Revision is the most crucial part of mathematics preparation in Civil Services Examination. For that purpose, I had prepared short notes on each topic of the syllabus which I am sharing here. I would suggest aspirants to use this as just a reference for quick revision and not as your primary source. This optional requires good understanding of the topic which cannot be derived from short notes.

It is the process of preparing such notes that gives a candidate confidence about his or her preparation and brings conceptual clarity. Hence I would encourage aspirants to do this exercise themselves even if it is time consuming.

All the best!

Regards,
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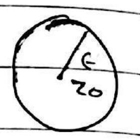
AIR 8 - CSE 2015

COMPLEX ANALYSIS

(1) Euler introduced symbol i with property $i^2 = -1$.
Complex plane is also called Gaussian or Argand plane.

Argument (amplitude) of $z \to \theta = \tan^{-1}\left(\frac{y}{x}\right)$ $\quad -\pi < \theta \leq \pi$
Principal value of argument is when θ lies in $(-\pi, \pi]$

$|z - z_0| = r \to$ Circle with centre z_0.



(2) ϵ neighbourhood is denoted by $N(z_0, \epsilon)$

A complex set is said to be bounded if it is contained in a neighbourhood of origin.

Interior Point :- z_1 is interior point of set S if $\exists$ a neighbourhood of z_1 contained in S.
Boundary Point :- If every neighbourhood of z_1 contained some points of set S & some not from S.
Exterior Point :- Neither internal nor boundary.

Limit Point :- z_0 is lt pt. of set S if every deleted neighbourhood of z_0 contains one point of S. Limit pt. may or may not belong to the set.
Closed Set :- Set that contains all its limit points.

Closure of a set :- Union of S & limit pts. of S is closure of S

(3) A set S is said to be connected if any 2 points can be connected by a continuous curve all of whose pts. are in S.

(4) In complex analysis, we can have both single valued & multiple valued functions.

(5) Limit of a function.

$\lim_{z \to z_0} f(z) = L$ if $\forall \epsilon > 0 \; \exists \delta > 0$ s.t. $\forall x$ s.t. $|x - z_0| < \delta$
$\Rightarrow |f(x) - \cancel{f(z_0)}\,L| < \epsilon$

If $f(z_0) = L$ then we say f is continuous at z_0.

(6) Thm :- A function $f(z) = u(x,y) + i\,v(x,y)$ is continuous at $z_0 = x_0 + iy_0$ iff $u(x,y)$ & $v(x,y)$ are both continuous at (x_0, y_0)

eg. $f(x,y) = \frac{xy}{x^2+y^2}$ $z \neq 0$ & 0 when $z = 0$. Continuous?
Generally 1st check continuity along $y = mx$ line.
$\rightarrow$ Limit $= \frac{m}{1+m^2}$ $\therefore$ dependent on m $\rightarrow$ not continuous.

(7) Differentiability
If $f(z)$ is single valued then derivative of $f(z)$ is defined as
$$f'(z) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$$

While proving a given function $f(z)$ is differentiable, need to show $\lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$ exists in any direction (curve) along which h approaches origin.

(8) CAUCHY - RIEMANN EQUATIONS
If $f'(z)$ exists for $f(z) = u(x,y) + i\,v(x,y)$, then following Cauchy-Riemann Equations are satisfied
$$u_x = v_y \quad \& \quad u_y = -v_x$$
Then we can write $f'(z) = u_x + i\,v_x$
or $f'(z) = v_y - i\,u_y$

Cauchy-Riemann equations is a necessary condition for differentiability

When you check if C-R are satisfied & find they are not satisfied in general, also check if for any single point (say origin) they can be satisfied.

e.g. $u = x^2 - y^2$ $v = -2xy$ $u_x = 2x$, $u_y = -2y$, $v_x = -2y$, $v_y = -2x$

C-R satisfied nowhere except at origin.

C-R are necessary conditions for differentiability but not sufficient.

(9) If you check diff. along $y = mx$ paths & find limits are same, it is not enough to show function is diff. there. So, $y = mx$ paths help in showing function is not diff.

Also if u_x, u_y, v_x, v_y have x^2+y^2 etc. in denominator, you have to find these partial derivatives at origin with basic limit definition.

(10) When you have been specifically asked to prove $f(z)$ is not differentiable at origin, start with parameter $y = mx^P$ & after you have reduced $\frac{f(z)-f(0)}{z}$ to nice form in m, x & P Then put reasonable P value that gives derivative dependent on m.

ANALYTIC FUNCTIONS

(1) A function $f(z)$ is said to be analytic at z_0 if it is differentiable everywhere in a neighbourhood of z_0. So analyticity is a region bound property.

A function $f(z)$ analytic at every point in complex plane is called an entire function.
If $f(z)$ is not analytic only at finite points in complex plane, $f(z)$ is said to be analytic & those points are called singularities of the function.

Regular or holomorphic function same as analytic.

(2) z_0 is called singular point of $f(z)$ if $f'(z_0)$ doesn't exist.
All polynomials are entire functions & $\frac{1}{1-z}$ is analytic everywhere except $z=1$.

(3) $f(z)$ is analytic in domain $D \iff$
$u_x = v_y$ & $u_y = -v_x$ & these partial derivatives are continuous throughout D. (necessary & sufficient condition)

e.g. P.T. An analytic $g(z)$ is independent of $\bar{z}$.
This means showing $\frac{dg}{d\bar{z}} = 0$ i.e. $\frac{dg(x+iy)}{d\bar{z}} = \frac{dg}{dx}\cdot\frac{dx}{d\bar{z}} + \frac{dg}{dy}\cdot\frac{dy}{d\bar{z}}$

(4) Using (r,θ) for finding limits at origin.
$\lim_{z\to 0}$ is same as $\lim_{r\to 0}$ irrespective of θ.

e.g. Prove continuity at origin of $f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}$ $z \neq 0$
& $f(0) = 0$

Here (r,θ) is helpful as numerator degree is higher than denominator & both of them are homogeneous.

(5) Polar form of Cauchy - Riemann Equation

$\frac{du}{dr} = \frac{1}{r}\frac{dv}{d\theta}$ $\quad \frac{dv}{dr} = \frac{-1}{r}\frac{du}{d\theta}$

These equations are not valid at origin $\because r=0$.

(6) L'Hospital Rule

Let $f(z)$ & $g(z)$ be diff. at z_0 with $f(z_0)=g(z_0)=0$

If $g'(z_0) \neq 0$ then $\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$

(7) Harmonic Functions

Def^n:- $u(x,y)$ is said to be harmonic if it satisfies Laplace's Equation i.e. $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

If $f(z)$ is analytic, then its real & imaginary parts u, v are harmonic.

(8) If $f(z) = u + iv$ is analytic then u & v are said to be conjugate harmonic functions.

v is called a harmonic conjugate of u.

Also $\Rightarrow$ u is a harmonic conjugate of $-v$.

This gives us a necessary condition for a function $u(x,y)$ to be real part of a complex analytic function

(9) Finding the Harmonic Conjugate Function

$\rightarrow$ u is given & we have to find v s.t. $f(z) = u + iv$ is analytic

Use $u_x = v_y$ Then you get $u_y = -v_x$ & with these 2 get v

$u = x^3 - 3xy^2$ $\quad u_x = 3x^2 - 3y^2$ $\quad \therefore v = 3x^2y - y^3 + \phi(x)$

$u_y = -6xy$ $\quad \therefore v_x = 6xy$ $\quad \therefore v = 3x^2y + \phi_2(y)$

$\therefore v = 3x^2y - y^3 + c$

Some useful methods to find $f(z)$ when u is given

10 (a) These methods directly give $f(z)$ & not v. So if they ask you to find harmonic conjugate, this won't help.

$$f(z) = 2u\left(\frac{z}{2}, \frac{z}{2i}\right) - u(0,0) + Ci$$

e.g.: find $f(z)$ when $u = \sin x \cosh y$

$$u\left(z/2, z/2i\right) = \sin(z/2)\cosh(z/2i)$$
$$= (\sin z/2)\cos(z/2)$$

$$\therefore f(z) = \sin z + Ci \qquad \& \ u(0,0) = 0$$

(b) Mine Thomson's Method.

Let $u(x,y)$ be given & let $\phi_1(x,y) = \frac{\partial u}{\partial x}$

& $\phi_2(x,y) = \frac{\partial u}{\partial y}$

Then $f(z) = \int [\phi_1(z,0) - i\,\phi_2(z,0)]\,dz + C$ $\left(\int (u_x^{(z,0)} - i u_y^{(z,0)})\,dz\right)$

(also $f(z) = \int (v_y(z,0) + i\,v_x(z,0))\,dz$ ← easier to remember

If $v(x,y)$ is given & $\psi_1 = \frac{\partial v}{\partial y}$ ← (remember ∂y), $\psi_2 = \frac{\partial v}{\partial x}$ then

$$f(z) = \int [\psi_1(z,0) + i\,\psi_2(z,0)]\,dz + C$$

e.g. $u(x,y) = e^x(x\cos y - y\sin y)$

$\phi_1 = e^x(x\cos y - y\sin y) + e^x\cos y$ & $\phi_2 = e^x(-x\sin y - y\cos y - \sin y)$

$\therefore f(z) = \int e^z(z+1) = e^z z + C$

(c) Trick when $u+v$ or $u-v$ is given & asked $f(z)$

Let $f(z) = u + iv$ $\therefore if(z) = iu - v$

$\therefore (1+i)f(z) = (u-v) + i(u+v) = F_1(z)$

& $(1-i)f(z) = (u+v) + i(v-u) = F_2(z)$

So find F_1 or F_2 as per if $(u-v)$ or $(u+v)$ is given & divide appropriately to get ~~1+i~~ $f(z)$.

COMPLEX INTEGRATION

Different curves

(1) Simple closed curve / Jorden curve

Simple open curve

Closed but not simple

not closed not simple

Smooth curve - If $z(t)$ has continuous derivative

Piece-wise or sectionally smooth :- A curve which is composed of finite no. of smooth arcs.

Simply Connected Domain :-

A domain D is called simply connected if any closed curve in D can be shrunk to a point without leaving D.

A domain which is not simply connected $\Rightarrow$ multiply connected.

Positive Orientation

If walking on boundary, region is on your left.

(2) If $f(z)$ is analytic in R & C curve lies in R then $f(z)$ is certainly integrable along C.

If $z(t) = x(t) + iy(t) \quad a \le t \le b$

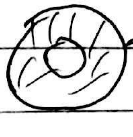
then $\int_C f(z) = \int_a^b f(z(t))\, z'(t)\, dt$

Eq. of line is given $z(t) = (a+bt) + i(c+dt)$

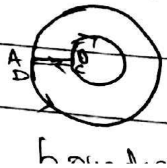
A curve integral can be expressed over parts of ~~its~~ the curve i.e.

$$\int_C f(z)dz = \int_{C_1+C_2+C_3} f(z)dz = \int_{C_1} f(z)dz + \int_{C_2} f(z)dz + \int_{C_3} f(z)dz$$

$\exists$ difference between convex region & simply connected domain.



but



This is simply connected domain if we consider boundary as ABCD

(3) Arc Length

Let a curve be defined parametrically by eq.
$x = \phi(t)$ & $y = \psi(t)$
The arc length L is given by $a \le t \le b$

$$L = \int_a^b \sqrt{(\phi_1'(t))^2 + (\psi'(t))^2}\, dt$$

$$= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

If $f(z)$ is integrable along a curve C having length L & if $\exists$ a +ve no. M s.t. $|f(z)| \le M$ on curve C then $\left|\int f(z)dz\right| \le ML$

(4) Cauchy's Fundamental Theorem

If $f(z)$ is analytic within & on a closed contour C then $\int_C f(z) = 0$

Green's Thm

Let $P(x,y)$ & $Q(x,y)$ have continuous partial derivatives in region R with boundary C.

$$\int_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

Cauchy's Weak Thm.

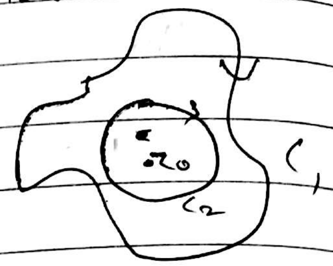
If $f(z)$ is analytic in a Simply Connected Domain D & contour C then $\int_C f(z) = 0$

This can be proven using Green's Thm. (put $f(z) = u + iv$, $dz = dx + i\,dy$)

Corollary :- Let C_1 & C_2 be curves with same initial & terminal points in analytic $f(z)$ $\rightarrow \int_{C_1} f(z) = \int_{C_2} f(z)$

e.g. Let C be curve $y = x^3 - 3x^2 + 4x - 1$ Joining $(1,1)$ & $(2,3)$

find $\int_C (12z^2 - 4iz^3)$ ← use fact that analytic ∴ integration depends only on endpoints.



e.g. $f(z)$ is analytic everywhere but only not at z_0. We are given a weired curve C_1 & asked $\int_{C_1} f(z)$.

We use fact that ~~an~~ if we construct nice circle C_2 with z_0 as centre, $\int_{C_1} f(z) = \int_{C_2} f(z)$

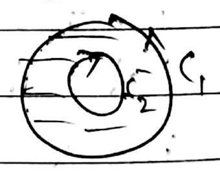
∵ On annulus, $f(z)$ is analytic & hence $\int$ is 0.

5 Cauchy - Goursat Theorem

If $f(z)$ is analytic in simply connected domain D & C is closed counter in D the $\int_C f(z) = 0$

Cauchy - Goursef Thm for Multiply Connected Region

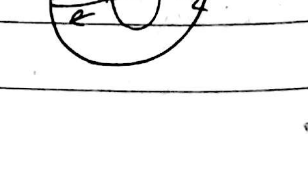
If $f(z)$ analytic in multiply connected domain D & its boundary C $\Rightarrow \int_C f(z) = 0$



Note :- Boundary C consists of $C_1 + C_2$.

Hence we are getting $\int_C f(z) = 0$

Proof :- Convert D to simply connected Dom. by $C = C_1 + AB + C_2 + BA$



APPLICATIONS OF CAUCHY'S THEOREM

(1) If $f(z)$ is analytic in domain D then $f(z)$ will have derivatives of all orders in D.

Cauchy's First Integral Formula

Let $f(z)$ be analytic in simply connected domain containing closed curve C. If z_0 is inside C then

$$f(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z - z_0} dz$$

If $f(z_0) = \frac{1}{2\pi i} \int \frac{f(z)}{z-z_0}$ for all points z_0 inside 'C', is f analytic inside & on C?

→ Yes. If $f(z)$ is not analytic, we can't get integral formula.

(2) Cauchy's General Integral Formula.

Let z_0 be inside C & let $f(z)$ be good function then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i} \int \frac{f(z)}{(z-z_0)^{n+1}} dz$$

e.g. Find $\int_{|z|=3} \frac{e^z \sin z}{(z-2)^2}$ → ~~use~~ $\int \frac{f(z)}{(z-z_0)^2} = 2\pi i\, f'(z_0)$

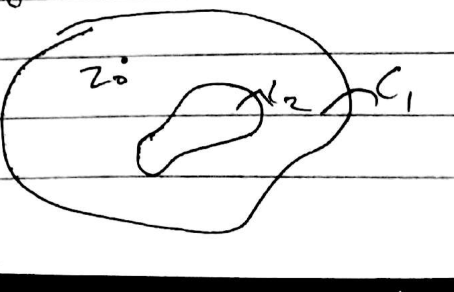
e.g. Find $\int_{|z|=2} \frac{z^3+3z-1}{(z-1)(z+3)} dz$ We use $f(z)$ as $\frac{(z^3+3z-1)}{z+3}$

Note we can't take $f(z)$ as $\frac{z^3+3z-1}{(z-1)}$ since -3 is not inside the curve $|z|=2$.

e.g. $\int_{|z|=2} \frac{1}{z^4-1} = \int \frac{1}{(z-1)(z+1)(z-i)(z+i)}$ becomes 0 after putting partial fractions.

(3) Cauchy's Integral Formula for Multiply-Connected Regions.

Let $f(z)$ be analytic in multiply connected Domain with boundary $C = C_1 \cup C_2$. Then



$$f(z_0) = \frac{1}{2\pi i}\left[\oint_{C_1} \frac{f(z)}{z-z_0} dz - \oint_{C_2} \frac{f(z)}{z-z_0} dz \right]$$

Similarly Cauchy's General formula remains valid for multiply-connected region.

(4) TAYLOR'S THEOREM

Let $f(z)$ be analytic in domain D with boundary C. z_0 be any point inside C, then $f(z)$ can be expressed as

$$f(z) = f(z_0) + \frac{(z-z_0)f'(z_0)}{1!} + \frac{(z-z_0)^2 f''(z_0)}{2!} + \ldots\ldots$$

The series converges for $|z-z_0| < \delta$ where δ is distance of z_0 from nearest point on C.

Conversely we can say that $f(z)$ is analytic at a point z_0 iff. $f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n$ in some δ neighbourhood of z_0. Here $a_n = \frac{1}{2\pi i} \int_{|z-z_0|=\delta} \frac{f(z)}{(z-z_0)^{n+1}} dz$

(5) MORERA'S THEOREM

Let $f(z)$ be continuous in a domain D & $\int_C f(z) = 0$ along every closed contour C contained in D, then $f(z)$ is analytic in D.

i.e. continuous + integral independent of path $\Rightarrow$ Analytic

Continuity is important condition here. Bcoz $\int_C \frac{1}{z^2} = 0$ if C is a closed curve containing origin. But $1/z^2$ is not continuous at origin.

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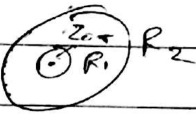
LAURENT'S THEOREM

(6) Let $f(z)$ be analytic in annulus $R_1 < |z-z_0| < R_2$.
Then the representation $f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$ is valid throughout the annulus.
& a_n is defined over integral over contour C contained in annulus:-

$$a_n = \frac{1}{2\pi i} \int \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta$$



Principal Part :- Series of negative powers of $(z-z_0)$ of the laurents series.
This part converges for $|z-z_0| > R_1$

Analytic Part :- Series of positive powers of $(z-z_0)$ in laurent's series. This part converges for $|z-z_0| < R$

(7) While calculating Laurent's series in problems, we don't actually find a_n by integration. We convert it to partial fraction & try to use $\frac{1}{1-z} = 1+z+z^2+z^3 \dots$ for $|z|<1$.

e.g. Expand $f(z) = \frac{1}{(z+1)(z+3)}$ valid for $1<|z|<3$

$\rightarrow f(z) = \frac{1}{2}\left(\frac{1}{z+1}\right) - \frac{1}{2}\left(\frac{1}{z+3}\right)$

(a) $1<|z|<3$ for $\frac{1}{z+1} = \frac{1}{z\left(1+\frac{1}{z}\right)} = \frac{1}{z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} \dots\right)$
$\rightarrow \because \frac{1}{|z|} < 1$.

(b) for $\frac{1}{z+3} = \frac{1}{3\left(1+\frac{z}{3}\right)} = \frac{1}{3}\left(1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} \dots\right)$

Here we take 3 common & not z like in first case bcoz $\left|\frac{z}{3}\right| < 1$ when $1<|z|<3$. So it all depends on what region you are finding Laurent's series for & accordingly you take common factors in denominator.

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(8) e.g. Find Laurent's expansion of $\frac{z^2}{z^4-1}$ for $0<|z-i|<\sqrt{2}$

→ Since we have been given region in terms of $|z-i|$, we should try to get series in those terms.

$$f(z) = \frac{z^2}{(z+i)(z-i)(z+1)(z-1)} = \frac{A}{z+i} + \frac{B}{z-i} + \frac{C}{z+1} + \frac{D}{z-1}$$

Consider $\frac{1}{z+i} = \frac{1}{(z-i+2i)} = \frac{1}{2i\left[1+\frac{z-i}{2i}\right]}$ Similarly ~~find~~ express $1/(z+1)$ & $1/z-1$ in terms of $(z-i)$. Keep $\frac{1}{(z-i)}$ as it is since it becomes the principle part.

e.g. Express $\sin(z)\cdot\sin(1/z)$ valid for $|z|>0$

$$\sin(z)\sin(1/z) = \left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\right)\left(\frac{1}{z} - \frac{1}{3!\,z^3} + \frac{1}{5!\,z^5} - \cdots\right)$$

So, don't be afraid of directly using taylor series that you already know of.

SINGULARITIES

(1) Def^n:- If $f(z)$ is not analytic at z_0, it is said to have a singularity there.
Isolated Singularity:- If $f(z)$ is analytic in deleted neighbourhood of z_0 then z_0 is called isolated singularity.

In this case, $f(z)$ has a laurent's expansion
$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n (z-z_0)^{-n}$$

(2) RIEMANN'S THEOREM
If $f(z)$ is bounded & has isolated singularity at z_0 then $f(z)$ can be redefined at z_0 s.t. $f(z)$ becomes analytic at z_0.

(3) Removable Singularity.
If $f(z)$ has an isolated singularity at z_0 then z_0 is said to be removable singularity if $\lim_{z \to z_0} f(z)$ exists.
e.g. $f(z) = \sin z / z$ has removable singularity at $z=0$ if we define $f(0)=1$.

(4) Pole of Order n.
If z_0 is isolated singularity & $f(z) \to \infty$ as $z \to z_0$ then $f(z)$ has a pole at z_0.
Let n be +ve integer s.t. $\lim_{z \to z_0} (z-z_0)^n f(z) = A \neq 0, \infty$

Then $f(z)$ has pole of order n at z_0.
If $n=1$ then we say it is simple pole at z_0.

If z_0 is pole of order K then $f(z)$ can be shown as

$$f(z) = \sum_{n=-K}^{\infty} b_n (z-z_0)^n .$$

e.g. $\dfrac{3z-3}{(z-1)^2(z+1)(z-4)}$ has a pole of order 2 at $z=1$
& simple poles at 4 & -1.

(5) Essential Singularity.
An isolated singularity that is neither a pole nor a removable singularity is said to be an isolated essential singularity.
e.g. $e^{1/z}$ has essential singularity at $z=0$.

(6) Casorati - Weistrass Theorem
If $f(z)$ has essential singularity at z_0, then $f(z)$ comes arbitrarily close to every complex value in each deleted neighbourhood of z_0.

i.e. $\forall\ \epsilon > 0, \delta > 0$ & complex number w, $\exists\ z'$ s.t.
$|z'-z_0| < \delta$ & $|f(z')-w| < \epsilon$

(7) Equivalent definition of essential singularity would be: we cannot find n s.t.
$$\lim_{z \to z_0} (z-z_0)^n f(z) = A \neq 0 \text{ or } \infty.$$

Here, the principal part of laurent's series has infinitely many terms.

(8) Nature of Singularity at $z=\infty$.
A singularity of $f(z)$ at $z=\infty$ is removable, a pole or essential accordingly singularity of $f\left(\frac{1}{z}\right)$ at $z=0$.

e.g. $f(z) = z^2+1 \rightarrow f\left(\frac{1}{z}\right) = \frac{1}{z^2}+1$
$\therefore$ $f(z)$ has pole of order 2 at $z=\infty$.

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(9) e.g. Find all finite isolated singularities of $\frac{1}{\sin z - \cos z}$.

→ It will have singularities when $\sin z = \cos z$ i.e. $z = n\pi + \frac{\pi}{4}$

But what is the order?

$$\frac{1}{\sin z - \cos z} = \frac{1/\sqrt{2}}{\frac{1}{\sqrt{2}}\sin z - \cos z \frac{1}{\sqrt{2}}} = \frac{1/\sqrt{2}}{\sin\left(z - \frac{\pi}{4}\right)}$$

$$\therefore \lim_{z \to n\pi + \frac{\pi}{4}} \frac{\frac{1}{\sqrt{2}}\left(z - \pi/4\right)}{\sin\left(z - \frac{\pi}{4}\right)} = 1/\sqrt{2}$$ $\therefore$ Pole is simple pole.

(10) DETERMINING SINGULARITY NATURE FROM PRINCIPAL PART OF LAURENT'S SERIES.

1. Principle Part is zero → Removable Singularity
2. Principle Part has finitely many terms → It is a pole.
3. Principle part has infinite terms → Essential Singularity

(11) If it is logarithmic, exponential, trigonometric functions then we use standard series to find Laurent series. If it is algebraic function then we use binomial expansion.

(12) e.g. Find nature & location of singularities of $f(z) = \frac{1}{z(e^z - 1)}$

P.T. it can be expanded in the form $\frac{1}{z^2} - \frac{1}{2z} + a_0 + a_2 z^2 + a_4 z^4 + \cdots$

→ $e^z - 1 = 0 \Rightarrow z = i2n\pi$

$\therefore z = 0$ is a pole now, $\lim z^2 f(z) = \lim \frac{z}{e^z - 1} = 1$

$\therefore$ double pole at $z = 0$.

Single pole at all other $i2n\pi$ since $\lim f(i2n\pi) \times (z - i n\pi 2) = \lim \frac{(z - i2n\pi z)}{(i n \pi 2)(e^z - 1)}$

$$= \frac{1}{2 i n \pi}$$

Now, double pole at $z = 0 \Rightarrow \frac{1}{z^2}$ will be there in Laurent series.

now, $f(z) = \dfrac{1}{z(e^z - 1)} = \dfrac{1}{z\left[1 + z + \frac{z^2}{2!} + \dots - 1\right]}$

$$= \frac{1}{z^2}\left[1 + \left(\frac{z}{2!} + \frac{z^2}{3!} + \dots\right)\right]^{-1}$$

$$= \frac{1}{z^2}\left[1 - \left(\frac{z}{2!} + \frac{z^2}{3!} + \dots\right) + \left(\frac{z}{2!} + \frac{z^2}{3!} \dots\right)^2 + (\ )^3 + (\ )^4 \dots\right]$$

On some work, we get req. eq.

(13) find order of pole at $z = 0$ for $\dfrac{z - \sin z}{z^3}$

$\longrightarrow \lim\limits_{z \to 0} \dfrac{z - \sin z}{z^3} = \lim \dfrac{1 - \cos z}{3z^2} = \lim \dfrac{\sin z}{6z} = \dfrac{1}{6}$

$\therefore$ no pole!!

(14) Find laurent series at $z = -2$ for $\dfrac{z}{(z+1)(z+2)}$ & give region of convergence.

$\longrightarrow$ let $z + 2 = u$

$\therefore \dfrac{z}{(z+1)(z+2)} = \dfrac{2-u}{u(1-u)} = \dfrac{2-u}{u}\left[1 + u + u^2 + u^3 + \dots\right] \quad (|u| < 1)$

$$= \frac{2}{(z+2)} + 1 + (z+2) + (z+2)^2 + \dots$$

convergence for $0 < |z+2| < 1$

So just get appropriate u & make life simple.

Residues

(1) Remember earlier formula $f^{(n)}(z_0) = \frac{n!}{2\pi i} \int \frac{f(z)}{(z-z_0)^{n+1}} dz$

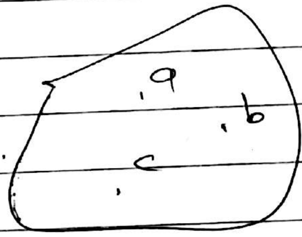
& then we had $a_{-1} = \frac{1}{2\pi i} \int \frac{f(z)}{1} dz$.

$\therefore$ We say a_{-1} is residue of $f(z)$ at z_0.

& $\int_C f(z)\, dz = 2\pi i\, a_{-1}$ where C includes z_0.
(z_0 is obv. singularity)

(2) Residue theorem

Let C curve contain singularities $a, b, c \ldots$ which have residues $a_{-1}, b_{-1}, c_{-1}, \ldots$



then $\int_C f(z) dz = 2\pi i (a_{-1} + b_{-1} + c_{-1} + \ldots)$

(3) e.g. Evaluate $\int_C \frac{1}{(z-1)(z-2)} dz$ along different C.

$\rightarrow$ At $z=1$ $\quad a_{-1} = \lim_{z \to 1} (z-1) \cdot f(z) = -1$

At $z=2$ $\quad b_{-1} = \lim_{z \to 2} (z-2) \cdot f(z) = 1$

$\therefore$ If C contains 1 & not 2 $\rightarrow \int_C = -2\pi i$

If C contain 2 & not 1 $\rightarrow \int_C = 2\pi i$

If C contains both 2 & 1 $\rightarrow \int_C = 0$.

(4) If simple pole at z_0; $\quad a_{-1} = \lim_{z \to z_0} (z - z_0) f(z)$

(5) Formula for calculating residue : Pole of order k

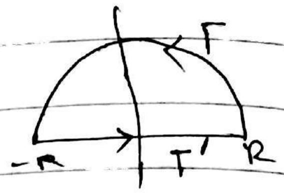
$$a_{-1} = \lim_{z \to z_0} \left(\frac{d^{(k-1)}}{dz^{(k-1)}} (z - z_0)^k f(z) \right) \times \frac{1}{(k-1)!}$$

(don't forget $\frac{1}{(k-1)!}$!!)

Contour Integration using Residues

∃ 3 types of contour integrals.

(1) Type I problem



We find $\int_{-\infty}^{\infty} f(x)\,dx$ by taking ~~region~~ curve

C as shown consisting Γ' & Γ.

We find integral around this curve using residues inside.

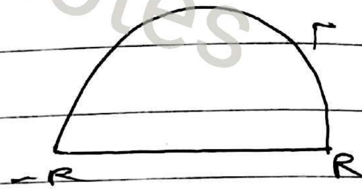
Then show that integral over Γ tends to 0 as $R \to \infty$,

This is our required integral.

for this use that if $|f(z)| < \frac{M}{R^k}$ $\forall z$, then $\int_\Gamma f(z)\,dz < \frac{M}{R^k} \cdot \pi R$ $\therefore$ As $R \to \infty$, $\int_\Gamma f(z)\,dz \to 0$ as $k > 1$.

(2) e.g. P.T. $\displaystyle\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)^2\,(x^2+2x+2)} = \frac{7\pi}{50}$

$\longrightarrow$



Consider $\displaystyle\int_C \frac{z^2\,dz}{(z^2+1)^2\,(z^2+2z+2)}$

Now poles of $f(z)$ are $\pm i$ & $-1 \pm i$.

Within our C, we have 2 poles $z = i$ & $z = -1+i$.

Residue at $z = i$ is of order 2.

$$\therefore a_{-1} = \frac{1}{1!} \lim_{z \to i} \frac{d}{dz} \left[(z-i)^2 \cdot \frac{z^2}{(z^2+1)^2\,(z^2+2z+2)} \right]$$

$$= \frac{9i-12}{100} \quad \text{(with some calculations)}$$

Residue at $z = -1+i$ is simple pole.

$$\therefore b_{-1} = \lim_{z \to (-1+i)} \frac{(z-(-1+i))\,z^2}{(z^2+1)^2\,(z^2+2z+2)} = \frac{3-4i}{25}$$

$$\therefore \int_C f(z)\,dz = 2\pi i\,(a_{-1} + b_{-1}) = \frac{7\pi}{50}.$$

Now, we show $\int$ over Γ tends to 0.

Remember to use L'Hospital's rule when calculating residue with single order pole. e.g. residue at $z=ae^{i3\pi/4}$ for $\int \frac{z^2}{z^4+a^4}$

$$\to \lim_{z\to ae^{i3\pi/4}} \frac{(z-ae^{i3\pi/4})}{(z^4+a^4)} = \lim \underset{R}{\frac{1}{4z^3}} = \frac{1}{4(a^3e^{i9\pi/4})}$$ so simple!!

$$\therefore \int_C f(z)\,dz = \int_{-R}^{R} \frac{x^2\,dx}{(x^2+1)^2(x^2+2x+2)} + \int_\Gamma \frac{z^2\,dz}{(z^2+1)^2(z^2+2z+2)}$$

Showing $\int_\Gamma \to 0$ $\left(\left|\int_C \frac{z^2dz}{(\ )(\ )}\right| < \int \frac{|z|^2|dz|}{|z^2+1|^2|z^2+2z+2|} < \int_0^{\pi} \frac{|z|^2|z|\,d\theta}{|z^4||z^2|} \le \frac{\pi}{R^3} \to 0\right)$ (use modulus; & $|dz| = R\,d\theta$)

We can take $z=re^{i\theta}$ & show as $r\to\infty$; 2nd integral $\to 0$.
& so on.

(3) Type II Problem.

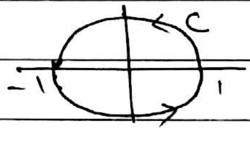
Here we have $\int_0^{2\pi} f(\theta)\,d\theta$. We use unit circle & on this circle $\frac{z+\frac{1}{z}}{2} = \cos\theta$ & $\frac{z-\frac{1}{z}}{2i} = \sin\theta$

the limit becomes curve of unit circle & we find residues lying in this circle. Remember $z=e^{i\theta} \Rightarrow dz = ie^{i\theta}d\theta$
i.e. $d\theta = \frac{dz}{iz}$.

(4) e.g. S.T. $\int_0^{2\pi} \frac{\cos\theta}{5+4\cos\theta}\,d\theta = \frac{-\pi}{3}$

→ taking $z=e^{i\theta}$ & with some work, we get

$$\int_0^{2\pi} \frac{\cos\theta}{5+4\cos\theta}\,d\theta = \int_C \frac{(z^2+1)}{2(z+2)(2z+1)}\,\frac{dz}{iz}$$



Now, poles at $z=0$, $z=-2$ & $z=-\frac{1}{2}$ lie in circle.
Residue at $z=-\frac{1}{2}$ given by

$$\lim_{z\to -\frac{1}{2}} \frac{(z+\frac{1}{2})\ (z^2+1)}{(2i)\ (z)\ (z+2)\ (2z+1)} = \frac{-5}{12i}$$

& so on.

Remember to properly put $(z-\alpha)f(z)$ & take limit. Don't directly take $f(\alpha)$ without $z-\alpha$ factor as here it would have given $\frac{-5}{6i}$.

eg. P.T. $\int_0^{\pi} \frac{a\,d\theta}{a^2+\sin^2\theta} = \frac{\pi}{\sqrt{1+a^2}}$

$$= \int_0^{\pi} \frac{2a\,d\theta}{2a^2+(1-\cos 2\theta)} = \int_0^{2\pi} \frac{a\,d\phi}{2a^2+(1-\cos\phi)}$$

Putting $2\theta=\phi$ reduces degree of eq. in z.

Also, limits $\int_0^{2\pi}$ are needed for complete circle.

(5) Note that in these type of problems, if you get $\cos n\theta$, then you have to use $\left(z^n + \frac{1}{z^n}\right)^n$ & thus number of poles will increase & thus calculation.

So when you have high n with $\cos n\theta$ or $\sin n\theta$, we represent it as real $(e^{i\theta n})$ or imag. $(e^{i\theta n})$ which means real (z^n) & thus reduces poles.

e.g. P.T. $$\int_0^{2\pi} \frac{\cos^2 3\theta \, d\theta}{1-2p\cos 2\theta + p^2} = \frac{\pi(1-p+p^2)}{1-p} \qquad 0<p<1$$

→ Now with std. approach, we get

$$f(z) = \frac{(z^6+1)^2}{z^5[z^2+\ldots]} \text{ ; thus 7 degree poles!!.}$$

instead we go

$$I = \frac{1}{2}\int_0^{2\pi} \frac{1+\cos 6\theta}{1-2p\cos 2\theta + p^2} = \text{real}\left\{\frac{1}{2}\int_0^{2\pi} \frac{1+e^{i6\theta}}{1-2p\cos 2\theta + p^2} d\theta\right.$$

$$= \text{real}\left\{\frac{1}{2}\int_C \frac{z^2(1+z^6)}{(z^2-p)(1-pz^2)}\right\}$$

Thus we reduced poles to degree 2.

(6) e.g. $$\int_0^{2\pi} e^{\cos\theta}\cos(n\theta - \sin\theta)\, d\theta = \text{real}\int_0^{2\pi} e^{\cos\theta} \cdot e^{-i(n\theta-\sin\theta)} d\theta = \text{real}\int e^{e^{i\theta}} \cdot e^{-in\theta}$$

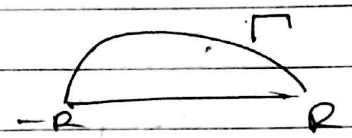
(note that we took $-i(n\theta - \sin\theta)$ to get e^z).

$$= \text{real}\int_C e^z z^{-n} \frac{dz}{iz} \quad \text{\& so on.}$$

(7) Type (3)

We get function of type $\int_{-\infty}^{\infty} f(x) \cdot \sin ax$ or $\cos ax$. Again we take semicircle from $-R$ to R & ~~show~~ get $\int_C f(z) e^{iaz}$. So here, We directly don't get $\int_{-\infty}^{\infty} f(x)\,dx$ but it will be real or imaginary part of $\int_c f(z)\,e^{iza}$ as per we having $\cos ax$ or $\sin ax$.

e.g. $\displaystyle\int_0^{\infty} \frac{\cos ax}{x^2+1}\,dx = \frac{\pi}{2} e^{-a}$; $a > 0$



$\longrightarrow$ $\displaystyle\int_0^{\infty} \frac{\cos ax}{x^2+1}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1}\,dx$

Consider $\displaystyle\int_c \frac{e^{iaz}}{z^2+1}\,dz$; note its real part for $\int_{-R}^{R}$ is our guy.

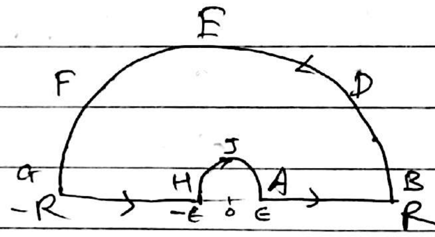
Basic calculations give residue & integral.

Then show $\displaystyle\int_{\Gamma} \frac{e^{iaz}}{z^2+1}\,dz \to 0$ as $R \to \infty$.

this could get complicated. Try to use $z = re^{i\theta}$ & $r \to \infty$ etc.

(8) Some special integrals.

P.T. $\displaystyle\int_0^{\infty} \frac{\sin x}{x}\,dx = \frac{\pi}{2}$.



$\rightarrow$ Given region has no residues.

$$\therefore \int_{-R}^{-\epsilon} \frac{e^{iz}}{z}\,dz + \int_{HJA} \frac{e^{iz}}{z}\,dz + \int_{\epsilon}^{R} \frac{e^{iz}}{z}\,dz + \int_{BDEFG} \frac{e^{iz}}{z}\,dz = 0.$$

$$\therefore 2i \int_{\epsilon}^{R} \frac{\sin x}{x}\,dx = -\int_{HJA} \frac{e^{iz}}{z}\,dz - \int_{BDEFG} \frac{e^{iz}}{z}\,dz$$

As $R \to \infty$ & $\epsilon \to 0$; 2^{nd} integral of RHS becomes 0.

$$\therefore I = \frac{1}{2i} \lim_{\epsilon \to 0} -\int_{HJA} \frac{e^{iz}}{z}\,dz = \frac{\pi i}{2i} = \frac{\pi}{2}$$

So, in questions having 0 as pole; go around it using ϵ radius semicircle & let $\epsilon \to 0$ & $R \to \infty$.

Complex Analysis :- Additional theorems

(first complete basic theory from next pages)

① If we are given that $f(z)$ has pole of multiplicity p at α and zero of multiplicity q at β. We use following form and then with manipulation try to arrive at required result.

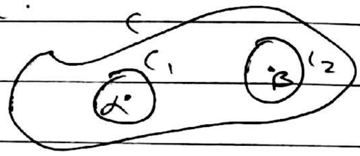
Pole of mul. p at α gives $f(z) = G(z)/(z-\alpha)^p$ $\left(\frac{G(z)}{(z-\alpha)^p}\right)$
where $G(z)$ is analytic at α.

Zero of mul. q at β means $f(z) = H(z) \cdot (z-\beta)^q$
$H(z)$ is analytic & nonzero at β.

② The Argument Theorem

→ Let $f(z)$ have pole of multiplicity p at α & zero of order n at β and no other pole or zero inside C.

then $\displaystyle \frac{1}{2\pi i}\int \frac{f'(z)}{f(z)}\,dz = n - p$



Proof :→ As above; $f(z) = \dfrac{F(z)}{(z-\alpha)^p}$ $\therefore \dfrac{f'(z)}{f(z)} = \dfrac{F'(z)}{F(z)} - \dfrac{p}{(z-\alpha)}$

$$\therefore \frac{1}{2\pi i}\oint_C \frac{f'(z)}{f(z)} = \frac{1}{2\pi i}\left[\oint_{C_1} \frac{F'(z)}{F(z)} - p\int_{C_1}\frac{dz}{z-\alpha}\right] = \frac{0 - 2\pi i p}{2\pi i} = -p$$

& $f(z) = (z-\beta)^n G(z)$ $\therefore \dfrac{f'(z)}{f(z)} = \dfrac{n}{z-\beta} + \dfrac{G'(z)}{G(z)}$

$$\therefore \frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)} = \frac{1}{2\pi i}\left[n\int_{C_2}\frac{dz}{z-\beta} + \oint_{C_2}\frac{G'(z)}{G(z)}\,dz\right] = n.$$

$\therefore$ given integral $= n - p$.

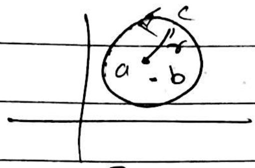
Just remember trick ①.

3) Liouville's Theorem

If for all z in entire complex plane (i) $f(z)$ is analytic
(ii) $f(z)$ is bounded.

Then $f(z)$ must be constant.

→ Consider circle of radius r with centre at a & another point b in it.



Now,

$$f(b)-f(a) = \frac{1}{2\pi i}\left[\oint_c \frac{f(z)}{z-b} - \oint_c \frac{f(z)}{z-a}\right]$$

$$= \frac{b-a}{2\pi i}\left[\oint_c \frac{f(z)\,dz}{(z-a)(z-b)}\right]$$

now, $|z-a| = r$ & for large enough r; we can have $|z-b| \geq r/2$. & let $|f(z)| < M$ ∵ bounded.

$$\therefore |f(b)-f(a)| = \left|\frac{b-a}{2\pi}\right| \left|\oint_c \frac{f(z)\,dz}{(z-a)(z-b)}\right| < \left(\frac{|b-a|}{2\pi}\right) \frac{M \cdot 2\pi r}{r \cdot r/2}$$

$$= \frac{2|b-a|M}{r}$$

let $r \to \infty$ ∴ $|f(b)-f(a)| = 0$ ∴ $f(a) = f(b)$

4) Rouche's Theorem

If $f(z)$ & $g(z)$ are analytic in C & on curve C $|g(z)| < |f(z)|$
then $f(z)+g(z)$ and $f(z)$ have same no. of zeroes inside C.

→ Very useful theorem.

Question :- P.T. Roots of $z^7 - 5z^3 + 12 = 0$ lie betn $|z| = 1$ & $|z| = 2$

→ On $|z| = 1$ take $f(z) = 12$ & $g(z) = z^7 - 5z^3$ $|g(z)| \leq |1| + |5| \leq 6 < 12 = |f(z)|$

$f(z)$ has no 0 ∴ $f(z)+g(z)$ has no 0 inside $|z| = 1$

on $|z| = 2$ take $f(z) = z^7$ $g(z) = -5z^3 + 12$ ∴ $|g(z)| < |5 \times 2^3| + 12 = 52 < 2^7 = |f(z)|$

∴ $f(z)+g(z)$ has 7 zeroes inside $|z| = 2$.

& $f(z)$ has 7 zeroes inside $|z| = 2$

Essentially you have to choose convenient $|z^k|$.