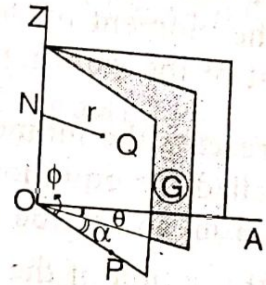
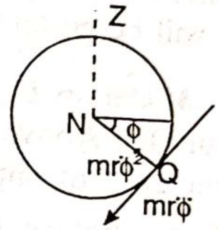
2

Motion About A Fixed Axis

2·01. *A rigid body is rotating about a fixed axis. To find the moment of the effective forces about the axis of rotation.*

Let the axis of rotation be OZ, perpendicular to the plane of the paper. Take a plane AOZ through OZ and fixed in space, cutting the plane of the paper along OA. Let this plane be taken as the plane of reference. Let θ be the angle, which another plane ZOG through the axis fixed in the body makes with the plane AOZ.





Take a particle of mass m at Q and let the plane through OZ and Q cut the plane of the paper along OP. Let the angle between ZOP and ZOG be α. When body rotates about OZ, α remains constant. Let the angle between the plane ZOP and the plane ZOA be ϕ. Now

$$\theta + \alpha = \phi \quad \therefore \quad \dot{\theta} = \dot{\phi} \text{ and } \ddot{\theta} = \ddot{\phi}$$

The accelerations of the particle of mass m are $r\dot{\phi}^2$ and $r\ddot{\phi}$ along QN and perpendicular to QN respectively.

Therefore effective forces on the particles are $mr\dot{\phi}^2$ and $mr\ddot{\phi}$ in the above said directions. Again $r\dot{\phi}^2 = r\dot{\theta}^2$ and $r\ddot{\phi} = r\ddot{\theta}$

The moment of the force $mr\dot{\phi}^2$ about OZ is zero and moment of the force $mr\ddot{\phi}$ about OZ (& NZ) is $r.mr\ddot{\phi} = mr^2\ddot{\phi} = mr^2\ddot{\theta}$

Hence the moment of the effective forces of the whole body about OZ is $\Sigma mr^2\ddot{\theta} = \ddot{\theta}\,\Sigma mr^2 = Mk^2\ddot{\theta}$, where k is the radius of gyration of the body about OZ.

Moment of momentum about the axis of rotation.

Velocity of the particle m is $r\dot{\phi}$ perpendicular to QN. Therefore the moment of momentum of the particle about OZ is $mr^2\dot{\phi}$ or $mr^2\dot{\theta}$.

Hence the moment of momentum of the whole body about OZ is

$$\Sigma m r^2 \dot{\phi} = (\Sigma m r^2)\dot{\theta} = \dot{\theta}\,\Sigma m r^2 = M k^2 \dot{\theta}$$

2·02. Kinetic Energy : The kinetic energy of the particles of mass m is

$$\frac{1}{2} m r^2 \dot{\phi}^2.$$

Hence K.E. of the whole body is

$$\Sigma \frac{1}{2} m r^2 \dot{\phi}^2 = \Sigma \frac{1}{2} m r^2 \dot{\theta}^2 = \frac{1}{2}\dot{\theta}^2\,\Sigma m r^2 = \frac{1}{2} M k^2 \dot{\theta}^2.$$

2·03. Equation of motion :

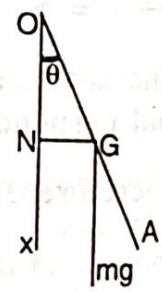
The impressed forces include besides the external forces, the reactions on the axis of rotation OZ. We take moment about OZ, so that this reaction could be avoided i.e. the moment of the effective forces about OZ will be equal to the moment of the external forces about OZ. Thus

$M k^2 \ddot{\theta} = L$, where L represents the moment of all external forces about OZ. Above equation is called the equation of motion of the body. In the case of impulsive forces if ω_1 and ω_2 be angular velocities of the body just before and just after the action of the impulses, L the moment of the impulses then we have $Mk^2(\omega_2 - \omega_1) = L$.

ILLUSTRATIVEEXAMPLES

Ex.1. *A straight uniform rod can turn freely about one end O, hangs from O vertically. Find the least angular velocity with which it must begin to move so that it may perform complete revolution in a vertical plane.*

Sol. Let the rod OA at any instant t make an angle θ with the initial vertical position OX. Let G be the centre of gravity and GN perp. to OX. Let $OA = a$ and mass of the rod be m. The equation of motion is



$$mk^2\ddot{\theta} = -mg\left(\frac{a}{2}\right)\sin\theta$$

$\because$ Moment of effective forces about the axis of rotation $= mk^2\ddot{\theta}$ and moment of external forces about the axis of rotation $= -mg(a/2)\sin\theta$

$$\Rightarrow 2a\ddot{\theta} = -3g\sin\theta \quad \left(\because\ k^2 = \frac{a^2}{3}\right)$$

Multiplying the above equation by $\dot{\theta}$ and integrating, we get

$$a\dot{\theta}^2 = 3g\cos\theta + C \qquad \text{...(1)}$$

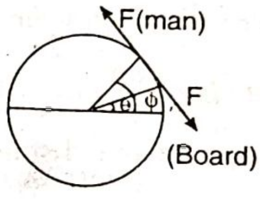
Let $\dot{\theta} = \omega$ when $\theta = 0$ $\therefore a\omega^2 = 3g + C$ ...(2)

Hence from (1) and (2), we get $a\dot{\theta} = a\omega^2 - 3g(1-\cos\theta)$

we require that $\dot{\theta} = 0$ when $\theta = \pi$ $\therefore 0 = a\omega^2 - 6g \Rightarrow \omega = \sqrt{(6g/a)}$.

Ex.2. *A perfectly rough circular horizontal board is capable of revolving freely round a vertical axis through the centre. A man whose weight is equal to that of the board walks on and around it at the edge. when he has completed the circuit, what will be his position in space.*

Sol. Let any time t, θ and ϕ be the angles described by the board and man respectively and let F be the action between the feet of the man and the board. Equation of motion for the man is $m a \ddot{\phi} = F$ ...(1)



Equation of motion for the board is

$$m k^2 \ddot{\theta} = -F a \qquad ...(2)$$

On eliminating F between (1) and (2), we get

$$a^2\ddot{\phi} + k^2\ddot{\theta} = 0 \Rightarrow 2\ddot{\phi} + \ddot{\theta} = 0 \left(\because\ k^2 = \frac{a^2}{2}\right)$$

Integrating twice the above equation and considering that initially both man and the board were at rest, we get $2\phi + \theta = 0$

Therefore when $\phi - \theta = 2\pi$ (after completing the circuit)

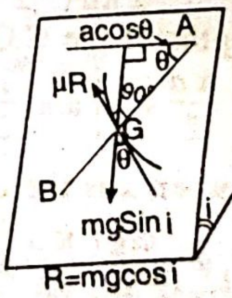
we get, $3\phi = 2\pi \Rightarrow \phi = 2\pi/3$.

This is the angle in space described by the man.

Ex.3. *A uniform rod AB is freely movable on a rough inclined plane whose inclination to the horizon is i and whose coefficient of friction is μ, about a smooth pin fixed through the end A; the bar is held in the horizontal position in the plane and allowed to fall from this position, if θ be the angle through which it falls from rest show that* $(\sin\theta/\theta) = \mu \cot i$.

[Agra 89,Raj. 80,Meerut 94,95]

Sol. Let any instant t, the position of the rod be AB, making an angle θ with the initial horizontal position. The external forces acting on the rod, perp. to the plane, are the normal reaction R and resolved part of its weight i.e. $mg\cos i$. External forces acting on the rod in the plane are, **(i)** the resolved part of its weight, $mg\sin i$ acting down the line of greatest slope through G (centre of gravity). **(ii)** the friction $\mu R = \mu\, mg\cos i$ acting perp. to AB through G; **(iii)** the reaction at A. We take moments about A to avoid reaction, so



$$m k^2\ddot{\theta} = mg\sin i\,.a\cos\theta - \mu\, mg\cos i\; a$$

$\Rightarrow k^2\ddot{\theta} = ga(\sin i\cos\theta - \mu\cos i)$, where $2a$ is length of the rod.

Multiplying the above equation by $2\dot{\theta}$ and integrating, we get

$$k^2\dot{\theta}^2 = 2ag\sin i\sin\theta - 2\mu ag\theta\cos i + D.$$

When $\theta = 0, \dot{\theta} = 0 \therefore D = 0$. Hence $k^2\dot{\theta}^2 = 2ag\sin i\sin\theta - 2\mu ag\theta\cos i$

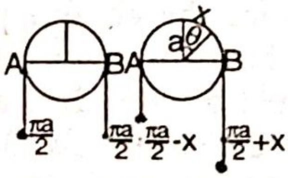
Rod will come to rest when $\dot{\theta} = 0$

$\therefore 0 = 2ag\sin i\sin\theta - 2\mu ag\cos i \Rightarrow (\sin\theta/\theta) = \mu\cot i.$

Ex.4. *A uniform vertical circular plate of radius a, is capable of revolving about a smooth horizontal axis through its centre; a rough perfectly flexible chain, whose mass is equal to that of the plate and whose length is equal to its circumference hangs over its rim in equilibrium, if one end be slightly displaced show that the velocity of the chain when the other end reaches the plate is* $\left(\dfrac{\pi ag}{6}\right)^{1/2}$ **[Meerut 1996,94,92,90]**

Sol. Let x be the distance described in time t. Let v be the velocity of the string and θ be the angular velocity of the plate, then $v = \dot{x} = a\dot{\theta}$

Let m be the mass of the plate and that of the string, then K.E. of the string $= \frac{1}{2}mv^2$. K.E. of the plate



$$= \frac{1}{2}mk^2\dot{\theta}^2 = \frac{1}{2}mk^2\frac{v^2}{a^2}$$

$$= \frac{1}{2}m\frac{a^2}{2}\cdot\frac{v^2}{a^2} = \frac{1}{4}mv^2$$

$$\left[\because k^2 = \frac{a^2}{2}\right]$$

Hence, the total K.E. generated $= \frac{1}{2}mv^2 + \frac{1}{4}mv^2 = \frac{3}{4}mv^2.$

At time t, length of the string hanging to the right is $\left(\dfrac{\pi a}{2} + x\right)$ and hanging to the left is $\left(\dfrac{\pi a}{2} - x\right)$. the weights of these two portion are respectively $\dfrac{mg}{2\pi a}\left(\dfrac{\pi a}{2} + x\right)$ and $\dfrac{mg}{2\pi a}\left(\dfrac{\pi a}{2} - x\right)$.

The depths of the C.G.'s of these portions below AB are $\dfrac{1}{2}\left(\dfrac{\pi a}{2} + x\right)$ and $\dfrac{1}{2}\left(\dfrac{\pi a}{2} - x\right)$.

Hence when x is the displacement, work function on the right is

$$W_1 = \frac{mg}{2\pi a}\left(\frac{\pi a}{2} + x\right)\cdot\frac{1}{2}\left(\frac{\pi a}{2} + x\right).$$

Work function of the left is $W_2 = \frac{mg}{2\pi a}\left(\frac{\pi a}{2} - x\right) \cdot \frac{1}{2}\left(\frac{\pi a}{2} - x\right)$

$\therefore$ Total work function

$$W = W_1 + W_2 = \frac{mg}{4\pi a}\left(\frac{\pi a}{2} + x\right)^2 + \frac{mg}{4\pi a}\left(\frac{\pi a}{2} - x\right)^2 \qquad ...(1)$$

In the initial position i.e. when $x = 0$

$$W_0 = 2\frac{mg}{4\pi a}\frac{\pi^2 a^2}{4} = \frac{1}{8} mg\pi a \qquad \text{[from (1)]}$$

Hence total work done

$$= W - W_0 = \frac{mg}{4\pi a}\left[\left(\frac{\pi a}{2} + x\right)^2 + \left(\frac{\pi a}{2} - x\right)^2 - \frac{1}{2}\pi^2 a^2\right] = \frac{mgx^2}{2\pi a}.$$

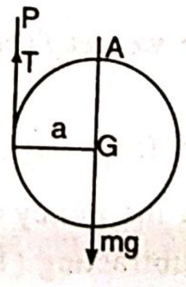
Therefore energy equation gives $\frac{3}{4}mv^2 = \frac{mgx^2}{2\pi a} \Rightarrow v^2 = \frac{2gx^2}{3\pi a}$.

When $x = \frac{\pi a}{2}$ (i.e. when other end reaches the plate)

$$v^2 = \frac{1}{6}\pi a g \Rightarrow v = \left(\frac{1}{6}\pi a g\right)^{1/2}$$

Ex.5. *One end of a light string is fixed to a point of the rim of a uniform circular disc of radius a and mass m and the string is wounded several times round the rim. The free end is attached to a fixed point and the disc is held so that the part of the string not in contact with it, is vertical. If the disc be let go, find the acceleration and the tension of the string.*

Sol. Let the free end be attached to the fixed point P. Let A be the initial position of the centre of gravity G. Let T be the tension of the string. There being no horizontal force the C.G. will move vertically downwards. Let x be the distance moved by G in time t and during this period, θ be the angle turned through some radius.



$$\therefore mg - T = m\ddot{x} \qquad ...(1)$$

and $$Ta = mk^2\ddot{\theta} = m\frac{a^2}{2}\ddot{\theta} \qquad ...(2)$$

Again $x = a\theta$, $\therefore \ddot{x} = a\ddot{\theta}$ ...(3)

On eliminating T and $\ddot{\theta}$ from (1), (2) and (3), we get

$$mga = ma\ddot{x} + m\frac{a}{2}\ddot{x} \Rightarrow \ddot{x} = \frac{2g}{3}$$

Substituting this value in (1), we get $T = \frac{1}{3}mg$.

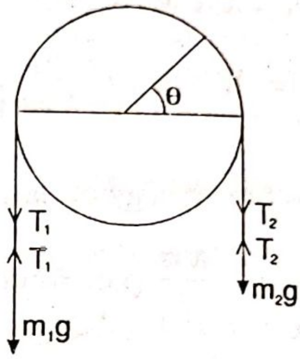
Ex.6. *Two unequal masses* m_1 and m_2 $(m_1 > m_2)$ *are suspended by a light* *string over a circular pulley of mass M and radius a. There is no slipping and the friction of the axis may be neglected. If f be the acceleration; show that this is constant, and if* k^2 *be the radius of gyration of the pulley about the axle, show that* $k^2 = \dfrac{a^2}{Mf}\left[(g-f)m_1-(g+f)m_2\right]$

Sol. Let in time t, m_1 move a distance x downwards and m_2 move a distance x upwards. Let θ be the angle through which the pulley has rotated in time t. Since $x = a\,\theta$, $\therefore \ddot{x} = a\,\ddot{\theta}$

Equations of motion of m_1 and m_2 are

$$m_1\ddot{x} = m_1 g - T_1 \qquad \text{...(1)}$$

and $$m_2\ddot{x} = T_2 - m_2 g \qquad \text{...(2)}$$



Equation of motion of the pulley is

$$Mk^2\ddot{\theta} = T_1 a - T_2 a$$

(Moment is taken about the axle)

$$\Rightarrow M\frac{k^2}{a^2}\ddot{x} = T_1 - T_2 \quad \left(\because\ \ddot{\theta} = \frac{\ddot{x}}{a}\right) \qquad \text{...(3)}$$

Adding (1), (2) and (3), we get $\ddot{x}\left(m_1+m_2+M\dfrac{k^2}{a^2}\right) = m_1 g - m_2 g$

$$\Rightarrow \ddot{x} = f = \frac{(m_1-m_2)g}{m_1+m_2+M\dfrac{k^2}{a^2}},\text{ which is constant.}$$

From above we get $f(m_1+m_2)+\dfrac{Mk^2}{a^2}f = (m_1-m_2)g$

$$\Rightarrow k^2 = \frac{a^2}{Mf}\left[(m_1-m_2)g-(m_1+m_2)f\right] = \frac{a^2}{Mf}\left[(g-f)m_1-(g+f)m_2\right].$$

Pressure on the pulley $= T_1 + T_2$.

Again on subtracting (1) from (2), we get

$$(m_2-m_1)\ddot{x} = T_2+T_1-(m_1+m_2)\,g$$

$$\Rightarrow T_2+T_1 = (m_2-m_1)\ddot{x}+(m_1+m_2)\,g = (m_2-m_1)f+(m_1+m_2)\,g.$$

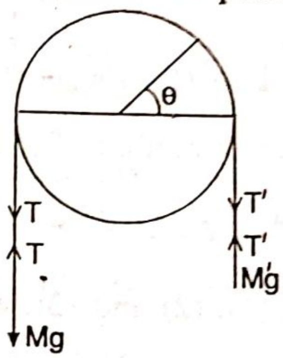
Ex.7. *A fine string has two masses M and M′ tied to its ends and passes over a rough pulley, of mass m, whose centre is fixed. If the string does not slip over the pulley, show that M will descend with acceleration*

$$\frac{M-M'}{M+M'+(mk^2/a^2)}\,g$$ **[Agra 85,89]**

where a is the radius and k the radius of gyration of the pulley.
If the pulley be not sufficiently rough to prevent sliding, and M be the descending mass, show that its acceleration is $\dfrac{M - M' e^{\mu\pi}}{M + M' e^{\mu\pi}} g$ *and that pulley will now spin with an angular acceleration equal to*

$$\frac{2MM' ga(e^{\mu\pi} - 1)}{mk^2 (M + M' e^{\mu\pi})}$$ **[Agra, 90]**

Sol. First part, when the pulley is rough enough to prevent sliding.
Proceeding like **Ex.6** the equations of motion of masses and pulley are



$$M\ddot{x} = Mg - T \qquad \text{...(1)}$$

and $M'\ddot{x} = T' - M'g$ ...(2)

and moment of effective forces about the axis of rotation $= mk^2 \ddot{\theta} = (T - T')a$ ...(3)

Again $x = a\theta, \ddot{x} = a\ddot{\theta}$,

$$\therefore mk^2 \frac{\ddot{x}}{a^2} = T - T' \qquad \text{...(4)}$$

Adding (1), (2) and (4), we get $\ddot{x}\,[M + M' + (mk^2/a^2)] = (M - M')\,g$

$\Rightarrow$Acceleration $= \ddot{x} = \dfrac{(M - M')\,g}{M + M' + (mk^2/a^2)}$.

Second Part. When the pulley is not sufficiently rough to prevent sliding, then we can not take $x = a\theta$. In this case, from Statics, we have

$T = T' e^{\mu\pi}$ ...(5). Solving (1), (2) and (5), we have

$$T = \frac{2MM' g e^{\mu\pi}}{M + M' e^{\mu\pi}},\ T' = \frac{2MM' g}{M + M' e^{\mu\pi}}$$

and $\ddot{x} = \dfrac{M - M' e^{\mu\pi}}{M + M' e^{\mu\pi}} g$.

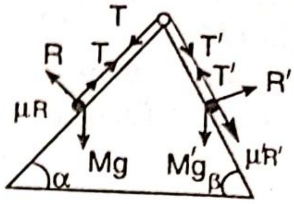
Further putting above values of T and T' in (3), we get

$$\ddot{\theta} = \frac{2ga\,(e^{\mu\pi} - 1)}{mk^2} \cdot \frac{MM'}{M + M' e^{\mu\pi}}.$$

Ex.8. *Two unequal masses, M and M′ rest on two rough planes inclined at an angles* α *and* β *to the horizon: they are connected by a fine string passing over a small pulley, of mass m and radius a, which is placed at the common vertex of the two planes; show that the acceleration of either mass is* $\dfrac{g\,[M(\sin\alpha - \mu\cos\alpha) - M'(\sin\beta + \mu'\cos\beta)}{M + M' + (mk^2/a^2)}$. **[Agra 1991]**

where μ *and* μ' *are the coefficients of friction, k is the radius of gyration of the pulley about its axis and M is the mass which moves downwards.*

Sol. Suppose in time t, the mass M moves a distance x downwards, and also M' moves a distance x upwards. Let the pulley turn through an angle θ, in the same time t.



$\therefore x = a\theta, \dot{x} = a\dot{\theta}, \ddot{x} = a\ddot{\theta}$. The equation of motion of the masses M and M' are

$$M\ddot{x} = Mg\sin\alpha - Mg\mu\cos\alpha - T \quad ...(1)$$

$$M'\ddot{x} = T' - M'g\sin\beta - M'g\mu'\cos\beta \quad ...(2)$$

Equation of motion of pulley is $mk^2\ddot{\theta} = (T - T')a$

$$\Rightarrow \frac{mk^2\ddot{x}}{a^2} = T - T' \qquad \left(\because \ddot{\theta} = \frac{\ddot{x}}{a}\right) \quad ...(3)$$

Adding (1), (2) and (3), we get

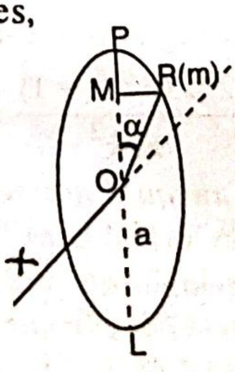
$$\left(\frac{mk^2}{a^2} + M + M'\right)\ddot{x} = g[M(\sin\alpha - \mu\cos\alpha) - M'(\sin\beta - \mu'\cos\beta)]$$

$$\Rightarrow \ddot{x} = \frac{g[M(\sin\alpha - \mu\cos\alpha) - M'(\sin\beta + \mu'\cos\beta)]}{M + M' + \dfrac{mk^2}{a^2}}.$$

Ex.9. *A uniform circular disc is free to turn about a horizontal axis through its centre perp. to its plane. A particle of mass m is attached to a point in the edge of the disc. If the motion starts from the position in which radius to the particle makes an angle* α *with the upward vertical, find the angular velocity when m is in its lowest position. Take the mass of the disc as M.* **[Meerut 1973]**

Sol. The circular disc is turning about the fixed horizontal axis OX, through its centre O. Let ω be the angular velocity when m is in its lowest position, say L then energy principle gives,

Change in K.E. = work done by forces.



$$\Rightarrow \left(\frac{1}{2}ma^2\omega^2 + \frac{1}{2}M\frac{a^2}{2}\omega^2\right) - 0$$

$$= mg(a + a\cos\alpha)$$

or $a\omega^2(2m + M) = 4g(1 + \cos\alpha)$

or $\omega = 2\dfrac{\sqrt{2}}{\sqrt{(2m+M)}}\sqrt{(g/a)}\cos\dfrac{\alpha}{2}$.

Remark : The weight of the disc does not work as its C.G. is fixed.

Supplementary Problems

1. *A uniform rod, of mass m and length 2a, can turn freely about one end which is fixed, it is started with angular velocity ω from the position in which it hangs vertically; find its angular velocity at any instant.*

Ans. $\dot{\theta}^2 = \omega^2 - (3g/2a)(1 - \cos\theta)$, where θ *is the angle which the rod makes with downward vertical.*

2. *In solved example 8 show that to prevent slipping the coefficient of friction must be greater than* $\frac{1}{\pi}\log_e \frac{M(2M'a^2 + mk^2)}{M'(2Ma^2 + mk^2)}$.

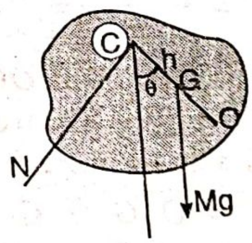
3. *A uniform string of length 20 feet and mass 40 lbs. hangs in equal length over a circular pulley, of mass 10 lbs, and small radius, the axis of the pulley being horizontal, masses of 40 and 35 lbs. are attached to the ends of the string and motion takes place. Show that the time taken by the smaller mass to reach the pulley is* $\frac{1}{4}\sqrt{(15)}\log_e(9 + 4\sqrt{5})$ *seconds.*

[Raj. 1982,Delhi Hons. 83]

2·04. The Compound Pendulum :

To determine the motion of a body acted on by the force of gravity only and moving about a fixed horizontal axis.

Let us take plane of the paper as the plane through the centre of gravity G of the body and perpendicular to the fixed axis. Let the plane meet the axis in C. Let θ be the angle between the vertical and CG *i.e.* θ is the angle between a plane fixed in space and a plane fixed in the body.



Let $CG = h$. The forces on the body are :

(i) its weight Mg acting downward through G.

(ii) the reaction at C of the fixed axis.

We take moments about the fixed axis to eliminate this reaction.

The equation of motion is $Mk^2\ddot{\theta} = -Mgh\sin\theta$

$$\Rightarrow \frac{d^2\theta}{dt^2} = -\frac{gh}{k^2}\sin\theta = -\frac{gh}{k^2}\theta, \quad (\theta \text{ being small}) \qquad ...(1)$$

Equation (1) shows that motion is S.H.M. Hence the time of complete oscillation of compound pendulum is $2\pi\sqrt{\left(\frac{k^2}{gh}\right)}$.

Simple Equivalent Pendulum. We know that equation of motion of a particle of any mass suspended by a string of length l is

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta = -\frac{g}{l}\theta \quad (\theta \text{ being small})$$

The time of complete oscillation is $2\pi\sqrt{\left(\dfrac{l}{g}\right)}$.

If $2\pi\sqrt{(l/g)} = 2\pi\sqrt{\{k^2/(gh)\}}$, then $l = (k^2/h)$.

This length (k^2/h) in the case of a compound pendulum is called the length of the simple equivalent pendulum.

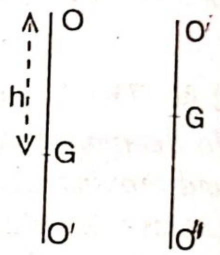
2·05. Centre of Suspension : **[Meerut 1995,94]**

Through C, if a line be drawn perpendicular to the axis of rotation cutting it at C, then C is called the **Centre of suspension.**

Centre of Oscillation. If O is the point on CG produced such that $CO = l = \dfrac{k^2}{h}$ (the length of the simple equivalent pendulum) then the point O is called the centre of oscillation.

2·06. To show that the centres of suspension and oscillation are convertible. **[Meerut 1993,85,83, Agra 86]**

Let us take O and O' as the centre of suspension and oscillation respectively. $\therefore\ OO' = \dfrac{k^2}{h}$ where $OG = h$, and k is radius of gyration of the body about the axis through O.



Now if K is the radius of gyration of the body about an axis through G parallel to the axis of rotation, then

$$Mk^2 = MK^2 + M.OG^2$$

$$\Rightarrow Mk^2 = MK^2 + Mh^2 \Rightarrow k^2 = K^2 + h^2$$

$$\therefore\ OO' = \frac{K^2 + h^2}{h} = \frac{K^2 + OG^2}{OG}$$

$$\Rightarrow OO'.OG = K^2 + OG^2 \Rightarrow K^2 = OG\,(OO' - OG) = OG.O'G. \quad ...(1)$$

Let O'' be the centre of oscillation when the body rotates about a parallel axis through O'. We can show as above that

$$K^2 = O'G.O''G \quad ...(2)$$

From (1) and (2), we observe that O'' is simply the point O. Thus if the body were suspended from a parallel axis through O', O is the centre of oscillation. This proves the theorem.

2·07. Minimum time of oscillation of a compound pendulum. **[Meerut 1980]**

If K is the radius of gyration of the body about an axis through G parallel to the axis of rotation, then $k^2 = K^2 + h^2$.

Therefore length of the simple equivalent pendulum is

$$l = \frac{k^2}{h} = \frac{K^2 + h^2}{h} = \frac{K^2}{h} + h.$$

The time of oscillation of a compound pendulum will be least when the length of the simple equivalent pendulum is minimum. For that

$$\frac{dl}{dh} = \frac{d}{dh}\left(h + \frac{K^2}{h^2}\right) = 0 \Rightarrow 1 - \frac{K^2}{h^2} = 0 \Rightarrow h = K.$$

The length of simple equivalent pendulum in this case

$$l = \frac{K^2 + h^2}{h} = \frac{K^2 + K^2}{K} = 2K.$$

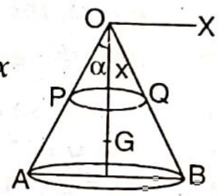
In case $h = 0$ or ∞ i.e. if the axis of suspension either passes through G or be at infinite, the corresponding simple equivalent pendulum is of infinite length, thus the time of oscillaion is infinite.

Ex.10. *A solid homogeneous cone of height h and vertical angle 2 α oscillates about a horizontal axis through its vertex. Show that the length of the simple equivalent pendulum is* $\frac{1}{5}h\,(4 + \tan^2\alpha)$. **[Meerut 1990,93]**

Sol. Let OX be the horizontal axis through the vertex O. Let us take a circular disc PQ of thickness δx at distance x from O. Moment of Inertia of disc about OX

$$= (\rho\,\pi x^2 \tan^2\alpha\,\delta x)\left(\tfrac{1}{4}x^2\tan^2\alpha + x^2\right).$$

Therefore M.I. of whole cone about OX



$$M k^2 = \rho\pi \tan^2\alpha\left(1 + \tfrac{1}{4}\tan^2\alpha\right)\int_0^h x^4\,dx$$

$$= \rho\pi \tan^2\alpha\left(1 + \tfrac{1}{4}\tan^2\alpha\right)\tfrac{1}{5}h^5$$

$$= \frac{1}{20}\rho\,\pi\tan^2\alpha\,(\tan^2\alpha + 4)\,h^5$$

$$= \frac{3}{20}M(\tan^2\alpha + 4)\,h^2 \qquad \left(\because\ M = \tfrac{1}{3}\pi h^3 \tan^2\alpha\,\rho\right)$$

$\therefore k^2 = \frac{3}{20}(\tan^2\alpha + 4)\,h^2$. Again $OG = \frac{3}{4}h$.

Therefore the length of the simple equivalent pendulum i.e.

$$l = \frac{k^2}{OG} = \frac{1}{5}(\tan^2\alpha + 4)\,h.$$

Ex.11. *A solid homogeneous cone of height h and semi-vertical angle α, oscillates about a diameter of its base. Show that the length of the simple equivalent pendulum is* $\frac{1}{5}h\,(2 + 3\tan^2\alpha)$.

Sol. Referring to the fig. of the **Example 10,** we observe that M.I. of the cone about AB

$$= \int_0^h \rho\,\pi x^2 \tan^2\alpha\,dx\left[\frac{x^2\tan^2\alpha}{4} + (h - x)^2\right]$$

$$= \int_0^h \frac{\rho\,\pi\tan^2\alpha}{4}\left[x^4\tan^2\alpha + 4x^2(h - x)^2\right]dx$$

$$= \frac{\rho \pi \tan^2\alpha}{4} \int_0^h (x^4 \tan^2\alpha + 4x^4 - 8hx^3 + 4h^2x^2)\, dx$$

$$= \frac{1}{4} \rho \pi \tan^2\alpha \left(\frac{h^5}{5} \tan^2\alpha + 4\frac{h^5}{5} - 8h\frac{h^4}{4} + 4h^2\frac{h^3}{3} \right)$$

$$= \frac{1}{4} \rho \pi \tan^2\alpha\, h^5 \left[\frac{1}{5} \tan^2\alpha + \frac{2}{15} \right] = \frac{1}{60} \rho \pi h^4 \tan^2\alpha\, [3 \tan^2\alpha + 2]$$

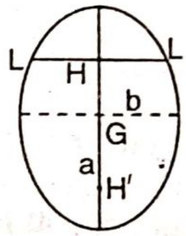
$$= \frac{1}{20} M h^2 (3 \tan^2\alpha + 2), \qquad \text{since } M = \frac{1}{3} \pi h^3 \tan^2\alpha \,.\, \rho$$

$\therefore M k^2 = \frac{1}{20} M h^2 (3 \tan^2\alpha + 2) \Rightarrow k^2 = \frac{1}{20} h^2 (3 \tan^2\alpha + 2)$, where k is the radius of gyration of the cone about AB. Hence length of the simple equivalent pendulum

$$= \frac{k^2}{\text{distance of } G \text{ from } AB} = \frac{k^2}{(h/4)} = \frac{1}{5} h (3 \tan^2\alpha + 2).$$

Ex.12. *An elliptical lamina is such that when it swings about one latus rectum as a horizontal axis, the other latus rectum passes through the centre of oscillation, prove that the eccentricity is* $\frac{1}{2}$.

Sol. When one of the focii say H, is the centre of suspension then the other focus H' is the centre of oscillation. LHL' is the latus rectum (horizontal axis) about which the elliptic lamina oscillates.



$\because$ The length of simple equivalent pendulum

$$l = HH' = 2ae, \qquad \text{...(1)}$$

Also $HG = ae$ and $M k^2 =$ Moment of Inertia of the body about the axis of the rotation

$$\text{'}LHL'\text{'} = M \left[\frac{a^2}{4} + a^2e^2 \right] \Rightarrow k^2 = \frac{1}{4} a^2 (1 + 4e^2)$$

$$\therefore l = \frac{k^2}{HG} = \frac{1}{4} \frac{a^2 (1 + 4e^2)}{ae} \qquad \text{...(2)}$$

From (1) and (2), we get $2ae = \frac{1}{4} \frac{a^2 (1 + 4e^2)}{ae}$

$\Rightarrow 8e^2 = 1 + 4e^2 \Rightarrow 4e^2 = 1 \Rightarrow e = \frac{1}{2}$.

Ex.13. *A uniform elliptic board swings about a horizontal axis at right angles to the plane of the board and passing through one focus. If the centre of oscillation be the other focus prove that its eccentricity is* $\sqrt{(2/5)}$.

Sol. Refer fig. **Ex.12,** here $M k^2 = M [\frac{1}{4} (a^2 + b^2) + a^2e^2]$

$\therefore$ length of simple equivalent pendulum

$$l = \frac{k^2}{HG} \Rightarrow \frac{k^2}{ae} = \frac{1}{4ae}(a^2 + b^2 + 4a^2e^2) \qquad ...(1)$$

Also $l = 2ae \therefore 2ae = \frac{1}{4ae}(a^2 + b^2 + 4a^2e^2)$

$$\Rightarrow 8a^2e^2 = a^2 + b^2 + 4a^2e^2 = a^2 + (1 - e^2)\,a^2 + 4a^2e^2$$

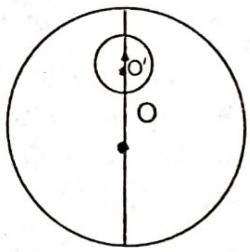
$$\Rightarrow 5a^2e^2 = 2a^2 \Rightarrow e = \sqrt{(2/5)}.$$

Ex.14. *A flat circular disc of radius a has a hole in it of radius b whose centre is at a distance c from the centre of the disc ($c < a - b$). The disc is free to oscillate in a vertical plane about a smooth horizontal circular rod of radius b passing through the hole. Show that the length of the equivalent pendulum is* $c + \frac{1}{2}\frac{a^4 - b^4}{a^2 c}$.

Sol. Let O' be the centre of the hole in the disc whose centre is O. $OO' = c$ (*given*). The disc is oscillated in a vertical plane about a smooth horizontal circular rod of radius b passing through O'.

If h be the depth of C.G. of the body from O',

then $h = \dfrac{\rho\pi a^2 c - \rho\pi b^2 .0}{\rho\pi a^2 - \rho\pi b^2} = \dfrac{a^2 c}{a^2 - b^2}$



Let k be the radius of gyration about the axis of rotation, then we have

$$(\rho\pi a^2 - \rho\pi b^2)\,k^2 = \rho\pi a^2\left(\frac{a^2}{2} + c^2\right) - \rho\pi b^2 . \frac{b^2}{2}$$

$$\Rightarrow k^2 = \frac{a^4 + 2a^2c^2 - b^4}{2\,(a^2 - b^2)}$$

$$\therefore l = \frac{k^2}{h} = \left[\frac{a^4 + 2a^2c^2 - b^4}{2\,(a^2 - b^2)}\right] \Big/ \left[\frac{a^2 c}{(a^2 - b^2)}\right]$$

$$= \frac{a^4 + 2a^2c^2 - b^4}{2\,a^2 c} = c + \frac{1}{2}\frac{a^4 - b^4}{a^2 c}.$$

Ex.15. *A bent lever, whose arms are of length a and b, the angle between them being α, makes small oscillations in its own plane about the fulcrum, show that the length of the corresponding simple pendulum is*

$$\frac{2}{3}\frac{a^3 + b^3}{\sqrt{(a^4 + 2a^2b^2\cos\alpha + b^4)}}.$$ **[Meerut 1993,85,84]**

Sol. Let G_1 and G_2 be the centres of gravity of the arms OA and OB of the lever. Let $OA = a$ and $OB = b$. Also let OA be the axis of x and a perp. line OY the axis of y. Then the co-ordinates of G_1 and G_2 will be $(\frac{1}{2}a, 0)$ and $(\frac{1}{2}b\cos\alpha, \frac{1}{2}b\sin\alpha)$ respectively.

Now if $(\bar{x}, \bar{y})$ is the C.G. of the lever, then

$$\bar{x} = \frac{a\omega \cdot \frac{1}{2}a + b\omega \cdot \frac{1}{2} b \cos \alpha}{a\omega + b\omega} = \frac{1}{2} \frac{a^2 + b^2 \cos \alpha}{a + b};$$

where ω is the weight of unit length of the rod.

$$\bar{y} = \frac{a\omega \cdot 0 + b\omega \cdot \frac{1}{2} b \sin \alpha}{a\omega + b\omega} = \frac{1}{2} \frac{b^2 \sin \alpha}{a + b}$$

Also the distance of C.G. $(\bar{x}, \bar{y})$ from $O\ (0, 0)$ is

$$\sqrt{(\bar{x}^2 + \bar{y}^2)} = \frac{1}{2\,(a+b)} \sqrt{\{a^4 + 2a^2b^2 \cos \alpha + b^4\}}$$

Now if k is the radius of gyration about the axis of rotation through O,

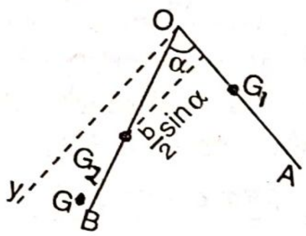
then we have $(a+b)\,\omega\, k^2 = a\,\omega . \frac{4}{3} (\frac{1}{2} a)^2 + b\,\omega . \frac{4}{3} (\frac{1}{2} b)^2 \Rightarrow k^2 = \frac{a^3 + b^3}{3(a+b)}$.

Hence the length of the simple pendulum

$$= \frac{k^2}{\text{Dist. of C.G. of the lever from } O}$$

$$= \frac{1}{3} \frac{a^3 + b^3}{a + b} \cdot \frac{2\,(a+b)}{(a^4 + 2a^2b^2 \cos \alpha + b^4)^{1/2}}$$

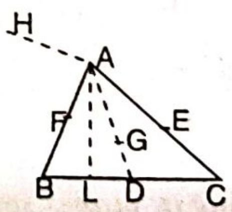
$$= \frac{2}{3} \frac{a^3 + b^3}{(a^4 + 2a^2b^2 \cos \alpha + b^4)^{1/2}}.$$



Ex.16. *A uniform triangular lamina can oscillate in its own plane about the angle A. Prove that the length of the simple equivalent pendulum is* $\dfrac{3\,(b^2 + c^2) - a^2}{4\sqrt{\{2\,(b^2 + c^2) - a^2\}}}$, *the axis through A being horizontal.*

[Agra 1990]

Sol. Let AH be perpendicular to the plane of the lamina so that it oscillates in its own plane about AH. Instead of the triangular lamina of mass m, we can have three particles each of mass $\frac{1}{3}m$ placed at the mid points D, E, F of the sides respectively. Distance of D from AH is



$$AD = [AL^2 + LD^2]^{1/2} = [AL^2 + (BD - BL)^2]^{1/2}$$

$$= [AL^2 + BD^2 + BL^2 - 2BD \cdot BL]^{1/2}$$

$$= [(AL^2 + BL^2) + BD^2 - 2BD \cdot BL]^{1/2}$$

$$= [(AB)^2 + (\tfrac{1}{2}BC)^2 - 2(\tfrac{1}{2}BC) \cdot AB \cos B]^{1/2}$$

$$= \left(c^2 + \frac{a^2}{4} - ac\cos B\right)^{1/2} = \left(c^2 + \frac{a^2}{4} - ac.\ \frac{a^2+c^2-b^2}{2ac}\right)^{1/2}$$

$$= \left(\frac{2b^2 + 2c^2 - a^2}{4}\right)^{1/2}$$

Distance of E from $AH = EA = b/2$

Distance of F from $AH = FA = c/2$

M.I. of the triangle about AH

$$= \frac{1}{3}m\left(\frac{2b^2 + 2c^2 - a^2}{4} + \frac{b^2}{4} + \frac{c^2}{4}\right) = \frac{1}{12}m(3b^2 + 3c^2 - a^2)$$

$$\therefore\ mk^2 = \frac{m}{12}[3b^2 + 3c^2 - a^2] \Rightarrow k^2 = \frac{3b^2 + 3c^2 - a^2}{12}$$

Hence length of the simple equivalent pendulum

$$= \frac{k^2}{\text{Dist. of C.G. from } AH} = \frac{k^2}{AG} = \frac{k^2}{\frac{2}{3}AD}$$

$$= \frac{k^2}{\frac{2}{3}\cdot\frac{1}{2}(2b^2 + 2c^2 - a^2)^{1/2}} = \frac{3k^2}{\sqrt{(2b^2 + 2c^2 - a^2)}}$$

$$= \frac{3(3b^2 + 3c^2 - a^2)}{12\sqrt{(2b^2 + 2c^2 - a^2)}} = \frac{3(b^2 + c^2) - a^2}{4\sqrt{(2b^2 + 2c^2) - a^2)}}.$$

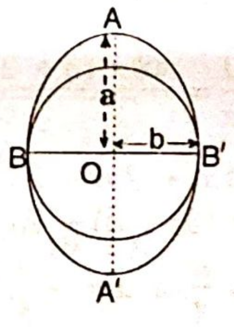
Ex. 17. *An ellipse of axis a, b and a circle of radius b are cut from the same sheet of thin uniform metal and are superposed and fixed together with their centres coincident. The figure is free to move in its own vertical plane about one end of the major axis. Show the length of the equivalent simple pendulum is* $\dfrac{5a^2 - ab + 2b^2}{4a}$.

Sol. Mass of the circle $= \pi b^2 \rho$

Mass of the ellipse $= \pi ab\,\rho$, where ρ is the mass of the sheet per unit area.

Mass of the system $= \pi b^2 \rho + \pi ab\,\rho$

Now taking k to be the radius of gyration of the body about a line through A perpendicular to lamina, we have



$$(\pi ab + \pi b^2)\rho k^2$$

$$= \pi b^2 \rho.\left(a^2 + \frac{b^2}{2}\right) + \pi ab\,\rho\left(a^2 + \frac{a^2 + b^2}{4}\right)$$

$$\Rightarrow 4\pi\, b\,(a + b)\,\rho k^2 = \pi b^2 \rho\,(4a^2 + 2b^2) + \pi ab\,\rho\,(5a^2 + b^2).$$

$$\Rightarrow k^2 = \frac{b(2b^2 + 4a^2) + a(5a^2 + b^2)}{4(a + b)} = \frac{5a^3 + 4a^2b + ab^2 + 2b^3}{4(a + b)}$$

$$= \frac{5a^2(a+b) - ab\,(a+b) + 2b^2(a+b)}{4(a+b)}$$

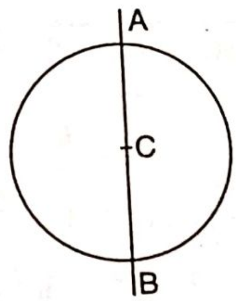
$$= \frac{(a+b)(5a^2 - ab + 2b^2)}{4(a+b)} = \frac{1}{4}(5a^2 - ab + 2b^2)$$

Hence length of the equivalent simple pendulum

$$= \frac{k^2}{\text{Dist. of C.G. of the system from } A} = \frac{k^2}{a} = \frac{5a^2 - ab + 2b^2}{4a}.$$

Ex. 18. *A uniform rod of mass m and length $2a$ can oscillate about a horizontal axis through one end. A circular disc of mass $24\,m$ and radius $\frac{1}{3}a$ can have its centre clamped to any point of the rod and its plane contains the axis of rotation. Show that for oscillations under gravity the length of the simple equivalent pendulum lies between $(a/2)$ and $2a$.*

Sol. Let AB be the rod axis of rotation pass through A. Let the centre C of the disc, be clamped at a distance x from A.



The distance of C.G. of the system i.e. of the rod and the disc together,

$$h = \frac{ma + 24\,m.x}{m + 24\,m} = \frac{2a^2 + 24x^2}{25}.$$

then if k is the radius of gyration then

$$(m + 24\,m)\,k^2 = m\frac{4}{3}a^2 + 24\,m \times \left[\left(\frac{1}{4}\cdot\frac{a}{3}\right)^2 + x^2\right]$$

$$\Rightarrow k^2 = \frac{4a^2 + 2a^2 + 72x^2}{3 \times 2} = \frac{2a^2 + 24x^2}{25}.$$

Hence length of the simple equivalent pendulum.

$$l = \frac{k^2}{h} = \left(\frac{2a^2 + 24x^2}{25}\right) \Big/ \left(\frac{a + 24x}{25}\right) \Rightarrow l = \frac{2a^2 + 24x^2}{a + 24x} \qquad \text{...(1)}$$

For maximum or minimum of l, $\dfrac{dl}{dx} = 0$

$$\Rightarrow \frac{dl}{dx} = \frac{48x\,(a + 24x) - 24\,(2a^2 + 24x^2)}{(a + 24x)^2} = 0$$

$$\Rightarrow (24x^2 + 2ax - 2a^2) = 0 \Rightarrow 24x^2 + 8ax - 6ax - 2a^2 = 0$$

$$\Rightarrow 8x\,(3x + a) - 2a\,(3x + a) = 0 \Rightarrow (3x + a)(8x - 2a) = 0$$

$$\Rightarrow x = \frac{a}{4} \text{ or } x = -\frac{a}{3}. \text{ Since } x \neq -\frac{a}{3}, \quad \text{we have } x = \frac{a}{4}.$$

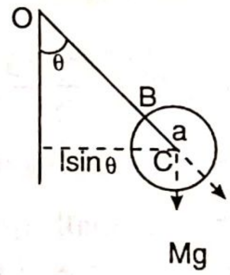
When $x = \dfrac{a}{4}$, we get $l = \dfrac{a}{2}$ [from (1)]

The other extreme value of l (*i.e.* $2a$) is given by putting $x = 0$ or $x = 2a$ in (1). Hence the length of the simple equivalent pendulum lies between $\frac{a}{2}$ and $2a$.

Ex.19. *A sphere of radius a, is suspended by a fine wire from a fixed point at a distance l from its centre. Show that the time of a small oscillation is given by* $2\pi\left(\frac{5l^2+2a^2}{5lg}\right)^{1/2}\left[1+\frac{1}{4}\sin^2\left(\frac{\alpha}{2}\right)\right]$ *where* α *represents the amplitude of the vibration.* **[Meerut 1988]**

Sol. Suppose that the axis of rotation is passing through O, where $OC = l$. Moment of inertia of sphere of mass M about the axis of rotation is $M\left(\frac{2}{5}a^2+l^2\right)$. Equation of motion is $M\left(\frac{2}{5}a^2+l^2\right)\ddot{\theta} = -Mgl\sin\theta$

$$\Rightarrow \ddot{\theta} = -\frac{5gl}{2a^2+5l^2}\sin\theta.$$



Integrating, we get $\dot{\theta}^2 = \frac{10\,gl}{2a^2+5l^2}\cos\theta+\lambda.$ ...(1)

Let when $\theta = \alpha$, $\dot{\theta} = 0$.
Hence (1) reduces to

$$\dot{\theta}^2 = \frac{10gl}{2a^2+5i^2}(\cos\theta - \cos\alpha)$$

$$\Rightarrow \frac{d\theta}{dt} = -\sqrt{\left(\frac{10gl}{5a^2+5l^2}\right)}\sqrt{(\cos\theta - \cos\alpha)}$$

($\because$ Sphere is coming in the direction of θ decreasing)

$$= -\sqrt{\left(\frac{10gl}{2a^2+5l^2}\right)}\sqrt{\left(1-2\sin^2\frac{\theta}{2}-1+2\sin^2\frac{\alpha}{2}\right)}$$

$$= -\sqrt{\left(\frac{10gl}{2a^2+5l^2}\right)}\sqrt{2}.\sqrt{\left(\sin^2\frac{\alpha}{2}-\sin^2\frac{\theta}{2}\right)}.$$

It t be the time from one extreme to the lowest point, then

$$t = -\frac{1}{\sqrt{2}}\sqrt{\left(\frac{2a^2+5l^2}{10gl}\right)}\int_{\alpha}^{0}\frac{d\theta}{\sqrt{\{\sin^2(\alpha/2)-\sin^2(\theta/2)\}}}$$

$$= \frac{1}{\sqrt{2}}\sqrt{\left(\frac{2a^2+5l^2}{10gl}\right)}\int_{0}^{\alpha}\frac{d\theta}{\sqrt{\{\sin^2(\alpha/2)-\sin^2(\theta/2)\}}}$$

Putting $\sin(\theta/2) = \sin(\alpha/2)\sin\phi$,

i.e. $\frac{1}{2}\cos\frac{\theta}{2}\,d\theta = \sin\frac{1}{2}\alpha\cos\phi\,d\phi$, we get

$$t = \sqrt{\left(\frac{2a^2+5l^2}{5gl}\right)}\int_{0}^{\pi/2}\frac{d\phi}{\cos(\theta/2)}$$

$$= \sqrt{\left(\frac{2a^2 + 5l^2}{5gl}\right)} \int_0^{\pi/2} \frac{d\phi}{\sqrt{\{(1 - \sin^2\frac{\alpha}{2} \cdot \sin^2\phi)\}}}$$

$$= \sqrt{\left(\frac{2a^2 + 5l^2}{5gl}\right)} \int_0^{\pi/2} \left[1 + \frac{1}{2}\sin^2\frac{\alpha}{2}\sin^2\phi + \ldots\right] d\phi$$

$$\left[\because \quad (1 - x)^{-1/2} = 1 + \tfrac{1}{2}x + \ldots\right]$$

$$= \sqrt{\left(\frac{2a^2 + 5l^2}{5gl}\right)} \left[\frac{\pi}{2} + \frac{1}{2}\sin^2\frac{\alpha}{2} \cdot \frac{\pi}{4} + \ldots\right]$$

$$\because \int_0^{\pi/2} \sin^2\phi \, d\phi = (\pi/4)$$

$$= (\pi/2)\sqrt{\left(\frac{2a^2 + 5l^2}{5gl}\right)} \left[1 + \frac{1}{4}\sin^2\frac{\alpha}{2}\right]$$

neglecting higher powers of $\sin\frac{\alpha}{2}$, since α is small.

$\therefore$ Time for one small oscillation is

$$4t = 2\pi\sqrt{\left(\frac{2a^2 + 5l^2}{5gl}\right)} \left[1 + \frac{1}{4}\sin^2\frac{\alpha}{2}\right].$$

Ex.20. *Three equal particles are attached to a weightless rod at equal distances a apart. The system is suspended, and is free to turn about a point of the rod distant x from the middle particle. Find the time of a small oscillation and show that it is least when $x = \cdot 82a$ nearly.*

Sol. Let the three particles each of mass m, be attached to the rod at the points A, B and C such that $AB = BC = a$.

Again let the system rotate about ON such that $OB = x$. Then M.I. of the three particles about ON

$$= m(a - x)^2 + mx^2 + m(a + x)^2$$

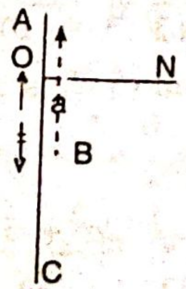
$$\Rightarrow 3mk^2 = m(a - x)^2 + mx^2 + m(a + x)^2$$

$$\Rightarrow k^2 = \frac{3x^2 + 2a^2}{3},$$

where k is the radius of gyration of the system about ON. Now if l is the length of the equivalent simple pendulum then we have

$$l = \frac{k^2}{\text{Dist of C.G. of the system from } O} = \frac{k^2}{x}$$

$$= \frac{3x^2 + 2a^2}{3x} = x + \frac{2a^2}{3x}.$$



$$\therefore \quad \frac{dl}{dx} = 1 - \frac{2a^2}{3x^2}$$

For max. or min. of l, we have $\frac{dl}{dx} = 0$

i.e. $1-\dfrac{2a^2}{3x^2}=0 \Rightarrow x=\dfrac{a}{3}\sqrt{6}=\cdot 816a=\cdot 82a$ nearly.

Further $\dfrac{d^2l}{dx^2}=\dfrac{4a^2}{3x^3}$, which is positive for $x=\cdot 82a$

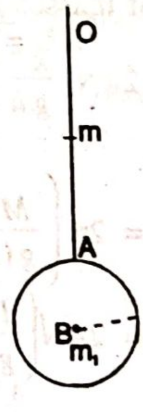
Hence min. value of l is given by $x=\cdot 82a$

Ex.21. *Find the time of oscillation of a compound pendulum consisting of a rod of mass m and length a, carrying at one end a sphere of mass m_1 and diameter 2b, the other end of the rod being fixed.*

Sol. Let $OA=a$ be the rod of mass m, and a sphere of mass m_1 be attached to it at A.

If h is the distance of the C.G. of the system from O, then

$$h=\frac{m.\frac{a}{2}+m_1(a+b)}{m+m_1} \qquad ...(1)$$



Also if k is the radius of gyration of the system about the axis through O, we have

$$(m+m_1)k^2=m.\frac{a^2}{3}+m_1\left[\frac{2}{5}b^2+(a+b)^2\right]$$

$$\Rightarrow k^2=\frac{m\frac{a^2}{3}+m_1\left[\frac{2}{5}b^2+(a+b)^2\right]}{m+m_1}$$

Hence length of equivalent simple pendulum

$$=\frac{k^2}{h}=\frac{m\frac{a^2}{3}+m_1\left[\frac{2}{5}b^2+(a+b)^2\right]}{m+m_1}\cdot\frac{m+m_1}{m\frac{a}{2}+m_1(a+b)}$$

$$=\frac{m\frac{a^2}{3}+m_1\left[\frac{2}{5}b^2+(a+b)^2\right]}{m\frac{a}{2}+m_1(a+b)}$$

and the time of complete oscillation is

$$=2\pi\left(\frac{k^2}{gh}\right)^{1/2}=\frac{2\pi}{\sqrt{g}}\cdot\left[\frac{m\frac{a^2}{3}+m_1\left\{\frac{2}{5}b^2+(a+b)^2\right\}}{m\frac{a}{2}+m_1(a+b)}\right]^{1/2}$$

Ex. 22. *A simple circular pendulum is formed of a mass M suspended from a fixed point by a weightless wire of length l, if a mass m, very small compared with M, be knotted on to the wire at a distance a from the point* *of suspension, show that the time of a small vibration of the pendulum is approximately diminished by*

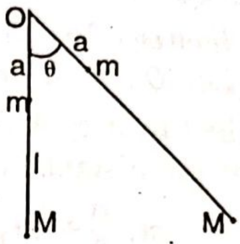
$\frac{m}{2M} \cdot \frac{a}{l}\left(1-\frac{a}{l}\right)$ *of itself.* **[Meerut 1993, Agra 84, 83]**

Sol. Let t be the period of simple pendulum before knotting the mass m, then $t = 2\pi\sqrt{\left(\frac{l}{g}\right)}$

Let k be the radius of gyration when mass m is attached to the wire at a distance a from the point of suspension O.

Then $(m+M)k^2 = Ml^2 + ma^2$

or $$k^2 = \frac{Ml^2+ma^2}{M+m}$$



Distance of C.G. of the system from O is

$$h = \frac{Ml+ma}{M+m}$$

If t' be period for the compound pendulum consisting of masses M and m, then

$$t' = 2\pi\sqrt{\left(\frac{k^2}{gh}\right)} = 2\pi\left[\frac{l}{g}\left(\frac{ml^2+ma^2}{M+m}\cdot\frac{M+m}{Ml+ma}\right)\right]^{1/2}$$

$$= 2\pi\left(\frac{Ml^2+ma^2}{g(Ml+ma)}\right)^{1/2} = 2\pi\sqrt{\left(\frac{l}{g}\right)}\left[1+\frac{ma^2}{Ml^2}\right]^{1/2}\left[1+\frac{ma}{Ml}\right]^{-1/2}$$

$$= 2\pi\sqrt{\left(\frac{l}{g}\right)}\left[1+\frac{ma^2}{2Ml^2}\right]\left[1-\frac{ma}{2Ml}\right],$$

neglecting higher powers of $\frac{m}{M}$.

$$= 2\pi\sqrt{\left(\frac{l}{g}\right)}\left[1-\frac{ma}{2Ml}\left(1-\frac{a}{l}\right)\right] = t\left[1-\frac{ma}{2Ml}\left(1-\frac{a}{l}\right)\right]$$

$$\Rightarrow t-t' = \frac{ma}{2Ml}\left(1-\frac{a}{l}\right)t.$$

Ex. 23. *A weightless straight rod ABC of length 2a is movable about the end A which is fixed and carries two particles of the same mass, one fastened to the middle point B and the other to the end C of the rod. If the rod be held in a horizontal position and then let go, show that its angular velocity when vertical is* $\left(\frac{6g}{5a}\right)^{1/2}$ and that $\frac{5a}{3}$ *is the length of the simple equivalent pendulum.*

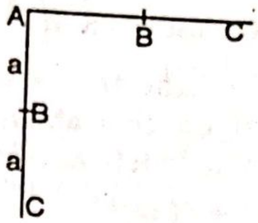
Sol. Let v, v' be the velocities of the masses at B and C when in vertical position. Let ω be the angular velocity of the rod in this position. Then we have energy equation as

$$\frac{1}{2}mv^2 + \frac{1}{2}mv'^2 = mg.a + mg.2a$$

Also $v = a\omega$ and $v' = 2a\omega$

$$\therefore \frac{1}{2}m\,(a^2 + 4a^2)\,\omega^2 = mga + 2mga$$

$$\Rightarrow \omega = \left(\frac{6g}{5a}\right)^{1/2}$$



Again $(m + m)\,k^2 = ma^2 + m\,(2a)^2 \Rightarrow k^2 = \frac{5a^2}{2}$.

Distance of C.G. from A

$$h = \frac{m.a + m.2a}{m + m} = \frac{3a}{2} \therefore l = \frac{k^2}{h} = \frac{\frac{5a^2}{2}}{\frac{3a}{2}} = \frac{5a}{3}$$

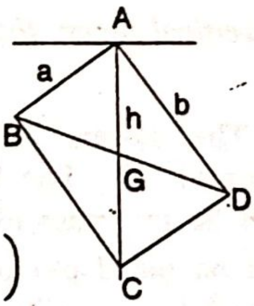
Ex.24. *A rectangular plate swings in a vertical plane about one of its corners. If its period is one second, find the length of the diagonal.*

[Meerut 1989]

Sol. Let k be the radius of gyration of the plate about the axis, through A and perpendicular to its plane; then we have

$$mk^2 = m\,\frac{a^2 + b^2}{4.3} + mh^2 \quad \text{[by parallel axis theorem]}$$

$$= \frac{mh^2}{3} + mh^2 = \frac{4mh^2}{3} \Rightarrow k^2 = \frac{4h^2}{3}$$



$BG = GD$. Further, distance of C.G. from A

$$= AG = h = \frac{1}{2}\sqrt{(a^2 + b^2)}$$

$$\therefore \text{period} = 2\pi\sqrt{\left(\frac{k^2}{hg}\right)} = 2\pi\sqrt{\left(\frac{4h^2}{3gh}\right)} = 4\pi\sqrt{\left(\frac{h}{3g}\right)}$$

But period $= 1 \Rightarrow 4\pi\sqrt{\left(\frac{h}{3g}\right)} = 1$ or $h = \frac{3g}{16\pi^2}$

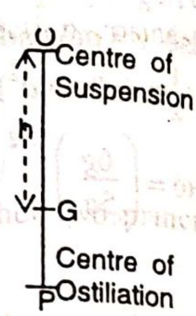
$\therefore$ Length of the diagonal $= 2h = \frac{3g}{8\pi^2}$.

Ex.25. *A pendulum is supported at O, and P is the centre of oscillation. Show that, if an additional weight is rigidly attached at P, the period of oscillation is unaltered.* **[Meerut 1986, 84]**

Sol. Let m be the mass of the body forming the compound pendulum and let h be the depth of its C.G. below the point of suspension O. Also let k be its radius of gyration about the horizontal axis through O; then we easily obtain

$$OP = (k^2/h)$$

$\Rightarrow$ Period of Oscillation $= 2\pi\sqrt{\left(\dfrac{k^2/h}{g}\right)} = T$, say.



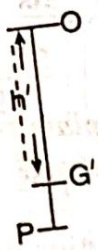
Let an additional weight M be knotted at P, then if k' is the radius of gyration about the horizontal axis through O, we immediately have

$$(M+m)\,k'^2 = mk'^2 + M.OP^2$$

$$= mk^2 + M.\left(\frac{k^2}{h}\right)^2 = k^2\left(m + M\frac{k^2}{h^2}\right) \quad ...(1)$$

and by well known C.G. formula,

$$(M+m)\,h' = mh + M.OP = mh + M.\frac{k^2}{h} = h\left(m + M.\frac{k^2}{h^2}\right) \quad ...(2)$$



(1) and (2) $\Rightarrow \dfrac{k'^2}{h'} = \dfrac{k^2}{h} \Rightarrow T'$

i.e. $2\pi\sqrt{\left(\dfrac{(k'^2/h')}{g}\right)} = 2\pi\sqrt{\left(\dfrac{(k^2/h)}{g}\right)} = T$

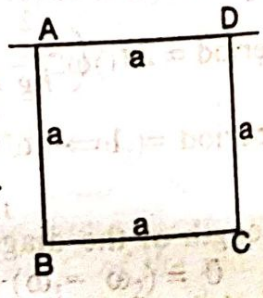
$\Rightarrow$ Period of oscillation is unaltered.

Ex. 26. *Three uniform rods AB, BC, CD each of length a, are freely jointed at B and C and suspended from the points A and D which are in the same horizontal line and a distance a apart. Prove that when the rods move in a vertical plane, the length of simple equivalent pendulum is $\dfrac{5a}{6}$.*

[Meerut 1990, '84]

Sol. The system form a compound pendulum swinging about the horizontal AD. The figure is self explanatory.

Let m be the mass of the each rod.



Let h be the depth of C.G. of the system from AD and k the radius of gyration of the system about the horizontal axis AD, then we easily obtain $3mk^2 =$ sum of the moments of inertia of three rods about AD

$$= m\frac{a^2}{3} + m\frac{a^2}{3} + ma^2 = \frac{5ma^2}{3} \Rightarrow k^2 = (5a^2/9)$$

$$\text{and } h = \frac{\left(m\dfrac{a}{2} + m\dfrac{a}{2} + ma\right)}{3m} = \frac{2ma}{3m} = \frac{2a}{3} \quad ...(2)$$

$$\Rightarrow (k^2/h) = (5a^2/9)/(2a/3) = \frac{5a}{6}$$

$\Rightarrow$ length of simple equivalent pendulum $= \dfrac{5a}{6}$.

Supplementary Problems

1. Find the length of the equivalent simple pendulum in the following cases, the axis being horizontal :

(i) Circular disc; axis a tangent to it. **Ans.** $(5a/4)$

(ii) Hemisphere; axis a diameter of the base. **Ans.** $(16a/15)$

(a) a diagonal of one face, (b) an edge.

Ans. (a) $(5a/3)$, (b) $4\sqrt{(2a/3)}$

2. Find the length of the simple equivalent pendulum for an elliptic lamina when the axis is a latus rectum. **[Meerut 74]**

Ans. $a\,[e+(\frac{1}{4}e)]$

3. A uniform wire, in the form of an arc of a circle of given radius, is swinging about a horizontal axis through the middle point of the arc perpendicular to the plane of the arc. Show that the time of a small oscillation is independent of the length of the arc, and length of an equivalent simple pendulum is equal to the diameter of the circle.

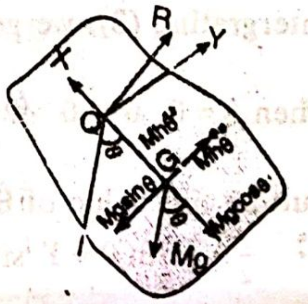
[Agra 1976, 69, Punjab 54]

4. Two bodies can move freely and independently under the action of gyration about the same horizontal axis, their masses are m, m' and the distance of their centres of gravity from the axis are h and h'. If the lengths of their equivalent simple pendulum be l, l', Prove that when fastened togther the length of the equivalent simple pendulum will be $\dfrac{mhl+m'h'l}{mh+m'h'}$.

2·08. Reactions of the axis of rotation.

A body moves about a fixed axis under the action of forces and both the body and the forces are symmetrical with respect to the plane through the C.G. perpendicular to the axis, find the reactions of the axis of rotation.

Let O be the point where the plane through G perpendicular to the axis of rotation meets this axis. By symmetry the actions on the axis reduce to a single force at O, the centre of suspension. Let the components of this single force be X and Y along and perpendicular to GO respectively.



Now G describes a circle round O as centre, its acceleration along and perpendicular to GO are $h\dot{\theta}^2$ and $h\ddot{\theta}$. Equations of motion of C.G. are

$$Mh\,\dot{\theta}^2 = X - Mg\cos\theta \qquad ...(1)$$

$$Mh\,\ddot{\theta} = Y - Mg\sin\theta \qquad ...(2)$$

By taking moments about O, $Mk^2\ddot{\theta} = -Mgh\sin\theta$ ...(3)

where k is the radius of gyration about the axis.

Y is obtained by eliminating $\ddot{\theta}$ from (2) amd (3), By integrating (3) and determining the constant from the initial conditions, and then from (1), we can find X.

Resultant reaction $R = \sqrt{(X^2 + Y^2)}$ and $\tan\phi = (X/Y)$
where ϕ is the angle which the direction of R makes with GO.

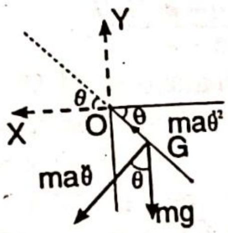
Note : On resolving X and Y horizontally and vertically.
The horizontal reaction $= X \sin\theta - Y \cos\theta$
Vertical reaction $= X \cos\theta + Y \sin\theta$

IIIUSTRATIVE EXAMPLES

Ex.27. *A thin uniform rod has one end attached to a smooth hinge and is allowed to fall from a horizontal position. Show that the horizontal strain on the hinge is greatest when the rod is inclined at an angle of 45° to the vertical, and that the vertical strain is then $\frac{11}{8}$ times the weight of the rod.*

[Meerut 1995]

Sol. Let $OA = 2a$, and let the rod make an angle θ with the horizontal after time t. Equations of motion of G along and perpendicular to GO are



$$ma\dot{\theta}^2 = Y \sin\theta + X \cos\theta - mg \sin\theta \quad ...(1)$$

$$ma\ddot{\theta} = -Y \cos\theta + X \sin\theta + mg \cos\theta \quad ...(2)$$

Since $k^2 = a^2 + \frac{a^2}{3} = \frac{4}{3}a^2$,

$\therefore$ moment equation about O is

$$m.\tfrac{4}{3}a^2\ddot{\theta} = mg.a \cos\theta \Rightarrow \ddot{\theta} = \frac{3g}{4a}\cos\theta. \quad ...(3)$$

Intergrating (3), we get $\dot{\theta}^2 = \frac{3g}{2a}\sin\theta + C$

when $\theta = 0$, $\dot{\theta} = 0$ $\therefore C = 0$, $\therefore$ $\dot{\theta}^2 = \frac{3g}{2a}\sin\theta$.

Putting this value of $\dot{\theta}^2$ in (1), we get

$$\tfrac{3}{2}mg \sin\theta = Y \sin\theta + X \cos\theta - mg \sin\theta$$

$$\Rightarrow Y \sin\theta + X \cos\theta = \tfrac{5}{2}mg \sin\theta \quad ...(4)$$

With the help of (3), the equation (2) becomes as

$$-Y \cos\theta + X \sin\theta + mg \cos\theta = \frac{3mg}{4}\cos\theta$$

$$\Rightarrow -Y \cos\theta + X \sin\theta = -\tfrac{1}{4}mg \cos\theta \quad ...(5)$$

Multiplying (4) by $\cos\theta$ and (5) by $\sin\theta$ and adding, we get

$$X = (\tfrac{5}{2} - \tfrac{1}{4})\, mg \sin\theta \cos\theta = \tfrac{9}{4}mg \sin\theta \cos\theta = \tfrac{9}{8}mg \sin 2\theta.$$

Similarly, we have $Y = mg\,(\tfrac{5}{2}\sin^2\theta + \tfrac{1}{4}\cos^2\theta)$.

We observe that X is maximum when $\sin 2\theta = 1$

i.e. when $2\theta = \dfrac{\pi}{2}$ or $\theta = \dfrac{\pi}{4}$.

when $\theta = (\pi/4)$, we have $Y = mg\ [\frac{5}{2}\sin^2(\pi/4) + \frac{1}{4}\cos^2(\pi/4)]$

$$= mg\ [\tfrac{5}{2}\cdot\tfrac{1}{2} + \tfrac{1}{4}\cdot\tfrac{1}{2}] = \tfrac{11}{8}\,mg = \tfrac{11}{8} \text{ times the weight of the rod.}$$

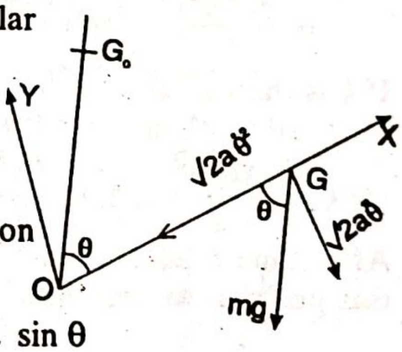
Ex.28. *A heavy homogeneous cube of weight W, can swing about an edge which is horizontal, it starts from rest being displaced from its unstable position of equilibrium. When the perpendicular from the centre of gravity upon the edge has turned through an angle θ, show that the components of the action at the hinge along and at right angles to this perpendicular are* $\frac{1}{2}W(3 - 5\cos\theta)$ *and* $\frac{1}{4}W\sin\theta$.

Sol. Let G_0 be the initial position of C.G. and G be the position of C.G. when the edge has turned through an angle θ.

$OG_0 = OG = \sqrt{(OL^2 + LG_0^2)} = \sqrt{(a^2 + a^2)} = a\sqrt{2}$,

where $2a$ is the length of the edge.

Equation of motion of G along and perpendicular to GO are



$$Ma\sqrt{2}\dot{\theta}^2 = mg\cos\theta - X \qquad \text{...(1)}$$

and $ma\sqrt{2}\ddot{\theta} = mg\sin\theta - Y$ ...(2)

where X, Y are the components of the reaction of the axis in this position.

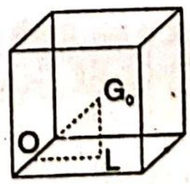
Moment equation about O is $mk^2\ddot{\theta} = mga\sqrt{2}\sin\theta$

$$\Rightarrow m\,(2a^2 + \tfrac{2}{3}a^2)\,\ddot{\theta} = \sqrt{2}amg\sin\theta \Rightarrow \ddot{\theta} = \frac{3}{8}\cdot\frac{\sqrt{2}}{a}\,g\sin\theta \qquad \text{...(3)}$$

Integrating, we get $\dot{\theta}^2 = -\dfrac{3}{4a}\sqrt{2}g\cos\theta + C$.

Initally $\dot{\theta} = 0$, when $\theta = 0$

$$\therefore \dot{\theta}^2 = \frac{3\sqrt{2}\,g}{4a}(1 - \cos\theta). \qquad \text{...(4)}$$



From (1) and (4), we have

$$\tfrac{3}{2}mg(1 - \cos\theta) = mg\cos\theta - X$$

$$\Rightarrow X = mg(\tfrac{3}{2}\cos\theta + \cos\theta - \tfrac{3}{2}) = \frac{mg}{2}(5\cos\theta - 3) = -\frac{mg}{2}(3 - 5\cos\theta)$$

$$= -\tfrac{1}{2}W(3 - 5\cos\theta) \qquad [\because mg = W],$$

where negative sign of X shows its opposite direction.

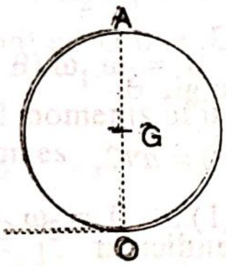
From (2) and (3), we have $\frac{3}{4} mg \sin\theta = mg\sin\theta - Y$

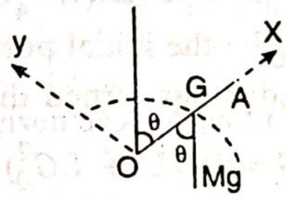
$\Rightarrow Y = mg\sin\theta - \frac{3}{4}mg\sin\theta = \frac{1}{4}mg\sin\theta.$

Ex.29. *A circular area can turn freely about a horizontal axis which passes through a point O of its circumference and is perp. to its plane. If motion commences when the diameter through O is vertically above, show that when the diameter has turned through an angle θ the components of the strain at O along and perp. to this diameter are respectively* $\frac{1}{3}W(7\cos\theta - 4)$ *and* $\frac{1}{3}W\sin\theta$. **[Agra 1986]**

Sol. Initially, when the diameter through O is vertically above O.

M.I. of the disc about an axis through O perp. to the disc





$$= M\frac{a^2}{2} + Ma^2$$

$$= \frac{3Ma^2}{2}.$$

If k is the radius of gyration, then

$$Mk^2 = \frac{3Ma^2}{2} \Rightarrow k^2 = \frac{3a^2}{2}$$

After time t, let the diameter OA makes an angle θ with the vertical. In this position we will have

$$Mk^2\frac{d^2\theta}{dt^2} = Mgh\sin\theta.$$

where h = distance of C.G. of the disc from $O = a$.

$$\therefore Mk^2\frac{d^2\theta}{dt^2} = Mga\sin\theta \Rightarrow \frac{3a^2}{2}\frac{d^2\theta}{dt^2} = ga\sin\theta \Rightarrow \frac{d^2\theta}{dt^2} = \frac{2g}{3a}\sin\theta.$$

Multiplying by $2\dot{\theta}$ on both sides and integrating it, we get

$$(d\theta/dt)^2 = -\frac{4g}{3a}\cos\theta + c.$$

Initially $\theta = 0$, $(d\theta/dt) = 0$. $\therefore 0 = -\frac{4g}{3a} + a \Rightarrow c = \frac{4g}{3a}$

Hence $\left(\frac{d\theta}{dt}\right)^2 = \frac{4g}{3a}(1 - \cos\theta)$. ...(2)

Now considering the motion of C.G., we have

$$M a\left(\frac{d\theta}{dt}\right)^2 = Mg\cos\theta - X \quad ...(3)$$

and $$M a\frac{d^2\theta}{dt^2} = M g\sin\theta - Y \quad ...(4)$$

where X, Y are the components of the reaction and perp. to GO.

Solving equation (3), we get

$$X = Mg\cos\theta - Ma\frac{4g}{3a}(1-\cos\theta) \Rightarrow X = \frac{Mg}{3}(7\cos\theta - 4)$$

$$= \frac{1}{3}W(7\cos\theta - 4) \quad ...(5)$$

Similarly, solving equations (1) and (4), we get

$$Y = Mg\sin\theta - Ma\frac{2g}{3a}\sin\theta = \frac{Mg}{3}\sin\theta = \frac{1}{3}W\sin\theta \quad ...(6)$$

Ex.30. *A circular disc of weight W can turn freely about a horizontal axis perp. to its plane which passes through a point O on its circumference. If it starts from rest with the diameter vertically above O, show that the resultant pressure on the axis when that diameter is horizontal and vertically below O are respectively* $\frac{1}{3}\sqrt{(17)}\,W$ *and* $\frac{11}{3}W$*, Further prove that the axis must be able to bear at least* $\frac{11}{3}$ *times the weight of the disc.*

Sol. This question is a particular case of the previous example.

When the diameter is horizontal, viz $\theta = \frac{\pi}{2}$, we have

$$X = \frac{W}{3}(0-4) = -\frac{4W}{3}, \quad Y = \frac{W}{3} \quad \left(\because \sin\frac{\pi}{2} = 1\right)$$

Hence resultant pressure in this case $= \sqrt{\left(\frac{16}{9}W^2 + \frac{W^2}{9}\right)} = \frac{W}{3}\sqrt{(17)}$

When the diameter is vertically below.

$$\theta = \pi, \quad \therefore X = \frac{W}{3}(-7-4) = -\frac{11W}{3}, \quad Y = \frac{1}{3}W\sin\pi = 0$$

Resultant pressure in this case $= \left\{\left(\frac{11W}{3}\right)^2 + 0\right\}^{1/2} = \frac{11}{3}W.$

in general, we have

$$\sqrt{(X^2+Y^2)} = \left[\left\{\frac{W}{3}(7\cos\theta - 4)\right\}^2 + \left\{\frac{W}{3}\sin\theta\right\}^2\right]^{1/2}$$

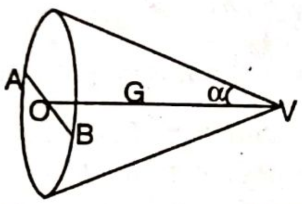
$$= \left\{\frac{W^2}{9}\left(48\cos^2\theta - 56\cos\theta + 17\right)\right\}^{1/2}$$

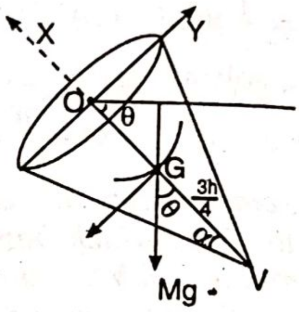
This is maximum when $\theta = \pi$ and its value is $\frac{11}{3}W$, which implies that the maximum pressure, that the axis must be able to bear is at least $\frac{11}{3}$ times the weight of the disc.

Ex.31 *A right cone of angle 2α can turn freely about an axis passing through the centre of its base and perpendicular to the axis; if the cone starts from rest with its axis horizontal, show that, when the axis is vertical, the thrust on the fixed axis is to the weight of the cone as* $1+\frac{1}{2}\cos^2\alpha : 1-\frac{1}{3}\cos^2\alpha$.

[Agra 1971]

Sol. Let initially the cone be as shown in fig.(i). After any time t, let the cone take the position as shown in fig.(ii). If the height of the cone i.e. $OV = h$ then $OG = \frac{1}{4}h$ where G denotes the centre of gravity of the cone.





Now since the C.G. of the cone i.e. point G is describing a circle of radius $h/4$, the equations of motion of G are

$$M.\frac{1}{4}h\,\dot{\theta}^2 = X - Mg\sin\theta \quad ...(1)\ ;\ M.\frac{1}{4}h\,\ddot{\theta} = Mg\cos\theta - Y \quad ...(2)$$

where X and Y denote the components of reaction at O along and perp. to OX. Taking moments about O, we have

$$Mk^2\ddot{\theta} = Mg\frac{1}{4}h\cos\theta \quad ...(3)$$

Also $Mk^2 =$ M.I. of the cone about $AB = M.\frac{1}{20}(2h^2 + 3h^2\tan^2\alpha)$

$$\Rightarrow k^2 = \frac{h^2}{20}(2 + 3\tan^2\alpha) \quad ...(4)$$

Substituting this value of k^2 in (3), we get

$$h\ddot{\theta} = \frac{5}{2+3\tan^2\alpha}g\cos\theta \quad ...(5)$$

Multiplying both sides by $2\dot{\theta}$ and integrating, we get

$$h\dot{\theta}^2 = \frac{10g}{2+3\tan^2\alpha}\sin\theta + C$$

Initially $\dot{\theta} = 0$, when $\theta = 0$, giving there by the constant $C = 0$.

Therefore, we have $h\dot{\theta}^2 = \dfrac{10g}{2+3\tan^2\alpha}\sin\theta$ ...(6)

Using (6) in (1), we get $M.\frac{1}{4}\dfrac{10g}{2+3\tan^2\alpha}\sin\theta = X - Mg\sin\theta$

$$\Rightarrow X = Mg \sin\theta \left(\frac{9 + 6\tan^2\alpha}{4 + 6\tan^2\alpha} \right)$$

Also using (5) in (2), we get $Y = Mg\cos\theta \left[\frac{3 + 6\tan^2\alpha}{8 + 12\tan^2\alpha} \right]$

When the axis is vertical i.e. when $\theta = \pi/2$, we have

$$X = Mg \left(\frac{9 + 6\tan^2\alpha}{4 + 6\tan^2\alpha} \right), Y = 0.$$

$\therefore$ Resultant pressure

$$= \sqrt{(X^2 + Y^2)} = X = Mg \left(\frac{9 + 6\tan^2\alpha}{4 + 6\tan^2\alpha} \right) = Mg \left(\frac{9\cos^2\alpha + 6\sin^2\alpha}{4\cos^2\alpha + 6\sin^2\alpha} \right)$$

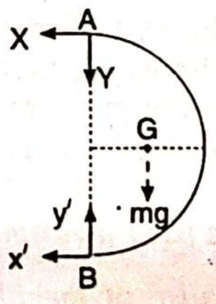
$$\Rightarrow \frac{X}{Mg} = \frac{6 + 3\cos^2\alpha}{6 - 2\cos^2\alpha} = \frac{1 + \frac{1}{2}\cos^2\alpha}{1 - \frac{1}{3}\cos^2\alpha}.$$

Note : If $2\alpha = \pi/2$ then in that case, we have

$$\frac{X}{Mg} = \frac{1 + \frac{1}{2}\cos^2(\pi/4)}{1 - \frac{1}{3}\cos^2(\pi/4)} = \frac{1 + \frac{1}{4}}{1 - \frac{1}{6}} = \frac{3}{2}.$$

Ex.32. *A uniform semi-circular arc, of mass m and radius a, is fixed at its ends to two points in the same vertical line and is rotating with constant angular velocity* ω. *Show that the horizontal thrust on the upper end is* $m\frac{g + \omega^2 a}{\pi}$. **[Meerut 1993]**

Sol. Let the uniform semi-circular arc with centre at O rotate about AB with constant angular velocity ω. If G is the C.G. of the arc, then $OG = \frac{2a}{\pi}$. As the arc rotate, the point G will describe a circle of radius $\frac{2a}{\pi}$ about the point O.



Let X and Y be the horizontal and vertical components of reactions at the point A and X' and Y' the horizontal and vertical reactions at the lower end B. Now since the arc is rotating with constant angular velocity ω about AB, the only effective force on it is $m\frac{2a}{\pi}\omega^2$ along GO.

Taking moments about the point B, we have

$$m\frac{2a}{\pi}\omega^2 a = -mg\frac{2a}{\pi} + X \cdot 2a$$

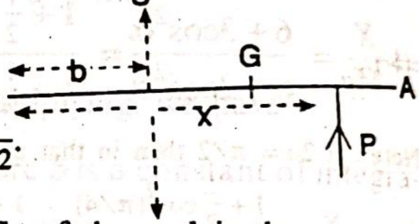
[$\because$ moment of the effective forces = moment of external forces].

$$\Rightarrow X = w\frac{(g + a\,\omega^2)}{\pi}.$$

Ex.33. *A uniform rod OA of mass M and length 2a rests on a smooth table and is free to turn about a smooth pivot at its end O; in contact with it at distance b from O is an inelastic particle of mass m, a horizontal blow of impulse P, is given to the rod at a distance x from O in a direction perp. to the rod ; find the resulting instantaneous angular velocity of the rod and the impulsive action at O and on the particle.* **[Agra 1994]**

Sol. Let OA be the rod of length $2a$ and let a horizontal blow of impulse P be given at a distance x from O. Further let S be the impulse of the action between the rod and inelastic particle of mass m. Then the moment equation about O is $M\,\frac{4}{3}a^2\omega = Px - Sb$ ...(1)

But $S = mb\,\omega$,(since velocity $b\,\omega$ is generated in mass m by the impulse S).



$$\therefore M\frac{4}{3}a^2\omega = Px - m\,b^2\omega$$

$$\Rightarrow \omega = \frac{Px}{\frac{4}{3}Ma^2 + m\,b^2} \text{ and } S = \frac{mPbx}{\frac{4}{3}Ma^2 + mb^2}.$$

Now since the change in the motion of C.G. of the rod is the same as if all the impulsive forces were applied there, so $Ma\,\omega = P - S - X$, where X is the impulsive action at O.

$$\therefore X = P - (Ma + mb)\,\omega = P\,[1 - (mb + Ma)\,x]/(M\tfrac{4}{3}a^2 + m\,b^2).$$

2·09. Motion about a fixed axis : Impulsive forces.

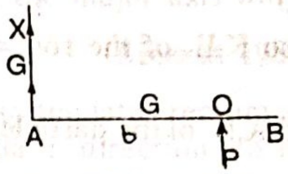
Consider a rigid body under the effect of impulsive forces. Let ω and ω' be the angular velocities about the axis just before and just after the action of impulsive forces. Now change in moment of momentum about the axis $= Mk^2(\omega' - \omega)$. Also let L the moment of external impulses about the axis of rotation, then we have $Mk^2(\omega' - \omega) = L$ (since change in moment of momentum of the body about the axis is equal to the moment of the impulsive forces about it).

Ex.34. *A rod, of mass m and length 2a, which is capable of free motion about one end A falls from a vertical position and when it is horizontal strikes a fixed inelastic obstacle at a distance b from the end A. Show that the impulse of the blow is* $m\,\frac{2a}{b}\sqrt{(2ga/3)}$ *and that the impulse of the reaction at A is* $m\,\sqrt{(3ga/2)}\left[1 - \frac{4a}{3b}\right]$ *vertically upwards.*

Sol. If ω is the angular velocity just before striking the obstacle then we have the energy equation as $\frac{1}{2}m\,.\,\frac{4}{3}a^2\omega^2 - 0 = mga$

[Change of K.E. = work done]. $\therefore \omega = \sqrt{(3g/2a)}$

Let the rod AB strike the inelastic obstacle at O such that $AO = b$ and the impulse of the blow be P and the impulsive reaction at A be X. Since the rod reduces to rest after striking the obstacle, therefore we get on taking moment about A



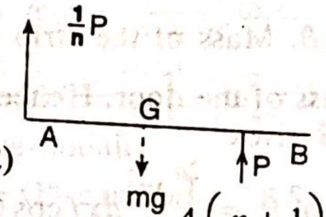
$$m\,\frac{4}{3}a^2(0-\omega) = -Pb$$

$$\Rightarrow P = \frac{4ma^2\omega}{3b} = \frac{2a}{b}m\,.\sqrt{(2ga/3)}$$

Also for G, we have $m\,(0-a\omega) = -P-X \Rightarrow X = m\sqrt{(3ga/2)}\left[1-\frac{4a}{3b}\right]$.

Ex.35. *A uniform beam AB can turn about its end A is in equlibrium; find the point of its length where a blow must be applied to it so that the impulses at A may be in each case $\frac{1}{n}$ th of that of the blow.*

Sol. Let AB be the uniform rod of mass m and length $2a$. Let an impulse P be applied at a distance x from A so as to produce an impulsive action $\frac{1}{n}P$ at A. If the angular velocity produced is ω, then the equations of motion are



$$m\,k^2\omega = Px \Rightarrow m\,\frac{4}{3}a^3\,\omega = Px \quad ...(1)$$

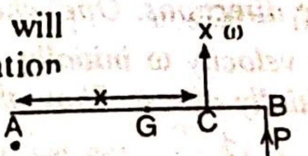
and $$m\,a\,\omega = P+\frac{1}{n}P = \frac{n+1}{n}P. \quad ...(2)$$

Eliminating P from these two equations, we get $x = \frac{4}{3}\left(\frac{n+1}{n}\right)a$

Note. If the direction of the impulsive action is opposite to that as shown in the fig., then in that case we will have $x = \frac{4}{3}\left(\frac{n-1}{n}\right)a$.

Ex.36. *A rod of mass nM is lying in a horizontal table and has one end fixed; a particle of mass M is in contact with it. The rod receives a horizontal blow at its free end; find the position of the particle so that it may start moving with the maximum velocity. In this case show that the kinetic energies communicated to the rod and mass are equal.*

Sol. Let AB be the rod, the end A of which is fixed. Let an impulse P be applied to the rod at the end B so as to give an angular velocity ω, if the particle of mass M is at C where $AC = x$ then the velocity V acquired by the particle will be $V = x\omega$. Thus we get the moment equation as $nM\,\frac{4}{3}a^2\omega + Mx\omega\,.\,x = P\,.\,2a$



$$\Rightarrow \omega = \frac{2aP}{M\frac{4}{3}(a^2n+x^2)}\,. \qquad \therefore V = x\omega = \frac{2aPx}{M\frac{4}{3}(na^2+x^2)}$$

For maximum V, we must have $\frac{dV}{dx} = 0 \Rightarrow \frac{2a\,P}{M}\left[\frac{\frac{4}{3}na^2 + x^2 - 2x^2}{(\frac{4}{3}na^2 + x^2)^2}\right] = 0$

$\Rightarrow \frac{4}{3}na^2 - x^2 = 0 \Rightarrow x = 2a\sqrt{(n/3)}$

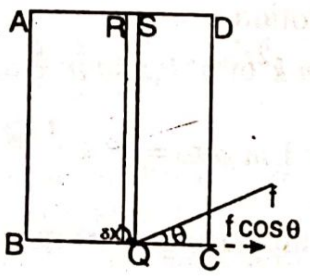
Also K.E. of the rod $= \frac{1}{2}n\,M\,.\frac{4}{3}a^2\omega^2 = \frac{2}{3}n\,M\,a^2\omega^2$ ...(1)

and K.E. of the particle $= \frac{1}{2}M\,x^2\,\omega^2 = \frac{1}{2}M\,\frac{4a^2\,n}{3}\,\omega^2 = \frac{2}{3}M\,na^2\,\omega^2.$ ...(2)

From (1) and (2), we observe that kinetic energies of the rod and mass are equal.

Ex.37. *The door of a railway carriage stands upon at right angles to the length of the train when the latter starts to move with an acceleration f; the door being supposed to be smoothly hinged to the carriage and to be uniform and of breadth 2a, show that its angular velocity when it has turned through an angle θ is* $\sqrt{\left(\frac{3f}{2a}\sin\theta\right)}$.

Sol. Let $ABCD$ be the door which can rotate about AB. If the train moves with acceleration f, then every element of the door will have the same acceleration f parallel to the rails. Now consider an elementary strip $PQRS$ at a distance x from AB. Mass of the strip $= \frac{M}{2a}\delta x$, where M is the mass of the door. Hence moment equation about AB gives



$$M\frac{4}{3}a^2\ddot{\theta} = \int_0^{2a}\frac{m}{2a}\,dx\,f\cos\theta\,x = m\,a\,f\cos\theta$$

$$\Rightarrow \ddot{\theta} = \frac{3f}{4a}\cos\theta$$

Multiplying both sides by $2\dot{\theta}$ and integrating it, we get

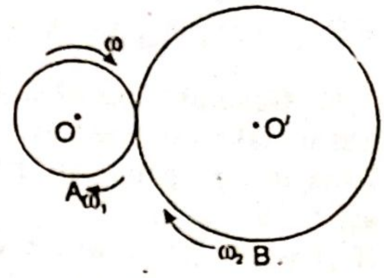
$$\dot{\theta}^2 = \frac{3f}{2a}\sin\theta + \lambda.$$ Initially $\dot{\theta} = 0$ when $\theta = 0 \;\therefore\; \lambda = 0$

Hence $\dot{\theta} = \sqrt{\left(\frac{3f}{2a}\sin\theta\right)}$.

Ex.38. *Two wheels on spindles in fixed bearings suddenly engage so that their angular velocities become inversely proportional to their radii and in opposite directions. One wheel, of radius a and moment of inertia I_1 has angular velocity ω initially, the other of radius b and moment of inertia I_2 is initially at rest. Show that their new angular velocities are*

$$\frac{I_1\,b^2}{I_1\,b^2 + I_2\,a^2}\,\omega \quad\text{and}\quad \frac{I_1\,ab\,\omega}{I_1\,b^2 + I_2\,a^2}.$$

Sol. Let A and B be the two wheels. The wheel A is of radius a and moment of inertia I_1 whereas the wheel B is of radius b and moment of inertia I_2. Initially A was rotating with angular velocity ω and the wheel B was at rest. Now let ω_1 and ω_2 be the angular velocity of A and B after the impact. Since the velocity of the point of contact is the same for each wheel, we have $a\,\omega_1 = b\,\omega_2$ ...(1)



Also $I_1(\omega - \omega_1) = R \times a$ (for the wheel A) ...(2)

$I_2(\omega - 0) = R \times b$ (for the wheel B) ...(3)

where R is the impulsive force.

From the last two equations, we get $I_1(\omega - \omega_1)\,b = I_2\,a\,\omega_2$ ...(4)

Now substituting the value of ω_2 from (1) in (4), we get

$$\omega_1 = \frac{I_1 b^2}{I_1 b^2 + I_2 a^2}\,\omega$$

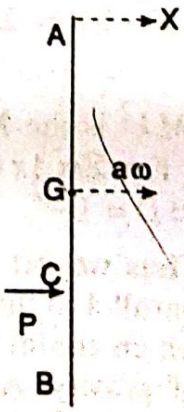
Substituting the value of ω_1 in (1), we have $\omega_2 = \dfrac{I_1 a b}{I_1 b^2 + I_2 a^2}\,\omega.$

2·10. Centre Of Percussion : **[Meerut 1990,94,93,82,80,76]**

If a body, rotating about a given axis, is so struck that there is no impulsive pressure on the axis, then any point on the line of action of the force is called a *centre of percussion*. If the line of action of the blow is known, the axis about which the body begins to turn is called the axis of *spontaneous rotation*. Obviously this combines with the position of the fixed axis in the first case.

2·11. Centre of Percussion of a rod : **[Meerut 1996,75]**

Consider a rod AB of length $2b$. Let it be suspended freely from one end A. Let a horizontal blow of impulse P be applied to it at the point C where $AC = x$. If X is the impulsive action at A and ω the angular velocity communicated to the rod, then the equations of motion are



$$M k^2 \omega = P x \quad (moment\ eq^n) \quad ...(1)$$

$$M(a\omega - 0) = P + X \quad ...(2)$$

where $a\,\omega$ is the velocity with which G moves.

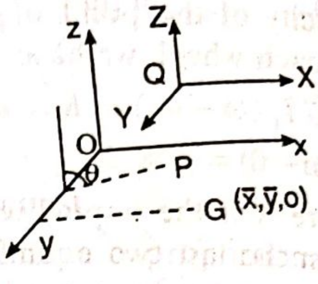
Now if the blow has been given through the centre of percussion then $X = 0$ and equation (2) becomes $M a\omega = P$.

Substituting this value of P in (1), we get

$$x = \frac{k^2}{a} = \text{length of the equivalent simple pendulum.}$$

2·12. General Case of Centre of Percussion :

Let us take the fixed axis as the axis of y. Also let centre of gravity G lie in the xy– plane, so that coordinates of G are $(\bar{x}, \bar{y}, 0)$.



If Q is the point where the blow is applied then take a plane through Q and perp. to xy-plane as the xz-plane so that coordinates of Q may be $(\xi, 0, \zeta)$. Now consider any other point P of mass m of the body at a distance r from Oy at any angle θ with z-axis. The coordinates of P will be

$$x = r\sin\theta,\ y = const.,\ z = r\cos\theta.$$

If before the blow, angular velocity is ω and the velocity component along the axes are u, v, w respectively, then we have

$$\dot{x} = u = r\cos\theta,\ \dot{\theta} = z\,\omega,\ \dot{y} = v = 0,\ \dot{z} = w = -r\sin\theta\,.\,\dot{\theta} = -x\,\omega.$$

If after the blow, the angular velocity is ω and velocity component along the axes becomes as u', v', w', then

$$u' = z\,\omega',\ v' = 0,\ w' = -x\,\omega'.$$

If X, Y, Z are the components of the blow at the point Q, then equations of motion will be

$$X = \Sigma m(u' - u) = \Sigma mz(\omega' - \omega) = (\omega' - \omega)\Sigma mz$$
$$= (\omega' - \omega)\bar{z}\,\Sigma m = M(\omega' - \omega)\,\bar{z} = 0 \quad (\text{since } \bar{z} = 0) \quad \text{...(1)}$$
$$Y = \Sigma m\,(v' - v) = 0 \qquad (\text{since } v' = 0 \text{ and } v = 0) \quad \text{...(2)}$$
$$Z = \Sigma m(w' - w) = -(\omega' - \omega)\Sigma mx = -(\omega' - \omega)\,\bar{x}\,\Sigma m$$
$$= -M(\omega' - \omega)\,\bar{x} \quad \text{...(3)}$$
$$-Y\zeta = \Sigma m\,\{y(w' - w) - z(v' - v)\} = -(\omega' - \omega)\Sigma mxy$$
$$= -(\omega' - \omega)\,F \quad \text{...(4)}$$
$$\Rightarrow F = 0 \qquad (\because\ Y = 0)$$
$$\zeta X - \xi Z = \Sigma m\,\{z(u' - u) - x(w' - w)\}$$
$$= -(\omega' - \omega)\,\Sigma m\,(z^2 + x^2) = M k^2(\omega' - \omega) \quad \text{...(5)}$$

[Mk^2 is the M.I. of the body about y– *axis*]

$$\xi Y = \Sigma m\,\{x(v' - v) - y(u' - u)\} = -(\omega' - \omega)\,\Sigma mzx = -(\omega' - \omega)\,D$$
$$\Rightarrow D = 0 \qquad [\because\ Y = 0] \quad \text{...(6)}$$

Thus we get $X = 0$, $Y = 0$, which implies that blow has no components parallel to the axes of x and y. Hence the blow must be perp. to xy– plane which contains the fixed axis and the instantaneous position of the centre of gravity. Also we see that $F = 0$ and $D = 0$ which implies that the y– *axis* which is also the axis of the body is a principal axis at the point where the plane through the line of action of the blow perp. to the fixed axis cuts it. This is a necessary condition for the existence of the centre of percussion. So if the fixed axis is not a principal axis at some point, then there is no centre of percussion.

Using equation (3) and (5), we get $\xi = \dfrac{k^2}{x}$ ...(7)

The obvious conclusion from the relation (7) is that the distance of the centre of percussion from the fixed axis is the same as that of the centre of oscillation.

Points to remember in finding out the centre of percussion of a body for fixed axis.

(i) *Find the point where the fixed axis is principal axis.*

(ii) *Take a distance* $\dfrac{k^2}{x}$.

(iii) *Draw an axis perp. to the plane containing the fixed axis and C.G. at a distance* $\dfrac{k^2}{x}$ *below the point where fixed axis is principal axis.*

(iv) *Any point on this line is a centre of percussion of the body for the fixed axis.*

Ex.39. *A pendulum is constructed of a solid sphere of mass M and radius a which is attached to the end of a rod of mass m and length b. Show that there will be no strain on the axis if the pendulum be struck at a distance* $\left[M.\left\{\frac{2}{5}a^2+(a+b)^2\right\}+\frac{1}{3}mb^2\right]\div\left[M(a+b)+\frac{1}{2}mb\right]$ *from the axis.*

Sol. Let $OA = b$ be the rod fixed at the point O. Let a sphere of radius a and mass M be attached to the other end A of the rod.

Distance of the C.G. of the pendulum from O

$$h = \frac{m\,(b/2) + M\,(b+a)}{m+M} \quad ...(1)$$

Let k be the radius of gyration of the pendulum about O, then we have

$$(m+M)\,k^2 = M\left[(b+a)^2 + \tfrac{2}{5}a^2\right] + \tfrac{4}{3}m\left(\frac{b}{2}\right)^2$$

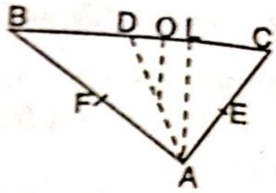
$$\Rightarrow k^2 = \frac{M}{m+M}\left[(b+a)^2 + \frac{2}{5}a^2\right] + \frac{4m}{3(m+M)}\left(\frac{k}{2}\right)^2$$

$\therefore$ Distance of centre of percussion from $O = \dfrac{k^2}{h}$

$$= \frac{M\left[\frac{2}{5}a^2 + (a+b)^2\right] + \frac{1}{3}mb^2}{\frac{1}{2}mb + M\,(a+b)}.$$

Ex.40. *Find the centre of percussion of a triangle ABC which is free to move about its side BC.* **[Meerut 1983,80,75]**

Sol. To find out the point where BC is a principal axis. Let us proceed like this. Draw AD, the median and AL the perp. from A on BC. Let O be the mid point of DL. Then, by the elementary knowledge of M.I. and P.I., BC is a principal axis at the point O. Let the mass of the ΔABC be m. The triangle of mass



m is kinetically equivalent to the particles each of mass $\frac{m}{3}$ placed at the mid points $D, E,$ and F. Let $AL = p$, then

$$m\,k^2 = \frac{m}{3}\left(\tfrac{1}{2}p\right)^2 + \frac{m}{3}\left(\tfrac{1}{2}p\right)^2 + \frac{m}{3}(0) = \tfrac{1}{6}m\,p^2 \Rightarrow k^2 = \tfrac{1}{6}p^2.$$

But the depth of C.G. below $BC = h = \frac{1}{3}p$.

Hence depth of the centre of percussion below BC along a vertical through $O = (k^2/h) = \frac{1}{2}p$.

Particular Case : If the triangle ABC is an equilateral triangle, then the point D and O coincide. In this case $k^2 = \frac{1}{6}p^2, h = \frac{1}{3}p$

Hence the depth of the centre of percussion below BC along the median bisecting BC is $= \frac{k^2}{h} = \frac{1}{2}p$.

Ex.41. *Find how an equilateral lamina must be struck that it may commence to rotate about a side.* **[Meerut 1974,73]**

Sol. Refer fig. **Ex.40.** The triangle ABC rotate about the side BC. The blow should be given at the centre of percussion when BC is the axis of rotation of the lamina. Here BC is the principal axis of triangle at its middle point (points D, O, L will coincide).

Again $k^2 = \frac{1}{6}p^2$; $h = \frac{1}{3}p$ where p is the height of the triangle.

$\therefore$ Depth of the centre of percussion below BC along the median bisecting BC is $\frac{k^2}{h}$ *i.e.* $\frac{1}{2}p$. Hence the blow should be given at the middle point of the median bisecting the side about which the lamina rotates.

Ex.42. *Find the position of the centre of percussion of a sector of a circle, axis in the plane of the sector, perp. to its symmetrical radius and passing through the centre of the circle.*

Sol. Consider the sector AOB of a circle of radius a. Let $\angle AOB = 2\alpha$. Let a line OY perp. to the plane of the sector be the axis of rotation.

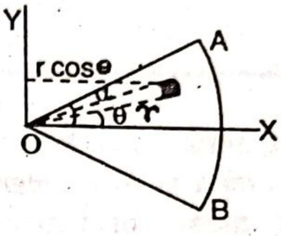
Then M.I. of the sector AOB about $OY = 2\int_0^{\alpha}\int_0^{\alpha} \rho\, r^2\cos^2\theta\, r\, d\theta\, dr$

$$= \rho \cdot \frac{a^4}{4} \int_0^{\alpha} (1 + \cos 2\theta)\, d\theta = \rho \cdot \frac{a^4}{4} (\alpha + \sin \alpha \cos \alpha)$$

$$= \frac{M a^2}{4\alpha} (\alpha - \sin \alpha \cos \alpha), \text{ since mass of the sector } M = \rho \,.\, \alpha a^2$$

$$\therefore M k^2 = \frac{M a^2}{4\alpha} (\alpha + \sin \alpha \cos \alpha) \Rightarrow k^2 = \frac{a^2}{4\alpha} (\alpha + \sin \alpha \cos \alpha)$$

Distance of C.G. from $O = h = \frac{2a}{3} \cdot \frac{\sin \alpha}{\alpha}$



Hence distance of centre of percussion from O

$$= \frac{k^2}{h} = \frac{3a}{8} \left(\frac{\alpha + \sin \alpha \cos \alpha}{\sin \alpha} \right).$$

☆☆☆