MAINS TEST SERIES – 2020

TEST – VI , PAPER – II

ANSWER KEY (Full Syllabus)

1(a) Prove that a non-abelian group of order 10 must have a subgroup of order 5.

Sol'n: Let $(G, \times)$ be a group such that $o(G) = 10$
$= 5 \times 2$
(and $5 > 2$)

Let us Prove that G has a one Subgroup of order 5.

If possible G has two subgroups H & K of order 5.
i.e. $o(H) = o(K) = 5$ and $H \neq K$.

Since $H \cap K \leq H$

$\therefore$ By Lagrange's theorem $\dfrac{o(H\cap K)}{o(H)}$

i.e. $\dfrac{o(H)}{o(H\cap K)} = \dfrac{5}{o(H\cap K)}$

$\Rightarrow o(H\cap K) = 1$ or 5

If $o(H\cap K) = 5$ then $H\cap K = H$
$\Rightarrow K = H$
which is a contradiction to $H \neq K$.

If $o(H\cap K) = 1$ then $o(HK) = \dfrac{o(H)o(K)}{o(H\cap K)}$

$\Rightarrow o(HK) = \dfrac{5\times 5}{1} = 25 > o(G)$

$\therefore$ which is impossible.

$\therefore$ our assumption that the two Subgroups H & K of order 5 is wrong.

$\therefore$ If G is a group of order 10,
then it cannot have two subgroups of order '5'.
(i.e. it has only subgroup of order '5')

(OR)

An alternate solution.

1(a) →

Let G be a group of order 10.
By Lagrange's theorem, if there exist a subgroup H of G, the
$O(H) = 1, 2, 5$ or 10.

Assume that there is no subgroup of order 5.

Let $e \neq a \in G$. Then we have $o(a) / o(G)$.
If every element of G is of order 2, then $a^2 = e$ i.e, $a = a^{-1}$ for all $a \in G$.
Then we can show that G is abelian.
This is a contradiction.
Therefore, every element of G is not of order 2.
Thus $o(a) = 5$ or 10.
If $o(a) = 5$, then $H = \langle a \rangle$ is a cyclic subgroup of order 5.

1(b) Let R be a ring with characteristic n. Suppose $ma=0$ for all $a \in R$ and for some positive integer m. Show that n divides m. Determine characteristic of Z_n.

Sol^n: Since char $R=n$, n is the least +ve integer such that

$na=0 \ \forall \ a \in R$.

It is given that $ma=0 \ \forall \ a \in R$ and for some +ve integer m.

By division algorithm,

there exist integers q and r such that $m=nq+r$

where $r=0$ (or) $0<r<n$.

Consider the case $0<r<n$

we have $0=ma=(nq+r)a = q(na)+ra = 0+ra = ra$

$\therefore ra=0, \ \forall \ a \in R$; where r is a +ve integer $<n$.

This is a contradiction to the fact that char $R=n$.

Consequently, $r=0$ and so $m=nq \Rightarrow n$ divides m.

(ii) we know $Z_n = \{0,1,2 \ldots, n-1\}$

clearly,

n is the least +ve integer such that $na=0$

in Z_n, $\forall \ a \in Z_n$.

Hence Char $Z_n = n$.

1(c). Let f be defined on $[-2,2]$ by $f(x) = 3x^2\cos\frac{\pi}{x^2} + 2\pi\sin\frac{\pi}{x^2}, x \neq 0$

$= 0 \quad , \quad x = 0$

Show that f is integrable on $[-2,2]$. Evaluate $\int_{-2}^{2} f$.

Sol^n: f is bounded on $[-2,2]$. f is continuous on $[-2,2]$ except at 0, Since f is continuous on $[-2,2]$ except at only one point, f is integrable on $[-2,2]$.

Let $\phi : [-2,2] \to \mathbb{R}$ be defined by

$$\phi(x) = x^3\cos\frac{\pi}{x^2} \quad , x \neq 0$$
$$= 0 \quad , x = 0.$$

Then $\phi'(x) = 3x^2\cos\frac{\pi}{x^2} + 2\pi\sin\frac{\pi}{x^2}$, for all $x(\neq 0) \in [-2,2]$.

$$\lim_{x\to 0}\frac{\phi(x)-\phi(0)}{x-0} = \lim_{x\to 0} x^2\cos\frac{\pi}{x^2}$$
$$= 0$$

$$\therefore \phi'(0) = 0.$$

Hence ϕ is an antiderivative of f on $[-2,2]$.

By the fundamental theorem,

$$\int_{-2}^{2} f(x)\,dx = \phi(2) - \phi(-2)$$
$$= 4\cos\frac{\pi}{4} + 8\cos\frac{\pi}{4}$$
$$= 8\sqrt{2}.$$

1(d). Prove that the function $f(z) = u+iv$, where

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} \quad (z \neq 0),\ f(0)=0.$$

is continuous and that Cauchy – Riemann equations are satisfied at the origin, yet $f'(z)$ does not exist there.

Sol'n: Here $u = \dfrac{x^3-y^3}{x^2+y^2}$, $v = \dfrac{x^3+y^3}{x^2+y^2}$ (where $z \neq 0$).

Here we see the both u and v are rational and finite for all values of $z \neq 0$, so u and v are continuous at all those points for which $z \neq 0$. Hence $f(z)$ continuous where $z \neq 0$.

At the origin $u=0, v=0$. [Since $f(0)=0$].
Hence u and v are both continuous at the origin; Consequently $f(z)$ is continuous at the origin.

Hence $f(z)$ is continuous everywhere.

At the origin $\dfrac{\partial u}{\partial x} = \lim\limits_{x\to 0} \dfrac{u(x,0)-u(0,0)}{x} = \lim\limits_{x\to 0}\left(\dfrac{x}{x}\right) = 1$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y} = \lim_{y\to 0}\left(\frac{-y}{y}\right) = -1$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0} \frac{v(x,0)-v(0,0)}{x} = \lim_{x\to 0}\left(\frac{x}{x}\right) = 1$$

$$\frac{\partial v}{\partial y} = \lim_{\to 0} \frac{v(0,y)-v(0,0)}{y} = \lim_{y\to 0}\left(\frac{y}{y}\right) = 1$$

Thus we see that $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$.

Hence Cauchy-Riemann conditions are satisfied at $z=0$.

Again $f'(0) = \lim\limits_{z\to 0} \dfrac{f(z)-f(0)}{z} = \lim\limits_{z\to 0}\left[\dfrac{x^3-y^3+i(x^3+y^3)}{x^2+y^2}\cdot\dfrac{1}{x+iy}\right]$

Now let $z\to 0$ along $y=x$, then

$$f'(0) = \lim_{x\to 0} \frac{x^3-y^3+i(x^3+y^3)}{x^2+y^2}\cdot\frac{1}{x+ix} = \lim_{x\to 0}\frac{2i}{2(1+i)} = \frac{1}{2}(1-i)$$

— along $y=0$, then

$$f'(0) = \lim_{x\to 0}\frac{x^3(1+i)}{x^3} = 1+i$$

So we that $f'(0)$ is not unique, i.e. the values of $f'(0)$ are not the same as $z\to 0$ along different curves.

1(e) Show that the following system of linear equations has two degenerate feasible basic solutions and the non-degenerate basic solution is not feasible:

$2x_1 + x_2 - x_3 = 2,\ 3x_1 + 2x_2 + x_3 = 3.$

Sol^n: The given system of equations can be written in the matrix form as

$$Ax = b$$

where $A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & 2 & 1 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$; $b = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

Since the rank of A is 2, the maximum no.of linearly independent columns of A is 2.

Thus we consider any of the 2×2 sub-matrices as basis matrix B.

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}, \begin{bmatrix} 2 & -1 \\ 2 & 1 \end{bmatrix} \ \& \begin{bmatrix} 2 & -1 \\ 3 & 1 \end{bmatrix}$$

The variables not associated with the columns of these submatrices are respectively x_3, x_1 & x_2.

$$\text{Let } B = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$$

A basic solution to the given system is obtained by setting $x_3 = 0$ and solving the system

$$\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow 2x_1 + x_2 = 2$$

$$3x_1 + 2x_3 = 3$$

$$\Rightarrow x_1 = 0,\ x_2 = 0$$

$\therefore$ The basic solution to the given Problem is

$$x_1 = 1,\ x_2 = 0 \text{ (Basic)} \text{ — } ①$$

$$x_3 = 0 \text{ (non-Basic)}$$

Similarly other two solutions are

$$x_2 = 5/3,\ x_3 = -1/3 \text{ (Basic) — } ③$$

$$x_2 = 0 \text{ (non-Basic)}$$

In each of the two basic solutions, atleast one of the basic variables is zero.

Hence two of the basic solutions (i.e. ① & ③) are degenerate solutions.

Note: The Non-degenerate solution $(0, 5/3, -1/3)$ is not feasible. Only basic feasible degenerate solutions are (i) $x_1=0$ & $x_2=0$
(ii) $x_1=1$ and $x_3=0$.

2(b) (i) Prove that $f(x) = \sin\frac{1}{x}, x \neq 0$
$= 0, x = 0$ is not uniformly continuous on $[0, \infty[$.

Sol'n: Let $\epsilon = \frac{1}{2}$ and $\delta > 0$ be such that

$$\frac{1}{n(2n\pi + \pi)} < \delta, \ \forall n \geq m.$$

Taking $x = \frac{2}{2m\pi}$ and $y = \frac{2}{(2m+1)\pi}$ be any two points of $[0, \infty[$,

then $|x - y| = \left|\frac{2}{2m\pi} - \frac{2}{(2m+1)\pi}\right| = \frac{1}{m(2m+1)\pi} < \delta$

Now $|x_1 - x_2| < \delta$

but $|f(x_1) - f(x_2)| = |\sin n\pi - \sin\frac{1}{2}(2n+1)\pi| = 1 > \epsilon$.

This shows that for this choice of ϵ, we are unable to find $\delta > 0$ such that $|f(x_1) - f(x_2)| < \epsilon$ whenever $|x_1 - x_2| < \delta \ \forall x_1, x_2 \in [0, \infty[$

Hence f is not uniformly continuous on $[0, \infty[$.

(ii) Define an open set, Prove that the union of a arbitrary family of open sets is open. Show also that the intersection of a finite family of open sets is open. Does it hold for an arbitrary family of open sets? Explain the reason for your answer by example.

Sol'n:
Open set: A subset S of $\mathbb{R}$ is said to be an open set if S is a nbd of each of its points. i.e., if for each $p \in S$ $\exists$ an $\epsilon > 0$ such that $(p - \epsilon, p + \epsilon) \subset S$.

(or)

If S is a subset of $\mathbb{R}$ is said to be open if every point of S is an interior point of S.

To show ~~that~~ the union of an arbitrary family of open sets is open.

Let $\{G_\lambda : \lambda \in \Lambda\}$ be an arbitrary family of open sets. Here Λ is an index set and is such that for every $\lambda \in \Lambda$, G_λ is an open set.

Let $G = \cup\{G_\lambda : \lambda \in \Lambda\}$. Then in order to show that G is an open set, we shall show that every point of G is an interior point of G.

Let $p \in G$. Since G is the union of the family $\{G_\lambda\}$, therefore $p \in G_\lambda$ for some $\lambda \in \Lambda$.

Since G_λ is an open set and $p \in G_\lambda$.

$\therefore \exists$ some $\epsilon > 0$ such that $(p-\epsilon, p+\epsilon) \subset G_\lambda \subset G$.

Thus $(p-\epsilon, p+\epsilon) \subset G$ and so p is an interior point of G.

Since every point of G is an interior point of G, therefore G is the open set.

Hence the union of an arbitrary family of open sets is an open set.

→ The intersection of a finite collection of open sets is an open set.

Let $G = \bigcap_{i=1}^{n} G_i$, where each G_i is an open set.

If $G = \phi$, then G is an open set and the proof is complete. So let $G \neq \phi$.

Let $p \in G$. Then $p \in G_i$ for each $i = 1, 2, \ldots n$.

Since each G_i is an open set, therefore for every $i = 1, 2, \ldots n$, $\exists\ \epsilon_i > 0$ such that

$$(p - \epsilon_i, p + \epsilon_i) \subset G_i$$

Let $\epsilon = \min[\epsilon_1, \epsilon_2, \ldots \epsilon_n]$

Then $(p-\epsilon, p+\epsilon) \subset (p-\epsilon_i, p+\epsilon_i) \subset G_i$ for each i.

$$\Rightarrow (p-\epsilon, p+\epsilon) \subset \bigcap_{i=1}^{n} G_i$$

$$\Rightarrow (p-\epsilon, p+\epsilon) \subset G.$$

$\Rightarrow p$ is an interior point of G.

Thus every point p of G is an interior point of G and so G is an open set.

Hence a finite intersection of open sets is an open set

The intersection of arbitrary family of open sets is not necessarily an open set.

For example: Let $G_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$, $n \in N$.

Then each G_n is an open set because every open interval is an open set.

Now $\bigcap_{n=1}^{\infty} G_n = \bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, \frac{1}{n}\right) = \{0\}$

which is not open because $\nexists$ no $\epsilon>0$ such that $(-\epsilon, \epsilon) \subset \{0\}$.

Hence the intersection of an arbitrary family of open sets is not necessarily an open set.

Q2(C). (i) Use Cauchy's integral formula to evaluate

$$\int_C \frac{e^z dz}{(z^2+\pi^2)^2}, \quad \text{where } C \text{ is } |z|=4.$$

(ii) Use the method of contour integration to prove that

$$\int_0^\infty \frac{\cos mx}{a^2+x^2}dx = \frac{\pi}{2a}e^{-ma}, \; (m>0)$$

Sol^n: (i) By n^{th} derivative formula of Cauchy's integral formula, we have

$$f^n(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^{n+1}} \quad \text{if } z=a$$

lies inside C and $f(z)$ is analytic inside C.

Here take $f(z)=e^z$ and so $f'(z)=e^z$.

$$(z^2+\pi^2)^2 = (z+\pi i)^2(z-\pi i)^2$$

Also $z=\pm\pi i = \pm 3.14i$ lies inside $C: |z|=4$.

Let $I = \int_C \frac{e^z dz}{(z^2+\pi^2)^2}$ ——(1)

$$\frac{1}{(z^2+\pi^2)^2} = \frac{1}{(z+\pi i)^2(z-\pi i)^2} = \frac{A}{z+\pi i} + \frac{B}{(z+\pi i)^2} + \frac{C}{z-\pi i} + \frac{D}{(z-\pi i)^2}$$

$$1 = A(z+\pi i)(z-\pi i)^2 + B(z-\pi i)^2 + C(z-\pi i)(z+\pi i)^2 + D(z+\pi i)^2$$

put $z=\pi i \Rightarrow 1=(2\pi i)^2 D \Rightarrow D = -\frac{1}{4\pi^2}$ and put $z=-\pi i \Rightarrow 1 = B(-2\pi i)^2 \Rightarrow B = -\frac{1}{4\pi^2}$

$z^3: A+C=0$
$z^2: -A\pi i + B + C\pi i + D = 0$
$\Rightarrow A+C=0, \; -A+C = -\frac{1}{2\pi^3 i}$
$\Rightarrow A = -\frac{1}{4\pi^3 i}$ and $C = \frac{1}{4\pi^3 i}$

$\therefore$ (1) $\equiv$ $$I = \int_C \left(\frac{Ae^z}{z+\pi i} + \frac{Be^z}{(z+\pi i)^2} + \frac{Ce^z}{z-\pi i} + \frac{De^z}{(z-\pi i)^2}\right)dz$$

$$= 2\pi i\left[Af(-\pi i) + Bf'(-\pi i) + Cf(\pi i) + Df'(\pi i)\right]$$

$$= 2\pi i\left[\frac{1}{4\pi^3 i}\left(-e^{-\pi i}+e^{\pi i}\right) - \frac{1}{4\pi^2}\left(e^{-\pi i}+e^{\pi i}\right)\right]$$

$$= 2\pi i\left[\frac{1}{4\pi^3 i}(2\sin\pi) - \frac{1}{4\pi^2}(2\cos\pi)\right]$$

$$= 2\pi i\left[0 + \frac{1}{2\pi^2}\right] = \frac{i}{\pi}$$

$\therefore I = \frac{i}{\pi}$

2(c)(ii) sol^n: consider the integral $\int_C f(z)dz$, where $f(z)=\frac{e^{imz}}{(a^2+z^2)^2}$,

taken round the closed contour C consisting of the upper half of a large circle $|z|=R$ and the real axis from $-R$ to R.

pole of $f(z)$ are given by $(a^2+z^2)^2=0$

i.e $z=ia$ and $z=-ia$ are the two sample poles, only $z=ia$ lies within the contour.

Residue at $z=ia = \lim_{z\to ia} (z-ia)f(z)$

$$= \lim_{z\to ia} (z-ia)\frac{e^{imx}}{(z-ia)(z+ia)} = \frac{e^{-ma}}{2ia}$$

Hence by Cauchy's residue theorem, we have

$\int_C f(z)dz = 2\pi i \times$ sum of the residue within the contour

i.e. $$\int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)dz = 2\pi i\,\frac{e^{-ma}}{2ia} \quad \text{---} (1)$$

$$\Rightarrow \int_{-R}^{R}\frac{e^{imx}}{a^2+x^2}dx + \int_{C_R}\frac{e^{imx}}{a^2+z^2}dz = \frac{1}{a}\pi e^{-ma}$$

Now, $$\left|\int_{C_R}\frac{e^{imx}}{a^2+z^2}dz\right| \le \int_{C_R}\frac{|e^{imx}|\,|dz|}{|a^2+z^2|}$$

$$\le \int_0^{\pi}\frac{e^{-mR\sin\theta}\cdot R\,d\theta}{(|z|^2-a^2)}$$

$$= \frac{R}{R^2-a^2}\int_0^{\pi} e^{-2mR\sin\theta}\,d\theta$$

$$\le \frac{R}{R^2-a^2}\,2\int_0^{\pi/2} e^{-2mR\theta/\pi}\,d\theta$$

$$= \frac{\pi}{m(R^2-a^2)}\left[1-e^{-mR}\right] \text{ which} \to 0 \text{ as } R\to\infty$$

Hence by making $R\to\infty$ relation ① becomes

$$\int_{-\infty}^{\infty} \frac{e^{imx}}{a^2+x^2}\,dx = \frac{\pi e^{-ma}}{a}$$

Equating real parts, we have

$$\int_{0}^{\infty} \frac{\cos mx}{a^2+x^2}\,dx = \frac{\pi e^{-ma}}{a}$$

$$\Rightarrow \int_{0}^{\infty} \frac{\cos mx}{a^2+x^2}\,dx = \frac{\pi e^{-ma}}{2a}$$

3(a) → (i) How many generators are there of the cylic group G of order 8 ? Explain.

(ii) Let R^c = Ring of all real valued continuous functions. on $[0,1]$, under the operations

$$(f+g)x = f(x)+g(x)$$

$$(fg)x = f(x)g(x)$$

Let $M = \left\{ f \in R^c \,/\, f(\tfrac{1}{2})=0 \right\}$.

Is M a maximal ideal of R ? Justify your answer.

Sol'n : (i) we have $G=\langle a\rangle$, $a^8=e$.

All the generators of G are a^1, a^3, a^5, a^7

i.e. $G=\langle a\rangle = \langle a^3\rangle = \langle a^5\rangle = \langle a^7\rangle$

[$\because$ 1, 3, 5, 7 are positive integers less than 8 and prime to 8]

(ii) Given R^c = Ring of all real valued continuous functions. on $[0,1]$, under the operations

$$(f+g)x = f(x)+g(x)$$

$$(fg)x = f(x)g(x)$$

To show that M is an ideal of R.

Let $f, g \in M$. Then $f(\tfrac{1}{2}) = 0 = g(\tfrac{1}{2})$.

we have $(f-g)(\tfrac{1}{2}) = f(\tfrac{1}{2}) - g(\tfrac{1}{2}) = 0$ and so $f-g \in M$.

Let $f \in M$ and $h \in R$. Then $f(\tfrac{1}{2})=0$

and $(fh)(\tfrac{1}{2}) = f(\tfrac{1}{2})g(\tfrac{1}{2}) = 0 \cdot h(\tfrac{1}{2}) = 0 \Rightarrow fh \in M$.

Similarly, $hf \in M$ and so M is an ideal of R. Finally, we show that M is a maximal ideal of R. Let U be any ideal of R such that

$M \subset U \subset R$ and $M \neq U$.

We need to show that $U = R$.

Since $M \subset U$ and $M \neq U$, there exists a function $g \in U$ such that $g \notin M$, i.e. $g(\frac{1}{2}) \neq 0$.

i.e., $\alpha \neq 0$, where $\alpha = g(\frac{1}{2})$ —— ①

[Notice that $g(\frac{1}{2}) = 0 \Rightarrow g \in M$, a contradiction]

We define a function $h : [0,1] \to R$ as

$$h(x) = g(x) - \alpha, \ \forall x \in [0,1] \text{ —— ②}$$

$$\Rightarrow h(\tfrac{1}{2}) = g(\tfrac{1}{2}) - \alpha = 0, \text{ using ①}$$

$$\Rightarrow h \in M \Rightarrow h \in U, \text{ since } M \subset U.$$

Since U is an ideal of R,

$\therefore$ $g \in U$ and $h \in U \Rightarrow g - h \in U \Rightarrow \alpha \in U$, by ②

Since $\alpha \neq 0$, $\alpha^{-1} \in R$ exists. The constant function

$\alpha^{-1} : [0,1] \to R$ defined as $\alpha^{-1}(x) = \alpha^{-1} \ \forall x \in [0,1]$

is a continuous function on $[0,1]$ and as such $\alpha^{-1} \in R$.

Since U is an ideal of R, so

$\alpha \in U$ and $\alpha^{-1} \in R \Rightarrow \alpha\alpha^{-1} \in U \Rightarrow 1 \in U$

$\Rightarrow 1.f \in U \ \forall f \in R \Rightarrow f \in U \ \forall f \in R \Rightarrow R = U$.

Hence M is a maximal ideal of R.

3(b) (i) Prove that between any two real roots of the equation $e^x \sin x + 1 = 0$ there is at least one real root of the equation $\tan x + 1 = 0$.

(ii) Prove that the integral $\int_0^\infty x^{m-1} e^{-x}\,dx$ is convergent if and only if $m > 0$.

Sol^n: (i) Let $x = a$ & $x = b$ be the roots of $e^x \sin x + 1 = 0$ then

$$e^a \sin a + 1 = 0 \text{ \& } e^b \sin b + 1 = 0 \quad \text{--- (1)}$$

Let $f(x) = e^x \sin x + 1 \ \forall x \in [a, b]$

Since e^x & $\sin x$ are continuous and differentiable for all reals.

$\therefore f(x) = e^x \sin x + 1$ is continuous and differentiable in $[a, b]$.

and $f(a) = f(b) = 0$ (by (1))

$\therefore f$ has been satisfied the conditions of Rolle's theorem.

$\therefore \exists$ atleast one $x \in (a, b)$ such that $f'(x) = 0$ --- (2)

$\because f'(x) = e^x \sin x + e^x \cos x$

$\Rightarrow f'(x) = e^x (\sin x + \cos x)$

$\therefore (1) \equiv e^x (\sin x + \cos x) = 0 \ \forall x \in (a, b)$

$\Rightarrow \sin x + \cos x = 0 \quad (\because e^x \neq 0 \ \forall x)$

$\Rightarrow \tan x + 1 = 0 ; x \in (a, b)$.

3(b)(ii) Sol'n: If $m \geq 1$, the integrand $x^{m-1}e^{-x}$ is continuous at $x=0$.
If $m<1$, the integrand $\frac{e^{-x}}{x^{1-m}}$ has infinite discontinuity at $x=0$.

Thus we have to examine the convergence at 0 and ∞ both. Consider any positive number, say 1, and examine the convergence of

$$\int_0^1 x^{m-1}e^{-x}\,dx \quad \text{and} \quad \int_1^\infty x^{m-1}e^{-x}\,dx$$

at 0 and ∞ respectively.

Convergence at 0, when $m<1$.

Let $f(x) = \frac{e^{-x}}{x^{1-m}}$

take $g(x) = \frac{1}{x^{1-m}}$

Then $\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} e^{-x} = 1$

which is non-zero, finite.

Also $\int_0^1 g(x)\,dx = \int_0^1 \frac{dx}{x^{1-m}}$ is convergent at $x=0$ if $m>0$.

$\therefore$ By comparison Test

$$\int_0^1 f(x)\,dx = \int_0^1 \frac{e^{-x}}{x^{1-m}}\,dx = \int_0^1 x^{m-1}e^{-x}\,dx$$

is convergent at $x=0$ if $m>0$.

Convergence at ∞:

we know $e^{x} > x^{m+1}$ whatever value m may have

$\therefore e^{-x} < x^{-m-1}$

and $x^{m-1}e^{-x} < x^{m-1}\cdot x^{-m-1} = \frac{1}{x^2}$

Since $\int_1^\infty \frac{1}{x^2}\,dx$ is convergent at ∞.

$\therefore \int_1^\infty x^{m-1} e^{-x}\,dx$ is convergent at ∞ for every value of m.

Now $\int_0^\infty x^{m-1} e^{-x}\,dx = \int_0^1 x^{m-1} e^{-x}\,dx + \int_1^\infty x^{m-1} e^{-x}\,dx$

$\therefore \int_0^\infty x^{m-1} e^{-x}\,dx$ converges iff $m > 0$.

3(c) Consider the following transportation problem:

Factory	Godowns 1	2	3	4	5	6	Stock available
A	7	5	7	7	5	3	60
B	9	11	6	11	–	5	20
C	11	10	6	2	2	8	90
D	9	10	9	6	9	12	50
Demand	60	20	40	20	40	40	

It is not possible to transport any quantity from factory B to Godown 5. Determine the optimum basic feasible solution by finding the initial solution by Vogel's approximation method.

Is the optimum solution unique? If not, find the alternative optimum basic feasible solution.

Sol^n: Since it is not possible to transport any quantity from factory B to Godown 5, we assign a very high cost to the cell (2,5), say M.

From the given table,

Total demand = 220 = Total availability

$\therefore$ Given T.P is balanced.

By using vogel's Approximation method, the IBFS is given below.

7	(20) 5	7	7	(ε) 5	(40) 3	60
(10) 9	11	(10) 6	11	M	5	20
11	10	(30) 6	(20) 2	(40) 2	8	90
(50) 9	10	9	6	9	12	50
60	20	40	20	40	40	

Since the no. of occupied cells is 8 i.e, less than $m+n-1 = 4+6-1 = 9$; there is degeneracy in the initial solution.

To overcome degeneracy, we allocated a small quantity $\epsilon(\to 0)$ in the cell $(1,5)$ being the unoccupied cell having the lowest transportation cost.

To test the initial solution for optimality.

Using U-V method find the values of u_i & v_j.

As the maximum no. of basic cells exists in the first and 3rd row. putting $u_1 = 0$ and the values of u_i's and v_j's and also the net evaluations $\Delta_{ij} = u_i + v_j - c_{ij}$ for all unoccupied cells exhibited as shown below.

						u_i
(5) θ 7	(20) 5	(2) 7	(-ve) 7	(ϵ) $-\theta$ 5	(40) 3	0
(10) $-\theta$ 9	(-*) 11	(10) $+\theta$ 6	(-ve) 11	(-ve) M	(-ve) 5	-3
(-ve) 11	(-ve) 10	(30) $-\theta$ 6	(20) 2	(40) $+\theta$ 2	(-ve) 8	-3
(50) 9	(-ve) 10	(-ve) 9	(-ve) 6	(-ve) 9	(-ve) 12	-3
v_j: 12	5	5	9	5	3	

Since $\Delta_{11}(=5)$ is the most +ve.

$\therefore$ the cell (1,1) enters the basis. So we allocate an unknown quantity θ to this cell and identify a closed loop involving basic cells around this entering cell. Making $\pm\theta$ adjustments in the corner cells of the loop, we observe that the maximum value that can admit is $\epsilon(=\theta)$

Thus the present occupied cell (1,5) becomes unoccupied in the next iteration.

						u_i
(ϵ) 7	(20) 5	(–) 7	(–) 7	(–) 5	(40) 3	0
(10) 9	(–) 11	(10) 6	(–) 11	(–) M	(0) 5	2
(–) 11	(–) 10	(30) 6	(20) 2	(40) 2	(–) 8	2
(50) 9	(–) 10	(–) 9	(–) 6	(–) 9	(–) 12	2
v_j: 7	5	4	0	0	3	

In the above transportation table, since all the net evaluations are non-positive, optimum solution has been obtained. Hence the optimum solution is

$x_{11} = 0$, $x_{12} = 20$, $x_{16} = 40$, $x_{21} = 10$, $x_{33} = 30$

$x_{34} = 20$, $x_{35} = 40$ and $x_{41} = 50$.

The minimum transportation cost is $20\times 2 + 40\times 2$

$20\times 5 + 40\times 3 + 10\times 9 + 10\times 6 + 30\times 6$

$+ 50\times 9 + 7\times \epsilon = 1120 + 7\epsilon$

$= 1120$ (as $\epsilon \to 0$)

Since $\Delta_{26} = 0$, there exists an alternative optimum solution. Letting the unoccupied cell (2,6) enter the basis it is easily seen that the currently occupied cell in the (1,1) becomes unoccupied in the next iteration. In the Revised transportation table, the following alternate optimum solution is obtained.

$x_{12} = 20$, $x_{16} = 40$, $x_{21} = 10$, $x_{23} = 10$

$x_{26} = 0$, $x_{33} = 30$, $x_{34} = 20$, $x_{35} = 40$

and $x_{41} = 50$ with minimum transportation cost as 1120

4(a) In a group G, if $a^5 = e$ and $a * b * a^{-1} = b^m$ for some positive integer m, and some $a, b \in G$, then Prove that $b^{m^5 - 1} = e$.

Sol'n:

4(b) Show that the sequence $\{f_n\}$ where $f_n(x) = \frac{x}{1+nx^2}$, $0 \le x \le 1$ converges uniformly to a function f but $\lim_{n\to\infty} f_n'(x) = f'(x)$ is true if $x \neq 0$.

Sol'n: The sequence $\langle f_n \rangle$ converges uniformly to zero for all real x.

$$\Rightarrow f(x) = 0 \ \forall x \in R$$
$$\Rightarrow f'(x) = 0 \ \forall x \in R$$

when $x \neq 0$,
$$f_n'(x) = \frac{(1+nx^2)\cdot 1 - x\cdot 2nx}{(1+nx^2)^2} = \frac{1-nx^2}{(1+nx^2)^2}$$

$$\lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty} \frac{1-nx^2}{(1+nx^2)^2} \quad \Big| \text{ form } \frac{\infty}{\infty}$$

$$= \lim_{n\to\infty} \frac{-x^2}{2(1+nx^2)\cdot x^2} = 0 = f'(x)$$

So that if $x \neq 0$, the formula $\lim_{n\to\infty} f_n'(x) = f'(x)$ is true.

At $x=0$
$$f_n'(0) = \lim_{h\to 0} \frac{f_n(0+h) - f_n(0)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{h}{1+nh^2} - 0}{h}$$

$$= \lim_{h\to 0} \frac{1}{1+nh^2} = 1$$

So that $\lim_{n\to\infty} f_n'(0) = 1 \neq f'(0)$.

Hence at $x=0$, the formula $\lim_{n\to\infty} f_n'(x) = f'(x)$ is false.

It is so because the sequence $\langle f_n' \rangle$ is not uniformly convergent in any interval containing zero.

4(c) Determine a function which shall be regular within the circle $|z|=1$ and shall have on the circumference of this circle the value $\dfrac{(a^2-1)\cos\theta + i(a^2+1)\sin\theta}{a^4-2a^2\cos 2\theta+1}$

where $a^2>1$, and θ is the vectorial angle at points on the circumference.

Sol^n: Let $f(z)$ be the required function. Since $f(z)$ is analytic inside the circle $|z|=1$, it can be expanded in a Taylor's series at any point z inside this circle.

Thus $f(z) = \sum_0^\infty a_n z^n$ Taylor's Expansion

where $a_n = \dfrac{1}{2\pi i}\int_C \dfrac{f(z)\,dz}{z^{n+1}}$ where C is the circle $|z|=1$.

Given that $f(z) = \dfrac{(a^2-1)\cos\theta + i(a^2+1)\sin\theta}{a^4-2a^2\cos 2\theta+1}$ on $|z|=1$.

$$= \frac{a^2(\cos\theta + i\sin\theta) - (\cos\theta - i\sin\theta)}{a^4 - a^2(e^{2i\theta}+e^{-2i\theta})+1}$$

$$= \frac{a^2 e^{i\theta} - e^{-i\theta}}{a^4 - a^2(e^{2i\theta}+e^{-2i\theta}+1)}$$

$$= \frac{a^2 z - \frac{1}{z}}{a^4 - a^2\left(z^2+\frac{1}{z^2}\right)+1}$$ since on $|z|=1$, $z=e^{i\theta}$

$$= \frac{z\left(a^2-\frac{1}{z^2}\right)}{\left(a^2-\frac{1}{z^2}\right)(a^2-z^2)} = \frac{z}{a^2-z^2}$$

$$a_n = \frac{1}{2\pi i}\int_C \frac{f(z)}{z^{n+1}}\,dz$$

$$= \frac{1}{2\pi i}\int_C \frac{z}{a^2-z^2}\cdot\frac{1}{z^{n+1}}\,dz$$

$$= \frac{1}{2\pi i}\int_C \frac{1}{a^2-z^2}\cdot\frac{1}{z^n}\,dz$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{1}{a^2-e^{2i\theta}}\cdot\frac{1}{e^{in\theta}}\cdot e^{i\theta}\cdot i\,d\theta \qquad \text{since } z=e^{i\theta}$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} e^{-i(n-1)\theta}\left[1-\frac{e^{2i\theta}}{a}\right]^{-1} d\theta$$

$$= \frac{1}{2\pi a^2}\int_0^{2\pi} e^{-i(n-1)\theta}\left[1+\frac{e^{2i\theta}}{a^2}+\frac{e^{4i\theta}}{a^4}+\cdots + \frac{e^{[2i(n-1)\theta]/2}}{a^{n-1}}+\cdots\right]d\theta$$

So $a_n = \dfrac{1}{2\pi a^2}\displaystyle\int_0^{2\pi}\frac{1}{a^{n-1}}\,d\theta$ if n is odd, all other integrals vanish being of the form $\displaystyle\int_0^{2\pi} e^{ik\theta}\,d\theta,\ k\neq 0.$

$$= \frac{1}{2\pi a^2}\cdot\frac{1}{a^{n-1}}\,2\pi = \frac{1}{a^{n+1}}$$

and $a_n = 0$ if n is even, because then all the integrals vanish.

Hence $f(z) = \displaystyle\sum_0^\infty a_n z^n$ (n odd)

$$= \sum_0^\infty \frac{1}{a^{n+1}} z^n$$

$$= \frac{1}{a}\left[\frac{z}{a}+\frac{z^3}{a^3}+\frac{z^5}{a^5}+\cdots\right]$$

even values of n do not contribute.

$$= \frac{z}{a^2}\left[1+\frac{z^2}{a^2}+\frac{z^4}{a^4}\cdots\right]$$

$$= \frac{z}{a^2}\left[1-\frac{z^2}{a^2}\right]^{-1}$$

$$= \frac{z}{a^2}\cdot\frac{1}{1-\frac{z^2}{a^2}} = \frac{z}{a^2-z^2}$$

4(d) Use two-phase simplex method to solve the problem:

Minimize $z = \frac{15}{2}x_1 - 3x_2$

Subject to the constraints:

$$3x_1 - x_2 - x_3 \geq 3$$
$$x_1 - x_2 + x_3 \geq 2$$
$$x_1, x_2, x_3 \geq 0.$$

Sol^n: Convert the objective function into the maximization form:

Maximize $z' = -\frac{15}{2}x_1 + 3x_2$.

Introducing the surplus variables $x_3, x_4 \geq 0$ and artificial variables $a_1 \geq 0$, $a_2 \geq 0$

$$\left.\begin{aligned} 3x_1 - x_2 - x_3 - x_4 \quad + a_1 &= 3 \\ x_1 - x_2 + x_3 \quad - x_5 + a_2 &= 2 \end{aligned}\right\} \text{--- (1)}$$

$$x_1, x_2, x_3, x_4, x_5, a_1, a_2 \geq 0.$$

Phase I: Assigning a cost -1 to artificial variables a_1, a_2 and cost 0 to all other variables, the new objective function for auxiliary problem becomes:

$$\text{Max } z_1^* = 0x_1 + 0x_2 + 0x_3 + 0x_4 + 0x_5 - a_1 - a_2.$$

Subject to

$$\left.\begin{aligned} 3x_1 - x_2 - x_3 - x_4 + 0x_5 + a_1 + 0a_2 &= 3 \\ x_1 - x_2 + x_3 + 0x_4 - x_5 + 0a_1 + a_2 &= 2 \end{aligned}\right\} \text{--- (2)}$$

$$x_1, x_2, x_3, x_4, x_5, a_1, a_2 \geq 0.$$

Now the IBFS is given by

setting $x_1 = x_2 = x_3 = x_4 = 0$ (Non-basic)

$a_1 = 3,\ a_2 = 2$ (basic)

and $z_1^* = -5\ (<0)$

$\therefore$ Initial simplex table is

	c_j	0	0	0	0	0	-1	-1		
C_B	Basis	x_1	x_2	x_3	x_4	x_5	a_1	a_2	b	θ
-1	a_1	(3)	-1	-1	-1	0	1	0	3	1 →
-1	a_2	1	-1	1	0	-1	0	1	2	2
	$z_{ij}^* = \sum C_B a_{ij}$	-4	2	0	1	1	-1	-1	-5	
	$C_j^\circ = c_j - z_{ij}^*$	4 ↑	-2	0	-1	-1	0	0		

from the above

x_1 is the entering variable,

a_1 is the outgoing variable

Here (3) is the key element and make it into unity and make all other elements in its column to zero.

$\therefore$ The new simplex table is:

	c_j	0	0	0	0	0	-1	-1	b	θ
C_B	Basis	x_1	x_2	x_3	x_4	x_5	a_1	a_2		
0	x_1	1	$-1/3$	$-1/3$	$-1/3$	0	$1/3$	0	1	-3
-1	a_2	0	$-2/3$	($4/3$)	$1/3$	-1	$-1/3$	1	1	$\frac{3}{4}$ →
	$z_{ij}^* = \sum C_B a_{ij}$	0	$2/3$	$-4/3$	$-1/3$	1	$1/3$	-1	-1	
	$C_j = c_j - z_{ij}^*$	0	$-2/3$	$4/3$ ↑	$1/3$	-1	$-1/3$	0		

From the above table,
x_3 is the entering variable,
a_2 is the outgoing variable
Here $(4/3)$ is the key element and we convert it into unity and all other elements in its column equal to zero
∴ The revised simplex table is

	c_j	0	0	0	0	0	-1	-1	
C_B	Basis	x_1	x_2	x_3	x_4	x_5	a_1	a_2	b
0	x_1	1	$-1/2$	0	$-1/4$	$-1/4$	$1/4$	$1/4$	$5/4$
0	x_3	0	$-1/2$	1	$1/4$	$-3/4$	$-1/4$	$3/4$	$3/4$
$z_{ij}^* = \Sigma a_{ij} c_B$		0	0	0	0	0	0	0	0
$C_j = c_j - z_{ij}^*$		0	0	0	0	0	-1	-1	

from the above table,
all C_j's ≤ 0.
this table gives the optimal solution
Also $\text{Max } Z_1^* = 0$
and no artificial variable appears in the basis. Therefore $x_1 = 5/4$, $x_3 = 3/4$ is an OBFS to the original problem.

we proceed to phase II

Phase-II : Considering the actual costs associated with the original variables, the objective function

$\text{Max } Z_1 = -7.5x_1 + 3x_2 + 0x_3 + 0x_4 + 0x_5 - 0a_1 - 0a_2$

subject to

$3x_1 - x_2 - x_3 - x_4 + 0x_5 + a_1 + 0a_2 = 3$

$x_1 - x_2 + x_3 + 0x_4 - x_5 + 0a_1 + a_2 = 2$

$x_i \geq 0,\ a_1, a_2 \geq 0. \quad i = 1, 2, 3, 4, 5.$

Using final table of phase-I, the initial simplex table of phase-II is as follows.

C_B	C_j / Basis	$-15/2$ x_1	3 x_2	0 x_3	0 x_4	0 x_5	b
$-15/2$	x_1	1	$-1/2$	0	$-1/4$	$-1/4$	$5/4$
0	x_3	0	$-1/2$	1	$1/4$	$-3/4$	$3/4$
	$Z_j = \Sigma C_B a_{ij}$	$-15/2$	$15/4$	0	$15/8$	$15/8$	$-75/8$
	$C_j = c_j - z_j$	0	$-3/4$	0	$-15/8$	$-15/8$	

From the above table all C_j's ≤ 0.

This gives optimal solution.

Hence an OBFS to the given LPP is

$x_1 = 5/4,\ x_2 = 0,\ x_3 = 3/4$

and $\text{Max } Z_1 = -75/8$

Hence $\text{Min } Z = \text{Max}(-Z)$

$= -\text{Max } Z = 75/8.$

5(a) solve $(D^2 + 2DD' + D'^2)z = 2\cos y - x\sin y$.

Sol^n: Given equation is $(D+D')^2 z = 2\cos y - x\sin y$

Its auxiliary equation is $(m+1)^2 = 0$

$\Rightarrow m = -1, -1.$

$\therefore$ C.F $= \phi_1(y-x) + x\phi_2(y-x)$, ϕ_1, ϕ_2 being arbitrary functions.

Now $P.I = \frac{1}{D+D'} \frac{1}{D+D'}(2\cos y - x\sin y)$

$= \frac{1}{D+D'}\int [2\cos(x+c) - x\sin(x+c)]dx$, where $c = y-x$

$= \frac{1}{D+D'}\left[2\int \cos(x+c)\,dx - \int x\sin(x+c)dx\right]$

$= \frac{1}{D+D'}\left[2\sin(x+c) - \left\{-x\cos(x+c) + \int \cos(x+c)dx\right\}\right]$

$= \frac{1}{D+D'}\left[2\sin(x+c) + x\cos(x+c) - \sin(x+c)\right]$

$= \frac{1}{D+D'}(\sin y + x\cos y)$, as $c = y-x$

$= \int [\sin(x+c') + x\cos(x+c')]dx$, where $c' = y-x$

$= -\cos(x+c') + x\sin(x+c') - \int \{1.\sin(x+c')\}dx$

$= -\cos(x+c') + x\sin(x+c') + \cos(x+c')$

$= x\sin y$, as $c' = y-x$

So the required solution is

$z = \phi_1(y-x) + x\phi_2(y-x) + x\sin y.$

5(b) Find a complete integral of $2(pq+yp+qx)+x^2+y^2=0$.

Soln: Given equation is $f(x,y,z,p,q) = 2(pq+yp+qx)+x^2+y^2$ —— ①

∴ Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

(or) $$\frac{dp}{2q+2x} = \frac{dq}{2p+2y} = \frac{dz}{-p(2q+2y)-q(2p+2x)} = \frac{dx}{-(2q+2y)} = \frac{dy}{-(2p+2x)}$$

Each of the above fractions $= \dfrac{dp+dq+dx+dy}{(2q+2x)+(2p+2y)-(2q+2y)-(2p+2x)}$

$= (dp+dq+dx+dy)/0$

This $\Rightarrow dp+dq+dx+dy=0$ so that $p+q+x+y=a$

$\Rightarrow (p+x)+(q+y)=a$ —— ②

Rewriting ①, $2(p+x)(q+y)+(x-y)^2=0$

$\therefore (p+x)(q+y) = -(x-y)^2/2$ —— ③

Now, $(p+x)-(q+y) = \sqrt{\{(p+x)+(q+y)\}^2-4(p+x)(q+y)}$

$\therefore (p+x)-(q+y) = \sqrt{a^2+2(x-y)^2}$ —— ④

Adding ② and ④, $2(p+x) = a+\sqrt{a^2+2(x-y)^2}$.

Substracting ④ from ②, $2(q+y) = a-\sqrt{a^2+2(x-y)^2}$.

These give $p = -x+\frac{a}{2}+\frac{1}{2}\sqrt{a^2+2(x-y)^2}$, $q = -y+\frac{a}{2}-\frac{1}{2}\sqrt{a^2+2(x-y)^2}$

∴ $dz = pdx+qdy$ becomes.

$dz = -(xdx+ydy)+\frac{a}{2}(dx+dy)+\frac{1}{2}\sqrt{a^2+2(x-y)^2}$

∴ $dz = pdx+qdy$ becomes

$dz = -(xdx+ydy)+\frac{a}{2}(dx+dy)+\frac{1}{2}\sqrt{a^2+2(x-y)^2}\,(dx-dy)$

$$\Rightarrow dz = -\tfrac{1}{2}d(x^2+y^2) + \tfrac{a}{2}d(x+y) + \sqrt{2}\times\tfrac{1}{2}\sqrt{\tfrac{a^2}{2}+(x-y)^2}\,d(x-y) \quad \text{--- (5)}$$

Putting $x-y=t$ so that $d(x-y)=dt$. Then (5) becomes

$$dz = -\tfrac{1}{2}d(x^2+y^2) + \tfrac{a}{2}d(x+y) + \tfrac{1}{\sqrt{2}}\sqrt{\left(\tfrac{a}{\sqrt{2}}\right)^2+t^2}\,dt.$$

Integrating
$$z = -\tfrac{1}{2}(x^2+y^2) + \tfrac{a}{2}(x+y) + \tfrac{1}{\sqrt{2}}\left[\tfrac{t}{2}\sqrt{\left(\tfrac{a}{\sqrt{2}}\right)^2+t^2} + \tfrac{(a/\sqrt{2})^2}{2}\log\left\{t+\sqrt{(a/\sqrt{2})^2+t^2}\right\}\right]+b$$

$$\Rightarrow z = \left(-\tfrac{1}{2}\right)(x^2+y^2) + \tfrac{a}{2}(x+y) + \tfrac{1}{2\sqrt{2}}\left[(x-y)\sqrt{\tfrac{a^2}{2}+(x-y)^2} + \tfrac{a^2}{2}\log\left\{x-y+\sqrt{\tfrac{a^2}{2}+(x-y)^2}\right\}\right]+b$$

5(c). The current i in an electric circuit is given by $i = 10e^{-t}\sin 2\pi t$ where t is in seconds. Using Newton's method, find the value of t correct to 3 decimal places for $i = 2$amp.

sol'n: Given function $i = 10e^{-t}\sin 2\pi t$.

we have to find t such that $i = 2$amp.

$$\Rightarrow 10e^{-t}\sin 2\pi t = 2$$

Let $f(t) = 5\sin 2\pi t - e^{t} = 0$

$$\therefore f'(t) = 10\pi\cos 2\pi t - e^{t}$$

By Newton's Method,

$$t_{n+1} = t_n - \frac{f(t_n)}{f'(t_n)}$$

$$\Rightarrow t_{n+1} = t_n - \frac{5\sin 2\pi t - e^{t}}{10\pi\cos 2\pi t - e^{t}}$$

taking $t_0 = 0$

$$t_1 = \frac{-(-1)}{10\pi - 1} = 0.03289$$

$$t_2 = 0.03289 - \frac{5\sin 2\pi(0.03289) - e^{0.03289}}{10\pi\cos 2\pi(0.03289) - e^{0.03289}}$$

$$= 0.03289 - \frac{(-0.0075)}{29.69837}$$

$$= 0.03263$$

$$t_3 = 0.03314$$

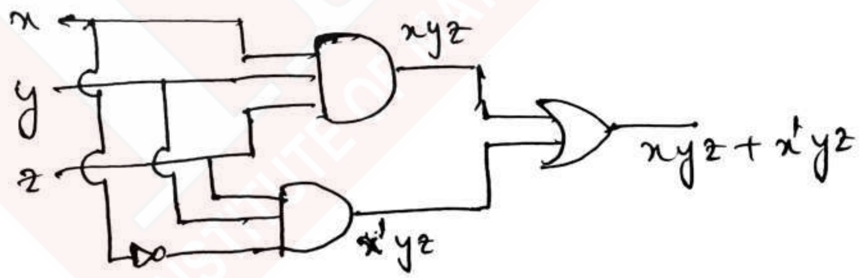
$\therefore$ The value of t correct to 3 decimal places is $t = 0.033$ sec.

5(d) Find the logic circuit that represents the following Boolean function. Find also an equivalent simpler circuit.

x	y	z	f(x, y, z)
1	1	1	1
1	1	0	0
1	0	1	0
1	0	0	0
0	1	1	1
0	1	0	0
0	0	1	0
0	0	0	0

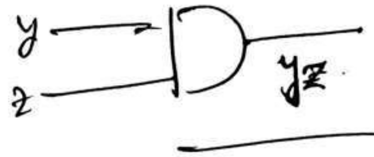
Sol: $f(x,y,z) = xyz + x'yz$.

The logic circuit is



$$f(x,y,z) = xyz + x'yz = (x+x')\,yz$$
$$= yz \quad (\because x+x'=1)$$

Simple circuit:



5(e) If the expression for stream function is described by $\psi = x^3 - 3xy^2$, determine whether flow is rotational or irrotational. If the flow is irrotational, then indicate the correct value of the velocity potential.

(i) $\phi = y^3 - 3x^2y$ (ii) $\phi = -3x^2y$.

Sol'n: Now $u = \frac{\partial\psi}{\partial y} = -6xy$, $v = -\frac{\partial\psi}{\partial x} = -3(x^2 - y^2)$ — ①

Hence $\frac{\partial v}{\partial x} = -6x$ and $\frac{\partial u}{\partial y} = -6x$ — ②

A two-dimensional flow in xy-plane will be irrotational if the vorticity vector component Ω_z in the direction is zero.

Here $\Omega_z = \partial v/\partial x - \partial u/\partial y = -6x - (-6x) = 0$, by ②

Hence flow is irrotational.

Now, $u = -\partial\phi/\partial x$ and $v = -\frac{\partial\phi}{\partial y}$ — ③

For an irrotational flow Laplace equation in ϕ must be satisfied,

i.e. $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$.

we now check the validity of each given value of ϕ.

(a) Given $\phi = y^3 - 3x^2y \Rightarrow \frac{\partial^2\phi}{\partial x^2} = -6y$ and $\frac{\partial^2\phi}{\partial y^2} = 6y$

$$\therefore \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = -6y + 6y = 0$$

(b) Given $\phi = -3x^2y \Rightarrow \frac{\partial^2\phi}{\partial x^2} = -6y$ and $\frac{\partial^2\phi}{\partial y^2} = 0$

$$\therefore \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = -6y + 0 \neq 0$$

Hence the correct value of ϕ is given by $\phi = y^3 - 3x^2y$.

6(a). Find a partial differential equation by eliminating a, b, c from $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

Sol'n: Given that $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ —— (1)

Differentiating (1) w.r.t x and y, we get

$$\frac{2x}{a^2} + \frac{2z}{c^2}\frac{\partial z}{\partial x} = 0 \Rightarrow c^2x + a^2z\frac{\partial z}{\partial x} = 0 \quad \text{—— (2)}$$

and $$\frac{2y}{b^2} + \frac{2z}{c^2}\frac{\partial z}{\partial y} = 0 \Rightarrow c^2y + b^2z\frac{\partial z}{\partial y} = 0 \quad \text{—— (3)}$$

Differentiating (2) w.r.t x and (3) w.r.t y, we have

$$c^2 + a^2\left(\frac{\partial z}{\partial x}\right)^2 + a^2z\frac{\partial^2 z}{\partial x^2} = 0 \quad \text{—— (4)}$$

& $$c^2 + b^2\left(\frac{\partial z}{\partial y}\right)^2 + b^2z\frac{\partial^2 z}{\partial y^2} = 0 \quad \text{—— (5)}$$

from (2), $c^2 = \frac{-a^2z}{x}\left(\frac{\partial z}{\partial x}\right)$

putting this value of c^2 in (4) and dividing by a^2,

we obtain

$$-\frac{z}{x}\frac{\partial z}{\partial x} + \left(\frac{\partial z}{\partial x}\right)^2 + z\frac{\partial^2 z}{\partial x^2} = 0 \quad \text{—— (6)}$$

Similarly, from (3) & (5)

$$zy\frac{\partial^2 z}{\partial y^2} + y\left(\frac{\partial z}{\partial y}\right)^2 - z\frac{\partial z}{\partial y} = 0 \quad \text{—— (7)}$$

Differentiating (2) partially w.r.t y, we get

$$a^2\left\{\left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial z}{\partial x}\right) + z\frac{\partial^2 z}{\partial x\partial y}\right\} = 0$$

i.e, $$\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} + z\frac{\partial^2 z}{\partial x\partial y} = 0 \quad \text{—— (8)}$$

∴ (6), (7) and (8) are three possible forms of the required partial differential equations.

6(b) Find the equation of the integral surface of the differential equation $2y(z-3)p+(2x-z)q = y(2x-3)$, which pass through the circle $z=0$, $x^2+y^2=2x$.

Sol'n: Given equation is $2y(z-3)p+(2x-z)q = y(2x-3)$ —— (1)

Given circle is $x^2+y^2=2x,\ z=0$ —— (2)

Here the Lagrange's auxiliary equations of (1) are

$$\frac{dx}{2y(z-3)} = \frac{dy}{2x-z} = \frac{dz}{y(2x-3)} \quad \text{—— (3)}$$

Taking the first and third fractions of (3), we have

$$(2x-3)dx - 2(z-3)dz = 0.$$

Integrating, $x^2-3x-z^2+6z = c_1$ —— (4)

Choosing $\frac{1}{2}, y, -1$ as multipliers, each fraction of (3)

$$= \frac{\frac{1}{2}dx + ydy - dz}{y(z-3)+y(2x-z)-y(2x-3)} = \frac{\frac{1}{2}dx+ydy-dz}{0}$$

Hence $\frac{1}{2}dx + ydy - dz = 0 \Rightarrow dx + 2ydy - 2dz = 0$.

Integrating, $x+y^2-2z = c_2$ —— (5)

Now, the parametric equations of given circle (2) are

$$x=t,\ y=(2t-t^2)^{1/2},\ z=0 \quad \text{—— (6)}$$

Substituting these values in (4) and (5), we have

$$t^2-3t = c_1 \text{ and } 3t-t^2 = c_2 \quad \text{—— (7)}$$

Eliminating t from the above equations (7), we have

$$c_1+c_2 = 0 \quad \text{—— (8)}$$

Substituting the values of c_1 and c_2 from (4) & (5) in (8) the desired integral surface is

$$x^2-3x-z^2+6z + x+y^2-2z = 0 \text{ (or) } x^2+y^2-z^2-2x+4z=0.$$

6(C) Reduce the equation $x^2r - 2xys + y^2t - xp + 3yq = 8y/x$ to canonical form.

Sol'n: Given $x^2r - 2xys + y^2t - xp + 3yq - 8y/x = 0$ —(1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = x^2$, $S = -2xy$, $T = y^2$ so that $S^2 - 4RT = 0$, showing that (1) is parabolic.

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to

$$x^2\lambda^2 - 2xy\lambda + y^2 = 0 \Rightarrow (x\lambda - y)^2 = 0 \text{ so that } \lambda = y/x, y/x.$$

The corresponding characteristic equation is

$$\frac{dy}{dx} + y/x = 0 \Rightarrow \frac{1}{y}dy + \frac{1}{x}dx = 0 \text{ so that } xy = c_1$$

Choose $u = xy$ and $v = x$ —(2)

where we have chosen $v = x$ in such a manner that u and v are independent functions as verified below.

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = -x \neq 0.$$

Now, $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = y\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$, by (2) —(3)

$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = x\frac{\partial z}{\partial u}$, by (2) —(4)

$$r = \frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(y\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right] + \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}$$

$$= y\left(y\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v\partial u}\right) + y\frac{\partial^2 z}{\partial u\partial v} + \frac{\partial^2 z}{\partial v^2} = y^2\frac{\partial^2 z}{\partial u^2} + 2y\frac{\partial^2 z}{\partial u\partial v} + \frac{\partial^2 z}{\partial v^2} \quad (5)$$

$$s = \frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial u}\right) = \frac{\partial z}{\partial u} + x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)$$

$$= \frac{\partial z}{\partial u} + x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right] = \frac{\partial z}{\partial u} + xy\frac{\partial^2 z}{\partial u^2} + x\frac{\partial^2 z}{\partial u\partial v} \quad (6)$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(x\frac{\partial z}{\partial u}\right) = x\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right), \text{ by (4)}$$

$$= x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}\right] = x^2\frac{\partial^2 z}{\partial u^2}, \text{ by } ② \quad — ⑦$$

using ②, ③, ④, ⑤, ⑥ and ⑦ in ①, we have

$$x^2\left[y^2\frac{\partial^2 z}{\partial u^2} + 2y\frac{\partial^2 z}{\partial u\partial v} + \frac{\partial^2 z}{\partial v^2}\right] - 2xy\left[\frac{\partial z}{\partial u} + xy\frac{\partial^2 z}{\partial u^2} + x\frac{\partial^2 z}{\partial u\partial v}\right]$$

$$+ y^2x^2\frac{\partial^2 z}{\partial x^2} - x\left[y\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right] + 3yx\frac{\partial z}{\partial u} - \frac{8y}{x} = 0$$

$$\Rightarrow x^2\frac{\partial^2 z}{\partial v^2} - x\frac{\partial z}{\partial v} = \frac{8y}{x}, \text{ where } D \equiv \frac{\partial}{\partial u},\ D' \equiv \partial/\partial v \quad — ⑧$$

To solve ⑧, let $u = e^X$ and $v = e^Y$ so that

$X = \log u$, $Y = \log v$ — ⑨

Then ⑧ becomes $\{D'(D'-1) - D'\}z = 8e^{X-2Y}$

$$\Rightarrow D'(D'-2)z = 8e^{X-2Y}$$

$$\text{C.F} = \phi(X) + e^{2Y}\psi(X)$$

$$= \phi(\log u) + v^2\psi(\log u), \text{ by } ⑨$$

$$= F(u) + v^2G(u) = F(xy) + x^2G(xy), \text{ by } ②$$

$$\text{P.I} = \frac{1}{D'(D'-1)}\, 8e^{X-2Y}$$

$$= 8e^{X-2Y}\frac{1}{(D'-2)(D'-2-2)}\cdot 1$$

$$= 8(e^X/e^{2Y})\frac{1}{8}\left(1-D'/2\right)^{-1}\left(1-D'/4\right)^{-1}\cdot 1 = \frac{u}{v^2} \text{ by } ⑨$$

$$= (xy)/x^2 = y/x \text{ using } ②$$

Hence the required general solution of ①, is given by $z = F(xy) + x^2G(xy) + y/x$, F, G being arbitrary functions.

6(d) The temperature at one end of a bar, 50cm long with insulated sides, is kept at 0°c and that the other end is kept at 100°c until steady state condition prevails. The two ends are then suddenly insulated, so that the temperature gradient is zero at each end thereafter. Find the temperature distribution.

Sol^n: Let the rod lie along the axis of x with one end which is kept at 0°c be at the origin. The temperature distribution $u(x,t)$ in the bar at any time t at any distance x is governed by one dimensional heat equation

$$\frac{\partial u}{\partial t} = K\frac{\partial^2 u}{\partial x^2} \quad \text{——— (1)}$$

Initially $u=0$ at $x=0$ and $u=100°c$ at $x=50$.

In the steady state, u is independent of t.

$$\therefore \frac{\partial u}{\partial t}=0$$

$\therefore$ from (1), we have $\frac{\partial^2 u}{\partial x^2}=0$

$$\Rightarrow u = Ax+B$$

Since at $x=0, u=0$ and $x=50$, $u=100°c$

$\therefore 0=0+B$ and $100=50A+B \Rightarrow B=0$ and $A=2$.

i.e, initial temperature distribution in the bar is given by

$u=2x$ i.e, $u(x,0)=2x$.

Now let $u(x,t)$ represent the temperature distribution in the rod at time t measured from the instant the two ends of the bar are suddenly insulated after the rod reaches to steady state.

Thus, we are required to solve (1) under the following conditions.

B.C. $u_x(0,t)=0$ and $u_x(50,t)=0$

I.C $u(x,0)=2x$.

we can prove that the solution of the heat equation

$$K\frac{\partial^2 u}{\partial x^2}=\frac{\partial u}{\partial t} \quad \text{——— (1)}$$

subject to boundary conditions are

$$u_x(0,t)=u_x(50,t)=0 \quad \forall\ t\geq 0 \quad \text{——— (2)}$$

and the initial condition $u(x,0)=2x,\ 0<x\leq 50$ (3)

Suppose (1) has solution of the form

$$u(x,t)=X(x)T(t) \quad \text{——— (A)}$$

where X is a function of x alone and T that of t alone.

Substituting this value of u in (1), we get

$$KX''T=XT' \Rightarrow \frac{X''}{X}=\frac{T'}{KT}$$

Since x and t are independent variables.

$$X''-\mu X=0 \text{ —— (4)} \quad \text{and} \quad T'-\mu KT=0 \text{ —— (5)}$$

Differentiating (A) partially w.r.t x, we get

$$u_x(x,t)=X'(x)T(t) \quad \text{——— (6)}$$

using (2), (6) gives $X'(0)T(t)=0$ & $X'(50)T(t)=0$ ——— (7)

Since $T(t)=0$ leads to $u=0$ so assume that $T(t)\neq 0$.

Hence from (7), we get

$$X'(0)=0 \text{ and } X'(50)=0 \quad \text{—— (8)}$$

Three cases arise :

Case 1 : Let $\mu=0$. Then solution of (4) is

$$X(x)=Ax+B \text{ —— (9)} \quad \text{which yields } X'(x)=A \text{ ——— (10)}$$

using B.C (8), (10) gives $A=0$.

Then (9) reduces to $X(x)=B$.

Again, corresponding to $\mu=0$, (5) yields

$$\frac{dT}{dt}=0 \Rightarrow T=\text{constant}=\frac{E_0}{2B} \text{ (say)}$$

$\therefore$ corresponding to $\mu=0$ a solution of the given boundary value problem from (A) is given by $u(x,t) = B \times \frac{E_0}{2B} = \frac{E_0}{2}$ —— (11)

Case (2): Let $\mu = \lambda^2$, $\lambda \neq 0$. The solution of (4) is

$$X(x) = Ae^{\lambda x} + Be^{-\lambda x}$$

which yields $X'(x) = A\lambda e^{\lambda x} - B\lambda e^{-\lambda x}$ —— (12)

using B.C. (8), (12) gives $A = B = 0$ so that $X(x) = 0$ and hence $u \equiv 0$. which does not satisfy (3). so reject $\mu = \lambda^2$.

Case 3: let $\mu = -\lambda^2$, $\lambda \neq 0$. Then the solution of (4) is

$$X(x) = A\cos\lambda x + B\sin\lambda x.$$

which yields $X'(x) = -A\lambda\sin\lambda x + B\lambda\cos\lambda x$ —— (13)

using B.C. (8); (13) gives $0 = B\lambda$

and $0 = -A\lambda\sin\lambda(50) + B\lambda\cos\lambda(50)$

$\Rightarrow B = 0$ and $A\lambda\sin(\lambda 50) = 0$

$\Rightarrow \sin(\lambda 50) = 0$ $(\because A \neq 0)$.

$\Rightarrow \lambda(50) = n\pi \Rightarrow \lambda = \frac{n\pi}{50}$, $n = 1, 2, 3 \cdots$ —— (14)

Hence non-zero solutions $X_n(x)$ of (4) are given by

$$X_n(x) = A_n\cos\left(\frac{n\pi x}{50}\right) \text{ —— (15)}$$

using (14), (5) reduces to $\frac{dT}{T} = -\frac{n^2\pi^2 k}{(50)^2}dt$ $\left(\because \mu = -\lambda^2 = \frac{n^2\pi^2}{(50)^2}\right)$

$\Rightarrow \frac{1}{T}dT = -C_n^2\,dt$ where $C_n^2 = \frac{n^2\pi^2 k}{(50)^2}$ —— (16)

solving (16),

$$T_n(t) = D_n e^{-C_n^2 t} \text{ —— (17)}$$

from (15) & (17) $u_n(x,t) = X_n(t)\,T_n(t)$

$$= E_n\cos\left(\frac{n\pi x}{50}\right)e^{-C_n^2 t} \text{ —— (18)}$$

are solutions of (1). For $n = 1, 2, 3 \cdots$ each one of these satisfy the boundary conditions (2).

Here $E_n = A_n D_n$

thus (11) and (18) constitute a set of infinite solutions of (1). To obtain a solution also satisfy the initial condition (3), we consider a linear combination of these solutions. Hence complete solution of ① may be taken in the following form.

$$u(x,t) = \frac{E_0}{2} + \sum_{n=1}^{\infty} u_n(x,t)$$

$$= \frac{E_0}{2} + \sum_{n=1}^{\infty} E_n \cos\left(\frac{n\pi x}{50}\right) e^{-C_n^2 t} \quad \text{—— (19)}$$

which is Fourier Cosine Series.

substituting $t=0$ in (19), and using ③,

we have $$2x = \frac{E_0}{2} + \sum_{n=1}^{\infty} E_n \cos\left(\frac{n\pi x}{50}\right)$$

where $$E_0 = \frac{2}{50}\int_0^{50} 2x\,dx = \frac{2}{50}\left[x^2\right]_0^{50} = \frac{2}{50}[2500] = 100$$

and $$E_n = \frac{2}{50}\int_0^{50} (2x)\cos\frac{n\pi x}{50}\,dx \quad \text{—— (20)}$$

$$= \frac{1}{25}\int_0^{50} 2x\cos\left(\frac{n\pi x}{50}\right)dx$$

$$= \frac{1}{25}\left[2x\,\frac{50}{n\pi}\sin\left(\frac{n\pi x}{50}\right) - 2\left(\frac{50}{n\pi}\right)^2\cos\left(\frac{n\pi x}{50}\right)\right]_0^{50}$$

$$= \frac{2}{25}\left(\frac{50}{n\pi}\right)^2(\cos n\pi - 1)$$

$$= \frac{200}{n^2\pi^2}\left((-1)^n - 1\right)$$

$$E_n = \begin{cases} 0 & \text{if } n = 2m \text{ (even)} \\ \frac{-400}{n^2\pi^2} & \text{when } n = 2m-1 \text{ (odd)} \end{cases}$$

i.e, $E_{2m} = 0$ and $$E_{2m-1} = \frac{-400}{(2m-1)^2\pi^2} \quad \text{—— (21)}$$

Now using (20) and (21) the required solution from (19) is given by

$$u(x,t) = \frac{100}{2} + \sum_{m=1}^{\infty} \frac{-400}{(2m-1)^2\pi^2}\cos\frac{(2m-1)\pi x}{50}\, e^{-C_{2m-1}^2 t}$$

$$= 50 - \frac{400}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}\cos\frac{(2m-1)\pi x}{50}\, e^{-\frac{(2m-1)^2\pi^2 k t}{2500}}$$

7(a) Using Gauss-Seidel iterative method, find the solution of the following system. $4x - y + 8z = 26$, $5x + 2y - z = 6$, $x - 10y + 2z = -13$ upto three iterations.

Ans: $x = 1$, $y = 2$, $z = 3$

7(b) A rocket is launched from the ground. Its acceleration is registered during the first 80 seconds and is given in the table below. Using Simpson's $\frac{1}{3}$rd rule, find the velocity of the rocket at $t = 80$ seconds.

t(sec) :	0	10	20	30	40	50	60	70	80
f(cm/sec^2) :	30	31.63	33.34	35.47	37.75	40.33	43.25	46.69	50.67

Sol'n: Since acceleration is defined as the rate of change of velocity.

we have $\frac{dv}{dt} = a$ (or) $v = \int_0^{80} a\,dt$

Using Simpson's $\frac{1}{3}$rd rule, we have

$$v = \frac{h}{3}\left[(y_0 + y_8) + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)\right]$$

$$= \frac{10}{3}\left[(30 + 50.67) + 4(31.63 + 35.47 + 40.33 + 46.69) + 2(33.34 + 37.75 + 43.25)\right]$$

$$= 3086.1 \text{ m/s}$$

$\therefore$ The required velocity is given by

$$v = 3.0861 \text{ km/sec}.$$

7(c) Using Runge-Kutta method of order 4, compute $y(0.2)$ and $y(0.4)$ from $10\frac{dy}{dx} = x^2+y^2$, $y(0)=1$, taking $h=0.1$.

Sol'n: Given that $\frac{dy}{dx} = \frac{x^2+y^2}{10}$

To find $y(0.1)$: $h=0.1$; $x_0=0$, $y_0=1$.

$K_1 = hf(x_0, y_0) = (0.1)\frac{1}{10} = 0.01$

$K_2 = hf\left(x_0+\frac{h}{2}, y_0+\frac{K_1}{2}\right) = 0.01012$

$K_3 = h\left[f\left(x_0+\frac{h}{2}, y_0+\frac{K_2}{2}\right)\right] = 0.01012$

$K_4 = hf(x_0+h, y_0+K_3) = 0.0103$

$y(0.1) = y_0 + \frac{K_1+2K_2+2K_3+K_4}{6} = 1.0101$

To find $y(0.2)$: $x_1 = x_0+h = 0+0.1 = 0.1$, $y_1 = 1.0101$, $h=0.1$

$K_1 = hf(x_1, y_1) = (0.1)f(0.1, 1.0101) = 0.0103$

$K_2 = 0.01053$, $K_3 = 0.01053$, $K_4 = 0.0108$

$y(0.2) = 1.0101 + \frac{1}{6}(K_1+2K_2+2K_3+K_4)$

$= 1.0206$

To find $y(0.3)$: $x_2 = 0.2$, $y_2 = 1.0206$, $h=0.1$

$K_1 = hf(x_2, y_2) = 0.0108$

$K_2 = hf\left(x_2+\frac{h}{2}, y_2+\frac{K_1}{2}\right) = 0.0111$

$K_3 = hf\left(x_2+\frac{h}{2}, y_2+\frac{K_2}{2}\right) = 0.0111$

$K_4 = hf(x_2+h, y_2+K_3) = 0.0115$

$y(0.3) = 1.0317$

To find $y(0.4)$: $x_3 = 0.3$, $y_3 = 1.0317$, $h=0.1$

$K_1 = 0.0115$, $K_2 = 0.01198$

$K_3 = 0.01199$, $K_4 = 0.01249$

$y(0.4) = 1.04369$.

7(d) (i) If $A \oplus B = AB' + A'B$, find the value of $x \oplus y \oplus z$.

(ii) find the hexadecimal equivalent of the decimal number $(587632)_{10}$.

Sol^n: (i) Given that $A \oplus B = AB' + A'B$.

$$x \oplus y \oplus z = (x \oplus y) \oplus z$$
$$= (x \oplus y)z' + (x \oplus y)'z$$
$$= (xy' + x'y)z' + (xy' + x'y)'z$$
$$= (xy' + x'y)z' + [(xy')' \cdot (x'y)']z$$
$$= (xy' + x'y)z' + [(x' + y) \cdot (x + y')]z$$
$$= (xy' + x'y)z' + (x'y' + xy)z$$
$$= xy'z' + x'yz' + x'y'z + xyz.$$

(ii)

16	587632
16	36727 — 0
16	2295 — 7
16	143 — 7
16	8 — 15
	0 — 8

$\therefore (587632)_{10} = (8F770)_{16}$.

8(a). A solid homogeneous sphere is rolling on the inside of a fixed hollow sphere, the two centres being always in the same vertical plane, show that the smaller sphere will make complete revolution if, when it is in its lowest position, the pressure on it is greater than 34/7 times its own weight.

Sol^n: Let O be the centre and a the radius of fixed hollow sphere. Let C be the centre, M the mass and b the radius of the sphere rolling inside this fixed sphere. A time t let the line CB fixed in moving sphere make an angle ϕ to the vertical then let the line OC joining centres make an angle θ to the vertical where initially B coincided with A.

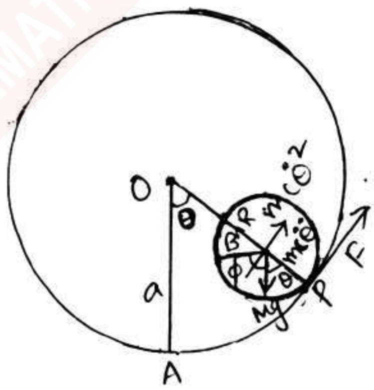
Since there is no slipping.

$$\therefore \text{Arc } AP = \text{Arc } PB$$

$$\Rightarrow a\theta = b(\phi+\theta)$$

$$\therefore b\dot{\phi} = (a-b)\dot{\theta} = c\dot{\theta} \quad \text{—— (1)}$$

where $c = a-b$.



Since C describes circle of radius $OC = a-b = c$ (say), about C,

$\therefore$ the equations of motion are

$$Mc\dot{\theta}^2 = R - Mg\cos\theta \quad \text{—— (2)}$$

$$\text{and} \quad Mc\ddot{\theta} = F - Mg\sin\theta \quad \text{—— (3)}$$

The coordinates (x_c, y_c) of C referred to the horizontal and vertical lines through O as axes are given by

$$x_c = c\sin\theta \quad \text{and} \quad y_c = c\cos\theta$$

$$\therefore v_c^2 = \dot{x}_c^2 + \dot{y}_c^2 = c^2\dot{\theta}^2,$$

At time t, K.E of the moving sphere

$$= \frac{1}{2}Mk^2\dot\phi^2 + \frac{1}{2}Mv_c^2 = \frac{1}{2}M.\frac{2}{5}b^2\dot\phi^2 + \frac{1}{2}Mc^2\dot\theta^2$$

$$= \frac{1}{5}Mc^2\dot\theta^2 + \frac{1}{2}Mc^2\dot\theta^2 = \frac{7}{10}Mc^2\dot\theta^2 \quad [\text{from } ①]$$

The coordinates (x_c, y_c) of C referred to the horizontal and vertical lines through O as axes are given by

$x_c = c\sin\theta$ and $y_c = c\cos\theta$

$$\therefore v_c^2 = \dot x_c^2 + \dot y_c^2 = c^2\dot\theta^2$$

At time t, K.E of the moving sphere

$$= \frac{1}{2}Mk^2\dot\phi^2 + \frac{1}{2}Mv_c^2 = \frac{1}{2}M.\frac{2}{5}b^2\dot\phi^2 + \frac{1}{2}Mc^2\dot\theta^2$$

$$= \frac{1}{5}Mc^2\dot\theta^2 + \frac{1}{2}Mc^2\dot\theta^2 = \frac{7}{10}Mc^2\dot\theta^2 \quad [\text{from } ①]$$

If ω is the initial angular velocity i.e. $\dot\theta = \omega$, then the initial K.E at $t=0$ is $\frac{7}{10}Mc^2\omega^2$.

$\therefore$ The energy equation gives

Change in K.E = workdone by the gravity

$$\frac{7}{10}Mc^2\dot\theta^2 - \frac{7}{10}Mc^2\omega^2 = -Mg(c - c\cos\theta) \Rightarrow c\dot\theta^2 = c\omega^2 - \frac{10}{7}g(1-\cos\theta) \quad — ④$$

$$\therefore \text{from } ②,\ R = Mg\cos\theta + M\left[c\omega^2 - \frac{10}{7}g(1-\cos\theta)\right] \quad — ⑤$$

The sphere will make complete revolution if $R=0$ when $\theta = \pi$.

$$\therefore \text{from } ⑤ \quad 0 = Mg\cos\pi + M\left[c\omega^2 - \frac{10}{7}g(1-\cos\pi)\right]$$

$$\Rightarrow c\omega^2 = g\,\frac{20}{7}g \Rightarrow \omega^2 = \frac{27g}{7c}.$$

$\therefore \omega = \sqrt{\left(\frac{27g}{7c}\right)}$ is the least velocity of ω to make the complete revolution.

Now at the lowest position when $\theta = 0$, $\dot\theta = \omega = \sqrt{\left(\frac{27g}{7c}\right)}$, then from ⑤,

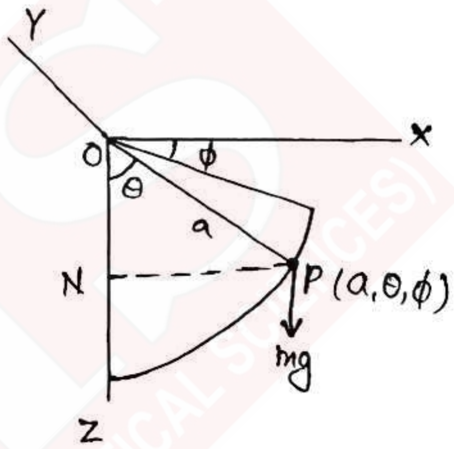
$$R = Mg + M\left[c.\frac{27g}{7c} - \frac{10}{7}g(1-1)\right] = \frac{34}{7}Mg.$$

$\therefore$ when the sphere makes complete revolution, then reaction at the lowest position is greater than $\frac{34}{7}$ times its own weight.

8(b) Determine the motion, of a spherical pendulum, by using Hamilton's equations.

Sol'n: Let m be the mass of the bob suspended by a light rod of length a. In a spherical pendulum of length a, the path of the motion of the bob is the surface of a sphere of radius a and centre at the fixed point O.

At time t, let $P(a,\theta,\phi)$ be the position of the bob. If (x,y,z) are the cartesian coordinates of P then



$x = a\sin\theta\cos\phi$, $y = a\sin\theta\sin\phi$

$z = a\cos\theta$

$\therefore$ K.E., $T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$

$= \frac{1}{2}ma^2(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta)$

and potential $V = -mgz = -mga\cos\theta$ (since m is below the fixed point O).

$\therefore L = T-V$

$= \frac{1}{2}ma^2(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta) + mga\cos\theta$

$\therefore p_\theta = -\frac{\partial L}{\partial\theta} = ma^2\dot{\theta}$ and

$p_\phi = -\frac{\partial L}{\partial\phi} = ma^2\dot{\phi}\sin^2\theta$ ——— (1).

Since L does not contain t explicity.

$\therefore H = T+V = \frac{1}{2}ma^2(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta) - mga\cos\theta$

Substituting the values of $\dot{\theta}$ and $\dot{\phi}$ from relations (1), we get

$$H = \frac{1}{2ma^2}\left(p_\theta^2 + \operatorname{cosec}^2\theta\, p_\phi^2\right) - mga\cos\theta.$$

Hence the four Hamilton's equations are

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = \frac{1}{ma^2}\operatorname{cosec}^2\theta \cot\theta\, p_\phi^2 - mga\sin\theta \quad \text{———} (H_1)$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{1}{ma^2} p_\theta \quad \text{————} (H_2)$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = \frac{1}{ma^2} p_\theta \quad \text{———} (H_3)$$

and $$\phi = \frac{\partial H}{\partial p_\phi} = \frac{1}{ma^2}\operatorname{cosec}^2\theta\, p_\phi \quad \text{——} (H_4)$$

$\therefore$ from (H_3), Integrating $p_\phi = C$ (constant).

$\therefore$ from (H_4), we have

$$\dot{\phi} = \frac{1}{ma^2} C \operatorname{cosec}^2\theta = A/\sin^2\theta \quad (\text{where } A = C/ma^2)$$

Also from (H_1) & (H_2), we have

$$\ddot{\phi} = \frac{1}{ma^2}\dot{p}_\theta = \frac{1}{ma^2}\left[\frac{1}{ma^2}\frac{\cos\theta}{\sin^3\theta}p_\phi^2 - mga\sin\theta\right]$$

$$= \frac{1}{(ma^2)^2} C^2 \frac{\cos\theta}{\sin^3\theta} - \frac{g}{a}\sin\theta, \quad \because p_\phi = C$$

$$= A^2\frac{\cos\theta}{\sin^3\theta} - \frac{g}{a}\sin\theta \quad (\because A = C/ma^2)$$

Multiplying bothsides by $2\dot{\theta}$ and integrating, we get

$$\dot{\theta}^2 = -\frac{A^2}{\sin^2\theta} + \frac{2g}{a}\cos\theta + B, \quad (B \text{ is Const}). \text{——} (3)$$

Equations (2) and (3) determine the required motion.

8(c) Show that the velocity field

$$u(x,y) = \frac{B(x^2-y^2)}{(x^2+y^2)^2}, \quad v(x,y) = \frac{2Bxy}{(x^2+y^2)^2}, \quad w=0$$

satisfies the equation of motion for an inviscid incompressible flow. Determine the pressure associated with this velocity field.

Sol^n: Euler's equation of motion in absence of external forces

is $\dfrac{dq}{dt} = -\dfrac{1}{\rho}\nabla p$

$$\Rightarrow \left(\frac{\partial}{\partial t} + q\cdot\nabla\right) q = -\frac{1}{\rho}\nabla p$$

But motion is two dimensional as $w=0$ and $q = ui + vj$

$$\therefore \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right) q = -\frac{1}{\rho}\left(i\frac{\partial p}{\partial x} + j\frac{\partial p}{\partial y}\right)$$

Putting the values.

$$\left[\frac{\partial}{\partial t} + \frac{B(x^2-y^2)}{(x^2+y^2)^2}\frac{\partial}{\partial x} + \frac{2Bxy}{(x^2+y^2)^2}\frac{\partial}{\partial y}\right](ui+vj) = -\frac{1}{\rho}\left(i\frac{\partial p}{\partial x} + j\frac{\partial p}{\partial y}\right)$$

As u, v are independent of t, by assumption.

$$\therefore \frac{\partial u}{\partial t} = 0 = \frac{\partial v}{\partial t}.$$

Hence the last gives

$$\frac{B}{(x^2+y^2)^2}\left[(x^2-y^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}\right](ui+vj) = -\frac{1}{\rho}\left(i\frac{\partial p}{\partial x} + j\frac{\partial p}{\partial y}\right)$$

This $\Rightarrow$ $$-\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{B}{(x^2+y^2)^2}\left[(x^2-y^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}\right]\frac{B(x^2-y^2)}{(x^2+y^2)^2} \quad \text{--- (1)}$$

and $$-\frac{1}{\rho}\frac{\partial p}{\partial y} = \frac{B}{(x^2+y^2)^2}\left[(x^2-y^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}\right]\frac{2Bxy}{(x^2+y^2)^2} \quad \text{--- (2)}$$

But $$\frac{\partial}{\partial x}\left\{\frac{x^2-y^2}{(x^2+y^2)^2}\right\} = \frac{2x(3y^2-x^2)}{(x^2+y^2)^3} \quad \text{--- (3)}$$

$$\frac{\partial}{\partial y}\left\{\frac{x^2-y^2}{(x^2+y^2)^2}\right\} = \frac{-2y(3x^2-y^2)}{(x^2+y^2)^3} \text{ --- } (4), \quad \frac{\partial}{\partial x}\left\{\frac{2y(y^2-x^2)}{(x^2+y^2)^3}\right\} \text{ --- } (5)$$

$$\frac{\partial}{\partial y}\left\{\frac{2xy}{(x^2+y^2)^2}\right\} = \frac{2x(x^2-y^2)}{(x^2+y^2)^3} \text{ --- } (6)$$

from equation (1), (3) & (4)

$$\frac{\partial p}{\partial x} = \frac{-2\rho B^2}{(x^2+y^2)^5}\left[(x^2-y^2)x(3y^2-x^2)-2xy^2(3x^2-y^2)\right]$$

$$\Rightarrow \frac{\partial p}{\partial x} = \frac{2\rho B^2 x}{(x^2+y^2)^3} \text{ --- } (7)$$

from (2), (5) & (6)

$$\frac{\partial p}{\partial x} = \frac{2\rho B^2}{(x^2+y^2)^5}\left[(x^2-y^2)y(y^2-x^2)+2x^2y(x^2-y^2)\right]$$

$$\Rightarrow \frac{\partial p}{\partial x} = \frac{2B^2 y}{(x^2+y^2)^4} \text{ --- } (8)$$

Differentiating (7) and (8) partially w.r.t y and x we find that

$$\frac{\partial^2 p}{\partial y\,\partial x} = \frac{\partial^2 p}{\partial x\,\partial y} \quad (\text{Prove it})$$

this proves that velocity field satisfies the equation of motion.

$$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy$$

Using (7) and (8)

$$dp = 2\rho B^2\left[\frac{x\,dx}{(x^2+y^2)^3} - \frac{y(x^2-y^2)}{(x^2+y^2)^4}dy\right]$$

$$= 2\rho B^2\left[M dx + N dy\right], \text{ say} \text{ --- } (9)$$

$$\frac{\partial M}{\partial y} = -\frac{6xy}{(x^2+y^2)^4} = \frac{\partial N}{\partial x}$$

$\therefore M dx + N dy$ is exact.

$$\int M dx + N dy = \int \frac{x\,dx}{(x^2+y^2)^3} + \int 0\,dy$$

$$= \frac{1}{2}\int 2x(x^2+y^2)^{-3}dx + C = -\frac{1}{4(x^2+y^2)^2} + C$$

In view of this (9) becomes.

$$p = -\frac{2\rho B^2}{4(x^2+y^2)^2} + C_1$$

$$\Rightarrow p = -\frac{\rho B^2}{2(x^2+y^2)^2} + C_1$$

This is the required expression for pressure.