1. Find DD. of $\vec{V}^2$, where $\vec{V} = xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k}$ at $(2,0,3)$ in outward normal to surface $x^2+y^2+z^2=14$ at point $(3,2,1)$.

Given $\vec{V} = xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k}$

$\vec{V}\cdot\vec{V} = x^2y^4 + z^2y^4 + x^2z^4$.

i.e.

Directional Derivative of any ~~f~~function f is given by $= \nabla f \cdot \hat{a}$ where $\vec{a}$ is unit vector in desired direction.

Any vector normal to surface $g(x,y,z) = x^2+y^2+z^2=14$ is given by $\frac{\nabla g}{|\nabla g|}$

$$= \frac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4(x^2+y^2+z^2)}} \quad \text{At } (3,2,1)$$

$$\boxed{\hat{a} = \frac{3\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{14}}}$$

$$\nabla V^2 = (2xy^4 + 2xz^4)\hat{i} + (4x^2y^3 + 4z^2y^3)\hat{j} + (2zy^4 + 4x^2z^3)\hat{k}$$

At $(2,0,3)$ it is $\nabla V^2 = 81\times4\,\hat{i} + 0\hat{j} + 16\times27\,\hat{k}$

So $DD = 27\times4\left[3\hat{i} + 0\hat{j} + 4\hat{k}\right]\cdot\frac{(3\hat{i} + 2\hat{j} + \hat{k})}{\sqrt{14}}$

$$= 108\left[\frac{9+4}{\sqrt{14}}\right] = \boxed{\frac{1404}{\sqrt{14}}}$$

2 Show $\vec{F} = (2xy+z^2)\hat{i} + x^2\hat{j} + 3z^2x\hat{k}$

(1) is a conservative field. Find its scalar potential and work done in moving a particle from $(1,-2,1)$ to $(3,1,4)$

(2) show $\nabla^2 f(r) = \left(\frac{2}{r}\right) f'(r) + f''(r)$,

$r = \sqrt{x^2+y^2+z^2}$.

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy+z^2 & x^2 & 3z^2x \end{vmatrix} = 0\hat{i} + (2z-3z)\hat{j} + 0\hat{k}$$

$$\vec{F} = \nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$$

$\frac{\partial\phi}{\partial x} = 2xy+z^2 \Rightarrow \phi = x^2y + z^2x + f_1(y,z)$ — (1)

$\frac{\partial\phi}{\partial y} = x^2 \Rightarrow \phi = x^2y + f_2(x,z)$ — (2)

$\frac{\partial\phi}{\partial z} = 3z^2x \Rightarrow \phi = z^3x + f_3(x,y)$ — (3)

Using (1), (2), (3), $\phi = x^2y + z^2x + z^3x$

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$\Rightarrow$

$$\nabla^2 f(r) \Rightarrow \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

$$\frac{\partial}{\partial x} f(r) = f'(r)\frac{\partial r}{\partial x} = f'(r)\frac{x}{r}$$

$$\left[\because r = \sqrt{x^2+y^2+z^2} \Rightarrow \boxed{\frac{\partial r}{\partial x} = \frac{x}{r}}\right]$$

$$\frac{\partial^2}{\partial x^2} f(r) = f''(r)\left(\frac{\partial r}{\partial x}\right)^2 + f'(r) \times \frac{\partial}{\partial x}\left(\frac{\partial r}{\partial x}\right)$$

$$= f''(r)\left(\frac{x}{r}\right)^2 + f'(r)\left[\frac{\partial}{\partial x}\left(\frac{x}{r}\right)\right]$$

$$= f''(r)\left(\frac{x}{r}\right)^2 + f'(r)\left[\frac{1}{r} - \frac{x}{r^2} \times \frac{\partial r}{\partial x}\right]$$

$$= f''(r)\frac{x^2}{r^2} + f'(r)\left[\frac{1}{r}\left(1 - \frac{x^2}{r^2}\right)\right]$$

similarly for $\frac{\partial^2}{\partial y^2} f(r)$, $\frac{\partial^2}{\partial z^2} f(r)$ and adding,

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) f(r) = f''(r)\left(\frac{x^2+y^2+z^2}{r^2}\right) + \frac{f'(r)}{r}\left[3 - \frac{x^2+y^2+z^2}{r^2}\right]$$

$$= f''(r) + 2\frac{f'(r)}{r} \quad \left[\because r = \sqrt{x^2+y^2+z^2}\right]$$

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3. Use divergence Th^m $\iint_S (x^3\,dydz + x^2y\,dzdx + x^2z\,dydx)$

S is sphere $x^2+y^2+z^2=1$.

Divergence Thm says that

$$\iint \vec{F}\cdot\hat{n}\,dS = \iiint \nabla\cdot\vec{F}\,dV$$

S = sphere $x^2+y^2+z^2=1$

$$\hat{n} = \frac{\nabla f}{|\nabla f|} = \frac{2x\hat{i}+2y\hat{j}+2z\hat{k}}{\sqrt{4(x^2+y^2+z^2)}}$$

$$= \frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}}$$

ie. $\vec{F}\cdot\hat{n}$

Here $\vec{F} = x^3\hat{i} + x^2y\hat{j} + x^2z\hat{k}$

so $\nabla\cdot\vec{F} = \frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(x^2y) + \frac{\partial}{\partial z}(x^2z)$

$= 3x^2 + x^2 + x^2 = 5x^2$

so; $\iiint \nabla\cdot\vec{F}\,dV = \iiint 5x^2\,dV$ — (1)

converting integral into spherical co-ordinate

$x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$

since S = sphere $x^2+y^2+z^2=1$ so,

$0\le r\le 1$; $0\le\theta\le\pi$, $0\le\phi\le 2\pi$

and $dV = (dr)(r\,d\theta)(r\sin\theta\,d\phi)$

$= r^2\sin\theta\,dr\,d\theta\,d\phi$ so

(1) becomes $I = \iiint 5r^2\sin^2\theta\cos^2\phi \times r^2\sin\theta\,dr\,d\theta\,d\phi$

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$$= \iiint 5r^4 dr \times (\sin^2\theta \times \sin\theta \, d\theta \times \cos^2\phi \, d\phi$$

$$= \int_0^1 5r^4 \, dr \times \int_0^\pi (1-\cos^2\theta)\sin\theta \, d\theta \times \int_0^{2\pi} \frac{1+\cos 2\phi}{2} d\phi$$

$$= \frac{5r^5}{5}\Big|_0^1 \times -\int_1^{-1} (1-t^2)\,dt \times \left(\frac{\phi}{2} + \frac{\sin 2\phi}{4}\right)\Big|_0^{2\pi}$$

Using $\cos\theta = t \rightarrow -\sin\theta \, d\theta = dt$ $\quad \theta = 0, t = +1$
$\theta = \pi, t = -1$
and $\int \cos a\theta = \frac{\sin a\theta}{a}$

$$= 5 \times \left(\frac{1}{5} - 0\right) \times t - \frac{t^3}{3}\Big|_{-1}^{1} \times \left(\frac{2\pi}{2} + 0 - 0 - 0\right)$$

$$= 1 \times \frac{4}{3} \times \pi = \frac{4}{3}\pi.$$

$\because -\int_b^a f(x)dx = \int_a^b f(x)dx$
if $f(x)$ is even

Q-4. $\vec{A} = 2y\hat{i} - z\hat{j} - x^2\hat{k}$

S = parabolic cylinder $y^2 = 8x$ if first octant bounded by planes $y=4, z=6$

Find $\iint \vec{A}\cdot\hat{n}\, dS$

$f(x,y) = 8x - y^2$

normal to surface of parabolic cylinder $= \frac{\nabla f}{|\nabla f|}$

$$= \frac{8\hat{i} - 2y\hat{j}}{\sqrt{64+4y^2}} = \frac{4\hat{i} - y\hat{j}}{\sqrt{16+y^2}}$$

$$\vec{A}\cdot\hat{n} = \frac{8y - zy}{\sqrt{16+y^2}}$$

$$\iint \vec{A}\cdot\hat{n}\, dS = \iint \vec{A}\cdot\hat{n}\, \frac{dy\,dz}{|\hat{n}\cdot\hat{i}|}$$ [taking projection on yz plane of S]

$$= \iint \frac{(8y - zy)}{\sqrt{16+y^2}} \frac{dy\,dz}{\frac{4}{\sqrt{16+y^2}}}$$

$$= \frac{1}{4}\iint y(8-z)\, dy\,dz = \frac{1}{4}\int_0^4 y\,dy \int_0^6 (8-z)\,dz$$

$$= \frac{1}{4} \times \left.\frac{y^2}{2}\right|_0^4 \times \left.\left(8z - \frac{z^2}{2}\right)\right|_0^6$$

$$= \frac{1}{4} \times 8 \times (48 - 18) = 2 \times 30 = \boxed{60}$$

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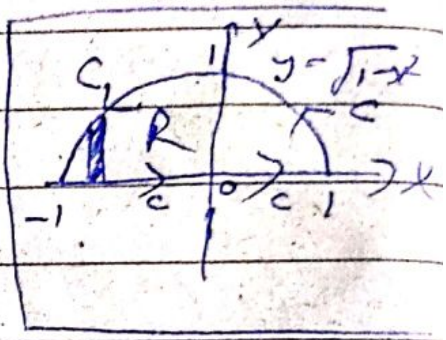
Q-5 Using Green's theorem in plane,
$\int [(2x^2-y^2)\,dx + (x^2+y^2)\,dy]$; C = boundary of surface in xy plane enclosed by $y=0$ and semicircle $y=\sqrt{1-x^2}$

Green's Th^m $\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

$P = 2x^2 - y^2 \Rightarrow \frac{\partial P}{\partial y} = -2y,$

$Q = x^2 + y^2 \Rightarrow \frac{\partial Q}{\partial x} = 2x$



$\int (2x^2 - y^2)\,dx + (x^2+y^2)\,dy$

$= \iint (2x + 2y)\,dx\,dy$

$= 2 \iint (x+y)\,dx\,dy$

$= 2 \int_{-1}^{1} \left[\int_{y=0}^{\sqrt{1-x^2}} (x+y)\,dy \right] dx$

$= 2 \int_{-1}^{1} \left[xy + \frac{y^2}{2} \right]_0^{\sqrt{1-x^2}} dx$

$= 2 \times \int_{-1}^{1} \left[x\sqrt{1-x^2} + \left(\frac{1-x^2}{2}\right) \right] dx$

$= 2 \int_{-1}^{1} x\sqrt{1-x^2}\,dx + 2 \int_{-1}^{1} \frac{1-x^2}{2}\,dx$

Now for $\left[\int_{-c}^{c} f(x)\,dx = \begin{cases} 0 & \text{if } f(-x) = -f(x) \\ 2\int_0^c f(x) & \text{if } f(-x) = f(x) \end{cases} \right.$

$= 2 \times 0 + 2 \times 2 \times \left. \frac{x}{2} - \frac{x^3}{6} \right|_0^1 = \boxed{\frac{4}{3}}$

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