Integrate the function $f(x,y) = xy(x^2+y^2)$ over the domain,
$R: \{-3 \le x^2-y^2 \le 3,\ 1 \le xy \le 4\}$

To find $\iint_R f(x,y)\,dx\,dy$

Let $u = x^2-y^2 \quad ; -3 \le u \le 3$

$v = xy \quad ; 1 \le v \le 4$

$$\iint_R f(x,y)\,dx\,dy = \iint f(u,v).J(u,v)\,du\,dv \quad \text{— ②}$$

We know that

$$\frac{\partial(x,y)}{\partial(u,v)} \times \frac{\partial(u,v)}{\partial(x,y)} = (-1)^2 = 1$$

$$\therefore \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ y & x \end{vmatrix} = 2(x^2+y^2) \quad \text{— ③}$$

$\Rightarrow$ from ② & ③

$$\Rightarrow \cancel{dx}\ (x^2+y^2)\,dx\,dy = \tfrac{1}{2}du.dv.$$

$$\therefore \iint_R f(x,y).dx\,dy = \frac{1}{2}\int_{v=1}^{4}\int_{u=-3}^{3} v.du.dv = \frac{1}{2}\int_{v=1} v[u]_{-3}^{3}\,dv$$

$$= \frac{1}{2}\int_{v=1}^{4} 6v.dv = \frac{3}{2}[v^3]_1^4 = \frac{3}{2}[16-1] = \frac{45}{2}$$

$$\iint_R f(x,y)\,dx\,dy = \frac{45}{2}$$

Find the volume of solid above the xy-plane and directly below the ~~position~~ portion of elliptic paraboloid $x^2+\frac{y^2}{4}=z$ which is cut off by the plane $z=9$.

$\Rightarrow$ Let $V=\iiint dx\,dy\,dz=\iint (z_2-z_1)\,dx\,dy.$

$$V=\iint \left(9-x^2-\frac{y^2}{4}\right)dx\,dy$$

for bounds of y; $y=0$ to $2\sqrt{9-x^2}$

for bounds of x; $x=0$ to 3.

$$\therefore V=\int_0^3\int_0^{2\sqrt{9-x^2}}\left(9-x^2-\frac{y^2}{4}\right)dx\,dy=\int_0^3\left[9y-x^2y-\frac{y^3}{12}\right]_0^{2\sqrt{9-x^2}}dx$$

$$=\int_0^3\left[18\sqrt{9-x^2}-2x^2\sqrt{9-x^2}-\frac{8(9-x^2)\sqrt{9-x^2}}{12}\right]dx$$

$$=\int_0^3\left[2(9-x^2)^{3/2}-\frac{2}{3}(9-x^2)^{3/2}\right]dx$$

$$V=\frac{4}{3}\int_0^3(9-x^2)^{3/2}dx \qquad \text{Put } x=3\sin\theta\ ;\ dx=3\cos\theta.d\theta.$$

$$V=\int_0^{\pi/2}\frac{4}{\cancel{3}}.\overset{9}{\cancel{27}}\cos^3\theta.3\cos\theta\,d\theta.=27\times4\int_0^{\pi/2}\cos^4\theta.d\theta$$

$$=27\times\cancel{4}.\frac{3\pi}{\cancel{4}\times2\times2}=\frac{81\pi}{4}$$

$$\text{Volume}=\frac{81\pi}{4}$$

Scanned with CamScanner

If $f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0), \end{cases}$

Calculate $\dfrac{\partial^2 f}{\partial x \partial y}$ and $\dfrac{\partial^2 f}{\partial y \partial x}$ at $(0,0)$

$$\Rightarrow \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} f(x,y)\right) = \underset{h\to 0}{Lt} \frac{f_y(x+h,y) - f_y(x,y)}{h} \quad \text{—①}$$

$$\frac{\partial^2 f}{\partial y . \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x} f(x,y)\right) = \underset{k\to 0}{Lt} \frac{f_x(x,y+k) - f_x(x,y)}{k} \quad \text{—②}$$

Now at $(x,y) = (0,0)$

$$f_y(0,0) = \underset{k\to 0}{Lt} \frac{f(0,k) - f(0,0)}{k} = 0$$

$$f_y(h,0) = \underset{k\to 0}{Lt} \frac{f(h,k) - f(0,0)}{k} = \underset{k\to 0}{Lt} \frac{\frac{hk(h^2-k^2)}{h^2+k^2} - 0}{k}$$

$$f_y(h,0) = \frac{h^3}{h^2} = h$$

$$\therefore \text{from ①}; \left.\frac{\partial^2 f}{\partial x . \partial y}\right|_{(0,0)} = \underset{h\to 0}{Lt} \frac{f_y(h,0) - f_y(0,0)}{h} = \frac{h-0}{h} = \frac{h}{h} = 1$$

Again
$$f_x(0,0) = \underset{h\to 0}{Lt} \frac{f(h,0) - f(0,0)}{h} = 0$$

$$f_x(0,k) = \underset{h\to 0}{Lt} \frac{f(h,k) - f(0,0)}{h} = \underset{h\to 0}{Lt} \frac{\frac{hk(h^2-k^2)}{h^2+k^2}}{k}$$

$$f_x(0,k) = -k$$

$$\therefore \left.\frac{\partial^2 f}{\partial y . \partial x}\right|_{(0,0)} = \underset{k\to 0}{Lt} \frac{f_x(0,k) - f_x(0,0)}{k} = \underset{k\to 0}{Lt} \frac{-k-0}{k} = -1$$

$$\frac{\partial^2 f}{\partial x . \partial y} \neq \frac{\partial^2 f}{\partial y . \partial x}.$$

Examine if the improper integral $\int_0^3 \frac{2x\,dx}{(1-x^2)^{2/3}}$ exists.

$\Rightarrow$

Let $I = \int_0^3 \frac{2x\,dx}{(1-x^2)^{2/3}}$

The only point of discontinuity shall be '1' which belongs to $(0,3)$.

$$I = \underset{\epsilon\to 0^-}{Lt}\int_0^{1-\epsilon} \frac{2x}{(1-x^2)^{2/3}}\,dx + \underset{\epsilon\to 0^+}{Lt}\int_{1+\epsilon}^{3} \frac{2x}{(1-x^2)^{2/3}}\,dx$$

$$= \underset{\epsilon\to 0^-}{Lt}\left[-3(1-x^2)^{1/3}\right]_0^{1-\epsilon} + \underset{\epsilon\to 0^+}{Lt}\left[-3(1-x^2)^{1/3}\right]_{1+\epsilon}^{3}$$

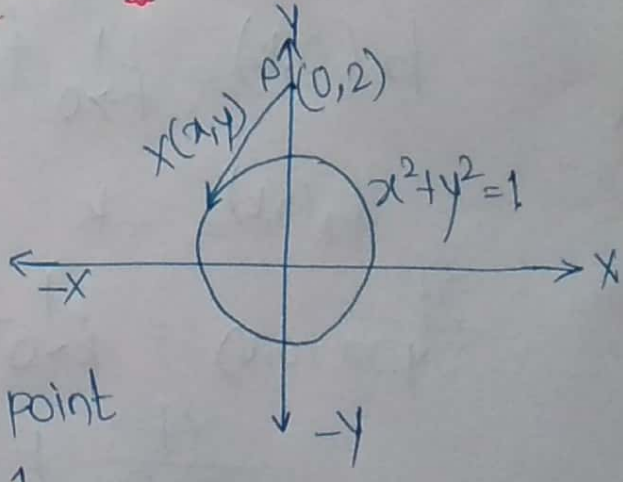
$$= \underset{\epsilon\to 0^-}{lt}\left[-3(1-(1-\epsilon)^2)^{1/3} + (-3(1-3^2)^{1/3} + 3(1-(1+\epsilon)^2)^{1/3}\right]$$

$$= -3[1-1]^{1/3} + (-3(-8)^{1/3}) + 3(1-1)^{1/3}$$

$$= -3\times -2 = 6 \quad \text{which is finite}$$

Hence, I exists.

Prove that $\frac{\pi}{3} \le \iint_D \frac{dx\,dy}{\sqrt{x^2+(y-2)^2}} \le \frac{\pi}{2}$ where D is the unit disc.



$\Rightarrow$ Let $f(x,y) = \sqrt{x^2+(y-2)^2}$ be a function; which given the distance between $(0,2)$ and any point (x,y) on the unit circle; $x^2+y^2=1$.

Here distance (max) OR (min) of PX is such that

$$1 \leq PX \leq 3$$

or $$1 \leq \sqrt{x^2+(y-2)^2} \leq 3$$

$$1 \geq \frac{1}{\sqrt{x^2+(y-2)^2}} \geq \frac{1}{3}$$

$$\iint_D \frac{1}{3}\, dx\, dy \leq \iint_D \frac{dx\,dy}{\sqrt{x^2+(y-2)^2}} \leq \iint_D dx\,dy$$

$$= \frac{1}{3}\left[\pi(1)\right] \leq \iint_D \frac{dx\,dy}{\sqrt{x^2+(y-2)^2}} \leq \pi(1)^2.$$

$$\Rightarrow \quad \frac{\pi}{3} \leq \iint_D \frac{dx\,dy}{\sqrt{x^2+(y-2)^2}} \leq \pi.$$

Hence Proved.