* The Riemann Integral *

(See videos of Dr. Ambedkar College Nagpur, Ms Renu Gupta)

Introduction :-

In elementary treatments the Process of integration is generally introduced as the inverse of differentiation.

If $F'(x) = f(x)$ for all x belonging to the domain of the function f, F is called an integral of the given function f.

Historically, however the Subject of integral arose in connection with the problem of finding areas of plane regions in which the area of a plane region is calculated as the limit of a sum. This notion of integral as summation is based on geometrical concepts.

A German mathematician G.F.B. Riemann gave the first rigorous arithmetic treatment of definite integral free from geometrical concepts.

Riemann's definition covered only bounded functions.

It was Cauchy who extended this definition to unbounded functions.

In the present chapter we shall study the Riemann integral of real valued bounded functions defined on some closed interval.

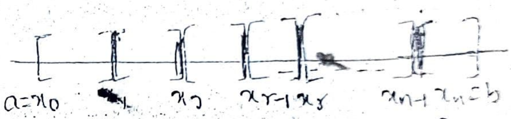
* Partition of a closed Interval

Let $I = [a,b]$ be a closed bounded interval.

If $a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$ then the finite ordered set

$P = \{x_0, x_1, x_2, \ldots, x_{r-1}, x_r, \ldots, x_n\}$ where $r = 1, 2, \ldots n$. is called a partition of I;

The $(n+1)$ Points $x_0, x_1, \ldots x_{n-1}, x_n$ are called Partition points of P.



The n closed Subintervals $I_1 = [x_0, x_1]$, $I_2 = [x_1, x_2]$, ... $I_r = [x_{r-1}, x_r]$ ... $I_n = [x_{n-1}, x_n]$ determined by P are called segments of the partition P.

Clearly $\bigcup_{r=1}^{n} I_r = \bigcup_{r=1}^{n} [x_{r-1}, x_r] = [a,b] = I$

(Or)

$P = \{[x_{r-1}, x_r]\}_{r=1}^{n}$

--- The length of the r^{th} Subinterval $I_r = [x_{r-1}, x_r]$ is denoted by δ_r.

i.e. $\delta_r = x_r - x_{r-1}$; $r = 1, 2, \ldots n$.

Note : (1) By changing the partition points, the partition can be changed and hence there can be an infinite number of partitions of the interval I.

we shall denote it by $P[a,b]$ the set (or family) of all partitions of $[a,b]$.

D. Partition is also known as dissection (or) net.

* Norm of a partition :-

The maximum of the lengths of the subintervals of a partition P is called norm (or) mesh of the partition P and is denoted by $\|P\|$ (or) $\mu(P)$.

i.e. $\|P\| = \text{Max}\{\delta_r / r = 1,2 \ldots n\}$

$= \text{Max}\{x_r - x_{r-1} / r = 1,2 \ldots n\}$

$= \text{Max}\{x_1 - x_0, x_2 - x_1, \ldots x_n - x_{n-1}\}$

Note (1): If $P = \{x_0, x_1, \ldots x_n\}$ is a partition of $[a,b]$ then

$\sum_{r=1}^{n} \delta_r = \delta_1 + \delta_2 + \ldots + \delta_n$

$= (x_1 - x_0) + (x_2 - x_1) + \ldots + (x_n - x_{n-1})$

$= x_n - x_0$

$= b - a$

* Refinement of a partition:-

If P, P' be two partitions of $[a,b]$ and $P \subset P'$ then the partition P' is called a refinement of partition P on $[a,b]$. we also say P' is finer than P.

i.e. If P' is finer than P, then every point of P is used in the construction of P' and P' has atleast one additional point.

→ If P_1, P_2 are two partitions of $[a,b]$ then $P_1 \subset P_1 \cup P_2$ and $P_2 \subset P_1 \cup P_2$.

Therefore $P_1 \cup P_2$ is called a common refinement of P_1 & P_2.

Note: If $P_1, P_2 \in P[a,b]$ and $P_1 \subset P_2$ then $\|P_2\| \leq \|P_1\|$.

* Upper and Lower Darboux Sums:

Let $f : [a,b] \to \mathbb{R}$ be a bounded function and

$P = \{a = x_0, x_1, \ldots x_n = b\}$ be a partition of $[a,b]$.

Since f is bounded on $[a,b]$, f is also bounded on each of the subintervals. (i.e. $I_r = [x_{r-1}, x_r], r = 1,2 \ldots n$)

Let M, m be the Supremum and infimum of f in $[a,b]$ and M_r, m_r be the Supremum and infimum of f in the rth Subintervals.

$I_r = [x_{r-1}, x_r]$; $r = 1,2, \ldots n$.

The Sum $M_1\delta_1 + M_2\delta_2 + \ldots + M_r\delta_r + \ldots + M_n\delta_n = \sum_{r=1}^{n} M_r\delta_r$.

is called the upper Darboux sum of f corresponding to the partition P and is denoted by $U(P,f)$ or $U(f,P)$

→ The sum $m_1\delta_1 + m_2\delta_2 + \cdots + m_r\delta_r + \cdots + m_n\delta_n = \sum_{r=1}^{n} m_r\delta_r$

is called the lower Darboux sum of f corresponding to the partition P and is denoted by $L(P,f)$ or $L(f,P)$.

i.e. $U(P,f) = \sum_{r=1}^{n} M_r\delta_r$

$L(P,f) = \sum_{r=1}^{n} m_r\delta_r$.

* Oscillatory Sum:

Let $f:[a,b]\to\mathbb{R}$ be a bounded function and $P=\{a=x_0, x_1, \ldots x_{n-1}, x_n=b\}$ be a partition of $[a,b]$.

Let m_r and M_r be the infimum and supremum of f on $I_r=[x_{r-1},x_r]$ $r=1,2\ldots n$. Then

$U(P,f) - L(P,f) = \sum_{r=1}^{n} M_r\delta_r - \sum_{r=1}^{n} m_r\delta_r$

$= \sum_{r=1}^{n}(M_r - m_r)\delta_r$

$= \sum_{r=1}^{n} O_r\delta_r$.

where $O_r = M_r - m_r$ denotes the oscillation of f on I_r

$U(P,f) - L(P,f) = \sum_{r=1}^{n} O_r\delta_r$ is called the oscillatory sum of f corresponding to the partition P and is denoted by $\omega(P,f)$.

i.e. $\omega(P,f) = \sum_{r=1}^{n} O_r\delta_r$

→ If $f:[a,b]\to\mathbb{R}$ is bounded function and $P\in P[a,b]$ then

$m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$

where m, M are the infimum and supremum of f on $[a,b]$

Proof: Let $P=\{a=x_0, x_1, \ldots x_n=b\}$ be partition of $[a,b]$.

Since f is bounded on $[a,b]$

$\Rightarrow$ f is bounded on each subinterval of $[a,b]$.

i.e. f is bounded on $I_r=[x_{r-1}, x_r]$, $r=1,2\ldots n$.

Let m_r and M_r be the infimum & Supremum of f on $I_r=[x_{r-1},x_r]$

$\therefore m \le m_r \le M_r \le M$.

$\Rightarrow m\delta_r \le m_r\delta_r \le M_r\delta_r \le M\delta_r$

$\Rightarrow \sum_{r=1}^{n} m\delta_r \le \sum_{r=1}^{n} m_r\delta_r \le \sum_{r=1}^{n} M_r\delta_r \le \sum_{r=1}^{n} M\delta_r$

$\Rightarrow m\sum_{r=1}^{n}\delta_r \le L(P,f) \le U(P,f) \le M\sum_{r=1}^{n}\delta_r$

$\Rightarrow m(b-a) \le L(P,f) \le U(P,f) \le M(b-a)$

$\left[\because \sum_{r=1}^{n}\delta_r = b-a\right]$

Note: The above theorem implies that $L(P,f)$ & $U(P,f)$ are bounded if f is bounded.

* Upper and Lower Riemann Integrals:–

Let $f:[a,b]\to\mathbb{R}$ be a bounded function and $P\in P[a,b]$ then we have

$m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a)$

where

m, M are infimum and supremum of f on $[a,b]$.

For every $P \in P[a,b]$,

we have

$L(P,f) \leq M(b-a)$ and

$U(P,f) \geq m(b-a)$

$\Rightarrow$ the set $\{L(P,f)\}_{P \in P[a,b]}$ of lower sums is bounded above by $M(b-a)$.

$\therefore$ It has the least upper bound. (lub)

The set $\{U(P,f)\}_{P \in P[a,b]}$ of the upper sums is bounded below by $m(b-a)$.

$\therefore$ It has the greatest lower bound (glb)

Now the $\sup\{L(P,f)\}_{P \in P[a,b]}$ is called lower Riemann Integral of f on $[a,b]$ and is denoted by $\underline{\int_a^b} f(x)\,dx$.

i.e. $\underline{\int_a^b} f(x)\,dx = \text{Lub}\{L(P,f)\}_{P \in P[a,b]}$

and the glb $\{U(P,f)\}_{P \in P[a,b]}$ is called upper Riemann Integral of f on $[a,b]$ and is denoted by $\overline{\int_a^b} f(x)\,dx$

i.e. $\overline{\int_a^b} f(x)\,dx = \text{glb}\{U(P,f)\}_{P \in P[a,b]}$

★ Riemann Integral:

A bounded f is said to be Riemann integrable (or R-integrable) on $[a,b]$ if its lower and upper Riemann integrals are equal.

i.e. if $\boxed{\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx}$

— The common value of these integrals is called the Riemann integral of f on $[a,b]$ and is denoted by $\int_a^b f(x)\,dx$.

i.e., $\int_a^b f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$

Note: (1) The interval $[a,b]$ is called the range of the integration. The numbers a and b are called the lower and upper limits of integration respectively.

(2) The family of all bounded functions which are R-integrable on $[a,b]$ is denoted by $R[a,b]$. If f is R-integrable on $[a,b]$ then $f \in R[a,b]$.

(3) ∵ f is R-integrable on $[a,b]$

$\Rightarrow$ (i) f is bounded on $[a,b]$

(ii) $\int_a^b f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx$

(4) A bounded function f on $[a,b]$ is such that

$$\underline{\int_a^b} f(x)\,dx \neq \overline{\int_a^b} f(x)\,dx$$

then f is not R-integrable on $[a,b]$

Problems:

→ Let $f(x) = x \ \forall\, x \in [0,1]$ and let $P = \{0, \frac{1}{3}, \frac{2}{3}, 1\}$ be a partition of $[0,1]$. Compute $U(P,f)$ and $L(P,f)$

Sol^n: Partition set P divides the interval $[0,1]$ into subintervals.

$I_1 = [0, \frac{1}{3}]$, $I_2 = [\frac{1}{3}, \frac{2}{3}]$, $I_3 = [\frac{2}{3}, 1]$

Now $\delta_1 = \frac{1}{3} - 0 = \frac{1}{3}$

$\delta_2 = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$

$\delta_3 = 1 - \frac{2}{3} = \frac{1}{3}$

Since $f(x) = x$ is an increasing function on $[0,1]$.

∴ $M_1 = \frac{1}{3}$, $m_1 = 0$

$M_2 = \frac{2}{3}$, $m_2 = \frac{1}{3}$

$M_3 = 1$, $m_3 = \frac{2}{3}$

∴ $U(P,f) = \sum_{r=1}^{3} M_r \delta_r$

$= M_1\delta_1 + M_2\delta_2 + M_3\delta_3$

$= \frac{1}{3}\cdot\frac{1}{3} + \frac{2}{3}\cdot\frac{1}{3} + 1\cdot\frac{1}{3}$

$= \frac{1}{3}\left(\frac{1}{3} + \frac{2}{3} + 1\right) = \frac{2}{3}$.

Now $L(P,f) = \sum_{r=1}^{3} m_r \delta_r = 0\cdot\frac{1}{3} + \frac{1}{3}\cdot\frac{1}{3} + \frac{2}{3}\cdot\frac{1}{3}$

$= \frac{1}{3}$

→ Compute $L(P,f)$ and $U(P,f)$ for the function f defined by $f(x) = x^2$ on $[0,1]$ and $P = \{0, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, 1\}$.

Sol^n: The partition set P divides $[0,1]$ into subintervals $I_1 = [0, \frac{1}{4}]$, $I_2 = [\frac{1}{4}, \frac{2}{4}]$, $I_3 = [\frac{2}{4}, \frac{3}{4}]$ and $I_4 = [\frac{3}{4}, 1]$

∴ $\delta_1 = \delta_2 = \delta_3 = \delta_4 = \frac{1}{4}$.

Since $f(x) = x^2$ is an increasing on $[0,1]$.

∴ $m_1 = 0$; $M_1 = \frac{1}{16}$

$m_2 = \frac{1}{16}$; $M_2 = \frac{4}{16}$

$m_3 = \frac{4}{16}$; $M_3 = \frac{9}{16}$

$m_4 = \frac{9}{16}$; $M_4 = 1$

$L(P,f) = \sum_{r=1}^{4} m_r \delta$

$= \frac{7}{32}$

and $U(P,f) = \sum_{r=1}^{n} M_r \delta_r = \frac{15}{32}$.

→ If f is defined on $[a,b]$ by $f(x) = K \ \forall\, x \in [a,b]$ where K is constant then $f \in R[a,b]$. and $\int_a^b f(x)\,dx = K(b-a)$. (or)

A constant function is R-integrable.

Sol'n:- Let $P=\{a=x_0, x_1, \dots x_n=b\}$ be any partition of $[a,b]$.

Let $I_r=[x_{r-1}, x_r]$, $r=1,2 \dots n$ be the r^{th} subinterval of $[a,b]$.

Since $f(x)=K$ (Constant).

$\therefore M_r = m_r = K$.

$$U(P,f) = \sum_{r=1}^{n} M_r \delta_r$$

$$= \sum_{r=1}^{n} K(x_r - x_{r-1})$$

$$= K\sum_{r=1}^{n} (x_r - x_{r-1})$$

Remember

$$= K(b-a)$$

and $L(P,f) = \sum_{r=1}^{n} m_r \delta_r$

$$= K(b-a)$$

Now $\underline{\int_a^b} f(x)dx = \text{lub}_{P\in P[a,b]}\{L(P,f)\}$

$$= K(b-a).$$

and $\overline{\int_a^b} f(x)dx = \text{glb}_{P\in P[a,b]}\{U(P,f)\}$

$$= K(b-a).$$

$\therefore \underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = K(b-a)$

$\therefore f \in R[a,b]$

and $\int_a^b f(x)dx = K(b-a)$.

Hence $f(x)=K$ is R-integrable

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Show that the function f defined by $f(x)=\begin{cases} 0 & \text{when } x \text{ is rational} \\ 1 & \text{when } x \text{ is irrational}\end{cases}$

is not Riemann integrable on any interval.

(or)

Show by an example that every bounded function need not be R-integrable.

Sol'n:- Let f be denoted defined on $[a,b]$ by

$$f(x) = \begin{cases} 0 & \text{when } x \text{ is rational} \\ 1 & \text{when } x \text{ is irrational}\end{cases}$$

Clearly $f(x)$ is bounded on $[a,b]$ because $0 \le f(x) \le 1 \;\forall\, x \in [a,b]$.

Let $P=\{a=x_0, x_1, \dots x_n=b\}$ be a partition of $[a,b]$.

Let $I_r=[x_{r-1}, x_r]$; $r=1,2 \dots n$.

be r^{th} subinterval of $[a,b]$.

$\therefore M_r = 1$; $m_r = 0$. {since $f(x)$ has only values for any $x \in [a,b]$ either 0 or 1.}

$$U(P,f) = \sum_{r=1}^{n} M_r \delta_r = \sum_{r=1}^{n} 1.\delta_r$$

$$= b-a$$

and $L(P,f) = \sum_{r=1}^{n} m_r \delta_r$

$$= \sum_{r=1}^{n} 0.\delta_r$$

$$=0$$

Now $\underline{\int_a^b} f(x)\,dx = \text{Lub}_{P\in P[a,b]}\{L(P,f)\}$

$$=0$$

and $\overline{\int_a^b} f(x)\,dx = \text{glb}_{P\in P[a,b]}\{U(P,f)\}$

$$= b-a.$$

$$\therefore \underline{\int_a^b} f(x)\,dx \neq \overline{\int_a^b} f(x)\,dx$$

$\therefore f$ is not Riemann integrable on $[a,b]$.

$\therefore$ Every bounded function need not be a Riemann integrable.

→ Let $f(x)$ be defined on $[0,1]$ as follows.

$$f(x) = \begin{cases} 1 & \text{when } x \text{ is rational} \\ -1 & \text{when } x \text{ is irrational} \end{cases}$$

show that f is not Riemann integrable over $[0,1]$.

Here $U(P,f) = (b-a) = 1$
$L(P,f) = (a-b) = -1$
Hence not R-integrable

Sol^n:- $P = \{0 = x_0, x_1, \ldots x_n = 1\}$ be a Partition of $[0,1]$

Let $I_r = [x_{r-1}, x_r]$, $r = 1, 2, \ldots n$.

→ Evaluate $\int_0^1 dx$ by applying the definition of Riemann-integral.

Sol^n: Let $f(x) = 1 \; \forall x \in [0,1]$.

→ If f is defined on $[0,1]$ by $f(x) = x \; \forall x \in [0,1]$. then $f \in R[0,1]$ and $\int_0^1 f(x)\,dx = 1/2$.

Sol^n: Let $P = \{0, \frac{1}{n}, \frac{2}{n}, \ldots \frac{r-1}{n}, \frac{r}{n}, \ldots \frac{n}{n} = 1\}$ be any partition of $[0,1]$.

Remember

$I_r = [x_{r-1}, x_r] = \left[\frac{r-1}{n}, \frac{r}{n}\right]$

$r = 1, 2, \ldots n$

$\delta_r = x_r - x_{r-1}$; $r = 1, 2, \ldots n$

$= \frac{r}{n} - \left(\frac{r-1}{n}\right)$

$= \cancel{\frac{r}{n}} - \cancel{\frac{r}{n}} + \frac{1}{n}$

$\boxed{\delta_r = 1/n}$

Since $f(x) = x$ is an increasing on $[0,1]$.

$\therefore m_r = \frac{r-1}{n}$; $M_r = \frac{r}{n}$ $\quad (\because f(x) = x)$

$\therefore U(P,f) = \sum_{r=1}^{n} M_r \delta_r$

$= \sum_{r=1}^{n} \frac{r}{n} \cdot \frac{1}{n}$

$= \frac{1}{n^2} \sum_{r=1}^{n} (r)$

$= \frac{1}{n^2}(1 + 2 + \cdots + n)$

$= \frac{1}{n^2}\left[\frac{n(n+1)}{2}\right]$ Remember

$= \frac{(n+1)}{n(2)}$

$= \frac{1 + 1/n}{2} = \frac{1}{2}\left[1 + \frac{1}{n}\right]$

$L(P,f) = \sum_{r=1}^{n} m_r \delta_r$

$= \sum_{r=1}^{n} \left(\frac{r-1}{n}\right) \cdot \frac{1}{n}$

$= \frac{1}{n^2} \sum_{r=1}^{n} (r-1)$

$= \frac{1}{n^2}\left[0 + 1 + 2 + \cdots + (n-1)\right]$

$= \frac{1}{n^2}\left[(1 + 2 + \cdots + n-1) + n - n\right]$ Note!

$= \frac{1}{n^2}\left[\frac{n(n+1)}{2} - n\right]$

$= \left[\frac{1}{n^2} \cdot \frac{n^2(1 + 1/n)}{2} - \frac{1}{n}\right]$

$= \frac{1}{2}\left(1 + \frac{1}{n}\right) - \frac{1}{n}$

Note! [Here we don't use glb]

Now $\int_0^1 f(x)\,dx = \underline{\text{Lub}}\{L(P,f)\}$ = lub of lower sum
$P \in P[0,1]$

Remember

$= \lim_{n\to\infty}\left[\frac{1}{2}\left(1+\frac{1}{n}\right)-\frac{1}{n}\right]$

$= \frac{1}{2}(1+0)-0$

$= \frac{1}{2}$

and $\overline{\int_0^1} f(x)\,dx = \text{glb}\{U(P,f)\}$
$P \in P[0,1]$

$= \lim_{n\to\infty}\left[\frac{1}{2}\left(1+\frac{1}{n}\right)\right]$

$= \frac{1}{2}(1+0)$

$= \frac{1}{2}$.

$\therefore \underline{\int_0^1} f(x)\,dx = \overline{\int_0^1} f(x)\,dx = \frac{1}{2}$.

$\therefore f \in R[0,1]$ and $\int_0^1 f(x)\,dx = \frac{1}{2}$

→ If f is defined on $[0,a]$; $a>0$ by $f(x)=x^2$ $\forall x \in [0,a]$, then $f \in R[0,a]$ and $\int_0^a f(x)\,dx = \frac{a^3}{3}$.

Sol^n: Let $P = \left\{0, \frac{a}{n}, \frac{2a}{n}, \frac{3a}{n}, \dots \right.$

$\left. \dots \left(\frac{r-1}{n}\right)a, \left(\frac{r}{n}\right)a, \dots \frac{na}{n} = a\right\}$

be any partition of $[0,a]$.

Let $I_r = \left[\frac{(r-1)a}{n}, \frac{ra}{n}\right]$; $r = 1,2,3,\dots n$

$\delta_r = \cancel{\frac{ra}{n}} - \cancel{\frac{ra}{n}} + \frac{a}{n}$

$\boxed{\delta_r = \frac{a}{n}}$

Since $f(x) = x^2$ is an increasing function on $[0,a]$; $a>0$.

$\therefore M_r = \left(\frac{ra}{n}\right)^2$; $m_r = \left[\frac{(r-1)a}{n}\right]^2$

$= \frac{r^2a^2}{n^2}$; $= \frac{(r-1)^2a^2}{n^2}$

$\{\because f(x) = x^2\}$

$\therefore U(P,f) = \sum_{r=1}^{n} M_r \delta_r$

$= \sum_{r=1}^{n} \frac{r^2a^2}{n^2}\cdot\frac{a}{n}$

$= \frac{a^3}{n^3}\sum_{r=1}^{n} r^2$

$= \frac{a^3}{n^3}\left[\frac{n(n+1)(2n+1)}{6}\right]$ Remember

$= \frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)a^3$

$L(P,f) = \sum_{r=1}^{n} m_r\delta_r$

$= \sum_{r=1}^{n} \frac{(r-1)^2a^2}{n^2}\cdot\frac{a}{n}$

$= \frac{a^3}{n^3}\sum_{r=1}^{n}(r-1)^2$

$= \frac{a^3}{n^3}\left[0+1^2+2^2+\dots+(n-1)^2\right]$

$= \frac{a^3}{n^3}\left[1^2+2^2+3^2+\dots+(n-1)^2+n^2-n^2\right]$

$= \frac{a^3}{n^3}\left[\frac{n(n+1)(2n+1)}{6} - n^2\right]$

$= \frac{a^3}{6}\left[\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)-\frac{1}{n}\right]$

Now $\underline{\int_0^a} f(x)\,dx = \text{Lub}\{L(P,f)\}$
$P \in P[0,a]$

$= \lim_{n\to\infty}\left[\frac{a^3}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)-\frac{1}{n}\right]$

$= \frac{a^3}{6}(1)(2)-0$

$= \frac{a^3}{3}$

and $\overline{\int_0^a} f(x)\,dx = \text{glb}\{U(P,f)\}$
$P \in P[0,a]$

$$= \lim_{n\to\infty} \left[\frac{a^3}{6}\left(1+\tfrac{1}{n}\right)\left(2+\tfrac{1}{n}\right)\right]$$

$$= \frac{a^3}{6}(1+0)(2+0)$$

$$= \frac{a^3}{6}(2) = \frac{a^3}{3}$$

$\therefore \underline{\int_0^a} f(x)\,dx = \overline{\int_0^a} f(x)\,dx = \frac{a^3}{3}$.

$\therefore f \in R[0,a]$ and $\int_0^a f(x)\,dx = \frac{a^3}{3}$.

→ If f is defined on $[0,a]$, $a>0$ by $f(x) = x^3 \ \forall\, x \in [0,a]$ then $f \in R[0,a]$ and $\int_0^a f(x)\,dx = \frac{a^4}{4}$.

Here $\sum_{r=1}^{n} (r^3) = \left[\frac{n(n+1)}{2}\right]^2$

→ Show that $f(x) = 3x+1$ is Riemann integrable on $[0,1]$ and $\int_0^1 (3x+1)\,dx = 5/2$.

Sol'n:- Let $f(x) = 3x+1 \ \forall\, x \in [0,1]$

Then f is bounded on $[0,1]$.

Let $P = \left\{0, \frac{1}{n}, \frac{2}{n}, \frac{3}{n}, \dots \dots \frac{(r-1)}{n}, \frac{r}{n}, \dots \frac{n}{n} = 1\right\}$.

$I_r = [x_{r-1}, x_r] = \left[\frac{r-1}{n}, \frac{r}{n}\right]$; $r = 1, 2 \dots n$

$\therefore \delta_r = \frac{r}{n} - \left(\frac{r-1}{n}\right)$

$= \cancel{\frac{r}{n}} - \cancel{\frac{r}{n}} + \frac{1}{n} = 1/n$

Since $f(x) = 3x+1$ is an increasing function on $[0,1]$.

$\therefore M_r = 3\left(\frac{r}{n}\right)+1$; $m_r = 3\left(\frac{r-1}{n}\right)+1$

$\{\because f(x) = 3x+1\}$

Now $U(P,f) = \sum_{r=1}^{n} M_r \delta_r$

$= \sum_{r=1}^{n} \left[3\left(\frac{r}{n}\right)+1\right]\frac{1}{n}$

$= \frac{1}{n}\sum_{r=1}^{n}\left[\frac{3r}{n}+1\right]$

$= \frac{1}{n}\left[\frac{3}{n}\sum_{r=1}^{n} r + \sum_{r=1}^{n}(1)\right]$

$= \frac{1}{n}\left[\frac{3}{n}(1+2+\dots+n) + (1+1+\dots+1)\right]$

$= \frac{1}{n}\left[\frac{3}{\cancel{n}}\left(\frac{\cancel{n}(n+1)}{2}\right) + n(1)\right]$

$= \frac{1}{n}\left[\frac{3}{2}(n+1) + n\right]$

$= \frac{3}{2}\left(1+\tfrac{1}{n}\right)+1$

$L(P,f) = \sum_{r=1}^{n} m_r \delta_r$

$= \sum_{r=1}^{n}\left[3\left(\frac{r-1}{n}\right)+1\right]\frac{1}{n}$

$= \frac{1}{n}\left[3\sum_{r=1}^{n}\frac{(r-1)}{n} + \sum_{r=1}^{n}(1)\right]$

$= \frac{1}{n}\left[\frac{3}{n}\sum_{r=1}^{n}(r-1) + \sum_{r=1}^{n}(1)\right]$.

$= \frac{1}{n}\left[\frac{3}{n}(0+1+2+\dots+(n-1)+n-n) + (1+1+\dots+1)\right]$

$= \frac{1}{n}\left[\frac{3}{n}(1+2+\dots+n) - 3 + n(1)\right]$

$= \frac{1}{n}\left[\frac{3}{\cancel{n}}\,\frac{\cancel{n}(n+1)}{2} - 3 + n\right]$

$= \frac{3}{2}\left(1+\frac{1}{n}\right) - \frac{3}{n} + 1$

Now $\int_0^1 f(x)\,dx = \text{lub}\{L(P,f)\}_{P \in P[0,1]}$

$= \lim_{n\to\infty}\left[\frac{3}{2}\left(1+\tfrac{1}{n}\right) - \tfrac{3}{n} + 1\right]$

$= 3/2 + 1 = 5/2$.

and $\int_0^1 f(x)\,dx = glb\{U(P,f)\}_{P\in P[0,1]}$

$= \lim_{n\to\infty}\left[\frac{3}{2}\left(1+\frac{1}{n}\right)+1\right]$

$= \frac{3}{2}+1 = \frac{5}{2}$

$\therefore \underline{\int_0^1} f(x)\,dx = \overline{\int_0^1} f(x)\,dx = \frac{5}{2}$

$\therefore f\in R[0,1]$ and $\int_0^1 (3x+2)\,dx = \frac{5}{2}$

→ show that $f(x) = 2x+1$ is integrable on $[1,2]$ and $\int_1^2 (2x+1)\,dx = 4$

Sol'n: Let $f(x) = 2x+1\ \forall\ x\in[1,2]$

then $f(x)$ is bounded on $[1,2]$

Let $P = \{1, 1+\frac{1}{n}, 1+\frac{2}{n}, 1+\frac{3}{n}, \dots$

Note!

$\dots 1+\frac{r-1}{n}, 1+\frac{r}{n}, \dots 1+\frac{n}{n} = 2\}$

be any partition of $[1,2]$.

Let $I_r = \left[1+\frac{r-1}{n}, 1+\frac{r}{n}\right]$

$r = 1, 2, \dots, n.$

→ Prove that $\int_1^2 f(x)\,dx = 7$

where $f(x) = 2x+4$.

→ Prove that $f(x) = 3x+1$ integrable on $[1,2]$ and $\int_1^2 (3x+1)\,dx = \frac{11}{2}$.

→ show that $f(x) = 2-3x$ is integrable on $[1,3]$ and $\int_1^3 (2-3x)\,dx = -8$.

→ Show that $f(x) = x$ is integrable on $[a,b]$ and $\int_a^b f(x)\,dx = \frac{1}{2}(b^2-a^2)$.

Sol'n: Let $f(x) = x\ \forall\ x\in[a,b]$ then $f(x)$ is bounded on $[a,b]$.

Let the partition $P = \{a = a, a+\frac{h}{n}, a+\frac{2h}{n}, \dots a+\frac{nh}{n}\}$

Note

where $h = b-a$; be dividing the interval $[a,b]$ into 'n' equal parts.

Let $I_r = \left[a+\frac{(r-1)h}{n}, a+\frac{rh}{n}\right]$,

$r = 1, 2, 3 \dots$

→ Let f be defined on $[0,1]$ by

$$f(x) = \begin{cases} \frac{1}{2} & \text{when } x\in\mathbb{Q} \\ \frac{1}{3} & \text{when } x\in\mathbb{R}-\mathbb{Q} \end{cases}$$

Then show that f is bounded but not Riemann integrable on $[0,1]$.

→ A function f is bounded on $[a,b]$. show that

(i) when k is a +ve constant:

$\overline{\int_a^b} kf\,dx = k\overline{\int_a^b} f\,dx$ and $\underline{\int_a^b} kf\,dx = k\underline{\int_a^b} f\,dx$

and

(ii) when k is a −ve constant:

$\overline{\int_a^b} kf\,dx = k\underline{\int_a^b} f\,dx$ and $\underline{\int_a^b} kf\,dx = k\overline{\int_a^b} f\,dx$

Also deduce that if f is integrable on $[a,b]$, then kf is also Riemann integrable where k is a constant.

and $\int_a^b kf\,dx = k\int_a^b f\,dx$.

Sol'n:- Let f be a bounded function on $[a,b]$ and Let $P = \{a = x_0, x_1, x_2, \dots$

$\dots x_{r-1}, x_r, \dots x_n = b\}$ be any partition of $[a,b]$.

Let $I_r = [x_{r-1}, x_r]$; $r = 1,2,\dots n$

Let m_r and M_r be the infimum and Supremum of f on $I_r = [x_{r-1}, x_r]$ $r = 1,2,\dots n$

Let m_r' & M_r' be the infimum and Supremum of Kf on

$I_r = [x_{r-1}, x_r]$; $r = 1,2 \dots n$.

(i) when K is +ve constant:

$m_r' = Km_r$ and $M_r' = KM_r$

$$U(P, Kf) = \sum_{r=1}^{n} M_r' \delta_r$$

$$= \sum_{r=1}^{n} KM_r \delta_r$$

$$= K \sum_{r=1}^{n} M_r \delta_r = KU(P,f)$$

Similarly $L(P, Kf) = KL(P,f)$.

$$\therefore \overline{\int_a^b} Kf\,dx = \text{glb}\{U(P,Kf)\}_{P\in P[a,b]}$$

$$= \text{glb}\{KU(P,f)\}_{P\in P[a,b]}$$

$$= K\,\text{glb}\{U(P,f)\}_{P\in P[a,b]}$$

$$= K \overline{\int_a^b} f(x)\,dx.$$

and
$$\underline{\int_a^b} Kf\,dx = \text{Lub}\{L(P,Kf)\}_{P\in P[a,b]}$$

$$= \text{Lub}\{KL(P,f)\}_{P\in P[a,b]}$$

$$= K\,\text{Lub}\{L(P,f)\}_{P\in P[a,b]}$$

$$= K \underline{\int_a^b} f(x)\,dx.$$

(ii) when K is −ve constant:

Note! $\{\because K < 0\}$

$m_r' = KM_r$ and $M_r' = Km_r$

$$\therefore U(P, Kf) = \sum_{r=1}^{n} M_r' \delta_r$$

$$= \sum_{r=1}^{n} Km_r \delta_r$$

$$= K \sum_{r=1}^{n} m_r \delta_r$$

$$= K\,L(P,f)$$

Similarly $L(P, Kf) = KU(P,f)$

$$\therefore \overline{\int_a^b} Kf\,dx = \text{glb}\{U(P,Kf)\}_{P\in P[a,b]}$$

$$= \text{glb}\{KL(P,f)\}_{P\in P[a,b]}$$

$$= K\,\text{lub}\{L(P,f)\}_{P\in P[a,b]}$$

$$= K \underline{\int_a^b} f\,dx.$$

Similarly $\underline{\int_a^b} Kf\,dx = K \overline{\int_a^b} f\,dx$

(iii) If f is integrable on $[a,b]$ then

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx.$$

$\therefore$ from parts (i) & (ii) we have

$$\underline{\int_a^b} Kf\,dx = \overline{\int_a^b} Kf\,dx = K \int_a^b f\,dx.$$

$\Rightarrow$ Kf is Riemann-Integrable on $[a,b]$ and $\int_a^b Kf\,dx = K \int_a^b f\,dx$.

→ Let $f(x) = \sin x \ \forall\, x \in [0, \pi/2]$

Let $P = \{0, \pi/2n, 2\pi/2n, \dots \frac{n\pi}{2n}\}$ Note! be a partition of $[0, \pi/2]$. Compute $U(P,f)$ and $L(P,f)$. Hence Prove that $f \in R[0, \pi/2]$.

Sol^n:- Let $f(x) = \sin x \ \forall x \in [0, \pi/2]$

then f is bounded on $[0, \pi/2]$

Let $P = \left\{0, \frac{\pi}{2n}, \frac{2\pi}{2n}, \frac{3\pi}{2n}, \cdots \frac{n\pi}{2n} = \frac{\pi}{2}\right\}$

be a partition of $[0, \pi/2]$.

$I_r = \left[\frac{(r-1)\pi}{2n}, \frac{r\pi}{2n}\right]$; $r = 1, 2, 3 \cdots$

Since f is increasing function on $[0, \pi/2]$

$\therefore m_r = \sin\left(\frac{(r-1)\pi}{2n}\right)$ and

$M_r = \sin\left(\frac{r\pi}{2n}\right)$

and $\delta_r = \frac{r\pi}{2n} - \frac{(r-1)\pi}{2n}$

$= \frac{\pi}{2n}$

Now $U(P,f) = \sum_{r=1}^{n} M_r \delta_r$

$= \sum_{r=1}^{n} \sin\left(\frac{r\pi}{2n}\right) \cdot \frac{\pi}{2n}$

$= \frac{\pi}{2n}\left[\sin\left(\frac{\pi}{2n}\right) + \sin\left(\frac{2\pi}{2n}\right) + \sin\left(\frac{3\pi}{2n}\right) + \cdots + \sin\left(\frac{n\pi}{2n}\right)\right]$

$= \frac{\pi}{2n} \cdot \frac{\sin\left[\frac{\pi}{2n} + \frac{n-1}{2} \cdot \frac{\pi}{2n}\right] \sin\left(\frac{n}{2} \cdot \frac{\pi}{2n}\right)}{\sin\left(\frac{1}{2} \cdot \frac{\pi}{2n}\right)}$

$\because \sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \cdots + \sin(\alpha + \overline{n-1}\beta) = \frac{\sin\left[\alpha + \frac{n-1}{2}\beta\right] \sin\frac{n\beta}{2}}{\sin \beta/2}$

[Here $\alpha = \frac{\pi}{2n}$, $\beta = \frac{\pi}{2n}$] $\therefore \alpha + \beta = \frac{2\pi}{2n}$

$= \frac{\pi}{2n} \cdot \frac{\sin\frac{(n+1)\pi}{4n} \cdot \sin\frac{\pi}{4}}{\sin(\pi/4n)}$

$= \frac{\pi}{2n} \cdot \frac{1}{\sqrt{2}} \cdot \frac{\sin\frac{(n+1)\pi}{4n}}{\sin\frac{\pi}{4n}}$

$= \frac{\pi}{2n\sqrt{2}} \cdot \frac{\sin\left(\frac{\pi}{4} + \frac{\pi}{4n}\right)}{\sin(\pi/4n)}$

$= \frac{\pi}{2n\sqrt{2}} \cdot \frac{\sin\left(\frac{\pi}{4}\right)\cos\left(\frac{\pi}{4n}\right) + \cos\left(\frac{\pi}{4}\right)\sin\left(\frac{\pi}{4n}\right)}{\sin\left(\frac{\pi}{4n}\right)}$

$= \frac{\pi}{2n\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \cdot \frac{\left[\cos\left(\frac{\pi}{4n}\right) + \sin\left(\frac{\pi}{4n}\right)\right]}{\sin\left(\frac{\pi}{4n}\right)}$

$= \frac{\pi}{4n}\left[\cot\left(\frac{\pi}{4n}\right) + 1\right]$

$L(P,f) = \sum_{r=1}^{n} m_r \delta_r$

$= \sum_{r=1}^{n} \sin\frac{(r-1)\pi}{2n} \cdot \frac{\pi}{2n}$

$= \frac{\pi}{2n}\left[0 + \sin\frac{\pi}{2n} + \sin\frac{2\pi}{2n} + \sin\frac{3\pi}{2n} + \cdots + \sin\frac{(n-1)\pi}{2n}\right]$

$= \frac{\pi}{2n}\left[\sin\frac{\pi}{2n} + \sin\left(\frac{\pi}{2n} + \frac{\pi}{2n}\right) + \sin\left(\frac{\pi}{2n} + \frac{2\pi}{2n}\right) + \cdots + \sin\left(\frac{\pi}{2n} + \frac{(n-2)\pi}{2n}\right)\right]$

$= \frac{\pi}{2n} \cdot \frac{\left[\sin\frac{\pi}{2n} + \frac{(n-2)}{2} \cdot \frac{\pi}{2n}\right] \sin\left(\frac{n-1}{2} \cdot \frac{\pi}{2n}\right)}{\sin\left(\frac{1}{2} \cdot \frac{\pi}{2n}\right)}$

$= \frac{\pi}{2n} \cdot \frac{\sin\left[\cancel{\frac{\pi}{2n}} - \cancel{\frac{\pi}{2n}} + \frac{n\pi}{4n}\right] \sin\left[\frac{\pi}{4} - \frac{\pi}{4n}\right]}{\sin\left(\frac{\pi}{4n}\right)}$

$= \frac{\pi}{2n} \cdot \frac{\sin\frac{\pi}{4} \sin\left(\frac{\pi}{4} - \frac{\pi}{4n}\right)}{\sin\left(\frac{\pi}{4n}\right)}$

$= \frac{\pi}{2n} \cdot \frac{1}{\sqrt{2}} \cdot \frac{\left[\sin\frac{\pi}{4}\cos\frac{\pi}{4n} - \cos\frac{\pi}{4}\sin\frac{\pi}{4n}\right]}{\sin\left(\frac{\pi}{4n}\right)}$

$= \frac{\pi}{2n} \cdot \frac{1}{\sqrt{2}}\left[\frac{\cos\frac{\pi}{4n} - \sin\frac{\pi}{4n}}{\sin\left(\frac{\pi}{4n}\right)}\right] \frac{1}{\sqrt{2}}.$

$$= \frac{\pi}{2n}\cdot\frac{1}{2}\left[\cot\left(\frac{\pi}{4n}\right)-1\right]$$

Now $\int_0^{\pi/2} f(x)\,dx = \sup\{L(P,f)\}\ P\in P[0,\frac{\pi}{2}]$

$$= \lim_{n\to\infty}\frac{\pi}{4n}\left[\cot\left(\frac{\pi}{4n}\right)-1\right]$$

$$= \lim_{\frac{\pi}{4n}\to 0}\frac{\pi/4n}{\tan(\pi/4n)} - \lim_{n\to\infty}\frac{\pi}{4n}$$

$$= 1-0$$

$$= 1$$

and $\int_0^{\pi/2} f(x)\,dx = \inf\{U(P,f)\}\ P\in P[0,\pi/2]$

$$= \lim_{n\to\infty}\frac{\pi}{4n}\left[\cot\frac{\pi}{4n}-1\right]$$

$$= \lim_{\frac{\pi}{4n}\to 0}\frac{\pi/4n}{\tan(\pi/4n)} + \lim_{n\to\infty}(\pi/4n)$$

$$= 1+0$$

$$= 1$$

$\therefore \int_0^{\pi/2} f(x)\,dx = \int_0^{\pi/2} f(x)\,dx = 1$

$\therefore f\in R[0,\pi/2]$ and $\int_0^{\pi/2} f(x)\,dx = 1$

→ If f be a function defined on $[0,\pi/4]$ by $f(x)=\begin{cases}\cos x & \text{if } x \text{ is rational}\\ \sin x & \text{if } x \text{ is irrational}\end{cases}$

then $f\notin R[0,\pi/4]$.

Solⁿ:- Let $f=\left\{0,\frac{\pi}{4n},\frac{2\pi}{4n},\frac{3\pi}{4n},\ldots\frac{(r-1)\pi}{4n},\frac{r\pi}{4n}\ldots\frac{n\pi}{4n}=\frac{\pi}{4}\right\}$

be a partition of $[0,\pi/4]$.

$$I_r = \left[\frac{(r-1)\pi}{4n},\frac{r\pi}{4n}\right];\ r=1,2,\ldots,n$$

Since $\cos x \geq \sin x$

i.e. $\sin x \leq \cos x$ in $[0,\pi/4]$

$$\therefore m_r = \sin\left(\frac{(r-1)\pi}{4n}\right)$$

$$M_r = \cos\left(\frac{(r-1)\pi}{4n}\right)$$

and $\delta_r = \frac{\pi}{4n}$

Now $U(P,f) = \sum_{r=1}^{n} M_r\delta_r$

$$= \sum_{r=1}^{n}\cos\frac{(r-1)\pi}{4n}\,\frac{\pi}{4n}$$

$$= \frac{\pi}{4n}\sum_{r=1}^{n}\cos\frac{(r-1)\pi}{4n}$$

$$= \frac{\pi}{4n}\left[\cos 0+\cos\frac{\pi}{4n}+\cos\frac{2\pi}{4n}+\cdots+\cos\frac{(n-1)\pi}{4n}\right]$$

$$= \frac{\pi}{4n}\left[\cos 0+\cos\left(0+\frac{\pi}{4n}\right)+\cos\left(0+\frac{2\pi}{4n}\right)+\cdots+\cos\left(0+\frac{(n-1)\pi}{4n}\right)\right]$$

$$= \frac{\pi}{4n}\left[\frac{\cos\left(0+\frac{n-1}{2}\cdot\frac{\pi}{4n}\right)\cdot\sin\left(\frac{n}{2}\cdot\frac{\pi}{4n}\right)}{\sin\left(\frac{1}{2}\cdot\pi/4n\right)}\right]$$

$$= \frac{\pi}{4n}\,\frac{\left[\cos\frac{(n-1)\pi}{8n}\cdot\sin\left(\frac{\pi}{8}\right)\right]}{\sin\left(\frac{\pi}{8n}\right)}$$

$$= \frac{\pi}{4n}\cdot\frac{1}{\sqrt{2}}\,\frac{\left[\cos\frac{(n-1)\pi}{8n}\cdot\sin\frac{\pi}{8}\right]}{\sin(\pi/8n)}$$

Similarly $L(P,f) = \frac{\pi}{4n}\cdot\frac{\sin\left[\frac{(n-1)\pi}{8n}\right]\cdot\sin\frac{\pi}{8}}{\sin(\pi/8n)}$

Now

$$\underline{\int_0^{\pi/4}} f(x)\,dx = \text{lub}\{L(P,f)\},\ P\in P[0,\pi/4]$$

$$= \lim_{n\to\infty}\left[\frac{\pi}{4n}\cdot\frac{\sin\left[\frac{(n-1)\pi}{8n}\right]\sin\left(\frac{\pi}{8}\right)}{\sin\left(\frac{\pi}{8n}\right)}\right]$$

$$= \lim_{n\to\infty}\left[\frac{(\pi/8n)}{\sin(\pi/8n)}\,2\sin\left(\frac{\pi}{8}-\frac{\pi}{8n}\right)\sin\frac{\pi}{8}\right]$$

$$= 1\times 2\sin(\pi/8)\sin(\pi/8)$$

$$= 2\sin^2(\pi/8)$$

$$= 1-\cos\pi/4 = 1-\frac{1}{\sqrt{2}}$$

and

$$\overline{\int_0^{\pi/4}} f(x)\,dx = \text{glb}\{U(P,f)\},\ P\in P[0,\tfrac{\pi}{4}]$$

$$= \lim_{n\to\infty}\left[\frac{\pi}{4n}\cdot\frac{\cos\frac{(n-1)\pi}{8n}\cdot\sin\frac{\pi}{8}}{\sin(\pi/8n)}\right]$$

$$= \lim_{n\to\infty}\left[\frac{\frac{\pi}{8n}}{\sin\frac{\pi}{8n}}\,2\cos\left(\frac{\pi}{8}-\frac{\pi}{8n}\right)\sin\frac{\pi}{8}\right]$$

$$= 2\cos\frac{\pi}{8}\sin\frac{\pi}{8}$$

$$= \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$$

$$\therefore \underline{\int_0^{\pi/4}} f(x)\,dx \ne \overline{\int_0^{\pi/4}} f(x)\,dx.$$

$$\therefore f\notin R[0,\pi/4]$$

* Some Theorem statements:-

→ If $f:[a,b]\to\mathbb{R}$ is a bounded function and $P\in P[a,b]$ then

(i) $L(P,f)\le U(P,f)$

(ii) $L(P,-f) = -U(P,f)$

(iii) $U(P,-f) = -L(P,f)$

→ If $f:[a,b]\to\mathbb{R}$ and $g:[a,b]\to\mathbb{R}$ are bounded functions and $P\in P[a,b]$ then

(i) $U(P,f+g)\le U(P,f)+U(P,g)$

(ii) $L(P,f+g)\ge L(P,f)+L(P,g)$

(iii) $\omega(P,f+g)\le \omega(P,f)+\omega(P,g)$.

→ If $f:[a,b]\to\mathbb{R}$ is a bounded function then

$$\underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx$$

i.e. Lower Riemann Integral cannot exceed upper Riemann integral.

→ If $f:[a,b]\to\mathbb{R}$ is a bounded function, then

$$m(b-a)\le \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx \le M(b-a)$$

where m & M are the infimum and supremum of f on $[a,b]$.

→ If $f\in R[a,b]$ then

(i) $m(b-a)\le \int_a^b f(x)\,dx \le M(b-a)$ if $b\ge a$

(ii) $m(b-a)\ge \int_a^b f(x)\,dx \ge M(b-a)$ if $b\le a$.

ST
where m and M are the infimum and supremum of f on $[a,b]$

Definition:

The meaning of $\int_a^b f(x)\,dx$ when $b<a$

If f is bounded and integrable on $[b,a]$ for $a>b$ i.e. $b<a$.

we define

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \text{ when } a>b$$

Also $\int_a^b f(x)\,dx = 0$ when $a=b$.

Darboux Theorem:-

If $f:[a,b]\to\mathbb{R}$ is a bounded function then for each $\epsilon>0$, there exists a $\delta>0$ such that

(i) $U(P,f) < \int_a^b f(x)\,dx + \epsilon$

(ii) $L(P,f) > \int_a^b f(x)\,dx - \epsilon$

for each $P\in P[a,b]$ with $\|P\|<\delta$.

A bounded function f is integrable on $[a,b]$ iff for each $\epsilon>0$, $\exists$ a paritition P of $[a,b]$ such that $U(P,f)-L(P,f)<\epsilon$.

Imp: If $f:[a,b]\to\mathbb{R}$ is continuous function on $[a,b]$ then f is integrable on $[a,b]$.

Imp: If $f:[a,b]\to\mathbb{R}$ is monotonic on $[a,b]$ then f is integrable on $[a,b]$.

Imp: If the set of points of discontinuity of bounded function $f:[a,b]\to\mathbb{R}$ is finite, then f is integrable on $[a,b]$.

Imp: If the set of points of discontinuity of a bounded function $f:[a,b]\to\mathbb{R}$ has a finite number of limit points then f is integrable on $[a,b]$

Riemann Sum:

Let f be a real valued function defined on $[a,b]$

Let $P=\{a=x_0, x_1, \ldots, x_n=b\}$ be a partition of $[a,b]$.

Let $\xi_r\in[x_{r-1}, x_r]$, $r=1,2,\ldots,n$

Then sum $\sum_{r=1}^{n} f(\xi_r)\,\delta_r$ is called a Riemann sum of f on $[a,b]$ relative to P.

It is denoted by $S(f,P)$ or $S(P,f)$

i.e. $S(P,f) = \sum_{r=1}^{n} f(\xi_r)\,\delta_r$ —— (1)

Note: (i)

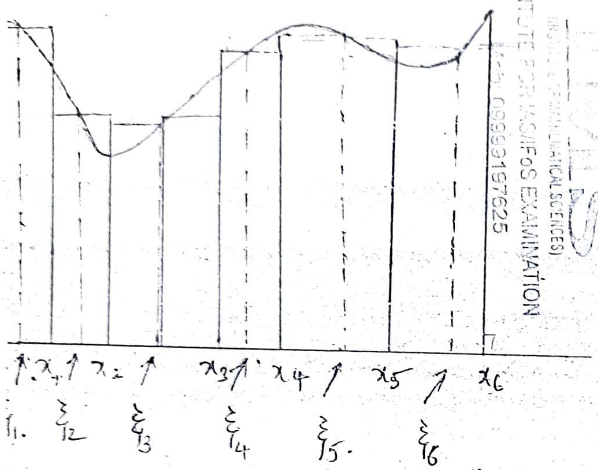
Since ξ_r is any arbitrary point of $[x_{r-1}, x_r]$, therefore corresponding to each partition P of $[a,b]$, there exist infinitely many Riemann sums.

Note (ii)

If the function f is +ve on $[a,b]$, then the Riemann sum (1) is the

Sum of the areas of 'n' rectangles whose bases are the subintervals $I_r = [x_{r-1}, x_r]$ and whose heights are $f(\xi_r)$.

A Riemann Sum



Integral as the limit of a Sum (Second definition of Riemann-Integral):-

Note: Earlier, we arrived at the integral of a function via the upper and the lower sums. The number M_r, m_r which appear in these sums are not necessarily the values of the function f (they are values of 'f' if 'f' is continuous) we shall now show that $\int f\,dx$ can also be considered as the limit of a sequence of sums in which M_r and m_r are replaced by the values of 'f'.

Corresponding to a partition P of $[a,b]$, let us choose points $\xi_1, \xi_2, \xi_3 \dots \xi_n$ such that $x_{r-1} \le \xi_r \le x_r$ $(r = 1, 2 \dots n)$ and

Consider the sum $S(P,f) = \sum_{r=1}^{n} f(\xi_r)\,\delta_r$

[The sum $S(P,f)$ is called Riemann sum of 'f' on $[a,b]$ relative to P.]

Definition:

We say that $S(P,f)$ converges to L as $\|P\| \to 0$. i.e. $\underset{\|P\|\to 0}{Lt} S(P,f) = L$

i.e. if for each $\epsilon > 0$, $\exists$ a $\delta > 0$ such that $|S(P,f) - L| < \epsilon$.

For every Partition

$P = \{a = x_0, x_1, \dots x_n = b\}$ with $\|P\| < \delta$ and for every choice of points $\xi_r \in [x_{r-1}, x_r]$

(or)

A function f is said to be integrable on $[a,b]$ if $\underset{\|P\|\to 0}{Lt} S(P,f)$ exists and

$$\underset{\|P\|\to 0}{Lt} S(P,f) = \int_a^b f(x)\,dx.$$

Note(1):

Since $\|P\| \to 0$ when $n \to \infty$ therefore $\underset{\|P\|\to 0}{Lt}$ can be replaced by $\underset{n\to\infty}{Lt}$ in the above definition.

Note (2):

$f \in R[a,b] \Rightarrow \lim S(P,f)$ exists.

Corollary (1): If f is integrable on $[a,b]$, then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{r=1}^{n} h f(a+rh)$$

where $h = \frac{b-a}{n}$.

Let $P = \{a, a+h, a+2h, \ldots, a+nh=b\}$ be a partition of $[a,b]$.

It divides the interval $[a,b]$ into n equal subintervals, each of length $h = \frac{b-a}{n}$

$\therefore \|P\| = \frac{b-a}{n}$

As $\|P\| \to 0$, $n \to \infty$

$I_r = [a+(r-1)h, a+rh]$

Let $\xi_r \in I_r$ such that

$a+(r-1)h \le \xi_r \le a+rh$; $r = 1,2,\ldots n$

Then $\int_a^b f(x)dx = \lim_{\|P\|\to 0} S(P,f)$

$$= \lim_{\|P\|\to 0} \sum_{r=1}^{n} f(\xi_r)\delta_r$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} f(a+rh)h \quad (\text{Taking } \xi_r = a+rh)$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} hf(a+rh)$$

Corollary-2:

If f is integrable on $[0,1]$, then

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} f\left(\frac{r}{n}\right)$$

Sol'n: Let $P = \left\{0, \frac{1}{n}, \frac{2}{n}, \ldots \frac{r-1}{n}, \frac{r}{n}, \ldots \frac{n}{n}=1\right\}$ be a partition of $[0,1]$.

It divides $[0,1]$ into 'n' equal subintervals, each of length $1/n$

$\therefore \|P\| = 1/n$.

As $\|P\| \to 0$, $n \to \infty$

$I_r = \left[\frac{r-1}{n}, \frac{r}{n}\right]$; $r = 1,2,\ldots n$

$\therefore \delta_r = 1/n$.

Let $\xi_r \in I_r$ such that

$\frac{r-1}{n} \le \xi_r \le \frac{r}{n}$, $r = 1,2,\ldots n$

then $\int_0^1 f(x)dx = \lim_{\|P\|\to 0} \sum_{r=1}^{n} f(\xi_r)\delta_r$ Remember

$$= \lim_{n\to\infty} \sum_{r=1}^{n} f\left(\frac{r}{n}\right)\frac{1}{n} \quad \left(\text{Taking } \xi_r = \frac{r}{n}\right)$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} f\left(\frac{r}{n}\right).$$

Note!

To evaluate the limit of a sum.

(i) write the limit of sum in the form

$$\lim_{n\to\infty} \sum_{r=1}^{n} f\left(\frac{r}{n}\right)\frac{1}{n}$$

(ii) replace $\frac{r}{n}$ by x and $\frac{1}{n}$ by dx.

(iii) replace $\lim_{n\to\infty} \sum_{r=1}^{n}$ by $\int_0^1$.

Note that the limits of integration are the values of $\frac{r}{n}$ for the first and last terms (as $n \to \infty$).

$$\therefore \lim_{n\to\infty} \sum_{r=1}^{n} f\left(\frac{r}{n}\right)\left(\frac{1}{n}\right) = \int_0^1 f(x)dx.$$

Corollary (3):

If f is integrable on $[a,b]$,

then $\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{r=1}^{n} (ah^r - ah^{r-1}) f(ah^r)$

where $h = \left(\frac{b}{a}\right)^{1/n}$.

Sol'n:- Let $P = \{a, ah, ah^2, \ldots ah^{r-1}, ah^r \ldots ah^n = b\}$ be a partition of $[a,b]$

$I_r = [ah^{r-1}, ah^r]$; $r = 1, 2 \ldots n$

$\delta_r = ah^r - ah^{r-1}$

As $||P|| \to 0$, $h \to 1$; $n \to \infty$.

Let $\xi_r \in I_r$ such that

$$ah^{r-1} \le \xi_r \le ah^r; \; r = 1, 2, \ldots, n.$$

Then $$\int_a^b f(x)dx = \lim_{||P|| \to 0} \sum_{r=1}^{n} f(\xi_r)\delta_r$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} f(ah^r)(ah^r - ah^{r-1})$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} (ah^r - ah^{r-1}) f(ah^r)$$

Problems :-

From definition, Prove that

$\int_1^2 f(x)\,dx = 6$ where $f(x) = 2x+3$.

Sol'n :- Let $f(x) = 2x+3 \; \forall \; x \in [1,2]$

Since f is bounded and continuous on $[1,2]$.

$\therefore$ f is integrable on $[1,2]$.

Let $P = \{1 = x_0, \ldots x_2 \ldots x_{r-1}, x_r, \ldots x_n = 2\}$

$= \{1, 1+1/n, 1+2/n, \ldots 1+\frac{r-1}{n}, 1+\frac{r}{n}, \ldots 1+\frac{n}{n} = 2\}$

Note!

be a partition of $[1,2]$ which divides $[1,2]$ into n equal subintervals.

Each of length $= \frac{b-a}{n} = \frac{2-1}{n} = \frac{1}{n}$

$\therefore$ $||P|| = 1/n$ and

$\therefore$ $||P|| \to 0$ as $n \to \infty$.

$I_r = [x_{r-1}, x_r] = \left[1+\frac{r-1}{n}, 1+\frac{r}{n}\right]$, $r = 1, 2 \ldots n$

Let $\xi_r \in I_r$ such that $1+\frac{r-1}{n} \le \xi_r \le 1+\frac{r}{n}$.

$r = 1, 2, \ldots n$.

i.e. $\xi_r \in I_r$ such that $x_{r-1} \le \xi_r \le x_r$

$$\therefore \int_1^2 f(x)\,dx = \lim_{||P|| \to 0} \sum_{r=1}^{n} f(\xi_r)\delta_r$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} f(x_r)\delta_r \text{ (taking } \xi_r = x_r)$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} f\left(1+\frac{r}{n}\right)\frac{1}{n}$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \left(2\left(1+\frac{r}{n}\right)+3\right)$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[5\sum_{r=1}^{n}(1) + \frac{2}{n}\sum_{r=1}^{n} r\right]$$

$$= \lim_{n\to\infty} \left[\frac{5}{n} n + \frac{2}{n^2} \cdot \frac{n(n+1)}{2}\right]$$

$$= \lim_{n\to\infty} \left[5 + \left(1+\frac{1}{n}\right)\right]$$

$= 6$.

→ $\int_1^2 x\,dx = 3/2$

→ $\int_0^1 (2x^2 - 3x + 5)\,dx = 25/6$.

→ Evaluate $\int_{-1}^{1} f(x)\,dx$

where $f(x) = |x|$.

Sol'n:- Let $f(x) = |x| \; \forall \; x \in [-1,1]$.

6I

$$f(x) = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

$\therefore f(x)$ is bounded and continuous on $[-1,1]$.

$\Rightarrow f(x)$ is integrable on $[-1,1]$.

Let $P = \{-1 = x_0, x_1, x_2, \cdots x_n = 0, x_{n+1}, x_{n+2}, \cdots x_{n+n} = x_{2n} = 1\}$

$= \{-1, -1+\frac{1}{n}, -1+\frac{2}{n}, \cdots -1+\frac{n+1}{n}, -1+\frac{n+2}{n} \cdots -1+\frac{n+n}{n} = 1\}$

be a partition of $[-1,1]$, which divides $[-1,1]$ into $2n$ equal subintervals, each of length $= \frac{b-a}{2n} = \frac{1-(-1)}{2n} = \frac{1}{n}$

$\therefore \|P\| = \frac{1}{n} \to 0$ as $n \to \infty$.

Let $I_r = [x_{r-1}, x_r]$; $r = 1, 2, \cdots 2n$.

Let $\xi_r \in I_r$ such that $x_{r-1} \leq \xi_r \leq x_r$; $r = 1, 2, \cdots n$

and $\delta_r = 1/n$

$$\therefore \int_{-1}^{1} f(x)\,dx = \lim_{\|P\| \to 0} \sum_{r=1}^{2n} f(\xi_r)\,\delta_r$$

$$\lim_{n\to\infty} \sum_{r=1}^{2n} f(x_r)\frac{1}{n} \quad (\text{taking } \xi_r = x_r)$$

$$= \lim_{n\to\infty} \sum_{r=1}^{2n} f\left(-1+\frac{r}{n}\right)\frac{1}{n}.$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\sum_{r=1}^{n}\left[-\left(-1+\frac{r}{n}\right)\right] + \sum_{r=n+1}^{2n}\left(-1+\frac{r}{n}\right)\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\cancel{n} - \frac{1}{\cancel{n}}\frac{n(n+1)}{2} - \cancel{n} + \frac{1}{n}\sum_{r=n+1}^{2n} r\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\frac{-(n+1)}{2} + \frac{1}{n}\left((n+1)+(n+2)+\cdots+(n+n) = 2n\right)\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[\frac{-(n+1)}{2} + \frac{1}{n}\left\{\frac{n(n+1+2n)}{2}\right\}\right]$$

[$\because$ in an AP $S_n = \frac{n}{2}(a+l)$ — first term, last term] Note!

$$= \lim_{n\to\infty} \frac{1}{n}\left[\frac{-(n+1)}{2} + \frac{(3n+1)}{2}\right]$$

$$= \lim_{n\to\infty}\left[-\tfrac{1}{2}(1+1/n) + (3+1/n)\tfrac{1}{2}\right]$$

$$= -\tfrac{1}{2} + \tfrac{3}{2} = \frac{2}{2} = 1$$

$\rightarrow$ Evaluate $\int_{-1}^{2} f(x)\,dx$ where $f(x) = |x|$

Sol'n: Since $f(x) = \begin{cases} -x & \text{if } x \leq 0 \\ x & \text{if } x > 0 \end{cases}$

$\therefore f(x)$ bounded and continuous function on $[-1,2]$.

Let $P = \{-1 = x_0, x_1, x_2 \cdots x_n = 0, x_{n+1}, x_{n+2} \cdots x_{2n} = 1, x_{2n+1}, x_{2n+2}, \cdots x_{2n+n} = x_{3n} = 2\}$.

$= \{-1, -1+\frac{1}{n}, -1+\frac{2}{n}, \cdots -1+\frac{n}{n} = 0,$

$-1+\frac{n+1}{n}, -1+\frac{n+2}{n}, \cdots -1+\frac{2n}{n} = 1,$ Note!

$-1+\frac{2n+1}{n}, -1+\frac{2n+2}{n}, \cdots -1+\frac{2n+n}{n} = 2\}$

be a partition of $[-1,2]$ which divides $[-1,2]$ into $3n$ equal sub intervals each sub interval length

$\frac{b-a}{3n} = \frac{2-(-1)}{3n} = \frac{1}{n}$

$\therefore \|P\| = \frac{1}{n} \to 0$ as $n \to \infty$.

$I_r = [x_{r-1}, x_r]$; $r = 1, 2, \cdots 3n$

Let $\xi_r \in I_r$ such that $x_{r-1} \le \xi_r \le x_r$; $r = 1, 2, \ldots 3n$.
and $\delta_r = \frac{1}{n}$

$$\therefore \int_{-1}^{2} f(x)\,dx = \lim_{\|P\| \to 0} \sum_{r=1}^{3n} f(\xi_r)\,\delta_r$$

Note!

$$= \lim_{n\to\infty} \sum_{r=1}^{3n} f(x_r)\frac{1}{n}$$

(Taking $\xi_r = x_r$)

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{3n} f\left(-1+\frac{r}{n}\right)$$

Note!

$$= \lim_{n\to\infty} \frac{1}{n}\left[\sum_{r=1}^{n}\left[-\left(-1+\frac{r}{n}\right)\right] + \sum_{r=n+1}^{3n}\left(-1+\frac{r}{n}\right)\right]$$

Note!

$$= \lim_{n\to\infty} \frac{1}{n}\left[\sum_{r=1}^{n}(1) - \sum_{r=1}^{n}\frac{r}{n} - \sum_{r=n+1}^{3n}(1) + \sum_{r=n+1}^{3n}\left(\frac{r}{n}\right)\right]$$

$$= \lim_{n\to\infty} \frac{1}{n}\left[n - \frac{1}{\cancel{n}}\frac{\cancel{n}(n+1)}{2} - 2n + \frac{1}{n}\{(n+1) + (n+2) + (n+3) + \cdots + 3n\}\right]$$

Note!

$$= \lim_{n\to\infty} \frac{1}{n}\left[-n - \frac{1}{2}(n+1) + \frac{1}{\cancel{n}}\cdot\frac{2\cancel{n}}{2}(n+1+3n)\right]$$

Note!

$\left(\because \text{in an A.P } S_{2n} = \frac{2n}{2}(a+l)\right)$

$$= \lim_{n\to\infty}\left[-1 - \frac{1}{2}\left(1+\frac{1}{n}\right) + \left(4+\frac{1}{n}\right)\right]$$

$$= -1 - \tfrac{1}{2} + 4$$

$$= -\tfrac{3}{2} + 4 = \frac{-3+8}{2}$$

$$= \tfrac{5}{2}$$

→ Evaluate $\int_{-1}^{1} f(x)\,dx$

where $f(x) = |x|$

→ show that $\int_0^a \sin x\,dx = 1 - \cos a$

where a is fixed Real number.

Sol^n:- Since $f(x) = \sin x$ is bounded and Continuous on $[0,a]$.

$\therefore f$ is Riemann integrable on $[0,a]$.

Let $P = \{0 = x_0, x_1, x_2, \ldots x_n = a\}$

$= \left\{0, \frac{a}{n}, \frac{2a}{n}, \ldots \frac{na}{n} = a\right\}$ be

Partition of $[0,a]$ which divides $[0,a]$ into n equal subinterval each of length $= \frac{b-a}{n} = \frac{a-0}{n} = \frac{a}{n}$.

$\therefore \|P\| \to 0$ as $n \to \infty$

$I_r = [x_{r-1}, x_r] = \left[\frac{(r-1)a}{n}, \frac{ra}{n}\right]$

$r = 1, 2, \ldots n$

$\delta_r = \frac{a}{n}$; $r = 1, 2 \ldots n$.

$$\therefore \int_0^a f(x)\,dx = \lim_{\|P\|\to 0} \sum_{r=1}^{n} f(\xi_r)\,\delta_r$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} f(x_r)\frac{a}{n}$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} f\left(\frac{ra}{n}\right)\cdot\frac{a}{n}$$

$$= \lim_{n\to\infty} \frac{a}{n}\sum_{r=1}^{n} \sin\left(\frac{ra}{n}\right).$$

$$= \lim_{n\to\infty} \frac{a}{n}\left[\sin\frac{a}{n} + \sin\frac{2a}{n} + \cdots + \sin\frac{na}{n}\right]$$

$$= \lim_{n\to\infty} \frac{\frac{a}{n}\left[\sin\left(\frac{a}{n} + \frac{n-1}{2}\frac{a}{n}\right)\sin\left(\frac{n}{2}\cdot\frac{a}{n}\right)\right]}{\sin\left(\frac{a}{2n}\right)}$$

$\left[\because \sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \cdots + \sin(\alpha+\overline{n-1}\,\beta) = \frac{\sin\left(\alpha + \left(\frac{n-1}{2}\right)\beta\right)\sin\frac{n\beta}{2}}{\sin\beta/2}\right]$

$$I_2 = \lim_{n\to\infty} \frac{\frac{a}{n}\left[\sin\left(\frac{a}{n}+\frac{a}{2}-\frac{a}{2n}\right).\sin\left(\frac{a}{2}\right)\right]}{\sin\left(\frac{a}{2n}\right)}$$

$$= \lim_{\frac{a}{2n}\to 0} 2.\frac{\frac{a}{2n}}{\sin\left(\frac{a}{2n}\right)} . \lim_{n\to\infty} \sin\left(\frac{a}{n}+\frac{a}{2}-\frac{a}{2n}\right)\sin\left(\frac{a}{2}\right)$$

$$= 2.(1)\sin^2\left(\frac{a}{2}\right)$$

$$= 1-\cos a$$

H.W. Show that $\int_0^a \cos x\,dx = \sin a$.

where a is a fixed number.

H.W. $\int_0^{\pi/2} \cos x\,dx = 1$.

$\boxed{2004}$ P-II Show that the greatest integer function $f(x)=[x]$ is integrable on $[0,4]$ and $\int_0^4 [x]dx = 6$.

Sol'n: $f(x)=[x]\ \forall x\in[0,4]$

$$\Rightarrow f(x) = \begin{cases} 0 & \text{when } 0\le x<1 \\ 1 & \text{when } 1\le x<2 \\ 2 & \text{when } 2\le x<3 \\ 3 & \text{when } 3\le x<4 \end{cases}$$

$\Rightarrow f$ is bounded and has only four points of finite discontinuity at 1, 2, 3, 4.

Since the points of discontinuity of f on $[0,4]$ are finite in number.

$\therefore f$ is integrable on $[0,4]$

$$\int_0^4 [x]dx = \int_0^1 [x]dx + \int_1^2 [x]dx + \int_2^3 [x]dx + \int_3^4 [x]dx$$

$$= \int_0^1 0\,dx + \int_1^2 1\,dx + \int_2^3 2\,dx + \int_3^4 3\,dx$$

$$= 1(2-1)+2(3-2)+3(4-3)$$

$$= 1+2+3$$

$$= 6$$

H.W. Prove that $f(x)=x[x]$ is integrable on $[0,2]$ and $\int_0^2 x[x]dx = \frac{3}{2}$.

H.W. Prove that $f(x)=x-[x]$ is integrable on $[1,10]$ and $\int_1^{10} f(x)dx = \frac{9}{2}$.

$\boxed{2004}$ P-I Show that the function f defined by

$f(x)=\frac{1}{2^n}$ when $\frac{1}{2^{n+1}} < x \le \frac{1}{2^n}$; $n=0,1,2\ldots$

$f(0)=0$ is integrable on $[0,1]$, although it has an infinite number of points of discontinuity. Show that $\int_0^1 f(x)dx = \frac{2}{3}$.

Sol'n:- $f(x)=\frac{1}{2^n}$ when $\frac{1}{2^{n+1}}<x\le\frac{1}{2^n}$

$n=0,1,2\ldots$

$$f(x) = \begin{cases} \frac{1}{2^0} = 1 & \text{when } \frac{1}{2^1}<x\le\frac{1}{2^0}=1 \\ \frac{1}{2} & \text{when } \frac{1}{2^2}<x\le\frac{1}{2^1} \\ \frac{1}{2^2} & \text{when } \frac{1}{2^3}<x\le\frac{1}{2^2} \\ \vdots & \\ \frac{1}{2^{n-1}} & \text{when } \frac{1}{2^n}<x\le\frac{1}{2^{n-1}} \\ \vdots & \\ 0 & \text{when } x=0. \end{cases}$$

$\Rightarrow f$ is bounded and continuous on $[0,1]$ except at the points

$\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots$

The set of points discontinuity of f on $[0,1]$ is

$\left\{\frac{1}{2}, \frac{1}{2^2}, \ldots\right\}$ which has only one limit point '0'.

Since the set of points of discontinuity of f on $[0,1]$ has a finite number of limit points.

$\therefore f$ is integrable on $[0,1]$.

Now $$\int_{1/2^n}^{1} f(x)\,dx = \int_{1/2}^{1} f(x)\,dx + \int_{1/2^2}^{1/2} f(x)\,dx + \int_{1/2^3}^{1/2^2} f(x)\,dx + \cdots + \int_{1/2^n}^{1/2^{n-1}} f(x)\,dx$$

$$= \left(1-\tfrac{1}{2}\right) + \tfrac{1}{2}\left(\tfrac{1}{2} - \tfrac{1}{2^2}\right) + \tfrac{1}{2^3}\left(\tfrac{1}{2^2} - \tfrac{1}{2^3}\right) + \cdots + \frac{1}{2^{n-1}}\left(\frac{1}{2^{n-1}} - \frac{1}{2^n}\right)$$

$$= \left(1-\tfrac{1}{2}\right) + \tfrac{1}{2}\left(\tfrac{1}{2^2}\right) + \frac{1}{2^2}\left(\frac{1}{2^3}\right) + \cdots + \frac{1}{2^{n-1}}\left(\frac{1}{2^n}\right).$$

$$= \frac{1}{2}\left[1 + \frac{1}{2^2} + \left(\frac{1}{2^2}\right)^2 + \cdots + \left(\frac{1}{2^2}\right)^{n-1}\right].$$

$$\frac{1}{2}\left[\frac{1-\left(\frac{1}{2^2}\right)^n}{1-\frac{1}{2^2}}\right] = \frac{2}{3}\left(1 - \frac{1}{4^n}\right)$$

Taking limit when $n \to \infty$, we get

$$\lim_{n\to\infty} \int_{1/2^n}^{1} f(x)\,dx = \lim_{n\to\infty} \frac{2}{3}\left(1 - \frac{1}{4^n}\right)$$

$$\Rightarrow \int_0^1 f(x)\,dx = 2/3.$$

$\rightarrow$ Show that f defined on $[0,1]$ by

$$f(x) = \begin{cases} \frac{1}{n}, & \frac{1}{n+1} < x \le \frac{1}{n}, \ (n = 0, 1, 2, \ldots) \\ 0; & x = 0 \end{cases}$$

is integrable on $[0,1]$. Also show that $\int_0^1 f(x)\,dx = \frac{\pi^2}{6} - 1$.

Sol'n:- $$f(x) = \begin{cases} \frac{1}{n}, & \frac{1}{n+1} < x \le \frac{1}{n}; \ n = 1, 2, \ldots \\ 0, & x = 0. \end{cases}$$

$= 1; \ \frac{1}{2} < x \le 1$

$= \frac{1}{2}; \ \frac{1}{3} < x \le \frac{1}{2}$

$= \frac{1}{3}; \ \frac{1}{4} < x \le \frac{1}{3}$

$\vdots$

$= \frac{1}{n-1}; \ \frac{1}{n} < x \le \frac{1}{n-1}$

$\vdots$

$= 0; \ x = 0.$

$\Rightarrow f(x)$ is bounded and continuous on $[0,1]$ except at the points.

$\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots \frac{1}{n+1}, \ldots$

The set of points of discontinuity

Set of f on $[0,1]$ is $\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\}$

which has one limit point '0'.

The set of points of discontinuity of f on $[0,1]$ has a finite number of limit points.

$\therefore$ f is integrable on $[0,1]$.

Now $\displaystyle\int_{\frac{1}{n+1}}^{1} f(x)\,dx = \int_{1/2}^{1} f(x)\,dx + \int_{1/3}^{1/2} f(x)\,dx + \int_{1/4}^{1/3} f(x)\,dx + \dots + \int_{\frac{1}{n+1}}^{1/n} f(x)\,dx$

$\displaystyle = \int_{1/2}^{1} 1\,dx + \int_{1/3}^{1/2} \frac{1}{2}\,dx + \int_{1/4}^{1/3} \frac{1}{3}\,dx + \dots + \int_{\frac{1}{n+1}}^{1/n} \frac{1}{n}\,dx$

$= (1-\frac{1}{2}) + \frac{1}{2}(\frac{1}{2}-\frac{1}{3}) + \frac{1}{3}(\frac{1}{3}-\frac{1}{4}) + \dots + \frac{1}{n}\left(\frac{1}{n}-\frac{1}{n+1}\right)$

$= \left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots+\frac{1}{n^2}\right) - \left(\frac{1}{2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\dots+\frac{1}{n(n+1)}\right)$

$= \left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots+\frac{1}{n^2}\right) - \left((1-\frac{1}{2})+(\frac{1}{2}-\frac{1}{3})+(\frac{1}{3}-\frac{1}{4})+\dots+\left(\frac{1}{n}-\frac{1}{n+1}\right)\right)$

$= \left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots+\frac{1}{n^2}\right) - \left(1-\frac{1}{n+1}\right)$

$\displaystyle\therefore \int_{\frac{1}{n+1}}^{1} f(x)\,dx = \left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots+\frac{1}{n^2}\right) - \left(1-\frac{1}{n+1}\right)$

Now taking limit as $n\to\infty$, we get.

$\displaystyle \lim_{n\to\infty} \int_{\frac{1}{n+1}}^{1} f(x)\,dx = \lim_{n\to\infty}\left[\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots+\frac{1}{n^2}\right) - \left(1-\frac{1}{n+1}\right)\right]$

$\displaystyle \Rightarrow \int_0^1 f(x)\,dx = \lim_{n\to\infty}\left(\frac{1}{1^2}+\frac{1}{2^2}+\dots+\frac{1}{n^2}\right) - \lim_{n\to\infty}\left(1-\frac{1}{n+1}\right)$

$= \frac{\pi^2}{6} - 1$

($\because$ The series $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\dots+\frac{1}{n^2}+\dots$ converges to $\frac{\pi^2}{6}$) Remember

HW show that the function f defined on $[0,1]$ as

$f(x) = 2r x$ if $\frac{1}{r+1} < x \le \frac{1}{r}$, $r = 1, 2, 3, \dots$

is integrable on $[0,1]$ and $\displaystyle\int_0^1 f(x)\,dx = \frac{\pi^2}{6}$.

* Properties of Riemann Integral:

→ If $f \in R[a,b]$ then $-f \in R[a,b]$

→ If $f \in R[a,b]$ then $|f| \in R[a,b]$

[Note: Converse is not true.

i.e. If $|f| \in R[a,b]$ then f need not be R-integrable on $[a,b]$.]

Ex:- $f: [a,b] \to \mathbb{R}$ defined as follows

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ -1, & \text{if } x \text{ is irrational.} \end{cases}$$

→ If $f, g \in R[a,b]$ then $f \pm g \in R[a,b]$

and $\displaystyle\int_a^b (f \pm g)(x)\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$.

→ If $f, g \in R[a,b]$ and $\alpha, \beta \in \mathbb{R}$ then $\alpha f + \beta g \in R[a,b]$

→ If $f \in R[a,b]$ then $f^2 \in R[a,b]$

→ If $f, g \in R[a,b]$ and there exists $t > 0$ such that $|g(x)|$ $\forall x \in [a,b]$ then $\frac{f}{g} \in R[a,b]$

→ If $f, g \in R[a,b]$ then $fg \in R[a,b]$.

Note:
Even though f, g are not integrable on $[a,b]$, fg may be integrable on $[a,b]$

Ex:- Let $f : [a,b] \to \mathbb{R}$; $g : [a,b] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 0, & x \in \mathbb{Q} \\ 1, & x \in \mathbb{R}-\mathbb{Q} \end{cases} \quad \text{and } g(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R}-\mathbb{Q} \end{cases}$$

Then $(fg)x = f(x).g(x)$

$= 0 \ \forall x \in \mathbb{R}$.

Since fg is constant function.

$\Rightarrow fg \in R[a,b]$

But f, g are not Riemann integrable on $[a,b]$.

→ If $f \in R[a,b]$ and $a < c < b$ then $f \in R[a,c]$, $f \in R[c,b]$.

and $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

→ If $f \in R[a,c]$, $f \in R[c,b]$ and $a < c < b$ then $f \in R[a,b]$.
$f \in R[a,b]$

→ If $f \in R[a,b]$ and $f(x) \geq 0$ $\forall x \in [a,b]$.

then $\int_a^b f(x)\,dx \geq 0$.

Sol'n: Since $f \in R[a,b]$

$\Rightarrow f$ is bounded on $[a,b]$

Let m & M be the infimum & Supremum of f on $[a,b]$.

Since $f(x) \geq 0$ $\forall x \in [a,b]$

Now for every $P \in P[a,b]$,

$L(P,f) \geq m(b-a)$ ✓
≥ 0

$\Rightarrow L(P,f) \geq 0$

Now $\int_a^b f(x)\,dx = \text{lub}\{L(P,f)\} \geq 0$ $\forall P \in P[a,b]$

But $\int_a^b f(x)\,dx = \int_a^b f(x)\,dx$. ($\because f \in R[a,b]$)

$\therefore \int_a^b f(x)\,dx \geq 0$.

2004 → If $f, g \in R[a,b]$ and $f(x) \geq g(x)$ $\forall x \in [a,b]$ then $\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx$.

Sol'n: $f, g \in R[a,b]$

$\Rightarrow f - g \in R[a,b]$.

Since $f(x) \geq g(x)$ $\forall x \in [a,b]$

$\Rightarrow f(x) - g(x) \geq 0$ $\forall x \in [a,b]$

$\Rightarrow (f-g)(x) \geq 0$ $\forall x \in [a,b]$

We know that

If $f \in R[a,b]$ and $f(x) \geq 0$ $\forall x \in [a,b]$ then $\int_a^b f(x)\,dx \geq 0$.

$$\int_a^b (f-g)(x)\,dx \geq 0$$

$$\Rightarrow \int_a^b (f(x)-g(x))\,dx \geq 0$$

$$\Rightarrow \int_a^b f(x)\,dx - \int_a^b g(x)\,dx \geq 0$$

$$\Rightarrow \int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx.$$

Corollary → If $f, g, h \in R[a,b]$ and $f(x) \geq g(x) \geq h(x)\ \forall x \in [a,b]$ then

$$\int_a^b f(x)\,dx \geq \int_a^b g(x)\,dx \geq \int_a^b h(x)\,dx.$$

Note: $0 < x < 1$

$\Rightarrow 0 < x^2 < 1$

Since exponential function is an increasing function on $\mathbb{R}$.

$\therefore 0 < x^2 < 1$

$\Rightarrow e^0 < e^{x^2} < e^1$

$\Rightarrow 1 < e^{x^2} < e$

Take $f(x) = 1$, $g(x) = e^{x^2}$, $h(x) = e$

$\therefore$ we have $f(x) < g(x) < h(x)\ \forall x \in [0,1]$

$$\Rightarrow \int_0^1 f(x)\,dx < \int_0^1 g(x)\,dx < \int_0^1 h(x)\,dx$$

$$\Rightarrow \int_0^1 1\,dx < \int_0^1 e^{x^2}\,dx < \int_0^1 e\,dx$$

$$\Rightarrow 1 < \int_0^1 e^{x^2}\,dx < e$$

→ If $f \in R[a,b]$ then

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx.$$

Sol^n:- Since $f \in R[a,b]$.

$\Rightarrow |f| \in R[a,b]$

$\Rightarrow -|f| \in R[a,b]$.

we have

$-|f(x)| \leq f(x) \leq |f(x)|\ \forall x \in [a,b]$

$$\Rightarrow \int_a^b -|f(x)|\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx$$

$$\Rightarrow -\int_a^b |f(x)|\,dx \leq \int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx.$$

$$\therefore \left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx.$$

→ $f \in R[a,b]$ and m, M are infimum and supremum of 'f' in $[a,b]$, then

$$m(b-a) \leq \int_a^b f(x)\,dx \leq M(b-a)$$

$$\int_a^b f(x)\,dx = \mu(b-a) \text{ where } \mu \in [m, M].$$

Sol^n:- For every $P \in P[a,b]$, we have

$$m(b-a) \leq L(P,f) \leq U(P,f) \leq M(b-a) \quad \text{— (1)}$$

Now $\underset{P \in P[a,b]}{\text{Lub}}\{L(P,f)\} = \underline{\int_a^b} f(x)\,dx$

$= \int_a^b f(x)\,dx$

$(\because f \in R[a,b])$

$$\Rightarrow L(P,f) \leq \int_a^b f(x)\,dx \quad \text{— (2)}$$

and $\underset{P \in P[a,b]}{\text{glb}}\{U(P,f)\} = \overline{\int_a^b} f(x)\,dx$

$= \int_a^b f(x)\,dx$

$(\because f \in R[a,b])$

$\Rightarrow U(P,f) \geq \int_a^b f(x)dx$ ——— (3)

$\therefore$ from ①, ② & ③ we have

$m(b-a) \leq L(P,f) \leq \int_a^b f(x)dx \leq U(P,f) \leq M(b-a)$

$\Rightarrow m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$

$\Rightarrow m \leq \frac{1}{b-a}\int_a^b f(x)dx \leq M$ for $a \neq b$.

$\Rightarrow \frac{1}{b-a}\int_a^b f(x)dx$ is a number μ (say) lying between the bounds m & M.

$\therefore \frac{1}{b-a}\int_a^b f(x)dx = \mu$

$\Rightarrow \int_a^b f(x)\,dx = \mu(b-a)$

where $m \leq \mu \leq M$.

For $a=b$, the result is trivially true.

Ex:- $f(x) = \sqrt{3+x^2}\ \forall x \in [1,3]$

$\Rightarrow f(x)$ bounded & Riemann integrable on $[1,3]$

Now $f'(x) = \frac{x}{\sqrt{3+x^2}} > 0\ \forall x \in [1,3]$.

$\therefore f(x)$ is an increasing.

$\therefore m = f(1) = \sqrt{3+1^2} = 2$ $\quad M = f(3) = \sqrt{3+3^2} = \sqrt{12}$

$\therefore 2(3-1) \leq \int_1^3 \sqrt{3+x^2}\,dx \leq \sqrt{12}\,(3-1)$

$m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$

$\Rightarrow 4 \leq \int_1^3 \sqrt{3+x^2}\,dx \leq 4\sqrt{3} < 7$

$\Rightarrow 4 \leq \int_1^3 \sqrt{3+x^2}\,dx < 7$

$\rightarrow$ If f is continuous on $[a,b]$, $\exists\, c \in [a,b]$ such that $\int_a^b f(x)dx = (b-a)f(c)$

Proof:- Since f is continuous on $[a,b]$

$\Rightarrow f \in R[a,b]$

$\therefore \exists\, \mu \in [m,M]$ such that $\int_a^b f(x)dx = \mu(b-a)$ ——— ①

Since f is continuous on $[a,b]$, it attains every value between its bounds m, M.

$\therefore \mu \in [m,M] \Rightarrow \exists$ a number $c \in [a,b]$ such that $f(c) = \mu$.

$\therefore$ ① $\equiv \int_a^b f(x)\,dx = (b-a)f(c)$.

$\rightarrow$ If $f \in R[a,b]$ and $|f(x)| \leq k\ \forall x \in [a,b]$ and $k \in R^+$ then

$\left|\int_a^b f(x)dx\right| \leq k(b-a)$.

Sol^n:- Since $|f(x)| \leq k\ \forall x \in [a,b]$

$\Rightarrow -k \leq f(x) \leq k\ \forall x \in [a,b]$.

If m, M are the infimum and supremum of f on $[a,b]$ then we have

$-k \leq m \leq f(x) \leq M \leq k\ \forall x \in [a,b]$

since $f \in R[a,b]$

$\therefore m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$

$\therefore -k(b-a) \leq m(b-a) \leq \int_a^b f(x)dx \leq M(b-a) \leq k(b-a)$

$\Rightarrow -k(b-a) \leq \int_a^b f(x)dx \leq k(b-a)$

$\Rightarrow \left|\int_a^b f(x)dx\right| \leq k(b-a)$

Ex:- $f(x) = \dfrac{\sin x}{1+x^8} \quad \forall x \in [10, 19]$

For $x \geq 10 \Rightarrow x^8 \geq 10^8$

$\Rightarrow 1+x^8 > 1+10^8$

$\Rightarrow |1+x^8| > |1+10^8|$

$\Rightarrow \dfrac{1}{|1+x^8|} < \dfrac{1}{1+10^8} < \dfrac{1}{10^8}$

Since $|\sin x| \leq 1 \; \forall x \in \mathbb{R}$

$\Rightarrow \left| \dfrac{\sin x}{1+x^8} \right| < \dfrac{1}{10^8} = 10^{-8} \; \forall x \in [10, 19]$

$\therefore$ By theorem,

$$\left| \int_a^b f(x)\,dx \right| \leq K(b-a).$$

where $|f(x)| \leq K$.

we have

$$\left| \int_{10}^{19} \frac{\sin x}{1+x^8}\,dx \right| \leq 10^{-8}(19-10)$$

$$= 10^{-8} \cdot 9$$

$$< 10^{-7}.$$

* Functions defined by integrals:

Let $f \in R[a,b]$. Then for each $t \in [a,b]$, $[a,t] \subset [a,b]$ and hence $f \in R[a,t]$.

Therefore $\int_a^t f(x)\,dx$ is well defined.

The function $\phi(t) = \int_a^t f(x)\,dx, \; t \in [a,b]$

The function ϕ is called an integral function or Indefinite integral of the integrable function f.

Note: The integral function of f may also be defined as

$$\phi(t) = \int_t^b f(x)\,dx, \; t \in [a,b].$$

→ **Continuity of the integral function :-**

If $f \in R[a,b]$ then $\phi(t) = \int_a^t f(x)\,dx$, $t \in [a,b]$ is continuous on $[a,b]$.

(or)

The integral function of an integrable is continuous.

Proof: Since $f \in R[a,b]$

$\Rightarrow f$ is bounded on $[a,b]$.

$\Rightarrow \exists K \in \mathbb{R}^+$ such that $|f(x)| \leq K \; \forall x \in [a,b]$

If $x_1, x_2 \in [a,b]$ such that $a \leq x_1 < x_2 \leq b$

then $|\phi(x_2) - \phi(x_1)| = \left| \int_a^{x_2} f(x)\,dx - \int_a^{x_1} f(x)\,dx \right|$

$$= \left| \int_a^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx - \int_a^{x_1} f(x)\,dx \right|$$

$$= \left| \int_{x_1}^{x_2} f(x)\,dx \right|.$$

$$\leq K(x_2 - x_1) \quad (\because f \in R[a,b] \text{ and } |f(x)| \leq K, K \in \mathbb{R}^+)$$

Now for each $\epsilon > 0$,

we have

$|\phi(x_2) - \phi(x_1)| < \epsilon$ whenever $|x_2 - x_1| < \dfrac{\epsilon}{K}$

choosing $\delta = \dfrac{\epsilon}{K}$

$\therefore |\phi(x_2) - \phi(x_1)| < \epsilon$ whenever $|x_2 - x_1| < \delta$.

$\therefore \phi(x)$ is uniformly continuous on $[a,b]$.

$\therefore \phi(x)$ is continuous on $[a,b]$

* Derivability of the integral function :-

If $f \in R[a,b]$ and f is continuous at $c \in [a,b]$ then $\phi(t) = \int_a^t f(x)\,dx$ $\forall\, t \in [a,b]$ is derivable at 'c' and $\phi'(c) = f(c)$.

Proof: Since f is continuous at $c \in [a,b]$.

$\therefore$ for each $\epsilon > 0$, $\exists$ a $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ whenever $|x-c| < \delta$; $x, c \in [a,b]$. —— (1)

Take h, so that $|h| < \delta$.

Now $\phi(c+h) - \phi(c) = \int_a^{c+h} f(x)\,dx - \int_a^{c} f(x)\,dx$

$$= \int_a^c f(x)\,dx + \int_c^{c+h} f(x)\,dx - \int_a^c f(x)\,dx$$

$$= \int_c^{c+h} f(x)\,dx \quad \text{—— (2).}$$

Since f is continuous on $[a,b]$.

$\therefore \exists\, c \in [a,b]$ such that

$$\int_a^b f(x)\,dx = (b-a) f(c).$$

$\therefore$ (2) $\equiv \phi(c+h) - \phi(c) = h \cdot f(c)$

$$\therefore \left| \frac{\phi(c+h) - \phi(c)}{h} - f(c) \right| = \left| \frac{1}{h}\int_c^{c+h} f(x)\,dx - \frac{1}{h}\int_c^{c+h} f(c)\,dx \right|$$

$$= \left| \frac{1}{h} \int_c^{c+h} \{f(x) - f(c)\}\,dx \right|$$

$$\leq \frac{1}{h} \int_c^{c+h} |f(x) - f(c)|\,dx$$

$$< \frac{1}{h} \int_c^{c+h} \epsilon\,dx \quad \text{(from (1))}$$

$= \epsilon$ whenever $0 < |x-c| < \delta$.

$\therefore \left| \frac{\phi(c+h) - \phi(c)}{h} - f(c) \right| < \epsilon$ whenever $0 < |x - c| < \delta$ (i.e. $0 < |h| < \delta$)

i.e. $\lim_{h \to 0} \frac{\phi(c+h) - \phi(c)}{h} = f(c)$.

i.e. $\phi'(c) = f(c)$

$\therefore \phi$ is derivable at 'c'

i.e. continuity of f at 'c' $\in [a,b]$ is derivability of ϕ at 'c'.

i.e. continuity of f on $[a,b]$ is derivability of ϕ on $[a,b]$.

Note:

(1) This theorem is sometimes referred to as the first fundamental theorem of integral calculus.

(2) The integral function ϕ defined by $\phi(t) = \int_t^b f(x)dx$ is continuous and derivable on $[a,b]$ under the conditions of the above two theorems.

(3) Since $\phi(t) = \int_t^b f(x)$ is continuous on $t \in [a,b]$

$$\phi(0) = \int_0^b f(x)\,dx = \lim_{t \to 0} \phi(t).$$

(4) If f is continuous on $[a,b]$ then $\phi(t) = \int_a^t f(x)dx \;\forall\; t \in [a,b]$ is derivable at every $x \in [a,b]$ and $\phi'(x) = f(x)$.

Definition

If $f \in R[a,b]$ and if $\exists\, \phi : [a,b] \to \mathbb{R}$ such that $\phi'(x) = f(x) \;\forall\; x \in [a,b]$ then ϕ is called a primitive or antiderivative of f.

Note: (1) If f is continuous on $[a,b]$ then f possesses a primitive

$$\phi(t) = \int_a^t f(x)dx \;\forall\; t \in [a,b].$$

(2) Primitive of f is not unique.

If ϕ is a primitive of f then $\phi + c$, $c \in \mathbb{R}$ is also a primitive of f.

Ex: $f : [a,b] \to \mathbb{R}$ be defined by $f(x) = \sin x$.

Since $f(x) = \sin x$ is continuous on $[a,b]$.

$\therefore$ Primitive of $\sin x$ exists on $[a,b]$

$\therefore$ If $\phi : [a,b] \to \mathbb{R}$ is defined by $\phi(x) = -\cos x$ then we know that $\phi'(x) = (-\cos x)' = \sin x \;\forall\; x \in [a,b]$

$\therefore -\cos x$ is the primitive of $\sin x$ on $[a,b]$.

(3) Continuity of a function is not a necessary condition for the existence of a primitive.

Ex:- Consider the function ϕ defined on $[0,1]$ as

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$$\phi(x) = \begin{cases} x^2 \sin 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

$$\text{Then } \phi'(x) = \begin{cases} 2x \sin\frac{1}{x} - \frac{\cos\frac{1}{x}}{x^2}; & x \neq 0 \\ 0, & x = 0 \end{cases}$$

we know that $\phi'(x)$ is not continuous at $x = 0$.

If $f(x) = \phi'(x)$ on $[0,1]$ then $f(x)$ is not continuous on $[0,1]$.

Even though $f(x)$ admits of a primitive $\phi(x)$ in $[0,1]$, it fails to be continuous in $[0,1]$.

* Fundamental theorem of integral Calculus:

Imp:

If $f \in R[a,b]$ and ϕ is primitive of f on $[a,b]$ (i.e $\phi'(x) = f(x)$ $\forall\, x \in [a,b]$)

then $\boxed{\int_a^b f(x)\,dx = \phi(b) - \phi(a).}$ $f(x) = \phi'(x)$

Proof: ϕ is a primitive of f on $[a,b]$.

$\therefore \phi'(x) = f(x)$ $\forall\, x \in [a,b]$ ——— (1)

Since $f \in R[a,b]$,

Consider a partition

$P = \{a = x_0, x_1, x_2, \ldots, x_{r-1}, x_r, \ldots, x_n = b\}$ of $[a,b]$.

and $I_r = [x_{r-1}, x_r]$; $r = 1, 2, \ldots n$.

let $\xi_r \in I_r$ such that $x_{r-1} \le \xi_r \le x_r$, $r = 1, 2 \ldots n$

we have

$$\int_a^b f(x)\,dx = \lim_{\|P\| \to 0} \sum_{r=1}^{n} f(\xi_r)\,\delta_r \quad \text{——— (2)}$$

norm of P

Since ϕ is derivable on $[a,b]$

$\Rightarrow \phi$ is continuous and derivable on $[x_{r-1}, x_r]$; $r = 1, 2 \ldots n$.

$\therefore$ By Lagranges Mean Value theorem,

$$\phi'(\xi_r) = \frac{\phi(x_r) - \phi(x_{r-1})}{x_r - x_{r-1}}$$

$r = 1, 2 \ldots n$

$\Rightarrow \phi(x_r) - \phi(x_{r-1}) = (x_r - x_{r-1})\phi'(\xi_r) = \delta_r \ldots$ $\xi_r \in I_r$

$r = 1, 2, \ldots n$

$$\Rightarrow \sum_{r=1}^{n} [\phi(x_r) - \phi(x_{r-1})] = \sum_{r=1}^{n} \phi'(\xi_r)\,\delta_r$$

$$= \sum_{r=1}^{n} f(\xi_r)\,\delta_r \quad \text{(from (1))}$$

$$\Rightarrow \sum_{r=1}^{n} f(\xi_r)\,\delta_r = [(\phi(x_1) - \phi(x_0)) + (\phi(x_2) - \phi(x_1)) + \cdots + (\phi(x_n) - \phi(x_{n-1}))]$$

$$= \phi(x_n) - \phi(x_0)$$

$$= \phi(b) - \phi(a).$$

$$\Rightarrow \sum_{r=1}^{n} f(\xi_r)\,\delta_r = \phi(b) - \phi(a)$$

$$\therefore \lim_{\|P\| \to 0} \sum_{r=1}^{n} f(\xi_r)\,\delta_r = \lim_{\|P\| \to 0} (\phi(b) - \phi(a))$$

$$\Rightarrow \int_a^b f(x)\,dx = \phi(b) - \phi(a).$$

Remember

Note: (1) The above theorem is sometimes called the second fundamental theorem of integral Calculus

(2) If ϕ' is continuous on $[a,b]$ then $\int_a^b \phi'(x)\,dx = \phi(b) - \phi(a)$.

(3) $\phi(b) - \phi(a)$ is denoted as $[\phi(x)]_a^b$.

Ex:- show that $\int_0^1 x^4\,dx = 1/5$.

sol'n: $f(x) = x^4$ is continuous on $\mathbb{R}$.

$\therefore$ It is continuous on $[0,1]$.

$\therefore \int_0^1 x^4\,dx$ exists.

Let $\phi(x) = \frac{x^5}{5}$ $\forall\, x \in [0,1]$

$\Rightarrow \phi'(x) = x^4$ exists on $[0,1]$.

If $\phi(x)$ is derivable on $[0,1]$

$\therefore \phi'(x) = x^4 = f(x) \ \forall x \in [0,1]$

$\therefore \phi(x)$ is a primitive of f on $[0,1]$.

$\therefore$ By fundamental theorem of Integral Calculus.

$$\int_0^1 x^4\,dx = \phi(1) - \phi(0)$$

$$= \tfrac{1}{5} - 0 = \tfrac{1}{5}$$

Ex 1 (2) show that $\int_a^b \cos x\,dx = \sin b - \sin a$.

$f(x) = \cos x$ is continuous on $[a,b]$

$\therefore \int_a^b \cos x$ exists.

Let $\phi(x) = \sin x \ \forall x \in [a,b]$

$\phi'(x) = \cos x \ \forall x \in [a,b]$

$\therefore \phi(x)$ is derivable on $[a,b]$

$\therefore \phi'(x) = \cos x = f(x) \ \forall x \in [a,b]$

$\phi(x)$ is a primitive of f on $[a,b]$.

By fundamental theorem

$$\int_a^b \cos x\,dx = \phi(b) - \phi(a)$$

$$= \sin b - \sin a$$

H.W. Prove that $\int_a^b e^x\,dx = e^b - e^a$.

* Mean Value theorems of Integral Calculus :-

If $f, g \in R[a,b]$ and $g(x) \geq 0$ (or) $g(x) \leq 0 \ \forall x \in [a,b]$ then there exists a number μ with $m \leq \mu \leq M$ such that

$$\boxed{\int_a^b f(x)\,g(x)\,dx = \mu \int_a^b g(x)\,dx.}$$

Proof: let $g(x) \geq 0 \ \forall x \in [a,b]$.

Since $f \in R[a,b]$

$\Rightarrow f$ is bounded on $[a,b]$

$\therefore$ If m, M are the infimum and supremum of f on $[a,b]$

then $m \leq f(x) \leq M \ \forall x \in [a,b]$.

Since $g(x) \geq 0 \ \forall x \in [a,b]$.

$\therefore m\,g(x) \leq f(x)\,g(x) \leq M g(x) \ \forall x \in [a,b]$

$$\Rightarrow \int_a^b m\,g(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq \int_a^b M g(x)\,dx$$

$$\Rightarrow m\int_a^b g(x)\,dx \leq \int_a^b f(x)g(x)\,dx \leq M\int_a^b g(x)\,dx.$$

$\Rightarrow \exists\ \mu \in [m, M]$ such that

$$\int_a^b f(x)\,g(x)\,dx = \mu \int_a^b g(x)\,dx \quad \text{(1)}$$

Now suppose that $g(x) \leq 0 \ \forall x \in [a,b]$.

$\Rightarrow -g(x) \geq 0 \ \forall x \in [a,b]$.

$\Rightarrow \exists\ \mu \in [m, M]$, we have

$$\int_a^b f(x)\,(-g(x))\,dx = \mu \int_a^b (-g(x))\,dx$$

(from (1))

$$\Rightarrow \int_a^b f(x).\,g(x)\,dx = \mu \int_a^b g(x)\,dx$$

Note: If $g(x) = 1 \ \forall x \in [a,b]$ then $g \in R[a,b]$ and $g(x) > 0 \ \forall x \in [a,b]$

∴ By Mean Value Theorem, we have

$$\int_a^b f(x).(1)\,dx = \mu\int_a^b 1\,dx$$

$$= \mu(b-a) \text{ where } \mu\in[m,M]$$

∴ $$\int_a^b f(x)\,dx = \mu(b-a).$$

Corollary: If f is continuous on $[a,b]$, $g\in R[a,b]$ and $g(x)\geq 0$ (or) $g(x)\leq 0$ $\forall x\in[a,b]$ then $\exists$ $c\in[a,b]$ such that

$$\boxed{\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx.}$$

∵ $f(c)=\mu$, $\forall$ $c\in[a,b]$, $\mu\in[m,M]$

Proof:- Since f is continuous on $[a,b]$

$\Rightarrow f\in R[a,b]$

∴ $\exists$ $\mu\in[m,M]$ such that

$$\int_a^b f(x)g(x)\,dx = \mu\int_a^b g(x)\,dx.$$

Since f is continuous on $[a,b]$, it attains every value between its bounds m, M.

∴ $\mu\in[m,M] \Rightarrow \exists$ $c\in[a,b]$ such that $f(c)=\mu$.

∴ $$\int_a^b f(x)g(x)\,dx = f(c)\int_a^b g(x)\,dx$$

Problems:-

→ show that $\exists$ $\xi\in[0,\pi/2]$ such that

$$\int_0^{\pi/2} x\sin x\,dx = \xi$$

Soln: Let $f(x)=x$, $g(x)=\sin x$

then $f(x)$ is continuous and $g(x)$ is integrable on $[0,\pi/2]$

and $g(x)=\sin x\geq 0$ $\forall x\in[0,\pi/2]$

Applying first mean value theorem, we get,

$$\int_0^{\pi/2} x.\sin x\,dx = \xi\int_0^{\pi/2}\sin x\,dx.$$

$$= \xi(1)$$

$$= \xi$$

→ Prove that $\frac{1}{\pi}\leq\int_0^1 \frac{\sin\pi x}{1+x^2}\,dx\leq\xi$

Sol'n: Let $f(x)=\frac{1}{1+x^2}$, $g(x)=\sin\pi x$ $\forall x\in[0,1]$

then f,g are continuous on $[0,1]$ and hence integrable on $[0,1]$.

and $g(x)\geq 0$ $\forall x\in[0,1]$

Since f is decreasing on $[0,1]$

∴ infimum $f = f(1) = \frac{1}{2}$

Supremum $f = f(0) = 1$.

∴ By first Mean Value theorem,

$\exists\ \mu \in [\tfrac{1}{2}, 1]$.

Such that $\int_0^1 f(x)g(x) = \mu \int_0^1 g(x)\,dx$

$$\Rightarrow \int_0^1 \frac{\sin \pi x}{1+x^2}\,dx = \mu \int_0^1 \sin \pi x\,dx$$

$$= \mu \left[\frac{-\cos \pi x}{\pi}\right]_0^1$$

$$= \mu\left[\frac{2}{\pi}\right] \quad \text{--- (1)}$$

Since f is continuous on $[0,1]$.

it attains every value between its bounds $\tfrac{1}{2}$ and 1.

$\therefore \mu \in [\tfrac{1}{2}, 1]$

$\Rightarrow$ a number $c \in [0,1]$ such that

$f(c) = \mu$

from (1)

$$f(c) = \frac{\pi}{2} \int_0^1 \frac{\sin \pi x}{1+x^2}\,dx \quad \text{--- (2)}$$

But $0 \le c \le 1$ and f is decreasing on $[0,1]$

$\Rightarrow f(0) \geq f(c) \geq f(1)$

$\Rightarrow 1 \geq f(c) \geq \tfrac{1}{2}$

$$\Rightarrow \tfrac{1}{2} \le \tfrac{\pi}{2} \int_0^1 \frac{\sin \pi x}{1+x^2}\,dx \le 1$$

$$\Rightarrow \frac{1}{\pi} \le \int_0^1 \frac{\sin \pi x}{1+x^2}\,dx \le \frac{2}{\pi}$$

H.W $\rightarrow$ Prove that $\dfrac{\pi^2}{9} \le \displaystyle\int_{\pi/6}^{\pi/2} \frac{x}{\sin x}\,dx \le \frac{2\pi^2}{9}$

Sol^n: Let $f(x) = \dfrac{1}{\sin x}$, $g(x) = x$

H.W. $\rightarrow$ Prove that $\dfrac{1}{3\sqrt{2}} \le \displaystyle\int_0^1 \frac{x^2}{\sqrt{1+x^2}}\,dx \le \frac{1}{3}$.

Sol^n: Let $f(x) = \dfrac{1}{\sqrt{1+x^2}}$, $g(x) = x^2$

H.W. Prove that $\dfrac{\pi^3}{24} \le \displaystyle\int_0^{\pi} \frac{x^2}{5+3\cos x}\,dx \le \frac{\pi^3}{6}$

Let $f(x) = \dfrac{1}{5+3\cos x}$, $g(x) = x^2$

H.W. $\rightarrow$ Prove that $\dfrac{\pi}{4} \le \displaystyle\int_0^{\pi/4} \sec x\,dx \le \frac{\pi}{2\sqrt{2}}$.

Let $f(x) = \sec x$ $\quad g(x) = 1$.

$\rightarrow$ By applying Mean value theorem of integral calculus, show that $e^{1/4} < 4/3 < e^{1/3}$ by considering $\displaystyle\int_3^4 \tfrac{1}{x}\,dx$.

Sol^n: Let $f(x) = \tfrac{1}{x}$, $g(x) = 1$ $\forall x \in [3,4]$

then f, g are continuous on $[3,4]$ and hence integrable on $[3,4]$.

and $g(x) = 1 > 0$ $\forall x \in [3,4]$.

Since $f(x)$ is decreasing function on $[3,4]$.

$\therefore$ infimum $f = f(4) = \tfrac{1}{4}$

Supremum $f = f(3) = \tfrac{1}{3}$.

$\therefore$ By first Mean value theorem

$$\int_3^4 f(x)\cdot g(x)\,dx = f(c) \int_3^4 g(x)\,dx$$

$$\Rightarrow \int_3^4 f(x)\,dx = f(c) \int_3^4 (1)\,dx$$

$$= f(c)(4-3)$$

$$= f(c) \quad \text{where } c \in [3,4]$$

Now. $3 \le c \le 4$

$\Rightarrow f(3) \ge f(c) \ge f(4)$

($\because f$ is decreasing)

$\Rightarrow \frac{1}{3} \ge f(c) \ge \frac{1}{4}$

$\Rightarrow \frac{1}{4} \le \int_3^4 f(x)\,dx \le \frac{1}{3}$

$\Rightarrow \frac{1}{4} \le (\log x)_3^4 \le \frac{1}{3}$ $\left[\because f(x) = \frac{1}{x}\right]$

$\Rightarrow \frac{1}{4} \le \log\left(\frac{4}{3}\right) \le \frac{1}{3}$

$\Rightarrow \frac{1}{4} \le \log(4/3) \le \frac{1}{3}$

$\Rightarrow e^{1/4} \le 4/3 \le e^{1/3}$.

Bonnet's Mean Value Theorem:-

Let $g \in R[a,b]$ and let f be monotonically decreasing and non-negative on $[a,b]$. then for some $\xi \in [a,b]$ such that

$$\int_a^b f(x)g(x)\,dx = f(a)\int_a^{\xi} g(x)\,dx$$

(or)

Let $g \in R[a,b]$ and let f be monotonically increasing and non-negative on $[a,b]$ then for some $\eta \in [a,b]$ such that

$$\int_a^b f(x)g(x)\,dx = f(b)\int_{\eta}^{b} g(x)\,dx$$

Problems:-

- If $0 < a < b$, show that $\left|\int_a^b \frac{\sin x}{x}\,dx\right| < \frac{2}{a}$

Sol^n: Let $f(x) = \frac{1}{x}$; $g(x) = \sin x$ $\forall x \in [a,b], a > 0$

Since $f(x) = \frac{1}{x}$ $\forall x \in [a,b], a>0$.

$\Rightarrow f(x)$ is monotonically decreasing on $[a,b]$.

and $f(x) > 0$ $\forall x \in [a,b], a > 0$.

and $g(x) = \sin x$ $\forall x \in [a,b]$

$\Rightarrow g(x)$ is continuous on $[a,b]$

$\Rightarrow g \in R[a,b]$.

$\therefore$ The conditions of Bonnet's Mean Value theorem are satisfied.

$\therefore \exists\, \xi \in [a,b]$ such that

$$\int_a^b f(x)g(x)\,dx = f(a)\int_a^{\xi} g(x)\,dx$$

$$\Rightarrow \int_a^b \frac{\sin x}{x}\,dx = \frac{1}{a}\int_a^{\xi} \sin x\,dx$$

$$= \frac{1}{a}\left[-\cos x\right]_a^{\xi}$$

$$= \frac{1}{a}\left[-\cos\xi + \cos a\right]$$

$$\left|\int_a^b \frac{\sin x}{x}\,dx\right| = \frac{1}{a}\left|\cos a - \cos\xi\right|$$

$$\le \frac{1}{a}\left[|\cos a| + |\cos\xi|\right]$$

$$\le \frac{1}{a}(1+1) \quad (\because |\cos\theta| \le 1)$$

$$= \frac{2}{a}$$

$$\therefore \left|\int_a^b \frac{\sin x}{x}\,dx\right| \le \frac{2}{a}.$$

5(i) If $f(x) = x$, $g(x) = e^x$ verify the first mean value theorem in $[-1,1]$

(ii) verify Bonnet's mean value theorem in $[-1,1]$ for the functions $f(x) = e^x$ and $g(x) = x$.

Sol'n :- (i) $f(x) = x$, $g(x) = e^x \ \forall x \in [-1,1]$

$\Rightarrow f, g$ are continuous on $[-1,1]$

$\Rightarrow f, g \in R[-1,1]$.

and $g(x) = e^x > 0 \ \forall x \in [-1,1]$.

$\therefore$ The conditions of first mean value theorem are satisfied.

Now $\int_{-1}^{1} f(x)\, g(x)\, dx = \int_{-1}^{1} x e^x\, dx$

$= \left[xe^x - e^x\right]_{-1}^{1}$

$= (e^1 - e^1) - (-e^{-1} - e^{-1})$

$= 0 + 2e^{-1}$

$= \frac{2}{e}$ —— (1)

Now $\int_{-1}^{1} g(x)\, dx = \int_{-1}^{1} e^x\, dx = \left[e^x\right]_{-1}^{1}$

$= e^1 - e^{-1}$

$= \frac{e^2 - 1}{e}$

Since f is continuous on $[-1,1]$; it attains every value between $f(-1) = -1$ and $f(1) = 1$.

If $\mu = \frac{2}{e^2 - 1}$ then $0 < \mu < 1$, as $e > 2$.

$\Rightarrow e^2 > 4$

$\Rightarrow e^2 - 1 > 3$.

Now $\exists\, c \in [-1,1]$ such that

$f(c) = \frac{2}{e^2 - 1}$

$\therefore$ we have $f(c) \int_{-1}^{1} g(x)\, dx = \frac{2}{e^2-1} \cdot \frac{e^2 - 1}{e}$ —— (2)

$\therefore$ from (1) & (2) we have

$$\int_{-1}^{1} f(x)\, g(x)\, dx = f(c) \int_{-1}^{1} g(x)\, dx$$

$\therefore$ The first mean value theorem is verified.

(ii) Since $g(x) = x \ \forall x \in [-1,1]$

$\Rightarrow$ It is continuous on $[-1,1]$

$\Rightarrow g \in R[-1,1]$

and $f(x) = e^x$ is monotonically increasing and +ve on $[-1,1]$.

$\therefore$ The conditions of Bonnet's mean value theorem are satisfied.

Now $\int_{-1}^{1} f(x)\, g(x)\, dx = \int_{-1}^{1} x e^x\, dx$

$= 2/e$

and $\int_{\eta}^{1} g(x)\, dx = \int_{\eta}^{1} x\, dx = \frac{1}{2}(1 - \eta^2)$ for $\eta \in [-1,1]$

$\therefore f(1) \int_{\eta}^{1} g(x)\, dx = e \cdot \frac{1}{2}(1 - \eta^2)$

Let us choose η such that

$\frac{2}{e} = \frac{e}{2}(1 - \eta^2)$

i.e. $\eta^2 = \frac{e^2 - 4}{e^2}$

$\Rightarrow \eta = \frac{\sqrt{e^2 - 4}}{e}$; $0 < \eta < 1$.

For this value of η, we have

$$\int_{-1}^{1} f(x)\, g(x)\, dx = f(1) \int_{\eta}^{1} g(x)\, dx.$$

$\therefore$ Bonnet's mean value theorem is verified.

→ Show that the Bonnet's Mean Value theorem doesnot hold on $[-1,1]$ for $f(x) = g(x) = x^2$.

Sol'n:- $f(x) = x^2$ is not monotonic on $[-1,1]$. Because for the interval $[-1,0]$, it is decreasing and for the interval $[0,1]$ it is increasing.

∴ The conditions of Bonnet's mean value theorem are not satisfied.

∴ Bonnet's mean value theorem doesnot hold on $[-1,1]$.

→ **Second Mean Value theorem :-**

Let $g \in R[a,b]$ and let f be bounded and monotonic on $[a,b]$ then

$$\int_a^b fg = f(a)\int_a^{\xi} g + f(b)\int_{\xi}^b g.$$

Proof:- Let f be monotonically decreasing on $[a,b]$ then $f(x)-f(b)$ is monotonically decreasing and non-negative on $[a,b]$

∴ By Bonnet's theorem $\exists\ \xi \in [a,b]$ such that

$$\int_a^b (f(x)-f(b))\, g(x)\,dx = (f(a)-f(b))\int_a^{\xi} g(x)\,dx$$

$$\Rightarrow \int_a^b f(x)g(x)\,dx - f(b)\int_a^b g(x)\,dx = f(a)\int_a^{\xi} g(x)\,dx - f(b)\int_a^{\xi} g(x)\,dx.$$

$$\therefore \int_a^b f(x)g(x)\,dx = f(a)\int_a^{\xi} g(x)\,dx + f(b)\left[\int_a^b g(x)\,dx - \int_a^{\xi} g(x)\,dx\right].$$

$$= f(a)\int_a^{\xi} g(x)\,dx + f(b)\left[\int_a^b g(x)\,dx + \int_{\xi}^a g(x)\,dx\right]$$

$$= f(a)\int_a^{\xi} g(x)\,dx + f(b)\int_{\xi}^b g(x)\,dx.$$

Now if f is monotonically increasing on $[a,b]$ then $-f$ is monotonically decreasing on $[a,b]$.

$$\therefore \int_a^b [-f(x)]\,g(x)\,dx = [-f(a)]\int_a^{\xi} g(x)\,dx + [-f(b)]\int_{\xi}^b g(x)\,dx.$$

$$\Rightarrow \int_a^b f(x)g(x)\,dx = f(a)\int_a^{\xi} g(x)\,dx + f(b)\int_{\xi}^b g(x)\,dx.$$

→* **Integral as the limit of a Sum!**

(Formula given in Pg.No.45)

→ Evaluate $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{n^2}{(n^2+r^2)^{3/2}}$.

Sol'n: The given limit changes in to

$$\lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n} \frac{1}{\left[1+\left(\frac{r}{n}\right)^2\right]^{3/2}} \rightarrow \int_0^1 \frac{dx}{(1+x^2)^{3/2}}$$

[As per formula ∵ $\frac{r}{n} = x$ & $\frac{1}{n} = dx$]

$$= \int_0^{\pi/4} \frac{\sec^2\theta}{\sec^3\theta}\,d\theta$$

[Putting $x = \tan\theta$ ⇒ $dx = \sec^2\theta\,d\theta$; limits for θ are 0 to $\pi/4$]

$$= \int_0^{\pi/4} \cos\theta \, d\theta = [\sin\theta]_0^{\pi/4}$$

$$= \frac{1}{\sqrt{2}} - 0 = \frac{1}{\sqrt{2}}$$

→ Evaluate

$$\lim_{n\to\infty} \left[\frac{1}{n} + \frac{n^2}{(n+1)^3} + \frac{n^2}{(n+2)^3} + \cdots + \frac{1}{8n} \right]$$

Sol'n: The given limit

$$\lim_{n\to\infty} \left[\frac{n^2}{n^3} + \frac{n^2}{(n+1)^3} + \frac{n^2}{(n+2)^3} + \cdots + \frac{n^2}{8n^3} \right]$$

$$= \lim_{n\to\infty} \left[\frac{n^2}{(n+0)^3} + \frac{n^2}{(n+1)^3} + \frac{n^2}{(n+2)^3} + \cdots + \frac{n^2}{(n+n)^3} \right]$$

$$= \lim_{n\to\infty} \sum_{r=0}^{n} \frac{n^2}{(n+r)^3}$$

$$= \lim_{n\to\infty} \sum_{r=0}^{n} \frac{n^2}{n^3\left(1+\left(\frac{r}{n}\right)\right)^3} = \lim_{n\to\infty} \sum_{r=0}^{n} \frac{1}{n} \cdot \frac{1}{\left(1+\frac{r}{n}\right)^3}$$

$$= \int_0^1 \frac{dx}{(1+x)^3} = -\frac{1}{2}\left[\frac{1}{(1+x)^2}\right]_0^1$$

$$= -\frac{1}{2}\left[\frac{1}{(1+1)^2} - \frac{1}{(1+0)^2}\right]$$

$$= -\frac{1}{2}\left[\frac{1}{4} - 1\right]$$

$$= -\frac{1}{2}\left[\frac{1-4}{4}\right] = -\frac{1}{2}\left(-\frac{3}{4}\right) = \frac{3}{8}$$

→ Find the limit $\left\{\frac{n!}{n^n}\right\}^{1/n}$ when $n \to \infty$

Sol'n: Let $A = \lim_{n\to\infty} \left\{\frac{n!}{n^n}\right\}^{1/n}$

$$= \lim_{n\to\infty} \left[\frac{1.2.3 \cdots n}{n.n.n. \cdots n}\right]^{1/n}$$

$$= \lim_{n\to\infty} \left[\frac{1}{n} \cdot \frac{2}{n} \cdot \frac{3}{n} \cdots \frac{n}{n}\right]^{1/n}$$

$$\log A = \lim_{n\to\infty} \frac{1}{n}\left[\log\frac{1}{n} + \log\frac{2}{n} + \cdots + \log\left(\frac{n}{n}\right)\right]$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n} \log\left(\frac{r}{n}\right)$$

$$= \int_0^1 \log x \, dx$$

$$= \int_0^1 (\log x) \cdot 1 \, dx$$

$$= [(\log x) x]_0^1 - \int_0^1 \frac{1}{x} \cdot x \, dx.$$

$$= 0 - [x]_0^1 = -1$$

$$A = e^{-1} = \frac{1}{e}$$

→ Evaluate $\lim_{n\to\infty} \left[\left(1+\frac{1}{n^4}\right)\left(1+\frac{2^4}{n^4}\right)^{1/2}\left(1+\frac{3^4}{n^4}\right)^{1/3} \cdots \left(1+\frac{n^4}{n^4}\right)^{1/n}\right]$

Sol'n: – Let $A = \lim_{n\to\infty} \left[\left(1+\frac{1}{n^4}\right)\left(1+\frac{2^4}{n^4}\right)^{1/2}\left(1+\frac{3^4}{n^4}\right)^{1/3} \cdots \left(1+\frac{n^4}{n^4}\right)^{1/n}\right]$

$$\log A = \lim_{n\to\infty} \log\left(1+\frac{1}{n^4}\right) + \frac{1}{2}\log\left(1+\frac{2^4}{n^4}\right) + \frac{1}{3}\log\left(1+\frac{3^4}{n^4}\right) + \cdots + \frac{1}{n}\log\left(1+\frac{n^4}{n^4}\right)$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{r} \log\left(1+\frac{r^4}{n^4}\right)$$

$$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n} \cdot \frac{n}{r} \log\left(1+\left(\frac{r}{n}\right)^4\right)$$

$$= \int_0^1 \frac{1}{x} \cdot \log(1+x^4) \, dx.$$

Note!

$$= \int_0^1 \frac{1}{x}\left[x^4 - \frac{x^8}{2} + \frac{x^{12}}{3} - \cdots\right] dx$$

$$= \int_0^1 \left[x^3 - \frac{x^7}{2} + \frac{x^{11}}{3} - \cdots\right] dx$$

$$= \left[\frac{x^4}{4} - \frac{x^8}{16} + \frac{x^{12}}{36} - \cdots\right]_0^1$$

$$= \frac{1}{4} - \frac{1}{16} + \frac{1}{36} + \cdots$$

$$= \frac{1}{4}\left[1 - \frac{1}{4} + \frac{1}{9} + \cdots\right]$$

$$= \frac{1}{4}\left[1 - \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right] = \frac{1}{4} \cdot \frac{\pi^2}{12}$$

$$\Rightarrow \log A = \pi^2/48$$

$$\Rightarrow \boxed{A = e^{\pi^2/48}}$$

→ show that

$$\lim_{n\to\infty} \frac{1}{n}\left(e^{3/n} + e^{6/n} + e^{9/n} + \cdots + e^{3n/n}\right) = \frac{1}{3}(e^3 - 1)$$

Sol'n: $\lim_{n\to\infty} \frac{1}{n}\left(e^{3/n} + e^{6/n} + \cdots + e^{3n/n}\right)$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} e^{3r/n}$$

$$= \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} e^{3(r/n)}$$

$= \lim_{n\to\infty} \sum_{r=1}^{n} \frac{1}{n} e^{3(r/n)}$

$= \int_0^1 e^{3x}dx = \left[\frac{e^{3x}}{3}\right]_0^1 = \frac{e^3}{3} - \frac{e^0}{3}$

$= \frac{e^3}{3} - \frac{1}{3}$

→ show that the function f defined by $f(x) = \begin{cases} 0, & \text{if } x \text{ is an integer} \\ 1 & \text{otherwise.} \end{cases}$ is integrable on $[0,m]$, m being a +ve integer.

Note!

Sol'n: $f(x) = \begin{cases} 0 & \text{if } x = 0,1,2 \ldots m \\ 1 & \text{if } r-1 < x < r, \; r = 1,2 \ldots n \end{cases}$

⇒ f is bounded and has only $m+1$ points of finite discontinuity at $0, 1, 2, \ldots, m-1, m$.

Since the points of discontinuity of f $[0,m]$ are finite in number.

f is integrable on $[0,m]$.

Now $\int_0^m f(x)dx = \int_0^1 f(x)dx + \int_1^2 f(x)dx + \cdots + \int_{m-1}^{m} f(x)dx$

$= \int_0^1 1dx + \int_1^2 1dx + \cdots + \int_{m-1}^{m} 1dx$

$= (1-0) + (2-1) + \cdots + [m-(m-1)]$

$= 1 + 1 + \cdots + 1$ (m times)

$= m$

→ Let f be a function on $[0,1]$ defined by $f(x) = \begin{cases} 1 & \text{if } x \neq 1/2 \\ 0 & \text{if } x = 1/2 \end{cases}$ show that $f \in R[0,1]$ and find $\int_0^1 f$.

Sol'n: clearly $0 \le f(x) \le 1 \;\forall x \in [0,1]$

⇒ $f(x)$ is bounded on $[0,1]$ and the function has only point of discontinuity $\frac{1}{2}$.

∴ $f \in R[0,1]$

Now $\int_0^1 f(x)dx = \int_0^{1/2} f(x)dx + \int_{1/2}^{1} f(x)dx$.

$= \int_0^{1/2} 1dx + \int_{1/2}^{1} 1dx = \frac{1}{2} + 1 - \frac{1}{2} = 1$

H.W.

→ A function f is defined on $[-1,1]$ as follows $f(x) = \begin{cases} K \text{ (+ve constant) when } x \neq 0 \\ 0 \text{ when } x = 0 \end{cases}$

show f is integrable on $[-1,1]$ and $\int_{-1}^{1} f(x)dx = 2K$.

→ If f is continuous on $[a,b]$, $f(x) \ge 0 \;\forall x \in [a,b]$ and $\int_a^b f(x)dx = 0$ then $f(x) = 0 \;\forall x \in [a,b]$.

Sol'n:- If possible suppose that $f(x) \neq 0 \;\forall x \in [a,b]$.

Then $\exists\, c \in [a,b]$ such that $f(c) \neq 0 \Rightarrow f(c) > 0$

Let $\epsilon = \frac{1}{2} f(c)$.

Suppose $a < c < b$

Since f is continuous at c $\exists$ a $\delta > 0$ such that $|f(x) - f(c)| < \epsilon$ whenever $|x - c| < \delta$

$\Rightarrow f(c) - \epsilon < f(x) < f(c) + \epsilon$

$\Rightarrow f(x) > f(c) - \frac{1}{2} f(c) = \frac{1}{2} f(c)$

$\therefore f(x) > \frac{1}{2} f(c)$

Now choosing δ such that

$a < c - \delta < c < c + \delta < b$

$\int_a^b f(x)dx = \int_a^{c-\delta} f(x)dx + \int_{c-\delta}^{c+\delta} f(x)dx + \int_{c+\delta}^{b} f(x)dx$

$\ge \int_{c-\delta}^{c+\delta} f(x)dx \quad \because f(x) \ge 0 \;\forall x \in [a,b]$

$> \int_{c-\delta}^{c+\delta} \frac{1}{2} f(c)\, dx = \frac{1}{2} f(c)(2\delta)$

$= \delta f(c) > 0$.

$\therefore \int_a^b f(x)dx > 0$.

which is contradiction to $\int_a^b f(x)dx = 0$

If $c = a$ or $c = b$ then also we get a contradiction.

$\therefore f(x) \neq 0 \;\forall x \in [a,b]$ is wrong.

$\therefore f(x) = 0 \;\forall x \in [a,b]$.

Miscellaneous content
of Riemann Integral.

D(i)

EXAMPLES

Example 1 *Compute $L(P, f)$ and $U(P, f)$ if*

(*i*) $f(x) = x^2$ on $[0, 1]$ and $P = \{0, 1/4, 2/4, 3/4, 1\}$ be a partition of $[0, 1]$

(*ii*) $f(x) = x$ on $[0, 1]$ and $P = \{0, 1/3, 2/3, 1\}$ be a partition of $[0, 1]$

(Meerut 2010; Pune 2010)

(*iii*) $f(x) = x$ on $[0, 1]$ and $P = \{0, 1/4, 2/4, 3/4, 1\}$ on $[0, 1]$ **(Meerut 2012)**

Solution. (*i*) Here partition P divides $[0, 1]$ into sub-intervals

$$I_1 = [0, 1/4], I_2 = [1/4, 2/4], I_3 = [2/4, 3/4] \text{ and } I_4 = [3/4, 1].$$

The length of these intervals are given by $\delta_1 = \delta_2 = \delta_3 = \delta_4 = 1/4$

Let M_r and m_r be the supremum and infimum of f in interval I_r, $r = 1, 2, 3, 4$,. Since $f(x) = x^2$ is increasing on $[0, 1]$, we have

$$m_1 = 0, M_1 = 1/16; m_2 = 1/16, M_2 = 4/16; m_3 = 4/16, M_3 = 9/16; m_4 = 9/16, M_4 = 1$$

$$\therefore \quad L(P, f) = \sum_{r=1}^{4} m_r \delta_r = m_1\delta_1 + m_2\delta_2 + m_3\delta_3 + m_4\delta_4$$

$$= 0 \times (1/4) + (1/16) \times (1/4) + (4/16) \times (1/4) + (9/16) \times (1/4) = 7/32$$

$$U(P, f) = \sum_{r=1}^{4} M_r \delta_r = M_1\delta_1 + M_2\delta_2 + M_3\delta_3 + M_4\delta_4$$

$$= (1/16) \times (1/4) + (4/16) \times (1/4) + (9/16) \times (1/4) + 1 \times (1/4) = 15/32$$

(ii). Ans. $L(P, f) = 1/3$; $U(P, f) = 2/3$ **(iii). Ans.** $L(P, f) = 3/8$, $U(P, f) = 5/8$

Example 2 *Show that a constant function is integrable*

(Agra 2002, 06; Calicut 2004: Purvanchal 1997)

Solution. Let $$f(x) = k, \ \forall\, x \in [a, b]. \qquad ...(1)$$

be a constant function, k being a constant. f is a bounded function. Let $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$ be any partition of $[a, b]$.

If M_r and m_r be respectively the supremum and infimum of f in $(x_{r-1}, x_r]$. Then

$$M_r = k \text{ and } m_r = k \text{ as } f(x) = k \ \forall\, x \in [a, b]$$

$$\therefore \quad U(P, f) = \sum_{r=1}^{n} M_r \delta_r = \sum_{r=1}^{n} k(x_r - x_{r-1}) = k \sum_{r=1}^{n} (x_r - x_{r-1})$$

$$= k[(x_1 - x_2) + (x_2 - x_1) + ... + (x_n - x_{n-1})]$$

$$= k(x_n - x_0) = k(b - a) = \text{constant}$$

and $$L(P, f) = \sum_{r=1}^{n} m_r \delta_r = \sum_{r=1}^{n} k(x_r - x_{r-1}) = k(b - a) = \text{constant, as before}$$

$$\therefore \quad \overline{\int_a^b} f(x)\,dx = \inf U(P,f) = \inf\{k(b-a)\} = k(b-a)$$

and $$\underline{\int_a^b} f(x)\,dx = \sup L(P,f) = \sup\{k(b-c)\} = k(b-a)$$

Since $$\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx,$$

f is Riemann integrable and $$\int_a^b f(x)\,dx = k(b-a)$$

Example 3 *If $f(x) = x^3$ is defined on $[0, a]$, show that $f \in R[a, 0]$ and $\int_0^a f(x)\,dx = \frac{a^4}{4}$*

(Agra 2000; Meerut 2002; Purvanchal 1998)

Solution Let $P = \left\{0, \frac{a}{n}, \frac{2a}{n}, \ldots, \frac{(r-1)a}{n}, \frac{ra}{n}, \ldots, \frac{na}{n} = a\right\}$ be any partition of $[0, a]$

Then, its rth sub-interval $= I_r = \left[\frac{(r-1)a}{n}, \frac{ra}{n}\right], r = 1, 2, \ldots, n$

If δ_r be the length of I_r, then

$$\delta_r = \frac{ra}{n} - \frac{(r-1)a}{n} = \frac{a}{n}.$$

Let M_r and m_r be respectively the supremum and infimum of f in I_r. Since f is increasing on $[0, a]$, we have

$$M_r = \frac{r^3a^3}{n^3} \text{ and } m_r = \frac{(r-1)^3a^3}{n^3}, \text{ as } f(x) = x^3 \ \forall x \in [0, a]$$

$$U(P,f) = \sum_{r=1}^{n} M_r\delta_r = \sum_{r=1}^{n} \frac{r^3a^3}{n^3}\cdot\frac{a}{n} = \frac{a^4}{n^4}\sum_{r=1}^{n} r^3$$

$$= \frac{a^4}{n^4}(1^3 + 2^3 + \cdots + n^3) = \frac{a^4}{n^4}\cdot\frac{n^2(n+1)^2}{4} = \frac{a^4}{4}\left(1+\frac{1}{n}\right)^2$$

and $$L(P,f) = \sum_{r=1}^{n} m_r\delta_r = \sum_{r=1}^{n} \frac{(r-1)^3}{n^3}\cdot\frac{a}{n} = \frac{a^4}{n^4}\sum_{r=1}^{n}(r-1)^3$$

$$= \frac{a^4}{n^2}\{(1^3 + 2^3 + \cdots + (n-1)^3\} = \frac{a^4}{n^2}\cdot\frac{(n-1)^2n^2}{4} = \frac{a^4}{4}\left(1-\frac{1}{n}\right)^2$$

$$\left[\text{Since } 1^3 + 2^3 + \ldots + n^3 = \frac{n^2(n+1)^2}{4} \Rightarrow 1^3 + 2^3 + \ldots + (n-1)^3 = \frac{(n-1)^2(n-1+1)^2}{4}\right]$$

$$\therefore \quad \overline{\int_a^b} f(x)\,dx = \lim_{n\to\infty} U(P,f) = \lim_{n\to\infty} \frac{a^4}{4}\left(1+\frac{1}{n}\right)^2 = \frac{a^4}{4}$$

and $$\underline{\int_a^b} f(x)\,dx = \lim_{n\to\infty} L(P,f) = \lim_{n\to\infty} \frac{a^4}{4}\left(1-\frac{1}{n}\right)^2 = \frac{a^4}{4}$$

Since $\displaystyle\int_0^a f(x)\,dx = \int_0^a f(x)\,dx,$

f is Riemann integrable and $\displaystyle\int_0^a f(x)\,dx = \frac{a^4}{4}.$

Example 4 *Show that $f(x) = \sin x$ is integrable on $[0, \pi/2]$ and $\displaystyle\int_0^{\pi/2} \sin x\,dx = 1$.*

(Meerut 1996)

Solution Let any partition P of $[0, \pi/2]$ be given by

$$P = \left\{0, \frac{\pi}{2n}, \frac{2\pi}{2n}, \dots, \frac{(r-1)\pi}{2n}, \frac{r\pi}{2n}, \dots, \frac{n\pi}{2n} = \frac{\pi}{2}\right\}$$

Then, its rth subinterval $= I_r = \left[\frac{(r-1)\pi}{2n}, \frac{r\pi}{2n}\right]$, $r = 1, 2, \dots, n$

and $\delta_r =$ length of $I_r = (r\pi)/n - (r-1)\pi/2n = \pi/2n$

Let M_r and m_r be respectively the supremum and infimum of f in I_r. Since f is increasing on $(0, \pi/2)$, we have

$$M_r = \sin\frac{r\pi}{2n},\ m_r = \sin\frac{(r-1)\pi}{2n}, \text{ as } f(x) = \sin x \text{ on } \left[0, \frac{\pi}{2}\right]$$

In what follows, we shall use the following result of Trigonometry :

$$\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \dots \text{ to } n \text{ terms} = \frac{\sin\left(\alpha + \frac{n-1}{2}\beta\right)\sin\frac{n\beta}{2}}{\sin(\beta/2)} \quad \dots(1)$$

Now, $$U(P,f) = \sum_{r=1}^{n} M_r\delta_r = \sum_{r=1}^{n} \sin\frac{r\pi}{2n}\cdot\frac{\pi}{2n} = \frac{\pi}{2n}\sum_{r=1}^{n}\sin\frac{r\pi}{2n}$$

$$= \frac{\pi}{2n}\left(\sin\frac{\pi}{2n} + \sin\frac{2\pi}{2n} + \dots + \sin\frac{n\pi}{2n}\right)$$

$$= \frac{\pi}{2n} \times \frac{\sin\left(\frac{\pi}{2n} + \frac{n-1}{2}\cdot\frac{\pi}{2n}\right)\sin\left(\frac{n}{2}\cdot\frac{\pi}{2n}\right)}{\sin(\pi/4n)}, \text{ using (1)}$$

$$= \frac{\pi}{2n} \times \frac{\sin\frac{(n+1)\pi}{4}\sin\frac{\pi}{4}}{\sin(\pi/4n)} = \frac{\pi}{2\sqrt{2}n} \times \frac{\sin(\pi/4 + \pi/4n)}{\sin(\pi/4n)}$$

$$= \frac{\pi}{2\sqrt{2}n} \times \frac{\sin(\pi/4)\cos(\pi/4n) + \cos(\pi/4)\sin(\pi/4n)}{\sin(\pi/4n)}$$

$$= \frac{\pi}{2\sqrt{2}n} \times \frac{1}{\sqrt{2}}\,\frac{\cos(\pi/4n) + \sin(\pi/4n)}{\sin(\pi/4n)} = \frac{\pi}{4n}\left(\cot\frac{\pi}{4n} + 1\right)$$

and $$L(P,f) = \sum_{r=1}^{n} m_r\delta_r = \sum_{r=1}^{n} \sin\frac{(r-1)\pi}{2n}\cdot\frac{\pi}{2n} = \frac{\pi}{2n}\sum_{r=1}^{n}\sin\frac{(r-1)\pi}{2n}$$

$$= \frac{\pi}{2n}\left\{\sin\frac{\pi}{2n} + \sin\frac{2\pi}{2n} + \ldots + \frac{\sin(n-1)\pi}{2n}\right\}$$

$$= \frac{\pi}{2n} \times \frac{\sin\left(\frac{\pi}{2n} + \frac{n-2}{2}\cdot\frac{\pi}{2n}\right)\sin\left(\frac{n-1}{2}\cdot\frac{\pi}{2n}\right)}{\sin(\pi/4n)}, \text{ using (1)}$$

$$= \frac{\pi}{2n} \times \frac{\sin\frac{\pi}{4}\sin\left(\frac{\pi}{4} - \frac{\pi}{4n}\right)}{\sin(\pi/4n)} = \frac{\pi}{2\sqrt{2}n} \times \frac{\sin(\pi/4 - \pi/4\pi)}{\sin(\pi/4n)}$$

$$= \frac{\pi}{2\sqrt{2}n} \times \frac{\sin(\pi/4)\cos(\pi/4n) - \cos(\pi/4)\sin(\pi/4n)}{\sin(\pi/4n)}$$

$$= \frac{\pi}{2\sqrt{2}n} \times \frac{1}{\sqrt{2}}\frac{\cos(\pi/4n) - \sin(\pi/4n)}{\sin(\pi/4n)} = \frac{\pi}{4n}\left(\cot\frac{\pi}{4n} - 1\right)$$

$$\therefore \quad \overline{\int_0^{\pi/2}} f(x)\,dx = \lim_{n\to\infty} U(P, f) = \lim_{n\to\infty}\frac{\pi}{4n}\left(\cot\frac{\pi}{4n} + 1\right) = \lim_{n\to\infty}\left\{\frac{(\pi/4n)}{\tan(\pi/4n)} + \frac{\pi}{4n}\right\} = 1$$

and
$$\underline{\int_0^{\pi/2}} f(x)\,dx = \lim_{n\to\infty} L(P, f) = \lim_{n\to\infty}\frac{\pi}{4n}\left(\cot\frac{\pi}{4n} - 1\right) = \lim_{n\to\infty}\left\{\frac{(\pi/4n)}{\tan(\pi/4n)} - \frac{\pi}{4n}\right\} = 1$$

Since
$$\overline{\int_0^{\pi/2}} f(x)\,dx = \underline{\int_0^{\pi/2}} f(x)\,dx,$$

f in integrable and
$$\int_0^{\pi/2} f(x)\,dx = 1$$

Example 5. *Show by an example that every bounded function need not be Riemonn integrable.* **(*Kanpur 2008*)**

OR

Let $f(x)$ be defind on $[a, b]$ as follows

$$f(x) = \begin{cases} 0, & \text{when } x \text{ is rational} \\ 1, & \text{when } x \text{ is irrational} \end{cases}$$

Show that f is not integrable on $[a, b]$. **[Agra 2007, 2011; Delhi Maths (Prog) 2008 Calicut 2004; Garhwal 1996, Kumaun 1995; I.A.S. 2000; Meerut 1993; Nagpur 2003]**

Solution. Clearly $f(x)$ is bounded on $[a, b]$ because $0 \le f(x) \le 1 \;\forall\, x \in [a, b]$

Let $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ be any partition of $[a, b]$ and let its sub-intervals be $I_r = [x_{r-1}, x_r]$, for $r = 1, 2, \ldots n$.

Here $\quad \delta_r =$ the length of $I_r = x_r - x_{r-1}$.

Let M_r and m_r be respectively the supremum and infimum of the function f in I_r. Since rational and irrational points are everywhere dense so every sub-interval I_r will contain rational and irrational numbers. Hence, by definition of $f(x)$, it follows that

$$m_r = 0 \text{ and } M_r = 1, \text{ for } r = 1, 2, \ldots, n.$$

$\therefore \quad U(P, f) = \sum_{r=1}^{n} M_r \delta_r = \sum_{r=1}^{n} 1.\delta_r = \sum_{r=1}^{n} \delta_r = \sum_{r=1}^{n} (x_r - x_{r-1})$ O(iii)

$$= (x_1 - x_0) + (x_2 - x_1) + \ldots + (x_n - x_{n-1}) = x_n - x_0 = b - a$$

and $$L(P, f) = \sum_{r=1}^{n} m_r \delta_r = \sum_{r=1}^{n} 0.\delta_r = 0$$

$\therefore \quad \overline{\int_a^b} f(x)\, dx = \lim_{n\to\infty} U(P, f) = 1$ and $\underline{\int_a^b} f(x)\, dx = \lim_{n\to\infty} L(P, f) = 0$

Since $\overline{\int_a^b} f(x)\, dx \neq \underline{\int_a^b} f(x)\, dx$, so f is not R-integrable.

Example 6. *If $f(x)$ be defined on $[0, 2]$ as follows,*

$$f(x) = x + x^2, \text{ when } x \text{ is rational.}$$
$$= x^2 + x^3, \text{ when } x \text{ is irrational}$$

then evaluate the upper and lower Riemann integrals of f over $[0, 2]$ and show that f is not R-integrable over $[0, 2]$. **(Purvanchal 2007; Agra 2001, Gorakhpur 1996, Kanpur 1994, Lucknow 1996, 97, I.A.S. 1993, Meerut 2010)**

Solution. Here $(x + x^2) - (x^2 + x^3) = x - x^3 = x(1 - x^2)$, so

$$(x + x^2) - (x^2 + x^3) > 0 \text{ when } 0 < x < 1.$$

and $$(x + x^2) - (x^2 + x^3) < 0 \text{ when } 1 < x < 2.$$

Let M_r and m_r be the supremum and infimum of the given function $f(x)$ in I_r where I_r is the rth usual sub-interval of any partition. Then for all values of n we have

$$M_r = x + x^2, \text{ if } 0 < x < 1$$
$$= x^2 + x^3, \text{ if } 1 < x < 2$$

and $$m_r = x^2 + x^3, \text{ if } 0 < x < 1$$
$$= x + x^2, \text{ if } 1 < x < 2$$

Now, by definition,

the upper Riemann integral $= \overline{\int_0^2} f(x)\, dx = \int_0^1 (x + x^2)\, dx + \int_1^2 (x^2 + x^3) dx = \dfrac{83}{12}$

and the lower Riemann integral $= \underline{\int_0^2} f(x)\, dx = \int_0^1 (x^2 + x^3)\, dx + \int_1^2 (x + x^2) dx = \dfrac{53}{12}$

Since $\overline{\int_0^2} f(x)\, dx \neq \underline{\int_0^2} f(x)\, dx$, so f is not R-integrable on $[0, 2]$

Example 7. *Find the upper and lower Riemann integrals for the function f defined on $[0, 1]$ as follows*

$$f(x) = (1 - x^2)^{1/2}, \text{ if } x \text{ is rational.}$$

and $$f(x) = 1 - x, \text{ if } x \text{ is irrational.}$$

Hence show that f is not Riemann integral on $[0, 1]$

(Meerut 2009; Agra 2007; Garhwal 1999; I.A.S. 1992,

Solution Here $f(x) = \sqrt{(1-x^2)} = \sqrt{(1-x)(1+x)}$, *if x is rational.*

and $$f(x) = 1 - x = \sqrt{(1-x)(1-x)}, \text{ if } x \text{ is irrational}$$

So, $$\sqrt{(1-x^2)} = \sqrt{(1-x)(1+x)} > \sqrt{(1-x)(1-x)} \text{ for } 0 < x < 1$$

Thus $$(1-x^2)^{1/2} > 1 - x, \text{ for } 0 < x < 1.$$

Let M_r and m_r be the supremum and infimum of the given function f in I_r where I_r is the usual sub-interval of any partition P of [0 1]. Then, for all values of r. we have

$$M_r = (1-x^2)^{1/2} \text{ and } m_r = 1 - x$$

Now, by definition, the upper Riemann integral

$$= \overline{\int_0^1} f(x)\,dx = \int_0^1 \sqrt{(1-x^2)}\,dx = \left[\frac{1}{2}x\sqrt{(1-x^2)} + \frac{1}{2}\sin^{-1} x\right]_0^1 = \frac{\pi}{4}$$

and the lower Riemonn integral

$$= \underline{\int_0^1} f(x)\,dx = \int_0^1 (1-x)dx = \left[x - \frac{1}{2}x^2\right]_0^1 = \frac{1}{2}$$

Since upper and lower Riemann integrals are not equal, hence the given function f is not R-integrable on [0, 1]

EXERCISES

1. If f is defined on [0, 1] by $f(x) = x \ \forall\ x \in [0, 1]$, then prove that f is Riemann integrable on [0, 1] and $\int_0^1 f(x)\,dx = \frac{1}{2}$. *(Rajasthan 2010; Agra 2003, Garhwal 1997)*

2. If f is defined on $[0, a]$, $a > 0$ by $f(x) = x^2 \ \forall\ x \in [0, a]$ then prove that f is Riemann integrable on $[0, a]$ and $\int_0^a f(x)\,dx = \frac{a^3}{3}$.

   (Agra 1999; Delhi Physics (H) 2000; Garhwal 2001; Meerut 2001)

3. Give an example to show that every bounded function need not be Riemann integrable.
   Hine Refer solved example 5, page 13.12.

4. Show that $(3x + 1)$ is R-integrable on [1, 2] and $\int_1^2 (3x+1)\,dx = \frac{11}{2}$.

   (G.N.D U. Amritsar 2010; Delhi Physics (H) 1993)

5. Show that the function $f : [1, 2] \to \mathbf{R}$ defined by $f(x) = \alpha x + \beta$, $x \in [1, 2]$, where α, β are constants, is integrable on [1, 2]. *(Delhi Physics(H) 2001)*

6. Show that the Dirichlet function $f : [0, 1] \to \mathbf{R}$ defined by

   $$f(x) = 1, \text{ when } x \text{ is rational.}$$
   $$= 0, \text{ when } x \text{ is irrational}$$

   is not integrable on [0, 1]. *(Purvanchal 1994, 96)*

O(iv)

7. (*a*) If $f(x)$ be defined on $[0, 1]$ as follows :
$f(x) = 1$, when x is rational and $f(x) = -1$, when x is irrational , then prove that f is not R-integrable over $[0, 1]$.
(*b*) If a real valued function f is defined on $[a, b]$ by $f(x) = -1$, if x is rational and $f(x) = 1$, if x is irrational then show that f is not R-integrable on $[a, b]$ **[Garhwal 1995]**

8. If $f(x)$ be a function defined on $[0, \pi/4]$ by $f(x) = \cos x$, if x is rational and $f(x) = \sin x$, if x is irrational, then prove that f is not Riemann-integrable over $[0, \pi/4]$. **[Agra 2008]**

9. Given $a > 0, b > 0$ and a function $f(x)$ defined on interval $[a, b]$ such that $f(x) = 1$, for rational x and $f(x) = 2$, for irrational x, then find $\underline{\int_a^b} f(x), dx$ and $\overline{\int_a^b} f(x)\, dx$. Also show that f is not Riemann integrable. *(Garhwal 1996, Purvanchal 98)*

10. Let f be a function defined on $[0, 1]$ as follows $f(x) = \begin{cases} 1, \text{ if } x \neq 1/2 \\ 0, \text{ if } x = 1/2 \end{cases}$

Show that f is R-integrable on $[0, 1]$ and $\int_0^1 f(x)\, dx = 1$.

11. Calculate the values of upper and lower integrals for the function f defined on $[0, 2]$ as follows :
$f(x) = x^2$ when x is rational and $f(x) = x^3$; when x is irrational. **[Ans. 49/12; 31/12]**

12. If $f(x) = 1 + x$, when x is rational
$= x + x^2$, when x is irrational,

then show that $\overline{\int_0^2} f(x)\, dx = \frac{16}{3}$ and $\underline{\int_0^2} f(x)\, dx = \frac{10}{3}$ *(Garhwal 1999)*

13. Show that $f(x) = x$ is integrable on $[a, b]$ and $\int_a^b f(x)\, dx = \frac{1}{2}(b^2 - a^2)$.

14. Prove that if $f \in R[a, b]$, then the value of the integral is uniqually determined. *(Calicut 2004)*

15. A function f is defined on $[0, 1/2]$ by $f(x) = \begin{cases} x, \text{ if } x \text{ is rational} \\ 1 - x, \text{ if } x \text{ is irrational} \end{cases}$

Prove that $\overline{\int_0^{1/2}} f = \frac{3}{8}, \underline{\int_0^{1/2}} f = \frac{1}{8}$ and f is not integrable.

16. Let $f : [0, 1] \to$ R be defined by $f(x) = 1$, if x is rational; $f(x) = 2$, if x is irrational

Show that $\int_0^1 f(x)\, dx$ does not exist. *(Garhwal 1994, Purvanchal 98)*

17. Using only the definition of integral, show that
$f(x) = \sin(1/x)$, x irrational, $0 \leq x \leq 1$
$= 0$, otherwise, $0 \leq x \leq 1$
is not Riemann integrable. *(Delhi Maths (H) 1995)*

18. Show that lower Reimann integral of $\int_0^1 (x^2 + 1)\, dx$ is 4/3. **(Delhi B.Sc. III (Prog.) 2010)**

18. Let

$$h(x) = \begin{cases} x+1 \text{ for rational } x \text{ in } [0,1] \\ 0 \text{ for irrational } x \text{ in } [0,1] \end{cases}$$

Prove that h is not Riemann integrable (*Calicut 20*

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Miscellaneous content of Riemann Integral 1

13.20 INTEGRATION BY PARTS

If $\int_a^b f(x)\,dx, \int_a^b g(x)\,dx$ both exist and

$$F(x) = A + \int_a^x f(x)\,dx, \quad G(x) = B + \int_a^x g(x)\,dx$$

where A, B are two constants, then

$$\int_a^b F(x)\,g(x)\,dx = \left|F(x)\,G(x)\right|_a^b - \int_a^b G(x)\,f(x)\,dx.$$

[Here $|F(x)\,G(x)|_a^b$ denotes the difference $[F(b)\,G(b) - F(a)\,G(a)]$.

Proof. Let $P = \{a = x_0, x_1, x_2, x_3, \ldots x_{r-1}, x_r, \ldots, x_n = b\}$ be a partition of $[a, b]$.

We have $\left|F(x)\,G(x)\right|_a^b = \sum_{r=1}^{r=n} \left[F(x_r)\,G(x_r) - F(x_{r-1})\,G(x_{r-1})\right]$.

$$= \Sigma\, F(x_r)\left[G(x_r) - G(x_{r-1})\right] + \Sigma\, G(x_{r-1})\left[F(x_r) - F(x_{r-1})\right]$$

$$= \Sigma\, F(x_r)\int_{x_{r-1}}^{x_r} g(x)\,dx + \Sigma\, G(x_{r-1})\int_{x_{r-1}}^{x_r} f(x)\,dx \qquad \ldots (1)$$

Let M_r, m_r, O_r denote the bounds and the oscillation of f and M'_r, m'_r, O'_r those of g in $I_r = [x_{r-1}, x_r]$. Now $\forall\, x \in I_r$, we have

$$|g(x) - g(x_r)| \le O'_r, \quad |f(x) - f(x_{r-1})| \le O_r,$$

$$\Rightarrow \begin{cases} g(x_r) - O'_r \le g(x) \le g(x_r) + O'_r; \\ f(x_{r-1}) - O_r \le f(x) \le f(x_{r-1}) + O_r, \end{cases}$$

It follows that

$$\left.\begin{aligned} [g(x_r) - O'_r]\,\delta_r &\le \int_{x_{r-1}}^{x_r} g(x) \le [g(x_r) + O'_r]\,\delta_r; \\ [f(x_{r-1}) - O_r]\,\delta_r &\le \int_{x_{r-1}}^{x_r} f(x)\,dx \le [f(x_{r-1}) + O_r]\,\delta_r \end{aligned}\right\} \qquad \ldots (2)$$

These give

$$\int_{x_{r-1}}^{x_r} g(x)\,dx = [g(x_r) + \theta'_r\, O'_r]\,\delta_r \quad \text{and} \quad \int_{x_{r-1}}^{x_r} f(x)\,dx = [f(x_{r-1}) + \theta_r O_r]\,\delta_r \qquad \ldots (3)$$

where

$$-1 \le \theta_r, \theta'_r \le 1.$$

From (1), (2) and (3), we obtain

$$|f(x)\,G(x)|_a^b = \Sigma\, F(x_r)\, g(x_r)\,\delta_r + \Sigma\, G(x_{r-1})\, f(x_{r-1})\,\delta_r + \sigma \qquad \text{... (4)}$$

where

$$\sigma = \Sigma\left[F(x_r)\,\theta_r'\, O_r' + G(x_{r-1})\,\theta_r O_r\right]\delta_r.$$

Since F, G are continuous, therefore, they are bounded. Let k be a number such that $\forall\, x \in [a, b]$, $|F(x)| \le k, |G(x)| \le k$,

$$\therefore \qquad |\sigma| \le k\,(\Sigma O_r + \Sigma O_r')\,\delta_r.$$

Let the norm of the partition $P \to 0$. Then $\sigma \to 0$.

From (4), we now obtain

$$|F(x)\,G(x)|_a^b = \int_a^b F(x)\, g(x)\, dx + \int_a^b G(x)\, f(x)\, dx.$$

Hence the result.

Cor. If a function g is bounded and integrable in $[a, b]$ and a function f is derivable in $[a, b]$ and its derivative f' is bounded and integrable, then

$$\int_a^b f(x)\, g(x)\, dx = \left| f(x) \int_a^x g(x)\, dx \right|_a^x - \int_a^b \left\{ f'(x) \int_a^x g(x)\, dx \right\} dx$$

$$= f(b) \int_a^b g(x)\, dx - \int_a^b \left\{ f'(x) \int_a^x g(x)\, dx \right\} dx.$$

Example. *Show that the second mean value theorem does not hold good in the interval* $[-1, 1]$ *for* $f(x) = \phi(x) = x^2$. *What about the validity of the generalised first mean value theorem in this case.*

Solution. Since $f(x)$ and $\phi(x)$ are both continuous on $[-1, 1]$, so they are both integrable on $[-1, 1]$. Here $\phi(x) = x^2$ is not monotonic in $[-1, 1]$ because $\phi(x)$ is monotonically decreasing on $[-1, 0]$ and monotonically increasing on $[0, 1]$.

If possible, suppose the second mean value theorem holds for $f(x) = x^2$ and $\phi(x) = x^2$. Then there exist same $\xi \in [-1, 1]$ such that

$$\int_{-1}^1 f(x)\,\phi(x)\, dx = \phi(-1) \int_{-1}^{\xi} f(x)\, dx + \phi(1) \int_{\xi}^1 f(x)\, dx$$

or

$$\int_{-1}^1 x^4 dx = \int_{-1}^{\xi} x^2\, dx + \int_{\xi}^1 x^2 dx$$

or

$$\frac{2}{5} = \frac{1}{3}\left(\xi^3 + 1\right) + \frac{1}{3}\left(1 - \xi^3\right) \qquad \text{so that} \qquad \frac{2}{5} = \frac{2}{3},$$

which is absurd. Hence the second mean value theorem in not true.

We now examine the validity of the generalised first mean value theorem. Here $f(x)$ and $\phi(x)$ are integrable, as before. Also, $\phi(x) = x^2 \Rightarrow \phi(x) \ge 0\ \forall\, x \in [-1, 1]$ *i.e.*, $\phi(x)$ keeps the same sign in $[-1, 1]$. If possible, suppose the generalised first mean value theorem is true for $f(x) = x^2$ and $\phi(x) = x^2$. Then there must exist a number, μ, lying between the bounds 0 and 1 of $f(x)$ such that

$$\int_{-1}^1 f(x)\,\phi(x)\, dx = \mu \int_{-1}^1 \phi(x)\, dx$$

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i.e., $\int_{-1}^{1} x^4 dx = \mu \int_{-1}^{1} x^2 dx$ or $\frac{2}{5} = \mu \times \frac{2}{3}$ or $\mu = \frac{3}{5}$.

Since $0 < 3/5 < 1$, it follows that the generalized first mean value theorem holds for the given functions defined in $[-1, 1]$.

EXERCISES

1. Taking $f(x) = x$ and $\phi(x) = e^x$, verify second mean value theorems for the interval $[-1, 1]$. **[Meerut 2011; Delhi Maths (H) 2006]**

2. Varity first mean value theorem (generalized form) for the function $f(x) = \sin x$ and $\phi(x) = e^x, x \in [0, 1]$ **[Meerut 2011]**

3. Show that the Bonnet's mean value theorem holds on $[-1, 1]$, for $f(x) = e^x$, $\phi(x) = x$.

4. Show that the Bonnet's theorem does not hold on $[-1, 1]$ for $f(x) = \phi(x) = x^2$

5. Show that for the validity of the second mean value theorem of Integral calculary ϕ must be necessarily monotonic by showing that the theorem does not hold it $\phi(x) = \cos x$, $f(x) = x^2$

6. Prove Bonnet's form of the second mean value theorem that if f' is continuous and of constant sign and $f(b)$ has the same sign as $f(b) - f(a)$, then

$$\int_a^b f(x)\,\phi(x)\,dx = f(a)\int_a^{\xi} \phi(x)\,dx, \text{ when } \xi \text{ lies between } a \text{ and } b.$$

7. Text the validity of the second mean value theorem in $[-1, 1]$ for the functions $f(x) = e^x$, $g(x) = x$. **[Delhi Maths (H) 2009]**

EXAMPLES

Example 1. *If f is non-negative continous function an $[a, b]$ such that $\int_a^b f(x)\,dx > 0$. then show that $f(x) = 0\ \forall\ x \in [a, b]$* *(Delhi Maths (H) 1999, 2004, Nagpur 2003)*

Solution. Let, if possible, for some $c \in [a, b], f(c) > 0$

Let $\epsilon = (1/2) \times f(c) > 0$. Since f is continuous at c, so for given $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(c)| < \epsilon \quad \forall\ x \in\ ]c - \delta, c + \delta[$$

$$\Rightarrow \quad f(c) - \epsilon < f(x) < f(c) + \epsilon \text{ if } c - \delta < x < c + \delta$$

$$\Rightarrow \quad f(x) > f(c) - \epsilon \text{ if } c - \delta < x < c + \delta$$

$$\Rightarrow \quad f(x) > (1/2) \times f(c), \text{ if } c - \delta < x < c + \delta$$

Now f is continuous an $[a, b] \Rightarrow f$ is integrable on $[a, b]$.

$$\therefore \quad \int_a^b f(x)\,dx = \int_a^{c-\delta} f(x)\,dx + \int_{c-\delta}^{c+\delta} f(x)\,dx + \int_{c+\delta}^{b} f(x)\,dx$$

$$\geq \int_{c-\delta}^{c+\delta} f(x)\,dx, \text{ as } f(x) \geq 0\ \forall\ x \in [a, b]$$

$$> \frac{1}{2} f(c) \int_{c-\delta}^{c+\delta} dx = \delta\, f(c) > 0$$

Thus, $\int_a^b f(x)\,dx > 0$, which controdicts the given hypothesis $\int_a^b f(x)\,dx = 0$

Hence $f(c) > 0$ cannot hold. Similarly, we can show that $f(c) < 0$ cannot hold. Hence $f(x) = 0$ $\forall\ x \in [a, b]$.

Example 2 (a) *Prove that the function f defined on* $[0, 4]$ *by* $f(x) = [x]$, *where* $[x]$ *denotes the greatest integer not greater then x, is integralble on* $[0, 4]$ *and* $\int_0^4 f(x)\,dx = 6$

(Agra 2009; Delhi B.Sc. Maths (H) 2004)

(*b*) Evaluate $\int_0^2 x\,[2x]\,dx$, *where* $[x]$ *denotes the greatest integer function.*

(Delhi B.Sc. Maths (H) 2002)

Solution. (*a*) We have, by definition of the function $[x]$

$$\begin{aligned} f(x) = [x] &= 0 \text{ if } 0 \le x < 1 \\ &= 1 \text{ if } 1 \le x < 2 \\ &= 2 \text{ if } 2 \le x < 3 \\ &= 3 \text{ if } 3 \le x < 4 \end{aligned}$$

Here $f(x)$ is bounded and has four points of discontinuity at $x = 1, 2, 3$ and 4. Since the points of discontinity of f on $[0, 4]$ are finite in number, so f is integrable on $[0, 4]$ and

$$\int_0^4 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx + \int_2^3 f(x)\,dx + \int_3^4 f(x)\,dx$$

$$= \int_0^1 0\,dx + \int_1^2 1\,dx + \int_2^3 2\,dx + \int_3^4 3\,dx = 6$$

(*b*) Here, we have

$$\begin{aligned} f(x) = x\,[2x] &= x \times 0 \text{ if } 0 \le x < 1/2 \\ &= x \times 1 \text{ if } 1/2 \le x < 1 \\ &= x \times 2 \text{ if } 1 \le x < 3/2 \\ &= x \times 3 \text{ if } 3/2 \le x < 2 \end{aligned}$$

As in part (*a*), $f(x)$ has only finite (3 here) number of points of discontinuity at $x = 1/2, 1, 3/2$ and so $f(x)$ is integrable and

$$\int_0^2 f(x)\,dx = \int_0^{1/2} f(x)\,dx + \int_{1/2}^1 f(x)\,dx + \int_1^{3/2} f(x)\,dx + \int_{3/2}^2 f(x)\,dx$$

$$= \int_0^{1/2} 0\,dx + \int_{1/2}^1 x\,dx + \int_1^{3/2} 2x\,dx + \int_{3/2}^2 3x\,dx = \frac{17}{4}$$

Example 3 *Show that the function f defined by*

$$f(x) = 1/2^n, \text{ when } 1/2^{n+1} < x \le 1/2^n, \ (n = 0, 1, 2, \ldots)$$
$$f(0) = 0$$

in integrable on $[0, 1]$, *although it has an infinite number of points of discontinuities. Also show that* $\int_0^1 f(x)\,dx = \dfrac{2}{3}$

(Delhi B.A. (Prog.) III 2012)

Solution. Here,

$$\begin{aligned} f(x) &= 1, \text{ when } 1/2 < x \le 1 \\ &= 1/2, \text{ when } 1/2^2 < x \le 1/2 \\ &= 1/2^2, \text{ when } 1/2^3 < x \le 1/2^2 \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &= 1/2^{n-1}, \text{ when } 1/2^n < x \le 1/2^{n-1} \\ &\ldots\ldots\ldots\ldots\ldots\ldots \\ &= 0, \text{ when } x = 0 \end{aligned}$$

Observe that f is bounded and continuous on $[0, 1]$ except at the points $0, 1/2, 1/2^2, \ldots$ which are infinite in number. The set of points has only one limit point, namely, 0 and hence f is integrable in $[0, 1]$. Now, we have

$$\int_{1/2^n}^{1} f(x)\,dx = \int_{1/2}^{1} f(x)\,dx + \int_{1/2^2}^{1/2} f(x)\,dx + \int_{1/2^3}^{1/2^2} f(x)\,dx + \ldots + \int_{1/2^n}^{1/2^{n-1}} f(x)\,dx$$

$$= \int_{1/2}^{1} 1dx + \int_{1/2^2}^{1/2} \frac{1}{2}dx + \int_{1/2^3}^{1/2^2} \frac{1}{2^2}dx + \ldots + \int_{1/2^n}^{1/2^{n-1}} \frac{1}{2^{n-1}}dx$$

$$= 1 - \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2^2}\right) + \frac{1}{2^2}\left(\frac{1}{2^2} - \frac{1}{2^3}\right) + \ldots + \frac{1}{2^{n-1}}\left(\frac{1}{2^{n-1}} - \frac{1}{2^n}\right)$$

$$= \frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2^2} + \frac{1}{2^2}\cdot\frac{1}{2^3} + \ldots + \frac{1}{2^{n-1}}\cdot\frac{1}{2^n} = \frac{1}{2}\left[1 + \frac{1}{2^2} + \left(\frac{1}{2^2}\right)^2 + \ldots + \left(\frac{1}{2^2}\right)^{n-1}\right]$$

$$= \frac{1}{2} \times \frac{1-(1/2^2)^n}{1-(1/2^2)} = \frac{2}{3}\left(1 - \frac{1}{4^n}\right)$$

Proceeding to the limit when $n \to \infty$, we get

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty} \int_{1/2^n}^{1} f(x)\,dx = \lim_{n\to\infty} \frac{2}{3}\left(1 - \frac{1}{4^n}\right) = \frac{2}{3}$$

Example 4. *Show that the function f defined by*

$$f(x) = \begin{cases} 0, \text{ if } x \text{ is an integer} \\ 1, \text{otherwise} \end{cases}$$

is integrable on $[0, m]$, m being an integer.

Solution. $f(x) = \begin{cases} 0, \text{if } x = 0, 1, 2, \ldots, m \\ 1, \text{if } r-1 < x < r, \text{ for } r = 1, 2, 3, \ldots \end{cases}$

Clearly f is bounded and is continuous everywhere except at $(m + 1)$ points $x = 0, 1, 2, 3, \ldots, m$. Since the points of discontinuity are finite, so f is integrable as $[0, m]$ and

$$\int_0^m f(x)\,dx = \sum_{r=1}^{m} \int_{r-1}^{r} f(x)\,dx = \sum_{r=1}^{m} \int_{r-1}^{r} 1\,dx = \sum_{r=1}^{n} \{r-(r-1)\} = \sum_{r=1}^{n} 1 = m$$

Example 5. Show that $$\frac{1}{\pi} \le \int_0^1 \frac{\sin \pi x}{1+x^2}dx \le \frac{2}{\pi}$$

Solution. Let $f(x) = 1/(1+x^2)$ and $\phi(x) = \sin \pi x$. Then f and ϕ are continuous on $[0, 1]$ and hence integrable on $[0, 1]$. Also, $\phi(x) = \sin \pi x \geq 0$ on $[0, 1]$.

Since f is decreasing on $[0, 1]$, $\inf f = f(1) = 1/2$ and $\sup f = f(0) = 1$. Hence by the generalised first mean value theorem, there exists $\mu \in [1/2, 1]$ such that

$$\int_0^1 f(x)\,\phi(x)\,dx = \mu\int_0^1 \phi(x)\,dx \quad \text{or} \quad \int_0^1 \frac{\sin \pi x}{1+x^2}dx = \mu\int_0^1 \sin \pi x\,dx$$

$$\int_0^1 \frac{\sin \pi x}{1+x^2}dx = \frac{2\mu}{\pi} \quad \text{or} \quad \mu = \frac{\pi}{2}\int_0^1 \frac{\sin \pi x}{1+x^2}dx \qquad \ldots(1)$$

Since f is continuous on [0, 1], it attains every value between its bounds 1/2 and 1. Since $1/2 < \mu < 1$, so there exists a number $c \in [a, b]$ such that $f(c) = \mu$.

Then (1) becomes $$f(c) = \frac{\pi}{2}\int_0^1 \frac{\sin \pi x}{1+x^2}\,dx$$

Now, $0 \le c \le 1$ and f is decreasing on [0, 1]

$$\Rightarrow \quad f(0) \ge f(c) \ge f(1) \qquad \Rightarrow \quad f(1) \le f(c) \le f(0)$$

$$\Rightarrow \quad \frac{1}{2} \le \frac{\pi}{2}\int_0^1 \frac{\sin \pi x}{1+x^2}\,dx \le 1 \quad \Rightarrow \quad \frac{1}{\pi} \le \int_0^{\pi} \frac{\sin \pi x}{1+x^2}\,dx \le \frac{2}{\pi}$$

Example 6. *By an example, prove that the equation* $\int_a^b f'(x)\,dx = f(b) - f(a)$ *is not always true.*

Solution. Consider the followin function :

$$f(x) = x^2 \sin(1/x^2), \text{ if } 0 < x \le 1$$
$$= 0, \text{ if } x = 0.$$

Then it can easily be proved that f is differentiable on [0, 1] and its derivative is given by

$$f'(x) = 2x \sin(1/x^2) - (2/x) \times \cos(1/x^2), \text{ if } 0 < x \le 1$$
$$= 0 \text{ if } x = 0$$

Since f' is not bounded, so f' is not Riemann integrable, *i.e.*, $\int_0^1 f'(x)\,dx$ does not exist and hence the given equation fails to hold.

Example 7. *If* $$G(x, \xi) = \begin{cases} x(\xi - 1), & \text{when } x \le \xi, \\ \xi(x-1), & \text{when } \xi < x, \end{cases}$$

and if f is a continuous function of x in [0, 1] and if

$$g(x) = \int_0^1 f(\xi)\, G(x, \xi)\, d\xi,$$

show that $g''(x) = f(x), \ \forall x \in [0, 1]$ and find $g(0)$ and $g(1)$.

Sol. We have $$g(x) = \int_0^x f(\xi)\,\xi(x-1)\,d\xi + \int_x^1 f(\xi)\,x(\xi-1)\,d\xi$$

$$= (x-1)\int_0^x \xi f(\xi)\,d\xi + x\int_x^1 (\xi - 1) f(\xi)\,d\xi$$

$$= x\int_0^1 \xi f(\xi)\,d\xi - \int_0^x \xi f(\xi)\,d\xi - x\int_x^1 f(\xi)\,d\xi$$

Hence $$g'(x) = \int_0^1 \xi f(\xi)\,d\xi - x f(x) + x f(x) - \int_x^1 f(\xi)\,d\xi.$$

and so $$g''(x) = f(x).$$

We may easily see that $g(0)$ and $g(1)$ are both zero.

Example 8. *Prove that if the functions f and ϕ are bounded and integrable in $[a, b]$, then*

$$\left[\int_a^b f(x)\,\phi(x)\,dx\right]^2 \le \int_a^b [f(x)]^2\,dx \int_a^b [\phi(x)^2]\,dx.$$

Under what conditions does the sign of equality hold ?

Sol. We have $$\left[\int_a^b f(x)\,\phi(x)\,dx\right]^2 = \left[\lim \Sigma\,(x_r - x_{r-1})\,f(\xi_r)\,\phi(\xi_r)\right]^2$$

$$\int_a^b [f(x)]^2\,dx = \lim \Sigma\left[\sqrt{(x_r - x_{r-1})}\,f(\xi_r)\right]^2$$

and $$\int_a^b [\phi(x)]^2\,dx = \lim\left[\Sigma\,\sqrt{(x_r - x_{r-1})}\,\phi(\xi_r)\right]^2$$

Now putting $a_r = \sqrt{(x_r - x_{r-1})}\,f(\xi_r),\ b_r = \sqrt{(x_r - x_{r-1})}\,\phi(\xi_r)$, in the Cauchy's inequality

$$(\Sigma\,a_r b_r)^2 \le \Sigma\,a_r^{\,2}\,\Sigma\,b_r^{\,2}$$

we get the required result.

The sign of equality holds only when

$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \ldots \quad \Leftrightarrow \quad \frac{f(\xi_1)}{\phi(\xi_1)} = \frac{f(\xi_2)}{\phi(\xi_2)} = \ldots \Leftrightarrow \quad f, \phi \text{ are both constant functions.}$$

Example 9. *If f is positive and monotonically decreasing in* $[1, \infty[$, *show that the sequence* $\{A_n\}$, *where* $A_n = \left\{f(1) + f(2) + \ldots + f(n) - \int_1^n f(x)\,dx\right\}$, *is convergent.*

Deduce the convergence of $\{1 + 1/2 + 1/3 + \ldots + 1/n - \log n\}$.

Sol. We have

$$A_n = \left[f(1) - \int_1^2 f(x)\,dx\right] + \left[f(2) - \int_2^3 f(x)\,dx\right] + \ldots + \ldots + \left[f(n-1) - \int_{n-1}^n f(x)\,dx\right] + f(n).$$

Now, because of the monotonic character of the function f, each of the expressions within brackets is positive. Also $f(n)$ is positive. Thus A_n is positive $\forall\ n$. Again $\forall\ n$

$$A_{n+1} - A_n = f(n+1) - \int_1^{n+1} f(x)\,dx + \int_1^n f(x)\,dx$$

$$= f(n+1) - \int_n^{n+1} f(x)\,dx < 0$$

$$\Rightarrow \qquad A_{n+1} < A_n \ \forall\, n.$$

Thus $\{A_n\}$ is monotonically decreasing. Also being positive, A_n is bounded below for each $n \in N$. Hence $\{A_n\}$ is convergent.

Taking $f(x) = 1/x$, we can now deduce that

$$\lim_{n \to \infty} \left(1 + 1/2 + 1/3 + \ldots + 1/n - \log n\right) \text{ exists.} \qquad \ldots (1)$$

Note. (1) is known as *Euler's constant* whose numerical valve is 0.577215

Example 10. *Show that the function F defined in the interval* $[0, 1]$ *by the condition that if r is a positive integer*

$$F(x) = 2rx \text{ when } 1/(r+1) < x < 1/r \text{ for each } r \in N.$$

is integrable over $[0, 1]$ *and that*

$$\int_0^1 F(x)\,dx = \frac{\pi^2}{6}. \qquad \textit{(Agra 2004, 08, 10; I.A.S. 1994, Patna 2003)}$$

Sol. The function F, as given, is not defined at the set of points $\{0, 1, 1/2, \ldots\ldots, 1/r, \ldots.\}$ ...(1)

We may, however, define F at these points in any manner we please provided F remains bounded.

Now, the only points of discontinuity of F are those given above in (1). The set formed by these points is infinite having only one limit point *viz*, 0. Thus the function is integrable.

Consider $$\psi(\varepsilon) = \int_\varepsilon^1 F(x)\,dx.$$

We know that ϕ is a continuous function so that

$$\psi(0) = \int_0^1 F(x)\,dx = \lim_{\varepsilon\to 0} \psi(\varepsilon).$$

We, now, find $\psi(\varepsilon)$. We take $\varepsilon = 1/n$ so that $\varepsilon \to 0$ as $n \to \infty$. we have

$$\int_{1/n}^1 F(x)\,dx = \int_{1/2}^1 F(x)\,dx + \int_{1/3}^{1/2} F(x)\,dx + \ldots \int_{1/(r+1)}^{1/r} F(x)\,dx + \ldots + \int_{1/n}^{1/(n-1)} F(x)\,dx. \quad \ldots(2)$$

Now, $$\int_{1/(r+1)}^{1/r} F(x)\,dx = \int_{1/(r+1)}^{1/r} 2r\,x\,dx = 2r\left[\frac{x^2}{2}\right]_{1/(r+1)}^{1/r} = r\left[\frac{1}{r^2} - \frac{1}{(r+1)^2}\right] = \frac{2r+1}{r(r+1)^2}$$

$\therefore$ from (2), $$\int_{1/n}^1 F(x)\,dx = \sum_{r=1}^{n-1} \frac{2r+1}{r(r+1)^2} = \sum_{r=1}^{n-1}\left(\frac{1}{r} - \frac{1}{r+1} + \frac{1}{(r+1)^2}\right)$$

$$= \sum_{r=1}^{n-1}\left(\frac{1}{r} - \frac{1}{r+1}\right) + \sum_{r=1}^{n-1}\frac{1}{(r+1)^2} = 1 - \frac{1}{n} + \sum_{r=1}^{n-1}\frac{1}{(r+1)^2}.$$

* Now the series $1/1^2 + 1/2^2 + 1/3^2 + 1/4^2 + \ldots$ is convergent and its sum is $\pi^2/6$

$$\therefore \quad \lim_{n\to\infty} \int_{1/n}^1 F(x)\,dx = 1 + \frac{\pi^2}{6} - 1 = \frac{\pi^2}{6} \quad \text{and so} \quad \int_0^1 F(x)\,dx = \frac{\pi^2}{6}$$

Example 11. *Show that when* $-1 < x \le 1$, $$\lim_{m\to\infty} \int_0^x \frac{t^m}{t+1}\,dt = 0$$

Sol. Now $\forall\, x \in [0,1]$, we have

$$0 \le \int_0^x \frac{t^m\,dt}{1+t} \le \int_0^x t^m\,dt = \frac{x^{m+1}}{m+1} < \frac{1}{m+1}$$

Let $-1 < x < 0$. Putting $t = -u$, we obtain

$$\left|\int_0^x \frac{t^m}{1+t}\,dt\right| = \left|\int_0^{-x} \frac{u^m}{1-u}\,du\right| < \left|\frac{1}{1+x}\int_0^{-x} u^m\,du\right| < \frac{1}{(m+1)(x+1)}.$$

Hence the result.

Example 12. *Show that, when* $|x| < 1$.

$$\int_0^x \frac{dt}{1+t^4} = x - \frac{1}{5}x^5 + \frac{1}{9}x^9 - \frac{1}{13}x^{13} + \ldots$$

* We assume that $\pi^2/6 = 1/1^2 + 1/2^2 + 1/3^2 + 1/4^2 + \ldots$

Sol. We have

$$\frac{1}{1+t^4}=1-t^4+t^8-t^{12}+\ldots+(-1)^{n-1}t^{4n-4}+\frac{(-1)^n t^{4n}}{1+t^4}$$

$$\Rightarrow \quad \int_0^x \frac{dt}{1+t^4}=x-\frac{1}{5}x^5+\frac{1}{9}x^9-\frac{1}{13}x^{13}+\ldots+\frac{(-1)^{n-1}x^{4n-3}}{4n-3}+(-1)^n\int_0^x \frac{t^{4n}}{1+t^4}\,dt.$$

Now, $\forall\, x\in[-1,1]$, we have

$$0\le\left|\int_0^x \frac{t^{4n}}{1+t^4}\,dt\right|<\left|\int_0^x t^{4n}\,dt\right|=\left|\frac{x^{4n+1}}{4n+1}\right|<\frac{1}{4n+1},$$

$$\lim_{n\to\infty}\int_0^x \frac{t^{4n}}{1+t^n}\,dt=0.$$

Hence the result.

Example 13. *If a function f is continuous in* $[0, 1]$, *show that*

$$\lim_{n\to\infty}\int_0^1 \frac{nf(x)}{1+n^2x^2}\,dx=\frac{\pi}{2}f(0)$$ **[I.A.S. 2008; Purvanchal 2006; Delhi Maths (H) 2003, 07]**

Sol. We write $\displaystyle\int_0^1 \frac{nf(x)}{1+n^2x^2}\,dx=\int_0^{1/\sqrt{n}} \frac{nf(x)}{1+n^2x^2}\,dx+\int_{1/\sqrt{n}}^1 \frac{nf(x)}{1+n^2x^2}\,dx.$

By the first mean value theorem, we have

$$\int_0^{1/\sqrt{n}} \frac{nf(x)}{1+n^2x^2}\,dx=f(\alpha_n)\int_0^{1/\sqrt{n}} \frac{n\,dx}{1+n^2x^2},\text{ where } 0\le\alpha_n\le 1/\sqrt{n}$$

$$=f(\alpha_n)\tan^{-1}\sqrt{n},\text{ which }\to f(0).\frac{\pi}{2}\text{ as } n\to\infty.$$

Again, $$\left|\int_{1/\sqrt{n}}^1 \frac{nf(x)}{1+n^2x^2}\,dx\right|=\left|f(\beta_n)\int_{1/\sqrt{n}}^1 \frac{n\,dx}{1+n^2x^2}\right|,\text{ where } 1/\sqrt{n}\le\beta_n\le 1$$

$$=f(\beta_n)(\tan^{-1}n-\tan^{-1}\sqrt{n})$$

$$\le M(\tan^{-1}n-\tan^{-1}\sqrt{n}),\text{ which }\to 0\text{ as }\to\infty,$$

M, being the supremum of $|f(x)|$. Hence the result.

Example 14. *If f is bounded and integrable in the interval* $[a, b]$, *show that*

$$\lim_{n\to\infty}\int_a^b f(x)\cos nx\,dx=0.$$

Sol. We write $$I_n=\int_a^b f(x)\cos nx\,dx.$$

Let ε be a positive number. Since f is bounded and integrable in $[a, b]$, there exists a partition

$$P=\{a=x_0<x_1<x_2<\ldots<x_{r-1}<x_r<\ldots<x_p=b\}$$

such that the corresponding oscillatory sum

$$\Sigma\,(x_r-x_{r-1})\,O_r<\varepsilon/2;$$

being the oscillation of f in $[x_{r-1}, x_r]$.

We have, $I_n = \Sigma \int_{x_{r-1}}^{x_r} f(x) \cos nx\, dx$

$$= \Sigma f(x_{r-1}) \int_{x_{r-1}}^{x_r} \cos nx\, dx + \Sigma \int_{x_{r-1}}^{x_r} \left[f(x) - f(x_{r-1})\right] \cos nx\, dx,$$

$$\Rightarrow \quad |I_n| \le \Sigma |f(x_{r-1})| \left|\int_{x_{r-1}}^{x_r} \cos nx\, dx\right| + \Sigma \left|\int_{x_{r-1}}^{x_r} \{f(x) - f(x_{r-1})\} \cos nx\, dx\right|.$$

We have $\forall\, x \in [x_{r-1}, x_r]$

$$|f(x) - f(x_{r-1})| \le O_r$$

$$\Rightarrow \quad |[f(x) - f(x_{r-1})] \cos nx| \le O_r.$$

Also $$\left|\int_{x_{r-1}}^{x_r} \cos nx\, dx\right| \le \frac{1}{n} \{|\sin nx_r|\} + |\sin nx_{r-1}|\} < \frac{2}{n}.$$

It follows that

$$|I_n| \le \frac{2}{n} \Sigma |f(x_{r-1})| + \Sigma (x_r - x_{r-1}) O_r \le \frac{2}{n} \Sigma |f(x_{r-1})| + \frac{\varepsilon}{2}.$$

Keeping the partition P fixed, we see that $\Sigma |f(x_{r-1})|$ is fixed. We now choose a positive integer m such that $\forall\, n \ge m$

$$\frac{2}{n} \Sigma |f(x_{r-1})| < \frac{\varepsilon}{2}.$$

Thus $\forall\, n \ge m$ we have $|I_n| < \varepsilon$.

Hence the result.

It may similarly be shown that

$$\lim_{n \to \infty} \int_a^b f(x) \sin nx\, dx = 0.$$

(*Delhi B. Sc. (H) 2004*)

Example 15. *Show that* $\lim \{I_n\}$, *where*

$$I_n = \int_0^h \frac{\sin nx}{x}\, dx,\ n \in \mathrm{I}$$

exists and that the limit is equal to $\pi/2$.

Sol. The integrand becomes continuous for every value of x, if we assign to it the value n for $x = 0$. The result will be proved in *three* steps.

I. Firstly, it will be proved that $\{I_n\}$ is convergent. Putting $nx = t$, we have

$$I_n = \int_0^{nh} \frac{\sin t}{t}\, dt.$$

$$\Rightarrow \quad |I_{n+p} - I_n| = \left|\int_{nh}^{(n+p)h} \frac{\sin t}{t}\, dt\right|.$$

As $1/t$ is positive and monotnicaly decreasing when $t \in]nh, (n+p)h[$, we have, by Bonnett's form of the second mean value theorem,

$$|I_{n+p} - I_n| = \frac{1}{nh} \left|\int_{nh}^{\alpha} \sin t\, dt\right| \le \frac{2}{nh} < \varepsilon \ \forall\, n > 2/\varepsilon h.$$

Hence, by Cauchy's principle of convergence, $\{I_n\}$ converges.

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II. It will now be proved that, when $n \to \infty$,

$$\lim I_n = \lim \int_0^{\pi/2} \frac{\sin nx}{\sin x}\, dx.$$

We write $$\int_0^{\pi/2} \frac{\sin nx}{x}\, dx = \int_0^{h} \frac{\sin nx}{x}\, dx + \int_h^{\pi/2} \frac{\sin nx}{x}\, dx$$

As proved in the preceding example,

$$\int_0^{\pi/2} \frac{\sin nx}{x}\, dx \to 0 \text{ as } n \to \infty.$$

$$\lim I_n = \lim \int_0^{\pi/2} \frac{\sin nx}{x}\, dx.$$

Again, taking $f(x) = (1/x - 1/\sin x)$ in the preceding example,

$$\lim \int_0^{\pi/2} \left(\frac{1}{x} - \frac{1}{\sin x}\right) \sin nx\, dx = 0,$$

for f is continuous in $[0, \pi/2]$, if we set $f(0) = 0$.

It follows that

$$\lim \int_0^{\pi/2} \frac{\sin nx}{x}\, dx = \lim \int_0^{\pi/2} \frac{\sin nx}{\sin x}\, dx$$

$$\lim I_n = \lim \int_0^{\pi/2} \frac{\sin nx}{x}\, dx = \lim \int_0^{\pi/2} \frac{\sin nx}{\sin x}\, dx.$$

To determine the actual value of the limit, we proceed by making $n \to \infty$ through odd integer values.

III. We have, as may be easily shown,

$$\frac{\sin(2n+1)x}{\sin x} = 2\left[\frac{1}{2} + \cos 2x + \cos 4x + \ldots + \cos 2nx\right]$$

so that $$\int_0^{\pi/2} \frac{\sin(2n+1)x}{\sin x}\, dx = \frac{\pi}{2}.$$

Hence the result.

EXERCISES

1. Give example of a function when upper Riemann integral is not equal to lower Riemann integral. Justify your answer. *(Delhi Maths (H) 2001)*
2. Give example of Riemann integral function defined on $[0, 2]$ and having discontinuity only at 1 and 2. *(Delhi Maths (H) 1999)*
3. Show that the function f defined by

$$f(x) = \lim_{n\to\infty} \frac{\log(2+x) - x^{2n}\sin x}{1 + x^{2n}}$$

in integrable on $[0, \pi/2]$ *(Delhi Maths (H) 2000)*

4. Let f be a real valued function defined on $[a, b]$. Define f^+ and f^- by $f^+(x) = \max\{f(x), 0\}$, $f^-(x) = \max\{-f(x), 0\}$. If $f \in R[a, b]$ then show that $f^+, f^- \in R[a, b]$ and that

$$\int_a^b f(x)\,dx = \int_a^b f^+(x)\,dx - \int_a^b f^-(x)\,dx.$$ *(Delhi Maths (H) 2004)*

5. (*a*) Give example of Riemann integrable function defined on [0, 1] having infinite number of discontinuties. *(Delhi Maths (H) 1999, 2004)*

   (*b*) Give example of an integrable function which has an infinite set of points of discontinuties having only finitely many limit points. *(Delhi Maths (H) 1998)*

6. State the class of Riemann integrable function and prove the result for one of them. *(Delhi Maths (H) 2003)*

7. A function f is defined on [0, 1] by

$$f(x) = 1/x \text{ for } 1/n > x \geq 1/(n+1),\ n = 1, 2, 3, \ldots$$

Prove that f is integrable on [0, 1] and $\int_0^1 f(x)\,dx = (\pi^2/6) - 1$

8. Show that the Oscillation ($M-m$) of a bounded function f defined on an interval $[a, b]$ is $\sup\{|f(x) - f(t)| : x, t \in [a, b]\}$ *(Delhi Maths (H) 2004)*

9. If a function f is bounded and integrable on $[a, b]$ prove that f^2 is also bounded and integrable. Is the converse of this result true ? *(Delhi Physice (H) 1999)*

10. Prove that $|f| \in R[a, b] \Rightarrow f \in R[a, b]$ *(Garhwal 1999)*

11. Let $f \in R[a, b]$. Put $F(x) = \int_a^x f(t)\,dt,\ a \leq x \leq b$. Prove that F is continuous on $[a, b]$. Also, if f is continuous at a point x_0 of $[a, b]$ then prove that F in differentiable at x_0 and $F'(x_0) = f(x_0)$. *(Nagpur 2003)*

12. If f is Riemann integrable over every interval of finite length and $f(x + y) = f(x) + f(y)$ for every pair of real numbers, show that $f(x) = cx$, where $c = f(1)$ *(IAS 1999)*

13. If f is Riemann integrable on $[a, b]$, show that the indefinite integral F of f given by

$F(x) = \int_a^x f(t)\,dt$ for all $x \in [a, b]$ is uniformly continuous. *(Delhi Maths (H) 1998)*

14. If $f(y, x) = 1 + 2x$ for y rational and $f(y, x) = 0$ for y irrational, calculate

$$F(y) = \int_0^1 f(y, x)\,dx.$$

15. Show that $\int_0^2 f(x)\,dx = 2$,

where $f(x) = 0$, when $x = n/(n+1),\ (n+1)/n,\ (n = 1, 2, 3, \ldots)$
$f(x) = 1$. elsewhere.

Examine for continuity the function f so defined at the point $x = 1$

[Delhi Maths (H) 2006]

16. A function f is defined for $x \geq 0$ by

$$f(x) = \int_{-1}^{1} \frac{dt}{\sqrt{(1 - 2tx + x^2)}}.$$

Prove that if $0 \leq x \leq 1$, $f(x) = 2$. What is the value of f for $x > 1$? Has the function f a differential coefficient for $x = 1$?

[For $x > 1$, $f(x) = 2/x$: f is not derivable for $x = 1$ even though it is continuous threat].

59 **17.** If for $x \geq 0$, φ is defined as

$$\varphi(x) = \lim \frac{x^n + 2}{x^n + 1} \text{ as } n \to \infty \text{ and } f(x) = \int_0^x \varphi(t)\, dt,$$

prove that f is continuous but not differentiable for $x = 1$.

18. If $f(x, y) = xy^2 e^{-xy} + x^2 y/(1+y)$ and a, b are positive, show that

$$\lim_{y \to \infty} \int_a^b f(x, y)\, dx = \int_a^b \left[\lim_{y \to \infty} f(x, y) \right] dx.$$

Also show that the equality does not hold for $a = 0$.

19. f is bounded and integrable in $[a, b]$; show that

$$\int_a^b [f(x)]^2\, dx = 0$$

if, and only if, $f(c) = 0$ at every point, c, of continuity of f.

20. If $a > 0$, $n > 0$, show that $\quad 0 < n \int_0^1 \frac{x^{n-1}}{(1+x^{-2})} a\, dx < 1.$

21. The functions f and g are bounded and integrable in $[a, b]$. If further

$$F(x) = \int_a^x f(t)\, dt \text{ and } H(x) = \int_a^x f(t)\, g(t)\, dt,$$

where $a \leq x \leq b$ and if $F' = f$ and g is continuous, show that

$$H' = f(x)\, g(x)$$

22. A function f is integrable in $[a-c, a+c]$ and $|f(x)| \leq M \ \forall\, x \in [a-c, a+c]$.

$$\int_{a-c}^{a+c} f(x)\, dt = 0 \text{ and } F(x) = \int_{a-c}^{a} f(t)\, dt.$$

Prove that
$$\left| \int_{a-c}^{a+c} f(x)\, dx \right| \leq Mc^2.$$

23. If f be defined in the interval $[0, 1]$ by the condition that if, r, is a positive integer,

$$f(x) = (-1)^{r-1}, \text{ when } 1/(r+1) < x < 1/r.$$

prove that $\quad \int_0^1 f(x)\, dx = \log 4 - 1.$ *(Agra 2012)*

24. A function f is defined in $[0, 1]$ as follows ;

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$$f(x) = \frac{1}{a^{r-1}} \text{ when } \frac{1}{a^r} < x \leq \frac{1}{a^{r-1}} \text{ for } r = 1, 2, 3, \ldots$$

where a is an integer greater than 2. Show that

$\int_0^1 f(x)\, dx$ exists and is equal to $\frac{a}{a+1}$. *(Agra 2008; Delhi Maths (H) 1997)*

25. If $f(x) = 0 \ \forall\, x \in [0,1]$ except at the set of points $\{x_1, x_2, \ldots, x_r, \ldots, x_n\}$, $n \in N$ and $f(x_n) = 1/\sqrt{n}$; show that f is integrable in $[0, 1]$. *(Delhi Maths (H) 2002)*

26. By repeadtly employing the method of integration by parts to the integral $\int_0^x e^{-t}\,dt$.

show that $$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + e^x \int_0^x \frac{t^n}{n!} e^{-t}\,dt,$$

and deduce the Maclaurin's infinite series for e^x.

27. Obtain by integration, from the identity

$$\frac{1}{1+t} = 1 - t + t^2 - t^3 + \ldots + (-1)^{n-1} t^n + (-1)^n \frac{t^n}{1+t},$$

the Maclaurin's infinite series for log $(1 + x)$ in $[-1, 1]$.

28. By applying the mean value theorem of Integral Calculus, show that

(i) $\frac{\pi^2}{9} \le \int_{\pi/6}^{\pi/2} \frac{x}{\sin x}\,dx \le \frac{2\pi^2}{9}$ (ii) $\frac{1}{3\sqrt{2}} \le \int_0^1 \frac{x^2}{\sqrt{1-x^2}}\,dx \le \frac{1}{3}$

(iii) $\frac{\pi}{6} < \int_0^{1/3} \frac{dx}{\sqrt{[(1-x^2)\,(1-k^2x^2)]}} \le \frac{\pi}{6} \cdot \frac{1}{\sqrt{1-(k^2/2)}}$.

29. If f is Riemann integrable on $[a, b]$, show that the indefinite integral $F(x) = \int_a^x f(t)\,dt$ $\forall\, x \in [a, b]$ is differentiable from the right at each x_0 such that $a \le x_0 \le b$ for which f is continuous from the right. (Delhi Maths (H) 1998)

30. By an example, show that the continuity assumption cannot be dropped in the first mean value theorem.

31. Let $f(x) \ge g(x)$ $\forall\, x \in [a, b]$ and f and g are both bounded and Riemann integrable on $[a, b]$. At a point $c \in [a, b]$, let f and g be continuous and $f(c) > g(c)$, then prove that $\int_a^b f(x)\,dx > \int_a^b g(x)\,dx$ and hence show that $-\frac{1}{2} < \int_a^b \frac{x^3 \cos 5x}{2+x^2}\,dx < \frac{1}{2}$

(Delhi Maths (H) 2004)

32. It f is Riemann integrable function as $[a, b]$, show that the function F defined by $F(x) = \int_a^x f(t)\,dt$ for all x in $[a, b]$ is uniformly continuous. (Delhi Maths (H) 2005, 09)

MISCELLANEOUS PROBELMS

1. Let $f(x) = \sin(1/x), 0 < x \le 1, f(0) = 7$. Is f Riemann integrable on [0, 1] Justify.

[Hint: 0 is the only point of discontinuity of f in [0, 1] and continuous elsewhere in [0,1]. Hence f is integrable by theorem II, page 13.25. **(Pune 2010]**

2. Prove that the function of defined as
$f(x) = \begin{cases} x & \text{when } x \text{ is rational} \\ -x & \text{when } x \text{ is irrational} \end{cases}$ is not integrable on [0, 1]. **[Kanpur 2005]**

3. Let f be defi ;d and bounded over an interval $[a, b]$ and P be a partition of $[a, b]$. If $\lim_{\|P\|\to 0} S(P, f)$ exits, then prove that f is integrable is $[a, b]$ and $\lim_{\|P\|\to 0} S(P, f) = \int_a^b f(x)dx$

[Delhi Maths (H) 2006]

4. Prove that the function f defined on [0, 1] as $f(x) = \begin{cases} 2n, & \text{if } x = 1/n, n = 1, 2, 3, \ldots \\ 0, & \text{other wise} \end{cases}$

is not Riemann integrable on [0, 1]. **[G.N.D.U. Amritsar 2010]**

Hint: $\lim_{x\to 0} f(x) = \infty$, f is not bounded above and hence not Riemann integrable on [0,1]

5. By using the generalised first mean value theorem, prove that $\frac{\pi^2}{24} \le \int_0^x \frac{x^2\,dx}{5+3\cos x} \le \frac{x^3}{6}$

[Meerut 2006]

6. *Let f be the function defined on* [0, 1] *by* $f(x) = \begin{cases} 1, & \text{when } x \text{ is rational} \\ 0, & \text{when } x \text{ is irrational} \end{cases}$

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Then calculate $\underline{\int_0^1} f$ and $\overline{\int_0^1} f$ and hence show that $f \notin R[0,1]$ **[Meerut 2007]**

7. *If* $f(x) = x$, $x \in [0, 3]$ *and* $P = [0, 1, 2, 3]$ *is the partition of* $[0, 3]$, *then show that* $U(P, f) = 6$, $L(P, f) = 3$. **[Meerut 2007]**

8. *Show that the function defined by* $f(x) = \begin{cases} 0, & \text{when } x \text{ is rational} \\ 2, & \text{when } x \text{ is irrational} \end{cases}$

is not Reimann integrable on $[1, 2]$. **[Purvanchal 2006]**

9. *Let* $f : [a, b] \to R$ *be a continuous function on interval* $[a, b]$, *then prove that*

$\overline{\int_a^b} f = \underline{\int_a^b} f$ **[Meerut 2007]**

10. *If* f_1, f_2 *are two R-integrable functions on* $[a, b]$ *then for* $k_1, k \in R$, prove that $k_1 f_1 + k_2 f_2$ *is also R-integrable.* **[Meerut 2007]**

11. *State and prove Darboux theorem involving upper Riemann integral.* **[Purvanchal 2006]**

12. *Show that a continuous function on a closed interval* $[a, b]$ *is integrable. Give example of a function which is integrable but not continuous on* $[a, b]$ **[Delhi Maths (Prog) 2007]**

13. State two classes of Riemann integrable functions. Give an example of a function which lies in neither of two of the classes stated above but still Riemann integrable. Justify your answer in the example with your mathematical argument. **[Delhi maths (H) 2008]**

14. Suppose f is Riemann integrable on the closed and bounded interval $[a, b]$. Define $f^+, f^- : [a, b] \to R$ by $f^+(x) = \max\{f(x), 0\}$ and $f^-(x) = \max\{-f(x), 0\}$. Show that f^+ and f^- are integrable on $[a, b]$ and $\int_a^b f = \int_a^b f^+ - \int_a^b f^-$. **[Delhi Maths (H) 2008]**

15. If a function f is bounded aand integrable on $[a, b]$, show that $|f|$ is also bounded and integrable on $[a, b]$. **[Delhi maths (H) 2008]**

16. Evaluate the integral $\int_0^1 x dx$ using Riemann concept of integration. **[Agra 2006]**

17. Evaluate the integral $\int_a^b x^2 dx$ by using the concept of Riemann integration. **[Agra 2007]**

18. Find the lower Riemann integral of $\int_0^1 (x^2 + 1) dx$. **[Delhi B.Sc. (Prog) III 2008]**

[Ans. 4/3]

19. Let $f(x)$ be a function defined on $[0, 1]$ as follows : $f(x) = \begin{cases} 0, & \text{when } x \text{ is rational} \\ 1, & \text{when } x \text{ is irrational} \end{cases}$

Calculate $\underline{\int_0^1} f$ and $\overline{\int_0^1} f$ and show that $f \notin R[a, b]$. **[Agra 2007]**

20. Let $f(x) = x$ on closed interval $[0,1]$. Calculate $\underline{\int_0^1} x dx$ and $\overline{\int_0^1} x dx$ by dividing $[0, 1]$ into n equal parts. Then prove that $f \in R[a, b]$. **[Agra 2003, 08]**

21. Prove the (*i*) $\dfrac{1}{3\sqrt{2}} \le \displaystyle\int_0^1 \frac{x^2}{\sqrt{1+x}}\,dx \le \frac{1}{3}$ **(Pune 2010)**

(*ii*) $2\sqrt{2} \le \displaystyle\int_1^3 \sqrt{1+x^3}\,dx \le 2\sqrt{28}$ **(G.N.D.U. Amritsar 2010)**

22. Let $P = \{0, 1, 2, 4\}$ be a partition of the interval $[0, 4]$. Let $f(x) = x^2$. Find

(*i*) norm P (*ii*) $U(P, f)$ (*iii*) $L(P, f)$. **[Delhi B.Sc III (Prog) 2009]**

Ans (*i*) Norm $P = 2$ (*ii*) $U(P, f) = 37$ (*iii*) $L(P, f) = 9$

23. (a) Show that the function f(x) defined on the interval [0, 5] as $f(x) = x[x]$ is integrable State clearly any result that you will use. $[x]$ denotes greatest integer less than or equal to x.

[Delhi B.A III (Prog) 2009]

(*b*) Show that the function defined by $f(x) = x[x],\ \forall\, x \in [0,3]$ is Riemann integrable on [0, 3], $[x]$ being the greatest integer function. Also evaluate $\int_0^3 f(x)dx$. **[Delhi B.A. (Prog) III 2010]**

Sol. (*a*) Since the given function is bounded and has only five points of discontinuity (namely $x = 1, 2, 3, 4, 5$), hence it is integrable (refer theorem II, Art. 13.8) and

$$\int_0^5 x[x]dx = \int_0^1 x[x]dx + \int_1^2 x[x]dx + \int_2^3 x[x]dx + \int_3^4 x[x]dx + \int_4^5 x(x)dx$$

$$= 0 + \int_1^2 x\,dx + 2\int_2^3 x\,dx + 3\int_3^4 x\,dx + 4\int_4^5 x\,dx$$

$$= [x^2/2]_1^2 + 2[x^2/2]_2^3 + 3[x^2/2]_3^4 + 4[x^2/2]_4^5 = 35, \text{ on simplification}$$

(*b*) **Hint.** Proceed as in part (a). The value of integral is 13/2.

24. If $f : [1, 2] \to \mathbf{R}$ is a non-negative Riemann integrable function such that $\displaystyle\int_1^2 \frac{f(x)}{\sqrt{x}}dx = k\int_1^2 f(x)dx \ne 0$, then k belongs to the

(*a*) [0, 1/3] (*b*) (1/3, 2/3] (*c*) (2/3, 1] (*d*) (1, 4/3] **(GATE 2010)**

25. Using fundamental theorem of Integral Calculus, compute $\displaystyle\int_0^{\pi/3} \cos x\,dx$ **[Meerut 2011]**

26. Let f be a monotonically increasing function on a closed and bounded interval $[a, b]$. Prove that f is Riemann integrable on $[a, b]$. **[Delhi B.A. (Prog.) III 2011]**

27. Let f be bounded function on $[a, b]$ so that there exists $B > 0$ such that $|f(x)| \le B\ \ \forall\, x \in [a,b]$. Show that $U(P, f^2) - L(P, f^2) < 2B\,[U(P, f) - L(P, f)]$ for all partitions. Hence show that if f is integrable then f^2 is also integrable. **[Delhi B.Sc. (Hons) II 2011]**

Hint. Refer theorem on page 13.34.

28. Suppose the functions f and g are integrable on $[a, b]$ and suppose that $f(x) \le g(x)$ for all $x \in [a,b]$. Show that $\int_a^b f \le \int_a^b g$. **[Delhi B.Sc. (Hons) II 2011]**

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[**Sol.** $g(x) \geq f(x) \Rightarrow |g(x) - f(x)| \geq 0$, for all $x \in [a, b]$

$\Rightarrow \int_a^b [g(x) - f(x)]dx \geq 0$, if $b \geq 0$ by theorem V, page 13.38

$\Rightarrow \int_a^b g(x)dx - \int_a^b f(x)dx \geq 0 \Rightarrow \int_a^b f(x)dx \leq \int_a^b g(x)dx \Rightarrow \int_a^b f \leq \int_a^b g$

Formulae

(i) $\sum_{r=1}^{n} (r) = \frac{n(n+1)}{2}$

(ii) $\sum_{r=1}^{n} (r^2) = \frac{n(n+1)(2n+1)}{6}$

(iii) $\sum_{r=1}^{n} (r^3) = \left\{\frac{n(n+1)}{2}\right\}^2 = \frac{n^2(n+1)^2}{4}$

(iv) $\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \dots + \sin(\alpha+\overline{n-1}\beta) = \left(\frac{\sin(\alpha+\frac{n-1}{2}\beta)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\right)$

(v) $\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \dots + \cos(\alpha+\overline{n-1}\beta) = \left(\frac{\cos(\alpha+\frac{n-1}{2}\beta)\cdot\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}\right)$

(vi) In A.P. $S_n = \frac{n}{2}(a + l)$

First term, Last term

(vii) In A.P. $S_{2n} = \frac{2n}{2}(a+l)$