CSE - 2017

**Q1⇒** Determine all Entire function $f(Z)$ such that $0$ is a removable singularity of $f\left(\frac{1}{Z}\right)$.

**Sol$^n$:-**

Given $f(Z)$ is entire function $\Rightarrow f(Z) = \sum_{n=0}^{\infty} a_n Z^n$

replace $Z \to \frac{1}{Z}$

$$f\left(\frac{1}{Z}\right) = \sum_{n=0}^{\infty} a_n \frac{1}{Z^n}$$

$$f\left(\frac{1}{Z}\right) = a_0 + \frac{a_1}{Z} + \frac{a_2}{Z^2} + \cdots + \frac{a_n}{Z^n} + \cdots$$

Given that → '0' is the removable singularity of $f\left(\frac{1}{Z}\right)$ and For removable singularity Principal part of the expansion must be zero.

$$\therefore a_1 = a_2 = a_3 = \cdots = a_n = 0 \quad \forall n \geq 1$$

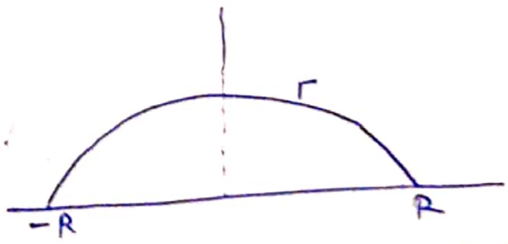
$\therefore \boxed{f(Z) = a_0}$ ; where $a_0$ is a constant.

**Q2⇒** Using contour integral method, Prove that

$$\int_0^{\infty} \frac{x \sin mx}{a^2 + x^2}\, dx = \frac{\pi}{2} e^{-ma}$$

**Sol$^n$** $\because \int_0^{\infty} \frac{x \sin mx}{a^2+x^2}\, dx$

replace $x$ with $Z$




$$\int_C \frac{Z \sin mZ}{a^2 + Z^2}\, dZ,$$ where 'C' is unit semi-circle.

$Z = \pm ai$ are two poles of $f(Z)$; where $f(Z) = \frac{Z \sin mZ}{a^2+Z^2}$.

$Z = +ai$ is the only Pole in defined region.

$$\int_C \frac{Z \sin mZ}{a^2+Z^2}\, dZ = 2\pi i \{\text{Residue at } Z = ai\}$$

$$f(z) = \frac{Z \sin mZ}{a^2+Z^2} = \frac{Z e^{imZ}}{a^2+Z^2} \text{ (imaginary Part).}$$

Residue at $z = ai$ = $\lim_{z \to ai} \cancel{(z-ai)} \dfrac{z e^{imz}}{\cancel{(z-ai)}(z+ai)}$

$$= \frac{ai \times e^{-ma}}{2ai} = \frac{e^{-ma}}{2}$$

$$\Rightarrow 2\pi i \times \frac{e^{-ma}}{2} = (\pi e^{-ma})\, i$$

$$\therefore \int_c \frac{z e^{imz}}{a^2+z^2} dz = (\pi e^{-ma})\, i$$

comparing imaginary part.

$$\int_c \frac{z \sin mz \, dz}{a^2+z^2} = \pi e^{-ma} \quad \text{—— (1)}$$

But

$$\int_c \frac{z \sin mz \, dz}{a^2+z^2} = \int_{-\infty}^{\infty} \frac{z \sin mz \, dz}{a^2+z^2} + \int_\Gamma \frac{z \sin mz}{a^2+z^2} dz \quad \text{—— (2)}$$

Now consider,

$$\left| \int_\Gamma \frac{z e^{imz}}{a^2+z^2} dz \right| = \left| \int_0^\pi \frac{z e^{imz}}{a^2+z^2} dz \right|$$

let $z = Re^{i\theta}$ ; $dz = iRe^{i\theta} d\theta$.

$$\therefore \left| \int_0^\pi \frac{z e^{imz}}{a^2+z^2} dz \right| \le \int_0^\pi \left| \frac{iRe^{i\theta} e^{imRe^{i\theta}} Re^{i\theta}}{a^2 + R^2 e^{i2\theta}} \right| d\theta$$

$$\le \int_0^\pi \frac{R^2 \left| e^{imRe^{i\theta}} \right|}{|R^2+a^2|} d\theta$$

$$\le \int_0^\pi \frac{R^2}{R^2-a^2} \left| e^{imR\cos\theta} \right| \left| e^{-mR\sin\theta} \right| d\theta$$

$\because |a+b| = |a-(-b)| \ge |a|-|b|$

$$\le \frac{R^2}{R^2-a^2} \int_0^\pi e^{-mR\sin\theta} d\theta$$

$$\le \frac{2R^2}{R^2-a^2} \int_0^{\frac{\pi}{2}} e^{-mR\sin\theta} d\theta$$

$\because \sin\theta \ge \frac{2\theta}{\pi}$ for $0 < \theta < \frac{\pi}{2}$

~~Hence~~ $mR\sin\theta \ge \frac{2aR\theta}{\pi}$, $a > 0$

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$$\Rightarrow \quad e^{mR\sin\theta} \geq e^{2aR\theta/\pi}$$

$$\bar{e}^{mR\sin\theta} \leq e^{\frac{-2aR\theta}{\pi}}$$

$$\frac{2R^2}{R^2-1}\int_0^{\frac{\pi}{2}} e^{-aR\sin\theta}\,d\theta \leq \frac{2R^2}{R^2-a^2}\int_0^{\frac{\pi}{2}} e^{-2mR\theta/\pi}\,d\theta$$

$$= \frac{-\pi R}{m(R^2-a^2)}\left[e^{-mR}-1\right]$$

$$= \frac{\pi R}{m(R^2-a^2)}\left[1-e^{-mR}\right]$$

this tends to $\longrightarrow 0$ as $R \longrightarrow \infty$.

$$\therefore \int_\Gamma \frac{z\sin mz}{a^2+z^2}\,dz = 0.$$

$\therefore$ from Eq$^n$ (2).

$$\int_c \frac{z\sin mz}{a^2+z^2}\,dz = \int_{-\infty}^{\infty} \frac{z\sin mz}{a^2+z^2}\,dz + 0$$

from Eq$^n$ (1).

$$\int_{-\infty}^{\infty} \frac{z\sin mz}{a^2+z^2}\,dz = \pi e^{-ma}$$

$$\int_0^{\infty} \frac{z\sin mz}{a^2+z^2} = \frac{\pi}{2}e^{-ma}$$

$$\boxed{\therefore \int_0^{\infty} \frac{x\sin mx}{a^2+x^2} = \frac{\pi}{2}e^{-ma}}$$

Q3 $\Rightarrow$ let $f = u+iv$ be an analytic function on the unit disc $D = \{z \in \mathbb{C} : |z| < 1\}$. show that

$$\frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0 = \frac{\delta^2 v}{\delta x^2} + \frac{\delta^2 v}{\delta y^2}$$

at all points at $D$.

Sol$^n$ Given that, $f(z) = u+iv$ is analytic on the unit disc.

$$\therefore f'(z) = \frac{\delta f}{\delta x} \quad \& \quad f'(z) = -i\frac{\delta f}{\delta y}$$

$\because$ Analytic function has derivates of all orders.

$$\therefore f''(z) = \frac{\delta}{\delta x} f'(z)$$

$$= \frac{\delta}{\delta x}\left(\frac{\delta f}{\delta x}\right)$$

$$= \frac{\delta^2 f}{\delta x^2} \quad \text{——} \quad (1)$$

$$\& \quad f''(z) = -i \frac{\delta}{\delta y} f'(z)$$

$$= -i \frac{\delta}{\delta y}\left(-i \frac{\delta f}{\delta y}\right)$$

$$= (-1) \frac{\delta^2 f}{\delta y^2} \quad \text{——} \quad (2)$$

$\therefore$ from $Eq^n$ (1) & (2).

$$\frac{\delta^2 f}{\delta x^2} = -\frac{\delta^2 f}{\delta y^2}$$

$$\Rightarrow \frac{\delta^2 f}{\delta x^2} + \frac{\delta^2 f}{\delta y^2} = 0$$

$$\therefore \Delta^2 f = 0 \quad \text{——} \quad (3)$$

$\because$ $f(z) = u + iv$ is analytic in a domain D.

then from (3).

$$\nabla^2 (u + iv) = 0$$

$$\nabla^2 u + i \nabla^2 v = 0$$

$$\Rightarrow \frac{\delta^2 u}{\delta x^2} + \frac{\delta^2 u}{\delta y^2} = 0 = \frac{\delta^2 v}{\delta x^2} + \frac{\delta^2 v}{\delta y^2}$$

Q4 ⇒ For a function $f: C \to C$ and $n \geq 1$, let $f^{(n)}$ denote the $n^{th}$ derivate of $f$ and $f^{(0)} = f$. let $f$ be an entire function such that for some $n \geq 1$, $f^{(n)}\left(\frac{1}{k}\right) = 0$ for all $k = 1, 2, 3, \ldots$ show that $f$ is a Polynomial.

Solⁿ

Given that $f$ is an entire function.

& for some $n \geq 1$, $f^{n}\left(\frac{1}{k}\right) = 0$

$$\because \underset{n \to \infty}{Lt} \frac{1}{k} = 0$$

$\therefore$ By Identity theorm:- $f^{n}(z) = 0$.

If $f^{n}\left(\frac{1}{k}\right) = 0$

& $\underset{n \to \infty}{Lt} \left(\frac{1}{k}\right) = 0$

then $f^{n}(z) = 0$

$$f^{(n-1)}(z) = \text{constant} = a_n$$

$$f^{(n-2)}(z) = a_n(z) + a_{n-1} z$$

$\vdots$

Similary.

$$f(z) = a_n \frac{z^n}{n!} + a_{n-1} \frac{z^{n-1}}{n-1!} + \cdots a_0$$

$\therefore$ $f$ is a Polynomial.