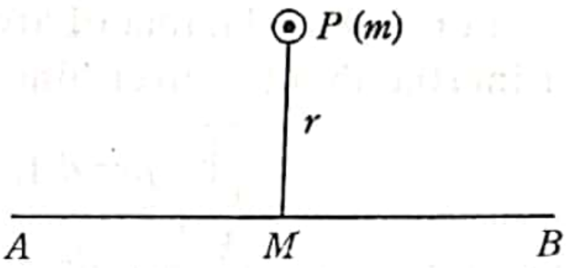
3
Moments of Inertia

§ 3.1. Definitions.

(a) **Rigid Body.** *A rigid body is a collection of particles such that the distance between any two particles of the body remains always the same.*

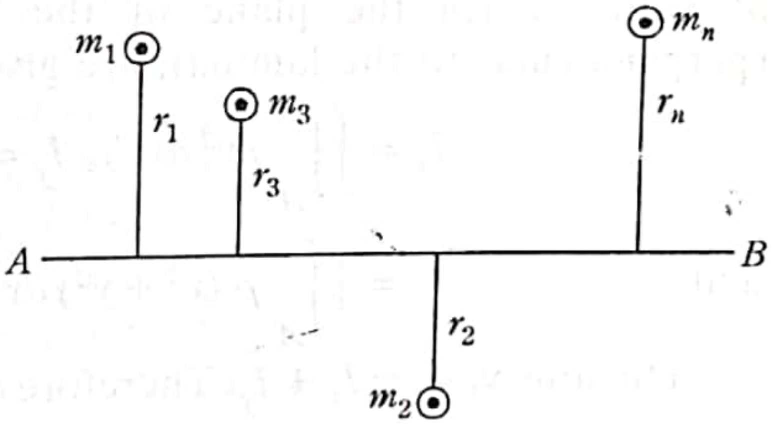
(b) **Moment of Inertia of a particle.** The moment of inertia of a particle of mass m at the point P, about the line AB is defined by



$$I = mr^2,$$

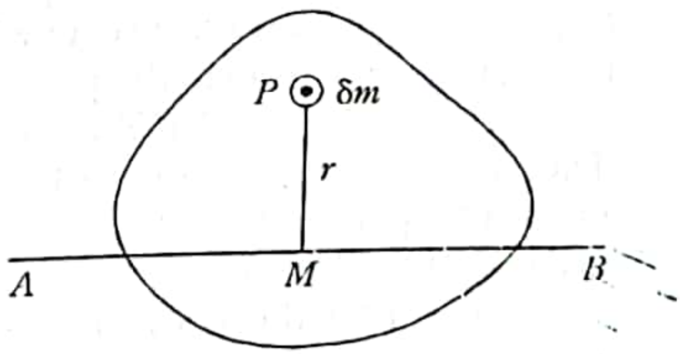
where r is the perpendicular distance of P from the line AB.

(c) **Moment of Inertia of a system of particles.** The moment of inertia of a system of particles of masses $m_1, m_2, \ldots, m_n$ at distances $r_1, r_2, \ldots, r_n$ respectively from the line AB, about the line AB is defined by



$$I = m_1 r_1^2 + m_2 r_2^2 + \ldots + m_n r_n^2$$
$$= \sum_{i=1}^{n} m_i r_i^2.$$

(d) **Moment of Inertia of a rigid body about a given line.** Let δm be the mass of an elementary portion of the body at the point P and r its distance from the line AB, then the moment of inertia of the mass δm about the line AB is $r^2 \delta m$.



∴ The moment of inertia of the body about the line AB is given by

$$I = \int r^2 \, dm,$$

where the integration is taken over the whole body.

(e) Radius of Gyration. The moment of Inertia of a body about the line AB is given by

$$I = \int r^2 \, dm.$$

If the total mass of the body is M and k a quantity such that

$$I = Mk^2,$$

then k is called the *radius of gyration* of the body about the line AB.

(f) Moment of Inertia of a plane lamina of area A about a given line.

For a plane lamina of area A and surface density ρ, the moment of inertia about a given line is

$$\iint_A \rho r^2 \, dA,$$

where r is the distance of the area element dA from the given line.

In particular, the moments of inertia I_x, I_y and I_z about the axes of x and y (in the plane of the lamina) and the axis of z (perpendicular to the lamina), are given by

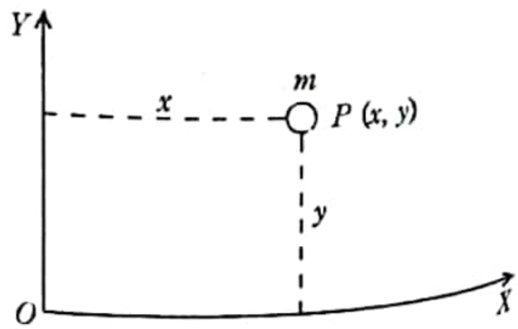
$$I_x = \iint_A \rho y^2 \, dx \, dy, \quad I_y = \iint_A \rho x^2 \, dx \, dy, \qquad \text{...(1)}$$

and

$$I_z = \iint_A \rho \, (x^2 + y^2) \, dx \, dy. \qquad \text{...(2)}$$

Obviously, $I_z = I_x + I_y$. Therefore *the moment of inertia of a plane lamina about any straight line OZ perpendicular to it and meeting it at O is equal to the sum of the moments of inertia of the lamina about any two perpendicular lines OX, OY lying in its plane.*

(g) Product of Inertia. Let m be a particle of mass m at the point P whose coordinates are (x, y) with respect to two mutually perpendicular lines OX and OY as axes. Then the product of inertia of mass m with respect to the lines OX and OY is defined by mxy.



If (x, y) be the coordinates of the mass m of an elementary portion of the body with respect to the perpendicular axes OX and OY, then the product of inertia of the body about these axes OX and OY is defined by $\Sigma\, mxy$.

§ 3.2 Moments and Products of Inertia with Respect to three mutually perpendicular axes.

Let (x, y, z) be the coordinates of the mass m of a body with respect to three mutually perpendicular axes OX, OY, OZ in space. Then the moments of inertia of the body about the coordinate axes OX, OY, OZ denoted by A, B, C respectively and the products of inertia about the axes OY, OZ ; OZ, OX and OX, OY denoted by D, E, F respectively are given by

$$A = \Sigma\, m\,(y^2 + z^2), \qquad B = \Sigma\, m\,(z^2 + x^2), \quad C = \Sigma\, m\,(x^2 + y^2),$$
$$D = \Sigma\, myz, \qquad E = \Sigma\, mzx, \qquad F = \Sigma\, mxy.$$

§ 3.3 Some Simple Propositions :

Prop. I. *If A, B, C denote the moments of inertia and D, E, F the products of inertia about three mutually perpendicular axes, the sum of any two of them is greater than the third.*

We have, $A = \Sigma\, m\,(y^2 + z^2), B = \Sigma\, m\,(z^2 + x^2), C = \Sigma\, m\,(x^2 + y^2)$.

Now, $A + B - C = \Sigma\, m\,(y^2 + z^2) + \Sigma\, m\,(z^2 + x^2) - \Sigma\, m\,(x^2 + y^2)$

$= 2\,\Sigma\, mz^2$, which is positive.

$\therefore A + B > C$.

Prop. II. *The sum of the moments of inertia about any three rectangular axes meeting at a given point is always constant and is equal to twice the moment of inertia about that point.*

Take the given point O as origin. We have

$A + B + C = \Sigma\, m\,(y^2 + z^2) + \Sigma\, m\,(z^2 + x^2) + \Sigma\, m\,(x^2 + y^2)$

$= 2\,\Sigma\, m\,(x^2 + y^2 + z^2) = 2\,\Sigma\, mr^2$, where $r = \sqrt{(x^2 + y^2 + z^2)}$ is the distance of the mass m at (x, y, z) from the given point O as origin

$= 2$ (M.I. of the body about the given point).

Thus, the sum $A + B + C$ is independent of the directions of axes and is equal to twice the moment of inertia about the given point.

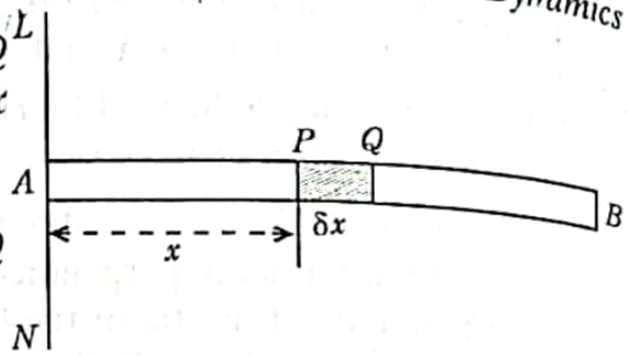
Moments of Inertia in Some Simple Cases

§ 3.4 Moment of Inertia of a Uniform Rod of Length $2a$.

(i) About a line through an end and perpendicular to the rod.

Let M be the mass of a uniform rod AB of length $2a$, then mass of the rod per unit length $= \rho = M/2a$.

Consider an element PQ of breadth δx at a distance x from the end A.



We have,

δm = mass of the element PQ

$= (M/2a)\,\delta x$.

M.I. of this element PQ about the line LN passing through the end A of the rod AB and perpendicular to it

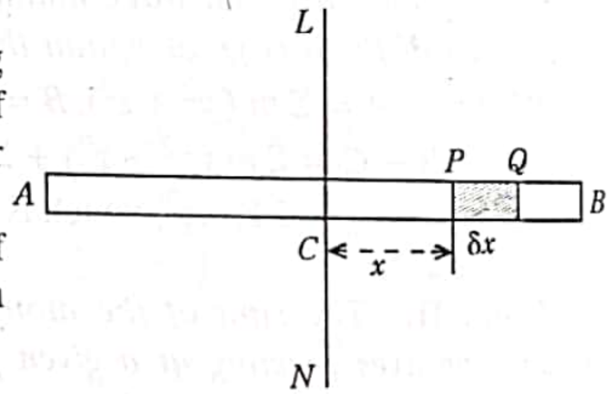
$$= x^2\,\delta m = (M/2a)\,x^2\,\delta x.$$

$\therefore$ M.I. of the rod AB about the line LN

$$= \int_0^{2a} \frac{M}{2a} x^2\,dx = \frac{M}{2a}\left[\frac{1}{3}x^3\right]_0^{2a} = \frac{4}{3}Ma^2.$$

(ii) About a line through the middle point and perpendicular to the rod.

Let LN be the line passing through the middle point C of the rod AB and perpendicular to it.



Consider an element PQ of breadth δx at a distance x from the middle point C.

We have,

δm = mass of the element $PQ = (M/2a)\,\delta x$. $[\because \rho = M/2a]$

M.I. of the element PQ about the line LN

$$= x^2\,\delta m = a\,(M/2a)\,x^2\,\delta x.$$

$\therefore$ M.I. of the rod AB about the line LN

$$= \int_{-a}^{a} \frac{M}{2a} x^2\,dx = \frac{M}{2a}\left[\frac{1}{3}x^3\right]_{-a}^{a} = \frac{1}{3}Ma^2.$$

§ 3.5 Moment of Inertia of a Rectangular Lamina.

(i) About a line through its centre and parallel to a side.

Let M be the mass of a rectangular lamina $ABCD$ such that $AB = 2a$ and $BC = 2b$.

$\therefore$ Mass per unit area of the rectangle $= \rho = M/4ab$.

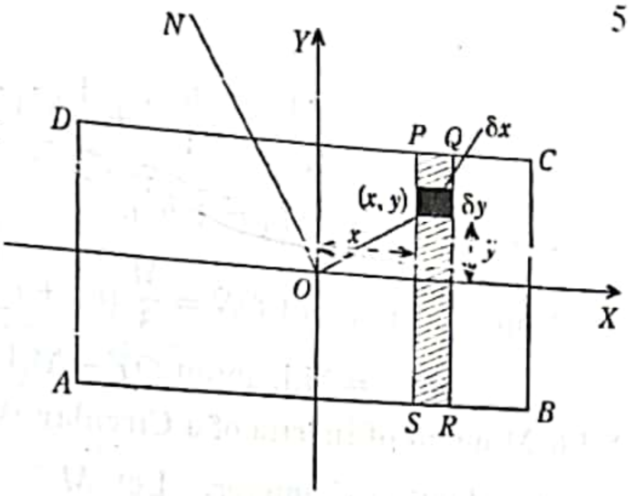
Let OX and OY be the lines parallel to the sides AB and BC of the rectangle through its centre O.

Consider an elementary strip $PQRS$ of breadth δx at a distance x from O and parallel to BC.

We have, δm = mass of the strip $PQRS$

$= \rho\,.\,2b\,\delta x$

$$= \frac{M}{4ab} 2b\,\delta x = \frac{M}{2a}\,\delta x.$$



M.I. of the strip about OX

$= \frac{1}{3} b^2\,\delta m$ [See § 3 part (ii)]

$$= \frac{1}{3} b^2 \cdot \frac{M}{2a}\,\delta x = \frac{1}{6}\frac{Mb^2}{a}\,\delta x.$$

$\therefore$ M.I. of the rectangle $ABCD$ about OX

$$= \int_{-a}^{a} \frac{Mb^2}{6a}\,dx = \frac{Mb^2}{6a}\Big[x\Big]_{-a}^{a} = \frac{1}{3}Mb^2.$$

Similarly, M.I. of the rectangle $ABCD$ about $OY = \frac{1}{3}Ma^2$.

Aliter. Consider an elementary area $\delta x\,\delta y$ at a point (x, y) of the lamina.

We have, δm = mass of the elementary area $\delta x\,\delta y$

$$= \rho\,\delta x\,\delta y = (M/4ab)\,\delta x\,\delta y.$$

M.I. of this elementary mass δm about OX

$$= y^2\,\delta m = (M/4ab)\,y^2\,\delta x\,\delta y.$$

Hence, M.I. of the rectangular lamina $ABCD$ about OX

$$= \int_{x=-a}^{a}\int_{y=-b}^{b} \frac{M}{4ab} y^2\,dx\,dy = \frac{M}{4ab}\int_{-a}^{a}\left[\frac{1}{3}y^3\right]_{-b}^{b} dx$$

$$= \frac{M}{4ab}\cdot\frac{2}{3}b^3\int_{-a}^{a} dx = \frac{M}{6a}b^2\Big[x\Big]_{-a}^{a} = \frac{1}{3}Mb^2.$$

(ii) About a line through its centre and perpendicular to its plane.

Let ON be the line through the centre O and perpendicular to the plane of the rectangular lamina $ABCD$.

Consider an elementary area $\delta x\,\delta y$ at a point (x, y) of the lamina.

We have, δm = mass of the elementary area $\delta x\,\delta y$

$$= \rho\,\delta x\,\delta y = (M/4ab)\,\delta x\,\delta y.$$

Distance of this elementary area from $ON = \sqrt{(x^2 + y^2)}$.

$\therefore$ M.I. of this elementary mass about ON

$$= ON^2 . \delta m = (x^2 + y^2)\,(M/4ab)\,\delta x\,\delta y.$$

Hence, M.I. of the rectangular lamina about ON

$$= \int_{x=-a}^{a}\int_{y=-b}^{b} \frac{M}{4ab}(x^2 + y^2)\,dx\,dy$$

$$= \frac{M}{4ab}\int_{-a}^{a}\left[x^2 y + \frac{1}{3}y^3\right]_{-b}^{b} dx = \frac{M}{4ab}\int_{-a}^{a} 2\left(bx^2 + \frac{1}{3}b^3\right) dx$$

$$= \frac{M}{4ab}\left[2\left(\frac{b}{3}x^3 + \frac{1}{3}b^3 x\right)\right]_{-a}^{a} = \frac{M}{4ab}\cdot\frac{4}{3}(ba^3 + b^3 a)$$

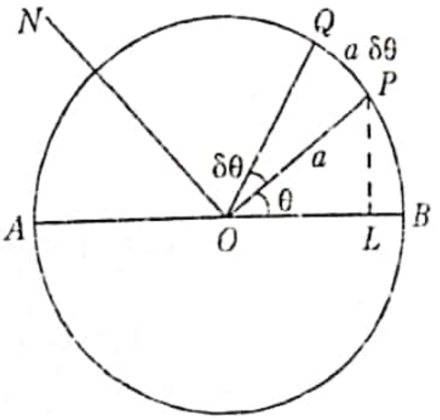
$$= \frac{1}{3}M(a^2 + b^2).$$

Note. M.I. about $ON = \frac{M}{3}(a^2 + b^2) = \frac{1}{3}Ma^2 + \frac{1}{3}Mb^2$

$= $ M.I. about $OY +$ M.I. about OX.

§ 3.6 Moment of Inertia of a Circular Wire. (Meerut 1995; Agra 91)

(i) About a diameter. Let M be the mass of the circular wire of centre O and radius a. Then, mass per unit length of the wire $= \rho = M/2\pi a$.



Consider an elementary arc $PQ = a\,\delta\theta$ of the wire. Its mass $= \rho a\,\delta\theta = \delta m$.

Distance of this element from the diameter $AB = PL = a\sin\theta$.

$\therefore$ M.I. of this element about the diameter AB

$$= (a\sin\theta)^2 . \delta m = a^2 \sin^2\theta . \rho a\,\delta\theta = \rho a^3 \sin^2\theta . \delta\theta.$$

Hence, M.I. of the circular wire about the diameter AB

$$= \int_0^{2\pi} \rho a^3 \sin^2\theta\, d\theta = \frac{1}{2}\rho a^3 \int_0^{2\pi} (1 - \cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\rho a^3 \left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{2\pi} = \frac{1}{2}\cdot\frac{M}{2\pi a}\cdot a^3 . 2\pi \qquad \left[\because \rho = \frac{M}{2\pi a}\right]$$

$$= \frac{1}{2}\boldsymbol{Ma^2}.$$

(ii) About a line through the centre and perpendicular to its plane. (Meerut 1988)

Let ON be the line through the centre O and perpendicular to the plane of the circular wire.

$\therefore$ M.I. of the elementary arc PQ about ON

$$= OP^2 . \delta m = a^2 . \rho a\,\delta\theta = \rho a^3\,\delta\theta.$$

Hence, M.I. of the wire about ON

$$= \int_0^{2\pi} \rho a^3\, d\theta = \frac{M}{2\pi a}\cdot a^3\left[\theta\right]_0^{2\pi} = \boldsymbol{Ma^2}. \qquad \left[\because \rho = \frac{M}{2\pi a}\right]$$

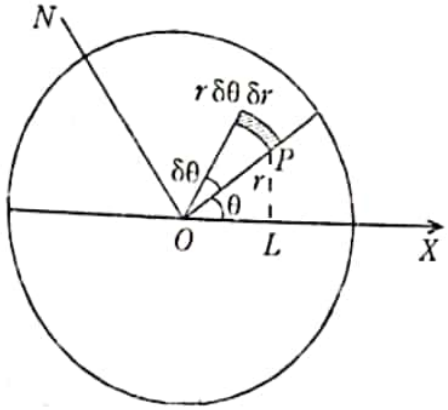
§ 3.7 Moment of Inertia of a Circular Disc.

(i) About a diameter. (Meerut 1990)

Let M be the mass of a circular disc, of centre O and radius a. Then.

mass per unit area of the disc $= \rho = M/\pi a^2$.

Consider an elementary area $r\,\delta\theta\,\delta r$ at the point $P(r, \theta)$ of the disc referred to the centre O as the pole and OX as the initial line.



Mass of this element

$$= \rho \,.\, r\,\delta\theta\,\delta r = \delta m.$$

Distance of this element from $OX = PL = r \sin\theta$.

$\therefore$ M.I. of this element about OX

$$= (r \sin\theta)^2 \,.\, \delta m$$

$$= r^2 \sin^2\theta \,.\, \rho r\,\delta\theta\,\delta r = \rho r^3 \sin^2\theta \,.\, \delta\theta\,\delta r.$$

Hence, M.I. of the circular disc about OX

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{a} \rho r^3 \sin^2\theta \, d\theta\, dr = \rho \int_0^{2\pi} \left[\frac{1}{4} r^4\right]_0^a \sin^2\theta \, d\theta$$

$$= \frac{1}{4}\rho a^4 \,.\, \int_0^{2\pi} \frac{1}{2}(1 - \cos 2\theta)\, d\theta = \frac{1}{8}\rho a^4 \left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{2\pi}$$

$$= \frac{1}{8} \cdot \rho a^4 \,.\, 2\pi = \frac{1}{4} \cdot \frac{M}{\pi a^2} \cdot \pi a^4 = \frac{1}{4} M a^2. \qquad [\because \rho = M/\pi a^2]$$

(ii) About a line through the centre and perpendicular to its plane. **(Meerut 1994)**

Let ON be the line through the centre O and perpendicular to the plane of the plate.

M.I. of the mass of the elementary area $r\,\delta\theta\,\delta r$ about ON

$$= OP^2 \,.\, \delta m = r^2 \,.\, \rho r\,\delta\theta\,\delta r = \rho r^3\,\delta\theta\,\delta r.$$

Hence, M.I. of the circular disc about ON

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{a} \rho r^3 \, d\theta\, dr = \rho \int_0^{2\pi} \left[\frac{1}{4} r^4\right]_0^a d\theta$$

$$= \frac{1}{4}\rho a^4 \left[\theta\right]_0^{2\pi} = \frac{1}{4} \cdot \frac{M}{\pi a^2} \cdot a^4 \,.\, 2\pi = \frac{1}{2} M a^2. \qquad [\because \rho = M/\pi a^2]$$

§ 3.8 Moment of Inertia of An Elliptic Disc.

(Meerut 1985)

(i) About the major axis.

Let M be the mass of an elliptic disc of axes $2a$ and $2b$. Then mass per unit area of the disc $= \rho = M/\pi ab$. ...(1)

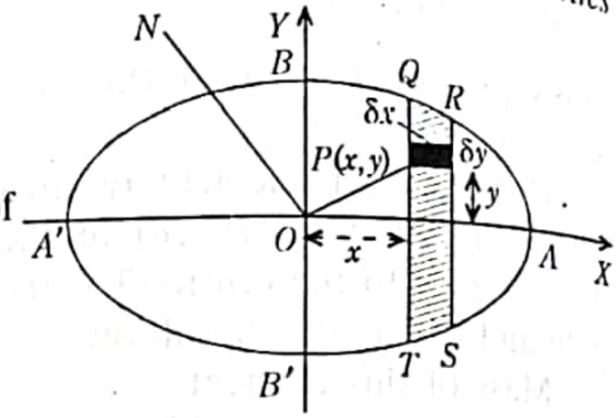
Equation of the ellipse is $x^2/a^2 + y^2/b^2 = 1$.

Consider an elementary area $\delta x\,\delta y$ at the point (x, y). Then its mass $\delta m = \rho\,\delta x\,\delta y$.

$\therefore$ M.I. of the elementary mass δm about OX

$= y^2\,\delta m = y^2 \rho\,\delta x\,\delta y.$

Hence moment of inertia of the elliptic disc about OX



$$= \int_{x=-a}^{a} \int_{y=-y_1}^{y_1} y^2 \rho\,dx\,dy,$$

where $y_1 = b\sqrt{\left(1 - \dfrac{x^2}{a^2}\right)}$, from(1)

$$= \rho \int_{-a}^{a} \left[\frac{1}{3} y^3\right]_{-y_1}^{y_1} dx = \frac{2}{3}\rho \int_{-a}^{a} y_1^3\,dx = \frac{2}{3}\rho \int_{-a}^{a} b^3 \left(1 - \frac{x^2}{a^2}\right)^{3/2} dx$$

$$= \frac{2}{3}\rho b^3 a \int_{-\pi/2}^{\pi/2} (1 - \sin^2\theta)^{3/2}\, a\cos\theta\,d\theta,$$

putting $x = a\sin\theta$, so that $dx = a\cos\theta\,d\theta$

$$= \frac{2}{3}\rho b^3 a \int_{-\pi/2}^{\pi/2} \cos^4\theta\,d\theta = \frac{2}{3}\rho b^3 a \,.\, 2\int_{0}^{\pi/2} \cos^4\theta\,d\theta$$

$$= \frac{4}{3} \cdot \rho b^3 a \,.\, \frac{3\,.\,1}{4\,.\,2} \cdot \frac{\pi}{2} = \frac{4}{3}\rho b^3 a \,.\, \frac{3}{16}\pi = \frac{1}{4}\rho\pi b^3 a$$

$$= \frac{1}{4}\frac{M}{\pi ab}\pi b^3 a = \frac{1}{4} Mb^2. \qquad [\because\ \rho = M/\pi ab]$$

(ii) About the minor axis.

Similarly, M.I. of an elliptic disc about the minor axis $BB' = \frac{1}{4} Ma^2$.

(iii) About the line ON through the centre O and perpendicular to the plane of lamina.

M.I. of the disc about the line ON through the centre and perpendicular to its plane

$= \text{M.I. about } OA + \text{M.I. about } OB$

$= \frac{1}{4}Mb^2 + \frac{1}{4}Ma^2 = \frac{1}{4}M(a^2 + b^2).$

Aliter. M.I. of element δm about ON

$= OP^2 \,.\, \delta m = \rho\,(x^2 + y^2)\,\delta x\,\delta y.$

$\therefore$ M.I. of the elliptic disc about ON

$$= \int_{x=-a}^{a} \int_{y=-y_1}^{y_1} \rho\,(x^2 + y^2)\,dx\,dy, \text{ where } y_1 = (b/a)\sqrt{(a^2 - x^2)},$$

from (1)

$$= \int_{x=-a}^{a} \rho \left[x^2 y + \frac{1}{3} y^3\right]_{-y_1}^{y_1} dx = 2\rho \int_{-a}^{a} \left(x^2 y_1 + \frac{1}{3} y_1^3\right) dx$$

$$= 2\rho \int_{-a}^{a} \left[x^2 \cdot (b/a) \sqrt{(a^2 - x^2)} + \frac{1}{3} (b^3/a^3) (a^2 - x^2)^{3/2}\right] dx$$

$$= 4\rho \int_{0}^{a} \left[(b/a) x^2 \sqrt{(a^2 - x^2)} + \frac{1}{3} (b^3/a^3) (a^2 - x^2)^{3/2}\right] dx$$

[$\because$ The integrand is an even function of x]

$$= 4\rho \int_{0}^{\pi/2} \left[(b/a) a^2 \sin^2\theta \cdot a\cos\theta + \frac{1}{3} (b^3/a^3) a^3 \cos^3\theta\right] a\cos\theta \, d\theta,$$

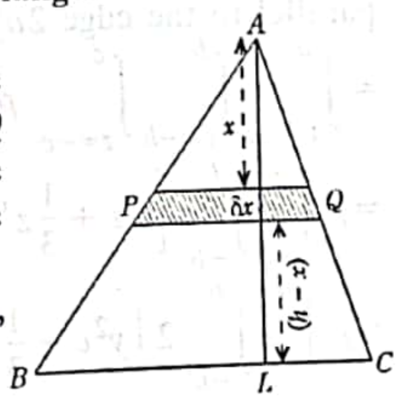
putting $x = a\sin\theta$, so that $dx = a\cos\theta \, d\theta$

$$= 4\rho ab \left[a^2 \int_{0}^{\pi/2} \sin^2\theta \cos^2\theta \, d\theta + \frac{1}{3} b^2 \int_{0}^{\pi/2} \cos^4\theta \, d\theta\right]$$

$$= 4\rho ab \left[a^2 \cdot \frac{\Gamma(3/2)\Gamma(3/2)}{2\Gamma(3)} + \frac{1}{3} b^2 \cdot \frac{\Gamma(5/2)\Gamma(1/2)}{2\Gamma(3)}\right]$$

$$= 4\rho ab \cdot \left(\frac{1}{16}\pi\right)(a^2 + b^2) = \frac{1}{4} M (a^2 + b^2).$$

§ 3.9 Moment of Inertia of a Uniform Triangular Lamina About one Side.



Let M be the mass and $h = AL$, the height of a triangular lamina ABC. Let PQ be an elementary strip parallel to the base BC, of breadth δx and at a distance x from the vertex A of the triangle.

From similar triangles APQ and ABC, we have $x/AL = PQ/BC$.

$\therefore$ $PQ = ax/h$, where $BC = a$.

$\therefore$ δm = mass of the elementary strip PQ

$= \rho \, PQ \, \delta x = \rho \, (ax/h) \, \delta x.$

$\therefore$ M.I. of this elementary strip about BC

$$= (x - h)^2 \delta m = \frac{\rho a}{h} (h - x)^2 x \, \delta x.$$

$\therefore$ M.I. of the triangle ABC about BC

$$= \int_{0}^{h} \frac{\rho a}{h} (h - x)^2 x \, dx = \left(\frac{\rho a}{h}\right) \int_{0}^{h} (h^2 x - 2hx^2 + x^3) \, dx$$

$$= \left(\frac{\rho a}{h}\right) \left[\frac{1}{2} h^2 x^2 - \frac{2}{3} hx^3 + \frac{1}{4} x^4\right]_0^h = \frac{1}{12} \rho a h^3 = \frac{1}{6} M h^2.$$

$\left[\because M = \text{mass of } \Delta ABC = \rho \cdot \left(\frac{1}{2} ah\right)\right]$

§ 3.10 Moment of Inertia of a Rectangular Parallelopiped About an Axis Through its Centre and Parallel to one of its Edges.

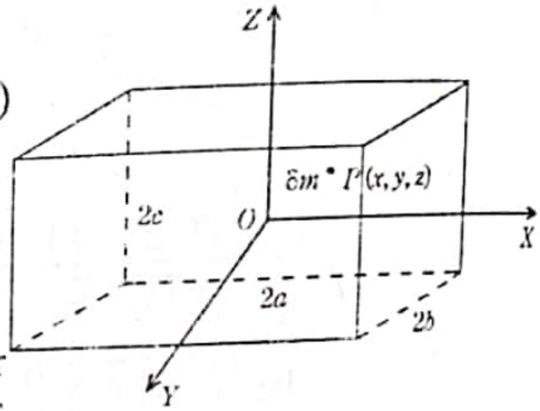
Let O be the centre and $2a, 2b, 2c$ the lengths of the edges of a rectangular parallelopiped. If M is the mass of the parallelopiped, the mass per unit volume

$$= \rho = \frac{M}{2a \,.\, 2b \,.\, 2c} = \frac{M}{8abc}.$$

Let OX, OY, OZ be the axes through the centre and parallel to the edges of the rectangular parallelopiped.

Consider an elementary volume $\delta x\, \delta y\, \delta z$ of the parallelopiped at the point $P(x, y, z)$. Its mass δm

$$= \rho\, \delta x\, \delta y\, \delta z.$$



Distance of the point $P(x, y, z)$ from OX is $\sqrt{(y^2 + z^2)}$.

$\therefore$ M.I. of the elementary volume of mass δm at P about OX

$$= \rho\,(y^2 + z^2)\, \delta x\, \delta y\, \delta z.$$

Hence, M.I. of the rectangular parallelopiped about OX (which is parallel to the edge $2a$)

$$= \int_{x=-a}^{a} \int_{y=-b}^{b} \int_{z=-c}^{c} \rho\,(y^2 + z^2)\, dx\, dy\, dz$$

$$= \rho \int_{-a}^{a} \int_{-b}^{b} \left[y^2 z + \frac{1}{3} z^3\right]_{-c}^{c} dx\, dy$$

$$= \rho \int_{-a}^{a} \int_{-b}^{b} 2\left(y^2 c + \frac{1}{3} c^3\right) dx\, dy = 2\rho \int_{-a}^{a} \left[\frac{1}{3} y^3 c + \frac{1}{3} c^3 y\right]_{-b}^{b} dx$$

$$= \frac{2}{3}\rho \int_{-a}^{a} 2\,(b^3 c + c^3 b)\, dx = \frac{4\rho}{3}\, bc\,(b^2 + c^2)\left[x\right]_{-a}^{a}$$

$$= \frac{4\rho}{3}\, bc\,(b^2 + c^2)\,.\,2a = \frac{4}{3} \cdot \frac{M}{8abc} \cdot bc\,(b^2 + c^2)\,.\,2a \qquad \left[\because\ \rho = \frac{M}{8abc}\right]$$

$$= \frac{1}{3} M\,(b^2 + c^2).$$

Similarly, the moments of inertia of the rectangular parallelopiped about the lines OY, OZ through the centre O and parallel to the edges $2b$ and $2c$ are $\frac{1}{3} M\,(c^2 + a^2)$ and $\frac{1}{3} M\,(a^2 + b^2)$ respectively.

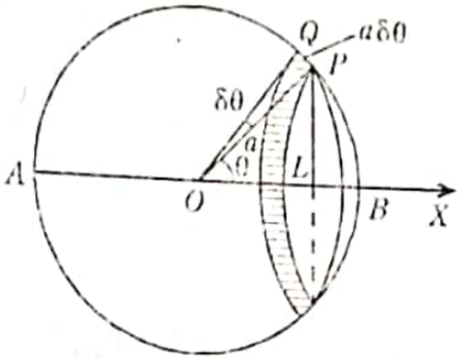
Note : For a cube of side $2a$, we have $2b = 2c = 2a$.

$\therefore$ M.I. of a cube about a line through its centre and parallel to any one of its edges $= \frac{2}{3} Ma^2$.

§3.11 M.I. of a Spherical Shell (*i.e.*, Hollow sphere) about a diameter.

(Meerut 1992)

A spherical shell (*i.e.*, hollow sphere) of radius a is formed by the revolution of a semi-circular arc of radius a about its diameter.



Consider an elementary arc $PQ = a\,\delta\theta$ at the point P of the semi-circular arc. A circular ring of radius $PL = a \sin\theta$ will be formed by the revolution of this arc PQ about the diameter AB.

Mass of this elementary ring

$= \delta m = \rho\,.\,2\pi PL\;a\,\delta\theta$

$= \rho\,2\pi\,a \sin\theta\;a\,\delta\theta = \rho\,2\pi a^2 \sin\theta\,\delta\theta,$ where $\rho = (M/4\pi a^2)$,

M is the mass of the shell.

The distance of each point of this ring from $AB = a \sin\theta$.

$\therefore$ M.I. of this elementary ring about AB (a line through the centre of the ring and perpendicular to its plane)

$= PL^2\,.\,\delta m$ [See § 3.6, part (ii)]

$= a^2 \sin^2\theta\,\rho\,2\pi a^2 \sin\theta\,\delta\theta = 2\pi\rho\,a^4 \sin^3\theta\,\delta\theta.$

$\therefore$ M.I. of the shell about the diameter AB

$$= \int_0^{\pi} 2\pi\rho a^4 \sin^3\theta\,d\theta = 2\pi\rho a^4 \int_0^{\pi} (1-\cos^2\theta)\sin\theta\,d\theta$$

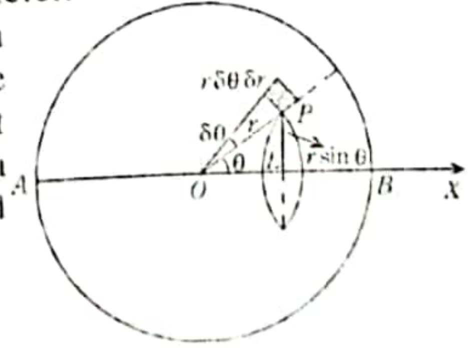
$$= -2\pi\rho a^4 \int_1^{-1} (1-t^2)\,dt, \text{ putting } \cos\theta = t, \text{ so that } -\sin\theta\,d\theta = dt$$

$$= -2\pi\cdot\frac{M}{4\pi a^2}a^4\left[t-\frac{1}{3}t^3\right]_1^{-1} = \frac{2}{3}Ma^2.$$

§3.12 M.I. of A Solid sphere about a diameter.

(Meerut 1985, 94)

A solid sphere of radius a is formed by the revolution of a semi-circular area of radius a about its diameter.



Consider an elementary area $r\,\delta\theta\,\delta r$ at the point $P\,(r, \theta)$ of the semi-circular area. When this element is revolved about the diameter AB, a circular ring of radius $PL = r \sin\theta$ and cross-section $r\,\delta\theta\,\delta r$ is formed.

Mass of this elementary ring

$= \delta m = \rho\,.\,2\pi r \sin\theta\,.\,r\,\delta\theta\,\delta r$

$$= \rho\, 2\pi r^2 \sin\theta\, \delta\theta\, \delta r, \text{ where } \rho = \frac{M}{\frac{4}{3}\pi a^3},$$

M is the mass of the sphere.

M.I. of this elementary ring about AB (a line through the centre of the ring perpendicular to its plane)

$= PL^2 . \delta m$ [See § 3.6, part (ii)]

$= r^2 \sin^2\theta . \rho\, 2\pi r^2 \sin\theta\, \delta\theta\, \delta r$

$= 2\pi\rho r^4 \sin^3\theta\, \delta\theta\, \delta r.$

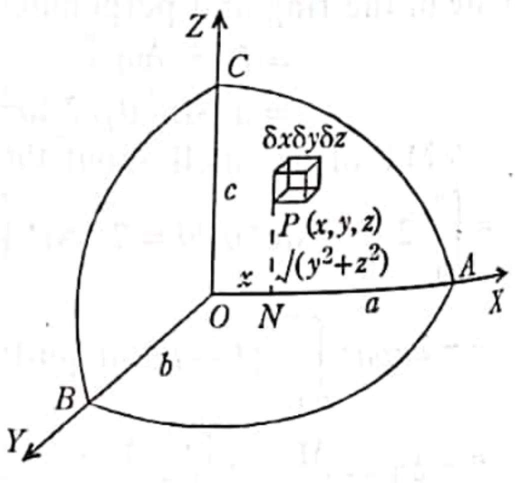
$\therefore$ M.I. of the sphere about the diameter AB

$$= \int_{\theta=0}^{\pi} \int_{r=0}^{a} 2\pi\rho r^4 \sin^3\theta\, d\theta\, dr = 2\pi\rho . \frac{1}{5} a^5 \int_0^{\pi} \sin^3\theta\, d\theta$$

$$= \frac{2\pi}{5}\rho a^5 . 2\int_0^{\pi/2} \sin^3\theta\, d\theta = \frac{4\pi}{5}\rho a^5 \cdot \frac{2}{3 . 1}, \text{ by Walli's formula}$$

$$= \frac{4\pi}{5} \cdot \frac{M}{\frac{4}{3}\pi a^3} \cdot a^5 . \frac{2}{3} = \frac{2}{5} Ma^2.$$

§ 3.13 M.I. of an Ellipsoid. (Meerut 1985, 93; Kanpur 90; Rohilkhand 94)



Let the equation of the ellipsoid be $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$

Consider an elementary volume $\delta x\, \delta y\, \delta z$ at the point $P(x, y, z)$ of the ellipsoid in the positive octant.

Mass of this element

$= \rho\, \delta x\, \delta y\, \delta z,$

where ρ = mass per unit volume

$$= \frac{M}{\frac{4}{3} . \pi abc} = \frac{3M}{4\pi abc}, \quad M \text{ is the mass of the ellipsoid.}$$

Distance of the point $P(x, y, z)$ from $OX = \sqrt{(y^2 + z^2)}$.

$\therefore$ M.I. of this elementary volume about $OX = (y^2 + z^2)\rho\, \delta x\, \delta y\, \delta z.$

$\therefore$ M.I. of the ellipsoid about OX

$$= \int_{x=-a}^{a} \int_{y=-y_1}^{y_1} \int_{z=-z_1}^{z_1} \rho\, (y^2 + z^2)\, dx\, dy\, dz$$

$$= 8\rho \int_{x=0}^{a} \int_{y=0}^{y_1} \int_{z=0}^{z_1} (y^2 + z^2)\, dx\, dy\, dz,$$

where $z_1 = c\sqrt{(1 - x^2/a^2 - y^2/b^2)}$ and $y_1 = b\sqrt{(1 - x^2/a^2)}$

$$= 8\rho \int_{x=0}^{a} \int_{y=0}^{y_1} \left[y^2 z + \frac{1}{3} z^3\right]_0^{z_1} dx\, dy = 8\rho \int_{x=0}^{a} \int_{y=0}^{y_1} \left(y^2 z_1 + \frac{1}{3} z_1^{\,3}\right) dx\, dy$$

$$= 8\rho \int_{x=0}^{a} \int_{y=0}^{y_1} \left[\{y^2 c \sqrt{(1 - x^2/a^2 - y^2/b^2)} \right.$$

$$\left. + \frac{1}{3} c^3 (1 - x^2/a^2 - y^2/b^2)^{3/2} \right] dx\, dy.$$

Put $y = b\sqrt{(1 - x^2/a^2)} \sin\theta$, so that $dy = b\sqrt{(1 - x^2/a^2)} \cos\theta\, d\theta$.

When $y = 0$, we have $b\sqrt{(1 - x^2/a^2)} \sin\theta = 0$, giving

$\sin\theta = 0$ *i.e.*, $\theta = 0$

and when $y = y_1 = b\sqrt{(1 - x^2/a^2)}$, we have

$b\sqrt{(1 - x^2/a^2)} \sin\theta = b\sqrt{(1 - x^2/a^2)}$, giving $\sin\theta = 1$ *i.e.*, $\theta = \pi/2$.

$\therefore$ M.I. of the ellipsoid about OX

$$= 8\rho \int_{x=0}^{a} \int_{\theta=0}^{\pi/2} \left[b^2 c \left(1 - \frac{x^2}{a^2}\right) \sin^2\theta \,.\, \sqrt{\left(1 - \frac{x^2}{a^2}\right)} \cos\theta \right.$$

$$\left. + \frac{1}{3} c^3 \left(1 - \frac{x^2}{a^2}\right)^{3/2} . \cos^3\theta \right] b \sqrt{\left(1 - \frac{x^2}{a^2}\right)} . \cos\theta\, dx\, d\theta$$

$$= 8\rho bc \int_{x=0}^{a} \int_{\theta=0}^{\pi/2} \left(1 - \frac{x^2}{a^2}\right)^2 \left[b^2 \sin^2\theta \cos^2\theta + \frac{1}{3} c^2 \cos^4\theta \right] dx\, d\theta$$

$$= 8\rho bc \int_{x=0}^{a} \left(1 - \frac{x^2}{a^2}\right)^2 \cdot \left[b^2 \frac{\Gamma(3/2)\,\Gamma(3/2)}{2\,\Gamma(3)} + \frac{1}{3} c^2 . \frac{\Gamma(5/2)\,\Gamma(1/2)}{2\,\Gamma(3)} \right] dx$$

$$= 8\rho bc \cdot \frac{1}{16} (b^2 + c^2)\, \pi \int_0^a \left(1 - \frac{2x^2}{a^2} + \frac{x^4}{a^4}\right) dx$$

$$= 8 \cdot \frac{3M}{4\pi abc} \cdot bc\, \frac{1}{16} (b^2 + c^2)\, \pi\, \frac{8}{15} a = \frac{1}{5} M (b^2 + c^2).$$

§ 3.14 Routh's Rule.

Moments of inertia of some symmetric rigid bodies may be remembered with the help of **Routh's Rule** which is as follows :

Moment of inertia about an axis of symmetry

$$= \text{mass} \times \frac{\text{sum of squares of perpendicular semi-axes}}{3, 4 \text{ or } 5},$$

the denominator is 3, 4 or 5 according as the body is rectangular (including rod), elliptical (including circular) or ellipsoid (including sphere).

Solved Examples

Ex. 1. *Find the M.I. of a thin rod of which the line density varies as the distance from one end about an axis passing through that end and at right angles to the rod.*

Sol. [Refer fig. of § 3.4 case (i)]

Let AB be a rod of mass $2a$ and LN the line through the end A and perpendicular to the rod. Consider an element PQ of breadth δx at a distance x from the end A.

If ρ is the density at P, then $\rho = k \, . \, AP = kx$.

$\therefore \delta m =$ mass of the element δx at $P = \rho \, \delta x = kx \, \delta x$.

If M is the mass of the rod AB, then $M = \int_{x=0}^{2a} kx \, dx = 2a^2 k.$

Now M.I. of the elementary mass δm at P about LN
$= AP^2 \, . \, \delta m = x^2 \, . \, kx \, \delta x = kx^3 \, \delta x.$

Hence, M.I. of the rod AB about $LN = \int_{x=0}^{2a} kx^3 \, dx = k \, . \, 4a^4$
$= 2 \, (2a^2 k) \, a^2 = 2Ma^2.$

Ex. 2. *Find the M.I. of the elliptic lamina $4x^2 + 9y^2 = 36$ about a straight line through its centre and perpendicular to the lamina, M being its mass.*

Sol. Here the equation of the lamina is $\dfrac{x^2}{3^2} + \dfrac{y^2}{2^2} = 1.$

Proceed as in § 3.8, case (iii), Aliter. Here $a = 3, b = 2$.

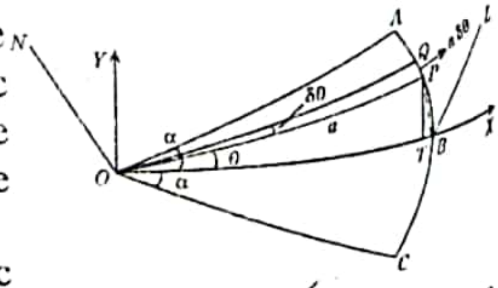
M.I. $= 13 \; M/4$.

Ex. 3. *Find the M.I. of the arc of a circle about*

(i) the diameter bisecting the arc **(Agra 1985; Kanpur 81)**

(ii) an axis through the centre, perpendicular to its plane **(Kanpur 1981)**

(iii) an axis through its middle point perpendicular to its plane. **(Agra 1996)**



Sol. Let OB be along the diameter bisecting the circular arc ABC subtending an angle 2α at the centre O. Let a be the radius of the arc.

Consider an elementary arc $PQ = a \, \delta\theta$ at the point P of the arc.

Its mass $\delta m = \rho a \, \delta\theta$, where ρ is mass per unit length of the arc

$= \dfrac{M}{2\alpha a}$, where M is the mass of the arc ABC.

(i) Distance of P from diameter $OB = PT = a \sin\theta$.

$\therefore$ M.I. of the elementary arc PQ about OB

$$= PT^2 \,.\, \delta m = (a \sin\theta)^2 \rho a \,\delta\theta = \rho a^3 \sin^2\theta \,\delta\theta.$$

Hence, M.I. of the arc ABC about the diameter OB.

$$= \int_{-\alpha}^{\alpha} \rho a^3 \sin^2\theta \, d\theta = \frac{1}{2}\rho a^3 \int_{-\alpha}^{\alpha} (1 - \cos 2\theta)\, d\theta = \frac{1}{2}\rho a^3 \left[\theta - \frac{1}{2}\sin 2\theta\right]_{-\alpha}^{\alpha}$$

$$= \frac{1}{2}\frac{M}{2\alpha a} a^3 [2\alpha - \sin 2\alpha] = \frac{Ma^2}{2\alpha}(\alpha - \sin\alpha\cos\alpha).$$

(ii) Distance of the point P from ON, an axis through the centre and perpendicular to the arc $= OP = a$.

$\therefore$ M.I. of the elementary mass δm at P about ON

$$= a^2 \,.\, \delta m = \rho a^3 \,\delta\theta.$$

$$\therefore \text{M.I. of the arc } ABC \text{ about } ON = \int_{-\alpha}^{\alpha} \rho a^3 \, d\theta = \rho a^3 \left[\theta\right]_{-\alpha}^{\alpha}$$

$$= (M/2\alpha a)\, a^3 \,.\, 2\alpha = Ma^2.$$

(iii) Distance of the point P from BL, an axis through the middle point B of the arc ABC and perpendicular to its plane

$$= PB = \sqrt{(OP^2 + OB^2 - 2OP\,.\,OB\cos\theta)} = \sqrt{(a^2 + a^2 - 2a^2\cos\theta)}$$

$$= a\sqrt{[2(1-\cos\theta)]} = a\sqrt{[2\,.\,2\sin^2\tfrac{1}{2}\theta]} = 2a\sin\tfrac{1}{2}\theta.$$

$\therefore$ M.I. of the elementary mass δm at P about $BL = PB^2 \,.\, \delta m$

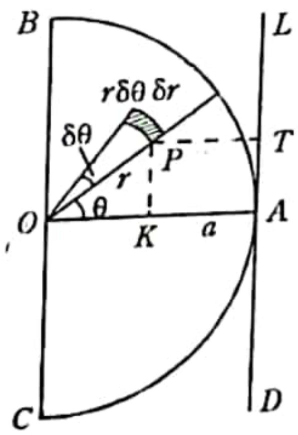
$$= \left(2a\sin\tfrac{1}{2}\theta\right)^2 \rho a\,\delta\theta = 4a^3\rho\sin^2\tfrac{1}{2}\theta\,\delta\theta.$$

Hence, M.I. of the arc ABC about BL

$$= \int_{-\alpha}^{\alpha} 4a^3\rho\sin^2\frac{1}{2}\theta\, d\theta = 2Ma^2(\alpha - \sin\alpha)/\alpha.$$

Ex. 4. *Show that the M.I. of a semi-circular lamina about a tangent parallel to the bounding diameter is* $Ma^2\left(\dfrac{5}{4} - \dfrac{8}{3\pi}\right)$ *where a is the radius and M is the mass of the lamina.*

(Meerut 1990, 95; Agra 91; Garhwal 95; Kanpur 90)



Sol. Let LD be the tangent parallel to the bounding diameter BC of a semi-circular lamina of radius a and mass M.

Consider an elementary area $r\,\delta\theta\,\delta r$ at the point $P(r,\theta)$ of the lamina; then its mass $\delta m = \rho r\,\delta\theta\,\delta r$, where

$$\rho = \text{mass per unit area} = \frac{M}{\frac{1}{2}\pi a^2} = \frac{2M}{\pi a^2}.$$

Distance of the point P from $LD = PT$

$$= KA = OA - OK = a - r\cos\theta.$$

$\therefore$ M.I. of the elementary mass δm at P about LD

$$= PT^2 . \delta m = (a - r\cos\theta)^2 . \rho r \,\delta\theta\, \delta r.$$

$\therefore$ M.I. of the lamina about LD

$$= \int_{\theta=-\pi/2}^{\pi/2} \int_{r=0}^{a} (a - r\cos\theta)^2 \rho r \, d\theta\, dr$$

$$= \rho \int_{-\pi/2}^{\pi/2} \int_{0}^{a} (a^2 r - 2ar^2\cos\theta + r^3\cos^2\theta)\, d\theta\, dr$$

$$= \rho \int_{-\pi/2}^{\pi/2} \left[\frac{a^2}{2} r^2 - \frac{2}{3} ar^3\cos\theta + \frac{1}{4} r^4\cos^2\theta\right]_{r=0}^{a} d\theta$$

$$= \rho \int_{-\pi/2}^{\pi/2} \left[\frac{1}{2} a^4 - \frac{2}{3} a^4\cos\theta + \frac{1}{4} a^4\cos^2\theta\right] d\theta$$

$$= 2\rho a^4 \int_{0}^{\pi/2} \left[\frac{1}{2} - \frac{2}{3}\cos\theta + \frac{1}{4}\cos^2\theta\right] d\theta$$

$$= 2\rho a^4 \left[\left\{\frac{1}{2}\theta - \frac{2}{3}\sin\theta\right\}_0^{\pi/2} + \frac{1}{4} \cdot \frac{\Gamma(3/2)\,\Gamma(1/2)}{2\Gamma(2)}\right]$$

$$= 2 \cdot \frac{2M}{\pi a^2} a^4 \left[\frac{1}{4}\pi - \frac{2}{3} + \frac{1}{4} \cdot \frac{\pi}{4}\right] = Ma^2 . \left(\frac{5}{4} - \frac{8}{3\pi}\right).$$

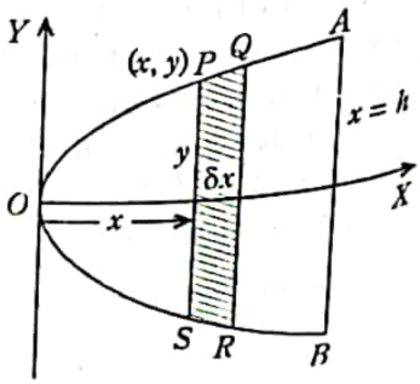
Ex. 5. *Show that the M.I. of a parabolic area (of latus rectum $4a$) cut off by an ordinate at a distance h from the vertex is $\frac{3}{7} Mh^2$ about the tangent at the vertex and $\frac{4}{5} Mah$ about the axis.* **(Meerut 1986, 89, 90)**

Sol. Let the equation of the parabola of latus rectum $4a$ be

$$y^2 = 4ax.$$

Let AOB be the portion of the parabola cut off by an ordinate at a distance h from the vertex.

Consider an elementary strip $PQRS$ parallel to OY at a distance x from the vertex O and of width δx.



The mass δm of this elementary strip $= \rho . 2y\,\delta x$, where ρ is the mass per unit area.

$\therefore$ M = mass of the portion AOB of the parabola

$$= \int_0^h \rho\, 2y\, dx$$

$$=2\rho\int_0^h 2\sqrt{(ax)}\,dx=4\rho\sqrt{a}.\tfrac{2}{3}h^{3/2}$$
$$=\tfrac{8}{3}\rho a^{1/2}h^{3/2}.$$

Now the distance of every point of the strip $PQRS$ from OY, the tangent at the vertex, is x.

$\therefore$ M.I. of the strip about $OY=x^2\,\delta m=\rho.2x^2y\,\delta x.$

$\therefore$ M.I. of the whole area AOB about $OY=\int_0^h 2\rho\,x^2y\,dx$

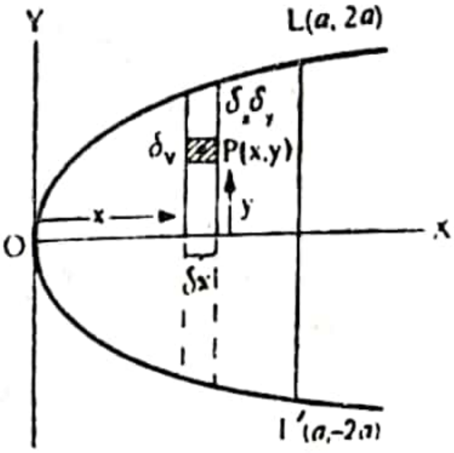
$$=2\rho\int_0^h x^2\,2\sqrt{(ax)}\,dx=4\rho a^{1/2}\int_0^h x^{5/2}\,dx=\frac{8}{7}\rho a^{1/2}\,h^{7/2}$$
$$=\tfrac{3}{7}\left(\tfrac{8}{3}\rho a^{1/2}\,h^{3/2}\right)h^2=\tfrac{3}{7}Mh^2.$$

Again M.I. of the strip $PQRS$ about $OX=\frac{1}{3}y^2\,\delta m$
$$=\tfrac{1}{3}y^2.\rho\,2y\,\delta x=\tfrac{2}{3}\rho\,y^3\,\delta x.$$

$\therefore$ M.I. of the whole area AOB about $OX=\int_0^h \frac{2}{3}\rho y^3\,dx$

$$=\tfrac{2}{3}\rho\int_0^h (4ax)^{3/2}\,dx=\tfrac{16}{3}a^{3/2}\,\rho.\tfrac{2}{5}h^{5/2}=\tfrac{4}{5}\left(\tfrac{8}{3}\rho a^{1/2}h^{3/2}\right)ah$$
$$=\tfrac{4}{5}Mah.$$

Ex. 6. *Show that the M.I. about the x-axis of the portion of the parabola $y^2=4ax$ bounded by the axis and the latus rectum, supposing the surface density at each point vary as the cube of the abscissa, is $\frac{12}{11}Ma^2$, where M is the mass of the portion.*



Sol. Equation of the parabola is $y^2=4ax$. ...(1)

Let $\delta x\,\delta y$ be an elementary area of the parabola at the point $P\,(x, y)$.

If ρ is the density at $P(x, y)$, then $\rho=kx^3$.

$\therefore$ δm=mass of the elementary area at $P=\rho\,\delta x\,\delta y=kx^3\,\delta x\,\delta y.$

$\therefore$ M=mass of the given area of the parabola bounded by the latus rectum LL'

$$=\int_{x=0}^{a}\int_{y=-\sqrt{(4ax)}}^{\sqrt{(4ax)}} kx^3\,dxdy=k\int_0^a x^3\Big[\,y\,\Big]_{y=-\sqrt{(4ax)}}^{\sqrt{(4ax)}}dx$$
$$=2k\int_0^a x^3\sqrt{(4ax)}\,dx=4k\sqrt{a}\int_0^a x^{7/2}\,dx=\frac{8}{9}ka^5.$$

Now the M.I. of the elementary mass δm at P about x-axis $=y^2\,\delta m=kx^3y^2\,\delta x\delta y.$

$\therefore$ M.I. of the given area about x-axis

$$=\int_{x=0}^{a}\int_{y=-\sqrt{(4ax)}}^{\sqrt{(4ax)}} kx^3y^2\,dxdy=k\int_0^a x^3\left[\frac{1}{3}y^3\right]_{y=-\sqrt{(4ax)}}^{\sqrt{(4ax)}} dx$$

$$=k.\frac{2}{3}\int_0^a x^3.(4ax)^{3/2}\,dx=\frac{16}{3}ka^{3/2}\int_0^a x^{9/2}\,dx$$

$$=\frac{32}{33}ka^7=\frac{12}{11}.\left(\frac{8}{9}ka^5\right).a^2=\frac{12}{11}Ma^2.$$

Ex. 7. *Find the moment of inertia of a quadrant of the elliptic disc $x^2/a^2+y^2/b^2=1$ of mass M about the line through its centre perpendicular to its plane, the density at any point is proportional to xy.* [Meerut 90S]

Sol. (For figure refer § 3·8)

Let $\delta x\delta y$ be an elementary area at the point (x, y) in the first quadrant of an elliptic disc of mass M and axes of lengths $2a$ and $2b$.

Then the density ρ at this point $(x, y)=kxy$, where k is a constant.

$\therefore$ δm=mass of the elementary area $\delta x\ \delta y$ at the point (x, y)
$=\rho\ \delta x\delta y=kxy\ \delta x\delta y.$

$\therefore$ M=mass of the quadrant of the disc

$$=\int_{x=0}^{a}\int_{y=0}^{b\sqrt{(1-x^2/a^2)}} kxy\,dx\,dy$$

$$=\tfrac{1}{2}kb^2\int_0^a x\,(1-x^2/a^2)\,dx=(1/8)\,ka^2b^2.$$

Now M.I. of the elementary area $\delta x\ \delta y$ at the point (x, y) about the line ON through the centre and perpendicular to the plane of the disc

$=(x^2+y^2)\ \delta m$

[$\because$ length of the perp. from (x, y) to $ON=\sqrt{(x^2+y^2)}$]

$=kxy\ (x^2+y^2)\ \delta x\delta y.$

$\therefore$ M.I. of the quadrant of the disc about ON

$$=\int_{x=0}^{a}\int_{y=0}^{b\sqrt{(1-x^2/a^2)}} kxy\,(x^2+y^2)\,dxdy$$

$$=k\int_0^a\left[\tfrac{1}{2}x^3.b^2\,(1-x^2/a^2)+\tfrac{1}{4}x.b^4\,(1-x^2/a^2)^2\right]dx$$

$$=\frac{1}{4}kb^2\int_0^a\left[b^2x+2\,(1-b^2/a^2)\,x^3-(2/a^2-b^2/a^4)\,x^5\right]dx$$

$$=\frac{1}{4}kb^2\left[\tfrac{1}{2}b^2x^2+2\,(1-b^2/a^2)\,\tfrac{1}{4}x^4-(2/a^2-b^2/a^4)\,\tfrac{1}{6}x^6\right]_0^a$$

$$=\frac{1}{24}ka^2b^2(a^2+b^2)=\frac{1}{3}M(a^2+b^2), \text{ from (1).}$$

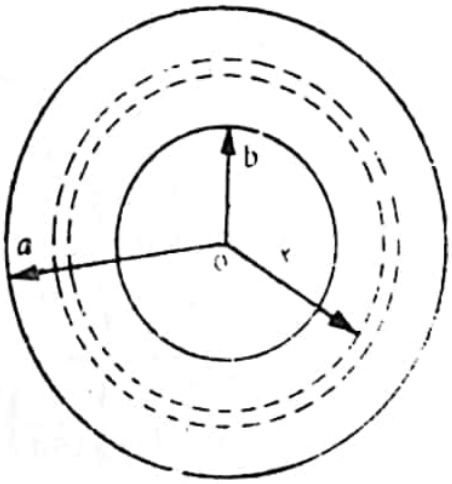
Ex. 8. *Find the M.I. of a hollow sphere about a diameter, its external and internal radii being a and b respectively.*

[Meerut 84, 86, 86S, 88S]

Sol. If M is the mass of the given hollow sphere, then the mass per unit volume

$$\rho=\frac{M}{\frac{4}{3}\pi a^3-\frac{4}{3}\pi b^3}=\frac{3M}{4\pi(a^3-b^3)}.$$

Consider a concentric spherical shell of radius x and thickness δx, where $a < x < b$.



Mass of this elementary shell

$$=\delta m=\rho . 4\pi x^2 \delta x.$$

M.I. of this shell about a diameter $=\frac{2}{3}x^2\,\delta m$

$$=\frac{2}{3}x^2 . \rho 4\pi x^2 \delta x=\frac{8}{3}\rho\pi x^4 \delta x.$$

$\therefore$ M.I. of the given hollow sphere about a diameter

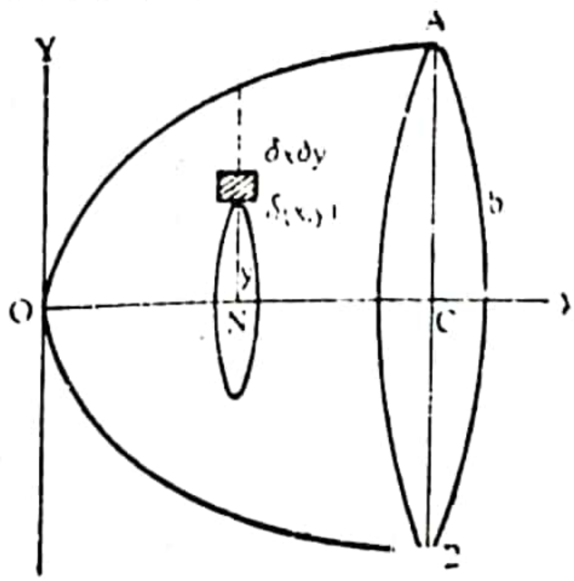
$$=\int_b^a \frac{8}{3}\rho\pi x^4 dx$$

$$=\frac{8}{3}\rho\pi . \frac{1}{5}(a^5-b^5)$$

$$=\frac{8}{15}\pi . \frac{3M}{4\pi(a^3-b^3)} . (a^5-b^5)=\frac{2M}{5}\frac{a^5-b^5}{a^3-b^3}.$$

Ex. 9. *Show that the M.I. of a paraboloid of revolution about its axis is $\frac{1}{3}M\times$ the square of the radius of its base.*

Sol. Let the paraboloid of revolution be generated by the revolution of the area bounded by the parabola $y^2=4ax$ and the x-axis about the axis OX of the parabola.



Let b be the radius of the base of this paraboloid of revolution.

Then for the point A,

$$y=AC=b.$$

$\therefore$ from $y^2=4ax$, for the point A,

$$x=b^2/4a=OC.$$

Consider an elementary area $\delta x\, \delta y$ at the point $P(x, y)$ of the area $OACO$. By the revolution of this elementary area $\delta x\, \delta y$ about OX, a circular ring of radius y and cross-section of area $\delta x\, \delta y$ is formed.

Mass δm of this elementary ring $=\rho 2\pi y\, \delta x\, \delta y$,

where ρ is the mass per unit volume.

$\therefore$ M=mass of the paraboloid of revolution under consideration

$$=\int_{x=0}^{b^2/4a}\int_{y=0}^{\sqrt{(4ax)}} \rho\, 2\pi y\, dx\, dy=2\pi\rho\int_0^{b^2/4a}\left[\tfrac{1}{2}y^2\right]_{y=0}^{\sqrt{(4ax)}} dx$$

$$=\pi\rho\int_0^{b^2/4a} 4ax\, dx=4\pi\rho a.\left[\tfrac{1}{2}x^2\right]_0^{b^2/4a}=\frac{\pi\rho b^4}{8a}. \qquad \ldots(1)$$

Now M.I. of the elementary ring of mass δm about OX (a line through its centre and perpendicular to its plane)

$$=y^2\delta m=y^2\, \rho\, 2\pi y\, \delta x\, \delta y=2\pi\rho\, y^3\, \delta x\, \delta y.$$

$\therefore$ M.I. of the paraboloid of revolution about OX

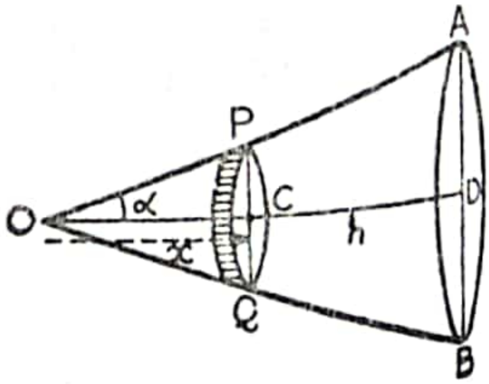
$$=\int_{x=0}^{b^2/4a}\int_{y=0}^{\sqrt{(4ax)}} 2\pi\rho\, y^3\, dx\, dy=\frac{2\pi\rho}{4}\int_0^{b^2/4a} 16a^2x^2\, dx$$

$$=8\pi\rho a^2.\frac{1}{3}\left(\frac{b^2}{4a}\right)^3=\frac{\pi\rho b^6}{24a}=\frac{1}{3}\left(\frac{\pi\rho b^4}{8a}\right)b^2$$

$$=\tfrac{1}{3}M.(\text{square of the radius of the base}).$$

Ex. 10. *Find the M.I. of a right solid cone of mass M, height h and radius of whose base is a, about its axis.*

Sol. Let O be the vertex of the right solid cone of mass M, height h and radius of whose base is a. If α is the semi-vertical angle and ρ the density of the cone, then



$$M=\tfrac{1}{3}\pi\rho h^3 \tan^2\alpha. \qquad \ldots(1)$$

Consider an elementary disc PQ of thickness δx, parallel to the base AB and at a distance x from the vertex O.

Mass δm of this disc $=\rho\pi x^2 \tan^2\alpha\, \delta x$.

M.I. of this elementary disc about the axis OD of the cone

$$=\tfrac{1}{2}\,\delta m CP^2=\tfrac{1}{2}(\rho\pi x^2\tan^2\alpha\delta x)\, x^2\tan^2\alpha=\tfrac{1}{2}\rho\pi x^4\tan^4\alpha\delta x.$$

$\therefore$ M.I. of the whole cone about the axis OD

$$=\int_0^h \tfrac{1}{2}\rho\pi\, x^4\tan^4\alpha\, dx=\frac{\rho\pi}{10}h^5\tan^4\alpha$$

$=\dfrac{3M}{\pi h^3 \tan^2\alpha}\cdot\dfrac{\pi}{10}h^5\tan^4\alpha$, substituting for ρ from (1)

$=\dfrac{3}{10}Mh^2\tan^2\alpha=\dfrac{3}{10}Ma^2$, since $\tan\alpha=a/h$.

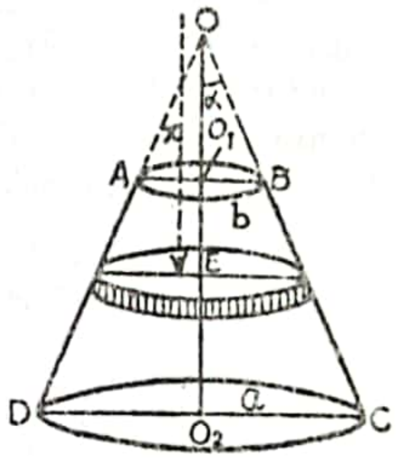
Ex. 11. *Find the M.I. of a truncated cone about its axis, the radii of its ends being a and b.*

[Kanpur 81, 83; Meerut 83, 89, 89S]

Sol. Let $ABCD$ be the truncated cone with the vertex at O and semi-vertical angle α. Also let $O_1B=b$ and $O_2C=a$.

Consider an elementary disc perpendicular to the axis of the cone at a distance x from the vertex O and of thickness δx.

Its mass$=\delta m=\rho\pi\ (x\tan\alpha)^2\ \delta x$.



If M is the total mass of the truncated cone, then

$$M=\int_{x=b\cot\alpha}^{a\cot\alpha}\rho\pi x^2\tan^2\alpha\,dx$$

[$\because$ $OO_1=b\cot\alpha$, $OO_2=a\cot\alpha$]

$=\frac{1}{3}\rho\pi\tan^2\alpha\,(a^3-b^3)\cot^3\alpha$

$=\frac{1}{3}\rho\pi\cot\alpha\,(a^3-b^3)$.

$$\therefore\quad \rho=\frac{3M\tan\alpha}{\pi(a^3-b^3)}. \qquad \ldots(1)$$

Now M.I. of the elementary disc about O_1O_2, a line through the centre of the disc and perpendicular to its plane

$=\frac{1}{2}(x\tan\alpha)^2\,\delta m=\frac{1}{2}x^2\tan^2\alpha.\rho\pi x^2\tan^2\alpha\delta x$

$=\frac{1}{2}\rho\pi x^4\tan^4\alpha\delta x$.

$\therefore$ M. I. of the truncated cone about its axis O_1O_2

$$=\int_{x=b\cot\alpha}^{a\cot\alpha}\tfrac{1}{2}\rho\pi x^4\tan^4\alpha\,dx=\frac{1}{10}\rho\pi\,(a^5-b^5)\cot^5\alpha.\tan^4\alpha$$

$$=\frac{1}{10}\cdot\frac{3M\tan\alpha}{\pi(a^3-b^3)}\pi(a^5-b^5)\cot\alpha=\frac{3M}{10}\cdot\frac{a^5-b^5}{a^3-b^3},\ \text{from (1).}$$

Ex. 12. *Find the M.I. of a circular disc about an axis through its centre and perpendicular to its plane, assuming that the density at any point varies as the square of its distance from the centre.*

Sol. Refer the figure of § 3·7.

Consider an elementary area $r\,\delta\theta\,\delta r$ at the point $P(r,\theta)$ of the circular disc of radius a referred to the centre O as the pole and OX as the initial line.

If ρ is density of the disc at P, then $\rho=k.OP^2=kr^2$.

$\therefore$ the mass δm of the elementary area $r\delta\theta\delta r$ at $P = \rho r\delta\theta\delta r = kr^3\delta\theta\delta r$.

If M is the mass of the disc, then

$$M = 2\int_{\theta=0}^{\pi} \int_{r=0}^{a} kr^3\, d\theta dr = \frac{1}{2}k\pi a^4$$

Now M.I. of the elementary mass δm at P about the line ON which passes through the centre O and is perpendicular to the plane of the disc

$$= OP^2 \cdot \delta m = r^2\delta m = kr^5\delta\theta\delta r$$

$\therefore$ M.I. of the disc about $ON = \int_{\theta=-\pi}^{\pi} \int_{r=0}^{a} kr^5\, d\theta dr$

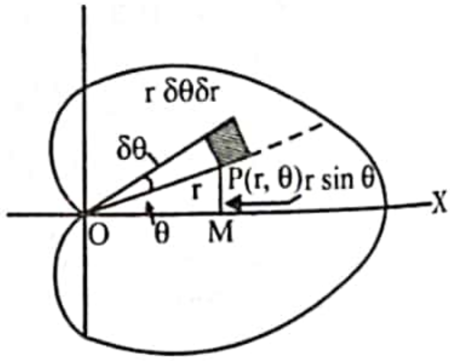
$$= \frac{1}{3}k\pi a^6 = \frac{1}{3}\cdot\frac{2M}{\pi a^4}\cdot\pi ab = \frac{2}{3}Ma^2$$

Ex. 13. *Find the M.I. of the cardioid $r = a(1+\cos\theta)$ of density ρ, about the initial line.* **[Meerut 90]**

Sol. Consider an elementary area $r\delta\theta\delta r$ at the point $P(r,\theta)$ of the cardioid $r = a(1+\cos\theta)$

Its mass $\delta m = \rho r\delta\theta\delta r$

If M is the mass of the cardioid, then



$$M = 2\int_{\theta=0}^{\pi} \int_{r=0}^{a(1+\cos\theta)} \rho r d\theta\, dr$$

$$= \rho a^2 \int_0^{\pi} (1+\cos\theta)^2 d\theta = \rho a^2 \int_0^{\pi} (1+2\cos\theta+\cos^2\theta)d\theta$$

$$= \rho a^2 \int_0^{\pi} \left[1+2\cos\theta+\frac{1}{2}(1+\cos 2\theta)\right]d\theta$$

$$= \rho a^2 \left[\frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi} \quad \text{or} \quad M = \frac{3}{2}\pi\rho a^2$$

Now M.I. of the elementary mass δm at P about OX

$$= PM^2 \cdot \delta m = (r\sin\theta)^2 \rho r\delta\theta\delta r = \rho r^3 \sin^2\theta\delta\theta\delta r.$$

$\therefore$ M.I. of the cardioid about OX

$$= \int_{\theta=-\pi}^{\pi} \int_{r=0}^{a(1+\cos\theta)} \rho r^3 \sin^2\theta\, d\theta dr = \frac{1}{4}\rho a^4 \int_{-\pi}^{\pi} (1+\cos\theta)^4 \sin^2\theta d\theta$$

$$= \frac{1}{4}\rho a^4 \int_{-\pi}^{\pi} \left(2\cos^2\frac{1}{2}\theta\right)^4 \cdot \left(2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta\right)^2 d\theta$$

$$= 32\rho a^4 \int_{-\pi/2}^{\pi/2} \sin^2 t\cos^{10} t\, dt, \text{putting } \frac{1}{2}\theta = t$$

$$=64\rho a^4 \int_0^{\pi/2} \sin^2 t \cos^{10} t\, dt,$$ since the integrand is an even function of t

$$=64\rho a^4 . \frac{\Gamma(3/2)\,\Gamma(11/2)}{2\Gamma(7)}=\frac{21}{32}\rho\pi a^4=\frac{7}{16}Ma^2.$$

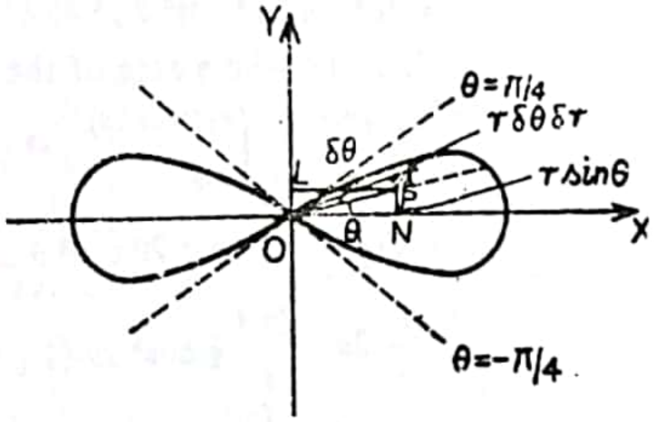
Ex. 14. *Find the M.I. of the area of the lemniscate $r^2=a^2\cos 2\theta$*

(*i*) *about its axis,* **[Rohilkhand 83]**

(*ii*) *about a line through the origin in its plane and perpendicular to its axis.* **[Meerut 90S]**

(*iii*) *about a line through the origin and perpendicular to its plane.*

Sol. One loop of the lemniscate is formed between the lines $\theta=-\pi/4$ and $\theta=\pi/4$. The curve is as shown in the figure. It consists of two symmetrical loops.



Consider an elementary area $r\delta\theta\delta r$ at any point $P(r, \theta)$ within the area of the curve; then its mass $\delta m=\rho r\, \delta\theta\, \delta r$.

$\therefore$ The mass of the whole area of the curve is given by

$$M=2\int_{\theta=-\pi/4}^{\pi/4}\int_{r=0}^{a\sqrt{(\cos 2\theta)}} \rho r\, d\theta\, dr=\rho\int_{-\pi/4}^{\pi/4} a^2 \cos 2\theta\, d\theta$$

$$=\rho a^2\left[\tfrac{1}{2}\sin 2\theta\right]_{-\pi/4}^{\pi/4}=\rho a^2. \qquad ...(1)$$

(i) M.I. of the elementary mass δm at P about the axis OX

$=PN^2.\delta m=(r\sin\theta)^2\, \rho r\, \delta\theta\, \delta r=\rho r^3 \sin^2\theta\, \delta\theta\, \delta r.$

$\therefore$ M.I. of the whole area of the lemniscate about OX

$$=2\int_{\theta=-\pi/4}^{\pi/4}\int_{r=0}^{a\sqrt{(\cos 2\theta)}} \rho r^3 \sin^2\theta\, d\theta\, dr$$

$$=2\rho\int_{-\pi/4}^{\pi/4} \tfrac{1}{4}a^4 \cos^2 2\theta \sin^2\theta\, d\theta$$

$$=2.\frac{2\rho a^4}{4}\int_0^{\pi/4} \tfrac{1}{2}\cos^2 2\theta\,(1-\cos 2\theta)\, d\theta$$

$$=\tfrac{1}{4}\rho a^4\int_0^{\pi/2} \cos^2 t\,(1-\cos t)\, dt,$$

putting $2\theta=t$, so that $d\theta=$

$$=\tfrac{1}{4}\rho a^4\left[\int_0^{\pi/2}\cos^2 t\,dt-\int_0^{\pi/2}\cos^3 t\,dt\right]$$

$$=\tfrac{1}{4}\rho a^4\left[\frac{\Gamma(\frac{3}{2})\,\Gamma(\frac{1}{2})}{2\Gamma(2)}-\frac{\Gamma(2)\,\Gamma(\frac{1}{2})}{2\Gamma(\frac{5}{2})}\right]=\frac{1}{4}Ma^2\left(\frac{\pi}{4}-\frac{2}{3}\right),\text{ from (1)}$$

$$=\frac{Ma^2}{16}\left(\pi-\frac{8}{3}\right).$$

(ii) Distance of the point $P(r, \theta)$ from OY a line through the origin in the plane of lemniscate and perpendicular to its axis $=PL=r\cos\theta$.

$\therefore$ M.I. of elementary mass δm at P about OY
$=PL^2.\delta m+r^2\cos^2\theta\,\rho^2\,\delta\theta\,\delta r=\rho r^3\cos^2\theta\,\delta\theta\,\delta r.$

$\therefore$ M.I. of the whole area of the lemniscate about OY

$$=2\int_{\theta=-\pi/4}^{\pi/4}\int_{r=0}^{a\sqrt{(\cos 2\theta)}}\rho r^3\cos^2\theta\,d\theta\,dr$$

$$=\frac{2\rho}{4}\int_{-\pi/4}^{\pi/4}a^4\cos^2 2\theta\cos^2\theta\,d\theta$$

$$=\frac{\rho}{2}\,2a^4\int_0^{\pi/4}\tfrac{1}{2}\cos^2 2\theta\,(1+\cos 2\theta)\,d\theta$$

$$=\tfrac{1}{2}Ma^2.\tfrac{1}{2}\int_0^{\pi/2}\cos^2 t\,(1+\cos t)\,dt,\qquad\text{putting } 2\theta=t$$

$$=\tfrac{1}{4}Ma^2.\,(\pi/4+2/3),$$

$$=\tfrac{1}{48}Ma^2\,(3\pi+8).$$

(iii) Let OT be the line through the origin and perpendicular to the plane of the lemniscate.

Distance of δm at P from $OT=OP=r$.

$\therefore$ M.I. of the elementary mass δm at P about OT
$=OP^2.\delta m=r^2.\rho r\,\delta\theta\,\delta r=\rho r^3\,\delta\theta\,\delta r.$

$\therefore$ M.I. of the whole area of the lemniscate about OT

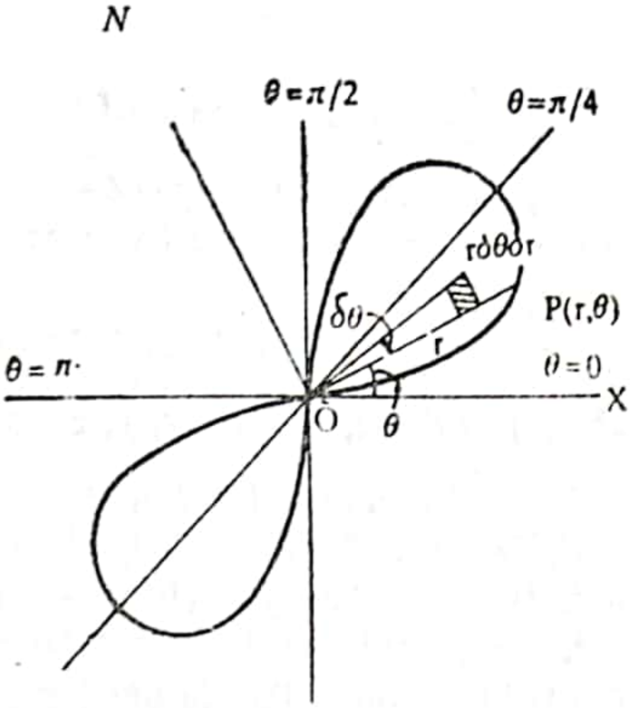
$$=2\int_{\theta=-\pi/4}^{\pi/4}\int_{r=0}^{a\sqrt{(\cos 2\theta)}}\rho r^3\,d\theta\,dr=\frac{2\rho}{4}\int_{-\pi/4}^{\pi/4}a^4\cos^2 2\theta\,d\theta$$

$$=\tfrac{1}{2}\rho a^4.2\int_0^{\pi/4}\tfrac{1}{2}\,(1+\cos 4\theta)\,d\theta=\tfrac{1}{2}Ma^2\left[\theta+\tfrac{1}{4}\sin 4\theta\right]_0^{\pi/4}=\tfrac{1}{8}\pi Ma^2.$$

Ex. 15. *Find the M.I. of one loop of the lemniscate $r^2=a^2\sin 2\theta$, about an axis perpendicular to its plane at the pole.*

Sol. The given curve is $r^2=a^2\sin 2\theta$. Obviously one loop of the curve lies between $0\leqslant\theta\leqslant\pi/2$.

Consider an elementary area $r\delta\theta\,\delta r$ of the loop at the point $P(r, \theta)$.



Its mass $\delta m=\rho r\delta\theta\ \delta r$.

$\therefore\quad M=$mass of one loop$=\int_{\theta=0}^{\pi/2}\int_{r=0}^{a\sqrt{(\sin 2\theta)}}\rho r\ d\theta\ dr$

$$=\tfrac{1}{2}\rho a^2\int_0^{\pi/2}\sin 2\theta\ d\theta=\tfrac{1}{2}\rho a^2.$$

Now M.I. of the elementary mass δm at P about ON (an axis perpendicular to the plane of the lemniscate at the pole O)

$$=OP^2\ \delta m=\rho r^3\ \delta\theta\ \delta r.$$

$\therefore\quad$ M.I. of one loop about $ON=\int_{\theta=0}^{\pi/2}\int_{r=0}^{a\sqrt{(\sin 2\theta)}}\rho r^3\ d\theta\ dr$

$$=\tfrac{1}{4}\rho a^4\int_0^{\pi/2}\sin^2 2\theta\ d\theta$$

$$=\tfrac{1}{8}\rho a^4\int_0^{\pi/2}(1-\cos 4\theta)\ d\theta=\tfrac{1}{16}\rho a^4\pi=\tfrac{1}{8}\pi Ma^2.$$

Ex. 16. *ABC is a uniform equilateral triangular plate of mass M and side 2a. Show that its M.I. about its side is $\tfrac{1}{2}Ma^2$.*

Sol. Proceed as in § 3·9. Here $h=2a\sin 60^\circ=a\sqrt{3}$.
M.I.$=\tfrac{1}{6}M(a\sqrt{3})^2=\tfrac{1}{2}Ma^2$.

Ex. 17. *Find the moment of inertia of the solid ellipsoid*

$$x^2/a^2+y^2/b^2+z^2/c^2=1$$

about the axis of z, given that M is the mass of the ellipsoid. **[Meerut 85]**

Sol. Refer fig. of § 3·13 on page 12.

Consider an elementary volume $\delta x\ \delta y\ \delta z$ at the point $P(x, y, z)$ of the ellipsoid in the positive octant.

Mass of this element $=\rho\ \delta x\ \delta y\ \delta z$,

where $\rho=\dfrac{M}{\frac{4}{3}\pi abc}=\dfrac{3M}{4\pi\ abc}$, M is the mass of the ellipsoid.

Distance of the point $P(x, y, z)$ from $OZ=\sqrt{(x^2+y^2)}$.

$\therefore$ M.I. of the elementary volume $\delta x\ \delta y\ \delta z$ about OZ
$=(x^2+y^2)\ \rho\ \delta x\ \delta y\ \delta z$.

$\therefore$ M.I. of the whole ellipsoid about OZ

$=8\iiint \rho\ (x^2+y^2)\ dx\ dy\ dz$, where x, y, z take all positive values subject to the condition $x^2/a^2+y^2/b^2+z^2/c^2 \leqslant 1$.

Put $x^2/a^2=u$, $y^2/b^2=v$ and $z^2/c^2=w$.

Then $x=au^{1/2}$, $dx=\frac{1}{2}au^{-1/2}\ du$; $y=bv^{1/2}$, $dy=\frac{1}{2}bv^{-1/2}\ dv$;

and $z=cw^{1/2}$, $dz=\frac{1}{2}cw^{-1/2}\ dw$.

$\therefore$ The required moment of inertia of the ellipsoid about OZ

$=8\rho\iiint \dfrac{abc}{8}\ (a^2u+b^2v)\ u^{-1/2}\ v^{-1/2}\ w^{-1/2}\ du\ dv\ dw$, where u, v, w take all positive values subject to the condition $u+v+w \leqslant 1$

$=abc\ \rho\iiint [a^2u^{(3/2)-1}\ v^{(1/2)-1}\ w^{(1/2)-1}+b^2u^{(1/2)-1}\ v^{(3/2)-1}\ w^{(1/2)-1}]\ du\ dv\ dw$

$=abc\ \rho\left[a^2.\dfrac{\Gamma(\frac{3}{2})\ \Gamma(\frac{1}{2})\ \Gamma(\frac{1}{2})}{\Gamma(1+\frac{3}{2}+\frac{1}{2}+\frac{1}{2})}+b^2.\dfrac{\Gamma(\frac{1}{2})\ \Gamma(\frac{3}{2})\ \Gamma(\frac{1}{2})}{\Gamma(1+\frac{1}{2}+\frac{3}{2}+\frac{1}{2})}\right]$, using Dirichlet's theorem

$=abc\ \rho\left[\dfrac{a^2.\frac{1}{2}\sqrt{\pi}.\sqrt{\pi}.\sqrt{\pi}}{\Gamma(\frac{7}{2})}+\dfrac{b^2.\frac{1}{2}\sqrt{\pi}.\sqrt{\pi}\ \sqrt{\pi}}{\Gamma(\frac{7}{2})}\right]$

$=\dfrac{3M}{4\pi}\left[\dfrac{\pi a^2.\frac{1}{2}\sqrt{\pi}}{(\frac{5}{2}).(\frac{3}{2}).(\frac{1}{2}).\sqrt{\pi}}+\dfrac{\pi b^2.\frac{1}{2}\sqrt{\pi}}{(\frac{5}{2}).(\frac{3}{2}).(\frac{1}{2}).\sqrt{\pi}}\right]$ $\left[\because\ \rho=\dfrac{3M}{4\pi abc}\right]$

$=\dfrac{3M}{4\pi}\cdot\dfrac{4\pi}{15}\left[a^2+b^2\right]=\dfrac{M}{5}\ (a^2+b^2)$.

Ex. 18. *Find the M. I. of the ellipsoid* $x^2/9+y^2/4+z^2/16=1$, *about the axis of x, M being its mass.* **[Meerut 84]**

Sol. Proceed as in § 3·13. Here $a=3$, $b=2$ and $c=4$.

$\therefore$ M.I.$=\frac{1}{5}M\ (2^2+4^2)=4M$.

§ 3·15. Theorem of parallel axis.

(i) *M. I. of a body about a line (say AB)*
$=$M. I of the body about the line through the centre of gravity 'G' of the body and parallel to the given line AB
$+$M. I. of the total mass M of the body at G, about the line AB.

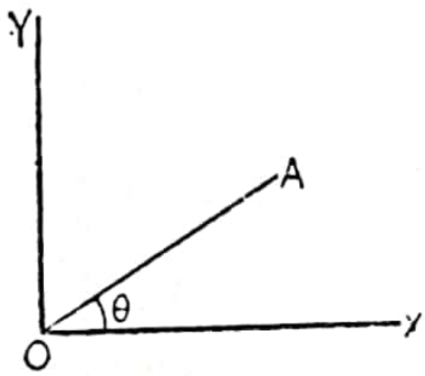
(ii) *P. I. of a body about the perpendicular lines say OX and OY*

=P. I. of the body about the lines GX′ and GY′ through the centre of gravity 'G' of the body and parallel to the given lines OX and OY

+P. I. of the total mass M of the body at G, about the lines OX and OY.

§ 3·16. Moment of inertia of a plane lamina about a line.

Let OX and OY be two mutually perpendicular lines and OA a line in the plane of the lamina passing through the meeting point O of OX and OY. If the line OA makes an angle θ with OX, A and B the moments of inertia of the lamina about OX and OY respectively and F its product of inertia about OX and OY, then

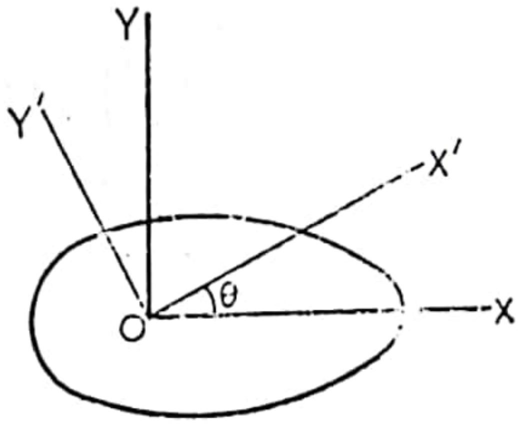


M. I. of the lamina about OA

$$= A \cos^2 \theta + B \sin^2 \theta - F \sin 2\theta.$$

§ 3·17. Principal Axes. *The lines OA, OB, OC are the principal axes of the system at the point O if the products of inertia of the system with reference to these axes taken two at a time vanish.*

Let OX and OY be two perpendicular axes in the plane of the lamina and OZ an axis perpendicular to its plane. Then OZ will be a principal axis at O and the other two principal axes at O will be the mutually perpendicular lines OX', OY' through O in the plane of the lamina such that the angle of inclination θ of one of these principal axes to OX is given by



$$\tan 2\theta = 2F/(B - A)$$

where A = M. I. of the lamina about OX,

B=M. I. of the lamina about OY

and F=P. I. of the lamina about OX and OY.

Solved Examples

Ex. 1. *Find the M. I. of a uniform rectangular lamina of sides 2a and 2b about its sides.*

Sol. (Refer figure of § 3·5 on page 5).

By the theorem of parallel axes, M. I. of the rectangular lamina about the side AB

=M. I. of the lamina about the parallel line OX through its centre of gravity 'O'+M. I. of the total mass M at O about AB

$$=\frac{1}{3}Mb^2+Mb^2=\frac{4}{3}Mb^2.$$

Similarly M. I. about the side $AD=\frac{1}{3}Ma^2+Ma^2=\frac{4}{3}Ma^2$.

Aliter. Consider an elementary area $\delta x\,\delta y$ at the point $P(x, y)$ of the lamina.

Its mass $\delta m=\rho\,\delta x\,\delta y$,

where $\rho=M/(2a.2b)=M/(4ab)$.

Now M. I. of this elementary mass δm about AB

$$=(y+b)^2.\delta m=\rho\,(y+b)^2\,\delta x\,\delta y.$$

$\therefore$ M. I. of the lamina about AB

$$=\int_{x=-a}^{a}\int_{y=-b}^{b}\rho\,(y+b)^2\,dx\,dy$$

$$=\rho\int_{-a}^{a}\frac{1}{3}\Big[(y+b)^3\Big]_{y=-b}^{b}dx=\frac{8}{3}\rho b^3\int_{-a}^{a}dx$$

$$=\frac{8}{3}\cdot\frac{M}{4ab}\cdot b^3.2a=\frac{4}{3}Mb^2.$$

Similarly M. I. of the lamina about AD

$$=\int_{x=-a}^{a}\int_{y=-b}^{b}\rho\,(x+a)^2\,dx\,dy=\frac{4}{3}Ma^2.$$

Ex. 2. *Find the M. I. of a uniform circular disc of radius 2 cm. and mass 10 gms. about a tangent.* **[Meerut 84]**

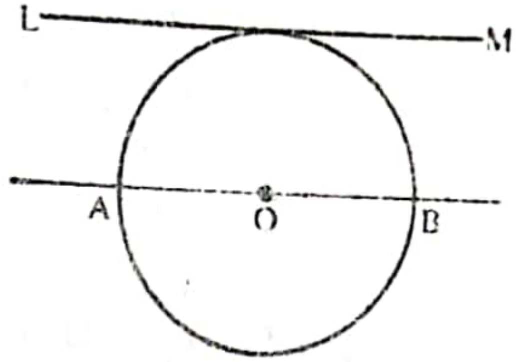
Sol. M. I. of the circular disc of mass M about the tangent

LM=M. I. of the disc about the parallel diameter AB
+M.I.of the mass M placed at O about LM

$=\frac{1}{4}Ma^2+Ma^2=\frac{5}{4}Ma^2$

$=\frac{5}{4}.(10).(2)^2=50.$

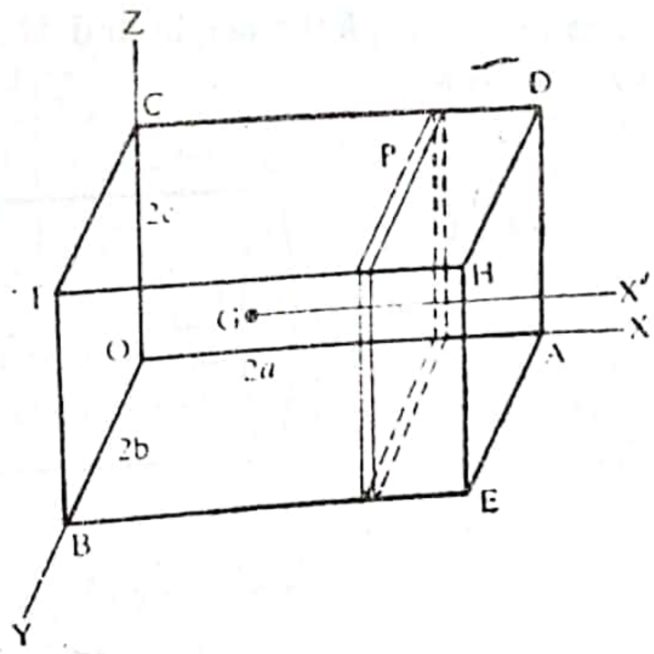
[$\because$ here $M=10$ gm, and $a=2$ cm.]



Ex. 3. *Find the M. I. of a rectangular parallelopiped about an edge.*

Sol. Let $2a, 2b, 2c$ be the lengths of the edges of a rectangular parallelopiped of mass M.

The M. I. of the rectangular parallelopiped about the edge OA=M. I. of the rectangular parallelopiped about a parallel axis GX' through its centre of gravity 'G'+M. I. of the total mass M placed at the centre of gravity 'G' about OA



$=\frac{M}{3}(b^2+c^2)$

$+M.$ (square of the perpendicular distance of G from OA)

$=\frac{1}{3}M(b^2+c^2)+M(b^2+c^2)=\frac{4}{3}M(b^2+c^2).$

Aliter. Consider an element of volume $\delta x\ \delta y\ \delta z$ of the parallelopiped at the point P whose coordinates referred to the rectangular axes along edges OA, OB, OC are (x, y, z).

Mass of this element$=\rho\delta x\ \delta y\ \delta z$.

Distance of $P\ (x, y, z)$ from $OA=\sqrt{(y^2+z^2)}$.

$\therefore$ M. I. of this elementary mass δm about OA

$$=(\rho\delta x\ \delta y\ \delta z).(y^2+z^2).$$

$\therefore$ M.I. of the rectangular parallelopiped about OA

$$=\int_{x=0}^{2a}\int_{y=0}^{2b}\int_{z=0}^{2c}\rho\ (y^2+z^2)\ dx\ dy\ dz$$

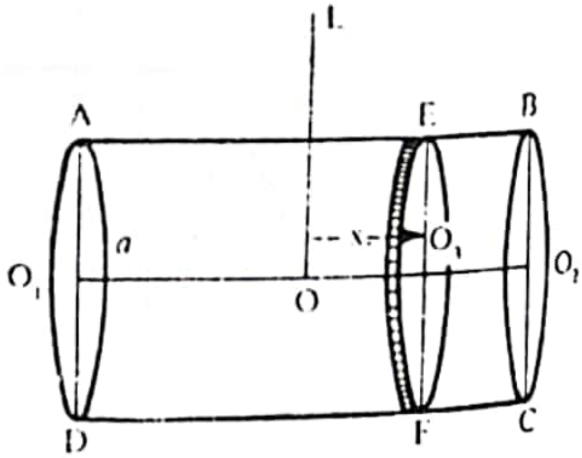
$$=\frac{32}{3}\rho\ abc\ (b^2+c^2)$$

$$=\frac{32}{3}\cdot\frac{M}{8\ abc}.abc\ (b^2+c^2) \qquad \left[\because\ \rho=\frac{M}{8\ abc}\right]$$

$$=\frac{4}{3}M\ (b^2+c^2).$$

Ex. 4. *Find the M. I. of a right circular cylinder about (i) its axis, (ii) a straight line through its C. G. and perpendicular to its axis.*

Sol. Let a be the radius, h the height and M the mass of a right circular cyliner. If ρ is the density of the cylinder, then $M=\rho\pi a^2h$.



Consider an elementary disc of breadth δx perpendicular to the axis O_1O_2 and at a distance x from the centre of gravity O of the cylinder.

Mass δm of this disc$=\rho.\pi a^2\delta x$.

M.I. of the disc about $O_1O_2=\frac{1}{2}a^2\delta m=\frac{1}{2}a^2.\rho\pi a^2\delta x=\frac{1}{2}\rho\pi a^4\delta x$.

$\therefore$ M.I. of the cylinder about O_1O_2

$$=\int_{-h/2}^{h/2}\tfrac{1}{2}\rho\pi a^4dx=\tfrac{1}{2}\rho\pi a^4h=\tfrac{1}{2}Ma^2. \qquad [\because\ M=\rho\pi a^2h]$$

(ii) Let OL be the line through the centre of gravity 'O' of the cylinder and perpendicular to the axis of the cylinder.

M.I. of the elementary disc about OL

=M.I. of the disc about the parallel line EF through its centre of gravity 'O_3'+ M.I. of the total mass δm of the disc placed at O_3 about OL

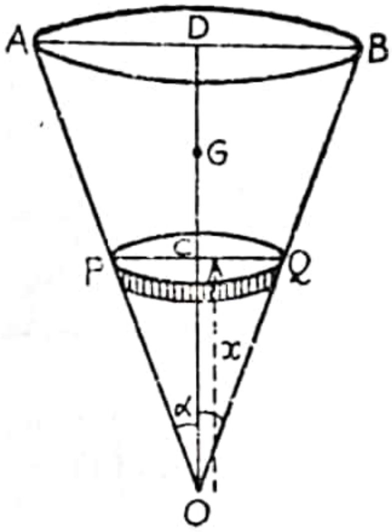
$$=\tfrac{1}{4}a^2\delta m+x^2\delta m=(\tfrac{1}{4}a^2+x^2)\ \delta m=(\tfrac{1}{4}a^2+x^2)\ \rho\pi a^2\delta x.$$

$\therefore$ M.I. of the cylinder about OL

$$=\int_{-h/2}^{h/2}\left(\tfrac{1}{4}a^2+x^2\right)\rho\pi a^2 dx=\rho\pi a^2\left[\tfrac{1}{4}a^2x+\tfrac{1}{3}x^3\right]_{-h/2}^{h/2}$$

$$=\tfrac{1}{4}\rho\pi a^2 h\left(a^2+\tfrac{1}{3}h^2\right)=\tfrac{1}{4}M\left(a^2+\tfrac{1}{3}h^2\right).$$

Ex. 5. *Prove that the M. I. of a uniform right circular solid cone of mass M, height h and radius a, about a diameter of its base is* $M(3a^2+2h^2)/20$.



Sol. Let O be the vertex of a right circular cone of mass M, height h and base-radius a. If α is the semi-vertical angle and ρ the density of the cone, then

$$M=\tfrac{1}{3}\pi h^3\tan^2\alpha\,\rho. \qquad \ldots(1)$$

Consider an elementary disc PQ of thickness δx, parallel to the base AB and at a distance x from the vertex O.

Mass of the disc

$=\delta m=\rho\pi x^2\tan^2\alpha\,\delta x.$

$\therefore$ M.I. of the disc about the diameter AB of the base of the cone

=Its M.I. about the parallel diameter PQ of the disc+M.I. of the total mass δm at centre C about AB

$=\tfrac{1}{4}\delta m.CP^2+\delta m.CD^2=\rho\pi x^2\tan^2\alpha\left[\tfrac{1}{4}x^2\tan^2\alpha+(h-x)^2\right]\delta x.$

$\therefore$ M. I. of the cone about the diameter of the base

$$=\int_0^h \rho\pi x^2\tan^2\alpha\left[\tfrac{1}{4}x^2\tan^2\alpha+(h-x)^2\right]dx$$

$$=\tfrac{1}{4}\rho\pi\tan^2\alpha\int_0^h\left(x^4\tan^2\alpha+4h^2x^2-8hx^3+4x^4\right)dx$$

$$=\tfrac{1}{4}\rho\pi\tan^2\alpha\left[\tfrac{1}{5}h^5\tan^2\alpha+\tfrac{4}{3}h^5-2h^5+\tfrac{4}{5}h^5\right]$$

$$=\tfrac{1}{60}\rho\pi h^5\tan^2\alpha\,(3\tan^2\alpha+2)=\tfrac{1}{20}Mh^2(3\tan^2\alpha+2),$$

[from (1)]

$$=\tfrac{1}{20}Mh^2\left(3.\frac{a^2}{h^2}+2\right)=M(3a^2+2h^2)/20. \qquad [\because \tan\alpha=a/h]$$

Ex. 6. *Find the moment of inertia of an elliptic area about a line, in its plane, passing through its centre and inclined at an angle θ to the major axis.* **[Meerut 86]**

Sol. Let M be the mass and $2a$, $2b$ the lengths of the axes of an elliptic area.

If PQ is a line passing through the centre O of the ellipse and θ is its inclination to the major axis OA, then the moment of inertia of the elliptic area about PQ

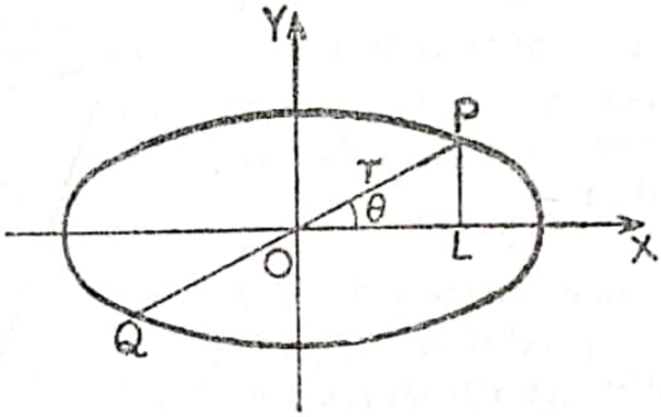
$= A\cos^2\theta + B\sin^2\theta - F\sin 2\theta$ [see § 3·16 on page 27]

where A = M.I. of the ellipse about $OA = \frac{1}{4}Mb^2$

B = M.I. of the ellipse about $OB = \frac{1}{4}Ma^2$

and F = P.I. of the ellipse about OA, OB

$=0$. [$\because$ ellipse is symmetrical about OA and OB]



$\therefore$ M.I. of the elliptic area about PQ

$$= \tfrac{1}{4}Mb^2\cos^2\theta + \tfrac{1}{4}Ma^2\sin^2\theta - 0$$
$$= \tfrac{1}{4}M(b^2\cos^2\theta + a^2\sin^2\theta).$$

Ex. 7. *Show that the M.I. of an elliptic area of mass M and semi-axes a and b about a diameter of length 2r is $Ma^2b^2/(4r^2)$.*

Sol. Refer figure of the preceding Ex. 6. Let POQ be the diameter of length $2r$ of an elliptic area of mass M and semi-axes a and b.

The equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \quad \text{...(1)}$$

If POQ makes an angle θ with OX, then coordinates of P are $(r\cos\theta, r\sin\theta)$.

Since P lies on (1), $\therefore$ $(r^2/a^2)\cos^2\theta + (r^2/b^2)\sin^2\theta = 1$

or $a^2\sin^2\theta + b^2\cos^2\theta = a^2b^2/r^2$. ...(2)

Now, M.I. of the elliptic area about $OX = A = \frac{1}{4}Mb^2$

M.I. of the elliptic area about $OY = B = \frac{1}{4}Ma^2$

and P.I. of the elliptic area about OX and $OY = F = 0$ (By symmetry)

$\therefore$ M. I. of the elliptic area about the diameter POQ

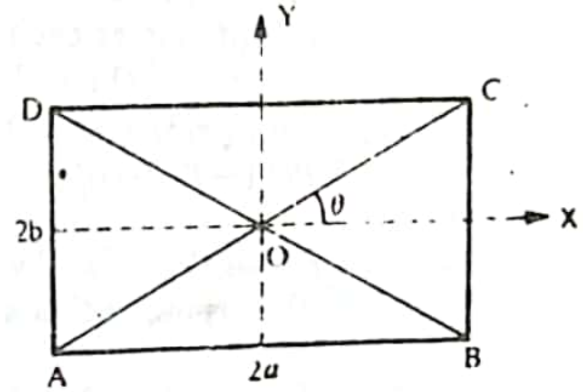
$= A\cos^2\theta + B\sin^2\theta - F\sin 2\theta$

$=\frac{1}{4} Mb^2 \cos^2 \theta + \frac{1}{4} Ma^2 \sin^2 \theta - 0$

$= \frac{1}{4} M (b^2 \cos^2 \theta + a^2 \sin^2 \theta) = \frac{1}{4} M (a^2b^2/r^2) = (Ma^2b^2)/(4r^2)$, from (2).

Ex. 8. *Show that the M.I. of a rectangle of mass M and sides 2a, 2b about a diagonal is* $\frac{2M}{3} \cdot \frac{a^2b^2}{a^2+b^2}$. *Deduce that in case of a square.*

Sol. Let $ABCD$ be a rectangle of mass M and sides $AB=2a$, $AD=2b$.



We have

$A=$ M.I. of the rectangle about $OX=\frac{1}{3} Mb^2$

$B=$ M.I. of the rectangle about $OY=\frac{1}{3}Ma^2$,

and $F=$ P.I. of the rectangle about OX and $OY=0$, since the rectangle is symmetrical about OX and OY.

If the diagonal AC makes an angle θ with AB, then

$$\cos \theta = AB/AC = 2a/\sqrt{(4a^2+4b^2)} = a/\sqrt{(a^2+b^2)}$$

and $$\sin \theta = BC/AC = b/\sqrt{(a^2+b^2)}.$$

$\therefore$ M.I. of the rectangle about the diagonal AC

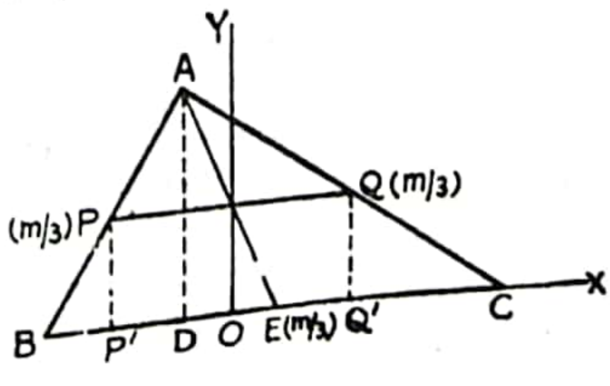
$$= A \cos^2 \theta + B \sin^2 \theta - F \sin 2\theta$$

$$= \frac{Mb^2}{3} \cdot \frac{a^2}{(a^2+b^2)} + \frac{Ma^2}{3} \cdot \frac{b^2}{(a^2+b^2)} - 0 = \frac{2M}{3} \cdot \frac{a^2b^2}{a^2+b^2}.$$

Deduction. For square, we have $2b=2a$ *i.e.*, $b=a$.

$\therefore$ M.I. of the square about a diagonal $= \frac{2M}{3} \cdot \frac{a^4}{a^2+b^2} = Ma^2/3$.

Ex. 9. *ABC is a triangular area and AD is perpendicular to BC and AE is a median, O is the middle point of DE, show that BC is a principal axis of the triangle at O.*



Sol. Let AD be the perpendicular to BC and AE the median of a triangle ABC. Let O be the middle point of DE and the lines OX, OY along and perpendicular to BC be the axes of reference.

If m is the mass of the $\triangle ABC$ then it can be replaced by three particles each of mass $m/3$ at the middle points E, P and Q of the sides BC, AB and AC respectively.

Then P.I. of the $\triangle ABC$ about OX and OY

=Sum of P.I. of masses each equal to $m/3$ at middle points E, P and Q, about OX and OY

$=(m/3).OE.0+(m/3).(-OP').P'P+(m/3)(OQ')Q'Q$

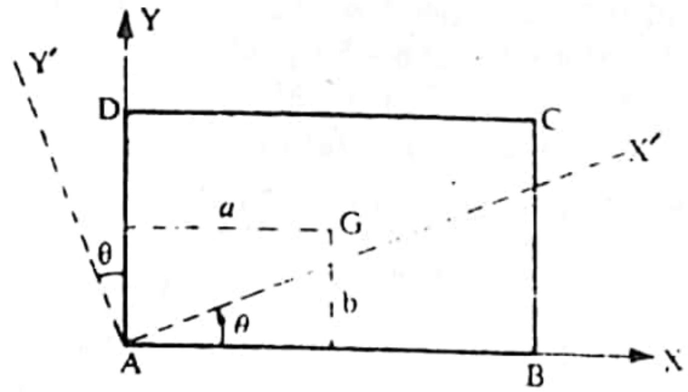
$=(m/3)\frac{1}{2}PQ(-P'P+Q'Q)$ $[\because \ OP'=OQ'=\frac{1}{2}PQ]$

$=0$ $[\because \ P'P=Q'Q]$

Thus the P.I. of the $\triangle ABC$ vanishes about BC and perpendicular to BC at O. Hence BC is a principal axis of the triangle ABC at O.

Ex. 10. *The lengths AB and AD of the sides of a rectangle ABCD are 2a and 2b; show that the inclination to AB of one of the principal axes at A is* $\frac{1}{2}\tan^{-1}\dfrac{3ab}{2(a^2-b^2)}$.

Sol. Let $ABCD$ be a rectangle of mass M and sides $AB=2a$, $AD=2b$.



We have

$A=$M.I. of the rectangle about $AB=\frac{1}{3}Mb^2+Mb^2=\frac{4}{3}Mb^2$,

$B=$M.I. of the rectangle about $AD=\frac{1}{3}Ma^2+Ma^2=\frac{4}{3}Ma^2$,

and $F=$P.I. of the rectangle about AB and AD

=P.I. of the rectangle about the axes parallel to AB and AD through its centre of gravity 'G'

+P.I. of the whole mass M at G about AB and AD

$=0+Mba=Mab$.

If a principal axis at A is inclined at an angle θ to AB, then

$$\tan 2\theta=\frac{2F}{B-A}=\frac{3ab}{2(a^2-b^2)}, \quad \therefore \quad \theta=\frac{1}{2}\tan^{-1}\frac{3ab}{2(a^2-b^2)}.$$

Ex. 11. *Find the angle of inclination of the principal axis to the side of a square at its corner.*

Sol. Proceed as in Ex. 10. Here $2a=2b$,

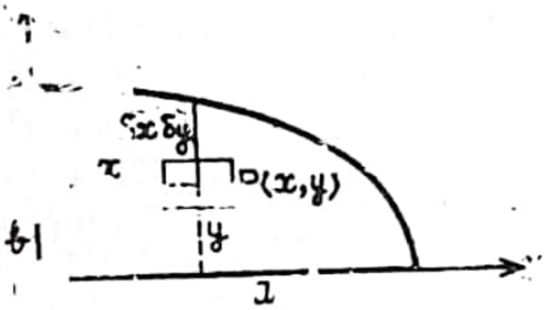
$$\theta=\tfrac{1}{2}\tan^{-1}\left\{\frac{3a^2}{2(a^2-a^2)}\right\}=\tfrac{1}{2}\tan^{-1}\infty=\pi/4.$$

Ex. 12. *Show that at the centre of a quadrant of an ellipse, the principal axes in its plane are inclined at an angle*

$\tfrac{1}{2}\tan^{-1}\left(\dfrac{4}{\pi}\cdot\dfrac{ab}{a^2-b^2}\right)$ *to the axes.* **[Kanpur [illegible], Rohilkhand 83]**

Sol. [illegible] the ellipse $x^2/a^2+y^2/b^2=1$.

If [illegible] mentary mass at the point (x, y) of the quadrant, then



$A=$M.I. of the quadrant [illegible] OX

$$=\int_{x=0}^{a}\int_{y=0}^{b\sqrt{(a^2-x^2)}/a} y^2 \rho\, dx\, dy$$

$$=\frac{\rho b^3}{3a^3}\int_{x=0}^{a}(a^2-x^2)^{3/2}\,dx$$

$$=\frac{\rho b^3}{3a^3}\cdot a^4\int_0^{\pi/2}\cos^4\theta\,d\theta=\frac{\rho b^3 a}{3}\cdot\frac{\Gamma(5/2)\,\Gamma(\frac{1}{2})}{2\Gamma(3)}$$ [Putting $x=a\sin\theta$]

$=\frac{1}{16}\rho\pi b^3 a=\frac{1}{4}Mb^2$, $\because$ $M=$mass of the quadrant$=\frac{1}{4}\rho\pi ab$]

$B=$M.I. of the quadrant about OY

$$=\int_{x=0}^{a}\int_{y=0}^{b\sqrt{(a^2-x^2)}/a} x^2 \rho\, dx\, dy=\frac{b\rho}{a}\int_{x=0}^{a} x^2\sqrt{(a^2-x^2)}\,dx$$

$=\frac{1}{4}Ma^2$, putting $x=$ [illegible] on simplification

$F=$P.I. of the quadrant about OX and OY.

$$=\int_{x=0}^{a}\int_{y=0}^{b\sqrt{(a^2-x^2)}/a} xy\,\rho\,dx\,dy=\tfrac{1}{2}\rho\frac{b^2}{a^2}\int_{x=0}^{a}x(a^2-x^2)\,dx$$

$$=\frac{Mab}{2\pi}.$$

$\therefore$ If one of the principal axes is inclined at an angle [illegible] OX, then