1(a) → Let A be a 3×2 matrix and B a 2×3 matrix. Show that $C = AB$ is a singular matrix

Sol Given that A be a 3×2 matrix
B be a 2×3 matrix

$\therefore \rho(A) \le 2$, & $\rho(B) \le 2$

and $\rho(AB) \le \min\{\rho(A), \rho(B)\}$.

Let $\rho(A) = r_1$; $\rho(B) = r_2$ and $\rho(AB) = r$

We know that $\exists$ a non-singular matrix P such that $PA = \begin{bmatrix} G \\ 0 \end{bmatrix}$,
where G is of order $r_1 \times 2$ and 0 is a zero matrix of order $(3-r_1)\times 2$

Now by postmultiplying both sides by B
$\therefore$, we have $PAB = \begin{bmatrix} G \\ 0 \end{bmatrix} B$

$\therefore \rho(PAB) = \rho(AB) = r$.

$\therefore$ rank of the matrix $\begin{bmatrix} G \\ 0 \end{bmatrix} B = r$

Since the matrix G has only r_1 non-zero rows.

$\therefore \begin{bmatrix} G \\ 0 \end{bmatrix} B$ can not have more than r_1 non-zero rows.

$\therefore$ Rank of the matrix $\begin{bmatrix} G \\ 0 \end{bmatrix} B \le r_1$

$\therefore r \le r_1$

i.e $\rho(AB) \le \rho(A)$ (i.e A is the pre-factor) — (1)

again $\rho(AB) = \left[\rho(AB)^T\right]$

$= \rho(B^T A^T)$

$\le \rho(B^T)$ (by using (1)) i.e $\rho(AB) \le \rho(A)$

$= \rho(B)$

$= r_2$

$\therefore \rho(AB) \le \rho(B)$ — (2).

From (1) & (2) we have

$\rho(AB) \le \rho(A)$ and $\rho(AB) \le \rho(B)$

Since A is of 3×2 order matrix

B is of 2×3 order matrix

$\therefore$ AB is of 3×3 order matrix

and $\rho(AB) \le 2$.

$\therefore$ AB is singular matrix.

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1(b) Express basis vectors $e_1=(1,0)$ and $e_2=(0,1)$ as linear combination of $\alpha_1=(2,-1)$ and $\alpha_2=(1,3)$.

Soln Basis vectors

$$e_1=(1,0) \text{ \& } e_2=(0,1)$$

given vectors.

$$\alpha_1=(2,-1) \text{ \& } \alpha_2=(1,3)$$

e_1 and e_2 can be represented as linear combination of given vectors.

$$e_1 = a\alpha_1+b\alpha_2$$
$$e_2 = c\alpha_1+b\alpha_2$$

$$e_1=(1,0) = (a)(2,-1)+(b)(1,3)$$

$$2a+b=1$$
$$-a+3b=0$$

by solving above eqn we get -

$$a=3/7,\ b=1/7 \text{ ——— (i)}$$

Similarly c and d

$$e_2=(0,1)=(c)(2,-1)+(d)(1,3)$$

$$2c+d=0$$
$$-c+3d=1$$

$$c=-1/7,\ d=2/7 \text{ ——— (ii)}$$

so from (i) and (ii) we get

$$\boxed{e_1=\frac{3\alpha_1+\alpha_2}{7}} \text{ \& } \boxed{e_2=\frac{-\alpha_1+2\alpha_2}{7}}$$

1(c) Determine if $\lim_{z\to 1}(1-z)\tan\frac{\pi z}{2}$ exists or not. If the limit exists, then find its value.

Sol^n Given function f to be continuous at $z=1$, we must have

$$\lim_{z\to 1} f(z) = f(1)$$

$$\therefore f(1) = \lim_{z\to 1}(1-z)\tan\left(\pi\frac{z}{2}\right) + \cdots \text{ (1)}$$

let $z = 1+h$, ie. $(z-1)=h$, such that as $z\to 1, h\to 0$

$\therefore$ by (1) –

$$f(1) = \lim_{h\to 0}\left\{-h\tan\left(\frac{\pi}{2}(1+h)\right)\right\}$$

$$= \lim_{h\to 0}\left\{-h\cdot\tan\left(\frac{\pi}{2}+\frac{\pi}{2}h\right)\right\}$$

$$= \lim_{h\to 0}\left\{(-h)\left(-\cot\left(\frac{\pi}{2}h\right)\right)\right\}$$

$$= \lim_{h\to 0}\left\{(-h)\left(-\cot\left(\frac{\pi}{2}h\right)\right)\right\}$$

$$= \lim_{h\to 0} h\cot\left(\frac{\pi}{2}h\right)$$

$$= \lim_{h\to 0}\left(\frac{h}{\sin\left(\frac{\pi}{2}h\right)}\right)\cdot\cos\left(\frac{\pi}{2}h\right)$$

$$= \left\{1\cdot\left(\frac{2}{\pi}\right)\right\}\cos 0 \quad \cdots \quad \left[\because \lim_{y\to 0}\frac{y}{\sin y} = 1\right]$$

$$\Rightarrow f(1) = \frac{2}{\pi}$$

Q1(d) Find the limit $\lim_{n\to\infty} \frac{1}{n^2} \sum_{r=0}^{n-1} \sqrt{n^2-r^2}$

Sol: We have, the limit of given function be l.

$$l = \lim_{n\to\infty} \frac{1}{n^2} \sum_{r=0}^{n-1} \sqrt{n^2-r^2}$$

$$= \lim_{n\to\infty} \frac{1}{n^2}(n) \sum_{r=0}^{n-1} \sqrt{1-\frac{r^2}{n^2}}$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=0}^{n-1} \sqrt{1-\frac{r^2}{n^2}}$$

$$l = \lim_{n\to\infty} \sum_{r=0}^{n-1} \cdot \frac{1}{n} \cdot \sqrt{1-\frac{r^2}{n^2}} \quad \text{---(1)}$$

$\sum_{r=0}^{n-1} \frac{1}{n}\sqrt{1-\frac{r^2}{n^2}}$ is in form of Reimon series.

S.t $\quad R(n) = \sum_{r=0}^{n-1} \sqrt{1-\frac{r^2}{n^2}} \times \frac{1}{n}$

$$\left[\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{r=1}^{n} \Delta x f(x_r) \right.$$

$$\Delta x = \frac{b-a}{n}$$

$$\left. x_r = a + (\Delta x) r \right]$$

Using above

$$\frac{1}{n} = \frac{b-a}{n}$$

$$\therefore b-a = 1.$$

Also

$$x_0 = 0 = a + \left(\frac{1}{n}\right)0$$

$$\therefore a = 0$$

$x_0 = \frac{r}{n}$

So

$$l = \int_0^1 \sqrt{1-x^2}\, dx$$

$$l = \frac{1}{2}\left[x\sqrt{1-x^2} + \sin^{-1}x\right]_0^1$$

$$l = \frac{1}{2} \times \frac{\pi}{2} = \frac{\pi}{4}$$

$$\boxed{\therefore l = \frac{\pi}{4}}$$

Hence; $\boxed{\lim_{n\to\infty} \frac{1}{n^2} \sum_{r=0}^{n-1} \sqrt{n^2-r^2} = \frac{\pi}{4}}$

Q.1(e) Find the projection of the straight line $\frac{x-1}{2} = \frac{y-1}{3} = \frac{z+1}{-1}$ on the plane $x+y+2z=6$.

Sol: Given line is

$$\frac{x-1}{2} = \frac{y-1}{3} = \frac{z+1}{-1} = r \text{ (say)}$$

any point on it is

$$x = 2r+1 \;;\; y = 3r+1 \;;\; z = -r-1.$$

$\therefore$ Co-ordinate of the point $P(2r+1, 3r+1, -(r+1))$

If it passes through the given plane, then it must intersect, then the point of intersection can be find as,

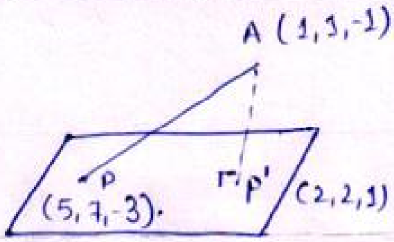
Given plane $\Rightarrow x+y+2z=6$

$$2r+1+3r+1+2(-r-1) = 6$$

$$5r+\cancel{2}-2r-\cancel{2} = 6$$

$$3r = 6$$

$$\boxed{r=2}$$



$\therefore$ The co-ordinate of P is

$$[2\times2+1,\; 3\times2+1,\; -(2+1)]$$

$$P = [5, 7, -3].$$

Now, equation of Normal of plane through A (1, 1, -1) is

$$\frac{x-1}{1} = \frac{y-1}{1} = \frac{z+1}{2} = s \text{ — say.}$$

$\therefore$ Co-ordinates of $P'(s+1, s+1, 2s-1)$

as P' is on Plane.

$$s+1+s+1+2(2s-1) = 6$$

$$2s+\cancel{2}+4s-\cancel{2} = 6$$

$$6s = 6$$

$$\boxed{s=1}$$

$\therefore$ Co-ordinates of $P'(2, 2, 1)$

and Direction ratios of $PP' = (3, 5, -4)$

$\therefore$ The Required Projection:–

$$\boxed{\frac{x-5}{3} = \frac{y-7}{5} = \frac{z+3}{-4}}$$

2(a) → Show that if A and B are similar $n\times n$ matrices then they have the same eigen values.

Sol Given that A and B are similar $n\times n$ matrices

$\therefore$ there exists invertible matrix P such that $B = P^{-1}AP$.

We have

$$B-\lambda I = P^{-1}AP-\lambda I$$
$$= P^{-1}AP - \lambda P^{-1}P$$
$$= P^{-1}AP - P^{-1}\lambda P$$
$$= P^{-1}AP - P^{-1}\lambda(IP)$$
$$= P^{-1}AP - P^{-1}(\lambda I)P$$
$$= P^{-1}(A-\lambda I)P.$$

$$\therefore |B-\lambda I| = |P^{-1}(A-\lambda I)P|$$
$$= |P^{-1}|\,|A-\lambda I|\,|P|$$
$$= |A-\lambda I|\,|P^{-1}P|$$
$$= |A-\lambda I|$$

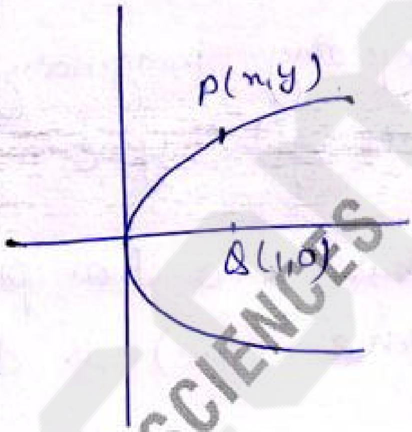
$\therefore |B-\lambda I| = |A-\lambda I|$.

$\therefore$ A and B have the same characteristic polynomial and hence same characteristic (eigen) roots (values).

2(b) Find the shortest distance from the point (1,0) to the parabola $y^2=4x$.

Sol^n The equation of the given curve is

$$y^2=4x$$



Let $P(x,y)$ be a point on the curve, which is nearest to point $(1,0)$.

Now, distance between the points P and Q is given by:-

$$PQ = \sqrt{(x-1)^2+y^2}$$

$$= \sqrt{\left(\frac{y^2}{4}-1\right)^2+(y^2)}$$

$$= \sqrt{\left[\frac{y^4}{16}-\frac{y^2}{2}+1\right]+(y^2)}$$

$$= \sqrt{\frac{y^4}{16}+\frac{y^2}{2}+1}$$

Let $z = PQ^2 = \frac{y^4}{16}+\frac{y^2}{2}+1$

Clearly z is maximum or minimum accordingly as PQ is maximum. Also Q will be nearest to the point P if PQ is minimum.

$$\frac{dy}{dz} = \frac{y^3}{4}+y=0$$

$\Rightarrow y(y^2+4)=0$

$\Rightarrow y=0, \; y=\pm 2i$

rejecting imaginary values we get $y=0$.

Thus $x=0$ (on putting $y=0$ in $y^2=4x$)

Hence, $(0,0)$ is closest to the point $(1,0)$.

2(c) → The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ revolves about the x-axis. find the volume of the solid of revolution.

Sol^n → If y is given as a function of x, volume of the solid obtained by rotating the portion of the curve between $x=a$ and $x=b$ about the x-axis is given by

$$\boxed{V=\int_a^b \pi y^2 dx.}$$

By rotating the ellipse around the x-axis, we generate a solid of revolution called an ellipsoid whose volume can be calculated using the disk method.

We revolve around x-axis a thin vertical strip of height $y=f(x)$ and thickness dx.

This generates a disk of radius y and thickness dx whose volume is dV.

$$dV = (\text{area of the disk})\, dx$$

$$dV = \pi r^2 dx$$

$$dV = \pi y^2 dx$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \Rightarrow y^2=b^2\left(1-\frac{x^2}{a^2}\right)$$

Substituting $y^2=b^2\left(1-\frac{x^2}{a^2}\right)$ in $dV=\pi y^2 dx$

$$dV=\pi b^2\left(1-\frac{x^2}{a^2}\right)$$

We get the volume of the ellipsoid by filling it with a very large number of very thin disks, that is by integrating dV from $x=-a$ to $x=a$.

Volume of the ellipsoid $= V = \int_{-a}^{a} dV = \int_{-a}^{a} \pi y^2 dx$

$$V = \int_{-a}^{a} \pi b^2 \left(1 - \frac{x^2}{a^2}\right) dx$$

On integrating we get —

$$V = \frac{4}{3}\pi a b^2$$

2(d) Find the shortest distance between the lines

$$a_1x + b_1y + c_1z + d_1 = 0$$
$$a_2x + b_2y + c_2z + d_2 = 0$$

and the z-axis.

Soln The plane through the given line is

$$(a_1x+b_1y+c_1z+d_1) + \lambda(a_2x+b_2y+c_2z+d_2) = 0$$

or $(a_1+\lambda a_2)x + (b_1+\lambda b_2)y + (c_1+\lambda c_2)z + (d_1+\lambda d_2) = 0$ — (1)

If this plane is parallel to z-axis, whose d.c's are 0, 0, 1 then the normal to the plane (i) is perpendicular to z-axis and we get

$$(a_1+\lambda a_2)\cdot 0 + (b_1+\lambda b_2)\cdot 0 + (c_1+\lambda c_2)\cdot 1 = 0$$

or, $\lambda = -c_1/c_2$

∴ from (1) the equation of the plane through the given line and parallel to z-axis is

$$(a_1x+b_1y+c_1z+d_1) - (c_1/c_2)(a_2x+b_2y+c_2z+d_2) = 0$$

or. $(c_2a_1-c_1a_2)x + (c_2b_1-b_2c_1)y + (d_1c_2-c_1d_2) = 0$ — (2)

Also any point on the z-axis can be taken as origin i.e. (0,0,0)

Required S.D. = length of perpendicular from (0,0,0) to the plane (2)

$$= \frac{(d_1c_2-c_1d_2)}{\sqrt{(c_2a_1-c_1a_2)^2+(c_2b_1-b_2c_1)^2}}.$$

3(a) → For the system of linear equations

$$x+3y-2z=-1$$

$$5y+3z=-8$$

$$x-2y-5z=7$$

determine which of the following statements are true and which are false.

(i) The system has no solution

(ii) The system has a unique solution

(iii) The system has infinitely many solutions.

Sol Given that

$$x+3y-2z=-1$$

$$0x+5y+3z=-8$$

$$x-2y-5z=7$$

We write single matrix equation

$$AX=B \quad \text{—— (1)}$$

where $A=\begin{bmatrix}1 & 3 & -2\\ 0 & 5 & 3\\ 1 & -2 & -5\end{bmatrix}$; $X=\begin{bmatrix}x\\ y\\ z\end{bmatrix}$; $B=\begin{bmatrix}-1\\ -8\\ 7\end{bmatrix}$

We have

$$[A|B]=\left[\begin{array}{ccc|c}1 & 3 & -2 & -1\\ 0 & 5 & 3 & -8\\ 1 & -2 & -5 & 7\end{array}\right]$$

$$\sim\left[\begin{array}{ccc|c}1 & 3 & -2 & -1\\ 0 & 5 & 3 & -8\\ 0 & -5 & -3 & 8\end{array}\right] \quad R_3\to R_3-R_1$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & -2 & -1 \\ 0 & 5 & 3 & -8 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad R_3 \to R_3 + R_2$$

Clearly it is in echelon form

$\therefore \rho(A) = \rho(A|B) = 2 <$ number of unknown variables x, y, z.

$\therefore$ The given system of equations are consistent and have infinitely many solution

$\therefore$ (i) and (ii) are false.

(iii) is true.

3(b) → Let $f(x,y) = xy^2$ if $y>0$
$= -xy^2$, if $y \le 0$

Determin which of $\frac{\partial f}{\partial x}(0,1)$ and $\frac{\partial f}{\partial y}(0,1)$ exists and which does not exist.

we have

$$\frac{\partial f}{\partial x}(0,1) = \underset{\delta x \to 0}{Lt} \frac{f(0+\delta x, 1) - f(0,1)}{\delta x}$$

$$= \underset{\delta x \to 0}{Lt} \frac{f(\delta x, 1) - f(0,1)}{\delta x}$$

$$= \underset{\delta x \to 0}{Lt} \frac{\delta x (1)^2 - 0}{\delta x} = 1.$$

we have

$$\frac{\partial f}{\partial y}(0,1) = \underset{\delta y \to 0}{Lt} \frac{f(0, 1+\delta y) - f(0,1)}{\delta y}$$

$$= \underset{\delta y \to 0}{Lt} \frac{0(1+\delta y)^2 - 0}{\delta y}$$

$$= 0$$

$\therefore$ $\frac{\partial f}{\partial x}(0,1)$ and $\frac{\partial}{\partial y}(0,1)$ both exist

3(c) P-I → Find the equations to the generating lines of the paraboloid $(x+y+z)(2x+y-z)=6z$ which pass through the point $(1,1,1)$.

Sol^n → The equation of the two generators of λ-μ system can be written as

$$x+y+z=6\lambda,\ 2x+y-z=z/\lambda \quad \text{——} (1)$$

and $$x+y+z=z/\mu,\ 2x+y-z=6\mu \quad \text{——} (2)$$

If these pass through the point $(1,1,1)$ then

$$3=6\lambda \text{ and } 2=6\mu \Rightarrow \lambda=\tfrac{1}{2},\ \mu=\tfrac{1}{3}$$

∴ from (1) & (2) the generators are given by:

$$x+y+z=3,\ 2x+y-z=2z$$

and $$x+y+z=3z,\ 2x+y-z=2$$

i.e. $$x+y+z=3,\ 2x+y-3z=0 \quad \text{——} (3)$$

and $$x+y-2z=0,\ 2x+y-z=2 \quad \text{——} (4)$$

We can find that direction ratio of the generators given by (3) and (4) are $4,5,-1$ and $1,-3,-1$ respectively and as they pass through the given point $(1,1,1)$ so their equations are

$$\frac{x-1}{4}=\frac{y-1}{-5}=\frac{z-1}{-1} \text{ and } \frac{x-1}{1}=\frac{y-1}{-3}=\frac{z-1}{-1}$$

3(d) → Find the equation of the sphere in xyz-plane passing through the points $(0,0,0)$, $(0,1,-1)$, $(-1,2,0)$ and $(1,2,3)$.

Sol: Let the required sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad \text{---(1)}$$

Since it is passing through the point $(0,0,0)$

$$\therefore d=0$$

$$x^2+y^2+z^2+2ux+2vy+2wz=0 \quad \text{---(2)}$$

Since it is passing through the points $(0,1,-1)$, $(-1,2,0)$ and $(1,2,3)$.

at $(0,1,-1)$ $\quad 1+1+2v-2w=0,$

at $(-1,2,0)$ $\quad 1+4-2u+4v=0$

at $(1,2,3)$ $\quad 1+4+9+2u+4v+6w=0$

Solving above three equations,

$$2+2v-2w=0$$

$$v-w=-1 \text{ or } w-v=1 \quad \text{---(I)}$$

$$4v-2u=-5 \quad \text{---(II)}$$

$$u+2v+3w=-7 \quad \text{---(III)}$$

Putting w from (1) in (3)

$$u+2v+3(v+1)=-7$$

$$u+5v=-10 \quad \text{---(IV)}$$

using equation (ii) & (iv)

$$4v - 2u = -5$$
$$u + 5v = -10$$
$$u = -10 - 5v$$

$$4v - 2(-10-5v) = -5$$
$$4v + 20 + 10v = -5$$
$$14v = -25$$
$$\boxed{v = \frac{-25}{14}}$$

$$u + 5 \times \frac{-25}{14} = -10$$
$$u = -10 + \frac{125}{14}$$
$$u = \frac{-140+125}{14}$$
$$\boxed{u = \frac{-15}{14}}$$

Hence; $u = -\frac{15}{14}$, $v = -\frac{25}{14}$

Put these values in eqn (iii)

$$-\frac{15}{14} + 2 \times \frac{-25}{14} + 3w = -7$$
$$3w = -7 + \frac{15}{14} + \frac{50}{14}$$
$$3w = \frac{-98+65}{14} \Rightarrow w = \frac{-\cancel{33}}{\cancel{3} \times 14} = \frac{-11}{14}$$

$\therefore u = \frac{-15}{14}$, $v = \frac{-25}{14}$, $w = \frac{-11}{14}$

Put these values in eqn (2)

$$x^2+y^2+z^2+2ux+2vy+2wz=0$$

$$x^2+y^2+z^2-2\times\frac{15}{14}x+2\times\frac{-25}{14}y+2\times\frac{-11}{14}z=0$$

$$14x^2+14y^2+14z^2-30x-50y-22z=0$$

is the required solution

or

$$7x^2+7y^2+7z^2-15x-25y-11z=0$$

or

$$7(x^2+y^2+z^2)-[15x+25y+11z]=0$$

is the required equation of Sphere.

Q.4(a) Find the minimum and maximum values of x^4-5x^2+4 on the interval $[2,3]$.

Sol: Given; $f(x) = x^4-5x^2+4$

$f'(x) = 4x^3-10x$

let $f'(x)=0$

$4x^3-10x=0$

$2x(2x^2-5)=0$

$x=0, \pm\sqrt{5/2}$

These are the critical points of the $f(x)$, which $\notin [2,3]$. Thus, $f(x)$ is monotonic on $[2,3]$

Now, to check, whether it is monotonic increasing or decreasing-

$f'(2) = 2\times 2(2\times 4-5) = 4[8-5] = 4\times 3 = 12>0$

$f'(3) = 2\times 3[2\times 9-5] = 6[18-5] = 6\times 13 = 78>0$

Since, $f(2)>0$ thus $f(x)$ is monotic increasing

Since; $f'(2), f'(3)>0$ and $f'(3)>f'(2)$.

Hence, $f(x)$ is monotonically increasing on $[2,3]$.

Thus; $f_{min} = f(x)\Big|_{x=2} = (2)^4-5(2)^2+4$

$= 16-20+4$

$=0$

$f_{max} = f(x)\Big|_{x=3} = (3)^4-5(3)^2+4$

$= 81-5\times 9+4$

$= 81+4-45$

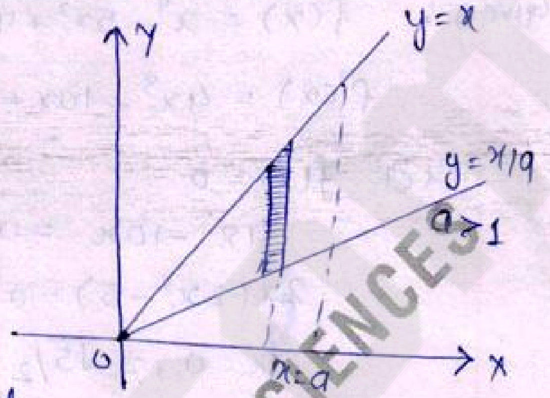
$= 85-45 = 40$

$\therefore f_{min}=0$ & $f_{max}=40$

Q.4 (b) Evaluate the Integral $\int_0^a \int_{x/a}^{x} \frac{x\,dy\,dx}{x^2+y^2}$?

Sol: Given integral is

$$\int_0^a \int_{x/a}^{x} \frac{x\,dy\,dx}{x^2+y^2}$$



as the integral limits are variable with respect to x, we integrate first with respect to y, (treating x as constant) from inside out.

Thus;

$$I = \int_0^a \left[\int_{x/a}^{x} \frac{x\,dy}{x^2+y^2}\right] dx$$

$$I = \int_0^a \left[x\left[\frac{1}{x}\tan^{-1}y/x\right]_{x/a}^{x}\right] dx$$

$$I = \int_0^a \left[\tan^{-1} x/x - \tan^{-1}\frac{x/a}{x}\right] dx$$

$$I = \int_0^a \left[\tan^{-1}1 - \tan^{-1}\frac{1}{a}\right] dx.$$

$$I = \int_0^a \left[\frac{\pi}{4} - \tan^{-1}(1/a)\right] dx = \left[\frac{\pi}{4} - \tan^{-1}\frac{1}{a}\right]\int_0^a dx$$

$$I = \left[\frac{\pi}{4} - \tan^{-1}\frac{1}{a}\right]\left[x\right]_0^a = \left[\frac{\pi}{4} - \tan^{-1}\frac{1}{a}\right][a-0]$$

$$\boxed{I = a\left[\frac{\pi}{4} - \tan^{-1}\frac{1}{a}\right]}$$

4(c) → Find the equation of the cone with $(0,0,1)$ as the vertex and $2x^2-y^2=4$, $z=0$ as the guiding curve.

Sol The given base conic is

$$2x^2-y^2=4,\ z=0 \quad \text{---(1)}$$

Now the equations of any line through $(0,0,1)$ is

$$\frac{x-0}{l}=\frac{y-0}{m}=\frac{z-1}{n} \quad \text{---(2)}$$

Since it meets the plane $z=0$ where $\frac{x-0}{l}=\frac{y-0}{m}=\frac{0-1}{n}$

$$\Rightarrow \frac{x}{l}=\frac{-1}{n};\quad \frac{y}{m}=\frac{-1}{n}$$

$$\Rightarrow x=-\frac{l}{n};\quad y=-\frac{m}{n}$$

$$\therefore (x,y,0)=\left(-\frac{l}{n},-\frac{m}{n},0\right)$$

Since it lies on the conic

$$2\left(\frac{-l}{n}\right)^2-\left(\frac{-m}{n}\right)^2=4.$$

$$\Rightarrow 2\frac{l^2}{n^2}-\frac{m^2}{n^2}=4. \quad \text{---(3)}$$

Now eliminating l,m,n from (2) & (3) we have

$$2\left(\frac{x^2}{(z-1)^2}\right)-\frac{y^2}{(z-1)^2}=4.$$

$$\Rightarrow 2x^2-y^2=4(z-1)^2$$ which is the required equation of the cone.

4(d) P-I Find the equation of the plane parallel to $3x-y+3z=8$ and passing through the point $(1,1,1)$.

Sol^n We know that any plane, parallel to $3x-y+3z-8$ is $3x-y+3z+k=0$

Since this plane passes to point $(1,1,1)$ then,

$$3\cdot 1-1+3\cdot 1+k=0$$

$$\boxed{k=-5}$$

$\boxed{3x-y+3z-5=0}$ is the required plane.

Section - B

5 (a). Solve. $y'' - y = x^2 \cdot e^{2x}$.

Sol:- Given Differential Equation is

$$y'' - y = x^2 \cdot e^{2x}.$$

It can be re-written as.

$$(D^2 - 1)y = x^2 . e^{2x}$$

where; $D = \frac{dx}{dy}$

The auxillary equation

$$m^2 - 1 = 0 \quad ; m = \pm 1$$

$$y_c = c_1 \cdot e^{x} + c_2 e^{-x}$$

Now for y_p -

$$y_p = \frac{1}{(D^2-1)} e^{2x} . x^2 .$$

$$y_p = e^{2x} . \frac{1}{((D+2)^2 - 1)} x^2 = e^{2x} . \frac{1}{(D^2 + 4D + 3)} \cdot x^2$$

$$y_p = e^{2x} \cdot \frac{1}{3}\left[\frac{1}{1 + \left(\frac{D^2}{3} + \frac{4D}{3}\right)}\right] x^2$$

$$y_p = \frac{e^{2x}}{3}\left[1 + \frac{D^2 + 4D}{3}\right]^{-1} x^2$$

$$y_p = \frac{e^{2x}}{3}\left[1 - \frac{D^2+4D}{3} + \frac{(D^2+4D)^2}{9} \cdots\right] x^2$$

$$y_p = \frac{e^{2x}}{3}\left[x^2 - \left(\frac{2+8x}{3}\right) + \frac{32}{9}\right] = \frac{e^{2x}}{3}\left[x^2 - \frac{8x}{3} + \frac{26}{9}\right]$$

General Solution -

$$\therefore y = y_c + y_p = c_1 e^{x} + c_2 e^{-x} + \frac{e^{2x}}{3}\left[x^2 - \frac{8x}{3} + \frac{26}{9}\right]$$

5(b) IAS 2018 P-I

Find the angle between the tangent at a general point of the curve whose equations are $x=3t$, $y=3t^2$, $z=3t^3$ and the line $y=z-x=0$

Sol^n

$$\vec{r_1} = 3t\hat{i} + 3t^2\hat{j} + 3t^2\hat{k}$$

$$\frac{d\vec{r_1}}{dt} = 3\hat{i} + 6t\hat{j} + 9t^2\hat{k}$$

$$y = z - x = 0$$

$$y = 0 \quad , \quad x = z$$

$$\frac{x}{1} = \frac{y}{0} = \frac{z}{1} \Rightarrow [1,0,1]$$

$$\vec{r_2} = \hat{i} + \hat{k} \quad , \quad |\vec{r_2}| = \frac{1}{\sqrt{1+1}} = \frac{1}{\sqrt{2}}$$

$$\vec{r} = \frac{1}{\sqrt{2}}(\hat{i} + \hat{k})$$

$$\therefore \cos\theta = \frac{\vec{r_1}\cdot\vec{r_2}}{|\vec{r_1}||\vec{r_2}|} = \frac{3\cdot 1 + 6t\times 0 + 9t^2\cdot 1}{\sqrt{9+36t^2+81t^4}\cdot\sqrt{2}}$$

$$= \frac{3+9t^2}{3\sqrt{2}\cdot\sqrt{1+4t^2+9t^4}}$$

$$= \frac{1+3t^2}{\sqrt{2}\cdot\sqrt{1+4t^2+9t^4}}$$

$$\Rightarrow \boxed{\theta = \cos^{-1}\left[\frac{1+3t^2}{\sqrt{2}\cdot\sqrt{1+4t^2+9t^4}}\right]}$$

5(c) Solve:- $y''' - 6y'' + 12y' - 8y = 12e^{2x} + 27e^{-x}$.

Sol: Given D.E.

$$y''' - 6y'' + 12y' - 8y = 12e^{2x} + 27e^{-x}.$$

It can be rewritten as:

$$(D^3 - 6D^2 + 12D - 8)y = 12e^{2x} + 27e^{-x}.$$

The Auxillary eqn.

$$m^3 - 6m^2 + 12m - 8 = 0$$

$$(m-2)(m^2 - 4m + 4) = 0$$

$$(m-2)(m-2)(m-2) = 0$$

$$m = 2, 2, 2.$$

$$y_c = C.F = (C_1 + C_2 x + C_3 x^2)e^{2x}.$$

$$y_p = \frac{1}{(D-2)^3}\left[12e^{2x} + 27e^{-x}\right]$$

$$y_p = \frac{12}{(D-2)^3} \cdot e^{2x} + \frac{27}{(D-2)^3} \cdot e^{-x}$$

$$y_p = 12 \cdot \frac{x^3}{3!} \cdot e^{2x} + 27 e^{-x} \cdot \frac{1}{(-1-2)^3}.$$

$$y_p = 2x^3 \cdot e^{2x} + \frac{27}{-27} \cdot e^{-x}$$

$$y_p = 2x^3 \cdot e^{2x} - e^{-x}.$$

$\therefore$ The solution for given D.E

$$y = y_c + y_p$$

$$\boxed{y = (C_1 + C_2 x + C_3 x^2)e^{2x} + 2x^3 e^{2x} - e^{-x}}$$

Que: 5)(d)(i) Find the laplace Transform of $f(t) = \frac{1}{\sqrt{t}}$.

Sol:- Given: $f(t) = \frac{1}{\sqrt{t}}$

$$L\left\{\frac{1}{\sqrt{t}}\right\} = \int_{t=0}^{\infty} e^{-st} \cdot \frac{1}{\sqrt{t}}\, dt$$

Put $u = st \Rightarrow t = u/s$

$du = s\,dt \Rightarrow dt = \frac{du}{s}$

$\sqrt{t} = \sqrt{\frac{u}{s}} \Rightarrow \frac{1}{\sqrt{t}} = \frac{\sqrt{s}}{\sqrt{u}}$.

$$= \int_{u=0}^{\infty} e^{-u} \cdot \frac{\sqrt{s}}{\sqrt{u}} \cdot \frac{1}{s}\, du$$

$$= \frac{\sqrt{s}}{s} \int_{u=0}^{\infty} e^{-u} \cdot u^{-1/2}\, du$$

$$= \frac{1}{\sqrt{s}} \int_{u=0}^{\infty} e^{-u} \cdot u^{-1/2}\, du.$$

put $\omega = \sqrt{u} \Rightarrow u = \omega^2$

$d\omega = \frac{1}{2\sqrt{u}}\, du.$

$du = 2\sqrt{u}\, d\omega$

$$= \frac{1}{\sqrt{s}} \int_{\omega=0}^{\infty} e^{-\omega^2} \cdot \frac{1}{\sqrt{u}} \cdot 2\cdot\sqrt{u}\, d\omega$$

$$= \frac{2}{\sqrt{s}} \int_{\omega=0}^{\infty} e^{-\omega^2} d\omega = \frac{2}{\sqrt{s}} \cdot \frac{\sqrt{\pi}}{2}$$

$$\boxed{L\left\{\frac{1}{\sqrt{t}}\right\} = \sqrt{\frac{\pi}{s}}} \qquad \left[\because \int_{x=0}^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}\right]$$

5(d)(ii) Find the Inverse Laplace transform of

$$\frac{5s^2+3s\mp 16}{(s-1)(s-2)(s+3)}.$$

Sol: Given $F(s) = \dfrac{5s^2+3s\mp 16}{(s-1)(s-2)(s+3)}$

$$L^{-1}\left\{\frac{5s^2+3s+6}{(s-1)(s\mp 2)(s+3)}\right\} = L^{-1}\left\{\frac{A}{s-1}+\frac{B}{s-2}+\frac{C}{s+3}\right\}$$

$$5s^2+3s+6 = A(s-2)(s+3)+B(s-1)(s+3)+C(s-1)(s\mp 2)$$

By equating

$$A+B+C=5$$
$$A+2B-3C=3$$
$$-6A-3B+2C=-16$$

By solving above three equation.

we get $A=2,\ B=2,\ C=1$

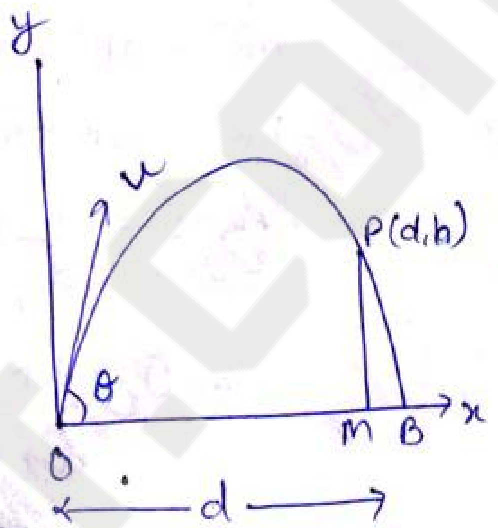
$$\therefore L^{-1}\left[\frac{2}{s-1}+\frac{2}{s-2}+\frac{1}{s+3}\right]$$

$$f(t) = 2e^{t}+2e^{2t}+e^{-3t}.$$

$$\boxed{f(t) = 2\left[e^{t}+e^{2t}\right]+e^{-3t}}$$

Q.5 (e) A particle projected from a given point on ground just clears a wall of height 'h' at a distance 'd' from the point of projection. If the particle moves in a vertical plane and if the horizontal range is 'R'. Find the elevation of projection?

Sol: Let a particle projected from O with a velocity u at an angle θ to the horizontal and vertical lines OX and OY in the plane of projection as the co-ordinate axes.



Here

$$x = u\cos\theta\, t \quad \text{—— (1)}$$

$$y = u\sin\theta\, t - \frac{1}{2}gt^2 \quad \text{—— (2)}$$

The equation of trajectory is –

$$y = x\tan\theta - \frac{1}{2}\frac{gx^2}{u^2\cos^2\theta} \quad \text{—— (3)}$$

$$h = x\tan\theta\left[1 - \frac{x}{R}\right]$$

Now $y = h$
$x = d$

[The particle just clears the wall PM of height h at a distance d from O and strikes ground at the point B at a distance R from O. Thus both the points (d, h) and $(R, 0)$ lie on the curve (3).

Therefore $h = d\tan\theta - \frac{1}{2}g\frac{d^2}{u^2\cos^2\theta}$ ——— (4)

and $0 = R\tan\theta - \frac{1}{2}g\frac{R^2}{u^2\cos^2\theta}$ ——— (5)

To eliminate u^2, we multiply (4) by R^2 and (5) by d^2 and subtract. Thus we get

$$hR^2 = dR^2\tan\theta - d^2R\tan\theta$$

or, $$hR^2 = dR\tan\theta(R-d)$$

$$\tan\theta = \frac{h}{d}\Big/\left[1-\frac{d}{R}\right]$$

$$\tan\theta = \frac{hR}{d[R-d]}$$

$$\boxed{\theta = \tan^{-1}\left[\frac{hR}{d[R-d]}\right]}$$

Q-6 (a) Solve: $\left(\frac{dy}{dx}\right)^2 y + 2\frac{dy}{dx}x - y = 0$

$\rightarrow$ i.e. $yp^2 + 2px - y = 0$ — (1) $\because p = dy/dx$

$\therefore 2px = y - yp^2$

$\therefore x = \frac{1}{2}\left(\frac{y}{p}\right) - \frac{1}{2}(yp)$ —— (2)

Clearly it is solvable for x

$\therefore$ diff. eqn (2) w.r.t. y

$\therefore \frac{dx}{dy} = \frac{1}{2}\left[\frac{p - y\frac{dp}{dy}}{p^2}\right] - \frac{1}{2}\left[p + y\frac{dp}{dy}\right]$

$\therefore \frac{2}{p} = \frac{1}{p} - \left(\frac{y}{p^2}\right)\frac{dp}{dy} - p - y\frac{dp}{dy}$

i.e. $\left(\frac{1}{p} + p\right) = \frac{-y}{p^2}\frac{dp}{dy} - y\frac{dp}{dy}$

$\therefore p\left(\cancel{\frac{1}{p^2}+1}\right) = -y\frac{dp}{dy}\left(\cancel{\frac{1}{p^2}+1}\right)$

$\therefore p = -y\frac{dp}{dy}$

$\therefore \frac{1}{p}dp = -\frac{1}{y}dy$

integrating both sides.

$\therefore \log p + \log y = \log c$

$\therefore py = c$

$\therefore \boxed{p = c/y}$

put value of p in eqn (2)

$\therefore x = \frac{1}{2}\left(y \times \frac{y}{c}\right) - \frac{1}{2}\left(y \times \frac{c}{y}\right)$

$\therefore \boxed{x = \frac{y^2}{2c} - \frac{c}{2}}$

SOLUTION : UPSC-IAS-MAINS-2018-MATHEMATICS-OPTIONAL-PAPER-1

6(b) A particle moving with simple harmonic motion in a straight line has velocities v_1 and v_2 at distances x_1 and x_2 respectively from the centre of its path. Find the period of its motion.

Sol^n Let the equation of the S.H.M. with centre O as origin be $d^2x/dt^2 = -\mu x$. Then the time period $T = 2\pi/\sqrt{\mu}$.

If a be the amplitude of the motion, we have

$$v^2 = \mu(a^2 - x^2) \quad \text{--- (1)}$$

where v is the velocity at a distance x from the centre.

But when $x = x_1$, $v = v_1$.

and when $x = x_2$, $v = v_2$

Therefore from (1), we have

$$v_1^2 = \mu(a^2 - x_1^2)$$

and $v_2^2 = \mu(a^2 - x_2^2)$

These give $v_2^2 - v_1^2 = \mu\{(a^2 - x_2^2) - (a^2 - x_1^2)\}$

$$= \mu(x_1^2 - x_2^2)$$

ie. $\mu = (v_2^2 - v_1^2)/(x_1^2 - x_2^2)$

Hence the time period $T = 2\pi/\sqrt{\mu}$

$$= 2\pi\sqrt{\left(\frac{x_1^2 - x_2^2}{v_2^2 - v_1^2}\right)}$$

6(c) Solve $y'' + 16y = 32\sec 2x$.

Sol: Given that $y''+16y = 32\sec 2x$ —— (1).

Its homogeneous equation is

$$y''+16y=0.$$

Auxillary Equation is

$$m^2+16=0$$

$$m=\pm 4i$$

$$\boxed{\therefore y_c(x) = C_1\cos 4x + C_2\sin 4x}$$

$$P.I = \frac{1}{D^2+16}\cdot 32\sec(2x)$$

$$= 32\cdot\frac{1}{(D+4i)(D-4i)}\sec 2x.$$

$$= 32\cdot\frac{1}{8i}\left[\frac{-1}{D+4i}+\frac{1}{D-4i}\right]\sec 2x$$

$$P.I = -4i\left[\frac{1}{D-4i} - \frac{1}{D+4i}\right]\sec 2x.$$

$$P.I = -4i\left[\underset{(i)}{\frac{1}{D-4i}\sec 2x} - \underset{(ii)}{\frac{1}{D+4i}\sec 2x}\right]$$

Now:

$$\frac{1}{D-4i}\sec 2x = e^{4ix}\int e^{-4ix}\cdot\sec 2x\,dx.$$

$$= e^{4ix}\int\left[\frac{\cos 4x - i\sin 4x}{\cos 2x}\right]dx.$$

$$= e^{4ix}\int\left[\frac{\cos^2 2x - \sin^2 2x - 2i\sin 2x\cos 2x}{\cos 2x}\right]dx$$

$$I_1 = e^{i4x}\int \frac{(\cos 2x - i\sin 2x)^2}{\cos 2x}\,dx.$$

$$I_1 = e^{i4x}\int \left(\frac{2\cos^2 2x - 1 - 2i\sin 2x\cos 2x}{\cos 2x}\right)dx.$$

$$I_1 = e^{i4x}\int (2\cos 2x - 2i\sin 2x - \sec 2x)\,dx$$

$$I_1 = e^{i4x}\left[\sin 2x + i\cos 2x - \log\left(\frac{\sec 2x + \tan 2x}{2}\right)\right] + c_1$$

Now

$$I_2 = \frac{1}{(D+4i)}\cdot \sec 2x = e^{-4ix}\int e^{4ix}\cdot \sec 2x\,dx.$$

$$I_2 = e^{-4ix}\int \frac{\cos 4x + i\sin 4x}{\cos 2x}\,dx.$$

$$I_2 = e^{-4ix}\int \left[\frac{2\cos^2 2x - 1 + 2i\sin 2x\cos 2x}{\cos 2x}\right]dx.$$

$$I_2 = e^{-4ix}\int [2\cos 2x - \sec 2x + 2i\sin 2x]\,dx$$

$$I_2 = e^{-4ix}\left[\sin 2x - i\cos 2x - \log\left(\frac{\sec 2x + \tan 2x}{2}\right)\right] + c_2$$

$$P.I = -4i[I_1 - I_2]$$

$$P.I = -4i\left[e^{i4x}\left[\sin 2x + i\cos 2x - \log\left[\frac{\sec 2x + \tan 2x}{2}\right]\right] - e^{-4ix}\left[\sin 2x - i\cos 2x - \log\left[\frac{\sec 2x + \tan 2x}{2}\right]\right]\right]$$

$$\therefore y = y_c + y_p$$

$$y = c_1\cos 4x + c_2\sin 4x - 4i\left[e^{i4x}\left[\sin 2x + i\cos 2x - \log\left[\frac{\sec 2x + \tan 2x}{2}\right]\right] - e^{-i4x}\left[\sin 2x - i\cos 2x - \log\left\{\frac{\sec 2x + \tan 2x}{2}\right\}\right]\right]$$

Q.6 (1). If 'S' is the surface of the sphere $x^2+y^2+z^2=a^2$; then evaluate

$$\iint_S [(x+z)\,dy\,dz + (y+z)\,dz\,dx + (x+y)\,dx\,dy]$$

using Gauss' Divergence Theorem?

Sol:- let 'S' is the surface of sphere

$$x^2+y^2+z^2=a^2$$

then

$$\iint_S [\underbrace{(x+z)}_{i}\,dy\,dz + \underbrace{(y+z)}_{j}\,dz\,dx + \underbrace{(x+y)}_{k}\,dx\,dy]$$

$$F = [x+z]\,dy\,dz + (y+z)\,dz\,dx + (x+y)\,dx\,dy.$$

or $f = [x+z]\hat{i} + (y+z)\hat{j} + (x+y)\hat{k}$

$$\iiint \nabla\cdot\vec{F}\,dv = \iint_S \vec{F}\cdot d\vec{s} \quad \left[\text{Gauss Divergence Theorem}\right]$$

$$\therefore \nabla\cdot\vec{F} = \nabla\cdot[(x+z)\hat{i} + (y+z)\hat{j} + (x+y)\hat{k}]$$

$$\nabla\cdot\vec{F} = 1+1+0 = 2.$$

$$\iiint \nabla\cdot\vec{F}\,dv = \iiint 2\,dv$$

$$= 2\iiint dv = 2\times\frac{4}{3}\pi a^3$$

$\left[\because \text{volume of sphere} = \frac{4}{3}\pi a^3\right]$

$$\therefore \iiint \nabla\cdot\vec{F}\,dv = \iint_S \vec{F}\cdot d\vec{s} = \frac{8}{3}\pi a^3$$

7(a) Solve $(1+x^2)y''+(1+x)y'+y = 4\cos(\log(1+x))$

Sol: Given D.E.

$$(1+x^2)y''+(1+x)y'+y = 4\cos(\log(1+x))$$

put $\log(1+x) = z$.

Hence, the given DE can be rewritten as

$$(D(D-1)y+Dy+y) = 4\cos z$$

$$(D(D-1)+D+1)y = 4\cos z$$

$$(D^2+\cancel{D}-\cancel{D}+1)y = 4\cos z$$

$$(D^2+1)y = 4\cos z$$

The Auxillary Eqn.

$$m^2+1=0 \Rightarrow m=\pm i$$

$\therefore y_c = \text{C.F} = C_1\cos z + C_2\sin z$

$$y_c = C_1\cos(\log(1+x)) + C_2\sin(\log(1+x)).$$

$$y_p = \frac{1}{D^2+1}4\cos z = 4\times\frac{z}{2}.\sin z$$

$$y_p = 2\log(1+x) \cdot \sin(\log(1+x))$$

$\therefore$ General Solution $\Rightarrow y = y_c + y_p$.

$$\boxed{y = C_1\cos\log(1+x) + C_2\sin\log(1+x) + 2\log(1+x)\sin(\log(1+x))}$$

$$y = C_1\cos\log(1+x) + C_2\sin\log(1+x) + \log(1+x)^2.\sin(\log(1+x))$$

is the required solution

7(b) Find the curvature and torsion of the curve.

$\vec{r} = a(u-\sin u)\vec{i} + a(1-\cos u)\vec{j} + bu\hat{k}$

Sol:- Given; $\vec{r} = a(u-\sin u)\vec{i} + a(1-\cos u)\hat{j} + bu\hat{k}$

$$\frac{d\vec{r}}{du} = (a - a\cos u)\hat{i} + a\sin u\hat{j} + bu\hat{k} \quad \text{———} (1)$$

$$\therefore \left|\frac{d\vec{r}}{du}\right| = \sqrt{a^2 - 2a^2\cos^2 u + a^2\cos^2 u + a^2\sin^2 u + b^2}$$

$$= \sqrt{2a^2(1-\cos u) + b^2}$$

$$= \sqrt{\frac{2a^2}{b^2}[1-\cos u + 1]}$$

$$\left|\frac{d\vec{r}}{du}\right| = \frac{a}{b}\sqrt{2(2-\cos u)} \quad \text{———} (2)$$

Also: $$\frac{d^2r}{du^2} = a\sin u\hat{i} + a\cos u\hat{j} \quad \text{———} (3)$$

& $$\frac{d^3r}{du^3} = a\cos u\hat{i} - a\sin u\hat{j} \quad \text{———} (4)$$

Now: $$\frac{d\vec{r}}{du} \times \frac{d^2\vec{r}}{du^2} = \begin{vmatrix} i & j & k \\ a - a\cos u & a\sin u & b \\ a\sin u & a\cos u & 0 \end{vmatrix}$$

$$= i[(a\sin u \times 0) - ab\cos u] \mp j[-ab\sin u] + k(a^2\cos u - a^2\cos^2 u - a^2\sin^2 u]$$

$$\frac{d\vec{r}}{du} \times \frac{d^2r}{du^2} = -ab\cos u\,\hat{i} + ab\sin u\hat{j} + a^2(\cos u - 1)\hat{k}$$

$$\left|\frac{d\vec{r}}{du} \times \frac{d^2r}{du^2}\right| = \sqrt{a^2b^2\cos^2 u + a^2b^2\sin^2 u + a^4[\cos u - 1]^2}$$

$$= \sqrt{(ab)^2[\cos^2 u + \sin^2 u] + a^4[\cos^2 u + 1 - 2\cos u]}$$

$$= \sqrt{a^2b^2 + a^4[\cos u - 1]^2}$$

$$\therefore \left|\frac{d\vec{r}}{du}\times\frac{d^2\vec{r}}{du^2}\right| = \sqrt{a^2b^2+a^4(\cos u-1)^2}$$
$$= a\sqrt{b^2+a^2(\cos u-1)^2}$$

$$\kappa = \frac{\left|\frac{d\vec{r}}{du}\times\frac{d^2\vec{r}}{du^2}\right|}{\left|\frac{d\vec{r}}{du}\right|^3} = \frac{\sqrt{a^2b^2+a^4(\cos u-1)^2}}{\left(2a^2(1-\cos u)+b^2\right)^{3/2}}$$

$$T = \frac{\left[\frac{d\vec{r}}{du}\cdot\frac{d^2\vec{r}}{du^2}\cdot\frac{d^3\vec{r}}{du^3}\right]}{\left|\frac{d\vec{r}}{du}\times\frac{d^2\vec{r}}{du^2}\right|} = \frac{[-ab\cos u\,\hat{i}+ab\sin u\,\hat{j}+a^2(\cos u-1)\hat{k}-(a\cos u\,\hat{i}-a\sin u\,\hat{j})]}{[a^2b^2+a^4(\cos u-1)^2}$$

$$T = \frac{-a^2b\cos^2u-a^2b\sin^2u}{(a^2b^2+a^4(\cos u-1)^2}$$

$$T = \frac{-a^2b(\cos^2u+\sin^2u)}{a^2b^2+a^4(\cos u-1)^2}$$

$$\boxed{T = \frac{-a^2b}{a^2b^2+a^4(\cos u-1)^2}}$$

7(c) Solve the initial value problem

$$y'' - 5y' + 4y = e^{2t}$$

$$y(0) = \frac{19}{12}\ ;\ y'(0) = \frac{8}{3}.$$

Sol:- Given; $y'' - 5y' + 4y = e^{2t}$ & $y(0) = \frac{19}{12}$; $y'(0) = \frac{8}{3}$

$$(D^2 - 5D + 4)y = e^{2t}$$

A.E = $m^2 - 5m + 4 = 0$

$(m-4)(m-1) = 0$

$m = 4, 1$

C.F = $c_1 e^t + c_2 e^{4t}$.

$$P.I. = \frac{1}{(D-1)(D-4)} e^{2t}$$

$$= \frac{1}{(D^2 - 5D + 4)} \cdot e^{2t}$$

$$= \frac{e^{2t}}{(2)^2 - 5\times 2 + 4} = \frac{e^{2t}}{4 - 10 + 4}$$

$$P.I = y_p = -\frac{1}{2} e^{2t}$$

$$\boxed{y = y_c + y_p = c_1 e^t + c_2 e^{4t} - \frac{1}{2} e^{2t}} \quad \text{—— (A)}$$

Now; $y(0) = \frac{19}{12}$.

Put $t = 0$ in (A) ;

$y(0) = c_1 + c_2 - \frac{1}{2} \Rightarrow c_1 + c_2 = \frac{19}{12} + \frac{1}{2} = \frac{25}{12}$ —(1)

$$\boxed{y' = c_1 e^t + 4c_2 e^{4t} - e^{2t}} \quad \text{—— (B)}$$

$y'(0) = \frac{8}{3}$

Put $t=0$ in eqn (B)

$$y'(0) = c_1 + 4c_2 - 1.$$

$$c_1 + 4c_2 = \frac{8}{3} + 1$$

$$c_1 + 4c_2 = \frac{11}{3} \quad \text{---} (2)$$

from (1) & (2)

$$\frac{11}{3} = \frac{25}{12} - c_2 + 4c_2$$

$$3c_2 = \frac{11}{3} - \frac{25}{12} = \frac{44-25}{12} = \frac{19}{12}$$

$$c_2 = 19/36$$

$$c_1 + c_2 = \frac{25}{12}$$

$$c_1 + \frac{19}{36} = \frac{25}{12}$$

$$c_1 = \frac{75-19}{36} = \frac{56}{36} = \frac{14}{9}$$

$$\therefore c_1 = \frac{14}{9} \ \& \ c_2 = \frac{19}{36}$$

$$\boxed{\therefore y = \frac{14}{9}e^{t} + \frac{19}{36}e^{4t} - \frac{1}{2}e^{2t}}$$ is the required solution.

7(d) → find α and β such that $x^\alpha y^\beta$ is an integrating factor of $(4y^2+3xy)dx-(3xy+2x^2)dy=0$ and solve the equation.

Sol Given that $(4y^2+3xy)dx-(3xy+2x^2)dy=0$ —— (1)

let $x^\alpha y^\beta$ be an integrating factor of (1)

then $x^\alpha y^\beta(4y^2+3xy)dx - x^\alpha y^\beta(3xy+2x^2)dy=0$

$\Rightarrow (4x^\alpha y^{\beta+2}+3x^{\alpha+1}y^{\beta+1})dx+(-3x^{\alpha+1}y^{\beta+1}-2x^{\alpha+2}y^\beta)dy=0$ —— (A)

clearly it is in the form of $Mdx+Ndy=0$

and is an exact. —— (2)

$\therefore \frac{\partial M}{\partial y}=\frac{\partial N}{\partial x}$ —— (3)

Here $\frac{\partial M}{\partial y}=4(\beta+2)x^\alpha y^{\beta+1}+3(\beta+1)x^{\alpha+1}y^\beta$

&

$\frac{\partial N}{\partial x}=-3(\alpha+1)x^\alpha y^{\beta+1}-2(\alpha+2)x^{\alpha+1}y^\beta$.

(3) $\equiv 4(\beta+2)x^\alpha y^{\beta+1}+3(\beta+1)x^{\alpha+1}y^\beta = -3(\alpha+1)x^\alpha y^{\beta+1}-2(\alpha+2)x^{\alpha+1}y^\beta$

$\Rightarrow 4(\beta+2)=-3(\alpha+1)$; $3(\beta+1)=-2(\alpha+2)$

$\Rightarrow -3\alpha-4\beta-11=0$; $-2\alpha-3\beta-7=0$

$\Rightarrow$ $3\alpha+4\beta+11=0$ —— (i)
$2\alpha+3\beta+7=0$ —— (ii) $\Big\} \Rightarrow \alpha=-5$ & $\beta=1$.

$\therefore$ (A) $\equiv x^{-5} y'(4y^2+3xy)\,dx - x^{-5}y(3xy+2x^2)\,dy = 0$

$$\Rightarrow \frac{y(4y^2+3xy)\,dx}{x^5} - \frac{y(3xy+2x^2)\,dy}{x^5}$$

$$\Rightarrow \left(\frac{4y^3}{x^5}+\frac{3y^2}{x^4}\right)dx - \left(\frac{3y^2}{x^4}+\frac{2y}{x^3}\right)dy = 0 \quad (1)$$

Since it is an exact

$\therefore$ The general solution is given by

$$\int_{y\text{-const}} \left(4\frac{y^3}{x^5}+\frac{3y^2}{x^4}\right)dx + \int 0\,dy = C$$

$$\Rightarrow \frac{+4y^3}{-4x^4} - \frac{3y^2}{3x^3} = C$$

$$\Rightarrow \boxed{\frac{y^3}{x^4} + \frac{y^3}{x^3} = C}$$

8(a) Let $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$. Show that ~~curl(curl v)=~~ $\text{curl}(\text{curl}\,\vec{v}) = \text{grad}(\text{div}\,\vec{v}) - \nabla^2 v$.

Sol let $\vec{v} = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$

Then $\nabla \times V = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ v_1 & v_2 & v_3 \end{vmatrix}$

$$\nabla \times V = \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\hat{i} + \left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\hat{j} + \left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right)\hat{k}$$

$$\therefore \nabla \times (\nabla \times V) = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} & \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} & \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \end{vmatrix}$$

$$= \Sigma\left[\left\{\frac{\partial}{\partial y}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\right\} i\right]$$

$$= \Sigma\left[\left\{\left(\frac{\partial^2 v_2}{\partial y\, \partial x} + \frac{\partial^2 v_3}{\partial z\, \partial x}\right) - \left(\frac{\partial^2 v_1}{\partial y^2} + \frac{\partial^2 v_1}{\partial z^2}\right)\right\} i\right]$$

$$= \Sigma\left[\left\{\frac{\partial}{\partial x}\left(\frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\right) - \left(\frac{\partial^2 v_1}{\partial y^2} + \frac{\partial^2 v_1}{\partial z^2}\right)\right\} i\right]$$

$$= \Sigma\left[\left\{\frac{\partial}{\partial x}\left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\right) - \left(\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial y^2} + \frac{\partial^2 v_1}{\partial z^2}\right)\right\} i\right]$$

$$= \Sigma\left[\left\{\frac{\partial}{\partial x}(\nabla \cdot v) - (\nabla^2 v_1)\right\} i\right]$$

$$= \Sigma\left[\left\{\frac{\partial}{\partial x}(\nabla \cdot v)\right\} i\right] - \nabla^2 \Sigma v_1 i$$

$$= \nabla(\nabla \cdot v) - \nabla^2 v.$$

$$\therefore \boxed{\nabla \times (\nabla \times V) = \text{grad}(\nabla \cdot V) - \nabla^2 V}$$ proved

8(b) Evaluate the line integral $\int_C -y^3dx + x^3dy + z^3dz$ using Stoke's theorem. Here 'C' is the intersection of the cylinder $x^2+y^2=1$ and the plane $x+y+z=1$. The orientation on 'C' corresponds to counterclockwise motion in the x-y plane?

Sol. let us evaluate $\int_C (-y^3dx + x^3dy + z^3dz)$

by using stokes theorem.

let us suppose that 'S' is the part of the plane cut by the cylinder.

The curve 'C' is oriented counter-clockwise when viewed from the end of the normal vector $\hat{n}$.

$$\therefore \hat{n} = \frac{i+j+k}{\sqrt{1^2+1^2+1^2}} = \frac{1}{\sqrt{3}}i + \frac{1}{\sqrt{3}}j + \frac{1}{\sqrt{3}}k$$

Let us apply stokes theorem

$$\oint_C (-y^3dx + x^3dy + z^3dz) = \iint_S (\nabla\times F)\cdot\hat{n}\, ds \quad \text{--- (1)}$$

let $F = Pi + Qj + Rk$.

where $P = -y^3$, $Q = x^3$, $R = z^3$

$$\therefore \nabla\times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & z^3 \end{vmatrix}$$

$$= i[0] - j[0] + k[3x^2+3y^2]$$

$$= 3(x^2+y^2)k.$$

we have

$$(\nabla\times\bar{F})\cdot\hat{n} = \frac{3}{\sqrt{3}}(x^2+y^2)$$

$$= \sqrt{3}(x^2+y^2)$$

$$\therefore \text{①} \equiv \oint_C (y^3 dx + x^3 dy + z^3 dz) = \iint_S \sqrt{3}(x^2+y^2)\,ds$$

$$= \sqrt{3}\iint_S (x^2+y^2)\,ds$$

$$= \sqrt{3}\iint_S ds \quad (\because x^2+y^2=1) \quad \text{--- ②}$$

The projection of the surface 'S' onto the xy-plane is circle $x^2+y^2=1$ of radius 1.

$\therefore$ Representing the equation of the plane in the form $z=1-x-y$ and using formula

$$\iint_S ds = \iint_R \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$$

$$= \iint_R \sqrt{1+(-1)^2+(-1)^2}\,dx\,dy$$

$$= \sqrt{3}\iint_R dx\,dy$$

$$= \sqrt{3}\,\pi(1)^2$$

$$= \sqrt{3}\,\pi$$

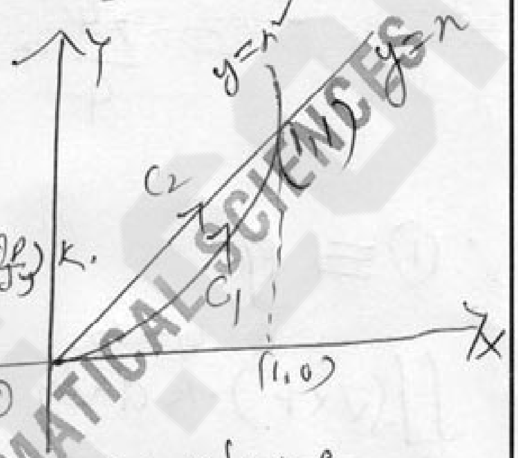
$$\therefore \text{②} \equiv \oint_C [-y^3 dx + x^3 dy + z^3 dz] = \sqrt{3}(\sqrt{3}\pi) = 3\pi$$

8(c) Let $\vec{F} = xy^2\hat{i} + (y+x)\hat{j}$. Integrate $(\nabla\times\vec{F})\cdot\vec{k}$ over the region in the first quadrant bounded by the curves $y = x^2$ and $y = x$ using Green's theorem.

P.I

Sol'n : Given that $\vec{F} = xy^2 i + (y+x) j$

Given curves: $y = x^2$ and $y = x$.



Let $\vec{F} = P i + Q j$

then $\nabla\times\vec{F} =$

$-\frac{\partial Q}{\partial z} i + \frac{\partial P}{\partial z} j + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) k.$

$\therefore (\nabla\times\vec{F})\cdot k = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$

Hence Green's theorem in plane can be written as:

$$\iint_R (\nabla\times\vec{F})\cdot\hat{n}\, dR = \oint_C \vec{F}\cdot d\vec{r}.$$

$$\Rightarrow \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \oint_C P\,dx + Q\,dy.$$

$$= \oint_C [xy^2 dx + (y+x)\, dy.] \qquad C: C_1 - C_2 \quad \text{——— (1)}$$

Along C_1 $y = x^2$: $dy = 2x\,dx$

limits: x: 0 to 1

$$\therefore \int_{C_1} [xy^2 dx + (y+x) dy] = \int_{x=0}^{1} [x^5 dx + (x^2+x)\, 2x\,dx]$$

$$= \int_{x=0}^{1} [x^5 dx + (2x^3 + 2x^2)\, dx]$$

$$= \left[\frac{x^6}{6} + \frac{2x^4}{4} + \frac{2x^3}{3}\right]_0^1 = \frac{1}{6} + \frac{1}{2} + \frac{2}{3} = \frac{4}{3}.$$

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Along C_2: $y = x \Rightarrow dy = dx$

limits x: 0 to 1.

$$\int_{C_2}[xy^2 dx + (y+x)dy] = \int_{x=0}^{1}[x^3 dx + (2x)dx]$$

$$= \frac{x^4}{4} + \frac{2x^3}{3}\Big]_0^1 = \frac{1}{4} + \frac{2}{3}$$

$$= \frac{3+8}{12} = \frac{11}{12}$$

$\therefore$ (1) $\equiv$

$$\iint_R (\nabla \times F)\cdot k\, dx\, dy = \oint_{C:\, C_1 + C_2}[xy^2 dx + (y+x)dy]$$

$$= \oint_{C_1}[xy^2 dx + (y+x)dy] - \int_{C_2}[xy^2 dx + (y+x)dy]$$

$$= \frac{4}{3} - \frac{11}{12} = \frac{1}{12}$$

8(d) Find $f(y)$ such that $(2xe^y + 3y^2)dy + (3x^2 + f(y))dx = 0$ is exact and hence solve.

Sol.: Given; $(2xe^y + 3y^2)dy + (3x^2 + f(y))dx = 0$

Given $Mdx + Ndy = 0$

and also this is exact

i.e $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$. —— (1)

$\dfrac{\partial N}{\partial x} = 2e^y$; $\dfrac{\partial M}{\partial y} = f'(y)$ —— (2)

Hence; from (1) & (2)

$f'(y) = 2e^y$.

$f(y) = 2e^y$

Thus, the given eqn transforms to

$[3x^2 + 2e^y]dx + [2xe^y + 3y^2]dy = 0$

Then, the given solution is

$\displaystyle\int M\,dx + \int N\,dy = 0$

treating y = constant   terms in N not containing x

$\Rightarrow x^3 + 2xe^y + y^3 = c$

$x^3 + y^3 + 2xe^y = c$