Mains Test Series - 2019

TEST NO. 14

Section-A

Ques: 1(a) How many elements of order 2 are there in the group of order 16 generated by a and b such that the order of 'a' is 8, the order of 'b' is 2 and $bab^{-1} = a^{-1}$?

Solution:-

Let G, be the group generated by a and b

$\therefore |a| = 8$ and $|b| = 2$

$\Rightarrow b^2 = e$

$\Rightarrow b = b^{-1}$

also $bab^{-1} = a^{-1}$

$ba = a^{-1}b$

$\therefore a^m b^n \in G.$

In the general form of elements of G as $b^n a^m$ can be written as:

$$ba = a^{-1}b$$

$$\therefore \underbrace{(b \ldots b)}_{n \text{ times}} \underbrace{(a \ldots a)}_{m \text{ times}} = \underbrace{(b \cdots b)}_{(n-1) \text{ times}} ba \underbrace{(a \ldots a)}_{(m-1) \text{ times}}$$

$$= (b \ldots b)(a^{-1}b)(a \ldots a)$$

$$= a^{-1} \underbrace{(bb \ldots b)}_{n \text{ times}} \underbrace{(a \ldots a)}_{(m-1) \text{ times}}$$

$= a^m b^n$

$G = \{e, a, a^2, a^3, a^4, a^5, a^6, a^7, b, ab, a^2b, a^3b, a^4b, a^5b, a^6b, a^7b\}$

elements of order 2 are:-

i) b

ii) a^4

iii) $ab \cdot ab^{-1} = abab^{-1} = aa^{-1} = e$

$\therefore |ab| = 2$

(iv) $a^2b \cdot a^2b = a^2b \cdot a^2b^{-1} = a^2(a^2)^{-1} = e$

$\therefore |a^2b| = 2$

(v) $|a^3b| = 2$

(vi) $|a^4b| = 2$

(vii) $|a^5b| = 2$

(viii) $|a^6b| = 2$

(ix) $|a^7b| = 2$

$(a^mb)(a^mb) = a^m(ba^mb)$
$= a^m(b \cdot a^m b^{-1})$
$= a^m(a^m)^{-1} = e$

$\therefore ab, a^2b, a^3b, a^4b, a^5b, a^6b, a^7b$ have order 2.

$\therefore$ 9 elements are of order 2.

Ques: 1(b)) Show that $Z[\sqrt{2}] = \{m+n\sqrt{2} : m,n \in Z\}$ is a Euclidean domain.

Solution:-

We know that $Z[2]$ is an integral domain with unity $1 = 1+\sqrt{2}\cdot 0$

Let us define a mapping

$$d : Z[2] - \{0\} \to Z \quad \text{by}$$

$$d(m+n\sqrt{2}) = |m^2 - 2n^2| \quad \forall\ m+n\sqrt{2} \in Z[\sqrt{2}] - \{0\}.$$

We have $m \neq 0$ or $n \neq 0$

$\therefore$ $d(m+n\sqrt{2})$ is a +ve integer for each $m+n\sqrt{3} \in Z[[\sqrt{2}]] - \{0\}$

$\therefore$ $d(m+n\sqrt{2}) \geq a$

Now, let $a = m+n\sqrt{2} \neq 0$
$b = m_1+n_1\sqrt{2} \neq 0$ in $Z[\sqrt{2}]$
$m \neq 0$ or $n \neq 0$; $m_1 \neq 0$ or $n_1 \neq 0$

Then we have

$$ab = (mm_1 + 2nn_1) + (mn_1 + m_1 n)\sqrt{2}$$

and $d(ab) = |(mm_1 + 2nn_1)^2 - 2(mn_1 + m_1 n)^2|$ (by definition)

$$d(ab) = |m^2m_1^2 + 4n^2n_1^2 - 2(m^2n_1^2 + m_1^2 n^2)|.$$

$$= |(m^2 - 2n^2)(m_1^2 - 2n_1^2)|$$

$$= |m^2 - 2n^2|\,|m_1^2 - 2n_1^2|$$

$$\geq |m^2 - 2n^2| \quad [\because |m_1^2 - 2n_1^2| \geq 1]$$

$$= d(a).$$

$$\therefore\ d(a) \leq d(ab)$$

Now, we have,

$$\frac{a}{b} = \frac{m+n\sqrt{2}}{m_1+n_1\sqrt{2}} = \frac{(m+n\sqrt{2})(m_1-n_1\sqrt{2})}{(m_1+n_1\sqrt{2})(m_1-n_1\sqrt{2})}$$

$$\frac{a}{b} = \left(\frac{mm_1-2nn_1}{m_1^2-2n_1^2}\right)+\left(\frac{m_1n-mn_1}{m_1^2-2n_1^2}\right)\sqrt{2} = p+q\sqrt{2}$$

Where, $p = \left(\frac{mm_1-2nn_1}{m_1^2-2n_1^2}\right)$ and $q = \left(\frac{m_1n-mn_1}{m_1^2-2n_1^2}\right)$ are rational number.

Corresponding to the rational number p and q, we can find two integers p' and q' such that $|p'-p| \leq 1/2$ and $|q'-q| \leq 1/2$.

Let, $t = p'+q'\sqrt{2}$; then $t \in Z[\sqrt{2}]$

We have; $\frac{a}{b} = \lambda$, where $\lambda = p+q\sqrt{2}$

$$\Rightarrow a = \lambda b = (\lambda-t)b + tb = tb + r$$

where $r = (\lambda-t)b$.

Now; $a, b, t \in Z[\sqrt{2}]$

$$\Rightarrow a-tb \in Z[\sqrt{2}] \Rightarrow r \in Z[\sqrt{2}]$$

$\therefore \exists\ t, r \in Z[\sqrt{2}]$, such that

$a = tb + r$, where $r = 0$ (or)

$$d(r) = d((\lambda-t)b) = d\{(p+q\sqrt{2})-(p'+q'\sqrt{2})\}d(b)$$

$$d(r) = d\{(p-p')+(q-q')\sqrt{2}\}d(b)$$

$$= |(p-p')^2-2(q-q')^2|\,d(b)$$

$$\leq |(p-p')^2+2(q-q')^2|\,d(b)$$

$$\leq \left(\frac{1}{4}+\frac{2}{4}\right)d(b) = \frac{3}{4}d(b). < d(b)$$

$\therefore Z[\sqrt{2}]$ is a Euclidean domain.

Ques: 1.(C) Let $f(x) = \begin{cases} \frac{x^2}{2}+4 & \text{if } x \geq 0 \\ -\frac{x^2}{2}+2 & \text{if } x<0 \end{cases}$

Is 'f' Reimann integrable in the intervae [-1,2]? why? Does there exist a function 'g' such that $g'(x) = f(x)$? Justify your answer?

Solution:-

given; $f(x) = \begin{cases} \frac{x^2}{2}+4 & ; x \geq 0 \\ -\frac{x^2}{2}+2 & ; x<0 \end{cases}$

To check the continuity of f(x) at $x=0$

$$\lim_{x\to 0^+} f(x+h) = \lim_{h\to 0^+}\left(\frac{0+h}{2}\right)^2+4 = 4$$

$$\lim_{x\to 0^-} f(x-h) = \lim_{h\to 0^-} -\left(\frac{0+h}{2}\right)^2+2 = 2$$

and $f(0) = 4$.

$\therefore f(x+h) = f(0) \neq f(x-h)$

Hence, f(x) is discontinuous at $x=0$ which is finite and countable. Therefore f(x) is Reimann Integrable in $[-1,2]$.

Suppose; g(x) exist such that $g'(x) = f(x)$

$$g(x) = \begin{cases} \int_0^x f(x)\,dx & ; \text{ if } x \geq 0 \\ \int_{-1}^x f(x)\,dx & ; \text{ if } x<0 \end{cases}$$

$$g(x) = \begin{cases} \int_0^x \left(\frac{x^2}{2}+4\right)dx & ; \ 0 \le x < 2 \\ \int_{-1}^x \left(-\frac{x^2}{2}+2\right)dx & ; \ -1 \le x < 0 \end{cases}$$

$$g(x) = \begin{cases} \frac{x^3}{6}+4x & ; \ 0 \le x < 2 \\ \left(-\frac{x^3}{6}+2x\right)_{-1}^{x} & ; \ -1 \le x < 0 \end{cases}$$

$$g(x) = \begin{cases} \frac{x^3}{6}+4x & ; \ 0 \le x < 2 \\ -\frac{x^3}{6}+2x+\frac{11}{6} & ; \ -1 \le x < 0 \end{cases}$$

But, $\lim\limits_{x \to 0} g(x)$ does not exist thus not continuous at $x=0$.

Therefore, there does not exist a function $g(x)$ such that $g'(x) = f(x)$.

Hence the result.

Q.1)d) Show that the function

$v(x,y) = \ln(x^2+y^2) + x + y$ is harmonic.

Find its conjugate harmonic function $u(x,y)$.
Also find the corresponding analytic function. $f(z) = u+iv$ in terms of z.

Solve:- Given; $v(x,y) = \ln(x^2+y^2) + x + y$

$$\frac{\partial v}{\partial x} = \frac{2x}{x^2+y^2} + 1 \quad ; \quad \frac{\partial v}{\partial y} = \frac{2y}{x^2+y^2} + 1$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{2(x^2+y^2) - 4x^2}{(x^2+y^2)^2} = \frac{-2x^2+2y^2}{(x^2+y^2)^2}$$

$$\frac{\partial^2 v}{\partial y^2} = \frac{2(x^2+y^2) - 4y^2}{(x^2+y^2)^2} = \frac{-2y^2+2x^2}{(x^2+y^2)^2}$$

Now, check

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{-2x^2+2y^2}{(x^2+y^2)^2} + \frac{(-2y^2)+2x^2}{(x^2+y^2)^2} = 0$$

Hence, it is a harmonic function.

$$\therefore \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} \Rightarrow \frac{\partial u}{\partial x} = \frac{2y}{x^2+y^2} + 1 = \phi_1(x,y) \text{ (say)}$$

Integrating w.r.t x.

$$u = 2y \cdot \frac{1}{y} \cdot \tan^{-1}(x/y) + x + f(y)$$

$$u = 2\tan^{-1}(x/y) + x + f(y)$$

$$\frac{\partial u}{\partial y} = \frac{-2 \cdot x}{y^2\left(\frac{x^2+y^2}{y^2}\right)} + f'(y) = \frac{-2x}{x^2+y^2} + f'(y).$$

$$f'(y) = \frac{\partial u}{\partial y} + \frac{2x}{x^2+y^2}$$

$$f'(y) = -\left(\frac{2x}{x^2+y^2}+1\right) + \frac{2x}{x^2+y^2}$$

$$f'(y) = \frac{\cancel{2x}}{\cancel{x^2+y^2}} - \frac{\cancel{2x}}{\cancel{x^2+y^2}} - 1 = -1.$$

$$\boxed{\therefore f'(y) = -1} \Rightarrow \boxed{f(y) = -y}$$

$$\frac{\partial u}{\partial y} = \frac{-2x}{x^2+y^2} - 1 = -\left(\frac{2x}{x^2+y^2}+1\right) = \phi_2(x,y) \text{ (say)}$$

$$\boxed{u = 2\tan^{-1}(x/y) + x - y}$$

$\therefore$ <u>By = Milne Thomson's method</u>

we have; $\boxed{x = z, \ y = 0}$

$$f'(z) = \phi_1(z,0) - i\phi_2(z,0)$$

$$f'(z) = \left(\frac{2\times 0}{z^2+0}+1\right) - i\left[-\left(\frac{2z}{z^2+0}+1\right)\right]$$

$$f'(z) = 1 + i\left(\frac{2}{z}+1\right)$$

$$f(z) = z + i(2\ln z + z) + c$$

$$\boxed{f(z) = z + i(\ln z^2 + z) + c}$$

which is required solution.

Ques: 1(c)) For each hour per day that Ashok studies mathematics it yields him 10 marks and for each hour that he studied physics, it yields him 5 marks. He can study at most 14 hours a day and He must get at least 40 marks in each. Determine graphically how many hours a day he should study mathematics and physics each, in order to maximize his marks?

Solution:-

Let 'x' represents no. of hours Ashok studies mathematics and

'y' represents no. of hours Ashok studies physics

and 'z' represents the function.

According to given:-

Maximize: $Z = 10x + 5y$

subject to $x + y \leq 14$ (atmost 14 hours a day)

$x \geq 4$, $y \geq 8$ } At least 40 marks in each subjects

and $x, y \geq 0$ ($\because$ marks can't be negative).

writing in equal form.

Maximize : $Z = 10x + 5y$

subject to : $x + y = 14$

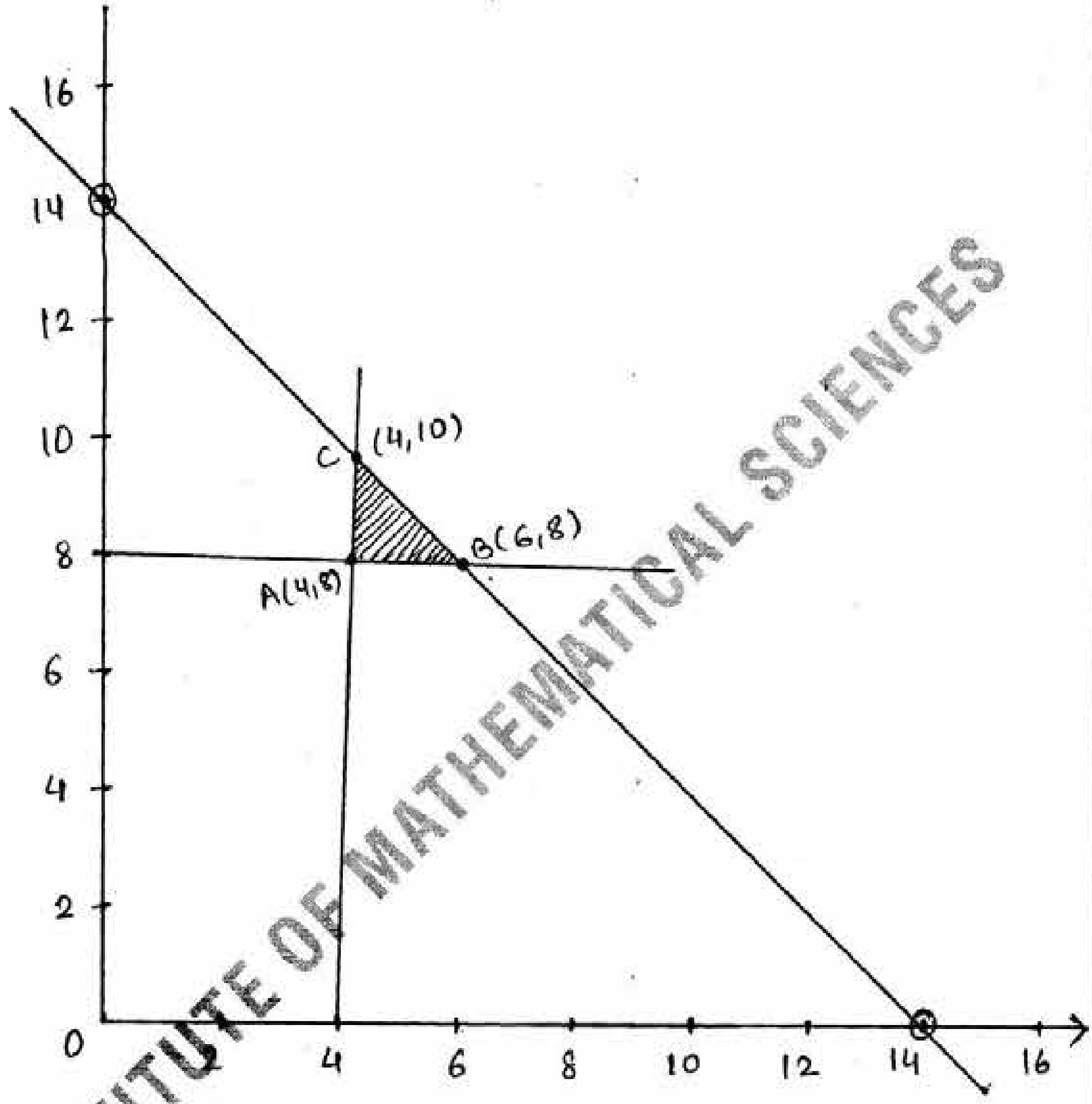
$x = 4$, $y = 8$ [$\because$ 40 mark – 10 marks each hour]

$x, y \geq 0$

writing basic solution for $x + y = 14$

X	0	14
Y	14	0

Considering only positive axes of x and y as
$x, y \geq 0$



At $(4,8) \Rightarrow Z = 10\times4 + 5\times8 = 40+40 = 80$

at $(6,8) \Rightarrow Z = 10\times6 + 5\times8 = 60+40 = 100$ (maximum)

at $(4,10) \Rightarrow Z = 10\times4 + 5\times10 = 40+50 = 90$.

So, the maximum marks, Ashok should study Mathematics for 6 hours and physics for 8 hours.

Q.2 (a) Show that the groups $Z_5 \times Z_7$ and Z_{35} are isomorphic.

Sol let $(Z_{35} = \{0, 1, 2, 3 \cdots 34\}, +)$ be a cyclic group of order 35;

let $Z_5 \times Z_7 = \{(a, b) / a \in Z_5, b \in Z_7\}$ be a cyclic ($\because \gcd(5, 7) = 1$)

where $(Z_5 = \{0, 1, 2, 3, 4\}, +)$ and $(Z_7 = \{0, 1, 2, 3, 4, 5, 6\}, +)$ are two cyclic groups such that $O(Z_5) = 5$ and $O(Z_7) = 7$

Define a mapping $f: Z_{35} \to Z_5 \times Z_7$ such that $f(a) = (a \oplus_5 0, a \oplus_7 0)$ — (1)

Now we shall show that f is well-defined:

let $a, b \in Z_5$ s.t $a = b$

$\Rightarrow a +_5 0 = b +_5 0$

$\therefore$ f is well-defined.

To S.T 1-1: let $(a +_5 0, a \oplus_7 0) = (b \oplus_5 0, b \oplus_7 0$

$\Rightarrow a = b$

$\therefore f$ is 1-1

To show onto:

for every $(a \oplus_5 0, a \oplus_7 0) \in Z_5 \times Z_7$

$\exists\, a \in Z_{35}$ such that

$f(a) = (a \oplus_5 0, a \oplus_7 0)$ by definition.

$\therefore f$ is onto.

To show homomorphism:

Let $a, b \in Z_{35}$, then

$f(a \oplus_{35} b) = \left((a \oplus_{35} b) \oplus_5 0, (a \oplus_{35} b) \oplus_7 0\right)$

$= (a \oplus_{35} b, a \oplus_{35} b)$

$= (a \oplus_5 0, a \oplus_7 0) \oplus_{35} (b \oplus_5 0, b \oplus_7 0)$

$= f(a) \oplus_{35} f(b)$

$\therefore f$ is a homomorphism

$\therefore Z_5 \times Z_7 \cong Z_{35}$.

Ques:-2 (b)(i) find the supremum and the infemum of $\frac{x}{\sin x}$ on the interval $\left(0, \frac{\pi}{2}\right]$.

Solution:-

Let $f(x) = \frac{x}{\sin x} \quad \forall\, x \in \left(0, \frac{\pi}{2}\right]$

then, $f'(x) = \frac{\sin x - x\cos x}{\sin^2 x}$ ——— (1)

Let $g(x) = \sin x - x\cos x$.

then $g'(x) = \cos x - (\cos x - x\sin x)$

$\boxed{g'(x) = x\sin x} > 0$

$\forall\, x \in (0, \pi/2]$

$\therefore$ g(x) is an increasing function on $(0, \pi/2]$

$\because x \in \left(0, \frac{\pi}{2}\right] \Rightarrow 0 < x \leq \frac{\pi}{2}$.

$\Rightarrow g(0) < g(x) \leq g\left(\frac{\pi}{2}\right)$.

$\Rightarrow 0 < g(x) \leq 1$.

$\therefore f'(x) > 0 \quad \forall\, x \in (0, \pi/2]$

Hence, f is an increasing on $\left(0, \frac{\pi}{2}\right]$.

$\therefore$ Infimum $f = \lim_{x \to 0} \frac{x}{\sin x} = 1$.

Supremum $f = f(\pi/2) = \frac{\pi/2}{\sin \pi/2} = \frac{\pi}{2}$.

Q.2 (b)(ii

Two sequences $\{x_n\}$ and $\{y_n\}$ are defined inductively by the following:

$x_1 = \frac{1}{2}, y_1 = 1$ and $x_n = \sqrt{x_{n-1} y_{n-1}}$, $n = 2, 3, 4, \ldots$

$$\frac{1}{y_n} = \frac{1}{2}\left(\frac{1}{x_n} + \frac{1}{y_{n-1}}\right), n = 2, 3, 4 \ldots$$

prove that $x_{n-1} < x_n < y_n < y_{n-1}$, $n = 2, 3, \ldots$

and deduce that both the sequences converge to the same limit 'l', where $\frac{1}{2} < l < 1$.

Sol Given that $x_1 = \frac{1}{2}, y_1 = 1$

$\therefore \frac{1}{2} = x_1 < y_1 = 1$ ——— (1)

We have $x_2 = \sqrt{x_1 y_1} < \sqrt{y_1 y_1} = y_1$

$\therefore x_2 < y_1$ ——— (2)

We have $\frac{1}{y_2} = \frac{1}{2}\left(\frac{1}{x_2} + \frac{1}{y_1}\right)$

$\Rightarrow \frac{1}{y_2} < \frac{1}{2}\left(\frac{1}{x_2} + \frac{1}{x_2}\right)$

$= \frac{1}{x_2}$ $\left(\because x_2 < y_1 \Rightarrow \frac{1}{x_2} > \frac{1}{y_1}\right)$

$\therefore \frac{1}{y_2} < \frac{1}{x_2} \Rightarrow y_2 > x_2$

$\therefore x_2 < y_2$ ——— (3)

let us suppose that

$x_{n-1} < y_{n-1}$ —— (4).

we have $x_n = \sqrt{x_{n-1} \cdot y_{n-1}}$

$< \sqrt{y_{n-1} \cdot y_{n-1}}$ (by (4))

$= y_{n-1} \ \forall \ n \geq 2.$

we have —— (5)

$$\frac{1}{y_n} = \frac{1}{2}\left(\frac{1}{x_n} + \frac{1}{y_{n-1}}\right)$$

$$< \frac{1}{2}\left(\frac{1}{x_n} + \frac{1}{x_n}\right)$$

$$= \frac{1}{x_n} \quad \left(\because x_n < y_{n-1} \Rightarrow \frac{1}{x_n} > \frac{1}{y_{n-1}} \Rightarrow \frac{1}{y_{n-1}} < \frac{1}{x_n}\right)$$

$$\therefore \frac{1}{y_n} < \frac{1}{x_n}$$

$\Rightarrow y_n > x_n$

$\Rightarrow x_n < y_n \ \forall \ n \geq 1$ —— (6)

we have $x_n = \sqrt{x_{n-1} \cdot y_{n-1}}$

$> \sqrt{x_{n-1} \, x_{n-1}}$ (by (4))

$= x_{n-1}$

$\therefore x_n > x_{n-1} \ \forall \ n \geq 2$ —— (7)

we have $\frac{1}{y_n} = \frac{1}{2}\left(\frac{1}{x_n} + \frac{1}{y_{n-1}}\right)$

$> \frac{1}{2}\left(\frac{1}{y_{n-1}} + \frac{1}{y_{n-1}}\right)$

$= \frac{1}{y_{n-1}}$ by (5)

$\therefore \dfrac{1}{y_n} > \dfrac{1}{y_{n-1}}$

$\Rightarrow y_n < y_{n-1} \;\; \forall\, n \geq 2$ ——— (8)

from (6), (7) & (8),
we have

$x_{n-1} < x_n < y_n < y_{n-1} \;\; \forall\, n \geq 2$

$\therefore \frac{1}{2} = x_1 < x_2 < x_3 < \cdots < y_3 < y_2 < y_1 = 1$

$\therefore (x_n)$ is an increasing and bounded above by 1

$\therefore (x_n)$ converges.

and (y_n) is a decreasing and bounded below by $\frac{1}{2}$.

$\therefore (y_n)$ converges.

Let $\lim x_n = l$ then $\lim x_{n-1} = l$.

and

$\lim y_n = m$ then $\lim y_{n-1} = m$.

$\therefore l = \sqrt{lm}$ and $\dfrac{1}{m} = \dfrac{1}{2}\left(\dfrac{1}{l} + \dfrac{1}{m}\right)$

$\therefore l = m$

and $\frac{1}{2} < l < 1$.

Ques:- 2(c)(i) using Cauchy integral formula,

Evaluate $\displaystyle\int_C \frac{z+2}{(z+1)^2(z-2)}\,dz$

where C is the circle $|z-i|=2$

Solution:-

Given; $\displaystyle\int_C \frac{z+2}{(z+1)^2(z-2)}\,dz$; where C is the circle $|z-i|=2$

$|z-i|=2$, circle with centre at i and radius 2.

Hence, $z-2=0$; i.e $z=2$ does not lies inside the circle, Hence.

$$\int_C \frac{z+2/z-2}{(z+1)^2}\,dz \quad \text{——} \ (1)$$

Comparing eqn (1) with $\displaystyle\int_C \frac{f(z)}{(z-z_0)^n}\,dz$.

where ; $z_0=-1$. and $n=2$.

and it is a point lies inside $|z-i|=2$.

$$f(z) = \frac{z+2}{z-2}.$$

$$\therefore \int_C \frac{f(z)\,dz}{(z-z_0)^2} = \frac{2\pi i}{(n-1)!}\cdot f^{n-1}(z_0) \quad \text{——} \ (2)$$

$$\Rightarrow \int_C \frac{f(z)\,dz}{(z-z_0)^2} = 2\pi i\cdot f'(z_0) \quad \text{——} \ (3)$$

Now, $f(z) = \dfrac{z+2}{z-2}$

$$f'(z) = \frac{(z-2)-(z+2)}{(z-2)^2} = \frac{-4}{(z-2)^2}$$

$$f'(z_0) = \frac{-4}{(z_0-2)^2}$$

$$f'(-1) = \frac{-4}{(-1-2)^2} = -\frac{4}{9}$$

$$\therefore \int_C \frac{f(z)\,dz}{(z-z_0)^2} = \int_C \frac{z+2}{(z+1)^2(z-2)}\,dz = \frac{2\pi i}{1} \cdot -\frac{4}{9}$$

$$\boxed{\therefore \int_C \frac{z+2}{(z+1)^2(z-2)}\,dz = -\frac{8\pi i}{9}}$$ required Solution;

Q.2 (c) (ii). Evaluate by contour integral Method – I:

$$\int_0^1 \frac{dx}{(x^2-x^3)^{1/3}}$$

Sol^n Given that,

$$\int_0^1 \frac{dx}{(x^2-x^3)^{1/3}}$$

$$= \int_0^1 \frac{dx}{x\left(\frac{1}{x}-1\right)^{1/3}}$$

Let $x=\frac{1}{t}$, $dx = -\frac{1}{t^2}dt$

$$\therefore \int_0^1 \frac{dx}{x\left(\frac{1}{x}-1\right)^{1/3}} = \int_\infty^1 \frac{t\left(-\frac{1}{t^2}\right)dt}{(t-1)^{1/3}} = \int_1^\infty \frac{dt}{t(t-1)^{1/3}}$$

Let $t = x+1$, $dt = dx$

$$\therefore \int_1^\infty \frac{dt}{t(t-1)^{1/3}} = \int_0^\infty \frac{dx}{x^{1/3}(x+1)} = \int_0^\infty \frac{x^{-1/3}}{(x+1)}dx$$

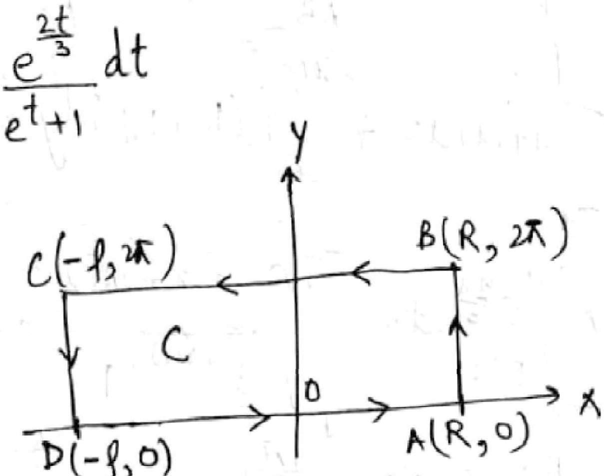
Thus, $$\boxed{I = \int_0^1 \frac{dx}{(x^2-x^3)^{1/3}} = \int_0^\infty \frac{x^{-1/3}}{(x+1)}dx} \quad \text{—— (1)}$$

Put $x = e^t$, we have $(dx = e^t dt)$

$$I = \int_{-\infty}^{\infty} \frac{e^{-t/3}e^t dt}{(e^t+1)} = \int_{-\infty}^{\infty} \frac{e^{\frac{2t}{3}}}{e^t+1}dt$$

Consider,

$$\int_C \frac{e^{\frac{2}{3}z}}{e^z+1}dz = \int_C f(z)\,dz$$



where C : Rectangle ABCD with vertices at

R, $R+2\pi i$, $-\rho+2\pi i$, $-\rho$.

$f(z)$ has simple poles given by

$$e^z = -1 = e^{\{(2K+1)\pi i\}}$$

$$\Rightarrow z = (2K+1)\pi i \; ; \; K = 0, \pm 1, \pm 2, \pm 3, \ldots$$

Now, Residue at $z = \pi i$ (inside contour) is

$$\left[\frac{e^{2/3 z}}{\frac{d}{dz}(e^z+1)}\right]_{z=\pi i}$$

$$= \left.\frac{e^{2/3 z}}{e^z}\right|_{z=\pi i}$$

$$= \frac{e^{\frac{2}{3}(\pi i)}}{e^{\pi i}}$$

$$= -e^{\left(\frac{2\pi i}{3}\right)}$$

By Residue theorem,

$$\int_C f(z) = 2\pi i \,[\text{Sum of residue}]$$

$$\Rightarrow \int_{DA} f(z)\,dz + \int_{AB} f(z)\,dz + \int_{BC} f(z)\,dz + \int_{CD} f(z)\,dz = -2\pi i\, e^{\left(\frac{2\pi i}{3}\right)}$$

$$\Rightarrow \int_{-\rho}^{R} f(x)\,dx + \int_0^{2\pi} f(R+iy)\,i\,dy + \int_R^{-\rho} f(x+2\pi i)\,dx + \int_{2\pi}^{0} f(-\rho+iy)\,i\,dy = -2\pi i\, e^{\frac{2\pi i}{3}}$$

$$\Rightarrow \int_{-\rho}^{R} \frac{e^{\frac{2x}{3}}}{e^x+1}\,dx + \int_R^{-\rho} \frac{e^{\frac{2}{3}(x+2\pi i)}}{e^{(x+2\pi i)}+1}\,dx + \int_0^{2\pi} f(R+iy)\,i\,dy + \int_{2\pi}^{0} f(-\rho+iy)\,i\,dy = -2\pi i\, e^{\frac{2\pi i}{3}}$$

$$\Rightarrow \int_{-\rho}^{R} \frac{e^{2/3 x}\left[1 - e^{\frac{4\pi i}{3}}\right]}{(e^x+1)}\,dx + \int_0^{2\pi} f(R+iy)\,i\,dy + \int_{2\pi}^{0} f(-\rho+iy)\,i\,dy = -2\pi i\, e^{\frac{2\pi i}{3}}$$

—— 2

$$\Rightarrow \left[1-e^{-\frac{4\pi i}{3}}\right] \int_{-\rho}^{R} \frac{e^{2/3\,x}}{e^{x}+1}\,dx = -2\pi i\, e^{\frac{2\pi i}{3}}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{2/3\,x}}{e^{x}+1}\,dx = \frac{-2\pi i\, e^{\frac{2\pi i}{3}}}{\left[1-e^{-\frac{4\pi i}{3}}\right]}$$

$$\Rightarrow I = \frac{-2\pi i\, \cancel{e^{\frac{2\pi i}{3}}}}{\cancel{e^{+\frac{2\pi i}{3}}}\left[e^{-\frac{2\pi i}{3}} - e^{+\frac{2\pi i}{3}}\right]}$$

$$= \frac{\pi}{\dfrac{e^{-\frac{2\pi}{3}i} - e^{\frac{2\pi}{3}i}}{-2i}}$$

$$= \frac{\pi}{\dfrac{e^{\frac{2\pi}{3}i} - e^{-\frac{2\pi}{3}i}}{2i}} = \frac{\pi}{\left(\sin\frac{2\pi}{3}\right)} = \frac{\pi}{\frac{\sqrt{3}}{2}} = \frac{2\pi}{\sqrt{3}}$$

$$\Rightarrow \boxed{I = \frac{2\pi}{\sqrt{3}}}$$ Ans

Ques: 3(a) Prove that every principal ideal domain is unique factorization domain. Show by an example that the converse is not true.

Solution:-

Let R be a P.I.D. Let 'a' be any non-zero, non-unit element of R. We proceed to show that 'a' is expressible as a finite product of irreducible elements of R. If a is irreducible, there is nothing to prove.

If 'a' is not irreducible,

then $a = a_1 a'$; where a_1 & a_1' are both units in R

We notice; that $a = a_1 a_1' \Rightarrow a_1 | a \Rightarrow (a) \subset (a_1)$

and $(a) \neq (a_1)$

for $(a) = (a_1) \Rightarrow a$ and a_1 are associates

$\Rightarrow a_1'$ is a unit, a contradiction.

If in the expression $a = a_1 a_1'$ both a_1 & a_1' are irreducible

The result is proved.

Otherwise, we may suppose that a_1 is not irreducable

Then $a_1 = a_2 a_2'$ where a_2 & a_2' are both non-units in R.

$(a_1) \subset (a_2)$ with $(a_1) \neq (a_2)$

Thus, we have $a = a_2 a_2' a_1'$.

If a_2, a_2', a_1' are all irreducible elements in R, the result is proved. otherwise proceeding in a similar manner, we obtain an ascending chains of ideals:

$(a) \subset (a_1) \subset (a_2) \subset (a_3) \cdots\cdots$

where no two of these ideals are equal.

By the above lemma, such a chain of ascending ideals in a P.I.D must be finite

i.e $(a) \subset (a_1) \subset (a_2) \subset (a_3) \subset \cdots \subset (a_m)$.

for some positive integer m.

consequently, after a finite number of steps, we arrive at an expression of 'a' as a product of finite number of irreducible elements of R of the form, $\boxed{a = p_1 p_2 \cdots p_m}$

where each p_i is irreducible

Uniqueness: We consider two representations of 'a' as products of finite number of irreducible (prime) elements of R as follow:

$$a = p_1 p_2 p_3 \cdots p_m = q_1 q_2 \cdots q_n$$

Then; $p_1 p_2 p_3 \cdots p_m = q_1 q_2 q_3 \cdots q_n$ —— (1)

It is clear that

$p_1 | p_1 p_2 p_3 \cdots p_m$ and so $p_1 | q_1 q_2 \cdots q_n$

$\because$ every irreducible element of R is a prime, so

$p_1 | q_1 q_2 q_3 \cdots q_n \Rightarrow p_1 | q_j$ for some j; $1 \le j \le n$.

without any loss of generality, we may assume that $p_1 | q_1$ (p_1 and q_1 both prime).

consequently, p_1 and q_1 are associates.

or $q_1 = u_1 p_1$, where u_1 is a unit in R.

Thus, (by 1) we have.

$p_1 p_2 \cdots p_m = u_1 p_1 q_2 \cdots q_n$.

$\boxed{p_2 p_3 \cdots p_m = u_1 q_2 q_3 \cdots q_n}$ $[\because p_1 \neq 0 \in R]$ —(2)

Since, $p_2 | p_2 p_3 \cdots p_m$,

we may take $p_2 | q_2$ and so p_2 and q_2 are associates.

[for otherwise $p_2 | u_1$ and $u_1 | 1 \Rightarrow p_2 | 1 \Rightarrow p_2$ is a unit which is a contradiction]

We can write; $p_2 = u_2 q_2$.

Putting this value in ② and cancelling out p_2 both the sides, we get

$$p_3 p_4 p_5 \cdots p_m = u_1 u_2 q_3 q_4 q_5 \cdots q_n \text{ and so on.}$$

After m steps of all p_i's are cancelled out and L.H.S becomes 1, but R.H.S contains some q_j's, It means that

number of p_i's $\le$ number of q_j's ie $m \le n$ —(A)

Repeating the above procedure with $q_1 | q_1 q_2 \cdots q_n$

$\Rightarrow q_1 | p_1 p_2 \cdots p_m \Rightarrow q_1 | p_1$ etc.

we get $n \le m$ —— (B)

from (A) & (B)

$$\boxed{n = m}$$

and in this process, we have also proved that p_i and q_j are associates for each i, $1 \le i \le m$.

Hence, R is U.F.D (Unique factorization Domain)

But the converse is not true:

We know that $Z[x]$ is a U.F.D.

as Z is a U.F.D, but $Z[x]$ is not P.I.D

Hence, the result.

Also,

$F[x,y]$ is a U.F.D which is not a P.I.D.

F being any field.

Ques: 3(b)) Show that the sequence of functions f_n defined on $[0,1]$ by $f_n(x) = n(1-nx)$, $0 \le x < \frac{1}{n}$

$= 0 \quad ; \frac{1}{n} \le x \le 1$

converges to the function 'f' given by $f(x)=0$, $x \in [0,1]$. Show that $\lim_{n\to\infty} \int_0^1 f_n(x)\,dx \ne \int_0^1 f(x)\,dx$.

Is the convergence of the sequence uniform?

Solution:-

$$\text{Given; } f_n(x) = \begin{cases} n(1-nx) & ; 0 \le x \le \frac{1}{n} \\ 0 & ; \frac{1}{n} \le x \le 1. \end{cases}$$

At $x=0$, the sequence is $\{1,1,1,\dots\}$

This converges to 'n'.

at $x=1$, the sequence ... is $\{0,0,0,0\}$. This converges to 0.

Let $c \in (0,1)$. By Archimedean property of $\mathbb{R}$ there exists a natural 'm' such that $0 < \frac{1}{m} < c$ and therefore $0 < \frac{1}{n} < c$ for all $n \ge m$.

$\therefore f_m(c) = 0$ and $f_n(c) = 0$ for all $n \ge m$.

This proves $\lim_{n\to\infty} f_n(c) = 0$.

There the sequence $\{f_n\}$ converges to the function f on $[0,1]$ given by.

$$f(x) = \begin{cases} 1 & , x=0 \\ 0 & , 0 < x \le 1 \end{cases}$$

Each f_n is continuous on $[0,1]$

But the limit function 'f' is not continuous on $[0,1]$.

Here; $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} n(1-nx)$ ($\infty \times 0$ form).

$= \lim_{n\to\infty} \frac{1-nx}{1/n}$ (Apply D.L rule.

$= \lim_{n\to\infty} xn^2 = 0$

Since; $n\to\infty$, $\frac{1}{n}\to 0$, as $x\to 0$

Hence; $\lim_{n\to\infty} f_n(x) = 0$ for $x\in[0,1]$

Now; $\int_0^1 f_n(x)\,dx = \int_0^1 n(1-nx)\,dx$.

Put $1-nx = t$ at $x=0$ $t=1$
$-n\,dx = dt$ $x=1$ $t=1-n$.

$\Rightarrow \int_1^{1-n} -t\,dt = \int_{1-n}^{1} t\,dt = \frac{1}{2}[t^2]_{1-n}^{1}$.

$= \frac{1}{2}[1-[1-n]^2] = \frac{1}{2}[2n-n^2]$

$\therefore \lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \infty$ —— (A)

but $\int_0^1 f(x)\,dx = \int_0^1 0\,dx = 0$ —— (B)

Clearly; from (A) and (B).

$$\boxed{\lim_{n\to\infty} \int_0^1 f_n(x)\,dx \neq \int_0^1 f(x)\,dx}$$

Therefore, the convergence of the sequence $\{f_n\}$ is not uniform on $[0,1]$, since uniform convergence of the sequence $\{f_n\}$ of continuous functions on $[0,1]$ implies continuity of f on $[0,1]$.

Ques:-3(c)) Solve the following transportation problem:

Destinations

FACTORIES	D_1	D_2	D_3	D_4	D_5	D_6	Availability
F_1	2	1	3	3	2	5	50
F_2	3	2	2	4	3	4	40
F_3	3	5	4	2	4	1	60
F_4	4	2	2	1	2	2	30
Demand	30	50	20	40	30	10	

finding the initial solution by Matrix Minima Method.

Solution:-

From the given table

Total Demand $= 30+50+20+40+30+10$
$= 180$

Total Availability $= 50+40+60+30 = 180$

Hence; $\boxed{\text{Demand} = \text{Availability}}$

$\therefore$ Given Transportation problem is balanced.

Hence, for initial solution - by Matrix Minima Mtd.

	D_1	D_2	D_3	D_4	D_5	D_6	Availability
F_1	× 2	50 1	× 3	× 3	× 2	× 5	50 ✓
F_2	20 3	0 2	20 2	× 4	× 3	× 4	40 ✓
F_3	10 3	× 5	× 4	10 2	30 4	10 1	60
F_4	× 4	× 2	× 2	30 1	× 2	× 2	30
Demand	30	50 ✓	20 ✓	40 ✓	30	10	

Here ; number of positive allocations = 8

and $m+n-1 = 6+4-1 = 9$

$\therefore$ solution degenerates.

Let , cell $a_{22} = (0)$ be a zero basic cell in the initial Basic feasible solution.

$\therefore$ Initial feasible solution $= 2\times0 + 1\times50 + 3\times20 + 2\times20 + 3\times10 + 2\times10 + 4\times30 + 1\times10 + 1\times30$

IBFS $= 0+50+60+40+30+20+120+10+30$

$\boxed{\text{IBFS} = 360}$ using Matrix minima method.

Now, for OBFS

By using MODI method:

For Basic cells ; $\Delta_{ij} = u_i + v_j - c_{ij} = 0$

$u_2 + v_2 - 2 = 0$

$\Rightarrow u_2 + v_2 = 2$

$\Rightarrow u_2 + v_1 = 3$

$u_3 + v_1 = 3$

$u_1 + v_2 = 1$

$u_2 + v_3 = 2$

$u_3 + v_4 = 2$

$u_4 + v_4 = 1$

$u_3 + v_5 = 4$

$u_3 + v_6 = 1$

Let $u_3 = 0$, $u_1 = -1$, $u_2 = 0$, $u_4 = -1$

$v_1 = 3$, $v_2 = 2$, $v_3 = 2$, $v_4 = 2$, $v_5 = 4$, $v_6 = 1$

Let us, calculate Δ_{ij} for Non-Basic Cells :-

$\Delta_{41} = u_4 + v_1 - 4 = -1+3-4 = -2$

$\Delta_{11} = u_1 + v_1 - 2 = -1+3-2 = 0$

$\Delta_{32} = u_3 + v_2 - 5 = 0+2-5 = -3$

$\Delta_{42} = u_4 + v_2 - 2 = -1 + 2 - 2 = -1.$

$\Delta_{13} = u_1 + v_3 - 3 = -1 + 2 - 3 = -2$

$\Delta_{33} = u_3 + v_3 - 4 = 0 + 2 - 4 = -2$

$\Delta_{43} = u_4 + v_3 - 2 = -1 + 2 - 2 = -1$

$\Delta_{15} = u_1 + v_5 - 2 = -1 + 4 - 2 = 1 > 0$

$\Delta_{25} = u_2 + v_5 - 3 = 0 + 4 - 3 = 1 > 0$

$\Delta_{45} = u_4 + v_5 - 2 = -1 + 4 - 2 = 1 > 0$

$\Delta_{16} = u_1 + v_6 - 5 = -1 + 1 - 5 = -5$

$\Delta_{26} = u_2 + v_6 - 4 = 0 + 1 + (-4) = -3$

$\Delta_{46} = u_4 + v_6 - 2 = -1 + 1 - 2 = -2$

As we observe, there are some positive values in non-basic cells; hence,

	D_1	D_2	D_3	D_4	D_5	D_6
F_1		50				
F_2	20	0	20			
F_3	10			10 $+\theta$	30 $-\theta$	10
F_4				30 $-\theta$	$+\theta$	

$+\theta$ 10	30 $-\theta$
$-\theta$ 30	$+\theta$

Here $\theta = 30$

$\Rightarrow$

40	0
0	30

$\therefore$ New feasible solution table is given by –

	D_1	D_2	D_3	D_4	D_5	D_6
F_1		50_1				
F_2	20_3	0_2	20_2			
F_3	10_3			40_2	$_4$	10_1
F_4				0_1	30_2	

$\because$ No. of allocations $= 9 = m+n-1$

$\therefore$ Using MODI Method

For Basic cells

$u_1+v_2 = 1$ $\quad$ $u_3+v_4 = 2$

$u_2+v_1 = 3$ $\quad$ $u_4+v_4 = 1$

$u_3+v_1 = 3$ $\quad$ $u_3+v_6 = 1$

$u_2+v_2 = 2$ $\quad$ $u_4+v_5 = 2$

$u_2+v_3 = 2$

Put

$u_3 = 0$ $\quad$ $v_1 = 3$

$u_4 = -1$ $\quad$ $v_4 = 2$

$u_1 = -1$ $\quad$ $v_5 = 3$

$u_2 = 0$ $\quad$ $v_6 = 1$

$v_2 = 2$

$v_3 = 2$

for non basic cells :- (Δ_{ij})

$\Delta_{11} = u_1+v_1-2 = -1+3-2 = 0$

$\Delta_{13} = u_1+v_3-3 = -1+2-3 = -2$

$\Delta_{14} = u_1+v_4-3 = -1+2-3 = -2$

$\Delta_{15} = u_1+v_5-2 = -1+3-2 = 0$

$\Delta_{16} = u_1+v_6-5 = -1+1-5 = -5$

$\Delta_{24} = u_2+v_4-4 = 0+2-4 = -2$

$\Delta_{25} = u_2+v_5-3 = 0+3-3 = 0$

$\Delta_{26} = u_2+v_6-4 = 0+1-4 = -3$

$\Delta_{32} = u_3+v_2-5 = 0+2-5 = -3$

$\Delta_{33} = u_3+v_3-4 = 0+2-4 = -2$

$\Delta_{41} = u_4+v_1-4 = -1+3-4 = -2$

$\Delta_{42} = u_4+v_2-2 = -1+2-2 = -1$

$\Delta_{43} = u_4+v_3-2 = -1+2-2 = -1$

$\Delta_{35} = u_3+v_5-4 = 0+3-4 = -1$

$\Delta_{46} = u_4+v_6-2 = -1+1-2 = -2$

By observing all the values of Δ_{ij} we get all $\boxed{\Delta_{ij} \leq 0}$ Hence optimality obtained

$\therefore$ optimal Transporation cost =

$1\times 50 + 3\times 20 + 0\times 2 + 2\times 20 + 3\times 10 + 2\times 40 + 1\times 10$
$+ 0\times 1 + 2\times 30$

$= 50 + 60 + 0 + 40 + 30 + 80 + 10 + 0 + 60$

$= 110 + 150 + 70 = \underline{\underline{330}}$

$\boxed{\therefore \text{ optimal cost for given Transportation} = \text{Rs: } 330}$

Q.4 (a) How many proper non-zero ideals does the ring Z_{12} have? Justify your answer. How many ideals does the ring $Z_{12} \oplus Z_{12}$ have? Why?

Solution: Consider, the ring $R = Z_{12}$. Ideals are the subgroup of the additive group of 'R' and subgroup of cyclic groups are cyclic, so the proper ideals are

$$\langle 2 \rangle = \{0, 2, 4, 6, 8, 10\}$$

$$\langle 3 \rangle = \{0, 3, 6, 9\}$$

$$\langle 4 \rangle = \{0, 4, 8\}$$

$$\langle 6 \rangle = \{0, 6\}.$$

Here, $\langle 2 \rangle$ & $\langle 3 \rangle$ are prime and maximal,

$\langle 4 \rangle$ & $\langle 6 \rangle$ neither prime nor maximal.

$\therefore$ Z_{12} has six ideals including $\{0\}$ and Z_{12} itself.

Thus, $Z_{12} \oplus Z_{12}$ has $6 \times 6 = 36$ ideals.

Ques: 4(b): $f(x)$ is defined as follows:

$$f(x) = \begin{cases} \frac{1}{2}(b^2-a^2) & \text{for } 0<x\le a \\ \frac{1}{2}b^2-\frac{x^2}{6}-\frac{a^3}{3x} & \text{for } a<x\le b \\ \frac{1}{3}\left(\frac{b^3-a^3}{x}\right) & \text{for } x>b \end{cases}$$

Prove that $f(x)$ and $f'(x)$ are continuous but $f''(x)$ is discontinuous.

Solution:-

Given; $$f(x) = \begin{cases} \frac{1}{2}(b^2-a^2) & \text{for } 0<x\le a \\ \frac{1}{2}b^2-\frac{x^2}{6}-\frac{a^3}{3x} & \text{for } a<x\le b \\ \frac{1}{3}\left(\frac{b^3-a^3}{x}\right) & \text{for } x>b. \end{cases}$$

for $f(x)$ to be continuous

L.H. limit at $x=a \Rightarrow \frac{1}{2}(b^2-a^2)$.

R.H. Limit at $x=a \Rightarrow \frac{1}{2}b^2-\frac{x^2}{6}-\frac{a^3}{3x}$.

$$= \lim_{x\to a^+} \frac{1}{2}b^2-\frac{x^2}{6}-\frac{a^3}{3x} = \frac{1}{2}b^2-\frac{a^2}{6}-\frac{a^2}{3}$$

$$= \frac{1}{2}(b^2-a^2) = \text{L.H limit}$$

At $x=b$) L.H.L $\Rightarrow \lim_{x\to b}\left[\frac{1}{2}b^2-\frac{x^2}{6}-\frac{a^3}{3x}\right]$

$$= \frac{1}{2}b^2-\frac{b^2}{6}-\frac{a^3}{3b} = \frac{1}{3}\left(\frac{b^3-a^3}{b}\right)$$

R.H.L $\Rightarrow \lim_{x\to b}\frac{1}{3}\left[\frac{b^3-a^3}{x}\right] = \frac{1}{3}\left[\frac{b^3-a^3}{b}\right]$.

$\therefore$ $f(x)$ is continuous at $x=a$ and $x=b$.

Now for $f'(x)$:-

$$f'(x) = \begin{cases} 0 & \text{for } 0 < x \le a \\ -\frac{x}{3} + \frac{a^3}{3x^2} & ;\ a < x \le b \\ -\frac{1}{3}\left(\frac{b^3 - a^3}{x^2}\right) & ;\ x > b. \end{cases}$$

Now at $x = a$.

$$\text{R.H.L} = \lim_{x \to a} -\frac{x}{3} + \frac{a^3}{3x^2} \Rightarrow -\frac{a}{3} + \frac{a^3}{3a^2} = 0 = \text{L.H.L.}$$

Now at $x = b$

$$\text{L.H.L} = \lim_{x \to b} -\frac{x}{3} + \frac{a^3}{3x^2} \Rightarrow -\frac{b}{3} + \frac{a^3}{3b^2} \Rightarrow \frac{1}{3}\left[\frac{a^3 - b^3}{b^2}\right].$$

$$\text{R.H.L.} = \lim_{x \to b} -\frac{1}{3}\left(\frac{b^3 - a^3}{x^2}\right) = \frac{1}{3}\left[\frac{a^3 - b^3}{b^2}\right] = \text{L.H.L.}$$

$\therefore f'(x)$ is continuous at $x = a$ and $x = b$.

Now for $f''(x) = \begin{cases} 0 & ;\ 0 < x \le a \\ -\frac{1}{3} + \frac{-2a^3}{3x^3} & ;\ a < x \le b \\ +\frac{2}{3}\left(\frac{b^3 - a^3}{x^3}\right) & ;\ b < x. \end{cases}$

at $x = a$

$$\text{R.H.L} = \lim_{x \to a} -\frac{1}{3} - \frac{2}{3}\frac{a^3}{x^3} = -\frac{1}{3} - \frac{2}{3} = -1 \ne 0 = \text{L.H.L.}$$

at $x = b$

$$\text{L.H.L} = \lim_{x \to b} -\frac{1}{3} - \frac{2a^3}{3x^3} \Rightarrow -\frac{1}{3} - \frac{2}{3}\frac{a^3}{b^3}.$$

$$\text{R.H.L} = \lim_{x \to b} +\frac{2}{3}\left(\frac{b^3 - a^3}{x^3}\right) = \frac{2}{3}\left[\frac{b^3}{b^3} - \frac{a^3}{b^3}\right] = \frac{2}{3} - \frac{2}{3}\frac{a^3}{b^3}$$

$$\text{L.H.L} \ne \text{R.H.L}$$

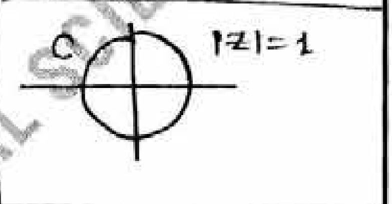
$\therefore f''(x)$ is not continuous at $x = a$ and $x = b$.

whereas $f(x)$ and $f'(x)$ are continuous at $x = a$ & $x = b$.

Ques: 4(C) Prove that if $be^{a+1} < 1$, where a and b are positive real number, then the function $z^n e^{-a} - be^z$ has 'n' zeroes in the unit circle.

Solution:-

Using Rouche's theorem that if $f(z)$ & $g(z)$ are analytic function in 'C', then $f(z)+g(z)$ has same number of zeroes as $f(z)$ has if $|g(z)| < |f(z)|$ on C.



Consider;

$$f(z) = z^n e^{-a}$$
$$g(z) = -be^z$$

$$\left|\frac{g(z)}{f(z)}\right| = \left|\frac{-be^z}{z^n e^{-a}}\right| \qquad \text{using } |z| = 1$$

$$\left|\frac{g(z)}{f(z)}\right| = \left|\frac{-be}{1^n \cdot e^{-a}}\right| = \left|\frac{be}{e^{-a}}\right|$$

$$\left|\frac{g(z)}{f(z)}\right| = |be^{a+1}| < 1.$$

$$\Rightarrow \boxed{|g(z)| < |f(z)| \quad \text{on} \quad z = 1}$$

Then, function $z^n e^{-a} - be^z$ will have same number of zeros as $f(z) = z^n e^{-a}$ in $|z| = 1$.

Clearly, $f(z)$ has 'n' zeroes in unit circle.

So; $z^n e^{-a} - be^z$ will have 'n' zeroes in unit circle.

Ques:-4(d)) Solve by Simplex method, the following LP problem:

Maximize, $Z = 5x_1 + 3x_2$

Constraints $3x_1 + 5x_2 \leq 15$

$5x_1 + 2x_2 \leq 10$

$x_1, x_2 \geq 0$

Solution:- Converting standard form:

Max. $Z = 5x_1 + 3x_2 + 0S_1 + 0S_2$

S.C: $3x_1 + 5x_2 + S_1 = 15$

$5x_1 + 2x_2 + S_2 = 10$

$x_1, x_2, S_1, S_2 \geq 0$

where S_1, S_2 are slack variables.

The IBFS is given by:-

$x_1 = x_2 = 0$; $S_1 = 15$; $S_2 = 10$

$\boxed{\therefore \text{Max}\cdot Z = 0}$

The initial simplex table is given as:-

	C_j	5	3	0	0		
C_B	Basis	x_1	x_2	S_1	S_2	b	θ
0	S_1	3	5	1	0	15	5
0	S_2	5	2	0	1	10	2 $\rightarrow$
	$Z_j = \sum C_B a_{ij}$	0	0	0	0	0	
	$C_j = C_j - Z_j$	5 $\uparrow$	3	0	0		

Here, x_1 is incoming variable and S_2 is outgoing variable.

'5'-Key element, making it unity and all other element in that column to 0.

The new Simplex Table :-

	C_j	5	3	0	0		
C_B	Basis	x_1	x_2	S_1	S_2	b	θ
0	S_1	0	(19/5)	1	3/5	9	$\frac{45}{19}$ →
5	x_1	1	2/5	0	1/5	2	5
$Z_j = \Sigma C_B a_{ij}$		5	2	0	1	10	
		0	1 ↑	0	-1		

Here x_2 is incoming variable and S_1 is the outgoing variable.
The key element here is (19/5), making it unity and all elements in that column to zero.

	C_j	5	3	0	0	
C_B	Basis	x_1	x_2	S_1	S_2	b
3	x_2	0	1	5/19	-3/19	$\frac{45}{19}$
5	x_1	1	0	-2/19	5/19	$\frac{20}{19}$
$Z_j = \Sigma C_B a_{ij}$		5	3	5/19	16/19	15
$C_j = C_j - Z_j$		0	0	-5/19	-16/19	

Since, all $C_j \leq 0$

Hence, optimality obtained

where, $x_1 = \frac{20}{19}$ and $x_2 = \frac{45}{19}$.

$$\text{Max. } Z = 5x_1 + 3x_2$$

$$\text{Max. } Z = 5 \times \frac{20}{19} + 3 \times \frac{45}{19}$$

$$\text{Max. } Z = \frac{100+135}{19} = \frac{235}{19} = 12.37 \approx \underline{\underline{12}}.$$

Section - B

Ques:- 5(a)) Find a complete and singular integrals of $2xz - px^2 - 2qxy + pq = 0$.

Solution:- Here given equation is

$$f = 2xz - px^2 - 2qxy + pq = 0 \quad \text{——} (1)$$

∴ Charpit's auxillary equations are:

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}.$$

By solving all the partial differential, we get.

$$\frac{dp}{2z - 2qy} = \frac{dq}{0} = \frac{dx}{x^2 - q} = \frac{dy}{2xy - p} = \frac{dz}{px^2 + 2xyq - 2pq}$$

The second fraction gives $dq = 0$

$$\Rightarrow \boxed{q = a} \text{ — by Integrating}$$

Putting $q = a$ in (1), we get

$$p = 2x(z - ay)/(x^2 - a)$$

Putting value of p and q in $dz = p\,dx + q\,dy$, we get

$$dz = \frac{2x(z - ay)}{x^2 - a}dx + a\,dy$$

$$\Rightarrow \quad \frac{dz - a\,dy}{z - ay} = \frac{2x\,dx}{x^2 - a}$$

Integrate the equation.

we get

$$\Rightarrow \log(z-ay) = \log(x^2-a) + \log b$$

$$\log(z-ay) = \log(x^2-a)b.$$

$$z-ay = (x^2-a)b$$

$$\boxed{z = (x^2-a)b + ay} \quad \text{—— (2)}$$

which is the complete integral, a & b being arbitrary constants.

Differentiating (2) partially w.r.t a and b, we get

$$\left.\begin{array}{l} 0 = y-b \Rightarrow y = b \\ 0 = x^2-a \Rightarrow x = \pm\sqrt{a} \Rightarrow a = x^2 \end{array}\right] \text{—(3)}$$

Substituting the values of a & b given by (3) in (2), we get

$$z = (x^2-x^2).y + x^2y$$

$$z = 0\cdot y + x^2y$$

$$\boxed{z = x^2y}$$

which is required singular integral.

Ques: 5(b) Solve $(D-D')(D+2D')z = (y+1)e^x$.

Solution:-

Given; $(D-D')(D+2D')z = (y+1)e^x$

put $D=m$ and $D'=1$

Hence; the auxillary equation is –

$$(m-1)(m+2) = 0$$

So that $m = 1, -2$

$\therefore$ C.F $= \phi_1(y+x) + \phi_2(y-2x)$

ϕ_1 and ϕ_2 being arbitrary functions

and P.I. $= \dfrac{1}{(D-D')(D+2D')}(y+1)e^x$

$$= \frac{1}{(D-D')}\left\{\frac{1}{D+2D'}(y+1)e^x\right\}$$

$$= \frac{1}{(D-D')}\int (c+2x+1)e^x\,dx$$

where $c = y-2x$

$$= \frac{1}{(D-D')}\left\{(c+2x+1)e^x - 2e^x\right\}$$

$$= \frac{1}{(D-D')}(y-1)e^x = \int (c'-x-1)e^x\,dx$$

where $c' = y+x \Rightarrow y = c'-x$

$$\Rightarrow (c'-x-1)e^x + e^x = ye^x \quad \text{as } c' = y+x.$$

$\boxed{\therefore \text{ General solution} = y = C.F + P.I = \phi_1(y+x) + \phi_2(y-2x) + ye^x}$

required solution

Ques:- 5(c)) The velocities of a car (running on a straight road) at intervals of 2 minutes are given below:-

Times in minutes	0	2	4	6	8	10	12
Velocity in km/hr	0	22	30	27	18	7	0

Applying Simpson's rule to find the distance covered by the car.

Solution:- We know velocity $v = \frac{ds}{dt}$ —— ①

where; s = distance
t = time.

① $\Rightarrow$ $ds = v\,dt$

$\therefore$ So distance covered by car in 12 min is

$$S = \int_0^S ds = \int_0^{12} v.dt \quad \text{—— ②}$$

Here given.

Time	$t_0=0$	$t_1=2$	$t_2=4$	$t_3=6$	$t_4=8$	$t_5=10$	$t_6=12$
Velocity	$V_0=0$	$V_1=22$	$V_2=30$	$V_3=27$	$V_4=18$	$V_5=7$	$V_6=0$

② $\Rightarrow$ $S = \int_0^{12} v\,dt$

Using Simpson's $\frac{1}{3}$ rule.

we get;

$$S = \frac{h}{3}\left[(V_0+V_6) + 4(V_1+V_3+V_5) + 2(V_2+V_4)\right]$$

$$S = \frac{h}{3}\left[(0+0) + 4(22+27+7) + 2(30+18)\right] \quad \text{—— ③}$$

Since; $\boxed{h = 2 \text{ min} = \frac{2}{60} = \frac{1}{30} \text{ hours}}$

(3) $\Rightarrow$ $S = \frac{1/30}{3}\left[4(56) + 2(48)\right]$

$$S = \frac{1}{90}\left[224 + 96\right] = \frac{1}{\cancel{90}}\left[32\cancel{0}\right]$$

$$\boxed{S = 3.556 \text{ km}}$$

Hence, distance covered by car is 3.556 km.

Ques:-5(d)) A committee of three approves proposal by majority vote. Each member can vote for the proposal by pressing a button at the side of their chairs. These three buttons are connected to a light bulb. For a proposal whenever the majority of votes takes place, a light bulb is turned on. Design a circuit as simple as possible so that the current passes and the light bulb is turned on only when the proposal is approved.

Solution:- Let, the member one represents $\rightarrow M_1$
Similarly for 2nd member $\rightarrow M_2$
and 3rd member $\rightarrow M_3$

and if a person approve it, then we'll represent $\rightarrow$ If as $-$ Y (yes)
If not $-$ N (No)

Now, using the truth table given below—

M_1	M_2	M_3	Result
N	N	N	N
N	N	Y	N
N	Y	N	N
N	Y	Y	Y - $\bar{A}BC$
Y	N	N	N
Y	N	Y	Y - $A\bar{B}C$
Y	Y	N.	Y - $AB\bar{C}$
Y	Y	Y	Y - ABC

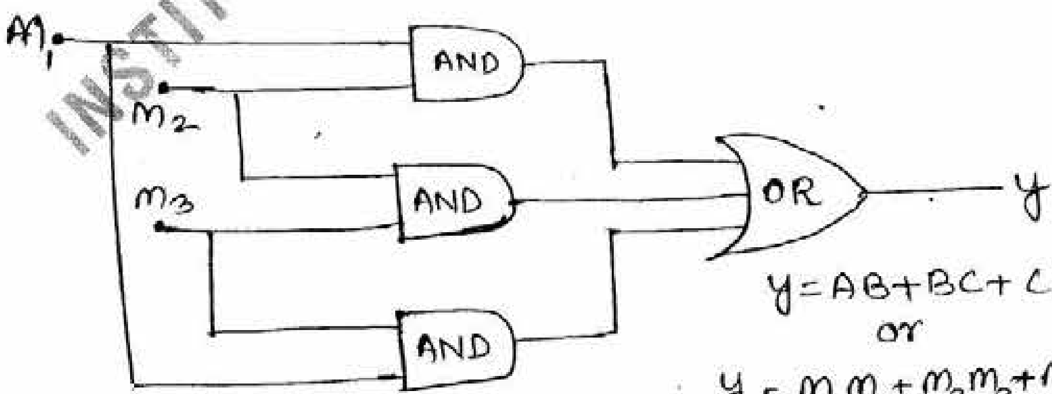
So solution is

$$y = \bar{A}BC + A\bar{B}C + AB\bar{C} + ABC$$

$$y = \bar{M}_1 M_2 M_3 + M_1 \bar{M}_2 M_3 + M_1 M_2 \bar{M}_3 + M_1 M_2 M_3$$

Simplifying we get; $\boxed{y = M_1 M_2 + M_2 M_3 + M_1 M_3}$

Its circuit diagram:-



$y = AB + BC + CA$
or
$y = M_1 M_2 + M_2 M_3 + M_3 M_1$

Ques:- 5(e)) Given $u = -Wy$, $v = Wx$, $w = 0$, show that the surfaces intersecting the streamlines orthogonally exist and are the planes through z-axis, although the velocity potential does not exist. Discuss the nature of flow.

Solution:-

Step-I) To show that liquid motion is possible we have to show that the equation of continuity-

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \text{ is satisfied.}$$

Here $u = -Wy \quad v = Wx \quad w = 0$

$$\frac{\partial u}{\partial x} = 0 \quad \frac{\partial v}{\partial y} = 0 \quad \frac{\partial w}{\partial z} = 0$$

Here $\therefore \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 + 0 + 0 = 0$

Hence, the result I.

Step-II) To show that the surface orthogonal to stream lines are planes through z-axis.

The required surface are solution of

$$u\,dx + v\,dy + w\,dz = 0$$

i.e $\quad -Wy\,dx + Wx\,dy + 0\,dz = 0$

or $\quad +\cancel{W}y\,dx = \cancel{+}\,\cancel{W}x\,dy$

$$\frac{dx}{x} = \frac{dy}{y} \Rightarrow \frac{dx}{x} - \frac{dy}{y} = 0$$

Integrating $\rightarrow \log x - \log y = \log a$

$$\Rightarrow \log \frac{x}{y} = \log a$$

$$\Rightarrow \frac{x}{y} = a$$

$$\Rightarrow \boxed{x = ay}$$

which is a plane through z-axis.

Step-III) To show that velocity potential ϕ does not exist.

By definition-

$$d\phi = -(u\,dx + v\,dy + w\,dz)$$

$$d\phi = -[-\omega y\,dx + \omega x\,dy + 0\,dz)$$

$$d\phi = \omega y\,dx - \omega x\,dy - 0\,dz$$

$$d\phi = \omega y\,dx - \omega x\,dy$$

$$d\phi = M\,dx + N\,dy \quad (\text{say}).$$

Here; $M = \omega y$ $\quad N = -\omega x$

$$\frac{\partial M}{\partial y} = \omega \quad ; \quad \frac{\partial N}{\partial x} = -\omega$$

Hence; $\dfrac{\partial M}{\partial y} \neq \dfrac{\partial N}{\partial x}$

Therefore, the equation is not exact so that $d\phi = \omega y\,dx - \omega x\,dy$ can be integrated so that ϕ does not exist.

Ques:- 6(a) (i) find a partial equation by eliminating a, b, c from $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Solution:-

Given; $$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad \text{--- (1)}$$

Differentiating (1) partially w.r.t x and y, we get

$$\frac{2x}{a^2}+\frac{2z}{c^2}\frac{\partial z}{\partial x}=0$$

$$\Rightarrow \quad c^2x + a^2z\frac{\partial z}{\partial x}=0 \quad \text{--- (2)}$$

$$\frac{2y}{b^2}+\frac{2x}{c^2}\frac{\partial z}{\partial y}=0$$

$$\Rightarrow \quad c^2y + b^2z\frac{\partial z}{\partial y}=0 \quad \text{--- (3)}$$

Differentiating (2) w.r.t x and (3) w.r.t y, we find;

$$c^2+a^2\left(\frac{\partial z}{\partial x}\right)^2+a^2z\frac{\partial^2 z}{\partial x^2}=0 \quad \text{--- (4)}$$

and; $$c^2+b^2\left(\frac{\partial z}{\partial x}\right)^2+b^2z\frac{\partial^2 z}{\partial x^2}=0 \quad \text{--- (5)}$$

from (2), $$\boxed{c^2=-\frac{a^2z}{x}\frac{\partial z}{\partial x}} \quad \text{--- (6)}$$

Putting this value of c^2 in (4) and dividing by a^2, we obtain

$$-\frac{z}{x}\frac{\partial z}{\partial x}+\left(\frac{\partial z}{\partial x}\right)^2+z\frac{\partial^2 z}{\partial x^2}=0$$

$$\Rightarrow \boxed{zx\frac{\partial^2 z}{\partial x^2}+x\left(\frac{\partial z}{\partial x}\right)^2-z\frac{\partial z}{\partial x}=0} \quad \text{--- (7)}$$

Similarly, from ③ and ⑤,

$$\boxed{zy\frac{\partial^2 z}{\partial y^2} + y\left(\frac{\partial z}{\partial y}\right)^2 - z\frac{\partial z}{\partial y} = 0} \quad \text{———} \quad 8$$

⑦ and ⑧ are two possible forms of the required equations.

Ques: 6(a)(ii) Find the integral surface of the linear partial differential equations

$x(y^2+z)p - y(x^2+z)q = (x^2-y^2)z$ which contains the straight line $x+y=0$ and $z=1$.

Solution:

given; $x(y^2+z)p - y(x^2+z)q = (x^2-y^2)z$ —— (1)

which contains the straight line

$x+y=0$ and $z=1$.

Lagrange's auxillary equations of (1) are.

$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{(x^2-y^2)z} \quad \text{—— (2)}$$

Now, for multipliers.

$$\frac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{y^2+\cancel{z}-\cancel{x^2}-\cancel{z}+\cancel{x^2}-y^2} \Rightarrow \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

$\Rightarrow \log x + \log y + \log z = \log c_1$ [By integrating]

$\Rightarrow \log(xyz) = \log c_1$

$$\boxed{xyz = c_1} \quad \text{—— (3)}$$

$$\frac{x\,dx + y\,dy - dz}{x^2(y^2+z) - y^2(x^2+z) - z(x^2-y^2)} =$$

$$= \frac{x\,dx + y\,dy - dz}{\cancel{x^2y^2} + \cancel{x^2z} - \cancel{x^2y^2} - \cancel{y^2z} - \cancel{x^2z} + \cancel{y^2z}}$$

$\Rightarrow x\,dx + y\,dy - dz = 0$

$\Rightarrow \frac{x^2+y^2}{2} - z = c_2 \Rightarrow \boxed{x^2+y^2-2z = c_2}$ —— (4)

Taking 't' as parameter, the given equation of the straight line $x+y=0$ and $z=1$ can be put in parameteric form.

$$x=t\ ,\ y=-t\ ,\ z=1 \quad \text{——} \quad (5)$$

Using (3), (4) and (5) may be re-written as.

$$-t^2 = c_1\ ,\quad 2t^2-2=c_2$$

$\Rightarrow$ These gives

$$2(-c_1)-2=c_2$$

$$\Rightarrow \boxed{2c_1+c_2+2=0} \quad \text{——} \quad (6)$$

Putting values of c_1 and c_2 from (3), (4) and (6), the desired integral surface is

$$\boxed{2xyz+x^2+y^2-2z+2=0}$$

required solution

Ques: 6(b)) Find the characteristic strips of the equation $xp + yq - pq = 0$ and then find the equation of the integral surface through the curve $z = x/2$, $y = 0$.

Solution:- Here $f(x, y, z, p, q) = xp + yq - pq = 0$ —— (1)

The integral surface passes through the curve $z = x/2$, $y = 0$, whose parametric equation can be written as →

$$x = f_1(\lambda) = \lambda \; ; \; y = f_2(\lambda) = 0 \; ; \; z = f_3(\lambda) = \lambda/2$$

λ being a parameter

∴ Initial values for x, y, z are $x = x_0 = \lambda$, $y = y_0 = 0$, $z = z_0 = \lambda/2$, when $t = 0$

And corresponding initial values p_0 and q_0 of p and q are determined by the relations.

$$f_3'(\lambda) = p_0 f_1'(\lambda) + q_0 f_2'(\lambda) \Rightarrow \frac{1}{2} = p_0 \cdot 1 + q_0 \cdot 0 \Rightarrow p_0 = \frac{1}{2}$$

and $f\{f_1(\lambda), f_2(\lambda), f_3(\lambda), p_0, q_0) = 0$

$$\Rightarrow x_0 p_0 + y_0 q_0 - p_0 q_0 = 0 \qquad \Rightarrow q_0 = \lambda.$$

The characteristic equations of the given partial Differential equation (1) are -

$$\frac{dx}{dt} = \frac{\partial f}{\partial p} = x - q \quad —— (2)$$

$$\frac{dy}{dt} = \frac{\partial f}{\partial q} = y - p \quad —— (3)$$

$$\frac{dz}{dt} = p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q} = p(x-q) + q(y-p) = px + qy - 2pq = -pq$$

using (1) —(4)

$$\frac{dp}{dt} = -\frac{\partial f}{\partial x} - p\frac{\partial f}{\partial z} = -p \quad —— (5)$$

$$\frac{dq}{dt} = -\frac{\partial f}{\partial y} - q\frac{\partial f}{\partial z} = -q \quad —— (6)$$

from (5) and (6), we get $p = Ae^{-t}$ and $q = Be^{-t}$

But initially, when $t=0$, $p = p_0 = \frac{1}{2}$ and $q = q_0 = \lambda$

$\therefore$ $A = p_0 = \frac{1}{2}$ and $B = q_0 = \lambda$

$\Rightarrow$ $\boxed{p = \frac{1}{2}e^{-t} \text{ and } q = \lambda e^{-t}}$ ——— (7)

using (7), from (2), we have $\frac{dx}{dt} - x = -\lambda e^{-t}$

which is a L.D.E, with I.F $= e^{\int -dt} = e^{-t}$.

$\therefore$ $x \cdot e^{-t} = C_1 + \int (-\lambda e^{-t})(e^{-t})\,dt = C_1 - \lambda \int e^{-2t}\,dt$

$= C_1 + \frac{1}{2}\lambda e^{-2t}$

But, when $t=0$, $x = x_0 = \lambda$

$\therefore$ $x_0 = \lambda = C_1 + \frac{1}{2}\lambda \Rightarrow \boxed{C_1 = \lambda/2}$

$\therefore$ $xe^{-t} = \frac{\lambda}{2} + \frac{\lambda}{2}e^{-2t}$

$\Rightarrow$ $\boxed{x = \frac{\lambda}{2}(1+e^{-2t})e^{t}}$ ——— (8)

Again using (7) from (3), we have $\Rightarrow \frac{dy}{dt} - y = -\frac{1}{2}e^{-t}$

which is L.D.E with I.F $= e^{\int -dt} = e^{-t}$

$\therefore$ $ye^{-t} = C_2 + \int (-\frac{1}{2}e^{-t})e^{-t}\,dt = C_2 - \frac{1}{2}\int e^{-2t}\,dt = C_2 + \frac{1}{4}e^{-2t}$

But when $t=0$, $y = y_0 = 0$, $\therefore$ $y_0 = 0 = C_2 + \frac{1}{4} \Rightarrow C_2 = -\frac{1}{4}$

$\therefore$ $ye^{-t} = -\frac{1}{4} + \frac{1}{4}e^{-2t} \Rightarrow \boxed{y = \frac{1}{4}(e^{-2t} - 1)e^{t}}$ ——— (9)

Now using (7), from (4), we have

$\frac{dz}{dt} = -\frac{\lambda}{2}e^{-2t}$

Integrating, we get

$\boxed{z = \frac{\lambda}{4}e^{-2t} + C_3}$

But, when $t = 0$

$z = z_0 = \lambda/2$

$\therefore z_0 = \frac{\lambda}{2} = \frac{\lambda}{4} + c_3 \Rightarrow \boxed{c_3 = \frac{\lambda}{4}}$

$\therefore z = \frac{\lambda}{4} e^{-2t} + \frac{\lambda}{4} \Rightarrow \boxed{z = \frac{\lambda}{4}(1 + e^{-2t})}$ —— (10)

Thus, the characteristic strips of the given equation are given by-

$x = \frac{\lambda}{2}(1 + e^{-2t})e^{t}$

$y = \frac{1}{4}(e^{-2t} - 1)e^{t}$ and

$z = \frac{\lambda}{4}(1 + e^{-2t})$

where, λ and t are two parameter.

The required integral surface is obtained by eliminating λ and t between x, y and z.

we have,

$\frac{x}{z} = 2e^{t} \Rightarrow e^{t} = \frac{x}{2z}$

$\therefore y = \frac{1}{4}(e^{-t} - e^{t}) = \frac{1}{4}\left(\frac{2z}{x} - \frac{x}{2z}\right)$

$\boxed{y = \frac{4z^2 - x^2}{8xz}}$

$\Rightarrow \boxed{4z^2 = x^2 + 8xyz}$

which is the required Integral Surface.

Ques: 6(c) A string of length 'l' has its ends $x=0$ and $x=l$ fixed. It is released from rest in the position $y=\{4\lambda x(l-x)\}/l^2$. Find the expression for the displacement of the string at any subsequent time.

Solve:- The displacement function $y(x,t)$ is the solution of the wave equation.

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\left(\partial^2 y/\partial t^2\right)} \quad \text{——} \; ①$$

subject to boundary conditions:

$$y(0,t) = y(l,t) = 0 \text{ for all } t \geq 0 \quad \text{——} \; ②$$

and initial condition, namely,

$$\text{initial velocity} = \left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \quad \text{for } 0 \leq x \leq l \text{ —} \; (3a)$$

and initial displacement $= y(x,0) = f(x)$

$$= [4\lambda x(l-x)]/l^2 \quad \text{——} \; (3b)$$

The solution of ① satisfying the above boundary and initial conditions is

$$y(x,t) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l} \cdot \cos\frac{n\pi ct}{l} \quad \text{——} \; ④$$

where; $\quad E_n = \dfrac{2}{l}\displaystyle\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx \quad$ —— ⑤

Substituting the value of $f(x)$, given by ③ in ⑤, we get

$$E_n = \left(\frac{2}{l}\right)\times\left(\frac{4\lambda}{l^2}\right)\int_0^l (lx - x^2)\sin\frac{n\pi x}{l}\,dx$$

$$E_n = \frac{8\lambda}{l^3}\left[(lx-x^2)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right)-(l-2x)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right) + (-2)\left[\frac{l^3}{n^3\pi^3}\cos\frac{n\pi x}{l}\right]\right]_0^l$$

$$E_n = \frac{8\lambda}{l^3}\left[-\frac{2l^3(-1)^n}{n^3\pi^3} + 2\cdot\frac{l^3}{n^3\pi^3}\right]$$

$$E_n = \frac{16\lambda}{n^3\pi^3}\left[1-(-1)^n\right] \quad ; \text{ as } \left[\cos n\pi = (-1)^n\right]$$

$$E_n = \begin{cases} 0 & , \text{ if } n = 2m \text{ (even) with } m = 1,2,3\ldots \\ \dfrac{32\lambda}{(2m-1)^3\pi^3} & ; \text{ if } n = 2m-1 \text{ (odd) with } m = 1,2,3\ldots \end{cases}$$

Substituting the value of E_n in ④, the required is

$$\boxed{y(x,t) = \frac{32\lambda}{\pi^3}\sum_{m=1}^{\infty}\frac{1}{(2m-1)^3}\sin\frac{(2m-1)\pi x}{l}\cos\frac{(2m-1)\pi ct}{l}}$$

Ques: 7(a) The equation $x^2+ax+b=0$ has two real roots α and β. Show that the iteration method $x_{k+1} = \frac{-(ax_k+b)}{x_k}$ is convergent near $x=\alpha$ if $|\alpha|>|\beta|$ and that $x_{k+1} = \frac{-b}{x_k+a}$ is convergent near $x=\alpha$ if $|\alpha|<|\beta|$. Show that iteration method $x_{k+1} = \frac{-(x_k^2+b)}{a}$ is convergent near $x=\alpha$ if $2|\alpha|<|\alpha+\beta|$?

Solution:- The iterations are given by

$$x_{k+1} = \frac{-(ax_k+b)}{x_k} = g(x_k) \text{ (say)}$$
$$k=0,1,2,\ldots$$

But the known theorem

If $g(x)$ and $g'(x)$ are continuous in an interval about a root α of the equation $x=g(x)$ and if $|g'(x)|<1$ for all x in the interval, then the successive approximations $x_1, x_2, \ldots$ given by

$$x_k = g(x_{k-1}) \quad , k=1,2,3\ldots$$

converges to the root α provided that the initial approximation x_0 is chosen in the interval.

$\therefore$ These iterations converges to α if

$$|g'(x)|<1 \quad \text{near } \alpha$$

i.e $|g'(x)| = \left|\frac{-b}{x^2}\right| < 1$

Note that $g'(x)$ is continuous near α.

If the iterations converge to $x=\alpha$, then we require $|g'(\alpha)| = \left|\frac{-b}{\alpha^2}\right| < 1.$

Thus; $|b| < |\alpha|^2$ i.e. $|\alpha|^2 > |b|$ —— (1)

Given, that α and β are roots of the equation
$$x^2 + ax + b = 0$$
then $\alpha + \beta = -a$ and $\alpha\beta = b \Rightarrow |b| = |\alpha||\beta|$ —— (2)

Substituting (2) in (1), we get
$$|\alpha|^2 > |b| = |\alpha||\beta|$$
$$\Rightarrow |\alpha|^2 > |\alpha||\beta|$$
$$\Rightarrow \boxed{|\alpha| > |\beta|}$$

Now, if $x = \dfrac{-b}{x+a}$

The iteration $x_{k+1} = \dfrac{-b}{x_k + a} = g(x_k)$ (say)
converges to α, if
$$|g'(x)| = \left|\frac{b}{(x+a)^2}\right| < 1 \text{ in a interval containing } \alpha.$$

In particular we require
$$|g'(\alpha)| = \left|\frac{b}{(\alpha+a)^2}\right| < 1.$$
$$\Rightarrow |(\alpha+a)^2| > |b|$$

But we have $\alpha + \beta = -a$ & $\alpha\beta = b$
$$\Rightarrow \beta^2 > |b| = |\alpha||\beta|$$
$$\Rightarrow |\beta^2| > |\alpha||\beta|$$
$$\Rightarrow \boxed{|\beta| > |\alpha|}$$

$\therefore x_{k+1} = \dfrac{-b}{x_k + a}$ is convergent near $x = \alpha$
if $|\beta| > |\alpha|$.

Ques: 7(b)) Using Newton's forward formula, estimate the number of persons earning wages between Rs. 60 and Rs. 70 from the following data:

Wages (Rs.)	Below 40	40-60	60-80	80-100	100-120
No. of persons (in thousands)	250	120	100	70	50

Solution:-

From the above given data:-

Newton's forward table:-

Wages x	No. of persons y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	250				
		120			
60	370		-20		
		100		-10	
80	470		-30		20
		70		+10	
100	540		-20		
		50			
120	590				

Newton forward Interpolation formulae

$$y(x_0+nh) = y(x_0) + nC_1 \Delta y_0 + \frac{n(n-1)}{2!} \Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!} \Delta^3 y_0 + \frac{n(n-1)(n-2)(n-3)}{4!} . \Delta^4 y_0 \quad \text{--- (1)}$$

$x_0 = 40$, $h = 20$

$\therefore$ $70 = 40 + n \times 20$ $\Rightarrow$ $\boxed{n = \frac{3}{2}}$

$$\therefore y(70) = 250 + \frac{3}{2}\times 120 + \frac{3/2(3/2-1)}{2}\times(-20)$$

$$+ \frac{3/2(3/2-1)(3/2-2)}{6}\times(-10) + \frac{3/2(3/2-1)(3/2-2)(3/2-3)}{24}\times 20$$

$$y(70) = 250 + 180 + (-7.5) + 0.625 + 0.46875$$

$$\boxed{y(70) = 423.59375}$$

$$\because \boxed{y(60) = 370}$$

$\therefore$ No. of persons with wages b/w Rs. 60 and Rs. 70 are:

$$= (423.59375 - 370)\times 1000$$

$$= 53.59375 \times 1000$$

$$= 53593.75 \simeq \underline{53593}$$

Ques-7(c)) Solve the initial value problem

$\frac{dy}{dx} = x(y-x)$, $y(2) = 3$ in the interval $[2, 2.4]$.

Using the Runge-kutta fourth-order method with step size $h = 0.2$.

Solution:- $\frac{dy}{dx} = x(y-x)$

$x_0 = 2,\ y_0 = 3,\ h = 0.2$

RK. 4th Order:-

$$y_{i+1} = y_i + \frac{1}{6}[K_1 + 2(K_2+K_3) + K_4]$$

$$K_1 = hf(x_i, y_i)$$

$$K_2 = hf\left(x_i+\frac{h}{2}, y_i+\frac{K_1}{2}\right)$$

$$K_3 = hf\left(x_i+\frac{h}{2}, y_i+\frac{K_2}{2}\right)$$

$$K_4 = hf(x_i+h, y_i+K_3)$$

Ist Iteration:-

$$y_1 = y_0 + \frac{1}{6}[K_1+K_4+2(K_2+K_3)]$$

$$K_1 = hf(x_0, y_0) = 0.2f(2,3) = 0.4$$

$$K_2 = hf\left(x_0+\frac{h}{2}, y_0+\frac{K_1}{2}\right) = 0.2f(2.1, 3.2) = 0.462$$

$$K_3 = hf\left(x_0+\frac{h}{2}, y_0+\frac{K_2}{2}\right) = 0.2f(2.1, 3.231) = 0.47502$$

$$K_4 = hf(x_0+h, y_0+K_3) = 0.3f(2.2, 3.47502) = 0.56101$$

$$\therefore y_1 = y_0 + \frac{1}{6}[K_1+K_4+2(K_2+K_3)]$$

$$y_1 = 3 + \frac{1}{6}[0.4 + 0.56101 + 2(0.462 + 0.47502)]$$

$$\boxed{y_1 = 3 + 0.47251 = 3.47251}$$

IInd Iteration:

at $x_1 = 2.2$; $y_1 = 3.47251$, $h = 0.2$

$$y_2 = y_1 + \frac{1}{6}\left[K_1 + K_4 + 2(K_2 + K_3)\right]$$

$$K_1 = hf(x_1, y_1) = 0.2 f(2.2, 3.47251) = 0.5599$$

$$K_2 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}\right) = 0.2 f(2.3, 3.75246)$$

$$K_2 = 0.66813$$

$$K_3 = hf\left(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}\right) = 0.2 f(2.3, 3.80658)$$

$$K_3 = 0.69303$$

$$K_4 = hf(x_1 + h, y_1 + K_3) = 0.2 f(2.4, 4.16554)$$

$$K_4 = 0.84746$$

$$y_2 = y_1 + \frac{1}{6}\left[K_1 + K_4 + 2(K_2 + K_3)\right]$$

$$y_2 = 3.47251 + \frac{1}{6}\left[0.5599 + 0.84746 + 2(0.66813 + 0.69303)\right]$$

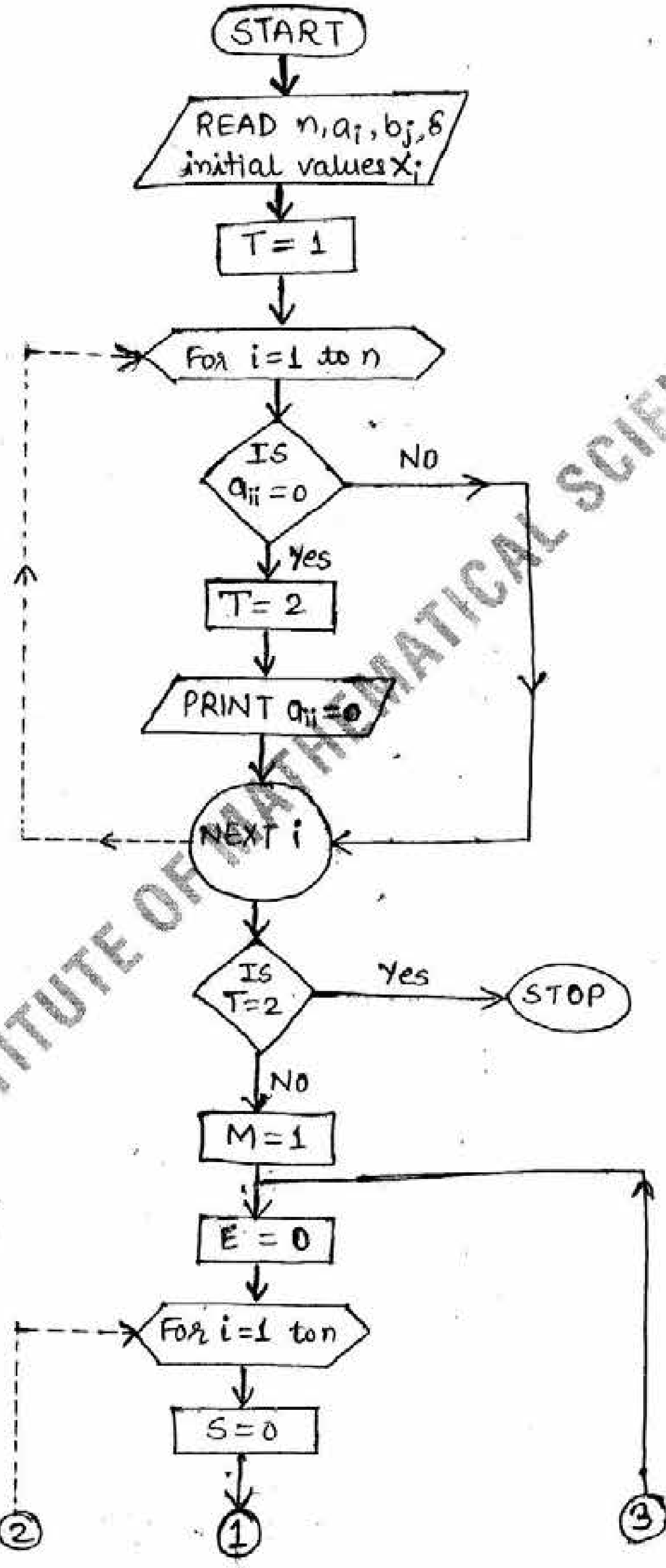
$$y_2 = 3.47251 + \frac{1}{6}\left[4.12968\right]$$

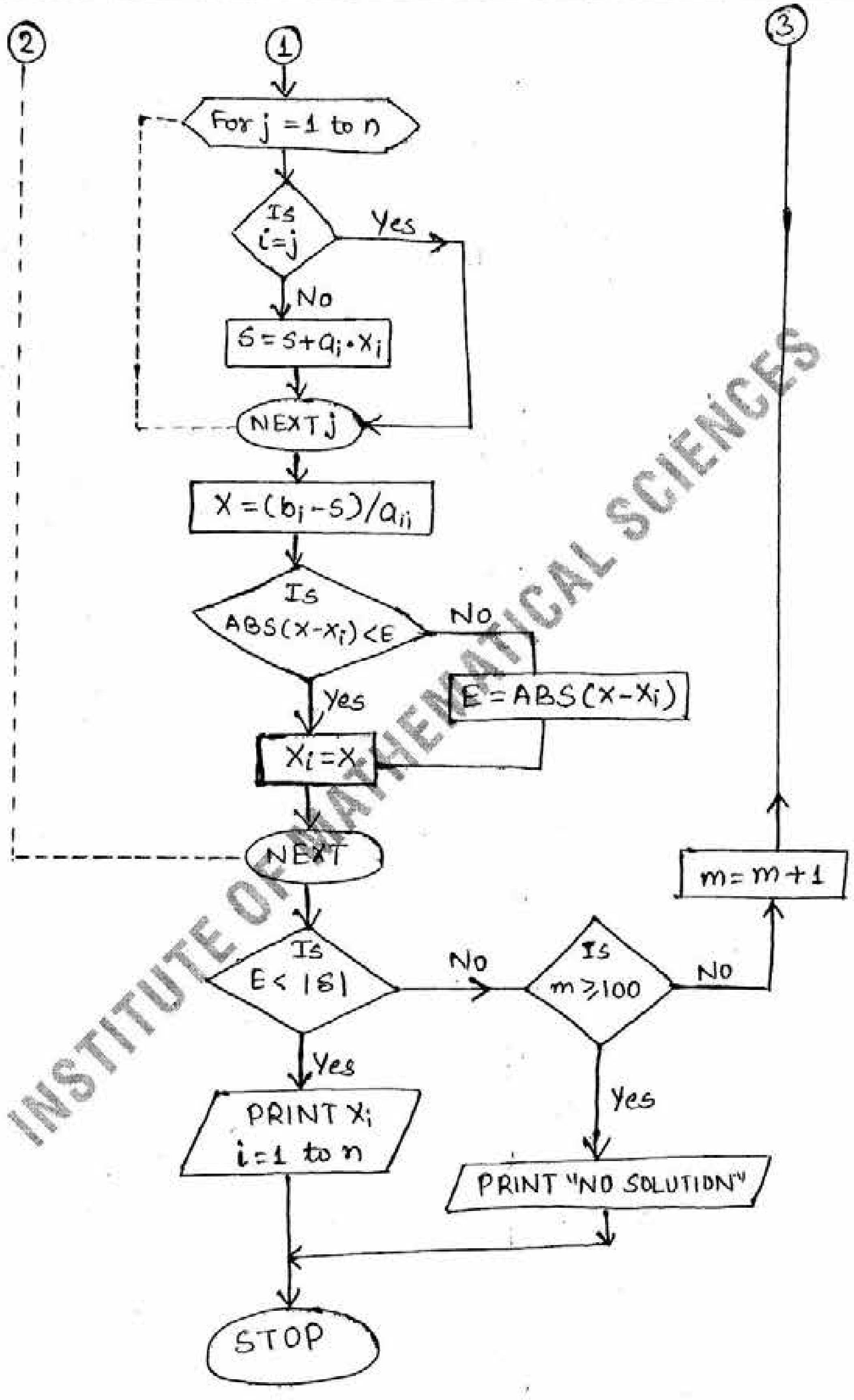
$$\boxed{y_2 = 4.16079}$$

$\boxed{\therefore \text{ At } x = 2.4,\ y = 4.16079}$ required solution.

Ques: 7 (d) } Draw a flow chart of Gauss-Seidel method.

Solve:-

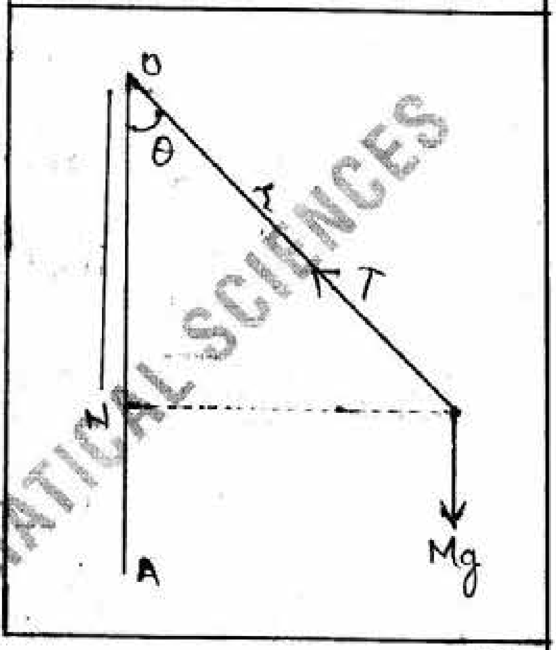




2
1
3
For j = 1 to n
Is i=j
Yes
No
S = S+a_i·x_i
NEXT j
X = (b_i - S)/a_ii
Is ABS(X-x_i) < E
No
E = ABS(X-x_i)
Yes
x_i = X
NEXT
Is E < |δ|
No
Is m ≥ 100
No
m = m+1
Yes
PRINT x_i i=1 to n
Yes
PRINT "NO SOLUTION"
STOP

Ques:-8(a) Use Hamilton's equations to write down the equations of motion of a pendulum bob suspended from a coil spril and allowed to swing in a vertical plane.

Solution:- At time 't', let 'r' be the stretched of length the spring to the vertical at this time 't', then then velocities of M at P $\dot{r}$ and $r\dot{\theta}$ along and perpendicular to OP respectively.



$\therefore$ Velocity of M at P

$$= \dot{r}^2 + (r\dot{\theta})^2$$

Thus, if T and V are the kinetic and the potential energies of the system, at time 't', then

$$T = \frac{1}{2}M(\dot{r}^2 + r^2\dot{\theta}^2)$$

and V = Workdone against the forces

V = workdone against the force Mg + Workdone against the tension in the string.

$$V = -Mg.r\cos\theta \int_0^r \lambda\left(\frac{r-r_0}{r_0}\right)dr \quad \left[\because \text{Tension} = \lambda\left(\frac{r-r_0}{r_0}\right)\right]$$

λ is the modulus of elasticity of the spring

$$V = -Mgr\cos\theta + (K/2)(r-r_0)^2 \quad , \text{ where } [k = \lambda/r_0]$$

$$\therefore L = T - V = \frac{1}{2}M(\dot{r}^2 + r^2\dot{\theta}^2) + Mgr\cos\theta - (K/2)(r-r_0)^2$$

Here r and θ are the generalised co-ordinates.

$$\boxed{\therefore\ p_r = \frac{\partial L}{\partial \dot{r}} = M\dot{r} \quad \text{and} \quad p_\theta = \frac{\partial L}{\partial \dot{\theta}} = Mr^2\dot{\theta}} \quad \text{—— (1)}$$

Since, L does not contain t explicitly.

$$\therefore\ H = T+V = \frac{1}{2}M(\dot{r}^2 + r^2\dot{\theta}^2) - Mgr\cos\theta + \left(\frac{K}{2}\right)(r-r_0)^2$$

Substituting $\dot{r} = p_r/M$ and $\dot{\theta} = p_\theta/Mr^2$ (from (1))

$$\boxed{H = \left(\frac{1}{2M}\right)\left[p_r^2 + \frac{p_\theta^2}{r^2}\right] + mgr\cos\theta + \left(\frac{K}{2}\right)(r-r_0)^2} \quad \text{—— (2)}$$

Hence, the four Hamilton's equations are :-

$$\dot{p}_r = -\frac{\partial H}{\partial r} \quad \text{i.e} \Rightarrow \dot{p}_r = \frac{p_\theta^2}{Mr^3} + Mg\cos\theta - K(r-r_0) \quad \text{—— } (H_1)$$

$$\dot{r} = \frac{\partial H}{\partial p_r} \quad \text{i.e} \Rightarrow \dot{r} = \frac{p_r}{M} \quad \text{—— } (H_2)$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} \quad \text{i.e} \Rightarrow \dot{p}_\theta = -Mgr\sin\theta \quad \text{—— } (H_3)$$

$$\text{and} \quad \dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{Mr^2} \quad \text{—— } (H_4)$$

Differentiating (H_2), we have

$$\ddot{r} = \frac{1}{M}\dot{p}_r \quad \text{or} \quad M\ddot{r} = \dot{p}_r = \frac{p_\theta^2}{Mr^3} + Mg\cos\theta - K(r-r_0)$$

[from (H_1)]

$$\therefore\ M\ddot{r} = \frac{(Mr^2\dot{\theta})^2}{Mr^3} + Mg\cos\theta - K(r-r_0)$$

[Substituting from H_4]

$$\text{or;} \quad \boxed{M\ddot{r} - MH_2r\dot{\theta}^2 - Mg\cos\theta + K(r-r_0) = 0} \quad \text{—— (3)}$$

Also, from (H_2), we have

$$Mr^2\dot{\theta} = \dot{P}_\theta$$

Differentiating —

$$M(r^2\ddot{\theta} + 2r\dot{r}\dot{\theta}) = \dot{P}_\theta$$

or, $M(r^2\ddot{\theta} + 2\dot{r}\dot{\theta}r) = -Mgr\sin\theta$

$$\Rightarrow \cancel{M}\cancel{r}(r\ddot{\theta} + 2\dot{r}\dot{\theta}) = -\cancel{M}g\cancel{r}\sin\theta$$

$$\Rightarrow r\ddot{\theta} + 2\dot{r}\dot{\theta} = -g\sin\theta$$

$$\Rightarrow \boxed{r\ddot{\theta} + 2\dot{r}\dot{\theta} + g\sin\theta = 0} \quad \text{—} (4)$$

$\therefore$ Equations (3) and (4) are required equations of motion.

Ques: 8(b)) A uniform lamina is bounded by a parabolic arc, of latus rectum $4a$, and a double ordinate at a distance b from the vertex. If $b = \frac{1}{3}a(7+4\sqrt{7})$, show that two of the principal axes at the end of a latus rectum are the tangent and normal there.

Solution:- Let, the equation of the parabola be

$$\boxed{y^2 = 4ax} \quad \text{——— (1)}$$

$\therefore$ Coordinates of the end L of L.R. LL' are $(a, 2a)$.

Differentiating (1), we get

$$\boxed{\frac{dy}{dx} = \frac{2a}{y}}$$

$\therefore$ At $L(a, 2a)$, $\frac{dy}{dx} = \frac{2a}{2a} = 1$.

$\therefore$ Equation of the tangent LT at L is

$$y - 2a = 1(x-a) \quad \text{or} \quad \boxed{y - x - a = 0} \quad \text{——— (2)}$$

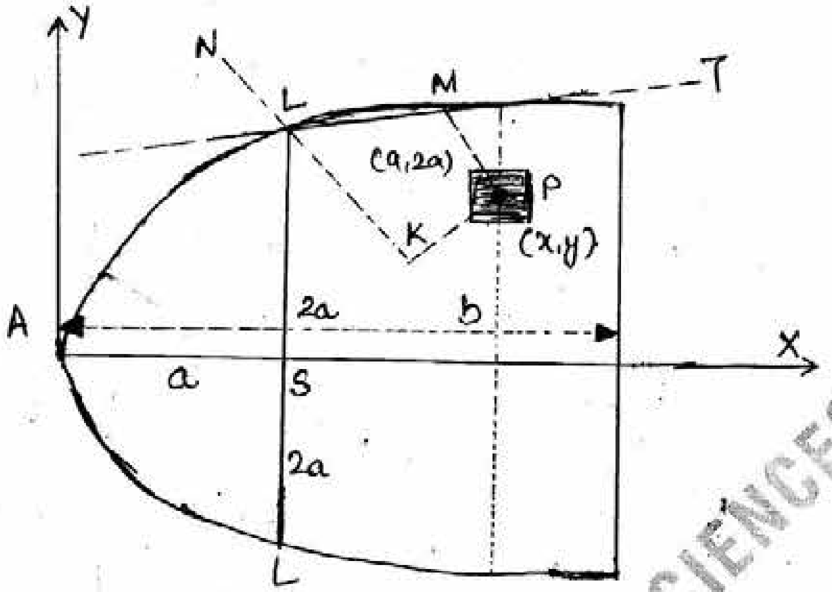
and the equation of the normal LN at L is

$$y - 2a = -\frac{1}{1}(x-a).$$

$$\Rightarrow \boxed{y + x - 3a = 0} \quad \text{——— (3)}$$

Consider an element $\delta x \delta y$ at the point $P(x,y)$ of the lamina, then

PM = length of Perpendicular from P on Tangent LT [given by (2)]



$$\Rightarrow \frac{y-x-a}{\sqrt{1+1}} = \frac{y-x-a}{\sqrt{2}}$$

and PK = length of the perpendicular from P on the normal LN given by (3)

$$\Rightarrow \frac{y+x-3a}{\sqrt{2}}$$

P.I. of the element about LT and LN.

$$PM.P.K.\delta m = \left(\frac{y-x-a}{\sqrt{2}}\right)\left(\frac{y+x-3a}{\sqrt{2}}\right)\rho\,\delta x\,\delta y$$

If the tangent and normal at L are the principal axes, then P.I. of the lamina about these will be zero.

i.e. P.I. of Lamina about LT and LN.

$$= \int_{x=0}^{b}\int_{y=-2\sqrt{ax}}^{2\sqrt{ax}} \left[\frac{y-x-a}{\sqrt{2}}\right]\left[\frac{y+x-3a}{\sqrt{2}}\right]\rho\,dx\,dy = 0$$

$$= \frac{\rho}{2}\int_0^b \int_{-2\sqrt{ax}}^{2\sqrt{ax}} (y^2 - 4ay + (3a^2 + 2ax - x^2))]\,dx\,dy = 0$$

$$= \frac{\rho}{2}\int_0^b \left[\frac{1}{3}y^3 - 2ay^2 + (3a^2 + 2ax - x^2)y\right]_{-2\sqrt{ax}}^{2\sqrt{ax}} dx = 0$$

$$= \frac{\rho \cdot \cancel{2}}{\cancel{2}}\int_0^b \left[\frac{8}{3}ax\sqrt{ax} + 2(3a^2 + 2ax - x^2)\sqrt{ax}\right] dx = 0$$

$$= \int_0^b \left[\frac{8}{3}a^{3/2}x^{3/2} + 6a^{5/2}x^{1/2} + 4a^{3/2}x^{3/2} - 2a^{1/2}x^{5/2}\right] dx = 0$$

$$= \left[\frac{16}{15}a^{3/2}b^{5/2} + 4a^{5/2}b^{3/2} + \frac{8}{5}a^{3/2}b^{5/2} - \frac{4}{7}a^{1/2}\cdot b^{7/2}\right] = 0$$

$$= \frac{16}{15}ab + 4a^2 + \frac{8}{5}ab - \frac{4}{7}b^2 = 0$$

$$\Rightarrow b^2 - \frac{14}{3}ab - 7a^2 = 0$$

$$\Rightarrow b = \frac{\frac{14}{3}a \pm \sqrt{\left(\frac{196}{9}\right)a^2 + 28a^2}}{2} = \frac{1}{2}\left(\frac{14}{3} \pm \frac{8}{3}\sqrt{7}\right)a.$$

$$\Rightarrow \boxed{b = \frac{a}{3}(7 + 4\sqrt{7})}$$

Leaving -ve sign, as b cannot be negative

Hence, if $b = \frac{a}{3}(7 + 4\sqrt{7})$

then, the principal axes at L are the tangents and normal there.

Ques: 8(c) Show that $\phi = x f(r)$ is a possible form for the velocity potential for an incompressible fluid motion. If the fluid velocity $\vec{q} \to 0$ as $r \to \infty$, find the surfaces of constant speed.

Solution:- given; $\phi = x f(r)$ —— ①

$$q = -\nabla\phi = -\nabla[x f(r)]$$

$$q = -[f(r)\nabla x + x\nabla f(r)] \quad \text{—— ②}$$

Now, $r^2 = x^2 + y^2 + z^2$

$$\Rightarrow 2r\left(\frac{\partial r}{\partial x}\right) = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} \quad \text{—— ③}$$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$ —— ④

Also; $\nabla x = \left[i\left(\frac{\partial}{\partial x}\right) + j\left(\frac{\partial}{\partial y}\right) + k\left(\frac{\partial}{\partial z}\right)\right] x = i$

and $\nabla f(r) = \left[i\left(\frac{\partial}{\partial x}\right) + j\left(\frac{\partial}{\partial y}\right) + k\left(\frac{\partial}{\partial z}\right)\right] f(r)$

$$\nabla f(r) = i f'(r)\left(\partial r/\partial x\right) + j f'(r)\left(\partial r/\partial y\right) + \hat{k} f'(r)\left(\frac{\partial r}{\partial z}\right)$$

$$\nabla f(r) = \hat{i} f'(r)\cdot\left(\frac{x}{r}\right) + \hat{j} f'(r)\left(y/r\right) + \hat{k} f'(r)\left(\frac{z}{r}\right)$$

$$\nabla f(r) = \frac{1}{r} f'(r)\left(\hat{i}x + \hat{j}y + \hat{k}z\right) = \frac{1}{r}\cdot f'(r)\cdot\vec{r}$$

$$\therefore ② \Rightarrow q = -f(r)\,i - \left(\frac{x}{r}\right) f'(r)\cdot\vec{r} \quad \text{—— ⑤}$$

For a possible motion of an incompressible fluid, we have —

$$\nabla \cdot q = 0 \quad \text{or} \quad \nabla \cdot (-\nabla\phi) = 0$$

$$\Rightarrow \nabla^2 \phi = 0$$

$$\Rightarrow \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right][x f(r)] = 0 \quad , (\text{using } ①) \quad \text{——} ⑥$$

Now, $\dfrac{\partial^2}{\partial x^2}[x f(r)] = \dfrac{\partial}{\partial x}\left[\dfrac{\partial}{\partial x}\{x f(r)\}\right]$

$$= \frac{\partial}{\partial x}\left[f(r) + x\frac{\partial f(r)}{\partial x}\right]$$

$$\therefore \frac{\partial^2}{\partial x^2}[x f(r)] = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial x} + x\frac{\partial^2 f(r)}{\partial x^2} = 2\frac{\partial f}{\partial x} + x\frac{\partial^2 f(r)}{\partial x^2}.$$

Also;
$$\frac{\partial^2}{\partial y^2}[x f(r)] = x\frac{\partial^2 f}{\partial y^2}$$
$$\frac{\partial^2}{\partial z^2}[x f(r)] = x\frac{\partial^2 f}{\partial z^2}$$

$\therefore$ ⑥ becomes

$$2\frac{\partial f}{\partial x} + x\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) = 0 \quad \text{——} ⑦$$

Now, $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial r} \cdot \dfrac{\partial r}{\partial x} = f'(x/r)$ — using ③

and $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial}{\partial x}\left(f'(x/r)\right) = \dfrac{f'}{r} + x\dfrac{\partial}{\partial x}\left(\dfrac{f'}{r}\right)$ — ⑧

$$= \frac{f'}{r} + x\frac{\partial}{\partial r}\left(\frac{f'}{r}\right)\frac{\partial r}{\partial x} = \frac{f'}{r} + x\left(\frac{\partial f'' - f'}{r^2}\right)\frac{x}{r}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{f'}{r} + \frac{x^2}{r^2}f'' - \frac{x^2}{r^3}f' \quad \text{---} (9)$$

Similarly,

$$\frac{\partial^2 f}{\partial y^2} = \frac{f'}{r} + \frac{y^2}{r^2}f'' - \frac{y^2}{r^3}f' \quad \text{---} (10)$$

and $$\frac{\partial^2 f}{\partial z^2} = \frac{f'}{r} + \frac{z^2}{r^2}f'' - \frac{z^2}{r^3}f' \quad \text{---} (11)$$

Adding (9), (10) and (11), we get:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{3f'}{r} + \left[\frac{x^2+y^2+z^2}{r^2}\right]f'' - \left(\frac{x^2+y^2+z^2}{r^3}\right)f'$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{3f'}{r} + \frac{r^2}{r^2}f'' - \frac{1}{r}\cdot f'$$

$$\boxed{\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{3f'}{r} + f'' - \frac{1}{r}f' = \frac{2}{r}f' + f''} \quad \text{---} (12)$$

Using (8) and (12), (7) reduces to

$$\frac{2f'(x)}{r} + x\left(\frac{2f'}{r} + f''\right) = 0$$

$$\Rightarrow f'' + \frac{4f'}{r} = 0$$

$$\Rightarrow \boxed{\frac{f''}{f'} + \frac{4}{r} = 0}$$

Integrating $\log f' + 4\log r = \log c_1$

So that $\Rightarrow \boxed{f' = c_1 r^{-4}}$ ——— (13)

Integrating (13); $f = -(c_1/3) \times r^{-3} + c_2$

C_2 being an arbitrary constant

Substituting the values of f' and f from (13) & (14) in (5), we get

$$\vec{q} = -\left\{(c_1/3r^3) - c_2\right\}\vec{i} - (c_1 x/r^5)\vec{r} \quad \text{--- (15)}$$

Given, that $\vec{q} \to 0$ as $r \to \infty$,

hence (15) shows that $C_2 = 0$

$\therefore$ from (15) ; $\boxed{\vec{q} = \dfrac{c_1}{3r^3}\left(\vec{i} - \dfrac{3x\vec{r}}{r^2}\right)}$ —— 16

Now; $q^2 = \vec{q}\cdot\vec{q} = \dfrac{c_1^2}{9r^6}\left(\vec{i} - \dfrac{3x\vec{r}}{r^2}\right)\cdot\left(\vec{i} - \dfrac{3x\vec{r}}{r^2}\right)$

$$= \frac{c_1^2}{9r^6}\left[\vec{i}\cdot\vec{i} - \frac{6x}{r^2}\vec{r}\cdot\vec{i} + \frac{9x^2 r^2}{r^4}\right]$$

as $\vec{r}\cdot\vec{r} = r^2$
& $\vec{r}\cdot\vec{i} = x$

$$= \frac{c_1^2}{9r^6}\left(1 - \frac{6x^2}{r^2} + \frac{9x^2 r^2}{r^{4}}\right)$$

$$= \frac{c_1^2}{9r^6}\left(1 + \frac{3x^2}{r^2}\right)$$

$$\boxed{q^2 = \frac{c_1^2}{9r^8}(r^2 + 3x^2)}$$

Hence, the required surfaces of constant speed are

$q^2 = $ constant (or) $\left(\dfrac{c_1^2}{9r^8}\right)(r^2 + 3x^2) = $ Constant

$\boxed{q^2 = (r^2 + 3x^2)r^{-8} = \text{constant}}$ required solution