LINEAR ALGEBRA

: IFoS - 2017 :

① Let A be a square matrix of order 3 such that each of its diagonal elements is 'a' and each of ~~th~~ its off-diagonal elts is 1. If $B = bA$ is orthogonal. Determine the values of a & b.

$\rightarrow$ $A = \begin{bmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{bmatrix}$ $\quad B = bA = \begin{bmatrix} ba & b & b \\ b & ba & b \\ b & b & ba \end{bmatrix}$

B is orthogonal $\Rightarrow B^T B = I_3$

$$\begin{bmatrix} ba & b & b \\ b & ba & b \\ b & b & ba \end{bmatrix} \begin{bmatrix} ba & b & b \\ b & ba & b \\ b & b & ba \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} b^2a^2+2b^2 & 2b^2a+b^2 & 2b^2a+b^2 \\ 2b^2a+b^2 & b^2a^2+2b^2 & 2b^2a+b^2 \\ 2b^2a+b^2 & 2b^2a+b^2 & b^2a^2+2b^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

i.e. $\quad b^2a^2+2b^2 = 1$ — ① and $\quad 2b^2a + b^2 = 0$ — ②

$\Rightarrow b^2(2a+1) = 0.$

$\Rightarrow a = -\frac{1}{2}$ or $b = 0.$

~~If but~~

but $b=0$ violates ①. $\therefore a = -\frac{1}{2}.$

$b^2a^2 + 2b^2 = 1 \Rightarrow b^2 \cdot \frac{1}{4} + 2b^2 = 1 \Rightarrow 9b^2 = 4 \Rightarrow b^2 = \frac{4}{9}.$

$\Rightarrow b = \pm\frac{2}{3}.$

$\therefore a = -\frac{1}{2}, \quad b = \pm\frac{2}{3}$

② let V be the vector space of all 2×2 matrices over field $\mathbb{R}$. Show that W is not a subspace of V where.

(i) W contains all 2×2 matrices with zero determinant

(ii) W consists of all 2×2 matrices such that $A^2 = A$

$\rightarrow$ (i) $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V \,/\, \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 0 \right\}$

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then $|A| = |B| = 0$. Hence $A, B \in W$.

$A+B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin W$ as $|A+B| = 1 \neq 0.$

Therefore, internal composition is violated. Hence, W is not a subspace of V.

(ii) $W = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V \;\middle|\; \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \right\}$

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. $B^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = A$ $\quad B^2 = B.$

Hence, $A, B \in W$. Now, $A+B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \notin W$

as $(A+B)^2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix} \neq (A+B)$

Hence, internal composition is violated.

Hence, W is not a subspace of V.

(3) State the Cayley-Hamilton theorem. Verify this theorem for the matrix $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Hence find A^{-1}.

→ Cayley-Hamilton Theorem states that every square matrix satisfies its characteristics equation..

Char. eqn of A is given by $|A - \lambda I| = 0$

i.e. $\begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & -1-\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0 \quad \Rightarrow (1-\lambda)[\lambda(1+\lambda)-1] = 0$

$\Rightarrow (1-\lambda)[\lambda^2+\lambda-1] = 0$

$\Rightarrow \lambda^2 - \lambda^3 + \lambda - \lambda^2 - 1 + \lambda = 0$

$\Rightarrow \lambda^3 - 2\lambda + 1 = 0$ —— (1)

Putting A in the LHS of (1)

$A^3 - 2A + I = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$

Hence A satisfies the char. equation.

Hence, Cayley-Hamilton Theorem is verified for A

Now $A^3 - 2A + I = 0$. Premultiplying by A^{-1} on both sides,

$A^{-1}A^3 - 2A^{-1}A + A^{-1}I = A^{-1}0 \Rightarrow A^2 - 2I + A^{-1} = 0$

$\Rightarrow A^{-1} = -A^2 + 2I = -\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -2 & -2 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$\therefore A^{-1} = \begin{bmatrix} 1 & -2 & -2 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

Q4 Reduce the following matrix to a row-reduced echelon form and hence find its rank $A = \begin{bmatrix} -1 & 2 & -1 & 0 \\ 2 & 4 & 4 & 2 \\ 0 & 0 & 1 & 5 \\ 1 & 6 & 3 & 2 \end{bmatrix}$

$\rightarrow$ $$A = \begin{bmatrix} -1 & 2 & -1 & 0 \\ 2 & 4 & 4 & 2 \\ 0 & 0 & 1 & 5 \\ 1 & 6 & 3 & 2 \end{bmatrix} \underset{\substack{R_2 \to R_2 + 2R_1 \\ R_4 \to R_4 + R_1}}{\sim} \begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & 8 & 2 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 8 & 2 & 2 \end{bmatrix} \underset{\substack{R_1 \to 4R_1 - R_2 \\ R_4 \to R_4 - R_2}}{\sim} \begin{bmatrix} -4 & 0 & -6 & -2 \\ 0 & 8 & 2 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_1 \to R_1 + 6R_3,\ R_2 \to R_2 - 2R_3$ $\qquad$ $R_1 \to R_1 \div -4,\ R_2 \to R_2 \div 8$

$$\sim \begin{bmatrix} -4 & 0 & 0 & 28 \\ 0 & 8 & 0 & -8 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

which is clearly in the row echelon form.

There are three non-zero rows in the final matrix.

Therefore, $\rho(A) = \underline{\underline{3}}$

Q5 Given the set (u, v, w) is linearly independent, examine the sets:

(i) $\{u+v, v+w, w+u\}$ $\qquad$ (ii) $\{u+v, u-v, u-2v+2w\}$ for linear independence.

$\rightarrow$ Given that $\{u, v, w\}$ is L.I. Then, if for some $a, b, c \in \mathbb{R}$,

$au + bv + cw = 0$, then $a = b = c = 0$ ——— (1)

Now:

(i) $\{u+v, v+w, w+u\}$.

Let $a_1, a_2, a_3 \in \mathbb{R}$ such that $a_1(u+v) + a_2(v+w) + a_3(w+u) = 0$

$\Rightarrow (a_1 + a_3)u + (a_1 + a_2)v + (a_2 + a_3)w = 0.$

Since $\{u, v, w\}$ are L.I. $\Rightarrow$ $a_1 + a_3 = 0,\ a_1 + a_2 = 0,\ a_2 + a_3 = 0.$

$\Rightarrow$ $a_1 = -a_3,\ a_1 = -a_2,\ a_2 = -a_3$

These give $a_1 = a_2 = a_3 = 0$.

Hence, the set $\{u+v, v+w, w+u\}$ is L.I.

(ii) $\{u+v, u-v, u-2v+2w\}$.

Let $b_1, b_2, b_3 \in \mathbb{R}$ such that $b_1(u+v) + b_2(u-v) + b_3(u-2v+2w) = 0$

$\Rightarrow (b_1 + b_2 + b_3)u + (b_1 - b_2 - 2b_3)v + 2b_3 w = 0$

Given that $\{u, v, w\}$ is a L.I. set, then

$b_1 + b_2 + b_3 = 0$, ⓵ $\quad b_1 - b_2 - 2b_3 = 0$, ⓶ $\quad 2b_3 = 0$ ⓷

$\Rightarrow b_3 = 0$.

② $\equiv b_1 - b_2 - 2b_3 = 0 \Rightarrow b_1 - b_2 = 0 \quad [b_3 = 0]$

$\Rightarrow b_1 = b_2$.

① $\equiv b_1 + b_2 + b_3 = 0 \Rightarrow b_1 + b_2 = 0 \quad [b_3 = 0]$

$\Rightarrow b_1 + b_1 = 0 \quad [b_1 = b_2]$

$\Rightarrow b_1 = 0 = b_2$.

$\therefore b_1 = b_2 = b_3 = 0$.

Hence the given set is L.I.

⑥ Find the eigen values & corresponding eigen vectors for the matrix $A = \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix}$. Examine whether A is diagonalizable. Obtain a matrix D if its diagonalizable such that $D = P^{-1}AP$.

$\rightarrow$ Char. eqn of A is given by $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix} = 0$

$\rightarrow -\lambda(3-\lambda) + 2 = 0 \Rightarrow \lambda^2 - 3\lambda + 2 = 0 \Rightarrow (\lambda-1)(\lambda-2) = 0$

$\lambda = 1, 2$.

Hence, eigen values of A are 2 and 3. Since both the eigen values are distinct, hence A is diagonalizable.

Now: Finding eigen vectors corresponding to the eigen value

(i) $\lambda = 1$: $(A - 1.I)X = 0$

$\rightarrow \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow R_2 \rightarrow R_2 + R_1$

$\begin{bmatrix} -1 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow -x - 2y = 0 \Rightarrow x = -2y$

$X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2y \\ y \end{bmatrix} = y \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

$\therefore X_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

(ii) $\lambda = 2$: $(A - 2.I)X = 0 \quad R_2 \rightarrow R_2 + \frac{1}{2}R_1$

$\Rightarrow \begin{bmatrix} -2 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} -2 & -2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$\Rightarrow -2x - 2y = 0 \Rightarrow x = -y$

$\therefore X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -y \\ y \end{bmatrix} = y \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

$\therefore X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Hence, the eigen vectors:

corr. to eigen value $\lambda = 1$ is $X_1 = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

corr. to eigen value $\lambda = 2$ is $X_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Now: Let $P = [X_1 \; X_2] = \begin{bmatrix} -2 & -1 \\ 1 & 1 \end{bmatrix}$ & $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$

Then, $P^{-1}AP = D$ ~~wt~~