

IAS-MATHEMATICS (Opt.) - 2018

PAPER - II : SOLUTIONS

11a),
 P-II
 Q18.

Let R be an integral domain with unit element. Show that any unit in $R[x]$ is a unit in R .

Soln

Let a_0 be a unit of R . Then a_0 divides 1

$$\text{i.e., } a_0/1$$

i.e. there exists some $b_0 \in R$ such that

$$a_0 b_0 = 1$$

$$\text{Let } f(x) = a_0 + a_1 x + a_2 x^2 + \dots$$

$$g(x) = b_0 + b_1 x + b_2 x^2 + \dots$$

Then $f(x), g(x) \in R[x]$ and

$$f(x)g(x) = a_0 b_0 + a_0 b_1 x + a_0 b_2 x^2 + \dots$$

$$\text{or } f(x)g(x) = 1 \quad (\because a_0 b_0 = 1)$$

$\Rightarrow f(1)/1$ (i.e. $f(x)$ divides 1) in $R[x]$

$\Rightarrow f(x)$ is a unit in $R[x]$

Hence $a_0 = f(x)$ is a unit in $R[x]$.

Conversely, let $f(x)$ be a unit of $R[x]$.

Then there exists some $g(x) \in R[x]$ such that

$$f(x)g(x) = 1 = 1 + 0 \cdot x + 0 \cdot x^2 \quad \text{--- (1)}$$

$$\Rightarrow \deg(f(x)g(x)) = \deg(1 + 0 \cdot x + 0 \cdot x^2 + \dots) = 0$$

$$\Rightarrow \deg f(x) + \deg g(x) = 0 \quad (\because R \text{ is ID.})$$

$$\deg(f(x)g(x))$$

$$= \deg f(x) + \deg g(x)$$

$$\Rightarrow \deg f(x) = 0 \text{ and } \deg g(x) = 0$$

⇒ $f(n)$ and $g(n)$ are constant polynomials
say $f(n) = \alpha$ ($\alpha \neq 0 \in R$), $g(n) = \beta$ ($\beta \neq 0 \in R$)

$$\Rightarrow \alpha\beta = 1 \text{ by (1)}$$

⇒ $\alpha | 1$ (i.e. α divides 1) in R

Hence $f(n) = \alpha$ is a unit of R .

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SOLⁿ

Prove the inequality: $\frac{\pi^2}{9} < \int_{\pi/6}^{\pi/2} \frac{x}{\sin x} dx < \frac{2\pi^2}{9}$.

$$1 \leq \frac{1}{\sin x} \leq 2 \text{ for all } x \in [\frac{\pi}{6}, \frac{\pi}{2}]$$

Therefore $x \leq \frac{x}{\sin x} \leq 2x$ for all $x \in [\pi/6, \pi/2]$

$$\text{Let } f(x) = \frac{x}{\sin x}$$

$$\phi(x) = x$$

$$\psi(x) = 2x, x \in [\pi/6, \pi/2]$$

f and ϕ are both bounded and integrable on $[\pi/6, \pi/2]$ and $f(x) \geq \phi(x)$ for all $x \in [\pi/6, \pi/2]$

Also f and ϕ are both continuous at $\pi/3$ and $f(\pi/3) > \phi(\pi/3)$.

$$\begin{aligned} \text{Hence, } \int_{\pi/6}^{\pi/2} f(x) dx &> \int_{\pi/6}^{\pi/2} \phi(x) dx \\ &= \int_{\pi/6}^{\pi/2} x dx \\ &= \frac{\pi^2}{9}. \end{aligned}$$

f and ψ are both bounded and integrable on $[\pi/6, \pi/2]$ and $f(x) \leq \psi(x)$ for all $x \in [\pi/6, \pi/2]$

Also f and ψ are both continuous at $\pi/3$ and $f(\pi/3) < \psi(\pi/3)$.

$$\begin{aligned} \text{Hence } \int_{\pi/6}^{\pi/2} f(x) dx &< \int_{\pi/6}^{\pi/2} \psi(x) dx \\ &= 2 \int_{\pi/6}^{\pi/2} x dx \\ &= \frac{2\pi^2}{9} \end{aligned}$$

$$\text{Consequently, } \frac{\pi^2}{9} < \int_{\pi/6}^{\pi/2} \frac{x}{\sin x} dx < \frac{2\pi^2}{9}.$$

111) prove that the function
 $u(x,y) = (x-1)^3 - 3xy^2 + 3y^2$ is harmonic
 and find its harmonic conjugate
 and the corresponding analytic function
 $f(z)$ in terms of z .

Sol Given that $u(x,y) = (x-1)^3 - 3xy^2 + 3y^2$

$$\therefore \frac{\partial u}{\partial x} = 3(x-1)^2 - 3y^2 ; \frac{\partial u}{\partial y} = 6xy \quad (1)$$

$$\frac{\partial u}{\partial y} = -6xy + 6y ; \frac{\partial y}{\partial y} = 6x + 6$$

$$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

$\therefore u(x,y)$ is a harmonic function.
 clearly $u(x,y)$ is a real part of

analytic function $f(z) = u + iv$.

$\therefore f(z)$ is an analytic function

$$\therefore u_y = v_x ; u_y = -v_x.$$

$$v_x = -u_y = -\frac{\partial u}{\partial y} = -(-6xy + 6y)$$

$$\therefore \frac{\partial v}{\partial x} = 6xy - 6y = 6xy - 6y.$$

$$\Rightarrow v(x,y) = 3x^2y - 6xy + f_1(y)$$

$$\Rightarrow \frac{\partial v}{\partial y} = 3x^2 - 6x + f_1'(y) \quad (1)$$

$$\Rightarrow \frac{\partial u}{\partial y} = 3x^2 - 6x + f_1'(y) \quad (\text{by C-R equations})$$

$$\Rightarrow 3(x-1)^2 - 3y^2 = 3x^2 - 6x + f_1'(y)$$

$$\Rightarrow 3(x^2 + 1 - 2x) - 3y^2 = 3x^2 - 6x + f_1'(y)$$

$$\Rightarrow 3x^2 - 6x + 3 - 3y^2 = 3x^2 - 6x + f_1'(y)$$

$$\Rightarrow f_1'(y) = 3 - 3y^2$$

$$\Rightarrow f_1(y) = 3y - y^3 + C$$

$$\therefore (1) \equiv v(x,y) = 3x^2y - 6xy + 3y - y^3 + C.$$

NOW let us find $f(z)$ in terms of z .

$$\therefore \frac{\partial u}{\partial x} = 3(z-1)^v - 3y^v = \phi_1(x, y) \text{ say.}$$

$$\text{and } \frac{\partial u}{\partial y} = -6xy + 6y = \phi_2(x, y) \text{ say.}$$

\therefore by Milne's method,

$$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$$

$$= 3(z-1)^v - 3(0)^v - i[6(z)^{(0)} - 6(0)]$$

$$\therefore f(z) = \underline{\underline{3(z-1)^v}} - (z-1)^3 + C.$$

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find the range of ($p > 0$) for which the series:

$$\frac{1}{(1+a)^p} - \frac{1}{(2+a)^p} + \frac{1}{(3+a)^p} - \dots, a > 0 \text{ is}$$

- (i) absolutely convergent and
- (ii) conditionally convergent.

Sol'n

Let $\sum u_n$ be the given series and $v_n = |u_n|$.

Then $\sum v_n$ is a series of positive real numbers and $v_n = \frac{1}{(n+a)^p}$.

Let $w_n = \frac{1}{n^p}$. Then $\lim \frac{v_n}{w_n} = 1$

By comparison test $\sum v_n$ is convergent if $p > 1$

$\sum v_n$ is divergent if $0 < p \leq 1$.

Case-1. $p > 1$

In this case $\sum u_n$ is an alternating series and $\sum |u_n|$ is convergent.

Therefore $\sum u_n$ is absolutely convergent.

Case-2 $0 < p \leq 1$

In this case $\{v_n\}$ is a monotone decreasing sequence of positive real numbers and

$$\lim v_n = 0$$

By Leibnitz's test, $\sum (-1)^{n+1} v_n$

i.e. $\sum u_n$ is convergent.

Since $\sum |u_n|$ is divergent, $\sum u_n$ is conditionally convergent.

Let $\sum u_n$ be a series of positive real numbers and let.

$$P_n = u_n \text{ if } u_n > 0, q_n = 0 \text{ if } u_n > 0 \\ = 0 \text{ if } u_n \leq 0, = u_n \text{ if } u_n < 0$$

Then $\sum P_n$ is a series of positive real numbers along with some 0's and $\sum q_n$ is a series of negative real numbers along with some 0's.

for example, for the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$

$$\sum P_n = 1 + 0 + \frac{1}{3} + 0 + \frac{1}{5} + \dots \text{ and}$$

$$\sum q_n = 0 - \frac{1}{2} + 0 - \frac{1}{4} + 0 - \dots$$

$$P_n = \frac{|u_n| + u_n}{2}, q_n = \frac{|u_n| - u_n}{2}$$

$$\text{and } u_n = P_n + q_n$$

(D)e)

An agricultural firm has 180 tons of nitrogen fertilizer, 250 tons of phosphate and 220 tons of potash. It will be able to sell a mixture of these substances in their respective ratio 3:3:4 at a profit of Rs. 1500 per ton and a mixture of these substances in the ratio 2:4:2 at a profit of Rs. 1200 per ton. Pose a linear programming problem to show how many tons of these two mixtures should be prepared to obtain the max. profit.

try.

	Nitrogen	Phosphate	Potash	Rs./ton
(w) Mix 1	3/10	3/10	4/10	₹1500
(y) Mix 2	2/8 = 1/4	4/8 = 1/2	4/8 = 1/4	₹1200
Tonnes	180	250	220	

Let x_1 tons of mix 1 and y tons of mix 2 should be prepared to obtain the max. profit.

Thus, LPP can be formulated as:

$$\text{Max } z = 1500x + 1200y$$

$$\frac{3x}{10} + \frac{4y}{4} \leq 180 \Rightarrow 6x + 5y \leq 3600$$

$$\frac{3x}{10} + \frac{4y}{2} \leq 250 \Rightarrow 3x + 5y \leq 2500$$

$$\frac{4x}{10} + \frac{4y}{4} \leq 220 \Rightarrow 8x + 5y \leq 4400$$

$$+ x_1 y_2 0$$

Adding the slack variables,

$$\text{Max } z = 1500x + 1200y + 0S_1 + 0S_2 + 0S_3$$

$$6x + 5y + S_1 + 0S_2 + 0S_3 = 3600$$

$$3x + 5y + 0S_1 + S_2 + 0S_3 = 2500$$

$$8x + 5y + 0S_1 + 0S_2 + S_3 = 4400$$

where, $s_1, s_2, s_3 \geq 0$ are slack variables and
 $x_1, y \geq 0$.
Now the initial basic feasible solution is given by, setting $x = y = 0$ (non-basic)

$$s_1 = 3600, s_2 = 2500, s_3 = 4400$$

∴ The I.B.F.S is $(0, 0, 3600, 2500, 4400)$ for which $z = 0$,

	C_j	1500	1200	0	0	0	0	0
C_B	Basis	x	y	s_1	s_2	s_3	b	θ
0	s_1	6	5	1	0	0	3600	600
0	s_2	3	5	0	1	0	2500	$2500/3$
0	s_3	8	5	0	0	1	4400	550

$$Z_j = \sum C_B a_{Bj}$$

$$Z_j = g - Z_j$$

$$\begin{array}{ccccccc} 1500 & 1200 & 0 & 0 & 0 & 0 & 0 \\ \uparrow & & \boxed{5/4} & 1 & 0 & -3/4 & 300 \\ 0 & s_1 & 0 & 5/4 & 1 & 0 & -3/4 & 300 & 240 \end{array} \rightarrow$$

$$\begin{array}{ccccccc} 0 & s_2 & 0 & 25/8 & 0 & 1 & -3/8 \\ 1500 & x & 1 & 5/8 & 0 & 0 & 1/8 \\ \uparrow & & \uparrow & & & & \end{array} \begin{array}{ccccccc} 1500 & 937.5 & 0 & 0 & 187.5 & 825,000 \\ 0 & 262.5 & 0 & 0 & -187.5 & - \end{array}$$

$$\begin{array}{ccccccc} 1200 & & 0 & 1 & 4/5 & 0 & -3/5 \\ 0 & s_2 & 0 & 0 & -5/2 & 1 & 3/2 \\ 1500 & x & 0 & 0 & -1/2 & 0 & 1/2 \\ \uparrow & & \uparrow & & & & \end{array} \begin{array}{ccccccc} 1500 & 1200 & 210 & 0 & 30 & 888,000 \\ 0 & 0 & -40 & 0 & -30 & - \end{array}$$

∴ As all $g - Z_j \leq 0$, thus the optimal condⁿ has been obtained.

Thus, $x = 400, y = 240$ tonnes of mix 1 and mix 2 are required for max. profit [Could also be solved by GRAPHICAL METHOD]

2(e) \rightarrow Show that the quotient group of $(\mathbb{R}, +)$ modulo \mathbb{Z} is isomorphic to the multiplicative group of complex numbers on the unit circle in the complex plane. Here \mathbb{R} is the set of real numbers and \mathbb{Z} is the set of integers.

Sol) Let $\frac{\mathbb{R}}{\mathbb{Z}} = \{z + a/a \in \mathbb{R}\}$ be a quotient group w.r.t. + modulo \mathbb{Z} .

Let T be a group of all complex numbers on the unit circle in the complex plane.

$$\begin{aligned} \text{Then } T &= \{z \in \mathbb{C} / |z| = 1\} \\ &= \{x + iy \in \mathbb{C} / x^2 + y^2 = 1\} \\ &= \{e^{i\theta} / \theta \in \mathbb{R}\} \end{aligned}$$

To show that $\frac{\mathbb{R}}{\mathbb{Z}}$ is an isomorphic to T .

$$i.e. \frac{\mathbb{R}}{\mathbb{Z}} \cong T.$$

Define a mapping $f: \mathbb{R} \rightarrow T$ such that
 $f(a) = e^{2a\pi i} / a \in \mathbb{R}$

To s.t. $f: \mathbb{R} \rightarrow T$ is well-defined!

Let $a, b \in \mathbb{R}$ such that

$$a = b$$

$$\Rightarrow 2a\pi i = 2b\pi i$$

$$\Rightarrow e^{2a\pi i} = e^{2b\pi i}$$

$$\therefore f(a) = f(b)$$

$\therefore f$ is well-defined.

To show that $f: \mathbb{R} \rightarrow T$ is onto:

For $e^{2a\pi i} \in T$ there exists a $a \in \mathbb{R}$,

i.e. by definition, $f(a) = e^{2a\pi i}$.

f is onto

To show that $f: \mathbb{R} \rightarrow T$

is a homomorphism:

Let $a, b \in \mathbb{R}$ s.t $f(a) = e^{2a\pi i}, f(b) = e^{2b\pi i}$

we have

$$f(a+b) = e^{2(a+b)\pi i} = e^{2a\pi i} \cdot e^{2b\pi i} \\ = f(a) \cdot f(b)$$

$\therefore f$ is homomorphism.

\therefore by fundamental theorem of homomorphism

$$\frac{\mathbb{R}}{K} \cong T$$

$$\text{where } K = \text{Ker } f = \{a \in \mathbb{R} \mid f(a) = 1\}$$

$$= \{a \in \mathbb{R} \mid e^{2a\pi i} = 1\}$$

$$= \{a \in \mathbb{Z} \mid e^{2a\pi i} = 1\}$$

$$= \mathbb{Z}$$

$$\therefore \frac{\mathbb{R}}{\mathbb{Z}} \cong T$$

(2)(b)

solve the following LPP by Big M-method and show that the problem has finite optimal solutions. Also find the value of the obj.

$$\text{function: } \text{Min } z = 3x_1 + 5x_2$$

$$\text{subject to } x_1 + 2x_2 \geq 8$$

$$3x_1 + 2x_2 \geq 12$$

$$5x_1 + 6x_2 \leq 60$$

$$x_1, x_2 \geq 0$$

Ans.

The objective function of the given LPP is of minimization type.

$$\text{Thus, } \text{Max } z' = \text{Min}(-z) \\ = -3x_1 - 5x_2$$

Thus, LPP in standard form can be written as:

$$\text{Max } z' = -3x_1 - 5x_2 + OS_1 + OS_2 + OS_3 - MA_1 - MA_2$$

subject to

$$x_1 + 2x_2 - S_1 + OS_2 + OS_3 + A_1 + MA_2 = 8$$

$$3x_1 + 2x_2 + OS_1 - S_2 + OS_3 + MA_1 + A_2 = 12$$

$$5x_1 + 6x_2 + OS_1 + OS_2 + S_3 + MA_1 + MA_2 = 60$$

$$x_1, x_2, S_1, S_2, S_3, A_1, A_2 \geq 0$$

where, S_1, S_2 are surplus var., S_3 is slack var.
and A_1, A_2 are artificial variables.

IBFS is given by,

$$x_1 = x_2 = 0, S_1 = S_2 = 0, A_1 = 8, A_2 = 12, S_3 = 60$$

$$\Rightarrow (x_1, x_2, S_1, S_2, A_1, A_2, S_3)$$

$$= (0, 0, 0, 0, 8, 12, 60)$$

(12)

		-3	-5	0	0	0	-M	-M		
C ₀	Basis	x ₁	x ₂	s ₁	s ₂	s ₃	A ₁	A ₂	b	θ
-M	A ₁	1	2	-1	0	0	1	0	8	8
-M	A ₂	3	2	0	-1	0	0	1	12	4 →
0	S ₃	5	6	0	0	1	0	0	60	10x5/60/5 = 12
	x _j	-4M	-4M	M	M	0	-M	-M	-20M	
	C _j - z _j	-3+4M	-5+4M	-M	-M	0	0	0	0	
-M	A ₁	0	4/3	-1	1/3	0	1	+ 4/3		→
-3	x ₁	1	4/3	0	-1/3	0	0	4/3	4/6	
0	S ₃	0	8/3	0	5/3	1	0	-40/3	15	
	x _j	-3	-2-4M/3	M	-M/3+1	0	-M	-	-4M-12	
	C _j - z _j	0	-3+4M/3	-M	M/3-1	0	0	-		
-5	x ₂	0	1	-3/4	1/4	0	3/4	-3		
-3	x ₁	1	0	1/2	-1/2	0	-1/2	-2		
0	S ₃	0	0	2	1	1	-	-32		
	x _j	-3	-5/2	1/4	0	-	-	-21		
	C _j - z _j	0	-2	-1/4	0	-	-			
As all C _j - z _j ≥ 0, thus the optimality has been reached and, x ₁ = 2, x ₂ = 3										
∴ Max Z' = -21										
⇒ Min Z = 21										
Thus, the value of the objective function is 21.										

(2)(c)

Show that if a function f defined on an open interval (a, b) of \mathbb{R} is convex, then f is continuous. Show by example if the condition of open interval is dropped, then the continuous convex function need not be cts .

Ans.

Suppose f is convex on (a, b) and let $[c, d] \subseteq (a, b)$. Choose a and d , such that

$$a < c < c < d < b$$

if $x, y \in [c, d]$ with $x \neq y$. Then, we know that,

$$\frac{f(y) - f(x)}{y - x} \geq \frac{f(x) - f(c)}{x - c} \geq \frac{f(c) - f(y)}{c - y}$$

and, $\frac{f(y) - f(x)}{y - x} \leq \frac{f(d) - f(y)}{d - y} \leq \frac{f(d) - f(d)}{d - d}$

showing the set

$$\left\{ \frac{f(y) - f(x)}{y - x} \mid c \leq x < y \leq d \right\}$$

is bounded by $M > 0$. It follows that

$$|f(y) - f(x)| \leq M |y - x|$$

and therefore f is uniformly continuous on $[c, d]$. Recalling that uniformly $ccts \Rightarrow$ continuity.

$\therefore f$ is $ccts$ on $[c, d]$.

Since the interval $[c, d]$ was arbitrary, f is continuous on (a, b) .

Example: Suppose $a > 0$ and define

$f: [a, b] \rightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 & \text{for } a \leq x \leq c \\ (x-c)^2 + c^2 & \text{for } c \leq x \leq b \end{cases}$$

Clearly f is cts at $x=0$ as

$$f(0) = 0^2$$

$$\lim_{x \rightarrow 0} f(x)$$

$$\lim_{x \rightarrow 0^-} f(x+0) = \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^+} x^2 = 0^2$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (x-0)^2 + 0^2 = 0^2$$

$\Rightarrow f$ is cts on $[a, b]$.

but f is not convex on $[a, b]$

$$\text{as, } f'(x) = 2x]_{x=0} = 2 \cdot 0$$

$$f^+(x) = 2(x-0)]_{x=0} = 0$$

$$\text{Now, } f'(x) \neq f^+(x) \quad (\because c > 0) \\ \forall x \in (a, b)$$

$\therefore f$ is not convex on $[a, b]$ as left hand derivative of $f(x)$ is not less than or equal to $f^+(x)$.

③@ Find all the proper subgroups of the multiplicative group of the field $(\mathbb{Z}'_{13}, +_{13}, \times_{13})$, where $+_{13}$ and \times_{13} represent addition modulo 13 and multiplication modulo 13 resp.

Ans.

First we shall show that $\mathbb{Z}'_{13} = \{1, 2, 3, \dots, 12\}$ under \times^n is cyclic with generator 2.

Since 13 is prime, $\mathbb{Z}'_{13} = \{1, 2, \dots, 12\}$ is a cyclic group.

($\because \mathbb{Z}_p$ is cyclic iff p is prime)

Now, 2 is a generator of \mathbb{Z}'_{13} as:

$$2^1 = 2$$

$$2^7 = 11$$

$$2^2 = 4$$

$$2^8 = 9$$

$$2^3 = 8$$

$$2^9 = 5$$

$$2^4 = 16 = 3$$

$$2^{10} = 10$$

$$2^5 = 32 = 6$$

$$2^{11} = 7$$

$$2^6 = 12$$

$$2^{12} = 1$$

$\Rightarrow \mathbb{Z}'_{13} = \langle 2 \rangle$ is cyclic of order 12, with generator 2,

Now, by Fundamental Theorem of Groups,

Order of every subgroup divides the order of the group (by Lagrange)

Also, subgroup of the cyclic group is also cyclic.

Since the divisors of 12 are 1, 2, 3, 4, 6 and 12.

Also, Every subgroup of the cyclic group of order n is a divisor of n and there is exactly one subgroup for each divisor.

This is called the fundamental theorem of cyclic groups.

Now, the subgroups of \mathbb{Z}_{13} are,

$$\langle 1 \rangle = \langle 2^{12} \rangle, \langle 2^6 \rangle = \langle 12 \rangle, \langle 2^4 \rangle = \langle 3 \rangle,$$

$$\langle 2^3 \rangle = \langle 8 \rangle$$

$$\langle 2^2 \rangle = \langle 4 \rangle$$

$$\langle 1 \rangle = \langle 2 \rangle = \mathbb{Z}_{13}.$$

Thus, the proper subgroups are:

$$\langle 12 \rangle$$

$$\langle 3 \rangle$$

$$\langle 8 \rangle$$

$$\langle 4 \rangle$$

Ans.

(3) (b)

Show by applying the residue theorem

that

$$\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}, a > 0$$

Ans.

$$\text{Consider } \int_C \frac{dz}{(z^2+a^2)^2} = \int_C f(z) dz,$$

$$\text{where } f(z) = \frac{1}{(z^2+a^2)^2}, C \text{ is the contour}$$

consisting of a large semi-circle Γ of radius R together with part of real axis from $x=-R$ to R .

By Cauchy's residue theorem,

$$\begin{aligned} \int_C f(z) dz &= \int_{-R}^R \frac{dx}{(a^2+x^2)^2} + \int_{\Gamma} \frac{dz}{(a^2+z^2)^2} \\ &= 2\pi i \sum_{R^+} \end{aligned} \quad \text{--- (1)}$$

$$\begin{aligned} \lim_{z \rightarrow \infty} z f(z) &= \lim_{z \rightarrow \infty} \frac{z}{(a^2+z^2)^2} \\ &= \lim_{z \rightarrow \infty} \frac{z}{z^4 \left(\frac{a^2}{z^2} + 1\right)^2} = 0 \end{aligned}$$

$$\therefore \lim_{R \rightarrow \infty} \int_{\Gamma} \frac{dz}{(a^2+z^2)^2} = 0$$

$$\text{and } \lim_{R \rightarrow \infty} \int_{-R}^R \frac{dx}{(x^2+a^2)^2} = \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)^2}$$

Taking $R \rightarrow \infty$ in (1), we get,

$$\int_{-\infty}^{\infty} \frac{dx}{(a^2+x^2)^2} + 0 = 2\pi i \sum_{R^+} \quad \text{--- (2)}$$

Poles of $f(z) = \frac{1}{(a^2+z^2)^2}$ are given by $(a^2+z^2)^2=0$

$\Rightarrow z = \pm ai$ (twice) out of which only pole $z = ai$ (order 2) lies inside C.

$$\therefore \text{Residue at } ai = \frac{\phi'(ai)}{2!} = \phi'(ai)$$

$$\begin{aligned} \text{Here, } \phi(z) &= (z-ai)^2 f(z) \\ &= (z-ai)^2 \times \frac{1}{(a^2+z^2)^2} \\ &= (z-ai)^2 \times \frac{1}{(z-ai)^2(z+ai)^2} = \frac{1}{(z+ai)^2} \end{aligned}$$

$$\therefore \phi'(z) = -\frac{2}{(z+ai)^3}$$

$$\begin{aligned} \text{Therefore, residue at } ai &= \phi'(ai) = \frac{-2}{(ai+ai)^3} \\ &= \frac{-2}{8a^3 i} = \frac{1}{4a^3 i} \end{aligned}$$

\therefore from ②, we have,

$$\int_{-\infty}^{\infty} \frac{dx}{(a^2+x^2)^2} = 2\pi i \times \frac{1}{4a^3 i} = \frac{\pi}{2a^3}$$

$$\Rightarrow 2 \int_0^{\infty} \frac{dx}{(a^2+x^2)^2} = \frac{\pi}{2a^3}$$

$$\therefore \int_0^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}, a>0.$$

(3) (c)

How many basic solutions are there in the following linearly independent set of equations? Find all of them.

Ane.

$$\text{Ans, } 2x_4 - x_2 + 3x_3 + x_4 = 6$$

$$4x_4 - 2x_2 - x_3 + 2x_4 = 10$$

The given system of eq's can be written in the matrix form as $Ax=b$

$$\text{where } A = \begin{bmatrix} 2 & -1 & 3 & 1 \\ 4 & -2 & -1 & 2 \end{bmatrix}, x = \begin{bmatrix} x_4 \\ x_2 \\ x_3 \\ x_1 \end{bmatrix}, b = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$$

Since the rank of A is 2 (i.e. $m=2$), the maximum number of linearly indep. columns of A is 2.

Thus, we consider any of the 2×2 submatrices as basic matrix B .

$$\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}, \begin{bmatrix} -1 & 3 \\ -2 & -1 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix},$$

$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$$

$$\text{Let } B = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$$

A basic soln to the system is obtained by taking $x_3 = x_4 = 0$ and solving the system

$$\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix} \begin{bmatrix} x_4 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$$

$$\Rightarrow 2x_4 - x_2 = 6$$

$$4x_4 - 2x_2 = 10$$

$$\text{As, } \begin{vmatrix} 2 & -1 \\ 4 & -2 \end{vmatrix} = -4 + 8 = 0$$

—————

$\Rightarrow B$ is not LI.

If, $B = \begin{bmatrix} -1 & 3 \\ -2 & -1 \end{bmatrix}$, then $x_4 = x_4 = 0$

$$\begin{bmatrix} -1 & 3 \\ -2 & -1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$$

$$-x_2 + 3x_3 = 6 \quad \Rightarrow -2x_2 + 6x_3 = 12$$

$$-2x_2 - x_3 = 10$$

$$7x_3 = 2 \quad \Rightarrow x_3 = \frac{2}{7}$$

$$\therefore x_2 = 3x_3 - 6 = \frac{6}{7} - 6 = -\frac{36}{7}$$

$\therefore (0, -\frac{36}{7}, \frac{2}{7}, 0)^T$ is one of the basic soln.

If $B = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, then $x_4 = x_2 = 0$

$$\therefore \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$$

$$3x_3 + x_4 = 6 \Rightarrow 6x_3 + 2x_4 = 12$$

$$-x_3 + 2x_4 = 10$$

$$7x_3 = 2 \Rightarrow x_3 = \frac{2}{7}$$

$$\Rightarrow x_4 = 6 - 3x_3 = 6 - \frac{6}{7} = \frac{36}{7}$$

$\therefore (0, 0, \frac{2}{7}, \frac{36}{7})^T$ is one of the basic soln.

If $B = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$, $x_2 = x_4 = 0$

$$\begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix} \begin{bmatrix} x_4 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$$

$$\Rightarrow 2x_4 + 3x_3 = 6$$

$$4x_4 - x_3 = 10 \Rightarrow 12x_4 - 3x_3 = 30$$

$$14x_4 = 36 \Rightarrow x_4 = \frac{18}{7}$$

$$x_3 = 4x_4 - 10 = \frac{72}{7} - 10 = \frac{2}{7}$$

$\therefore (\frac{18}{7}, 0, \frac{2}{7}, 10)^T$ is one of the basic soln

If $B = \begin{bmatrix} -1 & 1 \\ 2 & 2 \end{bmatrix}$,

$$\begin{bmatrix} -1 & 1 \\ 2 & 2 \end{bmatrix} = -2+2=0$$

$\Rightarrow B$ is not LI

If $B = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$ then $|B| = \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 4-4 = 0$

$\therefore B$ is not LI

Thus, $(0, 0, \frac{2}{7}, \frac{36}{7})^T$, $(\frac{18}{7}, 0, \frac{2}{7}, 0)^T$

and $(0, -\frac{36}{7}, \frac{2}{7}, 0)^T$
are the basic solutions.

4(a)

Suppose \mathbb{R} be the set of all real numbers and $f: \mathbb{R} \rightarrow \mathbb{R}$ is a function such that the following equation hold for all $x, y \in \mathbb{R}$.

$$(i) f(x+y) = f(x) + f(y)$$

$$(ii) f(x,y) = f(x)f(y)$$

Show that $\forall x \in \mathbb{R}$ either $f(x) = 0$ or $f(x) = x$.

SOLⁿ

Let us consider that $f(x) \neq 0$, then we shall show that $f(x) = x$.

$$\text{As } f(x+x) = f(x) + f(x) = 2f(x) \quad \dots \quad (1)$$

$$f(2x) = 2f(x) \quad \forall x \in \mathbb{R}.$$

From (1) -

$$f(2)f(x) = 2f(x)$$

$$\Rightarrow f(2)f(x) - 2f(x) = 0$$

$$\therefore f(x)(f(2)-2) = 0$$

$$\because f(x) \neq 0 \Rightarrow f(2) = 2$$

Similarly $\forall k \in \mathbb{R}$, $f(kx) = f(x + \dots + x)$

$$= f(x) + \dots + f(x) = kf(x)$$

$$\Rightarrow f(k)f(x) = kf(x) \quad \dots \quad (1)$$

$$\therefore f(k) = k \quad \forall k \in \mathbb{R}. \quad (\because f(x) \neq 0)$$

$$\Rightarrow f(x) = x \quad \forall x \in \mathbb{R}$$

Now let us assume that $f(x) \neq x$, for some x , then we need to show that $f(x) = 0$.

As, $f(k)f(x) = k f(x)$ (from ①)
 $\Rightarrow f(x)[f(k) - k] = 0$
Since, $f(x) = 0$ or $f(k) = k$
but $f(k) \neq k \Rightarrow f(x) = 0$

Hence we are done.

4(b)

find the Laurent's series which represents the function $\frac{1}{(1+z^2)(z+2)}$ when,

- (i) $|z| < 1$
- (ii) $1 < |z| < 2$
- (iii) $|z| > 2$

Soln

Solving $f(z)$ into partial fractions

$$\text{we obtain } f(z) = \frac{1}{5} \left[\frac{1}{z+2} - \frac{z-2}{z^2+1} \right]$$

$$\begin{aligned} \text{(i) for } |z| < 1, f(z) \text{ is analytic and so we have } \\ f(z) &= \frac{1}{5} \cdot \frac{1}{2} \left[1 + \frac{z}{2} \right]^{-1} - \frac{1}{5} (z-2) (1+z^2)^{-1} \\ &= \frac{1}{10} \left[1 - \frac{z}{2} + \frac{z^2}{2^2} + \dots + (-1)^n \frac{z^n}{2^n} + \dots \right] \\ &\quad - \frac{z-2}{5} \left[1 - z^2 + z^4 - z^6 + \dots - (-1)^n z^{2n} + \dots \right] \\ &= \frac{1}{10} \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{2^n} - \frac{(z-2)}{5} \sum_{n=0}^{\infty} (-1)^n z^{2n} \end{aligned}$$

This series being in the negative powers of z represents Taylor's expansion for $f(z)$.

(ii) $1 < |z| < 2$, we have.

$$\begin{aligned} f(z) &= \frac{1}{5} \cdot \frac{1}{2} \left(1 + \frac{1}{2} z \right)^{-1} - \frac{z-2}{5} \cdot \frac{1}{z^2} \left(1 + \frac{1}{z^2} \right)^{-1} \\ &= \frac{1}{10} \left[1 - \frac{z}{2} + \frac{z^2}{2^2} - \frac{z^3}{2^3} + \dots \right] - \frac{z-2}{5z^2} \left[1 - \frac{1}{z^2} + \frac{1}{z^4} + \dots \right] \\ &= \frac{1}{10} \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{2^n} - \frac{2-2}{5z^2} \sum_{n=0}^{\infty} (-1)^n \frac{1}{z^{2n}}. \end{aligned}$$

$$(\because |z| > 1, \Rightarrow \frac{1}{|z|} < 1 \Rightarrow \frac{1}{|z|^2} < 1)$$

so that binomial expansion of $(1 + \frac{1}{z^2})^{-1}$
is valid.

The above series being in positive and negative powers of z represents Laurent's expansion for $f(z)$ in the region $1 < |z| < 2$.

(iii) for $|z| > 2$ we have

$$\begin{aligned} f(z) &= \frac{1}{S} \frac{1}{z} \left(1 + \frac{2}{z}\right)^{-1} - \frac{1}{S} (z-2) \frac{1}{z^2} \left(1 + \frac{1}{z^2}\right)^{-1} \\ &= \frac{1}{Sz} \left[1 - \frac{2}{z} + \frac{2^2}{z^2} - \frac{2^3}{z^3} + \dots \right] - \frac{z-2}{Sz^2} \left[1 - \frac{1}{z^2} + \frac{1}{z^4} - \dots \right] \\ &= \frac{1}{Sz} \sum_{n=0}^{\infty} (-1)^n \left(\frac{2}{z}\right)^n - \frac{z-2}{Sz^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{z^{2n}}. \end{aligned}$$

4(c) →

Machine

	M_1	M_2	M_3	M_4	M_5
O_1	24	29	18	32	19
O_2	17	26	34	22	21
O_3	27	16	28	17	25
O_4	22	18	28	30	24
O_5	28	16	31	24	27

In a factory there are five operators O_1, O_2, O_3, O_4, O_5 and five machines M_1, M_2, M_3, M_4, M_5 . The operating costs are given when the O_i operator operates the M_j machine ($i, j = 1, 2, \dots, 5$). But there is a restriction that O_3 cannot be allowed to operate the third machine M_3 and O_2 cannot be allowed to operate the fifth machine M_5 . The cost matrix is given above. Find the optimal assignment and the optimal assignment cost also.

Sol →

Assigning 'm' to the restricted combinations or introducing 'm', where m is a large value.

P.T.O.

	m_1	m_2	m_3	m_4	m_5
O_1	24	29	18	32	19
O_2	17	26	34	22	M
O_3	27	16	M	17	25
O_4	22	18	28	30	24
O_5	28	16	31	24	27

Reducing the matrix row wise.

	m_1	m_2	m_3	m_4	m_5
O_1	6	11	0	14	1
O_2	0	9	17	5	M
O_3	11	0	M	1	9
O_4	4	0	10	12	6
O_5	12	0	15	8	11

Reducing column wise.

	m_1	m_2	m_3	m_4	m_5
O_1	6	11	0	13	0
O_2	0	9	17	4	M
O_3	11	0	M	0	8
O_4	4	0	10	11	5
O_5	12	0	15	7	10

All zeroes to be covered (since the no. of lines covered are less than 5, thus).

Subtracting smallest element 4 from the uncovered elements and adding at the initial.

	m_1	m_2	m_3	m_4	m_5
O_1	-6	15	0	13	0
O_2	0	13	17	4	11
O_3	11	4	M	0	8
O_4	0	0	6	7	1
O_5	8	0	11	3	6

Again $4 < 5$

	m_1	m_2	m_3	m_4	m_5
O_1	-7	16	0	13	0
O_2	0	13	16	3	M
O_3	12	5	M	0	8
O_4	0	0	5	6	9
O_5	8	0	10	2	

Now the zero's are covered.

Thus optional assignment is -

$O_1m_3, O_2m_1, O_3m_4, O_4m_5, O_5m_2$.

and cost is $18 + 17 + 17 + 24 + 16$

$$= 92 .$$

5(a) Find the partial differential equation of the family of all tangent plane to the ellipsoid: $x^2 + 4y^2 + 4z^2 = 4$, which are not perpendicular to the xy -plane.

Sol'n: Given that $x^2 + 4y^2 + 4z^2 = 4 \quad \text{--- (1)}$

Its tangent plane at a point $P(x_1, y_1, z_1)$ is

$$xx_1 + 4yy_1 + 4zz_1 = 4 \quad \text{--- (2)}$$

Let the plane be $lx + my + nz = P \quad \text{--- (3)}$

$$\text{then } \frac{x_1}{l} = \frac{4y_1}{m} = \frac{4z_1}{n} = \frac{4}{P}$$

$$\Rightarrow x_1 = \frac{4l}{P}, \quad y_1 = \frac{m}{P}, \quad z_1 = \frac{n}{P}$$

$$\text{--- (1)} = \frac{16l^2}{P^2} + \frac{4m^2}{P^2} + \frac{4n^2}{P^2} = 4$$

$$\Rightarrow 4l^2 + m^2 + n^2 = P^2$$

$$\therefore \text{--- (3)} = lx + my + nz = \pm \sqrt{4l^2 + m^2 + n^2} \quad \text{--- (4)}$$

Since it is not perpendicular to xy -plane,

$$\therefore n \neq 0 \quad \therefore \text{--- (4)} = \frac{l}{n}x + \frac{m}{n}y + z = \pm \sqrt{4\left(\frac{l}{n}\right)^2 + \left(\frac{m}{n}\right)^2 + 1}$$

$$\Rightarrow \alpha x + \beta y + z = \pm \sqrt{4\alpha^2 + \beta^2 + 1} \quad \text{--- (5)}$$

which is the required tangent plane

Differentiating (4) partially w.r.t x & y

we get -

$$x + \frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = -1 \Rightarrow P = -1 \Rightarrow \alpha = -P$$

$$\beta + \frac{\partial z}{\partial y} = 0 \Rightarrow \frac{\partial z}{\partial y} = -\beta \Rightarrow q = -\beta \Rightarrow \beta = -q$$

$$\therefore \text{--- (5)} =$$

$$-Px - Qy + z = \pm \sqrt{4P^2 + Q^2 + 1}$$

$$\Rightarrow (Px + Qy - z)^2 = (4P^2 + Q^2 + 1)$$

which is the required partial Differential Equation.

5.b) Using Newton's forward difference formula find the lowest degree polynomial U_x when it is given that $U_1 = 1$, $U_2 = 9$, $U_3 = 25$, $U_4 = 55$ and $U_5 = 105$.

Soln:

Here,

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
1	1				
2	9	8			
3	25	16	8	6	0
4	55	30	14	6	
5	105	50	20		

Here,

$$x_0 = 1, h = 1$$

$$\therefore u = \frac{x - x_0}{h} = \frac{x-1}{1} = (x-1)$$

Now, using Newton's forward formula,

$$f(x) = y_0 + u \Delta y_0 + \frac{u(u-1)}{2!} \Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!} \Delta^3 y_0 + \dots$$

$$= 1 + 8u + \frac{u(u-1)}{2} \times 8 + \frac{u(u-1)(u-2)}{6} \times 6 + 0$$

$$= 1 + 8(x-1) + (x-1)(x-2)4 + (x-1)(x-2)(x-3)$$

$$= 1 + 8x - 8 + 4(x^2 - 3x + 2) + (x^2 - 3x + 2)(x-3)$$

$$= 8x - 7 + 4x^2 - 12x + 8 + x^3 - 3x^2 + 2x - 3x^2 + 9x - 6$$

$$= x^3 - 2x^2 + 7x - 5$$

$$= x^3 - 2x^2 + 7x - 5$$

Q.5(c) For an incompressible fluid flow, two components of velocity (u, v, w) are given by

$$u = x^2 + 2y^2 + 3z^2 ; v = x^2y - y^2z + zx .$$

Determine the third component ' w ' so that they satisfy the equation of continuity. Also, find the z -component of acceleration.

Sol:

For an incompressible flow, the equation of continuity can be written in differential form as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \text{--- (1)}$$

$$\text{Given; } u = x^2 + 2y^2 + 3z^2$$

$$v = x^2y - y^2z + zx$$

Hence;

$$\frac{\partial u}{\partial x} = 2x \quad \text{and} \quad \frac{\partial v}{\partial y} = x^2 - 2yz$$

Substituting $\frac{\partial u}{\partial x}$ & $\frac{\partial v}{\partial y}$ in eq(1)

$$2x + x^2 - 2yz + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial w}{\partial z} = -x^2 - 2x + 2yz$$

Integrate w.r.t z , we have

$$w = -x^2z - 2xz + yz^2 + f(x, y)$$

There are infinite number of possible 'z' components of, since $f(x, y)$ is arbitrary, the simplest one would be found by setting $f(x, y) = 0$

z - component of acceleration

$$a_z = u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z}$$

and $w = -x^2 z - 2xz + yz^2 + f(x, y)$

$$\frac{\partial w}{\partial x} = -2xz - 2z + \frac{\partial f}{\partial x}$$

$$\frac{\partial w}{\partial y} = z^2 + \frac{\partial f}{\partial y}$$

$$\frac{\partial w}{\partial z} = -x^2 - 2x + 2yz$$

thus, substituting the values of u, v, w from the question,
we get the required z -component of acceleration.

$$a_z = (x^2 + 2y^2 + 3z^2) \left(-2xz - 2z + \frac{\partial f}{\partial x} \right) + (x^2 y - y^2 z + 2z) \left[z^2 + \frac{\partial f}{\partial y} \right] \\ + \left[-x^2 z - 2xz + yz^2 + f(x, y) \right] \left[-x^2 - 2x + 2yz \right]$$

5(d) →

Time (minutes)	2	4	6	8	10	12	14	16	18	20
Speed (km/hr)	10	18	25	29	32	20	11	5	2	8.5

Starting from rest in the beginning, the speed (in km/hr) of a train at different times (in minutes) is given by the above table:

Using Simpson's $\frac{1}{3}$ rd rule, find the approximate distance travelled (in km) in 20 minutes from the beginning.

Sol'n :

Time (minutes)	0	2	4	6	8	10	12	14	16	18	20
Speed (km/hr)	0	10	18	25	29	32	20	11	5	2	8.5

$$h = 2$$

If D km is the distance covered in time t minutes then

$$\begin{aligned} \frac{dD}{dt} &= v \\ D &= \int_0^{20} v dt \end{aligned}$$

$$v_0 = 0, v_1 = 0.17, v_2 = 0.30, v_3 = 0.42, v_4 = 0.48, v_5 = 0.53$$

$$v_6 = 0.33, v_7 = 0.18, v_8 = 0.08, v_9 = 0.03, v_{10} = 0.14$$

Applying Simpson's $\frac{1}{3}$ rd rule:

$$\int_0^{20} v dt = \frac{h}{3} \left[(v_0 + v_{10}) + 4(v_1 + v_3 + v_5 + v_7 + v_9) + 2(v_2 + v_4 + v_6 + v_8) \right]$$

$$= \frac{2}{3} \left[(0 + 0.14) + 4(0.17 + 0.42 + 0.53 + 0.18 + 0.03) + 2(0.30 + 0.48 + 0.33 + 0.08) \right]$$

$$= \frac{2}{3} [0.14 + 4(1.33) + 2(1.19)]$$

$$= \frac{2}{3} [0.14 + 5.32 + 2.38]$$

$$\int_0^{20} v dt = \underline{\underline{5.2267 \text{ kms}}}$$

5(e) Write down the basic algorithm for solving the equation : $x e^x - 1 = 0$ by bisection method, correct to 4 decimal places.

Sol: Basic Algorithm of Bisection Method:

1. Start

2. Read x_1, x_2, e

* Here x_1 and x_2 are initial guesses
 e is the absolute error
 i.e. the desired of accuracy

* Here $f(x) = x e^x - 1 = 0$

or

$f(x) = x + e^x - 1 = 0$

3. Compute: $f_1 = f(x_1)$ and $f_2 = f(x_2)$

4. If $(f_1 * f_2) > 0$, then display initial guesses are wrong and goto (1).
 Otherwise continue.

5. $x = (x_1 + x_2) / 2$

6. If $(|(x_1 - x_2)| / x) < e$, then display x and goto (1).
 * Here $[]$ refers to the modulus sign.*

7. Else ; $f = f(x)$

8. If $((f * f_1) > 0)$, then $x_1 = x$ and $f_1 = f$.

9. Else ; $x_2 = x$ and $f_2 = f$

10. Goto (5)

* Now the loop continues with new values
 upto correct to 4 decimal places *

11. Stop.

Its iterations are; when we enter the values.

Iteration No. 1 $x = 0.5$

Iteration No. 2 $x = 0.75$

Iteration No. 3 $x = 0.625$

Iteration No. 4 $x = 0.5625$

Iteration No. 5 $x = 0.59375$

Iteration No. 6 $x = 0.5781$

Iteration No. 7 $x = 0.5703$

Iteration No. 8 $x = 0.5664$

Iteration No. 9 $x = 0.5684$

Iteration No. 10 $x = 0.5674$

Iteration No. 11 $x = 0.5669$

Iteration No. 12 $x = 0.5672$

Iteration No. 13 $x = 0.5671 \underline{\quad} \underline{d}$

Q.6(a) Solve $(y^3x - 2x^4)p + (2y^4 - x^3y)q = g \geq (x^3 - y^3)$.

Sol:- Here Lagrange's Auxillary equations are given by

$$\frac{dx}{y^3x - 2x^4} = \frac{dy}{2y^4 - x^3y} = \frac{dz}{g \geq (x^3 - y^3)} \quad \text{--- (1)}$$

Taking first two fractions of (1), we have

$$(2y^4 - x^3y)dx = (y^3x - 2x^4)dy,$$

$$\left(\frac{2y}{x^3} - \frac{1}{y^2} \right)dx = \left[\frac{1}{x^2} - \frac{2x}{y^3} \right]dy$$

[\because By dividing it by x^3y^3]

$$\text{or } \left(\frac{1}{x^2}dy - \frac{2y}{x^3}dx \right) + \left(\frac{1}{y^2}dx - \frac{2x}{y^3}dy \right) = 0$$

$$\text{or } d\left(\frac{y}{x^2}\right) + d\left(\frac{x}{y^2}\right) = 0$$

Integrating, we get

$$\begin{aligned} \left(\frac{y}{x^2} \right) + \left(\frac{x}{y^2} \right) &= C_1 \\ \boxed{x^3 + y^3} &= x^2y^2C_1 \end{aligned} \quad \text{--- (2)}$$

Choosing $(1/x)$, $(1/y)$, $(1/3z)$ as multipliers of each fraction of (1).

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{3z}dz}{(y^3 - 2x^3) + (2y^3 - x^3) + 3(x^3 - y^3)}$$

$$= \frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{3z}dz}{0}$$

$$\Rightarrow \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{3z}dz = 0$$

$$\text{so that } \log x + \log y + \frac{1}{3} \log z = \log C_2$$

$$\log x + \log y + \log z^{1/3} = \log C_2$$

$$\log(xyz^{1/3}) = \log C_2$$

$$\boxed{\log(xyz^{1/3}) = \log C_2} - \textcircled{3}$$

from ② & ③, the required general solution is

$$\boxed{\phi(xyz^{1/3}, y/x^2 + x/y^2) = 0}$$

ϕ being the arbitrary function

6(b) →

find the equivalent of numbers given in a specified number system to the system mentioned against them.

- $(111011.101)_2$ to decimal system.
- $(100011110000.00101100)_2$ to hexadecimal system.
- $(C4F2)_{16}$ to decimal system.
- $(418)_{10}$ to binary system.

Soln →

$$(i) (111011.101)_2 \text{ to } (x)_{10}$$

$$= 1 \times 2^5 + 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^0 \cdot 1 \times 2^{-1} + 0 \times 2^{-2} + 1 \times 2^{-3}$$

$$= 32 + 16 + 8 + 0 + 2 + 1 \cdot 0.5 + 0 + 0.125$$

$$x = (59.625)_{10}$$

$$(ii) \underbrace{10001}_{0001} \underbrace{111}_{0001} \underbrace{0000}_{1111} \underbrace{0000}_{0000} \cdot \underbrace{0010}_{0010} \underbrace{1100}_{1100} \text{ to } (x)_{16}$$

$$\underbrace{0001}_{(1)} \underbrace{0001}_{(1)} \underbrace{111}_{(F)} \underbrace{0000}_{(0)} \cdot \underbrace{0010}_{(A)} \underbrace{1100}_{(C)}$$

$$(1 \ 1 \ F \ 0 \cdot A C)_{16}$$

$$x = (11F0.AC)_{16}.$$

(iii) $(C4F2)_6$ to decimal.

$$(C \times 16^3 + 4 \times 16^2 + F \times 16^1 + 2 \times 16^0)_{10}$$

C-12
F-15

$$(12 \times 16^3 + 4 \times 16^2 + 15 \times 16 + 2 \times 1)_{10}$$

$$= (49152 + 1024 + 240 + 2)_{10}$$

$$= (50,418)_{10}$$

(iv) $(418)_{10} \rightarrow (x)_2$

2	418	0
2	209	1
2	104	0
2	52	0
2	26	0
2	13	1
2	6	0
2	3	1
	1	

$$(110100010)_2$$

6(C) Suppose the Lagrangian of a mechanical system is given by $L = \frac{1}{2}m(\dot{x}^2 + 2b\dot{x}\dot{y} + c\dot{y}^2) - \frac{1}{2}K(a\dot{x}^2 + 2b\dot{x}\dot{y} + c\dot{y}^2)$ where $a, b, c, m (> 0), K (> 0)$ are constants and $b^2 \neq ac$. Write down the Lagrangian equations of motion and identify the system.

Soln: Given $L = \frac{1}{2}m(\dot{x}^2 + 2b\dot{x}\dot{y} + c\dot{y}^2) - \frac{1}{2}K(a\dot{x}^2 + 2b\dot{x}\dot{y} + c\dot{y}^2)$

To find Lagrangian equation of motion

Here variables involved x, y, \dot{x}, \dot{y}

$$\therefore \frac{\partial L}{\partial \dot{x}} = -\frac{1}{2}K(2ax + 2by) = -K(ax + by) \quad \textcircled{1}$$

$$\frac{\partial L}{\partial \dot{y}} = \frac{1}{2}m(2a\dot{x} + 2b\dot{y}) = m(a\dot{x} + b\dot{y}) \quad \textcircled{2}$$

and $\frac{\partial L}{\partial y} = -\frac{1}{2}K(2bx + 2cy) = -K(bx + cy) \quad \textcircled{3}$

$$\frac{\partial L}{\partial \dot{y}} = \frac{1}{2}m(2b\dot{x} + 2c\dot{y}) = m(b\dot{x} + c\dot{y}) \quad \textcircled{4}$$

Thus, Lagrangian equations of motion are

$$\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}} \right) - \frac{\partial L}{\partial x} = 0 \quad \textcircled{5}$$

and, $\frac{d}{dt} \left(\frac{\partial L}{\partial \dot{y}} \right) - \frac{\partial L}{\partial y} = 0 \quad \textcircled{6}$

$$\therefore \textcircled{5} \Rightarrow m(a\ddot{x} + b\ddot{y}) + K(ax + by) = 0 \quad \textcircled{7}$$

$$\textcircled{6} \Rightarrow m(b\ddot{x} + c\ddot{y}) + K(bx + cy) = 0 \quad \textcircled{8}$$

For equations of motion

$$\textcircled{7} \times c - \textcircled{8} \times b \text{ gives } m(ac - b^2)\ddot{x} + K(ac - b^2)x = 0$$

$$\Rightarrow \boxed{\ddot{x} = -\left(\frac{K}{m}\right)x} \text{ at } b^2 - ac \neq 0 \quad \textcircled{A}$$

Similarly $\textcircled{7} \times b - \textcircled{8} \times a$ gives.

$$m(b^2 - ac)\ddot{y} + k(b^2 - ac)y = 0$$

$$\Rightarrow \boxed{\ddot{y} = -\left(\frac{k}{m}\right)y} \quad \text{as } b^2 - ac \neq 0 \quad \textcircled{B}$$

from \textcircled{A} & \textcircled{B} are the required equations of motion, and the system is a 2-D Harmonic oscillator.

7(a) Solve the partial differential equation

$$(2D^2 - 5DD' + 2D'^2)Z = 5\sin(2x+y) + 24(y-x) + e^{3x+4y}$$

where; $D = \frac{\partial}{\partial x}$ and $D' = \frac{\partial}{\partial y}$

Sol. Given P.D.E. is

$$(2D^2 - 5DD' + 2D'^2)Z = 5\sin(2x+y) + 24(y-x) + e^{3x+4y}$$

The auxillary of the given equation is

$$2m^2 - 5m + 2 = 0$$

$$(2m-1)(m-1) = 0$$

$$m = \frac{1}{2}, 1$$

$$\therefore C.F. = \phi_1(y+x/2) + \phi_2(y+2x)$$

$$= \phi_1[\frac{1}{2}(2y+x)] + \phi_2(y+2x).$$

$$\boxed{C.F. = \phi_1(2y+x) + \phi_2(y+2x)}$$

where, ϕ_1, ϕ_2 being arbitrary functions.

$$\text{Now P.I.} = \frac{1}{2D^2 - 5DD' + 2D'^2} [5\sin(2x+y) + 24(y-x) + e^{3x+4y}]$$

$$\begin{aligned} \text{P.I.} &= \frac{1}{2D^2 - 5DD' + 2D'^2} [5\sin(2x+y)] + \frac{1}{2D^2 - 5DD' + 2D'^2} 24(y-x) \\ &\quad + \frac{1}{2D^2 - 5DD' + 2D'^2} \cdot e^{3x+4y}. \end{aligned} \quad \text{P.I.-2}$$

$$\text{P.I.}_2 = \frac{1}{2D^2 - 5DD' + 2D'^2} [24(y-x)]$$

$$= 24 \cdot \frac{1}{2D^2 - 5DD' + 2D'^2} (y-x)$$

$$= 24 \cdot \frac{1}{2(-1)^2 - 5 \times 1 \times -1 + 2 \times 1^2} \iint v dv dv \quad [\because v = y-x]$$

$$= \frac{24}{20} \frac{24}{20} \int \frac{v^2}{2} dv$$

$$P.I_2 = \frac{24}{20} \cdot \left(\frac{y^3}{6}\right) = \frac{1}{5}(y-x)^3$$

$$P.I_3 = \frac{1}{2D^2 - 5DD' + 2D'^2} \cdot e^{2x+3y}.$$

$$= e^{2x+3y} \cdot \frac{1}{2(2)^2 - 5 \times 2 \times 3 + 2 \times 9}$$

$$= e^{2x+3y} \cdot \frac{1}{8 - 30 + 18} = \frac{-e^{2x+3y}}{4}$$

$$P.I_1 = \frac{1}{2D^2 - 5DD' + 2D'^2} [5 \sin(2x+y)]$$

$$= 5 \left[\frac{1}{2D^2 - 5DD' + 2D'^2} \sin(2x+y) \right]$$

$$= 5 \cdot \frac{1}{(D-2D)} \left[\frac{5}{(2D-D')} \sin(2x+y) \right]$$

$$= 5 \cdot \frac{1}{(D-2D')} \left[\frac{1}{2 \cdot 2 - 1} \int \sin v dv \right] \text{ where } v = 2x+y$$

$$5 \cdot \frac{1}{(D-2D')} \times \frac{1}{3} (-\cos v)$$

$$P.I_1 = -\frac{5}{3} \cdot \frac{1}{D-2D'} \cos v = -\frac{5}{3} \cdot \frac{x}{1 \cdot 1} \cos(2x+y)$$

$$\therefore P.I = P.I_1 + P.I_2 + P.I_3$$

$$P.I = -\frac{5}{3} x \cos(2x+y) + \frac{1}{5} (y-x)^3 - \frac{e^{2x+3y}}{4}$$

$$\therefore G.S = y = C.F + P.I$$

$$y = \phi_1(2y+x) + \phi_2(y+2x) + \left(-\frac{5}{3}\right)x \cos(2x+y) \\ + \frac{1}{5}(y-x)^3 - \frac{1}{4} e^{2x+3y}$$

7(b) Find the values of the constants a, b, c such that the quadrature formula

$\int_0^h f(x) dx = h \left[af(0) + bf\left(\frac{h}{3}\right) + cf(h) \right]$ is exact for polynomials of as high degree as possible, and hence find the order of the truncation error.

Sol'n: Making the method exact for polynomials of degree upto 2, we obtain

$$f(x) = 1 : h = h(a+b+c) \text{ (or)} a+b+c=1$$

$$f(x) = x : \frac{h^2}{2} = h\left(\frac{bh}{3} + ch\right) \text{ (or)} \frac{1}{3}b + c = \frac{1}{2}$$

$$f(x) = x^2 : \frac{h^3}{3} = h\left(\frac{bh^2}{9} + ch^2\right) \text{ (or)} \frac{1}{9}b + c = \frac{1}{3}.$$

Solving the above equations, we get $a=0, b=\frac{3}{4}$
and $c=\frac{1}{4}$.

Hence the required formula is

$$\int_0^h f(x) dx = \frac{h}{4} [3f(h/3) + f(h)]$$

The truncation error of the formula is given by

$$T.E = \frac{C}{3!} f'''(\xi), 0 < \xi < h$$

$$\text{where } C = \int_0^h x^3 dx - h \left[\frac{bh^3}{27} + ch^3 \right] = -\frac{h^4}{36}$$

Hence we have.

$$T.E = -\frac{h^4}{216} f'''(\xi) = O(h^4).$$

7(c), The Hamiltonian of a mechanical system is given by,

$H = p_1 q_1 - aq_1^2 + bq_2^2 - p_2 q_2$, where a, b are the constants,
 Solve the Hamiltonian equations and show that

$$\frac{p_2 - bq_2}{q_1} = \text{constant.}$$

Sol'n: Let us consider the generalised coordinates $q_1, q_2 \dots q_n$ and the generalised components of momentum $p_1, p_2 \dots p_n$.

Now we have the Hamiltonian equation

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \quad \text{and} \quad \dot{q}_i = \frac{\partial H}{\partial p_i}$$

where $i = 1, 2 \dots$

$$\dot{p}_1 = -\frac{\partial H}{\partial q_1}, \quad \dot{q}_1 = \frac{\partial H}{\partial p_1}, \quad \dot{q}_2 = \frac{\partial H}{\partial p_2}, \quad \dot{p}_2 = -\frac{\partial H}{\partial q_2}$$

$$\dot{q}_1 = q_1 \Rightarrow \frac{dq_1}{dt} = q_1 \Rightarrow q_1 = C_1 e^t$$

$$\dot{p}_1 = -\frac{\partial H}{\partial q_1} = -[p_1, -2aq_1] = 2aq_1 - p_1$$

$$\frac{dp_1}{dt} = 2ac_1 e^t - p_1$$

$$\frac{dp_1}{dt} + p_1 = 2ac_1 e^t$$

Integrating factor = e^t

$$p_1 e^t = \int 2ac_1 e^t \cdot e^t dt$$

$$p_1 e^t = 2ac_1 \int e^{2t} dt$$

$$= ac_1 e^{2t} + C_2$$

$$P_1 = ac_1 e^t + c_2 e^{-t}$$

$$\dot{q}_2 = -q_2 \Rightarrow q_2 = c_3 e^{-t}$$

$$\frac{\partial P_2}{\partial t} = - [2bq_2 - P_2] = P_2 - 2bq_2$$

$$\frac{\partial P_2}{\partial t} - P_2 = -2bq_2 = -2bc_3 e^{-t}$$

Integrating factor = e^{-t}

$$P_2 \times e^{-t} = 2bc_3 \int e^{-2t} dt = bc_3 e^{-2t} + C_4$$

$$\Rightarrow P_2 = bc_3 e^{-t} + C_4 e^{+t}$$

Consider

$$\frac{P_2 - bq_2}{q_1} = \frac{bc_3 e^{-t} + C_4 e^{+t} - bc_3 e^{-t}}{c_1 e^t}$$

$$= \frac{C_4 e^{+t}}{c_1 e^t}$$

$$= \frac{C_4}{c_1}$$

= Constant

$\rightarrow \theta(a)$

Simplify the boolean expression:

$(a+b) \cdot (\bar{b}+c) + b \cdot (\bar{a}+\bar{c})$ by using the law of boolean algebra. From its truth table write it in minterm normal form.

$\rightarrow \text{Sol'n}$

$$(a+b)(\bar{b}+c) + b(\bar{a}+\bar{c})$$

$$= a\bar{b} + ac + b\bar{b} + bc + b\bar{a} + b\bar{c}$$

$$= a\bar{b} + ac + 0 + b[c+\bar{c}] + b\bar{a} \quad [\because b\bar{b}=0]$$

$$= a\bar{b} + ac + b[1] + b\bar{a} \quad [\because c+\bar{c}=1]$$

$$= a\bar{b} + ac + b[1+\bar{a}]$$

$$= a\bar{b} + ac + b$$

$$[\because 1+\bar{a}=1]$$

a	b	c	\bar{b}	$a\bar{b}$	ac	$a\bar{b}+ac$	$z=a\bar{b}+ac+b$	Minterm
0	0	0	1	0	0	0	0	$\bar{a}\bar{b}\bar{c}$
0	0	1	1	0	0	0	0	$\bar{a}\bar{b}c$
0	1	0	0	0	0	0	1	$\bar{a}b\bar{c}$
0	1	1	0	0	0	0	1	$\bar{a}bc$
1	0	0	1	1	0	1	1	$a\bar{b}\bar{c}$
1	0	1	1	1	1	1	1	$a\bar{b}c$
1	1	0	0	0	0	0	1	$ab\bar{c}$
1	1	1	0	0	1	1	1	abc

From Truth Table, Minterm normal form is

$$\bar{a}\bar{b}\bar{c} + \bar{a}\bar{b}c + a\bar{b}\bar{c} + ab\bar{c} + a\bar{b}c + abc$$

8(b)

for a two-dimensional potential flow, the velocity potential is given by $\phi = x^2y - xy^2 + \frac{1}{3}(x^3 - y^3)$. Determine the velocity components along the directions x and y . Also, determine the stream function ψ and check whether ϕ represents a possible case of flow or not.

Sol'n

Given velocity potential $\phi = x^2y - xy^2 + \frac{1}{3}(x^3 - y^3)$

The knowledge of velocity potential (ϕ) immediately gives the velocity components. Its gradient gives rise to the velocity vector (i.e.)

$$\vec{\nabla} = \vec{\nabla}\phi$$

$$(05) \quad u\vec{i} + v\vec{j} + \omega\vec{k} = \frac{\partial\phi}{\partial x}\vec{i} + \frac{\partial\phi}{\partial y}\vec{j} + \frac{\partial\phi}{\partial z}\vec{k}$$

thus, the velocity components can be written as.

$$u = \frac{\partial\phi}{\partial x}, \quad v = \frac{\partial\phi}{\partial y}, \quad \omega = \frac{\partial\phi}{\partial z}$$

Then,

$$u = \frac{\partial}{\partial x} \left[x^2y - xy^2 + \frac{1}{3}(x^3 - y^3) \right]$$

$$= 2xy - y^2 + x^2$$

$$v = \frac{\partial}{\partial y} \left[x^2y - xy^2 + \frac{1}{3}(x^3 - y^3) \right]$$

$$= x^2 - 2xy + \frac{1}{3}(-3y^2) = x^2 - y^2 - 2xy$$

Stream function ψ : Since $\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}$

$$x^2 - y^2 - 2xy = \frac{\partial\psi}{\partial y}$$

$$\Rightarrow \psi = x^2y - \frac{y^3}{3} - 2xy^2 + f(x)$$

ϕ will represent a case of flow if it satisfies
the Laplace equation.

i.e. $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$

$$\frac{\partial \phi}{\partial x} = 2xy - y^2 + \frac{1}{3}(3x^2) \Rightarrow \frac{\partial^2 \phi}{\partial x^2} = 2y + 2x$$

Similarly $\frac{\partial \phi}{\partial y} = x^2 - 2xy + \frac{1}{3}(-3y^2) \Rightarrow \frac{\partial^2 \phi}{\partial y^2} = -2x - 2y$

since $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$

Thus ϕ is a possible case of flow.

8(c) A thin annulus occupies the region $a \leq r \leq b$, $0 \leq \theta \leq 2\pi$. The faces are insulated. Along the inner edge the temperature is maintained at 0° , while along the outer edge the temperature is held at $T = K \cos \theta/2$, where K is a constant. Determine the temperature distribution in the annulus.

Solⁿ:

We are given a circular annulus whose inner and outer radii are a and b respectively.

The steady state temperature $T(r, \theta)$ at any point $P(r, \theta)$ of the annulus is the solution of the Laplace's equation in polar co-ordinates (r, θ) namely

$$r^2 \left(\frac{\partial^2 T}{\partial r^2} \right) + r \left(\frac{\partial T}{\partial r} \right) + \frac{\partial^2 T}{\partial \theta^2} = 0 \quad (1)$$

Since the temperatures along the inner ($r=a$) and outer boundary ($r=b$) are

maintained at 0° and $K \cos \theta/2$ respectively

$$\text{i.e., } T(a, \theta) = 0 \quad \text{and } T(b, \theta) = K \cos \theta/2 \quad (2)$$

Clearly the temperature function $T(r, \theta)$ must be periodic in θ of period 2π . Accordingly, we now proceed to solve ①.

Suppose ① has a solution of the form

$$T(r, \theta) = R(r) H(\theta) \quad \text{--- ③}$$

where R and H are functions of r and θ respectively

Using ③, ① reduces to

$$r^2 R'' H + rR' H + RH'' = 0$$

$$\Rightarrow (r^2 R'' + rR') H = -RH''$$

$$\Rightarrow \frac{r^2 R'' + rR'}{R} = -\frac{H''}{H} \quad \text{--- ④}$$

Since L.H.S. of ④ is a function of r only and the R.H.S. is a function of θ only, the two sides of ④ must be equal to the same constant say μ .

Then ④ gives

$$r^2 R'' + rR' - \mu R = 0 \quad \text{--- ⑤}$$

$$\text{and } H'' + \mu H = 0 \quad \text{--- ⑥}$$

As usual, we first reduce linear homogeneous differential equation ⑤ into a linear differential equation with constant coefficients.

$$\text{Re-writing ③, } (r^2 D^2 + rD - \mu) R = 0 \quad \text{--- ⑦}$$

$$\text{where } D = \frac{d}{dr}$$

Let $r = e^{\theta} \Rightarrow z = \log r$. and $D_1 = \frac{d}{dr}$

Then w.r.t $r D_1 = D_1$ and $r^{\gamma} D^r = D_1(D_1 - 1)$.

$$\textcircled{8} \quad (D_1(D_1 - 1) + D_1 - M)R = 0 \\ \Rightarrow (D_1^2 - M)R = 0 \quad \text{--- } \textcircled{8}$$

Again, let $D_2 = \frac{d}{d\theta}$. Then $\textcircled{6}$ may be
re-written as $(D_2^2 + M)H = 0 \quad \text{--- } \textcircled{9}$

The solutions of $\textcircled{8}$ and $\textcircled{9}$ depend on M .
Consider following cases:

Case(i): Let $M \neq 0$. Then $\textcircled{8}$ and $\textcircled{9}$ reduces to

$$\frac{d^2 R}{dr^2} = 0 \quad \text{and} \quad \frac{d^2 H}{d\theta^2} = 0$$

solving these,

$$R(r) = C_1 r^2 + C_2 = C_1 \log r + C_2$$

$$\text{and } H = C_3 \theta + C_4.$$

Hence, from $\textcircled{3}$, solution of $\textcircled{1}$ is of the form

$$T(r, \theta) = (C_1 \log r + C_2)(C_3 \theta + C_4) \quad \text{--- } \textcircled{10}$$

Since $T(r, \theta)$ is periodic in θ , we must take $C_3 = 0$. Then equation $\textcircled{10}$ becomes

$$\begin{aligned} T(r, \theta) &= (C_1 \log r + C_2) C_4 \\ &= \frac{1}{2}(a_0 \log r + b_0) \end{aligned} \quad \text{--- } \textcircled{11}$$

where $a_0 = 2C_1 C_4$ and $b_0 = C_2 C_4$ are new arbitrary constants.

Case ii): Let $\mu = \lambda^2$, where $\lambda \neq 0$. Then

(8) and (9) become

$$(D_1 - \lambda^2)R = 0 \text{ and } (D_2 + \lambda^2)H = 0. \quad (12)$$

Note that we cannot choose $\mu = -\lambda^2$ because it will lead to $(D_2 - \lambda^2)H = 0$ whose solution will not contain trigonometric functions and hence periodic nature of $T(r, \theta)$ will not be attained]

Solving (12)

$$\begin{aligned} R(r) &= (C_5 e^{r\lambda^2} + C_6 e^{-r\lambda^2}) \\ &= C_5 (e^{\lambda r})^\lambda + (C_6 e^{-\lambda r})^{-\lambda} = C_5 r^\lambda + C_6 r^{-\lambda} \end{aligned}$$

$$\text{and } H(\theta) = C_7 \cos \lambda \theta + C_8 \sin \lambda \theta.$$

Hence, from (3), a solution of (1) is of the form

$$T(r, \theta) = (C_5 r^\lambda + C_6 r^{-\lambda})(C_7 \cos \lambda \theta + C_8 \sin \lambda \theta) \quad (13)$$

Since $T(r, \theta)$ is periodic in θ with period 2π .

we must take $\lambda = n$, where $n = 1, 2, 3, \dots$

Hence (13) takes the form

$$T(r, \theta) = (C_5 r^n + C_6 r^{-n})(C_7 \cos n\theta + C_8 \sin n\theta) \quad (14)$$

$n = 1, 2, 3, \dots$

With the help of (1) and (ii), the most general solution of (1) is

$$T(r, \theta) = \frac{a_0 \log r + b_0}{2} + \sum_{n=1}^{\infty} (a_n r^n + b_n \bar{r}^n) \cos n\theta$$

which holds for $a \leq r \leq b$. (1)

Here $a_n = c_5 C_n$, $b_n = c_6 C_n$; $C_n = c_5 C_n$

$c_n = c_6 C_n$ are new arbitrary constants.

Putting $r=a$, and $r=b$ by turn in (1)

and B.C (2) we have

$$0 = \frac{a_0 \log a + b_0}{2} + \sum_{n=1}^{\infty} (a_n a^n + b_n \bar{a}^n) \cos n\theta + (c_n a^n + d_n \bar{a}^n) \sin n\theta \quad \text{--- (16)}$$

$$k \cos \frac{\theta}{2} = \frac{a_0 \log b + b_0}{2} + \sum_{n=1}^{\infty} (a_n b^n + b_n \bar{b}^n) \cos n\theta + (c_n b^n + d_n \bar{b}^n) \sin n\theta \quad \text{--- (17)}$$

(16) and (17) are usual expansions of $f_1(\theta) = 0$

and $f_2(\theta) = k \cos \frac{\theta}{2}$ as Fourier series in

$(0, 2\pi)$.

Hence we have

$$a_0 \log a + b_0 = \frac{1}{\pi} \int_0^{2\pi} (0) d\theta, \quad a_0 \log b + b_0 = \frac{1}{\pi} \int_0^{2\pi} (k \cos \frac{\theta}{2}) d\theta.$$

Solving these we get

$$a_0 \log a + b_0 = 0 \quad \text{and} \quad a_0 \log b + b_0 = 0$$

$$\rightarrow a_0 = 0 \quad \text{and} \quad b_0 = 0 \quad \text{--- (18)}$$

$$a_n a^n + b_n b^n = \frac{1}{\pi} \int_0^{2\pi} (0) \cos n\theta = 0 \Rightarrow b_n = -a_n^{2n}$$

$$\& a_n b^n + b_n a^n = \frac{1}{\pi} \int_0^{2\pi} k \cos \frac{n+1}{2}\theta \cos n\theta d\theta.$$

$$a_n b^n + b_n a^n = \frac{k}{2\pi} \int_0^{2\pi} \left[\cos(n+\frac{1}{2})\theta + \cos(n-\frac{1}{2})\theta \right] d\theta$$

$$= \frac{k}{2\pi} \left[\frac{\sin(n+\frac{1}{2})\theta}{n+\frac{1}{2}} + \frac{\sin(n-\frac{1}{2})\theta}{n-\frac{1}{2}} \right]_0^{\pi}$$

$$= 0$$

$$\Rightarrow i a_n b^n - a_n b^n = 0$$

$$\Rightarrow a_n [b^n - a^n b^n] = 0$$

$$\Rightarrow a_n = 0 \quad (a_n \neq 0 \text{ and } b_n \neq 0)$$

$$c_n a^n + d_n b^n = \frac{1}{\pi} \int_0^{2\pi} (0) \sin n\theta d\theta = 0$$

$$\& c_n b^n + d_n a^n = \frac{1}{\pi} \int_0^{2\pi} k \cos \frac{n+1}{2}\theta \sin n\theta d\theta$$

$$c_n b^n + d_n a^n = \frac{k}{2\pi} \int_0^{2\pi} \left[\sin(n+\frac{1}{2})\theta + \sin(n-\frac{1}{2})\theta \right] d\theta$$

$$= \frac{-k}{2\pi} \left[\frac{\cos(n+\frac{1}{2})\theta}{n+\frac{1}{2}} + \frac{\cos(n-\frac{1}{2})\theta}{(n-\frac{1}{2})} \right]_0^{\pi}$$

$$= \frac{-k}{2\pi} \left[-\frac{1}{n+\frac{1}{2}} - \frac{1}{n+\frac{1}{2}} - \frac{1}{n-\frac{1}{2}} - \frac{1}{n-\frac{1}{2}} \right]$$

$$= \frac{k}{\pi} \left(\frac{1}{n+\frac{1}{2}} + \frac{1}{n-\frac{1}{2}} \right)$$

$$= \frac{k}{\pi} \frac{2n}{(n^2-1)}$$

$$c_n b^n + d_n \bar{a}^n = \frac{8kn}{\pi (4n-1)} \quad \text{(20)}$$

Now we have

$$c_n a^n + d_n \bar{a}^{2n} = 0 \\ \Rightarrow d_n = -c_n a^{2n}$$

∴ from (20),

$$c_n b^n - c_n a^{2n} \bar{b}^{-n} = \frac{8kn}{\pi (4n-1)}$$

$$c_n [b^n - \bar{b}^{-n} a^{2n}] = \frac{8kn}{\pi (4n-1)}$$

$$\therefore \text{from } (2) \quad c_n = \frac{8kn}{\pi (4n-1)} \cdot \frac{1}{(b^n - \bar{b}^{-n} a^{2n})}$$

$$T(r, \theta) = (c_n r^n + d_n r^{-n}) \sin n\theta$$

$$= \sum_{n=1}^{\infty} (c_n r^n - c_n a^{2n} r^{-n}) \sin n\theta \\ (\because d_n = -c_n a^{2n})$$

$$= \sum_{n=1}^{\infty} c_n [r^n - a^{2n} r^{-n}] \sin n\theta$$

$$= \frac{8k}{\pi} \sum_{n=1}^{\infty} \frac{n}{4n-1} \frac{[r^n - a^{2n} r^{-n}]}{(b^n - \bar{b}^{-n} a^{2n})}$$

$$T(r, \theta) = \frac{8IC}{\pi} \sum_{n=1}^{\infty} \frac{n}{(4n^2-1)} \left[\frac{(r/a)^n - (a/b)^n}{(b/a)^n - (a/b)^n} \right] \sin n\theta$$

which is the required
temperature distribution in
the given annulus.

~~SCIENCE~~

NOTE: Q.No. 8.(c)
“STUDENTS ARE ADVISED TO
WRITE CONCISE ANSWER IN
EXAM”
