IAS
PREVIOUS YEARS QUESTIONS (2019-1983)
SEGMENT-WISE

MODERN ALGEBRA

2019

- Let G be a finite group, H and K subgroups of G such that $K \subset H$. Show that (G : K) = (G : H) (H : K). **[10]**
- If G and H are finite groups whose orders are relatively prime, then prove that there is only one homomorphism from G to H, the trivial one. **[10]**
- Write down all quotient groups of the group Z_{12}. **[10]**
- Let a be an irreducible element of the Euclidean ring R, then prove that R|(a) is a field. **[10]**

2018

- Let *R* be an integral domain with unit element. Show that any unit in *R*[*x*] is a unit in *R*. **(10)**
- Show that the quotient group of $(\mathbb{R}, +)$ modulo $\mathbb{Z}$ is isomorphic to the multiplicative group of complex numbers on the unit circle in the complex plane. Here $\mathbb{R}$ is the set of real numbers and $\mathbb{Z}$ is the set of integers **(15)**
- Find all the proper subgroups of the multiplicative group of the field $(\mathbb{Z}_{13}, +_{13}, \times_{13})$, where $+_{13}$ and $\times_{13}$ represent addition modulo 13 and multiplication modulo 13 respectively. **(20)**

2017

- Let G be a group of order n. Show that G is isomorphic to a subgroup of the permutation group S_n. **(10)**
- Let F be a field and F[X] denote the ring of polynomials over F in a single g variable X. For f[X], g(X) $\in$ F[X] with g(X) $\neq$ 0, show that there exist q(X), r(X) $\in$ F[X] such that degree (r(X))<degree (g(X)) and f(X)=q(X)·g(X)+r(X). **(20)**
- Show that the groups $\mathbb{Z}_5 \times \mathbb{Z}_7$ and $\mathbb{Z}_{35}$ are isomorphic. **(15)**

2016

- Let **K** be a field and **K**[X], be the ring of polynomials over **K** in a single variable X. For a polynomial $f \in \mathbf{K}[X]$, let (f) denote the ideal in **K**[X] generated by f. Show that (f) is a maximal ideal in **K**[X] if and only if f is an irreducible polynomial over **K**. **(10)**
- Let p be a prime number and z_p denote the additive group of integers modulo p. Show that every non-zero element of z_p generates z_p. **(15)**
- Let K be an extension of a field F, Prove that the elements of K, which are algebraic over F, form a subfield of K. Further, if $F \subset K \subset L$ are fields, L is algebraic over K and K is algebraic over F, then prove that L is algebraic over F. **(20)**
- Show that every algebraically closed field is infinite. **(15)**

2015

- How many generators are there of the cyclic group *G* of order 8 ? Explain. **(5)**
- Taking a group {e, a, b, c} of order 4, where *e* is the identity, construct composition tables showing that one is cyclic while the other is not. **(5)**
- Give an example of a ring having identity but a subring of this having a different identity. **(10)**
- If *R* is a ring with unit element 1 and ϕ is a homomorphism of *R* onto *R*', prove that $\phi(1)$ is the unit element of *R*'. **(15)**
- Do the following sets form integral domains with respect to ordinary addition and multiplication? If so, state if they are fields:
  - (i) The set of numbers of the form $b\sqrt{2}$ with *b* rational
  - (ii) The set of even integers
  - (iii) The set of positive integers **(5+6+4=15)**

2014

- Let G be the set of all real 2×2 matrices $\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$.

where $xz \neq 0$. Show that G is a group under matrix multiplication. Let N denote the subset $\left\{\begin{bmatrix}1 & a\\0 & 1\end{bmatrix} : a \in \mathbb{R}\right\}$. Is N a normal subgroup of G? Justify your answer. **(10)**

- Show that Z_7, is a field. Then find $([5]+[6])^{-1}$ and $(-[4])^{-1}$ in Z_7 **(15)**
- Show that the set $\{a + b\,\omega : \omega^3=1\}$, where a and b are real numbers, is a field with respect to usual addition and multiplication. **(15)**
- Prove that the set $Q(\sqrt{5}) = \{a + b\sqrt{5} :$ a, b $\in$ Q$\}$ is a commutative ring with identity. **(15)**

2013

- Show that the set of matrices $S = \left\{\begin{pmatrix}a & -b\\ b & a\end{pmatrix} \middle| a, b \in \mathbb{R}\right\}$ is a field under the usual binary operations of matrix addition and matrix multiplication. What are the additive and multiplicative identities and what is the inverse of $\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}$? Consider the map $f : \mathbb{C} \to S$ defined by $f(a+ib) = \begin{pmatrix}a & b\\ b & a\end{pmatrix}$. Show that f is an isomorphism. (Here $\mathbb{R}$ is the set of real numbers and $\mathbb{C}$ is the set of complex numbers.) **(10)**
- Give an example of an infinite group in which every element has finite order. **(10)**
- What are the orders of the following permutations in S_{10}?
$\begin{pmatrix}1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10\\ 1 & 8 & 7 & 3 & 10 & 5 & 4 & 2 & 6 & 9\end{pmatrix}$ and (1 2 3 4 5) (6 7). **(10)**
- What is the maximal possible order of an element in S_{10}? Why? Give an example of such an element. How many elements will there be in S_{10} of that order? **(13)**
- Let J = {a + bi | a, b $\in \mathbb{Z}$} be the ring of Gaussian integers (subring of $\mathbb{C}$).
Which of the following is J : Euclidean domain, principal ideal domain, unique factorization domain? Justify your answer. **(15)**
- Let R^C = Ring of all real valued continuous functions on [0, 1], under the operations
$(f+g)\,x = f(x) + g(x)$
$(fg)\,x = f(x)\,g(x)$.
Let $M = \left\{ f \in \mathbb{R}^c \middle| f\left(\frac{1}{2}\right) = 0 \right\}$.
Is M a maximal ideal of R? Justify your answer. **(15)**

2012

- How many elements of order 2 are there in the group of order 16 generated by a and b such that the order of a is 8, the order of b is 2 and $bab^{-1} = a^{-1}$. **(12)**
- How many conjugacy classes does the permutation group S_5 of permutations 5 numbers have? Write down one element in each class (preferably in terms of cycles). **(15)**
- Is the ideal generated by 2 and X in the polynomial ring Z(X) of polynomials in a single variable X with coefficients in the ring of integers Z, a principal ideal? Justify your answer. **(15)**
- Describe the maximal ideals in the ring of Gaussian integers $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$. **(20)**

2011

- Show that the set $G = \{f_1, f_2, f_3, f_4, f_5, f_6\}$ of six transformations on the set of Complex numbers defined by $f_1(z) = z,\ f_2(z) = 1 - z$
$f_3(z) = \frac{z}{(z-1)},\ f_4(z) = \frac{1}{z},$
$f_5(z) = \frac{1}{(1-z)}$ *and* $f_6(z) = \frac{(z-1)}{z}$
is a non-abelian group of order 6 with respect to composition of mappings. **(12)**
- (i) Prove that a group of prime order is abelian .
(ii) How many generators are there of the cyclic group $(G, \bullet)$ of order 8? **(12)**
- Give an example of a group G in which every proper subgroup is cyclic but the group itself is not cyclic. **(15)**
- Let F be the set of all real valued continuous functions defined on the closed interval [0, 1]. Prove that $(F, +, \bullet)$ is a commutative Ring with unity with respect to addition and multiplication of

functions defined pointwise as below :

$(f+g)(x) = f(x) + g(x)$

and $(f.g)(x) = f(x).g(x)$, $x \in [0,1]$

where $f, g \in F$. **(15)**

❖ Let a and b be elements of a group, with $a^2 = e$, $b^6 = e$ and $ab = b^4 a$. Find the order of ab, and express its inverse in each of the forms $a^m b^n$ and $b^m a^n$. **(20)**

2010

❖ Let IR -{-1} be the set of all real numbers omitting -1. Define the binary relation * on G by $a*b = a + b + ab$. Show (G, *) is a group and it is abelian. **(12)**

❖ Show that a cyclic group of order 6 is isomorphic to the product of a cyclic group of order 2 and a cyclic group of order 3. Can you generalize this? Justify. **(12)**

❖ Let $(IR^*, \bullet)$ be the multiplicative group of non – zero reals and $(GL(n,\mathbb{R}), X)$ be the multiplicative group of $n \times n$ non singular real matrices. Show that the quotient group $GL(n,\mathbb{R})/SL(n,\mathbb{R})$ and $(IR^*, \bullet)$ are isomorphic where SL (n, IR) = {A ∈ G L(n, IR)/ det A =1}.
What is the centre of GL (n, IR)? **(15)**

❖ Let $C = \{f : I = [0,1] \to IR \mid f \text{ is continuous}\}$.
Show C is a commutative ring with 1 under point wise addition and multiplication. Determine whether C is an integral domain. Explain. **(15)**

❖ Consider the polynomial ring Q[x]. show $p(x) = x^3 - 2$ is irreducible over Q. Let I be the ideal in Q[x] generated by p(x).Then show that a $Q[x]/I$ is a field and that each element of it is of the form $a_0 + a_1 t + a_2 t^2$ with a_0, a_1, a_2 in Q and $t = x + I$. **(15)**

❖ Show that the quotient ring Z [i]/ (1 + 3i) is isomorphic to the ring Z/ 10Z where Z [i] denotes the ring of Gaussian integers. **(15)**

2009

❖ If $\mathbb{R}$ is the set of real numbers and $\mathbb{R}_+$ is the set of positive real numbers, show that $\mathbb{R}$ under addition $(\mathbb{R}, +)$ and $\mathbb{R}_+$ under multiplication $(\mathbb{R}_+, \bullet)$ are isomorphic. Similarly if $\mathbb{Q}$ is the set of rational numbers and $\mathbb{Q}_+$ the set of positive rational numbers are $(\mathbb{Q}, +)$ and $(\mathbb{Q}_+, \bullet)$ isomorphic ? Justify your answer. **(12)**

❖ Determine the number of homomorphisms from the additive group Z_{15} to the additive group Z_{10}. (Z_n is the cyclic group of order n). **(12)**

❖ How many proper non – zero ideals does the ring Z_{12} have? Justify your answer. How many ideals does the ring $Z_{12} \oplus Z_{12}$ have? Why? **(15)**

❖ Show that the alternating group on four letters A_4 has no subgroup of order 6. **(15)**

❖ Show that Z[x] is a unique factorization domain that is not a principal ideal domain .(Z is the ring of integers). Is it possible to give an example of principal ideal domain that is not a unique factorization domain? (Z[x] is the ring of polynomials in the variable X with integer). **(15)**

❖ How many elements does the quotient ring $\frac{Z_5[x]}{x^2+1}$ have? Is it an integral domain? Justify your answers. **(15)**

2008

❖ Let R_0 be the set of all real numbers except zero. Define a binary operation '*' on R_0 as $a * b = |a|b$; Where |a| denotes absolute value of a. Does $(R_0, *)$ form a group? Examine. **(12)**

❖ Suppose that there is a positive even integer 'n' such that $a^n = a$ for all the elements 'a' of some ring R. show that a + a = 0, $\forall a \in R$ and $a + b = 0 \Rightarrow a = b \, \forall a, b \in R$. **(12)**

❖ Let R be ring with unity. If the product of any two non – zero elements is non – zero. Prove that a b = b a =1. Whether Z_6 has the above property or not explain. Is Z_6 an integral domain? **(15)**

❖ Show that any maximal ideal in the commutative ring F[x] of polynomials a field F is the principal ideal generated by an irreducible polynomial. **(15)**

❖ Prove that every Integral Domain can be embedded in a field. **(15)**

2007

- If in a group G $a^5 = e$, e is the identity element of G and $aba^{-1} = b^2$ for a, b $\in$ G then find the order of b. **(12)**
- Let $R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ where } a,b,c,d \in Z \right\}$.

  Show that R is ring under matrix addition and multiplication.

  Let $A = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} / a,b \in Z \right\}$. Then show that A itself ideal of R but not a right ideal of R. **(12)**
- Prove that there exists no simple group of order 48. **(15)**
- $1+\sqrt{-3}$ in $Z[-\sqrt{3}]$ is an irreducible element but not prime. Justify your answer. **(15)**

2006

- Let 'S' be the set of all real numbers except -1. Define $a*b = a+b+ab$. Is (S, *) a group? **(12)**

  Find the solution of the equation 2*x*3=7 in 'S'.
- If G is a group of real numbers under addition and N is the subgroup of G consisting of integers, prove that G/N is isomorphic to the group H of all complex numbers of absolute value 1 under X^n. **(12)**
- Let o(G) = 108. Show that there exists a normal subgroup of order 27 or 9. **(20)**
- Let G be the set of all those ordered pairs (a, b) of real numbers for which a $\neq$ 0 and defined in G, an operation $\otimes$ as follows: (a, b) $\otimes$ (c, d) = (a c, b c+ d)
  - (i) Examine whether G is a group with respect to the operation $\otimes$. If it is a group, is G abelian?
  - (ii) Is (H, $\otimes$ a subgroup of (G, $\otimes$) when H = { (1,b) / b $\in$ R}.
- Show that $Z\left[\sqrt{2}\right] = \left\{a + \sqrt{2}b / a, b \in Z\right\}$ is a Euclidean domain. **(10/1996)**

2005

- If M and N are normal subgroups of group G such that $M \cap N = \{e\}$, show that every element of M commutes with every element of N. **(12)**
- Show that (1+i) is a prime element in the ring R of Gaussian integers. **(12)**
- Let H and K be two subgroups of a finite group G such that $|H| > \sqrt{|G|}$ and $|K| > \sqrt{|G|}$. Prove that $H \cap K \neq \{e\}$. **(15)**
- If $f: G \to G'$ is an isomorphism, prove that the order of a $\in$ G is equal to the order of f (a). **(15)**
- Prove that any polynomial ring F[x] over a field F is UFD. **(30)**

2004

- Let G be a group such that for all a, b $\in$ G
  - (i) a b =b a
  - (ii) $\left(o(a), o(b)\right) = 1$

  then show that $o(ab) = o(a)o(b)$. **(12)**
- Verify that the set E of the four roots of $x^4 - 1 = 0$ forms a multiplicative group. Also prove that a transformation T, T (n) = i^n is a homomorphism from I_+ (group of all integers with addition) onto E under multiplication. **(10)**
- Prove that if the cancellation law holds for a ring a then $a(\neq 0) \in R$ is not a zero divisor and conversely. **(10)**
- The residue class ring $\frac{z}{(m)}$ is a field iff 'm' is a prime integer. **(15)**
- Define irreducible element and prime element in an integral domain D with units. Prove that every prime element in D is irreducible and converse of this is not (in general) true. **(25)**

2003

- If H is a subgroup of a group G such that $x^2 \in H$ for every $x \in G$, then prove that H is a normal subgroup of G. **(12)**
- Show that the ring $Z[i] = \left\{a+ib / a \in z, b \in z, i = \sqrt{-1}\right\}$ of Gaussian integers is a Euclidean domain. **(12)**
- Let R be the ring of all real valued continuous functions on the closed interval [0, 1].

  Let $M = \left\{ f(x) / f\left(\frac{1}{3}\right) = 0 \right\}$, show that M is a maximal ideal of R. **(10)**
- Let M and N be two ideals of a ring R.Show that $M \cup N$ is an ideal of R iff either $M \subseteq N$ or $N \subseteq M$. **(10)**

- Prove that x^2+x+4 is irreducible over F, the field of integers modulo 11. And prove further that $\frac{F[x]}{x^2+x+4}$ is a field having 121 elements. **(15/1992 & 1996)**
- If R is a unique factorization domain (UFD), then prove that R[x] is also a UFD. **(10)**

2002

- Show that a group of order 35 is cyclic. **(12)**
- Show that the polynomial $25x^4+9x^3+3x+3$ is irreducible over the field of rational numbers. **(12)**
- Show that a group of P^2 is abelian, where P is a prime number. **(10)**
- Prove that a group of order 42 has a normal subgroup of order 7. **(10)**
- Prove that in the ring F[x] of polynomials over a field F, the ideal $I=[P(x)]$ is maximal iff the polynomial P(x) is irreducible over F. **(20)**
- Show that every finite integral domain is a field. **(10)**

2001

- Let K be a field and G be a finite subgroup of the multiplicative group of none zero elements of G. show that G is a cyclic group. **(12)**
- Prove that the polynomial $1+x+x^2+........+x^{p-1}$ where P is a prime number, is irreducible over the field of rational numbers. **(12)**
- Let N be a normal subgroup of a group G. show that G/N is abelian iff for all $x,y\in G,\ xyx^{-1}y^{-1}\in N$ **(20)**
- If R is a commutative ring with unit element and M is an ideal of R, then show that M is a maximal ideal of R iff R/M is a field. **(20/1988)**

2000

- Let n be a fixed +ve integer and let Z_n be the ring of integers modulo n.
  Let
  $G=\left\{\overline{a}\in Z_n/\overline{a}\neq\overline{0}\ and\ a\ is\ relatively\ prime\ to\ 'n'\right\}$.
  Show that G is a group under multiplication defined in Z_n.
  Hence or otherwise, Show that $a^{\phi(n)}\equiv a(\bmod n)$ for all integers relatively prime to n where $\phi(n)$ denotes the number of positive integers that are less than n and are relatively prime to n. **(12)**
- Let M be a subgroup and N a normal subgroup of a group G. Show that MN is a subgroup of G and $\frac{MN}{N}$ is isomorphic to $\frac{M}{M\cap N}$. **(12)**
- Let F be a finite field. Show that the characteristic of F must be a prime integer p and the number of elements in F must be P^m for some positive integer m. **(20/1989)**
- Let F be a field and F[x] denote the set of all polynomials defined over F. If f(x) is an irreducible polynomial in F[x]; show that the ideal generated by f(x) in F[x] is maximal and $\frac{F[x]}{\langle f(x)\rangle}$ is a field. **(20)**
- Show that any finite commutative ring with no zero divisors must be field. **(20)**

1999

- If ϕ is a homomorphism of G into $\overline{G}$ with kernel K, then show that
  (i) K is a normal subgroup of G.
  (ii) $o(\phi(a))/o(a)$ **(20/2008)**
- If P is a prime number and $P^{\alpha}/o(G)$, then prove that G has a subgroup of order P^{α}. **(20)**
- Let R be a commutative ring with unit element whose only ideals are (0) and R itself. Show that R is a field. **(20)**

1998

- Prove that if a group has only four elements then it must be abelian. **(20)**
- If H and K are subgroups of a group G, then show that HK is a subgroup of G iff HK=KH. **(20)**
- Show that every group of order 15 has a normal subgroup of order 5. **(20)**
- Let $(R,+,\bullet)$ be a system satisfying all the axioms for a ring with unity with the possible exception of a + b = b + a. Prove that $(R,+,\bullet)$ is a ring. **(20)**
- If P is prime then prove that Z_p is a field. Discuss the case when P is not a prime number. **(20)**

- Let 'D' be a principal ideal domain. Show that every element that is neither Zero nor a unit in 'D' is a product of irreducibles. **(20)**

1997

- Show that a necessary and sufficient condition for a subset H of a group G to be a subgroup is $HH^{-1} = H$ **(20)**
- Show that the order of each subgroup of a finite group is a divisor of the order of the group. **(20)**
- In a group G, the commutator of (a, b) ; a, b$\in$G is the element $aba^{-1}b^{-1}$ and the smallest subgroup containing all commutators is called the commutator subgroup of G. Show that a quotient group $\frac{G}{H}$ is abelian iff H contains that the commutator subgroup of G. **(20)**
- If $x^2 = x \; \forall x$ in a ring R, show that R is commutative. Give an example to show that the converse is not true. **(20)**
- Show that an ideal 'S' of the ring of integers Z is maximal ideal iff 'S' generated by a prime integer. **(20)**
- Show that in an integral domain every prime element is irreducible. Give an example to show that the converse is not true. **(20)**

1996

- Let f be a homomorphism of group G onto a group G' with kernel H. For each subgroup K' of G' define K by $K = \{x \in G \mid f(x) \in K'\}$ prove that

  (i) K' is isomorphic to K/H

  (ii) G/K is isomorphic to $\frac{G'}{K'}$ **(20)**
- Prove that a normal subgroup H of a group G is maximal, iff the quotient group G/H is simple. **(20)**
- In a ring R, prove that cancellation laws hold, iff R has no zero divisors. **(20)**
- If S is an ideal of ring R and T any subring of R, then prove that S is an ideal of S+T = {s+ t |s $\in$ S, t $\in$ T}. **(20)**

1995

- Let G be a finite set closed under an associative binary operation such that

  $ab = ac \Rightarrow b = c$ & $ba = ca \Rightarrow b = c \; \forall a, b, c \in G$

  prove that G is a group. **(20)**
- Let G be a group of order P^n, where P is a prime number and n > 0. Let H be a proper subgroup of G and N (H) = $\{x \in G / x^{-1}hx \in H \; \forall h \in H\}$. Prove that N(H) $\neq$ H. **(20)**
- Show that a group of order 112 is not simple. **(20)**
- Let R be a ring with identity. Suppose there is an element 'a' of R which has more than one right inverse. Prove that 'a' has infinitely many right inverses. **(20)**
- Let F be a field and let P(x) be an irreducible polynomial over F. Let $\langle P(x) \rangle$ be the ideal generated by P(x). Prove that $\langle P(x) \rangle$ is a maximal ideal. **(20)**

1994

- If G is a group such that $(ab)^n = a^n b^n$ for three consecutive integers n for all a, b in G, then prove that G is abelian. **(20)**
- Can a group of order 42 be simple ? Justify your claim. **(20)**
- Show that the additive group of integers modulo 4 is isomorphic to the multiplicative group of the non Zero elements of integers modulo 5. State the two isomorphisms. **(20)**
- Find all the units of the integral domain of Gaussian integers. **(20)**
- Prove or disprove the statement: the polynomial ring $I[x]$ over the ring of integers is a principal ideal ring. **(20)**

1993

- Show that a group of order 56 cannot be simple. **(20)**
- If G is a cyclic group of order n and p divides n, then prove that there is a homomorphism of G onto a cyclic group of order p. what is the kernel of homomorphism ? **(20)**
- Suppose that H, K are normal subgroups of a finite group G with H a normal subgroup of K. If P=K/H, S= G/H, then prove that the quotient groups S/P and G/K are isomorphic. **(20)**
- If Z is the set of integers then show that $Z[\sqrt{-3}] = \{a + \sqrt{-3}\, b / a, b \in Z\}$ is not a unique factorization domain. **(20)**

- Construct the addition and multiplication table for $\frac{Z_3[x]}{\langle x^2+1\rangle}$ Where Z_3 is the set of integers modulo 3 and $\langle x^2+1\rangle$ is the ideal generated by (x^2+1) in $Z_3[x]$. **(20)**

1992

- If H is a cyclic normal subgroup of a group G then show that every subgroup of H is normal in G. **(20)**
- Show that no group of order 30 is simple. **(20)**
- If p is the smallest prime factor of the order of a finite group G, prove that any subgroup of index p is normal. **(20)**
- If R is a unique factorization domain, then prove that any $f \in R[x]$ is an irreducible element of R[x] iff either f is an irreducible element of R or f is an irreducible polynomial in R[x] . **(20)**
- Prove that x^2+1 and x^2+x+4 are irreducible over F, the field of integers modulo 11. Prove also that $\frac{F[x]}{\langle x^2+1\rangle}$ and $\frac{F[x]}{\langle x^2+x+4\rangle}$ are isomorphic fields each having 121 elements. **(20/1996 & 2002)**

1991

- If the group G has no non-trivial subgroups, show that G must be finite of prime order. **(20)**
- Show that a group of order 9 must be abelian. **(20)**
- If the integral domain D is of finite characteristic, show that the characteristic must be a prime number. **(20)**
- Find the greatest common divisor in $J(i)$ of
  $(i) 3+4i \text{ and } 4-3i$
  $(ii) 11+7i \text{ and } 18-i$
- Show that every maximal ideal of a commutative ring R with unit element must be a prime ideal.

1990

- Let G be a group having no proper subgroups. Show that G should be a finite group of order which is a prime number or unity. **(20)**
- If C is the centre of a group G and $\frac{G}{C}$ is cyclic, prove that G is abelian. **(20)**
- Show that the set of Gaussian integers is a Euclidean ring. Find an HCF of the two elements 5i and 3 + i.

1989

- Let G be a group of order 2p, p being a prime. Show that there exist a normal subgroup of G of order p. **(20)**
- Give an example of an infinite group in which every element is of finite order.
- Let G be a group. Consider the set of elements of the form $xyx^{-1}y^{-1}$ where x and y are in G. If H is the smallest subgroup of G containing all these elements, show that H is a normal subgroup of G and that the factor group G/H is abelian. **(20)**
- If each element of a ring is idempotent, show that the ring is commutative. **(20)**
- Let A be a ring and I be a two sided ideal generated by the subset of all elements of the form a b – b a; a, b $\in$ A, prove that the residue class ring $\frac{A}{I}$ is commutative. **(20)**
- If a finite field of characteristic p has q elements show that $q=p^n$ for some n. **(20/2000)**

1988

- If H and K are normal subgroups of a group G such that $H \cap K=\{e\}$, show that h k=k h for all $h \in H$ and $k \in K$. **(20)**
- Show that the set of even permutations on n symbols, n>1, is a normal subgroup of the symmetric group S_n and has order $\frac{n!}{2}$ **(20)**
- Show that the numbers 0, 2, 4, 6, 8 with addition and multiplication modulo 10 form a field isomorphic to J_5, the ring of integers modulo 5. Give the isomorphism explicity. **(20)**
- R is a commutative ring with identity and U is an ideal of R. show that the quotient ring R/U is a field if and only if U is maximal.

1987

- Let $f: X \to Y$ and $g: Y \to Z$ be both bijections, prove that gof is bijection and $(g \circ f)^{-1}=f^{-1} \circ g^{-1}$

- Prove that HoK is a subgroup of (G, o) if and only if $H \circ K = K \circ H$. **(20)**
- If G is a finite group of order g and H is a subgroup of G of order h , then prove that h is a factor of g. **(20)**

1986

- Prove that a map $f : X \to Y$ is injective iff f can be left cancelled in the sense that $f \circ g = f \circ h \Rightarrow g = h.$ f is subjective iff it can be right cancelled in the sense that $g \circ f = h \circ f \Rightarrow g = h$. **(20)**
- The product HK of two sub groups H, K of a group G is a sub group of G if and only if HK = KH. **(20)**
- Prove that a finite integral domain is a field. **(20)**

1985

- State and prove the fundamental theorem of homomorphism for groups.
- Prove that the order of each subgroup of a finite group divides the order of the group.
- Write if each of the following statements is true or false:
  (i) If a is an element of a ring (R,+,.) and m and $n \in N$, then $\left(a^m\right)^n = a^{mn}$
  (ii) Every sub group of an abelian group is not necessarily abelian.
  (iii) A semi group (G,.) in which the equations ax=b and x a=b are solvable (for any a,b) is a group.
  (iv) The relation of isomorphism in the set of all groups is not an equivalence relation.
  (v) There are only two abstract groups of order six. **(20)**

1984

- Prove that a non- void subset S of a ring R is a sub ring of R, if and only if, $a - b \in S$ and $ab \in S$ *for all* $a, b \in S$. **(20)**
- Prove that an integral domain can be embedded in a field. **(20/2008)**
- Prove that for any two ideals A and B of a ring R, the product AB is an ideal of R. **(20)**

1983

- Show that the set $I \times I$ of integers is commutative ring with respect to addition and multiplication defined as follows.
  (a, b) + (c, d) = (a+ c, b+ d)
  (a, b) . (c, d) = (ac, bd)
  Where, $a, b, c, d \in I$. **(20)**
- Prove that the relation of isomorphism in the set of all groups is an equivalence relation. **(20)**
- Prove that a polynomial domain $K[x]$ over a field K is a principal ideal domain. **(20)**

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IFoS
PREVIOUS YEARS QUESTIONS (2019-2000)
SEGMENT-WISE

MODERN ALGEBRA
(ACCORDING TO THE NEW SYLLABUS PATTERN) PAPER - II

2019

- ❖ Let R be an integral domain. Then prove that ch R (characteristic of R) is 0 or a prime. **(08)**
- ❖ Let I and J be ideals in a ring R. Then prove that the quotient ring $(I + J)/J$ is isomorphic to the quotient ring $I/(I \cap J)$. **(10)**
- ❖ If in the group G, $a^5 = e$, $aba^{-1} = b^2$ for some $a, b \in G$, find the order of b. **(10)**
- ❖ Show that the smallest subgroup V of A_4 containing (1, 2) (3, 4), (1, 3) (2, 4) and (1, 4) (2, 3) is isomorphic to the Klein 4-group. **(10)**

2018

- ❖ Prove that a non-commutative group of order $2n$, where n is an odd prime, must have a subgroup of order n. **(08)**
- ❖ Find all the homomorphisms from the group $(Z, +)$ to $(Z_4, +)$. **(10)**
- ❖ Let **R** be a commutative ring with unity. Prove that an ideal **P** of **R** is prime if and only if the quotient ring R/P is an integral domain. **(10)**
- ❖ Show by an example that in a finite commutative ring, every maximal ideal need not be prime. **(10)**
- ❖ Let H be a cyclic subgroup of a group G. If H be a normal subgroup of G, prove that every subgroup of H is a normal subgroup of G. **(10)**

2017

- ❖ Prove that every group of order four is abelian. **(8)**
- ❖ Let G be the set of all real numbers except -1 and define $a*b = a + b + ab \ \forall \ a, b \in G$. Examine if G is an Abelian group under *. **(10)**
- ❖ Let H and K are two finite normal subgroups of co-prime order of a group G. Prove that $hk = kh \ \forall \ h \in H$ and $k \in K$ **(10)**
- ❖ Let A be an ideal of a commutative ring R and $B = \{x \in R : x^n \in A \text{ for some positive integer } n\}$. Is B an ideal of R ? Justify your answer. **(10)**
- ❖ Prove that the ring $\mathbb{Z}[i] = \{a + ib : a, b \in \mathbb{Z}, i = \sqrt{-1}\}$ of Gaussian integers is a Euclidean domain. **(10)**

2016

- ❖ Prove that the set of all bijective functions from a non-empty set X onto itself is a group with respect to usual composition of functions. **(8)**
- ❖ Show that any non-abelian group of order 6 is isomorphic to the symmetric group S_3. **(15)**
- ❖ Let G be a group of order pq, where p and q are prime numbers such that $p > q$ and qx $(p - 1)$. Then prove that G is cyclic. **(15)**
- ❖ Show that in ring $R = \{a + b\sqrt{-5} \mid a, b \text{ are integers}\}$, the elements $\alpha = 3$ and $\beta = 1 + 2\sqrt{-5}$ are relatively prime, but $\alpha\gamma$ and $\beta\gamma$ have no g.c.d in R, where $\gamma = 7\left(1 + 2\sqrt{-5}\right)$. **(10)**

2015

- ❖ If in a group G there is an element a of order 360, what is the order of a^{220}? Show that if G is a cyclic group of order n and m divides n, then G has a subgroup of order m. **(10)**
- ❖ If p is a prime number and e a positive integer, what are the elements 'a' in the ring Z_{p^e} of integers module p^e such that $a^2 = a$? Hence (or otherwise) determine the elements in $\mathbf{Z}_{35}$ such that $a^2 = a$. **(14)**
- ❖ Let $X = (a, b]$. Construct a continuous function $f : X \to \mathbb{R}$ (set of real numbers) which is unbounded and not uniformly continuous on X. Would your function be uniformly continuous on $[a+\varepsilon, b]$, $a+\varepsilon<b$? Why? **(14)**
- ❖ What is the maximum possible order of a permutation in S_8, the group of permutations on the eight numbers $\{1,2,3,....,8\}$? Justify your

answer. (Majority of marks will be given for the justification). **(13)**

2014

- If G is a group in which $(a\cdot b)^4 = a^4\cdot b^4$, $(a\cdot b)^5 = a^5\cdot b^5$ and $(a\cdot b)^6 = a^6.b^6$, for all $a,b \in G$, then prove that G is Abelian. **(8)**
- Let J_n be the set of integers mod n. Then prove that J_n is a ring under the operations of addition and multiplication mod n. Under what conditions on n, J_n is a field? Justify your answer. **(10)**
- Let R be an integraldomain with unity. Prove that the units of R and R[x] are same. **(10)**

2013

- Prove that if every element of a group (G, 0) be its own inverse, then it is an abelian group. **(10)**
- Show that any finite integral domain is a field.**(13)**
- Every field is an integral domain — Prove it.**(13)**
- Prove that **(14)**
  (i) the intersection of two ideals is an ideal.
  (ii) a field has no proper ideals.

2012

- Show that every field is without zero divisor.**(10)**
- Show that in a symmetric group S_3, there are four elements σ satisfying σ^2 = Identity and three elements satisfying σ^3 = Identity. **(13)**
- If R is an integral domain, show that the polynomial ring R[x] is also an integral domain. **(14)**

2011

- Let G be a group and x and y be any two elements of G. If $y^5 = e$ and $yxy^{-1} = x^2$. then show that $O(x)=31$, Where e is the identity element of G and $x \neq e$. **(10)**
- Let Q be the set of all rational numbers show that $Q(\sqrt{2})=\{a+b\sqrt{2} : a,b \in Q\}$ is a field under the usual addition and multiplication. **(10)**
- Let G be the group of non- zero complex numbers under multiplication, and let N be the set of complex numbers of absolute value 1. Show that G/N is isomorphic to the group of all positive real numbers under multiplication. **(13)**
- Let G be a group of order 2p, p prime. Show that either G is cyclic or G is generated by $\{a, b\}$ with relations $a^p = e = b^2$ and $bab = a^{-1}$. **(10)**

2010

- Let $G=\left\{\begin{bmatrix} a & a \\ a & a \end{bmatrix} \middle| a \in \mathbb{R} , a \neq 0\right\}$ Show that G is a group under matrix multiplication. **(10)**
- Let F be a field of order 32. Show that the only subfields of F are F itself and $\{0, 1\}$. **(10)**
- Prove or disprove that $(\mathbb{R},+)$ and $(\mathbb{R}^+,.)$ are isomorphic groups where $\mathbb{R}^+$ denotes the set of all positive real numbers. **(13)**
- Show that zero and unity are only idempotents of Z_n if $n = p^r$, where p is a prime. **(13)**
- Let R be a Euclidean domain with Euclidean valuation d. Let n be an integer such that $d(1)+n \geq 0$. Show that the function $d_n : R-\{0\} \to S,$ where S is the set all negative intergers defined by $d_n(a)=d(a)+n$ for all $a \in R-\{0\}$ is a Euclidean valuation. **(13)**

2009

- Prove that a non-empty subset H of an group G is normal subgroup of $G \Leftrightarrow$ *for* all $x,y \in H$, $g \in G$, $(gx)(gy)^{-1} \in H$. **(10)**
- If G is a finite Abelian group, then show that $O(a,b)$ is a divisor of l.c.m of $O(a)$, $O(b)$. **(10)**
- Find the multiplicative inverse of the element $\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ of the ring, M , of all matrices of order two over the integers. **(13)**
- Show that $d(a) < d(ab)$, Where a, b be two non-zero elements of a Euclidean domain R and b is not a unit in R. **(13)**
- Show that a field is an integral domain and a non-zero finite integral domain is a field. **(13)**

2008

- If a group is such that $(ab)^2 = a^2b^2$ for all $a,b \in G$ then prove or disprove that G is abelian. **(10)**

- Prove or disprove that there exists an integral domain with six elements. **(10)**
- Prove or disprove that $(IR^*,.)$ is isomorphic to $(IR,+)$ **(13)**
- Find the sylow subgroups of the group Z_{24} (the additive group of modulo 24) .

2007

- **(i)** Prove or disprove that if H is a normal subgroup of a group G such that H and G/H are cyclic, then G is cylic.
  (ii) Show by counter example that the distributive laws in the definition of a ring is not redundant. **(10)**
- **(i)** In the ring of integers modulo 10 $(i.e\ z_{10}\ , \oplus_{10}\ , \odot_{10})$, find the subfields.
  (ii) Prove or disprove that only non-singular matrices form a group under matrix multiplication. **(10)**
- Show that there are no simple groups of order 63 & 56. **(14)**
- Prove that every Euclidean domain is PID. **(14)**

2006

- Prove that the set of all real numbers of the form, $(a+b\sqrt{2})$ where a and b are rational numbers , is a field under usual addition and multiplication. **(14)**

2005

- Show that the set of cube roots of unity is a finite Abelian group with respect to multiplication. **(10/2006)**
- Show that the set $S=\{1,2,3,4\}$ forms an Abelian group for the operation of multiplication modulo 5. **(14/2006)**
- Prove that the set of all real numbers of the form $a+b\sqrt{2}$, where a and b area real numbers, is a field under the usual addition and multiplication. **(13/2006)**
- If R is a commutative ring with unit element and M is an ideal in R, then show that M is maximal ideal if R/M is a field. **(13)**

2004

- If every element except the identity, of a group is of order 2, Prove that the group is abelian.
- Prove that the set $R=\{a+\sqrt{2}b, a,b\in I\}$ is a ring. Is it an integral domain ? Justify your answer. **(13)**
- Let G be a group of real numbers under addition and G′ be a group of +ve real numbers under multiplication . Show that the mapping $f:G\to G'$ defined by $f(x)=2^a\ \ \forall a\in G$ is a homomorphism. Is it an isomorphism too ? supply reasons. **(13)**

2003

- Let $G=\{a\in R: -1<a<b\}$. Define a binary operation* on G by $a*b=\dfrac{a+b}{1+ab}$ for all $a,b\in G$. Show that $(G,*)$ is a group . **(13)**
- Let R be the set of matrices of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, $a,b\in F$, Where F is a field with usual addition and multiplication as binary operations, Show that R is a commutative ring with unity. Is it a field if $F=z_2, z_5$?

2002

- Show that every group consisting of four or less than four elements is abelian.
- In the symmetric group S_n of Permutations of n symbols, find the number of even permutation. Show that the set A_n of even permutations forms a finite group. Identify S_n and A_n when $n=4.14$.
- If F is a finite field & α, β are two non-zero elements of F, then show that there exist elements a & b in F such that $1+\alpha a^2+\beta b^2=0$. **(14)**
- Show that in an integral domain every prime element is irreducible. Give an example to show that the converse is not true.

2001

- Write the elements of the symmetric group S_3 of degree 3, Prepare its multiplication table and find all normal subgroups of S_3.

- If every element of a group G is its own inverse, Prove that the group G is abelian. Is the converse true ? Justify your claim. **(14/2003)**
- Define a unique factorization domain. Show that $z\left[\sqrt{-5}\right]$ is an integral domain which is not a unique factorization domain. **(13)**

2000

- Show that an infinite cyclic group is isomorphic to the additive group of integers.
- Show that every finite integral domain is a field.
- Show that every finite field is a field extension of field of residues modulo a prime P.

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