IAS-2008-P-II

Ques: 3(b)) Show that any maximal ideal in the commutative ring $F[x]$ of polynomials over a field F is the principal ideal generated by an irreducible polynomial.

Solution:-

F is a field $\Rightarrow F[x]$ is an Integral Domain
$\Rightarrow F[x]$ is a Euclidean Ring
$\Rightarrow F[x]$ is a Principal Ideal Ring/Domain

Let $f, g \in F[x]$

$$f(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$$
$$g(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_m x^m$$

$$f(x) + g(x) = \sum_{r=0}^{\text{Max}(m,n)} (a_r + b_r) x^r.$$

$$f(x) \cdot g(x) = a_0 b_0 + (a_1 b_2 + b_1 a_2) x + \cdots + a_n b_m x^{n+m}$$

$\therefore$ All Ideal of $F[x]$ are generated by single element, say $\langle f(x) \rangle$

$\therefore$ maximal ideal is also a principal Ideal.

If F is a field then in $F[x]$, irreducible polynomials are irreducible elements.

$\therefore$ If $f(x)$ is irreducible element

$f(x) = g(x) \cdot h(x) \Rightarrow$ either $g(x)$ or $h(x)$ is a unit.

Condition is necessary:

Let M be a Maximal Ideal of $F[x]$, we have to prove that is $M = \langle a \rangle$; 'a' is irreducible element.

(i) $a \neq 0$; because if $\langle a \rangle = \langle 0 \rangle$ then some $\langle p \rangle$ be an ideal has '0' as its element.

$\langle 0 \rangle \subset \langle p \rangle \subset \langle R \rangle \Rightarrow \langle p \rangle \neq \langle 0 \rangle$

$\boxed{\therefore a \neq 0}$

(ii) 'a' cannot be unit.

because if 'a' is a unit, $\exists\, a^{-1} \in F[x]$, such that

$a^{-1}a \in M \Rightarrow I \in M$; $M = R$.

(iii) 'a' is non-zero; non-unit

$\therefore a = bc \quad / \, b, c \in R$

Let $B = \langle b \rangle$, let $r \in M = \langle a \rangle$

$r = ag = b \cdot c \cdot g \quad / \, g \in R$

$r = b(cg)$.

$\therefore r \in \langle b \rangle$

$\boxed{M \subseteq B \subseteq F[x]}$ Since, M is a maximal ideal

$\therefore M = B$ or $B = F[x]$

If $M = B$

$b = ah = bch \quad / \, h \in F[x]$

$ch = f(x) = 1$

$\therefore$ c is a unit element

$\therefore$ a is irreducable.

If $B = F[x]$

$f(x) = 1 \in B$ $\quad f(x) = 1 = bl \;/\; l \in F[x]$

$\therefore$ b is a unit $\Rightarrow$ a is irreducable.

Condition is sufficient:

If 'a' is irreducible polynomial in $F[x]$, the Ideal $\langle a \rangle = A$ is a Maximal Ideal.

To prove if $\langle a \rangle \subseteq I \subseteq F[x]$

either $I = \langle a \rangle$ or $I = F[x]$.

Since; $F[x]$ is a PID, therefore $I = \langle d \rangle$.

(i) Case 1 $\Rightarrow$ If $d \in A$ $\Rightarrow d = af \;/\; f \in F[x]$

Let $r \in \langle d \rangle \Rightarrow r = dg = a(fg)$

$\therefore r \in A \Rightarrow I \subseteq A$.

Also; $A \subseteq I$ $\quad \therefore \boxed{A = I}$

(ii) Case 2 $\Rightarrow$ If $d \notin A$

$A \subset I$

$\Rightarrow$ so $a = df$

Since, a is irreducible

$\therefore$ Either d or 'f' is a unit.

If f is a unit $\exists f^{-1} \Rightarrow d = af^{-1}$

$\Rightarrow d \in A$

contradictory.

If d is a unit $\exists d^{-1}$ in $F[x]$.

$\therefore 1 = d(d^{-1}) \in I$.

$\Rightarrow I = F[x]$

$\therefore A = \langle a \rangle$ is a Maximal Ideal.

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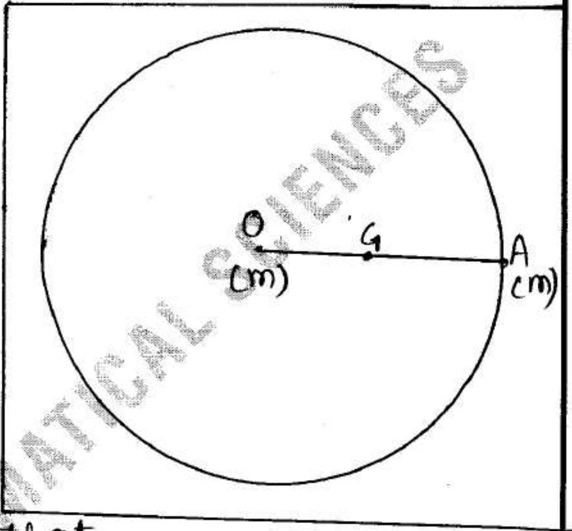
Ques: 5(e) A circular board is placed on a smooth horizontal plane and a boy runs round the edge of it at a uniform rate. What is the motion of the centre of the board? Explain what happens if the mass of boy and board are equal?



Solution:-

Let M be the mass and O the centre of the board. If initially the boy is at 'A' on the edge of the board then the C.G. 'G' of the system will be on the radius OA, such that

$$\boxed{OG = \frac{M.0 + m.a}{M+m} = \frac{ma}{M+m}}$$

Since the external forces, weight of the board and the boy act vertically downwards and the reaction of the smooth horizontal plane act vertically upwards, therefore there is no external force in the horizontal direction during the motion. Thus by D'Alembert's principle the C.G. 'G' of the system will remain at rest. Hence as the boy runs round the edge of the board with uniform spread, the centre 'O' of the board will describe a circle of radius $OG = ma/(M+n)$ round the centre at G.

If mass of the board and boy are equal

i.e $\boxed{M = m}$

Then; motion of board $= OG = \dfrac{ma}{M+m}$

$$\Rightarrow OG = \frac{\cancel{m}a}{2\cancel{m}}$$

$$\Rightarrow \boxed{OG = a/2}$$

The motion of board depends only on the position of boy. i.e. $\boxed{OG \propto a}$

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Ques: 8(b)) Let the fluid fills the region $x \geq 0$ (right half of 2d plane). Let a source α be at $(0, y_1)$ and equal sink at $(0, y_2)$, $y_1 > y_2$. Let the pressure be same as pressure at infinity i.e. P_0. Show that the resultant pressure on the boundary (y-axis) is $\pi\rho\alpha^2(y_1-y_2)^2/2y_1y_2(y_1+y_2)$, ρ being the density of the fluid.

Solution:- Here the image system w.r.t x-axis in z-plane consists of

i) a source m at $(0, y_1)$ i.e, at $z = y_1 i$

ii) a sink $-m$ at $(0, y_2)$ i.e at $z = y_2 i$

iii) a source m at $(0, -y_1)$ i.e at $z = -y_1 i$

iv) a sink $-m$ at $(0, -y_2)$ i.e at $z = -y_2 i$

Clearly the image system does away with the boundary $y=0$ (i.e, x-axis). Thus, the complex potential of this entire system is given by-

$$\therefore w = -m\log(z-y_1 i) + m\log(z-y_2 i) - m\log(z+y_1 i) + m\log(z+y_2 i)$$

$$w = -m\log(z^2+y_1^2) + m\log(z^2+y_2^2)$$

$$\therefore \text{velocity} = \left|\frac{dw}{dz}\right| = \left|-\frac{2zm}{z^2+y_1^2} + \frac{2zm}{z^2+y_2^2}\right|$$

The velocity q at a point on the boundary (i.e. $y=0$) is given by [setting $z = x+iy = x$ as $y=0$]

$$q = \left|\frac{-2xm}{x^2+y_1^2} + \frac{2xm}{x^2+y_2^2}\right| = \frac{2xm(y_1^2-y_2^2)}{(x^2+y_1^2)(x^2+y_2^2)}$$

Let P_0 be the pressure at infinity. Then by Bernoulli's theorem, the pressure p at any point is given by -

$$\frac{1}{2}q^2 + \frac{p}{\rho} = \frac{1}{2}\times 0^2 + \frac{P_0}{\rho}$$

(or) $$\boxed{\frac{P_0 - p}{\rho} = \frac{1}{2}q^2}$$

$\therefore$ The resultant pressure on the boundary

$$= \int_0^\infty (P_0 - p)\,dx = \frac{1}{2}\rho\int_0^\infty q^2\,dx = 2\rho m^2 \int_0^\infty \frac{x^2(y_1^2-y_2^2)^2\,dx}{(x^2+y_1^2)^2(x^2+y_2^2)^2}$$

$$= 2\rho m^2 \int_0^\infty \left[-\frac{y_1^2+y_2^2}{y_1^2-y_2^2}\left[\frac{1}{x^2+y_1^2} - \frac{1}{x^2+y_2^2}\right] - \frac{y_1^2}{(x^2+y_1^2)^2} - \frac{y_2^2}{(x^2+y_2^2)^2}\right]dx$$

[on resolving into partial fraction]

$$= 2\rho m^2 \left\{\frac{y_1^2+y_2^2}{y_1^2-y_2^2}\left[\frac{\pi}{2y_1} - \frac{\pi}{2y_2}\right] - \frac{\pi}{4y_1} - \frac{\pi}{4y_2}\right\}.$$

[on simplification]

$$= \frac{\pi\rho m^2}{2y_1y_2}\left[\frac{2(y_1^2+y_2^2) - (y_1+y_2)^2}{(y_1+y_2)}\right]$$

$$= \frac{\pi\rho m^2(y_1-y_2)^2}{2y_1y_2(y_1+y_2)}$$ Hence proved

$$\boxed{\therefore \text{ Resultant pressure on the boundary (y-axis)} = \frac{\pi\rho m^2(y_1-y_2)^2}{2y_1y_2(y_1+y_2)}}$$

2009-P.I Ques: 1(c) Suppose that f'' is continuous on $[1,2]$ and that f has three zeroes in the interval $(1,2)$. Show that f'' has at least one zero in the interval $(1,2)$.

Solution:-

Suppose $f(x)=0$ has three zeroes in the interval $(1,2)$.

To show that: $f''(x)=0$ has at least one zero in the interval $(1,2)$

Let the three zeroes be a_1, a_2 and a_3 and suppose that $1<a_1<a_2<a_3<2$ (three distinct zeroes).

$$\Rightarrow \quad f(a_1)=f(a_2)=f(a_3)=0$$

Since f is infinitely differentiable, it is continuous on $[a_1,a_2]$ and differentiable on (a_1,a_2)

$\therefore$ By Rolle's theorem, there exists b_1 in (a_1,a_2) such that $f'(b_1)=0$

likewise f is continuous on $[a_2,a_3]$ and differentiable on (a_2,a_3).

So, by Rolle's theorem, there exists b_2 in (a_2,a_3) such that $f'(b_2)=0$.

Since, $1<b_1<a_2<b_2<2$, they cannot coincide and, infact, $b_1<b_2$.

Since, f is infinitely differentiable, f' is continuous on $[b_1,b_2]$ and differentiable on (b_1,b_2).