and $\omega = |\vec{\omega}| = \sqrt{(\beta^2 + \kappa^2)} =$ constant.
This shows that the angular velocity is constant in magnitude equal to $\omega = \sqrt{(\beta^2 + \kappa^2)}$ and precesses round the k axis with frequency

$$f = \frac{\kappa}{2\pi} = \frac{|C - A|}{2\pi A} K$$

as shown in the adjoining diagram.

SUPPLEMENTARY PROBLEMS

1. Deduce Euler's Dynamical Equations
[Meerut 73, 77, 77(S), 80, 81(S), 83, 83(P), 84, 84(P), 85]

2. State Lagrange's equations and deduce Euler's equations from them, for a body turning about a fixed point.
[Meerut 79(S)]

3. State and prove Euler's Dynamical equations under the action of external forces.

4. A solid of revolution whose principal moments at the centre of inertia are A, A, C, (C > A) is set spinning with angular velocity ω about an axis passing through the centre of inertia and making an angle i with the figure. Prove that the instantaneous axis of rotation describes in body a right cone of semi-vertical angle i in period $\frac{2\pi A \sec i}{(C - A)\,\omega}$, and that the axis of the figure describes a right cone in space of semi-vertical angle θ where $\tan\theta\,(A/C)\tan i$, in the period $2\pi/\Omega$:
where $\Omega \sin\theta = \omega \sin i$ or $\Omega = (\omega/A)\sqrt{(A^2 \sin^2 i + C^2 \cos^2 i)}$.

5. A uniform circular disc free to turn about its centre is fixed, is set rotating with angular velocity ω about an axis which makes an angle 45° with the axis of the disc. Prove that in the subsequent motion the axis of the disc describes a right cone about an axis making an angle $\tan^{-1}\frac{1}{2}$ with the initial axis of rotation with constant velocity $\frac{1}{2}\omega^2\sqrt{(10)}$.

6. A uniform thin circular disc is set rotating with an angular velocity ω about an axis through the centre making an angle i with the normal. Prove that the semi-vertical angle θ of the cone described by the axis of the disc is given by $\tan\theta = (\frac{1}{2}) \tan i$.

If ω be the angular velocity, prove that the above cone is described in the period $2\pi/[\omega\sqrt{(1 + 3\cos^2 i)}]$.

7. An elliptic lamina acted on by an external force F is set rotating about a line through its centre which makes equal with the major and minor axes and with the normal to its plane. Prove that, if e be the eccentricity of the ellipse and $\omega_1, \omega_2, \omega_3$ the angular velocities about the major and minor axes and the lamina to the planes then.

$\omega_1 = F\cos\theta$, $\omega_2 = F\sin\theta$, $\omega_3 = \theta F\sqrt{\{(1 - e^2\sin^2\theta)/(2 - e^2)\}}$, where F is a constant.

☆☆☆

9
Hamiltonian Formulation and Variational Principles

9·00. Hamilton's form of the equations of motion.

Here we shall obtain the differential equations of motion of a conservative holonomic dynamical system in a form which constitutes the basis of most of the advanced theory of dynamics. Let $(q_1, q_2, \ldots, q_n)$ be the generalised co-ordinates and let $L(q_1, q_2, \ldots, q_n; \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n; t)$, the kinetic potential of the system, so that the equations of motion in the Lagrangian form are

$$\frac{d}{dt}(\partial L/\partial \dot{q}_i) - (\partial L/\partial q_i) = 0; \quad (i = 1, 2, \ldots, n) \qquad \ldots(1)$$

writting $p_i = (\partial L/\partial \dot{q}_i)$ we get $\dot{p}_i = (\partial L/\partial q_i)$ $(i = 1, 2, \ldots, n)$ ...(2)
hence from the former of these sets of equations we can regard either of the sets of quantities $(q_1, q_2, \ldots, a_n)$ or $(p_1, p_2, \ldots, p_n)$ as functions of the other set.

Now, let δ denote the increment in any function of the variables $(q_1, q_2, \ldots, q_n; p_1, p_2, \ldots, p_n)$ or $(q_1, q_2, \ldots, q_n; \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n)$; then we get

$$dL = \sum_{i=1}^{n}\left(\frac{\partial L}{\partial q_i} dq_i + \frac{\partial L}{\partial \dot{q}_i} d\dot{q}_i\right) + \frac{\partial L}{\partial t} \quad \text{(when } L \text{ contains } t \text{ explicitly)}$$

$$= \sum_{i=1}^{n} (\dot{p}_1\, dq_i + p_i\, d\dot{q}_i) + \frac{\partial L}{\partial t} dt$$

$$= d\left(\sum_{i=1}^{n} p_i \dot{q}_i\right) + \sum_{i=1}^{n} (\dot{p}_i\, dq_i - \dot{q}_i\, dp_i) + \frac{\partial L}{\partial t} dt$$

$$\Rightarrow d\left[\sum_{i=1}^{n} (p_i \dot{q}_i) - L\right] = \sum_{i=1}^{n} (\dot{q}_i\, dp_i - p_i\, dq_i) - (\partial L/\partial t)\, dt.$$

Thus if the quantity $\sum_{i=1}^{n} (p_i \dot{q}_i - L)$ when expressed in terms of $(q_1, q_2, \ldots, q_n; p_1, p_2, \ldots, p_n; t)$ be denoted by H, we have

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$$dH = \sum_{i=1}^{n} (\dot{q}_i \, dp_i - \dot{p}_i \, dq_i) - \frac{\partial L}{\partial t} dt$$

$$\Rightarrow \sum_{i=1}^{n} \frac{\partial H}{\partial q_i} dq_i + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i} dp_i + \frac{\partial H}{\partial t} dt = \sum_{i=1}^{n} (\dot{q}_i \, dp_i - \dot{p}_i \, dq_i) - \frac{\partial L}{\partial t} dt$$

$$\Rightarrow \dot{q}_i = \frac{\partial H}{\partial p_i}, \; \dot{p}_i = -\frac{\partial H}{\partial q_i}, \; \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

If H does not contain t explicity (i.e. does not contain t explicity) we have $\dot{p}_i = -(\partial H/\partial q_i)$ and $\dot{q}_i = (\partial H/\partial p_i)$ ...(4)

These equations are called as Hamilton's equations, or Hamilton's canoncial equations and the function H is called Hamiltonian.
The total order of Hamilton's equations is the same as the total order of Lagrange's equations, namely $2n$. But whereas Lagrange's equations present us with n equations each of the second order. Hamilton's equations are $2n$ equations, each of the first order. Hamilaton's equations can also be written as $\dfrac{dp_i}{-(\partial H/\partial q_i)} = \dfrac{dq_i}{(\partial H/\partial p_i)} = dt.$

9·01. Physical significance of the Hamiltonian. **[Meerut 1995]**
If the Hamiltonian H is independent of t explicity, prove that it is (a) *constant and* (b) *equal to the total energy of the system.*

Proof. (a) We have $\dfrac{dH}{dt} = \sum_{i=1}^{n} \dfrac{\partial H}{\partial q_i}\dfrac{dq_i}{dt} + \sum_{i=1}^{n} \dfrac{\partial H}{\partial p_i}\dfrac{dp_i}{dt}$

$$= \sum_{i=1}^{n} -(\dot{p}_i)\,\dot{q}_i + \sum_{i=1}^{n} \dot{q}_i\,\dot{p}_i \left(\because \; \dot{p}_i = -\frac{\partial H}{\partial q_i} \text{ and } \dot{q}_i = \frac{\partial H}{\partial p_i} \right) = 0$$

$\Rightarrow H =$ constant, say E.

(b) By Euler's theorem on homogeneous function, we have

$$\Sigma \dot{q}_i \frac{\partial T}{\partial \dot{q}_i} = 2T, \text{ where } T \text{ is the K.E. of the system.}$$

$$\text{But } L = T - V, \; \therefore \; \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial (T-V)}{\dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} \quad (\text{V does not depend on } \dot{q}_i)$$

$$\text{or } \Sigma \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} = 2T \Rightarrow \Sigma \dot{q}_i \, p_i = 2T \left(\because \; p_i = \frac{\partial L}{\partial \dot{q}_i} \right)$$

$$\therefore \; H = \Sigma p_i \dot{q}_i - L = 2T - (T - V) = T + V = E.$$

9·02. Passage from the Hamiltanian to the Lagrangian.
Suppose that we are given a function $^*H(q, p, t)$ and are told that the motion of the system satisfies the canoncial equations

$$\dot{p}_i = -(\partial H/\partial q_i) \text{ and } \dot{q}_i = (\partial H/\partial p_i) \quad ...(1)$$

Then we want to find a function

$L(p_1, p_2, \ldots, p_n; q_1, q_2, \ldots, q_n; t)$, i.e. $L(p, \dot{q}, t)$ such that the motion also satisfies the equations

$$(d/dt)(\partial L/\partial \dot{q}_i) - (\partial L/\partial q_i) = 0. \quad ...(2)$$

Solve the first set of equations in (1) for the p's in terms of the q's the $\dot{q}$'s and t.

Then write $L = \sum_{i=1}^{n} \dot{q}_i \, p_i - H$ and express L as a function of the q's, the $\dot{q}$'s and t. This is the required Lagrangian.

$$\therefore \; (\partial L/\partial \dot{q}_i) = p_i \left(\text{using } L = \sum_{i=1}^{n} p_i \dot{q}_i - H \right)$$

$$\Rightarrow (d/dt)(\partial L/\partial \dot{q}_i) = \dot{p}_i \text{ and } (\partial L/\partial q_i) = -(\partial H/\partial q_i)$$

$$\Rightarrow \frac{d}{dt}(dL/\partial \dot{q}_i) - (\partial L/\partial q_i) = \dot{p}_i + (\partial H/\partial q_i) = 0$$

i.e. L satisfies (2) assuming (1).

9·03. Variational Methods.

9·03·1. Techniques of Calculus of Variations. **(Meerut 1985, 90, 93)**
The calculus of variations arose out of the quest for the mathematical requirments in the solution of problems like the study of :
(i) The path followed by a body falling freely under gravity, (brachistochrone) first studied by Newton,
(ii) the equilibrium shape of a freely hanging homogeneous flexible cord between two horizontal points (catenary), first studied by Bernoulli.
Presently the technique of calculus of variations, serves as a mathamatical preliminary (in the form of Euler-Lagrange's equation) in the study of a wide range of physical problems such as geodesics and minimal surface in Riemannian and differential geometrics and the various *variational* (*minimal*) principles in the different branches of physics, The technique is co-ordinates invariant and there in lies its great power. First we develop here this technique in a purely mathematical form.
Suppose A and B are fixed points $(x_1, y_1), (x_2, y_2)$ in a cartesian plane. Also suppose that (x, y, y') is known functional form of the variables $x, y, y' \{= dy/dx\}$. Then if C is a curve joining A and B and having equation $y = y(x)$, Then the integral

$$I = \int_{x_1}^{x_2} f[x, y, y'] \, dx \quad ...(1)$$

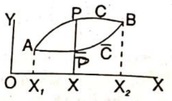
has a definite value whenever the function $y(x)$ is prescribed. The value of I will change as we vary the form of the curve C through A, B. Consequently we may consider that in general there will be some curve

$^*H(q_1, q_2, \ldots, q_n, p_1, \ldots, p_n, t)$ is also written as $H(q, p, t)$.

C through these fixed points such that the value of I *taken along it is stationary (in general a* maximum or minimum) compared with the value along neighbouring paths C. Calculus of variations, the branch of mathematics of concerned amongst other things with finding the form of $y(x)$ for which this stationary property holds. Let the path $\overline{C}$ have the equation $y = \overline{y}(x)$ and let the equation of a neighbouring curve C be $y = \overline{y}(x) + \varepsilon\eta(x)$, where ε is small and $\eta(x)$ is an arbitrary continuous differentiable function through A and B.
Now we have $PX = \overline{y}(x) + \varepsilon\eta(x)$, and $\overline{P}X = \overline{y}(x) \Rightarrow P\overline{P} = \varepsilon\eta(x)$.
Hence the values of I taken along is thus a function of ε of the form

$$I(\varepsilon) = \int_{x_1}^{x_2} f[x, \overline{y} + \varepsilon\eta, \overline{y}' + \varepsilon\eta']\,dx = \int_{x_1}^{x_2} f[x, y, y']\,dx \qquad ...(2)$$

where $y = \overline{y} + \varepsilon\eta$ and $y' = \overline{y}' + \varepsilon\eta'$
But the curve $\overline{C}$, for which $\varepsilon = 0$, makes I stationary, so we must have $I'(0) = 0$. Now differentiating (2) w.r.t. ε, we get



$$I'(\varepsilon) = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial x}\frac{\partial x}{\partial \varepsilon} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \varepsilon} + \frac{\partial f}{\partial y'}\frac{\partial y'}{\partial \varepsilon}\right] dx$$

$$\left[\because \quad f = f[x, y, y']\right] \qquad ...(3)$$

$$\Rightarrow I'(\varepsilon) = \int_{x_1}^{x_2} \left[\frac{\partial f}{\partial y}\cdot\eta(x) + \frac{\partial f}{\partial y'}\eta'(x)\right] dx$$

$$\left[\because \quad y = \overline{y} + \varepsilon\eta(x)\right]$$

$$= \int_{x_1}^{x_2} [f_y \cdot \eta(x) + f_{y'} \cdot \eta'(x)]\,dx$$

$$I'(\varepsilon) = \int_{x_1}^{x_2} [\eta f_y(x, \overline{y} + \varepsilon\eta, \overline{y}' + \varepsilon\eta')$$

$$+ \eta' f_{y'}(x, \overline{y} + \varepsilon\eta, \overline{y}' + \varepsilon\eta')]\,dx \qquad ...(4)$$

where $f_y(x, \overline{y} + \varepsilon\eta, \overline{y}' + \varepsilon\eta')$
$= (\partial/\partial y)\, f(x, y, y')$ when $y = \overline{y} + \varepsilon\eta, y' = \overline{y}' + \varepsilon\eta$,
with a similar meaning for $f_{y'}(x, \overline{y} + \varepsilon\eta, \overline{y}' + \varepsilon\eta')$. Hence the stationary condition gives $I'(0) = 0$.

$$\Rightarrow \int_{x_1}^{x_2} [\eta f_{y'}(x, \overline{y}, \overline{y}') + \eta' f_{y'}(x, \overline{y}, \overline{y}')]\,dx = 0 \qquad ...(5)$$

Now $\int_{x_1}^{x_2} \eta' f_{y'}(x, \overline{y}, \overline{y}')\,dx$

$$= \left[\eta f_{y'}(x, \overline{y}, \overline{y}')\right]_{x_1}^{x_2} - \int_{x_1}^{x_2} \eta \frac{d}{dx} f_{y'}(x, \overline{y}, \overline{y}')\,dx$$

$$= -\int_{x_1}^{x_2} \eta \frac{d}{dx} f_{y'}(x, \overline{y}, \overline{y}')\,dx \quad (\because \quad \eta(x_1) = \eta(x_2) = 0) \qquad ...(6)$$

Using (6), equation (5) implies

$$\int_{x_1}^{x_2} \eta(x) \left[f_y(x, \overline{y}, \overline{y}') - \frac{d}{dx} f_{y'}(x, \overline{y}, \overline{y}')\right] dx = 0 \qquad ...(7)$$

[using (5)]

But $\eta(x)$ is arbitrary, subject to its being differentiable and vanishing at A, B; (7) implies that

$$f_y(x, \overline{y}, \overline{y}') - (d/dx)\, f_{y'}(x, \overline{y}, \overline{y}') = 0 \qquad ...(8)$$

or

$$(\partial f/\partial y) - (d/dx)\,(\partial f/\partial y') = 0. \qquad ...(9)$$

This is called **Euler's Lagrange's equation.** Equation (9) can be proved from (7) by using the following lemma.

Lemma. *If* x_1 and $x_2 (> x_1)$ *are fixed constants and* $\phi(x)$ *is a particular continuous function for* $x_1 \leq \phi \leq x_2$ and *if*

$$\int_{x_1}^{x_2} \eta(x)\,\phi(x)\,dx = 0 \qquad ...(10)$$

for every choice of continuously differentiable function $\eta(x)$ *for which* $\eta(x_1) = \eta(x_2) = 0$ *then* $\phi(x) = 0$ *indentically in* $x_1 \leq x \leq x_2$.

Proof. Let the lemma be not true for all x *in* $x_1 \leq x \leq x_2$. Let x' be a point where $\phi(x') \neq 0$ and > 0. But $\phi(x)$ is continuous in $x_1 \leq x \leq x_2$ and in particular is continuous at $x = x'$. Hence there exists an interval $x_1' \leq x \leq x_0'$ around x' where $\phi(x) > 0$. For the other x's $\phi(x)$ may not vanish; then the equation $\int_{x_1}^{x_2} \eta(x)\,\phi(x)\,dx = 0$

could be written as $\int_{x_1'}^{x_2'} \eta(x)\,\phi(x)\,dx = 0$.

But $\eta(x)$ is at our disposal; we could choose it as greater than zero for $x_1' \leq x \leq x_2'$ and $\eta(x) = 0$ for other values of x. Thus $\eta(x)\,\phi(x) > 0$ for $x_1' \leq x \leq x_2'$; consequently integrand cannot be zero. Thus we arrive at the contradiction. Hence $\phi(x) = 0$ in $x_1 \leq x \leq x_2$.

Certain remarks about Euler-Lagrange's equation.

(A) We have $(\partial f/\partial y) - (d/dx)\,(\partial f/\partial y') = 0$
where $\quad f = f(x, y, y')$
which can be re-written as

$$(\partial f/\partial y) - (dp/dx) = 0 \text{ where } p = (\partial f/\partial y') \qquad ...(11)$$

But $\left(\frac{dp}{dx}\right) = \left(\frac{\partial p}{\partial x}\right) + \left(\frac{\partial p}{\partial y}\right)\left(\frac{dy}{dx}\right) + \left(\frac{\partial p}{\partial y'}\right)\left(\frac{dy'}{dx}\right) = \left(\frac{\partial^2 f}{\partial x \partial y'}\right) +$
$(\partial^2 f/\partial y\, \partial y')\,(dy/dx) + (\partial^2 f/\partial y'^2)\,(d^2y/dx^2)$. Hence (9) gives

$$(\partial f/\partial y) - [(\partial/\partial x)(\partial f/\partial y') + (\partial/\partial y)(\partial f/\partial y')(dy/dx) + (\partial/\partial y')(\partial f/\partial y')(dy'/dx)] = 0$$

or
$$(\partial f/\partial y) - [(\partial^2 f/\partial x/\partial y') + (\partial^2 f/\partial y\,\partial y')(dy/dx) + (\partial^2 f/\partial y'^2)(d^2y/dx^2)] = 0$$

or
$$(\partial^2 f/\partial y'^2)(d^2y/dx^2) + (\partial^2 f/\partial y\,\partial y')(dy/dx) - (\partial f/\partial y) + (\partial^2 f/\partial x\,\partial y') = 0 \qquad ...(12)$$

This is a second order differential equation for determining y as a function of x. The solution will contain two arbitrary constants which could be determined from the conditions :

$$y = y_1 \text{ at } x = x_1 \text{ and } y = y_2 \text{ at } x = x_2 \qquad ...(13)$$

(B) Consider $\frac{d}{dx}\left[f - y'\frac{\partial f}{\partial y'}\right] = \frac{\partial f}{\partial y'}$ ...(14)

i.e. $\frac{dQ}{dx} = \frac{\partial f}{\partial x}$ which $Q = f - y'\frac{\partial f}{\partial y'}$ and f is a function of x, y, y'

$$\Rightarrow \frac{\partial}{\partial x}\left[f - y'\frac{\partial f}{\partial y'}\right] + \frac{\partial}{\partial y}\left[f - y'\frac{\partial f}{\partial y'}\right]\frac{dy}{dx} + \frac{\partial}{\partial y'}\left[f - y'\frac{\partial f}{\partial y'}\right]\frac{d^2y}{dx^2} = \frac{\partial f}{\partial x}$$

$$\Rightarrow -y'\frac{\partial^2 f}{\partial x \partial y'} + \frac{\partial f}{\partial y}\frac{dy}{dx} - y'\frac{\partial^2 f}{\partial y \partial y'}\frac{dy}{dx} + \frac{\partial f}{\partial y'}\frac{d^2y}{dx^2} - \frac{\partial f}{\partial y'}\frac{\partial^2 y}{\partial x^2} - y'\frac{\partial^2 f}{\partial y'^2}\frac{d^2y}{dx^2} = 0$$

$$\Rightarrow y'\left[\frac{\partial f}{\partial y} - \frac{\partial^2 f}{\partial x \partial y'} - y'\frac{\partial^2 f}{\partial y \partial y'} - y''\frac{\partial^2 f}{\partial y'^2}\right] = 0$$

$$\Rightarrow f_y - f_{xy'} - y' f_{yy'} - y'' f_{y'y'} = 0 \text{ (if } y' \neq 0) \qquad ...(15)$$

which is same as (12) and shows that if Euler's equation is satisfied equation (14) is also satisfied. but if (14) is satisfied, Euler's equation may not be true.

(C) If f does not cotain x explicity, *i.e.* $I = \int_{x_1}^{x_2} f(y, y')\,dx$,

then from (14), we have **(Meerut 1985, 91)**

$$(d/dx)\{f - y'(\partial f/\partial y')\} = 0$$

$$\Rightarrow f - y'(\partial f/\partial y') = \text{constant} = C \text{ (say)}. \qquad \{\because (df/x) = 0\}$$

Obviously (14) is a first order differential equation. ...(16)

(D) If f does not contain y explicity, then we have $I = \int_{x_1}^{x_2} f(x, y')\,dx$.

$\therefore$ from the equation
$(\partial f/\partial y) - (d/dx)(\partial f/\partial y') = 0$, we obtain $(d/dx)(\partial f/\partial y') = 0$

$$\Rightarrow (\partial f/\partial y') = C \text{ (say)}.$$

(E) If we assume that the end points are not fixed. ...(17)

i.e. $\eta(x_1) \neq 0$ and $\eta(x_2) \neq 0$.

Euler's equation still holds in this case, if $(\partial f/\partial y') = 0$ when $x = x_1, x_2$

This becomes the boundary condition of the problem.

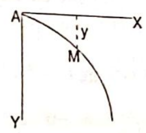
(F) Euler's equation gives only a necessary condition ; even if it is satisfied, there may be no extremum.

(G) The conditions to be imposed on f are such that its partial derivaties are continuous and differentiable.

(H) The conditions to be imposed on the curves are that they should be continuous curves.

9·03–2. Brachistochrone Problem.* **(Meerut 1994)**

Given two poits A and B in a vertical plane, to find for the moveable particle M, the path AMB, descending along which by its own gravity and beginning to be urged from the point A, it may in the shortest time reach the point B. It is a tacit in the statement that the particle decends without friction.



Let A be the origin, then velocity of the particle of mass M at any time t is given by

$$v = (ds/dt) = \sqrt{(2gy)}$$

$$\Rightarrow dt = (ds/\sqrt{(2gy)})$$

$$\Rightarrow t = \int_A^B \frac{ds}{\sqrt{(2gy)}} = \int_A^B \frac{\sqrt{(1+y'^2)}}{\sqrt{(2gy)}}\,dx$$

$$= \frac{1}{\sqrt{(2g)}}\int_A^B \frac{\sqrt{(1+y'^2)}}{\sqrt{y}}\,dx = \frac{1}{\sqrt{(2g)}}\int_A^B f(y, y')\,dx,$$

where $f(y, y') = \frac{\sqrt{(1+y'^2)}}{\sqrt{y}}$

As x is absent in f, so we have $f - y'(\partial f/\partial y') = A$

$$\Rightarrow \frac{\sqrt{(1+y'^2)}}{\sqrt{y}} - y'\frac{y'}{\sqrt{\{y(1+y'^2)\}}} = A$$

$$\Rightarrow \frac{1}{\sqrt{\{y(1+y'^2)\}}} = \text{constant} \Rightarrow y(1+y'^2) = \text{constant} = 2c, \text{ (say)}$$

But $(dy/dx) = \tan\psi \Rightarrow y' = \tan\psi$

$\therefore\ y\sec^2\psi = 2c$ or $y = 2c\cos^2\psi = c(1+\cos 2\psi)$

Also $dx = \cot\psi\, dy = -2c\sin 2\psi\cot\psi\, d\psi$

$$= -4c\cos^2\psi\, d\psi = -2c(1+\cos 2\psi)\,d\psi$$

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$$\Rightarrow x = a - 2c\left(\psi + \frac{\sin 2\psi}{2}\right) = a - 2c\,(\psi + \sin\psi\cos\psi)$$

Hence the required path is

$$x = a - c\,(2\psi + \sin 2\psi),\; y = c\,(1 + \cos 2\psi) \text{ (cycloid)}$$

9·03·3. Extension of the variational method.

Suppose the n-co-ordinates $q_1, q_2, \ldots, q_n$ are each functions of an independent variable t and that we require the solution of the variational problem $I = \int_{t_1}^{t_2} f(q_1, q_2, \ldots, q_n ; \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n ; t)\, dt$

$$= \int_{t_2}^{t_1} f\,[q, \dot{q}, t]\, dt = \text{stationary value}$$

where f is of known functional form, t_1 and t_2 are fixed and each q_i is to be determined.

Let $q_i = q_i(t)$ give the stationary value to I and let $q_i = q_i(t) + \varepsilon_i \eta_i(t)$ be the neighbouring path.

$$\Rightarrow I(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) = \int_{t_1}^{t_2} f\{q_i(t) + \varepsilon_i \eta_i(t) ; \dot{q}_i(t) + \varepsilon_i \eta_i(t), t\}\, dt$$

$$= \int_{t_1}^{t_2} f(q_1, \dot{q}_i ; t)\, dt + \Sigma\varepsilon_i \int_{t_1}^{t_2} \left\{\eta_i(t)\frac{\partial f}{\partial q_i} + \dot{\eta}_i(t)\frac{\partial f}{\partial \dot{q}_i}\right\} dt + 0(\varepsilon_i^2)$$

First variation $\delta I = \Sigma\varepsilon_i \int_{t_1}^{t_2} \left\{\eta_i(t)\frac{\partial f}{\partial q_i} + \dot{\eta}_i(t)\frac{\partial f}{\partial \dot{q}_i}\right\} dt$

$$= \Sigma\varepsilon_i \int_{t_1}^{t_2} \eta_i(t)\frac{\partial f}{\partial q_i}\, dt + \Sigma\varepsilon_i \int_{t_1}^{t_2} \dot{\eta}_i(t)\frac{\partial f}{\partial \dot{q}_i}\, dt$$

$$= \Sigma\varepsilon_i \int_{t_1}^{t_2} \eta_i(t)\frac{\partial f}{\partial q_i}\, dt + \Sigma\varepsilon_i \left[\left\{\eta_i(t)\frac{\partial f}{\partial \dot{q}_i}\right\}_{t_1}^{t_2} - \int_{t_1}^{t_2} \eta_i(t)\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{q}_i}\right) dt\right]$$

$$= \Sigma\varepsilon_i \int_{t_1}^{t_2} \eta_i(t)\left[\frac{\partial f}{\partial q_i} - \frac{d}{dt}\left(\frac{\partial f}{\partial \dot{q}_i}\right)\right] dt \quad \{\because\ 0 = \eta_i(t_1) = \eta_i(t_2)\}.$$

As η_i's are arbitrary and I is stationary, so we have

$$(\partial f/\partial q_i) - (d/dt)\,(\partial f/\partial \dot{q}_i) \qquad (i = 1, 2, \ldots, n)$$

9·03·4. Hamilton's variational principle. **[Meerut 1991, 94]**

(Proof based on calculus of variation)

Variational (mimimal) principles has been the greatest fascination of old generation physicists and in this pursuit several and varied variational

principles have been put forth in the varius branches of physics, such as, in optics, the Fermat's principles of *'lest time'*, in machanics, the Gauss's principles of *'least constraint'*, the Hertz's principle of *'lest curvature'* and most important of all the Hamilton's *variational principle'* which will be our main centre of attraction here in mechanics. The great value of these variational principles; lies in thier extreme economy of expression.

Let T, V be the kinetic and potential energies of a. Conservative holonomic dynamical system defined by n generalised co-ordinates $q_1, q_2, \ldots, q_n$ at time t. Writing $L = T - V$, we know that Lagrange's equations for the motion of the system are :

$$(\partial L/\partial q_i) - (d/dt)\,(\partial L/\partial \dot{q}_i) = 0 \qquad (i = 1, 2, \ldots, n)$$

Now applying previous article we can say that these are the n Euler-Lagrange's equation arising from the variational problem,

$$\int_{t_1}^{t_2} L\,(q_1, q_2, \ldots, q_n ; \dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n ; t)\, dt = \text{stationary (i.e. stat)}$$

where t_1, t_2 are fixed. Thus we have established that

During the motion of a 'conservative holonomic dynamical system over a fixed time interval, the time integral over that interval of the difference between the kinetic and potential energies is stationary. This is Hamilton's principle.

9·03·5. Derivation of Hamilton equations from the variational principle.

Hamilton's principle implies

$$\delta I = \delta \int_{t_1}^{t_2} L\, dt = 0 \text{ or } \delta \int_{t_1}^{t_2} [\Sigma_i\, p_i \cdot \dot{q}_i - H(q, p, t)]\, dt = 0$$

$$[\because\ L = \Sigma_i\, p_i\, \dot{q}_i - H(q, p, t)]$$

This implies $\delta \Sigma_i \int_{q_1}^{q_2} p_i\, dq_i - \delta \int_{t_1}^{t_2} H\, dt = 0.$ ...(1)

Equation (1) is called the modified Hamilton's principle.

Now $\delta I = \frac{\partial I}{\partial \alpha} d\alpha = d\alpha \frac{\partial}{\partial \alpha} \int_{t_1}^{t_2} (\Sigma_i\, p_i\, \dot{q}_i - H(q, p, t)\, dt = 0$

where $\delta \doteq d\alpha(\partial/\partial_\alpha)$

*This problem was set by the Johan Bernoulli June, 1696 before the scholars of his time. Although Newton had earlier considered at least one problem falling within, the province of the Calculus of variations, the proposal of Bernoulli brachistochrone problem marked the real beginning of general interest in the subject. Actually the term *Brachicstochrone* derives from the greek Brachistos, shortest and chronos, time

$$\Rightarrow d\alpha \int_{t_1}^{t_2} \frac{\partial}{\partial \alpha} [\sum_i p_i \cdot \dot{q}_i - H(q, p, t)]\, dt = 0$$

[$\because$ the times t_1, t_2 are not varied and so they are not functions of α, thus the differentiation can be interchanged].

$$\Rightarrow d\alpha \int_{t_1}^{t_2} \sum_i \left[\frac{\partial p_i}{\partial \alpha} \dot{q}_i + \frac{\partial \dot{q}_i}{\partial \alpha_i} p_i - \frac{\partial H}{\partial q_i} \frac{\partial q_i}{\partial \alpha} - \frac{\partial H}{\partial p_i} \frac{\partial p_i}{\partial \alpha} \right] dt = 0 \qquad \text{...(2)}$$

Further, we have $\int_{t_1}^{t_2} \frac{\partial \dot{q}_i}{\partial \alpha} p_i\, dt = \int_{t_1}^{t_2} p_i \frac{d}{dt}\left(\frac{\partial q_i}{\partial \alpha}\right) dt = p_i \left[\frac{\partial q_i}{\partial \alpha}\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \dot{p}_i \frac{\partial q_i}{\partial \alpha}\, dt.$

But all the varied paths have the same end points. Hence $\frac{\partial \dot{q}_i}{\partial \alpha}$ vanishes for t_1 and t_2, $\Rightarrow \int_{t_1}^{t_2} \frac{\partial q_i}{\partial \alpha} p_i\, dt = -\int_{t_1}^{t_2} \dot{p}_i \frac{\partial q_i}{\partial \alpha}\, dt = \int_{t_2}^{t_1} p_i \frac{\partial q_i}{\partial \alpha}\, dt.$

Making this substitution in equation (2), we have

$$d\alpha \int_{t_1}^{t_2} \sum \left[\frac{\partial p_i}{\partial \alpha} \dot{q}_i - \frac{\partial H}{\partial q_i} \frac{\partial q_i}{\partial \alpha} - \frac{\partial H}{\partial p_i} \frac{\partial p_i}{\partial \alpha} - \dot{p}_i \frac{\partial q_i}{\partial \alpha} \right] dt = 0 \qquad \text{...(2)}$$

$$\Rightarrow \int_{t_1}^{t_2} \sum \dot{q}_i \frac{\partial \dot{q}_i}{\partial \alpha} d\alpha - \frac{\partial H}{\partial q_i} \frac{\partial q_i}{\partial \alpha} d\alpha - \frac{\partial H}{\partial p_i} \frac{\partial p_i}{\partial \alpha} d\alpha - p_i \frac{\partial q_i}{\partial \alpha} d\alpha\, dt = 0 \qquad \text{...(3)}$$

But $\delta p_i = d\alpha\, (\partial p_i / \partial \alpha)$ and $\delta q_i = (\partial q_i / \partial \alpha)\, d\alpha$

$$\therefore (3) \Rightarrow \int_{t_1}^{t_2} \sum \left[\delta p_i \left(\dot{q}_i - \frac{\partial H}{\partial p_i} \right) + \delta q_i \left(\frac{\partial H}{\partial q_i} - \dot{p}_i \right) \right] dt = 0. \qquad \text{...(4)}$$

The variation δq_i and δp_i are independent to each other, hence equation (4) holds good only when the coefficients of δp_i and δq_i vanish separately.

This (4) $\Rightarrow \dot{q}_i = (\partial H / \partial p_i)$, $\dot{p}_i = -(\partial H / \partial q_i)$. These are Hamilton's equations.

9 03 6. Principal of Least Action.

Principle of least action states that if T is kinetic energy, at time t, of a conservative, holonomic dynamical system specified by the generalised co-ordinates, then the integral $I = \int_{t_1}^{t_2} 2T\, dt$

has necessary an extreme value, minimum or maximum, on actual path as compared with varied path as the systems passes from one configuration at time t_0 to another configuration at time t_1. **[Meerut 1992,93,94]**

We know that $L = T - V$, i.e. Lagrangian = K.E.– P.E. and $T + V = E$ (const), since system is conservative.
But by Hamilton's principle, we know that

$$\int_{t_0}^{t_1} \delta L\, dt = 0 \Rightarrow \int_{t_0}^{t_1} \delta(T - V)\, dt = 0 \quad \int_{t_0}^{t_1} \delta(2T - E)\, dt = 0$$

$$\Rightarrow \int_{t_0}^{t_2} \delta(2T)\, dt = 0 \quad \Rightarrow \delta \int_{t_0}^{t_1} (2T)\, dt = 0. \qquad \text{...(1)}$$

[$\because$ $\delta E = 0$ as E, the total energy is const.]

Result (1), is known as **Principle of least action.**

Equation (1) can also be written as $\delta A = 0$, where $A = \int_{t_0}^{t_1} 2T\, dt$ and is defined by action as follows :

This implies that principal of least action states that the action in the actual path is minimum compared with the varied path, as the system passes from one configuration to another.

9· 03– 7. Distinction between Hamilton's Principle and Principle of Least Action.

(Meerut 92,93,94)

In Hamilton's principle, the time of description $t_1 - t_0$ is prescribed (fixed) as the body moves from one configuration to another configuration, while in the principle of least action there is no such restriction on the time $t_1 - t_0$, but the total energy between the end points A and B is prescribed.

Deduction of Lagrange's Equation from Hamilton's Principle.

(Meerut 93)

By Hamilton's principle, we have $\int_{t_0}^{t_1} \delta L\, dt = 0$

$$\therefore (1) \Rightarrow \int_{t_0}^{t_1} \sum_{i=1}^{n} \left(\frac{\partial L}{\partial q_i} \delta q_i + \frac{\partial L}{\partial \dot{q}_i} \delta \dot{q}_i \right) dt = 0$$

$$\Rightarrow \int_{t_0}^{t_1} \sum_{i=1}^{n} \frac{\partial L}{\partial q_i} \delta q_i\, dt + \int_{t_0}^{t_1} \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \frac{d}{dt} (\delta q_i)\, dt = 0$$

Now integrating by parts, we have

$$\int_{t_0}^{t_1} \sum_{i=1}^{n} \frac{\partial L}{\partial q_i} \delta q_i\, dt + \left\{ \sum_{i=1}^{n} \frac{\partial L}{\partial \dot{q}_i} \delta q_i \right\}_{t_0}^{t_1} - \int_{t_0}^{t_1} \left\{ \sum_{i=1}^{n} \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) \delta q_i \right\} dt = 0$$

$$\Rightarrow \int_{t_0}^{t_1} \left[\sum_{i=1}^{n} \left\{ \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} \right\} \delta q_i \right] dt = 0$$

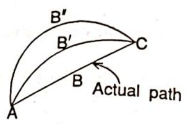
[$\because$ middle term vanishes as all $\delta q_i = 0$ at t_0 and t_1]

$$\Rightarrow \sum_{i=1}^{n} \left\{ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} \right\} \delta q_i = 0.$$

Here δq_i's are arbitrary and independent to each other, so equating to zero their coefficients, we readily obtain

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

9·03·8. Deduction of Hamilton's principle *(Proof based on D'Alembert's principle).* Assume that the conservative holonomic dyamical system moves from A to C, where A and C, are the initial and final configurations of the system at time t_1 and t_2 respectively. Let ABC be the actual path and $AB'C.AB''C$ the two neighbouring paths out of infinite number of possibilities. *In order to deduce the principle, the following two conditions must be satisfied.*



1. δt must be equal to zero at A and B *i.e.* at t_1 the particle must be at A, at t_2 the particle must be at C.
2. δr must be equal to zero at A and $C \Rightarrow$ that the points A, and C are fixed in space.

Now assume that the system is acted upon by a number of forces represented by **F**.

Let the ith particle of system be acted upon by a force $\mathbf{F}_i = m_i \ddot{r}_i$ where $\ddot{r}_i$ is acceleration vector.

Agian by D'Alembert's principle, we have

$$\sum_i (\mathbf{F}_i - m\ddot{\mathbf{r}}_i) \bullet \delta \mathbf{r}_i = 0, \text{ i.e. } \sum_i \mathbf{F}_i \bullet \delta \mathbf{r}_i - \sum m_i \ddot{\mathbf{r}}_i \bullet \delta \mathbf{r}_i = 0 \quad ...(1)$$

If there is a little variation along the actual and neighbouring paths, then $\delta \mathbf{r}_i = \mathbf{r}_i' - \mathbf{r}_i$ (say).

$$\Rightarrow (d/dt)(\delta \mathbf{r}_i) = (d/dt)(\mathbf{r}_i' - \mathbf{r}_i) = (d\mathbf{r}_i'/dt) - (d\mathbf{r}_i/dt)$$

$$= \delta(d\mathbf{r}_i/dt) = \delta(d\dot{\mathbf{r}}_i). \quad ...(2)$$

where dashes have been used for neighbouring paths.

$$\text{But } \ddot{\mathbf{r}}_i \bullet \delta_i = (d/dt)(\dot{\mathbf{r}}_i \bullet \delta \mathbf{r}_i) - \dot{\mathbf{r}}_i \bullet (d/dt)(\delta \mathbf{r}_i). \quad ...(3)$$

$$\text{Using (2), (3)} \Rightarrow \ddot{\mathbf{r}}_i \bullet \delta \mathbf{r}_i = (d/dt)(\dot{\mathbf{r}}_i \bullet \delta \mathbf{r}_i) - \dot{\mathbf{r}}_i\, \delta(\dot{\mathbf{r}}_i) \quad ...(4)$$

$$\therefore (1) \Rightarrow \sum_i \mathbf{F}_i \bullet \delta \mathbf{r}_i - \sum_i m_i [(d/dt)(\mathbf{r}_i \bullet \delta \mathbf{r}_i) - \dot{\mathbf{r}}_i \bullet \delta \dot{\mathbf{r}}_i] = 0$$

$$\text{or } \sum_i \mathbf{F}_i \bullet \delta \mathbf{r}_i - \sum_i m_i [(d/dt)(\dot{\mathbf{r}}_i \bullet \delta \mathbf{r}_i) - \tfrac{1}{2}\delta(\dot{\mathbf{r}}_i^2)] = 0$$

$$\text{or } \sum_i \mathbf{F}_i \bullet \delta \mathbf{r}_i + \sum_i \tfrac{1}{2} m_i\, \delta(\dot{\mathbf{r}}_i^2) = \sum_i (d/dt)(m_i \dot{\mathbf{r}}_i \bullet \delta \mathbf{r}_i)$$

$$\text{or } \sum_i \mathbf{F}_i \bullet \delta \mathbf{r}_i + \delta\left(\sum_i \tfrac{1}{2} m_i \dot{\mathbf{r}}_i^2\right) = \sum_i (d/dt)(m_i \dot{\mathbf{r}}_i \bullet \delta \mathbf{r}_i) \quad ...(5)$$

But $\sum_i \frac{1}{2} m_i \dot{\mathbf{r}}_i^2$ = kinetic energy of the system $= T$

and $\sum_i \mathbf{F}_i \bullet \delta \mathbf{r}_i$ = work done by the forces $\mathbf{F}_i$ during displacement $\delta \mathbf{r}_i = \delta W$ (say).

$$\therefore \text{ equation (5)} \Rightarrow \delta W + \delta T = \sum_i (d/dt)(m_i \dot{\mathbf{r}}_i \bullet \delta \mathbf{r}_i) \quad ...(6)$$

Intergrating (6) between the limits t_1 and t_2, we get

$$\int_{t_1}^{t_2} (\delta W + \delta T)\, dt = \int_{t_1}^{t_2} \sum_i \frac{d}{dt}(m_i \dot{\mathbf{r}}_i \bullet \delta \mathbf{r}_i)\, dt = \sum \int_{t_1}^{t_2} d(m_i \mathbf{r}_i \bullet \delta \mathbf{r}_i)$$

$$= \sum \left[m^2 \mathbf{r}_i^2 \bullet \delta \mathbf{r}^2 \right]_A^C = 0 \text{ [since } \delta \mathbf{r}_i^2 = 0 \text{ at the end points } A \text{ and } C.$$

But we know that, for a conservative system,

$$\delta W = -\delta V \text{ where } V \text{ is potential energy}$$

$$\Rightarrow \int_{t_1}^{t_2} (-\delta V + \delta T)\, dt = 0, \text{ i.e. } \delta \int_{t_1}^{t_2} (T - V)\, dt = 0$$

$$\text{i.e. } \delta \int_{t_1}^{t_2} (T - V)\, dt = 0 \text{ or } \delta \int_{t_1}^{t_2} L\, dt = 0$$

or $\int_{t_1}^{t_2} L\, dt$ = stat = extremum. This is Hamilton's principle.

Ex. 1. *Use Hamilton's principle to find the equation of motion of one-dimensional harmonic oscillator.*

Sol. The kinetic energy of harmonic oscillator is given by $T = m\dot{x}^2$ and the potential energy of the harmonic oscillator is given by $V = -\int F dx = \int kx\, dx = \frac{1}{2}kx^2$.

$\therefore$ The Lagrangian of the system $L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$.

Using Hamilton's Principle, we have $\delta \int_{t_1}^{t_2} L \, dt = 0.$

$$\Rightarrow \delta \int_{t_1}^{t_2} (\tfrac{1}{2} m\dot{x}^2 - \tfrac{1}{2} kx^2) \, dt = 0 \Rightarrow \int_{t_1}^{t_2} \delta (\tfrac{1}{2} m\dot{x}^2 - \tfrac{1}{2} kx^2) \, dt = 0$$

$$\Rightarrow \int_{t_1}^{t_2} (m\dot{x}\,\delta\dot{x} - kx\,\delta x) \, dt = 0 \qquad \text{But } \delta\dot{x} = (d/dt)(\delta x). \qquad ...(1)$$

$$\therefore (1) \Rightarrow \int_{t_1}^{t_2} m\dot{x} \frac{d}{dt}(\delta x) \, dt - \int_{t_1}^{t_2} kx\,\delta x \, dt = 0$$

$$\text{i.e., } \left[m\dot{x}\,\delta x \right]_{t_1}^{t_2} - \int_{t_1}^{t_2} m \frac{d}{dt}(\dot{x}) \, dt - \int_{t_1}^{t_2} kx\,\delta x \, dt = 0 \qquad ...(2)$$

But $\left[m\dot{x}\,\delta x \right]_{t_1}^{t_2} = 0$ [$\because$ $\delta x = 0$ at fixed points, *i.e.*, at instants t_1 and t_2]

$$\therefore (2) \Rightarrow -\int_{t_1}^{t_2} m \frac{d}{dt}(\dot{x})\,\delta x \, dt - \int_{t_1}^{t_2} kx\,\delta x \, dt = 0 \Rightarrow \int_{t_1}^{t_2} (m\ddot{x} + kx)\,\delta x \, dt = 0$$

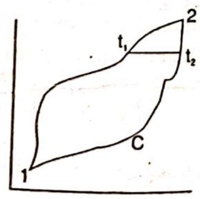
But δx is arbitrary, hence the above equation is only satisfied if $m\ddot{x} + kx = 0$. This is the equation of motion for one dimensional harmonic oscillator.

9·03·9. Extension of Hamiltion's principle to non-conservative and non-holonomic system.

We can generalise Hamilton's principle to include non-conservative forces as well, so that an alternative form of the Lagrange's equation can be obtained. the extension of the Hamilton's principle is

$$\delta I = \delta \int_1^2 (T + W) \, dt = 0 \quad ...(1); \quad \text{where} \quad W = \Sigma\, \mathbf{F}_i \bullet \mathbf{r}_i \qquad ...(2)$$

We know that the variations δq_i or $\delta \mathbf{r}_i$ are identical with virtual displacements of the co-ordinates as there is no variation of time. Thus we can consider the varied path in configuration space as built up by a succession of virtual displacements from the actual path of motion C. Again each virtual displacements takes place at some definite time and at that time the forces acting on the body have definite values. δW represents the amount of work done by the forces on the system during the period of virtual displacement from the actual to the varied path. Thus the Hamilton's principle given by (1) can be said as *the integral of the variation of kinetic energy togehter with the amount of virtual work involved in the variation must be zero.*



We can evaluate the variations $\delta \mathbf{r}_i$ in terms of δq_i by making use of the equations of transformation between r and q, where each q depends on the path chosen through a parameter say α :

$$\mathbf{r}_i = \mathbf{r} \,\{ q_k (\alpha, t) t \}$$

We can abbreviate the process by using the equivalance of $\delta \mathbf{r}_i$ with a virtual displacement. We know that

$$\sum_i \mathbf{F}_i \bullet \delta \mathbf{r}_i = \sum_k Q_k \, \delta q_k .$$

thus (1) may be put as $\delta \int_1^2 T \, dt + \int_1^2 \sum_k Q_k \, \delta q_k \, dt = 0.$ ...(3)

Now it can be shown easily that (3) reduces to ordinary form of Hamilton's principle, in case Q_k's are derivable form the generalised potential. Therefore the integral of virtual work, under these conditions becomes,

$$\int_1^2 \sum_k Q_k \, \delta q_k \, dt = -\int_1^2 \sum_k \delta q_k \left(\frac{\partial V}{\partial q_k} - \frac{d}{dt} \frac{\partial V}{\partial \dot{q}_k} \right) dt.$$

Now reversing by integration-paths procedure; the above integral can be put as $-\int_1^2 \sum_k \left(\frac{\partial V}{\partial q_k} \delta q_k + \frac{\partial V}{\partial \dot{q}_k} \delta \dot{q}_k \right) dt = -\delta \int_1^2 V \, dt$

Thus (3) reduces to

$$\delta \int_1^2 T \, dt = \int_1^2 V \, dt = \delta \int_1^2 (T - V) \, dt = \delta \int_1^2 L \, dt = 0,$$

which is Hamilton's principle.

For more general problem, the variation of the first integral in (3) can be written at once, as T like L for conservative systems is a function of q_k and $\dot{q}_k$

$$\therefore \delta \int_1^2 T \, dt = \int_1^2 \sum_b \left(\frac{\partial T}{\partial q_k} - \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_k} \right) \delta q_k \, dt \qquad ...(4)$$

$$\Rightarrow \int_1^2 \sum_k \left(\frac{\partial T}{\partial q_k} - \frac{d}{dt} \frac{\partial T}{\partial \dot{q}_k} + Q_k \right) \delta q_k \, dt = 0 \qquad ...(5)$$

Further it is assumed that the constraints are holonomic, so the integral (5) can vanish if and only if the seperate coefficients vanish, i.e.

$$\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_k} - \frac{\partial T}{\partial q_k} = Q_k$$

so that (1) represents the proper extension of Hamilton's principle, which yields that from of Lagrange's equation, in case the forces are not derived from a potential.

We can also extend the Hamilton's principle to cover certain cetogories of non-holonomic systems as well. While deriving Lagrange's equations from Hamilton's or D'Alemberts principle, application of holonomic constraints is made only in last step when the variation q_k are considered as independent of each other.

While in non-holonomic system the generalized co-ordinates are not independent to each other so they cannot be reduced further by means of equations of constrints of the form

$f\,[q_1, q_2, \ldots q_n\,;\,t] = 0$ Thus q_k's can not be as all independent.

We can further treat non-holonomic system, provided the equations of constriant can be puttin the form

$$\sum_j a_{ij}\, dq_j + a_{ij}\, dt = 0 \qquad [\,j = 1, 2, - \ldots m\,] \qquad \ldots(7)$$

which is a relation connecting the differentials of q's Since. in variation process used in Hamilton's principle, the time for each point on the path is taken constant, therefore the virtual displacements occurring in the variation must satisfy the equations of constraint of the form

$$\sum_j a_{ij}\, \delta q_j = 0,\ (j = 1, 2, \ldots m) \qquad \ldots(8)$$

The equations (8) can very well be used to reduce the number of virtual displacement to independent ones. The method used for eliminating these extra virtual displacements is that of Lagrange's *undetermined multipliers.*

Also from (8), we get

$$\lambda_l \sum_j a_{ij}\, \delta q_j = 0,\ (j = 1, 2, \ldots m) \qquad \ldots(9)$$

where λ_l are some undetermined constants, which in general, are functions of time.

Now first of all summing the equations (9) over l and then integrating the resulting equation from 1 to 2, we get

$$\int_1^2 \sum_{l\,j'i} \lambda_l a_{ij}\, \delta q_j\, dt = 0 \qquad \ldots(10)$$

Again the equations corresponding to (5) in the case of conservative system are given as

$$\int_1^2 dt \sum_j \left(\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} \right) \delta q_j = 0 \qquad \ldots(11)$$

Combining the equations (10) with (11), we get

$$\int_1^2 dt \sum_j^n \left(\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \sum_l \lambda_l a_{ij} \right) \delta q_j = 0 \qquad \ldots(12)$$

It is to be noted here that δq_j's are still not independent. They are connected by the m relations given by (8.) This is while the first $n-m$ of these equations may be chosen independently, the last m are then fixed by the equations (8). Since the values of λ_l's are at our disposal, we choose them to be such that

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \sum_l \lambda_l a_{ij} = 0,\ (j = n-m+1, \ldots n) \qquad \ldots(13)$$

which are in the nature of equations of motion for the last m of the q_j variable.

Using the value of λ_l obtained from (13) in (12), we get

$$\int_1 dt \sum_{j=1}^{n-m} \left(\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \sum_l \lambda_l a_{lj} \right) \delta q_j = 0 \qquad \ldots(14)$$

Since δq_j 's involved here are independent ones, therefore, we have

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} + \sum_l \lambda_l a_{lj} = 0\ (j = 1, 2, \ldots n-m) \qquad \ldots(15)$$

Now combining (13) with (15), we get

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j x} - \frac{\partial L}{\partial q_j} = \sum_l \lambda_l a_{lj},\ (j = 1, 2, \ldots n) \qquad \ldots(16)$$

which are known as *Lagrange's equations for non holonomic systems.*

We now observe that (16) gives us a total of only n equations, where as the unknowns involved are $(n+m)$ in number, namely then n co-ordinates q_j and the m, λ_l's. The additional equations required are exactly the equations of constraint i.e.

$$\sum_j a_{ij}\, \dot{q}_j + a_{ij} + 0 \qquad \ldots(17)$$

The equations (16) together with (17) constitute $(n+m)$ equations for $n+m$ unknowns.

It is to be noted here that equation (7) is not the most general type of non-holonomic constraint. For example, it does not include equations of constraints in the form of inequalities. More over it includes holonomic constraints. An equation of the form

$f\,(q_1, q_2, q_3 \ldots q_n\,;\,t) = 0$ is known as a holonomic equation of constraint.

This is equivalent to a differential equation

$$\sum_j \frac{\partial f}{\partial q_j}\, dq_j + \frac{\partial f}{\partial t}\, dt = 0 \qquad \ldots(18)$$

It is of the same form as the equation (7) with the coefficients

$$a_{lj} = \frac{\partial f}{\partial q_j} \text{ and } a_{ji} = \frac{\partial f}{\partial t} \qquad \text{...(19)}$$

It means that method of Lagrange's undetermined multipliers can be used also for holonomic constraints, when it is not easy to reduce all the q's to independent co-ordinates.

Illustrative Examples

Ex. 2. *A particle moves in the xy-plane under the influence of a central force depending only on its distance from the origin.*

(a) *Set up the Hamiltonian for the system.*

(b) *Write Hamilton's equations of motion.*

Sol. (a) Let the potential due to the central force be $V(r)$. Then, we have

$$T = \tfrac{1}{2} m v^2 = \tfrac{1}{2} m \text{ [(radial velocity)}^2 + \text{(transverse velocity}^2\text{)]}$$

$$= \tfrac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2),$$

$$L = T - V = \tfrac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) - V(r).$$

$$\therefore p_r = (\partial L/\partial \dot{r}) = m\dot{r}, p_\theta = (\partial L/\partial \dot{r}) = mr^2 \dot{\theta}$$

$$\Rightarrow \dot{r} = (p_r/m), \dot{\theta} = (p_\theta/mr^2)$$

Thus $H = \Sigma p_i \dot{q}_i - L = p_r \dot{q}_r + p_\theta \dot{q}_\theta - L$

$$= p_r . \dot{r} + p_\theta \dot{\theta} - \tfrac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2) + V(r)$$

$$= p_r (p_r/m) + p_\theta (p_\theta/mr^2)$$

$$- \tfrac{1}{2} m \{(p_r^2/m^2) + r^2 . (p_\theta^2/m^2 r^4)\} - V(r)\}$$

$$= (p_r^2/2m) + (p_\theta^2/2m_r^2) + V(r) = \text{total energy of the system.}$$

(b) Hamilton's equations are

$$\dot{p}_i = -(\partial H/\partial q_i), \dot{q}_i = (\partial H/\partial p_i)$$

$$\Rightarrow \dot{r} = (\partial H/\partial p_r) = (p_r/m), \dot{\theta} = (\partial H/\partial p_\theta) = (p_\theta/mr^2)$$

$$\dot{p}_i = -(\partial H/\partial r) = (p_\theta^2/mr^3) - V'(r), \dot{p}_\theta = -(\partial H/\partial \theta) = 0.$$

Ex. 3. *A particle of mass m moves in a force field of potential V. Write*

(a) the Hamiltonian and

(b) Hamilton's equations in spherical polar co-ordinates.

Sol. (a) K.E. is given by

$$T = \tfrac{1}{2} m (\dot{r}^2 + r^2 \dot{\theta}^2 + r^2 \sin^2 \theta \, \dot{\phi}^2) \qquad \text{...(1)}$$

$$\therefore L = T - V = \tfrac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2 \theta \, \dot{\phi}^2) - V \qquad \text{...(2)}$$

We have $p_r = (\partial L/\partial \dot{r}) = m\dot{r}, p_\theta = (\partial L/\partial \dot{\theta}) = mr^2 \dot{\theta},$

$$p_\phi = (\partial L/\partial \dot{\phi}) = mr^2 \sin^2 \theta \, \dot{\phi}$$

$$\Rightarrow \dot{r} = (p_r/m), \dot{\theta} = (p_\theta/mr^2), \dot{\phi} = p_\phi/(mr^2 \sin^2 \theta). \qquad \text{...(3)}$$

Now Hamiltonian is given by

$$H = \Sigma p_i \dot{q}_i - L = p_r \dot{r} + p_\theta \dot{\theta} + p_\phi \dot{\phi} - L$$

$$= \frac{p_r^2}{2m} + \frac{p_\theta^2}{2mr^2} + \frac{p_\theta^2}{2mr^2 \sin^2 \theta} + V(r, \theta, \dot{\phi})$$

$= $ total energy of the system.

(b) Hamilton's equations are given by

$$\dot{q}_i = (\partial H/\partial p_i), \dot{p}_i = -(\partial H/\partial q_i)$$

i.e. $\dot{r} = \dfrac{\partial H}{\partial p_r} = \dfrac{p_r}{m}$ $\qquad \dot{p}_r = -\dfrac{\partial H}{\partial r} = \dfrac{p_\theta^2}{mr^3} + \dfrac{p_\phi^2}{mr^3 \sin^2 \theta} - \dfrac{\partial V}{\partial r}$

$\dot{\theta} = \dfrac{\partial H}{\partial p_\theta} = \dfrac{p_\theta}{mr^2}$ $\qquad \dot{p}_\theta = -\dfrac{\partial H}{\partial \theta} = \dfrac{p_\phi^2 \cos \theta}{mr^2 \sin^3 \theta} - \dfrac{\partial V}{\partial \theta}$

$\dot{\phi} = \dfrac{\partial H}{\partial p_\phi} = \dfrac{p_\phi}{mr^2 \sin^2 \theta}$ $\qquad \dot{p}_\phi = -\dfrac{\partial H}{\partial \phi} = -\dfrac{\partial V}{\partial \phi}$

Ex. 4. *If H is the Hamiltonian, prove that if f is any function depending on position, momenta and time, then*

$$(df/dt) = (\partial f/\partial t) + [H, f].$$

Sol. We have

$$(df/dt) = (\partial f/\partial t) + \sum_i \{(\partial f/\partial q_i)(dq_i/dt) + (\partial f/\partial p_i)(dp_i/dt)\}$$

$$\Rightarrow (df/dt) = (\partial f/\partial t) + \sum_i \{(\partial f/\partial q_i)(\partial H/\partial p_i) - (\partial f/\partial p_i)(\partial H/\partial q_i)\}$$

{$\because$ By Hamilton's equations $\dot{q}_i = \partial H/\partial p_i, \dot{p}_i = -(\partial H/\partial q_i)$}

$\Rightarrow (df/dt) = (\partial f/\partial t) + [H, f]$ where $[H, f]$ is the Poisson Bracket)

Ex. 5. *A particle of mass m moves in a force field of potential V.*

(a) *Write the Hamiltonian and*

(b) *Hamilton's equations in cartesian co-ordinates.*

Sol. (a) We have

$$T = \tfrac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

$$\Rightarrow L = \tfrac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - V(x, y, z) \qquad \text{...(1)}$$

$\because \quad p_x = (\partial L/\partial \dot{x}) = m\dot{x}, p_y = (\partial L/\partial \dot{y}) = m\dot{y};\ p_z = (\partial L/\partial \dot{z}) = m\dot{z}$

$\Rightarrow \dot{x} = (p_x/m), \dot{y} = (p_y/m), \dot{z} = (p_z/m).$

Thus $H = \Sigma p_x \dot{q}_x - L = \dot{p}_x x + \dot{p}_y y + \dot{p}_z z - \tfrac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + V(x, y, z)$

$= p_x (p_x/m) + p_y (p_y/m) + p_z \cdot (p_z/m)$

$- \tfrac{1}{2} m [(p_x^2/m^2) + (p_y^2/m^2) + (p_z^2/m^2)] + V(x, y, z)$

$= (p_x^2/2m) + (p_y^2/2m) + (p_z^2/2m) + V(x, y, z)$

= total energy of the system.

(b) Hamilton's equations are :

$\dot{p}_x = -(\partial H/\partial x);\quad \dot{p}_y = -(\partial H/\partial y);\quad \dot{p}_z = -(\partial H/\partial z)$ and

$\dot{x} = (\partial H/\partial p_x);\quad \dot{y} = (\partial H/\partial p_y);\quad \dot{z} = (\partial H/\partial p_z).$

$\Rightarrow \dot{p}_x = -(\partial V/\partial x), \dot{p}_y = -(\partial V/\partial y), \dot{p}_z = -(\partial V/\partial z)$

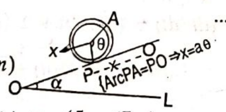
$\dot{x} = (p_x/m), \dot{y} = (p_y/m), \dot{z} = (p_z/m).$

Ex. 6. *A sphere rolls down a rough included plane ; if x be the distance of the point of contact of the sphere from a fixed point on the plane, find the acceleration.* **(Rajasthan 95; Meerut 93)**

Sol. We have $T = \tfrac{1}{2} m (\dot{x}^2 + k^2 \dot{\theta}^2) = \tfrac{1}{2} m (\dot{x}^2 + \tfrac{2}{5} a^2 \dot{\theta}^2) \quad (\because\ k^2 = \tfrac{2}{5} a^2)$

$= \tfrac{1}{2} m (\dot{x}^2 + \tfrac{2}{5}\dot{x}^2) = \tfrac{7}{10} m\dot{x}^2;$ ...(1) $V = -mgx \sin\alpha.$ ...(2)

$\therefore L = T - V = = \tfrac{7}{10} m\dot{x}^2 + mgx \sin\alpha.$ ...(3)



Now $p_x = (\partial L/\partial \dot{x}) = \tfrac{7}{5} m\dot{x} \Rightarrow \dot{x} = (5p_x/7m)$

Thus $H = -L + p_x \dot{x} = -\tfrac{7}{10} m\dot{x} - mgx \sin\alpha + p_x \cdot (5p_x/7m)$

$= \tfrac{5}{14}(p_x^2/m) - mgx \sin\alpha.$ ...(4)

$\therefore$ One of Hamilton's equations gives

$\dot{p}_x = -(\partial H/\partial x) = mg \sin\alpha \Rightarrow \tfrac{7}{5} m\ddot{x} = mg \sin\alpha \Rightarrow \ddot{x} = \tfrac{5}{7} g \sin\alpha$

Ex. 7. *If the Hamiltonian H is independent of time explicily, prove that it is*

(a) *a constant, and* **(b)** *equal to the total energy of the system.*

Sol. (*a*) $(dH/dt) = \sum_{i=1}^{n} (\partial H/\partial p_i)\, \dot{p}_i + \sum_{i=1}^{n} (\partial H/\partial q_i) \dot{q}_i$

$\sum_{i=1}^{n} \dot{q}_i\, \dot{p}_i + (-\dot{p}_i)\, \dot{q}_i = 0 \quad [\because\ (\partial H/\partial p_i) = \dot{q}_i, (\partial H/\partial q_i) = -\dot{p}_i]$

$\Rightarrow H =$ constant $= E$ say.

(b) By Euler's theorem on homogeneous functions, we have

$$\sum_{i=1}^{n} \dot{q}_i\ (\partial T/\partial \dot{q}_i) = 2T. \qquad \text{...(2)}$$

Put $p_i = (\partial L/\partial \dot{q}_i) = \{\partial(T - V)/\partial \dot{q}_i\} = (\partial T/\partial \dot{q}_i) - (\partial V/\partial \dot{q}_i) = (\partial T/\partial \dot{q}_i)$

$\{\because\ (\partial V/\partial \dot{q}_i) = 0$ as V is independent of $\dot{q}_i\}$

$\therefore (2) \Rightarrow \sum_{i=1}^{n} \dot{q}_i\, p_i = 2T$

Thus $H = \sum_{i=1}^{n} p_i \dot{q}_i - L = 2T - L = 2T - (T - V) = T + V = E.$

Ex. 8. *Write the Hamiltonian and equation of motion for a simple pendulum.*

Sol. We have $T = \tfrac{1}{4} ml^2 \dot{\theta}^2$ and $V = mgl(1 - \cos\theta)$,

$\therefore L = T - V = \tfrac{1}{2} ml^2 \dot{\theta}^2 - mgl(1 - \cos\theta)$ ...(1)

$\Rightarrow H = \Sigma p_i \dot{q}_i - L = p_\theta \dot{\theta} - L = ml^2 \dot{\theta}^2 - \{\tfrac{1}{2} ml^2 \dot{\theta}^2 - mgl(1 - \cos\theta)\}$

$= \tfrac{1}{2} ml^2 \dot{\theta}^2 + mgl(1 - \cos\theta) = T + V =$ total energy.

Now $p_\theta = (\partial L/\partial \dot{\theta}) = ml^2 \dot{\theta}^2 = (p_\theta/ml^2)$

$\therefore H = \tfrac{1}{2} ml^2 (p_\theta/ml^2)^2 + mgl(1 - \cos\theta) = (p_\theta^2/2ml^2) + mgl(1 - \cos\theta)$

$\Rightarrow (\partial H/\partial p_\theta) = (p_\theta/ml^2), (\partial H/\partial \theta) = mgl \sin\theta$

Now Hamilton's equation of motion for θ and p_θ are

$\dot{\theta} = (\partial H/\partial p_\theta), \dot{p}_\theta = -(\partial H/\partial \theta) \Rightarrow \dot{\theta} = (p_\theta/ml^2)$ and $\dot{p}_\theta = -mgl \sin\theta,$

These represent Hamilton's equations for a simple pendulum.

From above, we have $p_\theta = ml^2\dot{\theta}$, *i.e.* $\dot{p}_\theta = ml^2 \ddot{\theta}$

$\therefore ml^2 \ddot{\theta} = -mgl \sin\theta \Rightarrow \ddot{\theta} + (g/l) \sin\theta = 0$

This gives the equation of motion of the simple pendulum.

Ex. 9. *Write the Hamiltonian function and equation of motion for a compound pendulum.* **(Meerut 1995)**

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Sol. We have $L = \frac{1}{2} I \dot{\theta}^2 + mgh \cos\theta \Rightarrow p_\theta = (\partial L/\partial \theta) = I \theta.$

where $I = mk^2$

$$\therefore\ H = \Sigma p_i \dot{q}_i - L = p_\theta \dot{\theta} - L = I \dot{\theta}\dot{\theta} - \frac{1}{2} I \dot{\theta}^2 - mgh \cos\theta$$

$$= \frac{1}{2} I \dot{\theta}^2 - mgh \cos\theta$$

$$\Rightarrow \frac{1}{2} H = (p_\theta/I)^2 - mgh \cos\theta = (p_\theta^2/2I) - mgh \cos\theta \quad \{\because\ \dot{\theta} = (p_\theta/I)\}$$

$$\therefore\ (\partial H/\partial p_\theta) = (p_\theta/I), (\partial H/\partial \theta) = mgh \sin\theta$$

Thus the Hamilton's equations for $\dot{\theta}$ and $\dot{p}_\theta$ are given by

$$\dot{\theta} = (\partial H/\partial p_\theta), \dot{p}_\theta = -(\partial H/\partial \theta)$$

i.e. $\dot{\theta} = (p_\theta/I)$ and $\dot{p}_\theta = -mgh \sin\theta$. But $p_\theta = I\theta \Rightarrow \dot{p}_\theta = I\ddot{\theta}$.

$$\therefore\ I\ddot{\theta} = -mgh \sin\theta \Rightarrow \ddot{\theta} + \frac{mgh}{I} \sin\theta = 0.$$

This is exactly the same as obtained prevoiusly using Lagrange's equations.

Ex. 10. *Obtain Euler's equations from Hamilton's equations.*

Sol. We know that $2T = (A\omega_1^2 + B\omega_2^2 + C\omega_3^2)$,

$$\Rightarrow L = T - V = \frac{1}{2}(A\omega_1^2 + B\omega_2^2 + C\omega_3^2) - V \qquad ...(1)$$

Also Euler's geometrical relations give

$$\omega_1 = \dot{\theta} \sin\psi - \dot{\phi} \sin\theta \cos\psi;$$

$$\omega_2 = \dot{\theta} \cos\psi + \dot{\phi} \sin\theta \sin\psi; \text{ and}$$

$$\omega_3 = \psi + \phi \cos\theta.$$

Now $H = T + V = \frac{1}{2}(A\omega_1^2 + B\omega_2^2 + C\omega_3^2) + V$

$$\text{Again, } p_\phi = \frac{\partial L}{\partial \dot{\theta}} = \frac{\partial L}{\partial \omega_1}\frac{\partial \omega_1}{\partial \dot{\theta}} + \frac{\partial L}{\partial \omega_2}\frac{\partial \omega_2}{\partial \dot{\theta}} + \frac{\partial L}{\partial \omega_3}\frac{\partial \omega_3}{\partial \dot{\theta}}.$$

$$= A\omega_1 \sin\psi + B\omega_2 \cos\psi + C\omega_3 . 0 \qquad ...(1)$$

$$p_\phi = \frac{\partial L}{\partial \dot{\phi}} = -A\omega_1 \sin\theta \cos\psi + B\omega_2 \sin\theta \sin\psi + C\omega_3 \cos\theta \qquad ...(2)$$

and $\quad p_\psi = (\partial L/\partial \dot{\psi}) = C\omega_3$

Solving the three equations for $\omega_1, \omega_2, \omega_3$ we have

$$\omega_1 = \frac{1}{A}\left[p_\theta \sin\psi + (p_\psi \cos\theta - p_\phi)\frac{\cos\psi}{\sin\theta}\right]$$

$$\omega_2 = \frac{1}{A}\left[p_\theta \cos\psi - (p_\psi \cos\theta - p_\phi)\frac{\sin\psi}{\sin\theta}\right]; \text{ and } \omega_3 = \frac{1}{C} \cdot p_\psi$$

Also, Hamilton's equations are $\dot{p}_\psi = -\frac{\partial H}{\partial \psi}$ and $\dot{\psi} = \frac{\partial H}{\partial p_\psi}$

Now $\dot{p}_\psi = -\frac{\partial H}{\partial \psi}$

$$\Rightarrow C\dot{\omega}_3 = -\left[\frac{\partial H}{\partial \omega_1}\frac{\partial \omega_1}{\partial \psi} + \frac{\partial H}{\partial \omega_2}\frac{\partial \omega_2}{\partial \psi} + \frac{\partial H}{\partial \omega_3}\frac{\partial \omega_3}{\partial \psi}\right] - \frac{\partial V}{\partial \psi}$$

$$= -\left[A\omega_1 . \frac{1}{A} B\omega_2 + B\omega_2\left(\frac{A}{B}\omega_1\right) + C\omega_3 . 0\right] - \frac{\partial V}{\partial \psi}$$

$$= (A - B)\omega_1\omega_2 - \frac{\partial V}{\partial \psi}$$

$$\Rightarrow C\frac{d\omega_1}{dt} - (A - B)\omega_1\omega_2 = N\left(\because\ -\frac{\partial V}{\partial \phi} = N\right)$$

This is Euler's third familiar dynamical equation

$$\text{Also,} \quad \dot{\psi} = (\partial H/\partial p_\psi) = \frac{\partial H}{\partial \omega_1}\frac{\partial \omega_1}{\partial p_\psi} + \frac{\partial H}{\partial \omega_2}\frac{\partial \omega_2}{\partial p_\psi} + \frac{\partial H}{\partial \omega_3}\frac{\partial \omega_3}{\partial p_\psi}$$

$$= (\omega_1 \cos\psi - \omega_2 \sin\psi) \cot\theta + \omega_3 = -\phi \sin\theta \cot\theta + \omega_3$$

i.e. $\dot{\psi} = -\dot{\phi} \cos\theta + \omega_3 \Rightarrow \omega_3 = \dot{\psi} + \dot{\phi} \cos\theta$

This is Euler's third geometrical equation.

On the same lines, we can deduce Euler's other equations (dynamical and geometrical).

Ex. 11. *Prove that*

$$\left(\frac{dH}{dt}\right) = \left(\frac{\partial H}{dt}\right) \text{ where } H \text{ is the Hamilton's function.}$$

Sol. Let $q_1, q_2 ..., q_n$ be the generalised co-ordinates then Hamilton's equation are given by

$$\dot{p}_1 = -\frac{\partial H}{\partial q_i} \text{ and } \dot{q}_i = \frac{\partial H}{\partial p_i} (i = 1, 2, ..., n) \qquad ...(1)$$

But Hamiltonian H is a function of q's and ps'

$$\therefore\ \frac{dH}{dt} = \frac{\partial H}{\partial t} + \sum_{i=1}^{n} \frac{\partial H}{\partial q_i}\dot{q}_i + \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\dot{p}_i$$

$$= \frac{\partial H}{\partial t} + \sum_{i=1}^{n} (-\dot{p}_i)\dot{q}_i + \sum_{i=1}^{n} \dot{q}_i \dot{p}_i = \frac{\partial H}{\partial t}. \qquad \text{[using (1)]}$$

Ex. 12. *Use Hamilton's equations to find the equations of motion of a projectile in space.*

Solution. Let (x, y, z) be the co-ordinates of the projectile in space at time t, then we have

$$T = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2), V = mgz$$

$$\therefore\ L = T - V = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - m$$

$$\Rightarrow p_x = \frac{\partial L}{\partial \dot x} = m\dot x,\ p_y = \frac{\partial L}{\partial \dot y} = m\dot y,\ p_z = \frac{\partial L}{\partial \dot z} = m\dot z$$

But L does not involve t explicity therefore Hamiltonian H is given by

$$H = T + V = \tfrac{1}{2}(\dot x^2 + \dot y^2 + \dot z^2) + mgz$$

$$= \tfrac{1}{2}m\left(\frac{p_x^2}{m^2} + \frac{p_y^2}{m^2} + \frac{p_z^2}{m^2}\right) + mgz = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + mgz$$

Now Hamilton's equations are given by

$$\dot p_x = -\frac{\partial H}{\partial x} = 0 \quad ...(1), \qquad \dot x = \frac{\partial H}{\partial p_x} = \frac{p_x}{m} \quad ...(2),$$

$$\dot p_y = -\frac{\partial H}{\partial y} = 0 \quad ...(3), \qquad \dot y = \frac{\partial H}{\partial p_y} = \frac{p_y}{m} \quad ...(4),$$

$$\dot p_z = -\frac{\partial H}{\partial z} = -mg \quad ...(5), \qquad \dot z = \frac{H}{\partial p_z} = \frac{p_z}{m} \quad ...(6)$$

Using (1) and (2), we have $\ddot x = 0$ ...(7)

Using (3) and (4), we have $\ddot y = 0$ ...(8)

Again making use of (6) and (5), we have

$$m\ddot z = \dot p_z = -mg \text{ or } \ddot z = -g. \quad ...(9)$$

These (7, 8, 9) are the equations of motion of the projectile in space.

Ex. 13. *Using cylindrical coordinates (ρ, ϕ, z) write the Hamiltonian and Hamilton's equations for a particle of mass m moving in a force field of potential $V(\rho, \phi, z)$.*

Sol. In cylindrical coordinates, co-ordinates of any points are $x = \rho\cos\phi,\ y = \rho\sin\phi,\ z = z$ ...(1)

$$\therefore\ T = \tfrac{1}{2}m(\dot x^2 + \dot y^2 + \dot z^2) = \tfrac{1}{2}m(\dot\rho^2 + \rho^2\dot\phi^2 + \dot z^2) \quad ...(2)$$

$$\Rightarrow L = T - V = \tfrac{1}{2}m(\dot\rho^2 + \rho^2\dot\phi^2 + \dot z^2) - V(\rho, \phi, z) \quad ...(3)$$

$$\Rightarrow p_\rho = \frac{\partial L}{\partial\dot\rho} = m\dot\rho,\ p_\phi = \frac{\partial L}{\partial\dot\phi} = m\rho^2\dot\phi \text{ and } p_z = \frac{\partial L}{\partial\dot z} = m\dot z$$

Evidently, L does not involve t explicity, therefore Hamiltonian H is given by $H = T + V = \tfrac{1}{2}m(\dot\rho^2 + \rho^2\dot\phi^2 + \dot z^2) + V$

$$= \tfrac{1}{2}m\left[\frac{p_\rho^2}{m^2} + \frac{p_\phi^2}{m^2\rho^2} + \frac{p_z^2}{m^2}\right] + V = \frac{1}{2m}\left[p_\rho^2 + \frac{p_\phi^2}{\rho^2} + p_z^2\right] + V$$

Hence, Hamilton's are given by :

$$\dot p_\rho = -\frac{\partial H}{\partial\rho} = \frac{p_\phi^2}{m\rho^2} - \frac{\partial V}{\partial\rho};\ \dot\rho = \frac{\partial H}{\partial p_\rho} = \frac{p_\rho}{m_\phi}$$

$$\dot p_\phi = -\frac{\partial H}{\partial\phi} = -\frac{\partial V}{\partial\phi};\ \dot\phi = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{m\rho^2}$$

$$\dot p_z = -\frac{\partial H}{\partial z} = -\frac{\partial V}{\partial z};\ \dot z = \frac{\partial H}{\partial p_z} = \frac{p_z}{m}$$

Ex. 14. *Using cylindrical coordinates, write the Hamiltonian and Hamilton's equations for a particle of mass moving on the inside of a frictionless cone $x^2 + y^2 = z^2\tan^2\alpha$*

Sol. Like previous example, we have

$$T = \tfrac{1}{2}m(\dot x^2 + \dot y^2 + \dot z^2) = \tfrac{1}{2}m(\dot\rho^2 + \rho^2\dot\phi^2 + \dot\rho^2\cot^2\alpha)$$

$$[\because\ x = \rho\cos\phi,\ y = \rho\sin\phi,\ z = \rho\cot\alpha]$$

$$= \tfrac{1}{2}m(\dot\rho^2\operatorname{cosec}^2\alpha + \dot\rho^2\dot\phi^2)$$

and $V = -W = -mgz = mg\rho\cot\alpha,$ ...(1)

[$\because$ the particle is above the vertex (origin)].

$$\Rightarrow L = T - V = \tfrac{1}{2}m(\dot\rho^2\operatorname{cosec}^2\alpha + \rho^2\dot\phi^2) - mg\rho\cot\alpha \quad ...(2)$$

This gives, $p_\rho = \frac{\partial L}{\partial\dot\rho} = m\dot\rho\operatorname{cosec}^2\alpha,\ p_\phi = \frac{\partial L}{\partial\dot\phi} = m\rho^2\dot\phi$ ...(3)

Again, L does not involve t explicitly, therefore Hamiltonian H is given by

$$H = T + V = \tfrac{1}{2}m(\dot\rho^2\operatorname{cosec}^2\alpha + \rho^2\dot\phi^2) + mg\rho\cot\alpha$$

$$= \tfrac{1}{2}m\left[\frac{p_\rho^2}{m^2\operatorname{cosec}^2\alpha} + \frac{p_\phi^2}{m^2\rho^2}\right] + mg\rho\cot\alpha + \frac{1}{2m}\left[\frac{p_\rho^2}{\operatorname{cosec}^2\alpha} + \frac{p_\phi^2}{\rho^2}\right] + mg\rho\cot\alpha$$

Thus Hamilton's equations are given by :

$$\dot p_\rho = -\frac{\partial H}{\partial\rho} = \frac{p_\phi^2}{m\rho^2} - mg\cot\alpha;\ \dot\rho = \frac{\partial H}{\partial p_\rho} = \frac{p_\rho}{m\operatorname{cosec}^2\alpha}.$$

$$\dot p_\phi = -\frac{\partial H}{\partial\phi} = 0 \qquad ;\ \dot\phi = \frac{\partial H}{\partial p_\phi} = \frac{p_\phi}{m\rho^2}.$$

Ex. 15. *Use the variational method to show that the shortest curve joining two fixed is a straight line.* **(Meerut 1981, 90)**

Sol. We have

$$\frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}$$ for any curve, joining two fixed points say $A \equiv (x_1, y_1)$ and $B \equiv (x_2, y_2)$.

$$\Rightarrow s=\int_{x_1}^{x_2} \sqrt{(1+y'^2)}\,dx = \text{ length of the curve joining A to B.}$$

$$=\int_{x_1}^{x_2} f(y')\,dx \text{ where } f(y')=\sqrt{(1+y'^2)} \qquad \ldots(1)$$

In equation (1), y is absent from f, therefore for s to be stationary (here minimum), we have from Euler-Lagrange's equation

$$\frac{\partial f}{\partial y'}=(\text{constant}) \Rightarrow \frac{\partial}{\partial y'}[(\sqrt{1+y'^2})]=\text{const.} \quad \because \quad f=(1+y'^2)^{1/2}$$

$$\Rightarrow \frac{y'}{\sqrt{(1+y'^2)}}=(\text{constant}) \text{ i.e. } y'=b \text{ where } b \text{ is constant.}$$

$$\Rightarrow \frac{dy}{dx}=b \text{ or } y=a+bx \text{ where } a \text{ is a constant.} \qquad \ldots(2)$$

$\Rightarrow$ the shortest path joining A to B is the curve whose equation is $y=a+bx$ and this is evidently a line.

Ex. 16. *Show that the area of the surface of revolution of a curve $y=y(x)$ is*

$$2\pi\int_{x_1}^{x_2} y\sqrt{(1+y'^2)}\,dx.$$

Hence show that for this to be a minimum, the curve must be a catenary. **(Meerut 81 (P), 82 (P), 84 (P))**

Sol. Consider any curve, and let $A\equiv(x_1, y_1)$ and $B\equiv(x_2, y_2)$ be its extremities, then if this curve revolves about x-axis the surface of revolution is given by

$$S=\int 2\pi y\,ds=2\pi\int y\frac{ds}{dx}dx=2\pi\int_{x_1}^{x_2} y\sqrt{(1+y'^2)}\,dx$$

$$\Rightarrow S=2\pi\int_{x_1}^{x_2} f(y, y')dx, \text{ where } f=f(y, y')=y\sqrt{(1+y'^2)}$$

Evidently, x is absent from f, so for S to be stationary (here a minimum) we have :

$$f-y'\frac{\partial f}{\partial y'}=(\text{constant}) \text{ i.e. } y\sqrt{(1+y'^2)}-y'\frac{yy'}{\sqrt{(1+y')}}=\text{const.}$$

$$\Rightarrow \frac{y}{\sqrt{(1+y'^2)}}=c \text{ where } c \text{ is constant i.e. } y=c\sqrt{(1+\tan^2\psi)}$$

$$[\because \quad y'=\tan\psi]$$

$\therefore$ $y=c\sec\psi$ and this represents catenary.

Ex. 17. *A particle of unit mass is projected so that its total energy is h in a field of force of which the potential energy is $\phi(r)$ at a distance r from the origin.*

Deduce from the principle of energy and least action that the differential equation of the path is

$$c^2\left[r^2+\left(\frac{dr}{d\theta}\right)^2\right]=r^4[h-\phi(r)].$$

(Meerut 1995)

Sol. Let T and V be the kinetic are potential energies, then we have $T+V=h \Rightarrow T=h-V$

$$=h-\phi(r), \quad \because \quad V=\phi(r) \text{ given}$$

$$\Rightarrow \tfrac{1}{2}v^2=h-\phi(r) \text{ where v is velocity } (\because \quad T=\tfrac{1}{2}mv^2=\tfrac{1}{2}v^2)$$

$$\Rightarrow v=\sqrt{2}\{h-\phi(r)\}^{1/2}$$

Whence, the action $A=\int_{t_0}^{t_1} 2T\,dt$, by definition given earlier

$$=\int_{t_0}^{t_1} 2.\tfrac{1}{2}v^2\,dt=\int_{t_0}^{t_1} v\,ds \text{ since } v=(ds/dt)=\sqrt{2}\int_{t_0}^{t_1}\{h-\phi(r)\}^{1/2}ds$$

$$=\sqrt{2}\int_{t_2}^{t_1}\{h-\phi(r)\}^{1/2}\left\{1+r^2\left(\frac{d\theta}{dr}\right)^2\right\}^{1/2}dr \quad \because \quad \frac{ds}{dr}=\left\{1+r^2\left(\frac{d\theta}{dr}\right)^2\right\}^{1/2}$$

$$=\sqrt{2}\int_{t_0}^{t_1}\left[\{h-\phi(r)\}^{1/2}\{1+r^2\theta'^2\}^{1/2}\right]dr \text{ where } \theta'=\frac{d\theta}{dr}$$

$$=\sqrt{2}\int_{t_0}^{t_1} f(\theta', r)\,dr, \text{ where } f(\theta', r)=\{h-\phi(r)\}^{1/2}\{1+r^2\theta'^2\}^{1/2}$$

Evidently, θ is absent from f, therefore we must have

$$\frac{\partial f}{\partial\theta'}=c \text{ (const.)} \Rightarrow \frac{\partial}{\partial\theta'}[\{h-\phi(r)\}^{1/2}\{1+r^2\theta'^2\}^{1/2}]=c$$

$$\Rightarrow \{h-\phi(r)\}^{1/2}\frac{r^2}{\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}^{1/2}}=c$$

$$\Rightarrow c^2\left\{r^2+\left(\frac{dr}{d\theta}\right)^2\right\}=r^4\{h-\phi(r)\}.$$

This is the required equation.

Ex. 18. *A projectile is launched in a vertical plane with a velocity whose horizontal and vertical components are v_x and v_y respectively. Calculate the value of the integral $\int_0^{t_0} L\,dt$, where $t_0=\dfrac{n\pi}{\omega}$.*

Evaluate this integral for the varied path given by the equations

$$x = v_x t, y = v_y t - \tfrac{1}{2} g t^2 + \varepsilon \sin \omega t,$$

where ε is a small constant quantity.

Show that the integral $\int_0^{t_0} L\, dt$ is greater for the varied path than for the actual path, but the result is in agreement with Hamilton's principle. **(Meerut 1994)**

Sol. The path of the projectile is given by

$$x = v_x t, y = v_y t - \tfrac{1}{2} g t^2.$$

Also $V = mgy$, where V denotes the potential,

whence $L = T - V = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2) - mgy$

$$= \tfrac{1}{2} m [v_x^2 + (v_y - gt)^2] - mg(v_y t - \tfrac{1}{2} g t^2)$$

$$= \tfrac{1}{2} m [v_x^2 + v_y^2) - 4gt v_y + 2g^2 t^2] \quad ...(1)$$

$$\Rightarrow \int_0^{t_0} L\, dt = \int_0^{t_0} (T - V)\, dt = \tfrac{1}{2} m \int_0^{t_0} [(v_x^2 + v_y^2) - 4gt v_y + 2g^2 t^2] dt$$

$$= \tfrac{1}{2} m [(v_x^2 + v_y^2) t_0 - 2t_0^2 v_y + \tfrac{2}{3} g^2 t_0^3] \quad ...(1)$$

Also the varied path is given by : $x = v_x t, y = v_y t - \frac{1}{2} g t^2 + \varepsilon \sin \omega t$

Now after obtaining $\dot{x}, \dot{y}$; we have $T = \frac{1}{2} m [v_x^2 + (y_v - gt + \varepsilon\omega \cos \omega t)^2]$

and $V = mg(v_y t - \frac{1}{2} g t^2 + \varepsilon \sin \omega t) \Rightarrow \int_0^{t_0 = n\pi/\omega} L\, dt = \int_0^{t_0} (T - V)\, dt$

$$\int_0^{t_0} [\tfrac{1}{2} m \{v_x^2 + (v_y - gt + \varepsilon\omega \cos \omega t)^2\} - mg(v_y t - \tfrac{1}{2} g t^2 + \varepsilon \sin \omega t)\, dt$$

$$= \tfrac{1}{2} m [(v_x^2 + v_y^2) t_0 - 2t_0^2 g v_y + \tfrac{2}{3} g^2 t_0^3 + \tfrac{1}{2} \varepsilon^2 \omega^2 t_0] \quad (\because \text{ other integrals vanish})$$

$$= \tfrac{1}{2} m [(v_x^2 + v_y^2) t_0 - 2t_0^2 g v_y + \tfrac{2}{3} g^2 t_0^3] + m\pi n \omega \varepsilon^2 \quad ...(2)$$

From equations (1) and (2), we see that the integarl $\int_0^{t_0} L\, dt$ is greater for' the varied path then for the actual path and is minimum when $\varepsilon = 0$. Evidently, the minimum value in this case, the varied *path coincides with actual path.*

Ex. 19. *A particle moves in a straight line with central acceleration $\omega^2 x$ between two fixed points x_0 and x_1 in the prescribed time $t_1 - t_0$ ... that Hamilton's principal function S is*

$$\frac{\omega}{2 \sin \omega(t_1 - t_0)} \left[(x_1^2 + x_0^2) \cos \omega (t_1 - t_0) - 2x_1 x_0 \right]$$

(Meerut 92, 93; Agra 90)

Sol. Equation of motion is, $\ddot{x} = \omega^2 x$

Solving (1), the path is $x = A \cos \omega t + B \sin \omega t$ ...(1)

Now, $\dot{x} = \omega [-A \sin \omega t + B \cos \omega t] \Rightarrow T = \frac{1}{2} \dot{x}^2$. ...(2)

Also, we have potential V = work done against the force

$$= \int_0^{t_0} \omega^2 x\, dx = \tfrac{1}{2} \omega^2 x^2 \quad ...(3)$$

Again, we also have

$$A \cos \omega t_0 + B \sin \omega t_0 - x_0 = 0 \qquad (t = t_0;\ x = x_0)$$

and $A \cos \omega t_1 + B \sin \omega t_1 - x_1 = 0 \qquad (t = t_1;\ x = x_1)$

Solving these,

$$A = \frac{x_0 \sin \omega t_1 - x_1 \sin \omega t_0}{\sin \omega (t_1 - t_0)}, B = \frac{x_1 \cos \omega t_0 - x_0 \cos \omega t_1}{\sin \omega (t_1 - t_0)}$$

$$\Rightarrow B^2 - A^2 = [x_1^2 \cos 2\omega t_0 + x_0^2 \cos 2\omega t_1 - 2x_1 x_0 - \cos \omega (t_1 + t_0)] \frac{1}{\sin^2 \omega (t_1 - t_0)} \quad ...(4)$$

and $2AB = -[x_1^2 \sin 2\omega t_0 + x_0^2 \sin 2\omega t_1 - 2x_1 x_0 \sin \omega (t_1 + t_0)] \dfrac{1}{\sin^2 \omega(t_1 - t_0)}$

$\therefore S$ = Hamilton's principal function

$$= \int_{t_0}^{t_1} L\, dt = \int_{t_0}^{t_1} (T - V)\, dt \quad (\because L = T - V)$$

$$= \int_{t_0}^{t_1} (\tfrac{1}{2} \dot{x}^2 - \tfrac{1}{2} \omega^2 x^2)\, dt \qquad \text{[from above]}$$

$$= \tfrac{1}{2} \omega^2 \int_{t_0}^{t_1} [(-A \sin \omega t + B \cos \omega t)^2 - (A \cos \omega t + B \sin \omega t)^2]\, dt$$

$$= \tfrac{1}{2} \omega \left[(B - A)^2 \sin 2\omega t + 2AB \cos 2\omega t \right]_{t_0}^{t_1}$$

$$= \tfrac{1}{2} \omega [(B^2 - A^2) \cos \omega (t_1 + t_0) \sin \omega (t_1 - t_0) - 2AB \sin \omega (t_1 - t_0) \sin \omega (t_1 - t_0)]$$

Substituting for $B^2 - A^2$ and $2AB$, we obtain

$$S = \frac{\omega}{2 \sin \omega (t_1 - t_0)} [(x_1^2 \cos 2\omega t_0 + x_0^2 \cos 2\omega t_1 - 2x_1 x_0$$

$$\cos \omega (t_1 + t_0)\} \cos \omega (t_1 + t_0) + \{x_1^2 \sin 2\omega t_0 + x_0^2 \sin 2\omega t_1 - 2x_1 x_0 \sin \omega (t_1 + t_0)\} \sin \omega (t_1 + t_0)]$$

$$= \frac{\omega}{2 \sin \omega (t_1 - t_0)} [x_1^2 \cos \omega (t_1 - t_0) + x_0^2 \cos \omega (t_1 - t_0) - 2x_1 x_0]$$

$$= \frac{\omega}{2 \sin \omega (t_1 - t_0)} [(x_1^2 + x_0^2) \cos \omega (t_1 - t_0) - 2x_1 x_0]$$

This is the requried result.

Ex. 20. *A particle moves in a plane curve, under the central acceleration $\omega^2 r$, between two fixed points (x_0, y_0) and (x_1, y_1) in the prescribed time $t_1 - t_0$; prove that Hamilton's principle function S is*

$$= \frac{\omega}{2 \sin \omega (t_1 - t_0)} [(x_1^2 + y_1^2 + x_0^2 + y_0^2) \cos \omega (t_1 - t_0) - 2 (x_1 x_0 + y_1 y_0)]$$

(Meerut 1992; Agra 91)

Sol. Equations of motion of the particle are given by

$$\ddot{x} = -\omega^2 x \text{ and } \ddot{y} = -\omega^2 y$$

Solving these, we obtain

$$x = A \cos \omega t + B \sin \omega t \; ; \quad y = C \cos \omega t + D \sin \omega t \qquad \text{...(1)}$$

Now, kinetic energy $T = \frac{1}{2}(\dot{x}^2 + \dot{y}^2)$

$$= \tfrac{1}{2}\omega^2 [(-A \sin \omega t + B \cos \omega t)^2 + (-C \sin \omega t + D \cos \omega t)^2]$$

Potential $V = \frac{1}{2}\omega^2 r^2 = \frac{1}{2}\omega^2 (x^2 + y^2)$

Hence Hamilton's principal tunction S is given by

$$S = \int_{t_0}^{t_1} L\, dt = \int_{t_0}^{t_1} (T - V)\, dt \text{ since } L = T - V$$

$$= \int_{t_0}^{t_1} [\tfrac{1}{2}(\dot{x}^2 + \dot{y}^2) - \tfrac{1}{2}\omega^2 (x^2 + y^2)]\, dt$$

$$= \int_{t_0}^{t_1} (\tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}\omega^2 x^2)\, dt + \int_{t_0}^{t_1} (\tfrac{1}{2}\dot{y}^2 - \tfrac{1}{2}\omega^2 y^2)\, dt \qquad \text{...(2)}$$

But $\displaystyle\int_{t_0}^{t_1} (\tfrac{1}{2}\dot{x}^2 - \tfrac{1}{2}\omega^2 x^2)\, dt = \frac{\omega}{2 \sin \omega (t_1 - t_0)} [(x_1^2 + x_0^2) \cos \omega (t_1 - t_0) - 2x_1 x_0]$

and $\displaystyle\int_{t_0}^{t_1} (\tfrac{1}{2}\dot{y}^2 - \tfrac{1}{2}\omega^2 y^2)\, dt = \frac{\omega}{2 \sin \omega (t_1 - t_0)} \times [(y_1^2 + y_0^2) \cos \omega (t_1 - t_0) - 2y_1 y_0]$

$$\therefore\ S = \frac{\omega}{2 \sin \omega (t_1 - t_0)} [(x_1^2 + x_0^2 + y_1^2 + y_0^2) \cos \omega (t_1 - t_0) - 2 (x_1 x_0 + y_1 y_0)]$$

SUPPLEMENTARY PROBLEMS

1. Obtain Hamilton's canonical equations of motion for rigid body. **[Meerut 81;Raj. 77, 79]**
2. Define Hamiltonian function and derive Hamilton's equations of motion. **[Agra 77, 79]**
3. Write the Hamiltonian for the motion of a spherical pendulum and obtain Hamilton's equations. Solve them to show that Hamiltonian is a constant. **[Meerut 79(S)]**
4. Derive the Euler's dynamical equations from Hamilton's equations. **[Raj. 78, 81, 83]**
5. If **H** is the Hamiltonian, prove that if **f** is any function depending on position, momenta and time, then
   $(df/dt) = (\partial f/\partial t) + [H, f]$ where $[H, f]$ is the poisson's Bracket.
6. A heavy bead of mass **m** is freely movable on a smooth circular wire of radius a which is made to rotate about a vertical diameter with spin ω, prove that the action,
   $$A = ma^2 \int_{\theta_1}^{\theta_2} \left[\frac{2H}{ma^2} + \frac{2g}{a} \cos \theta + \omega^2 \sin^2 \theta \right]^{1/2} d\theta$$
   where **H** is the Hamiltonian and θ is the angle the radius through the bead makes with the downward vertical. **[Kanpur 1983]**
7. A particle of unit mass is projected so that its total energy is k, in a field of a force a of which the potential energy is $\phi(r)$ at a distance **r** from the origin. Find the differential equation of the path.
8. Apply the technique of calculus of variation to obtain Euler Lagrange's equations. **[Meerut 1982, 1984]**
9. State the variational principle of Hamilton. **[Meerut 1982]**
10. State & prove the principle of least action. **[Meerut 1985]**
11. State & prove the principle of least action for a conservative holonomic system. Is the action really least in all cases. **[Meerut 79(S)]**
12. Prove that for function $f(y, y', x)$
    $$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) = 0, \text{ where } y' = \frac{dy}{dx}$$ **[Meerut 78, 80, 80(S), 81, 82(P)]**

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