Date : 1|08|19

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A CONSOLIDATED QUESTION PAPER-CUM-ANSWER BOOKLET

IMS™
INSTITUTE OF MATHEMATICAL SCIENCES

MAINS TEST SERIES-2019

(JUNE-2019 *to* SEPT.-2019)

Under the guidance *of* K. Venkanna

MATHEMATICS

215/250

PAPER - II : FULL SYLLABUS

TEST CODE: TEST-8: IAS(M)/28-JULY.-2019

Time: 3 Hours **Maximum Marks: 250**

INSTRUCTIONS

1. This question paper-cum-answer booklet has **48** pages and has **30 PART/SUBPART** questions. Please ensure that the copy of the question paper-cum-answer booklet you have received contains all the questions.
2. Write your Name, Roll Number, Name of the Test Centre and Medium in the appropriate space provided on the right side.
3. A consolidated Question Paper-cum-Answer Booklet, having space below each part/sub part of a question shall be provided to them for writing the answers. Candidates shall be required to attempt answer to the part/sub-part of a question strictly within the pre-defined space. Any attempt outside the pre-defined space shall not be evaluated. "
4. Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
5. Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any **THREE** of the remaining questions selecting at least **ONE** question from each Section.
6. The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
7. Symbols/notations carry their usual meanings, unless otherwise indicated.
8. All questions carry equal marks.
9. All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
10. All rough work should be done in the space provided and scored out finally.
11. The candidate should respect the instructions given by the invigilator.
12. The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name: ABHISHEK GARG

Roll No.: 0818220

Test Centre: ONLINE

Medium: ENGLISH

Do not write your Roll Number or Name anywhere else in this Question Paper-cum-Answer Booklet.

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I have read all the instructions and shall abide by them

Signature of the Candidate

I have verified the information filled by the candidate above

Signature of the Invigilator

IMPORTANT NOTE:

Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed. Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

P.T.O.

DO NOT WRITE ON THIS SPACE

INDEX TABLE

QUESTION	No.	PAGE NO.	MAX. MARKS	MARKS OBTAINED	
1	(a)			08	42
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8	(a)				
	(b)				
	(c)				
	(d)				
Total Marks					

$\frac{215}{250}$

P.T.O.

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SECTION – A

1. (a) Give an example of two subgroups H, K which are not normal but HK is a subgroup. **[10]**

Let G be a group of 2×2 matrices, whose determinant is not equal to zero.

$\therefore$ let $A \in G$, such that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Big| \ ad-bc \neq 0$ —— (1)

Now, let $H = \left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \Big| \ ad \neq 0 \right\}$.

Check if H is normal $\left\{ \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} / ad \neq 0 \right\}$

$AHA^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} h_1 & 0 \\ 0 & h_2 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} \notin H$ $\notin H$.

$\therefore$ H is not normal, subgroup. —— (2)

let $K = \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix}$, clearly $AKA^{-1} \notin K$.

$\therefore$ K is not a normal subgroup. —— (3)

Now, $HK = \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} = \begin{bmatrix} 0 & ab \\ cd & 0 \end{bmatrix} \subseteq G$.

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and for $g_1, g_2 \in HK$, clearly,

$g_1 . g_2^{-1} \in HK \Rightarrow$ HK is a subgroup. —— (4).

1. (b) Show by means of an example that we can find $A \subseteq B \subseteq R$ where A is an ideal of B, B is an ideal of R, but A is not an ideal of R. [10]

Let $A = \begin{bmatrix} 0 & 0 & a \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 0 & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{bmatrix}$, and

$R = \begin{bmatrix} d & e & f \\ g & h & i \\ 0 & 0 & j \end{bmatrix}$. Clearly, $A \subseteq B \subseteq R$ —— (1).

Now, to prove A is an ideal of B.

Clearly, for $A_1, A_2 \in A$, $A_1 - A_2 \in A$.

and, for $B_1 \in B$, $A_1 B_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in A$.

$\therefore$ A is an ideal of B. —— (2).

Now, to prove B is an ideal of R.

for $B_1, B_2 \in B$, $B_1 - B_2 \in B$.

and for $R_1 \in R$, $B_1 R_1 = \begin{bmatrix} 0 & 0 & bj \\ 0 & 0 & cj \\ 0 & 0 & 0 \end{bmatrix} \in B$.

$\therefore$ B is an ideal of R —— (3).

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Now, for A and R, $A \subseteq R$,

but $A_1 R_1 = \begin{bmatrix} 0 & 0 & aj \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in A$, but $R_1 A_1 = \begin{bmatrix} 0 & 0 & da \\ 0 & 0 & ga \\ 0 & 0 & 0 \end{bmatrix}$

Clearly, $R_1 A_1 \notin A$, $\therefore$ A is not an ideal of R. —— (4)

1. (c) Prove that the function f defined on $\mathbb{R}$ by $f(x) = \frac{1}{x^2+1}, x \in \mathbb{R}$ is uniformly continuous of $\mathbb{R}$. **[10]**

Given $f(x) = \frac{1}{x^2+1}$ —— ①.

let $x_1, x_2 \in R$, such that $f(x_1) = \frac{1}{x_1^2+1}$; $f(x_2) = \frac{1}{x_2^2+1}$

Now,

$$|f(x_1) - f(x_2)| = \left| \frac{1}{x_1^2+1} - \frac{1}{x_2^2+1} \right| = \frac{|x_2^2 - x_1^2|}{(x_1^2+1)(x_2^2+1)}$$

$$\therefore |f(x_1) - f(x_2)| = \frac{|x_1+x_2||x_1-x_2|}{(x_1^2+1)(x_2^2+1)} \le |x_1-x_2| < \epsilon. \quad \text{—— ②}$$

let $\delta = \epsilon$, such that $|x_1 - x_2| < \delta$.

$\therefore$ By ②, we get,

$|f(x_1) - f(x_2)| < \epsilon$ (however small), whenever

$\exists$ a $\delta = \epsilon$, such that $|x_1 - x_2| < \delta$.

$\therefore$ $f(x)$ is uniformly continuous by $\epsilon - \delta$ method.

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1. (d) By using Riemann integrable definition evaluate $\int_a^b x^{99}\,dx$ where $0 < a < b$. [10]

let I be the interval, $I = \left[a, a+\frac{(b-a)}{n}, a+\frac{2(b-a)}{n}, \cdots b\right]$.

$\therefore I_r = \left[a + \frac{(b-a)(r-1)}{n}, a + \frac{(b-a)r}{n}\right]$.

The lower Darboux sum $= \sum m_r \cdot \delta_r$.

$$L(a,b) = \sum_{r=1}^{n}\left[a + \frac{(b-a)(r-1)}{n}\right]^{99} \cdot \left(\frac{b-a}{n}\right) \quad \text{—— (1)}$$

and upper Darboux sum $= \sum M_r \cdot \delta_r$.

$$U(a,b) = \sum\left[a + \frac{(b-a)r}{n}\right]^{99}\left[\frac{b-a}{n}\right] \quad \text{—— (2)}.$$

Now, $\underline{\int_a^b} x^{99}\,dx = \lim_{n\to\infty} L(a,b)$.

and $\overline{\int_a^b} x^{99}\,dx = \lim_{n\to\infty} U(a,b)$.

on solving, we shall get

$$\int_a^b x^{99}\,dx = \underline{\int_a^b} x^{99}\,dx = \overline{\int_a^b} x^{99}\,dx$$

$$= \frac{b^{100} - a^{100}}{100}$$

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1. (e) Prove that the function $u = e^x (x \cos y - y \sin y)$ satisfies Laplace's equation and find the corresponding analytic function $f(z) = u + iv$. **[10]**

$u = e^x (x\cos y - y\sin y)$, (given).

$\therefore \frac{\partial u}{\partial x} = e^x (x\cos y - y\sin y + \cos y). = \phi_1(x,y)$

$\frac{\partial^2 u}{\partial x^2} = e^x (x\cos y - y\sin y + \cos y + \cos y).$ — ①

Now, $\frac{\partial u}{\partial y} = e^x(-x\sin y - \sin y - y\cos y) = \phi_2(x,y).$

$\frac{\partial^2 u}{\partial y^2} = e^x(-x\cos y - \cos y + y\sin y - \cos y).$ — ②

Clearly, $\boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0}$; ie. u satisfies Laplace's equation.

II[nd] part, ~~let $\phi_1(x,y)$~~ BY MILNE-THOMSON METHOD

$f(z) = \int \phi_1(z,0) - \phi_2(z,0).dz \quad + c$

$f(z) = \int e^z (z+1-0).dz + c = \int e^z(z+1) + c.$

$\therefore$ The required analytic function,

$\boxed{f(z) = z.e^z + c.}$, where c is arbitrary constant.

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3. (a) Let G be defined as all formal symbols x^iy^j, $i = 0, 1$, $j = 0, 1, 2, \ldots, n-1$ where we assume

$x^iy^j = x^{i'}y^{j'}$ if and only if $i = i'$, $j = j'$

$x^2 = y^n = e$, $n > 2$

$xy = y^{-1}x$.

(i) find the form of the product $(x^iy^j)(x^ky^l)(x^\alpha y^\beta)$.

(ii) Using this, prove that G is a non-abelian group of order 2n.

(iii) If n is odd, prove that the center of G is (e), while if n is even the center of G is larger than (e). [This group G is known as a dihedral group.] **[18]**

(i) $(x^iy^j)(x^ky^l)(x^\alpha y^\beta) = P$ (say).

Powers of x can give result x or e. and $xy = y^{-1}x$

$\Rightarrow yxy = x \Rightarrow (yxy)(yxy) = x^2 = e$.

$\therefore$ P shall be of the form, $xy^jxy^lxy^\beta$ $\left\{\begin{matrix}\text{assuming}\\ i,k,\alpha \equiv \text{odd}\end{matrix}\right\}$

$xy^{j-1}\cdot(yxy)y^{l-2}\cdot yxy\cdot y^{\beta-1} = xy^{j-1}\cdot x\cdot y^{l-2}\cdot x\cdot y^{\beta-1}$

Solving this way, we shall get P as a power of x, even for other permutations and combinations of $i, j, k, l, \alpha, \beta$, $\boxed{P = x^{r} \text{ (or) } e \text{ (or) } y^{\lambda} \text{ (or) } xy^{\delta}}$

where $\lambda, \delta \leq n-1$

(ii) To prove that order of $G = 2n$ and G is non-abelian.

from first result, $O(G) = 1+1+n-1+n-1 = 2n$

for non-abelian → given that, $xy = y^{-1}x$ — ①

$\Rightarrow xyx^{-1} = y^{-1}x \cdot x^{-1} = y^{-1}$ (or) $xyx = y^{-1}$ $\left[\because x = x^{-1} \text{ as } x^2 = e\right]$

$\Rightarrow x x^{-1} y x = x^{-1}y^{-1}$ (or) $yx = x^{-1}y^{-1} = xy^{-1}$ — ②

Clearly, $xy \neq yx \Rightarrow G$ is non abelian group.

(iii) let $Z(G)$ be the centre of G; i.e.

$$Z(G) = \{a \in G \mid xa = ax \ \forall \ x \in G\}.$$

Case (i) $n \equiv$ odd

We know that, $y^{i} \cdot x \cdot y^{+i} = x$ $\left[\text{as } yxy = x\right]$

Now, if $y^{i} \in Z(G)$, then y^{i} commute with xy^{j}

i.e. $y^{i} \cdot x \cdot y^{j} = xy^{j} \cdot y^{i} \Rightarrow xy^{-i+j} = xy^{i+j}$

(or) $y^{2i} = e$ $\therefore y^i \in Z(G)$ if $y^{2i} = e$ (or) $\frac{2i}{n}$

Since n is odd, $i = n$ $\Rightarrow \boxed{Z(G) = \{e\}}$.

[as none of the elements of type $xy^j \in Z(G)$].

Case-ii $n \equiv$ even, now $\frac{2i}{n} \Rightarrow i$ can take other values, ~~...~~ & $y^{n/2}$ commutes with all elements of G.

$\therefore \boxed{Z(G) = \{e, y^{n/2}\}}$. & $O(Z(G)) = 2 > 1$

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3. (b) Let $f_n(x) = \frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}, x \in [0,1]$

Show that at $x = 0$, $\frac{d}{dx}\sum f_n(x) \neq \sum \frac{d}{dx} f_n(x)$. **[15]**

Limit function, $f(x) = \lim_{n\to\infty} f_n(x) = \lim \frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}$

using L-Hospital rule, differentiating Numerator and Denominator.

$f(x) = \lim \frac{x}{2nx^2} - \frac{x}{2(n-1)x^2} = \lim \frac{1}{2x}\left[\frac{-1}{n(n-1)}\right] = 0.$

Now, consider $\sum f_n(x)$, $\therefore S_n(x) = \frac{nx}{1+n^2x^2}$

$S(x) = \lim_{n\to\infty} S_n = 0.$ $\therefore S_n = \sum f_n(x)$ converges.

Now, to prove: $\frac{d}{dx}\sum f_n(x) \neq \sum \frac{d}{dx} f_n(x)$ at $x=0$.

$\underline{\text{L.H.S.}} \equiv \frac{d}{dx} S(x) = 0$

$\underline{\text{R.H.S.}} \equiv \sum \frac{d}{dx} f_n(x)\Big|_{x=0} = \sum \left[\frac{n(1+n^2x^2) - 2n^3x^2}{(1+n^2x^2)^2} - \frac{(n-1)(1+(n-1)^2x^2) - 2(n-1)^3x^2}{(1+(n-1)^2x^2)^2}\right]_{x=0} = \sum (n) - (n-1)$

$= \sum 1 = \infty$

$\therefore$ Clearly, L.H.S $\neq$ R.H.S. , ie.

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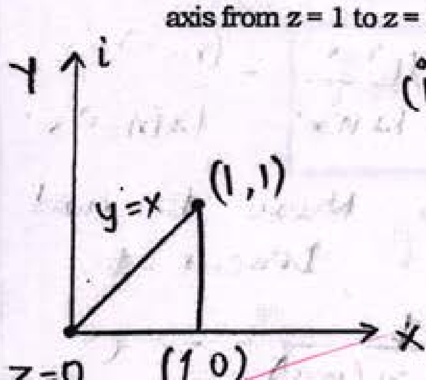
$\frac{d}{dx}\sum f_n(x) \neq \sum \frac{d}{dx} f_n(x)$ at $x=0$.

3. (c) Find the value of the intergral

$\int_0^{1+i}(x-y+ix)\,dz$

(i) Along the straight line from $z = 0$ to $z = 1 + i$.

(ii) Along the real axis from $z = 0$ to $z = 1$ and then along a line parallel to the imaginary axis from $z = 1$ to $z = 1 + i$. **[17]**



(i). let $I = \int_0^{1+i} x-y+ix \; dz$

$= \int_0^{1+i} (x-y+ix)(dx+idy)$

along $y=x$ $(z=0$ to $z=1+i)$.

$\therefore I = \int_0^1 (x-x+ix)(dx+idx) = \int_0^1 ix \cdot dx\,(1+i)$

$= \underline{\underline{i}}(1+i)\left[\frac{x^2}{2}\right]_0^1 = \boxed{\frac{-1+i}{2}}$

(ii) Now, to find $I = \int_0^{1+i} (x-y+ix)(dx+idy)$

first along $y=0$, x varies from $x=0$ to 1.

$$\int_0^1 (1+i)x \cdot dx = (1+i)\left[\frac{x^2}{2}\right]_0^1 = \frac{1+i}{2} = I_1 \text{(say)}.$$

Now, along, $x=1$, y varies from $y=0$ to 1.

$$\int_0^1 (1+i-y)i\,dy = i\left[(1+i)y - \frac{y^2}{2}\right]_0^1$$

$$= i\left[1+i-\frac{1}{2}\right] = i\left[\frac{1}{2}+i\right] = -1+\frac{1}{2}i = I_2 \text{ (say)}$$

Now, required result, $I = I_1 + I_2$

$$= \frac{1}{2}+\frac{i}{2}-1+\frac{i}{2} = \boxed{\frac{-1}{2}+i}$$

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4. (a) For any prime p, show that the polynomial $x^{p-1} + x^{p-2} + \ldots\ldots + x^2 + x + 1$ isirreducible over **Q**. **[15]**

let $f(x) = 1 + x + x^2 + x^3 \ldots \ldots x^{p-2} + x^{p-1} = \dfrac{x^p - 1}{x-1}$

Now, replacing x by $x+1$, we get

$$f(x+1) = \frac{(1+x)^p - 1}{x} = \frac{1}{x}\left[1 + px + \ldots x^p - 1\right]$$

$$\therefore f(x+1) = \frac{1}{x}\left[px + p(p-1)\frac{x^2}{2!} \ldots\ldots\right]$$

$$f(x+1) = p + p(p-1)\frac{x}{2} + p(p-1)(p-2)\frac{x^2}{6} \ldots + x^p$$

Now, let $a_0 = p$, $a_1 = \frac{p(p-1)}{2}$ ... and so on, $a_p = 1$.

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$\therefore p \mid a_0, a_1, \ldots\ldots a_{p-1}$, $p \nmid a_p$ and $p^2 \nmid a_0$.

By EINSTEIN'S CRITERIA OF IRREDUCIBILITY OVER $\mathbb{Q}$, $f(x+1)$ is irreducible over $\mathbb{Q}$. — (1)

To prove $f(x)$ is also irreducible over $\mathbb{Q}$.

Suppose, $f(x)$ is not irreducible and $f(x) = g(x) \cdot h(x)$.

Such $\deg(g(x))$ and $\deg(h(x)) > 0$.

Now, $f(x+1) = g(x+1) \cdot h(x+1)$ is not irreducible as $\deg(g(x+1)) > 0$, $\deg(h(x+1)) > 0$. > which is contradictory

$\therefore$ $\boxed{f(x) \text{ is irreducible over } \mathbb{Q}}$ [by wrong assumption]

4. (b) Prove that the series $e^{-x}-\frac{e^{-2x}}{2}+\frac{e^{-3x}}{3}-\frac{e^{-4x}}{4}+\ldots$ is uniformly convergent on [0,1]. [15]

Given series can be written as, $\sum f_n(x)$, where

$$f_n(x) = (-1)^n \cdot \frac{e^{-nx}}{n} \quad \text{——— (1)}$$

Limit function, $f(x) = \lim_{n\to\infty} f_n(x) = 0$.

Also, $\sum f_n(x) = \frac{e^{-x}}{1} - \frac{(e^{-x})^2}{2} + \frac{(e^{-x})^3}{3} \ldots$

$$\sum f_n(x) = \log(1+e^{-x}) \quad \text{——— (2)}$$ [By expansion of $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} \ldots$].

Now, clearly, since $\log(1+e^{-x})$ is a simple combination of log, e^x and ~~is~~ $\log(1+e^{-x})$ is bounded in [0,1].

$$0 \le \log(1+e^{-x}) \le \log\left(1+\frac{1}{e}\right).$$

$\therefore$ $\boxed{\text{given series is uniformly convergent.}}$

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4. (c) A job shop has purchased 5 new machines of different type. There are 5 available locations in the shop where a machine could be installed. Some of these locations are more desirable than others for particular machines because of their proximity to work centres which would have a heavy work flow to and from these machines. Therefore, the objective is to assign the new machines to the available locations in order to minimize the total cost of material handling. The estimated cost per unit time of materials handling involving each of the machines is given below for the respective locations. Locations 1, 2, 3, 4 and 5 are not considered suitable for machines A, B, C, D and E, respectively. find the optimal solution :

Location (Cost in Rs.)

Machine	1	2	3	4	5
A	×	10	25	25	10
B	1	×	10	15	2
C	8	9	×	20	10
D	14	10	24	×	15
E	10	8	25	27	×

How would the optimal solution get modified if location 5 is also unsuitable for machine A? [20]

	1	2	3	4	5
A	X	10	25	25	10
B	1	X	10	15	2
C	8	9	X	20	10
D	14	10	24	x	15
E	10	8	25	27	x.

Here, $n = 5$.

(i) Subtract lowest element in each row from all elements of that row and repeat same process with each column.

$\sim$

X	0	15	15	0
0	x	9	14	1
0	1	x	12	2
4	0	14	x	5
2	0	17	19	x

$\sim$

x	0	6	3	0
0	x	0	2	
0	1	x	0	2
4	0	5	x	5
2	0	8	7	x

(ii) Now, cover all rows / columns, containing zeroes with minimum number of lines (say) r.

$r = 4. < n(=5)$.

(iii) Find smallest number in uncovered elements and subtract this from all uncovered elements and add at point of intersection of lines.

X	~~0~~	4	1	0
~~0~~	X	0	2	3
~~0~~	3	X	0	4
2	0	3	X	5
0	~~0~~	6	5	X

(iv) Repeat step (ii), covering zeroes.

Now, $r = 5 = n$.

(v) Finding such rows with a single zero and cutting the other zeroes in that column and repeating till all zeroes are encircled/cut.

The optimal solution becomes.

$$\boxed{A \to 5,\ B \to 3,\ C \to 4,\ D \to 2,\ E \to 1,\ \text{Min. Cost} = 60.}$$

II^{nd} part location 5 becomes unsuitable for A.

New table

X	10	25	25	X
1	X	10	15	2
8	9	X	20	10
14	10	24	X	15
10	8	25	27	X

~

X	0	15	15	X
0	X	9	14	1
0	1	X	12	2
4	0	14	X	5
2	0	17	19	X

~

X	0	6	3	X
0	X	0	2	0
0	1	X	0	1
4	0	5	X	4
2	0	8	7	X

$r = 3$

~

X	0	4	1	X
0	X	0	2	0
0	3	X	0	1
2	0	3	X	2
0	0	6	5	X

$r = 4$

~

X	0	3	0	X
0	X	0	2	0
0	4	X	0	1
1	0	2	X	1
0	1	6	5	X

$r = 4$.

On further solving, new optimal sol.: $\boxed{A-2,\ B-3,\ C-4,\ D-5,\ E-1,\ \text{Min. Cost} = 65}$

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SECTION – B

5. (a) Solve

$$\left(\partial^2 z/\partial x^2\right)-\left(\partial^2 z/\partial y^2\right)+\left(\partial z/\partial x\right)+3\left(\partial z/\partial y\right)-2z=e^{x-y}-x^2y.$$ **[10]**

$$\left[D^2-D'^2+D+3D'-2\right]z = e^{x-y}-x^2y \quad ; \quad D=\frac{\partial}{\partial x},\ D'=\frac{\partial}{\partial y}.$$

The solution of this equation be, $z = z_c + z_p$. —— (1)

for complementary function, z_c, consider ~~let~~ homogeneous equation.

Auxillary equation : $(D+D'-1)(D-D'+2)=0$.

$$\therefore\ z_c = e^{x}\phi_1(y-x)+e^{-2x}\phi_2(y+x). \text{ —— (2)}$$

for Particular Integral, z_p., $z_p = \frac{1}{f(D,D')}\left[e^{x-y}-x^2y\right]$.

z_p w.r.t e^{x-y}

$$=\frac{1}{D^2-D'^2+D+3D'-2}\left(e^{x-y}\right)=\frac{1}{1^2-(-1)^2+1-3-2}e^{x-y}=\frac{-1}{4}e^{x-y}. \text{ —— (3)}$$

$$\left(\because\ \frac{1}{f(D,D')}\left(e^{ax+by}\right)=\frac{e^{ax+by}}{f(a,b)}\right).$$

Now, z_p, w.r.t x^2y $= \frac{1}{(D+D'-1)(D-D'+2)}\left(x^2y\right)$.

$$=\frac{-1}{2}\left[1-(D+D')\right]^{-1}\left[1+\frac{D-D'}{2}\right]^{-1}\left(x^2y\right)$$

on solving, we get, $=\frac{-1}{2}\left(x^2y+xy+\frac{3x^2}{2}+\frac{3y}{2}+3x+\frac{21}{4}\right)$ —— (4)

$\therefore$ The general solution becomes,

$$(1)\Rightarrow \boxed{z=z_c+z_p=e^{x}\phi_1(y-x)+e^{-2x}\phi_2(y+x)-\frac{1}{4}e^{x-y}+\frac{1}{2}\left(x^2y+xy+\frac{3x^2}{2}+\frac{3y}{2}+3x+\frac{21}{4}\right).}$$

ϕ_1,ϕ_2 ≡ arbitrary functions

09

5. (b) Reduce the following equation to a canonical form and hence solve it :

$yu_{xx} + (x + y)u_{xy} + xu_{yy} = 0$ **[10]**

Rewriting given equation, in $Rr + Ss + Tt + f(x,y,z,p,q) = 0$

$$yr + (x+y)s + xt = 0 \quad \text{——} \quad (1)$$

The quadratic $R\lambda^2 + S\lambda + T$ becomes.,

$$y\lambda^2 + (x+y)\lambda + x = 0 \Rightarrow \lambda = -1, \frac{-x}{y} \left[\begin{array}{l}\text{Real,} \\ \text{distinct}\end{array}\right]$$

Now, $\frac{dy}{dx} + \lambda_1 = 0$, $\frac{dy}{dx} + \lambda_2 = 0$. becomes.

$$\frac{dy}{dx} - 1 = 0, \quad \frac{dy}{dx} - \frac{x}{y} = 0. \quad \text{(or)}$$

$$y - x = 0, = u(\text{say}); \quad x^2 - y^2 = 0 = v(\text{say}).$$

Changing independent variable x, y to u, v.

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{2\partial^2 z}{\partial u \partial v}\left(\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x}\right) + \frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{\partial z}{\partial u}\left(\frac{\partial^2 u}{\partial x^2}\right) + \frac{\partial z}{\partial v}\left(\frac{\partial^2 v}{\partial x^2}\right)$$

$$\Rightarrow r = \frac{\partial^2 z}{\partial u^2} - 4x\frac{\partial^2 z}{\partial u \partial v} + 4x^2\frac{\partial^2 z}{\partial v^2} + \frac{2\partial z}{\partial v} \quad \text{——} \quad (2).$$

Similarly, $t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial^2 z}{\partial u^2} - 4y\frac{\partial^2 z}{\partial u \partial v} + 4y^2\frac{\partial^2 z}{\partial v^2} - \frac{2\partial z}{\partial v}$. —— (3)

Now, $s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial u^2}\cdot\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial y} + \frac{\partial^2 z}{\partial u \partial v}\left(\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}\right)$

$$+ \frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial x}\cdot\frac{\partial v}{\partial y}\right) + \frac{\partial z}{\partial u}\left(\frac{\partial^2 u}{\partial x \partial y}\right) + \frac{\partial z}{\partial v}\left(\frac{\partial^2 v}{\partial x \partial y}\right).$$

$$\therefore s = -\frac{\partial^2 z}{\partial u^2} + 2(x+y)\frac{\partial^2 z}{\partial u \partial v} - 4xy\frac{\partial^2 z}{\partial v^2} \quad \text{——} \quad (4).$$

Putting values of r, s, t from (2), (3), (4) in (1), eqⁿ :-

$$0 \cdot \frac{\partial^2 z}{\partial u^2} + 2(x-y)^2 \frac{\partial^2 z}{\partial u \partial v} + 0 \cdot \frac{\partial^2 z}{\partial v^2} + 2(y-x)\frac{\partial z}{\partial v} = 0$$

$$\Rightarrow \boxed{u^2 \frac{\partial^2 z}{\partial u \partial v} + u \frac{\partial z}{\partial v} = 0}$$, which is the canonical form. —— (5)

For solution : (5) $\equiv \left[u \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial z}{\partial v} = 0 \right] \times v$

$$uv \frac{\partial^2 z}{\partial u \partial v} + v \frac{\partial z}{\partial v} = 0 \Rightarrow \left[uvDD' + vD' \right] z = 0,$$

which is clearly Cauchy-Euler type. let $e^x = u$, $e^y = v$

$$(D_1 D_1' + D_1') z = 0 \Rightarrow z = e^{-x}\phi_1(y) + \phi_2(x).$$

09 (or) $\boxed{z = \frac{1}{u}\phi_1(\log v) + \phi_2(\log u)}$ → further put u, v in x, y terms.

5. (c) Using Newton's forward formula find the number of men getting wages between Rs. 10 and 15 from the following data : **[10]**

Wages in Rs. :	0 – 10	10 – 20	20 – 30	30 – 40
Frequency :	9	30	35	42

i	Wages (x_{i-1})	y_i	Δy	$\Delta^2 y$	$\Delta^3 y$
0	0–10	9			
			30		
1	10–20	39		5	
			35		2
2	20–30	74		7	
			42		
3	30–40	116			

Here, we use Cumulative frequency as y.

Finding for $x = 15$.

$y_0 = 9$, $\Delta y_0 = 30$, $\Delta^2 y_0 = 5$

$\Delta^3 y_0 = 2$, $u = \frac{15-10}{10} = 0.5$

Using NEWTON'S FORWARD FORMULA,

$$y(15) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0.$$

$$\therefore y(15) = 9 + 0.5(30) + \frac{(0.5)(-0.5)}{2}(5) + \frac{(0.5)(-0.5)(-1.5)}{6}(2).$$

$$\therefore y(15) = 23.5.$$

No. of men getting wages from 10 to 15, become $(y(15) - y(10)) = 23.5 - 9 = 14.5$

$$\boxed{\simeq 15 \text{ men}}$$

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5. (d) The velocity v is a particle at distance s from a point on its path is given by the table

s(ft.):	0	10	20	30	40	50	60
v(ft. / sec):	47	58	64	65	61	52	38

Estimate the time taken to travel 60 ft. by using Simpson's 1/3 rule. compare the result with Simpson's 3/8 rule. **[10]**

$v = \frac{ds}{dt} \Rightarrow dt = \frac{ds}{v}$ (or) $T = \int \frac{ds}{v}$; $h = 10$.

i	s_i	v_i
0	0	$47 = v_0$
1	10	$58 = v_1$
2	20	$64 = v_2$
3	30	$65 = v_3$
4	40	$61 = v_4$
5	50	$52 = v_5$
6	60	$38 = v_6$

Here, y_i becomes, v_i^{-1} $\left(y_i = \frac{1}{v_i}\right)$.

By SIMPSON'S $\frac{1}{3}^{rd}$ RULE,

$$T = \frac{h}{3}\left[y_0 + y_n + 2\left(\text{even ordinates}\right) + 4(\text{odd})\right]$$

$$T = \frac{10}{3}\left[47^{-1} + 38^{-1} + 2\left[64^{-1} + 61^{-1}\right] + 4\left[58^{-1} + 65^{-1} + 52^{-1}\right]\right].$$

$\boxed{T = 1.063521 \text{ sec.}}$ [By Simpson's $\frac{1}{3}$rd rule]. —— (1).

BY SIMPSON'S $\frac{3}{8}$ Rule, $T = \frac{3h}{8}\left[y_0 + y_n + 3(y_1 + y_2 + y_4 \ldots) + 2(y_3 + \ldots)\right]$.

$T = \frac{3 \times 10}{8}\left[47^{-1} + 38^{-1} + 3\left(58^{-1} + 64^{-1} + 61^{-1} + 52^{-1}\right) + 2(65^{-1})\right]$.

$\boxed{T = 1.064375 \text{ sec.}}$ [By Simpson's $\frac{3}{8}$ rule]. —— (2).

Comparing results by two methods,

Difference $= |1.063521 - 1.064375|$

$= 8.54 \times 10^{-4}$ sec.

08

5. (e) In an incompressible fluid the vorticity at every point is constant in magnitude and direction; prove that the components of velocity u, v, w are the solutions of Laplace equation.

[10]

Let velocity, $\vec{q} = u\hat{i} + v\hat{j} + w\hat{k}$ —— (1).

Vorticity, $\nabla \times \vec{q} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & w \end{vmatrix} = \Omega_1\hat{i} + \Omega_2\hat{j} + \Omega_3\hat{k}$

$\therefore \Omega_1 = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = a_1$ (say). = constant (given). —— (2).

Similarly, $\Omega_2 = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = a_2$; $\Omega_3 = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = a_3$.

Given fluid in incompressible, $\therefore \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$ —— (3)

Differentiating partially w.r.t x.,

$$③ \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} + \frac{\partial^2 w}{\partial x \partial z} = 0 \quad \text{———} ④$$

Now, using equations of ②,

$$\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 u}{\partial y^2} \text{ and } \frac{\partial^2 w}{\partial x \partial z} = \frac{\partial^2 u}{\partial z^2} \quad \text{———} ⑤$$

$$\therefore ④ \equiv \boxed{\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 = \nabla^2 u},$$

08

$\therefore$ u satisfies Laplace's equation

Similarly, v, w also satisfy Laplace's equation.

6. (a) (i) Form a partial differential equation by eliminating the arbitrary functions f and g from $z = y f(x) + x g(y)$.
(ii) Find a complete and the singular integral of $4xyz = pq + 2px^2y + 2qxy^2$. **[15]**

(i) $z = yf(x) + x(g(y))$

Partially differentiating w.r.t x, $\frac{\partial z}{\partial x} = yf'(x) + g(y)$.

Partially differentiating w.r.t y, $\frac{\partial z}{\partial y} = f(x) + xg'(y)$

Now, $\frac{\partial^2 z}{\partial x^2} = yf''(x)$ and $\frac{\partial^2 z}{\partial y^2} = xg''(y)$.

$$\frac{\partial^2 z}{\partial x \partial y} = f'(x) + g'(y)$$

Now, $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = xy\left(f'(x) + g'(y)\right) + xg(y) + yf(x)$

$\therefore \boxed{x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = xy\frac{\partial^2 z}{\partial x \partial y} + z}$, The required differential equation.

(ii) let $f = pq + 2px^2y + 2qxy^2 - 4xyz = 0$.

(or) $z = \frac{1}{2x}\frac{\partial z}{\partial x} \cdot \frac{1}{2y}\frac{\partial z}{\partial y} + x^2\left(\frac{1}{2x}\frac{\partial z}{\partial x}\right) + y^2\left(\frac{1}{2y}\frac{\partial z}{\partial y}\right) = 0$

let $2x\,dx = dX$ and $2y\,dy = dY$.

$\therefore \; x^2 = X \;,\; y^2 = Y$

and $z = XP + YQ + PQ$, where $P = \frac{\partial z}{\partial X}, Q = \frac{\partial z}{\partial Y}$

which is clearly Clairaut' form.

$\therefore$ solution becomes, $\boxed{z = x^2a + y^2b + ab.}$

Complete Integral

For singular Integral, put $\frac{\partial z}{\partial a}, \frac{\partial z}{\partial b} = 0$.

$\frac{\partial z}{\partial a} = x^2 + b = 0, \frac{\partial z}{\partial b} = y^2 + a = 0$.

$\Rightarrow b = -x^2, \; a = -y^2$.

Singular Integral : $z = x^2(-y^2) + y^2(-x^2) + (-x^2)(-y^2)$

$\boxed{z = -x^2y^2}$

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6. (b) Apply Gauss-Seidel iteration method to solve the equations
$20x + y - 2z = 17$; $3x + 20y - z = -18$; $2x - 3y + 20z = 25$. [12]

BY GAUSS - SEIDEL ITERATION METHOD,

$$x^{(k+1)} = \frac{1}{20}\left[17 - y^{(k)} + 2z^{(k)}\right]$$

$$y^{(k+1)} = \frac{1}{20}\left[-18 - 3x^{(k+1)} + z^{(k)}\right]$$

$$z^{(k+1)} = \frac{1}{20}\left[25 - 2x^{(k+1)} + 3y^{(k+1)}\right].$$

let $x^{\circ} = y^{\circ} = z^{\circ} = 0$.,

Using above formulae,

Iteration - 1 → $x^{(1)} = \frac{17}{20}$, $y^{(1)} = \frac{-411}{400}$, $z^{(1)} = \frac{8087}{8000}$

Iteration-2 → $x^{(2)} = 1.0024$, $y^{(2)} = -0.999$, $z^{(2)} = 0.999$

Iteration-3 → $x^{(3)} = 0.9999$, $y^{(3)} = -1.000$, $z^{(3)} = 1.0000$

Iteration-4 → $x^{(4)} = 1.000$, $y^{(4)} = -0.9999$, $z^{(4)} = 0.99999$

Clearly, the approximate values become,

$$\boxed{\begin{array}{l} x = 1 \\ y = -1 \\ z = 1 \end{array}}$$

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6. (c) Obtain the principal disjunctive and conjunctive normal forms of $p \to [(p \to q) \wedge \sim(\sim q \vee \sim p)]$. **[08]**

$p \to q \equiv \sim p \vee q$; $\sim(\sim p) = p$.

and $\sim(p \vee q) = \sim p \wedge \sim q$ and

$\sim(p \wedge q) = \sim p \vee \sim q$.

Now, $\sim(\sim q \vee \sim p) = \sim(\sim q) \wedge \sim(\sim p) = q \wedge p$. —— ①

$\therefore (p \to q) \wedge (\sim(\sim q \vee \sim p)) = (\sim p \vee q) \wedge (q \wedge p)$.

$\boxed{\cancel{\Leftrightarrow \sim p \vee (q \wedge q)}} = q \wedge p$ —— ②

Now, $p \to [(p \to q) \wedge \sim(\sim q \vee \sim p)]$ becomes.

$\sim p \vee \underline{(\sim p \vee q) \wedge (q \wedge p)}$. —— ③

$\sim p \vee (q \wedge p)$.

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$\Leftrightarrow \sim p \wedge (\sim q \vee q) \vee (q \wedge p)$

$\Leftrightarrow \boxed{(\sim p \wedge \sim q) \vee (\sim p \wedge q) \vee (p \wedge q)} \equiv$ Principal Disjunctive Normal form.

And similarly, given statement reduces to,

$\sim p \vee [(\sim p \vee q) \wedge (q \wedge p)]$

$\Leftrightarrow (\sim p \vee [\sim p \vee q]) \wedge (\sim p \vee [q \wedge p])$

$\Leftrightarrow (\sim p \vee q) \wedge (\sim p \vee q) \wedge (\sim p \vee p) \Leftrightarrow \boxed{\sim p \vee q}$ principal Conjunctive Normal form.

6. (d) Write Hamilton's equations for a particle of mass m moving in a plane under a force which is some function of distance from the origin. [15]

Mass of particle = m.

let co-ordinates of the particle at time t, be

$P(r,\theta)$., $r \equiv$ measured from origin O.

Velocity, $v = \dot{r}\hat{r} + r\dot{\theta}.\hat{\theta}$

$|v| = \left[\left(\dot{r}^2 + r^2\dot{\theta}^2\right)\right]^{1/2}$

$\therefore$ Kinetic Energy, $T = \frac{1}{2}m|v|^2 = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$. — (1)

For Potential energy, force $F = f(r)$.

$\therefore V = -\int F \cdot dr = V(r)$ (say). — (2)

Lagrangian: $L = T - V = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) - V(r)$. — (3)

$\therefore r, \theta$ becomes generalised co-ordinates.

$P_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$; $P_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$ — (4).

Hamilton's function, $H = \sum p_j \dot{q}_j - L$

$\therefore H = m\dot{r}^2 + mr^2\dot{\theta}^2 - \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + V(r)$.

$\therefore H = \frac{1}{2}m\left(\dot{r}^2\right) + \frac{1}{2}mr^2\left(\dot{\theta}^2\right) + V(r)$ — (5)

(or) $H = \frac{1}{2m}\left[P_r^2 + P_\theta^2/r^2\right] + V(r)$ — (6).

$\therefore$ Hamilton's equations are $\dfrac{\partial H}{\partial p_j} = \dot{q}_j \,;\, \dfrac{\partial H}{\partial q_j} = -\dot{p}_j$

i.e $\dot{p}_r = -\dfrac{\partial H}{\partial r} = \dfrac{p_\theta^2}{mr^3} - \dfrac{\partial V}{\partial r} \Longrightarrow H_1$

$\dot{p}_\theta = -\dfrac{\partial H}{\partial \theta} = 0 \Longrightarrow H_2$

$\dot{r} = \dfrac{\partial H}{\partial p_r} = \dfrac{p_r}{m} \Longrightarrow H_3$

$\dot{\theta} = \dfrac{\partial H}{\partial p_\theta} = \dfrac{p_\theta}{mr^2} \Longrightarrow H_4$

Hamilton's equations.

12.5

7. (a) A thin annulus occupies the region $0 < a \le r \le b$, $0 \le \theta \le 2\pi$, where $b > a$. The faces are insulated, and along the inner edge, the temperature is maintained at $0°$, while along the outer edge, the temperature is held at $100°$. Find the temperature distribution in the annulus. **[20]**

ROUGH SPACE

2 - 20(w) + 10(e) + 20 (No idea) = 30 (doable)

3 - 18 (t) + 15(w) + 17(easy). = ✓

4 - 15 (w) + 15(w) + 20 (w) = ✓

6 - 15(w) + 12(e) + 8 (e) + 15(w) = ✓

7 - 20(x) + 15(w) + 15(t) = x

8 - 18 (t) + 14(w) + 18(w) =

$$G = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} =$$

$$H = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}, \quad HG = \begin{bmatrix} a & b+kd \\ 0 & d \end{bmatrix} \neq$$

$$GH = \begin{bmatrix} a & ak+b \\ 0 & d \end{bmatrix}$$

$$\overset{H}{\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}} \overset{k}{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$$

x	0	3	0	x
0	x	0	2	0
0	4	x	0	1
1	0	2	x	1
0	1	6	5	x

~

x	0	2	0	x
1	x	0	3	x
~~0~~	4	x	0	~~0~~
1	~~0~~	1	x	0
0	1	5	5	x