Set – VI



# Eigen Values and Eigen Vectors

→ Introduction :–

Let $A = [a_{ij}]_{n\times n}$ be a given n-rowed square matrix

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$ be a column vector.

Now consider the equation

$$AX = \lambda X \quad \text{——— (1)}$$

where $\lambda$ is a scalar.

It is obvious that the zero vector $X=0$ is a solution of (1) for any value of $\lambda$.

→ If $I$ denotes the unit matrix of order $n$, then the equation (1) may be written as

$$AX = \lambda I X$$

$$\Rightarrow (A-\lambda I)X = 0 \quad \text{——— (2)}$$

The matrix equation (2) represents the following system of 'n' homogeneous equations in 'n' unknowns:

$$\left.\begin{array}{l} (a_{11}-\lambda)x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + (a_{22}-\lambda)x_2 + \cdots + a_{2n}x_n = 0 \\ \cdots \\ \cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + (a_{nn}-\lambda)x_n = 0 \end{array}\right\} \rightarrow (3)$$

The coefficient matrix of the equations (3) is $A-\lambda I$. The necessary and sufficient condition for equations (3) to possess a non-zero solution ($X \neq 0$) is that the coefficient matrix $A-\lambda I$ should be of rank less than the number of unknowns 'n'.

But this non-zero solution exists iff the matrix $A-\lambda I$ is singular i.e. iff $|A-\lambda I| = 0$.

→ Let $A = [a_{ij}]_{n\times n}$ be any n-rowed square matrix and $\lambda$ an indeterminate.

— The matrix $A-\lambda I$ is called the characteristic matrix of $A$, where $I$ is the unit matrix of order $n$.

— The determinant

$$|A-\lambda I| = \begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & & \cdots \\ \cdots & \cdots & & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix}$$

which is an ordinary polynomial in $\lambda$ of degree 'n', is called the characteristic polynomial of $A$.

— The equation $|A-\lambda I| = 0$ is called the characteristic equation of $A$ and the roots of this equation are called the characteristic roots (or) characteristic values (or) eigen values (or) latent roots (or) proper roots of the matrix A.

— The set of the eigen values of A is called the spectrum of A.

→ If $\lambda$ is a characteristic root of the matrix A, then $|A-\lambda I| = 0$. and the matrix $A-\lambda I$ is singular.

$\therefore \exists$ a non-zero vector x such that

$(A-\lambda I)x = 0$ (or) $AX = \lambda x$.

## * Characteristic Vectors :—

If $\lambda$ is a characteristic root of an $n\times n$ matrix A, then the non-zero vector x such that $Ax = \lambda x$ is called a characteristic vector (or) eigen vector of A corresponding to the characteristic root $\lambda$.

## → Certain relations b/w Characteristic roots and characteristic vectors :—

**Theorem :** $\lambda$ is a characteristic root of a matrix A iff $\exists$ a non-zero vector x such that $Ax = \lambda x$.

**Proof :-** Suppose $\lambda$ is a characteristic root of the matrix A.

$\therefore |A-\lambda I| = 0$.

and the matrix $A-\lambda I$ is singular.

$\therefore$ The matrix equation $(A-\lambda I)x = 0$ possess a non-zero solution.

i.e. $\exists$ a non-zero vector x such that $(A-\lambda I)x = 0$

$\Rightarrow$ $Ax = \lambda x$.

Conversely suppose that there exists a non-zero vector x such that $Ax = \lambda x$

i.e. $(A-\lambda I)x = 0$

Since the matrix equation $(A-\lambda I)x = 0$ possess a non-zero solution.

$\therefore$ The coefficient matrix $A-\lambda I$ is singular.

i.e. $|A-\lambda I| = 0$

$\therefore \lambda$ is a characteristic root of the matrix A.

**Theorem:** If x is a characteristic vector of a matrix A corresponding to the characteristic value $\lambda$, then $kx$ is also a characteristic vector of A corresponding to the same characteristic value $\lambda$. Here k is a non-zero scalar.

**Proof :-** Suppose $x$ is a charateristic vector of $A$ Corresponding to the characteristic value $\lambda$.

then $x \neq 0$ and $Ax = \lambda x$ ——(1)

If $K$ is a non-zero scalar then $Kx \neq 0$.

Now we have

$$A(Kx) = K(Ax)$$
$$= K(\lambda x)$$
$$= \lambda(Kx) \quad (\text{from } (1))$$

$\therefore A(Kx) = \lambda(Kx)$.

$\therefore$ a non-zero vector $Kx$ such that $A(Kx) = \lambda(Kx)$

$\therefore Kx$ is a charalterstic vector of $A$ Corresponding to the Characteristic value $\lambda$.

$\therefore$ Corresponding to a characteristic value $\lambda$, there Corresponds more than one Characteristic vectors.

**Theorem :-**

If $x$ is a characteristic vector of a matrix $A$, then $x$ cannot Correspond to more than one Characteristic value of $A$.

**Proof :-** Let $x$ be a characteristic vector of a matrix $A$ Corresponding to two characteristic values $\lambda_1$ and $\lambda_2$ then $Ax = \lambda_1 x$, &

$Ax = \lambda_2 x$

$$\Rightarrow \lambda_1 x = \lambda_2 x$$
$$\Rightarrow (\lambda_1 - \lambda_2)x = 0$$
$$\Rightarrow \lambda_1 - \lambda_2 = 0 \quad (\because x \neq 0)$$
$$\Rightarrow \lambda_1 = \lambda_2.$$

2003. **Linear Independence of characteristic vectors Corresponding to distinct characteristic roots:-**

**Statement :**

The characteristic vectors Corresponding to distinct characteristic roots of a matrix are linearly independent.

**Proof :-** Let $x_1, x_2, x_3, \ldots x_m$ be the Characteristic vectors of a matrix $A$ Corresponding to distinct Characteristic Values $\lambda_1, \lambda_2, \ldots \lambda_m$.

Then $Ax_i = \lambda_i x_i$ ; $i = 1, 2, \ldots m$ ——(1)

To Prove that the vectors $x_1, x_2 \ldots x_m$ are linearly independent.

If the vectors $x_1, x_2 \ldots x_m$ are linearly dependent.

then we can choose $r\,(1 \le r < m)$ Such that $x_1, x_2, \ldots x_r$ are L.I and $x_1, x_2 \ldots x_r, x_{r+1}$ are L.D.

$\therefore$ we can choose the Scalars $K_1, K_2, \ldots K_{r+1}$ not all zeros Such that

$K_1X_1 + K_2X_2 + \cdots + K_rX_r + K_{r+1}X_{r+1} = 0$ ——— ②

$\Rightarrow A(K_1X_1 + K_2X_2 + \cdots + K_rX_r + K_{r+1}X_{r+1}) = A(0)$.

$\Rightarrow K_1(AX_1) + K_2(AX_2) + \cdots + K_r(AX_r) + K_{r+1}(AX_{r+1}) = 0$

$\Rightarrow K_1(\lambda_1 X_1) + K_2(\lambda_2 X_2) + \cdots + K_r(\lambda_r X_r) + K_{r+1}(\lambda_{r+1}X_{r+1}) = 0 \longrightarrow$ ③ (by using ①)

Now ③ $- \lambda_{r+1}$② $\equiv$

$K_1(\lambda_1 - \lambda_{r+1})X_1 + \cdots K_r(\lambda_r - \lambda_{r+1})X_r = 0$ ——— ④

Since $X_1, X_2, \cdots X_r$ are L.I and $\lambda_1, \lambda_2, \cdots \lambda_r, \lambda_{r+1}$ are distinct

$\therefore K_1 = 0, K_2 = 0, \cdots K_r = 0$.

putting $K_1 = 0, K_2, \cdots K_r = 0$ in ②

we get $K_{r+1}X_{r+1} = 0$

$\Rightarrow K_{r+1} = 0 \; (\because X_{r+1} \neq 0)$

$\therefore$ from ②, $K_1 = 0, K_2 = 0, \cdots K_r = 0, K_{r+1} = 0$

$\therefore$ which is Contradiction to our assumption that the scalars $K_1, K_2, \cdots, K_r, K_{r+1}$ are not all zeros.

$\therefore$ our assumption that $X_1, X_2 \cdots X_m$ are L.D is wrong.

$\therefore X_1, X_2, X_3 \cdots X_m$ are L.I.

$\therefore X_1, X_2, \cdots X_m$ which corresponds to distinct characteristic roots of A are L.I.

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$\longrightarrow$ show that the characteristic roots of any diagonal matrix are same as its elements in the diagonal.

<u>sol'n</u> :- Let $A = \text{diag}(d_1, d_2, d_3 \cdots d_n)$ then

$$A = \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ 0 & 0 & d_3 & \cdots & 0 \\ - & - & - & \cdots & - \\ - & - & - & \cdots & - \\ 0 & 0 & 0 & \cdots & d_n \end{bmatrix}$$

Its characteristic equation

$|A - \lambda I| = 0$

i.e. $$\begin{vmatrix} d_1-\lambda & 0 & 0 & \cdots & 0 \\ 0 & d_2-\lambda & 0 & \cdots & 0 \\ 0 & 0 & d_3-\lambda & \cdots & 0 \\ - & - & - & \cdots & 0 \\ - & - & - & \cdots & 0 \\ 0 & 0 & 0 & \cdots & d_n-\lambda \end{vmatrix} = 0$$

$\Rightarrow (d_1 - \lambda)(d_2 - \lambda) \cdots (d_n - \lambda) = 0$

$\Rightarrow \lambda = d_1, d_2, \cdots d_n$.

$\therefore$ the elements in the diagonal of A are its characteristic roots.

$\longrightarrow$ Prove that the characteristic roots of a triangular matrix are just the diagonal elements of the matrix.

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ - & - & \cdots & - \\ - & - & \cdots & - \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$ be the triangular matrix.

Its characteristic equation is

$|A-\lambda I| = 0$.

i.e. $$\begin{vmatrix} a_{11}-\lambda & a_{12} & a_{13} \cdots a_{1n} \\ 0 & a_{22}-\lambda & a_{23} \cdots a_{2n} \\ - & - & - \cdots - \\ - & - & - \cdots - \\ 0 & 0 & 0 \cdots a_{nn}-\lambda \end{vmatrix} = 0$$

$\Rightarrow (a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda)=0$

$\Rightarrow \lambda = a_{11}, a_{22}, \cdots a_{nn}$

$\therefore$ The diagonal elements of $A$ are the characteristic roots of $A$.

→ Prove that the square matrices $A$ & $A^T$ have the same characteristic values.

Sol'n:- If $\lambda$ is any scalar then

$$(A-\lambda I)^T = A^T - \lambda I^T$$
$$= A^T - \lambda I \quad \text{--- (1)}$$

Now $|(A-\lambda I)^T| = |A^T - \lambda I|$ (by (1))

$\Rightarrow |A-\lambda I| = |A^T - \lambda I|$ ($\because |A| = |A^T|$)

$\Rightarrow |A-\lambda I| = 0$ iff $|A^T - \lambda I| = 0$

i.e $\lambda$ is a characteristic value of $A \Leftrightarrow \lambda$ is a characteristic value of $A^T$.

→ show that '0' is a characteristic root of a matrix iff the matrix is singular.

Sol'n:- 0 is a characteristic root of A.

$\Leftrightarrow \lambda = 0$ satisfies the equation $|A-\lambda I| = 0$

$\Leftrightarrow |A - 0I| = 0$

$\Leftrightarrow |A| = 0$

$\Leftrightarrow A$ is singular.

Note:-

(1) $\lambda$ is a characteristic root of a non-singular matrix.

$\Rightarrow \lambda \neq 0$.

(2). At least one characteristic root of every singular matrix is zero.

→ If $\lambda$ is a characteristic root of the matrix $A$, show that $K+\lambda$ is a characteristic root of the matrix $A+KI$

Sol'n:- Let $\lambda$ be a characteristic root of the matrix $A$ and $x$ be a corresponding characteristic vector.

Then $Ax = \lambda x$ --- (1)

$$\text{Now } (A+KI)x = Ax + K(Ix)$$
$$= \lambda x + Kx \text{ (by (1))}$$
$$= (\lambda + K)x$$

Since $x \neq 0$, $\lambda + K$ is a characteristic root of the matrix $A+KI$ and $x$ is a corresponding characteristic vector.

→ If $\lambda_1, \lambda_2 \dots \lambda_n$ are the characteristic values of n-rowed square matrix A then show that $K\lambda_1, K\lambda_2, K\lambda_3 \dots K\lambda_n$ are the characteristic values of KA.

Sol'n :- Let $K \neq 0$

Now $|KA-(\lambda K)I| = |K(A-\lambda I)|$

$$= K^n |A-\lambda I|$$

$\Rightarrow |KA-(K\lambda)I| = 0$ iff $|A-\lambda I| = 0$

i.e. $K\lambda$ is a charateristic value of KA iff $\lambda$ is a characteristic value of A.

$\therefore K\lambda_1, K\lambda_2, \dots K\lambda_n$ are the characteristic values of KA iff $\lambda_1, \lambda_2 \dots \lambda_n$ are the characteristic value of A

→ If $\lambda_1, \lambda_2, \dots \lambda_n$ are the characteristic roots of a n-rowed square matrix A and K is a scalar, show that the characteristic roots of $A-KI$ are $\lambda_1 - K, \lambda_2 - K, \dots \lambda_n - K$.

Sol'n :- Since $\lambda_1, \lambda_2 \dots \lambda_n$ are the characteristic roots of the n-rowed square matrix A.

$\therefore \lambda_1, \lambda_2 \dots \lambda_n$ are the n roots of

$$|A-\lambda I| = 0$$

which is a $n^{th}$ degree equation in $\lambda$.

$\therefore (\lambda_1-\lambda)(\lambda_2-\lambda)\dots(\lambda_n-\lambda) = 0$ ①

the characteristic equation of $A-KI$ is

$$|(A-KI)-\lambda I| = 0$$

i.e. $|A-(K+\lambda)I| = 0$

$\therefore$ from ①,

$$[\lambda_1-(K+\lambda)][\lambda_2-(K+\lambda)]\dots[\lambda_n-(K+\lambda)] = 0$$

$$\Rightarrow [(\lambda_1-K)-\lambda][(\lambda_2-K)-\lambda] \dots [(\lambda_n-K)-\lambda] = 0.$$

$\Rightarrow \lambda = (\lambda_1-K), (\lambda_2-K), \dots (\lambda_n-K)$.

which are the characteristic roots of $(A-KI)$.

→ If the characteristic roots of a n-rowed square matrix A are $\lambda_1, \lambda_2, \dots \lambda_n$ then Prove that the characteristic roots of $A^2$ are $\lambda_1^2, \lambda_2^2, \dots \lambda_n^2$.

Sol'n :- Let $\lambda$ be the roots of A and x be a corresponding characteristic vector of A.

Then $Ax = \lambda x$

$\Rightarrow A(Ax) = A(\lambda x)$

$\Rightarrow A^2x = \lambda(Ax)$

$\Rightarrow A^2x = \lambda(\lambda x)$ $(\because Ax = \lambda x)$

$\Rightarrow A^2x = \lambda^2 x$

$\therefore \lambda^2$ is a characteristic root of the matrix $A^2$ corresponding to the characteristic vector $x$ of $A^2$.

$\therefore$ If $\lambda_1, \lambda_2, \dots \lambda_n$ are the characteristic roots of $A$, then $\lambda_1^2, \lambda_2^2, \dots \lambda_n^2$ are the characteristic roots of $A^2$.

→ If the matrix $A$ is non-singular, then show that the eigen values of $A^{-1}$ are the reciprocals of the eigen values of $A$.

Sol'n :- Since $A$ is non-singular

$\therefore A^{-1}$ exists.

Let $\lambda$ be an eigen value of $A$ and $x$ be corresponding eigen vector of $A$.

then $Ax = \lambda x$

$\Rightarrow A^{-1}(Ax) = A^{-1}(\lambda x)$

$\Rightarrow x = \lambda(A^{-1}x)$

$\Rightarrow A^{-1}x = \frac{1}{\lambda}x$ ($\because \lambda \neq 0$; $A$ is non-singular)

$\Rightarrow \frac{1}{\lambda}$ is an eigen value of $A^{-1}$, and $x$ is a corresponding eigen vector.

Converse :-

Let $K$ be an eigen value of $A^{-1}$.

$A$ is non-singular.

$\Rightarrow A^{-1}$ is non-singular and $(A^{-1})^{-1} = A$.

$\therefore \frac{1}{K}$ is an eigen value of $A$.

$\therefore$ each eigen value of $A^{-1}$ is the reciprocal of the eigen value of $A$.

$\therefore$ The eigen values of $A^{-1}$ are nothing but the reciprocals of the eigen values of $A$.

Imp. ② show that the two matrices $A$, $C^{-1}AC$ have the same characteristic roots.

Sol'n :- Let $B = C^{-1}AC$ then

$$\begin{aligned} B - \lambda I &= C^{-1}AC - \lambda I \\ &= C^{-1}AC - C^{-1}\lambda I C \\ &= C^{-1}(A - \lambda I)C \end{aligned}$$

$$\begin{aligned} \therefore |B - \lambda I| &= |C^{-1}(A - \lambda I)C| \\ &= |C^{-1}|\,|A - \lambda I|\,|C| \\ &= |A - \lambda I| \quad \left(\because |C^{-1}| = \frac{1}{|C|}\right) \end{aligned}$$

$\therefore$ B and A

i.e. $C^{-1}AC$ and $A$ have the

same characteristic equations.

i.e. $A$ and $C^{-1}AC$ have the same characteristic roots.

→ The characteristic roots of a Hermitian matrix are real.

Proof :- Suppose $A$ is a Hermitian matrix.

$\lambda$ is a characteristic root of $A$.

and $X$ is a corresponding eigen vector then $AX = \lambda X$ —— (1)

Now pre-multiplying both sides of (1) by $X^\theta$, we get

$$X^\theta AX = \lambda X^\theta X \quad \text{—— (2)}$$

Taking the conjugate transpose of both sides of (2), we get

$$(X^\theta AX)^\theta = (\lambda X^\theta X)^\theta$$

$$\Rightarrow X^\theta A^\theta (X^\theta)^\theta = \bar{\lambda} X^\theta (X^\theta)^\theta$$

$$\Rightarrow X^\theta A^\theta X = \bar{\lambda} X^\theta X \quad (\because (X^\theta)^\theta = X)$$

$$\Rightarrow X^\theta AX = \bar{\lambda} X^\theta X \quad \text{—— (3)} \quad (\because A^\theta = A \text{ because } A \text{ is Hermitian})$$

$\therefore$ From (2) & (3), we have

$$\lambda X^\theta X = \bar{\lambda} X^\theta X$$

$$\Rightarrow (\lambda - \bar{\lambda}) X^\theta X = 0 \quad \text{—— (4)}$$

But $X \neq 0$

$\therefore X^\theta X \neq 0$

(4) $\equiv \lambda - \bar{\lambda} = 0$

$\Rightarrow \lambda = \bar{\lambda}$

$\therefore \lambda$ is real. ($\because \bar{z} = z$ $\therefore z$ is real)

→ Prove that the characteristic roots of a real symmetric matrix are all real.

Sol$^n$ :- Let $A$ be a real symmetric matrix.

$$\therefore A^T = A \quad \text{—— (1)}$$

Consider $A^\theta = (\bar{A})^T$

$= A^T$ ($\because$ the elements of $A$ are all real)

$= A$ (by (1))

$\therefore A^\theta = A$

$\Rightarrow A$ is a Hermitian matrix.

We know that the characteristic roots of a Hermitian matrix are real.

$\therefore$ the characteristic roots of $A$ are all real.

i.e. the characteristic roots of a real symmetric matrix are all real.

→ Prove that the eigen values (characteristic roots) of a skew-Hermitian matrix are either purely imaginary (or) zero.

Sol'n :- Let $A$ be a skew-Hermitian matrix.

$\therefore A^{\theta} = -A$ —— ①

Consider $(iA)^{\theta} = \bar{i}A^{\theta}$

$= -i(-A)$ [$\because \bar{i} = -i$ & by ①]

$= iA$

$\therefore iA$ is a Hermitian matrix.

Let $\lambda$ be an characteristic root of $A$.

$\therefore \exists$ a non-zero column matrix $X$ such that $AX = \lambda X$.

$\Rightarrow i(AX) = i(\lambda X)$

$\Rightarrow (iA)X = (i\lambda)X$

$\Rightarrow i\lambda$ is an eigen value of $iA$, which is Hermitian matrix.

we know that the roots Hermitian matrix are real.

$\therefore i\lambda$ is a real root of $iA$.

$\Rightarrow \lambda$ is either purely imaginary (or) zero.

→ Prove that the eigen values of a real skew-symmetric matrix are either purely imaginary (or) zero.

Sol'n :- Let $A$ be a skew-symmetric matrix.

$\therefore A^T = -A$ —— ①

Consider $A^{\theta} = (\bar{A})^T$

$= A^T$ ($\because$ the elements of $A$ are all real).

$= -A$ (by ①)

$\Rightarrow A$ is a skew-Hermitian matrix.

we know that the characteristic roots of a skew-Hermitian matrix are either purely imaginary (or) zero.

$\therefore$ The characteristic roots of $A$ are either purely imaginary (or) zero.

i.e. the characteristic roots of a real skew-symmetric matrix are either purely imaginary (or) zero.

→ Prove that the eigen values (characteristic roots) of a unitary matrix are of unit modules.

Sol'n :- Let $A$ be an unitary matrix

$\therefore A^{\theta}A = I$ —— ①.

Let $\lambda$ be a characteristic root of $A$.

$\therefore \exists$ a non-zero column matrix. i.e. characteristic vector $X$ such that

$AX = \lambda X$ —— ②

Taking conjugate transpose on the sides, we get

$(AX)^{\theta} = (\lambda X)^{\theta}$

$\Rightarrow X^{\theta}A^{\theta} = \bar{\lambda}X^{\theta}$ —— ③

from ② & ③, we have

$(X^\theta A^\theta) AX = (\bar{\lambda} X^\theta)(\lambda X)$

$\Rightarrow X^\theta (A^\theta A) X = (\bar{\lambda}\lambda)(X^\theta X)$

$\Rightarrow X^\theta (I) X = |\lambda|^2 (X^\theta X)$ (by ①)

$(\because \bar{\lambda}\lambda = |\lambda|^2)$

$\Rightarrow X^\theta X = |\lambda|^2 (X^\theta X)$

$\Rightarrow (1-|\lambda|^2)(X^\theta X) = 0$

$\Rightarrow 1-|\lambda|^2 = 0 \quad (\because X \neq 0 \Rightarrow X^\theta X \neq 0)$

$\Rightarrow |\lambda|^2 = 1$

$\Rightarrow |\lambda| = 1$

→ Prove that the eigen values of an orthogonal matrix are of unit modulus.

Sol'n:- We know that if the elements of a unitary matrix A are all real, then A is said to be an orthogonal matrix. and the eigen values of a unitary matrix are of unit modulus.

∴ Eigen values of an orthogonal matrix also of unit modulus.

→ A real matrix is unitary ⟺ it is orthogonal.

Sol'n:- A be a real matrix then

$A^\theta = (\bar{A})^T$

$= A^T$

Since A is unitary.

$\Leftrightarrow A^\theta A = I \Leftrightarrow A^T A = I$

$\Leftrightarrow$ A is orthogonal.

→ Determine the characteristic roots of the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

Sol'n:- The characteristic matrix of

$$A = A - \lambda I = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0-\lambda & 1 & 2 \\ 1 & 0-\lambda & -1 \\ 2 & -1 & 0-\lambda \end{bmatrix}$$

The characteristic polynomial of

$A = |A - \lambda I|$

$$= \begin{vmatrix} -\lambda & 1 & 2 \\ 1 & -\lambda & -1 \\ 2 & -1 & -\lambda \end{vmatrix} = -\lambda^3 + 6\lambda - 4.$$

∴ The characteristic equation of A is $|A - \lambda I| = 0$

$\Rightarrow -\lambda^3 + 6\lambda - 4 = 0$

$\Rightarrow (\lambda - 2)(\lambda^2 + 2\lambda - 2) = 0$

$\Rightarrow \lambda = 2, \; -1 \pm \sqrt{3}$

∴ The characteristic roots of A are $2, -1 \pm \sqrt{3}$.

→ Prove that ±1 can be the only eigen values of an orthogonal matrix.

Sol'n:- We know that the eigen values of an othogonal matrix are of unit modulus.

Also ±1 are the only real numbers of unit modulus.

$\therefore \pm 1$ are the only real numbers which can be the characteristic roots (or) eigen values of an orthogonal matrix.

→ If $A$ is both real symmetric and orthogonal, prove that all its eigen values are $+1$ or $-1$.

Sol'n :- If $A$ is a real symmetric matrix, then all its eigen values are real.

- If $A$ is orthogonal then all its eigen values must be of unit modulus.

Now $\pm 1$ are the only real numbers of unit modulus.

$\therefore$ If $A$ is both real symmetric and orthogonal then all its eigen values are $+1$ or $-1$.

H.W. → Find the characteristic roots of the 2-rowed orthogonal matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and verify that they are of unit modulus.

→ Show that the roots of the equation

$$\begin{vmatrix} a+x & h & g \\ h & b+x & f \\ g & f & c+x \end{vmatrix} = 0$$ are real

where $a, b, c, f, g, h$ are real numbers.

→ If $\lambda \in C$ is an eigen value of a square matrix $A$, then prove that $\bar{\lambda}$ is an eigen value of $A^\theta$ and conversely. (Or)

Prove that eigen values of $A^\theta$ are the conjugates of eigen values of $A$.

Sol'n :- We know that $\lambda$ is an eigen value of a square matrix $A$

iff $|A-\lambda I| = 0$.

i.e. iff $\overline{|A-\lambda I|} = 0$.

i.e. iff $|\overline{(A-\lambda I)}| = 0$ $(\because \overline{|P|} = |\bar{P}|)$

i.e. iff $|\overline{(A-\lambda I)}^T| = 0$ $(\because |P| = |P^T|)$

i.e. iff $|(A-\lambda I)^\theta| = 0$

i.e. iff $|A^\theta - \bar{\lambda} I| = 0$

i.e. iff $\bar{\lambda}$ is an eigen value of $A^\theta$.

## * The Construction of Orthogonal Matrices:—

Imp. Suppose $S$ is an $n$-rowed real skew-symmetric matrix and $I$ is the unit matrix of order $n$. Then show that

(i) $I-S$ is non-singular.

(ii) $A = (I+S)(I-S)^{-1}$ is orthogonal.

(iii) $A = (I-S)^{-1}(I+S)$

(iv) If $x$ is a characteristic vector of

S corresponding to the characteristic root $\lambda$, then $x$ is also a characteristic vector of $A$ and $\frac{(1+\lambda)}{1-\lambda}$ is the corresponding characteristic root.

Sol'n:- (i) Since $S$ is a real skew-symmetric matrix.

$\therefore$ The characteristic roots of $S$ are either pure imaginary (or) zero.

$\therefore$ The roots of the equation $|S-\lambda I|=0$ are pure imaginary (or) zero.

$\therefore$ 1 is not a root of the equation $|S-\lambda I|=0$.

$\therefore |S-I|\neq 0$

$\therefore (S-I)$ is non-singular.

$\Rightarrow I-S$ is non-singular.

(ii) Let $A=(I+S)(I-S)^{-1}$

then $A^T=\left[(I+S)(I-S)^{-1}\right]^T$

$$=\left[(I-S)^{-1}\right]^T\cdot(I+S)^T$$

$$=\left[(I-S)^T\right]^{-1}(I+S)^T \quad \text{——(1)}$$

Since $(I-S)^T=I^T-S^T$

$$=I+S$$

$(\because S$ is skew-symmetric $S^T=-S)$

and $(I+S)^T=I-S$

$\therefore (1) \equiv$

$$A^T=(I+S)^{-1}\cdot(I-S)$$

Now $A^TA=\left[(I+S)^{-1}(I-S)\right]\left[(I+S)(I-S)^{-1}\right]$

$$=(I+S)^{-1}(I+S)(I-S)(I-S)^{-1}$$

$(\because (I-S)(I+S)=(I+S)(I-S))$

$$=I\cdot I$$

$$=I.$$

$\therefore A$ is orthogonal.

(iii) Since $(I+S)(I-S)=(I-S)(I+S)$

Pre-multiplying throughout by $(I-S)^{-1}$ and post-multiplying throughout by $(I-S)^{-1}$, we get,

$$(I-S)^{-1}(I+S)(I-S)(I-S)^{-1}=(I-S)^{-1}(I-S)(I+S)(I-S)^{-1}$$

$\Rightarrow (I-S)^{-1}(I+S)\,I = I\,(I+S)(I-S)^{-1}$

$\Rightarrow (I-S)^{-1}(I+S)=(I+S)(I-S)^{-1}$

$\Rightarrow (I-S)^{-1}(I+S)=A$ (by (ii))

$\Rightarrow \boxed{A=(I-S)^{-1}(I+S)}$

(iv) Suppose $\lambda$ is a characteristic root of $S$ and $x$ is the corresponding characteristic vector, then

$$Sx=\lambda x$$

$\Rightarrow x+Sx=x+\lambda x$

$\Rightarrow (I+S)x=(1+\lambda)x$ —— (1)

Similarly $(I-S)x=(1-\lambda)x$ —— (2)

Pre multiplying ② throughout by $(I-S)^{-1}$, we get

$$(I-S)^{-1}(I-S)X = (1-\lambda)(I-S)^{-1}X$$

$$\Rightarrow X = (1-\lambda)(I-S)^{-1}X$$

$$\Rightarrow (1-\lambda)^{-1}X = (I-S)^{-1}X \quad (\because 1-\lambda \neq 0)$$

i.e $\lambda \neq 1$

$$\Rightarrow (I-S)^{-1}X = (1-\lambda)^{-1}X \quad \text{--- ③}$$

Now Pre-multiplying ① throughout by $(I-S)^{-1}$, we get

$$(I-S)^{-1}(I+S)X = (1+\lambda)(I-S)^{-1}X$$

$$\Rightarrow \left[(I-S)^{-1}(I+S)\right]X = \left[(1+\lambda)(1-\lambda)^{-1}\right]X$$

(from ③)

$\therefore X$ is a characteristic vector of $A = (I-S)^{-1}(I+S)$ and $(1+\lambda)(1-\lambda)^{-1}$ is the corresponding characteristic root.

H.W. If $S$ is a real skew-symmetric matrix then show that $(I+S)$ is non-singular and $(I-S)(I+S)^{-1}$ is orthogonal.

→ If $A$ is an orthogonal matrix with the property that $-1$ is not a characteristic root, then $A$ is expressible as $(I+S)(I-S)^{-1}$ for some suitable real skew-symmetric matrix '$S$'.

Sol'n :- Given that $A = (I+S)(I-S)^{-1}$ --- ①

Post-multiplying both sides of ① by $(I-S)$, we get

$$A(I-S) = (I+S)$$

$$\Rightarrow A - AS = I + S$$

$$\Rightarrow A - I = AS + S$$

$$\Rightarrow A - I = (A+I)S \quad \text{--- ②}$$

Since $-1$ is not a characteristic root of $A$.

$\therefore |A-(-1)I| \neq 0$.

i.e. $|A+I| \neq 0$

i.e. $A+I$ is non-singular.

$\therefore$ Pre-multiplying bothsides of ② by $(A+I)^{-1}$, we get

$$(A+I)^{-1}(A-I) = S \quad \text{--- ③}$$

Since $A$ is a real matrix

$\therefore S$ is also real matrix.

$\therefore$ we can easily show that '$S$' is a skew-symmetric matrix.

Now we have

$$S^T = \left[(A+I)^{-1}(A-I)\right]^T$$

$$= (A-I)^T\left[(A+I)^{-1}\right]^T$$

$$= (A^T - I^T)\left[A^T + I^T\right]^{-1}$$

$$= (A^T - I)\left[A^T + I\right]^{-1} \quad \text{--- ④}$$

Since $(A^T+I)(A^T-I) = (A^T-I)(A^T+I)$

Pre-multiplying throughout by $(A^T+I)^{-1}$ and Post multiplying throughout by $(A^T+I)^{-1}$, we get

$$(A^T+I)^{-1}(A^T+I)(A^T-I)(A^T+I)^{-1} = (A^T+I)^{-1}(A^T-I)(A^T+I)(A^T+I)^{-1}$$

$$\Rightarrow (A^T-I)(A^T+I)^{-1} = (A^T+I)^{-1}(A^T-I)$$

from (4), we get

$$S^T = (A^T-I)(A^T+I)^{-1}$$

$$= (A^T+A^TA)^{-1}(A^T-A^TA)$$

($\because$ A is orthogonal $\Rightarrow A^TA = I$).

$$\Rightarrow S^T = [A^T(I+A)]^{-1}[A^T(I-A)]$$

$$= (I+A)^{-1}(A^T)^{-1}A^T(I-A)$$

$$= (I+A)^{-1} I (I-A) \quad (\because (A^T)^{-1}A^T = I)$$

$$= (I+A)^{-1}(I-A)$$

$$= -(A+I)^{-1}(A-I)$$

$$= -S$$

$\therefore$ S is a skew-symmetric matrix

H.W. If A is an orthogonal matrix with the property that $-1$ is not a characteristic root, then A is expressible as $(I-S)(I+S)^{-1}$ for some suitable skew-symmetric matrix.

2005 → If S is a skew-Hermitian matrix, show that the matrices $(I-S)$ and $(I+S)$ are both non-singular.

Also show that $A = (I+S)(I-S)^{-1}$ is a unitary matrix.

Sol'n :- Given that 'S' is a skew-Hermitian matrix.

$$\therefore S^\theta = -S \quad \text{——— (1)}$$

we know that the eigen values of a skew-Hermitian matrix 'S' are either purely imaginary (or) zero.

$\therefore$ Neither 1 nor $-1$ is a root of the equation $|S-\lambda I| = 0$.

$\Rightarrow |S-I| \neq 0$ and $|S+I| \neq 0$.

$\Rightarrow |I-S| \neq 0$ and $|I+S| \neq 0$

($\because |-A| \neq 0 \Rightarrow |A| \neq 0$).

$\therefore$ $I-S$ and $I+S$ are both non-singular matrices.

Now given that $A = (I+S)(I-S)^{-1}$

Consider $A^\theta = [(I+S)(I-S)^{-1}]^\theta$

$$= [(I-S)^{-1}]^\theta (I+S)^\theta$$

$$= [(I-S)^\theta]^{-1}(I^\theta + S^\theta)$$

$$= (I^\theta - S^\theta)^{-1}(I-S) \quad (\text{by } (1))$$

$$= (I+S)^{-1}(I-S) \quad (\text{by } (1))$$

$\therefore A^{\theta}A = (I+S)^{-1}(I-S)(I+S)(I-S)^{-1}$

$= (I+S)^{-1}(I+S)(I-S)(I-S)^{-1}$

$[\because (I-S)(I+S) = (I+S)(I-S)]$

$= I.I$

$= I$

$\therefore A^{\theta}A = I$

$\therefore$ A is a unitary matrix.

→ show that every unitary matrix A can, by a suitable choice of skew-Hermitian matrix 'S' be expressed as $A=(I+S)(I-S)^{-1}$ provided that, $-1$ is not a characteristic root of A.

→ If H is a Hermitian matrix show that $A=(I+iH)^{-1}(I-iH)$ is a unitary matrix. Also show that $A=(I-iH)(I+iH)^{-1}$.

Further show that if $\lambda$ is an eigen value of H, then $\frac{(1-i\lambda)}{(1+i\lambda)}$ is an eigen value of A.

Sol'n :- Since H is a Hermitian matrix.

$\therefore H^{\theta}=H$ —①

we know that the characteristic roots of H are real.

$\therefore$ the roots of the equation $|H-\lambda I|=0$ are real.

$\therefore$ Neither $i$ nor $-i$ is a root of the equation $|H-\lambda I| = 0$.

$\Rightarrow |H-iI| \neq 0$ and $|H+iI| \neq 0$

$\Rightarrow (H-iI)$ is non-singular and $(H+iI)$ is non-singular.

$\Rightarrow (iH+I)$ & $(iH-I)$ are also non-singular.

$\Rightarrow (I+iH)$ & $(I-iH)$ also non-singular $(\because |A| \neq 0 \Rightarrow |-A| \neq 0)$

(i) Given that $A=(I+iH)^{-1}(I-iH)$

Consider $A^{\theta}=\left[(I+iH)^{-1}(I-iH)\right]^{\theta}$

$= (I-iH)^{\theta}\left((I+iH)^{-1}\right)^{\theta}$

$=\left[I^{\theta}-(iH)^{\theta}\right]\left[(I+iH)^{\theta}\right]^{-1}$

$= (I-\bar{i}H^{\theta})(I^{\theta}+\bar{i}H^{\theta})^{-1}$

$=(I-(-i)H)(I+(-i)H)^{-1}$ $[\because \bar{i}=-i$ & by ①$]$

$= (I+iH)(I-iH)^{-1}$

$A^{\theta}A = (I+iH)(I-iH)^{-1}(I+iH)^{-1}(I-iH)$

$= (I+iH)(I+iH)^{-1}(I-iH)^{-1}(I-iH)$

$(\because (I+iH)^{-1}(I-iH)^{-1} = (I-iH)^{-1}(I+iH)^{-1})$

$= I.I$

$= I$

$\therefore A^{\theta}A = I$

$\therefore$ A is a unitary matrix.

(ii) To show $A=(I-iH)(I+iH)^{-1}$

Since $(I-iH)(I+iH)=(I+iH)(I-iH)$

Pre-multiplying throughout by $(I+iH)^{-1}$ and post-multiplying throughout by $(I+iH)^{-1}$, we get

$$(I+iH)^{-1}(I-iH)(I+iH)(I+iH)^{-1}=(I+iH)^{-1}(I+iH).(I-iH)(I+iH)^{-1}$$

$$\Rightarrow (I+iH)^{-1}(I-iH)=(I-iH)(I+iH)^{-1}$$

$$\Rightarrow A=(I-iH)(I+iH)^{-1}.$$

(iii) Suppose $\lambda$ is an eigen value of $H$ and $x$ is the corresponding eigen vector of $H$. then

$$Hx=\lambda x$$

$$\Rightarrow iHx=i\lambda x$$

$$\Rightarrow x+iHx=x+i\lambda x$$

$$\Rightarrow (I+iH)x=(1+i\lambda)x \quad \text{—(a)}$$

similarly $(I-iH)x=(1-i\lambda)x$ —(b)

Pre-multiplying (a) throughout by $(I+iH)^{-1}$ we get

$$(I+iH)^{-1}(I+iH)x=(1+i\lambda)(I+iH)^{-1}x$$

$$\Rightarrow x=(1+i\lambda)(I+iH)^{-1}x$$

$$\Rightarrow (1+i\lambda)^{-1}x=(I+iH)^{-1}x$$

$$\Rightarrow (I+iH)^{-1}x=(1+i\lambda)^{-1}x \quad \text{—(c)}$$

Now pre-multiplying (b) throughout by $(I+iH)^{-1}$ we get

$$(I+iH)^{-1}(I-iH)x=(1-i\lambda)(I+iH)^{-1}x$$

$$\Rightarrow [(I+iH)^{-1}(I-iH)]x=(1-i\lambda)(1+i\lambda)^{-1}x \quad \text{(by (c))}$$

$$\Rightarrow [(I+iH)^{-1}(I-iH)]x=[(1-i\lambda)(1+i\lambda)^{-1}]x$$

$\therefore x$ is a characteristic vector of $A=(I+iH)^{-1}(I-iH)$ and $(1-i\lambda)(1+i\lambda)^{-1}$ is the corresponding characteristic root of $A$.

2003 → If $H$ is any Hermitian matrix, then $A=(H+iI)^{-1}(H-iI)=(H-iI)(H+iI)^{-1}$ is unitary and every unitary matrix can be thus expressed provided, $-1$, is not a characteristic root of $A$.

→ Find the eigen roots and corresponding eigen vectors of the matrix $A=\begin{bmatrix}1&4\\3&2\end{bmatrix}$.

Sol'n:– The characteristic matrix of $A=A-\lambda I$

$$=\begin{bmatrix}1&4\\3&2\end{bmatrix}-\lambda\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$$=\begin{bmatrix}1-\lambda&4\\3&2-\lambda\end{bmatrix}$$

The characteristic polynomial of $A=|A-\lambda I|$

$$=\begin{vmatrix}1-\lambda&4\\3&2-\lambda\end{vmatrix}$$

$= (1-\lambda)(2-\lambda) - 12$

$= 2 - 3\lambda + \lambda^2 - 12$

$= \lambda^2 - 3\lambda - 10$

$= (\lambda+2)(\lambda-5)$

The characteristic equation of A is $|A-\lambda I| = 0$

$\Rightarrow (\lambda+2)(\lambda-5) = 0$

$\Rightarrow \lambda = 5, -2$

$\therefore$ the required eigen roots of A are $-2, 5$.

<u>To find the eigen vectors associated with $-2$</u>:

Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be the eigen vector of A corresponding to the eigen value $-2$ then

$(A-(-2)I)X = 0$

$\Rightarrow \begin{bmatrix} 1+2 & 4 \\ 3 & 2+2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 3 & 4 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$R_2 \to R_2 - R_1$

$\Rightarrow \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ —— (1)

$\therefore$ clearly the coefficient is in echelon form.

$\therefore$ The rank of the coefficient matrix $= 1$.

$\therefore$ there is $2-1=1$ L.I eigen vector of A corresponding to eigen root $-2$.

Now from (1),

$3x_1 + 4x_2 = 0$

Let $x_2 = k$ where k is a non-zero parameter.

$\therefore x_1 = -4/3\,k$

$\therefore X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -4/3\,k \\ k \end{bmatrix}$

$= k\begin{bmatrix} -4/3 \\ 1 \end{bmatrix}$

$= kX_1$

Here $X_1 = \begin{bmatrix} -4/3 \\ 1 \end{bmatrix}$ is L.I eigen vector of A corresponding to eigen value $-2$. and the set of all eigen vectors of A corresponding to the eigen value $-2$ is given by $kX_1$.

where k is non-zero parameter.

<u>To find the eigen vectors associated with 5</u>:-

Let $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ be an eigen vector of A corresponding to the eigen value 5 then

$(A-5I)X = 0$

$\Rightarrow \begin{bmatrix} 1-5 & 4 \\ 3 & 2-5 \end{bmatrix} X = 0$

$$\Rightarrow \begin{bmatrix} -4 & 4 \\ 3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \to \frac{1}{3}R_2$

$$\Rightarrow \begin{bmatrix} -4 & 4 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \to 4R_2$

$$\Rightarrow \begin{bmatrix} -4 & 4 \\ 4 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + R_1$

$$\Rightarrow \begin{bmatrix} -4 & 4 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{——} (2)$$

Clearly the coefficient matrix is in echelon form.

$\therefore$ The rank of the coefficient matrix $= 1$.

$\therefore$ There is $2-1=1$ L.I eigen vector of A corresponding to eigen value 5.

Now from (2),

$$-4x_1 + 4x_2 = 0$$

$$\Rightarrow -x_1 + x_2 = 0$$

Let $x_2 = K$

where K is non-zero parameter.

then $x_1 = K$

$$\therefore X = \begin{bmatrix} K \\ K \end{bmatrix} = K\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Here $X = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is L.I eigen vector of A corresponding to eigen value 5.

and the set of all eigen vectors of A corresponding to the eigen value 5 is given by $KX_2$. where K is non-zero parameter.

→ Find all the eigen values and the eigen vectors of the matrix

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix}.$$

**Sol'n** :- The characteristic matrix of $A = A - \lambda I$

$$= \begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ 1 & 1 & 3 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 3-\lambda & 1 & 1 \\ 2 & 4-\lambda & 2 \\ 1 & 1 & 3-\lambda \end{bmatrix}$$

The characteristic polynomial $= |A - \lambda I|$

$$= \begin{vmatrix} 3-\lambda & 1 & 1 \\ 2 & 4-\lambda & 2 \\ 1 & 1 & 3-\lambda \end{vmatrix}$$

$$= (3-\lambda)\left[(4-\lambda)(3-\lambda) - 2\right] - 1\left[6 - 2\lambda - 2\right] + 1\left[2 - 4 + \lambda\right].$$

$$= (3-\lambda)\left[12 - 7\lambda + \lambda^2 - 2\right] - 4 + 2\lambda + \lambda - 2$$

$$= (3-\lambda)\left[\lambda^2 - 7\lambda + 10\right] + 3\lambda - 6$$

$= 3\lambda^2 - 21\lambda + 30 - \lambda^3 + 7\lambda^2 - 10\lambda + 3\lambda - 6$

$= -\lambda^3 + 10\lambda^2 - 28\lambda + 24$

The characteristic equation of A is

$|A - \lambda I| = 0$

$\Rightarrow \lambda^3 - 10\lambda^2 + 28\lambda - 24 = 0$

$\Rightarrow (\lambda-2)^2(\lambda-6) = 0$

$\Rightarrow \lambda = 2, 2, 6.$

<u>To find the eigen vectors associated with 2:</u>

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be an eigen vector of A corresponding to the eigen value 2 then $(A - 2I)X = 0$.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - R_1$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{---} (1)$$

Clearly the coefficient matrix is in echelon form.

$\therefore$ The rank of coefficient matrix $= 1$.

There is $3-1=2$ L.I eigen vectors corresponding to $\lambda = 2$.

from (1), $x_1 + x_2 + x_3 = 0$

Let $x_2 = K_1$, $x_3 = K_2$

where $K_1, K_2$ are parameters and not both zero simultaneously.

then $x_1 = -(K_1 + K_2)$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -K_1 - K_2 \\ 1K_1 + 0K_2 \\ 0K_1 + 1K_2 \end{bmatrix}$$

$$= K_1\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + K_2\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Here $X_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, $X_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ are L.I.

eigen vectors of A corresponding to eigen value $\lambda = 2$

and the set of all eigen vectors of A corresponding to the eigen value 2 is given by $K_1X_1 + K_2X_2$

where $K_1, K_2$ are parameters and both zero simultaneously.

Similarly we can easily find the eigen vectors corresponding to the eigen value $\lambda = 6$.

<u>H.W.</u> Determine the characteristic roots and the corresponding characteristic vectors of the matrix

$$A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$$

## * Matric Polynomial :—

An expression of the form

$F(x) = A_0 + A_1 x + A_2 x^2 + \dots + A_m x^m$,

where $A_0, A_1, A_2, \dots A_m$ are matrices each of order $n \times n$ over a field F, is called a matrix polynomial of degree m, provided $A_m \neq 0$.

— The symbol $x$ is called indeterminate

— the matrices themselves are matric polynomials of zero degree

— Two matric polynomials are equal iff the coefficients of like powers of $x$ are equal.

## * Addition and Multiplication of polynomials :—

Let $G(x) = A_0 + A_1 x + A_2 x^2 + \dots + A_k x^k + \dots + A_m x^m$ and

$H(x) = B_0 + B_1 x + B_2 x^2 + \dots + B_m x^m + \dots + B_k x^k$.

→ we define : — if $m > k$ then

$G(x) + H(x) =$

$(A_0 + B_0) + (A_1 + B_1) x + \dots + (A_k + B_k) x^k + A_{k+1} x^{k+1} + A_{k+2} x^{k+2} + \dots + A_m x^m$.

— if $m < k$ then

$G(x) + H(x) = (A_0 + B_0) + (A_1 + B_1) x + (A_2 + B_2) x^2 + \dots + (A_m + B_m) x^m + B_{m+1} x^{m+1} + \dots + B_k x^k$.

— if $m = k$ then $G(x) + H(x) =$

$(A_0 + B_0) + (A_1 + B_1) x^1 + (A_2 + B_2) x^2 + \dots + (A_k + B_k) x^k$.

→ $G(x) \cdot H(x) = A_0 B_0 + (A_0 B_1 + A_1 B_0) x + (A_0 B_2 + A_1 B_1 + A_2 B_0) + \dots + A_k B_m x^{k+m}$

Note :- (1). The degree of the product of two matric polynomials is less than or equal to the sum of their degrees.

Imp
(2) Every square matrix over a field F whose elements are ordinary polynomials in $x$ over F, can essentially be expressed as a matric polynomial in $x$ of degree m, where m is the index of the highest power of $x$ occuring in any element of the matrix.

Ex:— Let

$$A = \begin{bmatrix} 1+2x+3x^2 & x^2 & 6-4x \\ 1+x^3 & 3-4x^2 & 1-2x+4x^3 \\ 2-3x+2x^3 & 5 & 6x^2+7x \end{bmatrix}$$

then

$$A=\begin{bmatrix} 1+2x+3x^2+0x^3 & 0+0x+x^2+0x^3 & 6-4x+0x^2+0x^3 \\ 1+0x+0x^2+x^3 & 3+0x-4x^2+0x^3 & 1-2x+0x^2+4x^3 \\ 2-3x+0x^2+2x^3 & 5+0x+0x^2+0x^3 & 0+7x+6x^2+0x^3 \end{bmatrix}$$

$$=\begin{bmatrix} 1 & 0 & 6 \\ 1 & 3 & 1 \\ 2 & 5 & 0 \end{bmatrix}+\begin{bmatrix} 2 & 0 & -4 \\ 0 & 0 & -2 \\ -3 & 0 & 7 \end{bmatrix}x+\begin{bmatrix} 3 & 1 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 6 \end{bmatrix}x^2+\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 4 \\ 2 & 0 & 0 \end{bmatrix}x^3$$

which is a matrix polynomial of degree 8.

## * The Cayley-Hamilton Theorem

**Statement**: Every square matrix satisfies its Characteristic equation.

**Proof**:- Let

$$A=\begin{bmatrix} a_{11} & a_{12} \cdots a_{1n} \\ a_{21} & a_{22} \cdots a_{2n} \\ \cdots & \cdots\cdots \\ a_{n1} & a_{n2} \cdots a_{nn} \end{bmatrix}_{n\times n}, \quad I=\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{n\times n}$$

then the Characteristic matrix of A is

$$A-\lambda I=\begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{bmatrix}$$

the characteristic polynomial of A is $|A-\lambda I|$

$$=\begin{vmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22}-\lambda & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn}-\lambda \end{vmatrix}$$

$$=(-1)^n\left[\lambda^n+a_1\lambda^{n-1}+a_2\lambda^{n-2}+\cdots+a_n\right] \text{ (say)} \quad \text{--- (1)}$$

where $a_1, a_2 \cdots a_n \in F$

the characteristic equation of A is

$|A-\lambda I|=0$

i.e. $\lambda^n+a_1\lambda^{n-1}+a_2\lambda^{n-2}+\cdots+a_{n-1}\lambda+a_n=0$

Now we prove that

$A^n+a_1A^{n-1}+a_2A^{n-2}+\cdots+a_{n-1}A+a_nI=0$

Since all the elements of $A-\lambda I$ are at most of first degree in $\lambda$, all the elements of adj $(A-\lambda I)$ are polynomials in $\lambda$ of degree $(n-1)$ (or) less.

($\because$ elements of adj$(A-\lambda I)$ are Cofactors of the elements of $(A-\lambda I)$).

$\therefore$ adj $(A-\lambda I)$ Can be written as a matrix polynomials in $\lambda$ of degree $(n-1)$.

Let adj $(A-\lambda I)=B_0\lambda^{n-1}+B_1\lambda^{n-2}+\cdots+B_{n-2}\lambda^1+B_{n-1}$.

where $B_0, B_1, B_2 \cdots B_{n-1}$ are square matrices of order n. --- (2)

Now we have

$(A-\lambda I)\,\text{adj}\,(A-\lambda I)=|A-\lambda I|I$

($\because A\,\text{adj}A=|A|I$)

$\Rightarrow (A-\lambda I)(B_0\lambda^{n-1}+B_1\lambda^{n-2}+\cdots+B_{n-2}\lambda+B_{n-1})$.

$$= (-1)^n\left[\lambda^n + a_1\lambda^{n-1} + \cdots + a_n\right]I$$
(by ① & ②)

Comparing coefficients of like powers of $\lambda$, we obtain,

$$-B_0 = (-1)^n I$$
$$AB_0 - B_1 = (-1)^n a_1 I$$
$$AB_1 - B_2 = (-1)^n a_2 I$$
$$\cdots\cdots$$
$$-AB_{n-1} = (-1)^n a_n I.$$

Pre-multiplying the above equations. Successively by $A^n, A^{n-1}, \ldots I$ and adding, we obtain

$$0 = (-1)^n\left[A^n + a_1A^{n-1} + a_2A^{n-2} + \cdots + a_nI\right]$$
$$\Rightarrow A^n + a_1A^{n-1} + a_2A^{n-2} + \cdots + a_nI = 0 \quad \text{③}$$

$\therefore$ A satisfies the characteristic equation.

**Note** :— If A is a non-singular matrix then $|A| \neq 0$.

Also $|A| = (-1)^n a_n$

$\therefore a_n \neq 0$.

Now Pre-multiplying ③ by $A^{-1}$, we get

$$A^{-1}\left[A^n + a_1A^{n-1} + a_2A^{n-2} + \cdots + a_{n-1}A + a_nI\right] = 0$$
$$\Rightarrow A^{n-1} + a_1A^{n-2} + a_2A^{n-3} + \cdots + a_{n-1}I + a_nA^{-1}I = 0$$
$$\Rightarrow a_nA^{-1} = -\left[A^{n-1} + a_1A^{n-2} + \cdots + a_{n-1}I\right]$$
$$\Rightarrow \boxed{A^{-1} = \frac{(-1)}{a_n}\left[A^{n-1} + a_1A^{n-2} + \cdots + a_{n-1}I\right]}$$

**Problems** :-

2006 → state Cayley – Hamilton theorem and using it, find the inverse of $\begin{bmatrix}1 & 3\\ 2 & 4\end{bmatrix}$.

**Sol<sup></sup>n** :- **Statement** : Every square matrix satisfies its characteristic equation.

Let $A = \begin{bmatrix}1 & 3\\ 2 & 4\end{bmatrix}$, $I = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$

The characteristic matrix of A is

$$(A - \lambda I) = \begin{bmatrix}1-\lambda & 3\\ 2 & 4-\lambda\end{bmatrix}$$

The characteristic polynomial of A is $|A - \lambda I|$

$$= \begin{vmatrix}1-\lambda & 3\\ 2 & 4-\lambda\end{vmatrix}$$
$$= 4 - 5\lambda + \lambda^2 - 6$$
$$= \lambda^2 - 5\lambda - 2$$

The characteristic equation of A is

$$|A - \lambda I| = 0$$
$$\Rightarrow \lambda^2 - 5\lambda - 2 = 0$$

The given matrix A satisfies the characteristic equation.

$$\therefore A^2 - 5A - 2I = 0 \quad \text{①}$$

Now multiplying ① by $A^{-1}$, we get

$$A - 5I - 2A^{-1} = 0$$
$$\Rightarrow 2A^{-1} = A - 5I$$
$$\Rightarrow A^{-1} = \tfrac{1}{2}[A - 5I] \quad \text{②}$$

$$A-5I = \begin{bmatrix}1 & 3\\ 2 & 4\end{bmatrix} - 5\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}-4 & 3\\ 2 & -1\end{bmatrix}$$

$\therefore$ ② $\equiv$

$$A^{-1} = \frac{1}{2}\begin{bmatrix}-4 & 3\\ 2 & -1\end{bmatrix}$$

→ Find the characteristic equation of the matrix

$$A = \begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$$ and verify that it is satisfied by A and hence obtain $A^{-1}$.

Sol$^n$ :- Given that $A = \begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$

The characteristic matrix of A is

$$A-\lambda I = \begin{bmatrix}2-\lambda & -1 & 1\\ -1 & 2-\lambda & -1\\ 1 & -1 & 2-\lambda\end{bmatrix}$$

The characteristic polynomial of A is $|A-\lambda I| = \begin{vmatrix}2-\lambda & -1 & 1\\ -1 & 2-\lambda & -1\\ 1 & -1 & 2-\lambda\end{vmatrix}$

$$= -\lambda^3+6\lambda^2-9\lambda+4$$

$\therefore$ The characteristic equation of A is $|A-\lambda I| = 0$

i.e. $-\lambda^3+6\lambda^2-9\lambda+4 = 0$

$\Rightarrow \lambda^3-6\lambda^2+9\lambda-4 = 0$

By the cayley- Hamilton theorem

$$A^3-6A^2+9A-4I = 0$$

Now we verify that

$$A^3-6A^2+9A-4I = 0 \quad \text{—— ①}$$

$I = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$, $A = \begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$

$A^2 = \begin{bmatrix}6 & -5 & 5\\ -5 & 6 & -5\\ 5 & -5 & 6\end{bmatrix}$, $A^3 = A^2.A = \begin{bmatrix}22 & -21 & 21\\ -21 & 22 & -21\\ 21 & -21 & 22\end{bmatrix}$

$\therefore$ ① $\equiv$

$$A^3-6A^2+9A-4I = \begin{bmatrix}22 & -21 & 21\\ -21 & 22 & -21\\ 21 & -21 & 22\end{bmatrix} - 6\begin{bmatrix}6 & -5 & 5\\ -5 & 6 & -5\\ 5 & -5 & 6\end{bmatrix} + 9\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix} - 4\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$= 0$$

Now multiplying ① by $A^{-1}$, we get

$$A^2-6A+9I-4A^{-1} = 0$$

$$\Rightarrow A^{-1} = \frac{1}{4}\left[A^2-6A+9I\right] \quad \text{—— ②}$$

Now $A^2-6A+9I =$

$$\begin{bmatrix}6 & -5 & 5\\ -5 & 6 & -5\\ 5 & -5 & 6\end{bmatrix} - 6\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix} + 9\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

$\therefore$ from ②,

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$$

→ Show that the matrix $A = \begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$ satisfies Cayley-Hamilton theorem. (i.e. verification of C-H theorem.)

RPS 2014 Find the characteristic roots of the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and verify Cayley-Hamilton theorem for this matrix. Find the inverse of the matrix A and also express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$

2004 → Find the characteristic polynomial of the matrix $A = \begin{bmatrix} 1 & 1 \\ -3 & 3 \end{bmatrix}$. Hence find $A^{-1}$ and $A^6$.

2002 → Use Cayley-Hamilton theorem, to find the inverse of the following matrix.

$$\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$

→ If $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ then show that for every integer $n \geq 3$, $A^n = A^{n-2} + A^2 - I$. Hence determine $A^{50}$.

Sol'n :- If $n=3$ then

$$A^3 = A + A^2 - I \quad \text{——— ①}$$

Since $A^2 = A \cdot A$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Now $A^3 = A^2 \cdot A$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

Now $A + A^2 - I$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$\therefore A^n = A^{n-2} + A^2 - I$ is true for $n=3$

Suppose $A^n = A^{n-2} + A^2 - I$ is true for $n=k$

$\therefore A^k = A^{k-2} + A^2 - I$.

Now for $n = k+1$;

$$A^{k+1} = A \cdot A^{k}$$

$$= A \cdot [A^{k-2} + A^2 - I]$$

$$= A^{k-1} + A^3 - A$$

$$= A^{k-1} + A + A^2 - I - A$$

(from ①)

$$= A^{k-1} + A^2 - I$$

$\therefore A^n = A^{n-2} + A^2 - I$ is true for $n = k+1$.

$\therefore$ By mathematical induction, it is true for every integer $n \geq 3$,

$$A^n = A^{n-2} + A^2 - I \quad \text{— ②}$$

Now $A^3 = A + A^2 - I$

$$\boxed{A^4 = 2A^2 - I}$$

$$A^6 = A^4 + A^2 - I$$

$$\boxed{A^6 = 3A^2 - 2I}$$

$$A^8 = A^6 + A^2 - I$$

$$\boxed{A^8 = 4A^2 - 3I}$$

$$A^{10} = A^8 + A^2 - I$$

$$\boxed{A^{10} = 5A^2 - 4I}$$

Similarly $A^{12} = 6A^2 - 5I$

$\left(\text{i.e } A^{12} = \frac{12}{2}A^2 - \left(\frac{12}{2} - 1\right)I\right)$

$\vdots$

$$A^{50} = 25A^2 - 24I$$

$\left(\text{i.e. } A^{50} = \frac{50}{2}A^2 - \left(\frac{50}{2} - 1\right)I\right)$

$$\therefore A^{50} = 25\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} - 24\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{bmatrix}$$