2014 IFOS

1. Show that the general solution of the pde

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}$$

is of the form $Z(x,y) = F(x+ct) + G(x-ct)$, where F and G are arbitrary functions.

Solution: Given 1D wave equation

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2} \quad \text{--- (1)}$$

Let $u = x+ct$ and $v = x-ct$ so that z is a function of u and v and

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

Now, $$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$$

and $$\frac{\partial^2 z}{\partial t^2} = c^2\left(\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u\partial v} + \frac{\partial^2 z}{\partial v^2}\right)$$

$\therefore$ eqn (1) becomes

$$\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u\partial v} + \frac{\partial^2 z}{\partial v^2} = \frac{c^2}{c^2}\left(\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right)$$

$\longrightarrow$

contd

$$\Rightarrow \frac{\partial^2 z}{\partial u \partial v} = 0 \quad \text{--- (2)}$$

On integrating (2) w.r.t. v, we have

$$\frac{\partial z}{\partial u} = f(u) \quad \text{--- (3)}$$, where $f(u)$ is an arbitrary function of u.

Integrating (3) w.r.t. u gives

$$z = \int f(u)\,du + g(v),$$ where $g(v)$ is an arbitrary function of v.

Since the integral is a function of u alone, we can write

$$z = F(u) + G(v)$$

$\Rightarrow z(x,t) = F(x+ct) + G(x-ct)$, where F and G are arbitrary functions.

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Verify that the d.e.

$$(y^2+yz)dx + (xz+z^2)dy + (y^2-xy)dz = 0$$

is integrable and find its primitive.

Solution: Here, $(y^2+yz)dx + (xz+z^2)dy + (y^2-xy)dz = 0$

We have, $P = y^2+yz$, $Q = xz+z^2$ and $R = y^2-xy$

Now, $P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$

$= (y^2+yz)(x+2z-2y+x) + (xz+z^2)(-y-y) + (y^2-xy)(2y+z-z)$

$= 2(y^2+yz)(x+z-y) - 2y(xz+z^2) + 2y(y^2-xy)$

$= 2xy^2 + 2y^2z - 2y^3 + 2xyz + 2yz^2 - 2y^2z - 2xyz - 2yz^2 + 2y^3 - 2xy^2$

$= 0$

Thus the condition of integrability is satisfied.

Let $z =$ constant so that $dz = 0$

The given equation becomes

$(y^2+yz)dx + (xz+z^2)dy = 0$

$\Rightarrow y(y+z)dx + z(x+z)dy = 0$

contd.

$\Rightarrow \dfrac{dx}{x+z} + \dfrac{z\,dy}{y(y+z)} = 0$

$\Rightarrow \dfrac{dx}{x+z} + \left\{\dfrac{1}{y} - \dfrac{1}{y+z}\right\}dy = 0$

Integrating, we get

$\log(x+z) + \log y - \log(y+z) = \phi \quad \text{—— (1)}$

We know that

$\lambda P = \dfrac{\partial \phi}{\partial x}$

$\Rightarrow \lambda(y^2+yz) = \dfrac{\partial \phi}{\partial x}$

$\Rightarrow \lambda y(y+z) = \dfrac{1}{x+z}$

$\lambda = \dfrac{1}{y(x+z)(y+z)}$

Now $S = \lambda R - \dfrac{\partial \phi}{\partial z}$

$= \dfrac{y(y-x)}{y(x+z)(y+z)} - \dfrac{1}{x+z} + \dfrac{1}{y+z}$

$= \dfrac{y-x-y-z+x+z}{(x+z)(y+z)}$

$= 0$

contd

$\therefore$ The equation $d\phi + S\,dz = 0$ becomes

$$d\phi = 0$$

$$\Rightarrow \phi = \log C$$

Using (1), we have

$$\log(x+z) + \log y - \log(y+z) = \log C$$

$$\Rightarrow \log\left\{\frac{y(x+z)}{(y-z)}\right\} = \log C$$

$$\Rightarrow \frac{y(x+z)}{(y-z)} = C$$

$\Rightarrow y(x+z) = c(y-z)$ is the required solution.

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is integrable and find its primitive.

Solution: Here, $(y^2+yz)dx + (xz+z^2)dy + (y^2-xy)dz = 0$

We have, $P = y^2+yz$, $Q = xz+z^2$ and $R = y^2-xy$

Now, $P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$

$= (y^2+yz)(x+2z-2y+x) + (xz+z^2)(-y-y) + (y^2-xy)(2y+z-z)$

$= 2(y^2+yz)(x+z-y) - 2y(xz+z^2) + 2y(y^2-xy)$

$= 2xy^2 + 2y^2z - 2y^3 + 2xyz + 2yz^2 - 2y^2z - 2xyz - 2yz^2 + 2y^3 - 2xy^2$

$= 0$

Thus the condition of integrability is satisfied.

Let z = constant so that $dz = 0$

The given equation becomes

$$(y^2+yz)dx + (xz+z^2)dy = 0$$

$$\Rightarrow y(y+z)dx + z(x+z)dy = 0$$

$\rightarrow$

contd.

$$\Rightarrow \frac{dx}{x+z} + \frac{z\,dy}{y(y+z)} = 0$$

$$\Rightarrow \frac{dx}{x+z} + \left\{ \frac{1}{y} - \frac{1}{y+z} \right\} dy = 0$$

Integrating, we get

$$\log(x+z) + \log y - \log(y+z) = \phi \quad \text{--- (1)}$$

We know that

$$\lambda P = \frac{\partial \phi}{\partial x}$$

$$\Rightarrow \lambda(y^2 + yz) = \frac{\partial \phi}{\partial x}$$

$$\Rightarrow \lambda y(y+z) = \frac{1}{x+z}$$

$$\therefore \lambda = \frac{1}{y(x+z)(y+z)}$$

Now $S = \lambda R - \frac{\partial \phi}{\partial z}$

$$= \frac{y(y-x)}{y(x+z)(y+z)} - \frac{1}{x+z} + \frac{1}{y+z}$$

$$= \frac{y - x - y - z + x + z}{(x+z)(y+z)}$$

$$= 0$$