IFoS 2019

1. Given, $\psi = a(x^2 - y^2)$, a being Constant

$u = -\frac{\partial \psi}{\partial y} = 2ay$, $v = \frac{\partial \psi}{\partial x} = 2ax$

$\therefore \gamma = ui + vj = 2a(y\hat{i} + x\hat{j})$

$$\nabla \times \gamma = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2ay & 2ax & 0 \end{vmatrix}$$

$\boxed{\nabla \times \gamma = 0}$ → flow is irrotational.

Now $d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy$

$d\phi = -u\,dx - v\,dy$

$= -2a(y\,dx + x\,dy)$

$d\phi = -2a\,d(xy)$

$\therefore \boxed{\phi = -2axy + C}$

taking $C = 0$ as ϕ is a relative quantity.

$\therefore \boxed{\phi = -2axy}$ → Velocity potential.

Streamline Curve → $x^2 - y^2 = A$ — (1)

Equipotential Curve ⇒ $xy = B$ — (2)

from (1) $\left(\frac{dy}{dx}\right)_{m_1} = \frac{x}{y}$, from (2) $\left(\frac{dy}{dx}\right)_{m_2} = -\frac{y}{x}$

Since $m_1 \cdot m_2 = -1$

$\therefore$ Two curves are orthogonal.

2. Given $T = \frac{1}{2}\{(1+2K)\dot{\theta}^2 + 2\dot{\theta}\dot{\phi} + \dot{\phi}^2\}$.

$V = \frac{n^2}{2}\left[(1+K)\theta^2 + \phi^2\right\}$.

$L = T - V$ Lagrangian.

$$L = \frac{1}{2}\left\{(1+2K)\dot{\theta}^2 + 2\dot{\theta}\dot{\phi} + \dot{\phi}^2\right] - \frac{n^2}{2}\left[(1+K)\theta^2 + \phi^2\right].$$

Lagrangian Equation of Motion

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) - \frac{\partial L}{\partial q_j} = 0}$$

for $q_j = \theta$

$$\frac{\partial L}{\partial \dot{\theta}} = (1+2K)\dot{\theta} + \dot{\phi}, \quad \frac{\partial L}{\partial \theta} = -n^2(1+K)\theta$$

$$\therefore \frac{d}{dt}\left((1+2K)\dot{\theta} + \dot{\phi}\right) + n^2(1+K)\theta = 0$$

$$\boxed{(1+2K)\ddot{\theta} + \ddot{\phi} + n^2(1+K)\theta = 0} \quad -(1)$$

for $q_j = \phi$

$$\frac{\partial L}{\partial \dot{\phi}} = \dot{\phi} + \dot{\theta}, \quad \frac{\partial L}{\partial \phi} = -n^2\phi$$

$$\therefore \frac{d}{dt}(\dot{\phi} + \dot{\theta}) + n^2\phi = 0$$

$$\boxed{\ddot{\phi} + \ddot{\theta} + n^2\phi = 0} \quad -(2)$$

Solving ① & ②

$$(1+2K)\ddot{\theta} - \ddot{\theta} - n^2(1+K)\theta + n^2\phi = 0.$$

$$2K\ddot{\theta} + n^2\left[\phi - (1+K)\theta\right] = 0$$

$$\therefore \boxed{\ddot{\theta} + \frac{n^2}{2K}\left[\phi - (1+K)\theta\right] = 0} \quad -③.$$

Using in ②

$$\ddot{\phi} - \frac{n^2}{2K}\left[\phi - (1+K)\theta\right] - n^2\phi = 0.$$

$$\boxed{\ddot{\phi} + \frac{n^2}{2K}\left[(1+2K)\phi - (1+K)\theta\right] = 0} \quad -④.$$

Subtract ④ from ③

$$\ddot{\theta} - \ddot{\phi} + \frac{n^2}{2K}\left[2(K+1)\phi - 2(1+K)\theta\right] = 0$$

$$\boxed{\ddot{\theta} - \ddot{\phi} + \frac{n^2(K+1)}{K}\left[\theta - \phi\right] = 0} \quad -⑤$$

At $t=0$ $\theta = \phi$, $\dot{\theta} = \dot{\phi}$

$\therefore \ddot{\theta} - \ddot{\phi} = 0$ Putting $\theta = \phi$ in ⑤

$\dot{\theta} = \dot{\phi} + C$ now $\dot{\theta} = \dot{\phi}$ at $t=0$

$\therefore C = 0$

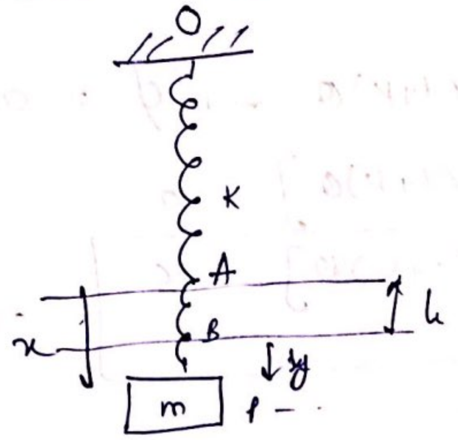
So $\dot{\theta} = \dot{\phi}$

$\therefore \theta = \phi + C'$ Now $\theta = \phi$ at $t=0$

$\therefore C' = 0$

So $\boxed{\theta = \phi}$ $\forall t$

3.



Let A be unstretched position.

$mg = Kh$

$\therefore \boxed{h = \frac{mg}{K}}$

B is equilibrium position.

Now at position P

$T = \frac{1}{2} m\dot{y}^2$, $V = -mgy + \frac{Kx^2}{2}$, $x = y + h$

So $L = T - V$

$= \frac{1}{2} m\dot{y}^2 + mgy - \frac{K(y+h)^2}{2}$

$p = \frac{\partial L}{\partial \dot{y}} = m\dot{y}$

$H = p\dot{y} - L$

$$\boxed{H = m\dot{y}^2 - \frac{m\dot{y}^2}{2} - mgy + \frac{K(y+h)^2}{2} = \frac{m\dot{y}^2}{2} - mgy + \frac{K(y+h)^2}{2}}$$

Also $\dot{y} = p/m$ $\therefore H = \frac{p^2}{2m} - mgy + \frac{K(y+h)^2}{2}$

Now $\dot{p} = -\frac{\partial H}{\partial y} = mg - K(y+h)$, $\dot{y} = \frac{\partial H}{\partial p} = \frac{p}{m}$

$\therefore \ddot{y} = \frac{\dot{p}}{m} = \frac{mg - K(y+h)}{m}$

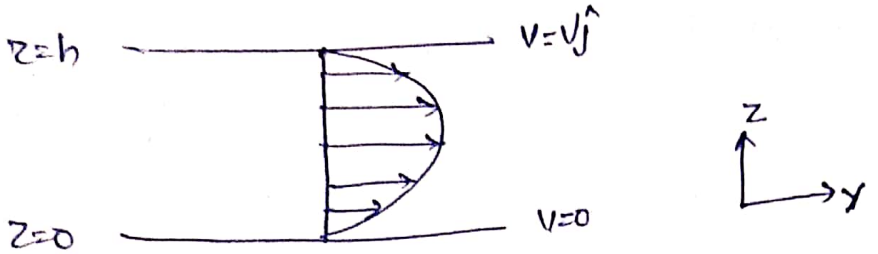
$\ddot{y} = g - \frac{K}{m}y - \frac{Kh}{m}$

$\ddot{y} = -\frac{K}{m}y$

Put $y = x - h$

$$\ddot{x} = -\frac{K}{m}(x - h)$$

$$\boxed{\ddot{x} = -\frac{K}{m}\left(x - \frac{mg}{K}\right)}$$ Equation of motion.



we have Velocity v in ĵ. ~~and along that we have~~ ~~pressure gradient (dp)~~

from eqn of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\frac{\partial v}{\partial y} = 0 \qquad [\because u = w = 0]$$

So no variation of v along y direction.

Again, from Navier's Stokes in all 3 directions we have

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\left(\frac{\partial P}{\partial x}\right) + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = -\frac{1}{\rho}\left(\frac{\partial P}{\partial y}\right) + \frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\left(\frac{\partial P}{\partial y}\right) + \frac{\mu}{\rho}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

For steady flow with $u = 0$, $w = 0$, $\frac{\partial v}{\partial y} = 0$,

we have $\frac{\partial P}{\partial x} = 0$, $\frac{\partial P}{\partial z} = 0$

$$\frac{1}{\rho}\left(\frac{dP}{dy}\right) = \frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial z^2}\right)$$

$$\frac{1}{\mu}\frac{dP}{dy} = \frac{\partial^2 v}{\partial z^2}$$

$$\frac{\partial v}{\partial z} = \frac{1}{\mu}\left(\frac{dp}{dy}\right)(z) + A$$

$$v = \frac{1}{2\mu}\left(\frac{\partial p}{\partial y}\right)z^2 + Az + B$$

Now

at $z = 0$ $\quad v = 0$

$B = 0$

at $z = h$ $\quad v = V$

$$V = \frac{1}{2\mu}\left(\frac{dp}{dy}\right)h^2 + Ah$$

$$A = \left[\frac{V}{h} - \frac{1}{2\mu}\left(\frac{dp}{dy}\right)h\right]$$

So

$$v = \frac{1}{2\mu}\left(\frac{dp}{dy}\right)\left[z^2 - hz\right] + \frac{Vz}{h} \quad \text{[parabolic]}$$

Tangential stress $= -\mu \frac{dv}{dz}$

$$= -\mu\left[\frac{1}{2\mu}\left(\frac{dp}{dy}\right)(2z-h) + \frac{V}{h}\right]$$

So drag per unit area for $z = 0$

$$= \frac{1}{2}\left(\frac{dp}{dy}\right)h - \frac{\mu V}{h}$$

for $z = h$

$$= \frac{-1}{2}\left(\frac{dp}{dy}\right)h - \frac{\mu V}{h}$$

• -ve sign indicating opp to flow direction.