MAINS TEST SERIES-2021

TEST-12 (BATCH-I)

FULL SYLLABUS (PAPER-II)

Answer Key

1(a) Let G be an infinite group. Prove that G has infinitely many proper subgroups.

Sol^n. Assume G has ~~in~~finitely many proper subgroups.

Let $H_1, H_2, \ldots H_n$ be all proper subgroups of finite order of G.

and let $D = \bigcup_{i=1}^{n} H_i$

Since G is infinite, there is an element $b \in G \setminus D$.

Since we know each element of G is of finite order, therefore $b \in G \setminus D$, ord(b) is finite and $\langle b \rangle$ is a proper subgroup of finite order of G and $\langle b \rangle \neq H_i$ for each $1 \leq i \leq n$.

$\Rightarrow$ contradiction of our assumption

$\Rightarrow G$ has infinitely many proper subgroups.

1(b) Show how to get all abelian groups of order $2^3 3^2 5$.

Solⁿ By the structure theorem of finite abelian groups the following list of abelian groups is a complete list of abelian groups of order $2^3 3^2 5$.

i.e. any abelian group of order $2^3 3^2 5$ is isomorphic to exactly one group of the following list-

$Z_{2^3} \times Z_{3^2} \times Z_5$

$Z_{2^2} \times Z_2 \times Z_{3^2} \times Z_5$

$Z_2 \times Z_2 \times Z_2 \times Z_{3^2} \times Z_5$

$Z_{2^3} \times Z_3 \times Z_3 \times Z_5$

$Z_{2^2} \times Z_2 \times Z_3 \times Z_3 \times Z_5$

$Z_2 \times Z_2 \times Z_2 \times Z_3 \times Z_3 \times Z_5$

1(c) Let $f(x)$, $(x \in (-\pi,\pi))$ be defined by $f(x) = \sin|x|$. Is continuous on $(-\pi,\pi)$? If it is continuous, then is it differentiable on $(-\pi,\pi)$?

Sol^n: Given that $f(x) = \sin|x|$, $x \in (-\pi,\pi)$

i.e. $f(x) = \begin{cases} \sin(-x) & \text{if } x \in (-\pi,0) \\ \sin x & \text{if } x \in (0,\pi) \end{cases}$

Clearly $f(x)$ is continuous and differentiable over each subinterval. The only doubtful point is the breaking point $x=0$.

At $x=0$, $f(x)=0$

Now LHL: $\underset{x\to 0-}{Lt}\, f(x) = \sin(-x) = 0$

RHL: $\underset{x\to 0+}{Lt}\, f(x) = \sin x = 0$

$\therefore \underset{x\to 0-}{Lt}\, f(x) = 0 = \underset{x\to 0+}{Lt}\, f(x)$

$\therefore f$ is continuous at $x=0$

Now RHD: $Rf'(0) = \underset{x\to 0+}{Lt}\, \dfrac{f(x)-f(0)}{x-0}$

$= \underset{x\to 0+}{Lt}\, \dfrac{\sin x - 0}{x}$

$= \underset{x\to 0+}{Lt}\, \dfrac{\sin x}{x} = 1$

LHD: $Lf'(0) = \underset{x\to 0-}{Lt}\, \dfrac{f(x)-f(0)}{x-0}$

$= \underset{x\to 0-}{Lt}\, \dfrac{\sin(-x)-0}{x-0}$

$= \underset{x\to 0-}{Lt}\, \dfrac{-\sin x}{x} = -1$

$\therefore Lf'(0) \neq Rf'(0)$

$\therefore f(x)$ is not differentiable at $x=0$

Hence f is continuous on $(-\pi,\pi)$

Also f is differentiable on $(-\pi,\pi)$ except at $x=0$.

1.(d) The only singularities of an analytic function $f(z)$ are poles of order 1 and 2 at $z=-1$ and $z=2$ with residues 1 and 2, respectively at these poles. Determine $f(z)$ if it also satisfies the conditions $f(0)=7/4$ and $f(1)=5/2$.

Solⁿ: Given that

(i) $z=-1$ is pole of order 1 with residue 1;
(ii) $z=2$ is a pole of order 2 with residue 2.

the above information leads to the formulation of the principal part as

$$\frac{1}{z+1}+\frac{2}{z-2}+\frac{b}{(z-2)^2}$$

Hence, $f(z)$ can be expressed as

$$f(z)=\sum_{n=0}^{\infty} a_n z^n+\frac{1}{z+1}+\frac{2}{z-2}+\frac{b}{(z-2)^2} \quad \text{—— (1)}$$

The function $f(z)$ has no singularity at $z=\infty$. This implies $f(1/z)$ is analytic at $z=0$. Thus, the regular part of $f(z)$ has no term except some constant. This means that

$$a_n=0, \quad n=1,2,3,\dots$$

with the data available so far, (1) turns out to be

$$f(z)=a_0+\frac{1}{z+1}+\frac{2}{z-2}+\frac{b}{(z-2)^2} \quad \text{—— (2)}$$

Now, using the given conditions $f(0)=7/4$ and $f(1)=5/2$ in (2) we have two linear equations

$$\frac{7}{4} = a_0 + \frac{b}{4}, \quad 4 = a_0 + b.$$

with solution $a_0 = 1$ and $b = 3$. Interesecting these values in (2), the desired function is

$$f(z) = 1 + \frac{1}{z+1} + \frac{2}{z-2} + \frac{3}{(z-2)^2}.$$

1.(c) Find all the basic feasible solutions of the following problem.

$2x_1+3x_2+x_3+x_4=8$

$x_1-2x_2+6x_3-7x_4=-3$

and Choose the one which maximize $z=2x_1+3x_2+4x_3+7x_4$.

Sol^n: Since there are four variables and two constraints, a basic solution can be obtained by setting any two variables equal to zero and then solving the resulting equations. Also the total number of basic solution $={}^4C_2=6$.

The characteristics of the various basic solutions are given below.

No. of basic solutions	Basic variables	Non-basic variables	values of basic variables	Is the solution feasible? (Are all $x_j\geq 0$?)	value of z	Is the solution optimal?
1	x_1, x_2	$x_3=0$, $x_4=0$	$2x_1+3x_2=8$, $x_1-2x_2=-3$, $x_1=1, x_2=2$	Yes	8	
2.	x_1, x_3	$x_2=0$, $x_4=0$	$2x_1+x_3=8$, $x_1+6x_3=-3$, $x_1=\frac{51}{11}, x_3=\frac{-14}{11}$	NO	—	—
3.	x_1, x_4	$x_2=0$, $x_3=0$	$2x_1+x_4=8$, $x_1-7x_4=-3$, $x_1=\frac{53}{15}$, $x_4=\frac{14}{15}$	Yes	$\frac{68}{5}$	
4	x_2, x_3	$x_1=0$, $x_3=0$	$3x_2+x_3=8$, $-2x_2+6x_3=-3$, $x_2=\frac{51}{20}, x_3=\frac{7}{20}$	Yes	$\frac{181}{20}$	
5.	x_2, x_4	$x_1=0$, $x_3=0$	$x_2=\frac{53}{19}, x_4=\frac{-7}{19}$	NO.	—	—
6	x_3, x_4	$x_1=0$, $x_2=0$	$x_3+x_4=8$, $6x_3-7x_4=-3$, $x_3=\frac{53}{13}, x_4=\frac{51}{13}$	Yes.	$\frac{569}{13}$	Yes

Hence the optimal basic feasible solution is $x_1=0$, $x_2=0$, $x_3=\frac{53}{13}$, $x_4=\frac{51}{13}$.

and the maximum value of $z=\frac{569}{13}=43.76$

2(a)(i) Give an example of a finite non abelian group G which contains a subgroup $H_0 \neq \{e\}$ such that $H_0 \subseteq H$ for all subgroups $H \neq \{e\}$ of G.

Sol^n Let $Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$ with multiplication $i^2 = j^2 = k^2 = ijk = -1$

$ij = k$, $jk = i$, $ik = -ki = -j$, $ji = -k$, $kj = -i$

$Z = \{1, -1\} = H_0$ is a subgroup of Q_8 and Z is contained in every subgroup $H_0 \neq \{e\}$ of Q_8

Indeed if $\{e\} \neq H$ is any subgroup of Q_8, then H contains at least one of the elements $\{-1, \pm i, \pm j, \pm k\}$ if it contains -1 then certainly it contains Z. If it contains one of $\pm i, \pm j, \pm k$, then

$i^2 = j^2 = k^2 = -1 \in H$

Hence $H \supseteq Z$

Subgroups of Q_8 are -

Q_8

$S_1 = \{1, -1, i, -i\}$

$S_2 = \{1, -1, j, -j\}$

$S_3 = \{1, -1, k, -k\}$

$Z = \{1, -1\}$

$\{1\}$

2(a)(ii) Give an example of a nonabelian group in which $(xy)^3 = x^3y^3$ for all x and y.

Sol^n Let $G = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & z & 1 \end{bmatrix} \middle| x, y, z \in Z_3 \right\}$

$$\begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & z & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & z & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2x & 1 & 0 \\ 2y+zx & 2z & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2x & 1 & 0 \\ 2y+zx & 2z & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ x & 1 & 0 \\ y & z & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence every element of G is of order 3. As if $x, y \in G$ then $xy \in G$

Therefore $(xy)^3 = x^3y^3$

But

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$$

Hence G is not abelian. Also $|G| = 27$

2.(b)(i) Let $f(x) = x^2$, $x \in \mathbb{R}$, show that f is uniformly continuous on any closed interval $[a,b]$, $a \geq 0$, but f is not uniformly continuous on $[a,\infty)$, $a \geq 0$.

Sol **first part:**

Let $\epsilon > 0$ be given however small.

f will be uniformly continuous on $[a,b]$ if we can find a $\delta > 0$ such that for any two points x_1, x_2 in $[a,b]$

$$|x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| < \epsilon$$

we have $|f(x_1) - f(x_2)| = |x_1^2 - x_2^2| = |(x_1 - x_2)(x_1 + x_2)|$

$$= |x_1 - x_2||x_1 + x_2|$$
$$\leq |x_1 - x_2|(|x_1| + |x_2|)$$
$$\leq |x_1 - x_2|(b+b) \quad (\because x_1, x_2 \in [a,b] \Rightarrow a \leq x_1, x_2 \leq b)$$
$$= 2b|x_1 - x_2|.$$
$$< \epsilon \text{ whenever } |x_1 - x_2| < \frac{\epsilon}{2b};\ b > 0$$

choosing $\delta = \frac{\epsilon}{2b}$, $b > 0$

$\therefore |f(x_1) - f(x_2)| < \epsilon$ whenever $|x_1 - x_2| < \delta$

$\therefore f$ is uniformly continuous on $[a,b]$

second part: To S.T f is not a uniformly continuous on $[a,\infty)$, $a \geq 0$.

for this, we are enough to show that

If $f: D \to \mathbb{R}$ be a continuous on D and $\exists$ two sequences $\{u_n\}$ & $\{v_n\}$ in D s.t $\lim_{n\to\infty} |u_n - v_n| = 0$ but $\lim_{n\to\infty} |f(u_n) - f(v_n)| \neq 0$ then f is not uniformly continuous.

let $u_n = \sqrt{n}$, $v_n = \sqrt{n+1}$ $\forall n \in \mathbb{N}$

s.t $\{u_n\}$ & $\{v_n\}$ in $[a,\infty)$, $a \geq 0$.

we have $|u_n - v_n| = |\sqrt{n} - \sqrt{n+1}|$

$$= \frac{1}{\sqrt{n} + \sqrt{n+1}} \longrightarrow 0 \text{ as } n \to \infty.$$

we have

$$|f(u_n) - f(v_n)| = |u_n^2 - v_n^2| = |(\sqrt{n})^2 - (\sqrt{n+1})^2|$$
$$= |n - (n+1)| = 1 \neq 0 \text{ as } n \to \infty.$$

2(b)(ii) Define an open set. Prove that the union of an arbitrary family of open sets is open. Show also that the intersection of a finite family of open sets is open. Does it hold for an arbitrary family of open sets? Explain the reason for your answer by example.

Sol'n: Open set: A subset S of $\mathbb{R}$ is said to be an open set if S is a nbd of each of its points i.e, if for each $p \in S$ $\exists$ an $\epsilon > 0$ such that

$$(p-\epsilon, p+\epsilon) \subset S.$$

(or) If S is a subset of $\mathbb{R}$ is said to be open if every point of S is an interior point of S

i.e, S is open $\Leftrightarrow S^0 = S$.

→ The union of an arbitrary family of opensets is an open set.

Sol'n: Let $\{A_\lambda\}_{\lambda \in \Lambda}$ be an arbitrary family of open sets.

Let x be any element of $S = \bigcup_{\lambda \in \Lambda} A_\lambda$.

$$x \in S$$

$\Rightarrow x \in \bigcup_{\lambda \in \Lambda} A_\lambda$

$\Rightarrow x \in A_\lambda$ for at least one $\lambda \in \Lambda$

$\Rightarrow$ at least on A_λ is a nbd of x [$\because$ each A_λ is open]

But $A_\lambda \subset S$ for all $\lambda \in \Lambda$

$\therefore$ S is a nbd of x.

$\Rightarrow$ S is a nbd of each of its points ($\because x$ is arbitrary)

Hence S is an open set.

2.(c)(i) prove that the function
$u(x,y) = (x-1)^3 - 3xy^2 + 3y^2$ is harmonic and find its harmonic conjugate and the corresponding analytic function $f(z)$ in terms of z.

Sol Given that $u(x,y) = (x-1)^3 - 3xy^2 + 3y^2$ ——(1)

$\therefore \frac{\partial u}{\partial x} = 3(x-1)^2 - 3y^2$; $\frac{\partial^2 u}{\partial x^2} = 6(x-1)$

$\frac{\partial u}{\partial y} = -6xy + 6y$; $\frac{\partial^2 u}{\partial y^2} = -6x+6$
$= -6(x-1)$.

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

$\therefore u(x,y)$ is a harmonic function.

clearly $u(x,y)$ is a real part of analytic function $f(z) = u+iv$.

$\because f(z)$ is an analytic function

$\therefore u_x = v_y$; $u_y = -v_x$.

$v_x = -u_y = -\frac{\partial u}{\partial y} = -(-6xy+6y)$
$= 6xy - 6y$.

$\therefore \frac{\partial v}{\partial x} = 6xy - 6y$

$\Rightarrow v(x,y) = 3x^2y - 6xy + f_1(y)$ ——(1)

$\Rightarrow \frac{\partial v}{\partial y} = 3x^2 - 6x + f_1'(y)$

$\Rightarrow \frac{\partial u}{\partial x} = 3x^2 - 6x + f_1'(y)$ (by C-R equations)

$\Rightarrow 3(x-1)^2 - 3y^2 = 3x^2 - 6x + f_1'(y)$

$\Rightarrow 3(x^2+1-2x) - 3y^2 = 3x^2 - 6x + f_1'(y)$

$\Rightarrow \cancel{3x^2} - \cancel{6x} + 3 - 3y^2 = \cancel{3x^2} - \cancel{6x} + f_1'(y)$

$\Rightarrow f_1'(y) = 3 - 3y^2$

$\Rightarrow \boxed{f_1(y) = 3y - y^3 + C}$

$\therefore (1) \equiv v(x,y) = 3x^2y - 6xy + 3y - y^3 + C$.

Now let us find $f(z)$ in terms of z.

$\because \frac{\partial u}{\partial x} = 3(x-1)^2 - 3y^2 = \phi_1(x,y)$ say.

and $\frac{\partial u}{\partial y} = -6xy + 6y = \phi_2(x,y)$ say.

$\therefore$ By Milne's Method,

$f'(z) = \phi_1(z,0) - i\,\phi_2(z,0)$
$= 3(z-1)^2 - 3(0)^2 - i[6(z)(0) + 6(0)]$
$= 3(z-1)^2$

$\therefore f(z) = (z-1)^3 + C$.

2.(C)(ii) Find all possible Taylor's and Laurent's series expansion of $f(z) = \frac{2z-3}{z^2-3z+2}$ about $z=0$?

Sol. $f(z) = \frac{2z-3}{z^2-3z+2} = \frac{2z-3}{(z-1)(z-2)}$

By using partial fraction

$$\frac{2z-3}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)}$$

$$= A(z-2) + B(z-1)$$

$\Rightarrow A+B = 2$ & $2A+B = +3$

$\therefore$ By solving ; we get

$A=1,\ B=1$

$$\therefore f(z) = \frac{1}{(z-1)} + \frac{1}{(z-2)}$$

$$f_1(z) = \frac{1}{(z-1)} = \frac{1}{z(1-1/z)} = \frac{1}{z}(1-1/z)^{-1}$$

$$f_1(z) = \frac{1}{z}\left[1+\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right] = \left[\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right]$$

Since $\left|\frac{1}{z}\right| < 1$ hence $|z|>1$. so valid for $|z|>1$.

Again;

$$f_2(z) = \frac{1}{(z-2)} = \frac{1}{z(1-2/z)} = \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}$$

$$f_2 = \frac{1}{z}\left[1+\frac{2}{z}+\frac{4}{z^2}+\cdots\cdots\right] = \left[\frac{1}{z}+\frac{2}{z^2}+\frac{4}{z^3}+\cdots\right]$$

Since ; $\left|\frac{2}{z}\right| < 1$; Hence valid for $|z|>2$.

Now, the Laurent's Series

$$(f_1+f_2)z = \left[\frac{1}{z}+\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right]+\left[\frac{1}{z}+\frac{2}{z^2}+\frac{4}{z^3}+\cdots\right]$$

$$(f_1+f_2)z = \frac{2}{z}+\frac{3}{z^2}+\frac{5}{z^3}+\frac{9}{z^4}+\cdots$$ is valid for $|z|>2$.

is the required Laurent's series.

For Taylor's Series

$$f_1(z) = \frac{1}{(z-1)} = \frac{-1}{(1-z)} = -1(1-z)^{-1}$$

$$f_1(z) = -(1+z+z^2+z^3+\cdots) \; ; \; |z|>1.$$

$$f_2(z) = \frac{1}{(z-2)} = \frac{-1}{2(1-z/2)} = \frac{-1}{2}(1-z/2)^{-1}$$

$$f_2(z) = -\frac{1}{2}\left[1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\right] \; ; \; |z|>2.$$

$$\therefore (f_1+f_2)z = -1(1+z+z^2+\cdots)-\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\cdots\right)$$

$$(f_1+f_2)z = -\left[(1+z+z^2+z^3+\cdots)+\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+\frac{z^3}{8}+\cdots\right)\right]$$

is the required Taylor's Series.

(3) (a) Let Z be the ring of integers, p a prime number and (P) the ideal of Z consisting of all multiples of p. Prove-

(i) $Z/(P)$ is isomorphic to Z_p the ring of integers mod p.

Sol^n Define a homomorphism

$\phi : Z \longrightarrow Z_p$ by

$\phi(a) = a \pmod P$

It is easy to check that this is a ring homomorphism, which is onto.

$Ker(\phi) = \{a \in Z \mid a \equiv 0 \pmod P\}$

$= \{a \in Z \mid a = KP \text{ for some } K \in Z\}$

$= (P)$

By isomorphism theorem that $Z/(P) \cong Z_p$

(3)(a)(ii) Prove that Z_p is a field.

Sol^n. Since Z is an integral domain, it is enough to show that (P) is a maximal ideal (which implies that $Z/(P)$ is a field)

Assume that I is another ideal of Z such that $I \neq (P)$ and $(P) \subset I$

Then there exist $a \in I \setminus (P)$.

Since P is prime and P does not divide a, then

$\gcd(a, P) = 1$

Therefore $1 = ax + Py$ for some $x, y \in Z$

Thus $1 \in I$ since $ax, Py \in I$

i.e. $I = Z$

Hence (P) is maximal and $Z/(P)$ is field $\Rightarrow$ Z_p is a field.

3.(b) $\rightarrow$ Show that the sequence of functions f_n defined on $[0,1]$ by $f_n(x) = n(1-nx)$, $0 \le x < \frac{1}{n}$
$= 0$; $\frac{1}{n} \le x \le 1$
converges to the function 'f' given by $f(x)=0$, $x \in [0,1]$. Show that $\lim_{n\to\infty} \int_0^1 f_n(x)dx \neq \int_0^1 f(x)dx$.

Is the convergence of the sequence uniform?

Solution:-

Given; $f_n(x) = \begin{cases} n(1-nx) & ; 0 \le x < \frac{1}{n} \\ 0 & ; \frac{1}{n} \le x \le 1. \end{cases}$

At $x=0$, the sequence is $\{1,1,1,\dots\}$

This converges to 'n'.

at $x=1$, the sequence is $\{0,0,0,0\}$. This converges to 0.

Let $c \in (0,1)$. By Archimedean property of $\mathbb{R}$ there exists a natural 'm' such that $0 < \frac{1}{m} < c$ and therefore $0 < \frac{1}{n} < c$ for all $n \ge m$.

$\therefore f_m(c) = 0$ and $f_n(c) = 0$ for all $n \ge m$.

This proves $\lim_{n\to\infty} f_n(c) = 0$.

There the sequence $\{f_n\}$ converges to the function f on $[0,1]$ given by.

$$f(x) = \begin{cases} 1 & , x=0 \\ 0 & , 0 < x \le 1 \end{cases}$$

Each f_n is continuous on $[0,1]$

But the limit function 'f' is not continuous on $[0,1]$.

Here; $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} n(1-nx)$ ($\infty \times 0$ form).

$$= \lim_{n\to\infty} \frac{1-nx}{1/n} \quad \text{(Apply D.I. rule.}$$

$$= \lim_{n\to\infty} xn^2 = 0$$

Since; $n\to\infty$, $\frac{1}{n}\to 0$, as $x\to 0$

Hence; $\lim_{n\to\infty} f_n(x) = 0$ for $x\in[0,1]$

Now; $\int_0^1 f_n(x)\,dx = \int_0^1 n(1-nx)\,dx$.

Put $1-nx = t$ at $x=0$ $t=1$

$-n\,dx = dt$ $x=1$ $t=1-n$.

$$\Rightarrow \int_1^{1-n} -t\,dt = \int_{1-n}^1 t\,dt = \frac{1}{2}[t^2]_{1-n}^1.$$

$$= \frac{1}{2}[1-[1-n]^2] = \frac{1}{2}[2n-n^2]$$

$$\therefore \lim_{n\to\infty}\int_0^1 f_n(x)\,dx = \infty \quad \text{—— (A)}$$

but $\int_0^1 f(x)\,dx = \int_0^1 0\,dx = 0$ —— (B)

Clearly; from (A) and (B).

$$\boxed{\lim_{n\to\infty}\int_0^1 f_n(x)\,dx \neq \int_0^1 f(x)\,dx}$$

Therefore, the convergence of the sequence $\{f_n\}$ is not uniform on $[0,1]$, since uniform convergence of the sequence $\{f_n\}$ of continuous functions on $[0,1]$ implies continuity of 'f' on $[0,1]$.

3.(C) → Nooh's Boats makes three different kinds of boats. All can be made profitably in this company, but the company's monthly production is constrained by the limited amount of labour, wood and screws available each month. The director will choose the combination of boats that maximizes his revenue in view of the information given in the following table:

Input	Row Boat	canoe	Keyak	Monthly Available
Labour (Hours)	12	7	9	1.260 hrs.
Wood (Board feet)	22	18	16	19,008 board feet
Screws (Kg.)	2	4	3	396 kg.
Selling price (in R.s.)	4,000	2,000	5,000	

(i) Formulate the above as a LPP.

(ii) Solve it by simplex method. From the optimal table of the solved LPP, answer the following questions:

(iii) How many boats of each type will be produced and what will be the resulting revenue?

(iv) which, if any, of the resources are not fully utilized? If so, how much of spare capacity is left?

(v) How much wood will be used to make all of the boats given in the optimal solution?

Solⁿ: (i) Let x_1, x_2 and x_3 be the number of Row Boats, Canoe and Keyak made every month. The LPP model can be formulated as follows:

Max. Revenue $z = 4000x_1 + 2000x_2 + 5000x_3$,

Subject to
$$12x_1 + 7x_2 + 9x_3 \leq 1260,$$
$$22x_1 + 18x_2 + 16x_3 \leq 19008$$
$$2x_1 + 4x_2 + 3x_3 \leq 396$$
$$x_1, x_2, x_3 \geq 0.$$

(ii) Adding slack variables s_1, s_2, s_3 the above formulated problem becomes

Max. $z = 4000x_1 + 2000x_2 + 5000x_3 + 0s_1 + 0s_2 + 0s_3$

Subject to
$$12x_1 + 7x_2 + 9x_3 + s_1 = 1260$$
$$22x_1 + 18x_2 + 16x_3 + s_2 = 19008$$
$$2x_1 + 4x_2 + 3x_3 + s_3 = 396$$
$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0.$$

The starting solution and subsequent simplex tables are given below:

Basic variables	Prog. CB	$C_j \rightarrow$ Qty XB	4000 x_1	2000 x_2	5000 x_3	0 S_1	0 S_2	0 S_3	Replacement Ratio. min (X_B/X_K)
S_1	0	1260	12	7	9	1	0	0	1260/9
S_2	0	19008	22	18	16	0	1	0	19008/16
S_3	0	396	2	4	$\boxed{3}$	0	0	1	396/3 ←
		$Z=0$	−4000	−2000	−5000↑	0	0	0↓	Δ_j (NER)
S_1	0	72	$\boxed{6}$	−5	0	1	0	−3	12 ←
S_2	0	16896	34/3	$-\frac{10}{3}$	0	0	1	−16/3	1491
x_3	5000	132	2/3	4/3	1	0	0	1/3	198
		$Z=660000$	$\frac{-2000}{3}$	$\frac{14000}{3}$	0	0↓	0	$\frac{5000}{3}$	Δ_j ←
x_1	4000	12	1	−5/6	0	1/6	0	−1/2	
S_2	0	16760	0	55/9	0	−17/9	1	1/3	
x_3	5000	124	0	17/9	1	−1/9	0	2/3	
		$Z=668000$	0	$\frac{37000}{9}$	0	$\frac{1000}{9}$	0	$\frac{4000}{3}$	Δ_j ←

Since all $\Delta_j \geq 0$, the optimal solution is given by $x_1 = 12$, $x_2 = 0$ and $x_3 = 124$.

(iii) The company should produce 12 Row boats and 124 Kayak boats only. The maximum revenue will be Rs. 6,68,000

(iv) Wood is not fully utilized. Its spare capacity is 16,760 board feet.

(v) The total wood used to make all of the boats given by the optimum solution is
$= 22 \times 12 + 16 \times 124 = 2248$ board feet.

4(a)(i) If R is commutative ring, let $N = \{x \in R \mid x^n = 0$ for some Integer $n\}$
prove : (a) N is an ideal of R
(b) In $\overline{R} = R/N$ if $(\overline{x})^m = 0$ for some m, then $\overline{x} = 0$

Sol^n. (a) Let $N = \{x \in R \mid x^n = 0$ for some integer $n\}$

Observe that n cannot be a negative integer because in order to talk negative power, x must be an invertible element.

But if $(x^{-1})^n = x^{-n} = 0$, then Multiplying by x^{n+1} we obtain $x = 0$ which is impossible

Since x is invertible.

Let $x, y \in N$, Then there exist $n, m > 0$ s.t. $x^n = 0, y^m = 0$

Since R is a commutative ring

$$(x-y)^{n+m} = \sum_{i=0}^{n+m} (-1)^i \binom{n+m}{i} x^{n+m-i} y^i$$

$= x^{n+m} - \binom{n+m}{1} x^{n+m-1} y + \cdots + (-1)^n \binom{n+m}{m} x^n y^m$

$+ (-1)^{m+1} \binom{n+m}{m+1} x^{n-1} y^{m+1} + \cdots + (-1)^{n+m} y^{n+m}$

all powers $x^{n+j} = 0$, $j = 0, \ldots m$

& all powers $y^{m+k} = 0$, $k = 0, \cdots n$

so we obtain $(x-y)^{n+m} = 0$

i.e. $x - y \in N$

For any $r \in R$, $x \in N$ we have

$(rx)^n = r^n x^n = 0$ since $x^n = 0$

Hence $rx \in N$

As R is commutative ring $rx = xr \in N$

Hence N is an ideal of R.

(b) By (a) $\bar{R} = R/N$ is a ring.

Let $\bar{x} \in \bar{R}$. Then $(\bar{x})^m = 0$ implies

$x^m \in N$, but this implies there

exists, n, such that $(x^m)^n = x^{mn} = 0$

Therefore $x \in N$ i.e. $\bar{x} = 0$

4.(b)(i) For $u_1>0$, the sequence u_n defined by
$u_{n+1} = 1+\frac{1}{u_n}\ \forall n$, Converges to $\left(\frac{\sqrt{5}+1}{2}\right)$.

Sol'n: Here for any $u_1>0$, $2\geq u_n \geq 3/2\ \forall\ n\geq 3$,

as $2\geq u_n\geq 3/2$

$\Rightarrow 2>5/3\geq u_{n+1} = 1+\frac{1}{u_n}\geq 3/2\ \forall\ n\geq 3$. So that $\forall\ n\geq 3$,

$$|u_{n+1}-u_n| = \frac{|u_n-u_{n-1}|}{u_nu_{n-1}} = \cdots = \frac{|u_4-u_3|}{u_nu_{n-1}^2\cdots u_3} \leq \frac{|u_4-u_3|}{(3/2)^{2(n-3)}}$$

$$\Rightarrow |u_{n+p}-u_n| \leq |u_{n+p}-u_{n+p-1}| + |u_{n+p-1}-u_{n+p-2}| + \cdots + |u_{n+1}-u_n|$$

$$\leq |u_4-u_3|\left(\frac{2}{3}\right)^{2(n-3)}\left\{1+\left(\frac{2}{3}\right)^2+\cdots+\left(\frac{2}{3}\right)^{2p}\right\}$$

$$< 2|u_4-u_3|\left(\frac{2}{3}\right)^{2(n-3)} \to 0 \text{ as } n\to\infty$$

$\Rightarrow u_n$ converges, say to l, then $u_{n+1} = 1+\frac{1}{u_n}$, as $n\to\infty$

$\Rightarrow l^2-l-1=0$, i.e. $l=\frac{\sqrt{5}+1}{2}$, as $l\geq 3/2$

$\Rightarrow u_n$ converges to $\frac{\sqrt{5}+1}{2}$

4.b(ii) Find the extreme values of the function

$$f(x,y) \equiv x^3+y^3-6(x^2+y^2)+12xy-75(x+y)$$

Sol: Here, f has f_x, f_y at all points. Also,

$$f_x \equiv 3x^2-12x+12y-75=0 \quad \text{—— (1)}$$

and $$f_y \equiv 3y^2-12y+12x-75=0 \quad \text{—— (2)}$$

on subtraction give $(x-y)(x+y-8)=0$

So that $y=x$, and $x+y=8$ along with (1) give the solutions

$$(-5,-5),(5,5),(7,1) \text{ and } (1,7)$$

Now, $f_{xx} \equiv 6x-12$, $f_{yy} \equiv 6y-12$, and $f_{yx} \equiv 12$;

at $(-5,-5)$ give $f_{xx} \equiv -42<0$ and

$$f_{xx}f_{yy}-f_{yx}^2 \equiv (-42)^2-12^2>0,$$

at $(5,5)$ give $f_{yy} \equiv 18>0$ and

$$f_{xx}f_{yy}-f_{yx}^2 \equiv (18)^2-(12)^2>0; \text{ and}$$

at $(7,1)$ and $(1,7)$

$$f_{xx}f_{yy}-f_{yx}^2 \equiv 30(-6)-(12)^2<0.$$

Therefore, the function $f(x,y)$ has extreme values only at $(-5,-5)$ and $(5,5)$.
Of these at $(-5,-5)$ it has a maxima, and a minima at $(5,5)$.

4.c(i) The function $f(z) = \dfrac{z^2+16}{(z-i)^2(z+3)}$ has singularities at $z=i$ and $z=-3$. find the residue at these singularities.

Soln: for $z=-3$,

$$\phi(z) = \frac{z^2+16}{(z-i)^2}$$

Here, $\phi(-3) \neq 0$ and $\phi(z)$ is analytic at $z=-3$. Hence the residue at $z=-3$ is given by

$$\phi(-3) = \frac{9+16}{(-3-i)^2} = 2 - \frac{3}{2}i$$

for residue at $z=i$,

$$\phi(z) = \frac{z^2+16}{z+3}$$

Here, $z=i$ is a pole of order 2. Hence the residue at $z=2$,

$$\phi'(z) = \frac{2z(z+3)-(z^2+16)}{(z+3)^2}$$

$$= \frac{z^2+6z-16}{(z+3)^2}$$

putting $z=i$,

$$\text{Res}_{z=i} f(z) = \phi'(i) = -1+(3/2)i.$$

4.c(ii) If $f(z) = (z-a)^{-n}(z-b)^{-m}$, where m, n are positive integers, show that

$$\text{Res}_{z=a} f(z) = -\text{Res}_{z=b} f(z).$$

Sol^n: The given function has a pole of order n at $z=a$. This suggests

$$\phi(z) = (z-b)^{-m}$$

$$\text{Res}_{z=a} f(z) = \frac{\phi^{(n-1)}(a)}{(n-1)!}$$

$$= \frac{(-m)(-m-1)\cdots(-m-n+2)}{(n-1)!}(a-b)^{-m-n+1}$$

or $$\text{Res}_{z=a} f(z) = (-1)^{n-1}\binom{m+n-2}{n-1}(a-b)^{-m-n+1}$$

Now, the result follows at once by interchanging the role of a and b; and m and n.

Note: for the function $f(z) = (\sin z)/z^4$, we cannot apply the above method, since $\phi(z) = \sin z = 0$ at $z=0$.

4(d) Solve the following assignment problem whose cost matrix is given below

	a	b	c	d
1	18	26	17	11
2	13	28	14	26
3	38	19	18	15
4	19	26	24	10

Solⁿ. The minimum elements of the rows 1, 2, 3, 4 are respectively (11, 13, 15, 10) subtracting these elements of the respective rows –

7	15	6	0
0	15	1	13
23	4	3	0
9	16	14	0

Minimum elements of column (0, 4, 1, 0) respectively. subtracting these element of the respective column

7	11	5	0
0	11	0	13
23	0	2	0
9	12	12	0

Minimum number of horizontal and vertical lines to cover all zeros (N) $= 3 < 4 = m$, order of matrix

Therefore, the smallest element

among the uncovered elements by the lines is 2.

Subtracting 2 from each uncovered elements and adding 2 to the elements of point of intersection of lines.

5	11	3	0
0	13	0	15
21	0	0	0
7	12	10	0

Again minimum line to cover zeros $(N) = 3$
Therefore repeating last step

minimum (smallest) element among uncovered elements $= 3$

2	8	0	0
0	13	0	18
22	0	0	3
4	9	7	0

Now $N = 4 =$ order of matrix
$\Rightarrow$ optimal assignment has been reached.

<u>Assignment</u> :- Single zero row in cell (4,4) assign 1st zero in that row under □ and cross all zero in corresponding column. Next single zero column in cell (2,1) assigns that and cross all zero in that row. Repeat this – zero in

	a	b	c	d
1			$\boxed{0}$	$\cancel{0}$
2	$\boxed{0}$		$\cancel{0}$	
3		$\boxed{0}$	$\cancel{0}$	
4				$\boxed{0}$

Assignment $\rightarrow (1 \rightarrow c, 2 \rightarrow a, 3 \rightarrow b, 4 \rightarrow d)$

Minimum cost $= 17 + 13 + 19 + 10$
$= 59$

5(a) Find a complete integral of $2(pq + yp + qx) + x^2 + y^2 = 0$.

Sol^n: Given equation is $f(x, y, z, p, q) = 2(pq + yp + qx) + x^2 + y^2$ — ①

∴ Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

(or) $$\frac{dp}{2q+2x} = \frac{dq}{2p+2y} = \frac{dz}{-p(2q+2y) - q(2p+2x)} = \frac{dx}{-(2q+2y)} = \frac{dy}{-(2p+2x)}$$

Each of the above fractions $= \dfrac{dp + dq + dx + dy}{(2q+2x) + (2p+2y) - (2q+2y) - (2p+2x)}$

$= (dp + dq + dx + dy)/0$

This $\Rightarrow dp + dq + dx + dy = 0$ so that $p + q + x + y = a$

$\Rightarrow (p+x) + (q+y) = a$ — ②

Rewriting ①, $2(p+x)(q+y) + (x-y)^2 = 0$

$\therefore (p+x)(q+y) = -(x-y)^2/2$ — ③

Now, $(p+x) - (q+y) = \sqrt{\{(p+x) + (q+y)\}^2 - 4(p+x)(q+y)}$

$\therefore (p+x) - (q+y) = \sqrt{a^2 + 2(x-y)^2}$ — ④

Adding ② and ④, $2(p+x) = a + \sqrt{a^2 + 2(x-y)^2}$.

Substracting ④ from ②, $2(q+y) = a - \sqrt{a^2 + 2(x-y)^2}$.

These give $p = -x + \frac{a}{2} + \frac{1}{2}\sqrt{a^2 + 2(x-y)^2}$, $q = -y + \frac{a}{2} - \frac{1}{2}\sqrt{a^2 + 2(x-y)^2}$

$\therefore dz = p\,dx + q\,dy$ becomes.

$dz = -(x\,dx + y\,dy) + \frac{a}{2}(dx + dy) + \frac{1}{2}\sqrt{a^2 + 2(x-y)^2}$

$\therefore dz = p\,dx + q\,dy$ becomes

$dz = -(x\,dx + y\,dy) + \frac{a}{2}(dx + dy) + \frac{1}{2}\sqrt{a^2 + 2(x-y)^2}\,(dx - dy)$

$$\Rightarrow dz = -\frac{1}{2}d(x^2+y^2)+\frac{a}{2}d\,(x+y)+\sqrt{2}\times\frac{1}{2}\sqrt{\frac{a^2}{2}+(x-y)^2}\,d(x-y) \quad \text{--- (5)}$$

Putting $x-y=t$ so that $d(x-y)=dt$. Then (5) becomes

$$dz = -\frac{1}{2}d\,(x^2+y^2)+\frac{a}{2}\,d(x+y)+\frac{1}{\sqrt{2}}\sqrt{\left(\frac{a}{\sqrt{2}}\right)^2+t^2}\,dt.$$

Integrating $z = -\frac{1}{2}(x^2+y^2)+\frac{a}{2}(x+y)$

$$+\frac{1}{\sqrt{2}}\left[\frac{t}{2}\sqrt{\left(\frac{a}{\sqrt{2}}\right)^2+t^2}+\frac{(a/\sqrt{2})^2}{2}\log\left\{t+\sqrt{(a/\sqrt{2})^2+t^2}\right\}\right]+b$$

$$\Rightarrow z = \left(-\frac{1}{2}\right)(x^2+y^2)+\frac{a}{2}(x+y)$$

$$+\frac{1}{2\sqrt{2}}\left[(x-y)\sqrt{\frac{a^2}{2}+(x-y)^2}\right.$$

$$\left.+\frac{a^2}{2}\log\left\{x-y+\sqrt{\frac{a^2}{2}+(x-y)^2}\right\}\right]+b$$

5.(b) → Solve the following differential equations:

$$(D^2 - 3DD' + 2D'^2)z = e^{2x-y} + e^{x+y} + \cos(x+2y).$$

Sol^n: Here auxiliary equation is $m^2 - 3m + 2 = 0$ so that $m = 1, 2$.

$$\therefore \text{C.F.} = \phi_1(y+x) + \phi_2(y+2x), \quad \text{——— (1)}$$

ϕ_1, ϕ_2 being arbitrary functions.

Now, P.I. corresponds to e^{2x-y}

$$= \frac{1}{D^2 - 3DD' + 2D'^2} e^{2x-y}$$

$$= \frac{1}{(2)^2 - 3\cdot 2(-1) + 2(-1)^2} \iint e^v \, dv \, dv,$$

where $v = 2x - y$ (Using formula)

$$= (1/12) \int e^v dv$$

$$= (1/12) e^v = (1/12) e^{2x-y} \quad \text{——— (2)}$$

Again, P.I. corresponding to e^{x+y}

$$= \frac{1}{D^2 - 3DD' + 2D'^2} e^{x+y} = \frac{1}{D - D'} \left\{ \frac{1}{D - 2D'} e^{x+y} \right\}$$

$$= \frac{1}{D - D'} \left\{ \frac{1}{1 - 2.1} \int e^v . dv \right\}, \text{ where } v = x + y.$$

(by using formula)

$$= -\frac{1}{D - D'} e^v = -\frac{1}{(D - D')^1} e^{x+y}$$

$$= -\frac{x}{(1)^1 . 1!} e^{x+y} = -xe^{x+y} \quad \text{——— (3)}$$

[by using formula, with $a = b = 1$, $m = 1$]

Finally, P.I. corresponding to $\cos(x+2y)$

$$= \frac{1}{D^2 - 3DD' + 2D'^2}\cos(x+2y)$$

$$= \frac{1}{(1)^2 - 3\cdot1\cdot2 + 2\cdot(2)^2}\iint \cos v \, dv \, dv,$$

where $v = x+2y$

$$= (1/3)\int \sin v \, dv = -(1/3)\cos v$$

$$= -(1/3)\cos(x+2y) \quad \text{—— (4)}$$

From (1), (2), (3) and (4),

the required solution is $z = C.F. + P.I.$

$$z = \phi_1(y+x) + \phi_2(y+2x) + (1/12)e^{2x-y} - xe^{x+y} - (1/3)\cos(x+2y).$$

5(c). Using modified - Euler's method, obtain the solution of $\frac{dy}{dt}=1-y$, $y(0)=0$ for the range $0\le t\le 0.2$ by taking $h=0.1$.

Sol'n: Here we have

$$f(x,y)=1-y$$

$$h=0.1,\ x_0=0,\ y_0=y(0)=0$$

$$x_1=0.1 \text{ and } x_2=0.2$$

At first we use Euler's method to get

$$y_1^{(0)}=y_0+hf(x_0,y_0)$$

$$=0+(0.1)(1-0)=0.1$$

Applying Euler's modified formula to find

$$y(0.1)=y_1^{(1)}=y_0+\frac{h}{2}\left[f(x_0,y_0)+f(x_1,y_1^{(0)})\right]$$

$$=0.1+\frac{0.1}{2}\left[(1-0)+(1-0.1)\right]$$

$$=0.05\left[1+0.9\right]$$

$$=0.05\left[1.9\right]$$

$$=0.095$$

$$y_1^{(2)}=y_0+\frac{h}{2}\left[f(x_0,y_0)+f(x_1,y_1^{(1)})\right]$$

$$=0+\frac{0.1}{2}\left[(1-0)+(1-0.095)\right]$$

$$=0.05\left[1+0.905\right]$$

$$=0.05(1.905)=0.09525$$

$$\therefore y_1^{(1)}=y_1^{(2)}=0.095$$

Similarly proceeding, we have from Euler's method

$$y_2^{(0)}=y_1+hf(x_1,y_1)$$

$$= 0.095 + (0.1)(1 - 0.095)$$

$$= 0.1855$$

$$y(0.2) = y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right]$$

$$= 0.095 + \frac{0.1}{2}\left[(1 - 0.095 + (1 - 0.1855)\right]$$

$$= 0.180975 \approx 0.181$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right]$$

$$= 0.095 + \frac{0.1}{2}\left[(1 - 0.095) + (1 - 0.180975)\right]$$

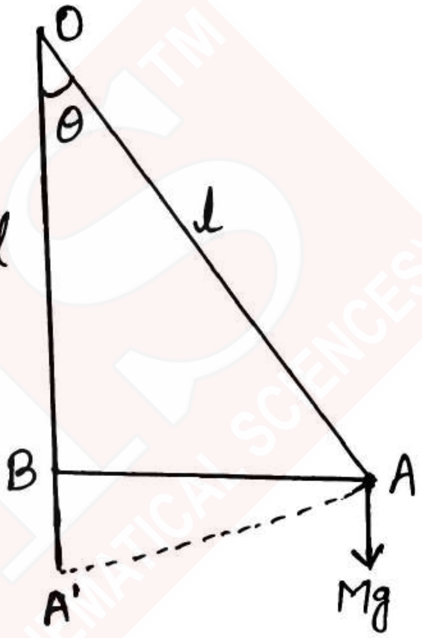
$$= 0.18120$$

$$\therefore\ y_2^{(1)} = y_2^{(2)} = 0.181$$

$$\therefore\ y(0.1) = 0.095 \text{ and } y(0.2) = 0.181.$$

5(d) For a simple pendulum (i) find the Lagrangian function and (ii) obtain an equation describing its motion.

Sol: Let l be the length of the simple pendulum and θ the angle made by the string with the vertical at time t. Thus θ is the only generalised coordinate. Then the velocity of mass M at A will be $v = l\dot{\theta}$.



$\therefore$ Total K.E.,

$$T = \frac{1}{2}Mv^2 = \frac{1}{2}Ml^2\dot{\theta}^2$$

And the potential function

$$V = Mg(A'B) = Mg(l - l\cos\theta)$$

$$= Mgl(1-\cos\theta)$$

(i) $\therefore$ The Lagrangian function

$$L = T - V$$

$$= \frac{1}{2}Ml^2\dot{\theta}^2 - Mgl(1-\cos\theta)$$

(ii) Lagrange's θ-equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

i.e. $\frac{d}{dt}(Ml^2\dot{\theta}) + Mgl \sin\theta = 0$

or $Ml^2\ddot{\theta} + Mgl\sin\theta = 0$

or $\ddot{\theta} = -(g/l)\sin\theta$

or $\ddot{\theta} = -(g/l)\theta$, since θ is small.

which is the required equation of motion.

5(e). In an incompressible fluid, the vorticity at every point is constant in magnitude and direction. Show that the components of velocity u, v, w are solutions of Laplace's equation.

Sol: Let $W = \xi i + \eta j + \zeta k$,

$$q = ui + vj + wk.$$

vorticity is constant in magnitude and direction

$\Rightarrow \xi, \eta, \zeta$ are constant.

$$\Rightarrow \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = \xi = \text{constant},$$

$$\frac{1}{2}\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) = \eta = \text{constant},$$

$$\frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \zeta = \text{constant}.$$

$$\therefore \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = \text{constant} \quad \text{---} (1)$$

$$\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} = \text{constant} \quad \text{---} (2)$$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \text{constant} \quad \text{---} (3)$$

Differentiating of (2) and (3) w.r.t. z and y

$$\frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 w}{\partial x \cdot \partial z},$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x\, \partial y}$$

Equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Observe that

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y\, \partial x} + \frac{\partial^2 w}{\partial x\, \partial z}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

$$= \frac{\partial}{\partial x}(0) = 0.$$

$\therefore \nabla^2 u = 0$. Similarly we can prove $\nabla^2 v = 0$, $\nabla^2 w = 0$. It means that components of velocity are solutions of Laplace's equation.

6(a) → Reduce the equation $\frac{\partial^2 z}{\partial x^2} + 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$ to canonical form and hence solve it.

Sol^n: Rewriting the given equation,

$$r + 2s + t = 0 \quad \text{--- (1)}$$

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$

here $R = 1$, $S = 2$, $T = 1$

and $S^2 - 4RT = 0$ showing that (1) is parabolic.

The λ-quadratic equation reduces to

$\lambda^2 + 2\lambda + 1 = 0 \Rightarrow \lambda = -1, -1$.

The corresponding characteristic equation is

$\frac{dy}{dx} - 1 = 0 \Rightarrow dx - dy = 0$

$\Rightarrow x - y = C$, C being an arbitrary constant.

Choose $u = x - y$ and $v = x + y$ --- (2)

where we have chosen $v = x + y$ in such a manner that u and v are independent functions.

Now $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$ --- (3)

$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = -\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$ --- (4)

From (3) & (4), $\frac{\partial}{\partial x} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}$ and $\frac{\partial}{\partial y} = -\frac{\partial}{\partial u} + \frac{\partial}{\partial v}$

$\therefore r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$

$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$

$= \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$ --- (5)

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left(-\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \quad \text{---} (6)$$

and $s = \dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right) = \left(\dfrac{\partial}{\partial u} + \dfrac{\partial}{\partial v}\right)\left(-\dfrac{\partial z}{\partial u} + \dfrac{\partial z}{\partial v}\right)$

$$= -\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} \quad \text{---} (7)$$

Using (5), (6) and (7) in (1), the required canonical form is

$$\frac{\partial^2 z}{\partial v^2} = 0$$

$$\Rightarrow \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right) = 0 \quad \text{---} (8)$$

Integrating (8) partially w.r.t 'v', we get

$$\frac{\partial z}{\partial v} = \phi(u), \quad \phi \text{ being an arbitrary function.} \quad \text{---} (10)$$

Integrating (10) partially, w.r.t v,

$$z = \int \phi(u)\,dv + \psi(u) = v\,\phi(u) + \psi(u).$$

$$\Rightarrow z = (x+y)\,\phi(x-y) + \psi(x-y), \quad \phi, \psi \text{ are arbitrary functions}$$

which is the required solution.

6.(b) → Find the characteristics of the equation $xp+yq-pq=0$ and then find the equation of the integral surface through the curve $z=x/2$, $y=0$.

Sol'n: Here $f(x,y,z,p,q)=xp+yq-pq=0$ —— ①.

The integral surface passes through the curve $z=x/2$, $y=0$, whose parametric equation can be written as

$x=f_1(\lambda)=\lambda$, $y=f_2(\lambda)=0$, $z=f_3(\lambda)=\lambda/2$, λ being a parameter.

∴ Initial values for x,y,z are $x=x_0=\lambda$, $y=y_0=0$, $z=z_0=\lambda/2$, when $t=0$.

And corresponding initial values p_0 and q_0 of p and q are determined by the relations.

$f_3'(\lambda)=p_0f_1'(\lambda)+q_0f_2'(\lambda) \Rightarrow \frac{1}{2}=p_0\cdot1+q_0\cdot0 \Rightarrow p_0=\frac{1}{2}$

and $f\{f_1(\lambda),f_2(\lambda),f_3(\lambda),p_0,q_0)=0 \Rightarrow x_0p_0+y_0q_0-p_0q_0=0$

$\Rightarrow q_0=\lambda$

The characteristic equations of the given partial differential equation ① are

$$\frac{dx}{dt}=\frac{\partial f}{\partial p}=x-q \quad \text{—— ②}.$$

$$\frac{dy}{dt}=\frac{\partial f}{\partial q}=y-p \quad \text{—— ③}$$

$$\frac{dz}{dt}=p\frac{\partial f}{\partial p}+q\frac{\partial f}{\partial q}=p(x-q)+q(y-p)=px+qy-2pq=-pq$$

using ① —— ④

$$\frac{dp}{dt}=-\frac{\partial f}{\partial x}-p\frac{\partial f}{\partial z}=-p \quad \text{—— ⑤}$$

$$\frac{dq}{dt}=-\frac{\partial f}{\partial y}-q\frac{\partial f}{\partial z}=-q \quad \text{—— ⑥}$$

from ⑤ and ⑥, we get $p=Ae^{-t}$ and $q=Be^{-t}$

But initially when $t=0$, $p=p_0=\frac{1}{2}$ and $q=q_0=\lambda$

$\therefore A=p_0=\frac{1}{2}$ and $B=q_0=\lambda$

$\Rightarrow p=(\frac{1}{2})e^{-t}$ and $q=\lambda e^{-t}$ —— ⑦

using ⑦, from ②, we have $\frac{dx}{dt}-x=-\lambda e^{-t}$

which is a L.D.E, with I.F $=e^{\int -dt}=e^{-t}$

$\therefore\ x.e^{-t} = c_1 + \int(-\lambda e^{-t})(e^{-t})dt = c_1 - \lambda\int e^{-2t}dt$
$= c_1 + \frac{1}{2}\lambda e^{-2t}$

But when $t=0$, $x = x_0 = \lambda$

$\therefore\ x_0 = \lambda = c_1 + \frac{1}{2}\lambda \Rightarrow c_1 = \lambda/2$

$\therefore\ xe^{-t} = \lambda/2 + (\lambda/2)e^{-2t}$

$\Rightarrow x = (\lambda/2)(1+e^{-2t})e^{t}$ ———— (8)

Again, using (7), from (3), we have

$$\frac{dy}{dt} - y = -\frac{1}{2}e^{-t}$$

which is L.D.E with I.F $= e^{\int -dt} = e^{-t}$,

$\therefore\ ye^{-t} = c_2 + \int(-\frac{1}{2}e^{-t})e^{-t}dt = c_2 - \frac{1}{2}\int e^{-2t}dt = c_2 + \frac{1}{4}e^{-2t}$

But when $t=0$, $y = y_0 = 0$, $\therefore\ y_0 = 0 = c_2 + \frac{1}{4} \Rightarrow c_2 = -\frac{1}{4}$

$\therefore\ ye^{-t} = -\frac{1}{4} + \frac{1}{4}e^{-2t} \Rightarrow y = \frac{1}{4}(e^{-2t}-1)e^{t}$ ———— (9)

Now using (7), from (4), we have

$$\frac{dz}{dt} = -\frac{\lambda}{2}e^{-2t},\ \text{Integrating } z = \frac{\lambda}{4}e^{-2t} + c_3$$

But when $t=0$, $z = z_0 = \lambda/2$, $\therefore\ z_0 = \lambda/2 = \lambda/4 + c_3 \Rightarrow c_3 = \lambda/4$

$\therefore\ z = \frac{\lambda}{4}e^{-2t} + \frac{\lambda}{4} \Rightarrow z = \frac{\lambda}{4}(e^{-2t}+1)$ ———— (10)

Thus, the characteristic strips of the given equation are given by

$x = \frac{\lambda}{2}(1+e^{-2t})e^{t}$, $y = \frac{1}{4}(e^{-2t}-1)e^{t}$ and $z = \frac{\lambda}{4}(e^{-2t}+1)$

where λ and t are two parameters.

The required integral surface is obtained by eliminating λ and t between x, y and z.

we have $\frac{x}{2} = ze^{t} \Rightarrow e^{t} = \frac{x}{2z}$

$$\therefore\ y = \frac{1}{4}(e^{-t} - e^{t}) = \frac{1}{4}\left(\frac{2z}{x} - \frac{x}{2z}\right) = \frac{4z^2 - x^2}{8xz}$$

$$\Rightarrow 4z^2 = x^2 + 8xyz$$

which is the required integral surface.

6(C) Obtain temperature distribution $y(x,t)$ in a uniform bar of unit length, whose one end is kept at $10°C$ and other end is insulated. Further, it is given that $y(x,0) = 1-x$, $0<x<1$.

Solution:-

Suppose the bar be placed along the x-axis with its one end (which is at $10°C$) at origin and the other end at $x=1$ (which is insulated, so the flux $-K(\partial y/\partial x)$ is zero there, K being the thermal conductivity).

Then we are to solve

$$\frac{\partial y}{\partial t} = k\left(\partial^2 y/\partial x^2\right). \quad \text{——— ①}$$

with B.C. $y_x(1,t) = 0$; $y(0,t) = 10$ ——②

with I.C. $y(x,0) = 1-x$; $0<x<1$ ——③

Let ; $y(x,t) = u(x,t) + 10$ ——④

i.e $u(x,t) = y(x,t) - 10$ ——⑤

using ④ or ⑤ in ①, ② and ③ reduce to

$$\frac{\partial u}{\partial t} = k\left(\partial^2 u/\partial x^2\right) \quad \text{—— ⑥}$$

$$u_x(1,t) = 0, \quad u(0,t) = 0 \quad \text{—— ⑦}$$

$$u(x,0) = y(x,0) - 10 = -(x+9) \quad \text{—— ⑧}$$

Suppose that ⑥ has solutions of the form

$$u(x,t) = X(x)T(t) \quad \text{——— ⑨}$$

substituting this value of 'u' in ⑥, we get

$$XT' = kX''T \text{ or } \boxed{\frac{X''}{X} = \frac{T'}{kT}} \quad \text{——— ⑩}$$

Since, x and t are independent variables, (5) can only be true if each side is equal to the same constant, say μ.

$$\therefore \quad X'' - \mu X = 0 \quad \text{——— (11)}$$

$$T' = \mu k T \quad \text{——— (12)}$$

using (7), (9) gives

$$X'(1)T(t) = 0 \text{ and } X(0)T(t) = 0 \quad \text{——— (13)}$$

Since, $T(t) = 0$, leads to $u = 0$,

so we suppose that $T(t) \neq 0$

$$\therefore \text{ from (13)} \quad X'(1) = 0 \text{ and } X(0) = 0 \quad \text{——— (14)}$$

We now solve (11) under B.C. (14).

Three cases arises.

Case(i) let $\mu = 0$. Then solution of (11) is

$$X(x) = Ax + B \quad \text{——— (15)}$$

from (15) $\quad X'(x) = A \quad$ ——— (15')

using B.C. (14), (15) and (15)' gives $0 = A$ & $0 = B$

so from (15), $X(x) \equiv 0$,

which lead to $u = 0$. So reject $\mu = 0$

Case ii let $\mu = \lambda^2$, $\lambda \neq 0$. Then solution of (11) is

$$X(x) = Ae^{\lambda x} + Be^{-\lambda x} \quad \text{——— (16)}$$

So that $X'(x) = A\lambda e^{\lambda x} - B\lambda e^{-\lambda x}$ ——— (16)'

using B.C. (14), (16) & (16)', give

$$0 = A\lambda e^{\lambda} - B\lambda e^{-\lambda} \text{ and } 0 = A + B$$

These give $A = B = 0$ so that $X(x) = 0$ and hence $u(x) \equiv 0$ and so we reject $\mu = \lambda^2$.

Case III) Let $\mu = -\lambda^2$, $\lambda \neq 0$. Then solution of (11) is

$$X(x) = A\cos\lambda x + B\sin\lambda x \quad \text{--- (17)}$$

So that $\quad X'(x) = -A\lambda\sin x + B\lambda\cos x \quad$ --- (17)'

using B.C. (14), (17) & (17)' give

$$0 = -A\lambda\sin x + B\lambda\cos x \text{ and } 0 = A$$

These gives ; $A = 0$ and $\cos\lambda = 0$ --- (18)

where we have taken $B \neq 0$, since otherwise $X(x) \equiv 0$ and hence $u = 0$

Now, $\cos\lambda = 0 \Rightarrow \lambda = \frac{1}{2}(2n-1)\pi$; $n = 1, 2, 3 \ldots$

$$\therefore \mu = -\lambda^2 = -\frac{1}{4}(2n-1)^2\pi^2 \quad \text{--- (19)}$$

Hence, non-zero solution $X_n(x)$ of (17) are given by

$$X_n(x) = B_n \sin\left\{\frac{1}{2}(2n-1)\pi x\right\}$$

Again using (19), (12) reduces to

$$\frac{dT}{dt} = -\frac{(2n-1)^2\pi^2 k}{4} T \text{ or } \frac{dT}{T} = -C_n^2 dt \quad \text{--- (20)}$$

where ; $C_n^2 = +\frac{1}{4}(2n-1)^2\pi^2 k$ --- (21)

Solving (20), $T_n(t) = D_n e^{-C_n^2 t}$ --- (22)

So; $u_n(x,t) = X_n T_n = E_n \sin\frac{(2n-1)\pi x}{2} e^{-C_n^2 t}$

are solutions of (6), satisfying (7). Here $E_n (= B_n D_n)$ is another arbitrary constant. In order to obtain a solution also satisfying (8), we consider more general solution.

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} E_n \sin\frac{(2n-1)\pi x}{2} e^{-C_n^2 t} \quad \text{--- (23)}$$

Putting $t=0$ in (23) and using (8), we have

$$-(x+9) = \sum_{n=1}^{\infty} E_n \sin \frac{(2n-1)\pi x}{2} \quad \text{--- (24)}$$

Multiply both sides of (24) by $\sin\left\{\frac{1}{2}(2m-1)\pi x\right\}$ and then integrating w.r.t 'x' from 0 to 1, we get.

$$\Rightarrow -\int_0^1 (x+9) \sin\left\{\frac{1}{2}(2m-1)\pi x\right\} dx.$$

$$= \sum_{n=1}^{\infty} E_n \int_0^1 \sin \frac{(2n-1)\pi x}{2} . \sin \frac{(2m-1)\pi x}{2} dx. \quad \text{--- (25)}$$

But $\int_0^1 \sin \frac{(2n-1)\pi x}{2} . \sin \frac{(2m-1)\pi x}{2} dx = \begin{cases} 0 & , \text{ if } m \neq n \\ 1 & ; \text{ if } m = n \end{cases}$ --- (26)

using (26), (25) gives.

$$-\int_0^1 (x+9) \sin \frac{(2m-1)\pi x}{2} dx = E_m$$

$$\therefore E_m = -\int_0^1 (x+9) \sin \frac{(2n-1)\pi x}{2} dx$$

$$\therefore E_m = -2\left[(x+9)\left\{\frac{-\cos\frac{(2n-1)\pi x}{2}}{\frac{(2n-1)\pi}{2}}\right\} - (1)\left\{\frac{-\sin\frac{(2n-1)\pi x}{2}}{\frac{(2n-1)^2\pi^2}{4}}\right\}\right]_0^1$$

[on using chain rule of integration by parts].

$$\therefore E_m = \frac{8(-1)^n}{(2n-1)^2\pi^2} - \frac{36}{(2n-1)\pi} \left\{ \begin{array}{l} \because \cos\frac{(2n-1)\pi}{2} = 0 \\ \text{and } \sin\frac{(2n-1)\pi}{2} = (-1)^{n-1} \end{array} \right\} \quad \text{--- (27)}$$

using (23) and (24), the required solution is given by

$$y(x,t) = 10 + \sum_{n=1}^{\infty} E_n \sin \frac{(2n-1)\pi x}{2} . e^{-c_n^2 t}$$

where C_n and E_n are given by (21) and (27) respectively.

7(a) Solve the equations $27x+6y-z=85$, $x+y+54z=110$, $6x+15y+2z=72$ by Gauss Seidal Method.

Sol^n: we write the given system in the form

$$\left.\begin{aligned} x &= \frac{1}{27}[85-6y+z] \\ y &= \frac{1}{15}[72-6x-2z] \\ z &= \frac{1}{54}[110-x-y] \end{aligned}\right\} \longleftarrow (1)$$

By Gauss-Seidal method, System (1) can be written as

$$x^{k+1} = \frac{1}{27}(85-6y^k+z^k)$$

$$y^{k+1} = \frac{1}{15}(72-6x^{k+1}-2z^k)$$

$$z^{k+1} = \frac{1}{54}(110-x^{k+1}-y^{k+1})$$

Now taking initial approximation as $x=y=z=0$.
we obtain the following iterations.

1st Iteration:

put $K=0$ $\quad x^{(1)} = \frac{1}{27}(85-6y^{(0)}+z^{(0)}) = 3.148$

$$y^{(1)} = \frac{1}{15}(72-6x^{(1)}-2z^{(0)}) = 3.541$$

$$z^{(1)} = \frac{1}{54}(110-x^{(1)}-y^{(1)}) = 1.9132$$

2nd Iteration: put $K=1$

$$x^{(2)} = 2.4321, \quad y^{(2)} = 3.572, \quad z^{(2)} = 1.9258$$

3rd Iteration: put $K=2$

$$x^{(3)} = 2.4257, \quad y^{(3)} = 3.573, \quad z^{(3)} = 1.9259$$

4th Iteration: put $K=3$

$$x^{(4)} = 2.4255, \quad y^{(4)} = 3.573, \quad z^{(4)} = 1.9259$$

$\therefore$ The solution is given by

$$x = 2.425, \quad y = 3.573, \quad z = 1.926$$

7(b) → The velocity v of a particle at distance s from a point on its path is given by the table.

s ft :	0	10	20	30	40	50	60
v ft/sec :	47	58	64	65	61	52	38

Estimate the time taken to travel 60 ft by using Simpson's 1/3 rule. Compare the result with Simpson's 3/8 rule.

Sol^n: we know $v = \frac{ds}{dt}$

$$dt = \int \frac{ds}{v}$$

$$\Rightarrow t = \int dt = \int \frac{ds}{v} \quad \text{--- (1)}$$

So,

S	0	10	20	30	40	50	60
$y = \frac{1}{v}$	0.2127	0.01724	0.015625	0.015385	0.016393	0.01923	0.026316

Using Simpson's 1/3 rule

(1) ⇒ $t = \frac{h}{3}\left[(y_0+y_6)+2(y_2+y_4)+4(y_1+y_3+y_5)\right]$

$$h = \frac{b-a}{n} = \frac{60-0}{6} = 10$$

$$\therefore t = \frac{10}{3}\Big[0.02127 + 0.02631 + 2(0.015625 + 0.01639) + 4(0.017241 + 0.01538 + 0.01923)\Big]$$

$t = 1.06538$ sec.

Simpson's 3/8 rule:

$$t = \int y\,ds = \frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2y_3\right]$$

$$= \frac{30}{8}\Big[(0.02127 + 0.02631) + (2\times 0.01538) \; 3(0.017241 + 0.015625 + 0.01639 + 0.01923)\Big]$$

$= 1.032615$ sec

∴ time 't' by Simpson's 1/3 rule = 1.06538 sec

time 't' by Simpson's 3/8 rule = 1.03615 sec

∴ Difference = 0.032765

7.(C) → using Runge-Kutta method, find an approximate value of y for $x=0.2$, if $\frac{dy}{dx} = x+y^2$, given that $y=1$ when $x=0$.

Solⁿ: Taking step-length $h=0.1$, we have

$$x_0 = 0,\ y_0 = 1,\ \frac{dy}{dx} = f(x,y) = x+y^2.$$

Now

$$K_1 = hf(x_0, y_0) = (0.1)(0+1) = 0.1$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$$
$$= (0.1)(0.05 + 1.1025)$$
$$= (0.1)(1.1525)$$
$$= 0.11525$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$$
$$= (0.1)(0.05 + 1.1185)$$
$$= (0.1)(1.1685)$$
$$= 0.11685$$

$$K_4 = hf(x_0 + h, y_0 + K_3)$$
$$= (0.1)(0.01 + 1.2474)$$
$$= (0.1)(1.3474)$$
$$= 0.13474$$

$$\therefore K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$
$$= \frac{1}{6}(0.1 + 2(0.11525) + 2(0.11685) + 0.13474)$$
$$= \frac{1}{6}(0.6991) = 0.1165.$$

We get

$$y_1 = y_0 + K = 1 + 0.1165$$

$$\therefore y(0.1) = 1.1165$$

For the second step, we have

$$x_0 = 0.1 \ , \ y_0 = 1.1165$$

$$K_1 = (0.1)(0.1 + 1.2466) = 0.1347$$

$$K_2 = (0.1)(0.15 + 1.4014)$$
$$= (0.1)(1.5514) = 0.1551$$

$$K_3 = (0.1)(0.15 + 1.4259)$$
$$= (0.1)(1.5759) = 0.1576$$

$$K_4 = (0.1)(0.2 + 1.6233)$$
$$= (0.1)(1.8233) = 0.1823$$

$$\therefore K = \frac{1}{6}(0.9424) = 0.1571$$

$$\therefore y(0.2) = 1.1165 + 0.1571 = 1.2736$$

$$\therefore y(0.1) = 1.1165 \text{ and } y(0.2) = 1.2736$$

4(d) (i) A NOR gate has three inputs A, B, C. which combination of inputs will give High output?

(ii) Implement the expression $Y = AB + CD$ using only NAND gates.

Soln: (i) Given that NOR gate has three inputs A, B, C the high output (Y) be 1

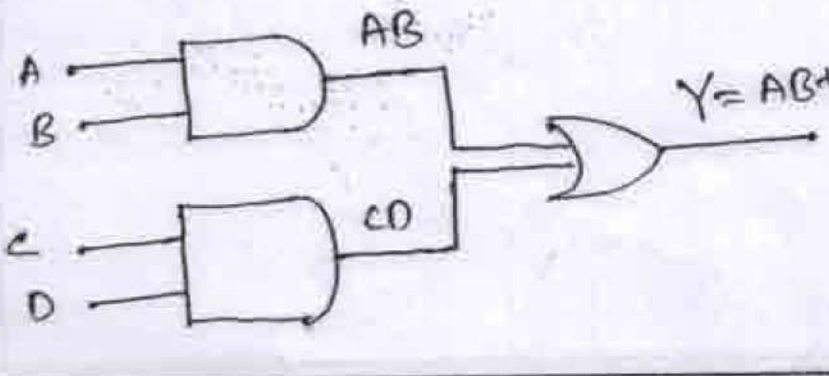
i.e $Y = 1 = \overline{A+B+C}$

Since $\overline{A+B+C} = 1$

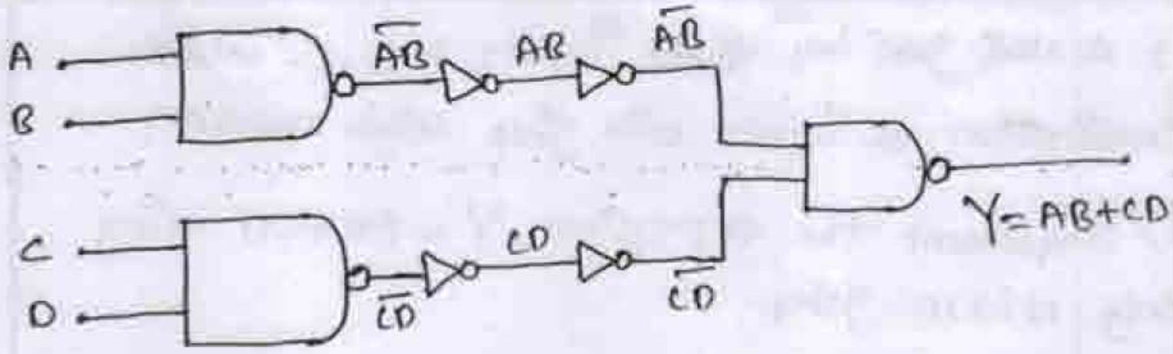
$\Rightarrow A+B+C = 0$

This is possible only if $A = B = C = 0$.

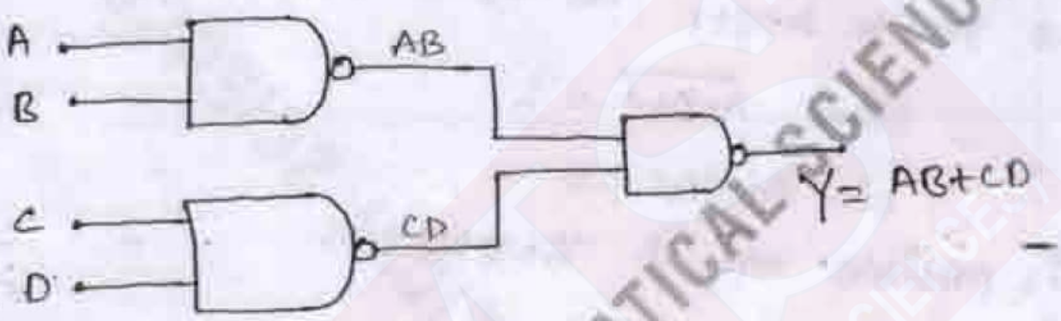
(ii) The straight forward implementation uses two AND gates and one OR gate as shown in Fig. 4.51 (a). Each AND gate can be replaced by a NAND gate and NOT gate in series. The OR gate can be also replaced by NAND gates. This is shown in Fig. 4.51(b) it is seen that NOT gates 1 and 2 are in series and can be eliminated (because $\overline{\overline{A}} = A$). Similarly NOT gates 3 and 4 are in series and can be eliminated. Thus we get the logic circuit shown in Fig 4.51(c)

(a) 

(b)


A
B
$\overline{AB}$
AB
$\overline{AB}$
C
D
$\overline{CD}$
CD
$\overline{CD}$
Y = AB+CD

(c)


A
B
AB
C
D
CD
Y = AB+CD

7(d)(iii) (i) using Runge – Kutta method of fourth order, solve $\frac{dy}{dx} = \frac{y^2-x^2}{y^2+x^2}$ with $y(0)=1$ at $x=0.2, 0.4$.

sol^n: Binary number can be converted into equivalent octal number by making groups of three bits starting from LSB (least significant digit) and moving towards MSB (most significant digit) for integer part of the number and then replacing each group of three bits by its octal representation.

For fractional part, the grouping of three bits are made starting from the binary point.

$$(1011101.1011)_2 = (\underset{1}{\underline{001}}\ \underset{3}{\underline{011}}\ \underset{5}{\underline{101}} \cdot \underset{5}{\underline{101}}\ \underset{4}{\underline{100}})_2$$

$$= (135.54)_8$$

Now, octal number can be converted to equivalent hex number by converting it to equivalent binary and then to hex number.

$$(135.54) = (001\ 011\ 101 \cdot 101100)_2$$

$$= (0\underline{0101}\ \underline{1101} \cdot \underline{1011})_2$$

$$= (5D.B)_{16}$$

Ques: 8(a)) A uniform lamina is bounded by a parabolic arc, of latus rectum $4a$, and a double ordinate at a distance b from the vertex. If $b = \frac{1}{3}a(7+4\sqrt{7})$, show that two of the principal axes at the end of a latus rectum are the tangent and normal there.

Solution:- Let, the equation of the parabola be

$$\boxed{y^2 = 4ax} \quad \text{——— (1)}$$

$\therefore$ Coordinates of the end L of L.R. LL' are $(a, 2a)$.

Differentiating (1), we get

$$\boxed{\frac{dy}{dx} = \frac{2a}{y}}$$

$\therefore$ At $L(a, 2a)$, $\frac{dy}{dx} = \frac{2a}{2a} = 1$.

$\therefore$ Equation of the tangent LT at L is

$$y - 2a = 1(x-a) \quad \text{or} \quad \boxed{y - x - a = 0} \quad \text{——— (2)}$$

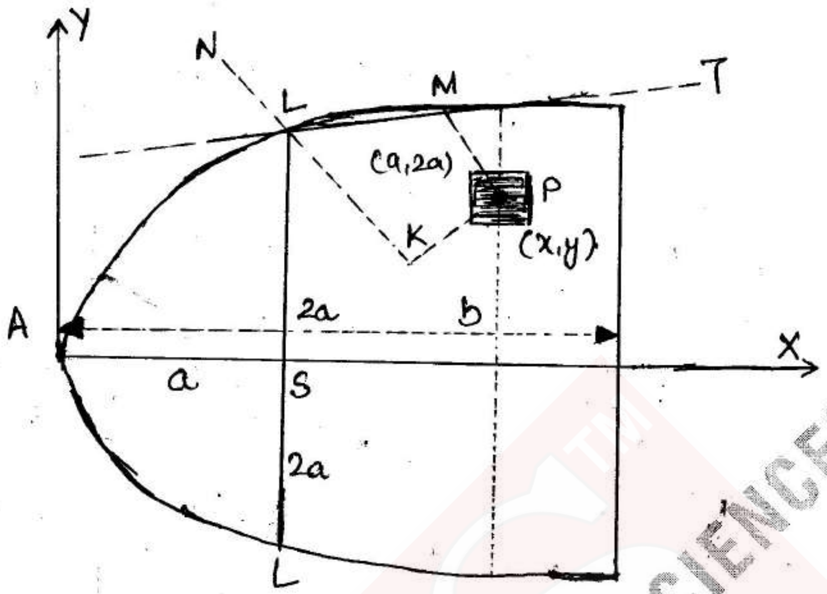
and the equation of the normal LN at L is

$$y - 2a = -\frac{1}{1}(x-a).$$

$$\Rightarrow \boxed{y + x - 3a = 0} \quad \text{——— (3)}$$

Consider an element $\delta x \delta y$ at the point $P(x, y)$ of the lamina, then

PM = length of Perpendicular from P on Tangent LT [given by (2)]



$$\Rightarrow \frac{y-x-a}{\sqrt{1+1}} = \frac{y-x-a}{\sqrt{2}}$$

and PK = length of the perpendicular from P on the normal LN given by (3)

$$\Rightarrow \frac{y+x-3a}{\sqrt{2}}$$

P.I. of the element about LT and LN.

$$PM \cdot PK \cdot \delta m = \left(\frac{y-x-a}{\sqrt{2}}\right)\left(\frac{y+x-3a}{\sqrt{2}}\right)\rho\,\delta x\,\delta y$$

If the tangent and normal at L are the principal axes, then P.I. of the lamina about these will be zero.

i.e. P.I. of Lamina about LT and LN.

$$= \int_{x=0}^{b}\int_{y=-2\sqrt{ax}}^{2\sqrt{ax}} \left[\frac{y-x-a}{\sqrt{2}}\right]\left[\frac{y+x-3a}{\sqrt{2}}\right]\rho\, dx\, dy = 0$$

$$= \frac{\rho}{2}\int_0^b \int_{-2\sqrt{ax}}^{2\sqrt{ax}} \left(y^2 - 4ay + (3a^2 + 2ax - x^2)\right] dx\,dy = 0$$

$$= \frac{\rho}{2}\int_0^b \left[\frac{1}{3}y^3 - 2ay^2 + (3a^2 + 2ax - x^2)y\right]_{-2\sqrt{ax}}^{2\sqrt{ax}} dx = 0$$

$$= \frac{\rho . \cancel{2}}{\cancel{2}}\int_0^b \left[\frac{8}{3}ax\sqrt{ax} + 2(3a^2 + 2ax - x^2)\sqrt{ax}\right] dx = 0$$

$$= \int_0^b \left[\frac{8}{3}a^{3/2}x^{3/2} + 6a^{5/2}x^{1/2} + 4a^{3/2}x^{3/2} - 2a^{1/2}x^{5/2}\right] dx = 0$$

$$= \left[\frac{16}{15}a^{3/2}b^{5/2} + 4a^{5/2}b^{3/2} + \frac{8}{5}a^{3/2}b^{5/2} - \frac{4}{7}a^{1/2}.b^{7/2}\right] = 0$$

$$= \frac{16}{15}ab + 4a^2 + \frac{8}{5}ab - \frac{4}{7}b^2 = 0$$

$$\Rightarrow b^2 - \frac{14}{3}ab - 7a^2 = 0$$

$$\Rightarrow b = \frac{\frac{14}{3}a \pm \sqrt{\left(\frac{196}{9}\right)a^2 + 28a^2}}{2} = \frac{1}{2}\left(\frac{14}{3} \pm \frac{8}{3}\sqrt{7}\right)a.$$

$$\Rightarrow \boxed{b = \frac{a}{3}(7 + 4\sqrt{7})}$$

Leaving -ve sign, as b cannot be negative

Hence, if $b = \frac{a}{3}(7 + 4\sqrt{7})$

then, the principal axes at L are the tangents and normal there.

8(b). A sphere of radius R, whose centre is at rest, vibrates radically in an infinite incompressible fluid of density ρ, which is at rest at infinity. If the pressure at infinity is Π, show that the pressure at the surface of the sphere at time t is

$$\Pi + \frac{1}{2}\rho\left\{\frac{d^2R^2}{dt^2} + \left(\frac{dR}{dt}\right)^2\right\}.$$

Sol: Equation of motion is

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = 0 - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

and equation of continuity is $x^2 v = F(t)$ so that

$$\frac{\partial v}{\partial t} = \frac{F'(t)}{x^2}$$

Hence $$\frac{F'(t)}{x^2} + \frac{\partial}{\partial x}\left(\frac{1}{2}v^2\right) = -\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right)$$

as ρ is constant.

Integrating w.r.t. 'x',

$$\frac{-F'(t)}{x} + \frac{1}{2}v^2 = -\frac{p}{\rho} + C \quad \text{——— (1)}$$

Boundary conditions are:

when $x = \infty$, $p = \Pi$, $v = 0$, ——— (2)

when $x = R$, $p = P$, $v = \dot{R}$. ——— (3)

Also $x^2 v = F(t) = R^2 \dot{R}$

$\therefore \; F'(t) = 2R(\dot{R})^2 + R^2 \ddot{R}$

Subjecting ① to the conditions ② and ③,

$$0 + 0 = -\frac{\Pi}{\rho} + c \quad \text{and}$$

$$\frac{-F'(t)}{R} + \frac{1}{2}(\dot{R})^2 = -\frac{P}{\rho} + c$$

$$= -\frac{P}{\rho} + \frac{\Pi}{\rho}$$

$$\frac{P}{\rho} = \frac{\Pi}{\rho} - \frac{1}{2}(\dot{R})^2 + \frac{1}{R}\left[2R(\dot{R})^2 + R^2 \ddot{R}\right]$$

$$P = \Pi + \frac{1}{2}\rho\left[3(\dot{R})^2 + 2R\ddot{R}\right] \quad \text{—— ④}$$

Now

$$\frac{d^2 R^2}{dt^2} + (\dot{R})^2 = \frac{d}{dt}(2R\dot{R}) + \dot{R}^2$$

$$= 2\dot{R}^2 + 2R\ddot{R} + \dot{R}^2$$

Now ④ becomes

$$P = \Pi + \frac{1}{2}\rho\left[\frac{d^2 R^2}{dt^2} + \dot{R}^2\right]$$

or $$P = \Pi + \frac{1}{2}\rho\left[\frac{d^2 R^2}{dt^2} + \left(\frac{dR}{dt}\right)^2\right]$$

8(c) Prove that in a steady motion of a liquid.

$$H = \frac{p}{\rho} + \frac{1}{2}q^2 + V = \text{constant along a stream line.}$$

If this constant has the same value every-where in the liquid, then prove that the motion must be either irrotational or the vortex lines must coincide with the stream lines.

Sol: $$\frac{\partial H}{\partial x} = 2(v\zeta - w\eta) \quad \text{——— (1)}$$

$$\frac{\partial H}{\partial y} = 2(w\xi - u\zeta), \quad \text{——— (2)}$$

$$\frac{\partial H}{\partial z} = 2(u\eta - v\xi) \quad \text{——— (3)}$$

where $H = V + \frac{1}{2}q^2 + \int \frac{dp}{\rho} = \frac{p}{\rho} + \frac{1}{2}q^2 + V$

Also $$u\frac{\partial H}{\partial x} + v\frac{\partial H}{\partial y} + w\frac{\partial H}{\partial z} = 0 \quad \text{——— (4)}$$

$$\xi\frac{\partial H}{\partial x} + \eta\frac{\partial H}{\partial y} + \zeta\frac{\partial H}{\partial z} = 0 \quad \text{——— (5)}$$

Equation (4) and (5) show that the surface $H = \text{const.}$ contains the stream lines and vortex lines.

If H has the value everywhere, then

$$\frac{\partial H}{\partial x}=0,\ \frac{\partial H}{\partial y}=0,\ \frac{\partial H}{\partial z}=0$$

so that
$$v\zeta - w\eta = 0,$$
$$w\xi - u\zeta = 0,$$
$$u\eta - v\zeta = 0.$$

This $\Rightarrow$ (i) $\dfrac{u}{\xi}=\dfrac{v}{\eta}=\dfrac{w}{\zeta}$

or (ii) $\xi=0,\ \eta=0,\ \zeta=0.$

(iii) $\Rightarrow$ motion is irrotational

(iv) $\Rightarrow$ stream lines are given by

$$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w}$$

coincide with vortex lines given by

$$\frac{dx}{\xi}=\frac{dy}{\eta}=\frac{dz}{\zeta}.$$