11 Years

Previous Years Solved Papers

Civil Services Main Examination

(2009-2019)

Mathematics

Paper-I

Topicwise Presentation

MADE EASY Publications

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Civil Services Main Examination Previous Solved Papers : Mathematics Paper-I

First Edition: 2020

Preface

B. Singh (Ex. IES)

Civil Service is considered as the most prestigious job in India and it has become a preferred destination by all engineers. In order to reach this estimable position every aspirant has to take arduous journey of Civil Services Examination (CSE). Focused approach and strong determination are the pre-requisites for this journey. Besides this, a good book also comes in the list of essential commodity of this odyssey.

I feel extremely glad to launch the first edition of such a book which will not only make CSE plain sailing, but also with 100% clarity in concepts.

MADE EASY team has prepared this book with utmost care and thorough study of all previous years papers of CSE. The book aims to provide complete solution to all previous years questions with accuracy.

I would like to acknowledge efforts of entire MADE EASY team who worked day and night to solve previous years papers in a limited time frame and I hope this book will prove to be an essential tool to succeed in competitive exams and my desire to serve student fraternity by providing best study material and quality guidance will get accomplished.

With Best Wishes
B. Singh (Ex. IES)
CMD, MADE EASY Group

Previous Years Solved Papers of

Civil Services Main Examination

Mathematics : Paper-I

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3 Analytic Geometry

1. Straight Lines

1.1 A line is drawn thrugh a variable point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $z=0$ to meet fixed lines $y=mx$, $z=c$ and $y=-mx$, $z=-c$. Find the locus of the line

(2009 : 12 Marks)

Solution:

Approach : Use general equation of line intersecting two lines given in planar form.

Given fixed lines are

$$y-mx = 0,\ z-c=0 \quad \text{...(i)}$$
$$y+mx = 0,\ z+c=0 \quad \text{...(ii)}$$

General equation of line intersecting both

$$(y-mx)+k_1(z-c) = 0 = (y+mx)+k_2(z+c) \quad \text{...(iii)}$$

If it meets ellipse we eliminate k_1 and k_2

Putting $z=0$ in (iii)

$$y-mx-k_1c = 0;\ y+mx+k_2c=0$$

$$\Rightarrow \quad \frac{y}{-k_2m+k_1m} = \frac{x}{-(k_1+k_2)}=\frac{c}{2m}$$

$$\Rightarrow \quad x = \frac{-(k_1+k_2)c}{2m};\ y=\frac{(k_1-k_2)c}{2}$$

Putting this in equation of ellipse

$$\frac{(k_1+k_2)^2c^2}{4m^2a^2}+\frac{(k_1-k_2)^2c^2}{4b^2} = 1$$

$$(k_1+k_2)^2c^2b^2+(k_1-k_2)^2c^2a^2m^2 = 4a^2b^2m^2$$

Substituting k_1 and k_2 from (iii)

$$\left\{\left(\frac{mx-y}{z-c}\right)+\left(-\frac{mx+y}{z+c}\right)\right\}^2c^2b^2+\left\{\left(\frac{mx-y}{z-c}\right)+\left(\frac{mx+y}{z+c}\right)\right\}^2\times c^2a^2 = 4a^2b^2m^2$$

$$\Rightarrow [(mx-y)(z+c)-(mx+y)(z-c)]^2c^2b^2+[(mx-y)(z+c)+(mx+y)(z-c)]^2m^2c^2a^2 = 4a^2b^2m^2(z^2-c^2)^2$$

$$\Rightarrow [cmx-yz]^2c^2b^2+[mxz-cy]^2m^2c^2a^2 = a^2b^2m^2(z^2-c^2)^2$$

which is required locus.

1.2 Prove that two of the straight lines represented by the equation

$$x^3+bx^2y+cxy^2+y^3 = 0$$

will be at right angles, if b + c = –2.

(2012 : 12 Marks)

Solution:

The given equation is a homogeneous equation of third degree and hence it represents three straight lines through the origin.

Let $y=mx$ be any of these lines.

Replacing $\frac{y}{x}$ by m in $x^3 + bx^2y + cxy^2 + y^3 = 0$ or $1 + b\frac{y}{x} + c\frac{y^2}{x^2} + \frac{y^3}{x^3} = 0$, we get

$$m^3 + cm^2 + bm + 1 = 0 \quad \text{...(i)}$$

Let m_1, m_2, m_3 be its roots, then

$$m_1 \cdot m_2 \cdot m_3 = -1$$

But, two of these lines, say with slopes, m_1 and m_2, are at right angles,

then, $\quad m_1 \cdot m_2 = -1$

Thus, $\quad (-m_3) = 1$ or $m_3 = 1$

But m_3 is a root of (i)

$\therefore \quad 1 + c + b + 1 = 0$

or $\quad b + c = -2$

1.3 Verify if the lines $\frac{x-a+d}{\alpha-\delta} = \frac{y-a}{\alpha} = \frac{z-a-d}{\alpha+\delta}$ and $\frac{x-b+c}{\beta-\gamma} = \frac{y-b}{\beta} = \frac{z-b-c}{\beta+\gamma}$ are coplanar. If yes, then find the equation of the plane in which they lie?

(2014 : 7 Marks)

Solution:

Two straight lines

$$\frac{x-x_1}{a_1} = \frac{y-y_1}{b_1} = \frac{z-z_1}{c_1} \text{ and } \frac{x-x_2}{a_2} = \frac{y-y_2}{b_2} = \frac{z-z_2}{c_2}$$

are coplanar if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

And equation of plane containing them, is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0$$

Here, in our case,

$$\begin{vmatrix} (b-c)-(a-d) & b-a & b+c-(a+d) \\ \alpha-\delta & \alpha & \alpha+\delta \\ \beta-\gamma & \beta & \beta+\delta \end{vmatrix} \begin{matrix} C_1 \to C_1 - C_2 \\ C_3 \to C_3 - C_2 \end{matrix}$$

$$= \begin{vmatrix} d-c & b-a & c-d \\ -\delta & \alpha & \delta \\ -\gamma & \beta & \gamma \end{vmatrix} = 0 \text{ as } C_1 = -C_3$$

Hence, the given lines are coplanar.

The equation of the plane containing them, is

$$\begin{vmatrix} x-(a-d) & y-a & z-(a+d) \\ \alpha-\delta & \alpha & \alpha+\delta \\ \beta-\gamma & \beta & \beta+\gamma \end{vmatrix} = 0. \text{ Applying } \begin{matrix} C_1 \to C_1 - C_2 \\ C_3 \to C_3 - C_2 \end{matrix}$$

$$\begin{vmatrix} x-y+d & y-a & z-y-d \\ -\delta & \alpha & \delta \\ -\gamma & \beta & \gamma \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x-2y+z & y-a & z-y-d \\ 0 & \alpha & \delta \\ 0 & \beta & \gamma \end{vmatrix} = 0 \text{ as } C_1 \to C_1 + C_3$$

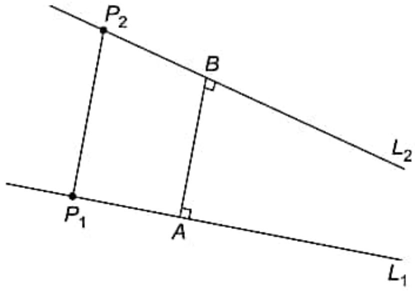
$\Rightarrow \quad x - 2y + z = 0$

2. Shortest Distance between Two Skew Lines

2.1 **Find the shortest distance between the lines $\frac{x-1}{2} = \frac{y-2}{4} = z-3$ and $y-mx = z = 0$. For what value of m will the two lines intersect?**

(2016 : 10 Marks)

Solution:



Lines are :

$$L_1 : \frac{k-1}{2} = \frac{y-2}{4} = \frac{z-3}{1}$$

$$L_2 : \frac{x}{1} = \frac{y}{m} = \frac{z}{0} \quad \begin{bmatrix} y-mx=0 \\ z=0 \end{bmatrix}$$

$P_1(1, 2, 3)$ on L_1; $P_2(0, 0, 0)$ on L_2

Shortest distance (SD) is the projection of P_1P_2 on AB which is perpendicular to both lines. Direction ratio's of AB :

$$\begin{vmatrix} i & j & k \\ 2 & 4 & 1 \\ 1 & m & 0 \end{vmatrix} = i(0-m) - j(0-1) + k(2m-4)$$

$$= -mi + j + (2m-4)k$$

$$SD = \frac{1}{\sqrt{m^2+1+(2m-4)^2}}[-m(1-0)+1(2-0)+(2m-4)(3-0)$$

$$= \left|\frac{5m-10}{\sqrt{5m^2-16m+17}}\right|$$

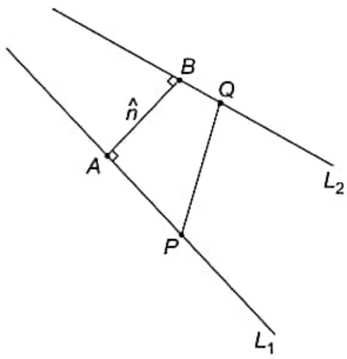
The lines will intersect if, $SD = 0$, *i.e.*, $5m-10=0 \Rightarrow m=2$.

2.2 **Find the shortest distance between the skew lines, $\frac{x-3}{3} = \frac{8-y}{1} = \frac{z-3}{1}$ and $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$.**

(2017 : 10 Marks)

Solution:

Shortest distance lies along a direction which is perpendicular to both lines and given by the cross-product of vectors along given two lines, L_1, L_2.



$$\vec{n} = \begin{vmatrix} i & j & k \\ 3 & -1 & 1 \\ -3 & 2 & 4 \end{vmatrix}$$

$$= i(-4-2) - j(12+3) + k(6-3)$$

$$= -6i - 15j + 3k$$

$$= -3(2i+5j-k)$$

$$\therefore \quad \hat{n} = \frac{1}{\sqrt{30}}(2i+5j-k)$$

S.D. is the projection of PQ along $\hat{n}$.

$$SD = \vec{AB}\cdot\hat{n}$$

$$= \frac{1}{\sqrt{30}}[(3-(-3))2+(8-(-7))5-(3-6)\cdot 1]$$

$$= \frac{1}{\sqrt{30}}(12+75+3) = \frac{90}{\sqrt{30}}$$
$$= 3\sqrt{30}$$

2.3 Find the shortest distance between the lines

$$a_1x + b_1y + c_1z + d_1 = 0$$
$$a_2x + b_2y + c_2z + d_2 = 0$$

and the z-axis.

(2018 : 12 Marks)

Solution:

The equation of z-axis is $x = y = 0$

$\therefore$ Any plane, P, through z-axis can be written as

$$x + \mu y = 0 \quad \text{...(i)}$$

Further, any plane P_2, through given set of planes is

$$a_1x + b_1y + c_1z + d_1 + \lambda(a_2x + b_2y + c_2z + d_2) = 0$$

i.e., $$(a_1 + \lambda a_2)x + (b_1 + \lambda b_2)y + (c_1 + \lambda c_2)z + d_1 + \lambda d_2 = 0 \quad \text{...(ii)}$$

For shortest distance P_1 and P_2 should be parallel.

$\therefore$ $$\frac{a_1+\lambda a_2}{1} = \frac{b_1+\lambda b_2}{\mu} = \frac{c_1+\lambda c_2}{0}$$

i.e., $$c_1 + \lambda c_2 = 0$$

$\Rightarrow$ $$\lambda = \frac{-c_1}{c_2}$$

$\therefore$ equation of P_2 is

$$\left(a_1 - \frac{c_1}{c_2}a_2\right)x + \left(b_1 - \frac{c_1}{c_2}b_2\right)y + \left(d_1 - \frac{c_1}{c_2}\right)d_2 = 0$$

Shortest distance, $$d = \frac{|d_1 + \lambda d_2 - 0|}{\sqrt{(a_1+\lambda a_2)^2 + (b_1+\lambda b_2)^2 + 0^2}}$$

$$d = \frac{|c_2d_1 + c_1d_2|}{\sqrt{(c_2a_1 - c_1a_2)^2 + (c_2b_1 - c_1b_2)^2}}$$

2.4 Show that the lines $\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z+2}{1}$ and $\frac{x}{1} = \frac{y-7}{-3} = \frac{z+7}{2}$ intersect. Find the coordinates of the point of intersection and the equation of the plane containing them.

(2019 : 10 Marks)

Solution:

Any point on the line

$$\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z+2}{1}$$

is $$(-1-3r, 3+2r, -2+r) \quad \text{...(1)}$$

Similarly, any part on the line

$$\frac{x}{1} = \frac{y-7}{-3} = \frac{z+7}{2}$$

is $$(r^2, 7-3r', -7+2r') \quad \text{...(2)}$$

If the two given lines intersect then for some value of r and r^2 teh two above points (i) and (ii) must coincide.

i.e.,
$$-1-3r = r^1;$$
$$3+2r = 7-3r^1;$$
$$-2+r = -7+2r^1$$

Solving the first two of these equations, we get
$$r = -1, r' = 2$$

These values of r and r' satisfy the third equation also. Hence, the given lines intersect.

Substituting these values r and r' in (1) or (2) we get the required coordinates of the point of intersection as (2, 1, –3). ...(I)

Also, the equation of the plane containing the given lines is
$$\begin{vmatrix} x+1 & y-3 & z+2 \\ -3 & 2 & 1 \\ 1 & -3 & 2 \end{vmatrix} = 0$$
$$\Rightarrow (x+1)(4+3)-(y-3)(-6-1)+(z+2)(9-2) = 0$$
$$x+y+z = 0$$

which is the required equation. ...(II)

3. Plane and its Properties

3.1 Find the equations of the straight line through the point (3, 1, 2) to intersect the straight line
$$x+4 = y+1 = 2(z-2)$$
and parallel to the plane $4x+y+5z=0$.

(2011 : 10 Marks)

Solution:

Let the required line intersects the given line
$$x+y = y+1 = 2(z-2) \quad ...(i)$$
at (x_1, y_1, z_1)

$\therefore$ The equations of the required line passing through (3, 1, 2) and (x_1, y_1, z_1) are
$$\frac{x-3}{x_1-3} = \frac{y-1}{y_1-1} = \frac{z-2}{z_1-1} \quad ...(ii)$$

The direction ratios of the line (ii) are x_1-3, y_1-1, z_1-1.

Because the line (ii) is parallel to the plane $4x+y+5z=0$, therefore the normal to the plane with direction ratios 4, 1, 5 is perpendicular to the line (ii).

$\therefore$ $4(x_1-3)+1(y_1-1)+5(z_1-1) = 0$

or $4x_1+y_2+5z_2-23 = 0$ ...(iii)

Because the point (x_1, y_1, z_1) lies on (i),

$\therefore$ $x_1+4 = y_1+1 = 2(z_1-2)$

$\Rightarrow$ $x_1 = 2z_1-8,\ y_1 = 2z_1-5$ ...(iv)

$\therefore$ from (iii), we get
$$4(2z_1-8)+2z_1-5+5z_1-23 = 0$$
$\Rightarrow$ $z_1 = 4$

$\therefore$ from (iv), $x_1 = 0,\ y_1 = 3$

$\therefore$ from (ii), the equations of the required line are :
$$\frac{x-3}{-3} = \frac{y-1}{2} = \frac{z_2-2}{2}$$

3.2 Find the equation of the plane which passes through the points (0, 1, 1) and (2, 0, –1) and is parallel to the line joining the points (–1, 1, –2), (3, –2, 4). Find also the distance between the line and the plane.

(2013 : 10 Marks)

Solution:

The general equation of plane through (0, 1, 1)

$$l(x-0)+m(y-1)+n(z-1) = 0$$

$$lx+m(y-1)+n(z-1) = 0 \quad \text{...(i)}$$

(2, 0, –1) lies on this plane.

$$\therefore \quad 2l-m-2n = 0 \quad \text{...(ii)}$$

Direction ratios of line passing through (–1, 1, –2), (3, –2, 4).

$$a_1 = 3-1=4,\ b_1=(-2-1)=-3,\ c_1=(4-(-2))=6$$

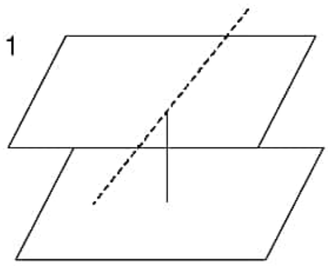
$\therefore$ Direction ratios are 4 :: –3 :: 6

This is parallel to plane.

$$\therefore \quad 4l-3m+6n = 0 \quad \text{...(ii)}$$

From (ii) and (iii)

$$\frac{l}{-12} = \frac{m}{-20}=\frac{n}{-2} \Rightarrow l:m:n::6:10:1$$



$\therefore$ Equation of plane is

$$6l+10(y-1)+(z-1) = 0$$

$$\Rightarrow \quad 6x+10y+z-11 = 0$$

Equation of plane parallel to this plane and passing through line

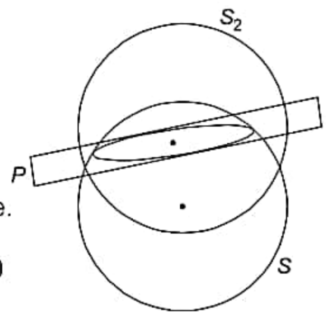
$$6(x+1)+10(y-1)+1(z+2) = 0$$

$$6x+10y+z-2 = 0$$

$\therefore$ distance between line and plane

= distance between two plane

$$= \frac{|d_1-d_2|}{\sqrt{l^2+m^2+n^2}} = \frac{9}{\sqrt{6^2+10^2+1^2}}$$



This must be on the plane for the intersection of this sphere to be a great circle.

$$\therefore \quad 5\left(\frac{5\lambda-3}{2}\right)-(4-2\lambda)+2(4\lambda-2)+7 = 0$$

$$\Rightarrow \quad 45\lambda = 17 \Rightarrow \lambda=\frac{17}{45}$$

$\therefore$ The given sphere is

$$x^2+y^2+z^2-\frac{10}{9}x+\frac{146}{45}y-\frac{22}{45}z-\frac{106}{45} = 0$$

$$\Rightarrow \quad 45(x^2+y^2+z^2)-50x+146y-22z-106 = 0$$

3.3 Obtain the equation of the plane passing through the points (2, 3, 1) and (4, –5, 3) parallel to x-axis.

(2015 : 6 Marks)

Solution:

The equation of any plane through (2, 3, 1) is

$$a(x-2)+b(y-3)+c(z-1) = 0 \quad \text{...(i)}$$

It passes through (4, –5, 3)

$$\therefore \quad a(4-2)+b(-5-3)+c(3-1) = 0$$

$$\text{i.e.,} \quad a-4b+c = 0 \quad \text{...(ii)}$$

If the plane (i) is parallel to x-axis, then it is perpendicular to yz-plane, i.e., $x = 0$, i.e.,

$$1x + 0y + 0z = 0$$

$$\therefore \quad 1a + 0b + 0c = 0 \Rightarrow a = 0$$

$\therefore$ from (ii), $\quad -4b + c = 0$, i.e., $c = 4b$

$$\therefore \quad \frac{a}{0} = \frac{b}{1} = \frac{c}{4}$$

Hence, (i) becomes $0 + 1(y - 3) + 4(z - 1) = 0$

$$y + 4z - 7 = 0$$

3.4 Find the surface generated by a line which intersects the lines $y = a = z$, $x + 3z = a = y + z$ and parallel to the plane $x + y = 0$.

(2016 : 10 Marks)

Solution:

Topic : Equation of a straight line intersecting two given lines.

Given lines are :

$$y - a = 0 = z - a \quad \text{...(i)}$$

$$x + 3z - a = 0 = y + z - a \quad \text{...(ii)}$$

Hence, the equation of a line intersecting the given lines (i) and (ii) will be

$$(y - a) + \lambda(z - a) = 0 \quad \text{...(i)*}$$

and $\quad (x + 3z - a) + \mu(y + z - a) = 0 \quad$ ...(ii)*

$$\Rightarrow \quad y + \lambda z - (a + \lambda a) = 0$$

and $\quad x + \mu y + (3 + \mu)z - (a + \mu a) = 0 \quad$ ...(iii)

Line (iii) is parallel to the plane $x + y = 0$...(iv)

If direction ratio's of line (iii) are l, m, n, then

$$\frac{l}{3+\mu-\mu\lambda} = \frac{m}{\lambda-0} = \frac{n}{0-1}$$

$\therefore$ (iv) $\Rightarrow \quad 1\cdot(3 + \mu - \mu\lambda) + 1\cdot\lambda + 0\cdot(-1) = 0$

$$3 + \mu + \lambda - \mu\lambda = 0 \quad \text{...(v)}$$

The required locus of the line is obtained by eliminating λ and μ between (i)*, (ii)* and (v).

$$3 - \frac{y-a}{z-a} - \frac{x+3z-a}{y+z-a} - \frac{y-a}{z-a}\cdot\frac{x+3z-a}{y+z-a} = 0$$

Solving and simplifying : $(y + z)(x + y) = 2a(x + z)$

3.5 Find the projection of straight line $\frac{x-1}{2} = \frac{y-1}{3} = \frac{z+1}{-1}$ on the plane $x + y + 2z = 6$.

(2018 : 10 Marks)

Solution:

Given line is $\quad \frac{x-1}{2} = \frac{y-1}{3} = \frac{z+1}{-1} = r$

Let this line meets given plane at $(2r + 1, 3r + 1, -r - 1)$.

The point lies on given plane, i.e.,

$$2r + r + 3r + 1 - 2r - z = 6$$

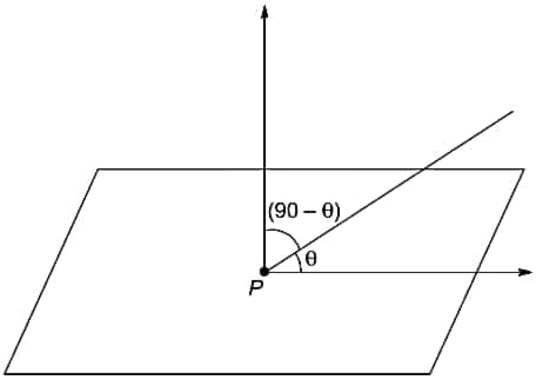
$$\Rightarrow \quad r = 2$$

$\therefore$ The point is $(5, 7, -3)$.

Let equation of line of projection is

$$\frac{x-5}{l} = \frac{y-7}{m} = \frac{n+3}{1}$$

Angle between given line and normal of plane is $(90 - \theta)$.



$$\therefore \qquad \cos(90-\theta) = \sin\theta = \frac{2\times1+3\times1-1\times2}{\sqrt{14}\sqrt{6}} = \frac{3}{\sqrt{84}}$$

$\therefore$ Angle between given line and its projection on given plane is θ.

$$\therefore \qquad \cos\theta = \frac{\sqrt{75}}{\sqrt{84}} = \frac{2l+3m-1}{\sqrt{l^2+m^2+1}\sqrt{14}} \qquad \text{...(i)}$$

Also, since projected line lies on given plane

$$\therefore \qquad l\times1 + m\times1 + 1\times2 = 0 \;\Rightarrow\; m = -l-2 \qquad \text{...(ii)}$$

Putting this value of m in eqn. (i), we get

$$\frac{25}{2} = \frac{(-l-7)^2}{2l^2+4l+5}$$

$$\Rightarrow \qquad l = \frac{-3}{7},\; m = \frac{-5}{4}$$

$\therefore$ equation of projected line is

$$\frac{x-5}{\left(\frac{-3}{4}\right)} = \frac{y-7}{\left(\frac{-5}{4}\right)} = \frac{z+3}{1}$$

or

$$\frac{x-5}{3} = \frac{y-7}{5} = \frac{z+3}{-4}$$

3.6 Find the equation of plane parallel to $3x - y + 3z = 8$ and passing through the point (1, 1, 1).

(2018 : 12 Marks)

Solution:

Given, equation of plane is $3x - y + 3z = 8$

Any plane parallel to given plane has equation

$$3x - y + 3z = P \qquad \text{...(i)}$$

Now (i) passing through (1, 1, 1)

$$\therefore \qquad 3\times1 - 1 + 3\times1 = P \;\Rightarrow\; P = 5$$

$\therefore$ Equation of plane is $3x - y + 3z = 5$.

4. Sphere and its Properties

4.1 **Find the equation of the sphere having its centre on the plane $4x - 5y - z = 3$ and passing through the circle**

$$x^2 + y^2 + z^2 - 12x - 3y + 4z + 8 = 0$$
$$3x + 4y - 5z + 3 = 0$$

(2009 : 12 Marks)

Solution:

Approach : General equation of sphere through any circle is used. The parameter can be found by the centre satisfying equation of plane.

General equation of a sphere passing through the circle is

$$S + \lambda P = 0$$

or $(x^2 + y^2 + z^2 - 12x - 3y + 4z + 8) + \lambda(3x + 4y - 5z + 3) = 0$

i.e., $x^2 + y^2 + z^2 + (3\lambda - 12)x + (4\lambda - 3)y + (4 - 5\lambda)z + 3\lambda + 8 = 0$

The centre of the sphere is $\left(\frac{3\lambda - 12}{2}\right), \left(\frac{4\lambda - 3}{2}\right), \left(\frac{4 - 5\lambda}{2}\right)$. This lies on the given plane if

$$-\left[4\left(\frac{3\lambda - 12}{2}\right) - 5\left(\frac{4\lambda - 3}{2}\right) - \left(\frac{4 - 5\lambda}{2}\right)\right] = 3$$

$\Rightarrow$ $$3\lambda + 37 = 6$$

$\Rightarrow$ $$\lambda = \frac{-31}{3}$$

$\therefore$ Required sphere is

$$x^2 + y^2 + z^2 - 43x - \frac{133}{3}y + \frac{167}{3}z - 23 = 0$$

4.2 **Show that the plane $x + y - 2z = 3$ cuts the sphere $x^2 + y^2 + z^2 - x + y = 2$ in a circle of radius 1 and find the equation of the sphere which has this circle as great cirlce.**

(2010 : 12 Marks)

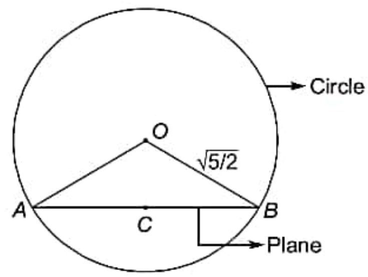
Solution:

Given : Equation of circle is $x^2 + y^2 + z^2 - x + y = 2$, Plane $\equiv x + y - 2z \equiv 3$

$$\text{Centre of given circle} \equiv \left(\frac{1}{2}, -\frac{1}{2}, 0\right)$$

$$\text{Radius} = \sqrt{\frac{1}{4} + \frac{1}{4} + 2} = \sqrt{2 + \frac{1}{2}}$$

$$= \sqrt{\frac{5}{2}}$$



Let O be the centre of this circle.

Distance of plane from centre

$$\frac{\left|\frac{1}{2} - \frac{1}{2} - 3\right|}{\sqrt{1^2 + 1^2 + 2^2}} = \frac{3}{\sqrt{6}}$$

$\therefore$ Radius of circle with BC as radius

$$\sqrt{\frac{5}{2}-\frac{9}{6}} = \sqrt{\frac{2}{2}} = 1$$

Equation of sphere with circle as great circle.

$$x^2 + y^2 + z^2 - x + y - 2 + \lambda(x + y - 2z - 3) = 0$$

$$\Rightarrow \quad x^2 + y^2 + z^2 - x + y - 2 + \lambda x + \lambda y - 2\lambda z - \lambda 3 = 0$$

$$\Rightarrow \quad x^2 + y^2 + z^2 + (1 - \lambda)x + (1 + \lambda)y - 2\lambda z - 3\lambda + 2 = 0$$

Radius of this sphere = 1

$$\Rightarrow \quad \left(\frac{1-\lambda}{2}\right)^2 + \left(\frac{1+\lambda}{2}\right)^2 + \lambda^2 + 3\lambda + 2 = 1^2$$

$$\Rightarrow \quad \frac{1+\lambda^2+2\lambda}{4} + \frac{1+\lambda^2-2\lambda}{4} + \lambda^2 + 3\lambda + 2 = 1$$

$$\Rightarrow \quad \frac{3\lambda^2}{2} + \frac{1}{2} + 3\lambda = -1$$

$$\Rightarrow \quad \frac{3\lambda^2}{2} + 3\lambda + \frac{3}{2} = 0$$

$$\Rightarrow \quad \lambda^2 + 2\lambda + 1 = 0$$

$$\Rightarrow \quad (\lambda + 1)^2 = 0$$

$$\Rightarrow \quad \lambda = -1$$

Using this value of λ, the equation of sphere is

$$x^2 + y^2 + z^2 - x + y - 2 - 1(x + y - 2z - 3) = 0$$

$$\Rightarrow \quad x^2 + y^2 + z^2 - x + y - z - x - y + 2z + 3 = 0$$

$$\Rightarrow \quad x^2 + y^2 + z^2 - 2x + 2z + 1 = 0$$

4.3 Show that the equation of the sphere which touches the sphere

$$4(x^2 + y^2 + z^2) + 10x - 25y - 2z = 0$$

at the point (1, 2, –2) and passes through the point (–1, 0, 0) is

$$x^2 + y^2 + z^2 + 2x - 6y + 1 = 0$$

(2011 : 10 Marks)

Solution:

The equation of the given sphere is

$$4(x^2 + y^2 + z^2) + 10x - 25y - 2z = 0 \quad \text{...(i)}$$

The equation of the tangent plane to the sphere (i) at the point (1, 2, –2) is

$$4(x\cdot 1 + y\cdot 2 + z(-2)) + 5\cdot(x+1) - \frac{25}{2}(y+2) - (z-2) = 0$$

or

$$18x - 9y - 18z + 14 = 0 \quad \text{...(ii)}$$

$\therefore$ The equation of the sphere which touches the sphere (i) at (1, 2, –2) is

$$4(x^2 + y^2 + z^2) + 10x - 25y - 2z + \lambda(18x - 9y - 18z + 14) = 0 \quad \text{...(iii)}$$

If (iii) passes through (–1, 0, 1) then,

$$4(1 + 0 + 0) + 10(-1) - 0 - 0 + \lambda(-18 - 0 - 0 + 14) = 0$$

$$4 - 10 + \lambda(-4) = 0$$

$$-6 = 4\lambda \Rightarrow \lambda = -\frac{3}{2}$$

Put $\lambda = -\frac{3}{2}$ in (iii),

$$4(x^2 + y^2 + z^2) + 10x - 25y - 2z - \frac{3}{2}(18x - 9y - 18z + 14) = 0$$

$$\Rightarrow \quad 8(x^2 + y^2 + z^2) + 20x - 50y - 4z - 54x + 27y + 54z - 42 = 0$$

$\Rightarrow 8(x^2 + y^2 + z^2) - 34x - 23y + 50z - 42 = 0$, which is the required equation of the sphere.

4.4 Show that three mutually perpendicular tangent lines can be drawn to the sphere $x^2 + y^2 + z^2 = r^2$ from any point on the sphere $2(x^2 + y^2 + z^2) = 3r^2$.

(2013 : 15 Marks)

Solution:

Let (α, β, γ) be any point. Equation of enveloping cone from this point to sphere

$$x^2 + y^2 + z^2 = r^2 \quad \text{...(i)}$$

is

$$SS_1 = T_1^2$$

$\Rightarrow (x^2 + y^2 + z^2 - r^2)(\alpha^2 + \beta^2 + \gamma^2 - r^2) = (\alpha x + \beta y + \gamma z - r^2)^2$

This cone will have three mutually perpendicular generators if coefficient of x^2 + coefficient of y^2 + coefficient of $z^2 = 0$.

i.e.,
$$a + b + c = 0$$
$$\Rightarrow \quad (\beta^2 + \gamma^2 - r^2) + (\alpha^2 + \gamma^2 - r^2) + (\alpha^2 + \beta^2 - r^2) = 0$$
$$\Rightarrow \quad 2(\alpha^2 + \beta^2 + \gamma^2) = 3r^2$$

Since this is also the condition that three tangent lines from (α, β, γ) to sphere are mutually perpendicular, so locus of (α, β, γ) is

$$2(x^2 + y^2 + z^2) = 3r^2$$

4.5 Find the co-ordinates of the points on the sphere $x^2 + y^2 + z^2 - 4x + 2y = 4$, the tangent planes at which are paralle to the plane $2x - y + 2z = 1$.

(2014 : 10 Marks)

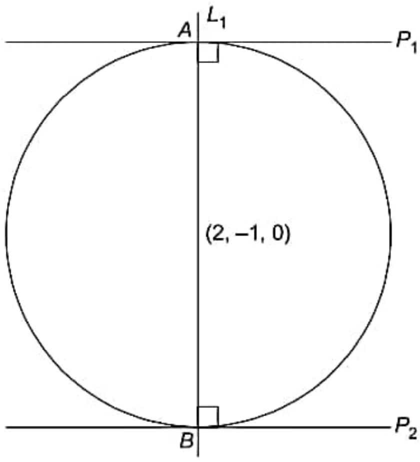
Solution:

Let, the equation of planes P_1 and P_2 parallel to

$$2x - y + 2z = 1$$
$$2x - y + 2z + \lambda = 0$$

Now, of
$$2x - y + 2z + \lambda = 0$$

be tangent to sphere length perpendicular to P_1 and P_2 = radius of sphere.



$$\left|\frac{2(2) - 1(-1) + 2(0) + \lambda}{\sqrt{4+1+4}}\right| = 3$$

So,
$$\lambda = 14, -4$$

Now, to find points of constant of tangent plane, P_1, P_2 and sphere (point A and B in diagram) equation of line 'L_1' normal to tangent plane and passing through centre (2, –1, 0) is

$$\frac{x-2}{2} = \frac{y+1}{-1} = \frac{z-0}{2} = r$$

For A and $B \Rightarrow r = \pm B$

So,
$$\frac{x-2}{2/3} = \frac{y+1}{-1/3} = \frac{z}{2/3} = \pm 3$$

Or
$$(x, y, z) \equiv (4, -2, 2), (0, 0, -2)$$

4.6 For what positive value of a, the plane $ax - 2y + z + 12 = 0$ touches the sphere $x^2 + y^2 + z^2 - 2x - 4y + 2z - 3 = 0$ and hence find the point of contact.

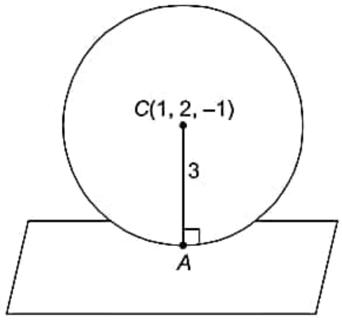
(2015 : 10 Marks)

Solution:

Plane : $ax - 2y + z + 12 = 0$ ...(i)

$x^2 + y^2 + z^2 - 2x - 4y + 2z - 3 = 0$ ...(ii)

Centre $(1, 2, -1)$



$$\text{Radius} = \sqrt{1+(2)^2+(-1)^2-(-3)} = 3$$

Since plane is a tangent plane.

$$CA = \text{radius} = \left|\frac{a\cdot 1-2\cdot 2+(-1)+12}{\sqrt{a^2+4+1}}\right| = 3$$

$\Rightarrow$ $(a+7)^2 = 9(a^2+5)$

i.e., $a^2 + 14a + 49 = 9a^2 + 45$

$\Rightarrow$ $4a^2 - 7a - 2 = 0$

$\Rightarrow$ $4a^2 - 8a + a - 2 = 0$

$\Rightarrow$ $4a(a-2) + (a-2) = 0$

$\Rightarrow$ $(a-2)(4a+1) = 0$

$\Rightarrow$ $a = 2 \text{ or } a = -\frac{1}{4}$

Now, equation of straight line CA is

$$\frac{x-1}{2} = \frac{y-2}{-2} = \frac{z+1}{1}$$ (perpendicular to given plane and taking $a = 2$)

Any point on this line $(2t + 1, -2t + 2, t - 1)$...(*)

It satisfies the equation of plane

$\therefore$ $2(2t+1) - 2(-2t+2) + (t-1) + 12 = 0$

$(4t + 4t + t) + 2 - 4 - 1 + 12 = 0$

$\Rightarrow$ $t = -1$

$\therefore$ Point of contact : $(-1, 4, -2)$ ...(from (*))

4.7 Find the equation of the sphere which passes through the circle $x^2 + y^2 = 4$; $z = 0$ and is cut by the plane $x + 2y + 2z = 0$ in a circle of radius 3.

(2016 : 10 Marks)

Solution:

Let the equation of sphere be

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ ...(i)

It passes through $x^2 + y^2 = 4$, $z = 0$

$z = 0 \Rightarrow x^2 + y^2 + 2ux + 2vy + d = 0$

$\Rightarrow$ $u = 0, v = 0, d = -4$

$\therefore$ (i) becomes

$x^2 + y^2 + z^2 + 2wz - 4 = 0$ ...(ii)

Plane, $x + 2y + 2z = 0$ cut the above sphere in the radius of 3.

$$OA^2 + AB^2 = OB^2$$

$$\left[\frac{0+2(0)+2(-W)}{\sqrt{1+4+4}}\right]^2 + (3)^2 = \left(\sqrt{0+0+W^2-(-4)}^2\right)$$

$$\frac{4W^2}{9} + 9 = W^2 + 4$$

$$\Rightarrow \qquad \frac{5W^2}{9} = 5 \Rightarrow W = \pm 3$$

Hence, the required equation of sphere

$$x^2 + y^2 + z^2 \pm 6z - 4 = 0$$

4.8 A plane passes through a fixed point (a, b, c) and cuts the axis at the points A, B, C respectively. Find the locus of the centre of the sphere which passes through the origin O and A, B, C.

(2017 : 15 Marks)

Solution:

Let points $A(x_1, 0, 0)$, $B(0, y_1, 0)$, $C(0, 0, z_1)$.

Eqn. of sphere is $x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$

As it passes through $O(0, 0, 0)$, A, B and C.

$$\therefore \qquad d = 0,\ u = -\frac{x_1}{2},\ v = -\frac{y_1}{2},\ w = -\frac{z_1}{2}$$

Centre of sphere, $\quad C(-u, -v, -w) \equiv C(\alpha, \beta, \gamma)$

i.e., $\quad C\left(\frac{x_1}{2}, \frac{y_1}{2}, \frac{z_2}{2}\right) =$ locus to be found.

Equation of plane through A, B, C is

$$\frac{x}{x_1} + \frac{y}{y_1} + \frac{z}{z_1} = 1$$

As it passes through fixed point (a, b, c)

$$\therefore \qquad \frac{a}{x_1} + \frac{b}{y_1} + \frac{c}{z_1} = 1$$

$$\text{i.e.,} \qquad \frac{a}{2\alpha} + \frac{b}{2\beta} + \frac{c}{2\gamma} = 1$$

Hence, the required locus of $C(\alpha, \beta, \gamma)$ is $\left[\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2\right]$.

4.9 Show that the plane $2x - 2y + z + 12 = 0$ touches the sphere $x^2 + y^2 + z^2 - 2x - 4y + 2z - 3 = 0$. Find the point of contact.

(2017 : 10 Marks)

Solution:

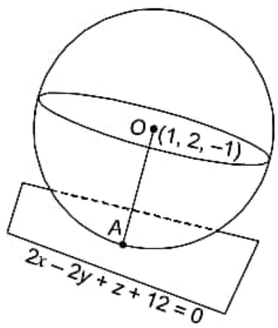
Centre of sphere, $O(1, 2, -1)$

$$\text{Radius} = \sqrt{1+4+1-(-3)} = 3$$

Perpendicular distance of point O from plane

$$d = \frac{2(1) - 2(2) + (-1) + 12}{\sqrt{4+4+1}}$$

$$= \frac{9}{3} = 3$$



which is equal to radius, hence plane touches the sphere, at A.

A line through $O(1, 2, -1)$ and perpendicular to given plane is

$$\frac{x-1}{2} = \frac{y-2}{-2} = \frac{z+1}{1}$$

A general point on it, $(2r + 1, -2r + 2, r - 1)$. If this lies on the plane, then

$$2(2r+1)-2(-2r+2)+(r-1)+12 = 0$$
$$9r+9 = 0 \Rightarrow r = -1$$

$\therefore$ Point of contact is $(-1, 4, -2)$.

4.10 Find the equation of sphere in xyz plane passing thrugh the points (0, 0, 0), (0, 1, –1), (–1, 2, 0) and (1, 2, 3).

(2018 : 12 Marks)

Solution:

Let the equation of sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d = 0$$

Since, it passes through (0, 0, 0), $\therefore \quad d = 0$

Also, (0, 1, –1), (–1, 2, 0) and (1, 2, 3) lie on this sphere. So, they satisfy equation of sphere.

i.e.,
$$v-w = -1 \qquad \text{...(i)}$$
$$2u-4v = 5 \qquad \text{...(ii)}$$
$$u+2v+3w = -7 \qquad \text{...(iii)}$$

Solving (i), (ii) and (iii), we get

$$v = \frac{-25}{14},\ w = \frac{-11}{14},\ u = \frac{-15}{14}$$

$\therefore$ Equation of given sphere is

$$x^2+y^2+z^2-\frac{15}{7}x-\frac{25}{7}y-\frac{11}{7}z = 0$$

or $\quad 7(x^2+y^2+z^2)-15x-25y-11z = 0$

4.11 The plane $x+2y+3z = 12$ cuts the axes of coordinates in A, B, C. Find the equations of the circle circumscribing the triangle ABC.

(2019 : 10 Marks)

Solution:

The given plane $\qquad x+2y+3z = 12 \qquad$ ...(1)

meets the x-axis, i.e., $y = 0$, $z = 0$ in the point A whose coordinates are (12, 0, 0).

Similarly, the co-ordinates of B and C where the given plane meets y-axis are (0, 6, 0) and z-axis are (0, 0, 4).

Thus, the point A, B, C are (12, 0, 0), (0, 6, 0) and (0, 0, 4) respectively.

Let the equation of the circle circumscribing the triangle ABC be

$$x^2+y^2+z^2+2ux+2vy+2wz+d = 0 \qquad \text{...(2)}$$

If it passes through A, B, C then we have

$$(12)^2+2u(12)+d = 0 \Rightarrow 24u+d+144 = 0 \qquad \text{...(3)}$$
$$(6)^2+2v(6)+d = 0 \Rightarrow 12v+d+36 = 0 \qquad \text{...(4)}$$
and
$$(4)^2+2w(4)+d = 0 \Rightarrow 8w+d+16 = 0 \qquad \text{...(5)}$$

From (3), (4) and (5), we get

$$2u = -\left[12-\left(\frac{d}{12}\right)\right]$$
$$2v = -\left[6-\left(\frac{d}{6}\right)\right] \text{ and}$$
$$2w = -\left[4-\left(\frac{d}{4}\right)\right]$$

Substituting these values of 2u, 2v, 2w in (2), we get the required equation as

$$x^2+y^2+z^2-\left[12-\left(\frac{d}{12}\right)\right]x-\left[6-\left(\frac{d}{6}\right)\right]y$$

$$-\left[4-\left(\frac{d}{4}\right)\right]z+d = 0$$

where i can take vany value. Hence the result.

5. Cone and its Properties

5.1 Prove that the normals from the point (α, β, γ) to the paraboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2z$ be on the cone

$$\frac{\alpha}{x-\alpha}-\frac{\beta}{y-\beta}+\frac{a^2-b^2}{z-\gamma} = 0$$

(Note : There is an error in the question of (+) sign instead of (–) before second term.)

(2009 : 20 Marks)

Solution:

Appraoch : From the general equation of normal passing through a point (α, β, γ) eliminate the direction cosines.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 2z \qquad \text{...(i)}$$

is given equation of paraboloid equation of tangent plane to paraboloid at (f, g, h) is

$$\frac{fx}{a^2}+\frac{gy}{b^2} = (z+h)$$

$\therefore$ Normal to paraboloid at (f, g, h) has direction cosines $\left(\frac{f}{a^2},\frac{g}{b^2},-1\right)$ and the equation of normal is

$$\frac{a^2(x-f)}{f} = \frac{b^2(y-g)}{g}=\frac{z-h}{-1}$$

It passes through a point (α, β, γ) if

$$\frac{a^2(\alpha-f)}{f} = \frac{b^2(\beta-g)}{g}=\frac{\gamma-h}{-1}=r \text{ (let)}$$

$$\Rightarrow \quad f = \frac{a^2\alpha}{a^2+r};\ g=\frac{b^2\beta}{b^2+r};\ h=\gamma+r$$

Now let any normal through (α, β, γ) be

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m}=\frac{z-y}{n} \qquad \text{...(ii)}$$

Then,
$$\frac{l}{f/a^2} = \frac{m}{g/b^2}=\frac{n}{-1} \text{ (any normal must have such d.c.'s)}$$

$$\Rightarrow \quad \frac{l(a^2+r)}{\alpha} = \frac{m(b^2+r)}{\beta}=\frac{n}{-1}$$

$$\Rightarrow \quad \frac{n}{-1} = \frac{a^2-b^2}{\frac{\alpha}{l}-\frac{\beta}{m}}$$

$$x^2+y^2+z^2-\left[12-\left(\frac{d}{12}\right)\right]x-\left[6-\left(\frac{d}{6}\right)\right]y$$

$$-\left[4-\left(\frac{d}{4}\right)\right]z+d = 0$$

where i can take vany value. Hence the result.

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Appraoch : From the general equation of normal passing through a point (α, β, γ) eliminate the direction cosines.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 2z \quad \text{...(i)}$$

is given equation of paraboloid equation of tangent plane to paraboloid at (f, g, h) is

$$\frac{fx}{a^2}+\frac{gy}{b^2} = (z+h)$$

$\therefore$ Normal to paraboloid at (f, g, h) has direction cosines $\left(\frac{f}{a^2}, \frac{g}{b^2}, -1\right)$ and the equation of normal is

$$\frac{a^2(x-f)}{f} = \frac{b^2(y-g)}{g} = \frac{z-h}{-1}$$

It passes through a point (α, β, γ) if

$$\frac{a^2(\alpha-f)}{f} = \frac{b^2(\beta-g)}{g} = \frac{\gamma-h}{-1} = r \text{ (let)}$$

$$\Rightarrow \quad f = \frac{a^2\alpha}{a^2+r};\ g = \frac{b^2\beta}{b^2+r};\ h = \gamma + r$$

Now let any normal through (α, β, γ) be

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \quad \text{...(ii)}$$

Then,

$$\frac{l}{f/a^2} = \frac{m}{g/b^2} = \frac{n}{-1} \text{ (any normal must have such d.c.'s)}$$

$$\Rightarrow \quad \frac{l(a^2+r)}{\alpha} = \frac{m(b^2+r)}{\beta} = \frac{n}{-1}$$

$$\Rightarrow \quad \frac{n}{-1} = \frac{a^2-b^2}{\frac{\alpha}{l}-\frac{\beta}{m}}$$

$$\Rightarrow \quad n\left(\frac{\alpha}{l}-\frac{\beta}{m}\right) = b^2-a^2$$

$\therefore$ Replacing l, m, n from (ii)

$$\frac{\alpha}{x-\alpha}-\frac{\beta}{y-\beta}+\frac{a^2-b^2}{z-\gamma} = 0$$

5.2 Show that the cone $yz + zx + xy = 0$ cuts the sphere $x^2+y^2+z^2=a^2$ in two equal circles, and find their area.

(2011 : 20 Marks)

Solution:

The given equations are

$$x^2+y^2+z^2 = a^2 \quad \text{...(i)}$$

and

$$yz+zx+xy = 0 \quad \text{...(ii)}$$

Multiply (ii) by 2 and add it to (i), we get

$$x^2+y^2+z^2+2(yz+2x+xy) = a^2$$

or

$$(x+y+z)^2 = x^2 \Rightarrow x+y+z = \pm a$$

$\therefore$ The equations of the required circles are

$$x^2+y^2+z^2 = a^2,\ x+y+z = a \quad \text{...(iii)}$$

and

$$x^2+y^2+z^2 = a^2,\ x+y+z = -a \quad \text{...(iv)}$$

Area of Circle (i)

Centre of the sphere $x^2+y^2+z^2=a^2$ is (0, 0, 0).

If l_1 is the length of perpendicular from the centre (0, 0, 0) of the sphere $x^2+y^2+z^2=a^2$ to the plane $x+y+z=a$, then

$$l_1 = \left|\frac{0+0+0-a}{\sqrt{1+1+1}}\right| = \frac{a}{\sqrt{3}}$$

$\therefore$ radius of the circle (iii) is

$$R_1 = \sqrt{a^2-l_1^2} = \sqrt{a^2-\frac{a^2}{3}} = \sqrt{\frac{2}{3}}a$$

$\therefore$ Area of circle (iii) $= \pi R_1^2$

$$= \pi\cdot\frac{2}{3}a^2 = \frac{2\pi}{3}a^2$$

Similarly area of circle (iv) is $\frac{2\pi}{3}a^2$.

5.3 A variable plane is parallel to the plane

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 0$$

and meets the axes in A, B, C respectively. Prove that the circle ABC lies on the cone

$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right) = 0$$

(2012 : 20 Marks)

Solution:

The equation of any plane parallel to the given plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$ is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = k \quad \text{...(i)}$$

It is given that the plane (i) meets the co-ordinate axes in A, B and C.

$\therefore$ A, B and C are $(ak, 0, 0)$, $(0, bk, 0)$ and $(0, 0, ck)$ respectively.

Equation of any sphere passing through the points O, A, B, C is

$$x^2 + y^2 + z^2 - kax - kby - kcz = 0$$

or $$x^2 + y^2 + z^2 - k(ax + by + cz) = 0 \quad \text{...(ii)}$$

The equation (i) and (ii) together represents the circle ABC.

Eliminating k from (i) and (ii), the required cone is :

$$x^2+y^2+z^2-\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)(ax+by+cz) = 0$$

or $$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right) = 0$$

5.4 **A cone has for its guiding curve the circle $x^2 + y^2 + 2ax + 2by = 0$, $z = 0$ and passes through a fixed point $(0, 0, C)$. If the section of the cone by the plane $y = 0$ is a rectangular hyperbola, prove that the vertex lies on the fixed circle.**

$$x^2 + y^2 + z^2 + 2ax + 2by = 0$$
$$2ax + 2by + cz = 0$$

(2013 : 15 Marks)

Solution:

Let $P(\alpha, \beta, \gamma)$ be the vertex.

Any line through P

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-y}{n} \quad \text{...(i)}$$

It passes through $z = 0$

$\Rightarrow$ $$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{-\gamma}{n}$$

$\Rightarrow$ $$x = \frac{-ly}{n}+\alpha y=\left(\frac{-my}{n}+\beta\right)$$

$\therefore$ Point of intersection with $z = 0$

$$\left(\alpha-\frac{ly}{n},\beta-\frac{my}{n},0\right)$$

It lies on $x^2 + y^2 + 2ax + 2by = 0$

$\Rightarrow$ $$\left(\alpha-\frac{ly}{n}\right)^2+\left(\beta-\frac{my}{n}\right)^2+2a\left(\alpha-\frac{ly}{n}\right)+2b\left(\beta-\frac{my}{n}\right) = 0$$

$\Rightarrow$ $$(\alpha n - ly)^2 + (n\alpha \div my)2 + 2an(\alpha n - ly) + 2bn(\beta n - my) = 0$$

Eliminating l, m, n from (i)

$$[\alpha(z-\gamma) - \gamma(x-\alpha)]^2 + [\beta(z-\gamma) - \gamma(y-\beta)]^2 + 2a(z-\gamma)[\alpha(z-\gamma) - (x-\alpha)\gamma] + 2b(z-\gamma)[\beta(z-\gamma) - (y-\beta)\gamma] = 0$$

$$\Rightarrow (\alpha z - \gamma x)^2 + (\beta z - \gamma y)^2 + 2a(z-\gamma)(\alpha z - x\gamma) + 2b(z-\gamma)(\beta z - y\gamma) = 0 \quad \text{...(ii)}$$

Intersection with $y = 0$ of (ii)

$$(\alpha z - \gamma x)^2 + (\beta z)^2 + 2a(z-\gamma)(\alpha z - \gamma x) + 2b(z-\gamma)(\beta z) = 0$$

This is rectangular hyperbola iff

Coefficient of x^2 + Coefficient of $z^2 = 0$

$\Rightarrow$ $$\gamma^2 + \alpha^2 + \beta^2 + 2a\alpha + 2b\beta = 0 \quad \text{...(iii)}$$

(ii) passes through fixed point $(0, 0, c)$

$$\therefore (\alpha c)^2 + (\beta c)^2 + 2a(c-\gamma)\gamma c + 2b(c-\gamma)\beta c = 0$$
$$\Rightarrow (\alpha^2 + \beta^2 + 2a\alpha + 2b\beta)c^2 - 2a\alpha\gamma c - 2b\beta\gamma c = 0 \quad \text{...(iv)}$$

Using (ii), (iii) & (iv) are equivalent to

$$-\gamma^2c^2 - 2(a\alpha + b\beta)\gamma c = 0$$
$$\gamma c(2a\alpha + 2b\beta + c\gamma) = 0$$

$\Rightarrow$ $$(2a\alpha + 2b\beta + c\gamma) = 0 \quad \text{...(v)}$$

as $c\gamma$ is not identically zero.

$\therefore$ (iii) and (iv) are required conditions.

Locus of $P(\alpha, \beta, \gamma)$ is

$$x^2 + y^2 + z^2 + 2ax + 2by = 0$$
$$2ax + 2by + cz = 0$$

5.5 **Examine whether the plane $x + y + z = 0$ cuts the cone $xy + zx + yz = 0$ in perpendicular lines.**

(2014 : 10 Marks)

Solution:

From the equation of plane and the cone it is clear that the lines of intensities passes through origin.

Let equation of lines be

$$\frac{x}{a} = \frac{y}{b}=\frac{z}{c} \quad \text{...(i)}$$

(i) must satisfy equation of plane and cone.

So, $$a + b + c = 0 \quad \text{...(ii)}$$

and $$ab + bc + ca = 0 \quad \text{...(iii)}$$

From (ii) and (iii)

$$ab + (a + b) \times -(a + b) = 0$$

$\Rightarrow$ $$(a + b)^2 - ab = 0$$

$\Rightarrow$ $$a^2 + b^2 + ab = 0$$

$\Rightarrow$ $$\left(\frac{a}{b}\right)^2+\left(\frac{a}{b}\right)+1 = 0 \quad \text{...(iv)}$$

$\therefore$ $$\frac{a_1a_2}{b_1b_2} = 1;\ \text{similarly}\ \frac{a_1a_2}{c_1c_2} = 1$$

$\therefore$ $$\Sigma a_1a_2 = 3a_1a_2 = 3b_1b_2 = 3c_1c_2$$

So, only those lines of intersection which are in yz, xz or xy planes will be perpendicular.

5.6 **Prove that the equation $ax^2 + by^2 + cz^2 + 2ux + 2vy + 2wz + d = 0$ represents a cone if**

$$\frac{u^2}{a}+\frac{v^2}{b}+\frac{w^2}{c} = d.$$

(2014 : 10 Marks)

Solution:

Let $$F(x, y, z, t) = ax^2 + by^2 + cz^2 + 2uxt + 2vyt + 2wzt + dt = 0$$

$\therefore$ $$\frac{\partial F}{\partial x} = 0 \text{ for } t = 1 \text{ gives}$$

$$2ax + 2u = 0 \Rightarrow x = -\frac{u}{a} \quad \text{...(i)}$$

Similarly $\frac{\partial F}{\partial y} = 0$ for t = 1 gives $y = -\frac{v}{b}$ ...(ii)

$$\frac{\partial F}{\partial z} = 0 \text{ for } t = 1 \text{ gives } z = -\frac{w}{c} \qquad \text{...(iii)}$$

and $\frac{\partial F}{\partial t} = 0$ for $t = 1$ gives $ux + vy + wz + d = 0$ ...(iv)

Substituting the values x, y, z from (i), (ii), (iii) in (iv), we get the required condition as

$$u\left(-\frac{u}{a}\right)+v\left(-\frac{v}{b}\right)+w\left(-\frac{w}{c}\right)+d = 0$$

$$\Rightarrow \quad \frac{u^2}{a}+\frac{v^2}{b}+\frac{w^2}{c} = d$$

which is the required result.

5.7 **Show that the lines drawn from the origin parallel to the normals to the central coincoid $ax^2 + by^2 + cz^2 = 1$ at its points of intersection with the plane $lx + my + nz = p$ generate the cone**

$$p^2\left(\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}\right) = \left(\frac{lx}{a}+\frac{my}{b}+\frac{nz}{c}\right)^2$$

(2014 : 15 Marks)

Solution:

Let (α, β, γ) be the point of intersection of the given conicoid and the given plane, then we have

$$a\alpha^2 + b\beta^2 + c\gamma^2 = 1 \qquad \text{...(i)}$$

and

$$l\alpha + m\beta + n\gamma = p \qquad \text{...(ii)}$$

Also, the equations of the normals to the given conicoid at (α, β, γ) are

$$\frac{x-\alpha}{a\alpha} = \frac{y-\beta}{b\beta} = \frac{z-\gamma}{c\gamma}$$

∴ The equations of the line thrugh the origin parallel to this line are

$$\frac{x}{a\alpha} = \frac{y}{b\beta} = \frac{z}{c\gamma} \qquad \text{...(iii)}$$

From (i) and (iii), we have

$$a\alpha^2 + b\beta^2 + c\gamma^2 = \left(\frac{l\alpha+m\beta+n\gamma}{P}\right)^2$$

$$\Rightarrow \quad P^2(a\alpha^2 + b\beta^2 + c\gamma^2) = (l\alpha + m\beta + n\gamma)^2$$

$$\Rightarrow \quad P^2\left(\frac{(a\alpha)^2}{a}+\frac{(b\beta)^2}{b}+\frac{(c\gamma)^2}{c}\right) = \left[\frac{l(a\alpha)}{a}+\frac{m(b\beta)}{b}+\frac{n(c\gamma)}{c}\right]^2$$

$$\Rightarrow \quad P^2\left[\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}\right] = \left(\frac{lx}{a}+\frac{my}{b}+\frac{nz}{c}\right)^2$$

from (iii), eliminating α, β, γ

Hence, the line (iii) generates the above cone. Henve Proved.

5.8 **If $6x = 3y = 2z$ represents one of the three mutually perpendicular generators of the cone $5yz - 8zx - 3xy = 0$ then obtain the equations of the other two generators.**

(2015 : 13 Marks)

Solution:

If $\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$ is one of the three mutually perpendicular generators, then it is normal to the plane thrugh the

vertex cutting the cone in two perpendicular generators and therefore the equation of the plane is

$$x + 2y + 3z = 0 \qquad \text{...(i)}$$

Now, we are to find the lines of intersection of this plane and the given cone.

Let one of the line be

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$$

$$\therefore \quad l + 2m + 3n = 0$$

and

$$5mn - 8nl \; 3lm = 0$$

Eliminating l between these,

$$5mn - (8n + 3n)[-(2m + 3n)] = 0 \qquad \text{...(ii)}$$

$$\Rightarrow \quad 24n^2 + 30mn + 6m^2 = 0$$

$$\Rightarrow \quad m^2 + 5mn + 4n^2 = 0 \text{ or } (m + n)(m + 4n) = 0$$

If $m = -n$, from (ii), $l = -n$

$$\therefore \quad \frac{l}{1} = \frac{m}{1} = \frac{n}{-1}$$

If $m = -4n$, from (ii), $l = 5n$

$$\therefore \quad \frac{l}{5} = \frac{m}{-4} = \frac{n}{1}$$

Hence, other two generators are :

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{-1} \text{ and } \frac{x}{5} = \frac{y}{-4} = \frac{z}{1}$$

And evidently these three form a set of mutually perpendicular generators.

5.9 **Show that the cone $3yz - 2zx - 2xy = 0$ has an infinite set of three mutually perpendicular generators. If $\frac{x}{1} = \frac{y}{1} = \frac{z}{2}$ is a generator belonging to one such set, find the other two.**

(2016 : 10 Marks)

Solution:

Condition for a cone to have three mutually perpendicular generators :

Cone, $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ has an infinite set of three mutually perpendicular generators, if

$$a + b + c = 0$$

Here, $a = 0$, $b = 0$, $c = 0$ ∴ $a + b + c = 0$

Part 2 :

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{2} \qquad \text{...(i)}$$

is one of three mutually perpendicular generators of cone.

$$3yz - 2zx - 2xy = 0 \qquad \text{...(ii)}$$

Let a line perpendicular to (i) be

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \qquad \text{...(iii)}$$

$$\therefore \quad l + m + 2n = 0 \qquad \text{...(iv)}$$

(iii) is generator of (ii)

$$\Rightarrow \quad 3mn - 2nl - 2lm = 0 \qquad \text{...(v)}$$

Eliminating l between (iv) and (v)

$$3mn - (n + m)[-(m + 2n)] = 0$$

or

$$4n^2 + 9m + 2m^2 = 0$$

or

$$(2m + n)(m + 4n) = 0$$

$$m = -4n \text{ or } m = -\frac{n}{2}$$

or

If $m = -4n$, from (iv), $l = 2n \Rightarrow \frac{l}{2} = \frac{m}{-4} = \frac{n}{1}$

If $m = -\frac{n}{2}$, from (iv), $2l = -3n \Rightarrow \frac{l}{3} = \frac{m}{1} = \frac{n}{-2}$

Hence, the other two generators are

$$\frac{x}{2} = \frac{y}{-4} = \frac{z}{1} \text{ and } \frac{x}{3} = \frac{y}{1} = \frac{z}{-2}$$

5.10 Find the locus of the point of intersection of three mutually perpendicular tangent planes to the conicoid $ax^2 + by^2 + cz^2 = 1$.

(2016 : 15 Marks)

Solution:

$$ax^2 + by^2 + cz2 = 1 \quad \text{...(i)}$$

The three equations of tangent planes to conicoid (i) are

$$l_1x + m_1y + n_1z = \left(\frac{l_1^2}{a} + \frac{m_1^2}{b} + \frac{n_1^2}{c}\right)^{1/2} \quad \text{...(ii)}$$

$$l_2x + m_2y + n_2z = \left(\frac{l_2^2}{a} + \frac{m_2^2}{b} + \frac{n_2^2}{c}\right)^{1/2} \quad \text{...(iii)}$$

$$l_3x + m_3y + n_3z = \left(\frac{l_3^2}{a} + \frac{m_3^2}{b} + \frac{n_3^2}{c}\right)^{1/2} \quad \text{...(iv)}$$

If the above three planes are at right angles to each other, then

$$l_1^2 + l_2^2 + l_3^2 = 1$$
$$m_1^2 + m_2^2 + m_3^2 = 1$$
$$n_1^2 + n_2^2 + n_3^2 = 1$$

and

$$l_1m_1 + l_2m_2 + l_3m_3 = 0$$
$$m_1n_1 + m_2n_2 + m_3n_3 = 0$$
$$n_1l_1 + n_2l_2 + n_3l_3 = 0 \quad \text{...(v)}$$

To find the locus of the point of intersection of three planes, we need to eliminate l_i, m_i, n_i from (ii), (iii), (iv) with the help of (v).

Squaring each of (ii), (iii), (iv) and adding

$$x^2\Sigma l_1^2 + y^2\Sigma m_1^2 + z^2\Sigma n_1^2 + 2xy\Sigma l_1m_1 + 2yz\Sigma m_1n_1 + 2zx\Sigma n_1l_1 = \frac{1}{a}\Sigma l_1^2 + \frac{1}{b}\Sigma m_1^2 + \frac{1}{c}\Sigma n_1^2$$

or

$$x^2 + y^2 + z^2 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

which is also called the **directors sphere.**

5.11 Find the equation of the tangent plane at point (1, 1, 1) to the coincoid $3x^2 - y^2 = 2z$.

(2017 : 10 Marks)

Solution:

The equation of the tangent plane to the coincoid, $ax^2 + by^2 = 2cz$ at point (α, β, γ)

$$a\alpha x + b\beta y = c(z + \gamma)$$

Here, equation of conicoid,

$$3x^2 - y^2 = 2z \quad \text{...(i)}$$

Tangent plane at (1, 1, 1)

$$3.1x - 1.1y = \frac{2}{2}(z+1)$$

i.e.,

$$3x - y - z = 1$$

Method II : Without using formula

Any line through point (1, 1, 1) is

$$\frac{x-1}{l} = \frac{y-1}{m} = \frac{z-1}{n} = r$$

Any point on it $(lr + 1, mr + 1, nr + 1)$. ...(ii)

If this line cut the paraboloid (i) at this point then it satisfies the equation of paraboloid (i).

$$3(lr + 1)^2 - (mr + 1)^2 = 2(nr + 1)$$
$$r^2(3l^2 - m^2) + 2r(3l - m - n) + 2 = 2$$

Line touches paraboloid if both values of r given by above equation are zero, for which $3l - m - n = 0$. The locus of all such lines gives the tangent plane. Eliminating l, m, n with help of eqn. (ii)

$$3(x - 1) - (y - 1) - (z - 1) = 0$$

$\Rightarrow$

i.e.,

$$3x - y - z = 1$$

5.12 Find the locus of the point of intersection of three mutually perpendicular tangent planes to $ax^2 + by^2 + cz^2 = 1$.

(2017 : 10 Marks)

Solution:

$$ax^2 + by^2 + cz2 = 1 \quad \text{...(i)}$$

The three equations of tangent planes to conicoid (i) are

$$l_1x + m_1y + n_1z = \left(\frac{l_1^2}{a} + \frac{m_1^2}{b} + \frac{n_1^2}{c}\right)^{1/2} \quad \text{...(ii)}$$

$$l_2x + m_2y + n_2z = \left(\frac{l_2^2}{a} + \frac{m_2^2}{b} + \frac{n_2^2}{c}\right)^{1/2} \quad \text{...(iii)}$$

$$l_3x + m_3y + n_3z = \left(\frac{l_3^2}{a} + \frac{m_3^2}{b} + \frac{n_3^2}{c}\right)^{1/2} \quad \text{...(iv)}$$

If the above three planes are at right angles to each other, then

$$l_1^2 + l_2^2 + l_3^2 = 1$$
$$m_1^2 + m_2^2 + m_3^2 = 1$$
$$n_1^2 + n_2^2 + n_3^2 = 1$$

and

$$l_1m_1 + l_2m_2 + l_3m_3 = 0$$
$$m_1n_1 + m_2n_2 + m_3n_3 = 0 \quad \text{...(v)}$$
$$n_1l_1 + n_2l_2 + n_3l_3 = 0$$

To find the locus of the point of intersection of three planes, we need to eliminate l_i, m_i, n_i from (ii), (iii), (iv) with the help of (v).

Squaring each of (ii), (iii), (iv) and adding

$$x^2\Sigma l_1^2 + y^2\Sigma m_1^2 + z^2\Sigma n_1^2 + 2xy\Sigma l_1 m_1 + 2yz\Sigma m_1 n_1 + 2zx\Sigma n_1 l_1 = \frac{1}{a}\Sigma l_1^2 + \frac{1}{b}\Sigma m_1^2 + \frac{1}{c}\Sigma n_1^2$$

or
$$x^2 + y^2 + z^2 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$$

which is also called the directors sphere.

5.13 Find the equation of the cone with (0, 0, 1) as the vertex and $2x^2 - y^2 = 4$, $z = 0$ as the guiding curve.

(2018 : 13 Marks)

Solution:

Given (0, 0, 1) is the vertex of cone.

Equation of line thrugh this vertex is

$$L_1 : \frac{x-0}{l} = \frac{y-0}{m} = \frac{z-1}{n} \quad ...(i)$$

Now, L_1 meets guiding curve.

$$\therefore \quad \frac{x}{l} = \frac{y}{m} = \frac{0-1}{n} \Rightarrow x = \frac{-l}{n}, y = \frac{-m}{n}, z = 0$$

Putting these values in equation of guiding curve, we get

$$2\left(\frac{-l}{n}\right)^2 - \left(\frac{-m}{n}\right)^2 = 4$$

$$\Rightarrow \quad \frac{2l^2}{n^2} - \frac{m^2}{n^2} = 4$$

From (i), putting values of $\frac{l}{n}$ and $\frac{m}{n}$, we have

$$2\left(\frac{x}{z-1}\right)^2 - \left(\frac{y}{z-1}\right)^2 = 4$$

$$\Rightarrow \quad 2x^2 - y^2 = 4(z^2 - 2z + 1)$$

$$\Rightarrow \quad 2x^2 - y^2 - 4z^2 + 8z = 4 \text{ is the equation of given cone.}$$

5.14 Prove that the plane $z = 0$ cuts the enveloping cone of the sphere $x^2 + y^2 + z^2 = 11$ which has the vertex at (2, 4, 1) in a rectangular hyperbola.

(2019 : 10 Marks)

Solution:

The equation of the sphere is

$$x^2 + y^2 + z^2 - 11 = 0$$

and the vertex is (2, 4, 1).

Here,
$$S = x^2 + y^2 + z^2 - 11, x_2 = 2, y_1 = 4, z_1 = 1$$
$$S_1 = x_1^2 + y_1^2 + z_1^2 - 11 = 4 + 16 + 1 - 11$$

i.e.,
$$S_1 = x_1^2 + y_1^2 + z_1^2 - 11 = 10$$

and
$$T = xx_1 + yy_1 + zz_1 - 11$$
$$= 2x + 4y + z - 11$$

$\therefore$ Equation of the enveloping cone is

$$(x^2 + y^2 + z^2 - 11)(10) = (2x + 4y + z - 11)^2 \text{ using } SS_1 = T^2$$

$$\Rightarrow \quad 10(x^2 + y^2 + z^2 - 11) - (2x + 4y + z - 11)^2 = 0$$

This meets the plane $z = 0$ in the curve

$$10(x^2 + y^2 - 11) - (2x + 4y - 11)^2 = 0$$

$$10(x^2 + y^2) - (4x^2 + 16y^2 + ...) = 0$$

This represents a rectangular hyperbola in the XY-plane if co-efficient of x^2 + coefficient of $y^2 = 0$.

if
$$(10 - 4) + (10 - 16) = 0 \text{ which is true. Hence the result.}$$

6. Paraboloid and its Properties

6.1 Show that the plane $3x + 3y + 7z + \frac{5}{2} = 0$ touches the paraboloid $3x^2 + 4y^2 = 10z$ and find the point of contact.

(2010 : 20 Marks)

Solution:

Given plane
$$P \equiv 3x + 3y + 7z + \frac{5}{2} = 0$$

and
$$\text{Paraboloid} \equiv 3x^2 + 4y^2 = 10z \quad ...(1)$$

Now, let point of contact be (x_1, y_1, z_1).

$\therefore$ If given plane touches paraboloid at this point of contact, then the plane should be tangent plane to the paraboloid.

Now, equation of tangent plane at (x_1, y_1, z_1) is

$$3xx_1 + 4yy_1 = \frac{10(z + z_1)}{2}$$

$$\Rightarrow \quad 3xx_1 + 4yy_1 - 5z - 5z_1 = 0 \quad ...(2)$$

Comparing (1) and (2), we get

$$\frac{3}{3x_1} = \frac{4}{4y_1} = \frac{7}{-5} = \frac{5/2}{-521}$$

$$\therefore \quad \frac{3}{3x_1} = \frac{-7}{5} \Rightarrow x_1 = \frac{-5}{7}$$

$$\frac{4}{4y_1} = \frac{7}{-5} \Rightarrow y_1 = \frac{-5}{7}$$

$$\frac{7}{-5} = \frac{5}{-5 \times 221} \Rightarrow z_1 = \frac{5}{14}$$

$$\therefore \quad \text{Point of contact } (x_1, y_1, z_1) \equiv \left(\frac{-5}{7}, \frac{-5}{7}, \frac{5}{14}\right)$$

6.2 Show that the locus of a point from which the three mutually perpendicular tangent lines can be drawn to the paraboloid $x^2 + y^2 + 2z = 0$ is

$$x^2 + y^2 + 4z = 1$$

(2012 : 20 Marks)

Solution:

Let $P(x_1, y_1, z_1)$ be the point from which three mutually perpendicular lines can be drawn to the paraboloid

$$x^2 + y^2 + 2z = 0 \quad ...(i)$$

Then, the enveloping cone of (i) with the vertex at $P(x_1, y_1, z_1)$ is

$$SS_1 = T^2 \quad ...(ii)$$

where
$$S = x^2 + y^2 + 2z$$
$$S_1 = x_1^2 + y_1^2 + 2z_1$$
$$T = xx_1 + yy_1 + z + z_1$$

$\therefore$ from (ii), we have

$$(x^2+y^2+2z)(x_1^2+y_1^2+2z_1) = (xx_1+yy_1+(z+z_1))^2$$

For three mutually perpendicular generators, coefficient of x^2 + coefficient of y^2 + coefficient of $z^2 = 0$

$$x_1^2+y_1^2+4z_1-1 = 0$$

or
$$x_1^2+y_1^2+4z_1 = 0$$

$\therefore$ Locus of (x_1, y_1, z_1) is

$$x^2+y^2+4z = 1$$

6.3 Two perpendicular tangent planes to the paraboloid $x^2+y^2=2z$ intersect in a straight line in the plane $x=0$. Obtain the curve to which this straight line touches.

(2015 : 13 Marks)

Solution:

Let the line of intersection of the two planes be :

$$my+nz = \lambda, x=0 \qquad \text{...(i)}$$

Since this lies on the plane $x=0$ (given).

$\therefore$ Equation of the plane through the line (i) is

$$(my+nz-\lambda)+kx = 0$$

or
$$kx+my+nz = \lambda \qquad \text{...(ii)}$$

If the plane (ii) touches the paraboloid, then

$$\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{2pn}{c} = 0 \text{ (condition)}$$

i.e.,
$$k^2+m^2+2\lambda n = 0 \qquad \text{...(iii)}$$

This being quadratic in k, gives two values of k, say k_1 and k_2 such that

$$k_1 \cdot k_2 = \frac{m^2+2\lambda n}{l} \qquad \text{...(iv)}$$

Also from (ii), the direction ratio's of the normal to the two tangent planes whose line of intersection is (ii) are k_1, m, n and k_2, m, n.

Also, as these two tangent planes are perpendicular

$$\therefore \quad k_1 \cdot k_2 + m \cdot m + n \cdot n = 0$$

$$\Rightarrow \quad (m^2+2\lambda n)+m^2+n^2 = 0 \qquad \text{\{from (iv)\}}$$

$$\Rightarrow \quad 2m^2+n^2+2\lambda n = 0 \qquad \text{...(v)}$$

Now, we are to prove that the line (i) touches a parabola (to be found). So, we are to find the envelope of (i) which satisfies the condition (v).

Eliminating λ between (i) and (v), the equations of the line of intersection of two tangent planes is :

$$2m^2+n^2+2(my+nz)n = 0, x=0$$

$$\Rightarrow \quad 2\left(\frac{m}{n}\right)^2+2y\left(\frac{m}{n}\right)+(1+2z) = 0, x=0$$

It is quadratic in $\frac{m}{n}$, so its envelope is given by :

$$B^2-4AC = 0, x=0$$

$$\Rightarrow \quad (2y)^2-4 \cdot 2(1+2z) = 0, x=0$$

$$\Rightarrow \quad y^2 = 2(2z+1), x=0$$

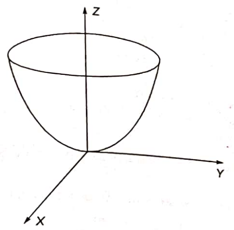
This is the required curve.

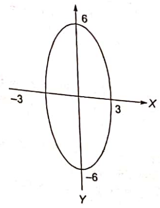
6.4 Find the volume of the solid above the xy-plane and directly below the portion of the elliptic paraboloid $x^2+\frac{y^2}{4}=z$ which is cut off by the plane $z=9$.

(2017 : 15 Marks)

Solution:

Equation of ϕ surface, cut off cut plane





$$x^2+\frac{y^2}{4} = 9; z=9$$

i.e.,
$$\frac{x^2}{9}+\frac{y^2}{36} = 1; z=9$$

Making the transformation,

$$x = 3r\cos\theta$$
$$y = 6r\sin\theta$$

$r: 0$ to 1; $\theta: 0$ to 2π

$$\frac{J(x,y)}{J(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} 3\cos\theta & -3r\sin\theta \\ 6\sin\theta & 6r\cos\theta \end{vmatrix}$$

$$= 18r$$

$$V = \iint_D z\,dx\,dy = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} (9r^2)18r\,dr\,d\theta \qquad \left(\because z = x^2+\frac{y^2}{4}\right)$$

$$= 9\times 18\int_0^{2\pi} d\theta \int_0^1 r^3 dr$$

$$= 9\times 18\times 2\pi \times \frac{1}{4} = 81\pi$$

6.5 Reduce the following equation to the standard form and hence determine the nature of the conicoid :

$$x^2+y^2+z^2-yz-zx-xy-3x-6y-9z+21=0$$

(2017 : 15 Marks)

Solution:

Comparing with

$$F(x, y, z) \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d = 0$$

The discriminating cubic is :

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \text{ or } \begin{vmatrix} 1-\lambda & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1-\lambda & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow$ $$4\lambda^3 - 12\lambda^2 + 9\lambda = 0 \text{ or } \lambda(2\lambda-3)^2 = 0$$

$\therefore$ $$\lambda = \frac{3}{2}, \frac{3}{2}, 0$$

As this discriminating cube has two roots equal and third root equal to zero, so it is either a paraboloid of revoltion or a right circular cylinder.

The d.r.'s of the axis are given by $al + hm + gn = 0$, $hl + bm + fn = 0$, $gl + fm + cn = 0$

i.e., $$l - \frac{m}{2} - \frac{n}{2} = 0,\ -\frac{l}{2} + m - \frac{n}{2} = 0,\ -\frac{l}{2} - \frac{m}{2} + n = 0$$

i.e., $$2l - m - n = 0,\ -l + 2m - n = 0,\ -l - m + 2n = 0$$

These gives: $$l = m = n = \frac{1}{\sqrt{3}}$$

Now, $$K = ul + vm + wn$$
$$= \left(-\frac{3}{2} + (-3) + \left(-\frac{9}{2}\right)\right)\frac{1}{\sqrt{3}} = -3\sqrt{3} \neq 0$$

$\therefore$ Reduced equation :
$$\lambda_1 x^2 + \lambda_2 y^2 + 2kz = 0$$
or $$x^2 + y^2 = 4\sqrt{3}z$$
which represents a paraboloid of revolution.

6.6 Find the equations to the generating lines of the paraboloid $(x + y + z)(2x + y - z) = 6z$ which pass through (1, 1, 1).

(2018 : 13 Marks)

Solution:

The given equation of paraboloid can be re-written in $\lambda - \mu$ form as
$$x + y + z = z\lambda,\ 2x + y - z = \frac{6}{\lambda} \quad \text{...(i)}$$
and $$x + y + z = \frac{6}{\mu},\ 2x + y - z = z\mu \quad \text{...(ii)}$$

Since these lines pass through (1, 1, 1), therefore
$$1 + 1 + 1 = 1\cdot\lambda \Rightarrow \lambda = 3$$

So for (1), the equation are
$$x + y + z = 3z \Rightarrow x + y - 2z = 0 \quad \text{...(iii)}$$
and $$2x + y - z = \frac{6}{3} \Rightarrow 2x + y - z = z \quad \text{...(iv)}$$

Similarly, for (ii), the equation are
$$1 + 1 + 1 = \frac{6}{\mu} \Rightarrow \mu = 2$$
i.e., $$x + y + z = 3 \quad \text{...(v)}$$
and $$2x + y - 3z = 0 \quad \text{...(vi)}$$

From (iii) and (iv), eqn. of line is symmetrical form
$$\frac{x-1}{1} = \frac{y-1}{-3} = \frac{z-1}{-1}$$

From (v) and (vi), eqn. of line in symmetrical form
$$\frac{x-1}{4} = \frac{y-1}{-5} = \frac{z-1}{1}$$

6.7 Prove that, in general, three normals can be drawn from a given point to the paraboloid $x^2 + y^2 = 2az$, but if the point lies on the surface
$$27a(x^2 + y^2) + 8(a - z)3 = 0$$
then two of the three normals coincide.

(2019 : 15 Marks)

Solution:

The equations of the normal at (x_1, y_1, z_1) to the paraboloid $x^2 + y^2 = 2az$ are
$$\frac{x - x_1}{x_1} = \frac{y - y_1}{y_1} = \frac{z - z_1}{-a}$$

This passes through a given point (α, β, γ) if
$$\frac{\alpha - x_1}{x_1} = \frac{\beta - y_1}{y_1} = \frac{\gamma - z_1}{z_1} = \lambda \text{ (say)}$$

These gives $$\alpha - x_1 = \lambda x_1 \Rightarrow x_1 = \frac{\alpha}{(1+\lambda)}$$

Similarly, $$y_1 = \frac{\beta}{(1+\lambda)},\ z_1 = \gamma + a\lambda \quad \text{...(1)}$$

Also, (x1, y1, z1) lies on the given paraboloid, so
$$x_1^2 + y_1^2 = 2az_1 \Rightarrow \left[\frac{\alpha}{1+\lambda}\right]^2 + \left[\frac{\beta}{1+\lambda}\right]^2 = 2a(\gamma + a\lambda) \quad \text{[from (1)]}$$

$\Rightarrow$ $$\alpha^2 + \beta^2 = 2a(\gamma + a\lambda)(1+\lambda)^2 \quad \text{...(2)}$$

This being a cubic in λ gives three values of l and so from (1) there are three points on the paraboloid normals at which pass through (α, β, γ).

The equation (2) can be rewritten as
$$f(\lambda) = 2a(1+\lambda)^2(\gamma + a\lambda) - (\alpha^2 + \beta^2) = 0 \quad \text{...(3)}$$

The condition that this equation has two equal roots is obtained by eliminating λ between $f(\lambda) = 0$ and $f'(\lambda) = 0$.

From (3), $f'(\lambda) = 0$ means $2a(1+\lambda)^2(a) + 4a(1+\lambda)(\gamma + a\lambda) = 0$
$$= a(1+\lambda) + 2(\gamma + a\lambda) = 0 \quad (\because 1 + \lambda \neq 0)$$
$$= (a + 2\gamma) + \lambda(3a) = 0$$

$\Rightarrow$ $$\lambda = \frac{-(a+2\gamma)}{(3a)}$$

Substituting this value of λ in (3), we get
$$2a\left[1 - \frac{a+2\gamma}{3a}\right]^2\left[\gamma - \frac{a(a+2\gamma)}{3a}\right] = \alpha^2 + \beta^2$$
$$2a[2(a-\gamma)]^2[a(\gamma - a)] = 27a^3(\alpha^2 + \beta^2)$$

$$27a(\alpha^2 + \beta^2) + 8(a - \gamma)^3 = 0$$

$\therefore$ Locus of the point (α, β, γ) is

$$27a(x^2 + y^2) + 8(a - z)^3 = 0. \text{ Hence, proved.}$$

7. Ellipsoid and its Properties

7.1 Three points P, Q, R are taken on the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$ so that the lines joining P, Q, R to the origin are mutually perpendicular. Prove that the plane PQR touches a fixed sphere.

(2011 : 20 Marks)

Solution:

Let the equation of the plane PQR be

$$lx + my + nz = 1 \quad \text{...(i)}$$

The equation of the given ellipsoid is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1 \quad \text{...(ii)}$$

The equation of the cone with vertex at (0, 0, 0) and the curve of intersection of (i) and the ellipsoid (ii) as the guiding curve is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = (lx + my + nz)^2 \quad \text{...(iii)}$$

If the cone (iii) has three mutually perpendicular generators then

Coefficient of x^2 + Coefficient of y^2 + Coefficient of $z^2 = 0$

$$\Rightarrow \quad \left(l^2-\frac{1}{a^2}\right)+\left(m^2-\frac{1}{b^2}\right)+\left(n^2-\frac{1}{c^2}\right) = 0$$

$$\Rightarrow \quad l^2 + m^2 + n^2 = \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{\lambda^2} \text{ (say)} \quad \text{...(iv)}$$

If the plane (i) touches the sphere $x^2 + y^2 + z^2 = \lambda^2$, then the length of the perpendicular from the centre (0, 0, 0) of the sphere to (i) must be equal to the radius λ of the sphere.

i.e.,
$$\frac{1}{\sqrt{l^2+m^2+n^2}} = \lambda$$

$$\Rightarrow \quad l^2 + m^2 + n^2 = \frac{1}{\lambda^2}, \text{ which is true by virtue of (iv).}$$

Hence, the plane (i) touches the sphere $x^2 + y^2 + z^2 = \lambda^2$.

7.2 Find the length of the normal chord through a point P of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$$

and prove that if it is equal to $4PG_3$, where G_3 is the point where the normal chord through P meets the xy-plane, then P lies on the cone

$$\frac{x^2}{a^6}(2c^2 - a^2)+\frac{y^2}{b^6}(2c^2 - b^2)+\frac{z^2}{c^4} = 0$$

(2019 : 15 Marks)

Solution:

Let P be (α, β, γ), then the equations of the normal to the given ellipsoid at $P(\alpha, \beta, \gamma)$ are

$$\frac{x-\alpha}{(P\alpha/a^2)} = \frac{y-\beta}{(P\beta/b^2)} = \frac{z-\gamma}{(P\gamma/c^2)} = \gamma \text{ (say)} \quad \text{...(1)}$$

where
$$\frac{1}{P^2} = \frac{\alpha^2}{a^4}+\frac{\beta^2}{b^4}+\frac{\gamma^2}{c^4} \quad \text{...(2)}$$

$\therefore$ The co-ordinates of any point Q on the normal (1) are $\left(\alpha+\frac{P\alpha}{a^2}\gamma, \beta+\frac{P\beta}{b^2}\gamma, \gamma+\frac{P\gamma}{c^2}\gamma\right)$ where g is the distance of Q from P.

If Q lies on the given ellipsoid i.e., PQ is the normal chord, then

$$\frac{1}{a^2}\left(\alpha+\frac{P\alpha}{a^2}\gamma\right)^2+\frac{1}{b^2}\left(\beta+\frac{P\beta}{b^2}\gamma\right)^2+\frac{1}{c^2}\left(\gamma+\frac{P\gamma}{c^2}\right)^2 = 1$$

$$= \gamma^2P^2\left(\frac{\alpha^2}{a^6}+\frac{\beta^2}{b^6}+\frac{\gamma^2}{c^6}\right)+2\gamma P\left(\frac{\alpha^2}{a^4}+\frac{\beta^2}{b^4}+\frac{\gamma^4}{c^4}\right)+\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}\right) = 1$$

$$= \gamma^2P^2\left(\frac{\alpha^2}{a^6}+\frac{\beta^2}{b^6}+\frac{\gamma^2}{c^6}\right)+2\gamma P\left(\frac{1}{P^2}\right) = 0$$

From (2) and $\Sigma\frac{\alpha^2}{a^2} = 1$ as $P(\alpha, \beta, \gamma)$ lies on the given coincoid.

$$\gamma = \frac{-2}{P^3\left(\frac{\alpha^2}{a^6}+\frac{\beta^2}{b^6}+\frac{\gamma^2}{c^6}\right)} = \text{length of normal chord } PQ \quad \text{...(3)}$$

Also, let the normal at $P(\alpha, \beta, \gamma)$ meets the coordinate planes viz. yz, zx and xy planes at G_1, G_2 and G_3 then putting $x = 0$, $y = 0$ and $z = 0$ in succession in the eqn. (1), we have respectively,

$$PG_1 = -\frac{a^2}{P}, PG_2 = -\frac{b^2}{P} \text{ and } PG_3 = \frac{c^2}{P} \quad \text{...(4)}$$

Given,
$$PQ = 4PG_3$$

$$PQ = 4\left(-\frac{c^2}{P}\right)$$

$$\Rightarrow \quad \frac{-2}{P^3\left(\frac{\alpha^2}{a^6}+\frac{\beta^2}{b^6}+\frac{\gamma^2}{c^6}\right)} = 4\left(-\frac{c^2}{P}\right)$$

$$\Rightarrow \quad 2c^2\left(\frac{\alpha^2}{a^6}+\frac{\beta^2}{b^6}+\frac{\gamma^2}{c^6}\right) = \frac{1}{P^2} = \frac{\alpha^2}{a^4}+\frac{\beta^2}{b^4}+\frac{\gamma^2}{c^4} \quad \text{...from (2)}$$

$$\Rightarrow \quad \frac{\alpha^2}{a^6}(2c^2 - a^2)+\frac{\beta^2}{b^6}(2c^2 - b^2)+\frac{\gamma^2}{c^6}(2c^2 - c^2) = 0$$

$\therefore$ The locus of $P(\alpha, \beta, \gamma)$ is

$$\frac{x^2}{a^6}(2c^2 - a^2)+\frac{y^2}{b^6}(2c^2 - b^2)+\frac{z^2}{c^4} = 0. \text{ Hence, Proved.}$$

8. Hyperboloid of One and Two Sheets and its Properties

8.1 Find the vertices of the skew quadrilateral formed by the four generators of the hyperboloid

$$\frac{x^2}{4}+y^2-z^2=49$$

passing through (10, 5, 1) and (14, 2, –2). **(2010 : 20 Marks)**

Solution:

Given, the equation of hyperboloid is

$$\frac{x^2}{4}+y^2-z^2 = 49.$$

It can be rewritten as

$$\left(\frac{x}{2}-z\right)\left(\frac{x}{2}+z\right) = (7-y)(7+y)$$

∴ The equation of two systems of generating lines are :

$$\left(\frac{x}{2}-z\right) = \lambda(7-y),\ \lambda\left(\frac{x}{2}+z\right) = (7+y) \quad \text{...(1)}$$

$$\left(\frac{x}{2}-z\right) = \mu(7+y),\ \mu\left(\frac{x}{2}+z\right) = (7-y) \quad \text{...(2)}$$

(1) and (2) pass through (10, 5, 1) and (14, 2, –2) for $\lambda = 2$, $\mu = \frac{1}{3}$ and $\lambda = \frac{9}{5}$, $\mu = 1$.

So, the two systems of generating lines are

$$\left(\frac{x}{2}-z\right) = 2(7-y),\ 2\left(\frac{x}{2}+z\right) = 7+y \quad \text{...(3)}$$

and

$$\left(\frac{x}{2}-z\right) = \frac{1}{3}(7+y),\ \frac{1}{3}\left(\frac{x}{2}+z\right) = 7-y \quad \text{...(4)}$$

$$\left(\frac{x}{2}-z\right) = \frac{9}{5}(7-y),\ \frac{9}{5}\left(\frac{x}{2}+2\right) = 7+y \quad \text{...(5)}$$

$$\left(\frac{x}{2}-z\right) = 1(7+y),\ \frac{x}{2}+z = 7-y \quad \text{...(6)}$$

Solving (3) and (6), we get the vertices as $\left(14, \frac{7}{3}, -\frac{7}{3}\right)$.

Solving (4) and (5), we get the vertices as $\left(\frac{21}{2}, \frac{77}{16}, \frac{21}{16}\right)$.

∴ Other two vertices are $\left(14, \frac{7}{3}, -\frac{7}{3}\right)$ and $\left(\frac{21}{2}, \frac{77}{16}, \frac{21}{16}\right)$.

8.2 Show that the generators through any one of the ends of an equiconjugate diameter of the principal elliptic section of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ are inclined to each other at an angle of 60° if $a^2+b^2=6c^2$. Find also the condition for the generators to be perpendicular to each other. **(2011 : 20 Marks)**

Solution:

Let ($a\cos\theta$, $b\sin\theta$, 0) be the given point on the diameter.

The equations of the two generators through this point are

$$\frac{x-a\cos\theta}{a\sin\theta} = \frac{y-b\sin\theta}{-b\cos\theta} = \frac{z}{\pm c}$$

The direction ratios of two generators are ($a\sin\theta$, $-b\cos\theta$, c) and ($a\sin\theta$, $-b\cos\theta$, $-c$) respectively.

Let α be the angle between two generators and let θ be the parameter of the end points of conjugate diameters.

$$\therefore \quad \cos\alpha = \frac{l_1l_2+m_1m_2+n_1n_2}{\sqrt{l_1^2+m_1^2+n_1^2}\sqrt{l_2^2+m_2^2+n_2^2}}$$

$$= \frac{a^2\sin^2\theta+b^2\cos^2\theta-c^2}{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta+c^2}\sqrt{a^2\sin^2\theta+b^2\cos^2\theta+c^2}}$$

$$= \frac{a^2\sin^2\theta+b^2\cos^2\theta-c^2}{a^2\sin^2\theta+b^2\cos^2\theta+c^2}$$

Putting $\alpha = 60°$ and $\theta = 45°$ (Q equi-conjugal diameters means equal length of conjugal diameters, i.e., $\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}$ which is equal to $\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}$ and is possible for $\theta = 45°$).

∴ from (i), we get

$$\cos 60° = \frac{\frac{a^2}{2}+\frac{b^2}{2}-c^2}{\frac{a^2}{2}+\frac{b^2}{2}+c^2}$$

$$\Rightarrow \quad \frac{1}{2} = \frac{a^2+b^2-2c^2}{a^2+b^2+2c^2}$$

$$\Rightarrow \quad a^2+b^2+2c^2 = 2a^2+2b^2-4c^2$$

$$\Rightarrow \quad a^2+b^2 = 6c^2$$

Again, put $\alpha = 90°$ and $\theta = 45°$ in (i), we get

$$0 = \frac{\frac{a^2}{2}+\frac{b^2}{2}-c^2}{\frac{a^2}{2}+\frac{b^2}{2}+c^2}$$

$$\Rightarrow \quad a^2+b^2 = 2c^2,$$

which is the required condition for the generators to be perpendicular to each other.

8.3 A variable generator meets two generators of the same system through the extremities B and B' of the minor axis of the principal elliptic section of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ in P and P'. Prove that $BP.B'P' = a^2+c^2$.

(Note : There is minor error in actual question. It must be $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ or $BP.B'P = a^2+\frac{1}{C^2}$. **(20 Marks)**

Solution:

The generator through any general point ($a\cos\theta$, $b\sin\theta$) on the principal elliptic section is

$$\frac{x-a\cos\theta}{a\sin\theta} = \frac{y-b\sin\theta}{-b\cos\theta} = \frac{z}{\pm c} \quad \text{...(i) (Both systems)}$$

Taking the positive system for the extremity of minor axis $\theta = \frac{\pi}{2}$ and $\frac{3\pi}{2}$

i.e.,
$$\frac{x-a}{a} = \frac{y-b}{0} = \frac{z}{c} \qquad \text{...(ii)}$$

and
$$\frac{x}{-a} = \frac{y+b}{0} = \frac{z}{c} \qquad \text{...(iii)}$$

Any point on these lines is
$$\frac{x}{a} = \frac{y-b}{a} = \frac{z}{c} = r \text{ and } \frac{x}{-a} = \frac{y+b}{0} = \frac{z}{c} = r_2$$

i.e.,
$$x = ar,\ y = b,\ z = cr_1 \text{ and } x = -ar,\ y = -b,\ z = cr_2$$

Note that the distance of such a point from B is $\sqrt{l^2+m^2+n^2}r_1 = \sqrt{a^2+c^2}r_1$.

For, P, P' this general point must be on the variable generator whose equation is given by (i) (taking the other system).

$\therefore$
$$\frac{ar_1 - a\cos\theta}{a\sin\theta} = \frac{b - b\sin\theta}{-b\cos\theta} = -r_1$$

and
$$\frac{-ar_1 - a\cos\theta}{a\sin\theta} = \frac{-b - b\sin\theta}{-b\cos\theta} = -r_2$$

$\Rightarrow$
$$r_1 = \frac{\cos\theta}{1+\sin\theta}.$$
$$r_2 = -\left(\frac{1+\sin\theta}{\cos\theta}\right)$$

$\therefore$
$$BP = \sqrt{a^2+c^2}\,|r_1|$$
$$B'P' = \sqrt{a^2+c^2}\,|r_2|$$

$\therefore$
$$BP \cdot B'P' = (a^2+c^2)\frac{\cos\theta}{1+\sin\theta} \times \frac{1+\sin\theta}{\cos\theta}$$
$$= a^2 + c^2$$

8.4 Find the equations of the two generating lines through any point ($a\cos\theta$, $b\sin\theta$, 0), of the principal elliptic section $\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$, $z = 0$, of the hyperboloid by the plane $z = 0$.

(2014 : 15 Marks)

Solution:

Any point on the elliptic section of the hyperboloid is ($a\cos\theta$, $b\sin\theta$, 0).

$\therefore$ Equations of any line through this point is
$$\frac{x - a\cos\theta}{l} = \frac{y - b\sin\theta}{m} = \frac{z-0}{n} = r(\text{say}) \qquad \text{...(i)}$$

Any point on this line is ($lr + a\cos\theta$, $mr + b\sin\theta$, and it lies on given hyperboloid, if
$$\frac{(lr+a\cos\theta)^2}{a^2} + \frac{(mr+b\sin\theta)^2}{b^2} - \frac{n^2r^2}{c^2} = 1$$

Or
$$\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}-\frac{n^2}{c^2}\right)r^2 + 2\left(\frac{l\cos\theta}{a}+\frac{m\sin\theta}{b}\right)r = 0 \qquad \text{...(ii)}$$

If the line (i) generator of given hyperboloid, then (i) lies wholly on the hyperboloid and the condition for which from (ii) are
$$\left(\frac{l^2}{a^2}\right)+\left(\frac{m^2}{b^2}\right)-\left(\frac{n^2}{c^2}\right) = 0 \qquad \text{...(iii)}$$
$$\frac{l\cos\theta}{a}+\frac{m\sin\theta}{b} = 0 \qquad \text{...(iv)}$$

from (iv) we get,
$$\frac{l}{a\sin\theta} = \frac{m}{-b\cos\theta} \text{ or } \frac{(l/a)}{\sin\theta} = \frac{(ml-b)}{\cos\theta}$$

$\Rightarrow$
$$\frac{(l/a)}{\sin\theta} = \frac{(ml-b)}{\cos\theta} = \sqrt{\frac{(l^2/a^2)+(m^2/b^2)}{\sqrt{\sin^2\theta+\cos^2\theta}}} = \frac{\sqrt{n^2/c^2}}{1} \qquad \text{from (iii)}$$

$\Rightarrow$
$$\frac{l}{a\sin\theta} = \frac{m}{-b\cos\theta} = \frac{n}{\pm c}$$

$\therefore$ The equation to the required generated from (i) are
$$\frac{x-\cos\theta}{a\sin\theta} = \frac{y-b\sin\theta}{-b\cos\theta} = \frac{z}{\pm c}$$

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