

MATHEMATICS

PAPER - I SECTION A

1. Answer any four of the following:

- (a) Let $V P_3(\mathbb{R})$ be the vector space of polynomial functions on reals of degree at most 3. Let $D: V \rightarrow V$ be the differentiation operator defined by

$$D(a_0 + a_1x + a_2x^2 + a_3x^3) = a_1 + 2a_2x + 3a_3x^2$$

- (i) Show that D is a linear transformation
- (ii) Find kernel and image of D .
- (iii) What are dimensions of V , $\ker D$ and image D ?
- (iv) Give relation among them of (iii).

(10)

- (b) Find the eigen values and the corresponding eigen vectors of $A = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}$

(10)

- (c) Let $f, g: [a, b] \rightarrow \mathbb{R}$ be functions such that $f'(x)$ and $g'(x)$ exist for all $x \in [a, b]$ and $g'(x) \neq 0$ for all x in (a, b) . Prove that for some $c \in (a, b)$

$$\frac{f(c) - f(a)}{g(b) - g(c)} = \frac{f'(c)}{g'(c)}$$

(10)

- (d) Let $f: \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by

$$f(x, y) = \frac{xy^2}{x^2 + y^2} \text{ for } (x, y) \neq (0, 0)$$

$$= 0 \text{ for } (x, y) = (0, 0)$$

Show that the partial derivatives $D_1 f(0, 0)$ and $D_2 f(0, 0)$ vanish but f is not differentiable at $(0, 0)$.

(10)

- (e) A line makes angles $\alpha, \beta, \gamma, \delta$ with the four diagonals of a cube. Show that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$$

(10)

2. (a) Show that the vectors $(1, 2, 1), (1, 0, -1)$ and $(0, -3, 2)$ form a basis for \mathbb{R}^3

(10)

- (b) Determine non-singular matrices P and Q such that the matrix PAQ is in canonical form, where

$$A = \begin{pmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{pmatrix}$$

Hence find the rank of A.

(10)

- (c) Find the minimum polynomial of the matrix.

$$A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix},$$

and use it to determine whether A is similar to a diagonal matrix.

(10)

- (d) Show that the quadratic form

$$2x_2 - 4xy + 3xy + 6y^2 + 6yz + 8z^2$$

in three variables is positive definite.

(10)

3. (a) Let $f(x) = e^{-1/x^2}$ ($x \neq 0$)

$= 0$ for $x = 0$

Show that $f'(0) = 0$ and $f''(0) = 0$.

Write $f^{(k)}(x)$ as $P\left(\frac{1}{x}\right)f(x)$ for $x \neq 0$, where P is a polynomial and $f^{(k)}$ denotes the kth derivation of f.

(10)

- (b) Using Lagrange multipliers, show that a rectangular box with lid of volume 1000 cubic units and of least surface area is a cube of side 10 units.

(10)

- (c) Show that the area of the surface of the solid obtained by revolving the arc of the curve $y = c \cosh\left(\frac{x}{c}\right)$ joining $(0, c)$ and (x, y) about the x-axis is

(10)

- (d) Define $\Gamma : (0, \infty) \rightarrow \text{IR}$ by

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t} dt. \text{ Show that this integral converges for all } x > 0 \text{ and that } \Gamma(x+1) = x \Gamma(x).$$

(10)

4. (a) Show that the equation $\sqrt{fx} + \sqrt{gy} + \sqrt{hz} = 0$ represents a cone that touches the co-ordinate planes and that the equation to its reciprocal cone is

$$fyz + gzx + hxy = 0$$

(10)

- (b) Show that any two generators belonging to the different system of generating lines of a hyperboloid of one sheet intersect.

(10)

- (c) Show that the locus of a point from which three mutually perpendicular tangent lines can be drawn to the paraboloid

$$ax^2 + by^2 + 2z = 0 \text{ is}$$

$$ab(x^2 + y^2) + 2(a + b)z = 1.$$

(10)

- (d) Show that the enveloping cylinder of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

whose generators are parallel to the line

$$\frac{x}{0} = \frac{y}{\pm\sqrt{a^2 - b^2}} = \frac{z}{c}$$

meet the plane $z = 0$ in circles.

(10)

SECTION B

5. Answer any four of the following:

- (a) Find the orthogonal trajectories of the family of co-axial circles

$$x^2 + y^2 + 2gx + c = 0$$

where g is a parameter

(10)

- (b) Find three solutions of

$$\frac{d^3y}{dx^3} - 2\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0$$

which are linearly independent on every real interval.

(10)

- (c) A particle moves with an acceleration which is always towards, and equal to μ divided by the distance from, a fixed point O . If it starts from rest at a distance a from O , show that it will arrive at O in time

$$a\sqrt{\frac{\pi}{2\mu}}$$

(10)

- (d) Show that the depth of the centre of pressure of the area included between the arc and the asymptote of the curve

$$(r-a)\cos\theta = b \text{ is } \frac{a}{4} \cdot \frac{3\pi a + 16b}{3\pi b + 4a},$$

the asymptote being in the surface and the plane of the curve being vertical.

(10)

- (e) Find expressions for curvature and torsion at a point on the curve $x = a \cos \theta$ $y = a \sin \theta$, $z = a \theta \cot \beta$.

(10)

6. (a) Solve and examine for singular solution:

$$y^2 - 2pxy + p^2(x^2 - 1) = m^2.$$

(10)

- (b) Solve

$$x^3 \frac{d^3y}{dx^3} + 2x^2 \frac{d^2y}{dx^2} + 2y = 10 \left(x + \frac{1}{x} \right).$$

(10)

- (c) Given $y = x$ is one solution of

$$(x^3 + 1) \frac{d^2y}{dx^2} - 2x \frac{dy}{dx} + 2y = 0.$$

find another linearly independent solution by reducing order and write the general solution.

(10)

- (d) Solve by the method of variation of parameters

$$\frac{d^2y}{dx^2} + a^2y = \sec ax, a \text{ is real.}$$

(10)

7. (a) A shell fired with velocity V at an elevation θ hits an airship at a height h from the ground, which is moving, horizontally away from the gun with velocity v . Show that if

$$(2V \cos \theta - v)(v^2 \sin^2 \theta - 2gh)^{1/2} = v V \sin \theta$$

the shell might have also hit the ship if the latter had remained stationary in the position it occupied when the gun was actually fired.

(10)

- (b) Assuming the eccentricity e of a planet's orbit is a small fraction, show that the ratio of the time taken by the planet to travel over the halves of its orbit separated by the minor axis is nearly $1 + \frac{4e}{\pi}$

(10)

- (c) A uniform rod AB of length $2a$ is hinged at A, a string attached to the middle point G of the rod passes over a smooth pulley at C at a height a , vertically above A, and supports a weight P having freely, find the positions of equilibrium and determine their nature as to stability or instability

(10)

- (d) A solid of cork bounded by the surface generated by the revolution of a quadrant of an ellipse about the major axis sinks in mercury up to the focus. If the centre of gravity of the cork coincides with the metacentre, prove that $2e^4 + 4e^3 + 2e^2 - e - 2 = 0$

(10)

8. (a) If \vec{r} is the position vector of the point (x, y, z) with respect to the origin, prove that

$$\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r).$$

Find $f(r)$ such that $\Delta^2 f(r) = 0$

(10)

- (b) If \vec{F} is solenoidal, prove that $\text{Curl Curl Curl Curl } \vec{F} = \nabla^4 \vec{F}$

(10)

(c) Verify Stoke's Theorem when

$$\vec{F} = (2xy - x^2) \vec{i} - (x^2 - y^2) \vec{d}$$

and C is the boundary of the region enclosed by the parabolas $y^2 = x$ and $x^2 = y$.

(10)

- (d) Express $\nabla \times \vec{F}$ and $\nabla^2 \Phi$ in cylindrical co-ordinates,

(10)

MATHEMATICS

PAPER - II

SECTION A

1. Answer any four parts. $(10 \times 4 = 40)$
- (a) Show that if every element of a group $(G, *)$ be its own inverse, then it is an Abelian group. Give an example to show that the converse is not true.
- (b) Evaluate
- $$I = \iint (a^2 - x^2 - y^2)^{1/2} dx dy$$
- over the positive quadrant of the circle $x^2 + y^2 = a^2$
- (c) If $w = f(z) = u(x, y) + iv(x, y)$, $z = x + iy$, is analytic in a domain, show that
- $$\frac{\partial w}{\partial z} = 0$$
- Hence or otherwise, show that $\sin(x + iy)$ cannot be analytic.
- (d) Investigate the continuity of the function $f(x) = |x|/x$ for $x \neq 0$ and $f(0) = -1$.
- (e) (i) Explain the following terms of an LPP:
1. Solution
 2. Basic solution
 3. Basic feasible solution
 4. Degenerate basic solution
- (ii) Give the dual of the following LPP:
- Maximize $Z = 2x_1 + 3x_2 + x_3$
Subject to $4x_1 + 3x_2 + x_3 = 6$
 $x_1 + 2x_2 + 3x_3 = 4$
 $x_1, x_2, x_3 \geq 0$
2. (a) Let $G = \{a \in \mathbb{R} : -1 < a < 1\}$. Define a binary operation $*$ on G by
- $$a * b = \frac{a+b}{1+ab} \text{ for all } a, b \in G.$$
- Show that $(G, *)$ is a group (13)
- (b) Let $f(x) = [x]$, $x \in [0, 3]$, where $[x]$ denotes the greatest integer not greater than x . Prove that f is Riemann integrable on $[0, 3]$ and evaluate

$$\int_0^3 f(x) dx$$

(13)

- (c) (i) Let (a, b) be any open interval, f a function defined and differentiable on (a, b) such that its derivative is bounded on (a, b) . Show that f is uniformly continuous on (a, b) .
- (ii) If f is a continuous function on $[a, b]$ and if

$$\int_a^b f^2(x) dx = 0$$

then show that $f(x) = 0$ for all x in $[a, b]$. Is this true if f is not continuous?

(14)

3. (a) Let R be the set of matrices of the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}, a, b \in F$$

where F is a field. With usual addition and multiplication as binary operations, show that R is a commutative ring with unity. Is it a field if $F = Z_2, Z_5$?

(14)

- (b) Discuss the transformation

$$w = z + \frac{1}{z}$$

and hence, show that—

- (i) a circle in z -plane is mapped on an ellipse in the w -plane;
 (ii) a line in the z -plane is mapped into a hyperbola in the w -plane.

(13)

- (c) Find the Laurent series expansion of the function

$$f(z) = \frac{z^2 - 1}{(z+2)(z+3)}$$

valid in the region $2 < |z| < 3$.

(13)

4. (a) Find the maximum and minimum distances of the point $(3, 4, 12)$ from the sphere $x^2 + y^2 + z^2 = 1$.

(13)

- (b) Find the maximum value of $Z = 2x + 3y$ subject to the constraints

$$\begin{aligned} x - y &\geq 0 \\ x + y &\leq 30 \\ y &\geq 3 \\ 0 \leq y &\leq 12 \text{ and} \\ 0 \leq x &\leq 20 \end{aligned}$$

by graphical method.

(13)

- (c) Apply simplex method to solve the following linear programming problem

(14)

Maximize $Z = 4x_1 + 3x_2$ subject to the constraints

$$\begin{aligned}3x_1 + x_2 &\leq 15 \\3x_1 + 4x_2 &\leq 24 \\x_1, x_2 &\geq 0\end{aligned}$$

SECTION B

5. Answer any four parts:

(10 x 4 = 40)

- (a) Find the general solution of the partial differential equation

$$(mz - ny) \frac{\partial z}{\partial x} + (nx - lz) \frac{\partial z}{\partial y} = ly - mx$$

(10)

- (b) Find the cube root of 10 using Newton-Raphson method, correct to 4 decimal places.

(10)

- (c) Apply modified Euler's method to determine $y(0.1)$, given that

$$\frac{dy}{dx} = x^2 + y$$

when $y(0) = 1$.

(10)

- (d) (i) Convert ABCD hex and 76543 octal to decimal

- (ii) Convert 39870 decimal to octal and hexadecimal

(10)

- (e) Show that the surface

$$\frac{x^2}{a^2 k^2 t^4} + k t^2 \left(\frac{y^2}{b^2} + \frac{z^2}{c^2} \right) = 1$$

is a possible form of boundary surface of a liquid at time t .

(10)

6. (a) Form the partial differential equation by eliminating the arbitrary function from

$$f(x^2 + y^2, z - xy) = 0, z = z(x, y)$$

(13)

- (b) Solve

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

given that

- (i) $u = 0$, when $t = 0$ for all x

- (ii) $u = 0$ when $x = l$ for all t

$$(iii) \begin{cases} u = x - \ln\left(0, \frac{l}{2}\right) \\ = l - x - \ln\left(\frac{l}{2}, l\right) \end{cases} \text{ at } t = 0$$

(14)

- (c) A rod of length $2a$ is suspended by a string of length l attached to one end; if the string and rod revolve about the vertical with uniform angular velocity and their inclinations to the vertical be θ and ϕ , respectively, show that

$$\frac{3l}{a} = \frac{(4 \tan \theta - 3 \tan \phi) \sin \phi}{(\tan \phi - \tan \theta) \sin \theta}$$

(13)

7. (a) The velocities of a car running on a straight road at intervals of 2 minutes are given below:

Time (in min) :	0	2	4	6	8	10	12
Velocity (in km/hr) :	0	22	30	27	18	7	0

Apply Simpson's one-third rule to find the distance covered by the car.

(13)

- (b) Consider the velocity field given by

$$\vec{v} = i(1 + At) + jx$$

Find the equation of the streamline at $t = t_0$ passing through the point (x_0, y_0) . Also, obtain the equation of the path line of a fluid element which comes to (x_0, y_0) at $t = t_0$. Show that, if $A = 0$, the streamline and path line coincide.

(13)

- (c) Write a program in BASIC to integrate

$$\int_0^{10} \left(1 - e^{-\frac{x}{2}}\right) dx$$

by trapezoidal rule for 20 equal sub-divisions of the interval $(0, 10)$. Indicate which lines are to be changed for a different integral.

(14)

8. (a) Draw a flowchart and write a program in BASIC for an algorithm to determine the greatest common divisor of two given positive integers.

(13)

- (b) Apply Runge-Kutta method of order 4 to find an approximate value of y when $x = 0.2$ given that

$$\frac{dy}{dx} = x + y, y = 1 \text{ when } x = 0.$$

(14)

- (c) A uniform sphere rolls down an inclined plane rough enough to prevent any sliding; discuss the motion. Hence, show that for pure rolling μ (coefficient of friction) is greater than $\frac{2}{7} \tan \alpha$ for a solid sphere, where α the inclination of the plane.

(13)