IFoS – 2019 → paper II

5) (b) The following table gives the values of $y = f(x)$ for certain equidistant values of x. Find the value of $f(x)$ when $x = 0.612$ using Newton's forward difference interpolation formula.

x	0.61	0.62	0.63	0.64	0.65
$y=f(x)$	1.840431	1.858928	1.877610	1.896481	1.915541

⇒

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.61	1.840431				
		0.018497			
0.62	1.858928		0.000185		
		0.018682		0.000004	
			0.000189		−0.000004
0.63	1.877610			0.000000	
		0.018871			
			0.000189		
0.64	1.896481				
		0.019060			
0.65	1.915541				

Here, $x = 0.612$, $x_0 = 0.61$, $h = 0.01$

$$\therefore u = \frac{0.612 - 0.61}{0.01} = 0.2$$

Now, by the Newton Forward formula,

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0$$

$$\Rightarrow f(0.612) = 1.840431 + 0.2 \times 0.018497 + \frac{0.2\times(-0.8)}{2}\times 0.000185 + \frac{0.2\times(-0.8)(-1.8)}{6}\times 0.000004 + \frac{0.2\times(-0.8)(-1.8)(-2.8)}{24}\times(-0.000004)$$

$$= 1.840431 + 0.0036994 - 0.0000148 + 0.00000019 - 0.00000013$$

$$= 1.84378620$$

5) (c) Following values of x_i and the corresponding values of y_i are given. Find $\int_0^3 y\,dx$ using Simpson's one-third rule.

x_i	0.0	0.5	1.0	1.5	2.0	2.5	3.0
y_i	0.0	0.75	1.0	0.75	0.00	−1.25	−3.0

⇒

x_i $i=0$ to 6	y_i $i=0$ to 6	y_i $i=0,6$	y_i $i=1,3,5$	y_i $i=2,4$
$x_0=0.0$	0.00	0.00	—	—
$x_1=0.5$	0.75	—	0.75	—
$x_2=1.0$	1.00	—	—	1.00
$x_3=1.5$	0.75	—	0.75	—
$x_4=2.0$	0.00	—	—	0.00
$x_5=2.5$	−1.25	—	−1.25	—
$x_6=3.0$	−3.00	−3.00	—	—
		$\Sigma y_i=-3.00(=Y_1)$	$\Sigma y_i=0.25(=Y_2)$	$\Sigma y_i=1.00(=Y_3)$

here $h=\frac{3-0}{6}=\frac{1}{2}=0.5$

∴ Now by Simpson's one-third rule,

$$\int_0^3 y\,dx=\frac{h}{3}\left[y_0+y_6+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$$=\frac{1}{6}\left[Y_1+4Y_2+2Y_3\right]$$

$$=\frac{1}{6}\left[-3.00+(4\times0.25)+(2\times1.00)\right]$$

$$=\frac{1}{6}\times(-3+1+2)$$

$$=0$$

6) (b) Solve the following system of equations by Gauss-Jordan elimination method:

$$x_1 + x_2 + x_3 = 3$$
$$2x_1 + 3x_2 + x_3 = 6$$
$$x_1 - x_2 - x_3 = -3$$

$\Rightarrow$ The augmented matrix is,

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{c|ccc|cc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & -2 & -2 & -1 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|ccc|c} 1 & 0 & 2 & 3 & -1 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -4 & -5 & 2 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 1 & 0 & -3/4 & 1/2 & -1/4 \\ 0 & 0 & 1 & +5/4 & -2/4 & -1/4 \end{array}\right]$$

Thus,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{3}{4} & -\frac{1}{2} & -\frac{1}{4} \\ \frac{5}{4} & -\frac{2}{4} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 3/2 \\ 3/2 \end{bmatrix}$$

$\therefore$ $x_1 = 0$, $x_2 = 3/2$, $x_3 = 3/2$ be the solution of the given system of equations.

7) (a) Given $\frac{dy}{dx} = x^2 + y^2$, $y(0) = 1$. Find $y(0.1)$ and $y(0.2)$ by fourth order Runge-Kutta method.

$\Rightarrow$ Here $x_0 = 0$, $y_0 = 1$, $f(x,y) = x^2 + y^2$, $h = 0.1$

Thus,

$$K_1 = hf(x_0, y_0) = 0.1 \times f(0,1) = 0.1$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = 0.1 f(0.05, 1.05) = 0.1105$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = 0.1 f(0.05, 1.05525) = 0.111605$$

$$K_4 = hf(x_0 + h, y_0 + K_3) = 0.1 f(0.1, 1.111605) = 0.124567$$

$$\therefore y(0.1) = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$= 1 + \frac{1}{6}(0.668777) = 1.111463$$

Now for $y(0.2)$, $x_1 = 0.1$, $y_1 = 1.111463$

$$K_1 = hf(x_1, y_1) = 0.1 f(0.1, 1.111463) = 0.124535$$

$$K_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}\right) = 0.1 f(0.15, 1.173730) = 0.140014$$

$$K_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}\right) = 0.1 f(0.15, 1.181470) = 0.141837$$

$$K_4 = hf(x_1 + h, y_1 + K_3) = 0.1 f(0.2, 1.25330) = 0.161076$$

$$\therefore y(0.2) = y_1 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$

$$= 1.111463 + \frac{1}{6}(0.849313)$$

$$= 1.253015$$

8) (c) Use Gauss quadrature formula of point six to evalute $\int_0^1 \frac{dx}{1+x^2}$ given

$x_1 = -0.23861919$, $\omega_1 = 0.46791393$

$x_2 = -0.66120939$, $\omega_2 = 0.36076157$

$x_3 = -0.93246951$, $\omega_3 = 0.17132449$

$x_4 = -x_1$, $x_5 = -x_2$, $x_6 = -x_3$, $\omega_4 = \omega_1$, $\omega_5 = \omega_2$, $\omega_6 = \omega_3$

$\Rightarrow$ For doing Gauss quadrature formula, we have to convert the limits -1 to 1.

Here, $f(x) = \frac{1}{1+x^2}$

We know that,

$$\int_a^b f(x)\,dx = \frac{(b-a)}{2}\int_{-1}^{1} f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right)dx$$

$$\therefore \int_0^1 f(x)\,dx = \frac{1-0}{2}\int_{-1}^{1} f\left(\frac{1-0}{2}x + \frac{1+0}{2}\right)dx$$

$$= \frac{1}{2}\int_{-1}^{1} f(0.5x + 0.5)\,dx$$

Given,

$x_1 = -0.23861919$, $\omega_1 = 0.46791393$

$x_2 = -0.66120935$, $\omega_2 = 0.36076157$

$x_3 = -0.93246951$, $\omega_3 = 0.71324490$

$x_4 = +0.23861919$, $\omega_4 = 0.46791393$

$x_5 = +0.66120935$, $\omega_5 = 0.36076157$

$x_6 = +0.93246951$, $\omega_6 = 0.7132449$

Now for Gauss quadrature's six point formula,

$$\int_{-1}^{1} f(x) = \omega_1 f(x_1) + \omega_2 f(x_2) + \omega_3 f(x_3) + \omega_4 f(x_4) + \omega_5 f(x_5) + \omega_6 f(x_6)$$

$$\therefore \frac{1}{2}\int_{-1}^{1} f(x) = \frac{1}{2}\Big[0.46791393 \times f(0.5\times -0.23861919 + 0.5)$$
$$+ 0.36076157 f(0.5\times -0.66120939 + 0.5) + 0.17132449 f(0.5\times -0.93246951 + 0.5)$$
$$+ 0.46791393 f(0.5\times 0.23861919 + 0.5) + 0.36076157 f(0.5\times 0.66120939 + 0.5)$$
$$+ 0.17132449 f(0.5\times 0.93246951 + 0.5)\Big]$$

$$= \frac{1}{2}\Big[0.46791393 f(0.380690405) + 0.36076157 f(0.169395305)$$
$$+ 0.17132449 f(0.033766524) + 0.46791393 f(0.619309595)$$
$$+ 0.36076157 f(0.830604695) + 0.17132449 f(0.966234755)\Big]$$

$$= \frac{1}{2}\Big[0.46791393 \times 0.8734195156 + 0.36076157 \times 0.97210565$$
$$+ 0.17132449 \times 0.9988611204 + 0.46791393 \times 0.7227812989$$
$$+ 0.36076157 \times 0.5917495347 + 0.17132449 \times 0.5171674774\Big]$$

$$= \frac{1}{2}\Big[0.408685158 + 0.3506983614 + 0.171129372$$
$$+ 0.3381994381 + 0.2134804912 + 0.5171674774\Big]$$

$$= \frac{1}{2} \times 1.999360298$$

$$\simeq 0.999680149$$