# Generating Lines.

Q. Find vertices of skew quadrilateral by 4 gens of $\frac{x^2}{4}+y^2-z^2=49$ through $(10,5,1)$, $(14,2,-2)$

A. $\dfrac{\frac{x}{2}-z}{7-y} = \dfrac{7+y}{\frac{x}{2}+z} = \lambda.$

- At $(10,5,1) \equiv \boxed{\lambda = +\frac{4}{2} = 2.}$
- $(14,2,-2) \equiv \boxed{\lambda = \frac{9}{5}}$

B. $\dfrac{\frac{x}{2}-z}{7+y} = \dfrac{7-y}{\frac{x}{2}+z} = \mu.$

- $(10,5,1) \equiv \mu = \frac{1}{3}.$
- $(14,2,-2) \equiv \mu = 1.$

Intersection $\equiv$

$\lambda=2,\ \mu=1 \equiv \quad \frac{x}{2}-z = 14-2y.$
$\qquad 7+y = x+2z.$

$-7+3y=0$
$(y=\frac{7}{3})$

$\equiv \quad \frac{x}{2}-z = 7+y.$
$\qquad 7-y = \frac{x}{2}+z$

$\Rightarrow \frac{3x}{2}+3z=14.$
$3\frac{x}{2}-3z = \frac{28}{3}.$
$\Downarrow$
$3x = 42 \Rightarrow x=14$

$\frac{28}{3} = 14+2z.$
$\boxed{-\frac{7}{3}=z}$

$A \equiv \left(14, \frac{7}{3}, \frac{-7}{3}\right).$

Q Show generators through ends of an equi-conjugate diameter of principal elliptic section of $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ are inclined at $60°$ if $a^2+b^2=6c^2$. Find condition for gens to be ⊥.

Generators are given by : $(a\cos\theta, b\sin\theta, \pm c)$

$$\frac{x-a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{-b\cos\theta}=\frac{z}{\pm c}.$$

C=0 for elliptic section

~~Other end : $(-a\cos\theta, -b\sin\theta, 0)$~~

$\cancel{\frac{x+a\cos\theta}{}}$

$$\frac{(a^2\sin^2\theta+b^2\cos^2\theta-c^2)}{\sqrt{(a^2\sin^2\theta+b^2\cos^2\theta+c^2)^2}}=\cos\alpha.$$

$60° \Rightarrow$ Put $\theta=45°$ (equi-conjugate)

$$2\left(\frac{a^2}{2}+\frac{b^2}{2}-2c^2\right)=\frac{1}{2}\left(\frac{a^2}{2}+\frac{b^2}{2}+2c^2\right)$$

$$\Rightarrow \boxed{a^2+b^2=6c^2}$$

$$90° \Rightarrow \boxed{a^2+b^2=2c^2}$$

(8) Show if gens through $P\ (a\cos\alpha, b\sin\alpha, 0)$, $Q\ (a\cos\beta, b\sin\beta, 0)$ are $\perp r$, their projection of $z=0$, find $\tan\theta$ of intersection of their projections

$\perp r$.

$$\frac{x - a\cos\alpha}{a\sin\alpha} = \frac{y - b\sin\alpha}{-b\cos\alpha} = \frac{z}{\pm c} \quad - (A)$$

$$(1)\ \perp r \Rightarrow \boxed{a^2\sin\alpha\sin\beta + b^2\cos\alpha\cos\beta - c^2 = 0}$$

Projection of A on $z=0$, is tangent at (Graid)

$(a\cos\alpha, b\sin\alpha)$ to $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

$$\Rightarrow \frac{a\cos\alpha\, x}{a^2} + \frac{b\sin\alpha\, y}{b^2} = 1$$

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2} \Rightarrow -\frac{b}{a}\cot\alpha,\ -\frac{b}{a}\cot\beta$$

$$= \frac{-\frac{b}{a}\left[\frac{\cos\alpha}{\sin\alpha} - \frac{\cos\beta}{\sin\beta}\right]}{1 + \frac{b^2}{a^2}\frac{\cos\alpha\cos\beta}{\sin\alpha\sin\beta}} = \boxed{\frac{ab\sin(\alpha-\beta)}{c^2}} \text{ Ans}$$

③ Locus of intersection of $\perp r$ generators of
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

① $$\frac{\frac{x}{a}-\frac{z}{c}}{1-\frac{y}{b}}=\frac{1+\frac{y}{b}}{\frac{x}{a}+\frac{z}{c}}=\lambda \quad \begin{cases} \frac{x}{a}-\frac{z}{c}+\frac{\lambda y}{b}=\lambda \\ \lambda\frac{x}{a}+\frac{\lambda z}{c}-\frac{y}{b}=1 \end{cases}$$

② $$\frac{\frac{x}{a}-\frac{z}{c}}{1+\frac{y}{b}}=\frac{1-\frac{y}{b}}{\frac{x}{a}+\frac{z}{c}}=\mu \quad \begin{cases} \frac{x}{a}-\frac{z}{c}-\frac{\mu y}{b}=\mu \\ \mu\frac{x}{a}+\frac{\mu z}{c}+\frac{y}{b}=1 \end{cases}$$

$$l_1, m_1, n_1 \equiv \left.\begin{aligned} &\frac{l_1}{a}+\frac{\lambda m_1}{b}-\frac{n_1}{c}=0 \\ &\frac{l_1\lambda}{a}-\frac{m_1}{b}+\frac{\lambda n_1}{c}=0 \end{aligned}\right\} \frac{l_1}{a\frac{(\lambda^2-1)}{bc}}=\frac{m_1}{b\frac{(-2\lambda)}{ac}}=\frac{n_1}{c\frac{(-1-\lambda^2)}{ab}}$$

$$l_2, m_2, n_2 \equiv \left.\begin{aligned} &\frac{l_2}{a}-\frac{\mu m_2}{b}-\frac{n_2}{c}=0 \\ &\frac{\mu l_2}{a}+\frac{m_2}{b}+\frac{\mu n_2}{c}=0 \end{aligned}\right\} \cancel{\frac{-l_2}{\frac{(\mu^2-1)}{bc}}=m_2}$$

$$\frac{l_2}{a(\mu^2-1)}=\frac{m_2}{2b\mu}=\frac{n_2}{-c(\mu^2+1)}$$

As $\perp r \Rightarrow \quad +a^2(\lambda^2-1)(\mu^2-1)-4b^2\lambda\mu+c^2(1+\lambda^2)(1+\mu^2)$

└ Try to form $x, y, z$

$$\left(\frac{a(1+\lambda\mu)}{\lambda+\mu}, \frac{b(\lambda-\mu)}{\lambda+\mu}, \frac{c(1-\lambda\mu)}{\lambda+\mu}\right)$$

$$\Rightarrow a^2(\lambda^2\mu^2-\lambda^2-\mu^2)-4b^2\lambda\mu+c^2(\lambda^2\mu^2+\lambda^2+\mu^2)+a^2+c^2=0$$

$$a^2\left[(1+\lambda\mu)^2-\lambda^2-\mu^2-2\lambda\mu\right]+b^2\left[\lambda^2+\mu^2-2\lambda\mu-\lambda^2-\mu^2-2\lambda\mu\right]$$
$$+c^2\left[(1-\lambda\mu)^2+2\lambda\mu+\lambda^2+\mu^2\right]=0$$

$$a^2\left[(1+\lambda\mu)^2-(\lambda+\mu)^2\right]+b^2\left[(\lambda-\mu)^2-(\lambda+\mu)^2\right]$$
$$+c^2\left[(1-\lambda\mu)^2+(\lambda+\mu)^2\right]=0$$

$$\Rightarrow \boxed{a^2\left(\frac{1+\lambda\mu}{\lambda+\mu}\right)^2+b^2\left(\frac{\lambda-\mu}{\lambda+\mu}\right)^2+c^2\left(\frac{1-\lambda\mu}{\lambda+\mu}\right)^2=a^2+b^2-c^2.}$$

$x^2/a^2 + y^2/b^2 - \frac{z^2}{c^2} = 1$

*

(1)

- Two generators $\lambda, \mu$

$$x = \frac{a(1+\lambda\mu)}{\lambda+\mu}, \quad y = \frac{b(\lambda-\mu)}{\lambda+\mu}, \quad z = \frac{c(1-\lambda\mu)}{\lambda+\mu}$$

- Whenever given,

Generators through $P$ intersect $z=0$ in $A$ and $B$.

→ Take $A \equiv (a\cos\alpha, b\sin\alpha, 0)$
$B \equiv (a\cos\beta, b\sin\beta, 0)$.

Then $P \equiv \left[ \frac{a\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}, \frac{b\sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}, \pm c\tan\left(\frac{\alpha-\beta}{2}\right) \right]$

## Paraboloid

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2z$$

$$(x, y, z) \equiv \left[ \frac{a(\lambda+\mu)}{\lambda\mu}, \frac{b(\mu-\lambda)}{\lambda\mu}, \frac{2}{\lambda\mu} \right]$$

$$\left[ (a, b, 2c\lambda), (a, -b, 2c\mu) \right]$$

$x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$

- Two generators $\lambda, \mu$

$$x = \frac{a(1+\lambda\mu)}{\lambda+\mu}, \quad y = \frac{b(\lambda-\mu)}{\lambda+\mu}, \quad z = \frac{c(1-\lambda\mu)}{\lambda+\mu}.$$

- Whenever given,

Generators through $P$ intersect $z=0$ in $A$ and $B$.

Take $A \equiv (a\cos\alpha, b\sin\alpha, 0)$

→ $B \equiv (a\cos\beta, b\sin\beta, 0)$

Then $P \equiv \left( a \dfrac{\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}, \dfrac{b\sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}, \pm c\tan\left(\frac{\alpha-\beta}{2}\right) \right)$

## Paraboloid

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2z$$

$$(x, y, z) \equiv \left[ \frac{a(\lambda+\mu)}{\lambda\mu}, \frac{b(\mu-\lambda)}{\lambda\mu}, \frac{2}{\lambda\mu} \right]$$

$$[(a, b, 2c\lambda), (a, -b, 2c\mu)]$$

# * Generating Lines of Conicoids

① No two generators of same system intersect

Two gens be
$$\begin{cases} \frac{x}{a}-\frac{z}{c}=\lambda_1\left(1-\frac{y}{b}\right) \quad (1); \quad 1+\frac{y}{b}=\lambda_1\left(\frac{x}{a}+\frac{z}{c}\right) \quad (2) \\ \frac{x}{a}-\frac{z}{c}=\lambda_2\left(1-\frac{y}{b}\right) \quad (3); \quad 1+\frac{y}{b}=\lambda_2\left(\frac{x}{a}+\frac{z}{c}\right) \quad (4) \end{cases}$$

$$(3)-(1) \Rightarrow (\lambda_2-\lambda_1)\left(1-\frac{y}{b}\right)=0 \Rightarrow y=b \text{ if } \lambda_1\neq\lambda_2$$

$$(4)-(2) \Rightarrow \left(\frac{1}{\lambda_2}-\frac{1}{\lambda_1}\right)\left(1+\frac{y}{b}\right)=0 \Rightarrow \underbrace{y=-b \text{ if } \lambda_1\neq\lambda_2}_{\text{inconsistent}}$$

② Gens of opposite systems intersect.

$$(1)\ \frac{x}{a}-\frac{z}{c}=\lambda\left(1-\frac{y}{b}\right) \quad ; \quad (2)\ 1+\frac{y}{b}=\lambda\left(\frac{x}{a}+\frac{z}{c}\right)$$

$$(3)\ \frac{x}{a}-\frac{z}{c}=\mu\left(1+\frac{y}{b}\right) \quad ; \quad (4)\ 1-\frac{y}{b}=\mu\left(\frac{x}{a}+\frac{z}{c}\right)$$

$$(1)-(3) \Rightarrow \lambda-\mu=\frac{y}{b}(\lambda+\mu) \Rightarrow \boxed{y=\frac{b(\lambda-\mu)}{(\lambda+\mu)}}$$

$$\frac{x}{a}-\frac{z}{c}=\lambda\left(\frac{2\mu}{\lambda+\mu}\right) \qquad \frac{x}{a}+\frac{z}{c}=\frac{2}{\lambda+\mu}.$$

$$\frac{x}{a}=\left(\frac{1+\lambda\mu}{\lambda+\mu}\right) \Rightarrow \boxed{x=\frac{a(1+\lambda\mu)}{(\lambda+\mu)}}, \cancel{y=} \boxed{z=\frac{c(1-\lambda\mu)}{\lambda+\mu}}$$

(8) Find generators to $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ on $z=0$

$$\frac{x-a\cos\alpha}{l}=\frac{y-b\sin\alpha}{m}=\frac{z-0}{n} \rightarrow (lr+a\cos\alpha,\ mr+b\sin\alpha,\ nr)$$

$$r^2\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}-\frac{n^2}{c^2}\right)+r\left(\frac{2al\cos\alpha}{a^2}+\frac{2bm\sin\alpha}{b^2}\right)=0$$

$$L\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}=\frac{n^2}{c^2}\right)$$

$$\frac{l/a}{\sin\alpha}=\frac{m/b}{\cos\alpha}=\pm\frac{\sqrt{l^2/a^2+m^2/b^2}}{1}=\pm\frac{n/c}{1}$$

$$\Rightarrow \boxed{\frac{l}{a\sin\alpha}=\frac{m}{b\cos\alpha}=\frac{n}{\pm c}}$$

Gens are : $\left(\frac{x-a\cos\alpha}{a\sin\alpha}=\frac{y-b\sin\alpha}{b\cos\alpha}=\frac{\pm z}{c}\right)$

(3) Points of intersection of gens of opposite systems through $(a\cos\theta, b\sin\theta, 0)$, $(a\cos\phi, b\sin\phi, 0)$

$A \qquad B$

$$\boxed{\left[a\frac{\cos\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)},\ b\frac{\sin\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)},\ \pm c\tan\left(\frac{\theta-\phi}{2}\right)\right]}$$

Line joining $AB$

$$\frac{x}{a}\cos\left[\frac{\alpha+\beta}{2}\right]+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$$

Q) Generators through any one of ends of equi-conjugate diameter of principal elliptic section of $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ are inclined to each other at an angle $60°$ if $a^2+b^2=6c^2$. Find also condition for generators to be perpendicular to each other.

2007/8

Equiconjugate diameter's end $\equiv (a\cos\theta, b\sin\theta, 0)$ $\boxed{\theta=\pi/4}$

Generators $\equiv \frac{x-a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{b\cos\theta}=\frac{z}{\pm c}$.

$\frac{\pi}{3}$

$$\frac{a^2\sin^2\theta+b^2\cos^2\theta-c^2}{\sqrt{(a^2\sin^2\theta+b^2\cos^2\theta+c^2)^2}}=\frac{1}{2}$$

$$\Rightarrow 2\left(\frac{a^2+b^2}{2}-c^2\right)=\frac{a^2}{2}+\frac{b^2}{2}+c^2$$

$$\Rightarrow \boxed{a^2+b^2=6c^2} \text{ Ans}$$

$\frac{\pi}{2}$

$$\frac{a^2}{2}+\frac{b^2}{2}=c^2 \Rightarrow \boxed{a^2+b^2=2c^2} \text{ Ans}$$

8 A variable generator meets two generators through ends of ~~minor~~ major axis of same system at $B, B'$; of principal elliptic section of hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ in $P, P'$. Show $\boxed{BP \cdot B'P' = b^2+c^2}$

2013

$\lambda, \mu$ generators intersect at :-

$$\left(\frac{a(1+\lambda\mu)}{\lambda+\mu}, \frac{b(\lambda-\mu)}{\lambda+\mu}, \frac{c(1-\lambda\mu)}{\lambda+\mu}\right) \quad -(1)$$

Consider $B \equiv (a,0,0)$; $B' \equiv (-a,0,0)$

Using $B, B'$, (1) gives

$$\lambda = \mu, \quad 1+\lambda\mu = 0 \Rightarrow \boxed{\lambda = \pm 1}$$

Consider gen through $B(a,0,0)$ be $\lambda = 1$.
Then its intersection with $\mu$-system at $P$:

$$P \Rightarrow \left[a\frac{\cancel{1+\mu}}{\cancel{1+\mu}}, \frac{b(1-\mu)}{1+\mu}, c\frac{(1-\mu)}{(1+\mu)}\right] = (a, bt, ct)$$

$$\boxed{AP^2 = (a-a)^2+(bt)^2+(ct)^2.} \quad \text{where } \boxed{t = \frac{1-\mu}{1+\mu}}$$

Illy, at $A'(-a, 0, 0) \equiv$ Take $\lambda = -1$

$$P' \equiv \left[-a, \frac{b}{t}, -\frac{c}{t}\right]$$

$$(A'P')^2 = \frac{b^2+c^2}{t^2}.$$

$$(AP)^2 (A'P')^2 = (b^2+c^2)^2 \Rightarrow \boxed{AP \cdot A'P' = b^2+c^2}$$

Ans

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(8) Show shortest distance between gens of same system at ends of diameters of principal elliptic section of $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ lie on surface $\frac{cxy}{x^2+y^2}=\pm\frac{abz}{a^2-b^2}$

Diameter $\equiv P(a\cos\theta, b\sin\theta, 0)$, $Q(-a\cos\theta, -b\sin\theta, 0)$

Gens are :

$$\frac{x-a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{-b\cos\theta}=\frac{z}{c}$$

$$\frac{\cancel{x+a\sin\theta}\; x+a\cos\theta}{-a\sin\theta}=\frac{y+b\sin\theta}{b\cos\theta}=\frac{z}{c}$$

S.D equations :

$$\begin{vmatrix} x-a\cos\theta & y-b\sin\theta & z \\ a\sin\theta & -b\sin\theta & c \\ \overset{L}{\cancel{-a\sin\theta}} & \overset{M}{\cancel{b\cos\theta}} & \overset{N}{\cancel{c}} \end{vmatrix} \oplus \begin{vmatrix} x+a\cos\theta & y+b\sin\theta & z \\ -a\sin\theta & b\cos\theta & c \\ \overset{L}{\cancel{a\sin\theta}} & \overset{M}{\cancel{b\cos\theta}} & \overset{N}{\cancel{c}} \end{vmatrix}$$

where $\left(\frac{L}{-2bc\cos\theta}=\frac{M}{-2ac\sin\theta}=\frac{N}{0}\right)$

$\Rightarrow$ Two equations (A) and (B)

$A+B \Rightarrow ax\sin\alpha = by\cos\alpha$.

$A-B \Rightarrow (a^2-b^2)c\tan\alpha + z(a^2\tan^2\alpha+b^2)=0$

$$\Rightarrow \frac{abz}{a^2-b^2}=\frac{cxy}{x^2+y^2}$$

||ly for other system $\boxed{\frac{abz}{a^2-b^2}=\frac{-cxy}{x^2+y^2}}$

D. Cs

- $\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{2z}{c}$

$\downarrow$

Systems

$$\frac{\frac{x}{a} - \frac{y}{b}}{2} = \frac{\frac{z}{c}}{\frac{x}{a} + \frac{y}{b}} = \lambda \qquad [a, b, 2c\lambda]$$

$$\frac{\frac{x}{a} + \frac{y}{b}}{2} = \frac{\frac{z}{c}}{\frac{x}{a} - \frac{y}{b}} = \mu \qquad [a, -b, 2c\mu]$$

Intersection of $\lambda, \mu \equiv \left[a\,\frac{\lambda+\mu}{\lambda\mu}, b\,\frac{\mu-\lambda}{\lambda\mu}, \frac{2c}{\lambda\mu}\right]$

Q) Locus of intersection of $\perp$r generator of

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2z$$

$\lambda$ & $\mu$ systems be $\Rightarrow$

$\frac{x}{a} - \frac{y}{b} = \lambda z.$

$\frac{x}{a} + \frac{y}{b} = \frac{2}{\lambda}$

$\Downarrow$

$l_1, m_1, n_1.$

$$\frac{l_1}{a\lambda} = \frac{m_1}{-b\lambda} = \frac{n_1}{2}$$

$\frac{x}{a} - \frac{y}{b} = 2\mu$

$\frac{x}{a} + \frac{y}{b} = \mu z.$

$\Downarrow$

$l_2, m_2, n_2$

$$\frac{l_2}{a\mu} = \frac{m_2}{b\mu} = \frac{n_2}{2.}$$

$a^2\lambda\mu - b^2\lambda\mu + 4 = 0$

$[l_1 l_2 + m_1 m_2 + n_1 n_2 = 0]$

$$\Rightarrow \boxed{(a^2 - b^2)\frac{2}{z} + 4 = 0}$$

Ans

# Analytic Geometry
(1992)



① locus of foot perpendicular from center to plane through extremities of three conjugate semi-diameters of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is ___?

→ let foot be $(p, q, r)$

Plane $\equiv \quad px+qy+rz = p^2+q^2+r^2$

$x_1(px_1+qy_1+rz_1) = (p^2+q^2+r^2)x_1$

$pa^2 = (p^2+q^2+r^2)(x_1+x_2+x_3)$

$(pa)^2 = (p^2+q^2+r^2)^2 \frac{(x_1+x_2+x_3)^2}{a^2}$ and adding others,

$$\sum(pa)^2 = (p^2+q^2+r^2)^2\left(\sum\frac{a^2+2(0)}{a^2}\right) = 3(p^2+q^2+r^2)^2$$

$$\boxed{a^2x^2+b^2y^2+c^2z^2 = 3(x^2+y^2+z^2)^2} \text{ Ans}$$

② Two spheres $(r_1, r_2)$ cut orthogonally. Find area of common circle.

→ Let circle be $x^2+y^2=a^2,\ z=0$.

$S_1 \equiv x_1^2+y_1^2-a^2+2k_1z=0 \quad S_2 \equiv x^2+y^2-a^2+2k_2z=0$

$\therefore r_1^2 = k_1^2+a^2 \quad r_2^2 = k_2^2+a^2$

$S_1 \perp S_2 \Rightarrow 2k_1k_2 = 2a^2.$

$\cancel{2}\sqrt{r_1^2-a^2}\sqrt{r_2^2-a^2} = \cancel{2}a^2 \Rightarrow (r_1^2-a^2)(r_2^2-a^2) = a^4$

$$\Leftrightarrow \frac{r_1^2r_2^2}{r_1^2+r_2^2} = a^2$$

$$\boxed{\text{Area} = \pi a^2 = \frac{\pi r_1^2r_2^2}{r_1^2+r_2^2}} \text{ Ans}$$

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(3) Show that a plane through one member of $\lambda$ and one of $\mu$ system is tangent at their intersection point.

$\rightarrow$ $$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

$$\frac{x}{a}-\frac{z}{c}=\lambda\left(1-\frac{y}{b}\right) \qquad \left(1+\frac{y}{b}\right)=\lambda\left(\frac{x}{a}+\frac{z}{c}\right)$$

$$\left(\frac{x}{a}-\frac{z}{c}\right)=\mu\left(1+\frac{y}{b}\right) \qquad \left(1-\frac{y}{b}\right)=\mu\left(\frac{x}{a}+\frac{z}{c}\right)$$

Plane through (1) and (2)

$$\rightarrow \left(\frac{x}{a}-\frac{z}{c}\right)-\lambda\left(1-\frac{y}{b}\right)+k_1\left[\left(1+\frac{y}{b}\right)-\lambda\left(\frac{x}{a}+\frac{z}{c}\right)\right]=0$$

$$\rightarrow \left(\frac{x}{a}-\frac{z}{c}\right)-\mu\left(1+\frac{y}{b}\right)+k_2\left[\left(1-\frac{y}{b}\right)-\mu\left(\frac{x}{a}+\frac{z}{c}\right)\right]=0.$$

$k_1=\mu$, $k_2=\lambda$ makes both equations same.

So, these lines are coplanar and intersect.

Plane through them $\equiv$

Put $k_1=\mu \longrightarrow$

$$\frac{x}{a}\left(\frac{1+\lambda\mu}{\lambda+\mu}\right)+\frac{y}{b}\left(\frac{\lambda-\mu}{\lambda+\mu}\right)+\left(-\frac{z}{c}\right)\left[\frac{1-\lambda\mu}{\lambda+\mu}\right]=1$$

$\frac{a(\lambda+\mu)}{(1+\lambda\mu)}$, $\frac{b(\lambda+\mu)}{b(\lambda-\mu)}$, $c\,\frac{(1-\lambda\mu)}{\lambda+\mu}$ is pt of intersection

(9) Pts of intersection of generators of opposite system drawn through points :- 3

$$(a\cos\theta,\ b\sin\theta,\ 0),\ (a\cos\phi,\ b\sin\phi,\ 0).$$

P Q.

of principal elliptic section of the hyperboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1.$$

---

→ Let $(\alpha, \beta, \gamma)$ be a point of intersection of generators

└ Tangent at pt ⇒ $\frac{x\alpha}{a^2}+\frac{y\beta}{b^2}+\frac{z\gamma}{c^2}=1$

⇓

Meets at $z=0$ as ⇒ $\boxed{\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}=1}$ — (1)

Same as

(1) ≡ line joining by P, Q., which is

$$\frac{x\cos\frac{1}{2}(\theta+\phi)}{a}+\frac{y\sin\frac{1}{2}(\theta+\phi)}{b}=\cos\tfrac{1}{2}(\theta-\phi),\quad z=0$$

└ (2)

Equating (1) and (2)

$$x_1 = a\,\frac{\cos\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}} \qquad y=b\,\frac{\sin\frac{\theta+\phi}{2}}{\cos\frac{\theta-\phi}{2}} \qquad z=\pm c\tan\tfrac{1}{2}(\theta-\phi)$$

* Intersection of line with Conicoid

$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ ; $ax^2+by^2+cz^2=1$

$a(lr+\alpha)^2+b(mr+\beta)^2+c(nr+\gamma)^2=1 \rightarrow$ two roots $r_1, r_2$.

Pt $P\equiv(\alpha,\beta,\gamma)$ intersect at $A, B$ with $ax^2+by^2+cz^2=1$

$|PA|=r_1$ , $|PB|=r_2$ [If l, m, n are d.cs]

* Tangent $ax^2+by^2+cz^2=1 \Rightarrow lx+my+nz=\underbrace{\pm\sqrt{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}}}_{P}$

Point of Tangency $\equiv\left(\frac{l}{ap},\frac{m}{bp},\frac{n}{cp}\right)$

* Normals

- From $(f,g,h)$ at $(\alpha,\beta,\gamma)$

$$\frac{f-\alpha}{a\alpha}=\frac{g-\beta}{b\beta}=\frac{h-\gamma}{c\gamma}=r \quad \boxed{\begin{aligned}\alpha&=f/_{1+ar}\\ \beta&=g/_{1+br}\\ \gamma&=h/_{1+cr}\end{aligned}}$$

general point on conicoid

→ Cone through normals from $(f,g,h)$

$$\frac{x-f}{l}=\frac{y-g}{m}=\frac{z-h}{n} \Rightarrow \frac{\frac{f}{1+ar}-f}{l}=\frac{\frac{g}{1+br}-g}{m}=\frac{\frac{h}{1+cr}-h}{n}.$$

$$\Rightarrow \frac{af}{l(1+ar)}=\frac{bg}{m(1+br)}=\frac{ch}{n(1+cr)}$$

Eliminate $r$ to get ⇒

$$\frac{af}{l}(b-c)+\frac{bg}{m}(c-a)+\frac{ch}{n}(a-b)=0$$

MUG IT

# Conicoid :-

① Tanget to $ax^2+by^2+cz^2=1$ meets axes in P, Q, R.
Find locus of centroid of PQR.

Let tangent be, $a\alpha x+b\beta y+c\gamma z=1$ $(\alpha,\beta,\gamma)$

Centroid $\equiv \left(\frac{1}{3a\alpha}, \frac{1}{3b\beta}, \frac{1}{3c\gamma}\right) \Leftarrow x_1=\frac{1}{3a\alpha} \Rightarrow \alpha=\frac{1}{(3ax_1)}$

$a\alpha^2+b\beta^2+c\gamma^2=1$

$$\frac{a}{(3ax_1)^2}+\frac{b}{(3by_1)^2}+\frac{c}{(3cz_1)^2}=1. \Rightarrow \boxed{\frac{1}{ax_1^2}+\frac{1}{by_1^2}+\frac{1}{cz_1^2}=9}$$

② Normals to Conicoid. $(\alpha,\beta,\gamma)\equiv$ ft normal.

$$\frac{x-\alpha}{a\alpha}=\frac{y-\beta}{b\beta}=\frac{z-\gamma}{c\gamma}$$

③ Enveloping Cylinder
– $ax^2+by^2+cz^2=1$ and // to $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$.

$SS_1=T^2$ (with S as $ax^2+by^2+cz^2-t^2=0$. at $(l,m,n,0)$)

$$(ax^2+by^2+cz^2-t^2)(al^2+bm^2+cn^2-0)=(alx+bmy+cnz-t\cdot 0)^2.$$

$$(ax^2+by^2+cz^2-1)(al^2+bm^2+cn^2)=(alx+bmy+cnz)^2 \quad \underline{\underline{\text{Ans}}}$$

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Q. Locus of feet of 6 normals to $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ from any pt $(x_1, y_1, z_1)$ lie of curve of intersection of ellipsoid and cone.

Let normal be $\Rightarrow$ $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$

Normal Equation at $\alpha,\beta,\gamma \Rightarrow$ $\frac{x_1-\alpha}{\alpha/a^2}=\frac{y_1-\beta}{\beta/b^2}=\frac{z_1-\gamma}{\gamma/c^2}=r_1$

$$x_1=\alpha\left(1+\frac{r}{a^2}\right) \Rightarrow \boxed{\frac{x_1 a^2}{(a^2+r)}=\alpha;\ \frac{y_1 b^2}{b^2+r}=\beta;\ \frac{z_1 c^2}{c^2+r}=\gamma}$$

$$r=\frac{x_1a^2-a^2\alpha}{\alpha}=\frac{y_1b^2-b^2}{\beta}=\frac{z_1c^2-c^2}{\gamma}$$

$$r(b^2-c^2)+r(c^2-a^2)+r(a^2-b^2)=0.$$

$$\left(\frac{x_1a^2}{\alpha}-a^2\right)(b^2-c^2)+\left(\frac{y_1b^2}{\beta}-b^2\right)(c^2-a^2)+\left(\frac{z_1c^2}{\gamma}-c^2\right)(a^2-b^2)=0$$

$$\frac{x_1a^2(b^2-c^2)}{\alpha}+\frac{y_1b^2(c^2-a^2)}{\beta}+\frac{z_1c^2(a^2-b^2)}{\gamma}=0.$$

$$\boxed{\frac{x_1a^2(b^2-c^2)}{x}+\frac{y_1b^2(c^2-a^2)}{y}+\frac{z_1c^2(a^2-b^2)}{z}=0}$$ – Cone

(3) $A, B, C = lx + my + nz = p$ and $A'B'C'$ are 6 ft of normals to $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$. Find plane $A'B'C'$.

Feet of normal $\equiv (\alpha, \beta, \gamma) \equiv \left(\frac{a^2 x_1}{a^2+\gamma}, \frac{b^2 y_1}{b^2+\gamma}, \frac{c^2 z_1}{c^2+\gamma}\right)$

Six ft are obtained from $\equiv$

$$\frac{a^2 x_1^2}{(a^2+\gamma)^2} + \frac{b^2 y_1^2}{(b^2+\gamma)^2} + \frac{c^2 z_1^2}{(c^2+\gamma)^2} = 1 \quad \equiv \text{Conic}_1$$

3 ft from $\Rightarrow$ $\frac{l a^2 x_1}{a^2+\gamma} + \frac{m b^2 y_1}{b^2+\gamma} + \frac{n c^2 z_1}{c^2+\gamma} = P \equiv \text{Plane}_1$

Let $A'B'C'$ be $\Rightarrow$

$$l_1 x + m_1 y + n_1 z = p_1$$

So, other 3 ft of normals.

$$\frac{l_1 a^2 x_1}{a^2+\gamma} + \frac{m_1 b^2 y_1}{b^2+\gamma} + \frac{n_1 c^2 z_1}{c^2+\gamma} = P_1 \equiv \text{Plane}_2$$

So, $\text{Plane}_1 \times \text{Plane}_2 = \text{Conic}_1$

Compare like powers of $x_1, y_1, z_1$ and constant

$$\frac{l l_1 a^2}{\cancel{(a^2+\gamma)^2}} = \frac{a^2}{\cancel{(a^2+\gamma)^2}} \Rightarrow \boxed{l_1 = \frac{1}{a^2 l}}$$

$$\boxed{m_1 = \frac{1}{b^2 m}} \qquad \boxed{n_1 = \frac{1}{c^2 n}}$$

$$P P_1 = -1 \Rightarrow \boxed{P_1 = -\frac{1}{P}}$$

(Q) Find cone through six normals from $(x_1, y_1, z_1)$ to $ax^2+by^2+cz^2=1$

$$\frac{x-\alpha}{a\alpha} = r, \Rightarrow \alpha = \frac{x_1}{1+ar} \quad \text{Ily} \quad \beta = \frac{y_1}{1+br}, \quad \gamma = \frac{z_1}{1+cr}$$

Let normal be $\Rightarrow$ We can have $\Rightarrow$

$$(1) - \quad \frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$$

$$l = a\alpha \qquad m = b\beta \qquad n = c\gamma$$

$$= \frac{ax_1}{1+ar} \qquad = \frac{by_1}{1+br} \qquad = \frac{cz_1}{1+cr}$$

$$\frac{ax_1}{l} = (1+ar) \qquad \frac{by_1}{m} = 1+br \qquad \frac{cz_1}{n} = 1+cr$$

$$\boxed{\frac{ax_1}{l}(b-c) + \frac{by_1}{m}(c-a) + \frac{cz_1}{n}(a-b) = 0} \Leftarrow \text{MUG IT UP!}$$

from (1) $\Rightarrow$

$$\frac{ax_1(b-c)}{x-x_1} + \frac{by_1(c-a)}{y-y_1} + \frac{cz_1(a-b)}{z-z_1} = 0$$

---

ALTER:
$$\frac{l}{a\alpha} = \frac{m}{b\beta}$$

$$\frac{l(1+ar)}{ax_1} = \frac{m(1+br)}{by_1}$$

$$\Rightarrow \frac{\frac{1}{a}+r}{x_1/l} = \frac{1/b+r}{y_1/m} \Rightarrow \frac{\frac{1}{a}-\frac{1}{b}}{\frac{x_1}{l}-\frac{y_1}{m}} = \frac{\frac{1}{b}-\frac{1}{c}}{\frac{y_1}{m}-\frac{z_1}{n}}$$

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(4) Conjugate diameters / Diametral Plane

$P(x_1, y_1, z_1)$ on ellipsoid

└ Plane bisecting chords ∥ to OP is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} + \frac{zz_1}{c^2} = 0$

- Some useful relations

- $\frac{x_1 x_2}{a^2} + \frac{y_1 y_2}{b^2} + \frac{z_1 z_2}{c^2} = 0$
- $x_1^2 + x_2^2 + x_3^2 = a^2$
- $x_1 y_1 + x_2 y_2 + x_3 y_3 = 0$
- Sum of (length)$^2$ $\equiv OP^2 + OQ^2 + OR^2 = a^2 + b^2 + c^2$

(5) locus of equal conjugate diameters.

$(x_1, y_1, z_1)$. $\quad \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} = 1.$ } Equating

Also equal ⇒ $x_1^2 + y_1^2 + z_1^2 = \frac{a^2+b^2+c^2}{3}.$

locus ⇒ $\boxed{\frac{3(x^2+y^2+z^2)}{a^2+b^2+c^2} = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}}$

$\boxed{Q}$ locus of foot of $\perp r$ from centre to plane through ends of 3 conjugate diameters of $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. is

$$a^2x^2+b^2y^2+c^2z^2=3(x^2+y^2+z^2)^2$$

Let plane be $\equiv$ $\alpha x+\beta y+\gamma z=\alpha^2+\beta^2+\gamma^2$

$(\alpha,\beta,\gamma)\equiv$ foot $\Downarrow$

$$\left.\begin{aligned}&\alpha(x_1^2+x_2^2+x_3^2)=(\alpha^2+\beta^2+\gamma^2)(x_1+x_2+x_3)\\&\alpha a^2=(\alpha^2+\beta^2+\gamma^2)(x_1+x_2+x_3)\\&\alpha a=(\alpha^2+\beta^2+\gamma^2)\left(\frac{x_1+x_2+x_3}{a}\right)\quad \text{lly for } b, c.\end{aligned}\right\}\text{Ignore}$$

Just put $(x_i,y_i,z_i)$

Squaring and adding $\Rightarrow$

$$\alpha^2a^2+\beta^2b^2+\gamma^2c^2=(\alpha^2+\beta^2+\gamma^2)^2[3]\quad \underline{\text{Ans}}$$

$\boxed{Q}$ Ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. Find equation of plane through 3 extremities of conjugate diameters. Show it touches a fixed sphere.

(5) Paraboloid

$\boxed{ax^2+by^2 = 2cz.}$ ← Form remember

- Tangent → $lx+my+nz=p$ if $\frac{l^2}{a}+\frac{m^2}{b}+\frac{2np}{c}=0$

  └ Better ⇒ $\boxed{2n(lx+my+nz)+c\left(\frac{l^2}{a}+\frac{m^2}{b}\right)=0}$

  └ IMPORTANT in this form

- Normal at $(\alpha,\beta,\gamma)$

  └ $\frac{x-\alpha}{a\alpha} = \frac{y-\beta}{b\beta} = \frac{z-\gamma}{-c}$

Revise:

Q. Two perpendicular tangent planes to paraboloid $\frac{x^2}{a}+\frac{y^2}{b} = 2z$ intersects in a straight line lying in $x=0$. Show that the line touches parabola $x=0,\ y^2=(a+b)(2z+a)$

Let the line be : $\boxed{\underbrace{my+nz=p\ ;\ x=0}_{①}}$ ⇒ $\boxed{\lambda x+my+nz=p}$

If tangent,

$\boxed{a\lambda^2+bm^2+2np=0.}$

- Perpendicularity ≡ $\left.\begin{matrix}\lambda_1 x+my+nz=p\\ \lambda_2 x+my+nz=p\end{matrix}\right\}$ If $\lambda_1\lambda_2+m^2+n^2=0.$

$\lambda_1\lambda_2 = \frac{bm^2+2np}{a}$ ⇒ $bm^2+\underset{①}{2np}+am^2+an^2=0$ ← ②

Parabola is envelope of ① subject to ②

$bm^2+2n(my+nz)+am^2+an^2=0$  $x=0$

$(a+b)\left(\frac{m}{n}\right)^2+2y\frac{m}{n}+(a+2z)=0$  $\left(\frac{m}{n}\right.$ so $\left. y, z\right)$

Envelope ≡ $b^2-4ac=0$ ⇒ $(2y)^2=4(a+2z)(a+b)$

Q Locus of pt of intersection of 3 $\perp r$ tangent planes.

Tangents be $\equiv 2n_i(l_i x + m_i y + n_i z) + c\left(\frac{l_i^2}{a} + \frac{m_i^2}{b}\right) = 0$.

Add 1,2,3 $\Rightarrow$ to get locus as $\equiv 2z + c\left(\frac{1}{a} + \frac{1}{b}\right) = 0$

Q Cone through normals through $(x', y', z')$

$ax^2 + by^2 = 2cz$.

$\equiv$ Ft of normal $\equiv [\alpha, \beta, \gamma] \equiv \left[\frac{x'}{1+a\lambda}, \frac{y'}{1+b\lambda}, z' + c\lambda\right]$

– $\frac{x-x'}{l} = \frac{y-y'}{m} = \frac{z-z'}{n}$ be normal.

$$\Rightarrow \frac{l(1+a\lambda)}{ax'} = \frac{m(1+b\lambda)}{by'} = \frac{n}{-c}$$

$$\Rightarrow \frac{\frac{1}{a}+\lambda}{x'/l} = \frac{\frac{1}{b}+\lambda}{y'/m} \Rightarrow \frac{\frac{1}{a}-\frac{1}{b}}{x'/l - y'/m} = -\frac{n}{c}.$$

$$\Rightarrow \frac{c}{a} - \frac{c}{b} = \frac{ny'}{m} - \frac{nx'}{l}.$$

$$\Rightarrow \frac{x'}{l} - \frac{y'}{m} + \frac{c}{n}\left(\frac{1}{a} - \frac{1}{b}\right) = 0$$

$$\Rightarrow \boxed{\frac{x'}{x-x'} - \frac{y'}{y-y'} + \frac{c}{z-z'}\left(\frac{1}{a} - \frac{1}{b}\right) = 0}$$ — Cone.

# ① Plane



- Through 3 points.

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

- Through P and normal OP $\equiv$ $\boxed{p(x-p)+q(y-q)+r(z-r)=0}$

- Angle Bisector Planes.

  * To find: Plane bisecting angle containing Origin.

    Planes : $\left.\begin{array}{l} ax+by+cz+d=0 \\ a_1x+b_1y+c_1z+d_1=0 \end{array}\right\}$ Make $d, d_1$ +ve in equations

    └ Take + sign plane

- General Equation represents 2 planes

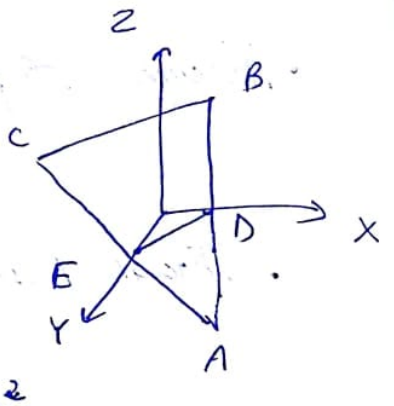
  $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$

  ↳ If $\boxed{abc+2fgh-af^2-bg^2-ch^2=0}$

  $$\tan\theta = \frac{2\sqrt{f^2+g^2+h^2-ab-bc-ca}}{a+b+c}$$

- Tetrahedron $\boxed{V = \frac{1}{3}\Delta P} = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$

Q Triangle a, b, c with sides so that m.p.s are on axes. Find equation of plane.




Let $BC = a$

$DE = a/2$.

Let plane $ABC \equiv \boxed{\frac{x}{\alpha} + \frac{y}{\beta} + \frac{z}{\gamma} = 1}$

$\alpha^2 + \beta^2 = a^2/4 \qquad \beta^2 + \gamma^2 = b^2/4 \qquad \gamma^2 + \alpha^2 = c^2/4$.

$\alpha^2 + \beta^2 + \gamma^2 = \frac{a^2 + b^2 + c^2}{8}$

$\rightarrow \alpha^2 = \frac{b^2 + c^2 - a^2}{8}, \quad \beta^2 =$ ---

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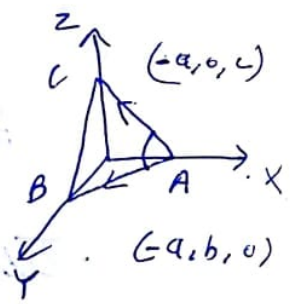
Q Triangle ABC with angles A, B, C intersects $\perp$r axes. Find angles made with principal planes.




Let plane be $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

$\cos A = \frac{\cancel{b^2}\, a^2}{\sqrt{a^2+c^2}\sqrt{a^2+b^2}} \leftarrow \left\{\frac{b^2+c^2-a^2}{2bc}\right\}$

$\cos B = \frac{b^2}{\sqrt{b^2+a^2}\sqrt{b^2+c^2}} \qquad \cos C = \frac{c^2}{\sqrt{a^2+c^2}\sqrt{c^2+b^2}}$

Angle with X-axis $\equiv \cos\theta = \frac{1/a}{\sqrt{1/a^2 + 1/b^2 + 1/c^2}} = \frac{bc}{\sqrt{a^2b^2 + b^2c^2 + c^2a^2}}$

$\boxed{\cos^2\theta = \cot B \cot C}$ — given in Quest

② Line

- $a_1 x + b_1 y + c_1 z + d_1 = 0$
  $a_2 x + b_2 y + c_2 z + d_2 = 0$

In Symmetrical form $\dfrac{l}{b_1c_2 - b_2c_1} = \dfrac{m}{c_1a_2 - c_2a_1} = \dfrac{n}{a_1b_2 - a_2b_1}$

④ Find a pt (Take $z=0$ and solve)

- Coplanar lines — $\dfrac{x-x_1}{l_1} = \dfrac{y-y_1}{m_1} = \dfrac{z-z_1}{n_1}$, $\dfrac{x-x_2}{l_2} = \dfrac{y-y_2}{m_2} = \dfrac{z-z_2}{n_2}$

— If $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$

— Plane $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$

- Intersection of Planes $\left[ a_r x + b_r y + c_r z + d_r = 0, \quad r = \{1,2,3\} \right.$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0 \xrightarrow{\text{No}} \text{Unique point}$$

$\downarrow$ Yes

$$\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} = 0 \xrightarrow{\text{No}} \text{Prism}$$

$\downarrow$ Yes

Line

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Q Plane through $(\alpha, \beta, \gamma)$ and $x = py+q = rz+S$ is

$$\begin{vmatrix} x & py+q & rz+S \\ \alpha & p\beta+q & r\gamma+S \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$$\frac{x}{1} = \frac{y+q/p}{1/p} = \frac{z+S/r}{1/r}$$ be the line, $Ax+By+Cz+D=0$ plane — ①

$\oplus$ $A.0 - B\frac{q}{p} - C\frac{S}{r} + D = 0$ — ② $\oplus$ $\boxed{A + \frac{B}{p} + \frac{C}{r} = 0}$ — ③

$\oplus$ $A\alpha + B\beta + C\gamma + D = 0$ — ④

① − ② $\equiv$ $\boxed{Ax + B\left(y+\frac{q}{p}\right) + C\left(z+S/r\right) = 0}$ — ⑤

④ − ② $\equiv$ $\boxed{A\alpha + B\left(\beta+\frac{q}{p}\right) + C\left(\gamma+S/r\right) = 0}$ — ⑥

Solving ③, ⑤, ⑥.

$$\begin{vmatrix} x & y+q/p & z+S/r \\ \alpha & \beta+q/p & \gamma+S/r \\ 1 & 1/p & 1/r \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} x & py+q & rz+S \\ \alpha & p\beta+q & r\gamma+S \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Q Find line through $(f, g, h)$, parallel to $lx+my+nz=0$ and intersects $ax+by+cz+d=0=a'x+b'y+c'z+d'$

$\Rightarrow$ Plane ∥ to given plane, through given point.

$$l(x-f) + m(y-g) + n(z-h) = 0 \quad — ①$$

Plane through given line $\equiv$ $ax+by+cz+d+\lambda(a'x+b'y+c'z+d')=0$

If it passes $(f, g, h) \Rightarrow \lambda = -\frac{(af+bg+ch+d)}{a'f+b'g+c'h+d'}$

So the plane is $\Rightarrow$ $ax+by+cz+d+\lambda(a'x+b'y+c'z+d')=0$ — ②

①, ② give the line

[S] Given d.r.s. 2, 1, 2 meets both lines.
① $x = y + a = z$ ; ② $x + a = 2y = 2z$.
Find co-ordinates of points of intersections.

→ Point on ① is $(r, r-a, r)$
② is $\left(r'-a, \frac{r'}{2}, \frac{r'}{2}\right)$

Line is $\left(r - r' + a, r - \frac{r'}{2} - a, r - \frac{r'}{2}\right)$ same d.r's $(2,1,2)$

$$\frac{r - r' + a}{2} = \frac{r - \frac{r'}{2} - a}{1} = \frac{r - \frac{r'}{2}}{2}$$

Solve to get → $r = 3a, r' = 2a$.

Points ≡ $(3a, 2a, 3a), (a, a, a)$ Ans

[Q] Find S.D between z-axis and line
$ax + by + cz + d = 0 = a'x + b'y + c'z + d'$

→ Find Plane through 2nd line // to z-axis
Find dist Plane from any random pt on z-axis

$ax + by + cz + d + \lambda(a'x + b'y + c'z + d') = 0$ is // to z-axis if

$(a + \lambda a') \cdot 0 + \ldots + (c + \lambda c') = 0 \Rightarrow \boxed{\lambda = -c/c'}$

Plane ≡ $(ac' - a'c)x + (bc' - b'c)y + dc' - d'c = 0$

⊥ dist from origin ≡ $\pm \frac{dc' - d'c}{\sqrt{(ac' - a'c)^2 + (bc' - b'c)^2}}$ Ans

---

* Orthocentre ≡ Where line found meets initial plane

Q) Planes $x = cy + bz$, $y = az + cx$, $z = bx + ay$ pass through a line if $a^2+b^2+c^2+2abc = 1$. Show equations of this line are $\frac{x}{\sqrt{1-a^2}} = \frac{y}{\sqrt{1-b^2}} = \frac{z}{\sqrt{1-c^2}}$.

$\Rightarrow$ Equation of line

All planes through origin, $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$

$$l - cm - bn = 0$$

$$m - an - cl = 0 \Rightarrow m - an - c^2m - cbn = 0$$

$$\Rightarrow \frac{m}{a+bc} = \frac{n}{1-c^2} = \frac{l}{ca+bc^2+b-bc^2} = \frac{l}{ac+b}$$

Line $\equiv \boxed{\frac{x}{ac+b} = \frac{y}{bc+a} = \frac{z}{1-c^2}}$

Also by $\begin{vmatrix} 1 & -c & -b \\ c & -1 & +a \\ b & a & -1 \end{vmatrix} = 0 \Rightarrow a^2+b^2+c^2+2abc = 1.$

$$(ac+b)^2 = a^2c^2+b^2+2abc = a^2c^2+\cancel{b^2}+1-a^2-c^2-\cancel{b^2}$$
$$= (1-a^2)(1-c^2)$$

$$(bc+a)^2 = (1-b^2)(1-c^2).$$

Thus we get required equation

Q Find surface by the line intersecting $\left\langle \begin{array}{l} y=b,\ z=-c \\ x=-a,\ z=c \\ x=a,\ y=-b \end{array}\right.$

Let the locus be $(\alpha, \beta, \gamma)$.

Line is $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$.

$\Rightarrow$ When constant lines

- $\frac{b-\beta}{m} = \frac{-c-\gamma}{n}$ · $\frac{-a-\alpha}{l} = \frac{c-\gamma}{n}$ · $\frac{a-\alpha}{l} = \frac{-b-\beta}{m}$

$$\frac{-c-\gamma}{b-\beta} = \frac{(c-\gamma)(a-\alpha)}{(+a+\alpha)(+b+\beta)}$$

$$(c-\gamma)(a-\alpha)(b-\beta) + (c+\gamma)(a+\alpha)(b+\beta) = 0$$

So locus is $\boxed{(a-x)(b-y)(c-z) + (a+x)(b+y)(c+z) = 0}$ Ans

Q Two skew lines $ax+by = z+c = 0$
$ax-by = z-c = 0$.

Find locus of line intersecting them and ⊥r to $l, m, n$.

$$\frac{x-\alpha}{\lambda} = \frac{y-\beta}{\mu} = \frac{z-\gamma}{\nu} \quad —— \quad l\lambda + m\mu + n\nu = 0 \quad - 1$$

$$z=c \rightarrow \quad a\left(\lambda\left(\frac{c-\gamma}{\nu}\right)+\alpha\right) + b\left(\mu\left(\frac{c-\gamma}{\nu}\right)+\beta\right) = 0.$$

$$\Rightarrow \lambda a(c-\gamma) + \mu b(c-\gamma) + \nu(a\alpha - b\beta) = 0. \quad - 2$$

$$z=-c \rightarrow a\lambda(\gamma+c) + b\mu(\gamma+c) - \nu(a\alpha + b\beta) = 0 \quad - 3.$$

$$\begin{vmatrix} a(\gamma-c) & -b(\gamma-c) & a\alpha - b\beta \\ a(\gamma+c) & b(\gamma+c) & a\alpha+b\beta \\ l & m & -n \end{vmatrix} = 0 \rightarrow \text{gives locus}$$

Both can be done by
$\lambda_1, \lambda_2 \ldots$

$\boxed{Q}$ Locus of line intersecting

$y = mx \quad z = c$

$y = -mx \quad z = -c$

$y = z \quad mx = -c.$

Plane through $L_1 \equiv y - mx + \lambda_1(z - c) = 0$

$L_2 \equiv y + mx + \lambda_2(z + c) = 0.$

As they meet third line $\Rightarrow$.

$L_1$ becomes $\equiv y + c + \lambda_1(y - c) = 0$

$L_2$ becomes $\equiv y - c + \lambda_2(y + c) = 0.$

$$\lambda_1 \lambda_2 = 1 \Rightarrow \left(\frac{y - mx}{c - z}\right)\left(\frac{y + mx}{-c - z}\right) = 1.$$

$$\Rightarrow \boxed{y^2 - m^2x^2 = z^2 - c^2} \text{ Ans}$$

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③ Sphere $x^2+y^2+z^2-2ux-2vy-2wz+d=0$ ← in LHS

$C \equiv (u, v, w)$ , $r^2 \equiv (u^2+v^2+w^2-d)$ IMP

1 Q. Locus of centres of spheres touching $y = mx,\ z = c$; $y = -mx,\ z = -c$;

$$\boxed{x^2+y^2+z^2+2ux+2vy+2wz+d=0}$$ ← START WITH THIS

$y = mx, z = c \Rightarrow x^2+m^2x^2+c^2+2ux+2vmx+2wc+d=0$

Touches → $(\cancel{2}u+\cancel{2}vm)^2 = \cancel{4}(1+m^2)(c^2+2wc+d)$. — ①

||ly → $(\cancel{2}u-\cancel{2}vm)^2 = \cancel{4}(1+m^2)(c^2-2wc+d)$ — ②

① − ② ⇒ $4uvm = 4wc(1+m^2)$.

Locus of Centre ≡ $(-u,-v,-w)$ by $(x,y,z)$

$$\boxed{xym + zc(1+m^2) = 0}$$

Q. Sphere of radius $r$ through origin and axes at $A, B, C$. Find locus of ft of pendicular from O to plane ABC.

⇒ ABC be $(a,0,0)\ (0,b,0)\ (0,0,c)$. $\boxed{\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1}$

$x^2+y^2+z^2-ax-by-cz=0$ $\boxed{a^2+b^2+c^2=4r^2}$

Line be ⇒ $\boxed{ax=by=cz} = \lambda$

Eliminate $a, b, c$ ⇒ ↳ $x = \lambda/a$ or $a = \lambda/x$ $b = \lambda/y$ $c = \lambda/z$.

So, ~~$\lambda\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)=1$~~ $\lambda^2\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right) = 4r^2$

$\frac{1}{\lambda}(x^2+y^2+z^2)=1$

↳ $$\boxed{(x^2+y^2+z^2)^2\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right) = 4r^2}$$

---

ALITER: Ft of ⊥r $= (l,m,n)$ → $l(x-l)+m(y-m)+n(z-n)=0$ be plane.

$l^2+m^2+n^2=k^2 \Rightarrow A = \left(\frac{k^2}{l}\right)$ $B = \left(0,\frac{k^2}{m},0\right)$, $C = \left(0,0,\frac{k^2}{n}\right)$ . . .

$$r^2 = \frac{k^4}{4l^2}+\frac{k^4}{4m^2}+\frac{k^4}{4n^2}.$$

Q A plane passes through $(a,b,c)$; show that locus of foot of $\perp r$ to it from origin is sphere $x^2+y^2+z^2-ax-by-cz=0$ 3

$\Rightarrow$ Any plane through $(a,b,c)$ is $\rightarrow$

$$l(x-a)+m(y-b)+n(z-c)=0 \quad -(1)$$

Line $\perp r$ to it from origin is $\Rightarrow$

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \quad -(2)$$

Foot of $\perp r$ (Eliminate $l, m, n$ between (1), (2))

$\hookrightarrow x(x-a)+y(y-b)+z(z-c)=0$ Ans

Q A is on OX, B on OY so tat $\angle OAB$ is constant $(\alpha)$ On AB as diameter, a circle is described as plane is parallel to OZ. Prove that as AB varies circle generates cone.

$\rightarrow$ $A \equiv (a,0)$, $B=(0,b)$ $\tan\alpha = b/a$

(1) Sphere on AB as diameter $\equiv x(x-a)+y(y-b)+z^2=0$.

(2) Plane as given $\equiv \frac{x}{a}+\frac{y}{b}=1$.

Eliminate $a, b$ from (1), (2)

$\Rightarrow \cancel{x^2}+\cancel{y^2}+z^2 = (ax+by)\left(\frac{x}{a}+\frac{y}{b}\right) = \cancel{x^2}+\cancel{y^2}+xy(\tan\alpha+\cot\alpha)$

$\Rightarrow z^2 = xy\left(\frac{1}{\sin\alpha\cos\alpha}\right)$ Ans

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* Sphere through ends of diameter.

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$$

* Circle ≡ Sphere through Circle.

$$S_1+\lambda_1 P_1=0 \quad -(1)$$

L Great circle ≡ Use the centre of (1) lying on given plane. to get $\lambda$.

* Orthogonality of spheres

$$2u_1u_2+2v_1v_2+2w_1w_2=d_1+d_2.$$

(5) Find the Sphere which touches $x^2+y^2+z^2-x+3y+2z=3$ at the point $(1,1,-1)$ and passes through origin.

$\Rightarrow$ Tangent at $(1,1,-1)$ $\Rightarrow$ $x+y-z-\frac{1}{2}(x+1)+\frac{3}{2}(y+1)+(z-1)=3.$

$\Rightarrow \frac{x}{2}+\frac{5}{2}y=3 \Rightarrow \boxed{x+5y=6}$

$$\boxed{\begin{array}{l}\text{Tangent} \Rightarrow \\ \alpha x+\beta y+\gamma z+\mu(x+\alpha)+\nu(y+\beta)+\omega(z+\gamma)+d=0.\end{array}}$$

$$x^2+y^2+z^2-x+3y+2z-3+\lambda(x+5y-6)=0$$

Through origin $\Rightarrow -3-6\lambda \Rightarrow \lambda=-\frac{1}{2} \rightarrow$ gives sphere reqd

(6) Two spheres cut orthogonally. Prove radius of common

# Cones, Cylinders.

⇒ Cones ≡ $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$.

⇒ Enveloping Cone of Sphere (Vertex ≡ $(x_1, y_1, z_1)$)

↳ $SS_1 = T^2$

⇒ Cone with vertex at origin is homogenous

⇒ General quadratic.

↳ Homogenise with $t$. → Solve $F_x, F_y, F_z, F_t = 0$.
(before put $\boxed{t=1}$)

↳ If values after solving 1st 3 satisfies $F_t = 0$ then Cone, also values obtained represent vertex.

Q. Two cones pass through $z^2 = 4ax, y=0;$ $x=0, z^2=4by$ and a common vertex. $z=0$ meets them in two conics that intersect in 4 concyclic points. Find locus of vertex.

Vertex $\equiv (\alpha, \beta, \gamma)$

$$\frac{x-\alpha}{l_i} = \frac{y-\beta}{m_i} = \frac{z-\gamma}{n_i}$$

(1) $y=0 \Rightarrow z^2 = 4ax \Rightarrow \left(\gamma - \frac{n_i}{m_i}\beta\right)^2 = 4a\left(\alpha - \frac{l_i\beta}{m_i}\right)$

$$\left(\gamma - \frac{(z-\gamma)}{(y-\beta)}\beta\right)^2 = 4a\left(\alpha - \beta\frac{(x-\alpha)}{(y-\beta)}\right)$$

$$(\gamma y - \cancel{\gamma\beta} - z\beta)^2 = 4a(\alpha y - \beta x)(y-\beta) \quad -(1)$$

$\boxed{\alpha \leftrightarrow \beta,\ y \leftrightarrow x,\ a \leftrightarrow b}$

$$\cancel{(\gamma x - z\beta)^2 = 4b(\alpha x - \beta y)(x-\beta)} - (2)$$

$$(\gamma x - z\alpha)^2 = 4b(\beta x - \alpha y)(x-\alpha) \quad -(2)$$

$\boxed{z=0} \to \gamma^2 y^2 = 4a(\alpha y - \beta x)(y-\beta)$ (+) $\gamma^2 x^2 = 4b(\beta x - \alpha y)(x-\alpha)$

$\to \gamma^2 y^2 - 4a(\alpha y - \beta x)(y-\beta) + \lambda\left(\gamma^2 x^2 - 4b(\beta x - \alpha y)(x-\alpha)\right) = 0$

Concyclic $\Rightarrow$ $x^2 = y^2 \Rightarrow \lambda(\gamma - 4b\beta) = \gamma^2 4a\alpha$
$xy = 0 \Rightarrow 4a\beta + 4\lambda ba$
} eliminate $\lambda$ to get locus

Q Find condition that angles between $x+y+z=0$, $ayz+bzx+cxy=0$ is $\frac{\pi}{3}$.

Let lines be $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$

(1) $l+m+n=0$ (2) $amn+bln+clm=0$.

$n=-(l+m) \Rightarrow am(-l-m)+bl(-l-m)+clm=0$

$\Rightarrow -bl^2-lm(a+b-c)\cancel{+l^2(-b)=0}$
$-am^2=0$

$\Rightarrow b\left(\frac{l}{m}\right)^2+\left(\frac{l}{m}\right)(a+b-c)+a=0$

$\frac{l_1 l_2}{m_1 m_2}=\frac{a}{b} \Rightarrow \boxed{\frac{l_1 l_2}{a}=\frac{m_1 m_2}{b}=\frac{n_1 n_2}{c}=(k)}$

$\frac{l_1}{m_1}+\frac{l_2}{m_2}=\frac{l_1 m_2+l_2 m_1}{m_1 m_2}=\frac{c-b-a}{b}=(k)$

$(l_1 m_2-l_2 m_1)^2=(c-b-a)^2k^2-4k^2ab$

$=k^2(a^2+b^2+c^2-2ab-2bc-2ca)$

$\tan\theta=\frac{\sqrt{\Sigma(l_1 m_2-l_2 m_1)^2}}{l_1 l_2+m_1 m_2+n_1 n_2}=\frac{\sqrt{3\cancel{k^2}(a^2+b^2+c^2-2ab-2bc-2ca)}}{\cancel{k}(a+b+c)}$

$\cancel{3}(a+b+c)^2=\cancel{3}(\ ) \Rightarrow ab+bc+ca=0$

$\Rightarrow \boxed{\frac{1}{c}+\frac{1}{b}+\frac{1}{a}=1}$

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* If $\frac{l}{m}$ is known,

then use $l+m+n=0$ eqation $\Rightarrow$ divide by $m$.

$\frac{l}{m}+1+\frac{n}{m}=0 \Rightarrow$ And get $\frac{n}{m}$ as well easily.

---

[Q] Show $x/l = y/m = z/n$ is line of intersection of tangent planes to cone

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0,$$ along lines in which it is cut by plane

$$x(al+hm+gn)+y(hl+bm+fn)+z(gl+fm+cn)=0$$

$\rightarrow$ At $(\alpha,\beta,\gamma)$ tangent to ~~plane~~ cone $\Rightarrow$

$$x(a\alpha+h\beta+g\gamma)+y(b\beta+f\gamma+h\alpha)+z(c\gamma+f\beta+g\alpha)=0 \quad -(1)$$

(1) contains given line if $\Rightarrow$

$$l(a\alpha+h\beta+g\gamma)+m(b\beta+f\gamma+h\alpha)+n(c\gamma+f\beta+g\alpha)=0$$

$$\Rightarrow \alpha(al+hm+gn)+\beta(lh+bm+nf)+\gamma(gl+fm+cn)=0$$

So, $(\alpha,\beta,\gamma)$ lies on given plane.

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* Reciprocal Cones

- Locus of lines through vertex of Cone $C_1$ which are $\perp r$ to its tangent planes.

$C_2$ is reciprocal of $C_1$

$\equiv$

Generating lines of $C_2$ are normals of tangent planes of $C_1$

$$C_2 \equiv Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0.$$

$$\begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix} \qquad A \equiv bc - f^2 \ \ldots$$

(Co-factor of a) ... similarly.

$C_2$ has 3 $\perp r$ generators $\Leftrightarrow A + B + C = 0$

$\equiv$

* Cylinder.

- Conic + A line to which generators are parallel.

[Q] Find cylinder with gens parallel to $x$-axis and passing through $ax^2+by^2+cz^2=1$, $lx+my+nz=p$.

$\Rightarrow$ As parallel to X-axis $\equiv$ Eliminate $x$.

$$x=\frac{p-my-nz}{l} \quad ; \quad a\left(\frac{p-my-nz}{l}\right)^2+by^2+cz^2=1$$

$$\rightarrow (bl^2+am^2)\cancel{x}y^2+(cl^2-am^2)z^2+2amnyz-2apmy$$
$$-2apnz+ap^2-l^2=0$$

This is given cylinder

(Remember as a concept)

$\downarrow$

whenever gens are parallel to any axis $\rightarrow$ eliminate that variable.

## Enveloping Cylinder

Cylinder touches $x^2+y^2+z^2=a^2$ and parallel to $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$

$$\boxed{(lx+my+nz)^2=(l^2+m^2+n^2)(x^2+y^2+z^2-a^2)}$$

$\boxed{SS_1=T^2}$ → (Remove $a^2$ from $T, S_1$)

Q Find enveloping cylinder of $x^2+y^2+z^2-2x+4y=1$ → $S_0$ with generators $x=y=z$. Find its guiding curve.

$\frac{x-\alpha}{1}=\frac{y-\beta}{1}=\frac{z-\gamma}{1}$ ≡ general point → $(r+\alpha, r+\beta, r+\gamma)$

$$(r+\alpha)^2+(r+\beta)^2+(r+\gamma)^2-2(r+\alpha)+4(r+\beta)=1.$$

$$3r^2+r(2\alpha+2\beta+2\gamma+2)+(\alpha^2+\beta^2+\gamma^2-2\alpha-4\beta-1)=0$$

If it touches sphere ⇒ $b^2=4ac$.

$$\cancel{4}(\alpha+\beta+\gamma+1)^2=\cancel{12}^{3}(\alpha^2+\beta^2+\gamma^2-2\alpha-4\beta-1)$$

$$\Rightarrow \alpha^2+\beta^2+\gamma^2+2\alpha\beta+2\beta\gamma+2\alpha\gamma+2\alpha+2\beta+2\gamma+1=3\alpha^2+3\beta^2+3\gamma^2-6\alpha-12\beta\ldots$$

Locus

Guiding curve ⇒ Plane (⊥r to generators, passing through centre of sphere) ∩ Sphere.

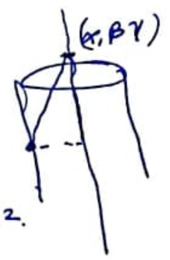
Plane ⇒ $l(x-1)+m(y+2)+n(z-0)=0$. ← IMPORTANT

$(l,m,n)\equiv(1,1,1)$

$$\Rightarrow \boxed{x+y+z+1=0} \text{ Ans } \cap S_0$$

* Right Circular Cylinder.

Axis is $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$.




$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2 - \frac{[l(x-\alpha)+m(y-\beta)+n(z-\gamma)]}{l^2+m^2+n^2} = r^2.$$

$\boxed{B}$ Find right circular cylinder circle $[(1,0,0),(0,1,0),(0,0,1)]$ as guiding curve.

Circle $C_1 \equiv [x^2+y^2+z^2-x-y-z=0] + [x+y+z=1]$

Centre $\equiv (\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$

Generator $\equiv \frac{x-\alpha}{1} = \frac{y-\beta}{1} = \frac{z-\gamma}{1} = r$.

$l$ meets circle at $(r+\alpha, r+\beta, r+\gamma)$ only once.

Good

$$\begin{cases} 3r^2 + r(2\alpha+2\beta+2\gamma-3) + \alpha^2+\beta^2+\gamma^2-\alpha-\beta-\gamma = 0 \\ b^2 = 4ac \end{cases}$$

* Find condition that $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ admits 3 $\perp r$ generators.

$\rightarrow L_1 \equiv \frac{x}{\lambda} = \frac{y}{\mu} = \frac{z}{\nu} \Rightarrow a\lambda^2+b\mu^2+c\nu^2+2f\mu\nu+2g\lambda\nu+2h\lambda\mu=0$ ... (1)

Let plane containing $L_1$ as normal and through O.

$P_1 \equiv \lambda x+\mu y+\nu z=0$.

Let $l_1, m, n$ be $\perp r$ to $L_1$ and lie in $P_1$

$$\begin{cases} al^2+bm^2+cn^2+2fmn+2gln+2hlm=0 \\ l\lambda+\mu m+\nu n=0 \end{cases}$$

$$a\left(\frac{\mu m+\nu n}{\lambda}\right)^2+bm^2+cn^2+2fmn+2gn\left[\frac{-\mu m-\nu n}{\lambda}\right]+2hm\left[\frac{-\mu m-\nu n}{\lambda}\right]=0$$

$$\left(\frac{m}{n}\right)^2\left[a\frac{\mu^2}{\lambda^2}+b-\frac{2h\mu}{\lambda}\right]+(\quad)+1\left(\frac{a\nu^2}{\lambda^2}+c-\frac{2g\nu}{\lambda}\right)=0$$

$$\frac{m_1m_2}{a\nu^2-2g\nu\lambda+c\lambda^2}=\frac{n_1n_2}{a\mu^2-2h\lambda\mu+b\lambda^2}=\frac{l_1l_2}{c\mu^2-2\lambda f\mu\nu+b\nu^2}$$

For $\perp r \Rightarrow$

$$a\nu^2+a\mu^2+b\lambda^2+b\nu^2+c\lambda^2+c\mu^2-2g\nu h-2h\lambda\mu-2f\mu\nu=0$$

(Using (1))

$$(a+b+c)(\lambda^2+\mu^2+\nu^2)$$

$\Rightarrow a+b+c=0$

* General Equation of a cone which touches the three co-ordinate planes is $\sqrt{fx} \pm \sqrt{gy} \pm \sqrt{hz} = 0$

– Reciprocal Cone passes through 3 principal axes

$$fyz + gzx + hxy = 0 \quad -①$$

Reciprocal of ① $\Rightarrow A \equiv -f^2, B = -g^2, C = -h^2$, $F = gh$, $G \equiv hf$, $H = fg$. $\begin{vmatrix} 0 & h & g \\ h & 0 & f \\ g & f & 0 \end{vmatrix}$

$$-f^2x^2 - g^2y^2 - h^2z^2 + 2ghyz + 2hfzx + 2fg\,xy = 0$$

$\Rightarrow (fx + gy - hz)^2 = 4fg\,xy \Rightarrow fx + gy - hz = \pm 2\sqrt{fgxy}$

$\Rightarrow (\sqrt{fx} \pm \sqrt{gy})^2 = hz$

$\Rightarrow \boxed{\sqrt{fx} \pm \sqrt{gy} \pm \sqrt{hz} = 0}$ Ans

IMP

* Locus of line of intersection of $\perp$r tangent planes to $ax^2 + by^2 + cz^2 = 0$

Let tangent plane be $lx + my + nz = 0 \Rightarrow$ Normal $= \frac{x}{l} = \frac{y}{m} = \frac{z}{n}$

Rec. Cone $\equiv \frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0$

Line be $\frac{x}{\lambda} = \frac{y}{\mu} = \frac{z}{\nu}$.

$$\left.\begin{array}{l} l\lambda + m\mu + n\nu = 0 \\ \frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} = 0 \end{array}\right\} \text{ Use } l_1l_2 + m_1m_2 + n_1n_2 = 0$$

* Cone with base $x^2+y^2+2ax+2by=0, z=0$, passes through $(0,0,c)$. If section of cone by $y=0$ is rec. hyperbola, Find locus of vertex. [Ans a circle-fixed]

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \Rightarrow \left(\alpha-\frac{l\gamma}{n}, \beta-\frac{m\gamma}{n}, 0\right)$$

$$\frac{(n\alpha-l\gamma)^2}{\cancel{n^2}}+\frac{(\beta n-m\gamma)^2}{\cancel{n^2}}+2a\frac{n(\alpha n-l\gamma)}{\cancel{n^2}}+2b\frac{n(\beta n-m\gamma)}{\cancel{n^2}}=0$$

$\cancel{(nx-lz)^2}$

$$\left(\alpha-\frac{\gamma(x-\alpha)}{(z-\gamma)}\right)^2+\left(\frac{\beta(z-\gamma)-\gamma(y-\beta)}{z-\gamma}\right)+2a\frac{(\alpha z-\gamma x)}{(z-\gamma)}+2b\frac{(\beta z-\gamma y)}{(z-\gamma)}=0$$

$$(\alpha z-\gamma x)^2+(\beta z-\gamma y)^2+2a(\alpha z-\gamma x)(z-\gamma)+2b(\beta z-\gamma y)(z-\gamma)=0$$

$\rightarrow$ Through $(0,0,c)$

$$\alpha^2c^2+\beta^2c^2+2a\alpha c(c-\gamma)+2b\beta c(c-\gamma)=0$$

$$\Rightarrow c(\alpha^2+\beta^2)+2a\alpha c-2a\alpha\gamma+2b\beta c-2b\beta\gamma=0 \quad -(1)$$

$\rightarrow$ By $y=0 \Rightarrow$ Coeff $z^2$ + Coeff $x^2=0$

$$\alpha^2+\beta^2+2\alpha a+2b\beta+\gamma^2=0 \quad -(2)$$

$$(1)\Rightarrow c(x^2+y^2)+2acx-2axz+2bcy-2byz=0$$

$$c(2)\Rightarrow c(x^2+y^2)+2acx+cz^2+2bcy=0$$

$$(1)-c(2)\Rightarrow cz^2+2axz+2byz=0$$

$$\Rightarrow 2by+2ax+cz=0 \leftarrow \text{Plane} \quad -(3)$$

(2), (3) give a circle.

• General Equation of 2nd degree.

① Find set of 3 $\perp r$ principal directions

$$|D-\lambda I| = 0 \to \lambda_i$$

└ Find eigenvectors $\equiv (l_i, m_i, n_i)$

② Find principal plane

└ $\lambda_i(l_i x + m_i y + n_i z) + (u l_i + v m_i + w n_i) = 0$

③ $8x^2 + 7y^2 + 3z^2 - 8yz + 4zx - 12xy + 2x - 8y + 1 = 0$

a b c. 2f 2g 2h 2u 2v d

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} \quad \begin{vmatrix} 8-\lambda & -6 & 2 \\ -6 & 7-\lambda & -4 \\ 2 & -4 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda^3 + 18\lambda^2 - 45\lambda = 0 \Rightarrow$$

$$\lambda = 0, 3, 15.$$

$\lambda = 0 \Rightarrow (1:2:2)$ or $(\frac{1}{3}\ \frac{2}{3}\ \frac{2}{3})$

So, principal plane $\equiv 0\,(\,) + \text{—} \Rightarrow$ (No such plane for $\lambda = 0$)

③ Find Centre

Find $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z} \equiv \frac{\cancel{(u, v, w)}}{\checkmark (\alpha, \beta, \gamma)}$

⑨ $x^2+y^2+z^2-2xy-2yz+2zx+x-y+z=0$

$$\left.\begin{aligned} \frac{\partial F}{\partial x} &\Rightarrow 2x-2y+2z+1=0 \\ \frac{\partial F}{\partial y} &\Rightarrow 2y-2x-2z-1=0 \\ \frac{\partial F}{\partial z} &\Rightarrow 2z-2y+2x+1=0 \end{aligned}\right\}$$

Same Equation.
|
So a plane of centres exists.

④ A. If centre at $(\alpha, \beta, \gamma)$.

A general equation can be reduced to
$$\Sigma(ax^2+2fyz)+(u\alpha+v\beta+w\gamma+d)=0 \quad -①$$
by shifting origin to centre.

B. (~~After A~~) → $3^{\perp r}_{\lambda}$ principal directions $(l_i, m_i, n_i)$ $i \in 1,2,3$
for 3 $\lambda_i$ from $|D-\lambda I|=0$.

No A
This is independent
↓
Can be done directly.

Now rotating axes to principal directions,
we transform original given ~~①~~ to →
$$\lambda_1 x^2+\lambda_2 y^2+\lambda_3 z^2+2x(ul_1+vm_1+wn_1)+$$
$$2y(ul_2+vm_2+wn_2)+2z(ul_3+vm_3+wn_3)+d=0$$

not previous constant.

- If $\lambda_1 x^2 + \lambda_2 y^2 + k = 0$ $k \neq 0$
  is a cylinder
  - Axis is line of intersection of principal planes of $\lambda_1, \lambda_2$.
  - The above axis is parallel to principal direction of $\lambda_0 (=0)$ and is also the line of centres
    $\rightarrow$ $\frac{x-\alpha}{l_0} = \frac{y-\beta}{m_0} = \frac{z-\gamma}{n_0}$ ($\alpha, \beta, \gamma$ is a centre) $\leftrightarrow$ Axis
- Homogenous 2-degree represents a pair of planes if $\Delta = 0$ $\left(\sum (ax^2 + 2fyz) = 0\right)$

- Surface of Revolution
  - About X-axis : $\sqrt{y^2+z^2} = f(x)$
  - Necessary to have atleast <u>two $\lambda$s equal</u>.
    (Not sufficient

## (5) Reducing to normal formal

- $|D-\lambda I|=0$ ; Compute Centre $\equiv (\alpha, \beta, \gamma)$.

(1) No $\lambda = 0 \rightarrow \lambda_1 x^2+\lambda_2 y^2+\lambda_3 z^2+(u\alpha+v\beta+w\gamma+d)=0$

1 (2) $\lambda_0 = 0 \rightarrow$ Find $(l_0, m_0, n_0)$.

A. If $ul_0+vm_0+wn_0 \neq 0$

$$\lambda_1 x^2+\lambda_2 y^2+2(ul_0+vm_0+wn_0)z=0$$

B. If $ul_0+vm_0+wn_0 = 0$

$$\lambda_1 x^2+\lambda_2 y^2+(u\alpha+v\beta+w\gamma+d)=0$$

(3) $\lambda_1, \lambda_2 = 0 \longrightarrow$ Find $(l_3, m_3, n_3)$ of $\lambda_3 \neq 0$

A. $\lambda_3 x^2+2\underbrace{(ul_3+vm_3+wn_3)}_{\gamma}z=0$
if $\gamma \neq 0$.

B. $\lambda_3 x^2+(u\alpha+v\beta+w\gamma+d)=0$
if $\gamma=0$.

(4) $2\lambda_i$ are equal $\rightarrow$ surface of revolution

IMP

Find $(l, m, n)$ for 3rd $\lambda_j$

$$\lambda_1 x^2+\lambda_1 y^2+2(ul+vm+wn)z=0$$

Discuss nature of the surface

(8) $4x^2-y^2-z^2+2yz+3x-4y+8z-2=0$.
and find co-ordinates of its vertex and its axis.

$$|D-\lambda I| \equiv \begin{vmatrix} 4-\lambda & 0 & 0 \\ 0 & -1-\lambda & 1 \\ 0 & 1 & -1-\lambda \end{vmatrix} = 0 \rightarrow (\lambda-4)\lambda(\lambda+2)=0$$

$0, 4, -2$

$$\lambda = 0 \Rightarrow \begin{pmatrix} 4 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{pmatrix}\begin{pmatrix} l \\ m \\ n \end{pmatrix} \Rightarrow \left(0, \tfrac{1}{\sqrt{2}}, \tfrac{1}{\sqrt{2}}\right)$$

$$\boxed{4y^2-2z^2+2\cdot\frac{2}{\sqrt{2}}x=0.}$$

$\lambda = -2 \Rightarrow \left(0, \tfrac{1}{\sqrt{2}}, -\tfrac{1}{\sqrt{2}}\right)$ $\quad \lambda = 4 \Rightarrow (1, 0, 0)$

Principal Plane $\Rightarrow \lambda(lx+my+nz) + ul+vm+wn = 0$ $[\lambda \neq 0]$

$$-\frac{2}{\sqrt{2}}(y-z) + \left(-\frac{2}{\sqrt{2}} + \frac{4}{\sqrt{2}}\right) = 0$$

Find both principal planes $\equiv$ Their intersection is the axis.

$\rightarrow$ The point on axis lying on given curve gives the vertex.

(8) Find the surface generated by line which intersects $y=a=z$, $x+3z=a=y+z$ and parallel to $x+y=0$.

2016

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}.$$

$$\frac{a-\beta}{m}=\frac{a-\gamma}{n} \;;\; l+m=0 \;;\; \begin{array}{l} lr+\alpha+3(nr+\gamma)=a \\ mr+\beta+nr+\gamma=a. \end{array}$$

$$r=\frac{a-\beta-\gamma}{m+n}.$$

$$\frac{(l+3n)}{(m+n)}(a-\beta-\gamma)=a-\alpha-3\gamma$$

$$(a-\beta-\gamma)(l+3n)=(a-\alpha-3\gamma)(n+m)$$

$$(a-\beta-\gamma)(3n-m)=(a-\alpha-3\gamma)(n+m).$$

$$(a-\beta-\gamma)\left(3-\left(\frac{a-\beta}{a-\gamma}\right)\right)=(a-\alpha-3\gamma)\left(1+\frac{a-\beta}{a-\gamma}\right).$$

$$(a-\beta-\gamma)\left(\frac{2a-3\gamma+\beta}{\cancel{a-\gamma}}\right)=\frac{(a-\alpha-3\gamma)}{\cancel{(a-\gamma)}}(2a-\gamma-\beta)$$

$$(a-y-z)(2a-3z+y)=(a-x-3z)(2a-z-y).$$

$$\cancel{2a^2}-\cancel{3az}+\cancel{2ay}-\cancel{2ay}+\cancel{3yz}-y^2+2az+\cancel{3z^2}-yz=$$

$$\cancel{2a^2}-\cancel{az}-\cancel{ay}-2ax+\cancel{xz}+xz+xy-\cancel{6az}+\cancel{3z^2}+\cancel{3yz}.$$

$$y^2+yz+xy+xz=2ax+2az$$

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ALITER

$$y-a+\lambda_1(z-a)=0$$

$$x+3z-a+\lambda_2(y+z-a)=0$$

Let $l, m, n$ be d.cs.

$$0l+m+\lambda_1 n=0$$

$$l+\lambda_2 m+(\lambda_2+3)n=0$$

Solving $\Rightarrow \dfrac{l}{\lambda_2+3-\lambda_1\lambda_2}=\dfrac{m}{\lambda_1}=\dfrac{n}{-1}$

$\lambda_2+3-\lambda_1\lambda_2+\lambda_1=0 \leftarrow$ Substitute here

(8) Line through a variable point of $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, z=0$ to meet $y=mx, z=c$ and $y=-mx, z=-c$. Find locus of line.

2009

Find $x, y$
OR
Find $l, m, n$

$$\frac{x-\alpha}{p}=\frac{y-\beta}{q}=\frac{z-\gamma}{r}.$$

$$\beta+q\left(\frac{c-\gamma}{r}\right)=m\left(\alpha+p\left(\frac{c-\gamma}{r}\right)\right) \Rightarrow \frac{q}{r}=\frac{m\left(\alpha+\frac{p}{r}(c-\gamma)\right)-\beta}{(c-\gamma)}$$

$$\beta+q\left(\frac{-c-\gamma}{r}\right)=-m\left(\alpha+p\left(\frac{-c-\gamma}{r}\right)\right) \Rightarrow \beta+(-c-\gamma)\frac{m\left(\alpha+\frac{p}{r}(c-\gamma)\right)-\beta}{(c-\gamma)}$$

$$\frac{\left(\alpha-\frac{p}{r}\gamma\right)^2}{a^2}+\frac{\left(\beta-\frac{q}{r}\gamma\right)^2}{b^2}=1.$$

$$y-mx+k_1(z-c)=0$$
$$y+mx+k_2(z+c)=0$$

Put $z=0 \Rightarrow$ $\left.\begin{array}{l} y-mx-k_1c=0 \\ y+mx+k_2c=0 \end{array}\right\}$ $y=\frac{(k_1-k_2)c}{2}$, $x=\frac{(k_1+k_2)c}{2m}$

Putting in Ellipse $\Rightarrow$

$$\frac{c^2}{4m^2a^2}(k_1+k_2)^2+\frac{c^2}{4b^2}(k_1-k_2)^2=1.$$

$$\frac{c^2}{4a^2m^2}\left(\frac{mx-y}{z-c}+\frac{(-mx-y)}{z+c}\right)^2+\frac{c^2}{4b^2}\left(\frac{mx-y}{z-c}+\frac{mx+y}{z+c}\right)^2=1.$$

Gives the locus