IFoS - 2013 → Paper II

5) (a) Use Newton-Raphson method and derive the iteration scheme $x_{n+1} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right)$ to calculate approximate value of the sequence root of the no. N. Show that the formula $\sqrt{N} = \frac{A+B}{4} + \frac{N}{A+B}$ where $AB = N$, can easily be obtained if the above scheme is applied two times. Assume $A = 1$ as an initial guess and use the formula twice to calculate the value of $\sqrt{2}$

⇒ Let $x = \sqrt{N} \Rightarrow x^2 - N = 0$

Now again let, $f(x) = x^2 - N$

$$\Rightarrow f'(x) = 2x$$

By Newton-Raphson formula,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - N}{2x_n}$$

$$= \frac{2x_n^2 - x_n^2 + N}{2x_n} = \frac{x_n^2 + N}{2x_n}$$

$$= \frac{x_n}{2} + \frac{N}{2x_n} = \frac{1}{2}\left(x_n + \frac{N}{x_n}\right) \quad \text{--- (1)}$$

let, $x_n = \frac{A+B}{2}$ in equation (1)

$$x_{n+1} = \frac{A+B}{4} + \frac{N}{A+B} \qquad \text{[proved]}$$

Now, to compute $\sqrt{2}$, putting $N = 2$, in the above formula, we get the iteration formula,

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)$$

Now taking $x_0 = 1$, we have successive approximatio,

| $n$ | $x_n$ | $x_{n+1}$ |
|---|---|---|
| 0 | 1 | 1.5 |
| 1 | 1.5 | 1.42 |
| 2 | 1.42 | 1.414 |
| 3 | 1.414 | 1.4142 |

$\therefore \sqrt{2} = 1.414$, correct upto 3-decimal places.

8) (b) Use the classical forth order Runge-Kutta method with $h=0.2$, to calculate a solution at $x=0.4$ for the initial value problem,

$$\frac{du}{dx} = 4-x^2+u, \quad u(0)=0 \text{ on the interval } (0,0.4)$$

$\Rightarrow$ For $(0.2)$: $x_0=0, u_0=0$, $f(x,u)=4-x^2+u$ and $h=0.2$

Thus,

$$K_1 = hf(x_0,u_0) = 0.2\times f(0,0) = 0.8$$

$$K_2 = hf\left(x_0+\frac{h}{2}, u_0+\frac{K_1}{2}\right) = 0.2\times f(0.1, 0.4) = 0.878$$

$$K_3 = hf\left(x_0+\frac{h}{2}, u_0+\frac{K_2}{2}\right) = 0.2\times f(0.1, 0.439) = 0.8858$$

$$K_4 = hf(x_0+h, u_0+K_3) = 0.2\times f(0.2, 0.8858) = 0.96916$$

For $u(0.4)$: here $x_1=0.2$, $u_1=0.96916$

$$K_1 = hf(x_1,u_1) = 0.2\times f(0.2, 0.96916) = 0.985832$$

$$K_2 = hf\left(x_1+\frac{h}{2}, u_1+\frac{K_1}{2}\right) = 0.2\times f(0.3, 1.462076) = 1.0744152$$

$$K_3 = hf\left(x_1+\frac{h}{2}, u_1+\frac{K_2}{2}\right) = 0.2\times f(0.3, 1.5063676) = 1.08327352$$

$$K_4 = hf(x_1+h, u_1+K_3) = 0.2\times f(0.4, 2.05243352) = 1.178486704$$

$\therefore u(0.1) = u_0 + \frac{1}{6}[K_1+2K_2+2K_3+K_4]$

$= 0 + \frac{1}{6}[5.29676] = 0.882793$

For $u(0.4)$: here $x_1=0.2$, $u_1=0.882793$

$$K_1 = hf(x_1,u_1) = 0.2\times f(0.2, 0.882793) = 0.9685586$$

$$K_2 = hf\left(x_1+\frac{h}{2}, u_1+\frac{K_1}{2}\right) = 0.2\times f(0.3, 1.3670723) = 1.05541446$$

$$K_3 = hf\left(x_1+\frac{h}{2}, u_1+\frac{K_2}{2}\right) = 0.2\times f(0.3, 1.41050023) = 1.064100046$$

$$K_4 = hf(x_1+h, u_1+K_3) = 0.2\times f(0.4, 1.946893046) = 1.157378609$$

$\therefore u(0.4) = u_1 + \frac{1}{6}[K_1+2K_2+2K_3+K_4]$

$= 0.882793 + \frac{1}{6}\times 6.36496622|$

$= 1.943620704$

$\therefore$ at $x=0.4$ the required solution is, 1.9436, Correct upto four decimal places.