1.(c) Evaluate the following limit :

$$\underset{x \to a}{\mathrm{Lt}} \left(2 - \frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$$

(10)

Sol : Let,

$$L = \underset{x \to a}{\mathrm{Lt}} \left(2 - \frac{x}{a}\right)^{\tan\left(\frac{\pi}{2a} \cdot x\right)}$$

$$\Rightarrow \log L = \underset{x \to a}{\mathrm{Lt}} \tan\left(\frac{\pi x}{2a}\right) \cdot \log\left(2 - \frac{x}{a}\right)$$

$$= \underset{x \to a}{\mathrm{Lt}} \frac{\log\left[2 - \frac{x}{a}\right]}{\cot\left(\frac{\pi x}{2a}\right)} \qquad \left[\frac{0}{0} \text{ form}\right]$$

$$= \underset{x \to a}{\mathrm{Lt}} \frac{-1/a}{(2 - x/a)} \cdot \frac{1}{\left[-\operatorname{cosec}^2\left(\frac{\pi x}{2a}\right) \cdot \frac{\pi}{2a}\right]}$$

(Using L-Hopital Rule)

$$= \frac{2}{\pi} \qquad \left[\because \operatorname{cosec}\left(\frac{\pi x}{2a}\right) \to 1 \text{ as } x \to a\right]$$

$$\therefore \boxed{L = e^{2/\pi}}$$

1.(d) Evaluate the following integral :

$$\int_{\pi/6}^{\pi/3} \frac{\sqrt[3]{\sin x}}{\sqrt[3]{\sin x}+\sqrt[3]{\cos x}}\,dx$$

(10)

Sol: Let

$$I = \int_{\pi/6}^{\pi/3} \frac{(\sin x)^{1/3}}{(\sin x)^{1/3}+(\cos x)^{1/3}}\cdot dx \quad \text{—①}$$

$$= \int_{\pi/6}^{\pi/3} \frac{\left[\sin\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)\right]^{1/3}}{\left[\sin\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)\right]^{1/3}+\left[\cos\left(\frac{\pi}{6}+\frac{\pi}{3}-x\right)\right]^{1/3}}\cdot dx$$

$$I = \int_{\pi/6}^{\pi/3} \frac{(\cos x)^{1/3}}{(\cos x)^{1/3}+(\sin x)^{1/3}}\,dx \quad \text{—②}$$

$$\left[\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx\right]$$

Adding ① and ②

$$2I = \int_{\pi/6}^{\pi/3} \frac{(\sin x)^{1/3}+(\cos x)^{1/3}}{(\sin x)^{1/3}+(\cos x)^{1/3}}\,dx$$

$$= \int_{\pi/6}^{\pi/3} dx = [x]_{\pi/6}^{\pi/3} = \left(\frac{\pi}{3}-\frac{\pi}{6}\right)$$

$$\therefore I = \frac{1}{2}\times\frac{\pi}{6} = \frac{\pi}{12}.$$

2.(b) A conical tent is of given capacity. For the least amount of canvas required, for it, find the ratio of its height to the radius of its base. (13)

Sol: Volume is fixed here,

$$\frac{1}{3}\pi r^2 h = V \quad \text{— (1)}$$

Surface Area (Lateral)

$$S = \pi r l = \pi r \sqrt{r^2+h^2}$$

S is minimized whenever (S^2) is minimized (for non-negative values).

Hence, we take

$$S^2 = \pi^2 r^2 (r^2+h^2)$$

$$= \pi^{\cancel{2}} \cdot \frac{3V}{\cancel{\pi} h}\left[\frac{3V}{\pi h} + h^2\right] \qquad \left[\text{from (1)} \quad r^2 = \frac{3V}{\pi h}\right]$$

$$= 3\pi V\left[\left(\frac{3V}{\pi}\right)\cdot\frac{1}{h^2} + h\right]$$

Differentiating w.r.t. h

$$\frac{d(S^2)}{dh} = 3\pi V\left[\frac{-6V}{\pi}\cdot\frac{1}{h^3} + 1\right]$$

$$\frac{d^2(S^2)}{dh^2} = 3\pi V\left[\frac{18V}{\pi}\cdot\frac{1}{h^4}\right] > 0$$

For critical points, $d(S^2)/dh = 0$

$$3\pi V\left[\frac{-6V}{\pi h^3} + 1\right] = 0 \Rightarrow 6V = \pi h^3$$

$$^2\cancel{6}\cdot\frac{1}{\cancel{3}}\cancel{\pi} r^2 h = \cancel{\pi} h^3$$

$$\Rightarrow \boxed{\frac{h}{r} = \sqrt{2}} \quad \text{and} \quad \frac{d^2(S^2)}{dh^2} > 0 \quad \therefore \text{Minima.}$$

3.(b) Which point of the sphere $x^2+y^2+z^2=1$ is at the maximum distance from the point $(2,1,3)$? (13)

Sol: (Calculus Approach): Let (x,y,z) be such point.

Then maximize, $u = (x-2)^2+(y-1)^2+(z-3)^2$ — (1)

such that, $x^2+y^2+z^2=1$ — (2)

Let, $u = x^2+y^2+z^2-1$

Consider, $F = u + \lambda u$

$$F = (x-2)^2+(y-1)^2+(z-3)^2+\lambda(x^2+y^2+z^2-1)$$

For critical points, $dF=0$.

$$2[(x-2)+\lambda x]dx + 2[(y-1)+\lambda y]dy + 2[(z-3)+\lambda z]dz = 0$$

$(\lambda+1)x = 2 \Rightarrow x = \frac{2}{\lambda+1}$, $y = \frac{1}{\lambda+1}$, $z = \frac{3}{\lambda+1}$

$(\lambda+1)y = 1$

$(\lambda+1)z = 3$ from (2), $\frac{4+1+9}{(\lambda+1)^2} = 1 \Rightarrow \lambda+1 = \pm\sqrt{14}$

Taking $\lambda+1 = \sqrt{14}$, $(x,y,z) = \left(\frac{2}{\sqrt{14}}, \frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}}\right)$

$$\therefore u = \left(\frac{2}{\sqrt{14}}-2\right)^2+\left(\frac{1}{\sqrt{14}}-1\right)^2+\left(\frac{3}{\sqrt{14}}-3\right)^2 = (\sqrt{14}-1)^2$$

Taking, $\lambda+1 = -\sqrt{14}$, $(x,y,z) = \left(\frac{-2}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{-3}{\sqrt{14}}\right)$.

$$\therefore u = \left(\frac{-2}{\sqrt{14}}-2\right)^2+\left(\frac{-1}{\sqrt{14}}-1\right)^2+\left(\frac{-3}{\sqrt{14}}-3\right)^2 = (\sqrt{14}+1)^2$$

Hence, the point $\left(\frac{-2}{\sqrt{14}}, \frac{-1}{\sqrt{14}}, \frac{-3}{\sqrt{14}}\right)$ of the sphere is at the maximum distance from point $(2,1,3)$

max distance $= \sqrt{u} = (\sqrt{14}+1)$

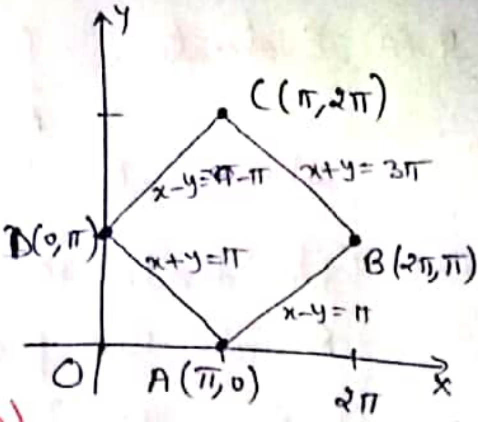
Geometrical Approach: The eqn of st line through Centre $(0,0,0)$ and point $(2,1,3)$ is $x/2 = y/1 = z/3$.

This line will cut the sphere in two point's (One max One min).

3(d) Evaluate the integral

$$\iint_R (x-y)^2 \cdot \cos^2(x+y)\, dx\, dy$$

where R is the rhombus with successive vertices as $(\pi,0),(2\pi,\pi),(\pi,2\pi),(0,\pi)$. (12)

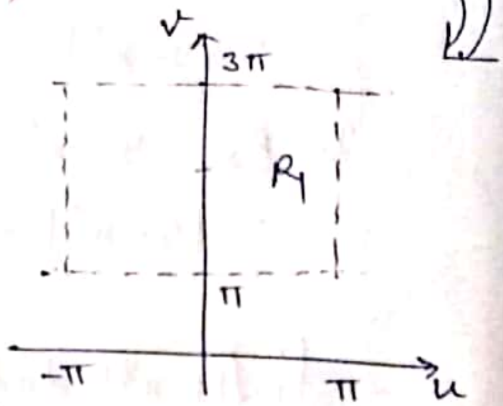


Sol: Using the transformation,

$x-y=u$ and $x+y=v$,

ie $x=\frac{u+v}{2}$, $y=\frac{-u+v}{2}$.

Our region of integration gets transformed to a square.



Jacobian,

$$\frac{J(x,y)}{J(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{1}{4}+\frac{1}{4}=\frac{1}{2}$$

$$\iint_R f(x,y)\,dx\,dy = \iint_{R_1} f_1(u,v)\,\|J(u,v)\|\,du\,dv$$

$$= \int_{-\pi}^{\pi}\int_{\pi}^{3\pi} u^2 \cdot \cos^2 v \cdot \frac{1}{2}\, du\, dv$$

$$= \frac{1}{2}\int_{-\pi}^{\pi} u^2\, du \cdot \int_{\pi}^{3\pi} \frac{1}{2}(1+\cos 2v)\, dv$$

$$= \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}\cdot\pi^3}{3} \cdot \frac{2\pi}{2} = \frac{\pi^4}{3}.$$

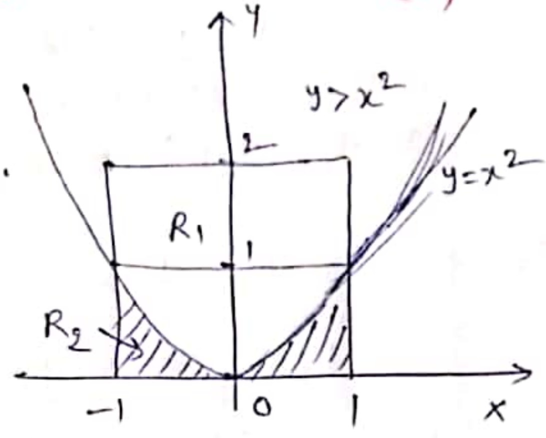
4. (a) Evaluate $\iint\limits_R \sqrt{|y-x^2|}\, dx\,dy$

where $R = [-1,1\,;\,0,2]$. (13)

Sol: We divide the domain R into two parts R_1 & R_2.

in R_1, $y > x^2$

$\therefore |y-x^2| = (y-x^2)$



In R_2, $y < x^2$

$\therefore |y-x^2| = -(y-x^2) = x^2-y$

$$\iint\limits_R \sqrt{|y-x^2|}\, dx\,dy = \iint\limits_{R_1} \sqrt{|y-x^2|}\, dx\,dy + \iint\limits_{R_2} \sqrt{|y-x^2|}\, dx\,dy$$

$$= \int_{x=-1}^{1}\int_{y=x^2}^{2} \sqrt{y-x^2}\, dx\,dy + \int_{x=-1}^{1}\int_{y=0}^{x^2} \sqrt{x^2-y}\, dx\,dy$$

$$= \int_{-1}^{1} \frac{2}{3}(y-x^2)^{3/2}\Big|_{y=x^2}^{y=2} dx + \int_{-1}^{1} -\frac{2}{3}(x^2-y)^{3/2}\Big|_{y=0}^{y=x^2} dx$$

$$= \frac{2}{3}\int_{-1}^{1}(2-x^2)^{3/2} dx + \left(-\frac{2}{3}\right)\int_{-1}^{1}(0-x^3)\,dx$$

$$= \frac{4}{3}\int_{0}^{1}(2-x^2)^{3/2} dx + \frac{2}{3}\left[\frac{x^4}{4}\right]_{-1}^{1} \quad \left(\because \int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx \text{ for even fn}\right).$$

$$= \frac{16}{3}\int_{0}^{\pi/4} \cos^4\theta\, d\theta, \text{ taking } x = \sqrt{2}\sin\theta$$

$$= \frac{4}{3}\int_{0}^{\pi/4}(1+\cos 2\theta)^2 d\theta = \frac{4}{3} + \frac{\pi}{2}.$$

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4d) For the function

$$f(x,y) = \begin{cases} \dfrac{x^2 - x\sqrt{y}}{x^2+y}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Examine the continuity and differentiability. (12 m)

First we check at $(0,0)$

Approaching $(0,0)$ along the curve, $y = mx^2$.

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{x^2 - x\sqrt{mx^2}}{x^2 + mx^2}$$

$$= \frac{1-\sqrt{m}}{1+m}$$

As limit is dependent on 'm', hence $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

$\Rightarrow$ $f(x,y)$ is not continuous at $(0,0)$

Since, continuity implies differentiability.

$\therefore$ $f(x,y)$ is not differentiable at $(0,0)$.

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$$f(x,y) = \begin{cases} \dfrac{x^2 - x\sqrt{y}}{x^2+y} ; & (x,y) \neq (0,0) \\ 0 ; & (x,y) = (0,0) \end{cases}$$

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$x^2 - x\sqrt{y}$ is defined for $y \geq 0$

~~Bn~~ $\dfrac{x^2 - x\sqrt{y}}{x^2+y}$ is defined for

the points where $x^2 + y \neq 0$

$\because y \geq 0 \Rightarrow$ the only

pt where $x^2 + y = 0$

is at $(0,0)$

At $x \in \mathbb{R}$ and $y > 0$:

$\phi(x,y) = x^2 - x\sqrt{y}$ is cont. & diff.

$\psi(x,y) = x^2 + y$ is cont. & diff.

So, $\dfrac{\phi(x,y)}{\psi(x,y)}$ is cont. & diff.