

MATHEMATICS

PAPER - I

SECTION A

1. Answer any four of the following:

- (a) Let V be the vector space of polynomials in x with real coefficients of degree at most 2.

Let t_1, t_2, t_3 be any 3 distinct real numbers. Define

$L_i : V \rightarrow \mathbb{R}$ by $L_i(f) = f(t_i)$, $i = 1, 2, 3$. Show that

(i) L_1, L_2, L_3 are linear functionals on V

(ii) $\{L_1, L_2, L_3\}$ is a basis for the dual space V^* of V .

(10)

- (b) In the notation of (a) above, find a basis $B = \{p_1, p_2, p_3\}$ for V which is dual to $\{L_1, L_2, L_3\}$ and also express each $P \in V$ in terms of elements of B .

- (c) Let f be a function defined on $[0, 1]$ by

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational} \\ \frac{1}{q}, & \text{if } x = \frac{p}{q}, q \neq 0 \end{cases}$$

and p, q , are relatively prime positive integers. Show that f is continuous at each irrational point and discontinuous at each rational point $\frac{p}{q}$.

(10)

- (d) Show that the function $[x]$, where $[x]$ denotes the greatest integer not greater than x , is integrable in $[0, 3]$. Also evaluate $\int_0^3 [x] dx$.

(10)

- (e) Prove that the polar of one limiting point of a coaxial system of circles with respect to any circle of the system passes through the other limiting point.

(10)

2. (a) Let V be the vector space of polynomials in x with complex coefficients. Define

$T: V \rightarrow V$ by $(Tf)(x) = xf(x)$ and

$$U: V \rightarrow V \text{ by } U\left(\sum_{i=0}^n c_i x^i\right) = \sum_{i=0}^{n-1} c_i + 1x^n$$

Find (i) $\ker T$ (ii) show U is linear (iii) show that $UT = I$ and $TU \neq I$, I = identity on V .

(10)

- (b) Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ show that for every integer $n \geq 3$, $A^n = A^{n-2} + A^2 - I$ and hence find the matrix A^8 .

(10)

- (c) Find the characteristic and minimal polynomials of

$$A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and determine whether A is diagonalizable.

(10)

- (d) Let $A = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$

Find an invertible 3×3 matrix P such that $P^{-1}AP = D$, D is diagonal matrix. Find D also.

(10)

3. (a) Examine the convergence of the integral

$$\int_0^1 x^{x-1} \log x dx$$

(10)

- (b) Examine the function

$$f(x, y) = \begin{cases} \frac{xy}{\sqrt{x^2 + y^2}} & (x, y) \neq (0, 0) \\ 0 & (x, y) = (0, 0) \end{cases}$$

For continuity, partial derivability of the first order and differentiability at $(0, 0)$.

(10)

- (c) Find the maximum and minimum values of the function $f(x, y, z) = xy + 2z$ on the circle which is the intersection of the plane $x + y + z = 0$ and the sphere $x^2 + y^2 + z^2 = 24$.

(10)

- (d) Find the volume of the region R lying below the plane $z = 3 + 2y$ and above the paraboloid $z = x^2 + y^2$.

(10)

4. (a) CP and CD are conjugate diameters of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. Prove that the locus of the orthocenter of the triangle CPD is the curve $2(b^2 y^2 + a^2 x^2)^3 = (a^2 - b^2)^2 (b^2 y^2 - a^2 x^2)^2$.

(13)

- (b) If $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ represents one of the three mutually perpendicular generators of the cone $5yz - 8zx - 3xy = 0$ Find the equations of the other two.

(13)

- (v) If the section of the enveloping cone of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

Whose vertex is P, by the plan = 0 is a rectangular hyperbola, prove that the locus of P is
 $\frac{x^2 - y^2}{a^2 + b^2} + \frac{z^2}{c^2} = 1.$

(14)

SECTION B

5. Answer any four of the following:

- (a) A tank of 100 liters capacity is initially full of water. Pure water is allowed to run into the tank at the rate of 1 liter per minute and at the same time salt water containing $\frac{1}{4}$ kg of salt per liter flows into the tank at the rate of 1 liter per minute. The mixture (there is perfect mixing in the tank at all times) flows out at the rate of 2 liters per minute. Form the differential equation and find the amount of salt in the tank after t minutes. Find this when $t = 50$ minutes.

(10)

- (b) A constant coefficient differential equation has auxiliary equation expressible in factored form as

$$P(m) = m^3 (m - 1)^2 (m^2 + 2m + 5)^2.$$

What is the order of the differential equation and find its general solution.

(10)

- (c) A particle rests in equilibrium under the reaction of two centers of forces which attract directly as the distance, their intensity being μ, μ' , the particle is displaced slightly towards one of them, show that the time of a small oscillation is

$$T = \frac{2\pi}{\sqrt{\mu + \mu'}}$$

(10)

- (d) A solid sphere rests inside a fixed rough hemispherical bowl of twice its radius. Show that however large a weight is attached to the highest point of the sphere, the equilibrium is stable.

(10)

- (e) Find an equation for the plane passing through the points $P_1(3, 1, -2)$, $P_2(-1, 2, 4)$, $P_3(2, -1, 1)$ by using vector method.

(10)

6. (a) Solve $x^2 \left(\frac{dy}{dx} \right)^2 + y(2x+y) \frac{dy}{dx} + y^2 = 0$

(10)

- (b) Using differential equations show that the system of confocal conics given by

$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} = 1, \lambda \text{ real is}$$

Self-orthogonal.

(10)

(c) Solve

$$(1-x^2) \frac{d^2y}{dx^2} - x \frac{dy}{dx} - a^2 y = 0$$

given that $y = e^{atx^{-1}}$ is one solution of this equation. (10)

(d) Find a general solution of $y'' + y = \tan x$, $-\frac{\pi}{2} < x < \frac{\pi}{2}$ by variation of parameters. (10)

7. (a) A particle moves with central acceleration $(\mu u^2 + \lambda u^3)$ and the velocity of projection at distance R is V, show that the particle will ultimately go off to infinity if

$$V^2 > \frac{2\mu}{R} + \frac{\lambda}{R^2}. \quad (13)$$

- (b) A smooth parabolic wire is fixed with its axis vertical and vertex downwards and in it is placed a uniform rod of length 2l with its ends resting on wire. Show that, for equilibrium, the rod is either horizontal or makes with horizontal an angle θ given by $\cos^2 \theta = \frac{2a}{l}$, 4a being the latus rectum of the parabola. (14)

- (c) Prove that if volume v and V of two different substances balance in vacuum and volumes v', V' balance when weighed in a liquid, the densities of the substances and the liquid are as

$$\frac{v'-V'}{v} = \frac{v'-v'}{V} \left(\frac{v'}{V} - \frac{V'}{V} \right) \quad (13)$$

8. (a) Prove that

$$\nabla \times (\nabla \times \vec{A}) = -\nabla^2 \vec{A} + \nabla(\nabla \cdot \vec{A}) \quad (10)$$

- (b) If

$$\nabla \cdot \vec{E}, \nabla \cdot \vec{H}, \nabla \times \vec{E} = -\frac{\partial \vec{H}}{\partial t}, \nabla \times \vec{H} = \frac{\partial \vec{E}}{\partial t}$$

Show that \vec{E} and \vec{H} satisfy

$$\nabla^2 \vec{u} = -\frac{\partial^2 \vec{u}}{\partial t^2} \quad (10)$$

- (c) Given the space curve $x = t, y = t^2, z = \frac{2}{3}t^3$. Find (i) the curvature ρ (ii) the torsion τ . (10)

- (d) If $\vec{F} = (y^2 + z^2 - x^2)i + (z^2 + x^2 - y^2)j + (x^2 + y^2 - z^2)k$, evaluate $\iint_S \text{curl } \vec{F} \cdot d\vec{s}$, taken over the portion of the surface,

$x^2 + y^2 - z^2 - 2ax + az = 0$ above the plane $z = 0$ and verify Stoke's theorem. (10)

MATHEMATICS

PAPER - II

SECTION A

1. Answer any four parts. (4 x 10 = 40)

(a) Write the elements of the symmetric group S_3 of degree 3, prepare its multiplication table and find all normal subgroups of S_3

(b) Change the order of integration in the integral

$$\int_{0}^{4x^2} \int_{y^2}^{x^2} dy dx$$

and evaluate it.

(c) Compute the Taylor series around $z = 0$ and give the radius of convergence for $\frac{z}{z-1}$

(d) By graphical method solve the linear programming problem

$$\begin{aligned} \text{Maximize } Z &= 100X_1 + 40X_2 \\ \text{subject to } 5X_1 + 2X_2 &\leq 1000 \\ 3X_1 - 2X_2 &\leq 900 \\ X_1 + 2X_2 &\geq 500 \\ X_1, X_2 &\geq 0 \end{aligned}$$

(e) If

$$a_n = \log\left(1 + \frac{1}{n^2}\right) + \log\left(1 + \frac{2}{n^2}\right) + \dots + \log\left(1 + \frac{n}{n^2}\right)$$

Find $\lim_{n \rightarrow \infty} a_n$

2. (a) If every element of a group G is its own inverse, prove that the group G is abelian. Is the converse true? Justify your claim. (14)

(b) Discuss the maxima and minima of $x^3 y^2 (1-x-y)$. (13)

(c) State the Weierstrass M-test for uniform convergence of an infinite series of functions. Prove that the series

$$\sum_{n=1}^{\infty} \frac{x}{n^2(1+nx^2)} \text{ with } \alpha < 0$$

is uniformly convergent on $(-\infty, \infty)$. (13)

3. (a) Define a field and prove that every finite integral domain is a field. (3 + 10)

(b) Show that the function $f(z) = \sqrt{xy}$ is not regular at the origin although the Cauchy-Riemann equations are satisfied. (13)

- (c) By using the Residue Theorem evaluate the integral

$$\int_0^{2\pi} \frac{d\theta}{1 - 2a \sin \theta + a^2} \quad \text{where } 0 < a < 1, \quad (14)$$

4. (a) Define a unique factorization domain. Show that $\mathbb{Z}[\sqrt{-5}]$ is an integral domain which is not a unique factorization domain. (5+10)

- (b) Using simplex method solve the linear programming problem:

Maximize $Z = X_1 + X_2 + 3X_3$

subject to $3X_1 + 2X_2 + X_3 \leq 3$

$2X_1 + X_2 + 2X_3 \leq 2$

$X_1, X_2, X_3 \geq 0.$

(13)

- (c) A company has three plants A, B and C, three warehouses X, Y and Z. The number of units available at the plants is 60, 70, 80 and the demands at X, Y, Z are 50, 80, 80 respectively. The unit cost of the transportation is given in the following table:-

	X	Y	Z
A	8	7	3
B	3	8	9
C	11	8	7

Find the allocation so that the total transportation cost is minimum.

(14)

SECTION B

5. Answer any four parts:-

(4 × 10 = 40)

- (a) Find the complete integral of the partial differential equation

$$x^2 p^2 + y^2 q^2 = z^2.$$

- (b) Find Lagrange's interpolation polynomial $P_2(x)$ which satisfies.

$$f(0) = P_2(0) = 1$$

$$f(-1) = P_2(-1) = 2$$

$$f(1) = P_2(1) = 3.$$

Find $f(0.5).$

- (c) Convert the decimal number $(1479.25)_{10}$ to the binary and the hexadecimal numbers.

- (d) Find the moment of inertia of the area bounded by $r^2 = a^2 \cos 2\theta$ about its axis.

- (e) Show that $\frac{x^2}{a^2} f(t) + \frac{y^2}{b^2} \cdot \frac{1}{f(t)} = 1$ is a possible form of the bounding surface of a liquid.

6. (a) Solve by Charpit's method

$$(p^2 + q^2) y = qz.$$

(13)

- (b) If $\varphi(x)$ is a continuous and bounded function of $-\infty < x < \infty$, prove that the function $u(x,t) = \frac{1}{2\sqrt{\pi k t}} \int_{-\infty}^{\infty} \varphi(\xi) e^{-(x-\xi)^2/4kt} d\xi$ is a solution of the initial value problem:
 $u_t - k u_{xx} = 0, \quad -\infty < x < \infty, \quad t > 0$
 $u(x,0) = \varphi(x) \text{ for } -\infty < x < \infty.$
- (c) Two equal masses m_1 and m_2 with $m_1 > m_2$ are suspended by a light string over a pulley of mass M and radius a . There is no slipping and the friction of axle may be neglected. If \ddot{x} be the acceleration, show that this is constant and if k^2 be the radius of gyration of the pulley about the axis, show that

$$k^2 = \frac{a^2}{Mg} [(g-f)m_1 - (g+f)m_2] \quad (13)$$

7. (a) Four equal rods, each of length $2a$, are hinged at their ends so as to form a rhombus ABCD. The angles B and D are connected by an elastic string and the lowest end A rests on a horizontal plane whilst the end C slides on a smooth vertical wire passing through A. In the position of equilibrium the string is stretched to twice its natural length and the angle BAD is 2α . Show that the time of small oscillation about this position is

$$2\pi \left\{ \frac{2a(1+3\sin^2 \alpha)\cos \alpha}{(3g \cos 2\alpha)} \right\}^{\frac{1}{2}} \quad (14)$$

- (b) If \mathbf{q} is the resultant velocity at any point of a fluid which is moving irrotationally in two dimensions, prove that

$$\left(\frac{\partial q}{\partial x} \right)^2 + \left(\frac{\partial q}{\partial y} \right)^2 = q \nabla^2 q$$

- (c) By applying the Newton-Raphson method to $f(x) = x^2 - a$ where $a > 0$, prove that

$$x_{n+1} = \frac{1}{2} \left(x_n + \frac{a}{x_n} \right)$$

For $n = 0, 1, 2, 3$ Apply this formula to find $\sqrt{2}$

8. (a) Write an algorithm for generating even integers ≤ 100 . Also draw the flow chart which executes this algorithm. (13)

- (b) Applying Simpson's one-third rule compute the value of the definite integral

$$\int_{0.2}^{0.6} \log x \, dx$$

with $h = 0.2$ and estimate the error. (13)

- (c) State the conditions under which the equations of motion can be integrated. Obtain Bernoulli's equation for the steady irrotational motion of an incompressible liquid. (14)