SuccessClap

Best Coaching for UPSC Mathematics

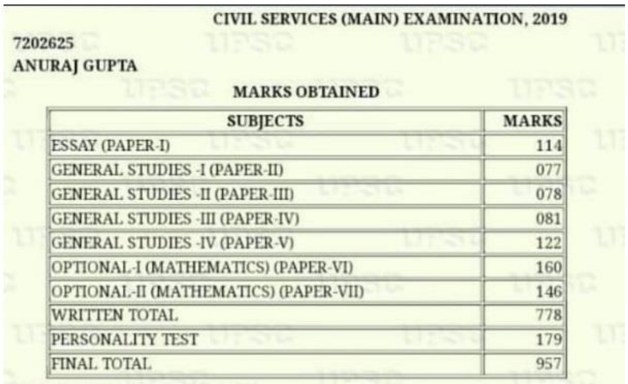
UPSC MATHEMATICS 2020

PAPER 2 SOLUTIONS

Solutions to ALL Questions

Our Successful Topper of 2019

Mr Anuraj Gupta .
Joined Test Series 2019



CIVIL SERVICES (MAIN) EXAMINATION, 2019

7202625
ANURAJ GUPTA

MARKS OBTAINED

SUBJECTS	MARKS
ESSAY (PAPER-I)	114
GENERAL STUDIES -I (PAPER-II)	077
GENERAL STUDIES -II (PAPER-III)	078
GENERAL STUDIES -III (PAPER-IV)	081
GENERAL STUDIES -IV (PAPER-V)	122
OPTIONAL-I (MATHEMATICS) (PAPER-VI)	160
OPTIONAL-II (MATHEMATICS) (PAPER-VII)	146
WRITTEN TOTAL	778
PERSONALITY TEST	179
FINAL TOTAL	957

UPSC Mathematics Marks 306

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Question Bank for UPSC Mathematics: Click Here

(1a) Let S_3 and Z_3 be permutation grp on 3 symbol and group of residue class modulo 3.
Show there is No homomorphism of S_3 in Z_3 except trivial homomorphism

Given S_3 : Permutation Grp on 3 symbol
Z_3 : Residue grp

To show : There is No homomorphism of S_3 in Z_3 except trivial homo
ie $f(x) = 0$ only homomorph

(Assumption)
→ If possible there is Homomorphism, which is Not trivial
then First theorem of Homomorphism

$f : S_3 \to Z_3$

$$\frac{S_3}{\ker f} \approx Z_3$$

$$\left|\frac{S_3}{\ker f}\right| = |Z_3|$$

$|S_3| = 3!$
$|Z_3| = 3$

$$\frac{6}{|\ker f|} = 3 \Rightarrow |\ker f| = 2$$

Also $\ker f$ is subgroup of S_3 and also Normal Subgroup of S_3.

But there is No Normal subgroup of order 2 in S_3

So our assumption is wrong.
So There is No homomorphism, except trivial

(1b) Let R be a principal ideal domain. Show that every ideal of a quotient ring of R is principal ideal and R/P is a Principal ideal domain for a prime ideal P of R.

Soln: Given R be PID

(a) To show: If P is prime ideal, then $\frac{R}{P}$ is PID

Let P is prime ideal, then $P=0$ or $P \neq 0$

(i) If $P=0$ $\frac{R}{P} = \{ r+0 \mid r \in R\}$
$= R$

Given R is PID $\Rightarrow \frac{R}{P}$ is PID ✓

(ii) If $P \neq 0$ $\frac{R}{P} = \{ r+P \quad r \in R\}$

Thm: Non zero Prime Ideal is Maximal Ideal

So P is Maximal Ideal

Thm: $\frac{R}{M}$ is field iff (if and only if) M is maximal

P is maximal so $\frac{R}{P}$ is field

Thm: Every field is PID

$\frac{R}{P}$ is field so $\frac{R}{P}$ is PID

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(1c) Prove sequence (a_n) satisfy

$$|a_{n+1}-a_n| \leq \alpha |a_n - a_{n-1}| \quad 0<\alpha<1 \text{ for } n \geq 2$$

natural number

is a Cauchy sequence.

Cauchy Sequence (Definition)

A sequence (a_n) is Cauchy sequence, if given $\varepsilon > 0$

we get $m, n > N$ (To find N)

such that $|a_m - a_n| < \varepsilon$

Note $\alpha < 1$

$\rightarrow |a_{n+1}-a_n| \leq \alpha |a_n - a_{n-1}|$

$|a_n - a_{n-1}| \leq \alpha |a_{n-1} - a_{n-2}|$

$|a_{n-1} - a_{n-2}| < \alpha |a_{n-2} - a_{n-3}|$

$\vdots$

$\alpha |a_1 - a_0|$

$|a_{n+1}-a_n| \leq \alpha^n |a_1 - a_0|$

$\rightarrow |a_n - a_m| \leq |a_{m+1}-a_m| + |a_{m+2}-a_{m+1}| + \cdots |a_n - a_{n-1}|$

$m \quad m+1 \quad m+2 \quad \cdots \quad n-2 \quad n-1 \quad n$

$$\leq \alpha^m (a_1 - a_0) + \alpha^{m+1}(a_1 - a_0) + \alpha^{m+2}(a_1 - a_0) + \cdots \alpha^n (a_1 - a_0)$$

$$= \alpha^m (1 + \alpha + \alpha^2 + \cdots \alpha^{n-m}) |a_1 - a_0|$$

when n is very large number $n \to \infty$

$$< \frac{\alpha^m}{1-\alpha} |a_1 - a_0|$$

$$< \varepsilon$$

Now choose m such that

$$\frac{\alpha^m}{1-\alpha} |a_1 - a_0| < \varepsilon$$

Hence Cauchy sequence.

Note: n is very large number

m : choosing suitably

(1d) Evaluate $\int_C (z^2+3z)\,dz$ counterclockwise from (2,0) to (0,2) along C, C is circle $|z|=2$

$|z|=2 \Rightarrow \begin{array}{l} x=2\cos\theta \\ y=2\sin\theta \end{array}$

$\rightarrow (2,0) \Rightarrow \left.\begin{array}{l} 2=2\cos\theta \\ 0=2\sin\theta \end{array}\right] \rightarrow \theta=0$

$\rightarrow (0,2) \rightarrow \left.\begin{array}{l} 0=2\cos\theta \\ 2=2\sin\theta \end{array}\right] \rightarrow \theta=\pi/2$

$I=\int_C (z^2+3z)\,dz=$

$$=\int_{\theta=0}^{\theta=\pi/2} \left(4e^{2i\theta}+6e^{i\theta}\right).2.e^{i\left(\frac{\pi}{2}+\theta\right)}\,d\theta$$

$z=x+iy$

$=2[\cos\theta+i\sin\theta]$

$dz=2[-\sin\theta+i\cos\theta]d\theta$

$z=2e^{i\theta}$

$dz=2ie^{i\theta}d\theta \quad i=e^{i\pi/2}$

$=2e^{i\theta}.e^{i\pi/2}$

$=2e^{i\left(\frac{\pi}{2}+\theta\right)}d\theta$

$z^2=4e^{2i\theta}$

$$=\int_0^{\pi/2} 8\,e^{i\left(\frac{\pi}{2}+3\theta\right)}d\theta + 12\int_0^{\pi/2} e^{i\left(\frac{\pi}{2}+2\theta\right)}d\theta$$

$$=\left. 8\,\frac{e^{i\left(\frac{\pi}{2}+3\theta\right)}}{3i}+\frac{12e^{i\left(\frac{\pi}{2}+2\theta\right)}}{2i}\right]_0^{\pi/2}$$

$$=\frac{8}{3i}\left(e^{2\pi i}-e^{\pi i/2}\right)+\frac{12\left(e^{i\left(\frac{\pi 3}{2}\right)}-e^{i\pi/2}\right)}{2i}$$

$$=\frac{8}{3i}(1-i)+\frac{6}{i}(-i-i)=\frac{8}{3i}-\frac{8}{3}-12$$

$$=-\frac{44}{3}-\frac{8i}{3}$$

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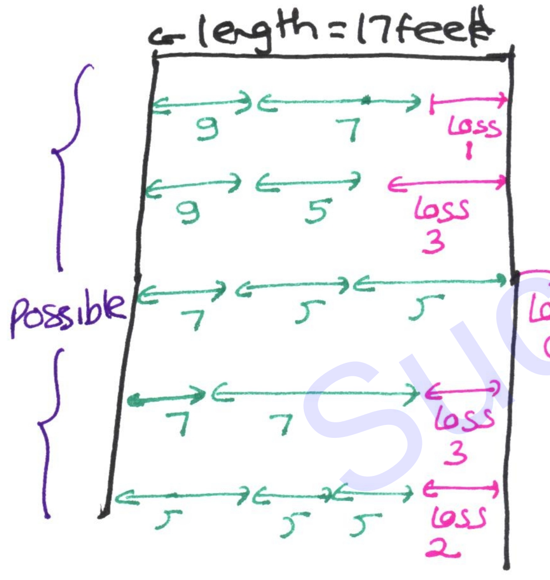
(1e) UPSC purchase cloth piece. The length of each piece is 17 feed. The repuirement to curtain length is

Curtain length (in feet)	Number repuired
5	700
9	400
7	300

Width of all curtains is same as available piece. Form LPP that decides the number of piece cut in different ways so that total trim loss is minimum. Also give a basic Feasible Soln



If we Consider

Quantity	9	7	5	Loss	Combination
	1	1	0	1	x_1
	1	0	1	3	x_2
	0	1	2	0	x_3
	0	2	0	3	x_4
	0	0	3	2	x_5
					4

Minimize loss $Z = 1.x_1 + 3x_2 + 0x_3 + 3x_4 + 2x_5$

If Consider this Combinations

5 feet : $0x_1 + 1.x_2 + 2.x_3 + 0.x_4 + 3x_5 \geq 700$

7 feet $1.x_1 + 0.x_2 + 1.x_3 + 2.x_4 + 0x_5 \geq 300$

9 feet $1.x_1 + 1.x_2 + 0.x_3 + 0.x_4 + 0x_5 \geq 400$

$x_i \geq 0 \quad i = 1,2,3,4,5$

5 Variable, 3 eqns] Let

Basic Soln (ANY) → (Not Optimal)

$x_3 = 0$, $x_4 = 0$] $\Rightarrow x_1 = 300 \Rightarrow x_2 = 100 \Rightarrow x_5 = 200$

Basic Soln $(300, 100, 0, 0, 200)$

(2a) Let G be finite Cyclic group of order n. Prove G has $\varphi(n)$ generators

This Qn is seen in all standard books

This is Basic qn

Let $G = \langle a \rangle$ be a cyclic grp of order n generated by a. Then

$o(a) = o(G) = n$.

we assert that a^m is generator of G iff m and n are relative prime

Ex: $n=8$ $\varphi(8)$ = Number of positive integers less than 8 and prime to 8 = {1,3,5,7} and so number of generators will be 4 and m can have values 1,3,5,7 ie

if a is generator of G, then other generators of G are a, a^3, a^5, a^7 and no other element can generate G.

Let a^m is generator of G for some m.

(m,n) are co-prime.

Since $a \in G$, $a = (a^m)^p$ for some $p \in \mathbb{Z}$

$\Rightarrow a = a^{mp} = a^{mp-1} = e$

$\Rightarrow o(a) \mid mp-1$

$\Rightarrow n \mid mp-1 \Rightarrow mp-1 = nq$ for some $q \in \mathbb{Z}$

$\Rightarrow mp - nq = 1 \Rightarrow m$ & n are coprimes

$\Rightarrow GCD(m,n) = 1$

Conversely, let m,n are coprimes.

Then there exists integer x & y s.t $mx+ny=1$

$\Rightarrow a^{mx+ny} = a$

$\Rightarrow a^{mx} a^{ny} = a$

$$\Rightarrow a^{mx} \cdot (a^n)^y = a$$

$$\Rightarrow a^{mx} \cdot e^y = a \quad \text{as } a^n = e$$

$$\Rightarrow a^{mx} \cdot e = a \Rightarrow a^{mx} = a$$

$$\Rightarrow a = (a^m)^x$$

Since every element of G is a power of a and a itself is a power of a^m, so a^m generates G.

This proves result

(26) Prove $f(x) = \sin x^2$ is Not uniformly continous on $[0, \infty[$

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This problem is present in Success Clap Question Bank (Qn-99)

Method - 1

Given in all books

Let $x_1 = \sqrt{\frac{n\pi}{2}}$ $\quad x_2 = \sqrt{(n+1)\frac{\pi}{2}}$

$|f(x_2) - f(x_1)| = |\sin x_2^2 - \sin x_1^2|$

$= \left|\sin(n+1)\frac{\pi}{2} - \sin\frac{n\pi}{2}\right|$

$= |0 - (\pm 1)| = 1$ if n is odd

$= |1 \pm 1 - 0| = 1$ if n is even

If I take $\varepsilon = \frac{1}{2}$ which is < 1
ie $\varepsilon < 1$

we have $|f(x_2) - f(x_1)| = 1 > \varepsilon$

& $|x_2 - x_1| = \left|\frac{x_2^2 - x_1^2}{x_2 + x_1}\right| = \frac{\pi/2}{\sqrt{n + \frac{1}{2}\pi} + \sqrt{\frac{n\pi}{2}}}$

$$< \frac{\pi}{2\left\{2\sqrt{\frac{n\pi}{2}}\right\}} < \frac{\pi}{\sqrt{n\pi}} = \sqrt{\frac{\pi}{n}} < \varepsilon$$

So we got $|f(x_2)-f(x_1)| > \varepsilon$

when $|x_2-x_1| < \varepsilon$

So Not uniform Convergence

BUT BUT

If we could not remember

$x_1 = \sqrt{\frac{n\pi}{2}}$ $\quad x_2 = \sqrt{\frac{(n+1)\pi}{2}}$

Then what ??

Solve by basic

Let given $\varepsilon > 0$ (Arbit

$$|f(x_2)-f(x_1)| = |\sin x_2^2 - \sin x_1^2|$$

$$\sin A - \sin B = 2\cos\frac{A+B}{2}\sin\frac{A-B}{2}$$

$$|f(x_2)-f(x_1)| = \left|2\cos\frac{x_1^2+x_2^2}{2}\sin\frac{x_2^2-x_1^2}{2}\right|$$

$$|\sin\theta| \leq \theta$$

$$|\cos\theta| \leq 1$$

$$|ab| < |a||b|$$

$$|f(x_2)-f(x_1)| \leq 2\left|\cos\frac{x_1^2+x_2^2}{2}\right|\sin\left(\frac{x_2^2-x_1^2}{2}\right)$$

$$\leq 2\times 1\times \frac{x_2^2-x_1^2}{2}$$

$$\leq |(x_2+x_1)(x_2-x_1)| \leq |x_1+x_2||x_2-x_1|$$

<u>My aim:</u>

Given $\varepsilon > 0$, get (x_1, x_2) s.t

$|f(x_2)-f(x_1)| < \varepsilon$

s.t $|x_2-x_1| < \delta$

in the given Range (Note)

Observe : inverval is $[0, \infty[$

$\downarrow$

Let us say interval is fixed at right side

ie $[0, \lambda]$ (say)

Then x_1 can have max λ value

x_2 can have max λ value

$x_1 + x_2$ can have max 2λ value

Then $|x_1 + x_2| \leq 2\lambda$

$$|f(x_2) - f(x_1)| \leq |x_1 + x_2| |x_2 - x_1|$$

$$\leq 2\lambda |x_2 - x_1|$$

Given $\varepsilon > 0$

what I do is, I will reduce my x_1, x_2 value such that

$$|x_2 - x_1| < \frac{\varepsilon}{2\lambda}$$

I will choose x_1, x_2, such that $|x_2 - x_1| < \frac{\varepsilon}{2\lambda}$

Now when $|x_2 - x_1| < \frac{\varepsilon}{2\lambda}$

we get

$$|f(x_2) - f(x_1)| < 2\lambda |x_2 - x_1|$$

$$< 2\lambda \cdot \frac{\varepsilon}{2\lambda} = \varepsilon$$

So if $|x_2 - x_1| < \frac{\varepsilon}{2\lambda}$

we get $|f(x_2) - f(x_1)| < \varepsilon$

Hence we get x_1, x_2 such that $|x_2 - x_1| < \delta$

where $\delta = \frac{\varepsilon}{2\lambda}$

and satisfy Cauchy criteria for uniform continuity

So we found that if

the interval is $[0, \lambda[$

if is Uniform Continous

Come to our problem

$[0, \infty[$

Given $\varepsilon > 0$, $\delta = \frac{\varepsilon}{2\lambda}$

Here (I donot know) what is that λ, because it extends to ∞, but No definite boundary,

So I could not choose δ

Given $\varepsilon > 0$, I cannot get δ

So Not Uniform Continous

(2c) Using Contour integration, Evaluate $\int_0^{2\pi} \frac{1}{3+2\sin\theta} d\theta$

$$I = \int_0^{2\pi} \frac{d\theta}{3+2\sin\theta}$$

$$\sin\theta = \frac{z-\frac{1}{z}}{2i} = \frac{z^2-1}{2iz}$$

$$d\theta = \frac{dz}{iz}$$

$$\frac{d\theta}{3+2\sin\theta} = \frac{1}{3+2\left(\frac{z^2-1}{2iz}\right)} \frac{dz}{iz}$$

$$= \frac{dz}{z^2+3iz-1}$$

$$= \int \frac{dz}{z^2+3iz-1}$$

$$= \int \frac{dz}{(z-\alpha)(z-\beta)}$$

$$\alpha, \beta = \frac{-3i \pm \sqrt{9i^2+4}}{2}$$

$$= \frac{-3i \pm \sqrt{5}i}{2}$$

$$\alpha = \frac{-3i+\sqrt{5}i}{2}$$

$$\beta = \frac{-3i-\sqrt{5}i}{2}$$

$|\beta| > 1$, α lie inside

$$I = 2\pi i \left(\text{Res}(z=\alpha)\right)$$

$$= 2\pi i \lim_{z\to\alpha} (z-\alpha) \frac{1}{(z-\alpha)(\alpha-\beta)}$$

$$= 2\pi i \frac{1}{\alpha-\beta}$$

$$\alpha-\beta = \sqrt{5}i$$

$$= \frac{2\pi}{\sqrt{5}} = \frac{2\pi}{\sqrt{5}}$$

Remember

$$I = \int_0^{2\pi} \frac{d\theta}{a+b\cos\theta} = \int_0^{2\pi} \frac{d\theta}{a+b\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

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(3a) Let R be a finite field of characteristic $p(>0)$. Show mapping $f: R \to R$ s.t $f(a) = a^p$ $\forall a \in R$ is a isomorphism.

→ Given characteristic p.

It means $pa = 0$ $\forall a \in R$

→ To show Iso

a) Show Homo : $f(a+b) = f(a) + f(b)$

(b) Show 1-1 : $f(a) = f(b) \Rightarrow a = b$

(c) Show Onto : For given y, there is x . s.t $f(x) = y$

(a) $f(a+b) = (a+b)^p = a^p + {}^pC_1 a^{p-1} b + {}^pC_2 a^{p-2} b^2 + \cdots b^p$

$= a^p + b^p + p\, a^{p-1} b + p(\,)\, a^{p-2} b^2 + p(\,) a^{p-3} b^3$

$= a^p + b^p + p(\,) + p(\,) + p(\,) +$

But $p(\,) = 0$ as p is characteristic

$= a^p + b^p$

$= f(a) + f(b)$ So Homo

(b) Let $f(a) = f(b) \Rightarrow a^p = b^p \Rightarrow a = b$ It is only

(c) For any given y, we get a, such that $y = a^p$, we have a, s.t $f(a) = a^p = y$

Hence it is Isomorphism

(3b) Solve by Simplex

$$\text{Minimize } z = -6x_1 - 2x_2 - 5x_3$$

$$\text{s.t } 2x_1 - 3x_2 + x_3 \le 14$$

$$-4x_1 + 4x_2 + 10x_3 \le 46$$

$$2x_1 + 2x_2 - 4x_3 \le 37$$

$$x_1 \geq 2,\ x_2 \geq 1\ \ x_3 \geq 1$$

$\rightarrow$ $x_1 \geq 2,\ x_2 \geq 1\ \ x_3 \geq 3 \Rightarrow$

$$y_1 = x_1 - 2 \quad y_2 = x_2 - 1 \quad y_3 = z - 3$$

$$\Rightarrow y_1 \geq 0 \quad y_2 \geq 0 \quad y_3 \geq 0$$

$\rightarrow$ $z'_{max} = -z = 6x_1 + 2x_2 + 5x_3$

$$= 6y_1 + 2y_2 + 5y_3 + [\text{Const}]$$

$$\text{s.t } 2y_1 - 3y_2 + y_3 \le 10$$

$$-2y_1 + 2y_2 + 5y_3 \le 10$$

$$2y_1 + 2y_2 - 4y_3 \le 43$$

$$y_1, y_2, y_3 \geq 0$$

$\rightarrow$

$$2y_1 - 3y_2 + y_3 + y_4 = 10$$

$$-2y_1 + 2y_2 + 5y_3 + 0y_4 + y_5 = 10$$

$$2y_1 + 2y_2 - 4y_3 + 0y_4 + 0y_5 + y_6 = 43$$

$$y_1 = y_2 = y_3 = 0 \Rightarrow y_4 = 10 \quad y_5 = 10 \quad y_6 = 43$$

IBFS $(0, 0, 0, 10, 10, 43)$

		6	2	5	0	0	0		
		y_1	y_2	y_3	y_4	y_5	y_6	b	θ
0	y_4	2	-3	1	1	0	0	10	$\boxed{5}$ →
0	y_5	-2	2	5	0	1	0	10	-ve
0	y_6	2	2	-4	0	0	1	43	43/2
z_j		0	0	0	0	0	0		
$\Delta = c_j - z_j$		$\boxed{6}$ ↑	2	5	0	0	0		

6	y_1	1	-3/2	1/2	1/2	0	0	5	-ve
0	y_5	0	-1	6	1	1	0	20	-ve
0	y_6	0	5	-5	-1	0	1	33	$\boxed{33/5}$ →
z_j		6	-9	3	3	0	0		
$\Delta = c_j - z_j$		0	$\boxed{11}$ ↑	2	-3	0	0		

6	y_1	1	0	-1	1/5	0	3/10	149/10	-ve
0	y_5	0	0	5	4/5	1	1/5	133/5	$\boxed{133/25}$ →
2	y_2	0	1	-1	-1/5	0	1/5	33/5	-ve
z_j		6	2	-8	4/5	0	22/10		
$\Delta = c_j - z_j$		0	0	$\boxed{13}$ ↑	-ve	0	-ve		

6	y_1	1	0	0	9/25	1/25	17/50	$\frac{1011}{50}$
5	y_3	0	0	1	4/25	1/5	1/25	133/25
2	y_2	0	1	0	-1/25	1/5	6/25	298/25
z_j		6	2	5	12/25	3/5	$\frac{24}{25}$	
$\Delta = c_j - z_j$		0	0	0	-ve	-ve	-ve	$\Delta_j \leq 0$

Optimal

$$y_1 = \frac{1011}{50}$$

$$y_2 = 298/25$$

$$y_3 = 133/25$$

$$x_1 = y_1 + 2 = \frac{1111}{50}$$

$$x_2 = y_2 + 1 = \frac{323}{25}$$

$$x_3 = y_3 + 3 = \frac{208}{5}$$

$$Z_{min} = -6x_1 - 2x_2 - 5x_3$$

$$= -\frac{5019}{25}$$

(3c) If $u = \tan^{-1} \dfrac{x^3+y^3}{x-y}$

Show $x^2\dfrac{\partial^2 u}{\partial x^2} + 2xy\dfrac{\partial^2 u}{\partial x \partial y} + y^2\dfrac{\partial^2 u}{\partial y^2} = (1-4\sin^2 u)\sin 2u$

Note: This qn is present in Question Bank of SuccessClap. Chapter 19 PD Euler - Qn-4

→ u is not Homogenous

→ $z = \tan u = \cancel{t}\ \dfrac{x^3+y^3}{x-y} = \dfrac{x^2\left[1+(y/x)^3\right]}{1-y/x}$

is Homogenous of ~~ad~~ degree 2

→ Formula $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = nu$ Euler theorem

→ $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = 2z$ $\qquad n=2$

But $\dfrac{\partial z}{\partial x} = \sec^2 u\dfrac{\partial z}{\partial x}$ $\qquad \dfrac{\partial z}{\partial y} = \sec^2 u\dfrac{\partial z}{\partial y}$

Put in eqn

$$\sec^2 u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right) = 2z = \tan u \cdot 2$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2\left(\frac{\sin u}{\cos u}\right) \times \cos^2 u = \sin 2u$$

→ $\dfrac{\partial^2 z}{\partial x^2} = \sec^2 u\dfrac{\partial^2 u}{\partial x^2} + 2\sec^2 u \tan u\left(\dfrac{\partial u}{\partial x}\right)^2$

$\dfrac{\partial^2 z}{\partial y^2} = \sec^2 u\dfrac{\partial^2 u}{\partial y^2} + 2\sec^2 u \tan u\left(\dfrac{\partial u}{\partial y}\right)^2$

$$\frac{\partial^2 z}{\partial x \partial y} = \sec^2 u \frac{\partial^2 u}{\partial x \partial y} + 2\sec^2 u \tan u \frac{\partial u}{\partial x}\frac{\partial u}{\partial y}$$

Euler 2nd theorem

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = 2(2-1)u$$

$$\sec^2 u\left(x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2}\right)$$

$$+ 2\sec^2 u \tan u\left(x^2\left(\frac{\partial u}{\partial x}\right)^2 + 2xy\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + y^2\left(\frac{\partial u}{\partial y}\right)^2\right]$$

$$= 2\tan u$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + 2\tan u\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)^2$$

$$= 2\sin u\cos u$$

$$\Rightarrow x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \sin 2u - 2\tan u \sin^2 2u$$

$$= (1 - 2\tan u \sin 2u)\sin 2u$$

$$= (1 - 4\sin^2 u)\sin 2u$$

(4a) If $v(r,\theta) = \left(r - \frac{1}{r}\right)\sin\theta$

Find Analytic function $f(z) = u(r,\theta) + iv(r,\theta)$

Cauchy Rieman eqn

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad , \quad \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r}$$

$$\rightarrow \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{1}{r}\left(r - \frac{1}{r}\right)\cos\theta = \left(1 - \frac{1}{r^2}\right)\cos\theta$$

Integrate $\quad u = \left(r + \frac{1}{r}\right)\cos\theta + F(\theta)$

$$\rightarrow \frac{1}{r}\frac{\partial u}{\partial \theta} = -\frac{\partial v}{\partial r} = (-1)\left(1 + \frac{1}{r^2}\right)\sin\theta$$

$$\frac{\partial u}{\partial \theta} = (-1)\left(r + \frac{1}{r}\right)\sin\theta$$

Integrate $\quad u = \left(r + \frac{1}{r}\right)\cos\theta + G(r)$

$\rightarrow F(\theta) = G(r) \Rightarrow F(\theta) = G(r) = 0$

$$u = \left(r + \frac{1}{r}\right)\cos\theta$$

$$\rightarrow f(z) = u + iv = \left(r + \frac{1}{r}\right)\cos\theta + i\left(r - \frac{1}{r}\right)\sin\theta$$

$$= r(\cos\theta + i\sin\theta) + \frac{1}{r}(\cos\theta - i\sin\theta)$$

$$= re^{i\theta} + \frac{1}{r}e^{-i\theta} = re^{i\theta} + \frac{1}{re^{i\theta}} \qquad z = re^{i\theta}$$

$$= z + \frac{1}{z}$$

(4a) Show $\int_0^{\pi/2} \frac{\sin^2\theta}{\sin x+\cos x}\,dx = \frac{1}{\sqrt{2}}\log_e(1+\sqrt{2})$

$$I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x+\cos x}\,dx \qquad \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

$$= \int_0^{\pi/2} \frac{\sin^2\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right)+\cos\left(\frac{\pi}{2}-x\right)}\,dx$$

$$I = \int_0^{\pi/2} \frac{\cos^2 x}{\sin x+\cos x}$$

$$2I = \int_0^{\pi/2} \frac{\sin^2 x}{\sin x+\cos x} + \frac{\cos^2 x}{\sin x+\cos x} = \int_0^{\pi/2} \frac{1}{\sin x+\cos x}\,dx$$

$$= \int_0^{\pi/2} \frac{dx}{\sqrt{2}\left[\sin x\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}\cos x\right]} = \int_0^{\pi/2} \frac{dx}{\sqrt{2}\cos\left(x-\frac{\pi}{4}\right)}$$

$$= \frac{1}{\sqrt{2}}\int_0^{\pi/2} \sec\left(x-\frac{\pi}{4}\right)dx$$

$$= \frac{1}{\sqrt{2}}\left[\ln\left|\sec\left(x-\frac{\pi}{4}\right)+\tan\left(x-\frac{\pi}{4}\right)\right|\right]_0^{\pi/2}$$

$$= \frac{1}{\sqrt{2}}\left[\ln|\sqrt{2}+1| - \ln|\sqrt{2}-1|\right]$$

$$= \frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2}+1}{\sqrt{2}-1}\right| \qquad \frac{\sqrt{2}+1}{\sqrt{2}-1} = \frac{\sqrt{2}+1}{\sqrt{2}-1}\times\frac{\sqrt{2}+1}{\sqrt{2}+1} = \frac{(\sqrt{2}+1)^2}{2-1} = (\sqrt{2}+1)^2$$

$$2I = \frac{1}{\sqrt{2}}\ln(\sqrt{2}+1)^2 = \frac{2}{\sqrt{2}}\ln(\sqrt{2}+1)$$

$$I = \frac{1}{\sqrt{2}}\ln(\sqrt{2}+1)$$

(4c) Find Initial Basic feasible soln of Transportation problem by Vogel and find optimal solution and Transportation Cost

Destinations

Sources	D1	D2	D3	D4	Availability
S1	10	0	20	11	15
S2	12	8	9	20	25
S3	0	14	16	18	10
Demand	5	20	15	10	

→ Balanced

	5	20	15	10	
→ 15	10	0	20	11	10−0=10
25	12	8	9	20	9−1=8
10	0	14	16	18	$\boxed{14-0=14}$ ←
	10−0 = 10	8−0 = 8	16−9 = 7	18−11 = 7	

Max value is 14
So Allot

	~~5~~ 0	20	15	10
15	10	0	20	11
25	12	8	9	20
~~10~~ 5	0 $\boxed{5}$	14	16	18	←

	20	15	10	
15	0	20	11	$\boxed{11-0=11}$ Max ← Allot
25	8	9	20	9-8=1
5	14	16	18	14-16=2
	8-0 =8	16-9 =7	11-18 =7	

	~~20~~ 5	15	10	
0 ~~15~~	0 $\boxed{15}$	20	11	←
25	8	9	20	
5	14	16	18	

	5	15	10	
25	8	9	20	8-9=1
5	14	16	18	14-16=2
	14-8 =6	$\boxed{16-9 =7}$ ↑ max	20-18 =2	

	5	~~15~~ 0	10
10 ~~25~~	8	9 $\boxed{15}$	20
5	14	16	18
		↑	

→

	5	10	
10	8	20	$\boxed{8-20=12}$ ← Max
5	14	18	14-18=4
	8-14 =6	20-18 =2	

	~~5~~ 0	10
5 ~~10~~	8 $\boxed{5}$	20
5	14	18

→

	10
5	20
5	18

→

	10
5	20 $\boxed{5}$
5	18 $\boxed{5}$

10	0 [15]	20	11	15
12	8 [5]	9 [15]	20 [5]	25
0 [5]	14	16	18 [5]	10
5	20	15	10	

$$\text{Cost} = 15\times 0 + 5\times 8 + 15\times 9 + 20\times 5 + 5\times 0 + 18\times 5 = 365$$

→ Check for optimal – UV method

	$v_1=$	$v_2=$	$v_3=$	$v_4=$
$u_1=0$		0		
$u_2=$		8	9	20
$u_3=0$	0			18

If we take $u_1=0 \Rightarrow v_2=0$

$\Rightarrow u_2=8 \rightarrow v_4=12$

$\Rightarrow v_3=1$

$u_3=6 \rightarrow v_1=-6$

	-6	0	1	12
0	-6		1	12
8	2			
6		6	7	

A

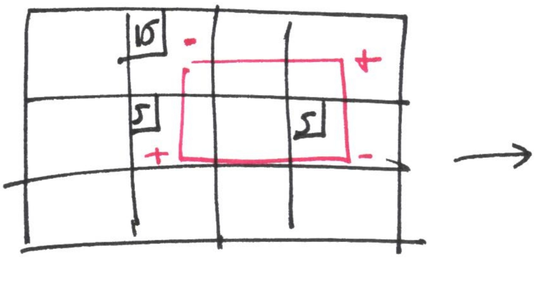
10		20	11
12			
	14	16	

B (original Matrix)

A – B →

-16		-19	1
-10			
	-8	-11	

Not all Negative
one positive
Not optimal



New Allotment

10	0 [10]	20	11 [5]	15
12	8 [10]	9 [15]	20	25
0 [5]	14	16	18 [5]	10
5	20	15	10	

Cost $= 10\times0 + 5\times11 + 8\times10 + 9\times15 + 0\times5 + 18\times5 = 360$

→ Check for optimal

	v_1	v_2	v_3	v_4
$u_1=0$		0		11
$u_2=$		8	9	
$u_3=$	0			18

$u_1=0 \Rightarrow v_2=0 \Rightarrow u_2=8$

$u_1=0 \downarrow v_4=11$, $u_2=8 \downarrow v_3=1$

$v_4=11 \hookrightarrow u_3=7$

$u_3=7 \downarrow v_1=-7$

	-7	0	1	11
0	-7		1	
8	1			19
7		7	8	

A

10		20	
12			20
	14	16	

B (original Matrix)

A−B =

−17		−19	
−11			−1
	−7	−8	

All Negative
So optimal

Cost is 360

(5a) Form PDE by eliminating $f(x), g(y)$ $z = yf(x) + xg(y)$ and specify its nature (Elliptic, hyperbolic, parabolic in region $x > 0$, $y > 0$

$$z = y f(x) + x g(y)$$

$$\frac{\partial z}{\partial x} = y f'(x) + g(y) \quad \text{— (1)}$$

$$\frac{\partial z}{\partial y} = f(x) + x g'(y) \quad \text{— (2)}$$

Diff w.r.t y

$$\frac{\partial^2 z}{\partial x \partial y} = f'(x) + g'(y)$$

Eliminate $f'(x), g'(y)$

Mul (1) by x $\quad x\frac{\partial z}{\partial x} = xy f'(x) + x g(y)$

Mul (2) by y $\quad y\frac{\partial z}{\partial y} = y f(x) + xy g'(y)$

Add

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \underbrace{[y f(x) + x g(y)]}_{z} + xy \underbrace{[f'(x) + g'(y)]}_{\frac{\partial^2 z}{\partial x \partial y}}$$

$$= z + xy \frac{\partial^2 z}{\partial x \partial y}$$

$$xy \frac{\partial^2 z}{\partial x \partial y} - x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} + z = 0$$

$Rr + Ss + Tt + f(x,y,z) = 0 \rightarrow R = 0 \quad T = 0$
$S = xy$

$S^2 - 4RT = x^2 y^2 > 0$ Hyperbolic

UPSC 2020

SuccessClap

(5b) Show eqn $f(x) = \cos\frac{\pi(x+1)}{8} + 0.148x - 0.9062 = 0$ has one root in the interval $(-1,0)$ and one in $(0,1)$.
Calculate negative root correct to four decimal places using Newton-Raphson method.

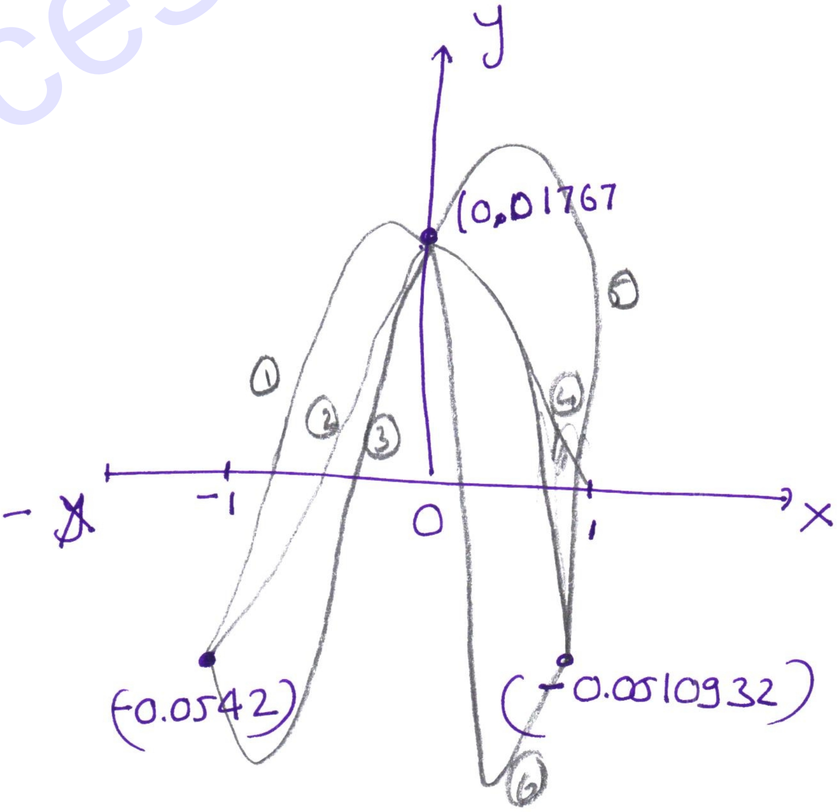
Note:
1) This problem is present in Iyengar book.
2) This question tests your CALCULATOR SKILLS. How fast and how efficient, you can use your Calculator.

→ $f(-1) = -0.0542$

$f(0) = 0.01767953$

$f(1) = -0.0510932$

(1, 2, 3, 4, 5, 6)
different possibilities



→ function change sign in $(-1,0)$ so there exist ATLEAST one root

→ Fn chage sign in $(0,1)$ so $\exists$ atleast one root

→ ATLEAST means there can be 1, 2, 3, 4 roots

→ To show there exists one root only

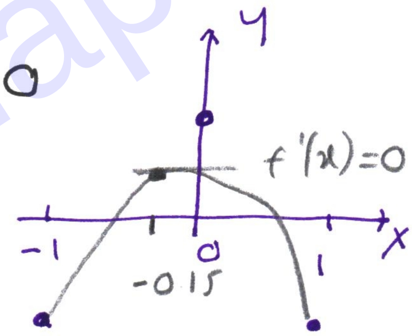
$$f'(x) = -\frac{\pi}{8}\sin\frac{\pi(x+1)}{8} + 0.148$$

At peak point $f'(x) = 0$

Use calculator to solve $f'(x) = 0$

gives $x = -0.15978$

$f(-0.15978) = 0.218$ (+ve)



There is only one point, where curve chapes monotonicity.

→ N.R $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

we want negative term : it lies between -1 and 0, So take $x_0 = \frac{-1+0}{2} = -0.5$

use Calculator and fix NR formula and get iteration values

$x_1 = -0.508199$

$x_2 = -0.5081295$

$x_3 = -0.5081285$

Root is

$x = -0.5081$

Verified with Iyengar book Soln.

UPSC 2020

SuccessClap

(5c) Let $g(w,x,y,z) = (w+x+y)(x+\bar{y}+z)(w+\bar{y})$

→ Obtain Conjunctive Normal form

→ Express as product of Max terms

$$g(w,x,y,z) = (w+x+y)\ (x+\bar{y}+z)\ (w+\bar{y})$$

$$w+x+y = (w+x+y+z\bar{z})$$
$$= (w+x+y+z)(w+x+y+\bar{z})$$

$$x+\bar{y}+z = (w\bar{w}+x+\bar{y}+z)$$
$$= (w+x+\bar{y}+z)(\bar{w}+x+\bar{y}+z)$$

$$A = A+B\bar{B}$$
$$= (A+B)(A+\bar{B})$$

to add B in it

$$(w+\bar{y}) = (w+x\bar{x}+\bar{y}+z\bar{z})$$

$$(w+x+\bar{y}+z\bar{z})(w+\bar{x}+\bar{y}+z\bar{z})$$

$$(w+x+\bar{y}+z)(w+x+\bar{y}+\bar{z}) \qquad (w+\bar{x}+\bar{y}+z)(w+\bar{x}+\bar{y}+\bar{z})$$

$$g(x,y,z) = \underbrace{(w+x+y+z)(w+x+y+\bar{z})}\ \underbrace{(w+x+\bar{y}+z)(\bar{w}+x+\bar{y}+z)}$$
$$(w+x+\bar{y}+z)(w+x+\bar{y}+\bar{z})(w+\bar{x}+\bar{y}+z)(w+\bar{x}+\bar{y}+\bar{z})$$

✓→ Same $AA = A$

$$= (w+x+y+z)(w+x+y+\bar{z})(\bar{w}+x+\bar{y}+z)(w+x+\bar{y}+z)$$
$$(w+x+\bar{y}+\bar{z})(w+\bar{x}+\bar{y}+z)(w+\bar{x}+\bar{y}+\bar{z})$$

0 → uncomplement

1 → Complement

	w	x	y	z		
$w+x+y+z$	0	0	0	0	→	$0 : M_0$
$w+x+y+\bar{z}$	0	0	0	1	→	$1 : M_1$
$w+x+\bar{y}+z$	0	0	1	0	→	$2 : M_2$
$\bar{w}+x+\bar{y}+z$	1	0	1	0	→	$8+2=10 : M_{10}$
$w+x+\bar{y}+\bar{z}$	0	0	1	$\cancel{0}$	→	$2+1=3 : M_3$
$w+\bar{x}+\bar{y}+z$	0	1	1	0	→	$4+2=6 \quad M_6$
$w+\bar{x}+\bar{y}+\bar{z}$	0	1	1	1	→	$4+2+1=7 \quad M_7$

Product of Max terms

$$= g = \Pi M(0,1,2,3,6,7,10)$$

$$= M_0 \times M_1 \times M_3 \times M_6 \times M_7 \times M_{10}$$

UPSC 2020

Successclap

(5d) Solve $(D^3 - 2D^2D' - DD'^2 + 2D'^3)z = e^{2x+y} + \sin(x-2y)$

$\rightarrow$ AE: $m^3 - 2m^2 - m + 2 = 0$

$m^2(m-2) - 1(m-2) = (m^2-1)(m-2) = (m+1)(m-1)(m-2) = 0$

$CF = \phi_1(y+2x) + \phi_2(y+x) + \phi_3(y-x)$

$\rightarrow PI_1 = \dfrac{1}{(D+D')(D-D')(D-2D')} e^{2x+y} = \dfrac{1}{(2+1)(2-1)(D-2D')} e^{2x+y}$

$= \dfrac{1}{3} \dfrac{e^{2x+y}}{(D-2D')} = \dfrac{1}{3} e^{2x+y} \dfrac{x^1}{1!} = \dfrac{xe^{2x+y}}{3}$

$\rightarrow PI_2 = \dfrac{1}{(D^2-D'^2)(D-2D')} \sin(x-2y)$ $\quad D^2 = -1$, $D'^2 = -4$

$= \dfrac{1}{D^3 - 2D^2D' - DD'^2 + 2D'^3} \sin(x-2y)$

$= \dfrac{1}{-D + 2D' + 4D' - 8D'} \sin(x-2y) = \dfrac{1}{-D-2D'} \sin(x-2y)$

$= \dfrac{1}{(-1)(D+2D')} \sin(x-2y) = \dfrac{(-1)(D-2D')}{D^2 - 4D'^2} \sin(x-2y)$

$= \dfrac{(-1)(D-2D')}{-1+16} = \left(\dfrac{-1}{15}\right)(D-2D') \sin x-2y)$

$= \left(\dfrac{-1}{15}\right)[\cos(x-2y) - 2\cos(x-2y)(-2)] = \dfrac{\cos(x-2y)}{3}$

(5e) Prove that moment of inertia of triangular lamina ABC about any axis through A in its plane is $\frac{M}{6}(\beta^2+\beta\gamma+\gamma^2)$

M is mass of lamina

β, γ are length of perpendiculars from B and C on the axis

Note: This question is direct Lift from Rigid Dynamics - Krishna series book.

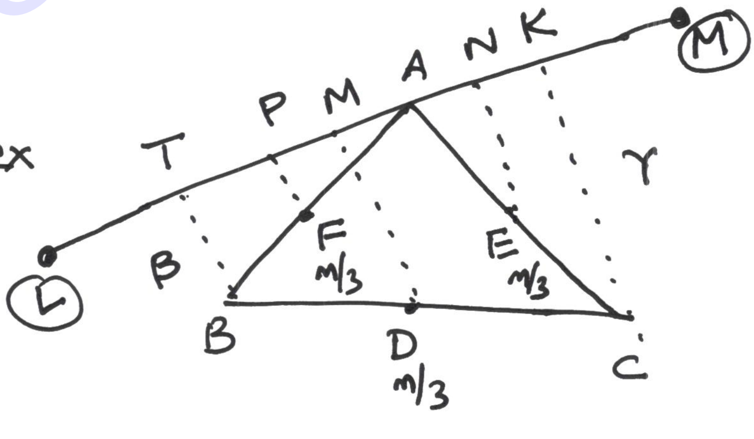
→ Let m be mass of triangle ABC,

→ the triangle is EQUIMOMENTAL to three particles each of mass $M/3$ placed at middle D, E, F of sides

→ β, γ distance of vertex B, C from Line LM

$BT = \beta$

$CK = \gamma$



→ ⊥ distance of D, E, F from LM are

$$DM = \frac{\beta+\gamma}{2}$$

$$EN = \frac{1}{2}CK = \frac{1}{2}\gamma$$

$$FP = \frac{1}{2}BT = \frac{1}{2}\beta$$

MI of $\triangle ABC$ alog LM

$= $ Sum of MI of masses $m/3$ each at D, E, F alog LM

$$= \frac{m}{3} DM^2 + \frac{m}{3} EN^2 + \frac{m}{3} FP^2$$

$$= \frac{m}{2}\left[\frac{1}{4}(\beta+\gamma)^2 + \frac{1}{4}\gamma^2 + \frac{1}{4}\beta^2\right]$$

$$= \frac{m}{6}\left(\beta^2 + \gamma^2 + \beta\gamma\right)$$

(6a) Find integral surface of

$$(x-y)y^2\frac{\partial z}{\partial x}+(y-x)x^2\frac{\partial z}{\partial y}=(x^2+y^2)z$$

Contains curve $xz=a^3, y=0$

Note: This qn is present in Question Bank of Success Clap

$$\frac{dx}{(x-y)y^2}=\frac{dy}{(y-x)x^2}=\frac{dz}{(x^2+y^2)z}$$

$$\rightarrow \frac{dx-dy}{(x-y)(x^2+y^2)}=\frac{dz}{(x^2+y^2)z} \Rightarrow \frac{d(x-y)}{x-y}=\frac{dz}{z}$$

$$\ln(x-y)=\ln z+c'$$

$$\boxed{\left(\frac{x-y}{z}\right)=c_1}$$

$$\rightarrow \frac{dx}{(x-y)y^2}=\frac{dy}{(y-x)x^2}$$

$$x^2dx+y^2dy=0 \rightarrow \frac{x^3+y^3}{3}=c''$$

$$\boxed{x^3+y^3=c_2}$$

$\rightarrow$ $xz=a^3, y=0$ $\rightarrow$ Parameters $x=t$ $z=\frac{a^3}{t}$ $y=0$

$$\rightarrow c_1=\frac{t}{a^3/t}=\frac{t^2}{a^3} \qquad c_2=t^3$$

$$\left.\begin{array}{l} c_1a^3=t^2 \rightarrow (c_1a^3)^3=t^6 \\ c_2=t^3 \rightarrow c_2^2=t^6 \end{array}\right] \rightarrow c_1^3a^9=c_2^2$$

$$\left(\frac{x-y}{z}\right)^3a^3=(x^3+y^3)^2$$

(6b) Set up Gauss Seidel scheme, and iterate 3 times starting with initial vector $X^0 = 0$.
Also find exact soln and compare it.

$$4x+y+2z=4$$
$$3x+5y+z=7$$
$$x+y+3z=3$$

Before Starting Gauss Seidel : Rearrange the order, s.t the coefficient of the eqn term is greatest. (Diagonally dominant system)

Eqn 1 : $x : 4$ ✓ $|4| > |1|, |2|$

Eq 2 : $y : 5$ ✓ $|5| > |3|, |1|$

Eq 3 : $z : 3$ ✓ $|3| > |1|, |1|$

$$x_{k+1} = \frac{4 - y_k - 2z_k}{4}$$

$$y_{k+1} = \frac{7 - 3x_{k+1} - z_k}{5}$$

$$z_{k+1} = \frac{3 - x_{k+1} - y_{k+1}}{3}$$

	k=0	k=1	k=2	k=3
x	0	1	0.6	0.52
y	0	0.8	0.96	0.992
z	0	0.4	0.48	0.496

Using Gauss Seidel 3 iterations

Exact Soln

Use any method

$x = 0.5$

$y = 1$

$z = 0.5$

UPSC 2020

Mechanics

SuccessClap

(6c) write Hamiltonian, Find the eqn of motion of a particle of mass m constrained on surface of cylinder by $x^2+y^2=R^2$, R is constant. The particle is subject to a force directed towards origin & proportional to distance r of the particle from origin given by $F=-kr$, k is const.

Note:

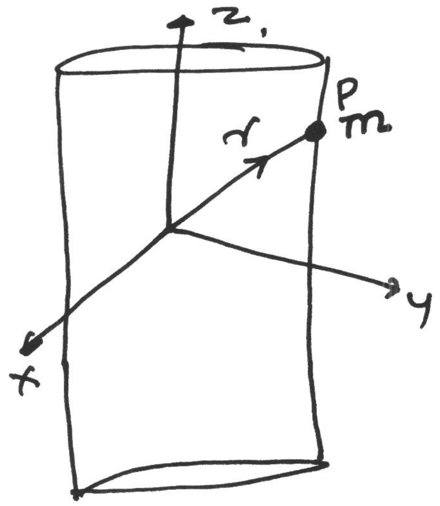
1) Same question is asked in UPSC 2006.
So solve all PYQs from 1992.
Donot stick PYQs for 10years only

2) checkout Video Soln of UPSC 2006 in Youtube of SuccessClap.

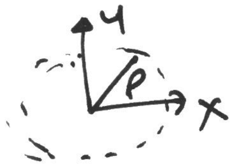
3) SAME Question is present in Question Bank of SuccessClap. So practice all Hamiltonian & Lag problems from Success Clap Question Bank.

→ Its Cylindrical Coordinate system

$v^2 = \dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2$ (Remember Formula)



ρ is distance from origin in xy plane



$\rho = R$ is constant in our problem

$\dot{R} = 0$

$v^2 = \dot{R}^2 + R^2\dot{\theta}^2 + \dot{z}^2$

$v^2 = R^2\dot{\theta}^2 + \dot{z}^2$

$T = \frac{1}{2}mv^2 = \frac{1}{2}m\left(R^2\dot{\theta}^2 + \dot{z}^2\right)$

→ We need V (Potential)

Given $F = -kr$

But $F = -\frac{\partial V}{\partial r}$

$-\frac{\partial V}{\partial r} = -kr$

$V = \int kr\,dr = \frac{kr^2}{2}$

$V = \frac{1}{2}kr^2$

$= \frac{1}{2}k\left(x^2 + y^2 + z^2\right)$

$r^2 = x^2 + y^2 + z^2)$

But $x^2 + y^2 = R^2$ const on Cylinder

$= \frac{1}{2}k\left(R^2 + z^2\right)$

→ $L = T - V$

$= \frac{1}{2}m\left(R^2\dot{\theta}^2 + \dot{z}^2\right) - \frac{1}{2}k\left(R^2 + z^2\right)$

→ Canonical momenta

$$p_\theta = \frac{\partial L}{\partial \dot\theta} = mR^2\dot\theta \Rightarrow \dot\theta = \frac{p_\theta}{mR^2}$$

$$p_z = \frac{\partial L}{\partial \dot z} = m\dot z \Rightarrow \dot z = \frac{p_z}{m}$$

→ Hamilton $= \sum p_j \dot q_j - L$

$$= (p_\theta\dot\theta + p_z\dot z) - \left[\frac{1}{2}m(R^2\dot\theta^2 + \dot z^2) - \frac{1}{2}k(R^2+z^2)\right]$$

$$= \frac{p_\theta^2}{mR^2} + \frac{p_z^2}{m} - \left[\frac{1}{2}m\left(\frac{R^2 p_\theta^2}{m^2R^4} + \frac{p_z^2}{m^2}\right)\right] + \frac{1}{2}k(R^2+z^2)$$

$$H = \frac{p_\theta^2}{2mR^2} + \frac{p_z^2}{2m} + \frac{1}{2}k(R^2+z^2)$$

→ Eqn of motion

$$\dot\theta = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{mR^2} \qquad \dot p_\theta = -\frac{\partial H}{\partial \theta} = 0$$

$$\dot z = \frac{\partial H}{\partial p_z} = \frac{p_z}{m} \qquad \dot p_z = -\frac{\partial H}{\partial z} = -kz$$

$\dot p_\theta = 0$ (Angular momentum conserved

$$\left.\begin{array}{l} \dot z = \frac{p_z}{m} \\ \dot p_z = -kz \end{array}\right] \Rightarrow \ddot z = \frac{\dot p_z}{m} = -\frac{kz}{m}$$

$$\boxed{\begin{array}{c}\ddot z + \left(\frac{k}{m}\right)z = 0 \\ \text{SHM}\end{array}}$$

UPSC 2020

SuccessClap

(7c) Find the Solution of

$$z = \frac{1}{2}(p^2+q^2) + (p-x)(q-y)$$ which pass through x-axis

This is Cauchy Strip problem.

This is present in Dr. Raisinghania book & SuccessClap Question Bank

This Solution is SIMPLER than the Textbook Soln

Step-by-step Procedure

→ pass thru x-axis ⇒

$$x_0 = \lambda, \quad y_0 = 0, \quad z_0 = 0$$

→ Find ~~Po~~ p_0, q_0 → use Main Eqn

→ use $dz = p\,dx + q\,dy$

→ Main eqn puts ⇒

$$0 = \frac{1}{2}(p_0^2+q_0^2) + (p_0 - x_0)(q - y_0)$$

$$= \frac{1}{2}(p_0^2+q_0^2) + (p_0 - \lambda)(q_0)$$

→ $dz_0 = p_0\,dx_0 + q_0\,dy_0$

$dz_0 = 0$

$dy_0 = 0$

$dx_0 = d\lambda$

$0 = p_0\,d\lambda + q_0(0)$

$\Rightarrow p_0\,d\lambda = 0$

$\Rightarrow p_0 = 0$

Put in previous eqn

$$0 = \frac{1}{2}(0^2 + q_0^2) + (0-\lambda) q_0$$

$$q_0^2 = 2\lambda q_0 \quad \Rightarrow q_0 = 2\lambda$$

$$\boxed{\begin{array}{l}\text{we got Initial values} \\ x_0 = \lambda,\ y_0 = 0,\ z = 0,\ p_0 = 0,\ q_0 = 2\lambda\end{array}}$$

→ <u>Stnp eqn:</u> $f = \frac{p^2+q^2}{2} + pq - py - qx + xy - z = 0$

$$\frac{dx}{dt} = \frac{\partial f}{\partial p} = p + q - y \quad -(1)$$

$$\frac{dy}{dt} = \frac{\partial f}{\partial q} = q + p - x \quad -(2)$$

$$\frac{dz}{dt} = p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q} \quad -(3)$$

$$= p(p+q-y) + q(q+p-x)$$

$$\frac{dp}{dt} = -\frac{\partial f}{\partial x} - p\frac{\partial f}{\partial z} = p + q - y \quad -(4)$$

$$\frac{dq}{dt} = -\frac{\partial f}{\partial y} - q\frac{\partial f}{\partial z} = p + q - x \quad -(5)$$

→ (1) & (4) $\Rightarrow \frac{dx}{dt} = \frac{dp}{dt} = [\ \ ]$

↓ Integrate

$$x = p + c_1$$

$\rightarrow$ ② & ⑤ $\Rightarrow \frac{dy}{dt} = \frac{dq}{dt} = [\ ]$

$\downarrow$ Integrate

$$y = q + c_2$$

$\rightarrow$ Find c_1, c_2 with intial value

$x_0 = p_0 + c_1 \Rightarrow \lambda = 0 + c_1 \Rightarrow c_1 = \lambda$

$y_0 = q_0 + c_2 \Rightarrow d = 2\lambda + c_2 \Rightarrow c_2 = -2\lambda$

$$\boxed{\begin{aligned} x &= p + \lambda \\ y &= q - 2\lambda \end{aligned}}$$

$\rightarrow$ Observe :

$$\frac{dp}{dt} + \frac{dq}{dt} - \frac{dy}{dt} = (p+q-y) + (p+q-x) - (q+p-x)$$

$$\frac{d(p+q-y)}{dt} = p+q-y$$

$\downarrow$ Integrate

$$p+q-y = Ae^{t}$$

Similarly

$$\frac{dp}{dt} + \frac{dq}{dt} - \frac{dx}{dt} = (p+q-y) + (p+q-x) - (p+q-y)$$

$$= p+q-x$$

$$\frac{d}{dt}(p+q-x) = p+q-x$$

$$(p+q-x) = Be^{t}$$

$$p+q-y=Ae^t \Rightarrow p_0+q_0-y_0=Ae^{(0)}$$

$$0+2\lambda-0=A \Rightarrow A=2\lambda$$

$$p+q-x=Be^t \Rightarrow p_0+q_0-x_0=Be^0$$

$$0+2\lambda-\lambda=B \Rightarrow B=\lambda$$

$$\rightarrow p+q-y=2\lambda e^t$$

we have

$$y=q-2\lambda$$
$$x=p+\lambda$$

$$p+q-q+2\lambda=2\lambda e^t$$

$$p=2\lambda(e^t-1)$$

$$\rightarrow p+q-x=\lambda e^t$$

$$p+q-p-\lambda=\lambda e^t$$

$$q=\lambda(e^t-1)$$

$$\rightarrow \frac{dz}{dt}=p(p+q-y)+q(p+q-x)$$

$$=p\cdot 2\lambda e^t+q\cdot\lambda e^t$$

$$=4\lambda^2e^t(e^t-1)+\lambda^2e^t(e^t+1)$$

$$=5\lambda^2e^{2t}-3\lambda^2e^t$$

$\downarrow$ Integrate

$$z=\frac{5\lambda^2}{2}e^{2t}-3\lambda^2e^t+C$$

$t=0 \Rightarrow z=z_0=0$ $\qquad 0=\frac{5\lambda^2}{2}e^0-3\lambda^2e^0+c$

$$c=\frac{\lambda^2}{2}$$

$$x = p + \lambda = \lambda(2e^t - 1)$$

$$y = q - 2\lambda = \lambda(e^t - 1)$$

$$z = \lambda^2\left(\frac{5}{2}e^{2t} - 3e^t + \frac{1}{2}\right)$$

$$\frac{x}{y} = \frac{2e^t - 1}{e^t - 1} \Rightarrow e^t = \frac{y - x}{2y - x}$$

we have ~~xy~~ $x - 2y = \lambda(2e^t - 1) - 2\lambda(e^t - 1)$

$= -\lambda + 2\lambda$

$= \lambda$

$$\boxed{x - 2y = \lambda}$$

$\downarrow$ Put in z

$$z = (x - 2y)^2\left[\frac{5}{2}\left(\frac{y - x}{2y - x}\right)^2 - 3\left(\frac{y - x}{2y - x}\right) + \frac{1}{2}\right]$$

UPSC 2020

SuccessClap

(7b) Find Quadrature formula

$$\int_0^1 \frac{f(x)\,dx}{\sqrt{x(1-x)}} = \alpha_1 f(0) + \alpha_2 f\left(\tfrac{1}{2}\right) + \alpha_3 f(1)$$

Use this to find $\int_0^1 \frac{dx}{\sqrt{x-x^3}}$

Note :

1) Similar model problems to practice is present in SuccessClap Question Bank

2) Such questions are Frequently present in ANY GRADUATION TEXT BOOKS

3) we must know the formula, to get direct solution.

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \beta(m,n)$$

$$\boxed{\Gamma(n) = (n-1)\Gamma(n-1)}$$

$$\beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} \qquad \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

Take $f(x) = 1$, $f(x) = x$, $f(x) = x^2$

$$f(x) = 1 \Rightarrow \int_0^1 \frac{dx}{\sqrt{x(1-x)}} = \alpha_1(1) + \alpha_2(1) + \alpha_3(1)$$

$$\left.\begin{aligned} f(0) &= 1 \\ f\left(\tfrac{1}{2}\right) &= 1 \\ f(1) &= 1 \end{aligned}\right\} f(x) = 1$$

$$\text{LHS} = \int_0^1 x^{-1/2}(1-x)^{-1/2}\,dx = \int_0^1 x^{1/2-1}(1-x)^{1/2-1}\,dx$$

$$= \beta\left(\tfrac{1}{2},\tfrac{1}{2}\right) \qquad \Gamma\left(\tfrac{1}{2}\right)=\sqrt{\pi}$$

$$= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} = \pi$$

$$\text{RHS} = \alpha_1+\alpha_2+\alpha_3$$

$$\boxed{\alpha_1+\alpha_2+\alpha_3=\pi}$$

$$\rightarrow f(x)=x \Rightarrow \int_0^1 \frac{x\,dx}{\sqrt{x(1-x)}} = \alpha_1(0)+\alpha_2\left(\tfrac{1}{2}\right)+\alpha_3(1)$$

$$\left.\begin{array}{l} f(0)=0 \\ f\left(\frac{1}{2}\right)=\frac{1}{2} \\ f(1)=1 \end{array}\right\}$$

$$= \int_0^1 x^{1/2}(1-x)^{-1/2}\,dx$$

$$= \int_0^1 x^{3/2-1}(1-x)^{1/2-1}\,dx = \beta\left(\tfrac{3}{2},\tfrac{1}{2}\right)$$

$$= \frac{\Gamma(3/2)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)} \qquad \Gamma\left(\tfrac{3}{2}\right)=\tfrac{1}{2}\Gamma\left(\tfrac{1}{2}\right) \qquad \Gamma(2)=1\,\Gamma(1)$$

$$= \frac{1}{2}\times\pi = \frac{\pi}{2}$$

$$\text{RHS} = \frac{\alpha_2}{2}+\alpha_3$$

$$\frac{\alpha_2}{2}+\alpha_3=\frac{\pi}{2} \Rightarrow \boxed{\alpha_2+2\alpha_3=\pi}$$

$$\rightarrow f(x)=x^2 \Rightarrow \int_0^1 \frac{x^2\,dx}{\sqrt{x(1-x)}} = \alpha_1(0)+\alpha_2\left(\tfrac{1}{4}\right)+\alpha_3(1)$$

LHS: $\int_0^1 x^{3/2}(1-x)^{-1/2}\,dx = \int_0^1 x^{5/2-1}(1-x)^{1/2-1}\,dx$

$= \beta\left(\frac{5}{2},\frac{1}{2}\right) = \frac{\Gamma(5/2)\,\Gamma\left(\frac{1}{2}\right)}{\Gamma(3)}$

$\Gamma\left(\frac{5}{2}\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\times\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3}{4}\times\sqrt{\pi}$

$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

$\Gamma(3) = 2\Gamma(2) = 2\times 1\times\Gamma(1) = 2$

$\beta\left(\frac{5}{2},\frac{1}{2}\right) = \frac{\frac{3}{4}\sqrt{\pi}\times\sqrt{\pi}}{2} = \frac{3\pi}{8}$

RHS: $\frac{\alpha_2}{4}+\alpha_3$

$\frac{\alpha_2}{4}+\alpha_3 = \frac{3\pi}{8} \Rightarrow \boxed{\alpha_2+4\alpha_3 = \frac{3\pi}{2}}$

$$\left.\begin{array}{l}\alpha_1+\alpha_2+\alpha_3=\pi \\ \alpha_2+2\alpha_3=\pi \\ \alpha_2+4\alpha_3=\frac{3\pi}{2}\end{array}\right\}$$

Solve 2 & 3 $\Rightarrow \alpha_3 = \frac{\pi}{4}$

$\alpha_2 = \pi/4 \quad \alpha_1 = \frac{\pi}{4}$

$$\boxed{\int_0^1 \frac{f(x)}{\sqrt{x(1-x)}} = \frac{\pi}{4}\left[f(0)+2f\left(\tfrac{1}{2}\right)+f(1)\right]}$$

<u>Problem</u>: Find $\int_0^1 \frac{dx}{\sqrt{x-x^3}} = I$

$x-x^3 = x(1-x^2)$
$= x(1-x)(1+x)$

$I = \int_0^1 \left(\frac{1}{\sqrt{1+x}}\right)\frac{1}{\sqrt{x(1-x)}}\,dx$

$f(x) = \frac{1}{\sqrt{1+x}}$

$\to f(0) = 1$

$f(1/2) = \sqrt{\frac{2}{3}}$

$f(1) = \frac{1}{\sqrt{2}}$

$= \frac{\pi}{4}\left[1+2\sqrt{\frac{2}{3}}+\frac{1}{\sqrt{2}}\right]$

UPSC 2020

SuccessClap

(7c) Velocity potential of 2D fluid is

$\phi(x,y) = xy + x^2 - y^2$.

Find stream function

Note:

This Question carry 15 Marks.

It is very very easy question.

$\rightarrow \dfrac{\partial \psi}{\partial x} = \dfrac{-\partial \phi}{\partial y} \qquad \dfrac{\partial \phi}{\partial x} = \dfrac{\partial \psi}{\partial y}$

$\rightarrow \dfrac{\partial \psi}{\partial y} = \dfrac{\partial \phi}{\partial x} = y + 2x \Rightarrow \psi = \dfrac{y^2}{2} + 2xy + f(x)$

$\rightarrow \dfrac{\partial \psi}{\partial x} = \dfrac{-\partial \phi}{\partial y} = -x + 2y \Rightarrow \psi = -\dfrac{x^2}{2} + 2xy + g(y)$

$\psi = -\dfrac{x^2}{2} + \dfrac{y^2}{2} + 2xy$

$f(x) = -\dfrac{x^2}{2}$

$g(y) = \dfrac{y^2}{2}$

(7c) Velocity potential of 2D fluid is
$\varphi(x,y) = xy + x^2 - y^2$.
Find stream function

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$$\psi = -\frac{x^2}{2} + \frac{y^2}{2} + 2xy$$

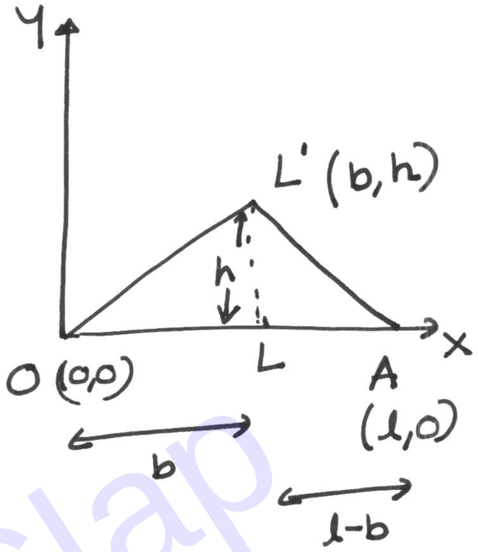
$$f(x) = -\frac{x^2}{2}$$

$$g(y) = \frac{y^2}{2}$$

(8a) One end of string of length l is fixed at origin and other at $x=l$.

It is plucked at $x=\frac{l}{3}$ and have shape of triangle of height h in xy plane.

Find displacement relation.



→ Plucked at L at distance

$b=\frac{l}{3}$

Lets take a generalized distance b

→ $L'(b,h)$

→ OL' eqn $\frac{y-0}{x-0}=\frac{h-0}{b-0}$

$O(0,0)$
$L(b,h)$

$$y=\frac{hx}{b}$$

→ $L'A$ eqn $\frac{y-0}{x-l}=\frac{h-0}{b-l}$

$A(l,0)$
$L'(b,h)$

$$y=-\left(\frac{h}{l-b}\right)(x-l)$$

$l>b$
$(l-b)>0$

→ Boundary Condition $y(0,t)=0$ $y(l,t)=0$

→ Initial Condition

$$y(x,0)=f(x)=\begin{cases}\frac{hx}{b} & 0\le x\le b\\ -\left(\frac{h}{l-b}\right)(x-l) & b\le x\le l\end{cases}$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0}=0$$

$$y(x,t) = \sum c_n \cos\frac{n\pi ct}{l} \sin\frac{n\pi x}{l}$$

This is the General Soln

If you know this is the soln
↳ write dow
↳ otherwise Derive using separable methods

$$c_n = \frac{2}{l}\int_0^l f(x) \sin\frac{n\pi x}{l}\, dx$$

$$= \frac{2}{l}\left[\int_0^b \frac{hx}{b} \sin\frac{n\pi x}{l} dx - \int_b^l \frac{h}{l-b}(x-l) \sin\frac{n\pi x}{l}\right.$$

$$= \frac{2}{l}\left[\frac{h}{b}\left\{-x\frac{l}{n\pi}\cos\frac{n\pi x}{l} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right\}_0^b\right.$$

$$\left. - \frac{h}{l-b}\left\{-(x-l)\frac{l}{n\pi}\cos\frac{n\pi x}{l} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right\}_b^l\right]$$

$$= \frac{2}{l}\left[\frac{h}{b}\left\{\frac{-bl}{n\pi}\cos\frac{n\pi b}{l} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi b}{l}\right\}\right.$$

$$\left. - \frac{h}{l-b}\left\{(b-l)\frac{l}{n\pi}\cos\frac{n\pi b}{l} - \frac{l^2}{n^2\pi^2}\sin\frac{n\pi b}{l}\right\}\right]$$

$$= \frac{2}{l}\left[\left(-\frac{hl}{n\pi} + \frac{hl}{n\pi}\right)\cos\frac{n\pi b}{l} + \frac{hl^2}{\pi^2 n^2}\left(\frac{1}{b} + \frac{1}{l-b}\right)\sin\frac{n\pi b}{l}\right]$$

$$= \frac{2hl^2}{\pi^2 b(l-b)n^2} \sin\frac{n\pi b}{l}$$

$$y(x,t) = \frac{2hl^2}{\pi^2 b(l-b)} \sum \frac{1}{n^2} \sin\frac{n\pi b}{l} \cos\frac{n\pi ct}{l} \sin\frac{n\pi x}{l}$$

Put $b = \frac{l}{3}$

$$y(x,t) = \frac{9h}{\pi^2} \sum \frac{1}{n^2} \sin\frac{n\pi}{3} \cos\frac{n\pi ct}{l} \sin\frac{n\pi x}{l}$$

Soln is Correct as per Book Ans Soln

UPSC 2020

Successclap

(8b) Write 3 point Lagrangian interpolation polynomial relative to x_0, $x_0+\varepsilon$, x_1

Take Limit $\varepsilon \to 0$, Establish

$$f(x) = \frac{(x_1-x)(x+x_1-2x_0)}{(x_1-x_0)^2} f(x_0) + \frac{(x-x_0)(x_1-x)}{(x_1-x_0)} f'(x_0)$$

$$+ \frac{(x-x_0)^2}{(x_1-x_0)} f(x_1) + E(x)$$

where $E(x) = \frac{1}{6}(x-x_0)^2(x-x_1)f'''(\varepsilon)$

is Error function

Note:

1. This problem is present in Iyengar Book
2. UPSC is exploring new question from different Indian books.
   Practice more from different book.
3. Check out SuccessClap Question Bank and Solutions for more practice
4. Print mistake in the question:
   It should be $\frac{(x-x_0)^2}{(x_1-x_0)^2} f(x_1)$

How to write 3-point Lag-Int-Polynomal for $x = a, b, c$

$$f(x) = \frac{(x-b)(x-c)}{(a-b)(a-c)} f(a) + \frac{(x-a)(x-c)}{(b-a)(b-c)} f(b) + \frac{(x-a)(x-b)}{(c-a)(c-b)} f(c)$$

$a = x_0 \qquad b = x_0 + \varepsilon \qquad c = x_1$

$$f(x) = \underbrace{\frac{(x-x_0-\varepsilon)(x-x_1)}{(-\varepsilon)(x_0-x_1)}}_{A} f(x_0) + \underbrace{\frac{(x-x_0)(x-x_1)}{\varepsilon(x_0+\varepsilon-x_1)}}_{B} f(x_0+\varepsilon) + \underbrace{\frac{(x-x_0)(x-x_0-\varepsilon)}{(x_1-x_0)(x_1-x_0-\varepsilon)}}_{C} f(x_1)$$

$\to$ C : $\dfrac{(x-x_0)(x-x_0-\varepsilon)}{(x_1-x_0)(x_1-x_0-\varepsilon)} f(x_1)$

when $\varepsilon \to 0$ C : becomes $\dfrac{(x-x_0)^2}{(x_1-x_0)^2} f(x_1)$

Got the term ✓
3rd term derived

$\to f(x_0+\varepsilon) = f(x_0) + \varepsilon f'(x_0) + \{\text{High order terms}\}$

$\underbrace{\dfrac{\varepsilon^2}{2!} f''(x_0) + \cdots}_{\text{Neglect as } \varepsilon \to 0}$

High terms

$$f(x_0+\varepsilon) = f(x_0) + \varepsilon f'(x_0)$$

$A+B \rightarrow$ terms $\rightarrow$

$$\frac{(x-x_0-\varepsilon)(x-x_1)}{-\varepsilon(x_0-x_1)} f(x_0) + \frac{(x-x_0)(x-x_1)}{\varepsilon(x_0+\varepsilon-x_1)}\left\{f(x_0) + \varepsilon f'(x_0)\right\}$$

$$= [\quad] f(x_0) + \frac{(x-x_0)(x-x_1)}{\varepsilon(x_0+\varepsilon-x_1)} \times \varepsilon f'(x_0)$$

When $\varepsilon \to 0$ Coeff of $f'(x_0)$ is

$$\frac{(x-x_0)(x-x_1)}{(x_0-x_1)} = \frac{(x-x_0)(x_1-x)}{x_1-x_0}$$

This term is

$$\frac{(x-x_0)(x_1-x)}{(x_1-x_0)} f'(x_0)$$

Got this term
2nd term derived

$$f(x_0)(x-x_1)\left\{\frac{x-x_0-\varepsilon}{-\varepsilon(x_0-x_1)} + \frac{x-x_0}{\varepsilon(x_0-x_1+\varepsilon)}\right\}$$

Solve

Let $x-x_0 = \alpha \quad x_0-x_1 = \beta$

$$\left\{\frac{\alpha-\varepsilon}{-\varepsilon\beta} + \frac{\alpha}{\varepsilon(\beta+\varepsilon)}\right\}$$

$$= \left(\frac{1}{\varepsilon}\right)\left\{\frac{\varepsilon-\alpha}{\beta} + \frac{\alpha}{\beta+\varepsilon}\right\}$$

$$= \left(\frac{1}{\varepsilon}\right) \left\{ \frac{\varepsilon\beta + \varepsilon^2 - \alpha\beta - \alpha\varepsilon + \alpha\beta}{\beta(\beta+\varepsilon)} \right\}$$

$$= \left(\frac{1}{\varepsilon}\right) \left\{ \frac{\varepsilon\beta + \varepsilon^2 - \alpha\varepsilon}{\beta(\beta+\varepsilon)} \right\}$$

$$= \left\{ \frac{\beta + \varepsilon - \alpha}{\beta(\beta+\varepsilon)} \right\}$$

When $\varepsilon \to 0$ ↳ $\dfrac{\beta-\alpha}{\beta^2}$

$$\left.\begin{aligned} \alpha &= x - x_0 \\ \beta &= x_0 - x_1 \end{aligned}\right] \quad \begin{aligned} \beta - \alpha &= x_0 - x_1 - x + x_0 \\ &= 2x_0 - x_1 - x \\ &= (-1)(x + x_1 - 2x_0) \end{aligned}$$

$$\frac{f(x_0)(x-x_1)(-1)(x+x_1-2x_0)}{(x_0-x_1)^2} = \frac{f(x_0)(x_1-x)(x_1+x-2x_0)}{(x_1-x_0)^2}$$

Ist terms also derived

we got the derived terms.

UPSC-2020

(8c) Two sources of strength $\frac{m}{2}$ are placed at $(\pm a, 0)$, Show that at any point on circle $x^2+y^2=a^2$, the velocity is parallel to y-axis and is inversely proportional to y

Note: Imaging Not required Simply put potential & get velocity

$$w = -\frac{m}{2}\log(z-a) - \frac{m}{2}\log(z+a)$$

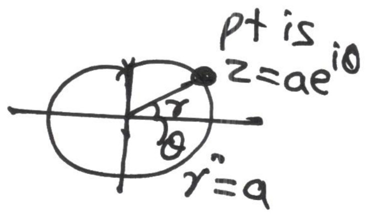
A B

$$q = \frac{dw}{dz} = \left(-\frac{m}{2}\right)\left[\frac{1}{z-a} + \frac{1}{z+a}\right] = \left(-\frac{m}{2}\right)\left[\frac{2z}{(z-a)(z+a)}\right]$$

$$= \frac{(-m)\,z}{(z^2-a^2)}$$

To find velocity on $x^2+y^2=a^2$ ie on $z = ae^{i\theta}$

$$q = (-m)\frac{ae^{i\theta}}{a^2e^{2i\theta}-a^2} = \frac{(-m)\,a}{(a^2)(e^{i\theta}-e^{-i\theta})}$$



$$= \frac{-m}{2ai\sin\theta}$$

$\cos\theta + i\sin\theta$
$-(\cos\theta - i\sin\theta)$
$= 2i\sin\theta$

$$= \frac{-m}{i2y} = \frac{-m}{i2y}\times\frac{i}{i}$$

$\left\{ \begin{array}{l} x = a\cos\theta \\ y = a\sin\theta \end{array} \right\}$

$$= 0 + i\,\frac{m}{2y}$$

No velocity in X-axis

Velocity in Y-axis is $\frac{m}{2y}$ → Inverse proportional to y