2019

1,2

(b) Show that the function $f(x) = \sin\left(\frac{1}{x}\right)$ is continuous and bounded in $(0, 2\pi)$, but it is not uniformly continuous in $(0, 2\pi)$.

(c) Test the Riemann integrability of the function f defined by

$$f(x) = \begin{cases} 0 & \text{when x is rational} \\ 1 & \text{when x is irrational} \end{cases}$$

on the interval [0, 1].

3

(b) Show that the integral $\int_0^{\pi/2} \log \sin x \, dx$ is convergent and hence evaluate it.

4

(b) Show that the sequence $\{\tan^{-1} nx\}$, $x \geq 0$ is uniformly convergent on any interval $[a, b]$, $a > 0$ but is only pointwise convergent on $[0, b]$.

IFoS
2019

Q-1 $f(x) = \sin\left(\frac{1}{x}\right)$ $(0, 2\pi)$

on $(0, 2\pi)$

$g(x) = \frac{1}{x}$, $h(x) = \sin x$

Then $f(x) = h \circ g(x)$

since $g(x)$, $h(x)$ both are continuous function on $(0, 2\pi)$ (being rational and trignometric function.)

so, $f(x)$ being a composition of 2 continuous function is continuous on $(0, 2\pi)$

Also sine function is by naturl bounded between -1 and 1 so,

$-1 \leq \sin\frac{1}{x} \leq 1$ is bounded.

But $\sin\frac{1}{x}$ is not uniform continuous.

Let us take $x_n = \frac{1}{(2n+1)\frac{\pi}{2}}$, $y_n = \frac{1}{2n\pi}$

Then $x_n, y_n \in (0, 2\pi)$ are sequences

Let δ be any number. Choose positive integeer n such that,

$\frac{1}{2n\left(n\pi + \frac{\pi}{2}\right)} < \delta$

ROUGH

consider $|x_1 - x_2| = \left| \frac{1}{n\pi} - \frac{1}{(2n+1)\frac{\pi}{2}} \right|$

$$= \left| \frac{\frac{(2n+1)\pi}{2} - n\pi}{(n\pi)(2n+1)\frac{\pi}{2}} \right| = \left| \frac{1}{\cancel{2}n(2n+1)\frac{\pi}{\cancel{2}}} \right| < \delta$$

$$|f(x_1) - f(x_2)| = \left| \sin (2n+1)\frac{\pi}{2} - \sin n\pi \right|$$

$$= |\cos n\pi - 0| = 1 > \epsilon \text{ if } \epsilon = \frac{1}{2}$$

Thus we have for any $\delta > 0$, there exists $\epsilon = \frac{1}{2}$ such that

$|f(x_1) - f(x_2)| \geq \epsilon$ for $|x_1 - x_2| < \delta$

so $\sin \frac{1}{x}$ is not unif. cts on $(0, \frac{2}{\pi})$

Q. 2 $f(x) = \begin{cases} 0 & , \text{when } x \text{ is rational}, \\ 1 & , \text{when } x \text{ is irrational} \end{cases}$

on $[0,1]$.

$f(x)$ is bounded as $0 \le f(x) \le 1 \; \forall x \in [0,1]$

Let $P = \{0 = x_0, x_1, x_2, \dots x_n = 1\}$ be a partition of $[0,1]$.

Let $I_r = [x_{r-1}, x_r]$; $r = 1, 2, \dots n$ be r^{th} sub-interval of $[0,1]$.

$\therefore M_r = 1, \; m_r = 0 \;$ & so

$$U(P,f) = \sum_{r=1}^{n} M_r \delta_r = \sum_{r=1}^{n} 1 \cdot \delta_r = b-a$$

$$L(P,f) = \sum_{r=1}^{n} m_r \delta_r = \sum_{r=1}^{n} 0 \cdot \delta_r = 0$$

$$\therefore \underline{\int_a^b} f(x)\,dx = \text{lub}\{L(P,f)\} \quad P \in P[a,1]$$
$$= 0$$

and

$$\overline{\int_a^b} f(x)\,dx = \text{glb}\{U(P,f)\} = b-a$$

$$\underline{\int_a^b} f(x)\,dx \ne \overline{\int_a^b} f(x)\,dx$$

$\therefore f$ is not Riemann integrable on $[0,1]$.

given integral is proper and
nce it is convergent.

n $n<0$:

hatever 'm' may be,
is the only point of infinite
scontinuity.

Let $f(x) = e^{-mx} \cdot x^n$

Let $g(x) = x^n = \frac{1}{x^{-n}}$

$$\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} e^{-mx} = 1$$

ce $\int_0^1 g(x)\,dx = \int_0^1 \frac{dx}{x^{-n}}$ converges if $-n<1$

i.e. if $n>-1$

By comparison test, $\int_0^1 f(x)\,dx$ also converges. if $-1<x<0$.

$\int_0^1 e^{-mx} \cdot x^n\,dx$ converges only for $-1<n<0$.

rrespective of the valeee of m.

Show that $\int_0^{\pi/2} \log \sin x\,dx$ is convergent.

Sol'n:- Let $f(x) = \log \sin x$

0 is the point of infinite discontinuity.

Since f is -ve on $[0, \pi/2]$

we consider

Take $g(x) = \frac{1}{x^n}$; $n>0$

$$\lim_{x\to 0+} \frac{-f(x)}{g(x)} = \lim_{x\to 0+} -x^n \log \sin x$$

$$= \lim_{x\to 0+} \frac{-\log \sin x}{\frac{1}{x^n}} \quad \left| \frac{\infty}{\infty} \right.$$

$$= \lim_{x\to 0+} \frac{\frac{\cot x}{n}}{x^{n+1}}$$

$$= \lim_{x\to 0+} \frac{x^n}{n} \cdot \frac{x}{\tan x}$$

$$= 0$$

Taking n b/w 0 & 1,

$\int_0^{\pi/2} g(x)\,dx$ is convergent

By Comparison test.

$\int_0^{\pi/2} -f(x)\,dx$ is convergent.

$\Rightarrow \int_0^{\pi/2} f(x)\,dx$ is convergent.

$\rightarrow$ Show that $\int_0^1 \frac{\operatorname{cosec} x}{x^n}\,dx$ is divergent if $n \geq 1$.

Sol'n: Let $f(x) = \frac{\operatorname{cosec} x}{x^n}$

Since $|\sin x| \leq 1 \;\forall x \in \mathbb{R}$

$\Rightarrow |\operatorname{cosec} x| \geq 1 \;\forall x \in \mathbb{R}$

$\Rightarrow \left|\frac{\operatorname{cosec} x}{x^n}\right| \geq \frac{1}{|x^n|}$ for all $x \in (0,1]$.

$\therefore f(x) \geq \frac{1}{x^n} \;\forall x \in (0,1]$.

Since $\int_0^1 \frac{1}{x^n}\,dx$ is divergent if $n \geq 1$.

Let $I = \int_0^{\pi/2} \log \sin x \, dx$ — (1)

$I = \int_0^{\pi/2} \log \sin\left(0 + \frac{\pi}{2} - x\right) dx$ $\left[\text{using } \int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right]$

$= \int_0^{\pi/2} \log \cos x \, dx$ — (2)

Adding (1) & (2),

$2I = \int_0^{\pi/2} (\log \sin x + \log \cos x) dx$

$= \int_0^{\pi/2} \log\left(\frac{\sin 2x}{2}\right) dx$ (using $\log mn = \log m + \log n$)

$= \int_0^{\pi/2} \log \sin 2x \, dx - \int_0^{\pi/2} \log 2 \, dx$

$= \frac{1}{2} \int_0^{\pi} \log \sin t \times dt - \frac{\pi}{2} \log 2$

$= \frac{1}{\cancel{2}} \times \cancel{2} \times \int_0^{\pi/2} \log \sin t \, dt - \frac{\pi}{2} \log 2$

(using $\int_0^a f(x) = 2\int_0^{a/2} f(x)$ if $f(x) = f(a-x)$)

$\Rightarrow \boxed{I = -\frac{\pi}{2} \log 2} \boxed{= \int_0^{\pi/2} \log \sin x \, dx}$

Q – $\{\tan^{-1} nx\}$, $x \geq 0$.

on $[a,b]$, $a > 0$.

$f_n(x) = \tan^{-1} nx$, $x \in [a,b]$, $a > 0$

$F(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \tan^{-1} nx = \tan^{-1}\left(\lim_{n\to\infty} nx\right)$

$= \tan^{-1} \infty = \frac{\pi}{2}$.

~~since tan inverse fun~~

so, $\lim_{n\to\infty} \sup |f_n(x) - f(x)| = M_n$

$\Rightarrow M_n = \sup \left| \lim_{n\to\infty} (f_n(x) - f(x)) \right|$

$= \sup \left| \frac{\pi}{2} - \frac{\pi}{2} \right| = 0$

i.e. $M_n = 0$ so by M_n-test,

$\{\tan^{-1} nx\}$ is uniformly cgt on $[a,b]$, $a>0$.

On $[0,b]$ take $x = \frac{1}{n}$, then

$f_n(x) = \tan^{-1} 1 = \frac{\pi}{2}$.

$f(x) = \begin{cases} 0, & \text{if } x = 0 \\ \frac{\pi}{2}, & \text{if } x \neq 0. \end{cases}$

so, for $x = \frac{1}{n}$, $M_n = \lim_{n\to\infty} \sup |f_n(x) - f(x)|$

$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4} \not\to 0$ so by M_n-test

$\{\tan^{-1} nx\}$ does not converge uniformly on $[0,b]$