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Rigid DYNAMICS

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1

D'Alembert's Principle

1·01. Motion of a particle and a rigid body.

Motion of a particle . The motion of a single particle under the action of given forces is determined by the Newton's second law of motion, which states that the rate of change of momentum in any direction is proportionate to the applied force in that direction .From this law it is deduced that $P = mf$ where f is the acceleration of particle m in the direction of the force P. This mf is called the *effective force* and P the applied force . If (x, y, z) be the co-ordinates of a moving particle of mass m at any time t referred to three rectangular axes fixed in space and X, Y, Z, be the components of the forces acting on the particle in directions parallel to the axes of x, y, z respectively , the motion is found by solving the following three simultaneous equations.

$$m\ddot{x} = X, \; m\ddot{y} = Y, \; m\ddot{z} = Z$$

Motion of a rigid body..If the rigid body is considered as the collection of material particles. we can write the equation of motion of all particles according to the above law but here the external forces include, over and above the applied forces, the mutual actions between the particles. As regards mutual actions between any two particles we assume that (1). The mutual action between two particles is along the line which joins them (2). The action and reaction beetwen them are equal and opposite. In order to find the motion of a rigid body or bodies, D' Alembert gave a method by which all the necessary equations may be obtained of the body. In doing so only the following consequence of the laws of motion is kept in view :

The internal actions and reactions of any system of rigid bodies in motion are in equilibrium amongst themselves.

1· 02. Impressed and effective forces.

Impressed forces. The external forces acting on a rigid body are termed as impressed forces *e.g.* weight of the body. If the body is tied to the string, then tension in the string is the impressed force on the body.

Effective forces. When a rigid body is in motion, each particle of it is acted upon by the external impressed forces and also by the molecular reactions of the other particles. If we assume that particle is separated from the rest of the body, and all these forces are removed, there is some force which would make it to move in the same direction as before. This force is termed as *effective force* on the particle, it is the resultant of the impressed and molecular forces on the particle.

1.03. D ' Alemberts Principle. *The reversed effective forces acting on each particle of the body and the external forces of the system are in equilibrium.*

[Meerut 1995, 88, 90, Agra 89, Nagpur 84, Delhi Hons 82]

Let (x, y, z) be the co-ordinates of a particle of mass m, of a rigid body at any time t. Let f be the resultant of its component accelerations $\ddot{x}, \ddot{y}, \ddot{z}$, so that the effective force on m is mf. Let F be the resultant of the impressed forces on m and R be the resultant of mutual actions, then mf is resultant of F and R.

In case mf is reversed, the mf (reversed), F and R are in equilibrium. So for all the other particles of the body. Thus the reversed effective forces $\Sigma(mf)$ acting on each particle of the body, the external forces (ΣF) and the internal actions and reactions (ΣR) of the rigid body form a system of forces in equilibrium.

But ΣR *i.e.* the internal actions and reactions of the body are itself in equilibrium i.e. $\Sigma R = 0$ Hence the forces ΣF and Σmf (reversed are in equilibrium

i.e. $\Sigma - (mf) + \Sigma F = 0$

Hence the reversed effective forces acting at each point of the system and the impressed (external) forces on the system are in equilibrium.

Note. This principle reduces the dynamical problem to the statical one.

Vector Method. Consider a rigid body in motion. Let at any time t, $\mathbf{r}$ be the position vector of a particle of mass m and F and R be the external and internal forces respectively acting on it. Now by Newton's second law $m(d^2\mathbf{r}/dt^2) = \mathbf{F} + \mathbf{R}$ or $\mathbf{F} + \mathbf{R} - m\,(d^2\mathbf{r}/dt^2) = 0$

i.e. the three forces, namely $\mathbf{F}, \mathbf{R}$ and $-\ m(d^2\mathbf{r}/dt^2)$ are in equilibrium.

Now applying the same argument to every particle of the rigid body, the forces $\Sigma\mathbf{F}, \Sigma\mathbf{R}$ and $\Sigma\left(-m\dfrac{d^2\mathbf{r}}{dt^2}\right)$ are in equilibrium, where the summation extends to all particles.

Since the internal forces acting on the rigid body form pairs of equal and opposite forces, thus their vector sum must be zero

i.e. $\Sigma\mathbf{R} = 0$

$\Rightarrow$ The forces $\Sigma\mathbf{F}$ and $-\ \Sigma m\,(d^2\mathbf{r}/dt^2)$ are in equilibrium. This proves the D' Alembert's Principle.

1.04. Angular momentum of a system of particles. If $\mathbf{r}$ be the position vector of a particle of mass m relative to a point O then the vector sum

$H = \Sigma r \times mv = \Sigma mr \times v$

is called *angular momentum (or moment of momentum)* of the system *about O.*

1.05. General equation of motion. *To deduce the general equation of motion of rigid body from D' Alembert's Principle.*

Cartesian method. Let (x, y, z) be the coordinates of a particle of mass m at any time t referred to a set of rectangular axes fixed in space. Let X, Y, Z represent the components, parallel to the axes of the external forces acting on it.

By D' Alembert's Principle of the forces

$$X - m\ddot{x}\ ,\ Y - m\ddot{y}\ ,\ Z - m\ddot{z}$$

together with similar forces acting on each particle of the body will be in equilibrium.

Hence as in statics, the six conditions of equilibrium are

$$\Sigma\,(X - m\ddot{x}) = 0, \Sigma(Y - m\ddot{y}) = 0, \Sigma(Z - m\ddot{z}) = 0.$$

$$\Sigma[y(Z - m\ddot{z}) - z\,(Y - m\ddot{y})] = 0$$

$$\Sigma[z\,(X - m\ddot{x}) - x\,(Z - m\ddot{z})] = 0$$

and $\Sigma[x(Y - m\ddot{y}) - y\,(X - m\ddot{x})] = 0$

where summations are to be taken over all the particles of the body. These equations give

$$\Sigma m\ddot{x} = \Sigma X,\ \Sigma m\ddot{y} = \Sigma Y,\ \Sigma\, m\ddot{z} = \Sigma Z$$

$$\Sigma m(y\ddot{z} - z\ddot{y}) = \Sigma(yZ - zY)$$

$$\Sigma m(z\ddot{x} - x\ddot{z}) = \Sigma(zX - xZ)$$

and $\Sigma m(x\ddot{y} - y\ddot{x}) = \Sigma(xY - yX)$

These are the six equations of motion of any rigid body.

The first three equations can be written as

$$\frac{d}{dt}\Sigma\, m\dot{x} = \Sigma X\,,\ \frac{d}{dt}\Sigma m\dot{y} = \Sigma Y, \frac{d}{dt}\Sigma m\dot{z} = \Sigma Z$$

and the other three equations are written as

$$\frac{d}{dt}\ \Sigma\, m(y\dot{z} - z\dot{y}) = \Sigma(yZ - zY)$$

$$\frac{d}{dt}\ \Sigma\, m\,(z\dot{x} - x\dot{z}) = (zX - xZ)$$

and $\dfrac{d}{dt}\ \Sigma\, m\,(x\dot{y} - y\dot{x}) = \Sigma\,(xY - yX)$

The first three equations show that the rate of change of linear momentum in any direction is equal to the total external force in that direction and the rest three equations express that the rate of change of the angular momentum about any given axis is equal to the total moment of all the external forces about that axis.

Vector Method. At time t let $\boldsymbol{r}$ be the position vector of a particle of mass m and $\boldsymbol{F}$ be the external force acting on it, then by D' Alembert's *Principle*

$$\Sigma\left(-m\frac{d^2r}{dt^2}\right)+\Sigma F = 0 \text{ or } \Sigma m\frac{d^2\boldsymbol{r}}{dt^2}=\Sigma \boldsymbol{F} \quad ...(1)$$

Taking cross product by $\boldsymbol{r}$, we get

$$\Sigma\, \boldsymbol{r}\times m\frac{d^2\boldsymbol{r}}{dt^2}=\Sigma\, \boldsymbol{r}\times \boldsymbol{F} \quad ...(2)$$

Equations (1) and (2) are in general, vector equations of motion of a rigid body.

Again $\boldsymbol{r} = x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k}$ ...(3) and $\boldsymbol{F} = X\boldsymbol{i}+Y\boldsymbol{j}+Z\boldsymbol{k}$ ...(4)

where X, Y, Z are the components of $\boldsymbol{F}$.

From (3) $(d^2\boldsymbol{r}/dt^2) = (d^2x/dt^2)\boldsymbol{i}+(d^2y/dt^2)\boldsymbol{j}+(d^2z/dt^2)\boldsymbol{k}$ ...(5)

Putting for $\boldsymbol{r}$, $\boldsymbol{F}$ and $(d^2\boldsymbol{r}/dt^2)$ from (3), (4) and (5) respectively in (1) and (2), we get

$$\Sigma m\,[(d^2x/dt^2)\,\boldsymbol{i}+(d^2y/dt^2)\,\boldsymbol{j}+(d^2z/dt^2)\,\boldsymbol{k}\,]=\Sigma\,(X\boldsymbol{i}+Y\boldsymbol{j}+Z\boldsymbol{k})$$

and
$$\Sigma m(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\times[(d^2x/dt^2)\,\boldsymbol{i}+(d^2y/dt^2)\,\boldsymbol{j}+(d^2z/dt^2)\,\boldsymbol{k}\,]$$
$$=\Sigma[(x\boldsymbol{i}+y\boldsymbol{j}+z\boldsymbol{k})\times(X\boldsymbol{i}+Y\boldsymbol{j}+Z\boldsymbol{k})]$$

Equating the coefficients of $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$, we get the six conditions of equilibrium as obtained earlier.

1.06. Linear Momentum. *The linear momentum in a given direction is equal to the product of the whole mass of the body and the resolved part of the velocity of its centre of gravity in that direction.*

Let $(\bar{x}, \bar{y}, \bar{z})$ be the co-ordinates of the C.G. of the system and M the whole mass, then

$M\bar{x} = \Sigma mx$, $M\bar{y} = \Sigma my$ and $M\bar{z} = \Sigma mz$.

Differentiating these relations, we get

$M\bar{x} = \Sigma\, mx$ etc. Hence the result.

1.07. Motion of the centre of inertia. *To prove that the centre of inertia (C.G.) of a body moves as if the whole mass of the body were collected at it, and as if all the external forces were acting at it in directions parallel to those in which they act.* **[Meerut 88, 80]**

Let $(\bar{x}, \bar{y}, \bar{z})$ be the co-ordinates of the C.G. of the body of mass M then

$M\bar{x} = \Sigma mx$, so that $M\ddot{\bar{x}} = \Sigma m\ddot{x}$.

But from the general equation of motion, we have $\Sigma\, mx = \Sigma X$

Therefore, $M\ddot{\bar{x}} = \Sigma X$ ...(1)

Similarly we have, $M\ddot{\bar{y}} = \Sigma Y$ ...(2) and $M\ddot{\bar{z}} = \Sigma Z$ ...(3)

The equation (1) is the equation of motion of a particle of mass M (placed at the centre of inertia) acted on by a force ΣX parallel to the original directions of the forces on different particles. Similarly the equations (2) and (3) can be interpreted.

Vector Method. Let $\bar{r}$ be the position vector of the centre of inertia and r be the position vector of mass m of a rigid body whose mass is M. Now by the definition of centroid, we have

$$\bar{r} = \frac{\Sigma m r^*}{\Sigma m} \quad \therefore \frac{d^2 \bar{r}}{dt^2} = \frac{\Sigma m \dfrac{d^2 r}{dt^2}}{\Sigma m} = \frac{\Sigma m \dfrac{d^2 r}{dt^2}}{M} \qquad ...(1)$$

Again vector equation of motion of a rigid body is

$$\Sigma m \frac{d^2 r}{dt^2} = \Sigma F. \qquad ...(2)$$

Putting the value of $\dfrac{d^2 r}{dt^2}$ from (1) in (2), we get $M \dfrac{d^2 \bar{r}}{dt^2} = \Sigma F$.

If the components of $\bar{r}$ be $(\bar{x}, \bar{y}, \bar{z})$ then $\bar{r} = \bar{x} i + \bar{y} j + \bar{z} k$;

$$\therefore \frac{d^2 \bar{r}}{dt^2} = \frac{d^2 \bar{x}}{dt^2} i + \frac{d^2 \bar{y}}{dt^2} j + \frac{d^2 \bar{z}}{dt^2} k$$

Equation (3) reduces to

$$M \left[\frac{d^2 \bar{x}}{dt^2} i + \frac{d^2 \bar{y}}{dt^2} j + \frac{d^2 \bar{z}}{dt^2} k \right] = \Sigma (X i + Y j + Z k),$$

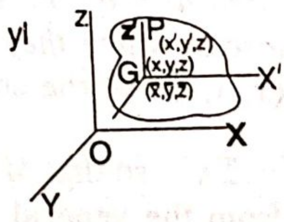
Equating coefficient of i, j, k, we get

$$M \frac{d^2 \bar{x}}{dt^2} = \Sigma x, \; M \frac{d^2 \bar{y}}{dt^2} = \Sigma y, \; M \frac{d^2 \bar{z}}{dt^2} = \Sigma z$$

1.08. Motion relative to centre of interia. *The motion of a body about its centre of inertia is the same as it would be if the centre of inertia were fixed and the same forces acted on the body.* **[Agra 1990, Meerut 90]**

Let $(\bar{x}, \bar{y}, \bar{z})$ be the co-ordinates of the centre of gravity G of the body with reference to the rectangular axes through a fixed point, say O.

Let (x', y', z') be the coordinates of a particle of mass m with G (centre of inertia) as original axes parallel to the original axes and (x, y, z) be its coordinates with reference to original axes. Then



$x = \bar{x} + x', y = \bar{y} + y'$ and $z = \bar{z} + z'$

Now consider the fourth equation of the general equation of motion of rigid body,

$$\text{viz } \Sigma m (y \ddot{z} - z \ddot{y}) = \Sigma (yZ - zY). \qquad ...(i)$$

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* If r is position vector of any particle of mass m of the system relative to a point O, the origin of vectors then the point with position vector $\bar{r} = (\Sigma m r / \Sigma m)$ is defined as the centroid of the system.

Again, $(y\ddot{z} - z\ddot{y}) = (\bar{y} + y')(\ddot{\bar{z}} + \ddot{z}') - (\bar{z} + z')(\ddot{\bar{y}} + \ddot{y}')$

$(\because \ddot{x} = \ddot{\bar{x}} + \ddot{x}'$ etc.)

Therefore from (1), we get

$$\Sigma m(y\ddot{z} - z\ddot{y}) = \Sigma m\bar{y}\ddot{\bar{z}} + \Sigma m\bar{y}\ddot{z}' + \Sigma m y'\ddot{\bar{z}} + \Sigma m y'.\ddot{z}'$$

$$- \Sigma m\bar{z}\ddot{\bar{y}} - \Sigma m\bar{z}\ddot{y}' - \Sigma m z'\ddot{\bar{y}} - \Sigma m z'\ddot{y}' \quad ...(2)$$

As G (the centre of inertia) is the origin of coordinates w. r. t. the new axis.

$$\therefore \Sigma m x' = \Sigma m y' = \Sigma m z' = 0 \quad \left(\therefore \frac{\Sigma m x'}{\Sigma m} = 0 \text{ etc}.\right)$$

Therefore $\Sigma m\ddot{x}' = 0 = \Sigma m\ddot{y}' = \Sigma m\ddot{z}'$, also $\Sigma m = M =$ total mass of the body. Again $\bar{x}, \bar{y}, \bar{z}$ and their differential coefficients are common to all particles of the body, so we can take them outside the sigma sign. Hence equation (2)

$$\Rightarrow \Sigma m(y\ddot{z} - z\ddot{y}) = M\bar{y}\ddot{\bar{z}} - M\bar{z}\ddot{\bar{y}} + \Sigma m(y'\ddot{z}' - \ddot{z}'y')$$

$\therefore$ Equation (1) becomes

$$M\bar{y}\ddot{\bar{z}} - M\bar{z}\ddot{\bar{y}} + \Sigma m(y'\ddot{z}' - z'\ddot{y}') = \Sigma\{(\bar{y} + y')Z - (\bar{z} + z')Y\}$$

$$= \Sigma\bar{y}Z + \Sigma y'Z - \Sigma\bar{z}Y - \Sigma z'Y.$$

Again by **1.07**, we know that $M\ddot{\bar{z}} = \Sigma Z,\ M\ddot{\bar{y}} = \Sigma Y$.

Hence $\Sigma m(y'\ddot{z}' - z'\ddot{y}') = \Sigma(y'Z - z'Y)$.

Similarly, we get other two equations.

But these equations are the same as would have been obtained had we regarded the C.G. to be a fixed point and same forces acted on the body.

Vector Method. Let $\boldsymbol{r}$ be the position vector of the centre of inertia of the rigid body at any time t (say). Mass of the rigid body is M and *let* m be the mass of particle whose position vector with respect to centre of inertia be $\boldsymbol{r}$ and position vector referred to a fixed origin be $\boldsymbol{r}'$ then $\boldsymbol{r} = \bar{\boldsymbol{r}} + \boldsymbol{r}'$.

$$\therefore \frac{d^2\boldsymbol{r}}{dt^2} = \frac{d^2\bar{\boldsymbol{r}}}{dt^2} + \frac{d^2\boldsymbol{r}'}{dt^2}.$$

Now moment vector equation of motion for a rigid body is

$$\Sigma m\boldsymbol{r} \times \frac{d^2\boldsymbol{r}}{dt^2} = = \Sigma\boldsymbol{r} \times \boldsymbol{F}$$

or $$\Sigma m\left(\bar{\boldsymbol{r}} + \boldsymbol{r}'\right) \times \left(\frac{d^2\bar{\boldsymbol{r}}}{dt^2} + \frac{d^2\boldsymbol{r}'}{dt^2}\right) = \Sigma\left(\bar{\boldsymbol{r}} + \boldsymbol{r}'\right) \times \boldsymbol{F}$$

$$\text{or } \bar{r}\times M\frac{d^2\bar{r}}{dt^2}+\bar{r}\,\Sigma mr\frac{d^2r'}{dt^2}+(\Sigma mr')\times\frac{d^2\bar{r}}{dt^2}+\Sigma mr'\times\frac{d^2r'}{dt^2}$$

$$=\bar{r}\times\Sigma F+\Sigma r'\times F \qquad ...(1)$$

Again $\dfrac{\Sigma mr'}{\Sigma m}=0$ (Since it is the position vector of the centre of inertia G referred to G as origin)

$\therefore\ \Sigma mr'=0$, so $\Sigma m\dfrac{d^2r'}{dt^2}=0$

Again we know that $M\dfrac{d^2\bar{r}'}{dt^2}=\Sigma F$,

$$\therefore\ \bar{r}\times M\frac{d^2\bar{r}}{dt^2}=\bar{r}\times\Sigma F$$

Thus equation (1) reduces to $\Sigma\, mr'\times\dfrac{d^2r'}{dt^2}=\Sigma r'\times F$ ...(2)

Equation (2) is the vector equation of motion of a rigid body when centre of gravity is regarded as a fixed point.

Again $r'=x'i+y'j+z'k$

$$\therefore\quad \frac{d^2r'}{dt^2}=\frac{d^2x'}{dt^2}i+\frac{d^2y'}{dt^2}j+\frac{d^2z'}{dt^2}k.$$

Thus equation (2) becomes

$$\Sigma m(x'i+y'j+z'k)\times\left(\frac{d^2x'}{dt^2}i+\frac{d^2y'}{dt^2}j+\frac{d^2z'}{dt^2}k\right)$$

$$=\Sigma(x'i+y'j+z'k)\times(Xi+Yj+Zk)$$

$$\text{or } \Sigma\Big[m\left(y'\frac{d^2z'}{dt^2}-z'\frac{d^2y'}{dt^2}\right)i+m\left(z'\frac{d^2x'}{dt^2}-x'\frac{d^2z'}{dt^2}\right)j$$

$$+m\left(x'\frac{d^2y'}{dt^2}-y'\frac{d^2x'}{dt^2}\right)k\Big]$$

$$=\Sigma[(y'Z-z'Y)i+(z'X-x'Z)j+(x'Y-y'X)k]$$

Equating coefficient of i, we get

$$\Sigma m(y'\ddot{z}'-z'\ddot{y}')=\Sigma(y'Z-z'Y)$$

Similarly, we get other two equations on equating the coefficients of j and k.

Note. 1. The two important properties discussed in **1.07** and **1.08** are respectively called the principle of conservation of motion of translation and rotation and together called the principle of independence of translation and rotation.

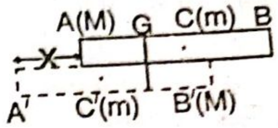
Note 2. 1.07 states that the motion of the C.G. is the same as if the whole mass were collected at the point and is therefore independent of rotation.

Note 3. 1.08 states that the motion round the C.G. is the same as if that point were fixed and is therefore independent of the motion of that point.

ILLUSTRATIVE EXAMPLES

Ex 1. *A rough uniform board, of mass m and length 2a, rests on a smooth horizontal plane, and a boy, of mass M, walks on it from one end to the other, show that the distance through which the board moves in this time is 2 Ma/(m + M).* **[Meerut 88]**

Sol. Here the weight of the boy and the board are downwards, the actions and reactions between the boy and the board vanish for the system.The reaction of the smooth plane is acting vertically upwards. Thus there are no external forces on the system in the horizontal direction. Thus by D ' Alembert's principle the C.G. of the system does not move. As the boy goes to left, the board comes to the right.



Let $\bar{x}$ be the distance of the C.G. of the system and x be the distance through which the board moves, when the boy goes from one end to the other.

Now in the initial position, $(M+m)\bar{x} = M\,2a + ma$

in the final position, $(M+m)\,\bar{x} = Mx + m(a+x)$.

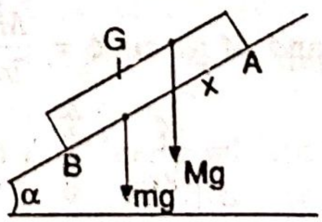
Therefore $M.2a + ma = Mx + m\,(a+x)$,

or $x = 2Ma/(M+m)$.

Ex.2. *A plank , of mass m and length 2a , is initially at rest along a line of greatest slope of a smooth plane inclined at an angle α to the horizon, and a man; of mass M, starting from the upper end walks downs the plank so that it does not move, show that he will reach the other end in time.*

$$\left[\frac{4Ma}{(m+M)\,g\sin\alpha}\right]^{1/2}.$$ **[Meerut 1990]**

Sol. Suppose that the man has come down a distance x in time t, starting from the end A of the plank. Since the plank does not move, its centre is fixed. If $\bar{x}$ be the distance of the C.G. of the system from A, then $(M+m)\,\bar{x} = am + M\,x$.



This gives $(M+m)\,\ddot{\bar{x}} = M\ddot{x}$. ...(1)

Again the motion of the C.G. of the system is given by

$(M+m)\,\ddot{\bar{x}} =$ Ext. forces acting parallel to the plank

$= \Sigma X = (m+M)g\sin\alpha$ ...(2)

From (1) and (2), we get

$$M\ddot{x} = (m+M)\,g\sin\alpha \text{ or } \ddot{x} = \frac{(m+M)g\sin\alpha}{M}.$$

Integrating twice and applying the condition that when we have

$$x = \frac{(m+M)\, g \sin \alpha}{M} \cdot \frac{1}{2} t^2.$$

Putting $x = 2a$, we get the time to reach the other end as

$$\left[\frac{4Ma}{(m+M)\, g \sin \alpha} \right]^{1/2}.$$

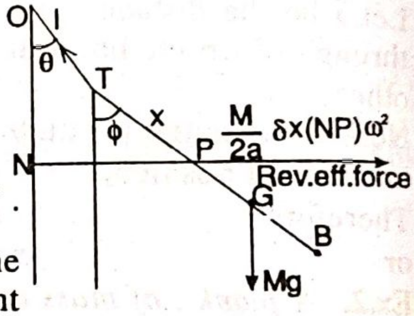
Ex.3. *A rod, of length 2a, is suspended by a string of length l, attached to one end, if the string and rod revolve about the vertical with uniform angular velocity, and their inclination to the vertical be* θ *and* ϕ *respectively, show that* $\dfrac{3l}{a} = \dfrac{(4 \tan \theta - 3 \tan \phi) \sin \phi}{(\tan \phi - \tan \theta) \sin \theta}$. **[Meerut 96,95,94]**

Sol. Take a small element δx of the rod AB at a distance x from A. Let ω be the uniform angular velocity of the rod. Mass of the element

$= \dfrac{M}{2a} \delta x$. Its reversed effective force

$= \dfrac{M}{2a} \delta x \, NP \, \omega^2$, along NP

$= \dfrac{M}{2a} \delta x \, (l \sin \theta + x \sin \phi) \, \omega^2$ along NP.



The external forces on the rod are (1) the tension T of the string and (2) of the weight Mg of the rod.

Resolving horizontally, vertically, and taking moments about A, we have

$$T \sin \theta = \frac{M}{2a.} \omega^2 \Sigma NP \, \delta x = \frac{M}{2a} \omega^2 \int_0^{2a} (l \sin \theta + x \sin \phi)\, dx$$

$$= \frac{M}{2a} \omega^2 (2al \sin \theta + 2a^2 \sin \phi) \qquad [\textit{Horizontally}] \qquad ...(1)$$

$$T \cos \theta = Mg \qquad [\textit{Vertically}] \qquad ...(2)$$

and $Mga \sin \phi = \dfrac{M}{2a} \omega^2 \, \Sigma \, NP. \, \delta x . x \cos \phi$ [*Moment equation*]

$$= \frac{M}{2a} \omega^2 \int_0^{2a} (l \sin \theta + x \sin \phi)\, x \cos \phi \, dx$$

$$= \frac{M}{2a} \omega^2 \left(l \sin \theta \cos \phi \, 2a^2 + \frac{8a^3}{3} \sin \phi \cos \phi \right)$$

$$\text{or } \omega^2 = \frac{3 g \sin \phi}{(3l \sin \theta + 4a \sin \phi) \cos \phi}. \qquad ...(3)$$

Diviiding (1) by (2), we have $\dfrac{\sin \theta}{\cos \theta} = \omega^2 \dfrac{(l \sin \theta + a \sin \phi)}{g}$. ...(4)

Putting value of ω^2 from (3) in (4), we get

$$\frac{\sin\theta}{\cos\theta} = \frac{3\sin\phi\,(l\sin\theta + a\sin\phi)}{(3l\sin\theta + 4a\sin\phi)\cos\phi}$$

or $\quad \sin\theta\cos\phi\,(3l\sin\theta + 4a\sin\phi) = \sin\phi\cos\theta\,(l\sin\theta + a\sin\phi)$

or $\quad 3l\sin\theta\,(\sin\theta\cos\phi - \sin\phi\cos\theta)$

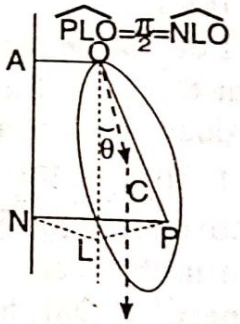
$$= a\sin\phi\,(3\sin\phi\cos\theta - 4\sin\theta\cos\phi)$$

or $\quad \dfrac{3l}{a} = \dfrac{\sin\phi\,(3\sin\phi\cos\theta - 4\sin\theta\cos\phi)}{\sin\theta\,(\sin\theta\cos\phi - \sin\phi\cos\theta)}$

$$= \frac{\sin\phi\,(3\tan\phi - 4\tan\theta)}{\sin\theta\,(\tan\theta - \tan\phi)} = \frac{(4\tan\theta - 3\tan\phi)\sin\phi}{(\tan\phi - \tan\theta)\sin\theta}.$$

Ex.4. *A thin circular disc of mass M and radius a, can turn freely about a thin axis OA, which is perp. to its plane and passes through a point O of its circumference. The axis OA is compelled to move in a horizontal plane with angular velocity ω about its end A. Show that the inclination θ to the vertical of the radius of the disc through O is $\cos^{-1}(g/a\omega^2)$ unless $\omega^2 < g/a$ and then θ is zero.*

Sol. Consider the circular disc in the vertical plane so that the axis OA about which it turns is horizontal. when the axis OA moves horizontally round A, the disc will be raised in its vertical plane and its radius OC makes an angle θ with the vertical. Consider an element δm at P. Let PL be perpendicular to the vertical through O and LN be perpendicular from L to the vertical through A so that PN is perp. to AN. Now P discribes a circle of radius PN with a constant angular velocity ω about N. Thus the reversed effective force along NP is $\delta m NP\omega^2$



Again $\vec{NP} = \vec{NL} + \vec{LP}$.

$\therefore\ \delta m.\,\omega^2\,\vec{NP} = \delta m\,\omega^2\,\vec{NL} + \delta m\,\omega^2\,\vec{LP}$ i.e. the force $\delta m\,\omega^2\,\vec{NP}$ is equivalent to forces $\delta m\,\omega^2\,LP$ and *other* $\delta m\,\omega^2 NL$ along NL. The external forces on the disc are its weight Mg and the reaction at O.

By D'Alemberts Principle, Rev. effective forces along with external forces form the system in equlibrium. Hence moment of Rev. effective forces + moment of external forces = 0 i.e. moment of effective forces about OA = moment of external forces (1).

In order to avoid reaction at O, we take moment about the line OA. Since NL and OA lie in one plane (they are parallel also) the shortest distance between them is zero.

$\therefore$ Moment of the force $\delta m\omega^2 \times NL$ about OA is zero. Further the shortest distance between OA and LP is OL and the shortest distance between OA and the vertical through C is $a \sin\theta$. Hence moment of the force $\delta m\omega^2 LP$ about OA is given by $\delta m\omega^2 LP \times OL$. Taking moments about OA, we get $Mga\sin\theta = \Sigma\delta m\omega^2 LP \times OL$

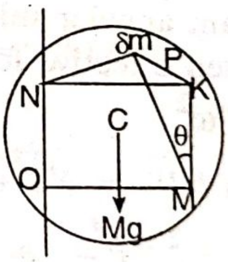
or $aMg\sin\theta \equiv \omega^2\Sigma(\delta m\, LP\,.\,OL)$. But, $\Sigma(\delta m\, LP\, OL) =$ Product of inertia of the disc about OL and horizontal line through $O =$ Product of inertia about the parallel lines through $C + Mx'y'$, where x', y' are the co-ordinates of C with respect of the vertical and horizontal through O $= 0 = Ma^2 \sin\theta\cos\theta$.

$\Rightarrow aMg\sin\theta = \omega^2 Ma^2 \sin\theta\cos\theta \Rightarrow \sin\theta = 0$ or $\cos\theta = (g/a\omega^2)$, where $a\omega^2 > g$.

But, $\omega^2 < (g/a) \Rightarrow \cos\theta > 1$, which is impossible and hence in this case $\cos\theta = 1$ *i.e.* $\theta = 0$.

Ex.5. *A thin heavy disc can turn freely about an axis in its own plane, and this axis revolves horizontally with a uniform angular velocity* ω *about a fixed point on itself. Show that the inclination* θ *of the plane of the disc to the vertical is given by* $\cos\theta = (gh/k^2\omega^2)$ *where h is the distance of the centre of inertia of the disc from the axis and k is the radius of the gyration of the disc about the axis. If* $\omega^2 < gh/k^2$, *prove that the plane of the disc is vertical.*

Sol. Let OM be the horizontal axis in the plane of the disc which, rotates about O so that the vertical line ON is the axis of rotation of the system. Consider an element of mass δm at P. Draw PN perp. to this vertical axis ON then effective force for δm is $\delta m\,\omega^2 PN$. Here PN is not in the plane of the disc. From P draw PM perp. to OM, here PM is in the plane of the disc. Through N draw NK perp. to OM and from P draw PK perp. to NK so that PK is perp. to KM. thus if $\angle PMK = \theta$, θ is the inclination of the disc to the vertical, KM being vertical.



Again $\overrightarrow{PN} = \overrightarrow{PK} + \overrightarrow{KN}$. Therefore $\delta m\,\omega^2\overrightarrow{PN} = \delta m\omega^2\overrightarrow{PK} + \delta m\,\omega^2\,\overrightarrow{KN}$.

Thus the effective force on δm are $\delta m\,\omega^2\overrightarrow{PK}$ and $\delta m\,\omega^2\overrightarrow{KN}$. Since KN is parallel to OM, the moment of the force $\delta m\,\omega^2\overrightarrow{KN}$ about OM will be zero and the moment of $\delta m\omega^2\,\overrightarrow{PK}$ about OM is $\delta m\omega^2\, PK\,.\,KM$. The moment of Mg about OM is $Mgh\sin\theta$. Hence taking moments about OM, we get $Mgh\sin\theta = \Sigma\delta m\,\omega^2\, PK\, KM$

$$= \Sigma\delta m\,\omega^2(PM\sin\theta)(PM\cos\theta) = \omega^2\sin\theta\cos\theta\,\Sigma\delta m\, PM^2$$

But $\Sigma \delta m PM^2 =$ M.I. of the disc about $OM = Mk^2$, where k is the radius of gyration. $\therefore Mgh \sin\theta = \omega^2 \sin\theta \cos\theta\, Mk^2$.

Hence either $\sin\theta = 0$ *i.e.* $\theta = 0$ or $\cos\theta = \dfrac{gh}{\omega^2 k^2}$

If $\omega^2 < \dfrac{gh}{k^2}$, as in that case $\cos\theta > 1$ the only possible value of θ is zero and then plane of the disc is vertical.

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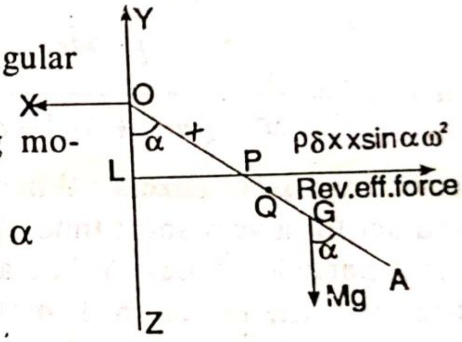
Ex.6. *A uniform rod OA, of length 2a, free to turn about its end O, revolves with uniform angular velocity ω about a vertical OZ through O. and is inclined at a constant angle α to OZ, show that the value of α is either zero or* $\cos^{-1}(3g/4a\omega^2)$. **[Meerut 84,75,73]**

Sol. Consider a small element $PQ = \delta x$ at a distance x from O. The point P will move in a horizontal circle whose radius is $PL = x\sin\alpha$. Here only effective force on the element PQ is

$\rho\, \delta x\, PL\omega^2 = \rho\delta x \,.\, x\sin\alpha\,\omega^2$,



where ρ is the density of the rod and angular velocity ω is constant.

Reversing the effective force and taking moments about O, we have

$$\Sigma(\rho\,\delta x\,.\, x\sin\alpha\,\omega^2) x\cos\alpha = Mga\sin\alpha$$

or $\rho\omega^2 \sin\alpha\cos\alpha \displaystyle\int_0^{2a} x^2\, dx = Mga\sin\alpha$

or $(M/2a)\omega^2 \sin\alpha\cos\alpha(8a^3/3) = Mga\sin\alpha \quad (\because\ 2a\rho = M)$

or $\sin\alpha\left(g - \dfrac{4a\omega^2\cos\alpha}{3}\right) = 0$. It implies either $\sin\alpha = 0$ *i.e.* $\alpha = 0$

or $\cos\alpha = (3g/4a\,\omega^2)$ *i.e.* $x = \cos^{-1}(3g/4a\,\omega^2)$.

Ex.7. *Two uniform spheres, each of mass M and radius a, are firmly fixed to the ends of two uniform thin rods, each of mass m and length l, and the other ends of the rods are freely hinged to a point O. The whole system revolves as in the Governor of a steam-Engine, about a vertical line through O with the angular velocity ω. Show that when the motion is steady, the rods are inclined to the vertical at an angle θ given by the equation*

$$\cos\theta = \frac{g}{\omega^2} \cdot \frac{M(l+a) + \frac{1}{2}ml}{M(l+a)^2 + \frac{1}{3}ml^2}.$$

Sol. Take an element δx in one of the rods at a distance x from O. Let PN, CM be the perpendiculars on the vertical line through O. Here C is

But $\Sigma \delta m PM^2 =$ M.I. of the disc about $OM = Mk^2$, where k is the radius of gyration. $\therefore Mgh \sin\theta = \omega^2 \sin\theta \cos\theta Mk^2$.

Hence either $\sin\theta = 0$ *i.e.* $\theta = 0$ or $\cos\theta = \dfrac{gh}{\omega^2 k^2}$

If $\omega^2 < \dfrac{gh}{k^2}$, as in that case $\cos\theta > 1$ the only possible value of θ is zero and then plane of the disc is vertical.

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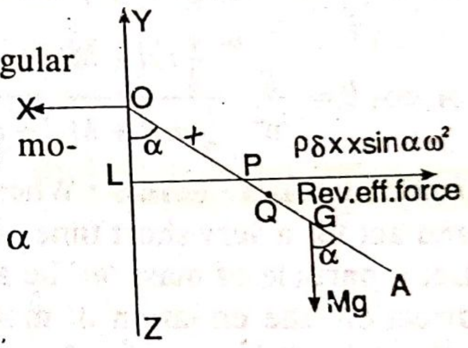
$\rho\,\delta x\,PL\omega^2 = \rho\delta x\,.\,x\sin\alpha\,\omega^2$,



where ρ is the density of the rod and angular velocity ω is constant.

Reversing the effective force and taking moments about O, we have

$$\Sigma(\rho\,\delta x\,.\,x\sin\alpha\,\omega^2)x\cos\alpha = Mga\sin\alpha$$

or $$\rho\omega^2\sin\alpha\cos\alpha\int_0^{2a} x^2\,dx = Mga\sin\alpha$$

or $$(M/2a)\omega^2\sin\alpha\cos\alpha(8a^3/3) = Mga\sin\alpha \quad (\because\ 2a\rho = M)$$

or $$\sin\alpha\left(g - \frac{4a\omega^2\cos\alpha}{3}\right) = 0.$$ It implies either $\sin\alpha = 0$ *i.e.* $\alpha = 0$

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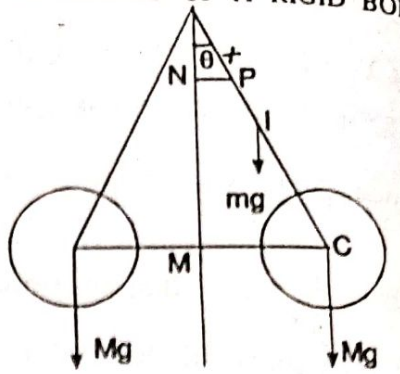
$$\cos\theta = \frac{g}{\omega^2}\cdot\frac{M(l+a)+\frac{1}{2}ml}{M(l+a)^2+\frac{1}{3}ml^2}.$$

Sol. Take an element δx in one of the rods at a distance x from O. Let PN, CM be the perpendiculars on the vertical line through O. Here C is



the centre of one of the spheres. The reversed effective force on the rod at P is $\delta x \frac{m}{l} \omega^2 x \sin\theta$ along NP and the reversed effective force on the sphere is $M\omega^2(a+l)\sin\theta$ along MC. On taking moments about O for the system of a rod and a sphere on one side of the vertical OM, we have

$$\Sigma\{\delta x(m/l)\omega^2 x \sin\theta\, x\cos\theta\} + M\omega^2(a+l)\sin\theta(a+l)\cos\theta$$
$$= Mg(a+l)\sin\theta + mg(l/2)\sin\theta$$

or $$\int_0^l \frac{m}{l}\omega^2 \sin\theta\cos\theta\, x^2\, dx + M\omega^2(a+l)^2 \sin\theta\cos\theta$$
$$= Mg(a+l)\sin\theta + \frac{1}{2} mgl\sin\theta$$

or $$\omega^2\{\tfrac{1}{3}ml^2 + M(a+l)^2\}\cos\theta = g(\tfrac{1}{2}ml + M(l+a))$$

or $$\cos\theta = \frac{g}{\omega^2}\cdot\frac{\frac{1}{2}ml + M(l+a)}{\frac{1}{3}ml^2 + M(l+a)^2}$$

1·09. Impulsive Forces : When the forces acting on a body are very large and act for a very short time, then their effects are measured by impulses. Let a particle of mass 'm' be acted upon by a force $\boldsymbol{F}$ always in the same direction, the equation of motion is $m\,(dv/dt) = \boldsymbol{F}$. ...(1)
where v is the velocity of the particle at time t. If τ be the time during which the force $\boldsymbol{F}$ acts and v_1, v_2 be the velocities before and after the action of the force, then on integrating (1), we have

$$m(v_2 - v_1) = \int_0^\tau \boldsymbol{F}\,dt \qquad ...(2)$$

Now if $\boldsymbol{F}$ increases indefinitely while τ decreases indefinitely, then the integral on the right hand side of (2) may have a definite finite limit. Let this finite limit be $\boldsymbol{I}$ then equation (2) may be written as

$$m\,(v_2 - v_1) = \boldsymbol{I} \qquad ...(3)$$

The velocity during the time τ has increased or decreased from v_1 to v_2. Supposing that the velocity have remained finite, let v be the greatest velocity during the interval. Then the space described is less than $v\tau$. Since $v\tau \to 0$ as $\tau \to 0$, hence we conclude that the particle has not moved during the action of the force $\boldsymbol{F}$. It could not have time to move, but its velocity has been changed from v_1 to v_2.

Thus in the case of finite forces which act on a body for indefinitely short *time, the change of place is zero and the change of velocity is the measure of these forces. A force so measured is called an impulse. We can define impulse as the limit of a force which is indefinitely greater but acts only for an indefinitely short time e.g. the below of a hammer is a force of this kind. In fact an impulsive force is measured by the whole momentum generated by the impulse.*

1·10. When impulsive force acts, the finite forces acting on the body may be neglected in calculating the effect.

Let F be the impulsive force and f a finite force acting simultaneously on the body. Then $m(v_1 - v_2) = \int_0^{\tau} F\,dt + \int_0^{\tau} f\,dt = P + f\tau$.

But since $f\tau \to 0$ as $\tau \to 0$, f may be neglected in forming the equations.

1·11. Application of D'Alembert's principle to impulsive forces, general equation of motion.

Scalar Method. let u, v, w be the velocities parallel to co-ordinate axes before the action of impulsive forces and u', v', w' be the velocities after the action of these forces. Let X', Y', Z' be the resolved parts of the impulsive forces parallel to the axes. Then from $\Sigma m\ddot{x} = \Sigma X$, on integrating with respect to t, we get

$$\left[\Sigma m \frac{dx}{dt}\right]_0^{\tau} = \int_0^{\tau} \Sigma X\,dt = \Sigma \int_0^{\tau} X\,dt = \Sigma X'$$

or $\Sigma m(u' - u) = \Sigma X'$.

Similarly, $\Sigma m(v' - v) = \Sigma Y'$ and $\Sigma m(w' - w) = \Sigma Z'$

Thus the change in the momentum parallel to any of the axes of the whole mass M, supposed collected at the centre of inertia and moving with it is equal to the impulse of the external forces parallel to the corresponding axis. Again we have the moment equation

$$\Sigma m(y\ddot{z} - z\ddot{y}) = \Sigma m(yZ - zY)$$

Integrating this we have $\left[\Sigma m(y\dot{z} - z\dot{y})\right]_0^{\tau} = \Sigma\left[y\int_0^{\tau} Z\,dt - z\int_0^{\tau} Y\,dt\right]$

Since the interval τ is so short that the body has not moved during this period, we may take x, y, z as constants, thus the above equation becomes

$$\Sigma m\{y(w' - w) - z(v' - v)\} = \Sigma(yZ' - zY')$$

Similarly we have other two equations

$$\Sigma m\{x(v' - v) - y(u' - u)\} = \Sigma(xY' - yX')$$

and $\Sigma m\{z(u' - u) - x(w' - w)\} = \Sigma(zX' - xZ')$

Hence the change in the moment of momentum about any of the axes is equal to the moment about that axis of the impulses of the external forces.

Ex.8. *A cannon of mass M, resting on a rough horizontal plane of coefficient of friction μ, is fired with such a charge that the relative velocity of the ball* *and cannon at the moment when it leaves the cannon is u. Show that the cannon will recoil a distance* $\left(\frac{mu}{M+m}\right)^2 \frac{1}{2\mu g}$ *along the plane, m being the mass of the ball.* **[Raj.88]**

Sol. Let I be the impulse between the cannon and the ball and V, v be their velocities. Since their relative velocity is u, we have

$$V + v = u \qquad \ldots(1) \quad \text{and} \quad mv = I = MV \qquad \ldots(2)$$

From (1) and (2), we have $(MV/m) + V = u$ or $V = \{mu/(m+M)\}$

Again on the rough plane, for the cannon the equation is

$$M\ddot{x} = -\mu R = -\mu Mg, \text{ where } x \text{ is the distance cannon has moved.}$$

$\therefore \ddot{x} = -\mu g$, , Multiplying by $2\dot{x}$ and integrating, we get

$$\dot{x}^2 = -2\mu g x + C$$

When $x = 0, \dot{x} = V$, so that $C = V^2$, $\quad \dot{x}^2 = V^2 - 2\mu g x$

when the cannon comes to rest $\dot{x} = 0$, $\quad \therefore x = (V^2/2\mu g)$

or $x = (mu/M+m)^2 (1/2\mu g)$ $\qquad [\because V = (mu/M+m)]$

Supplementary Problems

1. A light rod **OAB** can turn freely in a vertical plane about a smooth fixed hinge at **O**; two heavy particles of mass **m** and m' are attached to the rod A and B and oscillate with it. Find the motion.

2. Find the motion of the rod **OAB** with two masses **m** and m' attached it, when it moves round the vertical as a conical pendulum with uniform angular velocity, the angle θ which the rod makes with the vertical being constant.

3. A rod revolving on a smooth horizontal plane about one end, which is fixed, breaks into two parts. What is the subsequent motion of the two parts ?

4. A rod of length $2a$ revolves with uniform angular velocity ω about a vertical axis through a smooth joint at one extremity of the rod so that it describes cone of semi vertical angle α ; show that $\omega^2 = (3g/4a)\cos\alpha$. Prove also that the direction of reaction at the hinge makes with the vertical an angle $\tan^{-1}[(3/4)\tan\alpha]$. **[Agra 1971]**

5. Two persons are situated on a perfectly smooth horizontal plane at a distance a from each other. One of the persons, of mass M throws a ball of mass m towards the other which reaches him in time t. Prove that the first person will begin to slide along the plane with velocity (ma/Mt).

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