HYDRO DYNAMICS

(For Honours and Post-Graduate Students of all Universities)

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Published by Krishna Prakashan Mandir, Subhash Bazar, Meerut (U. P.) and Printed at Basant Printers, Meerut.

SD-121, PITAM PURA,
Delhi-88, 27311723

PREFACE TO FIRST EDITION

This book on **Hydrodynamics** has been specially designed to meet the requirements of the students preparing for Honour's and Post-graduate classes of all Indian Universities. Each chapter of the book contains a fairly large number of examples.

I do not claim any originality. All standard books on the subject have been consulted during the preparation of the book.

I shall be grateful to the readers for pointing out the errors, inspite of all care, might have crept in. Thanks are also due to the publishers and printers for their effective co-operation.

Any comment and suggestions for improvement by the readers of the book will be gratefully acknowledged.

N. A. S. College,
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—*S. Swarup*

PREFACE TO SECOND EDITION

In this second Edition the book has been revised thoroughly. Some new examples and topics has been added according to the latest syllabi and Question Papers of different Universities.

Any suggestion for further improvement will be highly appreciated.

N. A. S. College,
Meerut.

—*S. Swarup*

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Chapters *Pages*

1

Kinematics

The Science of **Hydrodynamics** may be classified into two different branches, *viz.* the motion of fluids and the motion of gases. Here we shall concentrate to the behaviour of the motion of fluids only. By the term fluid implies a substance that flows, it is defined to be an aggregation of molecules. A fluid is treated as an isotropic substance which means that the physical properties (*viz.* pressure, density, volume, temperature and viscosity etc.), are independent of direction. Fluid comprises both liquids and gases :

- **Fluid (Liquids and Gases)**
  - Newtonian
    - Non-Viscous (Inviscid or Perfect or Ideal)
    - Viscous
  - Non-Newtonian

The smallest lump of material having sufficient molecules to allow statistically of a continuum* interpretation is called a **fluid particle.**

Actual fluids fall into two categories, *e.g.* liquids and gases.

(i) **Liquids** : Which are incompressible *i.e.* their volume do not change when pressure changes or has a definite volume at constant temperature and pressure.

Although all known liquids are compressible to some extent but most of the practical purposes liquids are regarded as an incompressible fluids.

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* Continuum' means that the distance between fluid particles (or molecules) or the mean free path is small. By small we mean 'small compared to any physical dimensions of the problem.'

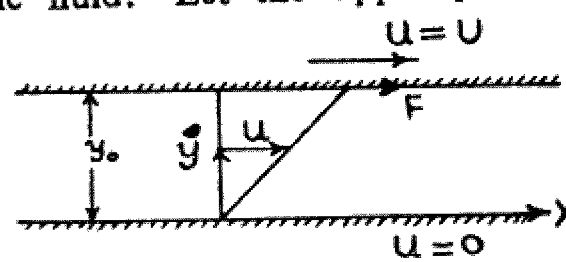
(ii) **Gases :** Which are compressible fluids, *i.e.* their volume change when the pressure changes.

Hydrodynamics is often applied to the Science of flowing frictionless and incompressible fluid.

An **Inviscid or Ideal fluid** is a continuous fluid substance exerting no tangential or shear stress between adjoining layers of fluid. The pressure at every point of an ideal fluid is equal in all directions, whether the fluid be at rest or in motion.

In general, **Actual fluids** (Real fluids) are viscous and compressible. In viscous fluids, both the tangential and normal forces exist. **Viscosity** of a fluid is that characteristic of real fluid which is capable to offer resistance to shearing stress, *e.g.* the resistance is small comparatively for fluid such as water and gases, but for other fluid such as oil, Glycerine etc., resistance is quite large. Viscosity is also known as an internal friction. Viscosity of a fluid is a measure of its resistance to shear or angular deformation.

Consider two parallel plates placed at a distance y_0 apart. The space between in is filled with the fluid. Let the upper plate is moving relative to the lower one with a velocity U by the application of a force F corresponding to some area A of the upper plate. The particles of fluid in contact with each plate will adhere to the surfaces. The velocity gradient will be a straight line if the distance y_0 is not too great or the velocity U too high. From similar triangles we see that U/y_0 can be replaced by velocity gradient du/dy. Let μ be the constant of proportionality, the shearing stress between any two thin sheets of fluid is



$$\tau=\frac{F}{A} \text{ (Force/unit area)}$$

$$\alpha \frac{U}{y_0}$$

or $$\tau=\mu \frac{U}{y_0}$$

or $$\tau=\mu \frac{du}{dy}$$

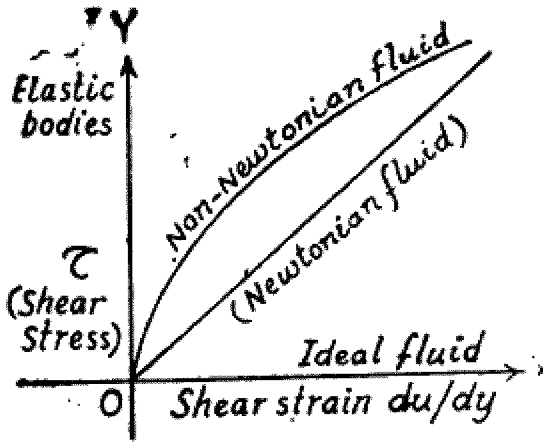
Which is known as Newton's Equation of viscosity.

$$\text{or} \qquad \mu = \frac{\tau}{du/dy} \qquad \ldots\text{(i)}$$

Where μ is the **coefficient of viscosity, Absolute Viscosity** or **Dynamic Viscosity.**

(I) When $\tau=0$ then $\mu=0$ *i.e.* (i) will represent an **ideal fluid** on the X-axis.

(II) When $\frac{du}{dy}=0$ then $\mu=\infty$ *i.e.* (i) will represent the elastic bodies along the Y-axis.



(III) A fluid for which the constant of proportionality (viscosity) does not change with the rate of deformation is said to be a **'Newtonian fluid'** and is represented by a straight line.

(IV) In Non-Newtonian fluid viscosity varies with the rate of deformation.

The motion of a fluid may be investigated by two different methods :

The one is the Lagrangian method, and the other is the Eulerian method.

1. Lagrangian Method (Individual time rate of change) : In Lagrange's method, **any particle of the fluid is selected** and pursue it on its onward course observing the change in velocity, density and pressure at each point and each instant **or in other sense we determine the history of every fluid particle.** If (x, y, z) are the co-ordinates of a particle of fluid at any time then $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ are the components of its velocity and $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}$ are the components of its acceleration along the axes. So (x, y, z) $(\dot{x}, \dot{y}, \dot{z})$ and $(\ddot{x}, \ddot{y}, \ddot{z})$ are the functions of t and of three independent parameters (initial co-ordinates) a, b, c (say) which states the position of a chosen particle at a particular instant. Thus x, y, z are the functions of **four independent variable** a, b, c and t.

i.e. $\quad x=f_1\,(a\ b\ c\ t), y=f_2\,(a\ b\ c\ t)$ and $z=f_3\,(a\ b\ c\ t)$.

2. Eulerian method (Local time rate of change) : In Euler's method (flux method) the individual fluid particles are not identified but a fixed position in space is considered. **We select any**

point fixed in space occupied by the fluid, and observe the change occurs in velocity, density and pressure as the fluid passes through this point. As the fluid is studied at all of its points at every instant t, So x, y, z and t are independent variables in this case The expressions $\frac{dx}{dt}, \frac{d^2x}{dt^2}$ etc., are meaningless in this method as x and t are independent.

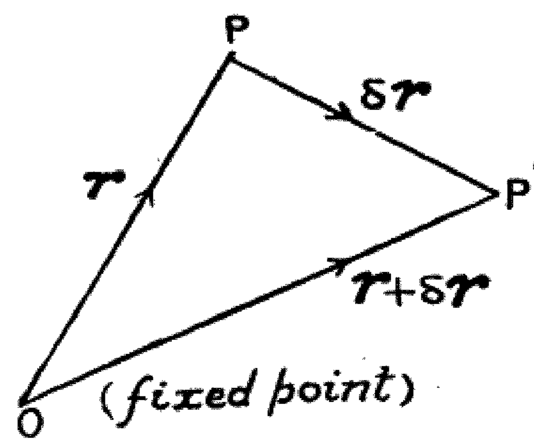
1·3. Velocity of a fluid particle at a point.

Consider the particle is at P at any instant t, such that

$\mathbf{OP}=\mathbf{r}$ (Where O is a fixed point)

and P' is the position of the particle at any instant $t+\delta t$, such that

$$\mathbf{OP}'=\mathbf{r}+\delta r$$



Let $\mathbf{q}$ be the velocity of the fluid particle at P.

Then $\mathbf{q}=\underset{\delta t\to 0}{\mathrm{Lt}}\dfrac{(\mathbf{r}+\delta\mathbf{r})-\mathbf{r}}{\delta t}$

$\mathbf{q}=\underset{\delta t\to 0}{\mathrm{Lt}}\dfrac{\delta\mathbf{r}}{\delta t}$

$\mathbf{q}=\dfrac{d\mathbf{r}}{dt}$

So $\mathbf{q}$ is in general dependent on both $\mathbf{r}$ and t, then we can write

$$\mathbf{q}=\mathbf{q}\ (\mathbf{r}\ t)$$

If the co-ordinates of P are $(x\ y\ z)$ referred to the fixed point O, then

$$\mathbf{q}=\mathbf{q}\ (x, y, z, t)$$

Let u, v, w are the cartesian components of the velocity at P, then

$$\mathbf{q}=u\mathbf{i}+v\mathbf{j}+w\mathbf{k}$$

$$\mathbf{q}=\frac{dx}{dt}\ \mathbf{i}+\frac{dy}{dt}\ \mathbf{j}+\frac{dz}{dt}\ \mathbf{k}$$

{ Since $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ }

So $u=\dfrac{dx}{dt},\ v=\dfrac{dy}{dt}$ and $w=\dfrac{dz}{dt}$.

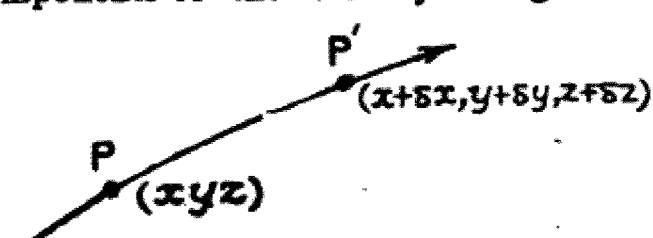
§ 1·4. Relation between the Local and Individual time rates.

Consider u, v, w be the components of the velocity along the co-ordinate axes, then

$$\mathbf{q}=u\mathbf{i}+v\mathbf{j}+w\mathbf{k}$$

Let $\phi=\phi\ (x, y, z, t)$

(where ϕ is a scalar point function)



or $\dfrac{d\phi}{dt}=\dfrac{\partial\phi}{\partial x}\cdot\dfrac{dx}{dt}+\dfrac{\partial\phi}{\partial y}\cdot\dfrac{dy}{dt}+\dfrac{\partial\phi}{\partial z}\cdot\dfrac{dz}{dt}+\dfrac{\partial\phi}{\partial t}$

or $\dfrac{d\phi}{dt}=\dfrac{\partial\phi}{\partial x}\cdot u+\dfrac{\partial\phi}{\partial y}\cdot v+\dfrac{\partial\phi}{\partial z}\cdot w+\dfrac{\partial\phi}{\partial t}$

or $\dfrac{d\phi}{dt}=\left(\mathbf{i}\dfrac{\partial\phi}{\partial x}+\mathbf{j}\dfrac{\partial\phi}{\partial y}+\mathbf{k}\dfrac{\partial\phi}{\partial z}\right).(\mathbf{i}u+\mathbf{j}v+\mathbf{k}w)+\dfrac{\partial\phi}{dt}$

or $\dfrac{d\phi}{dt}=\mathbf{q}.(\nabla\phi)+\dfrac{\partial\phi}{\partial t}$

or $\dfrac{d\phi}{dt}=(\mathbf{q}.\nabla)\,\phi+\dfrac{\partial\phi}{\partial t}.$

So $\dfrac{d}{dt}=\dfrac{\partial}{\partial t}+(\mathbf{q}.\nabla)$

This relation is also true for ϕ as a vector point function

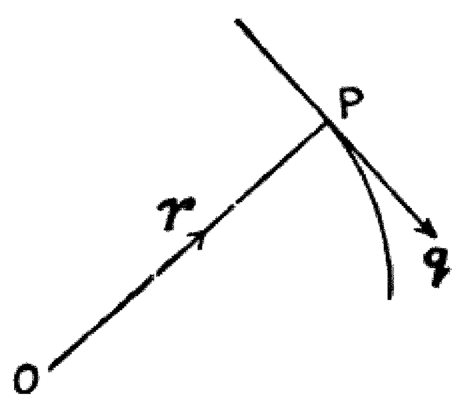
The operator $\dfrac{d}{dt}\equiv\dfrac{\partial}{\partial t}+(\mathbf{q}.\nabla)$ is known as the **differentiation following the motion of the fluid** or the substantive derivative.

§ 1·31. Acceleration.

Cartesian Co-ordinates. In § 1·3, if ϕ is replaced by the velocity vector $\mathbf{q}$, then

$$\frac{d\mathbf{q}}{dt}=\frac{\partial\mathbf{q}}{\partial t}+(\mathbf{q}.\nabla)\,\mathbf{q}=\vec{f}\ \text{(say)}$$

$$\left\{\text{Since } \mathbf{q}.\nabla=u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}\right\}$$



Let (u, v, w) be the components of velocity. The components of acceleration (f_x, f_y, f_z) are given by

$$f_x=\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}$$

$$f_y=\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}$$

and $$f_z=\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}$$

In this expression the first term is the rate at which the velocity increases at the point $(x\ y\ z)$ with regard to a fixed point. We shall denote the operator

$$\frac{\partial}{\partial t}+\mathbf{u}\frac{\partial}{\partial \mathbf{x}}+\mathbf{v}\frac{\partial}{\partial \mathbf{y}}+\mathbf{w}\frac{\partial}{\partial \mathbf{z}} \text{ by the symbol } \frac{\mathbf{D}}{\mathbf{Dt}}$$

Cor : Let the motion be along the curve S. Consider $\mathbf{q}$ be the velocity at a point P, then

$$\mathbf{q}=f(S, t)$$

and $\mathbf{q}+\delta\mathbf{q}$ be the velocity at any time $t+\delta t$, So

$$\mathbf{q}+\delta\mathbf{q}=f(S+\delta S,\ t+\delta t),$$

or
$$\mathbf{q}+\delta\mathbf{q}=f(S+q\delta t,\ t+\delta t) \qquad \{\text{since } \delta S=q\delta t\}$$

or
$$\mathbf{q}+\delta\mathbf{q}=f(S, t)+\left(q\frac{\partial f}{\partial S}+\frac{\partial f}{\partial t}\right)\partial t+\ldots.$$

Then
$$(\mathbf{q}+\delta\mathbf{q})-\mathbf{q}=\left\{f(S\ t)+\left(q\frac{\partial f}{\partial S}+\frac{\partial f}{\partial t}\right)\partial t+\ldots.\right\}-f(S\ t).$$

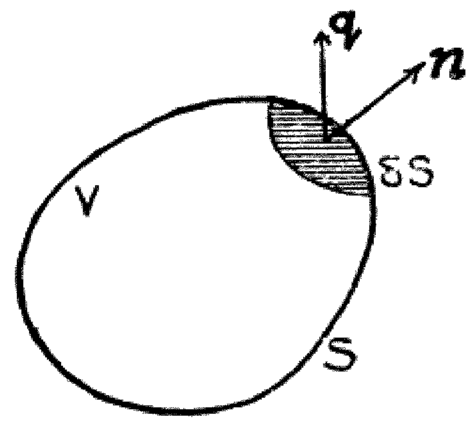
or
$$\underset{\delta t\to 0}{\text{Lt}}\ \frac{\delta\mathbf{q}}{\delta t}=\left(\mathbf{q}\frac{\partial f}{\partial S}+\frac{\partial f}{\partial t}\right)+\ldots\ldots$$

Thus acceleration is given by

$$\frac{d\mathbf{q}}{dt}=\frac{\partial f}{\partial t}+\mathbf{q}\frac{\partial f}{\partial S}=\frac{\partial \mathbf{q}}{\partial t}+\mathbf{q}\frac{\partial \mathbf{q}}{\partial S}$$

§ 1·4. Equation of Continuity.

When a region of fluid contain neither sources nor sinks *i.e.* there is no creation or annihilation of the fluid, then the amount of fluid within the region is conserved in accordance with the principle of conservation of matter. Now we shall develope the above principle mathematically by means of so-called Equation of continuity or conservation of mass.



Let ρ be the density of the fluid at any point (x, y, z) in V at any instant. Consider a fluid particle of infinitesimal volume dv and density $\rho\,(r, t)$ at any time t. The mass of this fluid particle cannot change as it moves about, therefore

$$\frac{d}{dt}(\rho\ dv)=0$$

this is one from of the **equation of continuity** or **conservation of mass.**

Consider a closed surface S in a fluid medium containing a volume V fixed in space. Let $\mathbf{n}$ is the unit outward drawn normal at a surface element δS. If $\mathbf{q}$ be the fluid velocity at the element δS, the normal component of $\mathbf{q}$ measured outward from V will be

$$=\mathbf{n}.\mathbf{q}$$

$L^{-3}\, L T^{-1}\, L^{2} = T^{-1}$

Rate of mass-flow across δS per unit mass

$$=\rho\ \mathbf{n}.\mathbf{q}\ \delta S$$

Total Rate of mass-flow out of V across δS

$$=\int_V \rho\ \mathbf{n}.\mathbf{q}\ dS$$

Total rate of mass-flow into V

$$=-\int_S \mathbf{n}.(\rho\mathbf{q})\ dS=-\int_V \nabla.(\rho\mathbf{q})\ dV \quad ...(i)$$

(By Gauss Theorem)

Also rate of **increase** of mass with in V

$$=\frac{\partial}{\partial t}\left\{\int_V \rho\ dV\right\}$$

$$=\int_V \frac{\partial\rho}{\partial t}\,dV \quad ...(ii)$$

Since the equation of continuity is based on the principle that the rate of increase of the mass of fluid with in the volume V is equal to the excess of the mass that flows-in-over the mass that flows out. Then from (i) and (ii), we have

$$\int_V \frac{\partial\rho}{\partial t}\,dV=-\int_V \nabla.(\rho\mathbf{q})\ dV$$

or
$$\int_V \frac{\partial\rho}{\partial t}\,dV+\int_V \nabla.(\rho\mathbf{q})\ dV=0$$

or
$$\int_V \left\{\frac{\partial\rho}{\partial t}+\nabla.(\rho\mathbf{q})\right\}dV=0 \quad ...(iii)$$

Since the surface S can be replaced by any arbitrary closed surface drawn within it, we must have, at every point

$$\frac{\partial\rho}{\partial t}+\nabla.(\rho\mathbf{q})=0 \quad ...(iv)$$

Which is known as **equation of continuity**, and is true at any point of a fluid free from sources and sinks.

Equation (iv) can be written in the following manner also.

Since $\nabla.(\rho\mathbf{q})=\rho\nabla.\mathbf{q}+\mathbf{q}.\nabla\rho$

Then (iv) reduces to

$$\frac{\partial\rho}{\partial t}+\rho\nabla.\mathbf{q}+\mathbf{q}.\nabla\rho=0$$

or
$$\frac{\partial\rho}{\partial t}+\rho\nabla.\mathbf{q}=0$$

or
$$\frac{d}{dt}(\log\rho)+\nabla.\mathbf{q}=0 \quad ...(v)$$

$\left\{ \text{where } \frac{d}{dt} \text{ denotes the differentiation following the fluid motion} \right.$

as $\left. \frac{d}{dt} \equiv \frac{\partial}{\partial t} + \mathbf{q}\cdot\nabla \right\}$

I. In case of **Steady flow :** the pattern of flow does not very with regard to time or the path of the fluid particle coincides with the stream line, So $\frac{\partial \rho}{\partial t}=0$ then (iv) reduces to

$$\nabla\cdot(\rho\mathbf{q})=0 \qquad \text{...(vi)}$$

II. In case of **non-homogeneous incompressible fluid :** the density of the fluid particle is invariable with time *i.e.* ρ=constant through out the entire region

$$\frac{\partial \rho}{\partial t}=0.$$

then from (iv), we have

$$\nabla\cdot\mathbf{q}=0$$

or
$$\textbf{Div } \mathbf{q}=0$$

or
$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0.$$

The quantity $\nabla\cdot\mathbf{q}$ gives the rate of volume expansion of a fluid element. It may be called **Dilatation** or **Expansion.**

III. If the flow is of the potential kind, then there exist a velocity potential ϕ *i.e.* for an irrotational motion

$$\mathbf{q}=-\nabla\phi$$

then from (II), we have

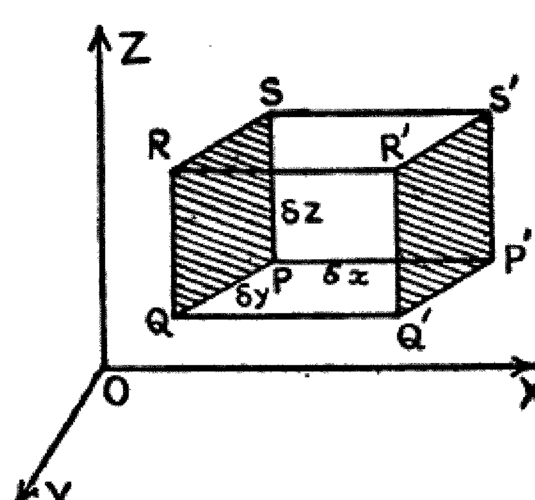
$$\nabla^2\phi=0.$$

Which is a **Laplace's Equation.**

§ 1·5. Alternative method : Equation of continuity,

(Cartesian Coordinates)

Consider ρ be the density of the fluid at P (x, y, z) and u, v, w be the components of the velocity parallel to the co-ordinate axes. Construct a small parallelopiped with edges of length $\delta x, \delta y, \delta z$ parallel to the axes.



Consider any closed surface drawn in the fluid then the increase in the mass of fluid within the surface in any time δt is equal to the excess of mass that flows in over the mass that flows out in the time δt.

Mass of the fluid that passes in through the face $PQRS$.

$$=\rho\ \delta y\ \delta z . u \quad \text{per unit time}$$

$$=f(x, y, z) \qquad \text{(let)} \qquad \ldots\text{(i)}$$

Mass of the fluid that passes out through the opposite face $P'Q'R'S'$.

$$=f(x+\delta x, y, z)$$

$$=f(x\ y\ z)+\delta x . \frac{\partial}{\partial x} f(x\ y\ z)+\ldots \qquad \ldots\text{(ii)}$$

(by Taylor's Theorem)

The excess of flow-in over flow out (in the direction of axis of X).

=Mass that enters through the face $PQRS$—Mass that leaves through the face $P'Q'R'S'$.

$$=f(x\ y\ z)-f(x\ y\ z)-\delta x \ . \ \frac{\delta}{\delta x} f(\ x\ y\ z)+\ldots$$

$$=-\delta x \ . \frac{\partial}{\partial x} f(x\ y\ z)$$

$$=-\delta x \frac{\partial}{\partial x}\left\{ \rho u\ \delta y\ \delta z \right\} \qquad \{\text{from } (i)\}$$

$$=-\frac{\partial(\rho u)}{\partial x} . \delta x\ \delta y\ \delta z \qquad \ldots\text{(iii)}$$

Similarly the excess of flow-in over flow out through the faces $QQ'R'R$ and $PP'S'S$

$$=-\ \delta y \frac{\partial}{\partial y}\left\{ \rho v\ \delta x\ \delta z \right\}$$

$$=-\frac{\partial(\rho v)}{\partial y} \delta x\ \ \delta y\ \delta z \qquad \ldots\text{(iv)}$$

and the excess of flow-in over flow out through the faces $PP'Q'Q$ and $SS'R'R$.

$$=-\delta z \frac{\partial}{\partial z}\left\{ \rho w\ \delta x\ \delta y \right\}$$

$$=-\frac{\partial(\rho w)}{\partial z} . \delta x\ \delta y\ \delta z. \qquad \ldots\text{(iv)}$$

Thus the total excess of flow-in over flow out from all the faces

$$=-\ \delta x\ \delta y\ \delta z \left\{ \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} \right\} \qquad \ldots\text{(v)}$$

Again total mass in the parallelopiped

$$=\rho.\delta x\ \delta y\ \delta z$$

Rate of increase in the mass of the parallelopiped

$$=\frac{\partial}{\partial t}(\rho\ \delta x\ \delta y\ \delta z)$$

$$=\delta x\ \delta y\ \delta z.\frac{\partial \rho}{\partial t}\text{ per unit time.} \qquad \text{...(vi)}$$

Since increase in mass = Total Excess of flow-in over flow-out through all the faces.

From (v) and (vi), we have

$$\delta x\ \delta y\ \delta z\ .\frac{\partial \rho}{\partial t}=-\delta x\ \delta y\ \delta z\ \left\{\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)\right\}$$

or
$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v)+\frac{\partial}{\partial z}(\rho w)=0. \qquad \text{...(vii)}$$

Which is known as the **equation of continuity in cartesian coordinates.**

Cor. (I) Equation (vii) can be written as

$$\frac{\partial \rho}{\rho t}+u\ \frac{\partial \rho}{\partial x}+v\ \frac{\partial \rho}{\partial y}+w\ \frac{\partial \rho}{\partial z}+\rho\ \left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0$$

or
$$\frac{D\rho}{Dt}+\rho\ \left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0.$$

{Refer § 1 . 31}

Cor. (II) If the fluid be non-homogeneous and incompressible, then

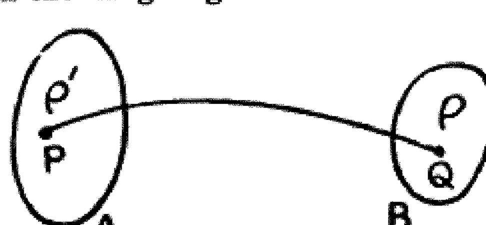
$$\rho=\text{cons.},\ \frac{D\rho}{Dt}=0$$

So
$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

measures the rate at which the volume of an element of fluid at $(x\ y\ z)$ is expanding and known as **dilatation** or the **expansion.**

§ 1·6. Equation of continuity in the Lagrangian Method.

Consider A be the region occupied by a fluid at the time $t=0$ and B the region occupied by the same fluid at any time t



Let $(a\ b\ c)$ be the initial co-ordinates of a particle P enclosed in this element and ρ' be the density.

Mass of the fluid element $=\rho'\ \delta a\ \delta b\ \delta c.$

Also the mass of the element enclosing the subsequent point Q is

$$=\rho\ \delta x\ \delta y\ \delta z$$

(where ρ is the density of the fluid at Q).

So the total mass inside the region A is equal to the total mass inside the region B, we have

$$\iiint_A \rho'\ da\ db\ dc = \iiint_B \rho\ dx\ dy\ dz$$

or
$$\iiint_A \rho'\ da\ db\ dc = \iiint_A \rho \cdot \frac{\partial (x\ y\ z)}{\partial (a\ b\ c)}\ da\ db\ dc$$

$$\left\{ \text{since } \delta x\ \delta y\ \delta z = \frac{\partial (x\ y\ z)}{\partial (a\ b\ c)} \delta a\ \delta b\ \delta c \right\}$$

or
$$\iiint_A \left\{ \rho' - \rho\ \frac{\partial (x\ y\ z)}{\partial (a\ b\ c)} \right\} da\ db\ dc = 0.$$

Since the region A is an arbitrary, then

$$\rho' - \rho\ \frac{\partial (x\ y\ z)}{\partial (a\ b\ c)} = 0$$

or
$$\rho' = \rho \frac{\partial (x\ y\ z)}{\partial (a\ b\ c)}$$

or
$$\rho' = \rho J \quad \text{where } J = \begin{vmatrix} \frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} & \frac{\partial x}{\partial c} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} & \frac{\partial y}{\partial c} \\ \frac{\partial z}{\partial a} & \frac{\partial z}{\partial b} & \frac{\partial z}{\partial c} \end{vmatrix}$$

which is called the equation of continuity in **Lagrangian form.**

§ 1·7. Equivalence of the two forms of the equation of continuity. (Lagrangian and Eulerian form)

We know that the equation of continuity in the Lagrangian form is

$$\rho J = \rho' \qquad \text{...(i)}$$

where $\qquad J = \dfrac{\partial (x\ y\ z)}{\partial (a\ b\ c)}$

Differentiating (i) w. r. to t, we have

$$\rho\ \frac{dJ}{dt} + J\ \frac{d\rho}{dt} = 0 \qquad \text{...(ii)}$$

These time-rates are variations due to the motion of a particle or the variability of x, y, z. Now we shall change from the

Lagrangian form to the Eulerian form of variables. We know that

$$\frac{du}{da}=\frac{\partial}{\partial a}\left(\frac{\partial x}{\partial t}\right)=\frac{d}{dt}\left(\frac{\partial x}{\partial a}\right) \text{ etc.}$$

$$\left\{\text{Since } u=\frac{dx}{dt},\ v=\frac{dy}{dt} \text{ and } w=\frac{dz}{dt}\right\}$$

Again $\dfrac{d\rho}{dt}=\dfrac{\partial\rho}{\partial t}+u\dfrac{\partial\rho}{\partial x}+v\dfrac{\partial\rho}{\partial y}+w\dfrac{\partial\rho}{\partial z}$

Sinc $J=\dfrac{\partial(x\,y\,z)}{\partial(a\,b\,c)}=\begin{vmatrix}\dfrac{\partial x}{\partial a} & \dfrac{\partial y}{\partial a} & \dfrac{\partial z}{\partial a}\\[2ex] \dfrac{\partial x}{\partial b} & \dfrac{\partial y}{\partial b} & \dfrac{\partial z}{\partial b}\\[2ex] \dfrac{\partial x}{\partial c} & \dfrac{\partial y}{\partial c} & \dfrac{\partial z}{\partial c}\end{vmatrix}$

or $\dfrac{dJ}{dt}=\begin{vmatrix}\dfrac{\partial u}{\partial a} & \dfrac{\partial y}{\partial a} & \dfrac{\partial z}{\partial a}\\[2ex] \dfrac{\partial u}{\partial b} & \dfrac{\partial y}{\partial b} & \dfrac{\partial z}{\partial b}\\[2ex] \dfrac{\partial u}{\partial c} & \dfrac{\partial y}{\partial c} & \dfrac{\partial z}{\partial c}\end{vmatrix}+\begin{vmatrix}\dfrac{\partial x}{\partial a} & \dfrac{\partial v}{\partial a} & \dfrac{\partial z}{\partial a}\\[2ex] \dfrac{\partial x}{\partial b} & \dfrac{\partial v}{\partial b} & \dfrac{\partial z}{\partial b}\\[2ex] \dfrac{\partial x}{\partial c} & \dfrac{\partial v}{\partial c} & \dfrac{\partial z}{\partial c}\end{vmatrix}+\begin{vmatrix}\dfrac{\partial x}{\partial a} & \dfrac{\partial y}{\partial a} & \dfrac{\partial w}{\partial a}\\[2ex] \dfrac{\partial x}{\partial b} & \dfrac{\partial y}{\partial b} & \dfrac{\partial w}{\partial b}\\[2ex] \dfrac{\partial x}{\partial c} & \dfrac{\partial y}{\partial c} & \dfrac{\partial w}{\partial c}\end{vmatrix}$

or $\dfrac{dJ}{dt}=\dfrac{\partial(u\,y\,z)}{\partial(a\,b\,c)}+\dfrac{\partial(x\,v\,z)}{\partial(a\,b\,c)}+\dfrac{\partial(x\,y\,w)}{\partial(a\,b\,c)}$. ...(iii)

But $\dfrac{\partial u}{\partial a}=\dfrac{\partial u}{\partial x}\cdot\dfrac{\partial x}{\partial a}+\dfrac{\partial u}{\partial y}\cdot\dfrac{\partial y}{\partial a}+\dfrac{\partial u}{\partial z}\cdot\dfrac{\partial z}{\partial a}$

$\dfrac{\partial u}{\partial b}=\dfrac{\partial u}{\partial x}\cdot\dfrac{\partial x}{\partial b}+\dfrac{\partial u}{\partial y}\cdot\dfrac{\partial y}{\partial b}+\dfrac{\partial u}{\partial z}\cdot\dfrac{\partial z}{\partial b}$

and $\dfrac{\partial u}{\partial c}=\dfrac{\partial u}{\partial x}\cdot\dfrac{\partial x}{\partial c}+\dfrac{\partial u}{\partial y}\cdot\dfrac{\partial y}{\partial c}+\dfrac{\partial u}{\partial z}\cdot\dfrac{\partial z}{\partial c}$

By eliminating $\dfrac{\partial u}{\partial y}$, $\dfrac{\partial u}{\partial z}$, we obtain

$$\frac{\partial u}{\partial x}\cdot\frac{\partial(x\,y\,z)}{\partial(a\,b\,c)}=\frac{\partial(u\,y\,z)}{\partial(a\,b\,c)}$$

or $\dfrac{\partial(u\,y\,z)}{\partial(a\,b\,c)}=J\dfrac{\partial u}{\partial x}$ etc. ...(iv)

The relation (iii) reduces to with the help of (iv)

$$\frac{dJ}{dt}=J\frac{\partial u}{\partial x}+J\frac{\partial v}{\partial y}+J\frac{\partial w}{\partial z}$$

or

$$\frac{dJ}{dt}=J\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right).$$

Substituting the value of $\frac{dJ}{dt}$ in (ii), we have

$$J\frac{d\rho}{dt}+\rho J\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial x}\right)=0.$$

or

$$\frac{d\rho}{dt}+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)-0,$$

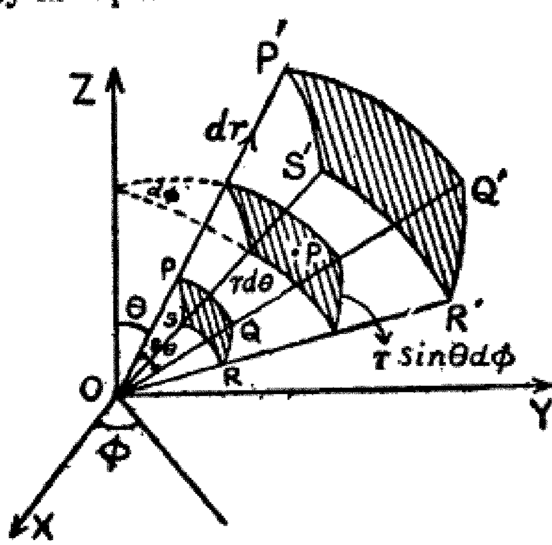
or

$$\frac{d\rho}{dt}+\rho \text{ div. } \mathbf{q}=0,$$

which is the Equation of Continuity in Eulerian form.

§ 1·8 Equation of Continuity in Spherical Polar Coordinates.

Let ρ be the density of the fluid at the point $P\ (r,\ \theta,\ \phi)$. Construct a parallelopiped with P as centre, the length of whose edges are dr, $r\ d\theta$ and $r \sin\theta\ d\phi$. Let q_r, q_θ and q_ϕ be the components of the velocity in the direction of the element respectively.



Mass of the fluid that passes through the face $PQRS$

$$=\rho\ (r\delta\theta . r \sin\theta\ \delta\phi) . q_r$$
$$=f(r,\ \theta,\ \phi) \text{ (say)}$$

and mass of the fluid that passes through the opposite face $P'Q'R'S'$

$$=f(r+\delta r,\ \theta,\ \phi)$$
$$=f(r,\ \theta,\ \phi)+\delta r . \frac{\partial}{\partial r} f(r,\ \theta,\ \phi)+\dots$$

...(To first approximation).

Excess of mass of flow-in over flow out through the faces $PQRS$ and $P'Q'R'S'$.

$$=f(r,\theta,\phi)-f(r,\theta,\phi)-\delta r\frac{\partial}{\partial r}f(r,\theta,\phi)\ldots\ldots$$

$$=-\delta r\frac{\partial}{\partial r}f(r,\theta,\phi)$$

$$=-\delta r.\frac{\partial}{\partial r}\{\rho\ r\ \delta\theta\cdot r\sin\theta\,\delta\phi . q_r\}\text{ per unit time}$$

$$=-\frac{\partial}{\partial r}\{\rho r^2 q_r\}\ \delta r\ \delta\theta\cdot\sin\theta\ \delta\phi .$$

Similarly excess of mass of flow-in over flow out through the faces $PSS'P'$ and $QRR'Q'$

$$=-r\ \delta\theta\cdot\frac{\partial}{r\,\partial\theta}\{\rho.\delta r\cdot r\sin\theta\ \delta\phi\cdot q_\theta\}\text{ per unit time}$$

$$=-\frac{\partial}{r\,\partial\theta}\{\rho\ q_\theta\sin\theta\}.\delta r\cdot r\delta\theta\cdot r\cdot\delta\phi$$

and Excess of mass flow in-over flow out through the face $PQQ'P'$ and $SRR'S'$

$$=-r\sin\theta\ \delta\phi.\frac{\partial}{r\sin\theta\ \partial\phi}\{\rho\ \delta r\cdot r\ \delta\theta\ q_\phi\}\text{ per unit time}$$

$$=-\frac{\partial}{r\sin\theta\,\partial\phi}\{\rho\ q_\phi\}\cdot\partial r\cdot r\ \delta\theta\cdot r\sin\theta\ \partial\phi .$$

Total excess of flow-in over flow out

$$=-\frac{\partial}{\partial r}\{\rho\ r^2 q_r\}\ \delta r\ \delta\theta\cdot\sin\theta\ \delta\phi$$

$$-\frac{\partial}{r\,\partial\theta}\{\rho\ q_\theta\sin\theta\}\cdot\delta r\cdot r\ \delta\theta\cdot r.\delta\phi$$

$$-\frac{\partial}{r\sin\theta\ \partial\phi}\{\rho\ q_\phi\}.\delta r\cdot r\ \delta\theta\cdot r\sin\theta\ \delta\phi .$$

Total mass in side the parallelopiped

$$=\rho\ \delta r\cdot r\delta\theta\cdot r\sin\theta\ \delta\phi$$

So increase in the mass inside the parallelopiped

$$=\frac{\partial}{\partial t}\{\rho\delta r . r\delta\theta\cdot r\sin\theta\ \delta\phi\}$$

$$=\frac{\partial\rho}{\partial t}\cdot r^2\sin\theta\ \delta r\ \delta\theta\ \delta\phi .$$

By the principle of equation of continuity, we have

Increase in mass=Excess of mass of flow-in over flow out

or $$\frac{\partial \rho}{\partial t}.r^2 \sin\theta\, \delta r\, \delta\theta\, \delta\phi = -\frac{\partial}{\partial r}\{\rho\, r^2 q_r\} \sin\theta\, \delta r\, \delta\theta\, \delta\phi - \frac{\partial}{r\,\partial\theta}\{\rho\, q_\theta \sin\theta\}\, r^2\, \delta r\, \delta\theta\, \delta\phi - \frac{\partial}{r \sin\theta\, \delta\phi}\{\rho\, q_\phi\}\cdot r^2 \sin\theta\, \delta r\, \delta\theta\, \delta\phi$$

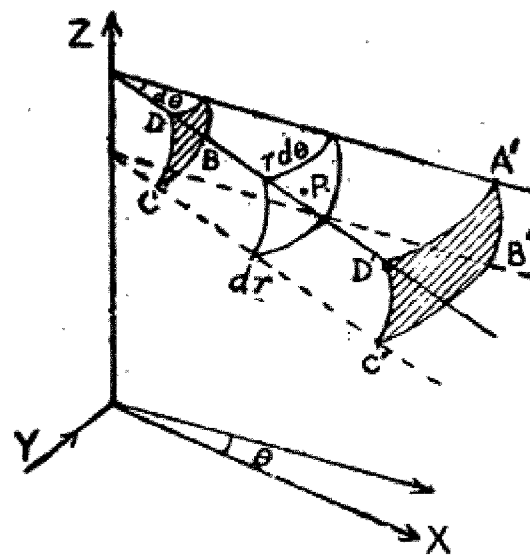
or $$r^2 \sin\theta\cdot\frac{\partial \rho}{\partial t} + \sin\theta\,\frac{\partial}{\partial r}(\rho r^2 q_r) + r^2\cdot\frac{\partial}{r\,\partial\theta}(\rho q_\theta \sin\theta) + r^2 \sin\theta\cdot\frac{\partial}{r\sin\theta\,\partial\phi}(\rho\, q_\phi) = 0$$

or $$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\cdot\frac{\partial}{\partial r}(\rho\, r^2 q_r) + \frac{1}{r\sin\theta}\cdot\frac{\partial}{\partial\theta}(\rho\, q_\theta \sin\theta) + \frac{1}{r\sin\theta}\cdot\frac{\partial}{\partial\phi}(\rho\, q_\phi) = 0$$

which determines the **equation of continuity** in **Spherical polar Coordinates.**

§ 1·81. Equation of continuity in cylindrical coordinates.

Let ρ be the density of the fluid at the point $P(r, \theta\, z)$. Construct a curvilinear parallelopiped with P as centre, the length of whose edges are dr, $rd\theta$ and dz.



Let q_r, q_θ and q_z be the components of the velocity in the direction of the elements respectively.

The mass of the fluid that passes through the face $ABCD$

$$=\rho\,(r\delta\theta.\delta z)\, q_r \text{ per unit time}$$
$$=f(r\,\theta\, z) \qquad \text{(let)}$$

and mass of the fluid that passes through the opposite face $A'B'C'D'$

$$=f(r+\delta r, \theta, z)$$
$$=f(r\;\theta\;z)+\delta r.\frac{\partial}{\partial r}f(r\,\theta\,z)+\ldots\ldots$$

...(To first approximation).

Excess of mass of flow-in over flow out through the faces $ABCD$ and $A'B'C'D'$

$$=f(r\,\theta\,z)-f(r\,\theta\,z)-\delta r.\frac{\partial}{\partial r}f(r\,\theta\,z)\ldots\ldots$$

$$=-\delta r\frac{\partial}{\partial r}f(r\,\theta\,z)$$

$$=-\delta r.\frac{\partial}{\partial r}\{\rho\;r\;q_r\;\delta\theta\;\delta z\}=-\frac{\partial}{\partial r}\{\rho\;r\;q_r\}.\delta r\;\delta\theta\;\delta z.$$

Similarly the excess of mass of flow-in over flow out through the other faces are

$$=-r\;\delta\theta.\;\frac{\partial}{r\,\partial\theta}\{\rho\;\delta r\;\delta z\;q_\theta\}\;=-\frac{\partial}{r\,\partial\theta}(\rho\;q_\theta).r\;\delta r\;\delta\theta\;\delta z$$

and $$=-\;\delta z.\frac{\partial}{\partial z}\{\rho\;\delta r.r\delta\theta\;q_z\}=-\frac{\partial}{\partial z}(\rho\;q_z).r\;\delta r\;\delta\theta\;\delta z.$$

Total excess of flow-in over flow out

$$=-\left\{\frac{\partial}{\partial r}(\rho r q_r)\;\delta r\delta\theta\delta z+\frac{\partial}{r\partial\theta}(\rho q_\theta).r\delta r\delta\theta\delta z+\frac{\partial}{\partial z}(\rho q_z).r\delta r\delta\theta\delta z\right\}$$

Total mass of fluid inside the parallelopiped

$$=\rho\delta r.r\delta\theta.\delta z.$$

Rate of increase in the mass inside the parallelopiped

$$=\frac{\partial}{\partial t}\{\rho\delta r.r\delta\theta.\delta z\}$$

$$=\frac{\partial\rho}{\partial t}.r\delta r\delta\theta\delta z$$

By the principle of equation of continuity, we have

Increase in mass=Excess of mass of flow in over flow out.

or $$\frac{\partial\rho}{\partial t}.(r\delta r\delta\theta\delta z)=-\left\{\frac{\partial}{\partial r}(\rho r q_r)+r.\frac{\partial}{r\partial\theta}(\rho q_\theta)+r.\frac{\partial}{\partial z}(\rho q_z)\right\}\delta r\delta\theta\delta z$$

or $$r.\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial r}(\rho r q_r)+r\frac{\partial}{r\partial\theta}(\rho q_\theta)+r\frac{\partial}{\partial z}(\rho q_z)=0.$$

or $$\frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(\rho r q_r)+\frac{1}{r}\frac{\partial}{\partial\theta}(\rho q_\theta)+\frac{\partial}{\partial z}(\rho q_z)=0$$

which is the **Equation of Continuity in Cylindrical Co-ordinates.**

§ 1·82. Method of writing the Continuity Equation.

Consider ρ be the density of the fluid. Construct a parallelopiped the length of whose edges are $dx\;dy\;dz$, then

(a) Cartesian Coordinates.

Length of elements	dx	dy	dz
Components of velocity	u	v	w

Now to calculate the flux* along the co-ordinate axes respectively, take the negative derivative with respect to the co-ordinate axes respect of the product,

density × velocity in the direction of axis of X etc. × product of the remaining length

add them and equate this sum to the rate of increase of mass inside the region.

i.e.
$$-\frac{\partial}{\partial x}(\rho u dy dz)\, dx - \frac{\partial}{\partial y}(\rho v dx dz)\, dy - \frac{\partial}{\partial z}(\rho w dx dy)\, dz = \frac{\partial}{\partial t}(\rho dx dy dz)$$

(b) Spherical Polar Co-ordinates.

Length of elements	dr	$rd\theta$	$r\sin\theta\, d\phi$
Components of velocity	q_r	q_θ	q_ϕ

Then flux along the co-ordinates axes

$$= -\frac{\partial}{\partial r}\{\rho r d\theta . r\sin\theta\, d\phi\, q_r\}\, dr - \frac{\partial}{r\partial\theta}\{\rho dr . r\sin\theta d\phi\, q_\theta\}\, r\delta\theta - \frac{\partial}{r\sin\theta\,\partial\phi}\{\rho dr . r d\theta . q_\phi\} . r\sin\theta\,\delta\phi$$

equate this sum to the rate of increase of mass in side the region

$$= \frac{\partial}{\partial t}\{\rho dr . r d\theta . r\sin\theta\, d\phi\}$$

(c) Cylindrical Co-ordinates.

Length of elements	dr	$rd\theta$	dz
Components of velocity	q_r	q_θ	q_z

$$= -\frac{\partial}{\partial r}(\rho q_r r d\theta dz)\, dr - \frac{\partial}{r\partial\theta}(\rho q_\theta dr dz)\, r d\theta - \frac{\partial}{\partial z}(\rho q_z dr . r d\theta)\, dz$$

equate this sum to the rate of increase of mass inside the region

$$= \frac{\partial}{\partial t}(\rho dr . r d\theta\, dz)$$

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* **Flux :** In any case of motion of an incompressible fluid the surface integral of the normal velocity taken over any surface, open or closed is known as the flux across the surface.

§ 1·83. General orthogonal curvilinear Coordinates.

A generalised coordinate system consists of a three fold family of surfaces whose equation in terms of the rectangular coordinates are,

$$\xi_1 (x\ y\ z) = \text{const.} = \lambda_1,$$
$$\xi_2 (x\ y\ z) = \text{const.} = \lambda_2$$
$$\text{and } \xi_3 (x\ y\ z) = \text{const.} = \lambda_3. \qquad \text{...(i)}$$

Where x, y, z are the cartesian coordinates.

The surfaces $\lambda_1 = \text{const.}$, $\lambda_2 = \text{const.}$ and $\lambda_3 = \text{const.}$ ...(ii) form an orthogonal system. The lines of intersection of these surfaces constitute three families of lines. A relation can be expressed in (i) and (ii).

$$x = x(\lambda_1\ \lambda_2\ \lambda_3),\ y = y(\lambda_1\ \lambda_2\ \lambda_3),\ z = z(\lambda_1\ \lambda_2\ \lambda_3)$$

The surfaces $\lambda_1 = \text{const.}$, $\lambda_2 = \text{const.}$, $\lambda_3 = \text{const.}$ are called **Coordinate surface.**

Consider $\mathbf{r}$ be the position vector of a point, then

$$\mathbf{r} = \mathbf{r}(\lambda_1\ \lambda_2\ \lambda_3) \qquad \{\text{Representation of relation (i) in vector form.}\}$$

Tangent vector at the point to the λ_1—curve

$$= \frac{\partial r}{\partial \lambda_1}$$

So the unit tangent vector in the direction of the λ_1—curve

or $$e_1 = \frac{\partial \mathbf{r}/\partial \lambda_1}{|\partial \mathbf{r}/\partial \lambda_1|}$$

or $$\frac{\partial \mathbf{r}}{\partial \lambda_1} = h_1 \mathbf{e}_1$$

$$\left\{ \begin{aligned} &\left| \frac{\partial \mathbf{r}}{\partial \lambda_1} \right| = h_1 \\ &\sqrt{\left[\left(\frac{\partial x}{\partial \lambda_1} \right)^2 + \left(\frac{\partial y}{\partial \lambda_1} \right)^2 + \left(\frac{\partial z}{\partial \lambda_1} \right)^2 \right]} = h_1 \end{aligned} \right.$$

Similarly the unit tangent vector in the direction of λ_2—curve and λ_3—curve is given by

$$\frac{\partial \mathbf{r}}{\partial \lambda_2} = h_2\, \mathbf{e}_2 \text{ and } \frac{\partial \mathbf{r}}{\partial \lambda_3} = h_3\, \mathbf{e}_3$$

where the quantity h_1, h_2, h_3 is known a **Scale factor.**

Since $$\mathbf{r} = \mathbf{r}(\lambda_1\ \lambda_2\ \lambda_3)$$

then $$d\mathbf{r} = \frac{\partial \mathbf{r}}{\partial \lambda_1} . d\lambda_1 + \frac{\partial \mathbf{r}}{\partial \lambda_2} . d\lambda_2 + \frac{\partial \mathbf{r}}{\partial \lambda_3} . d\lambda_3$$

$$d\mathbf{r} = h_1 d\lambda_1 . \mathbf{e}_1 + h_2 d\lambda_2 . \mathbf{e}_2 + h_3 d\lambda_3 . \mathbf{e}_3$$

and $$ds^2 = d\mathbf{r} . d\mathbf{r}$$

$$= h_1^2 d\lambda_1^2 + h_2^2 d\lambda_2^2 + h_3^2 d\lambda_3^2.$$

(i) **Volume of a curvilinear parallelopiped** whose edges are $h_1 d\lambda_1$, $h_2 d\lambda_2$ and $h_3 d\lambda_3$, is given by $=h_1 h_2 h_3 . d\lambda_1 d\lambda_2 d\lambda_3$
$\left\{ \begin{array}{ll} \text{Since} & dr=h_1 d\lambda_1 e_1 \\ \text{So} & ds=h_1 d\lambda_1 \end{array} \right.$

The vector definitions (in curvilinear coordinates) are as follows :

(ii) $$\text{grad. } \phi=\left(\frac{1}{h_1}\frac{\partial\phi}{\partial\lambda_1}, \frac{1}{h_2}\frac{\partial\phi}{\partial\lambda_2}, \frac{1}{h_3}\frac{\partial\phi}{\partial\lambda_3}\right)$$
$$=\frac{1}{h_1}\frac{\partial\phi}{\partial\lambda_1}\,\mathbf{a}_1+\frac{1}{h_2}\frac{\partial\phi}{\partial\lambda_2}\,\mathbf{a}_2+\frac{1}{h_3}\frac{\partial\phi}{\partial\lambda_3}.\mathbf{a}_3$$

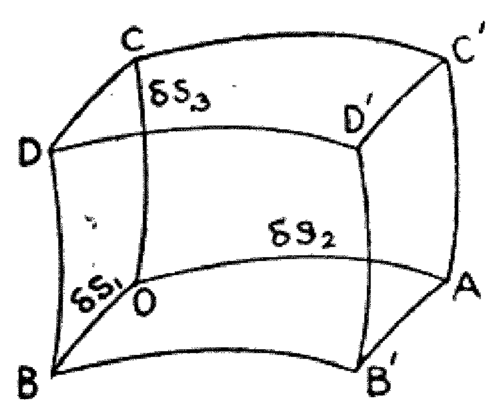
(iii) $$\text{div. } \vec{\mathbf{q}}=\frac{1}{h_1 h_2 h_3}\left\{\frac{\partial}{\partial\lambda_1}(q_1 h_2 h_3)+\frac{\partial}{\partial\lambda_2}(q_2 h_3 h_1)+\frac{\partial}{\partial\lambda_3}(q_3 h_1 h_2)\right\}$$

(iv) $$\text{curl. } \vec{\mathbf{q}}=\frac{1}{h_1 h_2 h_3}\begin{vmatrix} h_1\mathbf{a}_1 & h_2 a_2 & h_3 a_3 \\ \frac{\partial}{\partial\lambda_1} & \frac{\partial}{\partial\lambda_2} & \frac{\partial}{\partial\lambda_3} \\ h_1 q_1 & h_2 q_2 & h_3 q_3 \end{vmatrix}$$

(v) $$\nabla^2\phi=\frac{1}{h_1 h_2 h_3}\left\{\frac{\partial}{\partial\lambda_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial\phi}{\partial\lambda_1}\right)+\frac{\partial}{\partial\lambda_2}\left(\frac{h_3 h_1}{h_2}\frac{\partial\phi}{\partial\lambda_2}\right)+\frac{\partial}{\partial\lambda_3}\left(\frac{h_1 h_2}{h_3}.\frac{\partial\phi}{\partial\lambda_3}\right)\right\}$$

§ 1·84. **Equation to the conservation of mass in orthogonal curvilinear coordinates.**

Consider λ_1=const., λ_2=const. and λ_3=const. be the three families of surfaces that cut one another orthogonally at all their points of intersection (λ_1, λ_2, λ_3 represent the functions of rectangular coordinates x, y, z). Construct a curvilinear parallelopiped at O with edges $OB=\delta S_1$, $OA=\delta S_2$ and $OC=\delta S_3$.



Let q_1, q_2 and q_3 be the components of velocity in the dircetion of λ_1, λ_2 and λ_3 respectively.

Mass of the liquid that flows through the face $OAC'C$
$$=\rho q_1\,\delta S_2\,\delta S_3 \quad \text{per unit time}$$
Mass of the liquid that passes out through the face $BB'D'D$
$$=\rho q_1\,\delta S_2\,\delta S_3+\frac{\partial}{\partial S_1}(\rho q_1\,\delta S_2\,\delta S_3)\,\delta S_1$$

Excess of rate of flow in over flow out per unit time through the faces $OAC'C$ and $BB'D'D$

$$=-\frac{\partial}{\partial S_1}\{\rho q_1\,\delta S_2\,\delta S_3\}\,\delta S_1 \qquad \text{...(i)}$$

Similarly excess of rate of flow in over flow out per unit time through the other two opposite faces.

$$=-\frac{\partial}{\partial S_2}\left\{\rho q_2\,\delta S_1\,\delta S_3\right\}\delta S_2 \qquad \text{...(ii)}$$

and

$$=-\frac{\partial}{\partial S_3}\left\{\rho q_3\,\delta S_1\,\delta S_2\right\}\delta S_3 \qquad \text{...(iii)}$$

Now total mass inside the parallelopiped

$$=\rho\,\delta S_1\,\delta S_2\,\delta S_3$$

Rate of increase of mass of fluid inside the parallelopiped

$$=\frac{\partial}{\partial t}\left\{\rho\,\delta S_1\,\delta S_2\,\delta S_3\right\} \qquad \text{...(iv)}$$

Since by the principle of continuity,

Rate of increase of mass inside the parallelopiped=total excess of rate of flow-in over flow out through the parallelopiped

Then

$$\frac{\partial}{\partial t}\left\{\rho\,\delta S_1\,\delta S_2\,\delta S_3\right\}=-\frac{\partial}{\partial S_1}\left\{\rho q_1\,\delta S_2\,\delta S_3\right\}\delta S_1$$
$$-\frac{\partial}{\partial S_2}\left\{\rho q_2\,\delta S_1\,\delta S_3\right\}\delta S_2-\frac{\partial}{\partial S_3}\left\{\rho q_3\,\delta S_1\,\delta S_2\right\}\delta S_3$$

Substituting $\delta S_1=h_1 d\lambda_1$, $\delta S_2=h_2 d\lambda_2$ and $\delta S_2=h_3 d\lambda_3$

or $$h_1h_2h_3\,d\lambda_1\,d\lambda_2\,d\lambda_3.\frac{\partial\rho}{\partial t}=-\frac{\partial}{h_1\,\partial\lambda_1}\left\{\rho q_1\,h_2h_3\right\}h_1d\lambda_1d\lambda_2d\lambda_3$$
$$-\frac{\partial}{h_2\partial\lambda_2}\left\{\rho q_2\,h_1h_3\right\}h_2\,d\lambda_1\,d\lambda_2\,d\lambda_3$$
$$-\frac{\partial}{h_3\partial\lambda_3}\left\{\rho q_3\,h_1h_2\right\}.\quad h_3\,d\lambda_1\,d\lambda_2\,d\lambda_3.$$

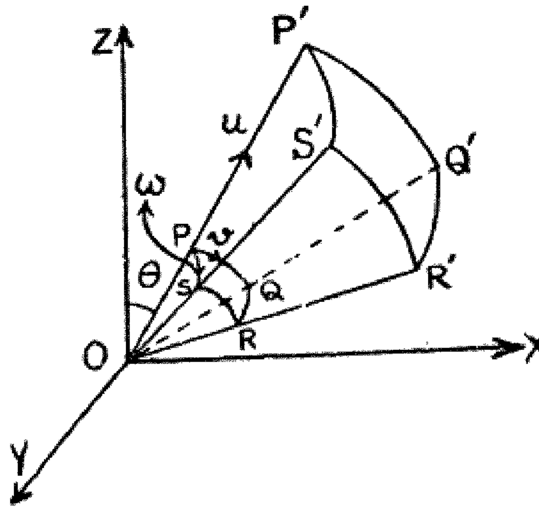
or $$\frac{\partial\rho}{\partial t}+\frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial\lambda_1}(\rho q_1\,h_2\,h_3)+\frac{\partial}{\partial\lambda_2}(\rho q_2\,h_1\,h_3)+\frac{\partial}{\partial\lambda_3}(\rho q_3\,h_1\,h_2)\right\}=0.$$

Which is known as the equation to continuity in **orthogonal curvilinear coordinates.**

Ex. 1. *The particles of a fluid move symmetrically in space with regard to a fixed centre ; prove that the equation of continuity is*

$$\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial r}+\frac{\rho}{r^2}\cdot\frac{\partial}{\partial r}(r^2u)=0$$

where u is the velocity at distance r.



Let $P(r, \theta, \omega)$ be a point in the fluid. Construct a parallelopiped with

PP' ($=\delta r$), PQ ($=r\delta\theta$) and PS ($=r\sin\theta\,\delta\omega$) as edges.

Consider u, v, w be the components of the velocity in the direction of the elements δr, $r\delta\theta$ and $r\sin\theta\,\delta\omega$.

Let the origin O be the fixed centre. Since the fluid particles move symmetrically in space with reqard to an origin (fixed centre) that means the motion is only along PP'. So there is no motion along other edges $r\delta\theta$ and $r\sin\theta\,\delta\omega$.

Therefore the excess of flow-in over flow out from the face $PQRS$ and the oppposite face $P'Q'R'S'$ along PP' in unit time

$$=-\delta r\frac{\partial}{\partial r}\{\rho u . r d\theta . r\sin\theta\,\delta\omega\}$$

Similarly the excess of mass of flow-in over flow out along PQ and PS vanishe as there is no motion along these directions.

Mass of the fluid inside the element

$$=\rho\delta r . r\delta\theta . r\sin\theta\,\delta\omega.$$

Increase in the mass of the element

$$=\frac{\partial}{\partial t}(\rho\delta r . r\delta\theta . r\sin\theta\delta\omega) \text{ per unit time.}$$

From the principle of equation of continuity, we have

Increase in mass=Excess of flow-in over flow out

$$\frac{\partial}{\partial t}(\rho\delta r . r\delta\theta . r\sin\theta\delta\omega)=-\delta r\frac{\delta}{\partial r}(\rho u . r\delta\theta . r\sin\theta\delta\omega)$$

$$\text{or}\quad \frac{\partial \rho}{\partial t}\{\delta r . r\delta\theta . r\sin\theta\,\delta\omega\}+\frac{\partial}{\partial r}(\rho u r^2)\,\delta r . \delta\theta\sin\theta\delta\omega=0.$$

$$\text{or}\quad \frac{\partial \rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(\rho u r^2)=0.$$

$$\text{or}\quad \frac{\partial \rho}{\partial t}+\frac{1}{r^2}(ur^2).\frac{\partial \rho}{\partial r}+\frac{1}{r^2}\rho.\frac{\partial}{\partial r}(ur^2)=0$$

or $\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial r}+\frac{\rho}{r^2}\cdot\frac{\partial}{\partial r}(ur^2)=0.$ **Proved.**

Ex. 2. *A mass of fluid is in motion so that the lines of motion lie on the surface of coaxial cylinders. Show that the equation of continuity is*

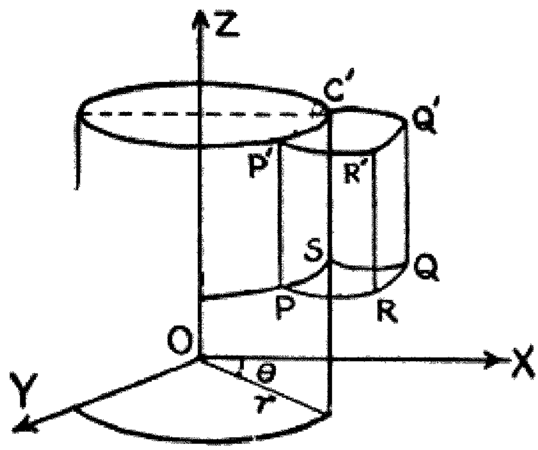
$$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial \theta}(\rho v_\theta)+\frac{\partial}{\partial z}(\rho v_z)=0$$

where v_θ v_z are the velocities perpendicular and parallel to z.

Let $P(r, \theta, z)$ be a point in the fluid. Construct a parallelopiped at P with edges

$PS\ (=r\delta\theta)$, $PR\ (=\delta r)$
and $PP'\ (=\delta z)$.

Let v_r, v_θ and v_z be the velocity components along PR, PS, PP'.



Since the lines of motion of the fluid lie on the surface of coaxial cylinders so there is no motion along PR.

Then excess of flow-in over flow out along PR vanishes.

Also excess of flow-in over flow out along PS

$$=-r\delta\theta\ \frac{\partial}{r\partial\theta}\{\rho v_\theta \delta r\cdot\delta z\}$$

and excess of flow-in over flow out alog PP'

$$=-\delta z\cdot\frac{\partial}{\partial z}\{\rho v_z\delta r\cdot r\delta\theta\}$$

Mass of the fluid in the element

$$=\rho r\delta\theta\cdot\delta r\cdot\delta z$$

Increase in the mass of the element

$$=\frac{\partial}{\partial t}\{\rho r\delta\theta\cdot\delta r\cdot\delta z\}\text{ per unit time.}$$

From the equation of continuity, we have

Increase in mass=change of flow-in over flow out.

or $\frac{\partial}{\partial t}\{\rho r\delta\theta\cdot\delta r\cdot\delta z\}=-r\delta\theta\cdot\frac{\partial}{r\partial\theta}\{\rho v_\theta\delta r\delta z\}-\delta z\frac{\partial}{\partial z}\{\rho v_z\delta r\cdot r\delta\theta\}$

or $\frac{\partial \rho}{\partial t}(r\delta\theta\cdot\delta r\cdot\delta z)+\frac{\partial}{r\partial\theta}(\rho v_\theta)\cdot(r\delta\theta\delta r\delta z)$

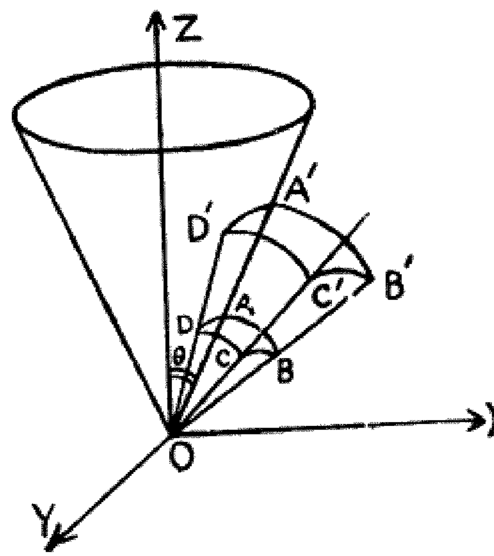
$$+\frac{\partial}{\partial z}(\rho v_z)\ (r\delta\theta\delta r\delta z)=0$$

or $\frac{\partial \rho}{\partial t}+\frac{\partial}{r\partial\theta}(\rho v_\theta)+\frac{\partial}{\partial z}(\rho v_z)=0.$ **Proved.**

Ex. 3. *If the lines of motions are curves on the surfaces of cones having their vertices at the origin and the axis of Z for common axis. Prove that the equation of continuity is*

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial r}\,(\rho q_r)+\frac{2\rho q_r}{r}+\frac{\text{cosec}\,\theta}{r}\,\frac{\partial}{\partial \omega}\,(\rho q_\omega)=0.$$

Let O be the vertex and OZ be the common axis of Z. Let $A\ (r, \theta, \omega)$ be a point on the surface of the cone and q_r, q_θ, q_ω be the components of the velocity along the edges of the parallelopiped AA', AB and AD respectively.



Since lines of motion are curves on the surfaces of cones so there will be no motion perpendicular to the surface of the cone. The length of the edges are

$$AA'=\delta r,\ AB=r\delta\theta \text{ and } AD=r\sin\theta\delta\omega.$$

Now excess of flow-in over flow out in the direction AA' *i.e.* from the face $ABCD$ and $A'B'C'D'$ in time δt

$$=-\frac{\partial}{\partial r}\left\{\rho q_r\cdot r\delta\theta\cdot r\sin\theta\delta\omega\right\}\delta r\cdot\delta t$$

and the excess of flow-in over flow out in the direction AD *i.e.* from the face $ABB'A$ and the opposite face in time δt

$$=-\frac{\partial}{r\sin\theta\delta\phi}\left\{\rho q_\omega\cdot r\delta\theta\delta r\right\}\cdot r\sin\theta\,\delta\omega\cdot\delta t$$

Total excess of flow-in over flow out in time δt

$$=-\frac{\partial}{\partial r}\left\{\rho q_r\, r\delta\theta\cdot r\sin\theta\delta\omega\right\}\delta r\cdot\delta t$$

$$-\frac{\partial}{r\sin\theta\,\delta\omega}\left\{\rho q_\omega\cdot r\delta\theta\,\delta r\right\} r\sin\theta\,\delta\omega\,\delta t$$

volume of an elementary parallelopiped ...(1)

$$=\delta r.r\delta\,\theta\cdot r\sin\theta\,\delta\omega$$

Increase in the mass of an element in time δt

$$=\frac{\partial}{\partial t}\left\{\rho\cdot r\,\delta\theta\cdot r\sin\theta\,\delta\omega\,\delta r\right\}\delta t \qquad ...(2)$$

Hence from the equation of continuity, we have

$$\frac{\partial \rho}{\partial t}\left\{r^2 \sin\theta\,\delta\theta\,\delta\omega\,\delta r\right\}\delta t=-\frac{\partial}{\partial r}\left\{\rho q_r\cdot r\,\delta\theta\cdot r\sin\theta\,\delta\omega\right\}\delta r\,\delta t$$
$$-\frac{\partial}{r\sin\theta\,\partial\omega}\left\{\rho q_\omega\cdot r\,\delta\theta\,\delta r\right\}r\sin\theta\,\delta\omega\,\delta t$$

or $$\frac{\partial\rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}\left(\rho q_r . r^2\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\omega}\left(\rho q_\omega\right)=0$$

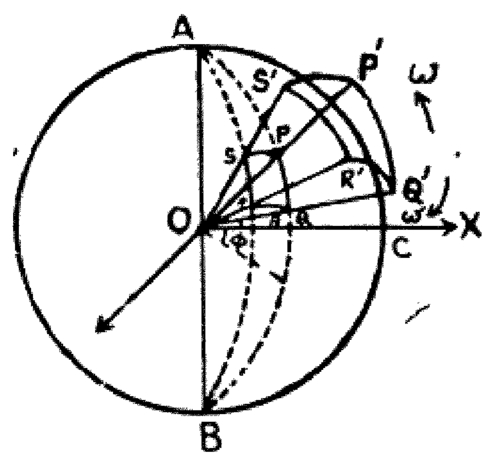
or $$\frac{\partial\rho}{\partial t}+\frac{1}{r^2}\left\{r^2.\frac{\partial}{\partial r}(\rho q_r)+(\rho q_r).2r\right\}+\frac{\operatorname{cosec}\theta}{r}\frac{\partial}{\partial\omega}(\rho q_\omega)=0$$

or $$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial r}(\rho q_r)+\frac{2\rho q_r}{r}+\frac{\operatorname{cosec}\theta}{r}\frac{\partial}{\partial\omega}(\rho q_\omega)=0.$$ Proved.

Ex. 4. *If every particle moves on the surface of a sphere, prove that the equation of contniuity is*

$$\frac{\partial\rho}{\partial t}\cos\theta+\frac{\partial}{\partial\theta}(\rho\omega\cos\theta)+\frac{\partial}{\partial\phi}(\rho\omega'\cos\theta)=0.$$

ρ being the density, θ, ϕ the latitude and longitude of any element, and ω and ω' the angular velocities of the element in latitude and longitude respectively.



Consider an elementary parallelopiped $PQRS$ and $P'Q'R'S'$ on the surface of the sphere, where

$PP'=\delta r$, $PQ=r\,\delta\theta$, $PS=r\cos\theta\,\delta\phi$ are the edges of the element. Since ω and ω' are the angular velocities along θ and ϕ respectively so $r\omega$ and $r\cos\omega'$ are the velocities along PQ and PS.

Since the particle moves on the surface of the sphere so there will be no velocity normal to the surface of the sphere that means the velocity along PP' vanishes.

Now excess of flow-in over flow out along the faces perpendicular to PQ *i.e.* from the face $QQ'R'R$ and $PP'S'S$ per unit time is

$$=-r\delta\theta.\frac{\partial}{r\partial\theta}\left\{\rho\,r\omega\cdot\delta r\cdot r\cos\theta\,\delta\phi\right\}$$

Similarly excess of flow-in over flow out along the faces perpendicular to PS *i.e.*, from the face $QQ'P'P$ and the opposite face per unit time is

$$=-r\cos\theta\,\delta\phi.\frac{\partial}{r\cos\theta\,\partial\phi}\left\{\rho r\cos\omega'.\delta r.r\,\delta\theta\right\}$$

and excess of flow-in over flow out along the faces perpendicular to PP' vanishes.

Total excess of flow-in over flow out

$$=\Big[-r\delta\theta.\frac{\partial}{r\partial\theta}\left\{\rho.r\omega.\delta r.r\cos\delta\phi\right\}$$
$$-r\cos\theta\delta\phi.\frac{\partial}{r\cos\theta\partial\phi}\left\{\rho\,r\cos\omega'.\delta r\cdot r\delta\theta\right\}\Big]$$
per unit time ...(1)

Volume of the parallelopiped

$$=\delta r.r\,\delta\theta\cdot r\cos\theta\,\delta\phi$$

Increase in the mass in unit time is

$$=\frac{\partial}{\partial t}\left\{\rho\,\delta r\cdot r\,\delta\theta\cdot r\cos\theta\,\delta\phi\right\} \quad ...(2)$$

Therefore the equation of continuity is given by, from (1) and (2)

$$\frac{\partial}{\partial t}\left\{\rho\delta r.r\,\delta\theta.r\cos\delta\phi\right\}=-r\,\delta\theta.\frac{\partial}{r\,\partial\theta}\left\{\rho\,r\omega.\delta r.r\cos\theta\,\delta\phi\right\}$$
$$-r\cos\theta\,\partial\phi.\frac{\partial}{r\cos\theta\,\partial\phi}\left\{\rho\,r\cos\theta\,\omega'\cdot\delta r\cdot r\,\delta\theta\right\}$$

or $$\frac{\partial\rho}{\partial t}\left\{r^2\cos\theta\,\delta r\,\delta\theta\,\delta\phi\right\}+r\,\delta\theta.\frac{\partial}{r\,\partial\theta}(\rho\omega\cos\theta)\,r^2\,\delta r\,\delta\phi$$
$$+r\cos\theta\,\delta\phi.\frac{\partial}{r\cos\theta\,\delta\phi}(\rho\omega'\cos\theta)\,r^2\,\delta r\,\delta\theta=0$$

or $$\frac{\partial\rho}{\partial t}+\frac{1}{\cos\theta}\cdot\frac{\partial}{\partial\theta}(\rho\omega\cos\theta)+\frac{1}{\cos\theta}\cdot\frac{\partial}{\partial\phi}(\rho\omega'\cos\theta)=0.$$

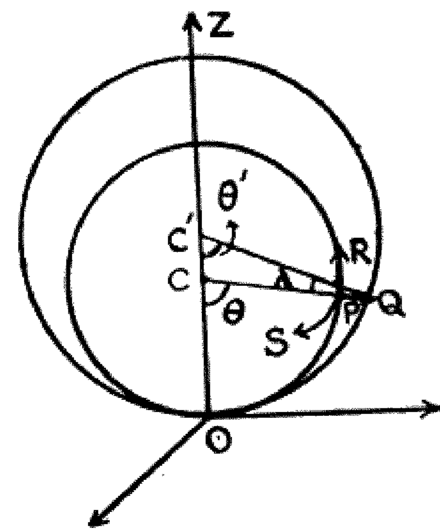
or $$\cos\theta\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial\theta}(\rho\omega\cos\theta)+\frac{\partial}{\partial\phi}(\rho\omega'\cos\theta)=0.$$

Proved.

Ex. 5. *If the lines of motion are curves on the surfaces of spheres all touching the plane of xy at the origin O, the equation of continuity is*

$$r\sin\theta\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial\phi}(\rho v)+\sin\theta\frac{\partial}{\partial\theta}(\rho u)+\rho u\,(1+2\cos\theta)=0$$

where r is the radius CP of one of the spheres, θ the angle PCO, u the velocity in the plane PCO, v the perpendicular velocity, and φ the inclination of the plane PCO to a fixed plane through the axis of Z.



Let C and C' be the centres on the Z-axis of the two consecutive spheres of radii r and $r+\delta r$ respectively $\qquad CC'=\delta r$

Join the line CP which meets the second sphere in Q. Consider P be a point on the inner sphere and PQ, PR and PS be the edges of the elementary parallelopiped.

Now the length of the edges is
$$PR=r\,\delta\theta,\ PS=r\sin\theta\,\delta\phi$$
where ϕ is the angle which the plane PCO makes with a fixed plane through the Z-axis *i.e.* with XOZ plane).

Now we shall determine the length of the edge PQ.

$$CP=r,\ C'Q=r+\delta r,\ CC'=\delta r,\ \angle PCO=\theta$$

Let $\qquad \angle CC'Q=\theta'$ and $\angle C'QC=\lambda$

Since $\qquad \theta'+\lambda=\theta$

Also $CQ^2=C'Q^2+CC'^2-2C'Q.CC\cos\theta'$

or $\quad (r+PQ)^2=(r+\delta r)^2+(\delta r)^2-2\ (r+\delta r).\delta r\cos(\theta-\lambda)$

or $\quad r^2+2r.PQ+PQ^2=r^2+2r\,\delta r+\delta r^2+\delta r^2$
$$-2\ (r\,\delta r+\delta r^2)\cos(\theta-\lambda)$$

Neglecting the squares of the quantities PQ and δr, being small, we have

$$2r.PQ=2r\,\delta r-2r\,\delta r.\cos(\theta-\lambda)$$

or $\quad PQ=\delta r\,\{1-\cos(\theta-\lambda)\}$

or $\quad PQ=\delta r\,\{1-(\cos\theta\cos\lambda+\sin\theta\sin\lambda)\}$

or $\quad PQ=\delta r\,\{1-\cos\theta-\lambda\sin\theta\}$ (To small app,)

or $\quad PQ=\delta r\,\{1-\cos\theta\}$ (Neglecting small quantities)

Since lines of motion are curves on the surface of sphere, so there will be no component of velocity along PQ, u and v be the velocity components along the edges PR and PS *i.e.* in the direction of θ and ϕ increasing.

Now excess of flow-in over flow out in the direction PR

$$=-\frac{\partial}{r\,\partial\theta}\left\{\rho u.r\sin\theta\,\delta\phi.(1-\cos\theta)\,\delta r\right\}.r\,\delta\theta$$

per unit time.

Similarly the excess of flow-in over flow out in the direction PS

$$=-\frac{\partial}{r\sin\theta\,\partial\phi}\left\{\rho v.r\,\delta\theta\,\delta r\,(1-\cos\theta)\right\}r\sin\theta\,\partial\phi$$

per unit time

and the excess of flow-in over flow out in the direction PQ vanishes.

Total excess of flow-in over flow out

$$=\left[-\frac{\partial}{r\,\partial\theta}\left\{\rho u\,r\sin\theta\,\delta\phi.(1-\cos\theta)\,\delta r\right\}r\,\delta\theta\right.$$
$$\left.-\frac{\partial}{r\sin\theta\,\partial\phi}\left\{\rho v\,r\,\delta\theta\,\delta r\,(1-\cos\theta)\right\}r\sin\theta\,\delta\phi\right]$$

per unit time. ...(1)

Volume of the parallelopiped

$$=(1-\cos\theta)\,\delta r.r\delta\theta.r\sin\theta\,\delta\phi$$

Increase in the mass of the parallelopiped

$$=\frac{\partial}{\partial t}\left\{\rho.(1-\cos\theta)\,\delta r.r\,\delta\theta.r\sin\theta\,\delta\phi\right\}\text{ per unit time.}\qquad ...(2)$$

Therefore, the equation of continuity is given by, from (1) and (2)

$$\frac{\partial}{\partial t}\left\{\rho\,(1-\cos\theta)\,\delta r.r\,\delta\theta.r\sin\theta\,\delta\phi\right\}$$
$$=-\frac{\partial}{r\,\partial\theta}\left\{\rho u.r\sin\theta\,\delta\phi.(1-\cos\theta)\,\delta r\right\}r\,\delta\theta$$
$$-\frac{\partial}{r\sin\theta\,\partial\phi}\left\{\rho v.r\,\delta\theta.\delta r\,(1-\cos\theta)\right\}r\sin\theta\,\delta\phi$$

or
$$\frac{\partial\rho}{\partial t}\left\{(1-\cos\theta)\,\delta r\,r\,\delta\theta.r\sin\theta\,\delta\phi\right\}$$
$$+\frac{\partial}{r\,\partial\theta}\left\{\rho u\sin\theta\,(1-\cos\theta)\right\}r^2\,\delta r\,\delta\phi\,\delta\theta$$
$$+\frac{\partial}{r\sin\theta\,\partial\phi}\left\{\rho v\right\}\delta r\,(1-\cos\theta).r\,\delta\theta.r\sin\theta\,\delta\phi=0$$

or
$$\frac{\partial\rho}{\partial t}+\frac{1}{r\sin\theta\,(1-\cos\theta)}\cdot\frac{\partial}{\partial\theta}\left\{\rho u.\sin\theta\,(1-\cos\theta)\right\}$$
$$+\frac{1}{r\sin\theta}\cdot\frac{\partial}{\partial\phi}(\rho v)=0$$

or
$$\frac{\partial\rho}{\partial t}+\frac{1}{r\sin\theta\,(1-\cos\theta)}\left[\sin\theta\,(1-\cos\theta)\frac{\partial}{\partial\theta}(\rho u)\right.$$
$$\left.+\rho u.\{\cos\theta\,(1-\cos\theta)+\sin^2\theta\}\right]+\frac{1}{r\sin\theta}\cdot\frac{\partial}{\partial\phi}(\rho v)=0$$

or $$\frac{\partial \rho}{\partial t}+\frac{1}{r}\cdot\frac{\partial}{\partial \theta}(\rho u)+\frac{\rho u\,(1+\cos\theta-2\cos^2\theta)}{r\sin\theta\,(1-\cos\theta)}$$
$$+\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(\rho v)=0$$

or $$\frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial \theta}(\rho u)+\frac{\rho u\,(1+2\cos\theta)}{r\sin\theta}+\frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(\rho v)=0$$

or $$r\sin\theta\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial \phi}(\rho v)+\sin\theta\frac{\partial}{\partial \theta}(\rho u)+\rho u\,(1+2\cos\theta)=0.$$

Proved.

Exercise

1. Show that the equation of continuity reduces to Laplace's equation when the liquid is incompressible and the motion is irrotational.

2. A mass of fluid moves in such a way that each particle describes a circle in one plane about a fixed axis, show that the equation of continuity is

$$\frac{\partial \rho}{\partial t}+\frac{\partial(\rho\omega)}{\partial \theta}=0$$

where ω is the angular velocity of a particle whose azimuthal angle is θ at time t.

3. Each particle of a mass of liquid moves in a plane through the axis of Z; determine the equation of continuity.

4. Homogeneous liquid moves so that the path of any particle P lies in the plane POX where OX is fixed axis. Prove that if $OP=r$ and the $\angle XOP=\theta$, the equation of continuity may be written as

$$\frac{\partial}{\partial r}(ur^2)-\frac{\partial}{\partial \mu}(vr\sin\theta)=0,$$

where u, v are the component velocities along and perpendicular to OP in the plane POX and $\mu=\cos\theta$.

Answers.

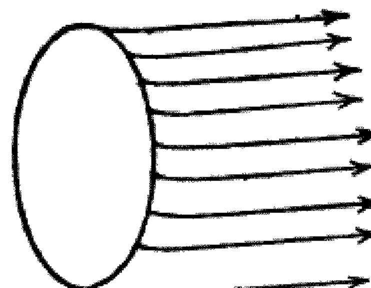
3. $$\frac{\partial \rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(\rho ur^2)+\frac{1}{r\sin\theta}\cdot\frac{\partial}{\partial \theta}(\rho v\sin\theta)=0.$$

§ 1·83. Stream lines.

In Eulerian method we have defined that the velocity as a function of time is given in direction and magnitude at every point in the space considered.

Let **q** $(u\ v\ w)$ be the velocity at each point $P\ (x\ y\ z)$ of the fluid at any point. We draw a curve C in the fluid for a fixed instant of time such that the direction of the tangent at P to C coincides with the direction of the ;velocity **q** at P. Then the curve C is called a **stream line.**

If we draw the stream lines through every point of a closed curve C in the fluid then we obtain the **stream tube.**



(Stream tube)

Let dr be the element of the arc length along the stream line and **q** be the fluid velocity. Then the equation of stream line is given by

$$\mathbf{q}\times d\,\mathbf{r}=0.$$

or
$$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w}$$

Where $\mathbf{q}=(u\ v\ w)$

and $\quad d\,\mathbf{r}=(dx\ dy\ dz)$

Path lines. From Lagrange method we conclude that the paths of the different fluid particle are as a function of time. Thus, a path line is a curve along which a specific fluid particle moves during its motion. Thus the differential equation of the path lines are

$$\mathbf{q}=\frac{d\mathbf{r}}{dt}$$

i.e. $\quad u\ (x, y, z, t)=\dfrac{dx}{dt},\ v\ (x, y, z\ t)=\dfrac{dy}{dt}$ and $w\ (x, y, z\ t)=\dfrac{dz}{dt}$

with initial conditions $x\ (t_0)=x_0, y\ (t_0)=y_0$ and $z\ (t_0)=z_0$

For a steady flow the velocity **q** is independent of time. It is evident that in a steady flow the path lines and the stream lines are identical.

§ 1·84. Velocity Potential.

Let **q** $(u\ v\ w)$ be the velocity of the fluid particle at any instant t. If the expression $u\ dx+v\ dy+w\ dz$ is a perfect differential

i.e. $$u\ dx+v\ dy+w\ dz=-d\phi \qquad \ldots\text{(i)}$$

or $$u\ dx+v\ dy+w\ dz=-\frac{\partial\phi}{\partial x}dx-\frac{\partial\phi}{\partial y}dy-\frac{\partial\phi}{\partial z}dz$$

Then $\quad u=-\dfrac{\partial\phi}{\partial x},\ v=-\dfrac{\partial\phi}{\partial y}$ and $w=-\dfrac{\partial\phi}{\partial z}$

or $$\mathbf{q}=-\nabla\phi \quad \ldots(2)$$

where ϕ is called the **velocity potential.**

The necessary and sufficient condition for (1) to be a perfect differential is

$$\nabla\times\mathbf{q}=0. \quad ..(3)$$

§ 1·85. **Irrotational Motion.**

If the relation (2) of § 1·84 holds *i.e.* the velocity potential ϕ exists or when the expressions $\dfrac{\partial w}{\partial y}-\dfrac{\partial v}{\partial z}$, $\dfrac{\partial u}{\partial z}-\dfrac{\partial w}{\partial x}$ and $\dfrac{\partial v}{\partial x}-\dfrac{\partial u}{\partial y}$ all vanish then the motion is said to be **irrotational.**

Ex. 1. *Show that*

$$u=-\frac{2xyz}{(x^2+y^2)^2},\ v=\frac{(x^2-y^2)z}{(x^2+y^2)^2},\ w=\frac{y}{x^2+y^2}$$

are the velocity components of a possible liquid motion. Is this motion irrotational?

If u, v and w satisfy the equation of continuity then it will be a possible liquid motion.

i.e. $$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0.$$

$$=2yz.\ \frac{3x^2-y^2}{(x^2+y^2)^3}+2yz.\frac{y^2-3x^2}{(x^2+y^2)^3}+0$$

$$=\text{zero}.$$

Which shows that the liquid motion is possible.

Again for irrotational motion, we have

$$\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}=0,\ \frac{\partial w}{\partial x}-\frac{\partial u}{\partial z}=0 \text{ and } \frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}=0.$$

So $$\frac{\partial v}{\partial z}-\frac{\partial w}{\partial y}=\frac{x^2-y^2}{(x^2+y^2)^2}-\frac{x^2-y^2}{(x^2+y^2)^2}=0.$$

also $$\frac{\partial w}{\partial x}-\frac{\partial u}{\partial z}=-\frac{2xy}{(x^2+y^2)^2}+\frac{2xy}{(x^2+y^2)^2}=0$$

and $$\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}=\frac{2xz\,(3y^2-x^2)}{(x^2+y^2)^2}-\frac{2xz\,(3y^2-x^2)}{(x^2+y^2)^2}=0.$$

Which satisfies the condition, hence the motion is irrotational.

Ex. 2. *Given* $u=-\dfrac{c^2y}{r^2}$, $v=\dfrac{c^2x}{r^2}$, $w=0$, *where r denotes the distance from Z-axis. Find the surfaces which are orthogonal to stream lines, the liquid being homogeneous.*

If u, v, w satisfy the equation of continuity then it will be a possible liquid motion.

i.e.
$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

or
$$\frac{2c^2\,xy}{r^4}-\frac{2c^2\,xy}{r^4}+0=0$$

which is an identity therefore the liquid motion is possible.

$$\frac{\partial u}{\partial x}=\frac{2c^2 y}{r^3},\ \frac{\partial r}{\partial x}=\frac{2c^2 y}{r^3}\cdot\frac{x}{r}$$

as $\qquad x^2+y^2=r^2$

$$\therefore \qquad \frac{\partial r}{\partial x}=\frac{x}{r}$$

The differential equation to the lines of flow are

$$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w}$$

or
$$\frac{dx}{-c^2\,y/r^2}=\frac{dy}{c^2x/r^2}=\frac{dz}{0}$$

or
$$\frac{dx}{-y}=\frac{dy}{x}=\frac{dz}{0}$$

or
$$x\,dx+y\,dy=0,\ dz=0$$

or
$$x^2+y^2=\text{const. and } z=\text{const.}$$

The surfaces which cut the stream lines orthogonally are

$$u\,dx+v\,dy+w\,dz=0$$

or
$$-\frac{c^2y}{r^2}\,dx+\frac{c^2x}{r^2}\,dy=0$$

or
$$-\frac{c^2y}{x^2+y^2}\,dx+\frac{c^2x}{x^2+y^2}\,dy=0$$

or
$$c^2\,\frac{x\,dy-y\,dx}{x^2+y^2}=0$$

By integrating, we have

$$c^2\tan^{-1}\left(\frac{y}{x}\right)=\text{const.}$$

or
$$\tan^{-1}\left(\frac{y}{x}\right)=\text{const}=\lambda$$

or
$$\frac{y}{x}=\tan\lambda=\mu\ \text{(let)}$$

or
$$y=\mu x$$

which represents a plane through Z-axis.

Ex. 3. *If the velocity of an incompressible fluid at the point* (x, y, z) *is given by* $\frac{3xz}{r^5}, \frac{3yz}{r^5}, \frac{3z^2-r^2}{r^5}$, *prove that the liquid motion is possible and that the velocity potential is* $\frac{\cos\theta}{r^2}$. *Also determine the stream lines.*

The liquid motion is possible

if $$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

or $$\frac{3z}{r^5}-\frac{15x^2z}{r^7}+\frac{3z}{r^5}-\frac{15y^2z}{r^7}+\frac{6z}{r^5}-\frac{15z^3}{r^5}+\frac{3z}{r^5}=0$$

or $$\frac{15z}{r^5}-\frac{15z\,(x^2+y^2+z^2)}{r^7}=0$$

or $$\frac{15z}{r^5}-\frac{15z}{r^5}=0,$$

which is an identity, hence the motion is a possible one.

$$\left[\begin{array}{ll} \text{Since} & r^2=x^2+y^2+z^2 \\ \text{then} & \frac{\partial r}{\partial x}=\frac{x}{r}\;\frac{\partial u}{\partial x} \\ & =\frac{3z}{r^5}+\left(-\frac{15xz}{r^6}\right)\frac{\partial r}{\partial x} \end{array}\right.$$

If ϕ be the velocity potential,

then $$d\phi=\frac{\partial\phi}{\partial x}\,dx+\frac{\partial\phi}{\partial y}\,dy+\frac{\partial\phi}{\partial z}\,dz$$

or $$d\phi=-(u\,dx+v\,dy+w\,dz)$$

or $$d\phi=-\frac{1}{r^5}\left\{3\,xz\,dx+3yz\,dy+(3z^2-r^2)\,dz\right\}$$

or $$d\phi=-\frac{1}{r^5}\left\{3z\,(x\,dx+y\,dy+z\,dz)-r^2\,dz\right\}$$

or $$d\phi=-\frac{3z}{2}\cdot\frac{d\,(x^2+y^2+z^2)}{r^5}+\frac{dz}{r^3}$$

or $$d\phi=-\frac{3z}{2}\cdot\frac{d\,(r^2)}{r^5}+\frac{dz}{r^3}$$

or $$d\phi=d\left(\frac{z}{r^3}\right)$$

By integrating, we have

$$\phi=\frac{z}{r^3}=\frac{r\cos\theta}{r^3}=\frac{\cos\theta}{r^2}$$

(Constant of integration vanishes as it has no significance)

Also the equation to stream lines are,

$$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w}$$

or
$$\frac{dx}{\frac{3xz}{r^5}}=\frac{dy}{\frac{3yz}{r^5}}=\frac{dz}{\frac{3z^2-r^2}{r^6}}$$

or
$$\underset{(1)}{\frac{dx}{3xz}}=\underset{(2)}{\frac{dy}{3yz}}=\underset{(3)}{\frac{dz}{3z^2-(x^2+y^2+z^2)}}=\underset{(4)}{\frac{x\,dx+y\,dy+z\,dz}{2z\,(x^2+y^2+z^2)}}$$

From (1) and (2), we have

$$\frac{dx}{x}=\frac{dy}{y}$$

or
$$\log x=\log y+\log c$$

or
$$\frac{x}{y}=c \qquad ...(5)$$

From (1) and (4), we have

$$\frac{dx}{3x}=\frac{x\,dx+y\,dy+z\,dz}{2(x^2+y^2+z^2)}$$

By integrating, we have

$$\tfrac{2}{3}\log x=\tfrac{1}{2}\log(x^2+y^2+z^2)+\log D$$

(where D is any arbitrary constant)

or
$$x^{2/3}=D\,(x^2+y^2+z^2)^{1/2} \qquad ...(6)$$

Hence equations (5) and (6) represent the stream lines.

Ans.

Ex. 4. *Given $u=-\omega y$, $v=\omega x$, $w=0$, show that the surfaces intersecting the stream lines orthogonally exist and are the planes through Z-axis, although the velocity potential daes not exist.*

The motion will be possible if it satisfies the equation of continuity.

i.e. $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$ which is true from the given relation. Hence the motion is a possible one.

The differential equation to the lines of flow are,

$$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w}$$

or
$$\frac{dx}{-\omega y}=\frac{dy}{\omega x}=\frac{dz}{0}$$

or
$$x\,dx+y\,dy=0 \quad \text{and} \quad dz=0$$

By integrating, we have

$$x^2+y^2=\text{const.} \quad \text{and} \quad z=\text{const.}$$

The surfaces which cut the stream lines orthogonally are

$$u\,dx+v\,dy+w\,dz=0$$

or
$$-\omega y\,dx+\omega x\,dy=0$$

or
$$\frac{dx}{x}-\frac{dy}{y}=0.$$

By integrating, we have

$$\log\left(\frac{x}{y}\right)=\log c \quad \text{(where } c \text{ is an arbitrary const).}$$

or $\frac{x}{y}=c$ which represent a plane through Z-axis and cuts the stream line orthogonally.

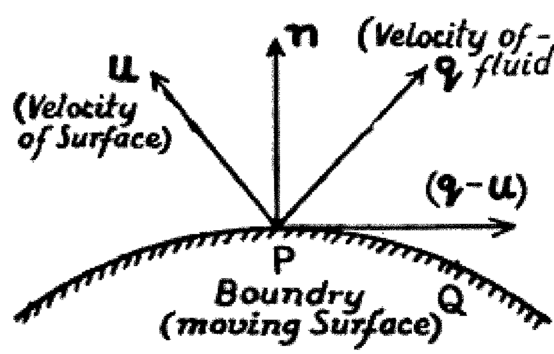
The velocity potential will exist if $u\,dx+v\,dx+w\,dz$ is a perfect differential

But $u\,dx+v\,dy+w\,dz$ is not a perfect differential. Hence the surfaces intersecting the stream lines orthogonally exist and are the planes through Z-axis, although the velocity potential does not exist.

§ 1·9. Boundary Surface.

At the boundary of the fluid, the equation of continuity is replaced by a special surface condition

When the fluid is in contact with a bounding surface then velocity of a fluid particle at any point of the boundary relative to the surface must be tangential to the boundary. Thus at a fixed boundary, the velocity of the fluid perpendicular to the surface must vanish and the normal component of the fluid velocity must be equal to the normal component of the surface.



Let **q** be the velocity of the fluid and **u** the velocity of the point P at the surface. Consider **n** be the unit normal vector drawn at the point P on the boundary surface $F(\mathbf{r}, t)=0$.

then $\quad \mathbf{q}.\mathbf{n}=\mathbf{u}.\mathbf{n} \qquad$ ...(i)

or $\quad (\mathbf{q}-\mathbf{u}).\mathbf{n}=0 \qquad$ {since $\mathbf{n}=\nabla F$

or $\quad (\mathbf{q}-\mathbf{u}).\nabla F=0. \qquad$ ...(ii)

Since the surface is in motion then the position of the point P at any instant $t+\delta t$ is given by

$$F(\mathbf{r}+\delta\mathbf{r},\ t+\delta t)=0 \qquad \{\text{ from (i)}$$

or
$$F(\mathbf{r},t)+\delta\mathbf{r}.\nabla F+\delta t.\frac{\delta F}{\delta t}=0 \qquad \text{(By Taylor's theorem)}$$

or
$$\frac{\delta F}{\delta t}+\frac{\delta\mathbf{r}}{\delta t}.\ \nabla F=0.$$

Now as $\delta\mathbf{r}\to 0$, $\delta t\to 0$, the above relation becomes

$$\frac{\partial F}{\partial t}+\mathbf{u}.\nabla F=0 \qquad \text{...(iii)}$$

$$\left\{ \text{as } \mathbf{u}=\frac{dr}{dt}\right.$$

or
$$\frac{\partial F}{\partial t}+q.\nabla F=0. \qquad \{\text{ from (ii)}$$

or
$$\frac{\partial F}{\partial t}+u\frac{\partial F}{\partial x}+v\frac{\partial F}{\partial y}+w\frac{\partial F}{\partial z}=0. \qquad \text{...(iv)}$$

Thus the equation of every boundary surface must satisfy the above differential equation.

If the surface is at rest, then $\frac{\partial F}{\partial t}=0$

Thus (iv) reduce to

$$u\frac{\partial F}{\partial x}+v\frac{\partial F}{\partial y}+w\frac{\partial F}{\partial z}=0.$$

or
$$\mathbf{q}.\nabla F=0$$

Which is the condition when the liquid is in contact with a rigid surface, in order that contact is maintained, the fluid and the surface must have the same velocity normal to the surface.

The **normal velocity** of the boundary is given by

$$\mathbf{u}.\mathbf{n}=\mathbf{u}.\ \frac{\nabla F}{|\nabla F|}$$

$$=-\frac{\partial F/\partial t}{|\nabla F|} \qquad \{\text{ from (iii)}\}$$

$$=-\frac{\partial F/\partial t}{\sqrt{\left\{\left(\frac{\partial F}{\partial x}\right)^2+\left(\frac{\partial F}{\partial y}\right)^2+\left(\frac{\partial F}{\partial z}\right)^2\right\}}}$$

Ex. 1. *Show that*

$$\frac{x^2}{a^2}\tan^2 t+\frac{y^2}{b^2}\cot^2 t=1,$$

is a possible form for the bounding surface of a liquid, and find an expression for the normal velocity.

We know that the surface $F(x, y, z, t)=0$ can be a possible boundary surface, if it satisfies the boundary condition

$$\frac{dF}{dt}=0$$

or
$$\frac{\partial F}{\partial t}+u\frac{\partial F}{\partial x}+v\frac{\partial F}{\partial y}+w\frac{\partial F}{\partial z}=0 \qquad \ldots(\text{i})$$

where u, v, w satisfy the equation of continuity

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0. \qquad \ldots(\text{ii})$$

Here, we have two-dimensional equation

$$F(x, y, t)\equiv\frac{x^2}{a^2}\tan^2 t+\frac{y^2}{b^2}\cot^2 t-1=0$$

$$\frac{\partial F}{\partial t}=\frac{2x^2}{a^2}\tan t\sec^2 t-\frac{2y^2}{b^2}\cot t\,\text{cosec}^2\, t$$

$$\frac{\partial F}{\partial x}=\frac{2x}{a^2}\tan^2 t \text{ and } \frac{\partial F}{\partial y}=\frac{2y}{b^2}\cot^2 t$$

From (1), we have

$$\frac{x\tan t}{a^2}(x\sec^2 t+u\tan t)+\frac{y\cot t}{b^2}(-y\,\text{cosec}^2\, t+v\cot t)=0. \qquad \ldots(\text{iii})$$

Now (i) will be the boundary condition, if

$$x\sec^2 t+u\tan t=0$$

or
$$u=-x\sec^2 t\cot t=-\frac{x}{\sin t\cos t}$$

and
$$-y\,\text{cosec}^2\, t+v\cot t=0$$

or
$$v=y\,\text{cosec}^2\, t\tan t=\frac{y}{\sin t\cos t}$$

which satisfies the (iii) relation.

Now $\dfrac{\partial u}{\partial x}=-\dfrac{1}{\sin t\cos t}$ and $\dfrac{\partial v}{\partial y}=\dfrac{1}{\sin t\cos t}$.

Hence equation of continuity

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \text{ is satisfied.}$$

Thus the given surface is a possible form for the boundary surface of a liquid with velocity components

$$u=-\frac{x}{\sin t\cos t} \text{ and } v=\frac{y}{\sin t\cos t}$$

Again, Normal veiocity$=\dfrac{u\dfrac{\partial F}{\partial x}+v\dfrac{\partial F}{\partial y}}{\sqrt{\left\{\left(\dfrac{\partial F}{\partial x}\right)^2+\left(\dfrac{\partial F}{\partial y}\right)^2\right\}}}$

$$=\frac{-\dfrac{x}{\sin t\cos t}\cdot\dfrac{2x}{a^2}\tan^2 t+\dfrac{y}{\sin t\cos t}\cdot\dfrac{2y}{b^2}\cot^2 t}{\sqrt{\left\{\left(\dfrac{2x}{a^2}\tan^2 t\right)^2+\left(\dfrac{2y}{b^2}\cot^2 t\right)^2\right\}}}$$

$$=\frac{a^2y^2\cot t\operatorname{cosec}^2 t-b^2x^2\tan t\sec^2 t}{\sqrt{(x^2b^4\tan^4 t+y^2a^4\cot^4 t)}}.$$

Ex. 2. *Show that the ellipsoid*

$$\frac{x^2}{a^2k^2t^{2n}}+kt^n\left\{\left(\frac{y^2}{b^2}\right)+\left(\frac{z^2}{c^2}\right)\right\}=1$$

is a possible form of the boundary surface of a liquid.

The surface $F(x\ y\ z\ t)=0$ can be a possible boundary surface, if it satisfies the boundary contdition.

$$\frac{dF}{dt}=0$$

or
$$\frac{\partial F}{\partial t}+u\frac{\partial F}{\partial x}+v\frac{\partial F}{\partial y}+w\frac{\partial F}{\partial z}=0 \quad \ldots\text{(i)}$$

where u, v, w satisfy the equation of continuity

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0. \quad \ldots\text{(ii)}$$

So
$$F(x\ y\ z\ t)\equiv\frac{x^2}{a^2k^2t^{2n}}+kt^n\left\{\left(\frac{y^2}{b^2}\right)+\left(\frac{z^2}{c^2}\right)\right\}-1=0.$$

$$\frac{\partial F}{\partial t}=-\frac{x^2}{a^2k^2}\cdot\frac{2n}{t^{2n+1}}+n\,k\,t^{n-1}\left\{\left(\frac{y^2}{b^2}\right)+\left(\frac{z^2}{c^2}\right)\right\}$$

$$\frac{\partial F}{\partial x}=\frac{2x}{a^2k^2t^{2n}},\ \frac{\partial F}{\partial y}=\frac{2kt^ny}{b^2}\text{ and }\frac{\partial F}{\partial z}=\frac{2kt^nz}{c^2}.$$

Now from (1), we have

$$-\frac{x^2}{a^2k^2}\frac{2n}{t^{2n+1}}+n\,k\,t^{n-1}\left\{\frac{y^2}{b^2}+\frac{z^2}{c^2}\right\}+\frac{2xu}{a^2k^2t^{2n}}$$
$$+\frac{2kt^ny.v}{c^2}+\frac{2kt^nz.w}{c^2}=0$$

or
$$\left(u\quad\frac{nx}{t}\right)\frac{2x}{a^2k^2t^{2n}}+\left(v+\frac{ny}{2t}\right)\frac{2k\,y\,t^n}{b^2}$$
$$+\left(w+\frac{nz}{2t}\right)\frac{2k\,z\,t^n}{c^2}=0$$

which will hold, if

$$u-\frac{nx}{t}=0,\ v+\frac{ny}{2t}=0 \quad \text{and } w+\frac{nz}{2t}=0$$

or
$$u=\frac{nx}{t},\quad v=-\frac{ny}{2t} \text{ and } \quad w=-\frac{nz}{2t}$$

which satisfies the equation of continuity

$$\frac{\partial u}{\partial x}=\frac{n}{t},\ \frac{\partial v}{\partial y}=-\frac{n}{2t} \text{ and } \frac{\partial w}{\partial z}=-\frac{n}{2t}$$

i.e.
$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0.$$

Thus (i) is a possible form for the boundary surface of a liquid with velocity components

$$u=\frac{nx}{t},\ v=-\frac{ny}{2t} \text{ and } w=-\frac{nz}{2t}.$$

Ex. 3. *Shew that all necessary conditions can be satisfied by a velocity potential of the form*

$$\phi=\alpha x^2+\beta y^2+\gamma z^2$$

and a bounding surface of the form

$$F\equiv ax^4+by^4+cz^4-\chi(t)=0$$

where $\chi(t)$ *is a given function of the time,* and $\alpha, \beta, \gamma, a, b, c$ *are the suitable functions of ihe time.*

The necessary conditions are :

(i) ϕ satisfies the Laplace's Equation *i. e.* $\nabla^2\phi=0$ for incompressible fluid flow.

(ii) F satisfies the condition for bounding surface, *i. e.*

$$\frac{\partial F}{\partial t}+u\frac{\partial F}{\partial x}+v\frac{\partial F}{\partial y}+w\frac{\partial F}{\partial z}=0$$

for the surface.

We have

$$\phi=\alpha x^2+\beta y^2+\gamma z^2 \qquad \ldots\text{(i)}$$

The Laplace's Equation

$$\nabla^2\phi=0$$

or
$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0$$

will be satisfied if

$$\alpha+\beta+\gamma=0 \qquad \{\text{from (i)}$$

where α, β, γ are some suitable functions of the time.

Again, $$F \equiv ax^4+by^4+cz^4-\chi(t) \qquad \ldots\text{(ii)}$$
can be a possible form for the bounding surface of a liquid, if
$$\frac{\partial F}{\partial t}+u\frac{\partial F}{\partial x}+v\frac{\partial F}{\partial y}+w\frac{\partial F}{\partial z}=0$$
or $$x^4\frac{\partial a}{\partial t}+y^4\frac{\partial b}{\partial t}+z^4\frac{\partial c}{\partial t}-\chi'(t)+4ax^3u+4by^3v+4cz^3w=0 \qquad \ldots\text{(iii)}$$

But $$u=-\frac{\partial \phi}{\partial x}=-2\alpha x \qquad v=-\frac{\partial \phi}{\partial y}=-2\beta y$$

and $$w=-\frac{\partial \phi}{\partial z}=-2\gamma z.$$

Substiting the value of u, v, w in (iii), we have
$$x^4\frac{\partial a}{\partial t}+y^4\frac{\partial b}{\partial t}+z^4\frac{\partial c}{\partial t}-\chi'(t)-8a\alpha x^4-8b\beta y^4-8c\gamma z^4=0$$
or $$x^4\left(\frac{\partial a}{\partial t}-8a\alpha\right)+y^4\left(\frac{\partial b}{\partial t}-8b\beta\right)+z^4\left(\frac{\partial c}{\partial t}-8c\gamma\right)-\chi'(t)=0$$

Comparing this with the equation of the bounding surface
$$F \equiv ax^4+by^4+cz^4-\chi(t)=0$$
we have
$$\frac{\frac{\partial a}{\partial t}-8a\alpha}{a}=\frac{\frac{\partial b}{\partial t}-8b\beta}{b}=\frac{\frac{\partial c}{\partial c}-8c\gamma}{c}=\frac{\chi'(t)}{\chi(t)}$$

The condition will hold if $a, b, c, \alpha, \beta, \gamma$ are some suitable function of time.

Hence ϕ and $F=0$ satisfy the necessary condition for velocity potential and boundary surface if $\alpha, \beta, \gamma, a, b, c$ are some suitable function of time.

Exercise

1. Show that the variable ellipsoid
$$\frac{x^2}{a^2k^2t^4}+kt^2\left\{\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2\right\}=1$$
is a possible form for the boundary surface of a liquid at any time t.

2. Show that a surface of the form
$$ax^4+by^4+cz^4-\mu(t)=0$$

is a possible form of a boundary surface of a homogeneous liquid at time t, the velocity potential of the liquid motion being $\phi=(\beta-\gamma)\, x^2+(\gamma-\alpha)\, y^2+(\alpha-\beta)\, z^2$

where μ, α, β, γ are given functions of time and a, b, c are suitable functions of time.

3. In the steady motion of homogeneous liquid if the surfaces $f_1=a_1, f_2=a_2$ define the stream lines, Prove that the most general values of the velocity components u, v, w, are

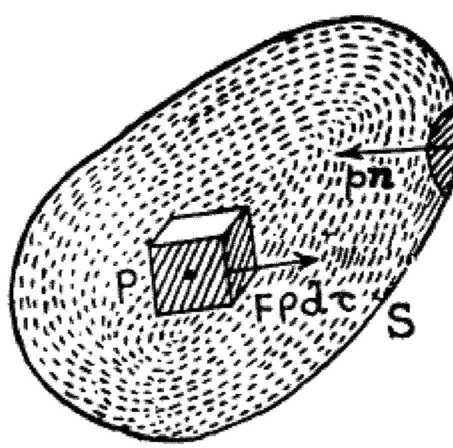
$$F(f_1 f_2)\frac{\partial(f_1, f_2)}{\partial(y, z)}, \; F(f_1 f_2)\frac{\partial(f_1, f_2)}{\partial(z, x)}, \; F(f_1 f_2)\frac{\partial(f_1 f_2)}{\partial(x, y)}$$

2

Equations of Motion

§ 2·0. The Equation of motion of an inviscid fluid.

Consider any arbitrary closed surface S drawn in the region occupied by the incompressible fluid and moving with it, so that it contains the same fluid particles at every instant. We know that the total force acting on this mass of fluid is equal to the rate of change of linear momentum. The forces are due to :



(i) The normal pressure thrusts on the boundary.

(ii) The external force (*e. g.* gravity) F (let) per unit mass.

Let ρ be the density of the fluid particle P within the closed surface, $d\tau$ be the volume enclosing P. The mass of the element $\rho d\tau$ will always remain constant. Let q be the velocity of fluid particle P. Then the momentum of the volume is

$$M=\int \mathbf{q}\ \rho\ d\tau \qquad \text{...(i)}$$

The time rate of change of momentum is given by differentiating (i) w.r. to t, we have

$$\frac{d\mathbf{M}}{dt}=\int\frac{d\mathbf{q}}{dt}\ (\rho d\tau)+\int \mathbf{q}\ \frac{d}{dt}\ (\rho d\tau)$$

$$\frac{d\mathbf{M}}{dt}=\int\frac{d\mathbf{q}}{dt}\ .\ \rho d\tau \qquad \text{...(ii)}$$

The second integral vanishes as the mass $\rho d\tau$ remains constant for all time.

Let F be the external force per unit mass acting on fluid particle P, then the total force on the volume is

$$=\int \mathbf{F}\ \rho\ d\tau \qquad \text{...(iii)}$$

Again let p be the pressure at a point on the surface along the outward drawn unit normal $\hat{\mathbf{n}}$

$$=\int p\,(-\hat{\mathbf{n}})ds \qquad \text{(Negative sign as the surface force acts inwards)}$$

$$=-\int p\,\hat{\mathbf{n}}\,ds$$

$$=-\int \nabla\, p d\tau \qquad \ldots \text{(iii)}$$

(By Gauss Theorem)

Since

Rate of change of momentum = Total force acting on the mass of the fluid

or $\displaystyle\int \frac{d\mathbf{q}}{dt}\,.\,\rho d\tau=\int \mathbf{F}\,\rho d\tau-\int \nabla\, p\; d\tau$

or $\displaystyle\int \left(\rho\,\frac{d\mathbf{q}}{dt}-\rho\mathbf{F}+\nabla p\right)\,d\tau=0$

Since the volume enclosed in the surface is arbitrary, then

$$\rho\,\frac{d\mathbf{q}}{dt}-\rho\;\mathbf{F}+\nabla p=0$$

or $\displaystyle\frac{d\mathbf{q}}{dt}=\mathbf{F}-\frac{1}{\rho}\nabla\; p \qquad \ldots(\text{v})$

which is known as **EULER'S EQUATION OF MOTION.**

or $\displaystyle\frac{\partial\mathbf{q}^*}{\partial t}+(\mathbf{q}\,.\,\nabla)\;\mathbf{q}=\mathbf{F}-\frac{1}{\rho}\nabla p \qquad \ldots(\text{v})$

or $\displaystyle\frac{\partial\mathbf{q}}{\partial t}+\nabla(\tfrac{1}{2}\mathbf{q}^2)-\mathbf{q}\times\text{curl }\mathbf{q}=\mathbf{F}-\frac{1}{\rho}\;\nabla p$

$$\left\{\begin{array}{l}\text{As } \nabla(\mathbf{q}\,.\,\mathbf{q})=2\,\{\mathbf{q}\times\text{curl }\mathbf{q}+(\mathbf{q}\,.\,\nabla)\mathbf{q}\}\\ \text{or } (\mathbf{q}\,.\,\nabla)\mathbf{q}=\nabla(\tfrac{1}{2}\mathbf{q}^2)-\mathbf{q}\times\text{curl }\mathbf{q}\end{array}\right\}$$

---

* We know that the velocity vector q is a function of position and time both

$$\mathbf{q}=\mathbf{q}\;(\mathbf{r},\,t)$$

let $\quad\mathbf{q}+\delta\mathbf{q}$ be the velocity to a neighbouring position at time $t+\delta t$.

then $\quad\delta\mathbf{q}=\mathbf{q}\;(\mathbf{r}+\delta\mathbf{r},\,t+\delta t)-\mathbf{q}(\mathbf{r},\,t) \qquad \ldots(\text{a})$

$\delta\mathbf{q}=\{\mathbf{q}\;(\mathbf{r}+\delta\mathbf{r},\,t+\delta t)-\mathbf{q}(\mathbf{r},\,t+\delta t)\}+\{\mathbf{q}\;(\mathbf{r},\,t+\delta r)-\mathbf{q}(\mathbf{r}\;t)\}$

$\delta\mathbf{q}=(\delta\mathbf{r}\,.\,\nabla)\;\mathbf{q}\;(\mathbf{r},\,t+\delta t)+\delta t.\partial\mathbf{q}\;(\mathbf{r}\;.\;t)/\partial t \qquad \ldots(\text{b})$

Dividing (b) by δt, we have

$$\frac{d\mathbf{q}}{dt}=\frac{\partial\mathbf{q}}{\partial t}+(\mathbf{q}\;.\;\nabla)\;\mathbf{q}$$

or $\quad \dfrac{\partial \mathbf{q}}{\partial t} - \mathbf{q} \times \text{curl } \mathbf{q} = \mathbf{F} - \dfrac{1}{\rho} \nabla p - \dfrac{1}{2} \nabla \mathbf{q}^2$

or $\quad \dfrac{\partial \mathbf{q}}{\partial t} + \omega \times \mathbf{q} = \mathbf{F} - \dfrac{1}{\rho} \nabla p - \dfrac{1}{2} \nabla \mathbf{q}^2$

or $\quad \dfrac{\partial \mathbf{q}}{\partial t} + \nabla \left(\tfrac{1}{2} \mathbf{q}^2\right) + \omega \times \mathbf{q} = \mathbf{F} - \dfrac{1}{\rho} \nabla p \qquad \text{...(vii)}$

$$\left\{ \begin{array}{l} \text{If } \mathbf{q} \text{ be the velocity vector of a fluid particle, then} \\ \qquad \omega = \nabla \times \mathbf{q} = \text{curl } \mathbf{q} \\ \text{or} \quad \text{then } \mathbf{q} \times \text{curl } \mathbf{q} = -\omega \times \mathbf{q} \end{array} \right.$$

which is known as **Lamb's Hydro-dynamical Equation.**

§ 2·01. Cartesian Co-ordinates.

Let $\mathbf{q}$ (u, v, w) be the velocity of the fluid particle and F $(X\ Y\ Z)$ be the external force.

Since $\qquad q = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$

$$\frac{d\mathbf{q}}{dt} = \frac{du}{dt}\,\mathbf{i} + \frac{dv}{dt}\,\mathbf{j} + \frac{dw}{dt}\,\mathbf{k}$$

and $\qquad \mathbf{F} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}$

also $\qquad \nabla p = \dfrac{\partial p}{\partial x}\mathbf{i} + \dfrac{\partial p}{\partial y}\mathbf{j} + \dfrac{\partial p}{\partial z}\mathbf{k}$

then from relation (iv), we have

$$\frac{d\mathbf{q}}{dt} = \mathbf{F} - \frac{1}{\rho}(\nabla p)$$

or $\left(\dfrac{du}{dt}\mathbf{i} + \dfrac{dv}{dt}\mathbf{j} + \dfrac{dw}{di}\mathbf{k}\right) = (X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k})$

$$-\frac{1}{\rho}\left(\frac{\partial p}{\partial x}\mathbf{i} + \frac{\partial p}{\partial y}\mathbf{j} + \frac{\partial p}{\partial z}\mathbf{k}\right)$$

Equating the coefficient of **i**, **j** & **k**, we get

$$\frac{du}{dt} = X - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{dv}{dt} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{\partial w}{\partial t} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

Again by equating the coefficients of **i**, **j**, **k** in relation (vi), we get

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = X - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\text{or}\quad \frac{\partial v}{\partial t}+u\,\frac{\partial v}{\partial x}+v\,\frac{\partial v}{\partial y}+w\,\frac{\partial v}{\partial z}=Y-\frac{1}{\rho}\frac{\partial p}{\partial y}$$

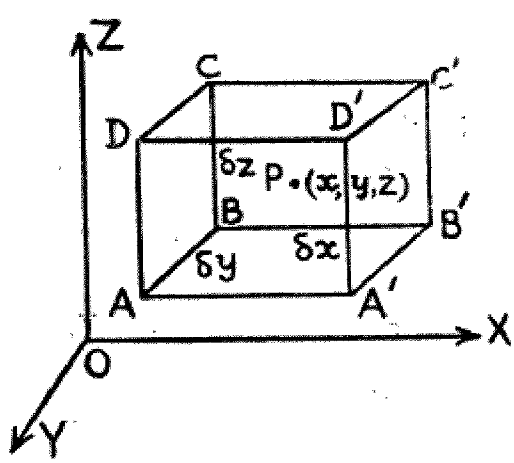
$$\text{and}\quad \frac{\partial w}{\partial t}+u\,\frac{\partial w}{\partial x}+v\,\frac{\partial w}{\partial y}+w\,\frac{\partial w}{\partial z}=Z=\frac{1}{\rho}\frac{\partial p}{\partial z}$$

$$\left\{ \text{as } (\mathbf{q}\cdot\nabla)\,\mathbf{q}=\left(u\,\frac{\partial}{\partial x}+v\,\frac{\partial}{\partial y}+w\,\frac{\partial}{\partial z}\right)\mathbf{q}\right.$$

Known as **Euler's Dynamical Equation** in **Cartesion form** along the co-ordinate axes respectively.

§ 2·1. Equation of Motion of a Perfect fluid. (Alternative method)

Consider a parallelopiped having centre as $P(x, y, z)$ and edges δx, δy, δz parallel to the rectangular co-ordinate axes. Let p be the pressure, ρ the density and X, Y, Z, be the components of the external forces per unit mass at any time t at P.



Pressure on a plane through P parallel to $ABCD$

$$=p\,\delta y\,\delta z$$
$$=f(x\,y\,z)\,\delta y\,\delta z \text{ (say)}$$

Pressure on the face $ABCD$ *i.e.* on yz-face

$$=f(x-\tfrac{1}{2}\delta x, y, z)\,\delta y\,\delta z$$
$$=\left[f(x\,y\,z)-\frac{1}{2}\,\delta x\,\frac{\partial p}{\partial x}\cdots\cdots\right]\delta y\,\delta z$$
$$=\left(p-\frac{1}{2}\,\frac{\partial p}{\partial x}\,\delta x\right)\delta y\,\delta z$$

Similarly pressure on the opposite face $A'B'C'D'$

$$=f(x+\tfrac{1}{2}\delta x, y, z)\,\delta y\,\delta z$$
$$=\left[f(x\,y\,z)+\frac{1}{2}\,\delta x\frac{\partial p}{\partial x}+\cdots\cdots\right]\delta y\,\delta z$$
$$=\left(p+\frac{1}{2}\,\frac{\partial p}{\partial x}\delta x.\right)\delta y\,\delta z$$

Thus the resultant pressure on the face

$$=\left(p-\frac{1}{2}\,\frac{\partial p}{\partial x}\delta x\right)\delta y\,\delta z-\left(p+\frac{1}{2}\frac{\partial p}{\partial x}\delta x\right)\delta y\,\delta z$$

$$=-\frac{\partial p}{\partial x}\,\delta x\,\delta y\,\delta z \text{ in the direction of } X \text{ positive.}$$

The pressure on the remaining faces are perpendicular to X. Again rate of increment of the momentum of the element in the direction of axis of X, is

$$=\rho\ \delta x\ \delta y\ \delta z.\frac{Du}{Dt}$$

By the equation of continuity, we have

$$\rho\ \frac{Du}{Dt}\ \delta x\ \delta y\ \delta z = X\rho\delta x\ \delta y\ \delta z - \frac{\partial p}{\partial x}\ \delta x\ \delta y\ \delta z$$

or $$\frac{Du}{Dt} = X - \frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \text{...(i)}$$

where $$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

Hence (i) becomes

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = X - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

Similarly

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y}$$

and $$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

Known as **Euler's Dynanical Equations.**

§ **2·12.** If Π be the external pressure upon its surface and p be the pressure of the liquid at the surface, then we have

$$p = \Pi$$

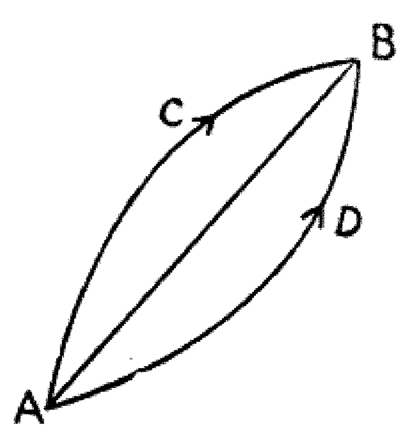
Thus at all points of the free surface

$$\frac{Dp}{Dt} = \frac{D\Pi}{Dt}$$

or $$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z} = \frac{\partial \Pi}{\partial t}$$

where Π depends on time only.

§ **2·2. Conservative field of force.**



If the work done by the force F of the field in taking a unit mass from one points A to another point B is independent of the path, then it is known as **conservative field of force.**

$$\int_{ACB} F.dr = \int_{ABD} F.dr = -\Omega \text{ (let)}$$

Where Ω is a scalar point function whose value depends on the initial and final position A and B.

or
$$F = -\nabla\Omega$$

Where Ω is known as force potential. It measures the potential energy of the field.

§ 2·3. Integration of Euler's Equation,

We know that the equation of motion is

$$\frac{\partial \mathbf{q}}{\partial t} + \nabla\left(\tfrac{1}{2}\mathbf{q}^2\right) - \mathbf{q}\times\omega = \mathbf{F} - \frac{1}{\rho}\nabla p \qquad \ldots(i)$$

{Ref. § 2·0}

and the external force form a conservative field of force then

$$\mathbf{F} = -\nabla\Omega. \qquad \ldots(ii)$$

From (i) and (ii), we have

$$\frac{\partial \mathbf{q}}{\partial t} - \mathbf{q}\times\omega = -\nabla\Omega - \frac{1}{\rho}\nabla p - \nabla\left(\tfrac{1}{2}\mathbf{q}^2\right)$$

$$\frac{\partial \mathbf{q}}{\partial t} - \mathbf{q}\times\omega = -\nabla\left\{\Omega + \int\frac{dp}{\rho} + \frac{1}{2}\mathbf{q}^2\right\}$$

Since $\mathbf{q} = -\nabla\phi$ where ϕ is the velocity potential.

or
$$\frac{\partial}{\partial t}(-\nabla\phi) - \mathbf{q}\times\textbf{Curl }\mathbf{q} = -\nabla\left\{\Omega + \int\frac{dp}{\rho} + \frac{1}{2}\mathbf{q}^2\right\} \qquad \ldots(iii)$$

Case I. It the motion be **irrotational,** then curl $\mathbf{q}=0$.

So (iii) reduces to

$$\nabla\left\{-\frac{\partial\phi}{\partial t} + \Omega + \int\frac{dp}{\rho} + \frac{1}{2}\mathbf{q}^2\right\} = 0$$

or
$$-\frac{\partial\phi}{\partial t} + \Omega + \int\frac{dp}{\rho} + \frac{1}{2}\mathbf{q}^2 = \chi(t) \qquad \ldots(iv)$$

The constant $\chi(t)$ will be a function of time only, $\chi(t)$ can be absorbed in $\frac{\partial\phi}{\partial t}$ then (iv) reduces to

$$-\frac{\partial\phi}{\partial t} + \Omega + \int\frac{dp}{\rho} + \frac{1}{2}\mathbf{q}^2 = \text{Constant}.$$

This is known as **Bernoulli's Equation for irrotational flow.**

Case II. If the motion be **Steady** as well as **irrotational** then $\frac{\partial\phi}{\partial t} = 0$ and **Curl** $\mathbf{q} = 0$.

Since $\mathbf{q}\times\text{Curl }\mathbf{q} = 0 \Rightarrow \mathbf{q}$ and **Curl** $\mathbf{q}$ are parallel that means streamlines and vortex lines coincide. For such a motion $\mathbf{q}$ is known a **Beltrami vector** and the flow is a **Beltrami flow** then (v) reduces to

$$\int\frac{dp}{\rho} + \frac{1}{2}\mathbf{q}^2 + \Omega = \text{Constant}. \qquad \ldots(vi)$$

Here the constant is an absolute constant *i.e.* independent of time.

Case III. If the fluid be **incompressible** and **homogeneous,** then

$$\rho=\text{Constant.} \qquad \left\{ \text{as } \int\frac{dp}{\rho}=\frac{p}{\rho}\right.$$

So (vi) reduces to,

$$\frac{p}{\rho}+\frac{1}{2}\,q^2+\Omega=\text{Const.}$$

where the const. depends upon the stream line chosen.

The equation (vi) is known as the **Bernoulli's equation for steady motion.**

§ 2·31. **Integration of Euler's Equation.** (Alternative method)

When a velocity potential ϕ exists, then

$$u=-\frac{\partial\phi}{\partial x},\ v=-\frac{\partial\phi}{\partial y} \text{ and } w=-\frac{\partial\phi}{\partial z}$$

and the extraneous forces are derivable from a potential function V

i.e.
$$X=-\frac{\partial V}{\partial x},\ Y=-\frac{\partial V}{\partial y} \text{ and } Z=-\frac{\partial V}{\partial z}$$

Substituting the values of u, v, w and X, Y, Z in Euler's Dynamical Equation, we have (Ref § 2·1)

$$-\frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial x}\right)+\frac{\partial\phi}{\partial x}\cdot\frac{\partial^2\phi}{\partial x^2}+\frac{\partial\phi}{\partial y}\cdot\frac{\partial^2\phi}{\partial y\partial x}+\frac{\partial\phi}{\partial z}\cdot\frac{\partial^2\phi}{\partial z\partial x}=-\frac{\partial V}{\partial x}-\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \text{...(i)}$$

or
$$-\frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial y}\right)+\frac{\partial\phi}{\partial x}\cdot\frac{\partial^2\phi}{\partial x\partial y}+\frac{\partial\phi}{\partial y}\cdot\frac{\partial^2\phi}{\partial y^2}+\frac{\partial\phi}{\partial z}\cdot\frac{\partial^2\phi}{\partial z\partial y}=-\frac{\partial V}{\partial y}-\frac{1}{\rho}\frac{\partial p}{\partial y} \qquad \text{...(ii)}$$

and
$$-\frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial z}\right)+\frac{\partial\phi}{\partial x}\cdot\frac{\partial^2\phi}{\partial x\partial z}+\frac{\partial\phi}{\partial y}\cdot\frac{\partial^2\phi}{\partial y\partial z}+\frac{\partial\phi}{\partial z}\cdot\frac{\partial^2\phi}{\partial z^2}=-\frac{\partial V}{\partial z}-\frac{1}{\rho}\frac{\partial p}{\partial z} \qquad \text{...(iii)}$$

Multiplying (i), (ii) and (iii) by dx, dy and dz respectively and adding, we have

$$-d\,\frac{\partial\phi}{\partial t}+\frac{1}{2}\,d\left\{\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right\}+dV+\frac{1}{\rho}\,dp=0$$

or
$$-d\,\frac{\partial\phi}{\partial t}+\frac{1}{2}\,dq^2+dV+\frac{1}{\rho}\,dp=0$$

$$\left\{ \text{as } q^2=\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2\right.$$

Considering a functional relationship between p and ρ.

By integrating, we get

$$-\frac{\partial \phi}{\partial t}+\frac{1}{2}q^2+V+\int\frac{dp}{\rho}=C \qquad \text{...(iv)}$$

where C is a constant depending on time only.

Case I. If the fluid be **homogeneous** and **inelastic** then ρ=Const.

The relation (iv) reduces to

$$-\frac{\partial \phi}{\partial t}+\frac{p}{\rho}+\frac{1}{2}q^2+V=C.$$

known as Bernoulli's Equation

Case II. If the motion be **Steady** *i.e.* $\frac{\partial \phi}{\partial t}=0$,

then (iv) reduces to

$$\frac{p}{\rho}+\frac{1}{2}q^2+V=C.,$$

here the constant C is an absolute const. *i.e.* independent from the time t.

Known as **Bernoulli's Equation for steady motion.**

§ 2·32. Bernoulli's Theorem (when velocity potential does not exist).

The pressure at a point for the **steady motion** *of a perfect fluid under conservative body forces is given by*

$$\int\frac{dp}{\rho}+\frac{1}{2}q^2+\chi=\text{Const.}$$

Consider the motion of a small cylinder of cross-sectional area A with its axis of length δs along a stream line, where S is the arc of the stream line on which the element lies. Since the pressure does not contribute anything to the resultant force in the direction of motion. Thus the resultant thurst in the direction of motion is

$$=\left[pA-\left\{p+\frac{\partial p}{\partial s}\delta s\right\}A\right]$$

$$=-\frac{\partial p}{\partial s}A\delta s$$

Let the component of the body force in the direction of motion be F. The total body force on this element of mass $\rho A\delta s$ is given by

$$=F\rho A\delta s.$$

Let q be the velocity, the equation of motion is

$$\rho A\delta s\left(\frac{dq}{at}\right)=F\rho A\delta s-\left(\frac{\partial p}{\partial s}\right)A\delta s \quad \text{...(i)}$$

(By Newton's Second Law of motion)
{ Ref. § 2·2

Since $\quad F=-\nabla\chi$

$$=-\frac{\partial\chi}{\partial s} \quad \text{...(ii)}$$

From (i), we have

$$\frac{dq}{dt}=F-\frac{1}{\rho}\frac{\partial p}{\partial s}$$

Since $\quad \frac{dq}{dt}=\frac{\partial q}{\partial t}+q\frac{\partial q}{\partial s} \quad \left(\frac{\partial q}{\partial t}=0 \text{ for steady flow}\right)$

or $\quad q\frac{\partial q}{\partial s}=-\frac{\partial\chi}{\partial s}-\frac{1}{\rho}\frac{\partial p}{\partial s}$ { from (ii)

or $\quad \frac{\partial}{\partial s}\left[\frac{1}{2}q^2+\chi+\int\frac{dp}{\rho}\right]=0.$

The rate of change along the stream line vanishes,

so $\quad \frac{1}{2}q^2+\int\frac{dp}{\rho}+\chi=\text{Const.} \quad \text{...(iii)}$

where constant varies from one streamline to the other

Case I. If ρ is constant then (iii) reduces to

$$\frac{1}{2}q^2+\frac{p}{\rho}+\chi=\text{Const.}$$

Case II. If the motion is irrotational, the velocity potential exists then const. is an absolute constant.

§ 2·4. Lagrange's Equations. Consider the independent variable $\mathbf{r}_0$ (initial position vector of the particle) at any instant t. Let the particle attains the position $\mathbf{r}$ at time t,

then $\quad \mathbf{r}=\mathbf{r}(\mathbf{r}_0, t).$

The equation of motion of the particle is given by

$$\frac{\partial^2\mathbf{r}}{\partial t^2}=\mathbf{F}-\frac{1}{\rho}\frac{\partial p}{\partial\mathbf{r}}$$

or $\quad \frac{\partial^2\mathbf{r}}{\partial t^2}=\mathbf{F}-\frac{1}{\rho}\frac{\partial\,;\,\mathbf{r}_0}{\partial\mathbf{r}}\cdot\frac{\partial p}{\partial\mathbf{r}_0}$*

---

*The indefinite or dyadic product of two vectors **a** and **b** is given by **ab** or **a : b**. The differential operator

$$\nabla=\frac{\partial}{\partial r}=\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z} \text{ and } \mathbf{r}=\mathbf{i}x+\mathbf{j}y+\mathbf{k}z$$

Then the dyadic product is

$$\nabla\,;\,\mathbf{r}=\frac{\partial}{\partial\mathbf{r}}\,;\,\mathbf{r}=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right);\,(\mathbf{i}x+\mathbf{j}y+\mathbf{k}z)=I$$

Again $\quad \frac{\partial}{\partial\mathbf{r}}=\frac{\partial\,;\,\mathbf{r}_0}{\partial\mathbf{r}_0}\cdot\frac{\partial}{\partial\mathbf{r}_0}$

$$\frac{\partial\,;\,\mathbf{r}_0}{\partial\mathbf{r}}\cdot\frac{\partial\,;\,\mathbf{r}}{\partial\mathbf{r}_0}=\frac{\partial\,;\,\mathbf{r}}{\partial\mathbf{r}}=I=\frac{\partial\,;\,\mathbf{r}}{\partial\mathbf{r}_0}\cdot\frac{\partial\,;\,\mathbf{r}_0}{\partial\mathbf{r}}$$

or $$\left(\frac{\partial^2 \mathbf{r}}{\partial t^2}-\mathbf{F}\right)+\frac{1}{\rho}\ \frac{\partial\ ;\ \mathbf{r}_0}{\partial \mathbf{r}}\cdot\ \frac{\partial p}{\partial \mathbf{r}_0}=0 \qquad \text{...(i)}$$

Pre-multiplying by $\frac{\partial\ ;\ \mathbf{r}}{\partial \mathbf{r}_0}$, (i) reduces to

$$\frac{\partial\ ;\ \mathbf{r}}{\partial \mathbf{r}_0}\cdot\ \left(\frac{\partial^2 \mathbf{r}}{\partial t^2}-\mathbf{F}\right)+\frac{1}{\rho}\frac{\partial\ ;\ \mathbf{r}}{\partial \mathbf{r}_0}\cdot\ \frac{\partial\ ;\ \mathbf{r}_0}{\partial \mathbf{r}}\cdot\ \frac{\partial p}{\partial \mathbf{r}_0}=0.$$

or $$\frac{\partial\ ;\ \mathbf{r}}{\partial \mathbf{r}_0}\cdot\ \left(\frac{\partial^2 \mathbf{r}}{\partial t^2}-\mathbf{F}\right)+\frac{1}{\rho}\ \frac{\partial p}{\partial \mathbf{r}_0}=0 \qquad \text{...(ii)}$$

$$\left\{\text{Since } \frac{\partial\ ;\ \mathbf{r}}{\partial \mathbf{r}_0}\cdot\ \frac{\partial \mathbf{r}_0}{\partial \mathbf{r}}=I\right.$$

which is known as **Lagrangian form** of the equation of motion.

Consider a fluid particle of infinitesimal volume dv, density ρ at any time t and an infinitesimal volume dv_0, density ρ_0 of the particle in its initial position. Then by equation of continuity, we have

$$\rho dv=\rho_0 dv_0 \qquad \text{...(iii)}$$

In cartesian coordinates, we have

$$dv=dxdydz,\ dv_0=dx_0dy_0dz_0$$

or $$dxdydz=Jdx_0dy_0dz_0$$

or $$J=\frac{\partial\ (x\ y\ z)}{\partial\ (x_0y_0z_0)}$$

where J being the Jacobian of the coordinates $(x\ y\ z)$ of $\mathbf{r}$ with regard to the coordinates $(x_0\ y_0\ z_0)$ of $\mathbf{r}_0$

Now (iii) reduces to with the help of Jacobian notation

$$\rho J=\rho_0$$

or $$\rho\ \frac{\partial\ (x\ y}{\partial\ (x_0y_0z_0)}=\rho_0 \qquad \text{...(iv)}$$

Known as **Lagrange's Equation of Continuity.**

Thus the equation (ii) together with the equation of continuity (iv) forms **Lagrange's Hydro-dynamical Equation.**

Cor. Representation of Lagrange Equation of motion (ii) in cartesian coordinates.

We know that

$$\mathbf{F}=-\nabla\Omega \text{ and } \ddot{\mathbf{r}}=-\frac{\partial^2 \mathbf{r}}{\partial t^2}$$

(ii) can be written as

$$(\nabla_0\ ;\ \mathbf{r})\ \ddot{\mathbf{r}}=-(\nabla_0\ ;\ \mathbf{r})\ \nabla_0\Omega-\frac{1}{\rho}\ \nabla_0 p \qquad \text{...(v)}$$

Since $\quad (\nabla_0 ; \mathbf{r})\, \nabla_0 \Omega = \dfrac{\partial ; \mathbf{r}}{\partial \mathbf{r}_0} \cdot \dfrac{\partial \Omega}{\partial \mathbf{r}} = \dfrac{\partial \Omega}{\partial \mathbf{r}_0}.$

Then (v) becomes*

$$(x_a\ddot{x}+y_a\ddot{y}+z_a\ddot{z})\,\mathbf{i}+\begin{matrix}\text{two similar}\\ \text{expressions}\end{matrix}=-\frac{\partial \Omega}{\partial \mathbf{r}_0}-\frac{1}{\rho}\nabla_0 p$$

Equating the coefficients of $\mathbf{i}, \mathbf{j}, \mathbf{k}$, we have

$$\ddot{x}x_a+\ddot{y}y_a+\ddot{z}z_a=-\Omega_a-\frac{1}{\rho}p_a$$

$$\ddot{x}\,x_b+\ddot{y}\,y_b+\ddot{z}\,z_b=-\Omega_b-\frac{1}{\rho}p_b$$

$$\ddot{x}\,x_c+\ddot{y}\,y_c+\ddot{z}\,z_c=-\Omega_c-\frac{1}{\rho}p_c$$

These suffix denotes partial derivatives

i.e. $\qquad p_a=\dfrac{\partial p}{\partial a}$ etc.

Known as **Lagrange's equation of motion in cartesian coordinates.**

§ 2·5. **Cauchy's Integral.**

Consider $\qquad Q=V+\displaystyle\int\frac{dp}{\rho}$ ...(i)

Let $(x\ y\ z)$ be the co-ordinates of a particle at any time t whose initial co-ordinats are $(a\ b\ c)$. Since ρ is a function of p, then from (i),

$$-\frac{\partial Q}{\partial a}=-\frac{\partial V}{\partial a}-\frac{1}{\rho}\frac{\partial p}{\partial a}.$$

Similarly we can write other two equations. ...(ii)

We know that the equations of motion are

$$\left.\begin{aligned}\frac{\partial^2 x}{\partial t^2}&=-\frac{\partial V}{\partial x}-\frac{1}{\rho}\frac{\partial p}{\partial x}\\ \frac{\partial^2 y}{\partial t^2}&=-\frac{\partial V}{\partial y}-\frac{1}{\rho}\frac{\partial p}{\partial y}\\ \frac{\partial^2 z}{\partial t^2}&=-\frac{\partial V}{\partial z}-\frac{1}{\rho}\frac{\partial p}{\partial z}\end{aligned}\right\}\text{(A)}$$

Multiplying these equations (A) by

$\dfrac{\partial x}{\partial a}, \dfrac{\partial y}{\partial a}$ and $\dfrac{\partial z}{\partial a}$ respectively and adding, we have

---

* $\mathbf{i} ; \mathbf{i}=\mathbf{ii}$; $\dfrac{\partial x}{\partial a}=x_a$ and $\ddot{\mathbf{r}}=\ddot{x}\mathbf{i}+\ddot{y}\mathbf{j}+\ddot{z}\mathbf{k}$

then $\nabla_0 ; \mathbf{r}=x_a\mathbf{ii}+y_a\mathbf{ij}+z_a\mathbf{ik}+x_b\mathbf{ji}+y_b\mathbf{jj}+z_b\mathbf{jk}+x_c\mathbf{ki}+y_c\mathbf{kj}+z_c\mathbf{kk}$

and $(\nabla_0 ; \mathbf{r})\,\ddot{\mathbf{r}}=(x_a\ddot{x}+y_a\ddot{y}+z_a\ddot{z})\,\mathbf{i}+(x_b\ddot{x}+y_b\ddot{y}+z_b\ddot{z})\,\mathbf{j}$
$+(x_c\ddot{x}+y_c\ddot{y}+z_c\ddot{z})\,\mathbf{k}$

or $$\frac{\partial^2 x}{\partial t^2}\cdot\frac{\partial x}{\partial a}+\frac{\partial^2 y}{\partial t^2}\cdot\frac{\partial y}{\partial a}+\frac{\partial^2 z}{\partial t^2}\cdot\frac{\partial z}{\partial a}=-\frac{\partial V}{\partial a}-\frac{1}{\rho}\frac{\partial p}{\partial a}$$

or $$\frac{\partial^2 x}{\partial t^2}\cdot\frac{\partial x}{\partial a}+\frac{\partial^2 y}{\partial t^2}\cdot\frac{\partial y}{\partial a}+\frac{\partial^2 z}{\partial t^2}\cdot\frac{\partial z}{\partial a}=-\frac{\partial Q}{\partial a}$$ { from (ii)

Similarly $$\frac{\partial^2 x}{\partial t^2}\cdot\frac{\partial x}{\partial b}+\frac{\partial^2 y}{\partial t^2}\cdot\frac{\partial y}{\partial b}+\frac{\partial^2 z}{\partial t^2}\cdot\frac{\partial z}{\partial b}=-\frac{\partial Q}{\partial b}$$

and $$\frac{\partial^2 x}{\partial t^2}\cdot\frac{\partial x}{\partial c}+\frac{\partial^2 y}{\partial t^2}\cdot\frac{\partial y}{\partial c}+\frac{\partial^2 z}{\partial t^2}\cdot\frac{\partial z}{\partial c}=-\frac{\partial Q}{\partial c}$$

$$\left\{\text{Since } \frac{\partial}{\partial b}\left(\frac{\partial Q}{\partial c}\right)=\frac{\partial}{\partial c}\left(\frac{\partial Q}{\partial b}\right)\right.$$

By eliminating Q in the last two equations, we obtain

$$\left(\frac{\partial^2 u}{\partial b\,\partial t}\frac{\partial x}{\partial c}-\frac{\partial^2 u}{\partial c\,\partial t}\frac{\partial x}{\partial b}\right)+\left(\frac{\partial^2 v}{\partial b\,\partial t}\frac{\partial y}{\partial c}-\frac{\partial^2 v}{\partial c\,\partial t}\frac{\partial y}{\partial b}\right)+\left(\frac{\partial^2 w}{\partial b\,\partial t}\cdot\frac{\partial z}{\partial c}-\frac{\partial^2 w}{\partial c\,\partial t}\frac{\partial z}{\partial b}\right)=0$$

or $$\left\{\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial b}\cdot\frac{\partial x}{\partial c}-\frac{\partial u}{\partial c}\cdot\frac{\partial x}{\partial b}\right)-\frac{\partial u}{\partial b}\cdot\frac{\partial^2 x}{\partial t\,\partial c}+\frac{\partial u}{\partial c}\cdot\frac{\partial^2 x}{\partial t\,\partial b}\right\}$$
$$+\text{two similar expressions}=0.$$

or $$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial b}\cdot\frac{\partial x}{\partial c}-\frac{\partial u}{\partial c}\cdot\frac{\partial x}{\partial b}\right)+\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial b}\cdot\frac{\partial y}{\partial c}-\frac{\partial v}{\partial c}\cdot\frac{\partial y}{\partial b}\right)+\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial b}\cdot\frac{\partial z}{\partial c}-\frac{\partial w}{\partial c}\cdot\frac{\partial z}{\partial b}\right)=0$$

By integrating w.r. to t, we have

$$\left\{\text{Since } \frac{\partial^2 x}{\partial c\,\partial t}=\frac{\partial u}{\partial c}\right. \text{ etc.}$$

$$\frac{\partial u}{\partial b}\cdot\frac{\partial x}{\partial c}-\frac{\partial u}{\partial c}\cdot\frac{\partial x}{\partial b}+\frac{\partial v}{\partial b}\cdot\frac{\partial y}{\partial c}-\frac{\partial v}{\partial c}\cdot\frac{\partial y}{\partial b}+\frac{\partial w}{\partial b}\cdot\frac{\partial z}{\partial c}-\frac{\partial w}{\partial c}\cdot\frac{\partial z}{\partial b}=\frac{\partial w_0}{\partial b}-\frac{\partial v_0}{\partial c} \quad \text{.. (iii)}$$

Where u_0, v_0, w_0 are initial values.

Initially $\frac{\partial x}{\partial a}=1$, $\frac{\partial x}{\partial b}=0$, $\frac{\partial x}{\partial c}=0$ etc.

Since $$\frac{\partial u}{\partial a}=\frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial a}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial a}+\frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial a}$$

Then relation (ii) becomes

$$\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)\cdot\frac{\partial\,(y\ z)}{\partial\,(b\ c)}+\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)\frac{\partial\,(z\ x)}{\partial\,(b\ c)}+\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\frac{\partial\,(x\ y)}{\partial\,(b\ c)}=\frac{\partial w_0}{\partial b}-\frac{\partial v_0}{\partial c}$$

or $$\xi\frac{\partial(y\,z)}{\partial(b\,c)}+\eta\frac{\partial(z\,x)}{\partial(b\,c)}+\zeta\frac{\partial(x\,y)}{\partial(b\,c)}=\xi_0 \qquad \ldots\text{(iv)}$$

Similarly other two expressions are,

$$\xi\frac{\partial(y\,z)}{\partial(c\,a)}+\eta\frac{\partial(z\,x)}{\partial(c\,a)}+\zeta\frac{\partial(x\,y)}{\partial(c\,a)}=\eta_0 \qquad \ldots\text{(v)}$$

and $$\xi\frac{\partial(y\,z)}{\partial(a\,b)}+\eta\frac{\partial(z\,x)}{\partial(a\,b)}+\zeta\frac{\partial(x\,y)}{\partial(a\,b)}=\zeta_0 \qquad \ldots\text{(vi)}$$

$\left\{\text{Where } \xi=\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z},\ \eta=\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x} \text{ and } \zeta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right.$

Also the equation of continuity is

$$\rho\frac{\partial(x\,y\,z)}{\partial(a\,b\,c)}=\rho_0 \qquad \ldots\text{(vii)}$$

Multiplying (iv), (v), (vi) by $\frac{\partial x}{\partial a}, \frac{\partial x}{\partial b}, \frac{\partial x}{\partial c}$ respectively and adding, we obtain with the help of (vii),

$$\frac{\xi}{\rho}=\frac{\xi_0}{\rho_0}\frac{\partial x}{\partial a}+\frac{\eta_0}{\rho_0}\frac{\partial x}{\partial b}+\frac{\zeta_0}{\rho_0}\frac{\partial x}{\partial c}$$

$$\frac{\eta}{\rho}=\frac{\xi_0}{\rho_0}\frac{\partial y}{\partial a}+\frac{\eta_0}{\rho_0}\frac{\partial y}{\partial b}+\frac{\zeta_0}{\rho_0}\frac{\partial y}{\partial c}$$

$$\frac{\zeta}{\rho}=\frac{\xi_0}{\rho_0}\frac{\partial z}{\partial a}+\frac{\eta_0}{\rho_0}\frac{\partial z}{\partial b}+\frac{\zeta_0}{\rho_0}\frac{\partial z}{\partial c}$$

These are known as **Cauchy's Integral.**

When velocity potential exists $\zeta=\eta=\zeta=0$, and from the above equations it conclude that these quantities are always zero if their initial values are zero.

When a velocity potential exists the motion is said to be irrotational, thus we can state that the **motion of a fluid under conservative forces, if once irrotational, is always irrotational.**

When a velocity potential does not exist, the motion is said to be **rotational.**

Ex. 1. *An elastic fluid, the weight of which is neglected, obeying Boyle's Law, is in motion in a uniform straight tube ; shew that on the hypothesis of parallel sections the velocity at any time t at a distance r from a fixed point in the tube is defined by the equation*

$$\frac{\partial^2 v}{\partial t^2}+\frac{\partial}{\partial r}\left(2v\frac{\partial v}{\partial t}+v^2\frac{\partial v}{\partial r}\right)=k\frac{\partial^2 v}{\partial r^2}$$

Since the fluid obeys Boyle's Law

i.e. $\quad p=k\rho \qquad \ldots(1)$

The equation of continuity is

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial r}(\rho v)=0. \qquad \ldots(2)$$

and equation of motion is

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}$$

$\left\{\begin{array}{l}\text{Since } \quad p=k\rho \\ \dfrac{\partial p}{\partial r}=k\dfrac{\partial \rho}{\partial r}\end{array}\right.$

or
$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial r}=-\frac{k}{\rho}\frac{\partial \rho}{\partial r} \qquad \ldots(3)$$

By differentiating (3) partially with regard to t, we have

$$\frac{\partial^2 v}{\partial t^2}+\frac{\partial}{\partial t}\left\{v\frac{\partial v}{\partial r}+\frac{k}{\rho}\frac{\partial \rho}{\partial r}\right\}=0.$$

or
$$\frac{\partial^2 v}{\partial t^2}+\frac{\partial}{\partial r}\left\{v\frac{\partial v}{\partial t}+\frac{k}{\rho}\frac{\partial \rho}{\partial t}\right\}=0$$

or
$$\frac{\partial^2 v}{\partial t^2}+\frac{\partial}{\partial r}\left\{v\frac{\partial v}{\partial t}+\frac{k}{\rho}\left(-\frac{\partial}{\partial r}(\rho v)\right)\right\}=0 \qquad \{\text{ from (2)}$$

$$\left[\begin{array}{rl}\text{as } \dfrac{\partial}{\partial t}\left(v\dfrac{dv}{dr}\right) & =\dfrac{\partial}{\partial t}\cdot\dfrac{\partial}{\partial r}\left(\tfrac{1}{2}v^2\right) \\ & =\dfrac{\partial}{\partial r}\cdot\dfrac{\partial}{\partial t}\left(\tfrac{1}{2}v^2\right) \\ & =\dfrac{\partial}{\partial r}\cdot\left(v\dfrac{\partial v}{\partial t}\right) \\ \text{and } \dfrac{\partial}{\partial t}\left(\dfrac{k}{\rho}\dfrac{\partial \rho}{\partial r}\right) & =k\dfrac{\partial}{\partial r}\cdot\dfrac{\partial(\log\rho)}{\partial t} \\ & =k\dfrac{\partial}{\partial t}\cdot\dfrac{\partial}{\partial r}(\log\rho) \\ & =\dfrac{\partial}{\partial r}\left\{\dfrac{k}{\rho}\dfrac{\partial \rho}{\partial t}\right\}\end{array}\right.$$

or
$$\frac{\partial^2 v}{\partial r^2}+\frac{\partial}{\partial r}\left\{v\frac{\partial v}{\partial t}-\frac{k}{\rho}\left(\rho\frac{\partial v}{\partial r}+v\frac{\partial \rho}{\partial r}\right)\right\}=0$$

or
$$\frac{\partial^2 v}{\partial t^2}+\frac{\partial}{\partial r}\left\{v\frac{\partial v}{\partial t}-k\frac{\partial v}{\partial r}-\frac{k}{\rho}\frac{\partial \rho}{\partial r}.v\right\}=0$$

or
$$\frac{\partial^2 v}{\partial t^2}+\frac{\partial}{\partial r}\left\{v\frac{\partial v}{\partial t}-k\frac{\partial v}{\partial r}+\left(\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial r}\right)v\right\}=0.$$
$\{\text{ from (3)}$

or
$$\frac{\partial^2 v}{\partial t^2}+\frac{\partial}{\partial r}\left\{2v\frac{v}{\partial t}+v^2\frac{\partial v}{\partial r}-k\frac{\partial v}{\partial r}\right\}=0$$

or
$$\frac{\partial^2 v}{\partial t^2}+\frac{\partial}{\partial r}\left\{2v\frac{\partial v}{\partial t}+v^2\frac{\partial v}{\partial r}\right\}=k\frac{\partial^2 v}{r^2}$$

Proved.

Ex. 2. *Air, obeying Boyle's Law, is in motion in a uniform tube of small section, prove that if ρ be the density and v the velocity at a distance x from a fixed point at time t*

$$\frac{\partial^2 \rho}{\partial t^2}=\frac{\partial^2}{\partial x^2}\{\rho(v^2+k)\}.$$

Let p be the pressure and v the velocity at a distance x from the end of the tube at any time t. The equation of motion is

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \ldots(1)$$

The equation of continuity is

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho v)=0 \qquad \ldots(2)$$

Since Air, obeys Boyle's Law,

$$p=k\rho \qquad \ldots(3)$$

from (1) and (3), we have

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=-\frac{k}{\rho}\frac{\partial \rho}{\partial x} \qquad \ldots(4)$$

$$\left\{ \text{Since } \frac{\partial p}{\partial x}=k\frac{\partial \rho}{\partial x}\right.$$

Differentiating (2) partially with regard to t, we have

$$\frac{\partial^2 \rho}{\partial r^2}+\frac{\partial}{\partial t}\left\{\frac{\partial}{\partial x}(\rho v)\right\}=0$$

or $$\frac{\partial^2 \rho}{\partial t^2}+\frac{\partial}{\partial x}\left\{\frac{\partial}{\partial t}(\rho v)\right\}=0$$

or $$\frac{\partial^2 \rho}{\partial t^2}+\frac{\partial}{\partial x}\left\{\rho\frac{\partial v}{\partial t}+v\frac{\partial \rho}{\partial t}\right\}=0$$

or $$\frac{\partial^2 \rho}{\partial t^2}+\frac{\partial}{\partial x}\left\{\rho\left(-v\frac{\partial v}{\partial x}-\frac{k}{\rho}\frac{\partial \rho}{\partial x}\right)-v\frac{\partial}{\partial x}(\rho v)\right\}=0$$

{ from (2) and (4),

or $$\frac{\partial^2 \rho}{\partial t^2}=\frac{\partial}{\partial x}\left\{\rho v\frac{\partial v}{\partial x}+k\frac{\partial \rho}{\partial x}+v\frac{\partial}{\partial x}(\rho v)\right\}$$

$$\frac{\partial^2 \rho}{\partial t^2}=\frac{\partial}{\partial x}\left\{\frac{\partial}{\partial x}(\rho v\cdot v)+k\frac{\partial \rho}{\partial x}\right\}=\frac{\partial^2}{\partial x^2}\{\rho(v^2+k)\}$$

or $$\frac{\partial^2 \rho}{\partial t^2}=\frac{\partial^2}{\partial x^2}\{\rho(v^2+k)\}$$

Proved.

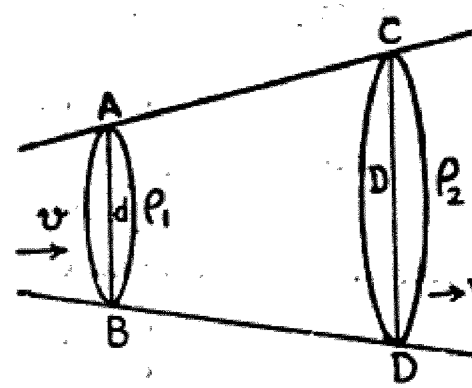
Ex. 3. *Steam is rushing from a boiler through a conical pipe, the diameters of the ends of which are D and d; If V and v be the corresponding velocities of the steam and if the motion be supposed to be that of divergence from the vertex of the cone,*

Prove that

$$\frac{v}{V}=\frac{D^2}{d^2}\, e^{\frac{v^2-V^2}{2k}}$$

where k is the pressure divided by the density, and supposed constant.

Let ρ_1 and ρ_2 be the densities of steam at the ends of the conical pipe AB and CD. By the principle of conservation of mass the masses of the steam that enters and leaves at the end AB and CD are the same. Thus, we have



$$\pi\left(\tfrac{1}{2}d\right)^2 v\,\rho_1=\pi\left(\tfrac{1}{2}D\right)^2 V\rho_2$$

or $$\frac{v}{V}=\frac{D^2}{d^2}\cdot\frac{\rho_2}{\rho_1} \qquad \ldots(1)$$

Let p be the pressure, ρ the density and u be the velocity at a distance r from AB. Then the equation of motion is given by,

$$u\frac{\partial u}{\partial r}=-\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad \text{Since} \quad p=k\rho$$

or $$u\frac{\partial u}{\partial r}=-\frac{k}{\rho}\cdot\frac{\partial \rho}{\partial r}$$

By integrating, we have

$$\tfrac{1}{2}u^2=-k\log\rho+k\log E$$

where E is an arbitrary constant.

or $$\log\frac{\rho}{E}=-\frac{u^2}{2k}$$

or $$\rho=E\,e^{-\frac{u^2}{2k}}$$

Since $\rho=\rho_1$ when $u=v$ then $\rho_1=E\,e^{-\frac{v^2}{2k}}$

and $\rho=\rho_2$ when $u=V$ then $\rho_2=E\,e^{-\frac{V^2}{2k}}$

$$\text{So} \quad \frac{\rho_1}{\rho_2}=\frac{e^{-v^2/2k}}{e^{-V^2/2k}} \quad \text{or} \quad \frac{\rho_2}{\rho_1}=e^{\frac{v^2-V^2}{2k}} \qquad \ldots(2)$$

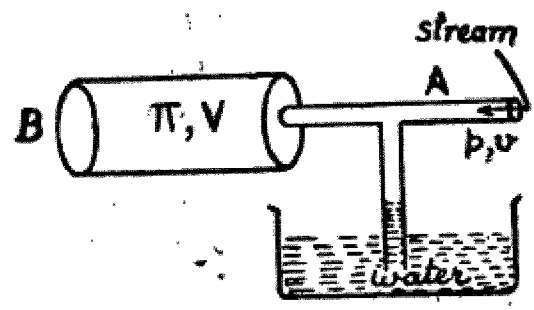
From (1) and (2), we have

$$\frac{v}{V}=\frac{D^2}{d^2}\, e^{\frac{v^2-V^2}{2k}}$$

Proved.

Ex. 4. *A steam in a horizontal pipe, after passing a contraction in the pipe at which its sectional area is A, is delivered at atmospheric pressure at a place where the sectional area is B. Show that if a side tube is connected with the pipe at the former place, water will be sucked up through it into the pipe from a reservoir at a depth* $\frac{S^2}{2g}\left(\frac{1}{A^2}-\frac{1}{B^2}\right)$ *below the pipe ; S being the delivery per second.*

Let v be the velocity, p be the pressure in the tube of section A and V be the velocity, Π the atmospheric pressure at the section B. The equation of motion is,



$$v\frac{\partial v}{\partial r}=-\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad \ldots(1)$$

By integrating, we have

$$\frac{1}{2}v^2=-\frac{p}{\rho}+C \qquad \ldots(2)$$

where C is an arbitrary constant.

Since $v=V$, $p=\Pi$ at the sectional area B

then $$\frac{1}{2}V^2+\frac{\Pi}{\rho}=C \qquad \ldots(3)$$

and $$\frac{1}{2}v^2+\frac{p}{\rho}=\frac{1}{2}V^2+\frac{\Pi}{\rho} \qquad \{\text{from (2) and (3)}$$

or $$\frac{1}{2}(v^2-V^2)=\frac{1}{\rho}(\Pi-p) \qquad \ldots(4)$$

From the equation of continuity, the delivery of stream per second in the tube of sectional area A and sectional area B, we have

$$Av=BV=S \text{ (given)} \qquad \ldots(5)$$

$$\frac{1}{\rho}(\Pi - p) = \frac{1}{2}\left(\frac{S^2}{A^2} - \frac{S^2}{B^2}\right) \quad ...(6)$$

or

{Eliminating v and V from (4) and (5)}

Let h be the height through which water is sucked up, then

$g\rho h$ = difference of pressure in the tube of section A and section B.

$$= \Pi - p \quad ...(7)$$

From (6) and (7), we have

$$\frac{1}{\rho}.\ g\rho h = \frac{S^2}{2}\left(\frac{1}{A^2} - \frac{1}{B^2}\right)$$

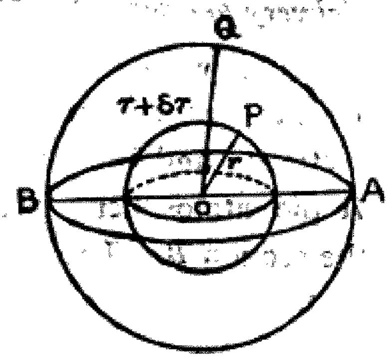
or

$$h = \frac{S^2}{2g}\left(\frac{1}{A^2} - \frac{1}{B^2}\right)$$

Proved.

§ 2·51. Some Symmetrical forms of the equation of continuity.

(i) Spherical Polar Co-ordinates. Consider two concentric spheres. Let **q** (**r**, t) be the velocity in the direction OP. Mass gained by the flow through the inner surface is



$$= 4\pi r^2 q_r . \rho$$

$$= f(r\ t) \quad \text{(let)}$$

and the mass lost by the flow through the outer surface

$$= f(r + \delta r,\ t)$$

The mass between the spheres at any time t

$$= 4\pi r^2 \rho . \delta r$$

Then by the principle of continuity, we have

$$\frac{\partial}{\partial t}\{4\pi r^2 . \rho\ \delta r\} = f(r,\ t) - f(r + \delta r,\ t)$$

or

$$\frac{\partial \rho}{\partial t}.4\pi r^2.\delta r = f(r,\ t) - \left\{f(r\ t) - \frac{\partial}{\partial r} f(r,\ t)\ \delta r + ...\right\}$$

or

$$\frac{\partial \rho}{\partial t}.\ 4\pi r^2\ \delta r = \frac{\partial}{\partial r} f(r\ t)\ \delta r$$

or

$$\frac{\partial \rho}{\partial t}.\ 4\pi r^2\ \delta r = \frac{\partial}{\partial r}\{4\pi r^2 . q_r\ \rho\}\ \delta r$$

or

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho q_r . r^2) = 0.$$

If the fluid be incompressible then ρ = constant

So $\qquad r^2\ q_r = f(t).$

In the **Steady Motion**, $f(t)$ shall be an absolute constant.

(ii) Cylindrical Coordinates. Let $q(r, t)$ be the velocity at any point P, which is perpendicular to a fixed axis OZ, r is the perpendicular distance of P from OZ. Consider two cylinders of radii r and $r+\delta r$ with OZ as axis.

Rate of flow across the inner surface $= \rho\, q . 2\pi r$

$= f(r, t)$ (let)

Rate of flow across the outer surface

$= f(r+\delta r, t)$

Mass between the two cylinders at any time t

$= 2\pi r . \rho\, \delta r$

Rate of increase of mass

$$= \frac{\partial}{\partial t}(2\pi r\rho . \delta r)$$

Then by the principle of continuity, we have

$$\frac{\partial}{\partial t}(2\pi r\rho . \delta r) = f(r, t) - f(r+\delta r, t)$$

or
$$\frac{\partial \rho}{\partial t} . 2\pi r\, \delta r = f(r\ t) - f(r\ t)\ - \delta r . \frac{\partial}{\partial r} f(r\ t) \ldots\ldots$$

or
$$\frac{\partial \rho}{\partial t} .\ 2\pi r\, \delta r = -\delta r . \frac{\partial}{\partial r}\{\rho q . 2\pi r\}$$

or
$$\frac{\partial \rho}{\partial t} .\ 2\pi r\, \delta r = -2\pi \delta\ r . \frac{\partial}{\partial r}(\rho\, rq).$$

or
$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\ \frac{\partial}{\partial r}(\rho\, rq) = 0.$$

If the fluid be **incompressible**, then $\rho =$ constant

i.e.
$$\frac{\partial \rho}{\partial t} = 0$$

or
$$\frac{1}{r}\ \frac{\partial}{\partial r}(\rho r q) = 0$$

$$rq = f(t)$$

In a **Steady flow**, $f(t)$ is an absolute constant.

Ex. 5. *A sphere is at rest in an infinite mass of homogeneous liquid of density ρ, the pressure at infinity being Π; shew that, if the radius R of the sphere varies in any manner, the pressure at the surface of the sphere at any time is*

$$\Pi + \tfrac{1}{2}\rho\left\{\frac{d^2}{dt^2}(R^2) + \left(\frac{dR}{dt}\right)^2\right\}.$$

In an incompressible liquid, the fluid velocity will be radial outside the sphere, so it will be a function of r (radial distance from the centre of the sphere which is origin) and the time t only. Thus the equation of continuity (in polar form) reduces to

$$\frac{1}{r^2}\cdot\frac{d}{dr}(r^2 v)=0$$

or $$r^2 v=\text{constant}=f(t) \text{ (let)} \qquad \ldots(1)$$

(where v is the velocity of a fluid particle at a distance r from the centre at any time t).

Equation of motion is

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}$$

or $$\frac{f'(t)}{r^2}+v\frac{\partial v}{\partial r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}$$

$\left\{\begin{array}{l}\text{Since } r^2 v=f(t) \\ \text{or } \dfrac{\partial v}{\partial t}=\dfrac{f'(t)}{r^2}\end{array}\right.$

Integrating with regard to r, we have

$$-\frac{f'(t)}{r}+\tfrac{1}{2}v^2=-\frac{p}{\rho}+B \qquad \ldots(2)$$

(where B is an arbitrary constant)

Initially when $r\to\infty$, $v=0$, $p=\Pi$

Then from (2) $\qquad B=\dfrac{\Pi}{\rho}$

or $$-\frac{f'(t)}{r}+\tfrac{1}{2}v^2=\frac{\Pi}{\rho}-\frac{p}{\rho}$$

or $$p=\Pi+\tfrac{1}{2}\rho\left\{\frac{2f'(t)}{r}-v^2\right\}$$

Let P be the pressure on the surface of the sphere of radius R and V be the velocity

then $$P=\Pi+\tfrac{1}{2}\rho\left\{\frac{2f'(t)}{R}-V^2\right\} \qquad \ldots(3)$$

Also $R^2V=f(t)$ $\qquad$ {from (1)

or $$f(t)=R^2\frac{dR}{dt}$$ $\left\{ \text{as } V=\dfrac{dR}{dt}\right.$

Differentiating with regard to t, we have

$$f'(t)=R^2\frac{d^2R}{dt^2}+2R\left(\frac{dR}{dt}\right)^2$$

Substituting the value of $f'(t)$ in (3), we get

$$P=\Pi+\tfrac{1}{2}\rho\left\{2R\frac{d^2R}{dt^2}+4\left(\frac{dR}{dt}\right)^2-V^2\right\}$$

or $$P=\Pi+\tfrac{1}{2}\rho\left\{2R\frac{d^2R}{dt^2}+4\left(\frac{dR}{dt}\right)^2-\left(\frac{dR}{dt}\right)^2\right\}$$

or $$P=\Pi+\tfrac{1}{2}\rho\left\{2\left[R\frac{d^2R}{dt^2}+\left(\frac{dR}{dt}\right)^2\right]+\left(\frac{dR}{dt}\right)^2\right\}.$$

or $$P=\Pi+\tfrac{1}{2}\rho\left\{\frac{d^2}{dt^2}(R^2)+\left(\frac{dR}{dt}\right)^2\right\}$$ **Proved.**

Ex. 6. *A centre of force attracting inversely as the square of the distance is at the centre of a spherical cavity within an infinite mass of incompressible fluid, the pressure on which at an infinite distance is* Π, *and is such that the work done by this pressure on a unit of area through a unit of length is one-half the work done by the attractive force on a unit volume of the fluid from infinity to the initial boundary of the cavity ; prove that the time of filling up the cavity will be*

$$\pi a\sqrt{\left(\frac{\rho}{\Pi}\right)}\left\{2-\left(\frac{3}{2}\right)^{3/2}\right\}$$

a being the initial radius of the cavity, and ρ *the density of the fluid.*

Let v' be the velocity at a distance r' at any time t and p be the pressure. The equation of continuity (in polar form) reduces

to $$\frac{1}{r^2}\frac{d}{dr}(r^2v)=0$$

(Since the fluid velocity will be radial outside the spherical cavity, so it will be a function of r and the time t only).

or $$r^2v=f(t)=\text{Const.}$$

or $$r'^2v'=r^2v=f(t) \qquad \text{.. (i)}$$

The equation of motion is

$$\frac{\partial v'}{\partial t}+v'\frac{\partial v'}{\partial r'}=-\frac{\mu}{r'^2}-\frac{1}{\rho}\frac{\partial p}{\partial r'} \qquad \text{...(ii)}$$

or $$\frac{f'(t)}{r'^2}+v'\frac{\partial v'}{\partial r'}=-\frac{\mu}{r'^2}-\frac{1}{\rho}\frac{\partial p}{\partial r'}$$

$\Bigg\{$ Here the external force is $P=\dfrac{\mu}{r^2}$

Integrating with regard to r, we have

$\Bigg\{$ Since $r'^2v'=f(t)$ or $\dfrac{\partial v'}{\partial t}=\dfrac{f'(t)}{r^2}$

$$-\frac{f'(t)}{r'}+\frac{1}{2}v'^2=\frac{\mu}{r'}-\frac{p}{\rho}+A$$

where A is an arbitrary constant.

Initially $r'=\infty$, $v'=0$ and $p=\Pi$

Then $$A=\frac{\Pi}{\rho}$$

or $$-\frac{f'(t)}{r'}+\frac{1}{2}v'^2=\frac{\mu}{r'}+\frac{\Pi-p}{\rho} \qquad \text{...(iii)}$$

Let r be the radius of the Spherical Cavity and v be the velocity at any time t. The pressure p vanishes on the surface of the cavity. Hence equation (iii) reduces to

$$-\frac{f'(t)}{r}+\frac{1}{2}v^2=\frac{\mu}{r}+\frac{\Pi}{\rho}$$

Since $r^2v=f(t)$

then $f'(t)=r^2\frac{dv}{dt}+2rv\frac{dr}{dt}$

$=r^2\frac{dv}{dr}\cdot\frac{dr}{dt}+2rv\frac{dr}{dt}$

$=r^2v\frac{dv}{dr}+2rv^2$

or $$-\frac{1}{r}\left\{r^2v\frac{dv}{dr}+2rv^2\right\}+\frac{1}{2}v^2=\frac{\mu}{r}+\frac{\Pi}{\rho}$$

or $$-rv\frac{dv}{dr}-2v^2+\tfrac{1}{2}v^2=\frac{\mu}{r}+\frac{\Pi}{\rho}$$

or $$rv\frac{dv}{dr}+\frac{3}{2}v^2=-\frac{\mu}{r}-\frac{\Pi}{\rho}$$

or $$2rv\,dv+3v^2\,dr=-2\left(\frac{\mu}{r}+\frac{\Pi}{\rho}\right)dr$$

Multiplying both the sides with r^2 and integrating, we get

$$2r^3v\,dv+3v^2r^2\,dr=-2\left(\mu r+\frac{\Pi}{\rho}r^2\right)dr$$

By integrating, we have

$$r^3v^2=-\left(\mu r^2+\frac{2\Pi}{3\rho}r^3\right)+B \qquad \text{...(iv)}$$

where B is an arbitrary constant.

Initially $r=a$ (Radius of the cavity), $v=0$.

Then $$B=\mu a^2+\frac{2\Pi}{3\rho}a^3$$

So equation (iv) becomes

$$r^3v^2=\mu(a^2-r^2)+\frac{2\Pi}{3\rho}(a^3-r^3) \qquad \text{...(v)}$$

Now the work done by the Pressure Π on a unit of area through a unit of length $=\frac{1}{2}\times$ work done by the attractive force on a unity volume of the fluid from infinity to the initial boundary of the cavity.

i.e.
$$\Pi\times 1\times 1=\tfrac{1}{2}\int_{\infty}^{a}\left(-\frac{\mu}{r^2}\right)\rho\,dr$$

or
$$\Pi=\frac{\mu\rho}{2a}\qquad\text{or}\qquad \mu=\frac{2a\Pi}{\rho}$$

Substituting the value of μ in (v), we get
$$r^3v^2=\frac{2a\Pi}{\rho}(a^2-r^2)+\frac{2\Pi}{3\rho}(a^3-r^3)$$

or
$$r^3v^2=\frac{2\Pi}{3\rho}\{3a(a^2-r^2)+(a^3-r^3)\}$$

or
$$v^2=\frac{2\Pi}{3\rho}\cdot\frac{\{3a(a^2-r^2)+(a^3-r^3)\}}{r^3}$$

or
$$\frac{dr}{dt}=\pm\sqrt{\left(\frac{2\Pi}{3\rho}\right)}\cdot\frac{\{3a(a^2-r^2)+(a^3-r^3)\}^{1/2}}{r^{3/2}}$$

or
$$dt=-\sqrt{\left(\frac{3\rho}{2\Pi}\right)}\int_a^0\frac{r^{3/2}.dr}{\{3a(a^2-r^2)+(a^3-r^3)\}^{1/2}}$$

(Negative sign is taken as r decreases when t increases.)

Let t be the time of filling up the cavity

then
$$t=\sqrt{\left(\frac{3\rho}{2\Pi}\right)}\int_0^a\frac{r^{3/2}.dr}{\sqrt{\{3a(a^2-r^2)+(a^3-r^3)\}}}$$

or
$$t=\sqrt{\left(\frac{3\rho}{2\Pi}\right)}\int_0^a\frac{r^{3/2}.dr}{(r+2a)\sqrt{(a-r)}}.\qquad\left\{\begin{array}{l}\text{Let } r=a\sin^2\theta\\ \quad dr=2a\sin\theta\cos\theta\,d\theta\end{array}\right.$$

or
$$t=\sqrt{\left(\frac{3\rho}{2\Pi}\right)}\int_0^{\pi/2}\frac{a^{3/2}\sin^3\theta.2a\sin\theta\cos\theta\,d\theta}{a(\sin^2\theta+2).a^{1/2}\sqrt{(1-\sin^2\theta)}}$$

or
$$t=2a\sqrt{\left(\frac{3\rho}{2\Pi}\right)}\int_0^{\pi/2}\frac{\sin^4\theta}{\sin^2\theta+2}\,d\theta$$

or
$$t=2a\sqrt{\left(\frac{3\rho}{2\Pi}\right)}\int_0^{\pi/2}\left(\sin^2\theta-2+\frac{4}{2+\sin^2\theta}\right)d\theta$$

or
$$t=2a\sqrt{\left(\frac{3\rho}{2\Pi}\right)}\left[\frac{\pi}{4}-2.\frac{\pi}{2}+\int_0^{\pi/2}\frac{4\sec^2\theta}{2\sec^2\theta+\tan^2\theta}\,d\theta\right]$$

or
$$t=2a\sqrt{\left(\frac{3\rho}{2\Pi}\right)}\left[-\frac{3\pi}{4}+\int_0^{\pi/2}\frac{4\sec^2\theta}{2+3\tan^2\theta}\,d\theta\right]$$

or
$$t=2a\sqrt{\left(\frac{3\rho}{2\Pi}\right)}\left[-\frac{3\pi}{4}+\frac{4}{3}.\sqrt{\left(\frac{3}{2}\right)}.\frac{\pi}{2}\right]$$

or
$$t=\pi a\sqrt{\left(\frac{\rho}{\Pi}\right)}\left\{2-\left(\frac{3}{2}\right)^{3/2}\right\}.$$ **Proved.**

Ex 7. *An infinite mass of homogeneous incompressible fluid is at rest subject to a uniform pressure* Π, *and contains a spherical* *cavity of radius a, filled with gas at a pressure m*Π *; prove that, if the inertia of the gas be neglected, and Boyle's low be supposed to hold throughout the ensuing motion, the radius of the sphere will oscillate between the values a and na, where n is determined by the equation*

$$1+3m \log n-n^3=0.$$

If m be nearly equal to 1 ; the time of an oscillation will be

$2\pi\sqrt{\left(\dfrac{a^2\rho}{3\Pi}\right)}$, *$\rho$ being the density of the fluid.*

In an incompressible fluid, the fluid velocity v will be radial outside the spherical cavity. So v will be a function of r (radial distance from the centre of cavity *i.e.* origin) and the time t only. The continuity equation (in polar form) reduce to,

$$\frac{1}{r}\frac{d}{dr}(r^2v)=0$$

or

$$r^2v=f(t) \qquad \text{(Const.)} \qquad \ldots\text{(i)}$$

Let v' be the velocity at a distance r' at any time t and p be the pressure, then from (i), we have

$$r'^2v'=r^2v=f(t) \qquad \ldots\text{(ii)}$$

The equation of motion is

$$\frac{\partial v'}{\partial t}+v'\frac{\partial v'}{\partial r'}=-\frac{1}{\rho}\frac{\partial p}{\partial r'}$$

or

$$\frac{f'(t)}{r'^2}+v'\frac{\partial v'}{\partial r'}=-\frac{1}{\rho}\frac{\partial p}{\partial r'}$$

$$\left\{\begin{array}{ll}\text{Since} & r'^2v'=f(t)\\ \text{or} & \dfrac{\partial v'}{\partial t}=\dfrac{f(t)}{r'^2}\end{array}\right.$$

Integrating with regard to r', we have

$$-\frac{f'(t)}{r'}+\frac{1}{2}v'^2=-\frac{p}{\rho}+A \qquad \ldots\text{(iii)}$$

(where A is an arbitrary constant.)

Initially when $r'=\infty$, $v'=0$, $p=\Pi$

Then $$A=\frac{\Pi}{\rho}$$

So equation (iii) becomes.

$$-\frac{f'(t)}{r'}+\frac{1}{2}v'^2=\frac{\Pi-p}{\rho} \qquad \ldots\text{(iv)}$$

Hence motion on the surface of the cavity is

$$-\frac{f'(t)}{r}+\frac{1}{2}v^2=\frac{\Pi-p}{\rho} \qquad \ldots\text{(v)}$$

Since Boyle's Law states that

$$pv=\text{Const.}$$

So $$\tfrac{4}{3}\pi r^3 . p = \tfrac{4}{3}\pi a^3 . m\Pi$$

or $$p = \frac{a^3}{r^3} . m\Pi$$

and $$r^2 v = f(t) \qquad \{ \text{ from (i)}$$

$$f'(t) = 2rv\frac{dr}{dt} + r^2\frac{dv}{dt}$$

$$f'(t) = 2rv^2 + r^2 v\frac{dv}{dr}.$$

Substituting the value of $f'(t)$ and p in (v), we get

$$-\frac{1}{r}\left(2rv^2 + r^2 v\frac{dv}{dr}\right) + \frac{1}{2}v^2 = \frac{\Pi}{\rho} - \frac{a^3}{\rho r^3} . m\Pi$$

or $$rv\frac{dv}{dr} + \frac{3}{2}v^2 = -\frac{\Pi}{\rho} + \frac{a^3}{\rho r^3} . m\Pi$$

Multiplying with $2r^2 . dr$ both the sides, we have

$$2r^3 v\, dv + 3r^2 v^2\, dr = \left(-\frac{2\Pi r^2}{\rho} + \frac{2a^3 m\Pi}{\rho r}\right) dr$$

By integrating, we have

$$r^3 v^2 = -\frac{2\Pi}{3\rho}r^3 + \frac{2a^3 m\Pi}{\rho}\log r + B.$$

To determine the value of the constant B, we have

$$r = a,\ v = 0$$

then $$B = \frac{2\Pi}{3\rho}a^3 - \frac{2a^3 m\Pi}{\rho}\log a$$

or $$r^3 v^2 = \frac{2\Pi}{3\rho}(a^3 - r^3) + \frac{2a^3 m\Pi}{\rho}\log\left(\frac{r}{a}\right). \qquad \text{...(vi)}$$

Since the radius of the sphere oscillates between a and na

i.e. at $r = a,\ v = 0$ and $r = na,\ v = 0$.

Substituting $r = na,\ v = 0$ in relation (vi), we get

$$0 = \frac{2\Pi}{3\rho}\left\{a^3 - n^3 a^3 + 3ma^3\log\left(\frac{na}{a}\right)\right\}$$

or $$1 + 3m\log n - n^3 = 0\ ; \quad a \neq 0$$

which proves the required condition.

Again if m be nearly equal to 1, let $r = a + x$ where x is small then from (vi), we have

$$(a+x)^3\dot{x}^2 = \frac{2\Pi}{3\rho}\left(a^3 - (a+x)^3\right) + \frac{2a^3\Pi}{\rho}\log\left(\frac{a+x}{a}\right)$$

$$\left\{\begin{array}{ll} \text{as} & r = a + x \\ \text{or} & \dfrac{dr}{dt} = \dfrac{dx}{dt} \\ \text{or} & v = \dot{x} \end{array}\right.$$

or $$a^3\left(1+\frac{x}{a}\right)^3\dot{x}^2=\frac{2\Pi}{3\rho}a^3\left\{1-\left(1+\frac{x}{a}\right)^3\right\}+\frac{2a^3\Pi}{\rho}\log\left(1+\frac{x}{a}\right)$$

or $$\left(1+\frac{x}{a}\right)^3\dot{x}^2=\frac{2\Pi}{3\rho}\left\{1-\left(1+\frac{3x}{a}+\frac{3x^2}{a^2}+\ldots\right)\right\}+\frac{2a^3\Pi}{\rho}\left\{\frac{x}{a}-\frac{1}{2}\frac{x^2}{a^2}+\ldots\ldots\right\}$$

or $$\dot{x}^2=\frac{2\Pi}{3\rho}\left[\left(-\frac{9}{2}\frac{x^2}{a^2}+\ldots\right)\left(1+\frac{x}{a}\right)^{-3}\right]$$

or $$\dot{x}^2=\frac{2\Pi}{3\rho}\left[\left(-\frac{9}{2}\frac{x^2}{a^2}+\ldots\right)\left(1-\frac{3x}{a}+\frac{6x^2}{a^2}\ldots\ldots\right)\right]$$

or $$\dot{x}^2=-\frac{3\Pi}{\rho a^2}x^2.$$

(Neglecting higher orders of x, being very small)

Now differenting with regard to t, we have

$$2\dot{x}\,\ddot{x}=-\frac{3\Pi}{\rho a^2}.\,2x\,\dot{x}$$

or $$\ddot{x}=-\frac{3\Pi}{\rho a^2}x$$

which is the standard equation of S. H. M. whose time period is

$$2\pi\sqrt{\left(\frac{\rho a^2}{3\Pi}\right)}.$$ **Proved.**

Ex. 8. *An infinite fluid in which as a spherical hollow of radius a is initially at rest under the action of no forces. If a constant pressure* Π *is applied at infinity, show that the time of filling up the cavity is*

$$\pi^2 a\left(\frac{\rho}{\Pi}\right)^{1/2}2^{5/6}\left\{\Gamma\left(\frac{1}{3}\right)\right\}^{-3}$$

In the incompressible fluid, the fluid velocity v will be radial out-side the hollow sphere. So it will be a function of r (radial distance from the centre of the sphere, *i.e.* origin) and the time t only.

The equation of continuity (in polar form) reduces to

$$\frac{1}{r^2}\frac{d}{dr}(r^2v)=0$$

or $$r^2v=f(t)\quad\text{(const)}.$$

Equations of Motion

Let v' be the velocity at a distance r' at any time t and p be the pressure there.

So
$$r'^2v'=r^2v=f(t). \quad ...(i)$$

The equation of motion is.

$$\frac{\partial v'}{\partial t}+v'\frac{\partial v'}{\partial r'}=-\frac{1}{\rho}\frac{\partial p}{\partial r'}$$

or
$$\frac{f'(t)}{r'^2}+v'\frac{\partial v'}{\partial r'}=-\frac{1}{\rho}\frac{\partial p}{\partial r'}$$

$$\left\{ \text{as} \quad r'^2v'=f(t), \quad \frac{\partial v'}{\partial t}=\frac{f'(t)}{r'^2} \right.$$

Integrating with regard to r', we have,

$$-\frac{f'(t)}{r'}+\tfrac{1}{2}v'^2=-\frac{p}{\rho}+A$$

(where A is arbitrary constant).

Initially

$$r'=\infty,\ v'=0,\ p=\Pi \text{ then } A=\frac{\Pi}{\rho}$$

So
$$-\frac{f'(t)}{r'}+\tfrac{1}{2}v'=\frac{\Pi-p}{\rho}. \quad ...(ii)$$

Let v be the velocity and r be the radius of Spherical cavity at any time t

$$v'=v,\ r'=r. \quad p=0 \text{ (being hollow)}$$

From (ii), we get

$$-\frac{f'(t)}{r}+\tfrac{1}{2}v^2=\frac{\Pi}{\rho} \quad ...(iii)$$

$$\left\{ \begin{aligned} &\text{as} && r^2v=f(t) \\ &\text{or} && f'(t)=r^2\frac{dv}{dt}+2rv\frac{dr}{dt} \\ &\text{or} && f'(t)=r^2v\frac{dv}{dr}+2rv^2 \end{aligned} \right.$$

or
$$-\frac{1}{r}\left(r^2v\frac{dv}{dr}+2rv^2\right)+\tfrac{1}{2}v^2=\frac{\Pi}{\rho}$$

or
$$rv\frac{dv}{dr}+\frac{3}{2}v^2=-\frac{\Pi}{\rho}$$

Multiplying both the sides by $2r^2dr$, we have

$$2r^3v\,dv+3r^2v^2\,dr=-\frac{2\Pi}{\rho}r^2\,dr.$$

By integrating, we have

$$r^3v^2=-\frac{2\Pi}{3\rho}r^3+B. \quad ...(iv)$$

Initially Radius of cavity $r=a$, $v=0$.

then
$$B=\frac{2\Pi}{3\rho}a^3.$$

So equation (iv) reduces to

$$r^3 v^2 = \frac{2\Pi}{3\rho}(a^3 - r^3)$$

or
$$v = \frac{dr}{dt} = -\sqrt{\left(\frac{2\Pi}{3\rho}\right)}\sqrt{\left(\frac{a^3 - r^3}{r^3}\right)}$$

(Negative sign is taken since v increases as r decreases)

Let t be the time of filling up the cavity, then

$$t = -\sqrt{\left(\frac{3\rho}{2\Pi}\right)}\int_a^0 \sqrt{\left(\frac{r^3}{a^3 - r^3}\right)}\,dr$$

$$= -\sqrt{\left(\frac{3\rho}{2\Pi}\right)}\int_0^{\pi/2} \frac{a^{3/2}\sin\theta}{a^{3/2}\cos\theta} \cdot \frac{2a}{3}\sin^{-1/3}\theta\cos\theta\,d\theta$$

$$\left\{\begin{array}{ll} \text{Let} & r = a\sin^{2/3}\theta \\ \text{or} & dr = \frac{2a}{3}\sin^{-1/3}\theta\cos\theta\,d\theta \end{array}\right.$$

$$= \frac{2a}{3}\sqrt{\left(\frac{3\rho}{2\Pi}\right)}\int_0^{\pi/2}\sin^{2/3}\theta\,d\theta$$

By applying Gamma function, we have

$$= \frac{2a}{3}\sqrt{\left(\frac{3\rho}{2\Pi}\right)} \cdot \frac{\Gamma\frac{5}{6}\,\Gamma\frac{1}{2}}{2\Gamma(\frac{5}{6}+\frac{1}{2})}$$

$$= \frac{2a}{3} \cdot \sqrt{\left(\frac{3\rho}{2\Pi}\right)} \cdot \frac{\Gamma\frac{5}{6}\,\Gamma\frac{1}{2}}{2\Gamma\frac{4}{3}}$$

$$= \frac{2a}{3} \cdot \sqrt{\left(\frac{3\rho}{2\Pi}\right)} \cdot \frac{\Gamma\frac{5}{6}\,\Gamma\frac{1}{2}}{2.\frac{1}{3}\,\Gamma\frac{1}{3}}$$

$$\left[\begin{array}{ll} \text{We know from Integral Calculus that} & \\ \Gamma n\,\Gamma(n+\frac{1}{2}) = \sqrt{\pi}.\frac{\Gamma 2n}{2^{2n-1}} & \ldots(X) \\ \text{and } \Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi} & \ldots(Y) \end{array}\right.$$

$$= \frac{2a}{3} \cdot \frac{3}{2}\sqrt{\left(\frac{3\rho}{2\Pi}\right)} \cdot \frac{\sqrt{\pi}.\Gamma(\frac{1}{3}+\frac{1}{2})}{\Gamma\frac{1}{3}}$$

$$= a\sqrt{\left(\frac{3\rho}{2\Pi}\right)}.\sqrt{\pi}.\frac{\Gamma(\frac{1}{3}+\frac{1}{2})\,\Gamma\frac{1}{3}}{(\Gamma\frac{1}{3})^2}$$

{Multiplying N^r and D^r with $\Gamma\frac{1}{3}$ and using the relation (X)}

$$= a\sqrt{\left(\frac{3\rho}{2\Pi}\right)}.\sqrt{\pi}.\frac{\sqrt{\pi}.\Gamma\frac{2}{3}}{(\Gamma\frac{1}{3})^2\,2^{2/3-1}}$$

$$= \pi a\sqrt{\left(\frac{3\rho}{2\Pi}\right)} \cdot \frac{}{(\Gamma\frac{1}{3})^2\,2^{-1/3}}$$

$$= \pi a\sqrt{\left(\frac{3\rho}{2\Pi}\right)} \cdot \frac{\Gamma(1-\frac{1}{3})\,\Gamma\frac{1}{3}}{2^{-1/3}(\Gamma\frac{1}{3})^3}$$

{Multiplying N^r and D^r with $\Gamma\frac{1}{3}$ and using the relation (Y)}

$$=\pi a\sqrt{\left(\frac{3\rho}{2\Pi}\right)}\cdot 2^{1/3}\cdot(\Gamma\tfrac{1}{3})^{-3}\cdot\frac{\pi}{\sin\frac{\pi}{3}}$$

$$=\pi^2 a\sqrt{\left(\frac{3\rho}{2\Pi}\right)}\cdot(\Gamma\tfrac{1}{3})^{-3}\cdot 2^{1/3}\cdot\frac{2}{\sqrt{3}}$$

$$=\pi^2 a\left(\frac{\rho}{\Pi}\right)^{1/2}2^{5/6}\,(\Gamma\tfrac{1}{3})^{-3}.$$

Proved.

Ex. 9. *An infinite mass of fluid is acted on by a force $\mu r^{-3/2}$ per unit mass directed to the origin. If initially the fluid is at rest and there is a cavity in the form of the sphere $r=c$ in it. Shew that the cavity will be filled up after an interval of time*

$$\left(\frac{2}{5\mu}\right)^{1/2}c^{5/4}.$$

In an incompressible fluid, the fluid velocity v will be radial outside the spherical cavity. The velocity will be a function of r (radial distance from the centre of the cavity, *i.e.* origin) and the time t only. The equation of continuity becomes

$$\frac{1}{r^2}\frac{d}{dr}(r^2 v)=0$$

or

$$r^2 v=f(t) \qquad \ldots(1)$$

Let v' be the velocity at a distance r' from the origin at any time t and p be the pressure. Then from (1) equation of continuity is

$$r'^2 v'=r^2 v=f(t)$$

The equation of motion is

$$\frac{\partial v'}{\partial t}+v'\frac{\partial v'}{\partial r'}=-\mu r'^{-3/2}-\frac{1}{\rho}\frac{\partial p}{\partial r'}$$

or

$$\frac{f'(t)}{r'^2}+v'\frac{\partial v'}{\partial r'}=-\mu r'^{-3/2}-\frac{1}{\rho}\frac{\partial p}{\partial r'}$$

$$\left\{\begin{aligned}\text{Since}\quad r'^2 v'&=f(t)\\ \frac{\partial v'}{\partial t}&=\frac{f'(t)}{r'^2}\end{aligned}\right.$$

By integrating with regard to r, we have

$$-\frac{f'(t)}{r'}+\tfrac{1}{2}v'^2=\frac{2\mu}{r'^{1/2}}-\frac{p}{\rho}+A \qquad \ldots(2)$$

Initially when $r'=\infty$, $v'=0$, $p=0$ then $A=0$

Then (2) reduces to

$$-\frac{f'(t)}{r'}+\tfrac{1}{2}v'^2=\frac{2\mu}{r'^{1/2}}-\frac{p}{\rho} \qquad \ldots(3)$$

Let r be the radius of the cavity and v be the velocity at any time t. The motion of the cavity is {putting $p=0$ in (3)},

$$-\frac{f'(t)}{r}+\tfrac{1}{2}v^2=\frac{2\mu}{r^{1/2}} \qquad \left[\begin{array}{l} \text{from (1) } r^2v=f(t) \\ \text{or} \quad f'(t)=r^2\dfrac{dv}{dt}+2rv\dfrac{dr}{dt} \\ \qquad\qquad =r^2v\dfrac{dv}{dr}+2rv^2 \end{array}\right.$$

or $$-\left(rv\frac{dv}{dr}+2v^2\right)+\tfrac{1}{2}v^2=\frac{2\mu}{r^{1/2}}$$

or $$rv\frac{dv}{dr}+\tfrac{3}{2}v^2=-\frac{2\mu}{r^{1/2}}$$

Multiplying both the sides with $2r^2dr$, we get

$$2r^3v\,dv+3r^2v^2dr=-4\mu.r^{3/2}.dr$$

By integrating, we have

$$r^3v^2=-\frac{8\mu}{5}r^{5/2}+B. \qquad ...(4)$$

Now $\qquad r=c,\ v=0$ then $B=\dfrac{8\mu}{5}c^{5/2}$

The equation (4) becomes

$$r^3v^2=\frac{8\mu}{5}(c^{5/2}-r^{5/2})$$

or $$v=\frac{dr}{dt}=\pm\sqrt{\left(\frac{8\mu}{5}\right)}.\sqrt{\left(\frac{c^{5/2}-r^{5/2}}{r^3}\right)}$$

(Negative sign is taken since velocity increases as r decreases).

If t be the time of filling the cavity

then $$t=-\sqrt{\left(\frac{5}{8\mu}\right)}\int_c^0\frac{r^{3/2}.dr}{\sqrt{(c^{5/2}-r^{5/2})}}$$

or $$t=\frac{4}{5}.\sqrt{\left(\frac{5}{8\mu}\right)}\int_0^{\pi/2}\frac{c^{5/2}\sin\theta\cos\theta}{c^{5/4}.\cos\theta}d\theta$$

$$\left\{\begin{array}{l} \text{Put} \quad r^{5/2}=c^{5/2}\sin^2\theta \\ \text{or} \quad \tfrac{5}{2}r^{3/2}\,dr=2c^{5/2}\sin\theta\cos\theta\,d\theta \end{array}\right.$$

$$=\frac{4}{5}c^{5/4}\sqrt{\left(\frac{5}{8\mu}\right)}\int_0^{\pi/2}\sin\theta\,d\theta$$

$$=\sqrt{\left(\frac{2}{5\mu}\right)}c^{5/4}$$ **Proved.**

Ex. 10. *A mass of fluid of density ρ and volume $\frac{4}{3}\pi c^3$ is in the form of a spherical shell. A constant pressure Π is exerted on the external surface of the shell. There is no pressure on the internal surface and no other forces act on the liquid. Initially the liquid is at rest and the internal radius of the shell is 2c. Prove that the velocity of the internal surface when its radius is c is*

$$\sqrt{\left(\frac{14\Pi}{3\rho}\cdot\frac{2^{1/3}}{2^{1/3}-1}\right)}.$$

In the incompressible fluid, the fluid velocity q will be radial outside the spherical shell, hence it will be a function of r (radial distance from the centre of the shell, which is origin) and the time t only. So the equation of continuity reduces to

$$\frac{1}{r^2}\frac{d}{dr}(r^2q)=0$$

or $\quad r^2q=f(t) \qquad$ (const.)

Let v' be the velocity at a distance r' at any time t and p be the pressure, then the equation of continuity is

$$r'^2v'=f(t) \qquad \text{...(1)}$$

The equation of motion is

$$\frac{\partial v'}{\partial t}+v'\frac{\partial v'}{\partial r'}=-\frac{1}{\rho}\frac{\partial p}{\partial r'} \qquad \text{...(2)}$$

ot $\quad \dfrac{f'(t)}{r'^2}+v'\dfrac{\partial v'}{\partial r'}=-\dfrac{1}{\rho}\dfrac{\delta p}{\partial r'} \qquad \left\{\begin{array}{ll}\text{Since} & r'^2v'=f(t)\\ \text{or} & \dfrac{\partial v'}{\delta t}=\dfrac{f'(t)}{r'^2}\end{array}\right.$

By integrating with regard to r', we have

$$-\frac{f'(t)}{r'}+\tfrac{1}{2}v'^2=-\frac{p}{\rho}+A \qquad \text{...(3)}$$

Let r and R be the internal and external radii of the shell and v and V be the velocities there at any time t.

Initially when

I $\quad r'=R,\ v'=V,\ p=\Pi$

II $\quad r'=r,\ v'=v,\ p=0.$

(Since there is no pressure on internal surface)

Now we shell determine the value of the constant A from (3) with the help of I and II condition.

or $\quad -\dfrac{f'(t)}{R}+\tfrac{1}{2}V^2=-\dfrac{\Pi}{\rho}+A$

and $\quad -\dfrac{f'(t)}{r}+\tfrac{1}{2}v^2=A$

By subtraction, we have

$$-f'(t)\left\{\frac{1}{r}-\frac{1}{R}\right\}+\tfrac{1}{2}\left\{v^2-V^2\right\}=\frac{\Pi}{\rho} \qquad \text{...(4)}$$

Since $\quad r^2v=R^2V=f(t) \qquad$ ...(5) $\quad$ { from (1)

or $\quad 2r^2dr=2R^2dR=2f(t)\,dt \qquad$ ...(6)

from (4) and (5), we get

$$-f'(t)\left\{\frac{1}{r}-\frac{1}{R}\right\}+\tfrac{1}{2}\left\{\frac{f^2(t)}{r^4}-\frac{f^2(t)}{R^4}\right\}=\frac{\Pi}{\rho}$$

Multiplying both sides by the relation (vi) and integrating

$$-2f(t)f'(t)\left\{\frac{1}{r}-\frac{1}{R}\right\}+\tfrac{1}{2}f^2(t)\left\{\frac{2r^2dr}{r^4}-\frac{2R^2dR}{R^4}\right\}=\frac{\Pi}{\rho}.2r^2dr$$

or $$-2f(t)f'(t)\left\{\frac{1}{r}-\frac{1}{R}\right\}+\tfrac{1}{2}f^2(t)\left\{\frac{2dr}{r^2}-\frac{2dR}{R^2}\right\}=\frac{2\Pi}{\rho}r^2dr$$

By integrating, we have

$$f^2(t)\left\{\frac{1}{r}-\frac{1}{R}\right\}=-\frac{2\Pi}{3\rho}r^3+B$$

Initially when $r=2c$, $v=0$ *i.e.* $f(t)=0$,

then $$B=\frac{2\Pi}{3\rho}.8c^3$$

or $$f^2(t)\left\{\frac{1}{r}-\frac{1}{R}\right\}=\frac{2\Pi}{3\rho}(8c^3-r^3)$$

or $$r^4v^2\left\{\frac{1}{r}-\frac{1}{R}\right\}=\frac{2\Pi}{3\rho}(8c^3-r^3)$$

again $\frac{4}{3}\pi R^3-\frac{4}{3}\pi r^3=\frac{4}{3}\pi c^3$
or $R^3-r^3=c^3$
and $r^2v=f(t)$

or $$r^4v^2\left\{\frac{1}{r}-\frac{1}{(r^3+c^3)^{1/3}}\right\}=\frac{2\Pi}{3\rho}(8c^3-r^3)$$

or $$v^2=\frac{2\Pi}{3\rho}.\frac{8c^3-r^3}{r^4\left\{\frac{1}{r}-\frac{1}{(r^3+c^3)^{1/3}}\right\}} \quad ...(7)$$

which gives the velocity in terms of the radius r at the inner surface of the cavity.

Now velocity of the internal surface when $r=c$, is given by from (7)

or $$v^2=\frac{2\Pi}{3\rho}\frac{7c^3}{c^4\left\{\frac{1}{c}-\frac{1}{c.2^{1/3}}\right\}}$$

or $$v=\sqrt{\left(\frac{14\Pi}{3\rho}.\frac{2^{1/3}}{2^{1/3}-1}\right)}.$$ **Proved.**

Ex. 11. *A solid sphere of radius a is surrounded by a mass of liquid whose volume is* $\frac{4}{3}\pi c^3$, *and its centre is a centre of attractive force varying directly as the square of the distance. If the solid* *sphere be suddenly annihilated, shew that the velocity of the inner surface when its radius is x, is given by*

$$\dot{x}^2 x^3 \left\{(x^3+c^3)^{1/3}-x\right\}=\left(\frac{2\Pi}{3\rho}+\frac{2\mu c^3}{9}\right)(a^3-x^3)(c^3+x^3)^{1/3}.$$

Let v' be the velocity at a distance r' from the centre of the spherical shell at any time t and p be the pressure there. The equation of continuity is

or
$$\frac{1}{r'^2}\cdot\frac{d}{dr}(r'^2 v')=0$$

or
$$r'^2 v'=f(t) \qquad ...(1)$$

The equation of motion is,

$$\frac{\partial v'}{\partial t}+v'\frac{\partial v'}{\partial r'}=-\mu r'^2-\frac{1}{\rho}\frac{\partial p}{\partial r'} \qquad ...(2)$$

{as $\mu r'^2$ is an external force}.

from (1) and (2), we have

$$\frac{f'(t)}{r'^2}+v'\frac{\partial v'}{\partial r'}=-\mu r'^2-\frac{1}{\rho}\frac{\partial p}{\partial r'}$$

By integrating with regard to r', we have

$$-\frac{f'(t)}{r'}+\frac{1}{2}v'^2=-\frac{1}{3}\mu r'^3-\frac{p}{\rho}+A \qquad ...(3)$$

Let r and R be the radii and v and V the velocities of the inner and outer surfaces of the spherical shell at any time t.

Then the condition are

I $r'=r,\ v'=v,\ p=0$

II $r'=R,\ v'=V,\ p=\Pi$

and III $\frac{4}{3}\pi R^3-\frac{4}{3}\pi r^3=\frac{4}{3}\pi c^3$

or $R^3-r^3=c^3$.

Now we shall determine the value of the constant A from (3) with the help of I and II condition.

or
$$-\frac{f'(t)}{r}+\frac{1}{2}v^2=-\frac{1}{3}\mu r^3+A$$

and
$$-\frac{f'(t)}{R}+\frac{1}{2}V^2=-\frac{1}{3}\mu R^3-\frac{\Pi}{\rho}+A$$

By subtracting, we can eliminate the constant A,

$$f'(t)\left\{\frac{1}{r}-\frac{1}{R}\right\}-\frac{1}{2}\{v^2-V^2\}=\frac{1}{3}\mu(r^3-R^3)-\frac{\Pi}{\rho}$$

or
$$f'(t)\left\{\frac{1}{r}-\frac{1}{R}\right\}-\frac{1}{2}\{v^2-V^2\}=-\frac{1}{3}\mu c^3-\frac{\Pi}{\rho} \qquad ...(4)$$

{from condition III}

Again $\qquad r^2 v=R^2 V=f(t) \qquad$ from (1)

or
$$f'(t)=\frac{d}{dt}\,(r^2 v)$$
$$=\frac{dr}{dt}\cdot\frac{d}{dr}(r^2 v)=v.\frac{d}{dr}(r^2 v)$$

Substituting the value of $f'(t)$ in (4), we have
$$v\,\frac{d}{dr}\,(r^2 v)\left\{\frac{1}{r}-\frac{1}{R}\right\}-\frac{1}{2}\,v^2\left\{1-\frac{V^2}{v^2}\right\}=-\,\frac{1}{3}\mu c^3-\frac{\Pi}{\rho}$$

or
$$v\,\frac{d}{dr}(r^2 v)\left\{\frac{1}{r}-\frac{1}{R}\right\}-\frac{1}{2}\,v^2\left\{1-\frac{r^4}{R^4}\right\}=-\,\frac{1}{3}\mu c^3-\frac{\Pi}{\rho}$$
$\{$as $r^2 v=R^2 V$

Multiplying both the sides by $2r^2$, we have
$$2r^2 v\,\frac{d}{dr}\,(r^2 v)\left\{\frac{1}{r}-\frac{1}{R}\right\}-r^4 v^2\left\{\frac{1}{r^2}-\frac{r^2}{R^4}\right\}$$
$$=-\left(\frac{2\mu c^2}{3}+\frac{2\Pi}{\rho}\right)r^2$$

or
$$2r^2 v\,\frac{d}{dr}(r^2 v)\left\{\frac{1}{r}-\frac{1}{R}\right\}-r^4 v^2\left\{\frac{1}{r^2}-\frac{1}{R^2}\,\frac{dR}{dr}\right\}$$
$$=-\left(\frac{2\mu c^3}{3}+\frac{2\Pi}{\rho}\right)r^2$$

$\left\{\begin{array}{l}\text{Since } R^3-r^3=c^3 \text{ from (III)}\\ \text{or}\quad 3R^2\,dR-3r^2\,dr=0\\ \text{or}\quad r^2\,dr=R^2\,dR\end{array}\right.$

or
$$\left\{\frac{1}{r}-\frac{1}{R}\right\}\frac{d}{dr}(r^4 v^2)+r^4 v^2.\frac{d}{dr}\left(\frac{1}{r}-\frac{1}{R}\right)$$
$$=-\left(\frac{2\mu c^3}{3}+\frac{2\Pi}{\rho}\right)r^2$$

By integrating, we have
$$r^4 v^2\left(\frac{1}{r}-\frac{1}{R}\right)=-\left(\frac{2\mu c^3}{9}+\frac{2\Pi}{3\rho}\right)r^3+B$$

Initially when $r=a$, $v=0$ then
$$B=\left(\frac{2\mu c^3}{9}+\frac{2\Pi}{3\rho}\right)a^3$$

or
$$r^4 v^2\left(\frac{1}{r}-\frac{1}{R}\right)=\left(\frac{2\mu c^3}{9}+\frac{2\Pi}{3\rho}\right)(a^3-r^3)$$

or
$$r^4 v^2\left\{\frac{1}{r}-\frac{1}{(c^3+r^3)^{1/3}}\right\}=\left(\frac{2\mu c^3}{9}+\frac{2\Pi}{3\rho}\right)(a^3-r^3)$$

$\left\{\begin{array}{l}\text{as}\quad R^3-r^2=c^3\\ \text{or}\quad R=(c^3+r^3)^{1/3}\end{array}\right.$

or $\quad r^3 v^2 \{(c^3+r^3)^{1/3}-r\} = \left(\frac{2\mu c^3}{9}+\frac{2\Pi}{3\rho}\right)(a^3-r^3)(c^3+r^3)^{1/3}$

Velocity of the inner surface, when $r=x$,

Then $\quad \frac{dr}{dt}=v=\dot{x}$

or $\quad x^3 \dot{x}^2 \{(c^3+x^3)^{1/3}-x\} = \left(\frac{2\mu c^3}{9}+\frac{2\Pi}{3\rho}\right)(a^3-x^3)(c^3+x^3)^{1/3}$

Proved.

Ex. 12. *A mass of uniform liquid is in the form of a thick spherical shell bounded by concentric spheres of radii a and b (a < b). The cavity is filled with gas the pressure of which varies according to Boyle's Law, and is initially equal to the atmospheric pressure* Π, *and the mass of which may be neglected. The outer surface of the shell is exposed to atmospheric pressure. Prove that if the system is symmetrically disturbed, so that each particle moves along the line joining it to the centre the time of a small oscillation is*

$$2\pi a\left\{\rho.\frac{b-a}{3\Pi b}\right\}^{1/2}$$

where ρ *is the density of the liquid.*

Let v' be the velocity at a distance r' from the centre of the spherical shell at any time t and p be the pressure there. The equation of continuity is

$$\frac{1}{r'^2}\cdot\frac{d}{dr}(r'^2 v')=0$$

or $$r'^2 v'=f(t) \qquad \ldots(1)$$

The equation of motion is

$$\frac{\partial v'}{\partial t}+v'\frac{\partial v'}{\partial r'}=-\frac{1}{\rho}\frac{\partial p}{\partial r'} \qquad \ldots(2)$$

or $$\frac{f'(t)}{r'^2}+v'\frac{\partial v'}{\partial r'}=-\frac{1}{\rho}\frac{\partial p}{\partial r'}$$

Integrating with regard to r', we get

$$-\frac{f'(t)}{r'}+\frac{1}{2}v'^2=-\frac{p}{\rho}+A \qquad \ldots(3)$$

Let r and R be the internal and external radii of the shell of liquid and v and V be the velocities there at any time t.

Initially, I $\quad r'=R,\ v'=V,\ p=\Pi$

II $\quad r'=r,\ v'=v,\ pr^3=\Pi a^3$

{By Boyle's Law pv=constant

So equation (3) becomes with the help of I and II

$$-\frac{f'(t)}{R}+\frac{1}{2}V^2=-\frac{\Pi}{\rho}+A$$

$$-\frac{f'(t)}{r}+\frac{1}{2}v^2=-\frac{\Pi}{\rho}\cdot\frac{a^3}{r^3}+A$$

By Subtracting, we can eliminate the constant A.

$$f'(t)\left\{\frac{1}{r}-\frac{1}{R}\right\}+\frac{1}{2}(V^2-v^2)=-\frac{1}{\rho}\left(\Pi-\frac{\Pi a^3}{r^3}\right)$$

Neglecting the terms of V^2 and v^2, we get

$$f'(t)\left\{\frac{1}{r}-\frac{1}{R}\right\}=-\frac{1}{\rho}\left(\Pi-\frac{\Pi a^3}{r^3}\right)$$

or $$f'(t)=\frac{\Pi(a^3-r^3)}{\rho r^3}\cdot\frac{rR}{R-r}$$

or $$r^2\frac{dv}{dt}=\frac{\Pi(a^3-r^3)}{\rho r^2}\cdot\frac{R}{R-r}$$

or $$\frac{d}{dt}\left(\frac{dr}{dt}\right)=\frac{\Pi(a^3-r^3)}{\rho r^4}\cdot\frac{R}{R-r}$$

$\left\{\begin{array}{l}\text{Since } r^2 v=f(t)\\ \text{or } f'(t)=r^2\dfrac{dv}{dt}\\ \text{neglecting the}\\ \text{other term.}\end{array}\right.$

or $$\frac{d^2r}{dt^2}=\frac{\Pi R}{\rho}\cdot\frac{a^3-r^3}{r^4(R-r)} \qquad \ldots(4)$$

Giving a small oscillation, we have

$$r=a+x \text{ and } R=b+x_1$$

and $$\tfrac{4}{3}\pi R^3-\tfrac{4}{3}\pi r^3=\tfrac{4}{3}\pi b^3-\tfrac{4}{3}\pi a^3$$

or $$R^3-r^3=b^3-a^3$$

or $$(b+x_1)^3-(a+x)^3=b^3-a^3$$

or $$x_1=\frac{a^2}{b^2}x$$

(Neglecting higher order of x and x_1, being very small)

From (4), we have

$$\frac{d^2x}{dt^2}=\frac{\Pi}{\rho}\cdot\frac{b+x_1}{(a+x)^4}\cdot\frac{a^3-(a+x)^3}{(b+x_1)-(a+x)}$$

or $$\frac{d^2x}{dt^2}=\frac{\Pi}{\rho}\cdot\frac{b+\frac{a^2}{b^2}x}{(a^4+4a^3x)}\cdot\frac{a^3-(a^3+3a^2x)}{b-a-x\frac{a^2}{b^2}x} \text{ app.}$$

{Neglecting higher order of x, being very small}

or $$\frac{d^2x}{dt^2}=\frac{\Pi}{\rho}\cdot\frac{\left(b+\frac{a^2}{b^2}x\right)(-3a^2x)}{(a^4+4a^3x)\left(b-a-x+\frac{a^2}{b^2}x\right)}$$

or $$\frac{d^2x}{dt^2}=-\frac{\Pi b}{\rho a^2\,(b-a)}\,x \text{ (first approximation)}$$

Which is a standard equation of S. H. M

Hence the time period of a small oscillation

$$=2\pi\sqrt{\left\{\frac{\rho a^2\,(b-a)}{3\Pi b}\right\}}$$

$$=2\pi a\sqrt{\left\{\rho.\left(\frac{b-a}{3\Pi b}\right)\right\}}.$$ **Proved.**

Ex. 13. *A volume $\frac{4}{3}\pi c^3$ of gravitating liquid, of density ρ, is initially in the form of a spherical shell of infinitely great radius. If the liquid shell contract under the influence of its own attraction, there being no external or internal pressure, show that when the radius of the the inner spherical surface is x, its velocity will be given by*

$$V^2=\frac{4\pi\gamma\rho z}{15\,x^3}\,(2z^4+2z^3x+2z^2x^2-3zx^3-3x^4)$$

where V is the constant of gravitation, and $z^3=x^3+c^3$.

Let v' be the velocity at a distance r' from the centre of the spherical shell at any time t and p be the pressure there. The equation of continuty is

$$\frac{1}{r'^2}.\frac{d}{dr}(r'^2v')=0$$

or $$r'^2v'=f(t) \qquad \ldots\text{(i)}$$

Let r be the radius of the inner surface. The attraction at the point distance r' from the centre due to liquid is given by

$$=\frac{\frac{4}{3}\gamma\pi\rho\,(r'^3-r^3)}{r'^2}=\frac{4}{3}\gamma\pi\rho\left(r'-\frac{r^3}{r'^2}\right)$$

The equation of motion is

$$\frac{\partial v'}{\partial t}+v'\frac{\delta v'}{\partial r'}=-\frac{4}{3}\gamma\pi\rho\left(r'-\frac{r^3}{r'^2}\right)-\frac{1}{\rho}\frac{\partial p}{\partial r'}$$

or $$\frac{f'(t)}{r'^2}+v'\frac{\partial v'}{\partial r'}=-\frac{4}{3}\gamma\pi\rho\left(r'-\frac{r^3}{r'^2}\right)-\frac{1}{\rho}\frac{\partial p}{\partial r'} \qquad \{\text{ from (i)}$$

Integrating with regard to r', we get

$$-\frac{f'(t)}{r'}+\frac{1}{2}v'^2=-\frac{4}{3}\gamma\pi\rho\left(\frac{1}{2}r'^2+\frac{r^3}{r'}\right)-\frac{p}{\rho}+A \qquad \ldots\text{(ii)}$$

Since initially, when

I $r'=r$, $v'=v$, $p=0$ (Since there being no pressure on internal surface)

II $r'=z$, $v'=V$ (let), $p=0$ (there being no pressure on external surface)

Now we shall determine the constant A with the help of I and II condition.

$$-\frac{f'(t)}{r}+\frac{1}{2}v^2=-\frac{4}{3}\gamma\pi\rho\left(\frac{1}{2}r^2+r^2\right)+A$$

$$-\frac{f'(t)}{z}+\frac{1}{2}V^2=-\frac{4}{3}\gamma\pi\rho\left(\frac{1}{2}z^2+\frac{r^3}{z}\right)+A$$

By subtracting, we can eliminate the const. A

$$-f'(t)\left\{\frac{1}{r}-\frac{1}{z}\right\}+\frac{1}{2}\left\{v^2-V^2\right\}$$
$$=-\frac{4}{3}\gamma\pi\rho\left\{\frac{3r^2}{2}-\frac{z^2}{2}-\frac{r^3}{z}\right\} \qquad \text{...(iii)}$$

or
$$-f'(t)\left\{\frac{1}{r}-\frac{1}{z}\right\}=\frac{1}{2}\left\{\frac{f^2(t)}{r^4}-\frac{f^2(t)}{z^4}\right\}$$
$$=-\frac{4}{3}\gamma\pi\rho\left\{\frac{3r^2}{2}-\frac{z^2}{2}-\frac{r^3}{z}\right\}$$

$$\left\{\begin{array}{ll}\text{as} & r^2v=z^2V=f(t) \text{ from (i)}\\ \text{or} & 2r^2\,dr=2z^2\,dz=2f(t)\,dt\end{array}\right. \qquad (X)$$

Multiplying both the sides of (iii) by the relation X and integrating we have

$$f^2(t)\left\{\frac{1}{r}-\frac{1}{z}\right\}=\frac{4}{3}\gamma\pi\rho\int 3r^4dr-z^4dz-\frac{r^3}{z}\cdot 2z^2dz$$

or
$$f^2(t)\left\{\frac{1}{r}-\frac{1}{z}\right\}=\frac{4}{3}\gamma\pi\rho\left\{\frac{3}{5}r^5-\frac{1}{5}z^5\right.$$
$$\left.-2\int z(z^3-c^3)\,dz\right\}$$

or
$$f^2(t)\left\{\frac{1}{r}-\frac{1}{z}\right\}=\frac{4}{3}\gamma\pi\rho\left\{\frac{3}{5}r^5-\frac{1}{5}z^5-\frac{2}{5}z^5\right.$$
$$\left.+c^3z^2\right\} \quad \left(\text{as } r^3+c^3=z^3\right)$$

or
$$f^2(t)\left\{\frac{1}{r}-\frac{1}{z}\right\}=\frac{4}{15}\gamma\pi\rho\left\{3(r^5-z^5)+5c^3z^2\right\} \qquad \text{...(iv)}$$

If the radius of the inner spherical surface be x

i.e.
$$r=x$$

equation of continuity becomes $x^2v=f(t)$

then (iv) reduces to

or
$$x^4v^2\left\{\frac{1}{x}-\frac{1}{z}\right\}=\frac{4}{15}\gamma\pi\rho\left\{3(x^5-z^5)+5c^3z^2\right\}$$

or
$$v^2=\frac{4}{15}\frac{\pi\gamma\rho z}{x^4}\left\{\frac{3(x^5-z^5)+5c^3z^2}{z-x}\right\}$$

or $$v^2=\frac{4}{15}\cdot\frac{\pi\gamma\rho z}{x^3}\left\{-3\,(x^4+x^3z+x^2z^2+xz^3+z^4)+5z^2\,(z^2+zx+x^2)\right\}$$

or $$v^2=\frac{4\pi\gamma\rho z}{15x^3}\left\{2z^4+2z^3x+2z^2x^2-3zx^3-3x^4\right\}$$ **Proved.**

Ex. 14. *A mass of gravitating fluid is at rest under its own attraction only; the free surface being a spere of radius b and inner surface a rigid concentric shell of radius a. Shew that if this shell suddenly disappear, the initial pressure at any point of the fluid at distance r from the centre is*

$$\frac{2}{3}\pi\rho^2\,(b-r)\,(r-a)\left(\frac{a+b}{r}+1\right)$$

Let v' be the velocity at a distance r' from the centre of the spherical shell at any time t and p be the pressure there. The equation of continuity is

$$\frac{1}{r'^2}\frac{d}{dr}\left(r'^2v'\right)=0$$

or $$r'^2v'=f\,(t) \qquad \ldots\text{(i)}$$

Let r be the radius of the internal spherical cavity, then attraction at the point at a distance r' from the centre of the liquid itself.

$$=\frac{4}{3}\pi\gamma\rho\left(\frac{r'^3-r^3}{r'^2}\right)=\frac{4}{3}\pi\gamma\rho\left(r'-\frac{r^3}{r'^2}\right)$$

Hence the equation of motion is,

$$\frac{\partial v'}{\partial t}+v'\frac{\partial v'}{\partial r'}=-\frac{4}{3}\pi\gamma\rho\left(r'-\frac{r^3}{r'^2}\right)-\frac{1}{\rho}\frac{\partial p}{\partial r'}$$

or $$\frac{f'(t)}{r'^2}+v'\frac{\partial v'}{\partial r'}=-\frac{4}{3}\pi\gamma\rho\left(r'-\frac{r^3}{r'^2}\right)-\frac{1}{\rho}\frac{\partial p}{\partial r'}$$ {from (i)

Integrating with regard to r', we have

$$-\frac{f'(t)}{r'}+\frac{1}{2}v'^2=-\frac{4}{3}\pi\gamma\rho\left(\frac{r'^2}{2}+\frac{r^3}{r'}\right)-\frac{p}{\rho}+A \qquad \ldots\text{(ii)}$$

Now at an internal surface of the spherical cavity,

$$r'=r,\ v'=v,\ p=0$$

from (ii), we have

$$-\frac{f'(t)}{r}+\frac{1}{2}v^2=-\frac{4}{3}\pi\gamma\rho\left(\frac{r^2}{2}+r^2\right)+A$$

or $$A=-\frac{f'(t)}{r}+\frac{1}{2}v^2+\frac{4}{3}\pi\gamma\rho.\frac{3r^2}{2}$$

Substituting the value of the constant A in (ii), we get

$$-f'(t)\left\{\frac{1}{r'}-\frac{1}{r}\right\}+\frac{1}{2}(v'^2-v^2)=\frac{4}{3}\pi\gamma\rho\left(\frac{3r}{2}-\frac{r'^2}{2}-\frac{r^3}{r'}\right)-\frac{p}{\rho}$$

or
$$-f'(t)\left\{\frac{1}{r'}-\frac{1}{r}\right\}+\frac{1}{2}(v'^2-v^2)=-\frac{2}{3}\pi\gamma\rho(r'^2-r^2)$$
$$-\frac{4}{3}\pi\gamma\rho\, r^3\left(\frac{1}{r'}-\frac{1}{r}\right)-\frac{p}{\rho}. \quad \ldots\text{(iii)}$$

I. Initially when $t=0$, $r=a$, the velocity of every particle is zero *i.e.* $v=v'=0$;

Then from (iii), we have

$$-f'(0)\left\{\frac{1}{r'}-\frac{1}{a}\right\}=-\frac{2}{3}\pi\gamma\rho(r'^2-a^2)$$
$$-\frac{4}{3}\pi\gamma\rho\, a^3\left(\frac{1}{r'}-\frac{1}{a}\right)-\frac{p}{\rho} \quad \ldots\text{(iv)}$$

II. Again when $t=0$, $r'=b$, $p=0$

then from (iv), we have

$$-f'(0)\left\{\frac{1}{b}-\frac{1}{a}\right\}=-\frac{2}{3}\pi\gamma\rho(b^2-a^2)-\frac{4}{3}\pi\gamma\rho\, a^3\left(\frac{1}{b}-\frac{1}{a}\right)$$

or
$$-f'(0)=-\frac{2}{3}\pi\gamma\rho\,\frac{b^2-a^2}{\frac{1}{b}-\frac{1}{a}}-\frac{4}{3}\pi\gamma\rho\, a^3$$

or
$$-f'(0)=\frac{2}{3}\pi\gamma\rho\,.\,ab(a+b)-\frac{4}{3}\pi\gamma\rho\, a^3 \quad \ldots\text{(v)}$$

Eliminating $f'(0)$ between (iv) and (v), we have

$$\left\{\frac{2}{3}\pi\gamma\rho\,.\,ab(a+b)-\frac{4}{3}\pi\gamma\rho\, a^3\right\}\frac{a-r'}{ar'}=-\frac{2}{3}\pi\gamma\rho(r'^2-a^2)$$
$$-\frac{4}{3}\pi\gamma\rho\, a^3\left(\frac{1}{r'}-\frac{1}{a}\right)-\frac{p}{\rho}$$

or
$$p=\frac{4}{3}\pi\gamma\rho^2\left\{a^2-\frac{1}{2}ab(a+b)\right\}\frac{a-r'}{ar'}$$
$$-\frac{2}{3}\pi\gamma\rho^2(r'^2-a^2)-\frac{4}{3}\pi\gamma\rho^2 a^3\,\frac{a-r'}{ar'}$$

or
$$p=\frac{2\pi\rho^2}{3}(r'-a)\left[-\left\{\frac{2a^3-ab(a+b)}{ar'}\right\}-(r'+a)+2a^3\,\frac{1}{ar'}\right]$$

or
$$p=\frac{2\pi\rho^2}{3}(r'-a)\left[-\frac{2a^3}{ar'}+\frac{b(a+b)}{r'}-(r'+a)+\frac{2a^3}{ar'}\right]$$

or
$$p=\frac{2\pi\rho^2}{3}(r'-a)\left[\frac{b(a+b)}{r'}-(r'+a)\right].$$

$$p=\frac{2\pi\rho^2}{3}(r'-a)\left[\frac{b^2-r'^2}{r'}+a\left(\frac{b}{r'}-1\right)\right]$$

Now, let P be the pressure at a distance r from the centre, then

$$p=P,\ r'=r$$

or

$$P=\frac{2\pi\rho^2}{3}(r-a)(b-r)\left[\frac{b+r}{r}+\frac{a}{r}\right].$$

$$P=\frac{2\pi\rho^2}{3}(r-a)(b-r)\left\{\frac{a+b}{r}+1\right\}.$$

Proved.

Ex. 15. *Liquid is contained between two parallel planes ; the free surface is a circular cylinder of radius a whose axis is perpendicular to the planes. All the liquid within a concentric circular cylinder of radius b is suddenly annihilated ; Prove that if Π be the pressure at the outer surface, the initial pressure at any point of the liquid distant r from the centre is*

$$\Pi\,\frac{\log r-\log b}{\log a-\log b}.$$

In the incompressible fluid, the fluid velocity will be radial out side the cylinder $|z'|=b$. It will be a function of r (radial distance from the centre of the cylinder *i.e.* origin $r<a$) and the time t only. The equation of continuity reduces to

$$\frac{1}{r}\cdot\frac{\partial}{\partial r}(r\,q)=0$$

or

$$r\,q=f(t)\quad \text{(let).}$$

Let v' be the velocity at a distance r' and p be the pressure there at any time t, then equation of continuity is, {from (i)

$$r'\,v'=f(t) \qquad \ldots\text{(i)}$$

and the equation of motion is

$$\frac{\partial v'}{\partial t}+v'\frac{\partial v'}{\partial r'}=-\frac{1}{\rho}\frac{\partial p}{\partial r'}$$

or

$$\frac{f'(t)}{r'}+v'\frac{\partial v'}{\partial r'}=-\frac{1}{\rho}\frac{\partial p}{\partial r'}$$

$$\left\{\begin{array}{l}\text{as}\quad r'\,v'=f(t)\\ \text{then } f'(t)=r'\dfrac{\partial v'}{\partial t}\end{array}\right.$$

Integrating with regard to r', we have

$$f'(t)\log r'+\frac{1}{2}v'^2=-\frac{p}{\rho}+A \qquad \ldots\text{(ii)}$$

Initially when $t=0$, $v'=0$, $r'=r$

then from (ii), we have

$$f'(0)\log r=-\frac{p}{\rho}+A. \qquad \ldots\text{(iii)}$$

Now I $\quad p=\Pi$ when $r=a$ (given)

II $\quad p=0$ when $r=b$.

By using the conditions I and II equations (iii) reduces to

$$f'(0)\log a=-\frac{\Pi}{\rho}+A \qquad \text{...(iv)}$$

and

$$f'(0)\log b=-A. \qquad \text{...(v)}$$

From (iii) and (v), we have

$$f'(0)\{\log r-\log b\}=-\frac{p}{\rho}\,.$$

Also from (iv) and (v), we have

$$f'(0)\{\log a-\log b\}=-\frac{\Pi}{\rho}$$

By dividing, we get

$$\frac{\log r-\log b}{\log a-\log b}=\frac{p}{\Pi}\,.$$

So the initial pressure at any point of the liquid at a distance r from the centre is,

or
$$p=\Pi\,\frac{\log r-\log b}{\log a-\log b}\,. \qquad \textbf{Proved.}$$

Ex. 16. *A mass of liquid of density ρ whose external surface is a long circular cylinder of radius a, which is subject to a constant pressure Π, surrounds a coaxial long circular cylinder of radius b. The internal cylinder is suddenly destroyed. Shew that if V is the velocity at the internal surface when the radius is r, then*

$$V^2=\frac{2\Pi\,(b^2-r^2)}{\rho r^2\,\log\left(\dfrac{r^2+a^2-b^2}{r^2}\right)}\,.$$

When the internal cylinder is destroyed, the motion of the liquid will be along the radii of the normal sections of the cylinder. The velocity will be a function of r and the time t only. Let v' be the velocity and p the pressure at any point distance r' from the axis of the cylinder. The Equation of continuity is

$$r'\,v'=f(t). \qquad \text{...(i)}$$

The equation of motion is

$$\frac{\partial v'}{\partial t}+v'\frac{\partial v'}{\partial r'}=-\frac{1}{\rho}\,\frac{\partial p}{\partial r'}$$

or $$\frac{f(t)}{r'}+v'\frac{\partial v'}{\partial r'}=-\frac{1}{\rho}\frac{\partial p}{\partial r'}$$ {from (i)}

Integrating with regard to r', we have

$$f'(t)\log r'+\tfrac{1}{2}v'^2=-\frac{p}{\rho}+A. \qquad \ldots\text{(ii)}$$

Let r and R be the radii of the internal and external surface of the cylinder and v and V, the velocities there at any time t.

When I $\quad r'=r,\ v'=v,\ p=0$

II $\quad r'=R,\ v'=V,\ p=\Pi$

then $$f'(t)\log r+\tfrac{1}{2}v^2=A$$

$$f'(t)\log R+\tfrac{1}{2}V^2=-\frac{\Pi}{\rho}+A.$$

To eliminate the constant A, by subtracting we have

$$f'(t)\{\log r-\log R\}+\tfrac{1}{2}(v^2-V^2)=\frac{\Pi}{\rho} \qquad \ldots\text{(iii)}$$

$$\left[\begin{array}{ll} \text{Again } rv=RV=f(t) & \text{from (i)} \\ \quad 2r\,dr=2R\,dR=2f(t)\,dt\ldots(X) & \\ \text{and} \quad R^2-r^2=a^2-b^2. & \end{array}\right.$$

Multiplying both the sides of (iii) by the relation (X), we have

$$2f(t)f'(t)\{\log r-\log R\}+\tfrac{1}{2}f^2(t)\left\{\frac{2r\,dr}{r^2}-\frac{2R\,dR}{R^2}\right\}$$
$$=\frac{2\Pi}{\rho}.\ r\,dr$$

or $$2f(t)f'(t)\{\log r-\log R\}+f^2(t)\left\{\frac{dr}{r}-\frac{dR}{R}\right\}=\frac{2\Pi}{\rho}.\ r\,dr$$

By integrating, we have

$$f^2(t)\{\log r-\log R\}=\frac{\Pi}{\rho}r^2+B. \qquad \ldots\text{(iv)}$$

When $\quad r=b,\quad v=0\quad$ *i.e.* $f(t)=0$,

then $$\frac{\Pi}{\rho}b^2+B=0 \quad \text{or} \quad B=-\frac{\Pi}{\rho}b^2.$$

Substituting the value of B in (iv), we have

$$f^2(t)\{\log r-\log R\}=\frac{\Pi}{\rho}(r^2-b^2)$$

or $$f^2(t)=\frac{\Pi}{\rho}\cdot\frac{r^2-b^2}{\log r-\log R} \qquad \left\{\begin{array}{ll}\text{as} & R^2-r^2=a^2-b^2\\ \text{and} & r\,v=f(t)\end{array}\right.$$

or $$v^2r^2=\frac{\Pi}{\rho}\ \frac{r^2-b^2}{\tfrac{1}{2}\log\left(\dfrac{r^2}{R^2}\right)}$$

or
$$v^2=\frac{2\Pi\,(r^2-b^2)}{\rho\, r^2 \log\left(\dfrac{r^2}{r^2+a^2-b^2}\right)}$$

or
$$v^2=\frac{2\Pi\,(b^2-r^2)}{\rho\, r^2 \log\left(\dfrac{r^2+a^2-b^2}{r^2}\right)}.$$

Proved.

Exercise

1. A pulse travelling along a fine straight uniform tube filled with gas causes the density at time t and distance x from the origin where the velocity is u_0 to become $\rho_0\ \phi\,(vt-x)$. Prove that the velocity u (at time t and distance x from the origin) is given by

$$v+\frac{(u_0-v)\,\phi\,(vt)}{\phi\,(vt-x)}$$

Hint. Let u be the velocity and ρ the density of the gas at a distance x, then

$$\rho=\rho_0\,\phi\,(vt-x) \qquad \ldots(1)$$

Equation to continuity is

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)=0.$$

or
$$\frac{\partial \rho}{\partial t}+\rho\frac{\partial u}{\partial x}+u\frac{\partial \rho}{\partial x}=0 \qquad \ldots(2)$$

Differentiating (1) w. r. to t and x respectively

$$\frac{\partial \rho}{\partial t}=\rho_0\, v\phi'\,(vt-x)$$

$$\frac{\partial \rho}{\partial x}=-\rho_0\,\phi'\,(vt-x)$$

Substituting the value of $\dfrac{\partial \rho}{\partial t}$ and $\dfrac{\partial \rho}{\partial x}$ in (2) and integrate,

$$\frac{du}{v-u}+\frac{\phi'\,(vt-x)}{\phi\,(vt-x)}dx=0$$

or
$$(v-u)\,\phi\,(vt-x)=\text{const.}$$

The constant can be determined by the conditions

$$x=0,\ u=u_0 \quad \text{(Given)}$$

2. A spherical hollow of radius a initially exists in an infinite fluid subject to constant pressure at infinity. Shew that the pressure at distance r from the centre when the radius of the cavity is x to pressure at infinity as

$$\{ 3x^2r^4+(a^3-4x^3)\ r^3-(a^3-x^3)\ x^3\} : 3x^2r^4.$$

Hint. Ref. Q. No. 8 Page.

$$v^2 x^2 = \frac{2\Pi}{3\rho}(a^3 - x^3)$$

(which gives the velocity of the inner surface of radius x which is hollow).

Substituting the value of v^2 in equation (iii), we have

$$F'(t) = x\left\{\tfrac{1}{2}v^2 - \frac{\Pi}{\rho}\right\}$$

or $$F'(t) = \frac{\Pi}{3\rho} \cdot \frac{a^3 - 4x^3}{x^2}$$

Put the value of $F'(t)$ in equation (ii), we have for the motion of any point in the fluid at a distance r',

or $$\frac{\Pi - p}{\rho} = -\frac{1}{r'}\left[\frac{\Pi}{3\rho} \cdot \frac{a^3 - 4x^3}{x^2}\right] + \frac{1}{2}v^2 \frac{x^4}{r'^4} \qquad \{\text{as } r'^2 v' = x^2 v$$

or $$\frac{p}{\rho} = \frac{\Pi}{\rho} + \frac{\Pi}{3\rho} \cdot \frac{a^3 - 4x^3}{x^2 r} - \frac{\Pi}{3\rho} \cdot \frac{x(a^3 - x^3)}{r^4} \qquad \{\text{at } r' = r$$

or $$\frac{p}{\Pi} = \frac{3x^2 r^4 + (a^3 - 4x^3) r^3 - (a^3 - x^3) x^3}{3x^2 r^4}$$

3. A sphere is at rest in an infinite mass of homogeous liquid of density ρ, the pressure at infinity being p. If the radius R of the sphere varies in such a way that $R = a + b \cos nt$ where $b < a$, shew that pressure at the surface of the sphere at any time is

$$p + \frac{b^2 n \rho}{4}(b - 4a \cos nt - 5b \cos 2nt)$$

Hint. Let v be the velocity at a distance r from the centre and p be the pressure at any time t. The equation to continuity is

$$r^2 v = f(t) \qquad \ldots(1)$$

Equation of motion is

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \qquad \ldots(2)$$

from (1) and (2), we have

$$\frac{f'(t)}{r^2} + v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

Integrating w. r. to r, we get

$$-\frac{f'(t)}{r} + \frac{1}{2}v^2 = A - \frac{p}{\rho}$$

when $r = \infty$, $p = \Pi$, $v = 0$, then $A = \dfrac{\Pi}{\rho}$

or $$-\frac{f'(t)}{r}+\frac{1}{2}v^2=\frac{\Pi-p}{\rho} \qquad ...(3)$$

Let R be the radius of the sphere and V be the velocity at any time t

or $$r^2v=R^2V=f(t)$$

So $v=\dfrac{R^2V}{r^2}$ and $f'(t)=\dfrac{d}{dt}(R^2V)=2RV^2+R^2\dfrac{dV}{dt}$

from (3), we have

$$\frac{\Pi-p}{\rho}=-\frac{1}{r}\left(2RV^2+R^2\frac{dV}{dt}\right)+\frac{1}{2}\frac{R^4r^2}{r^4}$$
$$=\frac{R}{r}\left\{-2V^2-R\frac{dV}{dt}+\frac{1}{2}\frac{R^3V^2}{r^3}\right\}$$

$$\left\{\begin{aligned} R&=b+a\cos nt\\ V&\equiv\frac{dR}{dt}=-an\sin nt\\ \frac{dV}{dt}&=-an^2\cos nt\end{aligned}\right.$$

Substituting the values of R, V and $\dfrac{dV}{dt}$ in the above expression. We can have the required result.

4. A mass of homogeneous liquid is moving so that the velocity at any point is proportional to the time, and that the pressure is given by

$$\frac{p}{\rho}=\mu xyz-\tfrac{1}{2}t^2(y^2z^2+z^2x^2+x^2y^2).$$

Prove that this motion may have been generated from rest by finite natural forces independent of the time; and shew that, if the direction of motion at every point coincide with the direction of the acting force, each particle of the liquid describes a curve which is the intersection of two hyperbolic cylinders.

Hint. consider q be the velocity at any instant t, then

$$q=\lambda t \qquad ...(1)$$

Also $$\frac{p}{\rho}=\mu xyz-\tfrac{1}{2}t^2(y^2z^2+z^2x^2+x^2y^2) \qquad ...(2)$$

Let the motion be generated from finite natural forces (conservative force), then

$$\frac{p}{\rho}-\frac{\partial\phi}{\partial t}+\tfrac{1}{2}q^2+V=f(t)$$

or $$\frac{p}{\rho}=\frac{\partial\phi}{\partial t}-\tfrac{1}{2}\lambda^2t^2-V+f(t) \qquad ...3)$$

By comparing (2) and (3), we have

$$\lambda^2=y^2z^2+z^2x^2+x^2y^2 \qquad ...(4)$$

So $$q^2=\lambda^2t^2=t^2(y^2z^2+z^2x^2+x^2y^2) \qquad \{\text{from (1) and (4)}$$

Also $q^2=\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2$

Consider $\phi=t\,xyz$

Thus the equation (3) reduces to

$$\frac{p}{\rho}=xyz-\tfrac{1}{2}t^2\,(y^2z^2+z^2x^2+x^2y^2)-V+f\,(t) \qquad \ldots(5)$$

from (2) and (5), we have

$$f\,(t)=0 \text{ and } xyz-V=\mu.xyz$$
$$V=xyz\,(1-\mu)$$

Let u, v, w are the component velocities and X, Y, Z be the component forces,

then $\qquad u=-\frac{\partial\phi}{\partial x}=-t\,yz$ etc.

and $\qquad X=-\frac{\partial V}{\partial x}=(\mu-1)\,yz$ etc.

Since the direction of motion conicides with the direction of acting force, so

$$\frac{u}{X}=\frac{v}{Y}=\frac{w}{Z}$$

Thus the equation to the path is given by

or $\qquad \frac{dx}{X}=\frac{dy}{Y}=\frac{dz}{Z}$

or $\qquad \frac{dx}{yz}=\frac{dy}{zx}=\frac{dz}{yx}$ $\qquad$ which gives $x^2-y^2=\text{const.}$ and $x^2-z^2=\text{const.}$

Hence each particle of the liquid will lie on a curve which is the intersection of two hyperbolic cylinders.

5. A sphere whose radius at time t is $b+a\cos nt$ is surrounded by liquid extending to infinity under no forces. Prove that the pressure at distance r from the centre is less than the pressure at an infinite distance by

$$\rho\,\frac{n^2a}{r}\,(b+a)\cos nt)$$

$$\left\{a\,(1-3\sin^2 nt)+b\cos nt+\frac{1}{2}\cdot\frac{a}{r^3}\sin^2 nt\,(b+a\cos nt)^3\right\}$$

6. A mass of perfect incompressible fluid, of density ρ, is bounded by concentric surfaces. The outer surface is contained by a flexible envelope which exerts continuously a uniform pressure Π and contracts from radius R_1 to Radius R_2. The hollow is filled with a gas obeying Boyle's Law, its radius contracts from C_1 to C_2,

and the pressure of the gas in initially p_1. Initially the whole mass is at rest. Prove that, neglecting the mass of the gas, the velocity v of the inner surface when the configuration $(R_2\ C_2)$ is reached is given by

$$\frac{1}{2}v^2=\frac{C_1^3}{C_2^3}\left\{\frac{1}{3}\left(1-\frac{C_2^3}{C_1^3}\right)\frac{\Pi}{\rho}-\frac{p_1}{p}\log\frac{C_1}{C_2}\right\}\Big/\left(1-\frac{C_2}{R_2}\right).$$

Let p be the pressure and v' be the velocity at a distance r' at any instant. The equation to continuity is

$$r'^2v'=f(t). \qquad \ldots(1)$$

The equation of motion is

$$\frac{\partial v'}{\partial t}+v'\frac{\partial v'}{\partial r'}=-\frac{1}{\rho}\frac{\partial p}{\partial r'} \qquad \ldots(2)$$

Eliminating $\frac{\partial v'}{\partial t}$ from (1) and (2) and integrating, we have

$$-\frac{f'(t)}{r'}+\frac{1}{2}v'^2=-\frac{p}{\rho}+D \qquad \ldots(3)$$

Let v be the velocity and r be the radius at the inner surface at any time t. Consider p_0 be the pressure, then

$$r'=r,\ v'=v \text{ and } r^3p=C_1^3p_1$$

(Since the gas obeys Boyle's Law)

from (3), we have

$$-\frac{f'(t)}{r}+\frac{1}{2}v^2=D-\frac{C_1^3p_1}{\rho r^3} \qquad \ldots(4)$$

Let R is the radius of the outer surface

then $\qquad r'=R,\ v'=V,\ p=\Pi$

Again from (3), we have

$$-\frac{f'(t)}{R}+\frac{1}{2}V^2=D-\frac{\Pi}{\rho} \qquad \ldots(5)$$

To eliminate D, by subtracting (4) and (5), we have

$$-f'(t)\left\{\frac{1}{r}-\frac{1}{R}\right\}+\frac{1}{2}(v^2-V^2)=\frac{\Pi}{\rho}-\frac{C_1^3p_1}{\rho}\cdot\frac{1}{r^3}$$

By integrating, we get

$$-f^2(t)\left\{\frac{1}{r}-\frac{1}{R}\right\}=\frac{2\Pi}{3\rho}r^3-\frac{2C_1^3p_1}{\rho}\log r+B$$

Since $\qquad r=C_1,\ v=0\ i.e\ f(t)=0,$

then $\qquad v^2r^4\left(\frac{1}{r}-\frac{1}{R}\right)=\frac{2\Pi}{3\rho}(C_1^3-r^3)-\frac{2C_1^3p_1}{\rho}\log\frac{C_1}{r}$

Now for the configuration $(R_2\ C_2)$ *i.e.* $R=R_2$, $r=C_2$, the velocity can be determined by the above relation

$$v^2 C_2^3\left(1-\frac{C_2}{R_2}\right)=\frac{2\Pi}{3\rho}\,C_1^3\left(1-\frac{C_2^3}{C_1^3}\right)-\frac{2C_1^3 p_1}{\rho}\log\left(\frac{C_1}{C_2}\right).$$

7. A homogeneous liquid is contained between two concentric spherical surfaces, the radius of the inner being a and that of the outer indefinitely great. The fluid is attracted to the centre of these surfaces by a force $\phi(r)$, and a constant pressure Π is exerted at the outer surface. Suppose $\int\phi(r)\,dr=\chi(r)$, and that $\chi(r)$ vanishes when r is infinite. Shew that if the inner surface is suddenly removed, the pressure at the distance r is suddenly diminished by

$$\Pi\,\frac{a}{r}-\frac{a\rho}{r}\,\chi(a).$$

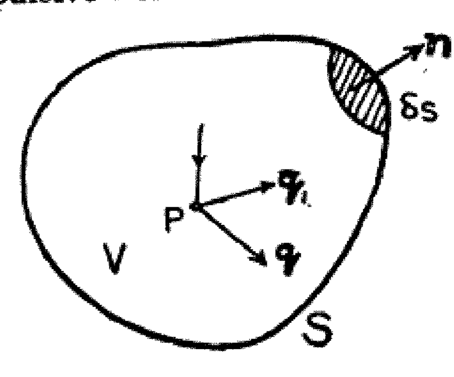
Find $\phi(r)$ so that the pressure immediately after the inner surface is removed may be the same as it would be if no attractive force existed.

Also with this value of $\phi(r)$, find the velocity of the inner boundary of the fluid at any period of the motion.

8. A sphere of radius a is alone in an unbounded liquid which is at rest at a great distance from the sphere and is subject to no external forces. The sphere is forced to vibrate radialy keeping its spherical shape, the radius r at any time being $r=a+b\cos nt$. Shew that if Π is the pressure in the liquid at a great distance from the sphere, the least pressure (assumed positive) at the surface of the sphere during the motion is

$$\Pi-n^2\rho b\,(a+b).$$

§ 26 Equations of motion under Impulsive Force.



Let the sudden changes in velocity are produced at the boundaries of an inviscid fluid or consider that the impulsive forces are applied to its interior than the disturbances produced are instantaneously propagated to every part of the fluid.

Consider a closed surface S moving with the fluid and enclosing a volume V. Let ϖ be the impulsive pressure and **I** the external impulsive body force per unit mass of fluid. The fluid particle is moving initially with velocity **q**, the impulse changes the velocity (of the particle) instantaneously from **q** to $\mathbf{q}_1$ or we can say that **q** and $\mathbf{q}_1$ be the velocities just before and just after the impulsive action.

Now change in momentum is

$$=\int_V \rho\,(\mathbf{q}_1-\mathbf{q})\,dV \qquad \ldots(1)$$

The total impulse applied to the fluid particle is

$$=\int_V \mathbf{I}\rho\,\;dV+\int_S (-\mathbf{n})\,\varpi\,dS \quad \left\{\begin{array}{l}\text{negative sign as } \mathbf{n} \text{ is unit}\\ \text{outward drawn normal.}\end{array}\right.$$

$$=\int_V \mathbf{I}\rho\,\;dV-\int_S \mathbf{n}\varpi\,dS$$

$$=\int_T \mathbf{I}\rho\,\;dV-\int_V \nabla\varpi\,dV \qquad \{\text{By Gauss Theorem}$$

$$=\int_V (\mathbf{I}\rho-\nabla\varpi)\,dV \qquad \ldots(2)$$

We know by Newton's Second Law for impulsive motion to the fluid with in the closed surface S that

Impulsive force=Change in Momentum

or $$\int_V \rho\,(\mathbf{q}_1-\mathbf{q})\,dV=\int_V (\mathbf{I}\rho-\nabla\varpi)\,dV \quad \{\text{ from (1) and (2)}$$

Since $\mathbf{V}$ is an arbitarily small volume, then

$$\rho\,(\mathbf{q}_1-\mathbf{q})=\mathbf{I}\rho-\nabla\varpi$$

or $$\mathbf{q}_1-\mathbf{q}=\mathbf{I}-\frac{1}{\rho}\,\nabla\varpi \qquad \ldots(3)$$

Case I. When there are no external impulsive body force but only impulsive pressure occurs *i. e.* $\mathbf{I}=0$

$$\mathbf{q}_1-\mathbf{q}=-\frac{1}{\rho}\;\nabla\varpi \qquad \ldots(4)$$

Let the motion be irrotational

then $\mathbf{q}=-\nabla\phi$ and $\mathbf{q}_1=-\nabla\phi'$ (where ϕ is the velocity potential just before the impulsive action.)

So (4) becomes

or $$-\nabla\phi'+\nabla\phi=-\frac{1}{\rho}\nabla\varpi$$

or $$\nabla\,(\phi'-\phi)=\frac{1}{\rho}\,\nabla\varpi$$

or $$\varpi=\rho\,(\phi'-\phi).$$

The constants may be omitted, as an extra pressure remains the same throughout the fluid which will not affect the motion.

Case II. If $\mathbf{q}_1=0$ implies that the motion is started from rest by applying the impulsive pressure at the boundries, then from (4), we have

$$\mathbf{q}=-\frac{1}{\rho}\nabla\varpi$$

or
$$\mathbf{q}=-\nabla\left(\frac{\varpi}{\rho}\right)$$

Since $\exists$ a velocity potential, then

$$\nabla\phi=-\nabla\left(\frac{\varpi}{\rho}\right)$$

or
$$\phi=-\frac{\varpi}{\rho}$$

or
$$\varpi=-\rho\phi+\text{Const.}$$

The constant may be omitted, as an extra pressure is constant throughout the fluid, produces no affect on the motion.

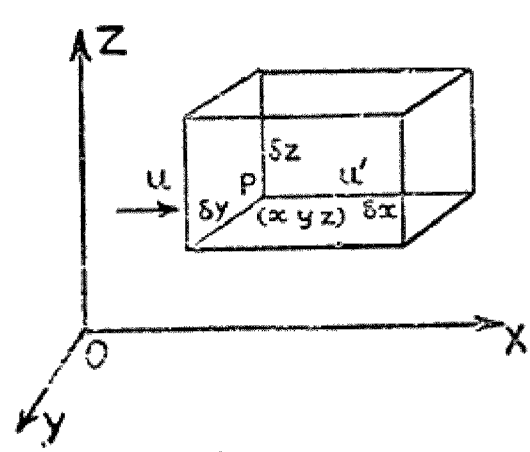
Case III. If the density ρ is const. $\mathbf{I}=0$. Taking the divergence of both the sides of (4), we have

$$\nabla^2\varpi=0. \qquad \left\{\begin{array}{c}\nabla\mathbf{q}_1=0=\nabla\mathbf{q}\\ \text{By equation of continuity.}\end{array}\right.$$

which is a **Laplace Equation.**

§ 2·7. Alternative Method : Equation of motion under impulsive forces.

Consider $P\ (x\ y\ z)$ be a point in the fluid and X', Y', Z' be the components of extraneous impulses parallel to the coordinate axes. Let ϖ be the impulsive pressure at P.



Consider an elementary parallelopiped. Pressure on faces parallel to $-YZ$ plane is given by

$$=\varpi\ \delta y\ \delta z$$

and to the opposite face is

$$=\left(\varpi+\frac{\partial\varpi}{\partial x}\ \delta x\right)\delta y\ \delta z.$$

Let $(u\ v\ w)$ and $(u'\ v'\ w')$ be the components of the velocity of P in the fluid before and after the impulse.

Change in the momentum $=\rho\ (u'-u)\ \delta x\ \delta y\ \delta z$

(in the direction of X-axis)

By Newton's Second Law for impulsive motion, we have

Change in Momentum = Impulsive force

$$\rho\,(u'-u)\,\delta x\,\delta y\,\delta z=\rho X'\,\delta x\,\delta y\,\delta z+\left\{\varpi\,\delta y\,\delta z-\left(\varpi+\frac{\partial\varpi}{\partial x}\,\delta x\right)\delta y\,\delta z\right\}$$

or $\quad \rho\,(u'-u)=\rho X'-\dfrac{\partial\varpi}{\partial x}$ (along the direction of X-axis)

Similarly $\quad \rho\,(v'-v)=\rho Y'-\dfrac{\partial\varpi}{\partial y}$ (along the direction of Y-axis)

and $\quad \rho\,(w'-w)=\rho Z'-\dfrac{\partial\varpi}{\partial z}$ (along the direction of Z-axis) ...(1)

Let there are no impulsive forces acting on the mass. Then $X'=0$, $Y'=0$ and $Z'=0$, So (1) reduces to

$$\rho\,(u'-u)=-\frac{\partial\varpi}{\partial x} \quad \text{...(1)}$$

$$\rho\,(v'-v)=-\frac{\partial\varpi}{\partial y} \quad \text{...(2)}$$

$$\rho\,(w'-w)=-\frac{\partial\varpi}{\partial z} \quad \text{...(3)}$$

We know that

$$d\varpi=\frac{\partial\varpi}{\partial x}.dx+\frac{\partial\varpi}{\partial y}.dy+\frac{\partial\varpi}{\partial z}.dz$$

or $\quad d\varpi=-\rho\,(u'-u)\,dx-\rho\,(v'-v)\,dy-\rho\,(w'-w)\,dz$ ...(4)

Let ϕ and ϕ' be the velocity potential just before and just after the impulsive action.

Then $d\phi=-(udx+vdy+wdz)$

and $\quad d\phi'=-(u'dx+v'dy+w'dz)$.

From (4), we have

$$d\varpi=\rho\,(d\phi'-d\phi)$$

By integrating, we have

$$\varpi=\rho\,(\phi'-\phi)+\text{Const.}$$

{ρ = Const. being an incompressible fluid

The constant may be neglected as an extra pressure, which is constant throughout, will not affect the motion.

Cor. 1. Let ρ is constant, differentiating (2), (3) and (4) with regard to dx, dy, dz respectively and adding, we have

$$\rho\left(\frac{\partial u'}{\partial x}+\frac{\partial v'}{\partial y}+\frac{\partial w'}{\partial z}\right)-\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)$$
$$=-\left(\frac{\partial^2\varpi}{\partial x^2}+\frac{\partial^2\varpi}{\partial y^2}+\frac{\partial^2\varpi}{\partial z^2}\right)$$

or $\quad \dfrac{\partial^2\varpi}{\partial x^2}+\dfrac{\partial^2\varpi}{\partial y^2}+\dfrac{\partial^2\varpi}{\partial z^2}=0$ $\quad\left[\begin{array}{l}\text{as } \dfrac{\partial u'}{\partial x}+\dfrac{\partial v'}{\partial y}+\dfrac{\partial w'}{\partial z}=0 \\ \text{and } \dfrac{\partial u}{\partial x}+\dfrac{\partial v}{\partial y}+\dfrac{\partial w}{\partial z}=0 \\ \text{Equation of continuity.}\end{array}\right.$

which is known as **Laplace Equation**.

§ 2·8. **Principle of Energy.**

The principle of energy enunciates that the change in energy $(K.E+P.E)$ is equal to work done by the extraneous forces. The potential due to the external forces is supposed to be independent of time.

§ 2·9. **Equation of Energy.**

We know that Euler's Equation of motion is

$$\frac{\partial \mathbf{q}}{\partial t}=\mathbf{F}-\frac{1}{\rho}\nabla p \qquad \{\text{Ref § 2·0}$$

or $$\rho\frac{\partial \mathbf{q}}{\partial t}=\rho\mathbf{F}-\nabla p \qquad ...(1)$$

Let the extraneous forces are conservative,

i.e. $\mathbf{F}=-\nabla\Omega$. ...(2)

[since potential is independent of time hence $\dfrac{d\Omega}{dt}=0$

From (1) and (2), we have

$$\rho\frac{\partial \mathbf{q}}{\partial t}=-\rho(\nabla\Omega)-\nabla p$$

Multiplying the above equation scalarly by $\mathbf{q}$, we have

or $$\rho\mathbf{q}\frac{\partial \mathbf{q}}{\partial t}=-\rho\mathbf{q}.(\nabla\Omega)-\mathbf{q}.\nabla p$$

or $$\rho.\frac{1}{2}\frac{d}{dt}(\mathbf{q}^2)+\rho\mathbf{q}.(\nabla\Omega)=-\mathbf{q}.\nabla p \quad \left\{\because \frac{d\Omega}{dt}=\frac{\partial\Omega}{\partial t}+(\mathbf{q}\,\nabla)\Omega = (\mathbf{q}.\nabla)\right.$$

or $$\rho.\frac{d}{dt}(\tfrac{1}{2}\mathbf{q}^2+\Omega)=-\mathbf{q}.\nabla p$$

By integrating over V, we have

$$\int_V \rho\frac{d}{dt}(\tfrac{1}{2}\mathbf{q}^2+\Omega)\,dv=-\int_V(\mathbf{q}.\nabla p)\,dv$$

or $$\frac{d}{dt}\left[\int_V \tfrac{1}{2}\rho q^2\,dv+\int_V \rho\Omega dv\right]=-\int_V(\mathbf{q}.\nabla p)\,dv$$

or $\quad \dfrac{d}{dt}(T+W)=-\displaystyle\int_V \nabla \cdot (\mathbf{q}p)\, dv+\int_V p\,(\nabla \cdot q)\, dv$

$$\left\{\begin{array}{l}\text{Since } \dfrac{d}{dt}(\rho dv)=0 \text{ equation of continuity} \\ \text{and } \nabla \cdot (p\mathbf{q}))=p\nabla \cdot \mathbf{q}+\mathbf{q} \cdot \nabla p\end{array}\right.$$

$$\frac{d}{dt}(T+W) = \int_S p\mathbf{q}.\mathbf{n}dS+\int_V p\nabla \cdot \mathbf{q}\, dv$$

$$\frac{d}{dt}(T+W) = \int_S p\mathbf{q}.\,\mathbf{n}\, dS-\frac{dI}{dt}$$

$$\left\{\begin{array}{l}\text{where } I=\displaystyle\int_V E\rho\, dv \\ E \text{ be the intrinsic energy per unit mass}\end{array}\right.$$

where T, W and I are the $K.E$, $P.E$ and intrinsic energy respectively

$$T=\int_V \tfrac{1}{2}\rho\mathbf{q}^2\, dv,\quad W=\int_V \rho\Omega d\Omega$$

and $\qquad I=\displaystyle\int_V E\,\rho\, dv$

or $\qquad \dfrac{d}{dt}\,(T+W+I)=\displaystyle\int_S p\mathbf{q}.\mathbf{n}\, dS$

which is known as **Energy Equation.**

Thus the rate of change of total energy (i. e. kinetic, potential and intrinsic energies) of a portion of perfect fluid is equal to the rate at which the work is done by the pressure forces on the boundary.

Ex. 1. *A portion of homogeneous fluid is confined between two concentric spheres of radii A and a, and is attracted towards their centre by a force varying inversely as a square of the distance. The inner spherical surface is suddenly annihilated, and when the radii of the inner and outer surface of the fluid are r and R, the fluid impinges on a solid ball concentric with their surfaces ; Prove that the impulsive pressure at any point of the ball for different values of R and r varies as*

$$\sqrt{\left\{(a^2-r^2-A^2+R^2)\left[\frac{1}{r}-\frac{1}{R}\right]\right\}}$$

In the incompressible fluid, the fluid velocity will be radial out side the inner spherical surface, it will be a function, of r' (radial distance from the centre of the spherical surface, *i. e.* origin) and the time t only. The equation of continuity is

$$\frac{1}{r'}\,\frac{d}{dr'}\,(r'^2\, v')=0$$

or $$r'^2 v' = f(t) \text{ (constant)} \qquad \ldots(i)$$

(where v' is the velocity at a distance r' from the centre)

and the equation of motion is

$$\frac{\partial v'}{\partial t} + v' \frac{\partial v'}{\partial r'} = -\frac{\mu}{r'^2} - \frac{1}{\rho}\frac{\partial p}{\partial r'} \qquad \ldots(ii)$$

or $$\frac{f'(t)}{r'^2} + v' \frac{\partial v'}{\partial r'} = -\frac{\mu}{r'^2} - \frac{1}{\rho}\frac{\partial p}{\partial r'} \qquad \{\text{from (i)}$$

By integrating with regard to r', we have

$$-\frac{f'(t)}{r'} + \tfrac{1}{2}v'^2 = \frac{\mu}{r'} - \frac{p}{\rho} + A \qquad \ldots(ii\ )$$

Let r and R be the internal and outer radii and v and V be the velocities there at any time t.

When I $\quad r' = r,\ \ v' = v,\ \ p = 0$

II $\quad r' = R,\ \ v' = V,\ \ p = 0$

Now we shall determine the value of the constant A with the help of I and II conditions.

or $$-\frac{f'(t)}{r} + \tfrac{1}{2}v^2 = A + \frac{\mu}{r}$$

and $$-\frac{f'(t)}{R} + \tfrac{1}{2}V^2 = A + \frac{\mu}{R}$$

By subtracting, we get

$$-f'(t)\left\{\frac{1}{r} - \frac{1}{R}\right\} + \tfrac{1}{2}(v^2 - V^2) = \mu\left(\frac{1}{r} - \frac{1}{R}\right) \qquad \ldots(iv)$$

$$\left\{\begin{array}{ll} \text{But} & r^2 v = R^2 V = f(t) \qquad \text{from (I)} \\ \text{or} & 2r^2\, dr = 2R^2\, dR = 2f(t)\, dt \qquad \ldots(X) \end{array}\right.$$

or $$-f'(t)\left\{\frac{1}{r} - \frac{1}{R}\right\} + \tfrac{1}{2} f^2(t)\left\{\frac{1}{r^4} - \frac{1}{R^4}\right\} = \mu\left(\frac{1}{r} - \frac{1}{R}\right)$$

Multiplying by $2f(t)\, dt$ both the sides, we have

$$-2f(t) f'(t)\, dt\left\{\frac{1}{r} - \frac{1}{R}\right\} + \tfrac{1}{2} f^2(t)\left\{\frac{2f(t)\, dt}{r^4} - \frac{2f(t)\, dt}{K^4}\right\}$$
$$= \mu\left\{\frac{2f(t)\, dt}{r} - \frac{2f(t)\, dt}{R}\right\}$$

or $$-2f(t) f'(t)\, dt\left\{\frac{1}{r} - \frac{1}{R}\right\} + f^2(t)\left\{\frac{dr}{r^2} - \frac{dR}{R^2}\right\}$$
$$= \mu\{\because r\, dr - 2R\, dR\}$$

By integrating, we have

$$-f^2(t)\left\{\frac{1}{r} - \frac{1}{R}\right\} = \mu(r^2 - R^2) + B \qquad \ldots(v)$$

Since, when $r=a$, $R=A$, $v=0$ i.e. $f(t)=0$,

$$0=\mu(a^2-A^2)+B$$

or $$B=-\mu(a^2-A^2).$$

So equation (v) becomes,

$$-f^2(t)\left\{\frac{1}{r}-\frac{1}{R}\right\}=\mu(r^2-a^2-R^2+A^2)$$

or $$f^2(t)\left\{\frac{1}{r}-\frac{1}{R}\right\}=\mu(a^2-r^2-A^2+R^2). \qquad \text{...(vi)}$$

Let ϖ be the impulsive pressure at a distance r' then

$$d\varpi=-\rho v'\,dr'$$

or $$d\varpi=-\rho\frac{f(t)}{r'^2}\,dr' \qquad \{\text{as } r'^2 v'=f(t)$$

By integrating, we have

$$\varpi=\frac{\rho f(t)}{r'}+C$$

where C is an arbitrary constant.

But $\varpi=0$, $r'=R$, then $C=\dfrac{\rho f(t)}{R}$

or $$\varpi=\rho f(t)\left\{\frac{1}{r'}-\frac{1}{R}\right\}$$

which determines the impulsive pressure at any distance r'

The impulsive pressure at any point of the ball where $r'=r$ is,

$$\varpi=\rho f(t)\left\{\frac{1}{r}-\frac{1}{R}\right\}.$$

Substituting the value of $f(t)$ from (vi), we have

$$\varpi=\rho\sqrt{\left\{\frac{\mu(a^2-r^2-A^2+R^2)}{\left(\frac{1}{r}-\frac{1}{R}\right)}\right\}}\left(\frac{1}{r}-\frac{1}{R}\right)$$

or $$\varpi=\rho\sqrt{\mu}\sqrt{\left\{(a^2-r^2-A^2+R^2)\left(\frac{1}{r}-\frac{1}{R}\right)\right\}}$$

Hence the impulsive pressure varies as

$$\sqrt{\left\{(a^2-r^2-A^2+R^2)\left(\frac{1}{r}-\frac{1}{R}\right)\right\}}. \qquad \textbf{Proved.}$$

Ex. 2. *A sphere of radius a is surrounded by infinite liquid of density* ρ, *the pressure at infinity being* Π. *The sphere is suddenly annihilated. Shew that the pressure at distance r from the centre immediately falls to* $\Pi\left(1-\dfrac{a}{r}\right)$.

Shew further that if the liquid is brought to rest by impinging on a concentric sphere of radius $\frac{1}{2}a$, the impulsive pressure sustained by the surface of the sphere is $\sqrt{\left(\frac{7}{6}\Pi\rho a^2\right)}$.

In the incompressible fluid, the fluid velocity will be radial outside the sphere, it will be a function of r (radial distance from the centre of the sphere *i.e.* origin) and the time t only. The equation of continuity is

$$\frac{1}{r^2}\frac{d}{dr}(r^2q)=0$$

or $$r^2q=f(t) \quad \text{(constant)}$$

Let v' be the velocity at a distance r' from the centre of the sphere at any time t and p be the pressure. The equation of continuity is

$$r'^2v'=f(t).$$

The equation of motion is

$$\frac{\partial v'}{\partial t}+v'\frac{\partial v'}{\partial r'}=-\frac{1}{\rho}\frac{\partial p}{\partial r'}$$

or $$\frac{f'(t)}{r'^2}+v'\frac{\partial v'}{\partial r'}=-\frac{1}{\rho}\frac{\partial p}{\partial r'} \qquad \{\text{from (i)}$$

By integrating with regard to r', we have

$$-\frac{f'(t)}{r'}+\tfrac{1}{2}v'^2=-\frac{p}{\rho}+A \qquad \ldots\text{(ii)}$$

where A is an arbitrary constant.

Since pressure at infinity is Π.

So $$r'=\infty,\quad p=\Pi,\quad v'=0,$$

then from (ii), we have

$$0=-\frac{\Pi}{\rho}+A$$

or $$A=\frac{\Pi}{\rho}.$$

Substituting the value of the constant A, we get

$$-\frac{f'(t)}{r'}+\tfrac{1}{2}v'^2=\frac{\Pi-p}{\rho}. \qquad \ldots\text{(iii)}$$

When the sphere is suddenly annihilated, we have

$$r'=a,\ v'=0,\ p=0,$$

or $$-\frac{f'(t)}{a}=\frac{\Pi}{\rho} \quad \text{or} \quad f'(t)=-\frac{\Pi a}{\rho}.$$

So just after annihilation, substituting the value of $f'(t)$ in (iii), it reduces to

$$-\frac{\Pi a}{\rho}\cdot\frac{1}{r'}+0=\frac{\Pi-p}{\rho}$$

{as $v'=0$ just after annihilation}

or $$\frac{a\Pi}{r'}=\Pi-p.$$

Thus pressure at the time of annihilation, since $r'=r$ is

or $$\frac{a\Pi}{r}=\Pi-p$$

or $$p=\Pi\left(1-\frac{a}{r}\right).$$ **Proved.**

Let ϖ be the impulsive pressure at a distance r',

then $d\varpi=-\rho\, v'\, dr'$ $\left\{\begin{array}{l}\text{from the equation of continuity} \\ r^2v=r'^2\, v'=f(t)\end{array}\right.$

or $$d\varpi=-\rho\, v\,\frac{r^2}{r'^2}\, dr'$$

(where r is the radius of the inner surface and v, the velocity there)

By integrating, we have

$$\varpi=B+\rho v\,\frac{r^2}{r'}$$

Now when $r'=\infty$, $\varpi=0$

Then $B=0$.

Thus $$\varpi=\rho v\,\frac{r^2}{r'} \qquad \text{...(iv)}$$

(Which determines the impulsive pressure ϖ at a distance r' and v, the velocity at the inner surface)

Since the liquid is brought to rest by impinging on a concentric sphere of radius $\frac{a}{2}$.

Substituting $r=\frac{a}{2}$ in (iv), we have

$$\varpi=\rho v\,\frac{a^2}{4r'} \qquad \text{...(v)}$$

Now we shall determine the velocity v at the inner surface of the sphere, $p=0$, then from (iii), we have

$$-\frac{f'(t)}{r}+\tfrac{1}{2}\, v^2=\frac{\Pi}{\rho}$$ {as $r'=r$, $v'=v$

or $$\left(rv\frac{dv}{dr}+2v^2\right)-\tfrac{1}{2}v^2=-\frac{\Pi}{\rho}$$

$$\left\{\begin{array}{l}\text{Since} \quad r^2v=f(t) \\ \text{or } r^2\dfrac{dv}{dt}+2rv\dfrac{dr}{dt}=f'(t)\end{array}\right.$$

or $$rv\frac{dv}{dr}+\frac{3}{2}v^2=-\frac{\Pi}{\rho}$$

$$\left\{\ \text{or } r^2v\frac{dv}{dr}+2rv^2=f'(t)\right.$$

Multiplying by $2r^2dr$ both the sides, we have

$$2r^3v\,dv+3v^2r^2dr=-\frac{2\Pi}{\rho}r^2dr$$

By integrating, we get

$$r^3v^2=-\frac{2\Pi}{3\rho}r^3+C$$

Since $r=a$, $v=0$; $C=\dfrac{2\Pi}{3\rho}a^3$

or $$r^3v^2=\frac{2\Pi}{3\rho}(a^3-r^3)$$

The velocity v at the surface of the sphere of radius $a/2$ on which the liquid strikes is

$$v^2=\frac{2\Pi}{3\rho}\cdot\frac{a^3-r^3}{r^3}$$

or $$v^2=\frac{2\Pi}{3\rho}\cdot\frac{a^3-a^3/8}{a^3/8} \qquad \left\{ \text{put } r=\frac{a}{2}\right.$$

or $$v^2=\frac{14}{3}\frac{\Pi}{\rho}$$

Substituting the value of v in relation (iv), we get

$$\varpi=\tfrac{1}{4}\rho\sqrt{\left(\frac{14}{3}\cdot\frac{\Pi}{\rho}\right)}.a^2\frac{1}{r'} \qquad \ldots\text{(vi)}$$

gives impulsive pressure at a distance r'.

Substitute $r'=\dfrac{a}{2}$, the relation (vi) gives the impulsive pressure at the surface of the sphere of radius $\dfrac{a}{2}$.

$$\varpi=\tfrac{1}{4}\rho\sqrt{\left(\frac{14}{3}\cdot\frac{\Pi}{\rho}\right)}.a^2.\frac{2}{a}=\sqrt{\left(\frac{7\Pi\rho a^2}{6}\right)} \qquad \textbf{Proved.}$$

Ex. 3. . *A mass of liquid surrounds a solid sphere of radius a, and its outer surface, which is a concentric sphere of radius b, is subject to a given constant pressure Π, no other force being in action on the liquid. The solid sphere suddenly shrinks into a concentric sphere ; it is required to determine the subsequent motion and the impulsive action on the sphere.*

Let v' be the velocity at a distance r' at any time t, and p be the pressure there. The equation of continuity is

$$r'^2 v' = f(t) \qquad ...(1)$$

The equation of motion is

$$\frac{\partial v'}{\partial t} + v' \frac{\partial v'}{\partial r'} = -\frac{1}{\rho}\frac{\partial p}{\partial r'}$$

or
$$\frac{f'(t)}{r'^2} + v' \frac{\partial v'}{\partial r'} = -\frac{1}{\rho}\frac{\partial p}{\partial r'} \qquad \{\text{from (1)}$$

By integrating with regard to r'. we have

$$-\frac{f'(t)}{r} + \frac{1}{2} v'^2 = -\frac{p}{\rho} + C \qquad ...(2)$$

{Where C is an arbitrary constant}.

Let r and R be the internal and external radii of the fluid and v and V the velocities there at any time t.

So initially I $r' = r$, $v' = v$, $p = 0$

II $r' = R$. $v' = V$, $p = \Pi$.

Then constant C can be determined with the help of I and II condition,

$$-\frac{f'(t)}{r} + \frac{1}{2} v^2 = C$$

and
$$-\frac{f'(t)}{R} + \frac{1}{2} V^2 = -\frac{\Pi}{\rho} + C$$

By subtracting, we have

$$-f'(t)\left\{\frac{1}{r} - \frac{1}{R}\right\} + \frac{1}{2}(v^2 - V^2) = \frac{\Pi}{\rho} \qquad ...(3)$$

or
$$-f'(t)\left\{\frac{1}{r} - \frac{1}{R}\right\} + \frac{1}{2} f^2(t)\left\{\frac{1}{r^4} - \frac{1}{R^4}\right\} = \frac{\Pi}{\rho}$$

$$\left\{\begin{array}{l}\text{Since } \quad r^2 v = R^2 V = f(t) \\ \text{or} \quad r^2 dr = R^2 dR = f(t)\, dt\end{array}\right.$$

Multiplying both the sides by $2f(t)\,dt$, we have

$$-2f(t)f'(t)\left\{\frac{1}{r} - \frac{1}{R}\right\} + \frac{1}{2} f^2(t)\left\{\frac{2f(t)\,dt}{r^4} - \frac{2f(t)\,dt}{R^4}\right\} = \frac{2\Pi}{\rho} f(t)\,dt$$

or
$$-2f(t)f'(t)\left\{\frac{1}{r} - \frac{1}{R}\right\} + f^2(t)\left\{\frac{dr}{r^2} - \frac{dR}{R^2}\right\} = \frac{2\Pi}{\rho} r^2\, dr$$

By integrating, we have

$$-f^2(t)\left\{\frac{1}{r} - \frac{1}{R}\right\} = \frac{2\Pi}{3\rho} r^3 + B$$

Initially, when $r=a$, $v=0$ *i.e.* $f(t)=0$.

$$B=-\frac{2\Pi}{3\rho}a^3$$

or $$f^2(t)\left\{\frac{1}{r}-\frac{1}{R}\right\}=\frac{2\Pi}{3\rho}(a^3-r^3)\qquad\left\{\begin{array}{ll}\text{as} & R^3-r^3=b^3-a^3\\ \text{and} & r^2v=f(t)\end{array}\right.$$

or $$r^4v^2=\frac{2\Pi}{3\rho}\cdot\frac{a^3-r^3}{\left(\frac{1}{r}-\frac{1}{R}\right)}$$

or $$v=\sqrt{\left[\frac{2\Pi}{3\rho}\cdot\left(\frac{a^3-r^3}{r^4\left\{\frac{1}{r}-\frac{1}{(r^3+b^3+a^3)^{1/3}}\right\}}\right)\right]}\qquad ...(4)$$

which determines the velocity of the fluid at a distance r.

Now we shall determine impulsive pressure on the sphere.

Let r be the radius of the solid sphere and ϖ be the impulsive pressure at distance r'.

Then $$d\varpi=-\rho v' dr'\qquad\left\{\begin{array}{l}\text{from the eqn. of continuity}\\ r^2v=r'^2v=f(t)\end{array}\right.$$

or $$d\varpi=-\rho\frac{r^2v}{r'^2}dr'$$

By integrating, we have

$$\varpi=\rho\frac{r^2v}{r'}+A$$

When $r'=R$, $\varpi=0$, then $A=-\dfrac{\rho r^2v}{R}$

So the impulsive pressure when $r'=r$ is

$$\varpi=\rho r^2v\left(\frac{1}{r}-\frac{1}{R}\right)$$

Thus impulsive action of the sphere

$$=4\pi r^2\varpi$$

$$=4\pi r^2.\rho r^2\,v\left(\frac{1}{r}-\frac{1}{R}\right)$$

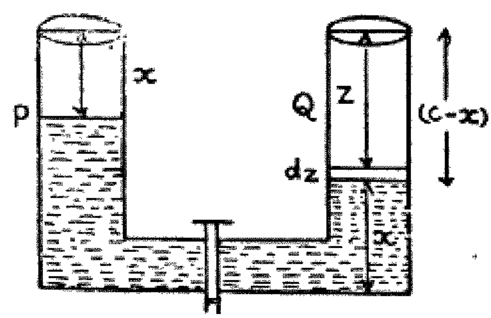
$$=4\pi r^3v\,\rho\left(\frac{R-r}{R}\right)$$

Thus we can calculate the impulsive action of the sphere by substituting the value of v from (4). **Answer.**

Ex. 4. *Two equal closed cylinders, of height c, with their bases in the same horizontal plane, are filled, one with water, and the other with air of such a density as to support a column h of water, h being less than c. If a communication be opened between* *them at their bases, the height x, to which the water rises, is given by the equation*

$$cx - x^2 + ch \log \frac{c-x}{x} = 0.$$

Let P, Q be two cylinders containing water and air respectively. Let K be the cross-section of each cylinder before and after the communication is set up, the air and water are at rest. Thus the initial and final kinetic energies are zero. The intrinsic energies also vanish, because of incompressibility.



So total work done = change in $K.E.$

$= \text{zero}$

The potential energy due to position of water in the cylinder P is

$$V_a = \int_0^c gP\,Kz\,dz = \tfrac{1}{2} g\rho^{*} Kc^2 \qquad \ldots(1)$$

The height x of water rises in cylinder Q and the height $(c-x)$ of water will remain in the cylinder P after communication is opened. Then the final potential energy is

$$V_b = \int_0^{c-x} g\,\rho\,k\,z\,d\,z + \int_0^{x} g\,\rho\,k\,z\,d\,z$$

$$V_b = \tfrac{1}{2} g\rho k\,[(c-x)^2 + x^2] \qquad \ldots(2)$$

Loss in potential energy of work done by gravity

$$V_a - V_b = \tfrac{1}{2} g\rho k\,[c^2 - (c-x)^2 - x^2]$$

or $$V_a - V_b = \tfrac{1}{2} g\rho k\,[2cx - 2x^2] = g\rho kx\,(c-x) \qquad \ldots(3)$$

Let p be the pressure when the water rises to a height z,

Then $$g\rho hkc = pk\,(c-z)$$

or $$p = \frac{g\rho hc}{c-z}.$$

Since the air has been compressed, then the work done in compressing this air in the cylinder Q is

$$= -g\rho hck \int_0^x \frac{dz}{c-z}$$

---

*Here we have considered the density ρ of water simply because that the density is such as to support a column h of water.

$$=g\rho hck\left[\log(c-z)\right]_0^x$$

$$=g\rho hck\log\left(\frac{c-x}{c}\right). \quad \ldots(4)$$

Now Total work done=zero.

Hence $g\rho kx(c-x)+g\rho hck\log\left(\frac{c-x}{c}\right)=0.$

or $$g\rho k\left[cx-x^2+ch\log\left(\frac{c-x}{c}\right)\right]=0.$$

or $$cx-x^2+ch\log\frac{c-x}{c}=0.$$ **Proved.**

Ex. 5. *A spherical mass of liquid of radius b has a concentric spherical cavity of radius a ; which contains gas at pressure p whose mass may be neglected ; at every point of the external boundary of the liquid an impulsive pressure* ϖ *per unit area is applied. Assuming that the gas obeys Boyle's Law, Shew that when the liquid first comes to rest, the radius of the internal spherical surface will be*

$$a\ exp.\left\{-\frac{\varpi^2 b}{2p\rho a^2(b-a)}\right\},$$

where ρ *is the density of the liquid.*

Let v' be the velocity at a distance r' from the centre of the spherical cavity at any time t. Equation of continuity is

$$r'^2v'=f(t) \quad \text{(Const.)} \quad \ldots(1)$$

Let ϖ' be the impulsive pressure at distance r'

Then $d\varpi'=-\rho v'dr'$ {as $r'^2v'=b^2V=f(t)$ from (1)

or $$d\varpi'=-\rho\frac{b^2V}{r'^2}dr'$$

By integrating with regard to r', we have

$$\varpi'=\frac{\rho b^2V}{r'}+C \quad \ldots(2)$$

where C is an arbitrary constant.

Since, when I $r'=a$, $\varpi'=0$

II $r'=b$, $\varpi'=\varpi$ (let)

Now we shall determine the value of the const. C from (2) with the help of the condition I and II.

$$0=\frac{\rho b^2V}{a}+C$$

$$\varpi = \frac{\rho b^2 V}{b} + C$$

$$\varpi = \rho b^2 V\left(\frac{1}{b} - \frac{1}{a}\right) = \frac{\rho b V}{a}(a-b) \qquad \ldots(3)$$

The initial kinetic energy is

$$= \tfrac{1}{2}\int_a^b (4\pi r'^2.\rho dr').v'^2$$

$$= 2\pi\rho \int_a^b r'^2.\frac{b^4V^2}{r'^4}.dr'$$

$$= 2\pi\rho b^4V^2 \int_a^b \frac{dr'}{r'^2}$$

$$= 2\pi\rho b^4V^2 \left(\frac{1}{a} - \frac{1}{b}\right) = \frac{2\pi\rho b^3V^2}{a}(b-a). \qquad \ldots(4)$$

Since the gas obeys Boyle's Law, let r be the radius of the internal spherical cavity and p_1 be the pressure of the gas there.

i.e. $\quad \frac{4}{3}\pi r^3 \,.\, p_1 = \frac{4}{3}\pi a^3.p \quad$ or $\quad p_1 = \frac{a^3p}{r^3}.$

Total work done

$$= \int_a^r 4\pi r^2.p_1 dr$$

$$= 4\pi a^3 p \int_a^r \frac{1}{r}dr = 4\pi a^3 p \log\left(\frac{r}{a}\right) \qquad \ldots(5)$$

Now change in K. E = Total work done

$$\frac{2\pi\rho b^3V^2}{a}(b-a) = 4\pi a^3 p \log\left(\frac{r}{a}\right) \qquad \{\text{ from (4) and (5)}$$

or

$$\log\left(\frac{r}{a}\right) = \frac{2\pi\rho b^3V^2(b-a)}{4\pi a^4 p}$$

$$= \frac{2\pi\rho b^3(b-a)}{4\pi p a^4}.\ \frac{a^2\varpi^2}{\rho^2 b^2(a-b)^2}$$

$$= -\frac{\varpi^2 b}{2p\rho a^2(b-a)}$$

or

$$r = a\exp.\left\{-\frac{\varpi^2 b}{2p\rho a^2(b-a)}\right\}$$

Exercises

1. Prove that if ϖ be the impulsive pressure, ϕ, ϕ' the velocity potentials immediately before and after an impulse acts, V the potential of the impulses

$$\varpi + \rho V + \rho(\phi' - \phi) = \text{Const.}$$

2. Show that in the absence of extraneous impulses, the impulsive pressure at any point of a liquid satisfies Laplace's Equation.

An explosion takes place at a point O at some distance below the surface of deep water. If O' is the image of O in the free surface. Show that the velocity potential of the initial motion at any point P varies as

$$\frac{1}{OP}-\frac{1}{O'P}.$$

Determine the initial velocity of the free surface at any point.

3. Find the equations of motion of a perfect fluid under extraneous impulses and impulsive pressure. Deduce that any actual irrotational motion of a liquid can be produced instantaneously from rest by a set of impulses properly applied.

4. A given quantity of liquid moves, under no forces, in a smooth conical tube having a small vertical angle, and the distance of its nearer and farther extremities from the vertex at the time are r and r'; prove that

$$2r\frac{d^2r}{dt^2}+\left(\frac{dr}{dt}\right)^2\left\{3-\frac{r}{r'}-\frac{r^2}{r'^2}-\frac{r^3}{r'^3}\right\}=0,$$

The pressures at the two surfaces being equal.

Show also that the preceeding equation results from supposing the vis-viva of the mass of liquid to be constant; and that the velocity of the inner surface is given by the equation

$$V^2=\frac{Cr'}{r^3(r'-r)},\ r'^3-r^3=c^3.$$

where C and c being constants.

Hint :— Equation to continuity is

$$vx^2\tan^2\alpha=F(t)$$

where α is the vertical angle of the cone

Total K.E. of the liquid

$$=\frac{1}{2}\int_r^{r'}\pi x^2\tan^2\alpha\,dx\,\rho v^2$$

$$=\frac{\pi}{2}\rho\tan^2\alpha.\,F^2(t)\left(\frac{1}{r}-\frac{1}{r'}\right)$$

Since K.E. is constant, it follows that

$$\frac{\pi\rho\tan^2\alpha}{2}F^2(t)\left(\frac{1}{r}-\frac{1}{r'}\right)=\text{const}$$

or
$$r^4\dot{r}^2\left(\frac{r'-r}{rr'}\right)=\text{const}$$

or
$$V^2=\frac{cr'}{r^3(r'-r)}$$

Since the total mass of the liquid is constant.

$$\rho \frac{\pi}{3} \tan^2 \alpha \, (r'^3 - r^3) = \text{const.}$$

$$\therefore \qquad r'^3 - r^3 = \text{const.}$$

5. Show that the rate per unit of time at which work is done by the internal pressures between the parts of a compressible fluid obeying Boyle's Law is

$$\iiint p \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) dx\, dy\, dz$$

where p is the pressure and (u, v, w) the velocity at any point, and the integration extends through the volume of the fluid.

Hint :—Let W is the work done, p be the pressure and dv an elementary volume. Work done in compressing the fluid is

$$W = \int p \, (-dv) \qquad \ldots(1)$$

Rate per unit time of work done is

$$\frac{DW}{Dt} = -\iiint \frac{Dp}{Dt} dv \qquad \ldots(2)$$

We know from the equation of continuity that

$$\frac{D\rho}{Dt} + \rho \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

or

$$\frac{Dp}{Dt} + p \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0 \qquad \ldots(3)$$

(as $p = \rho K$)

From (2) and (3), we have

$$\frac{DW}{Dt} = \iiint p \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) dv$$

Now $dv = dx\, dy\, dz$

Thus Rate per unit time of work done is given by

$$= \iiint p \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) dx\, dy\, dz.$$

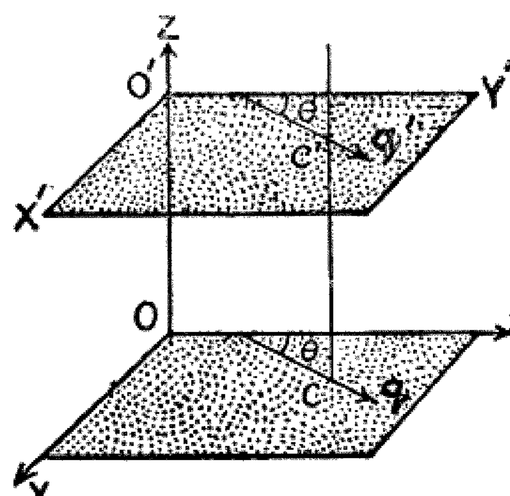
3

Motion In Two Dimensions

Here we shall develope special methods for solving the problems in two and three dimensions. The flow considered will be wholly incompressible which implies that the fluid are of constant density throughout the motion.

§ 3·0. Motion in two dimensions.

Consider the plane XOY as the fixed plane and C is a point in this plane ($i.\ e.\ Z=0$ plane.) Draw a perpendicular CC' to the XOY plane ($i\ e.$ parallel to OZ) where C' is the corresponding point of C in the plane $X'O'Y'$ (lying in the fluid). Let **q** be the velocity at C making an angle θ with the axis OY then the velocity at C' is equal in magnitude and direction to the velocity at C. The velocity at corresponding point C' is a function of x, y and the time t. Thus the velocity q will be a function of x, y and t only but not of Z.



Now it is often useful to consider that the fluid in two dimensional motion is confined between two planes parallel to the plane of motion at an unit distance apart. The reference plane of motion is parallel to and midway between the supposed fixed planes.

Thus the two dimensional motion is defined when the lines of motion are parallel to a fixed plane and the velocity at the corresponding points of all planes parallel to the fixed plane has the same magnitude and direction.

§ 3·1. Stream function.

We know that the velocity $\mathbf{q}$ is a function of x, y and t in two-dimensional motion, then the differential equation to the stream line is

or $$\frac{dx}{u}=\frac{dy}{v}$$

or $$v\,dx-u\,dy=0. \quad ...(1)$$

The equation of continuity (in two dimensions) is

$$\nabla\cdot\mathbf{q}=0$$

or $$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \quad ...(2)$$

Now (1) must be a perfect differential, $d\psi$

i.e. $$v\,dx-u\,dy=d\psi$$

or $$v\,dx-u\,dy=\frac{\partial \psi}{\partial x}dx+\frac{\partial \psi}{\partial y}dy$$

By comparing, we have

$$\mathbf{u}=-\frac{\partial \psi}{\partial \mathbf{y}} \text{ and } \mathbf{v}=\frac{\partial \psi}{\partial \mathbf{x}}$$

Then the function ψ is called the **Stream function** or **Current function**.

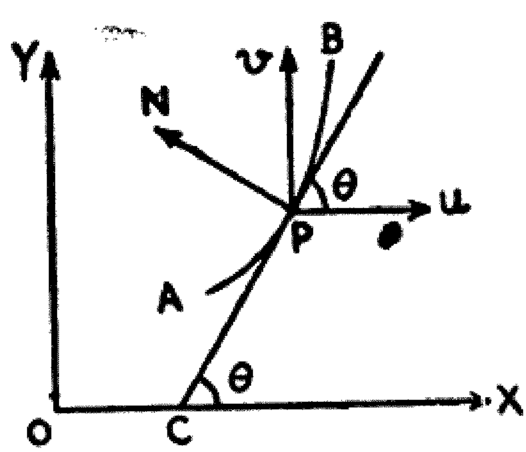
The stream lines are given by the solution of (1), *i.e.* ψ=cost.

So the stream function is obtained by taking the equation of continuity and the incompressibility of the fluid.

The stream function ψ always exists in all types of two dimensional motion : rotational or irrotational.

§ 3·11. Property of stream function.

Let ψ_1 and ψ_2 be the stream functions at, A and B respectively on the curve AB in the XOY plane.



Consider ds be an element at any point P on the curve, where the tangent to this point makes an angle θ with the axis of X. Let u and v be the components of velocity parallel to the axis of X and Y respectively.

Velocity along the inward drawn normal PN,

$$=v\cos\theta-u\sin\theta.$$

$$=\frac{\partial\psi}{\partial x}\cos\theta+\frac{\partial\psi}{\partial y}\sin\theta$$ {Ref. § 3·1

$$=\frac{\partial\psi}{\partial x}\cdot\frac{dx}{ds}+\frac{\partial\psi}{\partial y}\cdot\frac{dy}{ds}$$

Now the flow I across the curve from right to left, is given by

$$I=\int_{AB}\rho\,\mathbf{q}.\mathbf{n}\,ds$$

$$I=\rho\int_{AB}\left(\frac{\partial\psi}{\partial x}\cdot\frac{dx}{ds}+\frac{\partial\psi}{\partial y}\cdot\frac{dy}{ds}\right)ds$$

$$I=\rho\int_{\psi_1}^{\psi_2}d\psi$$

$$I=\rho\,(\psi_2-\psi_1)$$

So the difference between the value of stream function at any two points of a curve represents the flow across any line joining the points.

§ 3·2. Irrotational motion in two-dimensions.

We know that in an irrotational motion the velocity potential always exists, then we have

$$u=-\frac{\partial\phi}{\partial x}\text{ and }v=-\frac{\partial\phi}{\partial y}\qquad\ldots(1)$$

and, if ψ is the stream function, then

$$u=-\frac{\partial\psi}{\partial y}\text{ and }v=\frac{\partial\psi}{\partial x}\qquad\ldots(2)$$

From (1) and (2), we have

$$\frac{\partial\phi}{\partial x}=\frac{\partial\psi}{\partial y}\qquad\ldots(3)$$

and $$\frac{\partial\phi}{\partial y}=-\frac{\partial\psi}{\partial x}\qquad\ldots(4)$$

Which are known as **Cauchy-Riemann's equation.**

Such functions are called **conjugate functions.**

From (3) and (4), we get

$$\frac{\partial\phi}{\partial x}\cdot\frac{\partial\psi}{\partial x}+\frac{\partial\phi}{\partial y}\cdot\frac{\partial\psi}{\partial y}=0.\qquad\ldots(5)$$

The relation (5) shows the families of curves given by

$$\phi=\text{const.}\quad\text{and}\quad\psi=\text{const.}$$

which intersects orthogonally. Thus the curves of constant velocity potential cut the stream lines orthogonally

Remember.

(i) The stream function ψ exist whether the motion is irrotational or rotational.

(ii) The velocity potential ϕ exists only in the case of an irrotational motion.

(iii) If one part of the fluid is moving irrotationally and the other part rotationally, then the velocity potential exists only in the part having irrotational motion.

Again differentiating (3) and (4) with regard to x and y, we have

$$\frac{\partial^2\phi}{\partial x^2}=\frac{\partial^2\psi}{\partial x\,\partial y} \text{ and } \frac{\partial^2\phi}{}=-\frac{\partial^2\psi}{\partial x\,\partial y}$$

which gives

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0 \quad \text{or} \quad \nabla^2\phi=0^* \qquad \ldots(6)$$

Similarly, we have

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0 \quad \text{or} \quad \nabla^2\psi=0 \qquad \ldots(7)$$

(Condition for irrotational motion)

The velocity potential and stream function are said to be conjugate if they satisfy (5), (6) and (7).

§ 3·3. **Complex potential.**

Since, we have

$$\frac{\partial\phi}{\partial x}=\frac{\partial\psi}{\partial y} \text{ and } \frac{\partial\phi}{\partial y}=-\frac{\partial\psi}{\partial x} \qquad \ldots(A)$$

{Ref. § 3·2 (4)

then $\phi+i\psi$ must be a function of $x+iy$ *i.e.* z

or $$\phi+i\psi=f(x+iy) \qquad \ldots(1)$$

Differentiating partially w. r. to x and y respectively.

$$\frac{\partial\phi}{\partial x}+i\,\frac{\partial\psi}{\partial x}=f'\,(x+iy) \qquad \ldots(2)$$

or $$\frac{\partial\phi}{\partial y}+i\,\frac{\partial\psi}{\partial y}=-if'\,(x+iy) \qquad \ldots(3)$$

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*The equation of continuity is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad \text{Since } u=-\frac{\partial\phi}{\partial x},\ v=-\frac{\partial\phi}{\partial y}$$

or $$\frac{\partial}{\partial x}\left(-\frac{\partial\phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(-\frac{\partial\phi}{\partial y}\right)=0$$

or $$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0 \quad \text{or} \quad \nabla^2\phi=0. \quad \text{(Laplace's equation)}$$

o $$\frac{\partial\phi}{\partial y}+i\frac{\partial\psi}{\partial y}=i\left\{\frac{\partial\phi}{\partial x}+i\frac{\partial\psi}{\partial x}\right\}$$ {from (2) and (3)

Separating into real and imaginary parts, we have

$$\frac{\partial\phi}{\partial x}=\frac{\partial\psi}{\partial y} \text{ and } \frac{\partial\phi}{\partial y}=-\frac{\partial\psi}{\partial x}$$

which are the same as in relation (A). It follows that

$$\phi+i\psi=f(x+iy) \quad \text{or} \quad w=f(z)$$

i.e. w is a function of z then w is called the **complex potential** and an analytic function of z.

Converse : Similarly if w is an analytic function of z then the velocity potential is the real part and stream function is an imaginary part of a irrotational two dimensional fluid motion.

§ 3·31. Magnitude of the velocity.

Since $$w=f(z)$$

or $$\phi+i\psi=f(x+iy)$$

Differentiating w. r. to x partially, we obtain

$$\frac{dw}{dz}\cdot\frac{\partial z}{\partial x}=\frac{\partial\phi}{\partial x}+i\frac{\partial\psi}{\partial x}$$ $\left\{\frac{dw}{dz}=\frac{\partial w}{\partial z}\right.$ as w is a function of z only

or $$\frac{dw}{dz}=\frac{\partial\phi}{\partial x}-i\frac{\partial\phi}{\partial y}$$

or $$\frac{dw}{dz}=-u+iv$$

Magnitude of the velocity at any point is given by

$$\mathbf{q}=\left|\frac{dw}{dz}\right|=\left\{\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2\right\}^{1/2}$$

$$\mathbf{q}=(u^2+v^2)^{1/2}$$

When the velocity is zero, the points are called as **stagnation points.** Thus **for stagnation points** $\left(\frac{\mathbf{dw}}{\mathbf{dz}}\right)=\mathbf{0}$.

Ex. 1. *Prove that there is always a stream function whether the motion is rotational or irrotational.*

Equation of continuity in two dimension is

$$\textbf{div. } \mathbf{q}=0$$

or $$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0. \qquad \ldots(1)$$

The equation to stream line in two dimensions is given by

$$\frac{dx}{u}=\frac{dy}{v}$$

or $$v\,dx-u\,dy=0 \qquad \ldots(2)$$

Now (2) must be a perfect differen'ial $d\psi$. Where ψ is the stream function. Hence ψ exists for rotational as well as irrotational fluid motion.

Ex. 2. *Prove that the speed is everywhere the same, the stream lines are straight.*

The equation to the stream lines are given by

$$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w}$$

or $$v\,dx-u\,dy=0$$

or $$v\,dz-w\,dy=0$$

and $$u\,dz-w\,dx=0$$

which gives $$vx-uy=\text{constant},\ vz-wy=\text{constant}$$

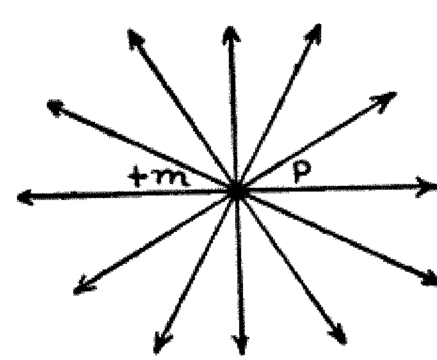
and $$uz-wx=\text{constant}.$$

Thus the intersection of these planes are straight lines.

Sources and Sinks.

§ 3·4. Two Dimensional Source.

If the two-dimensional motion of a liquid consists of **outward radial flow** from a point or fluid is imagined to flow uniformly in all directions, then the point is called a **simple source.**



A **source** is a point at which the fluid is continuously created. Infact source is a purely abstract conception which does not occur in nature.

If $2\pi m\rho$ is the rate of emission of volume per unit time, then $+m$ is called the **strength of source.***

A source of negative strength or inward radial flow is called a **sink** or it is a point at which the fluid continuously annihilates. The strength of the sink will be $-m$.

Let q_r be the radial velocity at a disiance r from the source, then the amount of fluid or flux out of the circle is

$$=2\pi r\rho q_r.$$

Then, by definition, we have

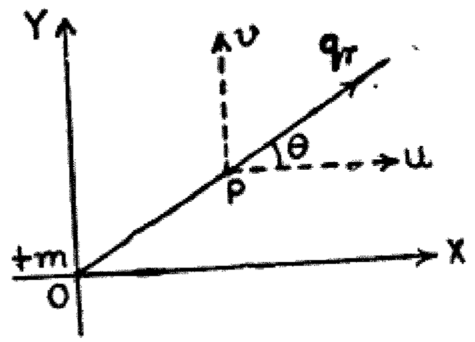
$$2\pi r\rho q_r=2\pi\rho m$$

*Some authors has defined the rate of emission of volume per unit time as m, then the strength of the source will be $\frac{m}{2\pi}$.

or $$q_r=\frac{m}{r}.$$

The source and sink are the points at which the velocity potential and stream function become infinite.

§ 3·5. Complex potential for a source.



Consider a source of strength $+m$ at an origin. Let q_r be the radial velocity at a distance r from the source, then

$$2\pi r\rho q_r=2\pi\rho m$$

or $$q_r=\frac{m}{r}$$

Consider u and v be the component velocity at the point P.

Then $u=\frac{m}{r}\cos\theta$ and $v=\frac{m}{r}\sin\theta$

Since $w=\phi+i\psi$

or $$\frac{dw}{dz}\cdot\frac{\partial z}{\partial x}=\frac{\partial\phi}{\partial x}+i\frac{\partial\psi}{\partial x}$$ $\left\{\text{as } \frac{\partial w}{\partial z}=\frac{dw}{dz}\right.$ w being a function of z only,

or $$\frac{dw}{dz}=\frac{\partial\phi}{\partial x}-i\frac{\partial\phi}{\partial y}$$ $\left\{\text{as } \frac{\partial z}{\partial x}=1 \text{ and } \frac{\partial\psi}{\partial x}=-\frac{\partial\phi}{\partial y}\right.$

or $$\frac{dw}{dz}=-u+iv$$

or $$\frac{dw}{dz}=-\frac{m}{r}(\cos\theta-i\sin\theta)$$

or $$\frac{dw}{dz}=-\frac{m}{re^{\theta i}}$$ {Since $z=re^{\theta i}$

or $$\frac{dw}{dz}=-\frac{m}{z}.$$

By integrating, we have

$$w=-m\log z. \quad ...(1)$$

{Integration constant will vanish at the origin}

which gives the complex potential due to source at origin, obviously the **velocity is infinite at z=0.**

If the source is situated at the point $z=z_1$, then by changing the origin, (1) becomes

$$w=-m\log(z-z_1).$$

Similarly, consider that the sources of strength $m_1, m_2, m_3...$

etc. are situated at the points z_1, z_2, z_3 ... etc., then the complex potential is given by

$$w=-m_1 \log (z-z_1)-m_2 \log (z-z_2)-m_3 \log (z-z_3)...$$

By separating real and imaginary parts, we have

$$\phi=-m_1 \log r_1-m_2 \log r_2-m_3 \log r_3-...$$

and $\quad \psi=-m_1\theta_1-m_2\theta_2-m_3\theta_3-...$ $\quad$ {Since $w=\phi+i\psi$

where $\quad r_n=|z-z_n|\quad n=1, 2, 3, ...$

and $\quad \theta_n=\arg.(z-z_n)$

§ 3·51. **Alternative Method.**

Complex potential for a source.

Consider a source of strength $+m$ be situated at the origin. Let q_r and q_θ be the radial and transverse velocities at the point P, then

$$2\pi r\rho q_r=2\pi\rho m$$

or
$$q_r=\frac{m}{r}$$

or
$$-\frac{\partial\phi}{\partial r}=\frac{m}{r} \qquad ...(1)$$

By integrating, we have

$$\phi=-m \log r \qquad ...(2)$$

and
$$q_\theta=0$$

or
$$-\frac{1}{r}\frac{\partial\phi}{\partial\theta}=0.$$

Since we know from the Cauchy Riemann Equations, that

$$\frac{\partial\phi}{\partial r}=\frac{1}{r}\frac{\partial\psi}{\partial\theta} \text{ and } \frac{1}{r}\frac{\partial\phi}{\partial\theta}=-\frac{\partial\psi}{\partial r}*$$

From (1)
$$-\frac{\partial\phi}{\partial r}=-\frac{1}{r}\frac{\partial\psi}{\partial\theta}=\frac{m}{r}$$

or
$$\frac{\partial\psi}{\partial\theta}=-m$$

By integrating, we get

$$\psi=-m\theta. \qquad ...(3)$$

$$* \frac{\partial\phi}{\partial r}=\frac{\partial\phi}{\partial x}\cos\theta+\frac{\partial\phi}{\partial y}\sin\theta=\frac{\partial\psi}{\partial y}\cos\theta-\frac{\partial\psi}{\partial x}\sin\theta$$

$$=\frac{1}{r}\left(\frac{\partial\psi}{\partial y}\cdot\frac{\partial y}{\partial\theta}+\frac{\partial\psi}{\partial x}\cdot\frac{\partial x}{\partial\theta}\right)=\frac{1}{r}\frac{\partial\psi}{\partial\theta}$$

$$\frac{1}{r}\frac{\partial\phi}{\partial\theta}=-\frac{\partial\phi}{\partial x}\sin\theta+\frac{\partial\phi}{\partial y}\cos\theta=-\frac{\partial\psi}{\partial y}\sin\theta-\frac{\partial\psi}{\partial x}\cos\theta=-\frac{\partial\psi}{\partial r}$$

Then the complex potential w is given by

$$w=\phi+i\psi$$

or $$w=-m\log r-im\theta$$

or $$w=-m\log(re^{i\theta})=-m\log z.$$

Let the source be placed at the point $z=z_1$, then the complex potential is given be

$$w=-m\log(z-z_1).$$

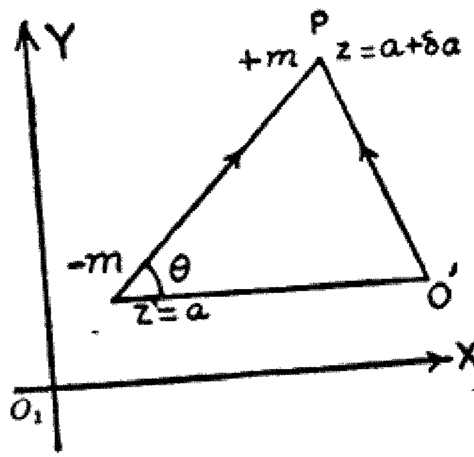
§ 3·6. Doublet in two-dimensions.

A combination of two equal and opposite sources of strength $\pm m$ (or a source of strength $+m$ and a sink of strength $-m$) at a distance δs apart, where δs is considered to be infinitely small and m infinitely great, such that the product $m\delta s$ is finite and equal to μ (say), is called a **doublet or double source or dipole of strength** μ.

The line δs drawn from $-m$ to $+m$ (*i.e.* from sink towards source) is called the **axis of the doublet.**

§ 3·61 Complex Potential for a Doublet.

Suppose a sink of strength $-m$ at the point O $(z=a)$ and a source of strength $+m$ at the point P $(z=a+\delta a)$, the line OP makes an angle θ with a line OO' parallel to the axis of X.



The complex potential is given by

$$w=m\log(z-a)-m\log\{z-(a+\delta a)\}$$

or $$w=-m\,[\log\{z-(a+\delta a)\}-\log(z-a)]$$

or $$w=-m\delta a.\left[\frac{\log\{z-(a+\delta a)\}-\log(z-a)}{\delta a}\right]$$

or $$w=-m\delta a.\frac{\partial}{\partial a}[\log(z-a)]$$ $\Bigg\{$ as $\dfrac{\log\{z-(a+\delta a)\}-\log(z-a)}{\delta a}=\dfrac{\partial}{\partial a}\log(z-a)$

or $$w=\frac{m\delta a}{z-a}$$

or $$w=\frac{m.OPe^{i\theta}}{z-a}$$ {Since $\delta a=OP\,e^{i\theta}$

or $\quad w = \dfrac{\mu e^{\theta i}}{z-a}$ ...(1)

$\left\{\begin{array}{l}\text{as } \ m.OP=\mu \\ \text{Strength of the doublet.}\end{array}\right.$

which gives the complex potential of the doublet.

Let the doublets of strength μ_1, μ_2, μ_3 ... etc. are placed at the points $z=a_1, a_2, a_3, \ldots$ etc. and their axes make an angles $\theta_1, \theta_2, \theta_3, \ldots$ etc. with the axis of X, then the complex potential is given by

$$w = \frac{\mu_1 e^{\theta_1 i}}{z-a_1} + \frac{\mu_2 e^{\theta_2 i}}{z-a_2} + \frac{\mu_3 e^{\theta_3 i}}{z-a_3} + \ldots\ldots$$

§ 3·62. **Particular Case.** Let the doublet of strength μ is situated at an origin and the axis of X be its axis, then the complex potential is given by

$$w = \frac{\mu}{z} \qquad \left\{\begin{array}{r}\text{From (1) as } \theta=0 \\ \text{and } a=0\end{array}\right.$$

or $\quad \phi + i\psi = \dfrac{\mu}{re^{\theta i}}$

or $\quad \phi + i\psi = \dfrac{\mu}{r} e^{-\theta i}.$

Separating real and imaginary parts, we have

$$\phi = \frac{\mu}{r} \cos\theta \text{ and } \psi = -\frac{\mu}{r} \sin\theta$$

§ 3·7. **Images in two dimensions.**

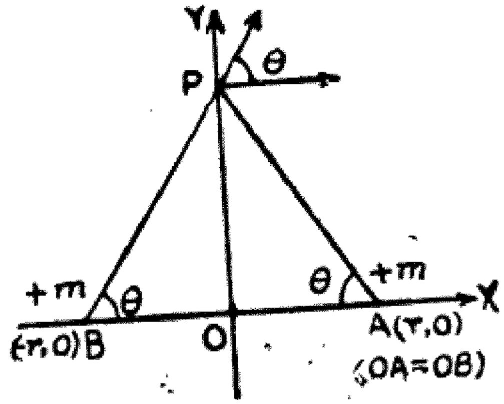
If a surface S can be drawn in a moving fluid in such a way that **there is no flow across that surface** then any system of sources, sinks and doublets on one side of the surface is said to be the **images** of the system of sources, sinks and doublets on the other side with regard to S.

Since there is no flow across the surface then it must be a stream line. If we introduce a rigid boundary in place of the surface then the fluid motion will remain unaltered and the fluid velocity at any point, normal to the rigid boundary must vanish.

This method of images is used to determine the complex potential due to sources, sinks and doublets in the presence of rigid boundary.

§ 3·8. Image of a source with regard to a line.

Consider two equal sources of strength $+m$ at A and B at an equidistant from the plane OP, on the opposite side. Then the velocity at P due to the source of strength $+m$ at A,



$$=+\frac{m}{r} \text{ along } AP$$

$$\left\{\begin{array}{l}\text{Since } \phi=-m\log r\\ -\dfrac{\partial\phi}{\partial r}=\dfrac{m}{r}\end{array}\right.$$

The velocity at P due to the source of strength $+m$ at B

$$=-\frac{m}{r} \text{ along } BP.$$

Thus the velocity at P along the normal

$$=\frac{m}{r}\cos\theta-\frac{m}{r}\cos\theta$$

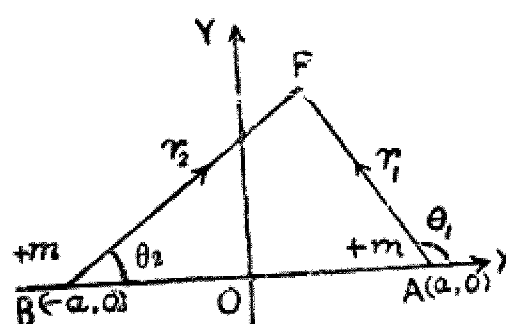
$$=\text{Zero}$$

which shows that there is no flow across the straight line OP.

Thus the image of a source is an equal source equidistant from the line on the opposite side with regard to a line.

§ 3·81. Alternative Method : Image of a source with regard to a plane.

Consider two equal sources of strength $+m$ at A and B at an equidistance from the plane OY. The complex potential is given by



$$w=-m\log(z-a)$$
$$-m\log(z+a)$$

or $\quad w=-m\log(r_1e^{i\theta_1})-m\log(r_2e^{i\theta_2})$

or $\quad w=-m\log\{r_1r_2.e^{i(\theta_1+\theta_2)}\}$

or $\quad w=-m\log r_1r_2-im(\theta_1+\theta_2)$

Thus stream lines are given by

$$\psi=\text{Constant.}$$
$$\theta_1+\theta_2=\text{Const.}$$

In fact Y-axis is a stream line, if P is on Y-axis,

Then $\theta_1+\theta_2=\pi$.

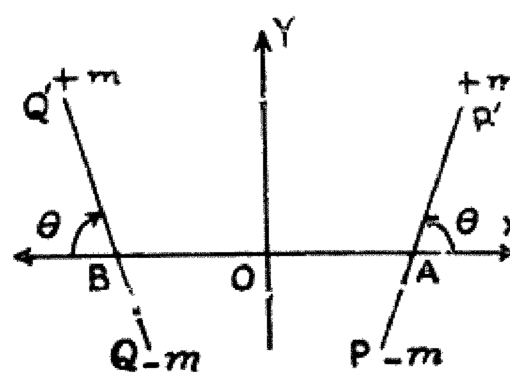
Thus there is no flow across the straight line OY.

Therefore the source of strength $+m$ at B is the image of the source $+m$ at A with regard to the line OY or the image system of a source at A consists an equal source at B. *i.e.* the optic image of A in the plane.

§ 3·82. Image of a doublet in a straight line.

We know that the doublet is a combination of two equal and opposite sources of strength $\pm m$, and the line drawn from $-m$ to $+m$ is called the axis of the doublet.

Let the axis of the doublet PP' makes an angle θ with the axis of X. Then the image system of a source $+m$ and a sink $-m$ with regard to a line OY consists at an equidistant (from OY) a source $+m$ and a sink $-m$ of the axis of the doublet QQ' (which is anti-parallel to that of the axis of the doublet PP') or in other sense the image of a doublet at A ($z=a$) with its axis PP' inclined at an angle θ with OX consists an equal doublet at B ($z=-a$) inclined with its axis QQ' (anti-parallel to that of PP') at an angle $(\pi-\theta)$ with OX. Then the complex potential is given by



$$w=\frac{\mu e^{\theta i}}{z-a}+\frac{\mu e^{(\pi-\theta)i}}{z+a}$$

or $$w=\frac{\mu e^{\theta i}}{z-a}-\frac{\mu e^{-\theta i}}{z+a}.$$

§ 3·83. Circle Theorem.

Consider $f(z)$ be the complex potential of a two-dimensional irrotational flow of an incompressible inviscid fluid with no rigid boundary. (The singularities of $f(z)$ are all at a greater distance than a from the origin). If a circular cylinder $|z|=a$ is inserted in the field of flow then the complex potential is given by

$$w=f(z)+\bar{f}\left(\frac{a^2}{z}\right).$$

Let C be the cross-section of the circular cylinder $|z|=a$.

Then $\bar{z}=\dfrac{a^2}{z}$ (on the circle)

or $z\bar{z}=a^2$.

Thus the complex potential is

$w=f(z)+\bar{f}(z)$ will be real quantity.

i.e. $\phi+i\psi$ is real.

So on the circle C $\psi=0$ *i.e.* the circle is a stream line.

Again, If the point z lies outside the circle, then the point $\dfrac{a^2}{z}$ will lie inside the circle C and vice—versa, since all the singularities of $f(z)$ are outside to $|z|=a$ and all the singularities of $\bar{f}\left(\dfrac{a^2}{z}\right)$ are inside $|z|=a$. In particular $\bar{f}\left(\dfrac{a^2}{z}\right)$ has no singularity at infinity, since $f(z)$ has none inside the circle.

Thus w has the same singularities as $f(z)$ outside $|z|=a$.

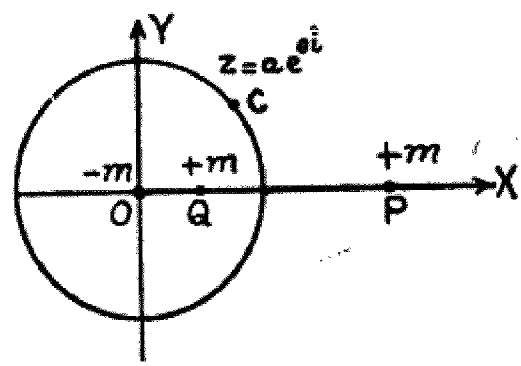
§ 3·84. Image of a source with regard to a circle.

A source of strength $+m$ is placed at the point P outside the circle. Let Q is an inverse point of the point P with regard to the circle such that

$$OP\ OQ=a^2$$

or
$$OQ=\frac{a^2}{f}$$

{as $OP=f$ (say)



Consider a source of strength $+m$ at $P\,(z=f)$ and an equal source $+m$ at an inverse point $Q\left(z=\dfrac{a^2}{f}\right)$. Then the complex potential is given by circle theorem. {Ref. § 3·83

$$w=-m\log(z-f)-m\log\left(\frac{a^2}{z}-f\right) \quad ...(1)$$

By adding a constant term $-\log(-f)$, the relation (1) reduces to

$$w=-m\log(z-f)-m\log\left(\frac{a^2}{z}-f\right)-m\log(-f)$$

$$w=-m\log(z-f)-m\log\left(z-\frac{a^2}{f}\right)+m\log z \quad ...(2)$$

which shows that the image system of a source $+m$ placed outside the circle consists of

(i) A source $+m$ at P $(z=f)$.

(ii) A source $+m$ at an inverse point Q $\left(z=\dfrac{a^2}{f}\right)$

(iii) A sink $-m$ at an origin $(z=0)$.

Let C be any point on the circle, $z=ae^{\theta i}$ (let)

Substitute $z=ae^{\theta i}$ in (2), it reduces to

$$w=-m\log\left\{(ae^{\theta i}-f)\left(ae^{\theta i}-\frac{a^2}{f}\right)\right\}+m\log(ae^{\theta i})$$

or $$\phi+i\psi=-m\log\left[\left\{(a\cos\theta-f)+ia\sin\theta\right\}\left\{\left(a\cos\theta-\frac{a^2}{f}\right)+ia\sin\theta\right\}\right]+m\left[\log a+\theta i\right]$$

Equating the imaginary part, we have

$$\psi=-m\tan^{-1}\left(\frac{a\sin\theta}{a\cos\theta-f}\right)-m\tan^{-1}\left(\frac{a\sin\theta}{a\cos\theta-\dfrac{a^2}{f}}\right)+m\theta$$

or $\psi=-m\tan^{-1}(\tan\theta)+m\theta$

or $\psi=-m\theta+m\theta$

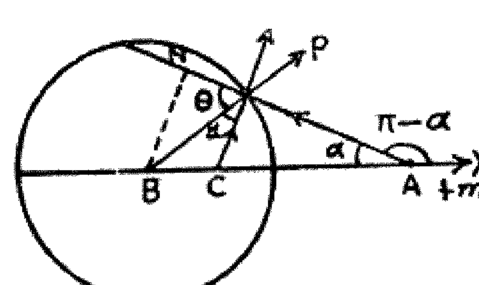
$=$Zero.

The above complex potential shows that the circle is a stream line. Thus the image system for a source of strength $+m$ placed outside the circle consists of an equal source $+m$ at an inverse point and a sink $-m$ at an origin.

Cor : A source of strength $+m$ inside a circle and an equal sink of strength $-m$ at the centre has for image system an equal source at an inverse point.

§ 3·85 Alternative Method : Image of a source with regard to a circle.

We shall determine the image system due to a source of strength $+m$ placed at A out side the circle.



Let C be an inverse point of A with regard to the circle such that

$$BC\cdot BA=a^2$$

(where a is the radius of the circle)

Let P be any point on the circle and an equal source $+m$ at an inverse point C.

Velocity at P due to the source of strength $+m$ at A

$$=+\frac{m}{AP} \text{ along } AP.$$

Velocity at P due to the source of strength $+m$ at C

$$=+\frac{m}{CP} \text{ along } CP.$$

Hence normal velocity at P

$$=\frac{m}{AP}\cos BPA+\frac{m}{CP}\cos BPC \qquad \text{(along } BP)$$

$$=-\frac{m}{AP}\cos\theta+\frac{m}{CP}\cos\alpha. \qquad \ldots\text{(i)}$$

$$\left\{\begin{array}{ll}\text{since} & \angle BPA=\pi-\theta \\ \text{and} & \angle BPC=\alpha\end{array}\right.$$

Again $\qquad \cos\alpha=\dfrac{AN}{AB}$

$$\cos\alpha=\frac{AP+PN}{AB}$$

$$\cos\alpha=\frac{AP+BP\cos\theta}{AB}=\frac{AP}{AB}+\frac{BP}{AB}\cos\theta$$

$$\left\{\begin{array}{l}\text{Since } \dfrac{BA}{BP}=\dfrac{BP}{BC}=\dfrac{AP}{CP} \\ \text{from similar triangles } BPC \text{ and } BPA\end{array}\right.$$

Substituting the value of $\cos\alpha$ in (i) it reduces to

$$=-\frac{m}{AP}\cos\theta+\frac{m}{CP}\left(\frac{AP}{AB}+\frac{BP}{AB}\cos\theta\right)$$

$$=-\frac{m}{AP}\cos\theta+\frac{m}{CP}\left(\frac{CP}{BP}+\frac{CP}{AP}\cos\theta\right)$$

$$=-\frac{m}{AP}\cos\theta+\frac{m}{BP}+\frac{m}{AP}\cos\theta$$

$$=\frac{m}{BP}.$$

It follows that if we place a sink of strength $-m$ at B, then the velocity due to the sink will be $\dfrac{m}{BP}$ along BP. The normal velocity of the system along BP vanishes.

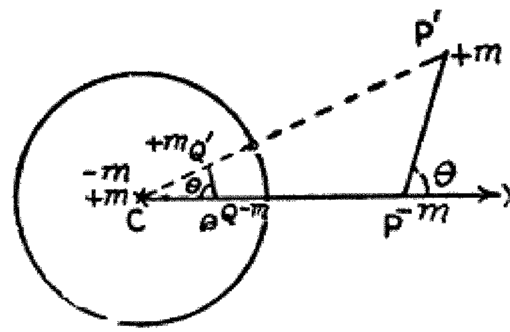
Thus the **image system of a source at A outside the circle** consists of

(i) A source of strength $+m$ at A.

(ii) A source of strength $+m$ at an inverse point C.

(iii) A sink of strength $-m$ at the centre B.

§ 3·86. Image of a doublet in a circle.



Let Q and Q' be the inverse points of P and P' with regard to the circular boundary, such that

$$CP \cdot CQ = a^2 = CP'.CQ'$$

(where a is the radius of the circle) from similar triangles CPP' and CQQ', we have

$$\frac{CP}{CP'} = \frac{CQ'}{CQ}.$$

Let PP' be the doublet outside the circle with its axis making an angle θ with X—axis, QQ' be the image of the doublet PP' with a sink $-m$ at Q and a source $+m$ at Q', the strength of the doublet is given by

$$\mu' = \text{Strength of the doublet } QQ' \qquad \text{(say)}$$

or $$\mu' = \underset{Q \to Q'}{\text{Lt}} (m.QQ')$$

or $$\mu' = \underset{P \to P'}{\text{Lt}} \left(m.PP' \frac{a^2}{CP.CP'} \right)$$

or $$\mu' = \frac{\mu a^2}{c^2}$$

$$\left\{ \begin{array}{ll} \text{as} & CP = CP' = c \\ \text{and} & \mu = m.PP' \\ \multicolumn{2}{l}{\text{strength of the doublet } PP'.} \end{array} \right.$$

The axis of the doublet QQ' is anti-parallel to the axis of the doublet PP',

The image system of a sink $-m$ at P consists an equal sink $-m$ at an inverse point Q and an equal source $+m$ at the centre C.

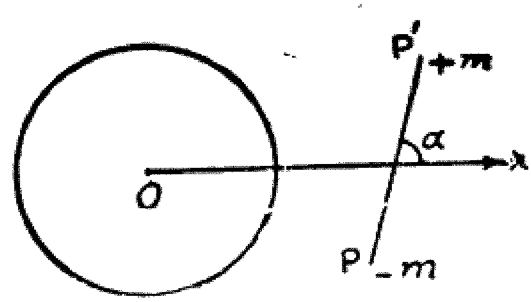
Again the image system of a source $+m$ at P' consists an equal souıce $+m$ at an inverse point Q' and an equal sink $-m$ at the centre C.

A source $+m$ and a sink $-m$ vanishes at the centre C.

Thus the image system of a doublet of strength μ at a distance c from the centre of a circle will give rise to a doublet of strength $\frac{\mu a^2}{c^2}$ at an inverse point, with its ax's anti-parallel to that of PP'.

§ 3·87. Alternative Method : Image of a doublet in a circle.

Consider a doublet of strength μ be placed at $z=c$, its axis makes an angle α with the X-axis. The complex potential due to the doublet is given by



$$f(z)=\frac{\mu e^{\alpha i}}{z-c}.$$

Let a circular cylinder $|z|=a$ be inserted, then by the circle theorem, the complex potential is given by

$$w=f(z)+\bar{f}\left(\frac{a^2}{z}\right)$$

or $$w=\frac{\mu e^{i\alpha}}{z-c}+\frac{\mu e^{-i\alpha}}{\left(\frac{a^2}{z}-c\right)}$$

{in the region $|z| \geqslant a$}

or $$w=\frac{\mu e^{\alpha i}}{z-c}-\frac{\mu e^{i(\pi-\alpha)}}{\frac{a^2}{z}-c}$$

or $$w=\frac{\mu e^{\alpha i}}{z-c}+\frac{\mu z\, e^{i(\pi-\alpha)}}{c\left(z-\frac{a^2}{c}\right)}$$

or $$w=\frac{\mu e^{\alpha i}}{z-c}+\frac{\mu e^{i(\pi-\alpha)}}{c}\cdot\frac{z-\frac{a^2}{c}+\frac{a^2}{c}}{z-\frac{a^2}{c}}$$

or $$w=\frac{\mu e^{\alpha i}}{z-c}+\frac{\mu a^2}{c^2}\cdot\frac{e^{i(\pi-\alpha)}}{\left(z-\frac{a^2}{c}\right)}+\frac{\mu}{c}e^{i(\pi-\alpha)}$$

or $$w=\frac{\mu e^{\alpha i}}{z-c}+\frac{\mu a^2}{c^2}\cdot\frac{e^{i(\pi-\alpha)}}{z-a^2/c}$$

{Neglecting the remaining terms, being constant}

If follows that the image system consists of

(i) A doublet of strength μ at $z=c$ inclined at an angle α with the X-axis.

(ii) A doublet of strength $\frac{\mu a^2}{c^2}$ at an inverse point $z=\frac{a^2}{c}$ inclined at an angle $(\pi-\alpha)$ to X-axis.

§ 3 88. Three-dimensional Source.

If the motion of a liquid consists of symmetrical radial flow in all directions proceeding from a point, then the point is called a three-dimensional **source**.

If $4\pi m$ is the total flow across a small surface surrounding the source, then m is called the **strength of the source.**

Let q_r be the radial velocity at a distance r from the centre, then

$$4\pi r^2 q_r = 4\pi m$$

or $$q_r = \frac{m}{r^2} \qquad \left\{ \text{Since } q_r = -\frac{\partial \phi}{\partial r} \right.$$

or $$-\frac{\partial \phi}{\partial r} = \frac{m}{r^2}$$

By integrating, we have

$$\phi = \frac{m}{r}.$$

A three-dimensional source with negative strength is called a three dimensional sink.

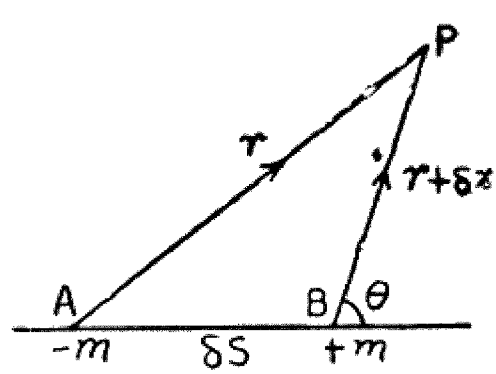
§ 3 89. Three-dimensional Doublet.

A three-dimensional doublet is a combination of a three-dimensional soure $+m$ and a sink $-m$ at a distance δs apart, where δs is taken to be infinitely small and m infinitely great such that the product $m\delta s$ is finite and is equal to μ (say), is called a **doublet of strength μ**

The line δs drawn in the sense from $-m$ to $+m$ is known the **axis of the doublet.**

§ 3 91. Velocity Potential due to a three dimensional doublet.

Consider AB be the doublet with a sink $-m$ at A and a source $+m$ at B. Let P be the point at a distance r from A and $r+\delta r$ from B.



Let ϕ be the velocity potential of the doublet, then

$$\phi = -\frac{m}{r} + \frac{m}{r+\delta r}$$

or $$\phi = -\frac{m}{r}\left\{ 1 - \left(1 + \frac{1}{r}\delta r \right)^{-1} \right\}$$

or $$\phi=-\frac{m}{r}\left(1-1+\frac{1}{r}\,\delta r-\dots\right)$$

to first approximation)

or $$\phi=-\frac{m\,\delta r}{r^2}$$ $\left\{\begin{array}{ll}\text{Since} & \cos\theta=-\dfrac{\partial r}{\partial s}\\ \text{or} & \delta r=-\delta s\cos\theta\end{array}\right.$

or $$\phi=\frac{m\,\delta s\cos\theta}{r^2}$$

or $$\phi=\frac{\mu\cos\theta}{r^2}=\mu\frac{\partial}{\partial s}\left(\frac{1}{r}\right)$$

(where $\mu=m\,\delta s$, strength of the doublet.)

$$\text{Radial velocity}=-\frac{\partial\phi}{\partial r}=\frac{2\mu\cos\theta}{r^3}$$

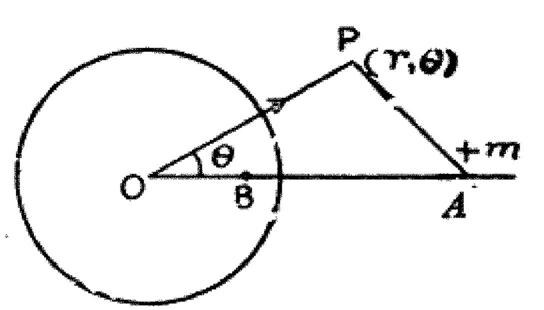
$$\text{Transverse velocity}=-\frac{1}{r}\frac{\partial\phi}{\partial\theta}=\frac{\mu\sin\theta}{r^3}.$$

§ 3·90. Images in three-dimensions.

The system of sources, sinks and doublets on one side of a surface S are said to be the image of the sources, sinks and doublets on the other side of it, if there is no flow of the fluid across the surface. The surface S is a stream surface.

§ 3·91. Image of a three-dimensional source with regard to a sphere.

Consider a source of strength $+m$ situated at a point A $(z=f)$ out side the sphere. Let ϕ_1 and ϕ_2 be the velocity potential at a point P due to the source $+m$ and the sphere respectively.



Then $$\phi_1=\frac{m}{AP}$$

or $$\phi_1=\frac{m}{(f^2+r^2-2fr\cos\theta)^{1/2}}$$

$$\left\{\text{as}\quad\cos\theta=\frac{r^2+f^2-AP^2}{2fr}\right.$$

or $$\phi_1=\frac{m}{f}\left(1-\frac{2r}{f}\cos\theta+\frac{r^2}{f^2}\right)^{-1/2}$$

or $$\phi_1=\frac{m}{f}\sum_{n=0}^{\infty}\left[\left(\frac{r}{f}\right)^n P_n(\mu)\right] \qquad \begin{cases}\text{as} & r<f\\ \text{and} & \mu=\cos\theta\end{cases}$$

or $$\phi_1=\frac{m}{f}\left[1+\sum_{n=1}^{\infty}\left(\frac{r}{f}\right)^n P_n(\mu)\right] \qquad \text{...(i)}$$

{as $P_0(\mu)=1$}

where P_n is Legendre's coefficient of order n, which exists for all values of $r<f$.

The motion is symmetric about OA and ϕ satisfies the Laplace's Equation. Let ϕ_2 is of the form

$$\phi_2=\Sigma\, A_n\frac{a_n}{r^{n+1}}\,P_n(\mu). \qquad \text{...(ii)}$$

Again, the velocity normal to the sphere is zero, it follows that

$$\frac{\partial}{\partial r}(\phi_1+\phi_2)=0 \qquad \text{at } r=a \text{ (radius of sphere)}$$

or $$\frac{m}{f}\sum_{1}^{\infty}\frac{na^{n-1}}{f^n}P_n(\mu)-\sum_{0}^{\infty}(n+1)\frac{A_n}{a^2}P_n(\mu)=0$$

So $$A_0=0 \quad \text{for} \quad n=0 \qquad (\forall \text{ values of } \theta)$$

and $$A_n=\frac{nm\,a^{n+1}}{(n+1)f^{n+1}}$$

Substituting the value of A_n in (ii), we have

$$\phi_2=m\sum_{1}^{\infty}\frac{n}{n+1}\cdot\frac{a^{2n+1}}{r^{n+1}f^{n+1}}P_n(\mu)$$

or $$\phi_2=m\sum_{1}^{\infty}\frac{(n+1)-1}{n+1}\cdot\frac{a^{2n+1}}{r^{n+1}f^{n+1}}P_n(\mu)$$

or $$\phi_2=m\sum_{1}^{\infty}\frac{a^{2n+1}}{r^{n+1}f^{n+1}}P_n(\mu)-m\sum_{1}^{\infty}\frac{a^{2n+1}}{r^{n+1}f^{n+1}}\cdot\frac{P_n(\mu)}{n+1} \qquad \text{...(iii)}$$

Since B is an inverse point of A with regard to the sphere,

such that $\qquad OA.OB=a^2$ or $OB=\dfrac{a^2}{f}=f'$ (let)

then (iii) can be written as $\left(\text{By adding and subtracting } \dfrac{ma}{f}\cdot\dfrac{P_0}{r}\right)$

$$\phi_2=\frac{ma}{f}\sum_{0}^{\infty}\frac{f'^n}{r^{n+1}}P_n(\mu)-\frac{ma}{f}\sum_{0}^{\infty}\frac{f'^n}{r^{n+1}}\cdot\frac{P_n(\mu)}{n+1}$$

or $$\phi_2=\frac{ma}{fr}\left(r-2\,\frac{f'}{r}\cos\theta+\frac{f'^2}{r^2}\right)^{-1/2}-\frac{ma}{f}\sum_{0}^{\infty}\frac{f'^n}{r^{n+1}}\cdot\frac{P_n(\mu)}{n+1}$$

or $$\phi_2=\frac{ma}{f(r^2+f'^2-2rf'\cos\theta)^{1/2}}-\frac{ma}{f}\sum_{0}^{\infty}\frac{f'^n}{r^{n+1}}\cdot\frac{P_n(\mu)}{n+1}$$

or $$\phi_2=\frac{(ma/f)}{BP}-\frac{ma}{f}\sum_{0}^{\infty}\frac{f'^n}{r^{n+1}}\cdot\frac{P_n(\mu)}{n+1}. \qquad \text{...(iv)}$$

The expression $\dfrac{(ma/f)}{BP}$ is the velocity potential due to a source of strength $+\dfrac{ma}{f}$ at B.

Again the velocity potential due to the source $+\dfrac{ma}{f}$ at any point on OB distance λ from the centre is given by

$$=\frac{ma}{f}(r^2+\lambda^2-2r\lambda\cos\theta)^{-1/2}$$
$$=\frac{ma}{f}\sum_0^\infty \frac{\lambda^n}{r^{n+1}} P_n(\mu)$$

Suppose the line OB formed of source of strength $\dfrac{ma}{ff'}$, then the velocity potential due to the line source OB of strength $\dfrac{ma}{ff'}$ is

$$=\frac{ma}{ff'}\int_0^{OB}\sum_0^\infty \frac{\lambda^n}{r^{n+1}} P_n.d\lambda$$
$$=\frac{ma}{ff'}\sum_0^\infty \int_0^{f'} \frac{\lambda^n}{r^{n+1}} P_n.d\lambda$$
$$=\frac{ma}{ff'}\sum_0^\infty \left(\frac{\lambda^{n+1}}{n+1}\right)_0^{f'} \cdot \frac{1}{r^{n+1}} P_n$$
$$=\frac{ma}{ff'}\sum_0^\infty \frac{f'^{n+1}}{r^{n+1}.(n+1)} P_n.$$

Thus the second term in (iv) is the velocity potential due to the **continuous line distribution of sinks** of strengths $\dfrac{ma}{ff'}$ or $-\dfrac{m}{a}$ per unit length extending from O to B.

Thus the image system of a source placed outside with regard to a sphere consists of a source of strength $+\dfrac{ma}{f}$ *at an inverse point and a line sink of strength* $-\dfrac{m}{a}$ *per unit length extending from the centre to an inverse point.*

Cor : In the same manner we can determine the image system of a doublet with regard to the sphere.

(Proceed as in two dimensions case Ref. § 3·86).

Ex. 1. *Find the lines of flow in the two-dimensional fluid motion given by*

$$\phi+i\,\psi=-\frac{1}{2}n\,(x+iy)^2 e^{2int}$$

Prove or verify that the paths of the particles of the fluid (in polar co-ordinates) may be obtained by eliminating t from the equations

$$r \cos (nt+\theta)-x_0=r \sin (nt+\theta)-y_0=nt\ (x_0-y_0)$$

Since $\phi+i\psi=-\frac{1}{2}n(x+iy)^2\, e^{2int}$

Put $x=r \cos \theta$ and $y=r \sin \theta$

then $\phi+i\psi=-\frac{1}{2}n\,(r \cos \theta+ir \sin \theta)^2\, e^{2int}$

or $\phi+i\psi=-\frac{1}{2}n\, r^2\, e^{2\theta i}\,.\,e^{2int}=-\frac{1}{2}nr^2\, e^{(\theta+nt)i}$

or $\phi+i\psi=-\frac{1}{2}nr^2\,\{\cos 2\,(\theta+nt)+i \sin 2(\theta+nt)\}$

Separating into real and imaginary parts, we get

$$\phi=-\tfrac{1}{2}\, nr^2 \cos 2\,(\theta+nt)$$

and $\psi=-\frac{1}{2}nr^2 \sin 2\,(\theta+nt)$

Since the motion is irrotational. The lines of flow are given by $\psi=$const.

i.e. $-\frac{1}{2}nr^2 \sin (2\theta+2nt)=$const.

or $r^2 \sin (2\theta+2nt)=$const.

Now, we shall determine the path of the particles of fluid

$$\frac{dr}{dt}=-\frac{\partial \phi}{\partial r}=nr \cos (2\theta+2nt)=nr \cos 2\lambda \quad \text{...(i)}$$

and $r\dfrac{d\theta}{dt}=-\dfrac{1}{r}\,\dfrac{\partial \phi}{\partial \theta}=-nr \sin (2\theta+2nt)=-nr \sin 2\lambda$ ...(ii)

{ where $\lambda=\theta+nt$

From (i), we have

$$nr \cos 2\lambda=\frac{dr}{dt}=\frac{dr}{d\lambda}\cdot\frac{d\lambda}{dt} \qquad \left\{\frac{d\lambda}{dt}=\frac{d\theta}{dt}+n\right.$$

or $nr \cos 2\lambda=\dfrac{dr}{d\lambda}\cdot\left(\dfrac{d\theta}{dt}+n\right)$

or $nr \cos 2\lambda=\dfrac{dr}{d\lambda}\,(n-n \sin 2\lambda)$

or $\dfrac{2dr}{r}=\dfrac{2 \cos 2\lambda}{1-\sin 2\lambda}\, d\lambda$

By integrating, we have

$$2 \log r=-\log (1-\sin 2\lambda)+\log C$$

or $r^2\,(1-\sin 2\lambda)=C$ where C is a constant

or $r^2\,\{\cos^2\lambda+\sin^2\lambda-2 \cos \lambda \sin \lambda\}=C$

or $r^2\,(\cos \lambda-\sin \lambda)^2=C$

or $r\,(\cos \lambda-\sin \lambda)=D$ ...(ii)

where D is also a constant.

when $t=0$, $\lambda=\theta_0$, $r=r_0$

then $D=x_0-y_0$

So (iii) reduces to

$$r\{\cos\lambda-\sin\lambda\}=x_0-y_0$$

or
$$r\cos(nt+\theta)-x_0=r\sin(nt+\theta)-y_0 \qquad \ldots\text{(iv)}$$

Again,
$$\lambda=nt+\theta$$

or
$$\frac{d\lambda}{dt}=n-n\sin 2\lambda \quad \{\text{ from (ii)}$$

Separating the variables and integrating, we have

$$\frac{d\lambda}{1-\sin 2\lambda}=ndt$$

or
$$\int\frac{d\lambda}{(\cos\lambda-\sin\lambda)^2}=n\int dt$$

or
$$\int\frac{\sec^2\lambda.d\lambda}{(1-\tan\lambda)^2}=n\int dt$$

or
$$\frac{1}{1-\tan\lambda}=nt+B$$

or
$$\frac{\cos\lambda}{\cos\lambda-\sin\lambda}=nt+B \qquad \ldots\text{(v)}$$

Since when $t=0$, $\lambda=\theta_0$, $r=r_0$

then
$$B=\frac{\cos\theta_0}{\cos\theta_0-\sin\theta_0}=\frac{r_0\cos\theta_0}{r_0\cos\theta_0-r_0\sin\theta_0}=\frac{x_0}{x_0-y_0}$$

Substituting the value of the constant B in (v), we have

$$\frac{\cos\lambda}{\cos\lambda-\sin\lambda}=nt+\frac{x_0}{x_0-y_0}$$

or
$$\frac{r\cos\lambda}{r\cos\lambda-r\sin\lambda}=nt+\frac{x_0}{x_0-y_0}$$

{Since $r\cos\lambda-r\sin\lambda=x_0-y_0$ by (iv)}

or
$$\frac{r\cos\lambda}{x_0-y_0}=nt+\frac{x_0}{x_0-y_0}$$

or
$$r\cos\lambda=nt\,(x_0-y_0)+x_0 \qquad \ldots\text{(vi)}$$

or
$$r\cos(nt+\theta)-x_0=nt\,(x_0-y_0)$$

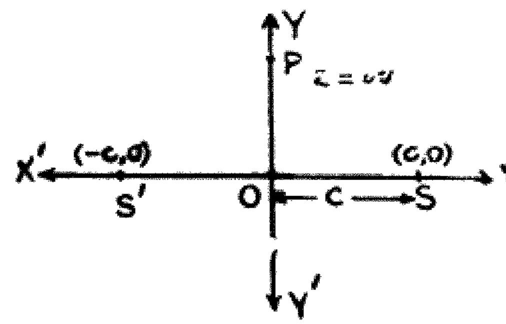
From (iv) and (vi), we have

$$r\cos(nt-\theta)-x_0=r\sin(nt+\theta)-y_0=nt\,(x_0-y_0)$$

Ex. 2. *In the case of the motion of liquid in a part of a plane bounded by a straight line due to a source in the plane, prove that if $m\rho$ is the mass of fluid (of density ρ) generated at the source per unit of time the pressure on the length $2l$ of the boundary immediately opposite to the source is less than that on an equal length at a great distance by*

$$\frac{1}{2}\ \frac{m^2\rho}{\pi^2}\left\{\frac{1}{c}\tan^{-1}\frac{l}{c}-\frac{l}{l^2+c^2}\right\}$$ *where c is the distance of the source from the boundary.*

Consider the axis of Y to be the bounding line. A source of strength $\frac{m}{2\pi}$ is placed at $S\,(c, o)$



Then the image system consists of

(i) A Source of strengh $\frac{m}{2\pi}$ at $S\,(c, o)$

(ii) A Source of strength $\frac{m}{2\pi}$ at $S'\,(-c, o)$

The complex potential w is

$$w=-\frac{m}{2\pi}\log(z-c)+\frac{m}{2\pi}\log(z+c)$$

$$w=-\frac{m}{2\pi}\log(z^2-c^2)$$

So the velocity is given by

$$q=\left|\frac{dw}{dz}\right|$$

$$=\frac{m}{2\pi}\left|\frac{2z}{z^2-c^2}\right|$$

Velocity at the point P, (Since P is a point on axis of Y, *i.e.* $z=iy$)

$$=\frac{m}{2\pi}\left|\frac{2\,iy}{-y^2-c^2}\right|=\frac{m}{\pi}\cdot\frac{y}{y^2+c^2} \qquad \ldots\text{(i)}$$

By Bernoullis theorem, we have

$$\frac{p}{\rho}=\frac{p_0}{\rho}-\tfrac{1}{2}q^2$$

Where p_0 is the pressure on Y-axis at a great distance from O.

$$\frac{p_0-p}{\rho}=\frac{1}{2}q^2$$

$$=\frac{1}{2}\ \frac{m^2}{\pi^2}\cdot\frac{y^2}{(y^2+c^2)^2}\ \Big\{\text{from (i)}$$

Let P be the pressure on the length $2l$ of the boundary

$$P=\int_{-l}^{l}(p_0-p)\,dy$$

$$P=\frac{1}{2}\,\frac{m^2\rho}{\pi^2}\int_{-l}^{l}\frac{y^2\,dy}{(y^2+c^2)^2}$$

Substitute $y=c\tan\theta$

$dy=c\sec^2\theta\,d\theta$

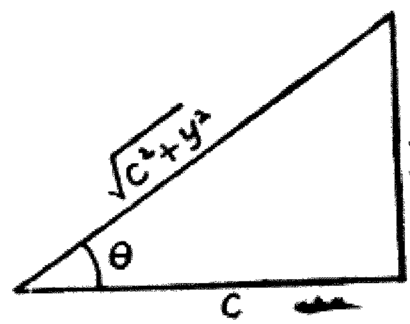
$$P=\frac{1}{2}\,\frac{m^2}{\pi^2}\cdot\rho\int\frac{c^2\tan^2\theta . c\sec^2\theta}{c^4 . \sec^4\theta}\,d\theta$$

$$P=\frac{1}{2}\,\frac{\rho m^2}{\pi^2 c}\int\frac{\tan^2\theta}{\sec^2\theta}\,d\theta$$

$$P=\frac{1}{2}\,\frac{\rho m^2}{\pi^2 c}\int\sin^2\theta\,d\theta$$

$$P=\frac{1}{4}\,\frac{\rho m^2}{\pi^2 c}\int(1-\cos 2\theta)\,d\theta$$

{as $y=c\tan\theta$



$$P=\frac{1}{4}\,\frac{\rho m^2}{\pi^2 c}(\theta-\sin\theta\cos\theta)$$

$$P=\frac{1}{4}\,\frac{\rho m^2}{\pi^2 c}\left\{\tan^1\left(\frac{y}{c}\right)-\frac{yc}{\sqrt{(c^2+y^2)}}\right\}_{-l}^{l}$$

$$\begin{cases}\text{then } \sin\theta=\dfrac{y}{\sqrt{(c^2+y^2)}}\\[2ex] \cos\theta=\dfrac{c}{\sqrt{(c^2+y^2)}}\end{cases}$$

$$P=\frac{1}{2}\,\frac{\rho m^2}{\pi^2}\left\{\frac{1}{c}\tan^{-1}\left(\frac{l}{c}\right)-\frac{l}{\sqrt{(c^2+l^2)}}\right\}$$

Proved.

Ex. 3. *What arrangement of sources and sinks will give rise to the function* $w=\log\left(z-\frac{a^2}{z}\right)$?

Draw a rough sketch of the stream lines in this curve and prove that two of them sub-divide into the circle $r=a$ and the axis of y.

The function is

$$w=\log\left(z-\frac{a^2}{z}\right)$$

or
$$w=\log\left\{\frac{(z-a)(z+a)}{z}\right\}$$

or
$$w=\log(z-a)+\log(z+a)-\log z \qquad \text{...(i)}$$

The complex potential (i), shows that there are two sinks of unit strength at distance $z=a$ and $z=-a$ and a source of unit strength at origin.

Further since $w=\phi+i\psi$, we have

$$\phi+i\psi=\log(x+iy-a)+\log(x+iy+a)-\log(x+iy)$$

or
$$\phi+i\psi=\log\{(x-a)+iy\}+\log\{(x+a)+iy\}-\log(x+iy).$$

$$\left\{\begin{array}{l}\text{We know that}\\ \log(x+iy)=\tfrac{1}{2}\log(x^2+y^2)+i\tan^{-1}\dfrac{y}{x}\end{array}\right.$$

Equating the imaginary part, we have

$$\psi=\tan^{-1}\frac{y}{x-a}+\tan^{-1}\frac{y}{x+a}-\tan^{-1}\frac{y}{x}$$

or
$$\psi=\tan^{-1}\left(\frac{\dfrac{y}{x-a}+\dfrac{y}{x+a}}{1-\dfrac{y}{x-a}\cdot\dfrac{y}{x+a}}\right)-\tan^{-1}\frac{y}{x}$$

or
$$\psi=\tan^{-1}\left(\frac{2xy}{x^2-y^2-a^2}\right)-\tan^{-1}\frac{y}{x}$$

or
$$\psi=\tan^{-1}\left(\frac{\dfrac{2xy}{x^2-y^2-a^2}-\dfrac{y}{x}}{1+\dfrac{2xy}{x^2-y^2-a^2}\cdot\dfrac{y}{x}}\right)$$

or
$$\psi=\tan^{-1}\frac{y\,(x^2+y^2+a^2)}{x\,(x^2+y^2-a^2)}$$

The lines of flow or the stream lines are given by

$$\psi=\text{constant}$$

or
$$\tan^{-1}\frac{y\,(x^2+y^2+a^2)}{x\,(x^2+y^2-a^2)}=\text{const.}$$

or
$$\frac{y\,(x^2+y^2+a^2)}{x\,(x^2+y^2-a^2)}=\text{const.}=k. \qquad \ldots\text{(iii)}$$

I. If the constant k is infinite, then from (ii)

$$\frac{y\,(x^2+y^2+a^2)}{x\,(x^2+y^2-a^2)}=\infty$$

or
$$x\,(x^2+y^2-a^2)=0,$$

or
$$x=0 \quad\text{and}\quad x^2+y^2=a^2.$$

So $x=0$ denotes that the axis of Y is a streamline.

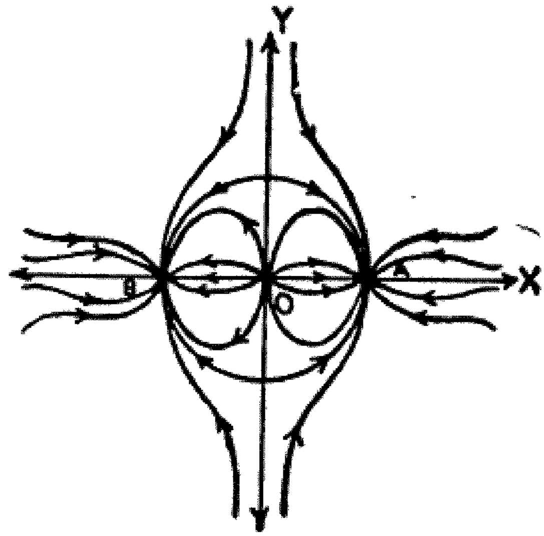
The equation $x^2+y^2=a^2$ or $r=a$ represents the circle with centre as arigin.

II. If the constant k is zero, then from (ii)

We have $y=0$ implies that axis of X is a stream line.

Thus the rough sketch of the lines is with a source of unit strength at origin and two sinks, of unit strength at the distance

$z=a$ and $z=-a$, as follows :



where

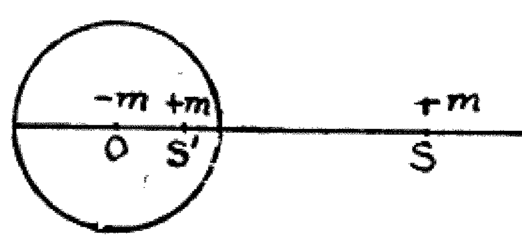
$$\begin{cases} A\ (a, 0) \\ B\ (-a, 0) \\ O\ (0, 0) \end{cases}$$

Ex. 4. *In the case of the two dimensional fluid motion produced by a source of strength m placed at a point S outside a rigid circular disc of radius a whose centre is O, show that the velocity of slip of the fluid in contact with the disc is greatest at the point where the lines joining S to the ends of the diameters at right angle to OS cut the circle ; and prove that its magnitude at these point is*

$$2m . \frac{OS}{OS^2-a^2}$$

Consider $OS=c$,

let S' be an inverse point of S with regard to the circular disc.



So $\quad OS . OS'=a^2$

$$OS'=\frac{a^2}{c}$$

Since a source of strength $+m$ is placed at a point S outside the circular disc. Then the image system consists of

(i) A source of strength $+m$ at S

(ii) A source of strength $+m$ at S'

(iii) A sink of strength $-m$ at O.

Consider O as origin and OS as real axis. The complex potential w for the motion of the fluid element at any point z is given by

$$w=-m \log (z-c)-m \log \left(z-\frac{a^2}{c}\right)+m \log z.$$

Differentiating with regard to z, we have

$$\frac{dw}{dz}=-m.\frac{1}{z-c}-m.\frac{1}{z-\frac{a^2}{c}}+m.\frac{1}{z}$$

So $$q=\left|\frac{dw}{dz}\right|=\left|-\frac{m}{z-c}-\frac{m}{z-\frac{a^2}{c}}+\frac{m}{z}\right|$$

or $$q=m\left|\frac{z\left(z-\frac{a^2}{c}\right)+z\,(z-c)+(z-c)\left(z-\frac{a^2}{c}\right)}{z\,(z-c)\left(z-\frac{a^2}{c}\right)}\right|$$

or $$q=m\left|\frac{(z-a)\,(z+a)}{z\,(z-c)\left(z-\frac{a^2}{c}\right)}\right|$$

The velocity at any point $z=x+iy=ae^{\theta i}$ on the boundary of the circular disc is

$$q=m\left|\frac{(ae^{\theta i}-a)\,(ae^{\theta i}+a)}{ae^{\theta i}\,(ae^{\theta i}-c)\left(ae^{\theta i}-\frac{a^2}{c}\right)}\right|$$

or $$q=mc\left|\frac{(e^{\theta i}-1)\,(e^{\theta i}+1)}{e^{\theta i}\,(ae^{\theta i}-c)\,(ce^{\theta i}-a)}\right|$$

$$=mc\left|\frac{(e^{2i\theta}-1)}{e^{\theta i}\,(ae^{\theta i}-c)\,(ce^{\theta i}-a)}\right|$$

or $$q=\frac{2mc\sin\theta}{a^2+c^2-2ac\cos\theta}. \qquad \text{...(i)}$$

For q to be maximum or minimum, differentiating (i) w. r. to θ, we have

$$\frac{dq}{d\theta}=0$$

or $$\frac{dq}{d\theta}=2mc.\frac{(a^2+c^2-2ac\cos\theta)\cos\theta-\sin\theta\,(2ac\sin\theta)}{(a^2+c^2-2ac\cos\theta)^2}$$

or $$\frac{dq}{d\theta}=2mc.\frac{(a^2+c^2)\cos\theta-2ac}{(a^2+c^2-2ac\cos\theta)^2}=0.$$

or $$(a^2+c^2)\cos\theta-2ac=0$$

or $$\cos\theta=\frac{2ac}{a^2+c^2}$$

Now $\theta=0$ gives minimum velocity as the expression vanishes at this point.

The maximum value of q will be obtained to the corresponding value of $\cos\theta=\dfrac{2ac}{a^2+c^2}$.

From (i) we have,

$$q=2mc.\frac{\dfrac{a^2-c^2}{a^2+c^2}}{(a^2+c^2)-\dfrac{4a^2c^2}{a^2+c^2}} \qquad \left\{\text{as} \quad \sin\theta=\frac{a^2-c^2}{a^2+c^2}\right.$$

$$q=2mc.\frac{a^2-c^2}{(a^2+c^2)^2-4a^2c^2}$$

$$=\frac{2mc}{c^2-a^2}$$

or $$q=\frac{2m.OS}{OS^2-a^2}. \qquad \{\text{as} \quad OS=c$$

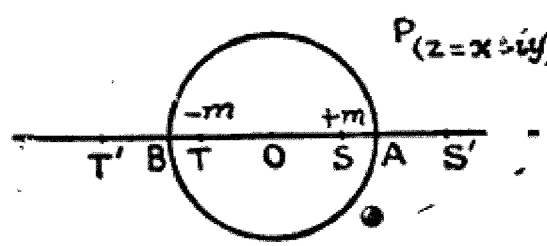
The velocity will be along the direction of the tangent to the boundary and is equal to the velocity of slip as the boundary of the circular disc is a stream line.

Ex. 5. *A source S and a sink T of equal strengths m are situated with in the space bounded by a circle whose centre is O. If S and T are at equal distances from O on opposite sides of it and on the same diameter AOB, Shew that the velocity of the liquid at any point P is*

$$2m.\frac{OS^2+OA^2}{OS}.\ \frac{PA\ PB}{PSPS'.PTPT'},$$

where S′ and T′ are the inverse points of S and T with regard to the circle.

Consider $OS=OT=f$ let S' and T' be the inverse points of S and T with regard to the circle, such that



$$OS.OS'=OT.OT'=a'$$

or $$OS'=OT'=\frac{a^2}{f}$$

Since a source of strength $+m$ and a sink of strength $-m$ are placed at the points S and T with in the circle. Then the image system consists of,

(i) *A* source of strength $+m$ at $S\ (f, o)$

(ii) *A* source of strength $+m$ at $S'\left(\dfrac{a^2}{f}, o\right)$

(iii) *A* sink of strength $-m$ at the origin O.

(iv) *A* sink of strength $-m$ at $T(-f, o)$

(v) *A* sink of strength $-m$ at $T'\left(-\frac{a^2}{f}, o\right)$

(vi) *A* source of strength $+m$ at the origin O.

The source and sink of same strength cancel each other at the origin. The complex potential w is given by

$$w=-m\log(z-f)-m\log\left(z-\frac{a^2}{f}\right)$$
$$+m\log(z+f)+m\log\left(z+\frac{a^2}{f}\right)$$

Differentiating w. r. to z, we have

$$\frac{dw}{dz}=-m.\frac{1}{z-f}-m.\frac{1}{z-\frac{a^2}{f}}+m\frac{1}{z+f}+m\frac{1}{z+\frac{a^2}{f}}$$

The velocity at any point P, is given by

$$q=\left|\frac{dw}{dz}\right|$$

$$q=m\left|-\frac{1}{z-f}-\frac{1}{z-\frac{a^2}{f}}+\frac{1}{z+f}+\frac{1}{z+\frac{a^2}{f}}\right|$$

$$q=m\left|-\frac{2f}{z-\ {}^{2}}-\frac{2\frac{a^2}{f}}{z^2-\frac{a^4}{f^2}}\right|=2m.\left|\frac{f}{z^2-f^2}+\frac{\frac{a^2}{f}}{z^2-\frac{a^4}{f^2}}\right|$$

$$q=2m.\frac{f^2+a^2}{f}\left|\frac{z^2-a^2}{\left(z^2-f^2\right)\left(z^2-\frac{a^4}{f^2}\right)}\right|$$

or $$q=2m.\frac{f^2+a^2}{f}\left|\frac{(z-a)(z+a)}{(z-f)(z+f)\left(z-\frac{a^2}{f}\right)\left(z+\frac{a^2}{f}\right)}\right|$$

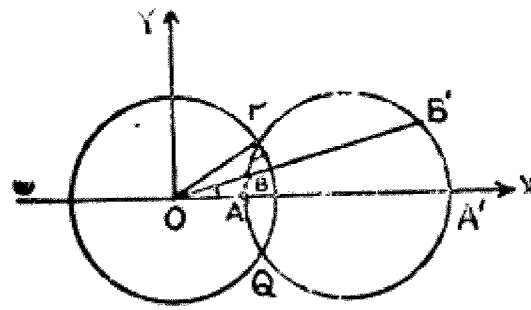
or $$q=2m.\frac{f^2+a^2}{f}\cdot\frac{|z-a||z+a|}{|z-f||z+f|\left|z-\frac{a^2}{f}\right||z+/\ |}$$

or $$q=2m.\frac{OS^2+OA^2}{OS}\cdot\frac{PA\cdot PB}{PS\ PS'.PT\ PT'}$$ **Proved.**

Ex. 6. *Find the velocity potential when there is a source and an equal sink inside a circular cavity and show that one of the stream lines is an arc of the circle which passes through the source and sink and cuts orthogonally the boundary of the cavity.*

Consider the source of strength $+m$ and a sink of strength $-m$ be placed at the point A and B with in the circular cavity with centre at origin.

The dynamical configuration does n t alter by placing a source of strength $+m$ and a sink of strength $-m$ at O.



Now a source of strength $+m$ at A and a sink of strength $-m$ at O will give rise to a source of strength m at an inverse point A'

Again a sink of strength $-m$ at B and a source of strength $+m$ at the origin O will give rise to a sink of strength m at an inverse point B'. Thus the image system consists of

(i) A source of strength $+m$ at $A\,(c, 0)$

(ii) A source of strength $+m$ at $A'\left(\frac{a^2}{c}, 0\right)$

(iii) A sink of strength $-m$ at $B\,(b, \alpha)$ i.e. $z=be^{\alpha i}$
$\left\{\begin{array}{l}\text{Here } OB=b \\ \text{and } \angle BoA=\alpha\end{array}\right.$

(iv) A sink of strength $-m$ at $B'\left(\frac{a^2}{b}, \alpha\right)$

i.e. at $z=\frac{a^2}{b}\,e^{\alpha i}$

{Here B' is an inverse pt. of B with regard to O}

Hence the complex potential w is given by

$$w=-m\log(z-c)-m\log\left(z-\frac{a^2}{c}\right)$$
$$+m\log(z-be^{\alpha i})+m\log\left(z-\frac{a^2}{b}\,e^{\alpha i}\right)$$

$$w=m\log\left[\frac{\left(z-be^{\alpha i}\right)\left(z-\frac{a^2}{b}e^{\alpha i}\right)}{\left(z-c\right)\left(z-\frac{a^2}{c}\right)}\right]$$

Equating real and imaginary parts we get the velocity potential and stream function, as follows

$$\phi+i\psi=m\log\left\{\frac{c}{b}\cdot\frac{(z-be^{\alpha i})(bz-a^2e^{\alpha i})}{(z-c)\,(c\,z-a^2)}\right\}.$$

Since the points A, A', B and B' are concyclic points. as $OA.OA'=OB.OB'=a^2$ thus the circle through these four points meet the original circle in P and Q.

Since $OA.OA'=a^2=OP^2$ (Being radius of the circle)

Therefore OP is a tangent at P to the circle through P and Q, hence two circles cuts orthogonally. The **circle A′B′PQ passes through A and B ; hence it must be a stream line.**

We can represent in other way also that the stream lines are circles for a source and an equal sink, so the circle through the above four points is a stream line and P and Q are known as stagnation points.

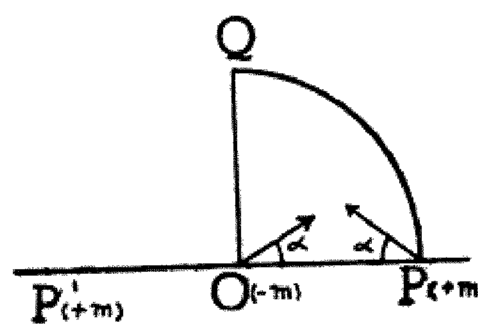
Ex. 7. *In a region by a fixed quadrantal arc and its radii, deduce the motion due to a source and an equal sink situated at the ends of one of the bounding radii, show that the stream line leaving either end at an angle α with the radius is*

$$r^2 \sin(\theta+\alpha)=a^2 \sin(\alpha-\theta)$$

Consider a source of strength $+m$ at P'. The image system consists of

(1) A source of strength $(+m)$ at P

(2) A source of strength $(+m)$ at P'



(3) A sink of strength $(-m)$ at O

The complex potential w for the motion of the fluid element at any point z is given by

$$w=-m\log(z-a)-m\log(z+a)+m\log z$$

or $$w=-m\log\left(\frac{z^2-a^2}{z}\right)$$

or $$z-\frac{a^2}{z}=e^{-\frac{1}{m}(\theta+i\psi)} \qquad \left\{ \text{Since } z=re^{\theta i}\right.$$

or $$(r\cos\theta+i\sin\theta)-\frac{a^2}{r}\left(\cos\theta-i\sin\theta\right)$$

$$=e^{-\phi/m}\left(\cos\frac{\psi}{m}-i\sin\frac{\psi}{m}\right)$$

Seperating into real and imaginary parts, we have

$$\left(r-\frac{a^2}{r}\right)\cos\theta=e^{-\phi/m}\cos\frac{\psi}{m}$$

and $$\left(r+\frac{a^2}{r}\right)\sin\theta=-e^{-\phi/m}\sin\frac{\psi}{m}$$

or $$\tan\frac{\psi}{m}=-\frac{r^2+a^2}{r^2-a^2}\tan\theta$$

At $r=a$ or $\theta=\pi/2$; $\dfrac{\psi}{m}=-\pi/2$

and $\theta=0$ gives $\psi=0$.

Thus OP is the stream line $\psi=0$, OQ and the arc PQ are the stream line $\psi=-\dfrac{\pi m}{2}$.

Hence the above relation determines the motion within the quadrant.

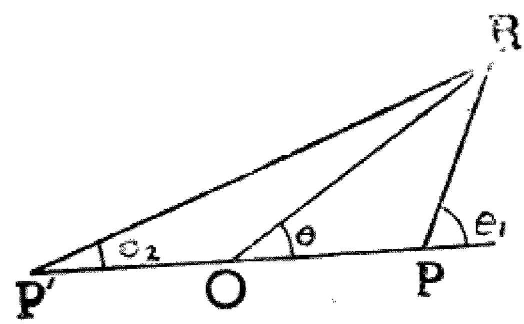
Again $\psi=-m\ (\theta_1+\theta_2-\theta)$

For a point very near to O, we have

$\theta_2=0$, $\theta_1=\pi$ and $\theta=\alpha$

then $\psi=-m\ (\pi-\alpha)$

i.e. for the stream line which leaves the point O making an angle α with OP



$$\psi=-m\ (\pi-\alpha)$$

For the stream line which leaves P making an angle α with PO,

$$\theta_1=\pi-\alpha,\quad \theta_2=0 \text{ and } \theta=0$$

Thus $\psi=-m\ (\pi-\alpha)$

So $\psi=-m\ (\pi-\alpha)$ gives the stream lines which make angle α at O and at P

So $$\tan(\pi-\alpha)=\frac{r^2+a^2}{r^2-a^2}\tan\theta$$

or $$-\tan\alpha=\frac{r^2+a^2}{r^2-a^2}\tan\theta$$

or $$(r^2+a^2)\sin\theta\cos\alpha=-(r^2-a^2)\sin\alpha\cos\theta$$

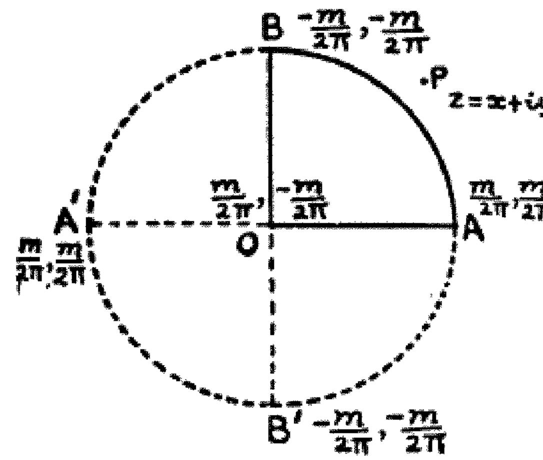
or $$r^2\sin(\alpha+\theta)=a^2\sin(\alpha-\theta)$$

Ex. 8. *In the parts of an infinite plane bounded by a circular quadrant AB and the productions of the radii OA and OB, there is a two-dimensional motion due to the production of liquid at A, and its absorption at B, at the uniform rate m. Find the velocity potential of the motion ; and shew that the fluid which issues from*

A in the direction making an angle μ with OA follows the path whose polar equation is

$$r=a \sin^{1/2} 2\theta \, [\cot \mu+\sqrt{\{\cot^2 \mu+\operatorname{cosec}^2 2\theta\}}]^{1/2}$$

the positive sign being taken for all the square-roots.



Since there is a production of liquid at the point A of the quadrant. Then the image system of the source at A with regard to the circular boundary consists of

(i) A source of strength $+\dfrac{m}{2\pi}$ at A.

(ii) A source of strength $+\dfrac{m}{2\pi}$ at A (since the point A is an inverse point of itself).

(iii) A sink of strength $-\dfrac{m}{2\pi}$ at the origin O.

The image system of the source at A with regard to the circular boundary and the radii OA and OB will give rise to

(i) Two sources of equal strength $+\dfrac{m}{2\pi}$ *at A i.e. a source of strength* $+\dfrac{m}{\pi}$ at A.

(ii) Two sources of strength $+\dfrac{m}{2\pi}$ *each at A' i.e. a source of strength* $+\dfrac{m}{\pi}$ *at A' (This is the image of the source at A).*

(iii) A sink of strength $-\dfrac{m}{2\pi}$ *at the origin O.*

Similarly there is an absorption of liquid at the point B of the quadrant. Then the image system of the sink at B with regard to the circular boundary consists of

(i) A sink of strength $-\dfrac{m}{2\pi}$ at B

(ii) A sink of strength $-\dfrac{m}{2\pi}$ at B (since the point B is an inverse point of itself).

(iii) A source of strength $+\frac{m}{2\pi}$ at the origin O.

Then the image system of the sink at B with regard to the circular boundary and the radii OA and OB will give rise to

(i) Two sinks of equal strength $-\frac{m}{2\pi}$ *at* B *i.e. a sink of strength* $-\frac{m}{\pi}$ *at* B.

(ii) Two sinks of equal strength $-\frac{m}{2\pi}$ *at* B' *i.e. a sink of strength* $-\frac{m}{\pi}$ *at* B' *(This is the image of the sink at B).*

(iii) A source of strength $+\frac{m}{2\pi}$ *at the origin* O.

Now a source of strength $+\frac{m}{2\pi}$ and a sink of strength $-\frac{m}{2\pi}$ at the origin O neutralize each other. Hence the image system consists of

(i) A source of strength $+\frac{m}{\pi}$ at A.

(at a distance $z=a$ *i.e.* on real axis).

(ii) A source of strength $+\frac{m}{\pi}$ at A'

(at a distance $z=-a$ *i.e.* on real axis).

(iii) A sink of strength $-\frac{m}{\pi}$ at B.

(at a distance $z=ai$ that is on imaginary axis).

(iv) A sink o strength $-\frac{m}{\pi}$ at B.

(at a distance $z=-ai$ *i.e.* on imaginary axis).

Let P $(z=x+iy)$ be any point in the fluid. The complex potential w at P due to the above system is

$$w=-\frac{m}{\pi}\log(z-a)-\frac{m}{\pi}\log(z+a)+\frac{m}{\pi}\log(z-ai)+\frac{m}{\pi}\log(z+ai)$$

or $$w=-\frac{m}{\pi}\log(z^2-a^2)+\frac{m}{\pi}\log(z^2+a^2). \qquad ...(1)$$

To determine the velocity potential of the motion, equating the real part from both the sides,

$$\phi=-\frac{m}{\pi}\log|z-a|-\frac{m}{\pi}\log|z+a|+\frac{m}{\pi}\log|z-ai|$$
$$+\frac{m}{\pi}\log|z+ai|$$

$$\phi=-\frac{m}{\pi}\log AP-\frac{m}{\pi}\log A'P+\frac{m}{\pi}\log BP+\frac{m}{\pi}\log B'P$$

$$\phi=\frac{m}{\pi}[\log BP.B'P-\log AP.A'P]$$

$$\phi=\frac{m}{\pi}\log\left(\frac{BP.B'P}{AP.A'P}\right)$$ **Answer.**

Since $P\,(z=x+iy)$ be any point in the fluid, substituting $z=re^{\theta i}$ in (1), we have

or $$\phi+i\psi=-\frac{m}{\pi}\log(r^2e^{2\theta i}-a^2)+\frac{m}{\pi}\log(r^2e^{2\theta i}+a^2)$$

or $$\phi+i\psi=-\frac{m}{\pi}\log\{(r^2\cos 2\theta-a^2)+ir^2\sin 2\theta\}$$
$$+\frac{m}{\pi}\log\{(r^2\cos 2\theta+a^2)+ir^2\sin 2\theta)\}$$

or $$\psi=-\frac{m}{\pi}\tan^{-1}\frac{r^2\sin 2\theta}{r^2\cos 2\theta-a^2}+\frac{m}{\pi}\tan^{-1}\frac{r^2\sin 2\theta}{r^2\cos 2\theta+a^2}$$

or $$=\frac{m}{\pi}\tan^{-1}\left[\frac{\dfrac{r^2\sin 2\theta}{r^2\cos 2\theta+a^2}-\dfrac{r^2\sin 2\theta}{r^2\cos 2\theta-a^2}}{1+\dfrac{r^2\sin 2\theta}{r^2\cos 2\theta+a^2}\cdot\dfrac{r^2\sin 2\theta}{r^2\cos 2\theta-a^2}}\right]$$

$$\left\{\text{Since } \log(x+iy)=\tfrac{1}{2}\log(x^2+y^2)+i\tan^{-1}\frac{y}{x}\right.$$

or $$\phi=-\frac{m}{\pi}\tan^{-1}\left(\frac{2a^2r^2\sin 2\theta}{r^4-a^4}\right) \quad ...(2)$$

Since the stream line that leaves A at an inclination μ with OA, is

$$\psi=-\frac{m}{\pi}\mu$$

From (2), we have

$$-\frac{m}{\pi}\tan^{-1}\frac{2a^2r^2\sin 2\theta}{r^4-a^4}=-\frac{m}{\pi}\mu$$

or $r^4-2a^2r^2\sin 2\theta\cot\mu-a^4=0.$

or $r^2=a^2\sin 2\theta\cot\mu\pm\sqrt{a^4\sin^2 2\theta\cot^2\mu+a^4}).$

Negleting the negative sign before the radical because it gives the relation $r^2 < 0_2$.

So $r=a(\sin 2\theta)^{1/2}[\cot\mu+\sqrt{\{(\cot^2\mu+\text{coses}^2 2\theta)\}}]^{1/2}$. **Proved.**

Ex. 9. *Within a rigid boundary in the form of the circle*

$$(x+\alpha)^2+(y-4\alpha)^2=8\alpha^2$$

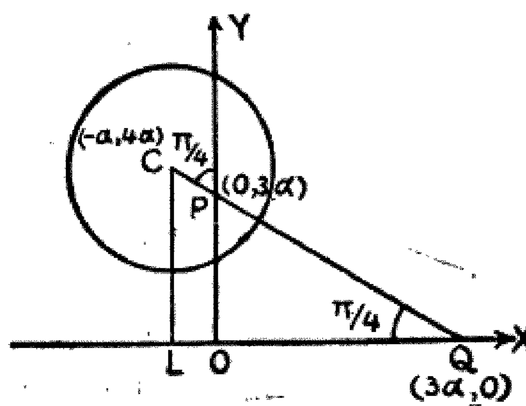
there is liquid motion due to a doublet of strength μ *at the point* $(0, 3\alpha)$ *with its axis along the axis of* Y. *Show that the velocity potential is*

$$\mu\left\{4.\frac{x-3\alpha}{(x-3\alpha)^2+y^2}+\frac{y-3\alpha}{x^2+(y-3\alpha)^2}\right\}$$

Equation to the rigid boundary is given by,

$$(x+\alpha)^2+(y-4\alpha)^2=8\alpha^2$$

whose centre is $(-\alpha, 4\alpha)$ and radius $2\sqrt{2}\,\alpha$.



The inverse point of $P(0, 3\alpha)$ with regard to the circle of centre C is $Q(3\alpha, 0)$.

The distance between two points C and Q

$$CQ=\sqrt{\{(-\alpha-3\alpha)^2+(4\alpha-0)^2}$$
$$=4\sqrt{2}\alpha.$$

So $CP.CQ=8\alpha^2$. (By the def. of the inverse points)

or $CP.(CP+PQ)=8\alpha^2$ $\quad\left\{\text{Since } CP=\sqrt{\{(-\alpha-0)^2+(4\alpha-3\alpha)^2\}}=\alpha\sqrt{2}.\right.$

or $\alpha\sqrt{2}.(\alpha\sqrt{2}+PQ)=8\alpha^2$

or $PQ=3\sqrt{2}\alpha$.

or $PQ=3\alpha\sec\dfrac{\pi}{4}$.

The point Q lies on the X-axis whose coordinates are $(3\alpha, 0)$.

Again gradiant of the line PC, $\tan\psi=\dfrac{3\alpha-4\alpha}{0+\alpha}=-1$

or $$\tan\psi=\tan\frac{3\pi}{4}$$

or $$\psi=\frac{3\pi}{4}.$$

which shows that the line PC makes an angle $\pi/4$ with the vertical axis OY.

Now the image of the doublet of strength μ at P with regard to a circle is a doublet of strength μ' (let) at an inverse point Q. The axis of Q will make an angle $\pi/4$ with PQ and will therefore be parallel to X-axis.

Then μ'=Strength of the doublet at Q.

$$=\mu\,\frac{(\text{Radius})^2}{(CP)^2}$$

or $$\mu'=\mu.\frac{8\alpha^2}{2\alpha^2}=4\mu \qquad ...(2)$$
{ from (1)

The complex potential function is

$$w=\frac{\mu e^{\frac{\pi}{2}i}}{z-3i\alpha}+\frac{\mu' e^{0i}}{z-3\alpha}$$

or $$w=\frac{\mu e^{\frac{\pi}{2}i}}{z-3i\alpha}+\frac{4\mu e^{0i}}{z-3\alpha} \qquad \{\text{From (2)}$$

or $$\phi+i\psi=\mu\left[i\,\frac{1}{x+i\,(y-3\alpha)}+\frac{4}{(x-3\alpha)+iy}\right]$$

or $$\phi+i\psi=\mu\left[\frac{i\,\{x-i\,(y-3\alpha)\}}{x^2+(y-3\alpha)^2}+\frac{4\,\{(x-3\alpha)-iy\}}{(x-3\alpha)^2+y^2}\right].$$

By equating the real part both the sides, we have the velocity potential as

$$\phi=\mu\left[\frac{(y-3\alpha)}{x^2+(y-3\alpha)^2}+\frac{4\,(x-3\alpha)}{(x-3\alpha)^2+y^2}\right]$$

ov $$\phi=\mu\left[4\,\frac{(x-3\alpha)}{(x-3\alpha)^2+y^2}+\frac{y-3\alpha}{x^2+(y-3\alpha)^2}\right].$$ **Proved.**

Ex. 10. *Shew that the velocity potential*

$$\phi=\tfrac{1}{2}\,log\,\frac{(x+a)^2+y^2}{(x-a)^2+y^2}$$

gives a possible motion. Determine the form of stream lines and curves of equal speed.

Since we have

$$\phi=\tfrac{1}{2}\log\frac{(x+a)^2+y^2}{(x-a)^2+y^2}$$

or $$\frac{\partial\phi}{\partial x}=\frac{x+a}{(x+a)^2+y^2}-\frac{x-a}{(x-a)^2+y^2} \qquad ...(A)$$

and $$\frac{\partial\phi}{\partial y}=\frac{y}{(x+a)^2+y^2}-\frac{y}{(x-a)^2+y^2} \qquad (B)$$

Again $$\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}\left\{-\frac{\partial\phi}{\partial x}\right\} \qquad \left\{\text{as } u=-\frac{\partial\phi}{\partial x}\right.$$

$$=\frac{\partial}{\partial x}\left\{\frac{x-a}{(x-a)^2+y^2}-\frac{x+a}{(x+a)^2+y^2}\right\}$$

$$=\frac{y^2-(x-a)^2}{[(x-a)^2+y^2]^2}-\frac{y^2-(x+a)^2}{[(x+a)^2+y^2]^2}$$

and $$\frac{\partial v}{\partial y}=\frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial y}\right) \qquad \left\{\text{as } v=-\frac{\partial \phi}{\partial y}\right.$$

$$=\frac{\partial}{\partial y}\left\{\frac{y}{(x-a)^2+y^2}-\frac{y}{(x+a)^2+y^2}\right\}$$

$$=\frac{(x-a)^2-y^2}{[(x-a)^2+y^2]^2}-\frac{(x+a)^2-y^2}{[(x+a)^2+y^2]^2}.$$

Thus $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$. *i.e.* equation of continuity in two dimensions is satisfied. Hence $\phi=\frac{1}{2}\log\frac{(x+a)^2+y^2}{(x-a)^2+y^2}$ gives a possible motion.

Now for stream lines, we have

$$\frac{\partial \phi}{\partial x}=\frac{\partial \psi}{\partial y} \text{ and } \frac{\partial \phi}{\partial y}=-\frac{\partial \psi}{\partial x} \qquad ...(1)$$

(property of conjugate functions).

or $$\frac{\partial \psi}{\partial y}=\frac{x+a}{(x+a)^2+y^2}-\frac{x-a}{(x-a)^2+y^2}.$$

By integrating with regard to y, we have

$$\psi=\tan^{-1}\frac{y}{x+a}-\tan^{-1}\frac{y}{x-a}+f(x) \qquad ...(2)$$

Now we shall determine $f(x)$.

Differentiating (2) *w. r.* to x, we have

$$\frac{\partial \psi}{\partial x}=-\frac{y}{(x+a)^2+y^2}+\frac{y}{(x-a)^2+y^2}+f'(x)$$

$$\therefore \quad \frac{\partial \psi}{\partial x}=-\frac{\partial \phi}{\partial y} \qquad \left\{\text{from (B)}\right.$$

So $f'(x)=0$

or $f(x)=$Const., which can be neglected.

or $$\psi=\tan^{-1}\frac{y}{x+a}-\tan^{-1}\frac{y}{x-a}$$

or $$\psi=\tan^{-1}\left[\frac{\frac{y}{x+a}-\frac{y}{x-a}}{1+\frac{y^2}{x^2-a^2}}\right]$$

or $$\psi=\tan^{-1}\left\{\frac{-2ay}{x^2+y^2-a^2}\right\}.$$

The lines of flow can be obtained by $\psi=$Const.

or $$\tan^{-1}\left\{-\frac{2ay}{x^2+y^2-a^2}\right\}=\text{Const.}$$

or $$\frac{2ay}{a^2-x^2-y^2}=\tan k=D=\text{Const.}$$

If $D=0$ then the stream line is $y=0$ *i. e.* real axis.

If $D=\infty$ then the stream line is $x^2+y^2=a^2$ *i, e.* a circle.

Answer.

Again $w=\phi+i\psi$

$$=\tfrac{1}{2}\log\{(x+a)^2+y^2\}-\tfrac{1}{2}\log\{(x-a)^2+y^2\})$$
$$+i\tan^{-1}\frac{y}{x+a}-i\tan^{-1}\frac{y}{x-a}$$

{from the given function.

or $$w=\left[\tfrac{1}{2}\log\{(x+a)^2+y^2\}+i\tan^{-1}\frac{y}{x+a}\right]$$
$$-\left[\tfrac{1}{2}\log\{(x-a)^2+y^2\}+\tan^{-1}\frac{y}{x+a}\right]$$

or $$w=\log\{(x+a)+iy\}-\log\{(x-a)+iy\}$$

or $$w=\log(z+a)-\log(z-a)$$ {where $z=x+iy$

So $$q=\left|\frac{dw}{dz}\right|=\left|\frac{1}{z+a}-\frac{1}{z-a}\right|=\frac{2a}{|z+a|\,|z-a|}$$

or $$q=\frac{2a}{rr'}\quad\text{(Speed)}$$ [where $|z+a|=r'$, $|z-a|=r$; r' and r are the distances from $(-a, 0)$ and $(a, 0)$

Now curves of equal speeds are given by

$$\frac{2a}{rr'}=\text{Const.}\qquad\text{or}\qquad rr'=\text{Const.}$$

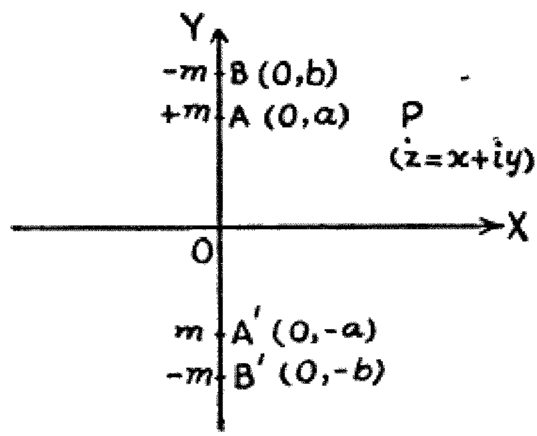
which are **Cassini Ovals.**

Ex. 11. *If the fluid fill the region of spaces on the positive side of X-axis, which is a rigid boundary, and if there be a source m at the point (0, a) and an equal sink at (0, b), and if the pressure on the negative side of the boundary, be the same as the pressure of the fluid at infinity. Shew that the resultant pressure on the boundary is*

$$\pi\rho m^2.\frac{(a-b)^2}{ab\,(a+b)},$$

where ρ is the density of the fluid.

Since a source of strength $+m$ is placed at the point $(0, a)$ *i. e.* on the positive side of the Y axis and a sink of strength $-m$ at the point $(0, b)$ on the same side. Hence the image system consists of



(i) A source of strength $+m$ at the point A $(0, a)$ *i. e.* at distance $z=ai$ from O.

(ii) A source of strength $+m$ at the other side of the surface at A' $(0, -a)$ *i. e.* at a distance $z=-ai$ from O.

(iii) A sink of strength $-m$ at the point B $(0, b)$ i. e. at a distance $z=bi$ from O.

(iv) A sink of strength $-m$ at the other side of the surface at B' $(0, -b)$ *i. e.* at a distance $z=-bi$ from O.

Thus the complex potentials is given by

$$w=-m \log (z-ai) -m \log (z+ai)+m \log (z-bi) +m \log (z+bi)$$

$$w=-m \log (z^2+a^2)+m \log (z^2+b^2).$$

or
$$\frac{dw}{dz}=-m.\frac{2z}{z^2+a^2}+m.\frac{2z}{z^2+b^2}$$

or
$$q=\left|\frac{dw}{dz}\right|=2m\left|-\frac{z}{z^2+a^2}+\frac{z}{z^2+b^2}\right|$$

$$=2m\left|\frac{z(a^2-b^2)}{(z^2+a^2)(z^2+b^2)}\right| \quad ...(i)$$

Consider a point R $(x, 0)$ on the X axis. Let q be the velocity at the point R, then substituting $z=x$ in (i), we have

$$q=2m(a^2-b^2)\left|\frac{x}{(x^2+a^2)(x^2+b^2)}\right|$$

$$=2m(a^2-b^2)\frac{x}{(x^2+a^2)(x^2+b^2)}.$$

By Bernoulli's theorem, the pressure p at any point is

$$\frac{p}{\rho}=C-\tfrac{1}{2}q^2 \quad \text{(where } C \text{ is a constant).}$$

Let $p_\infty=p_0$, $q=0$ then $C=\frac{p}{\rho}$

or
$$\frac{p}{\rho}=\frac{p_0}{\rho}-\tfrac{1}{2}q^2$$

or $$\frac{p_0-p}{\rho}=\tfrac{1}{2}q^2.$$

Thus the resultant pressure on the boundary is

$$P=\int_{-\infty}^{\infty}(p_0-p)\,dx$$

$$P=\tfrac{1}{2}\rho\int_{-\infty}^{\infty}q^2\,dx$$

$$P=\tfrac{1}{2}\rho\,.\,4m^2\,(a^2-b^2)^2\int_{-\infty}^{\infty}\frac{x^2\,dx}{(x^2+a^2)^2\,(x^2+b^2)^2}$$

$$P=4m^2\,(a^2-b^2)^2\,\rho\int_{0}^{\infty}\frac{x^2\,dx}{(x^2+a^2)^2\,(x^2+b^2)^2}$$

$$P=4m^2\rho\int_0^{\infty}\left[\frac{a^2+b^2}{b^2-a^2}\left\{\frac{1}{x^2+a^2}-\frac{1}{x^2+b^2}\right\}-\frac{a^2}{(x^2+a^2)^2}-\frac{b^2}{(x^2+b^2)^2}\right]dx$$

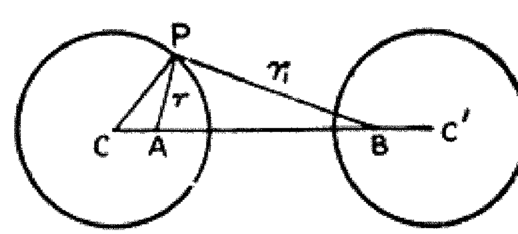
By integrating, we have

$$P=4m^2\rho\left[\frac{a^2+b^2}{b^2-a^2}\left(\frac{\pi}{2a}-\frac{\pi}{2b}\right)-\frac{\pi}{4a}-\frac{\pi}{4b}\right]$$

$$=\pi\rho m^2\,\frac{(a-b)^2}{ab\,(a+b)}.$$

Ex. 12. *Prove that for liquid circulating irrotationally in part of the plane between two non-intersecting circles the curves of constant velocity are Cassini's Ovals.*

Let the line of centres of the circles be CC', also A and B be the inverse points with regard to both the circles.



Consider a point P on one of the circles, such that

$$PA=r \quad\text{and}\quad PB=r_1$$

Since $$CA.CB=CP^2$$

The triangles CPA and CPB are similar

So $$\frac{PA}{PB}=\frac{CP}{CB}=\text{constant}.$$

i.e. $$\frac{r}{r_1}=\text{constant}.$$

Thus the equations of two circles are

$$\frac{r}{r_1}=C_1 \quad\text{and}\quad \frac{r}{r_1}=C_2 \qquad \text{(let)}$$

Since these two circles are the two stream lines, thus the stream function ψ must be of the form $f\left(\frac{r}{r_1}\right)$, but $f\left(\frac{r}{r_1}\right)$ is a plane harmonic.

then $\quad \psi = A \log \frac{r}{r_1} \quad \left\{ \text{because } \log r \text{ is the only function of } r \text{ which is plane harmonic.} \right.$

Let θ is the conjugate harmonic for $\log r$ then ψ contains log function. So the complex potential must be of the form

$$w = \psi - i\phi$$

$$w = c \log \frac{r}{r_1} + ic\,(\theta - \theta_1) \quad \left\{ \text{The expression } \psi - i\phi \text{ is an analytic function of } z \text{ as } \phi \text{ is a conjugate harmonic of } \psi. \right.$$

or
$$w = c\,[\log r + i\theta] - c\,[\log r_1 + i\theta_1]$$

or
$$w = c \log \left(\frac{re^{\theta i}}{r_1 e^{\theta_1 i}}\right)$$

or
$$w = c \log \left(\frac{z+a}{z-a}\right) \quad \left\{ \text{Taking } A \text{ to be } (-a, 0) \text{ and } B \text{ as } (a, 0) \right.$$

or
$$q = \left|\frac{dw}{dz}\right| = c\left|\frac{1}{z+a} - \frac{1}{z-a}\right|$$

$$= 2ac\left|\frac{1}{(z+a)(z-a)}\right| = \frac{2ac}{|z+a|\,|z-a|}$$

$$= \frac{2ac}{rr_1}.$$

Hence the curves of equal velocities are given by

$$q = \text{constant}$$

or
$$\frac{2ac}{rr_1} = \text{constant}$$

or
$$rr_1 = \text{constant}$$

which is known as Cassini's Ovals. **Proved.**

§ 3·92. Conformal Representation.

Consider the two curves C_1, C_2 in the domain D intersect at the point $P\,(x_0\, y_0)$ at an angle α, and if the two corresponding curves Γ_1, Γ_2 in the domain D' intersect at $P'\,(u_0\, v_0)$ at the same angle α_2, the transformation is said to be isogonal.

If the sense of rotation as well as the magnitude of the angle is preserved then the transformation is said to be **conformal.**

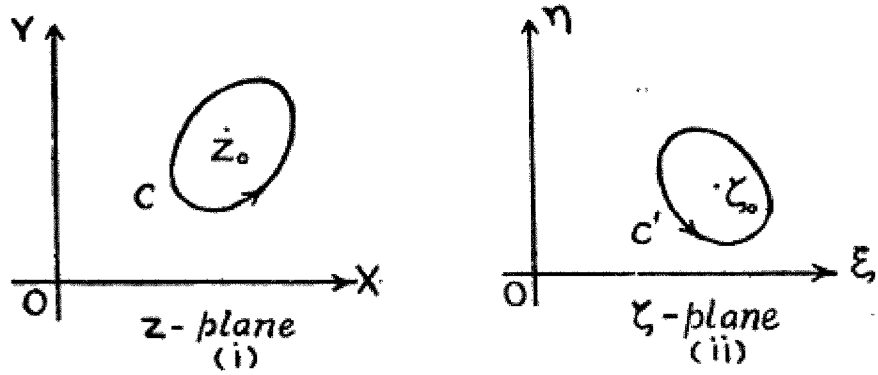
§ 3·93. Mapping.

Consider $f(z)$ be a function of $z\,(=x+iy)$, which is analytic inside and upon a simple closed contour C in the Z-plane.

Let the relation is

$$\zeta = f(z) \qquad \ldots(i)$$

where $\zeta = \xi + i\eta$.



With the help of the relation (i) with in or upon the curve in Z-plane there corresponds one point in the ζ-plane, since $f(z)$ being analytic and one valued. Thus the points of the curve in Z-plane and its interior are mapped into certain points in the ζ-plane.

§ 3·95. Application to Hydrodynamics.

Consider there is a two-dimensional source of strength $+m$ at the point $z = z_0$. Let the transformation be

$$\zeta = f(z)$$

and ζ_0 be the image of z_0. [Ref. Figure § 3·93]

Let C be the curve in the Z-plane surrounding the point z_0. Then C is mapped onto a closed curve C' in the ζ-plane. The complex pontential w, is given by

$$w = \phi + i\psi \qquad (Z\text{-plane})$$

$$w = \phi' + i\psi' \qquad (\zeta\text{-plane})$$

But at corresponding points $\psi = \psi'$

Then $$\int_C d\psi = \int_{C'} d\psi' \qquad \ldots(i)$$

Since $$w = -m \log (z - z_0) \qquad (\text{in } Z \text{ plane})$$

or $$dw = -m \cdot \frac{dz}{z - z_0}$$

or $$\int_C dw = -m \cdot \int_C \frac{dz}{z - z_0} \qquad \left\{ \begin{array}{l} \text{Since } w = \phi + i\psi \\ \quad dw = d\phi + i\,d\psi \end{array} \right.$$

or $$\int_C d\phi + i\,d\psi = -2\pi m i$$

or $$\int_C d\psi = -2\pi m$$

and $$\int_{C'} d\psi' = -2\pi m$$

If C' surrounds the point ζ_0 only once, then there is a source of strength $+m$ at the points ζ_0.

If C' sorrounds the point ζ_0 n times, then there is a source of strength $+\frac{m}{n}$ at the point ζ_0.

Ex. 13. *Use the method of images to prove that if there be a source m at the point z_0 in a fluid bounded by the lines $\theta=0$ and $\theta=\frac{\pi}{3}$, the solution is*

$$\phi+i\psi=-m \log \{(z^3-z_0^3)(z^3-z_0'^3)\}$$

where $z_0=x_0+iy_0$ *and* $z_0'=x_0-iy_0$.

Changing the motion from z-plane to t-plane by the transformation.

$$t=z^3 \quad \text{(let)} \qquad \ldots\text{(i)}$$

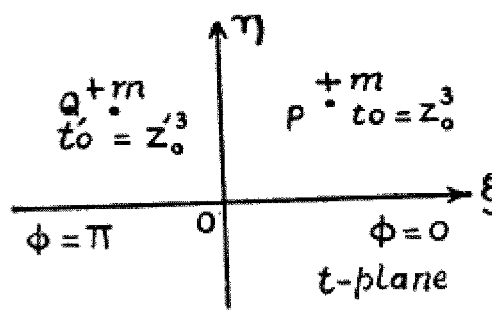
where $t=Re^{\phi i}$ and $z=re^{\theta i}$

or $Re^{\phi i}=r^3e^{3\theta i}$.

Equating the real and imaginary parts, we have

$$R=r^3 \quad \text{and} \quad \phi=3\theta$$

Thus the boundaries $\theta=0$ and $\theta=\frac{\pi}{3}$ in z-plane transforms to $\phi=0$ and $\phi=\pi$ in t-plane



Since a source of strength $+m$ is placed at the point z_0 in z-plane bounded by the line $\theta=0$ and $\theta=\pi/3$, which corresponds by transformation to the points $t_0=z_0^3$ bounded by the real axis $\phi=0$ and $\phi=\pi$ in t-plane. Then the image system consists of

(i) a source of strength $+m$ at $t_0=z_0^3$

(ii) a source of strength $+m$ at $t_0'=z_0'^3$.

Thus the complex potential is

$$w=-m \log (t-z_0^3)-m \log (t-z_0'^3)$$

or $$w=-m \log (z^3-z_0^3)-m \log (z^3-z_0'^3)$$

or $$\phi+i\psi=-m \log \{(z^3-z_0^3)(z^3-z_0'^3)\}.$$ **Proved.**

Ex. 14. *Between the fixed boundaries $\theta=\pi/4$ and $\theta=-\pi/4$, there is a two dimensional liquid motion due to a source of strength* *m at the point $r=a$, $\theta=0$ and an equal sink at the point $r=b$, $\theta=0$. Use the method of images to show that the stream function is*

$$-m\tan^{-1}\left\{\frac{r^4(a^4-b^4)\sin 4\theta}{r^8-r^4(a^4+b^4)\cos 4\theta+a^4b^4}\right\}$$

Shew also that the velocity at (r, θ) is

$$\frac{4m(a^4-b^4)r^3}{(r^8-2a^4r^4\cos^4\theta+a^4)^{1/2}(r^8-2b^4r^4\cos^4\theta+b^8)^{1/2}}$$

Let the transformation be

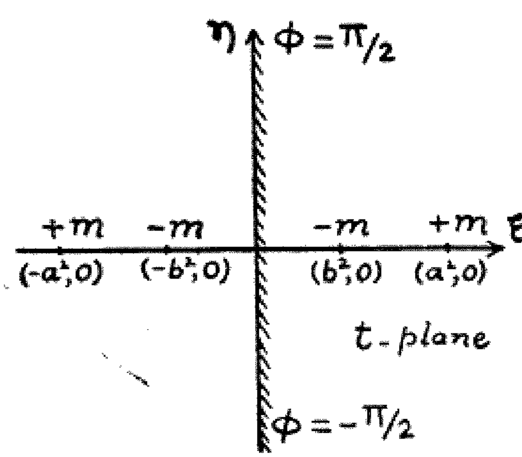
$$t=z^2 \qquad \text{where } t=Re^{\phi i} \text{ and } z=re^{\theta i}$$

or $$Re^{\phi i}=r^2e^{2\theta i}$$

which gives $$R=r^2 \text{ and } \phi=2\theta$$

Then the boundaries $\theta=\pi/4$ and $\theta=-\pi/4$ in Z-plane transformed to an imaginary axis $\phi=\pi/2$ and $\phi=-\pi/2$ in t-plane.

Since there is a source of strength $+m$ at $(r=a, \theta=0)$ and a sink of strength $-m$ at $(r=b, \theta=0)$ in Z-plane which corresponds by the transformation $(t=z^2)$ to a source of strength $+m$ at $(R=a^2, \phi=0)$ and a sink of strength $-m$ at $(R=b^2, \phi=0)$ in t-plane. Now the image system consists of



(i) *A* source of strength $+m$ at $(a^2, 0)$
(ii) *A* source of strength $+m$ at $(-a^2, 0)$
(iii) *A* sink of strength $-m$ at $(b^2, 0)$
(iv) *A* sink of strength $-m$ at $(-b^2, 0)$

Thus the complex potential is

$$w=-m\log(t-a^2)-m\log(t+a^2)+m\log(t-b^2)+m\log(t+b^2)$$

or $$w=-m\log(z^2-a^2)-m\log(z^2+a^2)+m\log(z^2-b^2)+m\log(z^2+b^2)$$

or $$w=-m\log(z^4-a^4)+m\log(z^4-b^4) \quad \text{.. (i)}$$

or $$\frac{dw}{dz}=-m\cdot\frac{4z^3}{z^4-a^4}+m\cdot\frac{4z^3}{z^4-b^4}$$

or $$\frac{dw}{dz}=-4m\left[\frac{z^3}{z^4-a^4}-\frac{z^3}{z^4-b^4}\right]$$

or $$\frac{dw}{dz}=-\frac{4m(a^4-b^4)z^3}{(z^4-a^4)(z^4-b^4)}$$

Then $q=\left|\frac{dw}{dz}\right|=4m\,(a^4-b^4)\left|\frac{z^3}{(z^4-a^4)\,(z^4-b^4)}\right|$ .. (ii)

Since $P\,(z=x+iy=re^{\theta i})$ be a point in the plane, then by substituting $z=re^{\theta i}$ in (ii), we can determine the velocity at the point (r, θ)

$$q=4m(a^4-b^4)\left|\frac{r^3\,e^{3\theta i}}{(r^4\,e^{4\theta i}-a^4)\,(r^4\,e^{4\theta i}-b^4)}\right|$$

$$q=4m(a^4-b^4)\left|\frac{r^3\,(\cos 3\theta+i\sin 3\theta)}{[(r^4\cos 4\theta-a^4)+i\,r^4\sin 4\theta]\,[(r^4\cos 4\theta-b^4)+i\,r^4\sin 4\theta]}\right|$$

$$q=\frac{4m\,(a^4-b^4)\,r^3}{(r^8-2a^4r^4\cos 4\theta+a^8)^{1/2}\,(r^8-2b^4r^4\cos 4\theta+b^8)^{1/2}}$$

Proved.

Again from (i), substituting $z=r\,e^{\theta i}$

$$\phi+i\psi=-m\log(r^4\,e^{4\theta i}-a^4)+m\log(r^4e^{\theta i}-b^4)$$

or $$\phi+i\psi=-m\log\{(r^4\cos 4\theta-a^4)+i\,r^4\sin 4\theta\}+m\log\{(r^4\cos 4\theta-b^4)+i\,r^4\sin 4\theta\}$$

Equating the imaginary part, we have

$$\psi=-m\tan^{-1}\frac{r^4\sin 4\theta}{r^4\cos 4\theta-a^4}+\tan^{-1}\frac{r^4\sin 4\theta}{r^4\cos 4\theta-b^4}$$

or $$\psi=-m\tan^{-1}\left\{\frac{\dfrac{r^4\sin 4\theta}{r^4\cos 4\theta-a^4}-\dfrac{r^4\sin 4\theta}{r^4\cos 4\theta-b^4}}{1+\dfrac{r^4\sin 4\theta}{r^4\cos 4\theta-a^4}\cdot\dfrac{r^4\sin 4\theta}{r^4\cos 4\theta-b^4}}\right\}$$

or $$\psi=-m\tan^{-1}\left\{\frac{r^4\,(a^4-b^4)\sin 4\theta}{r^8-r^4\,(a^4+b^4)\cos 4\theta+a^4\,b^4}\right\}$$ **Proved.**

Ex. 15. *Between the fixed boundaries $\theta=\pi/6$ and $\theta=-\pi/6$, there is a two dimensional liquid motion due to a source at the point $(r=c, \theta=\alpha)$ and a sink at the origin, absorbing water at the same rate as the source produces it. Find the stream function, and shew that one of the stream lines is a part of the curve*

$$r^3\sin 3\alpha=c^3\sin 3\theta.$$

Let the transformation be

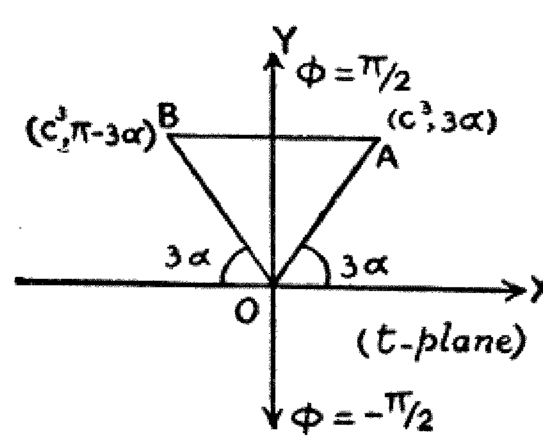
$$t=z^3$$

where $t=\rho\,e^{\phi i}$

and $z=re^{\theta i}$

or $\rho e^{\phi i}=r^3\,e^{3\theta i}$

So $\rho=r^3$ and $\phi=3\theta$



Now the boundaries $\theta=\pi/6$ and $\theta=-\pi/6$ in Z-plane transforms into $\phi=\pi/2$ and $\phi=-\pi/2$ *i.e.* an imaginary axis in t-plane

A source at the point $(r=c, \theta=\alpha)$ in Z-plane transforms to a source at the point $(\rho=c^3, \phi=3\alpha)$ in t-plane. Also a sink at the origin in Z-plane corresponds to a sink at the origin in t-plane. Let $+m$ be the strength of the source then strength of the sink will be $-m$ (as the rate of creation and absorption is same). The image system consist of

(i) The image of a source of strength $+m$ at the point $(c^3, 3\alpha)$ is an equal source of strength $+m$ at $\{c^3, (\pi-3\alpha)\}$.

(ii) The image of a sink of strength $-m$ at the origin O is a sink of strength $-m$ at the same point (*i.e.* at origin).

The complex potential is given by

$$w=2m \log t-m \log (t-c^3 e^{3\alpha i})-m \log (t-c^3 e^{(\pi-3\alpha)i})$$

$$w=2m \log t-m \log (t-c^3 e^{3\alpha i})-m \log (t+c^3 e^{-3\alpha i})$$

$$w=2m \log z^3-m \log (z^3-c^3 e^{3\alpha i})-m \log (z^3+c^3 e^{-3\alpha i})$$

Let $P\ (z=re^{\theta i})$ be a point in Z-plane, then

$$w=6m \log (re^{\theta i})-m \log (r^3 e^{3\theta i}-c^3 e^{3\alpha i})-m \log (r^3 e^{3\theta i}+c^3 e^{-3\alpha i})$$

or $$\phi+i\psi=6m \{\log r+\theta i\}-m \log (r^3 e^{3\theta i}-c^3 e^{3\alpha i})(r^3 e^{3\theta i}+c^3 e^{-3\alpha i})$$

or $$\phi+i\psi=6m (\log r+\theta i) -m \log \{(r^6 \cos 6\theta+2c^3 r^3 \sin 3\theta \sin 3\alpha-c^6) +i (r^6 \sin 6\theta-2c^3 r^3 \sin 3\alpha \cos 3\theta)\}$$

Equating the imaginary parts, we have

$$\psi=6m\theta-m \tan^{-1} \left\{\frac{r^6 \sin 6\theta-2c^3 r^3 \sin 3\alpha \cos 3\theta}{r^6 \cos 6\theta+2c^3 r^3 \sin 3\theta \sin 3\alpha-c^6}\right\}$$

Now one of the stream lines is when $\psi=0$, gives

$$6m\theta-m \tan^{-1}\left\{\frac{r^6 \sin 6\theta-2c^3 r^3 \sin 3\alpha \cos 3\theta}{r^6 \cos 6\theta+2c^3 r^3 \sin 3\theta \sin 3\alpha-c^6}\right\}=0.$$

or $$\tan 6\theta=\frac{r^6 \sin 6\theta-2c^3 r^3 \sin 3\alpha \cos 3\theta}{r^6 \cos 6\theta+2c^3 r^3 \sin 3\theta \sin 3\alpha-c^6}$$

or $$\frac{\sin 6\theta}{\cos 6\theta}=\frac{r^6 \sin 6\theta-2c^3 r^3 \sin 3\alpha \cos 3\theta}{r^6 \cos 6\theta+2c^3 r^3 \sin 3\theta \sin 3\alpha-c^6}$$

or $$r^6 \sin 6\theta \cos 6\theta+2c^3 r^3 \sin 3\theta \sin 3\alpha \sin 6\theta-c^6 \sin 6\theta =r^6 \sin 6\theta \cos 6\theta-2c^3 r^3 \sin 3\alpha \cos 3\theta \cos 6\theta$$

or $$c^6 \sin 6\theta=2r^3 c^3 \sin 3\alpha\, [\cos 6\theta \cos 3\theta+\sin 6\theta \sin 3\theta]$$

or $$c^6 \sin 6\theta=2r^3 c^3 \sin 3\alpha \cos (6\theta-3\theta)$$

or $$2c^6 \sin 3\theta \cos 3\theta=2r^3 c^3 \sin 3\alpha \cos 3\theta$$

or $$2c^3 \cos 3\theta\, [r^3 \sin 3\alpha-c^3 \sin 3\theta]=0.$$

either $\cos 3\theta=0$, gives $3\theta=\pm\pi/2$ or $\theta=\pm\pi/6$
which are the given boundaries.

or $\qquad r^3 \sin 3\alpha - c^3 \sin 3\theta = 0$

or $\qquad r^3 \sin 3\alpha = c^3 \sin 3\theta$

which is the required equation of curve. **Proved.**

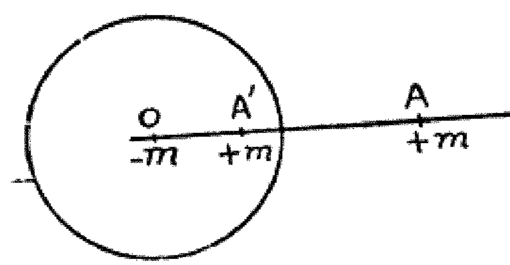
Ex. 16. *A source of fluid situated in space of two dimensions, is of such strength that $2\pi\rho m$ represents the mass of fluid of density ρ emitted per unit of time. Show that the force necessary to hold a circular disc at rest in the plane of the source is*

$$\frac{2\pi\rho m^2 a^2}{r(r^2-a^2)},$$

where a is the radius of the disc and r the distance of the source from its centre. In what direction is the disc urged by the pressure ?

Since $\quad OA=r.$

Let A' be an inverse point of A situated at a distance r from the centre with regard to the circular disc.



So $OA \cdot OA' = a^2$

or $\qquad OA' = \dfrac{a^2}{r}$

Since a source of strength $+m$ is situated at the point A, so the image system give rise to a source of strength $+m$ at an inverse point A' and a sink of strength $-m$ at the origin O.

The complex potential due to the above image system, is given by

$$w = -m \log(z-r) - m \log\left(z - \frac{a^2}{r}\right) + m \log z$$

or $$\frac{dw}{dz} = -m \cdot \frac{1}{z-r} - m \cdot \frac{1}{z - \dfrac{a^2}{r}} + m \cdot \frac{1}{z} \qquad \ldots(1)$$

The components of the forces are given by Blasius Theorem. If X and Y are the components parallel to the axes of the forces that act on the disc, then

$$X - iY = \tfrac{1}{2} i\rho \int_C \left(\frac{dw}{dz}\right)^2 dz$$

or $$X-iY=\tfrac{1}{2}i\rho\left\{2\pi i\times\text{Sum of the residues of }\left(\frac{dw}{dz}\right)^2\text{ with in the contour } i.\,e.\text{ the circular disc}\right\} \qquad \ldots(2)$$

Now from (1), we have

$$\left(\frac{dw}{dz}\right)^2=m^2\left[\frac{1}{(z-r)^2}+\frac{1}{\left(z-\frac{a^2}{r}\right)^2}+\frac{1}{z^2}+\frac{2}{(z-r)\left(z-\frac{a^2}{r}\right)}-\frac{2}{z(z-r)}-\frac{2}{z\left(\frac{a^2}{r}\right)}\right]$$

$$=m^2\left[\frac{1}{(z-r)^2}+\frac{1}{\left(z-\frac{a^2}{r}\right)^2}+\frac{1}{z^2}+\frac{2}{(z-r)\left(r-\frac{a^2}{r}\right)}+\frac{2}{\left(z-\frac{a^2}{r}\right)\left(\frac{a^2}{r}-r\right)}+\frac{2}{\frac{a^2}{r}z}-\frac{2}{\frac{a^2}{r}\left(z-\frac{a^2}{r}\right)}+\frac{2}{r(z-r)}+\frac{2}{zr}\right]$$

The poles inside the circular disc (contour) are

$$z=0 \text{ and } z=\frac{a^2}{r}.$$

So the sum of the residues can be obtained by equating the coefficient of $\frac{1}{z}$ and $\frac{1}{z-\frac{a^2}{r}}$.

Now residue at $z=0$ *i. e.* equating the coeff. of $\frac{1}{z}$

$$=m^2\left[\frac{2}{r}+\frac{2}{a^2/r}\right]=2m^2\left[\frac{1}{r}+\frac{r}{a^2}\right]$$

and residue at $z=\frac{a^2}{r}$ can be obtained by equating the coefficients of $\frac{1}{z-\frac{a^2}{r}}$

$$=m^2\left[\frac{2}{\frac{a^2}{r}-r}-\frac{2}{\frac{a^2}{r}}\right]$$

$$=m^2\left[\frac{2r}{a^2-r^2}-\frac{2r}{a^2}\right].$$

Sum of the residues

$$=2m^2\left[\frac{1}{r}+\frac{r}{a^2}+\frac{r}{a^2-r^2}-\frac{r}{a^2}\right]$$

$$=2m^2\left[\frac{a^2-r^2+r^2}{r\,(a^2-r^2)}\right]=-\frac{2m^2\,a^2}{r\,(r^2-a^2)}$$

Then from (2) we have

$$X-iY=\tfrac{1}{2}i\rho\,.2\pi i.\left[-\frac{2m^2a^2}{r\,(r^2-a^2)}\right]$$

or
$$X-iY=\frac{2\pi\rho m^2a^2}{r\,(r^2-a^2)}$$

Hence
$$X=\frac{2\pi\rho m^2a^2}{r\,(r^2-a^2)} \text{ and } Y=0.$$

Thus the circular disc is attracted towards the source and the pressure will be greater on the opposite side of the disc to that of the source. **Proved.**

Ex. 17. *Within a circular boundary of radius a there is a two dimensional liquid motion due to a source producing liquid at the rate m, at a distance f from the centre and an equal sink at the centre. Find the velocity potential, and show that the resultant of the pressure on the boundary is*

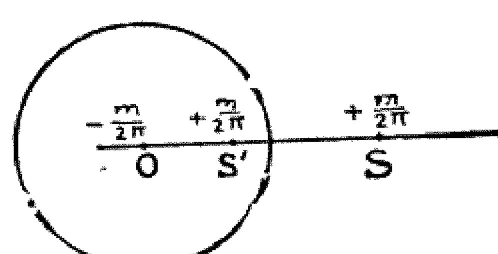
$$\frac{\rho m^2 f^3}{2a^2\pi(a^2-f^2)}.$$

where ρ is the density. Deduce, as a limit, the velocity potential due to the doublet at a centre.

Let S be an inverse point of S' with regard to the circular boundary, such that

or $OS.OS'=a^2$

or $OS=\dfrac{a^2}{f}$ $\Big\{$as $OS'=f$



Since the rate of emission of the liquid is m, then the strength of the source is $\dfrac{m}{2\pi}$.

The image of the source of strength $\dfrac{m}{2\pi}$ at S' give rise to an equal source at an inverse point S and an equal sink at the origin O. Also the image of sink at origin give rise to a sink at an inverse point of O which is at infinity and an equal source at the origin.

The source and the sink of equal strength at the origin O neutralize. Hence the image system consists of

(i) A source of strength $+\frac{m}{2\pi}$ at S'

(ii) A source of strength $+\frac{m}{2\pi}$ at S.

(iii) A sink of strength $-\frac{m}{2\pi}$ at O.

Thus the complex potential is given by

$$w=-\frac{m}{2\pi}\log(z-f)-\frac{m}{2\pi}\log\left(z-\frac{a^2}{f}\right)+\frac{m}{2\pi}\log z \qquad ...(1)$$

or $$\phi+i\psi=\frac{m}{2\pi}\left\{\log z-\log(z-f)-\log\left(z-\frac{a^2}{f}\right)\right\}$$

Equating the real part from both the sides, we have

$$\phi=\frac{m}{2\pi}\log OP-\frac{m}{2\pi}\log S'P-\frac{m}{2\pi}\log SP$$

or $$\phi=-\frac{m}{2\pi}\log\frac{S'P.SP}{OP}$$ **Answer.**

Again differentiating (1) w. r. to z, we have

$$\frac{dw}{dz}=\frac{m}{2\pi}\left[\frac{1}{z}-\frac{1}{z-f}-\frac{1}{z-a^2/f}\right] \qquad ...(2)$$

Let X and Y are the components, parallel to the axes of the force that acts on the circular boundary, then by Blasius's Theorem, we have

$$X-iY=\tfrac{1}{2}i\rho\int_C\left(\frac{dw}{dz}\right)^2 dz$$

$$=\tfrac{1}{2}i\rho\left\{2\pi i\times\text{Sum of the residues of }\left(\frac{dw}{dz}\right)^2\text{ with in the contour } i.e. \text{ the circular boundary}\right\}$$

So from (2), we have

$$\left(\frac{dw}{dz}\right)^2=\frac{m^2}{4\pi^2}\left[\frac{1}{z^2}+\frac{1}{(z-f)^2}+\frac{1}{\left(z-\frac{a^2}{f}\right)^2}-\frac{2}{z(z-f)}-\frac{2}{z\left(z-\frac{a^2}{f}\right)}+\frac{2}{(z-f)\left(z-\frac{a^2}{f}\right)}\right]$$

or $\left(\frac{dw}{dz}\right)^2=\frac{m^2}{4\pi^2}\left[\frac{1}{z^2}+\frac{1}{(z-f)^2}+\frac{1}{\left(z-\frac{a^2}{f}\right)^2}-\frac{2}{f(z-f)}+\frac{2}{zf}\right.$

$$-\frac{2}{\frac{a^2}{f}\left(z-\frac{a^2}{f}\right)}+\frac{2}{z.\frac{a^2}{f}}+\frac{2}{(z-f)\left(f-\frac{a^2}{f}\right)}$$

$$\left.+\frac{2}{\left(z-\frac{a^2}{f}\right)\left(\frac{a^2}{f}-f\right)}\right] \quad \ldots(3)$$

The poles inside the circular boundary are $z=0$ and $z=f$. So the residue at $z=0$ $\left(\text{equating the coefficients of } \frac{1}{z} \text{ from (3)}\right)$

$$=\left(\frac{2}{f}+\frac{2}{a^2/f}\right)=2\left\{\frac{1}{f}+\frac{f}{a^2}\right\}$$

and the residue at $z=f$ (equating the coefficients of $\frac{1}{z-f}$ from (3))

$$=\frac{2}{f-a^2/f}-\frac{2}{f}$$

$$=\frac{2f}{f^2-a^2}-\frac{2}{f}.$$

Hence the sum of the residues $\left(i.e. \text{ coeff. of } \frac{1}{z} \text{ and } \frac{1}{z-f}\right)$ is

$$=\frac{m^2}{4\pi^2}\left[\frac{2}{f}+\frac{2f}{a^2}+\frac{2f}{f^2-a^2}-\frac{2}{f}\right]$$

$$=\frac{m^2}{4\pi^2}\cdot\frac{2f^3}{a^2(f^2-a^2)}$$

So $\quad X-iY=\frac{1}{2}i\rho\,.2\pi i.\frac{m^2}{4\pi^2}\cdot\frac{2f^3}{a^2(f^2-a^2)}$.

So $\quad X-iY=\frac{1}{2}.\frac{\rho m^2 f^3}{\pi a^2(a^2-f^2)}$.

Hence $\quad X=\frac{\rho m^2 f^3}{2a^2\pi(a^2-f^2)}$ and $Y=0$. **Proved.**

Now we shall determine the velocity potential due to a doublet at the centre of the circular boundary.

The combination forms a doublet if f tends to zero or $\frac{a^2}{f}$ tends to infinity. Such that

$$\frac{m}{2\pi}f=\mu$$

Hence from (1), we have

$$w=-\frac{m}{2\pi}\log\left(1-\frac{f}{z}\right)-\frac{m}{2\pi}\log\left(1-\frac{z}{a^2/f}\right)+\text{Const.}$$

or $$w=-\frac{m}{2\pi}\left[-\frac{f}{z}-\frac{f^2}{2z^2}-\ldots\ldots\right]-\frac{m}{2\pi}\left[-\frac{z}{a^2/f}-\frac{z^2}{2\,(a^2/f\,)^2}-\ldots\ldots\right]$$

or $$w=\frac{m}{2\pi}\left[\frac{f}{z}+\frac{z}{a^2/f}\right]$$

or $$w=\frac{\mu}{z}+\frac{\mu z}{a^2} \qquad \left\{\text{as} \quad \mu=\frac{mf}{2\pi}\right.$$

or $$\phi+i\psi=\frac{\mu}{r}\,e^{-\theta i}+\frac{\mu}{a^2}\,re^{\theta i}$$

or $$\phi+i\psi=\frac{\mu}{r}\,(\cos\theta-i\sin\theta)+\frac{\mu r}{a^2}\,(\cos\theta+i\sin\theta)$$

Hence the velocity potential is, (equating the real part form both the sides)

$$\phi=\frac{\mu}{r}\cos\theta+\frac{\mu r}{a^2}\cos\theta$$

$$\phi=\mu\left(\frac{1}{r}+\frac{r}{a^2}\right)\cos\theta$$

Answer.

Ex. 18. *A line source is in the presence of an infinite plane on which is placed a semi circular cylindrical boss; the direction of the source is parallel to the axis of the boss, the source is at distance c from the plane and the axis of the boss, whose radius is a. Show that the radius to the point on the boss at which the velocity is a maximum makes an angle θ with the radius to the source where*

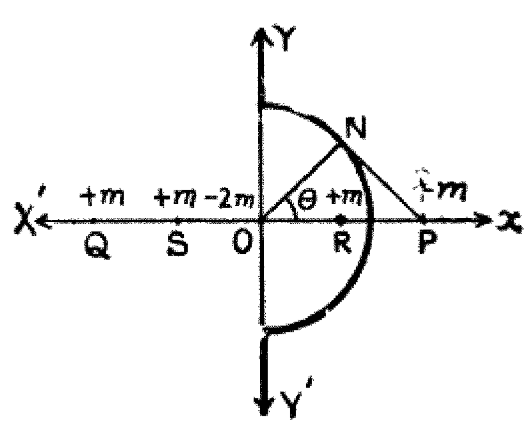
$$\theta=\cos^{-1}\left\{\frac{a^2+c^2}{\sqrt{\{2\,(a^4+c^4)\}}}\right\}.$$

Let R be the inverse point of P with regard to the circular boundary, such that

$$OR.OP=a^2$$

or $$OR=\frac{a^2}{c}$$

$$\left\{\begin{aligned}&\text{Since} \quad OP=c \text{ (Given)}\\ &\qquad ON=a\\ &\qquad OR=\frac{a^2}{c}\end{aligned}\right.$$



The image of a source of strength $+m$ at P will give rise to an equal source of strength $+m$ at an inverse point R and a sink of strength $-m$ at the origin with regard to the cylindrical boundary.

Again the image of source of strength $+m$ at P in the line $x=0$ will give rise to a source of strength $+m$ at Q $(z=-c)$, a source of strength $+m$ at S $\left(z=-\dfrac{a^2}{c}\right)$ and a sink of strength $-m$ at the origin.

Hence the complex potential is given by

$$w=2m \log z-m \log (z-c)-m \log (z+c)$$
$$-m \log \left(z-\frac{a^2}{c}\right)-m \log \left(z+\frac{a^2}{c}\right)$$

or $$w=2m \log z-m \log (z^2-c^2)-m \log\left\{z^2-\left(\frac{a^2}{c}\right)^2\right\}$$

or $$\frac{dw}{dz}=2m.\frac{1}{z}-m.\frac{2z}{z^2-c^2}-m.\frac{2z}{z^2-(a^2/c)^2}$$

or $$\frac{dw}{dz}=2m\left\{\frac{1}{z}-\frac{z}{z^2-c^2}-\frac{z}{z^2-a^4/c^2}\right\}$$

or $$\frac{dw}{dz}=-2m\frac{(z^4-a^4)}{z\,(z^2-c^2)\,(z^2-a^4/c^2)}.$$

The velocity at any point N on the circular boundary

$$q=\left|\frac{dw}{dz}\right|=2m\frac{|\,z^4-a^4\,|}{|\,z\,(z^2-c^2)\,(z^2-a^4/c^2)\,|}$$

Substituting $z=ae^{\theta i}$, we have

$$q=\left|\frac{dw}{dz}\right|=\frac{2ma^4\,|\,e^{4\theta i}-1\,|}{|\,ae^{\theta i}\,(a^2e^{2\theta i}-c^2)\,(a^2e^{2\theta i}-a^4/c^2)\,|}$$

or $$q=\frac{2ma^4c^2}{a^3}\cdot\frac{|\,(\cos 4\theta-1)+i\sin 4\theta\,|}{|\,(\cos\theta+i\sin\theta)\,\{(a^2\cos 2\theta-c^2)+ia^2\sin 2\theta)\}\,\{(c^2\cos 2\theta-a^2)+ic^2\sin 2\theta\}\,|}$$

or $$q=\frac{4mac^2.\sin 2\theta}{a^4+c^4-2a^2c^2\cos 2\theta}$$

or $$\frac{4mac^2}{q}=\frac{a^4+c^4-2a^2c^2\cos 2\theta}{\sin 2\theta}.$$

If q is maximum then $\dfrac{1}{q}$ must be minimum

$$\left\{\text{Let } \frac{4mac^2}{q}=\mu\right.$$

i. e. $\mu=(a^4+c^4)\operatorname{cosec} 2\theta-2a^2c^2\cot 2\theta$

or $$\frac{d\mu}{d\theta}=-2\,(a^4+c^4)\operatorname{cosec} 2\theta\cot 2\theta+4a^2c^2\operatorname{cosec}^2 2\theta. \qquad ...(1)$$

and $\dfrac{d^2\mu}{d\theta^2}=4\,(a^4+c^4)\operatorname{cosec}\theta\,(\operatorname{cosec}^2 2\theta+\cot^2 2\theta)$

$$-8a^2c^2\operatorname{cosec}^3 2\theta\cot 2\theta$$

$$=4\operatorname{cosec}\theta\,\{(a^2\operatorname{cosec}\theta-c^2\cot\theta)^2+a^4\cot^2\theta+c^4\operatorname{cosec}^2\theta\}$$

Since $\theta\leqslant\pi/2$ then $\dfrac{d^2\mu}{d\theta^2}>\theta$, so that μ should be minimum, then q (*i. e.* velocity) will be maximum.

From (1), $\dfrac{d\mu}{d\theta}=0$, we have

$(a^4+c^4)\operatorname{cosec}2\theta\cot 2\theta=2a^2c^2\operatorname{cosec}^2 2\theta$

or $$\cos 2\theta=\frac{2a^2c^2}{a^4+c^4}$$

or $$2\cos^2\theta-1=\frac{2a^2c^2}{a^4+c^4}$$

or $$2\cos^2\theta=\frac{a^4+c^4+2a^2c^2}{a^4+c^4}$$

or $$\cos^2\theta=\frac{(a^2+c^2)^2}{2\,(a^4+c^4)}$$

or $$\cos\theta=\frac{a^2+c^2}{\sqrt{2\,(a^4+c^4)}}$$

or $$\theta=\cos^{-1}\left\{\frac{a^2+c^2}{\sqrt{\{2\,(a^4+c^4)\}}}\right\}$$ **Proved.**

Ex 19. *An area A is bounded by that part of the X-axis for which $x>a$ and by that branch of $x^2-y^2=a^2$ which is in the positive quadrant. There is a two dimensional unit source at $(a, 0)$ which sends out liquid uniformly in all direction. Shew by means of the transformation $w=\log(z^2-a^2)$ that in steady motion the stream lines of the liquid within the area A are portions of rectangular hyperbolas. Determine the stream lines corresponding to $\psi=0, \pi/4$* and *$\pi/2$. If ρ_1 and ρ_2 are the distance of a point P within the fluid from the points $(\pm a, 0)$. Shew that the velocity of the fluid at P is measured by $\dfrac{2OP}{\rho_1\rho_2}$ O being the origin.*

Since $w=\log(z^2-a^2)$ ...(1)

or $\phi+i\psi=\log\{(x+iy)^2-a^2\}$ {as $z=x+iy$

or $\phi+i\phi=\log\{(x^2-y^2-a^2)+2ixy\}$.

To determine the stream function, equating the imaginary parts, we have

$$\psi = \tan^{-1} \frac{2xy}{x^2 - y^2 - a^2}$$

or $$\tan \psi = \frac{2xy}{x^2 - y^2 - a^2}. \qquad \ldots(2)$$

So stream lines are given by $\psi = \text{Const.}$

i. e. $$\frac{2xy}{x^2 - y^2 - a^2} = \text{Const.} = k \text{ (let).}$$

If k is zero, the stream lines will be

$$xy = 0 \Rightarrow x = 0 \text{ and } y = 0.$$

If k is infinite, the stream lines will be

$$x^2 - y^2 - a^2 = 0$$

$$x^2 - y^2 = a^2.$$

Thus the liquid flows in the area A bounded by $x = 0$, $y = 0$, and $x^2 - y^2 = a^2$ *i. e.* the portion of rectangular hyperbola in the positive quadrant.

Again (1) can be written as

$$w = \log (z - a) + \log (z + a).$$

Which shows that there is a unit source at the point $(a, 0)$ and the unit source at the point $(-a, 0)$ will be the image of the source at $(a, 0)$ with regard to Y-axis. **Proved.**

The velocity of the fluid at any point

$$q = \left| \frac{dw}{dz} \right| = \left| \frac{1}{z - a} + \frac{1}{z + a} \right|$$

$$\left| \frac{dw}{dz} \right| = \frac{|2z|}{|z - a| \; |z + a|}$$

$$\left| \frac{dw}{dz} \right| = \frac{2 \, OP}{\rho_1 \rho_2} \qquad \textbf{Proved.}$$

Again, the stream lines corresponding to $\psi = 0$ are,

$$\tan \psi = \frac{2xy}{x^2 - y^2 - a^2}$$

or $$= \frac{2xy}{x^2 - y^2 - a^2} = 0 \qquad \text{\{from (2)}$$

or $$2xy = 0$$

Which gives $x = 0$ and $y = 0$.

Also, the stream lines corresponding to $\psi = \pi/4$ are

$$\frac{2xy}{x^2 - y^2 - a^2} = \tan \pi/4 = 1.$$

or $$x^2 - y^2 - a^2 = 2xy$$

and the Stream lines Corresponding to $\psi=\pi/2$ are

$$\frac{2xy}{x^2-y^2-a^2}=\tan \pi/2=\infty$$

or $$x^2-y^2-a^2=0$$

or $$x^2-y^2=a^2.$$ **Answer.**

Ex. 20. *The space on one side of an infinite plane wall $y=0$, is filled with inviscid, incompressible fluid, moving at infinity with velocity U in the direction of the axis of X. The motion of the fluid is wholly two-dimensional, in the (x, y) plane. A doublet of strength μ is at a distance a from the wall and points in the negative direction of the axis of X. Shew that if μ is less than $4a^2U$, the pressure of the fluid on the wall is a maximum at points distant $\sqrt{3}a$ from O, the foot of the perpendicular from the doublet on the wall, and is minimum at O.*

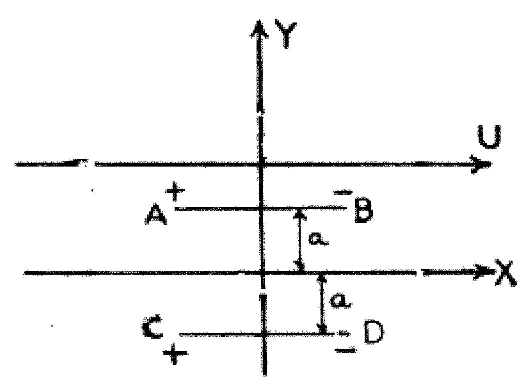
If μ is equal to $4a^2U$, find the points where the velocity of the fluid is zero, and shew that the stream lines include the circle

$$x^2+(y-a)^2=4a^2.$$

where the origin is taken at O.

We know that the complex potential for a doublet of strength μ at $z=z_0$ inclined at an angle α to the real axis is given by

$$w=e^{\frac{i\alpha}{z-z_0}}$$



The doublet points in the negative direction of the axis of X, it will make an angle π with the X-axis. Since a doublet of strength μ is placed at a distance a from the real axis, then the image system consists of an equal doublet on the other side of that axis.

The complex potential for the system

$$w=\frac{\mu e^{i\pi}}{z-ia}+\frac{\mu e^{i\pi}}{z+ia}-Uz$$

$$w=-\frac{2\mu z}{z^2+a^2}-Uz \qquad \ldots(1)$$

or $$\frac{dw}{dz}=-U-\frac{2\mu\,(a^2-z^2)}{(a^2+z^2)} \qquad \ldots(2)$$

Let q be the velocity on the wall,

Then $q=\left|\frac{dw}{dz}\right|=\left|U+\frac{2\mu\,(a^2-z^2)}{(a^2+z^2)^2}\right|$

Consider p be any point on the wall, substitute $z=x$ (real axis), we have

$$q=\left[2\mu\frac{a^2-x^2}{(a^2+x^2)^2}+U\right]$$

By Bernoullis theorem, we get

$$\frac{p}{\rho}+\frac{1}{2}q^2=\text{Const.}$$

Let $p=\Pi$, $q=U$. Then Const.$=\frac{\Pi}{\rho}+\frac{1}{2}U^2$

or $$\frac{p}{\rho}+\frac{1}{2}q^2=\frac{\Pi}{\rho}+\frac{1}{2}U^2$$

or $$\frac{\Pi-p}{\rho}=\frac{1}{2}(q^2-U^2)$$

or $$\frac{\Pi-p}{\rho}=\frac{1}{2}\left\{\left(2\mu\frac{a^2-x^2}{(a^2+x^2)^2}+U\right)^2-U^2\right\}$$

or $$\frac{\Pi-p}{\rho}=2\mu^2\cdot\frac{(a^2-x^2)^2}{(a^2+x^2)^4}+2\mu U\frac{a^2-x^2}{(a^2+x^2)^2}.$$

Since pressure will be maximum or minimum if $\frac{dp}{dx}=0$.

So $$2\mu^2\left\{\frac{4x\,(a^2-x^2)}{(a^2+x^2)^4}+\frac{8\,x(a^2-x^2)^2}{(a^2+x^2)^5}\right\}+2\mu U\left\{\frac{2x}{(a^2+x^2)^2}+\frac{4x(a^2+x^2)}{(a^2+x^2)^3}\right\}=0$$

or $$4\mu x\,[2\mu\,(a^2-x^2)+U\,(a^2+x^2)^2]\,\frac{3a^2-x^2}{(x^2+a^2)^5}=0.$$

either $\quad x\,(3a^2-x^2)=0$

or $\quad 2\mu\,(a^2-x^2)+U\,(a^2+x^2)^2=0$

If $\quad x\,(3a^2-x^2)=0$

Then $\quad x=0,\ \sqrt{3}a,\ -\sqrt{3}a.$

Now on the wall ($y=0$) at $x=+\sqrt{3}a$

$$\frac{dw}{dz}=-U-\frac{2\mu(a^2-3a^2)}{(a^2+3a^2)^2}$$

$$\frac{dw}{dz}=\frac{4\mu}{16a^2}-U=\frac{\mu}{4a^2}-U.$$

If $\mu<4a^2U$ then the value of $\left(\frac{d^2p}{dx^2}\right)_{x=a\sqrt{3}}$ is negative. Thus the pressure of the fluid at the wall is maximum. Also $\left(\frac{d^2p}{dx^2}\right)_{x=0}$ is positive, then the pressure at the wall is minimum.

If $\mu=4a^2U$, we have $\dfrac{dw}{dz}=0$.

Then from (2), we have

$$8a^2U\frac{a^2-z^2}{(a^2+z^2)^2}+U=0.$$

or $$z^4-6a^2z^2+9a^4=0.$$

or $$(z^2-3a^2)^2=0. \quad \text{or} \quad z=\pm a\sqrt{3}.$$

The stagnation points are $(a\sqrt{3}, 0)$ $(-a\sqrt{3}, 0)$.

Again to determine the stream function, equating the imaginary part from (1) and substitute $\mu=4a^2U$, we have

$$\psi=\frac{2\mu\,(x^2+y^2-a^2)}{(x^2+y^2)^2+2a^2\,(x^2-y^2)+a^4}-Uy$$

or $$\psi=\frac{8a^2Uy\,(x^2+y^2-a^2)}{(x^2+y^2)^2+2a^2\,(x^2-y^2)+a^4}-Uy$$

The stream lines are given by $\psi=0$, we have

$$-y\,\{(x^2+y^2)^2+2a^2\,(x^2-y^2)+a^4\}+8a^2y\,(x^2+y^2-a^2)=0$$

or $$(x^2+y^2)^2-6a^2x^2-10a^2y^2+9a^4=0.$$

or $$(x^2+y^2-2ay-3a^2)\,(x^2+y^2+2ay-3a^2)=0.$$

Obviously which includes the circle

$$x^2+y^2-2ay-3a^2=0$$

or $$x^2+(y-a)^2=4a^2.$$ **Proved.**

Exercises

1. Prove that in two dimensions :

   (i) There is always a stream function whether the motion is irrotational or rotational.

   (ii) If the speed is everywhere the same, the stream lines are straight.

2. If λ denotes a variable parameter, and f a given function ; find the condition that $f(x, y, \lambda)=0$ should be a possible system of stream lines for steady irrotational motion in two dimensions.

3. Two sources, each of strength m, placed at the points $(\pm a, 0)$ and a sink of strength $2m$ is placed at the origin. Shew that the stream lines are the curves,
   $(x^2+y^2)^2=a^2\,(x^2-y^2+\lambda xy)$ where λ is a variable parameter.

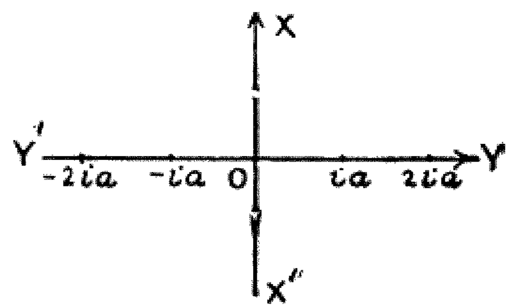
4. Parallel line sources (perp. to XY—plane) of equal strength m are placed at the points $z=nia$ where
   $n=\ldots\ldots-2, -1, 0, 1, 2, 3, \ldots\ldots$

Prove that the complex potential is $w=-m \log \sinh \frac{\pi}{a} z$.

Hence, shew that the complex potential for two dimensional doublets (line doublets), with their axes parallel to the X-axis, of strength μ at the same points is given by

$$w=\mu \coth \frac{\pi}{a} z$$

Hint : *Sources are given at the points $z=0, ia, -ia, 2ia, -2ia\ldots$ and so on.*



The complex potential is given by

$$w=-m \log z-m \log (z-ai)$$

$$-m \log (z+ai)-m \log (z-2ai)-m \log (z+2ai)\ldots\ldots$$

or $w=-m \log z\,(z^2+a^2)\,(z^2+2^2a^2)\,(z^2+3^2a^2)\ldots(z^2+n^2a^2)$

or $$w=-m \log\left\{\frac{\pi}{a} z \left(1+\frac{z^2}{a^2}\right)\left(1+\frac{z^2}{2^2a^2}\right)\ldots\right\}$$

$$-m \log\left\{\frac{a}{\pi}\, a^2\,(2^2a^2)\,(3^2a^2)\right\}$$

or $w=-m \log \sinh \frac{\pi}{a} z+\text{Const.}$

The complex potential for the doublets at the same points

$$w_1=-\frac{\partial w}{\partial z}=m.\frac{\pi}{a} \coth\left(\frac{\pi z}{a}\right)$$

$$=\mu \coth\left(\frac{\pi z}{a}\right).$$

5. With in a circular boundary of radius a there is a source of strength m at a distance f from the centre and an equal sink at the centre. Determine the resultant thurst on the boundary.

6. In two dimensional irrotational fluid motion show that, if the stream lines are confocal ellipses

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1.$$

$\psi=A \log \{\sqrt{(a^2+\lambda)}+\sqrt{(b^2+\lambda)}\}+B$ and the velocity at any point is inversely proportional to the square root of the rectangle under the focal radii of the point.

Hint : *Let the conformal transformation be $z=c \cos w$.*

or $$x+iy=c \cos (\phi+i\psi)$$

or $\quad x+iy=c\cos\phi\cosh\psi+i\,c\sin\phi\sinh\psi$

By eliminating ϕ, we have

$$\frac{x^2}{c^2\cosh^2\psi}+\frac{y^2}{c^2\sinh^2\psi}=1. \qquad ...(1)$$

Stream lines will be obtained by $\psi=$ Const., which are confocal ellipses. Comparing (1) with original equation of confocal ellipses

$c^2\cosh^2\psi=a^2+\lambda,\ c^2\sinh^2\psi=b^2+\lambda$ $\quad\left\{\begin{array}{l}\text{or } a^2-b^2=c^2.\\ \text{and } ae=c\end{array}\right.$

Also $\sqrt{(a^2+\lambda)}+\sqrt{(b^2+\lambda)}=c(\cosh\psi+\sinh\psi)$

or $\quad \sqrt{(a^2+\lambda)}+\sqrt{(b^2+\lambda)}=ce^{\psi}$

or $\quad \psi=\log\{\sqrt{(a^2+\lambda)}+\sqrt{(b^2+\lambda)}\}-\log c$

7. Shew that the force per unit length exerted on a circular cylinder, radius a, due to a source of strength m, at a distance c from the axes is

$$\frac{2\pi\rho m^2a^2}{c(c^2-a^2)}.$$

8. Prove that in two-dimensional liquid motion due to any number of sources at different points on a circle, the circle is a stream line provided that there is no boundary and that the algebraic sum of the strengths of the sources is zero.

9. If a homogeneous liquid is acted on by a repulsive force from the origin, the magnitude of which at a distance r from the origin is μr per unit mass. Shew that it is possible for the liquid to move steadily, without being contained by any boundaries, in the space between one branch of the hyperbola $x^2-y^2=a^2$ and the asymptotes. Also find the velocity potential.

10. A source and a sink, each of strength μ, exist in an infinite liquid on opposite sides of, and at equal distances c from the centre of a rigid sphere of radius a. Shew that the velocity potential V may be expressed in the form

$$V=\frac{2\mu}{c}\sum_{n=0}^{\infty}\left\{\left(\frac{r}{c}\right)^{2n+1}+\frac{2n+1}{2n+2}\cdot\frac{c}{a}\cdot\left(\frac{a^2}{rc}\right)^{2n+2}\right\}P_{2n+1}\ (\cos\theta)$$

θ being the vectorial angle measured from the diameter of the sphere on which the source and sink lie, and $r<c$; and find an expression for V when $r>c$.

Hint : Let ϕ_1 be the velocity potential due to a source of strength μ at A and sink of strength $-\mu$ at B, and ϕ_2 be the velocity potential due to sphere, then

$$r=\phi_1+\phi_2 \qquad ...(1)$$

Since $\phi_1=\dfrac{\mu}{AP}-\dfrac{\mu}{BP}$

$$\phi_1=\mu\,(r^2+c^2-2rc\cos\theta)^{-1/2}-\mu\,(r^2+c^2+2rc\cos\theta)^{-1/2}$$

$$\phi_1=\frac{\mu}{c}\left\{\left(1-\frac{2r}{c}\cos\theta+\frac{r^2}{c^2}\right)^{-1/2}-\left(1+\frac{2r}{c}\cos\theta+\frac{r^2}{c^2}\right)^{-1/2}\right\}$$

$$\phi_1=\frac{\mu}{c}\left[\left\{1+\sum_1^\infty\frac{r^n}{c^n}P_n(\cos\theta)\right\}-\left\{1+\sum_1^\infty\left(-\frac{r}{c}\right)^{-n}P_n(\cos\theta)\right\}\right.$$

$$\phi_1=\frac{2\mu}{c}\sum_0^\infty\left(\frac{r}{c}\right)^{2n+1}P_{2n+1}(\cos\theta) \qquad \{\text{for } r<c. \qquad ..(2)$$

$$\text{Let } \phi_2=\sum_{n=0}^\infty\frac{A_n}{r^{n+1}}P_n(\cos\theta) \qquad ..(3)$$

(The velocity potential ϕ_2 vanishes at infinity and satisfies Laplace's Equation).

Again the velocity normal to the sphere is zero,

$$\frac{\partial V}{\partial r}=0 \text{ where } r=a.$$

or

$$\left(\frac{\partial\phi_1}{\partial r}+\frac{\partial\phi_2}{\partial r}\right)_{r=a}=0.$$

or $$\frac{2\mu}{c}\sum_0^\infty\frac{(2n+1)}{c^{2n+1}}a^{2n}P_{2n+1}(\cos\theta)-\sum_0^\infty\frac{A_n\,(n+1)}{a^{n+2}}P_n(\cos\theta)=0$$

Equating to zero the coefficient of P_n', where

$$A_{2n}=0 \ \forall \text{ values of } n$$

and $$A_{2n+1}\frac{2n+2}{a^{2n+3}}-\frac{2\mu}{c}\,\frac{(2n+1)}{c^{2n+1}}a^{2n}=0$$

or $$A_{2n+1}=\frac{\mu\,(2n+1)\,a^{4n+3}}{c^{2n+2}\,(n+1)}$$

Since $(1-2xz+z^2)^{-1/2}=\sum_{n=0}^\infty P_n(x)\,z^n$

$$=1+\sum_1^\infty P_n(x)\,z^n$$

or $\quad A_{2n+1}=\frac{\mu}{a}\cdot\frac{2n+1}{n+1}\left(\frac{a^2}{c}\right)^{2n+1}.$

Therefore $\phi_2=\sum_0^{\infty}\frac{\mu}{a}\cdot\frac{2n+1}{n+1}\left(\frac{a^2}{rc}\right)^{2n+2}P_{2n+1}(\cos\theta)$

Substituting the values of ϕ_1 and ϕ_2 in (1), we have

$$V=\frac{2\mu}{c}\sum_{n=0}^{\infty}\left[\left(\frac{r}{c}\right)^{2n+1}+\frac{2n+1}{2n+2}\cdot\frac{c}{a}\left(\frac{a^2}{rc}\right)^{2n+2}\right]P_{2n+1}(\cos\theta)$$

Proved.

Similarly when $r > c$, we have

$$\phi_1=\frac{2\mu}{r}\sum_0^{\infty}\left(\frac{c}{r}\right)^{2n+1}P_{2n+1}(\cos\theta)$$

and $\quad \phi_2=\sum_0^{\infty}\frac{B_n}{r^{n+1}}P_n(\cos\theta).$ **Answer.**

4

General Theory of Irrotational Motion

Here we shall discuss the nature of the irrotational motion and the conditions under which it produces. At the first instance we shall consider, in general terms, the **motion of a small fluid element.**

§ 4·1. Consider (u, v, w) be the components of velocity of the fluid particle $P(x, y, z)$ then the components of relative velocities at an infinitely near point $Q\ (x+\delta x, y+\delta y, z+\delta z)$ are given by

$$\left.\begin{aligned}\delta u&=\frac{\partial u}{\partial x}\,\delta x+\frac{\partial u}{\partial y}\,\delta y+\frac{\partial u}{\partial z}\,\delta z\\ \delta v&=\frac{\partial v}{\partial x}\,\delta x+\frac{\partial v}{\partial y}\,\delta y+\frac{\partial v}{\partial z}\,\delta z\\ \text{and}\qquad \delta w&=\frac{\partial w}{\partial x}\,\partial x+\frac{\partial w}{\partial y}\,\delta y+\frac{\partial w}{\partial z}\,\delta z\end{aligned}\right\}\qquad \ldots\text{(i)}$$

Assuming

$$\left.\begin{aligned}&a=\frac{\partial u}{\partial x},\ b=\frac{\partial v}{\partial y},\ \ c=\frac{\partial w}{\partial z}\\ &f=\tfrac{1}{2}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right),\ g=\tfrac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right),\\ &h=\tfrac{1}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\\ \text{and}\qquad &\xi=\tfrac{1}{2}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right),\ \eta=\tfrac{1}{2}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right),\\ &\zeta=\tfrac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\end{aligned}\right]\qquad \ldots\text{(ii)}$$

The relation (i) can be written with the help of (ii),

$$\left.\begin{aligned}\delta u&=a\delta x+h\delta y+g\delta z+\eta\delta z-\zeta\delta y\\ \delta v&=h\delta x+b\delta y+f\delta z+\zeta\delta x-\xi\delta z\\ \text{and}\qquad \delta w&=g\delta x+f\,\delta y+c\delta z+\xi\delta y-\eta\delta x\end{aligned}\right\}\qquad \ldots\text{(iii)}$$

Thus the motion of a small fluid element (having the point $P(x\ y\ z)$ as its centre) consists of three parts.

The first part, whose components are u, v, w is **a motion of translation** of the fluid element.

The second part is represented by the first three terms on the R. H. S. of (iii), **a motion in the direction of the normal** to the surface (quadric of the system)

$$a(\delta x)^2+b(\delta y)^2+c(\delta z)^2+2f\,\delta y\delta z+2g\,\delta z\delta x+2h\,\delta x\delta y=\text{const.} \quad ...(iv)$$

Let the equation to the surface (iv) referred to its principal axis, then the corresponding parts of the velocities parallel to these coordinate axes be

$$\delta u'=a'\delta x',\ \delta v'=b'\delta y' \text{ and } \delta w'=c'\delta z'$$

If $$a'(\delta x')^2+b'(\delta y')^2+c'(\delta z')^2=\text{const.} \quad ...(v)$$

where a', b', c' are the time-rates of elongation of lines parallel to the axes X', Y', Z' at a uniform rate. Such a motion is called **pure strain** and the principal axes of the surface (iv) are called **the axes of the strain**. If there is no change of volume, then a', b', c' can not be independent (as in liquid), we have

$$a'+b'+c'=a+b+c$$
$$=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}$$
$$=\text{zero}.$$

Also the third part is represented by the last two terms on the R. H. S of (iii), a **rotation** of which the component angular velocities are ξ, η, ζ. The vector whose components are the angular velocities are called **vorticity** of the medium at the points (x, y, z).

Thus the most general motion of a fluid element consists of a pure strain together with a rotation and this resolution of a motion is unique.

Now ξ, η, ζ are the components of spin, when through out the finite portion of a fluid mass these components (ξ, η, ζ) all vanish then the **motion** is said to be **rotational** and the relative displacement of a fluid element consists of a pure strain only.

§ 4·2. **Flow and Circulation.**

Consider A and B be any two points in the fluid then the value of the integral

$$\int_A^B udx+vdy+wdz$$

or
$$\int_A^B \left(u\frac{dx}{ds}+v\frac{dy}{ds}+w\frac{dz}{ds}\right)ds$$

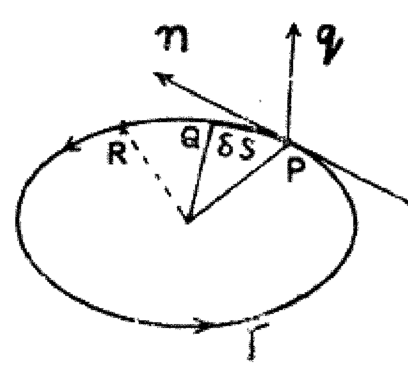
taken along any line from A to B, is called the **flow** of the fluid from A to B along that line.

When velocity potential exists, then the flow from A to B is given by

$$=-\int_A^B \frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz$$
$$=\phi_A-\phi_B$$

If A and B coincide, so that the curve along which the integration takes place is a closed curve, this line integral is called the **circulation** round the closed curve.

Consider a closed curved Γ in the moving fluid. Let $\mathbf{q}$ be the velocity at an arbitrary point P on the curve and $\mathbf{n}$ a unit vector drawn in the direction of the tangent at P. Consider a point Q on the curve, adjacent to P such that the arc PQ is of infinitesimal length δs.



The scalar product at the point $P=\mathbf{q}\ \delta s$

Similarly we can form the products at Q, R ... and so on round the curve and back again to P. Thus the **circulation** of the velocity vector round the curve Γ is given by

$$\underset{\delta s\to 0}{\mathrm{Lt}}\ \Sigma\ \mathbf{q}\ \delta s=\int_\Gamma \mathbf{q}\ ds$$

or
$$\text{Circ } \Gamma=\int_\Gamma \mathbf{q}\ ds=\int_\Gamma \mathbf{q}\ d\mathbf{r}$$

i. e. we can determine the circulation of any vector round a closed curve.

If the circulation vanishes round any closed curve then the velocity potential, ϕ, must be a single valued function.

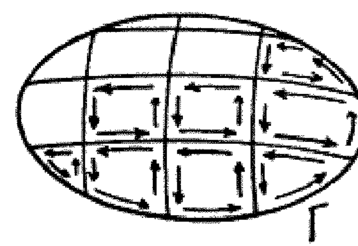
§ 4·3. Stoke's Theorem.

Stoke's theorem states that the circulation round any closed curve Γ drawn in a fluid is equal to the surface integral S of the normal component of spin taken over any surface, provided the surface lies wholly in the fluid.

$$\int_{\Gamma} \mathbf{q}\, d\mathbf{s} = \int_S \mathbf{n}\,(\nabla \times \mathbf{q})\, ds$$

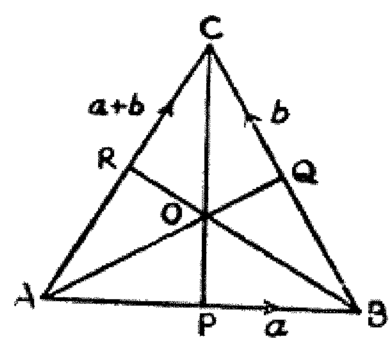
where **n** *is the unit normal vector at any point of S drawn in the sense in which a right handed screw could move when rotated in which Γ is described.*

Since any surface can be divided into small meshes by drawing a net work of lines across it, then the sum of the circulations round each mesh of the surface is equal to the circulation round the boundary Γ.



Any mesh can be divided into triangles, now we shall prove the theorem for a single triangular mesh *ABC* whose sides are of infinitesimal length. Let *P*, *Q*, *R* are the middle points of the sides of the triangle and *O* its centroid.

Let $\mathbf{q}_N$ represents the value of **q** at any point *N*.



Then $\int_{ABC} \mathbf{q}\,d\mathbf{s} = \overline{AB}\,\mathbf{q}_P + \overline{BC}\,\mathbf{q}_Q$

$$+\overline{CA}\,\mathbf{q}_R$$
$$= \mathbf{a}\mathbf{q}_P + \mathbf{b}\mathbf{q}_Q - (\mathbf{a}+\mathbf{b})\,\mathbf{q}_R$$

$$\int_{ABC} \mathbf{q}\,d\mathbf{s} = \mathbf{a}\,(\mathbf{q}_P - \mathbf{q}_R) + \mathbf{b}\,(\mathbf{q}_Q - \mathbf{q}_R) \qquad \ldots(1)$$

we know that

$$\mathbf{q}_P = \mathbf{q}_0 + (\overline{OP}\,\nabla)\,\mathbf{q}_0$$
$$\mathbf{q}_R = \mathbf{q}_0 + (\overline{OR}\,\nabla)\,\mathbf{q}_0$$

or $$\mathbf{q}_P - \mathbf{q}_R = (OP - \overline{OR}.\nabla)\,\mathbf{q}_0$$
$$= (\overline{RP}.\nabla)\mathbf{q}_0$$
$$= -\tfrac{1}{2}\,(\mathbf{b}.\nabla)\,\mathbf{q}_0$$

and $$\mathbf{q}_Q = \mathbf{q}_0 + (\overline{OQ}\,\nabla)\,\mathbf{q}_0$$
$$\mathbf{q}_R = \mathbf{q}_0 + (\overline{OR}\,\nabla)\,\mathbf{q}_0$$

or $$\mathbf{q}_Q-\mathbf{q}_R=(\overline{OQ}-\overline{OR}\ \nabla)\ \mathbf{q}_0$$
$$=(\overline{RQ}\ \nabla)\ \mathbf{q}_0$$
$$=\tfrac{1}{2}(\mathbf{a}\ \nabla)\ \mathbf{q}_0$$

Thus (1) reduces to,

$$\int_{ABC}\mathbf{q}\ d\mathbf{s}=-\tfrac{1}{2}\left[\mathbf{a}\ (\mathbf{b}\ \nabla)-\mathbf{b}\ (\mathbf{a}\ \nabla)\right]\mathbf{q}_0$$
$$=\tfrac{1}{2}\ [(\mathbf{a}\times\mathbf{b})\times\nabla]\ \mathbf{q}_0 \qquad \ldots(2)$$

Since $\mathbf{n}\ ds=\frac{1}{2}\ (\mathbf{a}\times\mathbf{b})$ where ds is the area of ABC

then $$\int_S \mathbf{n}\ (\nabla\times\mathbf{q})\ ds=\tfrac{1}{2}\ (\mathbf{a}\times\mathbf{b})\ (\nabla\times\mathbf{q}_0)$$
$$=\tfrac{1}{2}\ [(\mathbf{a}\times\mathbf{b})\times\nabla]\ \mathbf{q}_0 \qquad \ldots(3)$$

from (2) and (3), we have

$$\int_{ABC}\mathbf{q}\ d\mathbf{s}=\int_S\mathbf{n}\ (\nabla\times\mathbf{q})ds$$

Hence the theorem is proved for an infinitesimal triangle, which is the particular case of Stoke's theorem, in general, we can determine as follow

$$\int_S\left(\ ds\times\frac{\partial}{\partial\mathbf{p}}\right)X=\int_\Gamma X\ d\mathbf{s}.$$

Where X is any scalar or vector function and $d\mathbf{s}$ is the directed element of arc of Γ.

Since $$\int_C d\mathbf{s}\ X=-\tfrac{1}{2}\ \mathbf{a}\left(\ \mathbf{b}\frac{\partial}{\partial\mathbf{p}}\right)X+\tfrac{1}{2}\mathbf{b}\left(\ \mathbf{a}\frac{\partial}{\partial\mathbf{p}}\right)X$$

{as in equation (2)

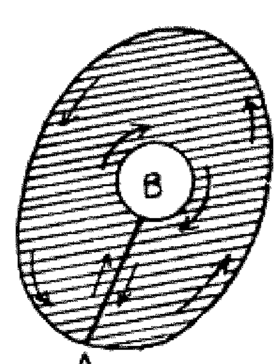
$$=\left\{\ \tfrac{1}{2}\ (\mathbf{a}\times\mathbf{b})\times\frac{\partial}{\partial\mathbf{p}}\right\}X \qquad \ldots(4)$$

Again direct area of the mesh$=\frac{1}{2}\ \mathbf{a}\times\mathbf{b}$

So $$\int\left(ds\times\frac{\partial}{\partial\mathbf{p}}\right)X=\left\{\ \tfrac{1}{2}\ (\mathbf{a}\times\mathbf{b})\times\frac{\partial}{\partial\mathbf{p}}\right\}X \qquad \ldots(5)$$

Thus (4) and (5) are equal which proves the Stoke's theorem in general.

Cor. This theorem is also true for a surface which is bounded by more than one closed curve *e. g.* the shaded area in the figure, considering the boundary as a continuous curve, the total flow along AB and BA is zero.



§ 4·31. Deduction from Stoke's theorem.

(i) We know by Stoke's theorem that

$$\int_{\Gamma} \mathbf{q}\, ds = \int_{S} (\mathbf{n} \times \nabla)\, \mathbf{q}\, ds \qquad ...(1)$$

$$= \int_{S} \mathbf{n}\, (\nabla \times \mathbf{q})\, ds$$

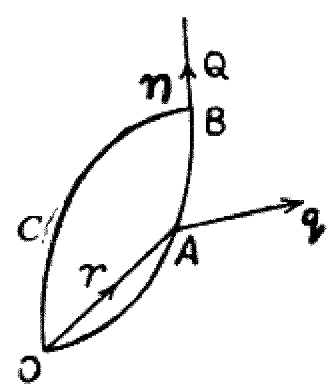
$$= \int_{S} \mathbf{n}\, \zeta\, ds = \int_{S} \zeta\, ds$$

Where ζ is the **vorticity**. In other words, we can define that the *circulation of the velocity in any circuit is equal to the integral of the normal component of the vorticity over any surface which encloses the circuit.*

(ii) $\int_{\Gamma} \phi\, ds = \int_{S} (\mathbf{n} \times \nabla) \phi\, ds$

§ 4·32. Irrotational Motion.

Let O be a fixed point and P an arbitrary point in a simply connected* region. Join O and B by two paths OAB and OCB, *i.e.* $OABCO$ is a closed circuit, then by Stoke's theorem, we have



$$\int_{OABCO} \mathbf{q}\, ds = \int_{S} \mathbf{n}\, (\nabla \times \mathbf{q})\, ds$$

where S is any surface lying wholly in the fluid.

$$\int_{OAB} \mathbf{q}\, ds + \int_{BCO} \mathbf{q}\, ds = \int_{S} \mathbf{n}\, (\nabla \times \mathbf{q})\, ds \qquad ...(1)$$

Since the motion is irrotational, so $\nabla \times \mathbf{q} = 0$.

i.e. the circulation round any closed curve is zero, provided the closed curve can be regarded as the boundary of a surfaces every part of which lies within the fluid. Then from (1), we have

$$\int_{OAB} \mathbf{q}\, ds + \int_{BCO} \mathbf{q}\, ds = -\phi_B \quad \text{(let)} \qquad ...(2)$$

Where ϕ_B is a scalar function whose value depends on the position of B and not on the path from O to B.

Consider a point Q near to B such that the velocity vector $\mathbf{q}$ is nearly constant along BQ. Let η be the position vector of Q with regard to B.

* A region in which every closed curve can be contracted to a point without passing out of the region is called a simply connected region.

$$-\eta\ \nabla\ \phi=-\phi_Q+\phi_B \qquad \{\text{let } \phi_B=\phi \text{ app.}$$

or $$-\eta\ \nabla\ \phi=\int_{BQ} \mathbf{q}\ ds=\mathbf{q}\ \eta$$

or $$\mathbf{q}=-\nabla\ \phi$$

So the motion is irrotational, the velocity vector is the gradient of scalar function of position $-\phi$. This scalar function is known the velocity potential. Thus **when the motion is irrotational the single valued velocity potential necessarily exists,** the motion is called **Acyclic Irrotational Motion.**

Converse. Let the velocity potential exists,

then $$\mathbf{q}=-\nabla\ \phi$$

But $$\nabla\times\mathbf{q}=-\nabla\times(\nabla\phi)$$

or $$\text{curl } \mathbf{q}=0$$

$\Rightarrow$ the motion is irrotational.

{Since the **curl** of the **grad.** of a scalar point function is zero.

§ 4·4. Kelvin's Circulation theorem.

Kelvin's theorem states that the constancy of circulation in a circuit moving with the fluid in an inviscid fluid in which the density is either constant or is a function of the pressure.

Let Γ be the closed circuit which moves with the fluid *i.e.* a circuit which always consists of the same fluid particles, then the circulation is given by

$$\int_\Gamma \mathbf{q}\ d\mathbf{r}$$

or $$\frac{d}{dt}\ (\text{circulation})=\frac{d}{dt}\int_\Gamma \mathbf{q}\ d\mathbf{r}$$

$$=\int_\Gamma\left\{\frac{d\mathbf{q}}{dt}\cdot d\mathbf{r}+\mathbf{q}\cdot d\left(\frac{d\mathbf{r}}{dt}\right)\right\} \qquad ...(1)$$

The equation of motion is

$$\frac{d\mathbf{q}}{dt}=-\nabla\Omega-\frac{1}{\rho}\nabla\ p \qquad ...(2)$$

{Since $F=-\nabla\Omega$ as the external forces are conservative

from (1) and (2), we have

$$\frac{d}{dt}\ (\text{circulation})=\int_\Gamma\left\{\left(-\nabla\Omega-\frac{1}{\rho}\nabla p\ \right)\cdot d\mathbf{r}+\mathbf{q}\cdot d\mathbf{q}\ \right\}$$

$$\left\{\text{as } \mathbf{q}=\frac{d\mathbf{r}}{dt}\right.$$

$$\frac{d}{dt}\,(\text{circulation})=\int_{\Gamma} -\frac{\partial}{\partial \mathbf{r}}\,\Omega.d\mathbf{r}-\frac{1}{\rho}\,\frac{\partial p}{\partial \mathbf{r}}\,.d\mathbf{r}+\mathbf{q}.d\mathbf{q}$$

$$\frac{d}{dt}(\text{circulation})=\int_{\Gamma} -d\Omega-\frac{dp}{\rho}+\mathbf{q}.d\mathbf{q}$$

$$\frac{d}{dt}(\text{circulation})=-\int_{\Gamma} d\left\{\Omega+\int\frac{dp}{\rho}-\frac{1}{2}\,q^2\right\} \qquad \{\text{ as } \mathbf{q}.\mathbf{q}=q^2$$

$$\frac{d}{dt}(\text{circulation})=-\left\{\Omega+\int\frac{dp}{\rho}-\frac{1}{2}\,q^2\right\}_{\Gamma}$$

$$\frac{d}{dt}(\text{circulation})=0 \quad \text{or} \quad \text{circ. } \Gamma=\text{constant}$$

Since the circuit is closed. Therefore the circ. Γ is independent of time t.

Cor. *When the external forces are conservative and derivable from a single valued potential, and pressure is a function of density only, then the motion of an inviscid fluid if once irrotational remain irrotational subsequently.*

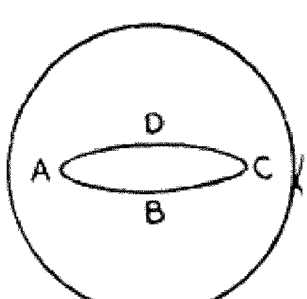
Let the motion is irrotational at any instant or in other words for every closed circuit circulation is zero, then

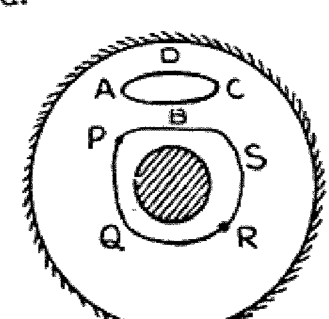
$$C=\int_{\Gamma} \mathbf{q}.d\mathbf{r}=\iint_{S} (\nabla\times\mathbf{q}).d\mathbf{s} \qquad \text{(By Stoke's theorem)}$$

Where Γ is a circuit moving with the fluid. Since the motion is irrotational, $\nabla\times\mathbf{q}=0 \Rightarrow C=0$. Thus the **irrotational motion is permanent.**

§ 4·5. Connectivity

If we can pass from any point of the region to any other point by moving along a path and every point of which lies in the given region then the region is said to be a **connected region** *e.g.* (a) the region interior to a sphere is connected.



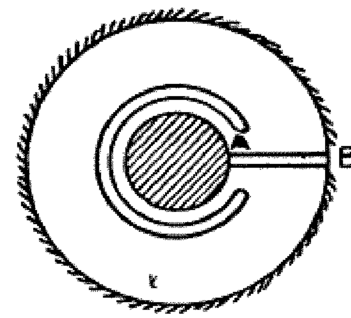


(b) the region between two coaxial infinitely long cylinders are connected.

In both the figures the circuit $ABCD$ is reducible but the circuit $PQRS$ is irreducible as it cannot be made smaller than the circumference of the inner cylinder.

Thus a region in which every circuit is **reducible** is said to be **simply connected** region (*e.g.* region interior to a sphere, region exterior to a sphere, region between two concentric spheres etc). otherwise the sphere is **multiply-connected.**

Since the region between the concentric cylinders contains irreducible circuits, so it is not simply connected. We can make this region simply connected by inserting a barrier or boundary. When the barrier is inserted, every circuit in the modified region is reducible and the modified region is known as simply connected.



A region is said to be **doubly connected**, if it can be made simply connected by the insertion of one barrier. A region is said to be **n-ply connected or of connectivity n** if by the insertion of $(n-1)$ barriers it can be made simply connected. The path joining two points P and Q of a region are said to be **reconcilable** or **Irreconcilable** according as it can or cannot be deformed so as to coincide with one another without going out of the region. **In simple connected space all circuits are reconcilable and reducible.** Two reconcilable paths taken together constitute a reducible circuit. In the above figure the path ABC and ADC are reconcilable.

The above properties of regions are termed **Topological** rather than geometrical as they donot depend on the particular shapes of the boundaries.

§ 4·51. **Acyclic and cyclic motion.**

When the region occupied by the fluid moving irrotationally is simply connected, the velocity potential will be single-valued. So velocity potential at B is {Ref. § 4·32}

$$\phi_B = -\int_{OAB} \mathbf{q}\ d\mathbf{r}. \qquad \{\forall \text{ paths from } O \text{ to } B\}$$

Thus the motion in which the velocity potential is single valued is called **acyclic.** In simply connected region the only possible irrotational motion is acyclic.

When the velocity potential is not single-valued the motion is said to be **cyclic** *i.e.* it is not possible to assign to every point of the original region a unique and definite value of ϕ.

§ 4·6. **Green's Theorem.**

If ϕ, ψ are both single-valued and continuously differentiable scalar point functions such that $\nabla\phi$ and $\nabla\psi$ are also continuously differentiable, then

$$\int_V (\nabla\phi . \nabla\psi)\, dv = -\int_V \phi\nabla^2\psi . dv - \int_S \phi \frac{\partial\psi}{\partial n}\, dS$$

$$= -\int_V \psi\nabla^2\phi . dv - \int_S \psi \frac{\partial\phi}{\partial n}\, dS$$

where S is a closed surface bounding any simply-connected region, δn is an element of the normal at any point on the boundary drawn into the region and V is the volume enclosed by S.

We know by Vector Calculus,

$$\nabla . (\phi \mathbf{a}) = \mathbf{a}\, (\nabla\phi) + \phi\, (\nabla \mathbf{a}) \qquad \text{...(i)}$$

where ϕ is a scalar point function and $\mathbf{a}$ is a vector point function.

Substituting $\nabla\psi$ for $\mathbf{a}$ in (i), we have

$$\nabla . (\phi\nabla\psi) = \nabla\psi\, (\nabla\phi) + \phi\, (\nabla\ \nabla\psi).$$

By integrating, it reduces to

$$\int_V \nabla . (\phi\ \nabla\psi)\, dv = \int_V \phi\, (\nabla . \nabla\psi)\, dv + \int_V \nabla\psi . \nabla\phi\, dv. \qquad \text{...(ii)}$$

By Gauss divergence theorem*, (ii) becomes

$$-\int_S \mathbf{n} . (\phi\nabla\psi)\, dS = \int_V \phi\nabla^2\psi\, dv + \int_V \nabla\phi . \nabla\psi\, dv$$

or
$$\int_V \nabla\phi . \nabla\psi\, dv = -\int_V \phi\nabla^2\psi\, dv - \int_S \mathbf{n} . (\phi\nabla\psi)\, dS$$

$$= -\int_V \phi\nabla^2\psi\, dv - \int_S \phi\, (\mathbf{n} . \nabla\psi)\, dS$$

or
$$\int_V \nabla\phi . \nabla\psi\, dv = -\int_V \phi\nabla^2\psi\, dv - \int_S \phi \frac{\partial\psi}{\partial n}\, dS \qquad \text{...(iii)}$$

$$\left\{ \text{as} \quad \mathbf{n} . \nabla\psi = \frac{\partial\psi}{\partial n} \right.$$

By interchanging ϕ and ψ, we have

$$\int_V \nabla\phi . \nabla\psi\, dv = -\int_V \psi\nabla^2\phi\, dv - \int_S \psi \frac{\partial\phi}{\partial n}\, dS \qquad \text{...(iv)}$$

Proved.

which is known as **Green's theorem** or **Green's first identity**.

§ 4·61. **An application to Green's theorem.**

(i) Let ψ is constant and $\nabla^2\phi = 0$ then from (iii) and (iv),

*Gauss divergence theorem

$$\int_V \text{div}\ F\, dv = -\int_S \mathbf{F}\, \mathbf{n}\, dS.$$

we have

$$\int_S \phi \frac{\partial \psi}{\partial n} dS = \int_S \psi \frac{\partial \phi}{\partial n} dS.$$

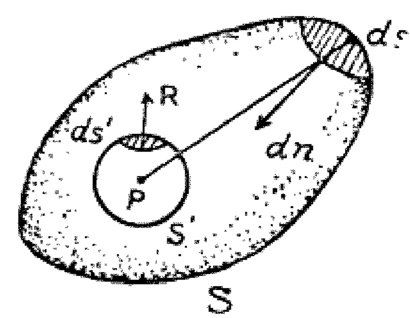
Since ψ is constant. So $\frac{\partial \psi}{\partial n}=0$ and $\nabla^2\phi=0$.

Thus $$\int_S \frac{\partial \phi}{\partial n} dS=0. \quad \text{...(v)}$$

{as ψ=constant.}

If follows that the **total flow of liquid into any closed region at any instant is zero.**

(ii) We shall determine the value of ϕ at any interior point in terms of its values on the boundary. The potential ϕ_P at the point P in a mass of liquid moving irrotationally with in the boundary is given by



$$4\pi\phi_P = \int_S \left\{ \phi \frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\frac{\partial \phi}{\partial n} \right\} dS$$

where r is the distance of P from the element of area dS.

Proof. Let P be any point with in the region, consider $\psi=\frac{1}{r}$

then $$\nabla^2\psi = \nabla^2\left(\frac{1}{r}\right)=0.$$

Since at P $\psi=\infty$ as r tends to zero, thus enclosing the point P by a sphere S' of radius λ (small quantity) such that S' lies entirely within S. By Green's Reciprocal theorem, we have

$$\int_S \left\{ \phi \frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\left(\frac{\partial \phi}{\partial n}\right) \right\} dS = -\int_{S'} \left[\phi \frac{\partial}{\partial \lambda}\left(\frac{1}{\lambda}\right) - \frac{1}{\lambda}\left(\frac{\partial \phi}{\partial \lambda}\right) \right] dS'$$

or $$\int_S \left\{ \phi \frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\left(\frac{\partial \phi}{\partial n}\right) \right\} dS = \frac{1}{\lambda^2}\int_{S'} \phi \, dS' \quad \text{...(vi)}$$

$\left\{ \text{Since } \int_{S'} \frac{\partial \phi}{\partial \lambda} dS' = \int_V \nabla^2\phi \, dv = 0 \right.$
By Gauss Theorem.

Consider λ so small that $\phi=\phi_P$ on the whole of the surface S',

then $$\frac{1}{\lambda^2}\int_{S'} \phi dS' = \frac{1}{\lambda^2}\phi_P \int_{S'} dS' = \frac{\phi_P}{\lambda^2}.4\pi\lambda^2$$
$$=4\pi\phi_P \quad \text{(app.)}$$

then (vi) becomes, as $\lambda \to 0$

$$4\pi\phi_P = \int_S \left\{ \phi \frac{\partial}{\partial n}\left(\frac{1}{r}\right) - \frac{1}{r}\left(\frac{\partial \phi}{\partial n}\right) \right\} dS.$$

§ 4·62. Kinetic energy of finite fluid.

The kinetic energy T is given by

$$T = \int_V \tfrac{1}{2}\rho \mathbf{q}^2 \, dv \qquad \text{...(i)}$$

where V is the volume occupied by the fluid when the motion is irrotational, then

$$\mathbf{q} = -\nabla\phi. \qquad \text{...(ii)}$$

Considering the fluid to be incompressible, the equation of continuity is

$$\text{div } \mathbf{q} = 0 \quad \text{or} \quad \nabla^2\phi = 0. \qquad \text{...(iii)}$$

Then for a simply connected region, by Gauss theorem, we have

$$\int_V \text{div } \mathbf{a} \, dv = \int_S \mathbf{a} . dS \qquad \text{...(iv)}$$

where $\mathbf{a}$ is a vector point function and S is a closed surface enclosing a volume V

Let $\quad \mathbf{a} = \phi \, \nabla\phi$ then from (iv), we have

$$\int_V (\phi\nabla^2\phi + \nabla\phi . \nabla\phi) \, dv = -\int_S \phi \, (\nabla\phi) . \mathbf{n} \, dS$$

or

$$\int_V (\nabla\phi)^2 \, dv = -\int_S \phi \frac{\partial \phi}{\partial n} \, dS$$

{where n is the unit inward drawn normal to area dS}

From (i), since

$$T = \int_V \tfrac{1}{2}\rho \, \mathbf{q}^2 \, dv$$

or

$$T = \int_V \tfrac{1}{2}\rho \, (\nabla\phi)^2 \, dv$$

or

$$T = -\tfrac{1}{2}\rho \int_S \phi \frac{\partial \phi}{\partial n} \, dS$$

which represents the kinetic energy of a given mass of liquid moving irrotationally in a simply connected region depending on the motion of the boundaries only.

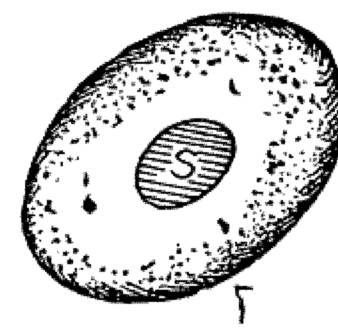
Cor. If $\dfrac{\partial \phi}{\partial n} = 0$ on the boundary

then $\quad T = \tfrac{1}{2}\rho \int_V q^2 \, dv = 0.$

Thus q vanishes everywhere in V. So the irrotational motion is not possible in a closed simply connected region with fixed boundaries.

§ 4 63. Kinetic energy of infinite liquid.

Consider the liquid moving irrotationally at rest at infinity, and bounded internally by a solid S and externally by a large surface Γ, the velocity potential ϕ is single-valued.



Kinetic energy of the finite liquid contained in the region between the solid S and surface Γ.

$$T=-\tfrac{1}{2}\rho\int_S \phi\frac{\partial\phi}{\partial n}\,dS-\tfrac{1}{2}\rho\int_\Gamma \phi\frac{\partial\phi}{\partial n}\,d\Gamma. \quad \text{...(i)}$$

Let P be the rate at which the mass flows into any surface γ through the boundary is

$$P=\int_\gamma \rho\,\mathbf{q}.\mathbf{n}\,d\gamma \qquad \left\{\begin{array}{l}\text{where } \mathbf{n} \text{ is the unit normal}\\ \text{drawn into the surface}\end{array}\right.$$

$$P=-\int_\gamma \rho\frac{\partial\phi}{\partial n}\,d\gamma \qquad \left\{\text{as } \mathbf{q}=-\nabla\phi=-\mathbf{n}\frac{\partial\phi}{\partial n}\right.$$

Since there is no flow into the region across S, the equation of continuity is of the form

$$-\int_S \rho\frac{\partial\phi}{\partial n}\,dS-\int_\Gamma \rho\frac{\partial\phi}{\partial n}\,d\Gamma=0. \quad \text{...(ii)}$$

Multiplying (ii) by $\tfrac{1}{2}c$ (constant) and subtracting from (i), we have

$$T_1=-\tfrac{1}{2}\rho\int_S (\phi-c)\frac{\partial\phi}{\partial n}\,dS-\tfrac{1}{2}\rho\int_\Gamma (\phi-c)\frac{\partial\phi}{\partial n}\,d\Gamma \quad \text{...(iii)}$$

where c is any constant.

Now from (ii), we have

$\int\frac{\partial\phi}{\partial n}\,d\Gamma$ is independent of Γ *i.e.* equal to zero. Since for a solid boundary $\int\frac{\partial\phi}{\partial n}\,dS=0$.

If at infinity ϕ tends to a constant c, then enlarge the surface Γ indefinitely in all directions, we have

$$\int_\Gamma (\phi-c)\frac{\partial\phi}{\partial n}\,d\Gamma=0$$

then (iii) reduce to

$$T_1=-\tfrac{1}{2}\rho \int_S (\phi-c)\frac{\partial \phi}{\partial n}\,\partial S$$

$$T_1=-\tfrac{1}{2}\rho \int_S \phi \frac{\partial \phi}{\partial n}\,dS.$$

§ 4·64 Kelvin's minimum energy theorem.

The irrotational motion of a liquid occupying a simple connected region has less kinetic energy than any other motion consistent with the same normal velocity of the boundary.

Let T be the kinetic energy of the irrotational motion of which ϕ is the velocity potential and $\mathbf{q}$ the fluid velocity, then

$$\mathbf{q}=-\nabla\phi \qquad \text{...(i)}$$

Again, let $\mathbf{q}_0$ be the fluid velocity of any other motion consistent with the same normal velocity of the boundary whose kinetic energy be T_0.

Then $\mathbf{n}.\mathbf{q}_0=\mathbf{n}.\mathbf{q}$

or $\mathbf{n}.(\mathbf{q}_0-\mathbf{q})=0$

or $\mathbf{n}.\mathbf{u}=0.$ ...(ii)

{where $\mathbf{u}=\mathbf{q}_0-\mathbf{q}$

Also, the equation of continuity is given by

$$\nabla.\mathbf{q}_0=0=\nabla.\mathbf{q}$$

or $\nabla.(\mathbf{q}_0-\mathbf{q})=0$

or $\nabla.\mathbf{u}=0.$ ...(iii)

Then $T_0=\tfrac{1}{2}\int_V \rho \mathbf{q}_0^2\,dv$

$$=\tfrac{1}{2}\int_V \rho(\mathbf{q}+\mathbf{u})^2\,dv$$

or $T_0=\tfrac{1}{2}\int_V \rho\,(q^2+u^2+2\mathbf{q}.\mathbf{u})\,dv$

$$\left[\begin{array}{l} \textit{Since}\quad \nabla.(\phi\mathbf{u})=\nabla\phi.\mathbf{u}+\phi\nabla.\mathbf{u} \\ \qquad\qquad\qquad\quad =\nabla\phi.\mathbf{u} \\ \textit{So}\quad \int_V \nabla\phi.\mathbf{u}\,dv=\int_V \nabla.(\phi\mathbf{u})\,dv \end{array}\right. \qquad \{\textit{from}\ \text{(iii)}$$

or $T_0=\tfrac{1}{2}\int_V \rho q^2\,dv+\tfrac{1}{2}\int_V \rho u^2 dv+\int_V \rho\,(\mathbf{q}.\mathbf{u})\,dv$

or $T_0=T+\tfrac{1}{2}\int_V \rho u^2\,dv-\rho\int_V (\nabla\phi.\mathbf{u})\,dv$

$$\begin{cases} =\int_V \phi \mathbf{n}.\mathbf{u}\, dS \\ =Zero. \end{cases}$$

$\left\{\begin{array}{l} \textit{(By Gauss Theorem} \\ \textit{Since } \mathbf{u}=\mathbf{q}_0-\mathbf{q} \textit{ on} \\ \text{the } \textit{surface S} \end{array}\right.$

Thus $\quad T_0-T=\frac{1}{2}\int \rho u^2\, dv$

$$>0$$

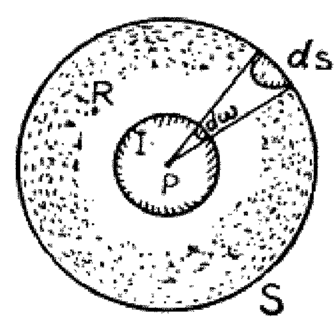
i. e. $\quad T_0-T>0$

or $\quad T<T_0.$

Hence for an irrotational motion of a liquid occupying a simple connected region has less kinetic energy than any other motion consistent with the same normal velocity of the boundary.

§ 4·7. Mean value of the velocity potential over a spherical surface.

If a region lying wholly in the liquid be bounded by a spherical surface the mean value of the velocity potential over the surface is equal to its value at the centre of the sphere.



Consider liquid at rest at infinity bounded internally by a closed surface R and unbounded externally. Describe a sphere S with P as centre of radius r large enough to enclose R. Let ϕ_S and ϕ_R denote the value of the velocity potential at P and the mean value of ϕ over a sphere S of radius r.

Let $d\omega$ be the solid angle subtended at the centre of the sphere by an element of area dS such that

$$\frac{dS}{d\omega}=\frac{r^2}{1^2} \quad \text{or} \quad d\omega=\frac{dS}{r^2} \qquad ..(1)$$

$\left\{\begin{array}{l}\text{Since the other concentric} \\ \text{sphere is of unit radius.}\end{array}\right.$

Now $\quad \phi_R=\frac{1}{4\pi r^2}\int_S \phi\, dS$

or $\quad \phi_R=\frac{1}{4\pi r^2}\int_R \phi r^2\, d\omega=\frac{1}{4\pi}\int_R \phi\, d\omega$

Then $\quad \frac{\partial \phi_R}{\partial r}=\frac{1}{4\pi}\int_R \frac{\partial \phi}{\partial r}\, d\omega$

$$=\frac{1}{4\pi r^2}\int_S \frac{\partial \phi}{\partial r}\, dS \qquad ...(2$$

Since $\int_S \frac{\partial \phi}{\partial r}\, dS=\int_S \frac{\partial \phi}{\partial n}\, dS=\int_V \mathbf{n}.\nabla\phi\, dS$

$$=\int_V \nabla^2\phi \, dv$$

(By divergence theorem)

$$=0$$ {Since $\nabla^2\phi=0$

Thus $\displaystyle\int_S \frac{\partial\phi}{\partial r}\,dS=0.$ ...(3)

From (2) and (3), we have

$\dfrac{\partial\phi_R}{\partial r}=0$, it follows that ϕ_R is independent of r.

i. e. $\phi_R=$Constant.

When S approaches to the point P.

Then $\phi_R=\phi_P$.

Thus the mean value of the velocity potential ϕ over the surface is equal to the value at the centre of the sphere.

Cor I. *In an irrotational motion the maximum values of the speed must occur on the boundary.*

Consider a point P to the fluid as origin and Q be any point nearer to P. Let q and q_1 are the speeds at the points P and Q. Let the direction of motion at the point P be the X-axis.

Then $q^2=\left(\dfrac{\partial\phi}{\partial x}\right)^2$ at the point P

$$q_1^2=\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2 \text{ at the point } Q.$$

Since $\nabla^2\left(\dfrac{\partial\phi}{\partial x}\right)=\dfrac{\partial}{\partial x}\,(\nabla^2\phi)=0.$

Now $\dfrac{\partial\phi}{\partial x}$ satisfies the Laplace's Equation. So there cannot be a maximum or minimum at P.

Thus there are points in the neighbourhood of P, such as Q, at which

$$\left(\frac{\partial\phi}{\phi x}\right)_Q^2>\left(\frac{\partial\phi}{\partial x}\right)_P^2$$

So $$q_1^2 > q^2.$$

Hence the velocity q at the point P cannot be maximum inside the fluid and its maximum value, if any, must occur on the boundary. Also q^2 may be minimum in the interior on the fluid, for $q=0$ at stagnation points.

Cor. II. *In steady irrotational motion the hydrodynamical pressure has its minimum values on the boundary.*

We know by Bernoulli's Theorem

$$\frac{p}{\rho}+\frac{1}{2}\,q^2=\text{Constant.}$$

$\left\{\begin{array}{l}\text{as } \dfrac{\partial\phi}{\partial t}=0 \text{ being steady motion and the effect of body force} \\ \text{are neglected.}\end{array}\right.$

Thus q^2 is greatest when the pressure p must be least, but this condition is not satisfied inside the fluid. Hence the minimum value of the pressure p occur on the boundary and the maximum value of p occur at the stagnation point.

Cor III. *The mean value of ϕ over any spherical surface, is equal to the value of ϕ at the centre of the sphere throughout whose interior $\nabla^2\phi=0$.*

Describe a sphere S of radius r about P as centre. Then

$$\phi=-\frac{1}{4\pi}\int_S \phi\frac{\partial}{\partial r}\left(\frac{1}{r}\right)dS+\frac{1}{4\pi r}\int_S\frac{\partial\phi}{\partial r}\,dS \quad \text{at the point } P$$

$\left\{\begin{array}{l}\text{Ref § 4·6 (ii)}\\ \text{Ref § 4·6 (i)}\end{array}\right.$

But the second integral vanishes.

Thus
$$\phi=-\frac{1}{4\pi}\int_S\phi\,\frac{\partial}{\partial r}\left(\frac{1}{r}\right)dS$$

$$\phi=\frac{1}{4\pi r^2}\int_S\phi\,dS$$

Cor. IV. *ϕ cannot be a maximum or minimum in the interior of any region throughout which $\nabla^2\phi=0$.*

If the velocity potential ϕ at the point P is maximum, then it will be greater than all the values of ϕ at the points of a small sphere of radius λ (where λ is small) with the same centre P. Thus the mean value of the velocity potential must be less than the velocity potential at P, which contradicts the above theorem. Hence ϕ at the point P cannot be maximum or minimum in the interior of any regions.

§ 4·8. Uniqueness Theorem.

Here we shall discuss some cases of acyclic irrotational motion of a liquid *i. e.* the motion in which the velocity potential is single-valued.

We know that the kinetic energy is given by

$$T=-\tfrac{1}{2}\rho\int_V q^2\,dv$$

$$=-\tfrac{1}{2}\rho\int_S\phi\,\frac{\partial\phi}{\partial n}\,dS.$$

Cor I. *Acyclic irrotational motion is impossible in a liquid bounded wholly by fixed rigid walls.*

Since $\dfrac{\partial\phi}{\partial n}=0$ at every points of the boundary, then

$$\int_V q^2\,dv=0.$$

Now q^2 cannot be negative, $q=0$ everywhere and the liquid is at rest.

Cor II. *There cannot be two different forms of acyclic irrotational motion for a given confined mass of liquid whose boundaries have prescribed velocities.*

Consider ϕ_1 and ϕ_2 be the velocity potential of two different motions subject to the conditions

$$\frac{\partial\phi_1}{\partial n}=\frac{\partial\phi_2}{\partial n} \text{ (at every points of the boundary)}$$

Let $\qquad \phi=\phi_1-\phi_2$

or $\qquad \nabla^2\phi=\nabla^2\phi_1-\nabla^2\phi_2=0.$

Then ϕ is solution of Laplace's Equation and represents a possible form of irrotational motion in which

$$\frac{\partial\phi}{\partial n}=\frac{\partial\phi_1}{\partial n}-\frac{\partial\phi_2}{\partial n}$$
$$=0.$$

Hence $\quad q=0 \;\Rightarrow\; \phi=\phi_1-\phi_2=\text{Const.}$

Thus the motion considered by ϕ_1 and ϕ_2 are essentially the same. So an acyclic irrotational motion is determined uniquely when the boundaries have prescribed velocities.

Cor. III. *There cannot be two different forms of irrotational motion for a given confined mass of inviscid fluid whose boundaries are subject to given impulse.*

Let ϕ_1 and ϕ_2 be the velocity potential of two different liquid motions in a region bounded by the surface S. The impulsive pressure for these motions are $\rho\phi_1$ and $\rho\phi_2$.

Then $\rho\phi_1=\rho\phi_2$ (at each point of the boundary).

Now $\phi=\phi_1-\phi_2$ satisfies Laplace's equation and is the velocity potential of a possible irrotational motion, such that $\phi=0$ at every point of the boundary.

i.e. $\qquad \rho\phi=\rho\,(\phi_1-\phi_2)$
$\qquad\qquad =0 \qquad$ (on the surface S)

or $\qquad$ K. E. $\quad T=\displaystyle\int_V \tfrac{1}{2}\rho q^2\,dv$

$$=-\tfrac{1}{2}\int_S \rho\phi\,\frac{\partial\phi}{\partial n}\,dS=0.$$

Then $q=0$ at each point of the region $\Rightarrow$ that $(\phi_1-\phi_2)$ is constant. Hence the two motions are essentially the same.

Cor. IV. *The acyclic irrotational motion of a liquid, at rest at infinity, due to the prescribed motion of an immersed solid, is uniquely determined by the motion of the solid.*

Let ϕ_1 and ϕ_2 be the velocity potentials of two different motions. The boundary conditions are

(a) $\dfrac{\partial\phi_1}{\partial n}=\dfrac{\partial\phi_2}{\partial n}$ (at the surface of the solid).

(b) $q_1=q_2=0$ (at infinity).

Thus $\phi=\phi_1-\phi_2$ is the velocity potential of a possible motion.

Since $\nabla^2\phi=\nabla^2\phi_1-\nabla^2\phi_2=0$

and $\dfrac{\partial\phi}{\partial n}=\dfrac{\partial\phi_1}{\partial n}-\dfrac{\partial\phi_2}{\partial n}=0$, at the surface of the solid, $q=0$ at infinity.

Thus $\phi_1-\phi_2=$Constant.

Hence the motions are essentially the same.

Ex. 1. *Shew that in the motion of a fluid in two dimensions if the coordinates (x, y) of an element at any time be expressed in terms of the initial coordinates $(a\ b)$ and the time, the motion is irrotational if*

$$\frac{\partial\,(\dot{x}\ x)}{\partial\,(a\ b)}+\frac{\partial\,(\dot{y}\ y)}{\partial\,(a\ b)}=0.$$

Since the motion is in two dimensions. Consider u and v are the components of velocity parallel to coordinate axes,

then $\dot{x}=\dfrac{dx}{dt}=u$ and $\dot{y}=\dfrac{dy}{dt}=v$

Again
$$\left.\begin{aligned}\frac{\partial u}{\partial a}&=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial a}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial a}\\ \frac{\partial u}{\partial b}&=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial b}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial b}\end{aligned}\right\}\qquad\ldots(1)$$

Similarly for $\dfrac{\partial v}{\partial a}$ and $\dfrac{\partial v}{\partial b}$.

Now
$$\frac{\partial\,(\dot{x}\ x)}{\partial\,(a\ b)}=\begin{vmatrix}\dfrac{\partial\dot{x}}{\partial a} & \dfrac{\partial\dot{x}}{\partial b}\\[2mm] \dfrac{\partial x}{\partial a} & \dfrac{\partial x}{\partial b}\end{vmatrix}$$

$$=\frac{\partial\dot{x}}{\partial a}\cdot\frac{\partial x}{\partial b}-\frac{\partial\dot{x}}{\partial b}\cdot\frac{\partial x}{\partial a}$$

$$=\left(\frac{\partial u}{\partial a}\cdot\frac{\partial x}{\partial b}-\frac{\partial u}{\partial b}\cdot\frac{\partial x}{\partial a}\right)$$

and $$\frac{\partial(\dot{y}\,y)}{\partial(a\,b)}=\begin{vmatrix} \dfrac{\partial\dot{y}}{\partial a} & \dfrac{\partial\dot{y}}{\partial b} \\ \dfrac{\partial y}{\partial a} & \dfrac{\partial y}{\partial b}\end{vmatrix}$$

$$=\left(\frac{\partial v}{\partial a}\cdot\frac{\partial y}{\partial b}-\frac{\partial v}{\partial b}\cdot\frac{\partial y}{\partial a}\right)$$

Hence from the given condition, the motion is irrotational, if

$$\frac{\partial(\dot{x}\,x)}{\partial(a\,b)}+\frac{\partial(\dot{y}\,y)}{\partial(a\,b)}=0$$

$$=\left(\frac{\partial u}{\partial a}\cdot\frac{\partial x}{\partial b}-\frac{\partial u}{\partial b}\cdot\frac{\partial x}{\partial a}\right)+\left(\frac{\partial v}{\partial a}\cdot\frac{\partial y}{\partial b}-\frac{\partial v}{\partial b}\cdot\frac{\partial y}{\partial a}\right)$$

From relation (1), we have

$$=\frac{\partial x}{\partial b}\left\{\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial a}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial a}\right\}-\frac{\partial x}{\partial a}\left\{\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial b}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial b}\right\}$$
$$+\frac{\partial y}{\partial b}\left\{\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial a}+\frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial a}\right\}-\frac{\partial y}{\partial a}\left\{\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial b}+\frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial b}\right\}$$

$$=\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\left\{\frac{\partial x}{\partial a}\cdot\frac{\partial y}{\partial b}-\frac{\partial y}{\partial a}\cdot\frac{\partial x}{\partial b}\right\}$$

$$=\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\frac{\partial(x,y)}{\partial(a,b)}$$

So $\dfrac{\partial(x\,y)}{\partial(a\,b)}\neq 0$ since x and y are independent, by Lagrangian equation of continuity.

Thus $$\frac{\partial(\dot{x}\,x)}{\partial(a\,b)}+\frac{\partial(\dot{y}\,y)}{\partial(a\,b)}=0$$

$$\Leftrightarrow\quad \frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=0$$

i.e. $\nabla\times\mathbf{q}=0 \Rightarrow$ the motion is irrotational.

Ex. 2. *Shew that if the velocity potential of an irrotational motion is equal to* $A\,(x^2+y^2+z^2)^{-3/2}\,z\,\tan^{-1}\left(\dfrac{y}{x}\right)$ *the lines of flow lie on the series of surfaces*

$$x^2+y^2+z^2=k^{2/3}\;(x^2+y^2)^{2/3}$$

The velocity potential is

$$\phi=A\,(x^2+y^2+z^2)^{-3/2}.z\;\tan^{-1}\left(\frac{y}{x}\right)$$

Let $(r\;\theta\;\omega)$ be the spherical coordinates,

Then $\phi=A.\dfrac{1}{r^3}.r\cos\theta\;\omega$

$$\phi=\frac{A}{r^2}\;\omega\cos\theta$$

Let u, v, w be the components of velocity along the coordinate axes respectively

$$u=-\frac{\partial\phi}{\partial r}=\frac{2A\omega\cos\theta}{r^3}$$

$$v=-\frac{\partial\phi}{r\partial\theta}=\frac{A\omega\sin\theta}{r^3}$$

and
$$w=-\frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\omega}=-\frac{A}{r^3}\cot\theta$$

The differential equation to the lines of flow is given by

$$\frac{dr}{u}=\frac{rd\theta}{v}=\frac{r\sin\theta d\omega}{w}$$

or
$$\frac{dr}{\frac{2A\omega\cos\theta}{r^3}}=\frac{rd\theta}{\frac{A\omega\sin\theta}{r^3}}=\frac{r\sin\theta\,d\omega}{\frac{A\cot\theta}{r^3}}$$

or
$$\underset{\text{I}}{\frac{r^3\,dr}{2\omega\cos\theta}}=\underset{\text{II}}{\frac{r^4\,d\theta}{\omega\sin\theta}}=\underset{\text{III}}{\frac{r^4\sin\theta\,d\omega}{-\cot\theta}}$$

From I and II, we get

$$\frac{r^3dr}{\omega\cos\theta}=\frac{r^4\,d\theta}{2\omega\sin\theta}$$

or
$$\frac{dr}{2r}=\cot\theta\,d\theta$$

By integrating, we have

$$\log r=2\log\sin\theta+\log k^2$$

(where k is an arbitrary constant).

or $$r=k^2\sin^2\theta$$

or $$r^3=k^2r^2\sin^2\theta$$

or $$(x^2+y^2+z^2)^{3/2}=k^2(x^2+y^2)$$

$$\left\{\begin{array}{l}\text{as } r^2=x^2+y^2+z^2\\ \text{and } r^2\sin^2\theta=x^2+y^2\end{array}\right.$$

or $$x^2+y^2+z^2=k^{2/3}(x^2+y^2)^{2/3}$$

Ex. 3. *A thin stratum of incompressible fluid is contained between two concentric spheres ; shew that the velocity at any point is equivalent to the components*

$$-\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\omega},\ \frac{\partial\psi}{\partial\theta}$$

along the meridian and parallel respectively. Also if the fluid be homogeneous and the motion irrotational, prove that

$$\frac{\partial\phi}{\partial\theta}=\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\omega},\ -\frac{1}{\sin\theta}\frac{\partial\phi}{\partial\omega}=\frac{\partial\psi}{\partial\theta},$$

and deduce that $\phi+i\psi=F(e^{i\omega}\tan\tfrac{1}{2}\theta)$

Since the fluid is incompressible then ρ is constant. There is no motion along the radius vector, the equation of continuity reduce to

$$\frac{\partial}{\partial\theta}(v \sin\theta)+\frac{\partial w}{\partial\omega}=0 \quad ...(i)$$

{Ref. equation of continuity in polar sp. coordinates

Now (i) is the condition that

$v \sin\theta\, d\omega - w\, d\theta=0$, is an exact differential $-d\psi$ (Let).

or $\quad v \sin\theta\, d\omega - w d\theta = -d\psi \quad$ (ii)

or $\quad v \sin\theta\, d\omega - w d\theta = -\left(\frac{\partial\psi}{\partial\omega}.\, d\omega+\frac{\partial\psi}{\partial\theta}.d\theta\right)$

Comparing the coefficients of $d\omega$ and $d\theta$, we have

$$v \sin\theta = -\frac{\partial\psi}{\partial\omega} \text{ and } w=\frac{\partial\psi}{\partial\theta}$$

$$v=-\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\omega} \text{ and } w=\frac{\partial\psi}{\partial\theta} \quad ...(iii)$$

determine the components of the velocity at any point.

Proved.

The differential equation to the lines of flow is

$$\frac{a d\theta}{v}=\frac{a \sin\theta d\omega}{w}$$

[consider an elementary element $a\delta\theta$ on the surface of the sphere.

or $\quad \frac{d\theta}{v}=\frac{\sin\theta d\omega}{w}$

or $\quad v \sin\theta\, d\omega - w d\theta=0.$

which gives from (ii)

$$d\psi=0$$

$$\psi=\text{const.}$$

Thus ψ is a stream function for the motion, **Proved.**

Since the motion is irrotational hence the velocity potential ϕ will exist.

$$v=-\frac{\partial\phi}{\partial\theta} \text{ and } w=-\frac{1}{\sin\theta}\frac{\partial\phi}{\partial\omega} \quad ...(iv)$$

then from (iii) and (iv), we have

$$\frac{\partial\phi}{\partial\theta}=\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\omega} \text{ and } \frac{\partial\psi}{\partial\theta}=-\frac{1}{\sin\theta}.\frac{\partial\phi}{\partial\omega} \quad ...(v)$$

Proved.

Also $\frac{\partial}{\partial\theta}(\phi+i\psi)=\frac{1}{\sin\theta}\frac{\partial}{\partial\omega}(\psi-i\phi)$ {from (v)

$\frac{\partial}{\partial\theta}(\phi+i\psi)=-\frac{i}{\sin\theta}\frac{\partial}{\partial\omega}(\phi-i\psi)$ { let $\zeta=\phi+i\psi$

or $\quad \frac{\partial\zeta}{\partial\theta}=-\frac{i}{\sin\theta}\frac{\partial\zeta}{\partial\omega}$

or $\dfrac{\partial \zeta}{\partial \theta}+\dfrac{i}{\sin \theta}\dfrac{\partial \xi}{\partial \omega}=0.$

Let the auxiliary equation is of the form

$$\frac{d\theta}{\sin \theta}=\frac{d\omega}{i}=\frac{\partial \zeta}{0}$$

or $\qquad \operatorname{cosec} \theta \, d\theta=-i \, d\omega \qquad$ and $d\zeta=0$

i.e. $\zeta=$Const.

By integrating, we have

$$\log \tan \theta/2+i\omega=\text{Const.} \qquad \ldots\text{(vi)}$$

$\{\phi+i\psi=$Const.

or $$\tan \theta/2 . e^{i\omega}=\text{Const.} \qquad \ldots\text{(vii)}$$

Then from (vi) and (vii). we have

$$\phi+i\psi=F\,(e^{i\omega} \tan \tfrac{1}{2} \theta).$$ **Proved.**

Ex. 4. *In the case of irrotational motion in two-dimensions on the surface of a sphere, shew that the velocity potential is of the form*

$$f\left(\frac{x+iy}{r+z}\right)+f\left(\frac{x-iy}{r+z}\right),$$

r being the radius of the sphere and x, y, z the coordinates of a point referred to rectangular axes through the centre of the sphere.

As in Ex. 3, we proved that

$\phi+i\psi=f(\tan \theta/2 . e^{i\omega})$

or $\phi+i\psi=f(\tan \theta/2 \cos \omega+i \tan \theta/2 \sin \omega).$ $\qquad \ldots(1)$

and $\phi-i\psi=f(\tan \theta/2 \cos \omega-i \tan \theta/2 \sin \omega).$ $\qquad \ldots(2)$

(Conjugate function of (1)

By adding (1) and (2), we have

$$2\phi=f\{\tan \theta/2\,(\cos \omega+i \sin \omega)\}+f\{\tan \theta/2\,(\cos \omega-i \sin \omega)\} \qquad \ldots(3)$$

We have

$$\frac{x+iy}{r+z}=\frac{r \sin \theta \cos \omega+ir \sin \theta \sin \omega}{r+r \cos \theta}$$

$$\left\{\begin{aligned} \text{as } x&=r \sin \theta \cos \omega \\ y&=r \sin \theta \sin \omega \\ z&=r \cos \theta \end{aligned}\right.$$

or $$\frac{x+iy}{r+z}=\frac{\sin \theta\,(\cos \omega+i \sin \omega)}{1+\cos \theta}$$

or $$\frac{x+iy}{r+z}=\tan \frac{\theta}{2}\,(\cos \omega+\sin \omega) \qquad \ldots(4)$$

Similarly $\dfrac{x-iy}{r+z}=\tan\dfrac{\theta}{2}(\cos\omega-i\sin\omega)$ ...(5)

Hence (3) becomes with the help of (4) and (5)

$$2\phi=f\left(\frac{x+iy}{r+z}\right)+f\left(\frac{x-iy}{r+z}\right).$$

Thus velocity potential is of the form

$$f\left(\frac{x+iy}{r+z}\right)+f\left(\frac{x-iy}{r+z}\right).$$

Proved

Ex. 5. *Prove that if*

$$\lambda=\frac{\partial u}{\partial t}-v\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+w\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)$$

and μ, ν *are two similar expressions, then* $\lambda dx+\mu dy+\nu dz$ *is a perfect differential, if the forces are conservative and the density is constant.*

Consider V be the potential function of the conservative force. Let ρ be the density of the fluid and p its pressure at any point $P(x, y, z)$ at time t.

The Euler's hydro-dynamical equations are,

$$\frac{Du}{Dt}=-\frac{\partial V}{\partial x}-\frac{1}{\rho}\frac{\partial p}{\partial x} \quad ...(i)$$

$$\frac{Dv}{Dt}=-\frac{\partial V}{\partial y}-\frac{1}{\rho}\frac{\partial p}{\partial y} \quad ...(ii)$$

$$\frac{Dw}{Dt}=-\frac{\partial V}{\partial z}-\frac{1}{\rho}\frac{\partial p}{\partial z} \quad ...(iii)$$

Multiplying (i), (ii) and (iii) by dx, dy and dz respectively and adding, we have

$$\frac{Du}{Dt}dx+\frac{Dv}{Dt}dy+\frac{Dw}{Dt}dz=-d\left(V+\frac{p}{\rho}\right) \quad ...(iv)$$

Since $\quad \lambda=\dfrac{\partial u}{\partial t}-v\left(\dfrac{\partial v}{\partial x}-\dfrac{\partial u}{\partial y}\right)+w\left(\dfrac{\partial u}{\partial z}-\dfrac{\partial w}{\partial x}\right)$

or $\quad \lambda=\left(\dfrac{\partial u}{\partial t}+u\dfrac{\partial u}{\partial x}+v\dfrac{\partial u}{\partial y}+w\dfrac{\partial u}{\partial z}\right)-\left(u\dfrac{\partial u}{\partial x}+v\dfrac{\partial v}{\partial x}+w\dfrac{\partial w}{\partial x}\right)$

or $\quad \lambda=\dfrac{Du}{Dt}-\dfrac{1}{2}\dfrac{\partial}{\partial x}(q^2)$ $\quad$ {where $q^2=u^2+v^2+w^2$

Therefore $\lambda dx+\mu dy+\nu dz$

$$=\left[\frac{Du}{Dt}-\frac{1}{2}\frac{\partial}{\partial x}(q^2)\right]dx+\left[\frac{Dv}{Dt}-\frac{1}{2}\frac{\partial}{\partial y}(q^2)\right]dy$$

$$+\left[\frac{Dw}{Dt}+\frac{1}{2}\frac{\partial}{\partial z}(q^2)\right]dz$$

$$=\left[\frac{Du}{Dt}\,dx+\frac{Dv}{Dt}\,dy+\frac{Dw}{Dt}\,dz\right]-\frac{1}{2}\left[\frac{\partial}{\partial x}(q^2)\,dx\right.$$
$$\left.+\frac{\partial}{\partial y}(q^2)\,dy+\frac{\partial}{\partial z}(q^2)\,dz\right]$$
$$=-d\left[V+\frac{p}{\rho}\right]-\frac{1}{2}\,d\,[q^2]$$
$$=-d\left[V+\frac{p}{\rho}-\frac{1}{2}\,q^2\right]$$

it follows that $\lambda dx+\mu dy+\nu dz$ is a perfect differential.

Ex. 6. *In irrotational motion in two dimensions, prove that*

$$\left(\frac{\partial q}{\partial x}\right)^2+\left(\frac{\partial q}{\partial y}\right)^2=q\nabla^2 q$$

Since the motion is irrotational thus velocity potential will exist and satisfies the Laplace Equation

$$\nabla^2\phi=0$$

or
$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0. \qquad ...(1)$$

We know that, if q be the velocity at a point

Then $$q^2=\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2 \qquad ...(2)$$

Differentiating (2) partially with regard to x and y respectively

$$q\,\frac{\partial q}{\partial x}=\frac{\partial\phi}{\partial x}\cdot\frac{\partial^2\phi}{\partial x^2}+\frac{\partial\phi}{\partial y}\cdot\frac{\partial^2\phi}{\partial x\partial y} \qquad ...(3)$$

$$q\,\frac{\partial q}{\partial y}=\frac{\partial\phi}{\partial x}\cdot\frac{\partial^2\phi}{\partial x\partial y}+\frac{\partial\phi}{\partial y}\cdot\frac{\partial^2\phi}{\partial y^2} \qquad ...(4)$$

Differentiating (2) partially w.r. to x and (4) w. r. to y, we have

$$q\,\frac{\partial^2 q}{\partial x^2}+\left(\frac{\partial q}{\partial x}\right)^2=\left(\frac{\partial^2\phi}{\partial x^2}\right)^2+\frac{\partial\phi}{\partial x}\cdot\frac{\partial^3\phi}{\partial x^3}+\left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2+\frac{\partial\phi}{\partial y}\cdot\frac{\partial^3\phi}{\partial x^2\partial y}$$

and $$q\,\frac{\partial^2 q}{\partial y^2}+\left(\frac{\partial q}{\partial y}\right)^2=\left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2+\frac{\partial\phi}{\partial x}\cdot\frac{\partial^3\phi}{\partial x\partial y^2}+\left(\frac{\partial^2\phi}{\partial y^2}\right)^2+\frac{\partial\phi}{\partial y}\cdot\frac{\partial^3\phi}{\partial y^3}$$

By adding these two, we get

$$q\left(\frac{\partial^2 q}{\partial x^2}+\frac{\partial^2 q}{\partial y^2}\right)+\left(\frac{\partial q}{\partial x}\right)^2+\left(\frac{\partial q}{\partial y}\right)^2=\left(\frac{\partial^2\phi}{\partial x^2}\right)^2+2\left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2+\left(\frac{\partial^2\phi}{\partial y^2}\right)^2$$
$$+\frac{\partial\phi}{\partial x}\frac{\partial}{\partial x}\left\{\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right\}+\frac{\partial\phi}{\partial y}\cdot\frac{\partial}{\partial y}\left\{\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right\} \qquad ...(5)$$

Hence (5) reduces to with the help of (1),

$$q.\nabla^2 q+\left\{\left(\frac{\partial q}{\partial x}\right)^2+\left(\frac{\partial q}{\partial y}\right)^2\right\}=\left(\frac{\partial^2\phi}{\partial x^2}\right)^2+2\left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2+\left(\frac{\partial^2\phi}{\partial y^2}\right)^2$$

$$=2\left(\frac{\partial^2\phi}{\partial x^2}\right)^2+2\left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2$$

$$\left\{\text{ as } \frac{\partial^2\phi}{\partial x^2}=-\frac{\partial^2\phi}{\partial y^2}\right.$$

or $$q.\nabla^2 q+\left\{\left(\frac{\partial q}{\partial x}\right)^2+\left(\frac{\partial q}{\partial y}\right)^2\right\}=2\left\{\left(\frac{\partial^2\phi}{\partial x^2}\right)^2+\left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2\right\} \quad ...(6)$$

Squaring and adding (3) and (4), we have

$$q^2\left[\left(\frac{\partial q}{\partial x}\right)^2+\left(\frac{\partial q}{\partial y}\right)^2\right]=\left(\frac{\partial\phi}{\partial x}\right)^2\left\{\left(\frac{\partial^2\phi}{\partial x^2}\right)^2+\left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2\right\}$$

$$+\left(\frac{\partial\phi}{\partial y}\right)^2\left\{\left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2+\left(\frac{\partial^2\phi}{\partial^2 y}\right)^2\right\}+2\,\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y}.\ \frac{\partial^2\phi}{\partial x\partial y}\left\{\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}\right\}$$

$$\{\text{from } (1)$$

or $$q^2\left[\left(\frac{\partial q}{\partial x}\right)^2+\left(\frac{\partial q}{\partial y}\right)^2\right]=q^2\left\{\left(\frac{\partial^2\phi}{\partial x^2}\right)^2+\left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2\right\}$$

or $$\left(\frac{\partial q}{\partial x}\right)^2+\left(\frac{\partial q}{\partial y}\right)^2=\left(\frac{\partial^2\phi}{\partial x^2}\right)^2+\left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2 \quad ...(7)$$

From (6) and (7), we have

$$q\nabla^2 q+\left(\frac{\partial q}{\partial x}\right)^2+\left(\frac{\partial q}{\partial y}\right)^2=2\left\{\left(\frac{\partial q}{\partial x}\right)^2+\left(\frac{\partial q}{\partial y}\right)^2\right\}$$

or $$q\nabla^2 q=\left(\frac{\partial q}{\partial x}\right)^2+\left(\frac{\partial q}{\partial y}\right)^2$$ **Proved.**

Ex. 7. *Show that the theorem that under certain conditions the motion of a frictionless fluid, if once irrotational, will always be so, is true also when each particle is acted on by a resistance varying as the velocity.*

Consider a point $P(x, y, z)$ and a point Q in its neighbourhood such that $PQ=\delta S$.

flow along $PQ=u\delta x+v\delta y+wdz$

therefore $\frac{D}{Dt}$ (flow along $PQ)=\frac{D}{Dt}(u\delta x+v\delta y+w\delta z)$

But $$\frac{D}{Dt}(u\delta x)=\delta x\,\frac{Du}{Dt}+u\,\frac{D}{Dt}\,\delta x$$

$$=\delta x\,\frac{Du}{Dt}+u\delta u$$

Components of the forces along the axes are

$$(-\lambda u,\ -\lambda v,\ -\lambda w)$$

from the equation of motion, we have

$$\frac{Du}{Dt}=-\frac{\partial V}{\partial x}-\frac{1}{\rho}\frac{\partial p}{\partial x}-\lambda u \quad \text{etc.}$$

or $$\frac{D}{Dt}(u\delta x)=\delta x\left[-\frac{\partial V}{\partial x}-\frac{1}{\rho}\frac{\partial p}{\partial x}-\lambda u\right]+u\delta u$$

or $$\frac{D}{Dt}(\text{flow along } PQ)=\delta x\left[-\frac{\partial V}{\partial x}-\frac{1}{\rho}\frac{\partial p}{\partial x}-\lambda u\right]$$

$$+\delta y\left[-\frac{\partial V}{\partial y}-\frac{1}{\rho}\frac{\partial p}{\partial y}-\lambda v\right]$$

$$+\delta z\left[-\frac{\partial V}{\partial z}-\frac{1}{\rho}\frac{\partial p}{\partial z}-\lambda w\right]+(u\delta u+v\delta v+w\delta w)$$

$$=-\delta V-\frac{\delta p}{p}+\tfrac{1}{2}\,\delta\,(u^2+v^2+w^2)-\lambda\,(u\delta x+v\delta y+w\delta z)$$

Let Γ be the circulation along APA

$$\Gamma=\int_A^A (u\delta x+v\delta y+w\delta z)$$

or $$\frac{D\Gamma}{Dt}=\left[-V-\int\frac{dp}{\rho}+\tfrac{1}{2}q^2\right]_A^A-\lambda\int_A^A(u\delta x+v\delta y+w\delta z)$$

$$=-\lambda\Gamma$$

or $$\Gamma=\Gamma_0\,e^{-\lambda t} \text{ where } \Gamma_0 \text{ is independent of } t.$$

If, initially $\xi=0$, $\eta=0$, $\zeta=0$ then $\Gamma=0$ when $t=0$.

i.e. $\Gamma_0=0$, thus $\Gamma=0$

Hence by Stoke's theorem, we have

$$\iint(l\xi+m\eta+n\zeta)\,dS=0 \text{ always}$$

therefore $\xi=0,\ \eta=0,\ \zeta=0$ always. **Proved.**

Ex. 8. *A rigid envelope is filled with homogeneous frictionless liquid ; Show that it is not possible, by any movements applied to the envelope, to set its contents into motion which will persist after the envelope has come to rest.*

The liquid motion is produced by the movements on the boundary. Equations of motion are

$$\left.\begin{aligned}u'-u&=-\frac{1}{\rho}\frac{\partial\omega}{\partial x}\\ v'-v&=-\frac{1}{\rho}\frac{\partial\omega}{\partial y}\\ w'-w&=-\frac{1}{\rho}\frac{\partial\omega}{\partial z}\end{aligned}\right\} \quad ...(1)$$

Where ω is the impulsive pressure. Here the components of the velocity u, v, w are zero.

So $\quad u'dx+v'dy+w'dz=-\frac{1}{\rho}\,d\omega=-d\phi$ (let)

when the density is constant, thus the motion is irrotational. Since the pressure at any point is single valued, ϕ is single valued, then motion is acyclic.

or $$\iiint q^2 dx\,dy\,dz=-\iint \phi\,\frac{\partial \phi}{\partial n}\,dS$$

If $\frac{\partial \phi}{\partial n}=0$ on the boundary $\Rightarrow$ q is zero every where. In other sense the liquid comes to rest.

Ex. 9. *Deduce from the principle that the kinetic energy set up is a minimum that, if a mass of incompressible liquid be given at rest, completely filling a closed vessel of any shape and if any motion of the liquid be produced suddenly by giving arbitrarily prescribed normal velocities at all points of its bounding surface subject to the condition of constant volume, the motion produced is irrotational.*

Let u, v, w be the components of velocity at any point and T be the kinetic energy.

Then $$T=\tfrac{1}{2}\iiint (u^2+v^2+w^2)\,dx\,dy\,dz \qquad \ldots(1)$$

Since T is minimum, $\delta T=0$

So $$\iiint u\delta u+v\delta v+w\delta w=0 \qquad (2)$$

on the boundary $lu+mv+nw$ is prescribed

thus $$l\delta u+m\delta v+n\delta w=0 \qquad \ldots(3)$$

Equation of continuity is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0.$$

$$\frac{\partial}{\partial x}\,\delta u+\frac{\partial}{\partial y}\,\delta v+\frac{\partial}{\partial z}\,\delta w=0. \qquad \ldots(4)$$

every where with in the liquid.

So $$\iiint \phi\left[\frac{\partial}{\partial x}\,\delta u+\frac{\partial}{\partial y}\,\delta v+\frac{\partial}{\partial z}\,\delta w\right] dxdydz=0.$$

or $$\iiint\left[\frac{\partial}{\partial x}\,(\phi\delta u)+\frac{\partial}{\partial y}\,(\phi\delta v)+\frac{\partial}{\partial z}\,(\phi\partial w)\right] dxdydz$$
$$-\iiint\left[\delta u\,\frac{\partial \phi}{\partial x}+\delta v\,\frac{\partial \phi}{\partial y}+\delta w\,\frac{\partial \phi}{\partial z}\right] dxdydz=0.$$

or $$-\iint \phi\,[l\delta u+m\delta v+n\delta w]\,dS$$
$$-\iiint\left[\frac{\partial\phi}{\partial x}\,\delta u+\frac{\partial\phi}{\partial y}\,\delta v+\frac{\partial\phi}{\partial z}\,\delta w\right]dxdydz=0$$

or $$\iiint\left[\frac{\partial\phi}{\partial x}\,\delta u+\frac{\partial\phi}{\partial y}\,\delta v+\frac{\partial\phi}{\partial z}\,\delta w\right]dxdydz=0. \qquad ...(5)$$

Adding (2) and (5), we have
$$\iiint\left[\left(u+\frac{\partial\phi}{\partial x}\right)\delta u+\left(v+\frac{\partial\phi}{\partial y}\right)\delta v+\left(w+\frac{\partial\phi}{\partial z}\right)\delta w\right]dxdydz=0$$

Hence
$$u+\frac{\partial\phi}{\partial x}=0,\ v+\frac{\partial\phi}{\partial y}=0,\ \ w+\frac{\partial\phi}{\partial z}=0.$$

i.e. u, v, w are derivable from ϕ hence the motion is irrotational.

Ex. 10. *If p denote the pressure, V the potential of the external forces and q the velocity of a homogeneous liquid moving irrotationally, Shew that ∇^2q^2 is positive and ∇^2p is negative provided $\nabla^2V=0$. Hence prove that the velocity cannot have a maximum value and the pressure cannot have a minimum value at a point in the interior of the liquid.*

Since the motion is irrotational, hence velocity potential ϕ exists.

and $$q^2=\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2$$

Now $$\nabla^2(UV)=V\,\nabla^2U+U\,\nabla^2V+2\left(\frac{\partial U}{\partial x}\cdot\frac{\partial V}{\partial x}+\frac{\partial U}{\partial y}\cdot\frac{\partial V}{\partial y}+\frac{\partial U}{\partial z}\cdot\frac{\partial V}{\partial z}\right)$$

or $$\nabla^2\left(\frac{\partial\phi}{\partial x}\right)^2=2\,\frac{\partial\phi}{\partial x}\,\nabla^2\left(\frac{\partial\phi}{\partial x}\right)+2\left[\left(\frac{\partial^2\phi}{\partial x^2}\right)^2+\left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2+\left(\frac{\partial^2\phi}{\partial x\partial z}\right)^2\right]$$

But $$\nabla^2\left(\frac{\partial\phi}{\partial x}\right)=\frac{\partial}{\partial x}\,\nabla^2\phi=0.$$

So $$\nabla^2\left(\frac{\partial\phi}{\partial x}\right)^2=2\left[\left(\frac{\partial^2\phi}{\partial x^2}\right)^2+\left(\frac{\partial^2\phi}{\partial x\partial y}\right)^2+\left(\frac{\partial^2\phi}{\partial x\partial z}\right)^2\right]$$
$$=\text{Positive}$$

Thus ∇^2q^2 is positive. **Proved.**

$$\frac{p}{\rho}-\frac{\partial\phi}{\partial t}+\tfrac{1}{2}\,q^2+V=f(t)$$

or $$\nabla\left(\frac{p}{\rho}\right)-\frac{\partial}{\partial t}\,\nabla^2\phi+\tfrac{1}{2}\nabla^2q^2+\nabla^2V=0$$

But $\nabla^2\phi=0=\nabla^2V$

$$\therefore \quad \frac{1}{\rho}\nabla^2 p = -\tfrac{1}{2}\nabla^2 q^2$$
$$= \text{Negative}$$

Thus $\nabla^2 p$ is negative. **Proved.**

From Green's theorem, we have

$$\iint (lU + mV + nW)\, dS = -\iiint \left[\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z}\right] dxdydz$$

Consider $\quad U = \frac{\partial \phi}{\partial x},\ V = \frac{\partial \phi}{\partial y},\ W = \frac{\partial \phi}{\partial z}$

$$\iint \frac{\partial \phi}{\partial n}\, dS = -\iiint \nabla^2 \phi \, dxdydz$$

Let $\quad \phi = q^2$

then $\quad \iint \frac{\partial}{\partial n}(q^2)\, dS = -\iiint \nabla^2 q^2 \, dxdydz$

Using this relation in the case of a liquid contained with in a small sphere, we have

$$\iint \frac{\partial q^2}{\partial n}\, dS = (\text{negative}) \quad (\text{as } \nabla^2 q^2 \text{ is positive})$$

but $\quad \delta n = -\delta r$

$$\therefore \quad \iint \frac{\partial q^2}{\partial r}\, dS = \text{positive}$$

If q^2 is maximum at a point within the sphere then $\frac{\partial q^2}{\partial r}$ is negative on the surface of a sphere surrounding that point.

or $\quad \iint \frac{\partial q^2}{\partial r}\, dS = \text{negative}$

which is contrary to our previous result, hence q^2 cannot have a maximum within the liquid. It can be maximum only on the boundary.

Similarly substitute $\phi = p$

$$\iint \frac{\partial p}{\partial n}\, dS = -\iiint \nabla^2 p \, dxdydz$$
$$= \text{positive.}$$

So p cannot have a maximum at a point with in the liquid ; the point of minimum pressure. It can occur only on the boundary.

Ex. 11. *A space is bounded by an ideal fixed surface S drawn in a homogeneous incompressible fluid, satisfying the conditions for the continued existence of a velocity potential ϕ under conservative* *forces. Prove that the rate per unit time at which energy flows across S into the space bounded by S is*

$$-\rho\iint\frac{\partial\phi}{\partial t}\cdot\frac{\partial\phi}{\partial n}\,dS$$

where ρ is the density and ∂n an element of the normal to dS drawn into the space considered.

Ideal surface is a surface which is free from any hydrodynamical singularity *i.e*, it is free from sources, sinks, doublets in the region enclosed by the surface.

Since the fluid is incompressible then the energy stored in the fluid by compression *i.e.* intrinsic energy vanishes.

The total energy is

$$=\text{K. E.}+\text{P. E.}+\text{Intrinsic Energy.} \qquad ...(1)$$

Let Ω be the force potential

i.e. $\quad F=-\nabla\Omega$ and $\dfrac{\partial\Omega}{\partial t}=0.$ $\qquad$ (*i.e.* Ω is independent of t).

$$\text{Potential Engergy}=\int\rho\Omega\,d_v$$

From (1), we have

$$\text{Total Energy}=\iiint_V \tfrac{1}{2}\rho q^2\,dv+\iiint_V\rho\Omega\,dv$$

(Since ρ is constant)

or $$\dot{E}=\tfrac{1}{2}\rho\iiint_V q^2dv+\rho\iiint_V\Omega dv \qquad ...(2)$$

where V is the volume bounded over fixed surface S.

Differentiating (2) partially w. r. to t, we have

$$\frac{\partial E}{\partial t}=\tfrac{1}{2}\rho\iiint_V\frac{\partial}{\partial t}\,(\nabla\phi.\nabla\phi)\,dv$$

{ Second integral vanish as Ω is independent of t

or $$\frac{\partial E}{\partial t}=\rho\iiint_V(\nabla\phi.\nabla\phi_t)\,dv \qquad ...(3)$$

{ where $\phi_t\Rightarrow\dfrac{\partial\phi}{\partial t}$

Again $\nabla\,(\phi_t.\nabla\phi)=\nabla\phi_t.\nabla\phi+\phi_t\nabla.\nabla\phi$

$$=\nabla\phi_t.\nabla\phi+\phi_t\nabla^2\phi=\nabla\phi_t\nabla\phi. \qquad ...(4)$$

{But $\nabla^2\phi=0$, by the equation of continuity.}

From (3) and (4), we have

$$\frac{\partial E}{\partial t}=\rho \iiint_V \nabla\cdot(\phi_t\nabla\phi)$$

$$\frac{\partial E}{\partial t}=-\rho \iint \phi_t\nabla\phi\cdot\hat{n}\, dS \qquad \{\hat{n}=\text{inward drawn normal}\}$$

Hence rate per unit time at which energy flows across the surface $=-\rho \iint \frac{\partial \phi}{\partial t}\cdot\frac{\partial \phi}{\partial n}\, dS$ **Proved.**

Ex. 12. *Shew that if* $\phi=-\frac{1}{2}(ax^2+by^2+cz^2)$

$$V=\tfrac{1}{2}(lx^2+my^2+nz^2)$$

where a, b, c ; l, m, n *are functions of time and* $a+b+c=0$, *irrotational motion is possible with a free surface of equi-pressure if*

$$(l+a^2+\dot{a})\, e^{2\int a\, dt}\ ;\ (m+b^2+\dot{b})\, e^{2\int b\, dt}\ ;\ (n+c^2+\dot{c})\, e^{2\int c\, dt}$$

are constants.

Since $\phi=-\frac{1}{2}(ax^2+by^2+cz^2)$

$$\frac{\partial \phi}{\partial x}=-(ax+by+cz)$$

$$\frac{\partial^2 \phi}{\partial x^2}=-(a+b+c)$$

So $\nabla^2\phi=-(a+b+c)$

$=0.$ (Given).

Thus, the irrotational motion is possible as velocity potential exists.

The equation of pressure for non-steady irrotational fluid motion is

$$\frac{p}{\rho}-\frac{\partial \phi}{\partial t}+\tfrac{1}{2}q^2+V=F(t)$$

or $$\frac{p}{\rho}=\frac{\partial \phi}{\partial t}-\tfrac{1}{2}\left\{\left(\frac{\partial \phi}{\partial x}\right)^2+\left(\frac{\partial \phi}{\partial y}\right)^2+\left(\frac{\partial \phi}{\partial z}\right)^2\right\}-V+F(t)$$

or $$\frac{p}{\rho}=F(t)-\tfrac{1}{2}(\dot{a}\,x^2+\dot{b}\,y^2+\dot{c}z^2)-\tfrac{1}{2}(a^2x^2+b^2y^2+c^2z^2)$$
$$-\tfrac{1}{2}(lx^2+my^2+nz^2)$$

or $$\frac{p}{\rho}=F(t)-\tfrac{1}{2}\left[(l+a^2+\dot{a})\,x^2+(m+b^2+\dot{b})y^2+(n+c^2+\dot{c})\,z^2\right]$$

For a free surface of equi-pressure, $p=$const, *i.e.* $\frac{dp}{dt}=0$.

$$\frac{dp}{dt}=\frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}+w\frac{\partial p}{\partial z}=0.$$

or $$\dot{F}(t)-\tfrac{1}{2}(\dot{l}+2a\dot{a}+\ddot{a})x^2-\tfrac{1}{2}(\dot{m}+2b\dot{b}+\ddot{b})y^2$$
$$-\tfrac{1}{2}(\dot{n}+2c\dot{c}+\ddot{c})z^2-ax^2(l+a^2+\dot{a})$$
$$-by^2(m+b^2+\dot{b})-cz^2(n+c^2+\dot{c})=0$$

or $$\dot{F}(t)=\tfrac{1}{2}x^2\{(\ddot{a}+2a\dot{a}+\dot{l})+2a(l+a^2+\dot{a})\}$$
$$+\tfrac{1}{2}y^2\{(\ddot{b}+2b\dot{b}+\dot{m})+2b(m+b^2+\dot{b})\}$$
$$+\tfrac{1}{2}z^2\{(\ddot{c}+2c\dot{c}+\dot{n})+2c(n+c^2+\dot{c})\}.$$

This equation is true for all values of t; So coefficients of x^2, y^2 and z^2 and the constant must vanish individualy, we have

$$(\ddot{a}+2a\dot{a}+\dot{l})+2a(l+a^2+\dot{a})=0 \qquad \ldots(i)$$

$$(\ddot{b}+2b\dot{b}+\dot{m})+2b(m+b^2+\dot{b})=0 \qquad \ldots(ii)$$

$$(\ddot{c}+2c\dot{c}+\dot{n})+2c(n+c^2+\dot{c})=0. \qquad \ldots(iii)$$

From (i), we have

$$\frac{(\ddot{a}+2a\dot{a}+\dot{l})}{\dot{a}+a^2+l}=-2a.$$

By integrating, we have

$$\log(l+a^2+\dot{a})+\int 2at\,dt=\text{constant}$$

or $$(l+a^2+\dot{a})\,e^{\int 2at\,dt}=\text{constant}.$$

Similarly other results can be obtained from (ii) and (iii).

Proved.

Ex. 13. *Prove that if the velocity potential at any instant be λxyz, the velocity at any point $(x+\xi,\ y+\eta,\ z+\zeta)$ relative to the fluid at the point (x, y, z) where ξ, η, ζ are small, is normal to the quadric $x\eta\zeta+y\zeta\xi+z\xi\eta=$constant with centre at (x, y, z).*

Let $q\,(u, v, w)$ and $q'\,(u', v', w')$ be the components of velocity of the fluid particle at the points $P\,(x\ y\ z)$ and $Q\,(x+\xi, y+\eta, z+\zeta)$.

Since $$\phi=\lambda\,x\,y\,z$$

then $$\left.\begin{aligned} u&=-\frac{\partial\phi}{\partial x}=-\lambda\,yz, \quad v=-\frac{\partial\phi}{\partial y}=-\lambda\,xz \\ \text{and}\quad w&=-\frac{\partial\phi}{\partial z}=-\lambda\,xy \end{aligned}\right] \qquad \ldots(i)$$

Also $$u'=u+\xi\frac{\partial u}{\partial x}+\eta\,\frac{\partial u}{\partial y}+\zeta\,\frac{\partial u}{\partial z}$$

where
$$\left.\begin{aligned} u'&=u-\lambda\,(\eta z+\zeta y)\\ v'&=v-\lambda\,(\zeta x+\xi z)\\ w'&=w-\lambda\,(\xi y+\eta x)\end{aligned}\right\}\qquad\text{...(ii)}$$
and

Velocity of Q relative to P is
$$(u'-u\,;\ v'-v,\ w-w')$$

i.e. $$\{-\lambda\,(\eta z+\zeta y)\,;\ -\lambda\,(\zeta x+\xi z)\ -\lambda\,;\ (\xi y+\eta x)\}\qquad\text{...(iii)}$$

{from (i) and (ii)}

Now the quadric is
$$F\equiv x\ \eta\xi+y\ \zeta\xi+z\ \xi\eta=\text{constant.}\qquad\text{...(iv)}$$

The direction ratios of the normal at the point
$$Q\ (x+\xi,\ y+\eta,\ z+\zeta)\ \text{are}$$
$$\left(\frac{\partial F}{\partial\xi},\ \frac{\partial F}{\partial\eta},\ \frac{\partial F}{\partial\zeta}\right)$$

i.e. $$(y\zeta+z\eta,\ x\zeta+z\xi,\ x\eta+y\xi)\qquad\text{...(iii)}$$

By comparing (iii) and (iv), we have that the velocity of Q relative to P is along the normal to the quadric (iv). **Proved.**

Ex. 14. *Prove that in a cyclic irrotional motion of a homogeneous fluid the total momentum of the fluid contained within the sphere of any radius is equivalent to a single vector through the centre of the sphere.*

Consider a spherical surface S enclosing the volume V of the fluid of density ρ. Let q be the fluid velocity and M the momentum contained by the fluid sphere.

$$M=\int_V \rho\ q\ dV.\qquad\text{...(i)}$$

Moment of Momentum about the centre of the sphere is

$$A=\int \rho r\times q\ dV$$

or $$A=\rho\int r\times\nabla\phi\ dV\qquad\text{...(ii)}$$

$\left\{\begin{array}{l}\text{Since the motion is irrotational,}\\ \quad\text{then}\qquad\qquad q=-\nabla\phi.\end{array}\right.$

Again $$\begin{aligned} r\times\nabla\phi&=\nabla\times(\phi r)\\ &=\phi\nabla\times r+\nabla\phi\times r\\ &=0-r\times\nabla\phi\end{aligned}$$

Hence (ii) becomes

$$A=\rho\int \nabla\times(\phi r)\,dV \quad \{\hat{n}=\text{inward drawn normal}$$

$$A=\rho\iint_S \hat{n}\times(\phi r)\,dS$$

$$\begin{cases}\text{as} \quad \hat{n}\times r=0\\ \text{on the surface of the sphere}\end{cases}$$

$$=\text{zero.}$$

Thus moment of momentum A about the centre of the sphere is zero. Hence M passes through the centre of the sphere.

Proved.

Ex. 15. *Incompressible fluid of density ρ is contained between two co-axial circular cylinders, af radii a and b $(a<b)$, and between two rigid planes perpendicular to the axis at a distance l apart. The cylinders are at rest and the fluid is circulating in irrotational motion, its velocity being V at the surface of the inner cylinder.*

Prove that the kinetic energy is

$$\pi\rho\, l\, a^2\, V^2 \log\left(\frac{b}{a}\right).$$

For irrotational two-dimensional fluid motion, we have

$$\nabla^2\psi=\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}=0. \qquad \ldots\text{(i)}$$

Since ψ is a function of r only,

So $$\frac{\partial^2\psi}{\partial\theta^2}=0,$$

then (i) reduces to

$$\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}=0,$$

or $$\frac{\partial\psi}{\partial r}=\frac{c}{r} \qquad \ldots\text{(ii)}$$

(Transverse velocity).

Now radial velocity is zero as $\frac{\partial\psi}{\partial\theta}=$ ˙.

Since $\frac{\partial\psi}{\partial r}=V$ when $r=a$ $\begin{cases}\text{as } V \text{ is the velocity at the}\\ \text{surface of cylinder } r=a.\end{cases}$

From (ii), we have $V=\frac{c}{a}$ or $c=aV$.

Thus $$q=\frac{aV}{r}$$

Hence the kinetic energy of the fluid

$$T=\int_a^b \tfrac{1}{2}\,(2\pi r\,dr\,l\rho).\left(\frac{aV}{r}\right)^2$$

or $$T=\pi l\rho\,a^2V^2\int_a^b \frac{1}{r}\,dr$$

or $$T=\pi l\rho\,a^2V^2\,\Big(\log r\Big)_a^b$$

or $$T=\pi\rho l\,a^2V^2\,\log\left(\frac{b}{a}\right).$$

Ex. 16. *Liquid of density ρ is flowing in two dimensions between the oval curves $r_1r_2=a^2$, $r_1r_2=b^2$ where r_1, r_2 are the distances measured from two fixed points ; if the motion is irrotational and quantity q per unit time crosses any line joining the bounding curves, then the kinetic energy is*

$$\pi\rho q^2 \Big/ \log\left(\frac{b}{a}\right).$$

The two-dimensional irrotational motion occurs in a doubly connected region. The equation to the curves are,

$r_1r_2=a^2$ (ψ=constant) and $r_1r_2=b^2$ (ψ=constant).

Assuming the complex potential w is of the form

$$w=iA\log(z-z_1)(z-z_2)$$

or $$\begin{aligned}\phi+i\psi&=iA\log\{r_1e^{i\theta_1}.r_2e^{i\theta_2}\}\\&=iA\log\{r_1r_2.e^{i(\theta_1+\theta_2)}\}\end{aligned}\quad\left\{\begin{aligned}&\text{as } z-z_1=r_1e^{i\theta_1}\\&\text{and } z-z_2=r_2e^{i\theta_2}\end{aligned}\right.$$

$$=iA\{\log r_1r_2+i(\theta_1+\theta_2)\}$$

Separating into real and imaginary parts, we have

$$\phi=-A(\theta_1+\theta_2),\quad \psi=A\log r_1r_2$$

Now $$q=\psi_b-\psi_a$$

or $$q=A\log b^2-A\log a^2=A\log\left(\frac{b^2}{a^2}\right)=2A\log\left(\frac{b}{a}\right)$$

or $$A=\frac{q}{2\log\left(\dfrac{b}{a}\right)}.$$

Since the region is doubly connected then

$$k=(\text{circulation})=A(2\pi+2\pi)=4\pi A$$

(difference on two sides of barrier)

$\left\{\begin{array}{l}\text{as decrease in } \phi \text{ on describing the}\\ \text{circuit completely once.}\end{array}\right.$

Thus kinetic energy of cyclic irrotational motion is given by

$$T=-\tfrac{1}{2}\rho\int \phi \frac{\partial \phi}{\partial n}\, dS-\tfrac{1}{2}\rho\, k \int \frac{\partial \phi}{\partial n}\, dS$$

$$T=2\pi\, A\rho \int_a^b d\psi \qquad \left\{ \begin{array}{l} \text{as} \int \phi \dfrac{\partial \phi}{\partial n}\, dS=0 \\ \text{on a rigid boundary} \end{array} \right.$$

$$T=2\pi\, A\rho\, (\psi_b-\psi_a)$$

$$T=\frac{\pi\rho q^2}{\log\left(\dfrac{b}{a}\right)}.$$

Proved.

Ex. 17. *Show that the curvature of a stream line in steady motion is* $\dfrac{1}{q^2}\cdot\dfrac{\partial}{\partial v}\left(\dfrac{p}{\rho}+V\right)$, *where* p, ρ, q *are the pressure, density and velocity of the liquid,* V *the potential of the external forces, and* δv *is an element of the principal normal to the stream line, and hence obtain the velocity potential of the two dimensional irrotational motion for which the stream line are confocal ellipse.*

Let r is the radius of curvature of the stream line at the point P. The normal acceleration of the fluid element at P is $\dfrac{q^2}{r}$. Equating the external forces along the normal to the stream line, we have

$$\frac{q^2}{r}=-\frac{\partial V}{\partial v}-\frac{1}{\rho}\frac{\partial p}{\partial v}.$$

or

$$-\frac{1}{r}=\frac{1}{q^2}\frac{\partial}{\partial v}\left(\frac{p}{\rho}+V\right)$$

which gives the curvature of a stream line in steady motion.

We know that

$$-\frac{\partial \phi}{\partial S}=q \qquad \text{.. (i)}$$

where δS is an element of length of the stream line passing through the point.

Equation to the confocal ellipse is,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

The velocity at the point P is

$$=A\,\{(a-ex)\,(a+ex)\}^{-1/2}$$

where A is an arbitrary constant and e is the eccentricity of the ellipse.

From (i), we have

$$\phi=-\int q\, dS$$

or $$\phi=-A\int\frac{dS}{\sqrt{(a^2-e^2x^2)}}$$

or $$\phi=-A\int\frac{1}{\sqrt{(a^2-e^2x^2)}}\,\frac{dS}{d\theta}\,.\,d\theta$$

or $$\phi=-A\int\frac{\sqrt{(a^2-e^2a^2\cos^2\theta)}}{\sqrt{(a^2-e^2a^2\cos\theta)}}\,d\theta$$

{Since the coordinates of a point on the ellipse is ($a\cos\theta$, $b\sin\theta$)}

or $$\phi=-A\theta$$

or $$\phi=-A\tan^{-1}\left(\frac{ay}{bx}\right)$$

which gives the velocity potential of the two dimensional irrotational motion for which the stream line are confocal ellipse.

Exercise

1. Prove that a motion under conservative system of forces, once rotational will always remains so.
2. Obtain the formula for circulating kinetic energy of a mass of homogeneous liquid moving irrotationally in a finite simply-connected space in the form
$$-\frac{1}{2}\rho\int\phi\frac{\partial\phi}{\partial n}\,dS.$$
3. Liquid moves irrotationally in two dimensions under the action of conservative forces whose potential Ω satisfies $\nabla^2\Omega=0$. Prove that the pressure satisfy the equation
$$\nabla^2\log(\nabla^2 p)=0.$$

5

Motion of Cylinders Elliptic Cylinders

(Problems of Irrotational Motion In Two Dimensions)

§ 5·0. The two-dimensional motion is defined by the fact that the stream lines are parallel to a fixed plane and the velocity at corresponding points of all planes parallel to the fixed plane has the same magnitude and direction. Here we shall discuss the two-dimensional irrotational motion produced by the motion of a cylinder in an infinite mass of liquid at rest at infinity or when a cylinder is inserted in a steady stream, the stream is disturbed by its presence and remains uniform only at a great distance from it. For convenience we shall consider the cylinder is of unit length and the liquid and the cylinder to be confined between two parallel planes at right angles to the generator of the cylinder.

§ 5·1. **Boundary condition for the stream function.**

The stream function ψ is to be determined by satisfying the following conditions :

(1) The stream function ψ satisfies the Laplace's equation

$\nabla^2\psi=0$, at all points of the liquid.

The boundary conditions are :

(i) Since the liquid is at rest at infinity *i.e.* the liquid will remain undisturbed at infinity so that

$$\frac{\partial\psi}{\partial x}=0 \quad \text{and} \quad \frac{\partial\psi}{\partial y}=0 \text{ at infinity.}$$

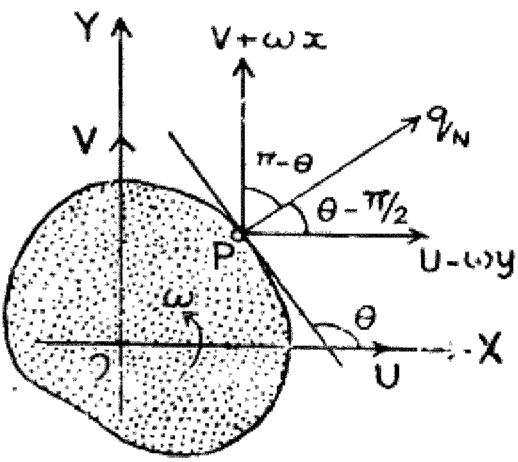
(ii) At any fixed boundary the normal velocity must vanish

$$i.e. \quad -\frac{\partial\psi}{\partial s}=0 \quad \text{or} \quad \psi=\text{const.}$$

so the boundary must coincide with a stream line $\psi=$const.

(iii) The normal component of the velocity of the liquid must be equal to the normal component of the velocity of the cylinder, at the boundary of the moving cylinder.

§ 5·2 General motion of a cylinder in two dimensions.



Consider a point of the cross-section of the cylinder as origin. Let a cylinder of any section be moving at right angles to the generators such that U and V are the velocities parallel to axes of X and Y and ω is the angular velocity of rotation of the cylinder. The cylinder is surrounded by incompressible inviscid liquid at rest at infinity.

Let $P\ (x, y)$ be any point on the boundary of the cylinder and θ be the inclination of the tangent with axis of X. The velocity components of the point P is given by

$$\left(U+\frac{dx}{dt}\ ,\ V+\frac{dy}{dt}\right)$$

Since $x=r\cos\theta$ and $y=r\sin\theta$

then $$\frac{dx}{dt}=-r\sin\theta\frac{d\theta}{dt}=-r\omega\sin\theta=-y\omega$$

and $$\frac{dy}{dt}=r\cos\theta\frac{d\theta}{dt}=r\omega\cos\theta=x\omega$$

Thus the velocity components of the point P are

$$U-\omega y,\ V+x\omega$$

The outward normal velocity at $P\ (x, y)$ **on the surface of the cylinder** (resolving the velocity component along the normal q_N).

$$=(U-\omega y)\cos\left(\theta-\frac{\pi}{2}\right)+(V+\omega x)\cos(\pi-\theta)$$

$$=(U-\omega y)\sin\theta-(V+\omega x)\cos\theta$$

$$=(U-\omega y)\frac{dy}{ds}-(V+\omega x)\frac{dx}{ds}\qquad\left\{\begin{array}{l}\text{as }\cos\theta=\dfrac{dx}{ds}\\[2mm]\quad\ \sin\theta=\dfrac{dy}{ds}\end{array}\right.$$

Also the normal velocity of the fluid at $P\ (x, y)$ on the surface of the cylinder

$$=-\frac{\partial\psi}{\partial s}$$

We know that the normal component of the velocity of the liquid must be equal to the normal component of the velocity of the cylinder at the boundary of the moving cylinder,

$$-\frac{\partial\psi}{\partial s}=(U-\omega y)\ \frac{dy}{ds}-(V+\omega x)\ \frac{dx}{ds}$$

or $$-\partial\psi=(U\ dy-V\ dx)-\omega\ (y\ dy+x\ dx)$$

By integrating, we have

$$\psi=Vx-Uy+\tfrac{1}{2}\omega\ (x^2+y^2)+\text{const.} \qquad ...(1)$$

which determines the current function for the most general type of motion of the cylinder,

§ 5·21. If there is **no rotation** then $\omega=0$ and $V=0$, so (1) reduces to

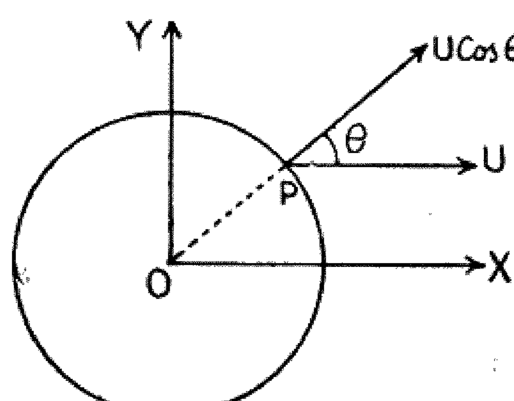
$$\psi=\text{const}-Uy.$$

§ 5·22. If there is **pure rotation** then $U=V=0$, so (1) reduces to

$$\psi=\tfrac{1}{2}\omega\ (x^2+y^2)+\text{const.}$$

§ 5·3. Motion of a circular cylinder in a uniform stream.

To determine the motion of a circular cylinder in an infinite mass of the liquid at rest at infinity, with velocity U along the X axis.



Consider the irrotational motion, originated from rest, The velocity potential ϕ will be single valued. Let the origin be taken in the axis of the circular cylinder and the co-ordinate axes be in a plane perpendicular to its length.

We know that

(I) The velocity potential ϕ satisfies the Laplace's equation

(Considering in Spherical polar coordinates in two dimensions)

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\ \frac{\partial\phi}{\partial r}+\frac{1}{r^2}\ \frac{\partial^2\phi}{\partial\theta^2}=0$$

Where the plane harmonics in polar form are $r^n \cos n\theta$, $r^n \sin n\theta$ for positive or negative integral values of n.

(II) The normal velocity of any point of the cylinder is equal to the normal velocity of the liquid at that point.

i.e. $$-\frac{\partial\phi}{\partial r}=U\cos\theta$$

(III) The liquid is at rest at infinity

$$\left(\frac{\partial\phi}{\partial r}\right)_{r\to\infty}=0$$

From the above conditions, consider the suitable form of the velocity potential ϕ as

$$\phi=Ar\cos\theta+\frac{B}{r}\cos\theta \qquad ...(1)$$

(where A and B are two arbitrary constants)

or $$\frac{\partial\phi}{\partial r}=\left(A-\frac{B}{r^2}\right)\cos\theta \qquad ...(2)$$

from (2) and the boundary condition (II), we have

$$U\cos\theta=-\left(A-\frac{B}{a^2}\right)\cos\theta \qquad ...(3)$$

(on the surface of the cylinder $r=a$)

Also from (2) and the boundary condition (III), we have

$$A\cos\theta=0$$

$$\Rightarrow \theta\neq\pi/2 \quad \text{hence} \quad \mathbf{A=0}$$

So from (3), we have

$$U\cos\theta=\frac{B}{a^2}\cos\theta$$

or $$\mathbf{B=Ua^2} \qquad (\text{as } \theta\neq\pi/2)$$

Then (1) reduces to

$$\phi=\frac{Ua^2}{r}\cos\theta$$

hence $$\psi=-\frac{Ua^2}{r}\sin\theta$$

Thus the complex potential will be

$$w=\phi+i\psi$$

or $$w=\frac{Ua^2}{r}\cos\theta+i\,\frac{Ua^2}{r}\sin\theta$$

or $$w=\frac{Ua^2}{r}(\cos\theta-i\sin\theta)$$

or $$w=\frac{Ua^2}{r}e^{-\theta i}$$

or $$\mathbf{w=\frac{Ua^2}{z}} \qquad \{\text{as } z=re^{\theta i}$$

The stream lines $\psi=$const. gives the circles

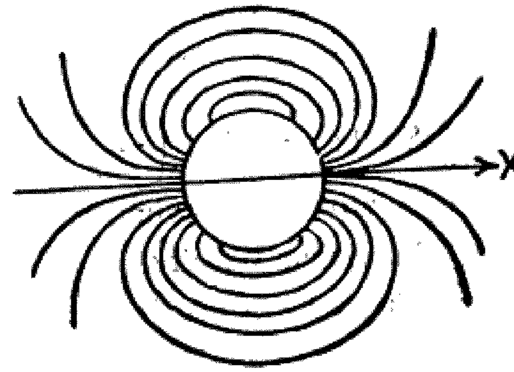
i.e. $\quad -\dfrac{Ua^2}{r}\sin\theta=$const.

or $\quad r\sin\theta=\dfrac{1}{k}\,r^2$

(where k is a constant)

or $\quad x^2+y^2-ky=0$

which represent the circles all touching X-axis at origin.



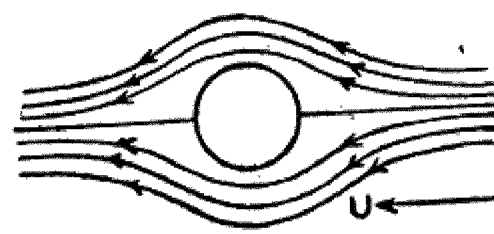
(Stream lines $\psi=$const.)

§ 5·4. **Liquid streaming past a fixed circular cylinder.**

We have determined that the complex potential due to a circular cylinder in an infinite mass of the liquid at rest at infinity, with velocity U along the X axis is

$$w_1=\frac{Ua^2}{z} \qquad \ldots(1)$$

Consider a stream flowing with uniform velocity U in the negative direction of the X-axis, so that the complex potential will be,



(Stream lines)

$$-\frac{dw_2}{dz}=-U$$

or $$w_2=Uz \qquad \ldots(2) \qquad \left\{\begin{array}{l}\text{as } -\dfrac{\partial\phi}{\partial x}=-U \\ \dfrac{\partial\phi}{\partial y}=0\end{array}\right.$$

Now the motion of a liquid streaming past a fixed circular cylinder can be obtained by super-imposing a velocity $-U$ parallel to the X-axis on the cylinder as well as to the liquid.

The complex potential will be

$$w=w_1+w_2$$

or $$w=Uz+\frac{Ua^2}{z} \qquad \ldots(3) \qquad \left\{\begin{array}{l}\text{as } w=\phi+i\psi \\ \quad z=r\,(\cos\theta \\ \qquad +i\sin\theta\end{array}\right.$$

or $$\phi+i\psi=U\,(r\cos\theta+i\sin\theta)+\frac{Ua^2}{r}\,(\cos\theta-i\sin\theta)$$

Equating real and imaginary parts, we get

$$\phi=U\left(r+\frac{a^2}{r}\right)\cos\theta$$

and $$\psi=U\left(r-\frac{a^2}{r}\right)\sin\theta$$

So the equation $\left(r-\dfrac{a^2}{r}\right)\sin\theta=\text{const.}$ represent the stream lines relative to the cylinder.

Maximum velocity.

The velocity distribution on a point of the circular cylinder *i.e.* at $r=a$, $z=ae^{\theta i}$

$$q=\left|\frac{dw}{dz}\right|$$

or $$q=|U|\left|1-\frac{a^2}{z^2}\right| \qquad \{\text{from (3)}$$

or $$q=|U|\ |1-e^{-2\theta i}|$$

or $$q=|U|\ |(1-\cos 2\theta)+i\sin 2\theta|$$

or $$q=|U|\{(1-\cos 2\theta)^2+(\sin 2\theta)^2\}^{1/2}$$

or $$q=2|U|\sin\theta \qquad ...(4)$$

Which shows that the fluid velocity vanishes at $\theta=0$ or $\theta=\pi$. These points are called **stagnation points of the flow** or the **critical points.**

So the maximum value of the velocity on the surface of the cylinder can be determined for $\theta=\pm\pi/2$

$$q_{max}=2|U|$$
$$=2\times\text{velocity of free stream.}$$

Pressure on the boundary of the cylinder.

We know from Bernoulli's Theorem, that

$$p+\tfrac{1}{2}\rho\, q^2=A\ (\text{const.})$$

since at infinity $$p=\Pi,\ q=U$$

then $$A=\Pi+\tfrac{1}{2}\rho\, U^2$$

or $$p+\tfrac{1}{2}\rho\, q^2=\Pi+\tfrac{1}{2}\rho\, U^2$$

or $$p-\Pi=\tfrac{1}{2}\rho\,(U^2-q^2)$$

or $$p-\Pi=\tfrac{1}{2}\rho\, U^2(1-4\sin^2\theta) \qquad ...(5)\ \{\text{from (4)}$$

Since the liquid will remain in contact with the boundary of the circular cylinder so long as the pressure at every point on it is positive it follows that the liquid cannot sustain a negative pressure. If the pressure is negative then the cavitation will occur which is opposite to our theory. **The formation of a vacuous space in a fluid is called cavitation.** So for p to be positive always at every point, we have

$$p>0$$

or $\quad \Pi+\frac{1}{2}\rho\, U^2 (1-4 \sin^2\theta) > 0$

or $\quad \Pi-\frac{3}{2}\, \rho\, U^2 > 0 \qquad$ {at $\theta=\pm\pi/2$ (on the sides).

or $\quad U^2 < \dfrac{2\Pi}{3\rho}$

If U exceeds the above value then the liquid will cavitate at the sides of the cylinder.

Ex. 1. *Shew that when a cylinder moves uniformly in a given straight line in an infinite liquid, the path of any point in the fluid is given by the equations*

$$\frac{dz}{dt}=\frac{Va^2}{(z'-Vt)^2}\,;\ \frac{dz'}{dt}=\frac{Va^2}{(z-Vt)^2}$$

when V=velocity of cylinder, a its radius, and z, z' are $x+iy$, $x-iy$ where x, y are the coordinetes measured from the starting point of the axis, along and perpendicular to its direction of motion.

Consider the motion of the cylinder be along X-axis with a velocity V. Let (x, y) be the centre of the cylinder, then after a time t, we have

$$x=Vt,\ y=0$$

Thus the complex potential is given by

$$w=\frac{A}{(z-Vt)} \qquad \ldots(\text{i})$$

(where A is an arbitrary constant.)

Let $z-Vt=re^{\theta i}$

or $\quad w=\dfrac{A}{r}\, e^{-\theta i}$

or $\quad w=\dfrac{A}{r}\,(\cos\theta-i\sin\theta)$

or $\quad \phi+i\psi=\dfrac{A}{r}\,(\cos\theta-i\sin\theta)$

Equating the real part, we get

$$\phi=\frac{A}{r}\cos\theta$$

or $\quad -\dfrac{\partial\phi}{\partial r}=\dfrac{A}{r^2}\cos\theta \qquad \ldots(\text{ii})$

We know that the normal velocity of the cylinder at its surface ($=V\cos\theta$) is equal to the velocity of the fluid at that point along the normal of the cylinder.

or $\quad V\cos\theta=\dfrac{A}{a^2}\cos\theta$

or $\quad A=a^2V,\ \cos\theta\neq 0$

From (i), we get

$$w=\frac{a^2V}{z-Vt} \qquad ...(iii)$$

Differentiating (iii) w. r. to z, we have

$$\frac{dw}{dz}=-\frac{Va^2}{(z-Vt)^2}$$

or $$-u+iv=-\frac{Va^2}{(z-Vt)^2}$$

or $$u-iv=\frac{Va^2}{(z-Vt)^2}$$

$$\left\{\begin{aligned} &\text{since } w=\phi+i\psi \\ &\frac{dw}{dz}=\frac{\partial\phi}{\partial x}+i\frac{\partial\psi}{\partial x} \\ &\quad=-u+iv \end{aligned}\right.$$

or $$\frac{dx}{dt}-i\frac{dy}{dt}=\frac{Va^2}{(z-Vt)^2} \quad \left\{\begin{aligned} &\text{Since } u=dx/dt \\ &v=dy/dt \end{aligned}\right.$$

or $$\frac{d}{dt}(x-iy)=\frac{Va^2}{(z-Vt)^2} \qquad ...(iv) \quad \left\{\begin{aligned} &\text{since } z=x+iy \\ &\text{and } z'=x-iy \end{aligned}\right.$$

or $$\frac{dz'}{dt}=\frac{Va^2}{(z-Vt)^2}$$ **Proved.**

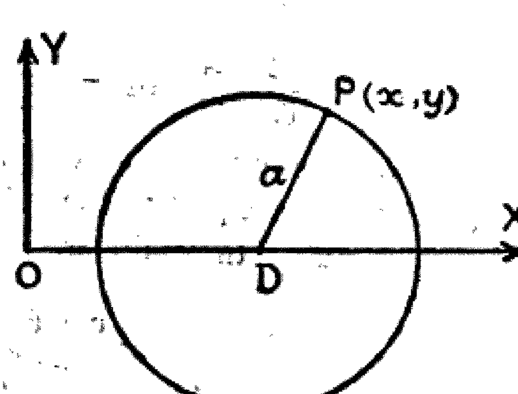
Substituting $-i$ for i in the equation (iv), we have

$$\frac{d}{dt}(x+iy)=\frac{Va^2}{(z'-Vt)^2}$$

or $$\frac{dz}{dt}=\frac{Va^2}{(z'-Vt)^2}$$ **Proved.**

Ex. 2. *If a long circular cylinder of radius a moves in a straight line at right angles to its length in liquid at rest at infinity, shew that when a particle of liquid in the plane of symmetry, initially at distance b in advance of the axis of the cylinder, has moved through a distance c, then the cylinder has moved through a distance*

$$c+\frac{b^2-a^2}{c+a\coth\dfrac{c}{a}}$$



Consider a line perpendicular to the line of motion as the coordinate axes. Let the cylinder moves with velocity V and P (x, y) be any point on it. After any time t it has described a distance

$$x=Vt,\ y=0.$$

Thus the complex potential at $z=x+iy=Vt$ is given by

$$w=\frac{Va^2}{z-Vt} \quad \ldots(\text{i})$$

or
$$w=\frac{Va^2}{x+iy-Vt}=\frac{Va^2[(x-Vt)-iy]}{(x-Vt)^2+y^2}$$

or
$$\frac{dw}{dz}=-\frac{Va^2}{(z-Vt)^2}$$

or
$$-u+iv=-\frac{Va^2}{(z-Vt)^2}$$
$$=-\frac{Va^2}{(x+iy-Vt)^2}$$

$$\left[\begin{array}{l} \text{since } w=\phi+i\psi \\ \text{or } \dfrac{dw}{dx}=\dfrac{\partial\phi}{\partial x}+i\dfrac{\partial\psi}{\partial x} \\ \text{or } \dfrac{dw}{dz}=\dfrac{\partial\phi}{\partial x}+i\dfrac{\partial\psi}{\partial x} \quad \text{as } \dfrac{\partial z}{\partial x}=1 \\ \dfrac{dw}{dz}=-u+iv \end{array}\right.$$

Substituting $y=0$, the velocity along the real axis, is given by

$$u=\frac{Va^2}{(x-Vt)^2}$$

or
$$\frac{dx}{dt}=\frac{Va^2}{(x-Vt)^2}$$

Let $x-Vt=\lambda$ or $\dfrac{dx}{dt}=V+\dfrac{d\lambda}{dt}$

or
$$V+\frac{d\lambda}{dt}=\frac{Va^2}{\lambda^2}$$

or
$$\frac{d\lambda}{dt}=\frac{Va^2}{\lambda^2}-V=\frac{V(a^2-\lambda^2)}{\lambda^2}$$

or
$$V\,dt=\frac{\lambda^2}{a^2-\lambda^2}\,d\lambda$$

or
$$V\,dt=-\left\{1+\frac{a^2}{\lambda^2-a^2}\right\}d\lambda$$

By integrating, we have

$$Vt+A=-\left\{\lambda+\frac{a}{2}\log\frac{\lambda-a}{\lambda+a}\right\}$$

or
$$Vt+A=-\left\{x-Vt+\frac{a}{2}\log\frac{x-Vt-a}{x-Vt+a}\right\}$$

or
$$x+A=-\frac{a}{2}\log\frac{x-Vt-a}{x-Vt+a}$$

Since $x=b$, $t=0$, then $b+A=-\dfrac{a}{2}\log\dfrac{b-a}{b+a}$

or
$$A=-b-\frac{a}{2}\log\frac{b-a}{b+a}$$

Substituting the value of the constant A, we have

$$x-b=-\frac{a}{2}\left\{\log \frac{x-Vt-a}{x-Vt+a}-\log\frac{b-a}{b+a}\right\}$$

Again, when $x=b+c$ at any time t, we have

$$-\frac{2c}{a}=\log\frac{b+c-Vt-a}{b+c-Vt+a}-\log\frac{b-a}{b+a}$$

or $$-\frac{2c}{a}=\log\frac{(b+a)\{(b-a)+c-Vt\}}{(b-a)\{(b+a)+c-Vt\}}$$

or $$e^{-2c/a}=\frac{(b^2-a^2)+(b+a)(c-Vt)}{(b^2+a^2)+(b-a)(c-Vt)}$$

or $$\frac{1+e^{-2c/a}}{1-e^{-2c/a}}=\frac{(b^2-a^2)+b(c-Vt)}{-a(c-Vt)}$$

or $$a.\frac{e^{c/a}+e^{-c/a}}{e^{c/a}-e^{-c/a}}=\frac{b^2-a^2}{Vt-c}-b$$

or $$a\coth\frac{c}{a}=\frac{b^2-a^2}{Vt-c}-b$$

or $$b+a\coth\left(\frac{c}{a}\right)=\frac{b^2-a^2}{Vt-c}$$

or $$Vt=c+\frac{b^2-a^2}{b+a\coth\dfrac{c}{a}}.$$

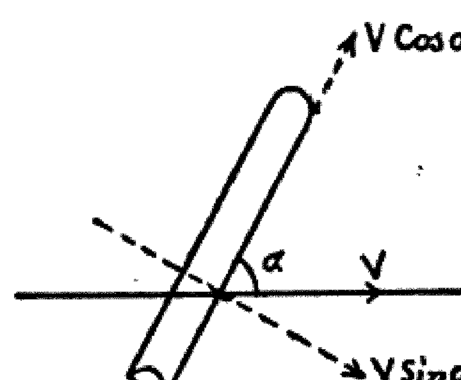
Proved.

Ex. 3. *A stream of water of great depth is flowing with uniform velocity V over a plane level bottom. An infinite cylinder, of which the cross-section is a semi-circle of radius a, lies on its flat side with its generating lines making an angle α with the undisturbed stream lines. Prove that the resultant fluid pressure per unit length on the curved surface is*

$$2a\Pi-\frac{5}{3}\rho\, a\, V^2\sin^2\alpha$$

where Π is the fluid pressure at a great distance from the cylinder.

Since V is the velocity of the stream over a plane level. The components of the velocity are $V\cos\alpha$ along the generator and $V\sin\alpha$ perpendicular to the generator of the cylinder. The component $V\cos\alpha$ will exert no pressure being parallel to the generator of the cylinder.



The component $V \sin \alpha$ will give rise a velocity potential ϕ.

$$\phi = V \sin \alpha \left(r + \frac{a^2}{r}\right) \cos \theta \qquad \{\text{Here } U = V \sin \alpha$$

or
$$\frac{\partial \phi}{\partial r} = V \sin \alpha \left(1 - \frac{a^2}{r^2}\right) \cos \theta$$

$$\frac{\partial \phi}{\partial \theta} = -V \sin \alpha \left(r + \frac{a^2}{r}\right) \sin \theta.$$

If q be the velocity at the point (r, θ)

then
$$q^2 = \left(\frac{\partial \phi}{\partial r}\right)^2 + \left(\frac{1}{r} \frac{\partial \phi}{\partial \theta}\right)^2$$

$$q^2 = \left\{V \sin \alpha \left(1 - \frac{a^2}{r^2}\right) \cos \theta\right\}^2 + \left\{-V \sin \alpha \left(1 + \frac{a^2}{r^2}\right) \sin \theta\right\}^2$$

$$q^2 = V^2 \sin^2 \alpha \left\{1 + \frac{a^4}{r^4} - \frac{2a^2}{r^2} (\cos^2 \theta - \sin^2 \theta)\right\}$$

or
$$q^2 = V^2 \sin^2 \alpha \left\{1 - \frac{2a^2}{r^2} \cos 2\theta + \frac{a^4}{r^4}\right\}$$

The pressure at any point is

$$\frac{p}{\rho} = C - \tfrac{1}{2} q^2$$

or
$$\frac{p}{\rho} = C - \tfrac{1}{2} V^2 \sin^2 \alpha \left\{1 - \frac{2a^2}{r^2} \cos 2\theta + \frac{a^4}{r^4}\right\}. \qquad \ldots \text{(i)}$$

Since $p = \Pi$ at a great distance *i.e.* r is infinite then from (i), we have

$$\frac{\Pi}{\rho} = C - \tfrac{1}{2} V^2 \sin^2 \alpha$$

or
$$C = \frac{\Pi}{\rho} + \tfrac{1}{2} V^2 \sin^2 \alpha.$$

So
$$\frac{p}{\rho} = \frac{\Pi}{\rho} + \tfrac{1}{2} V^2 \sin^2 \alpha - \tfrac{1}{2} V^2 \sin^2 \alpha \left\{1 - \frac{2a^2}{r^2} \cos 2\theta + \frac{a^4}{r^4}\right\}$$

or
$$p = \Pi + \tfrac{1}{2} \rho V^2 \sin^2 \alpha \left\{\frac{2a^2}{r^2} \cos 2\theta - \frac{a^4}{r^4}\right\} \qquad \ldots \text{(ii)}$$

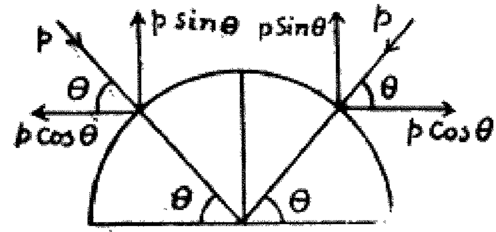
The pressure at a point (r, θ) on the cylinder, is given by substituting $r = a$ in (ii), we have

$$p = \Pi + \tfrac{1}{2} \rho V^2 \sin^2 \alpha \{2 \cos 2\theta - 1\}$$

$$p=\Pi+\tfrac{1}{2}\rho V^2 \sin^2 \alpha \{4 \cos^2 \theta-3\}$$

or
$$p=\Pi-\frac{3}{2}\rho V^2 \sin^2 \alpha+2\rho V^2 \sin^2 \alpha \cos^2 \theta$$

Now the component $p \cos \theta$ neutralize each other, as the pressure exerted on the curved surface of semi-circular cylinder is equal.



Now we shall determine the total resultant pressure on the semi-circular cylinder per unit length. Consider an elementary element $a\ \delta\theta$ on the surface

then
$$P=\int_0^\pi p \sin \theta . a\, d\theta$$
$$=a\rho\int_0^\pi \left(\frac{\Pi}{\rho}-\frac{3}{2} V^2 \sin^2 \alpha+2V^2 \sin^2 \alpha \cos^2 \theta\right) \sin \theta\, d\theta$$
$$=a\rho\left(\frac{\Pi}{\rho}-\frac{3}{2} V^2 \sin^2 \alpha\right)\left\{-\cos \theta\right\}_0^\pi$$
$$+2V^2 \sin^2 \alpha \left\{-\frac{\cos^3 \theta}{3}\right\}_0^\pi$$
$$=a\rho\left(\frac{\Pi}{\rho}-\frac{3}{2} V^2 \sin^2\alpha\right)(2)+2V^2 \sin^2 \alpha \left(\frac{2}{3}\right)$$
$$=2a\Pi-\frac{5}{3}\rho a V^2 \sin^2 \alpha.$$

Proved.

Ex. 4. *A circular cylinder of radius a is moving with velocity U along the axis of X; Shew that the motion produced by the cylinder in a mass of fluid at rest is given by the complex potential*

$$w=\phi+i\psi=\frac{a^2 U}{z-Ut} \qquad \textit{where } z=x+iy.$$

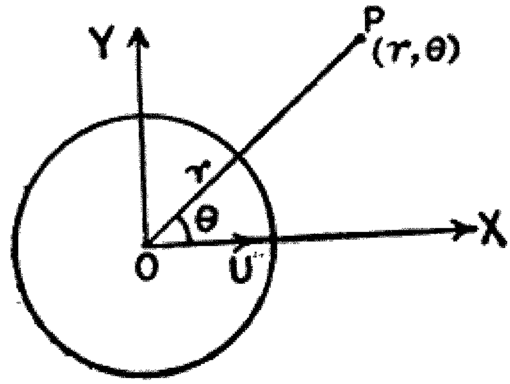
Find the magnitude and direction of the velocity in the fluid and deduce that for a marked particle of the fluid, whose polar coordinates are (r, θ) referred to the centre of the cylinder as origin,

$$\frac{1}{r}\frac{dr}{dt}+i\frac{d\theta}{dt}=\frac{U}{r^2}\left(\frac{a^2}{r^2}e^{i\theta}-e^{-i\theta}\right) \quad \textit{and}$$
$$\left(r-\frac{a^2}{r}\right)\sin \theta=b.$$

Hence prove that the path of such a particle is the elastic curve given by

$$\rho\,(y-\tfrac{1}{2}b)=\tfrac{1}{4}a^2.$$

where ρ is the radius of curvature of the path



Consider X-axis be the line of motion of the cylinder ; it moves a distance Ut along the line of motion at any time t. Taking the origin at $z=Ut$, then the complex potential for the motion produced by the cylinder is given by

$$w=\frac{Ua^2}{z-Ut}$$

or $$\phi+i\,\psi=\frac{Ua^2}{z-Ut}.$$ **Proved.**

Again, differentiating w. r. to z, we have

$$\frac{dw}{dz}=-\frac{Ua^2}{(z-Ut)^2}$$

$\left[\begin{array}{l}\text{Since } w=\phi+i\psi \\ \dfrac{dw}{dz}=\dfrac{\partial\phi}{\partial x}+i\dfrac{\partial\psi}{\partial x} \\ \qquad=-u+iv\end{array}\right.$

or $$u-iv=\frac{Ua^2}{(z-Ut)^2}$$

let $$z-Ut=re^{\theta i}$$

then $$u-iv=\frac{Ua^2}{r^2\,e^{2\theta i}}$$

$$u-iv=\frac{Ua^2}{r^2}\,(\cos 2\theta-i\sin 2\theta).$$

Equating real and imaginary parts, we have

$$u=\frac{Ua^2}{r^2}\cos 2\theta \quad\text{and}\quad v=\frac{Ua^2}{r^2}\sin 2\theta.$$

Magnitude of the velocity$=\sqrt{(u^2+v^2)}$

$$=\frac{Ua^2}{r^2}$$ **Answer.**

and the direction of the velocity is given by

$$\tan\alpha=\frac{v}{u}=\tan 2\theta$$

or $$\alpha=2\theta. \qquad \text{(i)}$$ **Answer.**

The complex potential of the motion at $P(r,\theta)$ relative to the cylinder is

$$w=Uz+\frac{Ua^2}{z}$$

$$\phi+i\psi=U\,(x+iy)+\frac{Ua^2}{x^2+y^2}\,(x-iy)$$

So $$\phi=Ux+\frac{Ua^2x}{x^2+y^2}$$

$$=Ur\cos\theta+\frac{Ua^2\cos\theta}{r} \qquad \text{...(ii)}$$

and
$$\psi = Uy - \frac{Ua^2 y}{x^2+y^2}$$
$$= Ur \sin\theta - \frac{Ua^2 \sin\theta}{r} \qquad \ldots\text{(iii)} \qquad \left\{\begin{array}{l}\text{as } x = r\cos\theta \\ \quad y = r\sin\theta\end{array}\right.$$

The line of flow through P is
$$\psi = \text{constant}$$
then from (iii), we have
$$U\left(r - \frac{a^2}{r}\right)\sin\theta = \text{constant}$$
or
$$\left(r - \frac{a^2}{r}\right)\sin\theta = B \qquad \ldots\text{(iv)}$$
where B is any constant. **Proved.**

Again, we know that
$$\frac{dr}{dt} = -\frac{\partial\phi}{\partial r} = -\left\{U\cos\theta - \frac{Ua^2\cos\theta}{r^2}\right\}$$
$$= -U\left(1 - \frac{a^2}{r^2}\right)\cos\theta$$
and
$$r\frac{d\theta}{dt} = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{1}{r}\left\{-Ur\sin\theta - \frac{Ua^2\sin\theta}{r}\right\}$$
$$= U\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$
then
$$\frac{1}{r}\frac{dr}{dt} + i\frac{d\theta}{dt} = -\frac{U}{r}\left(1 - \frac{a^2}{r^2}\right)\cos\theta + i\frac{U}{r}\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$
or
$$\frac{1}{r}\frac{dr}{dt} + i\frac{d\theta}{dt} = -\frac{U}{r}\left[(\cos\theta - i\sin\theta) - \frac{a^2}{r^2}(\cos\theta + i\sin\theta)\right]$$
$$= \frac{U}{r}\left[\frac{a^2}{r^2}e^{\theta} - e^{-\theta i}\right] \qquad \textbf{Proved.}$$

Let (x, y) be the coordinates of the point P referred to O as origin and the origin is at $z = Ut$, the coordinates of the fluid particle are
$$x = r\cos\theta + Ut, \; y = r\sin\theta$$

We know that $\rho = \dfrac{ds}{dx}$ $\qquad \left\{\text{Since } \alpha = 2\theta\right.$

or
$$\frac{1}{\rho} = \frac{d\alpha}{ds} = \frac{d\alpha}{dy}\cdot\frac{dy}{ds} = \frac{d\alpha}{dy}.\sin\alpha$$
$$\left[\begin{array}{l}\text{or} \quad \dfrac{1}{\rho} = \dfrac{d(2\theta)}{dy}.\sin 2\theta \\ \text{or} \quad \dfrac{1}{\rho} = 2\sin 2\theta\,\dfrac{d\theta}{dy}.\end{array}\right. \qquad \ldots\text{(v)}$$

Since from (iv), we have

$$\left[r-\frac{a^2}{r}\right]\sin\theta=B$$

or $$r\sin\theta-\frac{a^2}{r\sin\theta}.\sin^2\theta=B$$

or $$y-\frac{a^2}{y}\sin^2\theta=B. \qquad \text{...(vi)}$$

Differentiating w. r. to y, we have

$$1+\frac{a^2}{y^2}\sin^2\theta-\frac{2a^2}{y}\sin\theta\,\cos\theta\,\frac{d\theta}{dy}=0,$$

or $$\sin 2\theta\,\frac{d\theta}{dy}=\frac{y}{a^2}\left(1+\frac{a^2}{y^2}\sin^2\theta\right)$$

or $$\sin 2\theta\,\frac{d\theta}{dy}=\frac{y}{a^2}\left(1+\frac{y-b}{y}\right)\ \{\text{from (vi)}$$

Thus from (v) and (vi), we have

$$\frac{1}{2\rho}=\frac{y}{a^2}\left(1+\frac{y-b}{y}\right)$$

or $$\frac{1}{2\rho}=\frac{y}{a^2}\,(2y-b)$$

or $$\rho\left(y-\frac{b}{2}\right)=\frac{a^2}{4}.$$ **Proved.**

Ex. 5. *In the case of two-dimensional motion of a liquid streaming past a fixed circular disc, the velocity at infinity is u in a fixed direction where u is variable. Shew that the maximum value of the velocity at any point of the fluid is 2u. Prove that the force necessary to hold the disc at rest is* $2m\dot{u}$, *where m is the mass of the liquid displaced by the disc.*

The velocity potential of a liquid streaming past a fixed circular disc is given by

$$\phi=u\left(r+\frac{a^2}{r}\right)\cos\theta$$

or $$\frac{\partial\phi}{\partial r}=u\left(1-\frac{a^2}{r^2}\right)\cos\theta$$

or $$\frac{\partial\phi}{\partial\theta}=-u\left(r+\frac{a^2}{r}\right)\sin\theta$$

If q be the velocity at the point $P(r, \theta)$, then

$$q^2=\left(-\frac{\partial\phi}{\partial r}\right)^2+\left(-\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)^2$$

or $$q^2=u^2\left(1-\frac{a^2}{r^2}\right)^2\cos^2\theta+u^2\left(1+\frac{a^2}{r^2}\right)^2\sin^2\theta$$

or $$q^2=u^2\left\{1-\frac{2a^2}{r^2}\cos 2\theta+\frac{a^4}{r^4}\right\}.$$

Now q is maximum with regard to θ when $\cos 2\theta=-1$

or $$2\theta=\pi$$

then $$q^2=u^2\left\{1+\frac{2a^2}{r^2}+\frac{a^4}{r^4}\right\}$$

$$=u^2\left(1+\frac{a^2}{r^2}\right)^2$$

or $$q=u\left(1+\frac{a^2}{r^2}\right) \qquad \ldots\text{(i)}$$

Again, q is maximum with regard to r, when r is minimum *i e.* substituting $r=a$ in (i)

$$\text{max. value of } q_{r=a}=u\left(1+\frac{a^2}{a^2}\right)=2u.$$

Thus the maximum value of the velocity at any point of the fluid is $2u$. **Proved.**

• We know that

$$\frac{p}{\rho}=f(t)-\tfrac{1}{2}q^2+\frac{\partial\phi}{\partial t}$$

or $$\frac{p}{\rho}=f(t)-\tfrac{1}{2}u^2\left\{1-\frac{2a^2}{r^2}\cos 2\theta+\frac{a^4}{r^4}\right\}+\dot{u}\left(r+\frac{a^2}{r}\right)\cos\theta$$

At the boundary of the disc. $r=a$, we have

$$\frac{p}{\rho}=f(t)-2u^2\sin^2\theta+\dot{u}\,2a\cos\theta$$

Thus resultant pressure on the disc is

$$P=\int_0^{2\pi}(-p\cos\theta)\,a\,d\theta \text{ (let)} \quad \left\{\begin{array}{l}\text{Since } a\,\delta\theta \text{ is an element}\\ \text{on the surface of the disc.}\end{array}\right.$$

$$P=-\rho a\int_0^{2\pi}\left\{f(t)-2u^2\sin^2\theta+\dot{u}\,2a\cos\theta\right\}\cos\theta\,d\theta$$

$$P=-2\rho a^2\dot{u}\int_0^{2\pi}\cos^2\theta\,d\theta$$

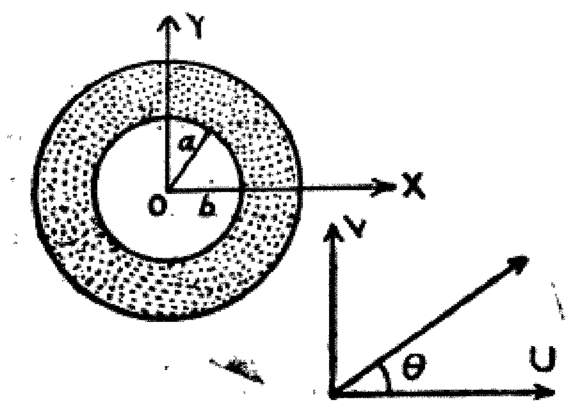
$$P=-2\pi a^2\rho\dot{u}$$

$$P=-2(\pi a^2\rho)\dot{u} \quad \left\{\begin{array}{l}\text{where } m=\pi a^2\rho\text{, mass of}\\ \text{the circular disc.}\end{array}\right.$$

$$P=-2m\dot{u}$$

Hence the force necessary to hold the disc at rest is $2m\dot{u}$. **Proved.**

§ 5·5. Two coaxial cylinders (Problem of initial motion)



To determine the velocity potential and stream function at any point of a liquid contained between two co-axial cylinders of radii a and b $(a < b)$, when the cylinders are moving suddenly parallel to themselve in directions at right angle with velocities U and V respectively.

Consider two co-axial cylinders of radii a and b $(a < b)$, the space between the cylinders being filled with liquid of density ρ. Let the cylinders are to be moved with velocities U and V respectively.

The boundary conditions for the velocity potential ϕ, are

(I) When $r=a$, $-\dfrac{\partial\phi}{\partial r}=U\cos\theta$

(II) When $r=b$, $-\dfrac{\partial\phi}{\partial r}=V\sin\theta$

So ϕ should be of the form such that $\dfrac{\partial\phi}{\partial r}$ contains $\cos\theta$ and $\sin\theta$ both and ϕ is also a solution of Laplace's equation. To satisfy above conditions, assuming that

$$\phi=\left(Ar+\frac{B}{r}\right)\cos\theta+\left(Cr+\frac{D}{r}\right)\sin\theta \qquad \text{...(i)}$$

where A, B, C and D are arbitrary constants.

or $$\frac{\partial\phi}{\partial r}=\left(A-\frac{B}{r^2}\right)\cos\theta+\left(C-\frac{D}{r^2}\right)\sin\theta \qquad \text{...(ii)}$$

from (I) and (ii), we have

$$-U\cos\theta=\left(A-\frac{B}{a^2}\right)\cos\theta+\left(C-\frac{D}{a^2}\right)\sin\theta \qquad \text{...(iii)}$$

Also from (II) and (ii), we get

or $$-V\sin\theta=\left(A-\frac{B}{b^2}\right)\cos\theta+\left(C-\frac{D}{b^2}\right)\sin\theta \qquad \text{...(iv)}$$

These being true for all values of θ, we have

$$A-\frac{B}{a^2}=-U,\quad A-\frac{B}{b^2}=0$$

and $\quad C-\dfrac{D}{a^2}=0$, and $C-\dfrac{D}{b^2}=-V$

given $\quad A=\dfrac{Ua^2}{b^2-a^2}$, $B=\dfrac{Ua^2b^2}{b^2-a^2}$

and $\quad C=-\dfrac{Vb^2}{b^2-a^2}$, $D=-\dfrac{Va^2b^2}{b^2-a^2}$

Substituting the value of the constants in (i), we have

$$\phi=\frac{Ua^2}{b^2-a^2}\left(r+\frac{b^2}{r}\right)\cos\theta-\frac{Vb^2}{b^2-a^2}\left(r+\frac{a^2}{r}\right)\sin\theta$$

Since ψ is a conjugate function of ϕ, then

$$\frac{\partial\phi}{\partial r}=\frac{1}{r}\frac{\partial\psi}{\partial\theta}$$

or $$\frac{\partial\phi}{\partial r}=\frac{Ua^2}{b^2-a^2}\left(1-\frac{b^2}{r^2}\right)\cos\theta-\frac{Vb^2}{b^2-a^2}\left(1-\frac{a^2}{r^2}\right)\sin\theta$$

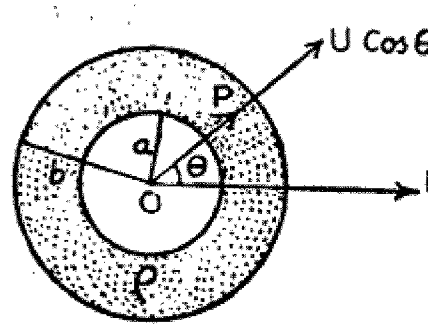
then $$\psi=\frac{Ua^2}{b^2-a^2}\left(r-\frac{b^2}{r}\right)\sin\theta+\frac{Vb^2}{b^2-a^2}\left(r-\frac{a^2}{r}\right)\cos\theta$$

which determines the velocity potential and stream function at any point of a liquid contained in an intervening space of the coaxial cylinders. These equations are valid iff the Cylinders are coaxial.

Ex. 6. *The space between two infinitely long co-axial cylinders of radii a and b respectively is filled with homogeneous liquid of density ρ and the inner cylinder is suddenly moved with velocity U perpendicular to the axis, the outer one being kept fixed. Shew that the resultant impulsive pressure on a length l of the inner cylinder is*

$$\pi\rho a^2 l\,\frac{b^2+a^2}{b^2-a^2}\,U$$

Liquid of density ρ is filled between the intervening space of two coaxial cylinder. The inner cylinder suddenly moves with velocity U perpendicular to the axis and the outer cylinder being kept fixed. The velocity potential, ϕ, must satisfy the Laplace's equation and the boundary conditions.



i.e. $\quad \nabla^2\phi=0$ (Laplace's equation in two dimensions)

and $\quad$ (I) $\quad \left(-\dfrac{\partial\phi}{\partial r}\right)_{r=a}=U\cos\theta$

(II) $\left(-\dfrac{\partial\phi}{\partial r}\right)_{r=b}=0$

Let the form of the potential be

$$\phi=\left(Ar+\frac{B}{r}\right)\cos\theta+\left(Cr+\frac{D}{r}\right)\sin\theta \quad \text{...(ii)}$$

where A, B, C and D are arbitrary constants.

or $$\frac{\partial\phi}{\partial r}=\left(A-\frac{B}{r^2}\right)\cos\theta+\left(C-\frac{D}{r^2}\right)\sin\theta \quad \text{...(iii)}$$

Now equation (iii) reduces to with the help of initial conditions I and II.

$$-U\cos\theta=\left(A-\frac{B}{a^2}\right)\cos\theta+\left(C-\frac{D}{a^2}\right)\sin\theta \quad \text{...(iv)}$$

$$0=\left(A-\frac{B}{b^2}\right)\cos\theta+\left(C-\frac{D}{b^2}\right)\sin\theta \quad \text{...(v)}$$

The equation (iv) and (v) hold for all values of θ, we have

$$A-\frac{B}{a^2}=-U, \quad C-\frac{D}{a^2}=0$$

$$A-\frac{B}{b^2}=0, \quad C-\frac{D}{b^2}=0$$

which gives $A=\dfrac{Ua^2}{b^2-a^2}$, $B=\dfrac{Ua^2b^2}{b^2-a^2}$, $C=D=0$.

substituting the value of the constants A, B, C and D in (i) we get

$$\phi=\frac{Ua^2}{b^2-a^2}\left(r+\frac{b^2}{r}\right)\cos\theta$$

The impulsive pressure on a point (a, θ) of the inner cylinder

$$=(\rho\phi)_{r=a}$$

$$=\rho\frac{Ua^2}{b^2-a^2}\left\{a+\frac{b^2}{a}\right\}\cos\theta$$

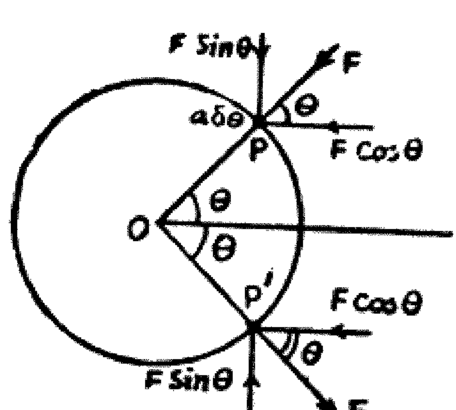
$$=\frac{\rho\, Ua\,(a^2+b^2)}{b^2-a^2}\cos\theta=F\text{ (say)}$$

The total impulsive pressure on the cylinder of length l

$$=\int_0^{2\pi}F\cos\theta . a\,d\theta\, l$$

$$=\frac{\rho\, Ua^2l\,(a^2+b^2)}{b^2-a^2}\int_0^{2\pi}\cos^2\theta\,d\theta$$

$$=\pi\,\rho a^2l\,U\frac{b^2+a^2}{b^2-a^2}$$

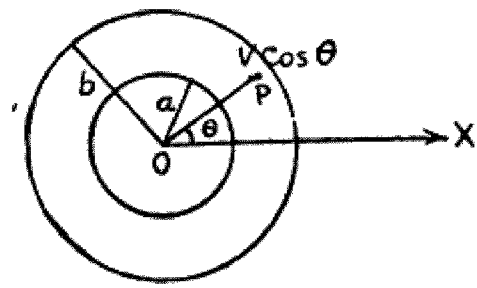


Proved.

Ex. 7. *An infinite cylinder of radius a and density σ is surrounded by a fixed concentric cylinder of radius b, and the intervening space is filled with liquid of density ρ. Prove that the impulse per unit length necessary to start the inner cylinder with velocity V is*

$$\frac{\pi a^2}{b^2-a^2}\left\{(\sigma+\rho)\,b^2-(\sigma-\rho)\,a^2\right\}V$$

The velocity potential ϕ must satisfy the Laplace's equation $\nabla^2\phi=0$ and the boundary conditions.



(i) $r=a,\quad -\dfrac{\partial\phi}{\partial r}=V\cos\theta$

(ii) $r=b,\quad -\dfrac{\partial\phi}{\partial r}=0$

To satisfy the conditions, consider

$$\phi=\left(Ar+\frac{B}{r}\right)\cos\theta+\left(Cr+\frac{D}{r}\right)\sin\theta \qquad \ldots(1)$$

where A, B, C, D are arbitrary constants.

$$\frac{\partial\phi}{\partial r}=\left(A-\frac{B}{r^2}\right)\cos\theta+\left(C-\frac{D}{r^2}\right)\sin\theta$$

By initial boundary conditions, we have

$$-V\cos\theta=\left(A-\frac{B}{a^2}\right)\cos\theta+\left(C-\frac{D}{a^2}\right)\sin\theta$$

or $$O=\left(A-\frac{B}{b^2}\right)\cos\theta+\left(C-\frac{D}{b^2}\right)\sin\theta$$

These equations being true for all values of θ, we have

$$A-\frac{B}{a^2}=-V,\quad C-\frac{D}{a^2}=0$$

$$A-\frac{B}{b^2}=0\quad,\quad C-\frac{D}{b^2}=0$$

or $$A=\frac{Va^2}{b^2-a^2}\text{ and }B=\frac{Va^2b^2}{b^2-a^2},\ C=D=0$$

Substituting the values of the constant in (i), we have

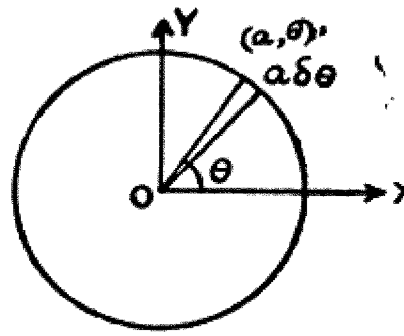
$$\phi=\frac{Va^2}{b^2-a^2}\left(r+\frac{b^2}{r}\right)\cos\theta$$

The impulsive pressure at a point on the inner cylinder, putting $r=a$

$$\varpi=\rho\phi=\frac{Va^2}{b^2-a^2}\left(a+\frac{b^2}{a}\right)\cos\theta$$

$$\varpi=\rho\phi=\frac{\rho Va\,(b^2+a^2)}{(b^2-a^2)}\cos\theta=P \text{ (say)}$$

The total impulsive force due to pressure on the cylinder in X-direction is



$$=-\int_0^{2\pi} P\cos\theta . a\,d\theta$$

$$=-\int_0^{2\pi}\left\{\rho Va\frac{b^2+a^2}{b^2-a^2}\cos\theta\right\}\cos\theta . a\,d\theta$$

$$=-\frac{\rho Va^2\,(b^2+a^2)}{(b^2-a^2)}.\pi \quad \left\{\begin{array}{l}\text{Since there is no compo-}\\ \text{nent perp. to } X\text{-axis.}\end{array}\right.$$

Also the impulse required to move the inner cylinder with a velocity V=mass of cylinder$\times V$

$$=\pi a^2\sigma V$$

We know that

Change in momentum=Sum of Impulsive force

or $\qquad \pi a^2\sigma(V-0)=I+P \qquad \left\{\begin{array}{l}\text{where } I \text{ represents the}\\ \text{impulsive force.}\end{array}\right.$

or $$\pi a^2\sigma V=I-\frac{\rho Va^2\,(b^2+a^2)}{b^2-a^2}\pi$$

or $$I=\pi a^2\sigma\,V+\frac{\rho Va^2\,(b^2+a^2)}{b^2-a^2}\pi$$

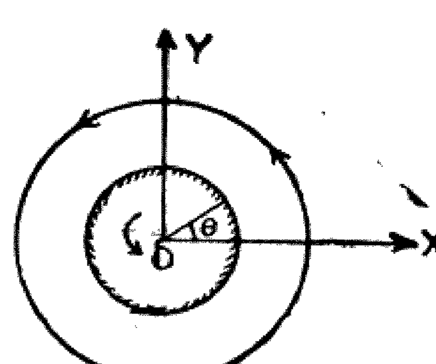
$$=\frac{\pi a^2 V}{b^2-a^2}\left\{\sigma\,(b^2-a^2)+\rho\,(b^2+a^2)\right\}$$

$$=\frac{\pi a^2 V}{b^2-a^2}\left\{(\sigma+\rho)\,b^2-(\sigma-\rho)\,a^2\right\}$$

Proved.

§ 5·6. Circulation about a circular cylinder.

Since the liquid occupies a doubly connected region, the cyclic motion is possible in an irrotational motion. Choosing a suitable form of the velocity potential ϕ, which may be obtained by equating to K (circulation round a circle), such that



$$-\frac{1}{r}\frac{\partial\phi}{\partial\theta}.2\pi r=K\ldots\text{(i)} \quad \text{and} \quad -\frac{\partial\phi}{\partial r}=0$$

(where K is the constant circulation about the cylinder) from (i), we get

$$\frac{\partial \phi}{\partial \theta} = -\frac{K}{2\pi}$$

By integrating, we have

$$\phi = -\frac{K\theta}{2\pi}$$

Since ϕ and ψ are conjugate functions, then

$$\psi = \frac{K}{2\pi} \log r$$

Thus the complex potential due to the circulation about a circular cylinder is given by

$$w = \phi + i\psi$$

or $$w = -\frac{K}{2\pi}\,\theta + i\,\frac{K}{2\pi} \log r$$

or $$w = \frac{iK}{2\pi}\left\{\log r + i\theta\right\} = \frac{iK}{2\pi} \log (re^{\theta i}) \qquad \{\text{Since } z = re^{\theta i}$$

or $$w = \frac{iK}{2\pi} \log z$$

§ 5·7. Streaming and circulation for a fixed circular cylinder.

We know that the streaming motion past a circular cylinder of radius a with velocity U is given by the complex potential,

$$w_1 = Uz + U\frac{a^2}{z} \qquad \ldots(1)$$

{Ref. § 5·4

Also the complex potential due to the circulation of strength k about the cylinder is

$$w_2 = \frac{ik}{2\pi} \log z \qquad \ldots(2)$$

Hence the complex potential for the combined motion is

$$w = U\left(z + \frac{a^2}{z}\right) + \frac{ik}{2\pi} \log z \qquad \ldots(3)$$

or $$\phi + i\psi = U\left\{(r \cos \theta + ir \sin \theta) + \frac{a^2}{r} (\cos \theta - i \sin \theta)\right\} + \frac{ik}{2\pi} (\log r + \theta i)$$

or $$\phi = U\left(r + \frac{a^2}{r}\right) \cos \theta - \frac{k}{2\pi}\,\theta \qquad \ldots(4)$$

and $$\psi = U\left(r - \frac{a^2}{r}\right) \sin \theta + \frac{k}{2\pi} \log r \qquad \ldots(5)$$

The velocity will be tangential at the boundary of the cylinder. Its magnitude is

$$q=\left|\frac{dw}{dz}\right|$$

or $$q=\left|\frac{1}{r}\ \frac{\partial\phi}{\partial\theta}\right| \text{ (Tangential velocity at } r=a).$$

or $$q=\left|2U\sin\theta+\frac{k}{2\pi a}\right| \qquad \{\text{from (4)}$$

If there is no circulation, the stagnation points are given by $\theta=0$ and $\theta=\pi$.

The stagnation points in the presence of circulation are

$$\sin\theta=-\frac{k}{4\pi aU} \qquad ...(6)$$

I. The relation (6) will exist only if $|k|<4\pi aU$ (as $|\sin\theta|$ cannot be greater than unity.)

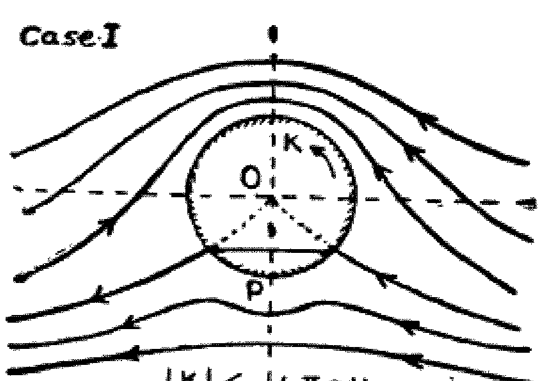
The stagnation points lie on the cylinder and on a line below the centre parallel to the X-axis.

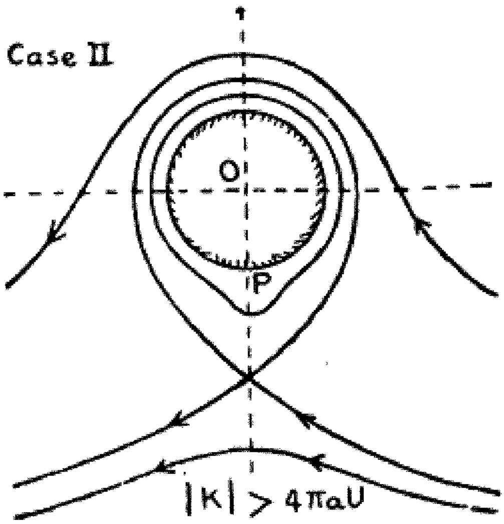
II. When $|k|>4\pi aU$, there exist no stagnation points on the cylinder. In this case the stagnation points are inverse points on the imaginary axis, therefore one is inside the cylinder and does not belong to the motion.

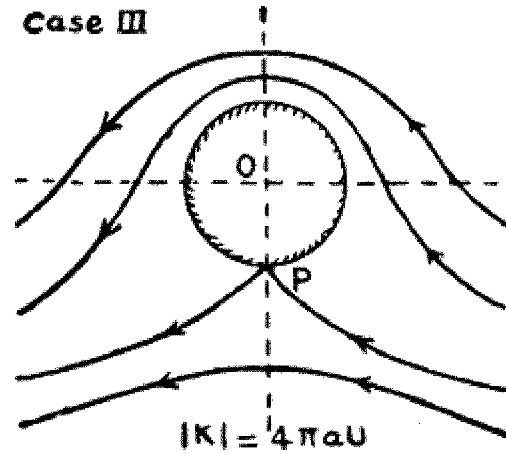
III. When $|k|=4\pi aU$. In this case the stagnation points coincide at the bottom P, of the cylinder.

The stream lines in three different cases are given as follows :

Case I







So any point on the circular cylinder can be made a stagnation point by the proper choice of $\frac{k}{U}$.

Ex. 8. *A circular cylinder is fixed across a stream of velocity U with circulation k round the cylinder. Shew that the maximum velocity in the liquid is* $2U+\frac{k}{2\pi a}$, *where a is the radius of the cylinder.*

The complex potential for the motion of the liquid streaming past a fixed circular cylinder of radius a with circulation k round the cylinder is given by

$$w=Uz+\frac{Ua^2}{z}+\frac{ik}{2\pi}\log z$$

$$\phi+i\psi=Ure^{\theta i}+\frac{Ua^2}{r}e^{-\theta i}+\frac{ik}{2\pi}\log(re^{\theta i})$$

$$\left\{\begin{array}{l}\text{Since } z=x+iy\\ \qquad =re^{\theta i}\end{array}\right.$$

$$=Ur(\cos\theta+i\sin\theta)+\frac{Ua^2}{r}(\cos\theta-i\sin\theta)$$

$$+\frac{ik}{2\pi}(\log r+\theta i)$$

or $$\phi=Ur\cos\theta+\frac{Ua^2}{r}\cos\theta-\frac{k\theta}{2\pi}.$$

If q be the velocity, then

$$q^2=\left(\frac{\partial\phi}{\partial r}\right)^2+\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)^2$$

or $$q^2=\left\{U\left(1-\frac{a^2}{r^2}\right)\cos\theta\right\}^2+\left\{U\left(1+\frac{a^2}{r^2}\right)\sin\theta+\frac{k}{2\pi r}\right\}^2$$

or $$q^2=U^2\left(1-\frac{2a^2}{r^2}\cos 2\theta+\frac{a^4}{r^4}\right)+\frac{2Uk}{2\pi r}\left(1+\frac{a^2}{r^2}\right)\sin\theta+\frac{k^2}{4\pi^2r^2}$$

$$\ldots(1)$$

Now the velocity q will be maximum when r is minimum *i.e.* at $r=a$ (1) reduces to

$$q^2=U^2(1-2\cos 2\theta+1)+\frac{2Uk}{2\pi a}(1+1)\sin\theta+\frac{k^2}{4\pi^2a^2}$$

or
$$q^2=4U^2\sin^2\theta+\frac{2Uk}{\pi a}\sin\theta+\frac{k^2}{4\pi^2a^2}$$

or
$$q^2=\left(2U\sin\theta+\frac{k}{2\pi a}\right)^2$$

or
$$q=2U\sin\theta+\frac{k}{2\pi a}.$$

Now θ will be maximum when $\theta=\pi/2$ *i.e.* $\sin\theta=1$.

Hence $\quad q_{\max}=2U+\dfrac{k}{2\pi a}.$

Hence the maximum velocity in the liquid is $2U+\dfrac{k}{2\pi a}$

Proved.

Ex. 9. *The space between two fixed coaxial circular cylinders of radii a and b, and between two planes perpendicular to the axes and distant c apart, is occupied by liquid of density* ρ. *Shew that the velocity potential of a motion whose kinetic energy shall equal to a given quantity T is given by* $A\theta$, *where* $\pi\rho A^2c\log(b/a)=T$.

The liquid moves in the intervening space between the two fixed co-axial circular cylinders, **the motion is purely of circulation.**

So, we have

$$\phi=A\theta \quad \text{(where } A \text{ is an arbitrary const.)} \qquad \ldots\text{(i)}$$

Let T be the kinetic energy of the liquid.

$$T=\tfrac{1}{2}\int_a^b(2\pi r\,dr\,\rho c).q^2$$

$$T=\pi\rho c\int_a^b rq^2\,dr$$

Where $q^2=\left(\dfrac{\partial\phi}{\partial r}\right)^2+\left(\dfrac{1}{r}\dfrac{\partial\phi}{\partial\theta}\right)^2$ $\qquad$ {from (i)

$$q^2=\frac{A^2}{r^2}$$

or
$$T=\pi\rho c\int_a^b\frac{A^2}{r^2}\cdot r\,dr$$

or
$$T=\pi\rho cA^2\int_a^b\frac{1}{r}\,dr$$

or
$$T=\pi\rho cA^2(\log b-\log a)$$

or
$$T=\pi\rho cA^2\log\left(\frac{b}{a}\right).$$

Proved.

§ 5·8. To determine the pressure at points on the cylinder.

If the circulation in a circuit is $2\pi k$, then k is known as the strength of the circulation.

$$\therefore \quad -\frac{1}{r}\frac{\partial \phi}{\partial \theta}2\pi r=2\pi k \text{ and } -\frac{\partial \phi}{\partial r}=0 \qquad \{\text{Ref. § 5·6}$$

By integrating, we have

$$\phi=-k\theta \text{ and } \psi=k \log r.$$

So $\quad w=\phi+i\psi=ik \log z$

The complex potential for a streaming and circulation of a fixed circular cylinder is

$$w=U\left(z+\frac{a^2}{z}\right)+ik \log z$$

or $$\frac{dw}{dz}=U\left(1-\frac{a^2}{z^2}\right)+\frac{ik}{z}$$

or $$\left(\frac{dw}{dz}\right)^2=U^2\left(1-\frac{a^2}{z^2}\right)^2-\frac{k^2}{z^2}+2i\,\frac{k}{z}\,U\left(1-\frac{a^2}{z^2}\right)$$

The pole inside the circular boundary is at $z=0$, so we shall calculate the sum of residues at $z=0$. Equating the coefficients of $1/z$, we have

$$\int_C \left(\frac{dw}{dz}\right)^2 dz=2\pi i\times(\text{sum of residues at } z=0)$$

$$\int_C \left(\frac{dw}{dz}\right)^2 dz=2\pi i\ \ (2ikU)=-4\pi kU$$

By Blasius Theorem, we have

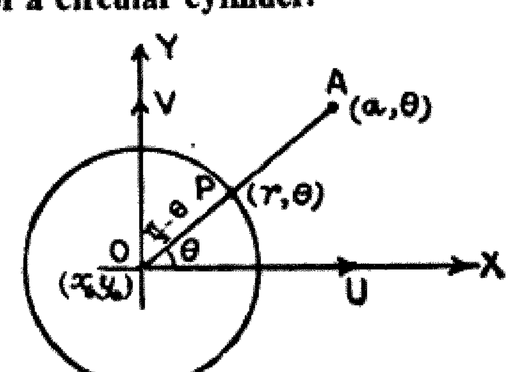
$$X-iY=\tfrac{1}{2}\ i\rho\int_C \left(\frac{dw}{dz}\right)^2 dz=\tfrac{1}{2}i\rho\,(-4\pi kU)$$
$$=-2\pi i\rho kU$$

Gives $\quad X=0 \quad$ and $\quad Y=2\pi\rho kX.$

Thus the cylinder experiences a force $2\pi\rho kU$. This force is usually called the **lift.**

§ 5·81. Equation of motion of a circular cylinder.

A cylinder is moving in a liquid at rest at infinity, we shall calculate the forces acting on the cylinder owing to the presence of the liquid.



Let the centre of the cross-section be (x_0, y_0) and U, V be the components of velocity of the cylinder.

Then $U=\dfrac{dx_0}{dt}=\dot{x}_0$

and $V=\dfrac{dy_0}{dt}=\dot{y}_0$ $\left\{\begin{array}{ll}\text{Since} & z=x+iy \\ \text{and} & z_0=x_0+iy_0\end{array}\right.$

Also $z-z_0=OP.e^{\theta i}=re^{\theta i}$.

We know that the normal component of velocity of the liquid at A $(r=a)$ is equal to the normal component of velocity of the cylinder at the point.

Then $$\left(-\frac{\partial\phi}{\partial r}\right)_{r=a}=(U\cos\theta+V\sin\theta) \quad \ldots(\text{i})$$

and $$\left(\frac{\partial\phi}{\partial r}\right)_{r=\infty}=0 \quad \ldots(\text{ii})$$

Assuming the velocity potential of the form

$$\phi=\left(Ar+\frac{B}{r}\right)\cos\theta+\left(Cr+\frac{D}{r}\right)\sin\theta \quad \ldots(\text{iii})$$

or $$\frac{\partial\phi}{\partial r}=\left(A-\frac{B}{r^2}\right)\cos\theta+\left(C-\frac{D}{r^2}\right)\sin\theta \quad \ldots(\text{iv})$$

From (i) and (iv), we get

or $$-(U\cos\theta+V\sin\theta)=\left(A-\frac{B}{a^2}\right)\cos\theta+\left(C-\frac{D}{a^2}\right)\sin\theta \quad \ldots(\text{v})$$

Again from (ii) and (iv), we get

$$0=A\cos\theta+C\sin\theta \quad \ldots(\text{vi})$$

These relations hold for all values of θ, we get

$$A-\frac{B}{a^2}=-U,\quad C-\frac{D}{a^2}=-V$$

$A=0$ and $C=0$.

So $B=a^2U$ and $D=a^2V$.

By substituting the value of the constants, (iii) reduce to.

$$\phi=\frac{a^2}{r}\left(U\cos\theta+V\sin\theta\right)$$

Since ϕ and ψ are conjugate functions, then

$$\psi=\frac{a^2}{r}\left(-U\sin\theta+V\cos\theta\right)$$

The complex potential is given by

$$w=\phi+i\psi$$

or $$w=\frac{a^2}{r}\left\{(U\cos\theta+V\sin\theta)+i\,(-U\sin\theta+V\cos\theta)\right\}$$

or $$w=\frac{a^2(U+iV)}{re^{\theta i}}=\frac{a^2(U+iV)}{z-z_0}\quad \left\{\text{as}\quad z-z_0=re^{i\theta}\right. \qquad \ldots\text{(vii)}$$

Again $$z_0=x_0+i\,y_0$$

or $$\dot{z}_0=\dot{x}_0+i\,\dot{y}_0=U+iV$$

Differentiating (vii) w. r. to t, we have

$$\frac{\partial w}{\partial t}=\frac{a^2(\dot{U}+i\,\dot{V})}{z-z_0}+\frac{a^2(U+iV)^2}{(z-z_0)^2}$$

or $$\frac{\partial\phi}{\partial t}+i\,\frac{\partial\psi}{\partial t}=\frac{a^2}{r}\,(\dot{U}+i\,\dot{V})\,(\cos\theta-i\sin\theta)$$
$$+\frac{a^2\,(U+iV)^2}{r^2}\,(\cos 2\theta-i\sin 2\theta)$$

Equating real and imaginary parts, we have

$$\frac{\partial\phi}{\partial t}=\frac{a^2}{r}\,(\dot{U}\cos\theta+\dot{V}\sin\theta)$$
$$+\frac{a^2}{r^2}\left\{(U^2-V^2)\cos 2\theta+2UV\sin 2\theta\right\}$$

or $$\left(\frac{\partial\phi}{\partial t}\right)_{r=a}=a\,(\dot{U}\cos\theta+\dot{V}\sin\theta)$$
$$+(U^2-V^2)\cos 2\theta+2UV\sin 2\theta \qquad \ldots\text{(viii)}$$

Also differentiating (vii) w.r. to z, we have

$$\frac{dw}{dz}=-\frac{a^2(U+iV)}{(z-z_0)^2}=-\frac{a^2(U+iV)}{r^2e^{2\theta i}}$$

or $$\frac{dw}{dz}=-\frac{a^2}{r^2}\,(U+iV)\,(\cos 2\theta-i\sin 2\theta)$$

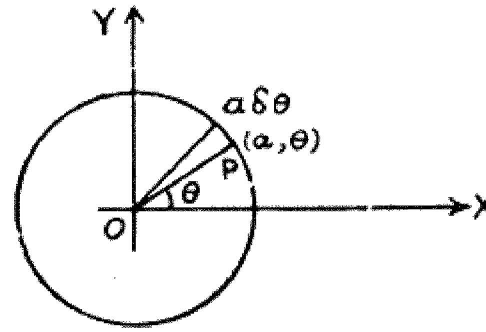
or $$\frac{dw}{dz}=-\frac{a^2}{r^2}\left\{(U\cos 2\theta+V\sin 2\theta)\right.$$
$$\left.+i\,(V\cos 2\theta-U\sin 2\theta)\right\}$$

then $$q^2=\left|\frac{dw}{dz}\right|^2=\left|-\frac{a^2\,(U+iV)}{(z-z_0)^2}\right|$$
$$=\frac{a^4\,(U^2+V^2)}{r^4}$$

Neglecting the extraneous forces, Bernoulli's equation reduces to

$$\frac{p}{\rho}=F(t)+\frac{\partial\phi}{\partial t}-\tfrac{1}{2}q^2 \qquad \ldots\text{(ix)}$$

Let $P(a, \theta)$ be any point on the cylinder, p be the pressure and X, Y are the components of forces on the cylinder.



Then $X = -\int_0^{2\pi} p \cos\theta . a\, d\theta$

and $Y = -\int_0^{2\pi} p \sin\theta . a\, d\theta$

$\Big\{$ as components of the pressure along the axes are $-p\cos\theta$ and $-p\sin\theta$

or $$X = -\int_0^{2\pi} \rho\left(F(t) + \frac{\partial\phi}{\partial t} - \frac{1}{2}q^2\right)_{r=a} . a\cos\theta\, d\theta$$

or $$X = -\int_0^{2\pi} \rho\Big\{F(t) + a(\dot{U}\cos + \dot{V}\sin\theta) + (U^2 - V^2)\cos\theta + 2UV\sin 2\theta - \frac{1}{2}\frac{a^4(U^2+V^2)}{a^4}\Big\}\, a\cos\theta\, d\theta$$

or $$X = -\rho a^2 \int_0^{2\pi} \dot{U}\cos^2\theta\, d\theta \qquad \text{(other integral vanishes)}$$

or $$X = -\pi\rho a^2\,\dot{U} = -M'\dot{U}$$

Similarly $$Y = -\pi\rho a^2\,\dot{V} = -M'\dot{V}$$

$\Big\{$ Since $M' = \pi a^2\rho$ mass of liquid displaced by the cylinder

Let X' and Y' be the components of the external forces on the cylinder of mass M, when there is no liquid. Let σ be the density of the cylinder and ρ the density of the liquid, then resultant force in X-direction (due to presence of the liquid)

$$= \pi a^2\sigma X_1 - \pi a^2\rho X_1$$

$$= \pi a^2\left(\frac{\sigma-\rho}{\sigma}\right). \sigma X_1$$

$\Big\{$ where X_1 is the acc. of the external forces on the liquid in X direction

$$= \frac{\sigma-\rho}{\sigma} X'$$

$\{$ as $X' = \pi a^2\sigma X_1$

Similarly the resultant external force along Y direction

$$= \frac{\sigma-\rho}{\sigma} Y'$$

So equations of motion are,

$$M\dot{U} = -M'\dot{U} + \frac{\sigma-\rho}{\sigma}X' \qquad \text{...(x)}$$

and $$M\dot{V} = -M'\dot{V} + \frac{\sigma-\rho}{\sigma}Y' \qquad \text{...(xi)}$$

From (x) we have

$$(M+M')\,\dot{U}=\frac{\sigma-\rho}{\sigma}X'$$

or
$$\dot{U}=\frac{\sigma-\rho}{\sigma\,(M+M')}X'$$

or
$$M\,\dot{U}=\frac{M}{M+M'}\cdot\frac{\sigma-\rho}{\sigma}X'$$

$$=\frac{\pi\sigma a^2}{\pi\sigma a^2+\pi\rho a^2}\cdot\frac{\sigma-\rho}{\sigma}X'$$

$$=\frac{\sigma-\rho}{\sigma+\rho}X' \qquad \text{...(A)}$$

From (xi), we have

$$(M+M')\,\dot{V}=\frac{\sigma-\rho}{\sigma}Y'$$

or
$$M\dot{V}=\frac{M}{M+M'}\cdot\frac{\sigma-\rho}{\sigma}Y'$$

$$=\frac{\pi\sigma a^2}{\pi\sigma\,a^2+\pi\rho\,a^2}\cdot\frac{\sigma-\rho}{\sigma}Y'$$

$$M\,\dot{V}=\frac{\sigma-\rho}{\rho+\rho}Y' \qquad \text{...(B)}$$

From (A) and (B) it shows that the effect of the presence of the liquid is to reduce the external force in the ratio

$$(\sigma-\rho):(\sigma+\rho)$$

Exercise

1. A circular cylinder is moving with velocity U parallel to OX in an infinite liquid ; show that the motion is the same as it would be if the cylinder were removed and a doublet placed at the centre of the cylinder with its axis pointing in the direction of motion of the liquid.

2. An infinite circular cylinder of radius a is in motion in a homogeneous fluid which extends to infinity and is at rest there. Shew that at any moment, the pressure at a point of the fluid at a distance r, from the axis of the cylinder exceeds the hydrostatic pressure by

$$\left[\frac{a^2}{r}f_1+\frac{a^2}{r^2}\left\{\left(1-\frac{a^2}{2r^2}\right)u_1^2-\left(1+\frac{a^2}{2r^2}\right)v_1^2\right\}\right]$$

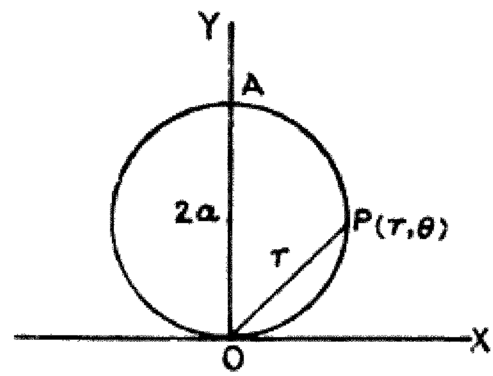
where f_1 is the component of the acceleration of the centre of the cylinder in the direction of r ; u_1 and v_1 are the component velocities of the centre in and perpendicular to that direction and ρ is the density of the liquid.

3. A circular cylinder of radius a and infinite length lies on a plane in an infinite depth of liquid. The velocity of the liquid at a great distance from the cylinder is U perpendicular to the generators, and the motion is ir-rotational and two dimensional. Verify that the stream function is the imaginary part of $w=\pi aU \coth \frac{\pi a}{z}$, where z is a complex variable zero on the line of contact and real on the plane.

Prove that the pressures at the two ends of the diameter of the cylinder normal to the plane differs by

$$\frac{1}{32}\pi^4 \rho U^2.$$

Hint : Consider $P(r, \theta)$ be any point on the cross-section of the circular cylinder



oe $$\frac{1}{z}=\frac{1}{re^{\theta i}}=\frac{1}{r^2}\, re^{-\theta i}$$

$$=\frac{1}{r^2}(r\cos\theta-ir\sin\theta)$$

$$=\frac{1}{r^2}(x-iy)$$

as $$w=\phi+i\psi=\pi aU\frac{\cosh\frac{\pi a}{z}}{\sinh\frac{\pi a}{z}}=\pi a\, U\frac{\cosh\frac{\pi a}{r^2}(x-iy)}{\sinh\frac{\pi a}{r^2}(x-iy)}$$

then $$\psi=\pi a\, U\frac{\sin\left(2\frac{\pi a}{r^2}y\right)}{\cosh\left(2\frac{\pi a}{r^2}x\right)-\cos\left(2\frac{\pi a}{r^2}y\right)}$$

Stream lines are given by $\psi=0$.

i.e. $$\sin\left(2\frac{\pi a}{r^2}y\right)=0 \quad i.e. \quad y=0 \quad \text{or} \quad y=\frac{r^2}{2a}$$

or $$r^2=2ay=2ar\sin\theta$$

or $$r=2a\sin\theta$$

Thus it represents a circle which is a stream line.

Let p_0 and p_1 be the pressure at O and A

$$p_0-p_1=\tfrac{1}{2}\rho v^2 \qquad \ldots(1)$$

and $$v^2=\left|\frac{dw}{dz}\right|^2 \quad at \quad x=0\ y=2a\ r=2a$$

Find the velocity v, thus pressure difference at the ends of the diameter can be determined from (1).

ELLIPITIC CYLINDERS

§ 5·82. Elliptic Coordinates.

Let $\qquad z=c \cosh \zeta$

Where $\qquad z=x+iy$ and $\zeta=\xi+i\eta$

or $\qquad x+iy=c \cosh (\xi+i\eta)$

or $\qquad x+iy=c (\cosh \xi \cos \eta-i \sinh \xi \sin \eta)$

Equating real and imaginary parts, we have

$$x=c \cosh \xi \cos \eta$$
$$y=c \sinh \xi \sin \eta$$

or $$\frac{x^2}{c^2 \cosh^2 \xi}+\frac{y^2}{c^2 \sinh^2 \xi}=1 \quad \forall \text{ values of } \eta. \qquad \text{...(i)}$$

and $$\frac{x^2}{c^2 \cos^2 \eta}-\frac{y^2}{c^2 . \sin^2 \eta}=1 \qquad \text{...(ii)}$$

Thus ξ=const. and η=const. represent confocal ellipses and confocal hyperbolas respectively, and the distance between the focii $(\pm c, 0)$ in each case is $2c$. **The parameter ξ and η are called the Elliptic Coordinates.**

When $\qquad \xi=\text{const}=\alpha$ (let)

From (i), we have $\qquad a=c \cosh \alpha,\ b=c \sinh \alpha$

then $$a+b=c (\cosh \alpha+\sinh \alpha)=ce^{\alpha} \qquad \text{...(iii)}$$
$$a-b=c (\cosh \alpha-\sinh \alpha)=ce^{-\alpha}$$

and $$a^2-b^2=c^2 \qquad \text{...(iv)}$$

By dividing (iii) and (iv), we get

$$\frac{a+b}{a-b}=\frac{ce^{\alpha}}{ce^{-\alpha}}=e^{2\alpha}$$

or $$\alpha=\tfrac{1}{2} \log \left(\frac{a+b}{a-b}\right)$$

and the distance between the focii of (i) is

{ since $b^2=a^2 (1-e^2)$ }

$$=2ae$$
$$=2\sqrt{(a^2-b^2)}$$
$$=2c\sqrt{(\cosh^2\alpha-\sinh^2\alpha)}$$
$$=2c$$

which shows that $x=c \cosh \alpha \cos \eta$ and $y=c \sinh \alpha \sin \eta$ are the coordinates of a point on (i), where η is an eccentric angle of the point.

Now the Laplace Equation $\dfrac{\partial^2\psi}{\partial\xi^2}+\dfrac{\partial^2\psi}{\partial\eta^2}=0$ has its solution of the type

$$\left.\begin{matrix}\cosh\\ \sinh\\ \exp.\end{matrix}\right\}(n\,\xi)\qquad \left.\begin{matrix}\cos\\ \sin\end{matrix}\right\}(n\,\eta)$$

for positive or negative integral values of n.

and that $e^{-n\xi}$ must be used when the property of vanishing at infinity is employed *i.e.* when the liquid is at rest at infinity. Also for confocal ellipses the form

$$(A \cosh n\,\xi+B \sinh n\,\xi)\left.\begin{matrix}\cos\\ \sin\end{matrix}\right\}(n\eta) \text{ may be used.}$$

§ 5·83. Motion of an elliptic cylinder.

(a) To determine the stream function and velocity potential when an elliptic cylinder moves in an infinite liquid with velocity u parallel to the major axis of the cross-section.

We know that the current function for the most general type of motion of the cylinder is

$$\psi=Vx-Uy+\tfrac{1}{2}\omega\,(x^2+y^2)=\text{const.} \qquad ...(i)$$

Here $V=0,\ \omega=0.$ {Ref. § 5·2

or $\psi=-Uy+\text{const.}$ ...(ii)

Let the cross-section of the cylinder be an ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

This is same as $\dfrac{x^2}{c^2\cosh^2\xi}+\dfrac{y^2}{c^2\sinh^2\xi}=1$

Where $a=c \cosh \alpha$, $b=c \sinh \alpha$ { at $\xi=\alpha$

Thus the elliptic coordinates are

$$x=c\cosh\alpha\cos\eta \text{ and } y=c\sinh\alpha\sin\eta \qquad ...(iii)$$

Since the effect is to vanish at infinity and $\sin\eta$ is the only variable factor in the boundary condition then ψ must be of the from $e^{-\xi}\sin\eta$.

Assuming the complex potential of the form

$$\phi+i\psi=Ae^{-(\xi+i\eta)}$$

So that $\qquad \psi=-Ae^{-\xi}\sin\eta \qquad$ ...(iv)

At the boundary $\xi=\alpha$ (ii) and (iv) are same for all values of η

i.e. $\qquad -Ae^{-\alpha}\sin\eta=-Uc\sinh\alpha\sin\eta+\text{const.} \qquad$ {from (iii)

Which gives that const $=0$ and $A=Uc\,e^{\alpha}\sinh\alpha$

From (iv), we have

$$\psi=-Uc\,e^{\alpha-\xi}\sinh\alpha\sin\eta. \qquad \text{...(v)}$$

the stream function which will make the boundary of the ellipse a stream line, when the cylinder moves parallel to its major axis with velocity U.

The relation (v) Can be written as

$$\psi=-Ub\,e^{\alpha}.e^{-\xi}\sin\eta \qquad \{\text{as } b=c\sinh\alpha$$

$$\psi=-Ub\sqrt{\left(\frac{a+b}{a-b}\right)}\,e^{-\xi}\sin\eta \qquad \{\text{Ref. § 5·82}$$

Similarly $\quad \phi=Ub.\sqrt{\left(\frac{a+b}{a-b}\right)}\,e^{-\xi}\cos\eta$

(b) To determine the stream function and velocity potential when an elliptic cylinder moves in an infinite liquid with a velocity V parallel to the minor axis of the cross-section.

The stream function for the most general type of motion of the cylinder is given by

$$\psi=Vx-Uy+\tfrac{1}{2}\omega\,(x^2+y^2)+\text{const.}$$

Here $U=0$, $\omega=0$, It reduces to

$$\psi=Vx+\text{const.} \qquad \text{...(i)}$$

The elliptic coordinates are

$$x=c\cosh\alpha\cos\eta \quad \text{and} \quad y=c\sinh\alpha\sin\eta$$

since the effect is to vanish at infinity and $\cos\eta$ is the only variable factor in the boundary condition then ψ must be of the form $e^{-\xi}\cos\eta$.

Assuming that $\psi=Ae^{-\xi}\cos\eta \qquad$ ...(ii)

At the boundary $\xi=\alpha$ (i) and (ii) are same for all values of η.

$$Ae^{-\alpha}\cos\eta=Vc\cosh\alpha\cos\eta+\text{const.}$$

Which gives const. $=0$ and $A=Vce^{\alpha}\cosh\alpha$.

Substituting the value of A in (ii), we get

$$\psi = Vce^{\alpha}\, e^{-\xi} \cosh \alpha \cos \eta \qquad \{\text{as } a = c \cosh \alpha$$

or
$$\psi = Va \sqrt{\left(\frac{a+b}{a-b}\right)}\, e^{-\xi} \cos \eta \qquad \text{and } e^{\alpha} = \sqrt{\left(\frac{a+b}{a-b}\right)}$$

Then $\phi = Va \sqrt{\left(\frac{a+b}{a-b}\right)}\, e^{-\xi} \sin \eta.$

§ 5·84. The Joukowski Transformation.

Let the transformation be

$$z = t + \frac{c^2}{4t}. \qquad \ldots(1)$$

Obviously when $|z|$ is large $t = z$ (approx.). So the corresponding distant parts of the two planes remains unaltered. Thus a uniform stream at infinity in z-plane corresponds to that of the same strength and direction in t-plane.

From (1) $\quad 4t^2 - 4zt + c^2 = 0$

Gives $\quad t = \frac{1}{2}\{z \pm \sqrt{(z^2 - c^2)}\}. \qquad \ldots(2)$

The positive sign outside the radical follows that the value of $\sqrt{(z^2 - c^2)}$ is to be taken which becomes real and positive when z is on the positive part of the real axis which lies outside the ellipse when $|z|$ is very large then from (2), we have

$$t = z \text{ or } t = 0 \qquad \left\{\begin{array}{l}\text{according as} + \text{ive or } - \text{ive sign is}\\ \text{considered before the radical}\end{array}\right.$$

Thus, taken positive sign before the radical

$$i.e. \qquad t = \tfrac{1}{2}\{z + \sqrt{(z^2 - c^2)}\}$$

the points outside the ellipse in the z-plane corresponds to the points outside the circle $t = \frac{1}{2}(a+b)$ in the t-plane.

§ 5·85. Streaming past a fixed elliptic cylinder.

Consider a stream whose complex potential in t-plane is $Ute^{-i\alpha}$. If we insert the circular cylinder $|t| = \frac{1}{2}(a+b)$, the complex potential becomes

$$w = Ute^{-i\alpha} + U\frac{(a+b)^2}{4t}\, e^{i\alpha} \qquad \ldots(1)$$

{Circle Theorem

We know by Joukowski's transformation

$$t = \tfrac{1}{2}\{z + \sqrt{(z^2 - c^2)}\},\ c^2 = a^2 - b^2. \qquad \ldots(2)$$

the region outside the ellipse in z-plane corresponds to the region outside the circle $|t| = \frac{1}{2}(a+b)$ in t-plane.

Thus the complex potential, for the flow past a fixed elliptic cylinder can be determined from (1) and (2),

$$w=\tfrac{1}{2}U\left[e^{-i\alpha}\{z+\sqrt{(z^2-c^2)}\}+\frac{(a+b)^2\,e^{i\alpha}}{z+\sqrt{(z^2-c^2)}}\right]$$

$$w=\tfrac{1}{2}U\left[e^{-i\alpha}\{z+\sqrt{(z^2-c^2)}\}+\frac{e^{i\alpha}}{c^2}(a+b)^2\{z-\sqrt{(z^2-c^2)}\}\right] \quad ...(3)$$

By using elliptic coordinates, Put $z=c\cosh\zeta$ in (3), we get

$$z+\sqrt{(z^2-c^2)}=c\cosh\zeta+\sqrt{(c^2\cosh^2\zeta-c^2)}$$
$$=c\cosh\zeta+c\sinh\zeta=ce^{\zeta}$$

or $$w=\tfrac{1}{2}U\left\{e^{-i\alpha}.ce^{\zeta}+\frac{e^{i\alpha}}{c^2}(a+b)^2.ce^{-\zeta}\right\}$$

or $$w=\tfrac{1}{2}U\left\{e^{-i\alpha}.e^{\zeta}\sqrt{(a^2-b^2)}+\frac{(a+b)^2}{\sqrt{(a^2-b^2)}}e^{i\alpha}.e^{-\zeta}\right\}$$

or $$w=\tfrac{1}{2}U\sqrt{(a^2-b^2)}\left\{e^{\zeta-i\alpha}+\frac{(a+b)^2}{(a^2-b^2)}e^{-(\zeta-i\alpha)}\right\}$$

or $$w=\tfrac{1}{2}U\sqrt{\left(\frac{a+b}{a-b}\right)}\left\{(a-b)\,e^{\zeta-i\alpha}+(a+b)\,e^{-(\zeta-i\alpha)}\right\}$$

or $$w=\tfrac{1}{2}U\sqrt{\left(\frac{a+b}{a-b}\right)}\left\{c\,e^{-\alpha_1}.e^{\zeta-i\alpha}+ce^{\alpha_1}.e^{-(\zeta-i\alpha)}\right\}$$

or $$w=\tfrac{1}{2}U\sqrt{\left(\frac{ce^{\alpha_1}}{ce^{-\alpha_1}}\right)}.c\left\{e^{\zeta-\alpha_1-i\alpha}+e^{i\alpha-\zeta+\alpha_1}\right\}$$

[Since on the ellipse $\zeta=\alpha_1$ (let)
$a=c\cosh\alpha_1$, $b=c\sinh\alpha_1$
then $a-b=ce^{-\alpha_1}$
and $a+b=ce^{\alpha_1}$]

or $$w=\tfrac{1}{2}Uce^{\alpha_1}\left\{e^{\zeta-\alpha_1-i\alpha}+e^{-(\zeta-\alpha_1-i\alpha)}\right\}$$

or $$w=U(a+b)\cosh(\zeta-\alpha_1-i\alpha). \quad ...(4)$$

Which is the complex potential for the streaming motion past a fixed elliptic cylinder.

Let the stream flows parallel to the real axis, then substituting $\alpha=0$ in (4), the complex potential is given by

$$w=U(a+b)\cosh(\zeta-\alpha_1). \quad ...(5)$$

Particular Case. Let the whole system is given a velocity U

inclined at an angle α with the real axis, then the stream is reduced to rest and the cylinder moves with velocity U.

The complex potential is

$$w=\frac{U(a+b)^2}{4t}\,e^{i\alpha}$$

or
$$w=\frac{U(a+b)^2}{2\{z+\sqrt{(z^2-c^2)}\}}\,e^{i\alpha}$$

or
$$w=\frac{U(a+b)^2}{2c^2}\,\{z-\sqrt{(z^2-c^2)}\}\,e^{i\alpha}$$

or
$$w=\frac{U(a+b)^2}{2c}\,e^{-\zeta}.e^{i\alpha}$$

or
$$w=\tfrac{1}{2}U(a+b)\,e^{\alpha_1}\,e^{-\zeta}.e^{i\alpha}$$

or
$$w=\tfrac{1}{2}U(a+b)\;e^{\alpha_1}\,.\,e^{-(\xi+i\eta)}\;e^{i\alpha}$$

or
$$w=\tfrac{1}{2}U(a+b)\,e^{\alpha_1-\xi}\,.\,e^{i(\alpha-\eta)}$$

or
$$\phi+i\psi=\tfrac{1}{2}U(a+b)\,e^{\alpha_1-\xi}\{\cos(\eta-\alpha)-i\sin(\eta-\alpha)\}$$

Equating real and imaginary parts, we get

$$\left.\begin{aligned}\phi&=\tfrac{1}{2}U(a+b)\,e^{\alpha_1-\xi}\cos(\eta-\alpha)\\ \psi&=-\tfrac{1}{2}U(a+b)\,e^{\alpha_1-\xi}\sin(\eta-\alpha)\end{aligned}\right\}\quad\ldots(1)$$

gives the velocity potential and stream function for an elliptic cylinder moving in an infinite liquid with velocity U inclined at an angle α with the real axis.

I. If the elliptic cylinder moves parallel to X-axis, then Putting $\alpha=0$ in (1), we have

$$\phi=\tfrac{1}{2}U(a+b)\,e^{\alpha_1-\xi}\cos\eta$$

and
$$\psi=-\tfrac{1}{2}U(a+b)\,e^{\alpha_1-\xi}\sin\eta.$$

Ex. 1. *Shew that with proper choice of units the motion of an infinite liquid produced by the motion of an elliptic cylinder parallel to one of its principal axes is given by the complex function*

$$w=e^{-\zeta}\text{ where } z=2\cosh\zeta.$$

Reduce the formulae

$$x=\phi\left(1+\frac{1}{\phi^2+\psi^2}\right);\;y=\psi\left(1-\frac{1}{\phi^2+\psi^2}\right)$$

The velocity potential and stream function for an elliptic cylinder moving parallel to major axis with a velocity U, is given by

$$\phi=Ub\sqrt{\left(\frac{a+b}{a-b}\right)}\,e^{-\xi}\cos\eta$$

and $\psi=-Ub\sqrt{\left(\frac{a+b}{a-b}\right)}\, e^{-\xi} \sin \eta$

So the complex potential is

$$w=\phi+i\psi=Ub\sqrt{\left(\frac{a+b}{a-b}\right)}e^{-\xi}\ (\cos \eta-i\ \sin \eta)$$

or $$w=\phi+i\psi=Ub\sqrt{\left(\frac{a+b}{a-b}\right)}\ e^{-(\xi+i\eta)}$$

or $$w=\phi+i\psi=Ub\sqrt{\left(\frac{a+b}{a-b}\right)}\ e^{-\zeta} \qquad \{\text{where } \zeta=\xi+i\eta$$

Substituting $a=c\cosh\alpha$, $b=c\sinh\alpha$, we have

$$w=Uc\sinh\alpha\sqrt{\left(\frac{\cosh\alpha+\sinh\alpha}{\cosh\alpha-\sinh\alpha}\right)}.e^{-\zeta}$$

or $$w=Uc\sinh\alpha.e^{\alpha}.e^{-\zeta}$$
or $$w=2U\sinh\alpha.e^{\alpha}.e^{-\zeta} \qquad \text{...(i)} \begin{cases}\text{Since } z=2\cosh\zeta\\ \text{So} \quad c=2\end{cases}$$

Thus by proper choice of U and α, we have

$$2U\sinh\alpha.e^{\alpha}=1$$

or $$2U\sinh\alpha=e^{-\alpha}, \text{ then (1) reduces to}$$
$$w=e^{-\zeta}$$ **Proved.**

Again $w=\phi+i\psi=e^{-(\xi+i\eta)}$ ...(2)

or $$z=2\cosh\zeta \qquad \text{(Given)}$$
or $$x+iy=2\cosh(\xi+i\eta) \qquad \text{...(3)}$$
or $$x+iy=e^{\xi+i\eta}+e^{-(\xi+i\eta)}$$
or $$x+iy=\left\{\frac{1}{\phi+i\psi}+(\phi+i\psi)\right\}$$
or $$x+iy=\left\{\frac{\phi-i\psi}{\phi^2+\psi^2}+(\phi+i\psi)\right\}$$

Separating into real and imaginary parts, we have

$$x=\frac{i\phi}{\phi^2+\psi^2}+\phi=\phi\left(1+\frac{1}{\phi^2+\psi^2}\right)$$

and $$y=-\frac{\psi}{\phi^2+\psi^2}+\psi=\psi\left(1-\frac{1}{\phi^2+\psi^2}\right).$$ **Proved.**

Ex. 2. *An elliptic cylinder, the semi-axes of whose cross-section are a and b, is moving with velocity U parallel to the major axis of its cross-section, through an infinite liquid of density* ρ *which is at rest at infinity, the pressure there being* Π. *Prove that in order that the pressure may every be positive*

$$\rho U^2 < \frac{2a^2\Pi}{2ab+b^2}.$$

Let p be the pressure and q be the velocity at any point, then by Bernoulli's theorem for steady motion ,we have

$$\frac{p}{\rho}+\tfrac{1}{2}q^2=C. \qquad \text{...(1)}$$

where C is any constant.

Since the motion reduces to a steady motion with cylinder at rest at infinity. By super-imposing a velocity $-U$ parallel to X-axis both on cylinder and liquid.

So the velocity at infinity $q_\infty=-U, p=\Pi$.

Then from (1) $\dfrac{\Pi}{\rho}+\dfrac{1}{2}U^2=C$

or $$\frac{p}{\rho}+\frac{1}{2}q^2=\frac{\Pi}{\rho}+\frac{1}{2}U^2$$

or $$\frac{p}{\rho}=\frac{\Pi}{\rho}+\frac{1}{2}U^2-\tfrac{1}{2}q^2$$

By the given condition, the pressure p will be positive every where If $p \geqslant 0$.

i.e. $\dfrac{\Pi}{\rho}+\tfrac{1}{2}U^2-\tfrac{1}{2}q^2$ is positive

or $$\frac{\Pi}{\rho}+\tfrac{1}{2}U^2>\text{max. value of } \tfrac{1}{2}q^2 \qquad \text{...(2)}$$

Now we shall determine the max. value of q^2.

The complex potential w is given by

$$w=U(a+b)\cosh(\zeta-\alpha)$$

or $$\frac{dw}{dz}=U(a+b)\sinh(\zeta-\alpha).\frac{d\zeta}{dz}$$

or $$\frac{dw}{dz}=U(a+b)\frac{\sinh(\zeta-\alpha)}{c\sinh\zeta} \qquad \left\{\begin{array}{l}\text{Since } z=c\cosh\zeta \\ \dfrac{dz}{d\zeta}=c\sinh\zeta\end{array}\right.$$

The stagnation points can be obtained by $\dfrac{dw}{dz}=0$

i.e. $\sinh(\zeta-\alpha)=0$

or $\zeta-\alpha=0$ or $i\pi$

or $(\xi+i\eta-\alpha)=0$ or $i\pi$

or $\xi-\alpha=0$ and $\eta=\pi$ ...(3)

or $\xi=\alpha$

or $$q=\left|\frac{dw}{dz}\right|=\left|\frac{U(a+b)}{c}\cdot\frac{\sinh(\zeta-\alpha)}{\sinh\zeta}\right|$$

or $$q=\left|U\frac{a+b}{\sqrt{(a^2-b^2)}}\cdot\frac{\sinh\{(\xi-\alpha)+i\eta\}}{\sinh(\xi+i\eta)}\right|$$

or $$q=U\sqrt{\left(\frac{a+b}{a-b}\right)}\cdot\frac{\sqrt{\{\sinh^2(\xi-\alpha)+\sin^2\eta\}}}{\sqrt{(\sinh^2\xi+\sin^2\eta)}}$$

$$\left\{\begin{array}{l}\text{as}\quad \sinh\zeta=\sinh(\xi+i\eta)\\ \qquad\sinh\zeta=\sinh\xi\cosh i\eta\\ \qquad\qquad+\cosh\xi\sinh i\eta\\ |\sinh\xi|=\sqrt{(\sinh^2\xi+\sin^2\eta)}\end{array}\right.$$

or $$q^2=U^2\,\frac{a+b}{a-b}\cdot\frac{\sinh^2(\xi-\alpha)+\sin^2\eta}{\sinh^2\xi+\sin^2\eta} \qquad ...(4)$$

Now for maximum value of q, (4) can be written

$$q^2=U^2\,\frac{a+b}{a-b}\left\{1-\frac{\sinh^2\xi-\sinh^2(\xi-\alpha)}{\sinh^2\xi+\sin^2\eta}\right\}$$

But $\sinh\xi>\sinh(\xi-\alpha)$, thus q^2 will be maximum, when $\sin\eta$ is maximum

i.e. $$\sin\eta=1\quad\text{or}\quad\eta=\frac{\pi}{2}$$

or $$q^2=U^2\frac{a+b}{a-b}\left\{\frac{\sinh^2(\xi-\alpha)+1}{\sinh^2\xi+1}\right\} \qquad \{\text{ from (3)}$$

$$q^2=U^2\frac{a+b}{a-b}\cdot\frac{\cosh^2(\xi-\alpha)}{\cosh^2\xi}$$

Since it is an elliptic cylindrical boundary surrounded by the liquid, minimum value of ξ is α. { from (3)

So q is maximum when $\xi=\alpha$ and $\eta=\pi/2$ (*i.e.* at the end of the minor axis).

So the max. value of velocity

$$q^2=U^2\frac{a+b}{a-b}\cdot\frac{1}{\cosh^2\alpha}$$

{ since $a=c\cosh\alpha$

$$q^2=U^2\frac{a+b}{a-b}\cdot\frac{c^2}{a^2}$$

$$q^2=U^2\frac{a+b}{a-b}\cdot\frac{a^2-b^2}{a^2}=U^2\cdot\frac{(a+b)^2}{a^2}$$

or $$(q)_{\max}=U\cdot\frac{(a+b)}{a}$$

So from (2), we have

$$\frac{\Pi}{\rho}+\tfrac{1}{2}U^2>\tfrac{1}{2}U^2\frac{(a+b)^2}{a^2}$$

or $$\frac{\Pi}{\rho}>\tfrac{1}{2}U^2\left\{-1+\frac{(a+b)^2}{a^2}\right\}$$

or $$\frac{\Pi}{\rho}>\tfrac{1}{2}U^2\left\{\frac{2ab+b^2}{a^2}\right\}$$

or $\quad 2\Pi a^2 > \rho U^2 \{2ab+b^2\}$

or $\quad \rho U^2 < \dfrac{2a^2\Pi}{2ab+b^2}$ **Proved.**

Ex. 3. *In the two-dimensional irrotational motion of a liquid streaming past a fixed elliptic disc* $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, *the velocity at infinity being parallel to the major axis and equal to V, prove that if*

$$x+iy=c \cosh (\xi+i\eta)$$

$$a^2-b^2=c^2 \text{ and } a=c\cosh\alpha,\ b=c\sinh\alpha$$

the velocity at any point is given by

$$q^2=V^2\frac{a+b}{a-b}\cdot\frac{\sinh^2(\xi-\alpha)+\sin^2\eta}{\sinh^2\xi+\sin^2\eta}$$

and that it has its maximum value $\dfrac{V(a+b)}{a}$ *at the end of the minor axis.*

Let an elliptic disc moves through a liquid at rest at infinity with a velocity V parallel to the major axis. The stream function is given by

$$\psi=Vy+\text{const.} \qquad \ldots(1)$$

or $\quad w=VC\, e^{\alpha}\{e^{-\alpha}\cosh\zeta+\sinh\alpha\, e^{-\zeta}\}$

or $\quad w=VC\, e^{\alpha}[(\cosh\alpha-\sinh\alpha)\cosh\zeta+\sinh\alpha\,(\cosh\zeta-\sinh\zeta)]$

$\{$ as $a+b=c\, e^{\alpha}$

or $\quad w=V(a+b)[\cosh\alpha\cosh\zeta-\sinh\alpha\sinh\zeta]$

or $\quad w=V(a+b)\cosh(\zeta-\alpha)$

Now velocity at any point

$$q=\left|\frac{dw}{dz}\right|=\left|\frac{dw}{d\zeta}\cdot\frac{d\zeta}{dz}\right|$$

$$q=\frac{V(a+b)}{c}\left|\frac{\sinh(\zeta-\alpha)}{\sinh\zeta}\right|$$

$$q=\frac{V(a+b)}{c}\left|\frac{\sinh\{(\xi-\alpha)+i\eta\}}{\sinh(\xi+i\eta)}\right|$$

$$q=\frac{V(a+b)}{\sqrt{(a^2-b^2)}}\left|\frac{\sinh(\xi-\alpha)\cos i\eta+\cosh(\xi-\alpha)\sin i\eta}{\sinh\xi\cos i\eta+\cosh\xi\sin i\eta}\right|$$

$$q=V\sqrt{\left(\frac{a+b}{a-b}\right)}\cdot\frac{[\sinh^2(\xi-\alpha)+\sin^2\eta]^{1/2}}{[\sinh^2\xi+\sin^2\eta]^{1/2}}$$

or $\quad q^2=V^2.\dfrac{a+b}{a-b}\cdot\dfrac{\sinh^2(\xi-\alpha)+\sin^2\eta}{\sinh^2\xi+\sin^2\eta}$ **Proved.**

Maximum value of the velocity q.

The velocity on the boundary of the elliptic disc is given by substituting $\xi=\alpha$

$$q^2=V^2.\frac{a+b}{a-b}\cdot\frac{\sin^2\eta}{\sinh^2\alpha+\sin^2\eta}$$

Differentiating w. r. to η, we have

$$2q\frac{dq}{d\eta}=V^2.\frac{a+b}{a-b}\cdot\frac{(\sinh^2\alpha+\sin^2\eta)\,2\sin\eta\cos\eta-2\sin\eta\cos\eta.\sin^2\eta}{(\sinh^2\alpha+\sin^2\eta)^2}$$

For maximum or minimum value of q, $\dfrac{dq}{d\eta}=0$.

$$(\sinh^2\alpha+\sin^2\eta).\sin 2\eta-\sin^2\eta\sin 2\eta=0$$

or $\quad \sin 2\eta\,\{\sinh^2\alpha+\sin^2\eta-\sin^2\eta\}=0$

or $\quad \sin 2\eta\sinh^2\alpha=0.$

Either $\sin 2\eta=0$ or $\sinh^2\alpha=0$, But $\sinh\alpha\neq 0$

or $\quad 2\eta=0,\ \pi$

i.e. $\quad \eta=0,\ \pi/2$

So $\eta=0$ gives minimum value of q^2

and $\quad (q^2)_{\eta=\pi/2}=V^2.\dfrac{a+b}{a-b}\cdot\dfrac{1}{1+\sinh^2\alpha}$ (Maximum value)

or $\quad q^2=V^2\dfrac{a+b}{a-b}\cdot\dfrac{1}{\cosh^2\alpha}$ $\quad\left[\begin{array}{l}\text{since } a=c\cosh\alpha\\ \text{and } c^2=a^2-b^2\end{array}\right.$

or $\quad q^2=V^2.\dfrac{a+b}{a-b}\cdot\dfrac{a^2-b^2}{a^2}$

or $\quad q^2=V^2\left(\dfrac{a+b}{a}\right)^2$

or $\quad (q)_{\max}=V.\dfrac{a+b}{a}$ **Proved.**

§ 5·86. **Elliptic cylinder rotating in an infinite mass of liquid at rest at infinity.**

We know that the stream function for the most general type of motion of the cylinder is

$$\psi=Vx-Uy+\tfrac{1}{2}\omega\,(x^2+y^2)+B \qquad \{\text{Ref. § 5·2}$$

where B is a constant.

Here $U=0=V$

then $\quad \psi=\frac{1}{2}\omega\,(x^2+y^2)+B$

or $\quad \psi=\frac{1}{2}\omega\,(c^2\cosh^2\xi\cos^2\eta+c^2\sinh^2\xi\sin^2\eta)+B$ $\quad\left\{\begin{array}{l}\text{Since } x=c\cosh\xi\cos\eta\\ \qquad y=c\sinh\xi\sin\eta\\ \text{Ref. § 5·82}\end{array}\right.$

On the boundary of the elliptic cylinder, $\xi=\alpha$

$$\psi=\tfrac{1}{4}\omega\,c^2\,(\cos 2\eta+\cosh 2\alpha)+B \qquad ...(1)$$

$$\left\{\begin{array}{l}\text{as } \cosh 2\alpha=2\cosh^2\alpha-1\\ \qquad\qquad =2\sinh^2\alpha+1\end{array}\right.$$

Since the velocity vanishes at infinity and η is the only variable, then ψ must be of the form $e^{-2\xi}\cos 2\eta$.

Let $\psi = A\,e^{-2\xi}\cos 2\eta$ (for all values of η) ...(2)

(1) and (2) must be same on the boundary $\xi=\alpha$

$$Ae^{-2\alpha}\cos 2\eta = \tfrac{1}{4}\omega c^2(\cos 2\eta+\cosh 2\alpha)+B$$

then $\quad Ae^{-2\alpha}=\tfrac{1}{4}\omega c^2 \quad$ and $\quad \tfrac{1}{4}\omega c^2\cosh 2\alpha + B=0$

or $\quad A=\tfrac{1}{4}\omega c^2 e^{2\alpha}$ and $\quad B=-\tfrac{1}{4}\omega c^2\cosh 2\alpha$

Substituting the value of A in (2), we have

$$\phi=\tfrac{1}{4}\omega c^2\, e^{2\alpha}.e^{-2\xi}\cos 2\eta \qquad \{\text{as } ce^{\alpha}=a+b$$

or $\quad \psi=\tfrac{1}{4}\omega\,(a+b)^2\,e^{-2\xi}\cos 2\eta$

Similarly $\quad \phi=\tfrac{1}{4}\omega\,(a+b)^2\,e^{-2\xi}\sin 2\eta$

{ as ϕ and ψ are conjugate functions

Thus the complex potential w is given by

$$w=\phi+i\psi=\tfrac{1}{4}\omega\,(a+b)^2\,e^{-2\xi}(\sin 2\eta+i\cos 2\eta)$$

$$\mathbf{w}=\tfrac{1}{4}\,i\omega\,(\mathbf{a}+\mathbf{b})^2\,\mathbf{e}^{-2\zeta} \qquad \{\text{ as } \zeta=\xi+i\eta$$

§ 5·87. Kinetic energy of rotating elliptic cylinder.

To determine the kinetic energy when an elliptic cylinder rotates in an infinite mass of liquid at rest at infinity.

Let the cylinder rotates with an angular velocity ω on the boundary of the elliptic cylinder $\xi=\alpha$, we have the stream function and velocity potential as follows,

$$\psi=\tfrac{1}{4}\omega\,(a+b)^2\,e^{-2\alpha}\cos 2\eta \qquad \{\text{Ref. § 5·86}$$

and $\quad \phi=\tfrac{1}{4}\omega\,(a+b)^2\,e^{-2\alpha}\sin 2\eta$

where η varies from 0 to 2π.

K. E. $\quad T=-\tfrac{1}{2}\rho\int\phi\,d\psi$

or $\quad T=\tfrac{1}{2}\rho\int_0^{2\pi}\tfrac{1}{4}\omega\,(a+b)^2\,e^{-2\alpha}\sin 2\eta\;\{\tfrac{1}{4}\omega\,(a+b)^2\,e^{-2\alpha}.2\sin 2\eta\}\,d\eta$

or $\quad T=\dfrac{1}{16}\rho\omega^2(a+b)^4\,e^{-4\alpha}\displaystyle\int_0^{2\pi}\sin^2 2\eta\,d\eta$

or $\quad T=\dfrac{1}{16}\rho\omega^2(a+b)^4\,e^{-4\alpha}.\tfrac{1}{2}\displaystyle\int_0^{2\pi}(1-\cos 4\eta)\,d\eta$

or $\quad T=\dfrac{1}{16}\rho\omega^2(a+b)^4\,e^{-4\alpha}.\pi$

or $\quad T=\dfrac{1}{16}\pi\rho\omega^2c^4 \qquad \left\{\begin{array}{ll}\text{as} & c^2=a^2-b^2\\ \text{and} & ce^{\alpha}=a+b\end{array}\right.$

or $\quad T=\dfrac{1}{16}\pi\rho\omega^2(a^2-b^2)^2.$

Now we shall discuss the cases of two dimensional motion of liquid contained in a cylinder moving parallel to itself.

§ 5·81. Kinetic energy when the liquid contained in a rotating elliptic cylinder.

Let $\qquad w=-iAz^2 \qquad \ldots$(i)

where $\qquad w=\phi+i\psi$ and $z=x+iy=(r\cos\theta+ir\sin\theta)$

or $\qquad \phi+i\psi=-i\,A\,r^2\,(\cos 2\theta+i\sin 2\theta)$.

Equating real and imaginary parts, we get

$$\phi=Ar^2\sin 2\theta \quad\text{and}\quad \psi=-Ar^2\cos 2\theta \qquad \ldots\text{(ii)}$$

or

$$\phi=2A\,xy \quad\text{and}\quad \psi=-A\,(x^2-y^2). \qquad \ldots\text{(iii)}$$

We know that the general expression of the stream function ψ is,

$$\psi=(Vx-Uy)+\tfrac{1}{2}\omega\,(x^2+y^2)+C. \qquad \ldots\text{(iv)}$$

Since $V=0=U$, (iv) reduces to

$$\psi=\tfrac{1}{2}\omega\,(x^2+y^2)+C. \qquad \ldots\text{(v)}$$

Equation (iii) and (v), we have

$$-A\,(x^2-y^2)=\tfrac{1}{2}\omega\,(x^2+y^2)+C$$

or

$$(\tfrac{1}{2}\omega+A)\,x^2+(\tfrac{1}{2}\omega-A)\,y^2=-C$$

or

$$\frac{x^2}{\dfrac{-C}{\frac{1}{2}\omega+A}}+\frac{y^2}{\dfrac{-C}{\frac{1}{2}\omega-A}}=1.$$

Comparing this with the equation to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

we get

$$a^2=-\frac{C}{\frac{1}{2}\omega+A} \quad\text{and}\quad b^2=-\frac{C}{\frac{1}{2}\omega-A}$$

or

$$C=-\frac{\omega a^2b^2}{a^2+b^2} \quad\text{and}\quad A=\tfrac{1}{2}\omega\,\frac{b^2-a^2}{b^2+a^2}.$$

Substituting the value of C in (v), we have

$$\psi=\tfrac{1}{2}\omega\,(x^2+y^2)-\omega\frac{a^2b^2}{a^2+b^2}$$

and

$$\phi=\omega\,.\,\frac{b^2-a^2}{b^2+a^2}\,xy \qquad \text{[from (iii)}$$

which determines the motion of the liquid in the rotating elliptic cylinder.

Let q denotes the velocity

then

$$q^2=\left(-\frac{\partial\phi}{\partial x}\right)^2+\left(-\frac{\partial\phi}{\partial y}\right)^2$$

$$q^2=\left\{-\frac{\omega\,(b^2-a^2)}{b^2+a^2}\,y\right\}^2+\left\{-\frac{\omega\,(b^2-a^2)}{b^2+a^2}\,x\right\}^2$$

$$q^2=\omega^2\left(\frac{b^2-a^2}{b^2+a^2}\right)^2.(x^2+y^2)$$

Let T be the kinetic energy,

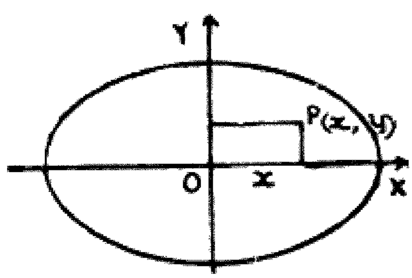
then $$T=\tfrac{1}{2}\rho\iint q^2.dx\,dy$$

$$T=\tfrac{1}{2}\rho\iint\omega^2\left(\frac{b^2-a^2}{b^2+a^2}\right)^2.(x^2+y^2)\,dx\,dy$$

$$T=\tfrac{1}{2}\rho\omega^2\left(\frac{a^2-b^2}{a^2+b^2}\right)^2\iint x^2\,dx\,dy+y^2\,dx\,dy$$

$$T=\tfrac{1}{2}\rho\omega^2.\left(\frac{a^2-b^2}{a^2+b^2}\right)^2\left\{\pi ab.\frac{a^2}{4}+\pi ab.\frac{b^2}{4}\right\}$$

$$T=\frac{1}{8}\,\pi\rho\,ab\omega^2\,\frac{(a^2-b^2)^2}{a^2+b^2}$$



$$\left\{\begin{aligned}\iint x^2\,dx\,dy&=\text{M. I. of an elliptic area about } OY\\&=\pi ab.\frac{a^2}{4}\\\iint y^2\,dx\,dy&=\text{M. I. of an elliptic area about } OX\\&=\pi ab.\frac{b^2}{4}\end{aligned}\right.$$

(a) To determine the K. E. when the liquid contained in a rotating prism whose section is an equilateral prism. Consider the complex potential w.

$$w=i\,A\,z^3.$$

(Proceed as in above case)

Ex. 4. *An infinite elliptic cylinder with semi-axes a, b is rotating round its axis with angular velocity ω, in an infinite liquid of density ρ which is at rest at infinity. Shew that if the fluid is* *under the action of no forces the moment of the fluid pressure on the cylinder round the centre is*

$$\frac{1}{8}\pi\rho c^4 \frac{d\omega}{dt} \qquad \textit{where} \quad c^2=a^2-b^2.$$

We know that for an elliptic cylinder rotating with an angular velocity ω,

$$\phi=\tfrac{1}{4}\omega\,(a+b)^2\,e^{-2\xi}\sin 2\eta \qquad \ldots\text{(i)}$$

and $$\psi=\tfrac{1}{4}\omega\,(a+b)^2\,e^{-2\xi}\cos 2\eta \qquad \ldots\text{(ii)}$$

so that
$$w=\phi+i\psi$$
$$=\tfrac{1}{4}i\omega\,(a+b)^2\,e^{-2\xi}.e^{-2\eta i}$$
$$=\tfrac{1}{4}i\omega\,(a+b)^2\,e^{-2\,(\xi+i\eta)} \qquad \{\text{since } \zeta=\xi+i\eta$$
$$w=\tfrac{1}{4}i\omega\,(a+b)^2\,e^{-2\zeta}$$

also $$z=c\cosh\zeta$$

Let q be the velocity at any point, then

$$q=\left|\frac{dw}{dz}\right|=\left|\frac{dw}{d\zeta}.\,\frac{d\zeta}{dz}\right|$$

$$q^2=\left|\frac{1}{4}\,i\omega\,(a+b)^2\,e^{-2\zeta}\,(-2).\frac{1}{c\sinh\zeta}\right|^2$$

or $$q^2=\frac{\omega^2\,(a+b)^4}{4c^2}\left|\frac{e^{-2\xi}.e^{-2i\eta}}{\sinh\,(\xi+i\eta)}\right|^2$$

$$q^2=\frac{\omega^2\,(a+b)^4.e^{-4\xi}}{4c^2\,(\sinh^2\xi+\sin^2\eta)}$$

Also from (i), we have

$$\frac{\partial\phi}{\partial t}=\frac{1}{4}\,(a+b)^2\,e^{-2\xi}\,\sin 2\eta\,\frac{d\omega}{dt}.$$

Let the pressure at any point, by Bernoulli's Theorem, we have

$$\frac{p}{\rho}=A-\tfrac{1}{2}q^2+\frac{\partial\phi}{\partial t}$$

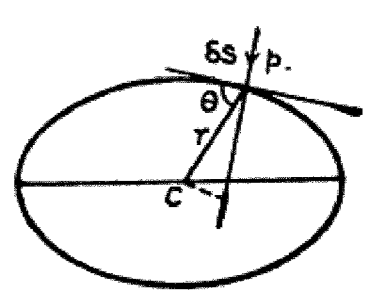
or $$\frac{p}{\rho}=A-\tfrac{1}{2}.\frac{\omega^2\,(a+b)^4}{4c^2}.\;\frac{e^{-4\xi}}{\sinh^2\xi+\sin^2\eta}$$
$$+\frac{1}{4}\,(a+b)^2\,e^{-2\xi}\,\sin 2\eta\,\frac{d\omega}{dt}$$

Consider an elementary arc δs on the disc then pressure

$$=p\;ds$$

So moment of the fluid pressure on the element ds about the centre C

$$=-p\;ds.r\cos\theta$$



$$\left\{\text{as} \quad \cos\theta=\frac{dr}{ds}\right.$$

$$= -p\,ds.r\frac{dr}{ds} \qquad \begin{cases}\text{Also} & r^2=a^2\cos^2\eta+b^2\sin^2\eta\\ \text{or} & r\,dr=-\tfrac{1}{2}(a^2-b^2)\sin 2\eta\,d\eta\end{cases}$$

$$=-pr\,dr$$

Hence moment of the fluid pressure on the elliptic cylinder (*i.e.* at $\xi=\alpha$) about the centre is

$$=-\int pr\,dr$$

$$=-\int_0^{2\pi} p.\{-\tfrac{1}{2}(a^2-b^2)\sin 2\eta\,d\eta\}$$

$$=\tfrac{1}{2}(a^2-b^2)\int_0^{2\pi}\rho\left\{A-\frac{\omega^2(a+b)^4}{8c^2}\cdot\frac{e^{-4\alpha}}{\sinh^2\alpha+\sin^2\eta}\right.$$

$$\left.+\frac{1}{4}(a+b)^2e^{-2\alpha}\sin 2\eta\frac{d\omega}{dt}\right]\sin 2\eta\,d\eta$$

$$=\tfrac{1}{2}(a^2-b^2)\rho\int_0^{2\pi}\frac{1}{4}(a+b)^2e^{-2\alpha}\sin 2\eta\frac{d\omega}{dt}\sin 2\eta.d\eta$$

$$=\frac{1}{8}(a^2-b^2)(a+b)^2\rho\,e^{-2\alpha}\int_0^{2\pi}\sin^2 2\eta\frac{d\omega}{dt}d\eta$$

$$=\frac{(a^2-b^2)(a+b)^2e^{-2\alpha}}{8}\rho\frac{d\omega}{dt}.\pi \qquad \begin{cases}\text{as} & ce^{\alpha}=a+b\\ \text{or} & c=(a+b)e^{-\alpha}\\ \text{and} & c^2=a^2-b^2\end{cases}$$

$$=\frac{1}{8}\pi\rho c^4\frac{d\omega}{dt}.$$

Ex. 5. *Prove that when an infinitely long cylinder of density σ whose cross section is an ellipse of semi-axes a, b is immersed in an infinite liquid of density ρ the square of its radius of gyration about it axis is effectively increased by the equation*

$$\frac{\rho}{8\sigma}\cdot\frac{(a^2-b^2)^2}{ab}.$$

The kinetic energy for an ellipic cylinder rotating about its axis with angular velocity ω

$$=\tfrac{1}{2}Mk^2\omega^2$$

$$=\tfrac{1}{2}M.\frac{a^2+b^2}{4}.\omega^2$$

$$=\tfrac{1}{2}\pi ab\sigma.\frac{a^2+b^2}{4}\omega^2 \qquad \ldots(\text{i})$$

$\Bigg\{$ k = M. I. of an elliptic plate about a line passing through the centre and perpendicular to its plane
M = Mass of the elliptic plate $=\pi ab.\sigma$.

Now the elliptic cylider is immersed in an infinite liquid of density ρ, we shall determine the K. E. of the liquid.

We know for an elliptic cylinder $\xi=\alpha$ rotating with an angular velocity ω, the velocity potential and stream function are, {Ref. § 5·26

$$\phi=\tfrac{1}{4}\omega\,(a+b)^2\, e^{-2\alpha}\sin 2\eta$$

$$\psi=\tfrac{1}{4}\omega\,(a+b)^2\, e^{-2\alpha}\cos 2\eta$$

K. E. $T=-\tfrac{1}{2}\rho\int \phi\,\dfrac{\partial\phi}{\partial\eta}\,dS=-\tfrac{1}{2}\rho\int\phi\,d\psi$ $\left\{\text{as } \dfrac{\partial\phi}{\partial\eta}=\dfrac{\partial\psi}{\partial S}\right.$

$$T=-\tfrac{1}{2}\rho\int_0^{2\pi}\frac{1}{4}\,\omega\,(a+b)^2\,e^{-2\alpha}\sin 2\eta\left\{-\tfrac{1}{4}\,\omega\,(a+b)^2\,e^{-2\alpha}\; 2\sin 2\eta\right\}d\eta.$$

$$T=\frac{1}{16}\rho\omega^2\,(a+b)^4\,e^{-4\alpha}\int_0^{2\pi}\sin^2 2\eta\,d\eta$$

or $T=\dfrac{1}{16}\,\rho\omega^2\,(a+b)^4\,e^{-4\alpha}\cdot\tfrac{1}{2}\displaystyle\int_0^{2\pi}(1-\cos 4\eta)\,d\eta$

or $T=\dfrac{1}{16}\,\pi\rho\omega^2\,(a+b)^4\,e^{-4\alpha}$ $\left\{\begin{array}{ll}\text{Since} & c\,e^{\alpha}=a+b\\ \text{or} & c=(a+b)\,e^{-\alpha}\\ \text{and} & c^2=a^2-b^2\end{array}\right.$

or $T=\dfrac{1}{16}\pi\rho\omega^2 c^4$

or $T=\dfrac{1}{16}\,\pi\rho\omega^2\,(a^2-b^2)^2$ ...(ii)

Total kinetic energy, adding (i) and (ii), we have

$$=\underbrace{\tfrac{1}{2}\,\pi\sigma ab\,\frac{a^2+b^2}{4}\omega^2}_{\text{K.E. of the cylinder}}+\underbrace{\frac{1}{16}\,\pi\rho\omega^2\,(a^2-b^2)^2}_{\text{K.E. of the liquid}} \qquad \text{...(iii)}$$

Let μ be the effective increase in the radius of gyration, then we have

$$=\tfrac{1}{2}\,\pi\sigma\,ab\,\frac{a^2+b^2}{4}\omega^2+\tfrac{1}{2}\,\pi\,\sigma ab\mu^2\,\omega^2$$

$$=\tfrac{1}{2}\,\pi\,ab\sigma\omega^2\left[\frac{a^2+b^2}{4}+\mu^2\right] \qquad \text{...(iv)}$$

Now the effective increase in the radius of gyration *i.e.* μ is determined by equation (iii) and (iv),

i.e. $\tfrac{1}{2}\,\pi\sigma ab.\,\dfrac{a^2+b^2}{4}\omega^2+\dfrac{1}{16}\,\pi\rho\omega^2\,(a^2-b^2)^2$

$$=\tfrac{1}{2}\,\pi\,ab\sigma\omega^2\left[\frac{a^2+b^2}{4}+\mu^2\right]$$

or $\tfrac{1}{2}\,\pi ab\sigma\omega^2.\mu^2=\dfrac{1}{16}\,\pi\,\rho\omega^2\,(a^2-b^2)^2$

or $\mu^2=\dfrac{\rho}{8\sigma}\cdot\dfrac{(a^2-b^2)}{ab}.$ **Proved.**

Ex. 6. *A thin shell in the form of an infinitely long elliptic cylinder, semi-axes a and b, is rotating about its axis in an infinite liquid otherwise at rest. It is filled with the same liquid. Prove that the ratio of the kinetic energy of the liquid inside to that of the liquid outside is*

$$2ab : a^2+b^2$$

Let the elliptic cylinder is rotating with angular velocity ω in an infinite liquid at rest at infinity then the complex potential is given by

$$=\tfrac{1}{4}\, i\omega\, (a+b)^2\, e^{-2\zeta} \qquad \{\text{where } \zeta=\xi+i\eta$$

Equating the real and imaginary parts, we have

$$\left.\begin{aligned}\phi&=\tfrac{1}{4}\,\omega\,(a+b)^2\, e^{-2\alpha}\sin 2\eta\\ \psi&=\tfrac{1}{4}\,\omega\,(a+b)^2\, e^{-2\alpha}\cos 2\eta\end{aligned}\right\} \quad \text{at } \xi=\alpha \text{ on the elliptic boundary}$$

Let T_1 be the K.E. of the liquid outside,

the $\quad T_1=-\tfrac{1}{2}\,\rho\int\phi\, d\psi$

or $\quad T_1=\dfrac{1}{16}\,\rho\,\omega^2\,(a+b)^4\, e^{-4\alpha}\displaystyle\int_0^{2\pi}\sin 2\eta\, d\eta$

{as in last example

or $\quad T_1=\dfrac{1}{16}\,\rho\omega^2\,(a^2-b^2).\pi \qquad \ldots(i)$

Now we shall determine the K.E. of the liquid contained in the rotating elliptic cylinder. The complex potential is given by

$$w=-iAz^2 \qquad \ldots(ii)$$

(where A is an arbitrary constant)

or $\quad \phi+i\,\psi=-iA\,(x+i\,y)^2$

or $\quad \phi+i\,\psi=-iA\,(x^2-y^2+2\,i\,xy)$

Equating imaginary part, we have

$$\psi=-A\,(x^2-y^2) \qquad \ldots(iii)$$

But for rotating cylinder, the stream function is given by

$$\psi=\tfrac{1}{2}\,\omega\,(x^2+y^2)-\text{constant} \qquad \ldots(iv)$$

Equating (iii) and (iv), we have

$$\tfrac{1}{2}\,\omega\,(x^2+y^2)-\text{constant}=-A\,(x^2-y^2)$$

or $\quad (\tfrac{1}{2}\omega+A)\,x^2+(\tfrac{1}{2}\,\omega-A)y^2=\text{constant} \qquad \ldots(v)$

Equation to the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \ldots(vi)$$

At the surface, comparing the coefficient of x^2 and y^2 from (v) and (vi) we have

$$a^2\,(\tfrac{1}{2}\,\omega+A)=b^2\,(\tfrac{1}{2}\,\omega-A)$$

or $$A\,(a^2+b^2)=\tfrac{1}{2}\,\omega\,(b^2-a^2)$$

or $$A=\tfrac{1}{2}\omega\cdot\frac{b^2-a^2}{a^2+b^2}$$

Substituting the value of the constant A in (ii), we get

$$w=\tfrac{1}{2}\,i\,\omega\,\frac{a^2-b^2}{a^2+b^2}\,z^2 \qquad \{\text{since } z=c\cosh\zeta$$

$$w=\tfrac{1}{2}\,i\,\omega\,\frac{a^2-b^2}{a^2+b^2}\cdot c^2\cosh^2(\xi+i\eta)$$

$$w=\tfrac{1}{2}\,i\,\omega\,\frac{a^2-b^2}{a^2+b^2}\cdot c^2\left\{\cosh\xi\cos\eta+i\sinh\xi\sin\eta\right\}^2$$

Equating the real and the imaginary part, we have

$$\phi+i\,\psi=\tfrac{1}{2}\,i\,\omega\,\frac{a^2-b^2}{a^2+b^2}c^2\left\{\cosh^2\xi\cos^2\eta-\sinh^2\xi\sin^2\eta+2i\cosh\xi\sinh\xi\cos\eta\sin\eta\right\}$$

or $$\phi=-\omega\,\frac{a^2-b^2}{a^2+b^2}\,c^2\cosh\xi\sinh\xi\cos\eta\sin\eta$$

and $$\psi=\tfrac{1}{2}\omega\,\frac{a^2-b^2}{a^2+b^2}\,c^2\left\{\cosh^2\xi\cos^2\eta-\sinh^2\xi\sin^2\eta\right\}$$

Let T_2 be the K. E. of the liquid inside the elliptic cylinder at $\xi=\alpha$

then $$T_2=-\tfrac{1}{2}\,\rho\int\phi\,d\psi$$

or $$T_2=\tfrac{1}{2}\,\rho\omega\cdot\frac{a^2-b^2}{a^2+b^2}\,c^2\cosh\alpha\sinh\alpha.$$
$$(\cosh^2\alpha+\sinh^2\alpha)\,\omega\,\frac{a^2-b^2}{a^2+b^2}c^2\int_0^{2\pi}\cos^2\eta\sin^2\eta\,d\eta$$

or $$T_2=\tfrac{1}{2}\,\rho\omega^2\left(\frac{a^2-b^2}{a^2+b^2}\right)^2 c^4\cosh\alpha\sinh\alpha\,(\cosh^2\alpha+\sinh^2\alpha)\int_0^{2\pi}\cos^2\eta\sin^2\eta\,d\eta$$

or $$T_2=\tfrac{1}{2}\,\rho\omega^2\left(\frac{a^2-b^2}{a^2+b^2}\right)^2 c^4\cosh\alpha\sinh\alpha\,(\cosh^2\alpha+\sinh^2\alpha)\cdot\frac{\pi}{4}$$

or $$T_2=\frac{1}{8}\,\pi\,\rho\omega^2\left(\frac{a^2-b^2}{a^2+b^2}\right)^2\cdot c\cosh\alpha\;c\sinh\alpha\,\{c^2\cosh^2\alpha+c^2\sinh^2\alpha\}$$

or $$T_2=\frac{1}{8}\pi\rho\omega^2\left(\frac{a^2-b^2}{a^2+b^2}\right)^2\cdot ab\,(a^2+b^2)$$

or $$T_2=\frac{1}{8}\pi\rho\omega^2\frac{(a^2-b^2)^2}{a^2+b^2}\cdot ab \qquad \text{...(vii)}$$

Dividing (i) and (vii), we have

$$\frac{T_2}{T_1}=\frac{16}{8}\cdot\frac{\pi\rho\omega^2(a^2-b^2)^2\cdot ab}{(a^2+b^2)\cdot\pi\rho\omega^2(a^2-b^2)^2}$$

or $$\frac{T_2}{T_1}=\frac{2ab}{a^2+b^2}$$

or $$T_2 : T_1 :: 2ab : a^2+b^2$$ **Proved.**

Ex. 7. *If an elliptic cylinder of semi-axes a, b filled with a liquid, rotates with a uniform angular velocity about its axis, shew that the kinetic energy of liquid contained is less than if it were moving as a solid in the ratio*

$$(a^2-b^2)^2 : (a^2+b^2)^2$$

Let T_1 be the K. E. when the liquid be moving as a solid with angular velocity ω.

Then $T_1=\frac{1}{2}Mk^2\omega^2$

$$T_1=\tfrac{1}{2}\cdot\pi ab\rho\cdot\frac{a^2+b^2}{4}\omega^2 \quad \text{...(i)} \quad \left\{\begin{array}{l}\text{as } M=\pi ab\rho\\ \text{and } k^2=\dfrac{a^2+b^2}{4}\end{array}\right.$$

Let T_2 be the K. E. of the liquid contained in the elliptic cylinder rotating with uniform angular velocity ω.

$$T_2=\tfrac{1}{8}\pi\rho\omega^2\frac{(a^2-b^2)^2}{a^2+b^2}\cdot ab \qquad \text{...(ii)}$$

{Ref. Ex. 5

Then $$\frac{T_2}{T_1}=\frac{\pi\rho\omega^2\cdot(a^2-b^2)^2\,ab}{(a^2+b^2)\pi\rho\omega^2\cdot(a^2+b^2)\,ab}$$

$$\frac{T_2}{T_1}=\frac{(a^2-b^2)^2}{(a^2+b^2)^2}$$

or $$T_2 : T_1 :: (a^2-b^2)^2 : (a^2+b^2)^2.$$ **Proved.**

Ex. 8. *The space between two confocal elliptic cylinders (a_0, b_0) and $(a_1\,b_1)$ and two planes perpendicular to their axes is filled with liquid. If both cylinders be made to rotate about their common axis with angular velocity ω, the kinetic energy of the motion set up is*

$$\frac{1}{8}\,\frac{M\omega^2c^4\,(b_1a_0-b_0a_1)}{(a_1a_0-b_1b_0)\,(a_1b_1-a_0b_0)}$$

M being the mass of the liquid, and 2c the distance between the focii

Let the elliptic cylinders $(a_0\, b_0)$ and $(a_1\, b_1)$ be given by $\xi=\alpha_0$ and $\xi=\alpha_1$ respectively, such that

$$\left.\begin{aligned} a_0&=c\cosh\alpha_0\\ b_0&=c\sinh\alpha_0\end{aligned}\right\} \quad \ldots\text{(i)}$$

and

$$\left.\begin{aligned} a_1&=c\cosh\alpha_1\\ b_1&=c\sinh\alpha_1\end{aligned}\right\} \quad \ldots\text{(ii)}$$

We know that for elliptic cylinder

$$z=c\cosh\zeta$$

$$z=c\cosh(\xi+i\eta)$$

or $$x+iy=c(\cosh\xi\cos\eta+i\sinh\xi\sin\eta)$$

Equating real and imaginary parts, we get

$$x=c\cosh\xi\cos\eta \text{ and } y=c\sinh\xi\sin\eta. \quad \ldots\text{(iii)}$$

Equation to general type of the motion of the cylinder

$$\psi=Vx-Uy+\tfrac{1}{2}\omega(x^2+y^2)+\text{Const.} \quad \ldots\text{(iv)}$$

Since there is a pure rotation and $U=V=0$. So (iv) reduces to,

$$\psi=\tfrac{1}{2}\omega(x^2+y^2)+\text{Const.}$$

or $$\psi=\tfrac{1}{2}\omega c^2(\cosh^2\xi\cos^2\eta+\sinh^2\xi\sin^2\eta)+\text{Const.}$$

or $$\psi=\tfrac{1}{4}\omega c^2(\cosh 2\xi+\cos 2\eta)+\text{Const.} \quad \ldots\text{(v)}$$

At the boundary $\xi=\alpha$, we must have

$$\psi=\tfrac{1}{4}\omega c^2(\cosh 2\alpha+\cos 2\eta)+\text{Const.} \qquad \forall \text{ values of } \eta.$$

or $$\psi_0=\tfrac{1}{4}\omega c^2(\cosh 2\alpha_0+\cos 2\eta)+\text{Const.} \quad \ldots\text{(vi)}$$

(at $\alpha=\alpha_0$, on inner elliptic boundary)

and $$\psi_1=\tfrac{1}{4}\omega c^2(\cosh 2\alpha_1+\cos 2\eta)+\text{Const.} \quad \ldots\text{(vii)}$$

(at $\alpha=\alpha_1$, on outer elliptic boundary)

Since ψ satisfy Laplace's equation then the suitable form of the stream function ψ is given by

$$\psi=(A\cosh 2\xi+B\sinh 2\xi)\cos 2\eta. \quad \ldots\text{(viii)}$$

at $\xi=\alpha=\alpha_0$ (vi) and (viii) should give the same value.

and $\xi=\alpha=\alpha_1$ (vii) and (viii) should also give the same value.

or $$\tfrac{1}{4}\omega c^2(\cosh 2\alpha_0+\cos 2\eta)+\text{Const.} =(A\cosh 2\alpha_0+B\sinh 2\alpha_0)\cos 2\eta$$

or $$\tfrac{1}{4}\omega c^2(\cosh 2\alpha_1+\cos 2\eta)+\text{Const.} =(A\cosh 2\alpha_1+B\sinh 2\alpha_1)\cos 2\eta$$

which gives $$A\cosh 2\alpha_0+B\sinh 2\alpha_0=\tfrac{1}{4}\omega c^2$$

and $$A\cosh 2\alpha_1+B\sinh 2\alpha_1=\tfrac{1}{4}\omega c^2$$

or $$A\{\cosh 2\alpha_0\sinh 2\alpha_1-\cosh 2\alpha_1\sinh 2\alpha_0\} =\tfrac{1}{4}\omega c^2(\sinh 2\alpha_1-\sinh 2\alpha_0)$$

or $\quad A \sinh (2\alpha_1-2\alpha_0)=\frac{1}{4}\omega c^2 . 2 \cosh (\alpha_1-\alpha_0) \sinh (\alpha_1-\alpha_0)$

or $$A=\tfrac{1}{4}\omega c^2 . \frac{2 \cosh (\alpha_1+\alpha_0) \sinh (\alpha_1-\alpha_0)}{2 \sinh (\alpha_1-\alpha_0) \cosh (\alpha_1-\alpha_0)}$$

or $$A=\tfrac{1}{4}\omega c^2 . \frac{\cosh (\alpha_1+\alpha_0)}{\cosh (\alpha_1-\alpha_0)}$$

and $$B=\tfrac{1}{4}\omega c^2 . \frac{\sinh (\alpha_1+\alpha_0)}{\cosh (\alpha_1-\alpha_0)}$$

Substituting the value of A and B in (viii), we get

$$\psi=\tfrac{1}{4}\omega c^2 \left\{\frac{\cosh (\alpha_1+\alpha_0)}{\cosh (\alpha_1-\alpha_0)} \cosh 2\xi + \frac{\sinh (\alpha_1+\alpha_0)}{\cosh (\alpha_1-\alpha_0)} \sinh 2\xi \right\} \cos 2\eta$$

or $$\psi=\tfrac{1}{4}\omega c^2 \left\{\frac{\cosh (\alpha_1+\alpha_0) \cosh 2\xi+\sinh (\alpha_1+\alpha_0) \sinh 2\xi}{\cosh (\alpha_1-\alpha_0)}\right\} \cos 2\eta$$

or $$\psi=\tfrac{1}{4}\omega c^2 \frac{\cosh (2\xi-\alpha_1-\alpha_0)}{\cosh (\alpha_1-\alpha_0)} \cos 2\eta$$

and $$\phi=-\tfrac{1}{4}\omega c^2 \frac{\sinh (2\xi-\alpha_1-\alpha_0)}{\cosh (\alpha_1-\alpha_0)} \sin 2\eta \qquad \left\{\text{as } \frac{\partial\psi}{\partial\eta}=\frac{\partial\phi}{\partial\xi}\right.$$

Let T be the K. E. of the liquid, then

$$T=-\tfrac{1}{2}\rho \int \phi d\psi$$

$$T=\underbrace{\left\{-\tfrac{1}{2}\rho \int_{\xi=\alpha_0} \phi \, d\psi\right\}}_{\substack{\text{K.E. on inner} \\ \text{cylindrical boundary}}}-\underbrace{\left\{-\tfrac{1}{2}\rho \int_{\xi=\alpha_1} \phi \, d\psi\right\}}_{\substack{\text{K.E. on outer} \\ \text{cylindrical bonndary}}}$$

$$T=\frac{1}{16} \rho \frac{\omega^2 c^4 \sinh (\alpha_1-\alpha_0)}{\cosh (\alpha_1-\alpha_0)} \int_0^{2\pi} \sin^2 2\eta \, d\eta + \frac{1}{16} \rho \frac{\omega^2 c^4 \sinh (\alpha_1-\alpha_0)}{\cosh (\alpha_1-\alpha_0)} \int_0^{2\pi} \sin^2 2\eta \, d\eta$$

$$T=\frac{1}{16} \rho\omega^2 c^4 \frac{\sinh (\alpha_1-\alpha_0)}{\cosh (\alpha_1-\alpha_0)} \left\{ \tfrac{1}{2} \int_0^{4\pi} \sin^2 t \, dt+\tfrac{1}{2}\int_0^{4\pi} \sin^2 t \, dt \right\}$$

$$T=\frac{1}{16} \rho\omega^2 c^4 \frac{\sinh (\alpha_1-\alpha_0)}{\cosh (\alpha_1-\alpha_0)} . 2\pi$$

$$=\frac{1}{8} \pi\rho\omega^2 c^4 \frac{\sinh \alpha_1 \cosh \alpha_0-\cosh \alpha_1 \sinh \alpha_0}{\cosh \alpha_1 \cosh \alpha_0-\sinh \alpha_1 \sinh \alpha_0}$$

$$=\frac{1}{8} \pi\rho\omega^2 c^4 . \frac{c \sinh \alpha_1 . c \cosh \alpha_0-c \cosh \alpha_1 . c \sinh \alpha_0}{c \cosh \alpha_1 . c \cosh \alpha_0-c \sinh \alpha_1 . c \sinh \alpha_0}$$

$$=\frac{1}{8}\pi\rho\omega^2 c^4 . \frac{b_1a_0-b_0a_1}{a_1a_0-b_1b_0}$$ { from (i) and (ii)

{ Let M be the mass of the liquid occupying the space between two confocal elliptic cylinders (a_0, b_0) and (a_1, b_1)
Then $M=\pi\rho\ a_1b_1-\pi\rho\ a_0b_0.$
$=\pi\rho\ (a_1b_1-a_0b_0)$

$$=\frac{1}{8}\omega^2c^4 . \frac{M}{a_1b_1-a_0b_0} . \frac{b_1a_0-b_0a_1}{a_1a_0-b_1b_0}$$

$$=\frac{1}{8}M\omega^2c^4 . \frac{b_1a_0-b_0a_1}{(a_1a_0-b_1b_0)\ (a_1b_1-a_0b_0)}$$ **Proved.**

Ex 9. *Liquid is contained in a rotating elliptic cylinder. By making use of the elliptic transformation $z=c\cosh\zeta$, Show directly that the stream function of the motion is*

$$\psi=\tfrac{1}{2}\omega\frac{a^2-b^2}{a^2+b^2}(x^2-y^2).$$

Hence prove that the paths of the particles are similar ellipses described in time $\frac{\pi(a^2+b^2)}{\omega ab}$.

Since the liquid is contained in a rotating elliptic cylinder, thus the complex potential is given by

$$w=-iAz^2 \quad \text{...(i)}$$

or $$w=-iA(x+iy)^2$$

or $$\phi+i\psi=-iA(x^2-y^2+2ixy)$$

then $$\psi=-A(x^2-y^2) \quad \text{...(ii)}$$

and $$\phi=2A\ xy. \quad \text{...(iii)}$$

But for rotating cylinders, we have

$$\psi=\tfrac{1}{2}\omega(x^2+y^2)-\text{Const.} \quad \text{...(iv)}$$

Equating (ii) and (iv), we have

$$\tfrac{1}{2}\omega(x^2+y^2)-\text{Const.}=-A(x^2-y^2)$$

or $$(\tfrac{1}{2}\omega+A)x^2+(\tfrac{1}{2}\omega-A)y^2=\text{Const.} \quad \text{...(v)}$$

Equation to the elliptic cylinder

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \text{...(vi)}$$

At the surface to the boundary, comparing (v) and (vi), we have

$$a^2(\tfrac{1}{2}\omega+A)=b^2(\tfrac{1}{2}\omega-A)$$

or $$A(a^2+b^2)=-\tfrac{1}{2}\omega(a^2-b^2)$$

or $$A=-\tfrac{1}{2}\omega\frac{a^2-b^2}{a^2+b^2}$$

Substituting the value of the constant A in (ii), we get

$$\psi=\tfrac{1}{2}\omega\cdot\frac{a^2-b^2}{a^2+b^2}\cdot(x^2-y^2)$$ **Proved.**

To determine the path of the particles, consider (x, y) be the coordinates of a point, then

Velocity of the fluid particle parallel to X-axis

$$=-\frac{\partial\phi}{\partial x}$$

i.e. $$-\frac{\partial\phi}{\partial x}=\dot{x}-\omega y \qquad \text{...(vii)} \qquad \{\text{Ref. § 5·2}$$

(Since the ellipse is rotating and the axis is also rotating with angular velocity ω).

and $$-\frac{\partial\phi}{\partial y}=\dot{y}+\omega x \qquad \text{...(viii)} \qquad \{\text{Ref. § 5·2}$$

So from (vii), we get

$$\dot{x}-\omega y=-\frac{\partial\phi}{\partial x}=\omega\,\frac{a^2-b^2}{a^2+b^2}\,y$$

$$\left\{\text{since}\quad \phi=-\omega\,\frac{a^2-b^2}{a^2+b^2}\,xy \qquad \text{from (13)}\right.$$

or $$\dot{x}=\omega y\left[\frac{a^2-b^2}{a^2+b^2}+1\right]$$

or $$\dot{x}=\frac{2a^2}{a^2+b^2}\,\omega y \qquad \text{...(ix)}$$

also $$\dot{y}+\omega x=-\frac{\partial\phi}{\partial y}=+\omega\,\frac{a^2-b^2}{a^2+b^2}\,x$$

or $$\dot{y}=-\omega x+\omega\frac{a^2-b^2}{a^2+b^2}\,x$$

or $$\dot{y}=-\,\frac{2b^2}{a^2+b^2}\,\omega x \qquad \text{...(x)}$$

Differentiating (ix) with regard to t, we have

$$\ddot{x}=\frac{2a^2\omega}{a^2+b^2}\,\dot{y}$$

or $$\frac{d^2x}{dt^2}=\frac{2a^2\omega}{a^2+b^2}\cdot\left\{-\frac{2b^2\omega}{a^2+b^2}\right\}x \qquad \{\text{from (xi)}$$

or $$\frac{d^2x}{dt^2}=-\frac{4a^2b^2\omega^2}{(a^2+b^2)^2}\cdot x \qquad \text{.. (xi)}$$

which is a standard equation of S. H. M. whose time period T is given by ;

or $$T=2\pi.\ \frac{a^2+b^2}{2ab\omega} \quad \left\{ \begin{array}{l} \text{as } \dfrac{d^2x}{dt^2}=-\mu x \\ T=2\pi.\ \dfrac{1}{\sqrt{\mu}} \end{array} \right.$$

or $$T=\frac{\pi\,(a^2+b^2)}{ab\omega}$$ **Proved.**

Solution of the differential equation (xi) is given by

$$x=A\cos\left\{\frac{2ab\omega}{a^2+b^2}\,t+C\right\} \qquad \text{...(xii)}$$

Similarly $$y=-A\,\frac{b}{a}\sin\left\{\frac{2ab\omega}{a^2+b^2}\,t+C\right\} \qquad \text{...(xiii)}$$

Eliminating t, we have

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{A^2}{a^2}$$

or $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \text{(let)}$$

Which is a similar ellipse. Hence path of the particles are similar ellipse.

Proved.

Ex. 10. *An elliptic cylinder of semi-axes a, b is filled with incompressible fluid and rotates about its axis with constant angular velocity ω. Prove that the velocity component (u, v) parallel to $0X$, $0Y$ (the axis of the ellipse) are given by*

$$u=\omega\,\frac{a^2-b^2}{a^2+b^2}\,y \quad ; \quad v=\omega\,\frac{a^2-b^2}{a^2+b^2}\,x$$

shew that the coordinates X, Y (relative to axes through O fixed in space) of a given particle at time t can be written as

$$X=\lambda\left[(a+b)\cos\left\{\frac{(a-b)^2\,\omega t}{a^2+b^2}\right\}+(a-b)\cos\left\{\frac{(a+b)^2\,\omega t}{a^2+b^2}\right\}\right]$$

$$Y=\lambda\left[(a+b)\sin\left\{\frac{(a-b)^2\,\omega t}{a^2+b^2}\right\}+(a-b)\sin\left\{\frac{(a+b)^2\,\omega t}{a^2+b^2}\right\}\right]$$

Where λ is a constant depending on the particle and $t=0$, when the particle crosses the axis OX.

We have proved in the last exercise, that

$$\phi=-\omega\,\frac{a^2-b^2}{a^2+b^2}\,xy \qquad \text{...(i)}$$

Since $$u=-\frac{\partial\phi}{\partial x} \quad \text{(parallel to } OX\text{)}$$

and $$v=-\frac{\partial\phi}{\partial y} \quad \text{(perpendicular to } OX\text{)}$$

Then from (i), we get

$$u=\omega\,\frac{a^2-b^2}{a^2+b^2}\,y \qquad \ldots\text{(ii)}$$

and $$v=\omega\,\frac{a^2-b^2}{a^2+b^2}\,x \qquad \ldots\text{(iii)}$$

Also from equation (xii) and (xiii) in the last exercise

$$x=A\cos\left\{\frac{2ab\omega}{a^2+b^2}\,t+C\right\} \qquad \ldots\text{(iv)}$$

and $$y=-\frac{b}{a}A\sin\left\{\frac{2ab\omega}{a^2+b^2}t+C\right\} \qquad \ldots\text{(v)}$$

{where C is a constant

Since λ is a constant depending on the particle $(x\ y)$ and $t=0$, when it crosses the axis of X, then

$$t=0,\ x=\lambda,\ y=0.$$

From (iv) and (v), we have

$$\lambda=A\cos C$$

and $$O=-\frac{b}{a}A\sin C \quad \text{gives } A=\lambda,\ C=0.$$

So (iv) and (v) becomes by $A=\lambda$, $C=0$.

$$\left.\begin{aligned} x&=\lambda\cos\frac{2ab\omega}{a^2+b^2}\,t \\ \text{and}\quad y&=-\frac{b}{a}\,\lambda\sin\frac{2ab\omega}{a^2+b^2}\,t \end{aligned}\right\} \qquad \ldots\text{(vi)}$$

These are the coordinates with regard to the rotating axes fixed in the body

Now we shall determine the coordinates (X, Y) with respect to fixed axis in space.

$$X+iY=e^{i\omega t}\,(x+iy)$$

or $$X+iY=(\cos\omega t+i\sin\omega t)\,(x+iy)$$

Equating real and imaginary parts, we have

$$X=x\cos\omega t-y\sin\omega t$$

and $$Y=x\sin\omega t+y\cos\omega t$$

Substituting the value of x and y from the (vi), we have

$$X=\lambda\cos\frac{2ab\omega t}{a^2+b^2}\cos\omega t+\frac{b}{a}\,\lambda\sin\frac{2ab\omega t}{a^2+b^2}\sin\omega t$$

or $$X=\frac{\lambda}{2a}\left\{2a\cos\omega t\cos\frac{2ab}{a^2+b^2}\,\omega t+2b\sin\omega t\sin\frac{2ab}{a^2+b^2}\,\omega t\right\}$$

or $X=\lambda'\left[a\left\{\cos\left(\frac{a^2+b^2+2ab}{a^2+b^2}\right)\omega t+\cos\left(\frac{a^2+b^2-2ab}{a^2+b^2}\right)\omega t\right\}\right.$

$\left.+b\left\{\cos\left(\frac{a^2+b^2-2ab}{a^2+b^2}\right)\omega t-\cos\left(\frac{a^2+b^2+2ab}{a^2+b^2}\right)\omega t\right\}\right]$

where $\lambda'=\dfrac{\lambda}{2a}$

or $X=\lambda'\left[(a+b)\cos\left\{\frac{(a-b)^2\,\omega t}{a^2+b^2}\right\}+(a-b)\cos\left\{\frac{(a+b)^2\,\omega t}{a^2+b^2}\right\}\right]$

Similarly

$Y=\lambda'\left[(a+b)\sin\left\{\frac{(a-b)^2\,\omega t}{a^2+b^2}\right\}+(a-b)\sin\left\{\frac{(a+b)^2\,\omega t}{a^2+b^2}\right\}\right]$

Proved.

Ex. 11. *If the liquid is contained between the elliptic cylinders* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ *and* $\frac{x^2}{a^2}+\frac{y^2}{b^2}=K^2$ *and the whole rotates about OZ with angular velocity* ω, *prove that the velocity potential* ϕ *referred to the axes OX, OY is given by*

$$\phi=-\omega\,\frac{a^2-b^2}{a^2+b^2}\,xy$$

and that the surfaces of equal pressure are the hyperbolic cylinders

$$\frac{x^2}{3a^2+b^2}-\frac{y^2}{a^2+3b^2}=C$$

For the first part see question No. 9.

Now when the axes are rotating the pressure is given by

$$\frac{dp}{\rho}-\frac{\partial\phi}{\partial t}+\tfrac{1}{2}q^2+\omega\left(x\frac{\partial\phi}{\partial y}-y\frac{\partial\phi}{\partial x}\right)+V=f(t) \qquad \ldots(i)$$

Since the fluid is incompressible and also there is steady motion *i.e.* $\frac{\partial\phi}{\partial t}=0$, $V=0$, then (i) reduces to

$$\frac{p}{\rho}+\tfrac{1}{2}q^2+\omega\left(x\frac{\partial\phi}{\partial y}-y\frac{\partial\phi}{\partial x}\right)=\text{const.}$$

or $$\frac{p}{\rho}+\tfrac{1}{2}\left[\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2\right]+\omega\left(x\frac{\partial\phi}{\partial y}-y\frac{\partial\phi}{\partial x}\right)=\text{const.}$$

$$\left\{\text{as }\ \phi=-\omega\,\frac{a^2-b^2}{a^2+b^2}\,xy\right.$$

or $$\frac{p}{\rho}+\tfrac{1}{2}\omega^2\left(\frac{a^2-b^2}{a^2+b^2}\right)^2(x^2+y^2)+\omega^2\left(\frac{a^2-b^2}{a^2+b^2}\right).(-x^2+y^2)=\text{const.}$$

or $$\frac{p}{\rho}+\tfrac{1}{2}\omega^2\left\{\frac{a^2-b^2}{a^2-b^2}(x^2+y^2)+2(y^2-x^2)\right\}\left(\frac{a^2-b^2}{a^2+b^2}\right)=\text{const.}$$

Since surfaces of equal pressure are obtained by substituting p=const.

i.e. $$\frac{a^2-b^2}{a^2+b^2}(x^2+y^2)+2(y^2-x^2)=\text{constant}$$

or $$x^2\left(\frac{a^2-b^2}{a^2+b^2}-2\right)+y^2\left(\frac{a^2-b^2}{a^2+b^2}+2\right)=\text{const.}$$

or $$x^2.\frac{a^2+3b^2}{a^2+b^2}-y^2.\frac{3a^2+b^2}{a^2+b^2}=\text{const.}$$

or $$\frac{x^2}{3a^2+b^2}-\frac{y^2}{a^2+3b^2}=\text{const.}$$

which are the hyperbolic cylinders. **Proved.**

Ex. 12. *Prove that if 2a, 2b are the axes of the cross-section of an elliptic cylinder placed across a stream in which the velocity at infinity is U parallel to the major axis of the cross-section, the velocity at a point (a cos η, b sin η) on the surface is*

$$U(a+b)\sin\eta\,(b^2\cos\eta+a^2\sin\eta)^{-1/2}$$

and that, in consequence of the motion of the liquid, the resultant thrust (per unit length) on that half cylinder on which the stream impinges is diminished by

$$\frac{2b^2\rho U^2}{a-b}\left\{1-\left(\frac{a+b}{a-b}\right)^{1/2}\tan^{-1}\left(\frac{a-b}{a+b}\right)^{1/2}\right\},$$ *where ρ is the density of the liquid.*

As in Ex. 3. the velocity at any point is given by

$$q^2=U^2\frac{a+b}{a-b}\cdot\frac{\sinh^2(\xi-\alpha)+\sin^2\eta}{\sinh^2\xi+\sin^2\eta}$$

on the boundary of the elliptic cylinder at $\xi=\alpha$ it reduces to

$$q^2=U^2\frac{a+b}{a-b}\cdot\frac{\sin^2\eta}{\sinh^2\alpha+\sin^2\eta}$$

or $$q^2=U^2\frac{(a+b)^2}{c^2}\cdot\frac{\sin^2\eta}{\sinh^2\alpha+\sin^2\eta}$$

$$\left[\text{as } \frac{a+b}{b-b}=\frac{(a+b)^2}{a^2-b^2}=\frac{(a+b)^2}{c^2} \text{ and } c^2=a^2-b^2\right.$$

or $$q^2=\frac{U^2(a+b)^2\sin^2\eta}{c^2\sinh^2\alpha+c^2\sin^2\eta}$$

or $$q^2=U^2.\frac{(a+b)^2\sin^2\eta}{b^2+(a^2-b^2)\sin^2\eta}$$

or $$q^2=U^2.\frac{(a+b)^2\sin^2\eta}{b^2\cos^2\eta+a^2\sin^2\eta} \quad \ldots(\text{i})$$

Hence the velocity at a point $(a \cos \eta, b \sin \eta)$ on the surface is

$$q^2 = U^2 (a+b)^2 \sin^2 \eta \, (b^2 \cos^2 \eta + a^2 \sin^2 \eta)^{-1}$$

or $$q = U (a+b) \sin \eta \, (b^2 \cos^2 \eta + a^2 \sin^2 \eta)^{-1/2}$$ **Proved.**

The pressure equation for steady motion is

$$\frac{p}{\rho} + \frac{1}{2} q^2 = \text{const.} \qquad \ldots\text{(ii)}$$

Again velocity at infinity is given U, parallel to major axis

i.e. $$p_\infty = \Pi, \; q = U \text{ (at infinity)}$$

from (ii), we have

$$\frac{\Pi}{\rho} + \tfrac{1}{2} U^2 = \text{const}$$

or $$\frac{p}{\rho} + \tfrac{1}{2} q^2 = \frac{\Pi}{\rho} + \tfrac{1}{2} U^2$$

or $$\frac{\Pi - p}{\rho} = \tfrac{1}{2} (q^2 - U^2) \qquad \ldots\text{(ii)}$$

So diminution in the pressure at a point on the surface of the the elliptic cylinder

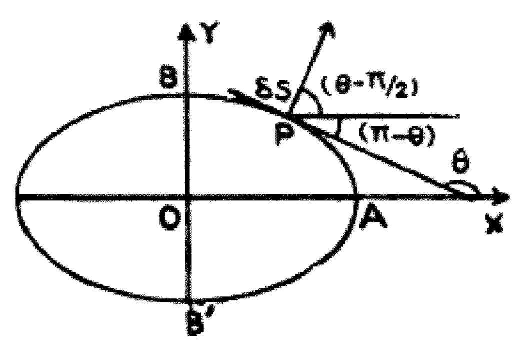
$$= (\Pi - p)$$

$$= \tfrac{1}{2}\rho \, (q^2 - U^2) \qquad \{\text{from (iii)}$$

$$= \tfrac{1}{2}\rho \left\{ \frac{U^2 (a+b)^2 \sin^2 \eta}{b^2 \cos^2 \eta + a^2 \sin^2 \eta} - U^2 \right\}$$

$$= \tfrac{1}{2}\rho U^2 \left\{ \frac{(a+b)^2 \sin^2 \eta - (b^2 \cos^2 \eta + a^2 \sin^2 \eta)}{b^2 \cos^2 \eta + a^2 \sin^2 \eta} \right\}$$

Consider an elementary element δs at any point P on the surface of the elliptic boundary. Let the tangent to point P makes an angle θ with major axis. Total diminution in the resultant thrust on the half cylinder (BAB') on which the stream impinges.



$$= \int (\Pi - p) \cos \left(\theta - \frac{\pi}{2} \right) ds$$

$$= \int (\Pi - p) \sin \theta \, ds \qquad \left\{ \begin{array}{l} \text{as } \sin \theta = \dfrac{dy}{ds} \\ \text{and } y = b \sin \eta \end{array} \right.$$

$$= \int (\Pi - p) \frac{dy}{ds} . ds$$

$$= \int (\Pi - p) \, dy$$

$$=\tfrac{1}{2}\rho U^2 \int_{-\pi/2}^{+\pi/2} \left\{\frac{(a+b)^2 \sin^2 \eta}{b^2 \cos^2 \eta+a^2 \sin^2 \eta}-1\right\} b \cos \eta \, d\eta$$

$$=\tfrac{1}{2}\rho U^2 b \int_{-\pi/2}^{+\pi/2} \left\{\frac{(a+b)^2 \sin^2 \eta}{b^2 (1-\sin^2 \eta)+a^2 \sin^2 \eta}-1\right\} \cos \eta \, d\eta$$

$$=\tfrac{1}{2}\rho U^2 b \int_{-1}^{+1} \left\{\frac{(a+b)^2 \lambda^2}{b^2+(a^2-b^2)\lambda^2}-1\right\} d\lambda \qquad \left\{\begin{array}{l}\text{Let } \sin \eta=\lambda \\ \cos \eta \, d\eta=d\lambda\end{array}\right.$$

$$=\rho U^2 b \int_0^1 \left\{\frac{2b}{a-b}-\frac{b^2 (a+b)}{a-b} \cdot \frac{1}{b^2+(a^2-b^2) \lambda^2}\right\} d\lambda$$

$$=\frac{2\rho U^2 b^2}{a-b}\left\{1-\frac{a+b}{2\sqrt{(a^2-b^2)}} \tan^{-1} \left(\frac{\sqrt{(a^2-b^2)}\, \lambda}{b}\right)\right\}_0^1$$

$$=\frac{2\rho U^2 b^2}{a-b}\left\{1-\tfrac{1}{2}\cdot \sqrt{\left(\frac{a+b}{a-b}\right)} \tan^{-1} \left(\sqrt{\frac{(a^2-b^2)}{b}}\right)\right\}$$

$$=\frac{2\rho U^2 b^2}{a-b}\left\{1-\sqrt{\left(\frac{a+b}{a-b}\right)} \tan^{-1} \sqrt{\left(\frac{a-b}{a+b}\right)}\right\}$$

$$\left\{\text{as } \tan^{-1}\left(\frac{\sqrt{(a^2-b^2)}}{b}\right)=2 \tan^{-1} \sqrt{\left(\frac{a-b}{a+b}\right)}\right\}$$

$$=\frac{2\rho U^2 b^2}{a-b}\left\{1-\left(\frac{a+b}{a-b}\right)^{1/2} \tan^{-1} \left(\frac{a-b}{a+b}\right)^{1/2}\right\}$$ **Proved**

§ 5·88. **Circulation about an elliptic cylinder.**

The elliptic coordinates are given by

$$z=c \cosh \zeta$$

$$x+iy=c \cosh (\xi+i\eta) \qquad \{\text{Ref. § 5·82}$$

or $$x=c \cosh \xi \cos \eta \text{ and } y=c \sinh \xi \sin \eta$$

We have

$$\frac{x^2}{c^2 \cosh^2 \xi}+\frac{y^2}{c^2 \sinh^2 \xi}=1 \quad \text{for all values of } \eta.$$

Now the stream function ψ should be a function of ξ, when the stream lines are confocal with a given ellipse.

Let $$\psi=f(\xi)$$

But ψ must satisfy the Laplace equation $\nabla^2\psi=0$.

or $$\frac{\partial^2\psi}{\partial\xi^2}=0 \qquad \left\{\begin{array}{l}\text{Since } \psi \text{ is independent of} \\ \eta \text{ then } \dfrac{\partial^2\psi}{\partial\eta^2}=0\end{array}\right.$$

By integrating, we have

$$\psi=A\xi$$

then $$\phi=-A\eta$$

or $$w=\phi+i\psi=iA\{\xi+i\eta\} \qquad ..\text{(i)}$$

Let k be the circulation, then

$$k=\int -\frac{\partial\phi}{\partial s}\, ds$$

or $$k=-\int_0^{2\pi} \frac{\partial \phi}{\partial \eta}\, d\eta$$

or $$k=A\int_0^{2\pi} d\eta=2\pi A \quad \text{or} \quad A=\frac{k}{2\pi}$$

Hence the complex potential is given by

$$w=\frac{ik}{2\pi}\,(\xi+i\eta)$$

We have already determined the hydrodynamic forces on a fixed cylinder due to the steady irrotational motion of a surrounding fluid. Now we shall discuss the general method, when the complex potential $w=\phi+i\psi$ for the fluid motion is known, given by Blasius.

§ 5·89. **Blasius Theorem.**

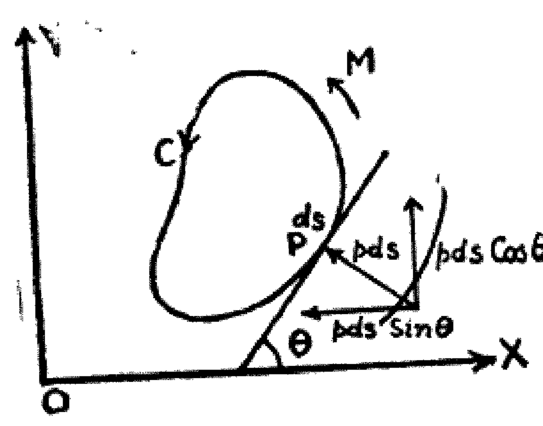
Consider a fixed cylinder by placed in a liquid which is moving steadily and irrotationally given by the relation $w=f(z)$, if the hydrodynamical pressure on the contour of a fixed cylinder are represented by a force (X, Y) and a couple M about the origin of coordinates, then

$$X-iY=\tfrac{1}{2}i\rho\int_c \left(\frac{dw}{dz}\right)^2 dz \quad \textit{(Neglecting extraneous forces)}$$

and $$M=\text{Real part of } -\tfrac{1}{2}\rho\int_c \left(\frac{dw}{dz}\right)^2 z\, dz$$

where the integrations are round any contour which surrounds the cylinder.

Proof. Consider an element of the arc ds at $P(x\,y)$ of the fixed cylinder. The tangent at the point P makes an angle θ with X-axis. The fluid thurst at $P(x\,y)$ (whose magnitude is $p\,ds$) will act along the inward normal to the cylinder, its components parallel to the coordinate axes are



$$-p \sin\theta\, ds,\ +p\cos\theta\, ds$$

The force acting on the element ds is

$$\begin{aligned} dF&=dX+i\,dY \\ &=-p\sin\theta\, ds+i\,p\cos\theta\, ds \\ &=ip\,(\cos\theta+i\sin\theta)\, ds=ip\,e^{i\theta}\, ds=ip\, dz \end{aligned}$$

By Bernoulli's pressure equation, we have

$$\frac{p}{\rho}=C-\tfrac{1}{2}q^2 \qquad \ldots\text{(i)}$$

(where q is the fluid velocity on the stream line)

$$\begin{cases} \text{Since} & x+iy=z \\ \text{or} & dx+i\,dy=dz \\ \text{or} & (\cos\theta+i\sin\theta)\,ds=dz \end{cases}$$

also $$\frac{dw}{dz}=-u+iv$$

$$=-q\,(\cos\theta-i\sin\theta)=-qe^{-i\theta} \qquad \ldots\text{(ii)}$$

Now, integrating over the contour, we have

$$F=X+iY=\int_c ip\,dz$$

$$=i\int_c (C-\tfrac{1}{2}\rho q^2)\,dz$$

$$=-\tfrac{1}{2}i\rho\int_c q^2\,dz \qquad \text{Since } C\int_c dz=0.$$

$$=-\tfrac{1}{2}i\rho\int_c q^2 e^{i\theta}\,ds \qquad \{\text{as } dz=e^{\theta i}\,ds$$

$$=-\tfrac{1}{2}i\rho\int_c (q^2 e^{2\theta i})\,(e^{-\theta i}\,ds)$$

or $$\bar{F}=X-iY=\tfrac{1}{2}i\rho\int_c (q^2 e^{-2\theta i})\,(e^{\theta i}\,ds)$$

$$X-iY=\tfrac{1}{2}i\rho\int_c\left(\frac{dw}{dz}\right)^2.dz \quad \{\text{from (ii)}$$

The moment about the origin of the fluid thrust on the element ds, is given by

$$dM=(p\,ds\sin\theta)\,y+(p\,ds\cos\theta)\,x$$

$$dM=p\left\{\frac{dy}{ds}.ds\;y+\frac{dx}{ds}.ds\;x\right\}$$

$$=p\,(y\,dy+x\,dx)$$

(clockwise moment taken to be positive)

The total moment is (by Bernoulli's theorem), given by

$$M=\int_c p\,(y\,dy+x\,dx)$$

or $$M=\int_c (C-\tfrac{1}{2}\rho q^2)\,(y\,dy+x\,dx) \quad \{\text{from (i)}$$

or $$M=C\int_c (y\,dy+x\,dx-\tfrac{1}{2}\rho\int_c q^2\,(y\,dy+x\,dx)$$

or $$M=-\tfrac{1}{2}\rho\int_c q^2\,(x\cos\theta+y\sin\theta)\,ds$$

{First integral vanish as $(y\,dy+x\,dx)$ is an exact differntial.

or $$M = \text{Real part of} -\tfrac{1}{2}\rho \int_c q^2 (x+iy)(\cos\theta - i\sin\theta)\, ds$$

or $$M = \text{Real part of } -\tfrac{1}{2}\rho \int_c q^2 z\, e^{-\theta i}\, ds$$

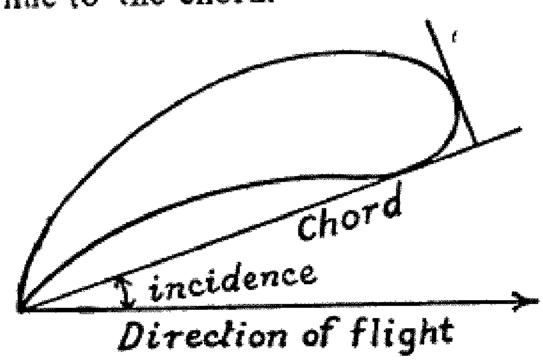
or $$M = \text{Real part of } -\tfrac{1}{2}\rho \int_c z.(q^2 e^{-2\theta i})(e^{\theta i}\, ds)$$

or $$M = \text{Real part of } -\tfrac{1}{2}\rho \int_c z \left(\frac{dw}{dz}\right)^2 dz$$

§ 5·89. **The aerofoil.**

The aerofoil has a **profile of fish type**. It is used in modern aeroplanes. Such an aerofoil has a **blunt leading edge** and a **sharp trailing edge**. The projection of the profile on the double tangent is the chord. The ratio of the span to the chord is the **aspect ratio.**

The locus of the point midway between the points in which an ordinate perpendicular to the chord meets the profile is known as the **camber line** of a profile. The **camber** is the ratio of the maximum ordinate of the camber line to the chord.



The theory of the flow round such an aerofoil is made on the following assumptions.

(i) The air behaves as an incompressible inviscid fluid.

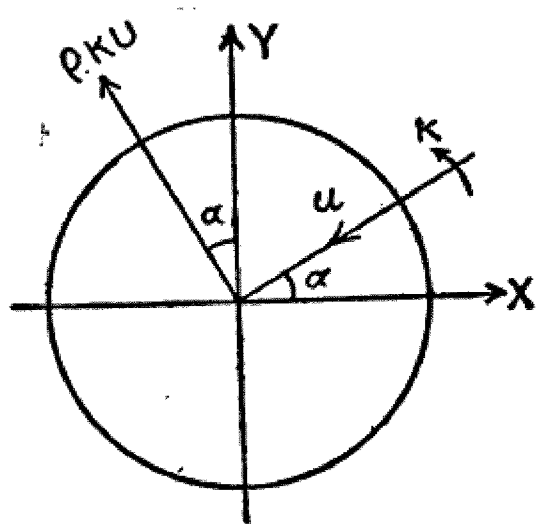
(ii) The aerofoil is a cylinder whose cross section is a curve of the above form.

(iii) The flow is two-dimensional irrotational cyclic motion.

The assumptions are simply approximation to the actual state of affairs. The profiles obtained by conformal transformation of a circle by the simple Joukowski transformation make good wing shapes, and the **lift** can be determined from the known flow with respect to a circular cylinder.

§ 5·9. **Theorem of Kutta and Joukowski.**

When a cylinder (an aerofoil) of any shape is placed in uniform stream (wind) of speed U, the resultant thrust on the cylinder is a lift of magnitude $k\rho U$ per unit length and is perpendicular to the stream, where k is the circulation around the cylinder.



Consider a fixed cylinder in a uniform stream U at a great distance (*i. e.* at infinity) making an angle α with the X-axis and directed towards the cylinder. Let k be the circulation round the cylinder.

The complex potential, due to uniform stream U,

$$w = Ue^{-i\alpha}.z \qquad \text{...(i)}$$

or $$u - iv = -\frac{dw}{dz} = -Ue^{-i\alpha}$$

Seperating into real and imaginary parts, we have

$$u = -U \cos \alpha \text{ and } v = -U \sin \alpha$$

The complex potential due to circulation

$$w = \frac{ik}{2\pi} \log z \qquad \text{...(iii)}$$

Since the presence of the cylinder produces some disturbance and this disturbance can be represented at an infinity from the cylinder by the following terms ;

$$w = \frac{A}{z} + \frac{B}{z^2} + \ldots\ldots$$

where A and B etc. are constants and depend on the uniform stream U and circulation k.

Thus the complex potential at a great distance from the origin

$$w = Ue^{-i\alpha} z + \frac{ik}{2\pi} \log z + \frac{A}{z} + \frac{B}{z^2} + \ldots$$

The force on the cylinder is given by

$$X - iY = \tfrac{1}{2} i\rho \int \left(\frac{dw}{dz}\right)^2 dz \qquad \text{\{By Blasius theorem}$$

$$X - iY = \tfrac{1}{2} i\rho \int \left\{ Ue^{-i\alpha} + \frac{ik}{2\pi z} - \frac{A}{z^2} - \ldots \right\}^2 dz$$

The function has the pole at $z=0$ inside the boundary, then the sum of the residues

$$= \tfrac{1}{2} i\rho \frac{iUk\, e^{-i\alpha}}{\pi}.\, 2\pi i$$

$$= -\rho k U e^{-i\alpha}.i$$

$$= -\rho k U i (\cos \alpha - i \sin \alpha)$$

Thus $\quad X = -\rho kU \sin\alpha$ and $Y = \rho kU \cos\alpha$

$\Rightarrow$ that X, Y are components of a force of magnitude.
$\sqrt{\{(-\rho kU \sin\alpha)^2 + (\rho kU \cos\alpha)^2\}} = \rho kU$ at right angles to the stream (wind), which is the maximum lift on the cylinder of magnitude ρkU.

Thus when a cylinder is inserted in a uniform stream U superposed by a circulation k, the cylinder experiences a lift *i. e.* a force perpendicular to the stream, of magnitude ρkU where U is the velocity of the stream and k the circulation. The lift depends only on the velocity of the stream and on the circulation k (which is independent of the shape of the cylinder). This is known as the **Theorem of Kutta and Joukowski.**

Cor. : The couple

$$N = \text{Real part of}\left\{-\tfrac{1}{2}\rho\int_c\left(\frac{dw}{dz}\right)^2 . z\,dz\right\}$$

or
$$\int\left(\frac{dw}{dz}\right)^2 z\,dz = \int_c\left\{-\frac{k^2}{4\pi^2 z} - \frac{2AUe^{-i\alpha}}{z} - \ldots\right\}dz$$

$$= -2\pi i\left(\frac{k^2}{4\pi^2} + 2AUe^{-i\alpha}\right) \qquad \left\{\begin{array}{l}\text{Sum of residues}\\ \text{at } z=0\end{array}\right.$$

Thus $\quad N = $ Real part of

$$= \pi\rho i\left\{\frac{k^2}{4\pi^2} + 2AU(\cos\alpha - i\sin\alpha)\right\}$$

$$= \pi\rho . 2AU\sin\alpha = 2\pi\rho AU\sin\alpha,$$

Since the result contains A (const) $\Rightarrow$ that the couple depends on the form of the cylinder.

Ex. 1. *A circular cylinder is placed in a uniform stream, find the forces acting on the cylinder.*

The complex potential for undisturbed motion is

$$w = (u - iv)\,z$$

By circle's theorem, we have

$$w = (u - iv)\,z + (u + iv)\frac{a^2}{z}$$

or
$$\frac{dw}{dz} = (u - iv) - (u + iv)\,\frac{a^2}{z^2}$$

By Blasius theorem, we get

$$X - iY = \tfrac{1}{2}i\rho\int_c\left(\frac{dw}{dz}\right)^2 dz$$

or
$$X - iY = \tfrac{1}{2}i\rho\int\left\{(u - iv) - (u + iv)\,\frac{a^2}{z^2}\right\}^2 . dz$$

or
$$X - iY = 0 \quad\Rightarrow X = 0 = Y$$

Now N=Real part of $-\frac{1}{2}\rho \int_{c} z \left(\frac{dw}{dz}\right)^2 dz$

$$N=\text{Real part of } -\tfrac{1}{2}\rho \int\left\{(u-iv)^2+2(u^2+v^2)\frac{a^2}{z^2}+\ldots\right\} z\, dz$$

$$N=\text{Real part of } -\tfrac{1}{2}\rho \{-2\,(u^2+v^2)\,a^2\}.2\pi i$$

$$N=\text{Real part of } \{2\pi\rho\, a^2 i\,(u^2+v^2)\}$$

$$N=\text{zero}$$

It follows that no force or couple acts on the cylinder.

Ex. 2. *The circle $(x+a)^2+y^2=a^2$ is placed in an oncoming wind of velocity U and there is a circulation $2\pi k$. Find the complex potential and show that the moment about the origin is $2\pi k\rho aU$.*

The complex potential for the uniform stream

$$w=Uz+\frac{Ua^2}{z-a}+ik\log(z-a) \qquad \text{\{By Circle's theorem}$$

or $$\frac{dw}{dz}=U-\frac{Ua^2}{(z-a)^2}+\frac{ik}{(z-a)}$$

or $$\frac{dw}{dz}=U-\frac{Ua^2}{z^2}\left(1-\frac{a}{z}\right)^{-2}+\frac{ik}{z}\left(1-\frac{a}{z}\right)^{-1}$$

$$=U-\frac{Ua^2}{z^2}\left(1+\frac{2a}{z}\right)+\frac{ik}{z}\left(1+\frac{a}{z}\right)$$

To determine the moment about the origin, equating the coefficients of $\frac{1}{z^2}$ in $\left(\frac{dw}{dz}\right)^2$

$$=(-k^2-2U^2a^2+2Uika)$$

or $$\int z\left(\frac{dw}{dz}\right)^2 dz=(-k^2-2U^2a^2+2Uika)\,2\pi i$$

Thus $$N=\text{Real part of } -\tfrac{1}{2}\rho \int_C z\left(\frac{dw}{dz}\right)^2 dz$$

$$=\text{Real part of } -\tfrac{1}{2}\rho\,[-k^2-2U^2a^2+2Ukai]\,2\pi i$$

$$=\tfrac{1}{2}\rho.2Uka.2\pi$$

$$=2\pi k\rho aU.$$

Ex. 13. *An elliptic cylinder, semi axis a and b, is held with its length perpendicular to, and its major axis making an angle θ with, the direction of a stream of velocity V. Prove that the magnitude of the couple per unit length on the cylinder due to the fluid pressure is*

$$\pi\rho\,(a^2-b^2)\,V^2\sin\theta\cos\theta.$$

and determine its sense.

The complex potential is given by

$$w=V(a+b)\cosh(\zeta-\zeta_0) \qquad \left\{\begin{array}{l}\text{where } \zeta=\xi+i\eta \\ \text{and } \quad \zeta_0=\alpha+i\theta \\ \text{also } \quad z=c\cosh\zeta\end{array}\right.$$

Now $\quad \dfrac{dw}{dz}=\dfrac{dw}{d\zeta}\cdot\dfrac{d\zeta}{dz}$

or $\quad \dfrac{dw}{dz}=V(a+b)\sinh(\zeta-\zeta_0)\cdot\dfrac{1}{c\sinh\zeta}$

or $\quad \dfrac{dw}{dz}=\dfrac{V(a+b)}{c}\cdot\dfrac{\sinh(\zeta-\zeta_0)}{\sinh\zeta}$

or $\quad \dfrac{dw}{dz}=\dfrac{V(a+b)}{c}\cdot\dfrac{\sinh\zeta\cosh\zeta_0-\cosh\zeta\sinh\zeta_0}{\sinh\zeta}$

or $\quad \dfrac{dw}{dz}=\dfrac{V(a+b)}{c}\cdot\left\{\cosh\zeta_0-\dfrac{z}{\sqrt{(z^2-c^2)}}\sinh\zeta_0\right\}$

$$\left[\begin{array}{l}\text{or } \dfrac{z}{\sqrt{(z^2-c^2)}}=\left(1-\dfrac{c^2}{z^2}\right)^{-1/2} \\ \qquad\qquad\qquad =1+\frac{1}{2}\cdot\dfrac{c^2}{z^2}+\dots \\ \text{when } z \text{ is large.}\end{array}\right.$$

$$\frac{dw}{dz}=\frac{V(a+b)}{c}\cdot\left\{\cosh\zeta_0-\left(1+\frac{c^2}{2z^2}\right)\sinh\zeta_0\right\}$$

$$\frac{dw}{dz}=\frac{V(a+b)}{c}\left\{(\cosh\zeta_0-\sinh\zeta_0)-\frac{c^2}{2z^2}\sinh\zeta_0\right\}$$

$$\frac{dw}{dz}=\frac{V(a+b)}{c}\left\{e^{-\zeta_0}-\frac{c^2}{2z^2}\sinh\zeta_0\right\}$$

or $\quad \left(\dfrac{dw}{dz}\right)^2=\dfrac{V^2(a+b)^2}{c^2}\left\{e^{-\zeta_0}-\dfrac{c^2}{2z^2}\sinh\zeta_0\right\}^2$

By Blasius theorem, if N is the couple about the origin then

$$N=\text{Real part of } -\tfrac{1}{2}\rho\int_C z\left(\frac{dw}{dz}\right)^2 dz$$

$$=\text{Real part of } -\tfrac{1}{2}\rho\left\{2\pi i\times\text{Sum of residues of } z\left(\frac{dw}{dz}\right)^2 \text{ in side the contour}\right\} \quad \ldots\text{(i)}$$

or $\quad z\left(\dfrac{dw}{dz}\right)^2=\dfrac{V^2(a+b)^2}{c^2}\,z\left(e^{-\zeta_0}-\dfrac{c^2}{2z^2}\sinh\zeta_0\right)^2 \quad \ldots\text{(ii)}$

The pole inside the contour is at origin *i.e.* residue at the origin $z=0$.

Equating the coefficients of $1/z$ in the expansion of (ii), we have

$$=-\frac{V^2(a+b)^2}{c^2}.c^2 \sinh \zeta_0\ e^{-\zeta_0}.$$

Now from (i), we get

$$N=\text{Real part of } -\tfrac{1}{2}\rho\ \{-2\pi i.V^2(a+b)^2 \sinh \zeta_0\ e^{-\zeta_0}\}$$

$$=\text{Real part of } \{\pi\rho\ i\ V^2(a+b)^2\ \sinh \zeta_0\ e^{-\zeta_0}\}$$

$\{$as $\zeta_0=\alpha+i\theta$

$$=\text{Real part of } \{\pi\rho i\ V^2(a+b)^2 \sinh(\alpha+i\theta)\ e^{-(\alpha+i\theta)}\}$$

$$=\text{Real part of } \{\pi\rho i\ V^2(a+b)^2 (\sinh\alpha\cos\theta+i\cosh\alpha\sin\theta)\ e^{-\alpha}(\cos\theta-i\sin\theta)\}$$

$$=-\pi\rho\ V^2(a+b)^2\ e^{-\alpha}(\cosh\alpha-\sinh\alpha)\sin\theta\cos\theta$$

$$=-\pi\rho\ V^2(a+b)^2\ e^{-2\alpha}\sin\theta\cos\theta$$

$$=-\pi\rho\ V^2(a^2-b^2)\sin\theta\cos\theta \quad \left\{\begin{array}{l}\text{since}\quad c=(a+b)\ e^{-\alpha}\\ \text{or}\quad c^2=(a+b)^2\ e^{-2\alpha}\\ \text{or } a^2-b^2=(a+b)^2\ e^{-2\alpha}\end{array}\right.$$

Proved.

negative sign shows that the couple tends to set the cylinder broadside to the stream. **Answer.**

Ex. 14. *Liquid of density* ρ *is circulating irrotationally between two confocal elliptic cylinders* $\xi=\alpha$, $\xi=\beta$, *where*

$$x+iy=c\cosh(\xi+i\eta).$$

Prove that, if k is the circulation, the kinetic energy per unit length of cylinder is

$$\frac{1}{4}\rho k^2\frac{\beta-\alpha}{\pi}.$$

The complex potential for irrotational cyclic motion of circulation k round the elliptic cylinder is

$$w=\frac{ik}{2\pi}(\xi+i\eta)$$

$$\phi+i\psi=\frac{ik}{2\pi}(\xi+i\eta)$$

Equating real and imaginary parts, we have

$$\phi=-\frac{k\eta}{2\pi}\quad\text{and}\quad\psi=\frac{k\xi}{2\pi}$$

The kinetic energy is, given by

$$=\tfrac{1}{2}\rho\int q^2\,ds$$

$$=\tfrac{1}{2}\rho \iint\left\{\left(\frac{\partial\phi}{\partial\xi}\right)^2+\left(\frac{\partial\psi}{\partial\eta}\right)^2\right\} d\xi\, d\eta$$

$$\left\{\begin{array}{ll}\text{as} & ds=d\xi\, d\eta \\ \text{and} & q^2=\left(\dfrac{\partial\phi}{\partial\xi}\right)^2+\left(\dfrac{\partial\psi}{\partial\eta}\right)^2\end{array}\right.$$

$$=\tfrac{1}{2}\rho\ \frac{k^2}{4\pi^2}\int_{\xi=\alpha}^{\beta}\int_{\eta=0}^{2\pi} d\xi\, d\eta$$

$$=\tfrac{1}{2}\rho\ \frac{k^2}{4\pi^2}.(\beta-\alpha).2\pi$$

$$=\frac{1}{4}.\rho k^2\frac{\beta-\alpha}{\pi}.$$

Proved.

Ex. 15. *Shew that the motion of a liquid streaming past the elliptic disc* $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, *the velocity at infinity being parallel to the axis of X and equal to U can be expressed by the relation*

$$\phi+i\psi=\frac{U}{a-b}\left\{az-b\sqrt{(z^2-c^2)}\right\}$$

where $c^2=a^2-b^2$ *and* $z=x+iy$.

We know that

$$\phi=Ux+Ub\sqrt{\left(\frac{a+b}{a-b}\right)}\, e^{-\xi}\cos\eta$$

and
$$\psi=Uy-Ub\sqrt{\left(\frac{a+b}{a-b}\right)}\, e^{-\xi}\sin\eta$$

Thus $\phi+i\psi=U(x+iy)+Ub\sqrt{\left(\dfrac{a+b}{a-b}\right)}\, e^{-\xi}(\cos\eta-i\sin\eta)$

or
$$\phi+i\psi=U(x+iy)+Ub\sqrt{\left(\frac{a+b}{a-b}\right)}\, e^{-(\xi+i\eta)}$$

$$=Uz+Ub\sqrt{\left(\frac{a+b}{a-b}\right)}\, e^{-\xi}$$

$$=Uz+Ub\sqrt{\left(\frac{a+b}{a-b}\right)}(\cosh\zeta-\sinh\zeta)$$

$$\left\{\begin{array}{ll}\text{as} & z=x+iy \\ \text{and} & \zeta=\xi+i\eta\end{array}\right.$$

$$=U\sqrt{\left(\frac{a+b}{a-b}\right)}\left\{\frac{a-b}{\sqrt{(a^2-b^2)}}z+b\cosh\zeta-b\sqrt{(\cosh^2\zeta-1)}\right\}$$

$$=U\sqrt{\left(\frac{a+b}{a-b}\right)}\left\{\frac{a-b}{c}z+b.\frac{z}{c}-b.\sqrt{\left(\frac{z^2}{c^2}-1\right)}\right\}$$

$$=U\sqrt{\left(\frac{a+b}{a-b}\right)}\left\{\frac{a-b}{c}z+b.\frac{z}{c}-\frac{b}{c}\sqrt{(z^2-c^2)}\right\}$$

$$=\frac{U}{c}\sqrt{\left(\frac{a+b}{a-b}\right)}\left\{az-b\ \sqrt{(z^2-c^2)}\right\}$$

$$=\frac{U}{a-b}\{az-b\ \sqrt{(z^2-c^2)}\}.$$ **Proved.**

Exercise

1. An elliptic cylinder is placed in a steady stream which at infinity makes an angle α with the major axis of the cylinder. Show that on the ellipse the pressure is greatest at the points where the stream divides, and least at the points where the fluid is moving parallel to the stream as it meets the ellipse.

2. The space between two confocal co-axial elliptic cylinders is filled with liquid which is at rest. Prove that if the outer cylinder be moved with a velocity U parallel to the major axis, the inner will begin to move in the same direction with a velocity

$$\frac{\rho\ U \sinh\beta\ \text{cosech}\ (\beta-\alpha)}{\rho\ \sinh\alpha\ \coth\ (\beta-\alpha)+\sigma\cosh\alpha}$$

where $c\cosh\alpha$, $c\sinh\alpha$ are semi-axes of the inner cylinder $c\cosh\beta$, $c\sinh\beta$ those of the outer and σ the density of the inner cylinder

3. An elliptic cylinder whose semi-axes are $c\cosh\alpha$, $c\sinh\alpha$ is divided in two by a plane through the axis of the cylinder and the major axis of its cross-section. An infinite liquid of density ρ streams past the cylinder, its velocity U at infinity being uniform and parallel to the major axis of the cross-section of the cylinder. Shew that in consequence of the motion of the liquid the pressure between the two portions of the cylinder is diminished by

$$\rho c\ U^2\ e^{a}\sinh\alpha\ \{2\cosh\alpha+e^{a}\sinh\alpha\log\tanh\tfrac{1}{2}\alpha\}$$

per unit length of the cylinder.

Hint : *Q. No. 12.*

$$q^2=\frac{U^2\ (a+b)^2\ \sin^2\eta}{b^2\ \cos^2\eta+a^2\ \sin^2\eta}$$

diminution of the liquid pressure on the portion ABA′

$$=\int(\Pi-p)\sin\left(\theta-\frac{\pi}{2}\right)ds$$

4. An elliptic cylinder, semi-axes a and b is held with its length perpendicular to, and its major axis making an angle ψ with,

the direction of a stream of velocity U. Prove that the magnitude of the couple per unit length on the cylinder due to the fluid pressure is

$$\pi\rho\, c^2 U^2 \sin\theta \cos\theta.$$

5. Verify the stream function for uniform streaming parallel to the axis past a solid, bounded by those parts of the circles

$$(x+1)^2+y^2=2\ ;\ (x-1)^2+y^2=2$$

which are external to each other, are

$$\phi=y\left\{1+\frac{1}{x^2+y^2}-\frac{2}{(x+1)^2+y^2}-\frac{2}{(x-1)^2+y^2}\right\}$$

and $$\psi=-x+\frac{x}{x^2+y^2}+\frac{2\,(x+1)}{(x+1)^2+y^2}+\frac{2\,(x-1)}{(x-1)^2+y^2}$$

Hint : We know that

$$|\,z-1\,|^2=(x-1)^2+y^2$$

and $$|\,z+1\,|^2=(x+1)^2+y^2.$$

Let the transformation be

$$t=\frac{2z}{z^2-1}$$

so that the circle $|\,t\,|=1$ in t-plane is same as

$$\left|\frac{2}{z^2-1}\right|=1 \text{ in } z \text{ plane}$$

or $$\frac{4\,|\,z\,|^2}{|\,z-1\,|^2\,|\,z+1\,|^2}=1 \text{ in } z\text{-plane}.$$

or $$\frac{4\,(x^2+y)^2}{\{(x-1)^2+y^2\}\,\{(x+1)^2+y^2\}}=1$$

or $$\{(x-1)^2+y^2-2\}\,\{(x+1)^2+y^2-2\}=0.$$

Thus the part of plane outside the given circles in z-plane transforms into part outside the circles $|\,t\,|=1$ in t-plane.

Let $$\phi+i\psi=i\left\{-(x+iy)+\frac{x-iy}{x^2+y^2}+2\,\frac{(x+1)-iy}{(x+1)^2+y^2}+2\,\frac{(x-1)-iy}{(x-1)^2+y^2}\right\} \quad \text{(Given)}$$

$$=i\left\{-z+\frac{1}{z}+\frac{2}{z+1}+\frac{2}{z-1}\right\}$$

$$=2i\left\{\frac{2z}{z^2-1}-\frac{z^2-1}{2z}\right\}=2i\left\{t-\frac{1}{t}\right\}$$

which is a complex potential at a point t when the liquid extending upto infinity.

6

Irrotational Motion In Three Dimensions

(MOTION OF A SPHERE)

Here we shall discuss the theory of irrotational motion in three dimensions with a particular emphasis to the motion of the sphere. The treatment is based more or less to the motion of a circular cylinder.

§ 6·0. Butler's Sphere Theorem.

Consider a rigid sphere $r=a$ be introduced into a field of an axi-symmetric irrotational flow in an incompressible inviscid (perfect) fluid with no rigid boundaries, characterised by the current function $\psi_0=\psi_0\ (r\ \theta)$ all of whose singularities are at a distance greater than a from the origin, where $\psi_0=0\ (r^2)$ at an origin, then the stream function becomes

$$\psi=\psi_0-\psi_1$$
$$=\psi_0\ (r,\theta)-\frac{r}{a}\,\psi_0\left(\frac{a^2}{r},\ \theta\right)$$

The following conditions are to be satisfied,

(i) The flow given by the current function ψ must be irrotational.

(ii) On the boundary of the sphere *i.e.* $r=a$, current function is constant.

(iii) ψ_1 (at origin) has no singularity outside the boundary of the sphere $r=a$.

(iv) The velocity due to ψ_1 (at origin) must tend to zero as r tends to infinity and must not introduce any net flux* over the sphere at infinity.

* The integral

$$\int_S \mathbf{F}.\mathbf{n}\ ds \quad \text{or} \quad \int_S F_n\ ds$$

is called the flux of F across the surface. Where F_n denotes the component of F in the direction of the outward drawn normal at the point P inside the surface.

Since ∃ a velocity potential ϕ in an irrotational motion then the velocity components in the direction of r and θ in terms of ϕ and ψ are given by

$$q_r = -\frac{1}{r\sin\theta} \cdot \frac{1}{r}\frac{\partial\psi}{\partial\theta}$$

and
$$q_\theta = \frac{1}{r\sin\theta} \cdot \frac{\partial\psi}{\partial r}$$

or
$$\frac{\partial\phi}{\partial r} = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} \text{ and } \frac{\partial\phi}{\partial\theta} = -\frac{1}{\sin\theta} \cdot \frac{\partial\psi}{\partial r}$$

or
$$\frac{\partial}{\partial\theta}\left(\frac{\partial\phi}{\partial r}\right) = \frac{\partial}{\partial\theta}\left(\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}\right)$$

or
$$\frac{\partial}{\partial r}\left(\frac{\partial\phi}{\partial\theta}\right) = -\frac{\partial}{\partial r}\left(\frac{1}{\sin\theta} \cdot \frac{\partial\psi}{\partial r}\right)$$

i.e.
$$r^2\frac{\partial^2\psi}{\partial r^2} + \sin\theta . \frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta} \cdot \frac{\partial\psi}{\partial\theta}\right) = 0 \qquad \ldots(v)$$

By direct differentiating the function ψ_0 and ψ_1 (at origin) the relation (v) is satisfied. Thus the flow given by the current function ψ is irrotational which satisfies the condition (i).

It is clear that at $r=a$ (on the boundary of the sphere) $\psi=0$, condition (ii) is satisfied.

Since r and $\frac{a^2}{r}$ are the inverse points with regard to the sphere $r=a$. If one point is in side the sphere, the other point lies outside. So all the singularities of ψ_0 lie outside the sphere, and all the singularities of ψ (at origin) will be inside. This satisfies the (iii) condition.

Since ψ_0 is analytic inside the sphere $r=a$ and near the origin $\psi_0 = O(r^2)$. Therefore at infinity ψ_1 (at origin)$=O\left(\frac{1}{r}\right)$.

Thus
$$q_r = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}$$

⇒ velocity at infinity due to ψ_1 (at origin) is $O\left(\frac{1}{r^3}\right)$ which tends to zero as r tends to infinity.

Flux $= \int q_r . ds = O\left(\frac{1}{r}\right)$ { which also tends to zero as r tend to infinity.

Thus the condition (iv) is also satisfied.

Similarly we can show that if all the singularities of $\psi_0\,(r\,\theta)$ are inside the sphere $r=a$, and if $\psi_0 = O\left(\frac{1}{r}\right)$ for large r, then

$\psi=\psi_0-\psi_1$ gives the flow inside the sphere when $r=a$ is a rigid boundary.

§ 6·01. Since the velocity potential ϕ satisfies the Laplace's equation in three dimensions.

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0.$$

This equation can be represented in spherical polar coordinates, as follows

$$\frac{\partial^2\phi}{\partial r^2}+\frac{2}{r}\cdot\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial\phi}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\phi}{\partial\omega^2}=0.$$

or
$$r^2\sin\theta\frac{\partial^2\phi}{\partial r^2}+2r\sin\theta\frac{\partial\phi}{\partial r}+\sin\theta\frac{\partial^2\phi}{\partial\theta^2}$$
$$+2\cot\theta\sin\theta\frac{\partial\phi}{\partial\theta}+\frac{1}{\sin\theta}\frac{\partial^2\phi}{\partial\omega^2}=0$$

or
$$\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial\phi}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right)$$
$$+\frac{\partial}{\partial\omega}\left(\frac{1}{\sin\theta}\cdot\frac{\partial\phi}{\partial\omega}\right)=0. \qquad \ldots\text{(i)}$$

let $\dfrac{\partial^2\phi}{\partial\omega^2}=0$ then (i) reduces to

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)+\frac{1}{\sin\theta}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right)=0. \qquad \ldots\text{(ii)}$$

Whose general solutions are of the type

$$\phi=\Sigma\,(A_nr^n+B_n\,r^{-n-1})\,P_n \qquad \forall \text{ integral values of } n$$

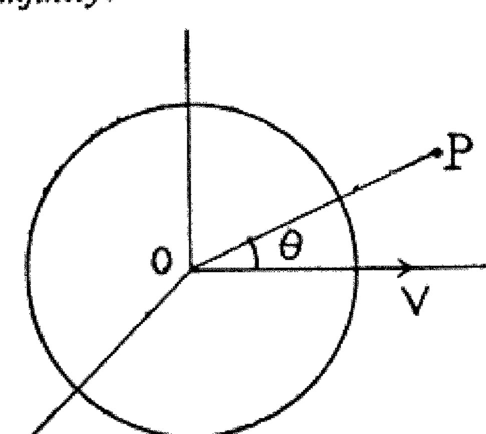
where **P_n is the Legendre's co-efficient of order n.**

and $P_0(\mu)=1,\ P_1(\mu)=\mu,\ P_2(\mu)=\frac{1}{2}(3\mu^2-1)$......etc.

§ 6·1. **Motion of a sphere through a liquid at rest at infinity.**

To determint the velocity potential and stream function, if a sphere is moving in a liquid at rest at infinity.

Consider the origin O be at the centre of the sphere, which is moving along a straight line with velocity V. The motion of the liquid will be symmetrical about this line. Now the velocity potential ϕ must satisfy the following boundary conditions.



(I) At every place the velocity potential ϕ satisfies the Laplace's condition $\nabla^2\phi=0$.

(II) $\dfrac{\partial \phi}{\partial r}=0$ at infinity.

(III) At the surface of the sphere $r=a$, we must have

$$-\frac{\partial \phi}{\partial r}=\text{Normal velocity}=V\cos\theta$$

If follows that ϕ must be a function of $\cos\theta$.* Consider the form of the function ϕ is given by,

$$\phi=\left(Ar+\frac{B}{r^2}\right)\cos\theta \qquad \text{...(i)}$$

So
$$\frac{\partial \phi}{\partial r}=\left(A-\frac{2B}{r^3}\right)\cos\theta$$

from II $\quad r=\infty,\ \dfrac{\partial \phi}{\partial r}=0.\quad$ which gives $A\cos\theta=0 \Rightarrow A=0.$

from III at the surface of the sphere $r=a$,

$$-\frac{\partial \phi}{\partial r}=V\cos\theta \qquad \text{...(ii)}$$

From (i) and (ii) we have

$$-\left(A-\frac{2B}{r^3}\right)\cos\theta=V\cos\theta$$

or
$$B=\tfrac{1}{2}a^3V \qquad \{\text{as } A=0$$

Then (i) reduce to

$$\phi=\frac{1}{2}\frac{a^3V}{r^2}\cos\theta \qquad \text{...(iii)}$$

Which determines the **velocity potential** for the motion.

Also
$$\psi=-\frac{1}{2}\frac{a^3V}{r}\sin^2\theta$$

Lines of flow. The differential equation of the lines of flow referring the centre of sphere as origin,

$$\frac{dr}{\partial\phi/\partial r}=\frac{r\,d\theta}{\partial\phi/r\,d\theta}$$

---

* From III condition we see that solution of (ii) § 6·01 must be of the from $\varphi=f(r)\cos\theta$. By substituting, we have

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right)-2f=0$$

or

$$r^2\frac{\partial^2 f}{\partial r^2}+2r\frac{\partial f}{\partial r}-2f=0$$

or

Its solution is

$$f=Ar+\frac{B}{r^2}$$

or $$\frac{dr}{-\frac{a^3V}{r^3}\cos\theta}=\frac{r\,d\theta}{-\frac{a^3V}{2r^3}\sin\theta}$$

$$\left\{ \text{as} \quad \frac{\partial\phi}{\partial r}=-\frac{a^3V}{r^3}\cos\theta\ \frac{\partial\phi}{\partial\partial}=-\frac{a^3V}{2r^2}\sin\theta \right.$$

or $$\frac{dr}{\cos\theta}=\frac{2r\,d\theta}{\sin\theta}$$ { from (iii)

or $$\frac{dr}{r}=2\cot\theta\,d\theta$$

By integrating, we have

$$\log r=\log\sin^2\theta+\log c$$

{ where $\log c$ is an arbitratry constant}

$$r=c\sin^2\theta$$

Which is the equation to the **lines of flow.**

§ 6·2. Liquid Streaming past a fixed sphere.

Consider the sphere to be fixed and the liquid streaming past it with velocity V. By superposing a velocity $-V$ on the sphere and the liquid, the velocity potential becomes

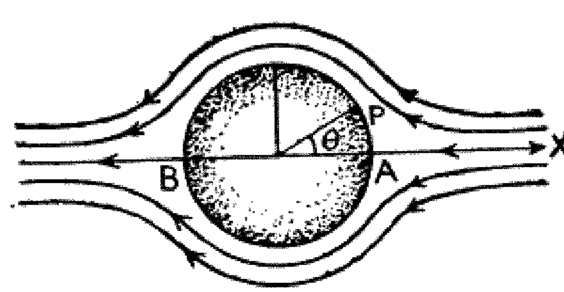
$$\phi=V\,r\cos+\frac{1}{2}\,\frac{V\,a^3}{r^2}\cos\theta$$

or $$\phi=V\left(r+\frac{1}{2}\,\frac{a^3}{r^2}\right)\cos\theta \qquad \text{...(i)}$$

Equation to the lines of flow, are given by

$$\frac{dr}{\partial\phi/\partial r}=\frac{r\,d\theta}{\partial\phi/r\partial\theta}$$

or $$\frac{dr}{V\left(1-\frac{a^3}{r^3}\right)\cos\theta}=-\frac{r\,d\theta}{V\left(1+\frac{1}{2}\frac{a^3}{r^3}\right)\sin\theta}$$



or $$\frac{2r^3+a^3}{r^3-a^3}\cdot\frac{1}{r}=-2\cot\theta\,d\theta$$

or $$\left(\frac{3r^2}{r^3-a^3}-\frac{1}{r}\right)dr=-2\cot\theta\,d\theta$$

By integrating, we have

$$\log (r^3-a^3)-\log r=-2 \log \sin \theta+\log C$$

{where $\log C$ is an arbitrary constant}

or
$$\frac{r^3-a^3}{r}=\frac{C}{\sin^2 \theta}$$

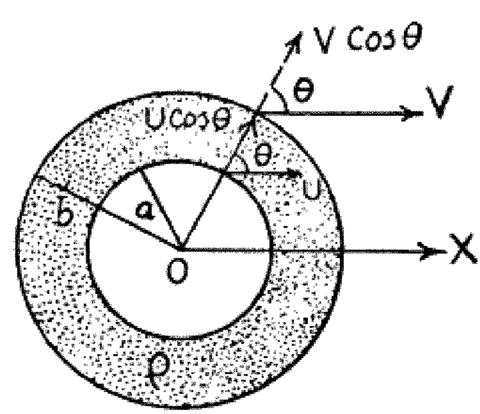
or
$$\sin^2 \theta \,(r^3-a^3)=Cr$$

Which gives the lines of flow relative to the sphere.

§ 6·3. Concentric Spheres. Initial Motion.

Consider two concentric spheres of radii a and b $(a<b)$, the intervening space being filled with liquid of density ρ. To determine the velocity potential of the initial motion, when the given impulsive force are so applied that the inner sphere starts moving with velocity U and the outer sphere with velocity V in the same direction.

Since the motion starts by applying the impulsive force, thus it is an irrotational motion. The velocity potential ϕ must satisfy the Laplace's function



i. e. $\nabla^2 \phi=0$ ...(i)

The bounday conditions are

I $-\dfrac{\partial \phi}{\partial r}=U \cos \theta$ when $r=a$

II $-\dfrac{\partial \phi}{\partial r}=V \cos \theta$ when $r=b$

Assuming that

$$\phi=\left(Ar+\frac{B}{r^2}\right) \cos \theta \qquad \text{...(ii)}$$

or
$$\frac{\partial \phi}{\partial r}=\left(A-\frac{2B}{r^3}\right) \cos \theta \qquad \text{...(iii)}$$

Now from I and (iii), we have

$$\left(A-\frac{2B}{a^3}\right) \cos \theta=-U \cos \theta \qquad \text{...(iv)}$$

Also from II and (iii), we have

$$\left(A-\frac{2B}{b^3}\right) \cos \theta=-V \cos \theta \qquad \text{...(v)}$$

These being true for all values of θ, we have

$$A-\frac{2B}{a^3}=-U$$

and $$A-\frac{2B}{b^3}=-V$$

or $$A=\frac{Ua^3-Vb^3}{b^3-a^3} \text{ and } B=\frac{(U-V)\,a^3\,b^3}{2\,(b^3-a^3)}$$

By substituting the value of the constants A and B in (ii), we have

$$\phi=\left\{\frac{U\,a^3-V\,b^3}{b^3-a^3}\right\}.\ r\cos\theta+\left\{\frac{(U-V)\,a^3\,b^3}{2\,(b^3-a^3)}\right\}.\ \frac{\cos\theta}{r^2} \quad \text{...(vi)}$$

Which determines the **velocity potential for initial motion.**

Particular Case : Let the **outer cylinder be at rest.** It follows that $V=0$ then the velocity potential ϕ reduces to

$$\phi=\frac{U\,a^3}{b^3-a^3}\ .\ r\cos\theta+\frac{U\,a^3\,b^3}{2(b^3-a^3)}.\ \frac{\cos\theta}{r^2} \quad \{\text{ from (vi)}$$

or $$\phi=\frac{U\,a^3}{b^3-a^3}\left\{\ r+\frac{b^3}{2r^2}\right\}\cos\ \theta$$

The velocity potential ϕ at the surface of the inner sphere $r=a$, is given by

$$\phi=\ \frac{U\,a^3}{b^3-a^3}\left\{\ a+\frac{b^3}{2a^2}\right\}\cos\theta$$

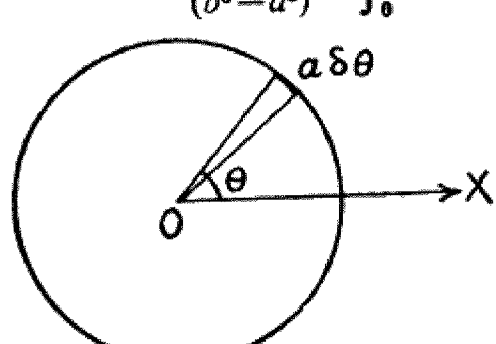
$$\phi=\tfrac{1}{2}\ \frac{U\,a\,(2a^3+b^3)}{(b^3-a^3)}\ .\cos\theta$$

Let M be the mass of the inner sphere and I the impulse necessary to produce the velocity U, then by the principle of momentum,

$$I-MU=\iint p\cos\theta\ ds$$

$\left\{\begin{array}{l}\text{Since } p=\rho\phi, \text{ the impulsive pressure of the liquid at } r=a, \text{ where}\\ \phi \text{ denote the velocity potential just before the impulsive action.}\end{array}\right\}$

or $$I-MU=\rho.\tfrac{1}{2}\ \frac{U\,a\,(2a^3+b^3)}{(b^3-a^3)}\int_0^{2\pi}\cos^2\theta\ .\ (2\pi a\sin\theta)\ a\ d\theta$$



or $$I-MU=\frac{\rho U\,(2a^3+b^3)\,.\,\pi a^3}{(b^3-a^3)}\int_0^{2\pi}\cos^2\theta\sin\theta\,d\theta$$

or $$I-MU=\frac{\rho U\,(2a^3+b^3)}{(b^3-a^3)}\,.\,\pi a^3\left(-\frac{\cos^3\theta}{3}\right)_0^{2\pi}$$

or $$I-MU=\frac{2\rho U\,(2a^3+b^3)}{3\,(b^3-a^3)}\,.\,\pi a^3$$

Let M' be the mass of the liquid displaced

then $$M'=\tfrac{4}{3}\,\pi a^3\,\rho$$

or $$I=MU+\tfrac{1}{2}\frac{M'U\,(2a^3+b^3)}{(b^3-a^3)}$$

$$=MU+\tfrac{1}{2}\,.\,M'U\left\{\frac{2\dfrac{a^3}{b^3}+1}{1-\dfrac{a^3}{b^3}}\right\}$$

If the radius of the outer sphere $(r=b)$ is increased indefinitely, (in other sense b tends to ∞,) then

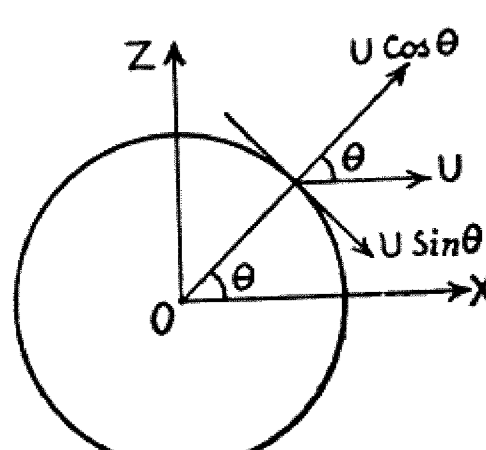
$$\underset{b\to\infty}{\mathrm{Lt}}\ \frac{2\dfrac{a^3}{b^3}+1}{1-\dfrac{a^3}{b^3}}=1$$

Thus the impulse necessary to give to a sphere of mass M placed in infinite liquid is given by

$$I=MU+\tfrac{1}{2}\,M'U$$

$$I=(M+\tfrac{1}{2}\,M')\,U$$

Ex. 1. *A solid sphere moves through quiescent frictionless liquid whose boundaries are at a distance from it great compared with its radius. Prove that at each instant the motion in the liquid depends only on the position and velocity of the sphere at that instant. Prove that the liquid streams past the sides of the sphere with half the velocity of the sphere.*



Let O be the centre of the sphere and U the velocity with which it moves. The motion being symmetrical to the X-axis and irrotational. The velocity potential ϕ must satisfy the following condition :

(i) $\nabla^2\phi=0.$

(Laplace's condition)

(ii) $$-\frac{\partial\phi}{\partial r}=U\cos\theta$$

when $r=a$ *i.e.* on the boundary of the sphere.

(iii) $$-\frac{\partial\phi}{\partial r}=\text{zero}$$

where r tends to infinity.

Since the solution of $\nabla^2\phi=0$ contains $\cos\theta$. So assuming the suitable form of the velocity potential.

$$\phi=\left(Ar+\frac{B}{r^2}\right)\cos\theta \quad \text{...(iv)}$$

or $$\frac{\partial\phi}{\partial r}=\left(A-\frac{2B}{r^3}\right)\cos\theta \quad \text{...(v)}$$

From (ii) and (v), we have

$$\left(A-\frac{2B}{a^3}\right)\cos\theta=-U\cos\theta \qquad \{\text{at } r=a \quad \text{...(vi)}$$

From (iii) and v), we have

$$A\cos\theta=0$$

$$\therefore \qquad A=0$$

The relation (vi) being true for all values of θ, we have

$$A-\frac{2B}{a^3}=-U$$

or $$B=\tfrac{1}{2}a^3U. \qquad \{\text{as } A=0$$

Substituting the value of A and B in (iv), we have

$$\phi=\tfrac{1}{2}.\frac{a^3U}{r^2}\cos\theta.$$

Thus at each instant the motion in the liquid depends only on the position of the sphere and velocity of the sphere at an instant. **Proved.**

Velocity with which liquid streams past the sides of the sphere

$$=-\left(\frac{1}{r}\frac{\partial\phi}{\partial r}\right)_{r=a}$$

$$=\left(\frac{1}{2}\frac{a^3U}{r^2}\sin\theta\right)_{r=a}$$

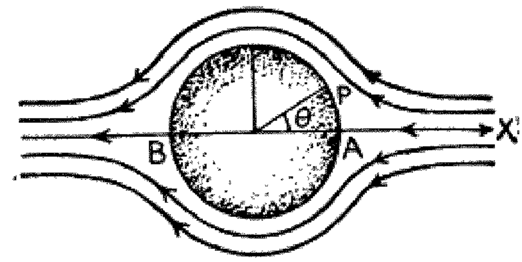
$$=\tfrac{1}{2}U\sin\theta$$

$=\tfrac{1}{2}\times$ the velocity of the sphere along the tangent. **Proved.**

Ex. 2. *An infinite ocean of an incompressible perfect liquid of density* ρ *is streaming past a fixed spherical obstacle of radius a. The velocity is uniform and equal to U except in so far as its disturbed by the sphere, and the pressure in the liquid at a great distance from the obstacle is* Π. *Shew that the thrust on that half* *of the sphere on which the liquid impinges is*

$$\pi a^2 \left\{ \Pi - \frac{1}{16} \rho U^2 \right\}.$$

The velocity potential when the liquid is streaming past the fixed sphere with velocity U is given by,



$$\phi = U \left\{ r + \tfrac{1}{2} \cdot \frac{a^3}{r^2} \right\} \cos\theta \qquad \text{...(i)}$$

The velocity at any point of the sphere is tangential, therefore

$$\left(-\frac{1}{r} \frac{\partial \phi}{\partial \theta} \right)_{r=a} = U \left\{ \left(1 + \tfrac{1}{2} \frac{a^3}{r^3} \right) \sin\theta \right\}_{r=a}$$

$$= \tfrac{1}{2} . 3U \sin\theta$$

Now the velocity will vanish when $\theta = 0$ or $\theta = \pi$. Thus the stagnation points will occur on the axis of X at A $(\theta = 0)$ and B $(\theta = \pi)$

and $$\left(\frac{\partial \phi}{\partial r} \right)_{\text{at } r=a} = \left[U \left\{ 1 - \frac{a^3}{r^3} \right\} \cos\theta \right]_{r=a}$$

$$= \text{Zero.}$$

Consider q be the velocity at a point on the boundary of the sphere.

So $$q^2 = \left\{ \left(\frac{\partial \phi}{\partial r} \right)^2 + \left(\frac{1}{r} \cdot \frac{\partial \phi}{\partial \theta} \right)^2 \right\}_{\text{at } r=a}$$

or $$q^2 = \frac{9}{4} U^2 \sin^2\theta.$$

Let p be the pressure then by Bernoulli's equation, we have

$$\frac{p}{\rho} + \tfrac{1}{2} q^2 = C. \qquad \text{...(ii)}$$

{In case of steady motion and absence of external forces}

Since $p = \Pi$, $q = U$

then $$C = \frac{\Pi}{\rho} + \tfrac{1}{2} U^2.$$

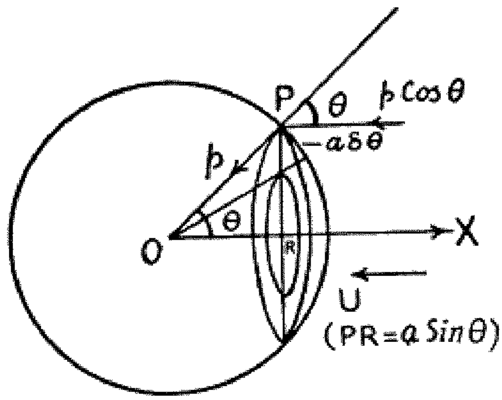
Substituting the values of constant C in (ii), we have

$$\frac{p}{\rho} + \tfrac{1}{2} q^2 = \frac{\Pi}{\rho} + \tfrac{1}{2} U^2$$

or $$\frac{p}{\rho} = \frac{\Pi}{\rho} + \tfrac{1}{2} U^2 - \frac{9}{8} U^2 \sin^2\theta$$

which gives the pressure at any point of the sphere.

Now the thrust on that half of the sphere on which the liquid impinges.



$$=\int_0^{\pi/2} p\cos\theta . 2\pi a\sin\theta . a\,d\theta$$

$$=2\pi a^2\rho\int_0^{\pi/2}\left\{\frac{\Pi}{\rho}+\tfrac{1}{2}U^2-\frac{9}{8}U^2\sin^2\theta\right\}.\sin\theta\cos\theta\,d\theta$$

$$=2\pi a^2\rho\left[\left\{\frac{\Pi}{\rho}+\tfrac{1}{2}U^2\right\}.\tfrac{1}{2}-\frac{9}{8}U^2.\frac{1}{4}\right]$$

$$=2\pi a^2\rho\left[\frac{\Pi}{2\rho}-\frac{1}{32}U^2\right]$$

$$=\pi a^2\left[\Pi-\frac{1}{16}\rho U^2\right].$$

Proved.

Ex. 3. *A stream of water of great depth is flowing with uniform velocity U over a plane level bottom. A hemi-sphere of weight w in water and of radius a, rests with its base on the bottom. Prove that the average pressure between the base of the hemisphere and the bottom is less than the fluid pressure at any point of the bottom at a great distance from the hemisphere, if*

$$U^2>\frac{32w}{11\pi a^2\rho}.$$

Consider the centre O of the hemisphere as origin and $(r\ \theta\ \phi)$ be the spherical polar coordinates of a point referred to the axes through the origin O.

The velocity potential when the liquid is streaming past the fixed sphere with velocity U is given by

$$\phi=U\left(r+\frac{a^3}{2r^2}\right)\cos\theta$$

where $x=r\sin\theta\cos\phi$, $y=r\sin\theta\sin\phi$ and $z=r\cos\theta$.

If q be the velocity at the point $P\ (r\ \theta\ \phi)$ on the sphere

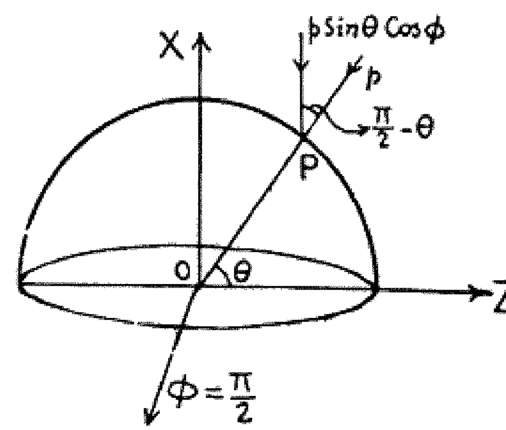
then $$q^2=\left(\frac{\partial\phi}{\partial r}\right)^2+\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)^2$$

$$=\frac{9}{4}U^2\sin^2\theta$$

{ Ref. Q. No. 2

Let Π be the pressure at a great distance *i.e.* at infinity, then

$$\frac{p}{\rho}=\frac{\Pi}{\rho}+\tfrac{1}{2}U^2-\frac{9}{8}U^2\sin^2\theta. \qquad \left\{ \text{Ref. Q. No. 2} \right.$$



Now the total thrust on the hemisphere due to liquid along the direction XO

$$=\int_{\theta=0}^{\pi}\int_{\phi=-\pi/2}^{\pi/2}\left[\rho\left\{\frac{\Pi}{\rho}+\tfrac{1}{2}U^2-\frac{9}{8}U^2\sin^2\theta\right\}\sin\theta\cos\phi\right].a\sin\theta\, d\phi .a\, d\theta.$$

$\left\{\begin{array}{l}\text{where } p\sin\theta\cos\phi \text{ is the component of the pressure}\\ \text{along } X\text{-axis and } (a\sin\theta\, d\phi\, a\, d\theta) \text{ the element on}\\ \text{surface of the sphere at } P.\end{array}\right.$

$$=a^2\rho\int_{\theta=0}^{\pi}\int_{\phi=-\pi/2}^{\pi/2}\left\{\frac{\Pi}{\rho}+\tfrac{1}{2}U^2-\frac{9}{8}U^2\sin^2\theta\right\}.\sin^2\theta\cos\phi\, d\theta\, d\phi$$

$$=2a^2\rho\int_{\theta=0}^{\pi}\left\{\frac{\Pi}{\rho}+\tfrac{1}{2}U^2-\frac{9}{8}U^2\sin^2\theta\right\}\sin^2\theta\, d\theta$$

$$=2a^2\rho\left[\left(\frac{\Pi}{\rho}+\tfrac{1}{2}U^2\right)\frac{\pi}{2}-\frac{9}{8}U^2.\frac{3\pi}{8}\right]$$

$$=\pi a^2\rho\left[\frac{\Pi}{\rho}-\frac{11}{32}U^2\right]$$

$$=\pi a^2\left[\Pi-\frac{11}{32}\rho U^2\right].$$

So total pressure on the base

$$=\pi a^2\left[\Pi-\frac{11}{32}\rho U^2\right]+w.$$

$$\text{Average pressure on the base}=\frac{\text{Pressure on the base}}{\text{area of the base}}$$

$$=\Pi-\frac{11}{32}\rho U^2+\frac{w}{\pi a^2}.$$

Since Average Pressure < Pressure at great distance.

or $$\Pi-\frac{11}{32}\rho U^2+\frac{w}{\pi a^2}<\Pi$$

or $$\frac{11}{32}\rho U^2>\frac{w}{\pi a^2}$$

or $$U^2>\frac{32w}{11\rho\pi a^2}.$$ **Proved.**

Ex. 4. *Liquid of density ρ fills the space between a solid sphere of radius a and density ρ' and a fixed concentric spherical* *envelope of radius b, prove that the work done by an impluse which starts the solid sphere with velocity U is*

$$\frac{1}{3}\pi a^3 U^2 \left\{ 2\rho' + \frac{2a^3+b^3}{b^3-a^3}\rho \right\}.$$

Ref. § 6·3 Particular case.

The impulse I necessary to start the solid sphere with velocity U is given by

$$I = MU + \tfrac{1}{2}M'U.\frac{2a^3+b^3}{b^3-a^3}. \quad \text{...(i)}$$

Since the impulse on the sphere of mass M

$$= MU$$
$$= \frac{4}{3}\pi a^3 \rho' U$$

and M' be the mass of the liquid displaced

i.e. $$M' = \frac{4}{3}\pi a^3 \rho U.$$

From (i), we have

$$I = \frac{4}{3}\pi a^3 \rho' U + \frac{2\pi a^3 \rho U}{3}\cdot\frac{2a^3+b^3}{b^3-a^3}$$
$$= \frac{2\pi a^3 U}{3}\left\{ 2\rho' + \rho\left(\frac{2a^3+b^3}{b^3-a^3}\right)\right\}$$

Now the work done by the impulse

$$= \text{Impulse} \times \text{Mean of initial and final velocities}$$
$$= I \times \frac{0+U}{2}$$
$$= \tfrac{1}{2} IU$$
$$= \frac{\pi a^3 U^2}{3}\left\{ 2\rho' + \frac{2a^3+b^3}{b^3-a^3}\rho \right\}.$$ **Proved.**

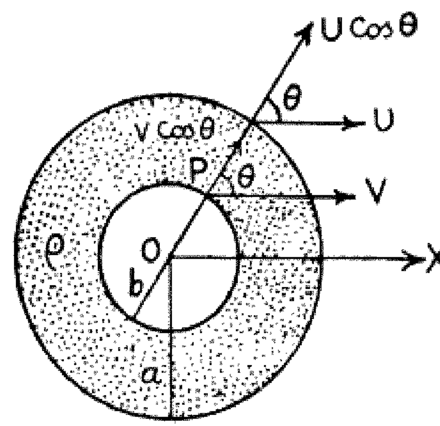
Ex. 5. *The space between two concentric spherical shells of radii a and b (a > b) is filled with an incompressible fluid of density ρ and the shells suddenly begin to move with velocities U, V in the some direction; Prove that the resultant implusive pressure on the inner shell is*

$$\frac{2\pi\rho b^3}{3(a^3-b^3)}\left\{ 3a^3U - (a^3+2b^3)V \right\}.$$

Ref. § 6·3, we have

$$\phi = -\frac{1}{a^3-b^3}\left\{ (Ua^3 - Vb^3)\, r + \frac{(U-V)\, a^3 b^3}{2r^2} \right\}\cos\theta$$

Let ϖ be the impulsive pressure at any point $P\ (b, \theta)$ on the inner shell.



$$\varpi = -\rho\phi$$

where ϕ is the velocity potential on the inner shell

$$\varpi = \frac{\rho b}{a^3-b^3}\left\{(Ua^3-Vb^3) + \tfrac{1}{2}(U-V)\,a^3\right\}\cos\theta$$

Thus the impulsive pressure on the surface of the inner spherical shell

$$= \int_0^\pi (-\varpi\cos\theta).2\pi b\sin\theta.b\,d\theta$$

$$= -2\pi b^2\int_0^\pi \varpi\sin\theta\cos\theta\,d\theta$$

$$= -\frac{2\pi\rho b^3}{a^3-b^3}\left\{(Ua^3-Vb^3)+\tfrac{1}{2}(U-V)\,a^3\right\}\int_0^\pi \cos^2\theta\sin\theta\,d\theta$$

$$= -\frac{4}{3}\cdot\frac{\pi\rho b}{a^3-b^3}\left\{(Ua^3-Vb^3)+\tfrac{1}{2}(U-V)\,a^3)\right\}$$

$$= -\frac{2\pi\rho b^3}{3\,(a^3-b^3)}\left\{3Ua^3-(a^3+2b^3)\,V\right\}.$$

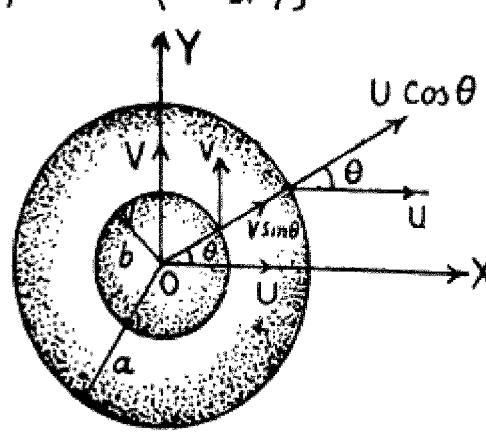
Negative sign is admissible. **Proved.**

Ex. 6. *Prove that for liquid contained between two instantaneously concentric spheres, when the outer (radius a) is moving parallel to the axis of X with velocity U and the inner (radius b) is moving parallel to the axis of Y with velocity V, the velocity potential is*

$$-\frac{1}{a^3-b^3}\left\{a^3\,Ux\left(1+\frac{b^3}{2r^3}\right)-b^3\,Vy\left(1+\frac{a^3}{2r^3}\right)\right\}.$$

and find the kinetic energy.

Since the motion is irrotational. So $\exists$ a velocity potential ϕ. The velocity potential ϕ must satisfy the Laplace condition $\nabla^2\phi=0$ and also the following boundary conditions;



$$\text{(I)} \ -\frac{\partial \phi}{\partial r}=U \cos \theta \quad \text{when} \quad r=a \qquad \ldots\text{(i)}$$

{Ref § 5.1 (II)

$$\text{(II)} -\frac{\partial \phi}{\partial r}=V \sin \theta \quad \text{when} \quad r=b. \qquad \ldots\text{(ii)}$$

Assuming the velocity potential ϕ be of the form

$$\phi=\left(Ar+\frac{B}{r^2}\right) \cos \theta+\left(Cr+\frac{D}{r^2}\right) \sin \theta \qquad \ldots\text{(iii)}$$

or $$\frac{\partial \phi}{\partial r}=\left(A-\frac{2B}{r^3}\right) \cos \theta+\left(C-\frac{2D}{r^3}\right) \sin \theta \qquad \ldots\text{(iv)}$$

from (I) and (iv), we have

or $$-U \cos \theta=\left(A-\frac{2B}{a^3}\right) \cos \theta+\left(C-\frac{2D}{a^3}\right) \sin \theta \qquad \ldots\text{(v)}$$

Also from (II) and (iv). we get

or $$-V \sin \theta=\left(A-\frac{2B}{b^3}\right) \cos \theta+\left(C-\frac{2D}{b^3}\right) \sin \theta \qquad \ldots\text{(vi)}$$

The relations (v) and (vi) being true for all values of θ, we have

$$A-\frac{2B}{a^3}=-U \quad \text{and} \quad C-\frac{2D}{a^3}=0$$

$$A-\frac{2B}{b^3}=0 \quad \text{and} \quad C-\frac{2D}{b^3}=-V$$

By solving these, we have

$$A=-\frac{Ua^3}{a^3-b^3}, \quad B=-\frac{Ua^3b^3}{2(a^3-b^3)}$$

$$C=\frac{Vb^3}{a^3-b^3}, \quad D=\frac{Va^3b^3}{2\,(a^3-b^3)}.$$

Substituting the values of A, B, C and D in (iii), the velocity potential ϕ reduces to

$$\phi=-\frac{Ua^3}{a^3-b^3}\left(r+\frac{b^3}{2r^2}\right) \cos \theta+\frac{Vb^3}{a^3-b^3}\left(r+\frac{a^3}{2r^2}\right) \sin \theta$$

or $$\phi=-\frac{1}{a^3-b^3}\left\{a^3U.r \cos \theta\left(1+\frac{b^3}{2r^3}\right)\right.$$
$$\left.-b^3V.r \sin \theta\left(1+\frac{a^3}{2r^3}\right)\right\}$$

or $$\phi=-\frac{1}{a^3-b^3}\left\{a^3Ux\left(1+\frac{b^3}{2r^3}\right)-b^3Vy\left(1+\frac{a^3}{2r^3}\right)\right\}$$

Proved.

Let T be the kinetic energy of the liquid at any instant.

Then $T=-\frac{1}{2}\rho \iint \phi \frac{\partial \phi}{\partial n}\, dS$

$$=\tfrac{1}{2}\rho \iint \phi \left(\frac{\partial \phi}{\partial r}\right)_{r=a} dS-\tfrac{1}{2}\rho \iint \left(\phi \frac{\partial \phi}{\partial r}\right)_{r=b} dS$$

the normal direction being the direction.

$$\left\{ \text{As } \left(\frac{\partial \phi}{\partial r}\right)_{r=a}=-U\cos\theta,\ \left(\frac{\partial \phi}{\partial r}\right)_{r=b}=-V\sin\theta \right.$$

$$=-\frac{1}{2(a^3-b^3)}\cdot \rho \iint_{r=a}\left\{a^3U\left(1+\frac{b^3}{2a^3}\right)x - b^3V\left(1+\frac{1}{2}\right)y\right\}(-U\cos\theta)\, dS$$

$$+\frac{1}{2}\cdot \rho\frac{1}{a^3-b^3}\iint_{r=b}\left\{a^3U(1+\tfrac{1}{2})x - b^3V\left(1+\frac{a^3}{2b^3}\right)y\right\}(-V\sin\theta)\, dS$$

$$=\tfrac{1}{4}\rho\frac{U^2(2a^3+b^3)}{(a^3-b^3)a}\iint_{r=a} x^2\, dS-\tfrac{3}{4}.\rho\frac{UVb^3}{a(a^3-b^3)}\iint_{r=a} xy\, dS$$

$$-\tfrac{3}{4}\rho.\frac{UVa^3}{b(a^3-b^3)}\iint_{r=b} xy\, dS+\tfrac{1}{4}\rho\frac{V^2(a^3+2b^3)}{(a^3-b^3)b}\iint_{r=b} y^2\, dS$$

$$=\tfrac{1}{4}\rho.\frac{U^2(2a^3+b^3)}{(a^3-b^3)a}.\tfrac{4}{3}\pi a^4+\tfrac{1}{4}\rho.\frac{V^2(a^3+2b^3)}{(a^3-b^3)b}.\tfrac{4}{3}\pi b^4$$

Thus kinetic energy is given by

$$=\frac{1}{3}\frac{\pi\rho}{a^3-b^3}\left\{2(U^2a^6+V^2b^6)+a^3b^3(U^2+V^2)\right\}$$

$$\left[\begin{array}{l} \text{as product of inertia } \iint xy\, dS=0 \\ \text{and } \iint_{r=a} x^2 dx=\tfrac{1}{2}\iint_{r=a}(x^2+y^2)\, dS \\ \quad =\text{M. I. of the hollow sphere of radius } a \text{ about } a \text{ diameter} \\ \quad =\frac{1}{2}\cdot\frac{2Ma^2}{3}=\frac{Ma^2}{3} \\ \quad =4\pi a^2.\frac{a^2}{3}=\frac{4\pi a^4}{3} \end{array}\right.$$

Ex. 6. *Incompressible fluid, of density* ρ, *is contained between two rigid concentric spherical surfaces, the outer one of mass* M_1 *and radius* a; *the inner one of mass* M_2 *and radius* b. *A normal blow* P *is given to the outer surface. Prove that the initial velocities of the two containing surfaces* (U *for the outer and* V *for the* *inner) are given by the equations.*

$$\left\{M_1+\frac{2\pi\rho a^3\,(2a^3+b^3)}{3\,(a^3-b^3)}\right\}\,U-\frac{2\pi\rho a^3b^3}{a^3-b^3}\,V=P$$

and
$$\left\{M_2+\frac{2\pi\rho b^3\,(2b^3+a^3)}{3\,(a^3-b^3)}\right\}\,V=\frac{2\pi\rho a^3b^3}{a^3-b^3}\,U$$

Ref. § 6·3 equation (vi).

$$\phi=\frac{1}{a^3-b^3}\left\{(Vb^3-Ua^3)\,r+\frac{(V-U)\,a^3b^3}{2r^2}\right\}\cos\theta \qquad \text{...(i)}$$

Since a normal blow P is given to the outer surface therefore,

$$M_1U=P-\iint_{r=a} p\cos\theta\,dS \qquad \text{...(ii)}$$

$$M_2V=P-\iint_{r=b} p\cos\theta\,dS \qquad \text{...(iii)}$$

[where p is an impulsive pressure on an element δs on the surface.

Also $\qquad p=\rho\phi \qquad$ ...(iv)

where ϕ is the velocity potential just before an impulsive action.

From (ii) and (iv), we have

$$M_1U=P-\iint_{r=a}\rho\phi\cos\theta\,dS$$

or
$$M_1U=P-\iint\rho.\,\frac{1}{a^3-b^3}\left\{(Vb^3-Ua^3)\,r+\frac{(V-U)\,a^3b^3}{2r^2}\right\}\cos\theta.\cos\theta\,dS$$

or
$$M_1U=P-\frac{\rho}{a^3-b^3}\int_0^{\pi}\left\{(Vb^3-Ua^3)\,r+\frac{(V-U)\,a^3b^3}{2r^2}\right\}\cos\theta.\cos\theta.2\pi a\sin\theta\,ad\theta$$

$$=P-\frac{2\pi\rho a^2}{2\,(a^3-b^3)}\left\{3Vab^3-aU\,(2a^3+b^3)\right\}_{r=a}\int_0^{\pi}\cos^2\theta\sin\theta.d\theta$$

$$=P+\frac{2\pi\rho a^3}{3\,(a^3-b^3)}\left\{3Vb^3-U\,(2a^3+b^3)\right\}$$

{ at $r=a$ *i. e.* surface of the outer sphere or $\left\{M_1+\frac{2\pi\rho a^3(2a^3+b^3)}{3\,(a^3-b^3)}\right\} U-\frac{2\pi\rho a^3 b^3}{a^3-b^3}\,V=P$

which proves the first result.

Now from (iii) and (iv), we have

$$M_2V=-\int\int \rho\phi \cos\theta\, dS$$

or $$M_2V=-\int_0^\pi \rho\,.\,\frac{1}{a^3-b^2}\left\{(Vb^3-Ua^3)\,r+\frac{(V-U)\,a^3b^3}{2r^2}\right\}\cos\theta.\cos\theta.2\pi b\sin\theta.bd\theta$$

or $$M_2V=-\frac{2\pi b^3\rho}{a^3-b^3}\left\{(Vb^3-Ua^3)+\tfrac{1}{2}\,(V-U)\,a^3\right\}\int_0^\pi \cos^2\theta\sin\theta.d\theta$$

$$=-\frac{2}{3}\cdot\frac{\pi b^3\rho}{a^3-b^3}\left\{V\,(2b^3+a^3)-3Ua^3\right\}$$

or $$\left\{M_2+\frac{2\pi\rho b^3\,(2b^3+a^3)}{3\,(a^3-b^3)}\right\}\;V=\frac{2\pi\rho a^3b^3}{a^2-b^3}\;U$$

which proves the second result.

Exercises

1. A sphere of radius a is moving with constant velocity U through an infinite liquid at rest at infinity. If p_0 be the pressure at infinity. Shew that the pressure at any point of the surface of the sphere, the radius to which point makes an angle θ with the direction of motion is given by

$$p=p_0+\tfrac{1}{2}\rho U^2\,(1-\tfrac{9}{4})\sin^2\theta$$ {Hint : Q. No. 2}

2. Prove that the thrust on that half of the sphere on which the liquid impinges is $\pi a^2\,(\Pi-\frac{1}{16}\rho U^2)$. Where Π is the pressure at infinity, U the undisturbed velocity of the liquid and ρ the density.

3. The space between two concentric spheres of radii a and b is filled with liquid. The spheres have velocities u and v in the same direction. Find the kinetic energy of the liquid.

{Hint : Q. No. 6}

4. A spherical shell of internal radius a contains a concentric sphere of radius λa and density σ, the intervening space being

filled with water (unit density) and the whole system is at rest. If a velocity V is suddenly communicated to the shell. Prove that the initial velocity U communicated to the sphere is

$$U=\frac{3V}{3\sigma(1-\lambda^2)+1+2\lambda^3}$$

Hint : Since the motion is irrotational $\exists$ a velocity potential. The velocity potential ϕ must satisfies $\nabla^2\phi=0$ and the following boundary condition,

(i) $-\dfrac{\partial\phi}{\partial r}=V\cos\theta$ when $r=a$

(ii) $-\dfrac{\partial\phi}{\partial r}=U\cos\theta$ when $r=\lambda a$

Assuming the velocity potential be of the form

$$\phi=\left(Ar+\frac{B}{r^2}\right)\cos\theta \qquad \left\{\begin{array}{l}\text{Find } A \text{ and } B \text{ with the help}\\ \quad\text{of (i) and (ii) conditions}\end{array}\right.$$

Then $\phi=\dfrac{1}{1-\lambda^3}\left\{(\lambda^3 U-V)\,r+\dfrac{a^3\lambda^3(U-V)}{2r^2}\right\}\cos\theta$

The impulsive pressure at a point of the sphere of radius λa

$$p=-\rho\phi \text{ on the sphere } r=a\lambda.$$

Impulsive pressure on the sphere

$$=\int_0^{\pi} p\cos\theta.(2\pi\lambda a\sin\theta).\lambda a\,d\theta \quad \left\{\begin{array}{l}\text{Substitute the value of } p\\ \text{and integrating}\end{array}\right.$$

$$=-\frac{2\pi\lambda^3a^3}{3(1-\lambda^3)}\left\{2\lambda^3U+U-3V\right\}$$

The sphere of density σ starts moving with velocity U. Equation of motion is given by

$$\tfrac{4}{3}\pi(\lambda a)^3\sigma U=-\frac{2\pi\lambda^3a^3}{3(1-\lambda^3)}\left\{2\lambda^3U+U-3V\right\}$$

Find the velocity component U, which is the required result.

5. A hollow spherical shell of inner radius a contains a concentric solid uniform sphere of radius b and density σ and the space between the two is filled with liquid of density ρ. If the shell is suddenly made to move with speed u, prove that a velocity v is imparted to the inner sphere where

$$v=\frac{3ua^3}{2(\sigma/\rho)(a^3-b^3)+a^3+2b^3}.$$

§ 6·4. Equations of motion of a sphere.

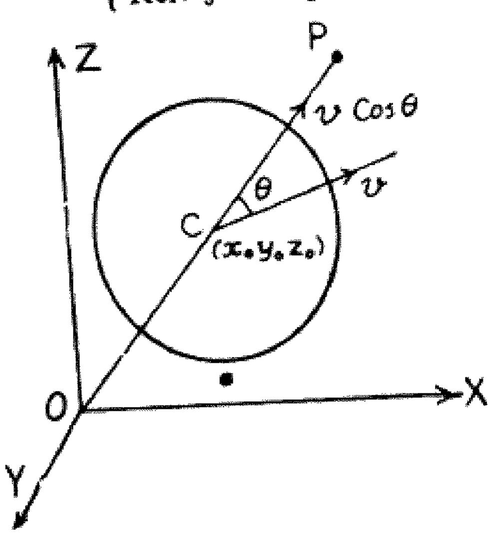
Consider the coordinates of the centre of the moving sphere referred to fixed point be $(x_0\ y_0\ z_0)$ at any instant t. Let U, V, W are the components of the velocity along the coordinate axes.

Then $U, V, W \equiv \dot{x}, \dot{y}, \dot{z}$ (velocities of the centre).

Assuming the velocity potential be of the form

$$\phi=\frac{1}{2}\frac{va^3}{r^2}\cos\theta \qquad ...(1)$$

{ Ref. § 6 1 equation (iii)



Since

$v\cos\theta$ = Resolved part of the velocity v along CP

$$=\left(U\frac{x-x_0}{r}+V\frac{y-y_0}{r}+W\frac{z-z_0}{r}\right)$$

where $\quad r^2=(x-x_0)^2+(y-y_0)^2+(z-z_0)^2$

or $\quad r\dot{r}=-(x-x_0)\,\dot{x}_0-(y-y_0)\,\dot{y}_0-(z-z_0)\,\dot{z}_0$

or $\quad r\dot{r}=-(x-x_0)\,U-(y-y_0)\,V-(z-z_0)\,W. \qquad ...(ii)$

Substituting the value of $v\cos\theta$ in (i), we get

$$\phi=\frac{1}{2}\cdot\frac{a^3}{r^2}\left\{U\frac{x-x_0}{r}+V\frac{y-y_0}{r}+W\frac{z-z_0}{r}\right\} \qquad ...(iii)$$

Now the pressure p is given by the equation, (neglecting the extraneous forces)

$$\frac{p}{\rho}=F(t)+\frac{\partial\phi}{\partial t}-\frac{1}{2}q^2 \qquad ...(iv)$$

From (iii), we have

$$\frac{\partial\phi}{\partial x}=\frac{a^3U}{2r^3}-\frac{3a^3}{2r^5}(x-x_0)\left\{U(x-x_0)+V(y-y_0)+W(z-z_0)\right\} \text{ etc.}$$

$$\left\{\text{as } \frac{\partial r}{\partial x}=\frac{x-x_0}{r}\right.$$

Since $\quad q^2=\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2+\left(\frac{\partial\phi}{\partial z}\right)^2$

$$q^2=\frac{a^6}{4r^6}(U^2+V^2+W^2)+\frac{3a^6}{4r^8}\left\{U(x-x_0)+\ldots+\ldots\right\}^2$$

and $\dfrac{\partial\phi}{\partial t}=\dfrac{a^3}{2r^3}\left\{\dot{U}(x-x_0)+\ldots+\ldots\right\}-\dfrac{a^3}{2r^3}\left\{U^2+V^2+W^2\right\}$
$$+\frac{3a^3}{2r^5}\left\{U(x-x_0)+\ldots+\ldots\right\}^2$$

Substituting the value of $\dfrac{\partial\phi}{\partial t}$ and q^2 in (iv), we have

$$\frac{p}{\rho}=F(t)+\frac{a^3}{2r^3}\left\{\dot{U}(x-x_0)+\ldots+\ldots\right\}-\frac{a^3}{2r^3}\left\{U^2+V^2+W^2\right\}$$
$$+\frac{3a^3}{2r^5}\left\{U(x-x_0)+\ldots+\ldots\right\}^2-\frac{a^6}{8r^6}(U^2+V^2+W^2)$$
$$-\frac{3a^6}{8r^8}\left\{U(x-x_0)+\ldots+\ldots\right\}^2$$

The pressure on the sphere $r=a$ is given by

$$\frac{p}{\rho}=F(t)+\tfrac{1}{2}\left\{\dot{U}(x-x_0)+\ldots+\ldots\right\}-\frac{5}{8}\left\{U^2+V^2+W^2\right\}$$
$$+\frac{9}{8a^2}\left\{U(x-x_0)+\ldots+\ldots\right\}^2 \quad \ldots(\text{v})$$

Consider f is the acceleration of the sphere and θ and θ_1 the angles that CP makes with the direction of the velocity v and acceleration f, then

$$v\cos\theta=U\frac{x-x_0}{a}+V\frac{y-y_0}{a}+W\frac{z-z_0}{a}$$

{On the surface of the sphere $r=a$

Also $f\cos\theta_1=\dot{U}\dfrac{x-x_0}{a}+\dot{V}\dfrac{y-y_0}{a}+\dot{W}\dfrac{z-z_0}{a}$

Substituting the value of $v\cos\theta$ and $f\cos\theta_1$ in (v), it reduces to

$$\frac{p}{\rho}=\frac{\Pi}{\rho}+\frac{1}{2}af\cos\theta_1-\frac{5}{8}v^2+\frac{9}{8}v^2\cos^2\theta$$

or $$\frac{p}{\rho}=\frac{\Pi}{\rho}+\frac{1}{2}af\cos\theta_1+\frac{1}{8}v^2(9\cos^2\theta-5) \quad \ldots(\text{vi})$$

{If pressure is Π at a great distance
i.e. $p=\Pi$, $r=\infty$
Then from (v) $F(t)=\dfrac{\Pi}{\rho}$

The resultant thrust on the sphere due to the motion is

$$=-\int p\cos\theta\,dS$$

$$=-\int_0^\pi \rho\left\{\frac{\Pi}{\rho}+\frac{1}{8}\,v^2\,(9\cos^2\theta-5)\right\}\cos\theta\;2\pi a\sin\theta\;ad\theta$$

$$-\int_0^\pi \tfrac{1}{2}af.\rho\cos\theta_1.\cos\theta_1.2\pi a\sin\theta_1.a\,d\theta_1$$

$$=0-\pi a^3\rho f\int_0^\pi \cos^2\theta_1\sin\theta_1\,d\theta_1$$

$$=-\tfrac{2}{3}\pi a^3\rho f \qquad \left\{\begin{array}{l}\text{Mass of the liquid displaced}\\ M'=\tfrac{4}{3}\pi a^3\rho\end{array}\right.$$

$$=-\tfrac{1}{2}M'f$$

Let $X\;Y\;Z$ are the components of resultant thrust

Then $X=-\frac{1}{2}M'\dot{U}=-\frac{2}{3}\pi a^3\rho\dot{U}$

Similarly $Y=-\frac{1}{2}M'\dot{V}$ and $Z=-\frac{1}{2}M'\dot{W}$

Again, consider $X'\;Y'\;Z'$ are the components of the extraneous force on the sphere when no liquid is present and M, the mass of the sphere of density σ. The equation of motion parallel to X-axis is given by

$$M\dot{U}=-\tfrac{1}{2}M\dot{U}+\frac{\sigma-\rho}{\sigma}X'$$

or $$M\dot{U}+\tfrac{1}{2}M'\dot{U}=\frac{\sigma-\rho}{\sigma}X'$$

or $$M\dot{U}=\frac{M}{M+\frac{1}{2}M'}\cdot\frac{\sigma-\rho}{\sigma}X'$$

or $$M\dot{U}=\frac{\frac{4}{3}\pi a^3\sigma}{\frac{4}{3}\pi a^3\sigma+\frac{1}{2}.\frac{4}{3}\pi a^3.\rho}\cdot\frac{\sigma-\rho}{\sigma}X' \qquad \left\{\begin{array}{l}\text{as } M=\frac{4}{3}\pi a^3\sigma\\ \text{and } M'=\frac{4}{3}\pi a^3\rho\end{array}\right.$$

or $$M\dot{U}=\frac{\sigma-\rho}{\sigma+\frac{1}{2}\rho}X'.$$

Thus the whole effect of the presence of the liquid is to reduce the extraneous force in the ratio $\sigma-\rho:\sigma+\frac{1}{2}\rho$.

Note : *Sometimes the above ratio is represented as* $S-1:S+\frac{1}{2}$, *where* $S=\frac{\sigma}{\rho}$ *is the* **specific gravity of the sphere** *compared with the liquid.*

Kinetic Energy.

Let T be the kinetic energy of the liquid, then

$$T=-\tfrac{1}{2}\rho\iiint q^2\,d\tau$$

(where $d\tau$ is an elementary volume)

Since $$q^2=\frac{a^6}{4r^6}\,v^2+\frac{3a^6}{4r^6}\,v^2\cos^2\theta$$

Then $T=-\frac{1}{2}\rho.\frac{1}{4}a^6v^2\int_{\theta=0}^{\pi}\int_{r=a}^{\infty}\left(\frac{1}{r^6}+\frac{3}{r^6}\cos^2\theta\right)\times 2\pi r\sin\theta . rd\theta\, dr$

$$=\frac{1}{4}\pi a^6\rho v^2.\ \frac{1}{3a^3}\int_0^{\pi}(1+3\cos^2\theta)\sin\theta\, d\theta$$

$$=\frac{1}{3}\pi\rho a^3v^2$$

$$=\frac{1}{4}M'v^2 \qquad \{\text{ where } M'=\tfrac{4}{3}\pi\rho a^3.$$

So the effect of the liquid is to increase the inertia of the system by half the mass of liquid displaced.

Ex. 8. *Prove that at a point on a sphere moving through an infinite liquid the pressure is given by the formula*

$$\frac{p-p_0}{\rho}=\tfrac{1}{2}af\cos\theta_1+\tfrac{1}{8}v^2\,(9\cos^2\theta-5)$$

where v is the velocity, f the acceleration of the sphere, and θ, θ_1 are the angles between the radius and the directions of v, f respectively and p_0 is the hydrostatic pressure.

Ref. § 6·4.

We have

$$q^2=\frac{a^6}{4r^6}(U^2+V^2+W^2)-\frac{3a^6}{2r^8}\Big\{U(x-x_0)+V(y-y_0)+W(z-z_0)\Big\}^2+\frac{9a^6}{4r^8}\Big\{U(x-x_0)+V(y-y_0)+W(z-z_0)\Big\}^2$$

or $$q^2=\frac{a^6}{4r^6}(U^2+V^2+W^2)+\frac{3a^6}{4r^8}\Big\{U(x-x_0)+V(y-y_0)+W(z-z_0)\Big\}^2$$

or $$q^2=\frac{a^6}{4r^6}v^2+\frac{3a^6}{4r^6}v^2\cos^2\theta$$

and $$\frac{\partial\phi}{\partial t}=\frac{1}{2}\frac{a^3}{r^2}f\cos\theta_1-\frac{1}{2}\frac{a^3}{r^3}(U^2+V^2+W^2)+\frac{3a^3}{2r^5}\Big\{U(x-x_0)+V(y-y_0)+W(z-z_0)\Big\}^2$$

Thus $$\frac{\partial\phi}{\partial t}=\frac{1}{2}\frac{a^3}{r^2}f\cos\theta_1-\frac{1}{2}\frac{a^3}{r^3}v^2+\frac{3a^3}{2r^3}v^2\cos^2\theta$$

The pressure equation is given by

$$\frac{p}{\rho}+\frac{1}{2}q^2-\frac{\partial\phi}{\partial t}+V=F(t) \qquad \ldots\text{(i)}$$

{where V is the potential due to the external forces.

Since p_0 is the hydrostatic pressure it follows that

$$q=0 \quad \text{and} \quad \frac{\partial \phi}{\partial t}=0.$$

From (i), we have $\frac{p_0}{\rho}+V=F(t)$ ...(ii)

From (i) and (ii), we have

$$\frac{p-p_0}{\rho}=\frac{\partial \phi}{\partial t}-\frac{1}{2}q^2$$

Substituting the value of $\frac{\partial \phi}{\partial t}$ and q^2, we get

$$\frac{p-p_0}{\rho}=\left(\frac{1}{2}\frac{a^3}{r^2}f\cos\theta_1-\frac{1}{2}\frac{a^3}{r^3}v^2+\frac{3a^3}{2r^3}v^2\cos^2\theta\right)$$
$$-\frac{1}{2}\left(\frac{a^6}{4r^6}v^2+\frac{3a^6}{4r^6}v^2\cos^2\theta\right)$$

Since the point is on the surface of the sphere *i.e.* $r=a$

$$\frac{p-p_0}{\rho}=\tfrac{1}{2}af\cos\theta_1-\tfrac{1}{2}v^2+\tfrac{3}{2}v^2\cos^2\theta-\tfrac{1}{8}v^2-\tfrac{3}{8}v^2\cos^2\theta$$

or $$\frac{p-p_0}{\rho}=\tfrac{1}{2}af\cos\theta_1+\tfrac{1}{8}v^2(9\cos^2\theta-5)$$

Proved.

Ex. 9. *When a sphere of radius a moves in an infinite liquid shew that the pressure at any point exceeds what would be the pressure if the sphere were at rest by*

$$\frac{a^3}{2r^2}f-\frac{a^3}{8r^6}(4r^3+a^3)\,q^2+\frac{3}{8}\frac{a^3}{r^6}(4r^3-a^3)\,q'^2$$

where q is the velocity of the sphere and q′ and f are the resolved parts of its velocity and acceleration in the direction of r and density of the liquid is unity.

Ref. 6·4.

Since q is the velocity of the sphere, then

$$q^2=U^2+V^2+W^2=\frac{a^6}{4r^6}q^2+\frac{3a^6}{4r^6}q'^2$$

$$q'=U\frac{x-x_0}{r}+V\frac{y-y_0}{r}+W\frac{z-z_0}{r} \qquad \left\{\begin{array}{l}\text{as } q=v\\ \text{and } q'=v\cos\theta\end{array}\right.$$

(where q' is the component velocity in the direction of r)

$$f=\dot{U}\frac{x-x_0}{r}+\dot{V}\frac{y-y_0}{r}+\dot{W}\frac{z-z_0}{r}$$

$$\frac{\partial \phi}{\partial t}=\frac{1}{2}\frac{a^3}{r^3}\left\{\dot{U}(x-x_0)+\dot{V}(y-y_0)+\dot{W}(z-z_0)\right\}$$
$$+\frac{1}{2}\frac{a^3}{r^3}\left\{U(-\dot{x}_0)+V(-\dot{y})+W(-\dot{z}_0)\right\}$$
$$+\frac{3a^3}{2r^5}\left\{U(x-x_0)+V(y+y_0)+W(z-z_0)\right\}^2$$

or $$\frac{\partial \phi}{\partial t}=\frac{1}{2}\frac{a^3}{r^2}f-\frac{1}{2}\frac{a^3}{r^3}q^2+\frac{3a^3}{2r^3}q'^2.$$

The pressure equation is given by

$$\frac{p}{\rho}+\frac{1}{\rho}q^2-\frac{\partial \phi}{\partial t}+V=F(t) \qquad \ldots(i)$$

{ $\rho=1$ (Given)

Let p_0 is the pressure when the sphere is at rest *i.e.* when the sphere is not moving.

Then $q^2=0, \frac{\partial \phi}{\partial t}=0,$

From (1) $p_0+V=F(t).$ ...(ii)

From (i) and (ii), we have

$$p-p_0=\frac{\partial \phi}{\partial t}-\frac{1}{2}q^2$$

or $$p-p_0=\frac{1}{2}\frac{a^3}{r^2}f-\frac{1}{2}\frac{a^3}{r^3}q^2+\frac{3a^3}{2r^3}q'^2-\frac{1}{2}\left(\frac{a^6}{4r^6}q^2+\frac{3a^6}{4r^6}q'^2\right)$$

$$=\frac{a^3}{2r^2}f-\frac{a^3}{8r^6}(4r^3+a^3)q^2+\frac{3}{8}\frac{a^3}{r^6}(4r^3-a^3)q'^2$$

which determines the pressure at any point. **Proved.**

Ex. 10. *A sphere of radius a is in motion in fluid, which is at rest at infinity, the pressure there being* Π; *determine the pressure at any point of the fluid, and shew that the pressure on the front hemisphere cut off by a plane perpendicular to the direction of motion is the resultant of pressure*

$$\pi a^2\left(\Pi-\frac{1}{16}\rho V^2\right) \text{ and } \tfrac{1}{3}\pi\rho a^3 f$$

in the direction respectively opposite to those of the velocity V, and the acceleration f, of the centre of the sphere.

Ref. § 6·4 equation (vi)

$$\frac{p}{\rho}=\frac{\Pi}{\rho}+\tfrac{1}{2}af\cos\theta_1+\frac{1}{8}v^2(9\cos^2\theta-5)$$

or $$\frac{p}{\rho}=\frac{1}{\rho}\left\{\Pi+\frac{1}{8}\rho V^2(9\cos^2\theta-5)\right\}+\frac{1}{\rho}\left\{\tfrac{1}{2}af\rho\cos\theta_1\right\}$$

or $$\frac{p}{\rho}=\frac{p_1}{\rho}+\frac{p_2}{\rho}$$

where $$p_1=\Pi+\frac{1}{8}\rho V^2(9\cos^2\theta-5)$$

and $$p_2=\tfrac{1}{2}af\rho\cos\theta_1$$

Let p_1 and p_2 be the pressures on the hemisphere along the direction opposite to the velocity V and acceleration f respectively.

Pressure on the hemisphere along the direction opposite to the velocity V

$$=\int_0^{\pi/2}(p_1 \cos\theta).2\pi a \sin\theta.a\,d\theta$$

$$=2\pi a^2\int_0^{\pi/2}\left\{\Pi+\frac{1}{8}\rho V^2(9\cos^2\theta-5)\right\}\sin\theta\cos\theta\,d\theta$$

$$=2\pi a^2\left[\left(\Pi-\frac{5}{8}\rho V^2\right).\tfrac{1}{2}+\frac{9}{8}\rho V^2.\tfrac{1}{4}\right]$$

$$=2\pi a^2\left[\tfrac{1}{2}\Pi-\frac{1}{32}\rho V^2\right]$$

$$=\pi a^2\left[\Pi-\frac{1}{16}\rho V^2\right]$$

which is the required result.

Also pressure on the hemisphere along the direction opposite to the acceleration f, we have

$$=\int_0^{\pi/2}(p_2\cos\theta_1).2\pi a\sin\theta_1.a\,d\theta_1$$

$$=2\pi a^2\int_0^{\pi/2}\tfrac{1}{2}a\rho f\cos\theta_1.\cos\theta_1\sin\theta_1\,d\theta_1$$

$$=\pi a^3\rho f\int_0^{\pi/2}\cos^2\theta_1\sin\theta_1\,d\theta$$

$$=\tfrac{1}{3}\pi a^3\rho f$$

which is the required result.

Ex. 11. *Prove that when the sphere is in motion with uniform velocity U, the pressure at the part of its surface where the radius makes an angle θ with the direction of motion is increased on account of the motion by the amount*

$$\frac{1}{16}\rho U^2(9\cos 2\theta-1)$$

where ρ is the density of the liquid

Assuming the velocity potential ϕ be of the form

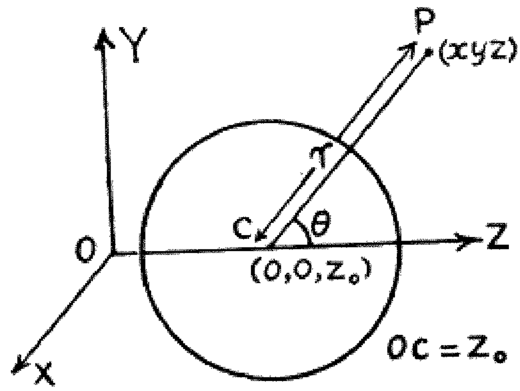
$$\phi=\tfrac{1}{2}\frac{Ua^3}{r^2}\cos\theta. \qquad \text{.. (i)}$$

Let q be the velocity of the fluid at P, then

$$q^2=\left(\frac{\partial\phi}{\partial r}\right)^2+\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)^2$$

or
$$q^2=\left(-\frac{Ua^3}{r^3}\cos\theta\right)^2+\left(-\tfrac{1}{2}\frac{Ua^3}{r^3}\sin\theta\right)^2$$
$$q^2=\frac{U^2a^6}{r^6}(\cos^2\theta+\tfrac{1}{4}\sin\theta)^2. \qquad \text{...(ii)}$$

Let the sphere of centre C be moving along the z-axis with velocity U, at any instant t.



then $\qquad U=\dot{z}_0 \qquad$ ...(iii)

and $\qquad r=CP=\sqrt{\{x^2+y^2+(z-z_0)^2\}}$

Also
$$\cos\theta=\frac{z-z_0}{\sqrt{\{x^2+y^2+(z-z_0)^2\}}}.$$

Substituting the value of $\cos\theta$ in (i), we have
$$\phi=\tfrac{1}{2}\frac{Ua^3(z-z_0)}{\{x^2+y^2+(z-z_0)^2\}^{3/2}}$$

or
$$\frac{\partial\phi}{\partial t}=\tfrac{1}{2}\frac{Ua^3(-\dot{z}_0)}{\{x^2+y^2+(z-z_0)^2\}^{3/2}}+\tfrac{1}{2}.\frac{Ua^3(-\frac{3}{2})\,2\,(z-z_0)^2(-\dot{z}_0)}{\{x^2+y^2+(z-z_0)^2\}^{5/2}}$$

or
$$\frac{\partial\phi}{\partial t}=-\tfrac{1}{2}\frac{U^2a^3}{r^3}+\frac{3}{2}\frac{U^2a^3\cos\theta}{r^3}$$

or
$$\frac{\partial\phi}{\partial t}=\frac{U^2a^3}{2r^3}\left\{(3\cos^2\theta-1)\right\}. \qquad \text{...(iv)}$$

Let p be the pressure at any point (a, θ) *i.e.* on the spherical boundary
$$\frac{p}{\rho}+\tfrac{1}{2}q^2-\frac{\partial\phi}{\partial t}=F(t)$$

or
$$\frac{p}{\rho}+\tfrac{1}{2}.\frac{U^2a^6}{r^6}(\cos^2\theta+\tfrac{1}{4}\sin^2\theta)-\frac{U^2a^3}{2r^3}(3\cos^2\theta-1)=F(t)$$

or
$$\frac{p}{\rho}+\tfrac{1}{2}U^2(\cos^2\theta+\tfrac{1}{4}\sin^2\theta)+\tfrac{1}{2}U^2(3\cos^2\theta-1)=F(t) \qquad \text{...(v)}$$
{at $r=a$

Let p_0 be the pressure on the surface of the sphere

then $\qquad p=p_0 \qquad$ when $U=0$ (*i.e.* there is no motion)

or $\qquad \dfrac{p_0}{\rho}=F(t)$ $\qquad$ {from (v)

Substituting the value of $F(t)$ in (v), we have

$$\frac{p-p_0}{\rho}=-\tfrac{1}{2}U^2(\cos^2\theta+\tfrac{1}{4}\sin^2\theta)+\tfrac{1}{2}U^2(3\cos^2\theta-1)$$

or $$\frac{p-p_0}{\rho}=\tfrac{1}{2}U^2(-\cos^2\theta-\tfrac{1}{4}\sin^2\theta+3\cos^2\theta-1)$$

or $$\frac{p-p_0}{\rho}=\frac{1}{8}U^2(-\sin^2\theta+8\cos^2\theta-4)$$

$$=\frac{1}{16}U^2(16\cos^2\theta-8-2\sin^2\theta)$$

$$=\frac{1}{16}U^2\{8(1+\cos 2\theta)-(1-\cos 2\theta)-8\}$$

$$=\frac{1}{16}U^2\{9\cos 2\theta-1\}$$

or $$p-p_0=\frac{1}{16}\rho U^2\{9\cos 2\theta-1\}.$$

Thus the pressure to the surface is increased on account of the motion by the amount.

$$\frac{1}{16}\rho U^2\{9\cos 2\theta-1\}.$$ **Proved.**

Ex. 12. *Find the pressure at any point of a liquid, of infinite extent and at rest at a great distance, through which a sphere is moving under no external forces with constant velocity U, and shew that the mean pressure over the sphere is a defect of the pressure* Π *at a great distance* $\frac{1}{4}\rho U^2$, *it being supposed that* Π *is sufficiently large for the pressure everywhere to be positive, that is, that*

$$\Pi>\frac{5}{8}\rho U^2.$$

Ref : As in Ex. 11.

$$\phi=\tfrac{1}{2}\frac{Ua^3}{r^2}\cos\theta.$$

If q be the velocity of fluid at the point P, then

$$q^2=\frac{U^2a^6}{r^6}(\cos^2\theta+\tfrac{1}{4}\sin^2\theta) \qquad ...(i)$$

and $$\frac{\partial\phi}{\partial t}=\frac{U^2a^3}{2r^3}(3\cos^2\theta-1) \qquad ...(ii)$$

The pressure p is given by the equation

$$\frac{p}{\rho}+\tfrac{1}{2}q^2-\frac{\partial\phi}{\partial t}=F(t)$$

or $$\frac{p}{\rho}+\frac{U^2a^6}{2r^6}\ (\cos^2\theta+\tfrac{1}{4}\sin^2\theta)-\frac{U^2a^3}{2r^3}\ (3\cos^2\theta-1)=F(t)$$

Since pressure is Π at a great distance

i.e. $$p=\Pi,\ r=\infty \quad \text{gives} \quad \frac{\Pi}{\rho}=F(t)$$

or $$\frac{p-\Pi}{\rho}=\tfrac{1}{2}\ \frac{U^2a^3}{r^3}\ (3\cos^2\theta-1)-\tfrac{1}{2}\ \frac{U^2a^6}{r^6}(\cos^2+\tfrac{1}{4}\sin^2\theta)$$

or $$\frac{p-\Pi}{\rho}=\tfrac{1}{2}U^2\ \{3\cos^2\theta-1-\cos^2\theta-\tfrac{1}{4}\sin^2\theta\}$$

{on the surface of the sphere $r=a$}

or $$\frac{p-\Pi}{\rho}=\frac{1}{8}\ U^2\ \{8\cos^2\theta-\sin^2\theta-4\}$$

$$=\frac{1}{8}\ U^2\ \{9\cos^2\theta-5\}.$$

or $$p=\Pi+\frac{1}{8}\ \rho U^2\ \{9\cos^2\theta-5\}. \qquad \text{...(iii)}$$

Now the mean pressure over the sphere.

$$=\frac{\int p\ dS}{\int dS}$$

$$=\frac{\int_0^\pi p\ 2\pi a\sin\theta . a\ d\theta}{\int_0^\pi 2\pi a\sin\theta\cdot a\ d\theta}$$

$$=\tfrac{1}{2}\int_0^\pi \left\{\Pi+\frac{1}{8}\ \rho U^2\ (9\cos^2\ \theta-5)\right\}\sin\theta\ d\theta$$

$$=\tfrac{1}{2}\left\{2\Pi+\frac{1}{8}\ \rho U^2\ (-4)\ \right\}$$

$$=\Pi-\tfrac{1}{4}\rho U^2$$

So $\quad \Pi-\text{mean pressure}=\tfrac{1}{4}\rho U^2$

which is the defect. **Proved.**

From (iii), minimum pressure is given

when $\qquad \cos\theta=0$

min. pressure $\qquad =\Pi-\dfrac{5}{8}\ \rho U^2.$

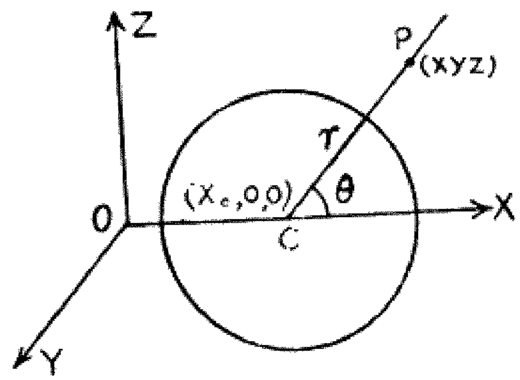
It will be positive everywhere

if $\qquad \Pi>\dfrac{5}{8}\ \rho U^2.$

Ex. 13. *A sphere of radius a is made to move to incompressible perfect fluid with non-uniform velocity u along the X-axis. If the pressure at infinity is zero. Prove that at a point x in advance of the centre*

$$p=\tfrac{1}{2}\rho a^3\left\{\frac{\dot{u}}{x^2}+u^2\left(\frac{2}{x^3}-\frac{a^3}{x^6}\right)\right\}$$

Since the sphere be moving with velocity u along the axis of X. Assuming the velocity potential ϕ be of the form



$$\phi=\tfrac{1}{2}\,\frac{ua^3}{r^2}\cos\theta \qquad \text{.. (i)}$$

where $r^2=CP^2=(X-X_0)^2+Y^2+Z^2$

and $$\cos\theta=\frac{X-X_0}{\sqrt{\{(X-X_0)^2+Y^2+Z^2\}}}$$

Substituting the value of r^2 and $\cos\theta$ in (i), we have

$$\phi=\tfrac{1}{2}\,\frac{ua^3\,(X-X_0)}{\{(X-X_0)^2+Y^2+Z^2\}^{3/2}}$$

again $$q^2=\left(\frac{\partial\phi}{\partial r}\right)^2+\left(\frac{1}{r}\cdot\frac{\partial\phi}{\partial\theta}\right)^2$$

or $$q^2=\left(-\frac{ua^3}{r^3}\cos\theta\right)^2+\left(-\tfrac{1}{2}.\frac{ua^3}{r^3}\sin\theta\right)^2$$

or $$q^2=\frac{u^2a^6}{r^6}\,(\cos^2\theta+\tfrac{1}{4}\sin^2\theta)$$

and $$\frac{\partial\phi}{\partial t}=\tfrac{1}{2}.\frac{\dot{u}a^3\,(X-X_0)}{\{(X-X_0)^2+Y^2+Z^2\}^{3/2}}$$

$$-\tfrac{1}{2}\,\frac{ua^3\dot{X}_0}{\{(X-X_0)^2+Y^2+Z^2\}^{3/2}}$$

$$+\tfrac{1}{2}\,\frac{ua^3\,(-\tfrac{3}{2}).2\,(X-X_0)^2\,(-\dot{X}_0)}{\{(X-X_0)^2+Y^2+Z^2\}^{5/2}}$$

or $$\frac{\partial\phi}{\partial t}=\tfrac{1}{2}-\frac{\dot{u}a^3}{r^2}\cos\theta-\tfrac{1}{2}\,\frac{u^2a^3}{r^3}+\tfrac{3}{2}\,\frac{u^2a^3\cos^2\theta}{r^3}$$

$\{$ since $u=\dot{X}$

If p be the pressure at any point, then

$$\frac{p}{\rho}+\frac{1}{2}q^2-\frac{\partial\phi}{\partial t}=F(t)$$

or
$$\frac{p}{\rho}+\frac{u^2a^6}{2r^6}(\cos^2\theta+\tfrac{1}{4}\sin^2\theta)$$
$$-\left(\tfrac{1}{2}\frac{\dot{u}a^3}{r^2}\cos\theta-\tfrac{1}{2}\frac{u^2a^3}{r^3}+\tfrac{3}{2}\frac{u^2a^3\cos^2\theta}{r^3}\right)=F(t)$$

Since pressure at infinity is zero

i.e. $p=0,\quad r=\infty\quad$ then $F(t)=0$

or
$$\frac{p}{\rho}=-\tfrac{1}{2}\frac{u^2a^6}{r^6}(\cos^2\theta+\tfrac{1}{4}\sin^2\theta)+\tfrac{1}{2}\frac{\dot{u}a^3}{r^2}\cos\theta$$
$$-\tfrac{1}{2}\frac{u^2a^3}{r^3}+\tfrac{3}{2}\frac{u^2a^3\cos^2\theta}{r^3}\qquad \text{...(ii)}$$

Now the point at a distance x in advance of the centre is given by substituting $\theta=0$, $r=x$ in (ii), we have

$$\frac{p}{\rho}=-\tfrac{1}{2}\frac{u^2a^6}{x^6}+\tfrac{1}{2}\frac{\dot{u}a^3}{x^2}-\tfrac{1}{2}\frac{u^2a^3}{x^3}+\tfrac{3}{2}.\frac{u^2a^3}{x^3}$$

or
$$\frac{p}{\rho}=\tfrac{1}{2}a^2\left[\frac{\dot{u}}{x^2}+u^2\left(\frac{2}{x^3}-\frac{a^3}{x^6}\right)\right]$$

or
$$p=\tfrac{1}{2}\rho a^3\left[\frac{\dot{u}}{x^3}+u^2\left(\frac{2}{x^3}-\frac{a^3}{x^6}\right)\right]$$

Which gives the pressure at a point x in advance of the centre.

Proved.

Ex. 14. *A rigid surface of radius a is moving in a straight line with velocity U and acceleration f through an infinite incompressible liquid, prove that the resultant fluid pressures over the two hemispheres, into which the sphere is divided by a diametral plane perpendicular to the direction of motion are*

$$\Pi\,\pi a^2\pm\tfrac{1}{4}Mf+\frac{3}{64}\frac{MU_2}{a},$$

where Π is the pressure at a great distance, and M is the mass of the fluid displaced by the sphere.

Assuming the velocity potential ϕ be of the from

{ Ref. Fig. Q. No. 11

$$\phi=\frac{\tfrac{1}{2}Ua^3}{r^2}\cos\theta\qquad \text{...(i)}$$

Let q be the velocity of the fluid at the point P, then

$$q^2=\left(\frac{\partial \phi}{\partial r}\right)^2+\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right)^2$$

$$q^2=\frac{U^2a^6}{r^6}\left(\cos^2\theta+\tfrac{1}{4}\sin^2\theta\right) \qquad \text{...(ii)}$$

Let the sphere of centre C (O, O, z_0) moves along Z-axis with velocity U at any instant of time t.

Now $$r^2=CP^2=x^2+y^2+(z-z_0)^2$$

and $$\cos\theta=\frac{z-z_0}{r}$$

Substituting the value of $\cos\theta$ in (i), we have

$$\phi=\tfrac{1}{2}\frac{Ua^3}{r^3}(z-z_0)$$

$$\frac{\partial \phi}{\partial t}=\frac{Ua^3}{2r^3}(-\dot{z}_0)+\frac{\dot{U}a^3}{2r^3}(z-z_0)+\frac{3U^2a^3}{2r^5}(z-z_0)^2$$

or $$\frac{\partial \phi}{\partial t}=-\frac{U^2a^3}{2r^3}+\frac{fa^3}{2r^2}\cos\theta+\frac{3U^2a^3}{2r^5}(z-z_0)^2$$

or $$\frac{\partial \phi}{\partial t}=\frac{U^2a^3}{2r^3}(3\cos^2\theta-1)+\frac{fa^3}{2r^2}\cos\theta$$

Since Π is the pressure at a great distance

i.e. $$p=\Pi,\ r=\infty$$

then pressure equation reduces to

$$\frac{p}{\rho}=\frac{\Pi}{\rho}-\tfrac{1}{2}q^2+\frac{\partial \phi}{\partial t}$$

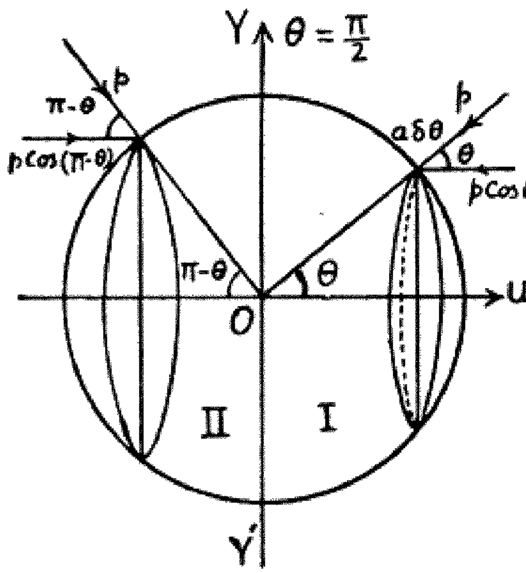
or $$\frac{p}{\rho}=\frac{\Pi}{\rho}-\frac{U^2a^6}{2r^6}\left(\cos^2\theta+\tfrac{1}{4}\sin^2\theta\right)$$
$$+\frac{U^2a^3}{2r^3}(3\cos^2\theta-1)+\frac{fa^3}{2r^2}\cos\theta$$

At the surface of the spherical boundary *i.e.* $r=a$

$$\frac{p}{\rho}=\frac{\Pi}{\rho}-\tfrac{1}{2}U^2\left(\cos^2\theta+\tfrac{1}{4}\sin^2\theta\right)$$
$$+\tfrac{1}{2}U^2(3\cos^2\theta-1)+\tfrac{1}{2}af\cos\theta$$

or $$\frac{p}{\rho}=\frac{\Pi}{\rho}+\frac{1}{8}U^2(9\cos^2\theta-5)+\tfrac{1}{2}af\cos\theta$$

Let YOY' be the diametral plane perpendicular to its direction of motion which divides it into two hemispheres.



Now pressure on the hemisphere at the I part.

$$=\int_0^{\pi/2} p\cos\theta \,.\, 2\pi a \sin\theta \,.\, a\,d\theta$$

$$=2\pi a^2\rho\int_0^{\pi/2}\left\{\frac{\Pi}{\rho}+\tfrac{1}{8}\,U^2\,(9\cos^2\theta-5)+\tfrac{1}{2}\,af\cos^2\theta\right\}\cos\theta\sin\theta\,d\theta$$

$$=2\pi a^2\rho\left\{\left(\frac{\Pi}{\rho}-\tfrac{5}{8}\,U^2\right)\frac{\sin^2\theta}{2}-\tfrac{9}{8}\,U^2\,\frac{\cos^4\theta}{4}-\tfrac{1}{2}\,af\,\frac{\cos^3\theta}{3}\right\}_0^{\pi/2}$$

$$=2\pi a^2\rho\left\{\frac{\Pi}{2\rho}+\frac{U^2}{8}\,\left(\tfrac{9}{4}-\tfrac{5}{2}\right)+\tfrac{1}{2}\,af\,.\,\tfrac{1}{3}\right\}$$

$$=\Pi\,\pi a^2+\tfrac{1}{4}\,Mf-\tfrac{3}{64}\,\frac{MU^2}{a} \qquad \left\{\text{ since } M=\tfrac{4}{3}\pi a^3\rho\right.$$

Also pressure on the hemisphere at the II part

$$=\int_{\pi/2}^{\pi} p\cos(\pi-\theta)\,.\,2\pi a\sin\theta\,.\,a\,d\theta$$

$$=-2\pi a^2\rho\int_{\pi/2}^{\pi}\left\{\frac{\Pi}{\rho}+\frac{U^2}{8}\,(9\cos^2\theta-5)+\tfrac{1}{2}\,af\cos\theta\right\}\cos\theta\sin\theta\,d\theta$$

$$=-2\pi a^2\rho\left\{\frac{\Pi}{\rho}\,.\,\tfrac{1}{2}+\frac{U^2}{8}\left(\tfrac{9}{4}-\tfrac{5}{2}\right)-\tfrac{1}{2}\,af\,.\,\tfrac{1}{3}\right\}$$

$$=\Pi\ \pi a^2-\tfrac{1}{3}\pi a^3 f-\frac{1}{16}\ \pi a^2\rho u^2$$

$$=\Pi\ \pi a^2-\tfrac{1}{4}\ Mf-\frac{3}{64}\ \frac{MU^2}{a}$$

Thus the resultant pressures over the two hemispheres is

$$\Pi\ \pi a^2\pm\tfrac{1}{4}\ Mf-\frac{3}{64}\ \frac{MU^2}{a}$$ **Proved.**

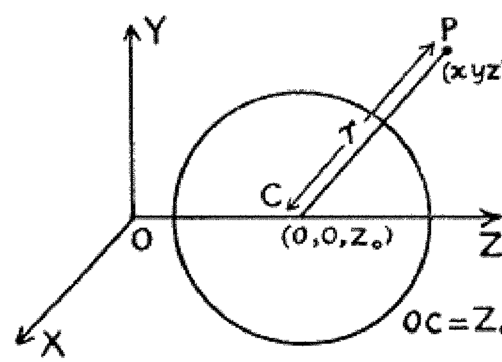
Ex. 15. *Shew that when a sphere of radius a moves with uniform velocity U through a perfect: incompressible, infinite liquid, the acceleration of a particle of liquid at (r, 0) is*

$$3U^2\left(\frac{a^3}{r^4}-\frac{a^6}{r^7}\right)$$

Assuming the velocity potential ϕ be of the form :

$$\phi=\tfrac{1}{2}\ \frac{Ua^3}{r^2}\cos\theta \qquad \ldots(i)$$

Let the sphere moves along the Z-axis with velocity U, having the coordinates of the centre as $(0, 0, z_0)$.



The relation (i) can be expressed in terms of x, y and z.

$$\phi=\tfrac{1}{2}\ \frac{Ua^3\ (z-z_0)}{\{x^2+y^2+(z-z_0)^2\}^{3/2}}$$

$$\left\{\begin{array}{l}\text{as } \cos\theta=\dfrac{z-z_0}{r}\\ \text{and}\quad r=x^2+y^2+(z-z_0)^2\end{array}\right.$$

or $$\frac{dz}{dt}=w=-\frac{\partial\phi}{\partial z}=-\tfrac{1}{2}Ua^3\left\{\frac{1}{\{x^2+y^2+(z-z_0)^2\}^{3/2}}-\frac{3\ (z-z_0)}{\{x^2+y^2+(z-z_0)^2\}^{5/2}}\right\}$$

or $$\frac{dz}{dt}=-\tfrac{1}{2}\ Ua^3\left\{\frac{1}{(z-z_0)^3}-\frac{3}{(z-z_0)^3}\right\}$$

at the centre of the sphere $C\ (0, 0, z_0)$

or $$\frac{dz}{dt}=\frac{Ua^3}{(z-z_0)^3} \qquad \ldots(ii)$$

Now to obtain the acceleration of a particle of fluid at the point $C\ (0, 0, z_0)$, differentiating (ii) with regard to t, we have

$$\frac{d^2z}{dt^2}=-\frac{3Ua^3}{(z-z_0)^4}\left\{\frac{dz}{dt}-\frac{dz_0}{at}\right\}$$

$$\left\{\text{ as } \frac{dz_0}{dt}=U \qquad \text{(Given)}\right.$$

or $$\frac{d^2z}{dt^2}=-\frac{3Ua^3}{(z-z_0)^4}\left\{\frac{Ua^3}{(z-z_0)^3}-U\right\}$$

or $$\frac{d^2z}{dt^2}=3U^2a^3\left\{\frac{1}{(z-z_0)^4}-\frac{a^3}{(z-z_0)^7}\right\}$$

or $$\frac{d^2z}{dt}=3U^2a^3\left\{\frac{1}{r^4}-\frac{a^3}{r^7}\right\} \qquad \{\text{as } r=z-z_0$$

or $$\frac{d^2z}{dt^2}=3U^2\left\{\frac{a^3}{r^4}-\frac{a^6}{r^7}\right\} \text{ at } (r, 0)$$

Which determines the acceleration of a particle of fluid at the point $(r, 0)$. **Proved.**

Ex. 16. *An infinite homogeneous liquid is flowing steadily past a rigid boundary consisting partly of the horizontal plane $y=0$, and partly of a hemispherical boss $x^2+y^2+z^2=a^2$, with irrotational motion which tends, at a great distance from the origin, to uniform velocity V parallel to the axis of z. Find the velocity potential and the surfaces of equal pressure.*

Assuming the velocity potential for the liquid streaming past a fixed sphere of radius a with velocity V parallel to axis of z.

$$\phi=Vr\cos\theta+\tfrac{1}{2}\frac{Va^3}{r^2}\cos\theta \qquad \{\text{Ref § 6·2}$$

Consider Y-axis be vertical then the velocity perpendicular to the plane $y=0$ vanishes. It follows that **the plane y=0 is a stream surface**, and the hemisphere also be a stream surface.

Thus the velocity potential ϕ for the hemispherical boss $x^2+y^2+z^2=a^2$ on the horizontal base $y=0$ is given by,

$$\phi=Vr\cos\theta+\tfrac{1}{2}\frac{Va^3}{r^2}\cos\theta \qquad \text{...(i)}$$

The pressure p is given by the equation

$$\frac{p}{\rho}+\tfrac{1}{2}q^2=\text{constant} \qquad \text{...(ii)}$$

Since the velocity V is constant (being uniform), the motion is steady i.e. $\frac{\partial\phi}{\partial t}=0$.

The surfaces of equal pressure can be obtained by

$$p=\text{constant}$$

Then from (ii), we have

$$q^2=\text{constant}$$

or $\quad \left(\dfrac{\partial\phi}{\partial r}\right)^2+\left(\dfrac{1}{r}\,\dfrac{\partial\phi}{\partial\theta}\right)^2=\text{const.}$

or $\quad \left\{V\left(1-\dfrac{a^3}{r^3}\right)\cos\theta\right\}^2+\left\{-V.\dfrac{1}{r}\left(r+\dfrac{a^3}{2r^2}\right)\sin\theta\right\}^2=\text{const.}$

or $\quad V^2\left\{\left(1-\dfrac{a^3}{r^3}\right)^2\cos^2\theta+\left(1+\dfrac{a^3}{2r^3}\right)^2\sin^2\theta\right\}=\text{const.}$

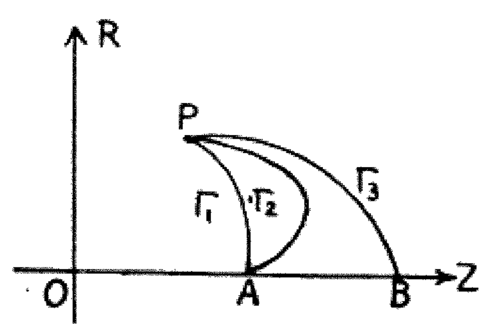
or $\quad \left(1-\dfrac{a^3}{r^3}\right)^2\cos^2\theta+\left(1+\dfrac{a^3}{2r^3}\right)^2\sin^2\theta=\text{const.}$

{as V=constant being uniform}

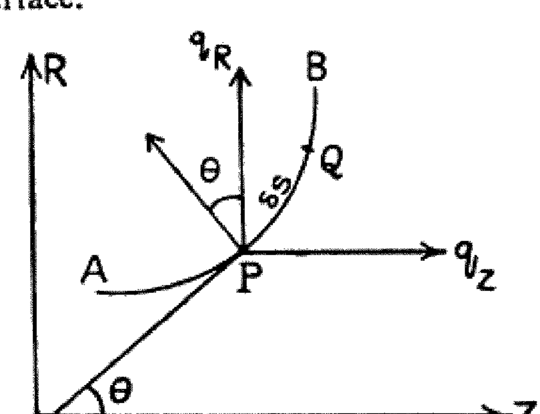
Which determines the surfaces of equal pressure.

§ 6·5. **Motion symmetrical about an axis. Stoke's stream function.**

So far we have discussed two-dimensional motion in terms of a single complex variable and a complex potential. Here we shall consider the motion in three dimensions *i.e.* the cases in which the motion remains same in every plane through a certain line. This line is called **axis.** Such types of motion occur, *e.g.* when a solid of revolution moves in the direction of its axis of revolution in a liquid otherwise at rest known as **axis-symmetrical** motion.



Consider an arbitrary point P and a fixed point A on the axis of symmetry. Join the point P to A by curves $A\Gamma_1P$ and $A\Gamma_2P$ lying in the **same plane** through the axis. This is known as **meridian plane.** The position of the point can be determined by **the coordinates** (Z R) (in cylindrical form). By rotating the meridian curves about the axis of symmetry, we obtain a closed surface.



Consider the Z-axis be taken as axis of symmetry. Let the coordinates of the point P be referred as (R, θ, Z) {in cylindrical coordinates} in the fluid and q_R and q_Z denote the components of velocity in the direction of axes of R and Z.

The equation of continuity for an incompressible flow is given by

$$\text{div. } \mathbf{q}=0$$

or
$$\frac{\partial}{\partial R}(Rq_R)+\frac{\partial}{\partial z}(Rq_z)=0 \qquad \ldots(i)$$

{The functions at the point P associated with the flow are independent of θ.

Now the condition that

$$Rq_R\, dz-Rq_z\, dR$$

may be an exact differential, let it be equal to $d\psi$.

$$Rq_R\, dz-Rq_z\, dR=d\psi$$

or
$$Rq_R\, dz-Rq_z\, dR=\frac{\partial\psi}{\partial z}dz+\frac{\partial\psi}{\partial R}dR.$$

Equating the coefficients of dz and dR, we have

$$Rq_R=\frac{\partial\psi}{\partial z} \text{ and } -Rq_z=\frac{\partial\psi}{\partial R}$$

or
$$\mathbf{q}_R=\frac{1}{\mathrm{R}}\frac{\partial\psi}{\partial \mathrm{z}} \text{ and } \mathbf{q}_z=-\frac{1}{\mathrm{R}}\frac{\partial\psi}{\partial \mathrm{R}} \qquad \ldots(ii)$$

By substituting the value of q_R and qz in (i) we see that the relation is identical. Such a function ψ is called the **Stoke's stream function**.

The stream lines are given by the equation

$$\frac{dR}{q_R}=\frac{dz}{q_z}$$

or
$$q_R\, dz-q_z\, dR=0$$

or
$$Rq_R\, dz-Rq_z\, dR=0$$

or
$$d\psi=0$$

or
$$\psi=\text{constant}.$$

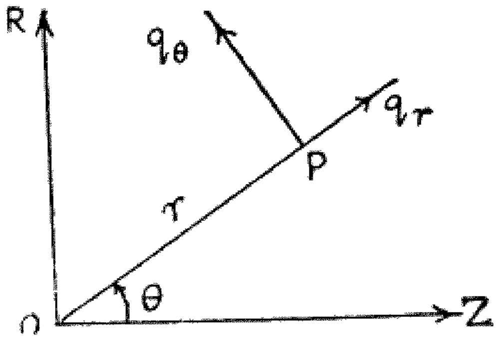
i.e. across such a line there is no flow *i.e.* no fluid crosses a stream line ψ=constant.

Further, we can also represent the velocity components in terms of polar coordinates. Consider q_r and q_θ be the velocity components of the fluid at the point P along OP and perpendicular to OP respectively. (In the sense of O increasing).

Here, let $\delta s=\delta r$, then the area of the surface of revolution of δs about OZ is $2\pi r \sin\theta\, \delta r$.

Also the volume of fluid crossing from right to left

$$=2\pi r \sin\theta\, \delta r . q_\theta \text{ per uint it time}$$



If ψ be the stream function at P,

then $\quad 2\pi\delta\psi = 2\pi r \sin\theta\, \delta r\, q_\theta$

or $$q_\theta = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}$$

Consider $\delta s = r\delta\theta$, then the area of the surface of revolution about OZ

$$= 2\pi r \sin\theta . r\delta\theta$$

The volume of the fluid crossing from right to left

$$= 2\pi r \sin\theta . r\delta\theta\,(-q_r)$$

or $$2\pi\delta\psi = -2\pi r\sin\theta\cdot r\delta\theta . q_r$$

or $$q_r = -\frac{1}{r^2\sin\theta}\cdot\frac{\partial\psi}{\partial\theta}$$

§ 6 51. Property of stoke's stream function.

A property of stoke's stream function is that 2π times the difference of its values at two points in the same meridian plane is equal to the flow across the annular surface obtained by the revolution round the axis of a curve joining the points.

The flow outwards across the surface of revolution

$$= \int_A^B (q_R\cos\theta - q_z\sin\theta)\, 2\pi R ds$$

$$= 2\pi \int \frac{\partial\psi}{\partial R} dR + \frac{\partial\psi}{\partial z} dz$$

$$= 2\pi \int_{\psi_1}^{\psi_2} d\psi$$

$$= 2\pi\,(\psi_3 - \psi_1)$$

which proves the property.

Again, we can also define the value of the stoke's stream function at any point P as $\frac{1}{2\pi}$ of the amount of flow across a surface obtained by revolving a curve $A\Gamma_1$ P round the symmetrical axis.

i.e. $$\psi = \frac{1}{2\pi}\int_A^P (q_R\cos\theta - q_z\sin\theta)\, 2\pi R ds$$

or $$\psi = \int_A (Rq_R\, dz - Rq_z\, dR)$$

Which gives

$$q_R = \frac{1}{R}\frac{\partial\psi}{\partial z} \quad \text{and} \quad q_z = -\frac{1}{R}\frac{\partial\psi}{\partial R} \qquad \text{.. (i)}$$

§ 6·6. Irrotational Motion.

When the motion is irrotational, a velocity potential ϕ always exists *i.e.* we have

$$q_R=-\frac{\partial\phi}{\partial R} \text{ and } q_z=-\frac{\partial\phi}{\partial z} \quad \text{.. (i)}$$

and also

$$q_R=-\frac{1}{R}\frac{\partial\psi}{\partial z} \text{ and } q_z=-\frac{1}{R}\frac{\partial\psi}{\partial R} \quad \text{...(ii)}$$

{Ref. § 6 5 eqn. (ii)

From (i) and (ii), we have

$$\frac{\partial\phi}{\partial R}=-\frac{1}{R}\frac{\partial\psi}{\partial z} \text{ and } \frac{\partial\phi}{\partial z}=\frac{1}{R}\frac{\partial\psi}{\partial R} \quad \text{...(iii)}$$

Since $$\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial R}\right)=\frac{\partial}{\partial R}\left(\frac{\partial\phi}{\partial z}\right)$$

or $$\frac{\partial}{\partial z}\left(-\frac{1}{R}\frac{\partial\psi}{\partial z}\right)=\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right)$$

or $$\frac{\partial^2\psi}{\partial R^2}+\frac{1}{R}\frac{\partial\psi}{\partial R}+\frac{\partial^2\psi}{\partial z^2}=0 \quad \text{...(iv)}$$

Again eliminating ψ from (iii), we have

$$\frac{\partial^2\phi}{\partial R^2}+\frac{\partial^2\phi}{\partial z^2}+\frac{1}{R}\frac{\partial\phi}{\partial R}=0 \quad \text{...(v)}$$

The relation (iv) and (v) shows that ψ is not a harmonic function.

The **equation (v) is the Laplace's equation in cylindrical coordinates** when **there is a symmetry about the axis**, on the other hand the stream function ψ does not satisfy the Laplace's Equation

§ 6·61 Polar coordinates (Irrotational motion).

Consider q_r and q_θ be the fluid velocity components at the point P along OP and perpendicular to OP and ψ be the stoke's stream function, then

{Ref. § 6·5

$$q_r=-\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}$$

and $$q_\theta=-\frac{1}{r\sin\theta}\cdot\frac{\partial\psi}{\partial r} \quad \text{...(i)}$$

Since the motion is irrotational, it follows that $\exists$ a velocity potential ϕ, then we have

$$q_r=-\frac{\partial\phi}{\partial r} \text{ and } q_\theta=-\frac{1}{r}\frac{\partial\phi}{\partial\theta} \quad \text{...(ii)}$$

From (i) and (ii), we have

$$\frac{1}{r^2 \sin\theta}\frac{\partial\psi}{\partial\theta}=-\frac{\partial\phi}{\partial r} \quad \text{...(iii)}$$

and

$$\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}=-\frac{1}{r}\frac{\partial\phi}{\partial\theta} \quad \text{...(vi)}$$

Since

$$\frac{\partial^2\phi}{\partial\theta\partial r}=\frac{\partial^2\phi}{\partial r\partial\theta}$$

or

$$\frac{\partial}{\partial\theta}\left(\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}\right)+\frac{\partial}{\partial r}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial r}\right)=0$$

or

$$\frac{1}{r^2}\cdot\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{\sin\theta}\cdot\frac{\partial^2\psi}{\partial r^2}=0$$

or

$$r^2\frac{\partial^2\psi}{\partial r^2}+\sin\theta\,\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right)=0 \quad \text{...(v)}$$

Suppose $\cos\theta=\mu$

or

$$\sin\theta\frac{\partial}{\partial\mu}=-\frac{\partial}{\partial\theta} \quad \text{...(vi)}$$

From (v) and (vi), we have

$$r^2\frac{\partial^2\psi}{\partial r^2}+\sin^2\theta\,\frac{\partial}{\partial\mu}\left(\frac{\partial\psi}{\partial\mu}\right)=0$$

or

$$r^2\frac{\partial^2\psi}{\partial r^2}+(1-\cos^2\theta)\frac{\partial}{\partial\mu}\left(\frac{\partial\psi}{\partial\mu}\right)=0.$$

or

$$r^2\frac{\partial^2\psi}{\partial r^2}+(1-\mu^2)\frac{\partial^2\psi}{\partial\mu^2}=0 \quad \text{...(vii)}$$

Also, eliminating the stream function ψ from (iii) and (iv), we have

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)+\frac{1}{\sin\theta}\cdot\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\phi}{\partial\theta}\right)=0$$

or

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial\phi}{\partial r}\right)+\frac{\partial}{\partial\mu}\left\{(1-\mu^2)\frac{\partial\phi}{\partial\mu}\right\}=0 \quad \text{...(viii)}$$

Which is a form of Laplace's equation and has the solution of the form

$$r^n P_n(\mu)=\phi_1 \text{ (say) and } r^{-n-1}P_n(\mu)=\phi_2 \text{ (say)}$$

Let ψ_1 and ψ_2 are the solutions of equation (vii), then from (iii) and (iv), we have

or

$$\frac{\partial\psi_1}{\partial\mu}=-r^2\frac{\partial\phi_1}{\partial r}$$

$$=-r^2\frac{\partial}{\partial r}\left\{r^n P_n(\mu)\right\}$$

$$=-n\,r^{n+1}P_n(\mu)$$

and $$\frac{\partial\psi_2}{\partial\mu}=-r^2\frac{\partial\phi_2}{\partial r}$$

$$=-r^2\frac{\partial}{\partial r}\left\{r^{-n-1}P_n(\mu)\right\}$$

$$=(n+1)r^{-n}P_n(\mu)$$

Also $$\frac{\partial\psi_1}{\partial r}=(1-\mu^2)\frac{\partial\phi_1}{\partial\mu}$$

$$=(1-\mu^2)\frac{\partial}{\partial\mu}\left\{r^nP_n(\mu)\right\}$$

$$=(1-\mu^2)r^n.\frac{\partial}{\partial\mu}\left\{P_n(\mu)\right\} \quad \ldots\text{(ix)}$$

and $$\frac{\partial\psi_2}{\partial r}=(1-\mu^2)\frac{\partial\phi_2}{\partial\mu}$$

$$=(1-\mu^2)\frac{\partial}{\partial\mu}\left\{r^{-n-1}P_n(\mu)\right\}$$

$$=(1-\mu^2)r^{-n-1}\frac{\partial}{\partial\mu}\left\{P_n(\mu)\right\} \quad \ldots\text{(x)}$$

By integrating (ix) and (x), we have

$$\psi_1=\frac{1-\mu^2}{n+1}r^{n+1}\frac{\partial}{\partial\mu}\left\{P_n(\mu)\right\}$$

and $$\psi_2=-\frac{1-\mu^2}{n}r^{-n}\frac{\partial}{\partial\mu}\left\{P_n(\mu)\right\}$$

Which gives the possible solution for the stream function ψ.

§ 6·7. **Values of Stoke's Stream function.**

Case I. **A Simple Source on the axis of X.**

A Simple Source is a point of outward radial flow symmetrically from a point.

We know that

$$\phi=\frac{m}{r} \quad \ldots\text{(i)}$$

{ Ref. § 3·4

But $$\frac{\partial\psi}{\partial\mu}=-r^2\frac{\partial\phi}{\partial r}$$

or $$\frac{\partial\psi}{\partial\mu}=-r^2.\left(-\frac{m}{r^2}\right) \quad \text{\{ from (i)}$$

or $$\frac{\partial\psi}{\partial\mu}=m$$

or $$d\psi=m\,d\mu$$

By integrating, we have

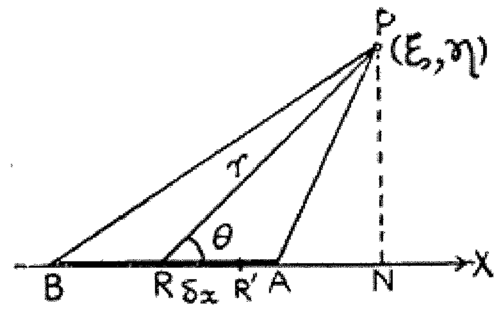
$$\psi = m\mu$$

ro $\psi = m\cos\theta$ $\{$ as $\mu = \cos\theta$

Thus $\psi(r,\theta) = m\cos\theta$

Case II. **A uniform line source along the axis.**

Consider BX be a uniform line source of fluid extending from B to X. Let RR' be a small element of length δx and RP equal to r. If $+m$ is the strength per unit length of the line source then the element RR' is effectively a simple source at R of strength $+m\delta x$.



Therefore $\delta\psi = m\delta x\cos\theta$

or $$\psi = \int_0^{BA} m\cos\theta . dx$$

or $$\psi = \int_0^{BA} \frac{m(\xi - x)}{\sqrt{\{(\xi-x)^2+\eta^2\}}} . dx$$

or $$\psi = -m\left[\left\{(\xi-x)^2+\eta^2\right\}^{1/2}\right]_0^{BA}$$

or $$\psi = m\left[\left(\xi^2-\eta^2\right)^{1/2} - \left\{(\xi-BA)^2+\eta^2\right\}^{1/2}\right]$$

or $$\psi = m\{BP - AP\}$$

Which shows that the stream surfaces ψ=constant are given by

$$BP - AP = \text{constant}$$

represents the confocal hyperboloids of revolution about BX having B and A as focii.

Case III. **A doublet along the X-axis.**

We know that

$$\phi = \frac{M\cos\theta}{r^2} \quad \text{.. (i)}$$

But $$\frac{\partial\psi}{\partial r} = (1-\mu^2)\frac{\partial\phi}{\partial\mu}$$ $\{$ Ref. § 6·61 .. (ii)

as $\mu = \cos\theta$ (say)

Then $$\phi = \frac{M\mu}{r^2}$$ $\{$ from (i)

or $$\frac{\partial\phi}{\partial\mu} = \frac{M}{r^2}$$

From (ii) we have

$$\frac{\partial\psi}{\partial r}=\frac{(1-\mu^2)\,M}{r^2}$$

By integrating, we get

$$\psi=-\frac{(1-\mu^2)\,M}{r}=-\frac{M\sin^2\theta}{r}$$

or $$\psi\,(r,\,\theta)=-\left(\frac{M}{r}\right)\sin^2\theta$$

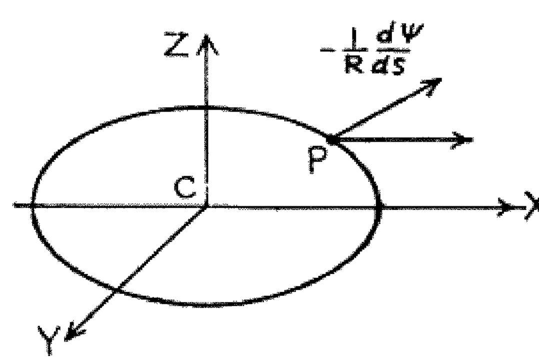
§ 6·8. Solid of revolution moving along their axes in an infinite mass of liquid.

Consider V is the velocity about the axis of revolution-X and ds an element of the meridian curve. Since the motion being symmetrical about X-axis, the Stoke's stream function ψ exists.

Normal velocity at any point$=V\,\dfrac{\partial R}{\partial S}$

and normal velocity of the liquid in contact with the surface

$$=-\frac{1}{R}\,\frac{\partial\psi}{\partial S}$$



Thus $$-\frac{1}{R}\,\frac{\partial\psi}{\partial s}=V\,\frac{\partial R}{\partial s}$$

or $$d\psi=-VR\,dR$$

By integrating, we have

$$\psi=-\tfrac{1}{2}VR^2+A$$

or $$\psi=-\tfrac{1}{2}\,Vr^2\sin^2\theta+A$$

$$\begin{cases}\text{Since} & R=r\sin\theta\\ \text{where} & \mu=\cos\theta\end{cases}$$

or $$\psi=-\tfrac{1}{2}\,Vr^2\,(1-\cos^2\theta)+A$$

or $$\psi=-\tfrac{1}{2}\,Vr^2\,(1-\mu^2)+A \qquad \text{...(i)}$$

Which is the boundary condition.

But the stream function ψ has to satisfy the equation

$$r^2\,\frac{\partial^2\psi}{\partial r^2}+(1-\mu^2)\,\frac{\partial^2\psi}{\partial\mu^2}=0$$

{ Ref. § 6·61 Equation (vii)

Whose solutions are of the from,

$$\frac{1-\mu^2}{n+1}\,r^{n+1}\,\frac{\partial}{\partial\mu}\left\{P_n\,(\mu)\right\}\text{ and }\frac{1-\mu^2}{nr^n}\,\frac{\partial}{\partial\mu}\left\{P_n\,(\mu)\right\}$$

Case of a sphere. Let the radius of the sphere be a, then from equation, (i), we have

$$\psi=-\tfrac{1}{2}\,Va^2\,(1-\mu^2)+A$$

Suppose $n=1$, then we have a solution of ψ of the form

$$\psi=A\,\frac{1-\mu^2}{r} \qquad \left\{\text{as } \frac{\partial}{\partial\mu}\left\{P_n\right\}=\text{constant}=1\right.$$

On the boundary, we have

$$A\,\frac{1-\mu^2}{a}=-\tfrac{1}{2}\,Va^2\,(1-\mu^2)+B \qquad \forall \text{ values of } \mu.$$

or $$A=-\tfrac{1}{2}\,Va^3, \quad B=0$$

Thus $$\psi=-\tfrac{1}{2}\,Va^3.\frac{1-\mu^2}{r}$$

or $$\psi=-\tfrac{1}{2}\,\frac{Va^3}{r}\sin^2\theta \qquad \text{...(ii)}$$

Again we know that

$$(1-\mu^2)\,\frac{\partial\phi}{\partial\mu}=\frac{\partial\psi}{\partial r}$$

or $$(1-\mu^2)\,\frac{\partial\phi}{\partial\mu}=\tfrac{1}{2}\,\frac{Va^3}{r^2}\sin^2\theta \qquad \{\text{from (ii)}$$

or $$\frac{\partial\phi}{\partial\mu}=\tfrac{1}{2}\,\frac{Va^3}{r^2}$$

By integrating, we have

$$\phi=\tfrac{1}{2}\,\frac{Va^3}{r^2}\,\mu$$

or $$\phi=\tfrac{1}{2}\,\frac{Va^3}{r}\cos\theta$$

Which is the same as the velocity potential obtained, if a sphere is moving in a liquid at rest at infinity. { Ref. § 6·1

It follows that the sphere of radius a moving with velocity V has the same effect as a doublet of strength $\frac{1}{2}Va^3$ at its centre.

Ex. 17. *If AB be a uniform line source, and A and B equal sinks of such strength that there is no total gain or loss of fluid. Shew that the stream function at any point is*

$$\lambda\left\{(r_1-r_2)^2-c^2\right\}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)$$

where c is the length of AB, r_1 and r_2 are the distances of the point considered from A and B.

Consider m be the strength per unit length and c the length of the line source. Then the total strength of the line source is $+mc$.

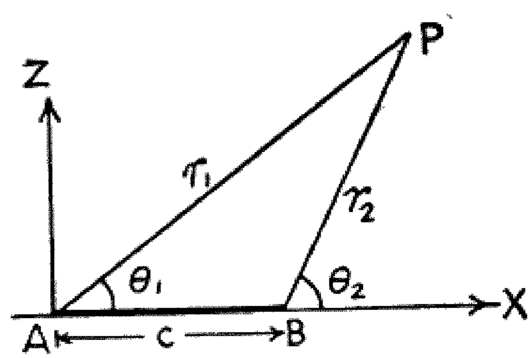
Let $\angle PAB=\theta_1$

and $\angle PBX=\theta_2$

then $\cos\theta_1=\dfrac{c^2+r_1^2-r_2^2}{2cr_1}$

and $\cos(\pi-\theta_2)=\dfrac{c^2+r_2^2-r_1^2}{2cr_2}$

or $\cos\theta_2=-\dfrac{c^2+r_2^2-r_1^2}{2cr_2}$



Now to neutralize the gain of fluid due to the line source, there must be two sinks of strength $-\frac{1}{2}mc$ each at A and B. If ψ is the stoke's stream function at P,

then $$\psi=m(r_1-r_2)-\tfrac{1}{2}mc\cos\theta_1-\tfrac{1}{2}mc\cos\theta_2$$

or $$\psi=-m\,r_1r_2\left(\frac{1}{r_1}-\frac{1}{r_2}\right)-\frac{m}{4}\left\{\frac{c^2+r_1^2-r_2^2}{r_1}+\frac{r_1^2-r_2^2-c^2}{v_2}\right\}$$

or $$\psi=-m\,r_1r_2\left(\frac{1}{r_1}-\frac{1}{r_2}\right)-\frac{m}{4}\left\{c^2\left(\frac{1}{r_1}-\frac{1}{r_2}\right)+(r_1-r_2)-\frac{r_2^2}{r_1}+\frac{r_1^2}{r_2}\right\}$$

or $$\psi=-m\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\left(r_1r_2+\frac{1}{4}c^2\right)-\frac{m}{4}\left\{(r_1-r_2)-\left(\frac{r_2^2}{r_1}-\frac{r_1^2}{r_2}\right)\right\}$$

$$=m\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\left\{-r_1r_2-\tfrac{1}{4}c^2+\tfrac{1}{4}(r_1+r_2)^2\right\}$$

$$=\tfrac{1}{4}m\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\left\{(r_1-r_2)^2-c^2\right\}$$

$$=\lambda\left\{(r_1-r_2)^2-c^2\right\}\left(\frac{1}{r_1}-\frac{1}{r_2}\right)\qquad\text{where } \lambda=\frac{m}{4}$$

Proved.

Ex. 18. *Discuss the motion for which Stoke's stream function is given by*

$$\psi=\tfrac{1}{2}V\{a^4r^{-2}\cos\theta-r^2\}\sin^2\theta$$

where r is the distance from a fixed point and θ is the angle this distance makes with a fixed direction.

Since $\psi=\frac{1}{2}V\,\{a^4 r^{-2}\cos\theta-r^2\}\sin^2\theta$

or $$\psi=\frac{1}{2}\frac{Va^4}{r^2}\sin^2\theta\cos\theta-\frac{1}{2}Vr^2\sin^2\theta \quad \text{...(i)}$$

Consider the liquid streaming past a fixed surface of revolution with velocity V along the axis of X. Then the stream function ψ, for this motion is

$$-\frac{1}{R}\frac{\partial\psi}{\partial R}=V$$

or $$d\psi=-VR\,dR$$

By intergrating, we have

$$\psi=\frac{1}{2}VR^2+\text{const.}$$

or $$\psi=-\frac{1}{2}VR^2 \qquad \{\text{const vanish as } \psi \text{ and } R \text{ vanish}\}$$

or $$\psi=-\frac{1}{2}Vr^2\sin^2\theta \qquad \text{(ii)}$$

$\{$as $R=r\sin\theta$

So the second term of (i) is due to the liquid streaming past a surface of revolution along the positive direction, which will give rise to the stream function

$$\psi=\frac{1}{2}\frac{Va^4}{r^2}\sin^2\theta\cos\theta \quad \text{...(iii)}$$

the surface of revolution moving in a liquid at rest at infinity.

But if a solid of revolution moves in an infinite liquid at rest at infinity parallel to X-axis (in negative direction) with velocity V then on the surface, we have

$$\psi=\frac{1}{2}Vr^2\sin^2\theta+\text{const.} \quad \text{...(iv)}$$

equation (iii) can be written as

$$\psi=\frac{1}{2}V\left(\frac{a^4}{r^4}\cos\theta\right)r^2\sin^2\theta \quad \text{...(v)}$$

From (iv) and (v), we have at the boundary

$$\frac{a^4}{r^4}\cos\theta=\text{const}=1 \quad (\text{say})$$

or $$r^4=a^4\cos\theta$$

Thus the motion is due to liquid streaming past the solid of revolution $r^4=a^4\cos\theta$ with velocity V along the X-axis in positive direction.

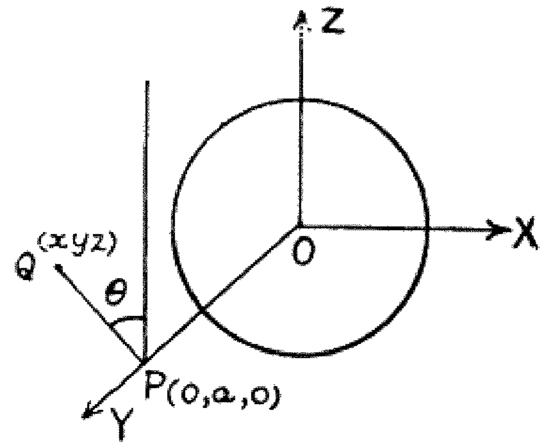
Ex. 19. *A doublet of strength M is placed at the point $(0, a, 0)$ with its axis parallel to the axis of z. Prove that at points close to the origin the velocity potential of the doublet is approximately*

$$\frac{Mz}{a^3}+\frac{3Myz}{a^4}$$

neglecting terms of the order $\frac{r^3}{a^3}$ *and higher powers. Deduce that if a small sphere of radius c is placed with its centre at the origin, the velocity potential is then increased by the terms.*

$$\frac{1}{2}\frac{Mc^3}{a^3}\cdot\frac{z}{r^3}+2\frac{Mc^5}{a^4}\cdot\frac{yz}{r^5}$$

A doublet of strength M is placed at the point P $(0, a, 0)$ with its axis parallel to the axis of Z. Consider a point P $(x\ y\ z)$ in the liquid. Let PQ makes an angle θ with a line parallel to Z-axis



or $\quad \cos\theta=\frac{z}{\sqrt{\{x^2+(y-a)^2+z^2\}}}$

Now the velocity potential of the doublet is given by

$$\phi=\frac{M\cos\theta}{PQ^2}$$

or
$$\phi=\frac{M.z}{\{x^2+(y-a)^2+z^2\}^{3/2}}$$

or
$$\phi=\frac{M.z}{\{r^2-2ay+a^2\}^{3/2}}\qquad \text{where } r^2=x^2+y^2+z^2$$

or
$$\phi=\frac{Mz}{a^3}\left\{1-\frac{2y}{a}+\frac{r^2}{a^2}\right\}^{-3/2}$$

or
$$\phi=\frac{Mz}{a^3}\left(1+\frac{3}{2}\cdot\frac{2y}{a}+\ldots\right)$$

(neglecting other terms of the order $\frac{r^3}{a^3}$ and higher powers)

or
$$\phi=\frac{Mz}{a^3}+\frac{3Myz}{a^4}$$

which proves the first part.

Again a small sphere of radius c is placed with its centre at the origin. Let ϕ' be the velocity potential at any point.

The velocity however remains unchaged at infinity. Assuming that the increase in the velocity potential be

$$\frac{Az}{r^3}+\frac{Byz}{r^5}$$

then $$\phi'=\frac{Mz}{a^3}+\frac{3Myz}{a^4}+\frac{Az}{r^3}+\frac{Byz}{r^5}$$
$$=\frac{Mr\cos\theta}{a^3}+\frac{3Mr^2\sin\theta\cos\theta\sin\omega}{a^4}+\frac{A\cos\theta}{r^2}$$
$$+\frac{B\sin\theta\cos\theta\sin\omega}{r^3}$$

or $$\frac{\partial\phi'}{\partial r}=\frac{M\cos\theta}{a^3}+\frac{6Mr\sin\theta\cos\theta\sin\omega}{a^4}-\frac{2A\cos\theta}{r^3}$$
$$-\frac{3B\sin\theta\cos\theta\sin\omega}{r^4}$$

But the velocity potential ϕ is such that
$$\frac{\partial\phi'}{\partial r}=0 \text{ at } r=c.$$

or $$\frac{M\cos\theta}{a^3}-\frac{2A\cos\theta}{c^3}+\frac{6Mc\sin\theta\cos\theta\sin\omega}{a^4}$$
$$-\frac{3B\sin\theta\cos\theta\sin\omega}{c^4}=0$$

or $$\left(\frac{M}{a^3}-\frac{2A}{c^3}\right)\cos\theta+\left(\frac{6Mc}{a^4}-\frac{3B}{c^4}\right)\sin\theta\cos\theta\sin\omega=0.$$

i.e. $$\frac{M}{a^3}-\frac{2A}{c^3}=0 \text{ and } \frac{6Mc}{a^4}-\frac{3B}{c^4}=0$$

or $$\text{A}=\frac{Mc^3}{2a^3} \text{ and } B=\frac{2Mc^5}{a^4}$$

Thus the increase in velocity potential ϕ is
$$=\frac{Az}{r^3}+\frac{Byz}{r^5}$$
$$=\frac{Mc^3z}{2a^3r^3}+\frac{2Mc^5\,yz}{a^4\,r^5}$$
$$=\tfrac{1}{2}\frac{Mc^3}{a^3}\cdot\frac{z}{r^3}+2\,\frac{Mc^5}{a^4}\cdot\frac{yz}{r^5}$$
Proved.

Exr 2. *A solid of revolution is moving along its axis in an infinite liquid, shew that the kinetic energy of the liquid is*
$$-\tfrac{1}{2}\pi\rho\int\frac{\psi}{\varpi}\frac{\partial\psi}{\partial n}\,ds$$
where ψ is the stoke's stream function of the time, ϖ the distance of a point from the axis and the integral is taken once round a meridian curve of the solid. Hence obtain the kinetic energy of infinite liquid due to the motion of a sphere through it with velocity V.

We know that the kinetic energy in an irrotational motion is given by

$$T=-\tfrac{1}{2}\rho\int\phi\,\frac{\partial\phi}{\partial n}\,dS \qquad \text{...(i)}$$

Consider an elementary arc ds of the meridian curve of the boundary. Since the solid of revolution moves about the axis, then

$$dS=2\pi\varpi\,ds \qquad \text{...(ii)}$$

or $$-\frac{\partial\phi}{\partial n}=\text{outward normal velocity}=\frac{1}{\varpi}\,\frac{\partial\psi}{\partial s} \qquad \text{...(iii)}$$

from (i), we have

$$T=-\tfrac{1}{2}\rho\int\phi\left\{-\frac{1}{\varpi}\,\frac{\partial\psi}{\partial s}\right\}.2\pi\varpi ds$$

$$T=\pi\rho\int\phi\,d\psi$$

By integrating the function, we have

$$T=\pi\rho\,\{\phi\psi-\int\psi\,d\phi\}$$

$$=-\pi\rho\int\psi\,d\phi \qquad \text{...(iii)}$$

$\left\{\right.$ The first term $\pi\rho\phi\psi$ vanishes as $\psi=$const along the curve in the meridian plane.

Again $$-\frac{\partial\phi}{\partial s}=-\frac{1}{\varpi}\,\frac{\partial\psi}{\partial n}$$

then from (iii), we have

$$T=-\pi\rho\int\psi\,\frac{\partial\phi}{\partial s}.ds$$

$$T=-\pi\rho\int\psi\,\frac{1}{\varpi}\,\frac{\partial\psi}{\partial n}\,ds$$

$$=-\tfrac{1}{2}\pi\rho\int\frac{\psi}{\varpi}\,\frac{\partial\psi}{\partial n}\,ds$$

Integration being taken round the whole boundary which is double of the meridian curve.

Kinetic energy of the moving sphere.

When a sphere moves with velocity V along the axis of rotation, then

$$\phi=\tfrac{1}{2}\,\frac{Va^3\cos\theta}{r^2}\text{ and }\psi=-\tfrac{1}{2}\,\frac{Va^3\sin^2\theta}{r}$$

Therefore kinetic energy, is given by

$$T=-\pi\rho\int\phi\,d\psi$$

or $$T=\tfrac{1}{2}\pi\rho V^2a^3\int_0^{\pi}\cos^2\theta\sin\theta\,d\theta \qquad \{\text{at } r=a$$

or $$T=\tfrac{1}{2}\pi\rho V^2a^3.\tfrac{1}{3}\left(-\cos^3\theta\right)_0^\pi$$

or $$T=\tfrac{1}{3}\pi\rho V^2a^3 \quad \text{where } M=\tfrac{4}{3}\pi a^3\rho \text{ mass of the liquid displaced by the sphere.}$$
$$=\tfrac{1}{4}MV^2$$

Thus the kinetic energy of infinite liquid due to the motion of a sphere through it with velocity V is

$$=\tfrac{1}{4}MV^2$$

§ 6·9. Motion of a liquid inside a rotating ellipsoidal shell.

The equation to the surface is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1. \tag{i}$$

Consider the coordinate axes fixed in space and coincident with the axes of the ellipsoid at any instant t, ω_x, ω_y, ω_z are the components of the angular velocity. Then the components of linear velocities of a point $P(x\ y\ z)$ of the ellipsoidal shell are given by

$$z\omega_y-y\omega_z,\ x\omega_z-z\omega_x \text{ and } y\omega_x-x\omega_y.$$

The direction cosines of the normal are proportional to

$$\frac{x}{a^2},\ \frac{y}{b^2} \text{ and } \frac{z}{c^2}.$$

Let ϕ be the velocity potential then the boundary condition is given by

$$-\frac{x}{a^2}\frac{\partial\phi}{\partial x}-\frac{y}{b^2}\frac{\partial\phi}{\partial y}-\frac{z}{c^2}\frac{\partial\phi}{\partial z}=\frac{x}{a^2}(z\omega_y-y\omega_z)$$
$$+\frac{y}{b^2}(x\omega_z-z\omega_x)+\frac{z}{c^2}(y\omega_x-x\omega_y) \qquad \text{...(ii)}$$

To satisfy the above relation assuming solution of Laplace's equation of the form

$$\phi=Ayz+Bzx+Cxy \qquad \text{...(iii)}$$

So the relation (ii) reduces to

$$-\frac{x}{a^2}(Bz+Cy)-\frac{y}{b^2}(Az+Cx)-\frac{z}{c^2}(Ay+Bx)$$
$$=\frac{x}{a^2}(z\omega_y-y\omega_z)+\frac{y}{b^2}(x\omega_z-z\omega_x)+\frac{z}{c^2}(y\omega_x-x\omega_y)$$

or $$Ayz\left(\frac{1}{b^2}+\frac{1}{c^2}\right)+Bzx\left(\frac{1}{c^2}+\frac{1}{a^2}\right)+Cxy\left(\frac{1}{a^2}+\frac{1}{b^2}\right)$$
$$=yz\omega_x\left(\frac{1}{b^2}-\frac{1}{c^2}\right)+zx\omega_y\left(\frac{1}{c^2}-\frac{1}{a^2}\right)+xy\omega_z\left(\frac{1}{a^2}-\frac{1}{b^2}\right)$$

Thus we have

$$A=-\frac{b^2-c^2}{b^2+c^2}\omega_x,\ B=-\frac{c^2-a^2}{c^2+a^2}\omega_y \text{ and } C=-\frac{a^2-b^2}{a^2+b^2}\omega_z$$

Substituting the values of A, B and C in (iii), we have

$$\phi=-\frac{b^2-c^2}{b^2+c^2}\omega_x\, yz-\frac{c^2-a^2}{c^2+a^2}\omega_y\, zx-\frac{a^2-b^2}{a^2+b^2}\omega_z\, xy \quad \text{...(iv)}$$

The velocity potential ϕ depends only on the mutual ratios of a, b, c and not on their absolute magnitudes, it means that the motion is the same in all ellipsoids of the same shape rotating with the same angular velocity.

Path of the particles relative to the ellipsoids.

Let $(\lambda\ \mu\ \nu)$ are the coordinates of a particle P referred to the axes of the ellipsoid. Then the velocity components of P referred to the axes fixed in space are

$$\dot{\lambda}-\mu\omega_z+\nu\omega_y,\ \dot{\mu}-\nu\omega_x+\lambda\omega_z,\ \dot{\nu}-\lambda\omega_y+\mu\omega_x$$

Therefore,

$$\dot{\lambda}-\mu\omega_z+\nu\omega_y=-\frac{\partial\phi}{\partial x}$$

$$=\frac{c^2-a^2}{c^2+a^2}\omega_y\nu+\frac{a^2-b^2}{a^2+b^2}\omega_z\mu$$

or $$\dot{\lambda}=\left(\frac{c^2-a^2}{c^2+a^2}-1\right)\omega_y\nu+\left(\frac{a^2-b^2}{a^2+b^2}+1\right)\omega_z\mu$$

or $$\dot{\lambda}=\frac{2a^2}{a^2+b^2}\omega_z\mu-\frac{2a^2}{c^2+a^2}\omega_y\nu$$

or $$\dot{\lambda}=a^2\,(\gamma\mu-\beta\nu)$$

similarly $$\dot{\mu}=b^2\,(\alpha\nu-\gamma\lambda)$$

and $$\dot{\nu}=c^2\,(\beta\lambda-\alpha\mu) \quad \text{...(v)}$$

$$\left\{\text{where}\quad \alpha=\frac{2\omega_x}{b^2+c^2},\ \beta=\frac{2\omega_y}{c^2+a^2} \text{ and } \gamma=\frac{2\omega_z}{a^2+b^2}\right\}$$

Multiplying (v) by $\frac{\alpha}{a^2}$, $\frac{\beta}{b^2}$ and $\frac{\gamma}{c^2}$ and adding, we have

$$\dot{\lambda}\frac{\alpha}{a^2}+\dot{\mu}\frac{\beta}{b^2}+\dot{\nu}\frac{\gamma}{c^2}=\alpha\,(\gamma\mu-\beta\nu)+\beta\,(\alpha\nu-\gamma\lambda)+\gamma\,(\beta\lambda-\alpha\mu)$$

or $$\dot{\lambda}\frac{\alpha}{a^2}+\dot{\mu}\frac{\beta}{b^2}+\dot{\nu}\frac{\gamma}{c^2}=\text{zero}$$

By integrating, we have

$$\frac{\lambda\alpha}{a^2}+\frac{\mu\beta}{b^2}+\frac{\nu\gamma}{c^2}=\text{constant.} \qquad \ldots\text{(vi)}$$

Again multiplying (v) by $\frac{\lambda}{a^2}$, $\frac{\mu}{b^2}$ and $\frac{\nu}{c^2}$ and adding, we get

$$\frac{\lambda\dot{\lambda}}{a^2}+\frac{\mu\dot{\mu}}{b^2}+\frac{\nu\dot{\nu}}{c^2}=\lambda\,(\gamma\mu-\beta\nu)+\mu\,(\alpha\nu-\gamma\lambda)+\nu\,(\beta\lambda-\alpha\mu)$$
$$=\text{zero.}$$

By integrating, we have

$$\frac{\lambda^2}{a^2}+\frac{\mu^2}{b^2}+\frac{\nu^2}{c^2}=\text{const.} \qquad \ldots\text{(vii)}$$

Thus the path of the particle lies on the plane (vi) and the ellipsoid (vii), so that it is an ellipse.

Again, assuming that the relation (v) have the solutions of the form

$$\lambda=Pe^{ipt},\ \mu=Qe^{ipt} \text{ and } \nu=Re^{ipt}.$$

Substituting the value of λ, μ, ν in (v) and eliminating P, Q, R, we have

$$\begin{vmatrix} \frac{ip}{a^2} & -\gamma & \beta \\ \gamma & \frac{ip}{b^2} & -\alpha \\ -\beta & \alpha & \frac{ip}{c^2} \end{vmatrix}=0$$

where $$p=abc\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}\right)^{1/2}$$

Thus every particle of the liquid describes an ellipse relative to the ellipsoid as a particle moving under a law of force varying as the distance from a fixed point.

Periodic time for each particle is

$$=\frac{2\pi}{p}$$

where $$p=2abc\left\{\left(\frac{\omega_x/a}{b^2+c^2}\right)^2+\left(\frac{\omega_y/b}{c^2+a^2}\right)^2+\left(\frac{\omega_z/c}{a^2+b^2}\right)^2\right\}^{1/2}$$

Particular Case. For a Sphere

$$a=b=c$$

then $$p=(\omega_x^2+\omega_y^2+\omega_z^2)^{1/2}.$$

i.e. the period of revolution of the liquid relative to the spherical shell is the same as the period of revolution of the shell $\Rightarrow$ that the shell is revolving alone and the liquid is left at rest in space.

§ 6·91. Motion of an ellipsoid in an infinite mass of liquid.

Equation to an ellipsoid,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1. \qquad \text{...(i)}$$

We know that the potential of a solid homogeneous ellipsoid of unit density at an external point $(x\ y\ z)$ is

$$V=\pi abc\int_\lambda^\infty\left(1-\frac{x^2}{a^2+u}-\frac{y^2}{b^2+u}-\frac{z^2}{c^2+u}\right)\frac{du}{(a^2+u)^{1/2}\,(b^2+u)^{1/2}\,(c^2+u)^{1/2}} \qquad \text{...(ii)}$$

where λ is the positive root of the equation

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}+\frac{z^2}{c^2+\lambda}-1=0. \qquad \text{...(iii)}$$

Assuming $V=\pi\,(\delta-\alpha x^2-\beta y^2-\gamma z^2)$ ...(iv)

where $\delta=abc\displaystyle\int_\lambda^\infty\frac{du}{\Delta}$, $\alpha=abc\displaystyle\int_\lambda^\infty\frac{du}{(a^2+u)\,\Delta}$

$\beta=abc\displaystyle\int_\lambda^\infty\frac{du}{(b^2+u)\,\Delta}$ and $\gamma=abc\displaystyle\int_\lambda^\infty\frac{du}{(c^2+u)\,\Delta}$

and $\Delta=(a^2+u)^{1/2}\,(b^2+u)^{1/2}\,(c^2+u)^{1/2}$.

Let X, Y, Z are the components of attraction at an external point.

where $$X=\frac{\partial V}{\partial x}=-2\pi\alpha x+\frac{\partial V}{\partial \lambda}\cdot\frac{\partial \lambda}{\partial x} \qquad \text{...(v)}$$
{from (iv)

But $$\frac{\partial V}{\partial \lambda}=0 \qquad \text{\{from (iii)}$$

then $$X=-2\pi\alpha x,\quad Y=-2\pi\beta y\quad \text{and}\quad Z=-2\pi\gamma z \qquad \text{...(vi)}$$

(α, β, γ are the functions of λ or x, y, z).

Since V is a solution of Laplace's Equation. Consider an ellipsoid moving with velocity U in the direction of X-axis.

Then the boundary condition is

or $$-\frac{x}{a^2}\frac{\partial\phi}{\partial x}-\frac{y}{b^2}\frac{\partial\phi}{\partial y}-\frac{z}{c^2}\frac{\partial\phi}{\partial z}=U\,\frac{x}{a^2} \qquad \text{...(1)}$$

$$\left\{\text{over the ellipsoid } \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \text{ i.e. } \lambda=0\right\}$$

To satisfy the above relation assuming a solution of the form

$$\phi = AX$$

or $$\phi = A\,(-2\pi\alpha x) \qquad \{\text{from (v)}$$

or $$\frac{\partial\phi}{\partial x} = -2\pi A\left(\alpha + x\,\frac{\partial\alpha}{\partial\lambda}\cdot\frac{\partial\lambda}{\partial x}\right) \qquad \text{.. (viii)}$$

where $$\lambda = 0,\ \frac{\partial\alpha}{\partial\lambda} = -\frac{1}{a^2}$$

By differentiating (iii) with regard to x, we have

$$\frac{2x}{a^2+\lambda} - \frac{\partial\lambda}{\partial x}\left\{\frac{x^2}{(a^2+\lambda)^2} + \frac{y^2}{(b^2+\lambda)^2} + \frac{z^2}{(c^2+\lambda)^2}\right\} = 0$$

or $$\frac{\partial\lambda}{\partial x} = \frac{2p^2x}{a^2+\lambda}$$

where $$\frac{1}{p^2} = \frac{x^2}{(a^2+\lambda)^2} + \frac{y^2}{(b^2+\lambda)^2} + \frac{z^2}{(c^2+\lambda)^2}\,.$$

Similarly $\dfrac{\partial\lambda}{\partial y} = \dfrac{2\,p^2y}{b^2+\lambda}$ and $\dfrac{\partial\lambda}{\partial z} = \dfrac{2\,p^2z}{c^2+\lambda}$.

Substituting the value of $\dfrac{\partial\lambda}{\partial z}$ and $\dfrac{\partial\alpha}{\partial\lambda}$ in (vii), we have

$$\frac{\partial\phi}{\partial x} = -2\pi A\left\{\alpha + \left(-\frac{1}{a^2}\right)\frac{2\,p^2x^2}{a^2+\lambda}\right\}$$

$$\frac{\partial\phi}{\partial x} = -2\pi A\left\{\alpha - \frac{2\,p^2x^2}{a^2(a^2+\lambda)}\right\}$$

when $\lambda = 0$, then

$$\frac{\partial\phi}{\partial x} = -2\pi A\left\{\alpha_0 - \frac{2\,p^2x^2}{a^4}\right\}$$

Similarly $$\frac{\partial\phi}{\partial y} = -2\pi A\left(-\frac{2\,p^2xy}{a^2b^2}\right)$$

and $$\frac{\partial\phi}{\partial z} = -2\pi A\left(-\frac{2\,p^2xz}{a^2c^2}\right).$$

Substituting the values of $\dfrac{\partial\phi}{\partial x}$, $\dfrac{\partial\phi}{\partial y}$ and $\dfrac{\partial\phi}{\partial z}$ in (vii), we have

or $$-\frac{x}{a^2}\cdot\left\{-2\pi A\left(\alpha_0 - \frac{2\,p^2x^2}{a^4}\right)\right\} - \frac{y}{b^2}\left\{-2\pi A\left(-\frac{2\,p^2xy}{a^2b^2}\right)\right\}$$
$$-\frac{z}{c^2}\left\{-2\pi A\left(-\frac{2\,p^2xz}{a^2c^2}\right)\right\} = U\cdot\frac{x}{a^2}$$

or $$2\pi A\left\{\frac{\alpha_0 x}{a^2} - \frac{2\,p^2x}{a^2}\left(\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4}\right)\right\} = \frac{Ux}{a^2}$$

or $$2\pi A\left\{\frac{\alpha_0 x}{a^2} - \frac{2\,p^2x}{a^2}\cdot\frac{1}{p^2}\right\} = \frac{Ux}{a^2}$$

or $$2\pi A\,(\alpha_0-2)\frac{x}{a^2}=\frac{Ux}{a^2}$$

or $$A=\frac{U}{2\pi\,(\alpha_0-2)}\,.$$

Thus $$\phi=-2\pi\alpha.\,\frac{U}{2\pi\,(\alpha_0-2)}\,x$$

or $$\phi=\frac{U\alpha x}{2-\alpha_0}$$

which gives the velocity potential of the liquid motion.

Hence if the ellipsoid have a velocity of which U, V, W are the components parallel to the axes, then the velocity potential ϕ becomes

$$\phi=\frac{U\alpha x}{2-\alpha_0}+\frac{V\beta y}{2-\beta_0}+\frac{Wyz}{2-\gamma_0}\,.$$

Ex. 21. *An ellipsoidal cavity (semi-axes a, b, c) in a solid initially at rest is filled with an incompressible frictionless fluid initially at rest. Prove that if the solid be moved with velocities u, v, w parallel to the axes of the cavity, and be rotated with angular velocities p, q, r round the semi-axis, the angular momentum of the fluid round the semi-axis a at any instant is*

$$\tfrac{4}{15}\pi\rho abc\frac{(b^2-c^2)^2}{b^2+c^2}\,p.$$

Since the ellipsoid is rotating with angular velocities

Here $\omega_x=p,\ \omega_y=q,\ \omega_z=r.$

The velocity potential is given by

$$\phi=-\frac{b^2-c^2}{b^2+c^2}pyz-\frac{c^2-a^2}{c^2+a^2}\,qzx-\frac{a^2-b^2}{a^2+b^2}\,rxy \qquad ...(1)$$

{Ref. § 6·9

By superimposing the velocities $-u$, $-v$, $-w$ along the axes, the ellipsoid can be reduced to simple rotation about the axes. Thus an expression $-ux-vy-wz$ should be added to the velocity potential ϕ.

i.e. $$\phi=-ux-vy-wz-\frac{b^2-c^2}{b^2+c^2}pyz-\frac{c^2-a^2}{c^2+a^2}qzx-\frac{a^2-b^2}{a^2+b^2}\,rxy \qquad ...(ii)$$

Consider U, V, W be the velocity components at $P\,(x, y, z)$

then $$U=-\frac{\partial\phi}{\partial x},\ V=-\frac{\partial\phi}{\partial y},\ W=-\frac{\partial\phi}{\partial z} \qquad ...(iii)$$

The angular momentum of the whole liquid about the axis of X, is given by

$$=\iiint (yW-zV)\,\rho\,dx\,dy\,dz$$

where $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \leqslant 1$

$$=-\rho\iiint\left(y\frac{\partial\phi}{\partial z}-z\frac{\partial\phi}{\partial y}\right)dx\,dy\,dz \qquad \{\text{from (iii)}$$

$$=-\rho\iiint\Big[y\left\{w+\frac{b^2-c^2}{b^2+c^2}py+\frac{c^2-a^2}{c^2+a^2}qx\right\}$$
$$-z\left\{v+\frac{b^2-c^2}{b^2+c^2}pz+\frac{a^2-b^2}{a^2+b^2}rx\right\}\Big]dx\,dy\,dz$$

$\left\{\text{Substituting the value of } \frac{\partial\phi}{\partial z} \text{ and } \frac{\partial\phi}{\partial y} \text{ from (ii)}\right.$

$$=\rho\left(\frac{b^2-c^2}{b^2+c^2}\right)p\iiint(y^2-z^2)\,dx\,dy\,dz$$

(other integrals vanish)

$$=\left(\frac{b^2-c^2}{b^2+c^2}\right)p\iiint\rho\,\{(x^2+y^2)-(x^2+z^2)\}\,dx\,dy\,dz$$

$$=\left(\frac{b^2-c^2}{b^2+c^2}\right)p\Big[\text{M. I. of the ellipsoid about } Z\text{-axis}$$
$$-\text{M. I. of the ellipsoid about } Y\text{-axis}\Big]$$

$$=\left(\frac{b^2-c^2}{b^2+c^2}\right)p\left[M\frac{a^2+b^2}{5}-M\frac{a^2+c^2}{5}\right]\text{where } M=\tfrac{4}{3}\pi\rho abc$$

$$=\tfrac{4}{3}\pi\rho abcp.\left(\frac{b^2-c^2}{b^2+c^2}\right).\frac{b^2-c^2}{5}$$

$$=\tfrac{4}{15}\pi\rho abc\,\frac{(b^2-c^2)^2}{b^2+c^2}\,p$$

which is the required result.

7

Vortex Motion

The most general type of motion of which a fluid is capable is one which is a combination of rotational and irrotational motion *i.e.* the component velocities may be regarded as consisting of two parts, the first as derivable from the velocity potential and the second depend upon the molecular rotation. This motion was first developed by Helmholtz and afterwards proof of some of his theorems were given by Kelvin.

§ 7·0.

I Let **q** represents the velocity of a fluid motion, then the **vorticity** vector $\vec{\omega}$ is defined as

$$\vec{\omega} = \text{curl } \mathbf{q}$$

In cartesian coordinates, let ξ, η, ζ are the components of the vorticity vector $\vec{\omega}$ and $\mathbf{q} = (u\ v\ w)$, then

$$\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z},\ \eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x},\ \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

II 'A vortex line is a line whose direction coincides with the direction of the instantaneous axis of molecular rotation'. In other words, we can define that a vortex line is a line drawn in the fluid such that the tangent to it is in the direction of the vorticity vector at that point.

The differential equation to the vortex line is given by

$$\vec{\omega} \times d\mathbf{r} = 0$$

or $$(\eta dz - \zeta\, dy)\ \mathbf{i} + (\zeta dx - \xi dz)\mathbf{j} + (\xi dy - \eta dx)\ \mathbf{k} = 0$$

or $$\frac{dx}{\xi} = \frac{dy}{n} = \frac{dz}{\zeta}$$

III A **vortex tube** is obtained if through every point of a small closed curve, the corresponding vortex lines are drawn.

The fluid contained within the vortex tube constitutes the **vortex filaments** or simply **vortices** and the boundary of a vortex filament is called a **vortex tube**.

§ **7·1.** Every vortex satisfies the following fundamental properties.

(a) Every vortex is always composed of the same elements of fluid.

Consider an element of fluid whose initial coordinates are (a, b, c) and at any instant t are (x, y, z). Then

$$\frac{da}{\xi_0}=\frac{db}{\eta_0}=\frac{dc}{\zeta_0}=\frac{dS_0}{\omega_0}=\lambda \quad \text{(say)}$$

Since
$$dx=\frac{dx}{da}\,da+\frac{dx}{db}\,db+\frac{dx}{dc}\,dc$$
$$=\lambda\left(\xi_0\frac{dx}{da}+\eta_0\frac{dx}{db}+\zeta_0\frac{dx}{dc}\right)$$
$$=\frac{\rho_0\,\xi.dS_0}{\rho\omega_0}$$

hence
$$\frac{\rho_0 dS_0}{\omega_0}=\frac{\rho dS}{\omega}=\epsilon \qquad \text{...(i)} \quad \left\{\text{§ 2·5 page 51}\right.$$

Let $(u\ v\ w)$ be the component velocities at $(x\ y\ z)$; and $(u+du,\ v+dv,\ w+dw)$ be the velocities at a neighbouring point $(x+dx, y+dy, z+dz)$ on the same vortex line.

Since
$$\frac{dx}{\xi}=\frac{dy}{\eta}=\frac{dz}{\zeta}=\frac{dS}{\omega}=\frac{\epsilon}{\rho}$$

But
$$du=\frac{du}{dx}\,dx+\frac{du}{dy}\,dy+\frac{du}{dz}\,dz$$
$$=\frac{\epsilon}{\rho}\left(\xi\,\frac{du}{dx}+\eta\,\frac{du}{dy}+\zeta\,\frac{du}{dz}\right)$$
$$=\epsilon\,\frac{\partial}{\partial t}\left(\frac{\xi}{\rho}\right)$$

The quantity du is the rate at which the projection of the element dS on the axis of X is increasing in length. The projection is equal to $\epsilon\,\frac{\partial}{\partial t}\left(\rho^{-1}\,\xi\right)$, the line dS still forms part of a vortex line. **Proved.**

(b) The product of the angular velocity of any vortex into its cross-section is constant with respect to the time, and is the same throughout its length.

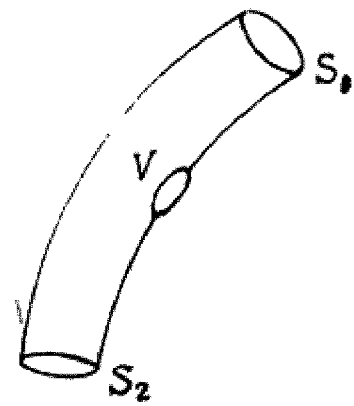
Consider σ be the area of the cross-section at any instant t, since the mass of the element remains unchanged,

$$\rho_0 \, \sigma_0 \, dS_0 = \rho \sigma dS$$

or $$\sigma_0 \omega_0 = \sigma \omega \qquad \text{\{from (i)}$$

$\Rightarrow$ that $\sigma\omega$ is independent of the time.

Consider S_1 and S_2 be any two cross-sectional surfaces of the vortex tube drawn in the fluid region at any instant t.



then $$\int_S \vec{\omega} . dS = \int_{S_1} \vec{\omega} . dS + \int_{S_2} \vec{\omega} . dS + \int_{\text{walls}} \vec{\omega} . dS \qquad \ldots\text{(i)}$$

where S is the closed surface enclosing the portion of the fluid region of volume V in the vortex tube and the cross-sectional surfaces S_1 and S_2.

Since on the walls of the tube, $\vec{\omega}$ is along the tube

So $$\vec{\omega} . dS = \vec{\omega} . \mathbf{n} \, dS$$

(Where $\mathbf{n}$ is a unit outward drawn normal vector)

$$= \text{zero}$$

Thus $$\int_{\text{walls}} \vec{\omega} . dS = 0 \qquad \ldots\text{(ii)}$$

Then from Gauss's theorem, we have

$$\int_S \vec{\omega} . dS = \int_V \text{div } \vec{\omega} . dv$$

$$= \int \text{div. (curl } \mathbf{q}) . dv$$

$$= \text{zero} \qquad \ldots\text{(iii)}$$

Now the relation (i) reduces to with the help of (ii) and (iii),

$$\int_{S_1} \vec{\omega} . dS + \int_{S_2} \vec{\omega} . dS = 0$$

or $$\int_{S_1} \vec{\omega} . \mathbf{n} \, dS + \int_{S_2} \vec{\omega} . \mathbf{n} \, dS = 0. \qquad \ldots\text{(iv)}$$

Consider $\mathbf{n}_1$, $\mathbf{n}_2$ be the unit outward drawn normal in the same sense on the surfaces S_1, S_2 then (iv) reduces to

$$\int_{S_1} \vec{\omega} . \mathbf{n}_1 \, dS = \int_{S_2} \vec{\omega} . \mathbf{n}_2 \, dS$$

$$\int_S \vec{\omega} . \mathbf{n} \, dS = \text{constant.}$$

(c) Every vortex must either form a closed curve or have its extremities in the boundaries of the fluid.

It a vortex did not form a closed curve or have its extremities in the boundary, it would be possible to draw a closed surface cutting the vortex once only, and the surface integral would not vanish.

§ 7·11. **Strength of the vortex tube :** Let k be the circulation around any closed curve.

Then $$k=\int_C \mathbf{q}.d\mathbf{r}$$

Consider the closed curve C encloses the vortex tube and lies on its walls, then by Stoke's theorem, we have

$$k=\int_C \mathbf{q}.d\mathbf{r}=\int_S \mathbf{n} \text{ curl } \mathbf{q}\, dS$$

$$=\int_S \vec{\omega}.\mathbf{n}\, dS$$

$$=\text{constant} \qquad \{\text{from (i)}$$

$\Rightarrow$ that the circulation round any closed curve enclosing a vortex tube is constant all along the tube. This constant is also known the **strength of the vortex tube**, or if ω denote the angular velocity and A the cross-section of the vortex tube (supposed small), the circulation round this section is $2\omega A$. This product is constant for all sections, we consider it as a measure of the **strength of the vortex.**

Let for a vortex filament of variable cross-section

$$k=\vec{\omega}.\mathbf{n}\, dS=\text{constant}.$$

where $\mathbf{n}\, dS$ is any cross-sectional area of the vortex filament and the outward drawn unit normal vector $\mathbf{n}$ is taken in the direction of $\vec{\omega}$.

So $$k=\vec{\omega}\, dS=\text{constant}$$

Thus vorticity at any section of a vortex filament is inversely proportional to its cross-sectional area.

§ 7·2. Rectilinear Vortices :

Now we shall consider two dimensional vortex motion. The vorticity vector is perpendicular to the plane of the motion. The vortex lines being straight and parallel, all vortex tubes are cylindrical, with generators perpendicular to the plane of the motion. Such vortices are known as **rectilinear vortices.**

Since the vortex filaments are all perpendicular to the plane of motion. Consider the axis of Z parallel to the filaments. The motion being similar in all planes parallel to XY-plane, we have

$$w=0, \frac{\partial u}{\partial z}=0 \text{ and } \frac{\partial v}{\partial z}=0$$

$\Rightarrow$ u and v are independent of z.

Thus the only component of vorticity (spin) is ζ.

$$\zeta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y} \quad \text{...(i)}$$

Equation of the lines of flow are

$$\frac{dx}{u}=\frac{dy}{v}$$

or $$vdx-udy=0$$

Equation of continuity is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \quad \text{...(iii)}$$

it follows from (iii) that $vdx-udx$ is a perfect differential $d\psi$, hence

$$vdx-udy=d\psi$$
$$=\frac{\partial \psi}{\partial x}dx+\frac{\partial \psi}{dy}dy$$

or $$u=-\frac{\partial \psi}{\partial y} \text{ and } v=\frac{\partial \psi}{\partial x}$$

Since $$\zeta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$$

or $$\zeta=\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right)-\frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial y}\right)$$

or $$\zeta=\frac{\partial^2 \psi}{\partial x^2}+\frac{\partial^2 \psi}{\partial y^2} \quad \text{...(iv)}$$

The lines of flow are given by $\psi=$constant.

Hence the stream function ψ satisfies the equation.

$$\frac{\partial^2 \psi}{\partial x^2}+\frac{\partial^2 \psi}{\partial y^2}=\zeta, \text{ on the vortex filament}$$
$$=0, \text{ outside the filament.}$$

The motion is irrotational except along the vortex filament and ζ is zero ; the stream function ψ may be regarded as the potential at any point of an infinite medium, the density of which is zero (except along the vortex filaments).

Consider P be a point outside the vortex filament, then we have

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0$$

In polar coordinates, since ψ being a function of r only *i.e.* independent of θ due to symmetry, we have

$$\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}=0$$

or $$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right)=0$$

or $$r\frac{d\psi}{dr}=\lambda \quad \text{(constant).}$$

By integrating, we have

$$\psi=\lambda \log r$$

Since the motion outside the vortex filament is irrotational, $\exists$ a velocity potential ϕ there, such that

$$\phi=-\lambda\theta$$

If w be the complex potential outside the filament then

$$w=\phi+i\psi$$

$$w=-\lambda\theta+i\lambda \log r$$

or $$w=i\lambda \{\log r+i\theta\}$$

or $$w=i\lambda \log (re^{\theta i})$$

or $$w=i\lambda \log z$$

Let k be the circulation in the circuit embracing the vortex

then $$k=\int_0^{2\pi}\left(-\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)r.d\theta$$

$$k=\int_0^{2\pi}\lambda\, d\theta \qquad =2\pi\lambda$$

or $$\lambda=\frac{k}{2\pi}$$

Thus $$\phi=-\frac{k}{2\pi}\theta \text{ and } \psi=\frac{k}{2\pi}\log r$$

So $$w=\frac{ik}{2\pi}\log z$$

Where k is called the **strength of vortex.**

Also the velocity components at any point in the direction of r and θ are

$$q_r = -\frac{\partial \phi}{\partial r} = -\frac{1}{r}\frac{\partial \psi}{\partial \theta} = 0$$

and
$$q_\theta = -\frac{1}{r}\frac{\partial \phi}{\partial \theta} = \frac{\partial \psi}{\partial r} = \frac{k}{2\pi r}$$

If there be a rectilinear vortex of strength k at z_0,

then
$$w = \frac{ik}{2\pi}\log(z - z_0)$$

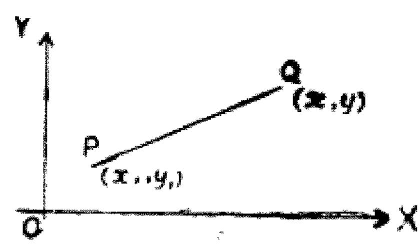
If there be a number of vortex filaments of strength $k_1, k_2, k_3 \ldots$ at $z_1, z_2, z_3 \ldots$ then the complex potential is given by

$$w = \frac{ik_1}{2\pi}\log(z - z_1) + \frac{ik_2}{2\pi}\log(z - z_2) + \frac{ik_3}{2\pi}\log(z - z_3) + \ldots$$

$$= \Sigma\,\frac{ik_s}{2\pi}\log(z - z_s)$$

§ 7·21. Velocity Components.

Consider a single vortex at $P(x_1\,y_1)$, the stream function ψ at any point $Q\,(x\,y)$ is given by



$$\psi = \frac{k}{2\pi}\log r_1$$

where $\quad r_1^2 = PQ^2 = (x - x_1)^2 + (y - y_1)^2$

Since
$$u = -\frac{\partial \psi}{\partial y} = -\frac{\partial \psi}{\partial r_1}\cdot\frac{\partial r_1}{\partial y}$$

$$= \frac{k}{2\pi r_1}\cdot\left\{-\frac{y - y_1}{r_1}\right\}$$

or
$$u = -\frac{k}{2\pi}\cdot\frac{y - y_1}{r_1^2}$$

and
$$v = \frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial r_1}\cdot\frac{\partial r_1}{\partial x}$$

$$= \frac{k}{2\pi}\cdot\frac{x - x_1}{r_1^2}$$

If there be any number of vortices of strength $k_1, k_2, k_3 \ldots$ at the points $(x_1\,y_1), (x_2\,y_2), (x_3\,y_3) \ldots$ then the velocity at any point outside is given by

$$u = -\Sigma\,\frac{k_s}{2\pi}\cdot\frac{y - y_s}{r_s^2}$$

and $$v=\Sigma \frac{k_s}{2\pi}\cdot \frac{x-x_s}{r_s^2}$$

where $$r_s^2=(x-x_s)^2+(y-y_s)^2$$

and $$u-iv=-\frac{dw}{dz}=-\Sigma \frac{ik_s}{2\pi}\cdot \frac{1}{(z-z_s)}$$

§ 7·22. **Centre of Vortices.**

Consider k_n be the strength of the vortex (x_n, y_n), so the motion of the nth vortex is given by

$$\dot{x}_n=-\frac{1}{2\pi}\Sigma k_s \frac{y_n-y_s}{r_{ns}^2}$$

and $$\dot{y}_n=\frac{1}{2\pi}\Sigma k_s \frac{x_n-x_s}{r_{ns}^2}$$

where $n\neq s$ and $r_{ns}^2=(x_n-x_s)^2+(y_n-y_s)^2$

or $\Sigma k_n \dot{x}_n=-\frac{1}{2\pi}\Sigma\Sigma\, k_n k_s \frac{y_n-y_s}{r_{ns}^2}=0$ as n and s can be interchanged, the denominator r_{ns}^2 remain positive.

i.e. $\Sigma k_n \dot{x}_n=0$

Similarly $\Sigma k_n \dot{y}_n=0$

i.e. k_n is independent of the time t, we have

$$\Sigma k_n x_n=\text{constant and } \Sigma k_n y_n=\text{constant}$$

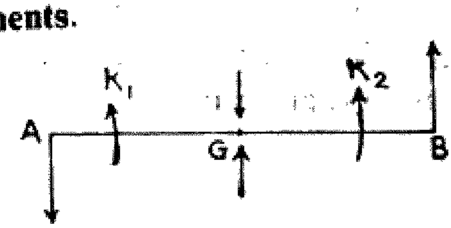
If $\bar{X}=\dfrac{\Sigma k_n x_n}{\Sigma k_n}$ and $\bar{Y}=\dfrac{\Sigma k_n y_n}{\Sigma k_n}$

$\Rightarrow$ $(\bar{X}, \bar{Y})$ are constant. Thus if $k_1, k_2, k_3 \ldots$ are supposed to be the masses then their centre of gravity is fixed through out the motion. This point is known as the **centre of vortices**, and the straight line parallel to Z may be called the **axis of the system**. If $\Sigma k=0$, the centre is at infinity or else indeterminate.

Hence if there be any number of rectangular vortices, then there tendency to move is in such a way that their **centre is stationary.**

If there be a single rectilinear vortex in an unlimited mass of liquid, it must remain fixed ; and when such a vortex is in the presence of other vortices, it has no tendency to move of itself but its velocity is due to the influence of other vortices.

§ 7·3. **A case of two vortex filaments.**



Consider two rectilinear vortex filaments of strengths k_1 and k_2 at A and B, each will produce a motion of the other perpendicular to the line joining them, *i.e.* the motion of B is due to the effect of A and is perpendicular to the line AB. Similarly the motion of A is due to the effect of B and is perpendicular to the line AB.

The complex potential due to the stationary system is given by

$$w=\frac{ik_1}{2\pi}\log(z-z_1)+\frac{ik_2}{2\pi}\log(z-z_2) \quad \ldots(\text{i})$$

let u and v be the components of velocity of A due to the vortex filament of strength k_2, then

$$u-iv=\left\{-\frac{dw}{dz}\right\}_{z=z_1}+\left\{\frac{ik_1}{2\pi}\cdot\frac{1}{z-z_1}\right\}_{z=z_1}$$

{ Ref § 7·22

$$=\left[-\frac{ik_1}{2\pi}\cdot\frac{1}{z-z_1}-\frac{ik_2}{2\pi}\cdot\frac{1}{z-z_2}+\frac{ik_1}{2\pi}\cdot\frac{1}{z-z_1}\right]_{z=z_1}$$

$$=-\frac{ik_2}{2\pi}\cdot\frac{1}{z_1-z_2} \quad \ldots(\text{ii})$$

Similarly if u_1 and v_1 be the components of velocity of B due to the vortex filament of strength k_1, then

$$u_1-iv_1=-\frac{ik_1}{2\pi}\cdot\frac{1}{z_2-z_1} \quad \ldots(\text{iii})$$

From (ii) and (iii), we have

$$\frac{u-iv}{k_2}=-\frac{u_1-iv_1}{k_1}$$

or $\quad k_1(u-iv)=-k_2(u_1-iv_1)$

or $\quad (k_1u+k_2u_1)-i(k_1v+k_2v_1)=0$

or $\quad k_1u+k_2u_1=0$ and $k_1v+k_2v_1=0$

Which shows the position of the centroid G of k_1, k_2 at z_1, z_2 moving with velocities $(u\ v)$ and $(u_1\ v_1)$, at rest. Thus the line AB rotates about G.

The velocity of A is,

$$|u-iv|=\frac{1}{2\pi}\left|\frac{ik_2}{z_1-z_2}\right|$$

$$=\frac{k_2}{2\pi(AB)}$$

$$=\frac{1}{2\pi}\cdot\frac{k_2\,AB}{k_1+k_2}\cdot\frac{k_1+k_2}{(AB)^2}$$

$$=GA\omega$$

Where the angular velocity of A is

$$\omega=\frac{k_1+k_2}{2\pi(AB)^2}$$

Similarly the angular velocity of B

$$=\frac{k_1+k_2}{2\pi(AB)^2} \text{ about } G.$$

So that the line AB revolves about G with uniform angular velocity $\dfrac{k_1+k_2}{2\pi(AB)^2}$

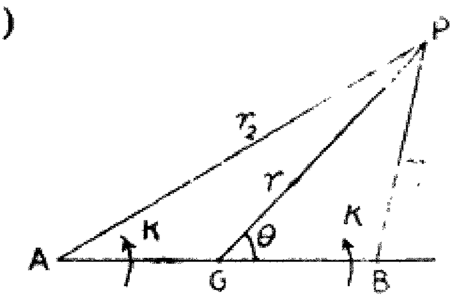
If k_1, k_2 are of opposite signs then at any intermediate point the velocities due to A, B are in the same direction and at an outside point the velocity may cancel so that the centre must be beyond AB. If $k_1 > k_2$, then the point G lies beyond B and if $k_1 < k_2$, then G lies beyond A.

The lines AB move round this point with angular velocity

$$\frac{k_1-k_2}{2\pi(AB)^2}$$

§ 7·4. To determine the stream function when the strength of the vortex filaments are equal.

Let k_1 and k_2 be the strengths of the vortex filaments.



Then $k_1=k_2$ (Given)

let $AG=GB=a$

i.e. $AB=2a$

or $$\psi=\frac{k}{2\pi}\log r_1+\frac{k}{2\pi}\log r_2$$

or $$\psi=\frac{k}{2\pi}\log r_1 r_2 \qquad \text{...(i)}$$

Since the centre G is the middle point of AB

then $$\cos\theta=\frac{r^2+a^2-r_1^2}{2ar} \text{ and } \cos(\pi-\theta)=\frac{r^2+a^2-r_2^2}{2ar}$$

or $$r_1^2=r^2+a^2-2ar\cos\theta \text{ and } r_2^2=r^2+a^2+2ar\cos\theta$$

From (i), we have

$$\psi=\frac{k}{4\pi}\log\{(r^2+a^2-2ar\cos\theta)(r^2+a^2+2ar\cos\theta)\}$$

or $$\psi=\frac{k}{4\pi}\log\{r^4+a^4+2a^2r^2-4a^2r^2\cos^2\theta\}$$

or $$\psi=\frac{k}{4\pi}\log\{r^4+a^4-2a^2r^2\cos 2\theta\} \qquad \text{...(ii)}$$

For stream lines, we have

$$\psi=\text{constant} \qquad \text{...(iii)}$$

Since the line AB revolves about G with angular velocity

$$=\frac{2k}{2\pi(AB)^2}$$

$$=\frac{k}{4\pi a^2}$$

The velocity at any point P due to this motion is

$$=\frac{kr}{4\pi a^2}$$

Now to reduce the vortex system to rest, by superposing a velocity $-\frac{kr}{4\pi a^2}$. let ψ' be the stream function in this case.

then $$\frac{\partial\psi'}{\partial r}=-\frac{1}{r}\frac{\partial\phi'}{\partial\theta}=-\frac{kr}{4\pi a^2}$$

or $$\psi'=-\frac{kr^2}{8\pi a^2} \quad \text{...(iv)}$$

Thus from (ii), we have

$$\psi=\frac{k}{4\pi}\log\{r^4+a^4-2a^2r^2\cos 2\theta)-\frac{kr^2}{8\pi a^2}$$

The stream lines are given by { from (iii)

$$\frac{k}{4\pi}\log(r^4+a^4-2a^2r^2\cos 2\theta)-\frac{kr^2}{8\pi a^2}=\text{constant}$$

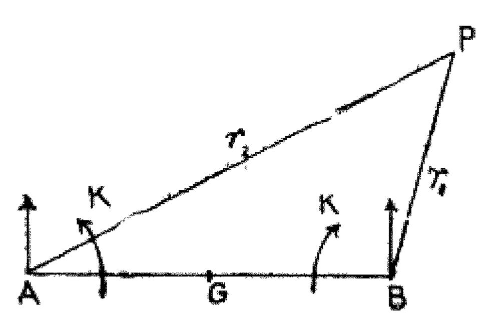
or $$\log(r^4+a^4-2a^2r^2\cos 2\theta)-\frac{r^2}{2a^2}=\text{constant}$$

§ 7·41. Stream function for two vortex filaments of strengths k and −k. (Vortex Pair)

consider $k_1=-k_2$

then $$AG=\frac{k_2}{k_1+k_2}.AB$$
$$=\infty$$

⇒ that the centre is at infinity.



Let the strengths of the vortex filaments be k and $-k$.

The velocity of A, due to B

$$=\frac{k}{2\pi.AB} \text{ perpendicular to } AB.$$

and the velocity of B, due to A

$$=\frac{k}{2\pi.AB} \text{ perpendicular to } AB \text{ in the same sense.}$$

So the vortices move in the same direction perpendicular to AB with uniform velocity $\frac{k}{2\pi.AB}$. The line may move backward or forward according to the direction of rotation.

Vortex Pair. **Thus a pair of vortices each of strength k but of opposite rotations is called a vortex pair.**

The stream function ψ at any point is given by

$$\psi = \frac{k}{2\pi} \log r_1 - \frac{k}{2\pi} \log r_2$$

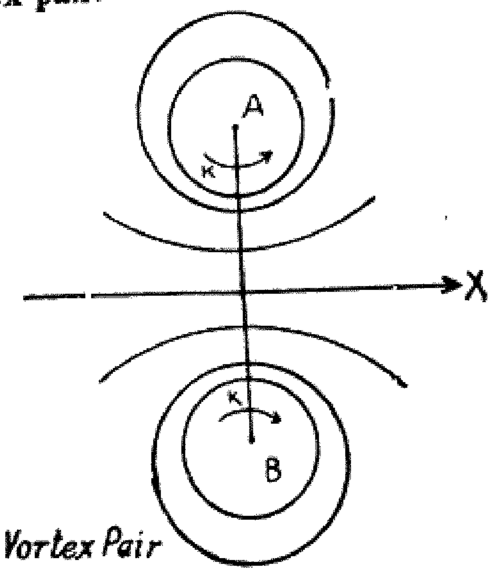
$$= \frac{k}{2\pi} \log \left(\frac{r_1}{r_2}\right).$$

The stream lines are given by

$$\psi = \text{constant}$$

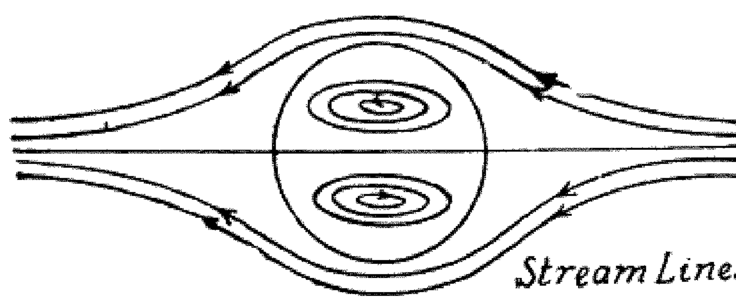
or $$\frac{k}{2\pi} \log \left(\frac{r_1}{r_2}\right) = \text{constant}$$

or $$\frac{r_1}{r_2} = \text{constant}$$



Vortex Pair

⇒ A system of co-axial circles having A and B as limiting points.



Stream Lines

Particular Case. Let Constant is equal to unity. then $r_1 = r_2$ which is the straight line bisecting AB at right angles. Thus the plane bisecting AB at right angles may be considered to be a **rigid boundary** as there being no flow across this plane and a single rectilinear vortex will move parallel to this plane boundary at a distance c from it with uniform velocity

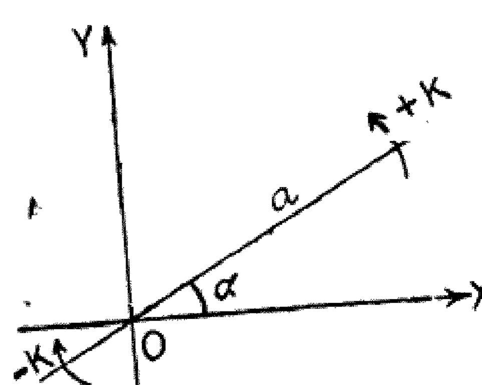
$$\frac{k}{2\pi c}.$$

§ 7·42. Vortex doublet.

Consider a vortex pair of strength $+k$ at $z = ae^{\alpha i}$ and of strength $-k$ at $z = -ae^{\alpha i}$. Let a tends to zero and k tends to infinity such that

$$2a \cdot \frac{k}{2\pi} = \mu,$$

where μ is known as the strength



of the vortex doublet and axis being inclined at an angle α to the X-axis.

The complex potential is given by

$$w=\frac{ik}{2\pi}\left[\log(z-ae^{i\alpha})-\log(z+ae^{i\alpha})\right]$$

or
$$w=\frac{ik}{2\pi}\left[\log z\left(1-\frac{ae^{i\alpha}}{z}\right)-\log z\left(1+\frac{a\,e^{i\alpha}}{z}\right)\right]$$

or
$$w=\frac{ik}{2\pi}\left[\log\left(1-\frac{ae^{i\alpha}}{z}\right)-\log\left(1+\frac{ae^{i\alpha}}{z}\right)\right]$$

or
$$w=-\frac{ik}{2\pi}\left[\frac{ae^{i\alpha}}{z}+\frac{a^2\,e^{2i\alpha}}{2z^2}+\frac{a^3\,e^{3i\alpha}}{3z^3}+\ldots\right.$$
$$\left.+\frac{ae^{i\alpha}}{z}-\frac{a^2e^{2i\alpha}}{2z^2}+\frac{a^3e^{3i\alpha}}{3z^3}\ldots\ldots\right]$$

or
$$w=-\frac{ik}{2\pi}\cdot\frac{2a\,e^{i\alpha}}{z}=-\frac{a\,ik}{\pi}\cdot\frac{e^{i\alpha}}{z}$$
$$=-\mu i\,.\,\frac{e^{i\alpha}}{z}$$
$$=-\mu i\,.\,\frac{e^{i\alpha}}{re^{\theta i}}$$
$$=-\frac{\mu i\;e^{i(\alpha-\theta)}}{r}\qquad\left\{\begin{array}{l}\text{as }\mu=\dfrac{ka}{\pi}\\ \text{and }z=re^{\theta i}\end{array}\right.$$

or
$$\phi+i\psi=-\frac{\mu i}{r}\left\{\cos(\alpha-\theta)+i\sin(\alpha-\theta)\right\}$$

or
$$\psi=-\frac{\mu\cos(\alpha-\theta)}{r}$$

If the vortex doublet be at O, then

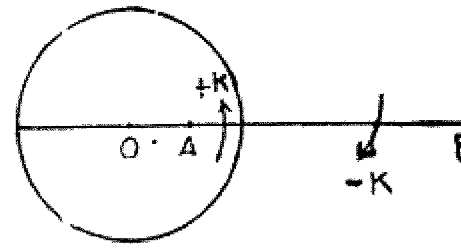
$$\psi=-\frac{\mu\sin\theta}{r}\qquad\{\;\mu=Ub^2\text{ (let)}$$

So
$$\psi=-\frac{Ub^2\sin\theta}{r}$$

which is the stream function for a circular cylinder of radius b moving with velocity U along the X-axis.

Thus the motion due to a circular cylinder is the same as that due to a suitable vortex doublet placed at the centre with axis perpendicular to the direction of motion.

§ 7·43. Vortex inside an infinite circular cylinder.



Consider the vortex of strength $+k$ be at the point A inside the circular cylinder of radius a with axis parallel to the axis of the cylinder. Put an equal and opposite vortex at an inverse point B, such that

$$OA.OB=a^2$$

or $$OB=\frac{a^2}{f} \qquad \left\{ \text{as } OA=f \right.$$

The circle is one of the co-axial system having A and B as limiting points hence it is a stream line.

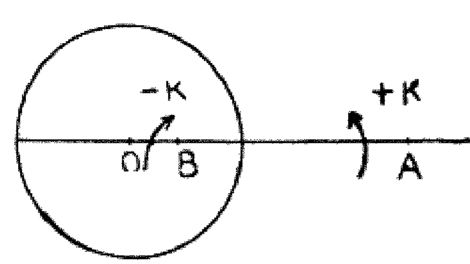
The velocity of A is given by

$$=\frac{k}{2\pi\cdot AB}=\frac{k}{2\pi\left(\frac{a^2}{f}-f\right)}=\frac{kf}{2\pi(a^2-f^2)}$$

(perpendicular to OA)

The velocity of B is same that of A so OAB will not remain a straight line at the next instant. If A describes a circle about O with the above velocity, then at every instant the circle will be a stream line.

§ 7·44. Vortex outside a circular cylinder.



Consider the vortex of strength $+k$ be at the point A outside the circular cylinder of radius a. Let there is a vortex of strength $-k$ at an inverse point A (*i.e.* B).

The velocity of A is given by

$$=\frac{k}{2\pi . AB}$$

$$=\frac{k}{2\pi\left(f-\frac{a^2}{f}\right)}=\frac{kf}{2\pi(f^2-a^2)} \qquad \left\{ \begin{array}{ll} \text{where} & OA=f \\ \text{and} & OA.OB=a^2 \\ \text{or} & OB=\frac{a^2}{f} \\ \text{or} & AB=f-\frac{a^2}{f} \end{array} \right.$$

Now the vortex of strength $-k$ at B gives a circulation $-k$ about the cylinder. Let k_1 be the circulation about the cylinder. So we have the vortex of strength $+k$ at A, $-k$ at B and $k+k_1$

at the centre O. Then the velocity of A due to above system is given by

$$=\frac{k+k_1}{2\pi . OA}-\frac{k}{2\pi . AB}$$

$$=\frac{k+k_1}{2\pi f}-\frac{k}{2\pi\left(f-\frac{a^2}{f}\right)}$$

$$=\frac{k+k_1}{2\pi f}-\frac{kf}{2\pi(f^2-a^2)}$$

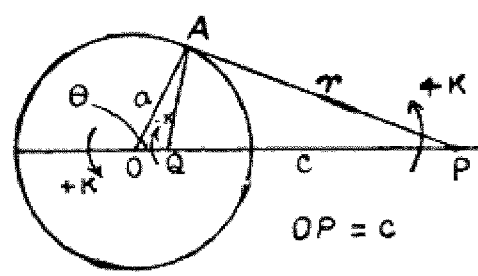
$$=\frac{k_1(f^2-a^2)-ka^2}{2\pi f(f^2-a^2)}$$

Thus A describes a circle with this velocity.

Ex. 1. *A long fixed cylinder of radius a is surrounded by infinite frictionless incompressible liquid, and there is a vortex filament of strength k in the liquid which is parallel to the axis of the cylinder at a distance c (c < a) from this axis. Given that there is no circulation round any circuit enclosing the cyliuder but not the filament. Show that the speed q of the fluid at the surface of the cylinder is*

$$\frac{k}{2\pi a}\left(1-\frac{c^2-a^2}{r^2}\right),$$

r being the distance of the point considered from the filament.



Since there is no circulation round the cylinder. The vortex of strength $+k$ at P gives rise to a vortex of strength $-k$ at an inverse point Q and a vortex of strength $+k$ at the centre O.

$$OP . OQ=a^2$$

or $$OQ=\frac{a^2}{c}$$

The complex potential is given by

$$w=\frac{ik}{2\pi}\log(z-c)-\frac{ik}{2\pi}\log\left(z-\frac{a^2}{c}\right)+\frac{ik}{2\pi}\log z$$

or $$\frac{dw}{dz}=\frac{ik}{2\pi}\left\{\frac{1}{z-c}-\frac{1}{z-\frac{a^2}{c}}+\frac{1}{z}\right\}$$

$$=\frac{ik}{2\pi}\left\{\frac{z^2-2\frac{a^2}{c}\,z+a}{z\,(z-c)\left(z-\frac{a^2}{c}\right)}\right\}$$

Thus $$q=\left|\frac{dw}{dz}\right|=\frac{k}{2\pi}\cdot\frac{\left|z^2-2\frac{a^2}{c}\,z+a^2\right|}{|z|\;|z-c|\left|z-\frac{a^2}{c}\right|}$$

$$=\frac{k}{2\pi}\cdot\frac{\left|z^2-2\frac{a^2}{c}\,z+a^2\right|}{OA.PA.QA}$$

Again from similar triangles,

$$\frac{QA}{a}=\frac{r}{c}$$

Then $$z^2-2\frac{a^2}{c}\,z+a^2=\left(z-\frac{a^2}{c}\right)^2+a^2-\frac{a^4}{c^2}$$

$$=\left(ae^{\theta i}-\frac{a^2}{c}\right)^2+\left(a^2-\frac{a^4}{c^2}\right)$$

$$=\left\{\left(a\cos\theta-\frac{a^2}{c}\right)+ia\sin\theta\right\}^2+\left(a^2-\frac{a^4}{c^2}\right)$$

$$=\left(a\cos\theta-\frac{a^2}{c}\right)^2-a^2\sin^2\theta$$

$$+2ia\sin\theta\left(a\cos\theta-\frac{a^2}{c}\right)+\left(a^2-\frac{a^4}{c^2}\right)$$

$$=a^2\cos^2\theta-\frac{2a^3}{c}\cos\theta+\frac{a^4}{c^2}-a^2\sin^2\theta+a^2-\frac{a^4}{c^2}$$

$$+2ia\sin\theta\left(a\cos\theta-\frac{a^2}{c}\right)$$

$$=2a^2\cos^2\theta-\frac{2a^3}{c}\cos\theta+2ia\sin\theta\left(a\cos\theta-\frac{a^2}{c}\right)$$

and $$\left|z^2-\frac{2a^2}{c}\,z+a^2\right|$$

$$=2a\sqrt{\left\{\left(a\cos^2\theta-\frac{a^2}{c}\cos\theta\right)^2\right.}$$

$$\left.+\sin^2\theta\left(a\cos\theta-\frac{a^2}{c}\right)^2\right\}$$

$$=2a\left\{\frac{a^2}{c}-a\cos\theta\right\}$$

$$=2a\left\{\frac{a^2}{c}-\frac{a}{2ac}(a^2+c^2-r^2)\right\}$$

$$\left\{\text{as } \cos\theta=\frac{a^2+c^2-r^2}{2ac}\right.$$

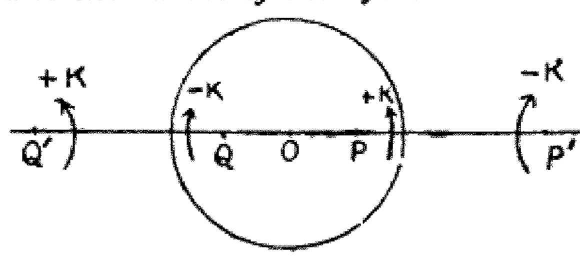
$$=\frac{a}{c}\left\{a^2-c^2+r^2\right\}$$

Hence $$q=\frac{k}{2\pi}\cdot\frac{\frac{a}{c}(a^2-c^2+r^2)}{a.r.\frac{ar}{c}}$$

$$=\frac{k}{2\pi}\cdot\frac{a^2-c^2+r^2}{ar^2}$$

$$=\frac{k}{2\pi a}\left\{1-\frac{c^2-a^2}{r^2}\right\}$$ **Proved.**

Ex. 2. *If a vortex pair is situated within a cylinder, show that it will remain at rest if the distance of either from the centre is given by* $(\sqrt{5}-2)^{1/2}\,a$, *where a is the radius of the cylinder.*



Let P, Q be the vortex-pair where $OP=OQ=r$. The image system give rise to a vortex of strength $-k$ at an inverse point of P (*i e.* P') and a vortex of strength $+k$ at an inverse point of Q (*i.e.* Q'), such that

$$OP.OP'=a^2=OQ.OQ'$$

or $$OP'=\frac{a^2}{r}=OQ'$$

The vortex will remain at rest if its velocity due to other three vortices be zero.

i.e. $$\frac{k}{2\pi}\left\{\frac{1}{PP'}-\frac{1}{PQ}+\frac{1}{Q'P}\right\}=0$$

or $$\frac{k}{2\pi}\left\{\frac{1}{\frac{a^2}{r}-r}-\frac{1}{2r}+\frac{1}{\frac{a^2}{r}+r}\right\}=0$$

or $$2r\left(\frac{a^2}{r}+r\right)-\left(\frac{a^2}{r}-r\right)\left(\frac{a^2}{r}+r\right)+2r\left(\frac{a^2}{r}-r\right)=0$$

or $$r^4+4a^2r^2-a^4=0$$

or $$\left(\frac{r^2}{a^2}\right)^2+4\left(\frac{r^2}{a^2}\right)-1=0$$

or $$\frac{r^2}{a^2}=(\sqrt{5}-2)$$

or $$r=a\sqrt{(\sqrt{5}-2)}$$

or $$r=a\,(\sqrt{5}-2)^{1/2}$$ **Proved.**

Ex. 3. *Investigate the nature of the motion of the liquid.*

$$u=\frac{ax-bv}{x^2+y^2}\quad v=\frac{ay+bx}{x^2+y^2}\ \textit{and}\ w=0.$$

Also determine the pressure at any point (x, y).

Since $w=0$ and u and v are functions of x and y only. So the motion is two-dimensional. The vorticity vector is

$$\zeta=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y} \quad \text{.. (i)}$$

and $$\xi=0,\ \eta=0.$$

or $$\frac{\partial u}{\partial y}=-\frac{b\,(x^2-y^2)+2axy}{(x^2+y^2)^2}$$

and $$\frac{\partial v}{\partial x}=-\frac{b\,(x^2-y^2)+2axy}{(x^2+y^2)^2}$$

$\Rightarrow$ that $\zeta=0$

Thus the motion is **irrotational in two dimensions.**

Also $$q^2=u^2+v^2$$

or $$q^2=\frac{(ax-by)^2+(ay+bx)^2}{(x^2+y^2)^2}$$

or $$q^2=\frac{a^2x^2+b^2y^2-2ab\ xy+a^2y^2+b^2x^2+2ab\ xy}{(x^2+y^2)^2}$$

or $$q^2=\frac{a^2+b^2}{x^2+y^2}$$

Thus the pressure is given by

$$\frac{p}{\rho}=\text{constant }-\tfrac{1}{2}\,q^2$$

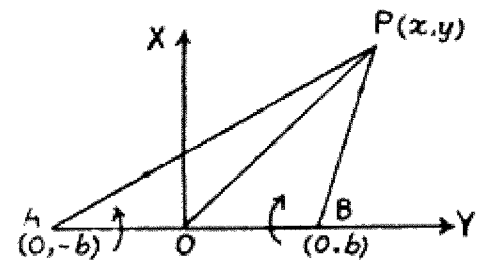
$$\frac{p}{\rho}=\text{constant }-\tfrac{1}{2}\,\frac{a^2+b^2}{x^2+y^2}$$

which gives the pressure at any point $(x\ y)$. **Ans.**

Ex. 4. *When an infinite liquid contains two parallel equal and opposite rectilinear vortices at a distance $2b$, prove that the stream lines relative to the vortices are given by the equation*

$$\log\left\{\frac{x^2+(y-b)^2}{x^2+(y+b)^2}\right\}+\frac{y}{b}=C,$$

the origin being the middle point of the join, which is taken for axis of Y.



Since the vortex system is moving with uniform velocity $\frac{k}{2\pi(AB)}$ perpendicular to the line AB Now to reduce the vortex system to rest, superposing a velocity $-\frac{k}{4\pi b}$. If ψ' be the stream function of the superposed system, we have

$$-\frac{\partial\psi'}{\partial y}=-\frac{k}{4\pi b}$$

or
$$\psi'=\frac{kv}{4\pi b}$$

If w_1 be the complex potential then

$$w_1=\frac{k}{4\pi b}z \qquad \ldots(i)$$

The complex potential at any point $P(x, y, z)$ due to fixed vortices of strength k and $-k$ is given by

$$w_2=\frac{ik}{2\pi}\log(z-ib)-\frac{ik}{2\pi}\log(z+ib) \qquad \ldots(ii)$$

Thus the total complex potential at a point is given by

$$w=\frac{ik}{2\pi}\log(z-ib)-\frac{ik}{2\pi}\log(z+ib)+\frac{k}{4\pi b}z$$

or
$$w=\frac{ik}{2\pi}\log\left\{x+i(y-b)\right\}-\frac{ik}{2\pi}\log\left\{x+i(y+b)\right\}+\frac{k}{4\pi b}(x+iy)$$

Equating the imaginary parts we have the stream function of the relative motion,

$$\psi=\frac{k}{2\pi}\log\{x^2+(y-b)^2\}^{1/2}-\frac{k}{2\pi}\log\{x^2+(y+b)^2\}^{1/2}+\frac{k}{4\pi b}y$$

$$\psi=\frac{k}{4\pi}\log\left\{\frac{x^2+(y-b)^2}{x^2+(y+b)^2}\right\}+\frac{k}{4\pi}\cdot\frac{y}{b}$$

Thus the relative stream lines are given by

$$\psi=\text{constant}$$

or
$$\frac{k}{4\pi}\log\left\{\frac{x^2+(y-b)^2}{x^2+(y+b)^2}\right\}+\frac{k}{4\pi}\cdot\frac{y}{b}=\text{constant}$$

or
$$\log\left\{\frac{x^2+(y-b)^2}{x^2+(y+b)^2}\right\}+\frac{y}{b}=C$$

Proved.

Ex. 5. *An in Ex. 2. If the vortices are of the same strength, and the spin is in the same sense in both, shew that the relative stream lines are given by*

$$\log\,(r^4+b^4-2b^2r^2\cos 2\theta)-\frac{r^2}{2b^2}=\text{constant}.$$

θ being measured from the join of the vortices, the origin being its middle point.

Ref. § 7 4.

Here the distance $AB=2b$.

then
$$\psi=\frac{k}{2\pi}\log\,(r^4+b^4-2b^2r^2\cos 2\theta-\frac{kr^2}{8\pi b^2}.$$

The stream lines are given by

$$\psi=\text{constant}.$$

or
$$\frac{k}{4\pi}\log\,(r^4+b^4-2b^2r^2\cos 2\theta)-\frac{kr^2}{8\pi b^2}=\text{constant}$$

or
$$\log\,(r^4+b^4-2b^2r^2\cos 2\theta-\frac{r^2}{2b^2}=\text{constant}.$$

Proved.

Ex. 6. *An infinite liquid contains two parallel, equal and opposite rectilinear vortex filaments at a distance 2b. Shew that the paths of the fluid particles relative to the vortices can be represented by the equation*

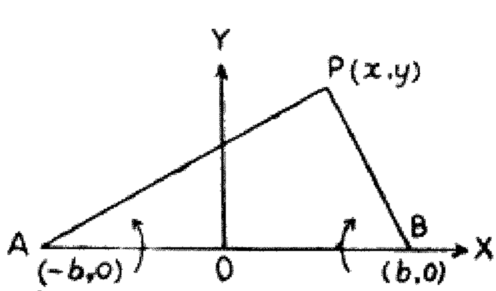
$$\log\frac{r^2+b^2-2br\cos\theta}{r^2+b^2+2br\cos\theta}+\frac{r\cos\theta}{b}=\text{constant}.$$

Consider X-axis as the line on which the two vortices of strength $+k$ and $-k$ lies. Ref. Q. No. 2.

The stream function ψ is given by

$$\psi=\frac{k}{4\pi}\left[\log\left\{\frac{y^2+(x-b)^2}{y^2+(x+b)^2}\right\}+\frac{x}{b}\right]$$

or
$$\psi=\frac{k}{4\pi}\left[\log\left\{\frac{x^2+y^2+b^2-2bx}{x^2+y^2+b^2+2bx}\right\}+\frac{x}{b}\right]$$



Changing it into Polar coordinates, we have

$$\psi=\frac{k}{4\pi}\left[\log \frac{r^2+b^2-2br\cos\theta}{r^2+b^2+2br\cos\theta}+\frac{r\cos\theta}{b}\right]$$

The stream lines are given by

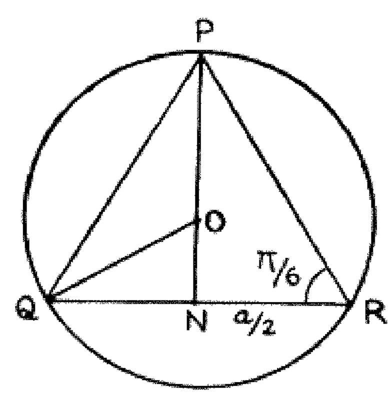
$$\psi=\text{constant}.$$

or
$$\frac{k}{4\pi}\left[\log \frac{r^2+b^2-2br\cos\theta}{r^2+b^2+2br\cos\theta}+\frac{r\cos\theta}{b}\right]=\text{constant}$$

or
$$\log \frac{r^2+b^2-2br\cos 2\theta}{r^2+b^2+2br\cos 2\theta}+\frac{r\cos\theta}{b}=\text{constant}.$$
Proved.

Ex. 7. *Three parallel rectilinear vortices of same strength k and in the same sense meet any plane perpendicular to them in an equilateral triangle of side a, Shew that the vortices all move round the same cylinder with uniform speed in time*

$$\frac{2\pi a^2}{3k}.$$



Let r be the radius of the circum-circle of equilateral triangle PQR.

Then
$$r=\frac{a}{\sqrt{3}}.$$

The complex potential of the vortices at the points

$$w=ik\,\{\log(z-re^{2\pi i/3})+\log(z-re^{4\pi i/3})+\log(z-re^{6\pi i/3})\}$$

or
$$w=ik\,[\log\{(z-re^{2\pi i/3})(z-re^{4\pi i/3})(z-re^{6\pi i/3})\}]$$

or
$$w=ik\log(z^3-r^3).$$

The velocity induced at $z=re^{2\pi i}=r$ by the other vortices is

$$w_1=u_1-iv_1=-\frac{d}{dz}\left\{ik\log(z^3-r^3)-ik\log(z-r)\right\}$$

or
$$u_1-iv_1=ik\left\{-\frac{3z^2}{z^3-r^3}+\frac{1}{z-r}\right\}$$

or
$$u_1-iv_1=-ik\,\frac{2z+r}{z^2+zr+r^2}$$

Thus
$$q_1=|u_1-iv_1|=k\left|\frac{2z+r}{z^2+zr+r^2}\right|_{z=r}$$
$$=\frac{k}{r}.$$

Hence the required time, is given by

$$=\frac{2\pi a}{\sqrt{3}}\Big/\frac{k}{r}=\frac{2\pi a}{\sqrt{3}}\cdot\frac{a}{\sqrt{3k}}$$

$$=\frac{2\pi a^2}{3k}.$$

Ex. 8. *Prove that the necessary and sufficient condition that the vortex lines may be at right angles to the stream lines are*

$$u,\, v,\, w=\mu\left(\frac{\partial\phi}{\partial x},\, \frac{\partial\phi}{\partial y},\, \frac{\partial\phi}{\partial z}\right)$$

where μ, ϕ are functions of x, y, z, t.

The stream lines are given by

$$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w} \qquad \ldots(1)$$

and the vortex lines are given by

$$\frac{dx}{\xi}=\frac{dy}{\eta}=\frac{dz}{\zeta} \qquad \ldots(2)$$

(1) and (2) will be at right angles, if

$$u\,\xi+v\,\eta+w\,\zeta=0$$

or
$$u\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)+v\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)+w\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)=0.$$

The condition that

$$u\,dx+v\,dy+w\,dz$$

is a perfect differential is

$$u\,dx+v\,dy+w\,dz=\mu\,d\phi$$

$$=\mu\left(\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz\right)$$

or
$$u=\mu\,\frac{\partial\phi}{\partial x},\quad v=\mu\frac{\partial\phi}{\partial y},\quad w=\mu\,\frac{\partial\phi}{\partial z}.$$ **Proved.**

Ex. 9. *In an incompressible fluid the velocity at every point is constant in magnitude and direction : Show that the components of velocity u, v, w are solutions of Laplace's equations.*

Let ξ, η, ζ are the components of velocity ω.

So that $\omega=\sqrt{(\xi^2+\eta^2+\zeta^2)}$.

The direction cosines of its direction are

$$\frac{\xi}{\omega},\, \frac{\eta}{\omega},\, \frac{\zeta}{\omega}.$$

The spin component ξ, η, ζ are each equal to constant

i.e.
$$2\xi=\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}, \qquad \ldots(\text{i})$$

$$2\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \qquad \ldots\text{(ii)}$$

$$2\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}. \qquad \ldots\text{(iii)}$$

Differentiating (ii) with regard to z and (iii) w. r. to y and subtracting, we have

$$\frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 w}{\partial x\, \partial z} - \frac{\partial^2 v}{\partial x\, \partial y} + \frac{\partial^2 u}{\partial y^2} = 0.$$

or
$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0.$$

But
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

then
$$\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = 0$$

or
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$$

which satisfies the Laplace's Equation. **Proved.**

Ex. 10. *If $u\,dx + v\,dy + w\,dz = d\theta + \lambda\, d\mu$ where θ, λ, μ are functions of x, y, z, t prove that the vortex lines at any time are the lines of intersection of the surfaces $\lambda = \text{constant}$ and $\mu = \text{constant}$.*

or
$$u\,dx + v\,dy + w\,dz = \frac{\partial \theta}{\partial x}dx + \frac{\partial \theta}{\partial y}dy + \frac{\partial \theta}{\partial z}\,dz + \frac{\partial \theta}{\partial t}\,dt$$
$$+\lambda\,\frac{\partial \mu}{\partial x}dx + \lambda\frac{\partial \mu}{\partial y}dy + \lambda\frac{\partial \mu}{\partial z}dz + \lambda\frac{\partial \mu}{\partial t}\,dt$$

then
$$u = \frac{\partial \theta}{\partial x} + \lambda\frac{\partial \mu}{\partial x}, \quad v = \frac{\partial \theta}{\partial y} + \lambda\,\frac{\partial \mu}{\partial y},$$
$$w = \frac{\partial \theta}{\partial z} + \lambda\,\frac{\partial \mu}{\partial z}, \quad 0 = \frac{\partial \theta}{\partial t} + \lambda\,\frac{\partial \mu}{\partial t}.$$

The components of spin are

$$2\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}$$

or
$$2\xi = \frac{\partial^2 \theta}{\partial y\,\partial z} + \lambda\,\frac{\partial^2 \mu}{\partial y\,\partial z} + \frac{\partial \lambda}{\partial y}\cdot\frac{\partial \mu}{\partial z} - \frac{\partial^2 \theta}{\partial y\,\partial z} - \lambda\,\frac{\partial^2 \mu}{\partial y\,\partial z} - \frac{\partial \lambda}{\partial z}\cdot\frac{\partial \mu}{\partial y}$$

or
$$2\xi = \frac{\partial \lambda}{\partial y}\cdot\frac{\partial \mu}{\partial z} - \frac{\partial \lambda}{\partial z}\cdot\frac{\partial \mu}{\partial y}$$

or
$$2\xi = \begin{vmatrix} \dfrac{\partial \lambda}{\partial y} & \dfrac{\partial \lambda}{\partial z} \\[2ex] \dfrac{\partial \mu}{\partial y} & \dfrac{\partial \mu}{\partial z} \end{vmatrix}$$

Similarly $\quad 2\eta = \begin{vmatrix} \frac{\partial \lambda}{\partial z} & \frac{\partial \lambda}{\partial x} \\ \frac{\partial \mu}{\partial z} & \frac{\partial \mu}{\partial x} \end{vmatrix}$ and $2\zeta = \begin{vmatrix} \frac{\partial \lambda}{\partial x} & \frac{\partial \lambda}{\partial y} \\ \frac{\partial \mu}{\partial x} & \frac{\partial \mu}{\partial y} \end{vmatrix}$

$$\therefore \quad 2\left(\xi \frac{\partial \lambda}{\partial x} + \eta \frac{\partial \lambda}{\partial y} + \zeta \frac{\partial \lambda}{\partial z}\right) = \begin{vmatrix} \lambda_x & \lambda_y & \lambda_z \\ \lambda_x & \lambda_y & \lambda_z \\ \mu_x & \mu_y & \mu_z \end{vmatrix} = 0.$$

So $\qquad \xi \lambda_x + \eta \lambda_y + \zeta \lambda_z = 0.$

Similarly $\qquad \xi \mu_x + \eta \mu_y + \zeta \mu_z = 0. \quad \Rightarrow$ that vortex lines lie on the surfaces λ=constant and μ=constant. **Proved.**

§ 7·5. Image of a vortex filament in a plane.

Consider the strengths of the vortices are k and $-k$ (equal but of opposite sign) at z_1 and z_2. The complex potential at any point is given by

$$w = \frac{ik}{2\pi} \log(z - z_1) - \frac{ik}{2\pi} \log(z - z_2)$$

$$w = \frac{ik}{2\pi} \log(r_1 e^{\theta_1 i}) - \frac{ik}{2\pi} \log(r_2 e^{\theta_2 i})$$

$$w = \frac{ik}{2\pi} (\log r_1 + \theta_1 i) - \frac{ik}{2\pi} (\log r_2 + \theta_2 i)$$

or $$\psi = \frac{k}{2\pi} \log\left(\frac{r_1}{r_2}\right).$$

If $r_1 = r_2$ then $\psi = 0$, $\Rightarrow$ that the flow across the plane is zero.

Thus the image of such a vortex with regard to a parallel plane is therefore an equal vortex symmetrically placed, the rotation of the two being in opposite sense.

Similarly the image of a vortex of strength $+k$ outside the circular cylinder consists of a vortex filament of strength $-k$ at an inverse point and a vortex filament of strength $+k$ at the centre of the cylinder.

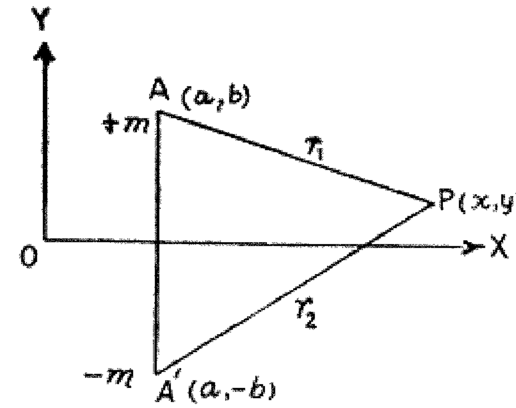
Ex. 11. *An infinitely long line vortex of strength m, parallel to the axis of Z; is situated in infinite liquid bounded by a rigid wall in the plane $y=0$. Prove that, if there be no field of force, the surfaces of equal pressure are given by*

$$\{(x-a)^2 + (y-b)^2\}\{(x-a)^2 + (y+b)^2\} = \lambda' \{y^2 + b^2 - (x-a)^2\}$$

where (a, b) are the coordinates of the vortex, and λ' is a parametric constant.

The image of the vortex of strength $+m$ at $A\ (a, b)$ is an equal and opposite vortex of strength $-m$ at the point $A'\ (a, -b)$. Two vortex together form a vortex pair with the line joining A and A' perpendicular to the X-axis.

$$AA'=2b.$$



The stream function for the steady motion is given by

$$\psi=\frac{m}{2\pi}\left\{\log\frac{r_1}{r_2}+\frac{y}{2b}\right\}$$

or
$$\psi=\frac{m}{4\pi}\left\{\log\frac{r_1^2}{r_2^2}+\frac{y}{b}\right\}$$

or
$$\psi=\frac{m}{4\pi}\Big[\log\{(x-a)^2+(y-b)^2\}$$
$$-\log\{(x-a)^2+(y+b)^2\}+\frac{y}{b}\Big]$$

The components of velocity at a point (x, y) are given by,

$$u=-\frac{\partial\psi}{\partial y}=-\frac{m}{2\pi}\left[\frac{y-b}{(x-a)^2+(y-b)^2}-\frac{y+b}{(x-a)^2+(y+b)^2}+\frac{1}{2b}\right]$$
$$=-\frac{m}{2\pi}\left[\frac{y-b}{r_1^2}-\frac{y+b}{r_2^2}+\frac{1}{2b}\right]$$

and
$$v=\frac{\partial\psi}{\partial x}=\frac{m}{2\pi}\left[\frac{x-a}{(x-a)^2+(y-b)^2}-\frac{x-a}{(x-a)^2+(y+b)^2}\right]$$
$$=\frac{m}{2\pi}\left[\frac{x-a}{r_1^2}-\frac{x-a}{r_2^2}\right]$$

If q be the velocity at the point $P\ (x, y)$, then

$$q^2=u^2+v^2$$

or
$$q^2=\frac{m^2}{4\pi^2}\Bigg[\frac{(x-a)^2+(y-b)^2}{r_1^4}+\frac{(x-a)^2+(y+b)^2}{r_2^4}$$
$$-\frac{2\{x-a)^2+(y^2-b^2)\}}{r_1^2\,r_2^2}+\frac{1}{b}\left(\frac{y-b}{r_1^2}-\frac{y+b}{r_2^2}\right)+\frac{1}{b^2}\Bigg]$$
$$=\frac{m^2}{4\pi^2}\Bigg[\frac{1}{r_1^2}+\frac{1}{r_2^2}-\frac{2\{(x-a)^2+y^2-b^2\}}{r_1^2r_2^2}$$
$$+\frac{1}{b}\left(\frac{y-b}{r_1^2}-\frac{y+b}{r_2^2}\right)+\frac{1}{4b^2}\Bigg] \qquad \text{...(i)}$$

Since the motion is steady, thus the pressure is given by the equation

$$\frac{p}{\rho}=C-\frac{1}{2}q^2 \qquad \ldots(ii)$$

where C is a constant.

The surfaces of equal pressure are given by

$$p=\text{constant}.$$

i.e. $$q^2=\text{constant}. \qquad \{\text{from (ii)}$$

or $$\frac{1}{r_1^2}+\frac{1}{r_2^2}-\frac{2\,\{(x-a)^2+y^2-b^2\}}{r_1^2r_2^2}+\frac{1}{b}\left(\frac{y-b}{r_1^2}-\frac{y+b}{r_2^2}\right)+\frac{1}{4b^2}=\text{const.}=\frac{1}{\lambda}\text{ (say).}$$

or $$\lambda\left[r_1^2+r_2^2-2\,\{(x-a)^2+y^2-b^2\}+\frac{y}{b}(r_2^2-r_1^2)-(r_1^2+r_2^2)\right]=r_1^2r_2^2$$

or $$\lambda\left[-2\,(x-a)^2-2\,(y^2-b^2)+\frac{y}{b}(4yb)\right]=r_1^2r_2^2$$

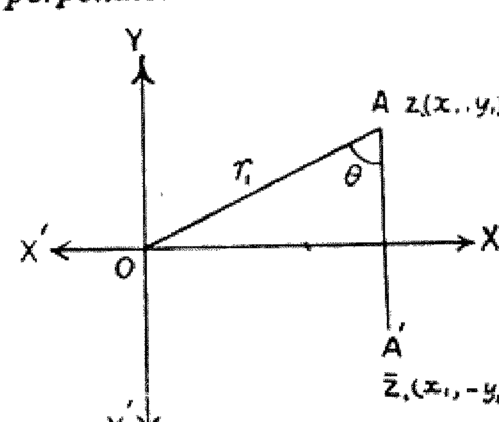
or $$\lambda\,[2\,\{y^2+b^2-(x-a)^2\}]=r_1^2r_2^2$$

or $$[\{(x-a)^2+(y-b)^2\}\,\{(x-a)^2+(y+b)^2\}]=\lambda'\,[y^2+b^2-(x-a)^2].$$

Proved.

Ex. 12. *Find the motion of a straight vortex filament in an infinite region bounded by an infinite plane wall to which the filament is parallel, and prove that the pressure defect at any point of the wall due to the filament is proportional to $\cos^2\theta\cos 2\theta$ where θ is the inclination of the plane through the filament and the point to the plane through the filament perpendicular to the wall.*

Let $A\,(x_1, y_1)$ be the position of the vortex of strength $+k$ at any instant with respect to the section of the plane XOX'. The vortex will have its image an equal and opposite vortex of strength $-k$ at the points $A'\,(x_1, -y_1)$. These two vortices together form a vortex pair.



The motion of the vortex A will be due to the vortex B which shall move with velocity

$$=\frac{k}{2\pi\,(AA')}=\frac{k}{4\pi y_1} \text{ parallel to the } X\text{-axis.}$$

Since the vortex A moves parallel to X-axis *i e.* x_1 will change with regard to time but y_1 remains constant

Let O be the fixed point on the plane. The pressure equation is given by

$$\frac{p}{\rho}=\frac{\partial\phi}{\partial t}-\tfrac{1}{2}q^2+f(t) \qquad \ldots\text{(i)}$$

Let p_0 be the pressure when there was no vortex,

then $$q=0 \text{ and } \frac{\partial\phi}{\partial t}=0.$$

So $$\frac{p_0}{\rho}=f(t). \qquad \ldots\text{(ii)}$$

From (i) and (ii), we have

$$\frac{p-p_0}{\rho}=\frac{\partial\phi}{\partial t}-\tfrac{1}{2}q^2. \qquad \ldots\text{(iii)}$$

Now the complex potential due to the vortex pair

$$w=\frac{ik}{2\pi}\log(z-z_1)-\frac{ik}{2\pi}\log(z-\bar{z}_1) \qquad \ldots\text{(iv)}$$

$$\frac{dw}{dz}=\frac{ik}{2\pi}\cdot\frac{1}{z-z_1}-\frac{ik}{2\pi}\cdot\frac{1}{z-\bar{z}_1}$$

Let (u_0, v_0) be the components of velocity of O,

then $$u_0-iv_0=-\left(\frac{dw}{dz}\right)_{z=0}=\frac{ik}{2\pi}\left[\frac{1}{z_1}-\frac{1}{\bar{z}_1}\right]$$

$$=\frac{ik}{2\pi}\cdot\frac{\bar{z}_1-z_1}{z_1\bar{z}_1}$$

$$=\frac{ik}{2\pi}\cdot\frac{-2iy_1}{x_1^2+y_1^2} \qquad \left\{\begin{array}{l}\text{as } z_1=x_1+iy_1\\ \bar{z}_1=x_1-iy_1\end{array}\right.$$

Equating real and imaginary parts, we have

$$u_0=\frac{ky_1}{\pi\,(x_1^2+y_1^2)} \quad \text{and} \quad v_0=0$$

or $$q^2=u_0^2+v_0^2$$

$$=\frac{k^2y_1^2}{\pi^2\,(x_1^2+y_1^2)^2} \qquad \left\{\begin{array}{l}\text{Since } y_1=r\cos\theta\\ x_1=r\sin\theta\end{array}\right.$$

$$=\frac{k^2r^2\cos^2\theta}{\pi^2r^4}=\frac{k^2\cos^2\theta}{\pi^2r^2}=\frac{k^2\cos^4\theta}{\pi^2y_1^2}. \qquad \ldots\text{(v)}$$

Also the velocity potential at $z=0$, is

$$(\phi)_{z=0}=\text{Real part of}\left[\frac{ik}{2\pi}\log(-z_1)-\frac{ik}{2\pi}\log(-\bar{z}_1)\right]$$

(from (iv))

or
$$\phi = \text{Real part of } \frac{ik}{2\pi}\left[\, i\tan^{-1}\frac{y_1}{x_1} + i\tan^{-1}\frac{y_1}{x_1}\right]$$

or
$$\phi = -\frac{k}{\pi}\tan^{-1}\left(\frac{y_1}{x_1}\right)$$

Therefore
$$\frac{\partial\phi}{\partial t} = -\frac{k}{\pi}\cdot\frac{1}{1+\dfrac{y_1^2}{x_1^2}}\left(-\frac{y_1}{x_1^2}\dot{x}_1\right)$$
$$= \frac{k}{\pi}\cdot\frac{x_1^2}{x_1^2+y_1^2}\quad\frac{y_1\dot{x}_1}{x_1^2} = \frac{k}{\pi}\cdot\frac{y_1\dot{x}_1}{x_1^2+y_1^2}$$
$$= \frac{k}{\pi}\cdot\frac{1}{r_1^2}y_1\cdot\frac{k}{4\pi y_1}$$
$$= \frac{k^2}{4\pi^2}\cdot\frac{1}{r_1^2}$$

or
$$\frac{\partial\phi}{\partial t} = \frac{k^2}{4\pi^2}\cdot\frac{\cos^2\theta}{r_1^2\cos^2\theta}$$
$$= \frac{k^2\cos^2\theta}{4\pi^2 y_1^2} \qquad \text{...(vi)}$$

Substituting the value of q^2 and $\dfrac{\partial\phi}{\partial t}$ in (iii) we have
$$\frac{p-p_0}{\rho} = \frac{k^2\cos^2\theta}{4\pi^2 y_1^2} - \frac{1}{2}\ \frac{k^2\cos^4\theta}{\pi^2 y_1^2}$$

or
$$\frac{p-p_0}{\rho} = -\frac{k^2}{4\pi^2 y_1^2}\{2\cos^4\theta - \cos^2\theta\}$$

or
$$\frac{p-p_0}{\rho} = -\frac{k^2}{4\pi^2 y_1^2}\{2\cos^2\theta - 1\}\cos^2\theta$$

or
$$p_0 - p = \frac{k^2\rho}{4\pi^2 y_1^2}\cos 2\theta\cos^2\theta.$$

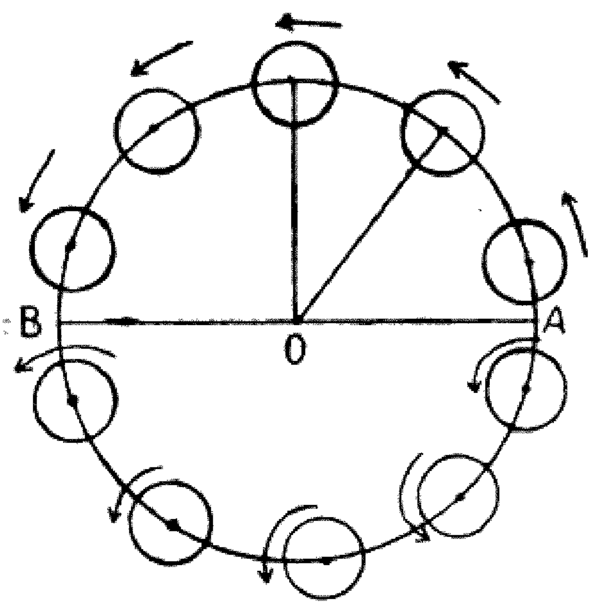
Thus the pressure defect at any point O of the wall due to the filament is proportional to $\cos 2\theta\cos^2\theta$. **Proved.**

Ex. 13. *If n rectilinear vortices of the same strength k are symmetrically arranged as generators of a circular cylinder of radius a in an infinite liquid prove that the vortices will move round the cylinder uniformly in time*
$$\frac{8\pi^2 a^2}{(n-1)\,k}\cdot$$
and find the velocity at any point of the liquid.

Since the n rectilinear vortices of strength k are symmetrically distributed then the angular distance between any two consecutive vortices is $\frac{2\pi}{n}$. Let the line through centre of the cylinder and one of vortices be taken as X-axis. Thus the vortices are at

$$z = ae^{0},\ ae^{2\pi i/n},\ ae^{4\pi i/n},\ \ldots\ ae^{2\pi ri/n},\ \ldots\ ae^{2\pi(n-1)i/n},$$

which are the n roots of the equation $z^n - a^n = 0$.



The complex potential due to n vortices is given by

$$w = \frac{ik}{2\pi}\ \{\log\ (z-a)\ (z-ae^{2\pi i/n})\ (z-ae^{4\pi i/n}) \ldots (z-ae^{2\pi\ (n-1)i/n})\}$$

or $\quad w = \frac{ik}{2\pi} \log\ (z^n - a^n)$

or $\quad w = \frac{ik}{2\pi} \log\ \{r^n\ (\cos n\theta + i \sin n\theta) - a^n\}$

or $\quad \phi + i\psi = \frac{ik}{2\pi} \log\ \{(r^n \cos n\theta - a^n) + ir^n \sin n\theta\}$

Equating imaginary parts, we have the stream function

$$\psi = \frac{k}{4\pi} \log\ \{(r^n \cos\ n\theta - a^n)^2 + (r^n \sin n\theta)^2\}$$

$$\psi = \frac{k}{4\pi} \log\ \{r^{2n} + a^{2n} - 2r^n a^n \cos n\theta\}$$

Let q be the velocity at any point of the liquid.

Then $\quad q^2 = \left(\frac{\partial\psi}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial\psi}{\partial\theta}\right)^2$

or $$q^2=\frac{k^2}{16\pi^2}\left\{\left(\frac{2nr^{2n-1}-2nr^{n-1}\,a^n\cos n\theta}{r^{2n}+a^{2n}-2r^n\,a^n\cos n\theta}\right)^2+\frac{1}{r^2}\cdot\left(\frac{2nr^n\,a^n\sin n\theta}{r^{2n}+a^{2n}-2r^n\,a^n\cos n\theta}\right)^2\right\}$$

or $$q^2=\frac{n^2k^2}{4\pi^2}\left\{\frac{(r^{2n-1}-r^{n-1}\,a^n\cos n\theta)^2+r^{2n-2}.a^{2n}\sin^2 n\theta}{(r^{2n}+a^{2n}-2r^n\,a^n\cos n\theta)^2}\right\}$$

or $$q^2=\frac{n^2k^2}{4\pi^2}\left\{\frac{r^{4n-2}+r^{2n-2}\,a^{2n}-2r^{3n-2}.a^n\cos n\theta}{(r^{2n}+a^{2n}-2r^na^n\cos n\theta)^2}\right\}$$

or $$q^2=\frac{n^2k^2}{4\pi^2}\left\{\frac{r^{2n-2}\,(r^{2n}+a^{2n}-2r^n\,a^n\cos n\theta}{(r^{2n}+a^{2n}-2r^na^n\cos n\theta)^2}\right\}$$

or $$q^2=\frac{n^2k^2}{4\pi^2}\cdot\frac{r^{2n-2}}{r^{2n}+a^{2n}-2r^n\,a^n\cos n\theta}$$

which determines the velocity at any point of the liquid.

Again the velocity of any one of the vortices at $z=a$.

$$w'=\frac{ik}{2\pi}\log(z^n-a^n)-\frac{ik}{2\pi}\log(z-a)$$

or $$\phi'+i\psi'=\frac{ik}{2\pi}[\log\{(r^n\cos n\theta-a^n)+ir^n\sin n\theta\}-\log\{(r\cos\theta-a)+ir\sin\theta\}]$$

or $$\psi'=\frac{k}{4\pi}[\log(r^{2n}+a^{2n}-2r^n\,a^n\cos n\theta)-\log(r^2+a^2-2ar\cos\theta)]$$

Let ψ_0 be the stream function of the vortex $(a, 0)$ lying on X-axis.

$$\left(\frac{\partial\psi'}{\partial r}\right)_{r=a}=\frac{k}{4\pi}\left[\frac{2nr^{2n-1}-2nr^{n-1}\,a^n\cos n\theta}{r^{2n}+a^{2n}-2r^n\,a^n\cos n\theta}-\frac{2r-2a\cos\theta}{r^2+a^2-2ar\cos\theta}\right]_{r=a}$$

$$\frac{\partial\psi_0}{\partial a}=\frac{k}{4\pi}\left[\frac{2na^{2n-1}\,(1-\cos n\theta)}{2a^{2n}\,(1-\cos n\theta)}-\frac{2a\,(1-\cos\theta)}{2a^2(1-\cos\theta)}\right].$$

$$\frac{\partial\psi_0}{\partial a}=\frac{k}{4\pi}\left\{\frac{n}{a}-\frac{1}{a}\right\}=\frac{k\,(n-1)}{4\pi a}$$

and $$\frac{\partial\psi_0}{a\partial\theta}=\left(\frac{1}{r}\,\frac{\partial\psi'}{\partial\theta}\right)_{r=a}\quad i.e.\quad\frac{1}{a}\,\frac{\partial\psi_0}{\partial\theta}=0$$

It follows that the velocity is only along the tangent, there being no velocity along the normal to the circle. Thus the vortices will move round the cylinder uniformly with velocity

$$=\frac{k\,(n-1)}{4\pi a}$$

The time of making one round

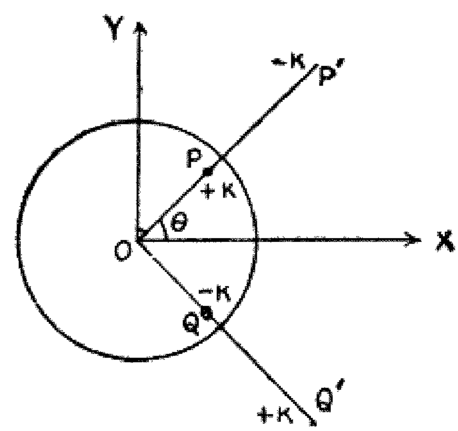
$$=\frac{2\pi a}{\frac{k\,(n-1)}{4\pi a}}$$

$$=\frac{8\pi^2 a^2}{k\,(n-1)}.\qquad\qquad \textbf{Proved.}$$

Ex. 14. *When a pair of equal and opposite rectilinear vortices are situated in a long circular cylinder at equal distances from its axis. Shew that the path of each vortex is given by the equation*

$$(r^2 \sin^2 \theta - b^2)\ (r^2 - a^2)^2 = 4a^2 b^2 r^2 \sin^2 \theta$$

θ being measured from the line through the centre perpendicular to the join of the vortices.



Let k be the strength of the vortex at the point $P\ (r, \theta)$ and an opposite and equal vortex of strength $-k$ at the point $Q\ (r, -\theta)$.

The image system consists of :

(i) A vortex of strength $-k$ at an inverse point $P'\ \left(\frac{a^2}{r}, \theta\right)$

(ii) A vortex of strength $+k$ at an inverse point $Q'\ \left(\frac{a^2}{r}, -\theta\right)$.

The complex potential at any point z is given by

$$w=\frac{ik}{2\pi}\log\,(z-re^{\theta i})-\frac{ik}{2\pi}\log\left(z-\frac{a^2}{r}e^{\theta i}\right)$$
$$-\frac{ik}{2\pi}\log\,(z-re^{-\theta i})+\frac{ik}{2\pi}\log\left(z-\frac{a^2}{r}e^{-\theta i}\right)$$

The motion of P is due to other vortices ; thus for the motion of P, the complex potential w_1 (let) is given by

$$w_1=\left[-\frac{ik}{2\pi}\log\left(z-\frac{a^2}{r}e^{\theta i}\right)-\frac{ik}{2\pi}\log\,(z-re^{-\theta i})\right.$$
$$\left.+\frac{ik}{2\pi}\log\left(z-\frac{a^2}{r}e^{-\theta i}\right)\right]_{z=re^{\theta i}}$$

$$=-\frac{ik}{2\pi}\left[\log\left(re^{\theta i}-\frac{a^2}{r}e^{\theta i}\right)+\log\,(re^{\theta i}-re^{-\theta i})\right.$$
$$\left.+\log\left(re^{\theta i}-\frac{a^2}{r}e^{-\theta i}\right)\right]$$

$$= -\frac{ik}{2\pi}\left[\log\left\{\left(r\cos\theta - \frac{a^2}{r}\cos\theta\right) + i\left(r\sin\theta - \frac{a^2}{r}\sin\theta\right)\right\} + \log(2ir\sin\theta) + \log\left\{\left(r\cos\theta - \frac{a^2}{r}\cos\theta\right) + i\left(r\sin\theta + \frac{a^2}{r}\sin\theta\right)\right\}\right]$$

or $$\psi = -\frac{k}{2\pi}\left[\log\left(r - \frac{a^2}{r}\right) + \log(2r\sin\theta) - \tfrac{1}{2}\log\left\{\left(r - \frac{a^2}{r}\right)^2\cos^2\theta + \left(r + \frac{a^2}{r}\right)^2\sin^2\theta\right\}\right]$$

or $$\psi = -\frac{k}{2\pi}\left[\log\left(r - \frac{a^2}{r}\right) + \log(2r\sin\theta) - \tfrac{1}{2}\log\left(r^2 + \frac{a^4}{r^2} - 2a^2\cos 2\theta\right)\right]$$

or $$\psi = -\frac{k}{4\pi}\left[\log\left\{\left(r - \frac{a^2}{r}\right).2r\sin\theta\right\}^2 - \log\left\{r^2 + \frac{a^4}{r^2} - 2a^2\cos 2\theta\right\}\right]$$

The stream lines are given by ψ=Constant.

or $$-\frac{k}{4\pi}\left[\log\{(r^2-a^2)^2.4\sin^2\theta\} - \log\left\{\frac{r^4+a^4-2a^2r^2\cos 2\theta}{r^2}\right\}\right] = \text{Constant}.$$

or $$\frac{(r^2-a^2)^2.r^2\sin^2\theta}{r^4+a^4-2a^2r^2\cos 2\theta} = b^2 \quad \text{(let)}$$

or $$b^2(r^4+a^4-2a^2r^2\cos 2\theta) = r^2(r^2-a^2)^2\sin^2\theta$$

or $$b^2\{(r^2-a^2)^2 + 2a^2r^2(1-\cos 2\theta)\} = r^2(r^2-a^2)^2\sin^2\theta$$

or $$2a^2b^2r^2(1-\cos 2\theta) = (r^2-a^2)^2(r^2\sin^2\theta - b^2)$$

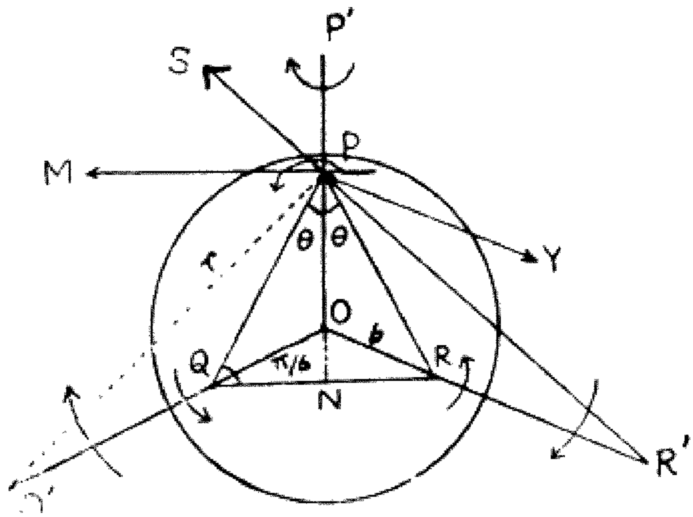
or $$4a^2b^2r^2\sin^2\theta = (r^2-a^2)^2(r^2\sin\theta - b^2).$$ **Proved.**

Ex. 15. *Three vortex filaments, each of strength k, are symmetrically placed inside a circular cylinder of radius a, and pass through the corners of an equilateral triangle of side $\sqrt{3}b$. If there is no circulation in the fluid other than that due to the vortices Shew that they will revolve about the axis of the cylinder with angular velocity*

$$\frac{k(a^6+2b^6)}{(a^6-b^6)}.$$

Let P', Q' and R' are the inverse points of P, Q and R with regard to the circular boundary. The vortex filament, each of strength k are placed at P, Q and R. $\left\{QN = NR = \frac{\sqrt{3}}{2}b.\right.$

Now $OP=OQ=OR=b$ {as side of the triangle is $\sqrt{3}b$}



By the definition of an inverse points, we have

or $$OP.OP'=a^2 \quad \text{so} \quad OP'=\frac{a^2}{b}=\lambda \quad \text{(let)}$$

Thus $PP'=QQ'=RR'=(b'-b)$.

The image system for the vortices, each of strength k at P, Q, R consists of the vortices, each of strength $-k$ at P', Q', R'

The vortex at P moves due to other vortices at Q, R, P', Q' and R' and describes a circle with an angular velocity about the centre. Let $PQ'=PR'=r$

and $\angle OPQ'=\theta=\angle OPR'$.

Also $$\cos\theta=\frac{OP^2+PQ'^2-OQ'^2}{2OP.PQ'}=\frac{b^2+r^2-\lambda^2}{2br}$$

or $$r^2=OQ'^2-OP^2+2OP.PQ'\cos\theta$$
$$=\lambda^2+b^2+a^2 \qquad \{\text{as } \theta=120°$$

Again, the velocities due to the vortex P are

(i) Velocity due to P' along PM

$$=\frac{k}{PP'}=\frac{k}{b'-b}=\frac{k}{\frac{a^2}{b}-b}$$

(ii) Velocity due to Q' along PY

$$=\frac{k}{r}$$

(iii) Velocity due to Q along PS

$$=\frac{k}{PQ}=\frac{k}{\sqrt{3}b}$$

(vi) Velocity due to R is $\dfrac{k}{PR}=\dfrac{k}{\sqrt{3}b}$

(v) Velocity due to R' is $\dfrac{k}{r}$

Thus the velocities perpendicular to OP are,

$$=k\left\{\frac{b}{a^2-b^2}-\frac{1}{r}\cos\theta+\frac{1}{\sqrt{3}b}\cos\pi/6+\frac{1}{\sqrt{3}b}\cos\pi/6-\frac{1}{r}\cos\theta\right\}$$

$$=k\left\{\frac{b}{a^2-b^2}-\frac{2r\cos\theta}{r^2}+\frac{1}{b}\right\}$$

$$=k\left\{\frac{b}{a^2-b^2}+\frac{1}{b}-\frac{1}{r^2}\left(b^2+r^2-\frac{a^4}{b^2}\right)\frac{1}{b}\right\}$$

$$=k\left\{\frac{b}{a^2-b^2}+\frac{a^4-b^4}{b^3r^2}\right\}$$

$$=k\left\{\frac{b}{a^2-b^2}+\frac{a^4-b^4}{b\,(a^4+b^4+a^2b^2)}\right\}$$

$$=k\left\{\frac{b^2\,(a^4+b^4+a^2b^2)+(a^2-b^2)\,(a^4-b^4)}{b\,(a^6-b^6)}\right\}$$

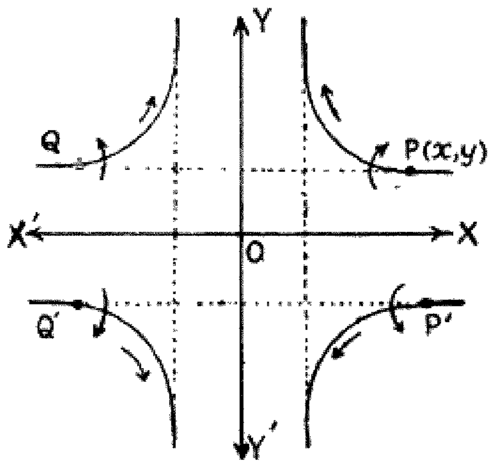
$$=k\,\frac{a^6+2b^6}{b\,(a^6-b^6)}$$

Thus $b\dot{\theta}=k\,\dfrac{a^6+2b^6}{b(a^6-b^6)}$ or $\dot{\theta}=k.\dfrac{a^6+2b^6}{b^2\,(a^6-b^6)}$ **Proved.**

§ 6·6. Four Vortices :

Consider four vortices in an infinite liquid such that the lines joining them form a rectangle $PQQ'P'$ at any instant. The strength being k for the vortices P' and Q and the strength $-k$ for the vortices P and Q', it is evident that the centres will always form a rectangle. Here we shall determine the motion of a vortex pair moving directly towards or from a parallel plane boundary between planes meeting at right angles. We see that the effect of the presence of the pair P, P' on Q, Q' is to separate them and at the same time to diminish their velocity perpendicular to the line joining them. Let the planes which bisect PQ, PP' at right angles may be considered as fixed rigid boundaries.

Let $(x\ y)$ be the co-ordinates of the vortex P relative to the planes of symmetry (*i.e.* due to other three vortices), we have the components



$$u=\frac{k}{2\pi.PP'}-\frac{k}{2\pi.PQ'}\cdot\frac{PP'}{PQ'}$$

$$=\frac{k}{4\pi}\cdot\frac{x^2}{y(x^2+y^2)}-\frac{k}{4\pi}\cdot\frac{x^2}{yr^2} \quad \text{...(i)}$$

where $r^2=x^2+y^2$

and $$v=-\frac{k}{2\pi.PQ}+\frac{k}{2\pi.PQ'}\cdot\frac{PQ}{PQ'}$$

$$=-\frac{k}{4\pi}\cdot\frac{y^2}{x(x^2+y^2)}$$

$$=-\frac{k}{4\pi}\cdot\frac{y^2}{xr^2} \quad \text{...(ii)}$$

For the path of the vortex P, we have

$$u=\dot{x} \text{ and } v=\dot{y}. \quad \text{...(iii)}$$

By dividing (i) and (ii), the differential equation of the path,

$$\frac{u}{v}=-\frac{x^2}{y}\cdot\frac{x}{y^2}=-\frac{x^3}{y^3}$$

or $$\frac{\dot{x}}{\dot{y}}=-\frac{x^3}{y^3} \quad \{\text{from (iii)}$$

or $$\frac{dx}{x^3}=-\frac{dy}{y^3}.$$

By integrating, we obtain

$$\frac{1}{x^2}+\frac{1}{y^2}=\frac{1}{a^2}=(\text{Const}).$$

or $$a^2(x^2+y^2)=x^2y^2$$

Transforming it into polar coordinates,

$$4a^2r^2=4r^4\sin^2\theta\cos^2\theta$$

or $$r^2\sin^2 2\theta=4a^2$$

or $$r\sin 2\theta=2a$$

Which represents a cotes spiral with asymptotes parallel to the axes.

Since $x\dot{y}-y\dot{x}=\dfrac{k}{4\pi}$ follows that the vortex moves as if under a centre of force at the origin. **This force is repulsive and varies as the inverse cube of the distance.**

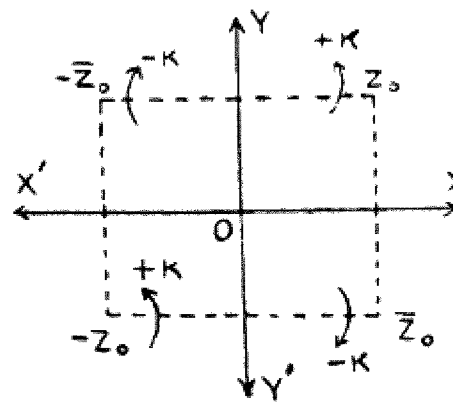
Alternative method of § 7·6.

Ex. 16. *A rectilinear vortex filament of strength k is in an infinite liquid bounded by two perpendicular infinite plane walls whose line of intersection is parallel to the filament. Show that the filament will trace out a curve (in a plane at right angles to the walls) r sin 2θ=Constant. Where r is the distance of the vortex from the line of intersection of the walls, and θ the angle between one of the walls and the plane containing the filament and the line of intersection.*

Consider the vortex of strength $+k$ be at z_0 where

$$z_0=x_0+iy_0.$$

Let a vortex of strength $-k$ at $\bar{z}_0$, $+k$ at $-z_0$ and $-k$ at $-\bar{z}_0$.



Thus the complex potential is given by

$$w=\frac{ik}{2\pi}\log(z-z_0)+\frac{ik}{2\pi}\log(z+z_0)$$

$$-\frac{ik}{2\pi}\log(z-\bar{z}_0)-\frac{ik}{2\pi}\log(z+\bar{z}_0)$$

$$w=\frac{ik}{2\pi}\{\log(z-z_0)(z+z_0)-\log(z-\bar{z}_0)(z+\bar{z}_0)\}$$

Now we shall determine the motion at z_0, (neglecting the part due to it,)

$$w_1=\frac{ik}{2\pi}\log\left\{\frac{z+z_0}{z^2-\bar{z}_0^2}\right\} \text{ (let)}$$

or $$u-iv=\underset{z\to z_0}{\text{Lt.}}\left(-\frac{dw_1}{dz}\right)$$

or $$u-iv=-\frac{ik}{2\pi}\underset{z\to z_0}{\text{Lt.}}\left\{\frac{1}{z+z_0}-\frac{2z}{z^2-\bar{z}_0^2}\right\}$$

or $$u-iv=-\frac{ik}{2\pi}\left\{\frac{1}{2z_0}-\frac{2z_0}{z_0^2-\bar{z}_0^2}\right\}$$

or $$u-iv=-\frac{ik}{2\pi}\left\{\frac{1}{2(x_0+iy_0)}-\frac{2(x_0+iy_0)}{(x_0+iy_0)^2-(x_0-iy_0)^2}\right\}$$

or $$u-iv=-\frac{ik}{4\pi}\left\{\frac{x_0-iy_0}{x_0^2+y_0^2}-\frac{x_0+iy_0}{ix_0y_0}\right\}$$

Separating into real and imaginary parts, we have

$$u=\frac{k}{4\pi}\left\{\frac{1}{y_0}-\frac{y_0}{x_0^2+y_0^2}\right\}$$

$$=\frac{k}{4\pi}\left\{\frac{x_0^2}{y_0\,(x_0^2+y_0^2)}\right\}$$

and
$$v=\frac{k}{4\pi}\left\{\frac{x_0}{x_0^2+y_0^2}-\frac{1}{x_0}\right\}$$

$$=-\frac{k}{4\pi}\left\{\frac{y_0^2}{x_0\,(x_0^2+y_0^2)}\right\}$$

Since $u=\dot{x}_0$ and $v=\dot{y}_0$

or
$$\frac{dx_0}{dy_0}=\frac{u}{v}$$

$$=-\frac{x_0^3}{y_0^3}$$

or
$$\frac{dx_0}{x_0^3}+\frac{dy_0}{y_0^3}=0.$$

By integrating, we have

$$\frac{1}{x_0^2}+\frac{1}{y_0^2}=\text{Const.}=\lambda$$

or $x_0^2+y_0^2=\lambda x_0^2y_0^2$ $\left\{\begin{array}{l}\text{Put } x_0=r\cos\theta\\ \quad\;\; y_0=r\sin\theta\end{array}\right.$

or $r^2=\lambda r^4\sin^2\theta\cos^2\theta$

or $r^2=\frac{1}{4}\lambda r^4(\sin 2\theta)^2$

or $r\sin 2\theta=\text{Const.}$ **Proved.**

Also $r^2\dot{\theta}=x_0\dot{y}_0-y_0\dot{x}_0$

or
$$r^2\dot{\theta}=-\frac{k}{4\pi}\,\frac{x_0y_0^2}{x_0\,(x_0^2+y_0^2)}-\frac{k}{4\pi}\,\frac{x_0^2y_0}{y_0\,(x_0^2+y_0^2)}$$

or
$$r^2\dot{\theta}=-\frac{k}{4\pi}\left\{\frac{y_0^2}{x_0^2+y_0^2}+\frac{x_0^2}{x_0^2+y_0^2}\right\}=-\frac{k}{4\pi}$$

or
$$\frac{\dot{\theta}}{\sin^2 2\theta}=\text{Const.}$$

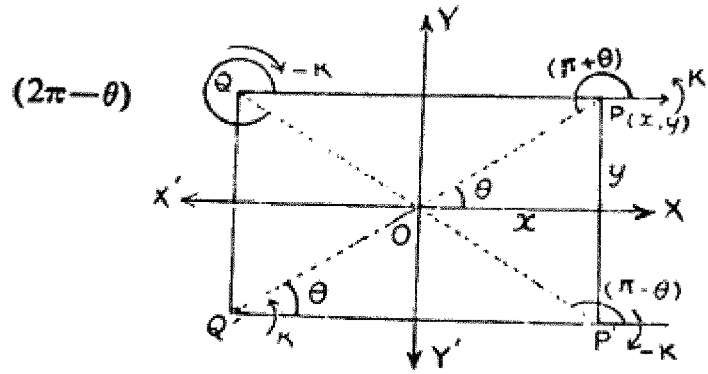
or
$$\frac{d\theta}{\sin^2 2\theta}=\text{Const. } dt$$

By integrating, we have

$$t\propto\cot 2\theta.$$

Ex. 17. *If a rectilinear vortex-moves parallel to two rigid planes which intersect at right angles, prove that on the line of intersection of the planes the excess of pressure due to the vortex varies inversely as the square of the distance of the vortex from the line of intersection.*

Since the velocities due to vortex P and Q cancel each other and also due to vortex Q and P' cancel. Thus the velocity at O is zero. Then



$$\phi = -\frac{k}{2\pi}(\pi+\theta) + \frac{k}{2\pi}(2\pi-\theta) - \frac{k}{2\pi}\theta + \frac{k}{2\pi}(\pi-\theta)$$

or $\phi = \text{constant} - \dfrac{2k}{\pi}\theta$

or
$$\dot{\phi} = -\frac{2k}{\pi}\dot{\theta} \qquad \text{...(i)}$$

Since $\tan\theta = \dfrac{y}{x}$

or
$$\sec^2\theta\,\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{x^2} \qquad \text{...(ii)}$$

Now the velocity of the vortex at $P\,(x\,y)$

$$\dot{x} = \frac{k}{2\pi.2y} - \frac{k}{2\pi.2r}\sin\theta$$

$$= \frac{k}{4\pi y} - \frac{k}{4\pi r}\cdot\frac{y}{r}$$

$$= \frac{k}{4\pi y} - \frac{kv}{4\pi r^2} = \frac{k}{4\pi}\cdot\frac{x^2}{y\,(x^2+y^2)}$$

{where $r^2 = x^2 + y^2$

and
$$\dot{y} = -\frac{k}{2\pi.2x} + \frac{k}{2\pi.2r}\cos\theta$$

$$= -\frac{k}{4\pi x} + \frac{k}{4\pi r}\cdot\frac{x}{r}$$

$$= -\frac{k}{4\pi x} + \frac{\kappa x}{4\pi r^2} = -\frac{k}{4\pi}\cdot\frac{y^2}{x\,(x^2+y^2)}$$

Substituting the value of $\dot{x}$ and $\dot{y}$ in (ii), we have

$$\dot{\phi} = \frac{k^2}{2\pi^2}\cdot\frac{\dfrac{y^2}{x^2+y^2} + \dfrac{x^2}{x^2+y^2}}{x^2+y^2}$$

$$\dot{\phi}=\frac{k^2}{2\pi^2}\cdot\frac{1}{x^2+y^2}=\frac{k^2}{2\pi^2 r^2}$$

Then the excess of pressure at origin

$$=p-p_0$$

$$=\rho\left(\frac{\partial\phi}{\partial t}-\tfrac{1}{2}q^2\right)$$

Since the velocity is zero at O *i.e.* $q=0$, thus the pressure varies as $\frac{\partial\phi}{\partial t}$ or $\dot{\phi}$.

i.e. pressure varies inversely as the square of the distance of the vortex from the line of intersection. **Proved.**

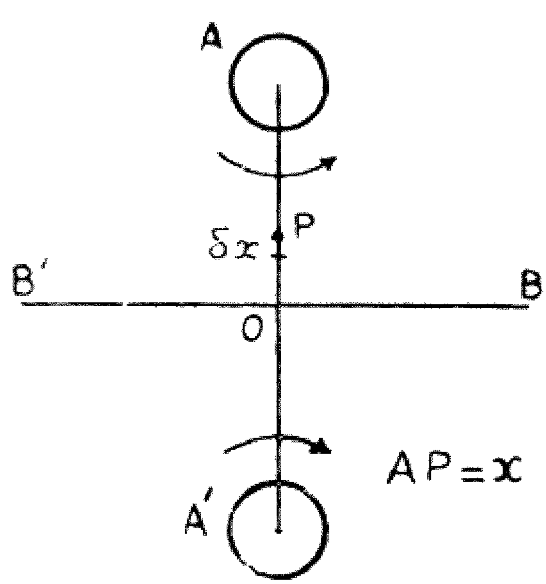
Ex. 18. *Prove that a thin cylindrical vortex of strength* σ*, running parallel to a plane boundary at distance a will travel with velocity* $\frac{\sigma}{4\pi a}$ *; and shew that a stream of fluid will flow past between the travelling vortex and the boundary of total amount*

$$\frac{\sigma}{2\pi}\left\{\log\left(\frac{2a}{c}\right)-\tfrac{1}{2}\right\}$$

per unit length along the vortex, when c is the (small) radius of the cross-section of the vortex.

Let A be the vortex of strength σ and BB' the plane boundary. The line AOA' as the perpendicular drawn from A on BB' intersecting at O.

A' is the image of the vortex at A. Velocity of the vortex at A



$$=\frac{k}{2\pi . AA'}$$

$$=\frac{k}{2\pi . 2a}=\frac{k}{4\pi a}$$

Consider an element δx at P at a distance x from A. The velocity at the point P,

$$=\frac{k}{2\pi}\cdot\frac{1}{x}+\frac{k}{2\pi}\cdot\frac{1}{2a-x} \quad \text{(perpendicular to } AA')$$

Then the fluid flow cross AO,

$$=\frac{k}{2\pi}\int_c^a\left(\frac{1}{x}+\frac{1}{2a-x}\right).dx$$

$$=\frac{k}{2\pi}\Big\{\log x-\log(2a-x)\Big\}_{c}^{a}$$

$$=\frac{k}{2\pi}\Big\{\log a-\log a-\log c+\log(2a-c)\Big\}$$

$$=\frac{k}{2\pi}\log\left(\frac{2a-c}{c}\right)$$

$$=\frac{k}{2\pi}\log\left(\frac{2a}{c}\right) \qquad \{\text{ Since } c \text{ is small}$$

But AA' moves with velocity $\frac{k}{4\pi a}$, so the vortex system can be reduced to rest by applying a velocity $\left(-\frac{k}{4\pi a}\cdot a\right)$

$$=-\frac{k}{4\pi}$$

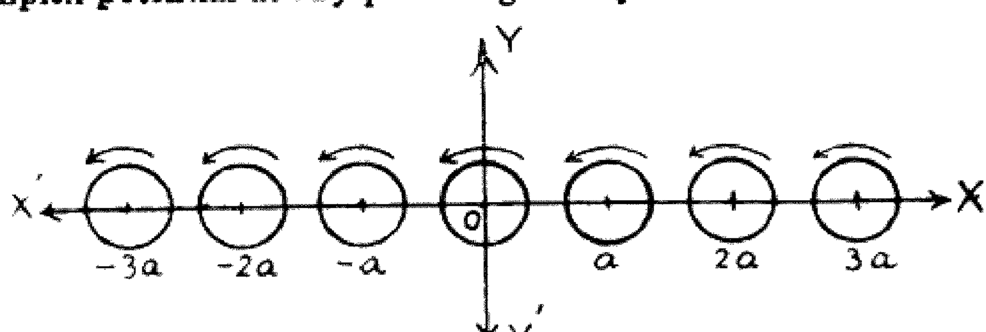
Hence the flow is

$$=\frac{k}{2\pi}\log\left(\frac{2a}{c}\right)-\frac{k}{4\pi}$$

$$=\frac{k}{2\pi}\Big\{\log\left(\frac{2a}{c}\right)-\tfrac{1}{2}\Big\} \qquad \textbf{Proved.}$$

§ 7·7. **An infinite single row of parallel rectilinear vortices of the same strength.**

Consider an infinite number of vortices of strength k each placed at a distance a apart. Let the line through the centres of their sections as the X-axis and origin at the middle one. The complex potential at any point is given by



$$w=\frac{ik}{2\pi}\Big\{\log z+\log(z-a)+\log(z+a)+\log(z-2a)+\log(z+2a)+\log(z-3a)+\log(z+3a)+\dots\Big\}$$

or

$$w=\frac{ik}{2\pi}\Big\{\log z(z^2-a^2)(z^2-2^2a^2)(z^2-3^2a^2)\dots(z^2-n^2a^2)\Big\}$$

or $$w=\frac{ik}{2\pi}\left[\log\left\{\frac{\pi z}{a}\left(1-\frac{z^2}{a^2}\right)\left(1-\frac{z^2}{2^2a^2}\right)\cdots\right\}\right.$$
$$\left.+\log\left(\frac{a}{\pi}.a^2.2^2a^2\ldots\right)\right]$$

or $$w=\frac{ik}{2\pi}\left(\log\left\{\frac{\pi z}{a}\left(1-\frac{z^2}{a^2}\right)\left(1-\frac{z^2}{2^2a^2}\cdots\right\}\right]+\text{constant.}$$

When $n\to\infty$ for an infinite row

$$w=\frac{ik}{2\pi}\log\sin\frac{\pi z}{a}^{*} \quad ..\text{(i)}$$

which gives tne complex potential

Let u_1 and v_1 be the velocity components of the vortex at the origin.

$$u_1-iv_1=-\frac{d}{dz}\left\{\frac{ik}{2\pi}\log\sin\frac{\pi z}{a}-\frac{ik}{2\pi}\log z\right\}\text{ at } z=0$$
$$=-\frac{ik}{2\pi}\left[\frac{\pi}{a}\cot\frac{\pi z}{a}-\frac{1}{z}\right]_{z=0}$$
$$=\text{zero}$$

Thus the vortex at origin is at rest. Similarly all other vortices at origin are at rest.

Let u and v are the velocity components at a point z,

or $$u-iv=-\frac{dw}{dz}=-\frac{ik}{2\pi}\cdot\frac{\pi}{a}\cot\frac{\pi z}{a}$$

or $$u-iv=-\frac{ik}{2a}\cdot\frac{\cos\frac{\pi z}{a}}{\sin\frac{\pi z}{a}}$$

or $$u-iv=-\frac{ik}{2a}\,\frac{\cos\frac{\pi(x+iy)}{a}\sin\frac{\pi(x-iy)}{a}}{\sin\frac{\pi(x+iy)}{a}\sin\frac{\pi(x-iy)}{a}}$$

Equating real and imaginary parts, we have

$$u=-\frac{k}{2a}\,\frac{\sinh\left(\frac{\pi y}{a}\right)\cosh\left(\frac{\pi y}{a}\right)}{\sin^2\frac{\pi x}{a}\cosh^2\frac{\pi y}{a}+\cos^2\frac{\pi x}{a}\sinh^2\frac{\pi y}{a}}$$

---

* as $\sin\theta=\theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\cdots$

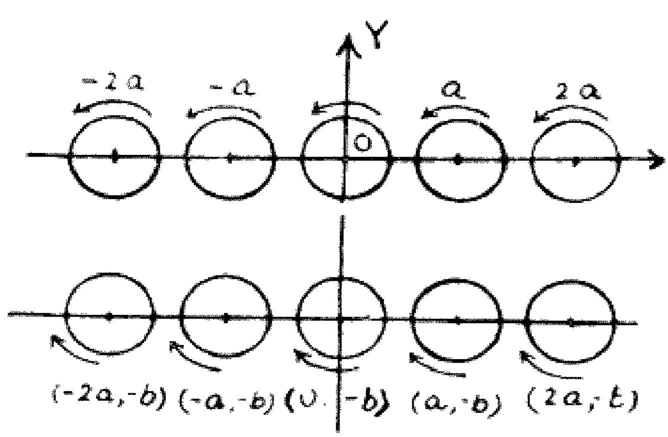
or
$$u=-\frac{k}{2a}\frac{\sinh\left(\frac{2\pi y}{a}\right)}{\cosh\left(\frac{2\pi x}{a}\right)-\cos\left(\frac{2\pi y}{a}\right)}$$

and
$$v=\frac{k}{2a}\frac{\sin\frac{2\pi x}{a}}{\cosh\left(\frac{2\pi y}{a}\right)-\cos\left(\frac{2\pi x}{a}\right)}$$

which will tend to zero as y tends to infinity. Thus the velocities along the distant stream lines are parallel to the row but in opposite directions. **Therefore the row behave like a vortex sheet as regard to distant points.**

§ 7·8. Two infinite rows of parallel rectilinear vortices.

Consider two infinite rows of vortices at a distance b apart, symmetrically placed with regard to the plane mid-way between them. The rotation in the two rows is in opposite sense. The upper row having vortices each of strength k and the lower row, each of strength $-k$.



Let the upper one be taken as X-axis and the lower one as $y=-b$. The vortices of strength k are each at $(0, 0)$, $(\pm a, 0)$ $(\pm 2a, 0)$ etc. and of strength $-k$ are each at $(0, -b)$ $(\pm a, -b)$, $(\pm 2a, -b)$ etc.

The complex potential at any point z is given by

$$\begin{aligned}w=\frac{ik}{2\pi}\Big\{&\log z+\log(z-a)+\log(z+a)\\&+\log(z-2a)+\log(z+2a)+\ldots\\&-\log z-\log(z-ae^{-bi})-\log(z+ae^{-bi})\\&-\log(z-2ae^{-bi})-\log(z+2ae^{-bi})-\ldots\Big\}\end{aligned}$$

As in last article, we have

$$w=\frac{ik}{2\pi}\log\sin\frac{\pi z}{a}-\frac{ik}{2\pi}\log\sin\frac{z+ib}{a}$$

Let u and v are the components of velocity at any point

then $$u-iv=-\frac{dw}{dz}=-\frac{ik}{2a}\cot\frac{\pi z}{a}+\frac{ik}{2a}\cot\frac{\pi\,(z+ib)}{a}$$

If u_0 and v_0 are the velocity components of vortex at origin, then

$$u_0-iv_0=-\frac{ik}{2\pi}\left[\frac{\pi}{a}\cot\frac{\pi z}{a}-\frac{1}{z}-\frac{\pi}{a}\cot\frac{\pi\,(z+ib)}{a}\right]_{z=0}$$

or $$u_0-iv_0=\frac{ik}{2a}\cot\frac{\pi bi}{a}$$

$$\left\{\text{as}\quad w_0=w-\frac{ik}{2\pi}\log z\right.$$

or $$u_0-iv_0=\frac{k}{2a}\coth\frac{\pi b}{a}$$

Equating real and imaginary parts, we get

$$u_0=\frac{k}{2a}\coth\frac{\pi b}{a}\quad\text{and}\quad v_0=0$$

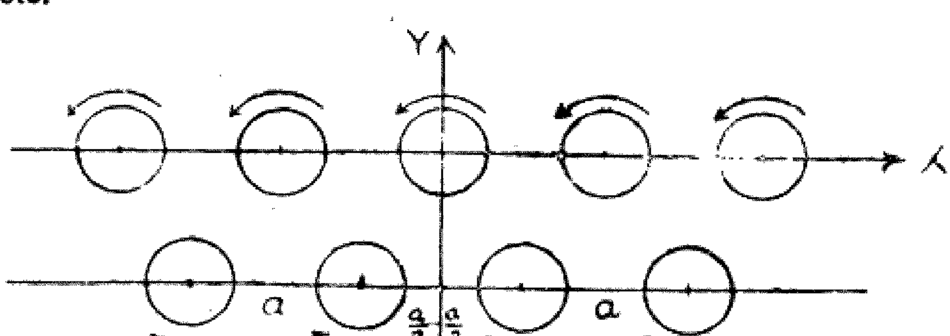
Thus the whole system will advance parallel to itself with a uniform velocity $$\frac{k}{2a}\coth\frac{\pi b}{a}$$

Where b is the distance between the two rows.

§ 7 81. ka'r ma'n vortex street.

This is a modified arrangement. Let there are two parallel rows of vortices at a distance b apart such that each vortex in one row is opposite the centre of the interval between two consecutive vortices in the other row.

The vortices of strength k are each at $(0, 0)$, $(\pm a, 0)$, $(\pm 2a, 0)$... and of strength $-k$ are each at $(\pm\frac{1}{2}a, b)$, $(\pm\frac{3}{2}a, b)$ ... etc.



Thus the complex potential at any point z at an instant $t=0$ is given by

$$w=\frac{ik}{2\pi}\log\sin\frac{\pi z}{a}-\frac{ik}{2\pi}\log\sin\frac{\pi}{a}(z+\tfrac{1}{2}a+ib)$$

{Ref. § 7.8.

Let u and v are the components of velocity of any point

then $$u-iv=-\frac{dw}{dz}=-\frac{ik}{2a}\cot\frac{\pi z}{a}+\frac{ik}{2a}\cot\frac{\pi}{a}(z+\tfrac{1}{2}a+ib)$$

Since neither row induces velocity in itself, the velocity of the vortex at origin is given by,

$$u_0-iv_0=-\frac{ik}{2\pi}\left[\frac{\pi}{a}\cot\frac{\pi z}{a}-\frac{1}{z}-\frac{\pi}{a}\cot\frac{\pi}{a}(z+\tfrac{1}{2}a+ib)\right]_{z=0}$$

$$=\frac{ik}{2a}\cot\left(\frac{\pi}{2}+\frac{i\pi b}{a}\right)$$

where u_0 and v_0 are the velocity components of vortex at origin.

$$=-\frac{k}{2a}\tan\frac{\pi bi}{a}\qquad\left\{\text{and } w_0=w-\frac{ik}{2\pi}\log z\right.$$

$$=\frac{k}{2a}\tanh\frac{\pi b}{a}$$

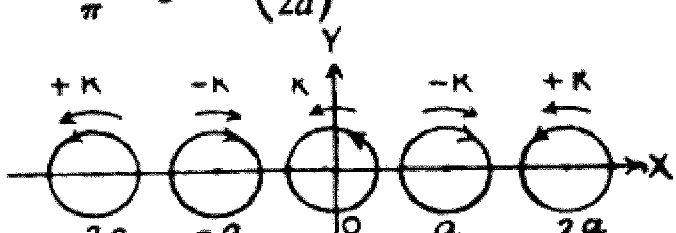
Thus the vortex system advances with the velocity

$$\frac{k}{2a}\tanh\frac{\pi b}{a}$$

This arrangement under certain condition is **stable** whereas the system discussed in § 7.7 and § 7.8 are **unstable.**

Ex. 19. *An infinite row of equidistant rectilinear vortices are at a distance a apart. The vortices are of the same numerical strength k but they are alternately of opposite signs. Find the complex function that determines the velocity potential and stream function. Shew that the vortices remain at rest Shew also that, if α be the radius of a vortex, the amount of flow between any vortex and the next is*

$$\frac{k}{\pi}\log\cot\left(\frac{\pi\alpha}{2a}\right)$$



Let the vortices be placed along the X-axis at an equidistant a apart at points $(0, 0)$ $(\pm a, 0)$, $(\pm 2a, 0)$... etc.

Thus the vortices of strength k are situated at $(0, 0)$, $(\pm 2a, 0)$... etc. and vortices of strength $-k$ are at $(\pm a, 0)$, $(\pm 3a, 0)$... etc. The complex potential is given by

$$w=\frac{ik}{2\pi}\log z-\frac{ik}{2\pi}\left\{\log(z-a)+\log(z+a)\right\}$$
$$+\frac{ik}{2\pi}\left\{\log(z-2a)+\log(z+2a)\right\}$$
$$-\frac{ik}{2\pi}\left\{\log(z-3a)+\log(z+3a)\right\}+\cdots$$

or $$w=\frac{ik}{2\pi}\,.\,\log\left\{\frac{z\,(z^2-2^2a^2)\,(z^2-4^2a^2)\ldots}{(z^2-a^2)\,(z^2-3^2a^2)\ldots}\right\}$$

or $$w=\frac{ik}{2\pi}\,.\,\log\left[\frac{\frac{z}{2a}\left\{1-\left(\frac{z}{2a}\right)^2\right\}\left\{1-\left(\frac{z}{4a}\right)^2\right\}\ldots\ldots}{\left\{1-\left(\frac{z}{a}\right)^2\right\}\left\{1-\left(\frac{z}{3a}\right)^2\right\}\ldots\ldots}\right]+\text{Constant.}$$

or $$w=\frac{ik}{2\pi}\log\frac{\sin\frac{\pi z}{2a}}{\cos\frac{\pi z}{2a}}\qquad\left\{\begin{array}{l}\text{as }\sin\theta=\theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\cdots\\ \qquad\cos\theta=\left(1-\frac{4\theta^2}{\pi^2}\right)\left(1-\frac{4\theta^2}{3^2\pi^2}\right)\cdots\end{array}\right.$$

or $$w=\frac{ik}{2\pi}\log\tan\frac{\pi z}{2a}\qquad\ldots\text{(i)}$$

which determines the complex potential.

Again $$\phi+i\psi=\frac{ik}{2\pi}\log\tan\frac{\pi z}{2a}$$

and $$\phi-i\psi=-\frac{ik}{2\pi}\log\tan\frac{\pi\bar{z}}{2a}.$$

By adding, we have

$$2\phi=\frac{ik}{2\pi}\left\{\log\tan\frac{\pi z}{2a}-\log\tan\frac{\pi\bar{z}}{2a}\right\}$$

By subtracting, we have

$$2i\psi=\frac{ik}{2\pi}\left\{\log\tan\frac{\pi z}{2a}+\log\tan\frac{\pi\bar{z}}{2a}\right\}$$

or $$\psi=\frac{k}{4\pi}\,.\,\log\left\{\frac{\sin\frac{\pi(x+iy)}{2a}\,\sin\frac{\pi(x-iy)}{2a}}{\cos\frac{\pi(x+iy)}{2a}\,\cos\frac{\pi(x-iy)}{2a}}\right\}$$

or $$\psi=\frac{k}{4\pi}\log\frac{\cosh\frac{\pi y}{a}-\cos\frac{\pi x}{a}}{\cosh\frac{\pi y}{a}+\cos\frac{\pi x}{a}} \qquad \text{...(ii)}$$

determines the stream function.

Now we shall consider the motion for the vortex system at origin

$$w_1=w-\frac{ik}{2\pi}\log z$$

$$=\frac{ik}{2\pi}\left\{\log\tan\frac{\pi z}{2a}-\log z\right\}_{z=0} \qquad \{\text{ from (i)}$$

Then $$u_0-iv_0=-\left(\frac{dw_1}{dz}\right)_{z=0}$$

$$=-\frac{ik}{2\pi}\left\{\frac{\sec^2\frac{\pi z}{2a}}{\tan\frac{\pi z}{2a}}\cdot\frac{\pi}{2a}-\frac{1}{z}\right\}_{z=0}$$

$$=\text{Zero}$$

which shows that the vortex remain at rest.

To dertermine the flow : Consider the stream function when $y=0$ *i. e* stream function at any points on X-axis is

$$\psi=\frac{k}{2\pi}\log\tan\frac{\pi x}{2a} \qquad \{\text{ from (ii), put } y=0$$

Thus the flow between two consecutive vortices

$$=2\times\text{flow across } (\alpha, 0) \text{ to } (\tfrac{1}{2}\alpha, 0)$$

$$=2\,(\psi_{1/2\alpha}-\psi_\alpha)$$

$$=\frac{2k}{2\pi}\left\{\log\tan\frac{\pi\alpha}{4\alpha}-\log\tan\frac{\pi\alpha}{2a}\right\}$$

$$=\frac{k}{\pi}\left\{\log 1-\log\tan\frac{\pi\alpha}{2a}\right\}$$

$$=-\frac{k}{\pi}\log\tan\frac{\pi\alpha}{2a}$$

$$=\frac{k}{\pi}\log\cot\frac{\pi\alpha}{2a}$$ where α is the radius of the vortex.

Proved.

Conformal transformation.

Ex. 20. *The space enclosed between the planes $x=0$, $x=a$ and $y=0$ on the positive sides of $y=0$ is filled with uniform incom-*

pressible liquid. A rectilinear vortex parallel to the axis of Z has coordinates (x_1, y_1). Determine the velocity at any point of the liquid and show that the path of the vortex is given by

$$\cot^2 \frac{\pi x}{a} + \coth^2 \frac{\pi y}{a} = Const.$$

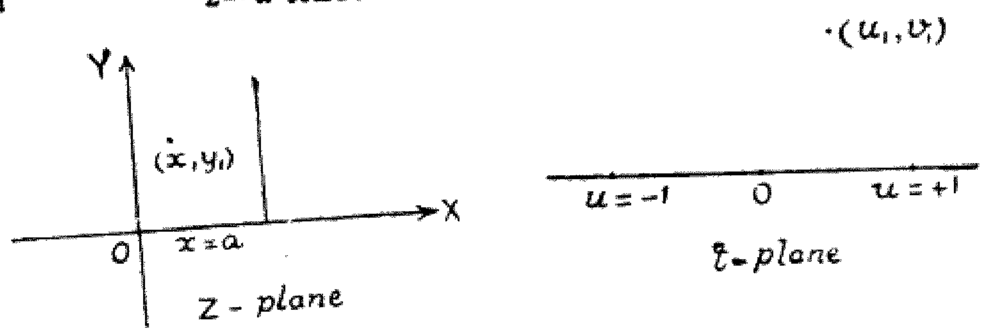
Consider the transformation be

$$t = -\cos \frac{\pi z}{a} \qquad \ldots(i)$$

which transforms the semi-infinite strip in z-plane to the upper half of the t-plane such that

$z=0$ tends to $t=-1$.

and $z=a$ tends to $t=+1$.



The stream function due to vortex at (u_1, v_1) in t-plane is

$$\psi = -\frac{k}{4\pi} \log v$$

or $$u_1 + iv_1 = -\cos \frac{\pi}{a}(x_1 + iy_1) \qquad \{ \text{from (i)}$$

or $$v_1 = \sin \frac{\pi x_1}{a} \sinh \frac{\pi y_1}{a}$$

Now $$\frac{dt}{dz} = \left| \frac{\pi}{a} \sin \frac{\pi z_1}{a} \right|$$

or $$\frac{dt}{dz} = \left| \frac{\pi}{a} \sin \frac{\pi}{a}(x_1 + iy_1) \right|$$

or $$\frac{dt}{dz} = \frac{\pi}{a} \left| \sin \frac{\pi x_1}{a} \cosh \frac{\pi y_1}{a} + i \cos \frac{\pi x_1}{a} \sinh \frac{\pi y_1}{a} \right|$$

or $$\frac{dt}{dz} = \frac{\pi}{a} \left\{ \sin^2 \frac{\pi x_1}{a} \cosh^2 \frac{\pi y_1}{a} + \cos^2 \frac{\pi x_1}{a} \sinh^2 \frac{\pi y_1}{a} \right\}^{1/2}$$

Thus $\lambda_2(x_1\, y_1)=\lambda_1(u_1\, v_1)+\dfrac{k}{4\pi}.$

$$\log\left\{\frac{\pi}{a}\sqrt{\left(\sin^2\frac{\pi x_1}{a}\cosh^2\frac{\pi y_1}{a}+\cos^2\frac{\pi x_1}{a}\sinh^2\frac{\pi y_1}{a}\right.}\right\}$$

$$=-\frac{k}{4\pi}\log\sin\frac{\pi x_1}{a}+\sinh\frac{\pi y_1}{a}$$

$$+\frac{k}{4\pi}\log\left\{\frac{\pi}{a}\sqrt{\left(\sin^2\frac{\pi x_1}{a}\cosh^2\frac{\pi y_1}{a}\right.}\right.$$

$$\left.+\cos^2\frac{\pi x_1}{a}\sinh^2\frac{\pi y_1}{a}\right\}$$

$$=\frac{k}{4\pi}\log\left\{\frac{\sin^2\frac{\pi x_1}{a}\cosh^2\frac{\pi y_1}{a}+\cos^2\frac{\pi x_1}{a}\sinh^2\frac{\pi y_1}{a}}{\sin^2\frac{\pi x_1}{a}\sinh^2\frac{\pi y_1}{a}}\right\}^{1/2}$$

(neglecting the constant terms).

$$\lambda_2(x_1\, y_1)=\frac{k}{4\pi}\log\left\{\coth^2\frac{\pi y_1}{a}+\cot^2\frac{\pi x_1}{a}\right\}^{1/2}$$

Thus the path of the vortex is given by

$$\lambda_2(x_1\, y_1)=\text{Const.}$$

or $$\coth^2\frac{\pi y_1}{a}+\cot^2\frac{\pi x_1}{a}=\text{Const.}$$ **Proved.**

§ 7 82. Rectilinear Vortex with Circular Section.

Let the section be a circle of radius a in an incompressible, inviscid liquid in a direction perpendicular to the XY plane. There is two-dimensional motion and let the vorticity vector ζ be uniform throughout the whole section. Its value being constant with in the tube and zero outside, the vortex being rectilinear.

The equation to continuity is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \qquad \text{...(i)}$$

Also the equation to stream-lines is

$$\frac{dx}{u}=\frac{dy}{v}$$

or $$v\,dx-u\,dy=0. \qquad \text{...(ii)}$$

If (ii) be a perfect differential, let it equal to $d\psi$

(where ψ is the stream function)

or $$v\,dx-u\,dy=\frac{\partial\psi}{\partial x}dx+\frac{\partial\psi}{\partial y}dy$$

or $$u=-\frac{\partial\psi}{\partial y}\text{ and }v=\frac{\partial\psi}{\partial x} \qquad \text{...(iii)}$$

The components of vorticity are

$$\zeta = \left(0, 0, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\zeta = \left(0, 0, \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right)$$

$$\zeta = (0, 0, \nabla^2 \psi).$$

.e The equations for the stream functions are

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 2\zeta \qquad \text{inside the vortex}$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \qquad \text{outside the vortex.}$$

By using the spherical-polar coordinates, we have

$$\frac{1}{r} \cdot \frac{d}{dr}\left(r \frac{d\psi}{dr}\right) = 2\zeta \quad \text{when } r < a$$

or $$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} = 2\zeta \qquad \text{...(iv)}$$

and $$\frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r} \frac{\partial \psi}{\partial r} = 0 \qquad \text{when } r > a. \qquad \text{...(v)}$$

{Since ψ is a function of r, being symmetrical about the origin, so $\frac{\partial^2 \psi}{\partial \theta^2} = 0$

By integrating (iv), we have

$$\frac{d}{dr}\left(r \frac{d\psi}{dr}\right) = 2\zeta r$$

or $$\psi = A \log r + \tfrac{1}{2}\zeta r^2 + C \qquad \text{when } r < a \qquad \text{...(vi)}$$

Also integrating (v), we have

$$\frac{d}{dr}\left(r \frac{d\psi}{dr}\right) = 0$$

or $$\psi = B \log r + D. \qquad \text{when } r > a. \qquad \text{...(vii)}$$

Let the constant D may be chosen to be zero. Since the motion outside the vortex tube is irrotational, so $\exists$ a velocity potential

$$\frac{\partial \psi}{\partial r} = -\frac{\partial \phi}{r \partial \theta}$$

or $$\frac{B}{r} = -\frac{1}{r} \frac{\partial \phi}{\partial \theta} \qquad \{\text{ from (vii)}$$

or $$\phi = -B\theta.$$

{Neglecting the integration constant.

The strength of the vortex is its circulation k.

But circulation = decrease in the value of ϕ on describing the circuit once only

or $$k=-B\{\theta-(\theta+2\pi)\}$$

or $$k=2\pi B$$

or $$B=\frac{k}{2\pi}$$

So $$\phi=-\frac{k}{2\pi}\theta \text{ and } \psi=\frac{k}{2\pi}\log r$$

or $$w=\phi+i\psi=ik\{\log r+i\theta\}$$
$$=ik\log z.$$

If the rectilinear vortex is placed at the point $z=z_0$, then the complex potential becomes by changing the origin

$$w=ik\log(z-z_0) \text{ etc.}$$

The strength k is positive when the circulation round the filament is counter clockwise, we may refer to such a filament as a **point vortex.**

Since the stream function ψ is not to be infinite when $r=0$, we must have $A=0$. If the motion is continuous at the surface $r=a$, we have the stream function ψ and the tangential velocity $\frac{\partial\psi}{\partial r}$ be continuous, so that

$$\tfrac{1}{2}\zeta r^2+C=B\log r+D \qquad \{\text{ from (vi) and (viii)}$$

or $$\tfrac{1}{2}\zeta a^2+C=B\log a+D \qquad \{\text{ at the surface } r=a$$

or $$\zeta a^2=B.$$

Neglecting the additive constant, we have

$$\psi=-\tfrac{1}{2}\zeta(a^2-r^2) \qquad \text{when } r<a \qquad \text{...(viii)}$$

and $$\psi=\zeta a^2\log\left(\frac{r}{a}\right) \qquad \text{when } r>a. \qquad \text{...(ix)}$$

The velocity is wholly transversal both inside and outside the vortex, values being ζr and $\zeta a^2.\frac{1}{r}$

Again the circulation in any circuit which does not enclose any point of the tube is zero, then the liquid inside is said to form a circular vortex of strength $k=\pi a^2\zeta$.

Now the stream function in terms of the strength of vortex becomes

$$\psi=-\frac{k}{2\pi a^2}(a^2-r^2) \qquad \text{when } r<a$$

and $$\psi = \frac{k}{\pi} \log\left(\frac{r}{a}\right) \qquad \text{when } r > a.$$

Pressure Distribution :

The motion is irrotational outside the vortex so $\exists$ a velocity potential ϕ Let the complex potential be of the form

$$w = i\zeta a^2 \log\left(\frac{z}{a}\right)$$

or $$\phi + i\psi = i\zeta\, a^2 \log (e^{\theta i})$$

or $$\phi + i\psi = -\zeta\, a^2\, \theta$$

or $$\phi = -\zeta\, a^2\, \theta$$

I. Pressure distribution outside the vortex tube ; $r > a$.

The pressure p is given by the equation

$$\frac{p}{\rho} = \frac{\partial \phi}{\partial t} - \tfrac{1}{2}q^2 + f(t). \qquad \ldots(1)$$

Since the motion is steady, so it reduces to

$$\frac{p}{\rho} = \text{constant} - \tfrac{1}{2}q^2.$$

Initially the pressure is Π when r is infinity

then $$\frac{p}{\rho} = \frac{\Pi}{\rho} - \tfrac{1}{2}q^2. \qquad \ldots(2)$$

{from (1)

Now the strength of vortex is given by

$$k = \pi a^2 . \zeta$$
$$= 2\pi\, a^2 . \omega$$

{ as $\zeta = 2\omega$, ω is called the spin velocity

Also the velocity outside the vortex

$$q^2 = \frac{\zeta a^2}{r} = \frac{k}{2\pi r} .$$

Substituting the value of q in (2), we have

$$\frac{p}{\rho} = \frac{\Pi}{\rho} - \tfrac{1}{2} . \frac{k^2}{4\pi^2\, r^2}$$

or $$\frac{p}{\rho} = \frac{\Pi}{\rho} - \frac{k^2}{8\pi^2\, r^2}$$

or $$p = \Pi - \frac{k^2 \rho}{8\pi^2\, r^2} \qquad \ldots(3)$$

determines the pressure outside the vortex *i.e.* at $r > a$.

II. Inside the vortex tube ; $r < a$.

The liquid is rotating uniformly with an angular velocity ω, so that

$$dp = \rho\, dr . r\omega^2$$

or $$\frac{dp}{\rho}=\omega^2 r\, dr.$$

By integrating, we have

$$\frac{p}{\rho}=\tfrac{1}{2}\omega^2 r^2+B. \quad \ldots(4)$$

Since initially $p=p_0$ (pressure at the centre of the vortex)

$$r=0, \quad \text{then} \quad B=\frac{p_0}{\rho}.$$

Substituting the value of the constant B in (3), we get

$$\frac{p}{\rho}=\tfrac{1}{2}\omega^2 r^2+\frac{p_0}{\rho}. \quad \ldots(5)$$

Also the pressure has to be continuous across the boundary, then from (3) and (5), we have

$$\frac{\Pi}{\rho}-\frac{k^2}{8\pi^2 a^2}=\tfrac{1}{2}\,\omega^2 a^2+\frac{p_0}{\rho}$$

or $$\frac{p_0}{\rho}=\frac{\Pi}{\rho}-\tfrac{1}{2}\,\omega^2 a^2-\frac{k^2}{8\pi^2 a^2} \qquad \left\{\text{Since } \omega=\frac{k}{2\pi a^2}\right.$$

or $$\frac{p_0}{\rho}=\frac{\Pi}{\rho}-\frac{k^2}{8\pi^2 a^2}-\frac{k^2}{8\pi^2 a^2}$$

or $$p_0=\Pi-\frac{k^2\rho}{4\pi^2 a^2}.$$

which shows if $\Pi<\dfrac{k^2\rho}{4\pi^2 a^2}$, then p_0 becomes negative $\Rightarrow$ that a cylinderical hollow must exist inside the vortex.

If $\Pi>\dfrac{k^2\rho}{4\pi^2 a^2}$, then p_0 is positive.

Now substituting the value of p_0 in (5), we have

$$\frac{p}{\rho}=\frac{\Pi}{\rho}-\frac{k^2}{4\pi^2 a^2}+\tfrac{1}{2}\omega^2 r^2$$

$$\frac{p}{\rho}=\frac{\Pi}{\rho}-\frac{k^2}{4\pi^2 a^2}+\frac{1}{8}\,\frac{k^2 r^2}{\pi^2 a^4}.$$

The necessary condition to have **cyclic irrotational motion** surrounding a hollow cylinderical space is given by

$$p=0 \quad \text{when} \quad r=a$$

then $$0=\frac{\Pi}{\rho}-\frac{k^2}{4\pi^2 a^2}+\frac{k^2}{8\pi^2 a^2}$$

or $$\Pi=\frac{k^2\rho}{8\pi^2 a^2}.$$

§ 7·83. *To prove that in an infinite mass of liquid at rest at infinity there can be only one type of motion when the components of spin are given.*

If possible let there are two sets of values of the velocity components u, v, w *e.g.* u_1, v_1, w_1 and u_2, v_2, w_2 each satisfying the equation of continuity and the equations

$$2\xi=\frac{\partial w_1}{\partial y}-\frac{\partial v_1}{\partial z},\quad 2\eta=\frac{\partial u_1}{\partial z}-\frac{\partial w_1}{\partial x},\quad 2\zeta=\frac{\partial v_1}{\partial x}-\frac{\partial u_1}{\partial y}$$

$$2\xi=\frac{\partial w_2}{\partial y}-\frac{\partial v_2}{\partial z},\quad 2\eta=\frac{\partial u_2}{\partial z}-\frac{\partial w_2}{\partial x},\quad .\zeta=\frac{\partial v_2}{\partial x}\quad\frac{\partial u_2}{\partial y}$$

at all points of space and vanishing at infinity.

Then the differences $u'=u_1-u_2$, $v'=v_1-v_2$, $w'=w_1-w_2$ also satisfy the equation of continuity and

$$\frac{\partial w'}{\partial y}-\frac{\partial v'}{\partial z}=0,\quad \frac{\partial u'}{\partial z}-\frac{\partial w'}{\partial x}=0,\quad \frac{\partial v'}{\partial x}-\frac{\partial u'}{\partial y}=0$$

at all point of space and vanishing at infinity.

Thus (u', v', w') are velocity components of irrotational motion of a liquid filling all space and vanishing at infinity. Hence we must have

$$u'=v'=w'=0 \quad \text{every where,}$$

i.e. $\qquad u_1=u_2,\quad v_1=v_2 \quad\text{and}\quad w_1=w_2$

$\Rightarrow$ there is only one motion satisfying the given conditions.

§ 7·84. **Steady motion.**

When the external forces have a potential function Ω, the general equations of motion are given by

$$\frac{Du}{Dt}=-\frac{\partial\Omega}{\partial x}-\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad ...(i)$$

and other two similar equations.

Since the motion is steady then (i) reduces to

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}=-\frac{\partial\Omega}{\partial x}-\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad ...(ii)$$

The above equation can be written as

$$u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial x}+v\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+w\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)$$

$$=-\frac{\partial\Omega}{\partial x}-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

or $$\frac{1}{2}\frac{\partial}{\partial x}(u^2+v^2+w^2)-2(v\zeta-w\eta)=-\frac{\partial\Omega}{\partial x}-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

or $$2(v\zeta-w\eta)=\frac{1}{2}\frac{\partial q^2}{\partial x}+\frac{\partial \Omega}{\partial x}+\frac{1}{\rho}\frac{\partial p}{\partial x}$$

or $$2(v\zeta-w\eta)=\frac{\partial}{\partial x}\left\{\tfrac{1}{2}q^2+\Omega+\int\frac{dp}{\rho}\right\}$$

or $$2(v\zeta-w\eta)=\frac{\partial \chi}{\partial x} \qquad \left\{\text{let } \chi=\tfrac{1}{2}q^2+\Omega+\int\frac{dp}{\rho}\right.$$

or $$\frac{\partial \chi}{\partial x}=2(v\zeta-w\eta)$$

Similarly $$\frac{\partial \chi}{\partial y}=2(w\xi-u\zeta)$$

and $$\frac{\partial \chi}{\partial z}=2(u\eta-v\xi).$$

Thus $$u\frac{\partial \chi}{\partial x}+v\frac{\partial \chi}{\partial y}+w\frac{\partial \chi}{\partial z}=0$$

and $$\xi\frac{\partial \chi}{\partial x}+\eta\frac{\partial \chi}{\partial y}+\zeta\frac{\partial \chi}{\partial z}=0$$

which shows that χ=constant represents a surface the normal to which at any point is at right angles to both the vortex line and the stream line through the point.

In an irrotational motion

$$\zeta=\eta=\xi=0$$

and χ is constant through out the liquid.

Consider the axis of X normal to the surface χ=constant, then $u=0$ and $\xi=0$. Let $\partial \nu$ is an element of the normal to the surface

$$\frac{\partial \chi}{\partial \nu}=\frac{\partial \chi}{\partial x}=2(v\zeta-w\eta)$$

and $$\frac{\partial \chi}{\partial y}=0=\frac{\partial \chi}{\partial z}.$$

Thus vortex lines and stream lines lie on a plane parallel to YZ-plane.

Let q is the resultant velocity at P, then

$$v=q\cos\theta_1, \quad w=q\sin\theta_1.$$

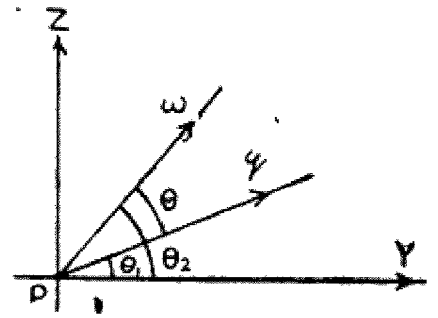
Since ω be the resultant velocity at P, then

$$\eta=\omega\cos\theta_2, \quad \zeta=\omega\sin\theta_2.$$

$$\therefore \quad \frac{\partial \chi}{\partial \nu}=2q\omega\,(\sin\theta_2\cos\theta_1 - \cos\theta_2\sin\theta_1)$$

$$=2q\omega\sin(\theta_2-\theta_1)$$

$$=2q\omega\sin\theta$$



where θ is the angle between the stream line and the vortex line.

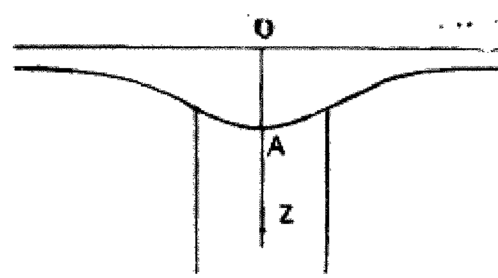
Hence

$$\delta\chi=2q\omega\sin\theta\,\delta\nu$$

$$=\text{constant} \qquad \forall \text{ points on the surface } \chi=\text{constant}.$$

Thus we have the condition for steady motion that it must be possible to draw a family of surfaces in the liquid each covered by a net work of stream lines and vortex lines such that at every point of the surface $q\omega\sin\theta\,\delta\nu$ is constant, where $\delta\nu$ is the normal distance between the two consecutive surfaces of the system.

§ 7·9. Rankine's Combined Vortex.



Rankine's combined vortex consists of a circular cylinderical vortex with its axis vertical in a liquid moving irrotationally under the action of gravity only the upper surface being exposed to atmospheric pressure Π. The external forces are derivable from the potential gz.

Consider the origin in the axis of the vortex and in the level of the liquid at infinity. We see that the kinematical conditions at the boundary are satisfied by considering the velocity system,

$$\frac{p}{\rho}=\text{constant}-\frac{k^2}{8\pi^2 r^2}-gz \quad \text{when} \quad r>a$$

(The motion is irrotational)

and
$$\frac{p}{\rho}=\text{constant}+\frac{k^2r^2}{8\pi^2a^4}-gz \quad \text{when} \quad r<a.$$

(where $-gz$ is the potential of the gravitational field)

The free surface has a depression over the top of the vortex. The equation of the free surfaces are given by

$$p=\text{constant}$$

or
$$z=\frac{k^2}{8\pi^2a^2g}-\frac{k^2}{8\pi^2r^2g}+\text{constant}$$

$$=\frac{k^2}{8\pi^2a^4g}\left(a^2-\frac{a^4}{r^2}\right)+\text{constant} \quad \text{...(i)}$$

when $r>a$

and $$z=\frac{k^2}{8\pi^2a^4g}(r^2-a^2)+\text{constant} \quad \text{...(ii)}$$

when $r<a$.

The constant can be determined to preserve the continuity at $r=a$.

Considering the origin in the general level of the free surface when $r>a$, such that $z=0$ when $r=\infty$.

So $$\text{constant}=-\frac{k^2}{8\pi^2a^2g} \quad \text{from (iv)}$$

And the depth of the central depression is given by putting $r=0$, we have

$$z=-\frac{k^2}{8\pi^2a^2g}.$$

§ 7·91. Rectilinear vortices with elliptic section.

To show that a rectilinear vortex whose cross section is an ellipse and whose spin is constant can maintain its form rotating as if it were a solid cylider in an infinite liquid.

We know if a rigid elliptic cylinder of semi-axes a, b rotates with uniform angular velocity ω in an infinite mass of liquid the complex potential for the cyclic irrotational motion with circulation k is

$$w=\tfrac{1}{4}i\omega\,(a+b)^2\,e^{-2\zeta}+\frac{ik}{2\pi}\,(\zeta) \qquad \left\{\begin{array}{l}\text{Ref. § 5·88}\\ \text{§ 5·88}\end{array}\right.$$

or $$\phi+i\psi=\tfrac{1}{4}\omega\,(a+b)^2\,e^{-2\xi}\,(\cos 2\eta-i\sin 2\eta)+\frac{ik}{2\pi}\,(\xi+i\eta)$$

or $$\psi=\tfrac{1}{4}\omega\,(a+b)^2\,e^{-2\xi}\cos 2\eta+\frac{k\xi}{2\pi}. \quad \text{...(i)}$$

Here $k=2\lambda ab.\omega'$ ...(ii)

where ω' be the constant spin.

{*i.e.* area of the cross section × angular velocity

The stream function ψ satisfies

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=2\omega' \text{ (Inside the vortex tube)} \quad \text{...(iii)}$$

and $$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0. \text{ (Outside the vortex tube)} \quad \text{...(iv)}$$

Assuming that

$$\psi=\omega'(Ax^2+By^2) \qquad ...(v)$$

Then from (iii), we have

$$2A\omega'+2B\omega'=2\omega'$$

or $$A+B=1 \qquad ...(vi)$$

Again the boundary condition is that the velocity of the liquid normal to the boundary is equal to that of the boundary,

i. e. $$\frac{ux}{a^2}+\frac{vy}{b^2}=-\omega y\ \frac{x}{a^2}+\omega x\ \frac{y}{b^2} \qquad ...(vii)$$

From (iii) and (vii), we have

$$Aa^2-Bb^2=\frac{\omega(a^2-b^2)}{2\omega'} \qquad ...(viii)$$

Now the condition of continuity of the tangential velocity at the boundary makes the values of $\frac{\partial\psi}{\partial\xi}$ from (i) and (v) must be the same.

Substituting $x=c\cosh\xi\cos\eta$

and $y=c\sinh\xi\sin\eta$ in equation (v)

At the boundary we have

$$-\frac{1}{2}\,\omega\,(a+b)^2\,e^{-2\xi}\cos 2\eta-\xi ab=\omega'\,c^2\cosh\xi\sinh\xi$$

$$\{A+B+(A-B)\cos 2\eta\}$$

$\forall$ values of η from o to 2π.

The relation being true for all values of η, we get

$$-\frac{1}{2}\,\omega\,(a+b)^2\,e^{-2\xi}=\omega' c^2\,(A-B)\cosh\xi\sinh\xi \qquad ...(ix)$$

On the boundary of the section

$$a=c\cosh\xi \quad b=c\sinh\xi$$

and $$a-b=c\,e^{-\xi},\ a^2-b^2=c^2$$

Then from (ix), we get

$$A-B=-\frac{\omega}{2\omega'}\cdot\frac{(a+b)^2\,e^{-2\xi}}{c^2\cosh\xi\sinh\xi}$$

or $$A-B=-\frac{\omega}{2\omega'}\cdot\frac{c^2 e^{2\xi} e^{-2\xi}}{c^2.\frac{a}{c}\cdot\frac{b}{c}}$$

or $$A-B=-\frac{\omega}{2\omega'}\cdot\frac{a^2-b^2}{ab} \qquad \text{...(x)}$$

From (viii) and (x), we have

$$Aa=Bb=\frac{ab}{a+b}$$

Substituting the value of A and B in (viii), we have

$$\frac{ba^2}{a+b}-\frac{ab^2}{a+b}=\frac{\omega(a^2-b^2)}{2\omega'}$$

or $$\frac{ab\ (a-b)}{a+b}=\frac{\omega(a^2-b^2)}{2\omega'}$$

or $$\omega=\frac{2ab}{(a+b)^2}\ \omega'.$$

or $$\omega=\frac{2a.a\sqrt{(1-e^2)}}{\{a+a\sqrt{(1-e^2)}\}^2}\ \omega'$$

or $$\omega=\frac{2\sqrt{(1-e^2)}}{\{1+\sqrt{(1-e^2)}\}^2}\ \omega' \qquad \{\text{as } b^2=a^2\ (1-e^2)$$

where e is the eccentricity of the ellipse.

Which determines the velocity of rotation of the cylinder as a whole in terms of the velocity and eccentricity of the section.

Path of the particles : Referred to the axes of the cross section, consider $(x\ y)$ be the co-ordinates of a particle of the vortex in the fluid, then

$$\dot{x}-\omega y=u=-\frac{\partial\psi}{\partial y}$$
$$=-2\omega'\ By$$
$$=-\frac{y\omega\ (a+b)}{b} \qquad \text{...(i)}$$

and $$\dot{y}+\omega x=v=\frac{\partial\psi}{\partial x}$$
$$=2\omega'\ Ax$$
$$=\frac{x\omega\ (a+b)}{a} \qquad \text{...(ii)}$$

From (i) and (ii), we obtain

$$\dot{x}=\omega y-\frac{(a+b)\ \omega y}{b}=-\frac{\omega ya}{b} \qquad \text{...(iii)}$$

and $\qquad \dot{y}=-\omega x+\dfrac{(a+b)\,\omega x}{a}=\dfrac{\omega x b}{a}$ ...(iv)

Differentiating (iii) $\omega . r$ to t, we have

$$\ddot{x}=-\frac{\omega a}{b}\dot{y}$$

or $\quad \ddot{x}=-\dfrac{\omega a}{b}\cdot\dfrac{\omega x b}{a}=-\omega^2 x$ ...(v)

also $\quad \ddot{y}=-\omega^2 y$ ...(vi)

which represents the S.H.M. whose time period is same *i.e.* the period of the relative motion is the same as that of the rotation of the cylinder.

Integrating (v) and (vi), we have

$$x=La\cos(\omega t+\epsilon)$$
$$y=Lb\sin(\omega t+\epsilon)$$

$\Rightarrow$ that the paths of the particles of the vortex parallel to the boundary are similar ellipse.

Ex. 22. *An elliptic cylinder is filled with liquid which has molecular rotation ω at every point, and whose particles move in planes perpendicular to the axis ; prove that the stream lines are similar ellipses described in periodic time*

$$\frac{\pi}{\omega}\cdot\frac{a^2+b^2}{ab}$$

Since an elliptic cylinder is filled with liquid, the stream function ψ satisfies

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=2\omega \quad \text{inside the boundary.}$$

...(i)

The normal velocity at any point of the boundary vanishes, being fixed.

$$u\,\frac{x}{a^2}+v\,\frac{y}{b^2}=0 \qquad \text{...(ii)}$$

Assuming the stream function ψ be of the form

$$\psi=\omega\,(Ax^2+By^2)$$

Then from (i), $\qquad 2\omega\,(A+B)=2\omega$

or $$A+B=1 \qquad \text{...(iii)}$$

and from (ii)

$$-2B\omega y \cdot \frac{x}{a^2} + 2A\omega x \cdot \frac{y}{b^2} = 0 \qquad \left\{ \begin{aligned} \text{as } u &= -\partial\psi/\partial y \\ &= -2B\omega y \\ v &= \partial\psi/\partial x \\ &= 2A\omega x \end{aligned} \right.$$

or

$$Aa^2 - Bb^2 = 0. \qquad \text{...(iv)}$$

Solving (iii) and (iv), we have

$$A = \frac{b^2}{a^2+b^2} \quad \text{and} \quad B = \frac{a^2}{a^2+b^2}$$

Thus $\psi = \frac{\omega}{a^2+b^2}(b^2x^2 + a^2y^2)$

Consider $P(x\ y)$ be any point in the fluid,

then

$$\dot{x} = -\frac{\partial\psi}{\partial y} = -\frac{2a^2y\omega}{a^2+b^2}$$

and

$$\dot{y} = -\frac{\partial\psi}{\partial x} = \frac{2b^2x\omega}{a^2+b^2}$$

Again $\ddot{x} = -\frac{2a^2\omega}{a^2+b^2}\dot{y}$

or

$$\ddot{x} = -\frac{4a^2b^2\,\omega^2}{(a^2+b^2)^2}x \qquad \text{...(v)}$$

Also

$$\ddot{y} = -\frac{4a^2b^2\,\omega^2}{(a^2+b^2)^2}y$$

By integrating, we have

$$x = L\,a\cos\left\{\frac{2ab\,\omega}{a^2+b^2}t + \lambda\right\}$$

and

$$y = L\,b\cos\left\{\frac{2ab\omega}{a^2+b^2}t + \lambda\right\}$$

$\Rightarrow$ that the stream lines are similar ellipses described in time period

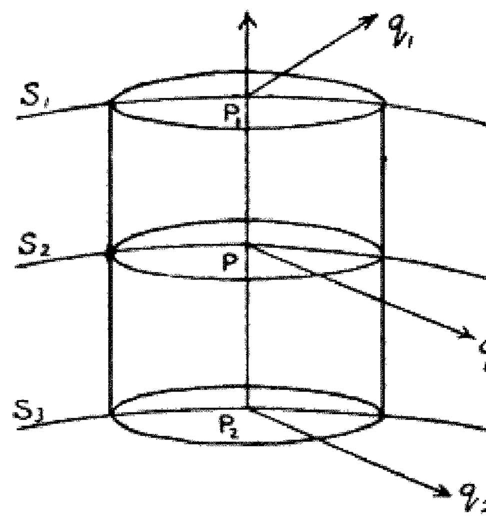
$$= \frac{2\pi}{\sqrt{\left(\frac{4a^2b^2\omega^2}{(a^2+b^2)^2}\right)}}$$

$$= \frac{\pi(a^2+b^2)}{\omega\,ab}$$

Proved.

§ 7·92. Vortex Sheets.

Consider the surfaces to be drawn in the fluid which is moving irrotationally every where except in that part which lies between S_1 and S_3 Let the point P of the surface S_2 is the point of centroid of an element dS_2 and the points P_1 and P_2 be taken on the normal at P, such that $PP_1 = \frac{1}{2}\epsilon\,\mathbf{n}$ and $PP_2 = -\frac{1}{2}\epsilon\,\mathbf{n}$ where $\mathbf{n}$ is the unit normal vector at the point P of a surface S_2 and ϵ be an infinitesimal positive Scalar. As P describes the surface S_2 the points P_1 and P_2 describe surfaces S_1 and S_3 parallel to the surface S_2 which is half way between them.



Consider an infinitesimal area dS_2 on S_2 whose centroid is P. The normals at the boundary of dS_2 together with dS_1 and dS_3 enclose a cylindrical element of volume

$$dv = \epsilon\ dS_2$$

Let ζ be the vorticity vector at P, then

$$\vec{\zeta}\,dv = \vec{\zeta}\,\epsilon dS_2$$

or $$\vec{\zeta}\,dv = \vec{\omega}\,dS_2$$

where $\vec{\omega} = \vec{\zeta}\,\epsilon$, now if ϵ tends to zero, ζ tends to infinity in such a manner that ω remains unaltered, then the surface S_2 is called a **vortex sheet** of vorticity ω per units area.

Note. *The normal components of velocity are continuous across the vortex sheet.*

The velocity will be continuous througout the fluid and if $\mathbf{q}$, $\mathbf{q}_1$, $\mathbf{q}_2$ are the velocities at P, P_1, P_2 then we have

$$\mathbf{q}_1 = \mathbf{q} + \tfrac{1}{2}\epsilon\,(\mathbf{n}\,\nabla)\,\mathbf{q}$$

and $$\mathbf{q}_2 = \mathbf{q} - \tfrac{1}{2}\epsilon\,(\mathbf{n}\,\nabla)\,\mathbf{q}$$

By addition, we have

$$\mathbf{q} = \tfrac{1}{2}(\mathbf{q}_1 + \mathbf{q}_2)$$

$\Rightarrow$ *that the velocity of a point P of a vortex sheet is the arithmetic mean of the velocities just above and just below P on the normal at P.*

By Gauss theorem.

$$\int_V \text{curl } \mathbf{q}\, dv = \int_S ds \times \mathbf{q}$$

or

$$\int_V \zeta\, dv = \int_S \mathbf{n} \times \mathbf{q}\, ds$$

Applying the above theorem to the cylinder of volume dv, we have

$$\zeta \epsilon\, ds = \mathbf{n} \times (\mathbf{q}_1 - \mathbf{q}_2)\, ds$$

or

$$\vec{\omega} = \mathbf{n} \times (\mathbf{q}_1 - \mathbf{q}_2) \qquad \{\text{as } \vec{\omega} = \xi \epsilon$$

It is evident that a non-zero value of ω is associated with a discountinuity of the velocity components $\mathbf{q}_1$, $\mathbf{q}_2$ perpendicular to $\mathbf{n}$.

$\Rightarrow$ *that a surface across which the tangential velocity changes abruptely is a vortex sheet.*

As $\vec{\omega} . \mathbf{n} = 0 \Rightarrow$ that $\vec{\omega}$ is perpendicular to $\mathbf{n}$ and is therefore tangential to the vortex sheet.

8

Waves

Waves *means the continuous transference of a particular state of form from one part of a medium to another.* This does not mean the transference of the medium itself from one place to another but simply the propagation through it of a particular form. We are acquainted with waterwaves, when a stone is thrown into a well, some disturbance occurs which travels radially over the surface of water. This disturbance is known as **waterwaves.** Similarly there are other examples of wave motion in liquids as high and low tides of the sea, standing waves in lakes etc. The wave motion is of great importance in physical investigations. The wave motion constitutes one of the principal modes of transmission of energy. The energy received from the Sun is transmitted by waves in the Ether, the energy of sound by air waves etc. Here we shall discuss water waves only.

§ 8·1. Wave motion.

Wave motion of a liquid acted upon by gravity and having a free surface is a motion in which the elevation of the free surface above a fixed horizontal plane varies.

Consider an arbitrary disturbance y which moves along X-axis with velocity c, so y is a function of the variables x and t *i.e.* $f(x, t)$ (let). At $t=0$, the curve $y=f(x)$ is generally known a **wave profile.** The wave profile will move a distance ct in the positive direction of X-axis, if the disturbance travels without any change of shape. The equation to the wave profile is given by

$$y=f(x-ct) \qquad \text{...(i)}$$

If we increase t by T and x by cT in (i), the wave profile will remain unchanged.

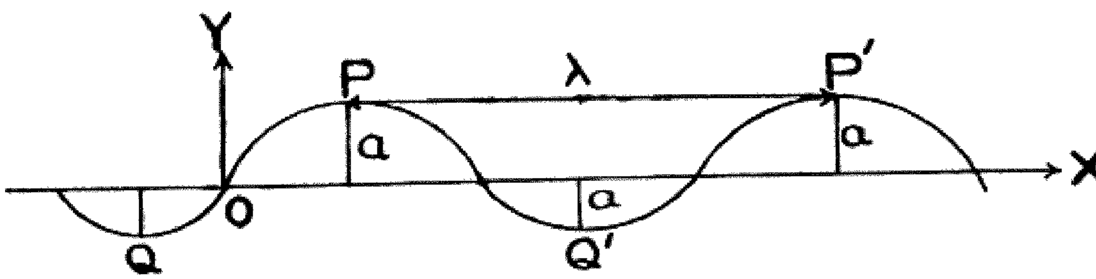
i.e. $$y=f\{x+cT-c(t+T)\}$$
$$=f(x-ct)$$

$\Rightarrow$ that the profile $y=f(x)$ moves with velocity c in the positive direction of X-axis.

A simple harmonic progressive wave is represented by a sine curve moving with definite velocity in the direction of its length at time t.

$$y=a \sin (mx-nt) \quad \ldots(ii)$$

or $$y=a \sin m\left(x-\frac{n}{m}t\right)$$



represents that the profile $y=a \sin mx$ at time $t=0$ moves with velocity $c=\frac{n}{m}$ along the positive direction of the X-axis, $c\left(=\frac{n}{m}\right)$ is called the **velocity of propogation.**

The maximum value of the disturbance y, *viz.* a, is called the **amplitude** of the wave.

The points P and P' of **maximum elevation** are called the **crests** and the points Q and Q' of **maximum depression** are called **troughs** of the wave.

The distance between successive crests is called the **wave length** λ *i.e.* $\lambda=\frac{2\pi}{m}$.

The aspect of free surface is exactly the same at time t and $t+\frac{2\pi}{n}$. Thus the **period** of the wave is $\frac{2\pi}{n}$ or $\frac{\lambda}{c}$ and $\frac{1}{T}$ be the frequency. It denotes the number of oscillations per second.

$$\lambda=cT.$$

Phase : Let the equation be taken as

$$y=a \sin (mx-nt+\epsilon) \quad \ldots(iii)$$

ϵ represents the **phase of the wave** at the instant from which t is measured. From (ii) and (iii), we see that wave motion have the same amplitude, wave length and period, but they differ in phase.

The angle $(mx-nt)$ in the equation (ii) is called the **phase angle** and n is called the **phase rate.**

§ 8·2. Standing or Stationary Waves.

Two simple harmonic progressive waves of the same amplitude, wave length and period travels in opposite directions are given by the equations.

$$y_1=a \sin (mx-nt) \quad \text{and} \quad y_2=a \sin (mx+nt)$$

or
$$\begin{aligned} y &= y_1+y_2 \\ &= a\{\sin (mx-nt)+\sin (mx+nt)\} \\ &= 2a \sin mx \cos nt \qquad \text{...(i)} \end{aligned}$$

A motion of this type is called a **standing or stationary wave.** At a given value of x the surface of the water moves up and down, also at any instant t the form of the surface (i) is a sine curve of amplitude $2a \cos nt$ which varies between 0 and $2a$. A wave of this type is not propagated.

The points for which $mx=p\pi$, $p=...-2, -1, 0, 1, 2...$ are always at rest in the mean surface *i.e.* the points of intersection of the curve with the X-axis are fixed points known as **Nodes.**

The points for which $mx=(2p+1)\dfrac{\pi}{2}$ are points of maximum displacement for a given value of t and are called **loops.** When $\cos nt=\pm 1$ the surface is in the form of the sine curve $y=\pm 2a \sin mx$, which shows the maximum departure from the mean level. When $\cos nt=0$ the free surface coincides with the mean level.

Similarly a **progressive wave** system is taken as the combination of two systems of stationary waves of the same amplitude, wave length and period, the crests and troughs of one system coincide with the nodes of the other.

e.g. if $y_1=a \sin mx \cos nt$ be one of the stationary waves the other must be

$$y_2=a \cos mx \sin nt.$$

then
$$y=y_1 \pm y_2=a\{\sin mx \cos nt \pm \cos mx \sin nt\}$$
or
$$y=a \sin (mx \pm nt)$$
represents a progressive wave.

§ 8·21. Progressive-type solutions of wave equation.

(i) One-dimensional wave equation.

$$\frac{\partial^2 y}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \qquad \text{...(i)}$$

Let $f=x-ct$ and $g=x+ct$.

then
$$\frac{\partial y}{\partial x}=\frac{\partial y}{\partial f}\cdot\frac{\partial f}{\partial x}+\frac{\partial y}{\partial g}\cdot\frac{\partial g}{\partial x}$$

or $$\frac{\partial y}{\partial x}=\frac{\partial y}{\partial f}+\frac{\partial y}{\partial g}$$

or $$\frac{\partial}{\partial x}\equiv\frac{\partial}{\partial f}+\frac{\partial}{\partial g}$$

and $$\frac{\partial^2 y}{\partial x^2}=\left(\frac{\partial}{\partial f}+\frac{\partial}{\partial g}\right)\left(\frac{\partial y}{\partial f}+\frac{\partial y}{\partial g}\right)$$

or $$\frac{\partial^2 y}{\partial x^2}=\frac{\partial}{\partial f}\left(\frac{\partial y}{\partial f}\right)+\frac{\partial}{\partial g}\left(\frac{\partial y}{\partial f}\right)+\frac{\partial}{\partial f}\left(\frac{\partial y}{\partial g}\right)+\frac{\partial}{\partial g}\left(\frac{\partial y}{\partial g}\right)$$

or $$\frac{\partial^2 y}{\partial x^2}=\frac{\partial^2 y}{\partial f^2}+2\frac{\partial^2 y}{\partial f\,\partial g}+\frac{\partial^2 y}{\partial g^2} \qquad \text{...(ii)}$$

also $$\frac{\partial y}{\partial t}=\frac{\partial y}{\partial f}\cdot\frac{\partial f}{\partial t}+\frac{\partial y}{\partial g}\cdot\frac{\partial g}{\partial t}$$

or $$\frac{\partial y}{\partial t}=-c\frac{\partial y}{\partial f}+c\frac{\partial y}{\partial g}$$

or $$\frac{\partial}{\partial t}\equiv c\left(\frac{\partial}{\partial g}-\frac{\partial}{\partial f}\right)$$

and $$\frac{\partial^2 y}{\partial t^2}=c^2\left(\frac{\partial}{\partial g}-\frac{\partial}{\partial f}\right)\left(\frac{\partial y}{\partial g}-\frac{\partial y}{\partial f}\right)$$

or $$\frac{\partial^2 y}{\partial t^2}=c^2\left\{\frac{\partial}{\partial g}\left(\frac{\partial y}{\partial g}\right)-\frac{\partial}{\partial g}\left(\frac{\partial y}{\partial f}\right)-\frac{\partial}{\partial f}\left(\frac{\partial y}{\partial g}\right)+\frac{\partial}{\partial f}\left(\frac{\partial y}{\partial f}\right)\right\}$$

or $$\frac{\partial^2 y}{\partial t^2}=c^2\left(\frac{\partial^2 y}{\partial f^2}-2\frac{\partial^2 y}{\partial f\,\partial g}+\frac{\partial^2 y}{\partial g^2}\right) \qquad \text{...(iii)}$$

Substituting the values of $\frac{\partial^2 y}{\partial x^2}$ and $\frac{\partial^2 y}{\partial t^2}$ in (i), we have

$$\frac{\partial^2 y}{\partial t^2}+2\frac{\partial^2 y}{\partial f\,\partial g}+\frac{\partial^2 y}{\partial g^2}=\frac{1}{c^2}\cdot c^2\left(\frac{\partial^2 y}{\partial f^2}-2\frac{\partial^2 y}{\partial f\,\partial g}+\frac{\partial^2 y}{\partial g^2}\right)$$

or $$\frac{\partial^2 y}{\partial f\,\partial g}=0.$$

whose general solution is given by

$$y=h(f)+k(g)$$

or $$y=h(x-ct)+k(x+ct)$$

where h and k are arbitrary functions.

(ii) Wave equation in Two Dimensions.

$$\nabla^2\psi=\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}$$

or $$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2} \qquad \text{...(i)}$$

Consider $f=lx+my-ct$ and $g=lx+my+ct$

where $$l^2+m^2=1.$$

Now $$\frac{\partial\psi}{\partial x}=\frac{\partial\psi}{\partial f}\cdot\frac{\partial f}{\partial x}+\frac{\partial\psi}{\partial g}\cdot\frac{\partial g}{\partial x}$$

$$=l\,\frac{\partial\psi}{\partial f}+l\,\frac{\partial\psi}{\partial g}$$

$$\frac{\partial}{\partial x}\equiv l\left(\frac{\partial}{\partial f}+\frac{\partial}{\partial g}\right)$$

and $$\frac{\partial^2\psi}{\partial x^2}=l^2\left(\frac{\partial}{\partial f}+\frac{\partial}{\partial g}\right)\left(\frac{\partial\psi}{\partial f}+\frac{\partial\psi}{\partial g}\right)$$

or $$\frac{\partial^2\psi}{\partial x^2}=l^2\left\{\frac{\partial^2\psi}{\partial f^2}+2\,\frac{\partial^2\psi}{\partial f\,\partial g}+\frac{\partial^2\psi}{\partial g^2}\right\}$$

Similarly, we have

$$\frac{\partial^2\psi}{\partial y^2}=m^2\left\{\frac{\partial^2\psi}{\partial f^2}+2\,\frac{\partial^2\psi}{\partial f\,\partial g}+\frac{\partial^2\psi}{\partial g^2}\right\}$$

and $$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=(l^2+m^2)\left\{\frac{\partial^2\psi}{\partial f^2}+2\,\frac{\partial^2\psi}{\partial f\,\partial g}+\frac{\partial^2\psi}{\partial g^2}\right\}$$

$$=\frac{\partial^2\psi}{\partial f^2}+2\,\frac{\partial^2\psi}{\partial f\,\partial g}+\frac{\partial^2\psi}{\partial g^2}\qquad\Big\{\text{as}\quad l^2+m^2=1$$

Also $$\frac{\partial\psi}{\partial t}=\frac{\partial\psi}{\partial f}\cdot\frac{\partial f}{\partial t}+\frac{\partial\psi}{\partial g}\cdot\frac{\partial g}{\partial t}\qquad\text{...(ii)}$$

$$=-c\,\frac{\partial\psi}{\partial f}+c\,\frac{\partial\psi}{\partial g}$$

and $$\frac{\partial^2\psi}{\partial t^2}=c^2\left\{\frac{\partial^2\psi}{\partial f^2}-2\,\frac{\partial^2\psi}{\partial f\,\partial g}+\frac{\partial^2\psi}{\partial g^2}\right\}\qquad\text{...(iii)}$$

Substituting the values of $\nabla^2\psi$ and $\frac{\partial^2\psi}{\partial t^2}$ in (i), we have

$$\frac{\partial^2\psi}{\partial f^2}+2\,\frac{\partial^2\psi}{\partial f\,\partial g}+\frac{\partial^2\psi}{\partial g^2}=\frac{1}{c^2}\cdot c^2\left\{\frac{\partial^2\psi}{\partial f^2}-2\,\frac{\partial^2\psi}{\partial f\,\partial g}+\frac{\partial^2\psi}{\partial g^2}\right\}$$

or $$\frac{\partial^2\psi}{\partial f\,\partial g}=0.$$

gives the general solution of the from

$$\psi=h\,(f)+k\,(g)$$

$$=h\,(lx+my-ct)+k\,(lx+my+ct)$$

where h and k are arbitrary functions.

Similarly we can determine the solution of the wave equation in three dimensions.

§ 8·22. Stationary type solutions of wave equation.

(i) One-dimensional wave equation. (Separation of variables).

$$\frac{\partial^2 \psi}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} \qquad ...(i)$$

Assuming the solution of the type

$$\psi=X(x)\, T(t) \qquad ...(ii)$$

from (i), we have

$$T\frac{d^2X}{dx^2}=\frac{1}{c^2}.X\frac{d^2T}{dt^2}$$

or $$\frac{1}{X}\frac{d^2X}{dx^2}=\frac{1}{c^2}.\ \frac{1}{T}\frac{d^2T}{dt^2}=-\lambda^2 \text{ (say)}$$

or $$\frac{1}{X}\frac{d^2X}{dx^2}=-\lambda^2 \quad \text{and} \quad \frac{1}{c^2}.\ \frac{1}{T}\frac{d^2T}{dt^2}=-\lambda^2$$

or $$\frac{d^2X}{dx^2}+\lambda^2X=0 \quad \text{and} \quad \frac{d^2T}{dt^2}+\lambda^2c^2T=0$$

which gives the solutions

$$X=A\cos\lambda x+B\sin\lambda x$$

and $$T=C\cos\lambda ct+D\sin\lambda ct$$

If we consider λ^2 in place of $-\lambda^2$ then it will give the solution of the type

$$X=e^{\pm\lambda x} \quad \text{and} \quad T=e^{\pm cpt}$$

(ii) Two-dimensional wave equation.

$$\frac{\partial^2 \psi}{\partial x^2}+\frac{\partial^2 \psi}{\partial y^2}=\frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2} \qquad ...(i)$$

Assuming the solution of the type

$$\psi=X(x)\, Y(y)\, T(t)$$

from (i), we have

$$YT\frac{d^2X}{dx^2}+XT\frac{d^2Y}{dy^2}=\frac{1}{c^2}XY\frac{d^2T}{dt^2}$$

Dividing both the sides by XYT, we get

$$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}=\frac{1}{c^2}\frac{1}{T}\frac{d^2T}{dt^2}=-\mu^2 \text{ (let)}$$

or $$\frac{1}{c^2}.\frac{1}{T}\frac{d^2T}{dt^2}=-\mu^2 \qquad ...(ii)$$

and $$\frac{1}{X}\frac{d^2X}{dx^2}+\frac{1}{Y}\frac{d^2Y}{dy^2}=-\mu^2 \qquad ...(iii)$$

or $$\frac{d^2T}{dt^2}+c^2\mu^2T=0$$

whose solution is given by

$$T=A\cos c\mu t+B\sin c\mu t \qquad ...(iv)$$

from (iii), we have

$$\left(\frac{X''}{X}+p^2\right)+\left(\frac{Y''}{Y}+q^2\right)=0$$

where $\mu^2=p^2+q^2$

Since the expressions with in the brackets are independent of each other, it given that

or $\frac{X''}{X}+p^2=0$ and $\frac{Y''}{Y}+q^2=0$

or $X''+p^2 X=0$ and $Y''+q^2 Y=0$.

whose solutions are given by

$$X=C\cos px+D\sin px \qquad \ldots(\text{v})$$

$$Y=E\cos qy+F\sin qy \qquad \ldots(\text{vi})$$

Thus the solution is of the type

$$\psi=\begin{matrix}\cos\\ \sin\end{matrix}\Big\}px\ \begin{matrix}\cos\\ \sin\end{matrix}\Big\}qy\ \begin{matrix}\cos\\ \sin\end{matrix}\Big\}c\mu t$$

where $p^2+q^2=\mu^2$

If we consider,

$$\mu^2=p^2-q^2$$

Then its solution will be of the form

$$\psi=\begin{matrix}\cos\\ \sin\end{matrix}\Big\}px\ \ exp\,(\pm qy)\ \begin{matrix}\cos\\ \sin\end{matrix}\Big\}\ c\mu t$$

Similarly we can determine the solution of the wave equation in three dimensions.

§ 9·3. Types of liquid waves.

The theory of wave motions in a liquid (water) is generally divided into two forms,

(i) Surface waves (ii) Tidal waves or long waves.

Surface waves *arises where the vertical acceleration of the fluid can not be neglected and the wave-length is small in comparison with the depth of water. The disturbance does not extend far below the surface.*

Tidal Waves or Long Waves *describe the alternative limit where the wave length is large in comparison with the depth. Here the vertical acceleration can be assumed negligible compared with the horizontal accelerations. The disturbance effects the motion of the whole of the liquid.*

§ 8·31. Surface Waves.

Consider the X-axis on the undisturbed surface and Y-axis vertically upwards. Since the motion generated originally from rest by the action of ordinary forces, is irrotationl, the velocity potential ϕ exists, we have

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0 \qquad \text{...(i)}$$

through out the liquid.

and
$$\frac{\partial\phi}{\partial\eta}=0 \qquad \text{...(ii)}$$

at a fixed boundary.

Surface condition.

Now we shall determine the condition which must be satisfied at the free surface *i.e.* p=const. Consider the origin O at the undistur bed level and OY be drawn vertically upwards. The pressure is given by Bernoull Bernoulli's equation

$$\frac{p}{\rho}=\frac{\partial\phi}{\partial t}-\Omega-\tfrac{1}{2}q^2+F(t) \qquad \text{...(iii)}$$

Substituting $\Omega=gy$ and neglecting the square of the velocity, we obtain.

$$\frac{p}{\rho}=\frac{\partial\phi}{\partial t}-gy+F(t) \qquad \text{...(iv)}$$

Let η denote the elevation of the surface at time t above the point $(x, 0)$. The equation to the surface is

$$y=\eta\ (x, t)$$

Let p_0 be the pressure on the surface, then from (iv),

$$\frac{p_0}{\rho}=\left(\frac{\partial\phi}{\partial t}\right)_{y=\eta}-g\eta$$

{Let $F(t)$ be marged in ϕ

Since the pressure is uniform

$$\eta=\frac{1}{g}\left(\frac{\partial\phi}{\partial t}\right)_{y=\eta} \qquad \text{...(v)}$$

Provided the function $F(t)$ and the constant be absorbed in the value of $\frac{\partial\phi}{\partial t}$.

Since the normal to the free surface makes an infinitely small angle $\frac{\partial\eta}{\partial x}$ with the vertical. We know that the normal component of the fluid velocity at the free surface must be equal to the normal velocity of the surface, which gives

$$\frac{\partial\eta}{\partial t}=\left(-\frac{\partial\phi}{\partial y}\right)_{y=\eta} \qquad \text{...(vi)}$$

from (v) and (vi), we have

$$\frac{1}{g}\left(\frac{\partial^2\phi}{\partial t^2}\right)_{y=\eta}=\left(-\frac{\partial\phi}{\partial y}\right)_{y=\eta}$$

or $$\left\{\frac{\partial^2\phi}{\partial t^2}+g\,\frac{\partial\phi}{\partial y}\right\}_{y=\eta}=0 \qquad \ldots\text{(vii)}$$

Since the oscilliations are small, (neglecting higher orders) so value on L. H. S. is considered at $y=0$ inplace of $y=\eta$. Thus the motion in a gravitational field can be determined by the following system of equations

$$\nabla^2\phi=0$$

$$\frac{\partial\phi}{\partial\eta}=0$$

$$\left(g\frac{\partial\phi}{\partial y}+\frac{\partial^2\phi}{\partial t^2}\right)_{y=0}=0 \qquad \text{(at the free surface)}$$

In the case of S. H. M. the time factor being $e^{i(\sigma t+\varepsilon)}$, the above condition becomes

$$\sigma^2\phi=g\frac{\partial\phi}{\partial y}$$

The relation (vii) can be obtained from $\frac{Dp}{Dt}=0$ at the free surface

$$\frac{\partial p}{\partial t}+u\,\frac{\partial p}{\partial x}+v\frac{\partial p}{\partial y}=0.$$

or $$\frac{\partial p}{\partial t}-\frac{\partial\phi}{\partial x}\cdot\frac{\partial p}{\partial x}-\frac{\partial\phi}{\partial y}\cdot\frac{\partial p}{\partial y}=0.$$

From (iv) substituting $\frac{p}{\rho}=\frac{\partial\phi}{\partial t}-gy$, we have

$$\rho\,\frac{\partial^2\phi}{\partial t^2}-\rho\,\frac{\partial\phi}{\partial x}\cdot\frac{\partial^2\phi}{\partial x\partial t}-\rho\,\frac{\partial\phi}{\partial y}\left(\frac{\partial^2\phi}{\partial y\partial t}-g\right)=0$$

or $$\frac{\partial^2\phi}{\partial t^2}+g\,\frac{\partial\phi}{\partial y}=0.$$

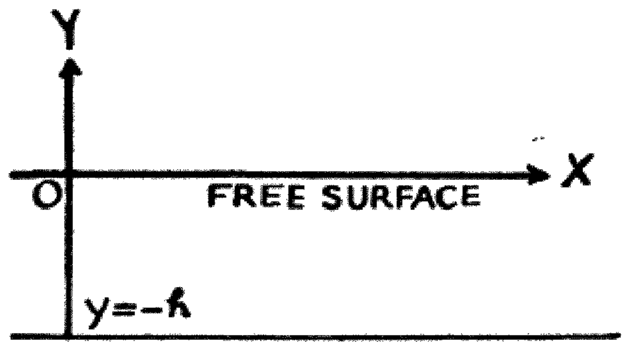
(neglecting other terms)

§ 8·32. Progressive waves on the surface of a canal.

Now we shall use the above conditions to determine the propagation of simple harmonic waves of the form

$$\eta=a\sin\,(mx-nt) \qquad \ldots\text{(i)}$$

at the surface of a canal of uniform depth h with parallel vertical walls.



Now the conditions are

(a) $$\nabla^2\phi=0$$

(b) $$\frac{\partial\phi}{\partial y}=0 \text{ at } y=-h$$

(c) $$g\frac{\partial\phi}{\partial y}+\frac{\partial^2\phi}{\partial t^2}=0 \text{ at } y=0$$

Again $$\left(\frac{\partial\eta}{\partial t}\right)_{y=0}=\left(-\frac{\partial\phi}{\partial y}\right)_{y=0}$$

or $$\left(\frac{\partial\phi}{\partial y}\right)_{y=0}=na\cos(mx-nt) \qquad \text{...(ii)}$$

from (i)

Assuming the solution be of the form

$$\phi=f(y)\cos(mx-nt) \qquad \text{...(iii)}$$

From the condition (a), we have

$$f''(y)\cos(mx-nt)-m^2f(y)\cos(mx-nt)=0.$$

or $$\frac{d^2f}{dy^2}-m^2f=0.$$

So $$f(y)=Ae^{my}+Be^{-my}$$

where A and B are arbitrary constants.

Substituting the value of $f(y)$ in (iii), we get

$$\phi=(Ae^{my}+Be^{-my})\cos(mx-nt) \qquad \text{...(D)}$$

From the condition (b), we have

$$Ae^{-mh}=Be^{mh}=\tfrac{1}{2}C \text{ (let)}$$

or $$\phi=\tfrac{1}{2}C\{e^{m(y+h)}+e^{-m(y+h)}\}\cos(mx-nt)$$

or $$\phi=C\cosh m(y+h)\cos(mx-nt) \qquad \text{...(iv)}$$

From the surface condition (c), we have

$$\{gmC\sinh m(y+h)-n^2c\cosh m(y+h)\}\cos(mx-nt)=0$$

or $$n^2 = gm \tanh m(y+h)$$

or $$n^2 = gm \tanh mh \text{ at } y=0 \qquad \text{...(v)}$$

If $c = \dfrac{n}{m}$ denote the velocity of propagation and

$\lambda = \dfrac{2\pi}{m}$ denote the wave length,

then $$c^2 = \frac{1}{m^2} gm \tanh mh$$

$$c^2 = \frac{g\lambda}{2\pi} \tanh \frac{2\pi h}{\lambda} \qquad \text{...(vi)}$$

The constant C in (iv) can be expressed in terms of the amptitude a of the wave. Thus from (ii), we have

$$mC \sinh m(y+h) \cos(mx-nt) = na \cos(mx-nt)$$

or $$mC \sinh mh = na \text{ at } y=0$$

Substituting the value of the constant C in (iv), we have

$$\left.\begin{aligned} \phi &= \frac{na}{m} \frac{\cosh m(y+h)}{\sinh mh} \cos(mx-nt) \\ \text{or} \quad \phi &= \frac{ga}{n} \frac{\cosh m(y+h)}{\cosh mh} \cos(mx-nt) \end{aligned}\right\} \qquad \text{...(vii)}$$

$\left[\text{as } m = \dfrac{n^2}{g \tanh mh}\right.$ from (v)

Let (X, Y) be the co-ordinates of a particle relative to the mean position $(x\ y)$, then

$$u = \dot{X} = -\frac{\partial \phi}{\partial x} = na \frac{\cosh m(y+h)}{\sinh mh} \sin(mx-nt) \qquad \text{...(viii)}$$

and $$v = \dot{Y} = -\frac{\partial \phi}{\partial y} = -na \frac{\sinh m(y+h)}{\sinh mh} \cos(mx-nt) \qquad \text{...(ix)}$$

Path of the particles

Integrating the relation (viii) and (ix) with regard to time t, we have

$$X = a \frac{\cosh m(y+h)}{\sinh mh} \cos(mx-nt)$$

and $$Y = a \frac{\sinh m(y+h)}{\sinh mh} \sin(mx-nt)$$

By eliminating t, we have

$$\frac{X^2}{\cosh^2 m(y+h)} + \frac{Y^2}{\sinh^2 m(y+h)} = \frac{a^2}{\sinh^2 mh}$$

Thus the particle describes the ellipse about its mean position. The distance between the focii is $2a$ cosech mh which is same for all such ellipses being independent of y. The major axis are horizontal. The major axis and the minor axis decrease as the depth of the particle increases, the minor axis vanishes when $y=-h$. Thus at the bottom, the ellipse degenerates into a straight line.

§ 8·33. Waves on a deep canal.

Consider the depth h of the canal be sufficiently great *i.e.* h tends to infinity (say). From Equation (D) § 8·32, we have

$$\phi=(Ae^{my}+Be^{-my})\cos(mx-nt)$$

Since h is great in comparison with λ for e^{-my} to be neglected then we must have $B=0$,

or $$\phi=Ae^{my}\cos(mx-nt) \qquad \ldots\text{(i)}$$

and $$n^2=mg$$

or $$c^2=\frac{g\lambda}{2\pi} \qquad \left\{\text{as } c=\frac{n}{m},\ \lambda=\frac{2\pi}{m}\right.$$

Let $\eta=a\sin(mx-nt)$ is the free surface

then $$-an\cos(mx-nt)=-Am\, e^{my}\cos(mx-nt)$$

or $$an\cos(mx-nt)=Am\cos(mx-nt) \qquad \text{at } y=0.$$

or $$an=Am$$

$$\left\{\left(\frac{\partial\eta}{\partial t}\right)_{y=0}=-\left(\frac{\partial\phi}{\partial y}\right)_{y=0}\right.$$

at free surface

Substituting the value of A in (i), we have

$$\phi=\frac{na}{m}e^{my}\cos(mx-nt) \qquad \ldots\text{(ii)}$$

Let $(X\,Y)$ be the co-ordinates of a particle relative to its mean position (x, y), then

$$u=\dot{X}=-\frac{\partial\phi}{\partial x}=na\,e^{my}\sin(mx-nt) \qquad \ldots\text{(iii)}$$

$$v=\dot{Y}=-\frac{\partial\phi}{\partial y}=-na\,e^{my}\cos(mx-nt) \qquad \ldots\text{(iv)}$$

Integrating the relation (iii) and (iv) with regard to time t, we have

$$X=ae^{my}\cos(mx-nt)$$

and $$Y=ae^{my}\sin(mx-nt).$$

By eliminating t, we have

$$X^2+Y^2=a^2e^{2my}$$

Thus the path of a particle is a circle described with uniform angular velocity n.

Here $\quad n=(gm)^{1/2}=\left(\dfrac{2\pi g}{\lambda}\right)^{1/2}$

§ 8·34. Energy of Progressive waves.

Assuming the progressive waves at the surface of water of depth h. Consider the wave profile be of the form

$$\eta=a \sin (mx-nt) \qquad \text{...(i)}$$

and

$$\phi=\frac{ga}{n}\frac{\cosh m(y+h)}{\cosh mh} \cos (mx-nt) \qquad \text{...(ii)}$$

{Ref. Equation (vii) § 8·32

Let V be the energy of the water between two verticle planes parallel to the direction of propagation at unit distance apart. Then the potential energy for a single wave length is given by

$$V=\tfrac{1}{2} g\rho \int_0^\lambda \eta^2 \, dx$$

or

$$V=\tfrac{1}{2} g\rho \, a^2 \int_0^\lambda \sin^2 (mx-nt) \, dx$$

or

$$V=\tfrac{1}{4} g\rho \, a^2 \int_0^\lambda \{1-\cos 2 (mx-nt)\} \, dx$$

$$V=\tfrac{1}{4} g \rho \, a^2 \lambda \qquad \text{...(iii)}$$

$$\left\{\text{Since } \lambda=\frac{2\pi}{m}\right.$$

Since the motion is irrotational The kinetic energy is given by

$$T=-\tfrac{1}{2} \rho \int \phi \frac{\partial \phi}{\partial n} \, ds \qquad \text{...(iv)}$$

integrated along the profile of a wave length and ∂n is measured along the normal into the water. To the order of small quantities (iv) can be written as

$$T=-\tfrac{1}{2} \rho \int_0^\lambda \phi \left(-\frac{\partial \phi}{\partial y}\right)_{y=0} dx$$

or

$$T=\tfrac{1}{2} \rho \int_0^\lambda \left(\phi \frac{\partial \phi}{\partial y}\right)_{y=0} dx$$

or

$$T=\tfrac{1}{2} g\rho \, a^2 \int_0^\lambda \cos^2 (mx-nt) \, dx$$

or

$$T=\tfrac{1}{4} g\rho \, a^2 \lambda$$

Thus the total energy E per wave length is given by

$$\begin{aligned} E &= V+T \\ &= \tfrac{1}{4} g\rho \, a^2 \lambda+\tfrac{1}{4} g \rho \, a^2 \lambda \\ &= \tfrac{1}{2} g\rho a^2\lambda \end{aligned}$$

§ 8·4. Progressive waves reduced to a steady motion.

Consider the progressive wave move towards the positive direction of X-axis with a velocity c without change in shape. By super-posing a velocity c in the negative direction of X-axis on the whole fluid, the wave-profile becomes fixed in space. Thus the problem reduces to one of the steady motion. The streamlines are same as the paths of the fluid particles.

Since the motion is irrotational $\exists$ a velocity potential ϕ. If there are no waves *i.e.* the free boundary coincide with the X-axis then there is a uniform flow with velocity c in the negative direction of X-axis. Let the complex potential be

$$\begin{aligned} w &= c\,(x+iy) \\ &= cz \end{aligned}$$

Consider the complex potential be of the form

$$w = cz + P \cos mz - i\,Q \sin mz \qquad \text{...(i)}$$

or
$$\phi + i\psi = c\,(x+iy) + P \cos m\,(x+iy) - i\,Q \sin m\,(x+iy)$$

Equating real and imaginary parts, we have

$$\phi = cx + (P \cosh my + Q \sinh my) \cos mx \qquad \text{...(ii)}$$

and
$$\psi = cy - (P \sinh my + Q \cosh my) \sin mx \qquad \text{...(iii)}$$

The velocity potential and stream function satisfy the Laplace Equation.

For the bottom to be a streamline, we have

$$\psi = \text{const. at } y = -h$$

so
$$-ch - (-P \sinh mh + Q \cosh mh) \sin mx = \text{constant}$$

{ from (iii)

or
$$-P \sinh mh + Q \cosh mh = 0$$

or
$$\frac{P}{\cosh mh} = \frac{Q}{\sinh mh} = \mu \text{ (let)}$$

Substituting the values of P and Q in (ii) and (iii), we have

$$\phi = cx + \mu\,(\cosh my \cosh mh + \sinh my \sinh mh) \cos mx$$

$$\phi = cx + \mu \cosh m\,(y+h) \cos mx \qquad \text{...(iv)}$$

and
$$\psi = cy - \mu\,(\sinh my \cosh mh + \cosh my \sinh mh) \sin mx$$

$$\psi = cy - \mu \sinh m\,(y+h) \sin mx \qquad \text{...(v)}$$

If the free surface be a simple sine curve

$$\eta = a \sin mx \qquad \text{...(vi)}$$

The equation (vi) will be a stream line provided

$$ca - \mu \sinh mh = 0 \qquad \text{...(vii)}$$

(neglecting squares of small quantities)

By Bernoulli's equation, pressure is given by

$$\frac{p}{\rho}+gy+\tfrac{1}{2}\,q^2=\text{constant}$$

or
$$\frac{p}{\rho}+gy+\tfrac{1}{2}\left\{\left(\frac{\partial\phi}{\partial x}\right)^2+\left(\frac{\partial\phi}{\partial y}\right)^2\right\}=\text{constant}$$

At the free surface, it reduces to

$$\frac{p}{\rho}+ga\sin mx+\tfrac{1}{2}\left\{\left(c-m\mu\cosh m\,(y+h)\sin mx\right)^2\right.$$
$$\left.+m^2\mu^2\sinh^2 m\,(y+h)\cos^2 mx\right\}=\text{constant}$$

or
$$\frac{p}{\rho}+ga\sin mx+\tfrac{1}{2}\,c^2\left\{1-2ma\coth mh\sin mx\right\}=\text{constant}$$

{ from (vii)

(Neglecting higher orders of a^2.)

But p=constant at the free surface

then $\quad ga\sin mx+\tfrac{1}{2}\,c^2\,(1-2ma\coth mh\sin mx)=\text{constant}$

or $\quad (ga-mac^2\coth mh)\sin mx=\text{constant}.$

The coefficient of $\sin mx$ must vanish

i.e. $\quad ga=ma\,c^2\coth mh$

or $\quad g=mc^2\coth mh$

or $\quad c^2=\dfrac{g\lambda}{2\pi}\tanh\dfrac{2\pi h}{\lambda}\qquad\left\{\text{as}\quad m=\dfrac{2\pi}{\lambda}\right.$

Superposing on this steady motion a uniform flow towards the positive direction of X-axis with velocity c, the fluid at great distance is brought to rest and the wave profile in the form of sine curve will move with *a* velocity c. Thus we obtain the progressive waves of wave length λ.

§ 8·41. **On Deep water** (Progressive waves).

Now we shall discuss progressive waves on deep water *i e.* when h tends to infinity. The conditions being that the upper surface is free and the total depth infinite.

Consider

$$\phi=cx+Ae^{my}\cos mx \qquad \ldots(\text{i})$$

and
$$\psi=cy-Ae^{my}\sin mx \qquad \ldots(\text{ii})$$

Let the equation to the free surface be

$$\eta=a\sin mx \qquad \ldots(\text{iii})$$

The free surface will be a streamline

if $\quad \psi=\text{constant}$

or $\quad ca\sin mx-Ae^{ma\sin mx}\sin mx=\text{constant}$

or $\qquad (ca - Ae^{ma \sin mx}) \sin mx = \text{constant}.$

This will hold if coefficients of $\sin mx$ must vanish

i.e. $\qquad ca = A$

then $\qquad \phi = cx + ca\, e^{my} \cos mx$

$\qquad \psi = cy - ca\, e^{my} \sin mx$

By Bernoulli's equation pressure is given by,

$$\frac{p}{\rho} + gy + \tfrac{1}{2}\left\{\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2\right\} = \text{constant}$$

At the free surface, it reduces to

$$gy + \tfrac{1}{2}\left\{\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2\right\} = \text{constant}$$

or $\qquad gy + \tfrac{1}{2}c^2\{1 - 2ma\, e^{my} \sin mx + m^2a^2\, e^{2my}\} = \text{constant}$

Neglecting the terms of a^2, we have

$$gy + \tfrac{1}{2}c^2\{1 - 3mae^{my} \sin mx\} = \text{constant}$$

or $\qquad ga \sin mx + \tfrac{1}{2}\, c^2\{1 - 2ma\, e^{ma \sin mx} . \sin mx\} + \text{constant}$

Equating the coefficients of $\sin mx$ to zero, we have

$$ga = \tfrac{1}{2}\, c^2\, (2ma)$$

or $\qquad c^2 = g/m \qquad$ {at free surface p=const. and $\psi = 0$

Which gives the velocity of propagation of two-dimensional waves of given length propagated over infinitely deep liquid.

This equation also represents that the pressure is constant along each streamline.

§ 8·5. Stationary or standing waves.

Assuming the equation for a stationary wave be

$$\phi = A \sin mx \sin nt \qquad \text{...(i)}$$

{ Ref. § 8·2

Where A is a function of y only

Then from the condition (a) § 8·32, we have

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

or $\qquad \left(\dfrac{d^2 A}{dy^2} - m^2 A\right)\phi = 0 \qquad$ from (i)

whose solution is given by

$$A = Be^{my} + Ce^{-my}$$

where B and C are arbitrary constants.

Substituting the value of A in (i), we have

$$\phi = (Be^{my} + Ce^{-my}) \sin mx \sin nt \qquad \text{...(ii)}$$

Again from condition (b) § 8·32,

$$\frac{\partial \phi}{\partial y} = 0 \text{ at } y = -h, \text{ we have}$$

or $$Bme^{-mh}-Cme^{+mh}=0$$

or $$Be^{-mh}=Ce^{mh}=\tfrac{1}{2}\mu \text{ (say)}$$

So equation (ii) reduces to

$$\phi=\tfrac{1}{2}\mu\,\{e^{m(y+h)}+e^{-m(y+h)}\}\sin mx \sin nt$$

or $$\phi=\mu \cosh m\,(y+h)\sin mx \cos nt \qquad \text{...(iii)}$$

By using the condition (c) § 8·32

$$g\,\frac{\partial\phi}{\partial y}+\frac{\partial^2\phi}{\partial t^2}=0 \text{ at } y=0, \text{ we have}$$

or $$\{g\mu m \sinh m\,(y+h)-\mu\, n^2 \cosh m\,(y+h)\}\sin mx \sin nt=0$$

which gives $$n^2=gm \tanh m\,(y+h)$$

The phase velocity c is given by

$$c^2=\frac{n^2}{m^2}=\frac{g}{m}\tanh m\,(y+h)$$

or $$c^2=\frac{g}{m}\tanh mh \qquad \{\text{at } \; y=0$$

or $$c^2=\frac{g\lambda}{2\pi}\tanh\frac{2\pi h}{\lambda} \qquad \text{...(iv)}$$

$$\left\{\text{as } \; m=\frac{2\pi}{\lambda}\right.$$

The constant μ can be determined in terms of wave amplitude by using the condition

$$\frac{\partial\phi}{\partial t}=\left(-\frac{\partial\phi}{\partial y}\right)_{y=0}$$

or $$-na=-m\mu \sinh mh$$

or $$\mu=\frac{na}{m \sinh mh}$$

Substituting the value of μ in (iii), we get

$$\phi=\frac{na}{m}\,\frac{\cosh\; m\,(y+h)}{\sinh mh}\sin mx \sin nt \qquad \text{...(v)}$$

Paths of Particles.

Consider (X, Y) be the co-ordinates of a fluid particle relative to its mean position (x, y).

$$\dot{X}=-\frac{\partial\phi}{\partial x}=-na\;\frac{\cosh m\,(y+h)}{\sinh\; mh}\cos mx \sin nt$$

and $$\dot{Y}=-\frac{\partial\phi}{\partial y}=-na\;\frac{\sinh m\,(y+h)}{\sinh mh}\sin mx \sin nt$$

By integrating with regard to t, we have

$$X=a\,\frac{\cosh m\,(y+h)}{\sinh mh}\cos mx \cos nt$$

and $$Y=a\frac{\sinh m\,(y+h)}{\sinh mh}\sin mx\cos nt$$

or $$\frac{Y}{X}=\tanh m\,(y+h)\tan mx$$

which is independent of the time t. The motion of each particle is rectilinear. The direction varying from vertical below the crests and troughs, to horizonal below the nodes.

§ 8·51. On deep water. (Stationary waves)

Now we shall discuss standing waves on deep water *i.e.* when h tends to infinity.

Since $\phi=(Be^{my}+Ce^{-my})\sin mx\sin nt$

{ Ref. equation (ii) § 8·5.

When h tends to infinity the constant C must vanish, so it reduces to

$$\phi=Be^{my}\sin mx\sin nt \qquad \text{...(i)}$$

By using the condition (c) § 8·32

$g\dfrac{\partial\phi}{\partial y}+\dfrac{\partial^2\phi}{\partial t^2}=0$ at $y=0$, we have

$$gm\,Be^{my}\sin mx\sin nt-Bn^2e^{my}\sin mx\sin nt=0$$

or $$B\,(gm-n^2)\,e^{my}\sin mx\sin nt=0$$

which gives

$$gm-n^2=0$$

or $$m=\frac{n^2}{g}$$

Thus $$c^2=\frac{n^2}{m^2}=\frac{gm}{m^2}=\frac{g}{m}=\frac{g\lambda}{2\pi}$$

The constant B can be determined in terms of the wave amplitude. Thus

$$na=Bm$$

or $$\phi=\frac{na}{m}e^{my}\sin mx\sin nt \qquad \text{\{from (i)}$$

and $$\psi=\frac{na}{m}e^{my}\cos mx\sin nt$$

Paths of the Particles.

We have

$$\dot{X}=+\frac{\partial\phi}{\partial x}=na\,e^{my}\cos mx\sin nt$$

and $$\dot{Y}=-\frac{\partial\phi}{\partial y}=na\,e^{my}\sin mx\sin nt$$

By integrating with regard to t, we have

$$X = ae^{my} \cos mx \cos nt$$
$$Y = ae^{my} \sin mx \cos nt$$

By dividing, we have

$$\frac{Y}{X} = \tan mx.$$

Thus the trajectory of the particle is a straight line with slope $\tan mx$. The particle oscillates in the vertical direction at the antinodes and oscillates in the horizontal direction at the nodes. The amplitude of the oscillations (ae^{my}) decreases as depth increases.

§ 8 52. Energy of Stationary Waves.

Consider the wave profile to be of the form

$$\eta = a \sin mx \cos nt \qquad \ldots(i)$$

and
$$\phi = \frac{ga}{n} \frac{\cosh m(y+h)}{\cosh mh} \sin mx \sin nt \qquad \ldots(ii)$$

{Ref. equation (v) § 8·5

Let the elevation be η. The mass of liquid standing above a base dx in the XY-plane is $\rho\eta dx$, its centre of mass is at a height $\frac{1}{2}\eta$. Let V be the potential energy*, then

$$V = \int_0^\lambda \tfrac{1}{2}\eta . g\rho\eta \, dx$$

or
$$V = \int_0^\lambda \tfrac{1}{2} g\rho . \eta^2 \, dx$$

or
$$V = \tfrac{1}{2} g\rho a^2 \int_0^\lambda \sin^2 mx \cos^2 nt \, dx$$

or
$$V = \tfrac{1}{4} g\rho a^2 \cos^2 nt \int_0^\lambda (1 - \cos 2mx) \, dx$$

or
$$V = \tfrac{1}{4}\lambda g\rho a^2 \cos^2 nt$$

when $\cos nt = 1$, the potential energy is $\frac{1}{4}\lambda g\rho a^2$.

Again, let T be the kinetic energy,

then
$$T = \tfrac{1}{2}\rho \int_0^\lambda q^2 \, dv$$

$$T = \tfrac{1}{2}\rho \int_0^\lambda \phi \frac{\partial \phi}{\partial n} \, ds \qquad \text{(See chapter 4)}$$

*The potential energy is due to elevated water in a wave length. So we measure the potential energy relative to the undisturbed system.

$$T=\tfrac{1}{2}\rho \int_0^\lambda \left(\phi\frac{\partial\phi}{\partial y}\right)_{y=0} dx$$

or
$$T=\tfrac{1}{2}\rho \cdot \frac{g^2a^2}{n^2} \cdot m \tanh mh \sin^2 nt \int_0^\lambda \sin^2 mx\, dx$$

or
$$T=\tfrac{1}{4}\rho g\, \frac{m^2}{n^2} \cdot a^2 \sin^2 nt \cdot \frac{g}{m} \tanh mx \int_0^\lambda (1-\cos 2mx)\, dx$$

or
$$T=\tfrac{1}{4}\lambda\rho g a^2 \sin^2 nt \cdot c^2 \cdot \frac{1}{c^2} \quad \left[\text{as } c^2=\frac{g}{m}\tanh mh \text{ and } c^2=\frac{n^2}{m^2}\right.$$

or
$$T=\tfrac{1}{4}g\rho a^2\, \lambda \sin^2 nt$$

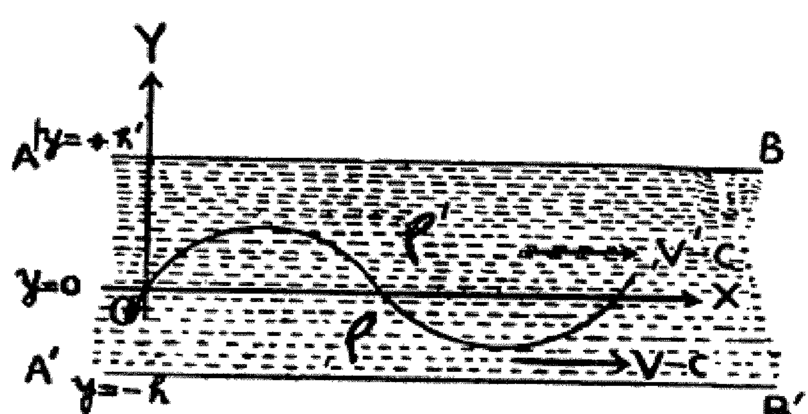
Thus the total energy E per wave length at any time t is

$$=V+T$$
$$=\tfrac{1}{4}g\rho a^2\lambda\, (\cos^2 nt+\sin^2 nt)$$
$$=\tfrac{1}{4}g\rho a^2\lambda$$

The kinetic energy and potential energy change continuously with the time.

§ 8·6. Waves at the common surface of two liquids.

Consider a liquid of density ρ' and depth h' move with velocity



V' over another liquid of density ρ and depth h with velocity V in the same direction.

Consider X-axis in the undisturbed interface and Y-axis to be vertically upwards. To make the motion steady, superposing on the whole mass a velocity (equal and opposite to that of propagation of waves) $-c$ thereby bringing the wave form to rest in space. Now velocities of the streams reduce to $V'-c$ and $V-c$.

Let the velocity potential and stream function related to lower liquid be

$$\phi=-(V-c)\,x+A\cosh m(y+h)\cos mx$$

and $$\psi=-(V-c)\,y-A\sinh m(y+h)\sin mx \qquad \ldots(\text{i})$$

{Ref. equation (iv) and (v) § 8·4 changing the velocities of the streams to $V'-c$ and $V-c$.

Similarly the velocity potential and stream function related to upper liquid are

$$\phi'=-(V'-c)\,x+A_1\cosh m(y-h')\cos mx$$

and $$\psi'=-(V'-c)\,y-A_1\sinh m(y-h')\sin mx \qquad \ldots(\text{ii})$$

Thus the complex potential for the lower liquid moving with velocity $-(V-c)$ in the negative direction of X is

$$w=\phi+i\psi=-(V-c)\,z-\frac{a\,(V-c)}{\sinh mh}\cos m\,(z+ih) \qquad \ldots(\text{iii})$$

{ as $\mu=\dfrac{(V-c)\,a}{\sinh mh}$ Ref. equation (vii) § 8·4

and the complex potential for the upper liquid moving with velocity $-(V'-c)$ in the negative direction of X is

$$w_1=\phi'+i\psi'=-(V'-c)z+\frac{a\,(V'-c)}{\sinh mh'}\cos m\,(z-ih') \qquad \ldots(\text{iv})$$

Let q be the speed in the lower liquid, then we have

$$q^2=\frac{dw}{dz}\cdot\frac{d\bar{w}}{d\bar{z}}$$

or $$q^2=\left\{(V-c)-\frac{am\,(V-c)}{\sinh mh}\sin m\,(z+ih)\right\}\left\{(V-c)-\frac{am\,(V-c)}{\sinh mh}\sin m\,(\bar{z}-ih)\right\}$$

or $$q^2=(V-c)^2-\frac{2ma\,(V-c)^2}{\sinh mh}\sin mx\cosh m\,(y+h)$$

(neglecting a^2)

Let q_0 be the speed at the interface, substituting $y=0$, we have

$$q_0^2=(V-c)^2-2ma\frac{(V-c)^2}{\sinh mh}\cosh mh\sin mx$$

or $$q_0^2=(V-c)^2\,\{1+2ma\coth mh\sin mx\}$$

The equation to the free surface is

$$\eta=a\sin mx \text{ (say)}$$

then $\quad q_0^2=(V-c)^2\{1-2m\eta \coth mh\}$ ...(v)

Similarly if q_0' be the speed at the interface due to upper liquid

then $\quad q_0'^2=(V'-c)^2\{1+2m\eta \coth mh'\}$ ...(vi)

[substituting $-h'$ for h in (v)]

The expressions for the pressure at the interface, are

$$\frac{p}{\rho}+\tfrac{1}{2}q_0^2+g\eta=\text{const}$$

$$\frac{p'}{\rho'}+\tfrac{1}{2}q_0'^2+g\eta=\text{const}$$

Since the pressure is to be continuous across the interface. $p=p'$, then

$$\tfrac{1}{2}\rho' q_0'^2-\tfrac{1}{2}\rho q_0^2+g\eta\,(\rho'-\rho)=\text{const}$$

or
$$\tfrac{1}{2}\rho'\,(V'-c)^2\{1+2m\eta \coth mh'\}$$
$$-\tfrac{1}{2}\rho\,(V-c)^2\{1-2m\eta \coth mh\}+g\eta\,(\rho'-\rho)=\text{const.}$$

Since R.H.S. is constant, the coefficient of the variable η must vanish. So

$$m\rho'(V'-c)^2 \coth mh'+m\rho\,(V-c)^2 \coth mh=g\,(\rho-\rho') \quad \text{...(vii)}$$

This equation determines the velocity of propagation c of waves of length $\frac{2\pi}{m}$ at the common surface of two streams whose velocities are V and V'.

Since the tangential velocities on opposite sides of the interface are different, even if V, $V'=0$, the interface must be a **Vortex Sheet.**

Case I. If the liquids are at rest *i.e.* $V=0$, $V'=0$,

then $\quad m\rho'c^2 \coth mh'+m\rho c^2 \coth mh=g\,(\rho-\rho')$

or
$$c^2=\frac{g}{m}\cdot\frac{\rho-\rho'}{\rho \coth mh+\rho' \coth mh'}$$

Case II. Consider the upper fluid to be air of specific gravity $s=\frac{\rho'}{\rho}$ at an infinite depth, substituting $V=0$, $V'=0$, we have

$$c^2=\frac{g}{m}\,\frac{\rho-\rho'}{\rho \coth mh+\rho' \coth mh'}$$

or
$$c^2=\frac{g}{m}\,\frac{1-\frac{\rho'}{\rho}}{\coth mh+\frac{\rho'}{\rho}\coth mh'}$$

or
$$c^2=\frac{g}{m}\,\frac{1-s}{\coth mh+s\coth mh'}$$

or
$$c^2=\frac{g}{m}\,\frac{1-s}{\coth mh+s}$$

{at an infinite depth

or
$$c^2=\frac{g}{m}\tanh mh\,(1-s)\,(1-s\tanh mh)$$

or
$$c^2=\frac{g}{m}\tanh mh\,\{1-s\,(1+\tanh mh)\}$$

{neglecting higher orders of s^2

Thus the pressure of atmosphere tends to decrease wave velocity c.

§ 8·61. Waves at an interface with upper surface free.*

Let a liquid of density ρ' and depth h' lie over another liquid of density ρ and depth h. Assuming the liquids to be at rest save for the wave motion. Consider a common velocity of wave propagation c at the free surface of the upper liquid and at the common surface. Superpose on the whole mass a velocity equal and opposite to that of propagation of waves, to make the motion steady. The wave profile gets fixed in space and the fluid flows with velocity c in the negative direction of X-axis.

The complex potentials of the lower and the upper liquids are given by

$$w=cz+\frac{ac}{\sinh mh}\cos m\,(z+ih) \qquad \text{...(i)}$$

and
$$w_1=cz+\frac{bc\cos mz}{\sinh mh'}-\frac{ac\cos m\,(z-ih')}{\sinh mh'}\,. \qquad \text{...(ii)}$$

{Ref. § 8·6

In the (ii) relation there is a term $\frac{bc\cos mz}{\sinh mh'}$ *which denotes complex potential of a simple wave* $\eta_2=b\sin mx$ *at* $t=0$.

Let q be the speed in the **lower liquid** ;

then
$$q^2=\frac{dw}{dz}\cdot\frac{d\bar{w}}{d\bar{z}}$$

* **Surface of the upper liquid is free** *e.g.* a layer of oil upon water of a layer of fresh water upon salt water.

or
$$q^2=c\left\{1-\frac{ma}{\sinh mh}\sin m\,(z+ih)\right\}$$
$$.\,c\left\{1-\frac{ma}{\sinh mh}\sin m\,(\bar{z}-ih)\right\}$$

or
$$q^2=c^2\left\{1-\frac{2ma}{\sinh mh}\sin mx\cosh m\,(y+h)\right\}$$

(neglecting the terms of a^2)

Let q_0 be the speed at the interface due to lower liquid, substituting $y=0$, we have

$$q_0^2=c^2\left\{1-\frac{2ma\sin mx\cosh mh}{\sinh mh}\right\} \qquad \text{...(iii)}$$

Consider q' to be the speed in the **upper liquid** :

then
$$q'^2=\frac{dw_1}{dz}\cdot\frac{d\bar{w}_1}{d\bar{z}}$$

or
$$q'^2=c^2\left\{1-\frac{bm\sin mz}{\sin mh'}+\frac{am\sin m\,(z-ih')}{\sinh mh'}\right\}$$
$$\times\left\{1-\frac{bm\sin m\bar{z}}{\sinh mh'}+\frac{am\sin m\,(\bar{z}+ih')}{\sinh mh'}\right\}$$

or
$$q'^2=c^2\left\{1-\frac{2bm\sin mx\cosh my}{\sinh mh'}\right.$$
$$\left.+\frac{2am\sin mx\cosh m\,(y-h')}{\sinh mh'}\right\} \qquad \text{...(iv)}$$

(neglecting the terms a^2, b^2 and ab etc.)

If q' be the speed at the interface due to upper liquid, substituting $y=0$, we have

$$q_0'^2=c^2\left\{1-\frac{2bm\sin mx}{\sinh mh'}+\frac{2am\sin mx\cosh mh'}{\sinh mh'}\right\} \qquad \text{...(v)}$$

The expressions for the pressure at the interface $\eta_1=a\sin mx$ are,

$$\frac{p'}{\rho'}+\frac{1}{2}q_0'^2+g\eta_1=\text{constant}$$

and
$$\frac{p}{\rho}+\frac{1}{2}q_0^2+g\eta_1=\text{constant}.$$

Since the pressure is continuous across the surface

So
$$p=p'$$

or
$$\tfrac{1}{2}\rho q_0^2+g\rho\eta_1+\text{constant}=\tfrac{1}{2}\rho' q_0'^2+g\rho'\eta_1+\text{constant}.$$

or
$$(\rho' q_0'^2-\rho q_0^2)+2g\eta_1(\rho'-\rho)=\text{const.} \qquad \text{...(vi)}$$

Substituting the values of q_0^2 and $q_0'^2$ from (iii) and (v) in (vi) we have

$$c^2\rho'\left(1-\frac{2bm \sin mx}{\sinh mh'}+\frac{2am \sin mx \cosh mh'}{\sinh mh'}\right)$$
$$-c^2\rho\left(1-\frac{2ma \sin mx \cosh mh}{\sinh mh}\right)$$
$$+2ga\,(\rho'-\rho) \sin mx=\text{constant.}$$

Since R. H. S. is constant, the coefficient of the variable sin mx must vanish.

So $$g\,(\rho-\rho')=c^2m\left\{\rho' \coth mh'+\rho \coth mh -b'\left(\frac{b}{a}\right) \operatorname{cosech} mh'\right\} \quad \ldots\text{(vii)}$$

Since p' is constant at free surface

then $$\eta_2=h'+b \sin mx'$$

or $$g\rho'\,\eta_2+\tfrac{1}{2}\rho' q'^2=\text{constant}$$

or $$g\rho'\,(h'+b \sin mx)+\tfrac{1}{2}\rho'\,c^2\left\{1-\frac{2bm \sin mx \cosh mh'}{\sinh mh'}+\frac{2am \sin mx}{\sinh mh'}\right\}=\text{constant.}$$

{Substituting $y=h'$ in (iv) we get q'^2}

The coefficients of sin mx must vanish,

So $$gb+\tfrac{1}{2}c^2\left\{\frac{2am}{\sinh mh'}-\frac{2bm \cosh mh'}{\sinh mh'}\right\}=0$$

or $$g=c^2m\left\{\coth mh'-\frac{a}{b} \operatorname{cosech} mh'\right\}$$

or $$\frac{b}{a}=\frac{c^2m}{c^2m \cosh mh'-g \sinh mh'}. \quad \ldots\text{(viii)}$$

This determines the ratio of the amplitudes of the waves.

By eliminating $\dfrac{b}{a}$ in (vii) and (viii), we have

$$g\,(\rho-\rho')=c^2m\left\{\rho' \coth mh'+\rho \coth mh -\frac{\rho'\,c^2m \operatorname{cosech} mh'}{c^2m \cosh mh'-g \sinh mh'}\right\}$$

or $$c^4m^2\,(\rho \coth mh \coth mh'+\rho') -c^2mg\rho\,(\coth mh+\coth mh')=g^2\,(\rho'-\rho). \quad \ldots\text{(ix)}$$

Thus we see that there are two possible velocities of propagation for a given length, if $\rho > \rho'$.

Particular Case. *When the lower liquid is deep :*

Substitute coth $mh=1$ in relation (ix), which reduces to

$$c^4m^2\,(\rho \coth mh'+\rho')-c^2mg\rho\,(1+\coth mh')$$
$$+g^2\,(\rho-\rho')=0$$

or $$(mc^2-g)\,\{mc^2\,(\rho \coth mh'+\rho')-g\,(\rho-\rho')\}=0$$

gives $$c^2=\frac{g}{m} \text{ and } c^2=\frac{g}{m}\cdot\frac{\rho-\rho'}{\rho \coth mh'+\rho'}.$$

§ 8·7. Group Velocity.

When waves are started by a local disturbance *e.g.* droping of a stone into a canal or the motion of a boat through water, the successive waves have different lengths and are propagated with different velocities. Since the velocity of propagation depends upon the wave length so the waves of slightly different wave lengths will be sorted out into groups. The velocity with which a group of waves advances is called the **Group velocity**.

Consider the case of two simple harmonic waves of same amplitude and slightly different wave lengths.

Let $$\eta_1=a \sin (mx-nt)$$

and $$\eta_2=a \sin \{(m+\delta m)\,x-(n+\delta n)\,t\}$$

where δm and δn are infinitesimals.

The elevation at any point, is given by

$$\eta=\eta_1+\eta_2$$

or $$\eta=a\,[\sin (mx-nt)+\sin \{(m+\delta m)\,x-(n+\delta n)\,t\}]$$

or $$\eta=2a \sin (mx-nt) \cos\tfrac{1}{2}\,(\delta m.x-\delta n.t)$$

$$\eta=A \sin (mx-nt) \qquad \text{..(i)}$$

$$\left\{\begin{array}{l}\text{as } \delta m << m\\ \quad\;\; \delta n << n\end{array}\right.$$

where $$A=2a \cos (\delta m.x-\delta n.t).$$

The relation (i) represents that the amplitude of the resulting disturbance, varies as a wave of velocity

$$c_g=\frac{\delta n}{\delta m}$$

known as **Group velocity**

So $$c_g=\frac{dn}{dm}$$

or $$c_g=\frac{d}{dm}\,(mc) \qquad \left\{\text{as } c=\frac{n}{m}\right.$$

or $$c_g = c + m\frac{dc}{dm}$$

or $$c_g = c\left\{1 + \frac{m}{c}\frac{dc}{dm}\right\}$$

$\Big\{$ on the surface of water of depth h, we have $c^2 = \frac{g}{m}\tanh mh$

or $$c_g = c\left\{1 + \frac{m}{2c^2}\frac{dc^2}{dm}\right\}$$

or $$c_g = c\left\{1 + \frac{m^2}{2g\tanh mh}\cdot\frac{g\,(hm\,\text{sech}^2\,mh - \tanh mh)}{m^2}\right\}$$

or $$c_g = c\left\{1 + \frac{1}{2}\left(\frac{hm\,\text{sech}^2\,mh}{\tanh mh} - 1\right)\right\}$$

or $$c_g = \frac{1}{2}\,c\left\{1 + \frac{2mh}{2\sinh mh\cosh mh}\right\}$$

or $$c_g = \frac{1}{2}\,c\left\{1 + \frac{2mh}{\sinh 2mh}\right\} \qquad \ldots\text{(ii)}$$

$\Rightarrow$ that in the case of waves on the surface of water of depth h, the ratio of the **group velocity** to the **wave velocity** is

$$\frac{1}{2} + \frac{mh}{\sinh mh}$$

where $\frac{2\pi}{m}$ is the wave length.

Particular Cases : (a) For shallow water, $\frac{h}{\lambda}$ is small.

Then $$\text{Lt}\,\frac{\sinh 2mh}{2mh} = 1.$$

From (ii), we have

$$c_g = \tfrac{1}{2}c\,(1+1)$$

or $$c_g = c$$

$\Rightarrow$ *that the group velocity c_g equals to the wave velocity c.*

(b) On deep water.

For the waves on the deep water, $\frac{h}{\lambda}$ is very large

Then $$\text{Lt}\,\frac{2mh}{\sinh 2mh}$$

From (ii), we have

$$c_g = \tfrac{1}{2}c$$

$\Rightarrow$ *that the group velocity c_g for deep water waves is half the wave velocity c.*

§ 8·71. Several Harmonic Waves.

Assuming the waves of different wave lengths and of very small amplitudes a, a_1, a_2 ... consider the wave groups of approximately same length (which differ but little from one another). Let one such group whose elevation at time t be given by

$$\eta = a \sin (mx - nt) + a_1 \sin \{(m+\delta m_1)\,x - (n+\delta n_1)\,t + \epsilon\} + \ldots$$

or
$$\eta = A \sin (mx - nt) + B \cos (mx - nt)$$
$$= C \sin (mx - nt - \epsilon)$$

where
$$A = a + a_1 \cos (x\delta m_1 - t\delta n_1 + \epsilon_1) + a_2 \cos (x\delta m_2 - t\delta n_2 + \epsilon_2) + \ldots$$
$$B = a_1 \sin (x\delta m_1 - t\delta n_1 + \epsilon_1) + a_2 \sin (x\delta m_2 - t\delta n_2 + \epsilon_2) + \ldots$$
$$C^2 = A^2 + B^2, \quad \tan \epsilon = \frac{B}{A}.$$

So
$$x\delta m_1 - t\delta n_1 + \epsilon_1 = \delta m_1 \left(x - t\,\frac{\delta n_1}{\delta m_1}\right) + \epsilon_1$$

or
$$x\delta m_1 - t\delta n_1 + \epsilon_1 = \delta m_1\,(x - c_g\,t) + \epsilon_1$$

and
$$x\delta m_2 - t\delta n_2 + \epsilon_2 = \delta m_2 \left(x - t\,\frac{\delta n_2}{\delta m_2}\right) + \epsilon_2$$

or
$$x\delta m_2 - t\delta n_2 + \epsilon_2 = \delta m_2\,(x - c_g\,t) + \epsilon_2 \qquad \text{and so on.}$$

By hypothesis
$$\frac{\delta n_1}{\delta m_1} = \frac{\delta n_2}{\delta m_2} = \ldots = \frac{dn}{am} = c_g.$$

Thus A, B and therefore C and ϵ are functions of $(x - c_g\,t)$. Thus the amplitude graph moves as a wave with group velocity c_g

§ 8·72. Dynamical significance of Group velocity.

The rate of transmission of enery is measured by taking a vertical section of the liquid at right angles to the direction of propagation.

We shall determine the rate at which the liquid on one side of this section is doing work upon the liquid on the other side. Let depth of the liquid be h. We know that

$$\phi = \frac{ga}{n}\,\frac{\cosh m\,(y+h)}{\cosh mh}\cos (mx - nt)$$

{ Ref. equation (vii) § 8·3

Neglecting squares of small quantities, the variable part of the pressure is given by

$$\delta p = \rho\,\frac{\partial \phi}{\partial t}$$

or $$\delta p = \rho g a \frac{\cosh m\,(y+h)}{\cosh mh} \sin\,(mx-nt).$$

The horizontal velocity is $-\dfrac{\partial \phi}{\partial x}$.

Thus the work done in unit time is

$$W = -\int_{-h}^{\eta} \delta p \,.\, \frac{\partial \phi}{\partial x}\, dy$$

or $$W = \rho g a \int_{-h}^{\eta} \frac{\cosh m\,(y+h)}{\cosh mh} \sin\,(mx-nt) \,.\, \frac{ga}{n} \frac{m \cosh m\,(y+h)}{\cosh mh} \cos\,(mx-nt)\,.\,dy$$

or $$W = \frac{g^2 \rho\, a^2 m}{n} \,.\, \frac{\sin^2 (mx-nt)}{\cosh^2 mh} \int_{-h}^{0} \cosh^2 m\,(y+h)\, dy$$

(Taking the upper limit as $\eta = 0$)

or $$W = \frac{g^2 \rho\, a^2 m}{n} \,.\, \frac{\sin^2 (mx-nt)}{\cosh^2 mh} \int_{0}^{h} \{1 + \cosh 2m\,(y+h)\}\, dy$$

or $$W = \frac{g^2 \rho\, a^2 m}{2n} \,.\, \frac{\sin^2 (mx-nt)}{\cosh^2 mh} \,.\, \frac{1}{2}\left(h + \frac{\sinh 2mh}{2m}\right)$$

or $$W = \frac{g^2 \rho\, a^2 m}{4n} \,.\, \frac{\sin^2 (mx-nt)}{\cosh^2 mh} \,.\, \frac{\sin 2mh}{2m}\left(1 + \frac{2mh}{\sinh 2mh}\right)$$

or $$W = \tfrac{1}{2} g \rho a^2\, \frac{n}{m}\left(1 + \frac{2mh}{\sinh 2mh}\right) \sin^2 (mx-nt)$$

$$\left\{ \text{as} \quad c^2 = \frac{g}{m} \tanh mh \right.$$

The above expression fluctuates with time. The average value of $\sin\,(mx-nt)$ is zero but that of $\sin^2 (mx-nt)$ is $\frac{1}{2}$.

So $$W = \tfrac{1}{4} g \rho a^2 c \left(1 + \frac{2mh}{\sinh 2mh}\right)$$

$$W = \tfrac{1}{2} g \rho\, a^2 c_g \qquad \left\{ \begin{array}{l} \text{as} \quad c_g = \tfrac{1}{2} c \left(1 + \dfrac{2mh}{\sinh 2mh}\right) \\ \qquad \text{Ref. § 8·7.} \end{array} \right.$$

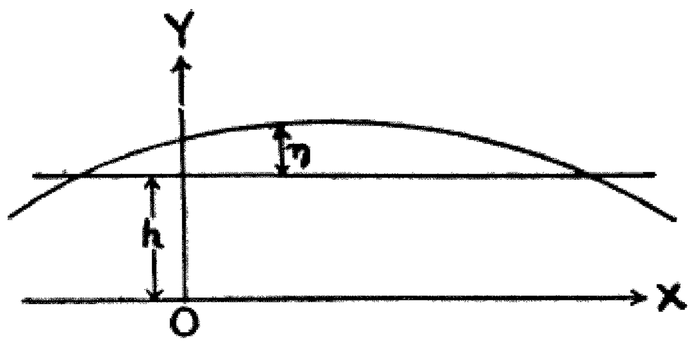
Thus the expression for the energy transmitted in unit time is equal to $\frac{1}{2} g \rho a^2 \times$ Group velocity.

But $\frac{1}{2} g \rho a^2$ is the whole energy per unit length at any instant. **Hence the energy is transmitted at a rate equal to the group velocity.**

This is known as **Dynamical Significance of Group Velocity.**

§ 8·8. Long Waves.

Long waves are the waves whose length is large compared with the depth of the water. Let h be the depth of water in a horizontal canal, the hypothesis is that $\frac{h}{\lambda}$ is small where λ is a typical wave length. Here the quantities $\frac{\eta}{h}$ and $\frac{d\eta}{dx}$ are small.



Let the condition to be satisfied by the long waves be,

$$\frac{\partial^2 \phi}{\partial t^2} - g\frac{\partial \psi}{\partial x} = 0 \quad \text{when } y = h \qquad (1)$$

{Ref. § 8·32 (c).

(Surface condition for wave profiles of small height and shape).

and $$\psi = 0 \quad \text{when} \quad y = 0 \qquad \text{...(ii)}$$

as there is no flow across the bottom.

Consider the complex potential to be,

$$V = W(z, t)$$

or $$V = \phi(x, y, t) + i\psi(x, y, t) \qquad \text{...(iii)}$$

From (ii), we see that V is real when $y=0$. By the principle of analytic continuation, V can be expressed in the region $-h \leqslant y < 0$.

So $$V(\bar{Z}, t) = \phi(x, y, t) - i\psi(x, y, t) \qquad \text{...(iv)}$$

From (iii) and (iv), we have

$$\phi(x, y, t) = \tfrac{1}{2}\{V(z, t) + V(\bar{Z}, t)\}$$

and $$\psi(x, y, t) = -\tfrac{1}{2}i\{V(z, t) - V(\bar{Z}, t)\}$$

Substituting $\frac{\partial^2 \phi}{\partial t^2}$ and $\frac{\partial \psi}{\partial x}$ in (i) at $y=h$, we have

$$\frac{\partial^2}{\partial t^2}\left\{V(x+ih, t)+V(x-ih, t)\right\}$$
$$+ig\frac{\partial}{\partial x}\left\{V(x+ih, t)-V(x-ih, t)\right\}=0. \qquad \text{...(v)}$$

Since V is an analytic function, the above relation must be true for any point in the domain. Write z for x in (v), we have

$$\frac{\partial^2}{\partial t^2}\left\{V(z+ih, t)+V(z-ih, t)\right\}$$
$$+ig\frac{\partial}{\partial z}\left\{V(z+ih, t)-V(z-ih, t)\right\}=0 \qquad \text{...(vi)}$$

or
$$\frac{\partial^2}{\partial t^2}(V_1+V_2)+ig\frac{\partial}{\partial z}(V_1-V_2)=0.$$

where
$$V_1=V(z+ih, t)$$
$$V_2=V(z-ih, t)$$

This is known as **Cisotti's equation.**

Let $V(z, t)$ be the complex potential of the long waves,

So
$$V(z+ih, t)=V(z, t)+ih\frac{\partial}{\partial z}V(z, t)+\ldots$$

and
$$V(z-ih, t)=V(z, t)-ih\frac{\partial}{\partial z}V(z, t)+\ldots$$

neglecting the terms containing h^2 and higher orders

$$V(z+ih, t)+V(z-ih, t)=2V(Z, t)$$

and
$$V(z+ih, t)-V(z-ih, t)=2ih\frac{\partial}{\partial z}V(z,t)$$

From equation (vi), we have

$$2\frac{\partial^2}{\partial t^2}V(z, t)+ig.2ih\frac{\partial}{\partial z}\left\{\frac{\partial}{\partial z}V(z, t)\right\}=0$$

or
$$\frac{\partial^2 V}{\partial z^2}=\frac{1}{gh}\frac{\partial^2 V}{\partial t^2}$$

or
$$\frac{\partial^2 V}{\partial z^2}=\frac{1}{c^2}\frac{\partial^2 V}{\partial t^2} \qquad \text{...(vii)}$$

{ if $gh=c^2$

which is the standard relation of wave equation whose general solution is given by

$$V=V_1(z+ct)+V_2(z-ct) \qquad \text{...(viii)}$$

where V_1 and V_2 are arbitrary analytic functions, subject to the condition that V is real when $y=0$.

Equating the real and imaginary parts, we have for the velocity potential and stream function,

$$\left.\begin{aligned}\phi=\phi\,(x\;y\;t)=\phi_1\,(x+ct,\,y)+\phi_2\,(x-ct,\,y)\\ \psi=\psi\,(x\;y\;t)=\psi_1\,(x+ct,\,y)+\psi_2\,(x-ct,\,y)\end{aligned}\right\}\qquad\ldots(\text{ix})$$

Since V is real when $y=0$ *i.e.* $\psi\,(x,\,o,\,t)=0$

By Maclaurin's theorem, we have

$$\phi=\phi\,(x\;o\;t)+y\left\{\frac{\partial}{\partial y}\,\phi\,(x\;y\;t)\right\}_{y=0}+\ldots\qquad\ldots(\text{x})$$

Since y lies between o and h, the other terms in the above expansion is infinitesimal as compared with the first. So, we have

$$\phi=\phi\,(x\;o\;t)$$

or

$$\phi\,(x\;y\;t)=\phi_1\,(x+ct)+\phi_2\,(x-ct)\qquad\ldots(\text{xi})$$

The same method applied to the stream function in relation (x), we have $\psi=0$

Thus relation (xi) is the complete solution of long waves.

It (relation xi) shows that all particles which are in the same vertical plane have the same horizontal velocity $-\dfrac{\partial\phi}{\partial x}$, to remain in a vertical plane.

Again from (x), we have

$$-\frac{\partial\phi}{\partial y}=-\left\{\frac{\partial}{\partial y}\,\phi\,(x\;y\;t)\right\}_{y=0}-y\left\{\frac{\partial^2}{\partial y^2}\,\phi\,(x\;y\;t)\right\}_{y=0}-\ldots$$

The first term on the R. H. S. is the vertical velocity at the bottom, which is zero

$\Rightarrow$ *that the vertical velocity is of the second order and is proportional to the height above the bottom.*

§ 8·81. **Pressure.** Let Π be the pressure at the free surface and η be surface elevation for given values of x and t, then

$$\frac{p}{\rho}+gy-\frac{\partial\phi}{\partial t}=\frac{\Pi}{\rho}+g\,(h+\eta)-\frac{\partial\phi}{\partial t}$$

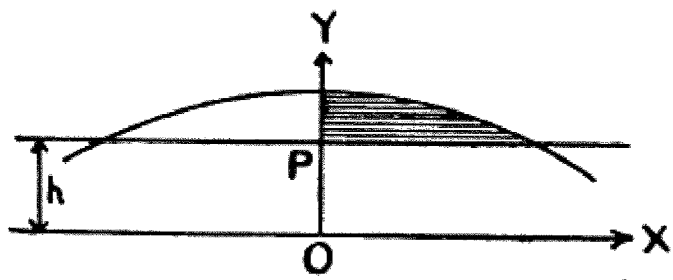
Since q^2 is neglected and $\dfrac{\partial\phi}{\partial t}$ is independent of y. (Considering wave length to be large in comparison with mean depth and vertical acceleration is negligible), then

$$p=\Pi+g\rho\,(h+\eta-y)$$

which shows that the pressure at the depth $(h+\eta-y)$ below the free surface is the same as that calculated by the laws of hydrostatics.

§ 8·82. Wave progressing in one direction only.

Let the waves move in one direction (X-direction) only. If u is the velocity of the liquid and η the surface elevation, we have



$$\eta = \frac{1}{g}\frac{\partial \phi}{\partial t}$$

or $$\eta = \frac{1}{g}\frac{\partial}{\partial t}\phi(x-ct)$$

or $$\eta = -\frac{c}{g}\phi'(x-ct) \qquad \text{..(i)}$$

and $$u = -\frac{\partial}{\partial x}\phi(x-ct)$$

or $$u = -\phi'(x-ct) \qquad \text{...(ii)}$$

From (i) and (ii), we get

$$\eta = \frac{c}{g}u \qquad \text{...(iii)}$$

So $$u = \frac{g\eta}{c} = \frac{c\eta}{h} \qquad \{\text{Since } c^2 = gh$$

To determine the motion of a particle originally at P in the undisturbed surface of water in a canal of depth h, we see that the displacement is

$$\int u\,dt = \frac{1}{h}\int c\eta\,dt$$

The second integral measures the shaded area. Thus the displacement of the particle is obtained by dividing the area of the profile which has passed P by the depth on the undisturbed water.

§ 8·83. Energy of a long wave.

For a wave in a canal of rectangular section the potential energy is due to the elevation or depression of the liquid above the mean level. The potential energy V for a unit breadth of the wave is

$$V=\tfrac{1}{2}g\rho\int\eta^2dx \qquad \text{...(i)}$$

where η is the elevation at x and the integration is over the whole length of the wave.

Similarly the kinetic energy T is

$$T=\tfrac{1}{2}\rho h\int u^2dx \qquad \text{...(ii)}$$

for the same range of integration.

But for a wave travelling in one direction

{Equation (ii) § 8·82

$$\eta=\frac{c}{g}u$$

or $\quad \eta^2g^2=c^2u^2$

or $\quad \eta^2g^2=gh\,.u^2$

or $\quad \eta^2g=hu^2 \qquad \text{...(iii)}$

From (ii), we have

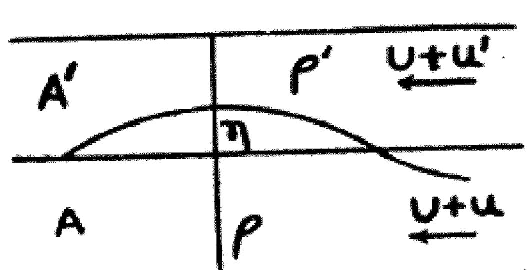
$$T=\tfrac{1}{2}\rho h\int\frac{\eta^2g}{h}dx \qquad \text{\{from (ii)}$$

or $$T=\tfrac{1}{2}\rho g\int\eta^2dx$$

Thus kinetic energy is equal to the potential energy and each equals to half the total energy at any instant.

§ 8·9. Long waves at the common surface of two liquids, bounded above and below by two fixed horizontal planes.

Consider ρ and ρ' be the densities of liquids, A and A' the cross-sections of the two liquid streams and b the breadth of the



common surface. Superpose a velocity equal and opposite to the velocity U of propagation of the wave to make the wave stationary. Let η be the elevation of the common surface due to the wave motion and u, u' the small additional velocities due to the wave motions in the two liquids.

The equations of continuity in the two liquids are

$$(A+b\eta)(U+u)=AU$$

and $$(A'-b\eta)(U+u')=A'U$$

Neglecting the quantities of second order, we have

$$Au+bU\eta=0 \qquad \text{...(i)}$$

and $$A'u'-bU\eta=0 \qquad \text{...(ii)}$$

If δp and $\delta p'$ denote increments of pressure close to the common surface in the two liquids due to the waves, then we have

$$\frac{\delta p}{\rho}+g\eta+\tfrac{1}{2}(U+u)^2=\tfrac{1}{2}U^2$$

and $$\frac{\delta p'}{\rho'}+g\eta+\tfrac{1}{2}(U+u')^2=\tfrac{1}{2}U^2$$

Neglecting the surface tension, so that $\delta p=\delta p'$, we have

$$g(\rho-\rho')\eta=(\rho'u'-\rho u)U$$

or $$g(\rho-\rho')\eta=\left(\rho'.\frac{bU\eta}{A'}+\rho.\frac{bU\eta}{A}\right)U$$

{ from (i) and (ii)

or $$g(\rho-\rho')=U^2b\left(\frac{\rho}{A}+\frac{\rho'}{A'}\right)$$

or $$U^2=\frac{g(\rho-\rho')}{b\left(\frac{\rho}{A}+\frac{\rho'}{A'}\right)}$$

Ans.

Ex. 1. *A fixed buoy in deep water is observed to rise and fall twenty times in a minute, prove that the velocity of the wave is about $10\frac{1}{2}$ miles per hour.*

Consider c be the velocity of the wave, then in deep water, we have

$$c^2=\frac{g\lambda}{2\pi} \qquad \text{...(i)}$$

Frequency of the wave

$=20$ times per minute

$=\frac{1}{3}$ per second.

Hence $\lambda=3c$

from (i), we have

$$c^2=\frac{3cg}{2\pi}$$

or $$c=\frac{3g}{2\pi}\text{ ft./sec.}$$

$$=\frac{3}{2}\times\frac{32\times 7}{22}=\frac{168}{11}\text{ ft /sec.}$$

$$=10\tfrac{1}{2}\text{ miles/hour.}$$ **Proved.**

Ex. 2. *The crests of rollers which are directly following a ship 220 ft. long are observed to overtake it at intervals of $16\frac{1}{2}$ seconds and it takes a crest 6 seconds to run along the ship. Find the length of the wave and the velocity of the ship.*

Let c represent the velocity of the wave and v that of ship. In deep water, we have

$$c^2=\frac{g\lambda}{2\pi} \quad \text{...(i)}$$

also $$(c-v)\,\frac{33}{2}=\lambda \quad \text{...(ii)}$$

and $$(c-v)\,6=220 \quad \text{...(iii)}$$

from (ii) and (iii). we have

$$\lambda=\frac{220}{6}\times\frac{33}{2}=605\text{ ft.}$$

Substituting the value of λ in (i), we have

$$c=\sqrt{\left(\frac{32\times 605\times 7}{2\times 22}\right)}\text{ ft/sec.}$$

or $$c=55\text{ ft./sec. (app.)}$$

or $$c=37\tfrac{1}{2}\text{ miles/hour.}$$ **Ans.**

and $$v=\left(c-\frac{110}{3}\right)\text{ ft/sec.}$$

$$=\left(55-\frac{110}{3}\right)\text{ ft/sec.}=\frac{55}{3}\text{ ft/sec.}$$

$$=12\tfrac{1}{2}\text{ miles/hour.}$$ **Ans.**

Ex. 3. *Find the type of waves that would travel on deep water at 30 knots. How much is the velocity of wave affected by the presence of atmosphere above the water, its density being ·0013 ?*

Let the fluids be at rest ; save for the wave motion the wave velocity c is given by

$$c^2=\frac{g}{m}\;\frac{\rho-\rho'}{\rho\coth mh+\rho'\coth mh'}$$

{Ref. Case I, § 8·6

where ρ and ρ' are the densities of the fluids and h and h', the depths of the fluids.

Since the wave is affected by the presence of atmosphere, so the wave velocity is given by

$$c^2 = \underset{\substack{h \\ h'} \to \infty}{\mathrm{Lt}} \frac{g(\rho-\rho')}{m\{\rho \coth mh + \rho' \coth mh'\}}$$

or $$c^2 = \frac{g(\rho-\rho')}{m(\rho+\rho')}$$

or $$c^2 = \frac{g\left(1-\dfrac{\rho'}{\rho}\right)}{m\left(1+\dfrac{\rho'}{\rho}\right)} \qquad \left\{ \text{as } \frac{\rho'}{\rho} = \sigma, \text{ specific gravity of air} \right.$$

or $$c^2 = \frac{g(1-\sigma)}{m(1+\sigma)}$$

If the wave is not affected by the presence of atmosphere then, let c_0 be the corresponding value of the wave velocity

$$c_0^2 = \frac{\lambda_0 g}{2\pi} \qquad \left\{ \begin{aligned} c_0 &= 30 \text{ nautical miles/hr,} \\ &= \frac{30 \times 6082}{60 \times 60} \text{ ft//sec.} \end{aligned} \right.$$

or $$\lambda_0 = \frac{2\pi c_0^2}{g}$$

or $$\lambda_0 = \frac{2 \times 22}{7 \times 32} \cdot \left(\frac{30 \times 6082}{60 \times 60}\right)^2$$

or $$\lambda_0 = 504 \text{ ft. (App.)}$$

If the atmosphere also be present

then $$\left(\frac{c}{c_0}\right)^2 = \frac{\lambda}{\lambda_0} = \frac{1-\sigma}{1+\sigma} = \frac{0.9987}{1.0013}$$

$$= 0\ 9974 \text{ (app.)}$$ **Ans.**

Ex. 4. *When simple harmonic waves of length λ are propagated over the surface of deep water, prove that, at a point whose depth below the undisturbed surface is h, the pressure at the instant when the disturbed depth of the point is $h+\eta$ bears to the undisturbed pressure at the same point the ratio,*

$$1 + \frac{\eta}{h} e^{-\frac{2\pi h}{\lambda}} : 1$$

atmospheric pressure and surface tension being neglected.

For a simple harmonic wave profile

$$\eta = a \sin(mx - nt) \qquad \ldots\text{(i)}$$

the velocity potential at any point $(x\ y)$ given by

$$\phi = \frac{an}{m} e^{my} \cos(mx - nt)$$

So $$\frac{\partial\phi}{\partial t}=\frac{an^2}{m}\,e^{my}\sin(mx-nt) \qquad \left\{\text{as } c^2=\frac{n^2}{m^2}=\frac{g}{m}\right.$$

$$=g\eta\,e^{my} \qquad \text{(ii)} \qquad \{\text{from (i)}$$

The pressure equation is

$$\frac{p}{\rho}+gy-\frac{\partial\phi}{\partial t}=\frac{\Pi}{\rho} \qquad \text{...(iii)}$$

where Π is the atmospheric pressure at the surface.

At a point which is at a depth h, we have $y=-h$ from (ii) and (iii), we have

$$\frac{p-\Pi}{\rho}=g\eta e^{my}-gy$$

or $$\frac{p-\Pi}{\rho}=g\,(\eta e^{-mh}+h) \qquad \{\text{at } y=-h \qquad \text{...(iv)}$$

Let p_0 be the un disturbed pressure at a depth h

then $$\frac{p_0-\Pi}{\rho}=gh \qquad \text{...(v)}$$

Neglecting the atmospheric pressure Π, we have from (iv) and (v),

$$\frac{p}{p_0}=\frac{h+\eta e^{-mh}}{h}$$

or $$\frac{p}{p_0}=\left\{1+\frac{\eta}{h}\,e^{-mh}\right\}$$

or $$\frac{p}{p_0}=1+\frac{\eta}{h}\,e^{-\frac{2\pi h}{\lambda}} \qquad \left\{\text{as } m=\frac{2\pi}{\lambda}\right.$$

Thus $$p:p_0::1+\frac{\eta}{h}\,e^{-\frac{2\pi h}{\lambda}}:1$$ **Proved.**

Ex. 5. *Two fluids of densities ρ_1, ρ_2 have a horizontal surface of separation but are otherwise unbounded. Shew that when waves of small amplitude are propagated at their common surface, the particles of the two fluids describe circles about their mean positions; and that at any point of the surface of the separation where the elevation is η, the particles on either side have a relative velocity*

$$\frac{4\pi c\eta}{\lambda}.$$

Consider that the wave propagated at the surface be

$$\eta=a\sin(mx-nt) \qquad \text{...(i)}$$

where $$m=\frac{2\pi}{\lambda}$$

Let ϕ and ϕ' be the velocity potentials in the lower and upper fluids and may be represented as

$$\phi = Ae^{my} \cos (mx+nt) \qquad \ldots\text{(ii)}$$

$$\phi' = Be^{-my} \cos (mx-nt) \qquad \ldots\text{(iii)}$$

Both the functions ϕ and ϕ' satisfy Laplace equation and the conditions

$$\frac{\partial \phi}{\partial y} \to 0 \text{ as } y \to -\infty \text{ (for lower fluid)}$$

$$\frac{\partial \phi'}{\partial y} \to 0 \text{ as } y \to +\infty \text{ (for upper fluid)}$$

At the surface

$$\frac{\partial \eta}{\partial t} = \left(-\frac{\partial \phi}{\partial y}\right)_{y=0} = \left(-\frac{\partial \phi'}{\partial y}\right)_{y=0}$$

(when η is small)

So from (i), (ii) and (iii), we have

$$-an \cos (mx-nt) = -\{Ame^{my} \cos (mx-nt)\}_{y=0}$$

$$= -\{-Bme^{-my} \cos (mx-nt)\}_{y=0}$$

or $$-an \cos (mx-nt) = -Am \cos (mx-nt)$$

$$= Bm \cos (mx-nt)$$

or $$A = \frac{an}{m} \text{ and } B = -\frac{an}{m}$$

Substituting the values of A and B in (ii) and (iii), we get

$$\phi = \frac{an}{m} e^{my} \cos (mx-nt) = ace^{my} \cos m (x-ct)$$

and $$\phi' = -\frac{an}{m} e^{-my} \cos (mx-nt) = -ace^{-my} \cos m (x-ct)$$

Let (x_0, y_0) be the co-ordinates in the equilibrium position of any point (x, y) in the lower fluid,

so $$x = x_0 + X \text{ and } y = y_0 + Y$$

or $$\frac{dX}{dt} = \dot{x} = -\frac{\partial \phi}{\partial x} = -ac\, me^{my_0} \sin m (x_0 - ct) \qquad \ldots\text{(iv)}$$

and $$\frac{dY}{dt} = \dot{y} = -\frac{\partial \phi}{\partial y} = -ac\, me^{my_0} \cos m (x_0 - ct) \qquad \ldots\text{(v)}$$

(upto first order)

Integrating (iv) and (v) with regard to t, we have

$$X = -ae^{my_0} \cos m (x_0 - ct)$$

and $$Y = ae^{my_0} \quad \sin m (x_0 - ct)$$

which gives { as $X = 0 = Y$ initially

$$X^2 + Y^2 = a^2 e^{2my_0} \{\cos^2 m (x_0 - ct) + \sin^2 m (x_0 - ct)\}$$

or $$X^2+Y^2=a^2e^{2my_0}$$
represents that the path is a circle with centre at $(x_0\ y_0)$.

Similarly we can prove that the path of a particle in the upper fluid is also a circle.

Again relative velocity at the interface

$$=\left\{\left(-\frac{\partial\phi}{\partial x}\right)-\left(-\frac{\partial\phi'}{\partial x}\right)\right\}_{y=0}$$

$$=\left\{acme^{my}\sin m\,(x-ct)+acme^{-my}\sin m\,(x-ct)\right\}_{y=0}$$

$$=2acm\sin m\,(x-ct)$$

$$=2cm\eta \qquad \{\text{ from (1)}$$

$$=\frac{4\pi c\eta}{\lambda} \qquad \left\{\text{as}\quad m=\frac{2\pi}{\lambda}\right.$$

and $$\left\{\left(-\frac{\partial\phi}{\partial y}\right)-\left(-\frac{\partial\phi'}{\partial y}\right)\right\}_{y=0}$$

$$=-\{amc\,e^{my}\cos m\,(x-ct)-amc\,e^{-my}\cos m\,(x-ct)\}$$

$$=\text{zero}$$

Hence the relative velocity at the surface is

$$\frac{4\pi c\eta}{\lambda}.$$ **Proved.**

Ex. 6. *If a canal of rectangular section contains a depth h of liquid of density* ρ *on which is superposed a depth* h' *of liquid of density* ρ', *the free surface of the latter being exposed to constant atmospheric pressure, prove that the velocities of propagation of waves of length* $\frac{2\pi}{m}$ *are given by* $c^2=\frac{gu}{m}$.

where

$$\rho\,(u\coth mh-1)\,(u\coth mh'-1)=\rho'\,(1-u^2)$$

Ref. Equation (ix) § 8·61.

or $$c^4m^2(\rho\coth mh\coth mh'+\rho')-c^2mg\rho(\coth mh+\coth mh')+g^2(\rho-\rho')=0 \qquad \text{...(i)}$$

since $$c^2=\frac{gu}{m}$$

or $$c^2m=gu \qquad \text{...(ii)}$$

from (i) and (ii), we have

$$g^2u^2(\rho\coth mh\coth mh'+\rho')-g^2\rho u(\coth mh+\coth mh')+g^2(\rho-\rho')=0$$

or $\rho\,\{u^2 \coth mh' - u(\coth mh + \coth mh') + 1\} = \rho'(1-u^2)$

or $\rho\,(u \coth mh - 1)(u \coth mh' - 1) = \rho'(1-u^2)$ **Proved.**

Ex. 7 *If there be two liquids in a straight canal of uniform section, of densities σ_1, σ_2 and depths l_1, l_2; shew that the velocity c of propagation of long waves is given by the equation*

$$\left(\frac{c^2}{l_1 g} - 1\right)\left(\frac{c^2}{l_2 g} - 1\right) = \frac{\sigma_1}{\sigma_2}$$

where $\sigma_2 < \sigma_1$. and it is assumed that the liquids do not mix.

Substituting $\rho = \sigma_2$, $\rho' = \sigma_1$, $h = l_2$ and $h' = l_1$ in the equation (ix) § 8·61, we have

$$c^4 m^2 (\sigma_2 \coth ml_2 \coth ml_1 + \sigma_1) - c^2 gm\sigma_2 (\coth ml_2 + \coth ml_1) + g^2(\sigma_2 - \sigma_1) = 0.$$

For long waves, we have

$$\coth mh \simeq \frac{1}{mh}$$

Thus

$$c^4 m^2 \left(\sigma_2 \frac{1}{ml_2} \cdot \frac{1}{ml_1} + \sigma_1\right) - c^2 gm\sigma_2 \left(\frac{1}{ml_2} + \frac{1}{ml_1}\right) + g^2(\sigma_2 - \sigma_1) = 0$$

or $$\frac{c^4 \sigma_2}{l_1 l_2} + c^4 m^2 \sigma_1 - c^2 g\sigma_2 \left(\frac{1}{l_1} + \frac{1}{l_2}\right) + g^2(\sigma_2 - \sigma_1) = 0.$$

Again for long waves $m = \dfrac{2\pi}{\lambda}$ is small; neglecting the terms containing m^2, we have

$$\frac{c^4 \sigma_2}{l_1 l_2} - c^2 g\sigma_2 \left(\frac{1}{l_1} + \frac{1}{l_2}\right) + g^2 \sigma_2 \left(1 - \frac{\sigma_1}{\sigma_2}\right) = 0.$$

or $$\frac{c^4}{g^2 l_1 l_2} - \frac{c^2}{g}\left(\frac{1}{l_1} + \frac{1}{l_2}\right) + 1 = \frac{\sigma_1}{\sigma_2}$$

or $$\left(\frac{c^2}{l_1 g} - 1\right)\left(\frac{c^2}{l_2 g} - 1\right) = \frac{\sigma_1}{\sigma_2}$$ **Proved.**

Ex. 8. *Two dimensional waves length $\dfrac{2\pi}{m}$ are produced at the surface of separation of two liquids which are of densities ρ, ρ' ($\rho > \rho'$) and depths h, h' confined between two fixed horizontal planes. Prove that, if the potential energy is reckoned zero in the position of equilibrium, the total energy of the lower liquid is to that of the upper in the ratio*

$$\rho\,\{(2\rho - \rho') \coth mh + \rho' \coth mh'\} : \rho'\,\{(\rho - 2\rho') \coth mh' - \rho \coth mh\}.$$

Consider the wave profile to be

$$\eta = a \sin (mx - nt)$$

or $$\eta = a \sin m (x - ct) \qquad \text{...(i)}$$

Let ϕ and ϕ' be the velocity potentials to the lower and upper liquids.

then $$\phi = \frac{ac}{\sinh mh} \cosh m(y+h) \cos m (x - ct) \qquad \text{...(ii)}$$

and $$\phi' = \frac{ac}{\sinh mh'} \cosh m(y-h') \cos m (x - ct) \qquad \text{...(iii)}$$

where $$c^2 (m\rho \coth mh + m\rho' \coth mh') = g (\rho - \rho') \qquad \text{...(iv)}$$

Let T be the kinetic energy of the lower liquid per wave length { Ref. § 8·6 case I

Then $$T = -\tfrac{1}{2} \rho \int \phi \frac{\partial \phi}{\partial n} ds$$

or $$T = \tfrac{1}{2} \rho \int_0^\lambda \left(\phi \frac{\partial \phi}{\partial y} \right)_{y=0} . \, dx$$

when η is small.

or $$T = \tfrac{1}{2} \rho \, m \, a^2 c^2 \coth mh \int_0^\lambda \cos^2 m (x - ct) \, dx$$

or $$T = \tfrac{1}{4} \rho \, \lambda a^2 c^2 \, m \coth mh \qquad \text{...(v)}$$

And the potential energy in the lower liquid per wave length of liquid is given by

$$V = \tfrac{1}{2} g\rho \int_0^\lambda \eta^2 \, dx$$

or $$V = \tfrac{1}{2} g\rho a^2 \int_0^\lambda \sin^2 m (x - ct) \, dx$$

or $$V = \tfrac{1}{4} g\rho \, a^2 \, \lambda \qquad \text{...(vi)}$$

Similarly kinetic energy of the upper liquid per wave length

$$= \tfrac{1}{4} \rho' \, \lambda \, a^2 c^2 \, m \coth mh' \qquad \text{...(vii)}$$

and the potential energy is given by

$$= -\tfrac{1}{4} g \, \rho' \, a^2 \, \lambda \qquad \text{...(viii)}$$

Thus the energy of lower liquid

$$= \tfrac{1}{4} g\rho a^2 \, \lambda + \tfrac{1}{4} \rho\lambda \, a^2 c^2 m \coth mh \quad \{\text{from (v) and (vi)}$$

$$= \tfrac{1}{4} g\rho a^2 \lambda \left\{ 1 + \frac{(\rho - \rho') \coth mh}{\rho \coth mh + \rho' \coth mh'} \right\}$$

{ from (iv)

$$=\frac{1}{4}\, g\rho a^2\lambda\ \frac{(2\rho-\rho')\coth mh+\rho'\coth mh'}{\rho\coth mh+\rho'\coth mh'} \quad ...(ix)$$

and the total energy of upper liquid

$$=\frac{1}{4}\,\rho'\lambda a^2c^2m\coth mh'-\frac{1}{4}\,g\rho' a^2\lambda$$

{from (vii) and (viii)

$$=\frac{1}{4}ga^2\lambda\rho'\left\{\frac{(\rho-\rho')\coth mh'}{\rho\coth mh+\rho'\coth mh'}-1\right\}$$

{from (iv)

$$=\frac{1}{4}\,ga^2\lambda\,\rho'\left\{\frac{(\rho-2\rho')\coth mh'-\rho\coth mh}{\rho\coth mh+\rho'\coth mh'}\right\} \quad ...(x)$$

Thus the ratio of the total energy of lower liquid to that of the upper liquid is given by

$\rho\,\{(2\rho-\rho')\coth mh+\rho'\coth mh'\}:\{(\rho-2\rho')\coth mh'-\rho\coth mh\}$

Proved.

Ex. 9. *An open rectangular box of length b contains two liquids of densities ρ, ρ' and depths h, h' respectively, that of density ρ being at the bottom. Prove that the periods of oscillation when the liquids are slightly disturbed so that there is no motion perpendicular to the sides of the box are determined by the equations of the type*

$$\left(p^2\coth\frac{n\pi h}{b}-\frac{gn\pi}{b}\right)\left(p^2\coth\frac{n\pi h'}{b}-\frac{gn\pi}{b}\right)+\frac{\rho}{\rho'}\left(p^4-\frac{g^2n^2\pi^2}{b^2}\right)=0$$

where n is an integer.

Let the stationary wave propagated at the common surface be

$$\eta=a\sin mx\sin pt \quad ...(i)$$

Let ϕ and ϕ' be the velocity potentials in the lower and upper liquids respectively. Since the upper surface of the upper liquid bears constant pressure assuming the velocity potentials of the form

$$\phi=c\cosh m\,(y+h)\cos mx\sin ht \quad ...(ii)$$

and
$$\phi'=(A\cosh my+B\sinh my)\cos mx\sin pt \quad ...(iii)$$

At the common surface, boundary conditions are

I. $\quad -\frac{\partial\eta}{\partial t}=\left(-\frac{\partial\phi}{\partial y}\right)_{y=0}=\left(-\frac{\partial\phi'}{\partial y}\right)_{y=0}$

II. $\left\{\left(-\frac{\partial\phi}{\partial t}+g\eta\right)\rho\right\}_{y=0}=\left\{\left(-\frac{\partial\phi'}{\partial t}+g\eta\right)\rho'\right\}_{y=0}$

III. $\left\{\frac{\partial^2\phi'}{\partial t^2}+g\frac{\partial\phi'}{\partial y}\right\}_{y=\lambda}=0$ $\left\{\begin{array}{l}\textit{Condition of constancy of the}\\ \textit{pressure at the upper surface.}\end{array}\right.$

From I, we have

$$-ap\sin mx\cos pt=-cm\sinh mh\cos mx\sin pt$$
$$=-B\,m\cos mx\sin pt$$

or $$ap=C\,m\sinh mh=B\,m$$

or $$C=\frac{ap}{m\sinh mh}\text{ and }B=\frac{ap}{m} \quad \text{...(iv)}$$

From II, we have

$$\rho\{-Cp\cosh mh+ga\}=\rho'\{-Ap+ga\} \quad \text{...(v)}$$

and from III, we have

$$p^2\{A\coth mh'+B\}=gm\{A+B\coth mh'\} \quad \text{...(vi)}$$

From (v), we have

$$A\rho'=\frac{pa\rho}{m}\coth mh-\frac{ga}{p}(\rho-\rho')$$

Substituting the values of A, B and C in (vi), we get

$$p^2\left\{\coth mh'\left(\frac{ap}{m}\rho\coth mh-\frac{ga}{p}(\rho-\rho')\right)+\frac{ap}{m}\rho'\right\}$$
$$=gm\left\{\frac{ap}{m}\rho\coth mh-\frac{ga}{p}(\rho-\rho')+\rho'\coth mh'\right\}$$

or $$\frac{p^4}{m}\rho\coth mh\coth mh'-gp^2\rho(\coth mh+\coth mh')$$
$$+\rho'\frac{p^4}{m}+g^2m(\rho-\rho')=0$$

or $$(p^2\coth mh-mg)(p^2\coth mh'-mg)$$
$$+\frac{\rho'}{\rho}(p^4-m^2g^2)=0 \quad \text{...(vii)}$$

But normal velocity of the fluid particles must vanish at the end $x=b$,

So $$\left(-\frac{\partial\phi}{\partial x}\right)_{x=b}=0$$

or $$\sin mb=0$$

ot $$mb=n\pi\quad\text{or}\quad m=\frac{n\pi}{b}$$

Substituting the value of m in (vii), we have

$$\left(p^2 \coth \frac{n\pi h}{b} - \frac{n\pi g}{b}\right)\left(p^2 \coth \frac{n\pi h'}{b} - \frac{n\pi g}{b}\right) + \frac{\rho'}{\rho}\left(p^4 - \frac{n^2 \pi^2 g^2}{b^2}\right) = 0$$

Proved.

Ex. 10. *If a horizontal rectangular canal of great depth has two vertical barriers at a distance l apart, prove that the periods of oscillation of the water are $2\sqrt{\left(\frac{\pi l}{sg}\right)}$, where s is a positive integer, and that corresponding to any mode, all the particles of fluid oscillate in straight lines of length inversely proportional to* exp. $\left(\frac{s\pi z}{l}\right)$, *where z is the depth.*

Let η be the elevation of the stationary waves at the surface

$$\eta = a \cos mx \cot nt \qquad \text{...(i)}$$

$$\phi = \frac{an}{m} e^{my} \sin nt \cos mx \qquad \text{...(ii)}$$

{Ref. § 8·51}

where $\qquad n^2 = mg \qquad$ ...(iii)

Now $\qquad \frac{\partial \phi}{\partial x} = 0$ for $x = 0$ and l

From (ii), we have

$$\sin ml = 0$$

or $\qquad ml = s\pi \quad$ where s is a positive integer.

or $\qquad m = \frac{s\pi}{l}$

From (iii), $\qquad n^2 = \frac{s\pi g}{l}$

Periods of oscillation are given by

$$= \frac{2\pi}{n}$$

$$= 2\pi \sqrt{\left(\frac{l}{s\pi g}\right)} = 2\sqrt{\left(\frac{\pi l}{sg}\right)}.$$ **Proved.**

Also $\qquad -\frac{\partial \phi}{\partial x} = u = \frac{dx}{dt} = an\, e^{my} \sin nt \sin mx \qquad$ ...(iv)

and $\qquad -\frac{\partial \phi}{\partial y} = v = \frac{dy}{dt} = -an\, e^{my} \sin nt \cos mx \qquad$ ...(v)

Let (x_0, y_0) be the equilibrium points

Substituting $\quad x = x_0 + X$

$$y = y_0 + Y$$

$$\left.\begin{aligned} \therefore \quad \dot{X} &= an\ e^{my_0} \sin nt \sin mx_0 \\ \text{and} \quad \dot{Y} &= -an\ e^{my_0} \sin nt \cos mx_0 \end{aligned}\right\} \qquad \ldots\text{(vi)}$$

Since the path is a straight line

$$\frac{Y-B}{X-A} = -\cot mx_0 = \text{const.} \qquad \ldots\text{(vii)}$$

and $\quad \sqrt{\{(X-A)^2+(Y-B)^2\}} = a \cos nt\ e^{my_0} \qquad \ldots\text{(viii)}$

The maximum value of (viii) is $\quad ae^{mY_0} \qquad \ldots$(ix)

Since z is the depth, hence $y_0 = -z$

From (ix), we have

$$= ae^{my_0}$$

$$= ae^{-mz}$$

$$= \frac{a}{e^{mz}}$$

$$= \frac{a}{e^{\frac{\pi s z}{l}}}$$

Thus all the particles of fluid oscillate in straight line of length inversely proportional to exp. $\left(\frac{\pi s z}{l}\right)$.

Ex. 11. *Shew that if the velocity of the wind is just great enough to prevent the propagation of length* λ *against it, the velocity of propagation of waves of wind is* $2c\left\{\frac{\sigma}{1+\sigma}\right\}^{1/2}$, *where* σ *is the specific gravity of the air and c the wave velocity when no wind is present.*

Let V be the velocity of the wave, U and U' the velocities of of lower and upper fluids of height h and h' respectively. Let ρ and ρ' be the densities of the fluids.

We know the relation

$$g(\rho-\rho') = m\{(U-V)^2 \rho \coth mh + (U'-V)^2 \rho' \coth mh'\}$$

$$\text{or} \quad g\left(1-\frac{\rho'}{\rho}\right)=m\left\{(U-V)^2 \coth mh+(U'-V)^2 \frac{\rho'}{\rho} \coth mh'\right\} \qquad \ldots(\text{i})$$

Since the sea is at rest, so $U=0$, $h=h'$ tends to ∞ and

$$\sigma=\frac{\rho'}{\rho} \text{ (let)}.$$

$$\text{or} \qquad g(1-\sigma)=m\{V^2+(U'-V)^2 \sigma\} \qquad \ldots(\text{ii})$$

If no wind is present $U'=0$, then $V=c$,

$$\text{So} \qquad g(1-\sigma)=m\{c^2+c^2 \sigma\}$$

$$\text{or} \qquad c^2=\frac{g}{m} \frac{1-\sigma}{1+\sigma} \qquad \ldots(\text{iii})$$

The velocity U' which prevents the propagation of waves is given by

$$g(1-\sigma)=mU'^2\sigma \qquad \ldots(\text{iv})$$

{Substituting $V=0$ in *(ii)*}.

Thus the velocity of propagation of the wind is

$$g(1-\sigma)=m\{V^2+(U'-V)^2 \sigma\}$$

$$\text{or} \quad \frac{g}{m}(1-\sigma)=V^2(1+\sigma)-2U'V\sigma+U'^2\sigma$$

$$\text{or} \quad U'^2\sigma=V^2(1+\sigma)-2U'V\sigma+U'^2\sigma \qquad \{\text{from (iv)}$$

$$\text{or} \quad V^2(1+\sigma)-2U'V\sigma=0$$

$$\text{or} \quad V=\frac{2U'\sigma}{1+\sigma}$$

$$\text{or} \quad V=\frac{2\sigma}{1+\sigma} \cdot \sqrt{\left(\frac{g}{m} \frac{1-\sigma}{\sigma}\right)} \qquad \{\text{From (iv)}\}$$

$$\text{or} \quad V=\frac{2\sigma}{1+\sigma}\, c\sqrt{\left(\frac{1+\sigma}{1-\sigma}\right)} \cdot \sqrt{\left(\frac{1-\sigma}{\sigma}\right)} \qquad \{\text{From (iii)}\}$$

$$\text{or} \quad V=2c\sqrt{\left(\frac{\sigma}{1+\sigma}\right)}.$$

Proved.

Ex. 12. *A canal, of infinite length and rectangular section, is of uniform depth h and breadth b in one part but changes gradually to uniform depth h' and breadth b' in another part. At infinite train of simple harmonic waves travelling in one direction only is propagated along the canal. Prove that, if a, a' are the heights and $\frac{2\pi}{m}$, $\frac{2\pi}{m'}$ the lengths of the waves in the two uniform portions.*

$$m \tanh mh = m' \tanh mh'$$

and $$a^2 b \operatorname{sech}^2 mh (\sinh 2mh + 2mh)$$
$$= a'^2 b' \operatorname{sech}^2 m'h' (\sinh 2m'h' + 2m'h')$$

Let the mean profiles of the wave in the two parts be

$$\eta = a \sin (mx - nt) = a \sin m (x - ct) \quad \text{.. (i)}$$

and $$\eta' = a' \sin (m'x - n't) = a' \sin m' (x - c't) \quad \text{...(ii)}$$

So that

$$c^2 = \frac{g}{m} \tanh mh \quad \text{...(iii)}$$

and $$c'^2 = \frac{g}{m'} \tanh m'h' \quad \text{...(iv)}$$

From (iii) and (iv), we have

$$\frac{c^2}{c'^2} = \frac{m'}{m} \cdot \frac{\tanh mh}{\tanh m'h'} \quad \text{. (v)}$$

The period of simple harmonic wave must remain the same all along the canal, which gives

$$\frac{2\pi}{n} = \frac{2\pi}{n'}$$

or $$\frac{m}{n} \cdot \frac{1}{m} = \frac{m'}{n'} \cdot \frac{1}{m'},$$

or $$\frac{1}{mc} = \frac{1}{m'c'} \quad \text{or} \quad \frac{m'}{m} = \frac{c}{c'}$$

or $$\frac{m'^2}{m^2} = \frac{m'}{m} \frac{\tanh mh}{\tanh m'h'}$$

or $$\frac{m'}{m} = \frac{\tanh mh}{\tanh m'h'}$$

or $$m' \tanh m'h' = m \tanh mh.$$ **Proved.**

Also the energy transmitted shall also be the same in either part of the canal. Then, we have

$$\tfrac{1}{2} g \rho a^2 bc (1 + 2 mh \operatorname{cosech} 2mh)$$
$$= \tfrac{1}{4} g \rho a'^2 b'c' (1 + m'h' \operatorname{cosech} 2m'h')$$

{Ref. § 8·72}

or $$\frac{a^2 b \sinh mh (2mh + \sinh 2mh)}{\cosh mh \sinh 2mh}$$
$$= \frac{a'^2 b' \sinh m'h' (2m'h' + \sinh 2m'h')}{\sinh 2m'h' \cosh m'h'}$$

{from (v)}

or $$a^2 b \operatorname{sech}^2 mh (\sinh 2mh + 2mh)$$
$$= a'^2 b' \operatorname{sech}^2 m'h' (\sinh 2m'h' + 2m'h')$$

Proved.

9

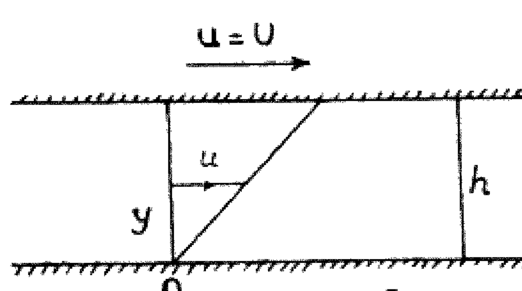
Viscosity

So far we have been concerned with perfect (ideal) fluids (frictionless and incompressible). In the motion of such perfect fluid, two contacting layers of the fluid experience no tangential forces (shearing stress) but act on each other with normal forces (pressure) only or in other sense we can define that a perfect fluid exerts no internal resistance to a change in shape. In this chapter we shall consider the cases of actual (real) fluids. In real fluids the inner layers of the fluid transmit tangential as well as normal stresses. **Viscosity** of the fluid is that property of actual fluids which exerts such resistance.

Because of the absence of tangential forces, a difference in relative tangential velocites exists on the boundary between a perfect fluid and a solid wall *i.e.* there is a slip, on the other hand, in actual fluids the existance of inter-molecular attraction causes the fluid to adhere to a solid wall and it gives rise to shearing stress. *The difference between a perfect and a real fluid is the existence of shearing stress and the condition of no slip.*

§ 9·1. Measurement of Viscosity.

Consider the motion of a fluid between two very long parallel plates, at a distance h apart. Let the lower plate be at rest and the upper plate is moving with a constant velocity U parallel to itself. The pressure being constant throughout the fluid. We see that the fluid adheres to both the walls, so that its velocity at the lower plate is zero and that at the upper plate is equal to the velocity U. Again, the velocity distribution in the fluid between the plates is linear, is



linear, so that the fluid velocity is proportional to the distance y from the lower plate (there being no slip on the walls).

Then $$u=U\frac{y}{h}$$

Since the tangential force to the upper plate be in equilibrium with the frictional forces in the fluid. Also the experiments shows that this force is proportional to the velocity U of the upper plate and inversely proportional to the distance h. Let τ denotes the frictional force per unit area

or $$\tau \propto \frac{U}{h}$$

$\left\{\right.$ In general $\frac{U}{h}$ can be replaced by the velocity gradient $\frac{du}{dy}$.

or $$\tau \propto \frac{du}{dy}$$

or $$\tau=\mu\frac{du}{dy} \qquad \text{.. (i)}$$

where μ is a constant of proportionality depending on the pressure and temperature. For gases μ is independent of the pressure at ordinary temperature. The relation (i) is known as **Newton's equation of viscosity**. By transformation. we have

$$\mu=\frac{\tau}{du/dy}$$

which is known as the **coefficient of viscosity** or **Absolute viscosity** or **Dynamic viscosity**. A fluid for which the constant of proportionality (*i.e.* viscosity) does not change with rate of deformation is said to be a **Newtonian fluid**. The coefficient of viscosity μ is small for the fluids such as water, alcohol or gases but it is not negligible, but large in the case of very viscous liquids such as oil, glycerine.

The physical dimensions of the coefficient of viscosity can be determined as follows :

$$\mu=\frac{\text{shearing stress}}{\text{velocity gradient}}$$

$$\left\{\begin{array}{l}\text{Shearing stress} \Rightarrow \text{Force/unit area}\\ \text{and velocity gradient}=\text{velocity/length}\end{array}\right.$$

or $$\mu \text{ (force per unit area/rate of shear)}$$

$$=\frac{\frac{ML}{T^2L^2}}{\frac{L}{TL}}=\frac{M}{LT}=ML^{-1}\,T^{-1}$$

In all fluid motions in which frictional and inertial forces interact, we consider the ratio of the viscosity to the density such as

$$\nu=\frac{\mu}{\rho}$$

which is known as **kinemetic Viscosity.**

9·2. Strain Analysis.

When the various elements of a system undergo relative displacements under the action of impressed forces, it is said to be **strained.** In other words strain is a non-dimensional deformation which measures the change of relative positions of the parts of a body under any cause. In a body under the influence of external forces, the displacements may consist of a translation, a rotation, and a distortion. Since a translation and a rotation represent rigid-body displacements which do not produce any relative displacement of the various elements of the system, consequently they do not constitute any strain. Strain is classified into two kinds, viz,

(a) **Normal Strain** is defined as the ratio of the change in length to the riginal length of a straight line element.

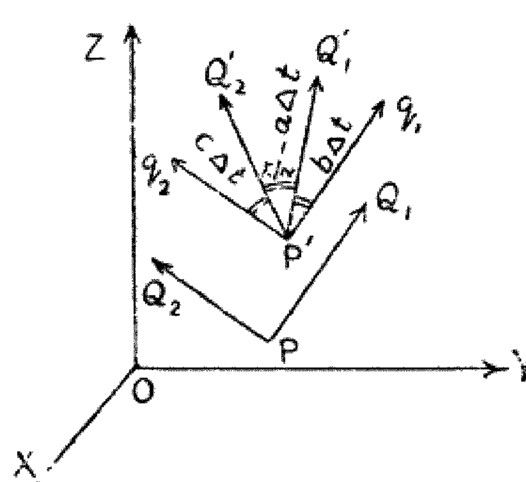
(b) **Shearing Strain** is defined as the change in angle between two linear elements from the unstrained state to the strained state.

Since the motion of the fluid is completely determined when the velocity vector **q** is given as a function of position and time

$$\mathbf{q}=\mathbf{q}\,(x\;y\;z\;t).$$

$\exists$ kinematic relations between the components of the rate of strain and this function. Let the velocity of an infinitesimal element at $P\,(x\;y\;z)$ at any time be $(u\;v\;w)$.

Let PQ_1 and PQ_2 be two perpendicular lines through P having infinitesimal length δS_1 and δS_2 and direction consines are $(l_1\;m_1\;n_1)$ and $(l_2\;m_2\;n_2)$ respectively. The coordinates and velocities at Q_1 and Q_2 be at the same time t are.



$$(x+\delta x_1,\,y+\delta y_1,\,z+\delta z_1;\;u+\delta u_1,\;v+\delta v_1,\,w+\delta w_1)$$

and $(x+\delta x_2, y+\delta y_2, z+\delta z_2; u+\delta u_2, v+\delta v_2, w+\delta w_2)$ respectively.

Evidently $$\delta S^2=\sum_{xyz} \delta x^2$$

or $$\delta x=l\delta S,\ \delta y=m\delta S,\ \delta z=n\delta S \qquad \text{...(i)}$$

Since PQ_1 and PQ_2 are perpendicular

then $$\sum_{xyz} \delta x_1\, \delta x_2=0 \qquad \text{...(ii)}$$

The relative velocity $(\delta u, \delta v, \delta w)$ of Q relative to the point P can be written as

$$\delta u=\sum_{xyz} u_x\, \delta x,\ \delta v=\sum_{xyz} v_x\, \delta x,\ \delta w=\sum_{xyz} w_x\, \delta x$$

Now assuming the following symbols,

$$\left.\begin{aligned} & e_{xx}=2u_x,\ e_{yy}=2v_y,\ e_{zz}=2w_z \\ \text{and}\quad & e_{yz}=e_{zy}=w_y+v_z \\ & e_{zx}=e_{xz}=u_z+w_x \\ & e_{xy}=e_{yx}=v_x+u_y \\ \text{also}\quad & \xi=w_y-v_z,\ \eta=u_z-w_x,\ \zeta=v_x-u_y \end{aligned}\right] \qquad \text{...(iii)}$$

Form the above assumed relation (iii), we have

$$\left.\begin{aligned} & u_x=\tfrac{1}{2}\, e_{xx}, && u_y=\tfrac{1}{2}\,(e_{xy}-\zeta), && u_z=\tfrac{1}{2}\,(e_{xz}+\eta) \\ & v_x=\tfrac{1}{2}\,(e_{xy}+\zeta), && v_y=\tfrac{1}{2}\, e_{yy}. && v_z=\tfrac{1}{2}\,(e_{yz}-\xi) \\ & w_x=\tfrac{1}{2}\,(e_{xz}-\eta), && w_y=\tfrac{1}{2}\,(e_{yz}+\xi), && w_z=\tfrac{1}{2}\, e_{zz} \end{aligned}\right\} \qquad \text{...(iv)}$$

Where ξ, η, ζ are the components of vorticity about the coordinate axes OX, OY, OZ.

Now the velocity of Q in terms of these symbols is

$$\left.\begin{aligned} u_Q &= u_P+\delta u \\ &= u_P+\tfrac{1}{2}\,(e_{xx}\, \delta x+e_{xy}\, \delta y+e_{xz}\, \delta z \\ &\qquad +\tfrac{1}{2}\,(\eta\delta z-\zeta\, \delta y) \\ v_Q &= v_P+\delta v \\ &= v_P+\tfrac{1}{2}\,(e_{yx}\, \delta x+e_{yy}\, \delta y+e_{yz}\, \delta z) \\ &\qquad +\tfrac{1}{2}\,(\zeta\, \delta x-\xi\, \delta z) \\ \text{and}\quad w_Q &= w_P+\delta w \\ &= w_P+\tfrac{1}{2}\,(e_{zx}\, \delta x+e_{zy}\, \delta y+e_{zz}\, \delta z) \\ &\qquad +\tfrac{1}{2}\,(\xi\, \delta y-\eta\, \delta x) \end{aligned}\right] \qquad \text{...(v)}$$

The velocity at Q consists of three parts :

(a) Velocity of translation (u_P) which is the same as that of P.

(b) Rate of deformation (Rate of component of strain) as

$\frac{1}{2}(e_{xx}\,\delta x+e_{xy}\,\delta y+e_{xz}\,\delta z)$, $\frac{1}{2}(e_{yx}\,\delta x+e_{yy}\,\delta y+e_{yz}\,\delta z)$

$\frac{1}{2}(e_{zx}\,\delta x+e_{zy}\,\delta y+e_{zz}\,\delta z)$

(c) Velocity produced by rigid body due to rotation of angular velocity $(\frac{1}{2}\xi, \frac{1}{2}\eta, \frac{1}{2}\zeta)$ about straight lines parallel to the axes of reference through P.

Velocity of Q relative to the point P is

$$\left.\begin{aligned} \delta u&=\tfrac{1}{2}\,\delta S\,\{le_{xx}+me_{xy}+ne_{xz}+(n\eta-m\zeta)\}\\ \delta v&=\tfrac{1}{2}\,\delta S\,\{le_{yx}+me_{yy}+ne_{yz}+(l\,\zeta-n\,\xi)\}\\ \delta w&=\tfrac{1}{2}\,\delta S\,\{le_{zx}+me_{zy}+ne_{zz}+(m\xi-l\eta)\}\end{aligned}\right\} \quad \ldots(\text{vi})$$

§ 9·21. Rate of elongation.

Consider P', Q_1', Q_2' be the position of P, Q_1, Q_2 respectively at time $t+\triangle t$. Evidently, the coordinates of P' are

$$(x+u\triangle t,\ y+v\triangle t,\ z+w\triangle t)$$

and that of Q'_1 $\{x+\delta x+(u+\delta u)\,\triangle t,\ y+\delta y+(v+\delta v)\,\triangle t,$

$$z+\delta z+(w+\delta w)\,\triangle t\}$$

then $(P'Q_1')^2=(\delta x+\delta u\,\triangle t)^2+(\delta y+\delta v\,\triangle t)^2+(\delta z+\delta w\,\triangle t)^2$

or $P'Q_1'=\{(\delta x+\delta u\,\triangle t)^2+(\delta y+\delta v\,\triangle t)^2+(\delta z+\delta w\,\triangle t)^2\}^{1/2}$

or $P'Q_1'=\{(\delta S)^2+2\triangle t\,(l\delta S\,\delta u+m\delta S\,\delta v+n\delta S\,\delta w)\}^{1/2}$

Using the relation (i) and (vi), we have

$$P'Q_1'=[1+\tfrac{1}{2}\triangle t\,\{l^2\,e_{xx}+m^2e_{yy}+n^2e_{zz}+2lm\,e_{xy}+2mn\,e_{yz}+2nl\,e_{zx}\}+0\,(\triangle t)^2] \quad \ldots(\text{i})$$

Rate of elongation

$$=\frac{P'Q_1'-PQ_1}{PQ_1}\cdot\frac{1}{\triangle t}$$

$$=\tfrac{1}{2}\,[l^2e_{xx}+m^2e_{yy}+n^2e_{zz}+2lm\,e_{xy}+2mn\,e_{yz}+2nl\,e_{zx}] \quad \ldots(\text{ii})$$

Which gives the **relative rate of elongation of PQ₁**.

Consider PQ_1, parallel to the X-axis then the direction cosines becomes (1, 0, 0), hence from (ii),

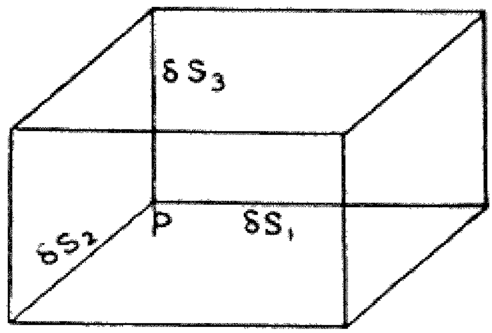
$\frac{1}{2}e_{xx}$ represents the relative rate of elongation in the direction of X-axis.

Similarly $\frac{1}{2}e_{yy}$ represents the relative rate of elongation in the direction of Y-axis.

and $\frac{1}{2}e_{zz}$ represents the relative rate of elongation in the direction of Z-axis.

Ex. 1. *Consider a rectangular parallelopiped with edges PQ_1, PQ_2 and PQ_3 parallel to the axis of reference of lengths δS_1,*

δS_2 and δS_3 respectively, then the relative rate of increase of its volume is given by



$$=\underset{\Delta t\to 0}{Lt}\ \frac{\delta S_1\left(1+\frac{1}{2}e_{xx}\Delta t\right).\delta S_2\left(1+\frac{1}{2}e_{yy}\Delta t\right).\delta S_3\left(1+\frac{1}{2}e_{zz}\Delta t\right)-\delta S_1\,\delta S_2\,\delta S_4}{\delta S_1\,\delta S_2\,\delta S_3\,\Delta t}$$

$$=\underset{\Delta t\to 0}{Lt}\ \left\{\left(1+\tfrac{1}{2}e_{xx}\Delta t\right)\left(1+\tfrac{1}{2}e_{yy}\Delta t\right)\left(1+\tfrac{1}{2}e_{zz}\Delta t\right)-1\right\}$$

$$=\tfrac{1}{2}\left\{e_{xx}+e_{yy}+e_{zz}\right\} \qquad \left\{\begin{array}{l}\text{neglecting the terms of}\\ \text{higher orders of } \Delta t.\end{array}\right.$$

$$=u_x+v_y+w_z$$

$$=\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}$$

$=$**div. q** (where **q** is the velocity vector)

The relative rate of increase in the volume is called **dilation** and generally is denoted by Δ.

Thus $\qquad \Delta=\frac{1}{2}\left(e_{xx}+e_{yy}+e_{zz}\right)$

If the rate of increase vanishes then it is known as equation of continuity.

§ 9·3. Rate of Shear.

Here we shall discuss the rate of change of the angle between the lines PQ_1 and PQ_2, which are orginally taken perpendicular to each other.

Let $P'q_1$ and $P'q_2$ be drawn parallel and equal to PQ_1 and PQ_2 respectively.

Consider,

$$\left.\begin{array}{rl} & Q'_2P'Q'_1=\frac{\pi}{2}-a\Delta t\\ & P'Q'_1q_2=\frac{\pi}{2}-b\Delta t\\ \text{and} & Q'_2P'q_1=\frac{\pi}{2}-c\Delta t\end{array}\right\} \qquad \text{...(i)}$$

(It is not necessary that $P'Q'_1$, $P'Q'_2$ will be in the plane of PQ_1 and PQ_2 or even in the plane parallel to this plane through P'.)

From (i), we have

$$\cos Q'_2P'Q'_1=\cos\left(\frac{\pi}{2}-a\triangle t\right)$$
$$=\sin(a\triangle t)=a\triangle t \qquad \text{...(ii)}$$

(to a small approximation).

Again, we have

$$P'Q_1'.P'Q_2'=\Sigma\,(\delta x_1+\delta u_1\triangle t)\,(\delta x_2+\delta u_2\triangle t)$$
$$=\triangle t\,\Sigma\,(\delta x_1\delta u_2+\delta x_2\delta u_1)$$

($\Sigma\ \delta x_1\delta x_2=0$, being at right angles)

$$=\delta s_1\delta s_2\triangle t\,\{l_1l_2\,e_{xx}+m_1m_2e_{yy}+n_1n_2e_{zz}+(l_1m_2+l_2m_1)\,e_{xy}+(m_1n_2+m_2n_1)\,e_{yz}+(n_1l_2+n_2l_1)\,e_{zx}\} \qquad \text{...(iii)}$$

{ from relation (vi) § 9·2

We can also determine this product in the following manner

$$P'Q'_1.P'Q'_2=\delta s_1\,(1+a_1\triangle t).\delta s_2\,(1+a_2\triangle t)\cos Q'_2P'Q'_1$$
$$=\delta s_1\delta s_2 a\triangle t \qquad \{\text{ from (ii)} \qquad \text{...(iv)}$$

neglecting the quantities of higher order of $\triangle t$ and considering the relative rate of elongation along PQ_1 and PQ_2 as a_1 and a_2.

Thus from (iii) and (iv), we have

$$a=l_1l_2e_{xx}+m_1m_2e_{yy}+n_1n_2e_{zz}+(l_1m_2+l_2m_1)e_{xy}+(m_1n_2+m_2n_1)\,e_{yz}+(n_1l_2+n_2l_1)\,e_{zx}. \qquad \text{...(v)}$$

Taking PQ_1 and PQ_2 parallel to OX and OY, then direction cosines reduces to

$$(l_1\ m_1\ n_1)=(1\ 0\ 0)$$
$$(l_2\ m_2\ n_2)=(0,\ 1,\ 0)$$

From (v), we have $\quad a=e_{xy}$.

Thus e_{xy} represents the rate of the decrease of the angle between the lines which were originally parallel to the axis of X and Y respectively *i.e.* $e_{xy}\Rightarrow$ *the rate of shear in the XY-plane.*

Similarly we can determine that e_{yz} and e_{zx} as the rates of shear in YZ-plane and in ZX-plane.

§ 9·31. Rate of strain tensor.

We shall prove that the rate of strain matrix

$$\begin{bmatrix} e_{xx} & e_{xy} & e_{xz} \\ e_{yx} & e_{yy} & e_{yz} \\ e_{zx} & e_{zy} & e_{zz} \end{bmatrix} \qquad \{\text{ Ref. relation (iv) § 9·2}$$

is a symmetric tensor, *i.e.*

$$e_{xy}=e_{yx},\ e_{yz}=e_{zy}\ \text{and}\ e_{zx}=e_{xz}. \qquad \text{...(i)}$$

Let x_i and v_i $(i=1, 2, 3)$ denote the coordinates and velocity components of a point. Consider another coordinate system x'_i obtained by change of axis according to the following manner

	x_1	x_2	x_3
x'_1	$l_{1,1}$	$l_{1,2}$	$l_{1,3}$
x'_2	$l_{2,1}$	$l_{2,2}$	$l_{2,3}$
x'_3	$l_{3,1}$	$l_{3,2}$	$l_{3,3}$

$$\left.\begin{array}{ll}\text{Then} & x'_i=l_{i,j}\,x_j \text{ and } x_i=l_{j,i}\,x'_j \\ \text{Also} & v'_i=l_{i,j}\,v_j \text{ and } v_i=l_{j,i}\,v'_j\end{array}\right\} \qquad \ldots\text{(ii)}$$

In this notation

$$\left.\begin{array}{l} e_{ij}=\dfrac{\partial V_j}{\partial x_i}+\dfrac{\partial V_i}{\partial x_j} \\ e'_{ij}=\dfrac{\partial V'_j}{\partial x'_i}+\dfrac{\partial V'_i}{\partial x'_j}\end{array}\right\} \qquad \ldots\text{(ii))}$$

Now $$\frac{\partial}{\partial x'_j}(v'_i)=\left\{l'_{jk}\frac{\partial}{\partial x_k}(l'_{ip}V_p)\right\}$$

$$=l'_{jk}\,l'_{ip}\frac{\partial V_p}{\partial x_k} \qquad \ldots\text{(iv)}$$

Thus from (iii),

$$e'_{ij}=l'_{ip}\,l'_{jk}\,e_{pk}=\frac{\partial x'_i}{\partial x_p}\cdot\frac{\partial x'_j}{\partial x_k}\,e_{pk}$$

$$=\frac{\partial x_p}{\partial x'_i}\cdot\frac{\partial x_k}{\partial x'_j}\,e_{pk}$$

which shows that e_{ij} is a **second order tensor.**

§ 9·32. **Rate of strain components.**

Let $(u\ v\ w)$ be the components of the velocity parallel to the coordinates axes at the point (x, y, z) at time t. The components of the relative velocity at an infinitely near point $(x+\delta x, y+\delta y, z+\delta z)$ are :

$$\delta u=\tfrac{1}{2}(e_{xx}\delta x+e_{xy}\delta y+e_{xz}\delta z)+\tfrac{1}{2}(\eta\delta z-\zeta\delta y)$$

$$\delta v=\tfrac{1}{2}(e_{yx}\delta x+e_{yy}\delta y+e_{yz}\delta z)+\tfrac{1}{2}(\zeta\delta x-\xi\delta z)$$

and $$\delta w=\tfrac{1}{2}(e_{zx}\delta x+e_{zy}\delta y+e_{zz}\delta z)+\tfrac{1}{2}(\xi\delta y-\eta\delta x)$$

where $$e_{xx}=2\frac{\partial u}{\partial x},\ e_{yy}=2\frac{\partial v}{\partial y},\ e_{zz}=2\frac{\partial w}{\partial z}$$

$$e_{yz}=e_{zy}=\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}$$

$$e_{xz}=e_{zx}=\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}$$

$$e_{xy}=e_{yx}=\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}$$

and $\xi=\dfrac{\partial w}{\partial y}-\dfrac{\partial v}{\partial z},\quad \eta=\dfrac{\partial u}{\partial z}-\dfrac{\partial w}{\partial x},\quad \zeta=\dfrac{\partial v}{\partial x}-\dfrac{\partial u}{\partial y}$

ξ, η, ζ are the components of the vorticity vector $\vec{\omega}$ and

$$\vec{\omega}=\text{Curl } V$$

where $V(u, v, w)$ is the velocity vector.

The quantities e_{xx}, e_{yy} etc. are called the rate of strain components and $\frac{1}{2}e_{xx}$ represents the rate of extension of a line element in the direction of the X-axis. e_{yx} is the rate of change of the angle between two lines along the axis of X and axis of Y.

Consider the general case of the rates of strain in three dimensions. The relations between coordinates and velocity components in the two systems are,

	x	y	z
x'	l_1	m_1	n_1
y'	l_2	m_2	n_2
z'	l_3	m_3	n_3

Then
$$u'=l_1u+m_1v+n_1w$$
$$v'=l_2u+m_2v+n_2w$$
$$w'=l_3u+m_3v+n_3w,$$

where (u', v', w') are the velocity components along $(x'\ y'\ z')$

Similarly
$$x=l_1x'+l_2y'+l_3z'$$
$$y=m_1x'+m_2y'+m_3z'$$
and
$$z=n_1x'+n_2y'+n_3z'$$

So $$e_{x'x'}=2\frac{\partial u'}{\partial x'}=2\left(l_1\frac{\partial}{\partial x}+m_1\frac{\partial}{\partial y}+n_1\frac{\partial}{\partial z}\right)(l_1u+m_1v+n_1w)$$
$$=l_1^2e_{xx}+m_1^2e_{yy}+n_1^2e_{zz}+2m_1n_1e_{yz}+2n_1l_1e_{zx}+2l_1m_1e_{xy}$$

and $$e_{y'z'}=\frac{\partial w'}{\partial y'}+\frac{\partial v'}{\partial z'}=l_2l_3e_{xx}+m_2m_3e_{yy}+n_2n_3e_{zz}$$
$$+(m_2n_3+m_3n_2)\,e_{yz}+(n_2l_3+n_3l_2)\,e_{zx}+(l_2m_3+l_3m_2)\,e_{xy}$$

Similarly we can have corresponding relation for the other quantities.

The quadric

$$e_{xx}(\delta x)^2+e_{yy}(\delta y)^2+e_{zz}(\delta z)^2+2e_{yz}\delta y\delta z+2e_{zx}\delta z\delta x+2e_{xy}\delta x\delta y=\text{Const.}$$

can be referred to its principal axes parallel to the axes, $x'\ y'\ z'$ then the quantities $e_{y'z'}, e_{z'x'}, e_{x'y'}$ will vanish. *Such motion is* *called one of pure strain and the principal axes of the quadric are called principal axes of the strain.*

§ 9·4. Stress Analysis.

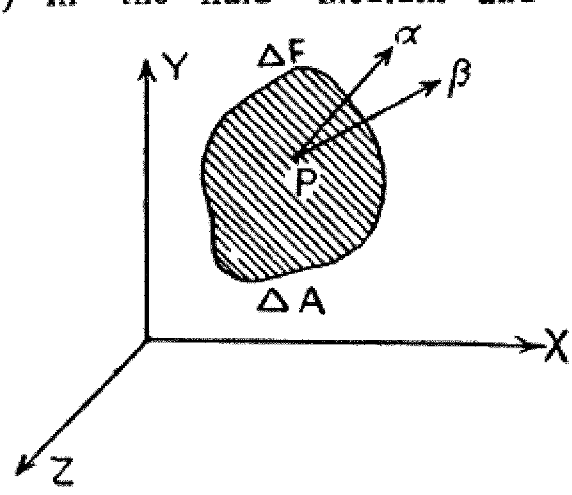
In general, the forces acting on an element of a media are of two types : (a) External forces and (b) Internal forces. The weight of the body and electromagnetic forces are called as the body forces which are distributed over the volume of the medium. The internal forces are regarded as acting on an element of volume through its bounding surface.

Assume a surface $\triangle A$ of a fluid in contact with a solid body. The forces exerted on the fluid across a portion of the surface $\triangle A$ are equal and opposite to the force exerted on the body across the portion of $\triangle A$ (By Newton's third law of motion). We know that a set of forces on the surface of the body can be resolved into a single resultant force P together with a couple G. A stress is the limit of the ratio of the force to the area on which it acts when the area reduces to a point, it follows that the ratio of the couple to the area must vanish.

Consider a point $P\ (x\ y\ z)$ in the fluid medium and take infinitesimal area $\triangle A$ surrounding the point P. The fluid on each side of the area exerts a force $\triangle F$ on it. Then the stress S of the fluid at P on the area A is defined as

$$S = \underset{\triangle A \to 0}{\text{Lt.}} \frac{\triangle F}{\triangle A}.$$

(This is finite and non-zero)



In other words the forces per unit area which two neighbouring elements of volume with a common surface exerts on each other are called stresses. For a fluid at rest, stress is normal to the surface and is in the nature of a pressure. When fluids are in motion, ∃ also shearing stress in addition to normal stress.

The stress components can be represented by $P_{\alpha\beta}$, where α denotes the direction of the normal to the area and β is the direction in which the stress component is taken.

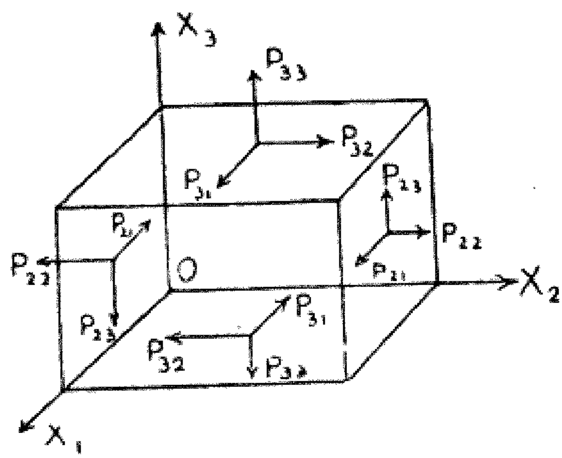
Considering the right handed system of coordinate axes, we define the stress matrix

$$P = P_{ij}$$

where P_{ij} is the component of stress acting on an area $\triangle A$ perpendicular to the axis x_i taken in the direction parallel to x_j-axis.

§ 9 41. Stress Tensor.

Now we shall prove that P is a symmetric tensor. We know by D' Alembert's principle that the reversed effective forces and the impressed forces acting on a dynamical system at any instant are in equilibrium, and the fact that the force on an infinitesimal area in any direction can be taken as the pr duct of the area and the stress acting at its centre in that direction.



Consider a rectangular parallelopiped with edges parallel to the coordinate axis and of infinitesimal length a, b, c having centre as $(x_1\, x_2\, x_3)$. The forces acting on the opposite faces of the parallelopiped can be taken equal in magnitude if we neglect the quantities of higher order than the second in linear dimension of the parallelopiped. The forces acting on the forces perpendicular to the axis-X_2 in the direction of X_3 are given by

$$= \int P_{23} \left((x_1, x_2 + \frac{b}{2}, x_3 \right) dx_1\, dx_3$$

$$= \int \left\{ P_{23}\ (x_1\, x_2\, x_3) + \frac{b}{2} \left(\frac{\partial P_{23}}{\partial x_2} \right) (x_1\, x_2\, x_3) + \ldots\ldots \right\} dx_1\, dx_3$$

$$= ac\ P_{23}\ (x_1\, x_2\, x_3) + 0\ (l^3) \qquad \ldots(\text{i})$$

and $$\int P_{23} \left(x_1, x_2 - \frac{b}{2}, x_3 \right) dx_1\, dx_3$$

$$=\iint\left\{P_{23}\,(x_1\,x_2\,x_3)-\frac{b}{2}\left(\frac{\partial P_{23}}{\partial x_2}\right)(x_1\,x_2\,x_3)+\ldots\ldots\right\}dx_1\,dx_3$$

$$=ac\,P_{23}\,(x_1\,x_2\,x_3)+0\,(l^3) \qquad \ldots\text{(ii)}$$

Similarly we can consider other components. Taking the moments about the axis X_1, we have

$$(P_{23}\,ac)\,\frac{b}{2}-(P_{32}\,ab)\frac{c}{2}+(P_{23}\,ac)\frac{b}{2}-(P_{32}\,ab)\,\frac{c}{2}+0\,(l^4)=0. \qquad \ldots\text{(iii)}$$

Where the terms of order l^4 arise on account of the body forces and reversed effective forces which are of the order l^3. Considering the stress on any force of the parallelopiped equal to the stress acting on a plane parallel to this force and passing through the centre of the parallelopiped.

Thus from (iii)

$$P_{23}=P_{32}$$

Similarly we can prove that

$$P_{ij}=P_{ji}$$

$\Rightarrow$ that P is symmetric, thus stress matrix is diagonally symmetric and contains only six unknown.

The three sets of stress components are given by

$$\begin{array}{ccc} p_{xx} & p_{xy} & p_{xz} \\ p_{yx} & p_{yy} & p_{yz} \\ p_{zx} & p_{zy} & p_{zz} \end{array}$$

The diagonal elements p_{xx}, p_{yy}, p_{zz} of this array are called normal of direct stresses. The remaining six elements are known shearing stress. For an inviscid fluid

$$p_{xx}=p_{yy}=p_{zz}=-p$$

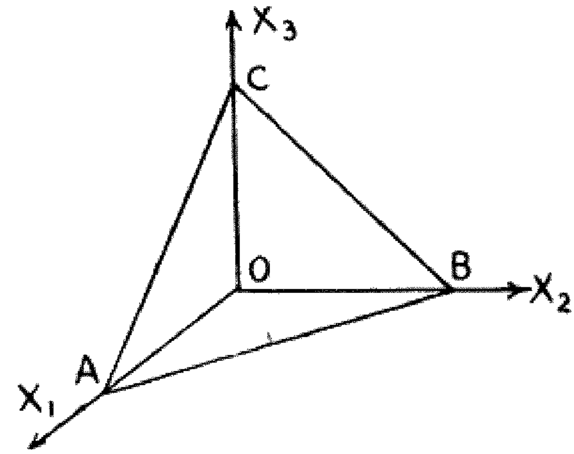
and $$p_{xy}=p_{xz}=\ldots=0$$

The matrix $\begin{bmatrix} p_{xx} & p_{xy} & p_{xz} \\ p_{yx} & p_{yy} & p_{yz} \\ p_{zx} & p_{zy} & p_{zz} \end{bmatrix}$ is called a *stress matrix*

The quantities p_{ij} where $i\,j=x, y, z$, are called the stress tensor which is a second order tensor.

§ 9·42. **Tensor character of P.**

Consider an element of fluid in the tetrahedron (O, ABC) having edges OA, OB, OC parallel to the axis of reference. Let the direction cosines of the normal to the plane ABC be $(l_{\nu 1}, l_{\nu 2}, l_{\nu 3})$. The fluid tetrahedron will be in equilibrium under the action of



(i) body forces.

(ii) reversed effective forces.

(iii) forces arising from the surface stresses.

Now the body forces and reversed effective forces will form a system of forces in equilibrium by *D' Alembert's* principle. Let the area of the triangle be $\triangle$.

or $$P_{\nu 1}\triangle=(\triangle\, l_{\nu 1})\, P_{11}+(\triangle l_{\nu 2})\, P_{21}+(\triangle\, l_{\nu 3})\, P_{31}$$

or $$P_{\nu 1}=l_{\nu 1}\, P_{11}+l_{\nu 2}\, P_{21}+l_{\nu 3}\, P_{31}.$$

In general, we can represent

$$P_{\nu\alpha}=l_{\nu\beta}\, P_{\beta\alpha} \qquad \ldots\text{(i)}$$

Consider the transformation of axis defined in § 9·31. Let $P_{i'j'}$ denotes the stress on the plane perpendicular to x_i'-axis in the direction of $x_{j'}$ axis. This is equal to the algebraic sum of the resolved parts of the stress acting on this plane in the directions of X_1, X_2, X_3 axis.

i.e. $$P_{i'j'}=l_{j\beta}\, P_{i'\beta} \qquad \ldots\text{(ii)}$$

The relation (i) can be written

$P_{i'\beta}=l_{i'\alpha}\, P_{\alpha\beta}$ (Taking ν-direction parallel to x_i' axis.) ...(iii)

From (ii) and (iii), we have

$$P_{i'j'}=l_{j'\beta}\, l_{i'\alpha}\, P_{\alpha\beta}$$

or $$P_{i'j'}=\frac{\partial x_{i'}}{\partial x_\alpha}\cdot\frac{\partial x_{j'}}{\partial x_\beta}P_{\alpha\beta}$$

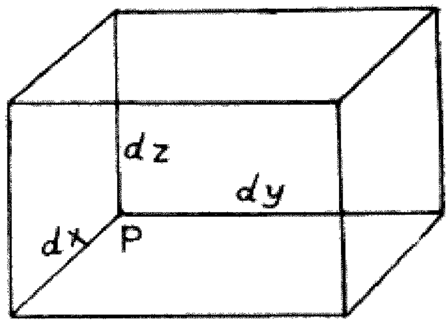
proves the tensor character of P_{ij}, the first invariant of the tensor P is

$$P_{11}+P_{22}+P_{33}=-3P$$

or $$P_{xx}=P_{yy}=P_{zz}=-p \qquad \ldots\text{(v)}$$

and $$p_{xy}=p_{yz}=p_{zx}=\ldots\text{etc}=0.$$

Consider a small parallelopiped of sides dx, dy, dz parallel to the axes with one corner of P (x y z).



Taking moment about X-axis, we have

$$(p_{xy}\ dy\ dz).dx=(p_{yx}\ dx\ dz).dy$$

or $\quad p_{xy}=p_{yx}$

Similarly $\quad p_{yz}=p_{zy}$

and $\quad p_{zx}=p_{xz}$

Thus the stress components are P_{xx}, P_{yy}, P_{zz}, P_{yz}, P_{zx} *and* P_{xy}.

§ 9·5. **Stress quadric.**

The quadric

$$\psi=P_{ij}\ X_i\ X_j \qquad \text{...(i)}$$
$$=\text{constant}$$

is called the stress quadric at the point (x_1, x_2, x_3) at which P_{ij} are the stress components and X_i are the co-ordinate relative to x_j.

§ 9·51. **Properties of Stress Quadric.**

(I) ψ **is invarient.**

$$\psi=P_{ij}\ X_i\ X_j$$
$$\psi=P_{ij}\ l_{k'i}\ X_{k'}\ .\ l_{p'j}\ X_{p'}$$

{Ref. § 9·31 relation (ii)

or $$\psi=P_{k'p'}\ X_{k'}\ X_{p'}$$

(II) Consider r be the length of the radius of the quadric in direction α $(l_{\alpha 1}, l_{\alpha 2}, l_{\alpha 3})$.

Then $\quad x_i=r\ l_{\alpha i} \qquad$ {where $i=1, 2, 3$

Now from (i) § 9·5

or $$P_{ij}\ X_i\ X_j=\text{constant}$$

or $$P_{ij}\ (r\ l_{\alpha i})\ (r\ l_{\alpha j})=\text{constant}$$

or $$P_{ij}\ (r^2\ l_{\alpha i}\ l_{\alpha j})=\text{constnnt}$$

or $$P_{\alpha\alpha}.r^2=\text{constant}$$

or $$P_{\alpha\alpha}=\frac{\text{constant}}{r^2}$$

$\Rightarrow$ *that the normal stress on a plane perpendicular to a direction is inversely proportional to the square of the radius of the quadric in that direction.*

(III) **Principal axis.**

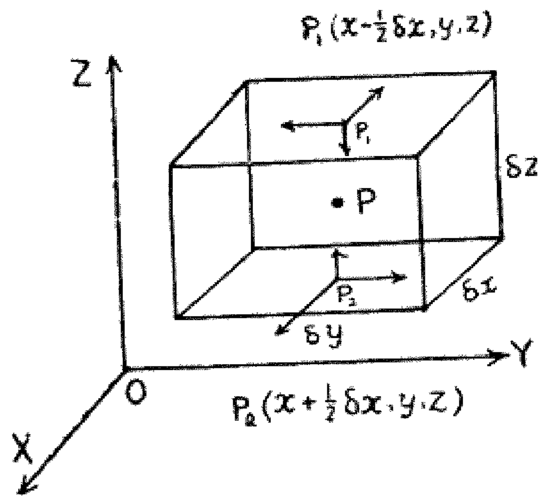
Referred the quadric to its principal axes, it will be of the from

$$P_1 X^2 + P_2 Y^2 + P_3 Z^2 = \text{constant}$$

$\Rightarrow$ *that in the principal planes there are no tangential stresses.*

§ 96. Translation motion of fluid element.

Consider the motion of a small rectangular parallelopiped of viscous fluid, having $P(x\,y\,z)$ as centre and its edges of lengths δx, δy, δz parallel to the fixed rectangular axes.



Mass of fluid element

$= \rho\, \delta x\, \delta y\, \delta z$

(which will remain constant)

Suppose the element move along with the fluid. The components of the forces parallel to the co-ordinate axes OX, OY, OZ on the surface of area $\delta y\, \delta z$ through the point $P(x, y, z)$ are

$$(p_{xx}\, \delta y\, \delta z,\ p_{xy}\, \delta y\, \delta z,\ p_{xz}\, \delta y\, \delta z)$$

{having $\mathbf{i}$ as the unit normal.

At the point $P_2\,(x+\frac{1}{2}\,\delta x, y, z)$, the corresponding force components across the parallel plane of area $\delta y\, \delta z$ are ($\mathbf{i}$ is the unit normal measured outwards from the fluid).

$$\left[\left\{ p_{xx} + \tfrac{1}{2}\, \delta x \left(\frac{\partial p_{xx}}{\partial x}\right)\right\} \delta y\, \delta z\,,\ \left\{ p_{xy} + \tfrac{1}{2}\, \delta x \left(\frac{\partial\, p_{xy}}{\partial x}\right)\right\} \delta y\, \delta z \right.$$
$$\left. \left\{ p_{xz} + \tfrac{1}{2}\, \delta x \left(\frac{\partial p_{xz}}{\partial x}\right)\right\} \delta y \delta z \right]$$

Similarly for the parallel plane through $P_1\,(x-\frac{1}{2}\,\delta x, y, z)$ the corresponding components are,

$$\left[-\left\{ p_{xx} - \tfrac{1}{2}\, \delta x \left(\frac{\partial\, p_{xx}}{\partial x}\right)\right\} \delta y\, \delta z,\ -\left\{ p_{xy} - \tfrac{1}{2} \delta x \left(\frac{\partial\, p_{xy}}{\partial x}\right)\right\} \delta y\, \delta z, \right.$$
$$\left. -\left\{ p_{xz} - \tfrac{1}{2}\, \delta x \frac{\partial\, p_{xz}}{\partial x}\right\} \delta y\, \delta z \right].$$

(Since $-\mathbf{i}$ is the unit normal drawn outwards from the fluid element).

The forces on the parallel planes through P_1 and P_2 are equivalent to a single force at P having components

$$\left\{\frac{\partial p_{xx}}{\partial x}, \frac{\partial p_{xy}}{\partial x}, \frac{\partial p_{xz}}{\partial x}\right\} \delta x\, \delta y\, \delta z.$$

together with the couple whose moments are

$$-p_{xz}\, \delta x\, \delta y\, \delta z \quad \text{about } OY.$$

and

$$+p_{xy}\, \delta x\, \delta y\, \delta z \quad \text{about } OZ.$$

Similarly the pair of faces perp. to Y axis give a force at P having components

$$\left[\frac{\partial p_{yx}}{\partial y}, \frac{\partial p_{yy}}{\partial y}, \frac{\partial p_{yz}}{\partial y}\right] \delta x\, \delta y\, \delta z$$

together with couples of moments

$$-p_{yx}\, \delta x\, \delta y\, \delta z \text{ about } OZ.$$

$$+p_{yz}\, \delta x\, \delta y\, \delta z \text{ about } OX.$$

and the pair of faces perp. to the Z-axis give a force at P having components

$$\left[\frac{\partial p_{zx}}{\partial z}, \frac{\partial p_{zy}}{\partial z}, \frac{\partial p_{zz}}{\partial z}\right] \delta x\, \delta y\, \delta z$$

together with couples of moments.

$$-p_{zy}\, \delta x\, \delta y\, \delta z \text{ about } OX.$$

$$+p_{zx}\, \delta x\, \delta y\, \delta z \text{ about } OY.$$

Thus the surface forces on all six faces of the cuboid reduce to a single force at P having components.

$$\left[\left(\frac{\partial p_{xx}}{\partial x}+\frac{\partial p_{yx}}{\partial y}+\frac{\partial p_{zx}}{\partial z}\right), \left(\frac{\partial p_{xy}}{\partial x}+\frac{\partial p_{yy}}{\partial y}+\frac{\partial p_{zy}}{\partial z}\right), \left(\frac{\partial p_{xz}}{\partial x}+\frac{\partial p_{yz}}{\partial y}+\frac{\partial p_{zz}}{\partial z}\right)\right] \delta x\, \delta y\, \delta z$$

together with a vector couple having cartesian components

$$\{(p_{yz}-p_{zy}), (p_{zx}-p_{xz}), (p_{xy}-p_{yx})\}\, \delta x\, \delta y\, \delta z$$

Consider the external body forces are $(X\ Y\ Z)$ per unit mass at the point P. Then the total body force on the element has components

$$(X\ Y\ Z)\ \rho\delta x\, \delta y\, \delta z.$$

The total force component acting on fluid element P along the $\mathbf{i}$—direction

$$=\left(\frac{\partial p_{xx}}{\partial x}+\frac{\partial p_{yx}}{\partial y}+\frac{\partial p_{zx}}{\partial z}\right) \delta x\, \delta y\, \delta z+\rho\, X\, \delta x\, \delta y\, \delta z.$$

Let $\mathbf{q}$ $(u\ v\ w)$ be the velocity at the point P at any time t, then the equation of motion along the $\mathbf{i}$ direction

$$\left(\frac{\partial p_{xx}}{\partial x}+\frac{\partial p_{yx}}{\partial y}+\frac{\partial p_{zx}}{\partial z}\right) \delta x\, \delta y\, \delta z+\rho\, X\, \delta x\, \delta y\, \delta z$$

$$=(\rho\, \delta x\, \delta y\, \delta z)\, \frac{du}{dt}$$

or $$\left(\frac{\partial p_{xx}}{\partial x}+\frac{\partial p_{yx}}{\partial y}+\frac{\partial p_{zx}}{\partial z}\right)+\rho X=\rho\frac{du}{dt}$$

Since $u=u(x\ y\ z\ t)$

and $$\frac{du}{dt}=\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}$$

Thus we have the equations of motion in the direction of **i, j** and **k**

or $$\frac{du}{dt}=X+\frac{1}{\rho}\left(\frac{\partial p_{xx}}{\partial x}+\frac{\partial p_{yx}}{\partial y}+\frac{\partial p_{zx}}{\partial z}\right)$$

or $$\frac{dv}{dt}=Y+\frac{1}{\rho}\left(\frac{\partial p_{xy}}{\partial x}+\frac{\partial p_{yy}}{\partial y}+\frac{\partial p_{zy}}{\partial z}\right)$$

or $$\frac{dw}{dt}=Z+\frac{1}{\rho}\left(\frac{\partial p_{xz}}{\partial x}+\frac{\partial p_{yz}}{\partial y}+\frac{\partial p_{zz}}{\partial z}\right)$$

Can be represented in tensor form

$$\frac{\partial u_i}{\partial t}+u_j\,u_{i,j}=X_i+\frac{1}{\rho}\,p_{ji,j}$$

$$(i\ j=1,2,3)$$

where $x_i \Rightarrow$ the co-ordinate

$u_i \Rightarrow$ the velocity components.

$X_i \Rightarrow$ the external body force components.

§ 9·61. Isotropic fluid.

A fluid is said to be **isotropic** if its physical properties at a point do not depend on direction *i.e.* they are invarient to orientation of the co-ordinate axes.

To prove that for an isotropic fluid the principal axis of the stress quadric and rate of strain quadric coincide.

Equation to the rate of strain quadric referred to its principal axis is,

$$e_1x^2+e_2y^2+e_3z^2=\text{constant.} \qquad \ldots(1)$$

Consider a rectangle in a principal plane of the rate of strain quadric. Since the fluid is isotropic then this rectangle will not be deformed hence there will be no shear in this plane. Consequently the tangential stress in this plane should vanish. Thus this plane plane is also a principal plane of the stress quadric.

Similarly in a principal plane of the stress quadric there can not be any tangential shear in an isotropic fluid. If tangential shear in a principal plane of stress quadric is zero, then there is no shear in it and hence it will be a principal plane of the rate of strain quadric.

§ 9·62. **Relation between rate of strain tensor and stress tensor.** *(Stoke's fluid)*

Stoke's made an assumption that the stress components are linear functions of rate of strain components. However, such a relation can be regarded as a first approximation to a more general one in which higher powers of rate of strain components may occur. Now we shall consider that the most general linear isotropic relations between the stress components and rate of strain components are of the following forms

$$\left.\begin{aligned} P_{11}&=-p+Ae_{11}+B\,(e_{22}+e_{33})\\ P_{22}&=-p+Ae_{22}+B\,(e_{33}+e_{11})\\ P_{33}&=-p+Ae_{33}+B\,(e_{11}+e_{22})\\ P_{ij}&=(A-B)\,e_{ij}\ i\neq j. \end{aligned}\right\} \quad \ldots\text{(i)}$$

Assuming that

$$P_{11}=A_0+Ae_{11}+Be_{22}+Ce_{33}+De_{23}+Ee_{31}+Fe_{12} \quad \ldots\text{(ii)}$$

where A, B, C, D, E, F are constants and A_0 is some isotropic function of coordinates.

The interchange of axis 2 and 3 will not change p_{11}, e_{11} and e_{23} *while e_{22} and e_{31} interchange with e_{33} and e_{12} respectively.*

So $\qquad B=C \quad \text{and} \quad E=F. \qquad \ldots\text{(iii)}$

(The axis 1 cannot be changed as we have to determine P_{11}).

Again the reversal of the direction of axis 2, will not change e_{11}, e_{22}, e_{33}, e_{31} and P_{11} while e_{23} and e_{12} change in sign.

So $\qquad D=E=0, \qquad \ldots\text{(iv)}$

consequently (ii) reduces to

$$p_{11}=A_0+Ae_{11}+B\,(e_{22}+e_{33}) \qquad \{\text{from (iii) and (iv)}$$

or $$p_{11}=A_0+(A-B)\,e_{11}+B\,(e_{11}+e_{22}+e_{33})$$

or $$p_{11}=A_0+(A-B)\,e_{11}+2B\Delta.$$

Consider $A-B=\mu$ and $2B=\lambda$ $\qquad \ldots\text{(v)}$

or $$p_{11}=A_0+\mu e_{11}+\lambda\Delta.$$

Similarly, we can have

$$p_{22}=A_0'+A'e_{22}+B'\,(e_{33}+e_{11})$$

$$p_{22}=A_0'+(A'-B')\,e_{22}+B'\,(e_{11}+e_{22}+e_{33}) \quad \ldots\text{(vi)}$$

and $$p_{33}=A_0''+(A''-B'')\,e_{33}+B''\,(e_{11}+e_{22}+e_{33}). \quad \ldots\text{(vii)}$$

For isotropy p_{11} is some function of e_{11} as p_{22} is of e_{22} and p_{33} is of e_{33}, which will give $A=A'=A''$.

Again p_{11} is some function of $(e_{22}+e_{33})$ as p_{22} is of $(e_{33}+e_{11})$ or p_{33} is of $(e_{11}+e_{22})$, which will give $B=B'=B''$.

Similarly isotropy of the fluid will require

$$A_0=A_0'=A_0''$$

then (vi) and (vii) reduces to ...(viii)

$$p_{22}=A_0+\mu e_{22}+\lambda\triangle \quad \text{...(ix)}$$

and

$$p_{33}=A_0+\mu e_{33}+\lambda\triangle.$$

Considering the fluid is at rest *i.e.* in this case there will be no component of velocity. Let each of the normal stresses coincides with $-p$

$$p_{11}=p_{22}=p_{33}=A_0=-p.$$

The relation (v), (viii) and (ix) can be written as

$$p_{xx}=-p+2\mu\frac{\partial u}{\partial x}+\lambda\triangle$$

$$p_{yy}=-p+2\mu\frac{\partial v}{\partial y}+\lambda\triangle$$

$$p_{zz}=-p+2\mu\frac{\partial w}{\partial z}+\lambda\triangle$$

where

$$\triangle=\textbf{div q}.$$

$$\begin{cases}\triangle=0 & \text{for incompressible flow}\\ \lambda=-\frac{2}{3}\mu & \text{for compressible flow}\end{cases}$$

Also, we have

$$p_{yz}=p_{zy}=\mu\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right)$$

$$p_{zx}=p_{xz}=\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)$$

$$P_{xy}=p_{yx}=\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)$$

which are true for compressible and incompressible fluids.

Again, assuming that

$$P_{23}=A_0+Ae_{11}+Be_{22}+Ce_{33}+De_{23}+Ee_{31}+Fe_{12}$$

where A, B, C, D, E and F are constants where A_0 is some isotropic function of the coordinates.

The interchange in axes 2 and 3 does not change P_{23}, e_{11}, e_{23} but e_{22} and e_{31} will interchange with e_{33} and e_{12} respectively.

Thus

$$B=C,\ E=F.$$

Also the reversal of the direction of axis 2, will not change e_{11}, e_{22}, e_{33}, e_{31} while P_{23}, e_{23}, e_{12} will change sign. Thus

$$A_0=A=B=C=E=F=0.$$

So

$$P_{23}\propto e_{23}.$$

Similarly, we can prove ...(x)

$$P_{ij}=k\,e_{ij}.\quad \text{If}\quad i\neq j$$

where k is some constant.

From (v), (viii) and (ix), we have

$$k=\mu$$

or $$p_{ii}=-p+\mu e_{ii}+\lambda\triangle \quad \text{if} \quad k=\mu. \qquad \text{...(xi)}$$

Now the isotropic linear relations between the stress components and the rate of strain components is given by

$$P_{\alpha\beta}=(-p+\lambda\triangle)\,\delta_{\alpha\beta}+\mu e_{\alpha\beta}$$
$$=(-p-\tfrac{2}{3}\mu\triangle)\,\delta_{\alpha\beta}+\mu e\triangle+\,\delta_{\alpha\beta}. \qquad \text{...(xii)}$$

where $\delta_{\alpha\beta}$ are kronecker delta with the property that

$$\delta_{\alpha\beta}=\begin{cases} 1 & \text{if} \quad \alpha=\beta \\ 0 & \text{if} \quad \alpha\neq\beta \end{cases}$$

and $$k=\lambda+\tfrac{2}{3}\mu \qquad \text{...(xiii)}$$

which determines $P_{\alpha\beta}$ assuming that A, B, C, D, E, F are constants in order to keep the stress-strain relationship linear. This relation is also true even if A, B, C, D, E, F are to be functions invariants of tensor.

Since $$P_{ij}=\mu\, e_{ij} \quad i\neq j$$

then μ can be determined with the coefficient of viscosity while *k is called the bulk modulus of Viscosity.*

If $k=0$, then $\lambda=-\frac{2}{3}\mu$ (for compressibe flow)
which is known as **Stoke's relation.**

When $k=0$, then $\frac{1}{3}\,p_{ii}=-p$

where $(-p)$ is the mean of the normal stresses.

Converse. If $(-p)$ is the mean of the normal stresses then $k=0$ that is $\exists$ an universal relation between the two coefficients of viscosity λ and μ.

§ 9·63. **Inviscid fluids.** When $\lambda=\mu=0$,

then $$p_{11}=p_{22}=p_{33}=-p$$

$$p_{ij}=0 \qquad \forall \quad i\neq j$$

known as inviscid fluid or an ideal fluid. There acts onlynormal stresses and they are same in all directions.

§ 9 64. **Newtonian fluids.** The fluids in which the stress components are linear functions of rate of strain components are called **Newtonian fluids.**

§ 9·7. Navier-Stokes equations of Motion of a Viscous fluid.

We know that the equation of translation motion of fluid element is

$$\frac{du}{dt}=X+\frac{1}{\rho}\left(\frac{\partial p_{xx}}{\partial x}+\frac{\partial p_{yx}}{\partial y}+\frac{\partial p_{zx}}{\partial z}\right). \qquad \ldots\text{(i)}$$

{Ref. § 9·6

We know that

$$p_{xx}=-p+2\mu\frac{\partial u}{\partial x}+\lambda\triangle$$

$$p_{yx}=\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right) \qquad \Big\{ \text{Ref. § 9·62}$$

$$p_{zx}=\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right).$$

Substituting the above values in (1), we have

$$\frac{du}{dt}=X+\frac{1}{\rho}\left[\frac{\partial}{\partial x}\left(-p+2\mu\frac{\partial u}{\partial x}+\lambda\triangle\right)+\frac{\partial}{\partial y}\left\{\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right\}+\frac{\partial}{\partial z}\left\{\mu\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right\}\right]$$

or $$\frac{du}{dt}=X+\frac{1}{\rho}\left[-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}\left(2\mu\frac{\partial u}{\partial x}+\lambda\triangle\right)+\mu\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)+\mu\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\right]$$

or $$\frac{du}{dt}=X-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\triangle^2 u+\left(\nu+\frac{\lambda}{\rho}\right)\frac{\partial\triangle}{\partial x}. \qquad \ldots\text{(ii)}$$

Since $\lambda=-\frac{2}{3}\mu$ for compressible fluid and $\triangle=0$ for an incompressible fluid, then (ii) reduces to

$$\frac{du}{dt}=X-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\triangle^2 u+\tfrac{1}{3}\nu\frac{\partial\triangle}{\partial x}$$

as $\nu=\frac{\mu}{\rho}$

or $\nu+\frac{\lambda}{\rho}=\frac{\mu}{\rho}+\frac{\lambda}{\rho}$

$=\frac{\mu}{\rho}-\frac{2}{3}\frac{\mu}{\rho}$

$=\frac{1}{3}\nu$

Thus the equations of motion along the co-ordinate axes are given by

$$\frac{du}{dt}=X-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\triangle^2 u+\tfrac{1}{3}\nu\frac{\partial\triangle}{\partial x}$$

Similarly $$\frac{dv}{dt}=Y-\frac{1}{\rho}\frac{\partial p}{\partial y}+\nu\triangle^2 v+\tfrac{1}{3}\nu\frac{\partial\triangle}{\partial y} \qquad \ldots\text{(iii)}$$

and $$\frac{dw}{dt}=Z-\frac{1}{\rho}\frac{\partial p}{\partial z}+\nu\triangle^2 w+\tfrac{1}{3}\nu\frac{\partial\triangle}{\partial z}$$

known as **Navier-Stoke's equations of motion.**

These equations can be written in tensor form as

$$\frac{du_i}{dt}=X_i-\frac{1}{\rho}\,p_{,i}+\nu\,u_{i,jj}+\tfrac{1}{3}\nu\,\nabla_{,i}.$$

The relation (iii), can also be represented in vectorial form

$$\frac{d\mathbf{q}}{dt}=\mathbf{F}-\nabla\int\frac{dp}{\rho}+\nu\nabla^2\,\mathbf{q}+\tfrac{1}{3}\nu\,\nabla\,(\nabla\cdot\mathbf{q}) \qquad \text{...(iv)}$$

where $\quad \mathbf{q}=(x\ y\ z)\quad$ and $\quad \mathbf{F}=(X, Y, Z)$.

$$\left\{\text{Since}\quad \frac{d\mathbf{q}}{dt}=\frac{\partial\mathbf{q}}{\partial t}+(\mathbf{q}\cdot\nabla)\,\mathbf{q}=\frac{\partial\mathbf{q}}{\partial t}+\nabla\,(\tfrac{1}{2}\mathbf{q}^2)-\mathbf{q}\times(\nabla\times\mathbf{q})\right.$$

$$\{\text{and}\quad \nabla\times(\nabla\times\mathbf{q})=\nabla\,(\nabla\cdot\mathbf{q})-\nabla^2\mathbf{q}$$

thus the equation (iv) reduces to,

$$\frac{\partial\mathbf{q}}{\partial t}+\nabla\,(\tfrac{1}{2}\mathbf{q}^2)-\mathbf{q}\times(\nabla\times\mathbf{q})=\mathbf{F}-\nabla\int\frac{dp}{\rho}+\nu\,[\nabla\,(\nabla\cdot\mathbf{q})$$
$$-\nabla\times(\nabla\times\mathbf{q})]+\tfrac{1}{3}\nu\nabla\,(\nabla\cdot\mathbf{q})$$

$$\frac{\partial\mathbf{q}}{\partial t}+\nabla\,(\tfrac{1}{2}\mathbf{q}^2)-\mathbf{q}\times(\nabla\times\mathbf{q})=\mathbf{F}-\nabla\int\frac{dp}{\rho}+\frac{4}{3}\nu\nabla\,(\nabla\cdot\mathbf{q})$$
$$-\nu\nabla\times(\nabla\times\mathbf{q}) \qquad \text{...(v)}$$

which is another form of **Navier-Stoke's equation of motion.**

For incompressible flow, the relations (iv) and (v) reduce to

$$\frac{d\mathbf{q}}{dt}=\mathbf{F}-\frac{1}{\rho}\nabla p+\nu\,\nabla^2\mathbf{q}$$
$$=\mathbf{F}-\frac{1}{\rho}\nabla p-\nu\nabla\times(\nabla\times\mathbf{q}). \qquad \text{...(vi)}$$

Boundary Conditions :

The equation (vi) represents that for an incompressible flow the equation of motion differs from Euler's equation of motion in inviscid flow by the form $-\nu\nabla\times(\nabla\times\mathbf{q})$. This term arises due to Viscosity which increases the order of differential equation and therefore an additional boundary condition is needed. This is satisfied by the condition that there must be *no slip between a viscous fluid and its boundary. So at fixed boundary* $\mathbf{q}=0$. *It follows that the normal and tangential velocity components both must vanish.*

§ 9·71. **Equations for vorticity and circulation.**

We know that the Navier-stoke's equation of motion is

$$\frac{\partial\mathbf{q}}{\partial t}+\nabla\,(\tfrac{1}{2}\mathbf{q}^2)-\mathbf{q}\times(\nabla\times\mathbf{q})=\mathbf{F}-\frac{1}{\rho}\nabla\,p+\nu\nabla^2\mathbf{q}$$

Let the external forces are conservative and density is a function of pressure only.

then $\qquad \vec{\zeta}=\nabla\times\mathbf{q}$

or $\qquad \dfrac{\partial\mathbf{q}}{\partial t}-\mathbf{q}\times\vec{\zeta}=-\nabla\left[\Omega+\int\dfrac{dp}{\rho}+\tfrac{1}{2}q^2\right]+\nu\nabla^2\mathbf{q}$

Taking curl of both the sides we have

$$\text{curl}\ \frac{\partial\mathbf{q}}{\partial t}-\text{curl}\ (\mathbf{q}\times\vec{\zeta})=\nu\ \text{curl}\ (\nabla^2\mathbf{q})$$

or $\qquad \dfrac{\partial\vec{\zeta}}{\partial t}+(\mathbf{q}\cdot\nabla)\ \vec{\zeta}-(\vec{\zeta}.\nabla)\ \mathbf{q}=\nu\nabla^2\vec{\zeta}$

$\left\{\begin{array}{l}\text{as}\quad \text{div.}\ \zeta\\ \quad =\text{div. curl}\ \mathbf{q}\\ \quad =0\end{array}\right.$

or $\qquad \dfrac{d\vec{\zeta}}{dt}=(\vec{\zeta}.\nabla)\ \mathbf{q}+\nu\nabla^2\vec{\zeta}$

Which is known the **equation to vorticity.**

Let Γ be the circulation round a closed circuit,

then $\qquad \Gamma=\int_C udx+vdy+wdz$

or $\qquad \dfrac{D\Gamma}{Dt}=\dfrac{D}{Dt}\int_C udx+vdy+wdz$

$$\frac{D\Gamma}{Dt}=\int_C\left(\frac{Du}{Dt}dx+\frac{Dv}{Dt}dy+\frac{Dw}{Dt}dz\right)+\int_C(udu+vdv+wdw)$$

{ The second integral vanishes as circuit being closed.

or $\qquad \dfrac{D\Gamma}{Dt}=\int_C\left[-\dfrac{\partial}{\partial x}\left(\dfrac{p}{\rho}+V\right)dx-\dfrac{\partial}{\partial y}\left(\dfrac{p}{\rho}+V\right)dy-\dfrac{\partial}{\partial z}\left(\dfrac{p}{\rho}+V\right)dz+\nu\ (\nabla^2\ udx+\nabla^2\ vdy+\nabla^2\ wdz)\right]$

$$=-\int_C d\left(\frac{p}{\rho}+V\right)+\nu\int_C\nabla^2\ (udx+vdy+wdz)$$

$$=\nu\ \nabla^2\int_C udx+vdy+wdz$$

$\left\{\text{as}\ \Gamma=\int_C udx+vdy+wdz\right.$

$=\nu\ \nabla^2\Gamma$ (other integral vanishes for a closed circuit)

§ 9·72. Equations of motion in cylindrical polar coordinates. We know that the Navier-stoke's equation is

$$\frac{\partial \mathbf{q}}{\partial t}+\nabla\,(\tfrac{1}{2}\mathbf{q}^2)-\mathbf{q}\times(\nabla\times\mathbf{q})$$

$$=\mathbf{F}-\frac{1}{\rho}\nabla\,p+\nu\nabla\times(\nabla\times\mathbf{q}) \qquad \ldots(\text{i})$$

Let (r, θ, z) be the coordinates of a point, then it reduces to

$$\frac{d}{dt}\,(q_r)-\frac{q_\theta^2}{r}=-\frac{\partial\Omega}{\partial r}-\frac{1}{\rho}\,\frac{\partial p}{\partial r}$$

$$+\nu\left(\nabla^2\,q_r-\frac{2}{r^2}\,\frac{\partial q_\theta}{\partial\theta}-\frac{q_r}{r^2}\right)$$

$$\frac{d}{dt}\,(q_\theta)+\frac{q_r q_\theta}{r}=-\frac{\partial\Omega}{r\partial\theta}-\frac{1}{\rho}\,\frac{\partial p}{r\partial\theta}$$

$$+\nu\left(\nabla^2 q_\theta+\frac{2}{r^2}\,\frac{\partial q_r}{\partial\theta}-\frac{q_\theta}{r^2}\right)$$

and $$\frac{d}{dt}\,(q_z)=-\frac{\partial\Omega}{\partial z}-\frac{1}{\rho}\,\frac{\partial p}{\partial z}+\nu\,\nabla^2\,q_z$$

where $$\frac{d}{dt}=\frac{\partial}{\partial t}+q_r\frac{\partial}{\partial r}+q_\theta\frac{\partial}{r\partial\theta}+q_z\frac{\partial}{\partial z}$$

and $$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\,\frac{\partial}{\partial r}+\frac{1}{r^2}\,\frac{\partial^2}{\partial\theta^2}+\frac{\partial^2}{\partial z^2}$$

(iii) Spherical Polar Coordinates.

$$\frac{dq_r}{dt}-\frac{1}{r}\,(q_\theta^2+q_\phi^2)=-\frac{\partial\Omega}{\partial r}-\frac{1}{\rho}\,\frac{\partial p}{\partial r}$$

$$+\nu\left(\nabla^2\,q_r-\frac{2q_r}{r^2}-2\,\frac{\cot\theta}{r^2}\,q_\theta-\frac{2}{r^2}\,\frac{\partial q_\theta}{\partial\theta}\right.$$

$$\left.-\frac{2}{r^2\sin\theta}\,\frac{\partial q_\phi}{\partial\phi}\right)$$

Navier stoke's equation can be written as

$$\frac{\partial\mathbf{V}}{\partial t}+\text{grad.}\,(\tfrac{1}{2}\mathbf{V}^2)-2\mathbf{V}\times\omega=-F-\frac{1}{\rho}\,\text{grad.}\,p+\nu\,\nabla^2\mathbf{V}$$

Now curl $\mathbf{V}=2\,(\xi,\eta,\zeta)$

$$=2\left(\frac{\partial\zeta}{\partial y}-\frac{\partial\eta}{\partial z}\right)$$

$$=\frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)-\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)$$

$$=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)-\nabla^2\,u=-\nabla^2\,u$$

Thus the equation reduces to, if $\mathbf{F}=-\text{grad. }\mathbf{V}$

or $$\frac{\partial\mathbf{V}}{\partial t}-2\mathbf{V}\times\omega=-\text{grad.}\left(\frac{p}{\rho}+V+\tfrac{1}{2}\mathbf{V}^2\right)-2\nu\,\text{curl}\,\vec{\omega}$$

and $$\frac{\partial q_\theta}{\partial t}-\frac{q_\phi^2 \cot\theta}{r}+\frac{q_r q_\theta}{r}=-\frac{1}{r}\frac{\partial\Omega}{\partial\theta}-\frac{1}{\rho}\frac{\partial p}{r\partial\theta}$$
$$+\nu\left(\nabla^2 q_\theta-\frac{q_\theta}{r^2\sin^2\theta}+\frac{2}{r^2}\frac{\partial q_r}{\partial\theta}-\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial q_\phi}{\partial\phi}\right)$$

and $$\frac{\partial q_\phi}{\partial t}+\frac{q_r q_\phi}{r}+\frac{q_\theta q_\phi\cot\theta}{r}=-\frac{\partial\Omega}{r\sin\theta\partial\phi}-\frac{1}{\rho}\frac{\partial p}{r\sin\theta\partial\phi}$$
$$+\nu\left(\nabla^2 q_\phi-\frac{q_\phi}{r^2\sin^2\theta}+\frac{2}{r^2\sin\theta}\frac{\partial q_r}{\partial\phi}\right.$$
$$\left.+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial q_\theta}{\partial\phi}\right)$$

where $$\frac{d}{dt}=\frac{\partial}{\partial t}+q_r\frac{\partial}{\partial r}+q_\theta\cdot\frac{\partial}{r\partial\theta}+q_\phi\frac{\partial}{r\sin\theta\partial\phi}$$

and $$\nabla^2=\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}$$
$$+\frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}$$

(iii) Orthogonal curvilinear coordinates.

We know
$$dS^2=(h_1\,d\lambda_1)^2+(h_2\,d\lambda_2)^2+(h_3\,d\lambda_3)^2$$

then $$\nabla\phi=\frac{1}{h_1}\frac{\partial\phi}{\partial\lambda_1}l_1+\frac{1}{h_2}\frac{\partial\phi}{\partial\lambda_2}m_1+\frac{1}{h_3}\frac{\partial\phi}{\partial\lambda_3}.n_1$$

$$\nabla\times F=\frac{1}{h_1h_2h_3}\begin{vmatrix} h_1l_1 & h_2m_1 & h_3n_1\\ \dfrac{\partial}{\partial\lambda_1} & \dfrac{\partial}{\partial\lambda_2} & \dfrac{\partial}{\partial\lambda_3}\\ h_1F_1 & h_2F_2 & h_3F_3\end{vmatrix}$$

§ 9·73. Energy Dissipation due to Viscosity.

Consider a particle of viscous fluid of fixed mass $\rho\delta\nu$ moving at any time t with velocity $\mathbf{q}$. The kinetic energy is
$$=\tfrac{1}{2}(\rho\ \delta\nu)\ \mathbf{q}^2$$

Rate of gain of kinetic energy at any time t
$$=\frac{d}{dt}\left\{\tfrac{1}{2}(\rho\delta\nu)\ \mathbf{q}^2\right\}$$
$$=\rho\delta\nu\ \mathbf{q}\cdot\frac{d\mathbf{q}}{dt}$$

Thus the total rate of gain of kinetic energy of the entire fluid of volume V is
$$=\int_V \rho\mathbf{q}\cdot\left(\frac{d\mathbf{q}}{dt}\right)d\nu$$
$$=\rho\int_V \mathbf{q}\cdot\left(\frac{d\mathbf{q}}{dt}\right)d\nu \qquad \text{for an incompressible fluid.}$$

We know that the Navier-stoke's equations for a viscous fluid is

$$\frac{d\mathbf{q}}{dt}=\mathbf{F}-\frac{1}{\rho}\nabla p-\nu\nabla\times(\nabla\times\mathbf{q}) \qquad \ldots\text{(i)}$$

Multiply both the sides of (i) scalarly by $\rho\mathbf{q}\,dv$ and integrating over the volume V of the fluid.

$$\int_V \rho\mathbf{q}.\frac{d\mathbf{q}}{dt}dv=\int\mathbf{q}.\mathbf{F}\,\rho dv-\int\nabla.(\rho\mathbf{q})\ dv-\nu.\rho\int\mathbf{q}.\{\nabla\times(\nabla\times\mathbf{q})\}dv$$

Thus the rate of energy dissipation (E) due to viscosity is

$$E=\mu\int_V\mathbf{q}.\{\nabla\times(\nabla\times\mathbf{q})\}\ dv \qquad \left\{\text{as}\quad \nu=\frac{\mu}{\rho}\right.$$

$$\left\{\begin{array}{l}\text{We know that}\\ \nabla.\{\mathbf{q}\times\text{curl }\mathbf{q})=(\text{curl }\mathbf{q})^2\\ \qquad\qquad -\mathbf{q}.\{\nabla\times(\nabla\times\mathbf{q})\}\end{array}\right.$$

or
$$E=\mu\int_V(\nabla\times\mathbf{q})^2\,dv-\mu\int_V\nabla.(\mathbf{q}\times\text{curl }\mathbf{q})\ dv$$

$$E=\mu\int_V(\nabla\times\mathbf{q})^2\,dv-\mu\int_S\mathbf{n}.(\mathbf{q}\times\text{curl }\mathbf{q})\ ds$$

{ Changing from volume integral to surface integral (where S is the total surface enclosing the volume V).

When the boundary S is at rest, and there is no slip between fluid and boundary

then $\qquad \mathbf{q}=0$ on the surface S

thus
$$E=\mu\int_V(\nabla\times\mathbf{q})^2\,dv$$

$$=\mu\int_V(\text{curl }\mathbf{q})^2\,dv=\mu\int_V\zeta^2\,dv$$

Ex. 1. *Prove that*

$$\left(\nu\nabla^2-\frac{\partial}{\partial t}\right)\nabla^2\psi=\frac{\partial\,(\psi,\nabla^2\psi)}{\partial\,(x,y)}$$

where ψ is a stream function for a two-dimensional motion of a viscous liquid.

We know that the Navier stoke's equations for compressible viscous fluid.

$$\frac{\partial\mathbf{q}}{\partial t}+(\mathbf{q}.\nabla)\ \mathbf{q}=-\frac{1}{\rho}\nabla p+\nu\nabla^2\mathbf{q}$$

(since external body forces are absent).

or
$$\frac{\partial\mathbf{q}}{\partial t}-\mathbf{q}\times(\nabla\times\mathbf{q})=-\nabla\left(\frac{p}{\rho}+\tfrac{1}{2}\mathbf{q}^2\right)+\nu\nabla^2\mathbf{q} \qquad \ldots\text{(i)}$$

Taking curl of the relation (i) both the sides, we have

$$\frac{\partial \vec{\zeta}}{\partial t}-\text{curl }(\mathbf{q}\times\vec{\zeta})=\nu \text{ curl } \nabla^2 \mathbf{q}$$

$$\{ \text{as } \vec{\zeta}=\nabla\times\mathbf{q}$$

or $$\frac{\partial \vec{\zeta}}{\partial t}+(\mathbf{q}.\nabla)\vec{\zeta}-(\vec{\zeta}.\nabla)\,\mathbf{q}=\upsilon\,\nabla^2\vec{\zeta} \quad ...(\text{i})$$

Since there is a two dimensional motion of a viscous fluid

then $$\mathbf{q}=(u, v, 0)$$

and $$\vec{\zeta}=(0, 0, \zeta)$$

Now (ii) can be written as

$$\left(\upsilon\,\nabla^2-\frac{\partial}{\partial t}\right)\vec{\zeta}=\left(u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}\right)\vec{\zeta} \quad ...(\text{iii})$$

The stream function ψ exist, then

$$u=-\frac{\partial\psi}{\partial y} \text{ and } v=\frac{\partial\psi}{\partial x}$$

or $$\zeta=\nabla^2\psi$$

Substituting the value of ζ in (iii), we have

$$\left(\upsilon\,\nabla^2-\frac{\partial}{\partial t}\right)\nabla^2\psi=v\,\frac{\partial\vec{\zeta}}{\partial y}+u\,\frac{\partial\vec{\zeta}}{\partial x}$$

or $$\left(\upsilon\,\nabla^2-\frac{\partial}{\partial t}\right)\nabla^2\psi=\frac{\partial\psi}{\partial x}\cdot\frac{\partial}{\partial y}\nabla^2\psi-\frac{\partial\psi}{\partial y}\cdot\frac{\partial}{\partial x}\nabla^2\psi$$

or $$\left(\upsilon\,\Delta^2-\frac{\partial}{\partial t}\right)\nabla^2\psi=\frac{\partial}{\partial x}(\psi)\cdot\frac{\partial}{\partial y}(\nabla^2\psi)-\frac{\partial}{\partial y}(\psi)\frac{\partial}{\partial x}(\nabla^2\psi)$$

or $$\left(\upsilon\,\nabla^2-\frac{\partial}{\partial t}\right)\nabla^2\phi=\frac{\partial\,(\psi,\,\nabla^2\psi)}{\partial\,(x\;y)}$$ **Proved.**

Ex. 2. *Prove that, in the slow steady motion of a viscous liquid in two dimensions*

$$\upsilon\,\nabla^4\psi=\frac{\partial X}{\partial x}-\frac{\partial Y}{\partial y}$$ *where (X, Y) is the impressed force per unit area.*

We know that the Navier-stoke's equation of motion is

$$\frac{\partial\mathbf{q}}{\partial t}+(\mathbf{q}.\nabla)\,\mathbf{q}=F-\frac{1}{\rho}\nabla p+\upsilon\nabla^2\mathbf{q} \quad ...(\text{i})$$

Here $\frac{\partial\mathbf{q}}{\partial t}=0$, motion being steady. Also the inertia term $(\mathbf{q}.\nabla)\,\mathbf{q}$ is negligible, that of slow motion. Since the motion of the liquid is in two dimensions, so

$\mathbf{F}=(X\,Y) \Rightarrow$ Impressed force or external body force

$\mathbf{q}=(u, v) \Rightarrow$ Components of the velocity.

The equation (1) reduces to

$$\mathbf{F}-\frac{1}{\rho}\nabla p+\nu\nabla^2\mathbf{q}=0 \qquad \text{...(ii)}$$

Taking curl of the above relation, we have

$$\text{Curl } \mathbf{F}+\nu\nabla^2 \text{ curl } \mathbf{q}=0$$

or
$$\text{Curl } \mathbf{F}+\nabla\nu^2\vec{\zeta}=0 \qquad \text{...(iii)}$$

Thus
$$\vec{\zeta}=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}.$$

Since $\exists$ a stream function ψ, therefore, we have

$$u=-\frac{\partial \psi}{\partial y},\ v=\frac{\partial \psi}{\partial x}$$

$$\therefore\quad \vec{\zeta}=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$$

$$\vec{\zeta}=\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=\nabla^2\psi$$

From (iii), we have

$$\text{Curl } \mathbf{F}=-\upsilon\nabla^4\psi$$

or
$$\nu\nabla^4\psi=-\text{Curl } \mathbf{F}$$

or
$$\upsilon\nabla^4\psi=\frac{\partial X}{\partial y}-\frac{\partial Y}{\partial x}.$$

Ex. 3. *Prove that for a liquid filling up a vessel in the form of surface of revolution which is rotating about its axis (Z-axis) with angular velocity ω, the rate of dissipation of energy has an additional term*

$$2\mu\omega\iint(lDu+mDv)\ dS$$

Where $D=\left(y\dfrac{\partial}{\partial x}-x\dfrac{\partial}{\partial y}\right)$, *$(l, m, n)$ are the direction cosines of the inward normal.*

Since the liquid rotates about Z-axis with an angular velocity ω.

Here $\quad u=-\omega y,\ v=\omega x,\ w=0.$

Consider the additional term is

$$=4\mu\iiint\left(\frac{\partial v}{\partial x}\cdot\frac{\partial u}{\partial y}-\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y}\right)dx\,dy\,dz \qquad \text{...(i)}$$

$$=4\mu \iiint \left\{ \frac{\partial}{\partial x}\left(v \frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial y}\left(v \frac{\partial u}{\partial x}\right) \right\} dx\, dy\, dz$$

$$=-4\mu \iint \left\{ lv \frac{\partial u}{\partial y} - mv \frac{\partial u}{\partial x} \right\} dS$$

$$=-4\mu \iint v\left(l \frac{\partial u}{\partial y} - m \frac{\partial u}{\partial x}\right) dS$$

$$=-4\mu \iint v\left(l \frac{\partial u}{\partial y} + m \frac{\partial v}{\partial y}\right) dS$$

$$=-4\mu w \iint x\left(l \frac{\partial u}{\partial y} + m \frac{\partial v}{\partial y}\right) dS \qquad \text{...(ii)}$$

{ as $v=\omega x$

{ from the equation of continuity

(i) can also be represented, as follows

$$=4\mu \iiint \left\{ \frac{\partial}{\partial y}\left(u \frac{\partial v}{\partial x}\right) - \frac{\partial}{\partial x}\left(u \frac{\partial v}{\partial y}\right) \right\} dx\, dy\, dz$$

$$=-4\mu \iint \left(mu \frac{\partial v}{\partial x} - lu \frac{\partial v}{\partial y}\right) dS$$

$$=-4\mu \iint u\left(m \frac{\partial v}{\partial x} - l \frac{\partial v}{\partial y}\right) dS$$

$$=-4\mu \iint u\left(m \frac{\partial v}{\partial x} + l \frac{\partial u}{\partial x}\right) dS$$

{ as $u=-\omega y$

$$=4\mu\omega \iint y\left(m \frac{\partial v}{\partial x} + l \frac{\partial u}{\partial x}\right) dS \qquad \text{...(iii)}$$

Taking the mean of (ii) and (iii), we get

$$=2\mu\omega \iint \left\{ y\left(m \frac{\partial v}{\partial x} + l \frac{\partial u}{\partial x}\right) - x\left(l \frac{\partial u}{\partial y} + m \frac{\partial v}{\partial y}\right) \right\} dS$$

$$=2\mu\omega \iint \left\{ l\left(y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}\right) u + m\left(m \frac{\partial}{\partial x} - x \frac{\partial}{\partial y}\right) v \right\} dS$$

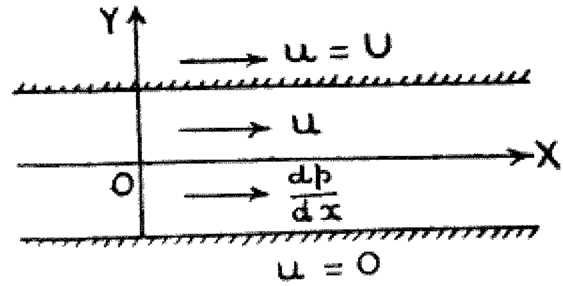
$$=2\mu\omega \iint (l.Du + m.Dv)\, dS.$$

Proved.

§ 9 8. Laminar flow between parallel plate.

By laminar flow we mean that the fluid moves in layers parallel to the plates. Consider the two-dimensional laminar flow of an incompressible fluid of constant viscosity between parallel straight plates. In order to maintain such a motion, the pressure difference in the direction of axis of X, *i e.* along the plates must be balanced by the shearing stress.

A flow is called parallel if only one velocity component is different from zero i.e. all fluid particles move in one direction.



Here for parallel flow, we have

$$u=u\ (x\ y\ t)$$

and $\qquad v=0,\ w=0$ every where.

Also $\qquad p=p\ (x, y, t).$

The equation of continuity is

$$\frac{\partial u}{\partial x}=0 \qquad \{ \text{ as } v=0=w$$

$\rightarrow$ that the velocity component u is independent from x.

or $\qquad u=u\ (y, t)$

The equation of motion is given by

$$\rho\frac{\partial u}{\partial t}=-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{\partial y^2} \qquad \text{...(i)}$$

Also $\qquad p=p\ (x, t).$

We see that $\frac{dp}{dx}$ must be a constant or a function of t, Since p is not a function of y and u is not a function of x.

Integrating (i) with regard to y for steady flow, we have

$$-\frac{\partial p}{\partial x}+\mu\frac{\partial^2 u}{\partial y^2}=0$$

or $$\frac{\partial^2 u}{\partial y^2}=\frac{1}{\mu}\ \frac{\partial p}{\partial x} \qquad \text{...(ii)}$$

or $$\frac{\partial u}{\partial y}=\frac{1}{\mu}\ \frac{\partial p}{\partial x}\ y+B$$

or $$u=\frac{1}{\mu}\cdot\frac{\partial p}{\partial x}\cdot\frac{y^2}{2}+By+A$$

or $$u=\frac{1}{2\mu}\ \frac{dp}{dx}\ y^2+By+A \qquad \text{...(iii)}$$

where A and B are arbitrary constants to be determined by the boundary conditions.

Case I. Plane Couette flow.

Here we shall determine the *solution of equation (ii) between two parallel plates when the upper plate is moving in its own* *plane with a velocity U and the lower plate is stationary i.e. at rest.*

Here $\frac{dp}{dx}=0$; one wall is at rest and other is in uniform motion. The boundary conditions are

$$y=-\frac{d}{2},\ u=0$$

$$y=+\frac{d}{2},\ u=U.$$

From relation (iii), we have

$$0=-\frac{Ad}{2}+B$$

and
$$U=+\frac{Ad}{2}+B$$

Solving these two, we get

$$A=\frac{U}{d} \text{ and } B=\frac{U}{2}$$

Substituting the values of the constants A and B in (iii), we have

$$u=\frac{U}{d}y+\frac{U}{2}$$

or
$$u=\frac{U}{2}\left(1+\frac{2y}{d}\right)$$

Such a flow is known a **plane couette flow or shear flow**, when the upper plate is moving with velocity U. The velocity distribution is linear.

Case. II. Plane Poiseuille flow.

In this case the pressure gradient $\frac{dp}{dx}$ is not equal to zero but both the plates are at rest *i.e.* $\frac{dp}{dx}=$Constant. The boundary conditions are

$$y=-\frac{d}{2},\ u=0$$

$$y=+\frac{d}{2},\ u=0.$$

From (iii), we have

$$0=\frac{1}{2\mu}\ \frac{dp}{dx}\cdot\frac{d^2}{4}+A\frac{d}{2}+B$$

and $$0=\frac{1}{2\mu}\frac{dp}{dx}\cdot\frac{d^2}{4}-A\frac{d}{2}+B$$

which gives $$A=0 \text{ and } B=-\frac{1}{8\mu}\frac{dp}{dx}.d^2$$

Substituting the values of the constants A and B in (iii), we have

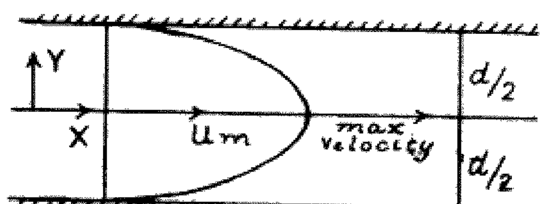
$$u=\frac{1}{2\mu}\frac{dp}{dx}y^2-\frac{1}{8\mu}\frac{dp}{dx}d^2$$

or $$u=-\frac{1}{8\mu}\frac{dp}{dx}d^2\left(1-\frac{4y^2}{d^2}\right)$$

or $$u=u_m\left(1-\frac{4y^2}{d^2}\right)$$

where $$u_m=-\frac{1}{8\mu}\frac{dp}{dx}d^2$$

is the maximum velocity in the flow occuring at $y=0$. The velocity distribution is parabolic in the interval between the two plates.



(Parallel flow with parabolic velocity-distribution)

Case III. Generalised plane Couette flow.

In this case the pressure gradient $\frac{dp}{dx}$ *is constant and one plate is at rest, the other plate is in motion.* The boundary conditions are given by

$$y=-\frac{d}{2},\ u=0$$

$$y=+\frac{d}{2},\ u=U$$

Then from (iii), we have

$$0=\frac{1}{2\mu}\frac{dp}{dx}\cdot\frac{d^2}{4}-A\frac{d}{2}+B$$

$$U=\frac{1}{2\mu}\frac{dp}{dx}\cdot\frac{d^2}{4}+A\frac{d}{2}+B$$

Which gives $A=\frac{U}{d}$

and $$B=A\frac{d}{2}-\frac{1}{2\mu}\ \frac{dp}{dx}\cdot\frac{d^2}{4}$$

or $$B=\frac{U}{2}-\frac{d^2}{8\mu}\ \frac{dp}{dx}$$

Substituting the values of the constants A and B in (iii), we have

$$u=\frac{1}{2\mu}\ \frac{dp}{dx}y^2+\frac{U}{d}\ y+\frac{U}{2}-\frac{d^2}{8\mu}\ \frac{dp}{dx} \qquad \text{...(iv)}$$

Total flux across a plane perpendicular to X is

$$\int_{-d/2}^{d/2} Udy=\int_{-d/2}^{d/2}\left\{\frac{1}{2\mu}\ \frac{dp}{dx}y^2+\frac{U}{d}\ y+\frac{U}{2}-\frac{d^2}{8\mu}\ \frac{dp}{dx}\right\}.dy$$

$$=\left[\frac{1}{2\mu}\ \frac{dp}{dx}\cdot\frac{y^3}{3}+\frac{U}{d}\cdot\frac{y^2}{2}+\frac{U}{2}\ y-\frac{d^2}{8\mu}\ \frac{dp}{dx}\ y\right]_{-d/2}^{d/2}$$

$$=\frac{1}{2\mu}\ \frac{dp}{dx}\cdot\frac{d^3}{12}+\frac{Ud}{2}-\frac{d^2}{8\mu}\ \frac{dp}{dx}\ d$$

$$\int_{-d/2}^{d/2} Udy=\frac{Ud}{2}-\frac{d^3}{12\mu}\cdot\frac{dp}{dx} \qquad \text{...(v)}$$

Differentiating (iv) with regard to y, we have

$$\frac{du}{dy}=\frac{1}{\mu}\ \frac{dp}{dx}\ y+\frac{U}{d}$$

At $$y=\pm\frac{d}{2}$$

$$\frac{du}{dy}=\frac{U}{d}\pm\frac{d}{2\mu}\ \frac{dp}{dx}.$$

Thus drag on the boundaries

$$=\left(\mu\ \frac{du}{dy}\right)y=\pm\frac{d}{2}$$

$$=\left(\frac{\mu U}{d}\pm\frac{d}{2}\ \frac{dp}{dx}\right) \text{ per unit area.}$$

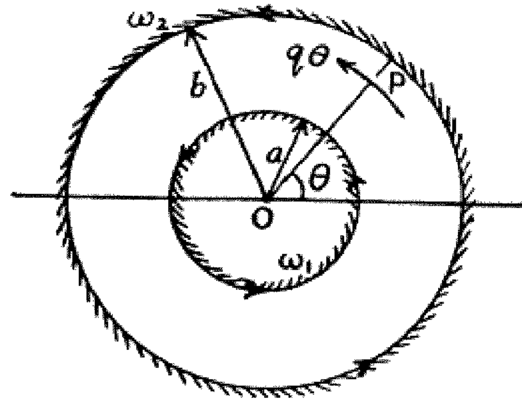
§ 9·9. Laminar flow between concentric rotating cylinders. Couette flow.

Consider the two-dimensional steady flow of an incompressible fluid between two concentric rotating cylinders. Let a and b be the radii of the inner and outer cylinder respectively, and ω_1 and ω_2 be their angular velocities.

Here the components of velocity in cylindrical coordinates are given by

$$\left.\begin{array}{r} u=0,\ v=v\ (r),\ w=0 \\ \text{and } p=p\ (r) \end{array}\right\} \qquad \text{...(i)}$$

Substituting these values in equations of motion, we have



$$-\frac{v^2}{r}=-\frac{1}{\rho}\frac{\partial p}{\partial r} \quad ...(ii)$$

and $$\frac{d^2v}{dr^2}+\frac{1}{r}\frac{\partial v}{\partial r}-\frac{v}{r^2}=0 \quad ...(iii)$$

and $$\frac{\partial p}{\partial z}=0 \quad ...(iv)$$

Let any point P the angular velocity be ω then

$$v=\omega r$$

or $$\frac{dv}{dr}=\omega+r\,\frac{d\omega}{dr}$$

and $$\frac{d^2v}{dr^2}=\frac{d\omega}{dr}+\frac{d\omega}{dr}+r\,\frac{d^2\omega}{dr^2}$$

$$=r\,\frac{d^2\omega}{dr^2}+2\,\frac{dw}{dr}$$

From (iii), we have

$$r\,\frac{d^2\omega}{dr^2}+2\,\frac{d\omega}{dr}+\frac{d\omega}{dr}+\frac{\omega}{r}-\frac{\omega}{r}=0$$

or $$r\,\frac{d^2\omega}{dr^2}+3\,\frac{d\omega}{dr}=0$$

or $$\frac{d^2\omega/dr^2}{d\omega/dr}=-\frac{3}{r}$$

By integrating, we have

$$\log\left(\frac{d\omega}{dr}\right)=-3\log r+\log A$$

or $$\frac{d\omega}{dr}=\frac{A}{r^3}$$

or $$\omega=B-\frac{A}{2r^2} \quad ...(v)$$

The boundary conditions are

I $\quad r=a,\ \omega=\omega_1$

II $\quad r=b,\ \omega=\omega_2$

or $$\omega_1=B-\frac{A}{2a^2}$$

and $$\omega_2=B-\frac{A}{2b^2}\ .$$

Solving the above equations, we have

$$A=\frac{2a^2b^2(\omega_1-\omega_2)}{a^2-b^2} \text{ and } B=\frac{\omega_2b^2-\omega_1a^2}{b^2-a^2}$$

Substituting the values of the constants A and B in (v), we have

$$\omega=\frac{\omega_2b^2-\omega_1a^2}{b^2-a^2}-\frac{a^2b^2(\omega_2-\omega_1)}{b^2-a^2}\cdot\frac{1}{r^2}$$

If the inner cylinder is at rest, then $\omega_1=0$.

So $$\omega=\frac{\omega_2b^2}{b^2-a^2}-\frac{a^2b^2\omega_2}{b^2-a^2}\cdot\frac{1}{r^2}$$

or $$\omega=\frac{\omega_2b^2}{r^2}\cdot\frac{r^2-a^2}{b^2-a^2}$$

There will be the tangential stress $p_{r\theta}$ only in the fluid

i.e. $$p_{r\theta}=\mu\left(\frac{dv}{dr}-\frac{v}{r}\right)$$

$$p_{r\theta}=\mu\left(\omega+r\frac{d\omega}{dr}-\omega\right)$$

$$p_{r\theta}=\mu r\frac{d\omega}{dr}$$

Its moment about the axis is given by

$$=2\pi r\,(p_{r\theta}).r$$

$$=2\pi r^2.\mu r\frac{d\omega}{dr}=2\pi\mu r^3\frac{d\omega}{dr}$$

$$=2\pi\mu r^3.\frac{A}{r^3}$$

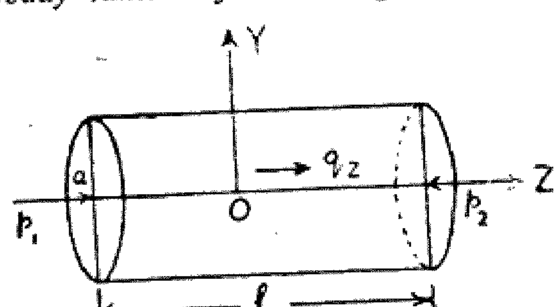
$$=2\pi\mu A$$

$$=4\pi\mu.\frac{a^2b^2}{a^2-b^2}(\omega_1-\omega_2)$$

$$=4\pi\mu\frac{a^2b^2}{b^2-a^2}\omega_2 \quad \left\{ \begin{array}{l}\omega_1=0 \text{ as the inner}\\ \text{cylinder is at rest.}\end{array}\right.$$

§ 9·91. Hagen-Poiseuille flow in a circular pipe.

Here we shall consider *the steady laminar flow through a long straight pipe of circular cross-section.* We know that the shearing force on the surface of any cylindrical shape of fluid must be balanced by the difference of pressure between the ends. Let Z-axis be choosen along the axis of the pipe.



Consider q_z be the component of velocity parallel to the axis of pipe which is a function of r only. The velocity component in the tangential and radial directions are zero.

Equation of continuity in cylindrical coordinates

$$\frac{\partial q_z}{\partial z}=0 \qquad \text{...(i)}$$

$\Rightarrow$ that q_z is independent of z or a function of r only.

Also the equations of motion in cylindrical coordinates are given by

$$\mu\left(\frac{d^2q_z}{dr^2}+\frac{1}{r}\frac{dq_z}{dr}\right)=\frac{dp}{dz} \qquad \text{...(ii)}$$

and
$$\frac{\partial p}{\partial r}=0\,;\ \frac{1}{r}\frac{\partial p}{\partial \theta}=0 \qquad \text{.. (iii)}$$

Since the velocity q_z is a function of r only and the pressure p is independent of r, therefore the pressure gradient $\frac{dp}{dz}$ must be a constant and let it be equal to $\frac{p_2-p_1}{l}$, from relation (ii), we have

$$\frac{d^2q_z}{dr^2}+\frac{1}{r}\frac{dq_z}{dr}=\frac{p_2-p_1}{\mu l}$$

or
$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dq_z}{dr}\right)=-\frac{p_1-p_2}{\mu l}$$

or
$$r\frac{dq_z}{dr}=-\frac{p_1-p_2}{2\mu l}r^2+A$$

or
$$\frac{dq_z}{dr}=-\frac{p_1-p_2}{2\mu l}r+\frac{A}{r}$$

or
$$q_z=-\frac{p_1-p_2}{4\mu l}r^2+A\log r+B \qquad \text{...(iv)}$$

The velocity is finite at $r=0$, so A must be zero. The boundary condition is

$$r=a,\ q_z=0.$$

Then (iv) reduces to

$$0=-\frac{p_1-p_2}{4\mu l}a^2+B$$

or
$$B=\frac{p_1-p_1}{4\mu l}a^2$$

Then
$$q_z=\frac{p_1-p_2}{4\mu l}(a^2-r^2)$$

Since the maximum velocity occurs on the axis, then

$$\left(q_z\right)_{max}=\frac{p_1-p_2}{4\mu l}a^2 \text{ (as } r=0 \text{ on the axis of the pipe)}$$

The volume V_0 of the fluid flowing through the pipe per unit time is

$$V_0=\frac{1}{2}\left(q_z\right)_{max}.\pi a^2$$

or
$$V_0=\frac{1}{2}\left(\frac{p_1-p_2}{4\mu l}a^2\right).\pi a^2$$

or
$$V_0=\frac{\pi a^4}{8\mu}\cdot\frac{p_1-p_2}{l}$$

This relation was obtained experimentally by Hagen and afterwards independently by poiseuille. With the help of this relation, the coefficient of visocosity of the fluid can be determined.

Again *total flux across any section*

$$=\int_0^a q_z\,2\pi r dr$$

$$=2\pi\,\frac{p_1-p_2}{4\mu l}\int_0^a (a^2-r^2).r\,dr$$

$$=\frac{p_1-p_2}{8\mu l}\cdot\pi a^4$$

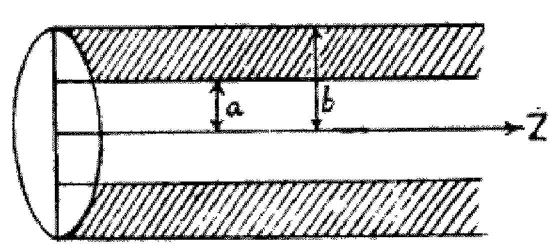
and the *drag on the cylinder is*

$$=2\pi a l\left(\mu\,\frac{dq_z}{dr}\right)_{r=a}$$

$$=2\pi a l\mu\left\{\frac{p_1-p_2}{4\mu l}(-2r)\right\}_{r=a}$$

$$=\pi a^2\,(p_1-p_2)$$

§ 9·92. **Steady flow between co-axial circular pipes.**



Let the flow take place between two co-axial cylinders of radii a and b $(b>a)$. Consider the inner boundary have a velocity V while the outer is at rest. The boundary conditions are

$$r=a,\quad q_z=V$$
$$r=b,\quad q_z=0$$

Then from relation (iv) § 9·91, we have

$$V=-\frac{p_1-p_2}{4\mu l}a^2+A\log a+B$$

and
$$0=-\frac{p_1-p_2}{4\mu l}b^2+A\log b+B$$

Substituting the values of the constants A and B in relation (iv) § 9·91.

$$q_z = V\,\frac{\log\left(\frac{r}{b}\right)}{\log\left(\frac{a}{b}\right)} - \frac{p_1-p_2}{4\mu l}\left\{r^2 - \frac{b^2\log\left(\frac{r}{a}\right) - a^2\log\left(\frac{r}{b}\right)}{\log\left(\frac{b}{a}\right)}\right\}$$

The flux relative to the fixed boundary is given by

$$= \int_a^b q_z \,.\, 2\pi r dr$$

$$= \pi V\left\{\frac{\frac{1}{2}(b^2-a^2)}{\log\left(\frac{b}{a}\right)} - a^2\right\} + \frac{\pi}{8\mu}\cdot\frac{p_1-p_2}{l}\left\{b^4 - a^4 - \frac{(b^2-a^2)^2}{\log\left(\frac{b}{a}\right)}\right\}$$

§ 9·93. Steady flow in tubes of cross-section other than circular.

Consider the axis of z along the axis of the tube. Let the component of velocity w is a function of x and y but not of z, and that $u=0=v$.

The equation to continuity reduces to

$$\frac{\partial w}{\partial z} = 0, \qquad \ldots(i)$$

$\Rightarrow$ that w is independent of z *i.e.* a function of x and y only.

There are no external forces and the inertia terms vanish in steady motion, then the equations of motion reduce to,

$$\frac{\partial p}{\partial x} = 0 \quad \text{and} \quad \frac{\partial p}{\partial y} = 0 \qquad \ldots(ii)$$

and
$$\mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = \frac{\partial p}{\partial z} \qquad \ldots(iii)$$

Since w is independent of z, p is independent of x and y then in steady flow along a tube the pressure gradient $\frac{\partial p}{\partial z}$ must be a constant, let it be equal to $(-P)$.

then
$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = -\frac{P}{\mu} \qquad \ldots(iv)$$

with a boundary condition $w=0$ on the surface of the tube.

Consider $w = \lambda - \frac{P}{4\mu}(x^2+y^2)$, then λ has to satisfy the equation

$$\frac{\partial^2 \lambda}{\partial x^2} + \frac{\partial^2 \lambda}{\partial y^2} = 0$$

with the boundary condition $\lambda=\frac{P}{4\mu}(x^2+y^2)$ on the surface of the tube.

Thus to solve the problem for a particular boundary we consider

$$w=\lambda-\frac{P}{4\mu}(x^2+y^2)+B \qquad \text{...(v)}$$

where B is an arbitrary constant, λ is a suitable solution of the two dimensional Laplace's equation. The constant B can be determined by applying the condition $w=0$ on the surface of the tube.

(a) Elliptic section.

Let $$\frac{x^2}{a^2}+\frac{y^2}{b^2}\leqslant 1$$

Consider $$w=A(x^2-y^2)+B-\frac{P}{4\mu}(x^2+y^2) \qquad \text{...(i)}$$

Since on the surface of the elliptic section

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \qquad \text{...(ii)}$$

on the boundary $w=0$

then $$\left(\frac{P}{4\mu}-A\right)x^2+\left(\frac{P}{4\mu}+A\right)y^2=B \qquad \text{...(iii)}$$

This requires that, from (ii) and (iii), we have

$$a^2\left(\frac{P}{4\mu}-A\right)=b^2\left(\frac{P}{4\mu}+A\right)=B$$

or $$A=\frac{P}{4\mu}\cdot\frac{a^2-b^2}{a^2+b^2}$$

and $$B=\frac{P}{2\mu}\cdot\frac{a^2b^2}{a^2+b^2}$$

Substituting the values of A and B in (i), we have

$$w=\frac{P}{4\mu}\cdot\frac{a^2-b^2}{a^2+b^2}(x^2-y^2)+\frac{P}{2\mu}\frac{a^2b^2}{a^2+b^2}-\frac{P}{4\mu}(x^2+y^2)$$

or $$w=\frac{P}{2\mu}\cdot\frac{a^2b^2}{a^2+b^2}\left\{1+\tfrac{1}{2}\frac{a^2-b^2}{a^2b^2}(x^2-y^2)-\tfrac{1}{2}\frac{a^2+b^2}{a^2b^2}(x^2+y^2)\right\}$$

or $$w=\frac{P}{2\mu}\cdot\frac{a^2b^2}{a^2+b^2}\left\{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right\}$$

Flux of the fluid over the area of the ellipse is given by

$$=\iint w\,dx\,dy$$

$$=\frac{P}{2\mu}\cdot\frac{a^2b^2}{a^2+b^2}\iint\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)dx\,dy$$

$$=\frac{P}{2\mu}\cdot\frac{a^2b^2}{a^2+b^2}\left\{\iint dx\,dy-\frac{1}{a^2}\iint x^2\,dx\,dy\right.$$

$$\left.-\frac{1}{b^2}\iint y^2\,dx\,dy\right\}$$

$$=\frac{P}{2\mu}\,\frac{a^2b^2}{a^2+b^2}\left\{\pi ab-\frac{1}{a^2}.\pi ab\,\frac{a^2}{4}-\frac{1}{b^2}.\pi ab\,\frac{b^2}{4}\right\}$$

$$=\frac{P}{2\mu}\,\frac{a^2b^2}{a^2+b^2}.\tfrac{1}{2}\,\pi ab$$

$$=\frac{\pi P}{4\pi}\cdot\frac{a^3b^3}{a^2+b^2}.$$

(b) Equilateral triangle.

Consider

$$w=A\,(x^3-3xy^2)+B-\frac{P}{4\mu}\,(x^2+y^2) \qquad \text{...(i)}$$

Since $w=0$ at all points of the boundary, then from (i), we have

$$A\,(x^3-3xy^2)+B-\frac{P}{4\mu}\,(x^2+y^2)=0 \qquad \text{...(ii)}$$

If $x=a$ be a part of the boundary, then

$$A\,(a^3-3ay^2)+B-\frac{P}{4\mu}\,(a^2+y^2)=0$$

or

$$Aa^3+B-\frac{Pa^2}{4\mu}=0$$

and

$$-3aA-\frac{P}{4\mu}=0$$

Thus

$$A=-\frac{P}{12a\mu}$$

and

$$B=\frac{Pa^2}{3\mu}$$

Substituting the values of A and B in (ii), we have

$$-\frac{P}{12a\mu}\,(x^3-3xy^2)+\frac{Pa^2}{3\mu}-\frac{P}{4\mu}\,(x^2+y^2)=0.$$

or

$$x^3-3xy^2+3ax^2+3ay^2-4a^3=0.$$

or

$$(x-a)\,(x+2a-\sqrt{3}y)\,(x+2a+\sqrt{3}y)=0.$$

Therefore the boundary consists of

$$x=a,\ x+2a-\sqrt{3}y=0 \quad \text{and} \quad x+2a+\sqrt{3}y=0$$

or $$x=a,\ y=\frac{1}{\sqrt{3}}\,x+\frac{2}{\sqrt{3}}\,a$$

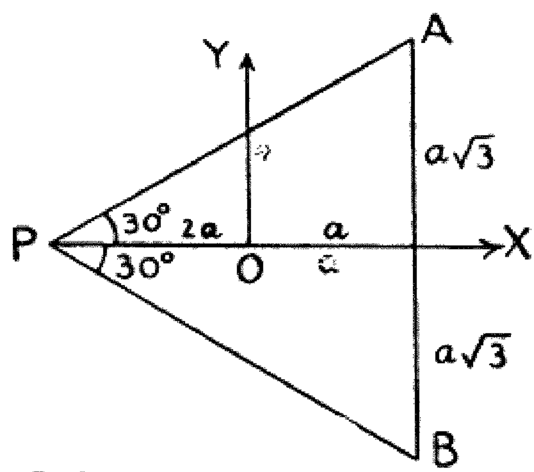
and $$y=-\frac{1}{\sqrt{3}}\,x-\frac{2a}{\sqrt{3}} \qquad \text{...(iii)}$$

Which forms an equilateral triangle

So $$w=-\frac{P}{12a\mu}\,(x^3-3xy^2+3ax^2+3ay^2-4a^3)$$

Flux of the fluid over the cross-section is

$$=\iint w\,dx\,dy$$



$$=-\frac{P}{12a\mu}\iint (x^3-3xy^2+3ax^2+3ay^2-4a^3)^*\,dxdy$$

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* (i) $$\iint x^3 dx\,dy=\int_{-2a}^{a} x^3\,(y)\Big|_{-\frac{x+2a}{\sqrt{3}}}^{\frac{x+2a}{\sqrt{3}}}.dx$$

$$=\frac{2}{\sqrt{3}}\int_{-2a}^{a} x^3\,(x+2a)\,dx=-\frac{9a^5}{5\sqrt{3}}$$

(ii) $$3\iint xy^2 dx\,dy=\int x\,(y^3)\Big|_{-\frac{x+2a}{\sqrt{3}}}^{\frac{x+2a}{\sqrt{3}}}.dx$$

$$=\frac{2}{3\sqrt{3}}\int_{-2a}^{a} x\,(x+2a)^3\,dx=\frac{27a^5}{5\sqrt{3}}$$

(iii) $$3a\iint (x^2+y^2)\,dx\,dy$$

$=3a$ (sum of the moments of Inertia)

$$=3a\left(\tfrac{1}{3}.3a.a\sqrt{3}\right)\left\{\frac{3a^2}{4}+\frac{3a^2}{4}+a^2+\frac{a^2}{4}+\frac{a^2}{4}\right\}=9\sqrt{3}\,a^5$$

(iv) $$4a^3\iint dx\,dy=4a^3.3a.a\sqrt{3}=12\sqrt{3}\,a^5$$

$$=-\frac{P}{12a\mu}\left\{-\frac{9a^5}{5\sqrt{3}}-\frac{27a^5}{5\sqrt{3}}+\frac{27a^5}{\sqrt{3}}-\frac{36a^5}{\sqrt{3}}\right\}$$

$$=\frac{27}{20\sqrt{3}}\cdot\frac{Pa^4}{\mu}$$

$$\text{average flow}=\frac{\text{Flux}}{\text{Area}}=\frac{\frac{27}{20\sqrt{3}}\ \frac{Pa^4}{\mu}}{\frac{1}{2}.3a.2a\sqrt{3}}=\frac{3}{20}\ \frac{Pa^2}{\mu}$$

§ 9·94. Steady motion due to a slowly rotating sphere.

Consider the component of velocity are

$$u=-\omega y,\ v=\omega x \text{ and } \omega=0$$

where ω is the angular velocity and is a function of r $(r^2=x^2+y^2+z^2)$ only.

The equations of motion are,

(neglecting the squares of velocities)

or $$0=-\frac{\partial p}{\partial x}+\mu\ \nabla^2 u \qquad \text{...(i)}$$

or $$0=-\frac{\partial p}{\partial y}+\mu\ \nabla^2 v \qquad \text{...(ii)}$$

or $$0=-\frac{\partial p}{\partial z} \qquad \text{...(iii)}$$

Since $\dfrac{\partial u}{\partial x}=-y\dfrac{\partial \omega}{\partial x}$ or $\dfrac{\partial^2 u}{\partial x^2}=-y\dfrac{\partial^2 \omega}{\partial x^2}$

or $\dfrac{\partial u}{\partial y}=-y\dfrac{\partial \omega}{\partial y}-\omega$ or $\dfrac{\partial^2 u}{\partial y^2}=-y\dfrac{\partial^2 \omega}{\partial^2 y}-2\ \dfrac{\partial \omega}{\partial y}$

and $\dfrac{\partial^2 u}{\partial z^2}=-y\dfrac{\partial^2 \omega}{\partial z^2}$

Thus $$\nabla^2 u=-y\left\{\nabla^2\omega+\frac{2}{y}\cdot\frac{\partial\omega}{\partial y}\right\}$$

or $$\nabla^2 u=-y\left\{\frac{d^2\omega}{dr^2}+\frac{2}{r}\cdot\frac{d\omega}{dr}+\frac{2}{r}\ \frac{d\omega}{dr}\right\}$$

$$\left\{\text{as } \frac{1}{y}\ \frac{\partial\omega}{\partial y}=\frac{1}{r}\cdot\frac{d\omega}{dr}\right.$$

or $$\nabla^2 u=-y\left\{\frac{d^2\omega}{dr^2}+\frac{4}{r}\ \frac{d\omega}{dr}\right\}$$

Now the equation (i), (ii) and (iii) reduce to

$$0=-\frac{\partial p}{\partial x}-\mu y\left(\frac{d^2\omega}{dr^2}+\frac{4}{r}\ \frac{d\omega}{dr}\right)$$

$$0=-\frac{\partial p}{\partial y}+\mu x\left(\frac{d^2\omega}{dr^2}+\frac{4}{r}\ \frac{d\omega}{dr}\right)$$

$$0=-\frac{\partial p}{\partial z}$$

These are satisfied by p=constant.

the $$\frac{d^2\omega}{dr^2}+\frac{4}{r}\frac{d\omega}{dr}=0.$$

By integrating, we have

$$r^4\frac{d\omega}{dr}=A$$

or $$\frac{d\omega}{dr}=\frac{A}{r^4}$$

or $$\omega=\frac{B}{r^3}+C \qquad ...(iv)$$

(where B and C are arbitrary constant).

Let the motion is produced by a solid sphere of radius a rotating with angular velocity Ω and the liquid extends to infinity, we have

$$C=0,\ B=a^3\,\Omega$$

So $$\omega=\frac{a^3}{r^3}\Omega$$

If there is an outer fixed concentric sphere of radius b, then the boundary conditions are

I $\quad r=a,\quad \omega=\Omega$

II $\quad r=b,\quad \omega=0$

From (iv), we have

$$\Omega=\frac{B}{a^3}+C$$

and $$0=\frac{B}{b^3}+C$$

or $$B=\frac{a^3b^3}{b^3-a^3}\Omega \text{ or } C=-\frac{a^3}{b^3-a^3}\Omega$$

Substituting the values of B and C in (iv), we have

$$\omega=\frac{a^3b^3}{b^3-a^3}\Omega\,\frac{1}{r^3}-\frac{a^3}{b^3-a^3}\Omega$$

$$\omega=\frac{a^3\Omega}{r^3}\cdot\frac{b^3-r^3}{b^3-a^3}.$$

Ex. 4. *One surface (nearly plane) is fixed and another near surface (plane) rotates with angular velocity ω about an axis perpendicular to its plane and there is a film of viscous fluid between them. Prove that the pressure p in the film satisfies the equation*

$$h^3\left(\frac{\partial^2 p}{\partial r^2}+\frac{1}{r}\frac{\partial p}{\partial r}+\frac{1}{r^2}\frac{\partial^2 p}{\partial\theta^2}\right)+\frac{\partial h^3}{\partial r}\frac{\partial p}{\partial r}+\frac{1}{r^2}\frac{\partial h^3}{\partial\theta}\frac{\partial p}{\partial\theta}$$
$$=6\,\mu\omega\frac{\partial h}{\partial\theta},$$

where (r, θ) are polar coordinates in the plane of the film, the origin being in the axis of rotation, and h is the thickness of the film.

Consider any point (x, y) on the upper surface

then $\qquad U=-\omega y,\ V=\omega x$

The total flux across a plane perpendicular to X-axis is

$$\int_0^h u dz=\tfrac{1}{2}hU-\frac{h^3}{12\mu}\frac{\partial p}{\partial x}$$

{Ref. equation (v) § 9·8 Case III

$$=-\tfrac{1}{2}h\omega y-\frac{h^3}{12\mu}\frac{\partial p}{\partial x} \qquad \text{...(ii)}$$

Similarly the total flux across a plane perpendicular to Y-axis,

$$\int_0^h v dz=\tfrac{1}{2}\ h\omega x-\frac{h^3}{12\mu}\frac{\partial p}{\partial y} \qquad \text{...(ii)}$$

Now from the equation of continuity, we have

$$\frac{\partial}{\partial x}\left\{-\tfrac{1}{2}h\omega y-\frac{h^3}{12\mu}\frac{\partial p}{\partial x}\right\}+\frac{\partial}{\partial y}\left\{\tfrac{1}{2}h\omega x-\frac{h^3}{12\mu}\frac{\partial p}{\partial y}\right\}=0$$

or
$$\frac{h^3}{12\mu}\left(\frac{\partial^2 p}{\partial x^2}+\frac{\partial^2 p}{\partial y^2}\right)+\frac{1}{12\mu}\left(\frac{\partial h^3}{\partial x}\cdot\frac{\partial p}{\partial x}+\frac{\partial h^3}{\partial y}\cdot\frac{\partial p}{\partial y}\right)$$
$$=\tfrac{1}{2}\omega\left(x\frac{\partial h}{\partial y}-y\frac{\partial h}{\partial x}\right) \qquad \text{...(iii)}$$

Since (r, θ) are the polar coordinates in the plane of the film,

$$\left.\begin{aligned}
&\text{then} && \frac{\partial}{\partial x}=\cos\theta\frac{\partial}{\partial r}-\frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\\
&\text{and} && \frac{\partial}{\partial y}=\sin\theta\frac{\partial}{\partial r}+\frac{\cos\theta}{r}\frac{\partial}{\partial\theta}\\
&\text{also} && \frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}=\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}
\end{aligned}\right\} \qquad \text{...(iv)}$$

Substituting the results of (iv) in (iii), we have

$$\frac{h^3}{12\mu}\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)p+\frac{1}{12\mu}\left\{\left(\cos\theta\frac{\partial h^3}{\partial r}-\frac{\sin\theta}{r}\frac{\partial h^3}{\partial\theta}\right)\right.$$
$$\left(\cos\theta\frac{\partial p}{\partial r}-\frac{\sin\theta}{r}\frac{\partial p}{\partial\theta}\right)+\left(\sin\theta\frac{\partial h^3}{\partial r}+\frac{\cos\theta}{r}\frac{\partial h^3}{\partial\theta}\right)$$
$$\left.\left(\sin\theta\frac{\partial p}{\partial r}+\frac{\cos\theta}{r}\frac{\partial p}{\partial\theta}\right)\right\}$$
$$=\tfrac{1}{2}\omega\left\{r\cos\theta\left(\sin\theta\frac{\partial h}{\partial r}+\frac{\cos\theta}{r}\frac{\partial h}{\partial\theta}\right)-r\sin\theta\left(\cos\theta\frac{\partial h}{\partial r}-\frac{\sin\theta}{r}\frac{\partial h}{\partial\theta}\right)\right\}$$

or $$h^3\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)p+\frac{\partial h^3}{\partial r}\cdot\frac{\partial p}{\partial r}+\frac{1}{r^2}\frac{\partial h^3}{\partial\theta}\cdot\frac{\partial p}{\partial\theta}=6\mu\omega\frac{\partial h}{\partial\theta}.$$

Proved.

Ex. 5. *A liquid occupying the space between two co-axial circular cylinders is acted upon by a force $\frac{c}{r}$ per unit mass, where r is the distance from the axis, the lines of force being circles round the axis. Prove that in the steady motion the velocity at any point is given by the*

$$\frac{C}{2\nu}\left\{\frac{b^2}{r^2}\frac{r^2-a^2}{b^2-a^2}\log\left(\frac{b}{a}\right)-r\log\frac{r}{a}\right\}$$

where ν is the coefficient of kinematric viscosity.

Consider the axis of the cylinder be the Z-axis. Here

$$q_r=0=q_z$$

and q_θ is independent of θ and z *i.e.* it is a function of r only.

So $\quad q_\theta=r\omega \quad \left\{ q_z=0 \text{ considering the cylinders to be sufficiently long.} \right.$

where ω is the angular velocity of the liquid at the point (r, θ, z).

Thus the equation of motion for viscous fluid reduces to

$$\nu\left(\nabla^2 q_\theta-\frac{q_\theta}{r^2}\right)+\frac{C}{r}=0.$$

or
$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}\right)(\omega r)-\frac{\omega}{r}=-\frac{C}{\nu r}$$

or
$$\frac{1}{r}\frac{d}{dr}\left(r^2\frac{d\omega}{dr}+r\omega\right)-\frac{\omega}{r}=-\frac{C}{\nu r}$$

or
$$r\frac{d^2\omega}{dr^2}+3\frac{d\omega}{dr}=-\frac{C}{\nu r}$$

Multiplying both the sides with r^2 and integrating, we have

$$r^3\frac{d^2\omega}{dr^2}+3r^2\frac{d\omega}{dr}=-\frac{C}{\nu}r$$

or
$$r^3\frac{d\omega}{dr}+\frac{Cr^2}{2\nu}=A$$

or
$$\frac{d\omega}{dr}=-\frac{C}{2\nu r}+\frac{A}{r^3}$$

or
$$\omega=-\frac{C}{2\nu}\log r-\frac{A}{2r^2}+B \quad \ldots\text{(i)}$$

where A and B are arbitrary constant.

The boundary conditions are,

I. $\omega=0, \quad r=a$

II. $\omega=0, \quad r=b.$

Now the relation (i) reduces to with the help of condition I and II,

$$0=-\frac{C}{2\nu}\log a-\frac{A}{2a^2}+B \quad \ldots\text{(ii)}$$

or $$0=-\frac{C}{2\nu}\log b-\frac{A}{2b^2}+B. \qquad \text{...(iii)}$$

By subtracting, we have

$$0=\frac{C}{2\nu}(\log b-\log a)-\frac{A}{2}\left(\frac{1}{a^2}-\frac{1}{b^2}\right)$$

or $$A=\frac{Ca^2b^2}{\nu(b^2-a^2)}\log\left(\frac{b}{a}\right). \qquad \text{...(iv)}$$

Form (i) and (ii), by subtracting, we have

$$\omega=-\frac{C}{2\nu}(\log r-\log a)-\frac{A}{2}\left(\frac{1}{r^2}-\frac{1}{a^2}\right)$$

or $$\omega=-\frac{C}{2\nu}(\log r-\log a)-\frac{Ca^2b^2}{2\nu(b^2-a^2)}\cdot\frac{a^2-r^2}{r^2a^2}\log\left(\frac{b}{a}\right)$$

{ from (iv)

or $$\omega=-\frac{C}{2\nu}\log\left(\frac{r}{a}\right)+\frac{C}{2\nu}\cdot\frac{r^2-a^2}{b^2-a^2}\cdot\frac{b^2}{r^2}\log\left(\frac{b}{a}\right).$$

Thus $q_\theta=rw$

$$=-\frac{Cr}{2\nu}\log\left(\frac{r}{a}\right)+\frac{C}{2\nu}\cdot\frac{r^2-a^2}{b^2-a^2}\cdot\frac{b^2}{r}\log\left(\frac{b}{a}\right)$$

$$=\frac{C}{2\nu}\left\{\frac{r^2-a^2}{b^2-a^2}\cdot\frac{b^2}{r}\log\left(\frac{b}{a}\right)-\log\left(\frac{r}{a}\right)\right\}.$$ **Proved.**

Ex. 6. *Incompressible viscous liquid is moving steadily under pressure between planes $y=0$, $y=h$. The plane $y=0$ has a constant velocity U in the direction of the axis x, and the plane $y=h$ is fixed. The planes are porous, and the liquid is sucked in uniformly over one and ejected uniformly over the other. Shew that a possible solution is given by*

$$u=\frac{(Ue^{h/a}+Ah)-(U+Ah)\,e^{y/a}}{e^{h/a}-1}+Ay,\quad v=\frac{\nu}{a}$$

where ν is the kinematic coefficient of viscosity. Determine the meaning of the constants A and a.

Since the planes $y=0$ and $y=h$ are taken infinitely large, the velocity components (u, v) at any point (x, y) will be independent from x. Thus the equation of continuity reduces to

$$\frac{\partial v}{\partial y}=0 \qquad \text{...(i)}$$

or $$v=\text{constant}=\frac{\nu}{a}.$$

The equations of motion are

$$v\frac{du}{dy}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\frac{\partial^2 u}{\partial y^2}. \qquad \text{...(ii)}$$

and $$0=-\frac{1}{\rho}\frac{\partial p}{\partial y}. \qquad \text{...(iii)}$$

The boundary conditions are,

$$\text{I.} \quad y=0, \quad u=U$$

$$\text{II.} \quad y=h, \quad u=0.$$

Substituting the given value of u and the value of $v=\frac{\nu}{a}$ in the equations of motion (ii), we have

$$\frac{\nu}{a}\left\{-\frac{1}{a}\frac{(U+Ah)\,e^{y/a}}{e^{h/a}-1}+A\right\}=-\frac{1}{\rho}\frac{\partial p}{\partial x}+\nu\left\{-\frac{1}{a^2}\frac{(U+Ah)\,e^{y/a}}{e^{h/a}-1}\right\}$$

or
$$\frac{A\nu}{a}=-\frac{1}{\rho}\frac{\partial p}{\partial x} \qquad \ldots(\text{iv})$$

or
$$\frac{\partial p}{\partial x}=-\frac{A}{a}\nu\rho$$

or
$$\frac{\partial p}{\partial x}=-\frac{A\mu}{a}$$

or
$$p=-\frac{A\mu}{a}x+B$$

which also satisfies the condition (iii). $\frac{\partial p}{\partial y}=0.$

Hence the given velocity components satisfy the equation of motion, and forms the possible solutions.

Consider the mass of liquid sucked per unit area per unit time at $y=0$ be m, then

$$m=\rho v$$

$$m=\rho\frac{\nu}{a}=\frac{\mu}{a} \qquad \{\text{as } \mu=\rho\nu$$

or
$$a=\frac{\mu}{m}.$$

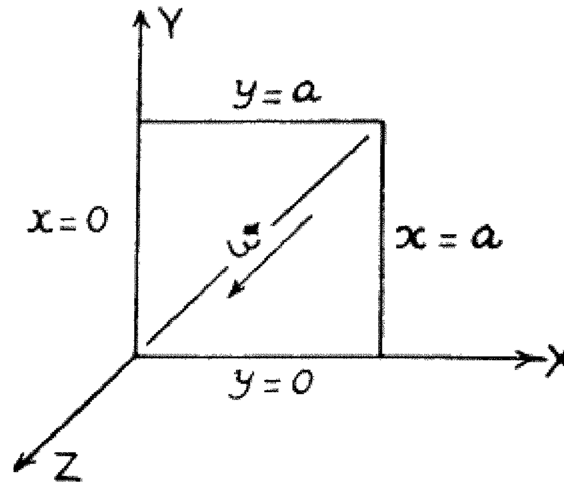
Substituting the value of a in (iv), we have

$$A=-\frac{1}{\rho}\cdot\frac{a}{\nu}\frac{\partial p}{\partial x}$$

$$A=-\frac{\mu}{m\rho\nu}\frac{\partial p}{\partial x}$$

$$A=-\frac{1}{m}\frac{\partial p}{\partial x}. \qquad \textbf{Ans.}$$

Ex. 7. *Viscous liquid is flowing steadily under pressure through an infinitely long rectangular tube whose axis is parallel to the axis of z. The sides $x=0$ and $x=a$ are smooth and the sides $y=0$, $y=a$ do not permit of slipping of liquid in contact with them. The pressure gradient maintaining the motion is suddenly annulled. Show that the total flux across any section is $\frac{Qa^2}{10\nu}$ where Q is the* *flux per unit time across a section in the initial steady motion, given that* $\sum\limits_{0}^{\infty} \dfrac{1}{(2n+1)^6} = \dfrac{\pi^6}{960}$.



Since the rectangular tube is infinitely long and the sides $x=0$ and $x=a$ are smooth. The velocity component w of an element at $(x\ y\ z)$ parallel to Z-axis is a function of y only, the other two components u and v are zero.

The equations of motion reduce to

$$\mu \frac{\partial^2 w}{\partial y^2} = \frac{\partial p}{\partial z} \qquad \text{...(i)}$$

and
$$\frac{\partial p}{\partial x} = 0 = \frac{\partial p}{\partial y}. \qquad \text{...(ii)}$$

Integrating (i), we have

$$\mu \frac{\partial w}{\partial y} = \frac{\partial p}{\partial z} y + A$$

or
$$\mu w = \tfrac{1}{2} \frac{\partial p}{\partial z} y^2 + Ay + B. \qquad \text{...(iii)}$$

The boundary conditions are

I. $w=0$, $y=0$ and II. $w=0$, $y=a$

which gives from (iii),

$$B=0 \quad \text{and} \quad A = -\tfrac{1}{2} a \frac{\partial p}{\partial z}$$

or
$$\mu w = \tfrac{1}{2}(y^2 - ay) \frac{\partial p}{\partial z}. \qquad \text{...(iv)}$$

Thus flux $Q = \displaystyle\int_0^a w\,(a\,dy)$

or
$$Q = \frac{a}{2\mu} \frac{\partial p}{\partial z} \int_0^a (y^2 - ay)\,dy$$

or
$$Q = \frac{a}{2\mu} \frac{\partial p}{\partial z} \left(\frac{a^3}{3} - \frac{a^2}{2} \right)$$

or
$$Q = -\frac{a^4}{12\mu} \frac{\partial p}{\partial z}$$

or
$$\frac{\partial p}{\partial z} = -\frac{12\mu Q}{a^4}. \qquad \text{...(v)}$$

So $$\mu w = -\tfrac{1}{2}(y^2 - ay) \cdot \frac{12\mu Q}{a^4}$$

or $$w = \frac{6Q}{a^4}\, y\,(a-y). \qquad \ldots\text{(vi)}$$

When the pressure gradient is suddenly annulled. the equation of motion becomes

$$\frac{\partial w}{\partial t} = -\nu \frac{\partial^2 w}{\partial y^2}. \qquad \ldots\text{(vii)}$$

{Here $u = 0 = v$

Consider $w = f(y)\, e^{-\nu k^2 t}$ be the solution of (vii), then

$$\frac{\partial^2 f(y)}{\partial y^2} = -k^2 f(y)$$

which shows that $f(y)$ is of the from $\cos ky$ or $\sin ky$

or $$w = \Sigma\, Ak\, e^{-\nu k^2 t} \begin{cases} \cos ky \\ \sin ky \end{cases} \qquad \ldots\text{(viii)}$$

At $t = 0$, we have

$$w = \frac{6Q}{a^4}\, y\,(a-y).$$

Expressing $y\,(a-y)$ in the form of Fourier Series $0 \leqslant y \leqslant a$, we have

$$y\,(a-y) = \frac{8a^2}{\pi^3} \sum_{n=1}^{\infty} \frac{\sin (2n+1)\frac{\pi y}{a}}{(2n+1)^3} \qquad \ldots\text{(ix)}$$

Consider $k = (2n+1)\dfrac{\pi}{a}$.

Thus w for any time t is given by

$$w = \frac{6Q}{a^4} \cdot \frac{8a^2}{\pi^3} \sum_{n=1}^{\infty} \frac{\sin (2n+1)\frac{\pi y}{a}}{(2n+1)^3} \cdot e^{-\nu t \left(\frac{2n+1}{a}\pi\right)^2}$$

Thus the total flux is

$$= \int_{t=0}^{\infty} \int_{y=0}^{a} (a\, dy)\, w \,.\, dt$$

$$= a . \frac{48Q}{\pi^3 a^2} \sum_{n=1}^{\infty} \frac{1}{(2n+1)^3} \int_0^{\infty} e^{-\nu t \left(\frac{2n+1}{a}\pi\right)^2} . dt \int_0^a \sin\left\{(2n+1)\frac{\pi y}{a}\right\} dy$$

$$= \frac{48Q}{\pi^3 a} \Sigma \frac{1}{(2n+1)^3} \left\{\frac{1}{\nu\left(\frac{2n+1}{a}\pi\right)^2}\right\} \left\{\frac{2}{(2n+1)\frac{\pi}{a}}\right\}$$

$$= \frac{48Q}{\pi^3 a} \cdot \frac{2a^2}{\nu\pi^2} \cdot \frac{a}{\pi} \Sigma \frac{1}{(2n+1)^6}$$

$$=\frac{96Qa^2}{\nu\pi^6}\Sigma\frac{1}{(2n+1)^6}$$

$$=\frac{96Qa^2}{\nu\pi^6}\cdot\frac{\pi^6}{960}$$

$$=\frac{Qa^2}{10\nu}.$$

Proved.

Exercise

1. In the case of steady flow of compressible fluid through a circular pipe of radius a, show that the mass which crosses any section per unit time is

$$\frac{\pi a^4(p_1-p_2)(\rho_1+\rho_2)}{16\mu l}$$

where p_1, ρ_1 and p_2, ρ_2 are the respective pressures and densities at the two section at distance l apart. It is assumed that the temperature is constant and the gradient of velocity in the direction of the axis may be neglected in comparison with its gradient in the direction of radius a.

2. The space between two co-axial cylinders a and b is filled with viscous fluid, and the cylinders are made to rotate with angular velocities ω_1, ω_2. Prove that in steady motion the angular velocity of the fluid at distance r from the axis is given by

$$\omega=\frac{a^2(b^2-r^2)\,\omega_1-b^2(r^2-a^2)\,\omega_2}{r^2(b^2-a^2)}$$

3. Incompressible liquid is flowing steadily through a circular pipe. Prove that the mean pressure is constant over the cross-section and that the rate of flow is

$$\frac{\pi a^4(p_1-p_2)}{8\mu l}$$

where p_1, p_2 are the pressures over sections at a distance l apart.

4. Prove that, for liquid filling a closed vessel which is at rest the rate of dissipation of energy due viscosity is

$$4\pi\iiint(\xi^2+\eta^2+\zeta^2)\,dx\,dy\,dz$$

where $\xi=\frac{1}{2}\left(\frac{\partial w}{\partial y}-\frac{\partial v}{\partial z}\right)$, $\eta=$, $\zeta=\ldots$ is the vorticity.

5. A liquid occuping the space between two co-axial circular cylinders is acted upon by a force $\frac{\lambda}{r}$ per unit mass, where r is th[illegible] distance from the axis, the lines of force being circles round th[illegible] ity at any point in the steady motion, μ [illegible] sity.

5 [illegible] $\frac{b}{a}-r\log\frac{r}{a}\Big\}$. **Ans.**