Previous Years' Paper (Solved)

IFS Main Examination, 2018

MATHEMATICS

PAPER-I

INSTRUCTIONS: *There are* **eight** *questions in all, out of which* **five** *are to be attempted. Question Nos. 1 and 5 are compulsory. Out of the remaining six questions,* **three** *are to be attempted selecting at least* **one** *question from each of the two Sections A and B. Answers must be written in* **English** *only.*

SECTION–A

1. (*a*) Show that the maximum rectangle inscribed in a circle is a square. **(8)**

(*b*) Given that Adj A = $\begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and det A = 2. Find the matrix A. **(8)**

(*c*) If $f : [a, b] \to R$ be continuous in $[a, b]$ and derivable in (a, b), where $0 < a < b$, show that for $c \in (a, b)$

$$f(b) - f(a) = cf'(c) \log (b/a).$$ **(8)**

(*d*) Find the equations of the tangent planes to the ellipsoid

$$2x^2 + 6y^2 + 3z^2 = 27$$

which pass through the line

$$x - y - z = 0 = x - y + 2z - 9.$$ **(8)**

(*e*) Prove that the eigenvalues of a Hermitian matrix are all real. **(8)**

2. (*a*) Find the equation of the cylinder whose generators are parallel to the line $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$ and whose guiding curve is $x^2 + y^2 = 4$, $z = 2$. **(10)**

(*b*) Show that the matrices

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 2 \\ 3 & 2 & 0 \end{bmatrix} \text{ are}$$

congruent. **(10)**

(*c*) If ϕ and ψ be two functions derivable in $[a, b]$ and $\phi(x)\,\psi'(x) - \psi(x)\,\phi'(x) > 0$ for any x in this interval, then show that between two consecutive roots of $\phi(x) = 0$ in $[a, b]$, there lies exactly one root of $\psi(x) = 0$. **(10)**

(*d*) Show that the vectors $\alpha_1 = (1, 0, -1)$, $\alpha_2 = (1, 2, 1)$, $\alpha_3 = (0, -3, 2)$ form a basis for R^3. Express each of the standard basis vectors as a linear combination of α_1, α_2, α_3. **(10)**

3. (*a*) Find the equation of the tangent plane that can be drawn to the sphere

$$x^2 + y^2 + z^2 - 2x + 6y + 2z + 8 = 0,$$

through the straight line

$$3x - 4y - 8 = 0 = y - 3z + 2.$$ **(10)**

(*b*) If $f = f(u, v)$, where $u = e^x \cos y$ and $v = e^x \sin y$, show that

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = (u^2 + v^2)\left(\frac{\partial^2 f}{\partial u^2} + \frac{\partial^2 f}{\partial v^2}\right).$$ **(10)**

(*c*) Let $T : V_2(R) \to V_2(R)$ be a linear transformation defined by $T(a, b) = (a, a + b)$. Find the matrix of T, taking $\{e_1, e_2\}$ as a basis for the domain and $\{(1, 1), (1, -1)\}$ as a basis for the range. **(10)**

(*d*) Evaluate $\iint_R (x^2 + xy)\,dx\,dy$ over the region R bounded by $xy = 1$, $y = 0$, $y = x$ and $x = 2$. **(10)**

4. (*a*) Find the equations of the straight lines in which the plane $2x + y - z = 0$ cuts the cone $4x^2 - y^2 + 3z^2 = 0$. Find the angle between the two straight lines. **(10)**

(*b*) Show that the functions $u = x + y + z$, $v = xy + yz + zx$ and $w = x^3 + y^3 + z^3 - 3xyz$ are dependent and find the relation between them. **(10)**

(*c*) Find the locus of the point of intersection of the perpendicular generators of the hyperbolic paraboloid $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 2z$. **(10)**

(*d*) If $(n + 1)$ vectors $\alpha_1, \alpha_2, ..., \alpha_n, \alpha$ form a linearly dependent set, then show that the vector α is a linear combination of $\alpha_1, \alpha_2, \alpha_n$; provided $\alpha_1, \alpha_2, ..., \alpha_n$ form a linearly independent set. **(10)**

SECTION–B

5. (*a*) Find the complementary function and particular integral for the equation

$$\frac{d^2y}{dx^2} - y = xe^x + \cos^2 x$$

and hence the general solution of the equation. **(8)**

(*b*) Solve $\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx} + y = xe^x \log x\,(x > 0)$ by the method of variation of parameters. **(8)**

(*c*) If the velocities in a simple harmonic motion at distances a, b and c from a fixed point on the straight line which is not the centre of force, are u, v and w respectively, show that the periodic time T is given by

$$\frac{4\pi^2}{T^2}(b-c)(c-a)(a-b) = \begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}.$$ **(8)**

(*d*) From a semi-circle whose diameter is in the surface of a liquid, a circle is cut out, whose diameter is the vertical radius of the semi-circle. Find the depth of the centre of pressure of the remainder part. **(8)**

(*e*) If $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $f(r)$ is differentiable, show that

$$\text{div}[f(r)\vec{r}] = rf'(r) + 3f(r).$$

Hence or otherwise show that $\text{div}\left(\dfrac{\vec{r}}{r^3}\right) = 0$. **(8)**

6. (*a*) Solve the differential equation $(y^2 + 2x^2y)\,dx + (2x^3 - xy)\,dy = 0$. **(10)**

(*b*) Let T_1 and T_2 be the periods of vertical oscillations of two different weights suspended by an elastic string, and C_1 and C_2 are the statical extensions due to these weights and g is the acceleration due to gravity.

Show that $g = \dfrac{4\pi^2(C_1 - C_2)}{T_1^2 - T_2^2}$. **(15)**

(*c*) Show that $\vec{F} = (2xy + z^3)\hat{i} + x^2\hat{j} + 3xz^2\hat{k}$ is a conservative force. Hence, find the scalar potential. Also find the work done in moving a particle of unit mass in the force field from (1, –2, 1) to (3, 1, 4). **(15)**

7. (*a*) The end links of a uniform chain slide along a fixed rough horizontal rod. Prove that the ratio of the maximum span to the length of the chain is

$$\mu \log \frac{1 + (1 + \mu^2)^{\frac{1}{2}}}{\mu}$$

where μ is the coefficient of friction. **(10)**

(*b*) Solve: $\dfrac{dy}{dx} = \dfrac{4x + 6y + 5}{3y + 2x + 4}$. **(10)**

(*c*) A frame ABC consists of three light rods, of which AB, AC are each of length a, BC of length $\frac{3}{2}a$, freely jointed together. It rests with BC horizontal, A below BC and the rods AB, AC over two smooth pegs E and F, in the same horizontal line, at a distance $2b$ apart. A weight W is suspended from A. Find the thrust in the rod BC. **(10)**

(*d*) Let α be a unit-speed curve in R^3 with constant curvature and zero torsion. Show that α is (part of) a circle. **(10)**

8. (*a*) A solid hemisphere floating in a liquid is completely immersed with a point of the rim joined to a fixed point by means of a string. Find the inclination of the base to the vertical and tension of the string. **(15)**

(*b*) A snowball of radius $r(t)$ melts at a uniform rate. If half of the mass of the snowball melts in one hour, how much time will it take for the entire mass of the snowball to melt, correct to two decimal places? Conditions remain unchanged for the entire process. **(15)**

(*c*) For a curve lying on a sphere of radius a and such that the torsion is never 0, show that

$$\left(\frac{1}{\kappa}\right)^2 + \left(\frac{\kappa'}{\kappa^2\tau}\right)^2 = a^2.$$ **(10)**

PAPER-II

INSTRUCTIONS: *There are* ***eight*** *questions in all, out of which* ***five*** *are to be attempted. Question Nos. 1 and 5 are compulsory. Out of the remaining* ***six*** *questions,* ***three*** *are to be attempted selecting at least* ***one*** *question from each of the two Sections A and B. Answers must be written in* ***English*** *only.*

SECTION–A

1. (*a*) Prove that a non-commutative group of order $2n$, where n is an odd prime, must have a subgroup of order n. **(8)**

(*b*) A function $f: [0, 1] \to [0, 1]$ is continuous on $[0, 1]$. Prove that there exists a point c in $[0, 1]$ such that $f(c) = c$. **(10)**

(*c*) If $u = (x-1)^3 - 3xy^2 + 3y^2$, determine v so that $u + iv$ is a regular function of $x + iy$. **(10)**

(*d*) Solve by simplex method the following Linear Programming Problem: **(12)**

Maximize $Z = 3x_1 + 2x_2 + 5x_3$

subject to the constraints

$$x_1 + 2x_2 + x_3 \le 430$$
$$3x_1 + 2x_3 \le 460$$
$$x_1 + 4x_2 \le 420$$
$$x_1, x_2, x_3 \ge 0.$$

2. (*a*) Find all the homomorphisms from the group $(\mathbb{Z}, +)$ to $(\mathbb{Z}_4, +)$. **(10)**

(*b*) Consider the function f defined by

$$f(x,y) = \begin{cases} xy\dfrac{x^2-y^2}{x^2+y^2}, & \text{where } x^2+y^2 \ne 0 \\ 0, & \text{where } x^2+y^2 = 0 \end{cases}$$

Show that $f_{xy} \ne f_{yx}$ at $(0, 0)$. **(10)**

(*c*) Prove that

$$\int_0^\infty \cos x^2 dx = \int_0^\infty \sin x^2 dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}.$$ **(10)**

(*d*) Let R be a commutative ring with unity. Prove that an ideal P of R is prime if and only if the quotient ring R/P is an integral domain. **(10)**

3. (*a*) Find the minimum value of $x^2 + y^2 + z^2$ subject to the condition $ax + by + cz = p$. **(10)**

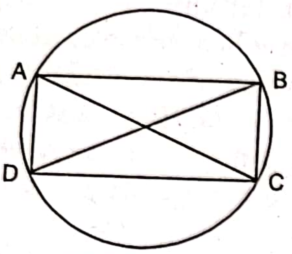
(*b*) Show by an example that in a finite commutative ring, every maximal ideal need not be prime. **(10)**

(*c*) Evaluate the integral $\int_0^{2\pi} \cos^{2n}\theta\, d\theta$, where n is a positive integer. **(10)**

Explanatory Answers

Paper-I

1.(*a*)



Let ABCD be the rectangle inscribed in a circle of radius r.

AC and BD are diameters of length $2r$ as angle in semicircle is always 90°.

Let x be length, y be breadth of rectangle.

$$x^2 + y^2 = (2r)^2$$

$$\Rightarrow \quad y^2 = 4r^2 - x^2$$

$$\Rightarrow \quad y = \sqrt{4r^2 - x^2}$$

Area of reactangle, A = xy

$$A = x\sqrt{4r^2 - x^2}$$

$$\Rightarrow \quad \frac{dA}{dx} = \sqrt{4r^2 - x^2} - \frac{2x^2}{2\sqrt{4r^2 - x^2}}$$

$$\Rightarrow \quad \frac{dA}{dx} = \frac{4r^2 - x^2 - x^2}{\sqrt{4r^2 - x^2}}$$

$$\Rightarrow \quad \frac{dA}{dx} = \frac{4r^2 - 2x^2}{\sqrt{4r^2 - x^2}}$$

For maximum area,

$$\frac{dA}{dx} = 0$$

$$\Rightarrow \quad \frac{4r^2 - 2x^2}{\sqrt{4r^2 - x^2}} = 0$$

$$\Rightarrow \quad 4r^2 - 2x^2 = 0$$

$$\Rightarrow \quad x = \sqrt{2}r$$

Also, $\quad y = r\sqrt{2}$

Since, length and breadth are same, it is a square.

1.(*b*) We Know that det(Adj(A)) = det(A)$^{(n-1)}$, where n is the number of rows and columns in your matrices.

Here we have,

$$\det(A) = 2, \text{ and } n = 3,$$

so $\quad \det(\text{Adj}(A)) = 2^2 = 4$

Adj(A)*Adj(Adj(A)) = det(Adj(A))*I

and, dividing this by det(Adj(A))$^{([n-2]/[n-1])}$, we get;

Adj(A)*[Adj(Adj(A))/det(Adj(A))$^{([n-2]/[n-1])}$]

= det (A)*I

Comparing this to the first equation above, out will pop;

A = Adj(Adj(A))/det(Adj(A))$^{([n-2]/[n-1])}$

$$\text{Adj}(A) = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\text{Adj}(\text{Adj}(A)) = \begin{bmatrix} 4 & -2 & 2 \\ -2 & 2 & -2 \\ 2 & -2 & 6 \end{bmatrix}$$

and $\det\left(\text{Adj}(A)\right)^{\left(\frac{3-2}{3-1}\right)} = (4)^{1/2} = 2$

$$\therefore \quad A = \frac{1}{2}\begin{bmatrix} 4 & -2 & 2 \\ -2 & 2 & -2 \\ 2 & -2 & 6 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 3 \end{bmatrix}.$$

1.(*c*) Let us construct a function ϕ on $[a, b]$ defined by

$$\phi(x) = f(x) + Ax,$$

where A is a constant to be determined by the condition that, $\phi(a) = \phi(b)$

$$\phi(a) = f(a) + Aa,$$
$$\phi(b) = f(b) + Ab$$
$\therefore$ $\phi(a) = \phi(b)$ implies A
$$= \frac{f(b)-f(a)}{a-b}$$

Now, $\phi(x)$ being the sum of two continuous functions on $[a, b]$ is itself continuous on $[a, b]$.

Also, $f(x)$ is derivable in (a, b) and Ax is also derivable everywhere.

$\therefore$ $\phi(x)$ is derivable in (a, b) and
$$\phi'(x) = f'(x) + A$$
Further, $\phi(a) = \phi(b)$ (chosen)

$\therefore$ $\phi(x)$ satisfies all conditions of Rolle's theorem.

$\therefore$ There exists at least one point c in (a, b) such that $\phi'(c) = 0$ or,
$$f'(c) + A = 0$$
or, $$-A = f'(c)$$
or, $$\frac{f(b)-f(a)}{b-a} = f'(c)$$
$$f(b) - f(a) = \text{c.f. } '(c) \,.\, \log(b/a).$$

1.(*d*) The given line is
$$x - y - z = 0 = x - y + 2z - 9$$
Any Plane through this line is
$$x - y - z + k(x - y + 2z - 9) = 0$$
$$x(1 + k) - y(1 + k) - z(1 - 2k) - 9k = 0 \quad ...(i)$$
This is of type $lx + my + nz - p = 0$

and the given ellipsoid
$$2x^2 + 6y^2 + 3z^2 = 27$$
or, $$\frac{2x^2}{27} + \frac{6}{27}y^2 + \frac{3}{27}z^2 = 1 \quad ...(ii)$$
The plane (*i*) touches (*ii*) if
$$\frac{(1+k)^2}{\left(\frac{2}{27}\right)} + \frac{(1+k)^2}{\left(\frac{6}{27}\right)} + \frac{(1-2k)^2}{\left(\frac{3}{27}\right)} = (9k)^2$$
$$\frac{(1+k^2+2k)}{2} + \frac{(1+k^2+2k)}{6} + \frac{(1+4k^2-4k)}{3} = 3k^2$$
$$3(1 + k^2 + 2k) + (1 + k^2 + 2k) + 2(1 + 4k^2 - 4k) = 6 \times 3k^2$$
$$12k^2 + 6 = 18k^2$$
$$k^2 = 1$$
$$\Rightarrow \quad k = \pm 1$$
Putting the value of k in equation (*i*), the required tangent planes are
$$x(1 + 1) - y(1 + 1) - z(1 - 2) = 9(1)$$
$$2x - 2y + z = 9$$
and $x(1 - 1) - y(1 - 1) - z(1 + 2) = 9(-1)$
$$\therefore \quad z - 3 = 0$$
Hence, equation of tangent plane to the ellipsoid are
$$2x - 2y + z - 9 = 0$$
and $$z - 3 = 0.$$

1.(*e*) Start with $Ac_j = \lambda_j c_j$.

Multiply both sides from the left by c_i^* to obtain $c_i^* A c_j = \lambda_j c_i^* c_j$, which we write in the notation
$$A_{ij} = \lambda_j c_i^* c_j \quad ...(i)$$
Now multiply $A^* c_i^* = \lambda_i^* c_i^*$ from the left by c_j to obtain $c_j A^* c_i^* = \lambda_i^* c_j c_i^*$, which we write as
$$A_{ji}^* = \lambda_i^* c_j c_i^* \quad ...(ii)$$
But A is Hermitian, so $A_{ij} = A_{ji}^*$. Furthermore $c_j c_i^* = c_i^* c_j$ because the dot product of two vectors is commutative. Comparing Equation (*i*) and (*ii*) gives
$$\left(\lambda_i^* - \lambda_j\right) c_i^* c_j = 0 \quad (iii)$$
If $i = j, c_i^* c_j \geq 0$, and so $\lambda_j = \lambda_j^*$, which says that the eigenvalues are real.

2.(*a*) Equation of line: $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$

Direction ratio of line

= Direction ratio of generator

= $\langle 1, -2, 3 \rangle$

Let (x_1, y_1, z_1) is any point on the cylinder, then the equation of generator through (x_1, y_1, z_1) is

$$\frac{x-x_1}{1} = \frac{y-y_2}{-2} = \frac{z-z_3}{3}$$

And, equation of guiding curve is

$$x^2 + y^2 = 4,\ z = 2 \text{ (given)}$$

The line must intersect the guiding curve. Taking $z = 2$, the line becomes,

$$\frac{x-x_1}{1} = \frac{y-y_1}{-2} = \frac{2-z_1}{3}$$

This yields, $\quad x = x_1 + \left(\frac{2-z_1}{3}\right)$

and $\quad y = y_1 - \left(\frac{4-2z_1}{3}\right)$

Putting these values in $x^2 + y^2 = 4$, we get

$$\left\{x_1 + \left(\frac{2-z_1}{3}\right)\right\}^2 + \left\{y_1 - \left(\frac{4-2z_1}{3}\right)\right\}^2 = 4$$

$$x_1^2 + y_1^2 + \frac{4}{9} + \frac{z_1^2}{9} + \frac{2x_1}{3}(2-z_1)$$

$$-\frac{4}{3}z_1 + \frac{16}{9} + \frac{4}{9}z_1^2 - \frac{16z_1}{3} - \frac{2y_1}{3}(4-2z_1) = 4$$

$$x_1^2 + y_1^2 + \frac{5}{9}z_1^2 + \frac{4}{3}x_1 - \frac{8y_1}{3}$$

$$-\frac{20z_1}{3} - \frac{2}{3}x_1z_1 + \frac{4y_1z_1}{3} - \frac{16}{9} = 0$$

$$9x_1^2 + 9y_1^2 + 5z_1^2 + 12x_1 - 24y_1$$

$$-60z_1 - 6x_1z_1 + 12y_1z_1 - 16 = 0$$

Therefore the locus of (x_1, y_1, z_1) is

$9x^2 + 9y^2 + 5z^2 + 12x - 24y - 60z - 6xz + 12yz - 16 = 0$

2.(*c*) Let α and β be two consecutive roots of $\phi(x) = 0$, in $[a, b]$ and $\alpha < \beta$. We have to prove that one and only one root of $\psi(x) = 0$ lies between α and β. If possible, $\psi(x) = 0$ has no root in (α, β). We consider the function

$$F(x) = \frac{\phi(x)}{\psi(x)}.$$

Now, $\quad F(\alpha) = \frac{\phi(\alpha)}{\psi(\alpha)} = 0$

$(\because \phi(\alpha) = 0;\ \psi(\alpha) \neq 0)$

$$F(\beta) = \frac{\phi(\beta)}{\psi(\beta)} = 0$$

$(\because \phi(\beta) = 0,\ \psi(\beta) \neq 0)$

$\psi(x) \neq 0$ in $[\alpha, \beta]$

$\therefore$ $F(x)$ is continuous in $[\alpha, \beta]$ since $\varphi(x)$ and $\psi(x)$ are so.

$$F'(x) = \frac{\phi'(x)\psi(x) - \psi'(x)\phi(x)}{\{\psi(x)\}^2}$$

exists in (α, β)

$\therefore$ $F(x)$ satisfies all conditions of Rolle's theorem in $[\alpha, \beta]$

$\therefore \quad F'(\gamma) = 0$ where $\alpha < \gamma < \beta$.

But by the given condition

$\phi'(x)\,\psi(x) - \psi'(x)\,\varphi(x) > 0$

$\therefore$ $F'(x) \neq 0$ in (α, β) and we arrive at a contradiction. Therefore, our assumption that $\psi(x) = 0$ has no root in (α, β) is wrong. In other words, $\psi(x) = 0$ has a root in (α, β). By like arguments it can be shown that between two roots of $\psi(x) = 0$, there is a root of $\phi(x) = 0$

To prove that there is exactly one root of $\psi(x) = 0$, argument's sake, let γ and δ be two roots of $\psi(x) = 0$ in (α, β) *i.e.*, $\alpha < \gamma < \delta < \beta$. Between γ and δ there would exist a roots of $\phi(x) = 0$ contradicting that α and β are the consecutive roots of $\phi(x) = 0$.

Therefore, there is one and only one root of $\psi(x) = 0$ between α and β.

2.(*d*) Let $S = \{\alpha_1, \alpha_2, \alpha_3\}$

First we shall show that the set S is linearly independent. Let a, b, c be scalars *i.e.*, real numbers such that

$$a\alpha_1 + b\alpha_2 + c\alpha_3 = 0$$

i.e., $a(1, 0, -1) + b(1, 2, 1) + c(0, -3, 2)$

$= (0, 0, 0)$

i.e., $(a + b + 0c,\ 0a + 2b - 3c,\ -a + b + 2c)$

$= (0, 0, 0)$

i.e., $\quad a + b = 0 \quad$ ...(1)

$2b - 3c = 0 \quad$ ...(2)

$-a + b + 2c = 0 \quad$ ...(3)

Adding both sides of the equations (1) and (3), we get

$$2b + 2c = 0 \quad ...(4)$$

Subtracting (2) from (4), we get

$$5c = 0 \text{ or } c = 0$$

Putting $c = 0$ in (2), we get $b = 0$ and then putting $b = 0$ in (1), we get $a = 0$.

Thus $a = 0$, $b = 0$, $c = 0$ is the only solution of the equations (1), (2) and (3) and so

$$a\alpha_1 + b\alpha_2 + c\alpha_3 = 0$$

$$\Rightarrow \quad a = 0, b = 0, c = 0$$

$\therefore$ The vectors α_1, α_2, α_3 are linearly independent.

We have dim $\mathbf{R}^3 = 3$.

Therefore the vectors α_1, α_2, α_3 form a basis of $\mathbf{R}^3$.

Now, let $\gamma = (p, q, r)$ be any vector in $\mathbf{R}^3$ and let

$$\gamma = (p, q, r)$$
$$= x\alpha_1 + y\alpha_2 + z\alpha_3, \quad x, y, z \in \mathbf{R}$$
$$= x(1, 0, -1) + y(1, 2, 1) + z(0, -3, 2) \quad ...(5)$$

Then,
$$x + y = p \quad ....(6)$$
$$2y - 3z = q \quad ...(7)$$
$$-x + y + 2z = r \quad ...(8)$$

Adding both sides of equations (6) and (8), we get

$$2y + 2z = p + r \quad ...(9)$$

Subtracting (7) from (9), we get

$$5z = p + r - q$$

or
$$z = \frac{1}{5}p - \frac{1}{5}q + \frac{1}{5}r$$

Then from (9), we get

$$y = -z + \frac{1}{2}p + \frac{1}{2}r$$
$$= \frac{3}{10}p + \frac{1}{5}q + \frac{3}{10}r$$

and from (6), we get

$$x = p - y$$
$$= p - \frac{3}{10}p - \frac{1}{5}q - \frac{3}{10}r = \frac{7}{10}p - \frac{1}{5}q - \frac{3}{10}r$$

Thus, the relation (5) expresses the vector $\gamma = (p, q, r)$ as a linear combination of α_1, α_2 and α_3 where $x, y, z \in \mathbf{R}$ are as found above.

The standard basis vectors are

$$e_1 = (1, 0, 0),$$
$$e_2 = (0, 1, 0)$$
and
$$e_3 = (0, 0, 1)$$

If $\gamma = e_1$, then $p = 1$, $q = 0$, $r = 0$ and so

$$x = \frac{7}{10}, \; y = \frac{3}{10} \text{ and } z = \frac{1}{5}$$

$$\therefore \quad e_1 = \frac{7}{10}\alpha_1 + \frac{3}{10}\alpha_2 + \frac{1}{5}\alpha_3$$

If $\gamma = e_2$, then $p = 0$, $q = 1$, $r = 0$ and so

$$x = -\frac{1}{5}, \; y = \frac{1}{5} \text{ and } z = -\frac{1}{5}$$

$$\therefore \quad e_2 = -\frac{1}{5}\alpha_1 + \frac{1}{5}\alpha_2 - \frac{1}{5}\alpha_3$$

Finally if $\gamma = e_3$, then $p = 0$, $q = 0$, $r = 1$ and so

$$x = -\frac{3}{10}, \; y = \frac{3}{10} \text{ and } z = \frac{1}{5}$$

$$\therefore \quad e_3 = -\frac{3}{10}\alpha_1 + \frac{3}{10}\alpha_2 + \frac{1}{5}\alpha_3.$$

3.(*a*) Equation of sphere is:

$$x^2 + y^2 + z^2 - 2x + 6y + 2z + 8 = 0 \quad ...(i)$$

Center: $(1, -3, -1)$

$$\text{Radius} = \sqrt{(1)^2 + (-3)^2 + (-1)^2 - 8}$$
$$= \sqrt{1+9+1-8} = \sqrt{3}$$

The equation of lines are

$$3x - 4y - 8 = 0 = y - 3z + 2$$

Any plane through this line is

$$(3x - 4y - 8) + k(y - 3z + 2) = 0$$
$$3x + (k - 4)y - 3kz + 2k - 8 = 0 \quad ...(ii)$$

It will touch the sphere (*i*), if perpendicular from centre on plane (*ii*)

= Radius of sphere (*i*)

$$\therefore \quad \frac{3(1) + (k-4)(-3) - 3k(-1) + 2k - 8}{\sqrt{(3)^2 + (k-4)^2 + (-3k)^2}} = \sqrt{3}$$

or, $$\frac{2k+7}{\sqrt{(3)^2+(k-4)^2+(3k)^2}}=\sqrt{3}$$

$$(2k+7)^2 = \left\{\sqrt{3}.\left(\sqrt{(3)^2+(k-4)^2+(3k)^2}\right)\right\}^2$$

$$4k^2 + 49 + 28k = 3\{25 + 10k^2 - 8k\}$$

$$26k^2 - 52k + 26 = 0$$

$$k^2 - 2k + 1 = 0$$

$$(k-1)^2 = 0$$

$$\Rightarrow \quad k = 1$$

Putting these values of k in (ii), we get

$$(3x - 4y - 8) + (y - 3z + 2) = 0$$

$$3x - 3y - 3z - 6 = 0$$

$$\Rightarrow \quad x - y - z - 2 = 0.$$

3.(*c*) Let $v_1 = (1, 1),\ v_2 = (1, -1)$

then, $Te_1 = T(1, 0) = (1, 1 + 0)$

$= (1, 1) = v_1 + 0v_2$

$Te_2 = T(0, 1) = (0, 0 + 1)$

$$= (0, 1) = \frac{1}{2}v_1 - \frac{1}{2}v_2$$

[working procedure : Let

$$(1, 1) = \lambda_1(1, 1) + \lambda_2(1, -1)$$

$$= (\lambda_1 + \lambda_2, \lambda_1 - \lambda_2)$$

$\therefore \quad \lambda_1 + \lambda_2 = 1$ and $\lambda_1 - \lambda_2 = 1$

solving, we get $\lambda_1 = 1$ and $\lambda_2 = 0$

$\therefore \quad (1, 1) = 1.\ v_1 + 0.\ v_2$

Next, let $(0, 1) = \mu_1 v_1 + \mu_2 v_2$

$= \mu_1(1, 1) + \mu_2(1, -1)$

$= (\mu_1 + \mu_2, \mu_1 - \mu_2)$

$\therefore \quad \mu_1 + \mu_2 = 0$ and $\mu_1 - \mu_2 = 1$

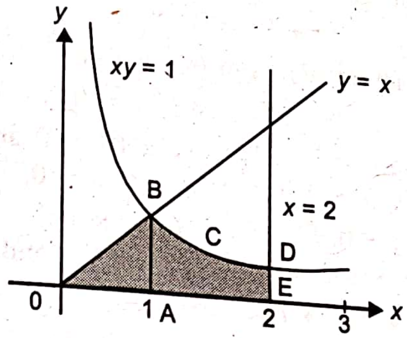
Solving we obtain,

$$\mu_1 = \frac{1}{2} \text{ and } \mu_2 = -\frac{1}{2}$$

Hence, $$(0, 1) = \left[\frac{1}{2}v_1 - \frac{1}{2}v_2\right]$$

So the matrix of T = $$\begin{bmatrix} 1 & 0 \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

3.(*d*) The shaded region is shown in the fig. is the region of integration.



Hence, $$\iint_R (x^2 + xy).dx.dy$$

$$= \iint_{OAB} (x^2+xy).dx.dy + \iint_{ABCDEA} (x^2+xy).dx.dy$$

$$= \int_0^1 dx \int_0^x (x^2+xy).dy + \int_1^2 dx. \int_0^{1/x} (x^2+xy).dy$$

$$= \int_0^1 \left[x^2.[y]_0^x + x.\left[\frac{y^2}{2}\right]_0^x\right].dx$$

$$+ \int_1^2 \left[x^2.[y]_0^{1/x} + x.\left[\frac{y^2}{2}\right]_0^{1/x}\right].dx$$

$$= \int_0^1 \left[x^3 + \frac{x^3}{2}\right]dx + \int_1^2 \left[x + \frac{1}{2x}\right]dx$$

$$= \frac{3}{8}\left[x^4\right]_0^1 + \left[\frac{x^2}{2} + \frac{1}{2}\ln(x)\right]_1^2$$

$$= \frac{3}{8} + \left(2 - \frac{1}{2}\right) + \frac{1}{2}\ln 2$$

$$= \frac{15}{8} + 0.34657 = 2.22157.$$

4.(*a*) Equation of plane : $2x + y - z = 0$

Equation of cone : $4x^2 - y^2 + 3z^2 = 0$

Let l, m, n be the direction cosines of any one line of section

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$$

Since, it lies in the plane and on the cone,

We have,

$$2l + m - n = 0 \qquad ...(i)$$

and $$4l^2 - m^2 + 3n^2 = 0 \qquad ...(ii)$$

From equation (*i*), $n = 2l + m$

Putting in equation (*ii*), we have

$$4l^2 - m^2 + 3(2l + m)^2 = 0$$
$$4l^2 - m^2 + 12l^2 + 3m^2 + 12lm = 0$$
$$8l^2 + m^2 + 6lm = 0$$
$$(4l + m)(2l + m) = 0$$
$$m = -4l \text{ or } -2l$$

On solving, we get

$l = 1, m = -2, n = 0$

and, $l = -1, m = 4, n = 2$

Hence, equation of lines:

$$\frac{x}{1} = \frac{y}{-2} = \frac{z}{0}$$

or $$\frac{x}{-1} = \frac{y}{4} = \frac{z}{2}$$

and angle between two lines:

$$\cos\theta = \frac{l_1.l_2 + m_1.m_2 + n_1.n_2}{\sqrt{l_1^2 + m_1^2 + n_1^2}.\sqrt{l_2^2 + m_2^2 + n_2^2}}$$

$$\therefore \cos\theta = \frac{1(-1) + 4(-2) + 0}{\sqrt{(1)^2 + (-2)^2 + 0}.\sqrt{(-1)^2 + (4)^2 + (2)^2}}$$

$$\cos\theta = \frac{-1-8}{\sqrt{5}.\sqrt{21}}$$

$$\theta = \cos^{-1}\left(\frac{-9}{\sqrt{105}}\right) = 151.74°.$$

4.(*b*) From question, we have

$$u = x + y + z$$
$$v = xy + yz + zx$$

and $$w = x^3 + y^3 + z^3 - 3xyz$$

Now, $$\frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial y} = 1, \frac{\partial u}{\partial z} = 1$$

$$\frac{\partial v}{\partial x} = (y + z)$$

$$\frac{\partial v}{\partial y} = (x + z)$$

$$\frac{\partial v}{\partial z} = (y + x)$$

$$\frac{\partial w}{\partial x} = 3x^2 - 3yz$$

$$\frac{\partial w}{\partial y} = 3y^2 - 3xz$$

$$\frac{\partial w}{\partial z} = 3z^2 - 3xy$$

$$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ (y+z) & (z+x) & (x+y) \\ 3(x^2 - yz) & 3(y^2 - zx) & 3(z^2 - xy) \end{vmatrix}$$

$$= (z + x)(3(z^2 - xy)) - 3(y^2 - zx)(x + y)$$
$$- 3(z^2 - xy)(y + z) + 3(x^2 - yz)(x + y)$$
$$+ 3(y^2 - zx)(y + z) - 3(x^2 - yz)(z + x)$$
$$= 3\{(z^2 - xy)(x - y) + (y^2 - zx)(z - x)$$
$$+ (x^2 - yz)(y - z)\}$$
$$= 3\{z^2x - x^2y - z^2y + x^2y + y^2z - z^2x - xy^2$$
$$+ zx^2 + x^2y - y^2z - x^2z + yz^2\}$$
$$= 0$$

Hence, u, v and w are functionally dependent.

Now, $u^3 = (x + y + z)^3$

$$= x^3 + y^3 + z^3 + 3x(xy + yz + 2x)$$
$$+ 3y(xy + yz + zx)$$
$$+ 3z(xy + yz + zx) - 3xyz$$
$$= x^3 + y^3 + z^3$$
$$+ 3[(x + y + z)(xy + yz + zx) - xyz]$$
$$3u.v = 3(x + y + z)(xy + yz + zx)$$

Now, $$u^3 - 3u.v = x^3 + y^3 + z^3 - 3xyz = w$$

Hence, Relation between u, v and w are

$$w = u^3 - 3u.v.$$

4.(*c*) Given equation of hyperbolic paraboloid:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2z \qquad ...(i)$$

The two systems of generators of the hyperbolic paraboloid (*i*) are

g_λ : $\dfrac{x}{a} - \dfrac{y}{b} = 2\lambda,$

$$\frac{x}{a} + \frac{y}{b} = \frac{z}{\lambda}, \qquad \text{...(ii)}$$

h_μ : $\dfrac{x}{a} + \dfrac{y}{b} = 2\mu,$

$$\frac{x}{a} - \frac{y}{b} = \frac{z}{\mu}, \qquad \text{...(iii)}$$

Clearly the direction ratios of the generator g_λ are a, b, 2λ and the direction ratios of the generator h_μ are a, $-b$, 2μ.

If g_λ is orthogonal to h_μ, it follows that

$$a^2 - b^2 + 4\lambda\mu = 0 \qquad \text{...(iv)}$$

The coordinates of the point of intersection, P, of g_λ and h_μ are

$$a(\lambda + \mu),\ b(\mu - \lambda),\ 2\lambda\mu \qquad \text{...(v)}$$

(*iv*) indicates that, if the generators g_λ and h_μ are orthogonal to each other, their point of intersection, P, lies on the plane

$$a^2 - b^2 + 2z = 0 \qquad \text{...(vi)}$$

As P also lies on the hyperbolic parabolic paraboloid (*i*), the locus of all such points P is the section of (*i*) by the plane (*vi*), which is the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = b^2 - a^2,$$

$$a^2 - b^2 + 2z = 0.$$

4.(*d*) Main Lemma. Any $n + 1$ vectors in $\mathbb{R}^n$ are linearly dependent.

Proof. Any two vectors on the line $\mathbb{R}^1$ are proportional and therefore linearly dependent. We intend to prove the lemma by deducing from this that any 3 vectors in $\mathbb{R}^2$ are linearly dependent, then deducing from this that any 4 vectors in $\mathbb{R}^3$ are linearly dependent, and so on. Thus we only need to prove that if any n vectors in $\mathbb{R}^{n-1}$ are linearly dependent *then* $n + 1$ vectors in $\mathbb{R}^n$ are linearly dependent too. To this end, consider n-vectors v_1, ... v_{n+1} as n-columns. If the last entry in each column is 0, then $v_1, ..., v_{n+1}$ are effectively $n - 1$ -columns, hence some nontrivial linear combination of $v_1, ..., v_n$ equals 0 and thus the set is linearly dependent. Now consider the case when at least one column has non-zero last entry. Reordering the vectors we may assume that it is the column v_{n+1}. Subtracting the column v_{n+1} with suitable coefficient $\alpha_1, ..., \alpha_n$ from $v_1, ..., v_n$ we form n new columns $u_1 = v_1 - \alpha_1 v_{n+1}, ..., u_n = v_n - \alpha_n v_{n+1}$ which all have the last entries equal to zero. Thus $u_1, ..., u_n$ are effectively $n - 1$-vectors and are therefore linearly dependent: $\beta_1 u_1 + ... + \beta_n u_n = 0$ for some $\beta_1, ..., \beta_n$ not all equal to 0. Thus $\beta_1 v_1 + ... + \beta_n v_n - (\alpha_1\beta_1 + ... + \alpha_n\beta_n)v_{n+1} = 0$ and hence $v_1, ..., v_{n+1}$ are linearly dependent too.

5.(*a*) $$\frac{d^2y}{dx^2} - y = x.e^x + \cos^2 x$$

The symbolic form of the given differential equation is $(D^2 - 1)y = x.e^x + \cos^2 x$

The auxilliary equation is $m^2 - 1 = 0$

$\therefore$ $m = \pm 1$

and so, $\text{C.F.} = C_1.e^x + C_2.e^{-x}$

The forcing function consists of terms $x.e^x$ and $\cos^2 x$

Their derivatives are $(x.e^x + e^x)$ and $(-\sin 2x)$

So, the general solution is

$$y(x) = C_1.e^x + C_2 e^{-x} + \left(\frac{x^2}{4} - \frac{x}{4}\right)e^x + \frac{\sin^2 x}{5} - \frac{3}{5}.$$

5.(*b*) Here A.E. is $(m - 1)^2 = 0$ *i.e.*, $m = 1, 1$

$\therefore$ C.F. is $y = (c_1 + c_2 x)e^x$

$$\text{The P.I. is } y_p = \frac{1}{(D-1)^2} xe^x \sin x$$

$$= e^x \frac{1}{D^2} x \sin x$$

$$= e^x \left[x - \frac{2D}{D^2}\right] \frac{\sin x}{D^2}$$

$$= e^x \left[x - \frac{2D}{D^2}\right] \frac{\sin x}{-1^2}$$

$$= e^x \left[-x\sin x + \frac{2D \sin x}{-1^2}\right]$$

$$= e^x [-x \sin x - 2 \cos x]$$

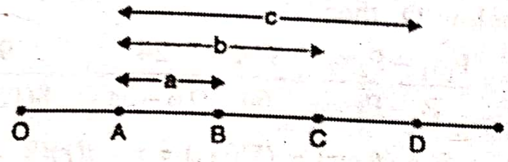
$\therefore$ The required complete solution of the given differential equation is

$$y = (c_1 + c_2 x - x \sin x - 2 \cos x)e^x.$$

5.(c) Let O be the centre of the force and A be the fixed point such that

AB $= a$, AC $= b$, AD $= c$

and let OA $= s$ and amplitude be A.



$\therefore$ $\qquad$ OB $= s + a$

$\qquad$ OC $= s + b$

and $\qquad$ OD $= s + c$

Velocities at B, C and D are u, v, w respectively

$\therefore$
$$u^2 = \mu[A^2 - (s+a)^2] \quad ...(1)$$
$$v^2 = \mu[A^2 - (s+b)^2] \quad ...(2)$$
$$w^2 = \mu[A^2 - (s+c)^2] \quad ...(3)$$

or
$$\frac{u^2}{\mu} = (A^2 - s^2) - 2as - a^2,$$
$$\frac{v^2}{\mu} = (A^2 - s^2) - 2bs - b^2,$$
$$\frac{w^2}{\mu} = (A^2 - s^2) - 2cs - c^2$$

or
$$\left(\frac{u^2}{\mu} + a^2\right) + 2as + s^2 - A^2 = 0 \quad ...(4)$$
$$\left(\frac{v^2}{\mu} + b^2\right) + 2bs + s^2 - A^2 = 0 \quad ...(5)$$
$$\left(\frac{w^2}{\mu} + b^2\right) + 2cs + s^2 - A^2 = 0 \quad ...(6)$$

From (4), (5) and (6) eliminating s and $s^2 - A^2$ using determinants, we get

$$\begin{vmatrix} \frac{u^2}{\mu} + a^2 & a & 1 \\ \frac{v^2}{\mu} + b^2 & b & 1 \\ \frac{w^2}{\mu} + c^2 & c & 1 \end{vmatrix} = 0$$

or
$$\begin{vmatrix} \frac{u^2}{\mu} & a & 1 \\ \frac{v^2}{\mu} & b & 1 \\ \frac{w^2}{\mu} & c & 1 \end{vmatrix} + \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} = 0$$

Property of determinant

$$\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} = -\frac{1}{\mu}\begin{vmatrix} u^2 & a & 1 \\ v^2 & b & 1 \\ w^2 & c & 1 \end{vmatrix} - \mu\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

$$= -\begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

Solving the determinant, we get

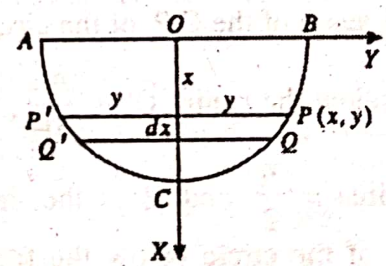
$$\mu(a-b)(b-c)(c-a) = \begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} \quad ...(7)$$

But, $\qquad T = \dfrac{2\pi}{\sqrt{\mu}} \Rightarrow \mu = \dfrac{4\pi^2}{T^2}$

Putting μ in (7), we get

$$\frac{4\pi^2}{T^2}(a-b)(b-c)(c-a) = \begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}.$$

5.(d) Let OC be the axis of x



By symmetry, it is evident that the C.P. lies on OX.

Consider an elementary strip PQQ'P' of width dx at a depth x below O.

Then dS = area of the strip = $2y\ dx$,

p = intensity of pressure at any point of the strip $= \rho g\, x$

If $\bar{x}$ be the depth of the C.P. of the semi-circular lamina below O, we have

$$\bar{x} = \frac{\int x\, p\, dS}{\int p\, dS} = \frac{\int_0^a x\rho g x.2y\,dx}{\int_0^a \rho g x.2y\,dx} = \frac{\int_0^a x^2 y dx}{\int_0^a xydx}$$

The parametric equations of the circle are

$$x = a \cos t,$$
$$y = a \sin t$$
$$\therefore \quad dx = -a \sin t\ dt$$

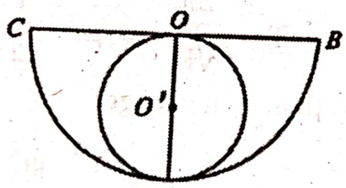
$$\therefore\ \bar{x} = \frac{\int_{\pi/2}^a a^2\cos^2 t.a\sin t(-a\sin t\,dt)}{\int_{\pi/2}^a a\cos t.a\sin t(-a\sin t\,dt)}$$

$$= \frac{a\int_0^{\pi/2}\cos^2 t\sin^2 t\,dt}{\int_0^{\pi/2}\cos t\sin^2 t\,dt} = \frac{a\left(\frac{1.1}{4.2}\frac{\pi}{2}\right)}{\frac{1}{3.1}} = \frac{3}{16}\pi a$$

Again,

x_1 = depth of the C.P. of the semi-circle below

$$O = \frac{3\pi}{16}a$$



P_1 = Pressure on the semi-circle

$$= w.\frac{1}{2}\pi a^2.\frac{4a}{3\pi} = \frac{2}{3}a^3 w$$

Again depth of the C.P. of the circle of radius $\frac{1}{2}a$ below the centre O′ is $\frac{A^2}{4H}$, where A is its radius $= \frac{a}{2}$ and H is the depth of the centre of the circle below the free surface

$$= OO' = \frac{a}{2}$$

$$\therefore \quad \frac{A^2}{4H} = \frac{(a/2)^2}{4(a/2)} = \frac{a}{8}$$

$\therefore\ x_2$ = depth of the C.P. of circle below

$$O = \frac{a}{2} + \frac{a}{8} = \frac{5a}{8}$$

P_2 = pressure on the circle

$$= w.\pi\left(\frac{a}{2}\right)^2.\frac{a}{2} = \frac{1}{8}w\pi a^3$$

If $\bar{x}$ be the depth of the C.P. of the remainder below O, then

$$\bar{x} = \frac{P_1x_1 - P_2x_2}{P_1 - P_2} = \frac{3a\pi}{64}.\frac{24}{(16-3\pi)} = \frac{9\pi a}{8(16-3\pi)}$$

5.(e) $\nabla \bullet (f(r)\underline{r}) = (\nabla f(r)) \bullet \underline{r} - f(r)(\nabla \bullet \underline{r})$

$$= f'(r)\frac{\underline{r}\bullet\underline{r}}{r} + 3f(r)$$
$$= rf'(r) + 3f(r)$$

If $f(r)\underline{r}$ is solenoidal, then $\nabla \bullet (f(r)\underline{r}) = 0$, so that $u = f(r)$ will satisfy

$$r\frac{du}{dr} + 3u = 0$$

$$\frac{du}{u} = -\frac{3dr}{r}$$

$$\ln|u| = -3\ln|r| + \ln|C|$$
$$u = Cr^{-3}$$
$$\Rightarrow \quad f(r) = Cr^{-3}$$

for an arbitrary constant C.

6.(a) We have

$$(y^2 + 2x^2y)dx + (2x^3 - xy)dy = 0 \quad ...(1)$$

Here, $\quad M = y^2 + 2x^2y$

and $\quad N = 2x^3 - xy$

$$\therefore \quad \frac{\partial M}{\partial y} = 2y + 2x^2$$

and $$\frac{\partial N}{\partial x} = 6x^2 - y$$

$$\therefore \quad \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$$

$\therefore$ The given equation is not exact.

$(1) \Rightarrow (y^2dx - xy\,dy) + (2x^2y\,dx + 2x^3dy) = 0$

$\Rightarrow y(ydx - xdy) + x^2(2y\,dx + 2x\,dy) = 0$

$\Rightarrow x^0y^1(1.ydx - 1.xdy) + x^2y^0(2y\,dx + 2x\,dy) = 0$

Comparing it with
$x^a y^b(my\,dx + nx\,dy) + x^c y^d(py\,dx + qx\,dy) = 0$,
we have $a = 0$, $b = 1$, $m = 1$, $n = -1$, $c = 2$, $d = 0$, $p = 2$, $q = 2$

Also $\quad \dfrac{m}{n} = \dfrac{1}{-1} = -1$

and $\quad \dfrac{p}{q} = \dfrac{2}{2} = 1$

$\therefore \quad \dfrac{m}{n} \neq \dfrac{p}{q}$

Let $\quad$ I.F. $= x^{\alpha} y^{\beta}$

$\therefore \quad \dfrac{a+\alpha+1}{m} = \dfrac{b+\beta+1}{n}$

and $\quad \dfrac{c+\alpha+1}{p} = \dfrac{d+\beta+1}{q}$

$\Rightarrow \quad \dfrac{0+\alpha+1}{1} = \dfrac{1+\beta+1}{-1}$

and $\quad \dfrac{2+\alpha+1}{2} = \dfrac{0+\beta+1}{2}$

$\Rightarrow \quad \alpha + 1 = -\beta - 2$

and $\quad \alpha + 3 = \beta + 1$

$\Rightarrow \quad \alpha + \beta = -3$

and $\quad \alpha - \beta = -2$

Solving, we get, $\alpha = -\dfrac{5}{2}$, $\beta = -\dfrac{1}{2}$

$\therefore \quad$ I.F. $= x^{\alpha} y^{\beta} = x^{-5/2} y^{-1/2}$

Multiplying (1) by $x^{-5/2}y^{-1/2}$, we get

$$x^{-5/2}y^{-1/2}(y^2 + 2x^2y)dx + x^{-5/2}y^{-1/2}(2x^3 - xy)dy = 0$$

$$\Rightarrow (x^{-5/2}y^{3/2} + 2x^{-1/2}y^{1/2})dx + (2x^{1/2}y^{-1/2} - x^{-3/2}y^{1/2})dy = 0$$

This equation is exact.

$\therefore$ The general solution is

$$\int^x (x^{-5/2}y^{3/2} + 2x^{-1/2}y^{1/2})dx = c$$

$$\Rightarrow \quad y^{3/2} \cdot \frac{x^{-3/2}}{-3/2} + 2y^{1/2} \cdot \frac{x^{1/2}}{1/2} = c$$

$$\Rightarrow \quad -\frac{2}{3}x^{-3/2}y^{3/2} + 4x^{1/2}y^{1/2} = c$$

$$\Rightarrow \quad -x^{-3/2}y^{3/2} + 6x^{1/2}y^{1/2} = c'.$$

$\left(\text{Putting } c' = \dfrac{3}{2}c\right)$

6.(b) Let one end of an elastic string of natural length a and modulus of elasticity λ be attached to the fixed point O and with the other end A, a mass m be attached (refer figure (*i*)).

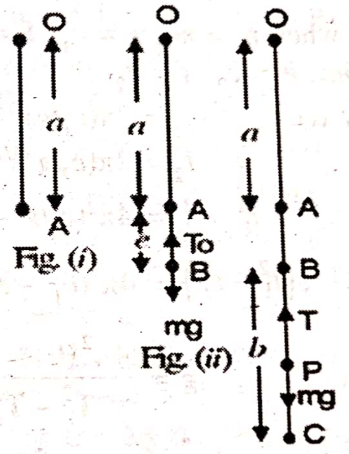


Fig. (*i*) Fig. (*ii*) Fig. (*iii*)

Due to weight mg of the particle the string OA is stretched and if B is the position of equilibrium of the particle such that AB = e, then tension T_0 in the string will balance the weight of the particle. Refer figure (*ii*).

Thus, at B, we get

$$mg = T_0$$

or $\quad mg = \lambda(e/a) \quad$ ...(1)

Let the particle be now pulled down a further distance BC(= b, say) and released. Let P be the position of the particle at any subsequent time t.

Let BP = x and let T be tension in the string. Then equation of motion of the particle is

$$m(d^2x/dt^2) = mg - T = mg - \lambda(e + x)/a$$
$$= mg - \lambda(e/a) - \lambda(x/a)$$

or $\quad m(d^2x/dt^2) = -\lambda(x/a)$, using (1)

or $\quad d^2x/dt^2 = -(\lambda/am)x \quad$ ...(2)

which is of standard form $d^2x/dt^2 = -\mu x$ of S.H.M., where $\mu = \lambda/am$.

Here centre of oscillation is B, from which x is measured and amplitude = BC = b. The periodic time T of S.H.M. represented by (2) is given by

$$T = 2\pi/\mu^{1/2} = 2\pi/(\lambda/am)^{1/2}$$
$$= 2\pi(am/\lambda)^{1/2}$$
$$= 2\pi(e/g)^{1/2}, \text{ by (1)} \quad ...(3)$$

Equation (3) by taking $e(=$ AB) as statical extension corresponding to mass m. Then, time period T is given by

$$T = 2\pi(e/g)^{1/2} \quad ...(i)$$

Here when $m = m_1$, $e = c_1$, $T = t_1$ and when $m = m_2$, $e = c_2$, $T = t_2$

So by (i), $\quad t_1 = 2\pi(c_1/g)^2$

and $\quad t_2 = 2\pi(c_2/g)^{1/2}$

Thus, $\quad t_1^2 - t_2^2 = 4\pi^2(c_1/g - c_2/g)$

or $\quad g\left(t_1^2 - t_2^2\right) = 4\pi^2(c_1 - c_2)$

Hence, $\quad g = \dfrac{4\pi^2(C_1 - C_2)}{T_1^2 - T_2^2}.$

6.(c) Here, $\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2xy+z^3 & x^2 & 3xz^2 \end{vmatrix}$

$$= \hat{i}\left(\frac{\partial}{\partial y}3xz^2 - \frac{\partial}{\partial z}x^2\right) + \hat{j}\left[\frac{\partial}{\partial z}(2xy+z^3) - \frac{\partial}{\partial x}3xz^2\right] + \hat{k}\left[\frac{\partial}{\partial x}x^2 - \frac{\partial}{\partial y}(2xy+z^3)\right]$$

$$= 0 + \hat{j}(3z^2 - 3z^2) + \hat{k}(2x - 2x) = 0$$

For a conservative force field, $\vec{\nabla}\times\vec{F} = 0$.

Work done

$$W = \int_A^B \vec{F}.d\vec{r} = \int_A^B \left(F_x.dx + F_y.dy + F_z.dz\right)$$

$$= \int_A^B (2xy + z^3)dx + x^2\,dy + 3xz^2\,dz$$

$$= \int_{(1,-2,1)}^{(3,1,4)} (2xydx + x^2dy) + (z^3dx + 3xz^2\,dz)$$

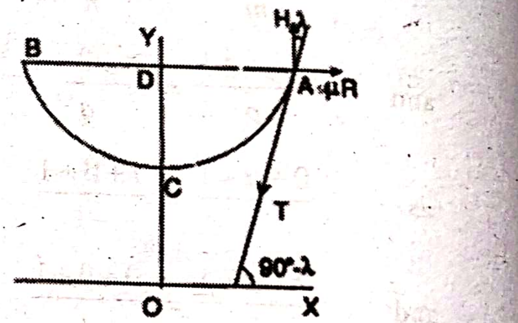
$$= \int_{(1,-2,1)}^{(3,1,4)} d(x^2y) + d(xz^3) = \int_{(1,-2,1)}^{(3,1,4)} d(x^2y + xz^3)$$

$$= \left[x^2y + xz^3\right]_{1,-2,1}^{3,1,4}$$

$$= 9 + 3.(4)^3 - 1(-2) - 1(1)^3$$

$$= 201 + 2 - 1 = 202.$$

7.(a) Let AB be the maximum span. Hence the end links A and B are in limiting equilibrium each under three forces namely the normal reaction R $\perp$ to AB (upwards) the force of friction μR along the fixed horizontal rod outwards and the tension T along the tangent at A (or B).



If S is the resultant of R and μR at A(say) inclined at λ(the angle of friction) to R, then the tension at A must balance S and therefore it is inclined at $(90° - \lambda)$ to the horizon. That is ψ at A = $(90° - \lambda)$,

and $\quad \tan\lambda = \dfrac{\mu R}{R} = \mu$

$\therefore$ Maximum span AB = $2x$

$$= 2c \log(\sec\psi + \tan\psi)$$
$$= 2c \log\{\sec(90 - \lambda) + \tan(90 - \lambda)\}$$
$$= 2c \log\{\operatorname{cosec}\lambda + \cot\lambda\}$$
$$= 2c\log\left\{\sqrt{\left(1+\frac{1}{\mu^2}\right)} + \frac{1}{\mu}\right\}$$
$$= 2c\log\left\{\frac{\sqrt{(\mu^2+1)}+1}{\mu}\right\} \quad ...(1)$$

And the length of chain

$$ACB = 2s = 2c \tan \psi = 2c \tan (90 - \lambda)$$

$$= 2c \cot \lambda = \frac{2c}{\mu} \qquad ...(2)$$

$$\Rightarrow \frac{\text{Maximum span AB}}{\text{Length of the chain ACB}}$$

$$= \mu \log \left\{ \frac{1+\sqrt{(1+\mu^2)}}{\mu} \right\}$$

[from (1) and (2) by division]

7.(b)

$$\frac{dy}{dx} = \frac{2(2x+3y)+5}{3y+2x+4}$$

let $\qquad 2x + 3y = t$

$$2+3\frac{dy}{dx} = \frac{dt}{dx}$$

$$\frac{dy}{dx} = \frac{1}{3}\left(\frac{dt}{dx}-2\right)$$

$$\frac{1}{3}\frac{dy}{dx}-\frac{2}{3} = \frac{2t+5}{t+4}$$

$$\frac{1}{3}\frac{dt}{dx} = \frac{2t+5}{t+4}+\frac{2}{3}$$

$$\Rightarrow \qquad = \frac{6t+15+2t+8}{3(t+4)}$$

$$\frac{1}{3}\frac{dt}{dx} = \frac{8t+23}{3(t+4)}$$

$$\int\left(\frac{t+4}{8t+23}\right)dt = \int dx$$

$$\frac{1}{8}\int\left(\frac{8t+32}{8t+32}\right)dt = x + c$$

$$= \frac{1}{8}\int\left(\frac{8t+23+9}{8t+23}\right)dt$$

$$= x + c$$

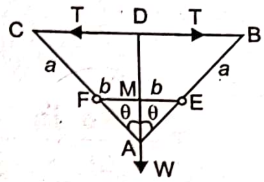
$$= \frac{1}{8}\int dt + \int\frac{9}{8t+23}dt$$

$$= x + c$$

$$= \frac{1}{8}|t|+\frac{9}{8}\int\frac{du}{u}$$

$$= \frac{1}{8}\left[2x+3y+\frac{9}{8}\ln(8t+23)\right] = x+c.$$

7.(c) ABC is framework consisting of three light rods AB, AC and BC. The rods AB and AC rest on two smooth pegs E and F which are in the same horizontal line and EF = $2b$. Each of the rods AB and AC is of length a.



Let T be the thrust in the rod BC which is given to be of length $\frac{3}{2}a$. A weight W is suspended from A. The line AD joining A to the middle point D of BC is vertical.

Let, $\qquad \angle BAD = \theta = \angle CAD.$

Replace the rod BC by two equal and opposite forces T as shown in the figure. Now give the system a small symmetrical displacement in which θ changes to $\theta + \delta\theta$. The line EF joining the pegs remains fixed, the lengths of the rods AB and AC do not change and the length BC changes.

The forces contributing to the sum of virtual works are: (*i*) the thrust T in the rod BC, and (*ii*) the weight W acting at A.

We have, $BC = 2BD = 2AB \sin \theta$

$$= 2a \sin \theta$$

Also the depth of the point of application A of the weight W below the fixed line EF

$$= MA = ME \cot \theta = b \cot \theta$$

The equation of virtual work is

$$T\delta(2a \sin \theta) + W\delta(b \cot \theta) = 0$$

or $\quad 2a\,T \cos \theta\, \delta\theta - bW \operatorname{cosec}^2 \theta\, \delta\theta = 0$

or $\quad (2a\,T \cos \theta - bW \operatorname{cosec}^2 \theta)\, \delta\theta = 0$

or $\quad 2a\,T \cos \theta - bW \operatorname{cosec}^2 \theta = 0$

$$[\because \delta\theta \neq 0]$$

or $\quad 2a\,T \cos \theta = bW \operatorname{cosec}^2 \theta$

or $\quad T = \frac{Wb}{2a}\operatorname{cosec}^2\theta \sec\theta$

But in the position of equilibrium,

$$BC = \frac{3}{2}a \text{ and so } BD = \frac{3}{4}a$$

Therefore, $\sin\theta = \dfrac{BD}{AB} = \dfrac{\frac{3}{4}a}{4} = \dfrac{3}{4}$

and $\cos\theta = \sqrt{(1-\sin^2\theta)}$

$$= \sqrt{\left(1-\frac{9}{16}\right)} = \frac{1}{4}\sqrt{7}$$

$\therefore$ $$T = \frac{Wb}{2a}\cdot\frac{16}{9}\cdot\frac{4}{\sqrt{7}} = \frac{32}{9\sqrt{7}}\frac{b}{a}W.$$

7.(*d*) The binormal b is a constant vector and γ is contained in a plane π, say, perpendicular to b. Now

$$\frac{d}{ds}\left(\gamma+\frac{1}{\kappa}n\right) = t+\frac{1}{\kappa}\dot{n} = 0$$

using the fact that the curvature κ is constant and the Frenet-Serret equation

$\dot{n} = -\kappa t + \tau b = -\kappa t$ (since $\tau = 0$)

Hence, $\gamma+\dfrac{1}{\kappa}n$ is a constant vector, say a, and we have

$$\|\gamma - a\| = \|-\frac{1}{\kappa}n\| = \frac{1}{\kappa}$$

This shows that γ lies on the sphere S, say, with centre a and radius $1/\kappa$. The intersection of π and S is a circle, say C, and we have shown that γ is a parametrization of part of C. If r is the radius of C, we have $\kappa = 1/r$, so $r = 1/\kappa$ is also the radius of S. It follows that C is a *great circle* on S, *i.e.*, that π passes through the centre a of S.

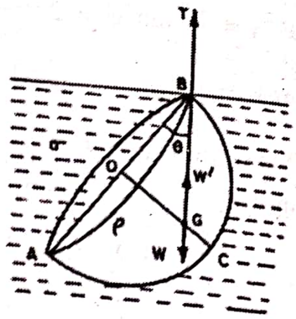
Thus, a is the centre of C and equation of π is $v \cdot b = a \cdot b$.

8.(*a*) ACB is the hemisphere of radius a and density ρ. Density of liquid is σ.

Since the hemisphere is completely immersed in the liquid, the weight of the body and force of buoyancy act at the same point G.

Here all the forces W, W′ and T act along the same vertical line BG.

$$OG = \frac{3a}{8};\ OB = a$$



(*i*) In ΔBOG, $\tan\theta = \dfrac{OG}{OB} = \dfrac{\frac{3a}{8}}{a} = \dfrac{3}{8}$

$\therefore$ $$\theta = \tan^{-1}\frac{3}{8}$$

(*ii*) Further T = W − W′

$$= \frac{2}{3}\pi a^3\rho g - \frac{2}{3}\pi a^3\sigma g = \frac{2}{3}\pi a^3(\rho-\sigma)g.$$

8.(*b*) Let $r(t)$ be the radius of the snowball at time t hours after the start of our "experiment," and let the initial radius of the snowball be $r(0)$ = R. The surface area of a sphere of radius r is $4\pi r^2$ and its volume is $4\pi r^3/3$. If we denote the uniform density of the snowball by ρ, then the mass of the snowball at any time t is

$$M(t) = \frac{4}{3}\pi\rho r^3(t) \quad ...(1)$$

The instantaneous rate of change of the mass of the snowball (the derivative of M(t) with respect to t) is then

$$\frac{dM}{dt} = 4\pi\rho r^2\frac{dr}{dt} \quad ...(2)$$

By assumption, dM/dt is proportional to S(t), the surface area at time t:

$$\frac{dM}{dt} = -4\pi r^2 k, \quad ...(3)$$

where k is the (positive) constant of proportionality, the negative sign implying that the mass is *decreasing* with time. By equating the last two expressions it follows that

$$\frac{dr}{dt} = -\frac{k}{\rho} = -\alpha, \text{ say} \quad ...(4)$$

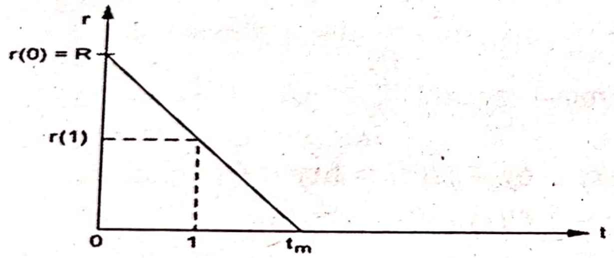
Note that this implies that according to this model, the radius of the snowball decreases uniformly with time. This means that the radius $r(t)$ is a linear function of t with slope $-\alpha$; since the initial radius is R, we must have

$$r(t) = R - \alpha t = R\left(1 - \frac{t}{t_m}\right) = 0$$

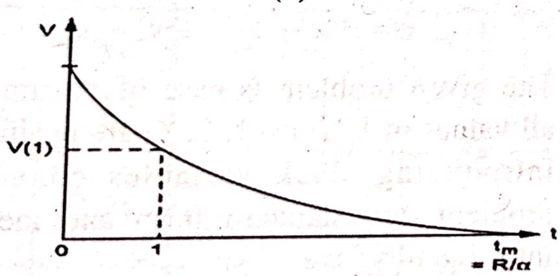
when $t = \frac{R}{\alpha} = t_m$ ...(5)

where t_m is the time for the original snowball to melt, which occurs when its radius is zero. However, we do not know the value of α since that information was not provided, but we are informed that after one hour, half the snowball has melted, so we have from equation (5) that $r(1) = R - \alpha$. A sketch of the linear equation in (5) and use of similar triangles (figure 1) shows that

$$t_m = \frac{R}{R - r(1)} \qquad \text{...(6)}$$



(*a*)



(*b*)

Fig. 1: ***Snowball radius r(t) and volume V(t)***

and furthermore

$$\frac{V(1)}{V(0)} = \frac{1}{2} = \frac{r^3(1)}{R^3} \qquad \text{...(7)}$$

so that, $\qquad r(1) = 2^{-1/3}\, R \approx 0.79\, R$

Hence, $t_m \approx 4.8$ hours, so that according to this model the snowball will take a little less than 4 more hours to melt away completely.

8.(*c*) For some constant $c \in \mathbb{R}$, where $R = 1/\kappa$, $T = 1/\tau$ and R' is the derivative of R with respect to s.

Let e_1, e_2, e_3 be the unit tangent, normal and binormal vectors of α, respectively.

Note that $\alpha(I)$ lies on a sphere if and only if there exists $p \in \mathbb{R}^3$ such that

$$\|\alpha(s) - p\|^2 \equiv \text{const} \Leftrightarrow \frac{d}{ds}\left(\|\alpha(s) - p\|^2\right) \equiv 0$$

$$\Leftrightarrow (\alpha(s) - p).\alpha'(s) \equiv 0$$

$$\Leftrightarrow (\alpha(s) - p).e_1(s) \equiv 0$$

$$\Leftrightarrow \quad \alpha(s) - p \equiv f(s)e_2(s) + g(s)e_3(s)$$

$$\Leftrightarrow \quad \alpha'(s) \equiv (f(s)e_2(s) + g(s)e_3(s))'$$

$$\Leftrightarrow \quad e_1(s) \equiv (f(s)e_2(s) + g(s)e_3(s))'$$

For some smooth functions f and g on I

So $\alpha(I)$ lies on a sphere if and only if

$$e_1 = f'e_2 + g'e_3 + fe_2' + ge_3'$$

On I for some smooth functions

$f, g : I \to \mathbb{R}$

By Frenet-Serret equations,

$$\begin{bmatrix} e_1' \\ e_2' \\ e_3' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & -\tau \\ 0 & \tau & 0 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix}$$

we have,

$$e_1 = f'e_2 + g'e_3 + fe_2' + ge_3'$$

$$\Leftrightarrow \quad e_1 = f'e_2 + g'e_3 + (-f\kappa e_1 - f\tau e_3) + g\tau e_2$$

$$\Leftrightarrow \quad e_1 = -f\kappa e_1 + (f' + g\tau)e_2 + (g' - f\tau)e_3$$

$$\Leftrightarrow \begin{cases} 1 = -f\kappa \\ 0 = f' + g\tau \\ 0 = g' - f'\tau \end{cases} \Leftrightarrow \begin{cases} 1 = -f\kappa \\ -\left(\dfrac{f'}{\tau}\right) = f\tau = g' \end{cases}$$

$$\Leftrightarrow \quad -\frac{1}{\kappa}\tau = \left(\frac{1}{\tau}\left(\frac{1}{\kappa}\right)'\right)'$$

$$\Leftrightarrow \quad -R = T(T\,R')'$$

$$\Leftrightarrow \quad -RR' = T\,R'(T\,R')'$$

$$\Leftrightarrow \quad -(R^2)' = (T\,R')^2)'$$

$$\Leftrightarrow \quad R^2 + (R')^2T^2 = \text{const}$$

Here, $\qquad$ constant = radius^2 = a^2.