1 Real Numbers

1. INTRODUCTION

In school algebra and arithmetic, we usually deal with two fundamental operations *viz. addition* and *multiplication* and their inverse operations, *subtraction* and *division* respectively. These operations are related to a certain class of 'numbers' which will be described more precisely in the following sections. The basic difference between 'elementary mathematics' and 'higher mathematics', which begins at the college level, is the introduction of the all important notion of **limit** which is very intimately related to the intuitive idea of *nearness* or *closeness* and which cannot be described in terms of the operations of addition and multiplication. The notion of limit comes into play in situations where one quantity depends on another varying quantity and we have to know the behaviour of the first when the second is *arbitrarily close* to a fixed given value. In order to illustrate our point in relation to a practical situation, consider the question of determining the velocity of the planet earth at a particular instant during its motion round the sun assuming that the path of its motion round the sun and its position on this path at any instant are known. We cannot determine the velocity of earth without taking recourse to the notion of limit and indeed we need the notion of limit even in defining the concept of 'velocity' of a moving object which is not moving with a uniform speed. The purpose of this illustration is simply to indicate that there are numerous situations where the methods of elementary algebra prove quite inadequate for the purpose of solving or even formulating a problem, and we are forced to evolve new concepts and methods. The notion of *limit* is one such concept.

For the proper understanding of the notion of limit and its importance, it is absolutely desirable that the reader should be familiar with the true nature and important properties of 'real numbers'. Starting with natural numbers, we shall briefly and intuitively introduce in the following sections the concept of *rational numbers* and *irrational numbers* which together form the system of *real numbers*, describing in the process the important properties possessed by these numbers. The branch of mathematics called *real analysis* deals with problems which are closely connected with the notion of 'limit' and some other notions, such as the operations of 'differentiation' and 'integration' which are directly dependent on the concept of limit when all these operations are confined to the domain of real numbers. It is difficult to say anything more precise at this stage. The interested reader will certainly have a clearer and precise understanding of this important branch of mathematics as he systematically studies this work.

1.1 Real Numbers

The system of real numbers has evolved as a result of a process of successive extensions of the system of **natural numbers** (*i.e.*, the positive whole numbers). It may be remarked here that the extension became absolutely inevitable as the science of Mathematics developed in the process of solving problems from other fields. Natural numbers came into existence when man first learnt counting which can also be viewed as adding successively the number 1 to unity. If we add two natural numbers, we get a natural number—but the inverse operation of subtraction is not always possible if we limit ourselves to the domain of natural numbers only. For instance, there is no natural number which added to 8 will give us 3. In other words, 8 cannot be subtracted from 3 within the system of natural numbers. In order that the operation of subtraction (*i.e.*, operation inverse of addition) be also performed without any restriction, it became necessary to enlarge the system of natural numbers by introducing the *negative integers* and the number *zero*. Thus to every natural number n corresponds a unique negative integer designated $-n$ and called the additive inverse of n, and there is a number zero, written 0, such that $n + (-n) = 0$, and $n + 0 = n$ for every natural number n. Also n is the inverse of $-n$. The negative integers, the number 0, and the natural numbers (*i.e.*, the positive integers) together constitute what is known as the **system of integers**. Similarly, to make division always possible, zero being an exception, the concept of fractions, positive and negative, was introduced. Division by zero, however, cannot be defined in a meaningful and consistent manner. The system so extended, including integers and fractions both positive and negative, and the number zero, is called the **system of rational numbers**. Thus, every rational number can be represented in the form p/q where p and q are integers and $q \neq 0$.

We know that the result of performing any one of the four operations of arithmetic (division by zero being, of course, excluded) in respect of any two rational numbers is again a rational number. So long as mathematics was concerned with these four operations only, the system of rational numbers was sufficient for all purposes but the process of extracting roots of numbers (*e.g.*, square-root of 2, cube-root of 7, etc.), as also the desirability of giving a meaning to non-terminating and non-recurring decimals, necessitated a further extension of the number system. There were lengths which could not be measured in terms of rational numbers, for instance—the length of the diagonal of a square whose sides are of unit length, cannot be measured in terms of rational numbers. In fact, this is equivalent to saying that there is no rational number whose square is equal to 2. In order to be able to answer such questions, the system of rational numbers had to be further enlarged by introducing the so called irrational numbers. It is beyond the scope of this book to discuss systematically the definition of irrational numbers in terms of rational numbers. Numbers like $\sqrt{2}$, $\sqrt[3]{7}$, π (*i.e.*, the ratio of the circumference to the diameter of a circle) with which the reader is already familiar are examples of irrational numbers. Rational numbers and irrational numbers together constitute what is known as the **system of real numbers**.

Though the real number system cannot be extended in a way in which a rational number system is extended but it can be used to develop another system, called the **system of complex numbers**. But since real analysis is not concerned with complex numbers, we have nothing to do with complex numbers in this book.

For the sake of brevity and clarity of exposition, and because the notion of set is fundamental to all branches of mathematics, we start with the algebra of sets.

1.2 Sets

A set is a well defined collection of objects. In other words, an aggregate or class of objects having a specified property in common enables us to tell whether any given object belongs to it or not. The individual objects of the set are called *members* or *elements* of the set. Capital letters A, B, C, etc. are generally used to denote the *sets* while small letters a, b, c, etc. for *elements*. If x is a member of a set A, then we write $x \in A$ and read it as 'x belongs to A' or 'x is an element of A' or 'x is a member of A' or simply 'x is in A'. If x is not a member of A, then we write $x \notin A$ and read it as 'x does not belong to A'.

Some Typical Sets

N: The set of natural numbers,

I : The set of integers,

$\mathbf{I}^+$: The set of positive integers,

$\mathbf{I}^-$: The set of negative integers,

Q: The set of rational numbers,

R: The set of real numbers.

There are two methods which are in common use to denote a set.

(*i*) A set may be described by listing all its elements.

(*a*) Set S has elements a, b, c, then we write

$$S = \{a, b, c\}$$

(*b*) Set V of vowels in the English alphabets

$$V = \{a, e, i, o, u\}$$

(*ii*) A set may be described by means of a property which is common to all its elements.

(*a*) The set S of all elements x which have the property $P(x)$

$$S = \{x: P(x)\}$$

(*b*) The set B of natural numbers

$$B = \{n: n \in \mathbf{N}\}$$

Null Set. A set having no element. Sometimes the defining property of a set is such that no object can satisfy it, so that the set remains empty. Such a set is called a **null set**, an **empty set** or a **void set**, and is generally denoted by the Danish letter ϕ or { }. Thus

$$\phi = \{n: n \text{ is a natural number less than } 1\}$$

$$\phi = \{x: x \neq x\}$$

1.3 Equality of Sets

Two sets are said to be equal when they consist of exactly the same elements. Thus, sets P and Q are equal ($P = Q$) if every element of P is an element of Q, and every element of Q is also an element of P. Thus,

$$\{a, b, c\} = \{b, a, c\}$$

$$\{4, 5, 6, 7, 8, 9\} = \{n: 3 < n < 10, n \in N\}$$

It is to be noted that while writing a set, an element occurs only once but the order in which the elements of a set are written is immaterial.

A set is **finite** or **infinite** according to the number of element in it is finite or infinite.

1.4 Notation

$$\forall, \exists, \Rightarrow, \Leftrightarrow, \wedge, \vee, \sim$$

These symbols borrowed from mathematical logic help in a neat and brief exposition of the subject and so we shall describe them briefly here.

(*i*) $\forall$ stands for 'for all' or 'for every'.

The statement $x < y,\ \forall x \in S$ means x is less than y for all members of S, *i.e.*, all members of S are less than y.

(*ii*) $\exists$ stands for 'there exists'.

(*iii*) $\Rightarrow$ stands for 'implies that'.

If P and Q are two statements, then $P \Rightarrow Q$ means that the statement P implies the statement Q, *i.e.*, if P is true then Q is also true. Thus

$$x = 5 \Rightarrow x^2 = 25$$

$$AB \parallel CD \text{ and } CD \parallel EF \Rightarrow AB \parallel EF$$

If the statements P and Q are such that P implies Q and Q implies P, then we write

$P \Leftrightarrow Q$ (both ways implication)

Thus for real numbers x, y

$$xy = 0 \Leftrightarrow x = 0 \text{ or } y = 0$$

(*iv*) $\wedge$ stands for 'and'

$\vee$ stands for 'or'

The statement $P \wedge Q$ holds when both the statements P and Q hold, but the statement $R \vee S$ can hold when either R holds or S holds, *i.e.*, $R \vee S$ holds when at least one of R and S holds. Thus,

$$(x-3)(x-5) < 0 \Rightarrow x > 3 \wedge x < 5$$

$$x^2 = 1 \Rightarrow x = 1 \vee x = -1$$

(*v*) Negation $\sim$ stands for 'not'.

If P is a statement then $\sim P$ is negation of P.

In other words, $\sim P$ denotes 'not P'.

Thus when P holds, $\sim P$ cannot hold and *vice versa*.

$P \wedge \sim P$ is always false, but $P \vee \sim P$ is always true.

1.5 Subsets

If A and B are two sets such that each member of A is also a member of B, *i.e.*, $x \in A \Rightarrow x \in B$, then A is called a **subset** of B (or is contained in B) and we write $A \subseteq B$.

This is sometimes expressed by saying that B is a **superset** of A (or contains A) and we write $B \supseteq A$.

Thus, if A is a subset of B, then there is no element in A which is not in B, *i.e.*, $y \notin B \Rightarrow y \notin A$. Consequently, the null set ϕ is a subset of every set and $A \subseteq A$, for every set A.

Thus, if $A \subseteq B$ and $B \subseteq A$, we write $A = B$

$A \subseteq B$ allows for the possibility that A and B might be equal. If A is a subset of B and is not equal to B, we say that A is a **proper subset** of B (or is properly contained in B) and we write $A \subset B$. Thus A is a proper subset of B if every member of A is a member of B and there exists at least one member of B which is not a member of A.

Two sets A and B are said to be **comparable** if either $A \supseteq B$ or $A \subseteq B$, otherwise they are not comparable.

1.6 Union and Intersection of Sets

Union. If A and B are two sets, then the set consisting of all those elements which belong to A or to B or to both, is called the union of A and B and is denoted by $A \cup B$.

Clearly

$$A \cup \phi = A,\ A \cup A = A \text{ and } A \cup B = B \cup A.$$

Intersection. If A and B are two sets, then the set consisting of all those elements which belong to both A and B is called the intersection of A and B and is denoted by $A \cap B$.

Clearly

$$A \cap \phi = \phi,\ A \cap A = A \text{ and } A \cap B = B \cap A.$$

Thus, $A \cap B$ consists of elements which are common to A and B.

Two sets A and B are said to be **disjoint** if they have no common element, *i.e.*, $A \cap B = \phi$.

1.7 Union and Intersection of an Arbitrary Family

The operations of forming unions and intersections are primarily binary operations, that is, each is a process which applies to a pair of sets and yields a third. We emphasize this by the use of parentheses to indicate the order in which the operations are to be performed, as in $(A_1 \cup A_2) \cup A_3$, where the parentheses direct us first to unite A_1 and A_2, and then to unite the result with A_3. Associativity makes it possible to dispense with parentheses in an expression like this and to write $A_1 \cup A_2 \cup A_3$, where we understand that these sets are to be united in any order and that the order in which the operations are performed is irrelevant. Similar remarks apply to $A_1 \cap A_2 \cap A_3$. Furthermore, if $\{A_1, A_2, ..., A_n\}$ is any **finite** class of sets, then we can form

$$A_1 \cup A_2 \cup ... \cup A_n \text{ and } A_1 \cap A_2 \cap ... \cap A_n$$

in much the same way without any ambiguity of meaning whatever. In order to shorten the notation, we let $I = \{1, 2, ..., n\}$ be the set of subscripts which index the set under consideration. I is called the **Index Set**. We then compress the symbols and write

$$\bigcup_{i \in I} A_i \text{ and } \bigcap_{i \in I} A_i \text{ or } \bigcup_{i=1}^{n} A_i \text{ and } \bigcap_{i=1}^{n} A_i$$

It is often necessary to form unions and intersections of large (really large) class of sets. Let Λ be a set and $\{A_\lambda : \lambda \in \Lambda\}$ an entirely arbitrary class or family **F** of sets which contains a set A_λ for each λ in Λ. Then

$$\bigcup_{\lambda \in \Lambda} A_\lambda = \{x : x \in A_\lambda \text{ for at least one } \lambda \text{ in } \Lambda\}$$

and

$$\bigcap_{\lambda \in \Lambda} A_\lambda = \{x : x \in A_\lambda \text{ for every } \lambda \text{ in } \Lambda\}$$

define the union and intersection of an **arbitrary family 'F'**.

Λ is called the **Index set**.

In particular, if $\Lambda = \{1, 2, 3, ...\}$ be the set of all natural numbers, then the union and intersection are often written in the form

$$\bigcup_{i=1}^{\infty} A_i \text{ and } \bigcap_{i=1}^{\infty} A_i$$

or simply $\bigcup A_i$ and $\bigcap A_i$.

1.8 Universal Set

In any discussion of sets, all sets are usually assumed to be subsets of a set, called the **universal set** (usually denoted by U). In our present discussion, however, the set **R** of real numbers can serve as the universal set.

1.9 Difference Set, Complement of a Set

If A and B are two sets, then the set consisting of those elements of B which do not belong to A is called the **difference set** of A and B and is denoted by $B - A$.

If, however, A is a subset of B then $B - A$ is called the **complement of A in B** or complement of A with respect to B.

Complement of A in the universal set U is called the **complement** of A and is denoted by A^c.

1.10 Functions

Let A and B be two sets and let there be a rule which associates to each member x of A, a member y of B.

Such a rule or a correspondence f under which to each element x of the set A there corresponds exactly one element y of the set B is called a *mapping* or a *function*.

Symbolically we write $f: A \to B$, *i.e.*, f is a mapping or a function of A *into* B.

The set A is called the *Domain* of the function.

The set B contains all the elements which correspond to the elements of A and is called the *co-domain* of f.

The unique element of B which corresponds to an element x of A is called the image of x or the value of the function at x and is denoted by $f(x)$; x is called the preimage of $f(x)$. It may be observed that while every element of the domain finds its image in B there may be some elements in B which are not the image of any element of the domain A. The set of all those elements of the co-domain B which are the images of the elements of the domain A is called the *range set* of the function f. If the co-domain B of f itself is the range set of f then we say that f is a function from A *onto* B. If members of the domain set are denoted by x and those of range set y then $y = f(x)$ is the value of the function f at x. We call a function $f: A \to B$ to be *one-one* if two different elements in A always have two different images under f, *i.e.*, $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$, for all $x_1, x_2 \in A$

If $f: A \to B$ is both *one-one* and *onto*, then we can define its inverse mapping $f^{-1}: B \to A$ as follows:

For each y in B, we find a unique element x in A such that $f(x) = y$ (x exists and is unique, since f is one-one and onto). We then define x to be $f^{-1}(y)$. The equation $x = f^{-1}(y)$ is the result of solving $y = f(x)$ for x.

If $f: A \to B$ is both *one-one* and *onto* then we say that f is a *one to one correspondence* between A and B. In this case $f^{-1}: B \to A$ is also a *one to one correspondence* between B and A.

If $A_1 \subseteq A$, then its image $f(A_1)$ is a subset of B defined by

$$f(A_1) = \{f(x) \in B: x \in A_1\}.$$

Similarly, if B_1 is a subset of B, then its inverse image $f^{-1}(B_1)$ is a subset of A defined by

$$f^{-1}(B_1) = \{x \in A: f(x) \in B_1\}$$

A function f is called an *extension* of a function g (and g is called a *restriction* of f) if the domain of f contains the domain of g

and $f(x) = g(x)$ for each x in the domain of g.

Just as we can combine sets to get a new set, we can combine given functions to construct a new function in the following way:

If $f: A \to B$ and $g: B \to C$ are two given functions, we define the *composite function* $g\,o\,f: A \to C$ by

$$(g\,o\,f)(x) = g(f(x)), \text{ for every } x \in A.$$

The function $I: A \to A$ defined by $I(x) = x$ for every $x \in A$ is called the identity function on A.

If $g\,o\,f = f\,o\,g = I$, then

$$g = f^{-1} \text{ or } f = g^{-1}.$$

The main properties of the function $f: A \to B$ and its inverse images are as follows:

(*i*) $f(\phi) = \phi$, where ϕ is an empty set.

(*ii*) $f(A) \subseteq B$

(*iii*) If $A_1 \subseteq A_2$, then $f(A_1) \subseteq f(A_2)$

(*iv*) $f\left(\bigcup_i A_i\right) = \bigcup_i f(A_i)$

(*v*) $f\left(\bigcap_i A_i\right) \subseteq \bigcap_i f(A_i)$

(*vi*) $f^{-1}(\phi) = \phi,\ f^{-1}(B) = A$

(*vii*) $f^{-1}\left(\bigcup_i B_i\right) = \bigcup_i f^{-1}(B_i),\ f^{-1}\left(\bigcap_i B_i\right) = \bigcap_i f^{-1}(B_i)$

(*viii*) $f^{-1}(B^c) = \left(f^{-1}(B)\right)^c$

1.11 Equivalent Sets

Two sets A and B are said to be equivalent (written as $A \sim B$) if there exists a one to one correspondence between their elements.

Let

$$A = \{a, e, i, o, u\} \text{ and } B = \{1, 2, 3, 4, 5\}$$

Then A is equivalent to B and the one-to-one correspondence can be seen as

$$a \leftrightarrow 1,\ e \leftrightarrow 2,\ i \leftrightarrow 3,\ o \leftrightarrow 4,\ u \leftrightarrow 5$$

Each of the two sets A and B have five elements which is a definite finite number and we call such sets as *finite sets*. Thus, if the sets are finite and have equal number of elements it is easy to see the one-to-one correspondence.

The positive integers are adequate for the purpose of counting any non-empty finite set; since all sets outside mathematics appear to be of this kind. But in mathematics we consider many sets which do not have a definite number of members. Such sets are called *infinite sets*.

The set $\mathbf{N}$ of all natural numbers, the set $\mathbf{I}$ of all integers, the set $\mathbf{Q}$ of rationals, the set $\mathbf{R}$ of reals, etc. are infinite sets.

The set $\mathbf{N}$ of naturals which is the same as the set of all positive integers seems to be larger than the set of all positive even integers $E = \{2, 4, 6, ...\}$, for $\mathbf{N}$ contains E as its proper subset. Does this mean that the set $\mathbf{N}$ has more elements than E? The answer is no. In dealing with infinite sets we must remember that the criterion for equivalent sets is whether there exists a one-to-one correspondence between these sets or not (irrespective of the fact which one is a proper subset of which). This function $f : \mathbf{N} \to E$ defined by $f(n) = 2n, n \in \mathbf{N}$ serves to establish a one-to-one correspondence between these sets. Thus, $\mathbf{N}$ is equivalent to E. Note that $\mathbf{N} \supset E$ but $\mathbf{N} \neq E$.

ILLUSTRATIONS

1. The set $\mathbf{N}$ of all natural numbers and the set S of all even integers are equivalent, a one-to-one correspondence is

$$1 \leftrightarrow 0,\ 2 \leftrightarrow 2,\ 3 \leftrightarrow -2,\ 4 \leftrightarrow 4,\ 5 \leftrightarrow -4, \ldots$$

2. The set $\mathbf{N}$ is equivalent to $\mathbf{I}$,

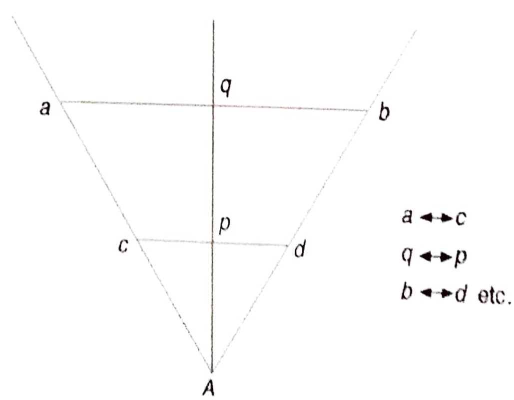
$$1 \leftrightarrow 0,\ 2 \leftrightarrow 1,\ 3 \leftrightarrow -1,\ 4 \leftrightarrow 2,\ 5 \leftrightarrow -2, \ldots$$

3. The set $\mathbf{N}$ and the set of all positive rationals are equivalent, the correspondence

$$1 \leftrightarrow 1,\ 2 \leftrightarrow \tfrac{1}{2},\ 3 \to 2,\ 4 \to \tfrac{1}{3},\ 5 \to 3,$$
$$6 \to \tfrac{1}{4},\ 7 \to \tfrac{2}{3},\ \text{and so on}$$

has been set up, by adding up the numerator and denominator where sum is $2 : \frac{1}{1} = 1$, where sum is $3 : \frac{1}{2}, \frac{2}{1}$ where sum is $4 : \frac{1}{3}, \frac{2}{2}, \frac{3}{1}$, etc. and omitting those already listed.

4. The two closed intervals $[a, b]$ and $[c, d]$ are equivalent. The figure establishes a one-to-one correspondence between them.



1.12 Compositions

We shall be dealing mainly with number sets and so we define only two types of compositions in the sets.

An **Addition Composition** is defined in a set S if to each pair of members a, b of S there corresponds a member $a + b$ of S.

Similarly, a **Multiplication Composition** is defined in S if to each pair of members a, b of S there corresponds a member ab of S.

A set is said to possess an **algebraic structure** if the two compositions of Addition and Multiplication are defined in the set.

Subtraction and *Division* may be defined as inverse operations of addition and multiplication respectively.

Let $a, b \in S$.

Subtraction: $a - b$ may be expressed as $a + (-b)$ when $-b \in S$.

Division: The quotient a/b $(b \neq 0)$ may be put as $a \cdot 1/b$ or ab^{-1} when $1/b$ or $b^{-1} \in S$.

2. FIELD STRUCTURE AND ORDER STRUCTURE

2.1 Field Structure

A set S is said to be a **field** if two compositions of *Addition* and *Multiplication* be defined in it such that $\forall\ a, b, c \in S$ the following properties are satisfied.

A-1. Set S is closed for addition,

$$a, b \in S \Rightarrow a + b \in S$$

A-2. Addition is commutative,

$$a + b = b + a$$

A-3. Addition is associative,

$$(a + b) + c = a + (b + c)$$

A-4. Additive identity exists, *i.e.*, $\exists$ a member 0 in S such that

$$a + 0 = a$$

A-5. Additive inverse exists, *i.e.*, to each element $a \in S$ there exists an element $-a \in S$ such that

$$a + (-a) = 0$$

M-1. S is closed for multiplication,

$$a, b \in S \Rightarrow ab \in S$$

M-2. Multiplication is commutative,

$$ab = ba$$

M-3. Multiplication is associative,

$$(ab)c = a(bc)$$

M-4. Multiplicative identity exists, *i.e.*, $\exists$ a member 1 in S such that

$$a \cdot 1 = a$$

M-5. Multiplicative inverse exists, *i.e.*, to each $0 \neq a \in S$, $\exists$ an element $a^{-1} \in S$ such that

$$aa^{-1} = 1$$

A-M. Multiplication is distributive with respect to addition, *i.e.*,

$$a(b + c) = ab + ac$$

Thus, a set S has a **field** structure if it possesses the two compositions of addition and multiplication and satisfies the eleven properties listed above.

2.2 Order Structure

Ordinarily the order relation does not exist between the members of a general field, but as we are to deal with the field of real numbers, we can speak of one number being 'greater than' (or less than) the other.

A *field* S is an **ordered field** if it satisfies the following properties:

O-1. *Law of Trichotomy*: For any two elements $a, b \in S$, one and only one of the following is true.

$$a > b,\ a = b,\ b > a$$

O-2. *Transitivity*: $\forall\ a, b, c \in S$,

$$a > b \wedge b > c \Rightarrow a > c$$

O-3. *Compatibility of Order Relation with Addition Composition*:

$$\forall\ a, b, c \in S,$$

$$a > b \Rightarrow a + c > b + c$$

O-4. *Compatibility of Order Relation with Multiplication Composition*:

$$\forall\ a, b, c \in S,$$

$$a > b \wedge c > 0 \Rightarrow ac > bc$$

2.3 It may be seen that the set **Q** of rational numbers and the set **R** of real numbers are ordered fields while the set **N** of natural numbers and the set **I** of integers are not fields.

(i) The Set N of Natural Numbers

We begin the development of real numbers with the set **N** of natural numbers: 1, 2, 3, ... we could certainly list many properties of natural numbers, however, the following are taken as axioms:

P_1: $1 \in \mathbf{N}$; that is, **N** is a non-empty set and contains an element we designate as 1.

P_2: For each element $n \in \mathbf{N}$ there is a unique element $n_0 \in \mathbf{N}$ called the successor of n.

P_3: For each $n \in \mathbf{N}$, $n_0 \neq 1$; that is, 1 is not the successor of any element in **N**.

P_4: For each pair $n, m \in \mathbf{N}$ with $n \neq m$, $n_0 \neq m_0$, that is, distinct elements in **N** have distinct successors.

P_5: If $A \subseteq \mathbf{N}$, $1 \in A$ and $p \in A$ implies $p_0 \in A$, then $A = \mathbf{N}$.

These five axioms are called *Peano's postulate* and all known properties of natural numbers can be shown to be the consequences of these.

P_5 is called the principle of *Mathematical Induction*. From the principle of Mathematical Induction it follows that "Every non-empty subset of natural numbers has a first element", this is called the *well-ordering principle* for **N**.

The sum or product of any two members is easily seen to be a member of **N**, so that the set possesses two compositions of addition and multiplication, *i.e.*, the set **N** possesses an *algebraic structure*. However, it does not satisfy all the properties of a field (it does not possess additive identity, additive inverse and multiplicative inverse) and hence the set of natural numbers is not a field. However, it has an *order structure* compatible with the *algebraic structure*.

(*ii*) **The Set I of Integers**

$$\ldots., -3, -2, -1, 0, 1, 2, 3, \ldots$$

It may be easily seen that the set possesses an *algebraic structure* but does not satisfy all the properties of a field. (M-5–Multiplicative inverses do not exist.) Hence, the set of integers is not a field. However, it has an *order structure* compatible with the *algebraic structure*.

(*iii*) **The Set Q of Rational Numbers**

A rational number is of the form p/q, where p and q are integers and $q \neq 0$. Evidently, the set **Q** of rational numbers includes the set of integers.

A real number which is not rational (*i.e.*, cannot be expressed as p/q) is called an **irrational number**.

The set **R** of **real numbers** consists of rational and irrational numbers.

The sets **Q** and **R** satisfy all the properties (§ 2.1) of a field and are, therefore, called Fields. In addition to this, both these fields satisfy the four properties 0–1 to 0–4 (§ 2.2) of order, and hence form **ordered fields**.

2.4 Upto this stage we have discussed two properties—*the field property* and *the order structure property*. We have found that both the sets, the set **R** of real numbers and the set **Q** of rational numbers possess these properties. However, there is a property called the property of **completeness** which is possessed only by the set of real numbers and this distinguishes it from other sets of numbers. Let us, now, consider some notions and examples which will facilitate the study of that property.

2.5 Example 1. Show that there is no rational number whose square is 2.

■ Let, if possible, there exist a rational number p/q, where $q \neq 0$ and p, q are integers prime to each other (*i.e.* having no common factor) whose square is equal to 2,

i.e.,
$$(p/q)^2 = 2 \text{ or } p^2 = 2q^2 \quad \ldots(1)$$

Now q is an integer and so is $2q^2$. Thus, p^2 is an integer divisible by 2. As such p must be divisible by 2, for otherwise p^2 would not be divisible by 2.

Let $p = 2m$, where m is an integer. Then, from (1),

$$2m^2 = q^2 \quad \ldots(2)$$

Thus, it follows that q is also divisible by 2. Hence, p and q are both divisible by 2 which contradicts the hypothesis that p and q have no common factor. Thus, there exists no rational number whose square is 2.

Example 2. Show that $\sqrt{8}$ is not a rational number.

■ Let, if possible, $\sqrt{8}$ be the rational number p/q, where $q \neq 0$ and p, q are positive integers prime to each other, so that $\sqrt{8} = p/q$.

But $2 < \sqrt{8} < 3$

$\therefore$
$$2 < p/q < 3 \Rightarrow 2q < p < 3q \text{ or } 0 < p - 2q < q$$

Thus, $p - 2q$ is a positive integer less than q, so that

$\sqrt{8}\,(p - 2q)$ or $p/q\,(p - 2q)$ is not an integer.

But $\sqrt{8}(p-2q) = p/q(p-2q) = \frac{p^2}{q} - 2p$

$$\Rightarrow \frac{p^2}{q^2}q - 2p = 8q - 2p, \text{ which is an integer.}$$

$\Rightarrow \sqrt{8}(p-2q)$ is an integer.

Thus, we arrive at a contradiction.

Hence, $\sqrt{8}$ is not a rational number.

Remark: We have considered $\sqrt{n}$ (n–not a perfect square), first when n was a prime and then n as a composite number. The procedures shown are typical and may be adopted under similar situations.

Ex. Show that there is no rational number whose square is

(*i*) 3, (*ii*) 5, (*iii*) 6.

2.6 Intervals – Open and Closed

A subset A of **R** is called an **interval** if A contains (*i*) at least two distinct elements and (*ii*) every element lies between any two members of A.

Open Interval. If a and b are two *real numbers* such that $a < b$, then the set

$$\{x : a < x < b\}$$

consisting of *all real numbers* between a and b (excluding a and b) is called an open interval and is denoted by $]a, b[$ or (a, b).

Closed Interval. The set

$$\{x : a \le x \le b\}$$

consisting of a, b and all real numbers lying between a and b is called a **closed interval** and is denoted by $[a, b]$.

Semi-closed or Semi-open Intervals.

$$]a, b] = \{x : a < x \le b\}$$

$$[a, b[= \{x : a \le x < b\}$$

The intervals are semi-closed or semi-open. The former is open at a and closed at b while the latter is closed at a and open at b.

3. BOUNDED AND UNBOUNDED SETS: SUPREMUM, INFIMUM

A subset S of real numbers is said to be **bounded above** if $\exists$ a real number K such that every member of S is less than or equal to K, *i.e.*,

$$x \le K, \ \forall x \in S$$

The number K is called an **upper bound** of S. If no such number K exists, the set is said to be **unbounded above** or **not bounded above**.

The set S is said to be **bounded below** if $\exists$ a real number k such that every member of S is greater than or equal to k, *i.e.*,

$$k \leq x, \ \forall \ x \in S$$

The number k is called a **lower bound** of S. If no such number k exists, the set is said to be **unbounded below** or **not bounded below.**

A set is said to be **bounded** if it is bounded above as well as below.

It may be seen that if a set has one upper bound, it has an infinite number of upper bounds. For if K is an upper bound of a set S then every number greater than K is also an upper bound of S. Thus every set S bounded above determines an infinite set—the set of its upper bounds. This set of upper bounds is bounded below in as much as every member of S is a lower bound thereof. Similarly, a set S bounded below determines an infinite set of its lower bounds, which is bounded above by the members of S.

A member G of a set S is called the **greatest** member of S if every member of S is less than or equal to G, *i.e.*,

(*i*) $G \in S$

(*ii*) $x \leq G, \ \forall \ x \in S$

Similarly, a member g of the set is its **smallest** (or the least) member if every member of the set is greater than or equal to g.

Clearly, a set may or may not have the greatest or the least member but an upper (lower) bound of the set, if it is a member of the set, is its greatest (least) member. A finite set always has the greatest as well as the smallest member.

If the set of all upper bounds of a set S has the smallest member, say M, then M is called the **least upper bound** (l.u.b.) or the **supremum** of S.

Clearly, the supremum of a set S may or may not exist and in case it exists, it may or may not belong to S. The fact that supremum M is the smallest of all the upper bounds of S may be described by the following two properties:

(*i*) M is the upper bound of S, *i.e.*,

$$x \leq M, \ \forall \ x \in S$$

(*ii*) No number less than M can be an upper bound of S, *i.e.*, for any positive number ε, however small, $\exists$ a number $y \in S$ such that

$$y > M - \varepsilon$$

Again it may be seen that *a set cannot have more than one supremum*. For, let if possible M and M' be two suprema of a set S, so that M and M' are both upper bounds of S.

Also M is the l.u.b. and M' is an upper bound of S.

$$\therefore \quad M \leq M' \quad \quad ...(1)$$

Again M' is the l.u.b. and M is an upper bound of S.

$$\therefore \quad M' \leq M \quad \quad ...(2)$$

From (1) and (2), it follows that $M = M'$.

If the set of all lower bounds of a set S has the greatest member, say m, then m is called the **greatest lower bound** (g.l.b.) or the **infimum** of S.

Like the supremum, the infimum of a set may or may not exist and it may or may not belong to S. It can be easily shown that a set cannot have more than one infimum.

The infimum m of a set S has the following two properties:

(*i*) m is the lower bound of S, *i.e.*,

$$m \leq x, \ \forall \ x \in S$$

(*ii*) No number greater than m can be a lower bound of S, *i.e.*, for any positive number however small, $\exists$ a number $z \in S$ such that

$$z < m + \varepsilon$$

ILLUSTRATIONS

1. The set **N** of natural numbers is bounded below but not bounded above. 1 is a lower bound.
2. The sets **I**, **Q** and **R** are not bounded.
3. Every finite set of numbers is bounded.
4. The set S_1 of all positive real numbers $S_1 = \{x : x > 0, x \in \mathbf{R}\}$ is not bounded above, but is bounded below. The infimum zero is not a member of the set S_1.
5. The infinite set $S_2 = \{x : 0 < x < 1, x \in \mathbf{R}\}$ is bounded with supremum 1 and infimum zero, both of which do not belong to S_2.
6. The infinite set $S_3 = \{x : 0 \leq x \leq 1, x \in \mathbf{Q}\}$ is bounded, with supremum 1 and infimum 0 both of which are members of S_3.
7. The set $S_4 = \left\{\frac{1}{n} : n \in \mathbf{N}\right\}$ is bounded. The supremum 1 belongs to S_4 while infimum 0 does not.
8. Each of the following intervals is bounded:

$$[a, b],]a, b], [a, b[,]a, b[.$$

Example 3. Prove that the greatest member of a set, if it exists, is the supremum (l.u.b.) of the set.

■ Let G be the greatest member of the set S.

Clearly,

$$x \leq G, \ \forall \ x \in S$$

so that G is an upper bound of S.

Again no number less than G can be an upper bound of S, for if y be any number less than G, there exists at least one member g of S which is greater than y.

Thus, G is the least of all the upper bounds of S, *i.e.*, G is the supremum of S.

EXERCISE

1. Give examples of sets which are:

(*i*) Bounded,

(*ii*) Not bounded,

(*iii*) Bounded below but not bounded above,

(*iv*) Bounded above but not bounded below.

2. Find the infimum and the supremum of the following sets. Which of these belongs to the set?

(i) $\{1, 3, 5, 7, 9\}$

(ii) $\left\{-1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \ldots\right\}$

(iii) $\left\{\frac{1}{n} ; n \in \mathbf{N}\right\}$

(iv) $\left\{\frac{(-1)^n}{n} ; n \in \mathbf{N}\right\}$

(v) $\left\{-2, -\frac{3}{2}, -\frac{4}{3}, -\frac{5}{4}, \ldots, -\frac{n+1}{n}, \ldots\right\}$

(vi) $\left\{1 + \frac{(-1)^n}{n} ; n \in \mathbf{N}\right\}$

(vii) $]a, b[$

(viii) $[a, b[$.

3. Which of the sets in question 2 are bounded?
4. Find the smallest and the greatest members (if they exist) for sets in question 2.
5. Show that the greatest (or the smallest) member of a set, in case it exists, is unique.
6. Show that the smallest member of a set, if it exists, is the infimum of the set.
7. Is the converse of the solved example 3, true?
8. If $S \subseteq T \subseteq \mathbf{R}$, where $S \neq \phi$, then show that

   (i) If T is bounded above, then $\sup S \leq \sup T$;

   (ii) If T is bounded below, then $\inf T \leq \inf S$.

ANSWERS

2. (i) 1, 9; both
   (ii) $-1, 0$; infimum
   (iii) 0, 1; supremum
   (iv) $-1, \frac{1}{2}$; both
   (v) $-2, -1$; infimum
   (vi) $0, \frac{3}{2}$; both
   (vii) a, b; none
   (viii) a, b; infimum.
3. All sets are bounded.
4. (i) 1, 9
   (ii) -1, does not exist
   (iii) does not exist, 1
   (iv) $-1, \frac{1}{2}$
   (v) -2, does not exist
   (vi) $0, \frac{3}{2}$
   (vii) do not exist
   (viii) a, does not exist

4. COMPLETENESS IN THE SET OF REAL NUMBERS

We have seen that all the properties—the properties of an ordered field, described so far, are possessed by the two sets, the set of real numbers $\mathbf{R}$ and the set of rational numbers $\mathbf{Q}$. We shall now state a property, the property of *completeness* (or *order-completeness*) which is possessed by $\mathbf{R}$ and not by $\mathbf{Q}$. This property not only distinguishes $\mathbf{R}$ from $\mathbf{Q}$, but together with the ordered field property, it characterises $\mathbf{R}$, *i.e.*, the set of real numbers is the only set which is a *Complete Ordered Field.*

4.1 Order-Completeness in R

(O-C) *Every non-empty set of real numbers which is bounded above has the supremum (or the least upper bound) in* $\mathbf{R}$.

In other words, the set of upper bounds of a non-empty set of real numbers bounded above has the smallest member.

If S is a set of real numbers which is bounded above, then by considering the set $T = \{x : -x \in S\}$ we may state the completeness property in the alternative form as:

Every non-empty set of real numbers which is bounded below has the infimum (or *g.l.b.*) *in* **R**. Or, equivalently the set of lower bounds of a non-empty set of real numbers bounded below has the greatest member.

We have thus completed the description of the set of real numbers as a **Complete Ordered Field**. We shall, however, show that the property of *completeness* does not hold good for the ordered field of rational numbers, *i.e.*, the ordered field **Q** of rationals is not order complete.

Theorem 1. *The set of rational numbers is not order-complete.*

To show that the set of rational numbers does not possess the property of completeness, it is suffice to show that there exists a non-empty set S of rationals (a subset of **Q**) which is bounded above but does not have a supremum in **Q**, *i.e.*, no rational number exists which can be the supremum of S.

Let S be the set (a subset of **Q**) of all those positive rational numbers whose square is less than 2.

i.e., $$S = \{x : x \in \mathbf{Q},\, x > 0 \wedge x^2 < 2\}$$

Since $1 \in S$, the set S is non-empty.

Clearly 2 is an upper bound of S, therefore, S is bounded above.

Thus, S is a non-empty set of rational numbers, bounded above. Let, if possible, the rational number K be its least upper bound. Clearly K is positive. Also by the law of trichotomy (0–1) which holds good in **Q**, one and only one of (*i*) $K^2 < 2$, (*ii*) $K^2 = 2$, (*iii*) $K^2 > 2$ holds.

(*i*) $K^2 < 2$. Let us consider the positive rational number

$$y = \frac{4+3K}{3+2K}$$

Then,

$$K - y = K - \frac{4+3K}{3+2K} = \frac{2(K^2-2)}{3+2K} < 0$$

$$\Rightarrow \qquad y > K \qquad \text{...(1)}$$

Also,

$$2 - y^2 = 2 - \left(\frac{4+3K}{3+2K}\right)^2 = \frac{2-K^2}{(3+2K)^2} > 0$$

$$\Rightarrow \qquad y^2 < 2 \Rightarrow y \in S \qquad \text{...(2)}$$

Thus, the member y of S is greater than K, so that K cannot be an upper bound of S and hence, there is a contradiction.

(*ii*) $K^2 = 2$. We have already shown that there exists no rational number whose square is equal to 2. Thus, this case is not possible.

(*iii*) $K^2 > 2$. Considering the positive rational number y as defined in case (*i*), we may easily deduce from (1) and (2) respectively that

$$y < K \text{ and } y^2 > 2$$

Hence, there exists an upper bound y of S smaller than the least upper bound K, which is a contradiction.

Thus, none of the three possible cases holds. Hence, our supposition that a rational number K is the least upper bound of S is wrong. Thus, no rational number exists which can be the least upper bound of S.

Note: If we admit K in **R** and regard **S** as a set of real numbers then by the order completeness property, the supremum K of S exists in **R**. Clearly $K > 0$ and

$$K^2 < 2 \Rightarrow y^2 < 2 \wedge y > K \Rightarrow K \neq \text{Sup } S$$

$$K^2 > 2 \Rightarrow y^2 > 2 \wedge y < K \Rightarrow K \neq \text{Sup } S$$

Thus by property 0–1, it follows that $K^2 = 2$, *i.e.*, the least upper bound K exists whose square is equal to 2. Further, since $K \notin \mathbf{Q}$, it follows that K is an irrational number. Similarly, it may be seen that there exist real numbers other than rational numbers whose squares are 2, 5, 7, ... etc. This establishes the *existence of irrational numbers*.

Ex. Show that the set of natural numbers is order-complete.

4.2 Archimedean Property of Real Numbers

The order-completeness property has important consequences, one of which is the Archimedean property of real numbers which we now proceed to prove.

Theorem 2. *The real number field is Archimedean, i.e., if a and b are any two positive real numbers then there exists a positive integer n such that na > b.*

Let a, b be any two positive real numbers and suppose, if possible, that for all positive integers $n(\in \mathbf{I}^+)$, $na \leq b$.

Thus, the set $S = \{na\colon n \in \mathbf{I}^+\}$ is bounded above, b being an upper bound. By the completeness property of the ordered-field of real numbers, set S must have the supremum M.

$$na \leq M, \quad \forall\, n \in \mathbf{I}^+$$

$$\therefore \quad (n+1)\, a \leq M, \quad \forall\, n \in \mathbf{I}^+$$

$$\Rightarrow \quad na \leq M - a, \quad \forall\, n \in \mathbf{I}^+$$

$\Rightarrow$ *i.e.*, $M - a$ is an upper bound of S.

Thus, a number $M - a$ less than the supremum M (l.u.b.) is an upper bound of S, which is a contradiction and hence our supposition is wrong.

Hence, the theorem.

Corollary 1. If a be a positive real number and b, any real number then there exists a positive integer n such that $na > b$.

Corollary 2. For any positive real number a there exists a positive integer n such that $n > a$.

The result follows by considering the two positive real numbers 1 and a.

Corollary 3. For any $\varepsilon > 0$ there exists a positive integer n such that $1/n < \varepsilon$.

The result follows by taking $a = 1/\varepsilon$ in Corollary 2.

Corollary 4. If a be any real number then there exists a positive integer n such that $n > a$.

For $a \le 0$, any positive integer $n > a$, and for $a > 0$, result follows by Corollary 2.

Theorem 3. *Every open interval* $]\,a, b\,[$ *contains a rational number.*

Case I. If $0 < a < b$, by Corollary 3 there is a $m \in \mathbf{N}$ such that $1/m < (b - a)$. Let $A = \left\{ n \in \mathbf{N} : \frac{n}{m} > a \right\}$

By Archimedean property $A \ne \phi$. Now by the well ordering principle for $\mathbf{N}$, A has a first element say n_0 and so $n_0 - 1 \notin A$.

$$\therefore \qquad \frac{n_0 - 1}{m} \le a$$

$$\Rightarrow \qquad \frac{n_0}{m} \le a + \frac{1}{m} < a + (b - a)$$

$$\Rightarrow \qquad \frac{n_0}{m} < b. \qquad \text{But } n_0 \in A, \qquad \therefore \frac{n_0}{m} > a$$

Hence, there exists a rational number n_0/m in the open interval $]a, b[$.

Case II. If $a \le 0 < b$. Again by Corollary 3 there is a $n \in \mathbf{N}$ with $1/n < b$

Clearly $1/n \in \,]\,a, b\,[$.

Case III. $a < b \le 0$, then $0 \le -b < -a$. By the previous cases there is a rational number $q \in \,]-b, -a\,[$ and so the rational number $-q \in \,]\,a, b\,[$.

Corollary 5. Every open interval $]\,a, b\,[$ contains infinitely many rational numbers.

4.3 Dedekind's Form of Completeness Property

We now state the completeness property of real numbers in another form, due to Dedekind, which states:

If all the real numbers be divided into two non-empty classes L and U such that every member of L is less than every member of U, then there exists a unique real number, say α, such that every real number less than α belongs to L and every real number greater than α belongs to U.

Clearly, the two classes L and U so defined are disjoint and the number α itself belongs either to L or U. The property of real numbers referred to above is known as *Dedekind's property*. We may restate **Dedekind's Property**.

If L and U are two subsets of $\mathbf{R}$ such that

(*i*) $L \ne \phi$, $U \ne \phi$ (each class has at least one member),

(*ii*) $L \cup U = \mathbf{R}$ (every real number has a class)

(*iii*) Every member of L is less than every member of U, *i.e.*,

$$x \in L \wedge y \in U \Rightarrow x < y,$$

then either L has the greatest member or U has the smallest member.

4.4 Let us now prove the **equivalence** of the two forms of completeness.

(*a*) First we show that the *order completeness property of real numbers implies Dedekind's property*. The set $\mathbf{R}$ has the order completeness property, *i.e.*, every non-empty subset of $\mathbf{R}$ which is bounded above (below) has the Supremum (Infimum).

Let L, U be two subsets of $\mathbf{R}$ such that

(*i*) $L \neq \phi, U \neq \phi$,

(*ii*) $L \cup U = \mathbf{R}$, and

(*iii*) Every member of L is less than every member of U.

We have to show that either L has the greatest member or U has the smallest.

By (*iii*) the non-empty set L is bounded above. If L has the greatest member, it establishes the result. If L has no greatest member, then by the order completeness property, the set of its upper bounds, which coincides with U, has the smallest member. Thus either L has the greatest member or U has the smallest member.

(*b*) Let, now, $\mathbf{R}$ *satisfy the Dedekind's property*. We shall show that $\mathbf{R}$ also satisfies the order completeness property.

Let S be a non-empty set of real numbers bounded above, then we have to prove that S has the supremum.

Let L and U be two sets of real numbers defined by

$L = \{x: x \text{ is not an upper bound of } S\}$,

$U = \{x: x \text{ is an upper bound of } S\}$.

It may be easily seen that

(*i*) $L \neq \phi, U \neq \phi$,

(*ii*) $L \cup U = \mathbf{R}$, and

(*iii*) $x \in L \wedge y \in U \Rightarrow x < y$.

Then by Dedekind property, either L has the greatest member or U has the smallest member.

We shall show that L cannot have the greatest member.

Let, if possible, L has the greatest member, say ξ. Then

$$\xi \in L \Rightarrow \xi \text{ is not an upper bound of } S$$

$$\Rightarrow \quad \exists \text{ an } a \in S \text{ such that } \xi < a.$$

Now the real number $\dfrac{\xi + a}{2}$ is such that

$$\xi < \frac{\xi + a}{2} < a$$

Since $\dfrac{\xi + a}{2}$ is greater than the greatest member ξ of L,

$$\therefore \quad \frac{\xi + a}{2} \in U$$

$$\Rightarrow \quad \frac{\xi + a}{2} \text{ is an upper bound of } S. \qquad \ldots(1)$$

Again, since $\dfrac{\xi + a}{2}$ is less than the member a of S,

$$\therefore \qquad \frac{\xi + a}{2} \in L$$

$$\Rightarrow \qquad \frac{\xi + a}{2} \text{ is not an upper bound of } S.$$

Thus, we arrive at contradictory conclusions, and as such L has no greatest member. Thus, it follows that U, the set of upper bounds of S, has the smallest member, *i.e.*, the set S has the supremum.

We have thus proved the equivalence of Dedekind's and the order completeness property of **R**.

4.5 Explicit Statement of the Properties of the Set of Real Numbers as a Complete-Ordered Field

The set **R** of real numbers is a *complete-ordered field* because for arbitrary members, a, b, c of R, it satisfies the following conditions:

A-1. $a, b \in \mathbf{R} \Rightarrow a + b \in \mathbf{R}$

A-2. $a + b = b + a$

A-3. $(a + b) + c = a + (b + c)$

A-4. $\exists$ a member 0 in **R** such that

$$a + 0 = a$$

A-5. To each $a \in \mathbf{R}, \exists$ an element $-a \in R$ such that

$$a + (-a) = 0$$

M-1. $a, b \in \mathbf{R} \Rightarrow ab \in \mathbf{R}$

M-2. $ab = ba$

M-3. $(ab)c = a(bc)$

M-4. $\exists$ a member 1 in **R** such that

$$a \cdot 1 = a$$

M-5. To each $a \neq 0 \in \mathbf{R}, \exists$ an element $a^{-1} \in \mathbf{R}$ such that

$$aa^{-1} = 1.$$

A-M. $a(b + c) = ab + ac.$

O-1. For any two elements a, b of **R**, one and only one of the following is true:

$$a > b, \quad a = b, \quad b > a$$

O-2. $a > b \wedge b > c \Rightarrow a > c$

O-3. $a > b \Rightarrow a + c > b + c$

O-4. $a > b \wedge c > 0 \Rightarrow ac > bc$

OC. Every non-empty subset of **R** which is bounded above (below) has the supremum (infimum) in **R**.

4.6 Representation of Real Numbers as Points on a Straight Line

Points on a line can be used to represent real numbers. This geometrical representation of real numbers is sometimes very useful and suggestive especially to the beginner. But this should not stop us from giving the proper proof of a theorem which may otherwise seem to be obvious.

Let $X'X$ be a straight line. Mark two points O and A on it such that A is to the right of O.

The point O divides the line XX' into two parts; the part to the right of O containing A, may be called positive and that to the left of O as negative. Such a line for which positive and negative sides are fixed is called a *directed line*.

Let us consider the points O and A to represent rational numbers zero and 1 respectively, so that the distance OA is unity on a certain scale. To represent a rational number m/n $(n > 0)$, take a point P on the right of O if m is positive and to the left of O if m is negative, such that OP in m times the nth part of the unit length OA. Of course, the point P coincides with O if m is zero. The point P thus represents the rational number m/n. We may say that the rational number m/n corresponds to the point P or the point P corresponds to the rational number m/n. This way any rational number can be made to correspond to a point on the line. If points on the line corresponding to rational numbers be termed as *rational points*. We see that infinite number of rational points lie between any two different rational points, *i.e.*, *between any two rationals, there lie infinitely many rationals*.

Even though the rational points seem to cover a straight line very closely, there remain points on the line which are not rational. For example, the point Q on the line such that OQ is equal to the diagonal of the square with side OA does not correspond to any rational number. Also a point R such that OR which is a rational multiple of OQ, is also such a point. In fact there are infinitely many such points on the line. Hence, the set **Q** of rational numbers is insufficient to provide a complete picture of the straight line.

Such points on the line which are not rational, and which may be supposed to fill up the gaps between rational points are called *irrational points* and these correspond to irrational numbers. In fact, there is at least one irrational between two rationals. Thus like rationals, there are infinitely many irrationals. Hence, every real number can be represented on the directed line and there seem to be as many points on the directed line as the real numbers. The same fact is expressed by **Dedekind-Cantor Axiom** which states:

To every real number there corresponds a unique point on a directed line and to every point on a directed line there corresponds a unique real number.

In view of the order completeness property, the set of real numbers **R** does not have gaps of the kind **Q** has, and thus forms a continuous system. On account of this characteristic, the set **R** is called the *Arithmetical Continuum* and the set of points on a line as the *Geometrical Continuum*. In view of the above axiom, we see that there is a one-one correspondence between the two continuum and accordingly we may use the word *point* for *real number*, and the *real line* for the *directed line*. It is evident that between any two real numbers, there exist infinitely many real numbers both rational and irrational. This is the property of *denseness* of the real number system.

5. ABSOLUTE VALUE OF A REAL NUMBER

The *absolute value*, the *numerical value* or the *modulus* of a real number x, denoted by $|x|$, is defined as

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

Thus we always have

$$|x| \geq 0$$

Also by definition

$$|-x| = |x|$$

Some theorems which are immediate consequences of the definition will now follow:

Theorem 4. $|x| = \max(x, -x)$

Now $|x| = x \geq -x$, if $x \geq 0$

Also $|x| = -x > x$, if $x < 0$

Thus in either case $|x|$ is greater of the two numbers, x, $-x$, *i.e.*,

$$|x| = \max(x, -x).$$

Corollary 1. $|-x| = \max(-x, -(-x))$

$$= \max(-x, x) = |x|$$

$\therefore$ $|-x| = |x|$.

Corollary 2. $|x| = \max(x, -x) \geq x$

$\therefore$ $|x| \geq x$.

Theorem 5. $-|x| = \min(x, -x)$

Now

$$-|x| = -x < x, \text{ if } x > 0$$

Also

$$-|x| = -(-x) = x < -x, \text{ if } x < 0$$

Thus in either case $-|x|$ is smaller of the two numbers x and $-x$, *i.e.*,

$$-|x| = \min(x, -x)$$

Corollary. $-|x| = \min(x, -x) \leq x$.

$\therefore$ $-|x| \leq x$

Theorem 6. If $x, y \in \mathbf{R}$, then

(*i*) $|x|^2 = x^2 = |-x|^2$

(*ii*) $|xy| = |x| \cdot |y|$

(*iii*) $\left|\dfrac{x}{y}\right| = \dfrac{|x|}{|y|}$, provided $y \neq 0$

(*i*) For $x \geq 0$,

$$|x| = x \Rightarrow |x|^2 = x^2$$

For $x < 0$,

$$|x| = -x \Rightarrow |x|^2 = (-x)^2 = x^2$$

Thus in either case $|x|^2 = x^2$

Similarly, $|-x|^2 = (-x)^2 = x^2$

Hence, $|x|^2 = x^2 = |-x|^2$

(*ii*) $|xy|^2 = (xy)^2 = x^2y^2 = |x|^2 \cdot |y|^2 = (|x| \cdot |y|)^2$

$\therefore \qquad |xy| = \pm |x| \cdot |y|$

But since $|xy|$ and $|x| \cdot |y|$ are both non-negative, we take only the positive sign.

$\therefore \qquad |xy| = |x| \cdot |y|$

(*iii*) $$\left|\frac{x}{y}\right|^2 = \left(\frac{x}{y}\right)^2 = \frac{x^2}{y^2} = \frac{|x|^2}{|y|^2} = \left(\frac{|x|}{|y|}\right)^2$$

But since $\left|\frac{x}{y}\right|$ and $\frac{|x|}{|y|}$ are both non-negative, therefore taking positive square root of both sides, we have

$$\left|\frac{x}{y}\right| = \frac{|x|}{|y|}, \text{ when } y \neq 0.$$

Theorem 7. *Triangle inequalities.* *For all real numbers x, y show that*

(*i*) $|x + y| \leq |x| + |y|$, and

(*ii*) $|x - y| \geq ||x| - |y||$.

(*i*) *First Method:*

$$|x + y|^2 = (x + y)^2 = x^2 + y^2 + 2xy$$

$$\leq |x|^2 + |y|^2 + 2|x| \cdot |y| \qquad [\because xy \leq |xy| = |x| \cdot |y|]$$

$$= (|x| + |y|)^2$$

Since $|x + y|$ and $|x| + |y|$ are both non-negative, therefore taking positive square roots on both sides, we have

$$|x + y| \leq |x| + |y|$$

Second Method: When $x + y \geq 0$.

$$|x+y| = x+y$$

$$\leq |x| + |y| \qquad [\because\ x \leq |x| \text{ and } y \leq |y|]$$

When $x + y < 0$,

$$|x+y| = -(x+y) = (-x) + (-y)$$

$$\leq |-x| + |-y| \qquad [\because\ -x \leq |-x| \text{ and } -y \leq |-y|]$$

But $|-x| = |x|$, $|-y| = |y|$

Thus in either case,

$$|x+y| \leq |x| + |y|.$$

(*ii*) *First Method*:

$$|x-y|^2 = (x-y)^2 = x^2 + y^2 - 2xy$$

$$\geq |x|^2 + |y^2| - 2|x|\cdot|y|$$

$$[\because\ -(xy) \geq -|xy| = -|x|\cdot|y|]$$

$$= (|x| - |y|)^2 = ||x| - |y||^2$$

Since $|x - y|$ and $||x| - |y||$ are both non-negative, therefore taking the positive square root of both sides, we have

$$|x-y| \geq ||x| - |y||.$$

Second Method: Now

$$|x| = |(x-y) + y| \leq |x-y| + |y| \qquad \text{[by part (}i\text{)]}$$

$$\therefore \quad |x-y| \geq |x| - |y| \qquad \ldots(1)$$

Again,

$$|y| = |(y-x) + x| \leq |y-x| + |x|$$

$$\therefore \quad |y-x| \geq |y| - |x| = -(|x| - |y|)$$

But $\quad |y-x| = |x-y|$

$$\therefore \quad |x-y| \geq -(|x| - |y|) \qquad \ldots(2)$$

From (1) and (2),

$$|x-y| \geq \max\{|x| - |y|, -(|x| - |y|)\}$$

$$= ||x| - |y||$$

Hence, $\quad |x-y| \geq ||x| - |y||$

Example 4. For real numbers $x, a, \varepsilon > 0$ show that

(*i*) $|x| < \varepsilon \Leftrightarrow -\varepsilon < x < \varepsilon$,

(*ii*) $|x - a| < \varepsilon \Leftrightarrow a - \varepsilon < x < a + \varepsilon$.

■ (*i*) $|x| = \max(x, -x) < \varepsilon$

$\Leftrightarrow \quad x < \varepsilon \wedge -x < \varepsilon$

$\Leftrightarrow \quad x < \varepsilon \wedge -\varepsilon < x$

$\Leftrightarrow \quad -\varepsilon < x < \varepsilon$

(*ii*) $|x - a| = \max\{(x - a), -(x - a)\} < \varepsilon$

$\Leftrightarrow \quad (x - a) < \varepsilon \wedge -(x - a) < \varepsilon$

$\Leftrightarrow \quad x < a + \varepsilon \wedge a - \varepsilon < x$

$\Leftrightarrow \quad a - \varepsilon < x < a + \varepsilon$

Example 5. Show that a set S of real numbers is bounded if there exists a real number $G > 0$ such that

$$|x| \leq G, \quad \forall\, x \in S.$$

■ Suppose that S is bounded, therefore it is bounded both above and below. Let K be an upper bound and k, a lower bound for S.

On taking a real number $G = \max(|K|, |k| + 1)$, we have

$$K \leq |K| \leq G \text{ and}$$

$$-k \leq |k| < |k| + 1 \leq G \qquad \textit{i.e.,} \quad k > -G$$

This implies

$$-G < k \leq x \leq K \leq G, \quad \forall\, x \in S$$

Hence,

$$|x| \leq G \quad \forall\, x \in S.$$

The converse is trivial.

Ex. If a and b are real numbers, then show that

$$\max(a, b) = \frac{a + b + |a - b|}{2}$$

and

$$\min(a, b) = \frac{a + b - |a - b|}{2}$$

Example 6. If $a, b \in \mathbf{R}$ such that $a < b + \varepsilon$ for each $\varepsilon > 0$, then $a \leq b$.

■ Suppose $a > b$. Then $a - b > 0$, so that

$$a < b + (a - b) \quad \text{(by taking } \varepsilon = a - b\text{)}$$

and so $a < a$

This is a contradiction. Hence our assumption $a > b$ must be false. Therefore $a \le b$.

Example 7. Let $a, b \in \mathbf{R}$. Show that if $a \le b + \frac{1}{n}$, for all $n \in \mathbf{N}$, then $a \le b$.

■ Assume $a \le b + \frac{1}{n}$, for all $n \in \mathbf{N}$ and $a > b$

Then $a - b > 0$ and by the Archimedean property, we have

$n_0(a - b) > 1$, for some $n_0 \in \mathbf{N}$

Then $a > b + \frac{1}{n_0}$, contrary to our assumption.

Example 8. If for any $\varepsilon > 0, |b - a| < \varepsilon$, then $b = a$

■ We have, for any $\varepsilon > 0$, $b < a + \varepsilon$ and $a - \varepsilon < b$. Since $b < a + \varepsilon$ for any $\varepsilon > 0$, it follows that $b \le a$. Since $a < b + \varepsilon$ for any $\varepsilon > 0$ this implies $a \le b$. Hence, $b = a$.

Example 9. If $a, b \in \mathbf{R}$ and $a < c$ *for each* $c > b$, then $a \le b$.

■ Assume that a and b satisfy the hypothesis but not the conclusion. Then $a > b$, and so there is a $c \in \mathbf{R}$ such that $a > c > b$. Now $c > b \Rightarrow a < c$ in contradiction to $a > c$.

EXERCISE

Prove the following (Qs.1-3):

1. $|x - y| \le |x| + |y|$.
2. $|x + y| \ge ||x| - |y||$.
3. (*i*) $\sqrt{x^2 + y^2} \le |x| + |y|$,

   (*ii*) $\sqrt{|x + y|} \le \sqrt{|x|} + \sqrt{|y|}$.
4. If $x_1, x_2, x_3, ..., x_n$ are real numbers, then show that

   (*i*) $|x_1 + x_2 + ... + x_n| \le |x_1| + |x_2| + ... + |x_n|$

   (*ii*) $|x_1x_2 \dots x_n| = |x_1| \cdot |x_2| \dots |x_n|$.
5. If x and y are real numbers, then show that

$$\frac{|x + y|}{1 + |x + y|} \le \frac{|x|}{1 + |x|} + \frac{|y|}{1 + |y|}.$$

6. Prove that

$$|x + y| = |x| + |y| \text{ iff } xy \ge 0$$

and $|x + y| < |x| + |y|$ iff $xy < 0$.