MAINS TEST SERIES-2021

TEST-14 (BATCH-I)

FULL SYLLABUS (PAPER-II)

Answer Key

(1)(a) Complete the partial Cayley group table given below.

*	1	2	3	4	5	6	7	8
1	1	2	3	4	5	6	7	8
2	2	1	4	3	6	5	8	7
3	3	4	2	1	7	8	6	5
4	4	3	1	2	8	7	5	6
5	5	6	8	7	1	2	4	3
6	6	5	7	8	2	1	3	4
7	7	8	5	6	3	4	1	2
8	8	7	6	5	4	3	2	1

← Answer in circle

Sol^n From table –
unity or Identity element is 1

Also
$1*1=1,\ 2*2=1,\ 3*4=1,\ 4*3=1$
$5*5=1,\ 6*6=1\ \ 7*7=1\ \ 8*8=1$

In Group – associativity property is satisfied.

So $(5*5)*6 = 5*(5*6)$

$\Rightarrow 1*6 = 5*(5*6)$

$\Rightarrow 6 = 5*(5*6)$

$\Rightarrow \boxed{5*6 = 2}$

Using same property (association) we get $6*5=2,\ 5*7=4,\ 5*8=3$

$6*7=3,\ 6*8=4,\ 7*5=3,\ 7*6=4,\ 7*8=2$

$8\times5=4,\ 8*6=3,\ 8*7=2$

(1)(b) Give an example of a Boolean ring with four elements. Give an example of an Infinite Boolean Ring.

Sol^n Let $R_1 = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

with multiplication.

Since $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Therefore $(R_1, \cdot)$ is example of Boolean ring with four element.

1(d) → Discuss the continuity of the following complex valued function at $z=0$,

$$f(z) = \begin{cases} \dfrac{1-\exp(-|z|^2)}{|z|^2} & ; f(z) \neq 0 \\ 1 & ; f(z) = 0 \end{cases}$$

Sol^n: let us approach 'o' along any coordinate axes path:

$$\underset{(x,y)\to(0,0)}{Lt} f(x,y) = \underset{(x,y)\to(0,0)}{Lt} \frac{1-e^{-|x+iy|^2}}{|x+iy|^2}$$

Along x-axis path:

$$\underset{x\to 0}{Lt} f(x,0) = \underset{x\to 0}{Lt} \frac{1-e^{-|x|^2}}{|x|^2}$$

$$= \underset{x\to 0}{Lt} \frac{1-e^{-x^2}}{x^2}$$

$$= \underset{x\to 0}{Lt} \frac{1-\left[1-\frac{x^2}{1!}+\frac{x^4}{2!}+\cdots\right]}{x^2}$$

$$= 1 = f(0)$$

It is going to become a continuous function.

please check it by ϵ-δ definition

1.(e) The standard weight of a special purpose brick is 5kg and it contains two basic ingredients B_1 and B_2. B_1 costs Rs 5 per kg and B_2 costs Rs. 8 per kg. Strength considerations state that the brick contains not more than 4kg of B_1 and minimum of 2kg of B_2. Since the demand for the product is likely to be related to the price of the brick, find out graphically minimum cost of the brick satisfying the above conditions.

Sol^n: The formulation of the given problem is.

Min (total cost) $Z = 5x_1 + 8x_2$

subject to the constraints

$x_1 \leq 4$

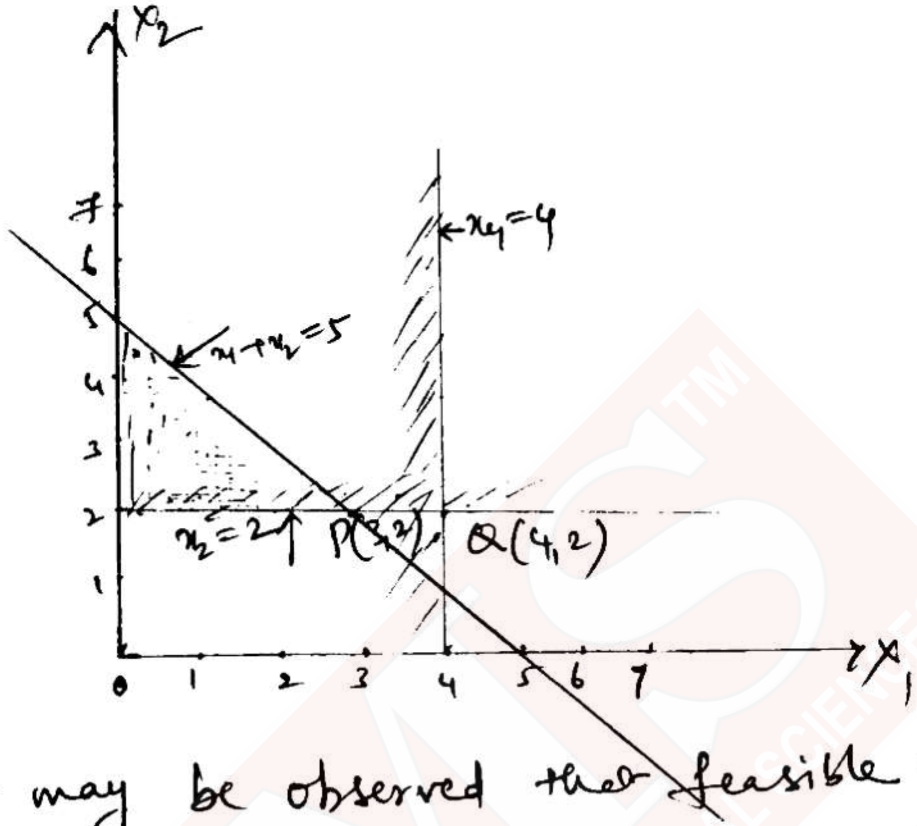
$x_2 \geq 2$

$x_1 + x_2 = 5$

$x_1, x_2 \geq 0$.

where x_1, x_2 are the amount of ingredients B_1 and B_2 (in kg) respectively

The given constraints are plotted on the graph as shown in the figure.



It may be observed that feasible region has two cornerpoints $P(3,2)$ and $Q(4,2)$.
The minimum value of Z is found at $P(3,2)$.

i.e, $x_1 = 3$, $x_2 = 2$,

Hence the optimum product mix is to have 3kg of ingredient B_1 and 2kg of ingredient B_2 of a special care brick in order to achieve the minimum cost of Rs. 31.

2(a) Suppose that $\phi : R \to S$ is a ring homomorphism and that the image of ϕ is not $\{0\}$. If R has a unity and S is an Integral domain. Show that ϕ carries the unity of R to unity of S. Give an example to show that the preceding statement need not be true if S is not an Integral domain.

Sol^n. Say 1_R is the unity of R

1_S is the unity of S

Pick $a \in R$ such that $\phi(a) \neq 0$

Then $1_S \phi(a) = \phi(1_R a) = \phi(1_R)\phi(a)$ [By Homomorphism]

$\Rightarrow 1_S \phi(a) = \phi(1_R)\phi(a)$

Using Right Cancellation Law -

$\Rightarrow 1_S = \phi(1_R)$ Hence proved

Example -

$\phi : Z_3 \to Z_6$

where $\phi(x) = 4x$, Z_6 is not ID.

2.(c) If the function $f(z)$ is analytic and one valued in $|z-a|<R$, Prove that for $0<r<R$

$$f'(a) = \frac{1}{\pi r}\int_0^{2\pi} P(\theta) e^{-i\theta} d\theta,$$ where $P(\theta)$ real part of $(a+re^{i\theta})$

Sol'n: Since $f(z)$ is regular in $|z-a|<R$, therefore it must be regular in $|z-a|=r$, $(r<R)$

Hence $f(z)$ can be expanded in a Taylor's Series within the Circle $|z-a|=r$

i.e $$f(z) = \sum_0^\infty a_m (z-a)^m$$
$$= \sum_0^\infty a_m r^m (e^{i\theta})^m \quad \text{Since } z-a = re^{i\theta}$$

So that $$\overline{f(z)} = \sum_0^\infty \bar{a}_m r^m (e^{i\theta})^m$$

Now, consider the integral $\int_C \overline{f(z)} \cdot \frac{dz}{(z-a)^{n+1}}$

$$\int_C \overline{f(z)} \cdot \frac{dz}{(z-a)^{n+1}} = \int_0^{2\pi} \sum_0^\infty \bar{a}_m r^m e^{-im\theta} \frac{re^{i\theta} \cdot id\theta}{r^{n+1} e^{i(n+1)\theta}}$$

$$= \sum_0^\infty \bar{a}_m \cdot r^{m-n} \cdot i \int_0^{2\pi} e^{-i(m+n)\theta} = 0 \text{ for all values of } n$$

Particularly $$\int_C \overline{f(z)} \frac{dz}{(z-a)^2} = 0 \quad \text{——— (1)}$$

we know that $$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^2}$$

$$= \frac{1}{2\pi i}\int_C \frac{f(z) + \overline{f(z)}}{(z-a)^2} dz \text{ from (1)}$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(a+re^{i\theta}) + \overline{f(a+re^{i\theta})}}{r^2 e^{2i\theta}} re^{i\theta} \cdot id\theta$$
since $z-a = re^{i\theta}$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{2\mathrm{Re} f(a+re^{i\theta})}{r^2 e^{2i\theta}} re^{i\theta} \cdot id\theta$$
since $z+\bar{z} = 2\mathrm{Re}(z)$

$$= \frac{1}{\pi r}\int_0^{2\pi} P(\theta) e^{-i\theta} d\theta. \quad [\text{Since } P(\theta) = \mathrm{Re} f(a+re^{i\theta})]$$

This Proves the result.

3(a) Let G be a group and H a subgroup. For any element g of G, define $gH = \{gh \mid h \in H\}$. If G is abelian and g has order 2, show that the set $K = H \cup gH$ is a subgroup of G. Is your proof valid if we drop the assumption that G is abelian and let $K = Z(G) \cup g\,Z(G)$?

Sol^n Since e is in $H \cup gH$, K is not empty.

Suppose $x, y \in K = H \cup gH$

Case I :- $x, y \in H$ then $xy^{-1} \in H$ as H is subgroup $\Rightarrow xy^{-1} \in K$.

Case II :- $x \in H$ and $y \in gH$

Then $xy^{-1} = h_1 (gh_2)^{-1}$ where $h_1 \in H$ & $h_2 \in H$

$\Rightarrow xy^{-1} = h_1 h_2^{-1} g^{-1}$ $[\because (ab)^{-1} = b^{-1}a^{-1}]$

$\Rightarrow xy^{-1} = (h_1 h_2^{-1}) g^{-1}$ [Since $o(g) = 2$ $\Rightarrow g^2 = e$ $\Rightarrow g = g^{-1}$]

$\Rightarrow xy^{-1} = (h_1 h_2^{-1}) g$

$\Rightarrow xy^{-1} = g (h_1 h_2^{-1})$ $[\because G$ is abelian$]$

$\Rightarrow xy^{-1} \in gH \Rightarrow xy^{-1} \in K$.

Similar case for $x \in gH$ and $y \in H$.

Case III :- $x \in gH$, $y \in gH$

Then $xy^{-1} = gh_1(gh_2)^{-1} = gh_1h_2^{-1}g^{-1}$

$\Rightarrow xy^{-1} = g(h_1h_2^{-1})g^{-1}$

$\Rightarrow xy^{-1} = gg^{-1}(h_1h_2^{-1})$ [G is abelian]

$\Rightarrow xy^{-1} = (h_1h_2^{-1})$

$\Rightarrow xy^{-1} \in H \Rightarrow xy^{-1} \in K$

Hence K is a subgroup.

The argument is valid for

$K = Z(G) \cup gZ(G)$.

3.(b)(i) Examine the convergence of the integral

$$\int_1^2 \frac{dx}{(1+x)\sqrt{2-x}}$$

Sol^n: Here $f(x) = \dfrac{1}{(1+x)\sqrt{2-x}}$

2 is the only point of infinite discontinuity of f on $[1,2]$.

Take $g(x) = \dfrac{1}{\sqrt{2-x}}$, then

$$\lim_{x\to 2-} \frac{f(x)}{g(x)} = \lim_{x\to 2-} \frac{1}{1+x} = \frac{1}{3}$$

which is non-zero and finite.

$\therefore$ By Comparison test,

$\int_1^2 f(x)\,dx$ and $\int_1^2 g(x)\,dx$ converge (or) diverge together.

But $\int_1^2 g(x)\,dx = \int_1^2 \dfrac{dx}{\sqrt{2-x}}$ converges $(\because n = \frac{1}{2} < 1)$

$\therefore \int_1^2 f(x)\,dx = \int_1^2 \dfrac{dx}{(1+x)\sqrt{2-x}}$ is convergent.

3.(b)(iii) Prove that $\prod_{n=1}^{\infty}\left(1-\frac{1}{n^{4/3}}\right)e^{\frac{1}{n^{2/3}}}$ is absolutely convergent

Soln: Here $1+a_n = \left(1-\frac{1}{n^{2/3}}\right)e^{\frac{1}{n^{2/3}}}$

$$= \left(1-\frac{1}{n^{2/3}}\right)\left(1+\frac{1}{n^{2/3}}+\frac{1}{2!\,n^{4/3}}+\frac{1}{3!\,n^{6/3}}+\cdots\right)$$

$$= 1-\frac{1}{n^{4/3}}+\frac{1}{2n^{4/3}}+\frac{1}{6n^{6/3}}-\frac{1}{2n^{6/3}}+\cdots$$

$$\Rightarrow a_n = -\frac{1}{2n^{4/3}}-\frac{1}{3n^{6/3}}-\cdots$$

$$= \frac{1}{n^{4/3}}\left(-\frac{1}{2}-\frac{1}{3n^{2/3}}+\cdots\right)$$

$$\Rightarrow |a_n| = \frac{1}{n^{4/3}}\left|-\frac{1}{2}-\frac{1}{3n^{2/3}}\cdots\right|$$

Comparing $\sum|a_n|$ with $\sum\frac{1}{n^{4/3}}$

we have $\lim_{n\to\infty}\frac{|a_n|}{\frac{1}{n^{4/3}}}=\frac{1}{2}$, a finite quantity.

But $\sum\frac{1}{n^{4/3}}$ is convergent.

$\therefore \sum_n |a_n|$ is convergent

$\Rightarrow \sum a_n$ is absolutely convergent

$\Rightarrow \prod_{n=1}^{\infty}(1+a_n) = \prod_{n=1}^{\infty}\left(1-\frac{1}{n^{2/3}}\right)e^{\frac{1}{n^{2/3}}}$ is

$\therefore$ absolutely convergent.

(3)(c) Use simplex method to solve the following Max $z = 5x_1 + 2x_2$

s.t

$6x_1 + x_2 \geq 6$

$4x_1 + 3x_2 \geq 12$

$x_1 + 2x_2 \geq 4$

$x_1, x_2 \geq 0$

Sol^n After Introducing surplus variable and artificial variable x_3, x_4, x_5 and A_1, A_2, A_3 we get :-

Max $Z = 5x_1 + 2x_2 - MA_1 - MA_2 - MA_3$

S.t.

$6x_1 + x_2 - x_3 + A_1 = 6$

$4x_1 + 3x_2 - x_4 + A_2 = 12$

$x_1 + 2x_2 - x_5 + A_3 = 4$

$x_i, A_j \geq 0$

c_B	Basis (c_j →)	x_1 (5)	x_2 (2)	x_3 (0)	x_4 (0)	x_5 (0)	A_1 (−M)	A_2 (−M)	A_3 (−M)	b	θ
−M	A_1	(6)	1	−1	0	0	1	0	0	6	6/6=1 →
−M	A_2	4	3	0	−1	0	0	1	0	12	12/4=3
−M	A_3	1	2	0	0	−1	0	0	1	4	4/1=4
	$z_j = \Sigma a_{ij} c_B$	−11M	−6M	M	M	M	−M	−M	M		
	$c_j = c_j - z_j$	5+11M	2+6M	−M	−M	−M	0	0	0		

x_1 is incoming variable, A_1 is outgoing.
$A_1 \to$ column can be omitted

	c_j	5	2	0	0	0	$-M$	$-M$		
C_B	Basis	x_1	x_2	x_3	x_4	x_5	A_2	A_3	b	θ
5	x_1	1	1/6	$-1/6$	0	0	0	0	1	6
$-M$	A_2	0	7/3	2/3	-1	0	1	0	8	$8\times3/7$
$-M$	A_3	0	$\boxed{11/6}$	1/6	0	-1	0	1	3	$18/11 \to$
$Z_j = \sum a_{ij} c_B$		5	$\frac{5}{6} - \frac{25}{6}M$	$-\frac{5}{6} - \frac{5}{6}M$	M	M	$-M$	$-M$		
$C_j = c_j - z_j$		0	$\frac{25}{6}M + \frac{7}{6}$ $\uparrow$	$\frac{5}{6}M + 5/6$	$-M$	$-M$	0	0		

A_3 is outgoing variable x_2 is incoming.
~~Dropping~~ A_3 column can be omitted.

	c_j	5	2	0	0	0	$-M$		
	Basis	x_1	x_2	x_3	x_4	x_5	A_2	b	θ
5	x_1	1	0	$-2/11$	0	1/11	0	8/11	(-4)
$-M$	A_2	0	0	5/11	-1	14/11	1	46/11	$46/5 \to$
2	x_2	0	1	1/11	0	$-6/11$	0	18/11	18/1
$z_j = \sum a_{ij} c_B$		5	2	$-\frac{5M}{11} - \frac{8}{11}$	M	$-\frac{7}{11}$	$-M$		
$C_j = c_j - z_j$		0	0	$\frac{5M}{11} + \frac{8}{11}$ $\uparrow$	$-M$	7/11	0		

x_3 is incoming variable, A_2 is outgoing variable.

A_2 column can be dropped.

	C_j	5	2	0	0	0	
	Basis	x_1	x_2	x_3	x_4	x_5	b
5	x_1	1	0	0	$-2/5$	$+3/5$	$12/5$
0	x_3	0	0	1	$-11/5$	$14/5$	$46/5$
2	x_2	0	1	0	$\boxed{1/5}$	$-4/5$	$4/5$ $\rightarrow$
$Z_j = \sum a_{ij} C_B$		5	2	0	$-8/5$	$7/5$	
$C_j = C_j - Z_j$		0	0	0	$8/5$ $\uparrow$	$-7/5$	

x_2 is outgoing variable and x_4 is incoming variable.

	C_j	5	2	0	0	0		
	Basis	x_1	x_2	x_3	x_4	x_5	b	θ
5	x_1	1	2	0	0	-1	4	-4
0	x_3	0	11	1	0	-6	18	$-18/6$
0	x_4	0	5	0	1	-4	4	-1
$Z_j = \sum a_{ij} C_B$		5	10	0	0	-5		
$C_j = C_j - Z_j$		0	-8	0	0	5 $\uparrow$		

Since all θs are negative and C_j for x_5 is positive, So given problem have unbounded solution.

$$I_3 = \int_{CA} z^2 dz = \frac{1}{3}\left[z^3\right]_{z=1+2i}^{z=0}$$

$$= \frac{1}{3}\left[0 - (1+2i)^3\right]$$

$$= -\frac{(+1 - 8i + 6i - 12)}{3} = -\frac{(-11-2i)}{3}$$

$$= \frac{11+2i}{3}$$

Adding $I_1 + I_2 + I_3$ we get

$$\int_C f(z)\, dz = \frac{1}{3} + \frac{1}{3}(-10-2i) + \frac{11+2i}{3}$$

$$= 2/3$$

$$\Rightarrow \boxed{\int_C f(z)\, dz = 2/3}$$

4(b) → Let $f(x)$, $(x \in (-\pi, \pi))$ be defined by $f(x) = \sin|x|$. Is continuous on $(-\pi, \pi)$? If it is continuous, then is it differentiable on $(-\pi, \pi)$?

Sol^n: Given that $f(x) = \sin|x|$, $x \in (-\pi, \pi)$

$$\text{i.e. } f(x) = \begin{cases} \sin(-x) & \text{if } x \in (-\pi, 0) \\ \sin x & \text{if } x \in (0, \pi) \end{cases}$$

Clearly $f(x)$ is continuous and differentiable over each subinterval. The only doubtful point is the breaking point $x = 0$.

At $x = 0$, $f(x) = 0$

Now LHL: $\underset{x \to 0-}{\text{Lt}} f(x) = \sin(-x) = 0$

RHL: $\underset{x \to 0+}{\text{Lt}} f(x) = \sin x = 0$

$\therefore \underset{x \to 0-}{\text{Lt}} f(x) = 0 = \underset{x \to 0+}{\text{Lt}} f(x)$

$\therefore f$ is continuous at $x = 0$

Now RHD: $Rf'(0) = \underset{x \to 0+}{\text{Lt}} \dfrac{f(x) - f(0)}{x - 0}$

$= \underset{x \to 0+}{\text{Lt}} \dfrac{\sin x - 0}{x}$

$= \underset{x \to 0+}{\text{Lt}} \dfrac{\sin x}{x} = 1$

LHD: $Lf'(0) = \underset{x \to 0-}{\text{Lt}} \dfrac{f(x) - f(0)}{x - 0}$

$= \underset{x \to 0-}{\text{Lt}} \dfrac{\sin(-x) - 0}{x - 0}$

$$= \lim_{x\to 0^-} \frac{-\sin x}{x} = -1$$

$\therefore\ Lf'(0) \neq Rf'(0)$

$\therefore\ f(x)$ is not differentiable at $x=0$

Hence f is continuous on $(-\pi, \pi)$

Also f is differentiable on $(-\pi, \pi)$ except at $x=0$.

4(c)(i) → Use Cauchy's theorem / Cauchy integral formula evaluate (i) $\int_C \frac{z-1}{(z+1)^2(z-2)} dz$ where $C: |z-i| = 2$

Sol^n: Let $\frac{z-1}{(z+1)^2(z-2)} = \frac{A}{z+1} + \frac{B}{(z+1)^2} + \frac{C}{z-2}$

$\therefore z-1 = A(z+1)(z-2) + B(z-2) + C(z+1)^2$

Putting $z=-1$, we have $-2 = B(-1-2)$

$\Rightarrow \boxed{B = 2/3}$

Putting $z=2$, we have $1 = C(2+1)^2$

$\Rightarrow \boxed{C = 1/9}$

Putting $z=0$, we have $-1 = A(-2) + B(-2) + C$

$\Rightarrow -1 = -2A - 4/3 + 1/9$

$\Rightarrow 2A = -2/9 \Rightarrow \boxed{A = -1/9}$

$\therefore \int_C \frac{z-1}{(z+1)^2(z-2)} dz = \frac{1}{9}\int_C \frac{1}{z+1} dz + \frac{2}{3}\int_C \frac{1}{(z+1)^2} dz + \frac{1}{9}\int_C \frac{1}{z-2} dz$

$|z-i| = 2$ is a circle with centre i and radius 2 i.e. Centre is $(0,1)$ and radius is 2

Consider $\int_C \frac{1}{z+1} dz$

Here $z=-1$ is within the circle $|z-i|=2$ and $f(z)=1$

$\therefore \int_C \frac{1}{z+1} dz = 2\pi i(1) = 2\pi i$

Consider $\int_C \frac{1}{(z+1)^2} dz = \frac{2\pi i(0)}{2!} = 0$

$\therefore f'(2) = 0$ and $f'(-1) = 0$

consider $\int_C \frac{1}{z-2}\,dz$

Here $z=2$ lies outside the circle $|z-i|=2$ and $\frac{1}{z-2}$ is analytic in the $|z-i|=2$.

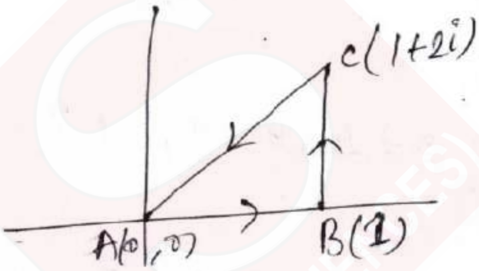
$\therefore$ By Cauchy's integral theorem,

$$\int_C \frac{1}{z-2}\,dz = 0$$

$$\therefore \int_C \frac{z-1}{(z+1)^2(z-2)}\,dz = -\frac{1}{9}(2\pi i) + 0 + 0 = \frac{-2\pi i}{9}$$

4 (c)(ii) Evaluate the line integral $\int_C f(z)\,dz$ where $f(z) = z^2$, C is the boundary of the triangle with the vertices $A(0,0)$, $B(1,0)$, $C(1,2)$ in that order.

Sol^n



$\int_C f(z)\,dz$

$= \int_{AB+BC+CA} f(z)\,dz$

$= \int_{AB+BC+CA} z^2\,dz = \int_{AB} z^2\,dz + \int_{BC} z^2\,dz + \int_{CA} z^2\,dz$
$\qquad\qquad (I_1) \qquad\quad (I_2) \qquad\quad (I_3)$

$I_1 = \int_{AB} z^2\,dz = \frac{1}{3}\left[z^3\right]_{z=0}^{z=1} = \frac{1}{3}\left[(1)^3 - 0\right] = \frac{1}{3}$

$I_2 = \int_{BC} z^2\,dz = \frac{1}{3}\left[z^3\right]_{z=i}^{z=1+2i}$

$= \frac{1}{3}\left[(1+2i)^3 - (1)^3\right] = \frac{1}{3}\left[1 - 8i + 6i - 12 + 1\right]$

$= \frac{1}{3}(-10 - 2i)$

4.(d) An automobile dealer wishes to put four repairmen to four different jobs. The repairmen have somewhat different kinds of skills and they exhibit different levels of efficiency from one job to another. The dealer has estimated the number of manhours that would be required for each job-man combination this is given in the matrix form in adjacent table:

Find the optimum assignment that will result in minimum manhours needed.

Man \ Job	A	B	C	D
1	5	3	2	8
2	7	9	2	6
3	6	4	5	7
4	5	7	7	8

Step(1):
(i) After subtracting the minimum of each row from all elements of that row the reduced matrix is given by

3	1	0	6
5	7	0	4
2	0	1	3
0	2	2	3

(ii) Subtracting the minimum elements of column from elements of that column, we get

3	1	0	3
5	7	0	1
2	0	1	0
0	2	2	0

Step(2): Cover all the zeros by minimum no of horizontal and vertical lines. A symmetric approach for this is to look for a row or column containing maximum no of zeros

we can cover all the zeros by 3 lines only
so, $r = 3 < 4 = n$
so, go to step (3)

3	1	0	3
5	2	0	1
2	0	1	0
0	2	1	0

Step(3):

1 is the least uncovered element. Subtract 1 from all the uncovered elements. Add 1 to elements at intersection of the covering lines namely 1 at position (3,3) and 1 at (4,3). Leave other uncovered elements unchanged and the reduced matrix so obtained is

2	0	0	2
4	6	0	0
2	0	2	0
0	2	3	0

Again, cover the zeros by minimum no. of horizontal and vertical lines to cover all the zeros we required exactly 4 lines
As $r = 4 = n$, optimal assignment can be made at this stage

2	0	✗	2
4	6	0	✗
2	✗	2	0
0	2	3	✗

It may be noted that an assignment problem can have more than one optimum solution.

optimum solution-I

2	✗	0	2
4	6	✗	0
2	0	2	✗
0	2	3	✗

optimum solution-I

Man	Job	Man hours
1	B	3
2	C	2
3	D	7
4	A	5

optimum solution II

Man	Job	Man hours
1	C	2
2	D	6
3	B	4
4	A	5

5.(a) Find the integral surface of the linear PDE $xp+yq=z$ which contains the circle defined by

$$x^2+y^2+z^2=4,\quad x+y+z=2.$$

Solⁿ: The integral surface of the given PDE is generated by the integral curves of the auxillary equation

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z} \quad \text{--- (1)}$$

Integration of the first two members of eq.(1) gives

$$\ln x=\ln y+\ln C$$

or $$\frac{x}{y}=C_1 \quad \text{--- (2)}$$

Similarly, integration of the last two members of eq.(1) yields

$$\frac{y}{z}=C_2 \quad \text{--- (3)}$$

Hence, the integral surface of the given PDE is

$$F\left(\frac{x}{y},\frac{y}{z}\right)=0 \quad \text{--- (4)}$$

If this integral surface also contains the given circle, then we have to find a relation between x/y and y/z.

The equation of the circle is

$$x^2+y^2+z^2=4 \quad \text{--- (5)}$$

$$x+y+z=2 \quad \text{--- (6)}$$

From eqs.(2) and (3), we have

$$y = x/c_1\ ,\ z = y/c_2 = x/c_1c_2$$

Substituting these values of y and z in eqs. ⑤ & ⑥, we find

$$x^2 + \frac{x^2}{c_1^2} + \frac{x^2}{c_1^2c_2^2} = 4$$

$$\text{or} \quad x^2\left(1 + \frac{1}{c_1^2} + \frac{1}{c_1^2c_2^2}\right) = 4 \quad \text{——— (7)}$$

$$\text{and} \quad x + \frac{x}{c_1} + \frac{x}{c_1c_2} = 2, \quad \text{or } x\left(1 + \frac{1}{c_1} + \frac{1}{c_1c_2}\right) = 2 \quad \text{——— (8)}$$

From eqs. ⑦ and ⑧ we observe

$$1 + \frac{1}{c_1^2} + \frac{1}{c_1^2c_2^2} = \left(1 + \frac{1}{c_1} + \frac{1}{c_1c_2}\right)^2,$$

Which on simplification gives us

$$\frac{2}{c_1} + \frac{2}{c_1c_2} + \frac{2}{c_1^2c_2} = 0$$

That is, $c_1c_2 + c_1 + 1 = 0$ ——— (9)

Now, replacing c_1 by x/y and c_2 by y/z, we get the required integral surface as

$$\frac{x}{y}\frac{y}{z} + \frac{x}{y} + 1 = 0,$$

$$\text{or} \quad \frac{x}{z} + \frac{x}{y} + 1 = 0$$

$$\text{or} \quad xy + xz + yz = 0.$$

5.(b) Show that the equation $f(x) = \cos\frac{\pi(x+1)}{8} + 0.148x - 0.9062 = 0$ has one root in the interval $(-1,0)$ and one in $(0,1)$. Calculate the negative root correct to four decimal places using Newton-Raphson method.

Sol: we have $f(x) = \cos\left(\frac{\pi(x+1)}{8}\right) + 0.148x - 0.9062$

$$f'(x) = -\frac{\pi}{8}\sin\left(\frac{\pi(x+1)}{8}\right) + 0.148.$$

Clearly $f(x)$ and $f'(x)$ are continuous everywhere.

we have

$f(-1) = -0.0542$

$f(0) = 0.01768$

$f(1) = -0.05109.$

Since $f(-1)\,f(0) < 0.$

$\Rightarrow$ The root lies between $(-1,0)$.

Similarly $f(0)\,f(1) < 0.$

$\Rightarrow$ The root lies between $(0,1)$.

To calculate the negative root:

Let $x_0 = -0.5$ be the initial approximation to the root.

The Newton's iteration formula is $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$, $n = 0,1,2\ldots$ —— (1)

putting $n=0$ in (1)

The first approximation is

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = -0.5 - \frac{f(-0.5)}{f'(-0.5)}$$

$$= -0.5 - 0.0138$$

$$= -0.5138$$

$$x_2 = -0.5138 - \frac{f(-0.5138)}{f'(-0.5138)}$$

$$= -0.5046.$$

$$x_3 = -0.5022.$$

$\therefore x = -0.5022$ which is the required root.

5.(c) Give a Boolean expression for the following statements.

(i) Y is a 1 only if A is a 1 and B is a 1 or if A is a 0 and B is a 0.

(ii) Y is a 1 only if A, B and C are all 1's or if only one of the variables is a 0.

Sol'n: (i) Truth table for given conditions.

A	B	Y
0	0	1
0	1	0
1	0	0
1	1	1

$\boxed{Y = A'B' + AB}$

(ii)

A	B	C	Y
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

$$Y = A'BC + AB'C + ABC' + ABC$$

Simplify the expression

$$= A'BC + ABC + AB'C + ABC + ABC' + ABC$$

$$= BC(A'+A) + AC(B'+B) + A(C'+C)$$

$$= BC + AC + AB$$

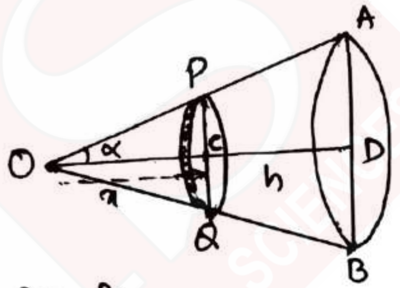
$$\therefore \boxed{Y = BC + AC + AB}$$

5.(d) Find the M.I of a right solid cone of mass M, height h and radius of whose base is a, about its axis

Sol'n: Let O be the vertex of the right solid cone of mass M, height h and radius of whose base is a, If α is the semi-vertical angle and ρ be density of the cone, then

$$M = \frac{1}{3}\pi\rho h^3 \tan^2\alpha \quad \text{——} \quad (1)$$

Consider an elementary disc PQ of thickness δx, Parallel to the base AB and at a distance x from the vertex O.



$\therefore$ Mass of the disc,

$$\delta m = \rho\pi x^2 \tan^2\alpha\, \delta x$$

M.I of this elementary disc about axis OD.

$$= \frac{1}{2}\delta m\, CP^2 = \frac{1}{2}(\rho\pi x^2 \tan^2\alpha\, \delta x)x^2\tan^2\alpha$$

$$= \frac{1}{2}\rho\pi x^4 \tan^4\alpha\, \delta x$$

$\therefore$ M.I of the cone about axis OD.

$$= \int_0^h \frac{1}{2}\rho\pi x^4 \tan^4\alpha\, dx$$

$$= \rho\frac{\pi}{10} h^5 \tan^4\alpha$$

$$= \frac{3}{10}Mh^2\tan^2\alpha \quad \text{from } (1)$$

$$= \frac{3}{10}Ma^2 \quad (\because \tan\alpha = a/h).$$

5.(e) Show that the velocity potential $\phi = \frac{a}{2} x(x^2+y^2-2z^2)$ satisfies the Laplace equation. Also determine the streamlines.

Sol^n: we know that the velocity q of the fluid is given by

$$q = -\nabla\phi = -\left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\left\{\frac{a}{2}(x^2+y^2-2z^2)\right\}$$

$$\Rightarrow q = -(a/2)\times(2xi+2yj+2zk) \quad \text{—— (1)}$$

But $q = ui + vj + wk$ —— (2)

Comparing (1) and (2) $u = -ax$, $v = -ay$, $w = 2az$

The equations of streamlines are given by $\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$

$$\frac{dx}{-ax} = \frac{dy}{-ay} = \frac{dz}{2az} \Rightarrow \frac{2dx}{x} = \frac{2dy}{y} = \frac{dz}{-z} \quad \text{—— (3)}$$

Taking the first two fractions of (3), $\frac{1}{x}dx = \frac{1}{y}dy$

Integrating, $\log x = \log y + \log c_1 \Rightarrow x = c_1 y$ —— (4)

Taking the last two fractions of (3),

$$(2/y)dy + (1/z)dz = 0.$$

Integrating, $2\log y + \log z = \log c_2$

$$\Rightarrow y^2 z = c_2 \quad \text{—— (5)}$$

(4) and (5) together give the equations of streamlines, c_1 and c_2 being arbitrary constants of integration.

Now, given that $\phi = a/2 \times (x^2+y^2-2z^2)$ —— (6)

from (6), $\frac{\partial\phi}{\partial x} = ax$, $\frac{\partial\phi}{\partial y} = ay$ and $\frac{\partial\phi}{\partial z} = -2az$

$$\Rightarrow \frac{\partial^2\phi}{\partial x^2} = a, \quad \frac{\partial^2\phi}{\partial y^2} = a \quad \text{and} \quad \frac{\partial^2\phi}{\partial z^2} = -2a$$

$$\therefore \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = a + a - 2a \Rightarrow \nabla^2\phi = 0.$$

showing that ϕ satisfies the Laplace equation.

6(a)(i) Form a PDE by eliminating arbitrary function f, g of the following

$$z = f(x-z) + g(x+y)$$

Solⁿ: Given $z = f(x-z) + g(x+y)$

Partially differentiating z w.r.t. x and y respectively –

$$\frac{\partial z}{\partial x} = f'(x-z)\left[1 - \frac{\partial z}{\partial x}\right] + g'(x+y)[1+0] \quad \text{--- (1)}$$

$$\frac{\partial z}{\partial y} = f'(x-z)\left[0 - \frac{\partial z}{\partial y}\right] + g'(x+y)[0+1] \quad \text{--- (2)}$$

From equation (1) and (2), we get

$$f'(x-z) = \frac{\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}}{1 - \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y}} \quad \text{--- (3)}$$

Also

$$\frac{\partial^2 z}{\partial x^2} = f''(x-z)\left[1 - \frac{\partial z}{\partial x}\right]^2 + f'(x-z)\left(-\frac{\partial^2 z}{\partial x^2}\right) + g''(x+y) \quad \text{--- (4)}$$

$$\frac{\partial^2 z}{\partial y^2} = f''(x-z)\left(-\frac{\partial z}{\partial y}\right)^2 + f'(x-z)\left(\frac{\partial^2 z}{\partial y^2}\right) + g''(x+y) \quad \text{--- (5)}$$

$$\frac{\partial^2 z}{\partial x \partial y} = f''(x-z)\left(1 - \frac{\partial z}{\partial x}\right)\left(-\frac{\partial z}{\partial y}\right) + f'(x-z)\left(-\frac{\partial^2 z}{\partial x \partial y}\right) + g''(x+y) \quad \text{--- (6)}$$

Subtracting eqⁿ ④ and eqⁿ ⑤ :-

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = f''(x-z)\left[(1-p)^2 - q^2\right] + f'\left(-\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2}\right) \quad -(7)$$

where $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$

from eqⁿ ④ and ⑥ :-

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = f''(x-z)\left[(1-p)^2 + (1-p)q\right] + f'\left(-\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y}\right) \quad -(8)$$

Putting value of f' in ⑦ and ⑧ :-

$$\left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2}\right)\left(\frac{1}{1-p+q}\right) = f''(x-z)\left[(1-p)^2 - q^2\right] \quad -(9)$$

$$\left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y}\right)\left(\frac{1}{1-p+q}\right) = f''(x-z)\left[(1-p)^2 + (1-p)q\right] \quad -(10)$$

Dividing ⑨ and ⑩ :-

$$\frac{\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial y^2}}{\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial x \partial y}} = \frac{(1-p)^2 - q^2}{(1-p)^2 + (1-p)q} = \frac{\cancel{(1-p+q)}(1-p-q)}{(1-p)\cancel{(1-p+q)}}$$

$$\Rightarrow \boxed{\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = \left(\frac{1-p-q}{1-p}\right)\left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y}\right)}$$

which is required PDE.

6.a(ii) Solve by charpit's method the partial diff. equation.

$$p^2x(x-1)+2pqxy+q^2y(y-1)-2pxz-2qyz+z^2=0.$$

Solⁿ: Let $f=p^2x(x-1)+2pqxy+q^2y(y-1)-2pxz-2qyz+z^2=0.$ —— (1)

Then the usual charpit's auxiliary equations are

$$\frac{dp}{f_x+pf_z}=\frac{dq}{f_y+qf_z}=\frac{dz}{-pf_p-qf_q}=\frac{dx}{-f_p}=\frac{dy}{-f_q} \quad \text{—— (2)}$$

from (1), $f_x=p^2(2x-1)+2pqy-2pz$, $f_y=2pqx+q^2(2y-1)-2qz$,

$f_z=-2px-2qy+2z$, $f_p=2px(x-1)+2qxy-2xz$;

$f_q=2pxy+2qy(y-1)-2yz$

and so, $f_x+pf_z=-p^2$, $f_y+qf_z=-q^2$. Then (2) becomes.

$$\frac{dp}{-p^2}=\frac{dq}{-q^2}=\frac{dz}{-p\{2px(x-1)+2qxy-2xz\}-q\{2pxy+2qy(y-1)-2yz\}}$$

$$=\frac{dx}{-(2px^2-2px+2qxy-2xz)}=\frac{dy}{-(2pxy+2qy^2-2qy-2yz)} \quad \text{—— (3)}$$

Each fraction of (3) =

$$\frac{(1/p)dp}{-p}=\frac{(1/q)dq}{-q}=\frac{(1/p)dp-(1/q)dq}{-p+q} \quad \text{—— (4)}$$

Also, each fraction of (3) $=\dfrac{(1/x)dx-(1/y)dy}{-2px+2p-2qz+2z+2px+2qy-2q-2z}$ —— (5)

$\therefore$ (4) and (5)

$$\Rightarrow \frac{(1/p)dp-(1/q)dq}{-(p-q)}=\frac{(1/x)dx-(1/y)dy}{2(p-q)}$$

or $(1/2)\{(1/x)dx-(1/y)dy\}=(1/q)dq-(1/p)dp$

Integrating, $(1/2)\{\log x - \log y\} = \log q - \log p + \log a$

or $(x/y)^{1/2} = aq/p$

or $p = (ay^{1/2}q)/x^{1/2}$, ——— (6)

(a being arbitrary constant.)

Re-writing (1), $(px+qy-z)^2 = p^2x + q^2y$

or $px+qy-z = \pm(p^2x+qy^2)^{1/2}$ ——— (7)

Taking positive sign in (7),

$$px+qy-z = (p^2x+q^2y)^{1/2} \quad \text{——— (8)}$$

[The case of negative sign in (7) can be discussed similarly.]

$$aqy^{1/2}x^{1/2}+qy-z = (a^2q^2y+q^2y)^{1/2}$$

or $q\{y+a(xy)^{1/2}-(1+a^2)^{1/2}y^{1/2}\} = z$

So that $q = z/y^{1/2}\{y^{1/2}+ax^{1/2}-(1+a^2)^{1/2}\}$ ——— (9)

Then (6) gives $p = az/x^{1/2}\{y^{1/2}+ax^{1/2}-(1+a^2)^{1/2}\}$ ——— (10)

putting these values of p and q in $dz = pdx + qdy$, we get,

$$dz = \frac{az\,dx}{x^{1/2}\{y^{1/2}+ax^{1/2}-(1+a^2)^{1/2}\}} + \frac{z\,dy}{y^{1/2}\{y^{1/2}+ax^{1/2}-(1+a^2)^{1/2}\}}$$

or $$\frac{dz}{z} = \frac{ay^{1/2}dx + x^{1/2}dy}{(xy)^{1/2}\{y^{1/2}+ax^{1/2}-(1+a^2)^{1/2}\}}$$

Integrating, $\log z = 2\log\{y^{1/2}+ax^{1/2}-(1+a^2)^{1/2}\}+\log b$.

or $z = b\{y^{1/2}+ax^{1/2}-(1+a^2)^{1/2}\}^2$,

'b' being an arbitrary constant.

6.b(i) Solve the equations

$$x_1+x_2+x_3=6$$
$$3x_1+(3+\epsilon)x_2+4x_3=20$$
$$2x_1+x_2+3x_3=13$$

using the Gauss elimination method, where 'ϵ' is small such that $1\pm\epsilon^2\approx 1$.

Soln: Given that
$$x_1+x_2+x_3=6$$
$$3x_1+(3+\epsilon)x_2+4x_3=20$$
$$2x_1+x_2+3x_3=1$$

Eliminating x_1 from last two equations, we get
$$x_1+x_2+x_3=6$$
$$\epsilon x_2+x_3=2$$
$$-x_2+x_3=1$$

Here, the pivot in the second is ϵ which is a very small number.

If we do not use pivoting, then we get.
$$x_1+x_2+x_3=6$$
$$\epsilon x_2+x_3=2$$
$$\left(1+\frac{1}{\epsilon}\right)x_3=1+\frac{2}{\epsilon}.$$

The solution is $x_3=\dfrac{1+(2/\epsilon)}{1+(1/\epsilon)}$, $x_2=\dfrac{1}{\epsilon}\left[2-\dfrac{1+(2/\epsilon)}{1+(1/\epsilon)}\right]$

and $x_1=6-x_2-x_3$.

However, this solution may be very inaccurate if ϵ is of the order of the round-off error. This situation can be avoided if pivoting is done at the second step. In this case we have
$$x_1+x_2+x_3=6$$
$$-x_2+x_3=1$$
$$(1+\epsilon)x_3=2+\epsilon.$$

The solution is
$$x_3=\frac{2+\epsilon}{1+\epsilon},\quad x_2=-1+\frac{2+\epsilon}{1+\epsilon}$$
and $x_1=6-x_2-x_3$.

6.b(ii) Convert:

(a) 46655 given to be in the decimal system into one in base 6.

Solve:-

6	46655	
	7775	5
	1295	5
	215	5
	35	5
	5	5

$$\boxed{\therefore \quad (46655)_{10} \longleftrightarrow (555555)_6}$$

(b) $(11110.01)_2$ into a number in the decimal system.

Solve:- $(11110.01)_2 \longleftrightarrow (30.25)_{10}$

$$= (1\times 2^4 + 1\times 2^3 + 1\times 2^2 + 1\times 2^1 + 0\times 2^0 \,.\, 0\times 2^{-1} + 1\times 2^{-2})_{10}$$

$$= \left[16+8+4+2+0 \,.\, 0+\tfrac{1}{4}\right]_{10}$$

$$= [30.25]_{10}$$

$$\boxed{\therefore (11110.01)_2 \longleftrightarrow (30.25)_{10}}$$

6.(c) → Write Hamilton's equations for a particle of mass 'm' moving in a plane under a force which is some function of distance from the origin.

Solution:-

Let, $P(r,\theta)$ be the co-ordinates of a particle of mass 'm' at time 't', referred to the pole at 'O' and OX as initial line.

The components of velocity at P along OP and $\perp^r$ to it are $\dot{r}$ and $r\dot{\theta}$ respectively.

$\therefore$ (vel)^2 of m at P; $\boxed{V^2 = \dot{r}^2 + (r\dot{\theta})^2}$

$\therefore$ Kinetic energy $\Rightarrow$ $\boxed{T = \frac{1}{2}mV^2 - \frac{1}{2}m(\dot{r}^2 + (r\dot{\theta})^2)}$

Since, the force at 'P' is some function of r, therefore the potential 'V' is function of 'r' alone and will be independent of 't'.

i.e $\boxed{V = V(r)}$

Thus, $L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$.

Here r and θ are the generalised coordinates.

$$\therefore\ P_r = \frac{\partial L}{\partial r} = m\dot{r} \quad \text{and}$$

$$P_\theta = \frac{\partial L}{\partial \theta} = mr^2\dot{\theta} \quad \text{——— (1)}$$

Since, L does not contain t explicitly.

$$\therefore\ H = T + V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r)$$

$$H = \frac{1}{2}m\left[\left(\frac{P_r}{m}\right)^2 + r^2\left(\frac{P_\theta}{mr^2}\right)^2\right] + V(r) \quad \text{— from (1)}$$

$$\therefore\ \boxed{H = \frac{1}{2m}\left[P_r^2 + \frac{P_\theta^2}{r^2}\right] + V(r)}$$

Hence, four Hamilton's equations are.

$$\boxed{\dot{P}_r = -\frac{\partial H}{\partial r} = \frac{P_\theta^2}{mr^3} - \frac{\partial V}{\partial r}} \quad \text{——— } H_1$$

$$\boxed{\dot{r} = \frac{\partial H}{\partial P_r} = \frac{P_r}{m}} \quad \text{——— } H_2$$

$$\boxed{\dot{P}_\theta = \frac{\partial H}{\partial \theta} = 0} \quad \text{——— } H_3$$

$$\boxed{\dot{\theta} = \frac{\partial H}{\partial P_\theta} = \frac{P_\theta}{mr^2}} \quad \text{——— } H_4.$$

7.a(i) Solve $(D^3+D^2D'-DD'^2-D'^3)z = e^y\cos 2x$.

Sol^n: Here $D^3+D^2D'-DD'^2-D'^3 = D^2(D+D') - D'^2(D+D')$

$= (D+D')(D^2-D'^2) = (D+D')(D+D')(D-D')$

$= (D+D')^2(D-D')$

So the given equation is

$$(D-D')(D+D')^2 z = e^y \cos 2x$$

$\therefore$ C.F. $= \phi_1(y+x) + \phi_2(y-x) + x\phi_3(y-x)$.

$$P.I = \frac{1}{(D-D')(D+D')}\,\frac{1}{D+D'}\,e^y\cos 2x$$

$$= \frac{1}{(D-D')(D+D')}\int e^{a+x}\cos 2x\,dx, \text{ where } y-x=a$$

$$= \frac{1}{(D-D')(D+D')}e^a\int e^x\cos 2x\,dx$$

$$= \frac{1}{(D-D')(D+D')}e^{y-x}\,\frac{1}{1^2+2^2}e^x(\cos 2x+2\sin 2x)^*$$

$$= \frac{1}{5}\,\frac{1}{(D-D')(D+D')}e^y(\cos 2x+2\sin 2x)$$

$$= \frac{1}{5}\,\frac{1}{D-D'}\int e^{x+a}(\cos 2x+2\sin 2x)\,dx, \text{ where } y-x=a$$

$$= \frac{1}{5}\,\frac{1}{D-D'}e^a\left\{\int e^x\cos 2x\,dx + 2\int e^x\sin 2x\,dx\right\}$$

$$= \frac{1}{5}\,\frac{1}{D-D'}e^{y-x}\left\{\frac{e^x}{1^2+2^2}(\cos 2x+2\sin 2x) + \frac{2e^x}{1^2+2^2}(\sin 2x-2\cos 2x)\right\}^*$$

$$= \frac{1}{25}\,\frac{1}{D-D'}e^y(4\sin 2x-3\cos 2x)$$

$$= \frac{1}{25}\int e^{b-x}(4\sin 2x-3\cos 2x)\,dx, \text{ where } b=y+x.$$

$$= \frac{1}{25} e^{b} \left\{ 4\int e^{-x} \sin 2x\, dx - 3\int e^{-x} \cos 2x\, dx \right\}$$

* we shall use the following results directly

$$\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b\cos bx),$$

$$\int e^{ax} \cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx)$$

$$= \frac{1}{25} e^{y+x} \left\{ \frac{4e^{-x}}{1^2+2^2}(-\sin 2x - 2\cos 2x) - \frac{3e^{-x}}{1^2+2^2}(-\cos 2x + 2\sin 2x) \right\}$$

$$= -(1/25) e^{y} (\cos 2x + 2\sin 2x)$$

$\therefore$ Solution is

$$z = \phi_1(y+x) + \phi_2(y-x) + x\,\phi_3(y-x) - (e^{y}/25)(\cos 2x + 2\sin 2x).$$

7.a(ii) Reduce to canonical form $\partial^2 z/\partial x^2 + x^2(\partial^2 z/\partial y^2) = 0$.

Solⁿ: Re-writing, $r + x^2 t = 0$ ——— (1)

Comparing (1) with $Rr + Ss + Tt + f(x,y,z,p,q) = 0$,
here $R = 1$, $S = 0$, $T = x^2$ so that

$S^2 - 4RT = -4x^2 < 0$, $x \neq 0$, showing that (1) is elliptic.

The λ-quadratic $R\lambda^2 + S\lambda + T = 0$ Reduces to

$\lambda^2 + x^2 = 0$ giving $\lambda = ix, -ix$.

The corresponding characteristic equations are given by.

$dy/dx + ix = 0$ and $dy/dx - ix = 0$

Integrating,

$y + i(x^2/2) = c_1$ and $y - i(x^2/2) = c_2$.

choose $u = y + i(x^2/2) = \alpha + i\beta$, $v = y - i(x^2/2) = \alpha - i\beta$,
where $\alpha = y$ and $\beta = x^2/2$ are now two ——— (2)
new independent variables.

Now, $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial \beta}\cdot\frac{\partial \beta}{\partial x} = x\frac{\partial z}{\partial \beta}$, by (2) ——— (3)

$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial y} + \frac{\partial z}{\partial \beta}\cdot\frac{\partial \beta}{\partial y} = y\frac{\partial z}{\partial \beta}$, by (2) ——— (4)

$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial \beta}\right) = \frac{\partial z}{\partial \beta} + x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \beta}\right)$, by (3)

$= \frac{\partial z}{\partial \beta} + x\left[\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \beta}{\partial x}\right] =$

$\frac{\partial z}{\partial \beta} + x^2\frac{\partial^2 z}{\partial \beta^2}$ ——— (5)

$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left[\frac{\partial z}{\partial y}\right] = \frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right) = \frac{\partial^2 z}{\partial \alpha^2}$, by (4) ——— (6)

using (5) and (6) in (1) the required canonical form is

$$\frac{\partial z}{\partial \beta} + x^2\frac{\partial^2 z}{\partial \beta^2} + x^2\frac{\partial^2 z}{\partial \alpha^2} = 0$$

or $\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = -\frac{1}{2\beta}\frac{\partial z}{\partial \beta}$, as $\beta = \frac{x^2}{2}$.

7.(b)(i) Using 4th order RK method find the solution of $x(dy+dx) = y(dx-dy)$, $y(0)=1$ at $x=0.1$ and 0.2 by taking $h=0.1$.

Sol^n Given $x(dy+dx) = y(dx-dy)$

$$\Rightarrow xdy + xdx = ydx - ydy$$

$$\Rightarrow xdy + ydy = ydx - xdx$$

$$\Rightarrow \frac{dy}{dx} = \frac{y-x}{y+x} = f(x,y)$$

$y(0)=1 \Rightarrow x_0=0,\ y_0=1,\ h=0.1$

For $x = 0.1$

$$y(0.1) = y(0) + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

Where

$K_1 = hf(x_0, y_0) = 0.1 f(0,1) = 0.1$

$K_2 = hf(x_0 + h/2, y_0 + k_1/2) = 0.1 \times f(0.05, 1.05)$
$= 0.0909$

$K_3 = hf(x_0 + h/2, y_0 + k_2/2)$
$= (0.1) f(0.05, 1.045) = 0.0908$

$K_4 = hf(x_0 + h, y_0 + K_3) = 0.1 f(0.1, 1.09)$
$= 0.083$

$y(0.1) = 1 + 0.0910 = 1.091$

At $\underline{x=0.2}$, $x_1 = 0.1$, $y_1 = 1.091$

$k_1 = hf(x_1, y_1) = 0.1 \times f(0.1, 1.091)$
$= 0.083$

$k_2 = hf(x_1 + h/2, y_1 + k_1/2) = 0.0766$

$k_3 = hf(x_1 + h/2, y_1 + k_2/2) = 0.0765$

$k_4 = hf(x_1 + h, y_1 + k_3) = 0.0707$

$y(0.2) = 1.091 + 0.0766 = \underline{\underline{1.1676.}}$

7.b(ii) Convert $(0.231)_5$, $(104.231)_5$ and $(247)_7$ to base 10.

Sol^n: (a) $(0.231)_5 = 2\times 5^{(-1)} + 3\times 5^{-2} + 1\times 5^{-3}$

$$= \frac{2}{5} + \frac{3}{25} + \frac{1}{125}$$

$$= 0.4 + 0.12 + 0.008$$

$$= (0.528)_{10}$$

(b) $(104.231)_5 = 1\times 5^2 + 0\times 5^1 + 4\times 5^0 + 2\times 5^{-1} + 3\times 5^{-2} + 1\times 5^{-3}$

$$= 25 + 0 + 4 + 0.4 + 0.12 + 0.008$$

$$= (29.528)_{10}$$

(c) $(247)_7$

This question is wrong

Since under base 7 the digit must be lies between 0 to 6.

7.(c) Prove that the velocity potentials $\phi_1 = x^2 - y^2$ and $\phi_2 = r^{1/2}\cos(\theta/2)$ are solutions of the Laplace equation and the velocity potential $\phi_3 = (x^2-y^2) + r^{1/2}\cos(\theta/2)$ satisfies $\nabla^2\phi_3 = 0$.

Sol: The Laplace's equation in cartesian and cylindrical polar co-ordinates are given by

$$\nabla^2\phi_1 = \frac{\partial^2\phi_1}{\partial x^2} + \frac{\partial^2\phi_1}{\partial y^2} = 0$$

and $$\nabla^2\phi_2 = \frac{\partial^2\phi_2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\phi_2}{\partial\theta^2} + \frac{1}{r}\frac{\partial\phi_2}{\partial r} = 0$$

Here $\partial^2\phi_1/\partial x^2 = 2$ and $\partial^2\phi_1/\partial y^2 = -2$.

So $$\nabla^2\phi_1 = 2-2 = 0 \quad \text{——— (1)}$$

Next,

$$\frac{\partial\phi_2}{\partial r} = \frac{1}{2}r^{-1/2}\cos\frac{\theta}{2},$$

$$\frac{\partial^2\phi_2}{\partial r^2} = -\frac{1}{4}r^{-3/2}\cos\frac{\theta}{2},$$

$$\frac{\partial^2\phi_2}{\partial\theta^2} = -\frac{r^{1/2}}{4}\cos\frac{\theta}{2}.$$

$$\therefore \nabla^2\phi_2 = -\frac{1}{4r^{3/2}}\cos\frac{\theta}{2} - \frac{1}{4r^{3/2}}\cos\frac{\theta}{2} + \frac{1}{2r^{3/2}}\cos\frac{\theta}{2} = 0$$

——— (2)

① and ② show that ϕ_1 and ϕ_2 satisfy Laplaces equation.

Now,
$$\phi_3 = (x^2-y^2) + r^{1/2}\cos(\theta/2) = \phi_1 + \phi_2$$

$$\Rightarrow \nabla^2\phi_3 = \nabla^2(\phi_1+\phi_2)$$
$$= \nabla^2\phi_1 + \nabla^2\phi_2$$
$$= 0 + 0 = 0. \quad \{\text{by ① and ②}\}.$$

Hence ϕ_3 satisfies $\nabla^2\phi_1 = 0$.

8.(a) $\rightarrow$ find the steady state temperature distribution in a thin rectangular plate bounded by the lines $x=0$, $x=a$, $y=0$, $y=b$. The edges $x=0$, $x=a$, $y=0$ are kept at temperature zero while the edge $y=b$ is kept at $100°c$.

Sol: The steady state temperature $u(x,y)$ is the solution of Laplace's equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \quad \text{——} (1)$$

subject to boundary conditions

$$u(0,y)=u(a,y)=0 \quad \text{——} (2)$$

$$u(x,0)=0 \quad \text{——} (3)$$

$$u(x,b)=100 \quad \text{——} (4)$$

suppose (1) has solution of the form

$$u(x,y)=X(x)\,Y(y) \quad \text{——} (5)$$

$\therefore$ from (1), $X''Y+XY''=0$

$\Rightarrow \frac{X''}{X}=-\frac{Y''}{Y}=\mu$ (say)

$\Rightarrow X''-\mu X=0$ —— (6) and $Y''+\mu Y=0$ —— (7)

using B.C. (2); (5) gives $X(0)Y(y)=0$ & $X(a)Y(y)=0$ —— (8)

$\rightarrow X(0)=0$ & $X(a)=0$

then $Y(y)\neq 0$ since otherwise $u=0$ which cannot satisfy (4)

we now solve (6) under B.C. (8):

Three cases arise.

case(i): Let $\mu=0$. Then (6) $\equiv X''=0 \Rightarrow X=Ax+B$

using B.C. (8) $A = B = 0$

$\Rightarrow X(x) = 0$ which leads to $u = 0$.

so reject $\mu = 0$

case(2): Let $\mu = \lambda^2$ where $\lambda \neq 0$

$\therefore$ from (6) $X'' - \lambda X = 0$

$\Rightarrow X(x) = Ae^{\lambda x} + Be^{-\lambda x}$

using B.C. (8); we get $A = B = 0$

$\Rightarrow X(x) = 0$

this leads to $u = 0$

so reject $\mu = \lambda^2$.

case(3): Let $\mu = -\lambda^2$, where $\lambda \neq 0$.

then solution of (6) is given by

$X(x) = A\cos\lambda x + B\sin\lambda x$

using B.C (8): $A = 0$ and $A\cos\lambda a + B\sin\lambda a = 0$

$\Rightarrow B\sin\lambda a = 0$

$\Rightarrow \sin\lambda a = 0.$ ($\because B \neq 0$) [otherwise we will get $X(x) = 0$ and hence $u = 0$]

$\Rightarrow \lambda a = n\pi$

$\Rightarrow \lambda = \frac{n\pi}{a}$, $n = 1, 2, \ldots$ — (9)

Hence non-zero solution $X_n(x)$ of (6) are given by

$$X_n(x) = B_n \sin\left(\frac{n\pi x}{a}\right) \quad \text{— (10)}$$

and $\mu = -\lambda^2 = -\frac{n^2\pi^2}{a^2}$.

$\therefore$ from (7): $Y'' - \frac{n^2\pi^2}{a^2} Y = 0$

whose general solution is — (11)

$$Y_n(y) = C_n e^{\frac{n\pi y}{a}} + D_n e^{-\frac{n\pi y}{a}}$$

Using (3), (5) gives $0 = X(x) Y(y)$ so that $Y(0) = 0$ ($\because X(x) \neq 0$, for otherwise we will get $u = 0$)

But $Y(0) = 0 \Rightarrow Y_n(y) = 0$ —

putting $y=0$ in (11), and using $Y_n(0)=0$,
we have $0 = C_n + D_n \Rightarrow D_n = -C_n$.

Then (16) reduces to
$$Y_n(y) = C_n\left(e^{n\pi y/a} - e^{-\frac{n\pi y}{a}}\right) = 2\sinh\left(\frac{n\pi y}{a}\right)$$

$$\therefore u_n(x,y) = X_n(x)\,Y_n(y)$$
$$u_n(x,y) = E_n \sin\left(\frac{n\pi x}{a}\right) \sinh \frac{n\pi y}{a}$$
are solutions of (1) satisfying (2) and (3)

where $E_n = 2B_nC_n$.

In order to satisfy (4), we now consider more general solution given by
$$u(x,y) = \sum u_n(x,y) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{a} \sinh\frac{n\pi y}{a} \qquad (12)$$
putting $y=b$ in (12) and using (4),
we get
$$f(x) = 100 = \sum_{n=1}^{\infty}\left(E_n \sinh\frac{n\pi b}{a}\right) \sin\frac{n\pi x}{a}.$$
where $E_n \sinh\left(\frac{n\pi b}{a}\right) = \frac{2}{a}\int f(x)\sin\frac{n\pi x}{a}\,dx$

$$\Rightarrow E_n = \frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)}\int (100)\sin\frac{n\pi x}{a}\,dx$$
$$= \frac{200}{a\sinh\left(\frac{n\pi b}{a}\right)}\left[\frac{-\cos\frac{n\pi x}{a}}{n\pi/a}\right]_0^a$$
$$\Rightarrow E_n = \frac{200}{n\pi}\left[1-(-1)^n\right]\operatorname{cosech}\left(\frac{n\pi b}{a}\right) = \begin{cases} 0, & \text{if } n=2m,\ m=1,2,3\ldots \\ \frac{400\operatorname{cosec}\{(2m-1)\pi b/a\}}{(2m-1)\pi}, & \text{if } n=2m-1,\ m=1,2\ldots \end{cases}$$
$\therefore$ (12) reduces to
$$u(x,y) = \sum_{m=1}^{\infty} E_{2m-1} \sin\frac{(2m-1)\pi x}{a}\sinh\frac{(2m-1)\pi y}{a}.$$
$$(13)\quad u(x,y) = \frac{400}{\pi}\sum_{m=1}^{\infty} \sin\frac{(2m-1)\pi x}{a}\sinh\frac{(2m-1)\pi y}{a}\operatorname{cosech}\frac{(2m-1)\pi b}{a}$$

8.(b) Provide a computer Algorithm to solve an ordinary differential equation $\frac{dy}{dx} = f(x, y)$ in the interval $[a, b]$ for n number of discrete points, where the initial value is $y(a) = \alpha$, using Euler's method?

Solution:-

Given; $\frac{dy}{dx} = f(x, y)$

have interval $[a, b]$.

$h = b - a$

$y(a) = \alpha$ i.e y_a, $\therefore x_a$ — initial values condition.

x_b is the required value.

n – number of discrete points

where $n = \frac{x_b - x_a}{h} + 1$

Hence, now the Algorithm for the solution of above ordinary differential equation using Euler's method:

1. Start
2. Define function, [i.e $\frac{dy}{dx} = f(x, y)$]
3. Get the values of x_a, y_a, h and x_b
4. $n = \frac{(x_b - x_a)}{h} + 1$.
5. Start loop from $i = 1$ to n
6. $y = y_a + h * f(x_a, y_a)$
   $x = x + h$
7. Print values of y_a and x_a
8. Check if $x < x_b$
   If yes, assign $x_a = x$ and $y_a = y$
   If no, go to 9.
9. End loop i
10. Stop.

8.(c) → If $u = \dfrac{ax-by}{x^2+y^2}$; $v = \dfrac{(ay+bx)}{x^2+y^2}$; $w=0$

investigate the nature of motion of the liquid.

Solve:- Given; $u = \dfrac{ax-by}{x^2+y^2}$; $v = \dfrac{(ay+bx)}{x^2+y^2}$; $w=0$ —①

from ① $\dfrac{\partial u}{\partial x} = \dfrac{a(x^2+y^2)-2x(ax-by)}{(x^2+y^2)^2} = \dfrac{ay^2-ax^2+2bxy}{(x^2+y^2)^2}$

$$\frac{\partial v}{\partial y} = \frac{a(x^2+y^2)-2x(ay+bx)}{(x^2+y^2)^2} = \frac{-ax^2+ay^2-2bxy}{(x^2+y^2)^2}$$

we see that $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0$ and

hence, the equation of continuity satisfied by ①

∴ ① represents a possible motion.

Moreover ① represents a 2-dimensional motion and hence vorticity components are given by—

$\Omega_x = 0$; $\Omega_y = 0$; $\Omega_z = (\partial v/\partial x) - (\partial u/\partial y)$ —②

from ① → $\dfrac{\partial u}{\partial y} = \dfrac{-b(x^2+y^2)-2y(ax-by)}{(x^2+y^2)^2} = \dfrac{-bx^2+by^2-2ayx}{(x^2+y^2)^2}$

$$\frac{\partial v}{\partial x} = \frac{b(x^2+y^2)-2x(ay+bx)}{(x^2+y^2)} = \frac{-bx^2+by^2-2axy}{(x^2+y^2)^2}$$

$\therefore \dfrac{\partial u}{\partial y} = \dfrac{\partial v}{\partial x}$

∴ from ①; $\Omega_z = 0$. Thus; $\Omega_x = \Omega_y = \Omega_z = 0$

∴ The motion is irrotational.