IFS MATHS PAPER-I (2016).

SEC-A

1(a) Let $T: \mathbb{R}^3 \to \mathbb{R}^4$ be given by
$T(x,y,z) = (2x-y,\ 2x+z,\ x+2z,\ x+y+z)$.
Find the matrix of T with respect to standard basis of $\mathbb{R}^3$ and $\mathbb{R}^4$.
Examine if T is a Linear mp. (8).

Sol: $T(1,0,0) = (2,2,1,1) = 2(1,0,0,0) + 2(0,1,0,0) + 1(0,0,1,0) + 1(0,0,0,1)$

$T(0,1,0) = (-1,0,0,1)$

$T(0,0,1) = (0,1,2,1)$

$$\therefore [T]_{\alpha}^{\beta} = \begin{bmatrix} 2 & 2 & 1 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 \end{bmatrix}^T = \begin{bmatrix} 2 & -1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

Let $a = (x_1, y_1, z_1)$, $b = (x_2, y_2, z_2)$ & k is constant.

$$T(a+b) = T(x_1+x_2,\ y_1+y_2,\ z_1+z_2)$$
$$= [2(x_1+x_2)-(y_1+y_2),\ 2(x_1+x_2)+(z_1+z_2),\ (x_1+x_2)+2(z_1+z_2),\ (x_1+x_2)+(y_1+y_2)+(z_1+z_2)]$$
$$= [(2x_1-y_1)+(2x_2-y_2),\ (2x_1+z_1)+(2x_2+z_2),\ (x_1+2z_1)+(x_2+2z_2),\ (x_1+y_1+z_1)+(x_2+y_2+z_2)]$$
$$= T(x_1,y_1,z_1) + T(x_2,y_2,z_2) = T(a)+T(b)$$

Similarly $T(kx_1) = k \cdot T(x_1)$.

Hence T is linear.

1.(b) Show that $\frac{x}{1+x} < \log(1+x) < x$ for $x > 0$. (8)

Consider the function, $f(x) = \log(1+x) - \frac{x}{1+x}$

$$f'(x) = \frac{1}{1+x} - \frac{(1+x)-x}{(1+x)^2} = \frac{1}{1+x^2} > 0$$

$\therefore$ $f(x)$ is increasing function.

$\therefore$ If $x > 0 \Rightarrow f(x) > f(0)$

ie $\log(1+x) - \frac{x}{1+x} > \log(1+0) - \frac{0}{1+0}$

ie $\log(1+x) > \frac{x}{1+x}$. — ①

Again, let $g(x) = x - \log(1+x)$

$$g'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x} > 0 \quad \forall\, x > 0$$

$\therefore$ $g(x)$ is increasing function

$\therefore$ for $x > 0 \Rightarrow f(x) > f(0)$

ie $x - \log(1+x) > 0 - \log(1+0)$

$x > \log(1+x)$ — ②

Combining ① & ②

$$\frac{x}{1+x} < \log(1+x) < x.$$

1.(c) Examine if the function $f(x,y) = \frac{xy}{x^2+y^2}, (x,y) \neq (0,0)$ and $f(0,0)=0$ is continuous at $(0,0)$.
Find $\frac{\partial F}{\partial x}$ and $\frac{\partial f}{\partial y}$ at points other than origin. (8)

$$f(x) = \begin{cases} \frac{xy}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

We show that limit does not exist at $(0,0)$
Along the curve $y = mx$.

$$\lim_{(x,y)\to(0,0)} \frac{x(mx)}{x^2+(mx)^2} = \frac{m}{1+m^2}$$

which is different for different values of x. Hence limit does not exist and so $f(x)$ is not cont at $(0,0)$.

For the points, other than origin

$$\frac{\partial F}{\partial x} = \frac{\partial}{\partial x}\left(\frac{xy}{x^2+y^2}\right) = \frac{y(x^2+y^2) - 2x(xy)}{(x^2+y^2)^2}$$

$$= \frac{y^3 - x^2y}{(x^2+y^2)^2} = \frac{y(y^2-x^2)}{(x^2+y^2)^2}$$

Similarly, $$\frac{\partial F}{\partial y} = \frac{x(x^2-y^2)}{(x^2+y^2)^2}.$$

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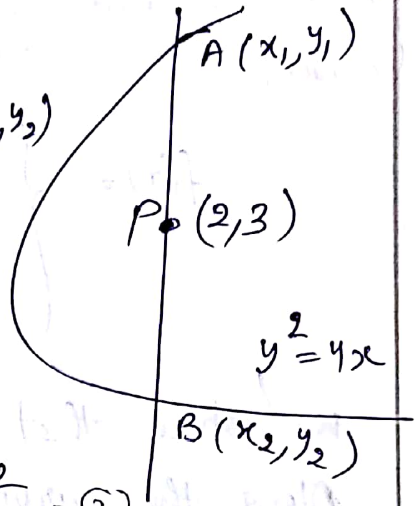
1(d) If the point $(2,3)$ is the mid-point of a chord of the parabola $y^2=4x$, then obtain the equation of the chord. (8)

Let two points on the parabola be $A(x_1,y_1)$ & $B(x_2,y_2)$ where chord cut the parabola & $P(2,3)$ be the mid-point.



$\therefore \; y_1^2=4x_1 \quad -(1)$, & $\; y_2^2=4x_2 \quad -(2)$

$$\frac{x_1+x_2}{2}=2, \qquad \frac{y_1+y_2}{2}=3$$

$$y_2^2-y_1^2=4x_2-4x_1$$

$$(y_1+y_2)(y_2-y_1)=4(x_2-x_1)$$

$$\frac{y_2-y_1}{x_2-x_1}=\frac{4}{y_1+y_2}$$

$$=\frac{4}{6}=\frac{2}{3}$$

Slope of AB $=\dfrac{y_2-y_1}{x_2-x_1}$

$$=\frac{2}{3}.$$

$\therefore$ Eqn of chord:

$$y-3=\frac{2}{3}(x-2)$$

$$3y-9=2x-4$$

$$2x-3y+5=0$$

1(e) For the matrix $A = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$,

obtain the eigen values and get the value of $A^4+3A^3-9A^2$. (8).

Sol: Here, $|A-\lambda I| = 0$ gives.

$$\begin{vmatrix} -1-\lambda & 2 & 2 \\ 2 & -1-\lambda & 2 \\ 2 & 2 & -1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda^3+3\lambda^2-9\lambda-27=0$$

$\therefore$ ~~$\lambda = 3, -3, -3$~~

$\Rightarrow (\lambda+3)(\lambda^2-9)=0$

$\therefore \lambda = -3, 3, 3$ are the eigen values.

By Cayley Hamilton Theorem.

$$A^3+3A^2-9A-27I=0$$

$$\Rightarrow A^4+3A^3-9A^2-27A=0$$

$$\therefore A^4+3A^3-9A^2 = 27A = \begin{bmatrix} -27 & 54 & 54 \\ 54 & -27 & 54 \\ 54 & 54 & -27 \end{bmatrix}$$

2(a) After changing the order of integral of $\int_0^\infty\int_0^\infty e^{-xy}\sin nx\,dx\,dy$, show that

$$\int_0^\infty \frac{\sin nx}{x}\,dx = \frac{\pi}{2}.$$ (10)

$$I = \int_0^\infty\int_0^\infty \sin nx \cdot e^{-xy}\cdot dy\,dx$$

$$= \int_0^\infty \sin nx \cdot \left[\frac{e^{-xy}}{-x}\right]_{y=0}^{\infty} dx$$

$$= \int_0^\infty \sin nx\left(0+\frac{1}{x}\right)dx = \int_0^\infty \frac{\sin nx}{x}\,dx \quad \text{—(1)}$$

Now first integrating w.r.t x,

$$I = \int_0^\infty\left[-\frac{1}{y}e^{-xy}\cdot\sin nx\Big|_{x=0}^{\infty} + \int_0^\infty \frac{1}{y}e^{-xy}\cdot n\cos nx\,dx\right]dy$$

$$= \int_0^\infty\left[\frac{n}{y}\left(-\frac{1}{y}e^{-xy}\cos nx\Big|_{x=0}^{\infty} - \int_0^\infty \frac{e^{-xy}}{y}n\sin nx\right]\right]dy$$

$$= \int_0^\infty \frac{n}{y}\left(0+\frac{1}{y}-\frac{n}{y}I'\right)dy$$

$$= \int_0^\infty\left(\frac{n}{y^2}-\frac{n^2}{y^2}I\right)dy$$

$$\therefore \quad \frac{n}{y^2}-\frac{n^2}{y^2}I = I \Rightarrow I\left(1+\frac{n^2}{y^2}\right)=\frac{n}{y^2}$$

$$I = \frac{n}{n^2+y^2}$$

$$\therefore \int_0^\infty \frac{n}{n^2+y^2}\,dy = \frac{1}{n}\cdot n\tan^{-1}\frac{y}{n}\Big|_0^\infty = \frac{\pi}{2}-0=\frac{\pi}{2}$$

$$\therefore I = \int_0^\infty \frac{\sin nx}{x} = \pi/2.$$

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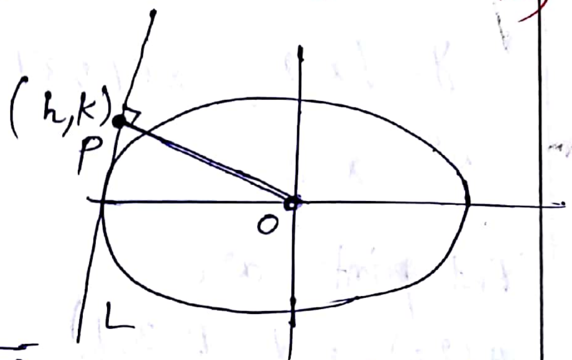
2.b) A perpendicular is drawn from the centre of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ to any tangent. Prove that the locus of the foot of the $\perp$s is given by $(x^2+y^2)^2 = a^2x^2+b^2y^2$. (14)

~~If line~~

The tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is



$$y = mx \pm \sqrt{a^2m^2+b^2} \quad \text{—①}$$ for any value of m.

Slope of line OP $= \frac{k-0}{h-0} = \frac{k}{h}$

Slope of tangent line $= -\frac{h}{k}$ $(OP \perp L)$

$\therefore$ Eqn of tangent line

$$y-k = -\frac{h}{k}(x-h)$$

$$y = -\frac{h}{k}x + \frac{h^2}{k}+k$$

$$y = -\frac{h}{k}x + \left(\frac{h^2+k^2}{k}\right) \quad \text{—②}$$

Comparing Eqn ① with ②

$$\pm\sqrt{a^2m^2+b^2} = \frac{h^2+k^2}{k}$$

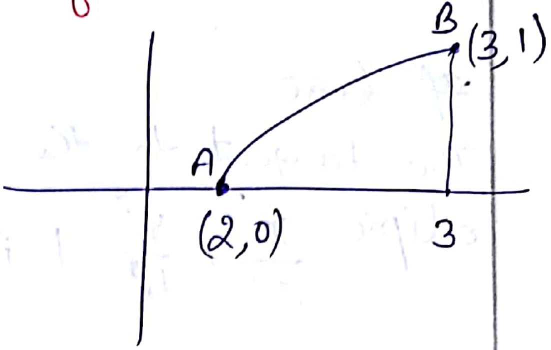
$$\left(a^2\left(\frac{-h}{k}\right)^2+b^2\right) = \left(\frac{h^2+k^2}{k}\right)^2 \quad \left(\because m = \frac{-h}{k}\right)$$

$$\therefore a^2h^2 + b^2k^2 = (h^2+k^2)^2$$

Hence required locus : $(x^2+y^2)^2 = a^2x^2+b^2y^2$

2.C) Using mean value theorem, find a point on the curve $y=\sqrt{x-2}$, defined on $[2,3]$, where the tagent is parallel to the chord joining the end points of the curve. (10)

$y=\sqrt{x-2}$, $x\in[2,3]$



$y^2=x-2$

End points are
A (2,0) and B(3,1)

$y=\sqrt{x-2}$ is continuous on $[2,3]$
$y=\sqrt{x-2}$ is differentiable on $(2,3)$.

Hence by lagrange's mean value theorem (LMVT)
there exist some $c\in(2,3)$ s.t.

$$f'(c)=\frac{f(b)-f(a)}{b-a}$$

$$\frac{1}{2\sqrt{c-1}}=\frac{f(3)-f(2)}{3-2}=\frac{1-0}{1}$$

$$\Rightarrow 2\sqrt{c-1}=1$$

ie. $c-2=\frac{1}{4} \Rightarrow c=\frac{9}{4}$

Hence, at $x=9/4$, $y=\sqrt{\frac{9}{4}-2}=\frac{1}{2}$, tangent to the curve is parallel to the chord joining the end points as slopes are equal there.

2.d) Let T be a L.T. s.t. $T: V_3 \to V_2$ defined by $T(e_1) = 2f_1 - f_2$, $T(e_2) = f_1 + 2f_2$, $T(e_3) = 0f_1 + 0f_2$. where e_1, e_2, e_3 and f_1, f_2 are standard basis in V_3 and V_2. Find the matrix of T relative to these basis. Further take two other basis $B_1 [(1,1,0), (1,0,1), (0,1,1)]$ and $B_2 [(1,1), (1,-1)]$. Obtained the Matrix T_1 relative to B_1 & B_2. (10).

$$T(e_1) = 2f_1 - f_2$$
$$T(e_2) = f_1 + 2f_2$$
$$T(e_3) = 0f_1 + 0f_2$$

$$T = \begin{bmatrix} 2 & -1 \\ 1 & 2 \\ 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & 0 \end{bmatrix}$$

$$T(a,b,c) = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2a+b \\ -a+2b \end{bmatrix}$$

$$T(1,1,0) = (3,1) = x_1(1,1) + y_1(1,-1)$$
$$T(1,0,1) = (2,-1) = x_2(1,1) + y_2(1,-1)$$
$$T(0,1,1) = (1,2) = x_3(1,1) + y_3(1,-1)$$

$\therefore$ $x_1 = 2$, $y_1 = 1$, $x_2 = \frac{1}{2}$, $y_2 = \frac{3}{2}$, $x_3 = \frac{3}{2}$, $y_3 = \frac{-1}{2}$

$$\therefore \quad [T]_{B_1}^{B_2} = \begin{bmatrix} 2 & 1 \\ \frac{1}{2} & 3/2 \\ 3/2 & -1/2 \end{bmatrix}^T = \begin{bmatrix} 2 & \frac{1}{2} & 3/2 \\ 1 & 3/2 & -1/2 \end{bmatrix}$$

3.a For the matrix, $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, find two non-singular matrices P and Q s.t. $PAQ = I$. Hence find A^{-1}. (10)

$$IAI = A$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$$

$R_2 \to R_2 - \frac{2}{3}R_1$
$$\begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -3 & 4 \\ 0 & -1 & 4/3 \\ 0 & -1 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$
$$\begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1 & 0 \\ 2/3 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -3 & 4 \\ 0 & -1 & 4/3 \\ 0 & 0 & -1/3 \end{bmatrix}$$

$C_2 \to C_2 + C_1$, $C_3 \to C_3 - 4/3\, C_1$
$$\begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1 & 0 \\ 2/3 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & -4/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 4/3 \\ 0 & 0 & -1/3 \end{bmatrix}$$

$C_3 \to C_3 + 4/3\, C_2$
$$\begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1 & 0 \\ 2/3 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4/3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1/3 \end{bmatrix}$$

$R_1 \to R_1/3$, $R_2 \to -R_2$, $R_3 \to -3R_3$
$$\begin{bmatrix} 1/3 & 0 & 0 \\ 2/3 & -1 & 0 \\ -2 & 3 & -3 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4/3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore PAQ = I$

$A = P^{-1}Q^{-1}$

$A^{-1} = QP$

$$A^{-1} = QP = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4/3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/3 & 0 & 0 \\ 2/3 & -1 & 0 \\ -2 & 3 & -3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

3.b) Using Lagrange's method of multipliers find the point on the plane $2x+3y+4z=5$ which is closest to the point $(1,0,0)$. (10)

Let the required point be (x,y,z).

Then our problem translates to

Maximize $f(x,y,z) = (x-1)^2 + y^2 + z^2$ — ①

Subject to $2x+3y+4z=5$ — ②

let $g(x,y,z) = 2x+3y+4z-5$

Let λ be the lagrange's multiplier

$$f+\lambda g = F(x,y,z)$$

For critical points, $\partial F = 0$ ~~ie $\partial f + \lambda \partial g = 0$~~

$\cancel{\partial F = 2(x-1) + 2y + 2z + \lambda(2}$

$dx = 2(x-1) + 2\lambda = 0 \Rightarrow x = -\lambda + 1$

$dy = 2y + 3\lambda = 0 \Rightarrow y = \frac{-3\lambda}{2}$

$dz = 2z + 4\lambda = 0 \Rightarrow z = -2\lambda$

Using Eqn ②

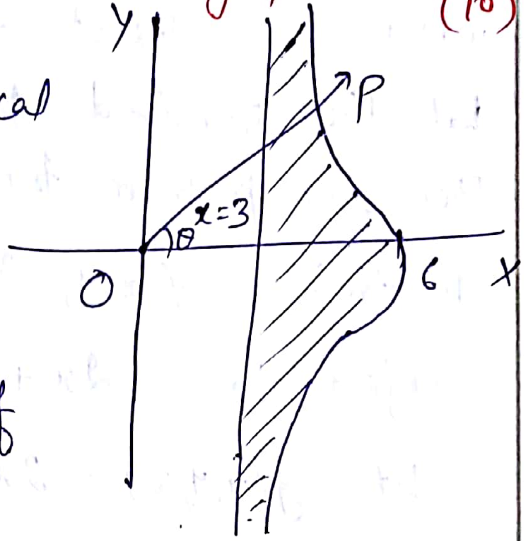
$$2(-\lambda+1) + 3\left(\frac{-3\lambda}{2}\right) + 4(-2\lambda) = 5$$

$$\frac{-29}{2}\lambda = 3 \Rightarrow \lambda = -\frac{6}{29}$$

$$\therefore x = \frac{6}{29} + 1 = \frac{35}{29}, \quad y = \frac{9}{29}, \quad z = \frac{12}{29}$$

Hence, the required point is $\left(\frac{35}{29}, \frac{9}{29}, \frac{12}{29}\right)$ (which is the foot of the $\perp$ also).

3.C) Obtain the area between the curves $r = 3(\sec\theta + \cos\theta)$ and its asymptote $x = 3$. (10)



The curve is symmetrical about the initial line and has an asymptote $r = 3\sec\theta$

In the upper half of the curve θ varies from 0 to $\pi/2$.

$\therefore$ The required area

$$= 2\int_0^{\pi/2}\int_{3\sec\theta}^{3(\sec\theta+\cos\theta)} r\,dr\,d\theta$$

$$= 2\int_0^{\pi/2} \left.\frac{r^2}{2}\right|_{3\sec\theta}^{3(\sec\theta+\cos\theta)} d\theta$$

$$= 2\cdot\frac{9}{2}\int_0^{\pi/2} (\sec\theta+\cos\theta)^2 - \sec^2\theta \, d\theta$$

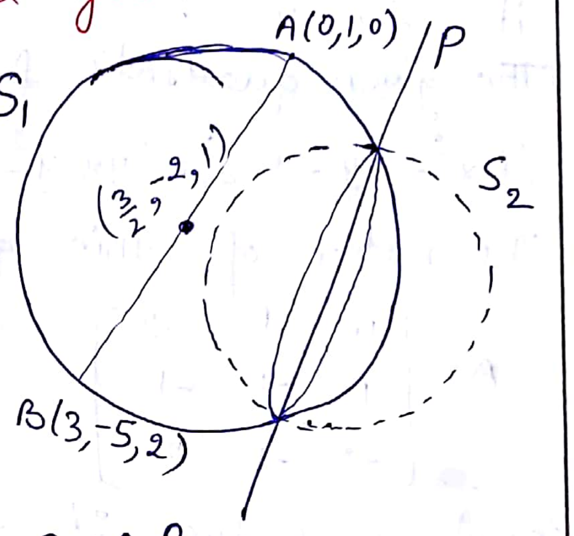
$$= 9\int_0^{\pi/2} (2+\cos^2\theta)\,d\theta = 9\int_0^{\pi/2}\left(2+\frac{1+\cos 2\theta}{2}\right)d\theta$$

$$= 9\left[(2\theta)_0^{\pi/2} + \frac{1}{2}\cdot\frac{\pi}{2}\right] = 9\cdot\frac{\pi}{2}\left(2+\frac{1}{2}\right)$$

$$= \frac{45\pi}{4} \text{ sq unit.}$$

3.d Obtain the eqn of the sphere on which the intersection of the plane $5x-2y+4z+7=0$ with the sphere which has $(0,1,0)$ and $(3,-5,2)$ as the end points of its diameter is a great circle.

$$r=\sqrt{\frac{9}{4}+9+1}=\frac{7}{2}$$



Equation of S_1

$$\left(x-\frac{3}{2}\right)^2+(y+2)^2+(z-1)^2=\frac{49}{4}$$

$$x^2+y^2+z^2-3x+4y-2z-5=0$$

Equation of S_2 is : $S_1+\lambda P=0$

$$(x^2+y^2+z^2-3x+4y-2z-5)+\lambda(5x-2y+4z+7)=0$$

~~Centre~~ $$x^2+y^2+z^2+(-3+5\lambda)x+(4-2\lambda)y+(-2+4\lambda)z-5+7\lambda=0$$

Centre $\left(\frac{3-5\lambda}{2}, -2+\lambda, 1-2\lambda\right)$ lies on P

$$5\left(\frac{3-5\lambda}{2}\right)-2(-2+\lambda)+4(1-2\lambda)+7=0$$

$$\boxed{\lambda=1}$$

$\therefore$ Eqn of S_2 :

$$x^2+y^2+z^2+2x+2y+2z+2=0$$

with centre $(-1,-1,-1)$ and radius 1.

4-a Examine whether the real quadratic form $4x^2-y^2+2z^2+2xy-2yz-4xz$ is a positive definite or not. Reduce it to its diagonal form and determine its signature. (10)

The given quadratic form can be written as–

$(4x^2+xy-2xz)+(yx-y^2-yz)+(-2zx-zy+2z^2)$

The matrix of this quadratic form is–

$$A=\begin{bmatrix}4 & 1 & -2\\ 1 & -1 & -1\\ -2 & -1 & 2\end{bmatrix}$$

which is a symmetric square matrix of order 3×3.

First we reduce it to its diagonal (canonical) form by writing $A = IAI$

$$\begin{bmatrix}4 & 1 & -2\\ 1 & -1 & -1\\ -2 & -1 & 2\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}A\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

To avoid fraction, $R_2\to 4R_2$, $R_3\to 2R_3$

$$\begin{bmatrix}4 & 1 & -2\\ 4 & -4 & -4\\ -4 & -2 & 4\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 2\end{bmatrix}A\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

Perform, corresponding column operation, $C_2\to 4C_2$, $C_3\to 2C_3$

$$\begin{bmatrix}4 & 4 & -4\\ 4 & -16 & -8\\ -4 & -8 & 8\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 2\end{bmatrix}A\begin{bmatrix}1 & 0 & 0\\ 0 & 4 & 0\\ 0 & 0 & 2\end{bmatrix}$$

Apply, $R_2\to R_2-R_1$, $R_3\to R_3+R_1$ & $C_2\to C_2-C_1$, $C_3\to C_3+C_1$

$$\begin{bmatrix}4 & 0 & 0\\ 0 & -20 & -4\\ 0 & -4 & 4\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ -1 & 4 & 0\\ 1 & 0 & 2\end{bmatrix}A\begin{bmatrix}1 & -1 & 1\\ 0 & 4 & 0\\ 0 & 0 & 2\end{bmatrix}$$

$R_3\to R_3-\frac{1}{5}R_2$, $C_3\to C_3-C_2/5$

$$\begin{bmatrix}4 & 0 & 0\\ 0 & -20 & 0\\ 0 & 0 & 24/5\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ -1 & 4 & 0\\ 6/5 & -4/5 & 2\end{bmatrix}A\begin{bmatrix}1 & -1 & 6/5\\ 0 & 4 & -4/5\\ 0 & 0 & 2\end{bmatrix}$$

(last page)

Diagonal form, $4x^2 - 20y^2 + \frac{24}{5}z^2$

Rank(r) of given quadratic form:

= No. of nonzero terms in diagonal form (canonical / normal form)

= 3

Signature (s) of given quadratic form:

= The no. of positive terms − No. of negative terms

= $2-1 = 1$

The index of the given quadratic form

= No. of positive terms in normal form

= 2

Since, $r \neq s$ here

The given quadratic form is not positive definite.

$\boxed{4\cdot b}$ ~~Sem-B~~

Show that the integral $\int_0^\infty e^{-x}\cdot x^{\alpha-1}\,dx$, $\alpha>0$ exists, by separately taking the cases for $\alpha \geq 1$ and $0<\alpha<1$

$$I=\int_0^\infty e^{-x} x^{\alpha-1}dx = \underbrace{\int_0^1 e^{-x}x^{\alpha-1}dx}_{(I_1)} + \underbrace{\int_1^\infty e^{-x}x^{\alpha-1}dx}_{(I_2)}$$

For $\alpha \geq 1$, I_1 is a proper integral while I_2 is improper

$I_2=\int_1^\infty e^{-x}\cdot\alpha^{\alpha-1}dx$, let $f(x)=x^{\alpha-1}\cdot e^{-x}$.

and take $g(x)=\frac{1}{x^2}$.

$$\therefore \lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{x^{\alpha-1}e^{-x}}{1/x^2} = \lim_{x\to\infty} x^{\alpha+1}\cdot e^{-x}$$

$$= \lim_{x\to\infty}\frac{x^{\alpha+1}}{e^x} \quad \left(\frac{\infty}{\infty}\text{ form}\right)$$

$$= \frac{(\alpha+1)!}{e^x} = 0. \Rightarrow \text{Convergent}$$

Hence I exists for $\alpha \geq 1$.

For $0<\alpha<1$,

I_1 is an improper integral & I_2 is an improper integral & point of non-convergence $x=0$

$I_1=\int_0^1 e^{-x}\,x^{\alpha-1}dx$, let $f(x)=\frac{e^{-x}}{x^{1-\alpha}}$

& $g(x)=\frac{1}{x^{1/2}}$ where $\int_0^1\frac{1}{x^{1/2}}du$ is convgt for $0<u<1$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{e^{-x}}{x^{1-\alpha}} \cdot x^{u} = \lim_{x\to\infty} \frac{e^{-x}}{x^{1-\alpha-u}}$$

$$= 0$$

$\therefore$ The integral is convergent

$$I_2 = \int_0^\infty e^{-x} \cdot x^{\alpha-1}\, dx \quad , \quad 0 < \alpha < 1$$

take $g(x) = \frac{1}{x^2}$

$$\lim_{x\to\infty} \frac{e^{-x}}{x^{1-\alpha}} \cdot x^2 = \frac{e^{-x}}{x^{1-\alpha-2}} = x^{1+\alpha} \cdot e^{-x}$$

$$= \frac{x^{1+\alpha}}{e^{x}} = \frac{(1+\alpha)x^{\alpha}}{e^{x}} = 0 \quad \left(\frac{0}{0} \text{ form}\right)$$

Hence we get it convergent by comparision Test.

Hence integral exist for $0 < \alpha < 1$

$\boxed{4\text{-}C}$ Prove that $\Gamma 2z = \dfrac{2^{2z-1}}{\sqrt{\pi}} \Gamma z \,\Gamma\left(z+\tfrac{1}{2}\right)$. (10)

We know that $\beta(m,n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$, $m>0, n>0$

Take $m=n$

$$\beta(n,n) = \frac{(\Gamma(n))^2}{\Gamma(2n)} = 2\int_0^{\pi/2} \sin^{2n-1}\theta \cos^{2n-1}\theta \, d\theta$$

$$= \int_0^1 x^{n-1}(1-x)^{n-1}dx \quad \left[\begin{array}{l} x = \sin^2\theta \\ dx = \sin 2\theta \, d\theta \end{array}\right]$$

$$\beta(n,n) = 2\int_0^{\pi/2} (\sin\theta \cdot \cos\theta)^{2n-1} d\theta = \frac{2}{2^{2n-1}} \int_0^{\pi/2} (\sin 2\theta)^{2n-1} d\theta$$

$$= \frac{1}{2^{2n-1}} \int_0^{\pi} (\sin\alpha)^{2n-1} d\alpha \quad \left[\begin{array}{l} \text{let } 2\theta = \alpha \\ 2d\theta = d\alpha \end{array}\right]$$

$$= \frac{2}{2^{2n-1}} \int_0^{\pi/2} \sin^{2n-1}\alpha \cdot d\alpha \quad \left[\int_0^{2a} f(x)dx = 2\int_0^a f(x)dx \text{ if } f(2a-x) = f(x)\right]$$

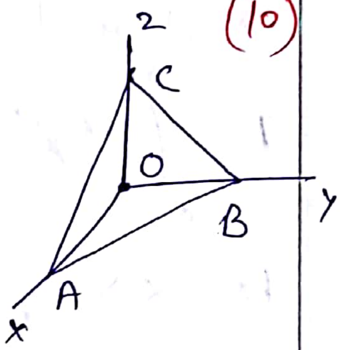
$$= \frac{1}{2^{2n-2}} \int_0^{\pi/2} \sin^{2n-1}\alpha \cdot \cos^0\alpha \, d\alpha$$

$$= \frac{1}{2^{2n-2}} \cdot \frac{\Gamma(n) \cdot \Gamma\left(\frac{1}{2}\right)}{2\,\Gamma\left(n+\frac{1}{2}\right)} \quad \left[2n-1=0 \Rightarrow n=\tfrac{1}{2}\right]$$

$$\therefore \frac{\Gamma(n) \cdot \cancel{\Gamma(n)}}{\Gamma(2n)} = \frac{1}{2^{2n-2} \cdot 2} \cdot \frac{\sqrt{\pi} \cdot \cancel{\Gamma(n)}}{\Gamma\left(n+\frac{1}{2}\right)}$$

$$\therefore \Gamma(2n) = \Gamma(n) \cdot \Gamma\left(n+\tfrac{1}{2}\right) \frac{2^{2n-1}}{\sqrt{\pi}}$$

4.d A plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ cuts the coordinate plane at A, B, C. Find the equation of the cone with vertex at origin and guiding curve as the circle passing through A, B, C. (10)



Let $A(a,0,0)$ $B(0,b,0)$, $C(0,0,c)$

Let Eqn of sphere passing through O, A, B, C be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0$$

$$\therefore d=0, \quad u=\frac{-a}{2}, \quad v=\frac{-b}{2}, \quad w=\frac{-c}{2}$$

$$\therefore \quad x^2+y^2+z^2-ax-by-cz=0 \quad \text{—} (1)$$

plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}$ — (2)

The equation of the required cone is obtained by making eqn (1) homogeneous with the help of of eqn (2)

$$x^2+y^2+z^2-(ax+by+cz)\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)=0$$

$$\cancel{x^2}+\cancel{y^2}+\cancel{z^2}-\left(\cancel{x^2}+\frac{a}{b}xy+\frac{a}{c}zx+\frac{b}{a}xy+\cancel{y^2}+\frac{b}{c}yz+\frac{c}{a}zx+\frac{c}{b}zy+\cancel{z^2}\right)$$

$$\Rightarrow xy\left(\frac{a}{b}+\frac{b}{a}\right)+yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{a}{c}+\frac{c}{a}\right)=0$$

Which is the required eqn of cone.