11 Years
Solved Papers
2009-2019

MADE EASY
Publications

Civil Services
Main Examination

TOPICWISE PREVIOUS YEARS' SOLVED PAPERS

Mathematics
Paper-II

2 Real Analysis

1. Real Number System

1.1 Show that every open subset of R is a countable union of disjoint open intervals.

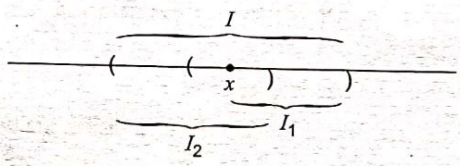
(2013 : 14 Marks)

Solution:

Let $U \leq R$ be an open and let $x \in U$. Then x is rational or irrational. If x is rational, define

$$I_x = \bigcup_{\substack{x \in I \subset U \\ I \text{ is open}}} I \text{ (Union of all open intervals in } U \text{ containing } x)$$

Note that I_x is simply the largest open interval containing x as it is union of disjoint open intervals (see figure below).



If x is irrational, as U is open, $\exists \in > 0$ such that $(x - \in, x + \in) \subset U$.

As every interval contains a rational $\exists y \in (x - \in, x + \in)$ and then $x \in I_y$

$\therefore$ $\forall x \in U$ $x \in I_q$ for some $q \in U \cup Q$, i.e., where q is a rational in U.

$\therefore$
$$U \subseteq \bigcup_{q \in U \cup Q} I_q$$

But
$$I_q \leq U \forall q$$

$\therefore$
$$U = \bigcup_{q \in U \cup Q} I_q$$

And as the number of rationals in U are countable U can be written as union of countable intervals. Also each of I_q is disjoint as it being maximal if two I_q overlap they will be same.

2. Sequences

2.1 Show that a bounded infinite subset of $\mathbb{R}$ must have a limit point.

(2009 : 15 Marks)

Solution:

Let A be an infinite and bounded set. There exist and interval $[k, K]$ such that $A \subseteq [k, K]$.
We define a set S as follows :
$x \in S$: It exceeds at the most a finite number of member of the set A.
Thus, while $k \in S$ and $K \in S$.
Also, S is bounded above in as much as K is an upper bounded of the same. Let ξ the least upper bound of S. Surely it exists by the order completeness properly of $\mathbb{R}$.

We show that ξ is a limit point of A.
Consider a neighbourhood say B of ξ,

$$\xi \in (a, b) \subset B$$

Now the numebr 'a' which is less than the least upper bound ξ of the set is not an upper bound of S. Thus there exist a number, η (sequence) of S such that

$$a < \eta \leq \xi, \eta \in S$$

Also η being a member of S exceeds at not a finite number of member of A : It follower and that the number 'a' also exceeds at the most a finite a finite number of member of A.
Again the number b which is greater than ξ is an upper bound of S without being a member of S. Thus must exceed an infinite number of member of A.
It follows that

(i) 'a' exceeds at the most a finite number of member of S.
(ii) 'b' exceeds an infinite number of members of S.

Thus, $[a, b]$ contains an infinite number of member of A. So, that ξ is the limit of point of S.

2.2 Discuss the convergence of the sequence $\{x_n\}$ where $x_n = \dfrac{\sin\frac{n\pi}{2}}{8}$.

(2010 : 12 Marks)

Solution:

Given : Sequence $\{x_n\} = \left\{\frac{1}{8}, 0, -\frac{1}{8}, 0, \frac{1}{8}, 0, -\frac{1}{8}, ...\right\}$

So, the given sequence $\{x_n\}$ assumes 3 values viz., 0, $-\frac{1}{8}$ and $\frac{1}{8}$ and is oscillatory in nature.

$\therefore$ $\{x_n\}$ does not converge.

2.3 Define $\{x_n\}$ by $x_1 = 5$ and $x_{n+1} = \sqrt{4 + x_n}$ for $n > 1$. Show that the sequence converges to $\dfrac{1+\sqrt{17}}{2}$.

(2010 : 12 Marks)

Solution:

Given : $x_1 = 5$ and $x_{n+1} = \sqrt{4+x_n}$ for $n > 1$

$\therefore$
$$x_2 = \sqrt{4+x_1} = \sqrt{4+5} = \sqrt{9} = 3$$
$$x_3 = \sqrt{4+x_2} = \sqrt{4+3} = \sqrt{7}$$

Let $n = 1$

$\therefore$ $$x_2 < x_1$$

$\therefore$ True for $n = 1$.
Let it is also true for $K \in N$.

$\therefore$
$$x_{k+1} < x_k$$
$$x_{k+1} + 4 < x_k + 4$$
$\therefore$
$$\sqrt{x_{k+1}+4} < \sqrt{x_k+4}$$
$\Rightarrow$
$$x_{k+2} < x_{k+1}$$

$\therefore$ True for $k + 1$ also.
So, by mathematical induction it is true for all $K \in N$.
$\therefore$ $\{x_n\}$ is monotonically decreasing sequence.

Now, $x_1 = 5 > 2$

$$x_2 = 3 > 2$$
$$x_3 = \sqrt{7} > 2$$
$$\vdots$$
$$x_{k+1} = \sqrt{4 + x_k} > 2 \text{ as } x_k > 0$$

$\therefore \{x_n\}$ is bounded below.

As $\{x_n\}$ is monotonically decreasing and bounded below, $\therefore$ it is convergent.

Let it converges to l. ($l > 0$).

$\therefore$ $$x_{k+1} = \sqrt{4 + x_k}$$

$\Rightarrow$ $$\lim_{k\to\infty} x_k + 1 = \lim_{k\to\infty} \sqrt{4 + x_k}$$

$\Rightarrow$ $$l = \sqrt{4+l}$$

$\Rightarrow$ $$l^2 = 4 + l$$

$\Rightarrow$ $$l^2 - l - 4 = 0$$

$\Rightarrow$ $$l = \frac{1 \pm \sqrt{1+16}}{2} = \frac{1 \pm \sqrt{17}}{2}$$

Now, $$l > 0$$

$\therefore$ $$l = \frac{1+\sqrt{17}}{2}$$

$\therefore \{x_n\}$ converges to $\frac{1+\sqrt{17}}{2}$.

2.4 Let $f_n(x) = x^n$ on $-1 < x \le 1$ for $n = 1, 2, \ldots\ldots$ Find the limit function. Is the convergence uniform? Justify your answer.

(2010 : 15 Marks)

Solution:

Given : $$f_n(x) = x^n$$

If $x \ne 1$:

$$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = 0 \text{ as } |x| < 1$$

$\therefore$ $$|f_n(x) - f(x)| = |x^n - 0| = |x^n|$$

At sup. $|f_n(x) - f(x)|$, $$\frac{d}{dx}(x^n) = 0$$

$\Rightarrow$ $$nx^{n-1} = 0 \text{ at } x = 0$$

At $x = 0$, sup. $|f_n(x) - f(x)| = 0$

So, limit function is 0 and is uniformly convergent.

At $x = 1$, $$f_n(x) = 1^n \to \infty \text{ as } n \to \infty$$

$\therefore$ At $n \to \infty$ $$f_n(x) = \begin{cases} 0 & \text{if } x \in (-1, 1) \\ 1 & \text{if } x = 1 \end{cases}$$

$\therefore$ In the interval $(-1, 1]$, $f_n(x)$ is discontinuous at $x = 1$ when $n \to \infty$.

So, $f_n(x)$ is not uniformly convergent on $(-1, 1]$.

2.5 Let $f_n(x) = nx(1-x)n$, $x \in [0, 1]$. Examine the uniform convergence of $\{f_n(x)\}$ on $[0, 1]$.

(2011 : 15 Marks)

Solution:

Given $$f_n(x) = nx(1-x)^n, x \in [0, 1]$$

At $x = 0$, $$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} 0 = 0$$

At $x = 1$, $$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} 0 = 0$$

For $0 < x < 1$, we have

$$-1 < -x < 0$$

$$\Rightarrow \quad 0 < 1-x < 1$$

$$\therefore \quad \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} nx(1-x)^n = 0 = f(x) \text{ (say)}$$

$$\therefore \quad \lim_{n\to\infty} f_n(x) = 0 = f(x) \ \forall \ x \in [0, 1]$$

Again, define $$M_n = \sup_{x\in[0,1]} |f_n(x) - f(x)|$$

Then, define $$M_n = \sup_{x\in[0,1]} |f_n(x) - f(x)|$$

Then, $$\{f_n\} \to f \text{ uniformly} \Rightarrow \lim_{n\to\infty} M_n = 0$$

Now, $$M_n = \sup_{x\in[0,1]} \left|nx(1-x)^n\right|$$

Let $$g(x) = nx(1-x)^n$$

$$\Rightarrow \quad g'(x) = n(1-x)^n - n^2x(1-x)^{n-1}$$

$$= n(1-x)^{n-1}(1-x-nx)$$

$$\therefore \quad g'(x) = 0$$

$$\Rightarrow \quad n(1-x)^{n-1}(1-x-nx) = 0$$

$$\Rightarrow \quad x = \frac{1}{n+1}$$

$$\therefore \quad M_n = \sup_{x\in[0,1]} \left|nx(1-x)^n\right|$$

$$= \left|n \cdot \frac{1}{n+1} \cdot \left(1 - \frac{1}{n+1}\right)^n\right|$$

$$= \left(\frac{n}{n+1}\right)^{n+1}$$

$$\therefore \quad \lim_{n\to\infty} M_n = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{n+1}$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n \cdot \left(\frac{n}{n+1}\right)$$

$$= \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}}\right)^n \cdot \left(\frac{1}{1+\frac{1}{n}}\right)$$

$$= \frac{1}{e}\cdot 1 = \frac{1}{e} \neq 0$$

$\therefore$ $[f_n(x)]$ is not uniformly convergent on $[0, 1]$.

2.6 Two sequences $\{x_n\}$ and $\{y_n\}$ are defined inductively by the following :

$$x_1 = \frac{1}{2},\ y_1 = 1 \text{ and } x_n = \sqrt{x_n + y_{n+1}},\ n = 2, 3, 4$$

$$\frac{1}{y_n} = \frac{1}{2}\left(\frac{1}{x_n} + \frac{1}{y_{n-1}}\right),\ n = 2, 3, 4, \ldots.$$

Prove that $x_{n-1} < x_n < y_n < y_{n-1}$, $n = 2, 3, 4, \ldots..$ and deduce that both the sequences comes to the same limit l, where $\frac{1}{2} < l < 1$.

(2016 : 10 Marks)

Solution:

Given :

$$x_1 = \frac{1}{2},\ y_1 = 1 \text{ and } x_n = \sqrt{x_{n-1}y_{n-1}} \quad \ldots(i)$$

$$\frac{1}{y_n} = \frac{1}{2}\left(\frac{1}{x_n} + \frac{1}{y_{n-1}}\right) \quad \ldots(ii)$$

Now, consider (ii)

$$\frac{1}{y_n} = \frac{1}{2}\left(\frac{1}{x_n} + \frac{1}{y_{n-1}}\right)$$

$$\Rightarrow \quad \frac{x_n}{y_n} = \frac{1}{2}\left(1 + \frac{x_n}{y_{n-1}}\right)$$

$$\Rightarrow \quad \frac{x_n}{y_n} = \frac{1}{2}\left(1 + \frac{\sqrt{x_{n-1}y_{n-1}}}{y_{n-1}}\right) \quad \text{(from (i))}$$

$$\Rightarrow \quad \frac{x_n}{y_n} = \frac{1}{2}\left(1 + \sqrt{\frac{x_n - 1}{y_{n-1}}}\right) \quad \ldots(iii)$$

Now,

$$\frac{x_1}{y_1} = \frac{\frac{1}{2}}{1} = \frac{1}{2} < 1$$

Let for K,

$$\frac{x_{K-1}}{y_{K+1}} < 1$$

$$\therefore \quad \frac{x_K}{y_K} = \frac{1}{2}\left(1 + \sqrt{\frac{x_{K-1}}{y_{K-1}}}\right) \quad \ldots\text{(iv) (from (iii))}$$

as

$$\frac{x_{K-1}}{y_{K-1}} < 1$$

$$\Rightarrow \quad \sqrt{\frac{x_{K-1}}{y_{K-1}}} < \sqrt{1} = 1$$

$$\Rightarrow \quad 1 + \sqrt{\frac{x_{K-1}}{y_{K-1}}} < 1 + 1$$

$$\Rightarrow \quad \frac{1}{2}\left(1+\sqrt{\frac{x_{K-1}}{y_{K-1}}}\right) < \frac{1}{2}\times 2$$

$$\Rightarrow \quad \frac{x_K}{y_K} < 1 \qquad \text{(from (iv))}$$

∴ By mathematical induction,

$$\frac{x_K}{y_K} < 1, \quad \forall K \in N$$

So,
$$x_n < y_n \qquad \text{...(v)}$$

Now, from (i)

$$x_n = \sqrt{x_{n-1}y_{n-1}}$$

$$\Rightarrow \quad x_n > \sqrt{x_{n-1}\cdot x_{n-1}} \qquad \text{(from (v))}$$

$$\Rightarrow \quad x_n > x_{n-1} \qquad \text{...(vi)} \quad \left\{\begin{array}{l} y_{n-1} > x_{n-1} \\ \Rightarrow x_{n-1}y_{n-1} > x_{n-1}\cdot x_{n-1} \end{array}\right\}$$

Now, from (ii)

$$\frac{1}{y_n} = \frac{1}{2}\left(\frac{1}{x_n}+\frac{1}{y_{n-1}}\right)$$

$$\Rightarrow \quad \frac{1}{y_n} > \frac{1}{2}\left(\frac{1}{y_n}+\frac{1}{y_{n-1}}\right)$$

$$\left\{\text{from (v)}, x_n < 1, \Rightarrow \frac{1}{x_n} > \frac{1}{y_n} \Rightarrow \frac{1}{2}\left(\frac{1}{x_n}+\frac{1}{y_{n-1}}\right) > \ldots\ldots\right\}$$

$$\Rightarrow \quad \frac{1}{2y_n} > \frac{1}{2y_n - 1}$$

$$\Rightarrow \quad \frac{1}{y_n} > \frac{1}{y_{n-1}}$$

$$\Rightarrow \quad y_n < y_{n-1} \qquad \text{...(vii)}$$

From (v), (vi) and (vii), it can be concluded that $x_{n-1} < x_n < y_n < y_{n-1}$.

Now, as $x_{n-1} < x_n \Rightarrow x_n$ is an monotonic increasing sequence but $x_n < y_n < y_{n-1}$.

$$\Rightarrow \quad x_n < y_1 \Rightarrow x_n < 1$$

Also,
$$x_n > x_{n-1} \Rightarrow x_n > \frac{1}{2}$$

As x_n is monotonically increasing and bounded above $\Rightarrow$ x_n is converging.

Also, $y_n < y_{n-1} \Rightarrow y_n$ is monotonically decreasing and $x_{n-1} < x_n < y_n < y_{n-1} \Rightarrow \frac{1}{2} < y_n < 1$

as y_n is monotonically decreasing and bounded below $\Rightarrow$ y_n is converging.

Now,
$$x_n < y_n$$

$\Rightarrow x_n$ and y_n converge to same limit, say l and $\frac{1}{2} < l < 1$.

2.7 Let $x_1 = 2$ and $x_{n+1} = \sqrt{x_n + 20}$, $n = 1, 2, 3, \ldots$ Show that the sequence $x_1, x_2, x_3, \ldots$ is convergent. (2017 : 10 Marks)

Solution:

Theorem A monotonically increasing sequence which is bounded above is convergent.

(i) $<x_n>$ is monotonically increasing

$$x_1 = 2,\ x_2 = \sqrt{x_1+20} = \sqrt{22} \Rightarrow x_2 > x_1$$

Assume, $x_{k+1} > x_k$

$\therefore$ $x_{k+1} + 20 > x_k + 20$, i.e., $\sqrt{x_{k+1}+20} > \sqrt{x_k+20}$

$\Rightarrow$ $x_{k+2} > x_{k+1}$

By PMI statement is true for all $n \in N$.

(ii) $<x_n>$ is bounded above (by 5)

$$x_1 = 2 < 5,\ x_2 = \sqrt{22} < 5$$

Assume, $x_k < 5$

$\therefore$ $\sqrt{x_k+20} < \sqrt{5+20}$, i.e., $x_{k+1} < 5$

By PMI, statement is true for all $x \in N$.

Hence, given sequence is convergent by 'Monotone-Sequence Theorem'. (*).

2.8 Find the range of $p(> 0)$ for which the sequation $\frac{1}{(1+a)^p} - \frac{1}{(2+a)^p} + \frac{1}{(3+a)^p} - \ldots$, $a > 0$ is

(i) absolutely convergent and

(ii) conditionally convergent

(2018 : 10 Marks)

Solution:

Given : $$S_n = \frac{1}{(1+a)^p} - \frac{1}{(2+a)^p} + \frac{1}{(3+a)^p} - \ldots,\ a > 0$$

Here, $$S_n = \Sigma(-1)^{n-1}u_n$$

$$u_n = \frac{1}{(n+a)^p}$$

$$u_{n+1} = \frac{1}{(n+a+1)^p}$$

Clearly, $$u_n > u_{n+1}$$

Also, $$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{(n+a)^p} = 0 \quad \text{(as } p > 0)$$

$\therefore$ By Leibnitz's test, S_n is convergent for $p > 0$ ...(i)

Now, $$\sum\left|(-1)^{n-1}u_n\right| = \sum\frac{1}{(n+a)^p} = \frac{1}{(1+a)^p} + \frac{1}{(2+a)^p} + \ldots$$

Now, $$\frac{1}{(n+a)^p} < \frac{1}{n^p} \text{ as } a > 0$$

Also, $\sum\frac{1}{n^p}$ is convergent if $p > 1$ (by p-test)

$\therefore$ $\sum\left|(-1)^{n-1}u_n\right|$ is convergent if $p > 1$ ...(ii)

From (i) and (ii), it can be concluded that

(i) Series is absolutely convergent if $p > 1$.

(ii) Series is conditionally convergent for $\alpha_p \leq 1$.

3. Series

3.1 Show that the series : $\left(\frac{1}{3}\right)^3 + \left(\frac{1\cdot 4}{3\cdot 6}\right)^2 + \ldots\ldots + \left(\frac{1\cdot 4\cdot 7\ldots\ldots(3n-2)}{3\cdot 6\cdot 9\ldots\ldots 3n}\right)^2 + \ldots\ldots$ converges.

(2009 : 15 Marks)

Solution:

$$\left(\frac{1}{3}\right)^2 + \left(\frac{1\cdot 4}{3\cdot 6}\right)^2 + \ldots\ldots + \left(\frac{1.47\ldots\ldots(3n+2)}{3.6.9\ldots\ldots 3n}\right)^2 + \ldots\ldots$$

$$a_n = \left(\frac{1.4.7\ldots\ldots(3n-2)}{3.6.9\ldots\ldots 3n}\right)^2$$

$$a_{n+1} = \left(\frac{1.4.7\ldots\ldots(3n-2)(3n+1)}{3.6.9\ldots\ldots(3n)(3n+3)}\right)^2$$

$$\frac{a_n}{a_{n+1}} = \frac{(a_n+3)^2}{(3n+1)^2}$$

$$\frac{a_n}{a_{n+1}} - 1 = \frac{9n^2+9+18n-9n^2-1-6n}{(3n+1)^2}$$

$$= \frac{12n+8}{(3n+1)^2}$$

$$\lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) = \lim_{n\to\infty} \frac{12n^2+8n}{(3n+1)^2}$$

$$= \frac{12}{9} > 1$$

By Rabee's test, series is convergent.

3.2 Show that : $\lim_{x\to 1} \sum_{n=1}^{\infty} \frac{n^2x^2}{n^4+x^4} = \sum_{n=1}^{\infty} \frac{n^2}{n^4+1}$. Justify all steps of your answer by quoting the theorems you are using.

(2009 : 15 Marks)

Solution:

To prove the required result, we use the following result : "The limit of the sum function of a series = the sum of the series of limits of functions", i.e.,

$$\lim_{x\to x_0} \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \lim_{x\to x_0} f_x(x)$$

where $\sum_{n=1}^{\infty} f_x(x)$ converges uniformly in $[a, b]$ and x_0 is a point in $[a, b]$.

So, as per the question, if we take $a = 0$ and $b = 2$ and $x_0 = 1$ i.e., x_0 is a point in $[0, 2]$, then

$$\lim_{x\to 1}\sum_{n=1}^{\infty}\frac{n^2x^2}{n^4+x^4} = \sum_{n=1}^{\infty}\frac{n^2}{n^4+x^4} \quad ...(i)$$

But for this result, we need to prove that $\sum_{n=1}^{\infty}\frac{n^2x^2}{n^4+x^4}$ converges uniformly in [0, 2] and where $x_0 = 1$ is a point in [0, 2].

Now to prove $\sum_{n=1}^{\infty}\frac{n^2x^2}{n^4+x^4}$ converges uniformly in [0, 2]. [$x_0 = 1$ is taken]

Proof : This theorem/result can be proved by Drichlet's test.

Let
$$\sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} g_x(x)h_x(x) = \sum_{n=1}^{\infty} n^2x^2 \times \frac{1}{n^4+x^4}$$

where
$$g_x(x) = n^2x^2$$

and
$$h_x(x) = \frac{1}{n^4+x^4} \quad ...(ii)$$

Drichlet's test states that

(i) if there exists a real number K such that

$$|S_n(x)| = \sum_{r=1}^{n} g_r(x) < K \ \forall \ x\in[a,b], x\in N \text{ and}$$

(ii) $<h_x(x)>$ is positive monotic decreasing sequence converging uniformly to zero on [a, b], then the series $\sum g_x(x)h_x(x)$ is uniformly convergent on [a, b].

From (ii), we know that

$$|S_n(x)| = \sum_{r=1}^{n} g_r(x) = \sum_{r=1}^{n} r^2x^2 = \frac{n(n+1)(2n+1)x^2}{6} < K \ \forall x\in[0, 2], n\in N$$

(where K is a real number)

Also,
$$<hx(x)> = \frac{1}{n^4+x^4}$$

is a positive monotonic decreasing sequence converges uniformly to zero on [0, 2].

$\therefore$
$$\sum_{n=1}^{\infty} g_x(x)h_x(x) = \sum_{n=1}^{\infty}\frac{n^2x^2}{n^4+x^4} \text{ converges uniformly in } [0, 2] \quad ...(iii)$$

From (i) and (iii), we get the desired result.

3.3 Consider the series $\sum_{n=0}^{\infty}\frac{x^2}{(1+x^2)^n}$. Find the values of x for which it is convergent and also the sum function. Is the convergence uniform? Justify your answer.

(2010 : 15 Marks)

Solution:

Given series is $\sum_{n=0}^{\infty}\frac{x^2}{(1+x^2)^n} = S_n$.

Case 1 : If $x = 0$

then
$$S_n = 0$$

So it uniformly converges to zero.

Case 2 : If $x \neq 0$

then
$$S_n = \sum_{n=0}^{\infty}\frac{x^2}{(1+x^2)^n}$$

Now,
$$\frac{u_n}{u_{n+1}} = \frac{\frac{x^2}{(1+x^2)^n}}{\frac{x^2}{(1+x^2)^{n+1}}} = 1 + x^2 > 1 \ \forall x \neq 0$$

∴ It will converge as per D'Alembert test.

∴ S_n converges for all values of x.

Now,
$$S_n = \sum_{n=0}^{\infty}\frac{x^2}{(1+x^2)^n} = x^2\sum_{n=0}^{\infty}\frac{1}{(1+x^2)^n}$$
$$= x^2 \times \frac{\left(1-\left(\frac{1}{1+x^2}\right)^n\right)}{1-\frac{1}{1+x^2}}$$
$$= (1 + x^2) \text{ as } n \to \infty$$

∴ S_n is finite.

So, S_n is uniformly convergent as $|u_n| = u_n$.

3.4 Show that the series for which the sum of first n terms
$$f_n(x) = \frac{nx}{1+n^2x^2}, 0 \le x \le 1$$
cannot be differentiated term-by-term at $x = 0$. What happens at $x \neq 0$?

(2011 : 15 Marks)

Solution:

Given
$$\sum_{k=1}^{\infty}\mu_k(x) = f_n(x) = \frac{nx}{1+n^2x^2}$$

$$\Rightarrow \quad \lim_{n\to\infty}\sum_{k=1}^{\infty}u_k(x) = \lim_{n\to\infty}f_n(x) = \lim_{n\to\infty}\frac{nx}{1+n^2x^2}$$
$$= \lim_{n\to\infty}\frac{\frac{x}{n}}{\frac{1}{n^2}+x^2} = 0, \ \forall\ x \in [0, 1]$$

If we differentiate term-wise, we get
$$\sum_{k=1}^{n}u_k'(x) = f_n'(x) = \frac{n(1-n^2x^2)}{(1+n^2x^2)^2} \quad \text{...(i)}$$

At $x = 0$,
$$\sum_{k=1}^{n}u_k'(x) = f_n'(0) = n, \text{ which clearly does not tend to 0 as } n \to \infty.$$

$$\therefore \quad \sum_{k=1}^{n}u_k(x) = f_n(x) = \frac{n_x}{1+n^2x^2} \text{ can not be differentiated term by term at } x = 0.$$

When $x \neq 0$, from (i)
$$\lim_{n\to\infty}\sum_{k=1}^{n}u_k'(x) = \lim_{n\to\infty}f_n'(x) = 0 = \lim_{n\to\infty}\sum_{k=1}^{n}u_k(x)$$

Hence, $\sum_{k=1}^{\infty}u_k(x)$ can be differentiated term by term when $x \neq 0$.

3.5 Show that if $S(x) = \sum_{n=1}^{\infty} \frac{1}{n^3 + n^4 x^2}$, then its derivative

$$S'(x) = -2x \sum_{n=1}^{\infty} \frac{1}{n^2(1+nx^2)^2}, \text{ for all } x.$$

(2011 : 20 Marks)

Solution:

Given :

$$S(x) = \sum_{n=1}^{\infty} \frac{1}{n^3 + n^4 x^2}$$

As $\frac{1}{n^3 + n^4 x^2} \le \frac{1}{n^3}$ $\forall x \in R, n \in N$

$\therefore$

$$M_n = \frac{1}{n^3} \text{ and } \sum_{n=1}^{\infty} M_n = \sum_{n=1}^{\infty} \frac{1}{n^3} \text{ converges.}$$

$\therefore$ By Weierstrass's M-test (A series of functions Σf_n will converge uniformly and absolutely on $[a, b]$ if there exists a convergent series ΣM_n of positive numbers such that for all $x \in [a, b]$.

$$|f_n(x)| \le M_n \text{ for all } n$$

$$S(n) = \sum_{n=1}^{\infty} S_n(x) \text{ converges uniformly.}$$

Let

$$g_n(x) = S_n'(x) = \frac{-2x}{n^2(1+nx^2)^2}$$

and

$$g_n'(x) = \frac{-2+6nx^2}{n^2(1+nx^2)^3}$$

For maximum and minimum,

$$g_n'(x) = 0$$

$\Rightarrow$

$$-2 + 6nx^2 = 0 \Rightarrow x^2 = \frac{1}{3n}$$

It can be verified that $g_n''(x) < 0$ for $x^2 = \frac{1}{3n}$.

$\therefore$ Maximum value of $|g_n(x)| = \frac{3\sqrt{3}}{8n^{5/2}}$

$\therefore$

$$|g_n(x)| < \frac{1}{n^{5/2}} \; \forall n$$

But $\sum \frac{1}{n^{5/2}}$ is convergent.

$\therefore$ By Weierstrass's M-test, $\Sigma g_n = \Sigma S_n'$ is uniformly convergent $\forall x \in R$.

Hence, by term by term differentiation of the series ΣS_n is justified.

$\therefore$

$$S'(x) = \sum_{n=1}^{\infty} S_n'(x)$$

$$= -2x \sum_{n=1}^{\infty} \frac{1}{n^2(1+nx^2)^2} \; \forall x$$

3.6 Show that the series $\sum_{n=1}^{\infty}\left(\frac{\pi}{\pi+1}\right)^n n^6$ is convergent.

(2012 : 12 Marks)

Solution:

Let
$$a_n = \left(\frac{\pi}{\pi+1}\right)^n n^6$$

$\Rightarrow$
$$a_{n+1} = \left(\frac{\pi}{\pi+1}\right)^{n+1} (n+1)^6$$

By Ratio Test, if $\lim_{n\to\infty}\frac{a_{n+1}}{a_n} = l$, then the series is convergent for $l < 1$.

Now,
$$\frac{a_{n+1}}{a_n} = \left(\frac{\pi}{\pi+1}\right)^{n+1} (n+1)^6 \times \left(\frac{\pi+1}{\pi}\right)^n \cdot \frac{1}{n^6}$$

$\Rightarrow$
$$\lim_{n\to\infty}\frac{a_n+1}{a_n} = \lim_{n\to\infty}\frac{\pi}{\pi+1}\times\left(\frac{n+1}{n}\right)^6$$
$$= \lim_{n\to\infty}\frac{\pi}{\pi+1}\left(1+\frac{1}{n}\right)^6$$
$$= \frac{\pi}{\pi+1} < 1$$

$\therefore$ By Ratio Test, $\sum_{n=1}^{\infty}\left(\frac{\pi}{\pi+1}\right)^n n$ is convergent.

3.7 Show that the series $\sum_{1}^{\infty}\frac{(-1)^{n-1}}{n+x^2}$ is uniformly convergent but not absolutely convergent for all real values of x.

(2013 : 13 Marks)

Solution:

Uniform Convergence :

Let
$$u_n(x) = (-1)^{n-1}$$

and
$$b_n(x) = \frac{1}{n+x^2}$$

Then
$$|\Sigma u_n(x)| \leq 1$$

and
$$b_n = \frac{1}{n+x^2} \text{ is monotonically on } n \text{ decreasing for all } x.$$

as $n_1 > n_2 \Rightarrow$
$$\frac{1}{n_1+x^2} < \frac{1}{n_2+x^2}$$

and
$$\lim_{n\to\infty}\frac{1}{n+x^2} = 0 \quad \forall x \in R$$

(**Note :** The limit must tend to zero for each fixed x and we can always find n big enough for any x).

$\therefore$ By dirichlet test,

$$\sum u_n(x) b_n(x) = \sum\frac{(-1)^{n-1}}{n+x^2}$$

3.8 Test the convergence and absolute convergence of the series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n^2+1}$.

(2015 : 10 Marks)

Solution:

Given, the series
$$S = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{n}{n^2+1} = \sum_{n=1}^{\infty}(-1)^n u_{n+1}$$

Now,
$$u_n = \frac{n}{n^2+1},\ u_{n+1} = \frac{n+1}{(n+1)^2+1}$$

So,
$$u_n - u_{n+1} = \frac{n}{n^2+1} - \frac{n+1}{(n+1)^2+1} = \frac{n(n^2+2n+2)-(n+1)(n+1)}{(n^2+1)(n+1)^2+1)}$$
$$= \frac{n^3+2n^2+2n-n^3-n^2-n-1}{(n^2+1)(n^2+2n+2)}$$
$$= \frac{n^3+n-1}{(n^2+1)(n^2+2n+2)} > 0$$

$\Rightarrow$
$$u_n > u_{n+1} \quad \text{...(i)}$$
$$\lim_{n\to\infty}\frac{n}{n^2+1} = \lim_{n\to\infty}\frac{1}{2n} = 0 \quad \text{...(ii) (By L'pital rule)}$$

From (i) and (ii), by Leibnitz test, $\Sigma(-1)^{n+1}u_n$ is convergent.

Now, consider series
$$S_1 = \sum_{n=1}^{\infty}\left|(-1)^n + u_n\right| = \sum_{n=1}^{\infty}\frac{n}{(n^2+1)} = \sum_{n=1}^{\infty}u_n$$
$$\frac{u_n}{u_{n+1}} = \frac{\left(\frac{n}{n^2+1}\right)}{\left(\frac{(n+1)}{(n+1)^2+1}\right)} = \frac{n(n^2+2n+2)}{(n+1)(n^2+1)}$$

$\Rightarrow$
$$\frac{u_n}{u_{n+1}} = \frac{n(n^2+2n+2)}{n^3+n^2+n+1}$$
$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{n^3+2n^2+2n}{n^3+n^2+n+1} = 1 \quad (\therefore \text{ D'Alembert's ratio test fails})$$

Applying Raabe's Test :
$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \lim_{n\to\infty} n\left(\frac{n^3+2n^2+2n-1}{n^3+n^2+n+1}\right)$$
$$= \lim_{n\to\infty} n\left(\frac{n^3+2n^2+2n-n^3-n^2-n-1}{n^3+n^2+n+1}\right)$$
$$= \lim_{n\to\infty} n\left(\frac{n^2+n-1}{n^3+n^2+n+1}\right) = 1$$

$\therefore$ Raabe's test also fails.

Applying De-Morgan's and Bertrand Test :
$$\lim_{n\to\infty}\left(n\left(\frac{u_n}{u_{n+1}}-1\right)-1\right)\log n = \lim_{n\to\infty}\left(\frac{n^3+n^2-n-1}{n^3+n^2+n+1}\right)\log n$$

$$= \lim_{n\to\infty} \frac{(-2n-1)}{(n^3+n^2+n+1)} \log n = 0 < 1$$

$\therefore$ By Bertrand test, series is divergent.
As, S is convergent, but S_1 is divergent.
$\therefore$ given series is conditionally convergent.

3.9 Test the series of functions $\sum_{n=1}^{\infty} \frac{nx}{1+n^2x^2}$ for uniform convergence.

(2015 : 15 Marks)

Solution:

Given :
$$\sum f_n(x) = \sum_{n-1}^{\infty} \frac{nx}{1+n^2x^2}$$

$$f(x) = \lim_{n\to\infty} \frac{nx}{1+n^2x^2} = 0$$

$$\therefore \quad |f_n(x) - f(x)| = \left|\frac{nx}{1+n^2x^2}\right| = M_n$$

Now, at sup $|M_n|$,
$$\frac{dM_n}{dx} = 0$$

$$\Rightarrow \quad \frac{(1+n^2x^2)x - nx \times 2n^2x}{(1+n^2x^2)^2} = 0 \Rightarrow \frac{1-n^3x^2}{(1+n^2x^2)^2} = 0$$

$$\Rightarrow \quad x^2n^3 = 1$$

$$\Rightarrow \quad x = \left(\frac{1}{n^3}\right)^{1/2}$$

$$\frac{d^2M_n}{dx^2} = \frac{(1+n^2x^2)^2(-2n^3x) - (1-x^3n^2)\times 2(1+n^2x^2)\times 2n^2x}{(1+n^2x^2)^n}$$

$$= \frac{-2n^3x(1+n^2x^2)^2}{(1+n^2x^2)^n} < 0 \text{ at } x^2 = \frac{1}{n^3}$$

as
$$\frac{dM_n}{dx} = 0 \text{ and } \frac{d^2M_n}{dx^2} < 0 \text{ at } x^2 = \frac{1}{n^3}$$

$\therefore$ the value gives supremum of M.

$$\therefore \quad \text{Sup. } M_n = \left.\frac{n\times x}{1+n^2x^2}\right|_{x=\frac{1}{n^{3/2}}} = \frac{n\times\frac{1}{x^{3/2}}}{1+n^2\times\frac{1}{x^3}}$$

$$= \frac{\frac{1}{x^{3/2}}}{1+\frac{1}{n}}$$

$$\lim_{n\to\infty} \text{Sup } M_n = \lim_{n\to\infty} \frac{\frac{1}{n^{1/2}}}{1+\frac{1}{n}} = 0$$

$\therefore$ By M_n-test, given series is uniformly convergent.

3.10 Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$ is conditionally convergent. (If you use any theorem(s) to show it then you must give a proof of that theorem(s)).

(2016 : 15 Marks)

Solution:

Given series : $$S_n = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$$

$\Rightarrow$ $$S_n = +\frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \ldots..$$

Let $$u_n = \frac{(-1)^{n+1}}{n+1}$$

So, $$S_n = \sum_{n=1}^{\infty} u_n$$

Now, $$|u_n| - |u_{n+1}| = \frac{1}{n+1} - \frac{1}{n+2} = \frac{1}{(n+1)(n+2)} \geq 0$$

$\Rightarrow$ $$|u_n| > |u_{n+1}|$$

Also, $$\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{(-1)^{n+1}}{n+1} = 0$$

$\therefore$ as $|u_n| > |u_{n+1}|$ and $\lim_{n\to\infty} u_n = 0$

So, by Leibnitz test, S_n is convergent.

Consider, $$\sum_{n=1}^{\infty} |u_n| = \sum_{n=1}^{\infty} \left(\frac{1}{n+1}\right) = S_n'$$

Let $$a_n = \sum_{n=1}^{\infty} \frac{1}{n} = \sum_{n=1}^{\infty} P_n$$

Now, $$\lim_{n\to\infty} \frac{|u_n|}{P_n} = \lim_{n\to\infty} \frac{\frac{1}{n+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n}{n+1}$$

$\therefore$ By Limit Comparison Test, S_n' and a_n converges and diverge together.

Now, $$a_n = \sum_{n=1}^{\infty} \frac{1}{n^p}$$

By p-test, we know that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges only if $p > 1$. $\therefore$ a_n diverges.

$\therefore$ S_n' also diverges.

As S_n is convergent and S_n' is divergent.

$\therefore$ $S_n = \sum \frac{(-1)^{n+1}}{n+1}$ is conditionally convergent.

Leibnitz Test Proof :

Let $$S_n = \Sigma(-1)^n b_n$$

Let $$b_n - b_{n+1} \geq 0$$

Now, taking a look at partial sums,

$$S_n = b_1 - b_2 > 0$$

$$S_4 = b_1 - b_2 + b_3 - b_4 = S_1 + b_3 - b_4 \geq S_2 \qquad \{b_3 \geq b_4\}$$
$$S_6 = S_4 + b_5 - b_6 \geq S_4 \qquad \{b_5 \geq b_6\}$$
$$\vdots$$
$$S_{2n} = S_{2n-2} + b_{2n-1} - b_{2n} \geq S_{2n-2} \qquad \{b_{2n-1} \geq b_2\}$$

So, S_{2n} is a increasing sequence.

We can also write the general term as

$$S_{2n} = b_1 - b_2 + b_3 - b_4 + ... - b_{2n-2} + b_{2n-1} - b_{2n}$$
$$= b_1 - (b_2 - b_3) - (b_4 - b_5) - ..... - (b_{2n-2} - b_{2n-1}) - .....$$

Each of the qualities in parenthesis are positive and by assumption, b_{2n} is also positive. So, we can conclude that $S_{2n} \leq b_1$ for all n.

Now, as S_{2n} is an increasing sequence which is bounded above ($S_{2n} \leq b_1$), so it must convergent.

So, let
$$\lim_{n\to\infty} S_{2n} = S$$

Next, we can determine limit of the sequence of odd partial sums, S_{2n+1} as follows.

$$\lim_{n\to\infty} S_{2n+1} = S + 0 = S$$

So, both S_{2n+1} and S_{2n} are convergent sequences and they both have same limit.

$\therefore$ S_n is a convergent sequence.

Limit Comparison Test Proof :

Let $\lim_{n\to\infty} \frac{a_n}{b_n} = c$, where a_n and b_n are series and $0 < c < \infty$.

$\therefore$ as $n \to \infty$, $\frac{a_n}{b_n} \to c$. $\therefore \exists\, n > N$ such that (N is a positive integer)

$$m < \frac{a_n}{b_n} < M$$

$$\Rightarrow \qquad mb_n < a_n < Mb_n$$

If Σb_n diverges then so does Σmb_n and since $\Sigma mb_n < a_n$ for all sufficiently large n, by comparison test Σa_n also diverges.

Similarly, if Σb_n converges, then so does ΣMb_n and $\because \Sigma a_n < \Sigma Mb_n$ for all sufficiently range n, by the comparison test Σa_n also converges.

P-Test Proof :

Let
$$S_n = \sum_{n=1}^{n} \frac{1}{n^p}, p > 1$$

Now, we know that $2^n > n \ \forall n \in N$

$$\therefore \qquad S_n < S_2n$$

Now,
$$S_2n = \frac{1}{1'} + \frac{1}{2'} + ... \frac{1}{(2^n)^p}$$

$$S_{2^{n+1}-1} = \frac{1}{1^p} + \frac{1}{2^p} + ... + \frac{1}{(2^{n+1}-1)^p}$$
$$= \frac{1}{1^p} + \left(\frac{1}{2^p} + \frac{1}{3^p}\right) + \left(\frac{1}{4^p} + \frac{1}{5^p} + \frac{1}{6^p} + \frac{1}{7^p}\right) + ...$$
$$+ \left[\frac{1}{(2^n)^p} + \frac{1}{(2^n+1)^p} + ... + \frac{1}{(2^{n+1}-1)^p}\right]$$

Now,
$$\frac{1}{2^p}+\frac{1}{3^p} < \frac{1}{2^p}+\frac{1}{2^p}$$

$$\Rightarrow \quad \frac{1}{2^p}+\frac{1}{3^p} < \frac{2}{2^p}=\frac{1}{2^{p-1}}$$

Similarly,
$$\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p} < \frac{4}{4^p}=\frac{1}{2^{2(p-1)}}$$

$$\frac{1}{(2^n)^p}+\frac{1}{(2^n+1)^p}+.....+\frac{1}{(2^{n+1}-1)^p} < \frac{2^{n+1}-2^n}{(2^n)^p}=\frac{2^n}{(2^n)^p}=\frac{1}{2^{n(p-1)}}$$

So,
$$S_{2^{n+1}-1} < \frac{1}{1^p}+\frac{1}{2^{p-1}}+\frac{1}{2^{2(p-1)}}+.....+\frac{1}{2^{n(p-1)}}$$

R.H.S. is a geometric series with common ratio $\frac{1}{2^{p-1}}$.

$\therefore$ its sum is

$$\frac{\left[1-\left(\frac{1}{2^{p-1}}\right)^{n+1}\right]}{1-\frac{1}{2^{p-1}}} < \frac{2^{p-1}}{2^{p-1}-1} \quad \forall n \in N$$

Thus,
$$S_{2^{n+1}-1} < \frac{2^{p-1}}{(2^{p-1}-1)} \quad \forall n \in N$$

as
$$2^{n+1}-1 > 2^n \Rightarrow S_{2^{n+1}-1} > S_{2^n}$$

So, we have $\forall n \in N$

$$S_n < S_{2^n} < S_{2^{n+1}-1} < \frac{2^{p-1}}{2^{p-1}-1}$$

and as the sequence $<S_n>$ is bounded above, the series $\frac{1}{x^p}$ is convergent if $p > 1$.

3.11 Let $\sum_{n=1}^{\infty} x_n$ be a conditionally convergent series of real numbers. Show that there is a rearrangement $\sum_{n=1}^{\infty} x_{\pi(n)}$ of the series $\sum_{n=1}^{\infty} x_n$ that converges to 100.

(2017 : 20 Marks)

Solution:

First we prove the following result :

Let $\sum x_n$ be a conditionally convergent series with real-valued terms. Let x and y be given numbers in the closed interval $[-\infty, +\infty]$, with $x \le y$. Then there exist a rearrangement Σb_n s.t. $\lim_{n\to\infty} \inf t_n = x$ and $\lim_{n\to\infty} \sup t_n = y$ where $t_n = b_1 + ... + b_n$.

Proof : Discarding those terms of a series which are zero does not affect its convergence or divergence. Hence, we might as well assume that no terms of Σa_n are zero.

Let p_n denote the n^{th} positive term of Σa_n and let $-q_n$ denote its n^{th} negative term. Then Σp_n and Σq_n are both divergent series of positive terms [$\because$ Σa_n is not absolutely cgt but converges conditionally only].

We construct two sequences of real numbers $<x_n>$ and $<y_n>$ so that

$$\lim_{n\to\infty} x_n = x; \lim_{n\to\infty} y_n = y, \text{ with } x_n < y_n, y_1 > 0.$$

The idea of the proof is that we take just enough (say K_1) positive terms so that

$$p_1 + p_2 + ... + p_{k1} > y_1$$

followed by just enough (say r_1) negative term so that

$$p_1 + ... + p_{k1} - q_1 - ..... - q_{r'} < x_1$$

Next we take just enough positive terms so that

$$p_1 + ... + p_{k1} - q_1 - ..... - q_{r1} + p_{k1+1} + ... + p_{k2} > y_2$$

followed by just enough further negative terms to satisfy the inequality.

$$p_1 + ... + p_{n1} - q_1 - ..... - q_{r1} + pk_{1+1} + ... + p_{k2} - q_{r1+1} - ..... - q_{r2} < x_2$$

These steps are possible since Σp_n and Σq_n are both divergent series of positive terms. If the process is continued in this way, we obviously obtain a rearrangement of Σa_n.

Now we need to show that partial sums of this rearrangement have limit superior y and limit inferior x.

Since Σa_n is convergent $\Rightarrow p_n, q_n \to 0$.

$y_n \to y$ $\therefore$ For any $\in > 0$, $\exists$ n_0 so that $y_n < y + \frac{E}{2} \forall n \geq n_0$.

We note that the selection of terms from the sequences p_n, q_n to make it just greater than y_n, now removing the last term of type p_n will make the sum less than y_n. Since $p_n \to 0$, $\therefore$ we will reach a point when this sum from y_n is less than $\frac{\varepsilon}{2}$. It follows that the sum itself will be less than $y_n + \frac{\varepsilon}{2}$. Thus, we can find the infinite sums of the type $\Sigma p - \Sigma q$ s.t. $y_n < \Sigma p_{k_i} - Sq_{k_j} < y_n + \frac{\varepsilon}{2} < y + \varepsilon$. Hence, we find a value n_1 such that $t_n < y +$ ε. We complete the question by taking $x = y = 100$.

4. Functions

4.1 State Rolle's theorem. Use it to prove that between two roots of $e^x \cos x = 1$ there will be a roof of $e^x \sin x = 1$.

IC

(2009 : 2+10=12 Marks)

Solution:

Rolle's theorem states that :

If a function f defined on $[a, b]$ such that f is continuous on $[a, b]$, differentiable on (a, b) and $f(a) = f(b)$. Then there exists at least one real no. 'c' between 'a' and 'b' $(a < c < b)$ such that $f'(c) = 0$.

As $\quad e^x \cos x = 1$

$\Rightarrow \quad \cos x = e^{-x}$

$\Rightarrow \quad e^{-x} - \cos x = 0$

Let $\quad g(x) = e^{-x} - \cos x$

$\quad g'(x) = -e^{-x} \sin x$

As g(x) is cts and differentiable.

$\therefore$ by rolles theorem

Between two zeros or roots of $g(x)$ there is atleast one zero of $g'(x)$.

And $\quad g'(x) = 0$

$\Rightarrow \quad -e^{-x} + \sin x = 0$

$\Rightarrow \quad e^{-x} \sin x - 1 = 0$

$\Rightarrow \quad e^x \sin x = 1$

Hence, between two roots of $e^x \cos x = 1$, there will be a root of $e^x \sin x = 1$.

1d

4.2 Let $f(x) = \begin{cases} \dfrac{|x|}{2}+1; & \text{if } x<1 \\ \dfrac{x}{2}+1; & \text{if } 1\le x<2 \\ \dfrac{|x|}{2}+1; & \text{if } 2\le x \end{cases}$. What are the points of discontinuity of f, if any? What are the points where f is not differentiable, if any? Justify your answer.

(2009 : 12 Marks)

Solution:

$$f(x) = \begin{cases} \dfrac{|x|}{2}+1; & x<1 \\ \dfrac{x}{2}+1; & 1\le x<2 \\ \dfrac{|x|}{2}+1; & 2\le x \end{cases}$$

We know
$$|x| = \begin{cases} x; & x\ge 0 \\ -x; & x<0 \end{cases}$$

$$f(x) = \begin{cases} \dfrac{-x}{2}+1; & x<0 \\ \dfrac{x}{2}+1; & 0\le x<1 \\ \dfrac{x}{2}+1; & 1\le x<2 \\ \dfrac{x}{2}+1; & x\ge 2 \end{cases}$$

$$f(x) = \begin{cases} \dfrac{-x}{2}+1; & x<0 \\ \dfrac{x}{2}+1; & x\ge 0 \end{cases}$$

We will find $\lim\limits_{x\to 0} f(x)$

$$\lim_{x\to 0} f(x) = \lim_{h\to 0} f(0-h)$$

$$= \lim_{h\to 0} -\left(\frac{-h}{2}\right)+1 = \lim_{h\to 0}\frac{h}{2}+1 = 1$$

$$\lim_{x\to 0} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0}\frac{h}{2}+1 = 1$$

$\therefore$
$$\lim_{x\to 0} f(x) = 1$$

And $f(0) = 1$

$\therefore$
$$\lim_{x\to\infty} f(x) = f(0)$$

$\therefore$ f is continuous every where in order to check the differentially of $f(x)$ at $x = 0$. We will construct

$$\phi(x) = \frac{f(x)-f(0)}{x-0}$$

$\lim_{x\to 0}\phi(x)$ exist finitely.

Then $f(x)$ is differentially at $x = 0$.

Now,

$$\phi(x) = \frac{f(x)-f(0)}{x-0}$$

$$= \frac{f(x)-1}{x}$$

$$\lim_{x\to 0}\phi(x) = \lim_{x\to 0}\frac{f(x)-1}{x} \approx \lim_{h\to 0}\frac{f(h)-1}{h}$$

$$= \lim_{h\to 0}\frac{\frac{-(-h)}{2}+1-1}{-h}$$

$$= \lim_{h\to 0} -\frac{h}{2\times h} = -\frac{1}{2}$$

$\therefore$

$$\lim_{x\to 0}\phi(x) \neq \lim_{x\to 0}\phi(x)$$

$\therefore f(x)$ is not differentiable $x = 0$.

4.3 Show that if $f : [a, b] \to \mathbb{R}$ is a continuous function then $f([a, b]) = [c, d]$ for some real numbers c and d, $c \le d$.

(2009 : 15 Marks)

Solution:

Let $f: [a, b] \to \mathbb{R}$ be a cts. function. As f is cts, function on closed interval $[a, b]$. Therefore, f is bounded on $[a, b]$. And attain its supremum and infimum.

Let

$$d = \text{least upper bound of } f \text{ on } [a, b]$$
$$c = \text{greatest lower bound of } f \text{ on } [a, b]$$

Then

$$c \le f(x) \le d \ \forall x \in [a,b]$$

$\Rightarrow f([a, b]) \subset [c, d]$ ...(1)

$\therefore f$ is cts on $[a, b]$

f assumes every value between 'c' and 'd'.

Hence, if $\xi \in [c, d]$

Then there exist some $x \in [a, b]$

Such that $f(x) = \xi$

Thus, $\xi \in [c, d]$

$\Rightarrow$ $c \in f([a, b])$

$\Rightarrow$ $[c, d] \subseteq f([a, b])$ ...(ii)

From (i) and (ii)

$$f([a, b]) = [c, d]$$

4.4 Define the function

$$f(x) = x^2 \sin\frac{1}{x}, \quad \text{if } x \neq 0$$
$$= 0 \quad \text{if } x = 0$$

Find $f'(x)$. Is $f'(x)$ continuous at $x = 0$? Justify your answer.

2 c

(2010 : 15 Marks)

Solution:

Given :
$$f(x) = \begin{cases} x^2 \sin\frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

for $x \neq 0$:
$$f(x) = x^2 \sin\frac{1}{x}$$

$\therefore$
$$f'(x) = 2x\sin\frac{1}{x} + x^2\cos\frac{1}{x} \times \left(\frac{-1}{x^2}\right)$$
$$= 2x\sin\frac{1}{x} - \cos\frac{1}{x}$$

for $x = 0$:
$$\left.\frac{df}{dx}\right|_{x=0} = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$
$$= \lim_{h\to 0}\frac{h^2\sin\frac{1}{h} - 0}{h}$$
$$= \lim_{h\to 0} h\sin\frac{1}{h} = 0$$

$\therefore$
$$f'(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x}, & \text{for } x \neq 0 \\ 0, & \text{for } x = 0 \end{cases}$$

Now,
$$\lim_{x\to 0} f'(x) = \lim_{x\to 0}\left(2x\sin\frac{1}{x} - \cos\frac{1}{x}\right)$$
$$= \lim_{x\to 0}\left(-\cos\frac{1}{x}\right),$$

which cannot be determine as it oscillates between [–1, 0]

$\therefore$ $f'(x)$ is not continuous at $x = 0$.

4.5 Let $S = (0, 1]$ and f be defined by $f(x) = \frac{1}{x}$ where $0 < x \leq 1$ (in $\mathbb{R}$). Is f uniformly continuous on S? Justify your answer.

(2011 : 12 Marks)

Solution:

Given :
$$f(x) = \frac{1}{x}, 0 < x \leq 1$$

A function f defined on an interval I is said to be uniformly continuous on I it for each $\in > 0$, $\exists\, \delta > 0$ such that $|f(x_1) - f(x_2)| < \in \ \forall\ |x_1 - x_2| < \delta$; $x_1, x_2 \in I$.

Taking $\in = \frac{1}{3}$ and δ as any positive number, then for $n > \frac{1}{\delta}$, we have, for $x_1 = \frac{1}{n}$ and $x_2 = \frac{1}{n+1}$ as any two points of (0, 1].

$$|x_1 - x_2| = \left|\frac{1}{n} - \frac{1}{n+1}\right| = \frac{1}{n(n+1)} < \frac{1}{n} < 8$$

$\therefore$
$$|f(x_1) - f(x_2)| = |n - (n+1)| = 1 > \in$$

Hence, f is not uniformly continuous on (0, 1].

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4.6 Let $f_n(x) = \begin{cases} 0, & \text{if } x < \frac{1}{n+1}, \\ \sin\frac{\pi}{x}, & \text{if } \frac{1}{n+1} \le x \le \frac{1}{n}, \\ 0, & \text{if } x > \frac{1}{n} \end{cases}$. Show that fn(x) converges to a continuous function but not uniformly.

(2012 : 12 Marks)

Solution:

Given :

$$f_n(x) = \begin{bmatrix} 0, & \text{if } x < \frac{1}{n+1} \\ \sin\frac{\pi}{x}, & \text{if } \frac{1}{n+1} \le x \le \frac{1}{n} \\ 0, & \text{if } x > \frac{1}{n} \end{bmatrix}$$

$$f_1(x) = \begin{cases} 0, & x < \frac{1}{2} \\ \sin\frac{\pi}{x}, & \frac{1}{2} \le x \le 1 \\ 0, & x > 1 \end{cases}$$

$$f_2(x) = \begin{cases} 0, & x < \frac{1}{3} \\ \sin\frac{\pi}{x}, & \frac{1}{3} \le x \le \frac{1}{2} \\ 0, & x > \frac{1}{2} \end{cases}$$

And so on.

$\therefore$ $$\lim_{x\to\infty} f_n(x) = \begin{cases} 0, & x < 0 \\ 0, & x \ge 0 \end{cases}$$

$\therefore$ $$f(x) = 0$$

$\Rightarrow$ $f_x(x)$ converges to $f(x)$ pointwise and $f(x)$ is continuous function.

Let $$M_n = \sup_x |f_n(x) - f(x)|$$

$$M_n = \sup_x |f_n(x)|$$

As $$\frac{1}{n+1} < \frac{2}{2n+1} < \frac{1}{n}$$

$\therefore$ $$\sin(2n+1)\frac{\pi}{2} = 1$$

$\Rightarrow$ $$M_n = 1$$

$\Rightarrow$ $$M_n \not\to 0 \text{ or } n \to \infty$$

$\Rightarrow$ $f_n(x)$ does not converge uniformly.

4.7 For the function $f : (0, \infty) \to R$ given by $f(x) = x^2 \sin\frac{1}{x}$, $0 < x < \infty$. Show that there is a differentiable function $g : R \to R$ that extends f.

(2016 : 10 Marks)

Solution:

Given : $$f(x) = x^2 \sin\frac{1}{x}, x \in (0, \infty)$$

$f(x)$ is not defined at $x = 0$

Let $$g(x) = \begin{cases} x^2 \sin\frac{1}{x} & \text{for } x > 0 \\ 0 & \text{for } x = 0 \end{cases}$$

Now, for $x > 0$, $$g'(x) = 2x\sin\frac{1}{x} + x^2\cos\frac{1}{x} + \frac{-1}{x^2}$$

$$= 2x\sin\frac{1}{x} - \cos\frac{1}{x}$$

which exists at all $x > 0$.

$$g'(0) = \lim_{n\to 0}\frac{n^2\sin\frac{1}{n} - 0}{n} = \lim_{n\to 0}\frac{n^2\sin\frac{1}{n} - 0}{n}$$

$$= \lim_{n\to 0} n\sin\frac{1}{n} = 0$$

$\therefore$ $g'(0)$ exists.

Now, as f is more restrictive than g as f is defined for $x > 0$ only, g is an extension of f. Also, g is differentiable. So, g is an extension of f which is differentiable.

4.8 Let $f : R \to R$ be a continuous function such that $\lim_{x\to+\infty} f(x))$ and $\lim_{x\to-\infty} f(x)$ exist and are finite. Prove that f is uniformly continuous on R.

(2016 : 15 Marks)

Solution:

We know that a function $f(x)$ is uniformly continuous if $|f(x) - f(y)| < \in, \in > 0, \exists\delta > 0$ such that $|x - y| < \delta$.

Taking the domain $[0, \infty)$

Let $$\lim_{x\to-\infty} f(x) = L \text{ as limit at infinity is finite (given)}$$

Also, f is a continuous function on R.

Case 1 :

Let $K > 0$, however large, such that $|f(x) - L| < K$ and $|f(y) - L| < K$.

$\therefore$ $$|f(x) - f(y)| = |f(x) - L + L - f(y)| \leq |f(x) - L| + |f(y) + L|$$

$\Rightarrow$ $$|f(x) - f(y)| \leq K + K = 2K$$

Choosing $N > 0$ such that $\forall K > N$,

$$|f(x) - L| < \frac{\in}{2}$$

$\Rightarrow$ $$|f(x) - f(y)| < \frac{\in}{2} + \frac{\in}{2} = \in$$

So, for $x > N$, $f(x)$ is uniformly continuous.

Case 2 :

If x and y both are less than N. Given f is continuous as R. $\therefore$ f is continuous on finite interval $[0, N]$.

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As f is continuous on finite interval, $\therefore$ f is uniformly continuus, i.e., if $|x - y| < \delta$, then $|f(x) - f(y)| < \in$.

Case 3 :

If $x \in [0, N]$ and $y \in (N, \infty)$ or vice-versa.

Let $|x - M < l$ and $|y - M < l$, l is however larger.

$$\therefore \qquad |x - y| = |x - N + N - y| \le |x - M + |y - M$$

$$\Rightarrow \qquad |x - y| \le l + l = 2l$$

$$\Rightarrow \qquad |x - y| \le 2l$$

If $f(x)$ and $f(y)$ are within some K of $f(N)$, they will be within $2K$ of each other.

Now, let δ_2 be such that if $|a - M < \delta_2$, then $|f(a) - f(N)| < \frac{\in}{2}$. Choosing x and y within δ_2 of N, we get $|f(x) - f(y)|$ $< \in$, i.e., $f(x)$ is uniformly continuous.

$\therefore$ From case (1), (2) and (3), it can be concluded that $f(x)$ is uniformly continuous on $[0, \infty)$. Similarly, it can also be inferred that $f(x)$ is uniformly continuous on $(-\infty, 0)$.

$\therefore$ f is uniformly continuous on R.

4.9 Find the supermum and the infimum of $\frac{x}{\sin x}$ on the interval $\left(0, \frac{\pi}{2}\right]$.

(2017 : 10 Marks)

Solution:

Let
$$f(x) = \frac{x}{\sin x}, x \in \left(0, \frac{\pi}{2}\right]$$

$$f'(x) = \frac{\sin x - x\cos x}{\sin^2 x} \qquad \text{...(i)}$$

Take
$$g(x) = \sin x - x \cos x, x \in \left[0, \frac{\pi}{2}\right]$$

$$g'(x) = \cos x - (\cos x - x \sin x)$$

$$= x \sin x > 0 \text{ on } \left[0, \frac{\pi}{2}\right]$$

Hence, $g(x)$ is increasing in $\left[0, \frac{\pi}{2}\right]$

Let $x \in \left[0, \frac{\pi}{2}\right]$ $\therefore$ $g(0) < g(x)$

$$\therefore \qquad g(x) = \sin x - x \cos x > 0$$

From (i),
$$f'(x) > 0 \text{ for } x \in \left(0, \frac{\pi}{2}\right]$$

$\therefore$ $f(x)$ is increasing on $\left(0, \frac{\pi}{2}\right]$

$$\therefore \qquad \text{Infimum} = \lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{x}{\sin x} = 1$$

$$\text{Supermum} = f\left(\frac{\pi}{2}\right) = \frac{\frac{\pi}{2}}{\sin\frac{\pi}{2}} = \frac{\pi}{2}$$

4.10 Let $f(t) = \int_0^t [x]dx$, where $[x]$ denotes the largest integer less than or equal to x.

(i) Determine all the real numbers t, at which f is differentiable.

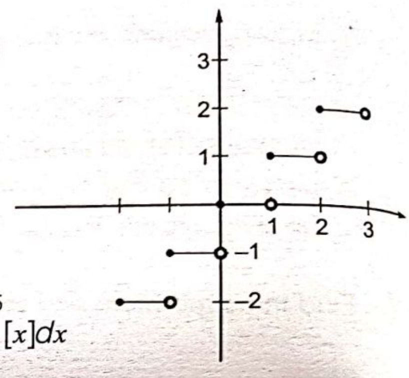
(ii) Determine all the real numbers t at which f is continuous but not differentiable.

(2017 : 15 Marks)

Solution:

Function, $[x]$ is discontinuous at every integer and continuous at non-integer points.

$$[x] = \begin{cases} 1, & 1 \le x < 2 \\ 0, & 0 \le x < 1 \\ -1, & -1 \le x < -2 \\ -2, & -2 \le x < -3 \end{cases} \qquad ...(i)$$



$$f(t) = \int_0^t [x]dx$$

It is defined by integrating the greatest integer function from 0 to 1.

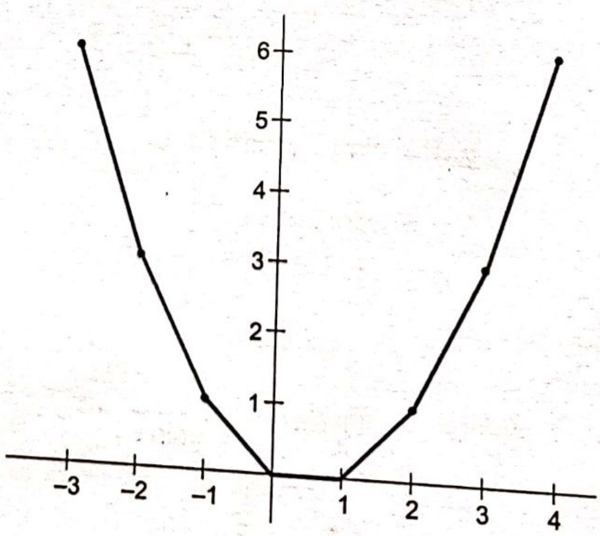
For example $\qquad f(3.5) = \int_0^1 [x]dx + \int_1^2 [x]dx + \int_2^3 [x]dx + \int_3^{3.5} [x]dx$

$$= \int_0^1 0dx + \int_1^2 1dx + \int_2^3 2dx + \int_2^{3.5} dx$$

$$= 0 + 1 + 2 + 3(0.5) = 4.5$$

$$f(x) = \begin{cases} \vdots \\ x-1 & \text{if } 1 \le x < 2 \\ 0, & \text{if } 0 \le x < 1 \\ -x, & \text{if } -1 \le x < 0 \\ -2x-1, & \text{if } -2 \le x < -1 \\ \vdots \end{cases} \qquad ...(ii)$$

Each piece is indeed an anti-derivative of the corresponding piece in the definition of $[x]$ in (i). Graphing this function, so the integral of the discontinuous function is, in fact, continuous.



We note that there are corners in this graph at integer values of x. Hence, the function

$$f(t) = \int_0^t [x]dx$$

is continuous for all $x \in \mathbb{R}$ but is not differentiable at the places where the integrand is discontinuous i.e., at all integer points.

4.11 Show that if a function f defined on open interval (a, b) of R is convex, then f is convex of open invows. Show by example, if the condition dropped, then the convex function need to be continuous.

(2018 : 15 Marks)

Solution:

Suppose f is convex on (a, b) and let ....

Choose c_1 and d_1 such that

$$a < c_1 < c < d < d_1 < b$$

If $x, y \in [c, d]$ with $x < y$, as f is ..... have

$$\frac{f(y)-f(x)}{y-x} \le \frac{f(d)-f(c)}{d-y} \le \frac{f(d_1)-f(d)}{d_1-d}$$

and
$$\frac{f(y)-f(x)}{y-x} \ge \frac{f(x)-f(c)}{x-c} \ge \frac{f(c)-f(c_1)}{c-c_1}$$

showing the given set

$$\left\{\left|\frac{f(y)-f(x)}{y-x}\right| : c \le x \le y \le d\right\}$$

is bounded by $M > 0$. It follows $|f(y) - f(x)| \le M|y - x|$ and therefore f is uniformly continuous on $[c, d]$. Now since uniform continuity implies continuity, we have shown that f is continuous on $[c, d]$. Since the interval $[c, d]$ was arbitrary. f is continuous on (a, b).

If interval is closed :

Let f be a convex function, defined in interval $[0, 1]$ such that

$$f(x) = 0; \text{ for } x \in (0, 1)$$
$$= 1; \text{ otherwise}$$

Clearly, f is discontinuous at 0 and 1.

4.12 Suppose R be a set of all real numbers. $f: R \to R$ is a function such that the following equations hold for all $xiy \in R$:

(i) $f(x) + f(y) = f(x + y)$

(ii) $f(xy) = f(x) \cdot f(y)$

Show that $\forall x \in R$ either $f(x) = 0$ or $f(x) = x$.

(2018 : 20 Marks)

Solution:

Given :
$$f(x + y) = f(x) + f(y) \quad \text{...(i)}$$
$$f(xy) = f(x) \cdot f(y) \quad \text{...(ii) } \forall\, xy \in R$$

Now consider :

Case 1 :

$$f(0 + 0) = f(0) + f(0) \Rightarrow f(0) = 0$$

for $x = 0$

Case 2 : $x \in N$

$$f(x) = f(1 + 1 + \text{..... } x \text{ times}) = f(1) + f(1) + \text{.....} \quad \text{(from (i))}$$

$\Rightarrow$ $f(x) = f(1) \cdot x = \alpha_1 \cdot x$ where α_1 is a constant.

Case 3 : If x is a negative integer.

Let $x = -y$ where y is positive integer.

$\therefore$ $$f(x) = f(-y) = f(-1 -1 \ldots, y \text{ times}) = f(-1) + f(-1) + \ldots y \text{ times} \quad \text{(from (i))}$$

$\Rightarrow$ $$f(x) = yf(-1) = -xf(-1) = -f(-1)\cdot x$$
$$= \alpha_2 \cdot x \text{ where } \alpha_2 \text{ is a constant.}$$

Case 4 : If x is a rational number.

Let $x = \frac{p}{q}$ where $q > 0$ and p & $q \in z$

$\therefore$ $$f(p) = f\left(q \cdot \frac{p}{q}\right) = f\left(\frac{p}{q} + \frac{p}{q} + \ldots q \text{ times}\right)$$
$$= f\left(\frac{p}{q}\right) + f\left(\frac{p}{q}\right) + \ldots q \text{ times} \quad \text{(from (i))}$$

$\Rightarrow$ $$f(p) = qf\left(\frac{p}{q}\right) \Rightarrow f\left(\frac{p}{q}\right) = \frac{1}{q} f(p) = \alpha \cdot \left(\frac{p}{q}\right)$$

$\Rightarrow$ $$f(x) = \alpha \cdot x \text{ where } \alpha \text{ is a constant.} \quad \text{(from Case 2 and Case 3)}$$

Case 5 : If x is an irrational number.

Consider $x_1, x_2, \ldots, x_n$ be a sequence, such that x_1 is a rational number and

$$\lim_{n \to \infty} x_n = x$$

$\therefore$ $$\lim_{n \to \infty} f(x_n) = \lim_{n \to \infty} \alpha x_n \quad \text{(from Case 4)}$$

$\Rightarrow$ $$\lim_{n \to \infty} f(x_n) = \alpha \lim_{n \to \infty} x_n = \alpha x = f(x) \text{ where } \alpha \text{ is a constant.}$$

$\therefore$ from all case (1), (2), (3), (4) & (5)

$$f(x) = \alpha x, \text{ where } \alpha \text{ is a constant.}$$

Now, $$f(x^2) = f(x) \cdot f(x)$$

$\Rightarrow$ $$\alpha x^2 = \alpha x \cdot \alpha x$$

$\Rightarrow$ $$(\alpha^2 - \alpha)x^2 = 0$$

$\Rightarrow$ $$\alpha(\alpha - 1)x^2 = 0$$

i.e., either $\alpha = 0$ or $\alpha = 1$

if $\alpha = 0$, then $$f(x) = 0$$

if $\alpha = 1$, then $$f(x) = 1 \cdot x = x$$

$\forall x \in R$

4.13 Show that the function $f(x, y) = \begin{cases} \dfrac{x^2 - y^2}{x - y}, & (x, y) \neq (1, -1), (1, 1) \\ 0, & (x, y) = (1, 1), (1, -1) \end{cases}$ **is continuous and differentiable at (1, –1).**

(2019 : 10 Marks)

Solution:

Given expression can be written as

$$f(x, y) = \begin{cases} x + y, & (x, y) \neq (1, -1), (1, 1) \\ 0, & (x, y) = (1, 1), (1, -1) \end{cases}$$

$\because$ $$\lim_{f(1,-1)} f(x, y) = \lim_{(x,y) \to (1,-1)} (x + y) = 1 + (-1) = 0$$

$\Rightarrow f(x, y)$ is continuous at $(1, -1)$

Since, $f_x(x, y) = 1$ and $f_y(x, y) = 1$ which are continuous everywhere including $(1, -1)$.

$\Rightarrow f(x, y)$ is differentiable at $(1, -1)$.

4.14 Using differentials, find an approximate value of $f(4.1, 4.9)$ where $f(x, y) = \left(x^3 + x^2y\right)^{\frac{1}{2}}$.

(2019 : 15 Marks)

Solution:

Given :
$$f(x, y) = (x^3 + x^2y)^{1/2}$$

then,
$$f(4.1, 4.9) = [(4.1)^3 + (4.1)^2 \times 4.9]^{1/2} \quad ...(1)$$

Let,
$$f(4.1, 4.9) = f(x) = y$$

$\therefore$
$$y = [x]^{1/2} \quad ...(2)$$
$$x = 151.29$$

which can be break into two parts.

$$x = 144 + 7.29$$
$$x = x + \Delta x$$
$$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} \text{ (by using (2))} \quad ...(3)$$

Also,
$$\Delta y = \frac{dy}{dx} \cdot \Delta x$$
$$\Delta y = \frac{1}{2\sqrt{x}} \cdot \Delta x$$
$$\Delta y = \frac{1}{2\sqrt{144}} \times 7.29 = \frac{1}{24} \times 7.29$$
$$\Delta y = 0.30375$$

as
$$\Delta x = 7.29, \Delta y = 0.30375$$
$$\Delta y = f(x + \Delta x) - f(x)$$
$$0.30375 = \sqrt{151.29} - \sqrt{144}$$
$$f(x + \Delta x) = \sqrt{151.29} = 12 + 0.30375$$
$$f(x + \Delta x) = \sqrt{151.29} = 12.30375$$

$\therefore$
$$f(x, y) = f(4.1, 4.9) = (x^3 + x^2y)^{1/2} = 12.30375$$

Hence, the approximate value of
$$f(4.1, 4.9) = 12.304$$

where
$$f(x, y) = (x^3 + x^2y)^{1/2}. \text{ Hence, the result.}$$

4.15 Discuss the uniform convergence of $f_n(x) = \dfrac{nx}{1+n^2x^2}, \forall x \in R(-\infty, \infty)$, $n = 1, 2, 3, ...$

(2019 : 15 Marks)

Solution:

Let
$$f_n(x) = \frac{nx}{1+n^2x^2}, n \in R$$

Suppose that $\{f_n\}$ is uniformly convergent in $(-\infty, \infty)$.

Also, the point-wise limit f is given as

$$\lim f(x) = \lim \frac{nx}{1+n^2x^2} = \lim_{n\to\infty} \frac{n}{nx^2 + \frac{3}{n}}$$

$$= 0, \forall\, n \in R$$

$$\Rightarrow \quad f(x) = 0 \;\forall\, n \in R$$

Now, from our assumption, $\{f_n\}$ is uniformly convergent in $(-\infty, \infty)$. So that we have the point wise limit f is also the uniform limit.

Let $E > 0$ be given. Then there exist in each that $\forall\, n \in (-\infty, \infty)$ and $\forall\, n \geq m$.

$$\left|\frac{nx}{1+n^2x^2} - 0\right| < \in$$

We take $\in = \frac{1}{4}$

Now, there exists an integer k such that $k \geq m$ and $\frac{1}{k} \in (-\infty, \infty)$.

Taking $n = k$ and $x = \frac{1}{k}$, we have

$$\frac{nx}{1+n^2x^2} = \frac{1}{2} \text{ which is not less than } \frac{1}{4}.$$

Thus, we arrive at a contradiction and so, the sequence $f_n(x) = \frac{nx}{1+n^2x^2}$, $\forall x \in R(-\infty, \infty)$, $n = 1, 2, 3, \ldots$ is not uniformly convergent with 0, in the interval $(-\infty, \infty)$, even though, it is point with convergent. Hence, the result.

5. Riemann Integral

5.1 Let $f(x)$ be differentiable on [0, 1] such that $f(1) = f(0) = 0$ and $\int_0^1 f^2(x)dx = 1$. Prove that $\int_0^1 xf(x)f'(x)dx = -\frac{1}{2}$.

(2012 : 15 Marks)

Solution:

Let $f(x)$ be differentiable on [0, 1] such that

$$f(1) = f(0) = 0$$

and

$$\int_0^1 f^2(x)dx = 1$$

Now

$$\int_0^1 xf(x)f'(x)dx = x\int_0^1 f(x)f'(x)dx - \int_0^1\left[\int_0^1\left(\frac{d}{dx}(x)\right)\int_0^1 f(x)f'(x)dx\right]dx \quad \ldots(i)$$

Consider

$$\int_0^1 f(x)f'(x)dx$$

Put $f(x) = t$ ...(ii)

On differentiating w.r.t. x, we have

$$f'(x)dx = dt$$

$$\therefore \qquad \int t dt = \left(\frac{t^2}{2}\right) = \left[\frac{1}{2}f^2(x)\right] \qquad \text{(from (ii))}$$

$\therefore$ From (i), we have,

$$\int_0^1 xf(x)f'(x)dx = (x)_0^1 \times \left[\frac{1}{2}f^2(x)\right]_0^1 - \int_0^1 \frac{f^2(x)}{-2}dx$$

$$= \frac{1}{2}(f(1))^2 - 0 \times \frac{(f(0))^2}{2} - \frac{1}{2}$$

$$= \frac{1}{2} \times 0 - 0 - \frac{1}{2} = \frac{-1}{2}$$

5.2 Give an example of a function $f(x)$, that is not Riemann integrable but $|f(x)|$ is Riemann integrable. Justify.

(2012 : 20 Marks)

Solution:

Let
$$f(x) = \begin{cases} 1; & x \in Q \cap [0,1] \\ -1; & x \in Q^C \cap [0,1] \end{cases}$$

Consider the partition P such that

$$P = \left\{0, \frac{1}{n}, \frac{1}{n}, \ldots\ldots, \frac{r}{n}, \ldots\ldots, \frac{n}{n} = 1\right\}$$

Let
$$m_r = \inf_{x \in [x_{r-1}, x_r]} f(x)$$

$$M_r = \sup_{x \in [x_{r-1}, x_r]} f(x)$$

Now, whatever interval we choose, it must contain rational as well irrational numbers.

$$\therefore \qquad m_r = -1 \text{ and } M_r = 1$$

Again,
$$L(f, p) = \sum m_r \Delta x_r = -1 \sum \Delta x_r = -1$$

$$U(f, p) = \sum m_r \Delta x_r = 1 \sum \Delta x_r = 1$$

$\therefore$ Upper Riemann integral,

$$\int_0^1 f(x)dx = \inf_p U(f,p) = 1$$

and Lower Riemann integral,

$$\int_0^1 f(x)dx = \sup_p L(f,p) = -1$$

Now,
$$f \in R[a, b] \text{ if } \int_a^b P(x)dx = \int_a^b f(x)dx$$

But here,
$$\int_0^1 f(x)dx \neq \int_0^1 f(x)dx$$

$\therefore$ f is not Riemann integrable.

But $$|f(x)| = 1 \ \forall \ x \in [0, 1]$$

$\Rightarrow |f(x)|$ being a constant function is continuous on $[0, 1]$.

$\therefore |f(x)|$ is Riemann integrable.

5.3 Let

$$f(x) = \begin{cases} \frac{x^2}{2}+4 & \text{if } x \geq 0 \\ \frac{-x^2}{2}+2 & \text{if } x < 0 \end{cases}$$

If f Riemann integrable in the interval $[-1, 2]$? Why? Does there exist a function g such that $g'(x) = f(x)$? Justify your answer.

(2013 : 10 Marks)

Solution:

Approach : The approach is by definition of Reimann integrability. But the second part is important for understanding.

$$f(x) = \begin{cases} \frac{x^2}{2}+4 & x \geq 0 \\ -\frac{x^2}{2}+2 & x < 0 \end{cases}$$

$$\lim_{x \to 0^-} f(x) = 2 \neq \lim_{x \to 0^+} f(x) = 4$$

$\therefore f(x)$ has a single point of discontinuity at $x = 0$ and is continuous in $(-1, 0)$ and $(0, 2)$. So, $f(x)$ is Reimann integrable.

There does not exist a function g such that

$$g'(x) = f(x)$$

because by Darboux theorem any derivative $g'(x)$ can not have jump discontinuities which $f(x)$ has one at $x = 0$.

(Note : By Darboux theorem g' has Intermediate Value Property, i.e., $a < b$ and $f(a) < c < f(b) \ \exists \ x \in (a, b)$ such that $f(c) = x$. But this means $g'(x)$ can not have jump discontinuity.)

Justification : Existence of primitive is a sufficient but not a necessary condition for Reimann integrability. So a Reimann integrable function may not be have an anti-derivative.

5.4 Let $[x]$ denote the integer part of the real number x, i.e., if n £ $x < n + 1$ where n is an integer, then $[x] = n$. Is the function $f(x) = [x]^2 + 3$ Reimann integrable in $[-1, 2]$? If not, explain why. If it is integrable compute $\int_{-1}^{2} ([x]^2 + 3)dx$.

Solution:

(2013 : 10 Marks)

$$f(x) = [x]^2 + 3; \ x \in [-1, 2]$$

$[x]$ has jump discontinuities at integers as

$$\lim_{x \to n^-} [x] = n - 1$$

and

$$\lim_{x \to n^+} [x] = n$$

$\therefore f(x)$ has jump discontinuities at integers.

e.g.,

$$\lim_{x \to n^-} [x]^2 + 3 = (n-1)^2 + 3 = n^2 - 2n + 4$$

$$\lim_{x\to n^+}[x]^2+3 = n^2+3$$

$$\lim_{x\to n^-} f(x) = \lim_{x\to n^+} f(x)$$

$\Rightarrow$

$$n^2-2n+4 = n^2+3 \Rightarrow n=\frac{1}{2} \text{ which is not possible.}$$

So, $f(x)$ has discontinuities at all integer, i.e., 2 discontinuities in $[-1, 2]$.
At other points $f(x)$ is constant and so continuous.
$\therefore$ $f(x)$ has finite number of discontinuities and so Reimann integrable.

$$\int_{-1}^{2}([x]^2+3)dx = \int_{-1}^{0}([x]^2+3)dx+\int_{0}^{1}([x]^2+3)dx+\int_{1}^{2}([x]^2+3)dx$$

$$= ((-1)^2+3)\int_{-1}^{0}dx+(0^2+3)\int_{0}^{1}dx+(1^2+3)\int_{1}^{2}dx = 11$$

5.5 $\int_0^1 f(x)dx$ **where** $f(x) = \begin{cases} 2x\sin\frac{1}{x}-\cos\frac{1}{x} & x\in\{0,1\} \\ 0, & x=0 \end{cases}$

2b

(2014 : 15 Marks)

Solution:

The function

$$f(x) = \begin{cases} 2x\sin\frac{1}{x}-\cos\frac{1}{x} & ; \quad x\in(0,1) \\ 0 & ; \quad x=0 \end{cases}$$

is not continuous on [0, 1]; (it is discontinuous at $x = 0$), but it is bounded and continuous on [0, 1] and thus Riemann-integral on [0, 1].

The function;

$$g(x) = \begin{cases} x^2\sin\frac{1}{x} & ; \quad x\in(0,1) \\ 0 & ; \quad x=0 \end{cases}$$

is differentiable on [0, 1] and satisfies

$$g'(x) = f(x);\ \forall x\in[0, 1]$$

$$\therefore \quad \int_0^1\left(2x\sin\frac{1}{x}-\cos\frac{1}{x}\right)dx = g(1)-g(0)$$

$$= (1)^2.\sin\frac{1}{1}-0 = 1.\sin 1-0 = \sin 1$$

$$\therefore \quad \int_0^1\left(2x\sin\frac{1}{x}-\cos\frac{1}{x}\right)dx = \sin 1$$

5.6 Is the function

$$f(x) = \begin{cases} \frac{1}{n}; & \frac{1}{n+1}<x\le\frac{1}{n} \\ 0; & x=0 \end{cases}$$

Riemann integrable? If yes, obtain the value of $\int_0^1 f(x)dx$.

(2015 : 15 Marks)

Solution:

Given :

$$f(x) = \begin{cases} 1; & \frac{1}{2} < x \le 1 \\ \frac{1}{2}; & \frac{1}{3} < x \le \frac{1}{2} \\ \frac{1}{3}; & \frac{1}{4} < x \le \frac{1}{3} \\ \vdots \\ 0; & x = 0 \end{cases}$$

Now, $f(x)$ is continuous in any interval $\left(\frac{1}{n}, \frac{1}{n+1}\right)$. However, it is discontinuous at $\left(\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}\right)$.

$\therefore$ Set of points of discontinuities,

$$D = \left\{\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}\right\}$$

Limit point of D is 0.

As limit point of set of discontinuities is zero, i.e., finite, $\therefore$ $f(x)$ is Riemann integrable.

Now,
$$\int_0^1 f(x)dx = \left(\int_{1/2}^{1} (1)dx + \int_{1/3}^{1/2}\left(\frac{1}{2}\right)dx + \int_{1/4}^{1/3}\left(\frac{1}{3}\right)dx + \ldots + \int_{1/n}^{1/n-1}\left(\frac{1}{n-1}\right)dx\right) \text{ with } n\to\infty$$

$$\int_0^1 f(x)dx = \lim_{n\to\infty}\sum_{r=1}^{n-1}\int_{\frac{1}{r+1}}^{\frac{1}{r}}\frac{1}{r}dx$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n-1}\frac{1}{r}\left(\frac{1}{r}-\frac{1}{r+1}\right)$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n-1}\left(\frac{1}{r^2}+\frac{1}{r+1}-\frac{1}{r}\right)$$

$$= \sum_{r=1}^{\infty}\frac{1}{r^2}+\sum_{r=1}^{\infty}\frac{1}{r+1}-\sum_{r=1}^{\infty}\frac{1}{r}$$

$$= \frac{\pi^2}{6}+\left(\frac{1}{2}+\frac{1}{3}+\ldots\right)-\left(1+\frac{1}{2}+\frac{1}{3}+\ldots\right)$$

$$= \frac{\pi^2}{6}-1$$

5.7 Prove the inequality : $\dfrac{\pi^2}{9} < \displaystyle\int_{\pi/6}^{\pi/2}\frac{x}{\sin x}dx < \frac{2\pi^2}{9}$

(2018 : 10 Marks)

Solution:

In the interval, $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$, $\sin x$ is positive and monotonically increasing.

$\therefore$
$$\sin\frac{\pi}{6} < \sin x < \sin\frac{\pi}{2}$$

$\Rightarrow$
$$\frac{1}{\sin\frac{\pi}{6}} > \frac{1}{\sin x} > \frac{1}{\sin\frac{\pi}{2}}$$

Multiplying by x, as $x > 0$. Signs do not change. Also, $\frac{x}{\sin x}$ is continuous in $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$.

$$\therefore \quad \frac{x}{\left(\frac{1}{2}\right)} > \frac{x}{\sin x} > 1 \cdot x$$

Integrating with respect to dx in $\left[\frac{\pi}{6}, \frac{\pi}{2}\right]$ get

$$\int_{\pi/6}^{\pi/2} 2x\,dx > \int_{\pi/6}^{\pi/2} \frac{x}{\sin x}dx > \int_{\pi/6}^{\pi/2} x\,dx$$

$$\Rightarrow \quad \left[x^2\right]_{\pi/6}^{\pi/2} > \int_{\pi/6}^{\pi/2} \frac{x}{\sin x}dx > \left[\frac{x^2}{2}\right]_{\pi/6}^{\pi/2}$$

$$\Rightarrow \quad \frac{\pi^2}{4} - \frac{\pi^2}{36} > \int_{\pi/6}^{\pi/2} \frac{x}{\sin x}dx > \frac{1}{2}\left(\frac{\pi^2}{4} - \frac{\pi^2}{36}\right)$$

$$\Rightarrow \quad \frac{2\pi^2}{9} > \int_{\pi/6}^{\pi/2} \frac{x}{\sin x}dx > \frac{\pi^2}{9}$$

$$\text{or} \quad \frac{\pi^2}{9} < \int_{\pi/6}^{\pi/2} \frac{x}{\sin x}dx < \frac{2\pi^2}{9}$$

5.8 Discuss the convergence of $\int_1^2 \frac{\sqrt{x}}{l_n x}dx$.

(2019 : 15 Marks)

Solution:

Let $$f(x) = \frac{\sqrt{x}}{\ln x}$$

1 is the only point of infinite discontinuity of 'f' on [1, 2].

Take $$g(x) = \frac{1}{(x-1)^n}$$

$$\therefore \quad \lim_{x\to 1^+} \frac{f(x)}{g(x)} = \lim_{x\to 1^+} \frac{(x-1)^n\sqrt{x}}{\ln x}\left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{z\to 1^+} \frac{n(x-1)^{n-1}\sqrt{x} + (x-1)^n \frac{1}{2\sqrt{x}}}{1/x}$$

$$= \lim_{z\to 1^+} (x-1)^{n-1}\left[nx^{3/2} + \left(\frac{x-1}{2}\right)\sqrt{x}\right]$$

$= 1$ if $n = 1$ ($\therefore$ a non-zero finite number)

$\therefore$ By comparison test

$\int_1^2 f(x)dx$ and $\int_1^2 g(x)dx$ are convergent or divergent together. But $\int_1^2 g(x)dx$ diverges ($\therefore n = 1$)

$\therefore \int_1^2 f(x)dx$ diverges, i.e., $\int_1^2 \frac{\sqrt{x}}{\ln x}dx$ diverges.

6. Improper Integral

6.1 Test the convergence of the improper integral $\int_1^\infty \frac{dx}{x^2(1+e^{-x})}$.

(2014 : 10 Marks)

Solution:

Given that : $\int_1^\infty \frac{dx}{x^2(1+e^{-x})}$

Let $$f(x) = \frac{1}{x^2(1+e^{-x})}$$

Let $$g(x) = \frac{1}{x^2}$$

$\therefore$ $$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{\lambda\to\infty}\frac{1}{1+e^{-x}} = 1 \neq a$$

Since $$\int_1^\infty g(x) = \int_1^\infty \frac{1}{x^2}dx \text{ is convergent.}$$

Here, $x = 2 > 1$ $\left(\because \int_a^\infty \frac{dx}{x^n} \text{ is convergent iff } n > 1\right)$

$\therefore$ By comparison test, $\int_1^\infty f(x)dx$ is convergent and $\int_1^\infty \frac{1}{x^2(1+e^{-x})}dx$ is convergent.

6.2 Evaluate : $\int_0^\infty \frac{\tan^{-1}(ax)}{x(1+x^2)}dx, a > 0, a \neq 1$

(2019 : 10 Marks)

Solution:

Let $$f(a) = \int_0^\infty \frac{\tan^{-1}(ax)}{x(1+x^2)}\cdot dx \quad \text{...(i)}$$

Differentiating both sides w.r.t. 'a', we get

$$f'(a) = \int_0^\infty \frac{2}{2a}\left[\frac{\tan^{-1}(ax)}{x(1+x^2)}\right]dx$$

$$= \int_0^\infty \frac{1}{x(1+x^2)}\cdot\frac{1}{1+a^2x^2}xdx$$

$$= \int_0^\infty \frac{dx}{(1+x^2)(1+a^2x^2)}$$

$$= \frac{1}{1-a^2}\int_0^\infty\left[\frac{1}{1+x^2} - \frac{a^2}{1+a^2x^2}\right]dx$$

$$= \frac{1}{1-a^2}\left[\tan^{-1}2\right]_0^\infty - \frac{a^2}{1-a^2}\int_0^\infty \frac{dx}{1+a^2x^2}$$

$$= \frac{1}{1-a^2}[\tan^{-1}\infty - \tan^{-1}0] - \frac{a^2}{1-a^2}\cdot\frac{1}{a^2}$$

$$= \int_0^\infty \frac{dx}{x^2+\frac{1}{a^2}}$$

$$= \frac{1}{1-a^2}\cdot\frac{\pi}{2} - \frac{1}{1-a^2}\cdot\frac{1}{\frac{1}{a}}\left[\tan^{-1}\frac{2}{\frac{-1}{a}}\right]_0^\infty$$

$$= \frac{1}{1-a^2}\cdot\frac{\pi}{2} - \frac{a}{1-a^2}\left[\tan^{-1}\infty - \tan^{-1}0\right]$$

$$= \frac{1}{1-a^2}\left[\frac{\pi}{2} - a;\frac{\pi}{2}\right]$$

$$f'(a) = \frac{1}{1-a^2}\cdot\frac{\pi}{2}[1-a]$$

Integrating both sides w.r.t. 'a'

$$f(a) = \frac{\pi}{2}\log(1+a) + C \quad \text{...(2)}$$

From (1), when $a = 0$, $\quad f(0) = 0$

$\therefore$ from (2), $\quad 0 = \frac{\pi}{2}\log 1 + C$

$\Rightarrow \quad 0 = 0 + C$

$\Rightarrow \quad C = 0$

$\therefore \quad f(a) = \frac{\pi}{2}\log(1+a)$

Thus, $\quad \displaystyle\int_0^\infty \frac{\tan^{-1}(ax)}{x(1+x^2)}dx = \frac{\pi}{2}\log(1+a)$

when $a > 0$, $a \neq 1$ is the required result.

7. Functions of Several (Two or Three) Variables

7.1 **Find the maxima, minima and saddle points of the surface $z = (x^2 - y^2)e^{(-x^2-y^2)/2}$.**

(2010 : 15 Marks)

Solution:

Given, the surface is $z = (x^2 - y^2)e^{(-x^2-y^2)/2}$

At extremum, $\quad z_x = z_y = 0$

$$\therefore \quad z_x = 2xe^{(-x^2-y^2)/2} + (x^2-y^2)e^{\left(\frac{-x^2-y^2}{2}\right)} \times \left(\frac{-2x}{2}\right)$$

$$\Rightarrow \quad e^{\left(\frac{-x^2-y^2}{2}\right)}\{2x+(x^2+y^2)(-x)\} = 0$$
$$\Rightarrow \quad 2x - x(x^2 - y^2) = 0$$
$$\Rightarrow \quad x(2 - x^2 + y^2) = 0 \quad \ldots(1)$$

$$z_y = -2ye^{(-x^2-y^2)/2} + (x^2 - y^2)e^{(-x^2-y^2)/2} \times \left(\frac{-2y}{2}\right) = 0$$

$$\Rightarrow \quad -2y + (x^2 - y^2)(-y) = 0$$
$$\Rightarrow \quad -y(2 + x^2 - y^2) = 0$$
$$\Rightarrow \quad y(2 + x^2 - y^2) = 0 \quad \ldots(2)$$

Solving (1) and (2), we get solutions as

$$(x, y) \equiv (0,0), \left(\pm\sqrt{2},0\right), \left(0,\pm\sqrt{2}\right)$$

Now,

$$z_{xx} = 2e^{(-x^2-y^2)/2} + 2xe^{\left(\frac{-x^2-y^2}{2}\right)} \times \left(\frac{-2x}{2}\right) - (3x^2 - y^2)e^{\left(\frac{-x^2-y^2}{2}\right)}$$
$$- (x^3 - y^2x)e^{\left(\frac{-x^2-y^2}{2}\right)} \times (-x)$$
$$= e^{\left(\frac{-x^2-y^2}{2}\right)}(2-2x^2-3x^2+y^2+x^4-y^2x^2)$$

$$z_{yy} = (-2y - x^2y + y^3)e^{\left(\frac{-x^2-y^2}{2}\right)} \times (-y) + e^{\left(\frac{-x^2-y^2}{2}\right)}(-2-x^2+3x^2)$$
$$= e^{\left(\frac{-x^2-y^2}{2}\right)}(2y^2+x^2y^2-y^4-2-x^2+3x^2)$$

$$z_{xy} = (-2y - x^2y + y^3)\left(e^{\frac{-x^2-y^2}{2}}\right) \times (-x) + e^{\left(\frac{-x^2-y^2}{2}\right)}(-2xy)$$
$$= e^{\left(\frac{-x^2-y^2}{2}\right)}(2xy + x^3y - xy^3 - 2xy)$$
$$= e^{\left(\frac{-x^2-y^2}{2}\right)}(x^3y - xy^3)$$

At $\left(\pm\sqrt{2},0\right)$:

$$z_{xx} = \frac{-y}{e}, \; z_{yy} = \frac{z}{e}, \; z_{xy} = 0$$

$\therefore$

$$z_{xx}.z_{yy} - z_{xy}^2 = \frac{-8}{e^2}$$

At $(0,\pm\sqrt{2})$:

$$z_{xx} = \frac{y}{e}, \; z_{yy} = \frac{-2}{e}, \; z_{xy} = 0$$

$\therefore$

$$z_{xx}.z_{yy} - z_{xy}^2 = \frac{-8}{e^2}$$

At all extremum points, $z_{xx}.z_{yy} - z_{xy}^2 < 0$.

$\therefore$ All extremum points viz. (0, 0), $(\pm\sqrt{2},0)$ and $(0,\pm\sqrt{2})$ are saddle points.

$(r\cos\theta, r\sin\theta)$ Min r

7.2 Find the shortest distance from the origin (0, 0) to the hyperbola

$$x^2 + 8xy + 7y^2 = 225$$

(2011 : 15 Marks)

Solution:

Let (x, y) be any point on the given hyperbola.

We need to minimize $\sqrt{x^2+y^2}$ or equivalently $(x^2 + y^2)$.

Consider $\quad F(x, y, \lambda) = x^2 + y^2 - \lambda(x^2 + 8xy + 7y^2 - 225)$

$\therefore \quad F_x(x, y, \lambda) = 2x - 2\lambda x - 8\lambda y = 0$

or $\quad (\lambda - 1)x + 4\lambda y = 0 \quad$ ...(i)

Also, $\quad F_y(x, y, \lambda) = 2y - 8\lambda x - 14\lambda y = 0$

or $\quad 4\lambda x + (7\lambda - 1)y = 0 \quad$ ...(ii)

Since $(x, y) \neq (0, 0)$ (as hyperbola does not pass through the origin), then solving for λ, we have

$$\begin{vmatrix} \lambda-1 & 4\lambda \\ 4\lambda & 7\lambda-1 \end{vmatrix} = 0 \Rightarrow 9\lambda^2 + 8\lambda - 1 = 0$$

$$\Rightarrow \quad \lambda = -1, \frac{1}{9}$$

If $\lambda = -1$, then $-2x - 4y = 0$ or $x = -2y$.

$\therefore$ From $x^2 + 8xy + 7y^2 = 225$, we have

$$-5y^2 = 225 \text{ for which no real solution exists.}$$

If $\lambda = \frac{1}{9}$, then from (i), $\quad y = 2x$

$\therefore$ From $x^2 + 8xy + 7y^2 = 225$, we have

$$x^2 = 5 \text{ and } y^2 = 20$$

$$\therefore \quad x^2 + y^2 = 25$$

Thus, the required shortest distance is $\sqrt{25} = 5$.

7.3 Let $f(x, y) = \begin{cases} \dfrac{(x+y)^2}{x^2+y^2}; & \text{if } (x,y) \neq (0,0) \\ 1, & \text{if } (x, y) = (0, 0) \end{cases}$. Show that $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ exist at (0, 0) though $f(x, y)$ is not continuous at (0, 0).

(2012 : 15 Marks)

Solution:

Given : $\quad f(x, y) = \begin{cases} \dfrac{(x+y)^2}{x^2+y^2}, & \text{if } (x, y) \neq (0,0) \\ 1, & \text{if } (x, y) \neq (0,0) \end{cases}$

$$f_x(0, 0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{h^2}{h^2} - 1}{h} = 0$$

$$f_y(0, 0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{\frac{k^2}{k^2}-1}{k} = 0$$

$\therefore$ $f_x(0, 0)$ and $f_y(0, 0)$ exists.

Now,
$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{(x+y)^2}{x^2+y^2}$$
$$= \lim_{(x,y)\to(0,0)} \frac{x^2+y^2+2xy}{x^2+y^2}$$

Taking $y = mx$

$$= \lim_{x\to 0} \frac{x^2+m^2x^2+2x\cdot mx}{x^2+m^2x^2}$$

$$= \lim_{x\to 0} \frac{1+m^2+2m}{1+m^2}$$, which is different for the different vaues of m.

$\therefore$ $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

7.4 Find the minimum distance of the line given by the planes $3x + 4y + 5z = 7$ and $x - z = 9$ from the origin, by the method of Lagrange's multipliers.

(2012 : 15 Marks)

Solution:

Let
$$W = f(x, y, z) = x^2 + y^2 + z^2 \quad \text{...(i)}$$
$$g(x, y, z) = 3x + 4y + 5z - 7 = 0 \quad \text{...(ii)}$$
$$h(x, y, z) = x - z - 9 = 0 \quad \text{...(iii)}$$
$$\nabla f = \lambda\nabla g + \mu\nabla h$$
$$\langle 2x, 2y, 2z\rangle = \lambda(3, 4, 5) + \mu(1, 0, -1)$$

$$\Rightarrow \left.\begin{aligned} 2x &= 3\lambda+\mu \\ 2y &= 4\lambda \\ 2z &= 5\lambda-\mu \end{aligned}\right] \quad \text{...(iv)}$$

$$\Rightarrow \quad x = \frac{3\lambda+\mu}{2},\ y = 2\lambda,\ z = \frac{5\lambda-\mu}{2}$$

From (ii),

$$3\left[\frac{3\lambda+\mu}{2}+2\lambda+\frac{5\lambda-\mu}{2}\right] = 7$$

$$\Rightarrow \quad 25\lambda - \mu = 7 \quad \text{...(v)}$$

From (iii),
$$\frac{3\lambda+\mu}{2}-\frac{5\lambda-\mu}{2} = 9$$

$$\Rightarrow \quad -\lambda + \mu = 9 \quad \text{...(vi)}$$

Solving (v) and (vi)

$$\lambda = \frac{2}{3},\ \mu = \frac{29}{3}$$

$$\therefore \quad x = \frac{35}{6},\ y = \frac{4}{3},\ z = \frac{-19}{6} \quad \text{(from (iv))}$$

$$\therefore \quad \text{Minimum distance} = \sqrt{x^2+y^2+z^2} = 5\sqrt{\frac{\pi}{6}}$$

7.5 Let $f(x, y) = y^2 + 4xy + 3x^2 + x^3 + 1$. At what points will $f(x, y)$ be maximum or minimum?

(2013 : 10 Marks)

Solution:

For stationary points

$$f(x, y) = y^2 + 4xy + 3x^2 + x^3 + 1$$

$$\frac{\partial f}{\partial x} = 0; \frac{\partial f}{\partial y} = 0$$

$\Rightarrow$

$$4y + 6x + 3x^2 = 0 \quad \text{...(i)}$$

$$2y + 4x = 0 \Rightarrow y = -2x \quad \text{...(ii)}$$

From (i) and (ii)

$$3x^2 - 2x = 0 \Rightarrow x = 0; x = \frac{2}{3}$$

So, (0, 0) and $\left(\frac{2}{3}, \frac{-4}{3}\right)$ are stationary points.

To test for maxima or minima we calculate 2nd partial derivative

$$\frac{\partial^2 f}{\partial x^2} = 6 + 6x; \quad \frac{\partial^2 f}{\partial y^2} = 2; \quad \frac{\partial^2 f}{\partial x \partial y} = 4$$

$$\frac{\partial^2 f}{\partial x^2} \times \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2 = 2(6 + 6x) - 4^2$$

$$= 12x - 4$$

$\therefore$

$$rt - s^2 = -4 < 0 \text{ at } (0, 0)$$

So, no extremum at this point and $rt - s^2 = 4 > 0$ at $\left(\frac{2}{3}, \frac{-4}{3}\right)$ and since $\frac{\partial^2 f}{\partial y^2} = 2 > 0$ so minima at $\left(\frac{2}{3}, \frac{-4}{3}\right)$.

7.6 Obtain $\frac{\partial^2 f(0,0)}{\partial x \partial y}$ and $\frac{\partial^2 f(0,0)}{\partial y \partial x}$ for the function

$$f(x, y) = \begin{cases} \dfrac{xy(3x^2 - 2y^2)}{x^2 + y^2}, & (x, y) \neq (0,0) \\ 0, & (x, y) = (0,0) \end{cases}$$

Also, discuss the continuity of $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ at (0, 0).

(2014 : 15 Marks)

Solution:

Given,

$$f(x, y) = \begin{cases} \dfrac{xy(3x^2 - 2y^2)}{x^2 + y^2} & : (x, y) \neq (0,0) \\ 0 & : (x, y) = (0,0) \end{cases}$$

Now,

$$f_{xy}(0, 0) = \lim_{h \to 0} \frac{f_y(h,0) - f_y(0,0)}{h}$$

$$f_y(0, 0) = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k \to 0} \frac{0 - 0}{k} = \frac{0}{k} = 0$$

$$f_y(h,0) = \lim_{k\to 0}\frac{f(h,k)-f(h,0)}{k} = \lim_{k\to 0}\frac{\frac{hk(3h^2-2k^2)}{h^2+k^2}-0}{k}$$

$$= \lim_{k\to 0}\frac{hk(3h^2-2k^2)}{k(h^2+k^2)} = \frac{h(3h^2)}{h^2} = 3h$$

$$f_{xy}(0,0) = \lim_{h\to 0}\frac{f_y(h,0)-f_y(0,0)}{h} = \frac{3h-0}{h}$$

$$f_{xy}(0,0) = 3$$

Again : $$f_{yx}(0,0) = \lim_{k\to 0}\frac{f_x(0,k)-f_x(0,0)}{k}$$

But, $$f_x(0,0) = \lim_{h\to 0}\frac{f(h,k)-f_x(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0$$

$$f_x(0,k) = \lim_{h\to 0}\frac{f(h,k)-f(0,0)}{h} = \lim_{h\to 0}\frac{hk(3h^2-2k^2)}{h(h^2+k^2)} - 0$$

$$= \lim_{h\to 0}\frac{k(3h^2-2k^2)}{h^2+k^2} = \frac{-k.2k^2}{k^2} = -2k$$

$\therefore$ $$f_{yx}(0,0) = \lim_{k\to 0}\frac{f_x(0,k)-f_x(0,0)}{k} = \frac{-2k}{k} = -2$$

$\therefore$ $$f_{xy}(0,0) \neq f_{yx}(0,0)$$

$\therefore$ $f(x, y)$ is not a continuous function.

7.7 Find the minimum value of $x^2 + y^2 + z^2$ subject to the condition $xyz = a^3$ by the method of Lagrange multipliers.

(2014 : 15 Marks)

Solution:

Let, $f_1 = x^2 + y^2 + z^2$; which is subject to

$f_2 = xyz = a^3$ or $f_2 = xyz - a^3 = 0$

Now, with the lagrange multipliers from:

$$F = f_1 + \lambda f_2 = 0$$

$$F = x^2 + y^2 + z^2 + \lambda(xyz - a^3)$$

Now, we need to find partial derivative of F w.r.t x, y, z respectively.

$$f_x = 2x + \lambda(yz) \qquad f_x = 0$$

$$f_y = 2y + \lambda(zx) \qquad f_y = 0$$

$$f_z = 2z + \lambda(xy) \qquad f_z = 0$$

We get

$$\lambda = -\frac{2x}{yz} \text{ for } F_x$$

$$\lambda = \frac{-2y}{zx} \text{ for } f_y \;\&\; \lambda = \frac{-2z}{xy} \text{ for } f_z$$

Equating λ values of f_x & f_y, we get

$$\frac{-2x}{yz} = \frac{2y}{zx}$$

$\Rightarrow$ $$x^2 = y^2 \Rightarrow x = y$$

[Since; we are ignoring the negative values]

Similarly,

$$y = z \quad \text{for} \quad fy \,\&\, fz$$
$$z = x \quad \text{for} \quad fz \,\&\, fx$$

Thus, $$x = y = z$$

Then from given condition $xyz = a^3$; we get $x = y = z = a^3$

So, minimum value of

$$x^2 + y^2 + z^2 = a^2 + a^2 + a^2 = 3a^2$$

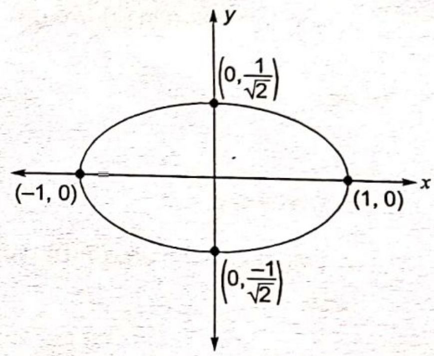
7.8 Find the absolute maximum and minimum values of the function $f(x, y) = x^2 + 3y^2 - y$ over the region $x^2 + 2y^2 \le 1$.

(2015 : 15 Marks)

Solution:

Given, $$f(x, y) = x^2 + 3y^2 - y$$

and region is $$x^2 + 2y^2 \le 1$$



Now, $$f(x, y) = x^2 + 3y^2 - y$$
$$x^2 \ge 0$$

$3y^2 - y$ is minimum and $y = \frac{1}{6}$ in $y \in$ .....

$\therefore$ $$f(x, y)_{min} = 0^2 + 3\times\left(\frac{1}{6}\right)^2 - \frac{1}{6}$$
$$= 0 + 3\times\frac{1}{36} - \frac{1}{6} = \frac{-1}{12}$$

Also, $$x^2 + 2y^2 \le 1$$

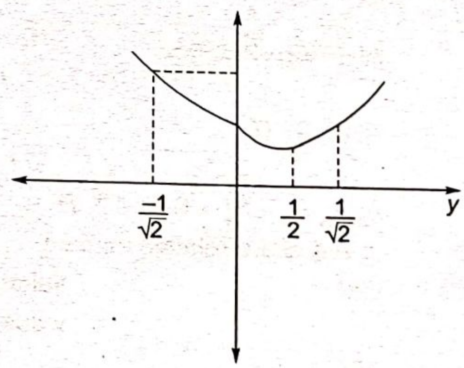
$\Rightarrow$ $$x^2 + 2y^2 + y^2 - y \le 1 + y^2 - y$$

$\Rightarrow$ $$x^2 + 3y^2 - y \le 1 + y^2 - y$$

Let $$g(y) = 1 + y^2 - y$$

$\therefore$ $$f(x, y) \le g(y)$$

In the interval, $$y \in \left[\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$$



$g(y)$ is maximum at $y = \frac{-1}{\sqrt{2}}$

$\therefore$ $f(x, y)_{max}$ is less than $g(y)_{max}$.

$$g(y)_{max} = 1 + \frac{1}{2} + \frac{1}{\sqrt{2}} - \frac{1}{2} + \frac{1}{\sqrt{2}}$$

If $y = \frac{-1}{\sqrt{2}}$, then $$x = 0$$

$\therefore$ $$(x, y) = \left(0, \frac{-1}{\sqrt{2}}\right) \text{ lies in the region}$$

$\therefore$ $$f(x, y)_{max} = g(y)_{max} = \frac{3}{2} + \frac{1}{\sqrt{2}}$$

So, $$\text{absolute minimum} = \frac{-1}{12}$$
$$\text{absolute maximum} = \frac{3}{2} + \frac{1}{\sqrt{2}}$$

7.9 Find the relative maximum and minimum values of the function

$$f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$$

(2016 : 15 Marks)

Solution:

Given : $f(x, y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

At extremum,

$f_x = 0 \Rightarrow \quad 4x^3 - 4x + 4y = 0 \Rightarrow x^3 - x + y = 0$ ...(i)

$f_y = 0 \Rightarrow \quad 4y^3 - 4y + 4x = 0 \Rightarrow y^3 - y + x = 0$ ...(ii)

Adding (i) and (ii), we get

$$x^3 + y^3 = 0$$

$$\Rightarrow \quad x^3 = -y^3$$

$$\Rightarrow \quad x = -y$$

Using this value in (ii), we get

$$y^3 - y - y = 0$$

$$\Rightarrow \quad y^3 - 2y = 0 \Rightarrow y(y^2 - 2) = 0$$

$$\Rightarrow \quad y = 0, y = \sqrt{2}, y = -\sqrt{2}$$

Accordingly, $x = 0, x = -\sqrt{2}, x = \sqrt{2}$

$\therefore$ Critical points are (0, 0), $(\sqrt{2},-\sqrt{2})$ and $(-\sqrt{2},\sqrt{2})$.

Now,

$$f_{xx} = 12x^2 - 4, \ f_{xy} = 4$$

$$f_{yy} = 12y^2 - 4$$

$$f_{xx} \cdot f_{yy} - f_{xy}^2 = (12x^2 - 4)(12y^2 - 4) - 16$$

At $(\sqrt{2},-\sqrt{2}),(A)$:

$$f_{xx} \cdot f_{yy} - f_{xy}^2 = (12 \times 2 - 4)(12 \times 2 - 4) - 4^2 = 384 > 0$$

At $(-\sqrt{2},\sqrt{2}),(B)$:

$$f_x \cdot f_{yy} - f_{xy}^2 = 384 > 0$$

At (0, 0), (C) :

$$f_{xx} \cdot f_{yy} - f_{xy}^2 = (0 - 4)(0 - 4) - 16 = 0$$

As $f_{xx} \cdot f_{yy} - f_{xy}^2 > 0$ at A & B

$\therefore$ A & B are relative minima

and value of f at $A = -8$

value of f at $B = -8$.

At C :

Let $n > 0$

$\therefore$

$$f(n, n) = n^4 + n^4 - 2n^2 + 4n^2 - 2n^2 = 2n^4$$

$$f(n, -n) = n^4 + n^4 - 2n^2 - 4n^2 - 2n^2 = 2n^4 - 8n^2 < 0, \text{ if } x \to 0$$

$\therefore$ C is neither minima nor maxima.

$\therefore$ Minimum value of $f(x, y)$ is -8 and maximum value does not exist.

7.10 Find the maximum value of $f(x, y, z) = x^2y^2z^2$ subject to the subsidiary condition $x^2 + y^2 + z^2 = c^2$, $(x, y, z > 0)$.

(2019 : 15 Marks)

Solution:

On the spherical surface $x^2 + y^2 + z^2 = 0^2$, the function must assume the greatest value. Since, the surface is a bounded and closed set.

According to the method of undetermined multipliers, we form the expression

$$f = x^2y^2z^2 + \lambda(x^2 + y^2 + z^2 - c^2)$$

and by differentiation, we obtain

$$2xy^2z^2 + 2\lambda x = 0$$
$$2x^2yz^2 + 2\lambda y = 0$$
$$2x^2y^2z + 2\lambda z = 0$$

The solution with $x = 0$, $y = 0$ or $z = 0$ can be excluded, for at these points the function takes on its least value, zero.

The other solution of the equation are $x^2 = y^2 = z^2$, $\lambda = x^4$. Using the subsidiary condition, we obtain the value.

$$x = \pm\frac{C}{\sqrt{3}},\ y = \pm\frac{C}{\sqrt{3}},\ z = \pm\frac{C}{\sqrt{3}}$$

for the required coordinates.

At all these points, the function assumes the same value $\frac{C^6}{Z_7}$, which accordingly is the maximum. Hence, any tried of numbers satisfies the relation

$$\sqrt[3]{x^2y^2z^2} \le \frac{C^2}{3} = \frac{x^2 + y^2 + z^2}{3}$$

which states that the geometric mean of three non-negative numbers x^2, y^2, z^2 is never greater than their arithmetic mean.

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