Set – V

Differentiability

Geometrical Meaning of Derivative at a point :

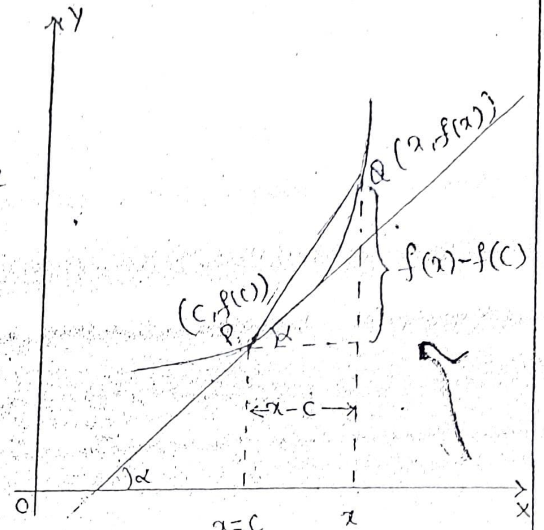
Consider the curve $y=f(x)$ defined in an open interval (a,b).

Let $x = c \in (a,b)$.

Let $y=f(x)$ be differentiable at $x=c$.

Let $P\,(c, f(c))$ be a point on the curve $y=f(x)$.

and let $Q\,(x, f(x))$ be a neighbouring point on the curve.



Now the slope of the chord $PQ = \dfrac{f(x)-f(c)}{x-c}$ $\left[\text{i.e. } \dfrac{y_2-y_1}{x_2-x_1}\right]$

Taking limit as $Q \to P$

i.e. $x \to c$, we get

$$\lim_{Q\to P}(\text{slope of the chord } PQ) = \lim_{x\to c}\frac{f(x)-f(c)}{x-c} \quad \text{——— (1)}$$

As $Q \to P$, chord PQ becomes tangent at P.

$\therefore$ from (1), we have

slope of the tangent at P.

$$= \lim_{x\to c}\frac{f(x)-f(c)}{x-c} = \left[\frac{d}{dx}(f(x))\right]_{x=c} \text{ (or) } f'(x)\Big\}_{x=c}$$

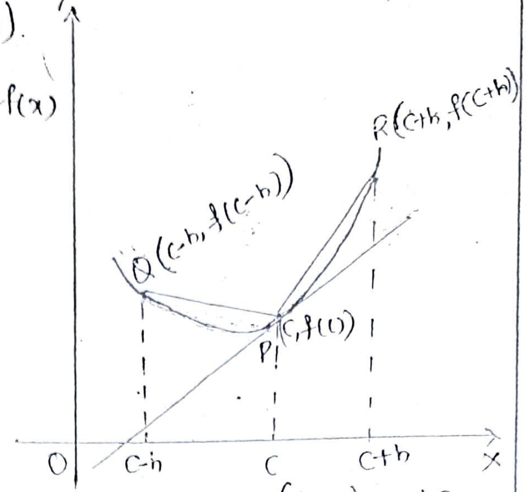
i.e., the derivative of a function at a point $x=c$ is the slope of the tangent to the curve $y=f(x)$ at the point $(c, f(c))$.

→ If a function is not differentiable at $x=c$ only if the point $(c, f(c))$ is a corner point of the curve $y=f(x)$ i.e., the curve suddenly changes its direction at a point $(c, f(c))$.

→ Consider the function $f(x)$ defined on (a,b).

Let $P(c, f(c))$ be a point on the curve $y = f(x)$

Let $Q(c-h, f(c-h))$ & $R(c+h, f(c+h))$ be two neighbouring points on the left hand side (LHS) and RHS respectively of the point 'P'.



Now slope of the chord PQ $= \dfrac{f(c-h)-f(c)}{(c-h)-c}$

$$= \frac{f(c-h)-f(c)}{-h}$$

and slope of chord PR $= \dfrac{f(c+h)-f(c)}{c+h-c}$

$$= \frac{f(c+h)-f(c)}{h}$$

Now taking limit as $Q \to P$
i.e. $h \to 0$

$\therefore \lim_{Q\to P}$ (slope of chord PQ) $= \lim_{h\to 0} \dfrac{f(c-h)-f(c)}{-h}$ — ①

Similarly $\lim_{R\to P}$ (slope of chord PR) $= \lim_{h\to 0} \dfrac{f(c+h)-f(c)}{h}$ — ②

As $Q \to P$ & $R \to P$, the chords PQ & PR become tangent at P.

$\therefore$ from ① & ②, we have the slope of the tangent at 'P'.

$$\lim_{h\to 0} \frac{f(c-h)-f(c)}{-h} = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$$

$\therefore f(x)$ is differentiable at $x = c$

$$\Leftrightarrow \lim_{h\to 0} \frac{f(c-h)-f(c)}{-h} = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$$

* **Derivative of a function at a point:**

Let $f:[a,b]\to\mathbb{R}$ be a function and $c\in(a,b)$, then f is said to be derivable (or differentiable) at c, if

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c} \text{ (or) } \lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$$

exists. The limit is called the derivative (or) the differential coefficient of the function 'f' at $x=c$ and is denoted by $f'(c)$.

i.e. $f'(c)=\lim_{x\to c}\dfrac{f(x)-f(c)}{x-c}$

(or)

Let $f:[a,b]\to\mathbb{R}$ and $c\in(a,b)$. Then we say that a real number L is the derivative of 'f' at 'c' if given any $\epsilon>0$, $\exists\ \delta(\epsilon)>0$ such that if $x\in I$ satisfies $0<|x-c|<\delta$

then $\left|\dfrac{f(x)-f(c)}{x-c}-L\right|<\epsilon$

In this case, we say that f is differentiable at 'c' and we write $f'(c)$ for L.

→ **Left-hand Derivative:-**

Let $f:[a,b]\to\mathbb{R}$ be a function and $c\in(a,b)$ if

$$\lim_{x\to c^-}\frac{f(x)-f(c)}{x-c} \text{ (or) } \lim_{h\to 0}\frac{f(c-h)-f(c)}{-h}$$

exists, then this limit is called the left-hand derivative of 'f' at 'c' and is denoted by $f'(c-0)$ (or) $f'(c-)$ (or) $Lf'(c)$.

→ **Right-hand Derivative:-**

Let $f:[a,b]\to\mathbb{R}$ be a function and $c\in(a,b)$.

If $\lim_{x\to c+}\dfrac{f(x)-f(c)}{x-c}$ (or) $\lim_{h\to 0}\dfrac{f(c+h)-f(c)}{h}$

exists, then this limit is called the right-hand derivative of 'f' at 'c' and denoted by $f'(c+)$ (or) $f'(c+0)$ (or) $Rf'(c)$.

Note:- The derivative $f'(c)$ exists $\iff Lf'(c)=Rf'(c)$.

→ **Derivability in an interval:**

* A function $f:[a,b]\to\mathbb{R}$ is said to be derivable in the open interval (a,b) if $f'(c)$ exists for each $c\in(a,b)$.

* A function $f:[a,b]\to\mathbb{R}$ is said to be derivable in $[a,b]$ if

(i) $f'(c)$ exists at $c\in(a,b)$

(ii) $Rf'(a)$ exists

(iii) $Lf'(b)$ exists.

* A function $f: I \to \mathbb{R}$ is said to be derivable on I if f is derivable at every point of I.

Ex:- (1) $f(x) = x^2 \; \forall x \in \mathbb{R}$

let $x = c \in \mathbb{R}$

then $f(c) = c^2$

Now $f'(c) = \underset{x \to c}{Lt} \dfrac{f(x)-f(c)}{x-c}$

$= \underset{x \to c}{Lt} \dfrac{x^2-c^2}{x-c}$

$= \underset{x \to c}{Lt} (x+c)$

$= 2c$ (exists)

$\therefore f(x)$ is derivable function at $x = c \in \mathbb{R}$.

$\therefore f'(x)$ is defined on $\mathbb{R}$ and $f'(x) = 2x \; \forall x \in \mathbb{R}$.

Ex:- (2) $f(x) = |x| \; \forall x \in \mathbb{R}$

Sol'n: $f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

at $x = 0$, $f(0) = 0$.

LHD $\quad Lf'(0) = \underset{x \to 0^-}{Lt} \dfrac{f(x)-f(0)}{x-0}$

$= \underset{x \to 0^-}{Lt} \dfrac{-x-0}{x}$

$= \underset{x \to 0^-}{Lt} (-1) = -1$

RHD :- $\quad Rf'(0) = \underset{x \to 0^+}{Lt} \dfrac{f(x)-f(0)}{x-0}$

$= \underset{x \to 0^+}{Lt} \dfrac{x-0}{x}$

$= \underset{x \to 0^+}{Lt} (1) = 1$

$Lf'(0) \neq Rf'(0)$

Theorem :- If $f: I \to \mathbb{R}$ has a derivative at $c \in I$ then f is continuous at 'c'.

Proof : Since f has a derivative at $c \in I$.

$\therefore f'(c) = \underset{x \to c}{Lt} \dfrac{f(x)-f(c)}{x-c} \; \forall c \in I.$ ——— (1)

Now for $x \in I$; $x \neq c$,

we have

$f(x) - f(c) = \left(\dfrac{f(x)-f(c)}{x-c} \right) * (x-c)$

Now applying limit on bothsides at $x \to c$, we get

$\underset{x \to c}{Lt} (f(x) - f(c)) = \underset{x \to c}{Lt} \left(\dfrac{f(x)-f(c)}{x-c} \right) (x-c)$

$= f'(c) \underset{x \to c}{Lt} (x-c)$ (by (1))

$= f'(c) \times (0)$

$= 0$

$\therefore \underset{x \to c}{Lt} f(x) - \underset{x \to c}{Lt} f(c) = 0.$

$\Rightarrow \underset{x \to c}{Lt} f(x) = f(c)$

$\therefore f(x)$ is continuous at $x = c$.

Note: (1) The converse of the above theorem need not be true.

Ex:- $f(x) = |x| \; \forall x \in \mathbb{R}$

$= \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$

At $x = 0$, $f(0) = 0$

LHL $\quad \underset{x \to 0^-}{Lt} f(x) = \underset{x \to 0^-}{Lt} (-x)$

$= 0$

RHL $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} (x) = 0.$

$\therefore$ LHL $=$ RHL $= f(0)$

$\therefore f$ is continuous at '0'.

LHD : $Lf'(0) = \lim_{x\to 0^-} \frac{f(x)-f(0)}{x-0}$

$= \lim_{x\to 0^-} \frac{-x-0}{x}$

$= -1$

RHD :- $Rf'(0) = \lim_{x\to 0^+} \frac{f(x)-f(0)}{x-0}$

$= \lim_{x\to 0^+} \frac{x-0}{x}$

$= +1$

$\therefore$ LHD $\neq$ RHD

$\therefore f$ is not differentiable at '0'.

Note(2): If f is not continuous at any point, it can not be derivable at that point.

Note(3): $I \subseteq \mathbb{R}$ be an interval, let $c \in I$ and let $f: I \to \mathbb{R}$ and $g: I \to \mathbb{R}$ be functions that are differentiable at 'c'. Then

(a) If $\alpha \in \mathbb{R}$ then the function αf is differentiable at 'c' and $(\alpha f)'(c) = \alpha f'(c)$

(b) The function $f+g$ is differentiable at 'c' and $(f+g)'(c) = f'(c) + g'(c)$

(c) The function fg is differentiable at 'c' and

$(fg)'(c) = f'(c)g(c) + f(c)g'(c).$

(d) If $g(c) \neq 0$ then the function f/g is differentiable at 'c' and $(f/g)'(c) = \frac{f'(c)g(c) - g'(c)f(c)}{(g(c))^2}$

Problems:

$\rightarrow$ use the definition to find the derivative of each of the following functions.

(a) $f(x) = x^3 \ \forall x \in \mathbb{R}$

(b) $g(x) = \frac{1}{x} \ \forall x \in \mathbb{R}; x \neq 0.$

(c) $h(x) = \sqrt{x} \ \forall x > 0.$

(d) $k(x) = \frac{1}{\sqrt{x}}$ for $x > 0.$

Sol'n (c) Let $x = c > 0$

then $h(c) = \sqrt{c}$

Now $h'(c) = \lim_{x\to c} \frac{h(x)-h(c)}{x-c}$

$= \lim_{x\to c} \frac{\sqrt{x}-\sqrt{c}}{x-c}$

$= \lim_{x\to c} \left(\frac{\sqrt{x}-\sqrt{c}}{x-c}\right) \times \left(\frac{\sqrt{x}+\sqrt{c}}{\sqrt{x}+\sqrt{c}}\right)$

$= \lim_{x\to c} \frac{\cancel{(x-c)}}{\cancel{(x-c)}(\sqrt{x}+\sqrt{c})}$

$= \lim_{x\to c} \frac{1}{\sqrt{x}+\sqrt{c}}$

$= \frac{1}{\sqrt{c}+\sqrt{c}} = \frac{1}{2\sqrt{c}}$ exists.

$\therefore f'$ is defined for all +ve value of $\mathbb{R}$ and $f'(x) = \frac{1}{2\sqrt{x}}$

→ show that $f(x) = x^{1/3}$; $x \in \mathbb{R}$ is not differentiable at $x=0$.

Sol'n : at $x=0$; $f(0)=0$

Now $f'(0) = \lim_{x \to 0} \frac{f(x)-f(0)}{x-0}$

$= \lim_{x \to 0} \frac{x^{1/3}-0}{x}$

$= \lim_{x \to 0} \left(\frac{1}{x}\right)^{2/3}$ does not exist.

$\therefore f$ is not differentiable at $x=0$.

→ Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 & \text{for } x \text{ rational.} \\ 0 & \text{for } x \text{ irrational.} \end{cases}$$

Show that f is differentiable at $x=0$ and find $f'(0)$.

Sol'n : Let $f'(0) = \lim_{x \to 0} \frac{f(x)-f(0)}{x-0}$

$= L$ — ①

(i) when x is rational number

then $f'(0) = \lim_{x \to 0} \frac{x^2-0}{x-0}$

$= \lim_{x \to 0} \frac{x^2}{x}$

$= \lim_{x \to 0} x = 0$

ii) when x is irrational number

then $f'(0) = \lim_{x \to 0} \frac{0-0}{x-0}$

$= \lim (0) = 0$

$\therefore f'(0) = \lim_{x \to 0} \frac{f(x)-f(0)}{x-0} = 0 = L$

Now we have

$\left| \frac{f(x)-f(0)}{x-0} - L \right| = \left| \frac{f(x)-f(0)}{x-0} - 0 \right|$

$= \left| \frac{x^2-0}{x} - 0 \right|$

($\because x$ is rational)

$= |x| < \epsilon$ whenever $|x| < \epsilon/1$

choosing $\delta = \epsilon/1$

$\therefore \left| \frac{f(x)-f(0)}{x-0} - L \right| < \epsilon$ whenever $0 < |x-0| < \delta$ — ⓐ

Now we have

$\left| \frac{f(x)-f(0)}{x-0} - L \right| = \left| \frac{f(x)-f(0)}{x-0} - 0 \right|$

$= \left| \frac{0-0}{x-0} - 0 \right|$

($\because x$ is irrational)

$= 0 < \epsilon$ whenever $0 < |x-0| < \delta$

$\therefore \left| \frac{f(x)-f(0)}{x-0} - L \right| < \epsilon$ whenever $0 < |x-0| < \delta$

$\therefore f$ is differentiable at $x=0$.

and $f'(0) = 0$.

P-I 2006 → Find 'a' & 'b' so that $f'(2)$ exists.

where

$$f(x) = \begin{cases} \frac{1}{|x|} & \text{if } |x| > 2 \\ a+bx^2 & \text{if } |x| \leq 2. \end{cases}$$

Hint : (i) It is continuous at $x=2$
(ii) It is derivable at $x=2$

Sol^n: Since f is derivable at $|x|=2$

i.e. at $x=2$.

$\therefore$ f is continuous at $|x|=2$

Now LHL

$$\lim_{|x|\to 2^-} f(x) = \lim_{|x|\to 2^-} (a+bx^2)$$

$$= \lim_{|x|\to 2^-} (a+b|x|^2)$$

$$= (a+4b)$$

Now RHL

$$\lim_{|x|\to 2^+} f(x) = \lim_{|x|\to 2^+} \left(\frac{1}{|x|}\right)$$

$$= \frac{1}{2}$$

and at $|x|=2$ i.e. $x=2$

$$f(2) = a+4b$$

Since f is continuous at $|x|=2$.
[$\because$ Given that $f'(x)$ exists]

$$\therefore \lim_{|x|\to 2^-} f(x) = \lim_{|x|\to 2^+} f(x) = f(2)$$

$$\Rightarrow a+4b = \frac{1}{2} = a+4b$$

$$\Rightarrow \boxed{a+4b = \frac{1}{2}} \quad \text{——} \quad (1).$$

Now LHD:

$$Lf'(2) = \lim_{|x|\to 2^-} \frac{f(x)-f(2)}{x-2}$$

$$= \lim_{|x|\to 2^-} \frac{(a+bx^2)-(a+4b)}{x-2}$$

$$= \lim_{|x|\to 2^-} \left(\frac{bx^2-4b}{x-2}\right)$$

$$= \lim_{|x|\to 2^-} \frac{b(x^2-4)}{x-2}$$

$$= \lim_{x\to 2^-} b(x+2)$$

$$= b(2+2) = 4b.$$

Now RHD:

$$Rf'(2) = \lim_{|x|\to 2^+} \frac{f(x)-f(2)}{x-2}$$

$$= \lim_{|x|\to 2^+} \left[\frac{\frac{1}{|x|} - (a+4b)}{x-2}\right]$$

$$= \lim_{|x|\to 2^+} \left[\frac{\frac{1}{|x|} - \frac{1}{2}}{x-2}\right] \text{(using (1))}$$

$$= \lim_{|x|\to 2^+} \left[\frac{2-|x|}{2|x|(x-2)}\right]$$

$$= -\lim_{|x|\to 2^+} \left[\frac{|x|-2}{2|x|(x-2)}\right]$$

$$= -\lim_{|x|\to 2^+} \left[\frac{\cancel{(x-2)}}{2|x|\cancel{(x-2)}}\right]$$

$$= -\lim_{|x|\to 2^+} \left[\frac{1}{2|x|}\right]$$

$$= -\frac{1}{2(2)} = -\frac{1}{4}$$

Since f is derivable at $|x|=2$.

$$\therefore Lf'(2) = Rf'(2)$$

$$\Rightarrow 4b = -\frac{1}{4}$$

$$\Rightarrow \boxed{b = -\frac{1}{16}}$$

$$(1) \equiv a + 4\left(-\frac{1}{16}\right) = \frac{1}{2}$$

$$\Rightarrow a - \frac{1}{4} = \frac{1}{2}$$

$$\Rightarrow \boxed{a = \frac{3}{4}}$$

$$\therefore a = \frac{3}{4} \;\&\; b = -\frac{1}{16}.$$

P-I
'2003, For all real numbers x, $f(x)$ is given as

$$f(x) = \begin{cases} e^{x} + a\sin x \;;\; x<0 \\ b(x-1)^2 + x - 2 \;;\; x \geq 0 \end{cases}$$

Find the values of a & b for which f is differentiable at $x=0$.

→ The function f defined by

$$f(x) = \begin{cases} x^2 + 3x + a & \text{if } x \leq 1 \\ bx + 2 & \text{if } x > 1 \end{cases}$$

is given to be derivable for every x. Find the values of a and b at $x=1$.

Sol'n:- Since f is derivable for every x

$\therefore$ f must be derivable at $x=1$

and hence f must be continuous at $x=1$.

→ For what choice of a & b, if any will the function

$$f(x) = \begin{cases} ax - 6 & \text{if } x > 1 \\ bx^2 & \text{if } x \leq 1 \end{cases}$$

become differentiable at $x=1$?

→ (i) Determine if $f(x)$ has a derivative at $x=0$ when

$$f(x) = \begin{cases} x^2 \sin 1/x \;;\; x \neq 0 \\ 0 \;;\; x=0 \end{cases}$$

) Examine the function

$$f(x) = \begin{cases} x^2 \cos 1/x \;;\; x \neq 0 \\ 0 \;;\; x=0 \end{cases}$$

for the existence of derivative at $x=0$.

→ Discuss the continuity and differentiability of the following functions at $x=a$.

(i) $f(x) = \begin{cases} (x-a)\cdot \sin\left(\frac{1}{x-a}\right) \;;\; x \neq a \\ 0 \;;\; x=a \end{cases}$

(ii) $f(x) = \begin{cases} (x-a)^2 \sin\left(\frac{1}{x-a}\right) \;;\; x \neq a \\ 0 \;;\; x=a \end{cases}$

Sol'n (i) : Since $x \to a^- \Rightarrow (x-a) \to 0^-$

$\Rightarrow \frac{1}{x-a} \to -\infty$

$x \to a^+ \Rightarrow (x-a) \to 0^+$

$\Rightarrow \frac{1}{(x-a)} \to +\infty$

Note

Continuous at $x=a$:-

at $x=a$

$f(a) = 0$

LHL $\lim_{x\to a^-} f(x) = \lim_{x\to a^-} (x-a)\sin\left(\frac{1}{x-a}\right)$

$= 0 \times l \quad (\because -1 \leq l \leq 1)$

$= 0$

RHL $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} (x-a)\sin\left(\frac{1}{x-a}\right)$

$= 0 \times l \quad (\because -1 \leq l \leq 1)$

$= 0$

$\therefore$ LHL = RHL = $f(0)$

$\therefore$ f is continuous at $x=a$.

Differentiable at $x=a$:

LHD : $\lim_{x\to a^-} \left(\frac{f(x) - f(a)}{x-a}\right)$

$= \lim_{x\to a^-} \left(\frac{(x-a)\sin\frac{1}{x-a} - 0}{x-a}\right)$

$\lim_{x\to a^-} \sin\left(\frac{1}{x-a}\right) = l$

$(\because -1 \leq l \leq 1)$

Here l is finite but not fixed because it rotates with -1 to $+1$.

$\therefore$ LHD does not exist.

Similarly RHD does not exist.

$\therefore$ f is not differentiable at $x=a$

P-I 2002 Let $f(x) = \begin{cases} x^p \sin 1/x ; & x \neq 0 \\ 0 ; & x=0 \end{cases}$

Obtain condition p so that

(i) f is continuous at $x=0$ and

(ii) f is differentiable at $x=0$.

Sol^n: (i) At $x=0$

$f(0)=0$

LHL $\lim_{x\to 0-} f(x) = \lim_{x\to 0-} x^p(\sin 1/x)$ — (1)

RHL $\lim_{x\to 0+} f(x) = \lim_{x\to 0+} x^p(\sin 1/x)$ — (2)

f is continuous at $x=0$

If the limits (1) & (2) both must be zero.

This is possible only when $P>0$.

$\therefore$ the required condition for continuity of f at $x=0$ is $P>0$.

(ii) LHD $Lf'(0) = \lim_{x\to 0-} \frac{f(x)-f(0)}{x-0}$

$= \lim_{x\to 0-} \frac{x^p \sin 1/x - 0}{x}$

$= \lim_{x\to 0-} x^{(P-1)} \sin 1/x$ — (3)

RHD $Rf'(0) = \lim_{x\to 0+} \frac{f(x)-f(0)}{x-0}$

$= \lim_{x\to 0+} \frac{x^P \sin 1/x - 0}{x}$

$= \lim_{x\to 0+} x^{(P-1)} \sin 1/x$ — (4)

f is differentiable at $x=0$ if the limits (3) & (4) both must be zero.

This is possible only when $(P-1)>0$.

$\therefore$ The required condition for differentiability of f at $x=0$ is $P>1$.

H.W. Let $f(x) = \begin{cases} x^m \sin 1/x ; & x \neq 0 \\ 0 ; & x=0 \end{cases}$

what conditions should be imposed on 'm' so that

(i) f may be continuous at $x=0$.

(ii) f may be differentiable at $x=0$

H.W. Show that the following function is continuous at $x=1$, for all values of P.

$f(x) = \begin{cases} Px+1 & \text{if } x \geq 1 \\ x^2+P & \text{if } x<1 \end{cases}$

find the left-hand & right-hand derivatives of $f(x)$ at $x=1$.

Hence find the condition for the existence of the derivative at that point.

H.W. Let $f(x) = \begin{cases} x\left[\frac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}\right] ; & x \neq 0 \\ 0 ; & x=0 \end{cases}$

show that f is continuous but not differentiable at $x=0$.

H.W. A function $f(x)$ is defined as follows,

$f(x) = \begin{cases} 1+\sin x & \text{for } 0<x<\pi/2 \\ 2+(x-\pi/2)^2 & \text{for } x \geq \pi/2 \end{cases}$

Examine its continuity and derivability at $x = \pi/2$.

→ Show that the function defined by $f(x) = |x| + |x-1|$ is continuous but not derivable at $x = 0$ and $x = 1$.

Sol'n -

$x < 0$; $0 \le x \le 1$; $1 < x$

if $x < 0$;

$|x| = -x$

$|x-1| = 1-x$

$\therefore \boxed{f(x) = 1-2x}$

if $0 \le x \le 1$; $|x| = x$,

$|x-1| = 1-x$

$\therefore \boxed{f(x) = 1.}$

if $x > 1$;

$|x| = x$

$|x-1| = x-1$

$\therefore \boxed{f(x) = 2x-1}$

$$\therefore f(x) = \begin{cases} 1-2x & \text{if } x < 0 \\ 1 & \text{if } 0 \le x \le 1 \\ 2x-1 & \text{if } x > 1 \end{cases}$$

Continuity at $x = 0$:

Derivability at $x = 0$; not

Continuity at $x = 1$;

Derivability at $x = 1$; not

11o → Show that the function $f(x)$ defined by $f(x) = |x-1| + 2|x-2|$. is continuous but not derivable at 1 and 2.

→ Discuss the continuity and differentiability of the function.

$f(x) = |x-1| + |x-2|$ in the interval $[0,3]$

Sol'n :-

$0 \le x \le 1$; $1 \le x \le 2$; $2 \le x \le 3$.

if $0 \le x \le 1$

$|x-1| = |1-x|$ &

$|x-2| = 2-x$

$\therefore \boxed{f(x) = 3-2x}$

$\therefore$ if $1 \le x \le 2$;

$|x-1| = x-1$ &

$|x-2| = 2-x$

$\therefore \boxed{f(x) = 1}$

if $2 \le x \le 3$;

$|x-1| = x-1$ &

$|x-2| = x-2$

$\therefore \boxed{f(x) = 2x-3}$

$$\therefore f(x) = \begin{cases} 3-2x & \text{if } 0 \le x \le 1 \\ 1 & \text{if } 1 \le x \le 2 \\ 2x-3 & \text{if } 2 \le x \le 3 \end{cases}$$

Since f is a linear (polynomial) function [illegible] function over the various subintervals.

$\therefore f$ is continuous and differentiable over each subinterval. The only doubtful points are the breaking points $x=1$ and $x=2$.

At $x=1$;

$$f(1) = 1$$

LHL $\lim_{x\to 1-} f(x) = \lim_{x\to 1-} (3-2x)$

$= 3-2(1)$

$= 1$

RHL $\lim_{x\to 1+} f(x) = \lim_{x\to 1} (1) = 1$

$\therefore$ LHL $=$ RHL $= f(1)$

$\therefore f$ is continuous at $x=1$.

Now LHD:

$Lf'(1) = \lim_{x\to 1-} \dfrac{f(x)-f(1)}{x-1}$

$= \lim_{x\to 1-} \dfrac{3-2x-1}{x-1}$

$= \lim_{x\to 1-} \left(\dfrac{-2x+2}{x-1}\right)$

$= -2 \lim_{x\to 1-} \dfrac{(x-1)}{x-1}$

$= -2(1)$

$= -2$

RHD: $Rf'(1) = \lim_{x\to 1+} \dfrac{f(x)-f(1)}{x-1}$

$= \lim_{x\to 1+} \dfrac{1-1}{x-1} = 0$

$\therefore Lf'(1) \neq Rf'(1)$.

$\therefore f$ is not differentiable at $x=1$.

Similarly we can easily show that f is continuous at $x=2$ but not differentiable at $x=2$.

$\therefore f$ is continuous on $[0,3]$. Also differentiable on $[0,3]$ except at $x=1$ and $x=2$.

H.W.: Discuss the continuity and differentiability of the function $f(x) = |x-2| + 2|x-3|$ in $[1,4]$.

Determine where each of the following functions from $\mathbb{R}\to\mathbb{R}$ is differentiable and find derivative.

(a) $f(x) = |x| + |x+1|$

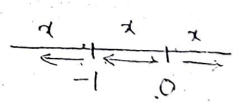
(b) $g(x) = 2x + |x|$

(c) $h(x) = x|x|$.

Sol'n: (a) $f(x) = |x| + |x+1|$ the value of f depends on

$x<0$, $x>0$;

$x+1>0$, $x+1<0$.



(or) $x+1<0$, $x+1>0$, $x<0$, $x>0$.

i.e. $x<-1$, $x>-1$, $x<0$, $x>0$.

i.e. $x<-1$; $-1<x<0$; $x>0$.

if $x<-1$; $|x| = -x$ & $|x+1| = -(x+1)$

$$\boxed{f(x) = -2x-1}$$

if $-1<x<0$; $|x| = -x$ & $|x+1| = x+1$

$\therefore \boxed{f(x) = 1}$

if $x>0$; $|x| = x$ & $|x+1| = x+1$

$\therefore \boxed{f(x) = 2x+1}$

$$f(x) = \begin{cases} -2x-1 , & x<-1 \\ 1 , & -1<x<0 \\ 2x+1 , & x>0 \end{cases}$$

Remember

$$f'(x)\Big|_{x<-1} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{-2(x+h)+1-(-2x+1)}{h}$$

$$= \lim_{h\to 0} \frac{-2h}{h}$$

$$= \lim_{h\to 0} (-2)$$

$$= -2$$

$$f'(x)\Big|_{-1<x<0} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \left(\frac{1-1}{h}\right) = 0$$

$$f'(x)\Big|_{x>0} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{2(x+h)+1-(2x+1)}{h}$$

$$= \lim_{h\to 0} \frac{2h}{h}$$

$$= \lim_{h\to 0} (2)$$

$$= 2$$

$$\therefore f'(x) = \begin{cases} -2 & \text{if } x<-1 \\ 0 & \text{if } -1<x<0 \\ 2 & \text{if } x>0 \end{cases}$$

$x<-1$
$-1<x<0$
$x>0$

(c) $\because f(x) = x|x|$

The value of f depends on $x<0$ and $x>0$.

if $x<0$ then $|x| = -x$

$\therefore \boxed{f(x) = -x^2}$

if $x>0$ then $|x| = x$

$\therefore f(x) = x^2$

$$\therefore f(x) = \begin{cases} -x^2 & \text{if } x<0 \\ x^2 & \text{if } x>0 \end{cases}$$

$$f'(x)\Big|_{x<0} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{-(x+h)^2-x^2}{h}$$

$$= -2x$$

$$\text{and } f'(x)\Big|_{x>0} = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$

$$= \lim_{h\to 0} \frac{(x+h)^2-x^2}{h}$$

$$= 2x$$

$$\therefore f'(x) = \begin{cases} -2x ; & x<0 \\ 2x ; & x>0 \end{cases}$$

$$= 2|x|.$$

→ Prove that if $f : \mathbb{R} \to \mathbb{R}$ is an even function (i.e. $f(-x) = f(x)$ $\forall x \in \mathbb{R}$) and has a derivative at every point then the derivative f′ is an odd function. (i.e. $f'(-x) = -f'(x)$ $\forall x\in\mathbb{R}$). Also

Prove if $g: \mathbb{R} \to \mathbb{R}$ is a differentiable odd function then g' is an even function.

Sol'n:- (I) Since f is even function

$\therefore f(-x) = f(x) \quad \forall x \in \mathbb{R}$

Let $x = c \in \mathbb{R}$ then

$f(-c) = f(c)$

Now $f'(c) = \lim_{x \to c} \frac{f(x)-f(c)}{x-c}$ ——(1)

Now $f'(-c) = \lim_{x \to c} \frac{f(-x)-f(-c)}{-x-(-c)}$

$= \lim_{x \to c} \frac{f(x)-f(c)}{-(x-c)}$

($\because f$ is even)

$= -\lim_{x \to c} \frac{f(x)-f(c)}{x-c}$

$= -f'(c)$ (from (1))

i.e. $f'(-x) = f'(x)$

$\therefore f'$ is an odd function.

(II) Since g is odd function

$\therefore g(-x) = -g(x) \;\forall x \in \mathbb{R}$.

Let $x = c \in \mathbb{R}$ then $g(-c) = -g(c)$

Now $g'(c) = \lim_{x \to c} \frac{g(x)-g(c)}{x-c}$ ——(2)

Now $g'(-c) = \lim_{x \to c} \frac{g(-x)-g(-c)}{-x-(-c)}$

$= \lim_{x \to c} \frac{-g(x)+g(c)}{-(x-c)}$

$= \lim_{x \to c} \frac{g(x)-g(c)}{x-c}$

$= g'(c)$

$\therefore g'(-c) = g'(c)$

i.e. $g'(-x) = g'(x)$

$\therefore g'$ is an even function.

P.I 2001 Let $f(x)$ $(x \in (-\pi, \pi))$ be defined by $f(x) = \sin|x|$. Is f Continuous on $(-\pi, \pi)$? If it is Continuous, then is it differentiable on $(-\pi, \pi)$?

Sol'n:- $f(x) = \sin|x| = \begin{cases} \sin x & x > 0 \\ \sin(-x) & x < 0 \end{cases} \quad \forall x \in (-\pi, \pi)$

$= \begin{cases} \sin x & x \geq 0 \\ -\sin x & x < 0 \end{cases} \quad \forall x \in (-\pi, \pi)$

→ Let $g: \mathbb{R} \to \mathbb{R}$ be defined by

$g(x) = \begin{cases} x^2 \sin(1/x^2) & \text{for } x \neq 0 \\ 0 & \text{for } x = 0 \end{cases}$

Show that g is differentiable for all $x \in \mathbb{R}$.

Also show that the derivative g' is not bounded on the interval $[-1, 1]$.

Sol'n:- $g'(x) = 2x \sin\left(\frac{1}{x^2}\right) + x^2\left(\frac{-2}{x^3}\right)\cos\frac{1}{x^2}$

$= 2x \sin\left(\frac{1}{x^2}\right) - \frac{2}{x}\cos\left(\frac{1}{x^2}\right)$ ——(2)

$\therefore g'(x)$ is well defined for $x \neq 0$.

Now at $x = 0$:

$g(0) = 0$

$x \neq 0$

$\therefore g'(0) = \underset{x\to 0}{\text{lt}} \dfrac{g(x) - g(0)}{x - 0}$

$g'(x)\Big|_{x=0}$ $= \underset{x\to 0}{\text{lt}} \dfrac{x^2 \sin\left(\frac{1}{x^2}\right) - 0}{x}$

$= \underset{x\to 0}{\text{lt}}\ x \sin\left(\frac{1}{x^2}\right) = 0$ ——— (B)

Now we have

$-1 \le \sin\left(\frac{1}{x^2}\right) \le 1 \quad \forall x \in \mathbb{R};\ x \neq 0.$

$\Rightarrow -x \le x \sin\left(\frac{1}{x^2}\right) \le x \quad \forall x > 0.$ is of the form

$f(x) \le t(x) \le h(x)$

Here $f(x) = -x$; $t(x) = x \sin\left(\frac{1}{x^2}\right)$

$h(x) = x$

$\therefore \underset{x\to 0}{\text{lt}} f(x) = 0 = \underset{x\to 0}{\text{lt}} h(x)$

$\therefore$ By Squeeze theorem

$\underset{x\to 0}{\text{lt}}\ x \sin\left(\frac{1}{x^2}\right) = 0.$

$\therefore g'(0) = 0$

$\therefore g$ is differentiable at $x=0$.

(A) $\equiv g'(x)$ is not bounded on $[-1,1]$ as $0 \in [-1,1]$.

Note!

→ If $\gamma > 0$ is a rational number let $f: \mathbb{R} \to \mathbb{R}$ be defined by

$f(x) = \begin{cases} x^{\gamma} \sin(1/x) & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$

Determine these values of γ for which $f'(0)$ exists.

Sol'n:— At $x=0$; $f(0)=0$

Now $f'(0) = \underset{x\to 0}{\text{lt}} \dfrac{f(x) - f(0)}{x - 0}$

$= \underset{x\to 0}{\text{lt}} \dfrac{x^{\gamma} \sin 1/x}{x}$

$= \underset{x\to 0}{\text{lt}}\ x^{\gamma-1} \sin(1/x)$

$= \underset{x\to 0}{\text{lt}}\ x^{\gamma-1}\ \underset{x\to 0}{\text{lt}} \sin(1/x)$

$= 0 \times \underset{x\to 0}{\text{lt}} \sin(1/x)$ ($\because \gamma > 0 \Rightarrow (\gamma-1) > 0$) must be if $f(0)$ exi[sts]

$= 0$

$\therefore f'(0)$ exists for $\gamma > 1$.

$\left| \dfrac{f(x) - f(c)}{x - c} - L \right| < \epsilon$

$0 < |x - c| < \delta$

$f'(c) = L$

$f'(x)\Big|_{x=c}$

* Extreme Value (Definition):

→ A real number 'x' is called an interior point of a set A if A is neighbourhood of x.

i.e. $\exists\ \epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subset A$.

Ex:- (1) Every point of (a,b) is its interior point.

(2). Every point $[a,b]$ is its interior point except a & b.

→ The function $f: I \to \mathbb{R}$ is said to have a relative maximum (or) maximum value (or) maxima at $c \in I$ if $f(c)$ is the greatest value of the function f in a small neighbourhood of $V = V_\delta(c)$ of c.

i.e. for all $x \in (c-\delta, c+\delta)$; $\delta > 0$ such that $f(x) \leq f(c)\ \forall\ x \in V \cap I$.

→ The function $f: I \to \mathbb{R}$ is said to have a relative minimum (or) minimum value (or) minima at $c \in I$ if $f(c)$ is the least value of the function in a small neighbourhood $V = V_\delta(c)$ [i.e. $(c-\delta, c+\delta)$] of c.

i.e. for all $x \in (c-\delta, c+\delta)$; $\delta > 0$ such that $f(x) \geq f(c)\ \forall\ x \in V \cap I$.

→ The function $f: I \to \mathbb{R}$ is said to have relative extremum (or) extreme value at $c \in I$, if f has either relative maximum (or) relative minimum at 'c'.

→ **Interior Extremum Theorem:**

Let c be an interior point of the interval I at which $f: I \to \mathbb{R}$ has a relative extremum at 'c'. If the derivative of f' at 'c' exists then $f'(c) = 0$.

Proof:- Since f has a relative extremum at 'c'

Suppose that f has a relative maximum at 'c'.

$\therefore f(x) \leq f(c)\ \forall\ x \in I \cap V_\delta(c)$. —— (1)

If possible let $f'(c) \neq 0$.

then $f'(c) > 0$ or $f'(c) < 0$.

Case(i): If $f'(c) > 0$

i.e. $\lim_{x \to c} \frac{f(x)-f(c)}{x-c} > 0$

$\therefore \frac{f(x)-f(c)}{x-c} > 0\ \forall\ x \in I \cap V_\delta(c)$; $x \neq c$.

Now if $x \in V_\delta(c)$ and $x > c$

then $f(x) - f(c) = \left(\frac{f(x)-f(c)}{x-c}\right)(x-c) > 0$

$\Rightarrow f(x) - f(c) > 0$

$\Rightarrow f(x) > f(c)$ —— (2)

But (1) & (2) are contradiction.

$\therefore f'(c) \not> 0$ —— (A)

Case (ii): If $f'(c) < 0$ then

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} < 0$$

$\therefore \frac{f(x) - f(c)}{x - c} < 0 \ \forall x \in N_\delta(c);$ $x \neq c$

If $x \in V_\delta(c)$ and $x < c$ then

$$f(x) - f(c) = \left[\frac{f(x) - f(c)}{(x - c)}\right] \times (x - c) > 0$$

$\therefore f(x) - f(c) > 0$

$\Rightarrow f(x) > f(c)$ ——— ③

But ① & ③ are contradiction.

$\therefore f'(c) \not< 0.$ ——— Ⓑ

from Ⓐ & Ⓑ

$\underline{\underline{f'(c) = 0}}$

Note: (1) If f has relative extremum at 'c' then $f'(c)$ may not exist. if it exists then $f'(c) = 0$.

Ex:- $f(x) = |x| \ \forall x \in [-1, 1]$

Sol'n: Let $x = c = 0 \in [-1, 1]$

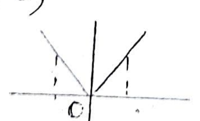
for $x = V_\delta(0)$ $-\delta \quad 0 \quad \delta$

$\Rightarrow x \in (-\delta, \delta)$

(i) $x \in (-\delta, 0)$

$\Rightarrow f(x) > f(0) = 0$

$\therefore f(x)$ has minimum at $x = 0$



$f(0) \le f(x)$ $\forall x \in (-\delta, \delta)$

(ii) $x \in (0, \delta)$

$\Rightarrow f(x) > f(0) = 0$

$\therefore f(x)$ has minimum at $x = 0$

f has relative extremum at $x = 0$.

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \to 0} \frac{|x| - 0}{x - 0}$$

$$= \lim_{x \to 0} \frac{|x|}{x}$$

Now $\lim_{x \to 0-} \frac{|x|}{x} = \lim_{x \to 0-} \frac{-(x)}{x}$

$= \lim_{x \to 0-} (-1) = -1$

Now $\lim_{x \to 0+} \frac{|x|}{x} = \lim_{x \to 0+} \frac{x}{x} = \lim_{x \to 0+} (1) = 1$

$\therefore f'(0)$ does not exist.

Note (2): (a) The converse of above theorem need not be true.

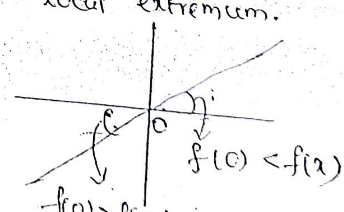
If $f'(c) = 0$ then $f(c)$ may not be an extreme value.

Ex:- $f(x) = x^3 \ \forall x \in \mathbb{R}$

$f'(x) = 3x^2$

At $x = 0$; $f'(0) = 0$.

But f is strictly increasing in $\mathbb{R}$ and has no local extremum.



Definition:-

The point 'c' is said to be stationary point and $f(c)$ the stationary value of the function f if $f'(c) = 0$

*Rolle's Theorem :- [Only Problems]

Suppose that f is continuous on $I = [a, b]$ that the derivative f'

LT

(i) f is cont. on $[a,b]$
(ii) f is derivable on (a,b)
(iii) $f(a) = f(b)$
$\Rightarrow f'(c) = 0$; $c \in (a,b)$.

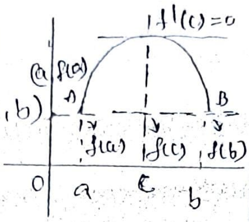
Eg 

exists at every point of (a,b) and $f(a) = f(b) = 0$. Then there exists at least one point $c \in (a,b)$ such that $f'(c) = 0$.

Proof: Case(i)

If $f(x) = 0$ on $I = [a,b]$ then $f'(x) = 0 \ \forall x \in [a,b]$

$\therefore f'(c) = 0 \ \forall x = c \in (a,b)$.



Case(ii):

If $f(x) \neq 0 \ \forall x \in [a,b]$ then $f(x) > 0$ or $f(x) < 0$.

Suppose that $f(x) > 0 \ \forall x \in [a,b]$

i.e. f assumes the +ve values in $I = [a,b]$

Since f is continuous on $I = [a,b]$

$\therefore f$ attains its supremum (lub) at least once in $[a,b]$.

i.e. let f attains its supremum at some point $c \in [a,b]$.

i.e $f(c) = \sup\{f(x) / x \in I = [a,b]\} > 0$ —— (A)

at $x = c \in I$

$\Rightarrow x \in (c-\delta, c+\delta)$

Since f takes some +ve values.

$\therefore f(x) \leq f(c)$

$\forall x \in I \cap (c-\delta$

$\therefore f$ has relative maximum at 'c'.

from (A),

$f(c) > 0$.

Since $f(a) = f(b) = 0$

and then $c \neq a$, $c \neq b$

$\Rightarrow c \in (a,b)$

Since f' exists at every point of (a,b).

$\therefore f'(c)$ exists.

$\therefore f$ has relative maximum at c.

and f has derivative at 'c'.

$\therefore$ By interior extremum theorem

$f'(c) = 0$

for at least one point $c \in (a,b)$.

Hence the theorem.

* **Failure of Rolle's Theorem :-**

Rolle's theorem fails to hold good for a function which does not satisfy all the three conditions of the theorem.

The theorem is not applicable if their (i) f is not continuous in $[a,b]$

(or) (ii) f is not derivable in (a,b)

(or) (iii) $f(a) \neq f(b)$.

Note! The converse of Rolle's theorem is not true i.e. $f'(x) = 0$ at $x = c \in (a,b)$ without $f(x)$ satisfying all the three conditions of Rolle's theorem —

Ex!—

$f(x) = \begin{cases} \ldots & \text{when } 0 \leq x \leq \ldots \\ x+1 & \text{when } 1 < x \leq 2 \end{cases}$

$\forall x \in [0,2]$.

Clearly f is not continuous and not

derivable at $x=1$.

$\therefore$ f is not continuous in $[0,2]$ and f is not derivable in $(0,2)$.

Also $f(0) \neq f(2)$.

But $f(x)=0 \ \forall \ x \in (0,1) \subset (0,2)$.

i.e. $f'(x)=0$ for at least one point $x \in (0,2)$.

Note(2): Another form of Rolle's theorem

If f is continuous on $[a, a+h]$ derivable on $(a, a+h)$ and $f(a)=f(a+h)=0$. then $\exists$ at least one real number $\theta \in (0,1)$ such that $f'(a+\theta h)=0$

Here $b=a+h$; $h>0$ and $c=a+\theta h$

Since $c \in (a,b)$

$\Rightarrow a<c<b$

$\Rightarrow a<a+\theta h<a+h$

$\Rightarrow 0<\theta h<h$

$\Rightarrow 0<\theta<1 \quad (\because h>0)$

$\Rightarrow \theta \in (0,1)$.

Problems:

→ Verify Rolle's theorem in the following cases:

(i) $f(x)=(x-a)^m (x-b)^n$

where m & n are +ve integers in the interval $[a,b]$.

Sol'n :- we have

$f(x)=(x-a)^m (x-b)^n$

(i) Since m & n are +ve integers.

$\therefore f(x)$ is polynomial in x (on expansion by binomial theorem).

Since every polynomial function is continuous function of x for all values of x.

$\therefore f(x)$ is continuous function for all values of x.

$\therefore$ It is continuous on $[a,b]$.

(ii) $f'(x)=m(x-a)^{m-1}(x-b)^n + n(x-a)^m (x-b)^{n-1}$

$=(x-a)^{m-1}(x-b)^{n-1}\left[m(x-b)+n(x-a)\right]$

$=(x-a)^{m-1}(x-b)^{n-1}\left[(m+n)x-(mb+na)\right]$

exists in (a,b)

$\therefore f(x)$ is derivable in (a,b).

(iii) $f(a)=f(b)=0$.

$\therefore f(x)$ satisfies all the three conditions of Rolle's theorem.

$\therefore \exists$ at least one value $x=c \in (a,b)$ such that $f'(c)=0$

$f'(c)=(c-a)^{m-1}(c-b)^{n-1}\left[c(m+n)-(mb+na)\right]$

$=0$

$\Rightarrow c(m+n)-(mb+na)=0$

$(\because c \neq a,\ c \neq b)$ Note!

$\Rightarrow c(m+n)=mb+na$

$\Rightarrow c=\dfrac{mb+na}{m+n} \in (a,b)$

$\therefore$ Rolle's theorem is verified.

11(iv) $f(x) = (x-a)^3(x-b)^4 \ \forall\ x \in [a,b]$

→ $f(x) = 2 + (x-1)^{2/3} \ \forall\ x \in [0,2]$

Sol'n :- Since $f'(x) = \frac{2}{3}(x-1)^{-1/3}$

$= \frac{2}{3(x-1)^{1/3}}$

which does not exist in $x = 1 \in (0,2)$

$\therefore f'(x)$ does not exist in $(0,2)$

$\therefore f$ is not derivable in $(0,2)$

$\therefore$ Rolle's theorem is not applicable to $f(x)$ in $[0,2]$

→ $f(x) = e^x \sin x \ \forall\ x \in [0,\pi]$

Sol'n :- (i) since e^x & $\sin x$ are both continuous functions for values of x.

$\therefore e^x \sin x$ is also continuous for all values of x.

$\therefore f(x)$ is continuous in $[0,\pi]$.

(ii) $f'(x) = e^x \cos x + e^x \sin x$
which exists in $(0,\pi)$.

$\therefore f(x)$ is derivable in $(0,\pi)$.

(iii) $f(0) = e^0 \sin(0)$
$= 0$

$f(\pi) = e^\pi \sin(\pi)$
$= 0$

$\therefore f(0) = f(\pi) = 0$

$\therefore$ The conditions of Rolle's theorem are satisfied.

$\therefore \exists$ atleast one value $c \in (0,\pi)$ such that $f'(c) = 0$

$f'(c) = e^c(\cos c + \sin c) = 0$

$\Rightarrow \cos c + \sin c = 0 \quad (\because e^c \neq 0)$

$\Rightarrow \cos c = -\sin c$

$\Rightarrow 1 = -\tan c$

$\Rightarrow \tan c = -1$

$\Rightarrow \tan c = -\tan(\pi/4)$

$= \tan(\pi - \pi/4)$ [Here we didn't take $(2n-1)\pi - \frac{\pi}{4}$ since $c \in (0,\pi)$]

$\Rightarrow c = \pi - \pi/4$

$\Rightarrow c = 3\pi/4 \in (0,\pi)$

$\therefore$ Rolle's theorem is verified.

11(vi) $f(x) = x(x+3)e^{-x/2} \ \forall\ x \in [-3,0]$

→ $f(x) = |x| \ \forall\ x \in [-1,1]$.

Sol'n :- (i) Since $f(x) = |x|$ is continuous for all values of x

$\therefore$ It is continuous in $[-1,1]$.

(ii) Since $f(x)$ is not derivable at $x = 0 \in (-1,1)$

$\therefore f$ is not derivable in $(-1,1)$

$\therefore$ The Rolle's is not applicable to $f(x) = |x|$ in $[-1,1]$

LHD at 0 = −1, RHD at 0 = 1, Hence LHD ≠ RHD, Hence not derivable at $x=0$

→ $f(x) = \log\left[\frac{x^2+ab}{x(a+b)}\right] \ \forall\ x \in [a,b]$

$0 \notin [a,b]$

Sol'n: (i) $f(x) = \log(x^2+ab) - \log[x(a+b)]$

$= \log(x^2+ab) - \log x - \log(a+b)$

It is continuous in $[a,b]$

(ii) $f'(x) = \frac{2x}{x^2+ab} - \frac{1}{x}$

$= \frac{x^2-ab}{x(x^2+ab)}$ exists in (a,b).

$\therefore f(x)$ is derivable in (a,b).

(iii) $f(a) = \log\left[\dfrac{(a^2+ab)}{a(a+b)}\right]$

$= \log\left(\dfrac{a^2+ab}{a^2+ab}\right)$

$= \log(1) = 0$

$f(b) = \log\left[\dfrac{b^2+ab}{b(a+b)}\right]$

$= \log(1) = 0$

$\therefore f(a) = f(b) = 0$

$\therefore$ The conditions of Rolle's theorem are satisfied.

$\therefore \exists$ atleast one point $c \in (a,b)$ such that $f'(c) = 0$.

$$f'(c) = \frac{c^2-ab}{c(c^2+ab)} = 0$$

$\Rightarrow c^2 - ab = 0$

$\Rightarrow c^2 = ab$

$\Rightarrow c = \pm\sqrt{ab}$

$\Rightarrow c = +\sqrt{ab} \in (a,b) \{\because -\sqrt{ab} \notin (a,b)\}$

$\therefore$ Rolle's is verified. (neglecting $-\sqrt{ab}$)

H.w.

$\rightarrow f(x) = \log\left(\dfrac{x^2+3}{4x}\right) \; \forall x \in [1,3]$

$\rightarrow f(x) = x^2 - 6x + 8 \; \forall x \in [2,4]$

$\rightarrow f(x) = 8x - x^2 \; \forall x \in [2,6]$

$\rightarrow f(x) = \begin{cases} x^2+1 & \text{for } 0 \le x \le 1 \\ 3-x & \text{for } 1 \le x \le 2 \end{cases}$

Sol^n:- Here $f(x)$ is defined in $[0,2]$.

Since $f(x) = x^2+1$ for $0 \le x \le 1$ i.e $x \in [0,1]$ is a polynomial.

$\therefore$ It is continuous & derivable in $[0,1]$.

Since $f(x) = 3-x$ for $1 \le x \le 2$ is a polynomial.

$\therefore$ It is continuous & derivable in $[1,2]$

Since the domain of function $f(x)$ is $[0,2]$ which is partioned at $x=1$ we are not sure about the continuity and derivability of $f(x)$ at $x=1$.

Now LHL $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-} (x^2+1)$

$= 2$

RHL $\lim\limits_{x\to 1^+} f(x) = 2$

at $x = 1$

$f(1) = 2$

$\therefore f$ is continuous at $x=1$.

LHD $Lf'(1) = \lim\limits_{x\to 1^-} \dfrac{f(x)-f(1)}{x-1}$

$= \lim\limits_{x\to 1^-} \dfrac{x^2+1-2}{x-1}$

$= \lim\limits_{x\to 1^-} \left(\dfrac{x^2-1}{x-1}\right)$

$= \lim_{x\to 1-} (x+1)$

$= 2.$

RHD $\quad Rf'(1) = \lim_{x\to 1+} \frac{f(x)-f(1)}{x-1}$

$= \lim_{x\to 1+} \frac{3-x-2}{x-1}$

$= \lim_{x\to 1+} \frac{1-x}{x-1}$

$= -1$

$\therefore$ LHD $\neq$ RHD

$\therefore$ f is not derivable at $x=1$

$\therefore$ f is not derivable in $(0,2)$.

$\therefore$ Rolle's theorem is not applicable to $f(x)$ in $[0,2]$.

Imp → Let $\frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \cdots + \frac{a_{n-1}}{2} + a_n = 0$

Show that the function $a_0x^n + a_1x^{n-1} + \cdots + a_n$ Vanishes $[i.e. = 0]$ atleast once in $(0,1)$.

Sol'n:- Let $f(x) = a_0 \frac{x^{n+1}}{n+1} + a_1 \frac{x^n}{n} + a_2 \frac{x^{n-1}}{n-1} + \cdots + \frac{a_{n-1}}{2}x^2 + a_n x$ $\forall x \in [0,1]$.

Since $f(x)$ is a polynomial. which is continuous & derivable for all x.

$\therefore$ f is continuous in $[0,1]$ & derivable in $(0,1)$.

Also $f(0) = 0$.

and $f(1) = \frac{a_0}{n+1} + \frac{a_1}{n} + \cdots + \frac{a_{n-1}}{2} + a_n$

$= 0$ (given)

$\therefore$ The conditions of Rolle's theorem is satisfied.

$\therefore \exists$ atleast one point $x \in (0,1)$ such that $f'(x) = 0$.

$\Rightarrow f'(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0.$

H.W: By considering the function $(x-4)\log x$, show that the equation $x\log x = 4-x$ is satisfied by at least one value of $x \in (1,4)$.

Sol'n: Let $f(x) = (x-4)\log x$

Show that between any two roots of $e^x\cos x = 1$, $\exists$ atleast one root of $e^x\sin x - 1 = 0$. It gives us hint to apply Rolle's Th.

i.e. $\sin x - e^{-x} = 0$

Sol'n: Let $x=a$ & $x=b$ be two distinct roots of the given equation $e^x\cos x = 1$

$\therefore e^a\cos a = 1$ & $e^b\cos b = 1$

$\Rightarrow \cos a = e^{-a}$ & $\cos b = e^{-b}$ —— (1)

Note!

Let $f(x) = -\cos x + e^{-x}$ $\forall x \in [a,b]$

(i) Since $\cos x$ & e^{-x} are continuous in $[a,b]$

$\therefore f(x)$ is continuous in $[a,b]$

ii, $f'(x) = \sin x - e^{-x}$

which exists for all $x \in (a,b)$.

$\therefore f$ is derivable in (a,b).

iii, $f(a) = -\cos a + e^{-a}$

$= 0$ (by ①)

& $f(b) = -\cos b + e^{-b}$

$= 0$ (by ①)

$\therefore f(a) = f(b) = 0$

$\therefore$ The Conditions of Rolle's theorem are satisfied.

$\therefore \exists$ at least one point $c \in (a,b)$ such that $f'(c) = 0$.

$\Rightarrow f'(c) = \sin c - e^{-c} = 0$

$\Rightarrow \sin c = e^{-c}$

$\Rightarrow e^{c}\sin c - 1 = 0$

$\Rightarrow x = c \in (a,b)$ is a root of the equation $e^{x}\sin x - 1 = 0$

$\therefore e^{x}\sin x - 1$ has at least one root b/w any two roots of the equation.

$e^{x}\cos x = 1$.

H.W. Prove that b/w any two roots of $e^{x}\sin x = 1$, $\exists$ atleast one real root of

$e^{x}\cos x + 1 = 0$.

V.V. Imp.

* Lagrange's Mean Value theorem :-

(first Mean Value theorem of Differential Calculus)

Statement : Suppose that f is continuous on $I=[a,b]$ and f has a derivative in (a,b). Then there exists at least one point $c\in(a,b)$ such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$ [in Rolle's Th., $f'(c)=0$]

i.e. $f(b)-f(a)=f'(c)(b-a)$

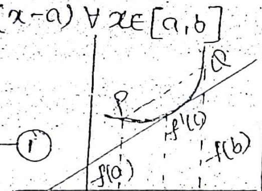
Prove it with the help of Rolle's Th.

Proof :- Consider the function

$\phi(x)=f(x)-f(a)-k(x-a)\ \forall\ x\in[a,b]$

where

$$k=\frac{f(b)-f(a)}{b-a} \quad \text{—(1)}$$



Since $f(x)$ is continuous on $I=[a,b]$

Since $(x-a)$ is polynomial it continuous on I and $f(a)$ & k are constants.

$\therefore \phi(x)$ is continuous on $[a,b]$.

Now $\phi'(x)=f'(x)-k$ exists in (a,b) —(2)

$[\because f'(x)$ exists in $(a,b)]$

Now $\phi(a)=0$

and $\phi(b)=f(b)-f(a)-k(b-a)$

$=(f(b)-f(a))-\left(\frac{f(b)-f(a)}{b-a}\right)(b-a)$

$=0$

→ (i) f is cont. on $[a,b]$
(ii) f is diff. on (a,b)
Note: In Rolle's Th., there was IIIrd condition i.e $f(a)=f(b)$

$\therefore \phi(a)=\phi(b)=0$

$\therefore \phi(x)$ satisfies the conditions of Rolle's theorem.

$\therefore \exists$ at least one $c\in(a,b)$ such that $\phi'(c)=0$.

$\phi'(c)=f'(c)-k=0$

$\Rightarrow f'(c)=k$

$$\Rightarrow f'(c)=\frac{f(b)-f(a)}{b-a}$$

Another statement :-

If a function f defined on $[a,b]$ is
(i) Continuous on $[a,a+h]$
(ii) derivable on $(a,a+h)$ then $\exists$ atleast one real number $\theta\in(0,1)$ such that $f(a+h)=f(a)+hf'(a+\theta h)$

Here $b=a+h$
& $c=a+\theta h$.

* Deductions from Lagrange's Mean Value theorem :-

→ If a function f is continuous on closed interval $I=[a,b]$ and derivable on (a,b) and $f'(x)=0\ \forall x\in(a,b)$ then f is constant on $I=[a,b]$

Sol^n : Let [illegible] distinct points of $[a,b]$ so that $[x_1,x_2]\subset[a,b]$.

Then f satisfies both conditions of Lagrange's mean value theorem on $[x_1, x_2]$.

$\therefore \exists\ c \in (x_1, x_2)$ such that

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(c) \quad \text{---} (1)$$

But $f'(x) = 0 \ \forall x \in (a,b)$ and $x_1 < c < x_2$

$\therefore f'(c) = 0$

from (1), $\dfrac{f(x_2)-f(x_1)}{x_2-x_1}$

$\Rightarrow f(x_2) - f(x_1) = 0$

$\Rightarrow f(x_1) = f(x_2)$

Since x_1 & x_2 are any two distinct points of $[a,b]$.

it follows that f keeps the same value for every $x \in [a,b]$.

$\therefore f(x)$ is constant on $[a,b]$.

→ If two functions f & g are continuous on $[a,b]$, differentiable on (a,b) and $f'(x) = g'(x) \ \forall x \in [a,b]$ then $f-g$ is a constant on $[a,b]$

Sol^n: Let us consider $\phi(x) = f(x) - g(x) \ \forall x \in [a,b]$

Since f & g continuous on $[a,b]$ and differentiable on (a,b).

$\therefore \phi$ is continuous on $[a,b]$ and differentiable on (a,b).

$\therefore \phi'(x) = f'(x) - g'(x)$ exists on (a,b).

Since $f'(x) = g'(x) \ \forall x \in [a,b]$

$\therefore \phi'(x) = 0 \ \forall x \in (a,b)$.

Since ϕ is continuous on $[a,b]$; differentiable on (a,b) and

$\phi'(x) = 0 \ \forall x \in (a,b)$

$\therefore \phi$ is a constant function on $[a,b]$.

i.e. $f-g$ is constant on $[a,b]$.

* Increasing and Decreasing Functions:

If in a part of the domain of the function $f(x)$,

→ $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$ then $f(x)$ is called monotonically increasing function in that part.

→ $x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$ then $f(x)$ is called strictly monotonically increasing function in that part

→ $x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$

monotonically decreasing.

→ $x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

Strictly Monotonically decreasing

Theorem:

Let $f: I \to \mathbb{R}$ be differentiable on I then

(a) f is increasing on I iff $f'(x) \geq 0 \; \forall x \in I$.

(b) f is decreasing on I iff $f'(x) \leq 0 \; \forall x \in I$.

Proof :- (a) Suppose that $f'(x) \geq 0 \; \forall x \in I$.

Let $x_1, x_2 \in I$ with $x_1 < x_2$, so that $[x_1, x_2] \subset I$.

Since f is differentiable on I

$\therefore$ It is differentiable on $[x_1, x_2]$ and therefore it is continuous on $[x_1, x_2]$.

$\therefore f$ satisfies both the conditions of Lagranges mean value theorem on $[x_1, x_2]$.

$\therefore \exists \; c \in (x_1, x_2)$ such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c).$$

$$\Rightarrow f(x_2) - f(x_1) = (x_2 - x_1) f'(c) \quad \text{--- (1)}$$

Since $x_1 < x_2 \Rightarrow x_2 - x_1 > 0$,

$f'(x) \geq 0 \; \forall \; x \in I$ and $x_1 < c < x_2$

$\Rightarrow f'(c) \geq 0$

$\therefore$ from (1), $f(x_2) - f(x_1) \geq 0$

$\Rightarrow f(x_1) \leq f(x_2)$.

Since $x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$

$\therefore f$ is an increasing on I.

Conversely suppose that f is differentiable on I and f is an increasing on I.

Now for $x \neq c \in I$ then $x > c$ or $x < c$

Case (i) if $x > c$ (i.e. $(x-c) > 0$)

then $f(x) \geq f(c)$ ($\because f$ is increasing on I).

$\Rightarrow f(x) - f(c) \geq 0$

$\Rightarrow \dfrac{f(x) - f(c)}{x - c} \geq 0 \quad (\because x - c > 0)$

$\Rightarrow \lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c} \geq 0$ --- (2)

Case (ii) If $x < c$ (i.e. $(x-c) < 0$)

then $f(x) \leq f(c)$ ($\because f$ is increasing on I)

$\Rightarrow f(x) - f(c) \leq 0$

$\Rightarrow \dfrac{f(x) - f(c)}{x - c} \geq 0 \quad (\because x - c < 0)$

$\Rightarrow \lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c} \geq 0$ --- (3)

Since f is differentiable on I.

Let f be differentiable at $c \in I$

$\therefore f'(c) = \lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c}$

$\therefore$ from (2) & (3)

we have

$f'(c) \geq 0$.

(b) The proof of part (b) is similar.

Problems:

* Verify Lagrange's mean value theorem for the following functions in the specified intervals:

→ $f(x) = x(x-1)(x-2)\ \forall x \in [0, \tfrac{1}{2}]$

Sol'n: $f(x) = x^3 - 3x^2 + 2x$ is a polynomial in x, which is continuous in $[0, \tfrac{1}{2}]$.

$f'(x) = 3x^2 - 6x + 2$ exists in $(0, \tfrac{1}{2})$

$\therefore f$ is differentiable in $(0, \tfrac{1}{2})$.

$\therefore f$ satisfies the conditions of Lagrange's Mean value Theorem.

$\therefore \exists\ c \in (0, \tfrac{1}{2})$ such that

$$f'(c) = \frac{f(\tfrac{1}{2}) - f(0)}{\tfrac{1}{2} - 0}$$

$$\Rightarrow 3c^2 - 6c + 2 = \frac{3/8 - 0}{1/2}$$

$$\Rightarrow 3c^2 - 6c + 2 = 3/4$$

$$\Rightarrow 12c^2 - 24c + 8 = 3$$

$$\Rightarrow 12c^2 - 24c + 5 = 0$$

$$\Rightarrow c = \frac{24 \pm \sqrt{576 - 240}}{24}$$

$$\Rightarrow c = \frac{24 \pm \sqrt{336}}{24}$$

$$\Rightarrow c = \frac{24 \pm 4\sqrt{21}}{24}$$

$$\Rightarrow c = \frac{6 \pm \sqrt{21}}{6}$$

Now the two values of c are

$1 + \frac{1}{6}\sqrt{21}$, $1 - \frac{1}{6}\sqrt{21}$

In these two values of c the second value $1 - \frac{1}{6}\sqrt{21} \in (0, \tfrac{1}{2})$.

$\therefore \exists$ atleast one value of c i.e. $c = 1 - \frac{1}{6}\sqrt{21} \in (0, \tfrac{1}{2})$ such that

$$\frac{f(\tfrac{1}{2}) - f(0)}{\tfrac{1}{2} - 0} = f'(c)$$

$\therefore$ The Lagranges Mean value theorem is verified.

H.W.

→ $f(x) = x^2 - 3x + 2\ \forall x \in [-2, 3]$

→ $f(x) = x^3 + x^2 - 6x\ \forall x \in [-1, 4]$

→ $f(x) = e^x$ on $[0, 1]$

→ $f(x) = \log x\ \forall x \in [1, e]$ where $e = 2.71828$.

Sol'n: Since $f(x) = \log x$ is continuous for all +ve values of x.

$\therefore$ It is continuous on $[1, e]$ and $f'(x) = 1/x$ exists in $(1, e)$.

$\therefore f$ is derivable in $(1, e)$

$\therefore f$ satisfies the conditions of Lagranges Mean value theorem.

$\therefore \exists$ at least one $c \in (1, e)$ such that

$$f'(c) = \frac{f(e) - f(1)}{e - 1}$$

$$\Rightarrow \frac{1}{c} = \frac{\log e - \log 1}{e - 1}$$

$$\Rightarrow \frac{1}{c} = \frac{1 - 0}{e - 1}$$

$\Rightarrow e-1 = c$

$\Rightarrow c = e-1 \in (1,e)$

$\therefore$ The Lagrange's Mean Value theorem is Satisfied.

H.W. $f(x) = \sqrt{x^2-4} \quad \forall x \in [2,4]$

$\rightarrow f(x) = \begin{cases} 2 & \text{if } x=1 \\ x^2 & \text{if } 1<x<2 \\ 1 & \text{if } x=2 \end{cases}$

Sol'n: Since $f(x) = x^2$ is a polynomial function in $1<x<2$ and every polynomial function is continuous for all values of x.

$\therefore$ It is continuous on $(1,2)$

Now at $x=1$;

$f(1) = 2$

Now $\lim_{x\to 1+} f(x) = \lim_{x\to 1+} x^2$

$= 1$

$\therefore \lim_{x\to 1+} f(x) \neq f(1)$.

At $x=2$:

$f(2) = 1$

Now $\lim_{x\to 2-} f(x) = \lim_{x\to 2-} x^2$

$= 4$

$\therefore \lim_{x\to 2-} f(x) \neq f(2)$

$\therefore f(x)$ is not at $x=1$ & 2

$\therefore f(x)$ is continuous in $(1,2)$ but not in $[1,2]$

$\therefore f(x)$ does not satisfy the conditions of Lagrange's Mean Value theorem.

$\therefore$ Lagrange's Mean value theorem is not applicable to $f(x)$.

$\rightarrow f(x) = |x| \quad \forall x \in [-1,2]$

Sol'n: It is continuous on $[-1,2]$ and it is differentiable at each point in $(-1,2)$ except at $x=0$.

$\therefore f(x)$ is not differentiable in $(-1,2)$

$\therefore f(x)$ does not satisfy the conditions of Lagranges Mean Value theorem.

$\therefore$ Lagranges Mean Value theorem is not applicable to $f(x)$.

$\rightarrow$ If $f(x) = (x-1)(x-2)(x-3)$; $a=0, b=4$ find c of Lagrange Mean Value theorem.

Sol'n: $f(x) = (x-1)(x-2)(x-3)$

$= x^3-6x^2+11x-6$

$f(a) = f(0)$

$= -6$

$f(b) = f(4)$

$= (3)(2)(1) = 6$

$f'(x) = 3x^2-12x+$

$f'(c) =$

$f'(c) = \frac{f(b)-f(a)}{b-a}$

$3c^2-12c+11 = \frac{6-(-6)}{4-0}$

$\Rightarrow 3c^2-12c+11 = 12/4$

$\Rightarrow 3c^2-12c+11 = 3$

$\Rightarrow 3c^2-12c+8 = 0$

$\Rightarrow c = \frac{12 \pm \sqrt{144-96}}{2\times 3}$

$\Rightarrow c = \frac{12 \pm \sqrt{48}}{6}$

$\Rightarrow c = \frac{12 \pm 4\sqrt{3}}{6}$

$\Rightarrow c = 2 \pm \frac{2}{\sqrt{3}} \in (0,4)$.

→ $f(x) = 1/x \ \forall x \in [-1,1]$

Sol'n: $f(0)$ is not finite while $0 \in [-1,1]$.

∴ LHL

$\lim_{x\to 0-} f(x) = -\infty$ &

RHL

$\lim_{x\to 0+} f(x) = \infty$ LHL $\neq f(x) =$ RHL

Hence ~~f(x)~~

∴ $f(x)$ is not continuous at $x=0$.

∴ $f(x)$ is not continuous on $[-1,1]$

∴ Lagranges Mean Value theorem is not applicable to $f(x)$.

H.W. $f(x) = x^{1/3}$ in $[-1,1]$.

Sol'n: $f'(x) = \frac{1}{3}x^{-2/3} = \frac{1}{3x^{2/3}}$

does not exist at $x=0 \in (-1,1)$

∴ Lagranges Mean value theorem is not applicable to $f(x)$.

However $\frac{f(1)-f(-1)}{1-(-1)} = f'(c)$

$\Rightarrow \frac{1-(-1)}{2} = \frac{1}{3c^{2/3}}$

$\Rightarrow 3c^{2/3} = 1$

$\Rightarrow c^{2/3} = 1/3$

$\Rightarrow c^{1/3} = 1/\sqrt{3}$

$\Rightarrow c = \frac{1}{3\sqrt{3}} \in (-1,1)$

∴ The hypothesis of Lagranges Mean Value theorem is not valid.

i.e., the two conditions of lagranges Mean Value theorem are sufficient but not necessary.

→ show that if $x>0$, $\log(1+x) > \frac{x}{1+x}$ and hence prove that $x^{-1}\log(1+x)$ decreases monotonically as x increases from 0 to ∞.

Sol'n:- Let $f(x) = \log(1+x) - \frac{x}{1+x}$

$f'(x) = \frac{1}{1+x} - \left[\frac{(1+x).1 - x}{(1+x)^2}\right]$

$= \cancel{\frac{1}{1+x}} - \cancel{\frac{1}{1+x}} + \frac{x}{(1+x)^2}$

$= \frac{x}{(1+x)^2} > 0 \quad (\because x>0)$

∴ $f'(x) > 0$ when $x>0$

i.e. $f(x)$ is an increasing when $x>0$.

∴ $f(x) > f(0)$.

~~if f'(x) ≥ 0~~ ,

Now $f(0) = \log(1+0) - \frac{0}{1+0}$

$= \log 1 - 0$

$= 0$

$\therefore f(x) > 0$

$\Rightarrow \log(1+x) - \frac{x}{1+x} > 0$

$\Rightarrow \log(1+x) > \frac{x}{1+x}$

Let $F(x) = x^{-1}\log(1+x)$

$= \frac{\log(1+x)}{x}$

$F'(x) = \frac{x\frac{1}{1+x} - \log(1+x)\cdot 1}{x^2}$

$= \frac{-\left[\log(1+x) - \frac{x}{1+x}\right]}{x^2}$

$= \frac{-f(x)}{x^2} < 0$ for $x>0$

$(\because f(x)>0)$

$\therefore F'(x) < 0$ for $x>0$

$\therefore F(x)$ is a decreasing function in $(0,\infty)$.

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→ Show that

$x - \frac{x^2}{2} < \log(1+x) < x - \frac{x^2}{2(1+x)}$; $x>0$

Sol^n: Let $f(x) = x - \frac{x^2}{2} - \log(1+x)$

$f'(x) = 1 - x - \frac{1}{1+x}$.

$= \frac{1-x^2-1}{1+x}$

$= \frac{-x^2}{1+x} < 0$ for $x>0$.

$\therefore f'(x) < 0$ for $x>0$.

$\therefore f(x)$ is a decreasing function for $x>0$.

$\therefore f(0) > f(x)$.

Now $f(0) = 0 - 0 - \log 1$

$= 0$

$\therefore f(x) < 0$

$\Rightarrow x - \frac{x^2}{2} - \log(1+x) < 0$

$\Rightarrow x - \frac{x^2}{2} < \log(1+x)$ —— (1)

Now let $g(x) = \log(1+x) - x + \frac{x^2}{2(1+x)}$,

$\Rightarrow g'(x) = \frac{1}{1+x} - 1 + \frac{1}{2}\left[\frac{(1+x)2x - x^2(1)}{(1+x)^2}\right]$

$= \frac{1}{1+x} - 1 + \frac{1}{2}\left[\frac{2x+x^2}{(1+x)^2}\right]$

$= \frac{1}{1+x} - 1 + \frac{1}{2}\frac{2x+x^2}{(1+x)^2}$

$= \frac{2(1+x) - 2(1+x)^2 + 2x + x^2}{2(1+x)^2}$

$= \frac{-x^2}{2(1+x)^2} < 0$ for $x>0$.

$\therefore g'(x) < 0$ for $x>0$.

$\therefore g(x)$ is a decreasing function for $x>0$.

$\therefore g(0) > g(x)$.

But $g(0) = 0$

$\therefore g(x) < 0$

$\Rightarrow \log(1+x) - x + \frac{x^2}{2(1+x)} < 0$

$\Rightarrow \log(1+x) < x - \frac{x^2}{2(1+x)}$ —— (2)

Combining (1) & (2),

$x - \frac{x^2}{2} < \log(1+x) < x - \frac{x^2}{2(1+x)}$

[2002 P-I] Show that

Imp

$$\frac{b-a}{\sqrt{1-a^2}} \le \sin^{-1}b - \sin^{-1}a \le \frac{b-a}{\sqrt{1-b^2}}$$

for $0<a<b<1$.

Here $\frac{(\sin^{-1}b - \sin^{-1}a)}{b-a}$ hints us to apply Lagrange's Th

Sol'n: Let $f(x) = \sin^{-1}x \ \forall x \in [a,b]$

where $a>0$; $b<1$

i.e. $0<a<b<1$.

$f(x)$ is continuous & derivable in $[a,b]$ and $f'(x) = \frac{1}{\sqrt{1-x^2}} \ \forall x \in (a,b)$

$\therefore$ By Lagrange's Mean value theorem, $\exists \ c \in (a,b)$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow \frac{1}{\sqrt{1-c^2}} = \frac{\sin^{-1}b - \sin^{-1}a}{b-a} \quad \text{—— (1)}$$

Since $c \in (a,b)$

$\Rightarrow a < c < b$

$\Rightarrow a^2 < c^2 < b^2$

$\Rightarrow -a^2 > -c^2 > -b^2$

$\Rightarrow 1-a^2 > 1-c^2 > 1-b^2$

$\Rightarrow \sqrt{1-a^2} > \sqrt{1-c^2} > \sqrt{1-b^2}$

$\Rightarrow \frac{1}{\sqrt{1-a^2}} < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-b^2}}$

$\Rightarrow \frac{1}{\sqrt{1-a^2}} < \frac{\sin^{-1}b - \sin^{-1}a}{b-a} < \frac{1}{\sqrt{1-b^2}}$

$$\Rightarrow \frac{b-a}{\sqrt{1-a^2}} < \sin^{-1}b - \sin^{-1}a < \frac{b-a}{\sqrt{1-b^2}} \quad \text{—— (I)}$$

→ Prove that $\frac{\pi}{6} + \frac{\sqrt{3}}{15} < \sin^{-1}0.6 < \frac{\pi}{6} + \frac{1}{8}$

Sol'n: Putting $b = 3/5$, $a = 1/2$ then from (I),

$$\frac{\frac{3}{5}-\frac{1}{2}}{\sqrt{1-1/4}} < \sin^{-1}3/5 - \sin^{-1}1/2 < \frac{\frac{3}{5}-\frac{1}{2}}{\sqrt{1-9/25}}$$

$\Rightarrow \frac{1}{10} \times \frac{2}{\sqrt{3}} < \sin^{-1}(0.6) - \pi/6 < \frac{1}{10} \times \frac{5}{4}$

$\Rightarrow \frac{1}{5\sqrt{3}} < \sin^{-1}(0.6) - \frac{\pi}{6} < \frac{1}{8}$

$\Rightarrow \frac{\sqrt{3}}{15} + \frac{\pi}{6} < \sin^{-1}(0.6) < \frac{\pi}{6} + 1/8$

[2008] [6M] → If $x>0$, show that

$$\frac{x}{1+x} < \log(1+x) < x.$$

Sol'n: Let $f(t) = \log(1+t) \ \forall t \in [0,x]$

where $x>0$.

It is continuous & differentiable in $[0,x]$.

and $f'(t) = \frac{1}{1+t} \ \forall \ t \in (0,x)$

By Lagrange's Mean value theorem

$\exists \ c \in (0,x)$ such that

$$f'(c) = \frac{f(x)-f(0)}{x-0}$$

$$\Rightarrow \frac{1}{1+c} = \frac{\log(1+x)-\log 1}{x}$$

$$\Rightarrow \frac{1}{1+c} = \frac{\log(1+x)-0}{x}$$

$$\Rightarrow \frac{1}{1+c} = \frac{\log(1+x)}{x} \quad \text{--- (1)}$$

Since $c \in (0,x)$

$$\Rightarrow 0 < c < x$$

$$\Rightarrow 1 < 1+c < 1+x$$

$$\Rightarrow 1 > \frac{\log(1+x)}{x} > \frac{1}{1+x} \quad \text{(by (1))}$$

$$\Rightarrow x > \log(1+x) > \frac{x}{1+x} \quad (\because x>0)$$

$$\Rightarrow \frac{x}{1+x} < \log(1+x) < x.$$

2000 P-I Use the Mean Value theorem to Prove that $\frac{2}{7} < \log(1.4) < \frac{2}{5}$.

Here $1.4 = (1+\frac{4}{10}) = (1+\frac{2}{5})$

Sol'n: Let $f(t) = \log(1+t)$ $\forall t \in [0,x]$ where $x>0$.

$f(t)$ is continuous & differentiable on $[0,x]$.

and $f'(t) = \frac{1}{1+t}$ $\forall t \in (0,x)$

By Lagrange's Mean Value theorem, $\exists\, c \in (0,x)$ such that

$$f'(c) = \frac{f(x)-f(0)}{x-0}$$

$$\Rightarrow \frac{1}{1+c} = \frac{\log(1+x)-\log 1}{x}$$

$$\Rightarrow \frac{1}{1+c} = \frac{\log(1+x)}{x} \quad \text{--- (1)}$$

Since $c \in (0,x)$

$$\Rightarrow 0 < c < x$$

$$\Rightarrow 1 < 1+c < 1+x$$

$$\Rightarrow 1 > \frac{1}{1+c} > \frac{1}{1+x}$$

$$\Rightarrow 1 > \frac{\log(1+x)}{x} > \frac{1}{1+x} \quad \text{(by (1))}$$

$$\Rightarrow \frac{1}{1+x} < \frac{\log(1+x)}{x} < 1$$

$$\Rightarrow \frac{x}{1+x} < \log(1+x) < x \quad (\because x>0)$$

Putting $x = 2/5$, we get.

$$\frac{2/5}{1+2/5} < \log(1+2/5) < 2/5$$

$$\Rightarrow \frac{2}{5} \times \frac{5}{7} < \log(7/5) < 2/5$$

$$\Rightarrow \frac{2}{7} < \log(1.4) < 2/5$$

H.W. show that

$$\frac{x^2}{2} > x - \log(1+x) > \frac{x^2}{2(1+x)} \text{ for } x>0$$

H.W. Prove that

$$x - \frac{x^2}{2} + \frac{x^3}{3(1+x)} < \log(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3} \text{ for } x>0.$$

→ Apply Lagrange's mean value theorem to the function $\log(1+x)$ to show that

$$0 < [\log(1+x)]^{-1} - x^{-1} < 1 \ \forall x>0$$

Sol'n: Let $f(t) = \log(1+t)$ $\forall t \in [0,x]$ where $x>0$.

which is continuous & differentiable on $[0,x]$.

and $f'(t) = \frac{1}{1+t}$ $\forall t \in (0,x)$.

By Lagranges mean Value theorem

$\exists c \in (0,x)$ such that $f'(c) = \frac{f(x)-f(0)}{x-0}$

$\Rightarrow \frac{1}{1+c} = \frac{\log(1+x) - \log(1)}{x-0}$

$\Rightarrow \frac{1}{1+c} = \frac{\log(1+x)}{x}$ —— (1)

Since $c \in (0,x)$; $x>0$

$\Rightarrow 0 < c < x$

$\Rightarrow 1 < 1+c < 1+x$.

$\Rightarrow 1 > \frac{1}{1+c} > \frac{1}{1+x}$

$\Rightarrow \frac{1}{1+x} < \frac{\log(1+x)}{x} < 1$ (by (1))

$\Rightarrow (1+x) > \frac{x}{\log(1+x)} > 1$.

$\Rightarrow \frac{1}{x} + 1 > \frac{1}{\log(1+x)} > \frac{1}{x}$

$\Rightarrow 1 > \frac{1}{\log(1+x)} - \frac{1}{x} > 0$

$\Rightarrow 0 < [\log(1+x)]^{-1} - x^{-1} < 0$ for $x>0$.

H.W. Use Lagranges mean Value theorem to prove that $1+x < e^x < 1+xe^x$ $\forall x>0$

Let $f(t) = e^t$ $\forall t \in [0,x]$ where $x>0$.

H.W. show that

$\frac{v-u}{1+v^2} < \tan^{-1}v - \tan^{-1}u < \frac{v-u}{1+u^2}$ if $0<u<v$ and deduce that

$\pi/4 + 3/25 < \tan^{-1} 4/3 < \pi/4 + 1/6$.

Sol^n: Let $f(x) = \tan^{-1}x$ $\forall x \in [u,v]$ where $0<u<v$.

→ Use the Mean Value theorem to Prove that $|\sin x - \sin y| \le |x-y|$ $\forall x,y \in \mathbb{R}$ V.V. Imp

Sol^n: If $x=y$ then there is nothing to Prove.

If $x>y$ then Consider the function $f(t) = \sin t$ $\forall t \in [y,x]$.

Clearly if f is continuous on $[y,x]$ and $f'(t) = \cos t$ exists on $[y,x]$

$\therefore$ By Mean Value theorem

$\exists c \in (y,x)$ such that

$f'(c) = \frac{f(x)-f(y)}{x-y}$

$\Rightarrow \cos c = \frac{\sin x - \sin y}{x-y}$

$\Rightarrow \left|\frac{\sin x - \sin y}{x-y}\right| = |\cos c|$

$\Rightarrow \frac{|\sin x - \sin y|}{|x-y|} = |\cos c| \le 1$

$\Rightarrow |\sin x - \sin y| \le |x-y|$

$\therefore$ $\forall x,y \in \mathbb{R}$

$|\sin x - \sin y| \le |x-y|$.

H.W. Use the Mean Value theorem to Prove that $\frac{x-1}{x} < \ln x < (x-1)$ for $x>1$

Let $f(t) = \ln t$ $\forall t \in [1,x]$ where $x>1$.

$f'(t) = \frac{1}{t}$ [illegible] $\log t$

2005 [P-II] V. Imp

Using Lagrange's Mean value theorem, show that $|\cos b - \cos a| \le |b-a|$

→ If a function f is such that its derivative f' is continuous on $[a,b]$ and derivable on (a,b), then show that

$f(b) = f(a) + (b-a)f'(a) + \frac{1}{2}(b-a)^2 f''(c)$.

Sol^n:- Let $\phi(x) = f(x) + (b-x) \cdot f'(x) + (b-x)^2 \cdot K \quad \forall x \in [a,b]$

where $K = \dfrac{f(b) - f(a) - (b-a)f'(a)}{(b-a)^2}$

(or) Let the function ϕ on $[a,b]$ defined by

$\phi(x) = f(x) + (b-x)f'(x) + (b-x)^2 K$

where K is a constant to be determined such that $\phi(a) = \phi(b)$

$\therefore f(a) + (b-a)f'(a) + (b-a)^2 K = f(b)$

$\Rightarrow K = \dfrac{f(b) - f(a) - (b-a)f'(a)}{(b-a)^2}$

Since f' is continuous on $[a,b]$

$\Rightarrow f'$ exists on $[a,b]$

$\Rightarrow f$ is derivable on $[a,b]$

$\Rightarrow f$ is continuous on $[a,b]$

$\therefore$ The functions f and f' are continuous function on $[a,b]$ and derivable on (a,b)

$(b-x), (b-x)^2$ and K are continuous on $[a,b]$ and derivable on (a,b).

$\therefore \phi(x)$ is continuous on $[a,b]$ and derivable on (a,b).

Now

$\phi(a) = f(a) + (b-a)f'(a) + (b-a)^2 K$.

$\Rightarrow \phi(a) = \cancel{f(a)} + \cancel{(b-a)f'(a)} + \cancel{(b-a)^2}\left[\dfrac{f(b) - \cancel{f(a)} - \cancel{(b-a)f'(a)}}{\cancel{(b-a)^2}}\right]$

$= f(b)$

and $\phi(b) = f(b)$

$\phi(a) = \phi(b)$

$\therefore \phi$ satisfies the conditions of Rolle's theorem.

$\therefore \exists\, c \in (a,b)$ such that $\phi'(c) = 0$ —— (1)

but

$\phi'(x) = f'(x) + (-1)f'(x) + (b-x)f''(x) + 2(b-x)(-1)K$

$\Rightarrow \phi'(c) = (b-c)f''(c) + 2(b-c)(-1)K$

$\Rightarrow 0 = (b-c)\left[f''(c) - 2K\right]$ (by (1))

$\Rightarrow f''(c) - 2K = 0$ ($\because b-c \neq 0$ i.e. $c \in (a,b) \Rightarrow a < c < b$)

$\Rightarrow f''(c) = 2K$

$\Rightarrow K = \frac{1}{2} f''(c)$

$\Rightarrow \dfrac{f(b) - f(a) - (b-a)f'(a)}{(b-a)^2} = \frac{1}{2} f''(c)$

$\Rightarrow f(b) - f(a) - (b-a)f'(a) = \frac{1}{2}(b-a)^2 f''(c)$

$\Rightarrow f(b) = f(a) + (b-a)f'(a) + \frac{1}{2}(b-a)^2 f''(c)$

→ If a function f is twice differentiable on $[a, a+h]$ then show that $f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a+\theta h)$ for some real number θ where $\theta \in (0,1)$.

sol'n :— Since f is twice differentiable on $[a, a+h]$

$\Rightarrow f', f''$ exist on $[a, a+h]$

$\Rightarrow f, f'$ are differentiable on $[a, a+h]$

$\Rightarrow f, f'$ are continuous on $[a, a+h]$

Let $\phi(x) = f(x) + (a+h-x)f'(x) + \dfrac{(a+h-x)^2}{2!} K$

where K is a constant to be determined such that $\phi(a) = \phi(a+h)$

$\Rightarrow f(a) + hf'(a) + \dfrac{h^2}{2!} K = f(a+h)$

$\Rightarrow K = \dfrac{f(a+h) - f(a) - hf'(a)}{\left(\frac{h^2}{2!}\right)}$ —— (1)

Since f & f' are continuous on $[a, a+h]$, $[a+h-x]$ and $\dfrac{(a+h-x)^2}{2!} K$ are continuous functions on $[a, a+h]$

$\Rightarrow \phi$ is continuous on $[a, a+h]$.

Since f & f' are derivable on $(a, a+h)$ and $(a+h-x)$, $\dfrac{(a+h-x)^2}{2!} K$ are derivable on $(a, a+h)$.

$\Rightarrow \phi$ is derivable on $(a, a+h)$.

Also $\phi(a) = \phi(b)$

$\therefore \phi$ satisfies the conditions of Rolle's theorem.

$\therefore \exists$ a real number $\theta \in (0,1)$ such that

$\phi'(a+\theta h) = 0$ —— (2)

But $\phi'(x) = f'(x) - f'(x) + (a+h-x)f''(x) - (a+h-x)K$

$= (a+h-x)\left[f''(x) - K\right]$.

$\Rightarrow \phi'(a+\theta h) = (a+h-a-\theta h)$

$[f''(a+\theta h)-k]$

$\Rightarrow 0 = (h-\theta h)[f''(a+\theta h)-k]$ (by ②)

$\Rightarrow f''(a+\theta h)-k = 0 \quad (\because h-\theta h \neq 0)$

$\Rightarrow f''(a+\theta h) = k$

$$\Rightarrow f''(a+\theta h) = \frac{f(a+h)-f(a)-hf'(a)}{\left(\frac{h^2}{2!}\right)}$$

$$\Rightarrow \frac{h^2}{2!} f''(a+\theta h) = f(a+h)-f(a)-hf'(a)$$

$$\Rightarrow f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a+\theta h)$$

P-II 2011
2006

A twice differentiable function f on $[a,b]$ is such that $f(a)=f(b)=0$ and $f(c)>0$ for $a<c<b$. Prove that there is atleast one value ξ, $a<\xi<b$ for which $f''(\xi)<0$.

Sol^n: f is twice differentiable on $[a,b]$

$\Rightarrow f', f''$ exist on $[a,b]$

$\Rightarrow f, f'$ are differentiable on $[a,b]$

f, f' are continuous functions on $[a,b]$.

Since $a<c<b$, applying Lagrange's Mean Value theorem to on the intervals $[a,c]$ and $[c,b]$ we get

$$\frac{f(c)-f(a)}{c-a} = f'(\xi_1)$$

where $a<\xi_1<c$ and

$$\frac{f(b)-f(c)}{b-c} = f'(\xi_2) \text{ where } c<\xi_2<b$$

But $f(a)=f(b)=0$

$$\therefore f'(\xi_1) = \frac{f(c)}{c-a} \text{ and}$$

$$f'(\xi_2) = \frac{-f(c)}{b-c} \text{ where}$$

$a<\xi_1<c<\xi_2<b$.

Again f' is continuous and derivable on $[\xi_1,\xi_2]$

$\therefore$ By Lagrange's Mean Value theorem we have

$$\frac{f'(\xi_2)-f'(\xi_1)}{\xi_2-\xi_1} = f''(\xi)$$

where $\xi_1<\xi<\xi_2$

Substituting the values of $f'(\xi_1)$ and $f'(\xi_2)$, we get

$$f''(\xi) = \frac{\frac{-f(c)}{b-c} - \frac{f(c)}{c-a}}{\xi_2-\xi_1}$$

$$= \frac{-f(c)}{\xi_2-\xi_1}\left[\frac{1}{b-c}+\frac{1}{c-a}\right]$$

$c-a+b-c$ / $b-a$

$$= \frac{-f(c)}{\xi_2-\xi_1}\left[\frac{b-a}{b-c}\right]$$

$a<\xi_1<c<\xi_2<b$ and $f(c)>0$

$\therefore f''(\xi) < 0$ where $a < \xi < b$.

Cauchy's Mean Value theorem (Second Mean Value theorem)

Statement : Let f and g be continuous on $[a,b]$ and differentiable on (a,b) and assume that $g'(x) \neq 0$ $\forall x \in (a,b)$ then $\exists$ $c \in (a,b)$ such that
$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

Proof :- Let $\phi(x) = f(x) - f(a) - K[g(x) - g(a)] \; \forall x \in [a,b]$

where $K = \dfrac{f(b)-f(a)}{g(b)-g(a)}$

If possible let $g(a) = g(b)$

Since $g(x)$ is continuous on $[a,b]$ and differentiable on (a,b)

g satisfies the conditions of Rolle's theorem.

$\therefore \exists$ $c \in (a,b)$ such that $g'(c) = 0$

which is contradiction to $g'(x) \neq 0$ $\forall x \in (a,b)$.

$\therefore g(a) \neq g(b)$

$\therefore \phi(x)$ is well defined.

Since $f(x)$ & $g(x)$ are continuous functions on $[a,b]$

and $f(a), g(a)$ and K are constants.

These are continuous for all x.

$\therefore \phi(x)$ is continuous on $[a,b]$

and $\phi'(x) = f'(x) - Kg'(x)$ exists on (a,b).

because f & g are differentiable functions on (a,b).

$\therefore \phi$ is differentiable function on (a,b)

Now $\phi(a) = 0$

and $\phi(b) = f(b) - f(a) - K[g(b) - g(a)]$

$= [f(b) - f(a)] - \dfrac{f(b)-f(a)}{g(b)-g(a)}[g(b) - g(a)]$

$= 0$

$\therefore \phi(a) = \phi(b)$.

$\therefore \phi(x)$ satisfies the conditions of Rolle's theorem.

$\therefore \exists$ $c \in (a,b)$ such that $\phi'(c) = 0$

But $\phi'(x) = f'(x) - Kg'(x)$ $\forall x \in (a,b)$

$\Rightarrow \phi'(c) = f'(c) - Kg'(c)$

$\Rightarrow 0 = f'(c) - Kg'(c)$ $(\because \phi'(c) = 0)$

$\Rightarrow f'(c) = Kg'(c)$

$\Rightarrow K = \dfrac{f'(c)}{g'(c)}$

$\Rightarrow \dfrac{f'(c)}{g'(c)} = \dfrac{f(b)-f(a)}{g(b)-g(a)}$

Another form of the statement:

If two functions f and g defined on $[a, a+h]$ are

(i) Continuous on $[a, a+h]$

(ii) differentiable on $(a, a+h)$

(iii) $g'(x) \neq 0$ for any $x \in (a, a+h)$

then $\exists$ atleast one real number $\theta \in (0,1)$ such that

$$\frac{f'(a+\theta h)}{g'(a+\theta h)} = \frac{f(a+h)-f(a)}{g(a+h)-g(a)}$$

$\rightarrow$ If f', g' are continuous and differentiable on $[a,b]$ then show that for $a<c<b$

$$\frac{f(b)-f(a)-(b-a)f'(a)}{g(b)-g(a)-(b-a)g'(a)} = \frac{f''(c)}{g''(c)}$$

Sol'n :- Let us consider

$$\phi(x) = f(x) + (b-x)f'(x) + k\{g(x)+(b-x)g'(x)\}$$

$\forall x \in [a,b]$

where k is a constant to be determined such that $\phi(a) = \phi(b)$.

$f(a) + (b-a)f'(a) + k[g(a)+(b-a)g'(a)]$

$= f(b) + kg(b)$.

$$\Rightarrow k = \frac{f(b)-f(a)-(b-a)f'(a)}{g(a)+(b-a)g'(a)-g(b)} \quad \text{(1)}$$

Since f', g' are continuous and differentiable functions on $[a,b]$

$\therefore \phi(x)$ is continuous and differentiable on $[a,b]$.

$\therefore \phi(x)$ satisfies the conditions Rolle's theorem on an interval $[a,b]$

$\therefore \exists\, c \in (a,b)$ such that $\phi'(c) = 0$

But $\phi'(x) = \cancel{f'(x)} + (b-x)f''(x) - \cancel{f'(x)}$
$\quad + k[\cancel{g'(x)} + (b-x)g''(x) - \cancel{g'(x)}]$

$= (b-x)f''(x) + k(b-a)g''(x)$

$\Rightarrow \phi'(c) = (b-c)f''(c) + k(b-c)g''(c)$

$\Rightarrow 0 = (b-c)f''(c) + k(b-c)g''(c)$

$(\because \phi'(c) = 0)$

$$\Rightarrow k = -\frac{f''(c)}{g''(c)} \quad (\because b-c \neq 0).$$

$$\Rightarrow \frac{f(b)-f(a)-(b-a)f'(a)}{g(a)+(b-a)g'(a)-g(b)} = -\frac{f''(c)}{g''(c)}$$

$$\Rightarrow \frac{f(b)-f(a)-(b-a)f'(a)}{g(b)-g(a)-(b-a)g'(a)} = \frac{f''(c)}{g''(c)}$$

P-II 2005

If $f'(x)$ and $g'(x)$ exist for all $x \in [a,b]$ and if $g'(x)$ doesnot vanish any where on (a,b) then prove that for some c between a and b.

$$\frac{f(c)-f(a)}{g(b)-g(c)} = \frac{f'(c)}{g'(c)}$$

Sol^n: Let us consider

$$\phi(x) = f(x)g(x) - f(a)g(x) - g(b)f(x) \; \forall x \in [a,b]$$

Remember

Since f' and g' exists in $[a,b]$.

$\therefore$ f and g are derivable functions on $[a,b]$.

$\therefore$ f and g are Continuous functions on $[a,b]$.

$\therefore$ $\phi(x)$ is Continuous and derivable on $[a,b]$.

and $\phi(a) = -f(a)g(b)$;

$\phi(b) = -f(a)g(b)$

$\therefore$ $\phi(a) = \phi(b)$

$\therefore$ $\phi(x)$ satisfies the conditions of Rolle's theorem on $[a,b]$.

$\therefore$ $\exists$ at least one point $c \in (a,b)$ such that $\phi'(c) = 0$

But $\phi'(x) = f'(x)g(x) + f(x)g'(x) - f(a)g'(x) - g(b)f'(x)$.

$\Rightarrow \phi'(c) = f'(c)g(c) + f(c)g'(c) - f(a)g'(c) - g(b)f'(c)$

$\Rightarrow 0 = f'(c)g(c) + f(c)g'(c) - f(a)g'(c) - g(b)f'(c)$ $(\because \phi'(c)=0)$

$g'(c)[f(c)-f(a)] + f'(c)[g(c)-g(b)] = 0$

$\Rightarrow g'(c)[f(c)-f(a)] = f'(c)[g(b)-g(c)]$

$$\Rightarrow \frac{f(c)-f(a)}{g(b)-g(c)} = \frac{f'(c)}{g'(c)}$$

$[\because g'(x) \neq 0 \; \forall x \in (a,b)]$

Generalised Mean Value Theorem:

If three functions f, g and h defined on $[a,b]$ are

(i) Continuous on $[a,b]$

(ii) Differentiable on (a,b)

there exists a real number $c \in (a,b)$

Such that $\begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0.$

Proof :- Consider the function ϕ on $[a,b]$ defined by

$$\phi(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix}$$

$$= f(x) \begin{vmatrix} g(a) & h(a) \\ g(b) & h(b) \end{vmatrix} - g(x) \begin{vmatrix} f(a) & h(a) \\ f(b) & h(b) \end{vmatrix} + h(x) \begin{vmatrix} f(a) & g(a) \\ f(b) & g(b) \end{vmatrix}$$

$$= A f(x) + B g(x) + C h(x)$$

where A, B, C are Constants.

Since f, g, h are continuous functions on $[a,b]$

$\therefore \phi(x)$ is continuous on $[a,b]$ and f, g, h are differentiable on (a,b)

$\therefore \phi(x)$ is differentiable on (a,b).

and $\phi(a) = \phi(b) = 0$.

$\therefore \phi$ satisfies the conditions of Rolle's theorem.

$\therefore \exists\, c \in (a,b)$ such that $\phi'(c) = 0$. — (1)

But $\phi'(x) = Af'(x) + Bg'(x) + Ch'(x)$ in (a,b)

$$= \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix}$$

$$\Rightarrow \phi'(c) = \begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix}$$

$$\Rightarrow 0 = \begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix}$$

where $c \in (a,b)$.

→ when h is a constant function the above theorem reduces to Cauchy's mean value theorem.

Let $h(x) = k$ (constant) then $h(a) = h(b) = k$ and $h'(c) = 0$.

Substituting generalised mean value theorem, we get,

$$\begin{vmatrix} f'(c) & g'(c) & 0 \\ f(a) & g(a) & k \\ f(b) & g(b) & k \end{vmatrix} = 0$$

$$\Rightarrow \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

which is the Cauchy's Mean value theorem.

→ when $g(x) = x$ and $h(x) = k$ (Constant) the above theorem (Generalised Mean Value) reduces to Lagrange's mean value theorem.

$g(x) = x$ and $h(x) = k$

$\Rightarrow g'(x) = 1$ and $h'(x) = 0$

$\Rightarrow g'(c) = 1$, $h'(c) = 0$ and

$g(a) = a$; $g(b) = b$; $h(a) = h(b) = k$

from generalised mean value theorem,

$$\begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0$$

$$\begin{vmatrix} f'(c) & 1 & 0 \\ f(a) & a & k \\ f(b) & b & k \end{vmatrix} = 0$$

$$\Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}.$$

which is the Lagrange's Mean Value theorem.

Problems :-

* → Verify Cauchy's Mean Value theorem for the following pairs of functions in the specified intervals.

$f(x) = x^2$ & $g(x) = x^3$ $\forall x \in [1,2]$

Sol'n: Since f & g are Continuous on $[1,2]$ and differentiable on $(1,2)$

Also $g'(x) = 3x^2 \neq 0$ for any $x \in (1,2)$.

$\therefore f$ & g Satisfy the conditions of Cauchy's Mean Value theorem.

$\therefore \exists\, c \in (1,2)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(2)-f(1)}{g(2)-g(1)} \quad \text{——— (1)}$$

But $g'(x) = 3x^2$ & $f'(x) = 2x$

$\therefore f'(c) = 2c$ & $g'(c) = 3c^2$

$$(1) \equiv \frac{2c}{3c^2} = \frac{4-1}{8-1}$$

$$\Rightarrow \frac{2}{3c} = \frac{3}{7}$$

$$\Rightarrow 9c = 14$$

$$\Rightarrow c = \frac{14}{9} \in (1,2)$$

$\therefore$ Cauchy's Mean Value theorem is verified.

Eg.

→ Find 'c' of Cauchy's Mean Value Theorem for the following pairs of functions.

(i) $f(x)=e^{x}$, $g(x)=e^{-x}$ $\forall x\in[a,b]$

Sol'n: $f(a)=e^{a}$; $f(b)=e^{b}$

$g(a)=e^{-a}$; $g(b)=e^{-b}$

$f'(x)=e^{x} \Rightarrow f'(c)=e^{c}$

$g'(x)=-e^{-x} \Rightarrow g'(c)=-e^{-c}$

$$\therefore \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

$$\Rightarrow \frac{e^{c}}{-e^{-c}} = \frac{e^{b}-e^{a}}{e^{-b}-e^{-a}}$$

$$\Rightarrow -e^{2c} = \frac{e^{b}-e^{a}}{\frac{1}{e^{b}}-\frac{1}{e^{a}}}$$

$$= \frac{-[-e^{b}+e^{a}]}{\frac{e^{a}-e^{b}}{e^{a}e^{b}}}$$

$$= -e^{a}.e^{b}$$

$$= -e^{(a+b)}$$

$\Rightarrow 2c = a+b$

$\Rightarrow c = \frac{a+b}{2} \in (a,b)$

H.W

(ii) $f(x)=x^{2}$, $g(x)=x$ $\forall x\in[a,b]$

(iii) $f(x)=\sin x$, $g(x)=\cos x$ $\forall x\in[-\pi/2, 0]$

→ Show that $\frac{\sin\alpha-\sin\beta}{\cos\beta-\cos\alpha} = \cot\theta$

where $0<\alpha<\theta<\beta<\pi/2$.

It is of the form $\frac{f(\alpha)-f(\beta)}{f(\alpha)-f(\beta)} = \cot\theta$

Sol'n: Let $f(x)=\sin x$

$g(x)=\cos x$ $\forall x\in[\alpha,\beta]$

Since f and g are both continuous on $[\alpha,\beta]$ and differentiable on (α,β).

$g'(x)=-\sin x \neq 0$ for any $x\in(\alpha,\beta)$

$\therefore$ By Cauchy's Mean value theorem $\exists\, \theta\in(\alpha,\beta)$ such that

$$\frac{f'(\theta)}{g'(\theta)} = \frac{f(\beta)-f(\alpha)}{g(\beta)-g(\alpha)} \quad \text{——} (1)$$

But $f'(x)=\cos x$; $g'(x)=-\sin x$

$\Rightarrow f'(\theta)=\cos\theta$; $g'(\theta)=-\sin\theta$.

$$\therefore (1) \equiv \frac{\sin\beta-\sin\alpha}{\cos\beta-\cos\alpha} = \frac{\cos\theta}{-\sin\theta} ; \theta\in(\alpha,\beta)$$

$$\Rightarrow \frac{\sin\alpha-\sin\beta}{\cos\alpha-\cos\beta} = \cot\theta, \ \theta\in(\alpha,\beta).$$

* Miscellaneous Problems :-

→ Assuming f'' to be continuous on $[a,b]$, show that

$$f(c)-f(a)\left(\frac{b-c}{b-a}\right)-\left(\frac{c-a}{b-a}\right)f(b)=\frac{1}{2}(c-a)(c-b)f''(\xi)$$

where c and ξ both lie in $[a,b]$ i.e. $c,\xi\in[a,b]$.

Sol^n: We have to show that

$$(b-a)f(c)-(b-c)f(a)-(c-a)f(b)=\frac{1}{2}(b-a)(c-a)(c-b)f''(\xi)$$

Let us consider the function for $x\in[a,b]$ defined by

$$\phi(x)=(b-a)f(x)-(b-x)f(a)-(x-a)f(b)-(b-a)(x-a)(x-b)K$$

where K is a constant to be determined such that $\phi(c)=0$.

$$0=(b-a)f(c)-(b-c)f(a)-(c-a)f(b)-(b-a)(c-a)(c-b)K$$

$$K=\frac{(b-a)f(c)-(b-c)f(a)-(c-a)f(b)}{(b-a)(c-a)(c-b)}$$

Clearly $\phi(a)=\phi(b)=0$ and $\phi(x)$ is differentiable in $[a,b]$

∴ The function ϕ satisfies all the conditions of Rolle's theorem on each intervals $[a,c]$ and $[c,b]$.

∴ ∃ two numbers ξ_1,ξ_2 in (a,c) and (c,b) such that $\phi'(\xi_1)=0$ and $\phi'(\xi_2)=0$

But $\phi'(x)=(b-a)f'(x)+f(a)-f(b)-(b-a)\{2x-(a+b)\}K$.

which is continuous on $[a,b]$ and derivable on (a,b).

∴ Continuous and derivable on $[\xi_1,\xi_2]$.

Also $\phi'(\xi_1)=\phi'(\xi_2)=0$

∴ By Rolle's theorem,

$\exists\ \xi\in(\xi_1,\xi_2)$ such that $\phi''(\xi)=0$

But $\phi''(x)=(b-a)f''(x)-2(b-a)K$.

∴ $f''(\xi)-2K=0$ (∵ $b-a\neq0$ & $\phi''(\xi)=0$)

$\Rightarrow K=\frac{1}{2}f''(\xi)$ where $a<\xi_1<\xi<\xi_2<b$ — (2)

from (1) & (2), we have

$$\frac{(b-a)f(c)-(b-c)f(a)-(c-a)f(b)}{(b-a)(c-a)(c-b)}=\frac{1}{2}f''(\xi)$$

$$\Rightarrow f(c)-\left(\frac{b-c}{b-a}\right)f(a)-\left(\frac{c-a}{b-a}\right)f(b)=\frac{1}{2}(c-a)(c-b)f''(\xi).$$

→ Let $\mathbb{R}$ be the set of real numbers and $f:\mathbb{R}\to\mathbb{R}$ such that for all x and y in $\mathbb{R}$, $|f(x)-f(y)|\le|x-y|^3$.
Prove that $f(x)$ is a constant function.

Sol^n:- Given $|f(x)-f(y)|\le|x-y|^3$

$$\forall x,y\in\mathbb{R} \quad \text{--- (1)}$$

Let $y\in\mathbb{R}$ and x be choosen arbitrarily close to y but not equal to y.

$$\therefore (1) \equiv \left|\frac{f(x)-f(y)}{x-y}\right|\le|x-y|^2$$

Taking limit when $x\to y$ we get

$$\lim_{x\to y}\left|\frac{f(x)-f(y)}{x-y}\right|\le\lim_{x\to y}|x-y|^2$$

$$\Rightarrow\left|\lim_{x\to y}\frac{f(x-f(y)}{x-y}\right|\le\left|\lim_{x\to y}(x-y)\right|^2$$

$$\Rightarrow|f'(x)|=0\left[\because\lim_{x\to y}\frac{f(x)-f(y)}{x-y}=f'(x) \text{ and } |f'(x)|\ge 0.\right]$$

$$\Rightarrow f'(x)=0$$

$f(x)$ is Constant.

[2004] → Prove that an equation of the form $x^n=\alpha$ where $n\in\mathbb{N}$ and $\alpha>0$ is a real number, has a positive root

(or)

show that $x^n-\alpha=0$ has at most one real +ve root if n is a +ve integer.

Sol^n. Let $f(x)=x^n-\alpha$

then $f'(x)=nx^{n-1}$

Since $f'(x)>0$ for $x>0$.

hence $f(x)$ is increasing on $(0,\infty)$.

Let $x_1,x_2\in(0,\infty)$ and $0<x_1<\gamma<x_2$

Such that $f(\gamma)=0$.

Then $f(x_1)<f(\gamma)<f(x_2)$ (or)

$$f(x_1)\le 0<f(x_2)$$

$\therefore$ This shows that if $x\ne\gamma, f(x)\ne 0$ on $(0,\infty)$.

i.e. $x^n-\alpha=0$ has at most one real +ve root

2008 → Prove that $\frac{\tan x}{x}>\frac{x}{\sin x}$ whenever $0<x<\pi/2$

Sol^n:- $\frac{\tan x}{x}-\frac{x}{\sin x}=\frac{\tan x\sin x-x^2}{x\sin x}$

Since $x\sin x>0\ \forall x\in(0,\pi/2)$

$\therefore$ we are enough to show that

$\tan x.\sin x-x^2>0\ \forall x\in(0,\pi/2)$

Let $f(x)=\tan x.\sin x-x^2\ \forall x\in(0,\pi/2)$

$\Rightarrow f'(x)=\sec^2x\sin x+\tan x\cos x-2x$

$$=\sin x(\sec^2x+1)-2x$$

We cannot decide about the sign of $f'(x)$ (because of the presence of $2x$ term)

Let $g(x) = f'(x)\ \forall\ x \in (0, \pi/2)$.

$\Rightarrow g'(x) = \cos x\,(\sec^2 x + 1) +$

$\sin x\,(2\sec^2 x\ \tan x) - 2$

$= \sec x + \cos x - 2 + 2\sin^2 x\ \sec^3 x$

$= \left(\sqrt{\sec x} - \sqrt{\cos x}\right)^2 + 2\sin^2 x\ \sec^3 x.$

Since $g'(x) > 0\ \forall\ x \in (0, \pi/2)$.

$\Rightarrow g(x)$ is an increasing function $(0, \pi/2)$.

$\Rightarrow g(0) < g(x)$ in $0 < x < \pi/2$.

Since $g(0) = 0$

$\therefore g(x) > 0$.

$\Rightarrow f'(x) > 0$ whenever $0 < x < \pi/2$

f is an increasing function in $0 < x < \pi/2$.

$\Rightarrow f(0) < f(x)$

$\Rightarrow 0 < f(x)$

$\Rightarrow \tan x \sin x - x^2 > 0$ in $(0, \pi/2)$.

$$\Rightarrow \frac{\tan x \sin x - x^2}{x \sin x} > 0$$

$$\Rightarrow \frac{\tan x}{x} - \frac{x}{\sin x} > 0$$

$$\Rightarrow \frac{\tan x}{x} > \frac{x}{\sin x} \text{ whenever } 0 < x < \pi/2$$

→ Prove that if f be defined for all real x such that $|f(x) - f(y)| < (x-y)^2$ then f is constant. [means its derivative will be zero (0)].

Sol^n:- Here we have to show that

$f'(x) = 0\ \forall\ x \in \mathbb{R}$.

Let $x = c \in \mathbb{R}$.

Now we have

$$\left|\frac{f(x) - f(c)}{x - c} - 0\right| \text{ for } x \neq c.$$

$$= \left|\frac{f(x) - f(c)}{x - c}\right|$$

$$= \frac{|f(x) - f(c)|}{|x - c|} < \frac{(x-c)^2}{|x-c|} = \frac{|x-c|^2}{x-c}$$

(by hyp.) (Given)

$= |x - c| < \epsilon$ whenever $|x - c| < \frac{\epsilon}{1}$

$$\therefore \left|\frac{f(x) - f(c)}{x - c} - 0\right| < \epsilon \text{ whenever}$$

$|x - c| < \delta$ by choosing $\delta = \frac{\epsilon}{1}$.

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = 0$$

i.e. $f'(c) = 0\ \forall\ c \in \mathbb{R}$.

$\Rightarrow f$ is constant function.

→ Find the interval in which the function $f(x) = \sin(\log_e x) - \cos(\log_e x)$ is strictly increases.

Sol^n: Given that

$f(x) = \sin(\log_e x) - \cos(\log_e x)$

Here domain is $x > 0$ as $\log_e x$

exists when $x>0$,

$$f'(x) = \frac{\cos(\log_e x) + \sin(\log_e x)}{x}$$

$$= \frac{\sqrt{2}\left\{\sin \pi/4 \cos(\log_e x) + \cos \frac{\pi}{4} \sin \log_e x\right\}}{x}$$

$$= \frac{\sqrt{2} \sin\left(\pi/4 + \log_e x\right)}{x}$$

Since $f(x)$ is strictly increases when $f'(x) \geq 0$.

i.e $\sin\left(\pi/4 + \log_e x\right) \geq 0$

$\Rightarrow 2n\pi \leq \frac{\pi}{4} + \log_e x \leq (2n+1)\pi \; \forall n \in Z$

$\Rightarrow 2n\pi - \pi/4 \leq \log_e x \leq 2n\pi + \pi - \pi/4$

$\Rightarrow e^{2n\pi - \pi/4} \leq x \leq e^{2n\pi + 3\pi/4}$

$\therefore f(x)$ is strictly increasing when $x \in \left[e^{2n\pi - \pi/4}, e^{2n\pi + 3\pi/4}\right]$

$\rightarrow$ Let $g(x) = f(x) + f(1-x)$ and $f''(x) > 0 \; \forall x \in (0,1)$. find the intervals of increase and decrease of $g(x)$.

Sol'n :- We have

$$g(x) = f(x) + f(1-x) \quad \text{—— (1)}$$

then $g'(x) = f'(x) - f'(1-x)$ —— (2)

Since $f''(x) > 0 \; \forall x \in (0,1)$

$\therefore f'(x)$ is increasing on $(0,1)$.

Hence two cases arise:

Case(i): $x > 1-x$ and $f'(x)$ is increasing for $\forall x > 1/2$ in $(0,1)$

$\Rightarrow f'(1-x) < f'(x) \; \forall x > \frac{1}{2}$

$\Rightarrow f'(x) - f'(1-x) > 0 \; \forall x > \frac{1}{2}$

| $x \in (0,1)$, $1-x \in (0,1)$ if f is $\uparrow$ fn $x_1 < x_2$ then $f(x_1) < f(x_2)$

$\therefore g'(x) > 0 \; \forall x > 1/2$ in $(0,1)$

i.e $g'(x) > 0 \; \forall x \in (1/2, 1)$

$\Rightarrow g(x)$ is increasing in $(1/2, 1)$.

Case(ii)

$x < 1-x$ and $f'(x)$ increasing for $0 < x < 1/2$ in $(0,1)$.

$\Rightarrow f'(x) < f'(1-x)$ for $0 < x < 1/2$

$\Rightarrow f'(x) - f'(1-x) < 0$ for $0 < x < 1/2$

$\Rightarrow g'(x) < 0$; $x \in (0, 1/2)$

$\Rightarrow g(x)$ is decreasing function in $(0, 1/2)$.

$\rightarrow$ show that

$$\frac{2}{\pi} < \frac{\sin x}{x} < 1, \quad 0 < x < \pi/2$$

Sol'n :- Let

$$f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases} \quad \forall x \in [0, \pi/2]$$

then f is continuous in $[0, \pi/2]$

and derivable in $(0, \pi/2)$

and $f'(x) = \frac{x\cos x - \sin x}{x^2}$;

$x \in (0, \pi/2)$ —— (i)

let $F(x) = x\cos x - \sin x$; $x \in (0, \pi/2)$

$F'(x) = \cos x - x\sin x - \cos x$

$= -x\sin x$

< 0 ; $x \in (0, \pi/2)$

$\therefore$ F is decreasing in $(0, \pi/2)$

$\therefore$ $F(x) < F(0)$ for $x > 0$ in $(0, \pi/2)$

$\Rightarrow$ $F(x) < 0$ for $x \in (0, \pi/2)$

$(\because F(0) = 0)$

$\Rightarrow f'(x) < 0$; $x \in (0, \pi/2)$

$\therefore$ $f(x)$ is decreasing in $(0, \pi/2)$

$\Rightarrow f(0) > f(x) > f(\pi/2)$ for

$0 < x < \pi/2$

$\Rightarrow 1 > \frac{\sin x}{x} > \frac{2}{\pi}$

$\Rightarrow \frac{2}{\pi} < \frac{\sin x}{x} < 1 \ \forall x \in (0, \pi/2)$

Taylor's Theorem:

Statement:

If a function f defined on $[a,b]$, is such that

(i) the $(n-1)^{th}$ derivative $f^{(n-1)}$ is continuous on $[a,b]$

(ii) the n^{th} derivative $f^{(n)}$ exists on (a,b)

then there exist atleast one real number $c \in (a,b)$

such that $f(b) = f(a) + (b-a) f'(a) + \frac{(b-a)^2}{2!} f''(a) + \cdots + \frac{(b-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{(b-a)^p (b-c)^{n-p}}{p\,(n-1)!} f^{n}(c)$

where p is a given +ve integer.

proof: Let $G(x) = F(x) - \left(\frac{b-x}{b-a}\right)^p F(a)$

where

$F(x) = f(b) - f(x) - (b-x) f'(x) - \frac{(b-x)^2}{2!} f''(x) - \cdots - \frac{(b-x)^{n-1}}{(n-1)!} f^{(n-1)}(x)$

Since the $(n-1)^{th}$ derivative $f^{(n-1)}$ is continuous on $[a,b]$

$f, f', f'', \ldots f^{(n-1)}$ are continuous on $[a,b]$

and $(b-x)^r$, $r = 1, 2, 3, \ldots n-1$ is continuous for all x.

$\therefore F(x)$ is continuous on $[a,b]$.

$\therefore G(x)$ is continuous on $[a,b]$

Since the n^{th} derivative $f^{(n)}$ exists on (a,b)

$\therefore f, f', f'', \ldots f^{(n-1)}$ are differentiable on (a,b)

and $(b-x)^r$; $r = 1,2,3, \ldots (n-1)$ is differentiable for all x.

$\therefore F(x)$ is differentiable on (a,b).

$\therefore G(x)$ is differentiable on (a,b)

Now

$$G(a) = F(a) - \left(\frac{b-a}{b-a}\right)^p F(a)$$

$$= 0$$

and $G(b) = F(b) - 0$

$$= f(b) - f(b) - (b-b) f'(b) - \cdots - \frac{(b-b)^{n-1}}{(n-1)!} f^{(n-1)}(b)$$

$$= 0$$

$\therefore G(a) = G(b) = 0$

$\therefore$ $G(x)$ satisfies the conditions of Rolle's theorem,

$\exists$ atleast one real number $c \in (a,b)$ such that $G'(c) = 0$.

But $G'(x) = F'(x) + p \dfrac{(b-x)^{p-1}}{(b-a)^p} F(a)$ — ①

Now

$$F'(x) = 0 - f'(x) + f'(x) - (b-x) f''(x) + (b-x) f''(x) - \frac{(b-x)^2}{2!} f'''(x) + \cdots + \frac{(n-1)(b-x)^{n-2}}{(n-1)!} f^{(n-1)}(x) - \frac{(b-x)^{n-1}}{(n-1)!} f^{(n)}(x)$$

$$F'(x) = -\frac{(b-x)^{n-1}}{(n-1)!} f^{(n)}(x) \quad \text{— ②}$$

① $\Rightarrow$

$$G'(c) = F'(c) + p \frac{(b-c)^{p-1}}{(b-a)^p} F(a)$$

$$\Rightarrow 0 = F'(c) + p \frac{(b-c)^{p-1}}{(b-a)^p} F(a) \qquad (\because G'(c) = 0)$$

$$\Rightarrow F'(c) = -\frac{p(b-c)^{p-1}}{(b-a)^p} F(a)$$

$$\Rightarrow -\frac{(b-c)^{n-1}}{(n-1)!} f^{(n)}(c) = -p \frac{(b-c)^{p-1}}{(b-a)^p} F(a) \qquad (\text{from ②})$$

$$\Rightarrow F(a) = \frac{(b-a)^p (b-c)^{n-p}}{p(n-1)!} f^{(n)}(c)$$

$$\Rightarrow f(b) - f(a) - \frac{(b-a)}{1!} f'(a) - \frac{(b-a)^2}{2!} f''(a) - \cdots - \frac{(b-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) - \frac{(b-a)^p (b-c)^{n-p}}{p(n-1)!} f^{(n)}(c).$$

$$\Rightarrow f(b) = f(a) + \frac{(b-a)}{1!} f'(a) + \frac{(b-a)^2}{2!} f''(a) + \cdots$$

$$\cdots + \frac{(b-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + \underbrace{\frac{(b-a)^p (b-c)^{n-p}}{p(n-1)!} f^{(n)}(c)}_{R_n(x)}$$

$$= P_n(x) + R_n(x)$$

Note: [1] After n terms $R_n(x) = T_{n+1}$

$$= \frac{(b-a)^p (b-c)^{n-p}}{p(n-1)!} f^{(n)}(c)$$

for some point $c \in (a,b)$.

This formula for R_n is referred to as the Roche's form (or derivative form) of the remainder.

[2].

(i) for $p=1$,

$R_n = \frac{(b-a)(b-c)^{n-1}}{(n-1)!} f^{(n)}(c)$ Called Cauchy's form of remainder.

(ii) for $p=n$,

$R_n = \frac{(b-a)^n}{n!} f^{(n)}(c)$ called Lagrange's form of remainder

[3]. Another form of Taylor's Theorem:

if a function f defined on $[a, a+h]$ is such that

(i) the $(n-1)^{th}$ derivative $f^{(n-1)}$ is continuous on $[a, a+h]$

(ii) the n^{th} derivative $f^{(n)}$ exists on $(a, a+h)$

then $\exists$ $\theta \in (0,1)$ such that

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a)$$

$$+ \frac{h^n (1-\theta)^{n-p}}{p(n-1)!} f^{(n)}(a+\theta h)$$

where p is +ve integer.

Here $c = (a+\theta h)$ where $0 < \theta < 1$)

4. Maclaurin's theorem:

putting $a=0$, $h=x$ in Taylor's theorem.

i.e, If a function f defined on $[0,x]$ is such that (i) the $(n-1)^{th}$ derivative $f^{(n-1)}$ is continuous on $[0,x]$

(ii) the n^{th} derivative $f^{(n)}$ exists on $(0,x)$

then $\exists\ \theta \in (0,1)$ such that

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) + \frac{x^n (1-\theta)^{n-p}}{p\,(n-1)!} f^{n}(\theta x).$$

Taylor's and Maclaurin's series:

Let a function f be continuous derivatives of every order in $[a, a+h]$ then for all $n \in N$ we have by Taylor's theorem

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{n!} f^{(n)}(a+\theta h)$$

where $\theta \in (0,1)$

Let $P_n = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \cdots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a)$

$R_n = \frac{h^n}{n!} f^{(n)}(a+\theta h)$ (which is Taylor's remainder after n terms)

Then $f(a+h) = P_n + R_n$.

if $R_n \to 0$ as $n \to \infty$,

we have $\lim_{n\to\infty} P_n = f(a+h)$

$\Rightarrow$ the infinite series

$f(a)+h f'(a)+\frac{h^2}{2!}f''(a)+\cdots\cdots+\frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a)+\cdots$

converges to $f(a+h)$

$\therefore$ $f(a)+h f'(a)+\frac{h^2}{2!}f''(a)+\cdots\cdots$ is called

Taylor's series which is cgs to $f(a+h)$

if $\lim_{n\to\infty} R_n=0$

Hence if $f:[a,a+h]\to\mathbb{R}$ possesses continuous derivatives of every order in $[a,a+h]$

and Taylor's remainder $R_n\to 0$ as $n\to\infty$

then $f(a+h)=f(a)+h f'(a)+\frac{h^2}{2!}f''(a)+\cdots$

$\cdots+\frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a)+\cdots\cdots$

$\rightarrow$ If we put $a=0,\ h=x$, we get

$f(x)=f(0)+x f'(0)+\frac{x^2}{2!}f''(0)+\cdots\cdots+\frac{x^n}{n!}f^{(n)}(0)$

$+\cdots$

This is called Maclaurin's series.

Note: This series is useful in finding the expansion of functions.

Problems:

If $f(x+h)=f(x)+hf'(x)+\frac{h^2}{2!}f''(x+\theta h)$, find the value of θ as $x\to a$ if $f(x)=(x-a)^{5/2}$

Sol: Given $f(x)=(x-a)^{5/2}$

$\Rightarrow f(x+h)=(x+h-a)^{5/2}$

and $f'(x)=\frac{5}{2}(x-a)^{3/2}$

$\Rightarrow f''(x)=\frac{15}{4}(x-a)^{1/2}$

$\Rightarrow f''(x+\theta h)=\frac{15}{4}(x+\theta h-a)^{1/2}$

$\therefore f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x+\theta h)$.

$\Rightarrow (a+h-a)^{5/2} = (x-a)^{5/2} + h\left(\frac{5}{2}\right)(x-a)^{3/2} + \frac{h^2}{2}\left(\frac{15}{4}\right)(x+\theta h-a)^{1/2}$

when $x \to a$, we get

$h^{5/2} = \frac{h^2}{2}\left(\frac{15}{4}\right)(\theta h)^{1/2}$

$\Rightarrow h^{5/2} = \frac{h^{5/2}}{2}\left(\frac{15}{4}\right)\theta^{1/2}$

$\Rightarrow \frac{8}{15} = \theta^{1/2}$

$\Rightarrow \theta = \frac{64}{225}$

HW $\Rightarrow$ If $f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(\theta x)$; find the value of θ, as $x \to 1$ if $f(x) = (1-x)^{5/2}$

$\rightarrow$ Using Taylor's theorem, Show that

(i) $\cos x \geq 1 - \frac{x^2}{2} \quad \forall x \in \mathbb{R}$.

(ii) $1 + x + \frac{x^2}{2} < e^x < 1 + x + \frac{x^2}{2} e^x, \quad x > 0$

(iii) $x - \frac{x^3}{3!} < \sin x < x, \quad x > 0$

(iv) $x - \frac{x^3}{3!} \leq \sin x \leq x - \frac{x^3}{3!} + \frac{x^5}{5!}; \quad x \geq 0$

Solⁿ (i) $\cos x \geq 1 - \frac{x^2}{2} \quad \forall x \in \mathbb{R}$

Case (1) Let $x = 0$

then $\cos x = 1$; $1 - \frac{x^2}{2} = 1$

$\therefore \cos x = 1 - \frac{x^2}{2}$

Case (2): Let $x > 0$ and $f(x) = \cos x$

$\Rightarrow f'(x) = -\sin x, \quad f''(x) = -\cos x$

Since $f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(\theta x)$

where $0 < \theta < 1$

$\therefore \cos x = 1 - \frac{x^2}{2} \cos \theta x$.

But $\cos \theta x < 1$; $\forall \theta x$; $x > 0$.

$\therefore 1 - \frac{x^2}{2} \cos \theta x > 1 - \frac{x^2}{2}$

$\Rightarrow \cos x > 1 - \frac{x^2}{2}$

Case(3):

Let $x<0 \Rightarrow -x>0$

put $y=-x$; $y>0$

By Case(2), $\cos y > 1-\frac{y^2}{2}$

$\Rightarrow \cos(-x) > 1-\frac{(-x)^2}{2}$

$\Rightarrow \cos x > 1-\frac{x^2}{2}$

$\therefore$ combining all cases,

$\cos x \geq 1-\frac{x^2}{2} \quad \forall x \in \mathbb{R}$.

(ii) $1+x+\frac{x^2}{2} < e^x < 1+x+\frac{x^2}{2}e^x$; $x>0$

Sol^n: Let $f(x)=e^x$; $x>0$

then $f'(x)=e^x=f''(x)$

Since $f(x)=f(0)+xf'(0)+\frac{x^2}{2!}f''(\theta x)$.

where $0<\theta<1$.

$\therefore e^x = 1+x+\frac{x^2}{2}e^{\theta x}$ —— (A)

Now $0<\theta<1$

$\Rightarrow 0<\theta x<x$; $x>0$

$\Rightarrow e^0 < e^{\theta x} < e^x$

$\Rightarrow 1<e^{\theta x}<e^x$

$\Rightarrow \frac{x^2}{2} < \frac{x^2}{2}e^{\theta x} < \frac{x^2}{2}e^x$

$\Rightarrow 1+x+\frac{x^2}{2} < 1+x+\frac{x^2}{2}e^{\theta x} < 1+x+\frac{x^2}{2}e^x$

$\Rightarrow 1+x+\frac{x^2}{2} < e^x < 1+x+\frac{x^2}{2}e^x$ (by (A))

→ Expand e^x as an infinite series

Sol^n: Let $f(x)=e^x$

$f(x)=e^x \Rightarrow f(0)=1$

$f'(x)=e^x \Rightarrow f'(0)=1$

$\vdots$

$f^n(x)=e^x \Rightarrow f^{(n)}(\theta x)=e^{\theta x}$

Clearly f and its derivatives exist and are continuous for every value of x.

$R_n(x) = \frac{x^n}{n!} e^{\theta x}$ (LFR)

$\lim_{n\to\infty} R_n = e^{\theta x} \lim_{n\to\infty} \frac{x^n}{n!}$ —— (1)

Now let $a_n = \frac{x^n}{n!}$; $\forall n \in N$

$\Rightarrow a_{n+1} = \frac{x^{n+1}}{(n+1)!}$ $\forall n \in N$

$\therefore \lim_{n\to\infty} \left(\frac{a_{n+1}}{a_n}\right) = \lim_{n\to\infty} \frac{x}{n+1}$

$= 0 < 1$

$\therefore \lim_{n\to\infty} a_n = 0$ $\left(\because \text{if } \lim_{n\to\infty}\left(\frac{a_{n+1}}{a_n}\right) = l < 1 \text{ then } \lim_{n\to\infty} a_n = 0\right)$

$\therefore \lim_{n\to\infty} R_n = 0$

$\therefore$ The conditions of Maclaurin's series are satisfied.

$\therefore \forall x \in R$,

$e^x = f(x)$
$= f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^n}{n!} f^{(n)}(0) + \cdots$

$= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$

→ Expand $\sin x$ as infinite series

Sol^n: Let $f(x) = \sin x$.

$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots$

Here $R_n(x)$ must be tend to '0'.

Now $f(x) = \sin x$; $f'(x) = \cos x$; $f''(x) = -\sin x$; $f'''(x) = -\cos x$

$\Rightarrow f(0) = 0$ $\therefore f'(0) = 1$ $\therefore f''(0) = 0$ $f'''(0) = -1$

$f^{iv}(x) = \sin x$; $f^{(v)}(x) = +\cos x$

$\therefore f^{iv}(x) = 0$ $\therefore f^{(v)}(0) = 1$.

Generally $f^{(n)}(x) = \sin\left(x + \frac{n\pi}{2}\right)$ Remember

$\therefore$ f and all its derivatives exist and continuous for every real value of x.

$$R_n(x) = \frac{x^n}{n!} \sin\left(\theta x + \frac{n\pi}{2}\right)$$

Now $\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{x^n}{n!} \times \lim_{n\to\infty} \sin\left(\theta x + \frac{n\pi}{2}\right)$

$= 0 \times l \quad (-1 \le l \le 1)$

$= 0$

$\therefore \lim_{n\to\infty} R_n = 0$

$\therefore$ The conditions of Maclaurin's series is satisfied $\forall x \in \mathbb{R}$.

$\therefore \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\cdots$

Similarly $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\cdots$

→ $f(x) = e^x$

→ $f(x) = a^x$

→ $f(x) = \log(1+x)$ etc

* Extreme Values of a function:

Maxima and Minima:–

[Some definitions discussed in pg NO.(8)]

* Theorem (First derivative Test):

→ Let f be continuous on $I=[a,b]$ and let c be an interior point on I. Assume that f is differentiable on (a,c) and (c,b). then

(i) If there is a neighbourhood $(c-\delta, c+\delta)\subseteq I$ such that $f'(x)\geq 0$ for $c-\delta<x<c$ and $f'(x)\leq 0$ for $c<x<c+\delta$ then f has maximum at c.

(ii) If there is neighbourhood $(c-\delta, c+\delta)\subseteq I$ such that $f'(x)\leq 0$ for $c-\delta<x<c$ and $f'(x)\geq 0$ for $c<x<c+\delta$ then f has a minimum at c.

→ **Theorem** :–

Let $I\subseteq \mathbb{R}$ be an interval, let $f: I\to\mathbb{R}$ let $c\in I$ and assume that f has a derivative at c then

(i) If $f'(c)>0$ then $\exists$ a $\delta>0$ such that $f(x)>f(c)$ for $x\in I$. such that $c<x<c+\delta$.

(ii) If $f'(c)<0$ then $\exists$ a $\delta>0$ such that $f(x)>f(c)$ for $x\in I$. such that $c-\delta<x<c$.

* Darboux's Theorem:–

If f is differentiable on $I=[a,b]$ and if k is a number between $f'(a)$ & $f'(b)$ then $\exists$ $c\in(a,b)$ such that $f'(c)=k$.

Proof:– Since k is number between $f'(a)$ & $f'(b)$.

Suppose that $f'(a)<k<f'(b)$

Now we define

$$g(x)=kx-f(x)\ \forall\ x\in[a,b] \quad \text{—(1)}$$

Since f is differentiable on I.

$\therefore f$ is continuous on I and kx is a polynomial which is continuous on I

$\therefore g(x)$ is continuous on I

$\therefore g(x)$ attains its supremum (infimum) atleast once on $[a,b]=I$

Since $g'(x)=k-f'(x)\ \forall\ x\in[a,b]$

$\Rightarrow g'(a)=(k-f'(a))>0$ $\quad (\because f'(a)<k<f'(b))$

$\Rightarrow g'(a)>0$.

we know that g has derivative at a and $g'(a)>0$. then $\exists$ a $\delta>0$ such that $g(x)>g(a)\ \forall\ x\in I$. such that $a<x<a+\delta$.

$\therefore g$ does not have the maximum at $x=a$.

Similarly g does not have the minimum at $x=b$.

$\therefore g$ has maximum at $c\in(a,b)$

$\therefore$ Interior extremum theorem

$g'(c) = 0 \ \forall \ c \in (a,b)$

$f'(c) = k \ \forall \ c \in (a,b)$.

* Generalised Test :-

Let I be an interval, let $x_0 \in I$ and let $n \geq 2$.

Suppose that the derivatives $f', f'', \dots f^{(n)}$ exist and are continuous in a neighbourhood of x_0 and that

$f'(x_0) = f''(x_0) = \dots = f^{(n-1)}(x_0) = 0$

but $f^n(x_0) \neq 0$.

(i) If n is even and $f^n(x_0) > 0$ then f has minimum at x_0

(ii) If n is even and $f^n(x_0) < 0$ then f has maximum at x_0

(iii) If n is odd, then f has neither a minimum nor maximum at x_0.

* First Method Working Rule For Finding Maxima and Minima :-

(1) Denote the given function by $f(x)$

(2) Find ~~$f'(x)$ and~~ equate it to zero.

Let its roots be $x_1, x_2, \dots$

(3) Find $f''(x)$, put $x = x_1$. If $f''(x_1) < 0$, $f(x)$ has a maximum at $x = x_1$.

If $f''(x_1) > 0$, $f(x)$ has a minimum at $x = x_1$.

(iv) If $f''(x_1) = 0$, find $f'''(x_1)$.

If $f'''(x_1) \neq 0$, there is neither maximum nor minimum at $x = x_1$.

If $f'''(x_1) = 0$, find $f^{iv}(x_1)$

If $f^{iv}(x_1) < 0$, $f(x)$ has a maxima at $x = x_1$.

If $f^{iv}(x_1) > 0$, $f(x)$ has a minimum at $x = x_1$, so on.

* Working Rule For Finding Maxima and Minima :-

(Second Method by First Derivative Test)

(1) Denote the given function by $f(x)$.

(2) Find $f'(x)$ and equate it to zero.

Let its roots be $x_1, x_2, x_3, \dots$

(3) Test these values in succession.

Consider $x = x_1$ (say)

If there is a neighbourhood $x \in (x_1 - \delta, x_1 + \delta)$ such that

- $f'(x) \geq 0$ for $x_1 - \delta < x < x_1$ and $f'(x) \leq 0$ for $x_1 < x < x_1 + \delta$ then f has maximum at x_1.
- $f'(x) \leq 0$ for $x_1 - \delta < x < x_1$ and $f'(x) \geq 0$ for $x_1 < x < x_1 + \delta$ then f has minimum at x_1.
- $f'(x) \leq 0$ (≥ 0 only) for $x_1 - \delta < x < x_1$ and $x_1 < x < x_1 + \delta$ then f is neither maximum nor minimum at x_1.

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(iv) Similarly test all these values of x obtained in (ii).

Problems:-

→ Examine the following function for extreme values. $(x-3)^5(x+1)^4$.

Sol^n:- Let $f(x) = (x-3)^5(x+1)^4$

$f'(x) = (x-3)^5\, 4(x+1)^3 + (x+1)^4\, 5(x-3)^4$

$= (x-3)^4(x+1)^3\left[(x-3)4 + (x+1)5\right]$

$= (x-3)^4(x+1)^3\left[4x-12+5x+5\right]$

$= (x-3)^4(x+1)^3\left[9x-7\right]$

For maximum or minimum $f'(x)=0$

$\Rightarrow (x-3)^4(x+1)^3(9x-7)=0$

$\Rightarrow \boxed{x=3,-1,7/9}$

Second Method:-

Take $x=3\in(3-\delta, 3+\delta)$; $\delta>0$ for $3-\delta<x<3 \Rightarrow f'(x)>0$ and for $3<x<3+\delta \Rightarrow f'(x)>0$.

$\therefore f'(x)>0$ for $3-\delta<x<3$ and $3<x<3+\delta$

$\therefore f(x)$ is neither minimum nor maximum at $x=3$.

Take $x=-1\in(-1-\delta, -1+\delta)$, $\delta>0$

for $-1-\delta<x<-1 \Rightarrow f'(x)>0$ and for
(e.g. -1.5), not (-0.5)
$-1<x<-1+\delta \Rightarrow f'(x)<0$.

$\therefore$ By first derivative test for extrema,

$f(x)$ has maximum at $x=-1$.

$\therefore f_{max} = f(-1) = 0$.

Take $x=7/9\in(7/9-\delta, 7/9+\delta)$

for $7/9-\delta<x<7/9$;

$\Rightarrow f'(x)<0$

for $7/9<x<7/9+\delta \Rightarrow f'(x)>0$

$\therefore f(x)$ has minimum at $x=7/9$.

(By first derivative test for extreme)

$\therefore f_{min} = f(7/9) = \dfrac{-4^{13}\cdot 5^5}{3^{18}}$

H.W → Examine for maxima and minima of the function defined by

$f(x) = x^2(1-x)^3$

→ Show that $\sin x(1+\cos x)$ is a maximum when $x=\pi/3$.

First Method:-

Sol^n: Let $f(x) = \sin x(1+\cos x)$

then $f'(x) = \cos x(1+\cos x) + \sin x(-\sin x)$

$= \cos^2 x - \sin^2 x + \cos x$

$= \cos 2x + \cos x$

and $\boxed{f''(x) = -2\sin 2x - \sin x}$

For maxima or minima $f'(x)=0$

$\Rightarrow \cos 2x + \cos x = 0$

$\Rightarrow 2\cos\left(\frac{3x}{2}\right)\cos\left(\frac{x}{2}\right)=0$

$\Rightarrow$ either $\frac{3x}{2}=\frac{\pi}{2}$ (or) $\frac{x}{2}=\frac{\pi}{2}$

$\Rightarrow x=\pi/3$ (or) $x=\pi$.

Here we consider only the point $x=\pi/3$.

$$f''(\pi/3) = -2\sin\left(\frac{2\pi}{3}\right) - \sin\left(\frac{\pi}{3}\right)$$
$$= -2\sin(120^\circ) - \sin 60^\circ$$
$$= -2\sin(180^\circ - 60^\circ) - \sin 60^\circ$$
$$= -2\sin 60^\circ - \sin 60^\circ$$
$$= -3\sin 60^\circ$$
$$= -3\sqrt{3}/2 < 0.$$

$\therefore f(x)$ has a maximum at $x=\pi/3$.

$$\therefore f_{max} = f(\pi/3) = \sin \pi/3\,(1+\cos \pi/3)$$
$$= \frac{\sqrt{3}}{2}\left(1+\tfrac{1}{2}\right)$$
$$= \frac{\sqrt{3}}{2}\left(\tfrac{3}{2}\right) = \frac{3\sqrt{3}}{4}$$

→ Find the maximum and minimum values if any of the function $(1-x)^2 e^x$.

Sol^n :- Let $f(x) = (1-x)^2 e^x$

then $f'(x) = (1-x)^2 e^x - 2(1-x)e^x$
$$= [1+x^2-2x-2+2x]e^x$$
$$= (x^2-1)e^x$$

For maximum or minimum,

$f'(x) = 0$

$\Rightarrow e^x(x^2-1) = 0$

$\Rightarrow x^2-1 = 0 \quad (e^x \neq 0)$

$\Rightarrow x = \pm 1$

When $x=1$: $f''(x) = e^x(x^2-1) + e^x(2x)$
$$= e^x[x^2+2x-1]$$
$$\therefore f''(1) = e^1(1+2-1)$$
$$= 2e > 0$$

$\therefore f$ is minimum at $x=1$.

$\therefore f_{min} = f(1) = 0$.

When $x=-1$: $f''(x) = e^x(x^2+2x-1)$
$$\therefore f''(-1) = e^{-1}(1-2-1)$$
$$= \frac{-2}{e} < 0$$

$\therefore f$ is maximum at $x=-1$

$$\therefore f_{max} = f(-1) = \frac{4}{e}.$$

→ Find the maximum value of $\frac{\log x}{x}$, $0 < x < \infty$.

Sol'n: Let $f(x) = \frac{\log x}{x}$

then $f'(x) = \frac{x\left(\frac{1}{x}\right) - \log x}{x^2}$
$$= \frac{1-\log x}{x^2}$$

and $f''(x) = \frac{x^2\left(-\frac{1}{x}\right) - (1-\log x)2x}{x^4}$
$$= \frac{-x-2x+2x\log x}{x^4}$$

For maximum or minimum.

$f'(x) = 0$

$\Rightarrow 1 - \log x = 0$

$\Rightarrow \log x = 1$

$\Rightarrow \boxed{x = e}$

When $x=e$:
$$f''(e) = \frac{-e-2e+2e\log e}{e^4}$$
$$= \frac{-3e+2e}{e^4}$$
$$= \frac{-e}{e^4}$$
$$= \frac{-1}{e^3} < 0$$

$\therefore f$ is maximum at $x = e$.

$\therefore f_{max} = f(e) = \frac{\log e}{e} = \frac{1}{e}$

→ Prove that the function $\left(\frac{1}{x}\right)^x$, $x>0$ has a maximum at $x = 1/e$.

Imp

Sol^n Let $f(x) = \left(\frac{1}{x}\right)^x$; $x>0$

$\Rightarrow \log f(x) = x \log 1/x$

$\Rightarrow \log f(x) = x[-\log x]$

$\Rightarrow \log f(x) = -x \log x$

$\Rightarrow \frac{1}{f(x)} f'(x) = -\left[x\left(\frac{1}{x}\right) + \log x\right]$

$\Rightarrow \boxed{f'(x) = -f(x)[1+\log x]}$

And $f''(x) = -f(x)\frac{1}{x} - f'(x)[1+\log x]$

For maximum (or) minimum $f'(x)=0$

$\Rightarrow -f(x)[\log x + 1] = 0$

$\Rightarrow 1+\log x = 0 \quad (\because f(x) \neq 0)$

$\Rightarrow \log x = -1$

$\Rightarrow \boxed{x = e^{-1}}$

Now when $x = e^{-1}$;

$f''(e^{-1}) = -f(e^{-1})\frac{1}{e^{-1}} - f'(e^{-1})[1+\log(e^{-1})]$

$= -(e)^{e^{-1}} \frac{1}{e^{-1}} - 0[1+\log(e^{-1})]$

$= -e^{e^{-1}} \cdot e$

$= -(e)^{1/e} \cdot e$

< 0

f is maximum at $x = 1/e$.

$f_{max} = f(1/e) = \left(\frac{1}{1/e}\right)^{1/e} = (e)^{1/e}$

H.W. → Prove that the function x^x, $x>0$ has a minimum at $x = 1/e$.

→ Find the maximum and minimum values of the following functions:

① $2x^3 - 9x^2 - 24x - 20$.

② $(x-1)(x-2)(x-3)$

→ For each of the following functions on $\mathbb{R} \to \mathbb{R}$ find points of extrema, the intervals on which the function is increasing, and those on it is decreasing.

(i) $f(x) = x^2 - 3x + 5$

(ii) $g(x) = 3x - 4x^2$

(iii) $h(x) = x^3 - 3x - 4$

Sol^n (i) $f(x) = x^2 - 3x + 5$

$f'(x) = 2x - 3$

for maximum (or) minimum $f'(x) = 0$

$x = 3/2$

Now $x = 3/2 \in (3/2 - \delta, 3/2 + \delta)$

For $3/2 - \delta < x < 3/2 \Rightarrow f'(x) < 0$

and $3/2 < x < 3/2 + \delta \Rightarrow f'(x) > 0$

$\therefore$ By first derivative test for extrema $f(x)$ has minimum at $x = 3/2$

$f_{min} = f(3/2) = (3/2)^2 - 3(3/2) + 5$

$= 9/4 - 9/2 + 5$

$= \frac{9 - 18 + 20}{4}$

$= 11/4$

Now $f'(x) = 2x - 3$

if $x < 3/2 \Rightarrow f'(x) < 0$.

$\therefore f(x)$ is an decreasing in $(-\infty, 3/2)$

if $x > 3/2$

$\Rightarrow f'(x) > 0$

$\therefore f(x)$ is an increasing in $(3/2, \infty)$.

(iii) $h(x) = x^3 - 3x - 4$

Sol^n: $h'(x) = 3x^2 - 3$

For maximum or minimum $h'(x) = 0$

$\Rightarrow 3x^2 - 3 = 0$

$\Rightarrow x^2 - 1 = 0$

$\Rightarrow x = \pm 1$

At $x = 1$: $h(x)$ has minimum.

At $x = -1$: $h(x)$ has maximum.

Now $h'(x) = 3x^2 - 3$

$= 3(x-1)(x+1)$ —— (1)

if $x < -1$

$\Rightarrow (x-1) < 0$; $(x+1) < 0$

$\therefore$ (1) $= h'(x) > 0$

$\therefore h(x)$ is increasing in $(-\infty, -1)$

if $-1 < x < 1 \Rightarrow (x-1) < 0$, $(x+1) > 0$

$\therefore$ (1) $= h'(x) < 0$

$\therefore h(x)$ is decreasing in $(-1, 1)$.

if $x > 1 \Rightarrow (x-1) > 0$; $(x+1) > 0$.

$\therefore$ (1) $\equiv h'(x) > 0$

$\therefore h(x)$ is increasing in $(1, \infty)$.

$\therefore$ In $(-\infty, -1) \cup (1, \infty)$,

$h(x)$ is increasing.

and in $(-1, 1)$, $\therefore h(x)$ is decreasing.

* L' Hospital's Rule *

* Indeterminate Forms :-

If $A = \lim_{x \to c} f(x)$ and $B = \lim_{x \to c} g(x)$.

and if $B \neq 0$ then $\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{A}{B}$

However, if $B = 0$ then it has no conclusion.

— If $B = 0$ and $A \neq 0$ then the limit is infinite (when it exists).

— If $A = 0$ & $B = 0$ then $\lim_{x \to c} \frac{f(x)}{g(x)}$ is said to be indeterminate form.

Ex :- (1) If α is any real number and if we define $f(x) = \alpha x$ and $g(x) = x$ then

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{\alpha x}{x} \quad \left| \frac{0}{0} \text{ form} \right.$$

$$= \lim_{x \to 0} (\alpha)$$

$$= \alpha$$

Ex :- (2) : If $f(x) = x^2 - 1$ and $g(x) = x - 1$ with $a = 1$

then we have

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to 1} \frac{x^2 - 1}{x - 1} \quad \left| \frac{0}{0} \text{ form} \right.$$

$$= \lim_{x \to 1} (x + 1)$$

$$= 2.$$

→ Other indeterminate form are represented by the symbols $\frac{\infty}{\infty}$, $0 \cdot \infty$, 0^0, 1^∞, ∞^0 and $\infty - \infty$. Remember

→ Our attention will be focused on the indeterminate forms $\frac{0}{0}$ and $\frac{\infty}{\infty}$.

The other indeterminate cases are usually reduced to the form $\frac{0}{0}$ (or) $\frac{\infty}{\infty}$ by taking logarithms, exponentials, or algebraic manipulations.

* We first establish an elementary result that is based simply on the definition of the derivative.

Theorem :-

Let f and g be defined on $[a, b]$. Let $f(a) = g(a) = 0$ and $g(x) \neq 0$ for $x \in (a, b)$ (i.e. $a < x < b$). If f and g are differentiable at a and if $g'(a) \neq 0$ then the limit of $\frac{f}{g}$ at a exists and is equal to $\frac{f'(a)}{g'(a)}$

i.e $\lim_{x \to a+} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$

* Working Rule for finding the value of $\lim_{x \to a} \frac{f(x)}{g(x)}$;

where $f(a) = 0 = g(a)$.

(1) Differentiate the numerator and denominator separately.

(2) Put $x = a$ and remove the word limit.

3) If the indeterminate form $\frac{0}{0}$ still persists, repeat the above process.

Problems:

Evaluate the following limits:

— $\underset{x\to 0}{Lt} \frac{(1+x)^n-1}{x}$

Sol^n: $\underset{x\to 0}{Lt} \frac{(1+x)^n-1}{x}$ | $\frac{0}{0}$ form

$= \underset{x\to 0}{Lt} \frac{n(1+x)^{n-1}}{1}$ (By differentiate numerator & denominator separately)

$= n$

H.W → $\underset{x\to 0}{Lt} \frac{xe^x - \log(1+x)}{x^2}$

H.W, $\underset{x\to 0}{Lt} \frac{x-\sin x}{x^3}$

H.W → $\underset{x\to 0}{Lt} \frac{\log(1-x^2)}{\log \cos x}$

→ $\underset{x\to 1}{Lt} \left(\frac{x^x - x}{1-x+\log x}\right)$

Sol^n: Let $u = x^x$

then $\log u = x\log x$

$\Rightarrow \frac{1}{u}\frac{du}{dx} = x\left(\frac{1}{x}\right) + \log x$

$\Rightarrow \frac{du}{dx} = u(1+\log x)$

$= x^x(1+\log x)$

Now $\underset{x\to 1}{Lt}\left(\frac{x^x-x}{1-x+\log x}\right)$ | form $\frac{0}{0}$

$= \underset{x\to 1}{Lt} \frac{x^x(1+\log x)-1}{-1+\frac{1}{x}}$ | form $\frac{0}{0}$

$= \underset{x\to 1}{Lt} \frac{x^x\left(\frac{1}{x}\right) + x^x(1+\log x)(1+\log x)}{-\frac{1}{x^2}}$

$= \frac{1^1\left(\frac{1}{1}\right) + 1^1(1+\log 1)^2}{\frac{-1}{1^2}}$

$= \frac{1+1(1+0)^2}{-1}$

$= \frac{1+1}{-1} = -2$

*L' Hospital's Rule - I :—

Let $-\infty \le a < b \le \infty$ and let f, g be differentiable on (a,b) such that $g'(x) \ne 0$ $\forall x \in (a,b)$

Suppose that

$\underset{x\to a+}{Lt} f(x) = 0 = \underset{x\to a+}{Lt} g(x)$

ⓐ If $\underset{x\to a+}{Lt} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$, then $\underset{x\to a+}{Lt} \frac{f(x)}{g(x)} = L$

ⓑ If $\underset{x\to a+}{Lt} \frac{f'(x)}{g'(x)} = L \in \{-\infty, \infty\}$ then

$\underset{x\to a+}{Lt} \frac{f(x)}{g(x)} = L.$

Problems:

Evaluate the following limits.

→ (a) $\underset{x\to 0}{Lt} \frac{e^x - 2\cos x + e^{-x}}{x\sin x}$

(b) $\underset{x\to 0}{Lt} \frac{\tan x - x}{x^2 \tan x}$

(c) $\underset{x\to a}{Lt} \frac{x^a - a^x}{x^x - a^a}$

(d) $\lim_{x \to 0} \dfrac{\cosh x - \cos x}{x \sin x}$

(e) $\lim_{x \to 0} \dfrac{x \cos x - \log(1+x)}{x^2}$

(f) $\lim_{x \to 0} \dfrac{e^x \sin x - x - x^2}{x^2 + x \log(1-x)}$

Sol'n (b): $\lim_{x \to 0} \dfrac{\tan x - x}{x^2 \tan x}$ $\left| \dfrac{0}{0} \text{ form} \right.$

$= \lim_{x \to 0} \dfrac{\tan x - x}{x^3} \cdot \dfrac{x}{\tan x}$

$= \lim_{x \to 0} \dfrac{\tan x - x}{x^3} \cdot \lim_{x \to 0} \dfrac{x}{\tan x}$

$= \lim_{x \to 0} \dfrac{\tan x - x}{x^3} \, (1)$

$= \lim_{x \to 0} \dfrac{\tan x - x}{x^3}$ $\left| \dfrac{0}{0} \text{ form} \right.$

$= \lim_{x \to 0} \dfrac{\sec^2 x - 1}{3x^2}$ $\left| \dfrac{0}{0} \text{ form} \right.$

$= \lim_{x \to 0} \dfrac{2 \sec x \sec x \tan x}{6x}$

$= \lim_{x \to 0} \dfrac{2 \sec^2 x}{6} \lim_{x \to 0} \dfrac{\tan x}{x}$

$= \dfrac{2(1)}{6} \cdot (1)$

$= \dfrac{1}{3}$

(c) $\lim_{x \to a} \dfrac{x^a - a^x}{x^x - a^a}$

Let $u = x^x$ then $\log u = x \log x$

$\Rightarrow \dfrac{1}{u} \dfrac{du}{dx} = x \dfrac{1}{x} + \log x$

$\Rightarrow \dfrac{du}{dx} = x^x (1 + \log x)$

$\lim_{x \to a} \left(\dfrac{x^a - a^x}{x^x - a^a} \right)$

$= \lim_{x \to a} \dfrac{a x^{a-1} - a^x \log a}{x^x (1 + \log a) - 0}$

$= \dfrac{a a^{a-1} - a^a \log a}{a^a (1 + \log a)}$

$= \dfrac{a \cdot a^{-1} - \log a}{1 + \log a}$

$= \dfrac{1 - \log a}{1 + \log a}$

→ what is wrong with the following application of L'Hospital rule:

$\lim_{x \to 1} \dfrac{x^3 + 3x - 4}{2x^2 + x - 3} = \lim_{x \to 1} \dfrac{3x^2 + 3}{4x + 1} = \lim_{x \to 1} \dfrac{6x}{4}$

$= 3/2$

Sol'n: $\lim_{x \to 1} \dfrac{x^3 + 3x - 4}{2x^2 + x - 3}$ $\left| \dfrac{0}{0} \text{ form} \right.$

$= \lim_{x \to 1} \dfrac{3x^2 + 3}{4x + 1}$

Now the expression $\dfrac{3x^2 + 3}{4x + 1}$ is not of the form $\dfrac{0}{0}$ as $x \to 1$

∴ It is not correct to apply L'Hospital's Rule to evaluate $\lim_{x \to 1} \dfrac{3x^2 + 3}{4x + 1}$

∴ $\lim_{x \to 1} \dfrac{3x^2 + 3}{4x + 1} = \dfrac{3(1) + 3}{4(1) + 1} = 6/5$.

→ what is wrong with the following use of L'Hospital's Rule:

$\lim_{x \to 1} \dfrac{x^4 - 4x^3 + 3}{3x^2 - x - 2} = \lim_{x \to 1} \dfrac{4x^3 - 12x^2}{6x - 1}$

$$= \underset{x\to 1}{\mathrm{lt}} \frac{12x^2-24x}{6}$$

$$= -2$$

→ For what value of a does $\dfrac{\sin 2x + a\sin x}{x^3}$ tend to a finite limit l as $x \to 0$? when 'a' has this value, what is the value of 'l'?

Sol'n: $\underset{x\to 0}{\mathrm{lt}} \dfrac{\sin 2x + a\sin x}{x^3}$ | form $\frac{0}{0}$

$$= \underset{x\to 0}{\mathrm{lt}} \frac{2\cos 2x + a\cos x}{3x^2} \quad \text{—— (1)}$$

The denominator (1) $\to 0$ as $x \to 0$

but (1) $\to$ a a finite limit l

∴ The numerator $(2\cos 2x + a\cos x)$ must be tend to zero as $x \to 0$.

∴ $2\cos(0) + a\cos(0) = 0$

$\Rightarrow 2 + a = 0$

$\Rightarrow \boxed{a = -2}$

with this value of a

$$(1) = \underset{x\to 0}{\mathrm{lt}} \frac{2\cos 2x - 2\cos x}{3x^2} \quad \Big|\ \frac{0}{0} \text{ form}$$

$$= \underset{x\to 0}{\mathrm{lt}} \frac{-4\sin 2x + 2\sin x}{6x} \quad \Big|\ \frac{0}{0} \text{ form}$$

$$= \underset{x\to 0}{\mathrm{lt}} \frac{-8\cos 2x + 2\cos x}{6}$$

$$= \frac{-8(1)+2(1)}{6} = -6/6 = -1$$

$$\therefore l = -1$$

→ Find the values of 'a' and 'b' in order that $\underset{x\to 0}{\mathrm{lt}} \dfrac{x(1-a\cos x)+b\sin x}{x^3}$ may be equal to $1/3$.

Sol'n: $\underset{x\to 0}{\mathrm{lt}} \dfrac{x(1-a\cos x)+b\sin x}{x^3}$ | form $\frac{0}{0}$

$$= \underset{x\to 0}{\mathrm{lt}} \frac{x(a\sin x) + (1-a\cos x) + b\cos x}{3x^2} \quad \text{—— (1)}$$

The denominator of (1) $\to 0$ as $x \to 0$

but (1) $\to 1/3$ as $x \to 0$

∴ The numerator of (1).

$x(a\sin x) + (1-a\cos x) + b\cos x$ tends to zero as $x \to 0$.

$\Rightarrow 0(0) + (1-a(1)) + b(1) = 0$

$\Rightarrow 1 - a + b = 0$ —— (2)

If the relation (2) holds then from (1)

$\underset{x\to 0}{\mathrm{lt}} \dfrac{ax\sin x + (1-a\cos x) + b\cos x}{3x^2}$

(is of the form $\frac{0}{0}$)

$$= \underset{x\to 0}{\mathrm{lt}} \frac{a\sin x + ax\cos x + a\sin x - b\sin x}{6x} \quad \Big|\ \text{form } \frac{0}{0}$$

$$= \underset{x\to 0}{\mathrm{lt}} \frac{a\cos x + a\cos x - ax\sin x + a\cos x - b\cos x}{6}$$

$$= \frac{a(1) + a(1) - a(0) + a(1) - b(1)}{6}$$

$$= \frac{3a - b}{6}$$

but the limit of (1) equal to $1/3$ (given)

$$\therefore \frac{3a-b}{6} = \frac{1}{3}$$

$\Rightarrow 3a-b=2$ —— (3)

From (2) & (3) we get

$a=\frac{1}{2}$, $b=-\frac{1}{2}$

H.W. Find the values p and q for which $\lim_{x\to 0}\frac{x(1+p\cos x)-q\sin x}{x^3}$ exists and equals 1.

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Find the values of 'a' and 'b' such that:

$$\lim_{x\to 0}\frac{a\sin^2 x+b\log\cos x}{x^4}=\frac{1}{2}$$

Sol^n: $\lim_{x\to 0}\frac{a\sin^2x+b\log\cos x}{x^4}$ $\left|\text{form }\frac{0}{0}\right.$

$$=\lim_{x\to 0}\frac{a(2\sin x\cos x)+\frac{b}{\cos x}(-\sin x)}{4x^3}$$

$$=\lim_{x\to 0}\frac{a\sin 2x-b\tan x}{4x^3} \quad \left|\frac{0}{0}\text{ form}\right.$$

$$=\lim_{x\to 0}\frac{2a\cos 2x-b\sec^2 x}{12x^2} \quad \text{—— (1)}$$

The denominator of (1) $\to 0$ as $x\to 0$.

but (1) $\to$ a finite limit value $\frac{1}{2}$

$\therefore$ The numerator of (1) must be zero as $x\to 0$.

$\therefore$ (1) $\equiv 2a\cos(0)-b\sec^2(0)=0$

$\Rightarrow \boxed{2a-b=0}$ —— (2)

with this form (1).

$$\lim_{x\to 0}\frac{2a\cos 2x-b\sec^2x}{12x^2} \quad \left|\frac{0}{0}\text{ form.}\right.$$

$$=\lim_{x\to 0}\frac{-4a\sin 2x-b[2\sec^2x\tan x]}{24x} \quad \left|\frac{0}{0}\text{ form}\right.$$

$$=\lim_{x\to 0}\left[\frac{-4a\sin 2x}{24x}-\frac{2b\sec^2x\tan x}{24x}\right]$$

$$=-\frac{a}{3}\lim_{2x\to 0}\frac{\sin 2x}{2x}-\frac{b}{12}\lim_{x\to 0}\sec^2x\lim_{x\to 0}\frac{\tan x}{x}$$

$$=-\frac{a}{3}(1)-\frac{b}{12}(1)(1)$$

$$=-\frac{a}{3}-\frac{b}{12}=\frac{-4a-b}{12}$$

but limit of (1) is equal to $\frac{1}{2}$

$\therefore \frac{-4a-b}{12}=\frac{1}{2}$

$\Rightarrow -4a-b=6$ —— (3)

(2) − (3) $\equiv 6a=6$

$\Rightarrow \boxed{a=1}$

(2) $\equiv 2(1)-b=0$

$\Rightarrow 2=b$

$\Rightarrow \boxed{b=2}$

$\therefore a=1;\ b=2$

*L' Hospital's Rule – 2:

Let $-\infty\le a<b\le\infty$ and let f, g be differentiable on (a,b) such that $g'(x)\ne 0\ \forall x\in(a,b)$

Suppose that $\lim_{x\to a^+} g(x)=\pm\infty$

(a) If $\underset{x\to a+}{Lt} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$ then

$$\underset{x\to a+}{Lt} \frac{f(x)}{g(x)} = L$$

(b) If $\underset{x\to a+}{Lt} \frac{f'(x)}{g'(x)} = L \in \{-\infty, \infty\}$, then

$$\underset{x\to a+}{Lt} \frac{f(x)}{g(x)} = L.$$

Note: In most of the Problems of the form $\frac{\infty}{\infty}$, it is necessary to change it into the form $\frac{0}{0}$ at the proper stage, otherwise the process will never end.

Problems:

Evaluate the following limits:

(a) $\underset{x\to 0}{Lt} \frac{\log x^2}{\cot x^2}$

Sol:

$$= \underset{x\to 0}{Lt} \frac{2\log x}{\cot x^2}$$ | form $\frac{\infty}{\infty}$ $\because \log 0 = -\infty$ & $\cot 0 = \infty$.

$$= \underset{x\to 0}{Lt} \frac{2\frac{1}{x}}{(-\operatorname{cosec}^2 x^2)(2x)}$$

$$= \underset{x\to 0}{Lt} \frac{-\cancel{2}}{\cancel{2}x^2 \operatorname{cosec}^2 x^2}$$

$$= \underset{x\to 0}{Lt} \left(\frac{-\sin^2(x^2)}{x^2}\right)$$ | $\frac{0}{0}$ form

$$= \underset{x\to 0}{Lt} \frac{-2\sin x^2 \cdot \cos x^2 \cdot (\cancel{2x})}{\cancel{2x}}$$

$$= \underset{x\to 0}{Lt} -\sin(2x^2)$$

$$= 0$$

→ $\underset{\theta\to \pi/2}{Lt} \frac{\log(\theta - \pi/2)}{\tan\theta}$

→ $\underset{x\to 0}{Lt} \frac{\operatorname{cosec} x}{\log x}$

→ $\underset{x\to \pi/2}{Lt} \frac{\tan 5x}{\tan x}$

→ $\underset{x\to 0+}{Lt} \frac{\log(\tan x)}{\log x}$

Sol^n: $\underset{x\to 0+}{Lt} \frac{\log(\tan x)}{\log x}$ | $\frac{\infty}{\infty}$ form

$$= \underset{x\to 0+}{Lt} \left[\frac{\frac{1}{\tan x}\sec^2 x}{1/x}\right]$$

$$= \underset{x\to 0+}{Lt} \frac{x}{\sin x \cos x}$$

$$= \underset{x\to 0+}{Lt} \frac{2x}{\sin 2x}$$ | $\frac{0}{0}$ form

$$= \underset{x\to 0+}{Lt} \frac{2}{2\cos 2x}$$

$$= \frac{2}{2(1)} = 1$$

* Other Indeterminate forms:

The indeterminate forms $\infty - \infty$, $0 \times \infty$, 1^∞, 0^0, ∞^∞ can be reduced to any one of the two indeterminate forms $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by algebraic manipulations and the exponential functions:

This is illustrated by the following examples.

Form $\infty - \infty$

→ $\lim_{x \to 0+} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$

Sol'n: $\lim_{x \to 0+} \left(\frac{1}{x} - \frac{1}{\sin x}\right)$ | $\infty - \infty$ form

$= \lim_{x \to 0+} \frac{\sin x - x}{x \sin x}$ | form $\frac{0}{0}$

$= \lim_{x \to 0+} \frac{\cos x - 1}{x \cos x + \sin x}$ | form $\frac{0}{0}$

$= \lim_{x \to 1+} \frac{\sin x}{-x \sin x + 2\cos x}$

$= \frac{0}{2} = 0$

H.W. $\lim_{x \to 0} \left(\frac{1}{x^2} - \frac{1}{x \tan x}\right)$

H.W. $\lim_{x \to 0} \left(\frac{1}{x} - \cot x\right)$

H.W. $\lim_{x \to \pi/2} (\sec x - \tan x)$

H.W. $\lim_{x \to 0} \left[\frac{1}{x} - \frac{1}{x^2} \log(1+x)\right]$

→ $\lim_{x \to 0} \left[\frac{\pi}{4x} - \frac{\pi}{2x(e^{\pi x}+1)}\right]$

→ $\lim_{x \to 1} \left[\frac{2}{x^2-1} - \frac{1}{x-1}\right]$

→ $\lim_{x \to 4} \left[\frac{1}{\log(x-3)} - \frac{1}{x-4}\right]$

→ $\lim_{x \to 0} \left[\frac{1}{e^x - 1} - \frac{1}{x}\right]$

→ $\lim_{x \to 0} \left[\cot^2 x - \frac{1}{x^2}\right]$

Sol'n: $\lim_{x \to 0} \left(\cot^2 x - \frac{1}{x^2}\right)$ | $\infty - \infty$ form

$= \lim_{x \to 0} \left(\frac{\cos^2 x}{\sin^2 x} - \frac{1}{x^2}\right)$

$= \lim_{x \to 0} \left(\frac{x^2 \cos^2 x - \sin^2 x}{x^2 \sin^2 x}\right)$

$= \lim_{x \to 0} \frac{x^2 \cos^2 x - \sin^2 x}{x^4} \cdot \frac{x^2}{\sin^2 x}$

$= \lim_{x \to 0} \frac{x^2 \cos^2 x - \sin^2 x}{x^4} \left(\lim_{x \to 0} \frac{x}{\sin x}\right)^2$

$= \lim_{x \to 0} \frac{x^2 \cos^2 x - \sin^2 x}{x^4} \cdot (1)^2$

$= \lim_{x \to 0} \frac{x^2 \left(\frac{1+\cos 2x}{2}\right) - \left(\frac{1-\cos 2x}{2}\right)}{x^4}$

$= \lim_{x \to 0} \frac{x^2(1+\cos 2x) - (1-\cos 2x)}{2x^4}$

$= \lim_{x \to 0} \frac{(x^2-1) + (x^2+1)\cos 2x}{2x^4}$ | $\frac{0}{0}$ form

$$= \lim_{x\to 0} \frac{2x-(x^2+1)\left(\frac{\sin 2x}{2}\right)+2x\cos 2x}{8x^3}$$

$$= \lim_{x\to 0}\left[\frac{2+2\cos 2x-4x\sin 2x-4x\sin 2x-4(x^2+1)\cos 2x}{24x^2}\right] \quad \Big|\ \frac{0}{0}\text{ form}$$

$$= \lim_{x\to 0}\left[\frac{-8\sin 2x-16x\cos 2x-8x\cos 2x+2(4x^2+2)\sin 2x}{48x}\right]$$

$$= \lim_{x\to 0}\frac{-24x\cos 2x+(8x^2-4)\sin 2x}{48x} \quad \Big|\ \text{form } \frac{0}{0}$$

$$= \lim_{x\to 0}\frac{-24\cos 2x+48x\sin 2x+16x\sin 2x+2(8x^2-4)\cos 2x}{48}$$

$$= \frac{-24+0+0-8}{48} = \frac{-32}{48} = -2/3$$

Form $0\times\infty$:

Evaluate the following limits:

$\rightarrow \lim_{x\to 0+} x\ln x$

Sol'n $\lim_{x\to 0+} x\ln x$ $\quad \Big|\ 0\times\infty$ form

$$= \lim_{x\to 0+}\frac{\log x}{1/x} \quad \Big|\ \frac{\infty}{\infty}\text{ form}$$

$$= \lim_{x\to 0+}\frac{1/x}{-1/x^2}$$

$$= \lim_{x\to 0+}\frac{-x^2}{x}$$

$$= \lim_{x\to 0+}(-x)$$

$$= 0$$

$\rightarrow \lim_{x\to 0} x^3\ln x$

$\rightarrow \lim_{x\to 0}\frac{1}{x(\log x)^2}$

Sol'n: $\lim_{x\to 0}\frac{1}{x(\log x)^2}$ $\quad \Big|\ 0\times\infty$ form

$$= \lim_{x\to 0}\frac{1/x}{(\log x)^2} \quad \Big|\ \frac{\infty}{\infty}\text{ form}$$

$$= \lim_{x\to 0}\left[\frac{-1/x^2}{2(\log x)\,1/x}\right]$$

$$= \lim_{x\to 0}\frac{-1/x}{2\log x} \quad \Big|\ \frac{\infty}{\infty}\text{ form}$$

$$= \lim_{x\to 0}\frac{1/x^2}{2(1/x)}$$

$$= \lim_{x\to 0}\frac{1}{2x}$$

$$= \infty$$

$\rightarrow \lim_{x\to\infty}\frac{x^3}{e^x} = l.$

Forms : $0^0,\ 1^\infty,\ \infty^\infty$

$\rightarrow \lim_{x\to 0+} x^x = ?$

Sol'n. Let $\lim_{x\to 0} x^x = l$ $\quad \Big|\ 0^0$ form

$$\Rightarrow \log\left[\lim_{x\to 0} x^x\right] = \log l$$

$$\Rightarrow \lim_{x\to 0+}\left[\log(x^x)\right] = \log l$$

$$\Rightarrow \log l = \lim_{x\to 0+}\left[x\log x\right] \quad \Big|\ 0\times\infty\text{ form}$$

$$= \lim_{x\to 0+}\frac{\log x}{1/x} \quad \Big|\ \frac{\infty}{\infty}\text{ form}$$

$= \lim_{x\to 0+} \dfrac{1/x}{-1/x^2}$

$= \lim_{x\to 0+} \left(\dfrac{-x^2}{x}\right)$

$= \lim_{x\to 0+} (-x) = 0$

$\therefore \log l = 0$

$\Rightarrow l = e^0$

$\Rightarrow \boxed{l = 1}$

→ $\lim_{x\to\infty} (1+1/x)^x = ?$

Sol'n: $\lim_{x\to\infty} \left(1+\dfrac{1}{x}\right)^x$ | 1^∞ form

Let $l = \lim_{x\to\infty} (1+1/x)^x$

$\log l = \lim_{x\to\infty} [x\log(1+1/x)]$ | $0\times\infty$ form

$= \lim_{x\to\infty} \dfrac{\log(1+1/x)}{1/x}$ | $\frac{0}{0}$ form

$= \lim_{x\to\infty} \left[\dfrac{\dfrac{1}{1+1/x}\left(-1/x^2\right)}{-1/x^2}\right]$

$= \lim_{x\to\infty} \dfrac{x}{1+x}$ | $\frac{\infty}{\infty}$ form

$= \lim_{x\to\infty} \dfrac{1}{1} = 1$

$\therefore \log l = 1$

$\Rightarrow \boxed{l = e^1}$

→ $\lim_{x\to 0+} (1+1/x)^x = ?$

Sol'n: $\lim_{x\to 0+} (1+1/x)^x$ | ∞^0 form

Let $l = \lim_{x\to 0+} (1+1/x)^x$

$\Rightarrow \log l = \lim_{x\to 0+} [\log(1+1/x)^x]$

$= \lim_{x\to 0+} [x\log(1+1/x)]$ | $0\times\infty$ form

$= \lim_{x\to 0+} \left[\dfrac{\log(1+1/x)}{1/x}\right]$

$= \lim_{x\to 0+} \left[\dfrac{\dfrac{1}{1+1/x}\cdot\dfrac{-1}{\cancel{x^2}}}{-1/\cancel{x^2}}\right]$

$= \left(\dfrac{1}{1+\frac{1}{0}}\right)$

$= 0$

→ (a) $\lim_{x\to\infty} \dfrac{\ln x}{x^2}$ $(0,\infty)$

(b) $\lim_{x\to\infty} \dfrac{\ln x}{\sqrt{x}}$ $(0,\infty)$

(c) $\lim_{x\to 0} x\ln\sin x$ $(0,\pi)$

(d) $\lim_{x\to\infty} \dfrac{x+\ln x}{x\ln x}$ $(0,\infty)$

→ (a) $\lim_{x\to 0+} x^{2x}$ $(0,\infty)$

(b) $\lim_{x\to 0} (1+3/x)^x$; $(0,\infty)$

(c) $\lim_{x\to\infty} (1+3/x)^x$; $(0,\infty)$

(a) $\lim_{x \to \infty} x^{1/x}$; $(0, \infty)$

(b) $\lim_{x \to 0+} (\sin x)^x$; $(0, \pi)$

(c) $\lim_{x \to 0+} x^{\sin x}$; $(0, \infty)$

(d) $\lim_{x \to \pi/2-} (\sec x - \tan x)$; $(0, \pi/2)$