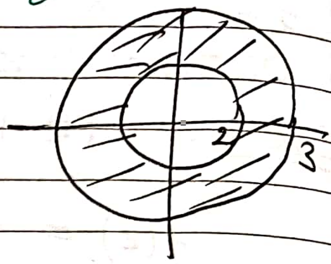
2011

Expand the function
$$f(z) = \frac{2z^2 + 11z}{(z+1)(z+4)}$$
in Laurent's series valid for $2 < |z| < 3$.

$$\begin{array}{r} 2 \\ z^2+5z+4 \overline{) \; 2z^2 + 11z} \\ \underline{2z^2 + 10z + 8} \\ z - 8 \end{array}$$



$$f(z) = 2 + \frac{z-8}{(z+1)(z+4)} = 2 + \frac{A}{z+1} + \frac{B}{z+4}$$

$$= 2 + \frac{-9}{3(z+1)} + \frac{(-4-8)}{(-3)(z+4)}$$

$$= 2 - \frac{3}{z+1} + \frac{4}{z+4}$$

$$= 2 - \frac{3}{z\left(1+\frac{1}{z}\right)} + \frac{4}{4\left(1+z/4\right)}$$

$$= 2 - \frac{3}{z}\left(1+\frac{1}{z}\right)^{-1} + \left(1+\frac{z}{4}\right)^{-1}$$

$\Big[$ As $2 < |z| \Rightarrow \frac{2}{|z|} < 1$ ie $\frac{1}{|z|} < 1$

$|z| < 3 \Rightarrow \frac{|z|}{4} < \frac{3}{4}$ ie $\frac{|z|}{4} < 1 \Big]$

$$f(z) = 2 - \frac{3}{z}\left(1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \dots\right) + \left(1 - \frac{z}{4} + \frac{z^2}{16} - \frac{z^3}{64}\right.$$

$$\cancel{= 2 + \frac{3}{z}}$$

2011

~~State Cauchy~~

Sketch the image of the infinite strip $1<y<2$ under the transformation $w=\frac{1}{z}$.

for $z=x+iy$, $w=1/z$

ie. $u+iv = \frac{1}{x+iy} = \frac{x-iy}{x^2+y^2}$

$\therefore u = \frac{x}{x^2+y^2}$, $v=\frac{-y}{x^2+y^2}$

Then, $\frac{u}{v} = \frac{-x}{y}$ ie. $x=-u\cdot\frac{y}{v}$

and, $v = \frac{-y}{\frac{u^2y^2}{v^2}+y^2} = \frac{-v^2}{(u^2+v^2)y}$ ie $y=\frac{-v}{u^2+v^2}$

If $y=1$, then $\frac{-v}{u^2+v^2}=1$

$\Rightarrow u^2+v^2+v=0$

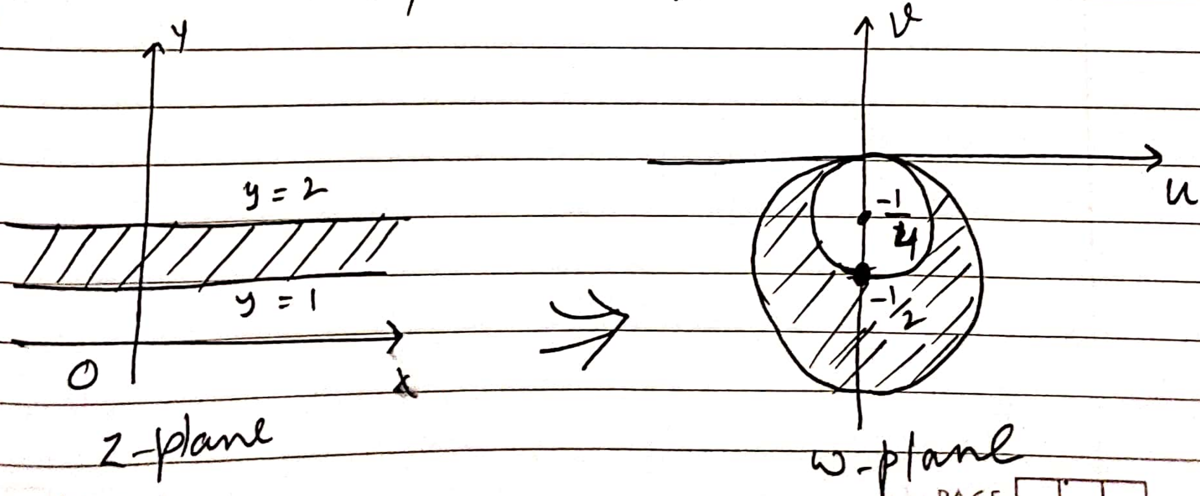
$\Rightarrow u^2+(v+\frac{1}{2})^2=\frac{1}{4}$ — ①

If $y=2$, then $\frac{-v}{u^2+v^2}=2$

$u^2+v^2+\frac{v}{2}=0$

$u^2+(v+\frac{1}{4})^2=\frac{1}{16}$ — ②

Hence infinite strip, $1<y<2$ is transformed into the region common to circles ① & ② in the w-plane as shown below



2011

State Cauchy's residue theorem. Using it, evaluate the integral

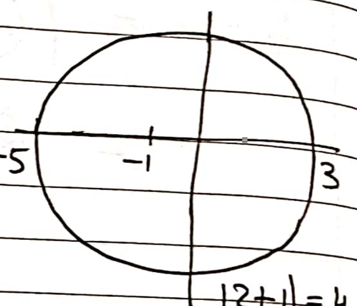
$$\int_C \frac{e^{z/2}}{(z+2)(z^2-4)}\,dz$$

counterclockwise around the circle, $C: |z+1|=4$.

$$I = \int \frac{e^{z/2}}{(z+2)(z^2-4)}\,dz = 2\pi i \sum R$$

$$(z+2)(z^2-4)=0 \Rightarrow z=-2,-2,2$$

$z=2$ is a simple pole lying inside circle, C



R_1 = Residue at $z=2$

$$= \lim_{z\to 2} (z-2)\cdot\frac{e^{z/2}}{(z+2)(z^2-4)}$$

$$= \lim_{z\to 2} \frac{e^{z/2}}{(z+2)^2} = \frac{e^1}{4^2} = \frac{e}{16}$$

$z=-2$ is pole of order 2.

R_2 = Residue at $z=-2$

$$= \frac{1}{1!}\lim_{z\to -2} \frac{d}{dz}\cancel{(z+2)^2}\,\frac{e^{z/2}}{\cancel{(z+2)^2}(z-2)}$$

$$= \lim_{z\to -2} \frac{\frac{1}{2}e^{z/2}\cdot(z-2) - 1\cdot e^{z/2}}{(z-2)^2}$$

$$= \frac{\frac{1}{2}e^{-1}(-4)-e^{-1}}{(-4)^2} = \frac{-3}{16e}$$

$$\therefore I = 2\pi i\left(\frac{e}{16} - \frac{3}{16e}\right)$$

classmate