Set-V

Applications of Partial Differentials

In physical problems, we always seek a solution of the differential equation which satisfies some specified conditions known as the boundary conditions.

The differential equation together with these boundary conditions, constitute a boundary value problem.

In problems involving ordinary differential equations, we may first find the general solution and then determine the arbitrary constants from the initial values. But the same process is not applicable to problems involving partial differential equations for the general solution of a partial differential equation contains arbitrary functions which are difficult to adjust so as to satisfy the given boundary conditions.

Most of the boundary value problems involving linear partial differential equations can be solved by the method of seperation of variables.

→ Seperation of variables: (or) product method

It involves a solution which breaks up into a product of functions each of which contains only one of the variables.

The following example explains this method.

→ Solve (by the method of seperation of variables):

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0.$$

Sol: Assume the trial solution $z = X(x)\,Y(y)$. —(i)

where X is a function of x alone and Y that of y alone.

Substituting this value of z in the given equation we have

$$X''Y - 2X'Y + XY' = 0$$

where $X' = \frac{dX}{dx}$, $Y' = \frac{dY}{dy}$ etc.

Seperating the variables, we get

$$(X'' - 2X')Y + XY' = 0$$

$$\Rightarrow \frac{X'' - 2X'}{X} = \frac{-Y'}{Y} \quad \text{—— (i)}$$

Since x and y are independent variables, therefore (i) can only true if each side is equal to the same constant, 'a' (say)

$\therefore \frac{X''-2X'}{X} = \frac{-Y'}{Y} = a$

$\Rightarrow \frac{X''-2X'}{X} = a$ and $\frac{-Y'}{Y} = a$

i.e, $X''-2X'-Xa=0$ —(ii) ie $Y'+aY=0$ —(iii)

To solve the equation (ii),

the auxiliary equation is

$m^2-2m-a=0$

$\Rightarrow m = 1 \pm \sqrt{1+a}$

$\therefore$ The solution of (ii) is

$$\boxed{X = c_1 e^{(1+\sqrt{1+a})x} + c_2 e^{(1-\sqrt{1+a})x}}$$

and the solution of (iii) is $\boxed{Y = c_3 e^{-ay}}$.

Substituting these values of X and Y in (i), we get

$$z = \left\{ c_1 e^{(1+\sqrt{1+a})x} + c_2 e^{(1-\sqrt{1+a})x} \right\} c_3 e^{-ay}.$$

i.e, $z = k_1 e^{(1+\sqrt{1+a})x} + k_2 e^{(1-\sqrt{1+a})x}$.

where $k_1 = c_1c_3$ and $k_2 = c_2c_3$.

which is the required complete solution.

→ Using the method of seperation of variables solve $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ where $u(x,0) = 6e^{-3x}$.

Sol^n: Assume the solution $u(x,t) = X(x)\,T(t)$. —(i)

Substituting in the given equation, we have

$$X'T = 2XT' + XT$$

$$\Rightarrow (X' - X)T = 2XT'$$

$$\Rightarrow \frac{X'-X}{2X} = \frac{T'}{T} = k \text{ (say)}.$$

$$\therefore \frac{X'-X}{2X} = k \quad \text{and} \quad \frac{T'}{T} = k \text{ —— (iii)}$$

$$\Rightarrow X' - X - 2kX = 0$$

$$\Rightarrow X' = (1+2k)X$$

$$\Rightarrow \frac{X'}{X} = (1+2k) \text{ —— (ii)}$$

solving (ii) $\log X = (1+2k)x + \log c$

$$\Rightarrow \boxed{X = ce^{(1+2k)x}}$$

From (iii) $\log T = kt + \log c'$

$$\Rightarrow \boxed{T = c'e^{kt}}$$

Thus from (i), we have

$$u(x,t) = XT = cc'e^{(1+2k)x} \cdot e^{kt} \text{ —— (iv)}$$

Given that $u(x,0) = 6e^{-3x}$.

$\therefore$ from (iv) $u(x,0) = 6e^{-3x} = cc'e^{(1+2k)x}$

$$\Rightarrow \boxed{cc' = 6} \text{ when } 1+2k = -3$$

$$\Rightarrow \boxed{k = -2}.$$

Substituting these values in (iv), we get

$u(x,t) = 6e^{-3x}e^{-2t}$ i.e., $\boxed{u = 6e^{-(3x+2t)}}$

which is the required solution.

Note: Suppose the given partial differential equation involves n independent variables

$x_1, x_2, \ldots x_n$ and one dependent variable 'u'. Then assume that the equation possesses product solution of the form 3.

$$u(x_1, x_2, \ldots x_n) = X_1(x_1)\, X_2(x_2) \ldots\ldots X_n(x_n) \quad \text{—— (1)}$$

where X_i is a function of x_i only $(i=1,2,\ldots n)$.

On substitution of (1) into the given equation, we shall obtain 'n' ordinary differential equations one in each of the unknown functions X_i $(i=1,2\ldots n)$.

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* Solve the following equations by the method of seperation of variables:

(1) $py^3 + qx^2 = 0$

(2) $4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u$, given $u = 3e^{-y} - e^{-5y}$ when $x=0$

(3) $3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$, $u(x,0) = 4e^{-x}$

(4) find a solution of the equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial x} + 2u$ in the form $u = f(x)\cdot g(y)$.

solve the equation subject to the conditions $u=0$ and $\frac{\partial u}{\partial x} = 1 + e^{-3y}$, when $x=0$ for all values of y.

$$u = \frac{1}{\sqrt{2}}\sinh\sqrt{2}x + e^{-3y}\sin x.$$

* The following are the well-known partial differential equations:

(i) one dimensional wave equation : $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ (or) $\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$

(ii) One dimensional heat flow equation:

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$$

(iii) Two dimensional heat flow equation which in steady state becomes the two dimensional Laplace's equation : $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

(iv) vibrating membrane : Two dimensional wave equation i.e, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$.

(v) Laplace's equation in three dimensions.

i.e, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$. etc.

Starting with the method of seperation of variables, we find their solutions subject to specific boundary conditions and the combination of such solutions gives the desired solution.. Quite often a certain condition is not applicable. In such cases, the most general solution is written as the sum of the particular solutions already found and the constants are determined using fourier series so as to satisfy the remaining conditions.

* Some basic definitions *

Rest: A body is said to be at rest if it doesnot change its position with time with respect to its surroundings.

Motion: A body is said to be in motion if it changes its position with time with respect to its surroundings.

Terms rest and motion are relative to each other.

for examples when a train is running, two passengers sitting in the train beside each other are at rest with respect to each other but they are in motion with respect to the person standing outside the train.

Displacement: The shortest distance between the starting point to the ending point is called 'displacement'. It is a vector

(or)

Displacement is a vector quantity representing a change of position.

Deflection: A sudden change in the direction that something is moving in.

Distance: Total length of the path covered by the body is called distance. It is a scalar.

velocity: Displacement of a body per unit time is said to be its velocity.

If 's' is displacement that takes place in time 't' then velocity of the body is given as

$$\text{velocity} = \frac{\text{displacement}}{\text{time taken}}.$$

Acceleration:- If velocity of a body changes with time (either due to change in magnitude or direction or both) it is said to have acceleration.

i.e, the rate of change of velocity is called acceleration.

Equilibrium: A system of forces acting on a particle is said to be in equilibrium if it is either at rest or moves with uniform motion in a straight line.

Mass: Mass of a body is the quantity of matter it contains.

force: force is an external agency which changes or tends to change the state of rest or of uniform motion in a straight line.

The effect of force acting on a rigid body depends not only on magnitude but also on its direction and point of application.

→ weight of body:

The force with which a body is attracted towards the centre of the earth due to the gravitational attraction is called the weight of the body on the earth.

$W = mg$.

where m is mass of body and g is acceleration due to gravity for earth.

Mass of the body remains constant at any place but weight of the body varies with changes in 'g'.

Gravitational force:

Gravitational force is a long range force and is responsible for the attraction between particles of different masses in universe.

Nature of gravity:

The gravitational force between two bodies is always attractive. It depends upon the masses of the bodies and the distance between them. The greater the masses the larger is the force, the greater the distance of seperation the lesser is the force. It is independent of the nature of the medium between the bodies.

→ Tension is a pulling force which is exerted on a body by means of a string or rod.

→ Thrust is a pushing force which is exerted on a body by means of a rod and not by means of string, because a string is flexible.

Newton's Laws of Motion

Newton's First Law:

Every body continues to be in its state of rest or of uniform motion along a straight line unless it is acted on by an external force to change its state.

Newton's Second Law:

The rate of change of momentum of a body is directly proportional to the external force applied and is takes place in the same direction in which the external force is acting.

$$\boxed{F = ma}$$

Momentum: The momentum (P) of a body is defined as the product of its mass (m) and velocity (v).

$$\boxed{P = mv}$$

Newton's third Law:

To every action there is always an equal and opposite reaction.

$$F_{(\text{action})} = F_{(\text{reaction})}$$

→ Action and reaction are equal in magnitude and opposite in direction.

→ They always occur in pairs.

principle of superposition of waves:

— principle of superposition of waves states that when two or more waves are simultaneously impressed on the particles of the medium, the resultant displacement of any particle is equal to the algebraic sum of displacements of all the waves.

If $y_1, y_2, y_3 \ldots$ etc are the displacements due to the overlapping waves, the resultant displacement of any particle is given by

$y = y_1 + y_2 + y_3 + \ldots$

Thus the resultant wave form can be obtained by the principle of superposition of waves.

— The general linear homogeneous partial differential equation of the second order is

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G \quad \text{--- (1)}$$

Suppose that (i) $u_1, u_2, u_3, \ldots u_n \ldots$ is an infinite set of solutions of (1) in a region R in xy-plane

(ii) the infinite series $u_1 + u_2 + \ldots + u_n + \ldots$ converges and is differentiable term by term in R. Then by principle superposition, the function u, defined by $u = \sum_{n=1}^{\infty} u_n$ is also a solution of (1) in R. Here R denotes the set of all real numbers.

Forier Sine Series: If it be required to expand $f(x)$ as a sine series in $0<x<l$, then its expansion will give the fourier sine series:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}, \text{ where } b_n = \frac{2}{l}\int_0^l f(x) \sin \frac{n\pi x}{l}\,dx$$

$n=1,2,3,\ldots$

Also known as half range sine series

Fourier cosine series: If it be required to expand $f(x)$ as a cosine series in $0<x<l$, then its expansion will give the fourier cosine series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l}; \text{ where } a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$a_n = \frac{2}{l}\int_0^l f(x) \cos \frac{n\pi x}{l}\,dx.$$

$n=1,2,3,\ldots$

Also known as half range cosine series.

Double fourier sine series:

If it be required to expand $f(x,y)$ as a sine series in rectangle $0\le x\le a$, $0\le y\le b$. then its expansion will give double fourier sine series.

$$f(x,y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}.$$

where $A_{mn} = \dfrac{4}{ab}\displaystyle\int_{x=0}^{a}\int_{y=0}^{b} f(x,y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\,dx\,dy$

$m=1,2,3,\ldots$; $n=1,2,3,\ldots$

Triple fourier sine series: If it be required to expand $f(x,y,z)$ as a sine series in parallelopiped $0\le x\le a$, $0\le y\le b$, $0\le z\le c$, then its expansion will give triple fourier sine series.

$$f(x,y,z) = \sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{lmn} \sin \frac{l\pi x}{a} \sin \frac{m\pi y}{b} \sin \frac{n\pi z}{c}, \text{ where}$$

where $A_{lmn} = \dfrac{8}{abc}\displaystyle\int_{x=0}^{a}\int_{y=0}^{b}\int_{z=0}^{c} f(x,y,z) \sin \frac{l\pi x}{a} \sin \frac{m\pi y}{b} \sin \frac{n\pi z}{c}\,dx\,dy\,dz.$

$l=1,2,3,\ldots$, $m=1,2,3,\ldots$, $n=1,2,3,\ldots$

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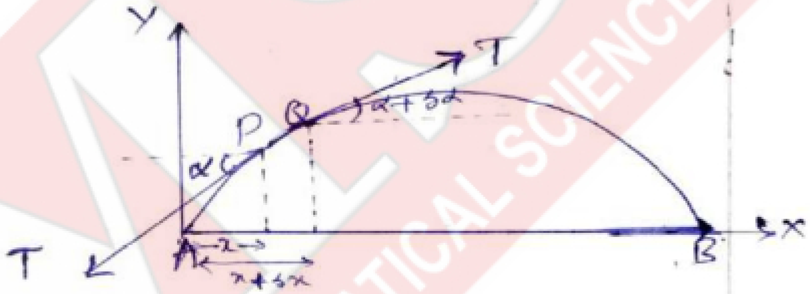
Vibrations of stretched elastic string - wave Equation:

Consider a tightly stretched elastic string of length l and fixed ends A and B subjected to constant tension T as shown in the figure. 4.

The tension T will be considered to be large as compared to the weight of the string so that the effects of gravity are negligible.



Let the string be released from rest and allowed to vibrate.

We shall study the subsequent motion of the string, with no external forces acting on it, assuming that each point of the string makes small vibrations at right angles to the equilibrium position AB, of the string entirely in one plane.

Taking the end A as the origin, AB as the x-axis and AY perpendicular to it as the y-axis; so that the motion takes place entirely in the xy-plane. The figure shows the string in the position APB at time t.

Consider the motion of the element PQ of the string between its points $P(x,y)$ and $Q(x+\delta x, y+\delta y)$, where the tangents makes angles α and $\alpha+\delta\alpha$ with x-axis.

Clearly the element is moving upwards with acceleration $\frac{\partial^2 y}{\partial t^2}$.

Since there is no motion in horizontal direction, we have

$$T\cos(\alpha+\delta\alpha) - T\cos\alpha = 0.$$

$$\Rightarrow T\cos(\alpha+\delta\alpha) = T\cos\alpha = T \text{ (say) constant.}$$

Also the vertical component of the force acting on this element PQ is $= T\sin(\alpha+\delta\alpha) - T\sin\alpha$

$$= T\{\sin(\alpha+\delta\alpha) - \sin\alpha\}$$

$$= T\{\tan(\alpha+\delta\alpha) - \tan\alpha\}, \ (\because \alpha \text{ is small})$$

$$= T\left\{\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right\}$$

If m be the mass per unit length of the string, then mass of the element PQ is $= m\delta x$.

Then by Newton's second law of motion we have

$$m\delta x \frac{\partial^2 y}{\partial t^2} = T\left\{\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right\}$$

i.e. $$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\,\frac{\left\{\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right\}}{\delta x}$$

Taking limits as $Q\to P$, i.e. $\delta x \to 0$,

we have $$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\frac{\partial^2 y}{\partial x^2}$$

i.e. $$\boxed{\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}} \quad \text{where } \boxed{c^2 = \frac{T}{m}}$$

This is the partial differential equation giving the transverse vibrations of the string. It is also called the one dimensional wave equation.

Solution of one dimensional wave equation. 8

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

Sol^n: Given $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ —— (1)

Let solution of (1) be of the form

$y(x,t) = X(x)T(t)$. —— (2)

where X is a function of x and T is a function of t only.

Then $\frac{\partial^2 y}{\partial t^2} = XT''$ and $\frac{\partial^2 y}{\partial x^2} = X''T$.

Substituting these values in (1), we get

$$XT'' = c^2 X'' T$$

$$\Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} \quad \text{—— (3)}$$

Clearly the left side of (3) is a function of x only and the right side is a function of t only.

Since x and t are independent variables, (3) can hold good if each side is equal to a constant k (say).

Then (3) leads to the ordinary differential equations:

$$\frac{d^2X}{dx^2} - kX = 0 \quad \text{—— (4)} \quad \text{and} \quad \frac{d^2T}{dt^2} - kc^2T = 0 \quad \text{—— (5)}$$

$\left(\because \text{from (3)}\ \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = k\right)$

Now we solve (4) and (5); Three cases arise

(i) when $k=0$. Then $X = a_1 x + a_2$; $T = a_3 t + a_4$

(ii) when k is positive.

Let $k = p^2$ (say)

Then $X = b_1 e^{px} + b_2 e^{-px}$; $T = b_3 e^{cpt} + b_4 e^{-cpt}$

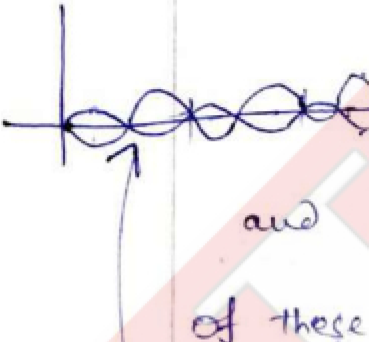
(iii) when k is negative.

Let $k = -p^2$ (say).

Then $\boxed{X = c_1 \cos px + c_2 \sin px}$;

$\boxed{T = c_3 \cos cpt + c_4 \sin cpt}$.

Thus the various possible solutions of wave equation (1) are



$$y(x,t) = (a_1 x + a_2)(a_3 t + a_4) \quad \text{--- (6)}$$

$$y(x,t) = (b_1 e^{px} + b_2 e^{-px})(b_3 e^{cpt} + b_4 e^{-cpt}) \quad \text{--- (7)}$$

and $$y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt). \quad \text{--- (8)}$$

Of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem. As we will be dealing with problems on vibrations, y must be a periodic function of x and t. Hence $y(x,t)$ must involve trignometric terms. Accordingly the solution given by (8) i.e,

$$y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$$

is the only suitable solution of the wave equation.

* **Boundary conditions and initial conditions of one dimensional wave equation:**

→ The boundary conditions which the solution has to satisfy are

(i) $y(x,t)=0$ when $x=0$

(ii) $y(x,t)=0$ when $x=l$ (where 'l' is the length of the stretched string)

These should satisfy for every value of t.

i.e, As the end points of the string are fixed, for all time.

$y(0,t)=0$ and $y(l,t)=0$

→ If the string is made to vibrate by pulling it in a curve $y=f(x)$ and then releasing it, the initial conditions are

$y(x,t)=f(x)$ when $t=0$ i.e, $y(x,0)=f(x)$ and

$\frac{\partial y}{\partial t}(x,t)=0$ when $t=0$ i.e, $\left(\frac{\partial y}{\partial t}\right)_t=0$

Here the initial velocity of the string is zero i.e, the string starts from the position of rest.

→ If the string is made to vibrate by giving its each point (when in equilibrium position) a specified velocity, the initial conditions are of the form

(displacement)

$y(x,t)=0$ when $t=0$ i.e, $y(x,0)=0$

$\frac{\partial y}{\partial t}(x,t) = g(x)$ when $t=0$ i.e, $\left(\frac{\partial y}{\partial t}\right)_{t=0} = g(x)$

→ If the String is given both a displacement and velocity initially, then the initial conditions are of the form

$y(x,t) = f(x)$ when $t=0$ i.e, $y(x,0) = f(x)$

$\frac{\partial y}{\partial t}(x,t) = g(x)$ when $t=0$ i.e, $\left(\frac{\partial y}{\partial t}\right)_{t=0} = g(x)$

General solution of one-dimensional wave equation satisfying the given boundary and initial conditions:

Ex-1 Show that the wave equation $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$.

under conditions $y(0,t)=0$, $y(l,t)=0$ $\forall t$.

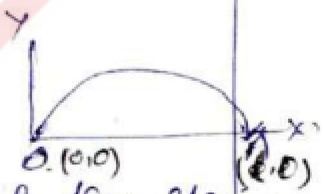
$y(x,0) = f(x)$, $\left(\frac{\partial y}{\partial t}\right)_{t=0} = g(x)$

has solution of the form:

$$y(x,t) = \sum_{n=1}^{\infty} \left(E_n \cos\frac{n\pi ct}{l} + F_n \sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}$$

where $E_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$

$$F_n = \frac{2}{n\pi c}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx.$$



Sol^n: Given $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ ——— (1)

where $u(x,t)$ is the deflection of the string.

Let it be stretched between fixed points $(0,0)$ and $(l,0)$. Then we are to find $y(x,t)$ under the following boundary condition (B.C) and initial conditions (I.C).

B.C. $y(0,t)=0$, $y(l,t)=0$ for all t ——— (2)

I.C. $y(x,0) = f(x)$ (Initial deflection) ——— (3)

$\left(\frac{\partial y}{\partial t}\right)_{t=0} = g(x)$ (Initial velocity) ——— (4)

Suppose that (1) has the solution of the form $y(x,t) = X(x)\,T(t)$ ——— (5)

Substituting this value of y in (1), we have

$$XT'' = c^2 X'' T$$

$$\Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = \mu \text{ (say)}$$

$$\Rightarrow X'' - \mu X = 0 \quad \text{——— (6)}$$

$$\text{and } T'' - \mu c^2 T = 0 \quad \text{——— (7)}$$

Using (2), (5) gives

$$X(0)\,T(t) = 0 \text{ and } X(l)\,T(t) = 0 \quad \text{——— (8)}$$

Since $T(t) \equiv 0$ leads to $y \equiv 0$. ~~$\forall t$~~

so suppose ~~that~~ $T(t) \not\equiv 0$.

Then (8) gives

$\boxed{X(0) = 0}$ and $\boxed{X(l) = 0}$ ——— (9) which are boundary condition.

We now solve (6) under B.C. (9).

Three cases arise.

<u>case (1)</u>: Let $\mu = 0$. Then solution of (6) is

$$X(x) = Ax + B. \quad \text{——— (10)}$$

Using B.C. (9), (10) gives

$$X(0) = A(0) + B \Rightarrow \boxed{0 = B}$$

$$\text{and } X(l) = Al + B \Rightarrow 0 = Al + 0$$

$$\Rightarrow \boxed{A = 0} \quad (\because l \text{ is the length of string})$$

$$\Longrightarrow \boxed{X(x) = 0}$$

This leads to $y \equiv 0$, which doesnot satisfy I.C. (3) and (4).

So we reject $\mu = 0$.

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Case②: Let $\mu = \lambda^2, \lambda \neq 0$. Then solution of ⑥ is
(i.e, positive)

$$X(x) = Ae^{\lambda x} + Be^{-\lambda x} \quad \text{——— (11)}$$

Using B.C. ⑨, ⑪ gives

$X(0) = 0 = A + B$ i.e, $\boxed{A + B = 0}$ —(i)

and $X(l) = 0 = Ae^{\lambda l} + Be^{-\lambda l}$

solving above we get

$Ae^{\lambda l} - Ae^{-\lambda l} = 0$

$\Rightarrow A(e^{\lambda l} - e^{-\lambda l}) = 0$ $(\because e^{\lambda l} - e^{-\lambda l} \neq 0$
$l, \lambda \neq 0)$

$\Rightarrow \boxed{A = 0}$

$\therefore$ from (i) $\boxed{B = 0}$

$\Rightarrow \boxed{X(x) = 0}$

This leads to $y = 0$ which doesnot satisfy ③ and ④.

So we reject $\mu = \lambda^2$.

Case③: Let $\mu = -\lambda^2, \lambda \neq 0$.
(i.e, μ is negative)

Then solution of ⑥

$$X(x) = A\cos\lambda x + B\sin\lambda x \quad \text{— (13)}$$

Using B.C. ⑨, ⑬ gives

$X(0) = 0 = A(1) + B(0) \Rightarrow \boxed{A = 0}$

and $X(l) = 0 = 0 + B\sin\lambda l$

$\Rightarrow B\sin\lambda l = 0$

$\Rightarrow \sin\lambda l = 0$

Here we taken $B \neq 0$,

Since otherwise $X = 0$ so that which doesnot satisfy ③ and ④

Now $\sin\lambda l = 0$

$\Rightarrow \lambda l = n\pi \quad , n=1,2,\ldots$

$\Rightarrow \boxed{\lambda = \frac{n\pi}{l}} \quad , n=1,2,3,\ldots$ —— $(\because)$

$\therefore$ From (13), we have

$$X(x) = B\sin\frac{n\pi x}{l} \quad ; n=1,2,3,\ldots$$

Hence non-zero solutions $X_n(x)$ of (6) are given by

$$\boxed{X_n(x) = B_n \sin\left(\frac{n\pi x}{l}\right)} \quad \text{—— (14)}$$

from (7)

$T'' - \mu c^2 T = 0$

$\Rightarrow T'' + \lambda^2 c^2 T = 0 \qquad (\because \mu = -\lambda^2)$

$\Rightarrow T'' + \frac{n^2\pi^2}{l^2} c^2 T = 0 \qquad (\because \lambda = \frac{n\pi}{l})$

whose general solution is

$$T_n(t) = C_n \cos\left(\frac{n\pi ct}{l}\right) + D_n \sin\left(\frac{n\pi ct}{l}\right)$$

$\therefore y_n(x,t) = X_n(x)\, T_n(t)$

$$= B_n \sin\frac{n\pi x}{l}\left[C_n \cos\frac{n\pi ct}{l} + D_n \sin\frac{n\pi ct}{l}\right]$$

$$= \left[E_n \cos\frac{n\pi ct}{l} + F_n \sin\frac{n\pi ct}{l}\right]\sin\frac{n\pi x}{l} \quad \text{—— (A)}$$

are solutions of (1) satisfying (2).

Here $E_n = B_nC_n$ and $F_n = B_nD_n$ are new arbitrary constants.

In order to obtain a solution also satisfying (3) and (4), we consider more general solution $y(x,t) = \sum_{n=1}^{\infty} y_n(x,t)$

i.e, $y(x,t) = \sum_{n=1}^{\infty}\left\{E_n \cos\frac{n\pi ct}{l} + F_n \sin\frac{n\pi ct}{l}\right\}\sin\frac{n\pi x}{l}$ ——— (15)

Differentiating (15) partially w.r.t 't', we get

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty}\left\{E_n\left(-\sin\frac{n\pi ct}{l}\right)\cdot\frac{n\pi c}{l} + F_n \cos\frac{n\pi ct}{l}\cdot\frac{n\pi c}{l}\right\}\sin\frac{n\pi x}{l}$$

$$= \sum_{n=1}^{\infty}\left\{-E_n \frac{n\pi c}{l}\sin\frac{n\pi ct}{l} + \frac{n\pi c F_n}{l}\cos\frac{n\pi ct}{l}\right\}\sin\frac{n\pi x}{l}$$ ——— (16)

putting $t=0$ in (15) and (16)

and using the I.C (3) and (4), we get

(15) ≡

$$f(x) = \sum_{n=1}^{\infty}\left\{E_n \cos\frac{n\pi c(0)}{l} + 0\right\}\sin\frac{n\pi x}{l}$$

$(\because y(x,0) = f(x))$

$$\Rightarrow f(x) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l}$$

and

(16) ≡ $g(x) = \sum_{n=1}^{\infty}\left\{0 + \frac{n\pi c}{l}F_n \cos\frac{n\pi c(0)}{l}\right\}\sin\frac{n\pi x}{l}$

$\left(\because \frac{\partial y}{\partial t} = g(x) \text{ at } t=0\right)$

$$\Rightarrow g(x) = \sum_{n=1}^{\infty}\frac{n\pi c F_n}{l}\sin\frac{n\pi x}{l}$$

which are fourier sine series expansion for $f(x)$ and $g(x)$ respectively.

According we get,

$$E_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$$ ——— (17)

and $\frac{n\pi c F_n}{l} = \frac{2}{l}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx \Rightarrow F_n = \frac{2}{n\pi c}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx.$ ——— (18)

Hence the required solution is given by (15)

i.e, $y(x,t) = \sum_{n=1}^{\infty}\left\{E_n \cos\frac{n\pi ct}{l} + F_n \sin\frac{n\pi ct}{l}\right\}\sin\frac{n\pi x}{l}$

where $E_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx$ & $F_n = \frac{2}{n\pi c}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx.$

Note:

particular case I : If initial velocity

$y_t(x,0) = g(x) = 0$ then $F_n = 0$ from (18)

$\therefore$ In this case the solution (15) reduces to

$$y(x,t) = \sum_{n=1}^{\infty} E_n \cos\frac{n\pi ct}{l} \sin\frac{n\pi x}{l}$$

where E_n is given by (17)

particular case II : If initial displacement

$y(x,0) = f(x) = 0$, then $E_n = 0$ by (16).

$\therefore$ In this case the solution (15) reduces to

$$y(x,t) = \sum_{n=1}^{\infty} F_n \sin\frac{n\pi ct}{l} \sin\frac{n\pi x}{l}$$

where F_n is given by (18).

→ A string is stretched between two fixed points at a distance l apart. Motion is started by displacing the string in the form $y = y_0 \sin\frac{\pi x}{l}$ from which it is released at time $t=0$. Find the displacement at any point at a distance x from one end at time t.

Soln: The vibration of the string is given by

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{——} (1)$$

As the end points of the string are fixed, for all time.

B.C $\quad y(0,t) = 0$ and $y(l,t) = 0$ ——(2)

I.C. $\quad$ Initial velocity $= \left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$ for $0 \le x \le l$ ——(3)

and initial displacement $= y(x,0) = y_0 \sin\frac{\pi x}{l}$; $0 \le x \le l$ — (4)

proceeding like as in Ex-(1) till equation (15).

i.e. $y(x,t) = \sum_{n=1}^{\infty} \left\{ E_n \cos\frac{n\pi ct}{l} + F_n \sin\frac{n\pi ct}{l} \right\} \sin\frac{n\pi x}{l}$ — (5)

Differentiating (5) partially w.r.t t, we get

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} \left\{ -E_n \frac{n\pi c}{l} \sin\frac{n\pi ct}{l} + \frac{n\pi c}{l} F_n \cos\frac{n\pi ct}{l} \right\} \sin\frac{n\pi x}{l} \quad \text{— (6)}$$

putting $t=0$ in (6) and (5) and using initial conditions (3) & (4) we get

(5) $\Rightarrow$ $y(x,0) = y_0 \sin\frac{\pi x}{l} = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l}$ — (7)

(6) $\Rightarrow$ $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 = \sum_{n=1}^{\infty} \frac{n\pi c}{l} F_n \sin\frac{n\pi x}{l}$

where $F_n = \frac{2}{n\pi c} \int_0^l (0) \sin\frac{n\pi x}{l}\, dx$

i.e, $F_n = 0$

$\therefore$ From (7), we have

$y_0 \sin\frac{\pi x}{l} = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l}$

comparing the coefficients of like terms on both sides, we have

$E_1 = y_0$ and $E_n = 0$ for $n \ne 1$.

$\therefore$ Equation (5) reduces to

$$\boxed{y(x,t) = y_0 \sin\frac{\pi x}{l} \cos\frac{\pi ct}{l}} \quad \text{Ans.}$$

H.W $\rightarrow$ for the vibrating string of length π if the initial shape is given by $y(x,0) = C\sin x$, C being constant and string is released from rest, then find the displacement $y(x,t)$.

Hint: Taking in above example $l=\pi$ and $y_0 = C$. Ans: $\boxed{y(x,t) = C \sin x \cos ct}$

→ A string of length l has its ends $x=0$ and $x=l$ fixed. It is released from rest in the position $y = \{4\lambda x(l-x)\}/l^2$. Find an expression for the displacement of the string at any subsequent time.

Sol:

The displacement function $y(x,t)$ is the solution of the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{—— (1)}$$

Subject to boundary conditions:

$$y(0,t) = y(l,t) = 0 \quad \text{for all } t \geq 0 \quad \text{—— (2)}$$

and initial conditions; namely

$$\text{Initial velocity} = \left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \quad \text{for } 0 \leq x \leq l \quad \text{—— (3)}$$

$$\text{Initial displacement} = y(x,0) = f(x) = \frac{4\lambda x(l-x)}{l^2} \quad \text{—— (4)}$$

proceeding like as in Ex-(1) till equation (15)

i.e., $$y(x,t) = \sum_{n=1}^{\infty}\left\{E_n \cos\frac{n\pi ct}{l} + F_n \sin\frac{n\pi ct}{l}\right\}\sin\frac{n\pi x}{l}. \quad \text{—— (5)}$$

Differentiating (5) partially w.r.t 't'.

we get

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty}\left\{-\frac{n\pi c}{l} E_n \sin\frac{n\pi ct}{l} + \frac{n\pi c}{l} F_n \cos\frac{n\pi ct}{l}\right\}\sin\frac{n\pi x}{l} \quad \text{—— (6)}$$

putting $t=0$ in equ (5) and (6).

and using initial conditions (3) and (4)

we get

$$y(x,0) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l} = \frac{4\lambda x(l-x)}{l^2}; \quad (\text{by } (3)) \quad \text{—— (7)}$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum_{n=1}^{\infty} \frac{n\pi c}{l} F_n \sin\frac{n\pi x}{l} = 0 \quad ; (\text{by } (3))$$

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where $F_n = \frac{2}{n\pi c}\int_0^l (0)\sin\frac{n\pi x}{l}\,dx = 0$

and $E_n = \frac{2}{l}\int_0^l \frac{4\lambda x(l-x)}{l^2}\sin\frac{n\pi x}{l}\,dx$

$\Rightarrow E_n = \frac{8\lambda}{l^3}\int_0^l (lx-x^2)\sin\frac{n\pi x}{l}\,dx$

$$= \frac{8\lambda}{l^3}\left[(lx-x^2)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - \int(l-2x)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{8\lambda}{l^3}\left[(lx-x^2)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) + (l-2x)\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l} + \int 2\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{8\lambda}{l^3}\left[(lx-x^2)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) + (l-2x)\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l} - 2\frac{l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{8\lambda}{l^3}\left[(0-0) + \left(\frac{2l^3}{n^2\pi^2}\sin n\pi - 0\right) - \frac{2l^3}{n^3\pi^3}(\cos n\pi - 1)\right]$$

$$= \frac{8\lambda}{l^3}\left[\frac{2l^3}{n^2\pi^2}\sin n\pi - \frac{2l^3}{n^3\pi^3}(\cos n\pi - 1)\right]$$

$$E_n = \begin{cases} \frac{8\lambda}{l^3}\left[0 - \frac{2l^3}{n^3\pi^3}(1-1)\right] = 0 & \text{if } n = 2m \text{ (even)}, m = 1,2,3,\ldots \\ \frac{8\lambda}{l^3}\left[0 - \frac{2l^3}{n^3\pi^3}(-1-1)\right] = \frac{8\lambda}{l^3}\left(\frac{4l^3}{(2m-1)^3\pi^3}\right) = \frac{32\lambda}{(2m-1)^3\pi^3} & \text{if } n = 2m-1 \text{ (odd)}, m = 1,2,3,\ldots \end{cases}$$

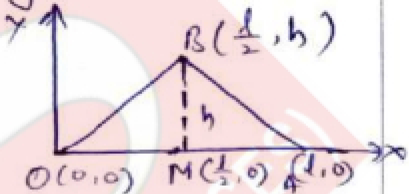
Substituting the values of E_n and F_n in (5), the required expression is

$$y(x,t) = \frac{32\lambda}{\pi^3}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^3}\sin\frac{(2m-1)\pi x}{l}\cos\frac{(2m-1)\pi ct}{l}.$$

H.W → A taut string of length l has its ends $x=0$ and $x=l$ fixed. The mid point is taken to a small height h and released from rest at time $t=0$. Find the displacement function $y(x,t)$.

Hint: B.C. $y(0,t) = y(l,t) = 0 \quad \forall t \geq 0$

Initial position of the string at $t=0$ is made up of two straight line segments OB and BA as shown in the figure and the string is released from rest.



The equation of OB is given by

$$y-0 = \frac{h-0}{\left(\frac{l}{2}-0\right)}(x-0) \quad \text{for } 0 \leq x \leq l/2$$

$$\Rightarrow y = \frac{2hx}{l} \quad \text{for } 0 \leq x \leq l/2$$

The equation of BA is given by

$$y-0 = \frac{h-0}{\left(\frac{l}{2}-l\right)}(x-l) \quad \text{for } l/2 \leq x \leq l.$$

$$\Rightarrow y = \frac{2h(l-x)}{l} \quad \text{for } \frac{l}{2} \leq x \leq l.$$

Hence, the initial displacement is given by

$$\underset{=f(x)}{y(x,t)} = \begin{cases} 2hx/l, & 0 \leq x \leq l/2 \\ \dfrac{2h(l-x)}{l}, & l/2 \leq x < l. \end{cases}$$

and the initial velocity $= \left(\partial y/\partial t\right)_{t=0} = 0$

Ans: $y(x,t) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l} \cos\frac{n\pi c t}{l}$.

where $E_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\,dx$.

$$= \frac{2}{l}\left[\int_0^{l/2} \frac{2hx}{l} \sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} \frac{2h(l-x)}{l} \sin\frac{n\pi x}{l}\,dx\right].$$

$$= \begin{cases} \dfrac{8(-1)^{m+1} h}{(2m-1)^2\pi^2}, & \text{if } n = 2m-1 \text{ (odd)} \quad m=1,2,\ldots \\ 0, & \text{if } n = 2m \text{ (even)} \quad m=1,2,\ldots \end{cases}$$

$$\therefore y(x,t) = \frac{8h}{\pi^2}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{(2m-1)^2}\sin\frac{(2m-1)\pi x}{l}\cos\frac{(2m-1)\pi ct}{l}$$

→ A tightly stretched elastic string of length l, with fixed end points $x=0$ and $x=l$ is initially in the position is given by $y = y_0\sin^3\frac{\pi x}{l}$, y_0 being constant. Find the displacement $y(x,t)$.

Hint: B.C. $y(0,t) = y(\pi,t) = 0$, $\forall t \geq 0$.

I.C. Initial velocity $= \left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$ for $0<x<\pi$ —— (1)

initial displacement $= y(x,0) = y_0\sin^3\frac{\pi x}{l}$ —— (2)

proceeding like as in ex-(1), till eqn (15),

we have

$$y(x,t) = \sum_{n=1}^{\infty}\left\{E_n\cos\frac{n\pi ct}{l} + F_n\sin\frac{n\pi ct}{l}\right\}\sin\frac{n\pi x}{l}. \quad \text{—— (3)}$$

Differentiating (3) partially w.r.t t, we get

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty}\left\{-\frac{n\pi c}{l}E_n\sin\frac{n\pi ct}{l} + \frac{n\pi c}{l}F_n\cos\frac{n\pi ct}{l}\right\}\sin\frac{n\pi x}{l} \quad \text{—— (4)}$$

putting $t=0$ in (3) and (4)

and using the I.C. (1) and (2), we get

(3) ≡ $y(x,0) = y_0\sin^3\frac{\pi x}{l} = \sum_{n=1}^{\infty}E_n\sin\frac{n\pi x}{l}$. —— (5)

(4) ≡ $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 = \sum_{n=1}^{\infty}\frac{n\pi c}{l}F_n\sin\frac{n\pi x}{l}$ —— (6)

where $F_n = \frac{2}{n\pi c}\int_0^l (0)\sin\frac{n\pi x}{l}dx = 0$

now from (5)

$$y_0\sin^3\frac{\pi x}{l} = \sum_{n=1}^{\infty}E_n\sin\frac{n\pi x}{l}$$

$\left\{\because \sin 3\theta = 3\sin\theta - 4\sin^3\theta \right.$

$\left.\Rightarrow \sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}\right\}$

$$\Rightarrow \frac{y_0}{4}\left[3\sin\frac{\pi x}{l} - \sin\frac{3\pi x}{l}\right] = E_1\sin\frac{\pi x}{l} + E_2\sin\frac{2\pi x}{l} + E_3\sin\frac{3\pi x}{l} + \cdots$$

comparing the coefficients of the like terms

we have $E_1 = \frac{3y_0}{4}$, $E_2 = 0$, $E_3 = -\frac{y_0}{4}$, $E_n = 0\ \forall n \geq 4$.

substituting these values in ③,
the required displacement is given by

$$\boxed{y(x,t) = \frac{3y_0}{4}\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l} - \frac{y_0}{4}\sin\frac{3\pi x}{l}\cos\frac{3\pi ct}{l}.}$$

→ A tightly stretched ~~elastic~~ string of length π, with fixed end points $x=0$ and $x=\pi$ is initially in the position is given by $y = y_0\sin^3 x$, y_0 being constant. find the displacement $y(x,t)$.

Ans: $y(x,t) = \frac{3y_0}{4}\sin x\cos ct - \frac{y_0}{4}\sin 3x\cos 3ct$

[putting $l=\pi$ in the above problem]

→ Solve the one dimensional wave equation $\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$, $0\le x\le 2\pi$, $t\ge 0$ subject to the following initial and boundary conditions

(i) $y(x,0) = \sin^3 x$, $0\le x\le 2\pi$ (ii) $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$, $0\le x\le 2\pi$

(iii) $y(0,t) = y(2\pi,t) = 0$, for $t\ge 0$.

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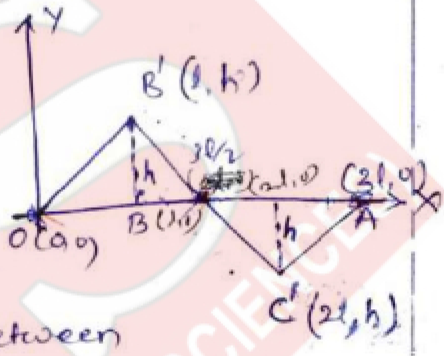
→ find the deflection $y(x,t)$ of the vibrating string (length $=\pi$, and $c^2=1$) corresponding to zero initial velocity and initial deflection (displacement) $f(x) = k(\sin x - \sin 2x)$.

Ans: $k(\cos t\sin x - \cos 2t\sin 2x.)$

2003 → The points of trisection of a string are pulled aside through the same distance 'h' on opposite sides of the position of equilibrium and the string is released from rest. Derive an expression for the displacement of the string at subsequent time and show that the mid point of the string always remains at rest.



(or)

Find the deflection $u(x,t)$ of vibrating string, stretched between fixed points $(0,0)$ and $(3l,0)$, corresponding to zero initial velocity and following initial deflection (displacement)

$$f(x) = \begin{cases} \frac{hx}{l} & \text{when } 0 \le x \le l \\ \frac{h(3l-2x)}{l} & \text{when } l \le x \le 2l \\ \frac{h(x-3l)}{l} & \text{when } 2l \le x \le 3l. \end{cases} \quad \text{— (A)}$$

where h is constant.

Sol^n: The displacement $y(x,t)$ of any point of the string is given by $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$.

B.C. $y(0,t) = y(3l,t) = 0 \quad \forall t \ge 0$ — (1)

I.C. $y(x,0) = f(x)$ — (2) (where $f(x)$ is given by (A))

and $\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$ — (3)

proceeding like as in Ex-(1) till equation (15) by replacing l by $3l$, we have,

$$y(x,t) = \sum_{n=1}^{\infty}\left\{E_n \cos\frac{n\pi ct}{3l} + F_n \sin\frac{n\pi ct}{3l}\right\}\sin\frac{n\pi x}{3l} \quad \text{——— (4)}$$

Differentiating (3) partially w.r.t t, we get

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty}\left\{-E_n \frac{n\pi c}{3l}\sin\frac{n\pi ct}{3l} + \frac{n\pi c}{3l}F_n \cos\frac{n\pi ct}{3l}\right\}\sin\frac{n\pi x}{3l} \quad \text{——— (5)}$$

putting $t=0$ in (4) and (5)
and using the I.C. (2) and (3), we get

(5) $\Rightarrow \left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum_{n=1}^{\infty} F_n \frac{n\pi c}{3l}\sin\frac{n\pi x}{3l} = 0$ (by (3))

where $F_n = \frac{2}{n\pi c}\int_0^{3l}(0)\sin\frac{n\pi x}{3l}\,dx = 0$

(4) $\Rightarrow y(x,0) = f(x) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{3l}$

where $E_n = \frac{2}{3l}\int_0^{3l} f(x)\sin\frac{n\pi x}{3l}\,dx$. ——— (6)

Now

$$E_n = \frac{2}{3l}\int_0^{3l} f(x)\sin\frac{n\pi x}{3l}\,dx$$

$$= \frac{2}{3l}\left[\int_0^{l} f(x)\sin\frac{n\pi x}{3l}\,dx + \int_l^{2l} f(x)\sin\frac{n\pi x}{3l}\,dx + \int_{2l}^{3l} f(x)\sin\frac{n\pi x}{3l}\,dx\right.$$

$$= \frac{2}{3l}\left[\int_0^{l}\frac{hx}{l}\sin\frac{n\pi x}{3l}\,dx + \int_l^{2l}\frac{h(3l-2x)}{l}\sin\frac{n\pi x}{3l}\,dx + \int_{2l}^{3l}\frac{h(x-3l)}{l}\sin\frac{n\pi x}{3l}\,dx\right.$$

Continuing in this way we get

$$E_n = \frac{18h}{n^2\pi^2}\left[\sin\frac{n\pi}{3} - \left\{\sin\left(n\pi - \frac{n\pi}{3}\right)\right\}\right]$$

$$= \frac{18h}{n^2\pi^2}\left[\sin\frac{n\pi}{3} - \left(\sin n\pi\cos\frac{n\pi}{3} - \cos n\pi\sin\frac{n\pi}{3}\right)\right]$$

$= \frac{18h}{n^2\pi^2}\left[\sin\frac{n\pi}{3} - 0 + \cos n\pi \sin\frac{n\pi}{3}\right]$ $(\because \sin n\pi = 0 \ \forall n)$

$= \frac{18h}{n^2\pi^2}\left[1+\cos n\pi\right]\sin\frac{n\pi}{3}$

$= \frac{18h}{n^2\pi^2}\left[1+(-1)^n\right]\sin\frac{n\pi}{3}$

Thus $E_n = 0$ if n is odd.

$E_n = \frac{36h}{n^2\pi^2}\sin\frac{n\pi}{3}$ if n is even.

put $n=2m$, $m=1,2,\ldots$

i.e, $= \frac{36h}{4m^2\pi^2}\sin\frac{2m\pi}{3}$, $m=1,2,\ldots$

$= \frac{9h}{m^2\pi^2}\sin\frac{2m\pi}{3}$

putting the values of E_n and F_n in (4), the required deflection is given by

$$y(x,t) = \sum_{m=1}^{\infty}\frac{9h}{m^2\pi^2}\sin\frac{2m\pi}{3}\sin\frac{n\pi x}{3l}\cos\frac{n\pi ct}{3l}.$$

$$\Rightarrow y(x,t) = \frac{9h}{\pi^2}\sum_{m=1}^{\infty}\sin\frac{2m\pi}{3}\cos\frac{n\pi ct}{3l}\sin\frac{n\pi x}{3l}. \quad \text{—— (7)}$$

putting $x=\frac{3l}{2}$ in (7), we find that the displacement of the midpoint of the string.

i.e, $y(\frac{3l}{2}, t) = 0$.

because $\sin m\pi = 0$, for all integral values of m

This shows that the mid-point of the string always rest.

→ A tightly stretched string of length l with fixed ends is initially in equilibrium position. It is set vibrating by giving each point a velocity $v_0\sin^3\frac{\pi x}{l}$. Find displacement. Ans: $y(x,t) = \frac{lv_0}{12\pi c}\left[9\sin\frac{\pi x}{l}\sin\frac{\pi ct}{l} - \sin\frac{3\pi x}{l}\sin\frac{3\pi ct}{l}\right]$

2004 → A uniform string of length l held tightly between $x=0$ and $x=l$ with no initial displacement, is struck at $x=a$, $0<a<l$ with velocity v_0. Find the displacement of the string at any time $t>0$.

Hint: B.C. $y(0,t)=y(l,t)=0 \ \forall t$.

I.C. initial displacement $y(x,0)=0, \quad 0\le x\le l$.

Initial velocity $= y_t(x,0)=v_0 \ ; \ 0\le x\le l$.

Ans: $$y(x,t)=\frac{4v_0 l}{c\pi^2}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^2}\sin\frac{(2m-1)\pi x}{l}\sin\frac{(2m-1)\pi ct}{l}$$

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2000 → A tightly stretched string with fixed end points $x=0$ and $x=l$ is initially at rest in its equilibrium position. If it is set vibrating giving each point a velocity $kx(l-x)$, find its displacement.

Ans: $$y(x,t)=\frac{8kl^3}{c\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2m-1)^4}\sin\frac{(2m-1)\pi x}{l}\sin\frac{(2m-1)\pi ct}{l}$$

2006 → A deflection of a vibrating string of length l, is governed by the partial differential equation $y_{tt}=c^2y_{xx}$. The ends of the string are fixed at $x=0$ and l. The initial velocity is zero. The initial displacement is given by

$$y(x,0)=\begin{cases}\frac{x}{l}, & 0<x<\frac{l}{2}\\ \frac{l-x}{l}, & \frac{l}{2}<x<l\end{cases}$$ Ans:

Find the deflection of the string at any instant of time.

Ans: $$y(x,t)=\frac{4l}{c\pi^3}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{(2m-1)^3}\sin\frac{(2m-1)\pi x}{l}\sin\frac{(2m-1)\pi ct}{l}.$$

1999 2009 IFS-2005 → A tightly stretched flexible string has its ends fixed at $x=0$ and $x=l$. At time $t=0$ the string is given a shape defined by $f(x)=\mu x(l-x)$, where μ is constant and then released. Find the displacement of any point x of the string at any time $t>0$.

Ans: $y(x,t)=\frac{8\mu l^2}{\pi^3}\sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}\sin\frac{(2n-1)\pi x}{l}\cos\frac{(2n-1)\pi ct}{l}$.

→ A string of length l is initially at rest in its equilibrium position and motion is started giving each of its points a velocity is given by $v=kx$ if $0\le x\le l/2$ and $v=k(l-x)$ if $\frac{l}{2}\le x\le l$ find the displacement function $y(x,t)$.

Ans: $\frac{4kl^2}{c\pi^3}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{(2m-1)^3}\sin\frac{(2m-1)\pi x}{l}\sin\frac{(2m-1)\pi ct}{l}$.

→ If the string of length l is initially at rest in equilibrium position and each of its points is given the velocity $v_0\sin\frac{3\pi x}{l}\cos\frac{2\pi x}{l}$ where $0<x<l$ at $t=0$. find the displacement function.

Ans: $y(x,t)=\frac{lv_0}{2\pi c}\sin\frac{\pi x}{l}\sin\frac{\pi ct}{l}+\frac{lv_0}{5\pi c}\sin\frac{5\pi x}{l}\sin\frac{5\pi ct}{l}$.

→ A string is stretched between the fixed points $(0,0)$ and $(l,0)$ and released at rest from the initial deflection given by $f(x)=\begin{cases}\frac{2kx}{l}, & \text{when } 0<x<l/2\\ \frac{2k(l-x)}{l}, & \text{when } l/2<x<l.\end{cases}$

find the deflection of the string at any time t.

Ans: $\frac{8k}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\cos\frac{(2n-1)\pi ct}{l}\sin\frac{(2n-1)\pi x}{l}$.

→ A taut string of length 20 cms. fastened at both ends is displaced from its position of equilibrium by imparting to each of its points an initial velocity is given by

$v = x$ in $0 \le x \le 10$

$= 20 - x$ in $10 \le x < 20$, x being the distance from one end. Determine the displacement at any subsequent time.