, with $M = \{0\}$, and function $F_1 : x \mapsto 2\sqrt{x}$, $x \geq 0$ is primitive to unbounded function

$$f_1 : x \mapsto \begin{cases} 5 & for \quad x = 0 \\ \frac{1}{\sqrt{x}}, & for \quad x \in (0, +\infty). \end{cases}$$

There is a similar statement on the uniqueness of the primitive function as there was for the strongly primitive function.

THEOREM 11.1.2. (i) Let F be the primitive function to function f on interval I and $C \in \mathbb{R}$. Then function $x \mapsto F(x) + C$, $x \in I$ is also a primitive function on I to function f.

(ii) Let F and G be primitive functions to function f on interval I. Then there exists one and only one constant $C \in \mathbb{R}$ such that

$$\forall x \in I: \ F(x) = G(x) + C.$$

(iii) If a function f has a primitive function on I then for any $x_0 \in I$, $y_0 \in \mathbb{R}$ there exists one and only one primitive function F to f on I such that $F(x_0) = y_0$.

Proof. Statement (i) is obvious.
(ii) There exist finite sets $M_1 \subset I$, $M_2 \subset I$ such that

$$\forall x \in I \setminus M_1: \ F'(x) = f(x), \quad \forall x \in I \setminus M_2: \ G'(x) = f(x).$$

The set $M = M_1 \cup M_2$ is a finite set in I and

$$\forall x \in I \setminus M: \ F'(x) = f(x) = G'(x).$$

Since $F \in C(I)$, $G \in C(I)$, according to part b) of example 2.8.4 there exists a constant $C \in \mathbb{R}$ such that $F(x) - G(x) = C$ for all $x \in I$. The uniqueness is obvious.
(iii) It is seen immediately that if G is a primitive function to f on I then $F: x \mapsto G(x) - G(x_0) + y_0$, $x \in I$ is the only primitive function to f on I with the required properties. $\square$

Sample Problem 11.1.1. Let $c_0, c_1, \ldots, c_m; l_1, \ldots, l_m$ be real numbers and let $x_0 = a < x_1 < x_2 < \ldots < x_{m-1} < x_m = b$. On interval $[a, b]$ we define the "step-like" function

$$f: x \mapsto \begin{cases} c_j, & x = x_j, \quad j = 0, 1, \ldots, m \\ l_i, & x \in (x_{i-1}, x_i), \quad i = 1, \ldots, m. \end{cases}$$

Prove that function

$$F: x \mapsto \begin{cases} l_1(x - x_0) & x \in [x_0, x_1) \\ l_2(x - x_1) + l_1(x_1 - x_0) & x \in [x_1, x_2) \\ l_m(x - x_{m-1}) + l_{m-1}(x_{m-1} - x_{m-2}) + \ldots + l_1(x_1 - x_0), & x \in [x_{m-1}, x_m] \end{cases}$$

is primitive to function f on interval [a,b].

`Solution`: Let M be the finite set of points $\{x_0, x_1, \ldots, x_m\} \subset [a, b]$. Then

$$\forall x \in [a, b] \setminus M: \ F'(x) = f(x)$$

and the statement is proved.

Note 11.1.4. 1. Note that the concepts of the strongly primitive function and primitive function have been defined on an interval. Both concepts may be extended to more general sets on $\mathbb{R}$. However, the crucial assumption that I is an interval should be kept in the uniqueness theorems (part (ii) of theorem 10.9.1 and parts (ii) and (iii) of theorem 10.9.2). For example, functions

$$f : x \mapsto \begin{cases} x, & x \in (0,1) \\ x+1, & x \in (1,2) \end{cases} \quad \text{and} \quad g : x \mapsto \begin{cases} x+1, & x \in (0,1) \\ x, & x \in (1,2) \end{cases}$$

are primitive (also strongly primitive) to function $x \mapsto 1,\ x \in (0,1) \cup (1,2)$, on $(0,1) \cup (1,2)$. Nevertheless the difference $f - g$ is not a constant.

2. Further generalisation of the concept of the primitive function can be obtained when set M in definition 10.9.2 is countable. In this case the primitive function to the Dirichlet function is any real constant (since Q, the rational number set, is countable). (See, for example, textbook [?], p. 359.)

Note 11.1.5. It is obvious that a function defined on $I \subset \mathbb{R}$ may not have a primitive function on I. For instance, function

$$f : x \mapsto \begin{cases} 5, & \text{for} \quad x = 0 \\ \frac{1}{x}, & \text{for} \quad x \in (0,1] \end{cases}$$

has no primitive function on $[0,1]$. On interval $(0,1]$, the primitive function to f is a function $x \mapsto \ln x,\ x \in (0,1]$.

The set of all functions $f : I \to \mathbb{R}$ whose primitive function on interval $I \subset \mathbb{R}$ exists will be denoted as $P(I)$.

Sample Problem 11.1.2. Let $f : (a,b) \to \mathbb{R}$ have a point of discontinuity of the first kind at point $x_0 \in (a,b)$. Furthermore, let F_1 (F_2) be the primitive function to restriction $f|_{(a,x_0)}$ $(f|_{(x_0,b)})$ such that proper limits $\lim\limits_{x \to x_0^-} F_1(x)$ and $\lim\limits_{x \to x_0^+} F_2(x)$ exist. Show then that a primitive function F to function f on interval (a,b) exists and

$$F(x) = \begin{cases} F_1(x) + k, & x \in (a, x_0), \quad (k \in \mathbb{R}) \\ F_2(x), & x \in (x_0, b) \\ \lim\limits_{x \to x_0^-} F_1(x) + k = \lim\limits_{x \to x_0^+} F_2(x), & x = x_0. \end{cases}$$

`Solution`: From the continuity of functions F_1 and F_2 on intervals (a,x_0) and (x_0,b), respectively, and from the choice of constant $k = \lim\limits_{x \to x_0^+} F_2(x) - \lim\limits_{x \to x_0^-} F_1(x)$ it follows that F is continuous on (a,b). That means that finite sets $M_1 \subset (a,x_0)$ and $M_2 \subset (x_0,b)$ exist such that $F_1'(x) = f(x)$ for all $x \in (a,x_0) \setminus M_1$ and $F_2'(x) = f(x)$ for all $x \in (x_0,b) \setminus M_2$. Hence $F'(x) = f(x)$ for all $x \in (a,b) \setminus (M_1 \cup M_2 \cup \{x_0\})$.

This example shows how to construct a primitive function to a discontinuous function with points of discontinuity of the first kind. This construction will be used frequently and will be referred to as "gluing".

Problems

1. Show that the existence of the primitive function to $f+g$ on interval I does not guarantee the existence of the primitive functions to f and g, separately, on I.

2. Let functions f and $f+g$ have primitive functions on $J \subset \mathbb{R}$. Find out if a primitive function to g on J exists.

3. Find out if primitive functions to f and g on $J \subset \mathbb{R}$ exist provided the primitive functions to $f+g$ and $f-g$ on J exist.

4. Prove that there exists no primitive function to function $x \mapsto \chi(x)$, $x \in I$, where χ is the Dirichlet function and $I \subset \mathbb{R}$ is an arbitrary interval in $\mathbb{R}$.

5. Find all primitive functions to the functions defined below together with their definition domains:

$$f_1 : x \mapsto \tfrac{1}{x},\ x \in (-\infty, 0); \qquad f_2 : x \mapsto \tfrac{1}{x},\ x \in (0, +\infty);$$

$$f_3 : x \mapsto \tfrac{1}{x},\ x \in (-\infty, 0) \cup (0, +\infty); \qquad f_4 : x \mapsto [x],\ x \in [0, n],\ n \in N;$$

$$f_5 : x \mapsto x\,|x|,\ x \in \mathbb{R}; \qquad f_6 : x \mapsto \begin{cases} \frac{1}{x}, & x \in \mathbb{R} \setminus \{0\} \\ 0, & x = 0. \end{cases}$$

6. Prove that $\tan^{-1} x + \cot^{-1} x = \pi$ for any $x \in \mathbb{R}$.

Answers

2. Yes, it does.

3. Yes, it does.

5. $F_1 : x \mapsto \ln(-x) + c,\ \ x \in (-\infty, 0)$;
$F_2 : x \mapsto \ln x + c,\ x \in (0, +\infty)$;
$F_3 : x \mapsto \begin{cases} \ln(-x) + c_1, & x \in (-\infty, 0) \\ \ln x + c_2, & x \in (0, +\infty) \end{cases}$
(a special case: $F_3 : x \mapsto \ln x + c,\ x \in \mathbb{R} \setminus \{0\}$);
$F_4 : x \mapsto x[x] - \frac{[x]([x]+1)}{2},\ x \in [0, n]$;
$F_5 : x \mapsto \begin{cases} \frac{x^3}{3} + c, & x \in [0, +\infty) \\ -\frac{x^3}{3} + c, & x \in (-\infty, 0); \end{cases}$

F_6 does not exist.

11.2 Riemann Integral and Its Properties

Inspired by the preceding section we are ready to claim that determining the trajectory of a moving particle out of its velocity given as a function of time effectively means finding the primitive function to the velocity function. In this section, besides the properties of the Riemann integral we shall show that when calculating the surface area one is again faced with the issue of finding the primitive function.

Riemann Integral

Before addressing the surface area question we shall study the following concept and its properties:

Definition 11.2.1. Let I be an interval in $\mathbb{R}$ and $f \in P(I)$. Let F be a primitive function to f on I and let $[a, b] \subset I$. Then the difference $F(b) - F(a)$ is called *a Riemann (definite) integral (R-integral) of function f from a to b*. It is denoted as $(N) \int_a^b f(x)\,\mathrm{d}x$. We also say that f is *Riemann integrable* on $[a, b]$. (The difference $F(b) - F(a)$ is denoted as $[F(x)]_a^b$ or $F(x)|_a^b$.)

Note 11.2.1. The definition of the Riemann integral is correct, *i.e.* its value does not depend on the choice of the primitive function.

Sample Problem 11.2.1. Calculate the Riemann integral from a to b of the following functions: a) $f_1 = \mathrm{sgn}$; b) $f_2 : x \mapsto \mathrm{sgn}x$, $x \in \mathbb{R} \setminus \{a, b\}$, $f_2(a) = c_1 \in \mathbb{R}$, $f_2(b) = c_2 \in \mathbb{R}$; c) the function defined in sample problem 2.10.1; d) an unbounded (in the neighbourhood of point 0) function

$$f_3 : x \mapsto \begin{cases} 2x \sin\frac{\pi}{x^2} - \frac{2\pi}{x}\cos\frac{\pi}{x^2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0, \end{cases}$$

for $a = -1$, $b = 1$.

`Solution`: For the first two cases

$$(N) \int_a^b \mathrm{sgn}x\,\mathrm{d}x = [|x|]_a^b = |b| - |a| = (N) \int_a^b f_2(x)\,\mathrm{d}x.$$

c)

$$(N) \int_a^b f(x)\,\mathrm{d}x = [F(x)]_a^b = \sum_{i=1}^{m} l_m(x_i - x_{i-1}).$$

d) It is easy to see that

$$F_3 : x \mapsto \begin{cases} x^2 \sin\frac{\pi}{x^2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

is a strongly primitive function to function f_3 on $\mathbb{R}$, and hence

$$(N) \int_{-1}^{1} f_3(x)\,\mathrm{d}x = F_3(x)|_{-1}^{1} = \sin\pi - \sin\pi = 0.$$

Note 11.2.2. In the case when $l_i > 0$ for $i = 1, \ldots, m$, the Riemann integral in example 1 c) gives a very intuitive geometrical interpretation — the surface area of the histogram-shaped object in figure 10.1.

Note 11.2.3. Note that the N-integral is defined on a closed interval. The extension of this concept to include an arbitrary (also unbounded) interval will be introduced in chapter 13.

Basic Properties of the N-integral

THEOREM 11.2.1. Let $J \subset \mathbb{R}$ be an interval, $a \in J$, $b \in J$, $f \in P(J)$ and $M \subset J$ is a finite set.

(i) If $f|_{J\setminus M} = g|_{J\setminus M}$ then

$$(N)\int_a^b g(x)\,\mathrm{d}x = (N)\int_a^b f(x)\,\mathrm{d}x.$$

(ii)

$$(N)\int_a^a f(x)\,\mathrm{d}x = 0, \qquad (N)\int_a^b f(x)\,\mathrm{d}x = -(N)\int_b^a f(x)\,\mathrm{d}x.$$

(iii) For function $k:\ x \mapsto c \in \mathbb{R},\ x \in J\setminus M$ the following is true:

$$(N)\int_a^b k(x)\,\mathrm{d}x = c(b-a).$$

Proof. (i) Taking the assumptions into account it follows that the primitive functions to functions f and g on interval J differ by a constant and the statement is then obvious.

(ii) Let F be a primitive function to f on J. Then

$$(N)\int_a^a f(x)\,\mathrm{d}x = F(a) - F(a) = 0,$$

$$(N)\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a) = -[F(a) - F(b)] = -(N)\int_b^a f(x)\,\mathrm{d}x.$$

(iii) According to example 2.10.1 the primitive function to function k on interval J is $F:\ x \mapsto c(x-a)$, $x \in J$. The result then follows trivially.

□

THEOREM 11.2.2. Let a and b be numbers from interval $J \subset \mathbb{R}$, $C_1 \in \mathbb{R}$, $C_2 \in \mathbb{R}$, and $f \in P(J)$ and $g \in P(J)$. Then $(C_1 f + C_2 g) \in P(J)$ and

$$(N)\int_a^b [C_1 f(x) + C_2 g(x)]\,\mathrm{d}x = C_1\,(N)\int_a^b f(x)\,\mathrm{d}x + C_2\,(N)\int_a^b g(x)\,\mathrm{d}x.$$

Proof. Let F and G be primitive functions to f and g, respectively, both on interval J. Then $C_1F + C_2G$ is a primitive function to $C_1f + C_2g$ and the following relations hold:

$$\begin{aligned}(N)\int_a^b [C_1 f(x) + C_2 g(x)]\,\mathrm{d}x &= C_1F(b) + C_2G(b) - C_1F(a) - C_2G(a) = \\ &= C_1[F(b) - F(a)] + C_2[G(b) - G(a)] = \\ &= C_1\,(N)\int_a^b f(x)\,\mathrm{d}x + C_2\,(N)\int_a^b g(x)\,\mathrm{d}x,\end{aligned}$$

and the proof is done. □

THEOREM 11.2.3. Let a, b and c be numbers from interval $J \subset \mathbb{R}$, and let $f \in P(J)$. Then

$$(N)\int_a^b f(x)\,\mathrm{d}x = (N)\int_a^c f(x)\,\mathrm{d}x + (N)\int_c^b f(x)\,\mathrm{d}x$$

Proof. Let F be a primitive function to f on interval J. Then

$$\begin{aligned}(N)\int_a^b f(x)\,\mathrm{d}x &= F(b) - F(a) = [F(b) - F(c)] + [F(c) - F(a)] \\ &= (N)\int_a^c f(x)\,\mathrm{d}x + (N)\int_c^b f(x)\,\mathrm{d}x\end{aligned}$$

and the proof is done. □

Using the method of mathematical induction and theorem 10.10.3 one can state

Corollary 11.2.1. Let x_0, x_1, $\ldots$, x_n be numbers from interval $J \subset \mathbb{R}$, and let $f \in P(J)$. Then

$$(N)\int_{x_0}^{x_n} f(x)\,\mathrm{d}x = (N)\int_{x_0}^{x_1} f(x)\,\mathrm{d}x + \ldots + (N)\int_{x_{n-1}}^{x_n} f(x)\,\mathrm{d}x.$$

THEOREM 11.2.4. Let $J \subset \mathbb{R}$ be an interval, $a \in J$, $b \in J$, $a < b$, and $f \in P(J)$ and $g \in P(J)$ be such that $f \geq g$ on interval $[a, b]$. Then

$$(N)\int_a^b f(x)\,\mathrm{d}x \geq (N)\int_a^b g(x)\,\mathrm{d}x$$

Proof. Let F (G) be a primitive function to f (g) on interval J. There exist finite sets $M_1 \subset J$ and $M_2 \subset J$ such that

$$\forall x \in J \setminus M_1 : F'(x) = f(x),$$

and

$$\forall x \in J \setminus M_2 : G'(x) = g(x).$$

Thus for any $x \in J \setminus (M_1 \cup M_2)$

$$(G - F)'(x) = (g - f)(x) \geq 0.$$

That means that the difference $G - F$ is a non-decreasing function on J (compare with sample problem 8.4.2 part a)) and hence

$$(N)\int_a^b (g-f)(x)\,\mathrm{d}x = [G(b)-F(b)] - [G(a)-F(a)] \geq 0.$$

The proof is then easily completed using theorem 10.10.2. □

Corollary 11.2.2. Let a, b, c and d be numbers from interval $J \subset \mathbb{R}$ such that $a \leq c \leq d \leq b$ and let $g \in P(J)$ be a non-negative function on interval $[a,b]$. Then the following relations hold:

$$(N)\int_a^b g(x)\,\mathrm{d}x \geq 0$$

and

$$(N)\int_c^d g(x)\,\mathrm{d}x \leq (N)\int_a^b g(x)\,\mathrm{d}x.$$

Proof. The first inequality follows from theorem 10.10.4 for $f : x \mapsto 0,\ x \in [a,b]$. The second inequality is obtained using the corollary to theorem 10.10.3 for this specific case:

$$(N)\int_a^b g(x)\,\mathrm{d}x = (N)\int_a^c g(x)\,\mathrm{d}x + (N)\int_c^d g(x)\,\mathrm{d}x + (N)\int_d^b g(x)\,\mathrm{d}x \geq (N)\int_c^d g(x)\,\mathrm{d}x.$$

□

The following theorem discusses the integral representation of the primitive function:

THEOREM 11.2.5. Let a and x be numbers from interval $J \subset \mathbb{R}$ and let $f \in P(J)$. Then function

$$F : x \mapsto (N)\int_a^x f(t)\,\mathrm{d}t, \qquad x \in J,$$

is the primitive function to function f on J with the property $F(a) = 0$.

Proof. Let us denote a primitive function to f on J as G. We then get

$$(N)\int_a^x f(t)\,\mathrm{d}t = G(x) - G(a).$$

Thus a finite set $M \subset J$ exists such that

$$\forall x \in J \setminus M : \ \frac{d}{\mathrm{d}x}(N)\int_a^x f(t)\,\mathrm{d}t = \frac{dG}{\mathrm{d}x}(x) = f(x),$$

which completes the proof. □

Surface Area as Primitive Function

The numerical evaluation of the surface area of two-dimensional objects has been one of the oldest mathematical tasks. The area's axiomatic definition for so-called elementary regions is introduced in section 12.2. There it is also shown that this definition of the surface area is correct (that means the surface area of an elementary region exists and is defined uniquely) and the surface area of elementary regions can be expressed using the Riemann integral.

Here we shall focus on the relation between the surface area $P(f;\, c,\, d)$ of a curved trapezoid

$$M(f;\, c,\, d) = \{(x,y) \in \mathbb{R}^2;\; x \in [c,d] \subset [a,b],\; 0 \le y \le f(x)\},$$

where function $f : [a,b] \to \mathbb{R}$ is continuous on interval $[a,b]$ (figure 10.2), and the primitive function to f.

THEOREM 11.2.6. Let function $f : [a,b] \to \mathbb{R}$ be continuous and non-negative on interval $[a,b]$ and let for every $x \in (a,b]$ a surface area of a curved trapezoid $M(f;\, a,\, x)$ exist. Then function

$$p_f : x \mapsto \begin{cases} P(f;\, a,\, x) & \text{for } x \in (a,b] \\ 0 & \text{for } x = a \end{cases}$$

is the strongly primitive function to function f on $[a,b]$.

Proof. Let $x_0 \in [a,b)$. Continuity of function f implies

$$\forall \epsilon > 0\; \exists 0 < \delta < b - x_0\; \forall x \in [x_0, x_0 + \delta) :\; f(x_0) - \epsilon < f(x) < f(x_0) + \epsilon.$$

From this observation and due to well-known (natural) properties of surface areas that agree with our intuition, and also based on the non-negativity of function f we obtain

$$[f(x_0) - \epsilon]\, \delta \le \max\{[f(x_0) - \epsilon]\, \delta, 0\} \le P(f;\, x_0,\, x_0 + \delta) \le [f(x_0) + \epsilon]\, \delta,$$

which means that the surface area of the largest rectangle placed inside a curved trapezoid $M(f;\, x_0,\, x_0 + \delta)$ is not greater than the surface area of the trapezoid, and the latter is in turn not greater than the surface area of the smallest rectangle containing the trapezoid. Furthermore we have

$$P(f;\, a,\, x_0) + P(f;\, x_0,\, x_0 + \delta) = P(f;\, a,\, x_0 + \delta),$$

which means that the sum of the surface areas of curved trapezoids $M(f;\, a,\, x_0)$ and $M(f;\, x_0,\, x_0 + \delta)$ (the two pieces we break trapezoid $M(f;\, a,\, x_0 + \delta)$ into) is equal to the surface area of trapezoid $M(f;\, a,\, x_0 + \delta)$.

From these relations we get

$$\left| \frac{p_f(x_0 + \delta) - p_f(x_0)}{\delta} - f(x_0) \right| < \epsilon.$$

If in this inequality one considers the limits $\delta \to 0^+$ and then $\epsilon \to 0^+$, then the relation $p'_{f+}(x_0) = f(x_0)$ emerges. The analogous statement, $p'_{f-}(x_0) = f(x_0)$ for $x_0 \in (a, b]$, for the left derivative is shown in the same way. Thus we can conclude that $p'_f(x) = f(x)$ for arbitrary $x \in [a, b]$ and the proof is done. □

Corollary 11.2.3. Let $J \subset \mathbb{R}$ be an interval and function $f : J \to \mathbb{R}$ be continuous and non-negative on J. If for any interval $[a, b] \subset J$ $(a \in \mathbb{R},\, b \in \mathbb{R})$ the surface area $P(f;\, a,\, b)$ of a curved trapezoid $M(f;\, a,\, b)$ exists then the strongly primitive function to f on J exists and the following relation holds:

$$P(f;\, a,\, b) = (N) \int_a^b f(x)\,\mathrm{d}x.$$

Proof. Based on theorem 10.10.6 a strongly primitive function $x \mapsto p_f(x)$, $x \in [a, b]$ to function f exists on any interval $[a, b] \subset J$, and therefore it exists on the whole interval J. Furthermore, from the definition of the N-integral it follows that

$$(N) \int_a^b f(x)\,\mathrm{d}x = p_f(b) - p_f(a) = P(f;\, a,\, b)$$

since $p_f(a) = 0$. □

Note 11.2.4. In section 11.5, we shall prove an even stronger statement than the one in the corollary above. Here we just quote the stronger statement:
Every continuous function on an interval in $\mathbb{R}$ has a strongly primitive function on this interval.

Problems

1. Find all $\alpha \in \mathbb{R}$ for which the Riemann integral $(N) \int_0^1 f(x)\,\mathrm{d}x$ exists for

$$f : x \mapsto \begin{cases} x^{-\alpha}, & x \neq 0 \\ c, & x = 0 \end{cases} \qquad \text{where } c \in \mathbb{R}.$$

2. Calculate the following N-integrals provided they exist:
   a) $(N) \displaystyle\int_{-2}^{2} \max\,(1, x^4)\,\mathrm{d}x$
   b) $(N) \displaystyle\int_{-\pi/2}^{+\pi/2} x \operatorname{sgn}(\sin x)\,\mathrm{d}x$
   c) $(N) \displaystyle\int_{-1}^{1} f(x)\,\mathrm{d}x$ for $f : x \mapsto \begin{cases} (1-x^2)^{-1/2}, & x \in (-1, 1) \\ c \in \mathbb{R}, & x = -1 \text{ or } x = +1, \end{cases}$
   d) $(N) \displaystyle\int_{0}^{2} [2^x]\,\mathrm{d}x.$

3. Find out if function

$$f : x \mapsto \begin{cases} |x|^{-1}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

   is Riemann integrable. Consider separately the following three intervals: $[-1, 1]$, $[-1, 0]$ and $[-2, -1]$. If it is, calculate the Riemann integral.

4. Prove the corollary that follows after theorem 10.10.3.

5. Show that the following statement is true:
   Let $a < c < b$ be real numbers, and $f \in P([a, c])$ and $f \in P([c, b])$. Then $f \in P([a, b])$ and
   $$(N)\int_a^c f(x)\,\mathrm{d}x + (N)\int_c^b f(x)\,\mathrm{d}x = (N)\int_a^b f(x)\,\mathrm{d}x.$$

6. Formulate and prove the theorem on the N-integral of a sum of functions (analogous to theorem 10.10.2) for an arbitrary finite number of terms.

7. Calculate
   a) $\dfrac{\mathrm{d}}{\mathrm{d}x}(N)\displaystyle\int_a^b e^{-x^2}\,\mathrm{d}x,\quad \dfrac{\mathrm{d}}{\mathrm{d}a}(N)\int_a^b e^{-x^2}\,\mathrm{d}x,\quad \dfrac{\mathrm{d}}{\mathrm{d}b}(N)\int_a^b e^{-x^2}\,\mathrm{d}x,$
   where $a \in \mathbb{R}$, $b \in \mathbb{R}$.
   b) $\dfrac{\mathrm{d}}{\mathrm{d}x}(N)\displaystyle\int_x^{(\sin x)^{1/3}} \sin^{-1} t^3\,\mathrm{d}t,\quad$ for $x \in [-\pi/2, +\pi/2]$,
   c) $\displaystyle\lim_{x\to 0}\frac{\int_0^x \sin^{-1} t\,\mathrm{d}t}{x},\qquad \lim_{x\to 0}\frac{\int_0^{\sin x}(\tan t)^{1/n}\,\mathrm{d}t}{\int_0^{\tan x}(\sin t)^{1/n}\,\mathrm{d}t},\qquad n \in N.$

8. Calculate the surface area of two-dimensional planar objects bounded by the following curves:
   a) the x axis, graph of function $x \mapsto |x|^3$ $(x \in \mathbb{R})$, and lines $x = -1$ and $x = 1$;
   b) the graphs of functions $x \mapsto x^2 - 2$ $(x \in \mathbb{R})$ and $x \mapsto -x^2 + 2$ $(x \in \mathbb{R})$.

Answers

1 $\alpha < 1$.

2 a) 12/5, b) $\pi^2/4$, c) π, d) $5 - \ln 3/\ln 2$.

3 f is Riemann integrable only on the last of the three intervals. The result is $\ln 2$.

7 a) 0, $-e^{-a^2}$, e^{-b^2}; b) $\frac{x}{3}(\sin x)^{-2/3}\cos x - \sin^{-1} x^3$; c) 1, 1.

8 a) $\frac{1}{2}$; b) $\frac{16}{3}\sqrt{2}$.

11.3 Indefinite Integral

In section 2 we saw that the hard part in the calculation of the N-integral consists of the determination of the primitive function to a given function on an interval under consideration. Therefore, in this section we shall focus on different alternative methods for finding the primitive function.

Concept of Indefinite Integral

We already know from theorem 10.9.1 that the primitive function to a given function is not uniquely determined. If a single primitive function to f on an interval I exists, then there are infinitely many of them, differing from each other by an additive constant.

Definition 11.3.1. The set of all primitive functions to $f : (J \subset \mathbb{R}) \to \mathbb{R}$ on an interval $I \subset J$ will be called an *indefinite integral of function* f *on* I and denoted by a symbol $\int f(x)\,\mathrm{d}x,\ x \in I$.

Note 11.3.1. 1. Function f is called an integrand, the symbol $\int$ is referred to as the integral sign, and letters $\mathrm{d}x$ denote the integration variable (x in this case). The meaning of the integration variable is as follows: calculating the derivative with respect to the integration variable x of any function from the set $\int f(x)\,\mathrm{d}x$ on $I \setminus M$ (M is a finite subset of I) one obtains restriction $f|_{I\setminus M}$.
2. If F is a primitive function to f on I we shall write

$$\int f(x)\,\mathrm{d}x \overset{\mathrm{c}}{=} F(x),\ x \in I \quad \text{or} \quad \int f(x)\,\mathrm{d}x = \{F(x) + c\}|_{c\in\mathbb{R}},\ x \in I.$$

For instance $\int e^{2x}\,\mathrm{d}x \overset{\mathrm{c}}{=} \frac{1}{2}e^{2x} =: F(x),\ x \in \mathbb{R}$ since $\frac{dF}{\mathrm{d}x}(x) = e^{2x}$ for $x \in \mathbb{R}$.
3. The indefinite integral will usually be sought on the so-called maximal set $A \setminus \mathbb{R}$ which does not have to be an interval but could also be a union of a finite number of intervals that satisfies the following statement: If f has a primitive function on set B then $B \subset A$. For example,

$$\int \frac{1}{x}\,\mathrm{d}x \overset{\mathrm{c}}{=} \ln|x|, \quad x \in (-\infty, 0) \cup (0, +\infty)$$
$$\int x^{-\frac{1}{3}}\,\mathrm{d}x \overset{\mathrm{c}}{=} \frac{3}{2}x^{\frac{2}{3}}, \quad x \in (-\infty, 0) \cup (0, +\infty).$$

Basic Indefinite Integrals

The following indefinite integrals can be derived from the rules for derivatives (section 8.2):

1. $\displaystyle\int 0\,\mathrm{d}x \overset{\mathrm{c}}{=} 0, \quad x \in \mathbb{R}$

2. $\displaystyle\int 1\,\mathrm{d}x =: \int \mathrm{d}x \overset{\mathrm{c}}{=} x, \quad x \in \mathbb{R}$

3. $\displaystyle\int x^n\,\mathrm{d}x \overset{\mathrm{c}}{=} \frac{x^{n+1}}{n+1}, \quad x \in \mathbb{R}\ (n \in N).$
   If $n \in \mathbb{R} \setminus \{-1\}$, the formula holds for $x \in \mathbb{R}^+$. For some values of n the formula is valid on an extended set (see, *e.g.* part 3 of the last note);

4. $\displaystyle\int \frac{1}{x}\,\mathrm{d}x \overset{\mathrm{c}}{=} \ln x, \quad x \in \mathbb{R}^+, \qquad \int \frac{1}{x}\,\mathrm{d}x \overset{\mathrm{c}}{=} \ln(-x), \quad x \in \mathbb{R}^-$
   $\displaystyle\left(\text{or} \quad \int \frac{1}{x}\,\mathrm{d}x \overset{\mathrm{c}}{=} \ln|x|, \quad x \in \mathbb{R} \setminus \{0\}\right)$

5. $\displaystyle\int \frac{1}{1+x^2}\,\mathrm{d}x \overset{c}{=} \tan^{-1} x, \quad x \in \mathbb{R}, \qquad \int \frac{1}{1+x^2}\,\mathrm{d}x \overset{c}{=} -\cot^{-1} x, \quad x \in \mathbb{R}$

6. $\displaystyle\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x \overset{c}{=} \sin^{-1} x, \quad x \in (-1,1), \qquad \int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x \overset{c}{=} -\cos^{-1} x, \quad x \in (-1,1)$

7. $\displaystyle\int e^x\,\mathrm{d}x \overset{c}{=} e^x, \quad x \in \mathbb{R}$

8. $\displaystyle\int a^x\,\mathrm{d}x \overset{c}{=} \frac{a^x}{\ln a}, \quad x \in \mathbb{R}\ (a \in \mathbb{R}^+ \setminus \{1\})$

9. $\displaystyle\int \sin x\,\mathrm{d}x \overset{c}{=} -\cos x, \quad x \in \mathbb{R}$

10. $\displaystyle\int \cos x\,\mathrm{d}x \overset{c}{=} \sin x, \quad x \in \mathbb{R}$

11. $\displaystyle\int \frac{1}{(\sin x)^2}\,\mathrm{d}x \overset{c}{=} -\cot x, \quad x \in \bigcup_{k\in Z} (k\pi, \pi + k\pi)$

12. $\displaystyle\int \frac{1}{(\cos x)^2}\,\mathrm{d}x \overset{c}{=} \tan x, \quad x \in \bigcup_{k\in Z} (k\pi - \frac{\pi}{2}, k\pi + \frac{\pi}{2})$

13. $\displaystyle\int \sinh x\,\mathrm{d}x \overset{c}{=} \cosh x, \quad x \in \mathbb{R}$

14. $\displaystyle\int \cosh x\,\mathrm{d}x \overset{c}{=} \sinh x, \quad x \in \mathbb{R}$

15. $\displaystyle\int \frac{1}{(\sinh x)^2}\,\mathrm{d}x \overset{c}{=} -\coth x, \quad x \in \mathbb{R} \setminus \{0\}$

16. $\displaystyle\int \frac{1}{(\cosh x)^2}\,\mathrm{d}x \overset{c}{=} \tanh x, \quad x \in \mathbb{R}$

In order to speed up calculations the following formulas are often useful:

17. $\displaystyle\int \frac{1}{\sqrt{x^2+1}}\,\mathrm{d}x \overset{c}{=} \ln(x + \sqrt{x^2+1}), \quad x \in \mathbb{R}$

18. $\displaystyle\int \frac{1}{\sqrt{x^2-1}}\,\mathrm{d}x \overset{c}{=} \ln(x + \sqrt{x^2-1}), \quad x \in (1, +\infty)$

19. $\displaystyle\int \frac{1}{x^2-1}\,\mathrm{d}x \overset{c}{=} \frac{1}{2} \ln\left|\frac{x-1}{x+1}\right|, \quad x \in \mathbb{R} \setminus \{-1, 1\}$

20. Let $f : (I \subset \mathbb{R}) \to \mathbb{R}$ (I is an interval) be differentiable on I. Then
$\displaystyle\int \frac{f'(x)}{f(x)}\,\mathrm{d}x \overset{c}{=} \ln|f(x)|, \quad x \in \{z \in I;\ f(z) \neq 0\}$

21. Let $f : (I \subset \mathbb{R}) \to \mathbb{R}$ (I is an interval) be differentiable on I. Then
$\displaystyle\int \frac{f'(x)}{\sqrt{f(x)}}\,\mathrm{d}x \overset{c}{=} 2\sqrt{f(x)}, \quad x \in \{z \in I;\ f(z) > 0\}$

Formulas 17-21 can be easily verified by performing the derivative on the right-hand sides, or with the use of forthcoming theorems 3 or 4.

Based on the formulas listed above one can calculate indefinite integrals for just a narrow class of functions. Hence in the following text we shall develop integration methods allowing us to transform integrals of a wide class of functions in such a way that the formulas above can be applied. We warn the reader right here that such a transformation may not be possible for every function $f \in P(I)$. For instance, indefinite integrals $\int \cos x^2 \, \mathrm{d}x,\ x \in \mathbb{R}$; $\int e^{-x^2} \, \mathrm{d}x,\ x \in \mathbb{R}$, although they do exist (see note 4.10.2), cannot be expressed in terms of the elementary functions (and, therefore, cannot be transformed into any of the formulas 1-21 for an elementary function f). Similarly, neither of the integrals $\int \frac{\sin x}{x} \, \mathrm{d}x,\ x \in \mathbb{R} \setminus \{0\}$ and $\int \frac{1}{\sqrt{x^3+1}} \, \mathrm{d}x,\ x \in \mathbb{R} \setminus \{-1\}$ belongs to the set of the elementary functions. (A complex proof of the last statement can be found in textbook [2].) There is no general solution to this problem.

Decomposition Method

THEOREM 11.3.1. Let $k_1, \ldots, k_m$ be real numbers and indefinite integrals $\int f_i(x) \, \mathrm{d}x \overset{c}{=} F_i(x)$ exist on interval $I \subset \mathbb{R}$ for $i = 1, \ldots, m$. Then the following indefinite integral exists

$$\int \left[\sum_{i=1}^{m} k_i f_i(x) \right] \mathrm{d}x \overset{c}{=} \sum_{i=1}^{m} k_i F_i(x), \quad x \in I.$$

Proof. The assumptions imply that finite sets $M_i \subset I,\ i = 1, \ldots, m$ exist such that

$$\forall x \in I \setminus M_i \ \forall i = 1, \ldots, m \ F_i'(x) = f_i(x) \quad (\in \mathbb{R}).$$

Set $M \cup_{i=1}^{m} M_i$ is also a finite subset of I and the following relation holds

$$\forall x \in I \setminus M \ \left(\sum_{i=1}^{m} k_i F_i \right)'(x) = \sum_{i=1}^{m} k_i F_i'(x) = \sum_{i=1}^{m} k_i f_i(x)$$

which means that the proof is done. □

Note 11.3.2. The statement in theorem 10.11.1 is usually written in the following form

$$\int \left[\sum_{i=1}^{m} k_i f_i(x) \right] \mathrm{d}x = \sum_{i=1}^{m} k_i \int f_i(x) \, \mathrm{d}x, \quad x \in J.$$

Sample Problem 11.3.1. Calculate the following indefinite integrals on their respective maximal sets

a) $\int \frac{2}{x^4-1} \, \mathrm{d}x$; b) $\int P_m(x) \, \mathrm{d}x$, where P_m is a polynomial of degree m $x \mapsto a_m x^m + \ldots + a_1 x + a_0$, $x \in \mathbb{R}$ and $a_i \in \mathbb{R}$ for $i = 0, 1, \ldots, m$; c) $\int \frac{1}{\sin x} \, \mathrm{d}x$.

`Solution`: a) Using formulas 5 and 19 one obtains

$$\int \frac{2}{x^4-1} \, \mathrm{d}x = \int \frac{1-x^2+1+x^2}{(x^2-1)(x^2+1)} \, \mathrm{d}x = \int \frac{1}{x^2-1} \, \mathrm{d}x - \int \frac{1}{x^2+1} \, \mathrm{d}x \overset{c}{=} \frac{1}{2} \ln \frac{x-1}{x+1} - \tan^{-1} x$$

for $x \in \mathbb{R} \setminus \{-1, 1\}$.

b) $\displaystyle\int P_m(x)\,\mathrm{d}x = \int \left[\sum_{i=0}^{m} a_i x^i\right] \mathrm{d}x = \sum_{i=0}^{m} a_i \int x^i\,\mathrm{d}x = \sum_{i=0}^{m} \frac{a_i}{i+1} x^{i+1}, \quad x \in \mathbb{R}.$

c) Using formula 20 one finds

$$\int \frac{\mathrm{d}x}{\sin x} = \int \frac{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}}{2\,\sin\frac{x}{2}\,\cos\frac{x}{2}}\,\mathrm{d}x = \int \frac{\frac{1}{2}\sin\frac{x}{2}}{\cos\frac{x}{2}}\,\mathrm{d}x + \int \frac{\frac{1}{2}\cos\frac{x}{2}}{\sin\frac{x}{2}}\,\mathrm{d}x \overset{\mathrm{c}}{=} \ln\left|\sin\frac{x}{2}\right| - \ln\left|\cos\frac{x}{2}\right| = \ln\left|\tan\frac{x}{2}\right|$$

for $x \in \bigcup_{k \in Z} (k\pi, k\pi + \pi)$.

Method of Integration By Parts

Let F and G be the primitive functions to functions f and g on interval $I \subset \mathbb{R}$. In general, it is not true that product FG is the primitive function to function fg. That is because $(FG)' = F'G + FG' = fG + Fg$ on I. However, the following theorem holds:

THEOREM 11.3.2. Let $f \in P(I)$ and $g \in P(I)$ where $\int f(x)\,\mathrm{d}x \overset{\mathrm{c}}{=} F(x)$, $\int g(x)\,\mathrm{d}x \overset{\mathrm{c}}{=} G(x)$, $x \in I$ and furthermore let $fG \in P(I)$ (or $Fg \in P(I)$). Then also $Fg \in P(I)$ (or $fG \in P(I)$) and

$$\int f(x)\,G(x)\,\mathrm{d}x + \int F(x)\,g(x)\,\mathrm{d}x \overset{\mathrm{c}}{=} F(x)\,G(x), \quad x \in I. \tag{11.1}$$

Proof. Let $fG \in P(I)$ and $\int f(x)G(x) \overset{\mathrm{c}}{=} H(x)$, $x \in I$. Then a finite set $M \subset I$ exists such that

$$\forall x \in I \setminus M : (FG - H)'(x) = f(x)G(x) + F(x)g(x) - f(x)G(x) = F(x)g(x).$$

That means that function $FG - H$ is primitive to Fg on I and $Fg \in P(I)$. The second part of the statement follows from theorem 10.11.1 and relation

$$\int (FG)'(x)\,\mathrm{d}x \overset{\mathrm{c}}{=} F(x)G(x), \quad x \in I$$

which completes the proof. □

Note 11.3.3. Theorem 10.11.2 on integration by parts is of large practical significance. However, when applied we shall use $\int f(x)G(x)\,\mathrm{d}x = F(x)G(x) - \int F(x)g(x)\,\mathrm{d}x$, $x \in I$, instead of (10.9).

Sample Problem 11.3.2. Calculate the following indefinite integrals on their maximal sets

$$\text{a)} \int x\,\cos x\,\mathrm{d}x; \quad \text{b)} \int \sin^{-1} x\,\mathrm{d}x;$$

`Solution`: a) If we choose $f : x \mapsto \cos x$, $x \in \mathbb{R}$ and $G : x \mapsto x$, $x \in \mathbb{R}$ then $F : x \mapsto \sin x$, $x \in \mathbb{R}$ and $g : x \mapsto 1$, $x \in \mathbb{R}$ and according to theorem 10.11.2

$$\int x\,\cos x\,\mathrm{d}x = x \sin x - \int \sin x\,\mathrm{d}x \overset{\mathrm{c}}{=} x\sin x + \cos x$$

for $x \in \mathbb{R}$.
b) Let us choose $f : x \mapsto 1, x \in (-1,1)$ and $G : x \mapsto \sin^{-1} x, x \in (-1,1)$. Then $F : x \mapsto x, x \in (-1,1)$ and $g : x \mapsto \frac{1}{\sqrt{1-x^2}}, x \in (-1,1)$. Now using formula 21 we have

$$\int \sin^{-1} x \,\mathrm{d}x = x \sin^{-1} x + \frac{1}{2} \int \frac{-2x}{\sqrt{1-x^2}} \,\mathrm{d}x \stackrel{\mathrm{c}}{=} x \sin^{-1} x + \sqrt{1-x^2}$$

for $x \in (-1,1)$. According to theorem 6.8.4 (see also problem 6a.8.4) the primitive function that has been found above can be extended to include the closed interval $[-1,1]$. Then one can write

$$\int \sin^{-1} x \,\mathrm{d}x \stackrel{\mathrm{c}}{=} x \sin^{-1} x + \sqrt{1-x^2}, \; x \in [-1,1].$$

Sample Problem 11.3.3. Prove that for indefinite integral $I_n(x) = \int \frac{1}{(1+x^2)^n} \,\mathrm{d}x$ on $\mathbb{R}$ the following recurrent relation holds

$$I_{n+1}(x) = \frac{1}{2n} \frac{x}{(1+x^2)^n} + \frac{2n-1}{2n} I_n(x), \quad x \in \mathbb{R}, \tag{11.2}$$

where $n \in N$.
`Solution`: Choosing $f : x \mapsto 1, x \in \mathbb{R}$ and $G : x \mapsto \frac{1}{(1+x^2)^n}, x \in \mathbb{R}$ one finds $F : x \mapsto x, x \in \mathbb{R}$ and $g : x \mapsto \frac{2nx}{(1+x^2)^{n+1}}, x \in \mathbb{R}$. Then according to theorem 10.11.2

$$\begin{aligned} I_n(x) &= \frac{x}{(1+x^2)^n} + 2n \int \frac{x^2}{(1+x^2)^{n+1}} \,\mathrm{d}x = \\ &= \frac{x}{(1+x^2)^n} + 2n \int \frac{x^2+1}{(1+x^2)^{n+1}} \,\mathrm{d}x - 2n \int \frac{1}{(1+x^2)^{n+1}} \,\mathrm{d}x = \\ &= \frac{x}{(1+x^2)^n} - 2n\, I_{n+1}(x) + 2n\, I_n(x), \quad x \in \mathbb{R}. \end{aligned}$$

This easily simplifies to formula (10.10)

Note 11.3.4. Note that the N-integral formula (10.9) can be written in the form

$$(N) \int_{x_0}^{x} f(t)\, G(t) \,\mathrm{d}t + (N) \int_{x_0}^{x} F(t)\, g(t) \,\mathrm{d}t = [F(x)G(x)]_{x_0}^{x}$$

for $x \in I$ and $x_0 \in I$ (compare with theorem 5.10.2).

Substitution Method

The basic theorems describing the substitution method are obtained from the theorem on the derivative of a composite function (the chain rule):

THEOREM 11.3.3. Let I_1 and I_2 be intervals in $\mathbb{R}$ and $f : I_1 \to \mathbb{R}$. Let function $\phi : I_2 \to I_1$ be differentiable on I_2. Then for $f \in P(I_1)$, and F being strongly primitive function to f on I_1 the composite function $F \circ f$ is strongly primitive function to function $(f \circ \phi)\phi'$ on I_2.

Proof. Since $F'(t) = f(t)$ for $t \in I_1$ then according to theorem 2.8.2 on the derivative of a composite function the following equality holds:

$$(F \circ \phi)'(x) = (F' \circ \phi)(x) \,.\, \phi'(x) = (f \circ \phi)(x) \,.\, \phi'(x)$$

for $x \in I_2$. $\square$

Note 11.3.5. 1. Using the indefinite integral the statement in theorem 3 can be expressed in the form

$$\int (f \circ \phi)(x)\phi'(x)\,\mathrm{d}x = \int f(t)\,\mathrm{d}t \tag{11.3}$$

for $t = \phi(x)$, $x \in I_2$. Alternatively, the N-integral can be used in the form

$$(N) \int_{x_0}^{x} (f \circ \phi)(s)\phi'(s)\,\mathrm{d}s = (N) \int_{\phi(x_0)}^{\phi(x)} f(t)\,\mathrm{d}t$$

for $x \in I_2$, $x_0 \in I_2$.

2. Theorem 3 can be applied in the following sense: when calculating the integral on the left side of equation (10.11) in the actual calculation one introduces $\phi(x) = t$ and symbol $\phi'(x)\mathrm{d}x$ is then replaced by $\mathrm{d}t$.

Sample Problem 11.3.4. Calculate

$$I := \int \frac{1}{x\,\ln x}\,\mathrm{d}x$$

on the maximal set.

`Solution`: Let $\phi : x \mapsto \ln x$, $x \in \mathbb{R}^+ \setminus \{1\}$. Restrictions $\phi|_{(0,1)}$ and $\phi|_{(1,\infty)}$ satisfy the conditions of theorem 3. Therefore $\left(\mathrm{d}t = \frac{1}{x}\mathrm{d}x\right)$

$$I = \int \frac{1}{t}\mathrm{d}t \overset{\mathrm{c}}{=} \ln|t| \quad \text{for } t \in (-\infty, 0) \cup (0, +\infty)$$

and hence

$$I \overset{\mathrm{c}}{=} \ln\,|\ln x|, \quad x \in (0,1) \cup (1, +\infty).$$

If the primitive function to function $(f \circ \phi) \,.\, \phi'$ is known then the primitive function to function f can be calculated using the following theorem:

THEOREM 11.3.4. Let I_1 and I_2 be intervals in $\mathbb{R}$ and $f : I_1 \to \mathbb{R}$. Let function $\phi : I_2 \to I_1$ be a bijection and has proper derivative $\phi'(t) \neq 0$ on interval I_2. If $(f \circ \phi) \,.\, \phi' \in P(I_2)$ and G is its primitive function on I_2, then function $G \circ \phi^{-1}$ (ϕ^{-1} is the inverse function to ϕ) is primitive to f on I_1.

Proof. A finite set $M_2 \subset I_2$ exists such that

$$\forall t \in I_2 \setminus M_2 : \; G'(t) = (f \circ \phi)(t)\phi'(t).$$

Since ϕ is a bijection set $M_1 := \phi(M_2)$ is finite. Then according to the theorem on the derivative of composite (theorem 2.8.2) and inverse (theorem 4.8.2) functions the following relation holds for all $x \in I_1 \setminus M_1$

$$(G\circ\phi^{-1})'(x) = (G'\circ\phi^{-1})(x)\,.\,(\phi^{-1})'(x) = [(f\circ\phi)\circ\phi^{-1}](x)\,.\,\frac{1}{\phi'\circ\phi^{-1})(x)} = f(x)$$

The last equality completes the proof. □

Note 11.3.6. 1. Using the indefinite integral the statement in theorem 4 can be expressed by the following formula

$$\int f(x)\,\mathrm{d}x = \int (f\circ\phi)(t)\phi'(t)\,\mathrm{d}t$$

for $t = \phi^{-1}(x)$, $x \in I_1$. Alternatively, the N-integral can be used in the form

$$(N)\int_{x_0}^{x} f(s)\,\mathrm{d}s = (N)\int_{\phi^{-1}(x_0)}^{\phi^{-1}(x)} (f\circ\phi)(t)\phi'(t)\,\mathrm{d}t$$

for $x \in I_1$, $x_0 \in I_1$.

2. In order to calculate integral $\int f(x)\mathrm{d}x$ theorem 4 can be used with substitutions $x = \phi(t)$ and $\mathrm{d}x = \phi'(t)\mathrm{d}t$.

Sample Problem 11.3.5. Calculate $\int \sqrt{1+x^2}\,\mathrm{d}x$ on $\mathbb{R}$.

`Solution`: 1. Consider $\phi : t \mapsto \sinh t$, $t \in \mathbb{R}$. ϕ is a bijection and for $t \in \mathbb{R}$ we have $\phi'(t) = \cosh t > 0$. The assumptions of theorem 4 are therefore satisfied. Hence

$$\int \sqrt{1+x^2}\mathrm{d}x = \int \sqrt{1+\sinh^2 t}\,\cosh t\,\mathrm{d}t = \int \cosh^2 t\,\mathrm{d}t$$

for $t = \sinh^{-1} x = \ln(x+\sqrt{x^2+1})$, $x \in \mathbb{R}$. (We have applied the identity $\cosh^2 x - \sinh^2 x = 1$, for $x \in \mathbb{R}$.) Furthermore we have

$$\int \cosh^2 t\,\mathrm{d}t = \frac{1}{4}\int e^{2t}\,\mathrm{d}t + \frac{1}{2}\int\,\mathrm{d}t + \frac{1}{4}\int e^{-2t}\,\mathrm{d}t \overset{\mathrm{c}}{=} \frac{t}{2} + \frac{1}{4}\sinh 2t, \quad t \in \mathbb{R}.$$

That implies

$$\int \sqrt{1+x^2}\,\mathrm{d}x \overset{\mathrm{c}}{=} \frac{1}{2}\ln(x+\sqrt{x^2+1}) + \frac{1}{4}\sinh[2\ln(x+\sqrt{x^2+1})] \text{ for } x \in \mathbb{R}.$$

2. Using integration by parts with $f : x \mapsto \sqrt{1+x^2}$, $x \in \mathbb{R}$ and $G : x \mapsto 1$, $x \in \mathbb{R}$ one gets

$$\int \sqrt{1+x^2}\,\mathrm{d}x \overset{\mathrm{c}}{=} \frac{1}{2}\ln(x+\sqrt{x^2+1}) + \frac{1}{2}x\sqrt{x^2+1}, \quad x \in \mathbb{R}.$$

Formally, the two methods give different results. However, one can work out the derivatives of the two results and easily see that both results are correct. As an independent check, one can recall theorem 2.10.1, part (ii):

$$\forall x \in \mathbb{R} :\ x\sqrt{x^2+1} = \frac{1}{2}\sinh[2\ln(x+\sqrt{x^2+1})] + c.$$

Choosing $x = 0$ in this formula we get $c = 0$ and hence the equality of the two results.

Note 11.3.7. Note that theorem 3 can be used with much weaker assumptions about function ϕ than theorem 4. In theorem 3, the bijection is not required and neither is the assumption $\phi'(t) \neq 0$.

Problems

1. Find recurrent relations for the following indefinite intergals:
   a) $I_n(x) = \int \sin^n x \, \mathrm{d}x$,
   b) $J_n(x) = \int \cos^n x \, \mathrm{d}x$.

2. Use both integration by parts and substitution method for the calculation of indefinite integral $\int \sqrt{x^2 - 1} \, \mathrm{d}x$ where $x \in (-\infty, -1) \cup (1, +\infty)$, and compare the two results.

3. Calculate (on $\mathbb{R}$)
   a) $\displaystyle\int f(x) \, \mathrm{d}x, \quad \text{where } f : x \mapsto \begin{cases} 1 - x^2, & |x| \leq 1 \\ 1 - |x|, & |x| > 1 \end{cases}$
   b) $\displaystyle\int \frac{x^2 + 1}{x^4 + 1} \, \mathrm{d}x.$

Answers

1. a) $I_n(x) = -\frac{1}{n} \sin^{n-1} x \, \cos x + \frac{n-1}{n} I_{n-2}(x)$ for $n \in N, \; n \geq 2$;
   b) $J_n(x) = \frac{1}{n} \cos^{n-1} x \, \sin x + \frac{n-1}{n} I_{n-2}(x)$ for $n \in N, \; n \geq 2$.

2. $\frac{x}{2}\sqrt{x^2 - 1} - \frac{1}{2} \ln |x + \sqrt{x^2 - 1}|$.

3. a) $x - \frac{x^3}{3}$ for $|x| \leq 1$; and $x - \frac{x}{2}|x| + \frac{1}{6} \operatorname{sgn} x$ for $|x| > 1$;
   b) $F : x \mapsto \frac{1}{\sqrt{2}} \tan^{-1} \frac{x^2 - 1}{x\sqrt{2}}, \; x \in \mathbb{R} \setminus \{0\}$; $F(0) = 0$. (Use the following change of variables: $x - \frac{1}{x} = t\sqrt{2}$.)

11.4 Integration of Special Types of Functions

The goal of this section is to show the ways how to find indefinite integrals of some important and frequent classes of real functions. For practical purposes we limit ourselves to show only the description of several useful algorithms without the ambition to provide the rigorous formulations. Hence in each particular case the validity conditions must be considered separately.

11.4.1 Integration of Rational Functions

Recall that in theorem 6.4.3 we have introduced the decomposition of a purely rational single variable function into partial fractions. According to this theorem and theorem 1.10.3 the problem of integration of such rational functions is reduced to the problem of integration of individual partial fractions.

Since for $k \in N$ and $k \neq 1$

$$\int \frac{1}{(x-a)^k} \mathrm{d}x \stackrel{c}{=} \frac{(x-a)^{1-k}}{1-k}, \quad x \in \mathbb{R} \setminus \{a\}$$

and

$$\int \frac{1}{x-a} \mathrm{d}x \stackrel{c}{=} \ln|x-a|, \quad x \in \mathbb{R} \setminus \{a\}$$

then the only remaining problem in the integration of rational single variable functions is the calculation of the primitive function to function $x \mapsto \frac{Ax+B}{(x^2+px+q)^k}$, $x \in \mathbb{R}$, where A, B, p and q are real numbers, $k \in N$, and the discriminant $p^2 - 4q$ of the quadratic expression in the denominator is negative. Taking the decomposition

$$\frac{Ax+B}{(x^2+px+q)^k} = \frac{A}{2} \frac{2x+p}{(x^2+px+q)^k} + \frac{B - \frac{Ap}{2}}{(x^2+px+q)^k}$$

into account the indefinite integral of the first term can easily be found based on theorem 3.10.3 (the first theorem on substitution) using the change of variables $x^2+px+q = t$ while

$$\frac{A}{2} \int \frac{2x+p}{(x^2+px+q)^k} \mathrm{d}x \stackrel{c}{=} \frac{A}{2(1-k)} (x^2+px+q)^{1-k}, \quad x \in \mathbb{R},\ k \neq 1$$

$$\frac{A}{2} \int \frac{2x+p}{x^2+px+q} \mathrm{d}x \stackrel{c}{=} \frac{A}{2} \ln(x^2+px+q), \quad x \in \mathbb{R}.$$

The denominator of the second term can be expanded according to

$$x^2+px+q = \left(x+\frac{p}{2}\right)^2 + q - \frac{p^2}{4} = \left(q - \frac{p^2}{4}\right) \left[\left(\frac{x+\frac{p}{2}}{\sqrt{q-\frac{p^2}{4}}} \right)^2 + 1 \right].$$

Then the primitive function to the second term can be obtained using the change of variable $t = \left(x+\frac{p}{2}\right)\left(q-\frac{p^2}{4}\right)^{-1/2}$ (theorem 3.10.3). This substitution transforms the original integral $\int \frac{\mathrm{d}x}{(x^2+px+q)^k}$ into integral $\int \frac{\mathrm{d}t}{(t^2+1)^k}$. The latter integral can be obtained using the recurrent formula (2.10.3) found in problem 3.10.3. The procedure described above can now be summarised in

THEOREM 11.4.1. The indefinite integral of an arbitrary rational single variable function on its definition domain belongs to the set of the elementary functions.

11.4.2 Integration of Trigonometric Functions

Next, we shall explicitly show the substitutions used to transform the integrals of type

$$\int R(\sin x, \cos x)\,\mathrm{d}x, \quad x \in M \subset \mathbb{R} \tag{11.4}$$

into integrals of rational functions. Function $R : (u,v) \mapsto R(u,v)$ from $\mathbb{R}^2$ to $\mathbb{R}$ is a rational function of two variables.

Integral (10.12) can be calculated using the substitutions shown below (and theorem 3.10.3). Each of the substitutions transforms the integrand into a rational single variable function.

1. If R is an odd function in the first of its two variables, $R(-\sin x, \cos x) = -R(\sin x, \cos x)$ then a substitution $\cos x = t,\ x \in \mathbb{R},$ can be used.

2. If $R(\sin x, -\cos x) = -R(\sin x, \cos x)$ then a substitution $\sin x = t,\ x \in \mathbb{R},$ can be used.

3. If $R(-\sin x, -\cos x) = R(\sin x, \cos x)$ then a substitution $\tan x = t$, $x \in \bigcup_{k \in Z} \left(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi\right),$ can be used.

4. For an arbitrary rational function R of two variables a substitution $\tan \frac{x}{2} = t$ can be used. (This substitution can also be used in the previous three cases although the calculation is most of the times technically more involved.)

The last substitution, $\tan \frac{x}{2} = t$, for $x \in (-\pi, \pi)$, $t \in \mathbb{R}$, leads to the formulas

$$\sin x = 2\,\sin\frac{x}{2}\,\cos\frac{x}{2} = \frac{2\,\tan\frac{x}{2}}{\tan^2\frac{x}{2} + 1} = \frac{2t}{t^2+1}$$

$$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{1-t^2}{1+t^2}$$

$$x = 2\,\tan^{-1} t, \quad \mathrm{d}x = \frac{2}{1+t^2}\mathrm{d}t.$$

It is now clear that integral (1) transforms into a rational function in terms of variable t.

Sample Problem 11.4.1. Calculate $\int \frac{1}{1+\cos^2 x}\,\mathrm{d}x$ on $\mathbb{R}$.

`Solution`: Based on the ideas above we can use either the change of variable $\tan x = t$ or $\tan\frac{x}{2} = t$. We shall use both substitutions:

a) With the change of variable $\tan x = t$ the conditions of theorem 3.10.3. are satisfied on every interval $\left(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi\right),\ k \in Z$. Since $\frac{\sin^2 x}{\cos^2 x} = t^2$ one gets $\cos^2 x = \frac{1}{1+t^2}$ and on any interval $\left(-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi\right),\ k \in Z,$ the following relations hold:

$$\begin{aligned}\int \frac{1}{1+\cos^2 x}\mathrm{d}x &= \int \frac{\cos^2 x}{1+\cos^2 x}\frac{\mathrm{d}x}{\cos^2 x} \\ &= \int \frac{\mathrm{d}t}{2+t^2} = \frac{1}{2}\int \frac{\mathrm{d}t}{1+\left(\frac{t}{\sqrt{2}}\right)^2} \stackrel{c}{=} \frac{2}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}},\ t \in \mathbb{R}.\end{aligned}$$

Thus one finds

$$\int \frac{1}{1+\cos^2 x}\mathrm{d}x \overset{\mathrm{c}}{=} \frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right)$$

on any interval $\left(-\frac{\pi}{2}+k\pi, \frac{\pi}{2}+k\pi\right)$, $k \in Z$.

The calculation is, however, not completely finished, since the function found at the end is not primitive to function $f:\, x \mapsto (1+\cos^2 x)^{-1}$, $x \in \mathbb{R}$ everywhere on set $\mathbb{R}$. Using the "gluing" construction of example 3.10.1 the primitive function (indefinite integral) obtained above can now be continuously extended at points $\frac{\pi}{2}+k\pi$, $k \in Z$ in such a way that new extension F will be the primitive function to function $f:\, x \mapsto (1+\cos^2 x)^{-1}$, $x \in \mathbb{R}$ on $\mathbb{R}$. For $k \in Z$ let us denote

$$F_{2k}:\, x \mapsto \frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right), \qquad x \in \left(-\frac{\pi}{2}+2k\pi, \frac{\pi}{2}+2k\pi\right)$$

$$F_{2k+1}:\, x \mapsto \frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right), \qquad x \in \left(-\frac{\pi}{2}+(2k+1)\pi, \frac{\pi}{2}+(2k+1)\pi\right)$$

and

$$C_k := \lim_{x\to\left(-\frac{\pi}{2}+(2k+1)\pi\right)^+} F_{2k+1}(x) - \lim_{x\to\left(\frac{\pi}{2}+2k\pi\right)^-} F_{2k}(x) = -\frac{\pi\sqrt{2}}{2}.$$

Then the primitive function F is

$$F:\, x \mapsto \begin{cases} \frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right) - \frac{\pi\sqrt{2}}{2}, & x \in \left(-\frac{\pi}{2}+2k\pi, \frac{\pi}{2}+2k\pi\right) \\ \frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right), & x \in \left(-\frac{\pi}{2}+(2k+1)\pi, \frac{\pi}{2}+(2k+1)\pi\right) \\ -\frac{\pi\sqrt{2}}{4}, & x = \frac{\pi}{2}+k\pi,\ \ k \in Z. \end{cases}$$

Besides that F also is the strongly primitive function to f on $\mathbb{R}$. Recalling theorem 6.8.4 this can be seen from

$$F'_+\left(\frac{\pi}{2}+k\pi\right) = \lim_{\left(\frac{\pi}{2}+k\pi\right)_+} F'(x) = \lim_{\left(\frac{\pi}{2}+k\pi\right)_+} \frac{1}{1+\cos^2 x} =$$

$$= 1 = F'_-\left(\frac{\pi}{2}+k\pi\right) \quad \text{and hence} \quad F'\left(\frac{\pi}{2}+k\pi\right) = 1 = \left(1+\cos^2\frac{\pi}{2}\right)^{-1}.$$

b) Substitution $\phi:\ x \mapsto \tan\frac{x}{2}$ also satisfies the conditions of theorem 3.10.3. on an interval $(-\pi+2k\pi, \pi+2k\pi)$, $k \in Z$. Then $\cos^2 x = \frac{(1-t^2)^2}{(1+t^2)^2}$, $\cos^2\frac{x}{2} = \frac{1}{1+t^2}$, $t \in \mathbb{R}$. For any $x \in (-\pi+2k\pi, \pi+2k\pi)$

$$\int \frac{1}{1+\cos^2 x}\mathrm{d}x = \int \frac{2\cos^2\frac{x}{2}}{1+\cos^2 x}\frac{\mathrm{d}x}{2\cos^2\frac{x}{2}} = \int \frac{t^2+1}{t^4+1}\mathrm{d}t, \quad t \in \mathbb{R}.$$

The rational function that has been obtained here is more complicated than in case a). The only benefit of this substitution is that we find primitive functions on larger intervals. However, since we are seeking the primitive function on $\mathbb{R}$ the obtained result must again be extended by "gluing".

11.4.3 Integration of Irrational Functions

a) An indefinite integral of type

$$\int R\left[x, \left(\frac{ax+b}{cx+d}\right)^{p/q}\right] \mathrm{d}x$$

where a, b, c and d are real numbers and p and q are integers, $q \neq 0$, can be transformed into an integral of rational functions using a change of variable

$$\left(\frac{ax+b}{cx+d}\right)^{1/q} = t.$$

Sample Problem 11.4.2. Transform $\int \frac{1+\sqrt{1+x}}{1-\sqrt[3]{1+x}}\,\mathrm{d}x$ on the definition domain of the integrand into an integral of rational functions.

`Solution`: The definition domain of the integrated function is $(-1,0)\cup(0,+\infty)$. Here we shall use the substitution $(1+x)^{1/6} = t$ that satisfies the conditions of theorem 3.10.3 on these intervals ($t \in (0,1)\cup(1,+\infty)$). According to the theorem, on each of the intervals

$$\int \frac{1+\sqrt{1+x}}{1-\sqrt[3]{1+x}}\mathrm{d}x = \int \frac{1+\sqrt{1+x}}{1-\sqrt[3]{1+x}}\,6(1+x)^{5/6}\,\frac{\mathrm{d}x}{6(1+x)^{5/6}} \overset{\mathrm{c}}{=} 6\int \frac{t^5(1+t^3)}{1-t^2}\,\mathrm{d}t.$$

b) The integration of an irrational function of type $R(x, \sqrt{ax^2+bx+c})$, where R is an irrational function of two variables, depends on the existence of the real roots of quadratic form $\sqrt{ax^2+bx+c}$ (a, b and c are real numbers).

1. If $x_1 < x_2$ are the real roots such that $ax^2+bx+c = a(x-x_1)(x-x_2)$ for $x \in \mathbb{R}$ then (assuming that $a<0$) one can write

$$\sqrt{ax^2+bx+c} = \sqrt{-a}\,(x-x_1)\sqrt{\frac{x_2-x}{x-x_1}} \quad \text{for } x \in (x_1, x_2)$$

and the calculation of the remaining integral can be accomplished using the method of part a).

2. If the quadratic term has no real roots then function $x \mapsto \sqrt{ax^2+bx+c}$ has a non-empty definition domain only for $a>0$ and $c>0$.

The integration of this function can be reformulated in terms of the integration of a rational function by one of the following Euler substitutions:

$$\sqrt{ax^2+bx+c} = \sqrt{a}x + t$$

or

$$\sqrt{ax^2+bx+c} = xt + \sqrt{c}.$$

These substitutions can also be used if ax^2+bx+c has real roots. In such a case, the first substitution is useful for $a>0$ and the second one for $c>0$.

When used in actual calculations it is necessary to check the intervals where the substitution theorems are satisfied in terms of the new variables.

11.4.4 Integration of Transcendent Functions

a) The integral of a transcendent function $x \mapsto R(e^{ax}),\ x \in A \subset (-\infty, +\infty)$ (R is a rational function of a real single variable) can be turned into the integral of a rational function using the change of variable $t = e^{ax}$.

b) The integral of a function $x \mapsto R(\ln x)\,\frac{1}{x},\ x \in A \subset (-\infty, +\infty)$ is transformed into an integral of a rational function using the change of variable $\ln x = t$.

Finally we note that mastering the methods of integration requires independent and honest calculation of a sufficient number of problems and exercices.

Problems

1. Prove in full detail that all substitutions suggested in this section transform the respective integrals into integrals of rational functions.

2. Calculate
   a) $\int \frac{\mathrm{d}x}{x^4-1}$ for $x \in \mathbb{R}$.
   b) $\int \frac{\sqrt{x}}{(1+\sqrt[3]{x})^2}\mathrm{d}x$ for $x \geq 0$.
   c) $\int \frac{\mathrm{d}x}{x+\sqrt{x^2+x+1}}$ for $x \in (-\infty, -1) \cup (-1, +\infty)$
   d) $I(x) := \int \frac{\sin^2 x \cos^2 x}{\sin^8 x + \cos^8 x}\mathrm{d}x$ for $x \in \mathbb{R}$
   e) $\int \sqrt{\frac{e^x-1}{e^x+1}}\mathrm{d}x$ for $x \in [0, +\infty)$.

Answers

2. a) $\dfrac{1}{4\sqrt{2}} \ln x^2 + x\sqrt{2} + 1x^2 - x\sqrt{2} + 1 + \dfrac{1}{2\sqrt{2}} \left[\tan^{-1}(x\sqrt{2}+1) + \tan^{-1}(x\sqrt{2}-1)\right];$

b) $\dfrac{6}{5}x^{\frac{5}{6}} - 4x^{\frac{1}{2}} + 18x^{\frac{1}{6}} + \dfrac{3x^{\frac{1}{6}}}{1+x^{\frac{1}{3}}} - 21\tan^{-1} x^{\frac{1}{6}}\,;$

c) $\dfrac{3}{2(2z+1)} + \dfrac{1}{2}\ln\dfrac{z^4}{|2z+1|^3}$ where $z = x + \sqrt{x^2+x+1}\,;$

d) $$I(x) = \frac{\sqrt{2+\sqrt{2}}}{4}\tan^{-1}\frac{\tan 2x}{\sqrt{4+2\sqrt{2}}} - \frac{\sqrt{2-\sqrt{2}}}{4}\tan^{-1}\frac{\tan 2x}{\sqrt{4-2\sqrt{2}}}$$

$$+\frac{\pi}{4}\left(\sqrt{2+\sqrt{2}} - \sqrt{2-\sqrt{2}}\right)\left[\frac{4x+\pi}{2\pi}\right],\quad x \neq \frac{\pi}{4} + \frac{k\pi}{2},\ k \in Z;$$

$$I\left(\frac{\pi}{4} + \frac{k\pi}{2}\right) = \lim_{x \to \frac{\pi}{4} + \frac{k\pi}{2}} I(x)\,;$$

e) $\ln(e^x + \sqrt{e^{2x}+1}) + \sin^{-1}(e^{-x})$.

The definition of the derivative of a function depending on a single variable (chapter 8) was motivated by two basic problems — a geometric one (construction of a tangential to the graph of the function) and a physical one (need to find the instantaneous velocity of a motion with known trajectory). Another two problems: the evaluation of a surface area and the calculation of a trajectory of a moving point whose velocity is known, despite being at first glance very different, lead to mathematical problems closely related to the concept of a function differentiability.

The goal of the next two chapters is to develop the constructs that are needed to solve problems of this type. Hence in this chapter we shall deal with the concept and basic propreties of a primitive function, Newton and indefinite integral of a real-valued function of a single variable, and we shall introduce some methods to calculate them.

In the next chapters, on the Riemann and generalised integral, further application and extension of these results will be presented.

11.5 Primitive Function

The task to find a primitive function is in some sense the inverse problem to the problem to find a proper derivative of a given function. (A function f is primitive to its derivative f'.) However, contrary to the calculation of derivatives there is no simple set of rules for the calculation of the primitive function in terms of the elementary functions.

Strongly Primitive Function

One of the basic types of the primitive function is introduced in the following definition:

Definition 11.5.1. Let $f:\ (J \subset \mathbb{R}) \to \mathbb{R}$ and $I \subset J$ is an interval (of any type). We say that *function* $F:\ I \to \mathbb{R}$ *is strongly primitive* (or is *a strong potential*) *to function* f *on interval* I if

$$\forall x \in I:\ F'(x) = f(x).$$

Note 11.5.1. 1. From the definition 10.9.1 it follows directly that function F is differentiable on interval I which, according to theorem **??**, means that F is continuous on I, $F \in C(I)$.
2. The *sine* function is strongly primitive to the *cosine* function on interval $(-\infty, +\infty)$. It can be seen from example 6.8.4 that the strongly primitive function on $\mathbb{R}$ to a discontinuous function

$$f:\ x \mapsto \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x}; & x \in \mathbb{R}\setminus\{0\} \\ 0, & x = 0 \end{cases}$$

is a function F

$$F:\ x \mapsto \begin{cases} x^2\sin\frac{1}{x}, & x \in \mathbb{R}\setminus\{0\} \\ 0, & x = 0. \end{cases}$$

Similarly, there exists a strongly primitive function on $(0,1]$ to a function $f:\ x \mapsto x^{-2/3},\ x \in (0,1]$, namely $F:\ x \mapsto 3x^{1/3},\ x \in (0,1]$.

Example 11.5.1. Prove that there exists no strongly primitive function on $\mathbb{R}$ to a function

$$sgn:\ x \mapsto \begin{cases} 1, & x \in \mathbb{R}^+ \\ 0, & x = 0 \\ -1, & x \in \mathbb{R}^-. \end{cases}$$

`Solution`: If the strongly primitive function F on $\mathbb{R}$ to function sgn existed then according to the corollary to theorem **??** the derivative $F'(x)$ would not have points of discontinuity of the first kind on $\mathbb{R}$. This is, however, in contradiction with the requirement that $F'(x) = \operatorname{sgn} x,\ x \in \mathbb{R}$. (Point 0 is a point of discontinuity of the first kind of the sgn function.)

The sufficient conditions for the existence of the strongly primitive function are listed in chapter 12.

Now let us focus on the problem of the correctness (uniqueness) of the strongly primitive function.

THEOREM 11.5.1. (i) Let F be the strongly primitive function to function f on interval I. Then for any $C \in \mathbb{R}$ the function $x \mapsto F(x) + C,\ x \in I,$ is also strongly primitive to function f on I.
(ii) Let F and G be strongly primitive functions to function f on interval I. Then there exists a unique constant $C \in \mathbb{R}$ such that

$$\forall x \in I:\ F(x) = G(x) + C.$$

Proof. (i) The statement follows directly from definition 10.9.1.
(ii) If both F and G are strongly primitive functions to f on I then $\forall x \in I : (F-G)'(x) = 0$. According to corollary 1 (part e$'$) to theorem 2.8.4 function $F - G$ is a constant function on I. The uniqueness of this constant can be shown indirectly. □

Note 11.5.2. 1. The strongly primitive function on an interval is not defined uniquely. If it exists then there are infinitely many of them, differing from each other by a constant.

2. It is, however, easy to see that if function f has a strongly primitive function on interval I then to every point $(x_0, y_0) \in I \times \mathbb{R}$ there exists one and only one strongly primitive function F to f on I such that $(x_0, y_0) \in F$. (The graph of F passes through the point (x_0, y_0).)

Primitive Function

In example 1 we saw that even such a simple function as the sgn function does not have a strongly primitive function on $\mathbb{R}$. Hence it is sensible to extend the concept to a wider class of functions.

Definition 11.5.2. Let $f\colon\ (J \subset \mathbb{R}) \to \mathbb{R}$ and $I \subset J$ is again an interval. We say that *function* $F:\ I \to \mathbb{R}$ *is primitive* (or *is a potential*) to function f on interval I if

(i) $F \in C(I)$,

(ii) a finite set $M \subset I$ exists such that

$$\forall x \in I \setminus M : \ F'(x) = f(x).$$

Note 11.5.3. 1. The concept of the primitive function is the generalisation of the concept of the strongly primitive function. In fact, definition 10.9.1 is obtained from definition 10.9.2 for $M = \emptyset$.

2. Function $f : \ x \mapsto |x|, \ x \in \mathbb{R}$ is a primitive function to the sgn function on $\mathbb{R}$, with $M = \{0\}$, and function $F_1 : \ x \mapsto 2\sqrt{x}, \ x \geq 0$ is primitive to unbounded function

$$f_1 : \ x \mapsto \begin{cases} 5 & for \quad x = 0 \\ \frac{1}{\sqrt{x}}, & for \quad x \in (0, +\infty). \end{cases}$$

There is a similar statement on the uniqueness of the primitive function as there was for the strongly primitive function.

THEOREM 11.5.2. (i) Let F be the primitive function to function f on interval I and $C \in \mathbb{R}$. Then function $x \mapsto F(x) + C, \ x \in I$ is also a primitive function on I to function f. (ii) Let F and G be primitive functions to function f on interval I. Then there exists one and only one constant $C \in \mathbb{R}$ such that

$$\forall x \in I : \ F(x) = G(x) + C.$$

(iii) If a function f has a primitive function on I then for any $x_0 \in I, \ y_0 \in \mathbb{R}$ there exists one and only one primitive function F to f on I such that $F(x_0) = y_0$.

Proof. Statement (i) is obvious.
(ii) There exist finite sets $M_1 \subset I, \ M_2 \subset I$ such that

$$\forall x \in I \setminus M_1 : \ F'(x) = f(x), \quad \forall x \in I \setminus M_2 : \ G'(x) = f(x).$$

The set $M = M_1 \cup M_2$ is a finite set in I and

$$\forall x \in I \setminus M : \ F'(x) = f(x) = G'(x).$$

Since $F \in C(I), \ G \in C(I)$, according to part b) of example 2.8.4 there exists a constant $C \in \mathbb{R}$ such that $F(x) - G(x) = C$ for all $x \in I$. The uniqueness is obvious.
(iii) It is seen immediately that if G is a primitive function to f on I then $F : \ x \mapsto G(x) - G(x_0) + y_0, \ x \in I$ is the only primitive function to f on I with the required properties. $\square$

Sample Problem 11.5.1. Let $c_0,\, c_1,\, \ldots,\, c_m;\, l_1,\, \ldots,\, l_m$ be real numbers and let $x_0 = a < x_1 < x_2 < \ldots < x_{m-1} < x_m = b$. On interval $[a, b]$ we define the "step-like" function

$$f : x \mapsto \begin{cases} c_j, & x = x_j, \quad j = 0, 1, \ldots, m \\ l_i, & x \in (x_{i-1}, x_i), \quad i = 1, \ldots, m. \end{cases}$$

Prove that function

$$F : x \mapsto \begin{cases} l_1(x - x_0) & x \in [x_0, x_1) \\ l_2(x - x_1) + l_1(x_1 - x_0) & x \in [x_1, x_2) \\ l_m(x - x_{m-1}) + l_{m-1}(x_{m-1} - x_{m-2}) + \ldots + l_1(x_1 - x_0), & x \in [x_{m-1}, x_m] \end{cases}$$

is primitive to function f on interval [a,b].

`Solution`: Let M be the finite set of points $\{x_0,\, x_1,\, \ldots,\, x_m\} \subset [a, b]$. Then

$$\forall x \in [a, b] \setminus M : \; F'(x) = f(x)$$

and the statement is proved.

Note 11.5.4. 1. Note that the concepts of the strongly primitive function and primitive function have been defined on an interval. Both concepts may be extended to more general sets on $\mathbb{R}$. However, the crucial assumption that I is an interval should be kept in the uniqueness theorems (part (ii) of theorem 10.9.1 and parts (ii) and (iii) of theorem 10.9.2). For example, functions

$$f : x \mapsto \begin{cases} x, & x \in (0,1) \\ x+1, & x \in (1,2) \end{cases} \quad \text{and} \quad g : x \mapsto \begin{cases} x+1, & x \in (0,1) \\ x, & x \in (1,2) \end{cases}$$

are primitive (also strongly primitive) to function $x \mapsto 1,\; x \in (0,1) \cup (1,2)$, on $(0,1) \cup (1,2)$. Nevertheless the difference $f - g$ is not a constant.

2. Further generalisation of the concept of the primitive function can be obtained when set M in definition 10.9.2 is countable. In this case the primitive function to the Dirichlet function is any real constant (since Q, the rational number set, is countable). (See, for example, textbook [**?**], p. 359.)

Note 11.5.5. It is obvious that a function defined on $I \subset \mathbb{R}$ may not have a primitive function on I. For instance, function

$$f : x \mapsto \begin{cases} 5, & \text{for} \quad x = 0 \\ \frac{1}{x}, & \text{for} \quad x \in (0, 1] \end{cases}$$

has no primitive function on $[0, 1]$. On interval $(0, 1]$, the primitive function to f is a function $x \mapsto \ln x,\; x \in (0, 1]$.

The set of all functions $f : I \to \mathbb{R}$ whose primitive function on interval $I \subset \mathbb{R}$ exists will be denoted as $P(I)$.

Sample Problem 11.5.2. Let $f : (a,b) \to \mathbb{R}$ have a point of discontinuity of the first kind at point $x_0 \in (a,b)$. Furthermore, let F_1 (F_2) be the primitive function to restriction $f|_{(a,x_0)}$ ($f|_{(x_0,b)}$) such that proper limits $\lim\limits_{x\to x_0^-} F_1(x)$ and $\lim\limits_{x\to x_0^+} F_2(x)$ exist. Show then that a primitive function F to function f on interval (a,b) exists and

$$F(x) = \begin{cases} F_1(x) + k, & x \in (a, x_0), \quad (k \in \mathbb{R}) \\ F_2(x), & x \in (x_0, b) \\ \lim\limits_{x\to x_0^-} F_1(x) + k = \lim\limits_{x\to x_0^+} F_2(x), & x = x_0. \end{cases}$$

`Solution`: From the continuity of functions F_1 and F_2 on intervals (a,x_0) and (x_0,b), respectively, and from the choice of constant $k = \lim\limits_{x\to x_0^+} F_2(x) - \lim\limits_{x\to x_0^-} F_1(x)$ it follows that F is continuous on (a,b). That means that finite sets $M_1 \subset (a,x_0)$ and $M_2 \subset (x_0,b)$ exist such that $F_1'(x) = f(x)$ for all $x \in (a,x_0) \setminus M_1$ and $F_2'(x) = f(x)$ for all $x \in (x_0,b) \setminus M_2$. Hence $F'(x) = f(x)$ for all $x \in (a,b) \setminus (M_1 \cup M_2 \cup \{x_0\})$.

This example shows how to construct a primitive function to a discontinuous function with points of discontinuity of the first kind. This construction will be used frequently and will be referred to as "gluing".

Problems

1. Show that the existence of the primitive function to $f+g$ on interval I does not guarantee the existence of the primitive functions to f and g, separately, on I.

2. Let functions f and $f+g$ have primitive functions on $J \subset \mathbb{R}$. Find out if a primitive function to g on J exists.

3. Find out if primitive functions to f and g on $J \subset \mathbb{R}$ exist provided the primitive functions to $f+g$ and $f-g$ on J exist.

4. Prove that there exists no primitive function to function $x \mapsto \chi(x)$, $x \in I$, where χ is the Dirichlet function and $I \subset \mathbb{R}$ is an arbitrary interval in $\mathbb{R}$.

5. Find all primitive functions to the functions defined below together with their definition domains:

   $f_1 : x \mapsto \frac{1}{x},\ x \in (-\infty, 0)$; $\quad f_2 : x \mapsto \frac{1}{x},\ x \in (0, +\infty)$;

   $f_3 : x \mapsto \frac{1}{x},\ x \in (-\infty, 0) \cup (0, +\infty)$; $\quad f_4 : x \mapsto [x],\ x \in [0, n],\ n \in N$;

   $f_5 : x \mapsto x\,|x|,\ x \in \mathbb{R}$; $\quad f_6 : x \mapsto \begin{cases} \frac{1}{x}, & x \in \mathbb{R} \setminus \{0\} \\ 0, & x = 0. \end{cases}$

6. Prove that $\tan^{-1} x + \cot^{-1} x = \pi$ for any $x \in \mathbb{R}$.

Answers

2. Yes, it does.

3. Yes, it does.

5. $F_1 : x \mapsto \ln(-x) + c,\ \ x \in (-\infty, 0);$
   $F_2 : x \mapsto \ln x + c,\ x \in (0, +\infty);$
   $F_3 : x \mapsto \begin{cases} \ln(-x) + c_1, & x \in (-\infty, 0) \\ \ln x + c_2, & x \in (0, +\infty) \end{cases}$
   (a special case: $F_3 : x \mapsto \ln x + c,\ x \in \mathbb{R} \setminus \{0\}$);
   $F_4 : x \mapsto x[x] - \frac{[x]([x]+1)}{2},\ x \in [0, n];$
   $F_5 : x \mapsto \begin{cases} \frac{x^3}{3} + c, & x \in [0, +\infty) \\ -\frac{x^3}{3} + c, & x \in (-\infty, 0); \end{cases}$

   F_6 does not exist.

11.6 Riemann Integral and Its Properties

Inspired by the preceding section we are ready to claim that determining the trajectory of a moving particle out of its velocity given as a function of time effectively means finding the primitive function to the velocity function. In this section, besides the properties of the Riemann integral we shall show that when calculating the surface area one is again faced with the issue of finding the primitive function.

Riemann Integral

Before addressing the surface area question we shall study the following concept and its properties:

Definition 11.6.1. Let I be an interval in $\mathbb{R}$ and $f \in P(I)$. Let F be a primitive function to f on I and let $[a, b] \subset I$. Then the difference $F(b) - F(a)$ is called *a Riemann (definite) integral (R-integral) of function f from a to b*. It is denoted as $(N)\ \int_a^b f(x)\,\mathrm{d}x$. We also say that f is *Riemann integrable* on $[a, b]$. (The difference $F(b) - F(a)$ is denoted as $[F(x)]_a^b$ or $F(x)|_a^b$.)

Note 11.6.1. The definition of the Riemann integral is correct, *i.e.* its value does not depend on the choice of the primitive function.

Sample Problem 11.6.1. Calculate the Riemann integral from a to b of the following functions: a) $f_1 = \mathrm{sgn}$; b) $f_2 : x \mapsto \mathrm{sgn} x,\ x \in \mathbb{R} \setminus \{a, b\},\ f_2(a) = c_1 \in \mathbb{R},\ f_2(b) = c_2 \in \mathbb{R};$

c) the function defined in sample problem 2.10.1; d) an unbounded (in the neighbourhood of point 0) function

$$f_3 : x \mapsto \begin{cases} 2x \sin \frac{\pi}{x^2} - \frac{2\pi}{x} \cos \frac{\pi}{x^2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0, \end{cases}$$

for $a = -1,\ b = 1$.

`Solution`: For the first two cases

$$(N) \int_a^b \operatorname{sgn} x \, \mathrm{d}x = [|x|]_a^b = |b| - |a| = (N) \int_a^b f_2(x) \, \mathrm{d}x.$$

c)

$$(N) \int_a^b f(x) \, \mathrm{d}x = [F(x)]_a^b = \sum_{i=1}^m l_m(x_i - x_{i-1}).$$

d) It is easy to see that

$$F_3 : x \mapsto \begin{cases} x^2 \sin \frac{\pi}{x^2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

is a strongly primitive function to function f_3 on $\mathbb{R}$, and hence

$$(N) \int_{-1}^1 f_3(x) \, \mathrm{d}x = F_3(x)|_{-1}^1 = \sin \pi - \sin \pi = 0.$$

Note 11.6.2. In the case when $l_i > 0$ for $i = 1, \ldots, m$, the Riemann integral in example 1 c) gives a very intuitive geometrical interpretation — the surface area of the histogram-shaped object in figure 10.1.

Note 11.6.3. Note that the N-integral is defined on a closed interval. The extension of this concept to include an arbitrary (also unbounded) interval will be introduced in chapter 13.

Basic Properties of the N-integral

THEOREM 11.6.1. Let $J \subset \mathbb{R}$ be an interval, $a \in J$, $b \in J$, $f \in P(J)$ and $M \subset J$ is a finite set.

(i) If $f|_{J \setminus M} = g|_{J \setminus M}$ then

$$(N) \int_a^b g(x) \, \mathrm{d}x = (N) \int_a^b f(x) \, \mathrm{d}x.$$

(ii)

$$(N) \int_a^a f(x) \, \mathrm{d}x = 0, \quad (N) \int_a^b f(x) \, \mathrm{d}x = -(N) \int_b^a f(x) \, \mathrm{d}x.$$

(iii) For function $k : x \mapsto c \in \mathbb{R},\ x \in J \setminus M$ the following is true:

$$(N) \int_a^b k(x) \, \mathrm{d}x = c(b - a).$$

Proof. (i) Taking the assumptions into account it follows that the primitive functions to functions f and g on interval J differ by a constant and the statement is then obvious.

(ii) Let F be a primitive function to f on J. Then

$$(N)\int_a^a f(x)\,\mathrm{d}x = F(a) - F(a) = 0,$$

$$(N)\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a) = -[F(a) - F(b)] = -(N)\int_b^a f(x)\,\mathrm{d}x.$$

(iii) According to example 2.10.1 the primitive function to function k on interval J is $F:\ x \mapsto c(x-a),\ x \in J$. The result then follows trivially.

□

THEOREM 11.6.2. Let a and b be numbers from interval $J \subset \mathbb{R}$, $C_1 \in \mathbb{R}$, $C_2 \in \mathbb{R}$, and $f \in P(J)$ and $g \in P(J)$. Then $(C_1 f + C_2 g) \in P(J)$ and

$$(N)\int_a^b [C_1 f(x) + C_2 g(x)]\,\mathrm{d}x = C_1\,(N)\int_a^b f(x)\,\mathrm{d}x + C_2\,(N)\int_a^b g(x)\,\mathrm{d}x.$$

Proof. Let F and G be primitive functions to f and g, respectively, both on interval J. Then $C_1 F + C_2 G$ is a primitive function to $C_1 f + C_2 g$ and the following relations hold:

$$\begin{aligned}(N)\int_a^b [C_1 f(x) + C_2 g(x)]\,\mathrm{d}x &= C_1 F(b) + C_2 G(b) - C_1 F(a) - C_2 G(a) = \\ &= C_1[F(b) - F(a)] + C_2[G(b) - G(a)] = \\ &= C_1\,(N)\int_a^b f(x)\,\mathrm{d}x + C_2\,(N)\int_a^b g(x)\,\mathrm{d}x,\end{aligned}$$

and the proof is done. □

THEOREM 11.6.3. Let a, b and c be numbers from interval $J \subset \mathbb{R}$, and let $f \in P(J)$. Then

$$(N)\int_a^b f(x)\,\mathrm{d}x = (N)\int_a^c f(x)\,\mathrm{d}x + (N)\int_c^b f(x)\,\mathrm{d}x$$

Proof. Let F be a primitive function to f on interval J. Then

$$\begin{aligned}(N)\int_a^b f(x)\,\mathrm{d}x &= F(b) - F(a) = [F(b) - F(c)] + [F(c) - F(a)] \\ &= (N)\int_a^c f(x)\,\mathrm{d}x + (N)\int_c^b f(x)\,\mathrm{d}x\end{aligned}$$

and the proof is done. □

Using the method of mathematical induction and theorem 10.10.3 one can state

Corollary 11.6.1. Let $x_0, x_1, \ldots, x_n$ be numbers from interval $J \subset \mathbb{R}$, and let $f \in P(J)$. Then

$$(N)\int_{x_0}^{x_n} f(x)\,\mathrm{d}x = (N)\int_{x_0}^{x_1} f(x)\,\mathrm{d}x + \ldots + (N)\int_{x_{n-1}}^{x_n} f(x)\,\mathrm{d}x.$$

THEOREM 11.6.4. Let $J \subset \mathbb{R}$ be an interval, $a \in J$, $b \in J$, $a < b$, and $f \in P(J)$ and $g \in P(J)$ be such that $f \geq g$ on interval $[a, b]$. Then

$$(N)\int_a^b f(x)\,\mathrm{d}x \geq (N)\int_a^b g(x)\,\mathrm{d}x$$

Proof. Let F (G) be a primitive function to f (g) on interval J. There exist finite sets $M_1 \subset J$ and $M_2 \subset J$ such that

$$\forall x \in J \setminus M_1 : F'(x) = f(x),$$

and

$$\forall x \in J \setminus M_2 : G'(x) = g(x).$$

Thus for any $x \in J \setminus (M_1 \cup M_2)$

$$(G - F)'(x) = (g - f)(x) \geq 0.$$

That means that the difference $G - F$ is a non-decreasing function on J (compare with sample problem 8.4.2 part a)) and hence

$$(N)\int_a^b (g - f)(x)\,\mathrm{d}x = [G(b) - F(b)] - [G(a) - F(a)] \geq 0.$$

The proof is then easily completed using theorem 10.10.2. □

Corollary 11.6.2. Let a, b, c and d be numbers from interval $J \subset \mathbb{R}$ such that $a \leq c \leq d \leq b$ and let $g \in P(J)$ be a non-negative function on interval $[a, b]$. Then the following relations hold:

$$(N)\int_a^b g(x)\,\mathrm{d}x \geq 0$$

and

$$(N)\int_c^d g(x)\,\mathrm{d}x \leq (N)\int_a^b g(x)\,\mathrm{d}x.$$

Proof. The first inequality follows from theorem 10.10.4 for $f : x \mapsto 0$, $x \in [a, b]$. The second inequality is obtained using the corollary to theorem 10.10.3 for this specific case:

$$(N)\int_a^b g(x)\,\mathrm{d}x = (N)\int_a^c g(x)\,\mathrm{d}x + (N)\int_c^d g(x)\,\mathrm{d}x + (N)\int_d^b g(x)\,\mathrm{d}x \geq (N)\int_c^d g(x)\,\mathrm{d}x.$$

□

The following theorem discusses the integral representation of the primitive function:

THEOREM 11.6.5. Let a and x be numbers from interval $J \subset \mathbb{R}$ and let $f \in P(J)$. Then function

$$F : x \mapsto (N) \int_a^x f(t)\,\mathrm{d}t, \qquad x \in J,$$

is the primitive function to function f on J with the property $F(a) = 0$.

Proof. Let us denote a primitive function to f on J as G. We then get

$$(N) \int_a^x f(t)\,\mathrm{d}t = G(x) - G(a).$$

Thus a finite set $M \subset J$ exists such that

$$\forall x \in J \setminus M : \frac{d}{\mathrm{d}x}(N) \int_a^x f(t)\,\mathrm{d}t = \frac{dG}{\mathrm{d}x}(x) = f(x),$$

which completes the proof. □

Surface Area as Primitive Function

The numerical evaluation of the surface area of two-dimensional objects has been one of the oldest mathematical tasks. The area's axiomatic definition for so-called elementary regions is introduced in section 12.2. There it is also shown that this definition of the surface area is correct (that means the surface area of an elementary region exists and is defined uniquely) and the surface area of elementary regions can be expressed using the Riemann integral.

Here we shall focus on the relation between the surface area $P(f;\, c,\, d)$ of a curved trapezoid

$$M(f;\, c,\, d) = \{(x, y) \in \mathbb{R}^2;\ x \in [c, d] \subset [a, b],\ 0 \le y \le f(x)\},$$

where function $f : [a, b] \to \mathbb{R}$ is continuous on interval $[a, b]$ (figure 10.2), and the primitive function to f.

THEOREM 11.6.6. Let function $f : [a, b] \to \mathbb{R}$ be continuous and non-negative on interval $[a, b]$ and let for every $x \in (a, b]$ a surface area of a curved trapezoid $M(f;\, a,\, x)$ exist. Then function

$$p_f : x \mapsto \begin{cases} P(f;\, a,\, x) & \text{for } x \in (a, b] \\ 0 & \text{for } x = a \end{cases}$$

is the strongly primitive function to function f on $[a, b]$.

Proof. Let $x_0 \in [a, b)$. Continuity of function f implies

$$\forall \epsilon > 0\ \exists 0 < \delta < b - x_0\ \forall x \in [x_0, x_0 + \delta) : f(x_0) - \epsilon < f(x) < f(x_0) + \epsilon.$$

From this observation and due to well-known (natural) properties of surface areas that agree with our intuition, and also based on the non-negativity of function f we obtain

$$[f(x_0) - \epsilon]\,\delta \le \max\{[f(x_0) - \epsilon]\,\delta, 0\} \le P(f;\, x_0,\, x_0 + \delta) \le [f(x_0) + \epsilon]\,\delta,$$

which means that the surface area of the largest rectangle placed inside a curved trapezoid $M(f;\, x_0,\, x_0+\delta)$ is not greater than the surface area of the trapezoid, and the latter is in turn not greater than the surface area of the smallest rectangle containing the trapezoid. Furthermore we have

$$P(f;\, a,\, x_0) + P(f;\, x_0,\, x_0+\delta) = P(f;\, a,\, x_0+\delta),$$

which means that the sum of the surface areas of curved trapezoids $M(f;\, a,\, x_0)$ and $M(f;\, x_0,\, x_0+\delta)$ (the two pieces we break trapezoid $M(f;\, a,\, x_0+\delta)$ into) is equal to the surface area of trapezoid $M(f;\, a,\, x_0+\delta)$.

From these relations we get

$$\left|\frac{p_f(x_0+\delta) - p_f(x_0)}{\delta} - f(x_0)\right| < \epsilon.$$

If in this inequality one considers the limits $\delta \to 0^+$ and then $\epsilon \to 0^+$, then the relation $p'_{f+}(x_0) = f(x_0)$ emerges. The analogous statement, $p'_{f-}(x_0) = f(x_0)$ for $x_0 \in (a, b]$, for the left derivative is shown in the same way. Thus we can conclude that $p'_f(x) = f(x)$ for arbitrary $x \in [a, b]$ and the proof is done. □

Corollary 11.6.3. Let $J \subset \mathbb{R}$ be an interval and function $f : J \to \mathbb{R}$ be continuous and non-negative on J. If for any interval $[a, b] \subset J$ $(a \in \mathbb{R},\, b \in \mathbb{R})$ the surface area $P(f;\, a,\, b)$ of a curved trapezoid $M(f;\, a,\, b)$ exists then the strongly primitive function to f on J exists and the following relation holds:

$$P(f;\, a,\, b) = (N) \int_a^b f(x)\, \mathrm{d}x.$$

Proof. Based on theorem 10.10.6 a strongly primitive function $x \mapsto p_f(x)$, $x \in [a, b]$ to function f exists on any interval $[a, b] \subset J$, and therefore it exists on the whole interval J. Furthermore, from the definition of the N-integral it follows that

$$(N) \int_a^b f(x)\, \mathrm{d}x = p_f(b) - p_f(a) = P(f;\, a,\, b)$$

since $p_f(a) = 0$. □

Note 11.6.4. In section 11.5, we shall prove an even stronger statement than the one in the corollary above. Here we just quote the stronger statement:
Every continuous function on an interval in $\mathbb{R}$ has a strongly primitive function on this interval.

Problems

1. Find all $\alpha \in \mathbb{R}$ for which the Riemann integral $(N)\int_0^1 f(x)\,\mathrm{d}x$ exists for
$$f : x \mapsto \begin{cases} x^{-\alpha}, & x \neq 0 \\ c, & x = 0 \end{cases} \qquad \text{where } c \in \mathbb{R}.$$

2. Calculate the following N-integrals provided they exist:
   a) $(N)\displaystyle\int_{-2}^{2} \max(1, x^4)\,\mathrm{d}x$
   b) $(N)\displaystyle\int_{-\pi/2}^{+\pi/2} x\,\mathrm{sgn}(\sin x)\,\mathrm{d}x$
   c) $(N)\displaystyle\int_{-1}^{1} f(x)\,\mathrm{d}x$ for $f : x \mapsto \begin{cases} (1-x^2)^{-1/2}, & x \in (-1,1) \\ c \in \mathbb{R}, & x = -1 \text{ or } x = +1, \end{cases}$
   d) $(N)\displaystyle\int_{0}^{2} [2^x]\,\mathrm{d}x.$

3. Find out if function
$$f : x \mapsto \begin{cases} |x|^{-1}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$
is Riemann integrable. Consider separately the following three intervals: $[-1,1]$, $[-1,0]$ and $[-2,-1]$. If it is, calculate the Riemann integral.

4. Prove the corollary that follows after theorem 10.10.3.

5. Show that the following statement is true:
Let $a < c < b$ be real numbers, and $f \in P([a,c])$ and $f \in P([c,b])$. Then $f \in P([a,b])$ and
$$(N)\int_a^c f(x)\,\mathrm{d}x + (N)\int_c^b f(x)\,\mathrm{d}x = (N)\int_a^b f(x)\,\mathrm{d}x.$$

6. Formulate and prove the theorem on the N-integral of a sum of functions (analogous to theorem 10.10.2) for an arbitrary finite number of terms.

7. Calculate
   a) $\dfrac{\mathrm{d}}{\mathrm{d}x}(N)\displaystyle\int_a^b e^{-x^2}\,\mathrm{d}x,\quad \dfrac{\mathrm{d}}{\mathrm{d}a}(N)\int_a^b e^{-x^2}\,\mathrm{d}x,\quad \dfrac{\mathrm{d}}{\mathrm{d}b}(N)\int_a^b e^{-x^2}\,\mathrm{d}x,$
   where $a \in \mathbb{R}$, $b \in \mathbb{R}$.
   b) $\dfrac{\mathrm{d}}{\mathrm{d}x}(N)\displaystyle\int_x^{(\sin x)^{1/3}} \sin^{-1} t^3\,\mathrm{d}t,$ for $x \in [-\pi/2, +\pi/2]$,
   c) $\displaystyle\lim_{x\to 0}\frac{\int_0^x \sin^{-1} t\,\mathrm{d}t}{x},\qquad \lim_{x\to 0}\frac{\int_0^{\sin x}(\tan t)^{1/n}\,\mathrm{d}t}{\int_0^{\tan x}(\sin t)^{1/n}\,\mathrm{d}t},\qquad n \in N.$

8. Calculate the surface area of two-dimensional planar objects bounded by the following curves:
   a) the x axis, graph of function $x \mapsto |x|^3$ $(x \in \mathbb{R})$, and lines $x = -1$ and $x = 1$;
   b) the graphs of functions $x \mapsto x^2 - 2$ $(x \in \mathbb{R})$ and $x \mapsto -x^2 + 2$ $(x \in \mathbb{R})$.

Answers

1 $\alpha < 1$.

2 a) 12/5, b) $\pi^2/4$, c) π, d) $5 - \ln 3/\ln 2$.

3 f is Riemann integrable only on the last of the three intervals. The result is $\ln 2$.

7 a) 0, $-e^{-a^2}$, e^{-b^2}; b) $\frac{x}{3}(\sin x)^{-2/3}\cos x - \sin^{-1} x^3$; c) 1, 1.

8 a) $\frac{1}{2}$; b) $\frac{16}{3}\sqrt{2}$.

11.7 Indefinite Integral

In section 2 we saw that the hard part in the calculation of the N-integral consists of the determination of the primitive function to a given function on an interval under consideration. Therefore, in this section we shall focus on different alternative methods for finding the primitive function.

Concept of Indefinite Integral

We already know from theorem 10.9.1 that the primitive function to a given function is not uniquely determined. If a single primitive function to f on an interval I exists, then there are infinitely many of them, differing from each other by an additive constant.

Definition 11.7.1. The set of all primitive functions to $f : (J \subset \mathbb{R}) \to \mathbb{R}$ on an interval $I \subset J$ will be called an *indefinite integral of function* f *on* I and denoted by a symbol $\int f(x)\,\mathrm{d}x$, $x \in I$.

Note 11.7.1. 1. Function f is called an integrand, the symbol $\int$ is referred to as the integral sign, and letters $\mathrm{d}x$ denote the integration variable (x in this case). The meaning of the integration variable is as follows: calculating the derivative with respect to the integration variable x of any function from the set $\int f(x)\,\mathrm{d}x$ on $I \setminus M$ (M is a finite subset of I) one obtains restriction $f|_{I\setminus M}$.
2. If F is a primitive function to f on I we shall write

$$\int f(x)\,\mathrm{d}x \overset{\mathrm{c}}{=} F(x),\ x \in I \quad \text{or} \quad \int f(x)\,\mathrm{d}x = \{F(x) + c\}|_{c\in\mathbb{R}},\ x \in I.$$

For instance $\int e^{2x}\,\mathrm{d}x \overset{\mathrm{c}}{=} \frac{1}{2}e^{2x} =: F(x)$, $x \in \mathbb{R}$ since $\frac{dF}{\mathrm{d}x}(x) = e^{2x}$ for $x \in \mathbb{R}$.
3. The indefinite integral will usually be sought on the so-called maximal set $A \setminus \mathbb{R}$ which does not have to be an interval but could also be a union of a finite number of intervals that

satisfies the following statement: If f has a primitive function on set B then $B \subset A$. For example,

$$\int \frac{1}{x}\,\mathrm{d}x \stackrel{c}{=} \ln|x|, \quad x \in (-\infty,0)\cup(0,+\infty)$$
$$\int x^{-\frac{1}{3}}\,\mathrm{d}x \stackrel{c}{=} \frac{3}{2}x^{\frac{2}{3}}, \quad x \in (-\infty,0)\cup(0,+\infty).$$

Basic Indefinite Integrals

The following indefinite integrals can be derived from the rules for derivatives (section 8.2):

1. $\int 0\,\mathrm{d}x \stackrel{c}{=} 0, \quad x \in \mathbb{R}$

2. $\int 1\,\mathrm{d}x =: \int \mathrm{d}x \stackrel{c}{=} x, \quad x \in \mathbb{R}$

3. $\int x^n\,\mathrm{d}x \stackrel{c}{=} \frac{x^{n+1}}{n+1}, \quad x \in \mathbb{R}\ (n \in N).$
   If $n \in \mathbb{R}\setminus\{-1\}$, the formula holds for $x \in \mathbb{R}^+$. For some values of n the formula is valid on an extended set (see, *e.g.* part 3 of the last note);

4. $\int \frac{1}{x}\,\mathrm{d}x \stackrel{c}{=} \ln x, \quad x \in \mathbb{R}^+, \qquad \int \frac{1}{x}\,\mathrm{d}x \stackrel{c}{=} \ln(-x), \quad x \in \mathbb{R}^-$
   $\Big($or $\quad \int \frac{1}{x}\,\mathrm{d}x \stackrel{c}{=} \ln|x|, \quad x \in \mathbb{R}\setminus\{0\}\Big)$

5. $\int \frac{1}{1+x^2}\,\mathrm{d}x \stackrel{c}{=} \tan^{-1}x, \quad x \in \mathbb{R}, \qquad \int \frac{1}{1+x^2}\,\mathrm{d}x \stackrel{c}{=} -\cot^{-1}x, \quad x \in \mathbb{R}$

6. $\int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x \stackrel{c}{=} \sin^{-1}x, \quad x \in (-1,1), \qquad \int \frac{1}{\sqrt{1-x^2}}\,\mathrm{d}x \stackrel{c}{=} -\cos^{-1}x, \quad x \in (-1,1)$

7. $\int e^x\,\mathrm{d}x \stackrel{c}{=} e^x, \quad x \in \mathbb{R}$

8. $\int a^x\,\mathrm{d}x \stackrel{c}{=} \frac{a^x}{\ln a}, \quad x \in \mathbb{R}\ (a \in \mathbb{R}^+\setminus\{1\})$

9. $\int \sin x\,\mathrm{d}x \stackrel{c}{=} -\cos x, \quad x \in \mathbb{R}$

10. $\int \cos x\,\mathrm{d}x \stackrel{c}{=} \sin x, \quad x \in \mathbb{R}$

11. $\int \frac{1}{(\sin x)^2}\,\mathrm{d}x \stackrel{c}{=} -\cot x, \quad x \in \bigcup_{k\in Z}(k\pi, \pi+k\pi)$

12. $\int \frac{1}{(\cos x)^2}\,\mathrm{d}x \stackrel{c}{=} \tan x, \quad x \in \bigcup_{k\in Z}(k\pi-\frac{\pi}{2}, k\pi+\frac{\pi}{2})$

13. $\int \sinh x\,\mathrm{d}x \stackrel{c}{=} \cosh x, \quad x \in \mathbb{R}$

14. $\displaystyle\int \cosh x \,\mathrm{d}x \stackrel{c}{=} \sinh x, \quad x \in \mathbb{R}$

15. $\displaystyle\int \frac{1}{(\sinh x)^2} \,\mathrm{d}x \stackrel{c}{=} -\coth x, \quad x \in \mathbb{R} \setminus \{0\}$

16. $\displaystyle\int \frac{1}{(\cosh x)^2} \,\mathrm{d}x \stackrel{c}{=} \tanh x, \quad x \in \mathbb{R}$

In order to speed up calculations the following formulas are often useful:

17. $\displaystyle\int \frac{1}{\sqrt{x^2+1}} \,\mathrm{d}x \stackrel{c}{=} \ln(x + \sqrt{x^2+1}), \quad x \in \mathbb{R}$

18. $\displaystyle\int \frac{1}{\sqrt{x^2-1}} \,\mathrm{d}x \stackrel{c}{=} \ln(x + \sqrt{x^2-1}), \quad x \in (1, +\infty)$

19. $\displaystyle\int \frac{1}{x^2-1} \,\mathrm{d}x \stackrel{c}{=} \frac{1}{2} \ln\left|\frac{x-1}{x+1}\right|, \quad x \in \mathbb{R} \setminus \{-1, 1\}$

20. Let $f : (I \subset \mathbb{R}) \to \mathbb{R}$ (I is an interval) be differentiable on I. Then
$\displaystyle\int \frac{f'(x)}{f(x)} \,\mathrm{d}x \stackrel{c}{=} \ln|f(x)|, \quad x \in \{z \in I;\ f(z) \neq 0\}$

21. Let $f : (I \subset \mathbb{R}) \to \mathbb{R}$ (I is an interval) be differentiable on I. Then
$\displaystyle\int \frac{f'(x)}{\sqrt{f(x)}} \,\mathrm{d}x \stackrel{c}{=} 2\sqrt{f(x)}, \quad x \in \{z \in I;\ f(z) > 0\}$

Formulas 17-21 can be easily verified by performing the derivative on the right-hand sides, or with the use of forthcoming theorems 3 or 4.

Based on the formulas listed above one can calculate indefinite integrals for just a narrow class of functions. Hence in the following text we shall develop integration methods allowing us to transform integrals of a wide class of functions in such a way that the formulas above can be applied. We warn the reader right here that such a transformation may not be possible for every function $f \in P(I)$. For instance, indefinite integrals $\int \cos x^2 \,\mathrm{d}x$, $x \in \mathbb{R}$; $\int e^{-x^2} \,\mathrm{d}x$, $x \in \mathbb{R}$, although they do exist (see note 4.10.2), cannot be expressed in terms of the elementary functions (and, therefore, cannot be transformed into any of the formulas 1-21 for an elementary function f). Similarly, neither of the integrals $\int \frac{\sin x}{x} \,\mathrm{d}x$, $x \in \mathbb{R} \setminus \{0\}$ and $\int \frac{1}{\sqrt{x^3+1}} \,\mathrm{d}x$, $x \in \mathbb{R} \setminus \{-1\}$ belongs to the set of the elementary functions. (A complex proof of the last statement can be found in textbook [2].) There is no general solution to this problem.

Decomposition Method

THEOREM 11.7.1. Let $k_1, \ldots, k_m$ be real numbers and indefinite integrals $\int f_i(x) \,\mathrm{d}x \stackrel{c}{=} F_i(x)$ exist on interval $I \subset \mathbb{R}$ for $i = 1, \ldots, m$. Then the following indefinite integral exists

$$\int \left[\sum_{i=1}^{m} k_i f_i(x)\right] \mathrm{d}x \stackrel{c}{=} \sum_{i=1}^{m} k_i F_i(x), \quad x \in I.$$

Proof. The assumptions imply that finite sets $M_i \subset I$, $i = 1, \ldots, m$ exist such that

$$\forall x \in I \setminus M_i \; \forall i = 1, \ldots, m \; F_i'(x) = f_i(x) \quad (\in \mathbb{R}).$$

Set $M \cup_{i=1}^m M_i$ is also a finite subset of I and the following relation holds

$$\forall x \in I \setminus M \; \left(\sum_{i=1}^m k_i F_i\right)' (x) = \sum_{i=1}^m k_i F_i'(x) = \sum_{i=1}^m k_i f_i(x)$$

which means that the proof is done. □

Note 11.7.2. The statement in theorem 10.11.1 is usually written in the following form

$$\int \left[\sum_{i=1}^m k_i f_i(x)\right] \mathrm{d}x = \sum_{i=1}^m k_i \int f_i(x)\, \mathrm{d}x, \quad x \in J.$$

Sample Problem 11.7.1. Calculate the following indefinite integrals on their respective maximal sets
a) $\int \frac{2}{x^4-1}\, \mathrm{d}x$; b) $\int P_m(x)\, \mathrm{d}x$, where P_m is a polynomial of degree m $x \mapsto a_m x^m + \ldots + a_1 x + a_0$, $x \in \mathbb{R}$ and $a_i \in \mathbb{R}$ for $i = 0, 1, \ldots, m$; c) $\int \frac{1}{\sin x}\, \mathrm{d}x$.
`Solution`: a) Using formulas 5 and 19 one obtains

$$\int \frac{2}{x^4-1}\, \mathrm{d}x = \int \frac{1 - x^2 + 1 + x^2}{(x^2-1)(x^2+1)}\, \mathrm{d}x = \int \frac{1}{x^2-1}\, \mathrm{d}x - \int \frac{1}{x^2+1}\, \mathrm{d}x \overset{c}{=} \frac{1}{2} \ln \frac{x-1}{x+1} - \tan^{-1} x$$

for $x \in \mathbb{R} \setminus \{-1, 1\}$.
b) $\displaystyle\int P_m(x)\, \mathrm{d}x = \int \left[\sum_{i=0}^m a_i x^i\right] \mathrm{d}x = \sum_{i=0}^m a_i \int x^i\, \mathrm{d}x = \sum_{i=0}^m \frac{a_i}{i+1} x^{i+1}, \quad x \in \mathbb{R}.$
c) Using formula 20 one finds

$$\int \frac{\mathrm{d}x}{\sin x} = \int \frac{\sin^2 \frac{x}{2} + \cos^2 \frac{x}{2}}{2 \sin \frac{x}{2} \cos \frac{x}{2}}\, \mathrm{d}x = \int \frac{\frac{1}{2} \sin \frac{x}{2}}{\cos \frac{x}{2}}\, \mathrm{d}x + \int \frac{\frac{1}{2} \cos \frac{x}{2}}{\sin \frac{x}{2}}\, \mathrm{d}x \overset{c}{=} \ln \left|\sin \frac{x}{2}\right| - \ln \left|\cos \frac{x}{2}\right| = \ln \left|\tan \frac{x}{2}\right|$$

for $x \in \bigcup_{k \in Z} (k\pi, k\pi + \pi)$.

Method of Integration By Parts

Let F and G be the primitive functions to functions f and g on interval $I \subset \mathbb{R}$. In general, it is not true that product FG is the primitive function to function fg. That is because $(FG)' = F'G + FG' = fG + Fg$ on I. However, the following theorem holds:

THEOREM 11.7.2. Let $f \in P(I)$ and $g \in P(I)$ where $\int f(x)\, \mathrm{d}x \overset{c}{=} F(x)$, $\int g(x)\, \mathrm{d}x \overset{c}{=} G(x)$, $x \in I$ and furthermore let $fG \in P(I)$ (or $Fg \in P(I)$). Then also $Fg \in P(I)$ (or $fG \in P(I)$) and

$$\int f(x)\, G(x)\, \mathrm{d}x + \int F(x)\, g(x)\, \mathrm{d}x \overset{c}{=} F(x)\, G(x), \quad x \in I. \tag{11.5}$$

Proof. Let $fG \in P(I)$ and $\int f(x)G(x) \stackrel{c}{=} H(x),\ x \in I$. Then a finite set $M \subset I$ exists such that

$$\forall x \in I \setminus M : (FG - H)'(x) = f(x)G(x) + F(x)g(x) - f(x)G(x) = F(x)g(x).$$

That means that function $FG - H$ is primitive to Fg on I and $Fg \in P(I)$. The second part of the statement follows from theorem 10.11.1 and relation

$$\int (FG)'(x)\,\mathrm{d}x \stackrel{c}{=} F(x)G(x), \quad x \in I$$

which completes the proof. □

Note 11.7.3. Theorem 10.11.2 on integration by parts is of large practical significance. However, when applied we shall use $\int f(x)G(x)\,\mathrm{d}x = F(x)G(x) - \int F(x)g(x)\,\mathrm{d}x,\ x \in I$, instead of (10.9).

Sample Problem 11.7.2. Calculate the following indefinite integrals on their maximal sets

$$\text{a)} \int x\,\cos x\,\mathrm{d}x; \quad \text{b)} \int \sin^{-1} x\,\mathrm{d}x;$$

`Solution`: a) If we choose $f : x \mapsto \cos x,\ x \in \mathbb{R}$ and $G : x \mapsto x,\ x \in \mathbb{R}$ then $F : x \mapsto \sin x,\ x \in \mathbb{R}$ and $g : x \mapsto 1,\ x \in \mathbb{R}$ and according to theorem 10.11.2

$$\int x\,\cos x\,\mathrm{d}x = x \sin x - \int \sin x\,\mathrm{d}x \stackrel{c}{=} x \sin x + \cos x$$

for $x \in \mathbb{R}$.
b) Let us choose $f : x \mapsto 1,\ x \in (-1,1)$ and $G : x \mapsto \sin^{-1} x,\ x \in (-1,1)$. Then $F : x \mapsto x,\ x \in (-1,1)$ and $g : x \mapsto \frac{1}{\sqrt{1-x^2}},\ x \in (-1,1)$. Now using formula 21 we have

$$\int \sin^{-1} x\,\mathrm{d}x = x\,\sin^{-1} x + \frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}}\,\mathrm{d}x \stackrel{c}{=} x \sin^{-1} x + \sqrt{1-x^2}$$

for $x \in (-1,1)$. According to theorem 6.8.4 (see also problem 6a.8.4) the primitive function that has been found above can be extended to include the closed interval $[-1,1]$. Then one can write

$$\int \sin^{-1} x\,\mathrm{d}x \stackrel{c}{=} x \sin^{-1} x + \sqrt{1-x^2},\ x \in [-1,1].$$

Sample Problem 11.7.3. Prove that for indefinite integral $I_n(x) = \int \frac{1}{(1+x^2)^n}\,\mathrm{d}x$ on $\mathbb{R}$ the following recurrent relation holds

$$I_{n+1}(x) = \frac{1}{2n}\,\frac{x}{(1+x^2)^n} + \frac{2n-1}{2n}\,I_n(x), \quad x \in \mathbb{R}, \tag{11.6}$$

where $n \in N$.

`Solution`: Choosing $f : x \mapsto 1$, $x \in \mathbb{R}$ and $G : x \mapsto \frac{1}{(1+x^2)^n}$, $x \in \mathbb{R}$ one finds $F : x \mapsto x$, $x \in \mathbb{R}$ and $g : x \mapsto \frac{2nx}{(1+x^2)^{n+1}}$, $x \in \mathbb{R}$. Then according to theorem 10.11.2

$$\begin{aligned} I_n(x) &= \frac{x}{(1+x^2)^n} + 2n \int \frac{x^2}{(1+x^2)^{n+1}}\,\mathrm{d}x = \\ &= \frac{x}{(1+x^2)^n} + 2n \int \frac{x^2+1}{(1+x^2)^{n+1}}\,\mathrm{d}x - 2n \int \frac{1}{(1+x^2)^{n+1}}\,\mathrm{d}x = \\ &= \frac{x}{(1+x^2)^n} - 2n\, I_{n+1}(x) + 2n\, I_n(x), \quad x \in \mathbb{R}. \end{aligned}$$

This easily simplifies to formula (10.10)

Note 11.7.4. Note that the N-integral formula (10.9) can be written in the form

$$(N) \int_{x_0}^{x} f(t)\, G(t)\,\mathrm{d}t + (N) \int_{x_0}^{x} F(t)\, g(t)\,\mathrm{d}t = [F(x)G(x)]_{x_0}^{x}$$

for $x \in I$ and $x_0 \in I$ (compare with theorem 5.10.2).

Substitution Method

The basic theorems describing the substitution method are obtained from the theorem on the derivative of a composite function (the chain rule):

THEOREM 11.7.3. Let I_1 and I_2 be intervals in $\mathbb{R}$ and $f : I_1 \to \mathbb{R}$. Let function $\phi : I_2 \to I_1$ be differentiable on I_2. Then for $f \in P(I_1)$, and F being strongly primitive function to f on I_1 the composite function $F \circ f$ is strongly primitive function to function $(f \circ \phi)\phi'$ on I_2.

Proof. Since $F'(t) = f(t)$ for $t \in I_1$ then according to theorem 2.8.2 on the derivative of a composite function the following equality holds:

$$(F \circ \phi)'(x) = (F' \circ \phi)(x)\,.\,\phi'(x) = (f \circ \phi)(x)\,.\,\phi'(x)$$

for $x \in I_2$. □

Note 11.7.5. 1. Using the indefinite integral the statement in theorem 3 can be expressed in the form

$$\int (f \circ \phi)(x)\phi'(x)\,\mathrm{d}x = \int f(t)\,\mathrm{d}t \tag{11.7}$$

for $t = \phi(x)$, $x \in I_2$. Alternatively, the N-integral can be used in the form

$$(N) \int_{x_0}^{x} (f \circ \phi)(s)\phi'(s)\,\mathrm{d}s = (N) \int_{\phi(x_0)}^{\phi(x)} f(t)\,\mathrm{d}t$$

for $x \in I_2$, $x_0 \in I_2$.

2. Theorem 3 can be applied in the following sense: when calculating the integral on the left side of equation (10.11) in the actual calculation one introduces $\phi(x) = t$ and symbol $\phi'(x)\mathrm{d}x$ is then replaced by $\mathrm{d}t$.

Sample Problem 11.7.4. Calculate

$$I := \int \frac{1}{x\,\ln x}\,\mathrm{d}x$$

on the maximal set.

`Solution`: Let $\phi:\ x \mapsto \ln x,\ x \in \mathbb{R}^+ \setminus \{1\}$. Restrictions $\phi|_{(0,1)}$ and $\phi|_{(1,\infty)}$ satisfy the conditions of theorem 3. Therefore $\left(\mathrm{d}t = \frac{1}{x}\mathrm{d}x\right)$

$$I = \int \frac{1}{t}\mathrm{d}t \overset{\mathrm{c}}{=} \ln|t| \quad \text{for } t \in (-\infty, 0) \cup (0, +\infty)$$

and hence

$$I \overset{\mathrm{c}}{=} \ln|\ln x|, \quad x \in (0,1) \cup (1, +\infty).$$

If the primitive function to function $(f\circ\phi)\,.\,\phi'$ is known then the primitive function to function f can be calculated using the following theorem:

THEOREM 11.7.4. Let I_1 and I_2 be intervals in $\mathbb{R}$ and $f:\ I_1 \to \mathbb{R}$. Let function $\phi:\ I_2 \to I_1$ be a bijection and has proper derivative $\phi'(t) \neq 0$ on interval I_2. If $(f\circ\phi)\,.\,\phi' \in P(I_2)$ and G is its primitive function on I_2, then function $G\circ\phi^{-1}$ (ϕ^{-1} is the inverse function to ϕ) is primitive to f on I_1.

Proof. A finite set $M_2 \subset I_2$ exists such that

$$\forall t \in I_2 \setminus M_2:\ G'(t) = (f\circ\phi)(t)\phi'(t).$$

Since ϕ is a bijection set $M_1 := \phi(M_2)$ is finite. Then according to the theorem on the derivative of composite (theorem 2.8.2) and inverse (theorem 4.8.2) functions the following relation holds for all $x \in I_1 \setminus M_1$

$$(G\circ\phi^{-1})'(x) = (G'\circ\phi^{-1})(x)\,.\,(\phi^{-1})'(x) = [(f\circ\phi)\circ\phi^{-1}](x)\,.\,\frac{1}{\phi'\circ\phi^{-1})(x)} = f(x)$$

The last equality completes the proof. □

Note 11.7.6. 1. Using the indefinite integral the statement in theorem 4 can be expressed by the following formula

$$\int f(x)\,\mathrm{d}x = \int (f\circ\phi)(t)\phi'(t)\,\mathrm{d}t$$

for $t = \phi^{-1}(x)$, $x \in I_1$. Alternatively, the N-integral can be used in the form

$$(N)\int_{x_0}^{x} f(s)\,\mathrm{d}s = (N)\int_{\phi^{-1}(x_0)}^{\phi^{-1}(x)} (f\circ\phi)(t)\phi'(t)\,\mathrm{d}t$$

for $x \in I_1,\ x_0 \in I_1$.

2. In order to calculate integral $\int f(x)\mathrm{d}x$ theorem 4 can be used with substitutions $x = \phi(t)$ and $\mathrm{d}x = \phi'(t)\mathrm{d}t$.

Sample Problem 11.7.5. Calculate $\int \sqrt{1+x^2}\,\mathrm{d}x$ on $\mathbb{R}$.

`Solution`: 1. Consider $\phi : t \mapsto \sinh t,\ t \in \mathbb{R}$. ϕ is a bijection and for $t \in \mathbb{R}$ we have $\phi'(t) = \cosh t > 0$. The assumptions of theorem 4 are therefore satisfied. Hence

$$\int \sqrt{1+x^2}\mathrm{d}x = \int \sqrt{1+\sinh^2 t}\,\cosh t\,\mathrm{d}t = \int \cosh^2 t\,\mathrm{d}t$$

for $t = \sinh^{-1} x = \ln(x+\sqrt{x^2+1})$, $x \in \mathbb{R}$. (We have applied the identity $\cosh^2 x - \sinh^2 x = 1$, for $x \in \mathbb{R}$.) Furthermore we have

$$\int \cosh^2 t\,\mathrm{d}t = \frac{1}{4}\int e^{2t}\,\mathrm{d}t + \frac{1}{2}\int \mathrm{d}t + \frac{1}{4}\int e^{-2t}\,\mathrm{d}t \stackrel{\mathrm{c}}{=} \frac{t}{2} + \frac{1}{4}\sinh 2t, \quad t \in \mathbb{R}.$$

That implies

$$\int \sqrt{1+x^2}\,\mathrm{d}x \stackrel{\mathrm{c}}{=} \frac{1}{2}\ln(x+\sqrt{x^2+1}) + \frac{1}{4}\sinh[2\ln(x+\sqrt{x^2+1})] \text{ for } x \in \mathbb{R}.$$

2. Using integration by parts with $f : x \mapsto \sqrt{1+x^2},\ x \in \mathbb{R}$ and $G : x \mapsto 1,\ x \in \mathbb{R}$ one gets

$$\int \sqrt{1+x^2}\,\mathrm{d}x \stackrel{\mathrm{c}}{=} \frac{1}{2}\ln(x+\sqrt{x^2+1}) + \frac{1}{2}x\sqrt{x^2+1}, \quad x \in \mathbb{R}.$$

Formally, the two methods give different results. However, one can work out the derivatives of the two results and easily see that both results are correct. As an independent check, one can recall theorem 2.10.1, part (ii):

$$\forall x \in \mathbb{R} :\ x\sqrt{x^2+1} = \frac{1}{2}\sinh[2\ln(x+\sqrt{x^2+1})] + c.$$

Choosing $x = 0$ in this formula we get $c = 0$ and hence the equality of the two results.

Note 11.7.7. Note that theorem 3 can be used with much weaker assumptions about function ϕ than theorem 4. In theorem 3, the bijection is not required and neither is the assumption $\phi'(t) \neq 0$.

Problems

1. Find recurrent relations for the following indefinite intergals:
   a) $I_n(x) = \int \sin^n x\,\mathrm{d}x$,
   b) $J_n(x) = \int \cos^n x\,\mathrm{d}x$.

2. Use both integration by parts and substitution method for the calculation of indefinite integral $\int \sqrt{x^2-1}\,\mathrm{d}x$ where $x \in (-\infty,-1) \cup (1,+\infty)$, and compare the two results.

3. Calculate (on $\mathbb{R}$)

   a) $\displaystyle\int f(x)\,\mathrm{d}x, \quad \text{where } f:\ x \mapsto \begin{cases} 1-x^2, & |x|\le 1 \\ 1-|x|, & |x|>1 \end{cases}$

   b) $\displaystyle\int \frac{x^2+1}{x^4+1}\,\mathrm{d}x.$

Answers

1. a) $I_n(x) = -\frac{1}{n}\sin^{n-1}x\,\cos x + \frac{n-1}{n}I_{n-2}(x)$ for $n\in N,\ n\ge 2$;
   b) $J_n(x) = \frac{1}{n}\cos^{n-1}x\,\sin x + \frac{n-1}{n}I_{n-2}(x)$ for $n\in N,\ n\ge 2$.

2. $\frac{x}{2}\sqrt{x^2-1} - \frac{1}{2}\ln|x+\sqrt{x^2-1}|$.

3. a) $x-\frac{x^3}{3}$ for $|x|\le 1$; and $x-\frac{x}{2}|x|+\frac{1}{6}\,\mathrm{sgn}x$ for $|x|>1$;
   b) $F:\ x\mapsto \frac{1}{\sqrt{2}}\tan^{-1}\frac{x^2-1}{x\sqrt{2}},\ x\in\mathbb{R}\setminus\{0\};\ F(0)=0$. (Use the following change of variables: $x-\frac{1}{x}=t\sqrt{2}$.)

11.8 Integration of Special Types of Functions

The goal of this section is to show the ways how to find indefinite integrals of some important and frequent classes of real functions. For practical purposes we limit ourselves to show only the description of several useful algorithms without the ambition to provide the rigorous formulations. Hence in each particular case the validity conditions must be considered separately.

11.8.1 Integration of Rational Functions

Recall that in theorem 6.4.3 we have introduced the decomposition of a purely rational single variable function into partial fractions. According to this theorem and theorem 1.10.3 the problem of integration of such rational functions is reduced to the problem of integration of individual partial fractions.

Since for $k\in N$ and $k\ne 1$

$$\int \frac{1}{(x-a)^k}\mathrm{d}x \overset{\mathrm{c}}{=} \frac{(x-a)^{1-k}}{1-k}, \quad x\in\mathbb{R}\setminus\{a\}$$

and

$$\int \frac{1}{x-a}\mathrm{d}x \overset{\mathrm{c}}{=} \ln|x-a|, \quad x\in\mathbb{R}\setminus\{a\}$$

then the only remaining problem in the integration of rational single variable functions is the calculation of the primitive function to function $x\mapsto \frac{Ax+B}{(x^2+px+q)^k}$, $x\in\mathbb{R}$, where A, B, p

and q are real numbers, $k \in N$, and the discriminant $p^2 - 4q$ of the quadratic expression in the denominator is negative. Taking the decomposition

$$\frac{Ax+B}{(x^2+px+q)^k} = \frac{A}{2}\frac{2x+p}{(x^2+px+q)^k} + \frac{B - \frac{Ap}{2}}{(x^2+px+q)^k}$$

into account the indefinite integral of the first term can easily be found based on theorem 3.10.3 (the first theorem on substitution) using the change of variables $x^2+px+q=t$ while

$$\frac{A}{2}\int \frac{2x+p}{(x^2+px+q)^k}\,\mathrm{d}x \stackrel{c}{=} \frac{A}{2(1-k)}(x^2+px+q)^{1-k}, \quad x \in \mathbb{R},\ k \neq 1$$

$$\frac{A}{2}\int \frac{2x+p}{x^2+px+q}\,\mathrm{d}x \stackrel{c}{=} \frac{A}{2}\ln(x^2+px+q), \quad x \in \mathbb{R}.$$

The denominator of the second term can be expanded according to

$$x^2+px+q = \left(x+\frac{p}{2}\right)^2 + q - \frac{p^2}{4} = \left(q-\frac{p^2}{4}\right)\left[\left(\frac{x+\frac{p}{2}}{\sqrt{q-\frac{p^2}{4}}}\right)^2 + 1\right].$$

Then the primitive function to the second term can be obtained using the change of variable $t = \left(x+\frac{p}{2}\right)\left(q-\frac{p^2}{4}\right)^{-1/2}$ (theorem 3.10.3). This substitution transforms the original integral $\int \frac{\mathrm{d}x}{(x^2+px+q)^k}$ into integral $\int \frac{\mathrm{d}t}{(t^2+1)^k}$. The latter integral can be obtained using the recurrent formula (2.10.3) found in problem 3.10.3. The procedure described above can now be summarised in

THEOREM 11.8.1. The indefinite integral of an arbitrary rational single variable function on its definition domain belongs to the set of the elementary functions.

11.8.2 Integration of Trigonometric Functions

Next, we shall explicitly show the substitutions used to transform the integrals of type

$$\int R(\sin x, \cos x)\,\mathrm{d}x, \quad x \in M \subset \mathbb{R} \tag{11.8}$$

into integrals of rational functions. Function $R : (u,v) \mapsto R(u,v)$ from $\mathbb{R}^2$ to $\mathbb{R}$ is a rational function of two variables.

Integral (10.12) can be calculated using the substitutions shown below (and theorem 3.10.3). Each of the substitutions transforms the integrand into a rational single variable function.

1. If R is an odd function in the first of its two variables, $R(-\sin x, \cos x) = -R(\sin x, \cos x)$ then a substitution $\cos x = t$, $x \in \mathbb{R}$, can be used.

2. If $R(\sin x, -\cos x) = -R(\sin x, \cos x)$ then a substitution $\sin x = t,\ x \in \mathbb{R}$, can be used.

3. If $R(-\sin x, -\cos x) = R(\sin x, \cos x)$ then a substitution $\tan x = t$, $x \in \bigcup_{k\in Z}\left(-\frac{\pi}{2}+k\pi, \frac{\pi}{2}+k\pi\right)$, can be used.

4. For an arbitrary rational function R of two variables a substitution $\tan\frac{x}{2} = t$ can be used. (This substitution can also be used in the previous three cases although the calculation is most of the times technically more involved.)

The last substitution, $\tan\frac{x}{2} = t$, for $x \in (-\pi, \pi)$, $t \in \mathbb{R}$, leads to the formulas

$$\sin x = 2\,\sin\frac{x}{2}\,\cos\frac{x}{2} = \frac{2\,\tan\frac{x}{2}}{\tan^2\frac{x}{2}+1} = \frac{2t}{t^2+1}$$

$$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{1-t^2}{1+t^2}$$

$$x = 2\,\tan^{-1} t, \quad \mathrm{d}x = \frac{2}{1+t^2}\mathrm{d}t.$$

It is now clear that integral (1) transforms into a rational function in terms of variable t.

Sample Problem 11.8.1. Calculate $\int \frac{1}{1+\cos^2 x}\,\mathrm{d}x$ on $\mathbb{R}$.

`Solution`: Based on the ideas above we can use either the change of variable $\tan x = t$ or $\tan\frac{x}{2} = t$. We shall use both substitutions:

a) With the change of variable $\tan x = t$ the conditions of theorem 3.10.3. are satisfied on every interval $\left(-\frac{\pi}{2}+k\pi, \frac{\pi}{2}+k\pi\right)$, $k \in Z$. Since $\frac{\sin^2 x}{\cos^2 x} = t^2$ one gets $\cos^2 x = \frac{1}{1+t^2}$ and on any interval $\left(-\frac{\pi}{2}+k\pi, \frac{\pi}{2}+k\pi\right)$, $k \in Z$, the following relations hold:

$$\begin{aligned}\int \frac{1}{1+\cos^2 x}\mathrm{d}x &= \int \frac{\cos^2 x}{1+\cos^2 x}\frac{\mathrm{d}x}{\cos^2 x}\\ &= \int \frac{\mathrm{d}t}{2+t^2} = \frac{1}{2}\int \frac{\mathrm{d}t}{1+\left(\frac{t}{\sqrt{2}}\right)^2} \stackrel{c}{=} \frac{2}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}},\ t \in \mathbb{R}.\end{aligned}$$

Thus one finds

$$\int \frac{1}{1+\cos^2 x}\mathrm{d}x \stackrel{c}{=} \frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right)$$

on any interval $\left(-\frac{\pi}{2}+k\pi, \frac{\pi}{2}+k\pi\right)$, $k \in Z$.

The calculation is, however, not completely finished, since the function found at the end is not primitive to function $f : x \mapsto (1+\cos^2 x)^{-1}$, $x \in \mathbb{R}$ everywhere on set $\mathbb{R}$. Using the "gluing" construction of example 3.10.1 the primitive function (indefinite integral) obtained above can now be continuously extended at points $\frac{\pi}{2}+k\pi$, $k \in Z$ in such a way that new extension F will be the primitive function to function $f : x \mapsto (1+\cos^2 x)^{-1}$, $x \in \mathbb{R}$ on $\mathbb{R}$.

For $k \in Z$ let us denote

$$F_{2k}\,:\, x \mapsto \frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right), \qquad x \in \left(-\frac{\pi}{2}+2k\pi, \frac{\pi}{2}+2k\pi\right)$$
$$F_{2k+1}\,:\, x \mapsto \frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right), \qquad x \in \left(-\frac{\pi}{2}+(2k+1)\pi, \frac{\pi}{2}+(2k+1)\pi\right)$$

and

$$C_k := \lim_{x\to\left(-\frac{\pi}{2}+(2k+1)\pi\right)^+} F_{2k+1}(x) - \lim_{x\to\left(\frac{\pi}{2}+2k\pi\right)^-} F_{2k}(x) = -\frac{\pi\sqrt{2}}{2}.$$

Then the primitive function F is

$$F\,:\, x \mapsto \begin{cases} \frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right) - \frac{\pi\sqrt{2}}{2}, & x \in \left(-\frac{\pi}{2}+2k\pi, \frac{\pi}{2}+2k\pi\right) \\ \frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right), & x \in \left(-\frac{\pi}{2}+(2k+1)\pi, \frac{\pi}{2}+(2k+1)\pi\right) \\ -\frac{\pi\sqrt{2}}{4}, & x = \frac{\pi}{2}+k\pi, \quad k \in Z. \end{cases}$$

Besides that F also is the strongly primitive function to f on $\mathbb{R}$. Recalling theorem 6.8.4 this can be seen from

$$F'_+\left(\frac{\pi}{2}+k\pi\right) = \lim_{\left(\frac{\pi}{2}+k\pi\right)_+} F'(x) = \lim_{\left(\frac{\pi}{2}+k\pi\right)_+} \frac{1}{1+\cos^2 x} =$$

$$= 1 = F'_-\left(\frac{\pi}{2}+k\pi\right) \quad \text{and hence} \quad F'\left(\frac{\pi}{2}+k\pi\right) = 1 = \left(1+\cos^2\frac{\pi}{2}\right)^{-1}.$$

b) Substitution $\phi\,:\, x \mapsto \tan\frac{x}{2}$ also satisfies the conditions of theorem 3.10.3. on an interval $(-\pi+2k\pi, \pi+2k\pi)$, $k \in Z$. Then $\cos^2 x = \frac{(1-t^2)^2}{(1+t^2)^2}$, $\cos^2\frac{x}{2} = \frac{1}{1+t^2}$, $t \in \mathbb{R}$. For any $x \in (-\pi+2k\pi, \pi+2k\pi)$

$$\int \frac{1}{1+\cos^2 x}\mathrm{d}x = \int \frac{2\cos^2\frac{x}{2}}{1+\cos^2 x}\frac{\mathrm{d}x}{2\cos^2\frac{x}{2}} = \int \frac{t^2+1}{t^4+1}\mathrm{d}t, \quad t \in \mathbb{R}.$$

The rational function that has been obtained here is more complicated than in case a). The only benefit of this substitution is that we find primitive functions on larger intervals. However, since we are seeking the primitive function on $\mathbb{R}$ the obtained result must again be extended by "gluing".

The definition of the derivative of a function depending on a single variable (chapter 8) was motivated by two basic problems — a geometric one (construction of a tangential to the graph of the function) and a physical one (need to find the instantaneous velocity of a motion with known trajectory). Another two problems: the evaluation of a surface area and the calculation of a trajectory of a moving point whose velocity is known, despite being at first glance very different, lead to mathematical problems closely related to the concept of a function differentiability.

The goal of the next two chapters is to develop the constructs that are needed to solve problems of this type. Hence in this chapter we shall deal with the concept and basic propreties of a primitive function, Newton and indefinite integral of a real-valued function of a single variable, and we shall introduce some methods to calculate them.

In the next chapters, on the Riemann and generalised integral, further application and extension of these results will be presented.

11.9 Primitive Function

The task to find a primitive function is in some sense the inverse problem to the problem to find a proper derivative of a given function. (A function f is primitive to its derivative f'.) However, contrary to the calculation of derivatives there is no simple set of rules for the calculation of the primitive function in terms of the elementary functions.

Strongly Primitive Function

One of the basic types of the primitive function is introduced in the following definition:

Definition 11.9.1. Let $f : (J \subset \mathbb{R}) \to \mathbb{R}$ and $I \subset J$ is an interval (of any type). We say that *function* $F : I \to \mathbb{R}$ *is strongly primitive* (or is *a strong potential*) *to function* f *on interval* I if

$$\forall x \in I : \; F'(x) = f(x).$$

Note 11.9.1. 1. From the definition 10.9.1 it follows directly that function F is differentiable on interval I which, according to theorem **??**, means that F is continuous on I, $F \in C(I)$.
2. The *sine* function is strongly primitive to the *cosine* function on interval $(-\infty, +\infty)$. It can be seen from example 6.8.4 that the strongly primitive function on $\mathbb{R}$ to a discontinuous function

$$f : x \mapsto \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x}; & x \in \mathbb{R} \setminus \{0\} \\ 0, & x = 0 \end{cases}$$

is a function F

$$F : x \mapsto \begin{cases} x^2 \sin \frac{1}{x}, & x \in \mathbb{R} \setminus \{0\} \\ 0, & x = 0. \end{cases}$$

Similarly, there exists a strongly primitive function on $(0,1]$ to a function $f : x \mapsto x^{-2/3}$, $x \in (0,1]$, namely $F : x \mapsto 3x^{1/3}$, $x \in (0,1]$.

Example 11.9.1. Prove that there exists no strongly primitive function on $\mathbb{R}$ to a function

$$sgn : x \mapsto \begin{cases} 1, & x \in \mathbb{R}^+ \\ 0, & x = 0 \\ -1, & x \in \mathbb{R}^-. \end{cases}$$

`Solution`: If the strongly primitive function F on $\mathbb{R}$ to function sgn existed then according to the corollary to theorem **??** the derivative $F'(x)$ would not have points of discontinuity of the first kind on $\mathbb{R}$. This is, however, in contradiction with the requirement that $F'(x) = \operatorname{sgn} x,\ x \in \mathbb{R}$. (Point 0 is a point of discontinuity of the first kind of the sgn function.)

The sufficient conditions for the existence of the strongly primitive function are listed in chapter 12.

Now let us focus on the problem of the correctness (uniqueness) of the strongly primitive function.

THEOREM 11.9.1. (i) Let F be the strongly primitive function to function f on interval I. Then for any $C \in \mathbb{R}$ the function $x \mapsto F(x) + C,\ x \in I,$ is also strongly primitive to function f on I.
(ii) Let F and G be strongly primitive functions to function f on interval I. Then there exists a unique constant $C \in \mathbb{R}$ such that

$$\forall x \in I:\ F(x) = G(x) + C.$$

Proof. (i) The statement follows directly from definition 10.9.1.
(ii) If both F and G are strongly primitive functions to f on I then $\forall x \in I : (F-G)'(x) = 0$. According to corollary 1 (part e′) to theorem 2.8.4 function $F - G$ is a constant function on I. The uniqueness of this constant can be shown indirectly. □

Note 11.9.2. 1. The strongly primitive function on an interval is not defined uniquely. If it exists then there are infinitely many of them, differing from each other by a constant.

2. It is, however, easy to see that if function f has a strongly primitive function on interval I then to every point $(x_0, y_0) \in I \times \mathbb{R}$ there exists one and only one strongly primitive function F to f on I such that $(x_0, y_0) \in F$. (The graph of F passes through the point (x_0, y_0).)

Primitive Function

In example 1 we saw that even such a simple function as the sgn function does not have a strongly primitive function on $\mathbb{R}$. Hence it is sensible to extend the concept to a wider class of functions.

Definition 11.9.2. Let $f\colon\ (J \subset \mathbb{R}) \to \mathbb{R}$ and $I \subset J$ is again an interval. We say that *function* $F : I \to \mathbb{R}$ *is primitive* (or *is a potential*) to function f on interval I if

(i) $F \in C(I)$,

(ii) a finite set $M \subset I$ exists such that

$$\forall x \in I \setminus M:\ F'(x) = f(x).$$

Note 11.9.3. 1. The concept of the primitive function is the generalisation of the concept of the strongly primitive function. In fact, definition 10.9.1 is obtained from definition 10.9.2 for $M = \emptyset$.

2. Function $f: \ x \mapsto |x|, \ x \in \mathbb{R}$ is a primitive function to the sgn function on $\mathbb{R}$, with $M = \{0\}$, and function $F_1 : \ x \mapsto 2\sqrt{x}, \ x \geq 0$ is primitive to unbounded function

$$f_1 : \ x \mapsto \begin{cases} 5 & for \quad x = 0 \\ \frac{1}{\sqrt{x}}, & for \quad x \in (0, +\infty). \end{cases}$$

There is a similar statement on the uniqueness of the primitive function as there was for the strongly primitive function.

THEOREM 11.9.2. (i) Let F be the primitive function to function f on interval I and $C \in \mathbb{R}$. Then function $x \mapsto F(x) + C, \ x \in I$ is also a primitive function on I to function f.
(ii) Let F and G be primitive functions to function f on interval I. Then there exists one and only one constant $C \in \mathbb{R}$ such that

$$\forall x \in I : \ F(x) = G(x) + C.$$

(iii) If a function f has a primitive function on I then for any $x_0 \in I, \ y_0 \in \mathbb{R}$ there exists one and only one primitive function F to f on I such that $F(x_0) = y_0$.

Proof. Statement (i) is obvious.
(ii) There exist finite sets $M_1 \subset I, \ M_2 \subset I$ such that

$$\forall x \in I \setminus M_1 : \ F'(x) = f(x), \quad \forall x \in I \setminus M_2 : \ G'(x) = f(x).$$

The set $M = M_1 \cup M_2$ is a finite set in I and

$$\forall x \in I \setminus M : \ F'(x) = f(x) = G'(x).$$

Since $F \in C(I), \ G \in C(I)$, according to part b) of example 2.8.4 there exists a constant $C \in \mathbb{R}$ such that $F(x) - G(x) = C$ for all $x \in I$. The uniqueness is obvious.
(iii) It is seen immediately that if G is a primitive function to f on I then $F : \ x \mapsto G(x) - G(x_0) + y_0, \ x \in I$ is the only primitive function to f on I with the required properties. $\square$

Sample Problem 11.9.1. Let $c_0, \ c_1, \ \ldots, \ c_m; \ l_1, \ \ldots, \ l_m$ be real numbers and let $x_0 = a < x_1 < x_2 < \ \ldots \ < x_{m-1} < x_m = b$. On interval $[a, b]$ we define the "step-like" function

$$f : \ x \mapsto \begin{cases} c_j, & x = x_j, \quad j = 0, 1, \ \ldots, \ m \\ l_i, & x \in (x_{i-1}, x_i), \quad i = 1, \ldots, \ m. \end{cases}$$

Prove that function

$$F : \ x \mapsto \begin{cases} l_1(x - x_0) & x \in [x_0, x_1) \\ l_2(x - x_1) + l_1(x_1 - x_0) & x \in [x_1, x_2) \\ l_m(x - x_{m-1}) + l_{m-1}(x_{m-1} - x_{m-2}) + \ldots + l_1(x_1 - x_0), & x \in [x_{m-1}, x_m] \end{cases}$$

is primitive to function f on interval [a,b].

`Solution`: Let M be the finite set of points $\{x_0, x_1, \ldots, x_m\} \subset [a,b]$. Then

$$\forall x \in [a,b] \setminus M : \ F'(x) = f(x)$$

and the statement is proved.

Note 11.9.4. 1. Note that the concepts of the strongly primitive function and primitive function have been defined on an interval. Both concepts may be extended to more general sets on $\mathbb{R}$. However, the crucial assumption that I is an interval should be kept in the uniqueness theorems (part (ii) of theorem 10.9.1 and parts (ii) and (iii) of theorem 10.9.2). For example, functions

$$f : x \mapsto \begin{cases} x, & x \in (0,1) \\ x+1, & x \in (1,2) \end{cases} \quad \text{and} \quad g : x \mapsto \begin{cases} x+1, & x \in (0,1) \\ x, & x \in (1,2) \end{cases}$$

are primitive (also strongly primitive) to function $x \mapsto 1, \ x \in (0,1) \cup (1,2)$, on $(0,1) \cup (1,2)$. Nevertheless the difference $f - g$ is not a constant.

2. Further generalisation of the concept of the primitive function can be obtained when set M in definition 10.9.2 is countable. In this case the primitive function to the Dirichlet function is any real constant (since Q, the rational number set, is countable). (See, for example, textbook [**?**], p. 359.)

Note 11.9.5. It is obvious that a function defined on $I \subset \mathbb{R}$ may not have a primitive function on I. For instance, function

$$f : x \mapsto \begin{cases} 5, & \text{for} \quad x = 0 \\ \frac{1}{x}, & \text{for} \quad x \in (0,1] \end{cases}$$

has no primitive function on $[0,1]$. On interval $(0,1]$, the primitive function to f is a function $x \mapsto \ln x, \ x \in (0,1]$.

The set of all functions $f : I \to \mathbb{R}$ whose primitive function on interval $I \subset \mathbb{R}$ exists will be denoted as $P(I)$.

Sample Problem 11.9.2. Let $f : (a,b) \to \mathbb{R}$ have a point of discontinuity of the first kind at point $x_0 \in (a,b)$. Furthermore, let F_1 (F_2) be the primitive function to restriction $f|_{(a,x_0)}$ $(f|_{(x_0,b)})$ such that proper limits $\lim\limits_{x \to x_0^-} F_1(x)$ and $\lim\limits_{x \to x_0^+} F_2(x)$ exist. Show then that a primitive function F to function f on interval (a,b) exists and

$$F(x) = \begin{cases} F_1(x) + k, & x \in (a,x_0), \quad (k \in \mathbb{R}) \\ F_2(x), & x \in (x_0,b) \\ \lim\limits_{x \to x_0^-} F_1(x) + k = \lim\limits_{x \to x_0^+} F_2(x), & x = x_0. \end{cases}$$

`Solution`: From the continuity of functions F_1 and F_2 on intervals (a, x_0) and (x_0, b), respectively, and from the choice of constant $k = \lim\limits_{x \to x_0^+} F_2(x) - \lim\limits_{x \to x_0^-} F_1(x)$ it follows that F is continuous on (a, b). That means that finite sets $M_1 \subset (a, x_0)$ and $M_2 \subset (x_0, b)$ exist such that $F_1'(x) = f(x)$ for all $x \in (a, x_0) \setminus M_1$ and $F_2'(x) = f(x)$ for all $x \in (x_0, b) \setminus M_2$. Hence $F'(x) = f(x)$ for all $x \in (a, b) \setminus (M_1 \cup M_2 \cup \{x_0\})$.

This example shows how to construct a primitive function to a discontinuous function with points of discontinuity of the first kind. This construction will be used frequently and will be referred to as "gluing".

Problems

1. Show that the existence of the primitive function to $f + g$ on interval I does not guarantee the existence of the primitive functions to f and g, separately, on I.

2. Let functions f and $f + g$ have primitive functions on $J \subset \mathbb{R}$. Find out if a primitive function to g on J exists.

3. Find out if primitive functions to f and g on $J \subset \mathbb{R}$ exist provided the primitive functions to $f + g$ and $f - g$ on J exist.

4. Prove that there exists no primitive function to function $x \mapsto \chi(x)$, $x \in I$, where χ is the Dirichlet function and $I \subset \mathbb{R}$ is an arbitrary interval in $\mathbb{R}$.

5. Find all primitive functions to the functions defined below together with their definition domains:

$$f_1 : x \mapsto \tfrac{1}{x},\ x \in (-\infty, 0); \qquad f_2 : x \mapsto \tfrac{1}{x},\ x \in (0, +\infty);$$

$$f_3 : x \mapsto \tfrac{1}{x},\ x \in (-\infty, 0) \cup (0, +\infty); \qquad f_4 : x \mapsto [x],\ x \in [0, n],\ n \in N;$$

$$f_5 : x \mapsto x\,|x|,\ x \in \mathbb{R}; \qquad f_6 : x \mapsto \begin{cases} \frac{1}{x}, & x \in \mathbb{R} \setminus \{0\} \\ 0, & x = 0. \end{cases}$$

6. Prove that $\tan^{-1} x + \cot^{-1} x = \pi$ for any $x \in \mathbb{R}$.

Answers

2. Yes, it does.

3. Yes, it does.

5. $F_1 : x \mapsto \ln(-x) + c, \;\; x \in (-\infty, 0)$;
   $F_2 : x \mapsto \ln x + c, \; x \in (0, +\infty)$;
   $F_3 : x \mapsto \begin{cases} \ln(-x) + c_1, & x \in (-\infty, 0) \\ \ln x + c_2, & x \in (0, +\infty) \end{cases}$
   (a special case: $F_3 : x \mapsto \ln x + c, \; x \in \mathbb{R} \setminus \{0\}$);
   $F_4 : x \mapsto x[x] - \frac{[x]([x]+1)}{2}, \; x \in [0, n]$;
   $F_5 : x \mapsto \begin{cases} \frac{x^3}{3} + c, & x \in [0, +\infty) \\ -\frac{x^3}{3} + c, & x \in (-\infty, 0); \end{cases}$

   F_6 does not exist.

11.10 Riemann Integral and Its Properties

Inspired by the preceding section we are ready to claim that determining the trajectory of a moving particle out of its velocity given as a function of time effectively means finding the primitive function to the velocity function. In this section, besides the properties of the Riemann integral we shall show that when calculating the surface area one is again faced with the issue of finding the primitive function.

Riemann Integral

Before addressing the surface area question we shall study the following concept and its properties:

Definition 11.10.1. Let I be an interval in $\mathbb{R}$ and $f \in P(I)$. Let F be a primitive function to f on I and let $[a, b] \subset I$. Then the difference $F(b) - F(a)$ is called *a Riemann (definite) integral (R-integral) of function f from a to b.* It is denoted as $(N) \int_a^b f(x)\,\mathrm{d}x$. We also say that f is *Riemann integrable* on $[a, b]$. (The difference $F(b) - F(a)$ is denoted as $[F(x)]_a^b$ or $F(x)|_a^b$.)

Note 11.10.1. The definition of the Riemann integral is correct, *i.e.* its value does not depend on the choice of the primitive function.

Sample Problem 11.10.1. Calculate the Riemann integral from a to b of the following functions: a) $f_1 = \operatorname{sgn}$; b) $f_2 : x \mapsto \operatorname{sgn} x, \; x \in \mathbb{R} \setminus \{a, b\}, \; f_2(a) = c_1 \in \mathbb{R}, \; f_2(b) = c_2 \in \mathbb{R}$; c) the function defined in sample problem 2.10.1; d) an unbounded (in the neighbourhood of point 0) function

$$f_3 : x \mapsto \begin{cases} 2x \sin \frac{\pi}{x^2} - \frac{2\pi}{x} \cos \frac{\pi}{x^2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0, \end{cases}$$

for $a = -1, \; b = 1$.

`Solution`: For the first two cases

$$(N)\int_a^b \operatorname{sgn} x \,\mathrm{d}x = [|x|]_a^b = |b| - |a| = (N)\int_a^b f_2(x)\,\mathrm{d}x.$$

c)

$$(N)\int_a^b f(x)\,\mathrm{d}x = [F(x)]_a^b = \sum_{i=1}^{m} l_m(x_i - x_{i-1}).$$

d) It is easy to see that

$$F_3 :\ x \mapsto \begin{cases} x^2 \sin \frac{\pi}{x^2} & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$$

is a strongly primitive function to function f_3 on $\mathbb{R}$, and hence

$$(N)\int_{-1}^{1} f_3(x)\,\mathrm{d}x = F_3(x)|_{-1}^{1} = \sin\pi - \sin\pi = 0.$$

Note 11.10.2. In the case when $l_i > 0$ for $i = 1, \ldots, m$, the Riemann integral in example 1 c) gives a very intuitive geometrical interpretation — the surface area of the histogram-shaped object in figure 10.1.

Note 11.10.3. Note that the N-integral is defined on a closed interval. The extension of this concept to include an arbitrary (also unbounded) interval will be introduced in chapter 13.

Basic Properties of the N-integral

THEOREM 11.10.1. Let $J \subset \mathbb{R}$ be an interval, $a \in J$, $b \in J$, $f \in P(J)$ and $M \subset J$ is a finite set.

(i) If $f|_{J\setminus M} = g|_{J\setminus M}$ then

$$(N)\int_a^b g(x)\,\mathrm{d}x = (N)\int_a^b f(x)\,\mathrm{d}x.$$

(ii)

$$(N)\int_a^a f(x)\,\mathrm{d}x = 0, \qquad (N)\int_a^b f(x)\,\mathrm{d}x = -(N)\int_b^a f(x)\,\mathrm{d}x.$$

(iii) For function $k :\ x \mapsto c \in \mathbb{R}$, $x \in J \setminus M$ the following is true:

$$(N)\int_a^b k(x)\,\mathrm{d}x = c(b-a).$$

Proof. (i) Taking the assumptions into account it follows that the primitive functions to functions f and g on interval J differ by a constant and the statement is then obvious.

(ii) Let F be a primitive function to f on J. Then

$$(N)\int_a^a f(x)\,\mathrm{d}x = F(a) - F(a) = 0,$$

$$(N)\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a) = -[F(a) - F(b)] = -(N)\int_b^a f(x)\,\mathrm{d}x.$$

(iii) According to example 2.10.1 the primitive function to function k on interval J is $F: x \mapsto c(x-a)$, $x \in J$. The result then follows trivially.

□

THEOREM 11.10.2. Let a and b be numbers from interval $J \subset \mathbb{R}$, $C_1 \in \mathbb{R}$, $C_2 \in \mathbb{R}$, and $f \in P(J)$ and $g \in P(J)$. Then $(C_1 f + C_2 g) \in P(J)$ and

$$(N)\int_a^b [C_1 f(x) + C_2 g(x)]\,\mathrm{d}x = C_1\,(N)\int_a^b f(x)\,\mathrm{d}x + C_2\,(N)\int_a^b g(x)\,\mathrm{d}x.$$

Proof. Let F and G be primitive functions to f and g, respectively, both on interval J. Then $C_1F + C_2G$ is a primitive function to $C_1 f + C_2 g$ and the following relations hold:

$$\begin{aligned}(N)\int_a^b [C_1 f(x) + C_2 g(x)]\,\mathrm{d}x &= C_1F(b) + C_2G(b) - C_1F(a) - C_2G(a) = \\ &= C_1[F(b) - F(a)] + C_2[G(b) - G(a)] = \\ &= C_1\,(N)\int_a^b f(x)\,\mathrm{d}x + C_2\,(N)\int_a^b g(x)\,\mathrm{d}x,\end{aligned}$$

and the proof is done. □

THEOREM 11.10.3. Let a, b and c be numbers from interval $J \subset \mathbb{R}$, and let $f \in P(J)$. Then

$$(N)\int_a^b f(x)\,\mathrm{d}x = (N)\int_a^c f(x)\,\mathrm{d}x + (N)\int_c^b f(x)\,\mathrm{d}x$$

Proof. Let F be a primitive function to f on interval J. Then

$$\begin{aligned}(N)\int_a^b f(x)\,\mathrm{d}x &= F(b) - F(a) = [F(b) - F(c)] + [F(c) - F(a)] \\ &= (N)\int_a^c f(x)\,\mathrm{d}x + (N)\int_c^b f(x)\,\mathrm{d}x\end{aligned}$$

and the proof is done. □

Using the method of mathematical induction and theorem 10.10.3 one can state

Corollary 11.10.1. Let $x_0, x_1, \ldots, x_n$ be numbers from interval $J \subset \mathbb{R}$, and let $f \in P(J)$. Then

$$(N)\int_{x_0}^{x_n} f(x)\,\mathrm{d}x = (N)\int_{x_0}^{x_1} f(x)\,\mathrm{d}x + \ldots + (N)\int_{x_{n-1}}^{x_n} f(x)\,\mathrm{d}x.$$

THEOREM 11.10.4. Let $J \subset \mathbb{R}$ be an interval, $a \in J$, $b \in J$, $a < b$, and $f \in P(J)$ and $g \in P(J)$ be such that $f \geq g$ on interval $[a, b]$. Then

$$(N) \int_a^b f(x)\,\mathrm{d}x \geq (N) \int_a^b g(x)\,\mathrm{d}x$$

Proof. Let F (G) be a primitive function to f (g) on interval J. There exist finite sets $M_1 \subset J$ and $M_2 \subset J$ such that

$$\forall x \in J \setminus M_1 : F'(x) = f(x),$$

and

$$\forall x \in J \setminus M_2 : G'(x) = g(x).$$

Thus for any $x \in J \setminus (M_1 \cup M_2)$

$$(G - F)'(x) = (g - f)(x) \geq 0.$$

That means that the difference $G - F$ is a non-decreasing function on J (compare with sample problem 8.4.2 part a)) and hence

$$(N) \int_a^b (g - f)(x)\,\mathrm{d}x = [G(b) - F(b)] - [G(a) - F(a)] \geq 0.$$

The proof is then easily completed using theorem 10.10.2. □

Corollary 11.10.2. Let a, b, c and d be numbers from interval $J \subset \mathbb{R}$ such that $a \leq c \leq d \leq b$ and let $g \in P(J)$ be a non-negative function on interval $[a, b]$. Then the following relations hold:

$$(N) \int_a^b g(x)\,\mathrm{d}x \geq 0$$

and

$$(N) \int_c^d g(x)\,\mathrm{d}x \leq (N) \int_a^b g(x)\,\mathrm{d}x.$$

Proof. The first inequality follows from theorem 10.10.4 for $f : x \mapsto 0$, $x \in [a, b]$. The second inequality is obtained using the corollary to theorem 10.10.3 for this specific case:

$$(N) \int_a^b g(x)\,\mathrm{d}x = (N) \int_a^c g(x)\,\mathrm{d}x + (N) \int_c^d g(x)\,\mathrm{d}x + (N) \int_d^b g(x)\,\mathrm{d}x \geq (N) \int_c^d g(x)\,\mathrm{d}x.$$

□

The following theorem discusses the integral representation of the primitive function:

THEOREM 11.10.5. Let a and x be numbers from interval $J \subset \mathbb{R}$ and let $f \in P(J)$. Then function

$$F : x \mapsto (N) \int_a^x f(t)\,\mathrm{d}t, \qquad x \in J,$$

is the primitive function to function f on J with the property $F(a) = 0$.

Proof. Let us denote a primitive function to f on J as G. We then get

$$(N)\int_a^x f(t)\,\mathrm{d}t = G(x) - G(a).$$

Thus a finite set $M \subset J$ exists such that

$$\forall x \in J \setminus M: \ \frac{d}{\mathrm{d}x}(N)\int_a^x f(t)\,\mathrm{d}t = \frac{dG}{\mathrm{d}x}(x) = f(x),$$

which completes the proof. □

Surface Area as Primitive Function

The numerical evaluation of the surface area of two-dimensional objects has been one of the oldest mathematical tasks. The area's axiomatic definition for so-called elementary regions is introduced in section 12.2. There it is also shown that this definition of the surface area is correct (that means the surface area of an elementary region exists and is defined uniquely) and the surface area of elementary regions can be expressed using the Riemann integral.

Here we shall focus on the relation between the surface area $P(f;\, c,\, d)$ of a curved trapezoid

$$M(f;\, c,\, d) = \{(x, y) \in \mathbb{R}^2;\ x \in [c, d] \subset [a, b],\ 0 \le y \le f(x)\},$$

where function $f : [a, b] \to \mathbb{R}$ is continuous on interval $[a, b]$ (figure 10.2), and the primitive function to f.

THEOREM 11.10.6. Let function $f : [a, b] \to \mathbb{R}$ be continuous and non-negative on interval $[a, b]$ and let for every $x \in (a, b]$ a surface area of a curved trapezoid $M(f;\, a,\, x)$ exist. Then function

$$p_f : x \mapsto \begin{cases} P(f;\, a,\, x) & \text{for } x \in (a, b] \\ 0 & \text{for } x = a \end{cases}$$

is the strongly primitive function to function f on $[a, b]$.

Proof. Let $x_0 \in [a, b)$. Continuity of function f implies

$$\forall \epsilon > 0\ \exists 0 < \delta < b - x_0\ \forall x \in [x_0, x_0 + \delta) :\ f(x_0) - \epsilon < f(x) < f(x_0) + \epsilon.$$

From this observation and due to well-known (natural) properties of surface areas that agree with our intuition, and also based on the non-negativity of function f we obtain

$$[f(x_0) - \epsilon]\,\delta \le \max\{[f(x_0) - \epsilon]\,\delta, 0\} \le P(f;\, x_0,\, x_0 + \delta) \le [f(x_0) + \epsilon]\,\delta,$$

which means that the surface area of the largest rectangle placed inside a curved trapezoid $M(f;\, x_0,\, x_0 + \delta)$ is not greater than the surface area of the trapezoid, and the latter is in turn not greater than the surface area of the smallest rectangle containing the trapezoid. Furthermore we have

$$P(f;\, a,\, x_0) + P(f;\, x_0,\, x_0 + \delta) = P(f;\, a,\, x_0 + \delta),$$

which means that the sum of the surface areas of curved trapezoids $M(f;\, a,\, x_0)$ and $M(f;\, x_0,\, x_0 + \delta)$ (the two pieces we break trapezoid $M(f;\, a,\, x_0 + \delta)$ into) is equal to the surface area of trapezoid $M(f;\, a,\, x_0 + \delta)$.

From these relations we get

$$\left| \frac{p_f(x_0 + \delta) - p_f(x_0)}{\delta} - f(x_0) \right| < \epsilon.$$

If in this inequality one considers the limits $\delta \to 0^+$ and then $\epsilon \to 0^+$, then the relation $p'_{f\,+}(x_0) = f(x_0)$ emerges. The analogous statement, $p'_{f\,-}(x_0) = f(x_0)$ for $x_0 \in (a, b]$, for the left derivative is shown in the same way. Thus we can conclude that $p'_f(x) = f(x)$ for arbitrary $x \in [a, b]$ and the proof is done. □

Corollary 11.10.3. Let $J \subset \mathbb{R}$ be an interval and function $f :\, J \to \mathbb{R}$ be continuous and non-negative on J. If for any interval $[a, b] \subset J$ $(a \in \mathbb{R},\, b \in \mathbb{R})$ the surface area $P(f;\, a,\, b)$ of a curved trapezoid $M(f;\, a,\, b)$ exists then the strongly primitive function to f on J exists and the following relation holds:

$$P(f;\, a,\, b) = (N) \int_a^b f(x)\, \mathrm{d}x.$$

Proof. Based on theorem 10.10.6 a strongly primitive function $x \mapsto p_f(x)$, $x \in [a, b]$ to function f exists on any interval $[a, b] \subset J$, and therefore it exists on the whole interval J. Furthermore, from the definition of the N-integral it follows that

$$(N) \int_a^b f(x)\, \mathrm{d}x = p_f(b) - p_f(a) = P(f;\, a,\, b)$$

since $p_f(a) = 0$. □

Note 11.10.4. In section 11.5, we shall prove an even stronger statement than the one in the corollary above. Here we just quote the stronger statement:
Every continuous function on an interval in $\mathbb{R}$ has a strongly primitive function on this interval.

Problems

1. Find all $\alpha \in \mathbb{R}$ for which the Riemann integral $(N) \int_0^1 f(x)\, \mathrm{d}x$ exists for

$$f :\, x \mapsto \begin{cases} x^{-\alpha}, & x \neq 0 \\ c, & x = 0 \end{cases} \qquad \text{where } c \in \mathbb{R}.$$

2. Calculate the following N-integrals provided they exist:
   a) $(N)\int_{-2}^{2} \max(1, x^4)\,\mathrm{d}x$
   b) $(N)\int_{-\pi/2}^{+\pi/2} x\,\mathrm{sgn}(\sin x)\,\mathrm{d}x$
   c) $(N)\int_{-1}^{1} f(x)\,\mathrm{d}x$ for $f : x \mapsto \begin{cases} (1-x^2)^{-1/2}, & x \in (-1,1) \\ c \in \mathbb{R}, & x = -1 \text{ or } x = +1, \end{cases}$
   d) $(N)\int_{0}^{2} [2^x]\,\mathrm{d}x.$

3. Find out if function
$$f : x \mapsto \begin{cases} |x|^{-1}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$
is Riemann integrable. Consider separately the following three intervals: $[-1, 1]$, $[-1, 0]$ and $[-2, -1]$. If it is, calculate the Riemann integral.

4. Prove the corollary that follows after theorem 10.10.3.

5. Show that the following statement is true:
Let $a < c < b$ be real numbers, and $f \in P([a, c])$ and $f \in P([c, b])$. Then $f \in P([a, b])$ and
$$(N)\int_a^c f(x)\,\mathrm{d}x + (N)\int_c^b f(x)\,\mathrm{d}x = (N)\int_a^b f(x)\,\mathrm{d}x.$$

6. Formulate and prove the theorem on the N-integral of a sum of functions (analogous to theorem 10.10.2) for an arbitrary finite number of terms.

7. Calculate
   a) $\dfrac{\mathrm{d}}{\mathrm{d}x}(N)\int_a^b e^{-x^2}\,\mathrm{d}x,\quad \dfrac{\mathrm{d}}{\mathrm{d}a}(N)\int_a^b e^{-x^2}\,\mathrm{d}x,\quad \dfrac{\mathrm{d}}{\mathrm{d}b}(N)\int_a^b e^{-x^2}\,\mathrm{d}x,$
   where $a \in \mathbb{R}$, $b \in \mathbb{R}$.
   b) $\dfrac{\mathrm{d}}{\mathrm{d}x}(N)\int_x^{(\sin x)^{1/3}} \sin^{-1} t^3\,\mathrm{d}t,$ for $x \in [-\pi/2, +\pi/2]$,
   c) $\lim\limits_{x\to 0} \dfrac{\int_0^x \sin^{-1} t\,\mathrm{d}t}{x},\quad \lim\limits_{x\to 0} \dfrac{\int_0^{\sin x} (\tan t)^{1/n}\,\mathrm{d}t}{\int_0^{\tan x} (\sin t)^{1/n}\,\mathrm{d}t},\quad n \in N.$

8. Calculate the surface area of two-dimensional planar objects bounded by the following curves:
   a) the x axis, graph of function $x \mapsto |x|^3$ $(x \in \mathbb{R})$, and lines $x = -1$ and $x = 1$;
   b) the graphs of functions $x \mapsto x^2 - 2$ $(x \in \mathbb{R})$ and $x \mapsto -x^2 + 2$ $(x \in \mathbb{R})$.

Answers

1 $\alpha < 1$.

2 a) $12/5$, b) $\pi^2/4$, c) π, d) $5 - \ln 3/\ln 2$.

3 f is Riemann integrable only on the last of the three intervals. The result is $\ln 2$.

7 a) $0,\ -e^{-a^2},\ e^{-b^2}$; b) $\frac{x}{3}(\sin x)^{-2/3}\cos x - \sin^{-1} x^3$; c) 1, 1.

8 a) $\frac{1}{2}$; b) $\frac{16}{3}\sqrt{2}$.

11.11 Indefinite Integral

In section 2 we saw that the hard part in the calculation of the N-integral consists of the determination of the primitive function to a given function on an interval under consideration. Therefore, in this section we shall focus on different alternative methods for finding the primitive function.

Concept of Indefinite Integral

We already know from theorem 10.9.1 that the primitive function to a given function is not uniquely determined. If a single primitive function to f on an interval I exists, then there are infinitely many of them, differing from each other by an additive constant.

Definition 11.11.1. The set of all primitive functions to $f:\ (J\subset\mathbb{R})\to\mathbb{R}$ on an interval $I\subset J$ will be called an *indefinite integral of function f on I* and denoted by a symbol $\int f(x)\,\mathrm{d}x,\ x\in I$.

Note 11.11.1. 1. Function f is called an integrand, the symbol $\int$ is referred to as the integral sign, and letters $\mathrm{d}x$ denote the integration variable (x in this case). The meaning of the integration variable is as follows: calculating the derivative with respect to the integration variable x of any function from the set $\int f(x)\,\mathrm{d}x$ on $I\setminus M$ (M is a finite subset of I) one obtains restriction $f|_{I\setminus M}$.
2. If F is a primitive function to f on I we shall write

$$\int f(x)\,\mathrm{d}x \overset{\mathrm{c}}{=} F(x),\ x\in I \ \text{ or } \ \int f(x)\,\mathrm{d}x = \{F(x)+c\}|_{c\in\mathbb{R}},\ x\in I.$$

For instance $\int e^{2x}\,\mathrm{d}x \overset{\mathrm{c}}{=} \frac{1}{2}e^{2x} =: F(x),\ x\in\mathbb{R}$ since $\frac{dF}{\mathrm{d}x}(x) = e^{2x}$ for $x\in\mathbb{R}$.
3. The indefinite integral will usually be sought on the so-called maximal set $A\setminus\mathbb{R}$ which does not have to be an interval but could also be a union of a finite number of intervals that satisfies the following statement: If f has a primitive function on set B then $B\subset A$. For example,

$$\int \frac{1}{x}\,\mathrm{d}x \overset{\mathrm{c}}{=} \ln|x|, \quad x\in(-\infty,0)\cup(0,+\infty)$$

$$\int x^{-\frac{1}{3}}\,\mathrm{d}x \overset{\mathrm{c}}{=} \frac{3}{2}x^{\frac{2}{3}}, \quad x\in(-\infty,0)\cup(0,+\infty).$$

Basic Indefinite Integrals

The following indefinite integrals can be derived from the rules for derivatives (section 8.2):

1. $\int 0 \, \mathrm{d}x \stackrel{c}{=} 0, \quad x \in \mathbb{R}$
2. $\int 1 \, \mathrm{d}x =: \int \mathrm{d}x \stackrel{c}{=} x, \quad x \in \mathbb{R}$
3. $\int x^n \, \mathrm{d}x \stackrel{c}{=} \dfrac{x^{n+1}}{n+1}, \quad x \in \mathbb{R} \ (n \in N).$
   If $n \in \mathbb{R} \setminus \{-1\}$, the formula holds for $x \in \mathbb{R}^+$. For some values of n the formula is valid on an extended set (see, *e.g.* part 3 of the last note);
4. $\int \frac{1}{x} \, \mathrm{d}x \stackrel{c}{=} \ln x, \quad x \in \mathbb{R}^+, \qquad \int \frac{1}{x} \, \mathrm{d}x \stackrel{c}{=} \ln(-x), \quad x \in \mathbb{R}^-$
   $\Big($or $\quad \int \frac{1}{x} \, \mathrm{d}x \stackrel{c}{=} \ln|x|, \quad x \in \mathbb{R} \setminus \{0\}\Big)$
5. $\int \frac{1}{1+x^2} \, \mathrm{d}x \stackrel{c}{=} \tan^{-1} x, \quad x \in \mathbb{R}, \qquad \int \frac{1}{1+x^2} \, \mathrm{d}x \stackrel{c}{=} -\cot^{-1} x, \quad x \in \mathbb{R}$
6. $\int \frac{1}{\sqrt{1-x^2}} \, \mathrm{d}x \stackrel{c}{=} \sin^{-1} x, \quad x \in (-1,1), \qquad \int \frac{1}{\sqrt{1-x^2}} \, \mathrm{d}x \stackrel{c}{=} -\cos^{-1} x, \quad x \in (-1,1)$
7. $\int e^x \, \mathrm{d}x \stackrel{c}{=} e^x, \quad x \in \mathbb{R}$
8. $\int a^x \, \mathrm{d}x \stackrel{c}{=} \dfrac{a^x}{\ln a}, \quad x \in \mathbb{R} \ (a \in \mathbb{R}^+ \setminus \{1\})$
9. $\int \sin x \, \mathrm{d}x \stackrel{c}{=} -\cos x, \quad x \in \mathbb{R}$
10. $\int \cos x \, \mathrm{d}x \stackrel{c}{=} \sin x, \quad x \in \mathbb{R}$
11. $\int \frac{1}{(\sin x)^2} \, \mathrm{d}x \stackrel{c}{=} -\cot x, \quad x \in \bigcup_{k \in Z} (k\pi, \pi + k\pi)$
12. $\int \frac{1}{(\cos x)^2} \, \mathrm{d}x \stackrel{c}{=} \tan x, \quad x \in \bigcup_{k \in Z} (k\pi - \frac{\pi}{2}, k\pi + \frac{\pi}{2})$
13. $\int \sinh x \, \mathrm{d}x \stackrel{c}{=} \cosh x, \quad x \in \mathbb{R}$
14. $\int \cosh x \, \mathrm{d}x \stackrel{c}{=} \sinh x, \quad x \in \mathbb{R}$
15. $\int \frac{1}{(\sinh x)^2} \, \mathrm{d}x \stackrel{c}{=} -\coth x, \quad x \in \mathbb{R} \setminus \{0\}$

16. $\displaystyle\int \frac{1}{(\cosh x)^2}\,\mathrm{d}x \overset{c}{=} \tanh x, \quad x \in \mathbb{R}$

In order to speed up calculations the following formulas are often useful:

17. $\displaystyle\int \frac{1}{\sqrt{x^2+1}}\,\mathrm{d}x \overset{c}{=} \ln(x+\sqrt{x^2+1}), \quad x \in \mathbb{R}$

18. $\displaystyle\int \frac{1}{\sqrt{x^2-1}}\,\mathrm{d}x \overset{c}{=} \ln(x+\sqrt{x^2-1}), \quad x \in (1,+\infty)$

19. $\displaystyle\int \frac{1}{x^2-1}\,\mathrm{d}x \overset{c}{=} \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right|, \quad x \in \mathbb{R}\setminus\{-1,1\}$

20. Let $f : (I \subset \mathbb{R}) \to \mathbb{R}$ (I is an interval) be differentiable on I. Then
$\displaystyle\int \frac{f'(x)}{f(x)}\,\mathrm{d}x \overset{c}{=} \ln|f(x)|, \quad x \in \{z \in I;\ f(z) \neq 0\}$

21. Let $f : (I \subset \mathbb{R}) \to \mathbb{R}$ (I is an interval) be differentiable on I. Then
$\displaystyle\int \frac{f'(x)}{\sqrt{f(x)}}\,\mathrm{d}x \overset{c}{=} 2\sqrt{f(x)}, \quad x \in \{z \in I;\ f(z) > 0\}$

Formulas 17-21 can be easily verified by performing the derivative on the right-hand sides, or with the use of forthcoming theorems 3 or 4.

Based on the formulas listed above one can calculate indefinite integrals for just a narrow class of functions. Hence in the following text we shall develop integration methods allowing us to transform integrals of a wide class of functions in such a way that the formulas above can be applied. We warn the reader right here that such a transformation may not be possible for every function $f \in P(I)$. For instance, indefinite integrals $\int \cos x^2\,\mathrm{d}x,\ x \in \mathbb{R}$; $\int e^{-x^2}\,\mathrm{d}x,\ x \in \mathbb{R}$, although they do exist (see note 4.10.2), cannot be expressed in terms of the elementary functions (and, therefore, cannot be transformed into any of the formulas 1-21 for an elementary function f). Similarly, neither of the integrals $\int \frac{\sin x}{x}\,\mathrm{d}x,\ x \in \mathbb{R}\setminus\{0\}$ and $\int \frac{1}{\sqrt{x^3+1}}\,\mathrm{d}x,\ x \in \mathbb{R}\setminus\{-1\}$ belongs to the set of the elementary functions. (A complex proof of the last statement can be found in textbook [2].) There is no general solution to this problem.

Decomposition Method

THEOREM 11.11.1. Let $k_1, \ldots, k_m$ be real numbers and indefinite integrals $\int f_i(x)\,\mathrm{d}x \overset{c}{=} F_i(x)$ exist on interval $I \subset \mathbb{R}$ for $i = 1, \ldots, m$. Then the following indefinite integral exists

$$\int \left[\sum_{i=1}^{m} k_i f_i(x)\right] \mathrm{d}x \overset{c}{=} \sum_{i=1}^{m} k_i F_i(x), \quad x \in I.$$

Proof. The assumptions imply that finite sets $M_i \subset I$, $i = 1, \ldots, m$ exist such that

$$\forall x \in I \setminus M_i\ \forall i = 1, \ldots, m\ F_i'(x) = f_i(x) \quad (\in \mathbb{R}).$$

Set $M \cup_{i=1}^{m} M_i$ is also a finite subset of I and the following relation holds

$$\forall x \in I \setminus M \ \left(\sum_{i=1}^{m} k_i F_i\right)'(x) = \sum_{i=1}^{m} k_i F_i'(x) = \sum_{i=1}^{m} k_i f_i(x)$$

which means that the proof is done. □

Note 11.11.2. The statement in theorem 10.11.1 is usually written in the following form

$$\int \left[\sum_{i=1}^{m} k_i f_i(x)\right] \mathrm{d}x = \sum_{i=1}^{m} k_i \int f_i(x)\,\mathrm{d}x, \quad x \in J.$$

Sample Problem 11.11.1. Calculate the following indefinite integrals on their respective maximal sets
a) $\int \frac{2}{x^4-1}\,\mathrm{d}x$; b) $\int P_m(x)\,\mathrm{d}x$, where P_m is a polynomial of degree m $x \mapsto a_m x^m + \ldots + a_1 x + a_0$, $x \in \mathbb{R}$ and $a_i \in \mathbb{R}$ for $i = 0, 1, \ldots, m$; c) $\int \frac{1}{\sin x}\,\mathrm{d}x$.
`Solution`: a) Using formulas 5 and 19 one obtains

$$\int \frac{2}{x^4-1}\,\mathrm{d}x = \int \frac{1-x^2+1+x^2}{(x^2-1)(x^2+1)}\,\mathrm{d}x = \int \frac{1}{x^2-1}\,\mathrm{d}x - \int \frac{1}{x^2+1}\,\mathrm{d}x \overset{c}{=} \frac{1}{2}\ln\frac{x-1}{x+1} - \tan^{-1} x$$

for $x \in \mathbb{R} \setminus \{-1, 1\}$.
b) $\displaystyle\int P_m(x)\,\mathrm{d}x = \int \left[\sum_{i=0}^{m} a_i x^i\right] \mathrm{d}x = \sum_{i=0}^{m} a_i \int x^i\,\mathrm{d}x = \sum_{i=0}^{m} \frac{a_i}{i+1} x^{i+1}, \quad x \in \mathbb{R}.$
c) Using formula 20 one finds

$$\int \frac{\mathrm{d}x}{\sin x} = \int \frac{\sin^2\frac{x}{2} + \cos^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}\,\mathrm{d}x = \int \frac{\frac{1}{2}\sin\frac{x}{2}}{\cos\frac{x}{2}}\,\mathrm{d}x + \int \frac{\frac{1}{2}\cos\frac{x}{2}}{\sin\frac{x}{2}}\,\mathrm{d}x \overset{c}{=} \ln\left|\sin\frac{x}{2}\right| - \ln\left|\cos\frac{x}{2}\right| = \ln\left|\tan\frac{x}{2}\right|$$

for $x \in \bigcup_{k \in Z} (k\pi, k\pi + \pi)$.

Method of Integration By Parts

Let F and G be the primitive functions to functions f and g on interval $I \subset \mathbb{R}$. In general, it is not true that product FG is the primitive function to function fg. That is because $(FG)' = F'G + FG' = fG + Fg$ on I. However, the following theorem holds:

THEOREM 11.11.2. Let $f \in P(I)$ and $g \in P(I)$ where $\int f(x)\,\mathrm{d}x \overset{c}{=} F(x)$, $\int g(x)\,\mathrm{d}x \overset{c}{=} G(x)$, $x \in I$ and furthermore let $fG \in P(I)$ (or $Fg \in P(I)$). Then also $Fg \in P(I)$ (or $fG \in P(I)$) and

$$\int f(x)\,G(x)\,\mathrm{d}x + \int F(x)\,g(x)\,\mathrm{d}x \overset{c}{=} F(x)\,G(x), \quad x \in I. \tag{11.9}$$

Proof. Let $fG \in P(I)$ and $\int f(x)G(x) \overset{c}{=} H(x),\ x \in I$. Then a finite set $M \subset I$ exists such that

$$\forall x \in I \setminus M : (FG - H)'(x) = f(x)G(x) + F(x)g(x) - f(x)G(x) = F(x)g(x).$$

That means that function $FG - H$ is primitive to Fg on I and $Fg \in P(I)$. The second part of the statement follows from theorem 10.11.1 and relation

$$\int (FG)'(x)\,\mathrm{d}x \overset{c}{=} F(x)G(x), \quad x \in I$$

which completes the proof. $\square$

Note 11.11.3. Theorem 10.11.2 on integration by parts is of large practical significance. However, when applied we shall use $\int f(x)G(x)\,\mathrm{d}x = F(x)G(x) - \int F(x)g(x)\,\mathrm{d}x,\ x \in I$, instead of (10.9).

Sample Problem 11.11.2. Calculate the following indefinite integrals on their maximal sets

$$\text{a)} \int x\,\cos x\,\mathrm{d}x; \quad \text{b)} \int \sin^{-1} x\,\mathrm{d}x;$$

`Solution`: a) If we choose $f : x \mapsto \cos x,\ x \in \mathbb{R}$ and $G : x \mapsto x,\ x \in \mathbb{R}$ then $F : x \mapsto \sin x,\ x \in \mathbb{R}$ and $g : x \mapsto 1,\ x \in \mathbb{R}$ and according to theorem 10.11.2

$$\int x\,\cos x\,\mathrm{d}x = x\sin x - \int \sin x\,\mathrm{d}x \overset{c}{=} x\sin x + \cos x$$

for $x \in \mathbb{R}$.
b) Let us choose $f : x \mapsto 1,\ x \in (-1,1)$ and $G : x \mapsto \sin^{-1} x,\ x \in (-1,1)$. Then $F : x \mapsto x,\ x \in (-1,1)$ and $g : x \mapsto \frac{1}{\sqrt{1-x^2}},\ x \in (-1,1)$. Now using formula 21 we have

$$\int \sin^{-1} x\,\mathrm{d}x = x\,\sin^{-1} x + \frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}}\,\mathrm{d}x \overset{c}{=} x\sin^{-1} x + \sqrt{1-x^2}$$

for $x \in (-1,1)$. According to theorem 6.8.4 (see also problem 6a.8.4) the primitive function that has been found above can be extended to include the closed interval $[-1,1]$. Then one can write

$$\int \sin^{-1} x\,\mathrm{d}x \overset{c}{=} x\sin^{-1} x + \sqrt{1-x^2},\ x \in [-1,1].$$

Sample Problem 11.11.3. Prove that for indefinite integral $I_n(x) = \int \frac{1}{(1+x^2)^n}\,\mathrm{d}x$ on $\mathbb{R}$ the following recurrent relation holds

$$I_{n+1}(x) = \frac{1}{2n}\,\frac{x}{(1+x^2)^n} + \frac{2n-1}{2n}\,I_n(x), \quad x \in \mathbb{R}, \tag{11.10}$$

where $n \in N$.

`Solution`: Choosing $f : x \mapsto 1,\ x \in \mathbb{R}$ and $G : x \mapsto \frac{1}{(1+x^2)^n},\ x \in \mathbb{R}$ one finds $F : x \mapsto x,\ x \in \mathbb{R}$ and $g : x \mapsto \frac{2nx}{(1+x^2)^{n+1}},\ x \in \mathbb{R}$. Then according to theorem 10.11.2

$$\begin{aligned} I_n(x) &= \frac{x}{(1+x^2)^n} + 2n \int \frac{x^2}{(1+x^2)^{n+1}}\,\mathrm{d}x = \\ &= \frac{x}{(1+x^2)^n} + 2n \int \frac{x^2+1}{(1+x^2)^{n+1}}\,\mathrm{d}x - 2n \int \frac{1}{(1+x^2)^{n+1}}\,\mathrm{d}x = \\ &= \frac{x}{(1+x^2)^n} - 2n\, I_{n+1}(x) + 2n\, I_n(x), \quad x \in \mathbb{R}. \end{aligned}$$

This easily simplifies to formula (10.10)

Note 11.11.4. Note that the N-integral formula (10.9) can be written in the form

$$(N)\int_{x_0}^{x} f(t)\,G(t)\,\mathrm{d}t + (N)\int_{x_0}^{x} F(t)\,g(t)\,\mathrm{d}t = [F(x)G(x)]_{x_0}^{x}$$

for $x \in I$ and $x_0 \in I$ (compare with theorem 5.10.2).

Substitution Method

The basic theorems describing the substitution method are obtained from the theorem on the derivative of a composite function (the chain rule):

THEOREM 11.11.3. Let I_1 and I_2 be intervals in $\mathbb{R}$ and $f : I_1 \to \mathbb{R}$. Let function $\phi : I_2 \to I_1$ be differentiable on I_2. Then for $f \in P(I_1)$, and F being strongly primitive function to f on I_1 the composite function $F \circ f$ is strongly primitive function to function $(f \circ \phi)\phi'$ on I_2.

Proof. Since $F'(t) = f(t)$ for $t \in I_1$ then according to theorem 2.8.2 on the derivative of a composite function the following equality holds:

$$(F \circ \phi)'(x) = (F' \circ \phi)(x)\,.\,\phi'(x) = (f \circ \phi)(x)\,.\,\phi'(x)$$

for $x \in I_2$. □

Note 11.11.5. 1. Using the indefinite integral the statement in theorem 3 can be expressed in the form

$$\int (f \circ \phi)(x)\phi'(x)\,\mathrm{d}x = \int f(t)\,\mathrm{d}t \tag{11.11}$$

for $t = \phi(x),\ x \in I_2$. Alternatively, the N-integral can be used in the form

$$(N)\int_{x_0}^{x} (f \circ \phi)(s)\phi'(s)\,\mathrm{d}s = (N)\int_{\phi(x_0)}^{\phi(x)} f(t)\,\mathrm{d}t$$

for $x \in I_2,\ x_0 \in I_2$.

2. Theorem 3 can be applied in the following sense: when calculating the integral on the left side of equation (10.11) in the actual calculation one introduces $\phi(x) = t$ and symbol $\phi'(x)\mathrm{d}x$ is then replaced by $\mathrm{d}t$.

Sample Problem 11.11.4. Calculate

$$I := \int \frac{1}{x \ln x}\, \mathrm{d}x$$

on the maximal set.

`Solution`: Let $\phi : x \mapsto \ln x,\ x \in \mathbb{R}^+ \setminus \{1\}$. Restrictions $\phi|_{(0,1)}$ and $\phi|_{(1,\infty)}$ satisfy the conditions of theorem 3. Therefore $\left(\mathrm{d}t = \frac{1}{x}\mathrm{d}x\right)$

$$I = \int \frac{1}{t}\mathrm{d}t \overset{c}{=} \ln|t| \quad \text{for } t \in (-\infty, 0) \cup (0, +\infty)$$

and hence

$$I \overset{c}{=} \ln|\ln x|, \quad x \in (0,1) \cup (1, +\infty).$$

If the primitive function to function $(f \circ \phi) \,.\, \phi'$ is known then the primitive function to function f can be calculated using the following theorem:

THEOREM 11.11.4. Let I_1 and I_2 be intervals in $\mathbb{R}$ and $f : I_1 \to \mathbb{R}$. Let function $\phi : I_2 \to I_1$ be a bijection and has proper derivative $\phi'(t) \neq 0$ on interval I_2. If $(f \circ \phi) \,.\, \phi' \in P(I_2)$ and G is its primitive function on I_2, then function $G \circ \phi^{-1}$ (ϕ^{-1} is the inverse function to ϕ) is primitive to f on I_1.

Proof. A finite set $M_2 \subset I_2$ exists such that

$$\forall t \in I_2 \setminus M_2 : \ G'(t) = (f \circ \phi)(t)\phi'(t).$$

Since ϕ is a bijection set $M_1 := \phi(M_2)$ is finite. Then according to the theorem on the derivative of composite (theorem 2.8.2) and inverse (theorem 4.8.2) functions the following relation holds for all $x \in I_1 \setminus M_1$

$$(G \circ \phi^{-1})'(x) = (G' \circ \phi^{-1})(x) \,.\, (\phi^{-1})'(x) = [(f \circ \phi) \circ \phi^{-1}](x) \,.\, \frac{1}{\phi' \circ \phi^{-1})(x)} = f(x)$$

The last equality completes the proof. □

Note 11.11.6. 1. Using the indefinite integral the statement in theorem 4 can be expressed by the following formula

$$\int f(x)\, \mathrm{d}x = \int (f \circ \phi)(t)\phi'(t)\, \mathrm{d}t$$

for $t = \phi^{-1}(x)$, $x \in I_1$. Alternatively, the N-integral can be used in the form

$$(N) \int_{x_0}^{x} f(s)\, \mathrm{d}s = (N) \int_{\phi^{-1}(x_0)}^{\phi^{-1}(x)} (f \circ \phi)(t)\phi'(t)\, \mathrm{d}t$$

for $x \in I_1,\ x_0 \in I_1$.

2. In order to calculate integral $\int f(x)\mathrm{d}x$ theorem 4 can be used with substitutions $x = \phi(t)$ and $\mathrm{d}x = \phi'(t)\mathrm{d}t$.

Sample Problem 11.11.5. Calculate $\int \sqrt{1+x^2}\,\mathrm{d}x$ on $\mathbb{R}$.
`Solution`: 1. Consider $\phi : t \mapsto \sinh t,\ t \in \mathbb{R}$. ϕ is a bijection and for $t \in \mathbb{R}$ we have $\phi'(t) = \cosh t > 0$. The assumptions of theorem 4 are therefore satisfied. Hence

$$\int \sqrt{1+x^2}\mathrm{d}x = \int \sqrt{1+\sinh^2 t}\,\cosh t\,\mathrm{d}t = \int \cosh^2 t\,\mathrm{d}t$$

for $t = \sinh^{-1} x = \ln(x+\sqrt{x^2+1}),\ x \in \mathbb{R}$. (We have applied the identity $\cosh^2 x - \sinh^2 x = 1$, for $x \in \mathbb{R}$.) Furthermore we have

$$\int \cosh^2 t\,\mathrm{d}t = \frac{1}{4}\int e^{2t}\,\mathrm{d}t + \frac{1}{2}\int \mathrm{d}t + \frac{1}{4}\int e^{-2t}\,\mathrm{d}t \overset{\mathrm{c}}{=} \frac{t}{2} + \frac{1}{4}\sinh 2t, \quad t \in \mathbb{R}.$$

That implies

$$\int \sqrt{1+x^2}\,\mathrm{d}x \overset{\mathrm{c}}{=} \frac{1}{2}\ln(x+\sqrt{x^2+1}) + \frac{1}{4}\sinh[2\ln(x+\sqrt{x^2+1})] \text{ for } x \in \mathbb{R}.$$

2. Using integration by parts with $f : x \mapsto \sqrt{1+x^2},\ x \in \mathbb{R}$ and $G : x \mapsto 1,\ x \in \mathbb{R}$ one gets

$$\int \sqrt{1+x^2}\,\mathrm{d}x \overset{\mathrm{c}}{=} \frac{1}{2}\ln(x+\sqrt{x^2+1}) + \frac{1}{2}x\sqrt{x^2+1}, \quad x \in \mathbb{R}.$$

Formally, the two methods give different results. However, one can work out the derivatives of the two results and easily see that both results are correct. As an independent check, one can recall theorem 2.10.1, part (ii):

$$\forall x \in \mathbb{R} :\ x\sqrt{x^2+1} = \frac{1}{2}\sinh[2\ln(x+\sqrt{x^2+1})] + c.$$

Choosing $x = 0$ in this formula we get $c = 0$ and hence the equality of the two results.

Note 11.11.7. Note that theorem 3 can be used with much weaker assumptions about function ϕ than theorem 4. In theorem 3, the bijection is not required and neither is the assumption $\phi'(t) \neq 0$.

Problems

1. Find recurrent relations for the following indefinite intergals:
   a) $I_n(x) = \int \sin^n x\,\mathrm{d}x$,
   b) $J_n(x) = \int \cos^n x\,\mathrm{d}x$.

2. Use both integration by parts and substitution method for the calculation of indefinite integral $\int \sqrt{x^2-1}\,\mathrm{d}x$ where $x \in (-\infty,-1) \cup (1,+\infty)$, and compare the two results.

3. Calculate (on $\mathbb{R}$)

   a) $\displaystyle\int f(x)\,\mathrm{d}x, \quad \text{where } f : x \mapsto \begin{cases} 1 - x^2, & |x| \le 1 \\ 1 - |x|, & |x| > 1 \end{cases}$

   b) $\displaystyle\int \frac{x^2+1}{x^4+1}\,\mathrm{d}x.$

Answers

1. a) $I_n(x) = -\frac{1}{n}\sin^{n-1}x\,\cos x + \frac{n-1}{n}I_{n-2}(x)$ for $n \in N,\ n \ge 2$;
   b) $J_n(x) = \frac{1}{n}\cos^{n-1}x\,\sin x + \frac{n-1}{n}I_{n-2}(x)$ for $n \in N,\ n \ge 2$.

2. $\frac{x}{2}\sqrt{x^2-1} - \frac{1}{2}\ln|x + \sqrt{x^2-1}|$.

3. a) $x - \frac{x^3}{3}$ for $|x| \le 1$; and $x - \frac{x}{2}|x| + \frac{1}{6}\,\mathrm{sgn}x$ for $|x| > 1$;
   b) $F : x \mapsto \frac{1}{\sqrt{2}}\tan^{-1}\frac{x^2-1}{x\sqrt{2}},\ x \in \mathbb{R}\setminus\{0\};\ F(0) = 0$. (Use the following change of variables: $x - \frac{1}{x} = t\sqrt{2}$.)

11.12 Integration of Special Types of Functions

The goal of this section is to show the ways how to find indefinite integrals of some important and frequent classes of real functions. For practical purposes we limit ourselves to show only the description of several useful algorithms without the ambition to provide the rigorous formulations. Hence in each particular case the validity conditions must be considered separately.

11.12.1 Integration of Rational Functions

Recall that in theorem 6.4.3 we have introduced the decomposition of a purely rational single variable function into partial fractions. According to this theorem and theorem 1.10.3 the problem of integration of such rational functions is reduced to the problem of integration of individual partial fractions.

Since for $k \in N$ and $k \ne 1$

$$\int \frac{1}{(x-a)^k}\mathrm{d}x \overset{\mathrm{c}}{=} \frac{(x-a)^{1-k}}{1-k}, \quad x \in \mathbb{R}\setminus\{a\}$$

and

$$\int \frac{1}{x-a}\mathrm{d}x \overset{\mathrm{c}}{=} \ln|x-a|, \quad x \in \mathbb{R}\setminus\{a\}$$

then the only remaining problem in the integration of rational single variable functions is the calculation of the primitive function to function $x \mapsto \frac{Ax+B}{(x^2+px+q)^k}$, $x \in \mathbb{R}$, where A, B, p

and q are real numbers, $k \in N$, and the discriminant $p^2 - 4q$ of the quadratic expression in the denominator is negative. Taking the decomposition

$$\frac{Ax+B}{(x^2+px+q)^k} = \frac{A}{2}\frac{2x+p}{(x^2+px+q)^k} + \frac{B-\frac{Ap}{2}}{(x^2+px+q)^k}$$

into account the indefinite integral of the first term can easily be found based on theorem 3.10.3 (the first theorem on substitution) using the change of variables $x^2+px+q=t$ while

$$\frac{A}{2}\int \frac{2x+p}{(x^2+px+q)^k}\,\mathrm{d}x \stackrel{\text{c}}{=} \frac{A}{2(1-k)}(x^2+px+q)^{1-k}, \quad x \in \mathbb{R},\ k \neq 1$$

$$\frac{A}{2}\int \frac{2x+p}{x^2+px+q}\,\mathrm{d}x \stackrel{\text{c}}{=} \frac{A}{2}\ln(x^2+px+q), \quad x \in \mathbb{R}.$$

The denominator of the second term can be expanded according to

$$x^2+px+q = \left(x+\frac{p}{2}\right)^2 + q - \frac{p^2}{4} = \left(q-\frac{p^2}{4}\right)\left[\left(\frac{x+\frac{p}{2}}{\sqrt{q-\frac{p^2}{4}}}\right)^2 + 1\right].$$

Then the primitive function to the second term can be obtained using the change of variable $t = \left(x+\frac{p}{2}\right)\left(q-\frac{p^2}{4}\right)^{-1/2}$ (theorem 3.10.3). This substitution transforms the original integral $\int \frac{\mathrm{d}x}{(x^2+px+q)^k}$ into integral $\int \frac{\mathrm{d}t}{(t^2+1)^k}$. The latter integral can be obtained using the recurrent formula (2.10.3) found in problem 3.10.3. The procedure described above can now be summarised in

THEOREM 11.12.1. The indefinite integral of an arbitrary rational single variable function on its definition domain belongs to the set of the elementary functions.

11.12.2 Integration of Trigonometric Functions

Next, we shall explicitly show the substitutions used to transform the integrals of type

$$\int R(\sin x, \cos x)\,\mathrm{d}x, \quad x \in M \subset \mathbb{R} \tag{11.12}$$

into integrals of rational functions. Function $R:\ (u,v) \mapsto R(u,v)$ from $\mathbb{R}^2$ to $\mathbb{R}$ is a rational function of two variables.

Integral (10.12) can be calculated using the substitutions shown below (and theorem 3.10.3). Each of the substitutions transforms the integrand into a rational single variable function.

1. If R is an odd function in the first of its two variables, $R(-\sin x, \cos x) = -R(\sin x, \cos x)$ then a substitution $\cos x = t,\ x \in \mathbb{R},$ can be used.

2. If $R(\sin x, -\cos x) = -R(\sin x, \cos x)$ then a substitution $\sin x = t,\ x \in \mathbb{R}$, can be used.

3. If $R(-\sin x, -\cos x) = R(\sin x, \cos x)$ then a substitution $\tan x = t$, $x \in \bigcup_{k\in Z}\left(-\frac{\pi}{2}+k\pi, \frac{\pi}{2}+k\pi\right)$, can be used.

4. For an arbitrary rational function R of two variables a substitution $\tan\frac{x}{2} = t$ can be used. (This substitution can also be used in the previous three cases although the calculation is most of the times technically more involved.)

The last substitution, $\tan\frac{x}{2} = t$, for $x \in (-\pi, \pi)$, $t \in \mathbb{R}$, leads to the formulas

$$\sin x = 2\,\sin\frac{x}{2}\,\cos\frac{x}{2} = \frac{2\,\tan\frac{x}{2}}{\tan^2\frac{x}{2}+1} = \frac{2t}{t^2+1}$$

$$\cos x = \cos^2\frac{x}{2} - \sin^2\frac{x}{2} = \frac{1-t^2}{1+t^2}$$

$$x = 2\,\tan^{-1} t, \quad \mathrm{d}x = \frac{2}{1+t^2}\mathrm{d}t.$$

It is now clear that integral (1) transforms into a rational function in terms of variable t.

Sample Problem 11.12.1. Calculate $\int \frac{1}{1+\cos^2 x}\,\mathrm{d}x$ on $\mathbb{R}$.

`Solution`: Based on the ideas above we can use either the change of variable $\tan x = t$ or $\tan\frac{x}{2} = t$. We shall use both substitutions:

a) With the change of variable $\tan x = t$ the conditions of theorem 3.10.3. are satisfied on every interval $\left(-\frac{\pi}{2}+k\pi, \frac{\pi}{2}+k\pi\right)$, $k \in Z$. Since $\frac{\sin^2 x}{\cos^2 x} = t^2$ one gets $\cos^2 x = \frac{1}{1+t^2}$ and on any interval $\left(-\frac{\pi}{2}+k\pi, \frac{\pi}{2}+k\pi\right)$, $k \in Z$, the following relations hold:

$$\begin{aligned}\int \frac{1}{1+\cos^2 x}\mathrm{d}x &= \int \frac{\cos^2 x}{1+\cos^2 x}\frac{\mathrm{d}x}{\cos^2 x}\\ &= \int \frac{\mathrm{d}t}{2+t^2} = \frac{1}{2}\int \frac{\mathrm{d}t}{1+\left(\frac{t}{\sqrt{2}}\right)^2} \overset{c}{=} \frac{2}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}},\ t \in \mathbb{R}.\end{aligned}$$

Thus one finds

$$\int \frac{1}{1+\cos^2 x}\mathrm{d}x \overset{c}{=} \frac{\sqrt{2}}{2}\tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right)$$

on any interval $\left(-\frac{\pi}{2}+k\pi, \frac{\pi}{2}+k\pi\right)$, $k \in Z$.

The calculation is, however, not completely finished, since the function found at the end is not primitive to function $f : x \mapsto (1+\cos^2 x)^{-1}$, $x \in \mathbb{R}$ everywhere on set $\mathbb{R}$. Using the "gluing" construction of example 3.10.1 the primitive function (indefinite integral) obtained above can now be continuously extended at points $\frac{\pi}{2}+k\pi$, $k \in Z$ in such a way that new extension F will be the primitive function to function $f : x \mapsto (1+\cos^2 x)^{-1}$, $x \in \mathbb{R}$ on $\mathbb{R}$.

For $k \in Z$ let us denote

$$F_{2k} : x \mapsto \frac{\sqrt{2}}{2} \tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right), \qquad x \in \left(-\frac{\pi}{2} + 2k\pi, \frac{\pi}{2} + 2k\pi\right)$$
$$F_{2k+1} : x \mapsto \frac{\sqrt{2}}{2} \tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right), \qquad x \in \left(-\frac{\pi}{2} + (2k+1)\pi, \frac{\pi}{2} + (2k+1)\pi\right)$$

and

$$C_k := \lim_{x \to \left(-\frac{\pi}{2}+(2k+1)\pi\right)^+} F_{2k+1}(x) - \lim_{x \to \left(\frac{\pi}{2}+2k\pi\right)^-} F_{2k}(x) = -\frac{\pi\sqrt{2}}{2}.$$

Then the primitive function F is

$$F : x \mapsto \begin{cases} \frac{\sqrt{2}}{2} \tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right) - \frac{\pi\sqrt{2}}{2}, & x \in \left(-\frac{\pi}{2} + 2k\pi, \frac{\pi}{2} + 2k\pi\right) \\ \frac{\sqrt{2}}{2} \tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right), & x \in \left(-\frac{\pi}{2} + (2k+1)\pi, \frac{\pi}{2} + (2k+1)\pi\right) \\ -\frac{\pi\sqrt{2}}{4}, & x = \frac{\pi}{2} + k\pi, \;\; k \in Z. \end{cases}$$

Besides that F also is the strongly primitive function to f on $\mathbb{R}$. Recalling theorem 6.8.4 this can be seen from

$$F'_+\left(\frac{\pi}{2} + k\pi\right) = \lim_{\left(\frac{\pi}{2}+k\pi\right)_+} F'(x) = \lim_{\left(\frac{\pi}{2}+k\pi\right)_+} \frac{1}{1+\cos^2 x} =$$

$$= 1 = F'_-\left(\frac{\pi}{2} + k\pi\right) \quad \text{and hence} \quad F'\left(\frac{\pi}{2} + k\pi\right) = 1 = \left(1 + \cos^2\frac{\pi}{2}\right)^{-1}.$$

b) Substitution $\phi : x \mapsto \tan\frac{x}{2}$ also satisfies the conditions of theorem 3.10.3. on an interval $(-\pi + 2k\pi, \pi + 2k\pi)$, $k \in Z$. Then $\cos^2 x = \frac{(1-t^2)^2}{(1+t^2)^2}$, $\cos^2\frac{x}{2} = \frac{1}{1+t^2}$, $t \in \mathbb{R}$. For any $x \in (-\pi + 2k\pi, \pi + 2k\pi)$

$$\int \frac{1}{1+\cos^2 x} \mathrm{d}x = \int \frac{2\cos^2\frac{x}{2}}{1+\cos^2 x} \frac{\mathrm{d}x}{2\cos^2\frac{x}{2}} = \int \frac{t^2+1}{t^4+1} \mathrm{d}t, \;\; t \in \mathbb{R}.$$

The rational function that has been obtained here is more complicated than in case a). The only benefit of this substitution is that we find primitive functions on larger intervals. However, since we are seeking the primitive function on $\mathbb{R}$ the obtained result must again be extended by "gluing".