Chapter 1

2020

1.1 Section-A

Question-1(a) If A is a skew-symmetric matrix and $I+A$ be a non-singular matrix, then show that $(I-A)(I+A)^{-1}$ is orthogonal.

[8 Marks]

Solution: Given A is a skew-symmetric matrix and $I+A$ is a non-singular matrix.

$$\begin{aligned}
\text{Let} \quad M &= (I-A)(I+A)^{-1} \\
\Rightarrow \quad M^T &= \left[(I-A)(I+A)^{-1}\right]^T \\
&= \left[(I+A)^{-1}\right)^T (I-A)^T \\
&= \left((I+A)^T\right]^{-1} \left(I^T - A^T\right) \\
&= (I-A)^{-1}(I+A) \\
&\qquad [\because \quad A \quad \text{is skew symmetric} \quad \Rightarrow A^T = -A] \\
\therefore \quad M^T M &= (I-A)^{-1}(I+A)(I-A)(I+A)^{-1} \\
&= (I-A)^{-1}(I-A)(I+A)(I+A)^{-1} \\
&= I \cdot I = I \\
&\begin{pmatrix} \because (I+A)(I-A) = I + A - A - A^2 = I - A^2 \\ (I-A)(I+A) = I - A + A - A^2 = I - A^2 \end{pmatrix}
\end{aligned}$$

Thus, $M = (I-A)(I+A)^{-1}$ is orthagonal.

Question-1(b) By applying elementary row operations on the matrix

$$A = \begin{bmatrix} -1 & 2 & -1 & 0 \\ 2 & 4 & 4 & 2 \\ 0 & 0 & 1 & 5 \\ 1 & 6 & 3 & 2 \end{bmatrix},$$

reduce it to a row-reduced echelon matrix. Hence find the rank of A.

[8 Marks]

Solution:

$$A = \begin{bmatrix} -1 & 2 & -1 & 0 \\ 2 & 4 & 4 & 2 \\ 0 & 0 & 1 & 5 \\ 1 & 6 & 3 & 2 \end{bmatrix}$$

$$R_2 \to R_2 + 2R_1, \quad R_4 \to R_4 + R_1$$

$$\sim \begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & 8 & 2 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 8 & 2 & 2 \end{bmatrix}$$

$$R_4 \to R_4 - R_2$$

$$\sim \begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & 8 & 2 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R_2 \to R_2 - 2R_3, \quad R_1 \to R_1 + R_3$$

$$\sim \begin{bmatrix} -1 & 2 & 0 & 5 \\ 0 & 8 & 0 & -8 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This is the row-reduced echelon form which has three non-zero rows.
Hence, $\text{rank}(A) = 3$.

Question-1(c) Given that $f(x+y) = f(x)f(y), f(0) \neq 0$, for all real x, y and $f'(0) = 2$.
Show that for all real $x, f'(x) = 2f(x)$. Hence find $f(x)$.

[8 Marks]

Solution: Given, $f(x+y) = f(x)f(y)$.

Let $x = 0$, $y = 0$.

$$\Rightarrow f(0) = f(0) \cdot f(0)$$
$$f(0)[f(0) - 1] = 0$$
$$f(0) = 1 \qquad \ldots (1)[\because f(0) \neq 0 \text{ given}]$$

By definition of derivative,

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \to 0} \frac{f(x) \cdot f(h) - f(x)}{h} \\ &= f(x) \quad \lim_{h \to 0} \frac{f(h) - 1}{h} \\ &= f(x) \lim_{h \to 0} \frac{f(h) - f(0)}{h} \qquad [\text{ using } (1)] \\ &= f(x) \cdot f'(0) = 2f(x) \qquad (\text{ given } f'(0) = 2) \end{aligned}$$
$$\therefore f'(x) = 2f(x)$$
$$\Rightarrow \int \frac{f'(x)}{f(x)} dx = \int 2dx \Rightarrow \log f(x) = 2x + c$$
$$f(0) = 1 \Rightarrow \log 1 = 2(0) + c \Rightarrow c = 0$$
$$\therefore \log f(x) = 2x \Rightarrow f(x) = e^{2x}$$

Question-1(d) Find the Taylor's series expansion for the function

$$f(x) = \log(1+x), -1 < x < \infty$$

about $x = 2$ with Lagrange's form of remainder after 3-terms.

[8 Marks]

Solution: **Taylor series**: If a function f is defined on $[a, a+h]$ such that
(i) $f, f', f'', \ldots f^{n-1}$ are continuous on $[a, a+h]$
(ii) $f^n(x)$ exists in $(a, a+h)$, then there exists at least one real number θ, $0 < \theta < 1$, such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{n-1}(a) + \frac{h^n}{n!} f^n(a + \theta h)$$

Here, $f(x) = \log(1+x) \Rightarrow f(2) = \log 3$.

$$f'(x) = \frac{1}{1+x} \qquad \Rightarrow f'(2) = \frac{1}{3}$$
$$f''(x) = \frac{-1}{(1+x)^2} \Rightarrow f''(2) = \frac{-1}{9}$$

$$f'''(x) = \frac{2}{(1+x)^3} \Rightarrow f'''(2) = \frac{2}{27}$$
$$f(x) = f(2) + f'(2)(x-2) + f''(2)\frac{(x-2)^2}{2!} + f'''(2+\theta h)\cdot\frac{(x-2)^3}{3!}$$
$$f(x) = \log 3 + \frac{(x-2)}{3} - \frac{(x-2)^2}{18} + \frac{f'''[2+\theta(x-2))}{81} \times (x-2)^3$$

where $0 < \theta < 1$.
Hence, it is the required form of Taylor expansion of $f(x)$ about $x = 2$ with Lagrange's form of remainder after 3-terms.

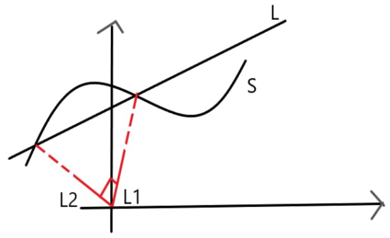
Question-1(e) If the straight lines, joining the origin to the points of intersection of the curve $3x^2 - xy + 3y^2 + 2x - 3y + 4 = 0$ and the straight line $2x + 3y + k = 0$, are at right angles, then show that $6k^2 + 5k + 52 = 0$.

[8 Marks]

Solution:

$$\text{Let } S \equiv 3x^2 - xy + 3y^2 + 2x - 3y + 4 = 0;$$
$$L \equiv 2x + 3y + k = 0$$

By homogenizing equation S with the help of Equation of line L, we get the equation of pair of lines L_1 and L_2 through origin.



$$3x^2 - xy + 3y^2 + (2x - 3y)\frac{(2x+3y)}{-k} + 4\left[\frac{(2x+3y)}{k}\right]^2 = 0$$

If these lines are at right angles then, sum of coefficients of x^2 and $y^2 = 0$

$$3 + 3 + \left(\frac{4-9}{-k}\right) + 4\left(\frac{4+9}{k^2}\right) = 0$$
$$\Rightarrow 6k^2 + 5k + 52 = 0$$

which is the required given condition.

$$[\text{ Pair of lines, } ax^2 + 2hxy + by^2 = 0$$
$$\Rightarrow \tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b}]$$

Question-2(a) **Let** $\mathrm{T} : \mathbb{R}^3 \to \mathbb{R}^3$ **be defined by** $\mathrm{T(x, y, z) = (2x, -3y, x+y)}$, **and** $B_1 = \{(-1, 2, 0), (0, 1, -1), (3, 1, 2)\}$ **be a basis of** $\mathbb{R}^3$. **Find the matrix representation of T relative to the basis** B_1**.**

[10 Marks]

Solution:

$$\begin{aligned} \mathrm{T(x, y, z)} &= \mathrm{(2x, -3y, x + y)} \\ B_1 &= \{(-1, 2, 0), (0, 1, -1), (3, 1, 2)\} \\ \therefore \quad T(-1, 2, 0) &= (-2, -6, 1) \\ &= -1(-1, 2, 0) - 3(0, 1, -1) - 1(3, 1, 2) \\ &\quad \text{[Using Calculator]} \\ T(0, 1, -1) &= (0, -3, 1) \\ &= \frac{-2}{3}(-1, 2, 0) - \frac{13}{9}(0, 1, -1) - \frac{2}{9}(3, 1, 2) \\ T(3, 1, 2) &= (6, -3, 4) \\ &= \frac{-5}{3}(-1, 2, 0) - \frac{10}{9}(0, 1, -1) + \frac{13}{9}(3, 1, 2) \end{aligned}$$

Hence, matrix representation of T is

$$[T]_{B_1} = \begin{bmatrix} -1 & -3 & -1 \\ \frac{-2}{3} & \frac{-13}{9} & \frac{-2}{9} \\ \frac{-5}{3} & \frac{-10}{9} & \frac{13}{9} \end{bmatrix}^{\top} = \begin{bmatrix} -1 & -\frac{2}{3} & -5/3 \\ -3 & \frac{-13}{9} & -10/9 \\ -1 & -2/9 & 13/9 \end{bmatrix}$$

Question-2(b) **Using Lagrange's multiplier, show that the rectangular solid of maximum volume which can be inscribed in a sphere is a cube.**

[15 Marks]

Solution: Let diameter of sphere be D, which is fixed. Let x, y, z be dimensions of rectangular solid inscribed in the sphere.

$$V = xyz$$
$$\text{such that } x^2 + y^2 + z^2 = (D)^2$$

(Diagonal of Rectangular solid = Diameter of the sphere)

Consider the function

$$F(x, y, z) = xyz + \lambda \left(x^2 + y^2 + z^2 - D^2\right)$$

We use Lagrange's multiplier method to maximize $V = xyz$ for critical points,

$$\partial F = 0 \quad \Rightarrow \quad F_x = 0, \quad F_y = 0, F_z = 0$$

$$\begin{aligned} \therefore \quad yz + \lambda(2x) &= 0 \quad \ldots(1) \\ xz + \lambda(2y) &= 0 \quad \ldots(2) \\ xy + \lambda(2z) &= 0 \quad \ldots(3) \end{aligned}$$

Subtracting equation (1) from (2) and (3), we get

$$\begin{aligned} z(x-y) + 2\lambda(y-x) &= 0 \quad \Rightarrow \quad x = y \\ y(x-z) + 2\lambda(z-x) &= 0 \quad \Rightarrow \quad x = z \\ &\therefore x = y = z \end{aligned}$$

As all the three dimensions x, y, z should be same for maximum volume of rectargular solid, hence it will be a cube.

$$\text{Let} \quad x = y = z = a$$

$$\therefore x^2 + y^2 + z^2 = 1^2 \Rightarrow 3a^2 = D^2$$

$$\Rightarrow \text{ Dimension of cube } = a = \frac{D}{\sqrt{3}}$$

Question-2(c) Prove that the angle between two straight lines whose direction cosines are given by $l+\mathrm{m}+\mathrm{n}=0$ and $\mathrm{fmn}+\mathrm{gn}l+\mathrm{hlm}=0$ is $\frac{\pi}{3}$, if $\frac{1}{\mathrm{f}}+\frac{1}{\mathrm{g}}+\frac{1}{\mathrm{h}}=0$.

[15 Marks]

Solution: Given $l+m+n=0 \quad \Rightarrow \quad n = -(1+m)$ Using it in,

$$\begin{aligned} fmn + gnl + hlm &= 0 \\ -(fm+gl)(l+m) + hlm &= 0 \\ flm + gl^2 + fm^2 + glm - hlm &= 0 \\ gl^2 + \ln(f+g-h) + fm^2 &= 0 \\ g\left(\frac{l}{m}\right)^2 + \frac{l}{m}(f+g-h) + f &= 0 \end{aligned}$$

Let $\frac{l_1}{m_1}$ and $\frac{l_2}{m_2}$ are two roots of the quadratic equation.

$$\therefore \left(\frac{l_1}{m_1}\right)\left(\frac{l_2}{m_2}\right) = \frac{f}{g} \Rightarrow \frac{l_1 l_2}{f} = \frac{m_1 m_2}{g}$$

By symmetry

$$\frac{l_1 l_2}{f} = \frac{m_1 m_2}{g} = \frac{n_1 n_2}{h} = k \quad \text{(Say)}$$

Again,

$$\begin{aligned} \frac{l_1}{m_1} + \frac{l_2}{m_2} &= \frac{-(f+g-h)}{g} \\ \frac{l_1 m_2 + m_1 l_2}{-(f+g+h)} &= \frac{m_1 m_2}{g} = k \end{aligned}$$

$$
\begin{aligned}
\therefore (l_1m_2 - l_2m_1)^2 &= (l_1m_2 + l_2m_1)^2 - 4(l_1m_2)(l_2m_1)\\
&= k^2(f+g-h)^2 - 4(kf)(kg)\\
&= k^2\left[f^2+g^2+h^2+2fg-2gh-2fh-4fg\right]\\
&= k^2\left[f^2+g^2+h^2-2(fg+gh+fh)\right]\\
&= k^2\left[f^2+g^2+h^2+2(fg+gh+fh)\right]\\
&\left(\because \frac{1}{f}+\frac{1}{g}+\frac{1}{h}=0 \Rightarrow fg+gh+hf=0\right)\\
&= [k(f+g+h)]^2
\end{aligned}
$$

$$
\begin{aligned}
\tan^2\theta &= \frac{\sum (l_1m_2 - l_2m_1)^2}{(l_1l_2+m_1m_2+n,n_2)^2}\\
&= \frac{3[k(f+g+h)]^2}{[k(f+g+h)]^2} = 3\\
\Rightarrow &\quad \tan\theta = \sqrt{3} \quad \Rightarrow \quad \theta = \frac{\pi}{3}
\end{aligned}
$$

Question-3(a) Find the asymptotes of the curve $x^3+3x^2y-4y^3-x+y+3=0$.

[10 Marks]

Solution: As the coefficients of highest powers of x and y are constant, hence asymptotes parallel to x-axis or y-axis do not exist.

Oblique Asymptotes:

$$
\begin{aligned}
&\text{Put } x=1, \quad y=m\\
&\phi_3(m) = 1+3m-4m^3 \text{ (Taking third degree terms)}\\
&\phi_2(m) = 0 \quad \text{(Taking second degree terms)}\\
&\phi_1(m) = -1+m \quad \text{(Taking first degree terms)}
\end{aligned}
$$

Slopes of the asymptotes are real roots of eqn, $\phi_3(m)=0$

$$
\begin{aligned}
\text{i.e., } 4m^3-3m-1&=0\\
\Rightarrow (m-1)(2m+1)^2&=0\\
\therefore m&=1,-1/2,-1/2\\
\text{for } m=1, \quad c=-\frac{\phi_2(m)}{\phi_3'(m)}&=0
\end{aligned}
$$

$\therefore y=mx+c$ i.e. $y=x$ is an asymptote.

For $m=-1/2$ (repeated root), the value of c is given by

$$
\begin{aligned}
&\frac{c^2}{2!}\phi_3''(m)+c\cdot\phi_2'(m)+\phi_1(m)=0\\
\text{i.e.}\quad &\frac{c^2}{2}(-24m)+c(0)+(m-1)=0
\end{aligned}
$$

$$-12\left(-\frac{1}{2}\right)c^2+\left(-\frac{1}{2}-1\right)=0$$
$$\rightarrow 12c^2-3=0$$
$$\Rightarrow c^2=\frac{1}{4}$$
$$\Rightarrow c=+\frac{1}{2},-\frac{1}{2}$$

$\therefore$ Thus, the asymptotes are $y=mx+c$

ie. $y=-\frac{1}{2}x+\frac{1}{2}$ and $y=\frac{-1}{2}x-\frac{1}{2}$

$\therefore$ Three asymptotes are $y-x=0$, $x+2y=1$ and $x+2y=-1$

Question-3(b) When is a matrix A said to be similar to another matrix B? Prove that
(i) if A is similar to B, then B is similar to A.
(ii) two similar matrices have the same eigenvalues.
Further, by choosing appropriately the matrices A and B, show that the converse of (ii) above may not be true.

[15 Marks]

Solution: **Similarity of Matrices:** Let A and B be square matrices of order n. Then A is said to be similar to B if there exists a non-singular matrix P such that

$$A=P^{-1}BP$$

i) If A is similar to B, therefore there exists an $n\times n$ non-singular matrix p such that

$$\begin{aligned} &A=P^{-1}BP\\ \Rightarrow &PAP^{-1}=P\left(P^{-1}BP\right)\\ \Rightarrow &\quad PAP^{-1}=B \end{aligned}$$

i.e. $B=\left(P^{-1}\right)^{-1}A\left(P^{-1}\right)$ P is invertible means P^{-1} is invertible and $\left(P^{-1}\right)^{-1}=P$. It implies that B is similar to A.

(ii) Suppose A and B are similar matrices

$$\text{Then, } B=P^{-1}AP$$

Characteristic polynomial of B is

$$\begin{aligned} |B-\lambda I| &= \left|P^{-1}AP-\lambda P^{-1}P\right|\\ &= \left|P^{-1}(A-\lambda I)P\right|\\ &= |P^{-1}||A-\lambda I||P|\\ &= |A-\lambda I| \quad \left(\because \left|P^{-1}\right||P|\left|P^{-1}P\right|=|I|=1\right) \end{aligned}$$

Which is same as characteristic bolymomial of A. Hence, A and B have same eigenvalues. Finally, consider matrices

$$A=\begin{bmatrix}0&0\\0&0\end{bmatrix} \text{ and } B=\begin{bmatrix}0&1\\0&0\end{bmatrix}$$

which clearly have the same eigenvalues, but they are not similar because B cannot be obtained by applying any sequence of elemantary transformtions on matrix A.
Hence, converse of (ii) is not true.

Question-3(c) A point P moves on the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, which is fixed. The plane through P and perpendicular to OP meets the axes in A, B, C respectively. The planes through A, B, C parallel to yz, zx and xy planes respectively intersect at Q. Prove that the locus of Q is $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{ax}+\frac{1}{by}+\frac{1}{cz}$.

[15 Marks]

Solution: The eqn of the plane is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \qquad \ldots(1)$$

Let the coordinates of the point P be (α, β, γ). Since the point P lies on plane (1), we have

$$\frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}=1 \qquad \ldots(2)$$

The d.r.'s of OP are $\alpha-0, \beta-0, \gamma-0$ i.e., α, β, γ.
Hence, the equation of plane passing through point $P(\alpha, \beta, \gamma)$ and perpendicular to OP is

$$\alpha(x-\alpha)+\beta(y-\beta)+\gamma(z-\gamma)=0$$

or

$$\alpha x+\beta y+\gamma z=\alpha^2+\beta^2+\gamma^2 \qquad \ldots(3)$$

The plane (3) meets the axes in the points A, B and c whose cordinates are

$$\left(\frac{\alpha^2+\beta^2+\gamma^2}{\alpha}, 0, 0\right), \left(0, \frac{\alpha^2+\beta^2+\gamma^2}{\beta}, 0\right), \left(0, 0, \frac{\alpha^2+\beta^2+r^2}{\gamma}\right)$$

Now the eqn of plane through A and parallel to $y=$ plane i.e. plane $x=0$ is

$$x=\frac{\alpha^2+\beta+\gamma^2}{\alpha}$$

Similarly, equations of the other two planes are

$$y=\frac{\alpha^2+\beta^2+\gamma^2}{\beta},$$

$$z=\frac{\alpha^2+\beta^2+\gamma^2}{\gamma}$$

Now, Q is the intersection point of above three planes.
The locus of point Q is obtained by eliminating α, β and γ from eqn (2) with the help of above three equations.

$$\frac{\alpha}{a}+\frac{\beta}{b}+\frac{\gamma}{c}=1 \text{ gives}$$

$$\frac{1}{a}\left(\frac{\alpha^2+\beta^2+\gamma^2}{x}\right)+\frac{1}{b}\left(\frac{\alpha^2+\beta^2+\gamma^2}{y}\right)+\frac{1}{c}\left(\frac{\alpha^2+\beta^2+\gamma^2}{z}\right)=1$$
$$\Rightarrow \frac{1}{ax}+\frac{1}{by}+\frac{1}{cz}=\frac{1}{\alpha^2+\beta^2+\gamma^2}$$

Also,

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{\alpha^2+\beta^2+\gamma^2}{(\alpha^2+\beta^2+\gamma^2)^2}=\frac{1}{\alpha^2+\beta^2+\gamma^2}$$

Hence, required locus is given by:

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{1}{ax}+\frac{1}{by}+\frac{1}{cz}$$

Question-4(a) Let P be the vertex of the enveloping cone of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

If the section of this cone made by the plane $z=0$ is a rectangular hyperbola, then find the locus of P.

[10 Marks]

Solution: Let $P(x,y,z)$ be the vertex of enveloping cone.
Eqn of ellipsoid is

$$S \equiv \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$$
$$\therefore \quad S_1=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z^2}{c^2}-1,$$
$$T=\frac{x_1x}{a^2}+\frac{y_1y}{b^2}+\frac{z_1z}{c^2}-1$$

Eqn of enveloping cone:

$$SS_1=T^2$$
$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)=\left(\frac{x_1x}{a^2}+\frac{y_1y}{b^2}+\frac{z_1z}{c^2}-1\right)^2$$

The section of enveloping cone by $z=0$ is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)=\left(\frac{x_1x}{a^2}+\frac{y_1y}{b^2}-1\right)^2$$

If it represents rectangular hyperbola, sum of coefficients of x^2 and y^2 will be zero.

$$\frac{1}{a^2}\left(\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)+\frac{1}{b^2}\left(\frac{x_1^2}{a^2}+\frac{z_1^2}{c^2}-1\right)=0$$

The required locus of $P\left(x_1,y_1,z_1\right)$ is

$$c^2\left(x^2+y^2\right)+\left(a^2+b^2\right)z^2=c^2\left(a^2+b^2\right)$$

Question-4(b) (i) Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$, hence find its inverse. Also, express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ as a linear polynominl in A.
(ii) Express the vector (1,2,5) as a linear combination of the vectors (1,1,1),(2,1,2) and $(3, 2, 3)$, if possible. Justify your answer.

[15 Marks]

Solution: (i) Given, $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$.
Characteristic polynomial is given by

$$\begin{aligned} |A - \lambda I| &= 0 \\ \Rightarrow \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} &= 0 \\ \Rightarrow (1-\lambda)(3-\lambda) - 8 &= 0 \\ \Rightarrow \lambda^2 - 4\lambda + 3 - 8 &= 0 \\ \Rightarrow \lambda^2 - 4\lambda - 5 &= 0 \end{aligned}$$

Cayley-Hamilton theorem states that every square matrix satisfies its characteristics eqn.

$$A^2 = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix}$$

$A^2 - 4A - 5I = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - 4\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 9-4-5 & 16-16 \\ 8-8 & 17-12-5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
Hence Cayley-Hamilton theorem is verified for matrix A. Now,

$$A^2 - 4A - 5I = 0$$

$$\begin{aligned} &\Rightarrow A^5 - 4A^3 - 7A^3 + 11A^2 - A - 10I \\ &= A^5 - 4A^3 - 5A^3 - 2A^3 + 11A^2 - A - 10I \\ &= A^3\left(A^2 - 4A - 5I\right) - 2A(4A + 5I) + 11A^2 - A - 10I \\ &= 0 - 8A^2 - 10 \cdot A + 11A^2 - A - 10I \\ &= 3A^2 - 11A - 10I \\ &= 3(4A + 5I) - 11A - 10I \\ &= 12A + 15I - 11A - 10I \\ &= A + 5I \end{aligned}$$

(ii) Let if possible

$$(1, 2, 5) = a(1, 1, 1) + b(2, 1, 2) + C(3, 3, 3)$$

i.e.

$$\begin{bmatrix} 1 & 2 & 3 & : & 1 \\ 1 & 1 & 2 & : & 2 \\ 1 & 2 & 3 & : & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & : & 1 \\ 0 & -1 & -1 & : & -1 \\ 0 & 0 & 0 & : & 4 \end{bmatrix}$$

If $AX = b$, then, Rank$(A) = 2$ and Rank $(A : b) = 3$, which are not equal. Hence, above system of equations is inconsistent.
Therefore, vector $(1, 2, 5)$ cannot be expressed as a linear combination of vector $(1, 1, 1)$,$(2, 1, 2)$ and $(3, 2, 3)$

Question-4(c) (i) Evaluate

$$\lim_{x\to 1}(x-1)\tan\frac{\pi x}{2}$$

(ii) Evaluate the following integral :

$$\int_{-\infty}^{\infty} xe^{-x^2}dx$$

[15 Marks]

Solution: Let

$$\begin{aligned} l &= \lim_{x\to 1}(x-1)\tan\frac{\pi x}{2} \\ &= \lim_{h\to 0} h\tan\left\{\frac{\pi}{2}(1+h)\right\} \\ &= \lim_{h\to 0}\left[-h\cot\left(\frac{\pi}{2}h\right)\right] \\ &= \lim_{h\to 0}\frac{-h}{\sin\left(\frac{\pi h}{2}\right)}\cdot\left(\cos\frac{\pi h}{2}\right) \\ &= \frac{-2}{\pi}\lim_{h\to 0}\frac{\pi h/2}{\sin\left(\frac{\pi h}{2}\right)}\cdot\lim_{h\to 0}\cos\left(\frac{\pi h}{2}\right) \\ &= \frac{-2}{\pi}\cdot 1\cdot 1 \\ &= \frac{-2}{\pi} \end{aligned}$$

(ii)

$$\int_{-\infty}^{\infty} xe^{-x^2}dx$$

$$= \lim_{a\to -\infty}\int_a^0 xe^{-x^2}dx + \lim_{b\to\infty}\int_0^b xe^{-x^2}dx$$

$$= \lim_{a\to -\infty}\left[-\frac{1}{2}e^{-x^2}\right]_a^0 + \lim_{b\to\infty}\left[-\frac{1}{2}e^{-x^2}\right]_0^b$$

$$= \frac{-1}{2}\left\{\left[e^{-0} - \lim_{a\to -\infty} e^{-a^2}\right] + \left[\lim_{b\to\infty} e^{-b^2} - e^{-0}\right]\right\}$$

$$= \left[-\frac{1}{2}\left(1-e^{-\infty}\right)+\left(e^{-\infty}-1\right)\right]$$
$$= -\frac{1}{2}[1-0+0-1] = 0$$

Hence, the given infinite integral is convergent.

1.2 Section-B

Question-5(a) Solve the initial value problem:

$$\left(2x^2+y\right)dx+\left(x^2y-x\right)dy=0, y(1)=2$$

[8 Marks]

Solution:

$$\text{Comparing with } Mdx+Ndy=0,$$
$$M=2x^2y \quad ; \quad N=x^2y-x$$
$$\therefore \frac{\partial M}{\partial y}=1 \quad ; \quad \frac{\partial N}{\partial x}=2xy-1$$

Since, $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, so ODE is not exact.
Integrating factor $(I.F.)$:

$$\frac{1}{N}\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)=\frac{1-(2xy-1)}{x^2y-x}$$
$$=\frac{2(1-xy)}{-x(1-xy)}=-\frac{2}{x}=f(x)$$

Which is a function of x only.

$$I.F=e^{\int f(x)dx}=e^{\int -\frac{2}{x}dx}$$
$$=e^{-2\log x}=e^{\log x^{-2}}=\frac{1}{x^2}$$

Multiplying the given Differential Equation with $I.F.=1/x^2$

$$\frac{1}{x^2}\left(2x^2+y\right)dx+\frac{1}{x^2}\left(x^2y-x\right)dy=0$$
$$\left(2+\frac{y}{x^2}\right)dx+\left(y-\frac{1}{x}\right)dy=0$$

Hence, the solution is $\int_{y-\text{ constant}} M dx + \int($ terms in N not containing $x) dy = C$

$$\int_{y-\text{constant}} \left(2 + \frac{y}{x^2}\right) dx + \int y dy = C$$
$$2x - \frac{y}{x} + \frac{y^2}{2} = C$$
$$\text{Now, when } x = 1, y = 2$$
$$\therefore \quad 2 - \frac{2}{1} + \frac{(2)^2}{2} = C \Rightarrow C = 2$$
$$\therefore \quad 2x - \frac{y}{x} + \frac{y^2}{2} = 2$$
$$\text{i.e.} \quad 4x^2 + xy^2 - 2y = 4x$$

is the required solution of the initial value problem.

Question-5(b) Solve the differential equation:

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 4y = 16x - 12e^{2x}$$

[8 Marks]

Solution: Given $D.E.$ can be written as

$$\left(D^2 - 3D - 4\right) y = 16x - 12e^{2x}$$

Auxiliary Equation:

$$m^2 - 3m - 4 = 0$$
$$m^2 - 4m + m - 4 = 0$$
$$(m - 4)(m + 1) = 0$$
$$\Rightarrow m = 4, -1$$

$$\therefore \quad C \cdot F = C_1 e^{4x} + C_2 e^{-x}$$
$$P \cdot I = \frac{1}{D^2 - 3D - 4}\left(16x - 12e^{2x}\right)$$
$$= 16\frac{1}{D^2 - 3D - 4} x - 12\frac{1}{D^2 - 3D - 4} e^{2x}$$
$$= -\frac{16}{-4}\left[1 - \frac{(D^2 - 3D)}{4}\right]^{-1} x - \frac{12}{(2)^2 - 3(2) - 4} e^{2x}$$
$$= -4\left(1 + \frac{D^2 - 3D}{4} + \dots\right) x - \frac{12}{(-6)} e^{2x}$$
$$= -4\left[x + \frac{1}{4}(-3)\right] + 2e^{2x}$$
$$= -4x + 2e^{2x} + 3$$

Hence, General Solution is

$$y = C.F. + P.I.$$
$$y = C_1e^{4x} + C_2e^{-x} - 4x + 2e^{2x} + 3$$

Question-5(c) If the radial and transverse velocities of a particle are proportional to each other, then prove that the path is an equiangular spiral. Further, if radial acceleration is proportional to transverse acceleration, then show that the velocity of the particle varies as some power of the radius vector.

[8 Marks]

Solution: Here it is given that radial velocity is proportional to transverse velocity

$$\therefore \quad \frac{dr}{dt} = k\left(r\frac{d\theta}{dt}\right) \quad \ldots(1)$$

where k is a constant of proportionality

$$\begin{aligned} \text{i.e.,} \quad \frac{dr}{r} &= kd\theta \\ \text{Integrating,} \quad \log r &= k\theta + \log c \\ \text{or} \quad \log r &= \log e^{k\theta} + \log c \\ \text{or} \quad \log r &= \log c \cdot e^{k\theta} \\ \therefore \quad r &= ce^{k\theta} \end{aligned}$$

Which is the equation of an equiangular spiral.
Further, radial acceleration is proportional to transverse acceleration. i.e.

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \mu\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) \quad \ldots(2)$$

where μ is a constant. Now, we shall eliminate $\frac{d\theta}{dt}$ between (1) and (2). From (1),

$$\frac{d\theta}{dt} = \frac{1}{kr}.\frac{dr}{dt}$$

Putting the value of $\frac{d\theta}{dt}$ in eq. (2), we get

$$\begin{aligned} \frac{d^2r}{dt^2} - r\left(\frac{1}{kr}\cdot\frac{dr}{dt}\right)^2 &= \frac{\mu}{r}\cdot\frac{d}{dt}\left[\frac{r^2}{kr}.\frac{dr}{dt}\right] \\ \text{or} \quad \frac{d^2r}{dt^2} - \frac{1}{k^2r}\left(\frac{dr}{dt}\right)^2 &= \frac{\mu}{k}\cdot\frac{1}{r}\left[r\frac{d^2r}{dt^2} + \left(\frac{dr}{dt}\right)^2\right] \\ \text{i.e.,} \quad \left(1 - \frac{\mu}{k}\right)\frac{d^2r}{dt^2} &= \left(\frac{1}{k^2} + \frac{\mu}{k}\right)\frac{1}{r}\left(\frac{dr}{dt}\right)^2 \\ \text{or} \quad \frac{d^2r}{dt^2} &= \frac{1+ku}{k(k-\mu)}\frac{1}{r}\left(\frac{dr}{dt}\right)^2 \end{aligned}$$

$$\text{which can be written as } \frac{\frac{d^2r}{dt^2}}{\frac{dr}{dt}} = A\frac{\frac{dr}{dt}}{r} \text{ where } A = \frac{1+k\mu}{k(k-\mu)}$$

$$\text{Integrating,} \qquad \log\frac{dr}{dt} = A\log r + \log c_1 = \log r^A + \log c_1$$

$$\text{Hence} \qquad \frac{dr}{dt} = c_1 r^A$$

$$\therefore \text{ From (1),} \qquad c_1 r^A = kr\frac{d\theta}{dt} \quad \text{i.e.,}\ r\frac{d\theta}{dt} = \frac{c_1}{k}r^{\mathrm{A}} \qquad \ldots(3)$$

$$\therefore \qquad \text{Velocity of the particle } = \sqrt{\left(\frac{dr}{dt}\right)^2 + \left(r\frac{d\theta}{dt}\right)^2} = \sqrt{(c_1r^{\mathrm{A}})^2 + \frac{1}{k^2}(c_1r^{\mathrm{A}})^2}$$

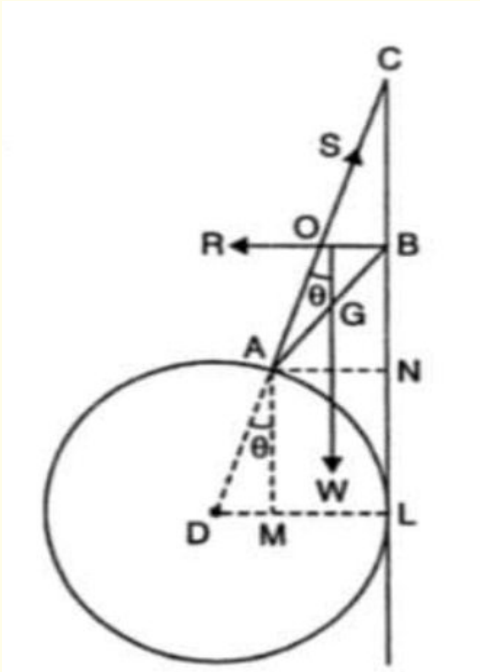
$$= cr^A \text{ where } c = \sqrt{c_1^2 + \frac{c_1^2}{k^2}} \text{ is a constant}$$

Hence the velocity of the particle varies as some power of radius vector.

Question-5(d) A cylinder of radius 'r', whose axis is fixed horizontally, touches a vertical wall along a generating line.
A flat beam of length l and weight $'W'$ rests with its extremities in contact with the wall and the cylinder, making an angle of $45°$ with the vertical.
Prove that the reaction of the cylinder is $\frac{\mathrm{W}\sqrt{5}}{2}$ and the pressure on the wall is $\frac{\mathrm{W}}{2}$.
Also, prove that the ratio of radius of the cylinder to the length of the beam is $5+\sqrt{5} : 4\sqrt{2}$.

[8 Marks]

Solution: Rod AB is in equilibrium on the cylinder and against the wall under the forces:
(i) Reaction R $\perp$ wall at B.
(ii) Reaction S at A passing through the centre D.
(iii) Weight W acting at middle point C of the rod vertically downwards.



Forces are concurrent at O,

$$\angle \text{ABL} = \angle \text{OGB} = 45^\circ \qquad \text{(Given)}$$

Let $\angle \text{AOG} = \theta$,
Applying $m:n$ theorem in $\triangle AOB$

$$(1+1)\cot 45^\circ = 1\cot\theta - 1\cot 90$$
$$\therefore \qquad \cot\theta = 2 \qquad \ldots(1)$$

Applying Lami's theorem

$$\frac{\text{R}}{\sin(180-\theta)} = \frac{\text{S}}{\sin 90^\circ} = \frac{\text{W}}{\sin(90+\theta)}$$
$$\text{or} \qquad \frac{\text{R}}{\sin\theta} = \frac{\text{S}}{1} = \frac{\text{W}}{\cos\theta}$$
$$\therefore \qquad R = W\tan\theta = \frac{W}{2} \qquad \text{(using 1)}$$
$$\text{S} = \text{W}\sec\theta = \text{W}\sqrt{1+\tan^2\theta}$$
$$= \text{W}\sqrt{1+\frac{1}{4}}$$
$$= \frac{1}{2}\,\text{W}\sqrt{5}$$
$$\text{Further} \qquad \text{DL} = \text{DM} + \text{ML} = \text{DM} + \text{AN} = r\sin\theta + l\sin 45^\circ$$
$$\therefore \qquad r = r\cdot\frac{1}{\sqrt{5}} + l\cdot\frac{1}{\sqrt{2}} \Rightarrow \frac{r(\sqrt{5}-1)}{\sqrt{5}} = \frac{l}{\sqrt{2}}$$
$$\Rightarrow \qquad \frac{r}{l} = \frac{(5+\sqrt{5})}{4\sqrt{2}}$$

Question-5(e) Prove that for a vector $\overrightarrow{a}$,

$$\nabla(\vec{a}\cdot\vec{r}) = \vec{a};$$

where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}, r = |\vec{r}|$ Is there any restriction on $\overrightarrow{a}$?
Further, show that

$$\vec{a}\cdot\nabla\left(\vec{b}\cdot\nabla\frac{1}{r}\right) = \frac{3(\vec{a}\cdot\vec{r})(\vec{b}\cdot\vec{r})}{r^5} - \frac{\vec{a}\cdot\vec{b}}{r^3}$$

Give an example to verify the above.

[8 Marks]

Solution: Let $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.
$\therefore \quad \vec{a}\cdot\vec{r} = a_1x + a_2y + a_3z.$

$$\begin{aligned}
\text{Now,}\nabla(\vec{a}\cdot\vec{r}) &= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(\vec{a}\cdot\vec{r}) \\
&= \hat{i}\frac{\partial}{\partial x}(a_1x + a_2y + a_3z) + \hat{j}\frac{\partial}{\partial y}(a_1x + a_2y + a_3z) + \hat{k}\frac{\partial}{\partial z}(a_1x + a_2y + a_3z) \\
&= \hat{i}a_1 + \hat{j}a_2 + \hat{k}a_3 = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}
\end{aligned}$$

$$\therefore \quad \nabla(\vec{a}\cdot\vec{r}) = \vec{a}$$

Condition: Above result is valid only when vector a is a constant vector.
Also,

$$\begin{aligned}
\nabla\left(\frac{1}{r}\right) &= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\left(\frac{1}{r}\right) \\
&= -\frac{1}{r^2}\left(\hat{i}\frac{\partial r}{\partial x} + \hat{j}\frac{\partial r}{\partial y} + \hat{k}\frac{\partial r}{\partial z}\right) \\
&= -\frac{1}{r^2}\left(\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right) \\
&\qquad \left[\because r^2 = |\vec{r}|^2 = x^2 + y^2 + z^2, \therefore \frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial r}{\partial z} = \frac{z}{r}\right] \\
&= -\frac{1}{r^2}\left(\frac{x\hat{i} + y\hat{j} + z\hat{k}}{r}\right) = -\frac{\vec{r}}{r^3}
\end{aligned}$$

$$\therefore \quad \vec{b}\cdot\nabla\left(\frac{1}{r}\right) = \vec{b}\cdot\left(\frac{-\vec{r}}{r^3}\right) = \left(-\frac{1}{r^3}\right)(\vec{b}\cdot\vec{r})$$

$$\begin{aligned}
\text{Also,}\quad \nabla\left(\vec{b}\cdot\nabla\frac{1}{r}\right) &= \left(-\frac{1}{r^3}\right)\nabla(\vec{b}\cdot\vec{r}) + \nabla\left(-\frac{1}{r^3}\right)(\vec{b}\cdot\vec{r}) \\
&= -\frac{1}{r^3}\vec{b} + \frac{3}{r^4}\left(\hat{i}\frac{\partial\vec{r}}{\partial x} + \hat{j}\frac{\partial\vec{r}}{\partial y} + \hat{k}\frac{\partial\vec{r}}{\partial z}\right)(\vec{b}\cdot\vec{r}) \\
&\qquad [\because \text{ For any vector } \vec{a}, \nabla(\vec{a}\cdot\vec{r}) = \vec{a}]
\end{aligned}$$

$$= -\frac{1}{r^3}\vec{b} + \frac{3}{r^4}\left(\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right)(\vec{b}\cdot\vec{r})$$
$$= -\frac{\vec{b}}{r^3} + \frac{3\vec{r}(\vec{b}\cdot\vec{r})}{r^5}$$
$$\text{Hence, } \vec{a}\cdot\nabla\left(\vec{b}\cdot\nabla\frac{1}{r}\right) = -\frac{\vec{a}\cdot\vec{b}}{r^3} + \frac{3(\vec{a}\cdot\vec{r})(\vec{b}\cdot\vec{r})}{r^5}$$

Verification of above Result: Let us take, $\vec{a} = \hat{i}, \vec{b} = \hat{j}$

$$\vec{r} = xi + yj + zk$$
$$\nabla\left(\frac{1}{r}\right) = i\frac{\partial}{\partial x}\left(\frac{1}{r}\right) + j\frac{\partial}{\partial y}\left(\frac{1}{r}\right) + \frac{k\partial}{\partial z}\left(\frac{1}{r}\right)$$
$$= i\left(\frac{-1}{r^2}\frac{\partial r}{\partial x}\right) + j\left(-\frac{1}{r^2}\frac{\partial r}{\partial y}\right) + k\left(\frac{-1}{r^2}\frac{\partial r}{\partial z}\right)$$
$$= \frac{-1}{r^2}\left(\frac{x}{r}i + \frac{y}{r}j + \frac{z}{r}k\right) = \frac{-\vec{r}}{r^3}$$
$$\left(\vec{b}\cdot\nabla\frac{1}{r}\right) = \left(\frac{-1}{r^3}\right)j\cdot(\vec{r}) = \frac{-y}{r^3}$$
$$\nabla\left(\frac{-y}{r^3}\right) = i\frac{\partial}{\partial x}\left(\frac{-y}{r^3}\right) + j\frac{\partial}{\partial y}\left(\frac{y}{r^3}\right) + k\frac{\partial}{\partial z}\left(\frac{-y}{r^3}\right)$$
$$= i\left(\frac{3y}{r^4}\right)\frac{x}{r} + j\left((-1)\frac{1}{r^3} + \frac{3y}{r^4}\cdot\frac{y}{r}\right) + k\left(\frac{3y}{r^4}\right)\frac{z}{r}$$
$$\vec{a}\cdot\nabla\left(\vec{b}\cdot\nabla\frac{1}{r}\right) = i\cdot\nabla\left(-\frac{y}{r^3}\right) = \frac{3xy}{r^5}$$
$$\frac{3\cdot(\vec{a}\cdot\vec{r})\cdot(\vec{b}\cdot\vec{r})}{r^5} - \frac{\vec{a}\cdot\vec{b}}{r^3} = \frac{3(x)(y)}{r^5} - \frac{ij}{r^3} = \frac{3xy}{r^5}$$

Question-6(a) Find one solution of the differential equation

$$\left(x^2+1\right)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$$

by inspection and using that solution determine the other linearly independent solution of the given equation. Obtain the general solution of the given differential equation.

[10 Marks]

Solution: The $D.E.$ is

$$\frac{d^2y}{dx^2} - \frac{2x}{(x^2+1)}\frac{dy}{dx} + \frac{2}{x^2+1}y = 0 \quad \ldots(1)$$

Which is in the form $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = 0$.
Here, we notice that

$$P + Qx = \frac{-2x}{x^2+1} + \frac{2}{x^2+1}\cdot x = 0$$

$\therefore y = x$ is a part of solution.

$$\text{Consider,} \quad y = vx$$
$$\therefore \frac{dy}{dx} = v.1 + x\frac{dv}{dx}$$
$$\frac{d^2y}{dx^2} = x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}$$

Putting these values in (1), we get

$$\left(x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}\right) - \frac{2x}{(x^2+1)}\left(v + x\frac{dv}{dx}\right) + \frac{2}{x^2+1}(vx) = 0$$
$$x\frac{d^2v}{dx^2} + \left(2 - \frac{2x^2}{1+x^2}\right)\frac{dv}{dx} = 0$$
$$x\frac{d^2v}{dx^2} + \frac{2}{1+x^2}\frac{dv}{dx} = 0 \qquad \ldots(2)$$
$$\text{Let} \quad \frac{dv}{dx} = p \quad \Rightarrow \quad \frac{d^2v}{dx^2} = \frac{dp}{dx}$$
$$\therefore x\frac{dp}{dx} + \frac{2}{1+x^2}p = 0 \text{ (from (2))}$$
$$\Rightarrow \frac{dp}{p} = -\frac{2}{x(1+x^2)}dx$$
$$\frac{dp}{p} = -2\left(\frac{1}{x} - \frac{x}{1+x^2}\right)dx$$
$$\text{Integrating,} \quad \log p = -2\log x + \log\left(1+x^2\right) + \log c_1$$
$$p = \frac{c_1(1+x^2)}{x^2}$$
$$\frac{dv}{dx} = c_1\frac{(1+x^2)}{x^2}$$
$$dv = c_1\left(\frac{1}{x^2} + 1\right)dx$$
$$\text{Integrating,} \quad v = c_1\left(\frac{-1}{x} + x\right) + c_2$$

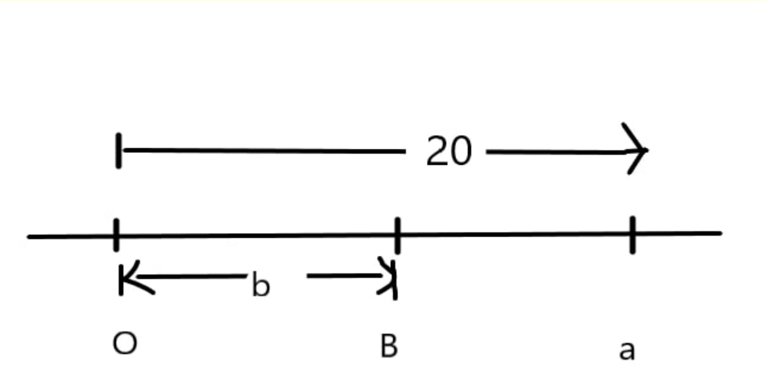
Hence, the complete solution is

$$y = vx$$
$$y = c_1\left(-1 + x^2\right) + c_2x$$

Question-6(b) A particle of mass 5 units moves in a straight line towards a centre of force and the force varies inversely as the cube of distance. Starting from rest at the point A distant 20 units from centre of force 0, **it reaches a point B distant** b ' **from** O.
Find the time in reaching from A **to** B **and the velocity at** B.
When will the particle reach at the centre?

[15 Marks]

Solution: Let O be the centre of force, and particle starts from rest at point A. Then it reaches point B.



Given $F = \frac{-k}{x^3}$. The differential equation of motion of particle is given by

$$m\frac{d^2x}{dt^2} = \frac{-k}{x^3}$$

Multiplying by $2\frac{dx}{dt}$ on both side and integrating, we get

$$m\left(\frac{dx}{dt}\right)^2 = \frac{k}{x^2} + C$$

$$\text{At } x = 20, v = \frac{dx}{dt} = 0$$

$$0 = \frac{k}{(20)^2} + C \Rightarrow \quad C = \frac{-k}{400}$$

$$\text{and, } m = 5 \text{ units}$$

$$5\left(\frac{dx}{dt}\right)^2 = k\left(\frac{1}{x^2} - \frac{1}{400}\right)$$

$$\frac{dx}{dt} = -\sqrt{\frac{k}{5}}\frac{\sqrt{400 - x^2}}{20x} = \frac{-\mu\sqrt{400 - x^2}}{20x} \qquad \left(\text{for } \mu = \sqrt{\frac{k}{5}}\right)$$

[Negative sign is taken because $v = \frac{dx}{dt}$ is decreasing (central force)]

$$\Rightarrow \int \frac{-20x}{\sqrt{400 - x^2}}dx = \int \mu dt$$

$$\Rightarrow 20\sqrt{400 - x^2} = \mu t + c_1$$

$$\text{When } t = 0, x = 20 \Rightarrow c_1 = 0$$

$$\therefore \quad t = \frac{20}{\mu}\sqrt{400 - x^2}, \mu = \sqrt{k/5}$$

Time taken in reaching from A to B,
i.e. when $x = b$

$$t_B = \frac{20}{\mu}\sqrt{400 - b^2}$$

$$V_B = \frac{-\mu\sqrt{400 - b^2}}{20b}$$

When particle reaches at centre, $x = 0$

$$\therefore \quad t = \frac{20}{\mu}\sqrt{400 - 0} = \frac{400}{\mu}, \mu = \sqrt{\frac{k}{5}}$$

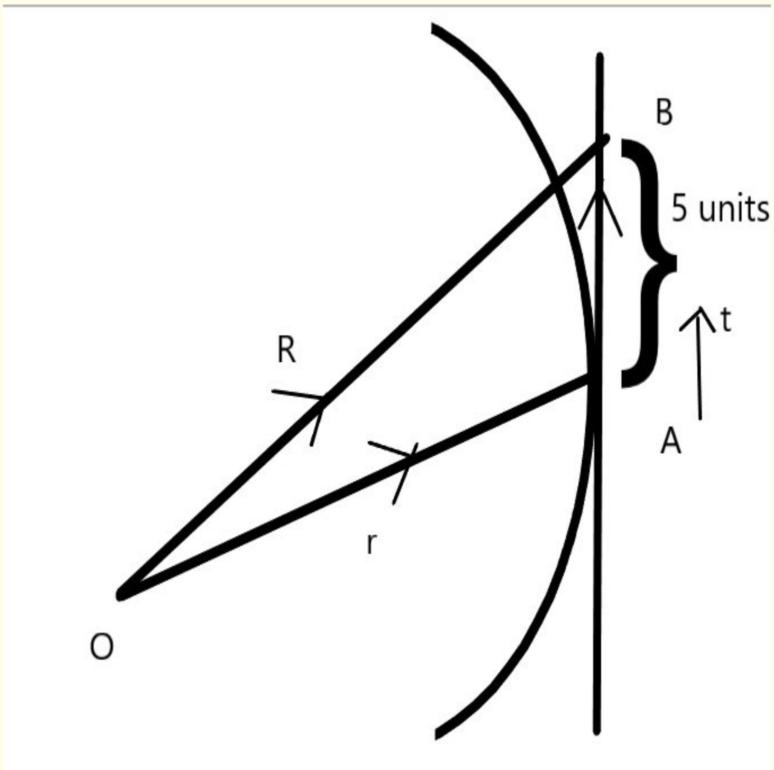
Question-6(c) A tangent is drawn to a given curve at some point of contact. B is a point on the tangent at a distance 5 units from the point of contact. Show that the curvature of the locus of the point B is

$$\frac{\left[25\kappa^2\tau^2\left(1+25\kappa^2\right)+\left\{\kappa+5\frac{d\kappa}{ds}+25\kappa^3\right\}\right]^{1/2}}{\left(1+25\kappa^2\right)^{3/2}}$$

Find the curvature and torsion of the curve $\overrightarrow{\mathrm{r}}=t\hat{\mathrm{i}}+\mathrm{t}^2\hat{\mathrm{j}}+\mathrm{t}^3\hat{\mathrm{k}}$.

[15 Marks]

Solution: Locus of point at distance 5 units from B,



$$
\begin{aligned}
R(s) &= r(t)+5t(s)\\
R'(s) &= r'(s)+5t'(s)\\
R''(s) &= r''(s)+5t''(s)\\
|R'(s)| &= |t(s)+5kn(s)| = \sqrt{1+25k^2} \qquad (\because \text{t(s) and n(s) are unit vectors})\\
\Rightarrow R'(s)\times R''(s) &= (r'(s)+5t'(s))\times(r''(s)+5t''(s))\\
&= r'(s)\times r''(s)+r'(s)\times 5t''(s)+5t'(s)\times r''(s)+25t'(s)\times t''(s)\\
&= t(s)\times kn(s)+t(s)\times 5k(-kt(s)+\tau b(s))+25kn(s)\times(-kt(s)+\tau b(s))
\end{aligned}
$$

$$\because t'(s) = kn(s)$$

$$\Rightarrow t''(s) = \frac{dk}{ds}n(s) + kn'(s)$$

$$t(s) \times kn(s) + t(s) \times 5k(-kt(s) + \tau b(s)) + 25kn(s) \times (-kt(s) + \tau b(s))$$

$$= kb(s) + (r'(s) + 5t'(s)) \times 5t''(s)$$

$$= kb(s) + (r'(s) + 5t'(s)) \times 5\left(\frac{dk}{ds}n(s) + k(-kt(s) + \tau b(s))\right)$$

$$= kb(s) + 5\left(r'(s) + 5t'(s)\right) \times \left(\frac{dk}{ds}n(s) - k^2 t(s) + \tau kb(s)\right)$$

$$= kb(s) + 5\left\{t(s) \times \frac{dk}{ds}n(s) - \tau kn(s) + 5k^3 b(s) + 5\tau k^2 t(s)\right\}$$

$$= kb(s) + 5\frac{dk}{ds}b(s) + 25k^3 b(s) - 5\tau kn(s) + 25\tau k^2 t(s)$$

$$= \left(k + 5\frac{dk}{ds} + 25k^3\right)b(s) - 5\tau kn(s) + 25\tau k^2 t(s)$$

$$|R'(s) \times R''(s)| = \left\{25\tau^2 k^2 + 625\tau^2 k^4 + \left(k + 5\frac{dk}{ds} + 25k^3\right)^2\right\}^{1/2}$$

$(\because k(s) \cdot t(s) = 0)$

$$= \left\{25\tau^2 k^2\left(1 + 25k^2\right) + \left(k + 5\frac{dk}{ds} + 25k^3\right)^2\right\}^{1/2}$$

$$\Rightarrow \frac{|R'(s) \times R''(s)|}{|R'(s)|^3} = \frac{\left\{25\tau^2 k^2\left(1 + 25k^2\right) + \left(k + 5\frac{dk}{ds} + 25k^3\right)\right\}^{1/2}}{\left\{1 + 25k^2\right\}^{3/2}}$$

Curvature and torsion of the curve $\overrightarrow{\mathrm{r}} = t\hat{\mathrm{i}} + \mathrm{t}^2\hat{\mathrm{j}} + \mathrm{t}^3\hat{\mathrm{k}}$

$$
\begin{aligned}
r(t) &= \left\langle t, t^2, t^3 \right\rangle \\
r'(t) &= \left\langle 1, 2t, 3t^2 \right\rangle \\
r''(t) &= \langle 0, 2, 6t \rangle \\
r'''(t) &= \langle 0, 0, 6 \rangle \\
r'(t) \times r''(t) &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = \langle 6t^2, -6t, 2 \rangle \\
\mid r'(t) \mid &= \sqrt{1 + 4 \cdot t^2 + 9t^4} \\
\textbf{Curvature} &= \frac{\sqrt{36t^4 + 36t^2 + 4}}{(1 + 4t^2 + 2t^4)^{3/2}} = \frac{2\sqrt{1 + 9t^2 + 9t^4}}{(1 + 4t^2 + 9t^4)^{3/2}} \\
\textbf{Torsion} &= \frac{[r' \quad r'' \quad r''']}{|r' \times r''|^2} = r' \cdot [r'' \times r'''] \\
r'' \times r''' &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 2 & 6t \\ 0 & 0 & 6 \end{vmatrix} = \langle 12, 0, 0 \rangle \\
\left[\ r' \quad r'' \quad r'''\ \right] &= \left\langle 1, 2t, 3t^2 \right\rangle \cdot \langle 12, 0, 0 \rangle = 12 \\
|r' \times r''|^2 &= \left(36t^4 + 36t^2 + 4\right) = 4\left(1 + 9t^2 + 9t^4\right) \\
\textbf{Torsion} &= \frac{12}{4\left(1 + 9t^2 + 9t^4\right)} = \frac{3}{1 + 9t^2 + 9t^4}
\end{aligned}
$$

Question-7(a) Derive intrinsic equation

$$\mathrm{x} = \mathrm{c} \log(\sec\psi + \tan\psi)$$

of the common catenary, where symbols have usual meanings.
Prove that the length of an endless chain, which will hang over a circular pulley of radius 'a' so as to be in contact with $\frac{2}{3}$ of the circumference of the pulley, is

$$a\left\{\frac{4\pi}{3} + \frac{3}{\log(2 + \sqrt{3})}\right\} \qquad (10)$$

[10 Marks]

Solution: Let the uniform flexible string ACB hang freely from two points A and B which are not in a vertical line. Let C be the lowest point of the common catenary. Let P be any point on the portion CA of the string such that are $CP = s$, measured along the curve.

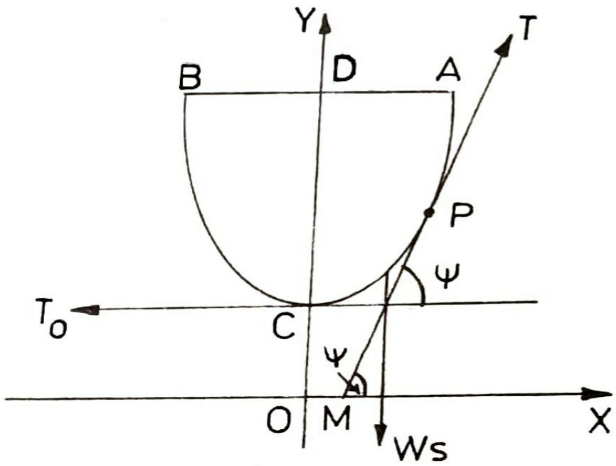


Fig. 4.1

Let w be the weight of the string per unit length of the string, then the weight of the portion $CP = ws$. This weight ws will act vertically downwards through S, the centre of gravity of the arc CP. Let the tangent at P make an angle ψ with the horizontal.
The portion CP of the string is in equilibrium under the action of the following forces:
(i) The tension T_0 at the lowest point C acting horizontally along the tangent to the curve at C.
(ii) The tension T along the tangent at P.
(iii) The weight ws to the portion CP acting vertically downwards through the centre of gravity of the are CP.

Since these three forces are in equilibrium, the line of action of the weight ws must pass through Q, the point of intersection of tangent at C and P.
Resolving these forces horizontally and vertically, we have

$$T\cos\psi = T_0 \qquad \ldots(1)$$

$$T\sin\psi = ws \qquad \ldots(2)$$

Dividing (2) by (1), we get

$$\tan\psi = \frac{ws}{T_0} \qquad \ldots(3)$$

Let the tension T_0 at the lowest point C be taken equal to the weight of a length c of the string, i.e., $T_0 = wc$.
Then from (3),

$$\tan\psi = \frac{ws}{wc} = \frac{s}{c}$$

Hence, $s = c\tan\psi$ is the required intrinsic equation of the catenary, where c is called the parameter of the catenary.

Relation between x and ψ:
We have

$$\frac{dx}{d\psi} = \frac{dx}{ds}\cdot\frac{ds}{d\psi} = \frac{dx}{ds}\cdot c\sec^2\psi$$

$$
\begin{aligned}
&= \cos\psi \cdot c\sec^2\psi \text{ (From differential calculus)}\\
&= c\sec\psi\\
\Rightarrow dx &= c\sec\psi \cdot d\psi\\
\text{Intergating, } x &= c\log(\sec\psi + \tan\psi) + c_2 \qquad \ldots (4)
\end{aligned}
$$

where c_2 is an arbitrary constant. At the lowest point $C, x = 0$ and $\psi = 0$

$$
\begin{aligned}
\therefore \qquad\qquad & c_2 = 0\\
\therefore \text{ From (4)}, \quad & x = c\log(\sec\psi + \tan\psi).
\end{aligned}
$$

Question-7(b) Solve the differential equation

$$x^2\frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = \frac{1}{(1-x)^2}$$

[15 Marks]

Solution:

$$\text{Put } x = e^z \Rightarrow z = \log x$$

$$\text{Let } D = x\frac{d}{dx} = \frac{d}{dz}$$

Then given $O.D.E$ reduces to

$$
\begin{aligned}
&[D(D-1) + 3D + 1]y = \frac{1}{(1-e^z)^2}\\
\text{or} \quad &\left(D^2 + 2D + 1\right)y = \frac{1}{(1-e^z)^2}
\end{aligned}
$$

Auxiliary Equation:

$$
\begin{aligned}
D^2 + 2D + 1 = 0 \quad &\text{i.e. } (D+1)^2 = 0\\
\Rightarrow D = -1, -1
\end{aligned}
$$

$$\begin{aligned}
\therefore C\cdot F\cdot &= (c_1 + c_2 z)\, e^{-z} \\
&= (c_1 + c_2 \log x)\, \frac{1}{x} \quad (\because e^z = x) \\
P.I. &= \frac{1}{(D+1)^2} \cdot \frac{1}{(1-e^z)^2} \\
&= \frac{1}{D+1} \left[\frac{1}{D+1} \cdot \frac{1}{(1-e^z)^2} \right] \\
&= \frac{1}{D+1} \cdot e^{-z} \int \frac{1}{(1-e^z)^2} \cdot e^z dz \\
&\left[\because \frac{1}{D-a} X = e^{ax} \int x e^{-ax} dx \right] \\
&= \frac{1}{D+1} \cdot e^{-z} \int (1-t)^{-2} dt, \qquad \text{where } e^z = t \\
&= \frac{1}{D+1} \cdot e^{-z}(1-t) \\
&= \frac{1}{D+1} \cdot \left(\frac{e^{-z}}{1-e^z} \right) \\
&= e^{-z} \int \frac{e^{-z}}{1-e^z} \cdot e^z dz \\
&= e^{-z} \int \frac{dz}{1-e^z} = e^{-z} \int \frac{e^{-z}}{e^{-z}-1} dz \\
&= -e^{-z} \int \frac{-e^{-z}}{e^{-z}-1} dz \\
&= -e^{-z} \log\left(e^{-z}-1\right) \\
&= -\frac{1}{x} \log\left(\frac{1}{x} - 1\right) \quad (\because e^z = x) \\
&= -\frac{1}{x} \log\left(\frac{1-x}{x}\right) = \frac{1}{x} \log\left(\frac{x}{1-x}\right)
\end{aligned}$$

Hence, the complete solution is,

$$y = (c_1 + c_2 \log x)\, \frac{1}{x} + \frac{1}{x} \log\left(\frac{x}{1-x}\right)$$

Question-7(c) Given a portion of a circular disc of radius 7 units and of height $1 \cdot 5$ units such that $x, y, z \geq 0$.
Verify Gauss Divergence Theorem for the vector field

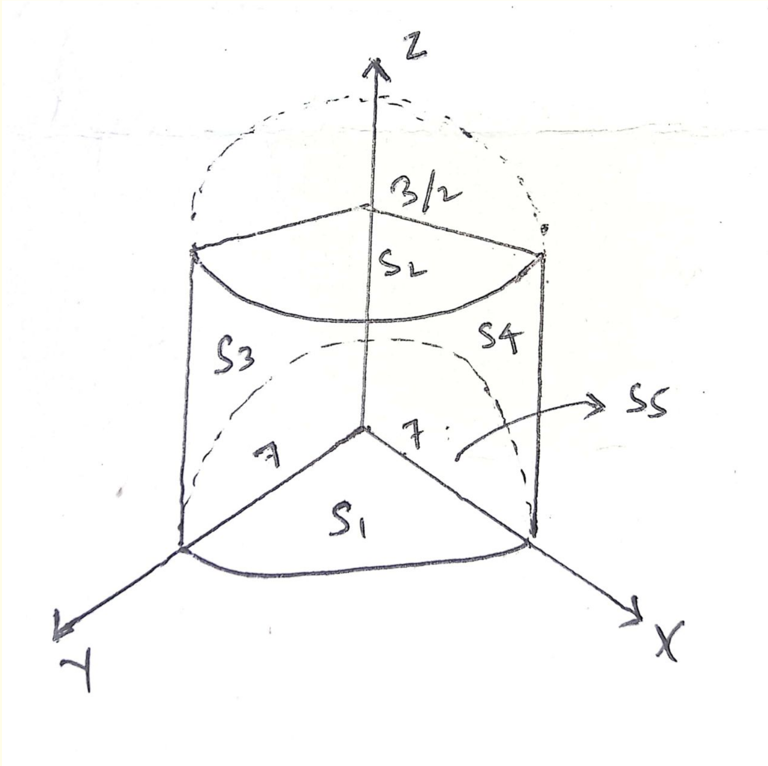
$$\overrightarrow{\mathrm{f}} = \left(z, x, 3\mathrm{y}^2 z\right)$$

over the surface of the above mentioned circular disc.

[15 Marks]

Solution: $\vec{F} = z\hat{i} + xj + 3y^2z\hat{k}$.
By Gauss Divergence Theorem,

$$\iint_S F \cdot \hat{n} dS = \iiint_V divF dV$$



Consider RHS,

$$\iiint_V divF dV = \iiint 3y^2 dV$$

$$= \int_0^{\pi/2} \int_0^7 \int_0^{3/2} 3 \text{ r}^3 \sin^2\theta dz dr d\theta \text{ (Changing to polar coordinates)}$$

V is given by:

$$x^2 + y^2 = 7^2$$
$$0 \leq z \leq 3/2$$
$$x \geq 0, y \geq 0$$

$$\therefore \text{ RHS} = 3\left(\frac{3}{2}\right)\left(\frac{1}{4}\right)(7)^4 \int_0^{\pi/2} \sin^2\theta d\theta$$
$$= \frac{9}{8}(7)^4\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right) = \frac{21609\pi}{32} \qquad \ldots(I)$$

Consider LHS

For S_1: $z = 0$, $\hat{n} = -\hat{k} \Rightarrow \vec{F} \cdot \hat{n} = -3y^2 z$

$$\Rightarrow \iint_{S_1} \vec{F} \cdot \hat{n} dS = \iint_{S_1} 0 dS = 0 \text{ (as } \vec{F} \cdot \hat{n} = 3y^2 z = 0 \because z = 0) \qquad \ldots(1)$$

For S_2: $z = 3/2$, $\hat{n} = \hat{k}$, $x^2 + y^2 = 7^2$, $x \geq 0$, $y \geq 0$

$$\Rightarrow \iint_{S_2} \vec{F} \cdot \hat{n} dS = \iint_{S_2} 3y^2 \left(\frac{3}{2}\right) dxdy = \frac{9}{2} \int_0^{\frac{\pi}{2}} \int_0^7 r^3 \sin^2 \theta d\theta$$

$$= \frac{9}{2} \left(\frac{1}{4}\right) 7^4 \times \left(\frac{1}{2}\right) \frac{\pi}{2} = \frac{21609\pi}{32} \qquad \ldots (2)$$

For S_3: $x = 0$, $\hat{n} = -i$, $\vec{F} \cdot \hat{n} = -z$, $0 \leq z \leq 3/2$, $0 \leq y \leq 7$

$$\Rightarrow \iint_{S_3} \vec{F} \cdot \hat{n} dS = \int_0^{3/2} \int_0^7 -zdydz = -\frac{9}{4 \times 2} \times 7 = \frac{-63}{8} \qquad \ldots (3)$$

For S_4: $y = 0$, $\hat{n} = -j$, $\vec{F} \cdot \vec{n} = -x$, $0 \leq z \leq 3/2$, $0 \leq y \leq 7$

$$\Rightarrow \int_{S_4} \vec{F} \cdot \hat{n} dS = \int_0^{1/2} \int_0^x -xdxdz = \left(\frac{3}{2}\right) \left(\frac{7^2}{2}\right) = \frac{-49 \times 3}{4} = \frac{-147}{4} \qquad \ldots (4)$$

For S_5: $\hat{n} = \frac{2xi + 2yj + 0k}{2 \times 7} = \frac{xi + yj}{7}$
Taking projection on yz plane, $0 \leqslant y \leqslant 2$, $0 \leqslant z \leqslant 3/2$.

$$\iint_{S_5} \vec{F} . \hat{n} dS = \int_0^{3/2} \int_0^7 x(z + y) \frac{dydz}{x}$$

$$= \frac{3}{2} \times \frac{7^2}{2} + 7 \left(\frac{9}{8}\right)$$

$$= \frac{147}{4} + \frac{63}{8} \qquad \ldots (5)$$

Adding (1), (2), (3), (4) and (5), we get:

$$0 + \frac{21609\pi}{32} - \frac{63}{8} - \frac{147}{4} + \frac{147}{4} + \frac{63}{8} = \frac{21609\pi}{32} = \text{RHS from (I)}$$

Therefore, Gauss Divergence Theorem is verified.

Question-8(a) Derive expression of ∇f in terms of spherical coordinates. Prove that

$$\nabla^2 (fg) = f \nabla^2 g + 2 \nabla f . \nabla g + g \nabla^2 f$$

for any two vector point functions f(r, θ, ϕ) **and** g(r, θ, ϕ)**. Construct one example in three dimensions to verify this identity.**

[10 Marks]

Solution: First we prove that

$$\nabla \phi = \frac{\hat{e}_1}{h_1} \frac{\partial \phi}{\partial u} + \frac{\hat{e}_2}{h_2} \frac{\partial \phi}{\partial v} + \frac{\hat{e}_3}{h_3} \frac{\partial \phi}{\partial w}$$

Consider any scalar point function $\phi(u, v, \omega)$ given in terms of orthogonal curvilinear

coordinates u, v, w.
Regarding u, v, w as functions of x, y, z, we have

$$\frac{\partial\phi}{\partial x} = \frac{\partial\phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial\phi}{\partial v}\frac{\partial u}{\partial x} + \frac{\partial\phi}{\partial w}\frac{\partial w}{\partial x} \qquad \ldots(1)$$
$$\frac{\partial\phi}{\partial y} = \frac{\partial\phi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial\phi}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial\phi}{\partial w}\frac{\partial w}{\partial y} \qquad \ldots(2)$$
$$\frac{\partial\phi}{\partial z} = \frac{\partial\phi}{\partial u}\frac{\partial u}{\partial z} + \frac{\partial\phi}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial\phi}{\partial w}\frac{\partial w}{\partial z} \qquad \ldots(3)$$

mutiplying (1) by $\hat{i}$, (2) by $\hat{j}$, (3) by $\hat{k}$ and adding, we get

$$\nabla\phi = \frac{\partial\phi}{\partial u}\nabla u + \frac{\partial\phi}{\partial v}\nabla v + \frac{\partial\phi}{\partial w}\nabla w$$
$$= \frac{\hat{e}_1}{h_1}\frac{\partial\phi}{\partial u} + \frac{\hat{e}_2}{h_2}\frac{\partial\phi}{\partial v} + \frac{\hat{e}_3}{h_3}\frac{\partial\phi}{\partial w} \qquad \ldots(4)$$
$$\left[\because \nabla u = \frac{\hat{e}_1}{h_1} etc\right]$$

The spherical coordinates of (x, y, z) are

$$\begin{aligned} x &= r\sin\theta\cos\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\theta \\ \therefore dx &= -r\sin\theta\sin\phi d\phi + r\cos\theta\cos\phi d\theta + \sin\theta\cos\phi dr \\ dy &= r\sin\theta\cos\phi d\phi + r\cos\theta\sin\phi d\theta + \sin\theta\sin\phi dr \\ dz &= -r\sin\theta d\theta + \cos\theta dr \end{aligned}$$

Element of Arc length

$$\begin{aligned} (ds)^2 &= (dx)^2 + (dy)^2 + (dz)^2 \\ &= (dr)^2 + r^2(d\theta)^2 + r^2\sin^2\theta(d\phi)^2 \end{aligned}$$

Thus we have scalar factors $h_1 = h_r = 1$

$$\begin{aligned} h_2 &= h_\theta = r \\ h_3 &= h\phi = r\sin\theta \end{aligned}$$

Using these in (4), we get

$$\nabla f = \frac{\hat{e}_r}{1}\frac{\partial f}{\partial r} + \frac{\hat{e}_\theta}{r}\frac{\partial f}{\partial\theta} + \frac{\hat{e}_\phi}{r\sin\theta}\frac{\partial f}{\partial\phi}$$

Now, we will calculate $\nabla^2(fg)$.

$$\nabla^2(fg) = \nabla\cdot(\nabla fg)$$
$$\Rightarrow \nabla\cdot(\nabla fg) = \nabla\cdot(f\nabla g + g\nabla f)$$
$$\begin{aligned} \nabla fg &= \left\langle \frac{\partial}{\partial x}fg, \frac{\partial}{\partial y}fg, \frac{\partial}{\partial z}fg \right\rangle \\ &= \left\langle g\frac{\partial f}{\partial x} + f\frac{\partial g}{\partial x}, g\frac{\partial f}{\partial y} + f\frac{\partial g}{\partial y}, g\frac{\partial f}{\partial z} + f\frac{\partial g}{\partial z} \right\rangle \\ &= g\left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle + f\left\langle \frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}, \frac{\partial g}{\partial z} \right\rangle \\ &= g\nabla f + f\nabla g \end{aligned}$$

$$\Rightarrow \nabla\cdot(f\nabla g+g\nabla f)=\nabla\cdot(f\nabla g)+\nabla\cdot(g\nabla f)$$
$$=(\nabla f)\cdot\nabla g+f(\nabla\cdot\nabla g)+(\nabla g)\cdot\nabla f+g(\nabla\cdot\nabla f)$$
$$=\nabla f\cdot\nabla g+f\nabla^2 g+\nabla g\cdot\nabla f+g\nabla^2 f$$
$$=f\nabla^2 g+2\nabla f\cdot\nabla g+g\nabla^2 f$$

Example to verify above result:
Let $f(r,\theta,\phi)=r;\quad g(r,\theta,\phi)=\theta$
We know that,

$$\nabla^2 f=\frac{\partial^2 f}{\partial r^2}+\frac{1}{r^2}\frac{\partial^2 f}{\partial\theta^2}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}+\frac{2}{r}\frac{\partial f}{\partial r}+\frac{\cot\theta}{r^2}\frac{\partial f}{\partial\theta}$$
$$\therefore\qquad \nabla^2 f=\nabla^2(r)=\frac{2}{r}$$
$$\nabla^2 g=\nabla^2(\theta)=\frac{\cot\theta}{r^2}$$
$$\nabla f=\frac{\partial f}{\partial r}\hat{r}+\frac{1}{r}\frac{\partial f}{\partial\theta}\hat{\theta}+\frac{1}{r\sin\theta}\frac{\partial f}{\partial\phi}\hat{\phi}$$
$$=\nabla(r)=\hat{r}$$
$$\nabla g=\nabla(\theta)=\frac{1}{r}\hat{\theta}$$
$$\nabla^2(fg)=\nabla^2(r\theta)$$
$$=\frac{2}{r}\theta+\frac{\cot\theta}{r^2}\cdot r=\frac{2\theta}{r}+\frac{\cot\theta}{r}$$
$$f\nabla^2 g+g\nabla^2 f+2\nabla f\cdot\nabla g=r\frac{\cot\theta}{r^2}+\theta\frac{2}{r}+2\hat{r}\cdot\hat{\theta}=\frac{2\theta}{r}+\frac{\cot\theta}{r}$$

Hence, $\nabla^2 fg=f\nabla^2 g+g\nabla^2 f+2\nabla f\cdot\nabla g$ is verified in spherical coordinates.

Question-8(b) Reduce the differential equation

$$xp^2-2yp+x+2y=0,\qquad \left(p=\frac{dy}{dx}\right)$$

to Clairaut's form and obtain its complete primitive. Also, determine a singular solution of the given differential equation.

[15 Marks]

Solution: Put $u=x^2$, $v=y-x$ in the given D.E.

$$\text{Let } P=\frac{dv}{du}=\frac{dv}{dx}\cdot\frac{dx}{du}=(p-1)\frac{1}{2x}$$
$$\Rightarrow p=1+2xP$$

using it in given $D \cdot E$, we get

$$\begin{aligned}
&x(1+2xP)^2 - 2y(1+2xP) + x + 2y = 0 \\
\Rightarrow &x\left(1+4x^2P^2+4xP\right) - 2y - 4xyP + x + 3y = 0 \\
\Rightarrow &1 + 4x^2P^2 + 4xP - 4yP + 1 = 0 \\
\Rightarrow &2 + 4uP^2 - 4Pv = 0 \\
\Rightarrow &v = Pu + \frac{1}{2P}
\end{aligned}$$

Which is in Clairaut's form, $y = px + f(p)$. To obtain complete primitive, we replace P with constant c.

$$\therefore \quad v = cu + \frac{1}{2c}$$

$$\text{i.e. } y - x = cx^2 + \frac{1}{2c} \Rightarrow 2c(y-x) = 2c^2x^2 + 1$$

For singular solution, p-discriminant =c-discriminant

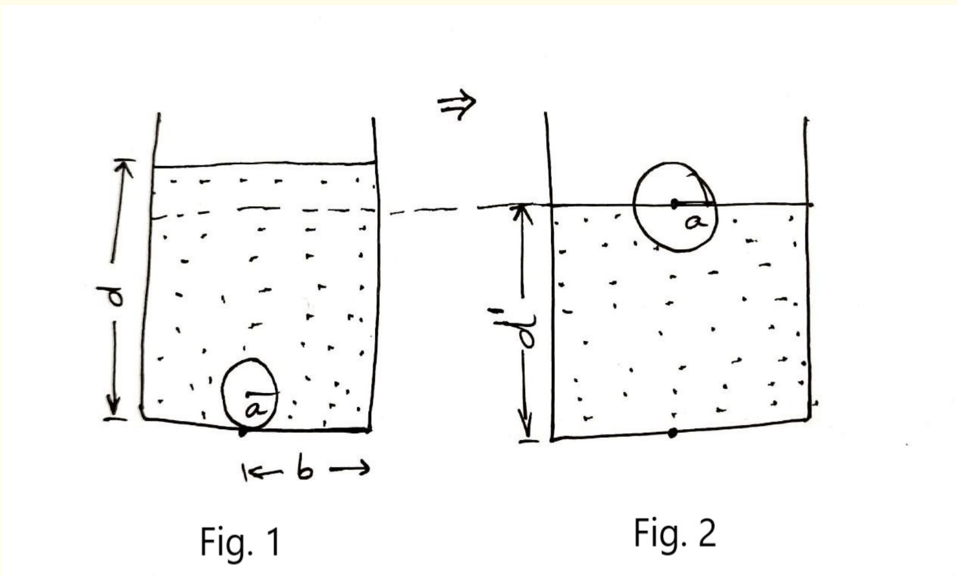
$$\begin{aligned}
\text{i.e.,} \quad & (-2y)^2 - 4x(x+2y) = 0 \\
\Rightarrow \quad & \quad y^2 - x^2 - 2xy = 0
\end{aligned}$$

Question-8(c) A sphere of radius $'a'$, and having density half of that of water, is completely immersed at the bottom of a circular cylinder of radius 'b', which is filled with water to depth 'd'. The sphere is set free and takes up its position of equilibrium. Show that the loss of potential energy this way is

$$\mathrm{W}\left(\mathrm{d} - \frac{11}{8}\mathrm{a} - \frac{\mathrm{a}^3}{3\ \mathrm{b}^2}\right)$$

where W is the weight of the sphere.

[15 Marks]



Solution:

Let d' be new height then:

Clearly for floating, height of submerged part of fluid displaced = weight of body

$$\Rightarrow \quad V_1\rho_1 g = V_2\rho_2 g$$

$$V_1\rho g = V_2\tfrac{\rho}{2}g \Rightarrow V_1 = \tfrac{1}{2}V_2$$

$\Rightarrow \frac{1}{2}$of sphere submereged in equilibrium as shown .
Now total vol. of sphere + water $= \pi b^2 d$ and it also equals

$$\pi b^2 d' + \frac{2}{3}\pi a^3$$

$$\Rightarrow \quad \pi b^2 d = \pi b^2 d' + \frac{2}{3}\pi a^3$$

$$\Rightarrow d = d' + \frac{2}{3}\frac{a^3}{b^2}$$

Now,

$$(\triangle PE)_{\text{sphere}} = w\left(d' - a\right)$$

$$= w\left(d - \frac{2a^3}{3b^2} - a\right)$$

, where w is weight of sphere.

Clearly for $(\triangle PE)_{\text{water}}$, water has moved from (α) and (β) in $fig.1$ to (α) and (β) in $fig.2$ leading to decerease in water level d' and enough space for sphere to keep floating in vacant space (β)

$$(\Delta PE) \text{ water due to } (\beta) = w_1'\left[\left(a - \frac{3a}{8}\right) - \left(d' - \frac{3a}{8}\right)\right]$$

$$= w_1'\left(a - d'\right)$$

$$= w\left(a - d'\right)$$

$$= w(a - d' + \frac{2}{3}\frac{a^3}{b^2})$$

[with w_1' = weight of the water in volume shaded by $(\beta) = \frac{1}{2}$ weight of a sphere of water of radius $'a' = \frac{1}{2}\cdot\left(\frac{4}{3}\pi \mathrm{a}^3\,\rho\cdot g\right) = \frac{1}{2}\left(\frac{4}{3}\pi a^3\frac{\rho}{2}g\right)\cdot 2\ \left(\because w = \frac{4}{3}\pi a^3\cdot\frac{\rho}{2}\cdot g\right) = \frac{1}{2}(w)\cdot 2 = w$]
Now, $(\triangle PE)_{water}$ due to $(x) = w_2'\left[\left(a + \frac{3a}{8}\right) - \frac{d+d'}{2}\right]$ but $w_1' + w_2'$ = weight of a sphere of water of radius a

$$= \frac{4}{3}\pi a^3\rho g = \left(\frac{4}{3}\pi \times a^3\frac{\rho}{2}\cdot g\right)\cdot 2 = 2w$$

and $w_1' = w$ as shown already.

$$\Rightarrow w_2' = w$$

$\therefore (\triangle PE)_{water}$ due to $(\beta) = w\left[\frac{11a}{8} - \frac{d+d'}{2}\right]$.
$\therefore$ Total $\triangle PE = (\Delta PE)_{\text{sphere}} + (\triangle PE)_{\text{water}}$ due to α and β

$$= w\left(d - \frac{2a^3}{3b^2} - a + a - d + \frac{2a^3}{3b^2} + \frac{11a}{8} - \frac{d+d'}{2}\right)$$

$$= w\left(\frac{11a}{8} - \frac{d+d'}{2}\right)$$

$$\begin{aligned}
\therefore \text{ Loss of PE} &= -\text{ Change in } PE \\
&= \omega\left(\frac{d+d'}{2} - \frac{11a}{8}\right) \\
&= \omega\left(\frac{d}{2} + \frac{d}{2} - \frac{1}{3}\frac{a^3}{b^2} - \frac{11a}{8}\right) \\
&= \omega\left(d - \frac{11a}{8} - \frac{a^3}{3b^2}\right)
\end{aligned}$$