ANALYTIC GEOMETRY

: CSE - 2014 :

① (e) Examine whether the plane $x+y+z=0$ cuts the cone $yz+zx+xy=0$ in perpendicular lines

→ let l, m, n be the dcs of the lines in which the plane cuts the cone.

Then, the line lies on the plane & on the cone. Therefore the line is a generator of the cone & cone eqⁿ is satisfied by l, m, n

$\therefore mn + nl + lm = 0$ and $l+m+n=0$

$mn + n(m+n)$ ⟵ $l = -(m+n)$.

$-m(m+n) = 0$

$\Rightarrow \cancel{mn} - \cancel{mn} - n^2 - m^2 - mn = 0$

$\Rightarrow m^2 + mn + n^2 = 0 \quad \Rightarrow \frac{m^2}{n^2} + \frac{m}{n} + 1 = 0$

$\cancel{\frac{m}{n} = \frac{-1 \pm \sqrt{1-4}}{2}} \quad \Rightarrow \frac{m_1}{n_1}\frac{m_2}{n_2} = 1$

$\therefore \frac{l_1 l_2}{1} = \frac{m_1 m_2}{1} = \frac{n_1 n_2}{1} = k$ (say) [By Symmetry]

$\therefore l_1 l_2 + m_1 m_2 + n_1 n_2 = 3k$

If the line lies on $xy, yz,$ or xz plane, then $n_1 n_2$, $l_1 l_2$ and $m_1 m_2$ are correspondingly zero.

Then, we can clearly say that the lines are perpendicular if it lies on xy, yz or zx planes.

④ (a) (i) Find the coordinates of the points on the sphere $x^2+y^2+z^2-4x+2y = 4$ the tangent planes at which are parallel to the planes $2x-y+2z=1$

→ Any plane parallel to the given plane is $2x-y+2z+d=0$ — ①

~~Tangent plane to the spheres are~~

Centre of the given sphere is $(2,-1,0)$

Radius of the given sphere is $r = \sqrt{4+1+4} = \sqrt{9} = 3$

If plane ① is tangent plane to the sphere, then the radius of sphere is equal to ⊥ar distance of plane ① from centre of the sphere

$\Rightarrow \quad 3 = \left| \dfrac{2.2 - 1(-1) + 2.0 + d}{\sqrt{2^2 + (-1)^2 + 2^2}} \right| \Rightarrow \left| \dfrac{5+d}{3} \right| = 3$

$\Rightarrow \quad 5 + d = \pm 9 \Rightarrow d = 4, -14$

$\therefore$ Reqd planes are $2x - y + 2z + 4 = 0$ — (2) and $2x - y + 2z - 14 = 0$ — (3)

Tangent plane to the given sphere at any point α, β, γ is

$\alpha . x + \beta . y + \gamma . z - 2(\alpha + x) + (\beta + y) - 4 = 0 \Rightarrow (\alpha - 2)x + (\beta + 1)y + \gamma z - 2\alpha + \beta - 4 = 0$

If plane (2) is tangent plane then

$\dfrac{\alpha - 2}{2} = \dfrac{\beta + 1}{-1} = \dfrac{\gamma}{2} = \dfrac{-2\alpha + \beta - 4}{4} \Rightarrow \dfrac{\alpha - 2}{2} = \dfrac{\beta + 1}{-1} \Rightarrow -\alpha = 2\beta$ — (4)

$\dfrac{\alpha - 2}{\cancel{2}} = -\dfrac{2\alpha + \beta - 4}{\cancel{4}2} \Rightarrow 2\alpha = -2\alpha + \beta \Rightarrow 4\alpha = \beta$ — (5)

(4) & (5) $\Rightarrow \beta = \alpha = 0$.

$\dfrac{\gamma}{2} = \dfrac{-2\alpha + \beta - 4}{4} = \dfrac{0 + 0 - 4}{4} = -1 \Rightarrow \gamma = -2 \Rightarrow (0, 0, -2)$ satisfies the sphere

$\therefore$ the point of contact of plane (2) with the sphere is $(0, 0, -2)$.

If plane (3) is a tangent plane, then

$\dfrac{\alpha - 2}{2} = \dfrac{\beta + 1}{-1} = \dfrac{\gamma}{2} = -\dfrac{2\alpha + \beta - 4}{-14}$

$\Rightarrow$ ~~$\alpha + \beta = 0$~~ $\quad \dfrac{\alpha - 2}{2} = \dfrac{\beta + 1}{-1} \Rightarrow -\alpha = 2\beta \Rightarrow \alpha + 2\beta = 0$ — (6)

$\dfrac{\alpha - 2}{2} = \dfrac{-2\alpha + \beta - 4}{-14} \Rightarrow -7\alpha + 14 = -2\alpha + \beta - 4$

$\Rightarrow 5\alpha + \beta = 18$ — (7)

(6) & (7) $\Rightarrow \alpha = 4, \beta = -2$

$\dfrac{\gamma}{2} = \dfrac{-2\alpha + \beta - 4}{-14} = \dfrac{-8 - 2 - 4}{-14} = 1 \Rightarrow \gamma = 2 \Rightarrow (4, -2, 2)$ satisfies the sphere

(2)

$\therefore$ The point of contact of plane ③ is $(4,-2,2)$

(4)(a)(ii): Prove that the equation $ax^2+by^2+cz^2+2ux+2vy+2wz+d=0$ represents a cone if $\frac{u^2}{a}+\frac{v^2}{b}+\frac{w^2}{c}=d$.

$\rightarrow$ Making the given equation homogeneous with the help of a new variable t,

$$F(x,y,z,t) = ax^2+by^2+cz^2+2uxt+2vyt+2wzt+dt^2=0$$

Now, partially diff $F(x,y,z,t)$ wrt x, y, z and t respectively and equating to zero.

$$\frac{\partial F}{\partial x} = 2ax+2ut=0, \quad \frac{\partial F}{\partial y} = 2by+2vt=0$$
$$= ax+ut=0 \qquad\qquad = by+vt=0$$

$$\frac{\partial F}{\partial z} = 2cz+2wt=0, \quad \frac{\partial F}{\partial t} = 2ux+2vy+2wz+2dt=0$$
$$= cz+wt=0 \qquad\qquad = ux+vy+wz+dt=0.$$

Putting $t=1$: $\quad ax+u=0,\ by+v=0,\ cz+w=0\} \ \&$ —— ①

$ux+vy+wz+d=0$ —— ②

① $\equiv$ $x=-\frac{u}{a},\ y=-\frac{v}{b},\ z=-\frac{w}{c}$.

Putting in ② $\quad u\left(-\frac{u}{a}\right)+v\left(-\frac{v}{b}\right)+w\left(-\frac{w}{c}\right)+d=0$

$$\Rightarrow \boxed{d=\frac{u^2}{a}+\frac{v^2}{b}+\frac{w^2}{c}.}$$

Hence, for the given equation to represent a cone, $\frac{u^2}{a}+\frac{v^2}{b}+\frac{w^2}{c}=d$ is the reqd condn.

4 ⑥ Show that the lines drawn from the origin parallel to the normals to the central conicoid $ax^2+by^2+cz^2=1$ at its points of intersection with the plane $lx+my+nz=p$ generate the cone:

$$p^2\left(\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}\right) = \left(\frac{lx}{a}+\frac{my}{b}+\frac{nz}{c}\right)^2.$$

→ Let (α,β,γ) be the point of intersection of the given conicoid & the plane. Then, $a\alpha^2+b\beta^2+c\gamma^2=1$ —(1) & $l\alpha+m\beta+n\gamma=p$. —(2)

Normal to the conicoid at (α,β,γ) is $\frac{x-\alpha}{a\alpha}=\frac{y-\beta}{b\beta}=\frac{z-\gamma}{c\gamma}$ —(3)

Line passing through the origin and parallel to line (3) is

$$\frac{x}{a\alpha}=\frac{y}{b\beta}=\frac{z}{c\gamma} \quad \text{—(4)}.$$

from (1) & (2): $a\alpha^2+b\beta^2+c\gamma^2=\left(\frac{l\alpha+m\beta+n\gamma}{p}\right)^2$

$\Rightarrow$ $p^2(a\alpha^2+b\beta^2+c\gamma^2)=(l\alpha+m\beta+n\gamma)^2$ —(5)

Eliminating (α,β,γ) between (4) & (5), the reqd locus is

$$p^2\left(a.\frac{x^2}{a^2}+b\frac{y^2}{b^2}+c\frac{z^2}{c^2}\right)=\left(l\frac{x}{a}+m\frac{y}{b}+n\frac{z}{c}\right)^2$$

$$\Rightarrow p^2\left(\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}\right)=\left(\frac{lx}{a}+\frac{my}{b}+\frac{nz}{c}\right)^2$$

4(c) Find the equations of two generating lines through any point $(a\cos\theta, b\sin\theta, 0)$ of the principal elliptic section $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, z=0$ of the hyperboloid by the plane $z=0$.

→ Eqn of any line through the given point is

$$\frac{x-a\cos\theta}{l}=\frac{y-b\sin\theta}{m}=\frac{z-0}{n}=r \text{ (say)} \quad \text{—(1)}$$

Any point on line (1) is $P(a\cos\theta+lr,\ b\sin\theta+mr,\ nr)$

If this point lies on hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, then

$$\frac{(a\cos\theta+lr)^2}{a^2}+\frac{(b\sin\theta+mr)^2}{b^2}-\frac{n^2r^2}{c^2}=1$$

$$\Rightarrow \frac{a^2\cos^2\theta}{a^2}+\frac{l^2}{a^2}r^2+\frac{2alr\cos\theta}{a^2}+\frac{b^2\sin^2\theta}{b^2}+\frac{m^2}{b^2}r^2+\frac{2bm\sin\theta\, r}{b^2}-\frac{n^2}{c^2}=1$$

$$\Rightarrow (\cos^2\theta+\sin^2\theta)+r^2\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}\right)+2r\left[\frac{l\cos\theta}{a}+\frac{m\sin\theta}{b}\right]-\frac{n^2}{c^2}r^2=1$$

$$\Rightarrow r^2\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}-\frac{n^2}{c^2}\right)+2r\left[\frac{l\cos\theta}{a}+\frac{m\sin\theta}{b}\right]=0 \quad \text{—(2)}$$

Since the line ① is a generator of the hyperboloid, it completely lies on it. Hence, the condⁿ is

$$\frac{l^2}{a^2}+\frac{m^2}{b^2}-\frac{n^2}{c^2}=0 \quad \text{③} \qquad \text{and} \qquad \frac{l\cos\theta}{a}+\frac{m\sin\theta}{b}=0 \quad \text{④}$$

$$\text{④} \equiv \frac{l\cos\theta}{a} = \frac{-m\sin\theta}{b} \Rightarrow \frac{l/a}{\sin\theta} = \frac{m/b}{-\cos\theta} = \frac{\sqrt{\frac{l^2}{a^2}+\frac{m^2}{b^2}}}{\sqrt{\sin^2\theta+\cos^2\theta}} = \frac{\sqrt{n^2/c^2}}{1} = \frac{\pm n/c}{1}.$$

$$\Rightarrow \frac{l}{a\sin\theta} = \frac{m}{-b\cos\theta} = \frac{n}{\pm c}$$

∴ Reqd generating lines are given by :

$$\frac{x-a\cos\theta}{a\sin\theta} = \frac{y-b\sin\theta}{-b\cos\theta} = \frac{z}{c} \quad \text{and}$$

$$\frac{x-a\cos\theta}{a\sin\theta} = \frac{y-b\sin\theta}{-b\cos\theta} = \frac{z}{-c}.$$