12 Uniform Convergence

Thus far we have considered, almost exclusively, sequences and series whose terms were numbers. It was only in particularly simple cases that the terms depended on a variable. We shall now consider sequences and series, whose terms depend on a variable, *i.e.*, those whose terms are real valued functions defined on an *interval* as domain. We, accordingly, denote the terms by $f_n(x)$ and consider sequences and series of the form $\{f_n\}$ and Σf_n respectively.

1. POINTWISE CONVERGENCE

Suppose $\{f_n\}$, $n = 1, 2, 3, ...$, is a sequence of functions, defined on an interval I, $a \leq x \leq b$. To each point $\xi \in I$, there corresponds a sequence of numbers $\{f_n(\xi)\}$ with terms

$$f_1(\xi), f_2(\xi), f_a(\xi), \ldots$$

Further, let us suppose that the sequence of numbers $\{f_n(\xi)\}$ converges for every $\xi \in I$.

Let $\{f_n(\xi)\}$ converges to $f(\xi)$.

This way let the sequences at (all) points $\xi, \eta, \zeta, \ldots,$ of I converge to

$$f(\xi), f(\eta), f(\zeta), \qquad \ldots(1)$$

We now define, in a natural way, a real valued function f with domain I and range the set defined by (1), so that its value $f(\eta)$ for $\eta \in I$ is $\lim \{f_n(\eta)\}$.

Thus

$$f(x) = \lim_{n\to\infty} \{f_n(x)\}, \ \forall \ x \in I \qquad \ldots(2)$$

The function f, so defined, is referred to as the *limit* or the *point-wise limit* of the sequence $\{f_n\}$ on $[a, b]$, and the sequence $\{f_n\}$ is said to be *pointwise convergent* to f on $[a, b]$.

Similarly, if the series Σf_n converges for every point $x \in I$, and we define

$$f(x) = \sum_{n=0}^{\infty} f_n(x), \ \forall \ x \in [a, b] \qquad \ldots(3)$$

the function f is called the *sum* or the *pointwise sum* of the series Σf_n on $[a, b]$.

Thus, if f is the point-wise limit of a sequence of functions $\{f_n\}$ defined on $[a, b]$, then to each $\varepsilon > 0$ and to each $x \in [a, b]$, there corresponds an integer m such that

$$|f_n(x) - f(x)| < \varepsilon, \quad \forall \ n \geq m \qquad \ldots(4)$$

1.1 For a sequence (series) of variable terms, the most important question will usually be whether, and to what extent, properties belonging to terms, viz., boundedness, continuity, integrability, differentiability, etc., are transferred to the limit function of the corresponding sequence (series). Let us consider a few examples.

1. The geometric series

$$1+x+x^2+x^3+\dots$$

converges to $(1-x)^{-1}$ in the interval $-1<x<1$.

All the terms are bounded without the sum being so.

2. Consider the series

$$\sum_{n=0}^{\infty} f_n, \text{ where } f_n(x)=\frac{x^2}{(1+x^2)^n} \ (x \text{ real})$$

At $x=0$, each $f_n(x)=0$, so that the sum of the series $f(0)=0$.

For $x\neq 0$, it forms a geometric series with common ratio $1/(1+x^2)$, so that its sum function $f(x)=1+x^2$.

Hence,

$$f(x)=\begin{cases}1+x^2, & x\neq 0\\ 0 \quad , & x=0\end{cases}$$

Each term of the series is continuous but the sum f is not.

3. The sequence $\{f_n\}$, where $f_n(x)=\dfrac{\sin nx}{\sqrt{n}}$ $(x$ real$)$, has the limit

$$f(x)=\lim_{n\to\infty} f_n(x)=0$$

$\therefore$ $$f'(x)=0, \text{ and so } f'(0)=0$$

But

$$f_n'(x)=\sqrt{n}\cos nx$$

so that

$$f_n'(0)=\sqrt{n}\to\infty \text{ as } n\to\infty$$

Thus at $x=0$, the sequence $\{f_n'(x)\}$ diverges whereas the limit function $f'(x)=0$, *i.e.*, the limit of differentials is not equal to the differential of the limit.

4. Consider the sequence $\{f_n\}$, where

$$f_n(x)=nx(1-x^2)^n,\ 0\le x\le 1, n=1,2,3,\dots$$

For $0<x\le 1$, $\lim_{n\to\infty} f_n(x)=0$

At $x=0$, each $f_n(0)=0$, so that $\lim_{n\to\infty} f_n(0)=0$

Thus the limit function $f(x) = \lim_{n\to\infty} f_n(x) = 0$, for $0 \le x \le 1$

$$\therefore \qquad \int_0^1 f(x)\, dx = 0$$

Again,

$$\int_0^1 f_n(x)\, dx = \int_0^1 nx(1-x^2)^n\, dx = \frac{n}{2n+2}$$

so that

$$\lim_{n\to\infty} \left\{ \int_0^1 f_n(x)\, dx \right\} = \frac{1}{2}$$

Thus,

$$\lim_{n\to\infty} \left\{ \int_0^1 f_n\, dx \right\} \neq \int_0^1 f\, dx = \int_0^1 [\lim_{n\to\infty} \{f_n\}] dx.$$

Thus, the limit of integrals is not equal to the integral of the limit. In other words, the sequence of integrals may not converge to the integral of the limit of the sequence.

These few examples should convince us that a quite new category of problems arises with the consideration of sequences (series) of variable terms. We have to investigate under what supplementary conditions these or other properties of the terms f_n are transferred to the limit function f. A concept of great importance in this respect is known as *Uniform convergence* of a sequence (series) in its domain of definition, $[a, b]$.

2. UNIFORM CONVERGENCE ON AN INTERVAL

A sequence of functions $\{f_n\}$ is said to *converge uniformly* on an interval $[a, b]$ to a function f if for any $\varepsilon > 0$ and for *all* $x \in [a, b]$ there exists an integer N (independent of x but dependent on ε) such that for all $x \in [a, b]$,

$$|f_n(x) - f(x)| < \varepsilon, \quad \forall\, n \ge N \qquad \text{...(5)}$$

It is clear that every uniformly convergent sequence is pointwise convergent, and the uniform limit function is same as the pointwise limit function.

The difference between the two concepts is this: In case of pointwise convergence, for each $\varepsilon > 0$ and for each $x \in [a, b]$ there exists an integer N (depending on ε and x both) such that (4) holds for $n \ge N$; whereas in uniform convergence, for each $\varepsilon > 0$, it is possible to find one integer N (dependent on ε alone) which will do for all $x \in [a, b]$.

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Notes:

1. If a sequence converges pointwise to f then for a given $\varepsilon > 0$, each point x_i of $[a, b]$ determines an integer N_i such that
$$|f_n(x_i) - f(x_i)| < \varepsilon, \quad \text{for } n \geq N_i$$
Consideration of all points of $[a, b]$ gives rise to a sequence of integers $N_1, N_2, N_3, \ldots$
In case the sequence $\{N_i\}$ is bounded above, with supremum N, say, then (4) holds for all points of $[a, b]$ when $n \geq N$ and so the given sequence $\{f_n\}$ converges uniformly on $[a, b]$.
If no such N exists, the sequence $\{f_n\}$ is not uniformly convergent.
2. *Uniform convergence* $\Rightarrow$ *pointwise convergence*
but not vice versa. However
Non-pointwise convergence $\Rightarrow$ *non-uniform convergence*
i.e., a sequence which is not pointwise convergent cannot be uniformly convergent.

2.1 A series of functions Σf_n is said to converge uniformly on $[a, b]$ if the sequence $\{S_n\}$ of its partial sums, defined by
$$S_n(x) = \sum_{i=1}^{n} f_i(x)$$
converges uniformly on $[a, b]$.

Thus, a series of functions Σf_n *converges uniformly* to f on $[a, b]$ if for $\varepsilon > 0$ and all $x \in [a, b]$ there exists an integer N (independent of x and dependent on ε) such that for all x in $[a, b]$
$$|f_1(x) + f_2(x) + \ldots + f_n(x) - f(x)| < \varepsilon, \quad \text{for } n \geq N.$$

2.2 Cauchy's Criterion for Uniform Convergence

Theorem 1. *A sequence of functions $\{f_n\}$ defined on $[a, b]$ converges uniformly on $[a, b]$ if and only if for every $\varepsilon > 0$ and for all $x \in [a, b]$, there exists an integer N such that*
$$|f_{n+p}(x) - f_n(x)| < \varepsilon, \quad \forall\, n \geq N,\ p \geq 1 \qquad \ldots(1)$$

Necessary. Let the sequence $\{f_n\}$ uniformly converges on $[a, b]$ to the limit function f, so that for a given $\varepsilon > 0$, and for all $x \in [a, b]$, there exist integers m_1, m_2 such that
$$|f_n(x) - f(x)| < \varepsilon/2, \quad \forall\, n \geq m_1$$
and
$$|f_{n+p}(x) - f(x)| < \varepsilon/2, \quad \forall\, n \geq m_2,\ p \geq 1$$
Let $N = \max\,(m_1, m_2)$.
$$\therefore \quad |f_{n+p}(x) - f_n(x)| \leq |f_{n+p}(x) - f(x)| + |f_n(x) - f(x)|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon, \quad \forall\, n \geq N,\ p \geq 1$$

Sufficient. Let now the given condition hold.

By Cauchy's general principle of convergence, $\{f_n\}$ converges for each $x \in [a, b]$ to a limit, say f. Thus the sequence converges pointwise to f. Let us now prove that the convergence is uniform.

For a given $\varepsilon > 0$, let us choose an integer N such that (1) holds. Fix n, and let $p \to \infty$ in (1). Since $f_{n+p} \to f$ as $p \to \infty$, we get

$$|f(x) - f_n(x)| < \varepsilon \quad \forall\, n \geq N, \text{ all } x \in [a, b]$$

which proves that $f_n(x) \to f(x)$ uniformly on $[a, b]$.

Note: In the statement of the theorem, (1) may be equivalently replaced by

$$|f_n(x) - f_m(x)| < \varepsilon, \quad \forall\, n, m \geq N$$

Theorem 2. *A series of functions Σf_n defined on $[a, b]$ converges uniformly on $[a, b]$ if and only if for every $\varepsilon > 0$ and for all $x \in [a, b]$, there exists an integer N such that*

$$|f_{n+1}(x) + f_{n+2}(x) + \ldots + f_{n+p}(x)| < \varepsilon, \quad \forall\, n \geq N,\ p \geq 1 \qquad \ldots(2)$$

The proof is left to the readers.

Note: Relation (2) in the statement may be replaced by

$$|f_{m+1}(x) + f_{m+2}(x) + \ldots + f_n(x)| < \varepsilon, \quad \forall\, n, m \geq N$$

2.3 Solved Examples

Example 1. Test for uniform convergence, the sequence $\{f_n\}$, where

$$f_n(x) = \frac{nx}{1 + n^2x^2}, \text{ for all real } x.$$

- The sequence converges pointwise to f, where

$$f(x) = 0, \quad \forall \text{ real } x$$

Let $\{f_n\}$ converges uniformly in any interval $[a, b]$, so that the point-wise limit is also the uniform limit. Therefore for given $\varepsilon > 0$, there exists an integer N such that for all $x \in [a, b]$

$$\left| \frac{nx}{1 + n^2x^2} - 0 \right| < \varepsilon, \quad \forall\, n \geq N$$

If we take $\varepsilon = \frac{1}{3}$, and m an integer greater than N such that $1/m \in [a, b]$, we find on taking $n = m$ and $x = 1/m$, that

$$\frac{nx}{1+n^2x^2} = \frac{1}{2} \not< \frac{1}{3} = \varepsilon$$

We, thus, arrive at a contradiction and so the sequence is not uniformly convergent in the interval $[a, b]$, which contains the point $1/m$. But since $1/m \to 0$, the interval $[a, b]$ contains 0. Hence, the sequence is not uniformly convergent on any interval $[a, b]$ containing 0.

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Example 2. Show that the sequence $\{f_n\}$, where

$$f_n(x) = x^n$$

is uniformly convergent on $[0, k]$, $k < 1$ and only pointwise convergent on $[0, 1]$.

■ Now

$$f(x) = \lim_{n \to \infty} f_n(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1 \end{cases}$$

Thus, the sequence converges pointwise to a discontinuous function on $[0, 1]$.

Let $\varepsilon > 0$ be given.

For $0 < x \le k < 1$, we have

$$|f_n(x) - f(x)| = x^n < \varepsilon$$

if
$$\left(\frac{1}{x}\right)^n > \frac{1}{\varepsilon}$$

or if
$$n > \log(1/\varepsilon)/\log(1/x)$$

This number, $\log(1/\varepsilon)/\log(1/x)$ increases with x, its maximum value being $\log(1/\varepsilon)/\log(1/k)$ in $]0, k]$, $k > 0$.

Let N be an integer $\ge \log(1/\varepsilon)/\log(1/k)$.

$$\therefore \qquad |f_n(x) - f(x)| < \varepsilon, \quad \forall\, n \ge N, 0 < x < 1$$

Again at $x = 0$,

$$|f_n(x) - f(x)| = 0 < \varepsilon, \quad \forall\, n \ge 1$$

Thus for any $\varepsilon > 0$, $\exists\, N$ such that for all $x \in [0, k]$, $k < 1$

$$|f_n(x) - f(x)| < \varepsilon, \quad \forall\, n \ge N$$

Therefore, the sequence $\{f_n\}$ is uniformly convergent in $[0, k]$, $k < 1$.

However, the number $\log(1/\varepsilon)/\log(1/x) \to \infty$ as $x \to 1$ so that it is not possible to find an integer N such that $|f_n(x) - f(x)| < \varepsilon$, for all $n \ge N$ and all x in $[0, 1]$.

Hence, the sequence is not uniformly convergent on any interval containing 1 and in particular on $[0, 1]$.

> **Note:** A point, like $x = 1$ which is such that the sequence is not uniformly convergent in any interval containing $x = 1$, is called a *point of non-uniform convergence.*

Example 3. Show that the sequence $\{f_n\}$, where

$$f_n(x) = \frac{1}{x + n}$$

is uniformly convergent in any interval $[o, b]$, $b > 0$.

▪ Here the sum function

$$f(x) = \lim_{n\to\infty} f_n(x) = 0 \quad \forall\, x \in [0, b]$$

so that the sequence converges pointwise to 0.

For any $\varepsilon > 0$,

$$|f_n(x) - f(x)| = \frac{1}{x+n} < \varepsilon$$

if $n > (1/\varepsilon) - x$, which decreases with x, the maximum value being $1/\varepsilon$.

Let N be an integer $\geq 1/\varepsilon$, so that for $\varepsilon > 0$, there exists N such that

$$|f_n(x) - f(x)| < \varepsilon, \quad \forall\, n \geq N$$

Hence, the sequence is uniformly convergent in any interval $[0, b]$, $b > 0$.

Example 4. Show that the sequence $\{f_n\}$, where

$$f_n(x) = \tan^{-1} nx,\ x \geq 0$$

is uniformly convergent in any interval $[a, b]$, $a > 0$, but is only pointwise convergent in $[0, b]$.

▪ The pointwise sum function

$$f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} \pi/2, & x > 1 \\ 0, & x = 0 \end{cases}$$

Let $\varepsilon > 0$ be given, so that for $x > 0$,

$$|f_n(x) - f(x)| = |\tan^{-1} nx - \pi/2| < \varepsilon$$

if

$$\cot^{-1} nx < \varepsilon$$

or if

$$n > \cot \varepsilon / x$$

which decreases when x increases, the maximum value being cot loss ε/a in $[a, b]$, $a > 0$. Let N be an integer $\geq \cot \varepsilon/a$.

Thus for $\varepsilon > 0$, $\exists\, N$ such that for all $x \in [a, b]$, $a > 0$.

$$|f_n(x) - f(x)| < \varepsilon, \quad \forall\, n \geq N$$

Hence, the sequence converges uniformly on $[a, b]$, $a > 0$.

However, $\cot \varepsilon/x \to \infty$ as $x \to 0$, so that no such integer N exists, such that $|f_n(x) - f(x)| < \varepsilon$, for all $n \geq N$, and hence the sequence is not uniformly convergent on $[0, b]$. It is only pointwise convergent in $[0, b]$.

Example 5. Show that the series Σf_n, whose sum to n terms, $S_n(x) = nxe^{-nx^2}$ is pointwise and not uniformly convergent on any interval $[0, k]$, $k > 0$.

- The pointwise sum $S(x) = \lim_{n\to\infty} S_n(x) = 0$, for all $x \geq 0$.

Thus the series converges pointwise to 0 on $[0, k]$.

Let us suppose, if possible, the series converges uniformly on $[0, k]$, so that for any $\varepsilon > 0$, there exists an integer N such that for all $x \geq 0$,

$$|S_n(x) - S(x)| = nxe^{-nx^2} < \varepsilon, \quad \forall\, n \geq N \quad \text{...(1)}$$

Let N_0 be an integer greater than N and $e^2\varepsilon^2$, then for $x = 1/\sqrt{N_0}$ and $n = N_0$, (1) gives

$$\sqrt{N_0}/e < \varepsilon \Rightarrow N_0 < e^2\varepsilon^2$$

so we arrive at a contradiction.

Hence, the series is not uniformly convergent on $[0, k]$.

Notes:

1. Choice of $x = 1/\sqrt{N_0}$ is admissible because the interval contains the origin.
2. The interval of uniform convergence is always be a closed interval, that is, it must include the end points. But the interval for pointwise or absolute convergence can be of any type.

Remark: These examples suggest that a discontinuity in the limit function implies a point of non-uniform convergence, although non-uniform convergence does not necessary involve discontinuity in the limit function.

Ex. 1. Show that the sequence $\{f_n\}$, where $f_n(x) = \dfrac{x}{n+x}$, is uniformly convergent in $[0, k]$, $k < \infty$ but only pointwise convergent when the interval extends to ∞.

Ex. 2. Show that the sequence $\{e^{-nx}\}$ is uniformly convergent in any interval $[a, b]$, where a and b are positive numbers but only pointwise in $[0, b]$.

Ex. 3. Show that the series Σf_n, the sum of whose n terms is $S_n(x) = x/(1 + nx^2)$, converges uniformly for all real x.

Ex. 4. Show that the sequence $\left\{\dfrac{nx}{1+n^3x^2}\right\}$ converges uniformly to zero for $0 \leq x \leq 1$.

Ex. 5. Show that the sequence $\{f_n\}$, where $f_n\{x\} = \dfrac{n^2x}{1+n^3x^2}$, is not uniformly convergent on $[0, 1]$.

Ex. 6. Show that the sequences $\{nx(1-x^2)^n\}$ and $\{n^2x(1-x^2)^n\}$ are not uniformly convergent on $[0, 1]$.

Ex. 7. Show that the series

$$(1-x)^2 + x(1-x)^2 + x^2(1-x)^2 + \ldots,$$

is not uniformly convergent on $[0, 1]$.

Ex. 8. Show that the series

$$\frac{x}{1+x} + \frac{x}{(1+x)(1+2x)} + \frac{x}{(1+2x)(1+3x)} + \dots$$

is uniformly convergent on $[a, b]$, $a > 0$ but only pointwise in $[0, b]$.

3. TESTS FOR UNIFORM CONVERGENCE

Now that we are acquainted with the meaning of the concept of uniform convergence, we shall naturally inquire how we can determine whether a given sequence or a series does or does not converge uniformly in a given interval. So far we have used merely the definition of uniform convergence for the purpose. This procedure is usually replaced by narrower tests which are more convenient in ordinary practice.

CHAPTER 12

3.1 A Test for Uniform Convergence of Sequences

Theorem 3. *Let $\{f_n\}$ be a sequence of functions, such that*

$$\lim_{n\to\infty} f_n(x) = f(x), \; x \in [a, b]$$

and let

$$M_n = \operatorname*{Sup}_{x\in[a,b]} |f_n(x) - f(x)|$$

Then $f_n \to f$ uniformly on $[a, b]$ if and only if $M_n \to 0$ as $n \to \infty$.

Necessary. Let $f_n \to f$ uniformly on $[a, b]$, so that for a given $\varepsilon > 0$, there exists an integer N such that

$$|f_n(x) - f(x)| < \varepsilon, \quad \forall\, n \geq N, \forall\, x \in [a, b]$$

$$\Rightarrow \quad M_n = \operatorname*{Sup}_{x\in[a,b]} |f_n(x) - f(x)| \leq \varepsilon, \quad \forall\, n \geq N$$

$$\Rightarrow \quad M_n \to 0, \quad \text{as } n \to \infty$$

Sufficient. Let $M_n \to 0$, as $n \to \infty$, so that for any $\varepsilon > 0$, $\exists$ an integer N such that

$$M_n < \varepsilon, \quad \forall\, n \geq N$$

$$\Rightarrow \quad \operatorname*{Sup}_{x\in[a,b]} |f_n(x) - f(x)| < \varepsilon, \quad \forall\, n \geq N$$

$$\Rightarrow \quad |f_n(x) - f(x)| < \varepsilon, \quad \forall\, n \geq N, \forall\, x \in [a, b]$$

$$\Rightarrow \quad f_n \to f \text{ uniformly on } [a, b].$$

Example 6. Show that the sequence $\{f_n\}$, where

$$f_n(x) = \frac{nx}{1 + n^2x^2}$$

is not uniformly convergent on any interval containing zero.

■ Here

$$\lim_{n\to\infty} f_n(x) = 0, \quad \forall\ x$$

Now $\dfrac{nx}{1+n^2x^2}$ attains the maximum value $\dfrac{1}{2}$ at $x = \dfrac{1}{n}$; $\dfrac{1}{n}$ tending to 0 as $n \to \infty$. Let us take an interval $[a, b]$ containing 0.

Thus

$$M_n = \operatorname*{Sup}_{x\in[a,b]} |f_n(x) - f(x)|$$

$$= \operatorname*{Sup}_{x\in[a,b]} \left|\frac{nx}{1+n^2x^2}\right| = \frac{1}{2} \text{ which does not tend to zero as } n \to \infty.$$

Hence, the sequence $\{f_n\}$ is not uniformly convergent in any interval containing the origin.

Example 7. Prove that the sequence $\{f_n\}$, where

$$f_n(x) = \frac{x}{1+nx^2},\ x \text{ being real}$$

converges uniformly on any closed interval I.

■ Here pointwise limit,

$$f(x) = \lim_{n\to\infty} f_n(x) = 0, \quad \forall\ x$$

$$M_n = \operatorname*{Sup}_{x\in I} |f_n(x) - f(x)| = \operatorname*{Sup}_{x\in I} \left|\frac{x}{1+nx^2}\right| = \frac{1}{2\sqrt{n}} \to 0 \text{ as } n \to \infty$$

Hence $\{f_n\}$ converges uniformly on I.

$$\left[\frac{x}{1+nx^2} \text{ attains the maximum value } \frac{1}{2\sqrt{n}} \text{ at } x = \frac{1}{\sqrt{n}}, \text{ i.e. at the origin}\right].$$

Example 8. Show that the sequence $\{f_n\}$, where

$$f_n(x) = nxe^{-nx^2},\ x \geq 0$$

is not uniformly convergent on $[0, k]$, $k > 0$

■
$$f(x) = \lim_{n\to\infty} f_n(x) = 0, \quad \forall\ x \geq 0$$

Also nxe^{-nx^2} attains maximum value $\sqrt{\dfrac{n}{2e}}$ at $x = \dfrac{1}{\sqrt{2n}}$

Now,

$$M_n = \operatorname*{Sup}_{x\in[0,k]} |f_n(x) - f(x)| = \operatorname*{Sup}_{x\in[0,k]} nxe^{-nx^2} = \sqrt{\frac{n}{2e}} \to \infty \text{ as } n \to \infty$$

Therefore, the sequence is not uniformly convergent on $[0, k]$.

Ex. 1. Show that the following sequences are not uniformly convergent on the intervals indicated:

(*i*) $\{x^n\}$ on $[0, 1]$ (*ii*) $\{e^{-nx}\}$ on $[0, k]$

Ex. 2. Test the following sequences for uniform convergence.

(*i*) $\left\{\dfrac{\sin nx}{\sqrt{n}}\right\}$, $0 \le x \le 2\pi$ (*ii*) $\left\{\dfrac{x}{n+x}\right\}$, $0 \le x \le k$

(*iii*) $\left\{\dfrac{x}{n+x}\right\}$, $0 \le x < \infty$ (*iv*) $\left\{\dfrac{n^2 x}{1+n^3x^2}\right\}$, $0 \le x \le 1$

(*v*) $\left\{\dfrac{nx}{1+n^3x^2}\right\}$, $0 \le x \le 1$

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3.2 Tests of Uniform Convergence of Series

Theorem 4. *Weierstrass's M-test.* *A series of functions Σf_n will converge uniformly (and absolutely) on $[a, b]$ if there exists a convergent series ΣM_n of positive numbers such that for all $x \in [a, b]$*

$$|f_n(x)| \le M_n, \quad \textit{for all } n$$

Let $\varepsilon > 0$ be a positive number.

Since ΣM_n is convergent, therefore there exists a positive integer N such that

$$|M_{n+1} + M_{n+2} + \ldots + M_{n+p}| < \varepsilon \quad \forall\, n \ge N,\ p \ge 1 \qquad \ldots(1)$$

Hence for all $x \in [a, b]$ and for all $n \ge N$, $p \ge 1$, we have

$$|f_{n+1}(x) + f_{n+2}(x) + \ldots + f_{n+p}(x)| \le |f_{n+1}(x)| + |f_{n+2}(x)| + \ldots + |f_{n+p}(x)| \qquad \ldots(2)$$

$$\le M_{n+1} + M_{n+2} + \ldots + M_{n+p} < \varepsilon \qquad \ldots(3)$$

Equations (2) and (3) imply that Σf_n is uniformly and absolutely convergent on $[a, b]$.

Remarks:

1. The converse is not asserted and is in fact, not true *i.e.*, non-convergence of ΣM_n does not imply anything as far as Σf_n is concerned.
2. Series which satisfy the *M*-test have been called *normally convergent* by *Baire*, to emphasize the fact that such series are uniformly as well as absolutely convergent. The terminology has the additional advantage of emphasising the fact that the test can be applied to nearly all series in ordinary everyday use.

ILLUSTRATIONS

1. The series $\Sigma r^n \cos n\theta$, $\Sigma r^n \sin n\theta$, $\Sigma r^n \cos n^2\theta$, $\Sigma r^n \sin (a^n\theta)$, $0 < r < 1$, converge uniformly for all real values of θ.

   The result follows by taking $M_n = r^n$.

2. The series $\Sigma \dfrac{a_n x^n}{1+x^{2n}}$, $\Sigma \dfrac{a_n x^{2n}}{1+x^{2n}}$ converge uniformly for all real values of x, if Σa_n is absolutely convergent.

3. $\Sigma \dfrac{\sin(x^2 + n^2 x)}{n(n+1)}$ is uniformly convergent for all real x.

$\left[\text{Take } M_n = \dfrac{1}{n(n+1)}\right]$.

4. $\Sigma \dfrac{\cos n\theta}{n^p}$ is uniformly and absolutely convergent for all real values of $\theta, p > 1$.

$\left[\text{Take } M_n = \dfrac{1}{n^p}\right]$.

5. $\Sigma \dfrac{(-1)^n x^{2n}}{n^p(1+x^{2n})}$ converges absolutely and uniformly for all real x if $p > 1$.

$\left[\text{Take } M_n = \dfrac{1}{n^p}\right]$.

6. Σn^{-x} is uniformly convergent in $[1+\delta, \infty[, \delta > 0$.

Example 9. Test for uniform convergence, the series

$$\frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8} + \ldots, -\frac{1}{2} \leq x \leq \frac{1}{2}$$

■ The nth term $f_n(x) = \dfrac{2^n x^{2n-1}}{1+x^{2n}}$

$$|f_n(x)| \leq 2^n (\alpha)^{2n-1}$$

where $|x| \leq \alpha \leq \dfrac{1}{2}$.

The series $\Sigma 2^n (a)^{2n-1}$ converges, and hence by M-test the given series converges uniformly on $\left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$.

Example 10. Show that the series $\Sigma \dfrac{x}{n^p + x^2 n^q}$ converges uniformly over any finite interval $[a, b]$, for

(i) $p > 1, q \geq 0$ (ii) $0 < p \leq 1, p + q > 2$

■ (i) When $p > 1, q \geq 0$

$$|f_n(x)| = \left|\frac{x}{n^p + x^2 n^q}\right| \leq \frac{\alpha}{n^p}$$

where $\alpha \geq \max\{|a|, |b|\}$.

The series $\Sigma(\alpha/n^p)$ converges for $p > 1$.

Hence by M-test, the given series converges uniformly over the interval $[a, b]$.

(*ii*) When $0 < p \leq 1$, $p + q > 2$.

$|f_n(x)|$ attains the maximum value $\dfrac{1}{2n^{\frac{1}{2}(p+q)}}$ at the point, where $x^2 n^q = n^p$

$$\therefore \qquad |f_n(x)| \leq \frac{1}{2n^{\frac{1}{2}(p+q)}}$$

The series $\Sigma \dfrac{1}{2n^{\frac{1}{2}(p+q)}}$ converges for $p + q > 2$. Hence by M-test, the given series converges uniformly over any finite interval $[a, b]$.

In spite of its great practical importance, Weierstrass's M-test is necessarily applicable to a restricted class of series—series which are absolutely convergent as well. When this is not the case, we have to make use of more delicate tests, which we construct by analogy with those for series of arbitrary terms—*Abel's* and *Dirichlet's* test.

Theorem 5. ***Abel's test.*** *If $b_n(x)$ is a positive, monotonic decreasing function of n for each fixed value of x in the interval $[a, b]$, and $b_n(x)$ is bounded for all values of n and x concerned, and if the series $\Sigma u_n(x)$ is uniformly convergent on $[a, b]$, then so also is the series $\Sigma b_n(x) u_n(x)$.*

Since $b_n(x)$ is bounded for all values of n and for x in $[a, b]$, therefore there exists a number $K > 0$, independent of x and n, such that for all $x \in [a, b]$,

$$0 \leq b_n(x) \leq K, \quad (\text{for } n = 1, 2, 3, \ldots) \qquad \ldots(1)$$

Again, since $\Sigma u_n(x)$ converges uniformly on $[a, b]$, therefore for any $\varepsilon > 0$, we can find an integer N such that

$$\left| \sum_{r=n+1}^{n+p} u_r(x) \right| < \frac{\varepsilon}{K}, \quad \forall\, n \geq N,\ p \geq 1 \qquad \ldots(2)$$

Hence using Abel's lemma (Ch. 9, § 13) we get

$$\left| \sum_{r=n+1}^{n+p} b_r(x) u_r(x) \right| \leq b_{n+1}(x) \max_{q=1,2,\ldots,p} \left| \sum_{r=n+1}^{n+q} u_r(x) \right|$$

$$< K \frac{\varepsilon}{K} = \varepsilon, \text{ for } n \geq N,\ p \geq 1,\ a \leq x \leq b$$

$\Rightarrow \Sigma b_n(x) u_n(x)$ is uniformly convergent on $[a, b]$.

Corollary 1. *A uniformly convergent series $\Sigma u_n(x)$ remains uniformly convergent on $[a, b]$, if its each term is multiplied by a function $a_n(x)$, $a \leq x \leq b$, provided that the sequence $\{a_n(x)\}$ is uniformly bounded on $[a, b]$ (i.e., $\exists\, K > 0$, such that $|a_n(x)| \leq K$, for all x in $[a, b]$ and for all n), and monotonic in n, for each $x \in [a, b]$.*

Under the given conditions, $\{a_n(x)\}$ converges pointwise. Let us write for each $x \in [a, b]$,

$$b_n(x) = \{\lim_{n\to\infty} a_n(x)\} - a_n(x), \text{ or } a_n(x) - \lim_{n\to\infty} a_n(x),$$

according as $\{a_n(x)\}$ is monotonic increasing or decreasing. With this function $b_n(x)$, we deduce as above that the series $\Sigma\, b_n(x)\, u_n(x)$ converges uniformly on $[a, b]$. Also, since $\Sigma\, u_n(x)$ and hence $\Sigma\, [\lim a_n(x)]\; u_n(x)$ $(\because |\lim a_n(x)| \le K, a \le x \le b)$ converges uniformly on $[a, b]$. The uniform convergence of $\Sigma\, a_n(x)\, u_n(x)$, then follows easily.

Corollary 2. *If $\sum_{n=1}^{\infty} a_n x^n$ is a (power) series which converges for all values of x, where $|x| < R$, then $\Sigma a_n x^n$ is uniformly convergent in $[0, R]$ if and only if $\Sigma a_n R^n$ is convergent.*

Let $\Sigma a_n R^n$, be convergent, so that being a series of real numbers, it is uniformly convergent in $[0, R]$.

Now, since $\Sigma a_n R^n$ is uniformly convergent, and $(x/R)^n$ is a positive monotonic decreasing bounded function of n, for each value of x in $[0, R]$, therefore by Abel's test, the series $\Sigma a_n\, R^n (x/R)^n < \Sigma a_n x^n$ is uniformly convergent on $[a, b]$.

If the series $\Sigma a_n x^n$ is uniformly convergent in $[0, R]$, it is obviously convergent at $x = R$.

Example 11. The series $\Sigma \dfrac{(-1)^n}{n} |x|^n$ is uniformly convergent in $-1 \le x \le 1$.

■ Since $|x|^n$ is positive, monotonic decreasing and bounded for $-1 \le x \le 1$, and the series $\Sigma \dfrac{(-1)^n}{n}$ is uniformly convergent, therefore $\Sigma \dfrac{(-1)^n}{n} |x|^n$ is also so in $-1 \le x \le 1$.

Example 12. Show that $\Sigma\, a_n/n^x$ converges uniformly in $[0, 1]$, if $\Sigma\, a_n$ converges.

■ Since $1/n^x$ is a positive, monotonic decreasing function and is bounded by 0 and 1 in $[0, 1]$, and the series $\Sigma\, a_n$ is convergent (and so uniformly), therefore $\Sigma\, a_n/n^x$ is uniformly convergent in $[0, 1]$.

Ex. If $\Sigma\, a_n$ is convergent, then show that each of the following series is uniformly convergent in $[0, 1]$.

$$\Sigma a_n x^n,\ \Sigma a_n \frac{x^n}{1+x^n},\ \Sigma \frac{a_n x^n}{1+x^{2n}},\ \Sigma \frac{n x^n (1-x)}{1+x^n} a_n,\ \Sigma \frac{2n a_n x^n (1-x)}{1+x^{2n}}$$

Theorem 6. *Dirichlet's test.* *If $b_n(x)$, is a monotonic function of n for each fixed value of x in $[a, b]$, and $b_n(x)$ tends uniformly to zero for $a \le x \le b$, and if there is a number $K > 0$ independent of x and n, such that for all values of x in $[a, b]$,*

$$\left| \sum_{r=1}^{n} u_r(x) \right| \le K,\ \forall\, n$$

then the series $\Sigma\, b_n(x) u_n(x)$ is uniformly convergent on $[a, b]$.

First method. We may assume that $b_n(x)$ is a positive monotonic decreasing function of n, for each $x \in [a, b]$, since the general case follows by the procedure given in the above cor. 1. Now $b_n(x)$ tends uniformly to zero, therefore for any $\varepsilon > 0$, we can find an integer N (independent of x) such that for all values of x in $[a, b]$.

$$0 \leq b_n(x) < \varepsilon/2K, \text{ for all } n \geq N$$

For such values of n and any integral value of $p \geq 1$, we have by Abel's Lemma,

$$\left|\sum_{r=n+1}^{n+p} b_r(x)u_r(x)\right| \leq b_{n+1}(x) \max_{q=1,2,\ldots,p}\left|\sum_{r=n+1}^{n+q} u_r(x)\right|$$

$$\leq b_{n+1}(x)\left\{\left|\sum_{r=1}^{n} u_r(x)\right| + \max_{q=1,2,\ldots,p}\left|\sum_{r=n}^{n+q} u_r(x)\right|\right\}$$

$$< \frac{\varepsilon}{2K}(K+K) = \varepsilon$$

Hence by Cauchy's criterion the series $\Sigma b_n(x)u_n(x)$ converges uniformly for $x \in [a, b]$.

Second method. Since $b_n(x)$ tends uniformly to zero, therefore for any $\varepsilon > 0$, there exists an integer N (independent of x) such that for all $x \in [a, b]$,

$$|b_n(x)| < \varepsilon/4K, \text{ for all } n \geq N.$$

Let $S_n(x) = \sum_{r=1}^{n} u_r(x)$, for all $x \in [a, b]$, and for all n,

$$\therefore \sum_{r=n+1}^{n+p} b_r(x)u_r(x) = b_{n+1}(x)\{S_{n+1} - S_n\} + b_{n+2}(x)\{S_{n+2} - S_{n+1}\} + \ldots$$

$$+ b_{n+p}(x)\{S_{n+p} - S_{n+p-1}\}$$

$$= -b_{n+1}(x) S_n + \{b_{n+1}(x) - b_{n+2}(x)\} S_{n+1} + \ldots$$

$$+ \{b_{n+p-1}(x) - b_{n+p}(x)\}S_{n+p-1} + b_{n+p}(x)S_{n+p}$$

$$= \sum_{r=n+1}^{n+p-1} \{b_r(x) - b_{r+1}(x)\} S_r(x) - b_{n+1}(x) S_n(x)$$

$$+ b_{n+p}(x) S_{n+p}(x)$$

$$\left|\sum_{r=n+1}^{n+p} b_r(x)\, u_r(x)\right| \leq \sum_{r=n+1}^{n+p-1} |b_r(x) - b_{r+1}(x)||S_r(x)| + |b_{n+1}(x)||S_n(x)| +$$

$$+ |b_{n+p}(x)||S_{n+p}(x)|$$

CHAPTER 12

Making use of the monotonicity of $b_n(x)$

$$\sum_{r=n+1}^{n+p-1} |b_r(x) - b_{r+1}(x)| = |b_{n+1}(x) - b_{n+p}(x)|, \text{ for } a \le x \le b,$$

and the relation $|S_n(x)| \le K$, for all $x \in [a, b]$ and for all $n = 1,2,3,\ldots$, we deduce that for all $x \in [a, b]$ and all $p \ge 1, n \ge N$

$$\left|\sum_{r=n+1}^{n+p} b_r(x)u_r(x)\right| \le K\,|b_{n+1}(x) - b_{n+p}(x)| + \frac{\varepsilon}{4K}\,2K$$

$$< K\frac{\varepsilon}{2K} + \frac{\varepsilon}{2} = \varepsilon.$$

Hence by Cauchy's criterion, the series $\Sigma b_n(x)u_n(x)$ converges uniformly on $[a, b]$.

Remark: The statement '$\left|\sum_{r=1}^{n} u_r(x)\right| \le k \;\; \forall\, x \in [a, b]$, and for all n' amounts to saying that the sequences of partial sums of $\Sigma u_n(x)$ are bounded for each value of $x \in [a, b]$, *i.e.*, for each point $x_i \in [a, b]$ there is a number K_i such that $\left|\sum_{r=1}^{n} u_r(x)\right| \le K_i$, and there exists a number K such that $K_t < K, \;\forall\, i$.

This fact is expressed by saying that 'the partial sums of the series are *uniformly bounded*'.

This in turn amounts to saying that 'the series $\Sigma u_n(x)$ either converges uniformly or oscillates finitely'.

So, *Dirichlet's test* can be stated also as:

If $b_n(x)$ is a monotonic function of n for each fixed value of x in $[a, b]$, and $b_n(x)$ tends uniformly to zero for $a \le x \le b$, and if $\Sigma u_n(x)$ either uniformly converges or oscillates finitely in $[a, b]$, then the series $\Sigma b_n(x)u_n(x)$ is uniformly convergent on $[a, b]$.

Example 13. Prove that the series $\Sigma(-1)^n \dfrac{x^2+n}{n^2}$, converges uniformly in every bounded interval, but does not converge absolutely for any value of x.

■ Let the bounded interval be $[a, b]$, so that $\exists$ a number K such that for all x in $[a, b]$, $|x| < K$.

Let us take $\Sigma u_n = \Sigma(-1)^n$ which oscillates finitely and

$$b_n = \frac{x^2+n}{n^2} < \frac{K^2+n}{n^2}$$

Clearly b_n is a positive, monotonic decreasing function of n for each x in $[a, b]$, and tends to zero uniformly for $a \le x \le b$.

Hence by Dirichlet's test, the series $\Sigma(-1)^n \dfrac{x^2+n}{n^2}$ converges uniformly on $[a, b]$.

Again $\sum\left|(-1)^n \frac{x^2+n}{n^2}\right| = \sum \frac{x^2+n}{n^2} \sim \sum \frac{1}{n}$, which diverges. Hence the given series is not absolutely convergent for any value of x.

Example 14. Prove that the series $\sum \frac{\cos n\theta}{n^p}$ and $\sum \frac{\sin n\theta}{n^p}$ converge uniformly for all values of $p > 0$ in an interval $[\alpha, 2\pi - \alpha[$, where $0 < \alpha < \pi$.

▪ When $p > 1$, Weierstrass's M-test at once proves that both the series converge uniformly for all real values of θ.

When $0 < p \leq 1$, both the series converge uniformly in any interval $[\alpha, 2\pi - \alpha]$, $\alpha > 0$. This can be proved by taking $b_n = (1/n^p)$ and $u_n = \cos n\theta$ (or $\sin n\theta$) in Dirichlet's test.

$(1/n^p)$ is positive monotonic decreasing and tends uniformly to zero for $0 < p \leq 1$, and

$$\left|\sum_{r=1}^{n} u_r\right| = \left|\sum_{r=1}^{n} \cos r\theta\right| = |\cos\theta + \cos 2\theta + \ldots + \cos n\theta|$$

$$= \left|\frac{\cos((n+1)/2)\theta \sin(n/2)\theta}{\sin(\theta/2)}\right| \leq \operatorname{cosec}(\alpha/2), \ \forall\, n,$$

for $\theta \in [\alpha, 2\pi - \alpha]$,

Thus, all the conditions are fulfilled and the series $\sum(\cos n\theta/n^p)$ and similarly $\sum(\sin n\theta/n^p)$ converge uniformly on $[\alpha, 2\pi - \alpha]$ where $0 < \alpha < \pi$.

Example 15. Show that the series $\sum\{\log(n+1)\}^{-x} \cos nx$ is uniformly convergent in $[\theta_1, \theta_2]$, where $0 < \theta_1 \leq x \leq \theta_2 < 2\pi$.

▪ When $x \in [\theta_1, \theta_2]$, $\{\log(n+1)\}^{-x}$ is a positive monotonic decreasing function of n. Also since $\{\log(n+1)\}^{-x} \leq \{\log(n+1)\}^{-\theta_1}$, the function $\{\log(n+1)\}^{-x}$ tends uniformly to zero as $n \to \infty$. Moreover, as in Example 12.14,

$$\left|\sum_{r=1}^{n} \cos rx\right| \leq \frac{1}{\sin(x/2)} \leq \max\left(\frac{1}{\sin(\theta_1/2)}, \frac{1}{\sin(\theta_2/2)}\right),$$

both are independent of x and n.

Thus by Dirichlet's test the series $\sum\{\log(n+1)\}^{-x} \cos nx$ is uniformly convergent in $[\theta_1, \theta_2]$.

Notes:

1. It is to be understood that θ_1 can be as close to zero and θ_2 as close to 2π as we please.
2. If $\{v_n\}$ is a monotonic sequence of real numbers that converges to zero, then each of the series $\sum v_n \sin n\theta$, $\sum v_n \cos n\theta$ is uniformly convergent with regard to θ in the interval $[\alpha, 2\pi - \alpha]$ where α is any fixed positive number less than π.

CHAPTER 12

EXERCISE

1. Prove that the series,

$$\text{(i) } \Sigma \frac{x^n}{n^2}, \qquad \text{(ii) } \Sigma \frac{x^n}{n(n+1)}, \qquad \text{(iii) } \Sigma \frac{x^{2n}}{n^2 + x^{2n}}$$

are uniformly convergent in $[-1, 1]$.

2. Prove that, if α is any fixed positive number less than unity, each of the series

$$\Sigma x^n, \Sigma (n+1)^{-1} x^n, \Sigma (n+1) x^n, \Sigma n^3 x^n$$

is uniformly convergent in $[-\alpha, \alpha]$.

3. (a) Show that each of the series

$$\Sigma \frac{1}{n^4 + n^2 x^2}, \Sigma \frac{1}{n^2 + n^4 x^2}$$

is uniformly convergent in $[-k, k]$, for real k.

(b) Show that $\Sigma 1/n^x$ converges uniformly for all real $x > 1$.

(c) Show that the series

$$1 + \frac{e^{-2x}}{2^2 - 1} - \frac{e^{-4x}}{4^2 - 1} + \frac{e^{-6x}}{6^2 - 1} - \ldots$$

converges uniformly for all real $x \geq 0$.

4. Show that the series $\sum\limits_{1}^{\infty} \dfrac{x}{n(1 + nx^2)}$ converges uniformly for all real x.

[*Hint*: The maximum value of $f_n(x)$ is $1/2n^{3/2}$ at $x^2 = 1/n$, apply M-test.]

5. Prove that each of the series, $\Sigma \dfrac{\sin nx}{n}, \Sigma \dfrac{\cos nx}{n}$ converges uniformly with respect to x in $[\alpha, 2\pi - \alpha]$, where α is any fixed positive number less than π.

Prove also that each of the series, $\Sigma \dfrac{\sin nx}{n^2}, \Sigma \dfrac{\cos nx}{n^2}$ is uniformly convergent in $[0, 2\pi]$.

6. Show that the series

$$\frac{1}{a} - \frac{2a}{a^2 - 1^2} \cos \theta + \frac{2a}{a^2 - 2^2} \cos 2\theta - \ldots$$

is uniformly convergent with respect to θ in any finite interval.

$$\left[\textit{Hint: } |a^2 - n^2| = |n^2 - a^2| > n^2/2, \text{ when } n \text{ exceeds a certain number } N, \text{ so that } \left|\frac{\cos n\theta}{a^2 - n^2}\right| \leq \frac{2}{n^2}\right].$$

7. Discuss the series $\Sigma(-x)^n / n(1 + x^n)$ for uniform convergence for real x.

8. Show that $\Sigma \dfrac{\log n}{n^x}$ converges uniformly for all real $x \geq 1 + \alpha > 1$.

$$\left[\textit{Hint: } |(\log n)/n^x| \leq \frac{1}{n^{1+\alpha/2}} \cdot \frac{\log n}{n^{\alpha/2}} < \frac{1}{n^{1+\alpha/2}}.\right]$$

9. Discuss the uniform convergence with respect to x of the series, $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{\sqrt{n}} \sin\left(1 + \dfrac{x}{n}\right)$, over any closed and bounded subset of $\mathbf{R}$.

4. PROPERTIES OF UNIFORMLY CONVERGENT SEQUENCES AND SERIES

Whereas we saw earlier that fundamental properties of the functions f_n do not in general hold for the pointwise limit function f, we shall now show that roughly speaking these properties hold for the limit function f when the convergence is uniform. In this connection we now give some theorems which become particularly important in application.

4.1. Theorem 7(A). *If a sequence $\{f_n\}$ converges uniformly in $[a, b]$ and x_0 is a point of $[a, b]$ such that*

$$\lim_{x \to x_0} f_n(x) = a_n,\ n = 1, 2, 3, \ldots$$

then (i) $\{a_n\}$ *converges,*

and (ii) $\lim_{x \to x_0} f(x) = \lim_{n \to \infty} a_n$

[The existence of $\lim_{x \to x_0} f(x)$ is a part of conclusion].

(i) Since the sequence $\{f_n\}$ converges uniformly on $[a, b]$, therefore for $\varepsilon > 0$, there exists an integer m (independent of x) such that for all $x \in [a, b]$,

$$| f_{n+p}(x) - f_n(x) | < \varepsilon/2, \quad \forall\ n \geq m,\ p \geq 1$$

Keeping n, p fixed and letting $x \to x_0$, we get

$$| a_{n+p} - a_n | \leq \varepsilon/2 < \varepsilon, \quad \forall\ n \geq m,\ p \geq 1$$

$\Rightarrow$ the sequence $\{a_n\}$ converges, say to A. ...(1)

(ii) Since $\{f_n\}$ converges uniformly to f, therefore for any $\varepsilon > 0$, there exists an integer N_1 such that for all $x \in [a, b]$,

$$| f_n(x) - f(x) | < \varepsilon/3, \quad \forall\ n \geq N_1 \qquad ...(2)$$

Similarly there exists an integer N_2, such that

$$| a_n - A | < \varepsilon/3, \quad \forall\ n \geq N_2 \qquad ...(3)$$

Let $N = \max(N_1, N_2)$.

Again, since $\lim_{x \to x_0} f_n(x) = a_n$, for all n, therefore $\lim_{x \to x_0} f_N(x) = a_N$ and so for $\varepsilon > 0$, there exists a $\delta > 0$, such that for $| x - x_0 | < \delta$, we have

$$| f_N(x) - a_N | < \varepsilon/3, \qquad ...(4)$$

Hence for $| x - x_0 | < \delta$, we have

$$| f(x) - A | \leq | f(x) - f_N(x) | + | f_N(x) - a_N | + | a_N - A |$$
$$< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon \quad \text{[using (2), (3), (4)]}$$

$\Rightarrow$ $\lim_{x \to x_0} f(x)$ exists and equals A

Thus, $\lim_{x \to x_0} f(x) = \lim_{n \to \infty} a_n$

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or equivalently

$$\lim_{x \to x_0} \lim_{n \to \infty} f_n(x) = \lim_{n \to \infty} \lim_{x \to x_0} f_n(x)$$

Theorem 7(B). *If a series $\sum_{n=1}^{\infty} f_n$ converges uniformly to f in $[a, b]$, and x_0 is a point in $[a, b]$ such that*

$$\lim_{x \to x_0} f_n(x) = a_n, \; (n = 1, 2, 3, ...)$$

then (i) $\sum_{n=1}^{\infty} a_n$ *converges,*

and (ii) $\lim_{x \to x_0} f(x) = \sum_{n=1}^{\infty} a_n$

[The existence of $\lim_{x \to x_0} f(x)$ is a part of conclusion.]

(i) Since the series $\sum f_n$ converges uniformly on $[a, b]$, for $\varepsilon > 0$, $\exists$ an integer m such that for all $x \in [a, b]$ and for any integer p,

$$\left| \sum_{r=n+1}^{n+p} f_r(x) \right| < \varepsilon/2, \;\; \forall \; n \geq m, \; p \geq 1$$

Keeping n, p fixed and letting $x \to x_0$, we get

$$\left| \sum_{r=n+1}^{n+p} a_r \right| \leq \varepsilon/2 < \varepsilon, \;\; \forall \; n \geq m, \; p \geq 1$$

$\Rightarrow$ The series $\sum a_n$ converges to A. ...(1)

(ii) Since $\sum f_n$ converges uniformly to f, therefore for $\varepsilon > 0$, $\exists$ N_1 such that for all $x \in [a, b]$,

$$\left| \sum_{r=1}^{n} f_r(x) - f(x) \right| < \frac{\varepsilon}{3}, \;\; \forall \; n \geq N_1 \qquad ...(2)$$

Similarly,

$$\left| \sum_{r=1}^{n} a_r - A \right| < \frac{\varepsilon}{3}, \;\; \forall \; n \geq N_2 \qquad ...(3)$$

Let $N = \max(N_1, N_2)$

Again, since

$$\lim_{x \to x_0} f_n(x) = a_n, \; n = 1, 2, 3, ...$$

Therefore for the above $\varepsilon > 0$, it is possible to choose $\delta > 0$ such that for $n = 1, 2, 3, ... N$, we have (taking $\delta = \min\{\delta_1, \delta_2, ..., \delta_N\}$.)

$$|f_n(x) - a_n| < \frac{\varepsilon}{3N}, \text{ for } |x - x_0| < \delta$$

$$\therefore \quad \left| \sum_{r=1}^{N} f_r(x) - \sum_{r=1}^{N} a_r \right| \le \sum_{r=1}^{N} | f_r(x) - a_r |$$

$$< N \cdot \frac{\varepsilon}{3N} = \frac{\varepsilon}{3}, \text{ for } | x - x_0 | < \delta \qquad ...(4)$$

Hence for $| x - x_0 | < \delta$, we have

$$| f(x) - A | \le \left| f(x) - \sum_{r=1}^{N} f_r(x) \right| + \left| \sum_{r=1}^{N} f_r(x) - \sum_{r=1}^{N} a_r \right| + \left| \sum_{r=1}^{N} a_r - A \right|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon, \text{ (Using 2, 3, 4)}$$

$\Rightarrow \quad \lim\limits_{x \to x_0} f(x)$ exists and equals A.

Remark: The result simply states, "the limit of the sum function of a series = the sum of the series of limits of functions", *i.e.*,

$$\lim_{x \to x_0} \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \lim_{x \to x_0} f_n(x)$$

We now prove a theorem which though of great significance appears to be a special case of the theorem proved above.

4.2 Uniform Convergence and Continuity

Theorem 8(A). *If $\{f_n\}$ is a sequence of continuous functions on an interval $[a, b]$, and if $f_n \to f$ uniformly on $[a, b]$, then f is continuous on $[a, b]$.*

(B). *If a series $\sum f_n$ converges uniformly to f in an interval $[a, b]$ and its terms f_n are continuous at a point x_0 of the interval, then the sum function f is also continuous at x_0.*

(B). Since $\sum f_n$ converges uniformly to f on $[a, b]$, therefore for $\varepsilon > 0$, we can choose N such that for all x in $[a, b]$.

$$\left| \sum_{r=1}^{n} f_r(x) - f(x) \right| < \frac{\varepsilon}{3}, \quad \forall\, n \ge N \qquad ...(1)$$

and in particular, at a point x_0 in $[a, b]$, and $n = N$

$$\left| \sum_{r=1}^{N} f_r(x_0) - f(x_0) \right| < \frac{\varepsilon}{3}, \qquad ...(2)$$

Again, since each f_n is continuous at x_0, the sum of a finite number of functions, $\sum\limits_{r=1}^{N} f_r$ is also continuous at $x = x_0$.

Therefore for $\varepsilon > 0, \exists\, \delta > 0$, such that

$$\left| \sum_{r=1}^{N} f_r(x) - \sum_{r=1}^{N} f_r(x_0) \right| < \frac{\varepsilon}{3}, \quad \text{for } |x - x_0| < \delta \qquad ...(3)$$

Hence for $|x - x_0| < \delta$, we have

$$|f(x) - f(x_0)| \le \left| f(x) - \sum_{r=1}^{N} f_r(x) \right| + \left| \sum_{r=1}^{N} f_r(x) - \sum_{r=1}^{N} f_r(x_0) \right|$$

$$+ \left| \sum_{r=1}^{N} f_r(x_0) - f(x_0) \right|$$

$$< \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon \quad \text{[using (1), (2) \& (3)]}$$

$$\Rightarrow \qquad f(x) \to f(x_0) \quad \text{when } x \to x_0$$

i.e., the sum function f is continuous at $x = x_0$.

Corollary. Since x_0 is an arbitrary point of $[a, b]$, the theorem holds for all points of $[a, b]$, and so we state:

"If a series $\sum f_n$ converges uniformly to f on an interval and if the functions f_n are all continuous throughout the interval, then so is the sum function f."

Remarks:

1. The converse of the theorem is neither asserted nor it is true, *i.e.*, series (sequence) of continuous terms exist which have a continuous sum (limit) but are not uniformly convergent. In other words, *the condition of uniform convergence is only sufficient, not necessary.*
2. *However, if the sum function (limit function) of a series (sequence) of continuous terms is not continuous on an interval, the convergence cannot be uniform.* This conclusion is very often used with the advantage in deciding that the convergence is not uniform.

ILLUSTRATIONS

1. The sequence $\{x^n\}$ or $\{\tan^{-1} nx\}$ of continuous functions has a discontinuous limit function on $[0, 1]$. Therefore, the convergence is not uniform on $[0, 1]$.
2. The sequence $\{nx/(1 + n^2x^2)\}$ or $\{nxe^{-nx^2}\}$ of continuous function has a continuous limit function on $[0, 1]$, although the convergence is not uniform.
3. The sum function $(1 + x)$ of the series $\Sigma(1 - x^2)x^n$ is continuous on $[0, 1]$ although the convergence is not uniform.
4. The sequence $\{1/(x + n)\}$ converges uniformly to the continuous function 0 for all real $x \ge 0$.
5. The sum function of the series $\sum_{n=0}^{\infty} (1 - x)x^n$ is $f(x) = \begin{cases} 1, & x \ne 1 \\ 0, & x = 1 \end{cases}$ which is discontinuous on $[0, 1]$. Therefore, the series is not uniformly convergent on $[0, 1]$.

Note: There is a special class of sequences (series) for which uniform convergence is equivalent to the continuity of the limit (sum) function of the sequence (series). In that connection, we do a theorem, due to **Dini**, an Italian mathematician.

Theorem 9. *Dini's Theorem on uniform convergence.* (A). *If a sequence of continuous functions* $\{f_n\}$, *defined on* $[a, b]$ *is monotonic increasing, and converges (pointwise) to a continuous function f, then the convergence is uniform on* $[a, b]$.

(B). *If the sum function of a series* $\sum f_n$, *with non-negative continuous terms defined on an interval* $[a, b]$ *is continuous on* $[a, b]$, *then the series is uniformly convergent on the interval.*

(A). Since the sequence $\{f_n\}$ is monotonic increasing, and converges to f on $[a, b]$, therefore, for any $\varepsilon > 0$ and each x in $[a, b]$ there is an integer N such that

$$0 \le f(x) - f_n(x) < \varepsilon \qquad ...(1)$$

Let $R_n(x) = f(x) - f_n(x)$, $n = 1, 2, 3, \ldots$

Clearly the sequence $\{R_n\}$ is monotonic decreasing, *i.e.*

$$R_1(x) \ge R_2(x) \ge \ldots \ge R_n(x) \ge \ldots \qquad ...(2)$$

and is bounded below by zero.

Thus, the sequence $\{R_n\}$ converges pointwise to zero on $[a, b]$.

However, if (1) and (2) hold for all x in $[a, b]$ and N is independent of x, then the convergence is uniform.

Suppose, if possible, that for a certain $\varepsilon_0 > 0$, no such N independent of x exists. Then for each $n = 1, 2, \ldots$, there is $x_n \in [a, b]$ such that

$$R_n(x_n) \ge \varepsilon_0 \qquad ...(3)$$

The sequence $\{x_n\}$ of points belonging to the interval $[a, b]$ is bounded and therefore has at least *one limit point* ξ in $[a, b]$.

Consequently we can assert that there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$, convergent to ξ *i.e.*, $x_{n_k} \to \xi$ as $k \to \infty$.

The function $R_n(x) = f(x) - f_n(x)$ being the difference of two continuous functions is continuous and therefore, we can write, for every fixed m, the relation

$$\lim_{k\to\infty} R_m(x_{n_k}) = R_m(\xi)$$

But for every m and any sufficiently large k we have $n_k \ge k > m$ and consequently, in view of (2) and (3), we get

$$R_m(x_{n_k}) \ge R_{n_k}(x_{n_k}) \ge \varepsilon_0$$

Proceeding to limits as $k \to +\infty$, we see that $R_m(\xi) \geq \varepsilon_0$, for any m, which contradicts the relation $\lim\limits_{m \to \infty} R_m(\xi) = 0$, implied by the pointwise convergence of the sequence $\{R_n\}$ on $[a, b]$. Hence the theorem.

(B). The partial sums $S_n(x) = \sum\limits_{r=1}^{n} f_r(x)$, with non-negative continuous terms f_r, form a non-decreasing sequence of continuous functions convergent point wise to a continuous function f. Therefore by (A), the sequence converges uniformly and thus the series is also uniformly convergent.

Example 16. Show that the series

$$x^4 + \frac{x^4}{1+x^4} + \frac{x^4}{(1+x^4)^2} + \frac{x^4}{(1+x^4)^3} + \dots$$

is not uniformly convergent on [0, 1].

▪ The terms of the series are continuous and the series converges pointwise of f, where

$$f(x) = \begin{cases} 1 + x^4, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

which is a discontinuous function on [0, 1].

Hence the series cannot converge uniformly on [0, 1].

Example 17. Show that the series $\sum \frac{x}{(nx+1)\{(n-1)x+1\}}$, is uniformly convergent on any interval $[a, b]$, $0 < a < b$, but only pointwise on $[0, b]$.

▪ Let

$$f_n(x) = \frac{x}{(nx+1)\{(n-1)x+1\}}$$

$$= \frac{1}{(n-1)x+1} - \frac{1}{nx+1}$$

$\therefore$ nth partial sum $S_n(x) = \sum\limits_{r=1}^{n} f_r(x) = 1 - \frac{1}{nx+1}$

$\therefore$ The sum function $f(x) = \begin{cases} 1, & x \neq 0 \\ 0, & x = 0 \end{cases}$

Thus f is discontinuous on $[0, b]$ and therefore convergence is not uniform on $[0, b]$, it is only pointwise.

When $x \neq 0$, let $\varepsilon > 0$ be given

$$|S_n(x) - f(x)| = \frac{1}{nx+1} < \varepsilon$$

When $n > \frac{1}{x}\left(\frac{1}{\varepsilon} - 1\right)$, but $\frac{1}{x}\left(\frac{1}{\varepsilon} - 1\right)$ decreases with x, let its maximum value $\frac{1}{a}\left(\frac{1}{\varepsilon} - 1\right) = m_0$ (independent of x) on $[a, b]$.

Thus for all $x \in [a, b]$, $\exists$ an integer m ($> m_0$), such that

$$|S_n(x) - f(x)| < \varepsilon, \quad \text{for } n \geq m$$

Hence, the series converges uniformly on $[a, b]$, $0 < a < b$.

Example 18. Show that the series $\sum_1^\infty \frac{(-1)^{n-1}}{n + x^2}$ is uniformly convergent but not absolutely for all real values of x.

▮ The given series converges by Leibnitz Test. However $\sum \frac{1}{n + x^2}$ behaves as $\sum \frac{1}{n}$ and is therefore divergent. Hence the series is not absolutely convergent for any value of x.

Again, let $S_n(x)$ denotes the nth partial sum and S(x), the sum of the series

$$S_{2n}(x) = \left(\frac{1}{1+x^2} - \frac{1}{2+x^2}\right) + \left(\frac{1}{3+x^2} - \frac{1}{4+x^2}\right) + \ldots + \left(\frac{1}{2n-1+x^2} - \frac{1}{2n+x^2}\right)$$

Since each bracket is positive, therefore $S_{2n}(x)$ is positive, increasing to its sum S(x).

$$\Rightarrow \; S(x) - S_{2n}(x) > 0$$

Also,

$$S(x) - S_{2n}(x) = \frac{1}{2n+1+x^2} - \frac{1}{2n+2+x^2} + \frac{1}{2n+3+x^2} - \ldots$$

$$= \frac{1}{2n+1+x^2} - \left(\frac{1}{2n+2+x^2} - \frac{1}{2n+3+x^2} - \ldots\right)$$

$$< \frac{1}{2n+1+x^2} < \frac{1}{2n+1}$$

$$\Rightarrow \qquad 0 < S(x) - S_{2n}(x) < \frac{1}{2n+1} \qquad \ldots(1)$$

Again,

$$S_{2n+1}(x) - S(x) = \frac{1}{2n+2+x^2} - \frac{1}{2n+3+x^2} + \ldots$$

$$= \left(\frac{1}{2n+2+x^2} - \frac{1}{2n+3+x^2}\right) + \left(\frac{1}{2n+4+x^2} - \frac{1}{2n+5+x^2}\right) + \ldots$$

$$> 0$$

Also

$$S_{2n+1}(x) - S(x) = \frac{1}{2n+2+x^2} - \left(\frac{1}{2n+3+x^2} - \frac{1}{2n+4+x^2}\right) - \ldots$$

$$< \frac{1}{2n+2+x^2} < \frac{1}{2n+2}$$

$$\Rightarrow \quad 0 < S_{2n+1}(x) - S(x) < \frac{1}{2n+2} \qquad \ldots(2)$$

Inequalities (1) and (2) imply that for any $\varepsilon > 0$, we can choose an integer m such that for all values of x,

$$|S(x) - S_n(x)| < \varepsilon, \quad \forall\, n \geq m$$

$\Rightarrow$ The series converges uniformly for all real values of x, and since each term of the series is continuous therefore the series will converge to a continuous sum function.

4.3 Uniform Convergence and Integration

Theorem 10 (A). *If a sequence $\{f_n\}$ converges uniformly to f on $[a, b]$, and each function f_n is integrable, then f is integrable on $[a, b]$, and the sequence $\left\{\int_a^x f_n\, dt\right\}$ converges uniformly to $\int_a^x f\, dt$ on $[a, b]$, i.e.,*

$$\int_a^x f\; dt = \lim_{n\to\infty} \int_a^x f_n\; dt, \quad \forall\, x \in [a, b] \qquad \ldots(1)$$

(B). *If a series Σf_n converges uniformly to f on $[a, b]$, and each term $f_n(x)$ is integrable, then f is integrable on $[a, b]$, and the series $\Sigma\left(\int_a^x f_n\; dt\right)$ converges uniformly to $\int_a^x f\; dt$ on $[a, b]$, i.e.,*

$$\int_a^x f\; dt = \sum_{n=1}^{\infty}\left(\int_a^x f_n\; dt\right), \quad \forall\, x \in [a, b] \qquad \ldots(2)$$

[We then say that the series is *integrable term by term*].

(A). Let $\varepsilon > 0$ be any number.

By the uniform convergence of the sequence, there exists an integer m such that for all $x \in [a, b]$

$$|f_n(x) - f(x)| < \frac{\varepsilon}{3(b-a)}, \quad \forall\, n \geq m$$

In particular,

$$|f_m(x) - f(x)| < \frac{\varepsilon}{3(b-a)} \quad ...(3)$$

For this fixed m, since f_m is integrable, we choose a partition P of $[a, b]$, such that

$$U(P, f_m) - L(P, f_m) < \varepsilon/3 \quad ...(4)$$

From equation (3),

$$f(x) < f_m(x) + \varepsilon/3(b-a)$$

$$\Rightarrow \quad U(P, f) < U(P, f_m) + \varepsilon/3 \quad ...(5)$$

Again from equation (3),

$$f(x) > f_m(x) - \varepsilon/3(b-a)$$

$$\Rightarrow \quad L(P, f) > L(P, f_m) - \varepsilon/3 \quad ...(6)$$

From equations (4), (5) and (6), we get

$$U(P, f) - L(P, f) < U(P, f_m) - L(P, f_m) + 2\varepsilon/3$$

$$< \varepsilon/3 + 2\varepsilon/3 = \varepsilon$$

$\Rightarrow$ f is integrable on $[a, b]$

We now proceed to prove relation (1).

Since the sequence $\{f_n\}$ converges uniformly to f, therefore for $\varepsilon > 0$, there exists an integer N such that for all $x \in [a, b]$,

$$|f_n(x) - f(x)| < \varepsilon/(b-a), \quad \forall\, n \geq N$$

Then for all $x \in [a, b]$ and for $n \geq N$, we have

$$\left| \int_a^x f\, dt - \int_a^x f_n\, dt \right| = \left| \int_a^x (f - f_n)\, dt \right| \leq \int_a^x |f - f_n|\, dt$$

$$< \frac{\varepsilon}{b-a}(x-a) \leq \varepsilon$$

$\Rightarrow$ $\left\{ \int_a^x f_n\, dt \right\}$ converges uniformly to $\int_a^x f\, dt$ over $[a, b]$, i.e.,

$$\int_a^x f\, dt = \lim_{n\to\infty} \int_a^x f_n\, dt, \quad \forall\, x \in [a, b]$$

(B). The proof may be supplemented by the reader himself.

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Remark: The converse is neither asserted nor true, *i.e.*, a series (sequence) may converge to an integrable limit without being uniformly convergent. On the other hand, if the limit is not integrable or if integrable, the integral is not equal to the limit of the series (sequence) of integrals, *the convergence cannot be uniform.*

If the terms f_n are continuous for all n, a much shorter and simpler proof is possible.

Theorem 11. *If a series Σf_n uniformly converges to f on [a, b] and each is f_n continuous on [a, b], then f is integrable on [a, b] and the series $\Sigma\left(\int_a^x f_n\, dt\right)$ converges uniformly to $\int_a^x f\, dt$, for all x in [a, b], i.e.,*

$$\int_a^x f\, dt = \sum_{n=1}^{\infty}\left(\int_a^x f_n\, dt\right), \quad \forall\, x \in [a, b].$$

Since Σf_n is uniformly convergent to f on $[a, b]$ and each f_n is continuous on $[a, b]$, therefore the sum function f is continuous and hence integrable on $[a, b]$.

Again, since all the functions f_n are continuous, therefore the sum of a finite number of functions, $\sum_{r=1}^{n} f_r$ is also continuous and integrable on $[a, b]$, and

$$\sum_{r=1}^{n}\int_a^x f_r\, dt = \int_a^x \sum_{r=1}^{n} f_r\, dt$$

By the uniform convergence of the series, for $\varepsilon > 0$, we can find an integer N such that for all x in $[a, b]$,

$$\left|f - \sum_{r=1}^{n} f_r\right| < \varepsilon/(b-a), \quad \forall\, n \geq N$$

For such values of n, and all x in $[a, b]$

$$\left|\int_a^x f\, dt - \sum_{r=1}^{n}\int_a^x f_r\, dt\right| = \left|\int_a^x \left(f - \sum_{r=1}^{n} f_r\right) dt\right|$$

$$\leq \int_a^x \left|f - \sum_{r=1}^{n} f_r\right| dt$$

$$< \frac{\varepsilon}{b-a}\int_a^x dt \leq \varepsilon$$

$$\Rightarrow \quad \sum_{n=1}^{\infty}\left(\int_a^x f_n\,dt\right) \text{ converges uniformly to } \int_a^x f\,dt \text{ on } [a, b],$$

i.e.,
$$\int_a^x f\,dt = \sum_{n=1}^{\infty}\int_a^x f_n\,dt, \text{ for all } x \in [a, b].$$

The corresponding theorem for sequences may be stated and proved in the same way.

4.4 Uniform Convergence and Differentiation

The series $\Sigma \dfrac{\sin nx}{n^2}$ converges uniformly for all values of x and for every term, without exception, is continuous and differentiable. The series of differentials $\Sigma \dfrac{\cos nx}{n}$, however, diverges at $x = 0$. The situation, therefore, seems to be different in the case of differentiation, and accordingly the theorem on term-by-term differentiation must be of a different stamp.

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Theorem 12 (A). *Let $\{f_n\}$ be a sequence of differentiable functions on $[a, b]$ such that it converges at least at one point $x_0 \in [a, b]$. If the sequence of differentials $\{f_n'\}$ converges uniformly to G on $[a, b]$, then the given sequence $\{f_n\}$ converges uniformly on $[a, b]$ to f and $f'(x) = G(x)$.*

(B). Let Σf_n be a series of differentiable functions on $[a, b]$ such that it converges at least at one point $x_0 \in [a, b]$. If the series of differentials $\Sigma f_n'$ converges uniformly to G on $[a, b]$, then the given series Σf_n converges uniformly on $[a, b]$ to f, and $f'(x) = G(x)$.

[The existence of f' is a part of the conclusion.]

(A). Let $\varepsilon > 0$ be any number.

By the convergence of $\{f_n(x_0)\}$ and uniform convergence of $\{f_n'\}$, for $\varepsilon > 0$, we can choose a positive integer N such that for all $x \in [a, b]$,

$$|f_{n+p}(x_0) - f_n(x_0)| < \varepsilon/2, \qquad \forall\, n \geq N,\ p \geq 1 \qquad ...(1)$$

$$|f_{n+p}'(x) - f_n'(x)| < \varepsilon/2(b-a), \quad \forall\, n \geq N,\ p \geq 1 \qquad ...(2)$$

Applying Lagrange's mean value theorem to the function $(f_{n+p} - f_n)$ for any two points x and t of $[a, b]$, we get for $x < \xi < t$, for all $n \geq N$, $p \geq 1$

$$|f_{n+p}(x) - f_n(x) - f_{n+p}(t) + f_n(t)| = |x - t|\,|f_{n+p}'(\xi) - f_n'(\xi)|$$

$$< \frac{|x-t|\,\varepsilon}{2(b-a)} \qquad ...(3)$$

$$< \varepsilon/2 \qquad ...(3A)$$

and

$$|f_{n+p}(x) - f_n(x)| \le |f_{n+p}(x) - f_n(x) - f_{n+p}(x_0) + f_n(x_0)| + |f_{n+p}(x_0) - f_n(x_0)|$$

$$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon \qquad \text{[using (1) \& (3A)]}$$

$\Rightarrow$ The sequence $\{f_n\}$ uniformly converges on $[a, b]$.

Let it converges to f, say.

For a fixed x on $[a, b]$ and for $t \in [a, b]$, $t \neq x$, let us define

$$\phi_n(t) = \frac{f_n(t) - f_n(x)}{t - x}, \; n = 1,2,3,\ldots \qquad ...(4)$$

$$\phi(t) = \frac{f(t) - f(x)}{t - x} \qquad ...(5)$$

Since each f_n is differentiable, therefore for each n

$$\lim_{t \to x} \phi_n(t) = f_n'(x) \qquad ...(6)$$

$\therefore$

$$|\phi_{n+p}(t) - \phi_n(t)| = \frac{1}{|t-x|}|f_{n+p}(t) - f_{n+p}(x) - f_n(t) + f_n(x)|$$

$$= \frac{1}{|t-x|}|\{f_{n+p}(t) - f_n(t)\} - \{f_{n+p}(x) - f_n(x)\}|$$

$$< \frac{\varepsilon}{2(b-a)}, \quad \forall\, n \ge N,\, p \ge 1 \qquad \text{[using (3)]}$$

so that $\{\phi_n(t)\}$ converges uniformly on $[a, b]$, for $t \neq x$.

Since $\{f_n\}$ also converges uniformly on f, therefore from (4),

$$\lim_{n \to \infty} \phi_n(t) = \lim_{n \to \infty} \frac{f_n(t) - f_n(x)}{t - x} = \frac{f(t) - f(x)}{t - x} = \phi(t)$$

Thus $\{\phi_n(t)\}$ converges uniformly to $\phi(t)$ on $[a, b]$, for $t \in [a, b]$, $t \neq x$.

Applying Theorem 7(A) to the uniformly convergent sequence $\{\phi_n(t)\}$ and using (6), we get

$$\lim_{t \to x} \phi(t) = \lim_{n \to \infty} f_n'(x) = G(x)$$

$\Rightarrow$

$$\lim_{t \to x} \phi(t) \text{ exists,}$$

and therefore (5) implies that f is differentiable and

$$\lim_{t \to x} \phi(t) = f'(x)$$

Hence.

$$f'(x) = G(x) = \lim_{n\to\infty} f_n'(x)$$

This completes the proof of the theorem.

If in addition to the above hypothesis of the theorem, continuity of the functions f_n' is also assumed, then a much shorter proof of the theorem exists.

We now prove a simple version of the theorem for series, that for sequence may be written down on the same lines.

Theorem 13. *Let a series Σf_n of differentiable functions converges pointwise to f on $[a, b]$ and each f_n' is continuous on $[a, b]$, and the series $\Sigma f_n'$ converges uniformly to G on $[a, b]$, then the given series Σf_n converges uniformly to f on $[a, b]$, and $f'(x) = G(x)$.*

Since the series $\Sigma f_n'$ of continuous functions converges uniformly to G on $[a, b]$, therefore its sum function G is continuous on $[a, b]$, and consequently the function

$$\int_a^x G(t)\,dt \text{ is differentiable, and}$$

$$\frac{d}{dx}\int_a^x G(t)\,dt = G(x), \quad \text{for all } x \in [a, b]. \qquad ...(1)$$

For every $x \in [a, b]$, $f(x) = \sum_{n=1}^{\infty} f_n(x)$.

Now, since each function f_n', being continuous, is integrable on $[a, b]$, and so by the fundamental theorem of calculus,

$$\int_a^x f_n'(t)\,dt = f_n(x) - f_n(a), \quad \text{for all } n \ge 1, x \in [a, b]$$

$$\therefore \quad \sum_{n=1}^{\infty}\int_a^x f_n'(t)\,dt = f(x) - f(a), \quad \text{for all } x \in [a, b] \qquad ...(2)$$

Again, since the series $\Sigma f_n'$, of integrable functions, converges uniformly to G on $[a, b]$, therefore term-by-term integration is valid, *i.e.*,

$$\int_a^x G(t)\,dt = \sum_{n=1}^{\infty}\int_a^x f_n'(t)\,dt, \quad \forall\, x \in [a, b] \qquad ...(3)$$

From equations (1), (2) and (3), it follows that

$$f'(x) = G(x), \quad \text{for all } x \in [a, b].$$

or equivalently

$$\frac{d}{dx}\sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty}\frac{d}{dx} f_n(x), \; a \le x \le b$$

i.e., the term-by-term differentiation of the series is valid.

4.5 Some Associated Examples

We now consider some examples to show that the limit functions of uniformly convergent series and sequences of continuous (integrable) functions are continuous (integrable) but there do exist series and sequences which though not uniformly convergent but still possess continuous (integrable) functions.

Example 19. The series

$$1 - x + x^2 - x^3 + K = \frac{1}{1+x}, \quad 0 < x < 1$$

■ Each term is integrable.

Integrating from 0 to 1, the right hand side gives

$$\int_0^1 \frac{dx}{1+x} = \log 2$$

while the other side gives

$$\left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right]_0^1 = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{2} + \ldots$$

But we know that

$$\log 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$

Thus the two sides are equal at $x = 1$, and so term by term integration is possible over [0, 1], even though the given series is not uniformly convergent on [0, 1].

Example 20. The sequence $\{f_n\}$, where

$$f_n(x) = nxe^{-nx^2}, n = 1, 2, 3, \ldots$$

converges pointwise to zero on [0, 1].

■ Here

$$\int_0^1 f \; dx = 0$$

and

$$\int_0^1 f_n dx = \frac{1}{2}\left[-e^{-nx^2}\right]_0^1 = \frac{1}{2}(1 - e^{-n})$$

$$\therefore \quad \lim_{n\to\infty} \int_0^1 f_n d_n = \lim_{n\to\infty} \frac{1}{2}\left(1 - e^{-n}\right) = \frac{1}{2} \neq \int_0^1 f \; dx$$

$\Rightarrow$ convergence cannot be uniform on [0, 1].

Note: If we, first, show that the sequence is non-uniformly convergent, then this is an example of a sequence which, though not uniformly convergent, yet has an integrable limit function.

Ex. Show although $f_n(x) = x \exp(-nx^2)$ is uniformly convergent in $[-1, 1]$ to a differentiable function, the limit and differentiation process cannot be interchanged.

Example 21. Show that for $-1 < x < 1$,

$$\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8} + \ldots = \frac{1}{1-x}$$

Consider the series

$$\log(1-x) + \log(1+x) + \log(1+x^2) + \log(1+x^4) + \ldots \qquad \ldots(1)$$

The nth partial sum

$$S_n = \log\{(1-x)(1+x)(1+x^2)(1+x^4)\ldots(1+x^{2n-1})\}$$

$$= \log(1-x^{2n}) \to 0 \text{ as } n \to \infty \text{ for } |x| < 1$$

Hence series (1) converges to zero.

The series of differentials of (1), ignoring the first two terms, is

$$\frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8} + \ldots + \frac{2^n x^{2n-1}}{1+x^{2n}} + \ldots \qquad \ldots(2)$$

Now

$$\left|\frac{2^n x^{2n-1}}{1+x^{2n}}\right| \le 2^n \rho^{2n-1}, \text{ for } |x| \le \rho < 1$$

The series $\Sigma\, 2^n \rho^{2n-1}$ is convergent and therefore by M-test, the series of differentials (2) converges uniformly for $|x| \le \rho < 1$, and therefore (by Theorem 12) its sum is the differential of the sum of series (1) without the first two terms.

But the sum of series (1) without the first two terms $= -\log(1-x) - \log(1+x)$.

Hence,

$$\frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^7}{1+x^8} + \ldots = \frac{d}{dx}\{-\log(1-x) - \log(1+x)\}$$

$$= \frac{1}{1-x} - \frac{1}{1+x}$$

Or

$$\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \ldots = \frac{1}{1-x}$$

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Example 22. The sequence $\{f_n\}$, where $f_n(x) = \dfrac{nx}{1+n^2x^2}$ converges to f and $f(x) = 0$, for all real x.

- Clearly $f'(x) = 0$

and
$$f_n'(x) = \frac{n(1-n^2x^2)}{(1+n^2x^2)^2}$$

Therefore, when $x \neq 0$, $f_n'(x) \to 0$, as $n \to \infty$ and so the formula
$$f'(x) = \lim_{n\to\infty} f_n'(x), \text{ is true.}$$

But at $x = 0$,
$$f_n'(0) = \lim_{x\to 0} \frac{n}{1+n^2x^2} = n, \text{ which tends to } \infty, \text{ as } n \to \infty$$

Thus at $x = 0$, $f'(x) = \lim\limits_{n\to\infty} f_n'(x)$ is false.

So here it is $\{f_n'\}$ that does not converge uniformly in an interval that contains zero.

Example 23. Show that the sequence $\{f_n\}$, where
$$f_n(x) = \frac{x}{1+nx^2}$$
converges uniformly to a function f on [0, 1], and that the equation
$$f'(x) = \lim_{n\to\infty} f_n'(x)$$
is true if $x \neq 0$ and false if $x = 0$. Why so?

- It may be easily shown (Example 7) that the sequence $\{f_n\}$ converges uniformly to zero for all real x.

  The limit function $f(x) = 0$.

  When $x \neq 0$,
$$f_n'(x) = \frac{1-nx^2}{(1+nx^2)^2} \to 0, \text{ as } n \to \infty$$

so that if $x \neq 0$ the formula $f'(x) = \lim\limits_{n\to\infty} f_n'(x)$, is true.

At $x = 0$,
$$f_n'(0) = \lim_{x\to 0} \frac{1}{1+nx^2} = 1$$
so that
$$\lim_{n\to\infty} f_n'(0) = 1 \neq f'(0)$$

Hence at $x = 0$, the formula $f'(x) = \lim_{n \to \infty} f_n'(x)$, is false.

It is so because, the sequence $\{f_n'\}$ is not uniformly convergent in any interval containing zero.

Example 24. Show that the sequence $\{f_n\}$, where

$$f_n(x) = \frac{\log(1 + n^3 x^2)}{n^2}$$

is uniformly convergent on the interval [0, 1].

▪ The sequence $\{\phi_n\}$, where $\phi_n(x) = \dfrac{2nx}{1 + n^3 x^2} \equiv f_n'(x)$, may be easily shown to be uniformly convergent to ϕ, where $\phi(x) = 0$ on [0, 1]. Also each function ϕ_n is continuous on the given interval.

Therefore, (by Theorem 10) the sequence of its integrals, $\{f_n\}$ converges uniformly to $\int_0^x \phi \, dt = 0$ on [0, 1].

Hence the result.

Example 25. Show that the sequence $\{f_n\}$, where

$$f_n(x) = \begin{cases} n^2 x, & 0 \le x \le 1/n \\ -n^2 x + 2n, & 1/n \le x \le 2/n \\ 0, & 2/n \le x \le 1 \end{cases}$$

is not uniformly convergent on [0, 1].

▪ The sequence converges to f, where $f(x) = 0$, for all $x \in [0, 1]$. Each function f_n and f are continuous on [0, 1].

Also $$\int_0^1 f_n \, dx = \int_0^{1/n} n^2 x \, dx + \int_{1/n}^{2/n} (-n^2 x + 2n) \, dx + \int_{2/n}^1 0 \, dx = 1$$

But $$\int_0^1 f \, dx = 0$$

$\therefore$ $$\lim_{n \to \infty} \int_0^1 f_n \, dx \neq \int_0^1 f \, dx$$

So (by Theorem 10) the sequence $\{f_n\}$ cannot converge uniformly on [0, 1].

EXERCISE

1. Show that the following series converge uniformly:

(i) $e^x + e^{2x} + e^{3x} + \ldots, |x| \le \frac{1}{4}$.

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(*ii*) $x - x^2 + x^3 - x^4 + \ldots, -\frac{1}{2} \le x \le \frac{1}{2}$.

2. Discuss the uniform convergence of

$$1 + \frac{e^{-2x}}{2^2 - 1} - \frac{e^{-4x}}{4^2 - 1} + \frac{e^{-6x}}{6^2 - 1} - \ldots, \text{ for all real } x \ge 0$$

3. Show that the sequence $\{f_n\}$, where $f_n(x) = x - \frac{x^n}{n}$, converges uniformly on [0, 1]. Show also that the sequence $\{f_n'\}$ of differentials does not converge uniformly on [0, 1].

4. Decide whether or not the sequences $\{f_n'\}$ and $\left\{\int_0^x f_n \, dt\right\}$ converge uniformly on [0, 1], where

(*i*) $f_n(x) = \frac{x}{1 + n^2 x}$, (*ii*) $f_n(x) = nxe^{-nx^2}$,

(*iii*) $f_n(x) = \frac{n^2 x}{1 + n^3 x^2}$, (*iv*) $f_n(x) = \frac{2 + nx^2}{2 + nx}$.

5. Show that the sequence $\{(\sin x)^{1/n}\}$ converges but not uniformly on $[0, \pi]$.

6. Show that the sequence $\left\{\left(\frac{\sin x}{x}\right)^{1/n}\right\}$ converges but not uniformly on $[0, \pi]$.

7. Show that the series $\sum_1^\infty f_n$, where $f_n(x) = \frac{x^2}{(1 + x^2)^n}$, does not converge uniformly for $x \ge 0$.

[*Hint:* Each f_n is continuous but the series converges to a discontinuous function].

8. Show that the sequence $\{f_n\}$, where $f_n(x) = nx\,(1 - x^2)^n$, converges but not uniformly to f, where $f(x) = 0$, for $0 \le x \le 1$, and that

$$\lim_{n\to\infty} \int_0^1 f_n \, dx \ne \int_0^1 f \, dx.$$

9. Show that the sequence $\{f_n\}$ defined on [0, 1] by

$$f_n(x) = \begin{cases} n(1 - nx), & 0 < x < 1/n \\ 0, & \text{otherwise} \end{cases}$$

converges pointwise, but not uniformly to f, where $f(x) = 0$ for $0 \le x \le 1$, and that

$$\lim_{n\to\infty} \int_0^1 f_n \, dx \ne \int_0^1 f \, dx.$$

10. Consider the sequence $\{f_n\}$ of functions defined by

(*i*) $f_n(x) = |x|^{1+1/n}$, $x \in [-1, 1]$,

(*ii*) $f_n(x) = (2x/\pi) \tan^{-n}(nx)$, $x \in \mathbf{R}$.

Show that $\{f_n\}$ converges uniformly to $|x|$, but $\{f_n'\}$ converges only pointwise to sgn (x), on the indicated interval.

11. Given $f(x) = \sum_{n=1}^{\infty} \frac{1}{n^3 + n^4 x^2}$, justify the validity of the equation

$$f'(x) = -2x \sum_{n=1}^{\infty} \frac{1}{n^2 (1 + nx^2)^2}.$$

12. Show that, if a sequence of $\{f_n\}$ bounded functions on $[a, b]$ converges uniformly to f on $[a, b]$, then f is bounded on $[a, b]$. Is the result still valid, if we have only pointwise convergence?

13. The sequence $\{f_n\}$ is defined for $x \geq 0$, as follows:

$$f_1(x) = \sqrt{x},\ f_{n+1}(x) = \sqrt{x + f_n(x)},\ n \geq 1.$$

Show that $\{f_n\}$ converges uniformly on $[a, b]$, $0 < a < b$. Is the convergence uniform on $[0, 1]$?

5. THE WEIERSTRASS APPROXIMATION THEOREM

We shall now study a very famous theorem, discovered originally by Weierstrass. The theorem is described by saying that every continuous function can be 'uniformly approximated' by polynomials to within any degree of accuracy. Many proofs of this classical theorem are known, and the one we give is perhaps as concise and instructive as most.

Theorem 14. *If f is a real continuous function defined on a closed interval* $[a, b]$ *then there exists a sequence of real polynomials* $\{P_n\}$ *which converges uniformly to* $f(x)$ *on* $[a, b]$, *i.e,* $\lim_{n\to\infty} P_n(x) = f(x)$, *converges uniformly on* $[a, b]$.

If $a = b$, the conclusion follows by taking $P_n(x)$ to be a constant polynomial, defined by $P_n(x) = f(a)$, for all n.

We may thus assume that $a < b$.

We next observe that a linear transformation $x' = (x - a)/(b - a)$ is a continuous mapping of $[a, b]$ onto $[0, 1]$. Accordingly, we assume without loss of generality that $a = 0$, $b = 1$.

Consider

$$F(x) = f(x) - f(0) - x\,[f(1) - f(0)], \quad \text{for } 0 \leq x \leq 1$$

Here $F(0) = 0 = F(1)$, and if F can be expressed as a limit of a uniformly convergent sequence of polynomials, then the same is true for f, since $f - F$ is a polynomial. So we may assume that $f(1) = f(0) = 0$.

Let us further define $f(x)$ to be zero for x outside $[0, 1]$. Thus, f is now uniformly continuous on the whole real line.

Let us consider the polynomial (non-negative for $|x| \leq 1$).

$$B_n(x) = C_n (1 - x^2)^n, \quad n = 1, 2, 3, \ldots \qquad \ldots(1)$$

where C_n, independent of x, is so chosen that

$$\int_{-1}^{1} B_n(x)\, dx = 1 \quad \text{for } n = 1, 2, 3, \ldots \qquad \ldots(2)$$

$$\therefore \quad 1 = \int_{-1}^{1} C_n(1-x^2)^n\,dx = 2C_n \int_0^1 (1-x^2)^n\,dx$$

$$\geq 2C_n \int_0^{1/\sqrt{n}} (1-x^2)^n\,dx$$

$$\geq 2C_n \int_0^{1/\sqrt{n}} (1-nx^2)\,dx = \frac{4C_n}{3\sqrt{n}} > \frac{C_n}{\sqrt{n}}$$

$$\Rightarrow \quad C_n < \sqrt{n} \qquad \text{...(3)}$$

which gives some information about the order of magnitude of C_n.

Therefore, for *any* $\delta > 0$, (3) gives

$$B_n(x) \leq \sqrt{n}\,(1-\delta^2)^n, \text{ when } \delta \leq |x| \leq 1 \qquad \text{...(4)}$$

so that $B_n \to 0$ uniformly, for $\delta \leq |x| \leq 1$.

Again, let

$$P_n(x) = \int_{-1}^{1} f(x+t)\,B_n(t)\,dt, \quad 0 \leq x \leq 1 \qquad \text{...(5)}$$

$$= \int_{-1}^{-x} f(x+t)\,B_n(t)\,dt + \int_{-x}^{1-x} f(x+t)\,B_n(t)\,dt + \int_{1-x}^{1} f(x+t)\,B_n(t)\,dt$$

For $|x| \leq 1, -1+x \leq x+t \leq 0$, for $-1 \leq t \leq -x$, so that $x+t$ lies outside [0, 1] and therefore $f(x+t)=0$, and hence the first integral on the right vanishes. Similarly, the third integral is also equal to zero. Hence

$$P_n(x) = \int_{-x}^{1-x} f(x+t)\,B_n(t)\,dt = \int_0^1 f(t)\,B_n(t-x)\,dt$$

which clearly is a real polynomial.

We now proceed to show that the sequence $\{P_n(x)\}$ converges uniformly to f on [0, 1].

Continuity of f on the closed interval [0, 1] implies that f is bounded and uniformly continuous on [0, 1].

Therefore, there exists M such that

$$M = \sup |f(x)| \qquad \text{...(6)}$$

and for any $\varepsilon > 0$, we can choose $\delta > 0$ such that for any two points x_1, x_2 in [0, 1],

$$|f(x_1) - f(x_2)| < \varepsilon/2, \text{ when } |x_1 - x_2| < \delta \leq 1 \qquad \text{...(7)}$$

For $0 \leq x \leq 1$, we have

$$|P_n(x) - f(x) = \left| \int_{-1}^{1} f(x+t)\,B_n(t)\,dt - f(x) \right|$$

$$= \left| \int_{-1}^{1} \{f(x+t) - f(x)\}\,B_n(t)\,dt \right| \qquad \text{[using (2)]}$$

$$\le \int_{-1}^{1} |f(x+t) - f(x)| B_n(t)\, dt \qquad (\because B_n(t) \ge 0)$$

$$= \int_{-1}^{-\delta} |f(x+t) - f(x)| B_n(t)\, dt + \int_{-\delta}^{\delta} |f(x+t) - f(x)| B_n(t)\, dt$$

$$+ \int_{\delta}^{1} |f(x+t) - f(x)| B_n(t)\, dt$$

$$\le 2M \int_{-1}^{-\delta} B_n(t)\, dt + \frac{\varepsilon}{2} \int_{-\delta}^{\delta} B_n(t)\, dt + 2M \int_{\delta}^{1} B_n(t)\, dt$$

[using equations (6) and (7)]

$$\le 2M \sqrt{n}\, (1-\delta^2)^n \left\{ \int_{-1}^{-\delta} dt + \int_{\delta}^{1} dt \right\} + \varepsilon/2$$

[using equations (2) and (4)]

$$\le 4M \sqrt{n} (1-\delta^2)^n + \varepsilon/2$$

$< \varepsilon$, for large values of n.

Thus for $\varepsilon > 0$, there exists N (independent of x) such that

$$|P_n(x) - f(x)| < \varepsilon, \quad \forall\, n \ge N$$

$$\Rightarrow \qquad \lim_{n \to \infty} P_n(x) = f(x), \text{ uniformly on } [0, 1].$$

Second method. If $a = b$, the conclusion follows by taking $P_n(x)$ to be a constant polynomial, defined by $P_n(x) = f(a)$, for all n.

We may thus assume that $a < b$.

We next observe that a linear transformation $x' = (x-a)/(b-a)$ is a continuous mapping of $[a, b]$ onto $[0, 1]$. Accordingly, we assume without loss of generality that $a = 0$, $b = 1$.

For positive integers n and k when $0 \le k \le n$, the binomial coefficient $\binom{n}{k}$ is defined by

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Let us define the polynomials B_n, where

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f(k/n),\ n = 1, 2, 3, \ldots, \text{ and } x \in [0, 1]$$

called the *Bernstein polynomials* associated with f.

Let us first consider some identities which will be our main tools to show that certain Bernstein polynomials exist which continuously converge to f on $[0, 1]$.

The first of the identities is a special case of the binomial theorem,

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$$\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} = [x + (1-x)]^n = 1 \qquad \text{...(1)}$$

Differentiating with respect to x, we get

$$\sum_{k=0}^{n} \binom{n}{k} [kx^{k-1}(1-x)^{n-k} - (n-k)\, x^k (1-x)^{n-k-1}] = 0$$

or
$$\sum_{k=0}^{n} \binom{n}{k} x^{k-1}(1-x)^{n-k-1} (k - nx) = 0$$

and multiplication by $x(1-x)$ gives

$$\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} (k - nx) = 0 \qquad \text{...(2)}$$

Differentiating again with respect to x, we get

$$\sum_{k=0}^{n} \binom{n}{k} [- nx^k (1-x)^{n-k} + x^{k-1}(1-x)^{n-k-1}(k-nx)^2] = 0$$

which on applying in (1), gives

$$\sum_{k=0}^{n} \binom{n}{k} x^{k-1}(1-x)^{n-k-1} (k-nx)^2 = n$$

and on multiplying by $x(1-x)$, we get

$$\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} (k-nx)^2 = nx(1-x)$$

or
$$\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} (x - k/n)^2 = \frac{x(1-x)}{n} \qquad \text{...(3)}$$

The maximum value of $x(1-x)$ in [0, 1] being $\frac{1}{4}$.

$\therefore$
$$\sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} (x - k/n)^2 \le \frac{1}{4n} \qquad \text{...(4)}$$

Continuity of f on the closed interval [0, 1] implies that f is bounded and uniformly continuous on [0, 1].

Hence, there exists $K > 0$, such that

$$| f(x) | \le K, \quad \forall\, x \in [0, 1]$$

and for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x \in [0, 1]$

$$|f(x) - f(k/n)| < \frac{1}{2}\varepsilon, \text{ when } |x - k/n| < \delta \qquad \ldots(5)$$

For any fixed but arbitrary x in $[0, 1]$, the values $0, 1, 2, 3, \ldots, n$ of k may be divided into two parts: Let A be the set of values of k for which $|x - k/n| < \delta$, and B the set of the remaining values, for which $|x - k/n| \geq \delta$.

For $k \in B$, using (4),

$$\sum_{k \in B} \binom{n}{k} x^k (1-x)^{n-k} \delta^2 \leq \sum_{k \in B} \binom{n}{k} x^k (1-x)^{n-k} (x - k/n)^2 \leq \frac{1}{4n}$$

$$\Rightarrow \qquad \sum_{k \in B} \binom{n}{k} x^k (1-x)^{n-k} \leq \frac{1}{4n\delta^2} \qquad \ldots(6)$$

Using (1), we see that for this fixed x in $[0, 1]$,

$$|f(x) - B_n(x)| = \left| \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} [f(x) - f(k/n)] \right|$$

$$\leq \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} |f(x) - f(k/n)|$$

Thus, summation on the right may be split into two parts, according as $|x - k/n| < \delta$ or $|x - k/n| \geq \delta$. Thus

$$|f(x) - B_n(x)| \leq \sum_{k \in A} \binom{n}{k} x^k (1-x)^{n-k} |f(x) - f(k/n)|$$

$$+ \sum_{k \in B} \binom{n}{k} x^k (1-x)^{n-k} |f(x) - f(k/n)|$$

$$< \frac{\varepsilon}{2} \sum_{k \in A} \binom{n}{k} x^k (1-x)^{n-k} + 2K \sum_{k \in B} \binom{n}{k} x^k (1-x)^{n-k}$$

$$\leq \varepsilon/2 + 2K/4n\delta^2 < \varepsilon, \text{ using equations (1), (5) and (6),}$$

for values of n greater than $K/\varepsilon\delta^2$.

Thus $\{B_n(x)\}$ converges uniformly to $f(x)$ on $[0, 1]$.

Corollary. *For any interval $[-a, a]$ there is a sequence of real polynomials P_n such that $P_n(0) = 0$, and that*

$$\lim_{n \to \infty} P_n(x) = |x|, \text{ uniformly on } [-a, a].$$

Since $|x|$ is a real continuous function on $[-a, a]$, therefore by 'Weierstrass approximation theorem' there exists a sequence $\{Q_n\}$ of real polynomials which converges uniformly to $|x|$ on $[-a, a]$.

In particular,

$$Q_n(0) \to 0, \text{ as } n \to \infty$$

Hence, the polynomials

$$P_n(x) - Q_n(x) - Q_n(0), \quad n = 1, 2, 3, \ldots$$

have the required property.

Example 26. If f is continuous on $[0, 1]$, and if

$$\int_0^1 x^n f(x)\, dx = 0, \quad \text{for } n = 0, 1, 2, \ldots \qquad \ldots(1)$$

then show that $f(x) = 0$ on $[0, 1]$.

■ From (1), it follows that the integral of the product of f with any polynomial is zero.

Now, since f is continuous on $[0, 1]$, therefore, by 'Weierstrass approximation theorem', there exists a sequence $\{P_n\}$ of polynomials, such that $P_n \to f$ uniformly on $[0, 1]$. And so $P_n f \to f^2$ uniformly on $[0, 1]$. Since f, being continuous, is bounded on $[0, 1]$. Therefore, by Theorem 10 (A),

$$\int_0^1 f^2\, dx = \lim_{n\to\infty} \int_0^1 P_n f\, dx = 0 \qquad \text{[using (1)]}$$

$$\therefore \qquad f^2 = 0 \text{ on } [0, 1]. \quad \text{Hence, } f = 0 \text{ on } [0, 1].$$

Ex. 1. Obtain the first six Bernstein polynomials for the following functions:

(*i*) $f(x) = |x|, [-1, 1]$ (*ii*) $f(x) = \sin x, [0, \pi]$

(*iii*) $f(x) = e^x, [-1, 1]$ (*iv*) $f(x) = \cos 6x, [0, \pi]$.

Ex. 2. Show that there does not exist a sequence of polynomials converging uniformly on **R** to f, where

(*i*) $f(x) = e^x$ (*ii*) $f(x) = \sin x$.

Ex. 3. Show that, if f is continuous on **R**, then there exists a sequence $\{P_n\}$ of polynomials converging uniformly to f on each bounded subset of **R**.

[*Hint*: Arrange for $|P_n(x) - f(x)| < 1/n$, for $|x| \leq n$].