Example. Consider the series $\sum \frac{(-1)^{n-1}}{(n+x^2)}$ for uniform convergence for all values of x.

Solution. Let $u_n = (-1)^{n-1}$, $v_n(x) \frac{1}{n+x^2}$

Since $f_n(x) = \sum_{r=1}^{n} u_r = 0$ or 1 according as n is even or odd, $f_n(x)$ is bounded for all n.

Also $v_n(x)$ is a positive monotonic decreasing sequence, converging to zero for all real values of x.

Hence by Dirichlets test, the given series is uniformly convergent for all real values of x.

Example . Prove that the series $\sum(-1)^n \frac{x^2+n}{n^2}$, converges uniformly in every bounded interval, but does not converge absolutely for any value of x.

Solution. Let the bounded interval be [a, b], so that $\exists$ a number K such that, for all x in [a, b], $|x| < K$.

Let us take $\sum u_n = \sum(-1)^n$, which oscillates finitely, and

$$a_n = \frac{x^2+n}{n^2} < \frac{K^2+n}{n^2}$$

Clearly a_n is a positive, monotonic decreasing function of n for each x in [a, b], and tends to zero uniformly for $a \le x \le b$.

Hence by Dirchlet's test, the series $\sum(-1)^n \frac{x^2+n}{n^2}$ converges uniformly on [a, b].

Again $\sum \left|(-1)^n \frac{x^2+n}{n^2}\right| = \sum \frac{x^2+n}{n^2} \sim \sum \frac{1}{n}$, which diverges. Hence the given series is not absolutely convergent for any value of x.

Example. Show that the series $\sum_{n=1}^{\infty}(-1)^{n-1}\ x^n$ converges uniformly in $0 \le x \le k < 1$.

Solution. Let $u_n = (-1)^{n-1}$, $v_n(x) = x^n$.

Since $f_n(x) = \sum_{n=1}^{n} u_r = 0$ or 1 according as n is even or odd, $f_n(x)$ is bounded for all n. Also $\{v_n(x)\}$ is a positive monotonic decreasing sequence, converging to zero for all values of x in $0 \le x \le k < 1$. Hence by Dirichlet's test, the given series is uniformly convergent in $0 \le x \le k < 1$.

Example 14. Prove that the series $\sum \frac{\cos n\theta}{n^p}$ converges uniformly for all values of $p > 0$ in an interval $[\alpha, 2\pi - \alpha[$, where $0 < \alpha < \pi$.

Solution. When $0 < p \le 1$, the series converges uniformly in any interval $[\alpha, 2\pi - \alpha]$, $\alpha > 0$. Take $a_n = (1/n^p)$ and $u_n = \cos n\theta$ in Dirchlet's test.

Now $(1/n^p)$ is positive monotonic decreasing and tending uniformly to zero for $0 < p \le 1$, and

$$\left|\sum_{t=1}^{n} u_t\right| = \left|\sum_{t=1}^{n} \cos t\theta\right| = |\cos\theta + \cos 2\theta + \ldots + \cos n\theta|$$

$$= \left|\frac{\cos((n+1)/2)\theta \sin(n/2)\theta}{\sin(\theta/2)}\right| \le \operatorname{cosec}(\alpha/2),\ \forall\ n,$$

$$\text{for } \theta \in [\alpha, 2\pi - \alpha]$$

Now by Dirchlet test , the series $\sum(\cos n\theta/n^p)$ converges uniformly on $[\alpha, 2\pi - \alpha]$ where $0 < \alpha < \pi$. When $p > 1$, Weierstrass's M-test , the series converges uniformly for all real values of θ.

Problem 4 (pg. 166 #5). *Let*

$$f_n(x) := \begin{cases} 0 & \left(x < \frac{1}{n+1}\right), \\ \sin^2\left(\frac{\pi}{x}\right) & \left(\frac{1}{n+1} \leq x \leq \frac{1}{n}\right), \\ 0 & \left(\frac{1}{n} < x\right). \end{cases}$$

Show that $\{f_n\}$ converges to a continuous function, but not uniformly. Use the series $\sum f_n$ to show that absolute convergence, even for all x, does not imply uniform convergence.

Solution. The first thing to do is decide what function these f_n's converge to. This is fairly simple, since if $x \leq 0$, $f_n(x) \equiv 0$ and if $x > 0$, $f_n(x) = 0$ for every large n. Hence, $f_n \to 0$ pointwise. However, this is not a uniform convergence. Fix $\epsilon = 1$. Let $N \in \mathbb{N}$, and set $x = \frac{1}{N+1/2}$. You can check that $|f_N(x) - f(x)| = |f_N(x)| = 1 > 0$ so the convergence cannot be uniform.

The series $\sum f_n$ converges to the function defined by

$$f^*(x) := \begin{cases} 0 & x < 0, \text{ or } x > 0; \\ \sin^2\left(\frac{\pi}{x}\right) & 0 < x < 1. \end{cases}$$

$\square$

Problem 5. *Prove that the series*

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2 + n}{n^2}$$

converges uniformly in every bounded interval, but does not converge absolutely for any value of x.

Solution. Fix an x, and Set $a_n := \frac{x^2+n}{n^2}$. Since

$$|(-1)^n a_n| = \frac{x^2 + n}{n^2} \geq \frac{1}{n},$$

it is easy to see that $\sum |(-1)^n a_n|$ diverges by the comparison test.

Now, if we fix a bounded interval, $|x| \leq M$ for some M and any x. Hence, $\lim a_n = 0$ (we need boundedness to be able to take the limit), so that $\sum (-1)^n a_n$ converges by the alternating series test.

Prove that $f(x) = f(n) = \begin{cases} x, & \text{if } n \in Q \\ -x, & \text{if } n \notin Q \end{cases}$

is not integrable on $[0, 1]$

Suppose that P is a partition of $[0, 1]$ with endpoints $x_0 = 0 < x_1 < \ldots < x_n = 1$. For $k = 1, \ldots, n$ let $I_k = [x_{k-1}, x_k]$. Then

$$\sup_{x \in I_k} f(x) = x_k \qquad \text{and} \qquad \inf_{x \in I_k} f(x) = -x_k \ ,$$

so

$$\begin{aligned} U(f, P) &= \sum_{k=1}^{n} x_k(x_k - x_{k-1}) && (1) \\ &\geq \sum_{k=1}^{n} \left(\frac{x_k + x_{k-1}}{2} \cdot (x_k - x_{k-1}) \right) && (2) \\ &= \frac{1}{2} \sum_{k=1}^{n} (x_k^2 - x_{k-1}^2) && (3) \\ &= \frac{1}{2} (x_n^2 - x_0^2) && (4) \\ &= \frac{1}{2} \ . \end{aligned}$$

The calculation showing that $L(f, P) \leq -\frac{1}{2}$ is entirely similar. Thus, for all partitions P we have

$$U(f, P) - L(f, P) \geq \frac{1}{2} - \left(-\frac{1}{2} \right) = 1 \ ,$$

and f is not Riemann integrable.

Example 10. *Show that the function f defined by* $f(x) = \begin{cases} 0, & \text{if } x \text{ is an integer} \\ 1, & \text{otherwise} \end{cases}$ *is integrable on* $[0, m]$, *m being a positive integer.* (M.D.U. 1990)

Sol. $$f(x) = \begin{cases} 0, & \text{if } x = 0, 1, 2, \ldots\ldots, m \\ 1, & \text{if } r - 1 < x < r, \quad r = 1, 2, \ldots\ldots, m \end{cases}$$

$\Rightarrow$ f is bounded and has only $m + 1$ points of finite discontinuity at $0, 1, 2, \ldots\ldots, m$.

Since the points of discontinuity of f on $[0, m]$ are finite in number, therefore, f is integrable on $[0, m]$.

Note. $$\int_0^m f(x)\, dx = \int_0^1 f(x)\, dx + \int_0^2 f(x)\, dx + \ldots\ldots + \int_{m-1}^m f(x)\, dx$$

$$= \int_0^1 1\, dx + \int_1^2 1\, dx + \ldots\ldots + \int_{m-1}^m 1\, dx$$

$$= (1 - 0) + (2 - 1) + \ldots\ldots + (m - (m - 1)) = 1 + 1 + \ldots\ldots + 1 = m.$$

Example 12. *Show that a function f defined on [0, 1] by* $f(x) = \begin{cases} \dfrac{1}{n}, & \dfrac{1}{n+1} < x \le \dfrac{1}{n}, \quad (n = 1, 2, \ldots\ldots) \\ 0, & x = 0 \end{cases}$

is integrable on [0, 1]. Also show that $\int_0^1 f(x)\, dx = \frac{\pi^2}{6} - 1$.

Sol.
$$\begin{aligned} f(x) &= 1, \text{ when } \tfrac{1}{2} < x \le 1 \\ &= \tfrac{1}{2}, \text{ when } \tfrac{1}{3} < x \le \tfrac{1}{2} \\ &= \tfrac{1}{3}, \text{ when } \tfrac{1}{4} < x \le \tfrac{1}{3} \\ &\;\vdots \\ &= \frac{1}{n}, \quad \text{when } \frac{1}{n+1} < x \le \frac{1}{n} \\ &\;\vdots \\ &= 0, \quad \text{when } x = 0 \end{aligned}$$

Example 11. *Show that the function f defined by*

$$f(x) = \frac{1}{2^n}, \quad \text{when } \frac{1}{2^{n+1}} < x \le \frac{1}{2^n}, \ (n = 0, 1, 2, ...)$$

$$f(a) = 0$$

is integrable on [0, 1], although it has an infinite number of points of discontinuity.

Also evaluate $\int_0^1 f(x)\,dx$ (M.D.U. 1995)

Sol.

$$\begin{aligned} f(x) &= 1, && \text{when } \frac{1}{2} < x \le 1 \\ &= \frac{1}{2}, && \text{when } \frac{1}{2^2} < x \le \frac{1}{2} \\ &= \frac{1}{2^2}, && \text{when } \frac{1}{2^3} < x \le \frac{1}{2^2} \\ &\vdots \\ &= \frac{1}{2^{n-1}}, && \text{when } \frac{1}{2^n} < x \le \frac{1}{2^{n-1}} \\ &\vdots \\ &= 0, && \text{when } x = 0 \end{aligned}$$

Thus we notice that f is bounded and continuous on [0, 1] except at the points $0, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, ...$

The set of points of discontinuity of f on [0, 1] is $\left\{0, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, ...\right\}$ which has only one limit point 0.

Since the set of points of discontinuity of f on [0, 1] has a finite number of limit points, therefore, f is integrable on [0, 1].

$$\text{Now} \int_{1/2^n}^1 f(x)\,dx = \int_{1/2}^1 f(x)\,dx + \int_{1/2^2}^{1/2} f(x)\,dx + \int_{1/2^3}^{1/2^2} f(x)\,dx + ... + \int_{1/2^n}^{1/2^{n-1}} f(x)\,dx$$

$$= \int_{1/2}^1 1dx + \int_{1/2^2}^{1/2} \frac{1}{2}dx + \int_{1/2^3}^{1/2^2} \frac{1}{2^2}dx + ... + \int_{1/2^n}^{1/2^{n-1}} \frac{1}{2^{n-1}}dx$$

$$= \left(1 - \frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2^2}\right) + \frac{1}{2^2}\left(\frac{1}{2^2} - \frac{1}{2^3}\right) + ... + \frac{1}{2^{n-1}}\left(\frac{1}{2^{n-1}} - \frac{1}{2^n}\right)$$

$$= \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2^2}\right) + \frac{1}{2^2}\left(\frac{1}{2^3}\right) + ... + \frac{1}{2^{n-1}}\left(\frac{1}{2^n}\right)$$

$$= \frac{1}{2}\left[1 + \frac{1}{2^2} + \left(\frac{1}{2^2}\right)^2 + ... + \left(\frac{1}{2^2}\right)^{n-1}\right] = \frac{1}{2} \cdot \frac{1 - \left(\frac{1}{2^2}\right)^n}{1 - \frac{1}{2^2}} = \frac{2}{3}\left(1 - \frac{1}{4^n}\right)$$

Proceeding to the limit when $n \to \infty$, we get $\int_0^1 f(x)\,dx = \frac{2}{3}$.

Example 14. *Show that* $\lim\limits_{n \to \infty} \left[\frac{1}{n} + \frac{n^2}{(n+1)^3} + \frac{n^2}{(n+2)^3} + \dots\dots + \frac{1}{8n} \right] = \frac{3}{8}$.

Sol. $\lim\limits_{n \to \infty} \left[\frac{1}{n} + \frac{n^2}{(n+1)^3} + \frac{n^2}{(n+2)^3} + \dots\dots + \frac{1}{8n} \right]$

$$= \lim_{n \to \infty} \left[\frac{n^2}{(n+0)^3} + \frac{n^2}{(n+1)^3} + \frac{n^2}{(n+2)^3} + \dots\dots + \frac{n^2}{(n+n)^3} \right]$$

$$= \lim_{n \to \infty} \sum_{r=0}^{n} \frac{n^2}{(n+r)^3} = \lim_{n \to \infty} \sum_{r=0}^{n} \frac{\frac{1}{n}}{\left(1 + \frac{r}{n}\right)^3}$$

$$= \int_0^1 \frac{dx}{(1+x)^3} \qquad \left[\text{replacing } \frac{r}{n} \text{ by } x \text{ and } \frac{1}{n} \text{ by } dx \right]$$

$$= \left[\frac{-1}{2(1+x)^2} \right]_0^1 = -\frac{1}{2}\left(\frac{1}{4} - 1\right) = \frac{3}{8}.$$

Example 9. *Show that the greatest integer function $f(x) = [x]$ is integrable on $[0, 4]$ and*

$$\int_0^4 [x]\, dx = 6.$$

(M.P.U. 1992)

Sol. $f(x) = [x]$ on $[0, 4]$ $\Rightarrow$ $f(x) = \begin{cases} 0 & \text{when } 0 \le x < 1 \\ 1 & \text{[illegible]} \\ 2 & \text{when [illegible]} \\ 3 & \text{when } 3 \le x < 4 \end{cases}$

$\Rightarrow$ f is bounded and has *only four* points of finite discontinuity at 1, 2, 3, 4.

Since the points of discontinuity of f on $[0, 4]$ are finite in number, therefore, f is integrable on $[0, 4]$ and

$$\int_0^4 [x]\, dx = \int_0^1 [x]\, dx + \int_1^2 [x]\, dx + \int_2^3 [x]\, dx + \int_3^4 [x]\, dx$$

$$= \int_0^1 0\, dx + \int_1^2 1\, dx + \int_2^3 2\, dx + \int_3^4 3\, dx$$

$$= 0 + (2-1) + 2(3-2) + 3(4-3) = 6.$$

Example 7. *Prove that* $\int_0^{\pi/2} \cos x\, dx = 1$.

Sol. Since $f(x) = \cos x$ is bounded and continuous on $\left[0, \frac{\pi}{2}\right]$, therefore, f is integrable on $\left[0, \frac{\pi}{2}\right]$.

Consider a partition $P = \left\{0 = x_0, x_1, x_2, \ldots\ldots, x_n = \frac{\pi}{2}\right\}$ of $\left[0, \frac{\pi}{2}\right]$ dividing it into n equal sub-inte

each of length $\frac{\frac{\pi}{2} - 0}{n} = \frac{\pi}{2n}$ so that $\|P\| \to 0$ as $n \to \infty$.

Also $\qquad x_r = 0 + \frac{r\pi}{2n} = \frac{r\pi}{2n}$ and $\delta_r = \frac{\pi}{2n}$, $\qquad r = 1, 2, \ldots\ldots, n$.

$$\therefore \quad \int_0^{\pi/2} f(x)\, dx = \lim_{\|P\| \to 0} \sum_{r=1}^{n} f(\xi_r)\delta_r = \lim_{n \to \infty} \sum_{r=1}^{n} f(x_r)\,\delta_r \qquad \text{(taking } \xi_r = x_r\text{)}$$

$$= \lim_{n \to \infty} \sum_{r=1}^{n} f\left(\frac{r\pi}{2n}\right) \cdot \frac{\pi}{2n} = \lim_{n \to \infty} \sum_{r=1}^{n} \frac{\pi}{2n} \cos \frac{r\pi}{2n}$$

$$= \lim_{n \to \infty} \frac{\pi}{2n} \left[\cos \frac{\pi}{2n} + \cos \frac{2\pi}{2n} + \ldots\ldots + \cos \frac{n\pi}{2n}\right]$$

$$= \lim_{n \to \infty} \frac{\pi}{2n} \cdot \frac{\cos\left(\frac{\pi}{2n} + \frac{n-1}{2} \cdot \frac{\pi}{2n}\right) \sin\left(\frac{n}{2} \cdot \frac{\pi}{2n}\right)}{\sin\left(\frac{1}{2} \cdot \frac{\pi}{2n}\right)}$$

$$\left[\because \quad \cos\alpha + \cos(\alpha + \beta) + \cos(\alpha + 2\beta) + \ldots\ldots \text{ to } n \text{ terms} = \frac{\cos\left(\alpha + \frac{n-1}{2}\beta\right) \sin \frac{n\beta}{2}}{\sin \frac{\beta}{2}}\right]$$

$$= \lim_{n \to \infty} 2 \cdot \frac{\frac{\pi}{4n}}{\sin \frac{\pi}{4n}} \cdot \cos \frac{\pi}{4}\left(\frac{2}{n} + \frac{n-1}{n}\right) \sin \frac{\pi}{4}$$

$$= \lim_{n \to \infty} 2 \cdot \frac{\frac{\pi}{4n}}{\sin \frac{\pi}{4n}} \cdot \cos \frac{\pi}{4}\left(1 + \frac{1}{n}\right) \sin \frac{\pi}{4} = 2 \times 1 \times \cos \frac{\pi}{4} \times \sin \frac{\pi}{4} = \sin \frac{\pi}{2} = 1.$$

Example 8. *Show by an example that every bounded function need not be R-integrable.*

(M.D.U. 1991)

Sol. Consider a function f defined on $[0, 1]$ by $f(x) = \begin{cases} 0, & \text{when } x \text{ is rational} \\ 1, & \text{when } x \text{ is irrational} \end{cases}$

Clearly, $f(x)$ is bounded in $[0, 1]$ because $0 \le f(x) \le 1 \quad \forall \quad x \in [0, 1]$

If $P = \{0 = x_0, x_1, x_2, \ldots\ldots, x_n = 1\}$ is any partition of $[0, 1]$, then for any sub-interval $I_r = [x_{r-1}, x_r]$, $r = 1, 2, \ldots, n$, we have $M_r = 1, m_r = 0$

$$\therefore \qquad U(P, f) = \sum_{r=1}^{n} M_r \delta_r = \sum_{r=1}^{n} 1 \, . \, (x_r - x_{r-1}) = x_n - x_0 = 1$$

and

$$L(P, f) = \sum_{r=1}^{n} m_r \delta_r = \sum_{r=1}^{n} 0(x_r - x_{r-1}) = 0$$

$$\therefore \qquad \underline{\int_{0}^{1}} f(x)\, dx = \sup \left\{ L(P, f) \right\}_{P \in P[0, 1]} = 0$$

and

$$\overline{\int_{0}^{1}} f(x)\, dx = \inf. \left\{ U(P, f) \right\}_{P \in P[0, 1]} = 1$$

Example 3. *Evaluate* $\int_{-1}^{2} f(x)\,dx$, *where* $f(x) = |x|$.

Sol. Since $f(x) = |x| = \begin{cases} -x, & \text{when } x \le 0 \\ x, & \text{when } x > 0 \end{cases}$

$\therefore$ f is bounded and continuous on $[-1, 2]$

$\Rightarrow$ f is integrable on $[-1, 2]$

Consider a partition $P = \{-1 = x_0, x_1, x_2, \ldots\ldots, x_n = 0, x_{n+1}, x_{n+2}, \ldots, x_{3n} = 2\}$ of $[-1, 2]$ dividing it into $3n$ equal sub-intervals, each of length $\frac{b-a}{3n} = \frac{2-(-1)}{3n} = \frac{1}{n}$ so that $\|P\| \to 0$ as $n \to \infty$.

Also $\quad x_r = -1 + \frac{r}{n}$ and $\delta_r = \frac{1}{n}$, $r = 1, 2, \ldots\ldots, 3n$.

$$\therefore \int_{-1}^{2} f(x)\,dx = \lim_{\|P\| \to 0} \sum_{r=1}^{3n} f(\xi_r)\delta_r = \lim_{n \to \infty} \sum_{r=1}^{3n} f(x_r)\delta_r \qquad \text{(taking } \xi_r = x_r\text{)}$$

$$= \lim_{n \to \infty} \left[\sum_{r=1}^{n} f(x_r)\delta_r + \sum_{r=n+1}^{3n} f(x_r)\delta_r \right]$$

$$= \lim_{n \to \infty} \left[\sum_{r=1}^{n} f\left(-1 + \frac{r}{n}\right) \cdot \frac{1}{n} + \sum_{r=n+1}^{3n} f\left(-1 + \frac{r}{n}\right) \cdot \frac{1}{n} \right]$$

$$= \lim_{n \to \infty} \left[\sum_{r=1}^{n} -\left(-1 + \frac{r}{n}\right) \cdot \frac{1}{n} + \sum_{r=n+1}^{3n} \left(-1 + \frac{r}{n}\right) \cdot \frac{1}{n} \right]$$

$$= \lim_{n \to \infty} \left[\sum_{r=1}^{n} \left(\frac{1}{n} - \frac{r}{n^2}\right) + \sum_{r=n+1}^{3n} \left(-\frac{1}{n} + \frac{r}{n^2}\right) \right]$$

$$= \lim_{n \to \infty} \left[\frac{1}{n} \cdot n - \frac{1}{n^2} \sum_{r=1}^{n} r + \left(-\frac{1}{n}\right) \cdot 2n + \frac{1}{n^2} \sum_{r=n+1}^{3n} r \right]$$

$$= \lim_{n \to \infty} \left[1 - \frac{1}{n^2} \cdot \frac{n(n+1)}{2} - 2 + \frac{1}{n^2} \left\{ (n+1) + (n+2) + \ldots + 3n \right\} \right]$$

$$= \lim_{n \to \infty} \left[-1 - \frac{1}{2}\left(\frac{n+1}{n}\right) + \frac{1}{n^2} \cdot \frac{2n}{2}(n + 1 + 3n) \right]$$

$$= \lim_{n \to \infty} \left[-1 - \frac{1}{2}\left(1 + \frac{1}{n}\right) + \left(4 + \frac{1}{n}\right) \right] = -1 - \frac{1}{2} + 4 = \frac{5}{2}.$$

4.10. THEOREM

Every convergent sequence has a unique limit.

Or

A sequence cannot converge to more than one limit. (D.U. 1987)

Proof. If possible, let a sequence $\{a_n\}$ converge to two distinct real numbers l and l'.

Let $\varepsilon = \frac{1}{2}|l - l'|$. Since $l \neq l'$, $|l - l'| > 0$ so that $\varepsilon > 0$.

Now the sequence $\{a_n\}$ converges to l

$\Rightarrow$ Given $\varepsilon > 0$, $\exists$ a positive integer m_1 such that $|a_n - l| < \dfrac{\varepsilon}{2} \quad \forall n \geq m_1$

Also the sequence $\{a_n\}$ converges to l'

$\Rightarrow$ Given $\varepsilon > 0$, $\exists$ a positive integer m_2 such that $|a_n - l'| < \dfrac{\varepsilon}{2} \quad \forall n \geq m_2$

Let $m = \max. \{m_1, m_2\}$

Then $|a_n - l| < \dfrac{\varepsilon}{2}$

and $|a_n - l'| < \dfrac{\varepsilon}{2} \quad \forall n \geq m$ ...(1)

Now
$$|l - l'| = |(l - a_n) + (a_n - l')| \leq |l - a_n| + |a_n - l'|$$
$$= |a_n - l| + |a_n - l'| \qquad [\because \ |-x| = |x|]$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \forall n \geq m \qquad \text{[Using (1)]}$$

$\therefore$ $|l - l'| < \varepsilon \quad \forall n \geq m$

which contradicts the assumption that $\varepsilon = \frac{1}{2}|l - l'|$

$\Rightarrow$ Our supposition is wrong. Hence $l = l'$.

Example 17. *Prove that the sequence $\{u_n\}$ defined by $u_1 = \sqrt{7}$, $u_{n+1} = \sqrt{7 + u_n}$ converges to the positive root of the equation $x^2 - x - 7 = 0$.* (K.U. 1984 S)

Sol. $u_1 = \sqrt{7}$, $u_{n+1} = \sqrt{7 + u_n}$

$\therefore$ $u_2 = \sqrt{7 + u_1} = \sqrt{7 + \sqrt{7}} > \sqrt{7} = u_1 \quad \Rightarrow \quad u_2 > u_1$

Suppose $u_n > u_{n-1}$

$\Rightarrow$ $7 + u_n > 7 + u_{n-1} \quad \Rightarrow \quad \sqrt{7 + u_n} > \sqrt{7 - u_{n-1}} \quad \Rightarrow \quad u_{n+1} > u_n$

$\therefore$ By mathematical induction, $u_{n+1} > u_n \quad \forall n$

$\Rightarrow$ $\{u_n\}$ is monotonically increasing.

Now $u_1 = \sqrt{7} < 7$

Suppose $u_n < 7$

then $7 + u_n < 7 + 7 \quad \Rightarrow \quad \sqrt{7 + u_n} < \sqrt{14} < \sqrt{49} = 7 \quad \Rightarrow \quad u_{n+1} < 7$

$\therefore$ By mathematical induction, $u_n < 7 \ \forall n$

$\Rightarrow$ $\{u_n\}$ is bounded above.

Since $\{u_n\}$ is monotonically increasing and bounded above, it is convergent.

Let $\quad \underset{n \to \infty}{\text{Lt}}\ u_n = l$

Now $\quad u_{n+1} = \sqrt{7 + u_n} \quad \Rightarrow \quad (u_{n+1})^2 = 7 + u_n$

$\Rightarrow \quad \underset{n \to \infty}{\text{Lt}}\ (u_{n+1})^2 = \underset{n \to \infty}{\text{Lt}}\ (7 + u_n) \quad \Rightarrow \quad l^2 = 7 + l$

$\Rightarrow \quad l^2 - l - 7 = 0 \quad \Rightarrow \quad l = \dfrac{1 \pm \sqrt{29}}{2}$

But $\dfrac{1 - \sqrt{29}}{2} < 0$ whereas $u_n > 0\ \forall n$

$\therefore \quad l \neq \dfrac{1 - \sqrt{29}}{2}$

Hence the sequence $\{u_n\}$ converges to $\dfrac{1 + \sqrt{29}}{2}$ which is the +ve root of the equation $x^2 - x - 7 = 0$.

Example 20. *If a_1, b_1 are two positive unequal numbers and a_n, b_n are defined as*

$$a_n = \frac{1}{2}(a_{n-1} + b_{n-1}),\ b_n = \sqrt{a_{n-1}\, b_{n-1}} \text{ for } n \geq 2,$$

prove that two sequences $\{a_n\}$ and $\{b_n\}$ are monotonic, one increasing and the other decreasing and that they tend to the same limit.

Sol. Let $a_1 > b_1$.

For any two +ve numbers, the arithmetic mean is greater than the geometric mean.

$\therefore \quad a_n > b_n$, for all n. ...(*i*)

Also $\quad a_{n+1} = \frac{1}{2}(a_n + b_n) < \frac{1}{2}(a_n + a_n) = a_n \quad [\because\ b_n < a_n]$

$\therefore \quad \{a_n\}$ is monotonic decreasing

$\Rightarrow \quad a_1 > a_2 > a_3 > a_4 \ldots\ldots$ ...(*ii*)

Again $\quad b_{n+1} = \sqrt{a_n \cdot b_n} > \sqrt{b_n \cdot b_n} = b_n \quad [\because\ a_n > b_n]$

$\therefore \quad \{b_n\}$ is monotonic increasing *i.e.* $\quad b_1 < b_2 < b_3 < b_4 \ldots\ldots$ ...(*iii*)

Now $\quad a_n = \frac{1}{2}(a_{n-1} + b_{n-1}) > \frac{1}{2}(b_{n-1} + b_{n-1}) = b_{n-1} \quad [\because$ of (*i*)]

i.e. $\quad a_n > b_{n-1} > b_{n-2} > \ldots\ldots > b_2 > b_1 \quad$ [from (*iii*)]

$\therefore \quad a_n > b_1$ for all n.

This implies that the sequence $\{a_n\}$ is bounded below and being monotonic decreasing is convergent.

Again, $\quad b_n = \sqrt{a_{n-1} \cdot b_{n-1}} < \sqrt{a_{n-1} \cdot a_{n-1}} = a_{n-1} \quad [\because$ of (*i*)]

i.e. $\quad b_n < a_{n-1} < a_{n-2} < \ldots\ldots < a_2 < a_1 \quad$ [from (*ii*)]

or $\quad b_n < a_1$ for all n.

$\therefore \quad \{b_n\}$ is bounded above and being monotonic increasing is convergent.

Now let $\quad \text{Lt}\ a_n = l \quad$ and $\quad \text{Lt}\ b_n = l'$.

Since $\quad a_n = \frac{1}{2}(a_{n-1} + b_{n-1}) \quad$ or $\quad 2a_n = a_{n-1} + b_{n-1}$

$\therefore \quad 2a_{n+1} = a_n + b_n$

or $\quad \text{Lt}\ (2a_{n+1}) = \text{Lt}\ a_n + \text{Lt}\ b_n \quad$ or $\quad 2l = l + l' \quad$ or $\quad l = l'$.

$\therefore \quad \{a_n\}$ and $\{b_n\}$ converge to the same limit.

Example 22. *A sequence $<a_n>$ is defined as* $a_1 = 1,\ a_{n+1} = \dfrac{4+3a_n}{3+2a_n},\ n \geq 1$. *Show that $<a_n>$ converges and find its limit.* (D.U. 1984, 87)

Sol. $a_1 = 1, \quad a_2 = \dfrac{4+3a_1}{3+2a_1} = \dfrac{7}{5} > 1 \quad \Rightarrow \quad a_2 > a_1.$

Let us assume that $a_{n+1} > a_n$

Then $a_{n+2} - a_{n+1} = \dfrac{4+3a_{n+1}}{3+2a_{n+1}} - \dfrac{4+3a_n}{3+2a_n} = \dfrac{a_{n+1}-a_n}{(3+2a_{n+1})(3+2a_n)} > 0$

$[\because \ a_{n+1} > a_n \ \text{and} \ a_n > 0 \ \forall n]$

$\Rightarrow \quad a_{n+2} > a_{n+1}.$

$\therefore$ By mathematical induction, $<a_n>$ is monotonically increasing.

Also $a_{n+1} = \dfrac{4+3a_n}{3+2a_n} = \dfrac{3}{2} - \dfrac{1}{2(3+2a_n)}$

$= \dfrac{3}{2} -$ (a positive quantity less than 1) $\quad [\because \ a_n > a_1 = 1 \ \forall n]$

$< \dfrac{3}{2} \quad \Rightarrow \quad a_{n+1} < \dfrac{3}{2} \quad \forall n.$

$\therefore$ The sequence $<a_n>$ is bounded above.

Since the sequence $<a_n>$ is monotonically increasing and bounded above, it is convergent.

Let the sequence $<a_n>$ converge to l, then $\lim\limits_{n\to\infty} a_n = l$

Now $a_{n+1} = \dfrac{4+3a_n}{3+2a_n} \quad \Rightarrow \quad \lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} \dfrac{4+3a_n}{3+2a_n}$

$\Rightarrow \quad l = \dfrac{4+3l}{3+2l} \quad \Rightarrow \quad 3l + 2l^2 = 4 + 3l$

$\Rightarrow \quad l^2 = 2 \quad \therefore \quad l = \pm\sqrt{2}$

But l cannot be negative. $\quad (\because \ a_n \geq 1 \ \forall n, \quad \therefore \quad l = \lim\limits_{n\to\infty} a_n \geq 1).$

$\therefore$ Rejecting $l = -\sqrt{2}$, we have $l = \sqrt{2}$.

Example 37. *Prove that the sequence $< a_n >$ where* $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots\ldots + \frac{1}{2n}$ *is convergent and its limit lies between $\frac{1}{2}$ and 1.* (M.D.U. 1993 ; D.U. 1984, 85)

Sol. $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots\ldots + \frac{1}{2n}$

$$a_{n+1} = \frac{1}{n+2} + \frac{1}{n+3} + \ldots\ldots + \frac{1}{2n} + \frac{1}{2n+1} + \frac{1}{2n+2}$$

$$\therefore \quad a_{n+1} - a_n = \frac{1}{2n+2} + \frac{1}{2n+1} - \frac{1}{n+1}$$

$$> \frac{1}{2n+2} + \frac{1}{2n+2} - \frac{1}{n+1} = \frac{2}{2n+2} - \frac{1}{n+1} = \frac{1}{n+1} - \frac{1}{n+1} = 0$$

$\Rightarrow \quad a_{n+1} - a_n > 0 \quad \forall n$

$\Rightarrow \quad a_{n+1} > a_n \quad \forall n$

$\Rightarrow$ $< a_n >$ is monotonically increasing.

Also $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots\ldots + \frac{1}{n+n}$

$$< \frac{1}{n+1} + \frac{1}{n+1} + \ldots\ldots + \frac{1}{n+1} = \frac{n}{n+1} < 1 \quad \forall n$$

$\Rightarrow$ $< a_n >$ is bounded above.

Since $< a_n >$ is monotonically increasing and bounded above, it is convergent.

Now $a_n = \frac{1}{n+1} + \frac{1}{n+2} + \ldots\ldots + \frac{1}{2n}$

$$> \frac{1}{2n} + \frac{1}{2n} + \ldots\ldots + \frac{1}{2n} = \frac{n}{2n} = \frac{1}{2} \quad \forall n$$

$\therefore \quad \frac{1}{2} < a_n < 1 \quad \forall n$

Hence $\frac{1}{2} \le \lim_{n \to \infty} a_n \le 1.$

a. Give an example of an infinite set that has no limit point.
As we saw in Exercise 1, the infinite set $\mathbb{Z}$ has no limit point.

b. Give an example of a bounded set that has no limit point.
A finite set like $\{2\}$ will not have any limit points. We could also look at the empty set $\emptyset$.

c. Give an example of an unbounded set that has no limit point.
As we saw in Exercise 1, the infinite set $\mathbb{Z}$ has no limit point.

d. Give an example of an unbounded set that has exactly one limit point.
The unbounded set $\mathbb{Z} \cup \left\{\frac{1}{n} \mid n \in \mathbf{Z}^+\right\}$ has only the limit point 0.

e. Give an example of an unbounded set that has exactly two limit points.
The set

$$\boxed{\mathbf{Z} \cup \left\{\frac{1}{n} \mid n \in \mathbf{Z}^+\right\} \cup \left\{1 + \frac{1}{n} \mid n \in \mathbf{Z}^+\right\}}$$

has the two limit points 0 and 1. We can see this directly or we can use the assertion proved in Exercise 6 below.