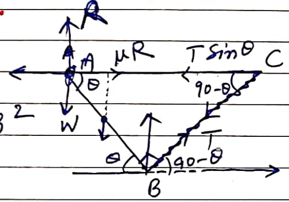
# CSE 2019 to 2010

STATICS.

2019/5(c)/10m

1. One end of a heavy uniform rod AB can slide along a rough horizontal rod AC, to which it is attached by a ring. B and C are joined by a string. When the rod is on the point of sliding, then $AC^2 - AB^2 = BC^2$.

If $\theta$ is the angle between AB and the horizontal line, then prove that the coefficient of friction is $\dfrac{\cot\theta}{2+\cot^2\theta}$.




As $AC^2 = BC^2 + AB^2$

$\therefore \angle B = 90^\circ$

Since system is in equilibrium, we compare the horizontal and vertical components of forces at A. Let weight of rod AB be W.

Horizontally,

$$T \cdot \sin\theta = \mu R$$

Vertically,

$$R + T\cos\theta = W$$

eliminating R,

$$\frac{T.\sin\theta}{\mu} + T.\cos\theta = W$$

$$T\left(\sin\theta + \mu\cdot\cos\theta\right) = \mu\cdot W \quad \text{—(1)}$$

Also, the moment at A is zero.

$$T\cdot(AB) - W\cdot\left(\frac{AB}{2}\right)\cdot\cos\theta = 0$$ clockwise

$$T = \frac{W\cos\theta}{2}$$

using it in (1)

$$\frac{\cancel{W}\cos\theta}{2}\left(\sin\theta + \mu\cos\theta\right) = \mu\cdot \cancel{W}$$

$$\sin\theta\cdot\cos\theta + \mu\cdot\cos^2\theta = 2\mu$$

$$\mu\left(2 - \cos^2\theta\right) = \sin\theta\cos\theta$$

$$\mu = \frac{\sin\theta\cos\theta}{2-\cos^2\theta}$$ dividing by $\sin^2\theta$

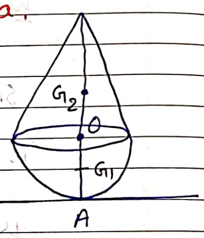
$$= \frac{\cot\theta}{2\cdot\operatorname{cosec}^2\theta - \cot^2\theta}$$

$$= \frac{\cot\theta}{2+2\cot^2\theta - \cot^2\theta}$$

$$\boxed{\mu = \frac{\cot\theta}{2+\cot^2\theta}}$$

2019/6(a)/15m

2. A body consists of a cone and underlying hemisphere. The base of the cone and the top of the hemisphere have same radius 'a'. The whole body rests on a rough horizontal table with hemisphere in contact with the table. Show that the greatest height of the cone, so that the equilibrium may be stable, is $\sqrt{3}a$.




Let us first try to find out the C.G. of the whole body.

As we know C.G. of a solid hemisphere is a point on its axis at a distance $3a/8$ from the centre of its flat base, where 'a' is radius of sphere.

$$x_1 = AG_1 = a - \frac{3a}{8} = \frac{5a}{8}$$

$w_1$ = weight of hemisphere

$$= \frac{2}{3}\pi a^3 \rho g$$

$x_2$ = distance of centre of gravity of cone from table

$$= AO + OG_2 = a + \frac{H}{4}$$

$w_2$ = weight of cone

$$= \frac{1}{3}\pi a^2 H\rho g, \quad H - \text{Height of cone}$$

$h$ = distance of C.G. of combined body from horizontal plane

$$= \frac{w_1 x_1 + w_2 x_2}{w_1 + w_2}$$

$$= \frac{\frac{2}{3}\pi a^3 \rho g \cdot \frac{5a}{8} + \frac{1}{3}\pi a^2 H\rho g\left(a + \frac{H}{4}\right)}{\frac{2}{3}\pi a^3 \rho g + \frac{1}{3}\pi a^2 H\rho g}$$

$$= \frac{\frac{5}{4}a^2 + H\left(a + \frac{H}{4}\right)}{2a + H}$$

$$h = \frac{5a^2 + H(4a + H)}{4(2a + H)}$$

Let $R$ = radius of lower surface $= \infty$

$r$ = radius of upper surface $= a$

For stable equilibrium, $\frac{1}{h} > \frac{1}{r} + \frac{1}{R}$

$$\frac{4(2a+H)}{5a^2 + H(4a+H)} > \frac{1}{a} + \frac{1}{\infty}$$

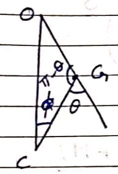
$$a(8a + \cancel{4H}) > 5a^2 + \cancel{4aH} + H^2$$

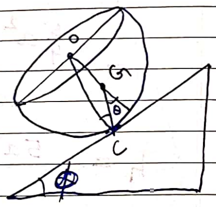
$$3a^2 > H^2$$

or $\boxed{H < \sqrt{3}a}$

2017/6(c)/17m

3. A uniform solid hemisphere rests on a rough plane inclined to the horizon at an angle $\phi$ with its curved surface touching the plane. find the greatest admissible value of the inclination $\phi$ for equilibrium. If $\phi$ be less than this value, is the equilibrium stable?




Let G be the centre of gravity of hemisphere.

$OG = 3r/8$.

In the position of equilibrium the line CG must be verticle.

Since $OC \perp$ inclined plane and $CG \perp$ horizontal,

$\therefore \angle OCG = \phi$

Suppose in equilibrium the axis of hemisphere makes an angle $\theta$ with the verticle.

Using Sine Rule in $\Delta OGC$,

$$\frac{OG}{\sin\phi} = \frac{OC}{\sin(\pi-\theta)}$$

ie. $$\frac{3r/8}{\sin\phi} = \frac{r}{\sin\theta}$$

$$\sin\theta = \frac{8}{3}\sin\phi \Rightarrow \theta = \sin^{-1}\left(\frac{8}{3}\sin\alpha\right) \quad \text{—(1)}$$

since $\sin\theta < 1$

$$\therefore \frac{8}{3}\sin\phi < 1 \quad \text{—(2)}$$

ie. $$\boxed{\phi < \sin^{-1}\frac{3}{8}}$$

Thus for the equilibrium to exist, we must have $\phi < \sin^{-1}\frac{3}{8}$

Let us check stability part.

Let $CG = h$,

$$\therefore \frac{h}{\sin(\theta-\phi)} = \frac{3r/8}{\sin\phi}$$

$$h = \frac{3r\sin(\theta-\phi)}{8\sin\phi}$$

Here, $r = r$, $R = \infty$

Equilibrium is stable if, $\left[\frac{\cos\phi}{h} > \frac{1}{r} + \frac{1}{R}\right]$

ie $$\frac{\cos\phi \cdot 8\sin\phi}{3\cancel{r}\sin(\theta-\phi)} > \frac{1}{\cancel{r}} + \frac{1}{\infty}$$

$$8\sin\phi\cdot\cos\phi > 3\sin\theta\cos\phi - 3\cos\theta\sin\phi$$

$$8\sin\phi\cdot\cos\phi > \cancel{3}\left(\frac{8}{\cancel{3}}\sin\phi\right)\cos\phi - 3\sin\phi\sqrt{1-\frac{64}{9}\sin^2\phi}$$

using (1)

$$\sin\phi\sqrt{9-64\sin^2\phi} > 0 \quad \text{—(3)}$$

But from (2) $9 - 64\sin^2\phi > 0$

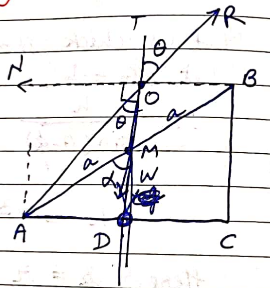
Hence, relation (3) is true. Hence the equilibrium is stable.

2016/ 7(a)/ 15m

4 A uniform rod AB of length $2a$ rests movable about a hinge at A rests with other end against a smooth vertical wall. If $\alpha$ is the inclination of the rod to the vertical, prove that the magnitude of reaction of the hinge is

$$\frac{1}{2}W\sqrt{4+\tan^2\alpha}$$

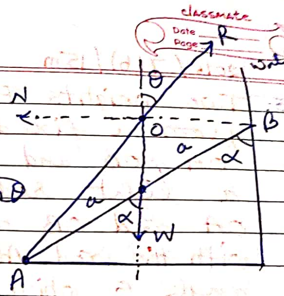
where W is the weight of the rod.




Let the line of action of ~~W~~ W and N meet in O.

Join OA. Reaction R at A must also pass through O.

Let $\angle TOR = \theta$




Comparing horizontal and vertical components at O.

$R\cos\theta = W$

$R\sin\theta = N$ —(1)

Balancing moments at A

$$W(\cancel{a}\sin\alpha) = N(2\cancel{a}\cos\alpha)$$

$$\cancel{R}\cos\theta\cdot\sin\alpha = \cancel{R}\sin\theta(2\cos\alpha) \quad \text{using (1)}$$

$$\tan\theta = \frac{1}{2}\tan\alpha$$

From (1) $R = \dfrac{W}{\cos\theta} = W\sec\theta$

$$R = W\sqrt{1+\tan^2\theta}$$

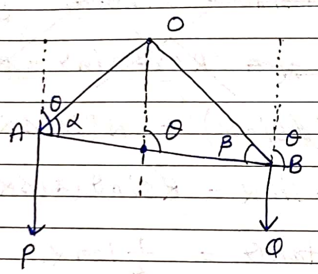
$$= W\sqrt{1+\frac{1}{4}\tan^2\alpha}$$

$$= \frac{W}{2}\sqrt{4+\tan^2\alpha}$$

2016/7(b)/15m

5. Two weights $P$ and $Q$ are suspended from a fixed point $O$ by strings $OA$, $OB$ and are kept apart by a light rod $AB$. If the strings $OA$ and $OB$ make angles $\alpha$ and $\beta$ with the rod $AB$, show that the angle $\theta$ which the rod makes with the vertical is given by

$$\tan\theta = \frac{P+Q}{P\cot\alpha - Q\cot\beta}$$




Let us balance the moments of rod $AB$ by forces $P$ and $Q$ ~~at~~ working about point $O$.

$$P\times(OA\sin(\theta-\alpha)) = Q\times(OB\sin(\pi-(\theta+\beta)))$$

$$P(OA)\sin(\theta-\alpha) = Q(OB)\sin(\theta+\beta) \quad \text{—(1)}$$

Applying Sine Rule in $\Delta OAB$

$$\frac{OA}{\sin\beta} = \frac{OB}{\sin\alpha} \quad \text{—(2)}$$

$$OA = \frac{\sin\beta}{\sin\alpha}OB$$

using in (2) in (1)

$$P\left(\frac{\sin\beta}{\sin\alpha}\cdot\cancel{OB}\right)\sin(\theta-\alpha) = Q\cdot(\cancel{OB})\sin(\theta+\beta)$$

$$P\sin\beta(\sin\theta\cos\alpha - \cos\theta\sin\alpha) = Q\sin\alpha\times(\sin\theta\cos\beta+\cos\theta\sin\beta)$$

Dividing both sides by $(\sin\alpha\cdot\sin\beta\cdot\cos\theta)$

$$\cancel{\sin\beta}P\left(\frac{\sin\theta\cos\alpha-\cos\theta\sin\alpha}{\sin\alpha\,\cancel{\sin\beta}\cdot\cos\theta}\right) = \frac{Q\cancel{\sin\alpha}(\sin\theta\cos\beta+\cos\theta\sin\beta)}{\cancel{\sin\alpha}\sin\beta\cdot\cos\theta}$$

$$P(\tan\theta\cdot\cot\alpha-1) = Q(\tan\theta\cot\beta+1)$$

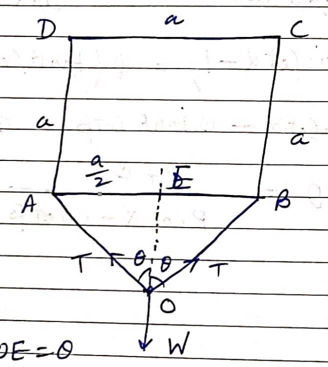
$$P\tan\theta\cdot\cot\alpha - Q\tan\theta\cot\beta = Q+P$$

$$\tan\theta = \frac{P+Q}{P\cot\alpha - Q\cot\beta}$$

2016 / 7(c) / 20m

6. A square ABCD, the length of whose sides is a, is fixed in a vertical plane with two of its sides horizontal. An endless string of length $l\ (>4a)$ passes over four pegs at the angles of the board and through a ring of weight W which is hanging vertically. Show that the tension of the string is

$$\frac{W(l-3a)}{2\sqrt{l^2-6la+8a^2}}.$$




Let $\angle AOE = \theta$

Balancing vertical force at point O

$$T\cos\theta + T\cos\theta = W$$

$$T = \frac{W}{2\cos\theta} \quad \text{--- (1)}$$

from $\Delta AOE$, $\cos\theta = \frac{OE}{AO}$

Total length of string $= l$

$\therefore\ 3a + AO + BO = l$

$2AO = l - 3a \qquad (AO = BO)$

$AO = \frac{1}{2}(l-3a)$

$OE^2 = AO^2 - AE^2$

$= \left(\frac{l-3a}{2}\right)^2 - \left(\frac{a}{2}\right)^2$

$= \frac{1}{4}\left[l^2 + 9a^2 - 6la - a^2\right]$

$= \frac{1}{4}\left[l^2 + 8a^2 - 6la\right]$

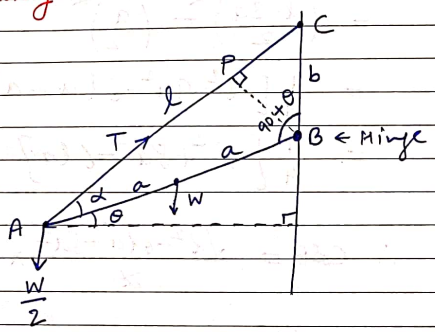
$\therefore\ \cos\theta = \frac{\sqrt{l^2-6la+8a^2}}{2} \times \frac{2}{l-3a}$

$\therefore$ from (1)

$$T = \frac{W}{2} \times \frac{(l-3a)}{\sqrt{l^2-6la+8a^2}}$$

2015/5(d)/10m

7. A rod of 8 kg is movable in a vertical plane about a hinge at one end, another end is fastned a weight equal to half of the rod, this end is fastned by a string of length $l$ to a point at a height 'b' above the hinge vertically. Obtain the tension in the string.




Balancing moments about point B:

$$\frac{W}{2}(2a\cos\theta) + W(a\cos\theta) = T(2a\sin\alpha)$$

$$\cancel{2a}W\cos\theta = \cancel{2a}T\sin\alpha$$

$$T = \left(\frac{\cos\theta}{\sin\alpha}\right)W \quad \text{—(1)}$$

Applying sine rule in $\Delta ABC$

$$\frac{l}{\sin(90+\theta)} = \frac{b}{\sin\alpha}$$

$$\frac{l}{b} = \frac{\cos\theta}{\sin\alpha}$$

using in (1)

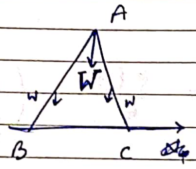
$$T = \left(\frac{\cos\theta}{\sin\alpha}\right)\cdot W$$

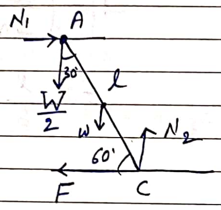
$$= \frac{l}{b}(8g) = \frac{8lg}{b}$$

($W = 8kg \times g$ given)

2015 | 6(b) | 13m

8. Two equal ladders of weight 4kg each are placed so as to lean at A against each other with their ends resting on a rough floor, given the coefficient of friction is $\mu$. The ladders at A make an angle $60°$ with each other. Find what weight on the top would cause them to slip.




Ladders AB and AC.

$w = 4kg$

Let W be the maximum weight which can be placed on top of them, A.

Taking equilibrium-diagram of one ladder and balancing forces in x and y-direction.

$$N_1 = F \quad \text{—(1)}$$

$$N_2 = \frac{W}{2} + 4g \quad \text{—(2)}$$

Also, balancing moment about C

$$N_1 \cdot (l \sin 60°) = \frac{W}{2}(l \cos 60) + 4g \cdot \left(\frac{l}{2} \cos 60\right)$$

$$\frac{\sqrt{3}}{2} N_1 = \frac{W}{2} \cdot \frac{1}{2} + 2g \cdot \frac{1}{2}$$

$$2\sqrt{3} N_1 = W + 4g \quad \text{—(3)}$$

In the situation of slip, $F = \mu N_2$ —(4)

We try to eliminate $N_1, N_2$

~~from (1) & (2)~~

$$\boxed{\begin{aligned} \mu N_2 &= N_1 \\ N_2 &= \frac{W}{2} + 8 \end{aligned}}$$

using (4) in (1)

$$F = N_1 \Rightarrow \mu N_2 = N_1$$

ie. $\mu\left(\frac{W}{2} + 4g\right) = N_1$ using (2)

Using it in (3)

$$2\sqrt{3} \cdot \mu \left(\frac{W}{2} + 4g\right) = W + 4g$$

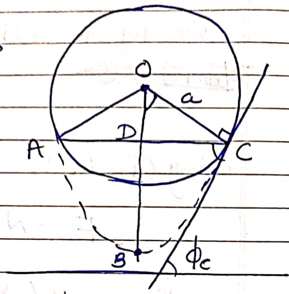
$$(\sqrt{3}\mu - 1) W = 4g - 8\sqrt{3}\mu \cdot g$$

$$\boxed{W = \frac{4g(1 - 2\sqrt{3}\mu)}{\sqrt{3}\mu - 1}}$$

2015/ 8(a) / 12m

9. Find the length of an endless chain which will hang over a circular pulley of radius 'a' so as to be in contact with the two-thirds of the circumference of the pulley.

Topic: Common Catenary




$\phi_c = \angle COD$

$= \frac{1}{2} \angle COA$

$= \frac{1}{2} \times \frac{2\pi}{3} = \frac{\pi}{3}$

$AD = DC$

$= a \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} a$

We know, Equation of a common catenary is

$$x = c \log(\tan\phi + \sec\phi)$$

At point C,

$$\frac{\sqrt{3}}{2} a = c \log\left(\tan\frac{\pi}{3} + \sec\frac{\pi}{3}\right)$$

$$\frac{\sqrt{3}}{2} a = c \log(\sqrt{3}+2)$$

$$\therefore \quad c = \frac{a\sqrt{3}}{2\log(\sqrt{3}+2)}$$

The intrinsic equation of common catenary is

$$s = c\tan\phi \quad , \text{where } \tan\phi = \frac{dy}{dx}$$

$$\therefore \text{Arc}(BC) = c \tan\left(\frac{\pi}{3}\right)$$

$$= \frac{3a}{2\log(\sqrt{3}+2)}$$

Hence length of the chain is

$$= \frac{2}{3}(2\pi a) + 2 \times \frac{3a}{2\log(\sqrt{3}+2)}$$

$$= \frac{4a\pi}{3} + \frac{3a}{\log(\sqrt{3}+2)}$$

2014 | 5(d) | 10m

10. Two equal uniform rods AB and AC, each of length $l$, are freely joined at A and rest on a smooth fixed vertical circle of radius $r$. If $2\theta$ is the angle between the rods, then find the relation between $l$, $r$ and $\theta$, by using the principle of virtual work.

Let weight of each rod be W

$AX = AY = l/2$

Weight 2W acts at point G.

$\sin\theta = \frac{r}{OA} \Rightarrow OA = r\operatorname{cosec}\theta$

Giving the system a small displacement along OA in which $\theta$ changes to $\theta + d\theta$.
Point O remains fixed.
Point G is slightly moved.

$OG = AO - AG = r\operatorname{cosec}\theta - \frac{l}{2}\cos\theta$

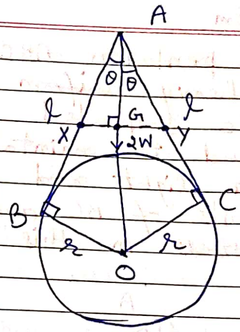
Appling the principle of virtual work.

$\delta W = 0$

$2W \times \delta(OG)$

$2W\,\delta\left(r\operatorname{cosec}\theta - \frac{l}{2}\cos\theta\right) = 0$

$\Rightarrow 2W\left(-r\operatorname{cosec}\theta\cot\theta + \frac{l}{2}\sin\theta\right)d\theta = 0.$




$\Rightarrow \frac{r}{\sin\theta}\cdot\frac{\cos\theta}{\sin\theta} = \frac{l}{2}\sin\theta$

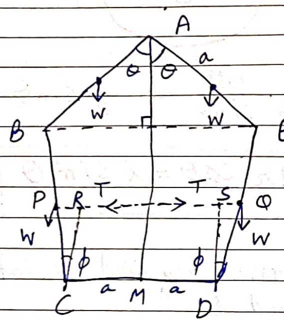
$\Rightarrow \boxed{2r\cdot\cos\theta = l\cdot\sin^3\theta}$

Which is the required relation between $l$, $r$ and $\theta$.

2014/7(c)/ 20m

11. A regular pentagon ABCDE, formed of equal heavy uniform bars jointed together, is suspended from the joint A, and is maintained in form by a light rod joining the middle points of BC and DE. Find the stress in this rod.

$2W\cot\frac{\pi}{10}$




Each side$=2a$

In the position of equilibrium, the pentagon is a regular one.
So each interior angle

$$= \frac{5(2\pi)-2\pi}{5} = \frac{3\pi}{5}$$

Point A is fixed. Give the system a small displacement about the vertical AM in which $\theta$ changes to $\theta+\delta\theta$ and $\phi$ changes to $\phi+\delta\phi$.

$PQ = PR + RS + SQ = a\sin\phi + 2a + a\sin\phi$
$= 2a(1+\sin\phi)$

The depth of the middle points of AB or AE below A $= a\cos\theta$

The depth of the middle points of BC or ED below A $= 2a\cos\theta + a\cos\phi$

The depth of the middle point of CD below A $= 2a\cos\theta + 2a\cos\phi$

The equation of ~~wo~~ virtual work is

$$T\cdot\delta[2a(1+\sin\phi)] + 2W\cdot\delta(a\cos\theta) + 2W\cdot\delta[a(2\cos\theta+\cos\phi)] + W\cdot\delta[2a(\cos\theta+\cos\phi)] = 0.$$

$$\Rightarrow 2Ta\cos\phi\,\delta\phi + 2Wa(-\sin\theta)\delta\theta + 4aW(-\sin\theta)\delta\theta + 2Wa(-\sin\phi)\delta\phi + 2aW(-\sin\theta)\delta\theta + 2aW(-\sin\phi)\delta\phi = 0.$$

$$\Rightarrow [T\cdot\cos\phi - 2W\sin\phi]\delta\phi = 4W\sin\theta\,\delta\theta \quad —(1)$$

From the figure, $BE = BE$

$$4a\sin\theta = 2a + 4a\sin\phi$$

$$\Rightarrow \cos\theta\,\delta\theta = \cos\phi\,\delta\phi \quad —(2)$$

using (2) in (1),

$$\frac{T\cdot\cos\phi - 2W\sin\phi}{\cos\phi} = \frac{4W\sin\theta}{\cos\theta}$$

$$\Rightarrow T = 2W(\tan\phi + 2\tan\theta)$$

In the position of equilibrium, $\theta = \frac{1}{2}\cdot\frac{3\pi}{5} = \frac{3\pi}{10}$

$$\phi = \frac{3\pi}{5} - \frac{\pi}{2} = \frac{\pi}{10}$$

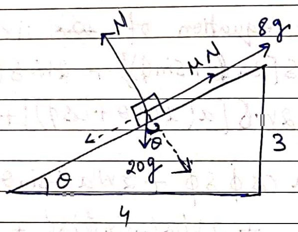
$$\therefore T = 2W\left(\tan\frac{\pi}{10} + 2\tan\frac{3\pi}{10}\right) = 2W\left(\tan\frac{\pi}{10} + 2\cot\frac{2\pi}{10}\right)$$

$$= 2W\left(\tan\frac{\pi}{10} + 2\cdot\frac{1-\tan^2\pi/10}{2\tan\frac{\pi}{10}}\right) = \boxed{2W\cot\frac{\pi}{10}}$$

Date 6/5/19

2013/ 5(d)/ 10m

12. The base of an inclined plane is 4 metres in length and the height is 3 meters. A force of 8 kg acting parallel to the plane will just prevent a weight of 20 kg from sliding down. Find the coefficient of friction between between the plane and the weight.




$\tan\theta = \frac{3}{4}$

Horizontally, $20g\sin\theta = \mu N + 8g$ — ①

Vertically, $N = 20g\cos\theta$ — ②

using ② in ①

$20g\sin\theta = \mu(20g\cos\theta) + 8g$

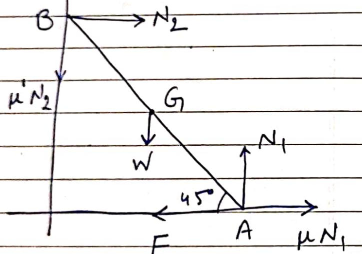
$\mu = \dfrac{20g\sin\theta - 8g}{20g\cos\theta}$

$\mu = \dfrac{20\times g\times\frac{3}{5} - 8g}{20g\times\frac{4}{5}} = \dfrac{4g}{16g} = \dfrac{1}{4}$

2013 | 7(b) | 15m

13. A uniform ladder rests at an angle of $45°$ with the horizontal with its upper extermity against a rough vertical wall and its lower extermity on the ground. If $\mu$ and $\mu'$ are the coefficients of limiting friction between the ladder and the ground and wall respectively, then find the minimum horizontal force required to move the lower end of the ladder towards the wall.

Let $F$ be the least force which will move the lower end of ladder towards the wall.




Balancing horizontal & vertical component of forces.

$$F = \mu N_1 + N_2$$

$$N_1 = W + \mu' N_2$$

$$\therefore \quad F = \mu(W + \mu' N_2) + N_2$$

i.e. $$N_2 = \frac{F - \mu W}{1 + \mu\mu'}$$

Taking moments about A, we get

$$N_2 (l \sin 45°) = \mu' N_2 (l \cos 45°) + W\left(\frac{l}{2}\cos 45°\right)$$

(where $AB = l$)

$$\Rightarrow \quad N_2 l = \mu' N_2 l + \frac{Wl}{2}$$

$$\frac{(F - \mu W)}{1 + \mu\mu'}(1 - \mu')\cancel{l} = \frac{W\cancel{l}}{2}$$

$$F - \mu W = \frac{W}{2}\,\frac{(1 + \mu\mu')}{1 - \mu'}$$

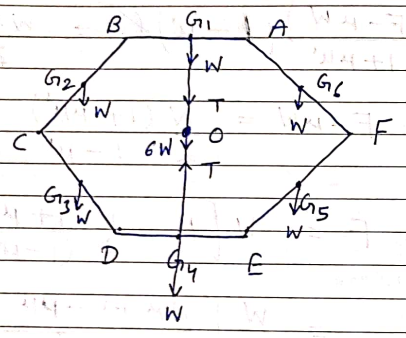
$$\therefore \quad F = W\left[\mu + \frac{1}{2}\cdot\frac{1 + \mu\mu'}{(1 - \mu')}\right]$$

$$= \frac{W}{2}\left[\frac{2\mu + 1 - \mu\mu'}{1 - \mu'}\right]$$

$\cancel{\mu - \mu\mu' + 1 + \mu}$

2013/15m/ 7(c).

14. Six equal rods AB, BC, CD, DE, EF and FA are each of weight W and are freely joined at their extremities so as to form a hexagon; the rod AB is fixed in a horizontal position and the middle points of AB and DE are joined by a string. Find the tension in the string.




Let $G_1, G_2, G_3, G_4, G_5, G_6$ be the middle points of the rods.

Let O be the centre of hexagon. Since AB is fixed in a horizontal position, therefore we measure the distance from AB.

Let $OG_1 = x$ $\quad\therefore\quad G_1G_4 = 2x = l$ (say)

The total weight 6W of the rods acts at O.

Let the system undergo a displacement in the vertical plane such that $x$ becomes $x+\delta x$ and $l$ becomes $l+\delta l$.

$\therefore$ By principle of virtual work.

$$6W\cdot\delta(OG_1) - T\cdot\delta(G_1G_4) = 0.$$

$$6W\cdot\delta(x) - T\cdot\delta(2x) = 0.$$

$$6W\cdot 1\cdot\delta x - T\cdot 2\cdot\delta x = 0$$

Or $\quad 6W - 2T = 0 \quad [\because \delta x \neq 0]$

$\therefore\quad \boxed{T = 3W}$

2012/20m/7(b)

15. A heavy hemispherical shell of radius 'a' has a particle attached to a point on the rim, and rests with the curved surface in contact with a rough sphere of radius 'b' at the highest point. Prove that if $b/a > \sqrt{5}-1$, the equilibrium is stable, whatever be the weight of the particle.

The centre of gravity $G_1$ of the hemispherical shell is on its symmetrical radius $O'D$ and $O'G_1 = a/2$

Let $G$ be the centre of gravity of the combined body consisting of the hemispherical shell and the weight at A. The G lies on line $AG_1$

For equilibrium the line $OCGO'$ must be vertical but $AG_1$ need not be horizontal.

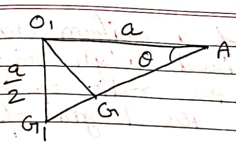
Let $CG = h$, $r = a$, $R = b$

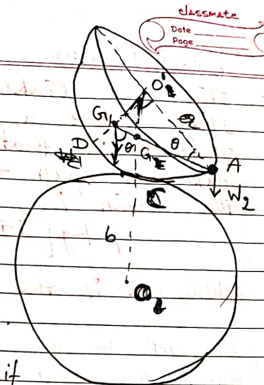
The equilibrium will be stable if

$$\frac{1}{h} > \frac{1}{r} + \frac{1}{R}$$

ie $\frac{1}{h} > \frac{1}{a} + \frac{1}{b}$ ie $h < \frac{ab}{a+b}$ (A)

The value of h depends on the weight of the particle attached at A. So the equilibrium will be stable, whatever be the weight of the particle attached at A, if eqn (A)




holds even for the maximum value of h.

h will be maximum if $O'G$ is minimum i.e. if $O'G$ is perpendicular to $AG_1$ or if $\Delta AO'G$ is right angled.

Let $\angle O'AG = \theta$, then from right angled $\Delta AO'G_1$

$$\tan\theta = \frac{a/2}{a} = \frac{1}{2}, \quad \therefore \sin\theta = \frac{1}{\sqrt{5}}$$

$\therefore$ min value of $O'G = O'A(\sin\theta)$

$$= \frac{a}{\sqrt{5}}$$

~~Also~~, max value of $h = a - \frac{a}{\sqrt{5}}$

$$= a\left(\frac{\sqrt{5}-1}{\sqrt{5}}\right)$$

Hence the equilibrium will be stable whatever be the weight of the particle at A if

$$\frac{\cancel{a}(\sqrt{5}-1)}{\sqrt{5}} < \frac{\cancel{a}b}{a+b} \text{ ie.}$$

ie. $(\sqrt{5}-1)(a+b) < \sqrt{5}\,b$

ie. $\boxed{\frac{b}{a} > \sqrt{5}-1}$

2012 / 7(c) / 20m

16. The end links of a uniform chain slide along a fixed rough horizontal rod. Prove that the ratio of the maximum span to the length of the chain is

$$\mu \log\left[\frac{1+\sqrt{1+\mu^2}}{\mu}\right]$$

where $\mu$ is the coeff of friction.

Let AB be the maximum span. Hence the end links A and B are in limiting equilibrium each under three forces namely

1) the normal reaction $R \perp$ to AB (upward)
2) the force of friction $\mu R$ along the fixed horizontal rod outwards.
3) the tension T along the tangent at A.

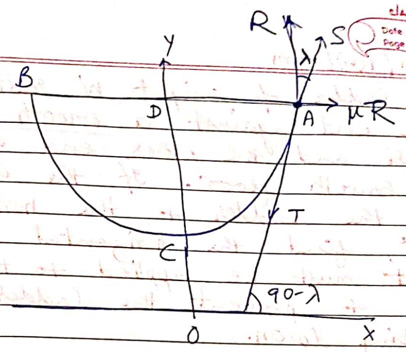
If S is the resultant of R and $\mu R$ at A inclined at $\lambda$ (the angle of friction) to R, then the tension at A must balance S, so it is inclined at $(90^\circ-\lambda)$ to the horizon.

$\psi$ at $A = 90^\circ-\lambda$, $\quad \tan\lambda = \frac{\mu R}{R} = \mu$

$\therefore$ Maximum span $AB = 2x$

$$= 2c\log(\sec\psi + \tan\psi)$$

$$= 2c\log\left[\sec(90-\lambda) + \tan(90-\lambda)\right]$$




$$= 2c\log(\operatorname{cosec}\lambda + \cot\lambda)$$

$$= 2c\log\left[\sqrt{1+\frac{1}{\mu^2}} + \frac{1}{\mu}\right]$$

$$= 2c\log\left[\frac{\sqrt{1+\mu^2}+1}{\mu}\right]$$

length of chain, $ACB = 2s = 2c\tan\psi$

$$= 2c\tan(90-\lambda) = 2c\cot\lambda = \frac{2c}{\mu}$$

$$\text{Ratio} = \frac{\text{Max Span } AB}{\text{length of chain}} = \mu\log\left[\frac{\sqrt{1+\mu^2}+1}{\mu}\right]$$

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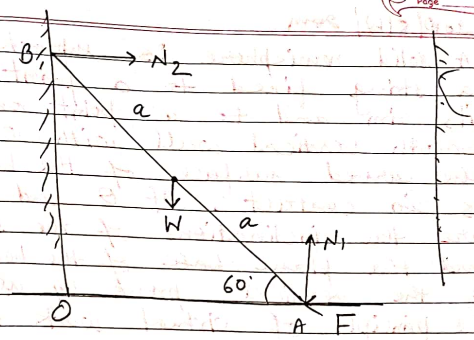
2011/7(b)/20m

A ladder of weight W rests with one end against a smooth vertical wall and the other end rests on a smooth floor. If the inclination of the ladder to the horizon is 60°, find the horizontal force that must be applied to the lower end to prevent the ladder from slipping down.




Let F be the least force which must be applied to the lower end A to prevent the ladder from slipping down.

As wall is given smooth, friction does not come into the picture.

Balancing horizontal and vertical component of forces

$N_2 = F$ ,

$N_1 = W$

Taking moments about point A.

$N_2 (2a \sin 60) = W (a \cos 60)$

$F \times 2 \times \frac{\sqrt{3}}{2} = W \times \frac{1}{2}$

$$\boxed{F = \frac{W}{2\sqrt{3}} = \frac{W\sqrt{3}}{6}}$$

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2010/8(b)/20m

A solid hemisphere is supported by a string fixed to a point on its rim and to a point on a smooth vertical wall with which the curved surface of the hemisphere is in contact. If $\theta$ and $\phi$ are the inclinations of the string and the plane base of the hemisphere to the vertical, prove by using the principle of virtual work that

$$\tan\phi = \frac{3}{8} + \tan\theta.$$

~~Let a small displacement~~

O is a fixed point on the wall. Let $l$ be the length of the string AO and 'a' be the radius of hemisphere.

The weight W of the hemisphere acts at its centre of gravity G which lies on the symmetrical radius CD and

$$CG = \frac{3}{8}a$$

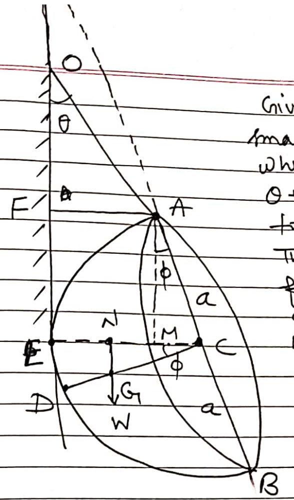
Hemisphere touches the wall at E.

$\therefore \angle OEC = 90^\circ \Rightarrow EC$ is horizontal.

The depth of G below O $= OF + AM + NG$

$$= l\cos\theta + a\cos\phi + \frac{3}{8}a\sin\phi$$

[Note, $\angle NCG = 90^\circ - \angle ACN = 90-(90-\phi) = \phi$]




Give the system a small displacement in which $\theta$ changes to $\theta+\delta\theta$, $\phi$ changes to $\phi+\delta\phi$.

The point O remains fixed, the length of string AO does not change so that the work done by its tension is zero and point G is slightly displaced.

The angle $\angle OEC$ remains $90^\circ$.

The only force which contributes to the equation of virtual work is weight W.

$$W\delta\left(l\cos\theta + a\cos\phi + \frac{3}{8}a\sin\phi\right) = 0.$$

$$-l\sin\theta\, d\theta - a\sin\phi\, d\phi + \frac{3}{8}a\cos\phi\, d\phi = 0.$$

or $\quad l\sin\theta\, d\theta = a\left(\frac{3}{8}\cos\phi - \sin\phi\right)d\phi \quad$ — (1)

from the figure, $EC = a$

Also, $EC = EM + MC = FA + MC = l\sin\theta + a\sin\phi$

$\therefore \quad a = l\sin\theta + a\sin\phi$

Differentiating, $\quad 0 = l\cos\theta\, d\theta + a\cos\phi\, d\phi$

or $\quad -l\cos\theta\, d\theta = a\cos\phi\, d\phi \quad$ — (2)

Dividing (1) by (2), we get

$$-\tan\theta = \frac{3}{8} - \tan\phi$$

or $\boxed{\tan\phi = \frac{3}{8} + \tan\theta}$