MAINS TEST SERIES - 2020

TEST-1 (PAPER-I) ANSWER KEY

LINEAR ALGEBRA, CALCULUS & 3-DIMENSIONAL GEOMETRY

1(a). If S and T are subspaces of $\mathbb{R}^4$ given by
$S = \{(x,y,z,w) \in \mathbb{R}^4 : 2x+y+3z+w=0\}$ and
$T = \{(x,y,z,w) \in \mathbb{R}^4 : x+2y+z+3w=0\}$, find $\dim(S\cap T)$.

Sol'n: $S\cap T$ consists of those vectors which satisfy the conditions defining S and the conditions defining T.

i.e, the two equations $2x+y+3z+w=0$ — ①
$x+2y+z+3w=0$ — ②

i.e, $S\cap T = \{(x,y,z,w) \in \mathbb{R}^4 / 2x+y+3z+w=0$ and $x+2y+z+3w=0\}$

Let us solve ① & ② for x,y,z,w.
We write single matrix equation $AX=O$

where $A = \begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 2 & 1 & 3 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ $O = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Convert A into echelon form by using elementary row transformations.

$$A = \begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 2 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 3 & 1 \\ 0 & 3/2 & -1/2 & 5/2 \end{bmatrix} \quad R_2 \to R_2 - \tfrac{1}{2}R_1$$

Rewrite the single matrix equation, we get

$$\begin{bmatrix} 2 & 1 & 3 & 1 \\ 0 & 3/2 & -1/2 & 5/2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 2x+y+3z+w=0$ — ③ and $\frac{3}{2}y - \frac{1}{2}z + \frac{5w}{2} = 0 \Rightarrow y = \frac{z-5w}{3}$ — ④

$\therefore$ from ③ & ④, we get $x = \frac{-5z+w}{3}$ — ⑤

$\therefore S\cap T = \left\{\left(-\frac{5}{3}z + \frac{w}{3}, \frac{1}{3}z - \frac{5w}{3}, z, w\right) / z, w \in \mathbb{R}\right\}$

$= \left\{ z\left(-\frac{5}{3}, -\frac{1}{3}, 1, 0\right) + w\left(\frac{1}{3}, -\frac{5}{3}, 0, 1\right) / z, w \in \mathbb{R}\right\}$

$\therefore \left\{\left(-\frac{5}{3}, \frac{1}{3}, 1, 0\right), \left(\frac{1}{3}, -\frac{5}{3}, 0, 1\right)\right\}$ is a basis of $S\cap T$

and $\dim(S\cap T) = 2$.

1(b) If $T : P_2(x) \rightarrow P_3(x)$ is such that $T(f(x)) = 5 + \int_0^x f(t)\,dt$, then choosing $\{1, 1+x, 1-x^2\}$ and $\{1, x, x^2, x^3\}$ as bases of $P_2(x)$ and $P_3(x)$ respectively, find the matrix of T.

sol^n: Given that

$$T(f(x)) = f(x) + 5\int_0^x f(t)\,dt \quad \text{——— (1)}$$

let $S = \{1, 1+x, 1-x^2\}$ and

$T = \{1, x, x^2, x^3\}$ be the bases of $P_2(x)$ & $P_3(x)$

$$\therefore T(1) = 1 + 5\int_0^x 1\,dt = 1 + 5x \quad \text{——— (a)}$$

$$T(1+x) = 1 + x + 5\int_0^x (1+t)\,dt$$

$$= 1 + x + 5\left[t + \tfrac{t^2}{2}\right]_0^x = 1 + x + 5x + \frac{5x^2}{2}$$

$$= 1 + 6x + \frac{5x^2}{2} \quad \text{——— (b)}$$

$$T(1-x^2) = 1 - x^2 + 5\int_0^x (1-t^2)\,dt$$

$$= 1 - x^2 + 5\left[t - \tfrac{t^3}{3}\right]_0^x$$

$$= 1 - x^2 + 5\left[x - \frac{x^3}{3}\right] = 1 - x^2 + 5x - \frac{5x^3}{3}$$

$$= 1 + 5x - x^2 - \frac{5x^3}{3} \quad \text{——— (c)}$$

$$\therefore T(1) = 1 + 5(x) + 0x^2 - 0x^3.$$

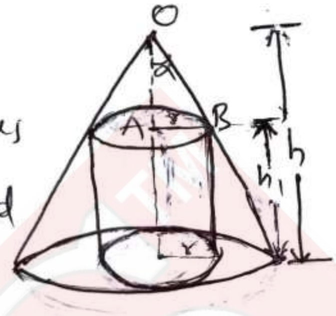
$$T(1+x) = 1(1) + 6(x) + \left(\tfrac{5}{2}\right)x^2 + 0x^3$$

$$T(1-x^2) = 1(1) + 5(x) + (-1)x^2 + \left(-\tfrac{5}{3}\right)x^3.$$

The required matrix representation:

$$[T : P_2, P_3] = \begin{bmatrix} 1 & 1 & 1 \\ 5 & 6 & 5 \\ 0 & 5/2 & -1 \\ 0 & 0 & -5/3 \end{bmatrix}.$$

1(c) Find the volume of the greatest cylinder that can be inscribed in a cone of height h and semi-vertical angle α.



Sol'n:

Let a cylinder of base radius 'r' and height h_1 is inscribed in a cone of height h and semivertical angle α.

In right-angled Δ^{le} AOB,

$$\tan\alpha = \frac{AB}{OA} = \frac{r}{h-h_1}$$

$$\Rightarrow r = (h-h_1)\tan\alpha.$$

$\therefore$ volume of the cylinder $V = \pi[(h-h_1)\tan\alpha]^2 h_1$. $\quad (\because V = \pi r^2 h)$

$$= \pi h_1 (h-h_1)^2 \tan^2\alpha \quad \text{--- (1)}$$

Diff. (1) w.r.t. h_1, we get

$$\frac{dv}{dh_1} = \pi\tan^2\alpha\,[h_1 \cdot 2(h-h_1)(-1) + (h-h_1)^2].$$

$$= \pi\tan^2\alpha\,(h-h_1)(h-3h_1). \quad \text{--- (2)}$$

For maximum volume V, $\frac{dv}{dh_1} = 0$

$$\Rightarrow \pi\tan^2\alpha\,(h-h_1)(h-3h_1) = 0$$

$$\Rightarrow h-h_1 = 0 \text{ (or) } h = 3h_1$$

$$\Rightarrow h = h_1 \text{ (or) } h_1 = \frac{h}{3}$$

$$\therefore h_1 = \frac{h}{3} \quad (\because h \neq h_1)$$

Again Diff (2) w.r.t. h_1,

$$\frac{d^2v}{dh_1^2} = \pi\tan^2\alpha\,[(h-h_1)(-3) + (h-3h_1)(-1)]$$

$$\left.\frac{d^2v}{dh_1^2}\right|_{h_1=\frac{h}{3}} = -2\pi h\tan^2\alpha < 0.$$

$\therefore$ volume is maximum for $h_1 = \frac{h}{3}$

$$\therefore V_{max} = \pi\tan^2\alpha\left(\frac{h}{3}\right)\cdot\left(h-\frac{h}{3}\right)^2 = \frac{4}{27}\pi h^3\tan^2\alpha.$$

1(d) Evaluate the following integral:

$$\int_{\pi/6}^{\pi/3} \frac{\sqrt[3]{\sin x}}{\sqrt[3]{\sin x}+\sqrt[3]{\cos x}}\,dx.$$

Sol^n: Given $I = \displaystyle\int_{\pi/6}^{\pi/3} \frac{\sqrt[3]{\sin x}}{\sqrt[3]{\sin x}+\sqrt[3]{\cos x}}\,dx$ — ①.

$$\int_a^b f(x)\,dx = \int_b^a f(a+b-x)\,dx$$

using this property

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt[3]{\sin(\pi/2-x)}\,dx}{\sqrt[3]{\sin(\pi/2-x)}+\sqrt[3]{\cos(\pi/2-x)}}$$

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt[3]{\cos x}}{\sqrt[3]{\sin x}+\sqrt[3]{\cos x}} \quad \text{— ②}$$

Add ① & ②

$$2I = \int_{\pi/6}^{\pi/3}\left[\frac{\sqrt[3]{\sin x}}{\sqrt[3]{\sin x}+\sqrt[3]{\cos x}}+\frac{\sqrt[3]{\cos x}}{\sqrt[3]{\sin x}+\sqrt[3]{\cos x}}\right]dx.$$

$$2I = \int_{\pi/6}^{\pi/3} dx.$$

$$I = \tfrac{1}{2}\left(\tfrac{\pi}{3}-\tfrac{\pi}{6}\right)$$

$$= \tfrac{1}{2}\left(\tfrac{\pi}{6}\right)$$

$$\boxed{I = \frac{\pi}{12}}$$

1(e) → A triangle, the length of whose sides are a, b and c is placed so that the middle points of the sides are on the axes. show that the lengths α, β, γ intercepted on the axes are given by

$8\alpha^2 = b^2+c^2-a^2$, $8\beta^2 = c^2+a^2-b^2$, $8\gamma^2 = a^2+b^2-c^2$.

and find the coordinates of its vertices.

sol^n: Let ABC be the given triangle which makes intercepts α, β, γ on the axes.

$\therefore$ the equation of its plane is

$$\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma} = 1 \quad \text{——— (1)}$$

The plane (1) meets the axes in $(\alpha,0,0)$, $(0,\beta,0)$ and $(0,0,\gamma)$

Also we know that the line joining the mid points of two sides of a triangle is parallel to the third side and half of it.

Let the mid-point of the side BC of the triangle ABC be on the x-axis, those of CA and AB on y and z-axes respectively.

Then we have $\sqrt{(\beta^2+\gamma^2)} = \frac{1}{2}a$ etc.

$$\Rightarrow \beta^2+\gamma^2 = \tfrac{1}{4}a^2, \quad \gamma^2+\alpha^2 = \tfrac{1}{4}b^2, \quad \alpha^2+\beta^2 = \tfrac{1}{4}c^2$$

Adding, $\alpha^2+\beta^2+\gamma^2 = \frac{1}{8}(a^2+b^2+c^2)$

$$\therefore \alpha^2 = (\alpha^2+\beta^2+\gamma^2)-(\beta^2+\gamma^2) = \tfrac{1}{8}(a^2+b^2+c^2)-\tfrac{1}{4}a^2$$
$$= \tfrac{1}{8}(b^2+c^2-a^2).$$

Similarly $\beta^2 = \frac{1}{8}(c^2+a^2-b^2)$ and $\gamma^2 = \frac{1}{8}(a^2+b^2-c^2)$

i.e. $8\alpha^2 = b^2+c^2-a^2$, $8\beta^2 = c^2+a^2-b^2$ and $8\gamma^2 = a^2+b^2-c^2$.

coordinates of the vertices A, B, C

Let A be (x_1, y_1, z_1). Then as mid-point of AB is $(0,0,\gamma)$

$\therefore$ B is $(-x_1, -y_1, 2\gamma - z_1)$.

Also the mid-point of AC is $(0, \beta, 0)$, so the co-ordinates of C are $(-x_1, 2\beta - y_1, -z_1)$

from these, we have the mid-point of BC as

$$(-x_1, \beta - y_1, \gamma - z_1)$$

But the mid-point of BC is $(\alpha, 0, 0)$, so we have

$-x_1 = \alpha$, $0 - \beta - y_1$ and $0 = \gamma - z_1$

i.e. $x_1 = -\alpha$, $y_1 = \beta$, $z_1 = \gamma$.

Hence A, B and C are $(-\alpha, \beta, \gamma)$; $(\alpha, -\beta, \gamma)$ and $(\alpha, \beta, -\gamma)$ respectively.

2(a) Find one vector in R^3 which generates the intersection of V and W, where V is the xy plane and W is the space generated by the vectors (1,2,3) and (1,−1,1).

Sol^n: Let $\mathbb{R}^3 = \{(x, y, z) / x, y, z \in \mathbb{R}\}$ be the given vectorspace.

Given that V is xy-plane

$\therefore V = \{(x, y, z) \in \mathbb{R}^3 / z = 0;\ x, y, z \in \mathbb{R}\}$ —— ①

and given that W is the space generated by the vectors (1,2,3) and (1,−1,1).

For this we find a homogeneous system whose solution set W is generated by

$$S = \{(1,2,3),\ (1,-1,1)\}$$

$$W = \{\alpha(1,2,3) + \beta(1,-1,1);\ \alpha, \beta \in \mathbb{R}^3\}$$

$$= \{(\alpha+\beta,\ 2\alpha-\beta,\ 3\alpha+\beta);\ \alpha, \beta \in \mathbb{R}^3\}$$

$$V = \{(x, y, 0);\ x, y \in \mathbb{R}^3\}$$

Now for $V \cap W$

$x = \alpha + \beta$, $y = 2\alpha - \beta$ and $3\alpha + \beta = 0$

$\Rightarrow \beta = -3\alpha$

$\Rightarrow x = -2\alpha$, $y = 5\alpha$ and $z = 0$

$\therefore V \cap W = \{-2\alpha, 5\alpha, 0;\ \alpha \in \mathbb{R}^3\}$

Clearly $V \cap W$ is spanned by $(-2, 5, 0) \in \mathbb{R}^3$.

2(b) Consider the singular matrix $A = \begin{bmatrix} -1 & 3 & -1 & 1 \\ -3 & 5 & 1 & -1 \\ 10 & -10 & -10 & 14 \\ 4 & -4 & -4 & 8 \end{bmatrix}$ Given that one eigen value of A is 4 and one eigen vector that does not correspond to this eigen value 4 is $(1\ 1\ 0\ 0)^T$. Find all the eigen values of A other than 4 and hence also find the real numbers p, q, r that satisfy the matrix equation $A^4 + pA^3 + qA^2 + rA = 0$.

Sol'n:

Given $A = \begin{bmatrix} -1 & 3 & -1 & 1 \\ -3 & 5 & 1 & -1 \\ 10 & -10 & -10 & 14 \\ 4 & -4 & -4 & 8 \end{bmatrix}$

First convert this matrix into echelon form using row operations.

$R \to -R_1$, $R_2 \to R_2 + 3R_1$, $R_3 \to R_3 - 10R_1$, $R_4 - 4R_1$

$$\sim \begin{bmatrix} 1 & -3 & 1 & -1 \\ 0 & -4 & 4 & -4 \\ 0 & 20 & -20 & 24 \\ 0 & 8 & -8 & 12 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 & 1 & -1 \\ 0 & -4 & 4 & -4 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix} \begin{array}{l} R_3 \to R_3 + 5R_2 \\ R_4 \to R_4 + 2R_2 \end{array}$$

$$\sim \begin{bmatrix} 1 & -3 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix} = A \quad R_2 \to -\tfrac{1}{4}R_2$$

$\therefore$ For Eigen values $|A - \lambda I| = 0$

$$\therefore A-\lambda I = \begin{bmatrix} 1 & -3 & 1 & -1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\therefore A-\lambda I = \begin{bmatrix} 1-\lambda & -3 & 1 & -1 \\ 0 & 1-\lambda & -1 & 1 \\ 0 & 0 & -\lambda & 4 \\ 0 & 0 & 0 & 4-\lambda \end{bmatrix}$$

$$\therefore |A-\lambda I| = (1-\lambda)(1-\lambda)(-\lambda)(4-\lambda) = 0$$

$$\Rightarrow \lambda(\lambda-1)(\lambda-1)(\lambda-4) = 0 \quad \text{——— (1)}$$

$$\lambda = 0, 1, 1, 4$$

By solving (2)

$$\lambda^4 - 6\lambda^3 + 9\lambda^2 - 4\lambda = 0$$

$$\Rightarrow A^4 - 6A^3 + 9A^2 - 4A = 0$$

compare it with given equation

$$A^4 + pA^3 + qA^2 + rA = 0$$

$$\Rightarrow p = -6, \; q = 9, \; r = -4$$

Eigen values other than $\lambda = 4$ are $\lambda = 0, 1, 1$

Now, to check the eigen vector at $\lambda = 4$

$$\begin{bmatrix} -3 & -3 & 1 & -1 \\ 0 & -3 & -1 & 1 \\ 0 & 0 & -4 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

from (iii) row $\rightarrow$ $-x_3 + x_4 = 0$

$$\Rightarrow x_3 = x_4$$

$$-3x_2 = 0 \Rightarrow x_2 = 0$$

$$\therefore -3x_1 = 0 \Rightarrow x_1 = 0$$

$\therefore$ Eigen vector at $\lambda = 4$ is $[0, 0, 1, 1]^T$

Hence, eigen vector $[1, 1, 0, 0]^T$ does not corresponds to eigen value $\lambda = 4$

$\therefore$ Eigen vector $[1, 1, 0, 0]^T$ must be from other eigen values.

2(C) (i) If $z = \tan(y+ax) + (y-ax)^{3/2}$, find the value of $\frac{\partial^2 z}{\partial x^2} - a^2\frac{\partial^2 z}{\partial y^2}$.

(ii) If $u = \tan^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{4}\sin 2u$.

Sol'n: (i) we have $\frac{\partial z}{\partial x} = a\sec^2(y+ax) - \frac{3a}{2}(y-ax)^{1/2}$

$$\frac{\partial^2 z}{\partial x^2} = 2a^2\sec(y+ax)\{\tan(y+ax)\sec(y+ax)\} + \frac{3a^2}{4}(y-ax)^{-1/2} \quad \text{—(1)}$$

$$\frac{\partial z}{\partial y} = \sec^2(y+ax) + \frac{3}{2}(y-ax)^{1/2}$$

$$\frac{\partial^2 z}{\partial y^2} = 2\sec(y+ax)\{\tan(y+ax)\sec(y+ax)\} + \frac{3}{4}(y-ax)^{-1/2} \quad \text{—(2)}$$

from (1) & (2), we obtain

$$\frac{\partial^2 z}{\partial x^2} - a^2\frac{\partial^2 z}{\partial y^2} = 0.$$

(ii) $u = \tan^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right) \Rightarrow \tan u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$

Let $z = \tan u$ —(1).

$$\therefore z = \frac{x+y}{\sqrt{x}+\sqrt{y}} = \frac{x(1+y/x)}{\sqrt{x}(1+\sqrt{y/x})} = x^{1/2} f(y/x) \quad \text{—(2)}$$

So z is a homogeneous function in x and y of degree $\frac{1}{2}$.

By Euler's theorem, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}z$

$\Rightarrow x\left(\sec^2 u\frac{\partial u}{\partial x}\right) + y\left(\sec^2 u\frac{\partial u}{\partial y}\right) = \frac{1}{2}\tan u$ by (1).

$$\Rightarrow x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\frac{\tan u}{\sec^2 u}$$

$$= \frac{1}{2}\sin u\cos u$$

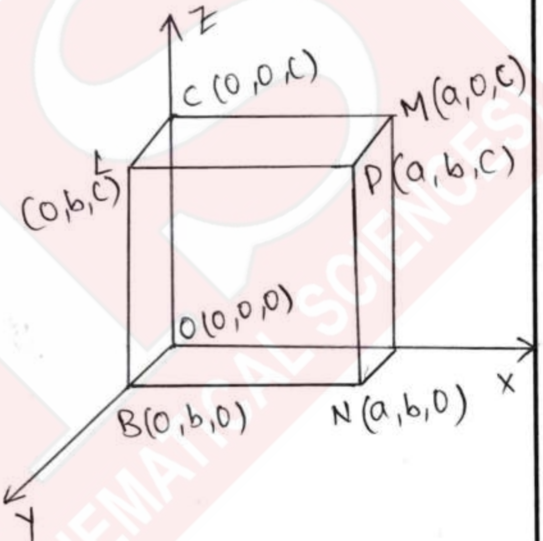
$$= \frac{1}{4}\sin 2u.$$

2(d) Prove that the S.D between the diagonals of rectangular parallelopiped and the edges not meeting it are $\frac{bc}{\sqrt{(b^2+c^2)}}$, $\frac{ca}{\sqrt{(c^2+a^2)}}$, $\frac{ab}{\sqrt{(a^2+b^2)}}$

where a, b, c are the lengths of the edges.

Sol'n: Let the continuous edges OA, OB and OC be taken as the axes of coordinates.

Then the coordinates of the various corners are as shown in the fig.



Now we find the S.D between the diagonal OP and the edge BL which doesnot meet OP.

∴ The equations of OP are

$$\frac{x-0}{a-0}=\frac{y-0}{b-0}=\frac{z-c}{c-0}$$

$$\Rightarrow \frac{x}{a}=\frac{y}{b}=\frac{z}{c} \quad \text{--- (1)}$$

and the equations of BL are

$$\frac{x-0}{a-0}=\frac{y-b}{b-b}=\frac{z-c}{0-0}$$

$$\Rightarrow \frac{x}{a}=\frac{y-b}{0}=\frac{z-c}{0} \quad \text{--- (2)}$$

Let l, m, n be the d.c's of the shortest distance between OP and BL.

Then, Since S.D is ⊥lar to both (1) & (2)

$$\therefore al+bm+cn=0$$

$$\text{and } al+0.m+0.n=0$$

on solving $\dfrac{l}{0-0} = \dfrac{m}{ac-0} = \dfrac{n}{0-ab}$

$$\Rightarrow \frac{l}{0} = \frac{m}{c} = \frac{n}{-b}$$

$\therefore$ dir's of the S.D are $0, c, -b$.

Dividing each by $\sqrt{0+c^2+(-b)^2} = \sqrt{b^2+c^2}$

$$0, \frac{c}{\sqrt{b^2+c^2}}, \frac{-b}{\sqrt{b^2+c^2}}.$$

$\therefore$ The length of S.D = projection of the join of $O(0,0,0)$ and $B(0,b,0)$ on the line of S.D.

$$= (0-0)0 + \frac{c}{\sqrt{b^2+c^2}}(b-0) + \frac{-b}{\sqrt{b^2+c^2}}(0-0)$$

$$= \frac{bc}{\sqrt{b^2+c^2}}$$

Similarly the S.D between OP and the other edges not meeting it (ie. AN and MC) are

$$\frac{ca}{\sqrt{c^2+a^2}} \text{ and } \frac{ab}{\sqrt{a^2+b^2}}.$$

3(a)(i) Let V be the vectorspace of all 2x2 matrices over the field of real numbers. Let W be the set consisting of all matrices with zero determinant. Is W a subspace of V? Justify your answer.

(ii) Find the dimension and a basis for the space W of all solutions of the following homogeneous system using matrix Solution: $x_1 + 2x_2 + 3x_3 - 2x_4 + 4x_5 = 0$

$$2x_1 + 4x_2 + 8x_3 + x_4 + 9x_5 = 0$$

$$3x_1 + 6x_2 + 13x_3 + 4x_4 + 14x_5 = 0.$$

Sol'n: Let $V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2} \middle| a, b, c, d \in \mathbb{R} \right\}$

$$\& \; W = \left\{ A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \middle| \det A = 0 \right\} \subseteq V$$

let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ & $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\det A = 0$ & $\det B = 0$

Clearly $A, B \in W$

$$A+B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\det(A+B) = 1 \neq 0$

$\Rightarrow A+B \notin W$.

Hence, W is not the subspace of V.

(ii) Given homogeneous system is:

$$x_1 + 2x_2 + 3x_3 - 2x_4 + 4x_5 = 0 \quad \text{——} (1)$$

$$2x_1 + 4x_2 + 8x_3 + x_4 + 9x_5 = 0 \quad \text{——} (2)$$

$$3x_1 + 6x_2 + 13x_3 + 4x_4 + 14x_5 = 0 \quad \text{——} (3)$$

Coefficient matrix of given system is

$$A = \begin{bmatrix} 1 & 2 & 3 & -2 & 4 \\ 2 & 4 & 8 & 1 & 9 \\ 3 & 6 & 13 & 4 & 14 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & -2 & 4 \\ 0 & 0 & 2 & 5 & 1 \\ 0 & 0 & 4 & 10 & 2 \end{bmatrix} \quad R_3 \to R_3 - 2R_2$$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & -2 & 4 \\ 0 & 0 & 2 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So, corresponding homogeneous system is

$$x_1 + 2x_2 + 3x_3 - 2x_4 + 4x_5 = 0 \quad \text{---} (4)$$

$$2x_3 + 5x_4 + x_5 = 0 \quad \text{---} (5)$$

Here 3 variables x_2, x_4, x_5 and values of x_1 and x_3 depend upon x_2, x_4, x_5

from (5) $x_3 = -\dfrac{(5x_4 + x_5)}{2}$ — (6)

from (4) $x_1 = -2x_2 - 3x_3 + 2x_4 - 4x_5$

$$x_1 = (-2x_2) + \frac{19x_4 - 5x_5}{2}$$

$$\Rightarrow x_1 = \frac{-4x_2 + 19x_4 - 5x_5}{2} \quad \text{---} (7)$$

Now, put $x_2 = 1$, $x_4 = 0$, $x_5 = 0$

$$\Rightarrow x_3 = 0, \; x_1 = -2$$

Then $(x_1, x_2, x_3, x_4, x_5) = (-2, 1, 0, 0, 0)$ — (8)

put $x_2 = 0$, $x_4 = 1$, $x_5 = 0$

$\Rightarrow x_3 = -5/2$, $x_1 = 19/2$

then $(x_1, x_2, x_3, x_4, x_5) = (19/2, 0, -5/2, 1, 0)$ — (9)

put $x_2 = 0$, $x_4 = 0$, $x_5 = 1$

$\Rightarrow x_3 = -1/2$, $x_1 = -5/2$

then $(x_1, x_2, x_3, x_4, x_5) = (-5/2, 0, -1/2, 0, 1)$ — (10)

Hence from (8), (9) & (10)

$B = \{(-2, 1, 0, 0, 0), (19/2, 0, -5/2, 1, 0), (-5/2, 0, -1/2, 0, 1)\}$

is the basis for the space W.

& $\boxed{\text{Dim } W = 3}$

3(b) Express $\int_0^1 x^m(1-x^n)^p\,dx$ interms of Gamma function and hence evaluate the integral.

$\int_0^1 x^6\sqrt{(1-x^2)}\cdot dx$.

Sol'n: put $x^n = z$ or $x = z^{1/n}$

So that $dx = \frac{1}{n} z^{1/n-1}\,dz$

$= \frac{1}{n} z^{\frac{1-n}{n}}\,dz$

when $x=0, z=0$ and when $x=1, z=1$

$$\therefore \int_0^1 x^m(1-x^n)^p\,dx = \int_0^1 z^{\frac{m}{n}}(1-z)^p \cdot \frac{1}{n} z^{\frac{1-n}{n}}\,dz$$

$$= \frac{1}{n}\int_0^1 z^{\frac{m+1-n}{n}}(1-z)^p\,dz$$

$$= \frac{1}{n} B\left(\frac{m+1-n}{n}+1, p+1\right)$$

$$= \frac{1}{n} B\left(\frac{m+1}{n}, p+1\right)$$

$$= \frac{1}{n}\frac{\Gamma\left(\frac{m+1}{n}\right)\Gamma(p+1)}{\Gamma\left(\frac{m+1}{n}+p+1\right)} \quad \text{--- (1)}$$

Comparing $\int_0^1 x^6(1-x^2)^{1/2}\,dx$ with $\int_0^1 x^m(1-x^n)^p\,dx$,

we have $m=6, n=2, p=1/2$

$\therefore$ from (1),

$$\int_0^1 x^6(1-x^2)^{1/2}\,dx = \frac{1}{2}\frac{\Gamma(7/2)\Gamma(3/2)}{\Gamma(5)}$$

$$= \frac{1}{2}\,\frac{5/2\cdot 3/2\cdot \frac{1}{2}\Gamma(1/2)\ \frac{1}{2}\Gamma(1/2)}{4!}$$

$$= \frac{15\pi}{32\times 4!}$$

$$= \frac{5\pi}{256}.$$

$\Gamma\left(\frac{m+1}{n}+p+1\right)$
$=\Gamma(7/2+3/2)$
$=\Gamma 5$

3(C) (i) Find the limiting points of the co-axial system of spheres determined by $x^2+y^2+z^2-20x+30y-40z+29=0$ and $x^2+y^2+z^2-18x+27y-36z+29=0$.

Sol^n: Let $S_1 = x^2+y^2+z^2-20x+30y-40z+29=0$

$S_2 = x^2+y^2+z^2-18x+27y-36z+29=0$

be two given spheres.

then $S_1-S_2 = -2x+3y-4z=0$

Now the equation of co-axial system of spheres is $S_1+\lambda(S_1-S_2)=0$

$\Rightarrow x^2+y^2+z^2-20x+30y-40z+29+\lambda(-2x+3y-4z)=0$

$\Rightarrow x^2+y^2+z^2+(-20-2\lambda)x+(30+3\lambda)y+(-40-4\lambda)z+29=0$

Its centre $\left(10+\lambda, \frac{-30-3\lambda}{2}, 20+2\lambda\right)$

and equating its radius to zero

we get $u^2+v^2+w^2-d=0$

$(10+\lambda)^2+\left(\frac{30+3\lambda}{2}\right)^2+(20+2\lambda)^2-29=0$

$\Rightarrow (10+\lambda)^2\left[1+\frac{9}{4}+4\right]=29$

$\Rightarrow (10+\lambda)^2\left[29/4\right]=29$

$\Rightarrow (10+\lambda)^2=4$

$\Rightarrow 10+\lambda=\pm 2 \Rightarrow \lambda=-8,-12$

For $\lambda=-8$, limiting point is $(2,-3,4)$

For $\lambda=-12$, limiting point is $(-2,3,4)$

3(C) (ii) Show that the plane $8x-6y-z=5$ touches the paraboloid $\left(\frac{x^2}{2}\right)-\left(\frac{y^2}{3}\right)=z$, and find the point of contact.

Sol^n: Let the plane $8x-6y-z=5$ —— (1)

touch the paraboloid $x^2/2 - y^2/3 = z \Rightarrow 3x^2-2y^2=6z$ —— (2)

at the point (α,β,γ).

The equation of the tangent plane to (2) at (α,β,γ) is

$$3\alpha x-2\beta y=3(z+\gamma) \text{ (or) } 3\alpha x-2\beta y-3z=3\gamma \quad \text{—— (3)}$$

If the plane (1) touches (2) at (α,β,γ), then (1) & (3) represent the same plane, and so comparing (1) & (3), we get

$$\frac{3\alpha}{8}=\frac{-2\beta}{-6}=\frac{-3}{-1}=\frac{3\gamma}{5}$$

which gives $\alpha=8,\ \beta=9,\ \gamma=5$

Also as (α,β,γ) given by (4) satisfy (5), so the plane (1) touches the paraboloid (2) at (α,β,γ).

Also from (4), the coordinates of the point of contact are

(α,β,γ) i.e. $(8,9,5)$.

4(a) (i) Using elementary row operations, find the inverse of $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix}$

(ii) If $A = \begin{bmatrix} 1 & 1 & 3 \\ 5 & 2 & 6 \\ -2 & -1 & -3 \end{bmatrix}$, then find $A^{14} + 3A - 2I$.

Sol'n: we know that

$$A = IA \Rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot A$$

Using Row transformation, Converting L.H.S to identity matrix;

$$R_2 \to R_2 - R_1 \quad , \quad R_3 \to R_3 - R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & -2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \cdot A$$

$$R_2 \leftrightarrow R_3$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \cdot A$$

$$R_3 \to R_3 + \tfrac{1}{2} R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \\ -3/2 & 1 & 1/2 \end{bmatrix} A$$

$$R_2 \to \frac{R_3}{-2} \text{ then } R_1 \to R_1 - 2R_2 - R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3/2 & -1 & 1/2 \\ 1/2 & 0 & -1/2 \\ -3/2 & 1 & 1/2 \end{bmatrix} A$$

$$= A^{-1} A$$

Hence

$$A^{-1} = \begin{bmatrix} 1.5 & -1 & 0.5 \\ 0.5 & 0 & -0.5 \\ -1.5 & 1 & 0.5 \end{bmatrix}$$

(ii) Sol^n: Characteristic equation of A is $|A-\lambda I| = 0$

$$\begin{bmatrix} 1-\lambda & 1 & 3 \\ 5 & 2-\lambda & 6 \\ -2 & -1 & -3-\lambda \end{bmatrix} = 0$$

$\Rightarrow (1-\lambda)\left[(2-\lambda)(3-\lambda)+6\right] - 1\left[5(-3-\lambda)+12\right] + 3\left[-5+2(2-\lambda)\right] = 0$

$\Rightarrow -\lambda^3 + 0\lambda^2 + 0\lambda + 0 = 0$

$\Rightarrow \lambda^3 = 0$ By Cayley-Hamilton theorem $A^3 = 0$.

$A^{14} + 3A - 2I$

$= (A^3)^4 \cdot A^2 + 3A - 2I = 0 + 3A - 2I$

$$= \begin{bmatrix} 1 & 3 & 9 \\ -15 & 4 & 18 \\ -6 & -3 & -11 \end{bmatrix}$$

$\therefore$ Required Result.

4(b) If $x>0$, show that

$$x-\frac{x^2}{2} < \log(1+x) < x-\frac{x^2}{2(1+x)}$$

Sol'n: Let $f(x) = x-\frac{x^2}{2}-\log(1+x)$

$$\therefore f'(x) = 1-x-\frac{1}{1+x} = \frac{1-x^2-1}{1+x}$$

$$= -\frac{x^2}{1+x} < 0 \text{ for } x>0.$$

$\Rightarrow f(x)$ is monotonic decreasing for $x>0$,

$\Rightarrow f(x) < f(0)$

But $f(0) = 0-0-\log 1 = 0$

$$\therefore f(x)<0 \Rightarrow x-\frac{x^2}{2}-\log(1+x)<0$$

$$\Rightarrow x-\frac{x^2}{2} < \log(1+x) \quad \text{——— (1)}$$

Now let $g(x) = \log(1+x) - x + \frac{x^2}{2(1+x)}$

$$\therefore g'(x) = \frac{1}{1+x} - 1 + \frac{1}{2}\cdot\frac{(1+x).2x-x^2}{(1+x)^2}$$

$$= \frac{1-1+x}{1+x} + \frac{1}{2}\frac{2x+x^2}{(1+x)^2} = -\frac{x}{1+x} + \frac{2x+x^2}{2(1+x)^2}$$

$$= \frac{-2x(1+x)+2x+x^2}{2(1+x)^2}$$

$$= -\frac{x^2}{2(1+x)^2} < 0 \text{ for } x>0$$

$\Rightarrow g(x)$ is monotonic decreasing for $x>0$

$\Rightarrow g(x) < g(0)$

But $g(0) = 0$ $\therefore g(x) < 0$

$$\Rightarrow \log(1+x) - x + \frac{x^2}{2(1+x)} < 0$$

$$\therefore \log(1+x) < x - \frac{x^2}{2(1+x)} \quad \text{——— (2)}$$

Combining (1) & (2) $x-\frac{x^2}{2} < \log(1+x) < x-\frac{x^2}{2(1+x)}$

4(C). Let ϕ be a function of two variables defined as

$$\phi(x,y) = (x^3+y^3)/(x-y), \quad \text{when } x \neq y$$
$$\phi(x,y) = 0, \quad \text{when } x = y.$$

Show that ϕ is discontinuous at the origin, but the first order partial derivatives exist at that point.

Sol'n: Suppose $(x,y) \to (0,0)$ along the curve

$$y = x - mx^3$$

Then $\displaystyle \lim_{x\to 0} \phi(x, x-mx^3) = \lim_{x\to 0} \frac{x^3 + (x-mx^3)^3}{mx^3}$

$$= \lim_{x\to 0} \frac{x^3\left[1+(1-mx^2)^3\right]}{mx^3}$$

$$= \lim_{x\to 0} \frac{\left[1+(1-mx^2)^3\right]}{m}$$

$$= \frac{2}{m}$$

which is different for different values of m.

Thus $\displaystyle \lim_{(x,y)\to(0,0)} \phi(x,y)$ does not exist and so the given function is discontinuous at $(0,0)$.

Now,

$$\phi_x(0,0) = \lim_{h\to 0} \frac{[\phi(0+h,0) - \phi(0,0)]}{h} = \lim_{h\to 0} \frac{h^2 - 0}{h} = 0$$

$$\phi_y(0,0) = \lim_{k\to 0} \frac{[\phi(0,0+k) - \phi(0,0)]}{k} = \lim_{k\to 0} \frac{-k^2 - 0}{k} = 0$$

$\therefore$ First order partial derivatives exist at the origin.

Hence the result.

4(d) → Show that the surface represented by the equation $x^2+y^2+z^2-yz-zx-xy-3x-6y-9z+21=0$ is a paraboloid of revolution the coordinates of the focus being $(1,2,3)$ and the equations to axis are $x=y-1=z-2$.

Sol^n: Here, the discriminating cube is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix}=0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1-\lambda & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1-\lambda \end{vmatrix}=0$$

$$\Rightarrow (1-\lambda)\left[(1-\lambda)^2-\frac{1}{4}\right]+\frac{1}{2}\left[-\frac{1}{2}(1-\lambda)-\frac{1}{4}\right]-\frac{1}{2}\left[\frac{1}{4}+\frac{1}{2}(1-\lambda)\right]=0$$

$$\Rightarrow (1-\lambda)\left[8(1-\lambda)^2-2\right]-2\left[2(1-\lambda)+1\right]=0$$

$$\Rightarrow -8\lambda^3+24\lambda^2-18\lambda=0$$

$$\Rightarrow 4\lambda^3-12\lambda^2+9\lambda=0$$

$$\Rightarrow \lambda\left[4\lambda^2-12\lambda+9\right]=0$$

$$\Rightarrow \lambda(2\lambda-3)^2=0$$

$$\Rightarrow \lambda=0,\ 3/2,\ 3/2$$

As, two roots of the discriminating cube are equal and third root is zero, so it is either a paraboloid of revolution or a right circular cylinder.

The dir's of the axis are given by

$$al+hm+gn=0,\quad hl+bm+fn=0,\quad gl+fm+cn=0$$

$\Rightarrow l - \frac{1}{2}m - \frac{1}{2}n = 0$; $-\frac{1}{2}l + m - \frac{1}{2}n = 0$; $-\frac{1}{2}l - \frac{1}{2}m + n = 0$

$\Rightarrow 2l - m - n = 0$; $-l + 2m - n = 0$; $-l - m + 2n = 0$

they give $l = m = n = \frac{1}{\sqrt{3}}$

Now $k = ul + vm + wn$

$\Rightarrow k = \left(\frac{-3}{2}\right)\left(\frac{1}{\sqrt{3}}\right) + (-3)\left(\frac{1}{\sqrt{3}}\right) + \left(-\frac{9}{2}\right)\left(\frac{1}{\sqrt{3}}\right)$

$\Rightarrow K = -3\sqrt{3} (\neq 0)$

$\therefore$ The required equation is $\lambda_1 x^2 + \lambda_2 y^2 + 2kz = 0$

$\Rightarrow \frac{3}{2}x^2 + \frac{3}{2}y^2 + 2(-3\sqrt{3})z = 0$

$\Rightarrow \underline{x^2 + y^2 = 4\sqrt{3}z}$ which represents a paraboloid of revolution.

Also, the coordinates of the vertex of the paraboloid are obtained by solving any two of the three equations (I) with the equation (II) (which are mentioned as follows):

$$\frac{\left(\frac{\partial F}{\partial x}\right)}{l} = \frac{\left(\frac{\partial F}{\partial y}\right)}{m} = \frac{\left(\frac{\partial F}{\partial z}\right)}{n} = 2k$$

$$\Rightarrow \frac{2x - y - z - 3}{\frac{1}{\sqrt{3}}} = \frac{2y - z - x - 6}{\frac{1}{\sqrt{3}}} = \frac{2z - y - x - 9}{\frac{1}{\sqrt{3}}} = -6\sqrt{3}$$

$\Rightarrow 2x - y - z - 3 = 2y - z - x - 6 = 2z - y - x - 9 = -6$

$\Rightarrow 2x - y - z + 3 = 0$; $x - 2y + z = 0$; $x + y - 2z + 3 = 0$ —— (I)

and $k(lx + my + nz) + ux + vy + wz + d = 0$

$\Rightarrow -3\sqrt{3}\left(\frac{1}{\sqrt{3}}x + \frac{1}{\sqrt{3}}y + \frac{1}{\sqrt{3}}z\right) + \left(-\frac{3}{2}\right)x + (-3)y + \left(-\frac{9}{2}\right)z + 21 = 0$

$\Rightarrow 3x + 4y + 5z - 14 = 0$ —— (II).

Now, solving $2x - y - z + 3 = 0$, $x - 2y + z = 0$,
$3x + 4y + 5z - 14 = 0$.

we get $x = 0$, $y = 1$, $z = 2$

$\therefore$ The required vertex is $(0, 1, 2)$

$\therefore$ Equations of the axis are $\frac{x-0}{1} = \frac{y-1}{1} = \frac{z-2}{1}$

$\Rightarrow x = y-1 = z-2$

Also, the focus will be a point on the axis whose actual d.c's are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$ and will be at a distance $(\frac{1}{4}) 4\sqrt{3}$ i.e. $\sqrt{3}$ from the vertex $(0,1,2)$.

Coordinates of the focus are given by

$$\frac{x-0}{1/\sqrt{3}} = \frac{y-1}{1/\sqrt{3}} = \frac{z-2}{1/\sqrt{3}} = \sqrt{3}$$

$\Rightarrow x=1, \ y=2, \ z=3$

$\therefore$ The required focus is $(1,2,3)$.

Hence the result.

5(a) Find the condition on a, b and c so that the following system in unknowns x, y and z has a solution.

$x+2y-3z=a$, $2x+6y-11z=b$, $x-2y+7z=c$.

Sol^n: The matrix form of the given system of equations is

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 6 & -11 \\ 1 & -2 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 2 & -5 \\ 0 & -4 & 10 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b-2a \\ c-a \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2-2R_1 \\ R_3 \to R_3-R_1 \end{array}$$

$$R_3 \to R_3+2R_2$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 0 & 2 & -5 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b-2a \\ c+2b-5a \end{bmatrix}$$

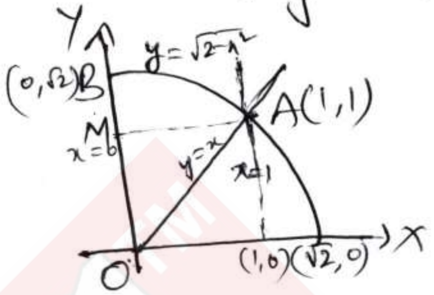
The system will have no solution if $c+2b-5a \neq 0$.

Thus the system will have atleast one solution if $c+2b-5a=0$ i.e. $5a=2b+c$ which is the required condition.

Note: In this case the system will have infinitely many solutions. In otherwords the system cannot have a unique solution.

5(d) Evaluate $\int_0^1\int_x^{\sqrt{2-x^2}} \frac{x\,dx\,dy}{\sqrt{x^2+y^2}}$ by changing the order of integration.

The limits of integration are given by $y=x$, $y=\sqrt{2-x^2}$ i.e, $x^2+y^2=(\sqrt{2})^2$; $x=0$ and $x=1$.



Clearly the area of integration is OABO. The strips parallel to x-axis change their character at the point A. Draw a straight line AM parallel to x-axis and this divides the are OABO into two portions OAM and MAB.

→ For the area OAM, the limits of x are from 0 to y and the limits of y are from 0 to 1.

→ For the area AMB, the limits of x are from 0 to $\sqrt{2-y^2}$ and the limits of y are 1 to $\sqrt{2}$.

Hence changing the order of integration, we have

$$\int_{x=0}^{1}\int_{y=x}^{\sqrt{2-x^2}} \frac{x\,dx\,dy}{\sqrt{x^2+y^2}} = \int_{y=0}^{1}\int_{x=0}^{y} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}} + \int_{y=1}^{\sqrt{2}}\int_{x=0}^{\sqrt{2-y^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}}$$

$$= \int_{y=0}^{1}\left[\sqrt{x^2+y^2}\right]_{x=0}^{y} dy + \int_{y=1}^{\sqrt{2}}\left(\sqrt{x^2+y^2}\right)_{x=0}^{\sqrt{2-y^2}} dy$$

Let $x^2+y^2=t^2$
$2x\,dx=2t\,dt$
$x\,dx=t\,dt$
$\int \frac{t\,dt}{\sqrt{t^2}}=\int dt$
$=\sqrt{x^2+y^2}$

$$= \int_0^1 (\sqrt{2}-1)y\,dy + \int_1^{\sqrt{2}}(\sqrt{2}-y)\,dy$$

$$= (\sqrt{2}-1)\left[\frac{y^2}{2}\right]_0^1 + \left(\sqrt{2}y-\frac{y^2}{2}\right)_1^{\sqrt{2}}$$

$$= (\sqrt{2}-1)\frac{1}{2} + (2-1) - \left(\sqrt{2}-\frac{1}{2}\right) = \frac{\sqrt{2}-1}{\sqrt{2}}$$ Ans

5(d) Show that the straight line whose direction cosines are given by the equations: $ul+vm+wn=0$, $al^2+bm^2+cn^2=0$ are (α) perpendicular if $u^2(b+c)+v^2(c+a)+w^2(a+b)=0$ and (β) parallel, if $u^2/a+v^2/b+w^2/c=0$.

Sol^n: The d.c's of the lines are given by

$$ul+vm+wn=0 \quad \text{and} \quad al^2+bm^2+cn^2=0$$

Eliminating n between these, we get

$$al^2+bm^2+c\left[-(ul+vm)/w\right]^2=0$$

$$\Rightarrow (aw^2+cu^2)l^2+(bw^2+cv^2)m^2+2cuvlm=0$$

$$\Rightarrow (aw^2+cu^2)(l/m)^2+2cuv(l/m)+(bw^2+cv^2)=0, \text{ ——— (1)}$$

dividing each term by m^2.

(i) Its two roots are l_1/m_1 and l_2/m_2, if the d.c's of the two lines be taken as (l_1, m_1, n_1) and (l_2, m_2, n_2).

∴ from (1), we have

$$\frac{l_1}{m_1}\cdot\frac{l_2}{m_2} = \text{Product of the roots} = \frac{bw^2+cv^2}{cu^2+aw^2}$$

$$\Rightarrow \frac{l_1 l_2}{bw^2+cv^2} = \frac{m_1 m_2}{cu^2+aw^2} = \frac{n_1 n_2}{av^2+bu^2}, \text{ by symmetry.}$$

∴ If the two lines are ⊥lar, then we have

$l_1l_2+m_1m_2+n_1n_2=0$ i.e. $(bw^2+cv^2)+(cu^2+aw^2)+(av^2+bu^2)=0$.

$$\Rightarrow u^2(b+c)+v^2(c+a)+w^2(a+b)=0 \quad \text{Hence proved.}$$

(ii) If the two lines are parallel, then their d.c's are equal are consequently the roots of (1) are equal, the condition for the same being $b^2=4ac$ i.e. $(2cuw)^2=4(aw^2+cu^2)(bw^2+cv^2)$

$$\Rightarrow c^2u^2v^2 = abw^4+acw^2v^2+bcu^2w^2+c^2u^2v^2$$

$$\Rightarrow abw^4+acw^2v^2+bcu^2w^2=0$$

$$\Rightarrow abw^2+acv^2+bcu^2=0$$

$$\Rightarrow \frac{w^2}{c}+\frac{v^2}{b}+\frac{u^2}{a}=0, \text{ dividing each term by } abc.$$

5(e) Find the equation of the Sphere for which the circle $x^2+y^2+z^2+7y-2z+2=0$, $2x+3y+4z=8$ is a great circle.

Sol^n: The equation of any sphere through the given circle is

$$(x^2+y^2+z^2+7y-2z+2)+\lambda(2x+3y+4z-8)=0 \quad \text{--- (1)}$$

$$\Rightarrow x^2+y^2+z^2+2\lambda x+(7+3\lambda)y+(4\lambda-2)z+(2-8\lambda)=0.$$

Its centre is $\left[-\lambda, -\frac{1}{2}(7+3\lambda), 1-2\lambda\right]$ --- (2)

Now if the given circle (which is the section of the sphere $x^2+y^2+z^2-7y-2z+2=0$ by the plane $2x+3y+4z=8$) is a great circle of the sphere (1), then the centre of the sphere (1) must lie on the plane of the circle i.e. on the plane $2x+3y+4z=8$.

$\therefore$ from (2) we get $2(-\lambda)+3\left[-\frac{1}{2}(7+3\lambda)\right]+4(1-2\lambda)=8$

$$\Rightarrow -2\lambda-\frac{21}{2}-\frac{9}{2}\lambda+4-8\lambda=8$$

$$\Rightarrow \lambda=-1$$

Substituting the value of λ in (1), we get the required equation as

$$(x^2+y^2+z^2+7y-2z+2)-(2x+3y+4z-8)=0$$

$$\Rightarrow x^2+y^2+z^2-2x+4y-6z+10=0.$$

6(a) Verify the Cayley-Hamilton theorem for the matrix $A=\begin{bmatrix}1&0&-1\\2&1&0\\3&-5&1\end{bmatrix}$. Using this, show that A is non-singular and find A^{-1}.

Sol^n: Let λ be an eigen value of matrix A, then characteristic matrix.

$$|A-\lambda I| = \begin{bmatrix}1&0&-1\\2&1&0\\3&-5&1\end{bmatrix} - \lambda\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = \begin{bmatrix}1-\lambda&0&-1\\2&1-\lambda&0\\3&-5&1-\lambda\end{bmatrix}$$

Characteristic polynomial

$$|A-\lambda I|=0 \Rightarrow \begin{vmatrix}1-\lambda&0&-1\\2&1-\lambda&0\\3&-5&1-\lambda\end{vmatrix}=0$$

$$\Rightarrow (1-\lambda)^3-\{-10-3(1-\lambda)\}=0$$

$$\Rightarrow (1-\lambda)^3+13-3\lambda=0$$

$$\Rightarrow \lambda^3-3\lambda^2+6\lambda-14=0 \quad \text{—— (1)}$$

$$A^2=\begin{bmatrix}1&0&-1\\2&1&0\\3&-5&1\end{bmatrix}\begin{bmatrix}1&0&-1\\2&1&0\\3&-5&1\end{bmatrix}=\begin{bmatrix}-2&5&-2\\4&1&-2\\-4&-10&-2\end{bmatrix} \quad \text{—— (2)}$$

$$A^3=\begin{bmatrix}-2&5&-2\\4&1&-2\\-4&-10&-2\end{bmatrix}\begin{bmatrix}1&0&-1\\2&1&0\\3&-5&1\end{bmatrix}=\begin{bmatrix}2&15&0\\0&11&-6\\-30&0&2\end{bmatrix} \quad \text{—— (3)}$$

Putting values of A, A^2, A^3 in equation (1)

$$\lambda^3-3\lambda^2+6\lambda-14=0$$

$$A^3-3A^2-6A-14I=0$$

$$A^3-3A^2-6A-14I=\begin{bmatrix}2&15&0\\0&11&-6\\-30&0&2\end{bmatrix}+\begin{bmatrix}6&-15&6\\-12&-3&6\\12&30&6\end{bmatrix}+\begin{bmatrix}6&0&-6\\12&6&0\\18&-30&6\end{bmatrix}+\begin{bmatrix}-14&0&0\\0&-14&0\\0&0&-14\end{bmatrix}$$

$\therefore A^3 - 3A^2 + 6A - 14I = 0$ —— (4)

$\Rightarrow$ A satisfies equation (1).

$\therefore$ Cayley - Hamilton theorem verified.

If $(-1)^n[\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n]$ is the characteristic equation of matrix A.

and $|A| = (-1)^n a_n$.

$\therefore$ A is non-singular if $a_n \neq 0$.

from (1), $a_n = -14 \neq 0$.

i.e, $|A| = 14 \neq 0$.

$\therefore$ A is non-singular.

Now multiplying equation (4) by A^{-1}

$\Rightarrow A^{-1}A^3 - 3A^{-1}A^2 + 6A^{-1}A - 14A^{-1}I = 0$

$\Rightarrow A^2 - 3A + 6I = 14A^{-1}$

$$\Rightarrow 14A^{-1} = \begin{bmatrix} -2 & 5 & -2 \\ 4 & 1 & -2 \\ -4 & -10 & -2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 5 & 1 \\ -2 & 4 & -2 \\ -13 & 5 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{14}\begin{bmatrix} 1 & 5 & 1 \\ -2 & 4 & -2 \\ -13 & 5 & 1 \end{bmatrix}$$

6(b) → show that the subspace of $\mathbb{R}^3$ spanned by two sets of vectors $\{(1,1,-1),(1,0,1)\}$ and $\{(1,2,-3),(5,2,1)\}$ are identical. Also find the dimension of this subspace.

Sol'n: Let $B_1 = \{(1,1,-1),(1,0,1)\}$

$B_2 = \{(1,2,-3),(5,2,1)\}$

Let W be subspace such that $W = L\{B_1\}$

V be subspace such that $V = L\{B_2\}$

The subspaces V and W are identical iff their row reduced echelon matrices have the same non-zero rows.

Consider a matrix A, where rows are vectors of B_1

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix} R_2 \to R_2 - R_1 \quad \text{——— (1)}$$

Now, consider: a matrix 'B', whose rows are vectors of B_2

$$\Rightarrow B = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 2 & 1 \end{bmatrix}$$

$$\sim B = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 8 & 16 \end{bmatrix} R_2 \to R_2 - 5R_1$$

$$\sim B = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & 2 \end{bmatrix} R_2 \to R_2(\tfrac{1}{8})$$

$$B \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix} R_1 \to R_1 + R_2 \quad \text{——— (2)}$$

from (1) & (2), clearly rowspace of 'A' and row space of 'B' are same.

Thus V and W are identical subspaces of $\mathbb{R}^3$.

∴ Dimension of row space A (i.e, dimension of subspace)

= Max. no of L.I. rows of A.

= Max no. of L.I. rows of echelon matrix of A

= no. of non-zero rows of echelon matrix of A

= 2.

∴ Basis for the row space is $\{(1,1,-1),(0,-1,2)\}$ and the dimension of row space is 2.

6(c) Let $L : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation defined by $L(x_1, x_2, x_3, x_4) = (x_3 + x_4 - x_1 - x_2,\ x_3 - x_2,\ x_4 - x_1)$

Then find the rank and nullity of L. Also, determine null space and range space of L.

Sol'n: Given that $L : \mathbb{R}^4 \to \mathbb{R}^3$ is a linear transformation such that

$$L(x_1, x_2, x_3, x_4) = (x_3 + x_4 - x_1 - x_2,\ x_3 - x_2,\ x_4 - x_1). \quad \text{—— (1)}$$

Range space of $L = \{\beta \in \mathbb{R}^3 \mid T(\alpha) = \beta \text{ for } \alpha \in \mathbb{R}^4\}$

$\therefore$ The range space consists of all vectors of the type $(x_3 + x_4 - x_1 - x_2,\ x_3 - x_2,\ x_4 - x_1)$ for all $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$

$$\therefore R(L) = \{(x_3 + x_4 - x_1 - x_2,\ x_3 - x_2,\ x_4 - x_1) \mid x_1, x_2, x_3, x_4 \in \mathbb{R}\} \text{ —— (A).}$$

Let $\beta = (x_3 + x_4 - x_1 - x_2,\ x_3 - x_2,\ x_4 - x_1) \in R(L)$

then $\beta = x_3(1,1,0) + x_4(1,0,1) + x_1(-1,0,-1) + x_2(-1,-1,0)$

$\in L'(S)$ (Linear span of S)

where $S = \{(1,1,0), (1,0,1), (-1,0,-1), (-1,-1,0)\} \subseteq R(L)$

$\therefore \beta \in R(L) \Rightarrow \beta \in L'(S)$

$\therefore R(L) \subseteq L'(S)$ —— (2)

Since $S \subseteq R(L)$

$\Rightarrow L'(S) \subseteq R(L)$ —— (3).

$\therefore$ from (2) and (3), we have

$$L'(S) = R(L)$$

i.e. S Spans $R(L)$.

Now we construct a matrix, whose rows are vectors of the subset 'S' of $R(L)$ and convert into echelon form by using E-row transformations.

$$A=\begin{bmatrix}1&1&0\\1&0&1\\-1&0&-1\\-1&-1&0\end{bmatrix}_{4\times 3} \sim \begin{bmatrix}1&1&0\\0&-1&1\\0&1&-1\\0&0&0\end{bmatrix} \begin{array}{l}R_2\to R_2-R_1\\R_3\to R_3+R_1\\R_4\to R_4+R_1\end{array} \sim \begin{bmatrix}1&1&0\\0&-1&1\\0&0&0\\0&0&0\end{bmatrix} R_3\to R_3+R_2$$

clearly which is in echelon form and the number of non-zero rows of echelon form is 2.

$\therefore$ The set $S'=\{(1,1,0),(1,0,1)\}$ forms a basis of $R(L)$ and the number of elements of $S'=2$.

$$\therefore \dim(R(L))=2.$$

$$\text{rank of } L=\rho(L)=2$$

we know that rank of L + nullity of $L=\dim \mathbb{R}^4$

$$\Rightarrow 2+\text{nullity of } L=4$$

$$\Rightarrow \boxed{\text{nullity of } L=2}$$

Now we find nullspace of L:

Null space of $L=N(L)$
$$=\{\alpha\in\mathbb{R}^4 \mid T(\alpha)=(0,0,0) \text{ in } \mathbb{R}^3\}\subseteq\mathbb{R}^4$$

Let $\alpha\in N(L)$

i.e. $(x_1,x_2,x_3,x_4)\in N(L)$

$$\Rightarrow L(x_1,x_2,x_3,x_4)=(0,0,0)$$

$$\Rightarrow (x_3+x_4-x_1-x_2;\ x_3-x_2,\ x_4-x_1)=(0,0,0)$$

$$\Rightarrow \left.\begin{array}{l}x_3+x_4-x_1-x_2=0 \text{ --- (i)}\\ x_3-x_2=0 \text{ --- (ii)}\\ x_4-x_1=0 \text{ --- (iii)}\end{array}\right\}$$

$$\Rightarrow \boxed{x_3=x_2}$$

$$\boxed{x_4=x_1}$$

$\therefore N(L)=\{(x_1,x_2,x_2,x_1) \mid x_1,x_2\in\mathbb{R}\}\subseteq\mathbb{R}^4.$

Clearly which is the required nullspace of L.

6(d) Let $H = \begin{bmatrix} 1 & i & 2+i \\ -i & 2 & 1-i \\ 2-i & 1+i & 2 \end{bmatrix}$ be a Hermitian matrix. Find a non-singular matrix P such that $D = P^T H \bar{P}$ is diagonal.

Sol^n: let us form the block matrix

$$[H|I] = \left[\begin{array}{ccc|ccc} 1 & i & 2+i & 1 & 0 & 0 \\ -i & 2 & 1-i & 0 & 1 & 0 \\ 2-i & 1+i & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & i & 2+i & 1 & 0 & 0 \\ 0 & 1 & i & i & 1 & 0 \\ 0 & -i & -3 & -(2-i) & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 + iR_1 \\ R_3 \to R_3 - (2-i)R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -i & -(2+i) \\ 0 & 1 & -i & i & 2 & -2i+1 \\ 0 & -i & -3 & -(2-i) & 2i+1 & -4 \end{array}\right] \quad \begin{array}{l} C_2 \to C_2 - iC_1 \\ C_3 \to C_3 - (2+i)C_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -i & -(2+i) \\ 0 & 1 & i & i & 2 & -(2i-1) \\ 0 & 0 & -4 & -3+i & 4i+1 & -2+i \end{array}\right] \quad R_3 \to R_3 + iR_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -i & -3-i \\ 0 & 1 & 0 & i & 2 & -4i+1 \\ 0 & 0 & -4 & -3+i & 4i+1 & 2 \end{array}\right] \quad C_3 \to C_3 - iR_2$$

Now H has been diagonalized.

$\therefore$ set $P = \begin{bmatrix} 1 & -i & -3-i \\ i & 2 & -4i+1 \\ -3+i & 4i+1 & 2 \end{bmatrix}$

and then $P^T H \bar{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix} = D$.

7(a) Find the values of a and b such that

$$\lim_{x\to 0} \frac{a\sin^2 x + b\log\cos x}{x^4} = \frac{1}{2}.$$

Sol'n: $\lim_{x\to 0} \frac{a\sin^2 x + b\log\cos x}{x^4}$ | form $\frac{0}{0}$

$$= \lim_{x\to 0} \frac{a(\sin x\cos x) + \frac{b}{\cos x}(-\sin x)}{4x^3} \quad \text{| by L' Hospitals Rule}$$

$$= \lim_{x\to 0} \frac{a\sin 2x - b\tan x}{4x^3} \quad \text{| } \frac{0}{0} \text{ form}$$

$$= \lim_{x\to 0} \frac{2a\cos 2x - b\sec^2 x}{12x^2} \quad \text{— (1)} \quad \text{| by L' Hospitals Rule}$$

The denominator of (1) $\to 0$ as $x\to 0$ but (1) $\to$ a finite limit value $\frac{1}{2}$.

$\therefore$ The numerator of (1) must be zero as $x\to 0$

$\therefore$ (1) $\equiv 2a\cos(0) - b\sec^2(0) = 0$

$$\Rightarrow \boxed{2a - b = 0} \quad \text{— (2)}$$

with this form (1)

$$\lim_{x\to 0} \frac{2a\cos 2x - b\sec^2 x}{12x^2} \quad \text{| } \frac{0}{0} \text{ form}$$

$$= \lim_{x\to 0} \frac{-4a\sin 2x - b\left[2\sec^2 x\tan x\right]}{24x} \quad \text{| } \frac{0}{0} \text{ form}$$

$$= \lim_{x\to 0} \left[\frac{-4a\sin 2x}{24x} - \frac{2b\sec^2 x\tan x}{24x}\right]$$

$$= -\frac{a}{3} \lim_{2x\to 0} \frac{\sin 2x}{2x} - \frac{b}{12} \lim_{x\to 0} \sec^2 x \lim_{x\to 0} \frac{\tan x}{x}$$

$$= -\frac{a}{3}(1) - \frac{b}{12}(1)(1)$$

$$= -\frac{a}{3} - \frac{b}{12} = \frac{-4a-b}{12}$$

But limit of ① is equal to $\frac{1}{2}$

$$\therefore \frac{-4a-b}{12} = \frac{1}{2}$$

$$\Rightarrow -4a-b = 6 \quad \text{--- ③}$$

② − ③ ≡ $6a = 6$

$$\Rightarrow \boxed{a=1}$$

② ≡ $2(1) - b = 0$

$$\Rightarrow 2 = b \Rightarrow \boxed{b=2}$$

$\therefore a=1,\ b=2.$

7(b) A function $f(x)$ is defined as follows:

$f(x) = 1+\sin x$ for $0 < x < \pi/2$

$f(x) = 2 + \left(x-\frac{\pi}{2}\right)^2$ for $x \geq \pi/2$

Examine its continuity and derivability at $x=\pi/2$.

Sol'n: Continuity at $x=\pi/2$.

$$L.H.L = \lim_{x\to \frac{\pi}{2}-0} f(x) = \lim_{x\to\frac{\pi}{2}-0} (1+\sin x) \quad \Big| \text{put } x = \tfrac{\pi}{2}-h$$

$$= \lim_{h\to 0}\left[1+\sin\left(\tfrac{\pi}{2}-h\right)\right]$$

$$= \lim_{h\to 0}(1+\cos h) = 1+1=2$$

$$R.H.L = \lim_{x\to\frac{\pi}{2}+0} f(x) = \lim_{x\to\frac{\pi}{2}+0}\left[2+\left(x-\tfrac{\pi}{2}\right)^2\right] \quad \Big| \text{put } x=\tfrac{\pi}{2}+h$$

$$= \lim_{h\to 0}\left[2+\left(\tfrac{\pi}{2}+h-\tfrac{\pi}{2}\right)^2\right]$$

$$= \lim_{h\to 0}(2+h^2) = 2$$

$\because$ L.H.L = R.H.L = 2

$\therefore \lim_{x\to\pi/2} f(x)$ exists and is 2

Also $f\left(\frac{\pi}{2}\right) = 2+\left(\frac{\pi}{2}-\frac{\pi}{2}\right)^2 = 2$

$\because \lim_{x\to\pi/2} f(x) = f\left(\frac{\pi}{2}\right)$

$\therefore f(x)$ is continuous at $x=\pi/2$.

Differentiability at $x=\pi/2$

$$L.H.D = \lim_{x\to\frac{\pi}{2}-0} \frac{f(x)-f(\pi/2)}{x-\pi/2} = \lim_{x\to\frac{\pi}{2}-0}\frac{1+\sin x-2}{x-\pi/2}$$

$$= \underset{x \to \frac{\pi}{2}-0}{\text{Lt}} \frac{\sin x - 1}{x - \pi/2} \qquad \Big| \text{put } x = \frac{\pi}{2} - h$$

$$= \underset{h \to 0}{\text{Lt}} \frac{\sin\left(\frac{\pi}{2}-h\right)-1}{\left(\frac{\pi}{2}-h\right)-\frac{\pi}{2}} = \underset{h \to 0}{\text{Lt}} \frac{\cos h - 1}{-h}$$

$$= \underset{h \to 0}{\text{Lt}} \frac{1-\cos h}{h} \qquad \Big| \text{form } \frac{0}{0}$$

$$= \underset{h \to 0}{\text{Lt}} \frac{\sin h}{1} \qquad \Big| \text{L' Hospital Rule.}$$

$$= 0$$

$$\text{R.H.D} = \underset{x \to \frac{\pi}{2}+0}{\text{Lt}} \frac{f(x) - f\left(\frac{\pi}{2}\right)}{x - \frac{\pi}{2}}$$

$$= \underset{x \to \frac{\pi}{2}+0}{\text{Lt}} \frac{2 + \left(x - \frac{\pi}{2}\right)^2 - 2}{x - \frac{\pi}{2}}$$

$$= \underset{x \to \frac{\pi}{2}+0}{\text{Lt}} \left(x - \pi/2\right) \qquad \Big| \text{put } x = \frac{\pi}{2} + h$$

$$= \underset{h \to 0}{\text{Lt}} \left(\frac{\pi}{2} + h - \frac{\pi}{2}\right) = \underset{h \to 0}{\text{Lt}}\ h = 0$$

$\because$ L.H.D = R.H.D = 0

$\therefore f(x)$ is differentiable at $x = \frac{\pi}{2}$ and $f'\left(\frac{\pi}{2}\right) = 0$.

7(c) Find the points on the sphere $x^2+y^2+z^2=4$ that are closet to and farthest from the point $(3,1,-1)$.

sol'n: Given sphere $\Rightarrow g(x,y) = x^2+y^2+z^2-4=0$ —— ①

we need to find the closest and farthest from the point $(3,1,-1)$.

Let any point on sphere (x,y,z) to determine using Lagrangian multiplier

$$d^2 = (x-3)^2+(y-1)^2+(z+2)^2 \quad \text{—— ②}$$

$$\text{Let}: f(x,y,z) = d^2 + \lambda g(x,y)$$

$$f(x,y,z) = (x-3)^2+(y-1)^2+(z+2)^2+\lambda(x^2+y^2+z^2-4) \quad \text{—— ③}$$

$$\left.\begin{aligned}\frac{\partial F}{\partial x} &= 2(x-3)+\lambda(2x)=0 \Rightarrow x=\frac{3}{1+\lambda}\\ \frac{\partial F}{\partial y} &= 2(y-1)+\lambda(2y)=0 \Rightarrow y=\frac{1}{1+\lambda}\\ \frac{\partial F}{\partial z} &= 2(z+1)+\lambda(2z)=0 \Rightarrow z=\frac{-1}{1+\lambda}\end{aligned}\right\} \leftarrow ④$$

put the values of x,y,z from ④ in equ. ①.

$$\left(\frac{3}{1+\lambda}\right)^2+\left(\frac{1}{1+\lambda}\right)^2+\left(\frac{-1}{1+\lambda}\right)^2=4$$

$$3^2+1^2+1^2=4(1+\lambda)^2$$

$$(1+\lambda)^2=\frac{11}{4}$$

$$\Rightarrow \boxed{1+\lambda = \pm\frac{\sqrt{11}}{2}}$$

$\therefore$ The stationary points are

$$\left(\frac{6}{\sqrt{11}},\frac{2}{\sqrt{11}},\frac{-2}{\sqrt{11}}\right) \text{ and } \left(\frac{-6}{\sqrt{11}},\frac{-2}{\sqrt{11}},\frac{2}{\sqrt{11}}\right)$$

from ④

when $\lambda+1=\sqrt{11}/2$

$x = \frac{6}{\sqrt{11}}, \; y = \frac{2}{\sqrt{11}}, \; z = \frac{-2}{\sqrt{11}}$

(2) $\equiv d^2 = 1.7325 \Rightarrow \boxed{d_{min} = 1.316}$

when $\lambda + 1 = -\sqrt{11}/2$

(2) $\equiv d^2 = 28.266 \Rightarrow \boxed{d_{max} = 5.316}$

again partially differentiating (4) w.r.t. x, y, z respectively.

$\boxed{F_{xx} = 1+\lambda = F_{yy} = F_{zz}}$

Similarly: $F_{yx} = 0 = F_{zx}$

$F_{xz} = F_{xy} = 0$

$F_{yz} = F_{zy} = 0$

$|F_{xx}|, \begin{vmatrix} F_{xx} & F_{xy} \\ F_{yx} & F_{yx} \end{vmatrix}, \begin{vmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{yx} & F_{yy} & F_{yz} \\ F_{zx} & F_{zy} & F_{zz} \end{vmatrix}$

retain same +ve sign; thus it is a closest point.

$\therefore$ Closest point $= d = 1.316 = \left[\frac{6}{\sqrt{11}}, \frac{2}{\sqrt{11}}, \frac{-2}{\sqrt{11}}\right]$

Farthest point $= d = 5.316 = \left[\frac{-6}{\sqrt{11}}, \frac{-2}{\sqrt{11}}, \frac{2}{\sqrt{11}}\right]$

which is required solution.

7(d) → Find the volume lying inside the cylinder $x^2+y^2-2x=0$ and outside the paraboloid $x^2+y^2=2z$, while bounded by xy-plane.

Sol'n:

$$\text{volume} = \iint z\,dx\,dy$$

$$= \iint \frac{x^2+y^2}{2}\,dx\,dy$$

changing into polar co-ordinates

$$x = r\cos\theta \quad y = r\sin\theta$$

$$\Rightarrow dx\,dy = r\,dr\,d\theta$$

$$\therefore V = 2\int_0 \int_{r=0}^{2\cos\theta} \frac{r^2}{2}\, r\,dr\,d\theta$$

$$= \frac{2}{2}\int_0^{\pi/2} \left(\frac{r^4}{4}\right)_0^{2\cos\theta} d\theta$$

$$= \frac{1}{4}\int_0^{\pi/2} 16\cos^4\theta\,d\theta$$

$$= 4\int_0^{\pi/2} \cos^4\theta\,d\theta$$

$$= 4\cdot\frac{\Gamma 5/2\,\Gamma 1/2}{2\,\Gamma 3}$$

$$= \frac{4}{2\times 2}\,\frac{3}{2}\,\frac{1}{2}\sqrt{\pi}\cdot\sqrt{\pi}$$

$$= \frac{3\pi}{4}.$$

$$\int_0^{\pi/2} \sin^p\theta\cos^q\theta\,d\theta = \frac{1}{2}B\left(\frac{p+1}{2},\frac{q+1}{2}\right)$$

$$B(m,n) = \frac{\Gamma m\,\Gamma n}{\Gamma m+n}$$

$$\Gamma 3 = (3-1)! = 2$$

$$\Gamma 1/2 = \sqrt{\pi}$$

8(a)(i) Find the surface generated by a line which intersects the lines $y=a=z$ and $x+3z=a=y+z$ and is parallel to the plane $x+y=0$.

Sol^n: Any line that intersects the given line is given by the planes

$$(y-a)+k_1(z-a)=0 \text{ and } (x+3z-a)+k_2(y+z-a)=0 \quad \text{——(1)}$$

If λ, μ, ν are direction ratios of this line, then

$$\mu+k_1\nu=0 \quad \text{and} \quad \lambda+k_2\mu+(3+k_2)\nu=0$$

solving these we get $\dfrac{\lambda}{(3+k_2)-k_1k_2}=\dfrac{\mu}{k_1}=\dfrac{\nu}{-1}$

$\therefore$ The direction ratios of the line intersecting the given lines are $3+k_2-k_2k_1,\ k_1,\ -1$

If this line is parallel to the plane $x+y=0$, then this line is $\perp$lar to the normal to this plane

we have $(3+k_2-k_1k_2).1+k_1.1=0$

$$\Rightarrow 3+k_1+k_2-k_1k_2=0 \quad \text{——(2)}$$

Also from (1), we get $k_1=\dfrac{a-y}{z-a}$, $k_2=\dfrac{a-x-3z}{y+z-a}$

$\therefore$ from (2), the required locus is

$$3+\frac{a-y}{z-a}+\frac{a-x-3z}{y+z-a}-\left\{\frac{a-y}{z-a}\right\}\left\{\frac{a-x-3z}{y+z-a}\right\}=0$$

$$\Rightarrow 3(z-a)(y+z-a)+(a-y)(y+z-a)+(a-x-3z)(z-a)-(a-y)(a-x-3z)=0$$

$$\Rightarrow (y+z-a)[3z-3a+a-y]=(a-x-3z)[a-y-z+a]$$

$\Rightarrow y^2+yz+xy+xz-2az-2ax=0$, on simplifying

$$\Rightarrow y^2+yz+xy+xz=2a(x+z)$$

8(a)(ii) Find the equation of the right circular cylinder whose axis is $x-2=z,\ y=0$ and passes through the point $(3,0,0)$.

Sol'n: The equations of the axis of the cylinder are $\frac{x-2}{1}=\frac{y}{0}=\frac{z}{1}$ —— ①

Also $(\text{radius of the cylinder})^2 = [\text{length of } \perp\text{lar from } (3,0,0) \text{ to the line ①}]^2$.

$$=\frac{1}{[1^2+0^2+1^2]}\left[(0.1-0.0)^2+\{0.1-1(3-2)\}^2+\{0.(3-2)-1.0\}^2\right]$$

$$=\frac{1}{2}[1]$$

$$=\frac{1}{2}$$

Note: $\begin{vmatrix} m & n \\ y'-\beta & z'-\gamma \end{vmatrix}^2+\begin{vmatrix} n & l \\ z'-\gamma & x'-\alpha \end{vmatrix}^2+\begin{vmatrix} l & m \\ x'-\alpha & y'-\beta \end{vmatrix}^2$

$=\begin{vmatrix} 0 & 1 \\ 0 & 0 \end{vmatrix}^2+\begin{vmatrix} 1 & 1 \\ 0 & 3-2 \end{vmatrix}^2+\begin{vmatrix} 1 & 0 \\ 3-2 & 0 \end{vmatrix}^2=1$

Here $(\alpha,\beta,\gamma)=(2,0,0)$
$(x',y',z')=(3,0,0)$
& $(l,m,n)=(1,0,1)$

i.e. the radius of the cylinder is $\frac{1}{\sqrt{2}}$.

Now let $P(x_1,y_1,z_1)$ be any point on the cylinder.

Then the length of $\perp$lar from $P(x_1,y_1,z_1)$ to the axis ① must be equal to radius $\frac{1}{\sqrt{2}}$ of the cylinder i.e.

$$\left(\frac{1}{2}\right)[1^2+0+1^2]=\{1.y_1-0.z_1\}^2+\{1.z_1-1.(x_1-2)\}^2+\{0(x_1-2)-1.y_1\}^2$$

$$\Rightarrow 1 = y_1^2+(z_1-x_1+2)^2+y_1^2$$

$$\Rightarrow x_1^2+2y_1^2+z_1^2-2z_1x_1-4x_1+2z_1+3=0$$

$\therefore$ The locus of $P(x_1,y_1,z_1)$ or the required equation of the cylinder is

$$x^2+2y^2+z^2-2zx-4x+2z+3=0$$

8(b) The sections of the enveloping cone of the surface $x^2/a^2+y^2/b^2+z^2/c^2=1$ whose vertex is $P(x_1,y_1,z_1)$ by the plane $z=0$ is
(i) Rectangular hyperbola, (ii) a parabola and (iii) a circle.
Find the locus of the vertex P.

sol'n: For the given surface we have

$$S=x^2/a^2+y^2/b^2+z^2/c^2-1;\ S_1=x_1^2/a^2+y_1^2/b^2+z_1^2/c^2-1.$$

and $T=(xx_1/a^2)+(yy_1/b^2)+(zz_1/c^2)-1$

$\therefore$ The enveloping cone is $SS_1=T^2$

i.e. $$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}-1\right)^2$$

Its section by the plane $z=0$ is given by

$$z=0,\ \left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2 \quad \text{—— (1)}$$

(i) If the equations (A) represent a rectangular hyperbola then the sum of the coefficients of x^2 and y^2 should be zero.

i.e. $$\frac{1}{a^2}\left(\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)+\frac{1}{b^2}\left(\frac{x_1^2}{a^2}+\frac{z_1^2}{c^2}-1\right)=0$$

$$\Rightarrow \frac{x_1^2+y_1^2}{a^2b^2}+\frac{1}{c^2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)z_1^2=\frac{1}{a^2}+\frac{1}{b^2}$$

$$\Rightarrow \frac{x_1^2+y_1^2}{a^2+b^2}+\frac{z_1^2}{c^2}=1,$$ dividing each term by a^2+b^2.

$\therefore$ The required locus of $P(x_1,y_1,z_1)$ is $\dfrac{x^2+y^2}{a^2+b^2}+\dfrac{z^2}{c^2}=1.$

(ii) If the equations (1) represent a parabola, then we should have $h^2=ab$

Here a = coefficient of $x^2=\frac{1}{a^2}\left(\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)$

$b' = \text{coeff. of } y^2 = \frac{1}{b^2}\left(\frac{x_1^2}{a^2}+\frac{z_1^2}{c^2}-1\right)$ & $h' = \text{coeff. of } 2xy = \frac{x_1y_1}{a^2b^2}$.

$\therefore$ If the equations ① represent a parabola, then $h'^2 = a'b'$

i.e. $$\frac{x_1^2y_1^2}{a^4b^4} = \frac{1}{a^2b^2}\left(\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{z_1^2}{c^2}-1\right)$$

$$\Rightarrow \frac{x_1^2}{a^2}\left(\frac{z_1^2}{c^2}-1\right)+\frac{y_1^2}{b^2}\left(\frac{z_1^2}{c^2}-1\right)+\left(\frac{z_1^2}{c^2}-1\right)^2=0$$

$$\Rightarrow \left(\frac{z_1^2}{c^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)=0$$

$\Rightarrow \frac{z_1^2}{c^2}-1=0$, since $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1 \neq 0$ as $P(x_1, y_1, z_1)$ does not lie on the given surface.

$\Rightarrow z_1^2 = c^2 \Rightarrow z_1 = \pm c$

$\therefore$ The locus of $P(x_1, y_1, z_1)$ is $z = \pm c$.

(iii) If ① represents a circle then the coefficients of x^2 & y^2 should be equal and coefficient of xy should be zero.

i.e. $$\frac{1}{a^2}\left(\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right) = \frac{1}{b^2}\left(\frac{x_1^2}{a^2}+\frac{z_1^2}{c^2}-1\right) \quad \text{— ②}$$

and $$x_1y_1/a^2b^2 = 0 \quad \text{— ③}$$

from ③ either $x_1 = 0$ (or) $y_1 = 0$.

If $x_1 = 0$, then from ② we have $\frac{1}{a^2}\left(\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right) = \frac{1}{b^2}\left(\frac{z_1^2}{c^2}-1\right)$

$\therefore$ The locus of $P(x_1, y_1, z_1)$ is $x=0$, $\frac{y^2}{b^2-a^2}+\frac{z^2}{c^2}=1$

If $y_1 = 0$, then from ② we have

$$\frac{1}{a^2}\left(\frac{z_1^2}{c^2}-1\right) = \frac{1}{b^2}\left(\frac{x_1^2}{a^2}+\frac{z_1^2}{c^2}-1\right)$$

$\therefore$ The locus of $P(x_1, y_1, z_1)$ is

$$y=0,\ \frac{x^2}{a^2-b^2}+\frac{z^2}{c^2}=1.$$

8(C) → Prove that the shortest distance between generators of the same system drawn at the ends of diameters of the principal elliptic section of the hyperboloid $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$ lie on the surfaces whose equations are $\frac{cxy}{x^2+y^2} = \pm\frac{abz}{a^2-b^2}$

Sol'n: Let any point on the elliptic section $P(a\cos\alpha, b\sin\alpha, 0)$, then the equations of the generator through it are

$$\frac{x-a\cos\alpha}{a\sin\alpha} = \frac{y-b\sin\alpha}{-b\cos\alpha} = \frac{z-0}{c} \quad \text{——— (1)}$$

The extremity of the diameter through this point P is $Q(-a\cos\alpha, -b\sin\alpha, 0)$ which is obtained by putting $\alpha+\pi$ for α in the coordinates of the point P and so the equations of the generator of the same system through Q is obtained by putting $\alpha+\pi$ for α in (1) and are

$$\frac{x+a\cos\alpha}{-a\sin\alpha} = \frac{y+b\sin\alpha}{b\cos\alpha} = \frac{z-0}{c} \quad \text{——— (2)}$$

If l, m, n be the direction cosines of the S.D then

$$la\sin\alpha - mb\cos\alpha + nc = 0$$

$$\text{and } -la\sin\alpha + mb\cos\alpha + nc = 0$$

Solving these simultaneously for l, m, n, we get

$$\frac{l}{-2bc\cos\alpha} = \frac{m}{-2ac\sin\alpha} = \frac{n}{0} \Rightarrow \frac{l}{b\cos\alpha} = \frac{m}{a\sin\alpha} = \frac{n}{0} \quad \text{——— (3)}$$

Also equation of plane containing the generator (1) and the line of S.D is

$$\begin{vmatrix} x-a\cos\alpha & y-b\sin\alpha & z-0 \\ a\sin\alpha & -b\cos\alpha & c \\ b\cos\alpha & a\sin\alpha & 0 \end{vmatrix} = 0$$

and the equation of the plane containing the generator (2) and the line of S.D is

$$\begin{vmatrix} x+a\cos\alpha & y+b\sin\alpha & z-0 \\ -a\sin\alpha & b\cos\alpha & c \\ b\cos\alpha & a\sin\alpha & 0 \end{vmatrix} = 0$$

Expanding the above determinants w.r.to 3rd column, we get

$$z(a^2\sin^2\alpha + b^2\cos^2\alpha) - c(a\sin\alpha(x-a\cos\alpha) - b\cos\alpha(y-b\sin\alpha)] = 0 \quad \text{--- (4)}$$

and $-z(a^2\sin^2\alpha + b^2\cos^2\alpha) - c[a\sin\alpha(x+a\cos\alpha) - b\cos\alpha(y+b\sin\alpha)] = 0$ --- (5)

Eliminating α between (4) & (5) we can find the locus of S.D. For this adding & Subtracting (4) & (5), we get

$$-2acx\sin\alpha + 2bcy\cos\alpha = 0 \Rightarrow \tan\alpha = (by)/(ax) \quad \text{--- (6)}$$

and $2z(a^2\sin^2\alpha + b^2\cos^2\alpha) + 2c(a^2-b^2)\sin\alpha\cos\alpha = 0$ --- (7)

from (7), $z(a^2\tan^2\alpha + b^2) + c(a^2-b^2)\tan\alpha = 0$

$$\Rightarrow z[a^2(b^2y^2/a^2x^2) + b^2] + c(a^2-b^2)(by/ax) = 0, \text{ from (6)}$$

$$\Rightarrow \frac{b^2z^2(x^2+y^2)}{x^2} + \frac{c(a^2-b^2)by}{ax} = 0$$

$$\Rightarrow abz(x^2+y^2) + cxy(a^2-b^2) = 0$$

$$\Rightarrow \frac{cxy}{x^2+y^2} = \frac{-abz}{(a^2-b^2)}$$

In a similar manner if we consider the generator of the other system, we can find that the locus of S.D is $\frac{cxy}{x^2+y^2} = \frac{abz}{a^2-b^2}$

Hence the required locus of S.D is

$$\frac{cxy}{x^2+y^2} = \frac{\pm abz}{a^2-b^2}$$