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7(a) State Stoke's theorem. Verify the stoke's theorem for the function

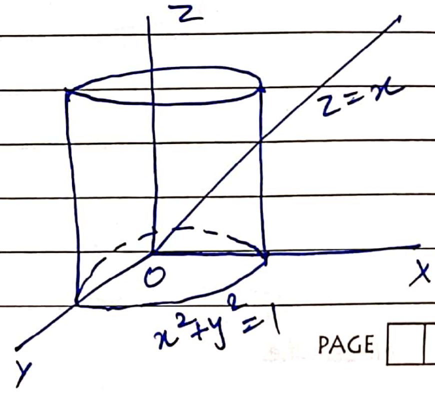
$$\vec{F} = xi + zj + 2yk,$$

where C is the curve obtained by the intersection of plane $z = x$ and the cylinder $x^2+y^2=1$ and S is the surface inside the intersected one. (15)

Stoke's Theorem: Let S be a closed surface, bounded by curve C, then

$$\oint_C \vec{F}\cdot d r = \iint_S (\nabla\times\vec{F})\cdot\hat{n}\, ds$$

$\hat{n}$ is outward unit normal to the surface.



Here, $\vec{F} = xi + zj + 2yk$

$r = xi + yj + zk$

$d\vec{r} = dxi + dyj + dzk$

$\vec{F}\cdot dr = xdx + zdy + 2ydz$

Surface S is intersection of cylinder $x^2+y^2=1$ and plane $x = z$ (passing through y-axis)

Boundary Curve C is : $x^2+y^2=1$ & $z=x$

parameterizing C : $x = \cos\theta$, $y = \sin\theta$

$0 \le \theta < 2\pi$

$$\oint_C F\cdot dr = \oint_C xdx + zdy + 2ydz$$

$$= \int_0^{2\pi} (\cos\theta)(-\sin\theta)d\theta + \cos\theta\cdot\cos\theta d\theta + 2\sin\theta(-\sin\theta)d\theta$$

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$$\oint \vec{F}\cdot d\vec{r} = \int_0^{2\pi}\left[-\frac{1}{2}\sin 2\theta + \left(\frac{1+\cos 2\theta}{2}\right) - 2\left(\frac{1-\cos 2\theta}{2}\right)\right]d\theta$$

$$= \int_0^{2\pi}\left(-\frac{1}{2}\sin 2\theta + \frac{3}{2}\cos 2\theta - \frac{1}{2}\right)d\theta$$

$$= \left[\frac{1}{4}\cos 2\theta + \frac{3}{4}\sin 2\theta - \frac{\theta}{2}\right]_0^{2\pi}$$

$$= -\pi$$

Now, $$\nabla\times\vec{F} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & z & 2y \end{vmatrix}$$

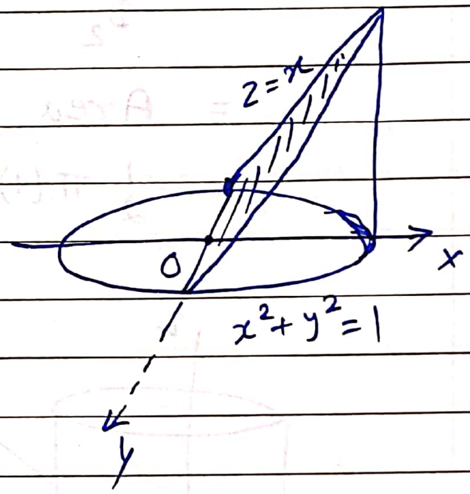
$$= i(2-1) + j(0-0) + k(0-0)$$

$$= i$$

$S:\ x - z = 0$

$$\hat{n} = \frac{\nabla S}{|\nabla S|}$$

$$= \frac{1}{\sqrt{2}}(i-k)$$



$$\iint_S (\nabla\times F)\cdot\hat{n}\,dS$$

$$= \iint_D i\cdot\left(\frac{i-k}{\sqrt{2}}\right)\frac{dx\,dy}{(\hat{n}\cdot k)}$$ (Taking Projection on xy-plane)

$D:\ x^2+y^2 \le 1$

$$= \iint_D \frac{1}{\cancel{\sqrt{2}}}\cdot\frac{dx\,dy}{-1/\cancel{\sqrt{2}}} = -\iint_D dx\,dy$$

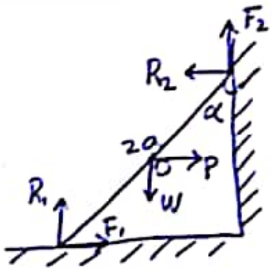
$= -$ Area of unit circle D

$$= -\pi(1)^2 = -\pi$$

⑦ (b)

$\underline{\mu = \tan\lambda}$

Let length $= 2a$ (say)



$F_1 = \mu R_1 \quad -①$

$F_2 = \mu R_2 \quad -②$

Forces:

$R_1 + F_2 = W \quad -③$

$F_1 + P = R_2 \quad -④$

Moments about O $\rightarrow R_1(a\sin\alpha) = R_2(a\cos\alpha) + F_1(a\cos\alpha) + F_2(a\sin\alpha)$

$\Rightarrow R_1(\sin\alpha - \mu\cos\alpha) = R_2(\cos\alpha + \mu\sin\alpha)$ (from ③ & ④)

$\Rightarrow R_2 = R_1 \times \dfrac{(\tan\alpha - \mu)}{(1+\mu\tan\alpha)}$

$\Rightarrow R_2 = R_1\tan(\alpha-\lambda) \quad -⑤ \quad (\because \mu = \tan\lambda)$

$③ \equiv R_1 + \mu R_2 = W$ &

$④ \equiv \mu R_1 + P = R_2$

Using ⑤,

$\Rightarrow R_1 + \mu\tan(\alpha-\lambda)R_1 = W \quad -⑥$

& $\mu R_1 + P = R_1\tan(\alpha-\lambda)$

$\Rightarrow P = R_1(\tan(\alpha-\lambda) - \mu) \quad -⑦$

$\dfrac{⑦}{⑥} \Rightarrow \dfrac{P}{W} = \dfrac{(\tan(\alpha-\lambda) - \mu)}{1 + \mu\tan(\alpha-\lambda)}$

$\Rightarrow \boxed{P = W\tan(\alpha - 2\lambda)} \quad (\because \mu = \tan\lambda)$

Condition is that P should be +ve,

$\Rightarrow \alpha > 2\lambda$.

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7(c) Obtain the singular solution of the DE

$$y^2 - 2pxy + p^2(x^2-1) = m^2, \quad p = \frac{dy}{dx}$$

(10)

$$y^2 - 2pxy + p^2x^2 = m^2 + p^2$$

$$(y - px)^2 = p^2 + m^2$$

$$y = px \pm \sqrt{p^2+m^2}$$

It is in Clairaut's form: $y = px + f(p)$

To get the solution, we replace p by arbitrary constant c.

$$y = cx \pm \sqrt{c^2+m^2}$$

or $$y^2 - 2cxy + c^2x^2 = c^2 + m^2$$

$$c^2(x^2-1) - 2cxy + y^2 - m^2 = 0$$

C-Discriminant : $B^2 - 4AC$

$$A = x^2 - 1, \quad B = -2xy, \quad C = y^2 - m^2$$

$$\therefore B^2 - 4AC = (-2xy)^2 - 4(x^2-1)(y^2-m^2)$$

$$= \cancel{4x^2y^2} - \cancel{4x^2y^2} + 4y^2 + 4x^2m^2 - 4m^2$$

$$= 4(y^2 + m^2(x^2-1))$$

$B^2 - 4AC = 0$ i.e. $y^2 + m^2(x^2-1)$ is the required singular solution of the given D.E.

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