Set - IV

* MATRICES *

Def → A matrix is a rectangular array of numbers (real or complex). The numbers are called the elements of the matrix or entries of the matrix.

Matrices are represented by the brackets ( ) or [ ].

Def → If a matrix A has m rows and n columns then the matrix is said to be of type or order or size m×n.

Ex:- If $A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & -2 & 5 \end{bmatrix}$ then the order of A is 2×3.

Def → A matrix A is said to be a square matrix if the number of rows in A is equal to the number of columns in A.

Ex:- $\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$

Note:- (1) An n×n square matrix is called a square matrix of type n.

(2) If A is a square matrix then the diagonal in A from the first element of the first row to the last element of the last row is called the principal diagonal of A. Ex: $A = \begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \\ 8 & 1 & 10 \end{bmatrix}$

Def → A matrix A is said to be a rectangular matrix if the number of rows in A is not equal to the number of columns in A.

Note: A is called a rectangular matrix if A is not a square matrix.

Ex:- $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$

Def → A matrix 'A' is said to be a zero matrix if every element of A is equal to zero.

An m×n zero matrix is denoted by $O_{m\times n}$ or O.

Ex:- $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Def → A matrix 'A' is said to be a row matrix if A contains only one row.

Ex:- $[23 \quad 24]$, $[23 \quad 24 \quad 25]$.

Def → A matrix A is said to be a column matrix if A contains only one column.

Ex:- $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$

Def → A square matrix $A = [a_{ij}]_{n\times n}$ is said to be an upper triangular matrix if $a_{ij} = 0$ whenever $i > j$.

Ex:- $\begin{bmatrix} 2 & -5 & 5 \\ 0 & 3 & 6 \\ 0 & 0 & 1 \end{bmatrix}$

Def → A square matrix $A = [a_{ij}]_{m\times n}$ is said to be a lower triangular matrix if $a_{ij} = 0$ whenever, $i < j$.

Ex:- $\begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 5 & 4 & 6 \end{bmatrix}$

Def → A square matrix A is said to be a triangular matrix if A is either an upper triangular matrix or lower triangular matrix.

Def → A square matrix A is said to be a diagonal matrix if A is both an upper triangular matrix and lower triangular matrix.

Ex:- $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$

Note: A square matrix in which every element is equal to '0', except those of principal diagonal of the matrix is a diagonal matrix.

Def → A diagonal matrix A is said to be a scalar matrix if all elements in the principal diagonal are equal.

Ex:- $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$, $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

Def → A diagonal matrix is said to be a unit matrix if every element in the principal diagonal is equal to unity.

A unit matrix of order n is denoted by I_n.

Ex:- $I_1 = [1]$, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Note:- $I_n = [x_{ij}]_{n\times n}$ where

$$x_{ij} = 1 \quad \text{if } i = j$$
$$\quad = 0 \quad \text{if } i \neq j.$$

*Equality of matrices:-

→ Two matrices A and B are said to be equal if

(i) A, B are of same-type and

(ii) the corresponding elements in A & B are equal.

It is denoted by $A = B$.

Ex:- If $A = \begin{bmatrix} a & b & c \\ x & y & z \end{bmatrix}$,

$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ then $A=B$

$\Leftrightarrow$ $a=1,\ b=2,\ c=3$
$x=4,\ y=5,\ z=6.$

*Matrix Addition :-

Definition :- If A and B are two matrices of the same type then their sum is defined to be the matrix obtained by adding the corresponding elements of A and B.

It is denoted by $A+B$.

If $A=[a_{ij}]_{m\times n}$, $B=[b_{ij}]_{m\times n}$ then

$A+B=[c_{ij}]_{m\times n}$ where $c_{ij}=a_{ij}+b_{ij}$

Ex:- If $A = \begin{bmatrix} 3 & 2 & -1 \\ 1 & 2 & 5 \end{bmatrix}$,

$B=\begin{bmatrix} 4 & -3 & 1 \\ 3 & 2 & -2 \end{bmatrix}$ then $A+B=\begin{bmatrix} 7 & -1 & 0 \\ 4 & 4 & 3 \end{bmatrix}$

Note :- If A and B are not of same order then $A+B$ and $A-B$ are not possible.

Def $\rightarrow$ If $A=[a_{ij}]_{m\times n}$ is a matrix and k is a scalar then their product is defined to be the matrix obtained by multiplying every element of the matrix A by the number k. It is denoted by kA.

Ex:- If $A=\begin{bmatrix} 2 & -3 & 1 \\ 1 & 3 & -2 \end{bmatrix}$ then

$3A=\begin{bmatrix} 6 & -9 & 3 \\ 3 & 9 & -6 \end{bmatrix}$

$\rightarrow$ Matrix Multiplication :-

Two matrices A and B are said to be conformable for multiplication if the number of columns in A is equal to the number of rows in B.

i.e. if $A=[a_{ij}]_{m\times n}$, $B=[b_{jk}]_{n\times p}$ are two matrices (which are conformable for multiplication) then their product is defined to be the matrix

$C=[c_{ik}]_{m\times p}$ where $c_{ik}=\sum_{j=1}^{n} a_{ij}\,b_{jk}$

It is denoted by AB.

Note :- If the product AB exists then it is not necessary that product BA exists.

$\rightarrow$ Idempotent Matrix :-

A square matrix A is said to be an idempotent matrix if $A^2=A$.

→ **Involutory Matrix :-**

A Square matrix A is said to be an involutory matrix if $A^2 = I$.

Ex: If $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ then $A^2 = I$

If $A = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$ then $A^2 = I$

If $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ then $A^2 = I$

→ **Nilpotent Matrix :-**

A square matrix A is said to be a nilpotent matrix if $\exists$ a +ve integer n such that $A^n = 0$. If n is the least +ve integer such that $A^n = 0$, then 'n' is called the index of the nilpotent matrix A.

Ex: show that $A = \begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$ is a nilpotent matrix.

Sol'n:- $A^2 = A.A$

$$= \begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix} \begin{bmatrix} 1 & -3 & -4 \\ -1 & 3 & 4 \\ 1 & -3 & -4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= O$$

$\therefore$ A is a nilpotent matrix of index 2.

→ **Trace of Matrix :**

Let $A = [a_{ij}]_{n \times n}$. The sum of the elements of A lying along the principal diagonal is called the trace of A. It is denoted by $\text{tr} A$.

i.e. if $A = [a_{ij}]_{n \times n}$ then

$$\text{tr} A = \sum_{i=1}^{n} a_{ii} = a_{11} + a_{22} + a_{33} + \cdots + a_{nn}.$$

Properties

If A and B are two square matrices of order n and λ be a scalar then

1. $\text{tr}(\lambda A) = \lambda \, \text{tr} A$.
2. $\text{tr}(A+B) = \text{tr} A + \text{tr} B$
3. $\text{tr}(AB) = \text{tr}(BA)$

→ **Transpose of Matrix :-**

Let $A = [a_{ij}]_{m \times n}$. Then the matrix $[a_{ji}]_{n \times m}$ obtained from A by changing its rows into columns and columns into rows is called the transpose of A and is denoted by A^T or A'.

Note:- 1. If A is $m \times n$ matrix then A^T is $n \times m$ matrix.

2. $(j,i)^{th}$ entry of $A^T = (i,j)^{th}$ entry of A.

Ex: Let $A = \begin{bmatrix} 3 & 2+i \\ -5 & 0 \\ \sqrt{2} & 1 \end{bmatrix}_{3\times 2}$ then

$$A^T = \begin{bmatrix} 3 & -5 & \sqrt{2} \\ 2+i & 0 & 1 \end{bmatrix}_{2\times 3}$$

Some important results are given below:-

1. $(A^T)^T = A$ and $(-A)^T = -A^T$
2. $(A+B)^T = A^T + B^T$
3. $(A-B)^T = A^T - B^T$
4. $(KA)^T = KA^T$ where K is any Scalar.
5. $(AB)^T = B^T A^T$

→ Symmetric Matrix :-

A square matrix $A=[a_{ij}]$ is said to be Symmetric if $A^T = A$.
i.e. $[a_{ji}] = [a_{ij}]$.

Ex: (1) Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix} = A$

(2). Let $A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$ then $A^T = A$.

→ Skew Symmetric (Anti Symmetric) Matrix :-

A square matrix $A=[a_{ij}]$ is said to be skew symmetric. If $A^T = -A$
i.e. $[a_{ji}] = [-a_{ij}]$.

Ex: Let $A = \begin{bmatrix} 0 & a & b \\ -a & 0 & c \\ -b & -c & 0 \end{bmatrix}$ then

$A^T = -A$.

Note: (1) The diagonal elements of a skew symmetric matrix must be zero.

(2). Below the diagonal and above the diagonal elements of a symmetric matrix must be equal.

(3). A zero matrix is both Symmetric as well as Skew-Symmetric.

* Some Properties of Symmetric & skew symmetric matrices :-

→ If A is symmetric matrix then KA is also symmetric matrix.

Sol'n :- Since A is symmetric matrix.

$$\therefore A^T = A \quad \text{——} ①$$

Now $(KA)^T = KA^T$
$= KA$ (from ①).

$\therefore$ KA is also Symmetric Matrix.

Similarly If A is skew symmetric matrix then KA is also skew-symmetric matrix.

→ If A, B are symmetric then A+B is also symmetric.

Sol'n: Since A, B are symmetric

$\therefore A^T = A$ & $B^T = B$ —— ①

Now $(A+B)^T = A^T + B^T$

$= A+B$ (from ①)

$\therefore$ A+B is Symmetric matrix.

→ If A, B are skew symmetric then A+B is also skew symmetric.

Sol'n:- Since A & B are skew symmetric

$\therefore A^T = -A$ & $B^T = -B$ —— ①

Now $(A+B)^T = A^T + B^T$

$= -A - B$

$= -(A+B)$

$\therefore$ A+B is a skew symmetric.

→ If A and B are symmetric matrices, show that AB+BA is symmetric and AB−BA is skew-symmetric.

Sol'n:- Since A & B are symmetric

$\therefore A^T = A$ & $B^T = B$.

Now we have

$(AB+BA)^T = (AB)^T + (BA)^T$ $(\because (A+B)^T = A^T + B^T)$

$= B^TA^T + A^TB^T$ $(\because (AB)^T = B^TA^T)$

$= BA + AB$

$= AB + BA$ $(\because A+B = B+A)$

$\therefore$ AB+BA is symmetric.

Similarly AB−BA is also skew-symmetric.

Imp → If A and B are symmetric matrices, then show that AB is symmetric if and only if A and B commute i.e. AB = BA.

Proof :- Given that A & B are symmetric.

$\therefore A^T = A$ & $B^T = B$ —— ①

Now suppose that AB = BA

To prove that AB is symmetric.

we have

$(AB)^T = B^TA^T$

$= BA$

$= AB$ $(\because AB = BA)$.

$\therefore$ AB is symmetric.

Now Conversely suppose that AB is a symmetric matrix.

To Prove that AB = BA

we have $AB = (AB)^T$ ($\because$ AB is symmetric)

$= B^TA^T$

$= BA$ $(\because B^T = B, A^T = A)$

→ If A be any matrix, then Prove that AA^T and A^TA are both symmetric matrices.

Sol'n :- Let A be any matrix. we have

$(AA^T)^T = (A^T)^T A^T$

$= AA^T$

$\therefore AA^T$ is symmetric.

we have $(A^TA)^T = (A^T)(A^T)^T$

$= A^TA$

$\therefore A^TA$ is symmetric.

Imp → show that the matrix B^TAB is symmetric (or) skew symmetric according as A is symmetric (or) skew symmetric

Sol'n :- Case(i)

Let A be a symmetric matrix then $A^T = A$ —— ①

Now we have

$(B^TAB)^T = B^TA^T(B^T)^T$ $(\because (A^T)^T = A)$

$= B^TAB$ (by ①)

$\therefore B^TAB$ is symmetric.

Case(ii) :- Let A be a skew-symmetric matrix.

Then $A^T = -A$

Now we have

$(B^TAB)^T = B^TA^T(B^T)^T$

$= B^T(-A)B$ $(\because A^T = A)$

$= -(B^TAB)$.

$\therefore B^TAB$ is skew-symmetric.

Imp. show that every square matrix is uniquely expressible as the sum of symmetric and skew-symmetric matrices.

Sol'n Let A be any square matrix.

Then $A = \frac{1}{2}A + \frac{1}{2}A$

$= \frac{1}{2}A + \frac{1}{2}A + \frac{1}{2}A^T - \frac{1}{2}A^T$

$= \frac{1}{2}(A+A^T) + \frac{1}{2}(A-A^T)$ —— ①

$= (P+Q)$ (say)

where $P = \frac{1}{2}(A+A^T)$ & $Q = \frac{1}{2}(A-A^T)$

since $P^T = [\frac{1}{2}(A+A^T)]^T$

$= \frac{1}{2}(A^T + (A^T)^T)$

$= \frac{1}{2}(A^T + A)$

$= \frac{1}{2}(A+A^T)$ $(\because A+B = B+A)$

$= P$

$\therefore P$ is symmetric.

since $Q^T = [\frac{1}{2}(A-A^T)]^T$

$= \frac{1}{2}(A^T - (A^T)^T)$

$= -\frac{1}{2}(A-A^T)$

$= -Q$

$\therefore Q$ is skew-symmetrix.

$\therefore A$ is the sum of a symmetric matrix P and skew-symmetric matrix Q i.e. $P+Q$. —— ①

Now to prove uniqueness i.e. we want to prove that the representation ① of A is unique.

If possible let $A = R+S$ be another representation, where R is symmetric & S is skew-symmetric.

Since R is symmetric & S is skew-symmetric.

$\therefore R^T = R, \quad S^T = -S.$

Now $A^T = (R+S)^T = R^T + S^T$

$= R-S$ —— ②

and also $A = R+S$ —— ③

from ② & ③ we have

$\therefore A + A^T = R+S+R-S$

$= 2R$

and $A - A^T = R+S-(R-S)$

$= 2S$

$\Rightarrow R = \frac{1}{2}(A+A^T) = P$ &

$S = \frac{1}{2}(A-A^T) = Q$.

$\Rightarrow$ The representation ① of A as the sum of a symmetric and skew-symmetric matrix is unique.

$\rightarrow$ show that all positive integral powers of a symmetric matrix is symmetric.

Sol^n:- Let A be a symmetric matrix of order n, then $A^T = A$ —— ①

Now $A^m = A \cdot A \cdot A \cdots A$ upto m times

where m is a +ve integer.

Now $(A^m)^T = (A \cdot A \cdots A \text{ upto } m \text{ times})^T$

$= (A^T \cdot A^T \cdots A^T \text{ upto } m \text{ times})$

$= (A \cdot A \cdots A \text{ upto } m \text{ times})$ (By ①)

$= A^m$

$\therefore A^m$ is also a symmetric matrix.

Imp $\rightarrow$ show that +ve odd integral powers of a skew-symmetric matrix are skew symmetric while +ve even integral powers are symmetric.

Sol^n:- Let A be a skew symmetric then $A^T = -A$.

Now let 'm' be a +ve integer.

we have

$A^m = (A \cdot A \cdot A \cdots A \text{ upto } m \text{ times})$

Now $(A^m)^T = (A \cdot A \cdots A \text{ upto } m \text{ times})^T$

$= A^T \cdot A^T \cdots A^T$ upto m times

$= (-A)(-A)\cdots(-A)$ upto m times $(\because A^T = A)$

$= (-1)^m A^m$

$= -A^m$ or A^m

according as m is odd or even.

$\therefore$ If m is odd +ve integer then $(A^m)^T = -A^m$.

$\therefore$ A^m is skew-symmetric.

If m is an even +ve integer then $(A^m)^T = A^m$.

$\therefore A^m$ is symmetric.

Imp → If U and V are two symmetric matrices, show that UVU is also symmetric. Is UV symmetric always? Explain and illustrate by an example.

Sol'n:- Since U & V are symmetric matrices.

$\therefore$ $U^T = U$ & $V^T = V$.

Now we have $(UVU)^T = U^T V^T U^T$

$= UVU$

$\therefore$ UVU is symmetric.

Since U & V are symmetric

$\therefore$ UV is symmetric iff $\underline{UV = VU}$.

If $UV \neq VU$ then UV is not symmetric.

Ex: Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$

then $AB = \begin{bmatrix} 8 & 11 \\ 13 & 18 \end{bmatrix}$ &

$BA = \begin{bmatrix} 8 & 13 \\ 11 & 18 \end{bmatrix}$.

$\therefore AB \neq BA$.

Now $(AB)^T = B^T A^T$

$= BA$

$\neq AB$ $(\because AB \neq BA)$

$\therefore$ AB is not symmetric.

→ Let A be a square matrix, Prove that

(i) $A + A^T$ is a symmetric matrix and

(ii) $A - A^T$ is a skew symmetric matrix

Sol'n:- (i) we have

$(A + A^T)^T = A^T + (A^T)^T$ $(\because (A+B)^T = A^T + B^T)$

$= A^T + A$ $(\because (A^T)^T)$

$= A + A^T$ $(\because A + B = B + A)$

$\therefore A + A^T$ is symmetric.

(ii) $(A - A^T)^T = A^T - (A^T)^T$ $(\because (A-B)^T = A^T - B^T)$

$= A^T - A$ $(\because (A^T)^T = A)$

$= -(A - A^T)$

$\therefore$ $A - A^T$ is skew-symmetric

* Conjugate of Matrix :-

Definition :- The matrix obtained from any given matrix A on replacing its elements by the corresponding conjugate complex numbers is called the conjugate of A and is denoted by $\bar{A}$.

i.e. If $A = [a_{ij}]_{m\times n}$ then

$$\bar{A} = [\bar{a}_{ij}]_{m\times n}.$$

Ex: Let $A = \begin{bmatrix} 1 & 1+i \\ 3 & 1-i \end{bmatrix}$ then

$$\bar{A} = \begin{bmatrix} 1 & 1-i \\ 3 & 1+i \end{bmatrix}$$

Note: If all the elements of A are purely real then $A = \bar{A}$.

→ **Properties of Conjugate :-**

Let $\bar{A}$ and $\bar{B}$ be the conjugates of A and B then

(i) $(\bar{\bar{A}}) = A$ (ii) $(\overline{A+B}) = \bar{A} + \bar{B}$

(iii) $(\overline{KA}) = \bar{K}\bar{A}$ where K is any complex number.

(iv) $(\overline{AB}) = \bar{A}\bar{B}$, A & B are conformable for multiplication.

* Tranjugate (or) Transposed Conjugate of matrix :-

The transpose of the conjugate of a matrix A is called transposed conjugate of A and is denoted by A^{θ} or A^{K}.

i.e. $A^{\theta} = (\bar{A})^{T}$

Ex:- $A = \begin{bmatrix} 2 & 1+i & 0 \\ 3 & 2 & i \end{bmatrix}$ then $A^{\theta} = \begin{bmatrix} 2 & 3 \\ 1-i & 2 \\ 0 & -i \end{bmatrix}$

Note: It is also possible that $(\bar{A})' = (\overline{A'})$.

→ **Properties of transpose Conjugate**

Let A^{θ} and B^{θ} be the transposed conjugates of A and B then

(i) $(A^{\theta})^{\theta} = A$

(ii) $(A+B)^{\theta} = A^{\theta} + B^{\theta}$, A & B are of the same order.

(iii) $(KA)^{\theta} = \bar{K}A^{\theta}$, where K is any complex number

(iv) $(AB)^{\theta} = B^{\theta}A^{\theta}$, A & B are conformable to multiplicati[on]

* Hermitian Matrix :-

A square matrix A is said to be a Hermitian matrix if the transpose of the conjugate matrix is equal to the matrix itself. i.e. $A^{\theta} = A$.

Note: The elements on the principal diagonal must be all real numbers. i.e. $\bar{a}_{ii} = a_{ii}$.

Ex: $\begin{bmatrix} 2 & 3+2i \\ 3-2i & 7 \end{bmatrix}$, $\begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}$ and

$$\begin{bmatrix} 2 & 1+i & -2+3i \\ 1-i & -1 & 5-6i \\ -2-3i & 5+6i & 0 \end{bmatrix}$$ are

Hermitian matrices.

→ Skew – Hermitian Matrix:–

A squre matrix A is said to be skew-Hermitian if $A^\theta = -A$.

Note: The elements on the principal diagonal must be purely imaginary number or zero.

Ex:– $\begin{bmatrix} 2i & 5+4i \\ -5+4i & 0 \end{bmatrix}$, $\begin{bmatrix} 4i & 2+i & 3 \\ -2+i & 0 & 4i \\ -3 & 4i & -3i \end{bmatrix}$

are skew-Hermitian matrices.

→ Some Properties of Hermitian & Skew – Hermitian matrices :–

→ If A is a Hermitian matrix, show that iA is skew Hermitian.

Sol^n :– Since A is Hermitian

$\therefore A^\theta = A$.

we have $(iA)^\theta = \bar{i}A^\theta$ $(\because (kA)^\theta = \bar{k}A^\theta)$

$= (-i)A^\theta$

$= -(iA^\theta)$

$= -(iA)$ $(\because A^\theta = A)$

$\therefore (iA)^\theta = -(iA)$

$\therefore$ iA is skew Hermitian.

→ If A is a skew-Hermitian matrix, then show that iA is Hermitian.

Sol^n:– Since A is skew-Hermitian.

$\therefore A^\theta = -A$

we have $(iA)^\theta = \bar{i}A^\theta$

$= -i(-A)$ $(\because A^\theta = -A)$

$= iA$

$\therefore (iA)^\theta = iA$

$\therefore$ iA is Hermitian.

→ If A, B are Hermitian or skew Hermitian then $A+B$ is also Hermitian or skew-Hermitian.

Sol^n:–(i) Given that A & B are Hermitian.

$\therefore A^\theta = A$, $B^\theta = B$

we have $(A+B)^\theta = A^\theta + B^\theta$

$= (A+B)$

$\therefore$ $A+B$ is Hermitian.

(ii) Similarly it can be easily done.

→ If A and B are two $n\times n$ matrices then show that

(i) $(-A)' = -(A')$ (ii) $(-A)^\theta = -(A^\theta)$

(iii) $(A-B)' = A'-B'$ (iv) $(A-B)^\theta = A^\theta - B^\theta$

Sol'n :- we have $(-A)' = \{(-1)A\}'$

$= (-1)A'$

$= -A'$

(ii) $(-A)^\theta = \{(-1)A\}^\theta = (\overline{-1})A^\theta$

$= -A^\theta \ (\because (\overline{-1}) = -1)$

(iii) we have

$(A-B)' = \{A+(-B)\}' = A' + (-B)'$

$= A' - B'$.

(iv) $(A-B)^\theta = \{A+(-B)\}^\theta$

$= A^\theta + (-B)^\theta$

$= A^\theta - B^\theta$

→ If A & B are Hermitian then show that $AB+BA$ is Hermitian and $AB-BA$ is Skew Hermitian.

Sol'n :- Given that A & B are Hermitian

$\therefore A^\theta = A$ & $B^\theta = B$ —— (1)

(i) Now we have $(AB+BA)^\theta = (AB)^\theta + (BA)^\theta$

$= B^\theta A^\theta + A^\theta B^\theta$

$= BA + AB$

$= AB + BA$

$\therefore AB+BA$ is Hermitian.

(ii) Now we have $(AB-BA)^\theta = (AB)^\theta - (BA)^\theta$

$= B^\theta A^\theta - A^\theta B^\theta$

$= BA - AB$

$= -(AB-BA)$

$\therefore AB-BA$ is Hermitian.

→ If A be any square matrix then prove that $A+A^\theta$, AA^θ, $A^\theta A$ are all Hermitian and $A-A^\theta$ is skew Hermitian.

Sol'n :- Given that A is any square matrix.

(i) we have $(A+A^\theta)^\theta = A^\theta + (A^\theta)^\theta$

$= A^\theta + A$

$= A + A^\theta$

$\therefore A+A^\theta$ is Hermitian.

(ii) we have $(AA^\theta)^\theta = (A^\theta)^\theta A^\theta$

$= AA^\theta$

$\therefore AA^\theta$ is Hermitian.

(iii) we have $(A^\theta A)^\theta = A^\theta (A^\theta)^\theta$

$= A^\theta A$

$\therefore A^\theta A$ is Hermitian.

(iv) $(A-A^\theta)^\theta = A^\theta - (A^\theta)^\theta$

$= A^\theta - A$

$= -(A - A^\theta)$

$\therefore A-A^\theta$ is skew - Hermitian.

Imp → Show that the matrix $B^\theta AB$ is Hermitian or skew Hermitian according as A is Hermitian (or) skew-Hermitian.

Sol'n :- Case (i)

Since A is Hermitian

$\therefore A^\theta = A$.

Now we have $(B^\theta AB)^\theta = B^\theta A^\theta (B^\theta)^\theta$

$= B^\theta AB$

$\therefore B^\theta AB$ is Hermitian.

Case (ii) :- Since A is skew-Hermitian

$\therefore A^\theta = -A$

we have $(B^\theta AB)^\theta = B^\theta A^\theta (B^\theta)^\theta$

$= B^\theta(-A)B$

$= -(B^\theta AB)$

$\therefore B^\theta AB$ is skew-Hermitian.

Imp → Prove that every square matrix A is uniquely expressed as the sum of a Hermitian and a skew-Hermitian matrix.

Sol'n :- Since A is any square matrix.

$\therefore A+A^\theta$ is Hermitian and

$A-A^\theta$ is skew-Hermitian.

$\therefore \frac{1}{2}(A+A^\theta)$ is Hermitian &

$\frac{1}{2}(A-A^\theta)$ is skew-Hermitian.

Now we have

$A = \frac{1}{2}(A+A^\theta) + \frac{1}{2}(A-A^\theta)$

$= P+Q$ (say) —— ①

where P is Hermitian &

Q is skew-Hermitian.

Now To Prove uniqueness.

i.e. we want to prove that the representation ① of A is unique.

If possible let $A = R+S$ be another representation, where R is Hermitian & S is skew-Hermitian.

Now since R is Hermitian & S is skew-Hermitian.

$\therefore R^\theta = R$, $S^\theta = -S$.

Now $A^\theta = (R+S)^\theta$

$= R^\theta + S^\theta$

$= R - S$ and also —— ②

$A = R+S$ —— ③

from ② & ③ we have

$\therefore A + A^\theta = R+S+R-S$

$= 2R$

and $A - A^\theta = R+S-(R-S)$

$= 2S$

$\Rightarrow R = \frac{1}{2}(A+A^\theta) = P$ and

$S = \frac{1}{2}(A-A^\theta) = Q$

$\Rightarrow$ The representation on ① of A as the sum of a symmetric and skew-symmetric matrix is unique.

Imp → Prove that $\bar{A}$ is Hermitian (or) skew-Hermitian according as A is Hermitian or skew-Hermitian.

Sol'n :- Case (i) : Given that A is Hermitian.

$\therefore A^{\theta} = A$

To Prove $\bar{A}$ is Hermitian.

Now we have

$$(\bar{A})^{\theta} = \left[\overline{(\bar{A})}\right]'$$

$$= A' \quad (\because \overline{(\bar{A})} = A)$$

$$= (A^{\theta})' \quad (\because A \text{ is Hermitian} \Rightarrow A = A^{\theta})$$

$$= ((\bar{A})')' \quad (\because A^{\theta} = (\bar{A})')$$

$$= \bar{A}.$$

$\therefore \bar{A}$ is Hermitian.

(ii) Given that A is skew-Hermitian.

$$\therefore A^{\theta} = -A.$$

we have $(\bar{A})^{\theta} = \left[\overline{(\bar{A})}\right]'$

$$= A'$$

$$= (-A^{\theta})' \quad (\because A^{\theta} = -A)$$

$$= -\left[(\bar{A})'\right]'$$

$$= -\bar{A}.$$

$\therefore \bar{A}$ is also skew-Hermitian.

Imp. Show that every square matrix A can be uniquely expressed as $P+iQ$, where P & Q are Hermitian matrices.

Sol'n :- Let $P = \frac{1}{2}(A+A^{\theta})$ and

$$Q = \frac{1}{2i}(A-A^{\theta})$$

Then $P+iQ = \frac{1}{2}(A+A^{\theta}) + i\frac{1}{2i}(A-A^{\theta})$

$$= \frac{1}{2}(A+A^{\theta}) + \frac{1}{2}(A-A^{\theta})$$

$$= \frac{1}{2}A + \frac{1}{2}A$$

$$= A$$

$\therefore P+iQ = A$ (or) $A = P+iQ$ —— ①

Now we prove that P & Q are Hermitian.

$$P^{\theta} = \left(\frac{1}{2}(A+A^{\theta})\right)^{\theta} = \frac{1}{2}(A+A^{\theta})^{\theta}$$

$$= \frac{1}{2}(A^{\theta}+A)$$

$$= \frac{1}{2}(A+A^{\theta}) = P$$

$\therefore P$ is Hermitian.

Now $Q^{\theta} = \left[\frac{1}{2i}(A-A^{\theta})\right]^{\theta}$

$$= \left(\overline{\frac{1}{2i}}\right)(A-A^{\theta})^{\theta}$$

$$= \frac{-1}{2i}(A^{\theta}-A)$$

$$= \frac{-1}{2i}\left[-(A-A^{\theta})\right]$$

$$= \frac{1}{2i}(A-A^{\theta})$$

$$\therefore Q^{\theta} = Q$$

$\therefore Q$ is Hermitian.

$\therefore$ we have expressed A in the form $P+iQ$.

where P & Q are Hermitian.

Now we prove that the expression ① is unique.

Let us suppose that $A = R + iS$ where R and S are Hermitian.

Since R & S are Hermitian,

$\therefore R^{\theta} = R$ & $S^{\theta} = S$.

Now $A^{\theta} = (R+iS)^{\theta}$

$= R^{\theta} + (iS)^{\theta}$

$= R + \bar{i}S^{\theta} = R - iS$ —— ②

Also $A = R + iS$ —— ③

from ② & ③ we have

$A + A^{\theta} = (R+iS) + (R-iS)$

$= 2R$

and $A - A^{\theta} = (R+iS) - (R-iS)$

$= 2iS$

$\Rightarrow \frac{1}{2}(A+A^{\theta}) = R$ and $\frac{1}{2i}(A-A^{\theta}) = S$

$\Rightarrow R = P$ and $S = Q$.

$\therefore$ The expression ① for A is unique.

→ If $AB = A$ and $BA = B$ then $B'A' = A'$ and $A'B' = B'$ and hence prove that A' and B' are idempotent.

Sol'n :- Since $AB = A \Rightarrow (AB)' = A'$

$\Rightarrow B'A' = A'$

and since $BA = B \Rightarrow (BA)' = B'$

$\Rightarrow A'B' = B'$

Now we prove that A' is idempotent.

We have $(A')^2 = A' \cdot A'$

$= A'(B'A')$

$= (A'B')A'$

$= B'A' \quad (\because A'B' = B')$

$= A'$

$\therefore A'$ is idempotent.

Now we prove that B' is idempotent

We have $(B')^2 = B' \cdot B'$

$= B'(A'B')$

$= (B'A')B'$

$= A'B' \quad (\because B'A' = A')$

$= B' \quad (\because A'B' = B')$

$\therefore B'$ is idempotent.

Determinants

Definition:-

To every square matrix, we associate a unique number called the determinant of matrix.

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots & a_{2n} \\ - & - & - \cdots & - \\ - & - & - \cdots & - \\ a_{n1} & a_{n2} & a_{n3} \cdots & a_{nn} \end{bmatrix}$

is any square matrix of order 'n' then the determinant of A is denoted by $|A|$ or $\det A$ or Δ.

i.e. $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} \cdots & a_{2n} \\ - & - & - \cdots & - \\ - & - & - \cdots & - \\ a_{n1} & a_{n2} & a_{n3} \cdots & a_{nn} \end{vmatrix}$

The numbers $a_{11}, a_{12}, \cdots$ are the elements of the determinant.

Note: (1) In determinant, number of rows must be equal to number of columns.

(2) The determinants has a value.

(3) Determinants are used for judging the invertibility of square matrices.

(4). Determinants are also used to solve the system of linear equations.

Ex:- $|7|$, $\begin{vmatrix} 3 & 5 \\ 7 & 9 \end{vmatrix}$, $\begin{vmatrix} 1 & 7 & -5 \\ 3 & -4 & 8 \\ 0 & 9 & 2 \end{vmatrix}$ are the determinants of order 1, 2, 3 respectively.

→ The determinant of 1×1 matrix $[a]$ is defined to be a.

→ **Determinant of order 2:**

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then the number $ad - bc$ is called the determinant of A: It is denoted by $|A|$.

i.e. $|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

*Minors and Cofactors :-

→**Minor** :- If a_{ij} is an element which is in the i^{th} row and j^{th} column of a square matrix A, then the determinant of the matrix obtained by deleting the i^{th} row and j^{th} column of A is called minor of A. It is denoted by M_{ij}.

Note:- If $A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ then

$$M_{11} = \text{minor of } a_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} = a_{22}a_{33} - a_{32}a_{23}.$$

$$M_{12} = \text{minor of } a_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} = a_{21}a_{33} - a_{31}a_{23}.$$

Cofactor :- If a_{ij} is an element which is in the ith row and jth column of a square matrix A, then the product of $(-1)^{i+j}$ and the minor of a_{ij} is called cofactor of a_{ij}.

It is denoted by A_{ij}.

Note : If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then

$$A_{11} = \text{Cofactor of } a_{11} = (-1)^{1+1} M_{11}$$

$$= +1 \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

$$= a_{22}a_{33} - a_{32}a_{23}$$

$$A_{12} = \text{Cofactor of } a_{12} = (-1)^{1+2} M_{12}$$

$$= -1 \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

$$= -(a_{21}a_{33} - a_{31}a_{23})$$

And so on.

→ If $\Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$ then

$$\Delta = \sum_{j=1}^{3} (-1)^{i+j} a_{ij} M_{ij} = \sum_{j=1}^{3} a_{ij} A_{ij}.$$

$$= a_{i_1}A_{i_1} + a_{i_2}A_{i_2} + a_{i_3}A_{i_3}$$

for either $i=1$ or $i=2$ or $i=3$

i.e. $\Delta = a_{11}A_{11} + a_{12}A_{12} + a_{13}A_{13}$,

$\Delta = a_{21}A_{21} + a_{22}A_{22} + a_{23}A_{23}$, (Row)

$\Delta = a_{31}A_{31} + a_{32}A_{32} + a_{33}A_{33}$.

(or)

$$\Delta = \sum_{i=1}^{3} (-1)^{i+j} a_{ij} M_{ij} = \sum_{i=1}^{3} a_{ij} A_{ij}.$$

$$= a_{1j}A_{1j} + a_{2j}A_{2j} + a_{3j}A_{3j}$$

for either $j=1$ or $j=2$ or $j=3$.

i.e. $\Delta = a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31}$,

$\Delta = a_{12}A_{12} + a_{22}A_{22} + a_{32}A_{32}$

$\Delta = a_{13}A_{13} + a_{23}A_{23} + a_{33}A_{33}$

(Column)

Note !— The determinant of a square matrix A is equal to the sum of the products of the elements of a row (or column) of A with their corresponding cofactors.

Problem

→ Find the value of the determinant of the matrix.

$$A = \begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{bmatrix}$$

Sol'n :— we have $|A| = \begin{vmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{vmatrix}$

$$= a \begin{vmatrix} b & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & d \end{vmatrix}$$

$$= ab \begin{vmatrix} c & 0 \\ 0 & d \end{vmatrix}$$

$$= ab(cd-0) = \underline{abcd}$$

Note: (1) The value of the determinant of diagonal matrix is equal to the product of the elements lying along its principal diagonal.

(2) Let I_n be a unit matrix of order 'n' then $|I_n| = 1$.

$\therefore$ The value of the determinant of a unit matrix is always equal to 1.

→ Find the value of the determinant of the matrix $A = \begin{bmatrix} a & h & g & f \\ 0 & b & c & e \\ 0 & 0 & d & K \\ 0 & 0 & 0 & l \end{bmatrix}$

Sol'n:- We have

$$|A| = \begin{vmatrix} a & h & g & f \\ 0 & b & c & e \\ 0 & 0 & d & K \\ 0 & 0 & 0 & l \end{vmatrix}$$

$$= a \begin{vmatrix} b & c & e \\ 0 & d & K \\ 0 & 0 & l \end{vmatrix}$$

$$= ab \begin{vmatrix} d & K \\ 0 & l \end{vmatrix}$$

$$= ab(dl - 0) = abdl.$$

Note: (1) The value of the determinant of an upper triangular matrix (i.e. in which all the elements below the principal diagonal are zero) is equal to the product of the elements along the principal diagonal.

(2) The value of the determinant of a lower triangular matrix is equal to the product of the elements along the principal diagonal.

* Properties of Determinants:-

→ The value of a determinant does not change when rows and columns are interchanged.

i.e. $$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{21} & \cdots & a_{n1} \\ a_{12} & a_{22} & \cdots & a_{n2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{vmatrix}$$

→ If A be an n-rowed square matrix then $|A| = |A^T|$.

→ If any two rows (or two columns) of a determinant are interchanged, then the value of the determinant is multiplied by -1.

→ If all the elements of one row (or one column) of a determinant are multiplied by the same number K, then the value of the new determinant is K times the value of the given determinant.

i.e. $$\begin{vmatrix} Ka_1 & b_1 & c_1 \\ Ka_2 & b_2 & c_2 \\ Ka_3 & b_3 & c_3 \end{vmatrix} = K \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

→ If A be an n-rowed square matrix, and K be any scalar then $|KA| = K^n|A|$.

i.e. $|KA| = \begin{vmatrix} Ka_{11} & Ka_{12} & \cdots & Ka_{1n} \\ Ka_{21} & Ka_{22} & \cdots & Ka_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ Ka_{n1} & Ka_{n2} & \cdots & Ka_{nn} \end{vmatrix}$.

$= K.K \cdots K(n\text{ times}) \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}$

$= K^n|A|$.

→ If two rows (or columns) of a determinant are identical, then the value of the determinant is zero.

i.e. $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \end{vmatrix} = 0$

→ In a determinant, the sum of the product of the elements of any row (column) with the cofactors of the corresponding elements of any other row (column) is zero.

Ex:- $\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & 4 \\ 6 & 7 & 8 \end{vmatrix} = 1(40-28) - 2(-24)+3(-30)$

Now $1(16-21)-2(8-18)+3(7-12)$

$= -5+20-15 = 0$

<u>Note</u>:- Let

$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ be a determinant of order 3.

Let A_1, B_1, C_1 etc be the cofactors of the elements a_1, b_1, c_1 etc in Δ.

Then we have

$a_1A_1 + b_1B_1 + c_1C_1 = \Delta$

$a_1A_2 + b_1B_2 + c_1C_2 = 0$

$a_1A_3 + b_1B_3 + c_1C_3 = 0$

$a_2A_2 + b_2B_2 + c_2C_2 = \Delta$

$a_2A_1 + b_2B_1 + c_2C_1 = 0$ etc.

→ Let A be a square matrix of order 'n' then show that

(i) $|\bar{A}| = \overline{|A|}$ (ii) $|A^\theta| = \overline{|A|}$

<u>Sol'n</u>:- Let $A = [a_{ij}]_{n\times n}$ then we have $\bar{A} = [\overline{a_{ij}}]_{n\times n}$

we have $|\bar{A}| = |\overline{a_{ij}}| = \overline{|a_{ij}|} = \overline{|A|}$

(ii) we have $A^\theta = (\bar{A}')$

$\therefore |A^\theta| = |(\bar{A}')| = \overline{|A'|}$

$= \overline{|A|}$ ($\because |A| = |A'|$)

→ Show that the determinant of a Hermitian matrix is always a real number.

<u>Sol'n</u>:- Let A be a Hermitian matrix.

Then $A^\theta = A$

$\therefore |A| = |A^\theta| = \overline{|A|}$

we know that if z is a complex number such that $z = \bar{z}$ then z is real.

$\therefore |A| = |\bar{A}|$

$\Rightarrow |A|$ is a real number.

→ Show that the value of the determinant of a skew-symmetric matrix of odd order is always zero.

Sol'n :- Let $A = \begin{bmatrix} 0 & -h & -g \\ h & 0 & -f \\ g & f & 0 \end{bmatrix}$ be a skew-symmetric matrix of order 3.

Then $|A| = \begin{vmatrix} 0 & -h & -g \\ h & 0 & -f \\ g & f & 0 \end{vmatrix}$

$= (-1)^3 \begin{vmatrix} 0 & h & g \\ -h & 0 & f \\ -g & -f & 0 \end{vmatrix}$

$= - \begin{vmatrix} 0 & -h & -g \\ h & 0 & -f \\ g & f & 0 \end{vmatrix}$ [By interchanging the rows & columns].

$= -|A|$

$\therefore |A| = -|A| \Rightarrow 2|A| = 0$

$\Rightarrow |A| = 0$

Note: If A, B are square matrices conformable for multiplication then $|AB| = |A|\,|B|$.

→ If A, B are square matrices each of order 'n' such that $|AB| = 0$ then prove that $|A| = 0$ or $|B| = 0$

Sol'n :- Since $|AB| = 0 \Rightarrow |A||B| = 0$

$\Rightarrow |A| = 0$ or $|B| = 0$.

* Adjoint Matrix :-

The transpose of the matrix obtained by replacing the elements of a square matrix A by the corresponding cofactors is called the adjoint matrix of A.

It is denoted by Adj A or adj A.

Note: (1)

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then

$$\text{Adj} A = \begin{bmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{bmatrix}$$

(2) If

$A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ then $\text{Adj} A = \begin{bmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{bmatrix}$

Imp → If A be any n-rowed square matrix then $(\text{adj} A)A = A(\text{adj} A) = |A| I_n$

Sol'n :-

Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ - & - & - & \cdots & - \\ - & - & - & \cdots & - \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}_{n\times n}$ then

$$\text{adj}A = \begin{bmatrix} A_{11} & A_{21} & A_{31} & \cdots & A_{n1} \\ A_{12} & A_{22} & A_{32} & \cdots & A_{n2} \\ - & - & - & \cdots & - \\ - & - & - & \cdots & - \\ A_{1n} & A_{2n} & A_{3n} & \cdots & A_{nn} \end{bmatrix}$$

Now we have

$$(\text{adj}A)A = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ - & - & \cdots & - \\ - & - & \cdots & - \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ - & - & \cdots & - \\ - & - & \cdots & - \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

The $(i,j)^{th}$ element in the matrix $(\text{adj}A)A$ is

$$A_{1i}a_{1j} + A_{2i}a_{2j} + A_{3i}a_{3j} + \cdots + A_{ni}a_{nj}$$
$$= |A|, \text{ if } i = j$$
$$= 0 \text{ if } i \neq j.$$

$$\Rightarrow (\text{adj}A)A = \begin{bmatrix} |A| & 0 & 0 & \cdots & 0 \\ 0 & |A| & 0 & \cdots & 0 \\ 0 & 0 & |A| & \cdots & 0 \\ - & - & - & \cdots & - \\ - & - & - & \cdots & - \\ 0 & 0 & 0 & \cdots & |A| \end{bmatrix}$$

$$= |A| \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ - & - & \cdots & - \\ - & - & \cdots & - \\ 0 & 0 & \cdots & 1 \end{bmatrix} = |A| I_n$$

Similarly $A(\text{adj}A) = |A| I_n$

$\therefore (\text{adj}A)A = A(\text{adj}A) = |A| I_n.$

Note: If $|A| \neq 0$ then

$$A\left(\frac{1}{|A|}(\text{adj}A)\right) = \left(\frac{1}{|A|}(\text{adj}A)\right)A = I.$$

$[\because A(\text{adj}A) = (\text{adj}A)A = |A| I_n]$

* Inverse of a Square Matrix:-

Let A be any square matrix, if there exists a square matrix B such that $AB = BA = I$. Then the matrix B is called inverse of A.

Note:- (1) For AB, BA to be both defined and equal, it is necessary that A and B are both square matrices of same order.

(2) A rectangular matrices cannot have inverse.

(3) Every square matrix cannot have inverse.

→ A matrix is said to be invertible if it has inverse.

Imp → Every Invertible matrix has unique inverse.

Sol^n:- Let A be an invertible matrix. Let B and C be two inverses of A.

Then $AB = BA = I$ —— (1)

and $AC = CA = I$ —— (2)

Now we have $B = BI$

$= B(AC)$ (By (2))

$= (BA)C$ (By associative property)

$= IC$ (By (1))

$= C$

$\therefore B = C$

$\therefore$ A has unique inverse.

Note :– (1) If A is an invertible matrix then its inverse is denoted by A^{-1}.

$\therefore AA^{-1} = A^{-1}A = I$.

(2) Since $I.I = I$, we have $I^{-1} = I$.
i.e. the inverse of a unit matrix is itself.

(3). Since $A^{-1}A = AA^{-1} = I \Rightarrow (A^{-1})^{-1} = A$.

(4). If A is invertible matrix and if $A = B$ then $A^{-1} = B^{-1}$.

Imp. The necessary and sufficient condition for a square matrix A to possess the inverse is that $|A| \neq 0$.

Proof :–

N.C. : Let A be a square matrix.
Let B be the inverse of A
Then $AB = I$

$\Rightarrow |AB| = |I|$

$\Rightarrow |A||B| = 1$

$\Rightarrow |A| \neq 0$.

S.C :– Let $|A| \neq 0$

To prove the matrix A possess the inverse.

we know that

$A(\text{adj} A) = (\text{adj} A)A = |A| I$.

$\Rightarrow A\left(\frac{1}{|A|}\text{adj} A\right) = \left(\frac{1}{|A|}\text{adj} A\right)A = I$

$(\because |A| \neq 0.)$

$\Rightarrow AB = BA = I$ where

$B = \frac{1}{|A|}\text{adj} A$.

$\Rightarrow$ B is the inverse of A.

$\Rightarrow$ A has the inverse.

Note: If $|A| \neq 0$ then $A^{-1} = \frac{1}{|A|}(\text{adj} A)$

* Singular and Non-Singular Matrices :–

→ A square matrix 'A' is said to be singular if $|A| = 0$.

→ A square matrix A is said to be Non-singular if $|A| \neq 0$.

→ Find the inverse of the matrix

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$$ Also verify your result.

sol'n :– we have $|A| = \begin{vmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{vmatrix}$.

$= 0(-1) - 1(-8) + 2(-5)$

$= 8 - 10$

$= -2 \neq 0$.

$\therefore A^{-1}$ exists.

Now the Co-factor matrix of

$$A = \begin{bmatrix} -1 & 8 & -5 \\ 1 & -6 & 3 \\ -1 & 2 & -1 \end{bmatrix} = B \text{ (say)}$$

Now $\text{adj} A = B^T$

$$= \begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$$

Now $A^{-1} = \dfrac{\text{adj} A}{|A|}$

$$= -\frac{1}{2}\begin{bmatrix} -1 & 1 & -1 \\ 8 & -6 & 2 \\ -5 & 3 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$$

Verification:–

$$AA^{-1} = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1/2 & -1/2 & 1/2 \\ -4 & 3 & -1 \\ 5/2 & -3/2 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

Similarly $A^{-1}A = I_3$

$\therefore A^{-1}A = AA^{-1} = I_3$

→ Find the inverse of the matrix $S = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ and show that SAS^{-1} is a diagonal matrix

where $A = \frac{1}{2}\begin{bmatrix} b+c & c-a & b-a \\ c-b & c+a & a-b \\ b-c & a-c & a+b \end{bmatrix}$

Sol'n :– $|S| = 0(-1) - 1(-1) + 1(1)$

$= 1 + 1 = 2$

$\therefore |S| \neq 0$

$\therefore S^{-1}$ exists.

Cofactors matrix of $S = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

$= B$ (say)

Now $\text{adj} A = B^T = \begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

Now $S^{-1} = \dfrac{\text{adj} S}{|S|} = \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$

Now $SA = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \times \frac{1}{2}\begin{bmatrix} b+c & c-a & b-a \\ c-b & c+a & a-b \\ b-c & a-c & a+b \end{bmatrix}$

$$= \frac{1}{2}\begin{bmatrix} 0 & 2a & 2a \\ 2b & 0 & 2b \\ 2c & 2c & 0 \end{bmatrix} = \begin{bmatrix} 0 & a & b \\ b & 0 & b \\ c & c & 0 \end{bmatrix}.$$

$$\therefore SAS^{-1} = \begin{bmatrix} 0 & a & a \\ b & 0 & b \\ c & c & 0 \end{bmatrix} \times \frac{1}{2}\begin{bmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} 2a & 0 & 0 \\ 0 & 2b & 0 \\ 0 & 0 & 2c \end{bmatrix} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}.$$

$\therefore SAS^{-1}$ is a diagonal matrix.

Imp* If A, B be two n-rowed non-singular matrices, then AB is also non-singular and $(AB)^{-1} = B^{-1}A^{-1}$.

Proof :- Let A & B be two n-rowed non-singular matrices.

$\therefore |A| \neq 0, \ |B| \neq 0.$

Now we have $|AB| = |A|\,|B|$

$\Rightarrow |AB| \neq 0 \quad (\because |A| \neq 0, |B| \neq 0)$

$\therefore$ AB is non-singular.

$\therefore$ AB has inverse.

$\therefore$ AB is invertible.

Let us define a matrix C by the relation $C = B^{-1}A^{-1}$.

Then $C(AB) = (B^{-1}A^{-1})(AB)$

$= B^{-1}(A^{-1}A)B$

$= B^{-1}IB = B^{-1}B = I$

Also $(AB)C = (AB)(B^{-1}A^{-1})$

$= A(BB^{-1})A^{-1}$

$= AIA^{-1}$

$= AA^{-1} = I$

$\therefore C(AB) = (AB)C = I$

$\therefore C = B^{-1}A^{-1}$ is the inverse of AB.

→ If A be an n×n non-singular matrix then $(A^T)^{-1} = (A^{-1})^T$

Sol'n:- Since $|A| \neq 0$

Now $|A^T| = |A| \neq 0$.

$\therefore A^T$ is non-singular.

We know that $AA^{-1} = A^{-1}A = I$

$\therefore (AA^{-1})^T = (A^{-1}A)^T = I^T$

$\Rightarrow (A^{-1})^T A^T = A^T(A^{-1})^T = I.$

$\therefore (A^{-1})^T$ is the inverse of A^T.

$\therefore (A^T)^{-1} = (A^{-1})^T.$

→ If A be an n×n non-singular matrix then $(A^{-1})^\theta = (A^\theta)^{-1}$.

Sol'n:- Since A is non-singular matrix

$\therefore |A| \neq 0.$

Now $|A^\theta| = \overline{|A|} \neq 0. \quad (\because |A| = |\bar{A}|)$

$\therefore A^\theta$ is non-singular.

We know that $AA^{-1} = A^{-1}A = I$.

$\Rightarrow (AA^{-1})^\theta = (A^{-1}A)^\theta = I^\theta$

$\Rightarrow (A^{-1})^\theta A^\theta = A^\theta (A^{-1})^\theta = I \quad (\because I^\theta = (\bar{I})' = I' = I).$

$\Rightarrow (A^{-1})^\theta$ is the inverse of A^θ.

$\therefore (A^\theta)^{-1} = (A^{-1})^\theta$

Note:- If A is a square matrix then $\text{adj}\,A^T = (\text{adj}\,A)^T$.

→ If A is a symmetric matrix then show that adj A is symmetric.

Sol'n:- Since A is symmetric.

$\therefore A^T = A.$

Now we have $(\text{adj}\,A)^T = \text{adj}\,A^T$

$= \text{adj}\,A \quad (\because A^T = A)$

$\therefore$ adj A is symmetric.

→ If the non-singular matrix A is symmetric then A^{-1} is also symmetric.

Sol'n :- Since A is non-singular.

i.e. $|A| \neq 0$.

$\therefore A^{-1}$ exists.

and A is symmetric.

$\therefore A^T = A$.

Now we have $(A^{-1})^T = (A^T)^{-1}$

$= A^{-1}$ $(\because A^T = A)$

$\therefore A^{-1}$ is symmetric.

→ Show that if A is a non-singular matrix, then $\det(A^{-1}) = (\det A)^{-1}$

Sol'n :- Since A is non-singular i.e. $|A| \neq 0$

$\therefore A^{-1}$ exists.

$\therefore A^{-1}A = AA^{-1} = I$

Now we have

$A^{-1}A = I \Rightarrow |A^{-1}A| = |I|$

$\Rightarrow |A^{-1}||A| = 1$

$\Rightarrow |A^{-1}| = \frac{1}{|A|}$

$\Rightarrow |A^{-1}| = |A|^{-1}$

$\Longrightarrow \det(A^{-1}) = (\det A)^{-1}$.

→ If B is non-singular, prove that the matrices A & $B^{-1}AB$ have the same determinant, A and B being both square matrices.

Sol'n :- Since B is non-singular

$\therefore |B| \neq 0$.

Now we have $|B^{-1}AB| = |B^{-1}||A||B|$

$= |B^{-1}||B||A|$

$= |B^{-1}B||A|$

$= |I||A|$

$= 1.|A|$

$\therefore |B^{-1}AB| = |A|$

→ If A be an $n \times n$ square matrix then prove that $|\text{adj} A| = |A|^{n-1}$.

Sol'n :- we have $A(\text{adj} A) = |A|.I_n$

$\Rightarrow |A(\text{adj} A)| = ||A|.I_n|$

$\Rightarrow |A||\text{adj} A| = ||A|||I_n|$

$\Rightarrow |A||\text{adj} A| = |A|^n.1$ $(\because |kA| = k^n|A|$ & $|I_n| = 1)$

$\Rightarrow |\text{adj} A| = |A|^{n-1}$ if $|A| \neq 0$

Note :- (1). From the above example if $|A| = 0$ then $|\text{adj} A| = 0$.

(2) From the above example. if $|A| \neq 0$ then $|\text{adj} A| \neq 0$. i.e. if A is non-singular then adj$A$ is non-singular.

→ If A is a non-singular matrix then show that

$\text{adj}(\text{adj} A) = |A|^{n-2} A$.

Sol'n :- Since A is non-singular.

$\therefore |A| \neq 0$ and A^{-1} exists.

we know that $A(\text{adj} A) = |A| I_n$

Taking adj A in place of A in (1), we get

$(\text{adj}A)(\text{adj}(\text{adj}A)) = |\text{adj}A|\, I_n$

$\Rightarrow (\text{adj}A)(\text{adj}(\text{adj}A)) = |A|^{n-1} I_n$

$(\because |\text{adj}A| = |A|^{n-1})$

Pre-multiplying bothsides by A, we get

$(A(\text{adj}A))(\text{adj}(\text{adj}A)) = |A|^{n-1}(A I_n)$

$\Rightarrow (|A| I_n)[\text{adj}(\text{adj}A)] = |A|^{n-1} A$

$(\because A I_n = A)$

$\Rightarrow |A|[I_n \,\text{adj}(\text{adj}A)] = |A|^{n-1} A.$

$\Rightarrow |A|[\text{adj}(\text{adj}A)] = |A|^{n-1} A$

Since $|A| \neq 0$.

$\therefore \boxed{\text{adj}(\text{adj}A) = |A|^{n-2} A}$

→ If A & B are two non-singular matrices of the same type then $(\text{adj}AB) = (\text{adj}B)(\text{adj}A)$.

Sol'n :- Since $|A| \neq 0$, $|B| \neq 0$

$\therefore |AB| \neq 0.$

we know that $(AB)(\text{adj}(AB)) = |AB|\, I$

$= (\text{adj}(AB))(AB)$ — (1)

Now $(AB)(\text{adj}B \cdot \text{adj}A) = A(B\,\text{adj}B)\text{adj}A$

(By matrix associative property)

$= A(|B| I)\text{adj}A$

$= |B|(AI)\text{adj}A$

$= |B|(A\,\text{adj}A)$

$= |B|(|A| I)$

$= (|B||A| I)$

$= (|A||B|)I$

$= |AB|\, I$

$= (AB)(\text{adj}(AB))$ (By (1))

$\Rightarrow \boxed{\text{adj}B\,\text{adj}A = \text{adj}(AB)}$

* Orthogonal and Unitary Matrix :-

→ **Orthogonal Matrix :-**

A matrix A is said to be orthogonal if $A^T A = I$.

Note : If A is an orthogonal matrix then $A^T A = I$.

$\Rightarrow |A^T A| = |I|$

$\Rightarrow |A^T||A| = 1$ $(\because |AB| = |A||B|)$

$\Rightarrow |A||A| = 1$ $(\because |A^T| = |A|)$

$\Rightarrow |A|^2 = 1$

$\Rightarrow |A| = \pm 1$

$\Rightarrow |A| \neq 0$

$\therefore A^{-1}$ exists.

$\therefore A$ is invertible.

and $A^T A = I \Rightarrow \boxed{A^T = A^{-1}}$

$\Rightarrow \boxed{AA^T = I}$

$\therefore A$ is orthogonal iff $A^T A = I = AA^T$

i.e. iff $A^T = A^{-1}$.

→ **Unitary matrix :-**

A matrix A is said to be unitary matrix if $A^\theta A = I$.

Note : If A is unitary then $A^\theta A = I$

$\Rightarrow |A^\theta A| = |I|$

$\Rightarrow |A^\theta||A| = 1$.

$\Rightarrow |\bar{A}||A| = 1$ $(\because |A^\theta| = |\bar{A}|)$

$\Rightarrow \overline{|A|}\,|A| = 1$

$\Rightarrow |A|^2 = 1$ $(\because z = x+iy,\ \bar{z} = x-iy$

$\Rightarrow |A| = \pm 1$ $\Rightarrow z\bar{z} = |z|^2)$

$\Rightarrow |A| \neq 0$

$\therefore A^{-1}$ exists.

$\therefore A$ is invertible.

and $A^\theta A = I \Rightarrow \boxed{A^\theta = A^{-1}}$

$\Rightarrow \boxed{AA^\theta = I}$

$\therefore A$ is unitary $\Leftrightarrow A^\theta A = I = AA^\theta$

i.e. $\Leftrightarrow A^\theta = A^{-1}$

* Some Properties of Orthogonal and unitary Matrices :-

→ If A, B are n-rowed orthogonal matrices then AB & BA are also orthogonal matrices.

Sol^n :- Since A & B are orthogonal.

$\therefore A^T A = I = AA^T$ & $B^T B = I = BB^T$ — (1)

Since A & B are n-rowed square matrices.

$\therefore AB$ is also n-rowed square matrix.

Now $(AB)^T (AB) = B^T A^T (AB)$

$= B^T (A^T A) B$ (By associative prop.)

$= B^T (I_n) B$ (By (1))

$= B^T (I_n B)$

$= B^T B$

$= I_n$ (By (1))

$\therefore AB$ is orthogonal.

Similarly we can prove BA is orthogonal.

→ If A, B be n-rowed unitary matrices then AB and BA are also unitary matrices.

Sol'n :- Since A & B are unitary matrices

$\therefore AA^\theta = I = A^\theta A$ & $BB^\theta = I = B^\theta B$ —— (1)

and since A, B are n-rowed square matrices.

$\therefore$ AB is also n-rowed square matrix.

Now we have

$$(AB)^\theta (AB) = (B^\theta A^\theta) AB$$

$$= B^\theta (A^\theta A) B$$

$$= B^\theta (IB) \quad (\text{By } (1))$$

$$= B^\theta B$$

$$= I \quad (\text{By } (1))$$

$\therefore$ AB is unitary.

Similarly we can prove BA is also unitary.

→ A real matrix is unitary $\Leftrightarrow$ it is orthogonal.

Sol'n :- Let A be a real matrix then

$$A^\theta = (\bar{A})' = A' \quad \text{—— (1)}$$

Since A is unitary.

$$\therefore A^\theta A = I$$

$$\Rightarrow A'A = I \quad (\text{by } (1))$$

$\Rightarrow$ A is orthogonal.

Conversely suppose that A is orthogonal.

$$\therefore A'A = I$$

$$\Rightarrow A^\theta A = I \quad (\text{By } (1))$$

$\Rightarrow$ A is unitary.

→ If P is orthogonal then P^T and P^{-1} are also orthogonal.

Sol'n :- P is orthogonal

$$\therefore P^T P = I \Rightarrow PP^T = I \quad \text{—— (1)}$$

Now we have

$$(P^T)^T P^T = PP^T$$

$$= I \quad (\text{by } (1))$$

$\therefore P^T$ is orthogonal.

Again $(P^{-1})^T (P^{-1}) = (P^T)^{-1} P^{-1}$

$$= (PP^T)^{-1} \quad (\because (AB)^{-1} = B^{-1}A^{-1})$$

$$= I^{-1} \quad (\text{By } (1))$$

$$= I$$

$\therefore P^{-1}$ is unitary.

→ If P is unitary then $\bar{P}$, P', P^θ and P^{-1} are also unitary.

Sol'n :- Since P is unitary.

$$\therefore P^\theta P = I = PP^\theta \quad \text{—— (1)}$$

(1) Now we have $(\bar{P})^\theta \bar{P} = [\overline{(\bar{P})}]' \bar{P}$

$$= [P'] \bar{P}$$

$$= [\overline{(\bar{P}')}] \bar{P}$$

$$= \overline{(P^\theta)} \bar{P} = \overline{(P^\theta P)}$$

$$= \bar{I} \quad (\text{by } (1))$$

$$= I.$$

$\therefore \bar{P}$ is unitary.

(ii) $(P')^{\theta} P' = [(\overline{P'})]' P'$

$= [(\overline{P})']' P' = [P^{\theta}]' P'$

$= [P P^{\theta}]' \quad (\because (AB)^T = B^T A^T)$

$= I'$

$= I$

$\therefore P^T$ is unitary.

(iii) we have $(P^{\theta})^{\theta} P^{\theta} = PP^{\theta} = I$

(iv) $(P^{-1})^{\theta} (P^{-1}) = (P^{\theta})^{-1} P^{-1}$

$= (PP^{\theta})^{-1}$

$= I^{-1} = I$

$\therefore P^{-1}$ is unitary.

→ A real skew-symmetric A satisfies the relation $A^2 + I = 0$ where I is the identity matrix. show that A is orthogonal.

Sol'n :- Given that A is real skew symmetric matrix.

$\therefore A^T = -A$ —①

Also given $A^2 + I = 0$

$\Rightarrow A^2 = -I$ —②

Now we have $AA^T = A(-A)$

(By ①)

$= -A^2$

$= -(-I)$

(By ②)

$= I$.

$\therefore$ A is orthogonal.

→ show that if A is Hermitian and P is unitary then $P^{-1}AP$ is Hermitian.

Sol'n :- Since A is Hermitian

$\therefore A^{\theta} = A$ —①

and P is unitary

$\therefore P^{\theta} P = I \Rightarrow P^{\theta} = P^{-1}$ —②

Now we have

$(P^{-1}AP)^{\theta} = P^{\theta} A^{\theta} (P^{-1})^{\theta}$

$= P^{\theta} A^{\theta} (P^{\theta})^{\theta}$ (By ②)

$= P^{\theta} A P$ (By ①)

$= P^{-1} A P$ (By ②)

$\therefore P^{-1}AP$ is Hermitian.

* Submatrix of Matrix :-

Suppose A is any matrix then a matrix obtained by leaving some rows and columns from A is called a Submatrix of A.

Note :- The matrix A itself is a Submatrix of A because it is obtained from A by leaving no rows or columns.

Minors of a Matrix :-

Let A be an $m \times n$ matrix then the determinant of every square submatrix of A is called a minor of the matrix A.

i.e. If we leave $m-p$ rows and $n-p$ columns from A then the square submatrix of A of order p.

The determinant of this square submatrix is called a p-rowed minor of A.

Ex :- $\begin{bmatrix} 2 & 4 & 1 & 9 & 1 \\ 0 & 5 & 2 & 5 & 2 \\ 3 & -2 & 8 & 1 & 8 \\ 1 & 9 & 7 & 3 & 4 \end{bmatrix}_{4\times 5}$

If we leave two columns and one row from A then we get square submatrix of A of order 3.

$\begin{vmatrix} 2 & 5 & 2 \\ 8 & 1 & 8 \\ 7 & 3 & 4 \end{vmatrix}$, $\begin{vmatrix} 1 & 9 & 1 \\ 2 & 5 & 2 \\ 8 & 1 & 8 \end{vmatrix}$ etc.

are 3-rowed minors of A.

* Rank of Matrix :-

A number r is said to be rank of a matrix A if

(i) there exists at least one minor of order r of the matrix A which is not zero and

(ii) Each minor of order $(r+1)$ of the matrix A is zero. (i.e. Vanishes)

Note :- (1) The rank of matrix A is any highest non-zero minor of order of the matrix A.

(2) Rank of A is denoted by $\rho(A)$ and is unique.

(3) Every matrix will have a rank.

(4) If A is a matrix of order $m\times n$, then $\rho(A) \le m$ or n (smaller of the two)

(5) If $\rho(A) = n$ then every minor of order $n+1, n+2$ etc is '0'.

(6) A is a square matrix order $n\times n$

$|A| \neq 0 \Leftrightarrow \rho(A) = n$.

(7) $\rho(I_n) = n$.

(8) A is a matrix of order $m\times n$.

If every k^{th} order minor $(k<m, k<n)$ is zero then $\rho(A) < k$.

(9) A is a matrix of order $m\times n$. If there is a minor of order K $(K<m, K<n)$ which is not zero then $\rho(A)\geq K$.

(10). If A is null matrix then $\rho(A)=0$. Since the rank of every non-zero matrix is ≥ 1, we agree to assign the rank, zero, to every null matrix.

* Elementary operations (or) Elementary transformations of a matrix :-

1) Interchange of the i^{th} and j^{th} rows : $R_i \longleftrightarrow R_j$ or R_{ij}.

2) Multiplying the i^{th} row by a non-zero scalar K : $R_i \longrightarrow KR_i$ or $R_i(K)$.

3) Adding to the i^{th} row k times the j^{th} row : $R_i \longrightarrow R_i + KR_j$. (or) $R_{ij}(K)$.

The corresponding column transformations are respectively.

$C_i \leftrightarrow C_j$ or C_{ij}, $C_i \rightarrow KC_i$ or $C_i(K)$, $C_i \rightarrow C_i + KC_j$ or $C_{ij}(K)$

Echelon Matrix :-

A matrix 'A' is said to be in echelon form iff the number of zeros preceding the non-zero elements of a rows increases row by row. The elements of the last row or rows may be all zeros.

(OR)

A matrix 'A' is said to be in echelon form if

(i) The number of zeros before the first non-zero element in a row is less than the number of such zeros in the next row.

(ii) The elements of the last row or rows may be all zero.

Note :- (1) the first non-zero elements in the rows of an echelon matrix A are called distinguished elements of A.

Ex :- $\begin{bmatrix} -3 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 4 \end{bmatrix}$, $\begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 1 & 5 & 0 & -7 \\ 0 & 3 & 1 & 5 \\ 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ are echelon matrices.

(2) Triangular matrix is also called echelon form.

Row Reduced Echelon Matrix :-

An echelon matrix is called a row reduced echelon matrix or row canonical form iff the distinguished elements are each equal to 1 and are the only non-zero elements in their respective columns.

Ex :- $\begin{bmatrix} 1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ is a row reduced echelon matrix.

Note :- The rank of a matrix in Echelon form is equal to the number of non-zero rows of the matrix.

Ex :-

$$A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

clearly which is in echelon form

∴ rank A = the number of non-zero rows of A = 2.

Note :- (1) The rank of the transpose of a matrix is equal to the rank of the original matrix

i.e. $\rho(A) = \rho(A^T)$.

(2). The rank of a matrix every element of which is unity is 1.

→ If A is a non-zero column and B is a non-zero row matrix then show that $\rho(AB) = 1$.

Sol'n :- Let $A = \begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \\ \vdots \\ a_{m1} \end{bmatrix}_{m\times 1}$ and

$B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \end{bmatrix}_{1\times n}$.

then

$$AB = \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & \cdots & a_{11}b_{1n} \\ a_{21}b_{11} & a_{21}b_{12} & \cdots & a_{21}b_{1n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1}b_{11} & a_{m1}b_{12} & \cdots & a_{m1}b_{1n} \end{bmatrix}_{m\times r}$$

Since A & B are non-zero matrices.

∴ AB is also non-zero matrix.

The matrix AB will have atleast one non-zero element obtained by multiplying corresponding non-zero elements of A & B.

All the two-rowed minors of AB is obviously zero.

But AB is a non-zero matrix.

∴ $\rho(AB) = 1$.

Ex :- Let $A = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$; $B = \begin{bmatrix} -4 & 5 & 6 \end{bmatrix}$

then $AB = \begin{bmatrix} -8 & 10 & 12 \\ -4 & 5 & 6 \\ -12 & 15 & 18 \end{bmatrix}$

Here all two-rowed minors are obviously zero.

But AB is non-zero matrix.

∴ $\rho(AB) = 1$.

→ If $U = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, find the values of U and U^2

Sol'n :- clearly U is in echelon form

The number of non-zero rows in echelon form = 3.

$\therefore\ \rho(U) = 3$

Now $U^2 = U \cdot U$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ which is in echelon form.

the number of non-zero rows in echelon form $= 2$.

$\therefore\ \underline{\underline{\rho(U^2) = 2.}}$

* Partitioning of Matrices:-

A matrix may be subdivided into sub-matrices by drawing lines parallel to its rows and columns.

So that the elements contained in rectangular blocks are the submatrix elements of the given matrix. This is called partitioning of matrices.

A matrix may be partitioned in many ways and it will be partitioned depending on a situation.

One useful partitioning is given below.

$$\text{Let } A = \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 7 & 8 & 10 & 11 & 12 \\ \hline 0 & 8 & 2 & 3 & 1 & 2 \\ \hline 4 & 5 & 1 & -3 & 0 & 0 \\ 2 & 4 & 5 & 1 & 1 & 3 \end{array}\right] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix}$$

where $A_{11}, A_{12}, A_{21}, A_{22}, A_{31}, A_{32}$ are the submatrices of orders 2×3, 2×3, 1×3, 1×3, 2×3, 2×3 respectively.

→ One more useful representation of a matrix products is given below.

Let $A = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{bmatrix}_{m\times1}$ where each of $R_1, R_2 \cdots R_m$ is a matrix of order $1\times n$. (or) Row vectors of A.

and $B = [C_1\ C_2 \cdots C_p]_{1\times p}$ where each of $C_1, C_2, \cdots C_p$ is a matrix of order $n\times1$ (or) column vectors of B.

then

$$AB = \begin{bmatrix} R_1C_1 & R_2C_2 & \cdots & R_1C_p \\ R_2C_1 & R_2C_2 & \cdots & C_2C_p \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ R_mC_1 & R_mC_2 & \cdots & C_mC_p \end{bmatrix}_{m\times p}$$

where $R_1C_1, R_1C_2 \cdots R_mC_p$ are all matrices each of order 1×1.

* Matrices Partitioned identically for addition :-

Let A, B be two matrices of the same order.

If the submatrices A_{ij} of A and B_{ij} of B are of the same order, then we say that the matrices are identically partitioned.

Ex: Let

$$A = \left[\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ \hline 9 & 10 & 11 & 12 \end{array}\right] \text{ and}$$

$$B = \left[\begin{array}{ccc|c} 0 & -1 & -2 & -3 \\ -4 & -5 & -6 & -7 \\ \hline -8 & -9 & -10 & -11 \end{array}\right]$$

then A & B are identically partitioned.

and $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}_{2\times2}$ $B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}_{2\times2}$

$$\therefore\ A+B = \begin{bmatrix} A_{11}+B_{11} & A_{12}+B_{12} \\ A_{21}+B_{21} & A_{22}+B_{22} \end{bmatrix}_{2\times 2}$$

* Matrices Partitioned Conformably for multiplication :—

Let $A=[a_{ij}]_{m\times n}$ and $B=[b_{ij}]_{n\times p}$ then AB exists.

Let A be partitioned in any way that the partitioning lines drawn parallel to the rows of B are in the same relative position as the partitioning lines drawn parallel to the columns of A.

The matrices A & B partitioned in the above manner are said to be conformably partitioned for multiplication.

Ex:— this line is connection with any rule.

$$A = \left[\begin{array}{ccc|cc} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 10 \\ \hline 11 & 12 & 13 & 14 & 15 \\ -1 & -2 & -3 & -4 & -5 \end{array}\right]_{4\times 5}$$

$$B = \left[\begin{array}{c|cc|cc} 0 & -1 & -2 & -3 & -4 \\ -5 & -6 & -7 & -8 & -9 \\ -10 & -11 & -12 & -13 & -14 \\ \hline -15 & -16 & -17 & -18 & -19 \\ -20 & -21 & -22 & -23 & -24 \end{array}\right]_{5\times 5}$$

then

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},\quad B = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \end{bmatrix}$$

Problems :—

→ If P, Q are non-singular matrices, show that if $A = \begin{bmatrix} P & O \\ O & Q \end{bmatrix}$ then $A^{-1} = \begin{bmatrix} P^{-1} & O \\ O & Q^{-1} \end{bmatrix}$

Let the inverse of $A = \begin{bmatrix} P & O \\ O & Q \end{bmatrix}$ partitioned conformably to pre-multiplication be denoted by $B = \begin{bmatrix} M & R \\ N & S \end{bmatrix}$

Then

$$AB = \begin{bmatrix} P & O \\ O & Q \end{bmatrix}\begin{bmatrix} M & R \\ N & S \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} PM+ON & PR+OS \\ OM+QN & OR+QS \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix}$$

$\Rightarrow PM+ON = I \Rightarrow PM = I$ ($\because O$ is null)

$PR+OS = O \Rightarrow PR = O$

$OM+QN = O \Rightarrow QN = O$

$OR+QS = I \Rightarrow QS = I$

Now since P is non-singular and $PR = O$.

$$\boxed{\therefore\ R = O}$$

Also P is non-singular and $PM = I$

$$\boxed{\therefore\ M = P^{-1}}$$

Similarly Q is Non-Singular & $QN = O$

$$\boxed{\therefore\ N = O}$$

Also Q is Non-Singular & $QS = I$

$$\boxed{\therefore\ S = Q^{-1}}$$

$$\therefore\ B = \begin{bmatrix} P^{-1} & O \\ O & Q^{-1} \end{bmatrix}$$

→ Find the inverse of $\begin{bmatrix} A & B \\ C & O \end{bmatrix}$ where B, C are non-singular.

Sol'n :- Let the inverse of $M = \begin{bmatrix} A & B \\ C & O \end{bmatrix}$ Partitioned conformably to pre-multiplication by M, be denoted by $A = \begin{bmatrix} P & R \\ Q & S \end{bmatrix}$

then $MA = \begin{bmatrix} A & B \\ C & O \end{bmatrix}\begin{bmatrix} P & R \\ Q & S \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix}$

$\Rightarrow AP + BQ = I$ —— ①

$AR + BS = O$ —— ②

$CP = O$ —— ③

$CR = I$ —— ④

Since C is non-singular & $CP = O$ & $CR = I$.

$\therefore \boxed{P = O}$ & $\boxed{R = C^{-1}}$

Now from ①, $AO + BQ = I$

$\Rightarrow \boxed{BQ = I}$

Since B is non-singular.

$\therefore \boxed{Q = B^{-1}}$

② $\equiv BS = -AR$

$\Rightarrow B^{-1}(BS) = -B^{-1}(AR)$ $(\because |B| \neq 0)$

$\Rightarrow (B^{-1}B)S = -(B^{-1}A)C^{-1}$ $(\because R = C^{-1})$

$\Rightarrow IS = -B^{-1}AC^{-1}$

$\Rightarrow \boxed{S = -B^{-1}AC^{-1}}$

$\therefore M^{-1} = A$

$\Rightarrow \begin{bmatrix} A & B \\ C & O \end{bmatrix}^{-1} = \begin{bmatrix} P & R \\ Q & S \end{bmatrix}$

$= \begin{bmatrix} O & C^{-1} \\ B^{-1} & -B^{-1}AC^{-1} \end{bmatrix}$.

→ If A, B, C are non-singular, but not necessarily of the same size, show that

$$\begin{bmatrix} A & H & G \\ O & B & F \\ O & O & C \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}HB^{-1} & A^{-1}HB^{-1}FC^{-1} - A^{-1}GC^{-1} \\ O & B^{-1} & -B^{-1}FC^{-1} \\ O & O & C^{-1} \end{bmatrix}$$

Sol'n :- Let the inverse of

$M = \begin{bmatrix} A & H & G \\ O & B & F \\ O & O & C \end{bmatrix}$ Partitioned conformably to pre-multiplication by A denoted by $\begin{bmatrix} P & S & V \\ Q & T & U \\ R & U & X \end{bmatrix}$

then

$$\begin{bmatrix} A & H & G \\ O & B & F \\ O & O & C \end{bmatrix}\begin{bmatrix} P & S & V \\ Q & T & W \\ R & U & X \end{bmatrix} = \begin{bmatrix} I & O & O \\ O & I & O \\ O & O & I \end{bmatrix}$$

$\Rightarrow AP + HQ + GR = I$ —— ①

$AS + HT + GU = O$ —— ②

$AV + HW + GX = O$ —— ③

$BQ + FR = O$ —— ④

$BT + FV = I$ —— ⑤

$BW + FX = O$ —— ⑥

$CR = O$ —— ⑦

$CU = O$ —— ⑧

$CX = I$ —— ⑨

Since C is non-singular & $CR=0, CU=0$

$CX=I$

$\therefore \boxed{R=0}, \boxed{U=0}, \boxed{X=C^{-1}}$

(4) $\equiv BQ=0$ — (10)

$BT=I$ — (11), $BW=-FC^{-1}$ — (12)

Since B is non-singular & $BQ=0$,

$BT=I$.

$\boxed{\therefore Q=0}$, $\boxed{T=B^{-1}}$

(12) $\equiv B^{-1}(BW)=-B^{-1}(FC^{-1})$ $(\because |B|\neq 0)$

$\Rightarrow \boxed{W=-B^{-1}FC^{-1}}$

(1) $\equiv AP+0+0=I$ $(\because Q=0, R=0)$

$\Rightarrow AP=I$

$\Rightarrow \boxed{P=A^{-1}}$ $(\because A$ is non-singular$)$

(2) $\equiv AS+HB^{-1}+0=0$

$\Rightarrow AS=-HB^{-1}$

$\Rightarrow \boxed{S=-A^{-1}HB^{-1}}$ $(\because |A|\neq 0)$

(3) $\equiv AV+H(-B^{-1}FC^{-1})+GC^{-1}=0$

$\Rightarrow AV=HB^{-1}FC^{-1}-GC^{-1}$

$\Rightarrow V=A^{-1}HB^{-1}FC^{-1}-A^{-1}GC^{-1}$

$(\because |A|\neq 0)$

$$\therefore \begin{bmatrix} A & H & G \\ 0 & B & F \\ 0 & 0 & C \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} & -A^{-1}HB^{-1} & A^{-1}HB^{-1}FC^{-1}-A^{-1}GC^{-1} \\ 0 & B^{-1} & -B^{-1}FC^{-1} \\ 0 & 0 & C^{-1} \end{bmatrix}$$

→ Show that the rank of each of $\begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix}$, $\begin{bmatrix} A_1 & B_1 \\ 0 & 0 \end{bmatrix}$ is at most r:

A_1 being an $r\times r$ order matrix.

Sol'n :- (i). Let $M=\begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix}$

Since A_1 is an $r\times r$ order matrix.

$\therefore$ The matrix A_2 has r columns.

Now every $(r+1)$-rowed square submatrix of the matrix M has at least one column of zeros.

$\therefore$ All minors of order $(r+1)$ of the matrix M are zero.

$\therefore \rho(M)\leq r$.

Ex:- $\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 3 & 2 & 0 \\ \hline 3 & 4 & 0 \\ 5 & 6 & 0 \\ 5 & 0 & 0 \end{array}\right] \Rightarrow M=\begin{bmatrix} A_1 & 0 \\ A_2 & 0 \end{bmatrix}$

Since A_1 is an 2×2 order matrix.

$\therefore$ The matrix A_2 has 2 columns

Now every $(2+1)$-rowed square submatrix of the matrix M has atleast one column of zero.

$\therefore$ All minors of order $(2+1)$ of the matrix M are zero.

$\therefore \rho(M)\leq 2$.

(ii) Let $M=\begin{bmatrix} A_1 & B_1 \\ 0 & 0 \end{bmatrix}$

Since A_1 is an $r\times r$ square matrix.

$\therefore$ The matrix B_1 has also 'r' rows

Now every $(r+1)$ rowed square submatrix of the matrix M has at least one row of zeros.

∴ All minors of order $(r+1)$ of the matrix M are zero.

$$\therefore \rho(M) \leq r.$$

Ex:- Let $M = \left[\begin{array}{cc|ccc} 1 & 2 & 3 & 5 & 6 \\ 3 & 2 & 4 & 6 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$$\Rightarrow M = \begin{bmatrix} A_1 & B_1 \\ 0 & 0 \end{bmatrix}$$

→ show that the rank of a matrix doesnot alter on affining any number of additional rows or columns of zeros.

Soln :- Let A be a matrix of rank 'r'.

Let M be the matrix obtained from the matrix A by affining some additional rows and columns of zeros.

$$\text{Let } M = \begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}$$

Now every $(r+1)$-rowed minor of the matrix M is either a minor of the matrix A or it will have at least one row (or) one column of zeros.

Since rank of A is r.

∴ Every $(r+1)$-rowed minor of the matrix A (if there is any) is equal to zero.

∴ Every $(r+1)$-rowed minor of the matrix M is equal to zero.

Since the matrix A has atleast one minor of order 'r' not equal to zero.

∴ At least one r-rowed minor of the matrix M is not equal to zero.

$$\therefore \rho(M) = r.$$

* Elementary Matrices :-

Definition :- A matrix obtained from a unit matrix by a single elementary transformation is called an elementary matrix or E-matrix.

Ex :- $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

are the elementary matrices obtained from I_3 by subjecting it to the elementary operations $C_1 \leftrightarrow C_3$, $R_2 \to 4R_2$, $R_1 \to R_1 + 2R_2$ respectively.

we shall use the following symbols to denote the elementry matrices of different types:

1) E_{ij} will denote the E-matrix obtained by interchanging i^{th} and j^{th} rows or i^{th} and j^{th} columns in I.

2) $E_i(k) \Rightarrow$ E-matrix obtained by

multiplying every element of i^{th} row or i^{th} column with K in I.

3). $E_{ij}(K) \Rightarrow$ Elementary matrix obtained by multiplying every element of j^{th} row with K and then adding them to the corresponding elements of i^{th} row in I.

4). $E'_{ij}(K) \Rightarrow$ E-matrix obtained by multiplying every element of j^{th} column with K and then adding them to the corresponding elements of i^{th} column in I.

* Properties of Elementary Matrices :-

→ 1. Every elementary matrix is a square matrix.

→ 2. $|E_{ij}^T| = -1$ $(\because |I| = 1)$.

→ 3. $|E_i(K)| = K$ where $K \neq 0$

$(\because |E_i(K)| = K|I| = K(1) = K)$.

Ex :- Let

$$E_i(K) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & K & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ then}$$

$$E_i(K) = K|I_3|$$
$$= K(1) = K.$$

→ 4. $E_{ij}(K) = 1$

Ex:- Let $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ then

$$E_{ij}(2) = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore E_{ij}(2) = 1$

→ 5. $|E^T_{ij}(k)| = 1$

→ 6. Every elementary matrix is non-singular (i.e $|E| \neq 0$).

$\therefore E^{-1}$ exists.

* Lemma :-

Every Elementary row transformation of a product $C = AB$ be effected by subjecting the pre-factor A to the same row operation.

Proof :- Let A and B be $m \times n$ and $n \times p$ matrices then AB is a matrix of order $m \times p$.

Now let $A = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{bmatrix}_{m\times 1}$, $B = [C_1\ C_2 \cdots C_p]_{1\times p}$

where $R_1, R_2 \cdots R_m$ denote the row vectors of the matrix A. and $C_1, C_2, C_3 \ldots C_p$ denote the column vectors of the matrix B.

$$\therefore AB = \begin{bmatrix} R_1 \\ R_2 \\ \vdots \\ R_m \end{bmatrix} \begin{bmatrix} C_1 & C_2 & \cdots & C_p \end{bmatrix}$$

$$= \begin{bmatrix} R_1C_1 & R_1C_2 & \cdots & R_1C_p \\ R_2C_1 & R_2C_2 & \cdots & R_pC_p \\ \vdots & \vdots & & \vdots \\ R_mC_1 & R_mC_2 & \cdots & R_mC_p \end{bmatrix}$$

Now if σ denotes any elementary row transformation, then

$(\sigma A)B = \sigma(AB)$.

Ex:- If σ denotes the elementary row-transformation $R_1 \longleftrightarrow R_2$ then $(\sigma A)B = \sigma(AB)$.

Theorem :- Every Elementary row transformation of a matrix can be obtained by pre-multiplication with the corresponding elementary matrix.

proof :- Let A be an $m\times n$ matrix and let I_m be a unit matrix.

Now $A = I_m A$

Now let σ be any elementary row transformation to be performed on A.

Then $\sigma A = \sigma(I_m A)$

$= \sigma(I_m)A$

$= EA$

where E is the elementary matrix corresponding to the row operation σ.

Ex :- Let $A = \begin{bmatrix} 1 & 4 & 2 \\ 2 & 7 & 1 \\ 3 & 8 & 4 \end{bmatrix}$

{E - row transformation $A \xrightarrow{\ \ } B$, $I \xrightarrow{\ \ } E$, i.e. $EA=B$}

Now the E-transformation $R_1 \longrightarrow R_1 + 2R_2$ transforms A int B.

$$\therefore B = \begin{bmatrix} 7 & 20 & 10 \\ 2 & 7 & 1 \\ 3 & 8 & 4 \end{bmatrix}$$

{E-column transformation $A \xrightarrow{\ \ } B$, $I \xrightarrow{\ \ } E$, i.e. $AE=B$}

Now Apply the same row-transformation $R_1 \longrightarrow R_1 + 2R_2$ to the unit matrix I_3. i.e. $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\therefore E = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\text{Now } EA = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 2 \\ 2 & 7 & 1 \\ 3 & 8 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & 20 & 10 \\ 2 & 7 & 1 \\ 3 & 8 & 4 \end{bmatrix} = B.$$

H.W.

Lemma(2) Every elementary Column transformation of a product $C=AB$ Can be effected by Subjecting the post-factor B to the same Column operation.

sol'n :- If σ denotes any column transformation then $A(\sigma B) = \sigma(AB)$.

Theorem :- Every elementary Column transformation of a matrix Can be obtained by post-multiplication with the corresponding elementary matrix.

Sol'n :- $A = AI_n$

If 'σ' is any column transformation then $\sigma(A) = \sigma(AI_n)$

$= A(\sigma I_n)$

$= AE_1$.

* Inverses of Elementary Matrices :-

Theorem → $E_{ij}^{-1} = E_{ij}$

Proof :- Let the given matrix E_{ij} be the elementary matrix obtained by interchanging the i^{th} and j^{th} rows of a unit matrix.

If we interchange the i^{th} & j^{th} rows of E_{ij} then we get the unit matrix. $\left\{\begin{matrix} E_{ij} \to I \\ I \to E_{ij} \\ \text{i.e. } E_{ij}\cdot E_{ij} = I \end{matrix}\right\}$

But we know that every elementary row transformation of a matrix can be obtained by pre-multiplying with corresponding elementary matrix.

$\therefore (E_{ij})(E_{ij}) = I$

$\Rightarrow E_{ij}^{-1} = E_{ij}$

$\therefore E_{ij}$ is its own inverse.

Similarly we can show that $E_{ij}^{-1} = E_{ij}$ by the Elementary column transformation.

Theorem: $[E_i(K)]^{-1} = E_i\left(\frac{1}{K}\right)$ where $K \neq 0$.

Proof :- The given matrix $E_i(K)$ is the elementary matrix obtained by multiplying i^{th} row by K of a unit matrix.

If we again multiply the i^{th} row of $E_i(K)$ by $\frac{1}{K}$ then we get a unit matrix. $\left\{\begin{matrix} E_i(K) \to I \\ I \to E_i(1/K) \end{matrix}\right\}$.

But we know that every E-row transformation of a matrix can obtained by pre-multiplication with corresponding elementary matrix.

Now Pre-multiplying the matrix $E_i(K)$ with the elementary matrix $E_i\left(\frac{1}{K}\right)$

$\therefore E_i\left(\frac{1}{K}\right) . E_i(K) = I$

$\therefore [E_i(K)]^{-1} = E_i\left(\frac{1}{K}\right) : K \neq 0$

Similarly we can show that $[E_i(K)]^{-1} = E_i\left(\frac{1}{K}\right)$ by the E-column transformation.

→ $[E_{ij}(K)]^{-1} = E_{ij}(-K)$ where $K \neq 0$.

Proof :- Let the given matrix $E_{ij}(K)$ is the elementary matrix obtained by multiplying element of j^{th} row by K

and adding to the corresponding element of ith row of a unit matrix.

If we again multiply every element of jth row by $(-K)$ and adding to the corresponding elements of ith row of $E_{ij}(K)$ then we get a unit matrix.

we know that every elementary row-transformation of a matrix can be obtained by pre-multiplying with corresponding elementary matrix.

Now pre-multiplying the matrix $E_{ij}(K)$ with the elementary matrix $E_{ij}(-K)$.

$\therefore E_{ij}(K)\,(E_{ij}(-K)) = I$

$\therefore [E_{ij}(K)]^{-1} = E_{ij}(-K);\ K \neq 0.$

Similarly we can show this by any column transformation.

Note :- (1) The inverse of an elementry matrix is also non-singular.

(2) The inverse any elementary is also an elementary matrix.

Theorem $\rightarrow$ Elementary transformations donot change the rank of matrix.

(or)

The rank of a matrix is invariant if the matrix is subjected to elementary transformations.

Proof :- Let A be a matrix of rank 'r'

i.e. $\rho(A) = r$

Let B be the matrix obtained from the matrix A by the E-transformation.

Since $\rho(A) = r$

$\therefore$ All the minors of order $r+1$ will be zero.

Let $|A_0|$ be any $(r+1)$-rowed minor of A.

and $|B_0|$ be any $(r+1)$-rowed minor of B having the same position as $|A_0|$.

Now Case(i) : Interchange the ith & jth rows of a matrix doesnot change the matrix.

Let $R_i \longleftrightarrow R_j$ be performed on A.

Then $|A_0|$ will be one of the following three types.

1) $|A_0|$ will remain unchanged.

2) Two of its rows will be interchanged.

3). One of its rows will be interchanged with a row not belong to $|A_0|$.

$\therefore$ Now in (1) $|B_0|=0$, $|A_0|=0$.

in (2) $|B_0|=-|A_0|=-0$
$=0$ and

in (3) $|B_0|$ will be equal in magnitude to some other minor of order $(r+1)$ of A.

$\therefore |B_0|=0$

i.e. all the minors of order $(r+1)$ of B are zero.

$\therefore \rho(B)\leq r \Rightarrow \rho(B)\leq \rho(A)$

Again we can obtain A from B by $R_i \leftrightarrow R_j$ and we can prove that $\rho(A)\leq \rho(B)$.

$$\therefore \rho(A)=\rho(B)$$

Case (ii) Multiplying the ith row by a non-zero scalar K doesnot change the rank of the matrix.

Let $R_i \longrightarrow KR_i$ be performed on A then $|A_0|$ will be one of the following two types:

(1) $|A_0|$ will remain unchanged.

(2) All the elements of one of the rows will be multiplied by K.

$\therefore$ Now in (1), $|B_0|=|A_0|=0$ and

in (2), $|B_0| = K|A_0| = K(0)$
$=0$

$\therefore$ All the minors of order $(r+1)$ of B will be zero.

$\therefore \rho(B)\leq r \Rightarrow \rho(B)\leq \rho(A)$.

Again we can obtain A from B by $R_i \longrightarrow \frac{1}{K}R_i$ and we can prove that

$$\rho(A)\leq \rho(B)$$

$$\therefore \rho(A)=\rho(B)$$

Case (iii) Adding to the ith row K times the jth row i.e. $R_i \rightarrow R_i + KR_j$.

Let $R_i \longrightarrow R_i + KR_j$ be performed on A then $|A_0|$ will be on of the following three types.

(1) $|A_0|$ will remain unchanged

(2) The elements of one row of $|A_0|$ will have addition of K times the corresponding elements of another row of $|A_0|$.

(3) The elements of one row of $|A_0|$ will have addition of K times the corresponding elements not belonging to $|A_0|$.

Theorem → The pre-multiplication or post-multiplication by an elementary matrix and as such by any series of elementary matrices do not change the rank of matrix.

Proof :- Let A be a given matrix. Let E be the elementary matrix, which is pre-multiply by A.

If λ be the row operation corresponding to the elementary matrix E then $EA = \lambda A$.

But λA does not change the rank of A.

$$\therefore \rho(EA) = \rho(A).$$

Let $E_1, E_2, E_3 \ldots E_n$ be 'n' elementary matrices, which are to pre-multiply the matrix A.

Let $\lambda_1, \lambda_2 \ldots \lambda_n$ be row operations corresponding to the elementary matrices $E_1, E_2 \ldots E_n$ respectively.

then $E_n . E_{n-1} \cdots E_2 . E_1 . A = \lambda_n . \lambda_{n-1} \cdots \lambda_2 \lambda_1 . A$

But $\lambda_n . \lambda_{n-1} \cdots \lambda_2 . \lambda_1 . A$ donot change the rank of A.

$$\therefore \rho(E_n . E_{n-1} \cdots E_2 . E_1 . A) = \rho(A)$$

$\therefore$ Now in ① $|B_0| = |A_0| = 0$ and

in ② & ③ $|B_0| = |A_0| + K$ times.

another $(r+1)^{th}$ order minor of 'A'.

$$\Rightarrow |B_0| = 0 + K(0)$$
$$= 0$$

$\therefore$ All the $(r+1)$-rowed minors of B are zero.

$$\therefore \rho(B) \le r \Rightarrow \rho(B) \le \rho(A)$$

Again we can obtain A from B by $R_i \to R_i - R_j(K)$ (or) $R_{ij}(-K)$ and we can prove that

$$\rho(A) \le \rho(B)$$

$$\therefore \rho(A) = \rho(B)$$

$\therefore$ By the cases (i), (ii) & (iii) we conclude that elementary row transformations on a matrix donot change the rank of the matrix.

Similarly we prove that elementary column transformations on a matrix donot change the rank.

$\therefore$ Elementary transformations on a matrix donot change the rank of the matrix.

Imp

* Reduction to Normal form (or) First Canonical form:—

Every non-zero matrix can be reduced to one of the following form I_r,

$\left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]$, $\left[I_r \mid 0\right]$, $\left[\begin{array}{c} I_r \\ \hline 0 \end{array}\right]$ by a finite number of elementary transformations, where I_r is the unit matrix of order r, called its normal form and r is called the rank of the matrix.

Note:— (1) Not every matrix A can be reduced to normal form by row (column) transformations alone.

Ex:— $\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ Cannot changed to normal form by row-transformation alone.

and $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 5 & 0 & 0 \end{bmatrix}$ Cannot be changed to normal form by column transformation alone.

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*Theorem:— If A be an $m\times n$ matrix of rank 'r', there exist non-singular matrices P & Q such that $PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$.

Proof:— Given that $\rho(A) = r$

The matrix A can be transformed to normal form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ by, say 's' number of elementary row-transformations and say, 't' number of elementary column transformations.

we know that every elementary row (column) transformation on A is equivalent to pre (post)— multiplication of A by a suitable elementary matrix.

Now there exist elementary matrices

$P_1, P_2, \dots P_s$ as pre-factors and $Q_1, Q_2, \dots Q_t$ as post-factors of A such that

$$P_s, P_{s-1} \cdots P_2 \cdot P_1 A \cdot Q_1 \cdot Q_2 \cdots Q_t = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \quad \text{——(1)}$$

we know that each elementary matrix is a non-singular matrix and the product of non-singular matrices is also non-singular.

Let $(P_s \cdot P_{s-1} \cdots P_2 \cdot P_1) = P$ and

$(Q_1 \cdot Q_2 \cdots Q_t) = Q$

$\therefore$ P & Q are non- singular.

$\therefore$ ① $\equiv PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$.

1984 Theorem → Every Non-singular matrix is a product of elementary matrices.

Proof :- let A be a non-singular matrix of order n×n.

$\therefore |A| \neq 0$.

$\therefore \rho(A) = n$.

$\therefore$ It can be reduced to the form I_n by a finite number of elementary row or column transformations.

we know that every elementary row (column) transformation on A is equivalent to pre (post) multiplication of A by a suitable elementary matrix.

Now there exist, say 's' elementary matrices $P_1, P_2 \cdots P_{s-1}, P_s$ as pre-factors and 't' elementary matrices $Q_1, Q_2 \cdots Q_t$ as post factors of A such that

$(P_s \cdot P_{s-1} \cdots P_2 \cdot P_1) A (Q_1, Q_2 \cdots Q_t) = I_n$ —①

Since $P_1, P_2, \cdots P_s$ & $Q_1, Q_2, \cdots Q_t$ are non-singular matrices.

$\therefore P_1^{-1}, P_2^{-1}, \cdots P_s^{-1}$ & $Q_1^{-1}, Q_2^{-1} \cdots Q_t^{-1}$ are exist.

Also these inverse matrices are elementary matrices.

Now pre-multiplying successively by $P_s^{-1}, P_{s-1}^{-1} \cdots P_2^{-1}, P_1^{-1}$ and post multiplying successively $Q_t^{-1}, Q_{t-1}^{-1} \cdots Q_2^{-1}, Q_1^{-1}$ from ① we get

$A = P_1^{-1} \cdot P_2^{-1} \cdot P_3^{-1} \cdot P_4^{-1} \cdots P_s^{-1} \cdot I_n \cdot Q_t^{-1} \cdot Q_{t-1}^{-1} \cdots Q_2^{-1} \cdot Q_1^{-1}$.

$= P_1^{-1} \cdot P_2^{-1} \cdots P_s^{-1} \cdot Q_t^{-1} \cdot Q_{t-1}^{-1} \cdots Q_2^{-1} \cdot Q_1^{-1}$.

= Product of elementary matrices.

Theorem → The rank of a matrix does not change by pre-multiplication or post multiplication with a non-singular matrix.

Sol^n :- Let A be a given matrix and P be a non-singular matrix such that PA is possible.

we know that the non-singular matrix P can be expressed as a product of elementary matrices.

Let $P = P_1 \cdot P_2 \cdots P_s$ where $P_1, P_2 \cdots P_s$ are elementary matrices.

$\therefore PA = P_1 \cdot P_2 \cdots P_{s-1} \cdot P_s A$

i.e. A is pre-multiplied by s elementary matrices.

i.e. Pre-multiplication of A by s elementary matrices is equivalent to s elementary row operations on A.

But elementary row operations on A donot change the rank of A.

$$\therefore \quad \rho(PA) = \rho(A)$$

Similarly if Q is a non-singular matrix such that AQ is possible. then we can prove that

$$\rho(AQ) = \rho(A)$$

Problems :-

→ Compute the matrix $E_{23} \cdot E_{34}(-1) \cdot E_2(-2) \cdot E_{12}$ for an elementary matrix of order 4.

Sol'n :- Consider $E_{23} E_{34}(-1) E_2(-2) E_{12} I_4 =$

$$E_{23} \cdot E_{34}(-1) E_2(-2) E_{12} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= E_{23} \cdot E_{34}(-1) E_2(-2) \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= E_{23} \cdot E_{34}(-1) \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= E_{23} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

→ Compute the following matrix. $E_{23}(-1) \cdot E_{31} E'_{24} E_3(2)$ for an elementary matrix of order 4.

Sol'n :- Consider ·

$$E_{23}(-1) \cdot E_{31} \cdot E'_{24} = E_{23}(-1) \cdot E_{31} I_4 \cdot E'_{24}$$

$$= E_{23}(-1) \cdot E_{31} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} E'_{24}$$

$$= E_{23}(-1) E_{31} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= E_{23}(-1) \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$\therefore E_{23}(-1) E_{31} E'_{24} E_3(2)$$

$$= \begin{bmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 2 & 0 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

→ Reduce the matrix $A = \begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$ into Echelon form and hence find its rank.

<u>Sol'n</u> :- $A = \begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$

$R_5 \longrightarrow 5R_5$

$$\sim \begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 5 & -5 & 10 & 0 \end{bmatrix}$$

$R_5 \longrightarrow R_5 - R_1$

$$\sim \begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & -8 & -4 & -4 \end{bmatrix}$$

$R_3 \longrightarrow R_3 + 8R_2$

$$\sim \begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 12 & 4 \end{bmatrix}$$

∴ This is in echelon form and the number of non-zero rowis is 3.

$$\therefore \rho(A) = 3.$$

→ Reduce the matrix $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$ to Canonical (normal) form and find its rank.

<u>Sol'n</u> :- $A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ -2 & 4 & 3 & 0 \\ 1 & 0 & 2 & -8 \end{bmatrix}$

$R_2 \longrightarrow R_2 + 2R_1$, $R_3 \longrightarrow R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & -2 & 1 & -8 \end{bmatrix}$$

$C_2 \longrightarrow C_2 - 2C_1$

$C_3 \longrightarrow C_3 - C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 8 & 5 & 0 \\ 0 & -2 & 1 & -8 \end{bmatrix}$$

$R_2 \longleftrightarrow R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -2 & 1 & -8 \\ 0 & 8 & 5 & 0 \end{bmatrix}$$

$C_2 \longleftrightarrow C_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & -8 \\ 0 & 5 & 8 & 0 \end{bmatrix}$$

$R_3 \longrightarrow R_3 - 5R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & -8 \\ 0 & 0 & 18 & 40 \end{bmatrix}$$

$C_3 \longrightarrow C_3 + 2C_2$, $C_4 \longrightarrow C_4 + 8C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 18 & 40 \end{bmatrix}$$

$R_3 \longrightarrow \frac{1}{18} R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{20}{9} \end{bmatrix}$$

$C_4 \longrightarrow C_4 - \frac{20}{9} C_3$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\sim [I_3 \mid 0]$$

$\therefore \ \rho(A) = 3$

$\longrightarrow$ Find the ranks of $A, B, AB, A+B$ & BA where

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}, \quad B = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$$

$\longrightarrow$ Reduce the following matrices to normal form and find their ranks.

(i) $$A = \begin{bmatrix} 1 & -1 & 3 & 6 \\ 1 & 3 & -3 & -4 \\ 5 & 3 & 3 & 11 \end{bmatrix}$$

(ii) $$A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 3 & 0 & 3 \\ 1 & -2 & -3 & -3 \\ 1 & -1 & 2 & 3 \end{bmatrix}$$

(iii) $$A = \begin{bmatrix} 1 & -1 & 2 & -3 \\ 4 & 1 & 0 & 2 \\ 0 & 3 & 0 & 4 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

$\longrightarrow$ Find two non-singular matrices P & Q such that PAQ is in the normal form (i.e. $PAQ = \left[\begin{array}{c|c} I_r & 0 \\ \hline 0 & 0 \end{array}\right]$) where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$

Also find the rank of the matrix A.

Sol^n:- we write $A = I_3 A I_3$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \longrightarrow R_2 - R_1, \quad R_3 \longrightarrow R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \longrightarrow C_2 - C_1, \quad C_3 \longrightarrow C_3 - C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & -2 \\ 0 & -2 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \longrightarrow -\frac{1}{2} R_2, \quad R_3 \longrightarrow -\frac{1}{2} R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ \frac{3}{2} & 0 & -\frac{1}{2} \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & -1/2 & 0 \\ 1 & 1/2 & -1/2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_3 \to C_3 - C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & -1/2 & 0 \\ 1 & 1/2 & -1/2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore PAQ = \left[\begin{array}{c|c} I_2 & 0 \\ \hline 0 & 0 \end{array}\right]$$

where $P = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & -1/2 & 0 \\ 1 & 1/2 & -1/2 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

and $\rho(A) = 2$.

Note:- P & Q are not unique.

→ Obtain non-singular matrices P & Q such that PAQ is of the form $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ where $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$.

→ Obtain non-singular matrices P, Q such that $PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ where $A = \begin{bmatrix} 3 & 2 & -1 & 5 \\ 5 & 1 & 4 & -2 \\ 1 & -4 & 11 & -19 \end{bmatrix}$

Imp → Express $\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$ as a product of elementary matrices.

Sol'n :- Given that

$$A = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

$R_{21}(-1)$ i.e. $R_2 \to R_2 - R_1$

$R_{31}(-1)$

$$\sim \begin{bmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_{21}(-3)$, $C_{31}(-3)$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

$\therefore \rho(A) = 3$

$\therefore |A| \neq 0$.

$\therefore$ A is non-singular matrix.

Now $R_{21}(-1)$, $R_{31}(-1)$ on A is equivalent to pre-multiplication of A by $E_{21}(-1)$ and $E_{31}(-1)$ elementary matrices.

Also $C_{21}(-3)$ and $C_{31}(-3)$ on A is equivalent to post-multiplication of A by $E'_{21}(-3)$ and $E'_{31}(-3)$ elementary matrices.

$$\therefore E_{31}(-1) \cdot E_{21}(-1) \cdot A \cdot E'_{21}(-3) \cdot E'_{31}(-3) = I_3.$$

$$\Rightarrow A = [E_{21}(-1)]^{-1} \cdot [E_{31}(-1)]^{-1} \cdot I_3 \cdot [E'_{31}(-3)]^{-1} [E'_{21}(-3)]^{-1}$$

$$= E_{21}(1) \cdot E_{31}(1) \cdot E'_{31}(3) \cdot E'_{21}(3).$$

$$\left(\because [E_{ij}(k)]^{-1} = E_{ij}(-k)\right)$$

$\therefore A =$ a product of Elementary matrices.

$$= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

→ Express the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & 3 \end{bmatrix}$ as a product of Elementary matrices.

* Equivalence of Matrices:-

Let A be an $m\times n$ and B be an $m\times n$ matrices. If B is obtained from A by finite number of elementary transformations of A then A is called equivalent to B. and is denoted by $A \sim B$.

Note :- The relation $\sim$ defined between two matrices called equivalence matrices.

the following three properties of the relation '$\sim$' in the set M of all $m\times n$ matrices are obvious.

1. **Reflexivity** : If $A \in M$ then $A \sim A$
2. **Symmetry** : If $A, B \in M$ such that $A \sim B$ then $B \sim A$.
3. **Transitivity** : If $A, B, C \in M$ such that $A \sim B$ and $B \sim C$ then $A \sim C$.

$\therefore$ The relation '$\sim$' in the set M of all $m\times n$ matrices is equivalence relation.

Row Equivalence :-

A matrix A is said to be row equivalence to B if B is obtained from A by a finite number of E-row transformations of A. and is denoted by $A \overset{R}{\sim} B$.

Column Equivalence :-

A matrix A is said to be column equivalent to B. if B is obtained from A by a finite number of E-column transformations of A and is denoted by $A \overset{C}{\sim} B$.

→ If $A \sim B$ then $\rho(A) = \rho(B)$

Proof :- Since $A \sim B$ i.e. $A \overset{R}{\sim} B$ or $A \overset{C}{\sim} B$

$\therefore$ B is obtained from A by a finite number of elementary transformations of A.

we know that E-transformations do not change the rank of the matrix.

$\therefore$ If $A \sim B$ then

$$\rho(A) = \rho(B)$$

Note! If A & B are equivalent matrices then there exist non-singular matrices P & Q such that $B = PAQ$.

→ If A and B are same order and $\rho(A) = \rho(B)$ then $A \sim B$.

Sol'n :– Let A & B be two $m \times n$ matrices of the same rank 'r' then

$$A \sim \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \text{ and } B \sim \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

By the symmetry of the equivalence relation.

$$B \sim \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \sim B$$

Now by the transitivity of the equivalence relation

$$A \sim \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \sim B.$$

$$\Rightarrow A \sim B.$$

→ Is the matrix $A = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & -3 \end{bmatrix}$ equivalent to I_3.

Sol'n :– Given that $A_3 = \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 2 \\ 2 & 1 & -3 \end{bmatrix}$

Now $|A| = 1(-2) + 1(-6-1) + 2(4)$

$= -2 - 7 + 8$

$= -1$

$\neq 0$

$\therefore \rho(A) = 3.$

But $\rho(I_3) = 3$

$\therefore A \& I_3$ are same order.

and $\rho(A) = \rho(I_3)$

$\therefore A \sim I_3.$

→ Is the pair of matrices $\begin{bmatrix} 4 & 0 & 2 \\ 3 & 1 & -2 \\ -5 & 0 & 0 \end{bmatrix}$; $\begin{bmatrix} 3 & 9 & 0 & 2 \\ 7 & -2 & 0 & 1 \\ 8 & 7 & 4 & 5 \end{bmatrix}$ equivalent.

Sol'n :– Given matrices are

$$\begin{bmatrix} 4 & 0 & 2 \\ 3 & 1 & -2 \\ -5 & 0 & 0 \end{bmatrix}_{3\times3} \text{ and } \begin{bmatrix} 3 & 9 & 0 & 2 \\ 7 & -2 & 0 & 1 \\ 8 & 7 & 4 & 5 \end{bmatrix}_{3\times4}$$

these two matrices are of different orders.

$\therefore$ They are not equivalent.

→ Reduce the matrix $A = \begin{bmatrix} 0 & 2 & 3 \\ 2 & 4 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ to a matrix B by only row transformations and obtain the rank of A by inspection of B

Sol'n :– $A = \begin{bmatrix} 0 & 2 & 3 \\ 2 & 4 & 0 \\ 3 & 0 & 1 \end{bmatrix}$

$R_1 \longleftrightarrow R_2$

$$\sim \begin{bmatrix} 2 & 4 & 0 \\ 0 & 2 & 3 \\ 3 & 0 & 1 \end{bmatrix}$$

$R_1 \to \frac{1}{2}R_1$,

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 3 \\ 3 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 3 \\ 0 & -6 & 1 \end{bmatrix}$$

$R_2 \to \frac{1}{2}R_2$

$$\sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 3/2 \\ 0 & -6 & 1 \end{bmatrix}$$

$R_1 \to R_1 - 2R_2$

$$\sim \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 3/2 \\ 0 & -6 & 1 \end{bmatrix}$$

$R_3 \to R_3 + 6R_2$

$$\sim \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 3/2 \\ 0 & 0 & 10 \end{bmatrix}$$

$R_3 \to \frac{1}{10}R_3$; $R_1 \to R_1 + 3R_3$, $R_2 \to R_2 - \frac{3}{2}R_3$

$$\sim \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 3/2 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = B \text{ (say)}$$

$\therefore \rho(A) = \rho(B) = 3.$

$\rightarrow$ obtain an equivalent matrix for the given matrix

$$A = \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$$ and hence find its rank.

Sol'n:-

$$A = \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$$

$C_1 \leftrightarrow C_2$

$$\sim \begin{bmatrix} 1 & 6 & 3 & 8 \\ 2 & 4 & 6 & -1 \\ 3 & 10 & 9 & 7 \\ 4 & 16 & 12 & 15 \end{bmatrix}$$

$C_2 \to C_2 - 6C_1$, $C_3 \to C_3 - 3C_1$

$C_4 \to C_4 - 8C_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -8 & 0 & -17 \\ 3 & -8 & 0 & -17 \\ 4 & -8 & 0 & -17 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$R_4 \to R_4 - 4R_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -8 & 0 & -17 \\ 0 & -8 & 0 & -17 \\ 0 & -8 & 0 & -17 \end{bmatrix}$$

$C_2 \to -\frac{1}{8}C_2$

$C_4 \to -\frac{1}{17}C_4$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$C_4 \to C_4 - C_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$R_4 \to R_4 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} = B \text{ (say)}$$

$$\underline{\underline{\rho(A) = 2 = \rho(B)}}$$

*Imp

Theorem :–

If A be an $m\times n$ matrix of rank 'r' then there exists a non-singular matrix P such that $PA = \begin{bmatrix} G \\ 0 \end{bmatrix}$, where G is an $r\times n$ matrix of rank 'r' and 0 is a zero-matrix of order $(m-r)\times n$.

Proof :– Given that A is an $m\times n$ matrix of rank 'r'

$\therefore \exists$ non-singular matrices P & Q such that $PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ —①

We know that every non-singular matrix can be expressed as a product of elementary matrices.

Now let $Q = Q_1 \cdot Q_2 \cdots Q_t$ where $Q_1, Q_2, \cdots Q_t$ are elementary matrices

$\therefore$ ① $\equiv PA \cdot Q_1 \cdot Q_2 \cdots Q_t = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ —②

Now every elementary column transformation of a matrix is equivalent to post-multiplication with the corresponding elementary matrix.

Clearly no column transformation can change the last $(m-r)$ rows of a matrix $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ in ②

($\because$ each element of $(m-r)$ rows is zero).

Now post-multiplying ② successively by elementary matrices $Q_t^{-1}, Q_{t-1}^{-1}, \cdots Q_2^{-1}, Q_1^{-1}$ we have $PA = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \cdot Q_t^{-1} \cdot Q_{t-1}^{-1} \cdots Q_2^{-1} Q_1^{-1}$.

$\Rightarrow PA = \begin{bmatrix} G \\ 0 \end{bmatrix}$

where G is of order $r\times n$ and 0 is of order $(m-r)\times n$.

Since elementary operations donot change the rank.

$$\rho(I_r) = r = \rho(G).$$

$\therefore$ There exists a non-singular matrix P such that $PA = \begin{bmatrix} G \\ 0 \end{bmatrix}$ where G is an $r\times n$ and is of rank 'r' and 0 is a zero matrix of order $(m-r)\times n$.

Ex:- Suppose $PAQ = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix}$ is obtained from $I_{3\times3} A_{3\times4} I_{4\times4} = A_{3\times4}$

$\Rightarrow PA = \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} Q_2^{-1} \cdot Q_1^{-1}$ where

$$Q_2^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad Q_1^{-1} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$= \begin{bmatrix} G \\ 0 \end{bmatrix}$ where G is of order 2×4 and 0 is of order $(3-2)\times 4$.

$\rho(I_2) = 2 = \rho(G)$.

*Theorem :- If A be an $m\times n$ matrix of rank 'r' then there exists a non-singular matrix Q such that $AQ = [H \ 0]$ where H is a matrix of order $m\times n$ and of rank 'r' and 0 is a zero matrix of order $m\times(n-r)$.

Proof :- Given that A is an $m\times n$ matrix of rank 'r'.

Now $\exists$ non-singular matrices P & Q such that $PAQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ —— ①

We know that every non-singular matrix can be expressed as a product of elementary matrices.

Hence let $P = P_s \cdot P_{s-1} \cdots P_2 \cdot P_1$ where $P_1, P_2 \cdots P_s$ are elementary matrices.

$\therefore$ ① $\equiv PAQ = P_s \cdot P_{s-1} \cdot P_{s-2} \cdots P_2 \cdot P_1 AQ = \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ —— ②

Now every elementary row transformation of a matrix is equivalent to pre-multiplication with corresponding elementary matrix.

Clearly no row-operation can change the last $(n-r)$ columns of a matrix $\begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$ in ②.

Now pre-multiplying ② successively by elementary matrices $P_s^{-1}, P_{s-1}^{-1} \cdots P_3^{-1}, P_2^{-1}, P_1^{-1}$.

we have

$$AQ = P_1^{-1} \cdot P_2^{-1} \cdot P_3^{-1} \cdots P_s^{-1} \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix}$$

$\Rightarrow AQ = [H \ 0]$

where H is of order $m\times r$ and 0 is a zero matrix of order $m\times(n-r)$.

Since E - transformations do not change the rank.

$$\rho(I_r) = r = \rho(H)$$

$\therefore \exists$ a non-singular matrix Q such that $AQ = [H \;\; O]$.

where H is an $m \times r$ matrix of rank 'r' and O is a zero-matrix of order $m \times (n-r)$.

* Rank of a Product of Matrices :—

Theorem :- The rank of a product of two matrices cannot exceed the rank of either matrix.

(or)

If A, B are matrices conformable for multiplication, then $\rho(AB) \le \rho(A)$ and $\rho(AB) \le \rho(B)$.

Proof :- Let A & B be two matrices of orders $m \times n$ and $n \times p$ respectively.

Let $\rho(A) = r_1$, $\rho(B) = r_2$ and $\rho(AB) = r$.

we know that $\exists$ a non-singular matrix P such that

$PA = \begin{bmatrix} G \\ O \end{bmatrix}$, where G is of order $r_1 \times n$ and O is a zero matrix of order $(m-r_1) \times n$.

Now by post-multiplying both sides by B we have

$$PAB = \begin{bmatrix} G \\ O \end{bmatrix} B$$

$\therefore \rho(PAB) = \rho(AB) = r$.

$\therefore$ rank of the matrix $\begin{bmatrix} G \\ O \end{bmatrix} B = r$.

Since the matrix G has only r_1 non-zero rows.

$\therefore \begin{bmatrix} G \\ O \end{bmatrix} B$ cannot have more than r_1 non-zero rows.

$\therefore$ Rank of the matrix $\begin{bmatrix} G \\ O \end{bmatrix} B \le r_1$.

$$\Rightarrow r \le r_1$$

i.e. $\rho(AB) \le \rho(A)$. (i.e. A is the pre-factor). —— ①

Again $\rho(AB) = [\rho(AB)']$

$$= \rho[B'A']$$

$$\le \rho(B') \text{ (by using ①)} \quad \text{i.e. } \rho(AB) \le \rho(A)$$

$$= \rho(B) \quad (\because \rho(B') = \rho(B))$$

$$= r_2$$

$\therefore r_1 \le r_2$

i.e. $\rho(AB) \le \rho(B)$ —— ②

$\therefore$ from ① & ② we have

$\rho(AB) \le \rho(A)$ and $\rho(AB) \le \rho(B)$.

Imp → **Working Rule for finding the inverse of a non-singular matrix by E-row transformations:**

Let $A_{n\times n}$ be a non-singular matrix then $A = I_n A$

Now we go on applying E-row transformations only to the matrix A and pre-factor I_n of the product $I_n A$ till we reach the result $I_n = BA$.

then B is the inverse of A.

Problems →

2005 → Find the inverse of the matrix given below using E-row operations only: $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

Sol'n :- Let $A = \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ then

$$A = I_3 A$$

$$\Rightarrow \begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \to 3R_1$

$$\sim \begin{bmatrix} 6 & 0 & -3 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \to R_1 - R_2$

$$\sim \begin{bmatrix} 1 & -1 & -3 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \to R_1 + R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_2 \to R_2 - 5R_1$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 15 & -6 & 6 \end{bmatrix} A$$

$R_3 \to \frac{1}{3}R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix} A$$

$$I_3 = BA$$

where $B = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$

$$\therefore A^{-1} = B = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

(or)

Now $I_3 A^{-1} = (BA) A^{-1}$

$$\Rightarrow A^{-1} = B(AA^{-1})$$

$$= BI_3$$

$$\boxed{A^{-1} = B}$$

$$\therefore A^{-1} = \begin{bmatrix} 3 & -1 & 1 \\ -15 & 6 & -5 \\ 5 & -2 & 2 \end{bmatrix}$$

→ Reduce the matrix

$A = \begin{bmatrix} 0 & 2 & 3 \\ 2 & 4 & 0 \\ 3 & 0 & 1 \end{bmatrix}$ to I_3 by

E-rowtransformations only.

→ Compute the inverses of matrices.

(i) $\begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 3 & 3 & 3 & 3 \end{bmatrix}$ (ii) $\begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$

using elementary operations.

→ Find the value of α for which the matrix $\begin{bmatrix} 0 & 1 & \alpha \\ 1 & \alpha & 0 \\ \alpha & 0 & 1 \end{bmatrix}$ is invertible and find its inverse.

Sol'n :- Let $A = \begin{bmatrix} 0 & 1 & \alpha \\ 1 & \alpha & 0 \\ \alpha & 0 & 1 \end{bmatrix}$

then $A = I_3 A$

$$\Rightarrow \begin{bmatrix} 0 & 1 & \alpha \\ 1 & \alpha & 0 \\ \alpha & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \longleftrightarrow R_2$

$$\Rightarrow \begin{bmatrix} 1 & \alpha & 0 \\ 0 & 1 & \alpha \\ \alpha & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_1 \longrightarrow R_1 - \alpha R_2$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -\alpha^2 \\ 0 & 1 & \alpha \\ \alpha & 0 & 1 \end{bmatrix} = \begin{bmatrix} -\alpha & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} A$$

$R_3 \longrightarrow R_3 - \alpha R_1$

$$\Rightarrow \begin{bmatrix} 1 & 0 & -\alpha^2 \\ 0 & 1 & \alpha \\ 0 & 0 & \alpha^3+1 \end{bmatrix} = \begin{bmatrix} -\alpha & 1 & 0 \\ 1 & 0 & 0 \\ \alpha^2 & -\alpha & 1 \end{bmatrix} A$$

If $-\alpha^2 = 0$, $\alpha = 0$

$\alpha^3 + 1 = 1$ then

$\alpha = 0$ and A is invertible.

when $\alpha = 0$

$$\therefore A^{-1} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Imp → Redue the matrix

$A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 3 & -5 \\ 1 & 1 & 5 \end{bmatrix}$ to I_3 by a finite sequence of E-row transformations and express A as a product of elementary matrices. Reduce A^{-1}

Sol'n :- $A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 3 & -5 \\ 1 & 1 & 5 \end{bmatrix} \overset{R_{21}(-4),\ R_{31}(-1)}{\sim} \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & -5 \\ 0 & 0 & 5 \end{bmatrix}$

$$R_{23}(-1) \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

$$\xrightarrow{R_{21}(1)} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix} \xrightarrow{R_2(-1),\ R_3(1/5)} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore A \sim I_3.$

But the E-matrices corresponding to

$R_{21}(-4)$, $R_{31}(-1)$, $R_{23}(1)$, $R_{12}(1)$, $R_2(-1)$, $R_3(1/5)$ and $E_{21}(-4)$, $E_{31}(-1)$, $E_{23}(1)$, $E_{12}(1)$, $E_2(-1)$, $E_3(1/5)$.

$$\therefore E_3(1/5)\cdot E_2(-1)\cdot E_{12}(1)\cdot E_{23}(1)E_3(-1).$$
$$E_{21}(-4)A = I_3 \quad \text{—— (1)}$$

$$\Rightarrow A = [E_{21}(-4)]^{-1}[E_{31}(-1)]^{-1}\cdot[E_{23}(1)]^{-1}[E_{12}(1)]^{-1}$$
$$[E_2(-1)]^{-1}\cdot[E_3(1/5)]^{-1} I_3.$$

$$\Rightarrow A = E_{21}(4)\cdot E_{31}(1)\cdot E_{23}(-1)\cdot E_{12}(-1)E_2(1).$$
$$E_3(5).I_3$$
$$\left(\because [E_{ij}(-k)]^{-1} = E_{ij}(k)\right).$$

$A = E_{21}(4)E_{31}(1).\ E_{23}(-1).E_{12}(-1).E_2(1)E_3(5)$

= Product of Elementary matrices.

$(1) \equiv E_3(1/5)E_2(-1)\ E_{12}(1)E_{23}(1)\ E_{31}(-1).E_{21}(-4)$

$$A.A^{-1} = I_3A^{-1}$$

$$\Rightarrow E_3(1/5)E_2(-1).E_{12}(1)E_{23}(1)E_{31}(-1)E_{21}(-4)$$
$$I_3 = A^{-1}$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/5 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/5 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/5 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/5 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/5 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} -4 & 1 & 1 \\ -5 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1/5 \end{bmatrix}\begin{bmatrix} -4 & 1 & 1 \\ 5 & -1 & -1 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -4 & 1 & 1 \\ 5 & -1 & -1 \\ -1/5 & 0 & 1/5 \end{bmatrix}$$

* Row space of a Matrix :-

Let $A = [a_{ij}]$ be an $m\times n$ matrix then

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

the rows of A are $R_1 = (a_{11}, a_{12}, \cdots a_{1n})$, $R_2 = (a_{21}, a_{22} \cdots a_{2n})$, ------ $R_m = (a_{m1}, a_{m2}, \cdots a_{mn})$. and each of these being an n-tuple over a field F, is a member of the vectorspace F^n (or $V_n(F)$).

The linearspan of these vectors i.e. $L(\{R_1, R_2, \cdots R_m\})$ is a subspace of F^n and is called the rowspace of A.

These vectors are called row vectors. and is denoted by row sp(A)

i.e. row sp(A) = span$(R_1, R_2, \cdots R_m)$

Similarly the space spanned by the column vectors

i.e. $L(\{C_1, C_2, \cdots C_n\})$ is a subspace of F^n and is called the column space of A.

where $C_1 = (a_{11}, a_{21}, \cdots a_{m1})$

$C_2 = (a_{12}, a_{22}, \cdots a_{m2})$

- - - - - - - - -

$C_n = (a_{1n}, a_{2n}, \cdots a_{mn})$ and

is denoted by col sp(A).

i.e. col sp(A) = span$(C_1, C_2, \cdots C_n)$.

Note:- (1). Column space of A is the same as the row space of A^T.

i.e. Col sp(A) = row sp(A^T).

(2). As the non-zero rows of an echelon matrix are L.I:

Dimension of row space of A = maximum number of L.I rows of A.

= maximum number of L.I rows of echelon matrix of A.

= number of non-zero rows of an echelon matrix of A.

*Row and Column rank of Matrix:

Let $A = [a_{ij}]_{m\times n}$ then the dimension of the row space of A is called the row rank of A. and the dimension of the column space of A is called the column rank of A.

→ Find the column of rank of the matrix.

$$A = \begin{bmatrix} 1 & 1 & 1 & 2 \\ 4 & 3 & 2 & 3 \\ 3 & 1 & 1 & -4 \\ -1 & -2 & -3 & -7 \\ -4 & -3 & -2 & -8 \end{bmatrix}$$

Sol'n :- we know that column of rank of A is same as row rank of A^T.

Now $A^T = \begin{bmatrix} 1 & 4 & 3 & -1 & -4 \\ 1 & 3 & 1 & -2 & -3 \\ 1 & 2 & -1 & -3 & -2 \\ 2 & 3 & -4 & -7 & -3 \end{bmatrix}$

$\sim \begin{bmatrix} 1 & 4 & 3 & -1 & -4 \\ 0 & -1 & -2 & -1 & 1 \\ 0 & -2 & -4 & -2 & 2 \\ 0 & -5 & -10 & -5 & 5 \end{bmatrix}$ $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$, $R_4 \to R_4 - 2R_1$

$\sim \begin{bmatrix} 1 & 4 & 3 & -1 & -4 \\ 0 & -1 & -2 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$ $R_3 \to R_3 - 2R_2$, $R_4 \to R_4 - 5R_2$

which is in echelon form.

The number of non-zero rows of this echelon form is 2.

$\therefore \rho(A^T) = 2$

$\therefore$ Column rank of A = 2.

Note :- 1. The row rank of A is the number of non-zero rows in echelon matrix of A.

2. The row rank and the column rank of a matrix are equal.

———