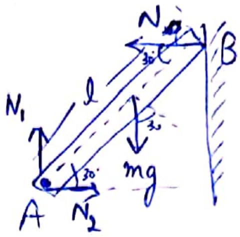
2009

[5(c)]

Q1 - A uniform rod AB hinges about A and rests with one end in contact with a smooth vertical wall. If rod is inclined at an angle of 30° with the horizontal, find reaction at hinge in magnitude and direction.



Soln - ~~Bala~~ Along x-direction,

$$N = N_2$$

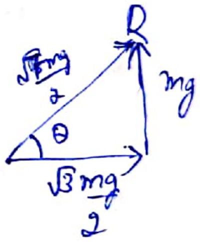
Along y - direction, $N_1 = mg$

Net moment about the ~~edge~~ hinge A = 0

$$\Rightarrow N \sin 30° \cdot l - mg \cos 30° \cdot \frac{l}{2} = 0$$

$$\Rightarrow N = \frac{mg\sqrt{3}}{2}$$

$$\therefore N_1 = mg, \quad N_2 = \frac{mg\sqrt{3}}{2}$$



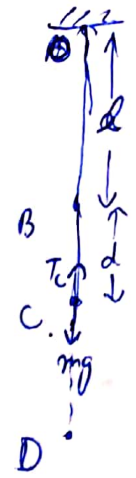
Net reaction at Hinge R is given by $\frac{\sqrt{7}\,mg}{2}$ at an angle $\theta = \tan^{-1}\left(\frac{2}{\sqrt{3}}\right)$ ~~from~~ with the horizontal.

[7(a)]

Q2 - One end of a light elastic string of natural length l and modulus of elasticity $2mg$ is attached to fixed point O and the other end to a particle of mass m, the particle initially held at rest at O is let fall. Find the greatest extension of the string during the motion and show that the particle will reach O again after a time $(\pi + 2 - \tan^{-1} 2)\sqrt{\frac{2l}{g}}$

Soln -

Let $OB = l$ and C be equilibrium position of particle of mass m. Then, $T_C = \lambda \frac{d}{l} = 2mg\frac{d}{l}$



$$\therefore\ mg = 2mg\frac{d}{l} \Rightarrow \underline{d = l/2}$$

Let velocity at point O be zero. Then at point B, velocity $= \sqrt{2gl}$ in downward direction.

Let D be point of maximum extension. Then the particle will come to rest instantaneously at D.

During the motion of particle, at a point P, at distance x below C,

$$T_P = \lambda\left(\frac{d+x}{l}\right) = 2mg\left(\frac{x+\frac{l}{2}}{l}\right) \text{ acting upwards.}$$

$$m\frac{d^2x}{dt^2} = mg - T_P = mg - 2mg\left(\frac{x+\frac{l}{2}}{l}\right) = -\frac{2mg}{l}x$$

$$\Rightarrow \frac{d^2x}{dt^2} = -\frac{2g}{l}x$$

Multiplying with $2\frac{dx}{dt}$ and integrating w.r.t 't', we get

$$\left(\frac{dx}{dt}\right)^2 = -\frac{2g}{l}x^2 + k$$

At the point B, velocity $= \sqrt{2ga} = dx/dt$

$$\Rightarrow k = 2gd + \frac{2g}{l}\left(\frac{d}{2}\right)^2 = \frac{5dg}{2}$$

$$\Rightarrow \left(\frac{dx}{dt}\right)^2 = \frac{5lg}{2} - \frac{2g}{l}x^2$$

At point D, $\frac{dx}{dt} = 0 \Rightarrow x = \frac{l\sqrt{5}}{2}$

$\therefore$ Greatest extension $= \frac{l}{2} + \frac{l\sqrt{5}}{2} = \frac{l}{2}(1+\sqrt{5})$

$$\therefore \left(\frac{dx}{dt}\right)^2 = \frac{2g}{a}\left[\frac{5}{4}l^2 - x^2\right] \Rightarrow \frac{dx}{dt} = \sqrt{\frac{2g}{a}}\sqrt{\frac{5}{4}l^2 - x^2}$$

$$\Rightarrow dt = \sqrt{\frac{l}{2g}} \frac{dx}{\sqrt{\frac{5}{4}l^2 - x^2}}$$

If t_1 is the time from B to D, then

$$\int_0^{t_1} dt = \sqrt{\frac{l}{2g}} \int_{-a/2}^{a\sqrt{5}/2} \frac{dx}{\sqrt{\frac{5}{4}l^2 - x^2}} = \sqrt{\frac{l}{2g}} \left[\frac{\pi}{2} + \sin^{-1}\frac{1}{\sqrt{5}}\right]$$

$$= \sqrt{\frac{l}{2g}} \left[\frac{\pi}{2} + \tan^{-1}\frac{1}{2}\right] = \sqrt{\frac{l}{2g}} \left[\pi - \tan^{-1}2\right]$$

If t_2 is time taken from O to B (during free fall)

$$l = 0 \cdot t_2 + \frac{1}{2} g t_2^2 \Rightarrow t_2 = \sqrt{\frac{2l}{g}}$$

Total time to reach O again $= 2(t_1 + t_2) = \sqrt{\frac{2l}{g}} \left[\pi - \tan^{-1}2 + 2\right]$

[7(b)]

Q3. A particle is projected with a velocity v from the cusp of a smooth inverted cycloid down the arc. Show that the time of reaching the vertex is $2\sqrt{\frac{a}{g}} \cot^{-1}\left(\frac{v}{2\sqrt{ag}}\right)$

Soln. The equation of motion is $\frac{d^2x}{dt^2} = -g\left(\frac{x}{4a}\right)$

i.e. $\left(D^2 + \frac{g}{4a}\right) x = 0$

$\therefore$ Solution is $x = A \cos\sqrt{\frac{g}{4a}}\, t + B \sin\sqrt{\frac{g}{4a}}\, t$

When $t = 0$, $x = 4a \Rightarrow A = 4a$

and velocity $= -v = \frac{dx}{dt} \Rightarrow B = -v\sqrt{\frac{4a}{g}}$

$$\therefore x = 4a \cos\left(\sqrt{\frac{g}{4a}}\, t\right) - v\sqrt{\frac{4a}{g}} \sin\left(\sqrt{\frac{g}{4a}}\, t\right)$$

When it comes to the vertex,

$$x = 0$$

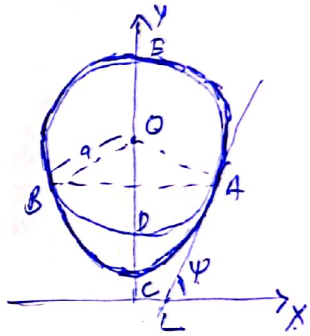
$$\Rightarrow \cot\left(\sqrt{\frac{g}{4a}}\, t\right) = V\sqrt{\frac{4a}{g}} \cdot \frac{1}{4a} = \frac{V}{\sqrt{4ag}}$$

$$\Rightarrow t = \sqrt{\frac{4a}{g}} \cot^{-1}\left(\frac{V}{\sqrt{4ag}}\right)$$

$$\boxed{t = 2\sqrt{\frac{a}{g}} \cot^{-1}\left(\frac{V}{2\sqrt{ag}}\right)}$$

[7(d)]

Q4. Find the length of an endless chain which will hang over a circular pulley of radius 'a' so as to be in contact with three-fourth circumference of the pulley.



Soln - Let AEBCA be endless chain of which

$$AEB = \frac{3}{4}(2\pi a)$$

The remaining portion not in contact is a catenary.

$$\angle AOB = \frac{1}{4} \cdot 2\pi = 90^\circ$$

$$\cancel{\Rightarrow \angle AOD = \frac{90^\circ}{2} = 45^\circ} \quad \Rightarrow \angle OAB = \angle OBA = \frac{90^\circ}{2} = 45^\circ$$

$$\Rightarrow \angle ALX = \psi = 45^\circ \quad \text{and } AM = \frac{a}{\sqrt{2}}$$

Now $x = c\log(\sec\psi + \tan\psi)$

$$\Rightarrow \frac{a}{2} = c\log\left(\sec 45^\circ + \tan 45^\circ\right)$$

$$\Rightarrow c = \frac{a}{\sqrt{2}\log(\sqrt{2}+1)}$$

$$\text{Arc } ACB = 2s = 2c\tan\psi = 2c\tan 45^\circ = \frac{2a}{\sqrt{2}\log(\sqrt{2}+1)}$$

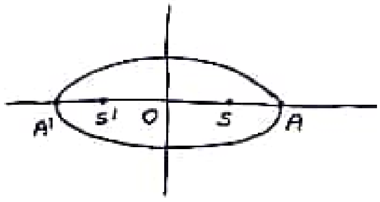
Total length of the chain $= \frac{3}{4}\cdot(2\pi a) + \frac{\sqrt{2}a}{\log(\sqrt{2}+1)}$

$$= a\left[\frac{3\pi}{2} + \frac{\sqrt{2}}{\log(\sqrt{2}+1)}\right]$$

Scanned with CamScanner

IAS 2009 → A body is describing an ellipse of eccentricity e under the action of a central force directed towards a focus and when at the nearer apse, the centre of force is transferred to the other focus. Find the eccentricity of the new orbit in terms of eccentricity of the original orbit.

Sol^n) Let S & S' be the focus of an ellipse of eccentricity e and length 2a (major axis). The force is directed towards the focus S



Velocity at distance r from S is given by ≡ (V)

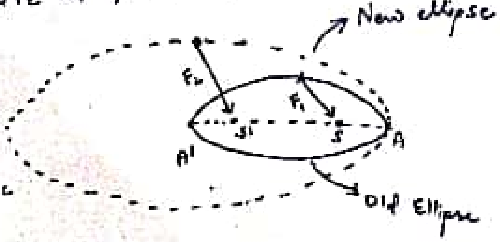
$$V^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right) \quad \text{—— ①}$$

Velocity at pt A ⇒ $V_A^2 = \mu\left[\frac{2}{AS} - \frac{1}{a}\right] = \mu\left[\frac{2}{a(1-e)} - \frac{1}{a}\right]$

$$= \frac{\mu}{a}\frac{(1+e)}{(1-e)} \quad \text{—— ②}$$

Now when the body reaches A, the centre of force get shifted to other focus S' (instantaneously). So the body will describe a new elliptic path with centre of force as S'.

$$V_A^2 = \mu\left[\frac{2}{a(1+e)} - \frac{1}{a'}\right] \quad \text{—— ③}$$



where 2a' is the length of the major axis of the new ellipse.

Using ② & ③ we get,

$$\frac{\mu}{a}\left(\frac{1+e}{1-e}\right) = \mu\left(\frac{2}{a(1+e)} - \frac{1}{a'}\right) \Rightarrow \frac{(1+e)}{a(1-e)} = \frac{2}{a(1+e)} - \frac{1}{a'} \quad \text{—— ④}$$

Let eccentricity of new ellipse be e' then $S'A = a'(1-e')$

$$\Rightarrow a'(1-e') = a(1+e) \Rightarrow a' = \frac{a(1+e)}{(1-e')}.$$

Putting the value of a' in (4) we get,

$$\frac{(1+e)}{a(1-e)} = \frac{2}{a(1+e)} - \frac{(1-e')}{a(1+e)}$$

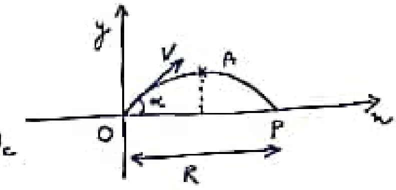
$$\Rightarrow \frac{1+e}{\cancel{a}(1-e)} = \frac{2-(1-e')}{\cancel{a}(1+e)} \Rightarrow 1+e' = \frac{(1+e)^2}{1-e}$$

$$\Rightarrow \boxed{e' = \frac{(1+e)^2}{1-e} - 1 = \frac{3e+e^2}{1-e}.}$$

IAS 2009 → A shot fired with a velocity V at an elevation α strikes at a point P in a horizontal plane through the pt of intersection. If the pt P is receding from the gun with velocity v, show that the elevation must be changed to θ where

$$\sin 2\theta = \sin 2\alpha + \frac{2v}{V}\sin\theta .$$

Soln) Let the point P be at $(R, 0)$ initially where R is the range of the projectile from O at an angle α.



Time period $\equiv T$ ⇒ will be find out by equating the velocity in y direction to zero at pt A.

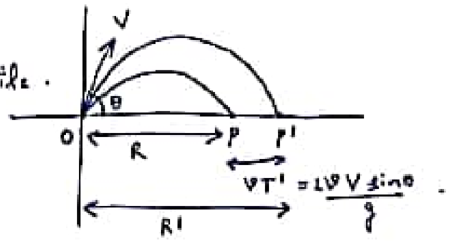
Using $v = u + at$ we get $0 = V\sin\alpha - g\frac{T}{2}$ ⇒ $\boxed{T = \frac{2V\sin\alpha}{g}}$ —①

∴ $\boxed{R = V\cos\alpha\, T = \frac{V^2 \sin 2\alpha}{g}}$ (It is also a direct formula) —②

Now as per the question the point P starts moving in x direction with a velocity v & the angle of the projectile be changed to θ such that it still hits the displaced P point (say P').

Now T' = new time period

$R' = OP'$ = New range of projectile.



Clearly $T' = \frac{2V\sin\theta}{g}$ (Using ①)

& $R' = \frac{V^2 \sin 2\theta}{g}$ (Using ②)

Now $OP + PP' = OP'$ ⇒ $\frac{\cancel{V^2}\sin 2\alpha}{\cancel{g}} + \frac{2v\cancel{V}\sin\theta}{\cancel{g}V} = \frac{\cancel{V^2}\sin 2\theta}{\cancel{g}}$

⇒ $\boxed{\sin 2\alpha + \frac{2v}{V}\sin\theta = \sin 2\theta}$ Ans