Mains Test Series - 2018
Test - 14, Paper - II, Answer Key

1(a) Let $G = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} / a, b, c, d \in Z \right\}$ under addition. Let $H = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G / a+b+c+d=0 \right\}$. Prove that H is a subgroup of G. What if '0' is replaced by '1'?

Sol Given that $G = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} / a, b, c, d \in Z \right\}$ is a group w.r.t addition.

and $H = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in G / a+b+c+d=0 \right\}$.

since $0+0+0+0=0$

$\therefore \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in H$.

$\therefore H \neq \phi$.

$\therefore$ H is non-empty subset of G

Let $A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$, $B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$ be any two elements of H such that $a_1+b_1+c_1+d_1=0$ & $a_2+b_2+c_2+d_2=0$.

Now we have

$$A - B = \begin{bmatrix} a_1-a_2 & b_1-b_2 \\ c_1-c_2 & d_1-d_2 \end{bmatrix} \longrightarrow (1).$$

since $(a_1-a_2)+(b_1-b_2)+(c_1-c_2)+(d_1-d_2)$

$= (a_1+b_1+c_1+d_1) - (a_2+b_2+c_2+d_2)$

$= 0 - 0$

$= 0$.

$\therefore$ From (1), we have

$A - B \in H$.

$\therefore$ H is a subgroup of G.

If 0 is replaced by 1 then

$H = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} / a+b+c+d=1 \right\}$

for example

if $A = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$ are any two elements in H then $A+B \notin H$

i.e H is not closed

because,

$$A+B = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \notin H \quad (\because 1+(-1)+0+2 \neq 1).$$

$\therefore$ H is not a subgroup of G.

1(b) Show that $\langle x+2\rangle$ is a maximal ideal of $Q[x]$ and hence $\frac{Q[x]}{\langle x+2\rangle}$ is a field.

Soln: Since Q is field, $Q[x]$ is a principal ideal domain.

We have $\langle x+2\rangle = \{(x+2)f(x) : f(x) \in Q[x]\}$.

W.K.T the ideal $A = \langle p(x)\rangle$ in $F[x]$ is a maximal ideal if $p(x)$ is an irreducible element of $F[x]$.

$\therefore \langle x+2\rangle$ is a maximal ideal of $Q[x]$, if we prove that $x+2$ is an irreducible element of $Q[x]$.

Let $x+2 = f(x)\,g(x)$, where $f(x), g(x) \in Q[x]$ — (1)

Then $\deg(f(x)\,g(x)) = \deg(x+2) = 1$

$\Rightarrow \deg f(x) + \deg g(x) = 1$

[$\because$ If $f(x)$, $g(x)$ are two non-zero polynomials in $F[x]$ (F being a field), then $\deg(f(x)\,g(x)) = \deg f(x) + \deg g(x)$]

This gives us two cases.

case (1): $\deg f(x) = 0$ and $\deg g(x) = 1$.

case (2): $\deg f(x) = 1$ and $\deg g(x) = 0$.

In case (1), we may take

$f(x) = a_0 \neq 0 \in Q$ and

$g(x) = b_0 + b_1 x$; $b_0 \in Q$, $b_1 \neq 0 \in Q$.

putting in ①, we get

$$x+2 = a_0(b_0+b_1x)$$

$\Rightarrow a_0b_0 = 2$ and $a_0b_1 = 1$, $a_0 \neq 0$, $b_1 \neq 0 \in \mathbb{Q}$.

Now $a_0b_1 = 1 \Rightarrow a_0 / 1$ (i.e. a_0 divides 1)

$\Rightarrow f(x) = a_0$ is a unit.

Thus $x+2$ is an irreducible element of $\mathbb{Q}[x]$.

Similarly, in case (2) we can prove that $g(x)$ is a unit and $x+2$ is an irreducible element of $\mathbb{Q}[x]$.

Hence $\langle x+2 \rangle$ is a maximal ideal of $\mathbb{Q}[x]$.

Since $\mathbb{Q}[x]$ is a commutative ring with unity, $\frac{\mathbb{Q}[x]}{\langle x+2 \rangle}$ is a field.

($\because$ If R is a commutative ring with unity, then an ideal M of R is maximal iff $\frac{R}{M}$ is a field).

1(C) Discuss the convergence of the series.

$$1+\frac{1}{2}x+\frac{1\cdot 3}{2\cdot 4}x^2+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3+\cdots\ (x>0)$$

Sol^n: Neglecting the first term $u_n = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}x^n$

$$u_{n+1} = \frac{1\cdot 3\cdot 5\cdots(2n-1)(2n+1)}{2\cdot 4\cdot 6\cdots(2n)(2n+2)}x^{n+1}$$

$$\frac{u_n}{u_{n+1}} = \frac{2n+2}{2n+1}\cdot\frac{1}{x} = \frac{1+\frac{1}{n}}{1+\frac{1}{2n}}\cdot\frac{1}{x} \longrightarrow \frac{1}{x} \text{ as } n\to\infty$$

$\therefore$ By Ratio test, Σu_n converges if $\frac{1}{x}>1$ i.e, $x<1$.

and diverges if $\frac{1}{x}<1$ i.e, $x>1$.

If $x=1$, Ratio test fails.

when $x=1$, $\frac{u_n}{u_{n+1}} = \frac{2n+2}{2n+1}$

$$n\left(\frac{u_n}{u_{n+1}}-1\right) = n\left(\frac{2n+2}{2n+1}-1\right) = \frac{n}{2n+1} = \frac{1}{2+\frac{1}{n}}$$

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty}\frac{1}{2+\frac{1}{n}} = \frac{1}{2}<1$$

$\therefore$ By Raabe's test Σu_n is divergent.

Hence Σu_n is convergent if $x<1$ and divergent if $x\geq 1$.

1(d) Use Cauchy's theorem and/or Cauchy integral formula to evaluate the following integrals.

(i) $\int_{|z|=1} \frac{z+3}{z^4+az^3}$; $|a|>1$ (ii) $\int_{|z-2|=2} \frac{\log(z+1)}{z-3} dz$

Sol^n: (i) Given that

$$\int_{|z|=1} \frac{z+3}{z^4+az^3} = \int_{|z|=1} \frac{z+3}{z^3(z+a)} dz$$

Comparing the given integral with $\int_C \frac{f(z)}{(z-z_0)^{n+1}} dz$ $\quad \int \frac{f(z)\,dz}{z^{2+1}} = \frac{2\pi i}{2!} f''(z_0)$

Since $f(z) = \frac{z+3}{z+a}$ is analytic in $|z|=1$.

Here $z_0=0$. and $z_0=0$ is a point inside $|z|=1$

$\therefore$ we can apply the Cauchy's integral formula

$$\int_{|z|=1} \frac{f(z)}{(z-z_0)^3} = \frac{2\pi i}{2!} f''(z_0) \quad \text{—— (1)}$$

$$f(z) = \frac{z+3}{(z+a)} \Rightarrow f'(z) = \frac{z+a-z-3}{(z+a)^2} = \frac{a-3}{(z+a)^2}$$

$$\Rightarrow f'(0) = \frac{a-3}{a^2}.$$

and $f''(z) = (a-3)\left[\frac{-2}{(z+a)^3}\right] = \frac{-2(a-3)}{(z+a)^3}$

$$\Rightarrow f''(z_0) = f''(0) = \frac{-2(a-3)}{(z+a)^3}$$

$\therefore$ from (1) $\int_{|z|=1} \frac{f(z)}{(z+a)^3} dz = \frac{2\pi i}{2!} \frac{(-2)(a-3)}{a^3}$

$$= -\frac{2\pi i(a-3)}{a^3} = \frac{2\pi i(3-a)}{a^3}$$

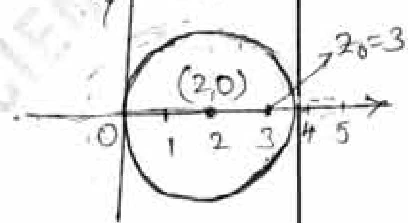
1(d) (ii) $\displaystyle\int_{|z-2|=2} \frac{\log(z+1)}{z-3}\,dz$

Sol'n: Given that $\displaystyle\int_{|z-2|=2} \frac{\log(z+1)}{z-3}\,dz$

comparing the given integral with $\displaystyle\int_C \frac{f(z)}{z-z_0}\,dz$

where $C: |z-2|=2$

Centre $(2,0)$, $r=2$



Since $f(z)=\log(z+1)$ is analytic in $|z-2|=2$ and $z_0=3$ is a point inside $|z-2|=2$

$\therefore$ we can apply Cauchy's integral formula

$$\int_{|z-2|=2} \frac{f(z)}{z-z_0}\,dz = 2\pi i f(z_0)$$

i.e. $$\int_{|z-2|=2} \frac{\log(z+1)}{z-3}\,dz = 2\pi i \log(3+1)$$

$$= 2\pi i \log 4$$

$$= 4\pi i \log 2$$

1(e) Write the dual of the following LPP

Maximize $Z = 5x_1 + 12x_2 + 4x_3$

subject to

$x_1 + 2x_2 + x_3 \leq 10$

$2x_1 - x_2 + 3x_3 = 8$

$x_1, x_2, x_3 \geq 0$

Solⁿ: The equality constraint can be written as $2x_1 - x_2 + 3x_3 \leq 8$

$2x_1 - x_2 + 3x_3 \geq 8$.

Since the problem is of maximization type all constraints should be of ($\leq$) type

Now multiply $2x_1 - x_2 + 3x_3 \geq 8$ by -1, we get

$-2x_1 + x_2 - 3x_3 \leq -8$

Now we may write the primal

$\text{Max } Z = 5x_1 + 12x_2 + 4x_3$

subject to:

$x_1 + 2x_2 + x_3 \leq 10$

$2x_1 - x_2 + 3x_3 \leq 8$

$-2x_1 + x_2 - 3x_3 \leq -8$

Let y_1, y_2 & y_3 be the dual variables associated with the above 3 constraints.

Then the dual problem is given by

$\text{Min } W = 10y_1 + 8y_2 - 8y_3$

subject to

$y_1 + 2y_2 - 2y_3 \geq 5$

$2y_1 - y_2 + y_3 \geq 12$

$y_1 + 3y_2 - 3y_3 \geq 4$

This can be written as

$\text{Min } W = 10y_1 + 8(y_2 - y_3)$

$y_1 + 2(y_2 - y_3) \geq 5$

$2y_1 - (y_2 - y_3) \geq 12$

$y_1 + 3(y_2 - y_3) \geq 4$

Here the new variable $y_2 - y_3 = y'$ becomes unrestricted in sign being the difference of two non-negative variables.

$\therefore$ The above dual problem, takes the form

$\text{Min } W = 10y_1 + 8y'$

subject to

$y_1 + 2y' \geq 5$

$2y_1 - y' \geq 12$

$y_1 + 3y' \geq 4.$

$y_1 \geq 0$, y' is unrestricted.

2(b) Given that β and γ are in S_4 with $\beta\gamma = (1\ 4\ 3\ 2)$ $\gamma\beta = (1\ 2\ 4\ 3)$ and $\beta(1) = 4$. determine β and γ.

Sol'n: let $\beta = (1\ 4\ 2\ 3)$, $\gamma = (2\ 3\ 4)$

To see this we compute $(\gamma\beta)(1)$ in two ways:

Using $\gamma\beta = (1\ 2\ 4\ 3)$, we obtain $\gamma\beta(1) = 2$

from this we have

$$\gamma\beta(1) = \gamma(\beta(1)) = \gamma(4)$$

$$\text{So } \gamma(4) = 2$$

Now continue in this way:

Another Correct answer is $\beta = (1\ 4)(2\ 3)$
$\gamma = (2\ 4)$

2(d) Prove that $\int_0^{2\pi} \frac{(1+2\cos\theta)^n \cos n\theta}{3+2\cos\theta} d\theta = \frac{2\pi}{\sqrt{5}}(3-\sqrt{5})^n$.

Sol'n: Let $I = \int_0^{2\pi} \frac{(1+2\cos\theta)^n \cos n\theta}{(3+2\cos\theta)} d\theta$

$= \text{real part of } \int_0^{2\pi} \frac{(1+2\cos\theta)^n e^{in\theta}}{(3+2\cos\theta)} d\theta$

putting $z = e^{i\theta}$, we get

$$\int_0^{2\pi} \frac{(1+2\cos\theta)^n}{3+2\cos\theta} e^{in\theta} d\theta = \int_C \frac{\left[1+2\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)\right]^n}{\left[3+2\left(\frac{e^{i\theta}+e^{-i\theta}}{2}\right)\right]} e^{in\theta} d\theta$$

$$= \int_C \frac{\left(1+z+\frac{1}{z}\right)^n z^n}{\left(3+z+\frac{1}{z}\right)} \cdot \frac{1}{iz} dz$$

where C is the circle $|z|=1$.

$$= \frac{1}{i}\int_C \frac{(1+z+z^2)^n}{(1+3z+z^2)} dz$$

$$= \frac{1}{i}\int_C \frac{(1+z+z^2)^n}{(z-\alpha)(z-\beta)} dz$$

$$= \frac{1}{i}\int_C f(z)\, dz$$

where α, β are the roots of the equation $z^2+3z+1=0$.

$z^2+3z+1=0 \Rightarrow z = \frac{-3\pm\sqrt{9-4}}{2}$

$\Rightarrow z = \frac{-3\pm\sqrt{5}}{2}$

$\alpha = \frac{-3\pm\sqrt{5}}{2}$; $\beta = \frac{-3-\sqrt{5}}{2}$

and $\alpha\beta = 1$

clearly $|\beta| > |\alpha|$ so that $|\alpha| < 1$.

$\therefore$ The only pole of $f(z)$ with in C is at $z=\alpha$.

Residue of $f(z)$ at $\alpha = \lim_{z\to\alpha} (z-\alpha)f(z)$

$$= \lim_{z\to\alpha} \frac{(1+z+z^2)^n}{z-\beta}$$

$$= \frac{(1+\alpha+\alpha^2)^n}{\alpha-\beta}$$

$$= \frac{\left(1-\frac{3}{2}+\frac{\sqrt{5}}{2}+\frac{7-3\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

$$= \frac{(3-\sqrt{5})^n}{\sqrt{5}}$$

Hence $\int_0^{2\pi} \frac{(1+2\cos\theta)^n}{(3+2\cos\theta)} e^{in\theta}\, d\theta = 2\pi i \frac{1}{i}\left(\frac{3-\sqrt{5}}{\sqrt{5}}\right)^n$

$$= \frac{2\pi\,(3-\sqrt{5})^n}{\sqrt{5}}$$

Equating real parts on bothsides, we get

$$I = \frac{2\pi}{\sqrt{5}}(3-\sqrt{5})^n$$

3(a)(i) Prove that A_5 has no subgroup of order 15 to 20

(ii) Find a permutation β such that $\beta^2 = (13579)(26t)$

(iii) Give an example of a finite non-commutative ring. Give an example of an infinite noncommutative ring that doesnot have a unity.

Sol^n:

(i) Suppose that H is a subgroup of A_5 of order 20.

we claim that H contains all 24 elements of A_5 that have order 5. To verify this assume that there is some α in A_5 of order 5 that is not in H.

Then $A_5 = H \cup \alpha H \cup \alpha^2 H$.

To see that the coset $\alpha^2 H$ is not the same as H note that $\alpha^2 H = H$ implies that $\langle \alpha^2 \rangle \subset H$ and $\langle \alpha \rangle = \langle \alpha^2 \rangle$.

Moreover, $\alpha^2 H$ is not the same as αH for then $\alpha \in H$.

It follows that $\alpha^3 H$ is equal to one of the cosets H, αH or $\alpha^2 H$. If $\alpha^3 H = H$ then $\alpha^3 \in H$ and therefore $\langle \alpha \rangle = \langle \alpha^3 \rangle \subseteq H$, which contradicts the assumption that α is not in H.

If $\alpha^3 H = \alpha H$ then $\alpha^2 \in H$ and therefore $\langle\alpha\rangle = \langle\alpha^2\rangle \subseteq H$, which contradicts the assumption that α is not in H.

If $\alpha^3 H = \alpha^2 H$ then $\alpha \in H$ which contradicts the assumption that α is not in H.

The same argument, shows that H must contain all 24 elements of order 5. Since $|H| = 20$ we have a contradiction.
An analogous arguement shows that A_5 has no subgroup of order 15.

(ii) Since $|\beta^2| = 15$, we know that $|\beta| = 15$ or 30.
But A_9 has no element of order 30,
so $|\beta| = 15$. Then
$\beta = \beta^{16} = (\beta^2)^8 = (1\,7\,3\,9\,5)(2\,8\,6)$

(iii) for any $n > 1$, the ring $M_2(Z_n)$ of 2×2 matrices with entries from Z_n is a finite non-commutative ring. The set $M_2(2Z)$ of 2×2 matrices with even integer entries is an infinite non-commutative ring that doesnot have a unity.

3(b) Let $f_n(x) = \frac{x}{1+nx^2}$ for all real x. Show that f_n converges uniformly to a function f. What is f? Show that for $x \neq 0$, $f_n'(x) \to f'(x)$ but $f_n'(0)$ does not converge to $f'(0)$. Show that the maximum value $|f_n(x)|$ can take is $\frac{1}{2\sqrt{n}}$.

Sol^n: Here $f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{1+nx^2} = 0 \ \forall x \in R$

$$|f_n(x) - f(x)| = \left|\frac{x}{1+nx^2} - 0\right| = \left|\frac{x}{1+nx^2}\right|$$

Let $y = \frac{x}{1+nx^2}$

then $$\frac{dy}{dx} = \frac{(1+nx^2)(1-\cancel{x})\cdot 2nx}{(1+nx^2)^2} = \frac{1-nx^2}{(1+nx^2)^2}$$

For max. (or) min $\frac{dy}{dx} = 0$

$$\Rightarrow 1-nx^2 = 0 \Rightarrow x = \frac{1}{\sqrt{n}}$$

Also $$\frac{d^2y}{dx^2} = \frac{(1+nx^2)^2(-2nx) - (1-nx^2)\cdot 2(1+nx^2)\cdot 2nx}{(1+nx^2)^4}$$

$$= \frac{-2nx(1+nx^2) - 4nx(1-nx^2)}{(1+nx^2)^3}$$

$$\left.\frac{d^2y}{dx^2}\right]_{x=\frac{1}{\sqrt{n}}} = \frac{-2\sqrt{n}(1+1)}{(1+1)^3} = -\frac{\sqrt{n}}{2} < 0.$$

$\Rightarrow$ y is maximum when $x = \frac{1}{\sqrt{n}}$ and maximum value of y

$$= \frac{1/\sqrt{n}}{1+1} = \frac{1}{2\sqrt{n}}$$

$\therefore M_n = \max_{x\in[a,b]} |f_n(x) - f(x)| = \max_{x\in[a,b]} \left| \frac{x}{1+nx^2} \right|$

$= \frac{1}{2\sqrt{n}} \to 0$ as $n \to \infty$

Hence $\langle f_n \rangle$ converges uniformly to f on $[a,b]$.

$\Rightarrow f(x) = 0 \quad \forall\, x \in R$

$\Rightarrow f'(x) = 0 \quad \forall\, x \in R$

when $x \neq 0$,

$f_n'(x) = \frac{(1+nx^2)\cdot(1-x)\cdot 2nx}{(1+nx^2)^2} = \frac{1-nx^2}{(1+nx^2)^2}$

$\lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty} \frac{1-nx^2}{(1+nx^2)^2}$. $\quad \left| \text{form } \frac{\infty}{\infty} \right.$

$= \lim_{n\to\infty} \frac{-x^2}{2(1+nx^2)\cdot x^2} = 0 = f'(x)$

So that if $x \neq 0$, the formula $\lim_{n\to\infty} f_n'(x) = f'(x)$ is true.

At $x = 0$,

$f_n'(0) = \lim_{h\to 0} \frac{f_n(0+h) - f_n(0)}{h}$

$= \lim_{h\to 0} \frac{\frac{h}{1+nh^2} - 0}{h} = \lim_{h\to 0} \frac{1}{1+nh^2} = 1$

So that $\lim_{n\to\infty} f_n'(0) = 1 \neq f'(0)$.

Hence at $x = 0$, the formula $\lim_{n\to\infty} f_n'(x) = f'(x)$ is false.

3(C) Solve the following LPP by Simplex method
Maximise $Z = x_1 + 2x_2$
subject to $-x_1 + 2x_2 \le 8$, $x_1 + 2x_2 \le 12$, $x_1 - 2x_2 \le 3$.
$x_1, x_2 \ge 0$. Obtain alternative optimal basic feasible solution, if it exists.

Sol^n:- The objective of the given LPP is of maximization type and the RHS of all constraints are ≥ 0.
Now we write the given LPP in the standard form

$$\text{Max}\,z = x_1 + 2x_2 + 0s_1 + 0s_2 + 0s_3$$

subject to

$$-x_1 + 2x_2 + s_1 = 8$$
$$x_1 + 2x_2 + s_2 = 12$$
$$x_1 - 2x_2 + s_3 = 3$$
$$x_1, x_2, s_1, s_2, s_3 \ge 0$$

where s_1, s_2, s_3 are slack variables. Now the initial basic feasible solution is given by setting $x_1 = x_2 = 0$ (Non basic)

$s_1 = 8$, $s_2 = 12$, $s_3 = 3$.

∴ The initial basic feasible solution (IBFS) is $(0, 0, 8, 12, 3)$ for which $Z = 0$

Now we move from the current basic feasible solution to the next better basic feasible solution. Put the above information in tableau form.

	C_j	1	2	0	0	0		
C_B	Basis	x_1	x_2	s_1	s_2	s_3	b	θ
0	s_1	-1	(2)	1	0	0	8	$8/2 = 4$ →
0	s_2	1	2	0	1	0	12	$12/2 = 6$
0	s_3	1	-2	0	0	1	3	—
	$Z_j = \sum a_{ij} C_B$	0	0	0	0	0	0	
	$C_j' = C_j - Z_j$	1	2 ↑	0	0	0		

from the table,

x_2 is the incoming variable as $C_j = 2$ is maximum and the corresponding column is known as key column.

The minimum +ve ratio θ occurs in the first row.

$\therefore$ s_1 is the outgoing variable and the common intersection element (2) is the key element.

Now convert the key element to unity and all other elements in its column to zero. Then we obtain a new iterated simplex tableau as

	c_j	1	2	0	0	0		
c_B	Basis	x_1	x_2	s_1	s_2	s_3	b	θ
2	x_2	$-1/2$	1	$1/2$	0	0	4	—
0	s_2	(2)	0	-1	1	0	4	$4/2 = 2 \rightarrow$
0	s_3	0	0	1	0	1	11	—
$Z_j = \sum a_{ij} c_B$		-1	2	1	0	0	8	
$C_j = c_j - Z_j$		2 $\uparrow$	0	-1	0	0		

From the above table, x_1 is the incoming variable, s_2 is the out-going variable and (2) is the key element. Now convert the key element to unity and all other elements in its column to zero. Then we get the new iterated simplex table as

C_B	Basis	x_1 (1)	x_2 (2)	s_1 (0)	s_2 (0)	s_3 (0)	b	θ
2	x_2	0	1	1/4	1/4	0	5	20
1	x_1	1	0	-1/2	1/2	0	2	
0	s_3	0	0	(1)	0	1	11	11 →
$z_j = \Sigma a_{ij} c_B$		1	2	0	1	0	12	
$C_j = c_j - z_j$		0	0	0	-1	0		

Since all $C_j \le 0$, an optimal solution has been reached
∴ The optimum basic feasible solution is
$x_1 = 2,\ x_2 = 5$ and $Z_{max} = 12$.

Existence of alternative optima

From the above table net evaluation for the non-basic variable s_1 is zero. Clearly this is an indication that the current solution is not unique. we can bring s_1 into the basis in place of s_3 which satisfies the exist criterian.

C_B	Basis	x_1 (1)	x_2 (2)	s_1 (0)	s_2 (0)	s_3 (0)	b
2	x_2	0	1	0	1/4	-1/4	9/4
1	x_1	1	0	0	1/2	-1/2	15/2
0	s_1	0	0	1	0	0	11
$Z_j = \Sigma a_{ij} c_B$		1	2	0	1	0	12
$C_j = z_j - c_j$		0	0	0	-1	0	

Therefore, an alternative optimum solution is
$x_1 = 15/2,\ x_2 = 9/4,\ Z_{max} = 12$

4(a) → Show that $Z[\sqrt{-5}] = \{a+b\sqrt{-5} / a, b \in Z\}$ is not a Euclidean domain.

Sol: Let, if possible, $Z[\sqrt{-5}]$ be a Euclidean domain. Consequently, an element in $Z[\sqrt{-5}]$ is prime if and only if it is irreducible.

forexample:
Here $3 \in Z[\sqrt{-5}]$ is irreducible but not prime.

To show that 3 is an irreducible element:

Let $3 = (a+b\sqrt{5}i)(c+d\sqrt{5}i)$ — ① ; $a, b, c, d \in Z[\sqrt{-5}]$

Taking conjugates on both sides, we get
$3 = (a-b\sqrt{5}i)(c-d\sqrt{5}i)$ — ②

multiplying ① & ②, we get
$9 = (a^2+5b^2)(c^2+5d^2)$

Both the sides of the above equation are positive integers.

Consequently, we have the following cases.

case (1): $a^2+5b^2 = 1$ and $c^2+5d^2 = 9$

case (2): $a^2+5b^2 = 9$ and $c^2+5d^2 = 1$

case (3): $a^2+5b^2 = 3$ and $c^2+5d^2 = 3$

It is clear that case (3) is not possible in Z.

case(1) is possible when $a = \pm 1$, $b = 0 \Rightarrow a+b\sqrt{5}i = \pm 1$ which are units in $Z[\sqrt{-5}]$.

Similarly, case (2) yields that $c+d\sqrt{5}i = \pm 1$

which are units in $Z[\sqrt{-5}]$.

Hence 3 is an irreducible element of $Z[\sqrt{-5}]$.

To show that 3 is not a prime element of $Z[\sqrt{-5}]$:

We know that
$Z[\sqrt{-5}] = \{a+b\sqrt{5}i \,/\, a,b \in Z \text{ and } i=\sqrt{-1}\}$.
is an integral domain with unity.

$2+\sqrt{5}i,\ 2-\sqrt{5}i \in Z[\sqrt{-5}]$ and
$(2+\sqrt{5}i)(2-\sqrt{5}i) = 9$.

obviously, 3 divides $(2+\sqrt{5}i)(2-\sqrt{5}i) = 9$ but doesnot divide $2+\sqrt{5}i$ and $2-\sqrt{5}i$.

For if 3 divides $2+\sqrt{5}i$, then $2+\sqrt{5}i = 3(a+b\sqrt{5}i)$ for some $a,b \in Z$

Consequently, $3a = 2$ $(a \in Z)$ which is impossible.

Similarly, 3 doesnot divide $2-\sqrt{5}i$.

Hence 3 is not prime in $Z[\sqrt{-5}]$.

$3 \in Z[\sqrt{-5}]$ is an irreducible but not prime.

$\therefore Z[\sqrt{-5}]$ is not an Euclidean domain.

4(b) → Show that $\prod_{n=0} (1+x^{2^n})$ converges to $\frac{1}{1-x}$ if $|x|<1$.

Sol^n: Given infinite product is

$$\prod_{n=0}^{\infty} (1+x^{2^n}) = (1+x)(1+x^2)(1+x^{2^2})\cdots\cdots$$
$$\cdots(1+x^{2^{n-1}})(1+x^{2^n})\cdots$$

Let

$$P_n = \prod_{n=0} (1+x^{2^n}) = (1+x)(1+x^2)(1+x^{2^2})\cdots(1+x^{2^{n-1}})$$

$$=\left(\frac{1}{1-x}\right)\left[(1-x^1)(1+x)(1+x^2)(1+x^{2^2})(1+x^{2^3})\cdots\cdots(1+x^{2^{n-1}})\right]$$

$$=\left(\frac{1}{1-x}\right)\left[1-x^2(1+x^2)(1+x^{2^2})\cdots(1+x^{2^{n-1}})\right]$$

$$=\frac{1}{1-x}\left[1-(x^2)^2)(1+x^{2^2})(1+x^{2^3})\cdots(1+x^{2^{n-1}})\right]$$

$$=\frac{1}{1-x}\left[1-x^{4(=2^2)}(1+x^4)(1+x^{2^3})\cdots(1+x^{2^{n-1}})\right]$$

$$=\frac{1}{1-x}\left[1-(x^4)^2(1+x^{2^3})\cdots(1+x^{2^{n-1}})\right]$$

$$=\frac{1}{1-x}\left[(1-x^{8(=2^3)}(1+x^8)\cdots(1+x^{2^{n-1}})\right]$$

$$=\frac{1}{1-x}\left[1-(x^8)^2(1+x^{2^4})\cdots\cdots(1+x^{2^{n-1}})\right]$$

$$=\frac{1}{1-x}\left[(1-x^{16})(1+x^{2^4})\cdots(1+x^{2^{n-1}})\right]$$

$$=\frac{1}{1-x}\left[(1-x^{2^4})(1+x^{2^4})\cdots(1+x^{2^{n-1}})\right]$$

$$=\frac{1}{1-x}(1-x^{2^n})$$

Now if $|x|<1$ (i.e. $-1<x<1$)

then $x^{2n}\to 0$ as $n\to\infty$

$\therefore P_n \to \frac{1}{1-x}$ as $n\to\infty$

$\therefore$ The infinite product $\prod_{n=0}^{\infty}(1+x^{2^n})$ Converges to $\frac{1}{1-x}$.

4(c) Show that $f(z) = \begin{cases} \dfrac{xy^2(x+iy)}{x^2+y^4} & \text{when } z \neq 0 \\ 0 & \text{when } z = 0 \end{cases}$.

is not differentiable.

Sol^n: Let $z \to 0$ along the radius vector $y = mx$.

Then we have

$$\underset{z\to 0}{Lt} \frac{f(z)-f(0)}{z} = \underset{z\to 0}{Lt} \frac{xy^2(x+iy)}{(x^2+y^4)z}$$

$$= \underset{z\to 0}{Lt} \frac{xy^2}{x^2+y^4}$$

$$= \underset{x\to 0}{Lt} \frac{x(mx)^2}{x^2+(mx)^4}$$

$$= \underset{x\to 0}{Lt} \frac{m^2x}{1+m^4x} = 0$$

Again suppose $z \to 0$ along $x = y^2$, then we have

$$\underset{z\to 0}{Lt} \frac{f(z)-f(0)}{z} = \underset{z\to 0}{Lt} \frac{x.x}{x^2+x^2}$$

$$= \frac{1}{2}, \text{ which is non-zero.}$$

So we see that $f'(0)$ is not unique, i.e. the values of $f'(0)$ are not same as $z \to 0$ along different curves.

Hence $f'(z)$ does not exist at the origin.

$\therefore f(z)$ is not differentiable at $z = 0$.

4(d) There are four men and each of them has to perform one of the four tasks. The men differ in their efficiency and ability to complete the tasks. The estimate of the time required by each person to complete each task as shown in the table below. Assign a task to each man so as to minimise the total time spent on the four assignments.

		Man			
		A	B	C	D
Tasks	a	18	26	17	11
	b	14	28	14	26
	c	38	19	18	15
	d	19	26	24	10

Sol: Step(i) After subtracting the minimum of each row from all elements of that row, the reduced matrix is given by

$$\begin{bmatrix} 7 & 15 & 6 & 0 \\ 0 & 14 & 0 & 12 \\ 23 & 4 & 3 & 0 \\ 9 & 16 & 14 & 0 \end{bmatrix}$$

(ii) subtracting the minimum elements of column from elements of that column, we get

$$\begin{bmatrix} 7 & 4 & 6 & 0 \\ 0 & 10 & 0 & 12 \\ 23 & 0 & 3 & 0 \\ 9 & 12 & 14 & 0 \end{bmatrix}$$

Step(2):

cover all the zeros by minimum no. of horizontal and vertical lines.

A symmetric approach for this is to look for a row or column containing the maximum number of zeros.

see that we can cover all the zeros by 3 lines only.

so, $r = 3 < 4 = n$

So, go to step (3)

$$\begin{bmatrix} 7 & 11 & 6 & 0 \\ 0 & 10 & 0 & 12 \\ 23 & 0 & 3 & 0 \\ 9 & 12 & 14 & 0 \end{bmatrix}$$

Step(3)

6 is the least uncovered element.

substract 6 from all the uncovered elements.

Add 6 to elements at intersection of the covering lines namely 12 at position (2,4) and 0 at position (3,4).

leave other covered elements unchanged and the reduced matrix so obtained is

$$\begin{bmatrix} 1 & 5 & 0 & 0 \\ 0 & 10 & 0 & 18 \\ 23 & 0 & 3 & 6 \\ 3 & 6 & 8 & 0 \end{bmatrix}$$

Again, cover the zeros by minimum number of horizontal and vertical lines. we required exactly 4 lines to cover all the zeros.

As $r = 4 = n$; optimal assignment can be made at this stage.

$$\begin{array}{c|cccc} & A & B & C & D \\ \hline a & 1 & 5 & 0 & \cancel{0} \\ b & 0 & 10 & \cancel{0} & 18 \\ c & 23 & 0 & 3 & 6 \\ d & 3 & 6 & 8 & 0 \end{array}$$

Assignments are made in the cell (4,4), (1,3), (2,1) & (3,2) for the fourth, second, first and third rows respectively, which contains only one zero in each. Cross of zeros lying on the corresponding columns marked zeros.

∴ The optimal assignment is

$C \to a$, $A \to b$, $B \to c$, $D \to d$

The minimum time assignment is

$10 + 17 + 14 + 19 = 60$

5(a) Solve $(x^2-y^2-yz)p+(x^2-y^2-zx)q = z(x-y)$

Sol'n: The Lagrange's auxillary equations for given equation are.

$$\frac{dx}{x^2-y^2-yz} = \frac{dy}{x^2-y^2-zx} = \frac{dz}{z(x-y)} \quad \text{——} (1)$$

Choosing $1,-1,0$ as multipliers each fraction of (1)

$$= \frac{dx-dy}{(x^2-y^2-yz)-(x^2-y^2-zx)} = \frac{dx-dy}{z(x-y)} \quad \text{——} (2)$$

Choosing $x,-y,0$ as multipliers each fraction of (1).

$$= \frac{xdx-ydy}{x(x^2-y^2-yz)-y(x^2-y^2-zx)}$$

$$= \frac{xdx-ydy}{(x-y)(x^2-y^2)} \quad \text{——} (3)$$

from (1), (2) and (3)

$$\frac{dz}{z(x-y)} = \frac{dx-dy}{z(x-y)} = \frac{xdx-ydy}{(x-y)(x^2-y^2)} \quad \text{——} (4)$$

$$\Rightarrow \frac{dz}{z} = \frac{dx-dy}{z} = \frac{2xdx-2ydy}{2(x^2-y^2)}$$

Taking the first two fractions of (4)

$$dz = dx-dy \Rightarrow z-(x+y) = c_1 \quad \text{——} (5)$$

Taking the 1st and 3rd fractions of (4)

$$\log(x^2-y^2) - 2\log z = c_2 \Rightarrow \frac{x^2-y^2}{z^2} = c_2 \quad \text{——} (6)$$

from (5) and (6), the solution is

$$\phi\left(z-x+y, \frac{x^2-y^2}{z^2}\right) = 0$$

where ϕ is an arbitrary function.

5(b) Reduce $\frac{\partial^2 z}{\partial x^2} + y^2 \frac{\partial^2 z}{\partial y^2} = y$ to canonical form.

Sol'n: Rewriting the given equation, we get

$$r + y^2 t - y = 0 \quad \text{---} \; (1)$$

comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = 1$, $S = 0$ and $T = y^2$ so that $S^2 - 4RT = -4y^2 < 0$ for $y \neq 0$, showing that (1) is elliptic.

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to

$$\lambda^2 + y^2 = 0 \Rightarrow \lambda = iy, -iy$$

the corresponding characteristic equations are given by

$$dy/dx + iy = 0 \text{ and } dy/dx - iy = 0$$

Integrating $\log y + ix = c_1$ and $\log y - ix = c_2$

choose $u = \log y + ix = \alpha + i\beta$ and $v = \log y - ix = \alpha - i\beta$

where $\alpha = \log y$ and $\beta = x$ — (2)

are now two independent variables.

Now $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial \beta}\cdot\frac{\partial \beta}{\partial x} = \frac{\partial z}{\partial \beta}$, using (2) — (3)

$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha}\cdot\frac{d\alpha}{dy} + \frac{\partial z}{\partial \beta}\cdot\frac{\partial \beta}{dy} = \frac{1}{y}\frac{\partial z}{\partial \alpha}$, using (2) — (4)

$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right) = \frac{\partial^2 z}{\partial \beta^2}$, by (3) — (5)

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{1}{y}\frac{\partial z}{\partial \alpha}\right) = -\frac{1}{y^2}\frac{\partial z}{\partial \alpha} + \frac{1}{y}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial \alpha}\right)$$

$$= -\frac{1}{y^2}\frac{\partial z}{\partial \alpha} + \frac{1}{y}\left\{\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right)\left(\frac{\partial \alpha}{\partial y}\right) + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \alpha}\right)\left(\frac{\partial \beta}{\partial y}\right)\right\}$$

$$= -\frac{1}{y^2}\frac{\partial z}{\partial \alpha} + \frac{1}{y}\left(\frac{\partial^2 z}{\partial \alpha^2}\cdot\frac{1}{y}\right)$$

$$= \frac{1}{y^2}\left(\frac{\partial^2 z}{\partial \alpha^2} - \frac{\partial z}{\partial \alpha}\right) \quad \text{---} \; (6)$$

using (5) and (6) in (1), the required canonical form is

$$\frac{\partial^2 z}{\partial \beta^2} + \frac{\partial^2 z}{\partial \alpha^2} - \frac{\partial z}{\partial \alpha} - y = 0 \text{ (or) } \frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = \frac{\partial z}{\partial \alpha} + e^{\alpha} \text{ using (2)}.$$

5(c) The velocity of a particle at distance S from a point on its path is given by the following table:

S(meters)	0	10	20	30	40	50	60
V(m/sec)	47	58	64	65	61	52	38

Estimate the time taken to travel the first 60 meters using Simpson's $\frac{1}{3}$ rule. compare the result with Simpon's 3/8 rule.

Sol'n: As we know, velocity $v = \frac{ds}{dt}$

$$\therefore dt = \frac{ds}{v}$$

$$\therefore t = \int dt = \int_{S_0}^{S} \frac{ds}{v} = \int_{S_0}^{S} y\, ds \text{ ——— (1)}. \quad \text{where } y = 1/v$$

S	0	10	20	30	40	50	60
V	47	58	64	65	61	52	38
$S = \frac{1}{V}$	0.0213	0.0172	0.0156	0.0154	0.0164	0.0192	0.0263
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Using Simpson's $\frac{1}{3}$rd rule

$$t = \frac{h}{3}\left[(y_0+y_6) + 2(y_2+y_4) + 4(y_1+y_3+y_5)\right]$$

$$\because h = \frac{60-0}{6} = 10$$

$$\therefore t = \frac{10}{3}\left[(0.0213+0.0263) + 2(0.0156+0.0164) + 4(0.0172+0.0154+0.0192)\right]$$

$$= 1.0635 \text{ sec}$$

Now, using Simpsons 3/8 rule:

$$t = \frac{3h}{8}\left[(y_0+y_6) + 2y_3 + 3(y_1+y_2+y_4+y_5)\right]$$

$$= \frac{3\times10}{8}\left[(0.0213+0.0263) + 2\times0.0154 + 3(0.0172 + 0.0156 + 0.0164 + 0.0192)\right]$$

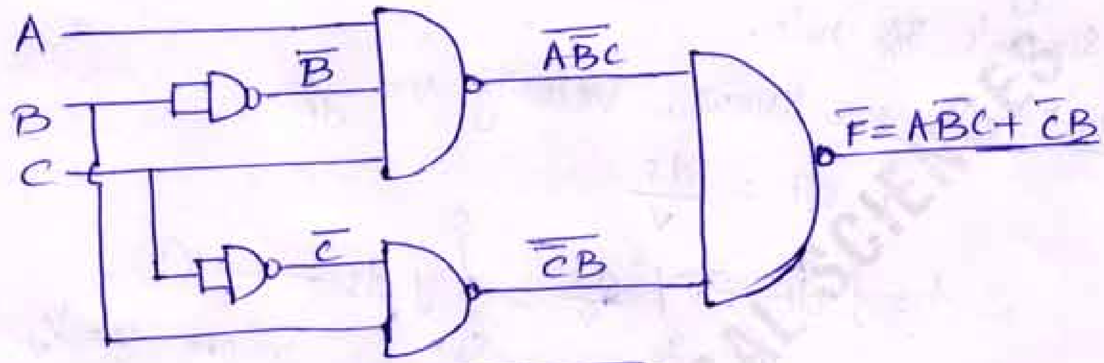
$$= 1.06437 \text{ sec}$$

and difference b/w $\frac{1}{3}$rd rule & $\frac{3}{8}$ rule 0.00087 sec

5(d) (i) Draw the circuit diagram for $\overline{F} = A\overline{B}C + \overline{C}B$ using NAND to NAND logic long.

(ii) In a Boolean algebra B, for any a and b prove that $ab' + a'b = 0$ if and only if $a = b$.

Sol'n: (i) $\overline{F} = A\overline{B}C + \overline{C}B = \overline{\overline{A\overline{B}C + \overline{C}B}} = \overline{\overline{A\overline{B}C} \cdot \overline{\overline{C}B}}$



(ii) Sufficient condition

consider when $a = b$, then

$$ab' + a'b = aa' + bb' = 0 + 0 = 0$$

$$\{\because aa' = 0\}$$

hence, if $a = b$, $ab' + a'b = 0$

Necessary condition

now consider

$$ab' + a'b = 0$$

a	b	ab'	$a'b$	$ab' + a'b$
0	0	0	0	0
0	1	0	1	1
1	0	1	0	1
1	1	0	0	0

$\Rightarrow ab' + a'b = 0 \Rightarrow \boxed{a = b}$

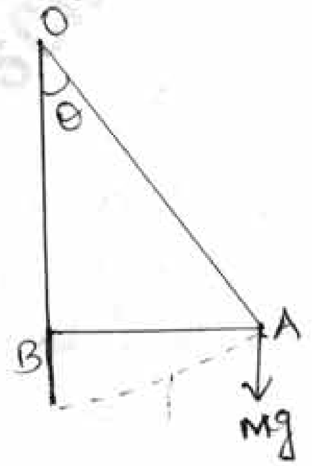
hence, $a'b + ab' = 0$ iff $a = b$.

5(c) Use Hamiltons equations to find the equation of motion of the simple pendulum.

Sol^n: Let l be the length of the pendulum and M the mass of the bob. At time t, let θ be the inclination of the string to the downward, vertical. Then, if T and V are the kinetic and potential energies of the pendulum, then

$$T = \tfrac{1}{2}M(l\dot{\theta})^2 = \tfrac{1}{2}Ml^2\dot{\theta}^2$$

and V = workdone against $Mg = MgA'B = Mgl(1-\cos\theta)$



$$\therefore L = T - V$$
$$= \tfrac{1}{2}Ml^2\dot{\theta}^2 - Mgl(1-\cos\theta) \quad \text{—(1)}$$

Here θ is the only generalised coordinate

$$\therefore p_\theta = \frac{\partial L}{\partial \dot{\theta}} = Ml^2\dot{\theta} \quad \text{—(2)}$$

Since L doesnot contain t explicity

$$\therefore H = T + V = \tfrac{1}{2}Ml^2\dot{\theta}^2 + Mgl(1-\cos\theta)$$

Here the two Hamilton's equations are

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} \quad \text{i.e} \quad \dot{p}_\theta = -Mgl\sin\theta \quad \text{—(3)}$$

and $$\dot{\theta} = \frac{\partial H}{\partial p_\theta} \quad \text{i.e.} \quad \theta = \frac{p_\theta}{(Ml^2)} \quad \text{—(4)}$$

Differentiating (4) we get

$$\ddot{\theta} = \dot{p}_\theta/(Ml^2) = -(Mgl\sin\theta)/Ml^2 \text{ from (3)}$$

$$\Rightarrow \ddot{\theta} = -(g/l)\sin\theta.$$

which is the equation of motion of simple pendulum.

6(a) solve $\{my(x+y)-nz^2\}\frac{\partial z}{\partial x} - \{lx(x+y)-nz^2\}\frac{\partial z}{\partial y} = (lx-my)z$

Sol'n: The Lagrange's auxiliary equations of the given equation are

$$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx} \quad \text{--- (1)}$$

Choosing x, y, z as multipliers, each fraction of (1)

$$= \frac{x\,dx + y\,dy + z\,dz}{x(mz-ny) + y(nx-lz) + z(ly-mx)}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{0}$$

$\therefore$ $x\,dx + y\,dy + z\,dz = 0$ (or) $2x\,dx + 2y\,dy + 2z\,dz = 0$

Integrating, $x^2 + y^2 + z^2 = c_1$ --- (2)

Again, choosing l, m, n as multipliers, each fraction of (1)

$$= \frac{l\,dx + m\,dy + n\,dz}{l(mx-ny) + m(nx-lz) + n(ly-mx)}$$

$$= \frac{l\,dx + m\,dy + n\,dz}{0}$$

$\therefore$ $l\,dx + m\,dy + n\,dz = 0$. so that $lx + my + nz = c_2$ --- (3)

from (2) and (3), the required general solution is given by

$\phi(x^2 + y^2 + z^2, lx + my + nz) = 0$,

ϕ being an arbitrary function.

6(b) Solve $(D^2+DD'-6D'^2)z = x^2\sin(x+y)$

Sol'n: Given $(D^2+DD'-6D'^2)z = x^2\sin(x+y)$

$$\Rightarrow (D+3D')(D-2D')z = x^2\sin(x+y)$$

let ϕ_1, ϕ_2 being arbitrary functions.

$$\therefore C.F = \phi_1(y-3x)+\phi_2(y+2x)$$

$$P.I = \frac{1}{(D+3D')(D-2D')}x^2\sin(x+y)$$

$$= \frac{1}{D+3D'}\left(\frac{1}{(D-2D')}x^2\sin(x+y)\right)$$

$$= \frac{1}{D+3D'}\int x^2\sin(x+c-2x)\,dx$$

$$= \frac{1}{D+3D'}\int x^2\sin(c-x)\,dx$$

Here $c = y+2x$

$$\therefore P.I = \frac{1}{D+3D'}\left[x^2\cos(c-x)-2\int x\cos(c-x)dx\right]$$

$$= \frac{1}{D+3D'}\left[x^2\cos(c-x)-\left\{-2x\sin(c-x)+\int 2\sin(c-x)dx\right\}\right]$$

$$= \frac{1}{D+3D'}\left[x^2\cos(c-x)+2x\sin(c-x)-2\cos(c-x)\right]$$

$$= \frac{1}{D+3D'}\left[(x^2-2)\cos(x+y)+2x\sin(x+y)\right] \quad [\because c = y+2x]$$

$$= \frac{1}{D+3D'}\left[(x^2-2)\cos(x+y)+2x\sin(x+y)\right]$$

$$= \int\left[(x^2-2)\cos(x+c'+3x)+2x\sin(x+c'+3x)\right]dx$$

Here $c' = y-3x$

$$\therefore P.I = \int (x^2-2)\cos(4x+c')\,dx + 2\int x\sin(4x+c')\,dx$$

$$= (x^2-2)\frac{\sin(4x+c')}{4} - \int 2x\,\frac{\sin(4x+c')}{4}\,dx + 2\int x\sin(4x+c')\,dx$$

$$= \frac{1}{4}(x^2-2)\sin(4x+c') + \frac{3}{2}\int x\sin(4x+c')\,dx$$

$$= \frac{x^2-2}{4}\sin(4x+c') + \frac{3}{2}\left[-\frac{x}{4}\cos(4x+c') + \int \frac{\cos(4x+c')}{4}\,dx\right]$$

$$= \frac{x^2-2}{4}\sin(4x+c') - \frac{3}{8}x\cos(4x+c') + \frac{3}{32}\sin(4x+c')$$

$$= \frac{1}{4}(x^2-2)\sin(4x+y-3x) - \frac{3}{8}x\cos(4x+y-3x) + \frac{3}{32}\sin(4x+y-3x) \quad (\because c' = y-3x)$$

$$= \left(\frac{x^2}{4} - \frac{13}{32}\right)\sin(x+y) - \frac{3x}{8}\cos(x+y)$$

Hence, the solution is

$$z = \phi_1(y-3x) + \phi_2(y-2x) + \left(\frac{x^2}{4} - \frac{13}{32}\right)\sin(x+y) - \frac{3x}{8}\cos(x+y).$$

6(c) Prove that for the equation $z+px+qy-1-pqx^2y^2=0$ the characteristic strips are given by

$x=(B+Ce^{-t})^{-1}$, $y=(A+De^{-t})^{-1}$, $z=E-(AC+BD)e^{-t}$

$p=A(B+Ce^{-t})^2$, $q=B(A+De^{-t})^2$.

where A, B, C, D and E are arbitrary constants. Hence, find the integral surface which passes through the line $z=0$, $x=y$.

Sol: Here $f=z+px+qy-1-pqx^2y^2=0$ —— (1)

The characteristic equations of the given partial differential equation (1) are given by

$$\frac{dx}{dt}=\frac{\partial f}{\partial p}=x-qx^2y^2 \quad \text{—— (2)}$$

$$\frac{dy}{dt}=\frac{\partial f}{\partial q}=y-px^2y^2 \quad \text{—— (3)}$$

$$\frac{dz}{dt}=p\frac{\partial f}{\partial p}+q\frac{\partial f}{\partial q}=p(x-qx^2y^2)+q(y-px^2y^2)$$
$$=px+qy-2pqx^2y^2 \quad \text{—— (4)}$$

$$\frac{dp}{dt}=-\frac{\partial f}{\partial x}-p\frac{\partial f}{\partial z}=-(p-2pqxy^2)-p\cdot 1=-2p(1-qxy^2) \quad \text{—— (5)}$$

$$\& \quad \frac{dq}{dt}=-\frac{\partial f}{\partial y}-q\frac{\partial f}{\partial z}=-(q-2pqx^2y)-q\cdot 1=-2q(1-px^2y) \quad \text{—— (6)}$$

from (2) & (5), we have

$$\frac{1}{x}\frac{dx}{dt}=-\frac{1}{2p}\frac{dp}{dt} \Rightarrow \frac{2}{x}\frac{dx}{dt}+\frac{1}{p}\frac{dp}{dt}=0$$
$$\Rightarrow 2\log x+\log p=\log A$$
$$\Rightarrow x^2p=A \quad \text{—— (7)}$$

Again from (3) & (6), we have

$$\frac{1}{y}\frac{dy}{dt}=-\frac{1}{2q}\frac{dq}{dt} \Rightarrow \frac{2}{y}\frac{dy}{dt}+\frac{1}{q}\frac{dq}{dt}=0$$
$$\Rightarrow 2\log y+\log q=B$$
$$\Rightarrow y^2q=B \quad \text{—— (8)}$$

from (2) & (8), we have $\frac{dx}{dt} = x - Bx^2$

$\Rightarrow \frac{1}{x^2}\frac{dx}{dt} = \frac{1}{x} - B$

putting $\frac{1}{x} = v \Rightarrow -\frac{1}{x^2}\frac{dx}{dt} = \frac{dv}{dt}$

$\Rightarrow \frac{dv}{dt} + v = B$. — which LDE with

$IF = e^{\int 1 dt} = e^t$.

$\therefore ve^t = c + \int Be^t dt = C + Be^t$.

$\Rightarrow \frac{1}{x}e^t = C + Be^t$.

$\Rightarrow x = (B + Ce^{-t})^{-1}$ ——— (9)

Again from (3) & (7),

we have $\frac{dy}{dt} = y - Ay^2 \Rightarrow \frac{1}{y^2}\frac{dy}{dt} = \frac{1}{y} - A$

putting $\frac{1}{y} = u \Rightarrow -\frac{1}{y^2}\frac{dy}{dt} = \frac{du}{dt}$

$\Rightarrow \frac{du}{dt} + u = A$. which is LDE

with $IF = e^{\int 1 dt} = e^t$,

$\therefore ue^t = D + \int Ae^t dt = D + Ae^t$

$\Rightarrow \frac{1}{y}e^t = D + Ae^t$

$\Rightarrow y = (A + De^{-t})^{-1}$.

Using (7), (8), (9) & (10) from (6), we have

$\frac{dz}{dt} = \frac{A}{x} + \frac{B}{y} - 2AB$

$= A(B + Ce^{-t}) + B(A + De^{-t}) - 2AB$

$= (AC + BD)e^{-t}$.

Integrating, we get $z = E - (AC+BD)e^{-t}$.

Thus, the characteristic strips for equation (1) are given by. $x=(B+Ce^{-t})^{-1}$, $y=(A+De^{-t})^{-1}$, $z=E-(AC+BD)e^{-t}$

$p=A(B+Ce^{-t})^{2}$, $q=B(A+De^{-t})^{2}$. ——— (11)

The parametric equation of the given line can be taken as

$x=f_1(\lambda)=\lambda$, $y=f_2(\lambda)=\lambda$, $z=0$, λ being a parameter.

Initial values of x, y, z are $x=x_0=\lambda$, $y=y_0=\lambda$, $z=z_0=0$ when $t=0$.

The corresponding initial values of p_0 and q_0 are given by

$f_3'(\lambda)=p_0 f_1'(\lambda)+q_0 f_2'(\lambda)$

$\Rightarrow 0=p_0(1)+q_0(1) \Rightarrow p_0+q_0=0 \Rightarrow q_0=-p_0$

and $z_0+p_0x_0+q_0y_0-1-p_0q_0x_0^2y_0^2=0$ (from ①)

$\Rightarrow 0+\lambda(p_0+q_0)-1-p_0q_0\lambda^4=0$

$\Rightarrow p_0q_0\lambda^4=-1. \Rightarrow p_0^2=\frac{1}{\lambda^4} \Rightarrow p_0=\frac{1}{\lambda^2}, q_0=-\frac{1}{\lambda^2}$

Using initial values in characteristic strip given by (11), we have

$x_0=\lambda=(B+C)^{-1}$, $y_0=(A+D)^{-1} \Rightarrow B+C=\frac{1}{\lambda}, A+D=\frac{1}{\lambda}$.

$\therefore p_0=\frac{1}{\lambda^2}=A(B+C)^2=A\left(\frac{1}{\lambda}\right)^2 \Rightarrow A=1$.

$q_0=-\frac{1}{\lambda^2}=B(A+D)^2=B\left(\frac{1}{\lambda}\right)^2 \Rightarrow B=-1$.

$\therefore C=\frac{1}{\lambda}+1$, $D=\frac{1}{\lambda}-1$.

$z_0=0=E-(AC+BD) \Rightarrow E=1\left(\frac{1}{\lambda}+1\right)-\left(\frac{1}{\lambda}-1\right)=2$

$\therefore x=\left\{-1+\left(\frac{1}{\lambda}+1\right)e^{-t}\right\}^{-1}$ $y=\left\{1+\left(\frac{1}{\lambda}-1\right)e^{-t}\right\}^{-1}$, $z=2(1-e^{-t})$

The required integral surface is obtained by eliminating parameters λ and t from x, y and z

Here $\frac{1}{x}+1=\left(\frac{1}{\lambda}+1\right)e^{-t}$, $\frac{1}{y}-1=\left(\frac{1}{\lambda}-1\right)e^{-t} \Rightarrow \frac{1}{x}-\frac{1}{y}+2=2e^{-t}$.

$\therefore z=2-\left(\frac{1}{x}-\frac{1}{y}+2\right)=-\frac{1}{x}+\frac{1}{y} \Rightarrow \boxed{xyz=x-y}$

which is the required integral surface.

6(d) → The deflection of a vibrating string of length l, is governed by the partial differential equation $y_{tt}=c^2 y_{xx}$. The initial velocity is zero. The initial velocity is zero. The initial displacement is given by

$$y(x,0)=\begin{cases} x/l, & 0<x<l/2 \\ (l-x)/l, & l/2<x<l \end{cases}$$ Here $y_{tt}=\partial^2 y/\partial t^2$ and

$y_{xx}=\partial y^2/\partial x^2$. Find the deflection of the string at any instant of time.

Sol'n: The required deflection $y(x,t)$ of the string is the solution of the one-dimensional wave equation.

$$y_{tt}=c^2 y_{xx} \quad \text{i.e.} \quad \frac{\partial^2 y}{\partial x^2}=\left(\frac{1}{c^2}\right)\times\frac{\partial^2 y}{\partial t^2} \quad \text{——(1)}$$

Subject to the boundary conditions:

$$y(0,t)=y(l,t)=0, \text{ for all } t \quad \text{——(2)}$$

and the given initial conditions, namely

$$\text{initial } \cancel{\text{displacement}}\ \text{velocity} = y(x,0)=f(x)=0,\ 0\le x\le l \quad \text{——3(a)}$$

$$\text{and initial } \cancel{\text{velocity}}\ \text{displacement} = y_t(x,t)=f(x)=\begin{cases} x/l, & 0<x<l/2 \\ (l-x)/l, & l/2<x<l \end{cases} \quad \text{——3(b)}$$

Let the solution of (1) be of the form

$$y(x,t)=X(x)\,T(t) \quad \text{——(4)}$$

Substituting this value of y in (1), we have

$$X''T=\frac{1}{c^2}XT'' \Rightarrow \frac{X''}{X}=\frac{1}{c^2}\frac{T''}{T}=\mu\ (\text{say}) \quad (5)$$

$$\Rightarrow X''-\mu X=0 \quad (6) \quad \text{and} \quad T''-\mu c^2 T=0 \quad (7)$$

Using (2), (4) gives $X(0)\,T(t)=0$ and $X(l)T(t)=0$ (8)

$$\Rightarrow X(0)=0 \ \& \ X(l)=0 \quad (9) \quad (\because T(t)\neq 0)$$

we now solve (6) under B.C. (9).

Three cases arise.

Case (1): Let $\mu=0$. Then solution of (6) is given by

$$X(x)=Ax+B \quad \text{——(10)}$$

Using B.C. (9), (10) give $B=0$, and $Al+B=0$
$\Rightarrow A=0=B.$
$\Rightarrow X(x)=0.$
This leads to $y\equiv 0$
which doesn't satisfy 3(c) & 3(b).
so we reject $\mu=0$

Case (2):
Let $\mu=\lambda^2$, $\lambda\neq 0$.
Then solution of (6) is $X(x)=Ae^{\lambda x}+Be^{-\lambda x}$ —— (11)
Using B.C (9), (11) gives $A+B=0$ and $Ae^{\lambda l}+Be^{-\lambda l}=0$
$\Rightarrow A=B=0$ so that $X(x)=0$.
This leads to $y\equiv 0$
which doesn't satisfy 3(c) & 3(b).
so we reject $\mu=\lambda^2$.

Case (3):
Let $\mu=-\lambda^2$, $\lambda\neq 0$.
Then solution of (6) is $X(x)=A\cos\lambda x+B\sin\lambda x$ —— (12)
Using B.C (9), (12) gives $A=0$ and $A\cos\lambda l+B\sin\lambda l=0$
$\Rightarrow A=0$ and $\sin\lambda l=0$ $(B\neq 0)$ otherwise $X=0$ so that $y\equiv 0$)

Now $\sin\lambda l=0 \Rightarrow \lambda l=n\pi$
$\Rightarrow \lambda=\frac{n\pi}{l}$ $n=1,2,\ldots$ —— (13)
Hence non-zero solutions $X_n(x)$ of (6) are given by
$X_n(x)=B_n\sin\left(\frac{n\pi x}{l}\right)$
Using (13), (7) reduces to
$$T''+\left(\frac{n^2\pi^2 a^2}{l^2}\right)T=0 \qquad \left(\because \mu=-\lambda^2=-\frac{n^2\pi^2}{l^2}\right)$$
whose general solution is
$$T_n(t)=C_n\cos\frac{n\pi ct}{l}+D_n\sin\frac{n\pi ct}{l} \quad \text{—— (14)}$$
$\therefore y_n(x,t)=X_n(t)\,T_n(t)$
$$y_n(x,t)=\sum_{n=1}^{\infty}\left(E_n\cos\left(\frac{n\pi ct}{l}\right)+F_n\sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l} \quad \text{—— (15)}$$
where $E_n=B_nC_n$ and $F_n=B_nD_n$.
Differentiating (15) partially w.r.t t, we get
$$\frac{\partial y}{\partial t}=\sum_{n=1}^{\infty}\left(-\frac{n\pi cE_n}{l}\left(\sin\frac{n\pi ct}{l}\right)+\frac{n\pi cF_n}{l}\cos\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l}=0 \quad \text{—— (16)}$$

putting $t=0$ in (15) and (16) and using 2(a) & 3(b)
we get

$$f(x) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l} \quad \text{and} \quad g(x) = \sum_{n=1}^{\infty} \frac{n\pi c F_n}{l} \sin\frac{n\pi x}{l}$$

where $F_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}dx = 0 \quad (\because f(x)=0)$

and $E_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}dx$

$$\Rightarrow E_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}dx$$

$$\Rightarrow E_n = \frac{2}{l}\int_0^{l/2} g(x) \sin\frac{n\pi x}{l}dx + \int_{l/2}^{l} g(x) \sin\frac{n\pi x}{l}dx$$

($\because$ from 3(b))

$$= \frac{2}{l}\int_0^{l/2} \frac{x}{l} \sin\frac{n\pi x}{l}dx + \frac{2}{n\pi c}\int_{l/2}^{l} \frac{l-x}{l} \sin\frac{n\pi x}{l}dx$$

$$= \frac{2}{l}\left[(x)\left(\frac{-l}{n\pi}\cos\frac{n\pi x}{l}\right) - (1)\left[\frac{-l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right]\right]_0^{l/2}$$

$$+ \frac{2}{l}\left[(l-x)\left[\frac{-l}{n\pi}\cos\frac{n\pi x}{l}\right] - (-1)\left(\frac{-l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right]_{l/2}^{l}$$

$$= \frac{2}{l}\left[\frac{-l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right] + \frac{2}{l}\left[\frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{1}{l}\left(\frac{4l}{n^2\pi^2}\right)\sin\frac{n\pi}{2} = \begin{cases} 0 & \text{if } n=2m,\ m=1,2,\dots \\ \frac{(-1)^{m+1}4}{\pi^2(2m-1)^2} & \text{if } n=2m-1 \text{ and } m=1,2,\dots \end{cases}$$

$$= \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2},$$

$\left[\because n=2m-1 \Rightarrow \sin\frac{n\pi}{2} = \sin(2m-1)\frac{\pi}{2} = \sin\left(m\pi-\frac{\pi}{2}\right)\right.$
$\left.= \sin m\pi\cos\frac{\pi}{2} - \cos m\pi\sin\frac{\pi}{2} = 0-(-1)^m = (-1)^{m+1}\right]$

substituting the above value of E_n and F_n in (15),
the required deflection is given by

$$y(x,t) = \frac{4l}{c\pi^3}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{(2m-1)^3}\sin\frac{(2m-1)\pi x}{l}\cos\frac{(2m-1)\pi ct}{l}.$$

7(a) → Solve the following system of equations.

$10x - 7y + 3z + 5w = 6$, $-6x + 8y - z - 4w = 5$, $3x + y + 4z + 11w = 2$

$5x - 9y - 2z + 4w = 7$ by Gauss Seidel method.

Sol'n: Let $(x^0, y^0, z^0, w^0) = (0,0,0,0)$ be the initial approximation.

Let us apply Gauss Seidel method by using Pivotal process.

$$x^{k+1} = \frac{1}{10}\left(6 + 7y^k - 3z^k - 5w^k\right)$$

$$y^{k+1} = \frac{1}{9}\left(-7 + 5x^{k+1} - 2z^k + 4w^k\right)$$

$$z^{k+1} = \frac{1}{4}\left(2 - 3x^{k+1} - y^{k+1} - 11w^k\right)$$

$$w^{k+1} = \frac{1}{4}\left(-5 - 6x^{k+1} + 8y^{k+1} - z^{k+1}\right)$$

where $k = 1, 2, 3 \ldots$

Proceed in this way, we get

Answer: $x = 5$, $y = 4$, $z = -7$, $w = 1$

7(b). Using Newton's forward formula, estimate the number of persons earning wages between RS 60 and RS. 70 from the following data.

Wages(Rs)	Below 40	40-60	60-80	80-100	100-120
No. of persons in thousands	250	120	100	70	50

Sol'n: Newtons forward formula:

wage x	No. of persons Y	ΔY	$\Delta^2 Y$	$\Delta^3 Y$	$\Delta^4 Y$
40	250				
		120			
60	370		-20		
		100		-10	
80	470		-30		20
		70		$+10$	
100	540		-20		
		50			
120	590				

Newton forward interpolation formulae

$$y(x_0+nh) = y(x_0) + nC_1 \Delta y_0 + \frac{n(n-1)}{2!}\Delta^2 y_0 + \frac{n(n-1)(n-2)}{3!}\Delta^3 y_0 + \frac{n(n-1)(n-2)(n-3)}{4!}\Delta^4 y_0 \quad \text{--- (1)}$$

$x_0 = 40$, $h = 20$

$\therefore 70 = 40 + n \times 20 \Rightarrow n = 3/2$

$$\therefore y(70) = 250 + \frac{3}{2}\times 120 + \frac{\frac{3}{2}\left(\frac{3}{2}-1\right)}{2}\times(-20) + \frac{\frac{3}{2}\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-2\right)}{6}\times(-10) + \frac{\frac{3}{2}\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-2\right)\left(\frac{3}{2}-3\right)}{24}\times 20$$

$$= 250 + 180 + (-7.5) + 0.625 + 0.46875$$

$y(70) = 423.59375$

$\because y(60) = 370$

$\therefore$ No. of persons with wages b/w Rs 60 & Rs. 70 are

$(423.59375 - 370) \times 1000$

$= 53593.75$

$\approx 53593.$

7(c) Convert (i) 46655 given to be in the decimal system into one in base 6.

(ii) $(11110.01)_2$ into a number in the decimal system.

Sol'n: (i)

		Remainder
6	46655	
6	7775	5
6	1295	5
6	215	5
6	35	5
6	5	5
	0	5

So $(46655)_{10} = (55555)_6$

(ii) Converting $(11110.01)_2$ into decimal system

$$(11110.01)_2 = (1\times 2^4 + 1\times 2^3 + 1\times 2^2 + 1\times 2^1 + 0.2^0) + (0\times 2^{-1} + 1\times 2^{-2})$$

$$= 16 + 8 + 4 + 2 + 0 + 0 + 1/4$$

$$= 30 + 0.25$$

So $(11110.01)_2 = (30.25)_{10}$

8(a) → A uniform lamina is bounded by parabolic arc, of latus rectum $4a$, and a double ordinate at a distance b from the vertex. If $b = \frac{1}{3}a(7+4\sqrt{7})$, show that two of the principal axes at the end of a latus rectum are the tangent and normal there.

Sol^n: Let the equation of the parabola be $y^2 = 4ax$ — ①

$\therefore$ coordinates of the end L of L.R. LL' are $(a, 2a)$

Differentiating ① we get $\frac{dy}{dx} = \frac{2a}{y}$

$\therefore$ At $L(a,2a)$, $\frac{dy}{dx} = \frac{2a}{2a} = 1$

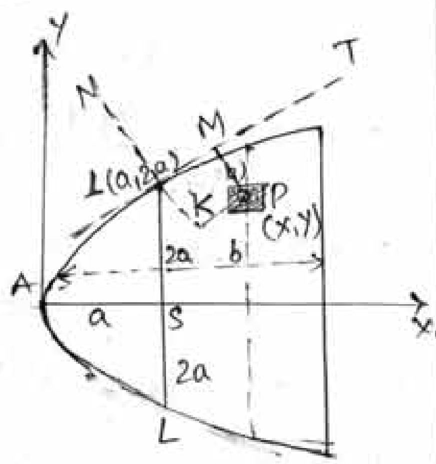
$\therefore$ Equation of the tangent LT at L is

$$y - 2a = 1.(x-a) \Rightarrow y - x - a = 0 \quad \text{—②}.$$

and the equation of the normal LN at L is

$$y - 2a = -\frac{1}{1}(x-a) \quad \text{—③}$$

$$\Rightarrow y + x - 3a = 0$$



Consider an element $\delta x \delta y$ at the point $P(x,y)$ of the lamina.

then PM = length of ⊥lar from P on tangent LT given by ②

$$\frac{y-x-a}{\sqrt{1+1}} = \frac{y-x-a}{\sqrt{2}}$$

and PK = length of ⊥lar from P on the normal LN given by ③

$$= \frac{y+x-3a}{\sqrt{2}}$$

P.I. of the element about LT and LN

$$PM.PK.\delta m = \left(\frac{y-x-a}{\sqrt{2}}\right)\left(\frac{y+x-3a}{\sqrt{2}}\right)\rho\,\delta x\,\delta y$$

If the tangent and normal at L are the principal axes, then the P.I of the lamina about these will be zero.

i.e. P.I of the lamina about LT and LN.

$$= \int_{x=0}^{b} \int_{y=-2\sqrt{ax}}^{2\sqrt{ax}} \left(\frac{y-x-a}{\sqrt{2}}\right)\left(\frac{y+x-3a}{\sqrt{2}}\right) \cdot \rho \, dx\, dy = 0$$

$$\Rightarrow \frac{\rho}{2}\int_0^b \int_{-2\sqrt{ax}}^{2\sqrt{ax}} \{y^2 - 4ay + (3a^2 + 2ax - x^2)\} dx\, dy = 0$$

$$\Rightarrow \int_0^b \left\{\tfrac{1}{3}y^3 - 2ay^2 + (3a^2+2ax-x^2)y\right\}_{-2\sqrt{ax}}^{2\sqrt{ax}} dx = 0$$

$$\Rightarrow 2\int_0^b \left\{\tfrac{8}{3}ax\sqrt{ax} + 2(3a^2+2ax-x^2)\sqrt{ax}\right\} dx = 0$$

$$\Rightarrow \int_0^b \left(\tfrac{8}{3}a^{3/2}x^{3/2} + 6a^{5/2}x^{1/2} + 4a^{3/2}x^{3/2} - 2a^{1/2}x^{5/2}\right) dx = 0$$

$$\Rightarrow \left[\frac{16}{15}a^{3/2}b^{5/2} + 4a^{5/2}b^{3/2} + \tfrac{8}{5}a^{3/2}b^{5/2} - \tfrac{4}{7}a^{1/2}b^{7/2}\right] = 0$$

$$\Rightarrow \frac{16}{15}ab + 4a^2 + \tfrac{8}{5}ab - \tfrac{4}{7}b^2 = 0$$

$$\Rightarrow b^2 - \frac{14}{3}ab - 7a^2 = 0$$

$$\Rightarrow b = \frac{\frac{14}{3}a \pm \sqrt{\left\{\frac{196}{9}a^2 + 28a^2\right\}}}{2} = \tfrac{1}{2}\left(\frac{14}{3} \pm \frac{8}{3}\sqrt{7}\right)a$$

$\Rightarrow b = \frac{a}{3}(7 + 4\sqrt{7})$, (leaving -ve sign. as b can not be -ve)

Hence if $b = \frac{a}{3}(7+4\sqrt{7})$
then the principal axes at L are the tangents & normal there.

8(b) A uniform rod, of length 2a, which has one end attached to a fixed point by a light inextensible string of length $5a/12$, is performing small oscillations in a vertical plane about its position of equilibrium. Find its position at any time, and show that the period of its principal oscillations are $2\pi\sqrt{(5a/3g)}$ & $\pi\sqrt{(a/3g)}$.

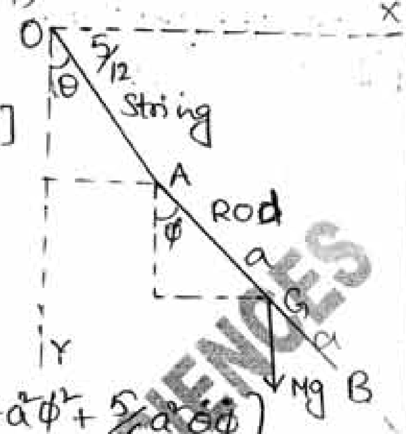
Sol^n: Let OA be the string of length $\frac{5}{12}a$ & AB the rod of mass M and length 2a. Let O be fixed point. At time t, let the string & the rod made angles θ & ϕ, to the vertical respectively. Referred to O as origin, horizontal & vertical lines OX & OY as axes. the coordinates of the C.G., G of the rod are given by

$$x_G = \tfrac{5}{12}a\sin\theta + a\sin\phi$$
$$y_G = \tfrac{5}{12}a\cos\theta + a\cos\phi$$

$\therefore V_G^2 = \dot{x}_G^2 + \dot{y}_G^2 = \left(\frac{5}{12}a\cos\theta\dot{\theta} + a\cos\phi\dot{\phi}\right)^2 + \left(-\frac{5}{12}a\sin\theta\dot{\theta} - a\sin\phi\dot{\phi}\right)^2$

$= \frac{25}{144}a^2\dot{\theta}^2 + a^2\dot{\phi}^2 + \frac{5}{6}a^2\dot{\theta}\dot{\phi}\cos(\theta-\phi)$

$= \frac{25}{144}a^2\dot{\theta}^2 + a^2\dot{\phi}^2 + \frac{5}{6}a^2\dot{\theta}\dot{\phi}$ [$\because \theta$ & ϕ are small]



If T be the total K.E and W the workfunction of the system, then.

$T = \frac{1}{2}M\cdot\frac{1}{3}a^2\dot{\phi}^2 + \frac{1}{2}M\cdot V_G^2$

$= \frac{1}{2}M\left[\frac{1}{3}a^2\dot{\phi}^2 + \frac{25}{144}a^2\dot{\theta}^2 + a^2\dot{\phi}^2 + \frac{5}{6}a^2\dot{\theta}\dot{\phi}\right]$

$\Rightarrow T = \frac{1}{2}Ma^2\left[\frac{25}{144}\dot{\theta}^2 + \frac{4}{3}\dot{\phi}^2 + \frac{5}{6}\dot{\theta}\dot{\phi}\right]$

and $W = Mg\,y_G + C = Mga\left(\frac{5}{12}\cos\theta + \cos\phi\right) + C$

Lagrange's θ-equation is

$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right) - \frac{\partial T}{\partial\theta} = \frac{\partial W}{\partial\theta}$

$\Rightarrow \frac{d}{dt}\left[\frac{1}{2}Ma^2\left(\frac{25}{72}\dot{\theta} + \frac{5}{6}\dot{\phi}\right)\right] - 0 = -\frac{5}{12}Mga\sin\theta$

$\Rightarrow 5\ddot{\theta} + 12\ddot{\phi} = -12c\theta,$ ($\because \theta$ is small) Taking $(g/a) = c$ — (1)

And Lagrange's ϕ-equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\phi}}\right) - \frac{\partial T}{\partial\phi} = \frac{\partial W}{\partial\phi}$

i.e. $\frac{d}{dt}\left[\frac{1}{2}Ma^2\left(\frac{8}{3}\dot{\phi} + \frac{5}{6}\dot{\theta}\right)\right] - 0 = -Mga\sin\phi$

$\Rightarrow 5\ddot{\theta} + 16\ddot{\phi} = -12c\phi,$ [$\because \phi$ is small & $(g/a) = c$] — (2)

Equations (1) & (2), can be written as

$(5D^2+12c)\theta + 12D^2\phi = 0$ and $5D^2\theta + (16D^2+12c)\phi = 0$

Eliminating ϕ b/w these two equations, we have

$\left[(5D^2+12c)(16D^2+12c) - 60D^4\right]\theta = 0$ — (3)

Eliminating ϕ between these two equations, we have

$$\left[(5D^2+12C)(16D^2+12C)-60D^4\right]\theta=0$$

$$\Rightarrow (5D^4+636D^2+36C^2)\theta=0$$

Let the solution of (3) be

$\theta=A\cos(pt+B)$. $\therefore D^2\theta=-p^2\theta$ and $D^4\theta=p^4\theta$

Substituting in (3), we get

$$(5p^4-63Cp^2+36C^2)\theta=0$$

$$\Rightarrow 5p^4-63Cp^2+36C^2=0 \quad \because \theta\neq 0$$

$$\Rightarrow (5p^2-3C)(p^2-12C)=0$$

$$\therefore p_1^2=3/5\,C=\frac{3g}{5a} \;\&\; p_2^2=12C=\frac{12g}{a}, \quad \because C=g/a$$

Hence period of oscillations are $\frac{2\pi}{p_1}$ & $\frac{2\pi}{p_2}$

$$\Rightarrow 2\pi\sqrt{\left(\frac{5a}{3g}\right)} \text{ and } 2\pi\sqrt{\left(\frac{a}{12g}\right)}$$

$$\Rightarrow 2\pi\sqrt{\left(\frac{5a}{3g}\right)} \text{ and } \cancel{2}\pi\sqrt{\left(\frac{a}{3g}\right)}$$

8(C) Show that $\phi=xf(r)$ is a possible form for the velocity potential for an incompressible fluid motion. If the fluid velocity $\vec{q}\to 0$ as $r\to\infty$, find the surfaces of constant speed.

Sol^n:

$$\phi=xf(r) \qquad \text{— (1)}$$

$$q=-\nabla\phi=-\nabla[xf(r)]=-\left[f(r)\nabla x+x\nabla f(r)\right] \text{ — (2).}$$

Now $r^2=x^2+y^2+z^2 \Rightarrow 2r\left(\frac{\partial r}{\partial x}\right)=2x \Rightarrow \frac{\partial r}{\partial x}=x/r$ — (3)

Similarly $\frac{\partial r}{\partial y}=y/r$ and $\frac{\partial r}{\partial z}=z/r$ — (4)

Also, $\nabla x=\left[i(\partial/\partial x)+j(\partial/\partial y)+k(\partial/\partial z)\right]x=i$

and $\nabla f(r)=\left[i(\partial/\partial x)+j(\partial/\partial y)+k(\partial/\partial z)\right]f(r)$

$$=if'(r)(\partial r/\partial x)+jf'(r)(\partial r/\partial y)+kf'(r)\left(\frac{\partial r}{\partial z}\right)$$

$= i f'(r)(x/r) + j f'(r)(y/r) + k f'(r)(z/r)$, by ③ & ④

$= (1/r) f'(r)\,(ix + jy + kz) = \frac{1}{r} f'(r)\vec{r}$.

$\therefore$ ② $\Rightarrow q = -f(r)i - (x/r) f'(r)\vec{r}$

For a possible motion of an incompressible fluid, we have

$\nabla\cdot q = 0$ (or) $\nabla\cdot(-\nabla\phi) = 0 \Rightarrow \nabla^2\phi = 0$.

$\Rightarrow \left[\frac{\partial^2}{\partial x^2} + \partial^2/\partial y^2 + \partial^2/\partial z^2\right][x f(r)] = 0$, using ①

Now, $\frac{\partial^2}{\partial x^2}[x f(r)] = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\{x f(r)\}\right] = \frac{\partial}{\partial x}\left[f(r) + x\frac{\partial f(r)}{\partial x}\right]$

$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial x} + x\frac{\partial^2 f}{\partial x^2} = 2\frac{\partial f}{\partial x} + x\frac{\partial^2 f}{\partial x^2}$

Also $\frac{\partial^2}{\partial y^2}[x f(r)] = x\frac{\partial^2 f}{\partial y^2}$ and $\frac{\partial^2}{\partial z^2}[x f(r)] = x\frac{\partial^2 f}{\partial z^2}$

$\therefore$ ⑥ becomes $2\frac{\partial f}{\partial x} + x\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) = 0$ —— (7)

Now $\partial f/\partial x = df/dr \cdot \partial r/\partial x = f'(x/r)$ using ③ —— (8).

and $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(f'\frac{x}{r}\right) = \frac{f'}{r} + x\frac{\partial}{\partial x}\left(\frac{f'}{r}\right)$

$= \frac{f'}{r} + x\frac{d}{dr}\left(\frac{f'}{r}\right)\frac{\partial r}{\partial x} = \frac{f'}{r} + x\cdot\frac{rf'' - f'}{r^2}\cdot\frac{x}{r}$

$$\frac{\partial^2 f}{\partial x^2} = \frac{f'}{r} + \frac{x^2}{r^2}f'' - \frac{x^2}{r^3}f' \quad \text{—— (9)}$$

Similarly $\frac{\partial^2 f}{\partial y^2} = \frac{f'}{r} + \frac{y^2}{r^2}f'' - \frac{y^2}{r^3}f'$ —— (10)

and $\frac{\partial^2 f}{\partial z^2} = \frac{f'}{r} + \frac{z^2}{r^2}f'' - \frac{z^2}{r^3}f'$ —— (11)

Adding (9), (10) and (11) we get

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{3f'}{r} + \frac{x^2+y^2+z^2}{r^2}f'' - \frac{x^2+y^2+z^2}{r^3}f'$$

$= \frac{3f'}{r} + f'' - \frac{f'}{r}$, as $x^2+y^2+z^2 = r^2$

$$\therefore \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 2f'/r + f'' \quad \text{—— (12)}$$

using (8) and (12), (7) reduces to

$$\frac{2f'(r)}{r} + x\left(\frac{2f'}{r} + f''\right) = 0 \Rightarrow f'' + \frac{4f'}{r} = 0$$

$$\Rightarrow f''/f' + 4/r = 0$$

Integrating $\log f' + 4\log r = \log c_1$ so that $f' = c_1 r^{-4}$ — (13)

Integrating (13) $f = -(c_1/3) \times r^{-3} + c_2$, c_2 being an arbitrary Constant. — (14)

Substituting the values of f' and f from (13) & (14) in (5), we get

$$\vec{q} = -\left\{(c_1/3r^3) - c_2\right\}\vec{i} - (c_1 x/r^5)\vec{r} \quad \text{— (15)}$$

Given that $\vec{q} \to 0$ as $r \to \infty$ hence (15) shows that

$$c_2 = 0$$

$$\therefore \text{ from (15)}, \quad \vec{q} = \frac{c_1}{3r^3}\left(\vec{i} - \frac{3x\vec{r}}{r^2}\right) \quad \text{— (16)}$$

Now $q^2 = \vec{q}\cdot\vec{q} = \frac{c_1^2}{9r^6}\left(\vec{i} - \frac{3x\vec{r}}{r^2}\right)\cdot\left(\vec{i} - \frac{3x\vec{r}}{r^2}\right)$

$$= \frac{c_1^2}{9r^6}\left[\vec{i}\cdot\vec{i} - \frac{6x}{r^2}\vec{r}\cdot\vec{i} + \frac{9x^2}{r^4}\vec{r}\cdot\vec{r}\right]$$

$$= \frac{c_1^2}{9r^6}\left(1 - \frac{6x^2}{r^2} + \frac{9x^2r^2}{r^4}\right) \text{ as } \vec{r}\cdot\vec{r} = r^2 \text{ \& } \vec{r}\cdot\vec{i} = x$$

$$= \frac{c_1^2}{9r^6}\left(1 + \frac{3x^2}{r^2}\right)$$

$$= \frac{c_1^2}{9r^8}(r^2 + 3x^2)$$

Hence the required surfaces of constant speed are

$$q^2 = \text{constant (or)} \left(\frac{c_1^2}{9r^8}\right)(r^2 + 3x^2) = \text{constant}$$

$$\Rightarrow (r^2 + 3x^2)r^{-8} = \text{Constant}$$