Example 2.3. *If* $\begin{vmatrix} a & a^2 & a^3-1 \\ b & b^2 & b^3-1 \\ c & c^2 & c^3-1 \end{vmatrix} = 0$ *in which a, b, c are different, show that abc = 1.*

Solution. As each term of C_3 in the given determinant consists of two terms, we express it as a sum of two determinants.

$$\begin{vmatrix} a & a^2 & a^3-1 \\ b & b^2 & b^3-1 \\ c & c^2 & c^3-1 \end{vmatrix} = \begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} + \begin{vmatrix} a & a^2 & -1 \\ b & b^2 & -1 \\ c & c^2 & -1 \end{vmatrix} = abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} a & a^2 & 1 \\ b & b^2 & 1 \\ c & c^2 & 1 \end{vmatrix}$$

[Taking common a, b, c from R_1, R_2, R_3 respectively of the first determinant and -1 from C_3 of the second determinant.]

$$= abc \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} - \begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix}$$

[Passing C_3 over C_2 and C_1 in the second determinant]

$\therefore$ $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} (abc-1) = 0$. Hence $abc = 1$, since $\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} \neq 0$ as a, b, c are all different.

VI. *If to each elements of a line be added equi-multiples of the corresponding elements of one or more parallel lines, the determinants remains unaltered.*

Let $$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Then $$\Delta' = \begin{vmatrix} a_1 + pb_1 - qc_1 & b_1 & c_1 \\ a_2 + pb_2 - qc_2 & b_2 & c_2 \\ a_3 + pb_3 - qc_3 & b_3 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} pb_1 & b_1 & c_1 \\ pb_2 & b_2 & c_2 \\ pb_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} -qc_1 & b_1 & c_1 \\ -qc_2 & b_2 & c_2 \\ -qc_3 & b_3 & c_3 \end{vmatrix}$$

$= \Delta + 0 + 0 = \Delta.$ [by *IV-Cor.*]

Example 2.6. *Prove that* $\begin{vmatrix} 1+a & 1 & 1 & 1 \\ 1 & 1+b & 1 & 1 \\ 1 & 1 & 1+c & 1 \\ 1 & 1 & 1 & 1+d \end{vmatrix} = abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$.

Solution. Let Δ be the given determinant. Taking a, b, c, d common from R_1, R_2, R_3, R_4 respectively, we get

$$\Delta = abcd \begin{vmatrix} a^{-1}+1 & a^{-1} & a^{-1} & a^{-1} \\ b^{-1} & b^{-1}+1 & b^{-1} & b^{-1} \\ c^{-1} & c^{-1} & c^{-1}+1 & c^{-1} \\ d^{-1} & d^{-1} & d^{-1} & d^{-1}+1 \end{vmatrix}$$

[Operate $R_1 + (R_2 + R_3 + R_4)$ and take out the common factor from R_1]

$$= abcd\,(1 + a^{-1} + b^{-1} + c^{-1} + d^{-1}) \begin{vmatrix} 1 & 1 & 1 & 1 \\ b^{-1} & b^{-1}+1 & b^{-1} & b^{-1} \\ c^{-1} & c^{-1} & c^{-1}+1 & c^{-1} \\ d^{-1} & d^{-1} & d^{-1} & d^{-1}+1 \end{vmatrix}$$

[Operate $C_2 - C_1, C_3 - C_1, C_4 - C_1$]

$$= abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right) \begin{vmatrix} 1 & 0 & 0 & 0 \\ b^{-1} & 1 & 0 & 0 \\ c^{-1} & 0 & 1 & 0 \\ d^{-1} & 0 & 0 & 1 \end{vmatrix} = abcd\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\right)$$

Example 2.27. *Using the partition method, find the inverse of* $\begin{bmatrix} 1 & 1 & 1 \\ 4 & 3 & -1 \\ 3 & 5 & 3 \end{bmatrix}$.

Solution. Let
$$A = \left[\begin{array}{cc:c} 1 & 1 & 1 \\ 4 & 3 & -1 \\ \hdashline 3 & 5 & 3 \end{array}\right] = \left[\begin{array}{c:c} A_1 & A_2 \\ \hdashline A_3' & \alpha \end{array}\right]$$

so that
$$A_1^{-1} = \begin{bmatrix} 1 & 1 \\ 4 & 3 \end{bmatrix}^{-1} = -\begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix}$$

Let
$$A^{-1} = \left[\begin{array}{c:c} X_1 & X_2 \\ \hdashline X_3' & x \end{array}\right] \text{ so that } AA^{-1} = I.$$

$$\alpha - A_3' A_1^{-1} A_2 = 3 + [3 \quad 5] \begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = -10$$

$\therefore$
$$x = (\alpha - A_3' A_1^{-1} A_2)^{-1} = -\frac{1}{10}$$

Also,
$$X_2 = -A_1^{-1} A_2 x = \begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \left(-\frac{1}{10}\right) = -\frac{1}{10} \begin{bmatrix} 4 \\ -5 \end{bmatrix}$$

Then
$$X_3' = -A_3' A_1^{-1} x = [3 \quad 5] \begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix} \left(-\frac{1}{10}\right) = -\frac{1}{10} [-11 \quad 2]$$

Finally,
$$X_1 = A_1^{-1}(I - A_2 X_3') = -\begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix} - \frac{1}{10} \begin{bmatrix} 3 & -1 \\ -4 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} [-11 \quad 2]$$

$$= \begin{bmatrix} -3 & 1 \\ 4 & -1 \end{bmatrix} - \frac{1}{10} \begin{bmatrix} -44 & 8 \\ 55 & -10 \end{bmatrix} = \begin{bmatrix} 1.4 & 0.2 \\ -1.5 & 0 \end{bmatrix}$$

Hence
$$A^{-1} = \begin{bmatrix} 1.4 & 0.2 & -0.4 \\ -1.5 & 0 & 0.5 \\ 1.1 & -0.2 & -0.1 \end{bmatrix}.$$

Example 2.37. *Find the values of λ for which the equations*

$$(\lambda - 1)x + (3\lambda + 1)y + 2\lambda z = 0$$
$$(\lambda - 1)x + (4\lambda - 2)y + (\lambda + 3)z = 0$$
$$2x + (3\lambda + 1)y + 3(\lambda - 1)z = 0$$

are consistent, and find the ratios of x : y : z when λ has the smallest of these values. What happens when λ has the greatest of these values. (*Kurukshetra, 2006 ; Delhi, 2002*)

Solution. The given equations will be consistent, if

$$\begin{vmatrix} \lambda - 1 & 3\lambda + 1 & 2\lambda \\ \lambda - 1 & 4\lambda - 2 & \lambda + 3 \\ 2 & 3\lambda + 1 & 3(\lambda - 1) \end{vmatrix} = 0 \qquad \text{[Operate } R_2 - R_1]$$

or if,
$$\begin{vmatrix} \lambda - 1 & 3\lambda + 1 & 2\lambda \\ 0 & \lambda - 3 & 3 - \lambda \\ 2 & 3\lambda + 1 & 3(\lambda - 1) \end{vmatrix} = 0 \qquad \text{[Operate } C_3 + C_2]$$

or if,
$$\begin{vmatrix} \lambda - 1 & 3\lambda + 1 & 5\lambda + 1 \\ 0 & \lambda - 3 & 0 \\ 2 & 3\lambda + 1 & 6\lambda - 2 \end{vmatrix} = 0 \qquad \text{[Expand by } R_2]$$

or if, $(\lambda - 3)\begin{vmatrix} \lambda - 1 & 5\lambda + 1 \\ 2 & 2(3\lambda + 1) \end{vmatrix} = 0$ or if, $2(\lambda - 3)\,[\,(\lambda - 1)(3\lambda - 1) - (5\lambda + 1)] = 0$

or if, $6\lambda(\lambda - 3)^2 = 0$ or if, $\lambda = 0$ or 3.

(*a*) *When* $\lambda = 0$, the equations become $-x + y = 0$ ...(*i*)

$$-x - 2y + 3z = 0 \qquad ...(ii)$$
$$2x + y - 3z = 0 \qquad ...(iii)$$

Solving (*ii*) and (*iii*), we get $\dfrac{x}{6-3} = \dfrac{y}{6-3} = \dfrac{z}{-1+4}$. Hence $x = y = z$.

(*b*) *When* $\lambda = 3$, equations becomes identical.

Example 2.50. *Find* e^A *and* 4^A *if* $A = \begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 3/2 \end{bmatrix}$. (*Mumbai, 2006*)

Solution. The characteristic equation of A is

$$\begin{vmatrix} 3/2-\lambda & 1/2 \\ 1/2 & 3/2-\lambda \end{vmatrix} = 0, \quad i.e., (3/2-\lambda)^2 - 1/4 = 0.$$

$\therefore$ $$\lambda^2 - 3\lambda + 2 = 0 \quad \text{whence } \lambda = 1, 2.$$

When $\lambda = 1$, $[A - \lambda I]\, X = 0$, gives

$$\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad [\text{By } 2R_1, 2R_2]$$

or $$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad [\text{By } R_2 - R_1]$$

$\therefore$ $x_1 + x_2 = 0$. If $x_2 = -1, x_1 = 1$, *i.e.*, the eigen vector is $[1, -1]'$.

When $\lambda = 2$, $[A - \lambda I]\, X = 0$, gives $\begin{bmatrix} -1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

or $$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} \text{By } 2R_1 \\ 2R_2 \end{bmatrix}$$

or $$\begin{bmatrix} -1 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad [\text{By } R_2 - R_1]$$

$\therefore$ $$-x_1 + x_2 = 0, \quad i.e., \quad x_1 = x_2$$

If $x_2 = 1, x_1 = 1$, *i.e.*, the eigen vector is $[1, 1]'$

Now $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ and $P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$

$\therefore$ $$P^{-1} = \frac{adj\, P}{|P|} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

If $$f(A) = e^A, f(D) = e^D = \begin{bmatrix} e^1 & 0 \\ 0 & e^2 \end{bmatrix}$$

$\therefore$ $$e^A = P\, f(D)\, P^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} e^1 & 0 \\ 0 & e^2 \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} e & e^2 \\ -e & e^2 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e+e^2 & -e+e^2 \\ -e+e^2 & e+e^2 \end{bmatrix}$$

Replacing e by 4, we get

$$4^A = \frac{1}{2}\begin{bmatrix} 20 & 12 \\ 12 & 20 \end{bmatrix} = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}.$$

Ex. 8. *For the 3-dimensional space* $\mathbf{R}^3$ *over the field of real numbers* $\mathbf{R}$, *determine if the set* $\{(2, -1, 0), (3, 5, 1), (1, 1, 2)\}$ *is a basis.*

Sol. We have dim $\mathbf{R}^3=3$. If the given set containing three vectors is linearly independent, it will form a basis of $\mathbf{R}^3$ otherwise not.

Let $a, b, c \in \mathbf{R}$ be such that

$$a(2, -1, 0)+b(3, 5, 1)+c(1, 1, 2)=(0, 0, 0)$$

$$\Rightarrow \quad (2a+3b+c, -a+5b+c, 0a+b+2c)=(0, 0, 0).$$

$$\therefore \quad 2a+3b+c=0, \quad \ldots(1)$$

$$-a+5b+c=0, \quad \ldots(2)$$

and

$$b+2c=0. \quad \ldots(3)$$

Now we shall solve these equations to get the values of a, b, c.

Multiplying (2) by 2 and adding to (1), we get

$$13b+3c=0. \quad \ldots(4)$$

Multiplying (3) by 13 and then subtracting (4) from it, we get

$$23c=0 \quad \text{or} \quad c=0.$$

Putting $c=0$ in (3), we get $b=0$.

Putting $b=0, c=0$ in (1), we get $a=0$.

Thus the only solution of the equations (1), (2) and (3) is

Example 8. *Let f be a linear transformation from a vector space U into a vector space V. If S is a subspace of U, prove that $f(S)$ will be a subspace of V.* **(Meerut 1974)**

Solution. $U(F)$ and $V(F)$ are two vector spaces over the same field F. The mapping f is a linear transformation of U into V *i.e.*,

$f: U \to V$ such that

$$f(a\alpha+b\beta)=af(\alpha)+bf(\beta) \ \forall \ a, b \in F \text{ and } \forall \ \alpha, \beta \in U.$$

Let S be a subspace of U. Then to prove that $f(S)$ is a subspace of V.

Let $a, b \in F$ and $f(\alpha), f(\beta) \in f(S)$ where $\alpha, \beta \in S$.

Since S is a subspace of U, therefore

$a, b \in F$ and $\alpha, \beta \in S \Rightarrow a\alpha+b\beta \in S$

$\Rightarrow f(a\alpha+b\beta) \in f(S)$

$\Rightarrow af(\alpha)+bf(\beta) \in f(S)$.

$$[\because \ f(a\alpha+b\beta)=af(\alpha)+bf(\beta)].$$

Thus $a, b \in F$ and $f(\alpha), f(\beta) \in f(S)$

$\Rightarrow af(\alpha)+bf(\beta) \in f(S)$.

Hence $f(S)$ is a subspace of V.

Ex. 1. *Show that the mapping* $T : V_3(\mathbf{R}) \to V_2(\mathbf{R})$ *defined as*

$$T(a_1, a_2, a_3) = (3a_1 - 2a_2 + a_3,\ a_1 - 3a_2 - 2a_3)$$

is a linear transformation from $V_3(\mathbf{R})$ *into* $V_2(\mathbf{R})$.

Solution. Let $\alpha = (a_1, a_2, a_3)$, $\beta = (b_1, b_2, b_3) \in V_3(\mathbf{R})$.

Then $T(\alpha) = T(a_1, a_2, a_3) = (3a_1 - 2a_2 + a_3,\ a_1 - 3a_2 - 2a_3)$

and $T(\beta) = (3b_1 - 2b_2 + b_3,\ b_1 - 3b_2 - 2b_3)$.

Also let $a, b \in \mathbf{R}$. Then $a\alpha + b\beta \in V_3(\mathbf{R})$. We have

$$T(a\alpha + b\beta) = T[a(a_1, a_2, a_3) + b(b_1, b_2, b_3)]$$

$$= T(aa_1 + bb_1,\ aa_2 + bb_2,\ aa_3 + bb_3)$$

$$= (3(aa_1 + bb_1) - 2(aa_2 + bb_2) + aa_3 + bb_3,$$
$$aa_1 + bb_1 - 3(aa_2 + bb_2) - 2(aa_3 + bb_3))$$

$$= (a(3a_1 - 2a_2 + a_3) + b(3b_1 - 2b_2 + b_3),$$
$$a(a_1 - 3a_2 - 2a_3) + b(b_1 - 3b_2 - 2b_3))$$

$$= a(3a_1 - 2a_2 + a_3,\ a_1 - 3a_2 - 2a_3) + b(3b_1 - 2b_2 + b_3,\ b_1 - 3b_2 - 2b_3)$$

$$= aT(\alpha) + bT(\beta).$$

Hence T is a linear transformation from $V_3(\mathbf{R})$ into $V_2(\mathbf{R})$.

Ex 2. *Show that the mapping* $T : V_3(\mathbf{R}) \to V_2(\mathbf{R})$ ...

Example 8 *Find the general solutions of the system whose augmented matrix is given by*

$$\begin{bmatrix} 1 & -3 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.3}$$

Solution:

$$\begin{bmatrix} 1 & -3 & 0 & -1 & 0 & -2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -3 & 0 & -1 & 9 & 2 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & -3 & 5 \\ 0 & 1 & 0 & 0 & -4 & 1 \\ 0 & 0 & 0 & 1 & 9 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Corresponding system:

$$\left\{ \begin{array}{ccccccc} x_1 & & & - & 3x_5 & = & 5 \\ & x_2 & & - & 4x_5 & = & 1 \\ & & x_4 & + & 9x_5 & = & 4 \\ & & & & 0 & = & = \end{array} \right.$$

Basic variables: x_1, x_2, x_4; free variables: x_3, x_5. General solution:

$$\left\{ \begin{array}{lcl} x_1 & = & 5 + 3x_3 \\ x_2 & = & 1 + 4x_5 \\ x_3 & = & \text{is free} \\ x_4 & = & 4 - 9x_5 \\ x_5 & = & \text{is free} \end{array} \right.$$

Note: A common error in this exercise is to assume that x_3 is zero. Another common error is to say *nothing* about x_3 and write only x_1, x_2, x_4, and x_5, as above. **To avoid these mistakes, identify the basic variables first. Any remaining variables are free.** □

Example 26 *Solve the following homogeneous system of linear equations by using Gauss–Jordan elimination.*

$$\left\{\begin{array}{rcrcrcrcrcr} 2x_1 & + & 2x_2 & - & x_3 & & & + & x_5 & = & 0 \\ -x_1 & - & x_2 & + & 2x_3 & - & 3x_4 & + & x_5 & = & 0 \\ x_1 & + & x_2 & - & 2x_3 & & & - & x_5 & = & 0 \\ & & & & x_3 & + & x_4 & + & x_5 & = & 0 \end{array}\right. \tag{6.1}$$

Solution: The augmented matrix for the system is

$$\begin{bmatrix} 2 & 2 & -1 & 0 & 1 & 0 \\ -1 & -1 & 2 & -3 & 1 & 0 \\ 1 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}$$

Reducing this matrix to reduced row-echelon form, we obtain

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The corresponding system of equations is

$$\begin{array}{rcrcrcrcrcr} x_1 & + & x_2 & & & & & + & x_5 & = & 0 \\ & & & & x_3 & & & + & x_5 & = & 0 \\ & & & & & & x_4 & & & = & 0 \end{array} \tag{6.2}$$

Solving for the leading variables yields

$$\begin{array}{rcrcrcr} x_1 & = & - & x_1 & - & x_2 \\ x_3 & = & - & x_5 & & \\ x_4 & = & & 0 & & \end{array}$$

Thus, the general solution is

$$\begin{array}{rcl} x_1 & = & -s-t \\ x_2 & = & s \\ x_3 & = & -t \\ x_4 & = & 0 \\ x_5 & = & t \end{array} .$$

Note that the trivial solution is obtained when $s = t = 0$. $\square$

Example 26 illustrates two important points about solving homogeneous systems of linear equations. First, none of the three elementary row operations

Example 66 *Find the inverse of the matrix*

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -3 & 1 & 4 \\ 2 & -3 & 4 \end{bmatrix}.$$

Solution:

$$\begin{bmatrix} A & I \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ -3 & 1 & 4 & 0 & 1 & 0 \\ 2 & -3 & 4 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & -3 & 8 & -2 & 0 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & -2 & 3 & 1 & 0 \\ 0 & 0 & 2 & 7 & 3 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 8 & 3 & 1 \\ 0 & 1 & 0 & 10 & 4 & 1 \\ 0 & 0 & 1 & 7/2 & 3/2 & 1/2 \end{bmatrix}$$

So:

$$A^{-1} = \begin{bmatrix} 8 & 3 & 1 \\ 10 & 4 & 1 \\ 7/2 & 3/2 & 1/2 \end{bmatrix}.$$

□

Example 5. *Prove that the set of all solutions (a, b, c) of the equation $a+b+2c=0$ is a subspace of the vector space V_3 (R).*

(Meerut 1989)

Sol. Let $W=\{(a, b, c) : a, b, c \in \mathbf{R}$ and $a+b+2c=0\}$.

To prove that W is a subspace of $V_3(\mathbf{R})$ or $\mathbf{R}^3$.

Let $\alpha=(a_1, b_1, c_1)$ and $\beta=(a_2, b_2, c_2)$ be any two elements of W. Then

$$a_1+b_1+2c_1=0 \quad \ldots(1)$$

and

$$a_2+b_2+2c_2=0. \quad \ldots(2)$$

If a, b be any two elements of $\mathbf{R}$, we have

$$\begin{aligned} a\alpha+b\beta &= a\,(a_1, b_1, c_1)+b\,(a_2, b_2, c_2) \\ &= (aa_1, ab_1, ac_1)+(ba_2, bb_2, bc_2) \\ &= (aa_1+ba_2, ab_1+bb_2, ac_1+bc_2). \end{aligned}$$

Now $(aa_1+ba_2)+(ab_1+bb_2)+2\,(ac_1+bc_2)$

$$\begin{aligned} &= a\,(a_1+b_1+2c_1)+b\,(a_2+b_2+2c_2) \\ &= a.0+b.0 \qquad \text{[from (1) and (2)]} \\ &= 0. \end{aligned}$$

$\therefore \quad a\alpha+b\beta=(aa_1+ba_2, ab_1+bb_2, ac_1+bc_2) \in W.$

Thus $\alpha, \beta \in W$ and $a, b \in \mathbf{R} \Rightarrow a\alpha+b\beta \in W$.

Hence W is a subspace of $V_3(\mathbf{R})$.

Example 6