ANALYTIC GEOMETRY

: CSE - 2010 :

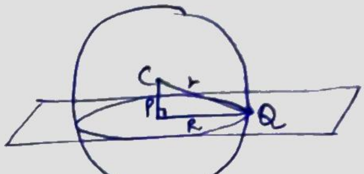
① (e) Show that the plane $x+y-2z=3$ cuts the sphere $x^2+y^2+z^2-x+y=2$ in a circle of radius 1 & find the equation of sphere which has this circle as a great circle.

$\longrightarrow$ Centre of the given sphere is $c\left(\frac{1}{2}, -\frac{1}{2}, 0\right)$

Radius of the given sphere is $r = \sqrt{\frac{1}{4}+\frac{1}{4}+2} = \sqrt{5/2}$

Distance of plane given from the centre of the sphere c is

$$p = \frac{\left|1\cdot\frac{1}{2} + 1\cdot\left(\frac{-1}{2}\right) + (-2)\cdot 0 - 3\right|}{\sqrt{1^2+1^2+(-2)^2}} = \frac{3}{\sqrt{6}}$$



Then, radius of circle in which the given plane cuts the given sphere is

$$R^2 = r^2 - p^2 = \frac{5}{2} - \frac{9}{6} = 1$$

$R = 1$.

$\therefore$ The given plane cuts the given sphere in a circle of radius 1.

Any ~~circle~~ sphere through the circle of intersection of the plane & the sphere is given by

$$x^2+y^2+z^2-x+y-2 + \lambda(x+y-2z-3) = 0 \quad \text{——} ①$$

$$\Rightarrow x^2+y^2+z^2+(\lambda-1)x + (\lambda+1)y - 2\lambda z - (2+3\lambda) = 0$$

Its centre is $\left(\frac{1}{2}(1-\lambda), \frac{-1}{2}(1+\lambda), \lambda\right)$.

Since the plane cuts the sphere in the great circle, its centre lies on the given plane. Then,

$$x+y-2z = 3 \Rightarrow \frac{1}{2}(1-\lambda) - \frac{1}{2}(1+\lambda) - 2\lambda = 3$$

$$\Rightarrow -3\lambda = 3 \Rightarrow \lambda = -1$$

$$\therefore ① \equiv x^2+y^2+z^2-x+y-2-(x+y-2z-3) = 0$$

$$\Rightarrow x^2+y^2+z^2-2x+2z+1 = 0$$ which is the reqd sphere

② (c) Show that the plane $3x+4y+7z+\frac{5}{2}=0$ touches the paraboloid $3x^2+4y^2=10z$ and find the point of contact.

→ Tangent plane to the given paraboloid at any point (α,β,γ) on it is

$3\alpha x+4\beta y=5(\gamma+z) \Rightarrow 3\alpha x+4\beta y-5z-5\gamma=0$ —— ①

The given plane is $3x+4y+7z+\frac{5}{2}=0$ —— ②

If the plane ② is tangent plane to the paraboloid at (α,β,γ), then the planes ① & ② are the same. Then

$$\frac{3\alpha}{3}=\frac{4\beta}{4}=\frac{-5}{7}=\frac{-5\gamma}{5/2}$$

$\Rightarrow \alpha=-\frac{5}{7},\quad \beta=-\frac{5}{7},\quad \gamma=\frac{5}{14}$

Putting $\left(-\frac{5}{7},-\frac{5}{7},\frac{5}{14}\right)$ in the LHS of paraboloid equation $3x^2+4y^2-10z=0$, we have

$$3\left(-\frac{5}{7}\right)^2+4\left(-\frac{5}{7}\right)^2-\frac{10\times5}{14}=\frac{25}{49}\times3+\frac{25\times4}{49}-\frac{25}{7}$$

$$=\frac{25}{7}-\frac{25}{7}=0.$$

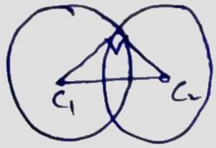
∴ the point lies on paraboloid.

∴ The given plane touches the paraboloid at the point

$\left(-\frac{5}{7},-\frac{5}{7},\frac{5}{14}\right)$

3 (c) Show that every sphere through the circle $x^2+y^2-2ax+r^2=0$, $z=0$ cuts orthogonally every sphere through the circle $x^2+z^2=r^2,\ y=0$

→ Any sphere through the circle $x^2+y^2-2ax+r^2=0$, $z=0$ is



$x^2+y^2+z^2-2ax+\lambda_1 z+r^2=0.$

Its centre is $C_1\left(a,0,-\frac{\lambda_1}{2}\right)$

Its radius is $r_1=\sqrt{a^2+\frac{\lambda_1^2}{4}-r^2}$

Any sphere through the circle $x^2+z^2=r^2,\ y=0$ is

$x^2+y^2+z^2+\lambda_2 y=r^2$

Its centre is $C_2(0, -\frac{\lambda_2}{2}, 0)$ & radius is $r_2 = \sqrt{\frac{\lambda_2^2}{4} + r^2}$

For the two spheres to cut orthogonally, the squared distance b/w their centres is equal to the sum of square of radius r_1 & r_2

$\therefore (C_1C_2)^2 = r_1^2 + r_2^2 \Rightarrow C_1C_2^2 = (a-0)^2 + \left(0 - \frac{\lambda_2}{2}\right)^2 + \left(\frac{\lambda_1}{2} - 0\right)^2$

$$= a^2 + \frac{\lambda_2^2}{4} + \frac{\lambda_1^2}{4}.$$

$$C_1C_2^2 = \frac{4a^2 + \lambda_1^2 + \lambda_2^2}{4}.$$

$$r_1^2 + r_2^2 = a^2 + \frac{\lambda_1^2}{4} - r^2 + \frac{\lambda_2^2}{4} + r^2$$

$$= \frac{4a^2 + \lambda_1^2 + \lambda_2^2}{4}.$$

$\therefore C_1C_2^2 = r_1^2 + r_2^2$

$\therefore$ Each sphere through the two given systems cut each other orthogonally

(4)(c) Find the vertices of the skew quadrilateral formed by the four generators of the hyperboloid $\frac{x^2}{4} + y^2 - z^2 = 49$ passing through $(10,5,1)$ and $(14,2,-2)$.

$\rightarrow$ Generators of the hyperboloid of the λ & μ system are

$$\left(\frac{x}{2} - z\right) = \lambda(7-y) \Big\} \text{①} \quad \left(\frac{x}{2} + z\right) = \frac{1}{\lambda}(7+y) \Big\} \text{③}$$

$$\left(\frac{x}{2} - z\right) = \mu(7+y) \Big\} \text{②} \quad \left(\frac{x}{2} + z\right) = \frac{1}{\mu}(7-y) \Big\} \text{④}$$

They pass through $(10,5,1)$

$$\left(\frac{10}{2} - 1\right) = \lambda(7-5) \quad \& \quad \left(\frac{10}{2} - 1\right) = \mu(7+1)$$

$$\Rightarrow \lambda = 2 \quad \& \quad \mu = \frac{1}{3}$$

They also pass through $(14, 2, -2)$. Therefore,

$\left(\frac{14}{2}+2\right) = \lambda(7-2)$ & $\left(\frac{14}{2}+2\right) = \mu(7+2)$

$\Rightarrow \lambda = \frac{9}{5}$, $\mu = 1$

$\therefore \lambda = 2, \frac{9}{5}$, $\mu = \frac{1}{3}, 1$

The given hyperboloid can be rewritten as $\frac{x^2}{(14)^2} + \frac{y^2}{7^2} - \frac{z^2}{7^2} = 1$

$\Rightarrow a = 14, b = 7, c = 7$

Then, the points of intersection of generators are given by

$$\left(\frac{a(1+\lambda\mu)}{\lambda+\mu}, \frac{b(\lambda-\mu)}{\lambda+\mu}, \frac{c(1-\lambda\mu)}{(\lambda+\mu)}\right)$$

(i) $\lambda = 2, \mu = \frac{1}{3} \Rightarrow \left(14\frac{(1+2/3)}{2+1/3}, 7\frac{(2-1/3)}{2+1/3}, 7.\frac{(1-2/3)}{(2+1/3)}\right)$

$= (10, 5, 1)$

(ii) $\lambda = 2, \mu = 1 : \Rightarrow \left(14\frac{(1+2)}{2+1}, 7\frac{(2-1)}{2+1}, 7\frac{(1-2)}{2+1}\right)$

$= \left(14, \frac{7}{3}, \frac{-7}{3}\right)$

(iii) $\lambda = \frac{9}{5}, \mu = \frac{1}{3} \Rightarrow \left(14.\frac{(1+\frac{9}{5}\times\frac{1}{3})}{\frac{9}{5}+\frac{1}{3}}, 7\frac{(\frac{9}{5}-\frac{1}{3})}{\frac{9}{5}+\frac{1}{3}}, 7\frac{(1-\frac{9}{5}.\frac{1}{3})}{\frac{9}{5}+\frac{1}{3}}\right)$

$= \left(\frac{21}{2}, \frac{77}{16}, \frac{21}{16}\right)$

(iv) $\lambda = \frac{9}{5}$ & $\mu = 1 : \Rightarrow \left(14\frac{(1+\frac{9}{5})}{1+9/5}, 7\frac{(\frac{9}{5}-1)}{\frac{9}{5}+1}, \frac{7(1-9/5)}{\frac{9}{5}+1}\right)$

$= (14, 2, -2)$

$\therefore$ The four vertices of quadrilateral are $(10, 5, 1)$, $\left(14, \frac{7}{3}, \frac{-7}{3}\right)$, $\left(\frac{21}{2}, \frac{77}{16}, \frac{21}{16}\right)$ and $(14, 2, -2)$.