IAS
PREVIOUS YEARS QUESTIONS (2019-1983) SEGMENT-WISE

LINEAR ALGEBRA

2019

- Let $T:\mathbb{R}^2 \to \mathbb{R}^2$ be a linear map such that $T(2, 1) = (5, 7)$ and $T(1, 2) = (3, 3)$. If A is the matrix corresponding to T with respect to the standard bases e_1, e_2, then find rank (A). **[10]**

- If $A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -4 & 1 \\ 3 & 0 & -3 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 1 & -1 \end{bmatrix}$

  then show that $AB = 6I_3$. Use this result to solve the following system of equations :
  $2x + y + z = 5$
  $x - y = 0$
  $2x + y - z = 1$ **[10]**

- Let A and B be two orthogonal matrices of same order and det A + det B = 0. Show that A + B is a singular matrix. **[15]**

- Let $A = \begin{bmatrix} 5 & 7 & 2 & 1 \\ 1 & 1 & -8 & 1 \\ 2 & 3 & 5 & 0 \\ 3 & 4 & -3 & 1 \end{bmatrix}$

  (i) Find the rank of matrix A.
  (ii) Find the dimension of the subspace

  $$V = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \;\middle/\; A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0 \right\}$$

  [15+5=20]

- State the Cayley-Hamilton theorem. Use this theorem to find A^{100}, where

  $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ **[15]**

2018

- Let A be a 3×2 matrix and B a 2×3 matrix. Show that $C = A\bullet B$ is a singular matrix. **(10)**
- Express basis vectors $e_1=(1,0)$ and $e_2=(0,1)$ as linear combinations of $\alpha_1= (2,-1)$ and $\alpha_2= (1, 3)$. **(10)**
- Show that if A and B are similar $n\times n$ matrices, then they have the same eigenvalues. **(12)**
- For the system of linear equations **(13)**
  $x + 3y - 2z = -1$
  $5y + 3z = -8$
  $x - 2y - 5z = 7$
  determine which of the following statements are true and which are false:
  (i) The system has no solution.
  (ii) The system has a unique solution.
  (iii) The system has infinitely many solutions.

2017

- Let $A = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$. Find a non-singular matrix P such that $P^{-1}AP$ is a diagonal matrix. **(10)**
- Show that similar matrices have the same characteristic polynomial. **(10)**
- Suppose U and W are distinct four dimensional subspaces of a vector space V, where dim V = 6. Find the possible dimensions of subspace $U \cap W$. **(10)**
- Consider the matrix mapping $A: R^4 \to R^3$, where

  $A = \begin{pmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & -2 \\ 3 & 8 & 13 & -3 \end{pmatrix}$.

  Find a basis and dimension of the image of A and those of the kernel A. **(15)**
- Prove that distinct non-zero eigenvectors of a matrix are linearly independent. **(10)**
- Consider the following system of equations in x,y,z:
  $x + 2y + 2z = 1$
  $x + ay + 3z = 3$
  $x + 11y + az = b.$

(i) For which values of a does the system have a unique solution?

(ii) For which pair of values (a, b) does the system have more than one solution? **(15)**

2016

❖ (i) Using elementary row operations, find the inverse of $A=\begin{bmatrix}1&2&1\\1&3&2\\1&0&1\end{bmatrix}$ **(6)**

(ii) If $A=\begin{bmatrix}1&1&3\\5&2&6\\-2&-1&-3\end{bmatrix}$, then find $A^{14}+3A-2I$. **(4)**

❖ (i) Using elementary row operations, find the condition that the linear equations

$x-2y+z=a$

$2x+7y-3z=b$

$3x+5y-2z=c$

have a solution. **(7)**

(ii) If

$W_1=\{(x,y,z)\mid x+y-z=0\}$

$W_2=\{(x,y,z)\mid 3x+y-2z=0\}$

$W_3=\{(x,y,z)\mid x-7y+3z=0\}$

then find dim $(W_1\cap W_2\cap W_3)$ and dim (W_1+W_2). **(3)**

❖ (i) If $M_2(R)$ is space of real matrices of order 2×2 and $P_2(x)$ is the space of real polynomials of degree at most 2, then find the matrix representation of $T: M_2(R)\to P_2(x)$, such that $T\left(\begin{bmatrix}a&b\\c&d\end{bmatrix}\right)=a+c+(a-d)x+(b+c)x^2$, with respect to the standard bases of $M_2(R)$ and $P_2(x)$. further find the null space of T. **(10)**

(ii) If $T: P_2(x)\to P_3(x)$ is such that $T(f(x))=f(x)+5\int_0^x f(t)dt$, then choosing $\{1, 1+x, 1-x^2\}$ and $\{1, x, x^2, x^3\}$ as bases of $P_2(x)$ and $P_3(x)$ respectively, find the matirx of T. **(10)**

❖ (i) If $A=\begin{bmatrix}1&1&0\\1&1&0\\0&0&1\end{bmatrix}$, then find the eigenvalues and eigenvectors of A. **(8)**

(ii) Prove that eigenvalues of a Hermitian matrix are all real. **(8)**

❖ If $A=\begin{bmatrix}1&-1&2\\-2&1&-1\\1&2&3\end{bmatrix}$ is the matrix representation of a linear transformation $T: P_2(x)\to P_2(x)$ with respect to the bases $\{1-x, x(1-x), x(1+x)\}$ and $\{1, 1+x, 1+x^2\}$, then find T. **(18)**

2015

❖ The vectors $V_1=(1,1,2,4)$, $V_2=(2,-1,-5,2)$, $V_3=(1,-1,-4,0)$ and $V_4=(2,1,1,6)$ are linearly independent. Is it true? Justify your answer. **(10)**

❖ Reduce the following matrix to row echelon form and hence find its rank: **(10)**

$$\begin{bmatrix}1&2&3&4\\2&1&4&5\\1&5&5&7\\8&1&14&17\end{bmatrix}.$$

❖ If matrix $A=\begin{bmatrix}1&0&0\\1&0&1\\0&1&0\end{bmatrix}$ then find A^{30}. **(12)**

❖ Find the eigen values and eigen vectors of the matrix:

$$\begin{bmatrix}1&1&3\\1&5&1\\3&1&1\end{bmatrix}.$$ **(12)**

❖ Let V=$\mathbf{R}^3$ and $T\in A(V)$, for all $a_i\in A(V)$, be defined by $T(a_1,a_2,a_3)=(2a_1+5a_2+a_3, -3a_1+a_2-a_3, -a_1+2a_2+3a_3)$

What is the matrix T relative to the basis

$V_1=(1,0,1)$ $V_2=(-1,2,1)$ $V_3=(3,-1,1)$? **(13)**

❖ Find the dimension of the subspace of $\mathbf{R}^4$, spanned by the set

$\{(1,0,0,0), (0,1,0,0), (1,2,0,1), (0,0,0,1)\}$

Hence find its basis. **(12)**

2014

❖ Find one vector in R^3 which generates the intersection of V and W, where V is the xy plane and W is the space generated by the vectors (1, 2, 3) and (1, –1, 1). **(10)**

❖ Using elementary row or column operations, find the rank of the matrix **(10)**

$$\begin{bmatrix} 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$$

- Let V and W be the following subspaces of R^4: $V = \{(a, b, c, d): b-2c+d=0\}$ and $W = \{(a,b,c,d): a= d, b = 2c\}$. Find a basis and the dimension of (i) V, (ii) W, (iii) $V \cap W$. **(15)**
- Investigate the values of λ and μ so that the equations $x + y + z = 6$, $x + 2y + 3z = 10$, $x + 2y + \lambda z = \mu$ have (1) no solution, (2) a unique solution, (3) an infinite number of solutions. **(10)**
- Verify Cayley - Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and hence find its inverse. Also, find the matrix represented by $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$. **(10)**
- (i) Let $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$. Find the eigen values of A and the corresponding eigen vectors. **(8)**

  (ii) Prove that the eigen values of a unitary matrix have absolute value 1. **(7)**

2013

- Find the inverse of the matrix

  $$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & -1 & 7 \\ 3 & 2 & -1 \end{bmatrix}$$

  by using elementary row operations. Hence solve the system of linear equations

  $$x + 3y + z = 10$$

  $$2x - y + 7z = 21$$

  $$3x + 2y - z = 4 \qquad \textbf{(10)}$$

- Let A be a square matrix and A^* be its adjoint, show that the eigenvalues of matrices AA^* and A^*A are real. Further show that trace (AA^*) = trace (A^*A). **(10)**
- Let P_n denote the vector space of all real polynomials of degree almost n and $T : P_2 \to P_3$ be a linear transformation given by

  $$T\left(p(x)\right) = \int_0^x p(t)\,dt, \;\; p(x) \in P_2.$$

  Find the matrix of T with respect to the bases $\{1, x, x^2\}$ and $\{1, x, 1+x^2, 1+x^3\}$ of P_2 and P_3 respectively. Also, find the null space of T. **(10)**
- Let V be an n-dimensional vector space and $T : V \to V$ be an invertible linear operator. If $\beta = \{X_1, X_2, ..., X_n\}$ is a basis of V, show that $\beta' = \{TX_1, TX_2, ...TX_n\}$ is also a basis of V. **(8)**
- Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix}$ where $\omega (\neq 1)$ is a cube root of unity. If $\lambda_1, \lambda_2, \lambda_3$ denote the eigenvalues of A^2, show that $|\lambda_1| + |\lambda_2| + |\lambda_3| \le 9$. **(8)**
- Find the rank of the matrix

  $$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 8 & 12 \\ 3 & 5 & 8 & 12 & 17 \\ 5 & 8 & 12 & 17 & 23 \\ 8 & 12 & 17 & 23 & 30 \end{bmatrix} \qquad \textbf{(8)}$$

- (i) Let A be a Hermetian matrix having all distinct eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$. If $X_1, X_2, ..., X_n$ are corresponding eigenvectors then show that the $n \times n$ matrix C whose k^{th} column consists of the vector X_k is non singular. **(8)**

  (ii) Show that the vecotors $X_1 = (1, 1+i, i)$, $X_2 = (i, -i, 1-i)$ and $X_3 = (0, 1-2i, 2-i)$ in $\mathbb{C}^3$ are linearly independent over the field of real numbers but are linearly dependent over the field of complex numbers. **(8)**

2012

- Prove or disprove the following statement: If $B = \{b_1, b_2, b_3, b_4, b_5\}$ is a basis for $\mathbb{R}^5$ and V is a two-dimensional subspace of $\mathbb{R}^5$, then V has a basis made of just two members of B.

- Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by $T(\alpha,\beta,\gamma)=(\alpha, 2\beta-3\gamma, 2\alpha+5\beta-4\gamma, \alpha+4\beta+\gamma)$

  Find a basis and the dimension of the image of T and the kernel of T.

- (i) Let V be the vector space of all 2×2 matrices over the field of real numbers. Let W be the set consisting of all matrices with zero determinant. Is W a subspace of V? Justify your answer.

  (ii) Find the dimension and a basis for the space W of all solutions of the following homogeneous system using matrix notation:

  $$x_1+2x_2+3x_3-2x_4+4x_5=0$$
  $$2x_1+4x_2+8x_3+x_4+9x_5=0$$
  $$3x_1+6x_2+13x_3+4x_4+14x_5=0$$

- (i) Consider the linear mapping $f:\mathbb{R}^2 \to \mathbb{R}^2$ by

  $f(x,y)=(3x+4y, 2x-5y)$

  Find the matrix A relative to the basis {(1, 0), (0, 1)} and the matrix B relative to the basis {(1, 2), (2, 3)}.

  (ii) If λ is a characteristic root of a non-singular matrix A, then prove that $\frac{|A|}{\lambda}$ is a characteristic root of Adj A.

- Let $H=\begin{pmatrix} 1 & i & 2+i \\ -i & 2 & 1-i \\ 2-i & 1+i & 2 \end{pmatrix}$ be a Hermitian matrix. Find a non-singular matrix P such that $D=P^T H \overline{P}$ is diagonal.

2011

- Let A be a non-singular, n × n square matrix. Show that A. (adj A) = |A| I_n. Hence show that

  |adj (adj A)| = $|A|^{(n-1)^2}$. **(10)**

- Let $A=\begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \\ 0 & 6 & 7 \end{bmatrix}, X=\begin{bmatrix} x \\ y \\ z \end{bmatrix}, B=\begin{bmatrix} 2 \\ 6 \\ 5 \end{bmatrix}$

  Solve the system of equations given by AX = B Using the above, also solve the system of equations $A^TX=B$ where A^T denotes the transpose of matrix A. **(10)**

- (i) Let $\lambda_1, \lambda_2, \ldots \lambda_n$ be the eigen values of a n × n square matrix A with corresponding eigen vectors $X_1, X_2, \ldots X_n$. If B is a matrix similar to A show that the eigen values of B are same as that of A. Also find the relation between the eigen vectors of B and eigen vectors of A. **(10)**

  (ii) Verify the Cayley-Hamilton theorem for the matrix $A=\begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \\ 3 & -5 & 1 \end{bmatrix}$

  Using this, show that A is non-singular and find A^{-1}. **(10)**

- (i) Show that the subspaces of IR^3 spanned by two sets of vectors $\{(1,1,-1),(1,0,1)\}$ and $\{(1,2,-3),(5,2,1)\}$ are identical. Also find the dimension of this subspace. **(10)**

  (ii) Find the nullity and a basis of the null space of the linear transformation $A:R^{(4)} \to R^{(4)}$ given by the matrix

  $$A=\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}.$$ **(10)**

- Show that the vectors (1, 1, 1), (2, 1, 2) and (1, 2, 3) are linearly independent in $R^{(3)}$. Let $T: IR^{(3)} \to IR^{(3)}$ be a linear transformation defined by $T(x,y,z)=(x+2y+3z, x+2y+5z, 2x+4y+6z)$

  Show that the images of above vectors under T are linearly dependent. Give the reason for the same. **(10)**

- Let $A=\begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ and C be a non-singular matrix of order 3×3. Find the eigen values of the matrix B^3 where $B = C^{-1}AC$. **(10)**

2010

- If $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of the matrix

  $$A=\begin{bmatrix} 26 & -2 & 2 \\ 2 & 21 & 4 \\ 4 & 2 & 28 \end{bmatrix}$$

  Show that $\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2} \le \sqrt{1949}$ **(12)**

- What is the null space of the differentiation transformation $\frac{d}{dx}: P_n \to P_n$

  Where P_2 is the space of all polynomials of degree $\leq$ n over the real numbers? What is the null space of the second derivative as a transformation of P_n? What is the null space of the kth derivative? **(12)**
- Let $M = \begin{pmatrix} 4 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$. Find the unique linear transformation $T: IR^3 \to IR^2$ so that M is the matrix of T with respect to the basis. $\beta = \{v_1 = (1,0,0), v_2 = (1,1,0), v_3 = (1,1,1)\}$ of IR^3 and $\beta = \{w_1 = (1,0), w_2 = (1,1)\}$ of IR^2. Also find T (x, y, z) **(20)**
- Let A and B be n × n matrices over reals. Show that I-BA is invertible if I-AB is invertible. Deduce that AB and BA have the same eigenvalues. **(20)**
- (i) In the n-space IR^n, determine whether or not the set $\{e_1 - e_2, e_2 - e_3 ..., e_{n-1} - e_n, e_n - e_1\}$ is linearly independent.

  (ii) Let T be a linear transformation from a vector space V over reals into V such that $T - T^2 = I$. Show that T is invertible. **(20)**

2009

- Find a Hermitian and a skew - Hermitian matrix each whose sum is the matrix

  $$\begin{bmatrix} 2i & 3 & -1 \\ 1 & 2+3i & 2 \\ -i+1 & 4 & 5i \end{bmatrix}$$ **(12)**
- Prove that the set V of the vectors (x_1, x_2, x_3, x_4) in IR^4 which satisfy the equations $x_1 + x_2 + 2x_3 + x_4 = 0$ and $2x_1 + 3x_2 - x_3 + x_4 = 0$, is a subspace of IR^4. What is the dimension of this subspace? Find one of its bases. **(12)**
- Let β = {(1, 1, 0), (1, 0, 1), (0, 1,1)} and β′ = {(2, 1, 1), (1, 2, 1), (-1, 1, 1)} be the two ordered bases of IR^3. Then find a matrix representing the linear transformation $T: IR^3 \to IR^3$ which transforms β into β′. Use this matrix representation to find $T(\overline{x})$, where $\overline{x} = (2,3,1)$. **(20)**
- Find a 2 × 2 real matrix A which is both orthogonal and skew-symmetric. Can there exist a 3×3 real matrix which is both orthogonal and skew - symmetric? Justify your answer. **(20)**
- Let $L: IR^4 \to IR^3$ be a linear transformation defined by

  $$L(x_1, x_2, x_3, x_4) = (x_3 + x_4 - x_1 - x_2, x_3 - x_2, x_4 - x_1)$$

  Then find the rank and nullity of L. Also, determine null space and range space of L. **(20)**
- Prove that the set V of all 3 × 3 real symmetric matrices forms a linear subspace of the space of all 3 × 3 real matrices. What is the dimension of this subspace? Find at least one of the bases for V. **(20)**

2008

- Show that the matrix A is invertible if and only if the adj(A) is invertible. Hence find $|adj(A)|$.
- Let S be a non empty set and let V denote the set of all functions from S into R. show that V is a vector space with respect to the vector addition $(f+g)(x) = f(x) + g(x)$ and scalar multiplication $(cf)(x) = cf(x)$ **(12)**
- Show that $B = \{(1,0,0), (1,1,0), (1,1,1)\}$ is a basis of R^3. Let $T: R^3 \to R^3$ be a linear transformation such that T(1, 0, 0) = (1, 0, 0), T (1, 1, 0) = (1, 1, 1) and T (1, 1, 1) = (1, 1, 0). Find T (x, y, z). **(20)**
- Let A be a non-singular matrix. Show that if $I + A + A^2 + ... + A^n = 0$ then $A^{-1} = A^n$. **(20)**
- Find the dimension of the subspace of R^4 spanned by the set {(1, 0, 0, 0), (0, 1, 0, 0), (1, 2, 0, 1), (0, 0, 0, 1)}. Hence find a basis for the subspace. **(20)**

2007

- Let S be the vector space of all polynomials P(x), with real coefficients, of degree less than or equal to two considered over the real field IR, such that P(0)=0 and P(1)=0. Determine a basis for S and hence its dimension. **(12)**
- Let T be the linear transformation from IR^3 to IR^4 defined by

$(2x_1 + x_2 + x_3, x_1 + x_2, x_1 + x_3, 3x_1 + x_2 + 2x_3)$

for each $(x_1, x_2, x_3) \in IR^3$

Determine a basis for the Null space of T. What is the dimension of the Range space of T. **(12)**

- Let W be the set of all 3×3 symmetric matrices over IR. Does it form a subspace of the vector space of the 3×3 matrices over IR? In case it does, construct a basis for this space and determine its dimension. **(20)**
- Consider the vector space $X = \{p(X)/p(X)$ is a polynomial of degree less than or equal to 3 with real coefficients$\}$, over the real field IR. Define the map $D : X \to X$ by

$(Dp)(X) := p_1 + 2p_2X + 3p_3X^2$

where $p(X) = p_0 + p_1X + p_2X^2 + p_3X^3$

Is D a linear tr-transformation on X? If it is, then construct the matrix representation for D with respect to the basis $\{1, X, X^2, X^3\}$ for X. **(20)**

2006

- Let V be the vector space of all 2 × 2 matrices over the field F. Prove that V has dimension 4 by exhibiting a basis for V. **(12)**
- State cayley Hamilton theorem and using it, find the inverse of $\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$. **(12)**
- If $T : IR^2 \to IR^2$ is defined by $T(x, y) = (2x - 3y, x + y)$ compute the matrix of T relative to the basis $\beta = \{(1,2),(2,3)\}$. **(15)**
- Using elementary row operations, find the rank of the matrix $\begin{bmatrix} 3 & -2 & 0 & -1 \\ 0 & 2 & 2 & 1 \\ 1 & -2 & -3 & -2 \\ 0 & 1 & 2 & 1 \end{bmatrix}$. **(15)**
- Investigate for what values of λ and μ the equations,

$x + y + z = 6,\ x + 2y + 3z = 10,\ x + 2y + \lambda z = \mu$

have (i) no solution (ii) a unique solution (iii) infinitely many solutions. **(15)**

2005

- Find the values of k for which the vectors (1, 1, 1, 1,), (1, 3, -2, k), (2, 2k-2, -k-2, 3k-1) and (3, k+2, -3, 2k+1) are linearly independent in R^4. **(12)**
- Let V be the vector space of polynomials in x of degree ≤n over R. Prove that the set $\{1, x, x^2, ..., x^n\}$ is a basis for V. Extend this basis so that it becomes a basis for the set of all polynomials in x. **(12)**
- Let T be a linear transformation on R^3, whose matrix relative to the standard basis of R^3 is $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 2 \\ 3 & 3 & 4 \end{bmatrix}$ find the matrix of T relative to the basis $\beta = \{(1,1,1),(1,1,0),(0,1,1)\}$. **(15)**
- Find the inverse of the matrix given below using elementary row operations only: $\begin{bmatrix} 2 & 0 & -1 \\ 5 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$ **(15)**
- If S is skew - Hermitian matrix,then show that A = (I+S) $(I–S)^{-1}$ is a unitary matrix. Also show that every unitary matrix can be expressed in the above form provided –1 is not an eigen value of A. **(15)**

2004

- Let S be space generated by the vectors $\{(0,2,6),(3,1,6),(4,-2,-2)\}$.

What is the dimension of the space S? find a basis for S. **(12)**

- Show that $f : IR^3 \to IR$ is a linear transformation, where $f(x, y, z) = 3x + y - z$. What is the dimension of the kernel? find a basis for the kernel. **(12)**
- Show that the linear transformation from IR^3 to IR^4 which is represented by the matrix $\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -2 \\ 2 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix}$ is one to one. Find a basis for its image. **(15)**
- Verify whether the following system of equations is consistent?

$x + 3z = 5$

$-2x + 5y - z = 0$

$-x + 4y + z = 4$ **(15)**

- Find the characteristic polynomial of the matrix $A=\begin{pmatrix}1 & 1\\ -1 & 3\end{pmatrix}$, Hence find A^{-1} and A^6. **(15)**

2003

- Let S be any non-empty subset of a vector space V over the field F.
Show that the set
$\{a_1\alpha_1 + a_2\alpha_2 + ... + a_n\alpha_n : a_1, a_2, ..., a_n \in F,$
$\alpha_1, \alpha_2, ...\alpha_n \in S, n \in N\}$ is the subspace generated by S. **(12)**
- If $A=\begin{bmatrix}2 & 1 & 1\\ 0 & 1 & 0\\ 1 & 1 & 2\end{bmatrix}$, then find the matrix represented by: $2A^{10}-10A^9+14A^8-6A^7-3A^6+15A^5-21A^4+9A^3+A-I$. **(12)**
- Prove that the eigen vectors corresponding to distinct eigen values of a square matrix are linearly independent. **(15)**
- If H is a Hermitian matrix. Then show that A=(H +iI)–1 (H–iI) is a unitary matrix. Also show that every unitary matrix can be expressed in this form, provided 1 is not an eigen value of A. **(15)**
- If $A=\begin{bmatrix}6 & -2 & 2\\ -2 & 3 & -1\\ 2 & -1 & 3\end{bmatrix}$, then find a diagonal matrix D and a matrix B such that A = BDB' where B' denotes the transpose of B. **(15)**

2002

- Show that the mapping $T: R^3 \to R^3$ where $T(a,b,c)=(a-b, b-c, a+c)$ is linear and non-singular. **(12)**
- A square matrix A is non-singular if and only if the constant term in its characteristic polynomial is different from zero. **(12)**
- Let $T: R^5 \to R^5$ be a linear mapping given by $T(a,b,c,d,e)=(b-d, d+e, b, 2d+e, b+e)$.
Obtain bases for its null space and range space. **(15)**
- Let A be a real 3×3 symmetric matrix with eigen values 0, 0 and 5. If the corresponding eigen - vectors are (2, 0, 1), (2, 1, 1) and (1, 0, -2), then find the matrix A. **(15)**
- Solve the following system of linear equations:
$x_1 - 2x_2 - 3x_3 + 4x_4 = -1$
$-x_1 + 3x_2 + 5x_3 - 5x_4 - 2x_5 = 0,$
$2x_1 + x_2 - 2x_3 + 3x_4 - 4x_5 = 17.$ **(15)**
- Use cayley - Hamilton theorem to find the inverse of the following matrix $\begin{bmatrix}0 & 1 & 2\\ 1 & 2 & 3\\ 3 & 1 & 1\end{bmatrix}$. **(15)**

2001

- Show that the vectors (1, 0, -1), (0, -3, 2) and (1, 2, 1) form a basis for the vector space $R^3(R)$. **(12)**
- If λ is a characteristic root of a non-singular matrix A, then prove that $\frac{|A|}{\lambda}$ is a characteristic root of AdjA. **(12)**
- If $A=\begin{bmatrix}1 & 0 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}$, show that for every integer $n \geq 3$, $A^n = A^{n-2} + A^2 - I$. Hence, determine A^{50}. **(15)**
- When is a square matrix A said to be congruent to a square matrix B? Prove that every matrix congruent to a skew symmetric matrix is skew symmetric.
- Determine an orthogonal matrix P such that $P^{-1}AP$ is a diagonal matrix, where $A=\begin{bmatrix}7 & 4 & -4\\ 4 & -8 & -1\\ -4 & -1 & -8\end{bmatrix}$. **(15)**

2000

- Let V be a vector space over R and Let $T=\{(x,y) / x, y \in V\}$
Define addition in T componentwise and scalar multiplication by a complex number $\alpha + i\beta$ by
$(\alpha+i\beta)(x,y)=(\alpha x-\beta y, \beta x+\alpha y) \quad \forall \alpha,\beta \in R.$
show that T is a vector space over C. **(12)**
- Show that if λ is a characteristic root of a non-singular matrix A, then λ^{-1} is a characteristic root of A^{-1}. **(12)**
- Prove that a system AX=B of n non-homogeneous equations n unknowns has a unique solution provided the coefficient matrix is non singular. **(15)**

- Prove that two similar matrices have the same characteristic roots. Is its converse true? Justify your claim. **(15)**

1999

- Let V be the vector space of functions from R to IR (the real numbers). Show that f, g, h in V are linearly independent where $f(t)=e^{2t}$, $g(t)=t^2$ and h(t) = t.
- Diagonalize the matrix $A=\begin{bmatrix}4&2&2\\2&4&2\\2&2&4\end{bmatrix}$
- Test for congruency of the matrices $A=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$ and $B=\begin{bmatrix}0&i\\-i&0\end{bmatrix}$. Prove that $A^{2n}=B^{2m}=I$ when m and n are positive integers.
- If A is a skew symmetric matrix of order n, prove that $(I-A)(I+A)^{-1}$ is orthogonal.**(1983&1987)**

1998

- Given two linearly independent vectors (1, 0, 1, 0) and (0, -1, 1, 0) of IR^4, find a basis of IR^4 which includes these two vectors:
- Let $T = IR^3 \to IR^3$ defined by $T(x_1,x_2,x_3)=(x_2,x_3-cx_1,-bx_2-ax_3)$, where a, b, c are fixed real numbers. Show that T is a linear transformation of IR^3 and that $A^3+aA^2+bA+cI=0$, where A is the matrix of T with respect to standard basis of IR^3.
- If A and B are two matrices of order 2×2 such that A is skew Hermitian and AB=B, then show that B=0.
- If T is a complex matrix of order 2×2 such that $tr.T=trT^2=0$, then show that $T^2=0$.
- Let A be an m×n matrix. Then show that the sum of the rank and nullity of A is n.
- Find all real 2×2 matrices A whose characteristic roots are real and which satisfy $AA'=I$

1997

- Let V be the vector space of polynomials over IR. Find a basis and dimension of the subspace w of v spanned by the polynomials

$v_1=t^3-2t^2+4t+1$,

$v_2=2t^3-3t^2+9t-1$, $v_3=t^2+6t-5$,

$v_4=2t^3-5t^2+7t+5$,

- Verify that the transformation defined by $T(x_1,x_2)=(x_1+x_2,x_1-x_2,x_2)$ is a linear transformation from IR^2 into IR^3. Find its range, rank, null space and nullity.
- Let V be the vector space of 2×2 matrices over IR. Determine whether the matrices $A,B,C\in V$ are dependent where $A=\begin{bmatrix}1&2\\3&1\end{bmatrix}$, $B=\begin{bmatrix}3&-1\\2&2\end{bmatrix}$, $C=\begin{bmatrix}1&-5\\-4&0\end{bmatrix}$.
- Show that $A=\begin{bmatrix}2&-1&0\\-1&2&0\\2&2&3\end{bmatrix}$ is diagonalisable over IR and find a matrix P such that $P^{-1}AP$ is diagonal. Hence determine A^{25}.
- Find the characteristic roots and their corresponding vectors for the matrix $\begin{bmatrix}6&-2&2\\-2&3&-1\\2&-1&3\end{bmatrix}$.

1996

- In IR^4, let w_1 be the space generated by (1, 1, 0, -1), (2, 4, 6, 0) and (-2, -3, -3, 1) and let w_2 be the space generated by (-1, -2, -2, 2), (4, 6, 4, -6) and (1, 3, 4, -3). Find a basis for the space w_1+w_2.
- Let V= IR^3 and v_1, v_2, v_3 be a basis of IR^3. Let $T:V\to V$ be linear transformation such that $T(v_i)=v_1+v_2+v_3$, 1 ″ i ″ 3. By writing the matrix of T w.r.t another basis, show that the matrix $\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix}$ is similar to $\begin{bmatrix}3&0&0\\0&0&0\\0&0&0\end{bmatrix}$
- Let $V=IR^3$ and $T:V\to V$ be the linear map defined by $T(x,y,z)=(x+z,-2x+y,-x+2y+z)$.

What is the matrix of T w.r.t the basis (1, 0, 1), (-1, 1, 1) and (0, 1, 1)? Using this matrix, write down the matrix of T w.r.t the basis (0, 1, 2), (-1, 1, 1) and (0, 1, 1).

- Solve x + y –2z = 1
  2x-7z=3
  x+y-z=5 by using cramer's rule.
- Find the inverse of the matrix $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ by computing its characteristic polynomial.
- Let A and B be n×n matrices such that AB=BA. Show that A and B have a common characteristic vector.

1995

- Let T be the linear operator in R^3 defined by T $(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_1+2x_2+4x_3)$. What is the matrix of T in the standard ordered basis for R^3? what is a basis of range space of T and a basis of null space of T?
- Let A be a square matrix of order n. prove that AX=b has a solution iff $b \in IR^n$ is orthogonal to all solutions Y of the system $A^TY = 0$.
- Show that $A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$ is diagonalisable and hence determine A^5.
- Let A and B be matrices of order n. Prove that if (I–AB) is invertible, then (I–BA) is also invertible and $(I-BA)^{-1}=I+B(I-AB)^{-1}A$. Show that AB and BA have precisely the same characteristic values.
- If a and b are complex numbers such that $|b|=1$ and H is a Hermitian matrix, show that the eigen values of aI+bH lie on a straight line in the complex plane.
- Let A and B be square matrices of order n. Show that AB–BA can never be equal to unit matrix.

1994

- Show that $f_1(t) = 1$, $f_2(t) = t-2$, $f_3(t) = (t-2)^3$ form a basis of P_3, the space of polynomials with degree ≤2. Express $3t^2-5t+4$ as a linear combination of f_1, f_2, f_3.
- If T: $V_4(R) \to V_3(R)$ is a linear transformation, defined by T(a, b, c, d) = (a–b+c+d, a+2c–d, a + b + 3 c –3d) for a, b, c, d $\in$ IR , then verify that $\text{Rank}T + \text{Nullity}T = \dim V_4(R)$.
- If T is an operator on IR^3 whose basis is B={(1, 0, 0), (0, 1, 0), (0, 0, 1)} such that
  $$[T:B] = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$$
  find a matrix of T w.r.t a basis.
- Prove that the eigen vectors corresponding to distinct eigen values of a square matrix are linearly independent.
- If A = $[a_{ij}]$ is an n×n matrix such that $a_{ii} = n$, $a_{ij} = r$ if $i \neq j$, show that
  $$[A-(n\times r)I][A-(n-r+nr)I]=0$$
  Hence find the inverse of the n×n matrix B = $[b_{ij}]$ where $b_{ii} = 1$, $b_{ij} = \rho$ where $i \neq j$ and , $\rho \neq 1$, $\rho \neq \frac{1}{1-n}$.
- Determine the eigen values and eigen vectors of the matrix $\begin{bmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{bmatrix}$.
- Show that a matrix congruent to a skew – symmetric matrix is skew – symmetric. Use the result to prove that the determinant of a skew-symmetric matrix of even order is the square of a rational function of its elements.
- Find the rank of the matrix $\begin{bmatrix} 0 & c & -b & a' \\ -c & 0 & a & b' \\ b & -a & 0 & c' \\ -a' & -b' & -c' & 0 \end{bmatrix}$ where $aa'+bb'+cc'=0$ and a, b, c are all positive integers.
- Reduce the following symmetric matrix to a diagonal form and interpret the result in terms of quadratic forms:
  $$A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 2 & 3 \\ -1 & 3 & 1 \end{bmatrix}$$

1993

- Show that the set S = {(1,0,0), (1,1,0), (1,1,1), (0,1,0)} spans the vector space $R^3(R)$ but it is not a basis set.
- Define rank and nullity of a linear transformation T. If V be a finite dimensional vector space and T a linear operator on V such that rank T^2 = rank T,

then prove that the null space of T = the null space of T^2 and the intersection of the range space and null space of T is the zero subspace of V.

- If the matrix of a linear operator T on IR^2 relative to the standard basis $\{(1, 0), (0, 1)\}$ is $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, what is the matrix of T relative to the basis $B = \{(1, 1), (1, -1)\}$? **(1999)**
- Prove that the inverse of $\begin{bmatrix} A & 0 \\ B & C \end{bmatrix}$ is $\begin{bmatrix} A^{-1} & 0 \\ -C^{-1}BA^{-1} & C^{-1} \end{bmatrix}$ where A, C are non-singular matrices and hence find the inverse of $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.
- If A be an orthogonal matrix with the property that –1 is not an eigen value, then show that A is expressible as $(I-S)(I+S)^{-1}$ for some suitable skew – symmetric matrix S.
- Show that any two eigen vectors corresponding to two distinct eigen values of
  (i) Hermitian matrix
  (ii) unitary matrix are orthogonal
- A matrix B of order n×n is of the form λA, where λ is a scalar and A has unit elements everywhere except in the diagonal which has elements μ. Find λ and μ so that B may be orthogonal. **(1997)**
- Find the rank of the matrix $A = \begin{bmatrix} 1 & -1 & 3 & 6 \\ 1 & 3 & -3 & -4 \\ 5 & 3 & 3 & 11 \end{bmatrix}$ by reducing it to canonical form.

1992

- Let V and U be vector spaces over the field K and let V be of finite dimension. Let $T:V \to U$ be a linear map. Prove that dim V = dim R (T) + dim N (T). **(1984&1985)**
- Let $S = \{(x,y,z) / x+y+z=0\}$, x,y,z being real, prove that S is a subspace of IR^3. Find a basis of 'S'.
- Verify which of the following are linear transformations.
  (i) $T:IR \to IR^2$ defined by $T(x) = (2x, -x)$
  (ii) $T:IR^2 \to IR^3$ defined by $T(x, y) = (xy, y, x)$
  (iv) $T:IR^2 \to IR^3$ defined by $T(x,y) = (x+y, y, x)$
  (iii) $T:IR \to IR^2$ defined by $T(x) = (1, -1)$
- Let $T:M_{2\times1} \to M_{2\times3}$ be a linear transformation defined by $T\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 2 & 1 & 3 \\ 4 & 1 & 5 \end{pmatrix}$, $T\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. Find $T\begin{pmatrix} x \\ y \end{pmatrix}$
- For what values of λ do the following equations
  $x + y + z = 1$
  $x + 2y + 4z = \lambda$
  $x + 4y + 10z = \lambda^2$
  have a solution?Solve them completely in each case.
- State cayley-Hamilton theorem and use it to calculate the inverse of the matrix $\begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$
- Transform the following to the diagonal forms and give the transformation employed:
  $x^2 + 2y,\ 8x^2 - 4xy + 5y^2$
- Prove that the characteristic roots of a Hermitian matrix are all real and a characteristic root of a skew-Hermitian matrix is either zero or a pure imaginary number.

1991

- Let V(IR) be the real vector space of all 2×3 matrices with real entries. Find a basis for V(IR). what is the dimension of V(IR)?
- Let C be the field of complex numbers and let T be the function from $\mathbb{C}^3$ into $\mathbb{C}^3$ defined by
  $T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3, 2x_1 + x_2, -x_1 - 2x_2 + 2x_3)$
  (i) verify that T is a linear transformation.
  (ii) If (a, b, c) is a vector in $\mathbb{C}^3$, what are the conditions on a, b and c so that the vector be in the range of T? what is the rank of T?
  (iii) What are the conditions on a, b and c so that (a, b, c) is in the null space of T? what is the nullity of T?
- If $A = \begin{pmatrix} 1 & 2 \\ -1 & 3 \end{pmatrix}$, express $A^6 - 4A^5 + 8A^4 - 12A^3 + 14A^2$ as a linear polynomial in A.
- Let T be the linear transformation from IR^2 to itself defined by $T(x_1, x_2) = (-x_2, x_1)$.
  (i) What is the matrix of T in the standard ordered basis for R^2.
  (ii) What is the matrix of T in the ordered basis $\beta = \{\alpha_1, \alpha_2\}$ where $\alpha_1 = (1, 2)$ and $\alpha_2 = (1, \square 1)$?
- Determine a non-singular matrix P such that $P^t AP$ is a diagonal matrix, where $A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{pmatrix}$

Is the matrix A congruent to a diagonal matrix?(Justify)

❖ Reduce the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 4 & -5 \\ -2 & -5 & -10 & 16 \\ 5 & 9 & 33 & -68 \\ 4 & 7 & 30 & -78 \end{bmatrix}$$

to echelon form by elementary row operations.

❖ U is an n-rowed unitary matrix such that $|I-U| \neq 0$. show that the matrix H defined by setting iH=(I+U) $(I-U)^{-1}$ is hermitian. Also show that if $e^{i\alpha_1}, e^{i\alpha_2}, \ldots\ldots\ldots e^{i\alpha_n}$ be the eigen values of U, the eigen values of H are $\cot\left(\alpha_1/2\right)$, $\cot\left(\alpha_2/2\right)$, ..., $\cot\left(\alpha_n/2\right)$.

❖ Let A be an n-rowed square matrix with distinct eigenvalues. Show that if A is non-singular, then there exists 2^n matrices X such that $X^2 = A$. What happens in case A is a singular matrix?

1989

❖ Find a basis for the null space of the matrix.

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

❖ If w is a subspace of a finite dimensional space V, then prove that

dim v/w = dimv -dimw

❖ Show that all vectors (x_1, x_2, x_3, x_4) in the vector space $V_4(R)$ which obey $x_4-x_3=x_2-x_1$ form a subspace V. Show that further that V is spanned by (1,0,0,-1),(0,1,0,1),(0,0,1,1).

❖ Let P be a real skew symmetric matrix and I, the corresponding unit matrix. Show that the matrix I-P is non-singular. Also show that the matrix Q = (I+P) $(I-P)^{-1}$ is orthogonal.

❖ Show that an n×n matrix A is similar to a diagonal matrix if and only if the set of characteristic vectors of A includes a set of n linearly independent vectors.

❖ Let r_1 and r_2 be distinct characteristic roots of a matrix A, and let ξ_i be a characteristic vector of A corresponding to r_i (i = 1, 2). If A is Hermitian, then show that $\xi_i \xi_2 = 0$.

❖ Show that

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} 1 & -\tan\theta/2 \\ \tan\theta/2 & 1 \end{bmatrix} \begin{bmatrix} 1 & \tan\theta/2 \\ -\tan\theta/2 & 1 \end{bmatrix}^{-1}$$

❖ Verify cayley – Hamilton theorem for the matrix

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix}$$

1988

❖ Show that a linear transformation of a vector space V_m of dimension m into a vector space V_n of dimension n over the same field can be represented by a matrix.

T is linear transformation of v_2 into v_4 such that

T(3, 1) = (4, 1, 2, 1)

T (–1, 2) = (3, 0, –2, 1) find the matrix of T.

❖ Determine a basis of the subspace spanned by the vectors

$v_1 = (1, 2, 3)$; $v_2 = (2, 1, -1)$

$v_3 = (1, -1, -4)$; $v_4 = (4, 2, -2)$

❖ Show that it is impossible for any 2×2 real symmetric matrix of the form $\begin{bmatrix} a_1 & b \\ b & a_2 \end{bmatrix}$ ($b \neq 0$) to have identical eigen values.

❖ Prove that the eigen values of a Hermitian matrix are all real and that an eigen value of a skew Hermitian matrix is either zero or a pure imaginary number.

❖ By converting A to an echelon matrix, determine the rank of

$$A = \begin{bmatrix} 0 & 0 & 1 & 2 & 8 & 9 \\ 0 & 0 & 4 & 6 & 5 & 3 \\ 0 & 2 & 3 & 1 & 4 & 7 \\ 0 & 3 & 0 & 9 & 3 & 7 \\ 0 & 0 & 5 & 7 & 3 & 1 \end{bmatrix}$$

❖ Given AB = AC does it follow that B = C? Can you provide a counter example?

❖ Find a non- singular transformation of three variables which simultaneously diagonalizes.

$$A = \begin{bmatrix} 0 & -1 & 0 \\ -1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 & -2 \\ 1 & 2 & -2 \\ -2 & -2 & 3 \end{bmatrix}.$$

Give the diagonal form of A.

1987

❖ (i) Find all matrices which commute with the matrix $\begin{bmatrix} 7 & -3 \\ 5 & -2 \end{bmatrix}$

(ii) Prove that the product of two n×n symmetric matrices is a symmetric matrix, if and only if the matrices commute.

- Show that the rank of the product of two square matrices A, B each of order n, satisfies the inequality

$r_A + r_B - n \le r_{AB} \le \min\left(r_A, r_B\right)$

where r_0 stands for the rank of a square matrix C.

- If 1″ a″ 5, find the rank of the matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & a \\ 2 & 2a-2 & -a-2 & 3a-1 \\ 3 & a+2 & -3 & 2a+1 \end{pmatrix}$$

- If the eigen values of a matrix A are λj, j = 1, 2.....n and if f(x) is a polynomial in x, show that the eigen values of the matrix, f(A) are f(λj), j = 1, 2 ....n
- If A is a skew symmetric matrix and I, the corresponding unit matrix, show that $(I-A)(I+A)^{-1}$, is orthogonal. Hence construct an orthogonal matrix, if $A = \begin{pmatrix} 0 & a/b \\ -a/b & 0 \end{pmatrix}$
- (i) If A, B are two arbitrary square matrices of which A is non-singular, show that AB and BA have the same characteristic polynomial.

(ii) Show that a real matrix A is orthogonal, if and only if $|Ax| = |x|$ for all real vectors x.

- Show that a necessary and sufficient condition for a system of Linear equations to be consistent is that the rank of the coefficient matrix is equal to the rank of the augmented matrix.

Hence show that the system

$x + 2y + 5z + 9 = 0$, $x - y + 3z - 2 = 0$,

$3x - 6y - z - 25 = 0$

is consistent and has a unique solution. Determine this solution.

- In an n – dimensional vector space the system of vectors a_j, j = 1, 2 ...., r are linearly independent and can be linearly expressed to terms of the vectors β_k, k = 1, 2, ..., s. show that r ″ s.

Find a maximal linearly independent subsystem of the system of linear forms

$f_1 = x + 2y + z + 3t$, $f_2 = 4x - y - 5z - 6t$

$f_3 = x - 3y - 4z - 7t$, $f_4 = 2x + y - z$

- Let $T: V \to W$ be a linear transformation. If V is finite dimensional then, show that rank (T) + nullity (T) = dim V.
- Prove that two finite dimensional vector spaces V and W over the same field F are isomorphic if they are of the same dimension n.
- (a) Prove that every square matrix is a root of its characteristic polynomial.

(b) Determine the eigen values and the corresponding eigen vectors of

$$A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$$

- Let A and B be n - square matrices over F. show that AB and BA have the same eigen values.

1986

- If A, B, C are three n×n matrices, show that A(BC) = (AB)C. Show by an example that the matrix multiplication is non-commutative.
- Examine the correctness or otherwise of the statements"

(i) The division law is not valid in matrix algebra.

(ii) If A, B are square matrices each of order 'n' and I is the corresonding unit matrix ,the equation AB-BA=I can never hold.

- Find a 3×3 matrix X such that

$$\begin{bmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{bmatrix} X = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$ may hold.

- Find a maximal linearly independent subsystem in the system of vectors:

$V_1 = (2, -2, -4)$; $V_2 = (1, 9, 3)$; $V_3 = (-2, -4, 1)$ and $V_4 = (3, 7, -1)$

- Show that the system of equations

$4x+y-2z+w = 3$

$x-2y-z+2w = 2$

$2x+5y-w = -1$

$3x + 3y - z - 3w = 1$

although consistent, is not uniquely solvable. Determine the general solution by choosing x as a parameter.

- Show that

(i) a square matrix is singular, iff, atleast one of its characteristic roots (i.e., eigen values) is zero.

(ii) the rank of an n×n matrix A remains unchanged, if A is pre-multiplied or post - multiplied by a non-singular n×n matrix X and that rank $\left(X^{-1}AX\right)$ = rank A. Prove that V is a vector space over K.

- Find all the eigenvalues and a basis for each eigen space for the matrix $\begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$.

- Prove that every square matrix satisfies its own characteristic equation.
- Prove that the rank of the product of two matrices cannot exceed the rank of either of them.
- If M, N are two subspaces of a vector space S, show that their dimensions satisfy.
 $\dim(M)+\dim(N)=\dim\ (M\cap N)+\dim\ (M+N)$.

- Show that every square matrix satisfies its characteristic equation.
 Using this result or otherwise, show that, if
 $A=\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ then $A^4-2A^3-2A^2+6A-2I=A$,

 where I is the 3×3 unit matrix.
- If V_1 and V_2 be subspaces of a vector space V then show that
 $\dim(V_1+V_2) = \dim V_1+\dim V_2-\dim(V_1\cap V_2)$

- Let V and W be vector spaces over the same field F and dimV=n, let $\{e_1,e_2,\ldots,e_n\}$ be a basis of V.Show that a map $f:\{e_1,e_2,\ldots,e_n\}\to W$ can be uniquely extended to a linear transformation $T:V\to W$

 whose restriction to the given basis $\{e_j\}$ is f, that is $T(e_j) = f(e_j)$; j = 1, 2, ..., n.
- (i) If A and B are two linear transformations on the space, and if A^{-1} and B^{-1} exist, then show that $(AB)^{-1}$ exists and $(AB)^{-1}=B^{-1}A^{-1}$.

 (ii) Prove that similar matrices have the same characteristic polynomial and hence the same eigen values.
 (iii) Prove that the eigen values of a Hermitian matrix are real.
- Reduce $2x^2+4xy+5y^2+4x+13y-½=0$ to

 cannonical form
 (i) Find reciprocal of the matrix
 $T=\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$,

 show that the transform of the matrix
 $A=\frac{1}{2}\begin{bmatrix} b+c & c-a & b-a \\ c-b & c+a & a-b \\ b-c & a-c & a+b \end{bmatrix}$ by T, i.e. TAT^1

 is the diagonal matrix. Determine the eigen values of the matrix A.

1985

- Let $M_1=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $M_2=\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $M_3=\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$,

 $M_4=\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ prove that the set $\{M_1, M_2, M_3, M_4\}$

 form the basis of the vector space of 2×2 matrices.
- Find the inverse of the matrix $\begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix}$

- Consider the basis $S = \{V_1, V_2, V_3\}$ of $\mathbb{R}^3$ where $V_1 = (1,1,1)$, $V_2 = (1, 1, 0)$, $V_3 = (1,0,0)$. Express (2,–3, 5) interns of the basis V_1, V_2, V_3.
 Let $T:\mathbb{R}^3\to\mathbb{R}^2$ be defined as $T(V_1) = (1, 0)$,

 $T(V_2)=(2,-1)$, $T(V_3)=(4,3)$. Find T (2, –3, 5)
- Reduce the matrix to echelon form $\begin{bmatrix} 6 & 3 & -4 \\ -4 & 1 & -6 \\ 1 & 2 & -5 \end{bmatrix}$

- Show that if λ is an eigen value of matrix A then λ^n is an eigen value of A^n if n is a positive integer.
- Determine the vectors (1, –2, 1), (2, 1, –1), (7, –4, 1) in $\mathbb{R}^3$ are linearly independent.

- Solve
 $$2x_1+3x_2+x_3=9$$
 $$x_1+2x_2+3x_3=6$$
 $$3x_1+x_2-2x_3=8$$
- Let V be the vector space of all functions from IR in to IR; let v_e be the subset of even functions f, f(-x)= f(x), and v_0 be the subset of odd functions f, f(–x) = -f(x). Prove that
 (i) v_e and v_0 are subspaces of v.
 (ii) $v_e+v_0=v$
 (iii) $v_e\cap v_0=\{0\}$

- Find the dimension and a basis of the solution space S of the system
 $x_1+2x_2+2x_3-x_4+3x_5=0$

$x_1+2x_2+3x_3+x_4+x_5 = 0$
$3x_1+6x_2+8x_3+x_4+5x_5 = 0$

- Let w_1 and w_2 be subspaces of a finite dimensional vector space V. Prove that $(w_1+w_2)^0 = w_1^0 \cap w_2^0$
- Prove that every matrix is a zero of its characteristic polynomial.
- If A is an orthogonal matrix and if B=AP, where P is non-singular, then prove that PB^{-1} is orthogonal.

1984

- If w_1 and w_2 are finite - dimensional subspaces of a vector space v then show that w_1+w_2 is finite dimensional and $\dim w_1 + \dim w_2 = \dim(w_1 \cap w_2) + \dim(w_1+w_2)$.

(1985, 1986)

- If A and B are n-rowed square matrices, not zero matrices, such that AB=0, then show that both A and B are non - singular. If both A and B are singular and AB=0, does it follow that BA = 0? Justify your answer.
- Show that row equivalent matrices have the same rank.
- A linear transformation T of a vector space V with finite basis $\{\alpha_1, \alpha_2, ...\alpha_n\}$ is non-singular iff the vectors $\alpha_1 T, ...\alpha_n T$ are linearly independent in V. When this is the case, show that T has a linear inverse, T^{-1}, with $TT^{-1}=T^{-1}T=I$.
- Solve the following system of equations"
  $3x_1 + 2x_2 + 2x_3 - 5x_4 = 8$
  $2x_1 + 5x_2 + 5x_3 - 18x_4 = 9$
  $4x_1 - x_2 - x_3 + 8x_4 = 7$
- Let A be a square matrix and T be non-singular. Let $\tilde{A}=T^{-1}AT$. Show that
  (i) A and $\tilde{A}$ have the same eigen values
  (ii) Trace A =T trace $\tilde{A}$
  (iii) If X be an eigen vector of A corresponding to an eigen value then $T^{-1}X$ is a eigen vector of $\tilde{A}$ corresponding to the same eigen value.
- A 3×3 matrix has the eigen values $\lambda_1 = 6$, $\lambda_2 = 2$, and $\lambda_3 = -1$ and the corresponding eigen vectors are $\begin{pmatrix} 2 \\ 3 \\ -2 \end{pmatrix}, \begin{pmatrix} 9 \\ 5 \\ 4 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \\ -1 \end{pmatrix}$. Find the matrix.
- Let V be the set of all functions from a non-empty set X into a field K, for any functions $f, g \in v$ and any scalar $k \in K$, let $(f+g)$ and kf be the functions in v defined as follows:
  $(f+g)(x) = f(x) + g(x)$, $(kf)(x) = kf(x)$ for every $x \in X$.
  prove that V is a vertorspace over K.
- Find all the eigenvalues and a basis for each eigen space for the matrix $\begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$
- Prove that every square matrix satisfies its own characteristic equation. ***(1986)***
- Prove that the rank of the product of two matrices cannot exceed the rank of either of them.

1983

- Let V be a finitely generated vector space. Show that V has a finite basis and any two bases of V have the same number of vectors.
- Let V be the vector space of polynomials of degree ≤3. Determine whether the following vectors of V are linearly dependent or independent:
  $u = t^3 - 3t^2 + 5t + 1$, $v = t^3 - t^2 + 8t + 2$,
  $w = 2t^3 - 4t^2 + 9t + 5$.
- For any linear transformation $T: V_1 \to V_2$, prove that rank (T) $\leq \min(\dim V_1, \dim V_2)$.
- Show that every non-singular matrix can be expressed as a product of elementary matrices.
- Reduce the matrix $\begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 3 & 1 & 1 & 3 \end{bmatrix}$ to its normal form and hence or otherwise determine its rank
- Show that the equations
  $x + y + z = 3$
  $3x - 5y + 2z = 8$
  $5x - 3y + 4z = 14$ are consistent and solve them.

- Prove that every square matrix satisfies its characteristic equation. Use this result to find the inverse of $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 1 & 1 \end{bmatrix}$

- Find the eigen values and eigen vectors of the matrix
$\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$

- Show that the characteristic roots of an upper or lower triangular matrix are just the diagonal elements of the matrix.

❖❖❖

IFoS
PREVIOUS YEARS QUESTIONS (2019-2000)
SEGMENT-WISE

LINEAR ALGEBRA
(ACCORDING TO THE NEW SYLLABUS PATTERN) PAPER - I

2019

- Let $T : R^3 \to R^3$ be a linear operator on R^3 defined by $T(x, y, z) = (2y + z, x - 4y, 3x)$. Find the matrix of T in the basis $\{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$. **(08)**
- The eigenvalues of a real symmetric matrix A are –1, 1 and –2. The corresponding eigenvectors are $\frac{1}{\sqrt{2}}\left(-1\ 1\ 0\right)^T, \left(0\ 0\ 1\right)^T$ and $\frac{1}{\sqrt{2}}\left(-1\ -1\ 0\right)^T$ respectively. Find the matrix A^4. **(08)**
- Consider the singular matrix

$$A = \begin{bmatrix} -1 & 3 & -1 & 1 \\ -3 & 5 & 1 & -1 \\ 10 & -10 & -10 & 14 \\ 4 & -4 & -4 & 8 \end{bmatrix}$$

  Given that one eigenvalue of A is 4 and one eigenvector that does not correspond to this eigenvalue 4 is $(1\ 1\ 0\ 0)^T$. Find all the eigenvalues of A other than 4 and hence also find the real numbers p, q, r that satisfy the matrix equation $A^4 + pA^3 + qA^2 + rA = 0$ **(15)**
- Consider the vectors $x_1 = (1, 2, 1, -1)$, $x_2 = (2, 4, 1, 1)$, $x_3 = (-1, -2, 0, -2)$ and $x_4 = (3, 6, 2, 0)$ in $\mathbb{R}^4$. Justify that the linear span of the set $\{x_1, x_2, x_3, x_4\}$ is a subspace of $\mathbb{R}^4$, defined as $\{(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4 : 2\xi_1 - \xi_2 = 0, 2\xi_1 - 3\xi_3 - \xi_4 = 0\}$ Can this subspace be written as $\{(\alpha, 2\alpha, \beta, 2\alpha - 3\beta) : \alpha, \beta \in \mathbb{R}\}$? What is the dimension of this subspace ? **(15)**
- Using elementary row operations, reduce the matrix

$$A = \begin{bmatrix} 2 & 1 & 3 & 0 \\ 3 & 0 & 2 & 5 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 3 \end{bmatrix}$$

  to reduced echelon form and find the inverse of A and hence solve the system of linear equations $AX = b$, where $X = (x, y, z, u)^T$ and $b = (2, 1, 0, 4)^T$. **(15)**

2018

- Given that Adj $A = \begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and det A=2. Find the matrix A. **(08)**
- Prove that the eigen, values of a Hermitian matrix are all real. **(08)**
- Show that the matrices

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 2 \\ 3 & 2 & 0 \end{bmatrix}$$

  are congruent.
- Show that the vectors $\alpha_1 = (1, 0, -1)$, $\alpha_2 = (1, 2, 1)$, $\alpha_3 = (0, -3, 2)$ form a basis for $\mathbf{R}^3$. Express each of the standard basis vectors as a linear combination of $\alpha_1, \alpha_2, \alpha_3$. **(10)**
- Let $T : V_2(\mathbf{R}) \to V_2(\mathbf{R})$ be a linear transformation defined by $T(a, b) = (a, a + b)$. Find the matrix of T, taking $\{e_1, e_2\}$ as a basis for the domain and $\{(1, 1)\ (1, -1)\}$ as a basis for the range. **(10)**
- If $(n+1)$ vectors $\alpha_1, \alpha_2, ...\alpha_n, \alpha$ form a linearly dependent set, then show that the vector α is a linear combination of $\alpha_1, \alpha_2, ...\alpha_n$; provided $\alpha_1, \alpha_2, ...\alpha_n$ form a linearly independent set. **(10)**

2017

- Let A be a square matrix of order 3 such that each of its diagonal elements is 'a' and each of its off-diagonal elements is 1. If B = bA is orthogonal determine the values of a and b. **[8]**
- Let V be the vector space of all 2×2 matrices over the field R. Show that W is not a subspace of V, where
  (i) W contains all 2×2 matrices with zero determinant.
  (ii) W consists of all 2×2 matrices A such that $A^2 = A$. **[8]**
- State the Cayley-Hamilton theorem. Verify this theorem for the matrix

$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Hence find A^{-1}. **[10]**

❖ Reduce the following matrix to a row-reduced echelon form and hence find its rank.

$$A = \begin{bmatrix} -1 & 2 & -1 & 0 \\ 2 & 4 & 4 & 2 \\ 0 & 0 & 1 & 5 \\ 1 & 6 & 3 & 2 \end{bmatrix}$$ **[10]**

❖ Given that the set {u, v, w} is linearly independent, examine the sets
(i) {u + v, v + w, w + u}
(ii) {u + v, u – v, u – 2v + 2w} for lienar independence. **[10]**

❖ Find the eigenvalues and the corresponding eigenvectors for the matrix $A = \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix}$.

Examine whether the matrix A is diagonalizable. Obtain a matrix D (if it is diagonalizable) such that $D = P^{-1} A P$. **[10]**

2016

❖ Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be given by

$T(x, y, z) = (2x - y, 2x + z, x + 2z, x + y + z)$.
Find the matrix of T with respect to standard basis of $\mathbb{R}^3$ and $\mathbb{R}^4$ (i. e., (1, 0, 0), (0, 1, 0), etc.).

Examine if T is a linear map. **(8)**

❖ For the matrix $A = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$, obtain the

eigen value and get the value of $A^4+3A^3-9A^2$. **(8)**

❖ Let T be a linear map such that $T : V_3 \to V_2$

defined by $T(e_1) = 2f_1 - f_2$, $T(e_2) = f_1 + 2f_2$, $T(e_3) = 0f_1 + 0f_2$, where e_1, e_2, e_3 and f_1, f_2 are standard basis in V_3 and V_2. Find the matrix of T relative to these basis. Further take two other basis B_1 [(1, 1, 0) (1, 0, 1) (0, 1, 1)] and B_2 [(1, 1) (1, – 1)]. Obtain the matrix T_1 relative to B_1 and B_2. **(10)**

❖ For the matrix $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, find two non-singular matrices P and Q such that PAQ = I. Hence find A^{-1}. **(10)**

❖ Examine whether the real quadratic form $4x^2 - y^2 + 2z^2 + 2xy - 2yz - 4xz$ is a positive definite or not. Reduce it to its diagonal form and determine its signature. **(10)**

2015

❖ Find an upper triangular matrix A such that
$A^3 = \begin{bmatrix} 8 & -57 \\ 0 & 27 \end{bmatrix}$ **(8)**

❖ Let G be the linear operator on $\mathbb{R}^3$ defined by $G(x, y, z) = (2y + z, x - 4y, 3x)$. Find the matrix representation of G relative to the basis
S= {(1, 1, 1), (1, 1, 0), (1, 0 ,0)}. **(8)**

❖ Suppose U and W are distinct four-dimensional subspaces of a vector space V, where dim V = 6. Find the possible dimensions of $U \cap W$. **(10)**

❖ Find the condition on *a, b,* and *c* so that the following system in unknowns *x, y* and *z* has a solution.
$x + 2y - 3z = a$
$2x + 6y - 11z = b$
$x - 2y + 7z = c$ **(10)**

❖ Find the minimal polynomial of the matrix
$A = \begin{bmatrix} 4 & -2 & 2 \\ 6 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$ **(10)**

❖ Find a 3 × 3 orthogonal matrix whose first two rows are $\left[\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right]$ and $\left[0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right]$. **(10)**

2014

❖ Show that u_1=(1,–1,0), u_2=(1,1,0) and u_3=(0,1,1) form a basis for $\mathbb{R}^3$. Express (5, 3, 4) in terms of u_1, u_2 and u_3. **(8)**

❖ For the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Prove that $A^n = A^{n-2} + A^2 - I$, $n \geq 3$. **(8)**

❖ Let $B = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$. Find all eigen values and corresponding eigen vectors of B viewed as a matrix over: **(10)**

(i) The real field R.
(ii) The complex field C.

- Show that the mapping $T:V_2(\bar{R})\to V_3(\bar{R})$ defined as $T(a,b)=(a+b,a-b,b)$ is a linear transformation. Find the range, rank and nullity of T. **(10)**

- Examine whether the matrix $A=\begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{bmatrix}$ is diagonalizable. Find all eigen values. Then obtain a matrix P such that $P^{-1}AP$ is a diagonal matrix. **(10)**

- Consider the linear mapping $F:\mathbb{R}^2\to\mathbb{R}^2$ given as $F(x,y)=(3x+4y,\ 2x-5y)$ with usual basis. Find the matrix associated with the linear transformation relative to the basis $S=\{u_1,u_2\}$ where $u_1=(1,2)$, $u_2=(2,3)$. **(10)**

2013

- Find the dimension and a basis of the solution space W of the system
$$x+2y+2z-s+3t=0,$$
$$x+2y+3z+s+t=0,$$
$$3x+6y+8z+s+5t=0$$ **(8)**

- Find the characteristic equation of the matrix $A=\begin{bmatrix}2 & 1 & 1\\ 0 & 1 & 0\\ 1 & 1 & 2\end{bmatrix}$ and hence find the matrix represented by
$$A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I.$$ **(8)**

- Let V be the vector space of 2×2 matrices over $\mathbb{R}$ and let $M=\begin{bmatrix}1 & -1\\ -2 & 2\end{bmatrix}$. Let $F:V\to V$ be the linear map defined by $F(A)=MA$. Find a basis and the dimension of
(i) The kernel of W of F
(ii) The image U of F. **(10)**

- Let F be a subfield of complex numbers and T a function from $F^3\to F^3$ defined by
$$T(x_1,x_2,x_3)=(x_1+x_2+3x_3,2x_1-x_2,-3x_1+x_2-x_3).$$
what are the conditions on (a,b,c) such that (a,b,c) be in the null space of T? Find the nullity of T. **(10)**

- Let $H=\begin{bmatrix}1 & i & 2+i\\ -i & 2 & 1-i\\ 2-i & 1+i & 2\end{bmatrix}$ be a Hermitian matrix. Find a non-singular matrix P such that $P'H\bar{P}$ is diagonal and also find its signature. **(10)**

- Find an orthogonal transformation of co-ordinates which diagonalizes the quadratic form $q(x,y)=2x^2-4xy+5y^2$. **(10)**

- Discuss the consistency and the solutions of the equations
$x+ay+az=1,\ ax+y+2az=-4,\ ax-ay+4z=2.$
for different values of a. **(10)**

2012

- Let $V=\mathbb{R}^3$ and $\alpha_1=(1,1,2)$, $\alpha_2=(0,1,3)$, $\alpha_3=(2,4,5)$ and $\alpha_4=(-1,0,-1)$ be the elements of V. Find a basis for the intersection of the subspace spanned by $\{\alpha_1,\alpha_2\}$ and $\{\alpha_3,\alpha_4\}$. **(8)**

- Show that the set of all functions which satisfy the differential equation $\dfrac{d^2f}{dx^2}+3\dfrac{df}{dx}=0$ is a vector space. **(8)**

- Let $f:\mathbb{R}\to\mathbb{R}^3$ be a linear transformation defined by $f(a,b,c)=(a,a+b,0)$.
Find the matrices A and B respectively of the linear transformation f with respect to the standard basis $(e_1,\ e_2,\ e_3)$ and the basis (e_1',e_2',e_3') where
$e_1'=(1,1,0),\ e_2'=(0,1,1),\ e_3'=(1,1,1).$
Also, show that there exists an invertible matrix P such that $B=P^{-1}AP$ **(10)**

- Verify Cayley–Hamilton theorem for the matrix $A=\begin{pmatrix}1 & 4\\ 2 & 3\end{pmatrix}$ and find its inverse. Also express $A^5-4A^4-7A^3+11A^2-A-10I$ as a linear polynomial in A. **(10)**

- Find the matrix representation of linear transformation T on $V_3(\mathbb{R})$ defined as $T(a,b,c)=(2b+c,a-4b,3a)$ corresponding to the basis $B=\{(1,1,1),(1,1,0),(1,0,0)\}$ (10)

2011

- Let V be the vector space of 2 × 2 matrices over the field of real numbers $\mathbb{R}$. Let $W=\{A\in V \mid Trace\ \ A=0\}$. Show that W is a subspace of V. Find a basis of W and dimension of W.
- Find the linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^3$ which has its range the subspace spanned by $(1,0,-1),(1,2,2)$
- Let $V=\{(x,y,z,u)\in\mathbb{R}^4 : y+z+u=0\}$,

  $W=\{(x,y,z,u)\in\mathbb{R}^4 : x+y=0\ ,\ z=2u\}$

  be two subspaces of $\mathbb{R}^4$. Find bases for V, W $V+W$ and $V\cap W$.
- Find the characteristic polynomial of the matrix $A=\begin{pmatrix}3&1&1\\2&4&2\\-1&-1&1\end{pmatrix}$ and hence compute A^{10}.
- *Let* $A=\begin{pmatrix}1&-3&3\\0&-5&6\\0&-3&4\end{pmatrix}$ Find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.
- Find an orthogonal transformation to reduce the quadratic form $5x^2+2y^2+4xy$ to a canonical form.

2010

- Show that the set $P[t]=\{at^2+bt+c\ /\ a,b,c\in\mathbb{R}\}$ forms a vector space over the field $\mathbb{R}$. Find a basis for this vector space. What is the dimension of this vector space ? **(10)**
- Show that the vectors $\alpha_1=(1,0,-1)$, $\alpha_2=(1,2,1)$, $\alpha_3\,(0,-3,2)$ form a basis for $\mathbb{R}^3$. Find the components of (1, 0, 0) w.r.t the basis $\{\alpha_1,\alpha_2,\alpha_3\}$. **(10)**
- Find the characteristic polynomial of $\begin{pmatrix}0&0&1\\1&0&2\\0&1&3\end{pmatrix}$.

  Verify Cayley-Hamilton theorem for this matrix and hence find its inverse. **(10)**
- Let $A=\begin{pmatrix}5&-6&-6\\-1&4&2\\3&-6&-4\end{pmatrix}$. Find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.
- Find the rank of the matrix $\begin{pmatrix}1&2&1&1&2\\2&4&3&4&7\\-1&-2&2&5&3\\3&6&2&1&3\\4&8&6&8&9\end{pmatrix}$

2009

- Let V be the vector space of polynomials over R. Let U and W be the subspaces generated by $\{t^3+4t^2-t+3\ ,\ t^3+5t^2+5\ ,\ 3t^3+10t^2-5t+5\}$ and $(t^3+4t^2+6\ ,\ t^3+2t^2-t+5\ ,\ 2t^3+2t^2-3t+9)$ recpectively. Find

  (i) $\dim(U+W)$

  (ii) $\dim(U\cap W)$. **(10)**
- Find a linear map $T:R^3\to R^4$ Whose image is generated by $(1,2,0,-4)$ *and* $(2,0,-1,-3)$ **(10)**
- Let T be the linear operator defined by $T(x,y,z)=(2x,4x-y\ ,\ 2x+3y-z)$

  (i) Show that T is invertible.

  (ii) Find a formula for T^{-1}. **(10)**
- Find the rank of the matrix: $A=\begin{pmatrix}1&3&1&-2&-3\\1&4&3&-1&-4\\2&3&-4&-7&-3\\3&8&1&-7&-8\end{pmatrix}$ **(10)**
- Let $A=\begin{pmatrix}1&-3&3\\3&-5&3\\6&-6&4\end{pmatrix}$. Is A similar to a diagonal

matrix ? If so, find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix. **(10)**

2008

- Determine a,b and c so that the matrix $A=\begin{pmatrix}0 & 2b & c\\ a & b & -c\\ a & -b & c\end{pmatrix}$ is orthogonal. **(10)**
- If S and T are subspace of IR^4 given by $S=\{(x,y,z,w)\in IR^4 : 2x+y+3z+w=0\}$ and $T=\{(x,y,z,w)\in IR^4 : x+2y+z+3w=0\}$, find $\dim(S\cap T)$. **(10)**
- Obtain the characteristic equation of the matrix $A=\begin{pmatrix}1 & 2\\ 2 & -1\end{pmatrix}$ and show that A satisfies its characteristic equation. Hence determine the inverse of A. **(10)**
- If S be a real skew-symmetric matrix of order prove that the matrix $P=(I_n+S)^{-1}(I_n-S)$ is orthogonal where I_n stands for the identity matrix of order 'n' **(10)**
- Find the row rank and column rank of the matrix $A=\begin{pmatrix}2 & 1 & 4 & 3\\ 3 & 2 & 6 & 9\\ 1 & 1 & 2 & 6\end{pmatrix}$. Hence determine the rank of A. **(10)**
- Reduce the equation $2y^2-2xy-2yz+2zx-x-2y+3z-2=0$ into canonical form and determine the nature of the quadric. ***(10)***

2007

- Suppose U and W are subspaces of the vector space $IR^4(IR)$ generated by the sets $B_1=\{(1,1,0,-1),(1,2,3,0),(2,3,3,-1)\}$ $B_2=\{(1,2,2,-2),(2,3,2,-3),(1,3,4,-3)\}$ recpectively. Determine $\dim(U+W)$. **(10)**
- Find the characteristic equation of the matrix $A=\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$ and verify that it satisfies by A. **(10)**
- Show that the solutions of the differential equation $2\frac{d^2y}{dx^2}-9\frac{dy}{dx}+2y=0$ is a subspace of the vector space of all real valued continuous functions.
- Show that vectors $(0,2,-4),(1,-2,-1),(1,-4,3)$ are linearly dependent. Also express $(0,2,-4)$ as a linear combination of $(1,-2,-1)$ and $(1,-4,3)$. **(10)**
- Is the matrix $A=\begin{bmatrix}6 & -3 & 2\\ 4 & -1 & -2\\ 10 & -5 & -3\end{bmatrix}$ Similar over the field $\mathbb{R}$ to a diagonal matrix ? Is A similar l over the field C to a diagonal matrix ? **(10)**
- Determine the definiteness of the following quadratic form $q(x_1,x_2,x_3)=[x_1\ x_2\ x_3]\begin{bmatrix}2 & 0 & -1\\ 1 & 5 & 2\\ -2 & 1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$

2006

- Let W be the subspace of IR^4 generated by the vectors: $(1,-2,5-3),(2,3,1,-4),\ and\ (3,8,-3,-5)$. Find a basis and dimension of W. **(10)**
- Find the symmetric matrix which corresponds to the following quadratic polynomial: $q(x,y,z)=x^2-2yz+xz$. **(10)**
- Let $A=\begin{pmatrix}1 & 4\\ 2 & 3\end{pmatrix}$. Find all eigen values of A and the corresponding eigen vectors. **(10)**
- $Let\ H=\begin{bmatrix}1 & 1+i & 2i\\ 1-i & 4 & 2-3i\\ -2i & 2+3i & 7\end{bmatrix}$ be a Hemitian matrix. Find a non-singular matrix P such that $P^t A\overline{P}$ is diagonal **(10)**

❖ Let $L: IR^3 \to IR^2$ be linear transformation for which we know that $L(1,0,0)=(2,-1), L(0,1,0)=(3,1)$ and $L(0,0,1)=(-1,2)$. Find $L(-3,4,2)$. **(10)**

❖ Let $L: IR^3 \to IR^2$ be defined by

$$L\left[\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right] = \begin{pmatrix} x+y \\ x-z \end{pmatrix}$$

Let $S=\{v_1, v_2, v_3\}$ and $T=\{w_1, w_2\}$ be bases for IR^3 and IR^2 respectively

where

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, w_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \ \& \ w_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Find the matrix of L w.r.t S and T.

2005

❖ Let $V = P_3(IR)$ be the vector space of polynomial functions over reals of degree atmost 3. Let $D: V \to V$ be the differentiation operator defined by $D(a_0 + a_1x + a_2x^2 + a_3x^3) = a_1 + 2a_2x + 3a_3x^2, x \in IR$.

(i) Show that $B = \{1, x, x^2, x^3\}$ is a basis for V.

(ii) Find the matrix $[D]_B$ with respect to B of D.

(iii) Show that $B' = \{1, (x+1), (x+1)^2, (x+1)^3\}$ is a basis for V.

(iv) Find the matrix $[D]_{B'}$ with respect B' of D.

(v) Find the matrix $[D]_{B'B}$ of D relative to B' and B.

❖ Find the eigen values and the corresponding eigen vectors of $A = \begin{pmatrix} 2 & 1 \\ 2 & 3 \end{pmatrix}$. **(10)**

❖ Show that the vectors $v_1 = (1,1,1)$, $v_2 = (0,1,1)$, $v_3 = (0,0,1)$ form a basis for IR^3. Express $v = (3,1,-4)$ as a linear combination of $v_1, v_2 and\ v_3$. Is the set $S = \{v, v_1, v_2, v_3\}$ linearly independent ? **(10)**

❖ Determine a non-singular matirx P such that $P^t AP$ is a diagonal matrix, where P^t denotes the transpose of P, and $A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{bmatrix}$ **(10)**

❖ If A be an orthogonal matrix with the properly that -1 is not a characteristic root of A, then show that A is expressible as $(I-S)(I+S)^{-1}$ for some suitable skew-symmetric matrix S. **(10)**

❖ Show that the real quadratic form $\phi = n(x_1^2 + x_2^2 + ... + x_n^2) - (x_1 + x_2 + ... + x_n)^2$. in n variables is positive semi-definite. **(10)**

2004

❖ Let U and W be subspaces of IR^3 for which $\dim U = 1$, $\dim W = 2$. and $U \not\subset W$. Show that $IR^3 = U + W$ and $U \cap W = \{0\}$.

❖ Let $\{e_1, e_2, e_3\}$ be the standard basis of IR^3 and T be a linear transformation from IR^3 into IR^2 defined by

$T(e_1) = (2,3)^T, T(e_2) = (1,2)^T$ *and* $T(e_3) = (-1,-4)^T$

(i) What is T $(1,-2,-1)$

(ii) What is the matrix of T with respect to the standard bases of IR^3 and IR^2 ?

❖ Find a linear map $T: IR^3 \to IR^4$, Whose range is generated by $(1,2,0,-4)$ *and* $(2,0,-1,-3)$. Also find a basis and the dimension of the

(i) range U of T.

(ii) kernel W of T.

❖ Find the eigen values and their corresponding eigen vectors of the matrix $\begin{pmatrix} 2 & 0 & 3 \\ 2 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$. Is the matrix diagonalisable? **(10)**

❖ For what values of α has the systme of equations $x+2y+z=1$, $\alpha x+4y+2z=2$, $4x-2y+2\alpha z=-1$.

(i) a unique solution
(ii) infinitely many solutions
(iii) no solution. **(10)**

❖ Determine an orthogonal matrix which reduces the quadratic form $Q(x_1, x_2, x_3) = 2x_1^2 + x_2^2 - 4x_2x_3 + x_3^2$ to a canonical form. Also, identify the surface respresented by $Q(x_1, x_2, x_3) = 7$. **(10)**

2003

❖ Let $V = P_3(IR)$ be the vector space of polynomial fucntions on reals of degree at most 3. Let $V = P_3(IR)$ be the differentiation operator defined by $D(a_0 + a_1x + a_2x^2 + a_3x^3) = a_1 + 2a_2x + 3a_3x^2$.

(i) Show that D is a linear transformation.
(ii) Find kernel and image of D.
(iii) What are dimensions of V, ker D and image D ?
(iv) Give relation among them of (iii) **(10)**

❖ Find the eigen values and the corresponding eigen, vectors of $A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$. **(10)**

❖ Show that the vectors $(1,2,1), (1,0,-1)$ *and* $(0,-3,2)$ form a basis for IR^3. **(10)**

❖ Determine non-singular matrices P and Q such that the matrix PAQ is in canonical form, Where $A = \begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$. Hence find the rank of A. **(10)**

❖ Find the minimum polynomial of the matrix $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$, and use it to determine whether A is similar to a diagonal matrix. **(10)**

❖ Show that the quadratic form $2x^2 - 4xy + 3zx + 6y^2 + 6yz + 8z^2$ in three variables is positive definite. **(10)**

2002

❖ Let $S_1, S_2......S_k$ be spaces of a vector space V(F). Show that the following statements are equivalent.

(i) $S_1+S_2+.......+S_k$ is a direct sum of V(F).
(ii) $(S_1+S_2+.....+S_k) \cap S_{i+1} = \{0\}$, for i = 1, 2,.........k–1.
(iii) $x_1 + x_2 + ...x_k = 0$, $x_i \in S_i$, $i = 1,2,......k$.
$\rightarrow x_i = 0$ for $i = 1,2,3........k$.
(iv) $d(S_1 + S_2 + ... + S_k) =$
$d(S_1) + d(S_2) + ... + d(S_k)$.

❖ Show that if λ is a non-zero eigen value of the non singular n-square matrix A, then $\frac{|A|}{\lambda}$ is an eigen value of adjA.
Also give an example to prove that the eigen values of AB are not necessarily the product of eigen values of A and that of B. **(10)**

❖ Given the linear transformation $Y = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 3 \\ -2 & 3 & 5 \end{bmatrix} X$,
Show that it is singular and the images of the linearly independent vectors,
$X_1 = [1,1,1]^T$, $X_2 = [2,1,2]^T$, $X_3 = [1,2,3]^T$ are linearly dependent. **(10)**

❖ Calculate $f(A) = e^A - e^{-A}$ for $A = \begin{bmatrix} 2 & 4 & 0 \\ 6 & 0 & 8 \\ 0 & 3 & -2 \end{bmatrix}$ **(5)**

❖ The n × n A matrix A satisfies $A^4 = -1.6A^2 - 0.641$ show that $\underset{n \to \infty}{Lt} A^n$ exists and determine this limit. **(5)**

❖ Reduce $A = \begin{bmatrix} 1 & 3 & 6 & -1 \\ 1 & 4 & 5 & 1 \\ 1 & 5 & 4 & 3 \end{bmatrix}$ to normal form N and compute the matices P & Q such that PAQ=N. **(10)**

2001

❖ Let V be the vector space of polynomials in x with real coefficients of degree almost 2. Let t_1, t_2, t_3 be any 3 distinct real numbers. Define $L_1 : V \to IR$

by $L_i(f) = f(t), i = 1, 2, 3.$ Show that

(i) L_1, L_2, L_3 are linear functional on V.

(ii) $\{L_1, L_2, L_3\}$ is a basis for the dual space V of V.

(10)

❖ In the notation of (a) above, find a basis $B = \{P_1, P_2, P_3\}$ for V which is dual to $\{L_1, L_2, L_3\}$

and also express each $P \in V$ interms of elements

of B. **(10)**

❖ Let V be the vector space of polynomials in x with complex coefficients.
Define $T : V \to V$ by $(Tf)(x) = xf(x)$ and

$$U : V \to V \text{ by } U\left[\sum_{i=0}^{n} c_i x^i\right] = \sum_{i=0}^{n-1} c_{i+1} x^i$$

(i) Find ker T
(ii) Show that U is linear
(iii) Show that UT = I and TV ≠ I, I = Identity on V.
(10)

❖ Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

Show that for every intergern n≥3 $A^n = A^{n+2} + A^2 - I$

and hence find the matrix A^8. **(10)**

❖ Find the characteristic and minimal polynomials of $A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ and determine whether A is

diagonalisable **(10)**

❖ Let $A = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$. Find an invertible 3 × 3

matrix P such that P′Ap = D, D is diagonal matrix. Find D also. **(10)**

2000

❖ Let C(IR) be the vector space of complex numbers over the field of real numbers. Under what conditions of the real numbers $\alpha, \beta, \gamma, \delta$ in the set

$S = \{\alpha + i\beta, \gamma + i\delta\},\ i = \sqrt{(-1)}$

do we have $L(s) = C(IR)$

where L(s) denotes the linear spans of S?
Justify your answer **(10)**

❖ Sketch the conic $[x\ y\ 1]\begin{bmatrix} 5 & 3 & -5 \\ 3 & 5 & -3 \\ -5 & -3 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$

❖ Let $D : V(IR) \to V(IR) : f(x) \to \frac{df(x)}{dx}$

$T : V(IR) \to V(IR) : f(x) \to xf(x)$

be linear transformations on V(IR), the vector space of all polynomials in an indeterminate X with real coefficients.
Show that
(i) DT – TD = I the identity operator
(ii) $(TD)^2 = TD^2 + TD$

find A^{-2} when $A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

by using caley-Hamilton theorem. **(20)**

❖ Show that every square matrix can be expressed uniquely as A + iB where A, B are Hemitian. **(20)**

❖ Reduce the quadratic form
$x^2 + 4y^2 + 9z^2 + t^2 - 12yz + 6zx \; □ 4xy - 2xt - 6zt$

to canonical form and find its rank and signature.

❖❖❖