MAINS TEST SERIES – 2020

TEST – IV (Paper – II)

Answer Key

PDE, NUMERICAL ANALYSIS & COMPUTER PROG. AND MECHANICS & FD

1(a) Show that the differential equation of all cones which have their vertex at the origin is $px+qy=z$. Verify that $yz+zx+xy=0$ is a surface satisfying the above equation.

Sol^n: The equation of any cone with vertex at origin is $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ —— ①

where a,b,c,f,g,h are parameters. Differentiating ① partially w.r.t x and y by turn, we have (nothing that $p=\frac{\partial z}{\partial x}$ & $q=\frac{\partial z}{\partial y}$)

$$2ax+2czp+2fyp+2g(px+z)+2hy=0$$

$$ax+gz+hy+p(cz+gx+fy)=0 \quad \text{—— ②}$$

and $$2by+2czq+2f(yq+z)+2gxq+2hx=0$$

$$by+fz+hx+q(cz+fy+gx)=0 \quad \text{—— ③}$$

Multiplying ② by x and ③ by y and adding, we have

$$(ax^2+by^2+gzx+fyz+2hxy)+(cz+fy+gx)(px+qy)=0$$

$$-(cz^2+fyz+gxz)+(cz+fy+gx)(px+qy)=0,$$

$$\Rightarrow (cz+fy+gx)(px+qy-z)=0 \Longrightarrow px+qy-z=0 \quad \text{—— ④}$$

which is required differential equation.

Given surface is $yz+zx+xy=0$ —— ⑤

Differentiating ⑤ partially w.r.t x and y by turn, we get

$$yp+px+z+y=0 \text{ and } z+qy+xq+x=0 \quad \text{—— ⑥}$$

solving ⑥ and p and q, $p=-(z+y)/(x+y)$, $q=-(z+x)/(x+y)$.

$$\therefore px+qy-z=-\frac{x(z+y)}{x+y}-\frac{y(z+x)}{x+y}-z=-\frac{2(xy+yz+zx)}{x+y}=0 \text{ by ⑤}$$

Hence ⑤ is a surface satisfying ④.

1(b). Solve $(D^2 - DD' - 2D'^2)z = (2x^2 + xy - y^2)\sin xy - \cos xy$.

Sol'n: Here A.E is $m^2 - m - 2 = 0$ so that $m = 2, -1$.

So. $C.F = \phi_1(y+2x) + \phi_2(y-x)$, ϕ_1, ϕ_2 being arbitrary functions.

$$P.I = \frac{1}{(D-2D')}\frac{1}{(D+D')}\{(2x^2+xy-y^2)\sin xy - \cos xy\}$$

$$= \frac{1}{(D-2D')}\frac{1}{D+D'}\{(2x-y)(x+y)\sin xy - \cos xy\}$$

$$= \frac{1}{D-2D'}\int\{(x-c)(2x+c)\sin x(c+x) - \cos x(c+x)\}dx$$

(Taking $c = y-x$)

$$= \frac{1}{D-2D'}\int\{(x-c)(2x+c)\sin(cx+c^2) - \cos((cx+x^2)\}dx$$

$$= \frac{1}{D-2D'}\left[-(x-c)\cos(cx+x^2) + \int\cos(cx+x^2)dx - \int\cos(cx+x^2)dx\right]$$

$$= \frac{1}{D-2D'}(y-2x)\cos xy \quad \text{as } c = y-x$$

$$= \int(c'-4x)\cos(c'x-2x^2)dx, \text{ where } c' = y+2x$$

$$= \int \cos t\, dt = \sin t, \quad \text{putting } c'x - 2x^2 = t \text{ so that } (c'-4x)dx = dt$$

$$= \sin(cx - 2x^2)$$

$$= \sin xy, \text{ as } c' = y+2x$$

So solution is $z = \phi_1(y+2x) + \phi_2(y-x) + \sin xy$.

1(C) By using Newton's forward interpolation formula, find the number of men getting wages between Rs. 10 and 15 from the following data.

wages in Rs.	0–10	10–20	20–30	30–40
No. of Men	9	30	35	42

Sol^n: First we prepare the cumulative frequency table, as follows.

wages in Rs. less than	10	20	30	40
No. of Men	9	39	74	116

Now the difference table is

x	y_x	Δy_x	$\Delta^2 y_x$	$\Delta^3 y_x$
10	9			
		30		
20	39		5	
		35		2
30	74		7	
		42		
40	116			

we shall find y_{15} i.e, the no. of men with wages less than 15.

Taking $x_0 = 10$, $x = 15$, we have

$$p = \frac{x - x_0}{h} = \frac{5}{10} = \frac{1}{2}$$

using Newton's forward interpolation formula

$$y_{15} = y_{10} + p\Delta y_{10} + \frac{p(p-1)}{2!}\Delta^2 y_{10} + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_{10}.$$

$$= 9 + \tfrac{1}{2}(30) + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2!}(5) + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{3!}(2)$$

$$= 9 + 15 + \tfrac{1}{2}\left(-\tfrac{1}{2}\right)\tfrac{1}{2}(5) + \tfrac{1}{2}\left(-\tfrac{1}{2}\right)\left(-\tfrac{3}{2}\right) \times \tfrac{2}{6}$$

$$= 24 - \tfrac{5}{8} + \tfrac{1}{8}$$

$$= 24 - 0.625 + 0.125$$

$= 24 - 0.5$

$= 23.5 \approx 24$

The no. of men with wages less than 15 is 24.

But the no. of men with wages less than 10 is 9.

Hence the no. of men getting wages between 10 and 15 $= 24 - 9 = 15$.

1(d). Give a Boolean expression for the following statements.
(i) Y is a 1 only if A is a 1 and B is a 1 or if A is a 0 and B is a 0.
(ii) Y is a 1 only if A, B and C are all 1's or if only one of the variables is a 0.

Sol'n: (i) Truth table for given conditions.

A	B	Y
0	0	1
0	1	0
1	0	0
1	1	1

$\boxed{Y = A'B' + AB}$

(ii)

A	B	C	Y
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	1
1	0	0	0
1	0	1	1
1	1	0	1
1	1	1	1

$$Y = A'BC + AB'C + ABC' + ABC$$

Simplify the expression

$$= A'BC + ABC + AB'C + ABC + ABC' + ABC$$

$$= BC(A'+A) + AC(B'+B) + A(C'+C)$$

$$= BC + AC + AB$$

$$\therefore \boxed{Y = BC + AC + AB}$$

1(e) → Find the M.I of a right solid cone of mass M, height h and radius of whose base is a, about its axis.

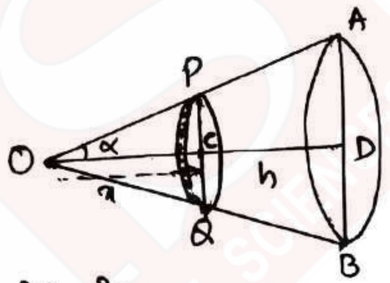
Sol'n: Let O be the vertex of the right solid cone of mass M, height h and radius of whose base is a, If α is the semi-vertical angle and ρ be density of the cone, then

$$M = \frac{1}{3}\pi\rho h^3 \tan^2\alpha \quad \text{——} \quad (1)$$

Consider an elementary disc PQ of thickness δx, parallel to the base AB and at a distance x from the vertex O.

∴ Mass of the disc,

$$\delta m = \rho\pi x^2 \tan^2\alpha\, \delta x$$



M.I of this elementary disc about axis OD.

$$= \frac{1}{2}\delta m\, CP^2 = \frac{1}{2}(\rho\pi x^2 \tan^2\alpha\, \delta x)x^2\tan^2\alpha$$

$$= \frac{1}{2}\rho\pi x^4 \tan^4\alpha\, \delta x$$

∴ M.I of the cone about axis OD.

$$= \int_0^h \frac{1}{2}\rho\pi x^4 \tan^4\alpha\, dx$$

$$= \rho\frac{\pi}{10}h^5\tan^4\alpha$$

$$= \frac{3}{10}Mh^2\tan^2\alpha \quad \text{from (1)}$$

$$= \frac{3}{10}Ma^2 \quad (\because \tan\alpha = a/h).$$

2(a) The ends A and B of a rod, 10 cm in length are kept at temperatures 0°C and 100°C until steady state condition prevails. Suddenly the temperature at the end A is increased to 20°C, and the end B is decreased to 60°C. Find the temperature distribution in the rod at time t.

Sol^n:

The problem is described by

PDE : $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}$ —— (1)

B.Cs : $u(0,t) = 0$
$u(10,t) = 100$

Prior to change in temperature at the ends of the rod, the heat flow in the rod is independent of time as steady state condition prevails.

For steady state,

$$\frac{d^2u}{dx^2} = 0 \quad \text{—— (2)}$$

whose solution is $u(x) = C_1x + C_2$.

when $x=0$, $u=0 \Rightarrow C_2 = 0$
when $x=10$, $u=100 \Rightarrow C_1 = 10$

Thus, the initial steady temperature distribution in the rod is

$u(x,0) = u(x) = 4x+20$ —— (3)

Similarly, when the temperature at the ends A and B are changed to 20 and 60,

i.e, BCs $u(0,t) = 20 \;\forall t$ —— (i)

$u(10,t) = 60 \;\forall t$ —— (ii)

from (2) $u(x,t) = c_1 x + c_2 = u_1(x)$ say.

$u(0,t) = 20 = c_1(0) + c_2 \Rightarrow c_2 = 20$

$u(10,t) = 60 = c_1(10) + 20 \Rightarrow c_1 = 4$

$\therefore \boxed{u_1(x) = 4x + 20}$ —— (4)

which is the final steady temperature in the rod and which will be obtained after a long time.

To get the temperature distribution $u(x,t)$ in the intermediate period. we split the temperature function $u(x,t)$ into two parts as

$u(x,t) = u_1(x) + u_2(x,t)$ —— (5)

where $u_1(x)$ is given by (4) and $u_2(x,t)$ may be treated as a transient part of the solution, which decreases with increase of t.

putting $x=0$ in (5) and using (i), we get

$u_2(0,t) = u(0,t) - u_1(0) = 20 - 20 = 0$ —— (iii)

$u_2(10,t) = u(10,t) - u_1(10) = 60 - 60 = 0$ —— (iv)

Again $u_2(x,0) = u(x,0) - u_1(x)$

$= 10x - (4x + 20)$

$u_2(x, 0) = 6x - 20.$ —— (v)

Hence the boundary condition and initial condition to the transient solution $u_2(x,t)$ are given by (iii), (iv) & (v).

i.e, So we now solve

$$\frac{\partial u_2}{\partial t} = k\frac{\partial^2 u_2}{\partial x^2} \quad \text{—— (6)}$$

subject to the boundary conditions (iii) & (iv) and initial condition (v).

Suppose that (6) has solution of the form

$u_2(x,t) = X(x)T(t)$. —— (7)

substituting (7) in (6), we get

$kX''T = XT' \Rightarrow X''/X = T'/kT = \mu$. —— (8)

$\Rightarrow X'' - \mu X = 0$ —— (9) and $T' = kT\mu$. —— (10)

using (iii) & (iv), (9) gives

$X(0)T(t) = 0$ and $X(10)T(t) = 0$ —— (11)

Since $T(t) = 0$ leads to $u_2 = 0$, so suppose that $T(t) \neq 0$.

$\therefore$ from (11), $X(0) = 0$, $X(10) = 0$ —— (12)

we now solve (9) under B.C (12). Three cases arise.

Case 1: Let $\mu = 0$, then soln of (9) is

$X(x) = Ax + B$

Case 2: Let $\mu = \lambda^2$, $\lambda \neq 0$. then the solution of (9) is $X(x) = Ae^{\lambda x} + Be^{-\lambda x}$.

Using the B.C. (12), from the case 1 & 2, we get $A = B = 0$ so that $X(x) = 0$ and hence $u_2 = 0$, which doesn't satisfy (v).

Case 3: Let $\mu = -\lambda^2$, $\lambda \neq 0$. Then solution of (6) is $X(x) = A\cos\lambda x + B\sin\lambda x$. —— (13)

Using B.C (12), (13) gives

$0 = A$ and $0 = A\cos\lambda a + B\sin\lambda a$, where $(a = 10)$

$\Rightarrow B\sin\lambda a = 0$

$\Rightarrow \sin\lambda a = 0$ $(\because B \neq 0$ otherwise $X=0$ so that $u_2 = 0)$

$\Rightarrow \lambda a = n\pi$

$\Rightarrow \lambda = \frac{n\pi}{10}$ $n = 1, 2, 3, \ldots$

Hence non-zero solutions $X_n(x)$ of (6) are given by $X_n(x) = B_n \sin\left(\frac{n\pi x}{a}\right)$. —— (14)

Using $\lambda = \frac{n\pi}{10}$, (10) reduces to

$$\frac{dT}{T} = \frac{-n^2\pi^2 k}{a^2}\,dt \Rightarrow \frac{dT}{T} = -C_n^2\,dt \quad \text{where } C_n = \frac{n^2\pi^2 k}{100}$$

$$\Rightarrow T_n(t) = D_n e^{-C_n^2 t}$$

$$\therefore u_{2n}(x,t) = X_n(x)\,T_n(t) = E_n \sin\frac{n\pi x}{10}\, e^{-C_n^2 t}.$$

are solutions of (6), satisfying (iii) & (iv).

Here $E_n (= B_n D_n)$ is another arbitrary constant.

In order to obtain a solution also satisfying (v), we consider more general solution

$$u_2(x,t) = \sum_{n=1}^{\infty} u_{2n}(x,t) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{10}\right) e^{-C_n^2 t} \quad \text{—— (15)}$$

where E_n is given by $E_n = \frac{2}{10}\int_0^{10} (6x-20)\sin\frac{n\pi x}{10}\,dx$

$$= \frac{4}{10}\int_0^{10}(3x-10)\sin\frac{n\pi x}{10}\,dx$$

$$= \frac{2}{5}\left[(3x-10)\left(-\cos\frac{n\pi x}{10}\right)\frac{10}{n\pi} - \left\{-3\sin\frac{n\pi x}{10}\cdot\frac{(n\pi)^2}{10^2}\right\}\right]_0^{10}$$

$$= \frac{2}{5}\left[\frac{200}{n\pi}\left(-\cos n\pi\right) - (-10)\frac{10}{n\pi}(-1) + \left(\frac{n\pi}{10}\right)^2\left[3\sin n\pi - 0\right]\right]$$

$$=$$

$$= \frac{2}{5}\left[\frac{100}{n\pi}\left(-2\cos n\pi - 1\right)\right] + 0$$

$$= \frac{2}{5}\left[\frac{100}{n\pi}\left(-1-2(-1)^n\right)\right]$$

$$= -\frac{1}{5}\left[\frac{200}{n\pi} + (-1)^n\frac{400}{n\pi}\right]$$

$\therefore$ from (15)

$$u_2(x,t) = \sum_{n=1}^{\infty}\left(-\frac{1}{5}\right)\left[\frac{200}{n\pi} + (-1)^n\frac{400}{n\pi}\right]\sin\left(\frac{n\pi x}{10}\right)e^{-c_n^2 t} \quad \text{— (16)}$$

combining (4) & (16),

from eqn (5)

$$u(x,t) = 4x+20+\sum_{n=1}^{\infty}\left(\frac{-1}{5}\right)\left[\frac{200}{n\pi} + (-1)^n\frac{400}{n\pi}\right]\sin\left(\frac{n\pi x}{10}\right)e^{-c_n^2 t}.$$

where $c_n^2 = \frac{n^2\pi^2 k}{100}$

2(b) Find the solution of the following system of equations

$$x_1 - \tfrac{1}{4}x_2 - \tfrac{1}{4}x_3 = \tfrac{1}{2}, \quad -\tfrac{1}{4}x_1 + x_2 - \tfrac{1}{4}x_4 = \tfrac{1}{2}$$

$$-\tfrac{1}{4}x_1 + x_3 - \tfrac{1}{4}x_4 = \tfrac{1}{4}, \quad -\tfrac{1}{4}x_2 - \tfrac{1}{4}x_3 + x_4 = \tfrac{1}{4}$$

using Gauss-Seidel method and perform the first five iterations.

Sol'n: The given system of equations can be written as

$$\left.\begin{aligned} x_1 &= 0.5 + 0.25x_2 + 0.25x_3 \\ x_2 &= 0.5 + 0.25x_1 + 0.25x_4 \\ x_3 &= 0.25 + 0.25x_1 + 0.25x_4 \\ x_4 &= 0.25 + 0.25x_2 + 0.25x_3 \end{aligned}\right\} \longleftarrow (1)$$

By Gauss-Seidel method, system (1) can be written as

$$x_1^{K+1} = 0.5 + 0.25x_2^{(K)} + 0.25x_3^{(K)}$$

$$x_2^{K+1} = 0.5 + 0.25x_1^{K+1} + 0.25x_4^{(K)}$$

$$x_3^{K+1} = 0.25 + 0.25x_1^{K+1} + 0.25x_4^{(K)}$$

$$x_4^{K+1} = 0.25 + 0.25x_2^{K+1} + 0.25x_3^{K+1}$$

where $K = 0, 1, 2, 3 \ldots$

Now taking $x^{(0)} = 0$ (i.e. $x_2^{(0)} = 0, x_3^{(0)} = 0, x_4^{(0)} = 0$) which is initial solution.

K=0

$$x_1^{(1)} = 0.5 + 0.25x_2^{(0)} + 0.25x_3^{(0)} = 0.5 + 0 + 0 = 0.5$$

$$x_2^{(1)} = 0.5 + 0.25x_1^{(1)} + 0.25x_4^{(0)} = 0.5 + (0.25)(0.5) + 0 = 0.625$$

$$x_3^{(1)} = 0.25 + 0.25x_1^{(1)} + 0.25x_4^{(0)} = 0.25 + (0.25)(0.5) + (0.25)(0) = 0.375$$

$$x_4^{(1)} = 0.25 + 0.25x_1^{(1)} + 0.25x_3^{(1)} = 0.25 + (0.25)(0.625) + (0.25)(0.375) = 0.5$$

K=1

$$x_1^{(2)} = 0.5 + 0.25x_2^{(1)} + 0.25x_3^{(1)} = 0.5 + (0.25)(0.625) + (0.25)(0.375) = 0.75$$

$$x_2^{(2)} = 0.5 + 0.25x_1^{(2)} + 0.25x_4^{(1)} = 0.5 + (0.25)(0.75) + (0.25)(0.5) = 0.8125$$

$$x_3^{(2)} = 0.25 + 0.25x_1^{(2)} + 0.25x_4^{(2)} = 0.25[1 + 0.75 + 0.5] = 0.5625$$

$$x_4^{(2)} = 0.25 + 0.25x_2^{(2)} + 0.25x_3^{(2)} = (0.25)[1 + 0.8125 + 0.5625]$$

$$= 0.59375$$

K=2

$$x_1^{(3)} = 0.5 + 0.25x_2^{(2)} + 0.25x_3^{(2)} = 0.5 + (0.25)(0.8125) + (0.25)(0.5625) = 0.84375$$

$$x_2^{(3)} = 0.5 + 0.25x_1^{(3)} + 0.25x_4^{(2)} = 0.5 + (0.25)(0.84375) + (0.25)(0.59375) = 0.85938$$

$$x_3^{(3)} = 0.25 + 0.25x_1^{(3)} + 0.25x_4^{(2)} = 0.25[1 + 0.84375 + 0.59375] = 0.60938$$

$$x_4^{(3)} = 0.25 + 0.25x_2^{(3)} + 0.25x_3^{(3)} = 0.25[1 + 0.85938 + 0.60938] = 0.61719$$

K=3

$$x_1^{(4)} = 0.5 + 0.25x_2^{(3)} + 0.25x_3^{(3)} = 0.5 + (0.25)x_2^{(3)} + 0.25x_3^{(3)}$$
$$= 0.5 + (0.25)(0.85938) + (0.25)(0.60938)$$
$$= 0.86719$$

$$x_2^{(4)} = 0.5 + 0.25x_1^{(4)} + 0.25x_4^{(3)} = 0.5 + 0.25(0.86719) + 0.25(0.61719)$$
$$= 0.87110$$

$$x_3^{(4)} = 0.25 + 0.25x_1^{(4)} + 0.25x_4^{(3)} = 0.25[1 + 0.46719 + 0.61719] = 0.62110$$

$$x_4^{(4)} = 0.25 + 0.25x_2^{(4)} + 0.25x_{(3)}^{(4)} = 0.25[1 + 0.8711 + 0.6211] = 0.62305$$

K=4

$$x_1^{(5)} = 0.5 + 0.25x_2^{(4)} + 0.25x_3^{(4)} = 0.5 + (0.25)(0.8715) + (0.25)(0.6211)$$
$$= 0.87305$$

$$x_2^{(5)} = 0.5 + 0.25x_1^{(5)} + 0.25x_4^{(4)} = 0.5 + 0.25(0.87305) + 0.25(0.62305)$$
$$= 0.87402$$

$$x_3^{(5)} = 0.25 + 0.25x_1^{(5)} + 0.25x_4^{(4)} = 0.25[1 + 0.87305 + 0.62305]$$
$$= 0.62402$$

$$x_4^{(5)} = 0.25 + 0.25x_2^{(5)} + 0.25x_3^{(5)} = 0.25[1 + 0.87402 + 0.62402]$$
$$= 0.62451$$

The solution is given by

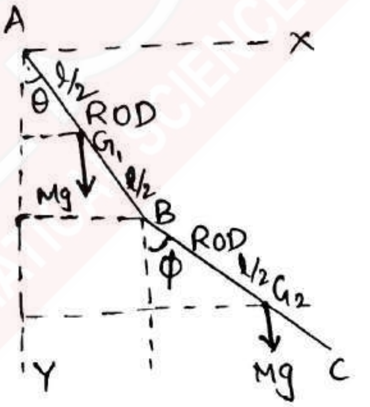
$x_1 = 0.87305$, $x_2 = 0.87402$, $x_3 = 0.62402$

$x_4 = 0.62451$.

2(c). Two equal rods AB and BC, each of length l smoothly joined at B are suspended from A and oscillate in a vertical plane through A. show that the periods of normal oscillations are $\frac{2\pi}{n}$, where $n^2 = \left(3 \pm \frac{6}{\sqrt{7}}\right)\frac{g}{l}$.

Sol^n: Let AB and BC be the rods of equal length l and mass M. At time t, let the two rods make angles θ and ϕ to the vertical respectively.

Referred to A as origin horizontal and vertical lines AX and AY as axes the coordinates of C.G. G_1 of rod AB and that of C.G. G_2 of rod BC are given by



$x_{G_1} = \frac{1}{2}l\sin\theta$, $y_{G_1} = \frac{1}{2}l\cos\theta$

$x_{G_2} = l\sin\theta + \frac{1}{2}l\sin\phi$,

$y_{G_2} = l\cos\theta + \frac{1}{2}l\cos\phi$.

$\therefore$ If v_{G_1} and v_{G_2} are velocities of G_1 & G_2, then

$$v_{G_1}^2 = \dot{x}_{G_1}^2 + \dot{y}_{G_1}^2 = \left(\tfrac{1}{2}l\cos\theta\dot{\theta}\right)^2 + \left(-\tfrac{1}{2}l\sin\theta\dot{\theta}\right)^2$$
$$= \frac{1}{4}l^2\dot{\theta}^2$$

$$v_{G_2}^2 = \dot{x}_{G_2}^2 + \dot{y}_{G_2}^2 = \left(l\cos\theta\dot{\theta} + \tfrac{1}{2}\cos\phi\dot{\phi}\right)^2 + \left(-l\sin\theta\dot{\theta} - \tfrac{1}{2}l\sin\phi\dot{\phi}\right)^2$$
$$= l^2\left[\dot{\theta}^2 + \tfrac{1}{4}\dot{\phi}^2 + \dot{\theta}\dot{\phi}\cos(\theta-\phi)\right]$$
$$= l^2\left[\dot{\theta}^2 + \tfrac{1}{4}\dot{\phi}^2 + \dot{\theta}\dot{\phi}\right] \quad (\because \theta, \phi \text{ are small})$$

If T be the total kinetic energy and W the work function of the system, then

$$T = \text{K.E of rod AB} + \text{K.E of rod BC}$$
$$= \left[\tfrac{1}{2}M\cdot\tfrac{1}{3}\left(\tfrac{1}{2}l\right)^2\dot{\theta}^2 + \tfrac{1}{2}M\cdot v_{G_1}^2\right] + \left[\tfrac{1}{2}m\cdot\tfrac{1}{3}\left(\tfrac{1}{2}l\right)\dot{\phi}^2 + \tfrac{1}{2}M\cdot v_{G_2}^2\right]$$
$$= \tfrac{1}{2}M\left[\tfrac{1}{12}l^2\dot{\theta}^2 + \tfrac{1}{4}l^2\dot{\theta}^2\right] + \tfrac{1}{2}M\left[\tfrac{1}{12}l^2\dot{\phi}^2 + l^2\left(\dot{\theta}^2 + \tfrac{1}{4}\dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\right)\right]$$
$$= \tfrac{1}{2}Ml^2\left(\tfrac{4}{3}\dot{\theta}^2 + \tfrac{1}{3}\dot{\phi}^2 + \dot{\theta}\dot{\phi}\right)$$

and $W = Mg\,y_{G_1} + Mg\,y_{G_2} + C$

$= Mg\left[\frac{1}{2}l\cos\theta + l\cos\theta + \frac{1}{2}l\cos\phi\right] + C$

$= \frac{1}{2}Mgl(3\cos\theta + \cos\phi).$

$\therefore$ Lagrange's θ - equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right) - \frac{\partial T}{\partial\theta} = \frac{\partial W}{\partial\theta}$

i.e $\frac{d}{dt}\left[\frac{1}{2}Ml^2\left(\frac{8}{3}\dot\theta + \dot\phi\right)\right] - 0 = \frac{1}{2}Mgl(-3\sin\theta) = -\frac{3}{2}Mgl\theta$ ($\because \theta$ is small)

$\Rightarrow 8\ddot\theta + 3\ddot\phi = -9c\theta$, (where $c = g/l$) ——— (1)

Equations (1) and (2) can be written as

$(8D^2+9c)\theta + 3D^2\phi = 0$ and $3D^2+\theta+(2D^2+3c)\phi = 0$

Eliminating ϕ between these two equations, we get

$[(2D^2+3c)(8D^2+9c) - 9D^4] = 0$

$\Rightarrow (7D^4 + 42cD^2 + 27c^2)\theta = 0$

If the periods of normal oscillations are $2\pi/n$, then the solution of (3), must be

$\theta = A\cos(nt+B)$

$\therefore D^2\theta = -n^2\theta$ and $D^4\theta = n^4\theta.$

Substituting in (3), we get

$(7n^4 - 42cn^2 + 27c^2)\theta = 0$

$\Rightarrow 7n^4 - 42cn^2 + 27c^2 = 0 \quad \because \theta \neq 0.$

$\therefore n^2 = \frac{42 \pm \sqrt{[(42c)^2 - 4.7.27c^2]}}{2.7}$

$\Rightarrow n^2 = \left(3 \pm \frac{6}{\sqrt7}\right)c = \left(3+\frac{6}{\sqrt7}\right)\frac{g}{l} \quad (\because c = g/l)$

3(a) Find the characteristic of the equation $z=\frac{1}{2}(p^2+q^2)+(p-x)(q-y)$ and determine the integral surface which passes through the x-axis.

Sol'n: Given equation is $z=\frac{1}{2}(p^2+q^2)+(p-x)(q-y)$ —— ①

we are to find its integral surface which passes through x-axis which is given by equations $y=0,\ z=0$. —— ②

Re-writing ② in parametric form, we have

$x=\lambda,\ y=0,\ z=0$, λ being the parameter —— ③

Let initial values x_0, y_0, z_0, p_0, q_0 of $x, y, z; p, q$ be taken

as $x_0=x_0(\lambda)=\lambda$, $y_0=y_0(\lambda)=0$, $z_0=z_0(\lambda)=0$ —— (4A)

Let p_0, q_0 be the initial values of p, q corresponding to the initial values x_0, y_0, z_0. Since initial values $(x_0, y_0, z_0, p_0, q_0)$ satisfy ①, we have

$$z_0=\frac{1}{2}(p_0^2+q_0^2)+(p_0-x_0)(q_0-x_0)$$

$$\Rightarrow 0=\frac{1}{2}(p_0^2+q_0^2)+q_0(p_0-\lambda) \text{ by (4A)}.$$

$$\Rightarrow p_0^2+q_0^2+2q_0p_0-2q_0\lambda=0 \quad \text{—— ⑤}$$

Also, we have $z_0'(\lambda)=p_0x_0'(\lambda)+q_0y_0'(\lambda)$.

so that $0=p_0\times 1+q_0\times 0 \Rightarrow p_0=0$ by (4A) —— ⑥

Solving ⑤ and ⑥ $p_0=0$ and $q_0=2\lambda$ —— (4B)

Collecting relations (4A) and (4B) together, initial values of x_0, y_0, z_0, p_0, q_0 are given by

$x_0=\lambda,\ y_0=0,\ z_0=0,\ p_0=0,\ q_0=2\lambda$ when $t=t_0=0$ —— ⑦

Let $f(x,y,z,p,q)=\frac{1}{2}(p^2+q^2)-pq-py-qx+xy-z=0$ —— ⑧

The usual characteristic equations of ⑧ are given by

$$dx/dt=\partial f/\partial p=p+q-y \quad \text{—— ⑨}$$

$$dy/dt=\partial f/\partial q=q+p-x \quad \text{—— ⑩}$$

$dz/dt = p(\partial f/\partial p) + q(\partial f/\partial q) = p(p+q-y) + q(q+p-x)$ — (11)

$dp/dt = -(\partial f/\partial x) - p(\partial f/\partial z) = p+q-y$ — (12)

and $dq/dt = -(\partial f/\partial y) - q(\partial f/\partial z) = p+q-x$ — (13)

from (9) & (12), $(dx/dt) - (dp/dt) = 0$ so that $x - p = c_1$ — (14)

where c_1 is an arbitrary constant. Using initial conditions (7) (14) gives $\lambda - 0 = c_1 \Rightarrow c_1 = \lambda$. Hence (14) reduces to

$$x - p = \lambda \Rightarrow x = p + \lambda \quad \text{— (15)}$$

from (10) and (13), $(dy/dt) - (dq/dt) = 0$ so that $y - q = c_2$ — (16)

where c_2 is an arbitrary constant.

Using initial conditions (7), (16) gives

$$0 - 2\lambda = c_2 \Rightarrow c_2 = 2\lambda.$$

Hence (16) reduces to

$$y - q = 2\lambda \Rightarrow y = q - 2\lambda \quad \text{— (17)}$$

Hence $\dfrac{d(p+q-x)}{dt} = \dfrac{dp}{dt} + \dfrac{dq}{dt} - \dfrac{dx}{dt} = p+q-y+p+q-x-(p+q-y)$
[using (9) (12) & (13)]

$$\Rightarrow \frac{d(p+q-x)}{dt} = p+q-x \Rightarrow \frac{d(p+q-x)}{p+q-x} = dt.$$

Integrating, $\log(p+q-x) - \log c_3 = t$ (or) $p+q-x = c_3 e^t$, — (18)

where c_3 is an arbitrary constant. Using initial conditions (7), (18) gives $0 + 2\lambda - \lambda = c_3 \Rightarrow c_3 = \lambda$. Hence (18) reduces to

$$p+q-x = \lambda e^t \quad \text{— (19)}$$

$$\frac{d(p+q-y)}{dt} = \frac{dp}{dt} + \frac{dq}{dt} - \frac{dy}{dt} = p+q-y+p+q-x-(q+p-x)$$
[using (10) (12) & (13)]

$$\Rightarrow \frac{d(p+q-y)}{dt} = p+q-y \Rightarrow \frac{d(p+q-y)}{p+q-y} = dt.$$

Integrating, $\log(p+q-y) - \log c_4 = t \Rightarrow p+q-y = c_4 e^t$ — (20)

where C_4 is an arbitrary constant. Using initial conditions (7), (20) gives $0+2\lambda-0=C_4 \Rightarrow C_4=2\lambda$. Hence (20) reduces to $p+q-y=2\lambda e^t$ ——— (21)

from (9) & (21), $dx/dt = 2\lambda e^t$ so that $x=2\lambda e^t + C_5$ — (22)

where C_5 is an arbitrary constant. Using initial conditions (7), (22) gives. $\lambda = 2\lambda + C_5 \Rightarrow C_5 = -\lambda$.

Hence (22) reduces to

$$x = 2\lambda e^t - \lambda \Rightarrow x = \lambda(2e^t - 1) \text{ ——— } (23)$$

from (10) & (7), $dy/dt = \lambda e^t$ so that $y=\lambda e^t + C_6$ — (24)

where C_6 is an arbitrary constant. Using initial conditions (7), (24) gives. $0 = \lambda + C_6 \Rightarrow C_6 = -\lambda$. Hence (24) reduces to

$$y = \lambda e^t - \lambda \Rightarrow y = \lambda(e^t - 1) \text{ ——— } (25)$$

Substituting value of y from (17) in (12), we get

$$dp/dt = p+q-(q-2\lambda) \Rightarrow \frac{dp}{dt} - p = 2\lambda \text{ ——— } (26)$$

whose integrating factor $= e^{\int(-1)dt} = e^{-t}$ and solution is

$$pe^{-t} = \int (2\lambda)e^{-t}dt + C_7 = -2\lambda e^{-t} + C_3$$

$$\Rightarrow p = -2\lambda + C_3 e^t \text{ ——— } (27)$$

where C_7 is an arbitrary constant. Using initial condition (7), (27) gives $0 = -2\lambda + C_7 \Rightarrow C_7 = 2\lambda$.

Hence (27) reduces to

$$p = -2\lambda + 2\lambda e^t \Rightarrow p = 2\lambda(e^t - 1) \text{ ——— } (28)$$

Substituting value of x from (23) in (13), we get

$$dq/dt = p+q-(p+\lambda).$$

$$\Rightarrow dq/dt - q = -\lambda \text{ ——— } (29)$$

whose integrating factor $= e^{\int(-1)dt} = e^{-t}$ and solution is

$$qe^{-t} = \int(-\lambda)\, e^{-t}dt + c_8 = \lambda e^{-t} + c_8 \Rightarrow q = \lambda + c_8 e^{t} \quad \text{---} (30)$$

where c_8 is an arbitrary constant. Using initial condition (7), (30) gives $2\lambda = \lambda + c_8$

$$\Rightarrow c_8 = \lambda.$$

Hence (30) reduces to $q = \lambda + \lambda e^{t}$

$$\Rightarrow q = \lambda(1+e^{t}) \quad \text{---} (31)$$

Substitutions the values of $p+q-x$ and $p+q-y$ from (13) and (24) respectively in (1), we have

$$\frac{dz}{dt} = p(2\lambda e^{t}) + q(\lambda e^{t}) = 2\lambda(e^{t}-1)(2\lambda e^{t}) + \lambda(1+e^{t})(\lambda e^{t})$$

[on putting values of p & q with help of (28) & (31)]

$$\Rightarrow \frac{dz}{dt} = 5\lambda^2 e^{2t} - 3\lambda^2 e^{t} \Rightarrow dz = (5\lambda^2 e^{2t} - 3\lambda^2 e^{t})dt.$$

Integrating $z = \frac{5}{2}\lambda^2 e^{2t} - 3\lambda^2 e^{t} + c_9$ --- (32)

where c_9 is an arbitrary constant. Using initial conditions (7), namely $z=0$, where $t=0$, (32) gives $0 = \frac{5}{2}\lambda^2 - 3\lambda^2 + c_9$

$$\Rightarrow c_9 = 3\lambda^2 - \frac{5}{2}\lambda^2$$

Hence (32) reduces to $z = \frac{5}{2}\lambda^2 (e^{2t}-1) - 3\lambda^2(e^{t}-1)$ --- (33)

solving (23) and (25) for λ and e^{t}, we have

$$\lambda = x-2y \quad \text{and} \quad e^{t} = (x-y)/(x-2y) \quad \text{---} (34)$$

Eliminating λ and e^{t} from (33) and (34), we have

$$z = \frac{5}{2}(x-2y)^2 \left\{\left(\frac{x-y}{x-2y}\right)^2 - 1\right\} - 3(x-2y)^2\left(\frac{x-y}{x-2y} - 1\right)$$

$$z = \frac{5}{2}\left\{(x-y)^2 - (x-2y)^2\right\} - 3\left\{(x-2y)(x-y) - (x-2y)^2\right\}$$

$$z = \frac{1}{2}y(4x-3y) \quad \text{(on Simplification)}$$

3(b) Solve the following differential equation $\frac{dy}{dx} = x+y$ with the initial condition $y(0)=1$, using fourth-order Runge-Kutta method from $x=0$ to $x=0.4$ taking $h=0.1$.

Sol^n: The fourth order Runge-Kutta method is

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where $k_1 = hf(x_n, y_n)$; $k_2 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$

$$k_3 = hf\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right); \quad k_4 = hf(x_n + h, y_n + k_3)$$

In this problem

$$f(x,y) = x+y, \quad h=0.1, \quad x_0 = 0, \quad y_0 = 1$$

$$k_1 = hf(x_0, y_0) = 0.1$$

$$k_2 = hf(x_0 + 0.05, y_0 + 0.05) = 0.11$$

$$k_3 = hf(x_0 + 0.05, y_0 + 0.055) = 0.1105$$

$$k_4 = hf(x_0 + 0.1, y_0 + 1.1105) = 0.12105$$

$$y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1.11034$$

Proceeding in this way, we get

$y_2 = y(0.2)$, $x_1 = 0.1$, $y_1 = 1.11034$

$\therefore y_2 = 1.2428$

$y_3 = y(0.3)$, $x_2 = 0.2$, $y_2 = 1.2428$

$\therefore y_3 = 1.399711$

$y_4 = y(0.4)$, $x_3 = 0.3$, $y_3 = 1.349711$

$\therefore y_4 = 1.58363.$

3(C) Steam is rushing from a boiler through a conical pipe, the diameters of the ends of which are D and d; if V and v be the corresponding velocities of the stream, and if the motion be supposed to be that of divergence from the vertex of the cone, Prove that $\frac{v}{V} = \frac{D^2}{d^2} e^{(v^2-V^2)/2k}$ where k is the pressure divided by the density and supposed to be constant.

Sol^n: Let u be the velocity at a distance x from the end A, the equation of motion is

$$u\frac{du}{dx} = 0 - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

(Since the motion is steady)

$$\Rightarrow \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = -\frac{k}{\rho}\frac{\partial \rho}{\partial x} \quad \text{as } p = k\rho$$

Integrating, $\frac{1}{2}u^2 = -k\log\rho + c$

$$\log\rho - \log A_1 = -\frac{u^2}{2k} \Rightarrow \rho = A_1 e^{-u^2/2k} \quad \text{--- (1)}$$

Boundary Conditions are

(i) $\rho = \rho_1$ when $u = v$ (ii) $\rho = \rho_2$ when $u = V$.

Subjecting (1) to (i) and (ii) we obtain $\rho_1 = A_1 e^{-v^2/2k}$ and

$$\rho_2 = A_1 e^{-V^2/2k}$$

$$\text{This} \Rightarrow \frac{\rho_1}{\rho_2} = e^{(V^2-v^2)/2k} \quad \text{--- (2)}$$

By the equation of continuity

flux at A = flux at B

$$\pi\left(\frac{d}{2}\right)^2 v\rho_1 = \pi\left(\frac{D}{2}\right)^2 V\rho_2$$

$$\Rightarrow \frac{\rho_1}{\rho_2} = \frac{V}{v}\frac{D^2}{d^2}$$

Now (2) becomes $\frac{V}{v}\cdot\frac{D^2}{d^2} = e^{(V^2-v^2)/2k}$

$$\Rightarrow \frac{v}{V} = \frac{D^2}{d^2}e^{(v^2-V^2)/2k}$$

4(a) → Find the integral surface of the partial differential equation $(x-y)p+(y-x-z)q=z$ through the circle $z=1$, $x^2+y^2=1$.

Sol^n: Given $(x-y)p+(y-x-z)q=z$ —— ①

Lagrange's auxiliary equations of ① are

$$\frac{dx}{x-y}=\frac{dy}{y-x-z}=\frac{dz}{z} \quad \text{—— ②}$$

Choosing 1, 1, 1 as multipliers, each fraction on ②

$$=\frac{dx+dy+dz}{0}$$

$\therefore dx+dy+dz=0$ so that $x+y+z=c_1$ —— ③

Taking the last two fractions of ② and using ③ we get

$$\frac{dy}{y-(c_1-y)}=\frac{dz}{z} \Rightarrow \frac{2dy}{2y-c_1}-\frac{2dz}{z}=0$$

Integrating it, $\log(2y-c_1)-2\log z=\log c_2$

$$\Rightarrow (2y-c_1)/z^2=c_2$$

$$\Rightarrow (2y-x-y-z)/z^2=c_2$$

$$\Rightarrow (y-x-z)/z^2=c_2 \quad \text{—— ④}$$

The given curve is given by $z=1$, $x^2+y^2=1$ —— ⑤

putting $z=1$ in ③ & ④, we get

$$x+y=c_1-1 \text{ and } y-x=c_2+1 \quad \text{—— ⑥}$$

But $2(x^2+y^2)=(x+y)^2+(y-x)^2$ —— ⑦

Using ⑤ and ⑥, ⑦ becomes

$$2=(c_1-1)^2+(c_2+1)^2 \Rightarrow c_1^2+c_2^2-2c_1+2c_2=0 \quad \text{—— ⑧}$$

putting the values of c_1 and c_2 from ③ & ④ in ⑧;

required integral surface is

$$(x+y+z)^2+(y-x-z)^2/z^4-2(x+y+z)+2(y-x-z)/z^2=0$$

$$\Rightarrow z^4(x+y+z)^2+(y-x-z)^2-2z^4(x+y+z)+2z^2(y-x-z)=0$$

4(c). For given equidistant values u_{-1}, u_0, u_1 and u_2, a value is interpolated by Lagranges formula. show that it may be written in the form

$$u_x = yu_0 + xu_1 + \frac{y(y^2-1)}{3!}\Delta^2 u_{-1} + \frac{x(x^2-1)}{3!}\Delta^2 u_0, \text{ where } x+y=1.$$

Solⁿ: R.H.S.

$$yu_0 + xu_1 + \frac{y(y^2-1)}{3!}\Delta^2 u_{-1} + \frac{x(x^2-1)}{3!}\Delta^2 u_0$$

$$= (1-x)u_0 + xu_2 + \frac{(1-x)[(1-x)^2-1]}{3!}\Delta^2 u_{-1} + \frac{x(x^2-1)}{3!}\Delta^2 u_0$$

$$= u_0 - xu_0 + xu_2 + \frac{(1-x)[x^2-2x]}{3!}\Delta(\Delta u_{-1}) + \frac{x(x^2-1)}{3!}\Delta(\Delta u_0)$$

$$= u_0 - xu_0 + xu_2 + \frac{x(1-x)(x-2)}{3!}\Delta(u_0 - u_{-1}) + \frac{x(x^2-1)}{3!}\Delta(u_1 - u_0)$$

$$= u_0 - xu_0 + xu_2 + \frac{x(1-x)(x-2)}{3!}[(u_1-u_0)-(u_0-u_{-1})] + \frac{x(x^2-1)}{3!}[(u_2-u_1)-(u_1-u_0)]$$

$$= u_0 - xu_0 + xu_2 + \frac{x(1-x)(x-2)}{3!}(u_1 - 2u_0 + u_{-1}) + \frac{x(x^2-1)}{3!}(u_2 - 2u_1 + u_0)$$

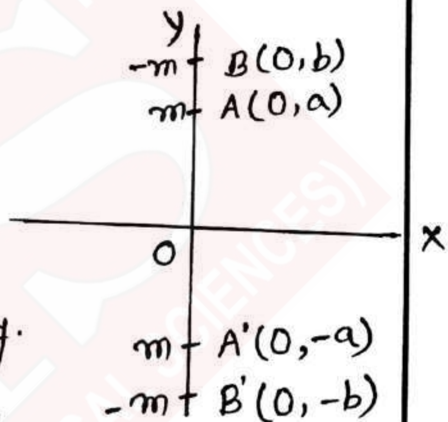
$$= u_0\left[(1-x) - \tfrac{1}{3}(1-x)(x-2) + \frac{x(x^2-1)}{3!}\right] + u_2\left[x + \frac{x(x^2-1)}{3!}\right] + u_1\left[\frac{x(1-x)(x-2)}{3!}\right] - \frac{1}{3}x(x^2-1) + \frac{x(1-x)(x-2)}{3!}u_{-1}.$$

$$= \frac{x(1-x)(x-2)}{3!}u_{-1} + \frac{[x^3-2x^2-x+2]}{2}u_0 + \frac{x^2(1-x)}{2}u_1 + \frac{x(x^2+5)}{6}u_2$$

$$= u_x.$$

4(d) If the fluid fills the region of space on the positive side of x-axis, is a rigid boundary, and if there be a source +m at the point $(0,a)$, and an equal sink at $(0,b)$, and if the pressure on the negative side of the boundary be the same as the pressure of the fluid at infinity, show that the resultant pressure on the boundary is $\pi\rho m^2(a-b)^2/ab(a+b)$, where ρ is the density of the fluid.

Sol^n: The object system consists of source $+m$ at $A(0,a)$, i.e. at $z=ia$ and sink $-m$ at $A'(z=-ia)$ and sink $-m$ at $B'(z=-ib)$ w.r.t. the positive link OX which is rigid boundary.



The complex potential due to object system with rigid boundary is equivalent to the object system and its image system with no rigid boundary

$$\therefore\ w = -m\log(z-ia) + m\log(z-ib) - m\log(z+ia) + m\log(z+ib)$$

or $$w = -m\log(z^2+a^2) + m\log(z^2+b^2)$$

$$\frac{dw}{dz} = -2mz\left[\frac{1}{z^2+a^2} - \frac{1}{z^2+b^2}\right]$$

$$= \frac{2mz(a^2-b^2)}{(z^2+a^2)(z^2+b^2)}$$

$$\therefore\ q = \left|\frac{dw}{dz}\right| = \frac{2m(a^2-b^2)\,|z|}{|z^2+a^2|\,|z^2+b^2|}$$

for any point on x-axis, we have $z=x$ so that

$$q = \frac{2mx(a^2-b^2)}{(x^2+a^2)(x^2+b^2)}$$

This is expression for velocity at any point on x-axis.
Let p_0 be the pressure at $x=\infty$. By Bernoulli's equation for steady motion.

$$\therefore \frac{p}{\rho} + \frac{1}{2}q^2 = C$$

In view of $p=p_0$, $q=0$ when $x=\infty$, we get, $C = p_0/\rho$.

$$\frac{p_0 - p}{\rho} = \frac{1}{2}q^2$$

Required pressure P on boundary is given by

$$P = \int_{-\infty}^{\infty}(p_0 - p)\,dx = \int_{-\infty}^{\infty}\frac{1}{2}\rho q^2\,dx$$

$$= \frac{1}{2}\rho\int_{-\infty}^{\infty}\frac{4m^2x^2(a^2-b^2)^2}{(x^2+a^2)^2(x^2+b^2)^2}\,dx$$

$$= 4\rho m^2(a^2-b^2)^2\int_0^{\infty}\frac{x^2\,dx}{(x^2+a^2)^2(x^2+b^2)^2}$$

$$= 4m^2\rho\int_0^{\infty}\left[\frac{a^2+b^2}{a^2-b^2}\left\{\frac{1}{x^2+b^2} - \frac{1}{x^2+a^2}\right\} - \left[\frac{a^2}{(x^2+a^2)^2} - \frac{b^2}{(x^2+a^2)^2}\right]\right]dx$$

$$= 4m^2\rho\left[\frac{a^2+b^2}{a^2-b^2}\left\{\frac{\pi}{2b} - \frac{\pi}{2a}\right\} - \frac{\pi}{4a} - \frac{\pi}{4b}\right]$$

$$= \frac{\pi\rho m^2(a^2-b^2)}{ab(a+b)}$$

for $\displaystyle\int_0^{\infty}\frac{dx}{x^2+a^2} = \left[\frac{1}{a}\tan^{-1}\frac{x}{a}\right]_0^{\infty} = \frac{\pi}{2a}$

$$\int_0^{\infty}\frac{dx}{(x^2+a^2)^2} = \frac{1}{a^3}\int_0^{\infty}\cos^2\theta\,d\theta,\quad x = a\tan\theta$$

$$= \frac{1}{2a^3}\int_0^{\pi/2}(1+\cos 2\theta)\,d\theta = \frac{\pi}{2}$$

$$= \frac{1}{2a^3} = \frac{\pi}{4a^3}.$$

5(a) Use Lagrange's method to solve the equation

$$\begin{vmatrix} x & y & z \\ \alpha & \beta & \gamma \\ \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} & -1 \end{vmatrix} = 0 \qquad \text{where } z = z(x, y).$$

Sol^n: The given PDE can be written as

$$x\left[-\beta - \gamma\frac{\partial z}{\partial y}\right] - y\left[-\alpha - \gamma\frac{\partial z}{\partial x}\right] + z\left[\alpha\frac{\partial z}{\partial y} - \beta\frac{\partial z}{\partial x}\right] = 0$$

$$\Rightarrow (\gamma y - \beta z)\frac{\partial z}{\partial x} + (\alpha z - \gamma x)\frac{\partial z}{\partial y} = \beta x - \alpha y \quad \text{——— (1)}$$

The corresponding auxiliary equations are

$$\frac{dx}{(\gamma y - \beta z)} = \frac{dy}{(\alpha z - \gamma x)} = \frac{dz}{(\beta x - \alpha y)} \quad \text{——— (2)}.$$

Using multipliers x, y and z we find that each fraction is

$$= \frac{x\,dx + y\,dy + z\,dz}{0}$$

$$\therefore\ x\,dx + y\,dy + z\,dz = 0$$

which on integration yields $x^2 + y^2 + z^2 = c_1$ ——— (3)

Similarly, using multipliers α, β and γ, we find from Eq (2) that each fraction is equal to

$$\alpha\,dx + \beta\,dy + \gamma\,dz = 0.$$

which on integration gives

$$\alpha x + \beta y + \gamma z = c_2 \quad \text{—— (4)}$$

Thus, the general solution of the given equation is found to be $F(x^2 + y^2 + z^2, \alpha x + \beta y + \gamma z) = 0$

5(b) → Find the complete integral of $(x^2+y^2)(p^2+q^2)=1$.

Sol'n: put $x = r\cos\theta$ and $y = r\sin\theta$ —— ①

Then $r^2 = x^2+y^2$ and $\theta = \tan^{-1}(y/x)$ —— ②

Differentiating ② partially w.r.to x and y, we get

$$2r(\partial r/\partial x) = 2x \text{ and } 2r(\partial r/\partial y) = 2y$$

$$\Rightarrow \partial r/\partial x = \frac{r\cos\theta}{r} = \cos\theta \text{ and } \partial r/\partial y = \frac{r\sin\theta}{r} = \sin\theta. \quad —— ③$$

and $\partial\theta/\partial x = \dfrac{1}{1+(y/x)^2}\left(-\dfrac{y}{x^2}\right) = -\dfrac{y}{x^2+y^2} = -\dfrac{r\sin\theta}{r^2} = \dfrac{-\sin\theta}{r}$ —— ④

$$\partial\theta/\partial y = \frac{1}{1+(y/x)^2}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r} \quad —— ⑤$$

Given equation is $(x^2+y^2)(p^2+q^2)=1$ —— ⑥

Now, $p = \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial r}\dfrac{\partial r}{\partial x} + \dfrac{\partial z}{\partial \theta}\dfrac{\partial \theta}{\partial x} = \cos\theta\dfrac{\partial z}{\partial r} - \dfrac{\sin\theta}{r}\dfrac{\partial z}{\partial\theta}$, by ③&④

and $q = \dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial r}\dfrac{\partial r}{\partial y} + \dfrac{\partial z}{\partial \theta}\dfrac{\partial \theta}{\partial y} = \sin\theta\dfrac{\partial z}{\partial r} + \dfrac{\cos\theta}{r}\dfrac{\partial z}{\partial\theta}$; by ③&⑤

Hence $p^2+q^2 = (\partial z/\partial r)^2 + \dfrac{1}{r^2}(\partial z/\partial\theta)^2$ —— ⑦

∴ ⑥ becomes $r^2[(\partial z/\partial r)^2 + (1/r^2)(\partial z/\partial\theta)^2] = 1$ using ②&⑦

$$\Rightarrow \left(r\frac{\partial z}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial\theta}\right)^2 = 1 \quad —— ⑧$$

Let R be a new variable such that $\frac{1}{r}dr = dR$ so that

$\log r = R$ —— ⑨

Then ⑧ becomes $\left(\dfrac{\partial z}{\partial R}\right)^2 + \left(\dfrac{\partial z}{\partial\theta}\right)^2 = 1 \Rightarrow P^2+Q^2=1$ —— ⑩

where $P = \dfrac{\partial z}{\partial R}$ and $Q = \dfrac{\partial z}{\partial\theta}$. ⑩ is of form $f(P,Q)=0$.

∴ solution of ④ is $z = aR + b\theta + c$ —— ⑪

where $a^2+b^2=1 \Rightarrow b = \sqrt{1-a^2}$, Putting a for P and b for Q in ⑩

∴ from ⑪, the required complete integral is

$z = aR + \theta\sqrt{1-a^2} + c \Rightarrow z = a\log r + \theta\sqrt{1-a^2} + c$

$\Rightarrow z = a\log(x^2+y^2)^{1/2} + \tan^{-1}(y/x)\cdot\sqrt{1-a^2} + c$ by (2)

$\Rightarrow z = \frac{a}{2}\log(x^2+y^2) + \sqrt{1-a^2}\,\tan^{-1}(y/x) + c.$

5(C) Solve the following system of equations by Gauss-Jordan elimination method.

$x_1+x_2+x_3=3$, $2x_1+3x_2+x_3=6$, $x_1-x_2-x_3=-3$.

Sol^n: Given system of equations are

$$x_1+x_2+x_3=3$$
$$2x_1+3x_2+x_3=6$$
$$x_1-x_2-x_3=-3$$

from this using Gauss Jordan Elimination.

$$A=\begin{bmatrix}1&1&1\\2&3&1\\1&-1&-1\end{bmatrix} \quad \& \quad B=\begin{bmatrix}3\\6\\-3\end{bmatrix}, \quad X=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

$$\therefore [A|B]=\left[\begin{array}{ccc|c}1&1&1&3\\2&3&1&6\\1&-1&-1&-3\end{array}\right]$$

$R_2\to R_2-2R_1$ & $R_3\to R_3-R_1$

$$\sim\left[\begin{array}{ccc|c}0&1&1&3\\0&1&-1&0\\0&-2&-2&-6\end{array}\right] \begin{array}{l}R_3\to -R_3/2\\R_3\to R_3-R_2\end{array} \sim\left[\begin{array}{ccc|c}1&1&1&3\\0&1&-1&0\\0&0&2&3\end{array}\right] \begin{array}{l}R_3\to \frac{R_3}{2}\\R_2\to R_2+R_3\\R_1\to R_1-R_3\end{array}$$

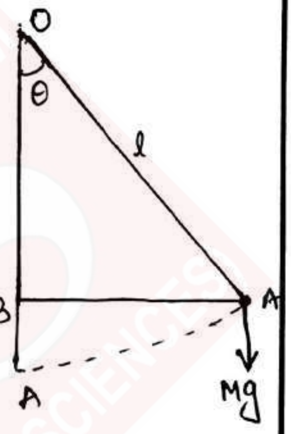
$$\sim\left[\begin{array}{ccc|c}1&1&0&3/2\\0&1&0&3/2\\0&0&1&3/2\end{array}\right] \xrightarrow{R_1\to R_1-R_2} \sim\left[\begin{array}{ccc|c}1&0&0&0\\0&1&0&3/2\\0&0&1&3/2\end{array}\right]$$

$\therefore$ By using Gauss-Jordan elimination method, we get the required solution of linear system

i.e. $$X=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\3/2\\3/2\end{bmatrix}$$

5(d) Use Hamilton's equations to find the equation of motion of the simple pendulum.

Sol'n: Let l be the length of the pendulum and M the mass of the bob. At time t, let θ be the inclination of the string to the downward vertical. Then, if T and V are the kinetic and potential energies of pendulum,



then $T = \frac{1}{2}M(l\dot{\theta})^2 = \frac{1}{2}Ml^2\dot{\theta}^2$

and V = workdone against Mg = $Mg\,A'B$

$= Mgl(1-\cos\theta)$

$\therefore L = T - V = \frac{1}{2}Ml^2\dot{\theta}^2 - Mgl(1-\cos\theta)$ — ①

Here θ is the only generalised coordinate

$\therefore p_\theta = \frac{\partial L}{\partial \dot{\theta}} = Ml^2\dot{\theta}$ —— ②

Since L doesnot contain t explicity,

$\therefore H = T + V = \frac{1}{2}Ml^2\dot{\theta}^2 + Mgl(1-\cos\theta)$

$\Rightarrow H = p_\theta^2/(2Ml^2) + Mgl(1-\cos\theta)$. (from ②)

Here the Hamilton's equations are

$\dot{p}_\theta = -\frac{\partial H}{\partial \theta}$ i.e $\dot{p}_\theta = -Mgl\sin\theta$

and $\dot{\theta} = \frac{\partial H}{\partial p_\theta}$ i.e. $\dot{\theta} = p_\theta/(Ml^2)$

Differentiating ④, we get

$\ddot{\theta} = \dot{p}_\theta/(Ml^2) = -(Mgl\sin\theta)/(Ml^2)$ from ③

$\Rightarrow \ddot{\theta} = -(g/l)\sin\theta$

which is the equation of motion of a simple pendulum.

5(e). Show that the velocity potential $\phi = \frac{a}{2}x(x^2+y^2-2z^2)$ satisfies the Laplace equation. Also determine the streamlines.

Sol^n: we know that the velocity q of the fluid is given by

$$q = -\nabla\phi = -\left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\left\{\frac{a}{2}(x^2+y^2-2z^2)\right\}$$

$$\Rightarrow q = -(a/2)\times(2xi+2yj+2zk) \quad \text{——— (1)}$$

$$\text{But } q = ui+vj+wk \quad \text{——— (2)}$$

Comparing (1) and (2) $u=-ax$, $v=-ay$, $w=2az$

The equations of streamlines are given by $\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w}$

$$\frac{dx}{-ax} = \frac{dy}{-ay} = \frac{dz}{2az} \Rightarrow \frac{2dx}{x} = \frac{2dy}{y} = \frac{dz}{-z} \quad \text{——— (3)}$$

Taking the first two fractions of (3), $\frac{1}{x}dx = \frac{1}{y}dy$

Integrating, $\log x = \log y + \log c_1 \Rightarrow x = c_1 y$ ——— (4)

Taking the last two fractions of (3),

$$(2/y)dy + (1/z)dz = 0.$$

Integrating, $2\log y + \log z = \log c_2$

$$\Rightarrow y^2 z = c_2 \quad \text{——— (5)}$$

(4) and (5) together give the equations of streamlines, c_1 and c_2 being arbitrary constants of integration.

Now, given that $\phi = a/2 \times (x^2+y^2-2z^2)$ ——— (6)

from (6), $\frac{\partial\phi}{\partial x} = ax$, $\frac{\partial\phi}{\partial y} = ay$ and $\frac{\partial\phi}{\partial z} = -2az$

$$\Rightarrow \frac{\partial^2\phi}{\partial x^2} = a, \quad \frac{\partial^2\phi}{\partial y^2} = a \quad \text{and} \quad \frac{\partial^2\phi}{\partial z^2} = -2a$$

$$\therefore \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = a+a-2a \Rightarrow \nabla^2\phi = 0.$$

showing that ϕ satisfies the Laplace equation.

6(a) obtain the partial differential equation governing the equations $\phi(u,v)=0$, $u=xyz$, $v=x+y+z$.

sol'n: $\phi(u,v)=0$, $u=xyz$, $v=x+y+z$

$\therefore$ partial differential equation is given by

$$\begin{vmatrix} p & q & -1 \\ u_x & u_y & u_z \\ v_x & v_y & v_z \end{vmatrix} = 0$$

$$\begin{vmatrix} p & q & -1 \\ yz & xz & xy \\ 1 & 1 & 1 \end{vmatrix} = 0$$

$$p(xz-xy)-q(yz-xy)-1(yz-xz)=0$$

$$xp(z-y)-yq(z-x)-z(y-x)=0$$

$$px(z-y)+qy(x-z)=z(y-x)$$

6(b). Solve $(D^2 - DD' - 2D)z = \sin(3x+4y) - e^{2x+y} + x^2y$.

Sol^n: the given equation can be re-written as

$$D(D - D' - 2)z = \sin(3x+4y) - e^{2x+y} + x^2y.$$

$\therefore$ C.F $= \phi_1(y) + e^{2x}\phi_2(y+x)$, ϕ_1, ϕ_2 being arbitrary functions.

P.I corresponding to $\sin(3x+4y)$

$$= \frac{1}{D^2 - DD' - 2D}\sin(3x+4y) = \frac{1}{(-3)^2 - (-3\cdot4) - 2D}\sin(3x+4y)$$

$$= \frac{1}{3-2D}\sin(3x+4y)$$

$$= \frac{3+2D}{9-4D^2}\sin(3x+4y) = \frac{3+2D}{9-4(-3^2)}\sin(3x+4y)$$

$$= \frac{1}{45}\left[3\sin(3x+4y) + 2D\sin(3x+4y)\right]$$

$$= \frac{1}{45}\left[3\sin(3x+4y) + 6\cos(3x+4y)\right]$$

$$= \frac{1}{15}\left[\sin(3x+4y) + 2\cos(3x+4y)\right]$$

and corresponding to $(-e^{2x+y})$

$$= -\frac{1}{D(D-D'-2)}e^{2x+y} = -\frac{1}{2(2-1-2)}e^{2x+y} = \frac{1}{2}e^{2x+y}$$

and P.I corresponding to x^2y

$$= \frac{1}{D(D-D'-2)}x^2y = \frac{-1}{2D}\left\{1 - \left(\frac{D-D'}{2}\right)\right\}^{-1}x^2y$$

$$= -\frac{1}{2D}\left\{1 + \frac{D-D'}{2} + \left(\frac{D-D'}{2}\right)^2 + \left(\frac{D-D'}{2}\right)^3 + \cdots\right\}x^2y$$

$$= -\frac{1}{2D}\left[1 + \frac{D}{2} - \frac{D'}{2} + \frac{D^2}{4} - \frac{DD'}{2} - \frac{3D^2D'}{8} + \cdots\right]x^2y$$

$$= \frac{-1}{2D}\left[x^2y + xy - \frac{x^2}{2} + \frac{y}{2} - x - \frac{3}{4}\right]$$

$$= -\frac{1}{2}\left[\frac{x^3y}{3}+\frac{x^2y}{2}-\frac{x^3}{6}+\frac{xy}{2}-\frac{x^2}{2}-\frac{3x}{4}\right]$$

Hence the required general solution is

$$z = C.F + P.I.$$

i.e. $$z = \phi_1(y)+e^{2x}\phi_2(y+x)+\frac{1}{15}\left[\sin(3x+4y)+2\cos(3x+4y)\right]$$
$$+\frac{1}{2}e^{2x+y}-\frac{1}{6}x^3y-\frac{1}{4}x^2y+\frac{1}{12}x^3-\frac{1}{4}xy-\frac{1}{4}x^2+\frac{3}{8}x.$$

6(C). Reduce $r+2xs+x^2t=0$ to canonical form.

Sol^n: Given $r+2xs+x^2t$ —— (1)

Comparing (1) with $Rr+Ss+Tt+f(x,y,z,p,q)=0$

here $R=1$, $S=2x$ and $T=x^2$.

So that $S^2-4RT=0$, showing that (1) is parabolic.

The λ-quadratic equation $R\lambda^2+S\lambda+T=0$ reduces to

$\lambda^2+2\lambda x+x^2=0 \Rightarrow (\lambda+x)^2=0$ so that $\lambda=-x,-x$.

The corresponding characteristic equation is $\frac{dy}{dx}-x=0$

$$\left.\begin{aligned} &\text{Integrating,} \quad y-\frac{x^2}{2}=c_1 \\ &\text{Choose} \quad u=y-\frac{x^2}{2} \text{ and } v=x \end{aligned}\right\} \text{—— (2)}$$

where we have chosen $v=x$ in such a manner that u & v are independent functions as verified below.

$$J=\frac{\partial(u,v)}{\partial(x,y)}=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}=-1\neq 0$$

Now, $p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=-x\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}$, by (2) —— (3)

$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\frac{\partial z}{\partial u}$, using (2) —— (4)

$r=\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(-x\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)$, using (3)

$=-\frac{\partial z}{\partial u}-x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)+\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$

$=-\frac{\partial z}{\partial u}-x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right]+\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}$

$=-\frac{\partial z}{\partial u}-x\left[-x\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v\partial u}\right]-x\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}$

$=x^2\frac{\partial^2 z}{\partial u^2}-2x\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}-\frac{\partial z}{\partial u}$ —— (5)

$s=\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)$, using (4)

$$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\left(\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x} = -x\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v \partial u} \quad \text{--- (6)}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) \quad ; \text{using (4)}$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y} = \frac{\partial^2 z}{\partial u^2} \quad \text{--- (7)}$$

using (5), (6) and (7) in (1) ; we finally obtain

$$\frac{\partial^2 z}{\partial v^2} = \frac{\partial z}{\partial u}$$

which is required canonical form.

6(d) → A string of length l is initially at rest in its equilibrium position and motion is started by giving each of its points a velocity v given by $v = kx$ if $0 \le x \le l/2$ and $v = k(l-x)$ if $l/2 \le x \le l$. Find the displacement function $y(x,t)$.

Sol'n: The required displacement function $y(x,t)$ is the solution of the wave equation.

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \quad \text{--- (1)}$$

Subject to boundary conditions

$$y(0,t) = y(l,t) = 0 \quad \forall\, t \geq 0 \quad \text{--- (2)}$$

and the given initial conditions, initial displacement

$$= y(x,0) = f(x) = 0 \quad \text{--- (3)}$$

and initial velocity.

$$= \left(\frac{\partial y}{\partial t}\right)_{t=0} = g x = \begin{cases} kx, & 0 \le x \le l/2 \\ k(l-x), & l/2 \le x \le l \end{cases} \quad \text{--- (4)}$$

Suppose (1) has the solution of the form

$$y(x,t) = X(x)T(t) \quad \text{--- (5)}$$

Substituting this value of y in (1), we have

$$XT'' = c^2 X'' T$$

$$\Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = \mu \text{ (say)}$$

$$\Rightarrow X'' - \mu X = 0 \quad \text{--- (6)} \quad \text{and} \quad T'' - \mu c^2 T = 0 \quad \text{--- (7)}$$

using (2), (5) gives

$$X(0)T(t) = 0 \quad \text{and} \quad X(l)T(t) = 0$$

Suppose that $T(t) \neq 0$ ($\because T(t) = 0$ leads to $y = 0 \ \forall t$)

$$\therefore \boxed{X(0) = 0} \quad \text{and} \quad \boxed{X(l) = 0} \quad \text{--- (8)}$$

which are boundary conditions.

we now solve (6) under boundary conditions (8)

Three cases arise

Case (1): Let $\mu = 0$

The solution of ⑥ is $X(x) = Ax + B$

using B.C ⑧, we get $A = 0$, $B = 0$

$\Rightarrow X(x) = 0$ This leads to $y \equiv 0$

which doesnot satisfy I.C.

so we reject $\mu = 0$ ③ and ④.

Case(2): Let $\mu = \lambda^2$, $\lambda \neq 0$. Then the solutions of

⑥ is $X(x) = Ae^{\lambda x} + Be^{-\lambda x}$.

using B.C ⑧, we get $A = 0$, $B = 0$

$\Rightarrow X(x) = 0$

This leads to $y \equiv 0$ which doesnot satisfy I.C ③ & ④

so reject $\mu = \lambda^2$.

Case(3): Let $\mu = -\lambda^2$, $\lambda \neq 0$

The solution of ⑥ is

$X(x) = A\cos\lambda x + B\sin\lambda x$

using B.C ⑧, we get

$X(0) = 0 = A(1) + B(0) \Rightarrow A = 0$

and $X(l) = 0 = 0 + B\sin\lambda l \Rightarrow B = \sin\lambda l$

$\Rightarrow \sin\lambda l = 0 \quad (\because B \neq 0)$

$\Rightarrow \lambda l = n\pi$

$\Rightarrow \lambda = \frac{n\pi}{l}$, $n = 1, 2, 3 \ldots$

$\therefore X(l) = B\sin\frac{n\pi x}{l}$, $n = 1, 2, 3 \ldots$

Hence non-zero solutions $X_n(x)$ of ⑥ are given by

$X_n(x) = B_n\sin\left(\frac{n\pi x}{l}\right)$ —— ⑨.

from ⑦, $T'' - \mu c^2 T = 0$

$\Rightarrow T'' + \lambda^2 c^2 T = 0 \quad (\because \mu = -\lambda^2)$

$\Rightarrow T'' + \frac{n^2\pi^2}{l^2} c^2 T = 0 \quad \left(\because \lambda = \frac{n\pi}{l}\right)$

whose general solution is $T_n(t) = C_n\cos\left(\frac{n\pi ct}{l}\right) + D_n\sin\left(\frac{n\pi ct}{l}\right)$

$$\therefore y_n(x,t) = X_n(x)\, T_n(t)$$

$$= B_n \sin\left(\frac{n\pi x}{l}\right)\left[C_n \cos\left(\frac{n\pi ct}{l}\right) + D_n \sin\left(\frac{n\pi ct}{l}\right)\right]$$

$$= \left[E_n \cos\frac{n\pi ct}{l} + F_n \sin\frac{n\pi ct}{l}\right] \sin\frac{n\pi x}{l}$$

are solutions of (1) satisfying (2).

Here $E_n = B_n C_n$ and $F_n = B_n D_n$.

In order to obtain a solution also satisfying (3) and (4), we consider more general solution.

$$y(x,t) = \sum_{n=1}^{\infty} y_n(x,t)$$

i.e. $$y(x,t) = \sum_{n=1}^{\infty}\left[E_n \cos\frac{n\pi ct}{l} + F_n \sin\frac{n\pi ct}{l}\right]\sin\frac{n\pi x}{l} \quad \text{—— (10)}$$

Putting $t=0$ in (10)

$$y(x,0) = 0 = \sum_{n=1}^{\infty} E_n \cos\frac{n\pi ct}{l}\cdot \sin\frac{n\pi x}{l}$$

where $$E_n = \frac{2}{l}\int_0^l (0)\cos\left(\frac{n\pi ct}{l}\right)\sin\left(\frac{n\pi x}{l}\right) = 0$$

from (10),

$$y(x,t) = \sum_{n=1}^{\infty} F_n \sin\left(\frac{n\pi ct}{l}\right)\sin\left(\frac{n\pi x}{l}\right) \quad \text{—— (11)}$$

where $$F_n = \frac{2}{n\pi c}\int_0^l g(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{n\pi c}\left[\int_0^{l/2} g(x)\sin\frac{n\pi x}{l}\,dx + \int_{l/2}^{l} g(x)\sin\frac{n\pi x}{l}\,dx\right]$$

$$= \frac{2k}{n\pi c}\int_0^{l/2} x\sin\frac{n\pi x}{l}\,dx + \frac{2k}{n\pi c}\int_{l/2}^{l}(l-x)\sin\frac{n\pi x}{l}\,dx$$

$$= \frac{2k}{n\pi c}\left[(x)\left(\frac{-l}{n\pi}\cos\frac{n\pi x}{l}\right) - (1)\left(\frac{-l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right]_0^{l}$$

$$+ \frac{2k}{n\pi c}\left[(l-x)\left(\frac{-l}{n\pi}\cos\frac{n\pi x}{l}\right) - (-1)\left(\frac{-l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right]_{l/2}^{l}$$

$$= \frac{2k}{n\pi c}\left[\frac{-l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{4kl^2}{cn^2\pi^3}\sin\frac{n\pi}{2}$$

$$= \begin{cases} 0, & \text{if } n=2m \text{ and } m=1,2,3\ldots \\ (-1)^{m+1}\left[\frac{4kl^2}{c\pi^3(2m-1)^3}\right], & \text{if } n=2m-1 \text{ and } m=1,2,3\ldots \end{cases}$$

when $n=2m-1$, $\sin\frac{n\pi}{2} = \sin\frac{\pi}{2}(2m-1)$

$$= \sin\left(m\pi - \frac{\pi}{2}\right)$$

$$= \sin m\pi \cos\frac{\pi}{2} - \cos m\pi \sin\frac{\pi}{2}$$

$$= 0 - (-1)^m = (-1)^{m+1}$$

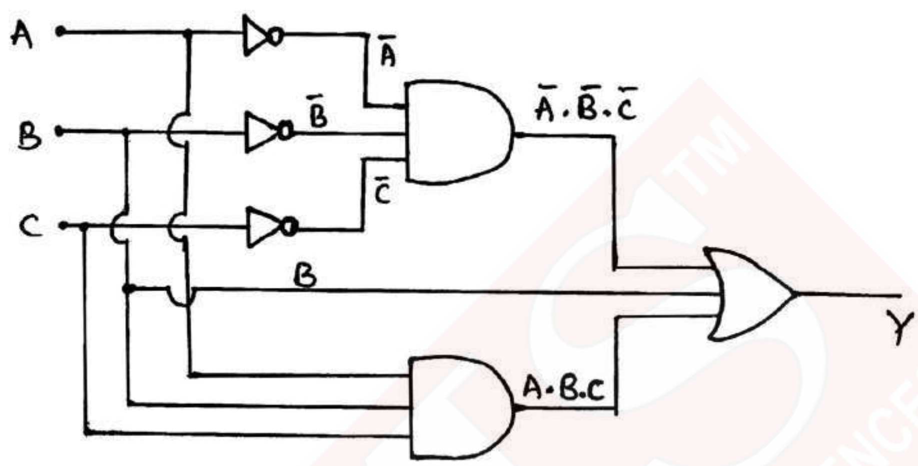
$\therefore$ from (11), the required displacement function is given by

$$y(x,t) = \frac{4kl^2}{c\pi^3}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{(2m-1)^3}\sin\frac{(2m-1)\pi x}{l}\sin\frac{(2m-1)\pi ct}{l}$$

7(a) (i) Draw a logic circuit to realize the function
$Y = A\cdot B\cdot C + \bar{A}\cdot\bar{B}\cdot\bar{C} + B$.

(ii) Simplify the expression and draw a logic for the simplified expression.

Sol'n: (i)

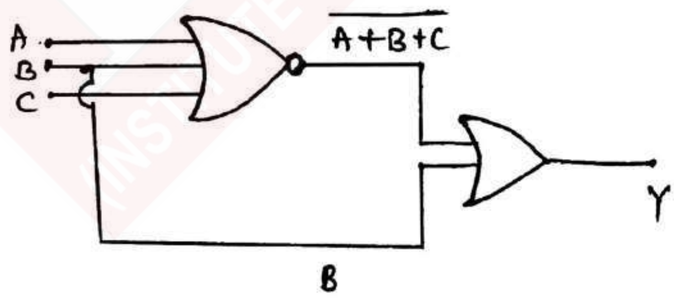


(ii) $Y = A\cdot B\cdot C + \bar{A}\cdot\bar{B}\cdot\bar{C} + B$

$= A\cdot B\cdot C + \overline{A+B+C} + B$ (using Demorgans theorem)

$= B + B\cdot A\cdot C + \overline{A+B+C}$ (by Commutative law)

$= B + \overline{A+B+C}$



7(b) The velocity v of a particle at distance s from a point on its path is given by the table:

s ft :	0	10	20	30	40	50	60
v ft/sec :	47	58	64	65	61	52	38

Estimate the time taken to travel 160 ft by using Simpson's $\frac{1}{3}$ rule. Compare the result with Simpson's $3/8$ rule.

Sol'n: We know,

$$\text{velocity } v = \frac{ds}{dt}$$

$$\therefore dt = \frac{ds}{v}$$

$$\therefore t = \int dt = \int_{s_0}^{s} \frac{ds}{v} = \int_{s_0}^{s} y\, ds \quad \text{--- (1)}$$

where $y = \frac{1}{v}$

s	0	10	20	30	40	50	60
v	47	58	64	65	61	52	38
$y=\frac{1}{v}$	0.0213	0.0172	0.0156	0.0154	0.0164	0.0192	0.0263
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Using Simpson's $\frac{1}{3}$rd rule.

$$I = \frac{h}{3}\left[y_0 + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4 + y_6)\right]$$

$$= 1.06 \text{ sec}$$

Using Simpson's $3/8$th rule.

$$I = \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + 2y_3 + 3y_4 + 3y_5 + y_6)\right]$$

$$= 1.064 \text{ sec}.$$

& Difference between $\frac{1}{3}$rd rule & $3/8$ rule $= 0.00087$ sec

7(c). Using modified Euler's method; obtain the solution of the differential equation $\frac{dy}{dt} = t+\sqrt{y} = f(t,y)$ with the initial condition $y_0 = 1$ at $t_0 = 0$ for the range $0 \le t \le 0.6$ in steps of 0.2.

Sol^n: At first, we use Euler's method to get

$$y_1^{(1)} = y_0 + hf(t_0, y_0) = 1 + (0.2)(0+1) = 1.2$$

then, we use modified Eulers method to find

$$y(0.2) = y_1 = y_0 + h\,\frac{f(t_0,y_0)+f(t_1,y_1^{(1)})}{2}$$

$$= 1.0 + 0.2\left[\frac{1+0.2+\sqrt{1.2}}{2}\right] = 1.2295$$

Similarly, proceeding, we have from Euler's method

$$y_2^{(1)} = y_1 + hf(t_1, y_1) = 1.2295 + 0.2\,(0.2+\sqrt{1.2295})$$

$$= 1.4913$$

Using Modified Eulers method, we get

$$y_2 = y_1 + h\left[\frac{f(t_1,y_1)+f(t_2,y_2^{(1)})}{2}\right]$$

$$= 1.2295 + (0.2)\,\frac{(0.2+\sqrt{1.2295})+(0.4+\sqrt{1.4913})}{2}$$

$$= 1.5225$$

Finally,

$$y_3^{(1)} = y_2 + hf(t_2, y_2)$$

$$= 1.5225 + (0.2)\,(0.4+\sqrt{1.5225})$$

$$= 1.8493$$

Now, Modified Euler's method gives

$$y(0.6) = y_3 = y_2 + h\,\frac{[f(t_2,y_2)+f(t_2,y_3^{(1)})]}{2}$$

$$= 1.5225 + (0.1)\left[0.4 + \sqrt{1.5225}\right] + \left(0.6 + \sqrt{1.8493}\right)$$

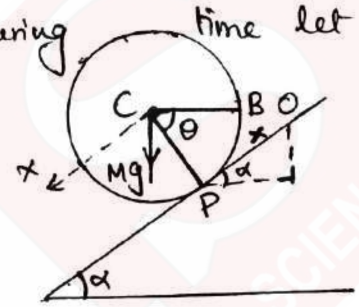
$$y(0.6) = 1.8819$$

Hence, the solution to the given problem is given by

t	0.2	0.4	0.6
y	1.2295	1.5225	1.8819

8(a) A sphere of radius a and mass M rolls down a rough plane inclined at an angle α to the horizontal. If x be the distance of the point of contact of the sphere from a fixed point on the plane, find the acceleration by using Hamilton's equation.

Sol'n: Let a sphere of radius a and mass M roll down a rough plane inclined at an angle α starting initially from a fixed point O of the plane. In time t, let the sphere roll down a distance x and during time let it turn through an angle θ.



Since there is no slipping

$$\therefore x = OA = \text{arc } AB = a\theta.$$

So that $\dot{x} = a\dot{\theta}$

If T and V are the kinetic and potential energies of the sphere, then $T = \frac{1}{2}MK^2\dot{\theta}^2 + \frac{1}{2}M\dot{x}^2 = \frac{1}{2}M\frac{2}{5}a^2\dot{\theta}^2 + \frac{1}{2}M(a\dot{\theta})^2$

$$\Rightarrow T = \frac{7}{10}M\dot{x}^2$$

and $V = -MgOL = -Mgx\sin\alpha$ (Since the sphere moves down the plane)

$$\therefore L = T - V = \frac{7}{10}M\dot{x}^2 + Mgx\sin\alpha$$

Here x is the only generalised coordinate.

$$\therefore p_x = \frac{\partial L}{\partial \dot{x}} = \frac{7}{5}M\dot{x} \quad \text{——— (1)}$$

Since L does not contain t explicitly,

$$\therefore H = T + V = \frac{7}{10}M\dot{x}^2 - Mgx\sin\alpha$$

$$\Rightarrow H = \frac{7}{10}M\left(\frac{5}{7M}p_x\right) - Mgx\sin\alpha$$

$$= \frac{5}{14M}p_x^2 - Mgx\sin\alpha \quad \cdot \text{ from (1)}$$

Hence the two Hamilton's equations are

$$\dot{p}_x = -\frac{\partial H}{\partial x} = Mg\sin\alpha \quad \text{---}(H_1), \quad \dot{x} = \frac{\partial H}{\partial p_x} = \frac{5}{7M}p_x \quad \text{---}(H_2)$$

Differentiating (H_2) and using (H_1), we get

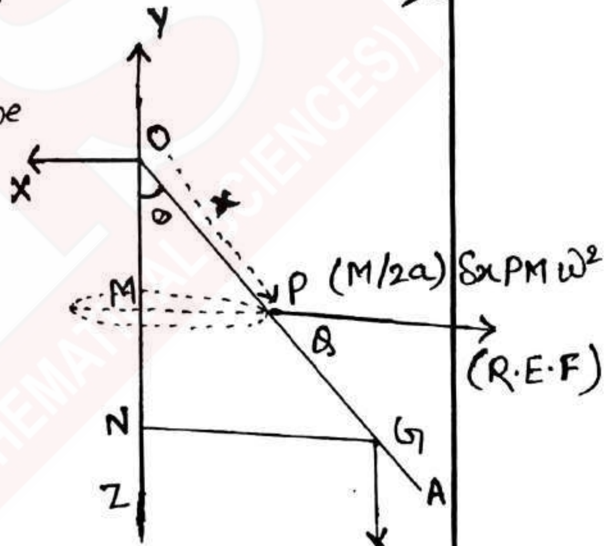
$$\ddot{x} = \frac{5}{7M}\dot{p}_x = \frac{5}{7M}Mg\sin\alpha$$

$$\Rightarrow \boxed{\ddot{x} = \frac{5}{7}g\sin\alpha}$$

which gives the required acceleration.

8(b). A uniform rod OA, of length 2a, free to turn about its end O, revolves with uniform angular velocity ω about the vertical OZ through O, and is inclined at a constant angle α to OZ, find the value of α.

Soln: Let the rod OA of length 2a and mass M revolve with uniform angular velocity ω about the vertical OZ through O, making a constant angle α to OZ. Let PQ $= \delta x$ be an element of the rod at a distance x from O. The mass of the element PQ is $\frac{M}{2a}\delta x$



This element PQ will make a circle in the horizontal plane with radius PM ($= x\sin\alpha$) and centre at M. Since the rod revolve with uniform angular velocity, the only effective force on this element is $\frac{M}{2a}\delta x \cdot PM \cdot \omega^2$ along PM.

Thus the reversed effective force on the element PQ is $\frac{M}{2a}\delta x \cdot x\sin\alpha \cdot \omega^2$ along MP.

Now by D'Alembert's principle all the reversed effective forces acting at different points of the rod, and the external forces, weight mg and reaction at O are in equilibrium. To avoid reaction at O, taking moment about O, we get

$$\Sigma\left(\frac{M}{2a}\delta x \cdot \omega^2 \cdot \sin\alpha\right)\cdot OM - Mg\cdot NG = 0$$

(or) $$\int_0^{2a} \frac{M}{2a}\omega^2 x^2 \sin\alpha\cos\alpha\, dx - Mg\cdot a\sin\alpha = 0, \quad (\because OM = x\cos\alpha)$$

(or) $\frac{M}{2a}\omega^2 \cdot \left\{\frac{1}{3}(2a)^3\right\} \cdot \sin\alpha\cos\alpha - Mg\, a\sin\alpha = 0$

(or) $Mg\, a\sin\alpha\left(\frac{4a}{3g}\omega^2\cos\alpha - 1\right) = 0$

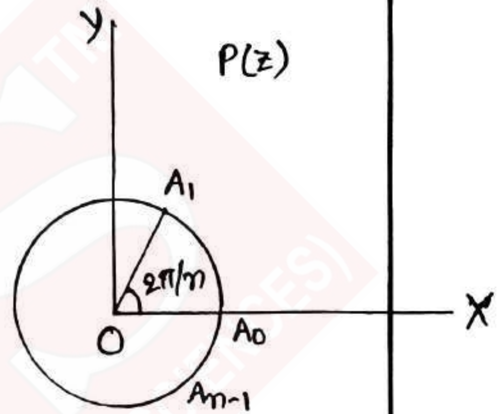
$\therefore$ either $\sin\alpha = 0$ i.e. $\alpha = 0$

or $\frac{4a}{3g}\omega^2\cos\alpha - 1 = 0$

i.e. $\cos\alpha = \frac{3g}{4a\omega^2}$

Hence, the rod is inclined at an angle zero or $\cos^{-1}\left(\frac{3g}{4a\omega^2}\right)$.

8(c) → If n rectilinear vortices of the same strength K are symmetrically arranged along generators of a circular cylinder of radius a in an infinite liquid, prove that the vortices will move round the cylinder uniformly in time $\frac{8\pi^2 a^2}{(n-1)K}$, and find the velocity at any point of the liquid.



Sol: The figure is self explanatory.

The n vortices are at.

$A_0, A_1, A_2, \ldots\ldots A_{n-1}$ s.t.

$\angle A_0OA_1 = \angle A_1OA_2 = \ldots\ldots = \angle A_{n-1}OA_1 = \frac{2\pi}{n}$.

The co-ordinates of the points A_r are given by $z = z_r = ae^{(2\pi/n)ir}$ where $r = 0, 1, 2, \ldots, n-1$

These are n roots of the equation $z^n - a^n = 0$

$$\left[\text{for } z^n - a^n = 0 \Rightarrow z^n = a^n e^{2\pi r i}\right]$$

Hence $z^n - a^n = (z-z_0)(z-z_1)\ldots\ldots(z-z_{n-1})$

The complex potential due to n vortices at P is given by

$$W = \frac{iK}{2\pi}\left[\log(z-z_0) + \log(z-z_1) + \ldots\ldots + \log(z-z_{n-1})\right]$$

$$= \frac{iK}{2\pi}\log(z-z_0)(z-z_1)\ldots\ldots(z-z_{n-1})$$

$$= \frac{iK}{2\pi}\log(z^n - a^n)$$

For the point A_0, $z = a$ so that $r = a$, $\theta = 0$

If w' is the complex potential at A_0, then

$$w' = w - \frac{iK}{2\pi}\log(z-a)$$

$$= \frac{iK}{2\pi}\left[\log(z^n - a^n) - \log(z-a)\right]$$

$$\phi' + i\psi' = \frac{iK}{2\pi}\left[\log(r^n e^{in\theta} - a^n) - \log(re^{i\theta} - a)\right]$$

$$\therefore\ \psi' = \frac{K}{4\pi}\left[\log(r^{2n}+a^{2n}-2r^n a^n\cos n\theta)-\log(r^2+a^2-2ra\cos\theta)\right]$$

$$\frac{\partial\psi'}{\partial r} = \frac{K}{4\pi}\left[\frac{2nr^{2n-1}-2nr^{n-1}a^n\cos n\theta}{r^{2n}+a^{2n}-2r^n a^n\cos n\theta}-\frac{2r-2a\cos\theta}{r^2+a^2-2ra\cos\theta}\right]$$

$$\frac{\partial\psi'}{\partial\theta} = \frac{K}{4\pi}\left[\frac{2nr^n a^n\sin n\theta}{r^{2n}-2r^n a^n\cos n\theta+a^{2n}}-\frac{2ra\sin\theta}{r^2+a^2-2ra\cos\theta}\right]$$

$$\left(\frac{\partial\psi'}{\partial r}\right)_{r=a} = \frac{K}{4\pi a}\left[n\left(\frac{1-\cos n\theta}{1-\cos n\theta}\right)-\left(\frac{1-\cos\theta}{1-\cos\theta}\right)\right] = \frac{K}{4\pi a}(n-1)$$

$$\left(\frac{\partial\psi'}{\partial\theta}\right)_{r=a} = \frac{K}{4\pi}\left[\frac{n\sin n\theta}{1-\cos n\theta}-\frac{\sin\theta}{1-\cos\theta}\right]$$

Since $\lim\limits_{x\to0}\frac{F(x)}{G(x)} = \lim\limits_{x\to0}\frac{F'(x)}{G'(x)} = \lim\limits_{x\to0}\frac{F''(x)}{G''(x)}$, $\left[\text{form } \frac{0}{0}\right]$

$$\left(\frac{\partial\psi'}{\partial\theta}\right)_{r=a} = \frac{K}{4\pi}\left[\frac{n^2\cos n\theta}{n\sin n\theta}-\frac{\cos\theta}{\sin\theta}\right] \text{ as } \theta\to0$$

$$= \frac{K}{4\pi}\left[\frac{-n^3\sin n\theta}{n^2\cos n\theta}-\frac{(-\sin\theta)}{\cos\theta}\right] \text{ as } \theta\to0$$

$$= \frac{K}{4\pi}[0+0] = 0$$

Finally, $\frac{\partial\psi'}{\partial\theta} = \frac{K}{4\pi a}(n-1)$, $\frac{\partial\psi'}{\partial\theta} = 0$ as $r\to a, \theta\to0$

Consequently, the velocity q_0 of the vortex A_0 is given by

$$q_0 = \left[\left(\frac{\partial\psi'}{\partial r}\right)^2+\left(\frac{1}{r}\frac{\partial\psi'}{\partial\theta}\right)^2\right]^{1/2} = \frac{K(n-1)}{4\pi a}$$

This proves that the whole of velocity is along the tangent and there is no velocity along the normal to the circle. Hence the vortices will move round the cylinder with uniform velocity $K(n-1)/4\pi a$.

The time of one complete revolution

$$= \frac{\text{distance}}{\text{velocity}} = \frac{2\pi a}{K(n-1)/4\pi a} = \frac{8\pi^2 a^2}{(n-1)K}.$$