Previous Years' Papers (Solved)

IFS MATHEMATICS EXAM., 2010

PAPER-I

Instructions : *Candidates should attempt questions 1 and 5 which are compulsory, and any THREE of the remaining questions, selecting at least ONE question from each Section. All questions carry equal marks. Marks allotted to parts of a question are indicated against each. Answers must be written in ENGLISH only. Assume suitable data, if considered necessary, and indicate the same clearly. Unless indicated otherwise, symbols & notations carry their usual meaning.*

Section-A

1. Answer any **five** of the following:

A. Show that the set
$P[t] = \{at^2 + bt + c \,/\, a, b, c \in \mathbb{R}\}$
forms a vector space over the field $\mathbb{R}$. Find a basis for this vector space. What is the dimension of this vector space? 8

B. Determine whether the quadratic form
$q = x^2 + y^2 + 2xz + 4yz + 3z^2$
is positive definite. 8

C. Prove that between any two real roots of $e^x \sin x = 1$, there is at least one real root of $e^x \cos x + 1 = 0$. 8

D. Let f be a function defined on $\mathbb{R}$ such that $f(x + y) = f(x) + f(y)$, $x, y \in \mathbb{R}$.
If f is differentiable at one point of $\mathbb{R}$, then prove that f is differentiable on $\mathbb{R}$. 8

E. If a plane cuts the axes in A, B, C and (a, b, c) are the coordinates of the centroid of the triangle ABC, then show that the equation of the plane is 8

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3.$$

F. Find the equations of the spheres passing through the circle 8
$x^2 + y^2 + z^2 - 6x - 2z + 5 = 0,\ y = 0$
and touching the plane $3y + 4z + 5 = 0$.

2. A. Show that the vectors
$\alpha_1 = (1, 0, -1), \alpha_2 = (1, 2, 1),$
$\alpha_3 = (0, -3, 2)$
form a basis for $\mathbb{R}^3$. Find the components of $(1, 0, 0)$ w.r.t. the basis $\{\alpha_1, \alpha_2, \alpha_3\}$. 10

B. Find the characteristic polynomial of
$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}$. Verify Cayley – Hamilton theorem for this matrix and hence find its inverse. 10

C. Let $A = \begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix}$. Find an invertible matrix P such that $P^{-1} A P$ is a diagonal matrix. 12

D. Find the rank of the matrix

$$\begin{pmatrix} 1 & 2 & 1 & 1 & 2 \\ 2 & 4 & 3 & 4 & 7 \\ -1 & -2 & 2 & 5 & 3 \\ 3 & 6 & 2 & 1 & 3 \\ 4 & 8 & 6 & 8 & 9 \end{pmatrix}$$ 8

3. A. Discuss the convergence of the integral

$$\int_0^\infty \frac{dx}{1+x^4\sin^2 x}$$ 10

B. Find the extreme value of xyz if $x + y + z = a$. 10

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C. Let $f(x,y)=\begin{cases}\dfrac{xy(x^2-y^2)}{x^2+y^2}, & \text{if } (x,y)\neq(0,0)\\ 0, & \text{if } (x,y)=(0,0)\end{cases}$

Show that:

(i) $f_{xy}(0, 0) \neq f_{yx}(0, 0)$

(ii) f is differentiable at (0, 0) 10

D. Evaluate $\iint_D (x+2y)\ dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$. 10

4. A. Prove that the second degree equation

$$x^2 - 2y^2 + 3z^2 + 5yz - 6zx - 4xy + 8x - 19y - 2z - 20 = 0$$

represents a cone whose vertex is (1, −2, 3). 10

B. If the feet of three normals drawn from a point P to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ lie in the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, prove that the feet of the other three normals lie in the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}+1=0$. 10

C. If $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ represents one of the three mutually perpendicular generators of the cone $5yz - 8zx - 3xy = 0$, find the equations of the other two. 10

D. Prove that the locus of the point of intersection of three tangent planes to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, which are parallel to the conjugate diametral planes of the ellipsoid $\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=1$ is

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}.$$ 10

Section-B

5. Answer any ***five*** of the following:

A. Show that cos (x + y) is an integrating factor of $y\,dx + [y + \tan(x + y)]\,dy = 0$. Hence solve it. 8

B. Solve

$$\frac{d^2y}{dx^2}-2\frac{dy}{dx}+y=xe^x\sin x$$ 8

C. A uniform rod AB rests with one end on a smooth vertical wall and the other on a smooth inclined plane, making an angle α with the horizon. Find the positions of equilibrium and discuss stability. 8

D. A particle is thrown over a triangle from one end of a horizontal base and grazing the vertex falls on the other end of the base. If θ_1 and θ_2 be the base angles and θ be the angle of projection, prove that, $\tan\theta = \tan\theta_1 + \tan\theta_2$. 8

E. Prove that the horizontal line through the centre of pressure of a rectangle immersed in a liquid with one side in the surface, divides the rectangle in two parts, the fluid pressure on which, are in the ratio, 4 : 5. 8

F. Find the directional derivation of $\vec{V}^2$, where, $\vec{V}=xy^2\,\vec{i}+zy^2\,\vec{j}+xz^2\,\vec{k}$ at the point (2, 0, 3) in the direction of the outward normal to the surface $x^2 + y^2 + z^2 = 14$ at the point (3, 2, 1). 8

6. A. Solve the following differential equation

$$\frac{dy}{dx}=\sin^2(x-y+6)$$ 8

B. Find the general solution of

$$\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+(x^2+1)y=0$$ 12

C. Solve

$$\left(\frac{d}{dx}-1\right)^2\left(\frac{d^2}{dx^2}+1\right)^2 y=x+e^x$$ 10

D. Solve by the method of variation of parameters the following equation

$$\left(x^2-1\right)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=\left(x^2-1\right)^2 \qquad 10$$

7. A. A uniform chain of length $2l$ and weight W, is suspended from two points A and B in the same horizontal line. A load P is now hung from the middle point D of the chain and the depth of this point below AB is found to be h. Show that each terminal tension is,

$$\frac{1}{2}\left[P.\frac{l}{h}+W.\frac{h^2+l^2}{2hl}\right] \qquad 14$$

B. A particle moves with a central acceleration $\frac{\mu}{(\text{distance})^2}$, it is projected with velocity V at a distance R. Show that its path is a rectangular hyperbola if the angle of projection is,

$$\sin^{-1}\left[\frac{\mu}{VR\left(V^2-\frac{2\mu}{R}\right)^{1/2}}\right] \qquad 13$$

C. A smooth wedge of mass M is placed on a smooth horizontal plane and a particle of mass m slides down its slant face which is inclined at an angle a to the horizontal plane. Prove that the acceleration of the wedge is,

$$\frac{mg\sin\alpha\cos\alpha}{M+m\sin^2\alpha} \qquad 13$$

8. A. (*i*) Show that

$$\vec{F}=\left(2xy+z^3\right)\vec{i}+x^2\,\vec{j}+3z^2x\,\vec{k}$$

is a conservative field. Find its scalar potential and also the work done in moving a particle from (1, −2, 1) to (3, 1, 4). 5

(*ii*) Show that, $\nabla^2 f(r)=\left(\frac{2}{r}\right)f'(r)+f''(r)$, where $r=\sqrt{x^2+y^2+z^2}$ 5

B. Use divergence theorem to evaluate,

$$\iint_S\left(x^3dy\,dz+x^2y\,dz\,dx+x^2z\,dy\,dx\right),$$

where S is the sphere, $x^2+y^2+z^2=1$. 10

C. If $\vec{A}=2y\,\vec{i}-z\,\vec{j}-x^2\,\vec{k}$ and S is the surface of the parabolic cylinder $y^2 = 8x$ in the first octant bounded by the planes $y = 4$, $z = 6$, evaluate the surface integral,

$$\iint_S \vec{A}.\hat{n}\,d\vec{S}. \qquad 10$$

D. Use Green's theorem in a plane to evaluate the integral, $\int_C\left[\left(2x^2-y^2\right)dx+\left(x^2+y^2\right)dy\right]$, where C is the boundary of the surface in the xy-plane enclosed by, $y = 0$ and the semi-circle, $y=\sqrt{1-x^2}$. 10

PAPER-II

Instructions : *Candidates should attempt Question Nos. 1 and 5 which are compulsory, and any THREE of the remaining questions, selecting at least ONE question from each Section. All questions carry equal marks. The number of marks carried by each part of a question is indicated against each. Answers must be written in ENGLISH only. Assume suitable data, if considered necessary, and indicate the same clearly. Symbols and notations have their usual meanings, unless indicated otherwise.*

Section-A

2. Answer any **four** parts from the following:

A. Let $G=\left\{\begin{bmatrix} a & a \\ a & a\end{bmatrix} \middle| a\in R, a\neq 0\right\}$

Show that G is a group under matrix multiplication. 10

B. Lef F be a field of order 32. Show that the only subfields of F are F itself and $\{0, 1\}$. 10

C. If $f : \mathbb{R} \rightarrow \mathbb{R}$ is such that

$f(x + y) = f(x) f(y)$

for all x, y in $\mathbb{R}$ and $f(x) \neq 0$ for any x in $\mathbb{R}$, show that $f'(x) = f(x)$ for all x in $\mathbb{R}$ given that $f'(0) = f(0)$ and the function is differentiable for all x in $\mathbb{R}$. 10

D. Determine the analytic function $f(z) = u + iv$ if $v = e^x (x \sin y + y \cos y)$. 10

E. A captain of a cricket team has to allot four middle-order batting positions to four batsmen. The average number of runs scored by each batsman at these positions are as follows. Assign each batsman his batting position for *maximum* performance:

Batting Position / Batsman	IV	V	VI	VII
A	40	25	20	35
B	36	30	24	40
C	38	30	18	40
D	40	23	15	33

2. A. A rectangular box open at the top is to have a surface area of 12 square units. Find the dimensions of the box so that the volume is maximum. 13

B. Prove or disprove that $(\mathbb{R}, +)$ and $(\mathbb{R}^+, .)$ are isomorphic groups where $\mathbb{R}^+$ denotes the set of all positive real numbers. 13

C. Using the method of contour integration, evaluate

$$\int_{-\infty}^{\infty} \frac{x^2 dx}{\left(x^2+1\right)^2\left(x^2+2x+2\right)}$$ 14

3. A. Show that zero and unity are only idempotents of Z_n if $n = p^r$, where p is a prime. 13

B. Evaluate

$$\iint_{R} (x-y+1)\, dx\, dy$$

where R is the region inside the unit square in which $x+y \geq \frac{1}{2}$. 13

C. Solve the following linear programming problem by the simplex mothod: 14

Maximize $Z = 3x_1 + 4x_2 + x_3$

subject to

$$x_1 + 2x_2 + 7x_3 \leq 8$$
$$x_1 + x_2 - 2x_3 \leq 6$$
$$x_2, x_2, x_3 \geq 0$$

4. A. Let R be a Euclidean domain with Euclidean valuation d. Let n be an integer such that $d(1) + n \geq 0$. Show that the function $d_n : R - \{0\} \rightarrow S$, where S is the set of all negative integers, defined by $d_n(a) = d(a) + n$ for all $a \in R - \{0\}$ is a Euclidean valuation. 13

B. Obtain Laurent's series expansion of the function

$$f(z) = \frac{1}{(z+1)(z+3)}$$

in the region $0 < |z + 1| < 2$. 13

C. *ABC* Electricals manufactures and sells two models of lamps, L_1 and L_2, the profit per unit being Rs 50 and Rs 30, respectively. The process involves two workers W_1 and W_2, who are available for 40 hours and 30 hours per week, respectively. W_1 assembles each unit of L_1 in 30 minutes and that of L_2 in 40 minutes. W_2 paints each unit of L_1 in 30 minutes and that of L_2 in 15 minutes. Assuming that all lamps made can be sold, determine the weekly production figures that maximize the profit. 14

Section-B

5. Answer any *four* parts from the following:

A. Find the general solution of

$x(y^2 + z)\, p + y\, (x^2 + z)q = z(x^2 - y^2)$ 10

B. Solve $x\log_{10} x = 1.2$ by regula falsi method. 10

C. Convert the following: 10
(i) $(736.4)_8$ to decimal number
(ii) $(41.6875)_{10}$ to binary number
(iii) $(101101)_2$ to decimal number
(iv) $(AF63)_{16}$ to decimal number
(v) $(1011110111111)_2$ to hexadecimal number

D. Show that the sum of the moments of inertia of an elliptic area about any two tangents at right angles is always the same. 10

E. A two-dimensional flow field is given by $\psi = xy$. Show that–
(i) the flow is irrotational;
(ii) ψ and ϕ satisfy Laplace equation.
Symbols ψ and ϕ convey the usual meaning. 10

6. A. Using Lagrange interpolation, obtain an approximate value of sin(0.15) and a bound on the truncation error for the given data: 12
sin (0.1) = 0.09983, sin(0.2) = 0.19867

B. Draw a flow chart for finding the roots of the quadratic equation $ax^2 + bx + c = 0$. 12

C. Solve

$$\frac{\partial u}{\partial t} = 4\frac{\partial^2 u}{\partial x^2}$$

given the conditions
(i) $u(0, t) = u(\pi, t) = 0, t > 0$
(ii) $u(x, 0) = \sin 2x, 0 < x < \pi$ 16

7. A. Find the general solution of $(D - D' - 1)(D - D' - 2) z = e^{2x - y} + \sin(3x + 2y)$ 13

B. Show that $\phi = (x - t).(y - t)$ represents the velocity potential of an incompressible two-dimensional fluid. Further show that the streamlines at time t are the curves $(x - t)^2 - (y - t)^2 = \text{constant}$ 13

C. Find the interpolating polynomial for (0, 2), (1, 3), (2, 12) and (5, 147). 14

8. A. A mass m_1, hanging at the end of a string, draws a mass m_2 along the surface of a smooth table. If the mass on the table be doubled, the tension of the string is increased by one-half.
Show that $m_1 : m_2 : 2 : 1$. 13

B. Solve the initial value problem

$$\frac{dy}{dx} = \frac{y-x}{y+x},\ y(0) = 1$$

for $x = 0.1$ by Euler's method. 13

C. Show that the vorticity vector $\vec{\Omega}$ of an incompressible viscous fluid moving under no external forces satisfies the differential equation

$$\frac{D\vec{\Omega}}{Dt} = (\vec{\Omega}.\nabla)\vec{q} + \nu\nabla^2\vec{\Omega}$$

where ν is the kinematic viscosity. 14

ANSWERS

PAPER-I

Section-A

1.(a): From question $P(t) = \{at^2 + bt + c\}$.

Let, $f(t)$ & $g(t) \in p(t)$

then, $f(t) = a_1t^2 + b_1t + c_1$.

$g(t) = a_2t^2 + b_2t + c_2$

then, $f(t) + g(t) = (a_1 + a_2)t^2 + (b_1 + b_2)t + (c_1 + c_2)$

$\Rightarrow f(t) + g(t) \in p(t)$

$\because a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}$

Also, $f(t) = g(t)$ iff $a_1 = a_2, b_1 = b_2, c_1 = c_2$.

Also, $kf(t) = (ka_1)t^2 + (kb_1)t + kc_1 = i\in p(t)$.

Also, $f(t) + g(t) = (a_1 + a_2)t^2 + (b_1 + b_2)t + (c_1 + c_2) = (a_2 + a_1)t^2 + (b_2 + b_1)t + (c_2 + c_1) = g(t) + f(t)$

$\Rightarrow$ set is commutative.

Also, if $h(t) = a_3t^2 + b_3t + c_3$

then $f(t) + \{g(t) + h(t)\} = \{f(t) + g(t)\} + h(t)$.

Existence of identity

$0 = 0.t^2 + 0.t + 0$

i.e., $0 \in p(t) \Rightarrow 0 + f(t) = f(t)$.

Existence of additive inverse of each member as $f(t) \in p(t)$

then $-f(t) \in p(t)$

then $-f(t) + f(t) = 0$ *i.e* the zero polynomial

$\therefore$ $-f(t)$ is the additive inverse of $f(t)$

i.e. $P(t)$ is an abelian group w.r.t. addition of polynomial of less than or equal to degree.

Hence: $p(t)$ is vector space.

(*b*) The associated symmetric matrix of the given quadratic form can be written as

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \quad i.e.\ q = \begin{bmatrix} x & y & z \end{bmatrix} A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

to ascertain the positive definite, we have to apply the congruent operation in the above matrix.

$$i.e. \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Apply congruent operation $R_3 \to R_3 - R_1$ & $C_3 \to C_3 - C_1$, we get,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Apply congruent operation $R_3 \to R_3 - 2R_2$ & $C_3 \to C_3 - 2C_2$, we get,

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

As all the roots of scalar matrix in the left hand side are not positive. Hence, the given quadratic form is not positive.

(c) The given function is $f(x) = e^x \sin x - 1 = 0$

or, $\sin x - e^{-x} = 0$.

Now, $f(x) = \sin x - e^{-x} = 0$

Now, if x_1 and x_2 are two roots of $f(x) = 0$

then by Rolb's theorem $\exists$ at least one real root of $f'(x) = 0$ lies between x_1 and x_2.

Now $f'(x) = \cos x + e^{-x} = 0$

i.e. $e^x \cos x + 1 = 0$ has a root lies between two real roots of $e^x \sin x = 1$.

(*d*) $\because$ $f(x + y) = f(x) + f(y)\ \forall\ x, y \in R$

putting $x = y = 0 \Rightarrow f(0 + 0) = f(0) + f(0)$

$\Rightarrow$ $f(0) = 0$

putting $x = -y$, we get, $f(x - x) = f(x) + f(-x)$

$\Rightarrow$ $f(-x) = -f(x)\ \forall\ x \in R$.

Let, c be the point at which c is differentiable then f is continuous at point c also.

Let, $<c_n>$ be sequence of real number such that $c_n \to 0$

the $fo(c_n + c) \to o + c$

$\Rightarrow f(c_n + c) \to f(o + c) = f(c)$

$\Rightarrow f(c_n + c) \to f(c) \Rightarrow f(c_n) \to 0$

i.e. $f(c_n) \to 0$ provided $c_n \to 0$.

Now, if $<x_n>$ be sequence such that $x_n \to x$

$\Rightarrow x_n - x \to 0 \Rightarrow f(x_n - x) \to f(o)$

$\Rightarrow f(x_n) \to f(x)$

i.e. f is continuous at all points $\in \mathbb{R}$.

Now, f is differentiable at c.

$$\Rightarrow f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h} \to \lim_{h \to 0} \frac{f(h)}{h}$$

$= \ell$ (say).

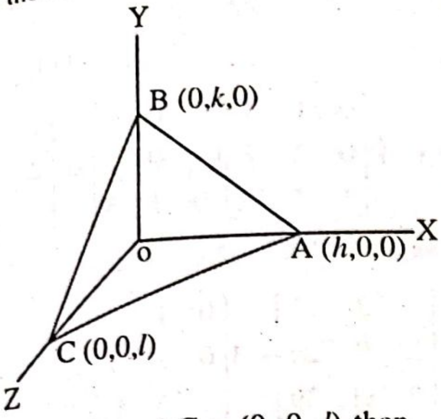
clearly $\frac{f(h)}{h}$ is defined since it is differentiable at c

$$\text{Now, } f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \to \lim_{h \to 0} \frac{f(h)}{h} = l$$

$\Rightarrow f$ is differentiable on $\mathbb{R}$.

(e) Refer to figure

Let the co-ordinate of $A \equiv (h, 0, 0)$



$B \equiv (0, k, 0)$ and $C \equiv (0, 0, l)$ then

equation of plane ABC is $\frac{x}{h}+\frac{y}{k}+\frac{z}{l}=1.$

Now, (a, b, c) is the centroid of ΔABC then

$$a=\frac{h+0+0}{3}, b=\frac{0+k+0}{3}, c=\frac{0+0+l}{3}$$

or, $h = 3a,\ k = 3b,\ l = 3c.$

i.e. equation of the plane ABC can be rewritten

as $\frac{x}{3a}+\frac{y}{3b}+\frac{z}{3c}=1$ or, $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=3.$

(f) The equation of the given circle is

$$\left.\begin{aligned} x^2+y^2+z^2-6x-2z+5=0 \\ y=0 \end{aligned}\right\} \quad \text{... (1)}$$

Equation of any sphere passing through the circle (1) is given by

$$x^2+y^2+z^2-6x-2z+5+\lambda y=0 \quad \text{... (2)}$$

centre of sphere (2) is $\left(3,-\frac{\lambda}{2},1\right)$

and radius of this sphere is $\sqrt{\frac{\lambda^2}{4}+5}$

Now, if the plane $3y + 4z + 5 = 0$ ... (3) is a tangent plane to (2), then,

$$\frac{\left|3\left(\frac{-\lambda}{2}\right)+4+5\right|}{5}=\sqrt{\frac{\lambda^2+20}{4}}$$

or, $\left|\frac{9-\frac{3\lambda}{2}}{5}\right|=\sqrt{\frac{\lambda^2+20}{4}}$

or, $\frac{3(6-\lambda)}{10}=\sqrt{\frac{\lambda^2+20}{4}}$

or, $\frac{9(6-\lambda)^2}{100}=\frac{\lambda^2+20}{4}$

or, $9(\lambda^2 - 12\lambda + 36) = 25(\lambda^2 + 20)$

or, $25\lambda^2 + 500 = 9\lambda^2 - 108\lambda + 324$

or, $16\lambda^2 + 108\lambda + 176 = 0$

or, $4\lambda^2 + 27\lambda + 44 = 0$

or, $4\lambda^2 + 11\lambda + 16\lambda + 44 = 0$

or, $\lambda(4\lambda + 11) + 4(4\lambda + 11) = 0$

or, $(\lambda + 4)(4\lambda + 11) = 0$

$\Rightarrow \lambda = -4$ or, $\lambda=-\frac{11}{4}$

Hence, the equation of sphere in given by

$x^2 + y^2 + z^2 - 6x - 2z + 5 - 4y = 0$

& $4(x^2 + y^2 + z^2 - 6x - 2z + 5) - 11y = 0$

2. (a): To show that $\alpha_1, \alpha_2, \alpha_3$ form a basis of $\mathbb{R}^3$. It is sufficient to show that they are linear independent.

i.e. $\exists$ a $x, y, z \in \mathbb{R}$ such that

$$x\alpha_1 + y\alpha_2 + z\alpha_3 = (0, 0, 0)$$

then $x = y = z = 0$

Now, $x(1,0,-1) + y(1,2,1) + z(0,-3,2) = (0,0,0)$

or $(x + y,\ 2y - 3z\ -x + y + 2z) = (0,0,0)$

Comparing the co-efficient we get,

$x + y = 0$ ...(1)

$2y - 3z = 0$ ...(2)

$-x + y + 2z = 0$ ...(3)

(1) and (3) $\Rightarrow$ $2y + 2z = 0$ ...(4)

(2) and (4) $\Rightarrow$ $5z = 0$ or $z = 0$

$\Rightarrow y = 0$ *i.e.* $x = y = z = 0$

Hence, $\{\alpha_1, \alpha_2, \alpha_3\}$ are linear independent.

Also dimension = 3, hence, they form a basis of $\mathbb{R}^3$.

Now, let $(1, 0, 0) = a\alpha_1 + b\alpha_2 + c\alpha_3$

then, $(1, 0, 0) = a(1, 0, -1) + b(1, 2, 1) + c(0, -3, 2)$

or, $(1,0,0) = (a + b, 2b - 3c, -a + b + 2c)$

$\Rightarrow a + b = 1, \quad 2b - 3c = 0 \quad -a + b + 2c = 0$

$\therefore a + b = 1 \Rightarrow a = (1 - b)$

or, $2b = 3c \Rightarrow c = \frac{2}{3}b$

also $-a + b + 2c = 0$

$\Rightarrow b - 1 + b + \frac{4}{3}b = 0$ or, $2b + \frac{4}{3}b = 1$

$\frac{10b}{3} = 1$ or $b = \frac{3}{10}$

$\therefore a = 1 - \frac{3}{10} = \frac{7}{10} \quad \therefore C = \frac{2}{3} \cdot \frac{3}{10} = \frac{1}{5}$

$\therefore (1, 0, 0) = \frac{7}{10}\alpha_1 + \frac{3}{10}\alpha_2 + \frac{1}{5}\alpha_3$

(b) The given matrix is $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} = A$ (let) then

the characteristic equation of A is given by,

$|A - \lambda I| = 0$

or, $\begin{bmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & 2 \\ 0 & 1 & 3-\lambda \end{bmatrix} = 0.$

or, $-\lambda\{\lambda(\lambda-3) - 2\} + 1(1) = 0$

or, $-\lambda\{\lambda^2 - 3\lambda - 2\} + 1 = 0$

or, $-\lambda^3 + 3\lambda^2 + 2\lambda + 1 = 0$

or, $\lambda^3 - 3\lambda^2 - 2\lambda - 1 = 0$

Now, by Cayley Hamilton theorem, it should also satisfy the matrix A.

i.e. $A^3 - 3A^2 - 2A - I = 0$ ...(1)

To prove identity (1), we'll calculate A^3 and A^2.

$\because A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} \quad A^2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$

or, $A^2 = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 2 & 7 \\ 1 & 3 & 11 \end{bmatrix}$

$\& A^3 = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 2 & 7 \\ 1 & 3 & 11 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 11 \\ 2 & 7 & 25 \\ 3 & 11 & 39 \end{bmatrix}$

Now, $A^3 - 3A^2 - 2A - I$

$= \begin{bmatrix} 1 & 3 & 11 \\ 2 & 7 & 25 \\ 3 & 11 & 39 \end{bmatrix} - 3\begin{bmatrix} 0 & 1 & 3 \\ 0 & 2 & 7 \\ 1 & 3 & 11 \end{bmatrix} - 2\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Hence, Cayley Hamilton Theorem is verified.

Now, $A^3 - 3A^2 - 2A - I = 0$

or, $I = A^3 - 3A^2 - 2A$

multiply both the sides by A^{-1}, we get

$A^{-1} = A^2 - 3A - 2I.$

or, $A^{-1} = \begin{bmatrix} 0 & 1 & 3 \\ 0 & 2 & 7 \\ 1 & 3 & 11 \end{bmatrix} - 3\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$\Rightarrow A^{-1} = \begin{bmatrix} -2 & 1 & 0 \\ -3 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}.$

(c) The such invertible matrix can be formed with the help of eigen vectors of matrix A.

The characteristic root of matrix is given by

$|A - \lambda I| = 0$

or, $\begin{vmatrix} 5-\lambda & -6 & -6 \\ -1 & 4-\lambda & 2 \\ 3 & -6 & -4-\lambda \end{vmatrix} = 0$

or, $(5-\lambda)\{(\lambda-4)(\lambda+4)+12\}$

$+6\{\lambda+4-6\}-6\{6-3(4-\lambda)\}=0$

or, $4-8\lambda+5\lambda^2-\lambda^3=0$

$\Rightarrow (1-\lambda)(2-\lambda)^2=0$

Hence, eigen values of matrix A is given by

$$\lambda = 1, 2, 2.$$

Now, Corresponding to $\lambda = 2$, the eigen vector is obtained through

$$[A-2I]X=0 \begin{bmatrix}5-2 & -6 & -6\\ -1 & 2 & 2\\ 3 & -6 & -6\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\text{or.}\begin{bmatrix}3 & -6 & -6\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\Rightarrow x_1-2x_2-2x_3=0.$

This imply there are two free variables

put $x_2=0, x_3=1$ we get eigen vector [2, 0, 1]

and $x_2=1, x_3=0$ we get eigen vector [2, 1, 0].

Hence, the two eigen vector corresponding to $\lambda = 2$ are [2, 0, 1] and [2, 1, 0]

Now, the eigen vector corresponding to $\lambda = 1$ is given by

$$\begin{bmatrix}4 & -6 & -6\\ -1 & 3 & 2\\ 3 & -6 & -5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\begin{bmatrix}1 & -3/2 & -3/2\\ 1 & 3 & 2\\ 3 & -6 & -5\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Apply $R_1 \to \frac{1}{4}R_1$

Apply $R_2 \to R_2 - R_1$ & $R_3 \to R_3 - 3R_1$, we get

$$\begin{bmatrix}1 & -3/2 & -3/2\\ 0 & 3/2 & 1/2\\ 0 & -3/2 & -1/2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\Rightarrow 2x_1-3x_2-3x_3=0$

$3x_2+x_3=0$

There is only one free variable say $x_2=1$

then $x_3=-3$

& $2x_1-3+9=0 \quad x_1=-3$

$\therefore \quad (-3, 1, -3).$

Hence, the invertible matrix P can be written as

$$P=\begin{bmatrix}2 & 2 & -3\\ 0 & 1 & 1\\ 1 & 0 & -3\end{bmatrix}$$

and $|P| = -6+2+3=-1$

$$\therefore \quad P^{-1}=-\begin{bmatrix}-3 & 1 & -1\\ 6 & -3 & 2\\ 5 & -2 & 2\end{bmatrix}^T=\begin{bmatrix}3 & -6 & -5\\ -1 & 3 & 2\\ 1 & -2 & -2\end{bmatrix}$$

Hence,

$$P^{-1}AP=\begin{bmatrix}3 & -6 & -5\\ -1 & 3 & 2\\ 1 & -2 & -2\end{bmatrix}\begin{bmatrix}5 & -6 & -6\\ -1 & 4 & 2\\ 3 & -6 & -4\end{bmatrix}\begin{bmatrix}2 & 2 & -3\\ 0 & 1 & 1\\ 1 & 0 & 3\end{bmatrix}$$

$$=\begin{bmatrix}2 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 1\end{bmatrix} \text{ which is a diagonal matrix.}$$

(d): The rank of any matrix is equal to number of non-zero rows in the echelon form of the given matrix.

$$\text{Now, let} \quad A=\begin{bmatrix}1 & 2 & 1 & 1 & 2\\ 2 & 4 & 3 & 4 & 7\\ -1 & -2 & 2 & 5 & 3\\ 3 & 6 & 2 & 1 & 3\\ 4 & 8 & 6 & 8 & 9\end{bmatrix}$$

Apply $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 + R_1$,

$R_4 \to R_4 - 3R_1$, $R_5 \to R_5 - 4R_1$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 6 & 5 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & 2 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \to -\frac{1}{4} R_3$ & $R_5 \to R_5 - 5R_3$

Apply $R_3 \to R_3 - 3R_2$, $R_4 \to R_4 + R_2$.
& $R_5 \to R_5 - 2R_2$

No. of non zero rows in echelon form = 3
i.e. Rank of the given matrix = 3.

3. (a): Consider the integral $I = \int_0^{n\pi} \frac{dx}{1+x^4\sin^2 x}$

or, $$I = \sum_{r=1}^{n} \int_{(r-1)\pi}^{r\pi} \frac{dx}{1+x^4\sin^2 x}$$

Now, $\int_{(r-1)\pi}^{r\pi} \frac{dx}{1+x^4\sin^2 x}$.

Let $x = (r-1)\pi + y$
then $dx = dy$.
$\therefore$ Above integral reduces to

$$\int_0^{\pi} \frac{dy}{1+\left[(r-1)\pi+y\right]^4 \sin^2\left[(r-1)\pi+y\right]}$$

$$= \int_0^{\pi} \frac{dy}{1+\left\{(r-1)\pi+y\right\}^4 \sin^2 y}$$

$$< \int_0^{\pi} \frac{dy}{1+\left\{(r-1)\pi\right\}^4 \sin^2 y}$$

or, $$2\int_0^{\pi/2} \frac{\operatorname{cosec}^2 y\,dy}{\operatorname{cosec}^2 + (r-1)^4\pi^4}$$

$$= 2\int_0^{\pi/2} \frac{\operatorname{cosec}^2 y\,dy}{1+(r-1)^4\pi^4+\cot^2 y}$$

$$= 2.\frac{1}{\sqrt{1+(r-1)^4\pi^4}}\cot^{-1}\frac{\cot y}{\sqrt{1+(r-1)^4\pi^4}}\Bigg|_0^{\pi/2}$$

$$= \frac{2}{\sqrt{1+(r-1)^4\pi^4}}\cdot\frac{\pi}{2}$$

i.e. $$\int_{(r-1)\pi}^{r\pi} \frac{dx}{1+x^4\sin^2\alpha} < \frac{\pi}{\sqrt{1+(r-1)^4\pi^4}}$$

$$\sim \frac{\pi}{(r-1)^2\pi^2} \sim \frac{1}{r^2\pi^2}$$

i.e. $$\sum_{r=1}^{n}\int \frac{dx}{1+x^4\sin^2\alpha} < \sum_{r=1}^{n}\frac{1}{\pi^2 r^2}$$

$$\therefore \operatorname*{Lt}_{n\to\infty}\int_0^{n\pi} \frac{dx}{1+x^4\sin^2\alpha} < \sum\frac{1}{r^2}$$

which is convergent

Hence, $\int_0^{\infty} \frac{dx}{1+x^4\sin^2 x}$ is convergent.

(b) Define a largrangian function $F(x, y, z, \lambda)$ $= xyz + \lambda\,(x + y + z - a)$.
Then for extremum value

$$dF = 0$$

$\Rightarrow yzdx + xzdy + xydz + \lambda(dx + dy + dz) = 0$
$\Rightarrow (yz + \lambda)\,dx + (xz + \lambda)dy + (xy + \lambda)\,dz = 0$
Equating the co-efficient we get,
$yz + \lambda = 0$; $xz + \lambda = 0$; $xy + \lambda = 0$
or, $yz + \lambda - xz - \lambda = 0 \Rightarrow z(x - y) = 0$
$\Rightarrow z = 0$ or, $x = y$
However, $z = 0 \Rightarrow \lambda = 0$
which further led to

$$x = y = 0$$

Hence, $x - y$ is the acceptable solution
Similarly from $xz + \lambda = 0$ and $xy + \lambda = 0$
we get $y = z$
ie. $x = y = z$ is the condition for extremum of Lagrangian function.

Also, $x + y + z = 0 \Rightarrow 3x = a$ or $x = a/3$

$\Rightarrow \quad y = z = a/3$

Hence, the extremum value of $xyz = \dfrac{a^3}{27}$

(c) $$f_{xy}(0,0) = \underset{k\to 0}{Lt}\,\frac{f_x(0,k)-f_x(0,0)}{k}$$

& $$f_{yx}(0,0) = \underset{h\to 0}{Lt}\,\frac{f_y(h,0)-f_y(0,0)}{h}$$

Now. $$f_x(0,k) = \underset{h\to 0}{Lt}\,\frac{f(h,k)-f(h,0)}{h}$$

$$= \underset{h\to 0}{Lt}\,\frac{\dfrac{hk\left(h^2-k^2\right)}{h^2+k^2}-0}{h}$$

$$\Rightarrow \quad f_x(0,k) = \underset{h\to 0}{Lt}\,\frac{k\left(h^2-k^2\right)}{h^2+k^2} = -k.$$

$$f_x(0,0) = \underset{h\to 0}{Lt}\,\frac{f(h,0)-f(0,0)}{h}$$

$$= \underset{h\to 0}{Lt}\,\frac{0}{h} = 0$$

$$\Rightarrow \quad f_{xy}(0,0) = \underset{k\to 0}{Lt}\,\frac{f_x(0,k)-f_x(0,0)}{k}$$

$$= \underset{k\to 0}{Lt}\,-\frac{k-0}{k} = -1.$$

Also, $$f_y(h,0) = \underset{k\to 0}{Lt}\,\frac{f(h,k)-f(h,0)}{k}$$

$$= \underset{k\to 0}{Lt}\,\frac{h\dfrac{k\left(h^2-k^2\right)}{h^2+k^2}-0}{k} = h.$$

$$f_y(0,0) = \underset{k\to 0}{Lt}\,\frac{f_y(h,0)-f_y(0,0)}{k}$$

$$= \underset{k\to 0}{Lt}\,\frac{0-0}{k} = 0.$$

$$\therefore \quad f_{yx}(0,0) = \underset{h\to 0}{Lt}\,\frac{f_y(h,0)-f_y(0,0)}{h}$$

$$= \underset{h\to 0}{Lt}\,\frac{h-0}{h} = 1$$

i.e. $f_{yx}(0,0) = 1$ also $f_{xy}(0,0) = -1$

Hence, $f_{yx}(0,0) \neq f_{xy}(0,0)$

further $f_x(0,0) = 0 = f_y(0,0)$

Also, when $x^2 + y^2 \neq 0$

then $$|f_x| = \frac{\left|x^4y+4x^2y^3-y^5\right|}{\left(x^2+y^2\right)^2} \leq \frac{6\left(x^2+y^2\right)^{5/2}}{\left(x^2+y^2\right)^2}$$

$$= 6\left(x^2+y^2\right)^{1/2}$$

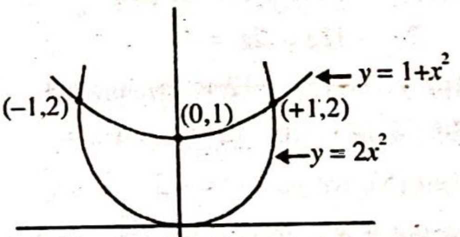
Evidently

$$\lim_{(x,y)\to(0,0)} f_x(x,y) = f_x(0,0)$$

Thus, f_x is continuous at (0,0) and $f_y(0,0)$ exists

$\Rightarrow f$ is differentiable at (0,0).

(d) We have to calculate



$$\iint (x+2y)\,dA = \int_{y=0}^{1}\int_{-1}^{1}(x+2y)\,dx\,dy$$

$$= \int_{y=0}^{1}\left.\frac{x^2}{2}+2xy\right|_{-1}^{1} dy$$

$$= 4\int_{y=0}^{1} y\,dy = 4\times\frac{1}{2} = 2 \text{ units}$$

4.(a) The given equation is

$f(x, y, z) = x^2 - 2y^2 + 3z^2 + 5yz - 6zx - 4xy + 8x - 19y - 2z - 20 = 0.$

Making homogeneous with the help of new variable t, to calculate the vertex of cone.

i.e. $F(x, y, z, t) = x^2 - 2y^2 + 3z^2 + 5yz - 6zx - 4xy + 8xt - 19xt - 2zt - 20t^2 = 0$

Now, differentiating partially with respect to x, y, z, t and then putting $t = 1$, we get,

$F_x = 2x - 6z - 4y + 8 = 0$

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or, $x - 2y - 3z + 4 = 0$ ... (1)

$F_y = -4y + 5z - 4x - 19 = 0$

or, $4x + 4y - 5z + 19 = 0$ ... (2)

$F_z = 6z - 6x + 5y - 2 = 0$

or, $6x - 5y - 6z + 2 = 0$ ... (3)

$F_t = 8x - 19y - 2z - 40 = 0$

or, $8x - 19y - 2z - 40 = 0$ ... (4)

Now, if $f(x, y, z) = 0$ represent a cone the value of x, y, z obtained from solving (1), (2) and (3) should satisfy (4) and that value represent the vertex of the cone.

Apply (2) – 4 × (1), we get,

$12y + 7z + 3 = 0$ ... (5)

Apply (3) – 6 × (1) we get,

$7y + 12z - 2z = 0$ ... (6)

Apply 7 × (5) – 12 × (6), we get,

$-95z + 285 = 0 \Rightarrow z = 3$

from (5), we get, $y = -2$.

putting y & z in (1), we get $x = 1$

i.e. $(x,y,z) = (1, -2,3)$

Now putting this in (4) we get,

$8 + 38 - 6 - 40 = 0$

Hence, the given second degree equation represent a cone with vertex $(1, -2,3)$.

(b) Let the co-ordinates of the given point be (x_1,y_1,z_1). Now the co-ordinates (α,β,γ) of the feet of six normals from (x_1,y_1,z_1) to given ellipsoid then

$$\alpha = \frac{a^2x_1}{a^2+\lambda},\ \beta = \frac{b^2y_1}{b^2+\lambda},\ \gamma = \frac{c^2z_1}{c^2+\lambda}$$

(standard result)

where λ is a parameter

Now, (α,β,γ) lie on ellipsoid.

$$\Rightarrow \frac{a^2x_1^2}{(a^2+\lambda)^2}+\frac{b^2y_1^2}{(b^2+\lambda)^2}+\frac{c^2z_1^2}{(c^2+\lambda)^2}=1 \quad ...(1)$$

Which gives six values of λ.

Now, if three of six lie on plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

then, $$\frac{ax_1}{a^2+\lambda}+\frac{by_1}{b^2+\lambda}+\frac{cz_1}{c^2+\lambda}-1=0 \quad ...(2)$$

(satisfied by three value of λ)

Let the other three feet lie on

$$\frac{x}{a'}+\frac{y}{b'}+\frac{z}{c'}-p'=0$$

then $$\frac{a^2x_1}{a'(a'+\lambda)}+\frac{b^2y_1}{b'(b^2+\lambda)}+\frac{c^2z_1}{c'(c^2+\lambda)}-p'=0$$

...(3)

(2) and (3) is combined form represent a conic passing through the feet of six normals. Which is represented by equation (1) also, comparing coefficient we get

$$\frac{a^3}{a'(a^2+\lambda)^2} = \frac{a^2}{(a^2+\lambda)^2}$$

$$\Rightarrow \quad \frac{1}{a'} = \frac{1}{a}$$

similarly $b' = b$ & $c' = c$

and $p' = -1$

$\Rightarrow$ the equation of other plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}+1=0$.

(c) Let $\frac{x}{l}=\frac{y}{m}=\frac{z}{h}$ represent one of other two generator as this is perpendicular to given generator $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$.

Hence, $l + 2m + 3n = 0$ ...(1)

Also $5mn - 8ln - 3lm = 0$

or, $5mn - l(3m + 8n) = 0$

or, $5mn + (2m + 3n)(3m + 8n) = 0$ <using (1)>

or, $6m^2 + 30mn + 24n^2 = 0$

or, $m^2 + 5mn + 4n^2 = 0$

or, $m^2 + mn + 4mn + 4n^2 = 0$

or, $m(m + n) + 4n(m + n) = 0$

or, $(m + n)(m + 4n) = 0$

$m + n = 0 \Rightarrow \frac{m}{1}=\frac{n}{-1}$

then, $l + 2 - 3 = 0 \Rightarrow l = 1$

i.e. $\frac{x}{1} = \frac{y}{1} = \frac{z}{-1}$ represent one generator

if $m + 4n = 0$, then, $\frac{m}{-4} = \frac{n}{1}$

then, $l - 8 + 3 = 0 \Rightarrow l = 5$

$\Rightarrow \frac{x}{5} = \frac{y}{-4} = \frac{z}{1}$ represent other generator.

Hence, the equation of two other generators are

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{-1} \;\&\; \frac{x}{5} = \frac{y}{-4} = \frac{z}{1}$$

(d) Let (x_1, y_1, z_1) (x_2, y_2, z_2) & (x_3, y_3, z_3) be the end point of conjugate diametrical planes of ellipsoid $\frac{x^2}{\alpha^1} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = 1$

then equation of plane parallel to these conjugate diameterical planes are given by,

$$\frac{xx_1}{\alpha^2} + \frac{yy_1}{\beta^2} + \frac{zz_1}{\gamma^2} = d_1; \quad \frac{xx_2}{\alpha^2} + \frac{yy_2}{\beta^2} + \frac{zz_2}{\gamma^2} = d_2$$

and $\frac{xx_3}{\alpha^2} + \frac{yy_3}{\beta^2} + \frac{zz_3}{\gamma^2} = d_3.$

Now, three planes are tangent planes to $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

then, by the properties of tangent planes.

$$\left.\begin{aligned} \frac{a^2x_1^2}{\alpha^4} + \frac{b^2y_1^2}{\beta^4} + \frac{c^2z_1^2}{\gamma^4} &= d_1^2 \\ \frac{a^2x_2^2}{\alpha^4} + \frac{b^2y_2^2}{\beta^4} + \frac{c^2z_2^2}{\gamma^4} &= d_2^2 \\ \frac{a^2x_3^2}{\alpha^4} + \frac{b^2y_3^2}{\beta^4} + \frac{c^2z_3^2}{\gamma^4} &= d_3^2 \end{aligned}\right| \begin{aligned} &\text{Adding,} \\ &\frac{a^2}{\alpha^4}(x_1^2 + x_2^2 + x_3^2) \\ &+ \frac{b^2}{\beta^4}(y_1^2 + y_2^2 + y_3^2) \\ &+ \frac{c^2}{\gamma^4}(z_1^2 + z_2^2 + z_3^2) = d_1^2 + d_2^2 + d_3^2 \end{aligned}$$

or, $\frac{a^2}{\alpha^4}\alpha^2 + \frac{b^2}{\beta^4}\beta^2 + \frac{c^2}{\gamma^4}\gamma^2 = d_1^2 + d_2^2 + d_3^2$

(By properties of conjugate diametrical planes)

$$\Rightarrow \frac{a^2}{\alpha^2} + \frac{b^2}{\beta^2} + \frac{c^2}{\gamma^2} = d_1^2 + d_2^2 + d_3^2$$

Also, $\left(\frac{xx_1}{\alpha^2} + \frac{yy_1}{\beta^2} + \frac{zz_1}{\gamma^2}\right)^2 + \left(\frac{xx_2}{\alpha^2} + \frac{yy_2}{\beta^2} + \frac{zz_2}{\gamma^2}\right)^2$

$$+\left(\frac{xx_3}{\alpha^2} + \frac{yy_3}{\beta^2} + \frac{zz_3}{\gamma^2}\right)^2 = d_1^2 + d_2^2 + d_3^2$$

$$\Rightarrow \frac{x^2}{\alpha^4}\Sigma x_1^2 + \frac{y^2}{\beta^4}\Sigma y_1^2 + \frac{z^2}{\gamma^4}\Sigma z_1^2 = d_1^2 + d_2^2 + d_3^2$$

(other term of equation vanishes)

$$\Rightarrow \frac{x^2}{\alpha^4}\alpha^2 + \frac{y^2}{\beta^4}\beta^2 + \frac{z^2}{\gamma^4}\gamma^2 = \frac{a^2}{\alpha^2} + \frac{b^2}{\beta^2} + \frac{c^2}{\gamma^2}$$

$$\Rightarrow \frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = \frac{a^2}{\alpha^2} + \frac{b^2}{\beta^2} + \frac{c^2}{\gamma^2}$$

Which is the locus of the point of intersection of tanqent planes of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} + 1.$

Section-B

5.(a): The given differential equation is

$y dx + [y + \tan(x + y)]\, dy = 0$ ... (1)

Now, if $\cos(x + y)$ is an I.F. of the above equation, then it should reduce it into exact form.

i.e. $y\cos(x + y)\, dx + [y\cos(x + y) + \sin(x + y)]\, dy = 0$

Now, if it is exact then

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

where, $M = y\cos(x + y)$
$N = y\cos(x + y) + \sin(x + y)$

$$\frac{\partial M}{\partial y} = \cos(x+y) - y\sin(x+y)$$

& $\frac{\partial N}{\partial x} = -y\sin(x+y) + \cos(x+y)$

ie. (1) becomes exact after multiplication by $\cos(x + y)$

Hence, solution of the equation is given by

$$\int y\cos(x+y)dx + \int\{y\cos(x+y) + \sin(x+y)\}dy$$

$y \sin(x+y) + 0 = c$

as there is no term independent of x is contained in second integral.

(b) For complementary function, the auxiliary equation is given by $m^2 - 2m + 1 = 0$

$\Rightarrow \quad m = 1, 1$

Hence, complementary function $y = (c_1 + c_2 x)e^x$

where, c_1, c_2 are arbitrary constant.

Now, the particular integral is given by,

$$y = \frac{1}{(D-1)^2} xe^x \sin x$$

$$= e^x . \frac{1}{(D+1-1)^2} x \sin x \qquad \text{[using property]}$$

$$= e^x \frac{1}{D^2} x \sin x = e^x \frac{1}{D} \int x \sin x dx$$

$$= e^x \frac{1}{D} [-x \cos x + \sin x]$$

$$= e^x \left[\int (\sin x - x \cos x) dx \right]$$

$$= e^x [-\cos x - \{x \sin x + \cos x\}]$$

$$= -xe^x \sin x - 2 \cos x$$

Hence, General solution is given by,

$$y = (c_1 + c_2 x)e^x - xe^x \sin x - 2 \cos x$$

(f) The unit normal vector at point (3,2,1) of the surface $x^2 + y^2 + z^2 = 14$ is given by

$$\frac{3\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{14}} = \hat{n} \text{ (say)}$$

Now, $\quad \bar{V} = xy^2\hat{i} + zy^2\hat{j} + xz^2\hat{k}$

then, $\quad \bar{V}^2 = (x^2y^4 + z^2y^4 + x^2z^4)$

then, $\quad \nabla\bar{V}^2 = (2xy^4 + 2xz^4)\hat{i} + (4x^2y^3 + 4y^3z^2)\hat{j} + (2y^4z + 4x^2z^3)\hat{k}$

Hence, required directional derivative at point (2,0,3) is given by $\nabla\bar{V}^2\hat{n}\Big|_{(2,0,3)}$

$$= \left[(2xy^4 + 2xz^4)\hat{i} + (4x^2y^3 + 4y^3z^2)\hat{j} + (2y^4z + 4x^2z^3)\hat{k} \right] . \left[\frac{3\hat{i} + 2\hat{j} + \hat{k}}{\sqrt{14}} \right]_{(2,0,3)}$$

$$= \frac{81 \times 4 \times 3 + 16 \times 27 \times 4}{\sqrt{14}} = \frac{2700}{\sqrt{14}}$$

6.(a) Let, $z = x - y + 6$; then, $\frac{dz}{dx} = 1 - \frac{dy}{dx}$

or, $\quad \frac{dy}{dx} = 1 - \frac{dz}{dx}$

$$1 - \frac{dz}{dx} = \sin^2 z \quad \text{or,} \quad \frac{dz}{dx} = \cos^2 z$$

or, $\quad \sec^2 z \, dz = dx$

integrating, we get,

$$\tan z = x + c$$

or, $\tan(x - y + 6) = x + c$

where c = arbitrary constant.

(b) The above equation is solved by reducing it to normal form.

i.e. (removal of 1st derivative).

Let, $y = uv$ be the solution of above equation then.

The above equation can be reduced to

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx} \right) \frac{dv}{dx} + \left(\frac{d^2u}{dx^2} + 2x\frac{du}{dx} + u \right) v = 0 \qquad \text{...(1)}$$

Now, to remove 1st derivative, we should equate

$$P + \frac{2}{u}\frac{du}{dx} = 0 \quad \text{or,} \quad \frac{du}{u} + x dx = 0$$

then, (1) is reduced to

$$\frac{d^2v}{dx^2} + \text{I}v = 0$$

where, $\quad \text{I} = \text{Q} - \frac{1}{4}\text{P}^2 - \frac{1}{2}\frac{dp}{dx}$

here, $Q = (x^2 + 1),\ \ P = 2x$

$\therefore$ $I = (x^2 + 1) - x^2 - 1 = 0$

or. $\frac{d^2v}{dx^2} = 0 \Rightarrow v = (c_1 + c_2x)$

where, c_1 and c_2 are arbitrary constant

Hence. $y = (c_1 + c_2x)\,e^{-x^2/2}$

$$y = c_1e^{-x^2/2} + c_2xe^{-x^2/2}$$

is the general solution of the given equation.

(c) The complementary function is given by $y = (c_1 + c_2x)e^x + (c_3 + c_4x)\sin x + (c_5 + c_6x)\cos x$.

The particular integral is given by.

$$y = \frac{1}{(D-1)^2(D^2+1)^2}(x+e^x)$$

$$= \frac{1}{(1-D)^2(1+D^2)^2}x + \frac{1}{(D-1)^2(D^2+1)^2}e^x$$

$$= [1+2D+3D^2+\cdots]\,(1-2D^2+3D^4-\ldots\ldots)x + \frac{x^2e^x}{2.4}$$

$$= (x+2) + \frac{x^2e^x}{8}$$

Hence, the general solution is given by

$$y = (c_1 + c_2x)\,e^x + (c_3 + c_4x)\sin x + (c_5 + c_6x)\cos x + (x + 2) + \frac{x^2e^x}{8}$$

(d) The above equation can be written as

$$y_2 - \frac{2x}{x^2-1}y_1 + \frac{2}{x^2-1}y = (x^2-1) \quad \ldots (1)$$

Clearly, x and $x^2 + 1$ is solution of reduced differential equation (*i.e.* making right hand side to zero).

Let, $y = Ax + B(x^2 + 1)$ be the solution of (1) where A and B are function of x.

put a condition $A_1x + B_1(x^2 + 1) = 0$

Now, $y = Ax + B(x^2 + 1)$

$$y_1 = A + 2\,Bx$$

$$y_2 = A_1 + 2B_1x + 2B$$

Putting y, y_1 & y_2 in equation (1) we get,

$$A_1 + 2B_1x + 2B - \frac{2x}{x^2-1}(A+2Bx) + \frac{2}{x^2-1}\left[Ax+B(x^2+1)\right] = (x^2-1)$$

or,, $A_1 + 2B_1\,x = x^2 - 1$

also, $A_1x + B_1(x^2 + 1) = 0$

or, $B_1(2x^2 - x^2 - 1) = x(x^2 - 1)$

or, $B_1 = x \Rightarrow B = \frac{x^2}{2} + c_1$

also, $A_1 + 2x^2 = x^2 - 1$

$\Rightarrow$ $A_1 = -(x^2 + 1)$

$\therefore$ $A = -\frac{x^3}{3} - x + c_2$

$\therefore$ $y = Ax + B(x^2 + 1)$

$$= \left(c_2 - x - \frac{x^3}{3}\right)x + \left(\frac{x^2}{2} + c_1\right)(x^2+1)$$

$$= c_1(x^2+1) + c_2x - x^2 - \frac{x^4}{3} + \frac{x^4}{2} + \frac{x^2}{2}$$

$$= c_1(x^2+1) + c_2x - \frac{x^2}{2} + \frac{x^4}{6}$$

i.e. the general solution is

$$y = c_1(x^2+1) + c_2x - \frac{x^2}{2} + \frac{x^4}{6}$$

7.(a) Let a chain of length 21 and weight w suspended from two points A and at the same level hang freely under gravity in form of ANB. (in catenary form).

Now when a weight P is attached at middle point of string it'll descend downwards to C, & two portions AC & BC of string each of length $\frac{1}{2}(21)$ will be parts of two equal catenaries.

(b) If the particle describes a hyperbola under the central acceleration $\frac{\mu}{(\text{distance})^2}$, then the velocities V of the particle at distance r from centre of force is given by,

$$V^2 = \mu\left(\frac{2}{r}+\frac{1}{a}\right) \quad \text{... (1)}$$

where $2a$ = transverse axis

As particle is projected with velocity V at distance R, then from (1), we have,

$$V^2 = \mu\left(\frac{2}{R}+\frac{1}{a}\right) \quad \text{or} \quad \frac{\mu}{a} = V^2 - \frac{2\mu}{R} \quad \text{...(2)}$$

If α is required angle of projection to describe a rectangular hyperbola, then at the point of projection from the relation $h = vp$,

we have $h = Vp = VR\sin\alpha$. ... (3)

[$\because p = r\sin\phi$ & initially $r = R$, $\phi = \alpha$]

Also, $h = \sqrt{\mu l} = \sqrt{\mu . b^2/a} = \sqrt{\mu a}$ ... (4)

[$b = a$ for rectangular hyperbola]

from (3) and (4) we have,

$$VR\sin\alpha = \sqrt{\mu a}$$

$$\Rightarrow \sin\alpha = \frac{\sqrt{\mu a}}{VR} = \frac{\mu\sqrt{a}}{VR\sqrt{\mu}} = \frac{\mu}{VR\sqrt{\mu a}}$$

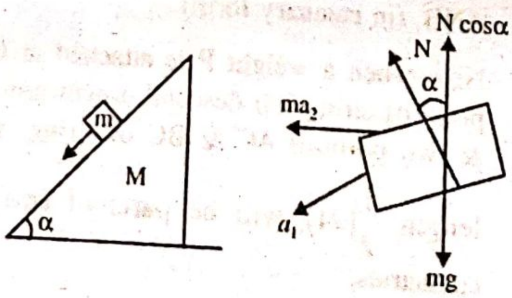
from (2)

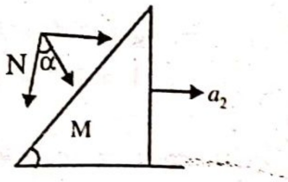
$$\Rightarrow \quad \sin\alpha = \frac{\mu}{VR\sqrt{V^2 - \frac{2\mu}{R}}}$$

$$\Rightarrow \quad \alpha = \sin^{-1}\left\{\frac{\mu}{VR\sqrt{V^2 - \frac{2\mu}{R}}}\right\}$$

which is required angle of projection.

(c) Drawing the free body diagram





Let a_1 and a_2 be the acceleration of m and M respectively.

Then from free body diagram.

$$mg - N\cos\alpha = ma_1\sin\alpha \quad \text{... (1)}$$

$$ma_2 + N\sin\alpha = ma_1\cos\alpha \quad \text{... (2)}$$

Also, $\quad N\sin\alpha = Ma_2$ ... (3)

(1) × cos α – (2) × sin α we get,

$$mg\cos\alpha - N - ma_2\sin\alpha = 0$$

putting N from (3), we get

$$\Rightarrow mg\cos\alpha - \frac{Ma_2}{\sin\alpha} - ma_2\sin\alpha = 0$$

$$\Rightarrow a_2(M + m\sin^2\alpha) = mg\sin\alpha\cos\alpha$$

$$\therefore a_2 = \frac{mg\sin\alpha\cos\alpha}{M + m\sin^2\alpha}$$

8.(a): Field F will be conservative then $\bar{\nabla}\times\bar{F} = 0$

$$i.e. \quad \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = 0$$

$$\text{Now } \bar{\nabla}\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy+z^3 & x^2 & 3z^2x \end{vmatrix}$$

$$= \hat{i}.0 - \hat{j}.(3z^2 - 3z^2) + \hat{k}(2x - 2x) = 0$$

i.e. $\bar{\nabla}\times\bar{F} = 0 \Rightarrow \bar{F}$ is conservative field.

Hence, $\bar{F}$ can be written as $\bar{F} = \nabla U$ where U is scalar function.

Now, $\frac{\partial U}{\partial x} = 2xy + z^3 \Rightarrow U = x^2y + xz^3 + f_1(y, z)$

$$\frac{\partial U}{\partial y} = x^2 \quad \Rightarrow U = x^2y + f_2(x, z)$$

$\frac{\partial U}{\partial z} = 3z^2x \quad \Rightarrow \quad U = xz^3 + f_3(x,y)$

from above three expression which represent same potential function, we get, $U = x^2y + xz^3$.

Now, work done in moving a particle from (1,–2,1) to (3,1,4).

$\Rightarrow U(3,1,4) - U(1,-2,1)$

$= 3.^2 1 + 3.4^3 - \{1(-2) + 1\}$

$= 202$ units.

(a)-ii. $\nabla^2 f(r) = \bar{\nabla}.(\nabla f(r))$

$$= \bar{\nabla}.\left(f'(r)\frac{\bar{r}}{r}\right) = \bar{\nabla}.\left(\frac{f'(r)}{r}\bar{r}\right)$$

$$= \left(\bar{\nabla}\frac{f'(r)}{r}\right).\bar{r} + \frac{f'(r)}{r}(\bar{\nabla}.\bar{r})$$

$$\left[\frac{f''(r)}{r}\frac{\bar{r}}{r} + f'(r)\left(-\frac{1}{r^2}\right)\frac{\bar{r}}{r}\right].\bar{r} + 3\frac{f'(r)}{r}$$

$$= \frac{f''(r)}{r^2}(\bar{r}.\bar{r}) - \frac{f'(r)}{r} + \frac{3f'(r)}{r}$$

$$= f''(r) + \frac{2f'(r)}{r}$$

i.e. $\nabla^2 f(r) = f''(r) + \frac{2f'(r)}{r}$

(b) By divergence theorem, we have

$$\iint_S F_1 dydz + F_2\, dzdx + F_3 dxdy$$

$$= \iiint\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)dxdydz$$

$$\Rightarrow \int\left(x^3dydz + x^2ydxdz + x^2zdxdy\right)$$

$$= \iiint\left\{\frac{\partial x^3}{\partial x} + \frac{\partial(x^2y)}{\partial y} + \frac{\partial(x^2z)}{\partial z}\right\}dxdydz$$

$$= \iiint_{x^2+y^2+z^2=1} 5x^2dxdydz$$

converting the above integral into polar form, we get,

$$\int_{r=0}^{1}\int_{\theta=0}^{\pi}\int_{\phi=0}^{2\pi}\left(5r^2\cos^2\theta\cos^2\phi\right)\left(r^2\sin\theta\, drd\theta d\phi\right)$$

$$= \int_{r=0}^{1} 5r^4dr\int\cos^2\theta\sin\theta\, d\theta\int_0^{2\pi}\cos^2\phi\, d\phi.$$

$$= 5.\frac{1}{5}.\frac{2}{3}.\frac{\pi}{2} = \frac{\pi}{3}$$

(c) A vector normal to the parabolic cylinder is given by.

$$\nabla\left(8x - y^2\right) = 8\bar{i} - 2y\hat{j}$$

$$\Rightarrow \hat{n} = \frac{8\hat{i} - 2y\hat{j}}{\sqrt{64 + 4y^2}} = \frac{4\bar{i} - y\bar{j}}{\sqrt{16 + y^2}}$$

$$\Rightarrow \iint_S \bar{A}.\hat{n}\, d\bar{S} = \iint\left(2y\bar{i} - z\bar{j} + x^2\hat{k}\right).\frac{(4\bar{i} - y\bar{j})}{\sqrt{16 + y^2}}.\frac{dydz}{|\hat{i}.\hat{n}|}$$

$$= \iint\left(2y\bar{i} - z\hat{j} + x^2\hat{k}\right).\frac{(4\bar{i} - y\hat{j})}{\sqrt{16 + y^2}}.\frac{dydz}{\frac{4}{\sqrt{16 + y^2}}}$$

$$= \frac{1}{4}\int(8 + z)y\, dydz = \frac{1}{4}\int_{Z=0}^{6}(8 + z)\frac{16}{2}dz$$

$$= \frac{1}{4}\int_0^6(64 + 8z)dz = \frac{1}{4}\left[64z + 4z^2\right]_{z=0}^{6}$$

$$= \frac{4}{4}\left[16z + z^2\right]_0^6 = 96 + 36 = 132 \text{ Units.}$$

(d) The green Theorem in a plane is defined as

$$\int Mdx + Ndy = \iint\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dxdy$$

$$\Rightarrow \int\left(2x^2 - y^2\right)dx + \left(x^2 + y^2\right)dy = \iint(2x + 2y)dxdy$$

$$\Rightarrow 2\int_{x=-1}^{1}\int_0^{\sqrt{1-x^2}}(x + y)dxdy = 2\int_{-1}^{1}\left[\left(x\sqrt{1 - x^2} + \frac{1 - x^2}{2}\right)\right]dx$$

$$= \frac{2\times 2}{2}\int_0^1\frac{1 - x^2}{0}dx \text{ [other integral vanishes]}$$

$$= 2\left(x - \frac{x^3}{3}\right)_0^1 = \frac{4}{3}.$$