LINEAR ALGEBRA

CLASS TEST

ANSWER KEY

Q.1(i) Show that the vectors $v = (1+i, 2i)$ and $w = (1, 1+i)$ in $\mathbb{C}^2$ are linearly dependent over the complex field $\mathbb{C}$ but are linearly independent over the real field $\mathbb{R}$.

Solution

We know that, the two vectors are dependent iff one is a multiple of the other. Since, the first co-ordinate of w is 1,

v can be multiple of w iff $v = (1+i)w$

But $1+i \notin \mathbb{R}$

Hence; v and w are independent over $\mathbb{R}$

Since;
$$(1+i)w = (1+i)(1, 1+i)$$
$$(1+i)w = (1+i, 2i) = v$$
and $(1+i) \in \mathbb{C}$.

They are dependent over $\mathbb{C}$.

Q.1(ii) Let W be the subspace of $\mathbb{R}^3$ defined by $W = \{(a, b, c) : a+b+c = 0\}$. Find a basis and dimension of W.

Solution :-

Let W be the subspace of $\mathbb{R}^3$ defined by $W = \{(a, b, c) : a+b+c = 0\}$.

Note ; $W \neq \mathbb{R}^3$

Since, for example, $(1, 2, 3) \notin W$

Thus ; $\dim W < 3$

$u_1 = (1, 0, -1)$ and $u_2 = (0, 1, -1)$
are two independent vectors in W.

Thus; $\boxed{\dim W = 2}$
and u_1 & u_2 are basis of W.
i.e $\boxed{W = \{(1, 0, -1), (0, 1, -1)\}}$

Q.1(iii) Suppose U and W are distinct four dimensional subspaces of a vector space V of dimension 6. Find the possible dimensions of $U \cap W$?

Solution:- Since, U and W are distinct four dimensional subspaces of a vector space
$\Rightarrow$ $U+W$; properly contains U and W
Hence; $\dim(U+W) > 4$
But $\dim(U+W)$ cannot be greater than the vector space V of dimension 6.
Since, $\dim(V) = 6$.
Hence, we have two possibilities
(i) $\dim(U+W) = 5$
(ii) $\dim(U+W) = 6$

If Case I> $\dim(U+W) = 5$
$$\therefore \dim(U \cap W) = \dim U + \dim W - \dim(U+W)$$
$$= 4 + 4 - 5 = 3$$

Case II> $\dim(U+W) = 6$
$$\therefore \text{Dim}(U \cap W) = \dim U + \dim W - \dim(U+W)$$
$$= 4 + 4 - 6 = 2$$

$\therefore$ $\boxed{\begin{array}{l} \text{When } \dim(U+W) = 5 \Rightarrow \dim(U\cap W) = 3 \\ \quad\quad\;\; \dim(U+W) = 6 \Rightarrow \text{Dim}(U \cap W) = 2 \end{array}}$

which is required solution.

Q.2(i) Let T be a linear operator on $\mathbb{R}^3$. which is represented in the standard ordered basis by the matrix.

$$A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$$

Prove that T is diagonalizable by exhibiting a basis for $\mathbb{R}^3$ each vector of which is characteristic vector of T.

Solution:-

$$A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$$

The characteristic equation of T is $\det(A - xI) = 0$

i.e. $$\begin{vmatrix} -9-x & 4 & 4 \\ -8 & 3-x & 4 \\ -16 & 8 & 7-x \end{vmatrix} = 0$$

Or $$\begin{vmatrix} -1-x & 4 & 4 \\ -1-x & 3-x & 4 \\ -1-x & 8 & 7-x \end{vmatrix} = 0 \qquad C_1 \to C_1 + C_2 + C_3$$

$$\Rightarrow -(1+x)\begin{vmatrix} 1 & 4 & 4 \\ 1 & 3-x & 4 \\ 1 & 8 & 7-x \end{vmatrix} = 0$$

Now, $R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$

$$= -(1+x)\begin{vmatrix} 1 & 4 & 4 \\ 0 & -(1+x) & 0 \\ 0 & 8 & 3-x \end{vmatrix} = 0$$

$$= -(1+x)\cdot 1\cdot[-(1+x)(3-x)-0]$$

$$\Rightarrow +(1+x)[1+x][3-x] = 0$$

$$(1+x)^2(3-x) = 0$$

Hence, the characteristics values of T are 3, -1, -1.

The characteristics vector corresponding to $x=3$ is given by -

$$(A-3I)X = 0$$

$$\begin{bmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \to R_1 - R_2$

$$\begin{bmatrix} -4 & 4 & 0 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 4R_1$

$$\begin{bmatrix} -4 & 4 & 0 \\ 0 & -8 & 4 \\ 0 & -8 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 4 & 0 \\ 0 & -8 & 4 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

The above matrix equation yields

$$-x_1 + x_2 = 0 \quad ; \quad -2x_2 + x_3 = 0$$

These equation are satisfied by ; $x_1 = 1, x_2 = 1$
$x_3 = 2$.

An eigen vector corresponding to eigen value $x=3$ is $\Rightarrow x_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$

The eigen vector corresponding to eigen value $x = -1$, is given by

$$(A+I)X = 0$$

$$\begin{bmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$; $R_3 \to R_3 - 2R_1$

$$\begin{bmatrix} -8 & 4 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

From, the above matrix equation, we get

$$-2x_1 + x_2 + x_3 = 0$$

Taking; $x_2 = 0$; $x_1 = 1, x_3 = 2$

taking; $x_3 = 0$; $x_1 = 1$, $x_2 = 2$

Hence, two L.I. characteristic vectors corresponding to the characteristic value $x = -1$

$$X_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

It is easy to verify that X_1, X_2, X_3 are linearly independent over $\mathbb{R}$ and so the set $\{X_1, X_2, X_3\}$ constitutes a basis of $\mathbb{R}^3$.

Hence, T is diagonalizable. Indeed.

$$P^{-1}AP = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}; \text{ where } \boxed{P = [X_1 \; X_2 \; X_3] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 2 & 2 & 0 \end{bmatrix}}$$

Required solution.

Q.2(ii) If A is non-singular, prove that the eigenvalues of A^{-1} are reciprocals of the eigenvalues of A.

Solution:- Let, λ be an eigen value of A and X be corresponding eigen vector. Then,

$$AX = \lambda X \Rightarrow X = A^{-1}(\lambda X) = \lambda(A^{-1}X)$$

$$\Rightarrow \frac{1}{\lambda} X = A^{-1} X \quad \left[\because A \text{ is non-singular} \Rightarrow \lambda \neq 0\right]$$

$$\Rightarrow A^{-1}X = \frac{1}{\lambda} X$$

$\Rightarrow \frac{1}{\lambda}$ is an eigenvalue of A^{-1}

and X is a corresponding eigenvector.

Conversely, suppose that K is an eigenvalue of A^{-1}. Since, A is non-singular

$\Rightarrow A^{-1}$ is non-singular and $(A^{-1})^{-1} = A$,

Therefore, it follows from the first part that $1/K$ is an eigenvalue of A.

Thus, each eigenvalue of A^{-1} is reciprocal of some eigenvalue of A.

Q.3

(i) Find the Kernel and range of the linear operator $T(x, y, z) = (x, y, 0)$ and describe transformation geometrically.

(ii) If α, β are any scalars, then prove that $A^2 - (\alpha+\beta)A + \alpha\beta I = (A - \alpha I)(A - \beta I)$, where A is any square matrix of order n and $I = I_n$.

Solution:

(i) Given $T: \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$T(x, y, z) = (x, y, 0)$$

Also, the Kernel and range will both be subspaces of $\mathbb{R}^3$.

Kernel:

Ker (T) is the subset that is mapped into $(0, 0, 0)$

i.e. $\text{Ker } T = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,/\, T(x_1, x_2, x_3) = (0, 0, 0)\}$

i.e. $T(x, y, z) = (0, 0, 0)$

$\Rightarrow (x, y, 0) = (0, 0, 0)$ if $x = 0, y = 0$.

$\therefore \text{Ker } T = \{(0, 0, z) \,/\, z \in \mathbb{R}\}$

Geometrically, Ker (T) is the set of all vectors that lie on the z-axis.

Range :

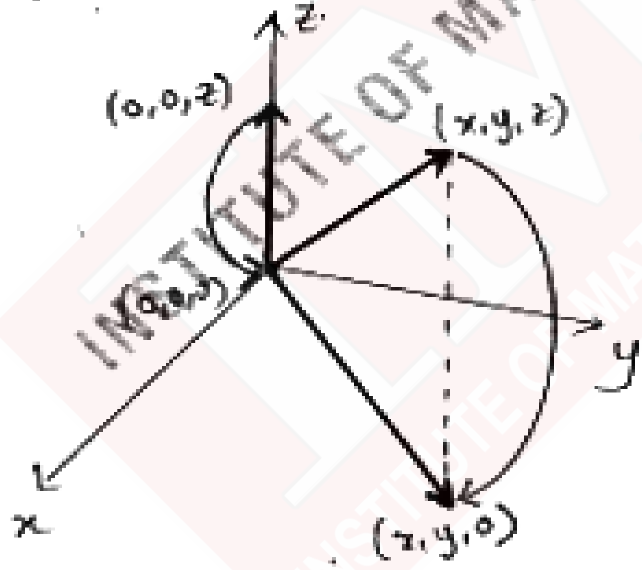
Range space of $T = \{ \beta \in \mathbb{R}^3 / T(\alpha) = \beta$ for $\alpha \in \mathbb{R}^5 \}$

i.e. the range space consists of all vectors of the type $(x, y, 0)$ for all $(x, y, z) \in \mathbb{R}^3$.

i.e. $\text{Range}(T) = \{ (x, y, 0) \in \mathbb{R}^3 / T(x, y, z) = (x, y, 0) \ \forall (x, y, z) \in \mathbb{R}^3 \}$

Geometrically, Range (T) is the set of all vectors that lie in the xy-plane.

Geometric interpretation of the transformation :



T projects the vector (x, y, z) into the vector $(x, y, 0)$ in the xy-plane & also T project all vectors onto xy plane.

[T is an example of projection operator.]

(ii) Given A is any square matrix of order n and $I = I_n$.

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n \times n}$, $I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & & \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{n \times n}$

then,

$$A^2 = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \cdot \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

$$\therefore A^2 = \begin{bmatrix} \sum a_{1i}a_{i1} & \sum a_{1i}a_{i2} & \cdots & \sum a_{1i}a_{in} \\ \sum a_{2i}a_{i1} & \sum a_{2i}a_{i2} & \cdots & \sum a_{2i}a_{in} \\ \vdots \\ \sum a_{ni}a_{i1} & \sum a_{ni}a_{i2} & \cdots & \sum a_{ni}a_{in} \end{bmatrix}$$

Now, for any α, β as scalars, consider

L.H.S.

i.e. $A^2 - (\alpha+\beta)A + \alpha\beta I$

$$= \begin{bmatrix} \sum a_{1i}a_{i1} & \sum a_{1i}a_{i2} & \cdots & \sum a_{1i}a_{in} \\ \sum a_{2i}a_{i1} & \sum a_{2i}a_{i2} & \cdots & \sum a_{2i}a_{in} \\ \vdots \\ \sum a_{ni}a_{i1} & \sum a_{ni}a_{i2} & \cdots & \sum a_{ni}a_{in} \end{bmatrix}$$

$$\begin{bmatrix} (\alpha+\beta)a_{11} & (\alpha+\beta)a_{12} & \cdots & (\alpha+\beta)a_{1n} \\ (\alpha+\beta)a_{21} & (\alpha+\beta)a_{22} & \cdots & (\alpha+\beta)a_{2n} \\ \vdots \\ (\alpha+\beta)a_{n1} & (\alpha+\beta)a_{n2} & \cdots & (\alpha+\beta)a_{nn} \end{bmatrix} +$$

$$\begin{bmatrix} \alpha\beta & 0 & \cdots & 0 \\ 0 & \alpha\beta & \cdots & 0 \\ \vdots \\ 0 & 0 & \cdots & \alpha\beta \end{bmatrix}$$

$$= \begin{bmatrix} \sum a_{1i}a_{i1} - (\alpha+\beta)a_{11} + \alpha\beta & \sum a_{1i}a_{i2} - (\alpha+\beta)a_{12} & \cdots & \sum a_{1i}a_{in} - (\alpha+\beta)a_{1n} \\ \sum a_{2i}a_{i1} - (\alpha+\beta)a_{21} & \sum a_{2i}a_{i2} - (\alpha+\beta)a_{22} + \alpha\beta & \cdots & \sum a_{2i}a_{in} - (\alpha+\beta)a_{2n} \\ \vdots & & & \\ \sum a_{ni}a_{i1} - (\alpha+\beta)a_{n1} & \sum a_{ni}a_{i2} - (\alpha+\beta)a_{n2} & \cdots & \sum a_{ni}a_{in} - (\alpha+\beta)a_{nn} + \alpha\beta \end{bmatrix} \quad \text{———} \ (1)$$

Now consider R.H.S,

i.e. $(A - \alpha I)(A - \beta I)$

$$= \begin{bmatrix} a_{11}-\alpha & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\alpha & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\alpha \end{bmatrix} \begin{bmatrix} a_{11}-\beta & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\beta & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\beta \end{bmatrix}$$

$$= \begin{bmatrix} \sum a_{1i}a_{i1} - (\alpha+\beta)a_{11} + \alpha\beta & \sum a_{1i}a_{i2} - (\alpha+\beta)a_{12} & \cdots & \sum a_{1i}a_{in} - (\alpha+\beta)a_{1n} \\ \sum a_{2i}a_{i1} - (\alpha+\beta)a_{21} & \sum a_{2i}a_{i2} - (\alpha+\beta)a_{22} + \alpha\beta & \cdots & \sum a_{2i}a_{in} - (\alpha+\beta)a_{2n} \\ \vdots & & & \\ \sum a_{ni}a_{i1} - (\alpha+\beta)a_{n1} & \sum a_{ni}a_{i2} - (\alpha+\beta)a_{n2} & \cdots & \sum a_{ni}a_{in} - (\alpha+\beta)a_{nn} + \alpha\beta \end{bmatrix} \quad \text{———} \ (2)$$

From (1) & (2), we conclude that,

L.H.S = R.H.S

i.e. $A^2 - (\alpha+\beta)A + \alpha\beta I = (A - \alpha I)(A - \beta I)$

for any square matrix A of order n and $I = I_n$ where α, β are any scalars.

Hence, proved.

Q.4 (i) A square matrix A is said to be involutory if $A^2 = I$. Prove that the matrices $\begin{bmatrix} 1 & \alpha \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ \alpha & -1 \end{bmatrix}$ are involutary for all scalars α. Determine all 2x2 involutary matrices

(ii) Determine the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ -1 & 3 & 0 & -4 \\ 2 & 1 & 3 & -2 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$

Solution:

Let $A = \begin{bmatrix} 1 & \alpha \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ \alpha & -1 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 1 & \alpha \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 & \alpha \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \quad \text{--- (i)}$$

$$B^2 = \begin{bmatrix} 1 & 0 \\ \alpha & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ \alpha & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \quad \text{--- (ii)}$$

from (i) & (ii), we conclude that A and B are involutary matrices for all scalars α.

Now, consider $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that M is an involutary matrix.

i.e. $M^2 = I$.

$$M^2 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

— [$\because$ M is an involutory matrix]

$\Rightarrow a^2+bc = 1$ —— (1)

$bc+d^2 = 1$ —— (2)

$b(a+d) = 0$ —— (3)

$c(a+d) = 0$ —— (4)

from (1) − (2), we have, $a^2 - d^2 = 0 \Rightarrow a = \pm d$.

(3) $\Rightarrow$ $b = 0$ or $a = -d$ } —— (5)

(4) $\Rightarrow$ $c = 0$ or $a = -d$.

Considering following cases for enumerating the required matrices:-

Case (i) :- $a = d \Rightarrow a^2 = 1 \Rightarrow a = \pm 1$

[$\because b = 0 = c$ from (5)]

$\therefore M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $M_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ —— (*)

Case (ii) :- $a = -d \Rightarrow bc = 1 - a^2$ $(b, c \neq 0)$ [from (5)]

Let $a = \alpha$, $b = \beta \Rightarrow d = -\alpha$, $c = \frac{1-\alpha^2}{\beta}$

$\therefore M_3 = \begin{bmatrix} \alpha & \beta \\ \frac{1-\alpha^2}{\beta} & -\alpha \end{bmatrix}$ —— (**).

from (*) & (**), we conclude that M_1, M_2, M_3 are all 2×2 involutory matrices where M_3 includes Pauli's matrices $\sigma_1, \sigma_2, \sigma_3$.

(ii) The given matrix is

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ -1 & 3 & 0 & -4 \\ 2 & 1 & 3 & -2 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$

$R_2 \to R_2 + R_1$, $R_3 \to R_3 - 2R_1$, $R_4 \to R_4 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 5 & -1 & -4 \\ 0 & -3 & 5 & -2 \\ 0 & -1 & 2 & -1 \end{bmatrix}$$

$R_2 \leftrightarrow R_4$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & -3 & 5 & -2 \\ 0 & 5 & -1 & -4 \end{bmatrix}$$

$R_2 \to R_2 \times (-1)$

$$\sim \begin{bmatrix} 1 & +2 & -1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & -3 & 5 & -2 \\ 0 & 5 & -1 & -4 \end{bmatrix}$$

$R_3 \to R_3 + 3R_2$, $R_4 \to R_4 - 5R_2$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 9 & -9 \end{bmatrix}$$

$R_4 \to R_4 + 9R_3$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \to R_3 \times (-1)$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & -2 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_2 \to R_2 + 2R_3$, $R_1 \to R_1 + R_3$

$$\sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1 - 2R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

clearly which is in row-reduced echelon form.

The number of non-zero rows in the echelon matrix is 3.

$\therefore$ $\boxed{\text{Rank of } A = 3.}$

Q.5(a) Find a basis for the column space of the following matrix A.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & -2 \\ -1 & -4 & 6 \end{bmatrix}$$

Solution:

The transpose of A is

$$A' = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & -4 \\ 0 & -2 & 6 \end{bmatrix}$$

The column space of A becomes the row space of A'.

Now, we proceed further to find a basis for the row space of A' by obtaining the reduced echelon form of A'.

$$\begin{bmatrix} 1 & 2 & -1 \\ 1 & 3 & -4 \\ 0 & -2 & 6 \end{bmatrix} \underset{R_2 \to R_2 - R_1}{\sim} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -3 \\ 0 & -2 & 6 \end{bmatrix} \underset{\substack{R_1 \to R_1 - 2R_2 \\ R_3 \to R_3 + 2R_2}}{\sim} \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

Clearly which is in reduced echelon form.

The non-zero row vectors of this echelon form are $(1, 0, 5)$, $(0, 1, -3)$ which form the basis for the row space of A'.

$\therefore$ These vectors in column form represent a basis for the column space of A.

Thus, $\begin{bmatrix}1\\0\\5\end{bmatrix}, \begin{bmatrix}0\\1\\-3\end{bmatrix}$ are a basis for the column space of A.

Hence, the result.

Q.5(b) Find a basis for the subspace V of $\mathbb{R}^4$ spanned by the vectors (1, 2, 3, 4), (−1, −1, −4, −2), (3, 4, 11, 8)

Solution:

We construct a matrix A having the given vectors as row vectors.

$$A = \begin{bmatrix}1 & 2 & 3 & 4\\-1 & -1 & -4 & -2\\3 & 4 & 11 & 8\end{bmatrix}$$

Now, we proceed ahead to compute the reduced echelon form of A.

$$\begin{bmatrix}1 & 2 & 3 & 4\\-1 & -1 & -4 & -2\\3 & 4 & 11 & 8\end{bmatrix} \sim \begin{bmatrix}1 & 2 & 3 & 4\\0 & 1 & -1 & 2\\0 & -2 & 2 & -4\end{bmatrix} \quad R_2 \to R_2 + R_1,\ R_3 \to R_3 - 3R_1$$

$R_1 \to R_1 - 2R_2,\ R_3 \to R_3 + 2R_2$

$$\sim \begin{bmatrix}1 & 0 & 5 & 0\\0 & 1 & -1 & 2\\0 & 0 & 0 & 0\end{bmatrix}$$

Clearly, this is in echelon form.

The non-zero vectors of this reduced echelon form, namely (1, 0, 5, 0) and (0, 1, −1, 2) are a basis for the subspace V of $\mathbb{R}^4$.

The dimension of this subspace is two.

Hence, the result.

Q.5(c) Determine the Kernel and the Range of the transformation defined by the following matrix.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \\ 1 & 1 & 4 \end{bmatrix}$$

Solution :

A is a 3×3 matrix.

Thus, A defines a linear operator $T: \mathbb{R}^3 \to \mathbb{R}^3$.

$$T(\mathbf{x}) = A\mathbf{x}$$

The elements of $\mathbb{R}^3$ are written in column matrix form for the purpose of matrix multiplication. For convenience, we express the elements of $\mathbb{R}^3$ in row form at all other times.

(i) Kernel:

The Kernel will consist of all vectors $\mathbf{x} = (x_1, x_2, x_3)$ in $\mathbb{R}^3$, such that $T(\mathbf{x}) = \mathbf{0}$.

Thus,

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \\ 1 & 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This matrix equation corresponds to the following system of linear equations.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 0 \\ -x_2 + x_3 &= 0 \\ x_1 + x_2 + 4x_3 &= 0 \end{aligned}$$

$\Rightarrow \quad x_2 = x_3$

$x_1 = -x_2 - 4x_3 = -x_3 - 4x_3 = -5x_3$

$\therefore$ The Kernel is thus the set of vectors of the form $(-5x_3, x_3, x_3)$.

$$\therefore \boxed{\text{Ker } T = \{(-5x_3, x_3, x_3) / x_3 \in \mathbb{R}\}}$$

Ker T is one-dimensional subspace of $\mathbb{R}^3$ with basis $(-5, 1, 1)$.

(ii) Range :

The range is spanned by the column vectors of A. We write these column vectors as rows of a matrix and compute an echelon form of the matrix. The non-zero ~~need~~ row vectors will give a basis for the range.

We get

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \\ 3 & 1 & 4 \end{bmatrix} \underset{\substack{R_2 \to R_2 - 2R_1,\\ R_3 \to R_3 - 3R_1}}{\sim} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix} \underset{R_2 \to (-1) R_2}{\sim} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Clearly it is in echelon form and the vectors $(1, 0, 1)$ and $(0, 1, 1)$ span the range of T.

An arbitrary vector in the range will be a linear combination of these vectors, $s(1, 0, 1) + t(0, 1, 1)$.

Thus, the range of T is

$$\boxed{\text{Range}(T) = \{(s, t, s+t) / s, t \in \mathbb{R}\}}$$

The vectors $(1, 0, 1)$ and $(0, 1, 1)$ are also linearly independent. Range (T) is a two-dimensional subspace of $\mathbb{R}^3$ with basis $\{(1, 0, 1), (0, 1, 1)\}$.

Hence the result.

Q.6(i)

Consider the linear transformation $T: P_2 \to P_1$ defined by $T(ax^2+bx+c) = (a+b)x - c$.

Find a matrix of T with respect to the bases $\{u_1, u_2, u_3\}$ and $\{u_1', u_2'\}$ of P_2 and P_1 where $u_1 = x^2$, $u_2 = x$, $u_3 = 1$ and $u_1' = x$, $u_2' = 1$.

Use this matrix to find the image of $u = 3x^2 + 2x - 1$.

Solution:

Consider the effect of T on each basis vector of P_2.

$$T(u_1) = T(x^2) = x = 1x + 0(1) = 1u_1' + 0u_2'$$
$$T(u_2) = T(x) = x = 1x + 0(1) = 1u_1' + 0u_2'$$
$$T(u_3) = T(1) = -1 = 0x + (-1)(1) = 0u_1' + (-1)u_2'$$

The coordinate vectors of $T(x^2)$, $T(x)$, and $T(1)$ are $\begin{bmatrix}1\\0\end{bmatrix}$, $\begin{bmatrix}1\\0\end{bmatrix}$, and $\begin{bmatrix}0\\-1\end{bmatrix}$.

The matrix of T is thus,

$$\boxed{A = \begin{bmatrix}1 & 1 & 0\\0 & 0 & -1\end{bmatrix}.} \quad \text{---(i)}$$

Now, we find the image of $u = 3x^2 + 2x - 1$ using the matrix A.

The coordinate vector of u relative to the basis $\{x^2, x, 1\}$ is $a = \begin{bmatrix}3\\2\\-1\end{bmatrix}$.

We get,

$$\mathbf{b} = A\mathbf{a}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

$$\therefore \boxed{T(u) = 5u_1' + 1\,u_2' = 5x + 1.} \quad \text{---(ii)}$$

Hence the result.

Q.6(ii) Consider the linear operator $T(x, y) = (3x + y, x + 3y)$ on $\mathbb{R}^2$. Find a diagonal matrix representation of T. Determine the basis for this representation and give a geometrical interpretation of T.

Solution:

First of all, we find a matrix representation A relative to the standard basis $B = \{(1,0), (0,1)\}$ of $\mathbb{R}^2$.

We get

$$T(1,0) = (3,1) = 3(1,0) + 1(0,1)$$

$$T(0,1) = (1,3) = 1(1,0) + 3(0,1)$$

The co-ordinate vectors of $T(1,0)$ and $T(0,1)$ relative to B are $\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

The matrix representation of T relative to the standard basis B is thus

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}.$$

Now, we have to compute eigenvalues and eigenvectors of A.

The eigenvalues are obtained by $|A - \lambda I| = 0$

$\Rightarrow \lambda^2 - 6\lambda + 8 = 0$

$\Rightarrow \lambda = 4, 2$

The eigenvectors corresponding to eigenvalues

$\lambda_1 = 4$ is $v_1 = r\begin{bmatrix} 1 \\ 1 \end{bmatrix}$;

$\lambda_2 = 2$ is $v_2 = s\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ on solving.

The following matrix A' is thus a diagonal matrix representation of T.

$$\boxed{A' = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}} \quad \text{—— (i)}$$

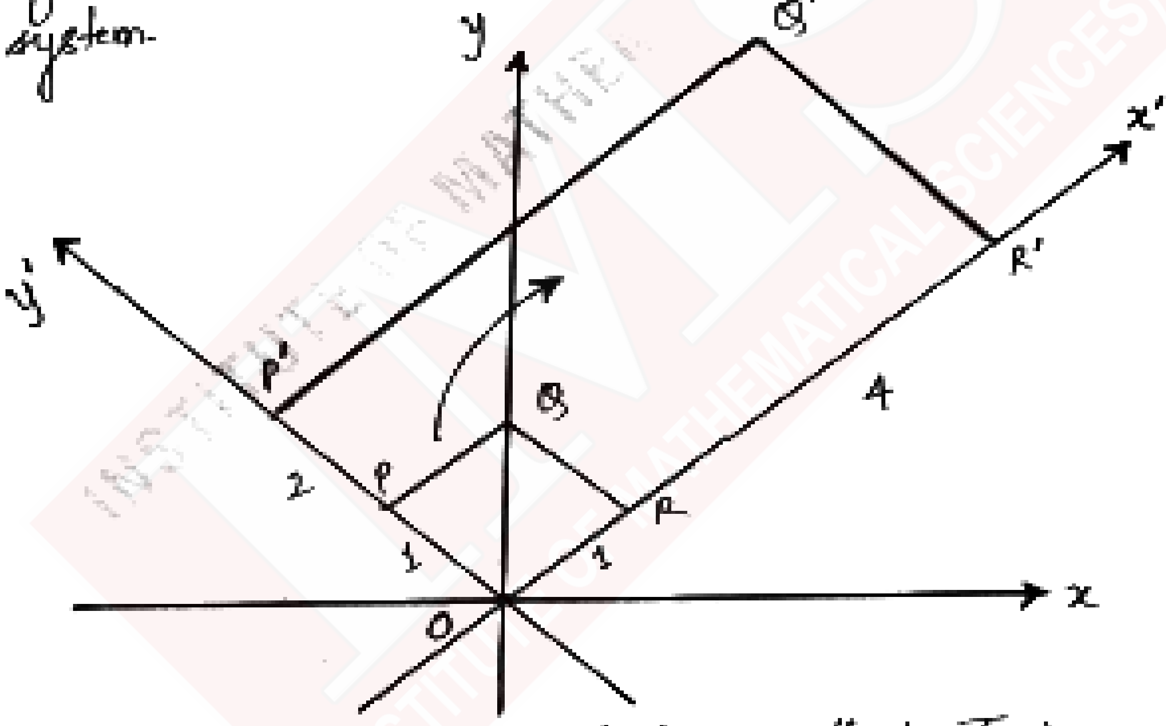
Now, we find the basis B' which gives this representation A'. Since A is a symmetric matrix, we select unit orthogonal eigenvectors for the coordinate vectors of B' relative to B. The transition matrix from B to B' will then be orthogonal, and the geometry will be preserved.

Let $\boxed{B' = \left\{\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right), \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)\right\}}$

Also, the basis B' is obtained from the basis B by rotation through $\pi/4$. ——— (ii)

Geometrical Interpretation:

The standard basis B defines an xy co-ordinate system. Let the basis B' define an x'y' coordinate system.



The matrix A' tells us that T is a scaling in the x'y' co-ordinate system, with factor 4 in the x' direction and factor 2 in the y' direction.

Thus, for example, T maps the square PQRO into the rectangle P'Q'R'O.

Q.7

Determine the conditions for the consistency of the equations.

$ax+by+cz=p$; $bx+cy+az=q$ & $cx+ay+bz=r$ where a, b, c are not all zero; solve completely in case of consistency.

Solution:- Given equations are

$$ax+by+cz=p$$
$$bx+cy+az=q$$
$$cx+ay+bz=r$$

It is in the form of $AX=B$

$$A=\begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix} \quad X=\begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad B=\begin{bmatrix} p \\ q \\ r \end{bmatrix}$$

$$|A|=\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 3abc-a^3-b^3-c^3$$

for consistency $|A|\neq 0$

$$|A|\neq 3abc-a^3-b^3-c^3$$
$$|A|\neq a^3+b^3+c^3-3abc\neq 0$$

$$\Rightarrow (a+b+c)(a^2+b^2+c^2-ab-bc-ca)\neq 0$$

$$\boxed{a+b+c\neq 0} \qquad a^2+b^2+c^2-ab-bc-ac\neq 0$$
$$a^2+b^2+c^2\neq ab+bc+ca.$$

$$\boxed{\therefore\ a\neq b\neq c}$$

These are the required conditions for consistency.

Hence, $\Delta = |A| = 3abc - a^3 - b^3 - c^3$

$$\Delta_x = \begin{vmatrix} p & b & c \\ q & c & a \\ r & a & b \end{vmatrix} = p(bc - a^2) - b(qb - ar) + c(aq - cr).$$

$$\boxed{\Delta_x = (ac - b^2)q + (ab - c^2)r + (bc - a^2)p}$$

$$\Delta_y = \begin{vmatrix} a & p & c \\ b & q & a \\ c & r & b \end{vmatrix} = a(qb - ar) - p(b^2 - ac) + c(br - qc)$$

$$\boxed{\Delta_y = -\left[(c^2 - ab)q + (a^2 - bc)r + (b^2 - ac)p\right]}$$

$$\Delta_z = \begin{vmatrix} a & b & p \\ b & c & q \\ c & a & r \end{vmatrix} = a(cr - aq) - b(br - cq) + p(ba - c^2)$$

$$\boxed{\Delta_z = (bc - a^2)q + (ac - b^2)r + (ab - c^2)p}$$

$$x = \frac{\Delta_x}{\Delta} = \frac{(b^2 - ac)q + (c^2 - ab)r + (a^2 - bc)p}{a^3 + b^3 + c^3 - 3abc}$$

$$y = \frac{\Delta_y}{\Delta} = \frac{(c^2 - ab)q + (a^2 - bc)r + (b^2 - ac)p}{a^3 + b^3 + c^3 - 3abc}$$

$$z = \frac{\Delta_z}{\Delta} = \frac{(a^2 - bc)q + (b^2 - ac)r + (c^2 - ab)p}{a^3 + b^3 + c^3 - 3abc}.$$

with consistency conditions ; $a + b + c \neq 0$

$a \neq b \neq c$

Hence the result.

Q.8(i)

Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the linear mapping defined by $T(x,y,z) = (x+2y-z, y+z, x+y-2z)$. Find a basis and the dimension of the image 'U' of 'T'.

Solution:- given; $T(x,y,z) = (x+2y-z, y+z, x+y-2z)$

The image of vectors which span the domain $\mathbb{R}^3$

$T(1,0,0) = (1,0,1)$

$T(0,1,0) = (2,1,1)$

$T(0,0,1) = (-1,1,-2)$

The images span the image U of T; hence form the matrix whose rows are the images vectors and rows reduce to echelon form:

$$U = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & -2 \end{bmatrix} \sim \underset{\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 + R_1}}{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix}} \sim \underset{R_3 \to R_3 - R_2}{\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}}$$

Thus;

Basis of $U = \{(1,0,1), (0,1,-1)\}$

and Dimension $= 2$.

Q.8(ii)

If $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, find out the values of α, β such that $(\alpha I + \beta A)^2 = A$.

Solution:- Given; $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I.$$

Given; $(\alpha I + \beta A)^2 = A$

$$\Rightarrow \alpha^2 I^2 + \beta^2 A^2 + 2\alpha\beta AI = A$$

L.H.S $\Rightarrow \alpha^2 I + \beta^2(-I) + 2\alpha\beta A = (\alpha^2-\beta^2) I + 2\alpha\beta A$

$$= (\alpha^2-\beta^2)\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + 2\alpha\beta\begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$$

$$= \begin{bmatrix}\alpha^2-\beta^2 & 2\alpha\beta\\ -2\alpha\beta & \alpha^2-\beta^2\end{bmatrix} = A = \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}$$

$\Rightarrow \alpha^2-\beta^2 = 0$ and $2\alpha\beta = 1$

$\alpha^2 = \beta^2 \Rightarrow \alpha = \pm\beta$; $\alpha\beta = \frac{1}{2}$.

$\Rightarrow \pm\beta^2 = \frac{1}{2}$ or $\beta^2 = \frac{1}{2}, -\frac{1}{2}$

$$\Rightarrow \boxed{\beta = \pm\frac{1}{\sqrt{2}}, \pm\frac{i}{\sqrt{2}}}$$

$$\therefore \boxed{\alpha = \pm\frac{1}{\sqrt{2}}, \pm\frac{i}{\sqrt{2}}}$$

which is required result

Q.9(i) Show that the vectors $v = (1+i, 2i)$ and $w = (1, 1+i)$ in C^2 are linearly dependent over the complex field 'C' but are linearly independent over the real field $\mathbb{R}$.

Solution

We know that, the two vectors are dependent iff one is a multiple of the other. Since, the first co-ordinate of w is 1,

v can be multiple of w iff $v = (1+i)w$

But $1+i \notin \mathbb{R}$

Hence; v and w are independent over $\mathbb{R}$

Since; $(1+i)w = (1+i)(1, 1+i)$

$$(1+i)w = (1+i, 2i) = v$$

and $(1+i) \in C$.

They are dependent over 'C'.

Q.9(ii) Let W be the subspace of $\mathbb{R}^3$ defined by $W = \{(a, b, c) : a+b+c = 0\}$. Find a basis and dimension of W.

Solution :-

Let W be the subspace of $\mathbb{R}^3$ defined by $W = \{(a, b, c) : a+b+c = 0\}$.

Note ; $W \neq \mathbb{R}^3$

since, for example, $(1, 2, 3) \notin W$

Thus ; $\dim W < 3$

$u_1 = (1, 0, -1)$ and $u_2 = (0, 1, -1)$

are two independent vectors in W.

Thus; $\boxed{\dim W = 2}$

and u_1 & u_2 are basis of W.

i.e $\boxed{W = \{(1, 0, -1), (0, 1, -1)\}}$

Q.9(iii) Suppose U and W are distinct four dimensional subspaces of a vector space V of dimension 6. Find the possible dimensions of $U \cap W$?

Solution:– Since, U and W are distinct four dimensional subspaces of a vector space

$\Rightarrow$ U+W; properly contains U and W

Hence; $\dim(U+W) > 4$

But $\dim(U+W)$ cannot be greater than the vector space V of dimension 6.

Since, $\dim(V) = 6$

Hence, we have two possibilities

(i) $\dim(U+W) = 5$

(ii) $\dim(U+W) = 6$

If Case I) $\dim(U+W) = 5$

$\therefore \dim(U \cap W) = \dim U + \dim W - \dim(U+W)$
$= 4 + 4 - 5 = 3$

Case II) $\dim(U+W) = 6$

$\therefore \dim(U \cap W) = \dim U + \dim W - \dim(U+W)$
$= 4 + 4 - 6 = 2$

$\therefore$ When $\dim(U+W) = 5 \Rightarrow \dim(U \cap W) = 3$
$\dim(U+W) = 6 \Rightarrow \dim(U \cap W) = 2$

which is required solution.

Q.10 (i). Let $A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$. Is A diagonalizable? If yes, find P such that $P^{-1}AP$ is diagonal.

(b) If interchanging the eigen vectors of P, does P still diagonalize A?

Soln: we have $|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)(3-\lambda) - 2 = 0$$

$$\Rightarrow \lambda^2 - 5\lambda + 4 = 0$$

$$\Rightarrow \lambda = 1, 4.$$

which are the eigenvalues of A.

Let us find the eigenvector corresponding to $\lambda = 1$

i.e. $(A - \lambda I)X = 0$

$$\Rightarrow \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix} X = 0$$

$$\Rightarrow x + 2y = 0$$

$\therefore X_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ is a non-zero solution of the system and so is an eigenvector of A corresponding to $\lambda = 1$.

Let us find the eigen vector corresponding to $\lambda = 4$

ie $(A - I\lambda)X = 0$

$$\Rightarrow \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} X = 0$$

$\Rightarrow x - y = 0$

$\therefore X_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a non-zero solution and so is an eigenvector of A corresponding to $\lambda = 4$.

Since A has two independent eigenvectors.

A is diagonalizable.

Let $P = \begin{pmatrix} 2 & 1 \\ -1 & 1 \end{pmatrix}$

Then $P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$.

(b) If interchanging the eigenvectors of P.

ie, $P = \begin{pmatrix} 1 & 2 \\ 1 & -1 \end{pmatrix}$, then P still diagonalize A.

However, now $P^{-1}AP = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$.

In other words, the order of the eigen values in $P^{-1}AP$ corresponds to the eigenvectors in P.

Rough work

$$P^{-1}AP = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & -1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 4 \\ -1 & 4 \end{bmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$$

$$P^{-1}AP = \frac{1}{-3}\begin{bmatrix} -1 & -2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ 4 & -1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$$

(iii) Show that no skew-symmetric matrix can be of rank 1.

Sol^n: Let

$$A = \begin{bmatrix} 0 & h & g & l \\ -h & 0 & f & m \\ -g & -f & 0 & n \\ -l & -m & -n & 0 \end{bmatrix}$$

be an 4x4 skew-symmetric matrix

If h,g,l,m,n are all equal to zero, the matrix A will be of rank zero. If at least one of these elements, say, g is not equal to zero, then at least one 2-rowed minor of the matrix A, i.e. the minor $\begin{vmatrix} 0 & g \\ -g & 0 \end{vmatrix}$ is not equal to zero as its value is g^2 which is not equal to zero.

$\therefore$ the rank of the matrix A is ≥ 2.

Thus in either case the rank of the matrix A is not equal to one.

$\boxed{\text{Note}}$: The method of proof can be given in the case of a skew symmetric matrix of any order.

Q.11 If S and T are subspaces of $\mathbb{R}^4$ given by

$S = \{(x, y, z, w) \in \mathbb{R}^4 : 2x + y + 3z + w = 0\}$ and

$T = \{(x, y, z, w) \in \mathbb{R}^4 : x + 2y + z + 3w = 0\}$, find $\dim(S \cap T)$.

Sol^n: $S \cap T$ consists of those vectors which satisfy the conditions defining S and the conditions defining T.

i.e, the two equations $2x + y + 3z + w = 0$ —(1)

$x + 2y + z + 3w = 0$ —(2)

i.e, $S \cap T = \{(x, y, z, w) \in \mathbb{R}^4 / 2x + y + 3z + w = 0$ and $x + 2y + z + 3w = 0\}$.

Let us solve (1) & (2) for x, y, z, w.

We write single matrix equation $AX = O$

where $A = \begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 2 & 1 & 3 \end{bmatrix}$ $X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ $O = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Convert A into echelon form by using elementary row transformations.

$$A = \begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 2 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 & 3 & 1 \\ 0 & 3/2 & -1/2 & 5/2 \end{bmatrix} \quad R_2 \to R_2 - \tfrac{1}{2}R_1$$

Rewrite the single matrix equation, we get

$$\begin{bmatrix} 2 & 1 & 3 & 1 \\ 0 & 3/2 & -1/2 & 5/2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 2x + y + 3z + w = 0$ —(3) and $\frac{3}{2}y - \frac{1}{2}z + \frac{5w}{2} = 0 \Rightarrow y = \frac{z - 5w}{3}$ —(4)

$\therefore$ from (3) & (4), we get $x = \frac{-5z + w}{3}$ —(5)

$$\therefore S \cap T = \left\{\left(-\frac{5}{3}z + \frac{w}{3}, \frac{1}{3}z - \frac{5w}{3}, z, w\right) / z, w \in \mathbb{R}\right\}$$

$$= \left\{ z\left(-\frac{5}{3}, -\frac{1}{3}, 1, 0\right) + w\left(\frac{1}{3}, -\frac{5}{3}, 0, 1\right) / z, w \in \mathbb{R}\right\}$$

$\therefore \left\{\left(-\frac{5}{3}, \frac{1}{3}, 1, 0\right), \left(\frac{1}{3}, -\frac{5}{3}, 0, 1\right)\right\}$ is a basis of $S \cap T$

and $\dim(S \cap T) = 2$.

Q.13 (i) Let V be the vectorspace of all 2x2 matrices over the field of real numbers. Let W be the set consisting of all matrices with zero determinant. Is W a subspace of V? Justify your answer.

(ii) Find the dimension and a basis for the space W of all solutions of the following homogeneous system using matrix solution:

$$x_1 + 2x_2 + 3x_3 - 2x_4 + 4x_5 = 0$$
$$2x_1 + 4x_2 + 8x_3 + x_4 + 9x_5 = 0$$
$$3x_1 + 6x_2 + 13x_3 + 4x_4 + 14x_5 = 0.$$

Sol'n: Let $V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times 2} \middle| a, b, c, d \in \mathbb{R} \right\}$

$$\& \; W = \left\{ A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \middle| \det A = 0 \right\} \subseteq V$$

let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ & $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\det A = 0$ & $\det B = 0$

Clearly $A, B \in W$

$$A+B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\det(A+B) = 1 \neq 0$

$\Rightarrow A+B \notin W$.

Hence, W is not the subspace of V.

(ii) Given homogeneous system is:

$$x_1 + 2x_2 + 3x_3 - 2x_4 + 4x_5 = 0 \quad \text{——} (1)$$
$$2x_1 + 4x_2 + 8x_3 + x_4 + 9x_5 = 0 \quad \text{——} (2)$$
$$3x_1 + 6x_2 + 13x_3 + 4x_4 + 14x_5 = 0 \quad \text{——} (3)$$

Coefficient matrix of given system is

$$A = \begin{bmatrix} 1 & 2 & 3 & -2 & 4 \\ 2 & 4 & 8 & 1 & 9 \\ 3 & 6 & 13 & 4 & 14 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & -2 & 4 \\ 0 & 0 & 2 & 5 & 1 \\ 0 & 0 & 4 & 10 & 2 \end{bmatrix} \quad R_3 \to R_3 - 2R_2$$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & -2 & 4 \\ 0 & 0 & 2 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

So, corresponding homogeneous system is

$$x_1 + 2x_2 + 3x_3 - 2x_4 + 4x_5 = 0 \quad \text{---} (4)$$

$$2x_3 + 5x_4 + x_5 = 0 \quad \text{---} (5)$$

Here 3 variables x_2, x_4, x_5 and values of x_1 and x_3 depend upon x_2, x_4, x_5

from (5) $x_3 = -\dfrac{(5x_4 + x_5)}{2}$ --- (6)

from (4) $x_1 = -2x_2 - 3x_3 + 2x_4 - 4x_5$

$$x_1 = (-2x_2) + \frac{19x_4 - 5x_5}{2}$$

$$\Rightarrow x_1 = \frac{-4x_2 + 19x_4 - 5x_5}{2} \quad \text{---} (7)$$

Now, put $x_2 = 1$, $x_4 = 0$, $x_5 = 0$

$$\Rightarrow x_3 = 0, \; x_1 = -2$$

Then $(x_1, x_2, x_3, x_4, x_5) = (-2, 1, 0, 0, 0)$ —— (8)

put $x_2 = 0$, $x_4 = 1$, $x_5 = 0$

$\Rightarrow x_3 = -5/2$, $x_1 = 19/2$

then $(x_1, x_2, x_3, x_4, x_5) = \left(19/2, 0, -5/2, 1, 0\right)$ —— (9)

put $x_2 = 0$, $x_4 = 0$, $x_5 = 1$

$\Rightarrow x_3 = -1/2$, $x_1 = -5/2$

then $(x_1, x_2, x_3, x_4, x_5) = \left(-5/2, 0, -1/2, 0, 1\right)$ —— (10)

Hence from (8), (9) & (10)

$B = \left\{(-2, 1, 0, 0, 0), \left(19/2, 0, -5/2, 1, 0\right), \left(-5/2, 0, -1/2, 0, 1\right)\right\}$

is the basis for the space W.

& $\boxed{\text{Dim } W = 3}$

Q.14 (i) Using elementary row operations, find the inverse of $A=\begin{bmatrix}1&2&1\\1&3&2\\1&0&1\end{bmatrix}$

(ii) If $A=\begin{bmatrix}1&1&3\\5&2&6\\-2&-1&-3\end{bmatrix}$, then find $A^{14}+3A-2I$.

Sol'n: we know that

$$A=IA \Rightarrow \begin{bmatrix}1&2&1\\1&3&2\\1&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\cdot A$$

Using Row transformation, Converting L.H.S to identity matrix;

$$R_2\to R_2-R_1, \quad R_3\to R_3-R_1$$

$$\sim\begin{bmatrix}1&2&1\\0&1&1\\0&-2&0\end{bmatrix}=\begin{bmatrix}1&0&0\\-1&1&0\\-1&0&1\end{bmatrix}\cdot A$$

$$R_2\longleftrightarrow R_3$$

$$\sim\begin{bmatrix}1&2&1\\0&-2&0\\0&1&1\end{bmatrix}=\begin{bmatrix}1&0&0\\-1&0&1\\-1&1&0\end{bmatrix}\cdot A$$

$$R_3\to R_3+\tfrac{1}{2}R_2$$

$$\sim\begin{bmatrix}1&2&1\\0&-2&0\\0&0&1\end{bmatrix}=\begin{bmatrix}1&0&0\\-1&0&1\\-3/2&1&\frac{1}{2}\end{bmatrix}A$$

$$R_2\to\frac{R_3}{-2} \text{ then } R_1\to R_1-2R_2-R_3$$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3/2 & -1 & 1/2 \\ 1/2 & 0 & -1/2 \\ -3/2 & 1 & 1/2 \end{bmatrix} A$$

$$= A^{-1}A$$

Hence

$$A^{-1} = \begin{bmatrix} 1.5 & -1 & 0.5 \\ 0.5 & 0 & -0.5 \\ -1.5 & 1 & 0.5 \end{bmatrix}$$

(ii) Sol^n: Characteristic equation of A is $|A-\lambda I| = 0$

$$\begin{bmatrix} 1-\lambda & 1 & 3 \\ 5 & 2-\lambda & 6 \\ -2 & -1 & -3-\lambda \end{bmatrix} = 0$$

$$\Rightarrow (1-\lambda)\left[(2-\lambda)(3-\lambda)+6\right] - 1\left[5(-3-\lambda)+12\right] + 3\left[-5+2(2-\lambda)\right] = 0$$

$$\Rightarrow -\lambda^3 + 0\lambda^2 + 0\lambda + 0 = 0$$

$$\Rightarrow \lambda^3 = 0$$ By Cayley. Hamilton. theorem $A^3 = 0$.

$$A^{14} + 3A - 2I$$

$$= (A^3)^4 . A^2 + 3A - 2I = 0 + 3A - 2I$$

$$= \begin{bmatrix} 1 & 3 & 9 \\ -15 & 4 & 18 \\ -6 & -3 & -11 \end{bmatrix}$$

$\therefore$ Required Result.

Q.16(a) Verify the Cayley-Hamilton theorem for the matrix $A=\begin{bmatrix}1&0&-1\\2&1&0\\3&-5&1\end{bmatrix}$ Using this, show that A is non-singular and find A^{-1}.

Sol^n: Let λ be an eigen value of matrix A, then characteristic matrix.

$$|A-\lambda I| = \begin{bmatrix}1&0&-1\\2&1&0\\3&-5&1\end{bmatrix} - \lambda\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix} = \begin{bmatrix}1-\lambda&0&-1\\2&1-\lambda&0\\3&-5&1-\lambda\end{bmatrix}$$

Characteristic polynomial

$$|A-\lambda I|=0 \Rightarrow \begin{vmatrix}1-\lambda&0&-1\\2&1-\lambda&0\\3&-5&1-\lambda\end{vmatrix}=0$$

$$\Rightarrow (1-\lambda)^3-\{-10-3(1-\lambda)\}=0$$

$$\Rightarrow (1-\lambda)^3+13-3\lambda=0$$

$$\Rightarrow \lambda^3-3\lambda^2+6\lambda-14=0 \quad \text{——— (1)}$$

$$A^2=\begin{bmatrix}1&0&-1\\2&1&0\\3&-5&1\end{bmatrix}\begin{bmatrix}1&0&-1\\2&1&0\\3&-5&1\end{bmatrix}=\begin{bmatrix}-2&5&-2\\4&1&-2\\-4&-10&-2\end{bmatrix} \quad \text{——— (2)}$$

$$A^3=\begin{bmatrix}-2&5&-2\\4&1&-2\\-4&-10&-2\end{bmatrix}\begin{bmatrix}1&0&-1\\2&1&0\\3&-5&1\end{bmatrix}=\begin{bmatrix}2&15&0\\0&11&-6\\-30&0&2\end{bmatrix} \quad \text{——— (3)}$$

Putting values of A, A^2, A^3 in equation (1)

$$\lambda^3-3\lambda^2+6\lambda-14=0$$

$$A^3-3A^2+6A-14I=0$$

$$A^3-3A^2-6A-14I=\begin{bmatrix}2&15&0\\0&11&-6\\-30&0&2\end{bmatrix}+\begin{bmatrix}6&-15&6\\-12&-3&6\\12&30&6\end{bmatrix}+\begin{bmatrix}6&0&-6\\12&6&0\\18&-30&6\end{bmatrix}+\begin{bmatrix}-14&0&0\\0&-14&0\\0&0&-14\end{bmatrix}$$

$\therefore A^3 - 3A^2 + 6A - 14I = 0$ —— (4)

$\Rightarrow$ A satisfies equation (1).

$\therefore$ Cayley - Hamilton theorem verified.

If $(-1)^n [\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \cdots + a_n]$ is the Characteristic equation of matrix A.

and $|A| = (-1)^n a_n$.

$\therefore$ A is non-singular if $a_n \neq 0$.

from (1), $a_n = -14 \neq 0$.

i.e, $|A| = 14 \neq 0$.

$\therefore$ A is non-singular.

Now multiplying equation (4) by A^{-1}

$\Rightarrow A^{-1}A^3 - 3A^{-1}A^2 + 6A^{-1}A - 14A^{-1}I = 0$

$\Rightarrow A^2 - 3A + 6I = 14A^{-1}$

$$\Rightarrow 14A^{-1} = \begin{bmatrix} -2 & 5 & -2 \\ 4 & 1 & -2 \\ -4 & -10 & -2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix} + 6\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 5 & 1 \\ -2 & 4 & -2 \\ -13 & 5 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{14}\begin{bmatrix} 1 & 5 & 1 \\ -2 & 4 & -2 \\ -13 & 5 & 1 \end{bmatrix}$$

Q.16(b) show that the subspace of $\mathbb{R}^3$ spanned by two sets of vectors $\{(1,1,-1),(1,0,1)\}$ and $\{(1,2,-3),(5,2,1)\}$ are identical. Also find the dimension of this subspace.

Sol'n: Let $B_1 = \{(1,1,-1),(1,0,1)\}$

$B_2 = \{(1,2,-3),(5,2,1)\}$

Let W be subspace such that $W = L\{B_1\}$

V be subspace such that $V = L\{B_2\}$

The subspaces V and W are identical iff their row reduced echelon matrices have the same non-zero rows.

Consider a matrix A, where rows are vectors of B_1

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix} R_2 \to R_2 - R_1 \quad \text{——— (1)}$$

Now, consider a matrix B, whose rows are vectors of B_2

$$\Rightarrow B = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 2 & 1 \end{bmatrix}$$

$$\sim B = \begin{bmatrix} 1 & 2 & -3 \\ 0 & 8 & 16 \end{bmatrix} R_2 \to R_2 - 5R_1$$

$$\sim B = \begin{bmatrix} 1 & 2 & -3 \\ 0 & -1 & 2 \end{bmatrix} R_2 \to R_2(\tfrac{1}{8})$$

$$B \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 2 \end{bmatrix} R_1 \to R_1 + R_2 \quad \text{(2)}$$

from (1) & (2), clearly rowspace of A and row space of B are same. Thus V and W are identical subspaces of $\mathbb{R}^3$.

Dimension of row space A (i.e, dimension of subspace)

= Max. no of L.I. rows of A.

= Max no of L.I. rows of echelon matrix of A

= no. of non-zero rows of echelon matrix of A

= 2.

$\therefore$ Basis for the row space is $\{(1,1,-1),(0,-1,2)\}$ and the dimension of row space is 2.

Q.17

Let $H = \begin{bmatrix} 1 & i & 2+i \\ -i & 2 & 1-i \\ 2-i & 1+i & 2 \end{bmatrix}$ be a Hermitian matrix. Find a non-singular matrix P such that $D = P^T H \bar{P}$ is diagonal.

Sol^n: let us form the block matrix

$$[H|I] = \left[\begin{array}{ccc|ccc} 1 & i & 2+i & 1 & 0 & 0 \\ -i & 2 & 1-i & 0 & 1 & 0 \\ 2-i & 1+i & 2 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & i & 2+i & 1 & 0 & 0 \\ 0 & 1 & i & i & 1 & 0 \\ 0 & -i & -3 & -(2-i) & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 + iR_1 \\ R_3 \to R_3 - (2-i)R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -i & -(2+i) \\ 0 & 1 & -i & i & 2 & -2i+1 \\ 0 & -i & -3 & -(2-i) & 2i+1 & -4 \end{array}\right] \quad \begin{array}{l} C_2 \to C_2 - iC_1 \\ C_3 \to C_3 - (2+i)C_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -i & -(2+i) \\ 0 & 1 & i & i & 2 & -(2i-1) \\ 0 & 0 & -4 & -3+i & 4i+1 & -2+i \end{array}\right] \quad R_3 \to R_3 + iR_2$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -i & -3-i \\ 0 & 1 & 0 & i & 2 & -4i+1 \\ 0 & 0 & -4 & -3+i & 4i+1 & 2 \end{array}\right] \quad C_3 \to C_3 - iR_2$$

Now H has been diagonalized.

$\therefore$ set $P = \begin{bmatrix} 1 & -i & -3-i \\ i & 2 & -4i+1 \\ -3+i & 4i+1 & 2 \end{bmatrix}$

and then $P^T H \bar{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -4 \end{bmatrix} = D$