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Date 12/10/20

A CONSOLIDATED QUESTION PAPER-CUM-ANSWER BOOKLET

IMS

INSTITUTE OF MATHEMATICAL SCIENCES

MAINS TEST SERIES-2020

(JULY to DEC.-2020)

IAS/IFoS

MATHEMATICS

187/250

Under the guidance of K. Venkanna

FULL SYLLABUS (PAPER-I)

TEST CODE: TEST-5: IAS(M)/09-AUG.-2020

Time: 3 Hours

Maximum Marks: 250

INSTRUCTIONS

1. This question paper-cum-answer booklet has 50 pages and has 37 PART/SUBPART questions. Please ensure that the copy of the question paper-cum-answer booklet you have received contains all the questions.
2. Write your Name, Roll Number, Name of the Test Centre and Medium in the appropriate space provided on the right side.
3. A consolidated Question Paper-cum-Answer Booklet, having space below each part/sub part of a question shall be provided to them for writing the answers. Candidates shall be required to attempt answer to the part/sub-part of a question strictly within the pre-defined space. Any attempt outside the pre-defined space shall not be evaluated.
4. Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
5. Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any THREE of the remaining questions selecting at least ONE question from each Section.
6. The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
7. Symbols/notations carry their usual meanings, unless otherwise indicated.
8. All questions carry equal marks.
9. All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
10. All rough work should be done in the space provided and scored out finally.
11. The candidate should respect the instructions given by the invigilator.
12. The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any page from this booklet.

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name: Divyanshu Choudhary

Roll No.

Test Centre: Jaipur

Medium: English

Do not write your Roll Number or Name anywhere else in this Question Paper-cum-Answer Booklet.

I have read all the instructions and shall abide by them

Signature of the Candidate
Divyanshu

I have verified the information filled by the candidate above

Signature of the invigilator

IMPORTANT NOTE:

Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed. Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

P.T.O.

INDEX TABLE

QUESTION	No.	PAGE NO.	MAX. MARKS	MARKS OBTAINED	
1	(a)			05	
	(b)			08	
	(c)			08	
	(d)			08	
	(e)			08	37
2	(a)			10	
	(b)			05	
	(c)			13	
	(d)			13	41
3	(a)				
	(b)				
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4	(a)				
	(b)				
	(c)				
	(d)				
5	(a)			03	
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6	(a)			10	
	(b)			08	
	(c)			10	
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	(b)				
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8	(a)			14	
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	(c)			08	
	(d)			13	41
			Total Marks		

SECTION – A

1. (a) Find a basis for a subspace U of V in the following

(i) $U = \left\{(x_1, x_2, x_3, x_4, x_5) \in V_5 \,/\, \begin{matrix} x_1 + x_2 + x_3 = 0 \\ 3x_1 - x_4 + 7x_5 = 0 \end{matrix}\right\}, V = V_5$

(ii) $U = \{p \in \rho_4 \,/\, p(x_0) = 0\}$, $V = \rho_4$, where ρ_4 is the set of all polynomials of degree ≤ 4.

[10]

(i) We have

$$U = \begin{Bmatrix} x_1 + x_2 + x_3 = 0 \\ 3x_1 - x_4 + 7x_5 = 0 \end{Bmatrix}$$

Converting into matrix form & then RRE, we get

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 3 & 0 & 0 & -1 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 0 & 3 & 3 & 1 & 7 \end{bmatrix} (R_2 \to 3R_1 - R_2)$$

$\Rightarrow x_1 + x_2 + x_3 = 0$ & $3x_2 + 3x_3 + x_4 + 7x_5 = 0$

$\Rightarrow x_1 = -x_2 - x_3$ & $x_4 = -3x_2 - 3x_3 - 7x_5$

$\Rightarrow U = (-x_2 - x_3, x_2, x_3, -3x_2 - 3x_3 - 7x_5, x_5)$

$\Rightarrow U = x_2(-1, 1, 0, -3, 0) + x_3(-1, 0, 1, -3, 0) + x_5(0, 0, 0, -7, 1)$

So basis of U is $[(-1, 1, 0, -3, 0), (-1, 0, 1, -3, 0), (0, 0, 0, -7, 1)]$

– 05 –

(ii) We have

$$U = \{p \in \rho_4 \mid p(x_0) = 0\}$$

$\Rightarrow a_0 + a_1 x_0 + a_2 x_0^2 + a_3 x_0^3 + a_4 x_0^4 = 0 \;\forall\; a_i \in R$

So basis of U is $\{1, x_0, x_0^2, x_0^3, x_0^4\}$ or $\{1, t_0, t_0^2, t_0^3, t_0^4\}$ where $a_0 + a_1 t_0 + a_2 t_0^2 + a_3 t_0^3 + a_4 t_0^4 = 0$ & t_0 is a general term

1. (b) If $A = \begin{bmatrix} 1 & 0 & 0 \\ i & \dfrac{-1+i\sqrt{3}}{2} & 0 \\ 0 & 1+2i & \dfrac{-1-\sqrt{3}\,i}{2} \end{bmatrix}$ then find trace of A^{102}.

08

[10]

We have

$$A = \begin{bmatrix} 1 & 0 & 0 \\ i & \omega & 0 \\ 0 & 1+2i & \omega^2 \end{bmatrix} \text{ where } \omega = \frac{-1+\sqrt{3}i}{2} \;\&\; \omega^2 = \frac{-1-\sqrt{3}i}{2}$$

$\Rightarrow$ Trace of $A = 1 + \omega + \omega^2 = 0$

$\Rightarrow$ Trace of $A^{102} = 1^{102} + \omega^{102} + (\omega^2)^{102}$

$= 1 + 1 + 1 = 3$ [$\because$ In Trace, same number will get multiplied 102 times]

$\therefore$ Trace of $A^{102} = 3$

1. (c) Show that

$$\int_0^{\pi/2} \log\left(\alpha^2 \cos^2\theta + \beta^2 \sin^2\theta\right) d\theta = \pi \log \frac{\alpha+\beta}{2}.$$

[10]

Let $I = \int_0^{\pi/2} \log(\alpha^2 \cos^2\theta + \beta^2 \sin^2\theta)\, d\theta$ — (1)

$\Rightarrow I = \int_0^{\pi/2} \log\left[\alpha^2 \cos^2(\pi/2 - \theta) + \beta^2 \sin^2(\pi/2-\theta)\right] d\theta$

$\Rightarrow I = \int_0^{\pi/2} \log(\alpha^2 \sin^2\theta + \beta^2 \cos^2\theta)\, d\theta$ — (2)

Adding (1) & (2), we get

$I = \frac{1}{2}\int_0^{\pi/2} \log\left[(\alpha^2\cos^2\theta + \beta^2\sin^2\theta)(\alpha^2\sin^2\theta + \beta^2\cos^2\theta)\right] d\theta$

$\Rightarrow I = \frac{1}{2}\int_0^{\pi/2} \log\left[(\alpha^4+\beta^4)\sin^2\theta\cos^2\theta + (\alpha^2\beta^2)(\sin^4\theta + \cos^4\theta)\right] d\theta$

$$\Rightarrow I = \frac{1}{2}\int_0^{\pi/2} \log\left[(\alpha^4+\beta^4)\sin^2\theta\cos^2\theta + \alpha^2\beta^2(1-2\sin^2\theta\cos^2\theta)\right] d\theta$$

$$\Rightarrow I = \frac{1}{2}\int_0^{\pi/2} \log\left[(\alpha^2-\beta^2)^2\sin^2\theta\cos^2\theta + \alpha^2\beta^2\right] d\theta$$

$$\Rightarrow I = \frac{1}{2}\left[2\pi\log(\alpha+\beta) - 2\pi\log 2\right]$$

$$\left[\because \int_0^{\pi/2}\log(\sin^2\theta\cos^2\theta)\,d\theta = 2\log\sin\theta + 2\log\cos\theta = -\pi\log 2 - \pi\log 2 = -2\pi\log 2\right]\left[\because \int_0^{\pi/2}\log\sin\theta\,d\theta = -\frac{\pi}{2}\log 2\right]$$

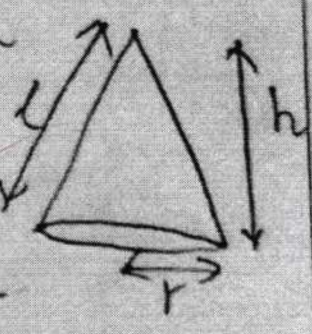
$$\Rightarrow I = \pi\log(\alpha+\beta) - \pi\log 2$$

$$= \boxed{\pi\log\left(\frac{\alpha+\beta}{2}\right)}$$ Hence proved

1. (d) Prove that a conical tent of a given capacity will require the least amount of canvas when the height is $\sqrt{2}$ times the radius of the base. **[10]**

We have

Let Volume (V) of tent $= \frac{1}{3}\pi r^2 h$ — ①



And amount of canvas required $S = \pi r l = \pi r\sqrt{r^2+h^2}$

From ①, we get

$$S = \pi r\sqrt{r^2+\left(\frac{3V}{\pi r^2}\right)^2} \Rightarrow S^2 = \pi^2 r^2\left[r^2+\frac{9V^2}{\pi^2 r^4}\right]$$

For greatest S, we put $\frac{dS}{dr} = 0$

$$\Rightarrow S = \pi^2 r^4 + \frac{9V^2}{r^2}$$

$$\Rightarrow \frac{dS}{dr} = 4\pi^2 r^3 - \frac{18V^2}{r^3} = 0 \Rightarrow 2\pi^2 r^3 = \frac{9V^2}{r^3}$$

$$\Rightarrow r^6 = \frac{9}{2}\frac{V^2}{\pi^2}$$

From ①, we get

$$r^6 = \frac{\cancel{9}}{2} \times \frac{1}{\cancel{\pi^2}} \times \frac{1}{\cancel{9}} \cancel{\pi^2} r^4 h^2$$

$$\Rightarrow r^2 = \frac{h^2}{2} \Rightarrow r = \frac{h}{\sqrt{2}} \Rightarrow h = \sqrt{2}r$$

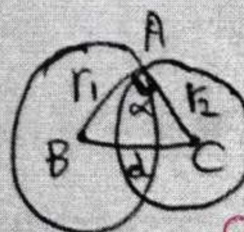
So height of conical tent should be $\sqrt{2}$ times radius of its base

08

1. (e) If d be the distance between the centres of two spheres of radii r_1 and r_2, prove that the angle between them is
$\cos^{-1}\left[(r_1^2 + r_2^2 - d^2)/2r_1 r_2\right]$.

Hence find the angle of intersection of the sphere $x^2 + y^2 + z^2 - 2x - 4y - 6z + 10 = 0$ with the sphere, the extremities of whose diameter are $(1, 2, -3)$ and $(5, 0, 1)$. **[10]**

We have,



Radii of 2 spheres $= r_1$ & r_2

& distance b/w centres $= d$

08

Let α be angle b/w them

$\Rightarrow$ By applying cosine formula in ΔABC, we get

$$\cos\alpha = \frac{r_1^2 + r_2^2 - d^2}{2r_1 r_2} \Rightarrow \alpha = \cos^{-1}\left[\frac{(r_1^2 + r_2^2 - d^2)}{2r_1 r_2}\right]$$

Now, equation of second sphere is

$(x-1)(x-5) + (y-2)y + (z+3)(z-1) = 0$

$\Rightarrow x^2+y^2+z^2-6x-2y+2z+2=0$

& other sphere is $x^2+y^2+z^2-2x-4y-6z+10=0$

Here $r_1 = 3$ & $r_2 = 2$

& $d = \sqrt{(3-1)^2+(1-2)^2+(-1-3)^2} = \sqrt{21}$

$\Rightarrow \alpha = \cos^{-1}\left[\dfrac{9+4-21}{12}\right] = \cos^{-1}\left(\dfrac{-2}{3}\right)$

2. (a) (i) Determine whether or not $v = (3,9,-4,-2)$ in $\mathbf{R}^4$ is a linear combination of $u_1=(1,-2,0,3)$, $u_2=(2,3,0,-1)$, and $u_3=(2,-1,2,1)$, that is, whether or not $v \in$ span (u_1, u_2, u_3).

(ii) Find conditions on a, b and c so that $(a, b, c) \in \mathbf{R}^3$ belongs to the space spanned by $u = (2,1,0)$, $v = (1,-1,2)$ and $w = (0,3,-4)$. **[12]**

(i) We have

$u_1 = (1,-2,0,3)$, $u_2 = (2,3,0,1)$ & $u_3 = (2,-1,2,1)$

$\Rightarrow v = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3$ if $v \in$ span(u_1, u_2, u_3)

$\Rightarrow (3,9,-4,-2) = \alpha_1(1,-2,0,3) + \alpha_2(2,3,0,-1) + \alpha_3(2,-1,2,1)$

$\Rightarrow \alpha_1 + 2\alpha_2 + 2\alpha_3 = 3$ — (1)

$-2\alpha_1 + 3\alpha_2 - \alpha_3 = 9$ — (2)

$2\alpha_3 = -4$ — (3)

$3\alpha_1 - \alpha_2 + \alpha_3 = -2$ — (4)

From (3), $\alpha_3 = -2$

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Substituting in ① & ②, we get

$\alpha_1 + 2\alpha_2 = 7$ & $-2\alpha_1 + 3\alpha_2 = 7$

$\Rightarrow \alpha_1 = 1$ & $\alpha_2 = 3$, $\alpha_3 = -2$

Substituting in ④, we get $0 = 0$

$v \in \text{span}(u_1, u_2, u_3)$ where

$v = u_1 + 3u_2 - 2u_3$

10

ii) We have

$u = (2,1,0)$, $v = (1,-1,2)$, $w = (0,3,-4)$

& $(a,b,c) \in \text{span}(u,v,w)$

So to find the required condition, we construct the following matrix & reduce it to RRE form

$$\Rightarrow \begin{bmatrix} 2 & 1 & 0 \\ 1 & -1 & 2 \\ 0 & 3 & -4 \\ a & b & c \end{bmatrix} = \begin{bmatrix} 1 & 1/2 & 0 \\ 1 & -1 & 2 \\ 0 & 3 & -4 \\ a & b & c \end{bmatrix} (R_1 \to R_1/2)$$

$$\Rightarrow \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & -3/2 & 2 \\ 0 & 3 & -4 \\ 0 & b - a/2 & c \end{bmatrix} \begin{pmatrix} R_2 \to R_2 - R_1 \\ R_4 \to R_4 - aR_1 \end{pmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & -3/2 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & c + \frac{2}{3}(2b-a) \end{bmatrix} \begin{pmatrix} R_3 \to R_3 + 2R_2 \\ R_4 \to R_4 + \frac{2b-a}{3} R_2 \end{pmatrix}$$

$\Rightarrow c + 2/3(2b - a) = 0 \Rightarrow \boxed{2a = 3c + 4b}$ is the required condition

(b) Determine whether the following matrices have the same row space

$$A=\begin{pmatrix}1&1&5\\2&3&13\end{pmatrix}, B=\begin{pmatrix}1&-1&-2\\3&-2&-3\end{pmatrix}, C=\begin{pmatrix}1&-1&-1\\4&-3&-1\\3&-1&3\end{pmatrix}$$ [07]

We have

$$A=\begin{pmatrix}1&1&5\\2&3&13\end{pmatrix} \quad B=\begin{pmatrix}1&-1&-2\\3&-2&-3\end{pmatrix}$$

$$C=\begin{pmatrix}1&-1&-1\\4&-3&-1\\3&-1&3\end{pmatrix}$$

We reduce all these to RRE form to check if they have same row space

$$\Rightarrow A=\begin{pmatrix}1&1&5\\2&3&13\end{pmatrix}=\begin{pmatrix}1&1&5\\0&1&3\end{pmatrix}(R_2\to R_2-2R_1) \sim \begin{pmatrix}1&0&2\\0&1&3\end{pmatrix}$$

$$\Rightarrow B=\begin{pmatrix}1&-1&-2\\3&-2&-3\end{pmatrix}=\begin{pmatrix}1&-1&-2\\0&1&3\end{pmatrix}(R_2\to R_2-3R_1)$$

$$\Rightarrow C=\begin{pmatrix}1&-1&-1\\4&-3&-1\\3&-1&3\end{pmatrix}=\begin{pmatrix}1&-1&-1\\0&1&3\\0&2&6\end{pmatrix}\begin{pmatrix}R_2\to R_2-4R_1\\R_3\to R_3-3R_1\end{pmatrix}$$

$$=\begin{pmatrix}1&-1&-1\\0&1&3\\0&0&0\end{pmatrix}(R_3\to R_3-2R_2) \sim \begin{pmatrix}1&0&2\\0&1&3\end{pmatrix}$$

Since ~~reduced~~ RRE forms are not same, $\therefore$ A, B, C do ~~not~~ have the same row space.

A & C

(c) Find the maximum and minimum values of $x^2+y^2+z^2$ subject to the conditions $\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}=1$ and $x+y-z=0$. **[15]**

We have

$$\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}=1 \text{ \& } x+y-z=0$$

We have to find max. & min value of

$x^2+y^2+z^2 = F$ (say)

So by Lagrange multiplier condition

$$\Rightarrow F+\lambda\left(\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}-1\right)+\mu(x+y-z)=0 \quad \text{--- (1)}$$

Differentiating (1) wrt x, y, z we get

$$2x+\lambda\left(\frac{2x}{4}\right)+\mu=0 \quad \text{--- (2)}$$

$$2y+\lambda\left(\frac{2y}{5}\right)+\mu=0 \quad \text{--- (3)}$$

$$2z+\lambda\left(\frac{2z}{25}\right)-\mu=0 \quad \text{--- (4)}$$

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Multiplying (2), (3), (4) by x, y, z respectively & adding, we get

$$2(x^2+y^2+z^2)+2\lambda(1)+\mu(0)=0$$

$$\Rightarrow F+\lambda=0 \Rightarrow \lambda=-F \quad \text{--- (5)}$$

Substituting values of x, y, z from (2), (3), (4) in $x+y-z=0$, we get

$$-\frac{\mu}{2}\left(\frac{4}{4+\lambda}\right) - \frac{\mu}{2}\left(\frac{5}{5+\lambda}\right) - \frac{\mu}{2}\left(\frac{25}{25+\lambda}\right) = 0$$

$$\Rightarrow \frac{4}{4+\lambda} + \frac{5}{5+\lambda} + \frac{25}{25+\lambda} = 0 \quad [\because \mu \neq 0]$$

$$\Rightarrow 4\left[\lambda^2+125+30\lambda\right] + 5\left[\lambda^2+29\lambda+100\right] + 25\left[\lambda^2+9\lambda+20\right] = 0$$

$$\Rightarrow \cancel{39\lambda^2}\ 34\lambda^2 + 490\lambda + 1500 = 0$$

$$\Rightarrow \lambda = -10,\ -75/17$$

$$\Rightarrow F = -\lambda \text{ (from ③)} \Rightarrow F = 10,\ 75/17$$

So max value of F is 10 and min value of F is 75/17

2. (d) (i) A variable plane is at a constant distance p from the origin and meets the axes in A, B and C. Show that the locus of the centroid of the tetrahedron OABC is $x^{-2} + y^{-2} + z^{-2} = 16p^{-2}$.

(ii) If $x/1 = y/2 = z/3$ represent one of a set of three mutually perpendicular generators of the cone $5yz - 8zx - 3xy = 0$, find the equations of the other two. **[15]**

(i) Let the plane ~~by~~ be given by

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \quad \text{where } A=a,\ B=b,\ C=c$$

Now by given condition

$$p = \frac{|-1|}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}} \quad \text{(distance from origin)}$$

$$\Rightarrow \frac{1}{p^2} = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \quad \text{—— ①}$$

Now coordinates of centroid of tetrahedron be (x, y, z) $\Rightarrow (x, y, z) = (a/4, b/4, c/4)$ ②

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Substituting ② in ①, we get

$$\frac{1}{p^2} = \frac{1}{(4x)^2} + \frac{1}{(4y)^2} + \frac{1}{(4z)^2} \Rightarrow \frac{16}{p^2} = \frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}$$

$$\Rightarrow 16p^{-2} = x^{-2} + y^{-2} + z^{-2}$$

ii) We have

$\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ is one of the generators

Let l, m, n be drs of other 2 generators

$\Rightarrow$ By condition of mutually perpendicular

$$\Rightarrow l + 2m + 3n = 0 \quad \text{—— ①}$$

$$\Rightarrow 5mn - 8ln - 3lm = 0 \quad \text{—— ②}$$

Substituting l from ① in ②, we get

$$\Rightarrow 5mn - 8n(-2m-3n) - 3m(-2m-3n) = 0$$

$$\Rightarrow 6m^2 + 24n^2 + 30mn = 0$$

$$\Rightarrow m^2 + 5mn + 4n^2 = 0 \Rightarrow (m+4n)(m+n) = 0$$

$$\Rightarrow m = -n \quad \text{—— ③} \quad \& \quad m = -4n \quad \text{—— ④}$$

Substituting these values in ①, we get

$l = \cancel{-2m}\ -n$ & $l = 5n$

So other 2 systems are $\frac{x}{1} = \frac{y}{1} = \frac{z}{-1}$

& $x/5 = y/-4 = \cancel{z/3}\ z/1$

SECTION - B

5\. (a) Find the differential equation of the family of circles $x^2 + y^2 + 2cx + 2c^2 - 1 = 0$ (c arbitrary constant). Determine singular solution of the differential equation. **[10]**

We have

Given family of circles is $x^2 + y^2 + 2cx + 2c^2 - 1 = 0$

$\Rightarrow$ Differentiating w.r.t. x, we get

$\cancel{2x\frac{dy}{dx}}\ 2x + 2yy' + 2c = 0$

Again differentiating, we get

$2 + 2(y')^2 + 2yy'' = 0 \Rightarrow y + y$

$1 + \left(\frac{dy}{dx}\right)^2 + \left(\frac{dy}{dx}\right)\left(\frac{d^2y}{dx^2}\right) = 0$ is the required differential eq^n

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For determining singular solution, we find c discriminant

$\Rightarrow (2x)^2 - 4(2)(x^2+y^2 \mp 1) = 0$

$\Rightarrow x^2 - 2(x^2+y^2 \mp 1) = 0$

$\Rightarrow x^2 + 2y^2 \mp 2 = 0$ is the required singular solution of given family of circles

-03

5. (b) Solve $(D^2 + 1) y = x^2 \sin 2x$. [10]

We have

$(D^2+1)\, y = x^2 \sin 2x$

We find CF & P.I of given equation

For finding CF $\Rightarrow D^2+1=0 \Rightarrow D = \pm i$

$\Rightarrow CF = c_1 \cos x + c_2 \sin x$

-04

Now for finding PI

$\Rightarrow PI = \frac{1}{D^2+1} x^2 \sin 2x \Rightarrow$ Imaginary part of $\frac{1}{D^2+1} x^2 e^{i2x} \Rightarrow$ Im of $e^{i2x} \frac{1}{(D+2i)^2+1} x^2$

$\Rightarrow$ Im of $e^{2ix}\dfrac{1}{D^2+2Di-3}x^2 \Rightarrow$ Im of $\dfrac{e^{i2x}}{-3}\left(1-\dfrac{D^2}{3}-\dfrac{2Di}{3}\right)^{-1}x^2 \Rightarrow$ Im of $\dfrac{e^{i2x}}{-3}\times$

$\left[1+\dfrac{D^2}{3}+\dfrac{2Di}{3}-\dfrac{4D^2}{9}+\cdots\right]x^2 \Rightarrow$ Im of

$\dfrac{e^{2ix}}{-3}\left[1+\dfrac{2Di}{3}-\dfrac{D^2}{9}\right]x^2 \Rightarrow$ Im of $\dfrac{e^{i2x}}{-3}\times$

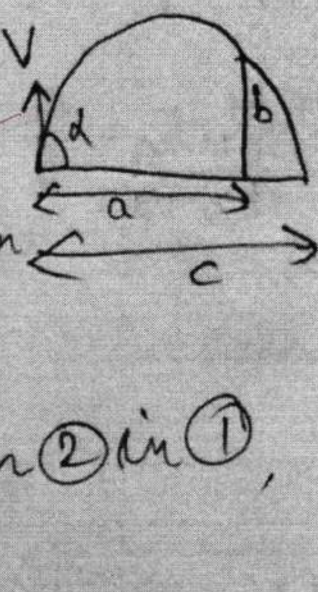
$\left(x^2+\tfrac{4}{3}ix-\tfrac{2}{9}\right) = -\tfrac{1}{3}\left(\tfrac{4}{3}x\cos 2x+x^2\sin 2x - \tfrac{2}{9}\sin 2x\right)$

$\Rightarrow y = CF + PI = C_1\cos x + C_2\sin x - \tfrac{4}{9}x\cos 2x + \sin 2x\left(\tfrac{2}{27}-\tfrac{1}{3}x^2\right)$

5. (c) A particle just clears a wall of height b at a distance a and strikes the ground at a distance c from the point of projection. Prove that the angle of projection is $\tan^{-1}\left\{\dfrac{bc}{a(c-a)}\right\}$, and the velocity of projection V is given by $\dfrac{2V^2}{g}=\dfrac{a^2(c-a)^2+b^2c^2}{ab(c-a)}$. [10]

By equation of trajectory, we have

$y = x\tan\alpha - \dfrac{gx^2}{2V^2\cos^2\alpha}$ — ①



Here c is Range of projection

$\Rightarrow c = \dfrac{V^2\sin 2\alpha}{g}$ — ②

Substituting value of V^2 from ② in ①,

$\Rightarrow y = x\tan\alpha\left(1-\dfrac{x}{c}\right)$

$\Rightarrow b = a\tan\alpha\left(1-\dfrac{a}{c}\right) \Rightarrow bc = (c-a)a\tan\alpha$

$\Rightarrow \tan\alpha = \frac{bc}{a(c-a)} \Rightarrow \alpha = \tan^{-1}\left[\frac{bc}{a(c-a)}\right]$ — ③

Substituting value of ③ in ①, we get

$$b = a\left[\frac{bc}{a(c-a)}\right] - \frac{ga^2}{2V^2}\left[1+\left(\frac{bc}{a(c-a)}\right)^2\right]$$

$$\Rightarrow b = \frac{bc}{c-a} - \frac{g}{2V^2}\left[\frac{a^2(c-a)^2+b^2c^2}{(c-a)^2}\right]$$

$$\Rightarrow \frac{ab}{c-a} = \frac{g}{2V^2}\left[\frac{a^2(c-a)^2+b^2c^2}{(c-a)^2}\right]$$

$$\Rightarrow \boxed{\frac{2V^2}{g} = \frac{a^2(c-a)^2+b^2c^2}{ab(c-a)}}$$

Hence proved

(d) Find the directional derivative of $f = x^2 y z^3$ along $x = e^{-t}$, $y = 1 + 2\sin t$, $z = t - \cos t$ at $t = 0$ **[10]**

We have

$f = x^2 y z^3$

Directional derivative of $f = \nabla \cdot f$

$\Rightarrow \nabla \cdot f = 2xyz^3\hat{i} + x^2z^3\hat{j} + 3x^2yz^2\hat{k}$ — ①

Now $x = e^{-t}$, $y = 1+2\sin t$ & $z = t - \cos t$

At $t=0$

$\Rightarrow x = 1$, $y = 1$ & $z = -1$ — ②

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Substituting these values in ①, we get

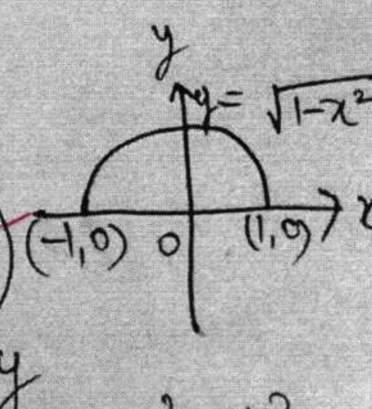
$\nabla \cdot f = -2\hat{i} - \hat{j} + 3\hat{k}$

So directional derivative of f along given coordinates is $-2\hat{i} - \hat{j} + 3\hat{k}$

?

5. (e) Apply Green's theorem in the plane to evaluate $\int_C \left[\left(2x^2 - y^2\right)dx + \left(x^2 + y^2\right)dy\right]$, where C is the boundary of the surface enclosed by the x-axis and the semi-circle $y = (1 - x^2)^{1/2}$. **[10]**

By Green's Theorem,



$$\int_C M\,dx + N\,dy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

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$\Rightarrow$ Here $M = 2x^2 - y^2$ & $N = x^2 + y^2$

Let $I = \int_C (2x^2 - y^2)\,dx + (x^2 + y^2)\,dy$

$\Rightarrow I = \iint (2x + 2y)\,dx\,dy$

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$$\Rightarrow I = \int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} (2x+2y)\, dy\, dx$$

$$\Rightarrow I = \int_{-1}^{1} \left(2x\sqrt{1-x^2} + 1 - x^2\right) dx$$

$$\Rightarrow I = \left(-\frac{2}{3}(1-x^2)^{3/2}\right)_{-1}^{1} + \left(x - \frac{x^3}{3}\right)_{-1}^{1}$$

$$\Rightarrow I = 1 - \frac{1}{3} + 1 - \frac{1}{3} = \frac{4}{3}$$

So value of given integration is $\frac{4}{3}$

6(a) (i) Obtain Laplace Inverse transform of

$$\left\{\ln\left(1+\frac{1}{s^2}\right)+\frac{s}{s^2+25}\right\}e^{-\pi s}.$$

(ii) Solve $(1+y^2)+\left(x-e^{\tan^{-1}y}\right)\frac{dy}{dx}=0.$ **[13]**

We have to find

$$L^{-1}\left[\ln\left(1+\frac{1}{s^2}\right) + \frac{s}{s^2+25}e^{-\pi s}\right]$$

$$\Rightarrow L^{-1}\ln\left(1+\frac{1}{s^2}\right) + L^{-1}\left(\frac{s}{s^2+25}\right)e^{-\pi s}$$

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$$\Rightarrow L^{-1}\ln\left(1+\frac{1}{s^2}\right) + u(t-\pi)\cos 5(t-\pi)$$

$$\Rightarrow L^{-1}\ln\left(\frac{s^2+1}{s^2}\right) + u(t-\pi)\cos 5t$$

Now to find $L^{-1}\ln\left(\frac{s^2+1}{s^2}\right) = L^{-1}\left[\ln(s^2+1) - 2\ln s\right]$

$\Rightarrow -t f(t) = \cancel{\frac{dx}{dp}} \frac{d}{ds}\left[\ln(s^2+1) - 2\ln s\right]$

$\Rightarrow -t f(t) = L^{-1}\left[\frac{2s}{s^2+1} - \frac{2}{s}\right] = \cancel{2\sin t}\ 2\cos t - 2$

$\Rightarrow f(t) = \frac{2 - 2\cos(t-\pi)}{(t-\pi)}$

$\Rightarrow$ Total inverse Laplace Transform is

$\frac{2 - 2\cos(t-\pi)}{t-\pi} u(t-\pi) + u(t-\pi)\cos 5t$

(ii) $(1+y^2) + \left(x - e^{-\tan^{-1}y}\right)\frac{dy}{dx} = 0$

$\Rightarrow (1+y^2)\frac{dx}{dy} = e^{-\tan^{-1}y} - x$

$\Rightarrow \frac{dx}{dy} + \frac{x}{1+y^2} = \frac{e^{-\tan^{-1}y}}{1+y^2}$

$\Rightarrow IF = e^{\int \frac{1}{1+y^2}dy} = e^{\tan^{-1}y}$

$\Rightarrow x \cdot e^{\tan^{-1}y} = \int \frac{1}{1+y^2}dy$ (Multiplying & integrating by IF)

$\Rightarrow x e^{\tan^{-1}y} = \tan^{-1}y + c$

$\Rightarrow x = e^{-\tan^{-1}y}\tan^{-1}y + ce^{-\tan^{-1}y}$

(where c is an arbitrary constant)

is the required solution

6 (b) Find the orthogonal trajectories of the system of circles touching a given straight line at a given point. [10]

Let system of circles be given by $(x-a)^2 + y^2 = a^2$.

$\Rightarrow x^2 + y^2 - 2ax = 0$ — ① which touches y axis at $(0,0)$

$\Rightarrow 2x + 2yy' - 2a = 0$ (Differentiating w.r.t x)

Now for finding orthogonal trajectories, we replace y' by $\frac{-1}{y'}$

$x - \frac{y}{y'} - a = 0 \Rightarrow a = x - \frac{y}{y'}$

Substituting value of a from ①, we get

$\frac{x^2+y^2}{2x} = \frac{xy'-y}{y'}$

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$\cancel{x^2 \frac{dy}{dx}}\ (x^2+y^2)\frac{dy}{dx} = 2x^2\frac{dy}{dx} - 2xy$

$\frac{dy}{dx} = \frac{-2xy}{y^2-x^2} = \frac{2xy}{x^2-y^2} = \frac{\frac{2y}{x}}{\cancel{\left(\frac{x}{y}\right)^2 - 1}\ 1-\left(\frac{y}{x}\right)^2}$

$y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$ — ②

$v + x\frac{dv}{dx} = \frac{2v}{1-v^2} \Rightarrow x\frac{dv}{dx} = \frac{v+v^3}{1-v^2}$

Integrating both sides & substituting $v = \frac{y}{x}$

we get $\boxed{x^2 + y^2 = cy}$ (c is a constant)

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$$\Rightarrow \int \frac{dx}{x} = \int \frac{1-v^2}{v(1+v^2)}\, dv$$

$$\Rightarrow \log x = \int \left[\frac{1+3v^2}{v+v^3} - \frac{4v^2}{v+v^3} \right] dv$$

$$\Rightarrow \log x = \log(v+v^3) - 2\log(1+v^2) + \log C$$

Substituting $v = \frac{y}{x}$ from (2), we get

$$\boxed{x^2 + y^2 = Cy} \quad [C \text{ is a constant}]$$

as the desired orthogonal trajectory

(c) Apply the method of variation of parameters to solve

$x^2y_2 + xy_1 - y = x^2 \log x,\ x > 0$ **[12]**

We have

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^2 \log x$$

First we calculate CF of above eq^n

Substituting $x = e^z$, (1) we get

$$\{D_1(D_1 - 1) + D_1 - 1\} = 0$$

$$\Rightarrow D_1^2 - 1 = 0 \Rightarrow D_1 = \pm 1$$

So $y = c_1 e^z + c_2 e^{-z} \Rightarrow c_1 x + c_2 x^{-1}$ (From (1))

Converting above eq^n to standard form, we get $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = \log x$

$$W = \begin{vmatrix} x & 1/x \\ 1 & -1/x^2 \end{vmatrix} = -2/x$$

ere $u = x$ & $v = 1/x$) & $R = \log x$)

$$\Rightarrow f(x) = -\int \frac{vR}{W} = \int \frac{\frac{1}{x}\log x}{2/x} dx = \frac{1}{2}(x\log x - x)$$

$$g(x) = \int \frac{uR}{W} = \int \frac{x \log x}{-2/x} dx = \frac{-1}{2}\int x^2 \log x \, dx$$

$$= \frac{x^3}{18} - \frac{x^3}{6}\log x$$

$$P.I = uf(x) + vg(x) = \frac{1}{2}(x^2\log x - x^2) + \left(\frac{x^2}{18} - \frac{x^2}{6}\log x\right)$$

$$y = c_1 x + c_2 x^{-1} + \cancel{\tfrac{1}{2}x^2(\log x - 1)}\ \frac{x^2}{3}\log x - \frac{4}{9}x^2$$

b(d) e the following initial value problem using Laplace transform:

$+ 9y = r(x), y(0) = 0, y'(0) = 4$

re $r(x) = \begin{cases} 8\sin x & \text{if } 0 < x < \pi \\ 0 & \text{if } x \geq \pi \end{cases}$ **[15]**

have

$$= \begin{cases} 8\sin x, & 0 < x < \pi \\ 0, & x \geq \pi \end{cases}$$

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ying Laplace transform, we get

$$(x)] = \int_0^{\pi} e^{-st}\, 8\sin t \, dt = I$$

$$= \int_0^{\pi} 8\cos t \frac{e^{-st}}{s} dt = \frac{1}{s}\left[\left(\frac{-8}{s}\cos t\, e^{-st}\right)_0^{\pi}\right.$$

$$+ \frac{1}{s^2}) = \frac{8}{s^2}(1 + e^{-\pi s}) - \int 8\sin t \frac{e^{-st}}{s} dt$$

$$\Rightarrow I = \frac{8}{s^2+1}\left(1+e^{-\pi s}\right)$$

Now taking laplace transform both sides, we get

$$s^2 L(y) - 4 + 9L(y) = \frac{8}{s^2+1}\left(1+e^{-\pi s}\right)$$

$$\Rightarrow L(y) = \frac{8}{(s^2+1)(s^2+9)}\left(1+e^{-\pi s}\right) + \frac{4}{s^2+9}$$

$$\Rightarrow L(y) = \frac{8}{(s^2+1)(s^2+9)} + \frac{8e^{-\pi s}}{(s^2+1)(s^2+9)} + \frac{4}{s^2+9}$$

$$\Rightarrow y = L^{-1}\left[\frac{1}{s^2+1} - \frac{1}{s^2+9} + \frac{e^{-\pi s}}{s^2+1} - \frac{e^{-\pi s}}{s^2+9} + \frac{4}{s^2+9}\right]$$

$$\Rightarrow y = \sin x - \frac{1}{3}\sin 3x + u(x-\pi).\sin(x-\pi) - \frac{1}{3}u(x-\pi)\sin 3(x-\pi) + \frac{4}{3}\sin 3x$$

$$\Rightarrow y = \begin{cases} \sin x + \sin 3x & 0 < x < \pi \\ \frac{4}{3}\sin 3x & x \geq \pi \end{cases}$$

8(a) (i) The position vector of a moving point at time t is $\bar{r} = \sin t\,\hat{i} + \cos 2t\,\hat{j} + (t^2 + 2t)\hat{k}$.
Find the components of acceleration $\bar{a}$ in the directions parallel to the velocity vector $\bar{v}$ and perpendicular to the plane of $\bar{r}$ and $\bar{v}$ at time t = 0.

(ii) Prove that vector f(r) **r** is irrotational.

(iii) Prove that curl $(\psi\nabla\phi) = \nabla\psi \times \nabla\phi = -$ curl $(\phi\nabla\psi)$. **[17]**

$$\vec{r} = \sin t\,\hat{i} + \cos 2t\,\hat{j} + (t^2+2t)\,\hat{k}$$
$$= \hat{j} \quad (\text{at } t=0)$$
$$\vec{v} = \frac{d\vec{r}}{dt} = \cos t\,\hat{i} - 2\sin 2t\,\hat{j} + (2t+2)\hat{k}$$
$$= \hat{i} + 2\hat{k} \quad (\text{at } t=0)$$
$$\vec{a} = \frac{d\vec{v}}{dt} = -\sin t\,\hat{i} - 4\cos 2t\,\hat{j} + 2\hat{k}$$
$$= -4\hat{j} + 2\hat{k} \quad (\text{at } t=0)$$

Component of $\vec{a}$ in direction parallel to $\vec{v}$

$$\frac{\vec{a}\cdot\vec{v}}{|\vec{v}|}\,\hat{v} = \cdot\,\frac{\hat{i}+2\hat{k}}{\sqrt{5}} = \frac{4}{5}(\hat{i}+2\hat{k})$$

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Now $\vec{r} \times \vec{v} = \begin{vmatrix} i & j & k \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 2\hat{i} - k$

$\Rightarrow \vec{a}$ component parallel to $\vec{r} \times \vec{v}$ =

$$\frac{\vec{a} \cdot (\vec{r} \times \vec{v})}{|\vec{r} \times \vec{v}|} (\widehat{\vec{r} \times \vec{v}}) = 0$$

2

(ii) We have

$$\text{curl}\,(f(r)\,\vec{r}) = [\text{grad}\, f(r)] \times \vec{r} + f(r)\,\text{curl}\,\vec{r}$$

Since $\text{curl}\,\vec{r} = 0$

$$\Rightarrow \text{curl}\,[f(r)\,\vec{r}] = [\text{grad}\, f(r)] \times \vec{r} =$$

$$f'(r)\,\frac{\vec{r}}{r} \times \vec{r} = 0 \quad [\because \vec{r} \times \vec{r} = 0]$$

$\therefore \text{curl}\,[f(r)\,\vec{r}] = 0 \Rightarrow f(r)\,\vec{r}$ is irrotational

(iii) We have

$$\text{curl}\,(\psi \nabla \phi) = (\text{grad}\,\psi) \times \nabla\phi + \psi\,\text{curl}\,(\nabla\phi)$$

$[\because \text{curl}\,(AB) = \cancel{\text{curl}+}\,(\text{grad}\,A) \times B + A\,\text{curl}\,B]$

Since $\text{curl}\,(\nabla\phi) = 0$

$$\Rightarrow \text{curl}\,(\psi\nabla\phi) = \nabla\psi \times \nabla\phi$$

$$\Rightarrow \nabla\psi \times \nabla\phi = -\nabla\phi \times \nabla\psi = -\text{curl}\,(\phi\nabla\psi)$$

$$\therefore \text{curl}\,(\psi\nabla\phi) = \nabla\psi \times \nabla\phi = -\text{curl}\,(\phi\nabla\psi)$$

(b) Show that F = (sin y + z) **i** + (x cos y − z) **j** + (x − y) **k** is a conservative vector field and find a function φ such that F = ∇φ. [08]

If F is a conservative field, then curl F = 0

$$\Rightarrow \text{curl } F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \sin y + z & x\cos y - z & x - y \end{vmatrix}$$

$$= i[-1+1] + j[1-1] + k[\cos y - \cos y] = 0$$

∴ F is a conservative vector field

Now $F = \nabla\phi$

$$\Rightarrow \frac{\partial\phi}{\partial x} = \sin y + z \quad \text{—(1)}, \quad \frac{\partial\phi}{\partial y} = x\cos y - z \quad \text{—(2)}$$

$$\& \quad \frac{\partial\phi}{\partial z} = x - y \quad \text{—(3)}$$

From (1), (2), (3) we get

$$\phi = \cancel{x\cos y + xz + c_1}, \quad \phi = x\sin y - yz + c_2 \quad \text{—(5)}$$

$$x\sin y + xz + c_1 \quad \text{—(4)}$$

$$\phi = xz - yz + c_3 \quad \text{—(6)}$$

Comparing like terms in (4), (5) & (6), we get

$$\phi = x\sin y + xz - yz + C$$

where C is ~~a~~ an arbitrary constant

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(c) By using divergence theorem evaluate

$\iint_S (a^2x^2 + b^2y^2 + c^2z^2)^{1/2} dS$

over the ellipsoid $ax^2 + by^2 + cz^2 = 1$.

[10]

We have

Let $I = \iint_S (a^2x^2 + b^2y^2 + c^2z^2)^{1/2} dS$

& given ellipsoid is $ax^2 + by^2 + cz^2 = 1$

By Divergence Theorem

$$\iiint \nabla \cdot F \, dV = \iint F \cdot n \, ds$$

Here $F \cdot n = (a^2x^2 + b^2y^2 + c^2z^2)^{1/2}$

& $\hat{n} = \dfrac{2ax\hat{i} + 2by\hat{j} + 2cz\hat{k}}{2\sqrt{a^2x^2 + b^2y^2 + c^2z^2}}$

$\Rightarrow F = ax\hat{i} + by\hat{j} + cz\hat{k}$

$\left[\because (ax\hat{i} + by\hat{j} + cz\hat{k}) \cdot \dfrac{2ax\hat{i} + 2by\hat{j} + 2cz\hat{k}}{2\sqrt{a^2x^2 + b^2y^2 + c^2z^2}} = \sqrt{a^2x^2 + b^2y^2 + c^2z^2} \right]$

08

So by Divergence Theorem

$$I = \iiint \left[\frac{\partial}{\partial x}(ax) + \frac{\partial}{\partial y}(by) + \frac{\partial}{\partial z}(cz) \right] dV$$

$$= \iiint (a+b+c) \, dV = (a+b+c) \iiint dV$$

Since volume of ellipsoid is $\dfrac{4\pi}{3} \dfrac{1}{\sqrt{abc}}$

$$\Rightarrow \boxed{I = \frac{4\pi}{3} \frac{a+b+c}{\sqrt{abc}}}$$

theorem for

$) i + 3xy j + (2xz + z^2) k$

upper half of the sphere $x^2 + y^2 + z^2 = 16$ and C is its boundary.

[15]

e

$y - 4) i + 3xy\, j + (2xz + z^2) k$

e's Theorem

$= \iint \text{curl} F \cdot n\, ds$

e calculate $F \cdot dr$

on surface $z = \cancel{dz = 0}$

$\cdot dr = \int_0^{2\pi} (x^2 + y - 4)\, dx + 3xy\, dy$

$x = 4\cos t$ & $y = 4\sin t$

$dr = \int_0^{2\pi} (16\cos^2 t + 4\sin t - 4)(-4\sin t)dt + 3(\cos t)(\sin t)(64\cos t)dt$

$F \cdot dr = \int_0^{2\pi} (-64\cos^2 t \sin t - 16\sin^2 t + 16\sin t + 192\cos^2 t \sin t)\, dt$

$\int^{2\pi} \cos^2 t \sin t = \int_0^{2\pi} \sin t = 0$

$= -16 \int_0^{2\pi} \sin^2 t\, dt = -16 \int_0^{2\pi} \left(\frac{1 - \cos 2t}{2}\right) dt$

$\times 2\pi = \boxed{-16\pi}$

$F = \begin{vmatrix} \hat{i} & \hat{j} & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 + y - 4 & 3xy & 2xz + z^2 \end{vmatrix} = j(2z) + (3y - 1)k$

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Since $\hat{n} = \hat{k}$

$\Rightarrow \iint (\text{curl } F_0) \cdot n \, ds = \iint (3y - 1) \, ds$

$= -\iint ds$ [$\because$ $3y$ is an odd function of y & hence $\iint 3y \, ds = 0$]

$= -\pi (4)^2 = \boxed{-16\pi}$

So value of given integration is -16π & hence Stokes Theorem is verified

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