IFoS - 2014 - Paper II

5) (a) Use Lagrange's formula to find the form of $f(x)$ from the following table:

x	0	2	3	6
$f(x)$	648	704	729	792

$\Rightarrow$ We know Lagrange's Interpolation Formula is,

$$L(x) = \omega(x) \sum_{r=0}^{n} \frac{f(x_r)}{(x-x_r)\,\omega'(x_r)} = \omega(x) \sum_{r=0}^{n} \frac{y_r}{D_r}$$

where,

$$\omega(x) = (x-x_0)(x-x_1) \cdots (x-x_{r-1})(x-x_r)(x-x_{r+1}) \cdots (x-x_n)$$

$$D_r = (x-x_r)(x_r-x_0) \cdots (x_r-x_{r-1})(x_r-x_{r+1}) \cdots (x_r-x_n)$$

Now, we have the computational scheme as follows:

				D_r	y_r	y_r/D_r
$(x-x_0)=x$	$x_0-x_1=-2$	$x_0-x_2=-3$	$x_0-x_3=-6$	$-36x$	648	$-18/x$
$(x_1-x_0)=2$	$x-x_1=x-2$	$x_1-x_2=-1$	$x_1-x_3=-4$	$8(x-2)$	704	$88/(x-2)$
$(x_2-x_0)=3$	$x_2-x_1=1$	$x-x_2=x-3$	$x_2-x_3=-3$	$-9(x-3)$	729	$-81/(x-3)$
$(x_3-x_0)=6$	$x_3-x_1=4$	$x_3-x_2=3$	$x-x_3=x-6$	$72(x-6)$	792	$11/(x-6)$

and $\omega(x) = x(x-2)(x-3)(x-6)$

$$\therefore L(x) = x(x-2)(x-3)(x-6)\left[-\frac{18}{x} + \frac{88}{x-2} - \frac{81}{x-3} + \frac{11}{x-6}\right]$$

$$= -18(x-2)(x-3)(x-6) + 88x(x-3)(x-6) - 81x(x-2)(x-6) + 11x(x-2)(x-3)$$

$$= -18(x^3-11x^2+36x-36) + 88(x^3-9x^2+18x) - 81(x^3-8x^2+12x) + 11(x^3-5x^2+6x)$$

$$= (-18+88-81+11)x^3 + (198-792+648-55)x^2 + (-648+1584-972+66)x + 648$$

$$= 0\cdot x^3 - x^2 + 30x + 648$$

$$= -x^2 + 30x + 648$$

6) (e) The values of $f(x)$ for different values of x are given as $f(1)=4$, $f(2)=5$, $f(7)=5$ and $f(8)=4$. Using Lagrange interpolation formula, find the value of $f(6)$. Also find the value of x for which $f(x)$ is optimum

$\Rightarrow$ we know Lagrange's Interpolation formula is,

$$L(x) = \omega(x) \sum_{r=0}^{n} \frac{f(x_r)}{(x-x_r)\omega'(x_r)} = \omega(x) \sum_{r=0}^{n} \frac{y_r}{D_r}$$

where, $\omega(x) = (x-x_0)(x-x_1) \cdots (x-x_{r-1})(x-x_r)(x-x_{r+1}) \cdots (x-x_n)$

$D_r = (x-x_r)(x_r-x_0) \cdots (x_r-x_{r-1})(x_r-x_{r+1}) \cdots (x_r-x_n)$

				D_r	y_r	y_r/D_r
$x-1$	-1	-6	-7	$-42(x-1)$	4	$-\frac{2}{21(x-1)}$
1	$x-2$	-5	-6	$30(x-2)$	5	$\frac{1}{6(x-2)}$
6	5	$x-7$	-1	$-30(x-7)$	5	$-\frac{1}{6(x-7)}$
7	6	1	$x-8$	$42(x-8)$	4	$\frac{2}{21(x-8)}$

$\omega(x) = (x-1)(x-2)(x-7)(x-8)$

$$\therefore f(x) = (x-1)(x-2)(x-7)(x-8)\left[-\frac{2}{21(x-1)} + \frac{1}{6(x-2)} - \frac{1}{6(x-7)} + \frac{2}{21(x-8)}\right]$$

$$= \frac{1}{42}\left[-4(x-2)(x-7)(x-8) + 7(x-1)(x-7)(x-8) - 7(x-1)(x-2)(x-8) + 4(x-1)(x-2)(x-7)\right]$$

$$= \frac{1}{42}\left[4(x-2)(x-7)(-x+8+x-\cancel{1}) + 7(x-1)(x-8)(x-7-x+2)\right]$$

$$= \frac{1}{42}\left[28(x-2)(x-7) - 35(x-1)(x-8)\right]$$

$$= \frac{1}{42} \times 7\left[4(x^2-9x+14) - 5(x^2-9x+8)\right]$$

$$= \frac{1}{6}\left[4x^2 - 36x + 56 - 5x^2 + 45x - 40\right]$$

$$= \frac{1}{6}\left[-x^2 + 9x + 16\right]$$

$$\therefore f(6) = \frac{1}{6}\left[-36+54+16\right] = 5.67$$

For optimum value of $f(x)$, $f'(x) = 0$

$$\Rightarrow \frac{1}{6}(-2x+9) = 0$$

$$\Rightarrow x = 9/2 = 4.5$$

$\therefore$ $f(x)$ is optimum for the value of $x = 4.5$.

7) (b) Solve the following system of equations,

$$2x_1+x_2+x_3-2x_4=-10$$
$$4x_1+2x_3+x_4=8$$
$$3x_1+2x_2+2x_3=7$$
$$x_1+3x_2+2x_3-x_4=-5$$

⇒ The given system of equation is not diagonally dominant. So, we solve the system of equations by Gauss-Jordan's Matrix inversion method.

The augmented matrix is

$$\left[\begin{array}{cccc|cccc} 2 & 1 & 1 & -2 & 1 & 0 & 0 & 0 \\ 4 & 0 & 2 & 1 & 0 & 1 & 0 & 0 \\ 3 & 2 & 2 & 0 & 0 & 0 & 1 & 0 \\ 1 & 3 & 2 & -1 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{c|cccc|ccc} 1 & 1/2 & 1/2 & -1 & 1/2 & 0 & 0 & 0 \\ 0 & -2 & 0 & 5 & -2 & 1 & 0 & 0 \\ 0 & 1/2 & 1/2 & 3 & -3/2 & 0 & 1 & 0 \\ 0 & 5/2 & 3/2 & 0 & -1/2 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cc|cccc|cc} 1 & 0 & 1/2 & 1/4 & 0 & -1/4 & 0 & 0 \\ 0 & 1 & 0 & -5/2 & 1 & -1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 17/4 & -2 & -1/4 & 1 & 0 \\ 0 & 0 & 3/2 & 25/4 & -3 & -3/4 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|cccc|c} 1 & 0 & 0 & -1/4 & -17/4 & 7/4 & -1 & 0 \\ 0 & 1 & 0 & -5/2 & 1 & -1/2 & 0 & 0 \\ 0 & 0 & 1 & 1 & 17/2 & -4 & -1/2 & 0 \\ 0 & 0 & 0 & 19/4 & -63/4 & 21/4 & +1/2 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -193/38 & 77/38 & -37/38 & 1/19 \\ 0 & 1 & 0 & 0 & -277/38 & 43/19 & 5/19 & 10/19 \\ 0 & 0 & 1 & 0 & 449/38 & -97/19 & -23/38 & -4/19 \\ 0 & 0 & 0 & 1 & -63/19 & 21/19 & 2/19 & 4/19 \end{array}\right]$$

Thus,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -193/38 & 77/38 & -37/38 & 1/19 \\ -277/38 & 43/19 & 5/19 & 10/19 \\ 449/38 & -97/19 & -23/38 & -4/19 \\ -63/19 & 21/19 & 2/19 & 4/19 \end{bmatrix} \begin{bmatrix} -10 \\ 8 \\ 7 \\ -5 \end{bmatrix}$$

$$= \begin{bmatrix} 2277/38 \\ 1714/19 \\ -6163/38 \\ 792/19 \end{bmatrix} = \begin{bmatrix} 59.92 \\ 90.21 \\ -162.18 \\ 41.68 \end{bmatrix}$$

$\therefore x_1 = 59.92$, $x_2 = 90.21$, $x_3 = -162.18$, $x_4 = 41.68$

8) (a) using Runge-kutta 4th order method, find y from $\frac{dy}{dx} = \frac{y^2-x^2}{y^2+x^2}$, with $y(0)=1$ at $x=0.2, 0.4$.

$\Rightarrow$ Here, in this problem step length is not given, so we choose, $h=0.2$

$\therefore$ For $y(0.2) \Rightarrow x_0 = 0$, $y_0 = 1$, $f(x,y) = \frac{y^2-x^2}{y^2+x^2}$, $h=0.2$

$K_1 = hf(x_0, y_0) = hf(0,1) = 0.2$

$K_2 = hf(x_0+\frac{h}{2}, y_0+\frac{K_1}{2}) = 0.2f(0.1, 1.1) = 0.1967$

$K_3 = hf(x_0+\frac{h}{2}, y_0+\frac{K_2}{2}) = 0.2f(0.1, 1.09835) = 0.1967$

$K_4 = hf(x_0+h, y_0+K_3) = 0.2f(0.2, 1.1967) = 0.1891$

$y_1 = y(0.2) = y_0 + \frac{1}{6}[K_1+K_4+2K_2+2K_3]$

$= 1 + \frac{1}{6}[1.1759] = 1.1960$

For $y(0.4) \Rightarrow x_1 = 0.2,\ y_1 = 1.1960$, $h = 0.2$

$$K_1 = hf(x_1, y_1) = 0.2 f(0.2, 1.1960) = 0.1891$$

$$K_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}\right) = 0.2 f(0.3, 1.2901) = 0.1795$$

$$K_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}\right) = 0.2 f(0.3, 1.2858) = 0.1793$$

$$K_4 = hf(x_1 + h, y_1 + K_3) = 0.2 f(0.4, 1.3753) = 0.1688$$

$$\therefore y_2 = y(0.4) = y_1 + \frac{1}{6}\left[K_1 + 2K_2 + 2K_3 + K_4\right]$$

$$= 1.1960 + \frac{1}{6} \times 1.0755$$

$$= 1.3752$$