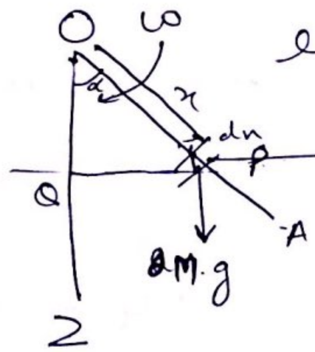
CSE 2019

1. A rod length $= 2a$ (OA)



let mass $= m$

let take an element dx at distance x from O.

Now <u>mass of element dx</u>

i.e. $dM = \frac{m}{2a} \times dx$

$QP = OP \sin\alpha = x \sin\alpha$.

Since rod is revolving with angular velocity ω

Centripetal force $F = \left(\frac{m}{2a} dx\right) \omega^2 \times QP$

$= \left(\frac{m}{2a} \omega^2 \sin\alpha\right) x \, dx$

Weight of rod $= mg$, acting at mid-point of OA

Now taking moment about O of $F = F \cdot OQ$

$= (F \cdot x \cos\alpha)$

Small moment $dI = \frac{m}{2a} \omega^2 \sin\alpha \cos\alpha \; x^2 dx$

Moment for whole rod $= \int_0^{2a} \frac{m}{2a} \omega^2 \sin\alpha \cos\alpha \; x^2 dx$

$= \frac{m}{4a} \omega^2 \sin 2\alpha \left[\frac{x^3}{3}\right]_0^{2a}$

$= \frac{2ma^2\omega^2 \sin 2\alpha}{3}$

Moment of weight $= mg \times a \times \sin\alpha$

Now these moments balance when angle α is constant

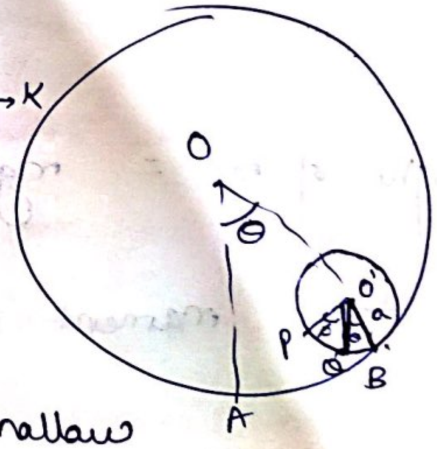
$$\therefore \cancel{m}g\, \cancel{a} \sin\cancel{\alpha} = \frac{2}{3}\cancel{m}a^2\omega^2 \times 2\cancel{\sin\alpha}\cos\alpha$$

$$\Rightarrow \cos\alpha = \frac{3g}{4a\omega^2}$$

$$\therefore \boxed{\alpha = \cos^{-1}\left(\frac{3g}{4a\omega^2}\right)}$$

2. Given $O'B = a$ Radii of Gyration $\rightarrow K$

$OA = b$.



let $\angle AOB = \theta = \angle OO'B$

Also let $PO'Q = \phi$

Given Cylinder rolls inside hollow cylinder $\therefore AB = PB$ P being position of A after some time t.

$AB = b\theta$

$PB = (\theta + \phi)a$

$$\therefore b\theta = (\theta+\phi)a \Rightarrow (b-a)\theta = a\phi \quad -(1)$$

Coordinates of O', Taking O as fixed point

$x_{0'} = (h-a)\sin\theta$, $y_{0'} = (h-a)\cos\theta$

$\therefore$ Velocity $O' = (\dot{x}_{0'}^2 + \dot{y}_{0'}^2)^{1/2}$

$\dot{x}_{0'} = (h-a)\cos\theta\,\dot{\theta}$

$\dot{y}_{0'} = -(h-a)\sin\theta\,\dot{\theta}$

$$\boxed{V_{0'} = (h-a)\dot{\theta}}$$

Writing the energy equation let mass of inner cylinder be 'm'.

$$\frac{1}{2} I_{cm}\omega^2 + \frac{1}{2} m V_{0'}^2 = -mg(h-a)(1-\cos\theta)$$

$$\frac{1}{2}\cancel{m}k^2\omega^2 + \frac{1}{2}\cancel{m}(h-a)^2\dot{\theta}^2 = -\cancel{m}g(h-a)(1-\cos\theta).$$

Also $\omega^2 = \dot{\phi}^2$

$$\therefore k^2\dot{\phi}^2 + (h-a)^2\dot{\theta}^2 = -2g(h-a)(1-\cos\theta)$$

Using ① $(h-a)\dot{\theta} = a\dot{\phi} \Rightarrow \dot{\phi} = \frac{(h-a)\dot{\theta}}{a}$

$$\therefore k^2\frac{(h-a)^{\cancel{2}}}{a^2}\dot{\theta}^2 + (h-a)^{\cancel{2}}\dot{\theta}^2 = -2g(\cancel{h-a})(1-\cos\theta)$$

$$(h-a)\dot{\theta}^2\left[1+\frac{k^2}{a^2}\right] = -2g(1-\cos\theta)$$

Differentiating both sides

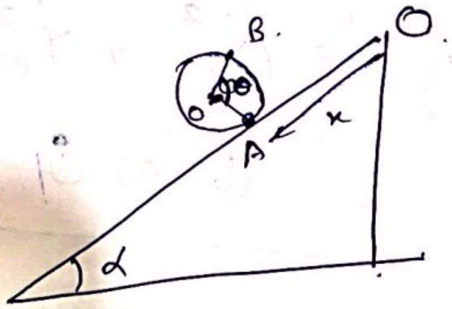
$$\cancel{2}\cancel{\dot{\theta}}\ddot{\theta}(h-a)\left[1+\frac{k^2}{a^2}\right] = -\cancel{2}g\sin\theta\,\cancel{\dot{\theta}}$$

$$\therefore \boxed{\ddot{\theta} = -\frac{g}{(h-a)\left[1+\frac{k^2}{a^2}\right]}\theta}$$

Since θ is small $\sin\theta \approx \theta$

$\therefore$ length of Circular pendulum $= (l-a)\left(1+\dfrac{K^2}{a^2}\right)$

3.



Let inclination be α.

Let radii $\rightarrow$ a.

$OA = x$.

Let O be the origin.   • Sphere is rolling.

$\therefore$ Now $AB = OA$ ($\because$ rolling).

$\therefore x = a\theta \Rightarrow \dot{x} = a\dot{\theta}$ — ①

$T = \frac{1}{2} I_{cm} \dot{\theta}^2 + \frac{1}{2} m v^2$ I_{cm} - Moment of Inertia about C.M.

$= \frac{1}{2} \times \frac{2}{5} m a^2 \dot{\theta}^2 + \frac{1}{2} m \dot{x}^2$

$= \frac{m\dot{x}^2}{5} + \frac{1}{2} m \dot{x}^2$ (Using ①)

$T = \frac{7m\dot{x}^2}{10}$

$V = -mg x \sin\alpha$

$\therefore L = T - V$ (Lagrangian)

$L = \frac{7m\dot{x}^2}{10} + mg x \sin\alpha$

$H = p_j \dot{q}_j - L$ here $q_j = x$

$p_j = \frac{\partial L}{\partial \dot{q}_j} = \frac{\partial L}{\partial \dot{x}} = \frac{7}{5} m \dot{x}$ — ②

$\therefore H = \frac{7}{5} m \dot{x} \times \dot{x} - \frac{7}{10} m \dot{x}^2 - mg x \sin\alpha$

$$\therefore H = \frac{7}{10} m\dot{x}^2 - mgx\sin\alpha.$$

Now $\dot{p}_j = -\frac{\partial H}{\partial q_j}$, $\dot{q}_j = \frac{\partial H}{\partial p_j}$

$$\dot{p}_x = mg\sin\alpha \quad (3)$$

Now $\dot{x} = \frac{5p_j}{7m}$ (from (2))

$$\therefore H = \frac{\cancel{7}}{\cancel{10}\,2} \times m \times \frac{\cancel{25}^{\,5}}{\cancel{49}_{\,7}} \times \frac{p_j^2}{m^2} - mgx\sin\alpha.$$

$$H = \frac{5}{14}\frac{p_j^2}{m} - mgx\sin\alpha$$

$$\therefore \dot{x} = \frac{\partial H}{\partial p_j} = \frac{5}{7}\frac{p_j}{m}$$

Now $\ddot{x} = \frac{5}{7}\frac{\dot{p}_x}{m}$

Using (3) $\dot{p}_x = mg\sin\alpha$

$$\therefore \boxed{\ddot{x} = \frac{5}{7} g\sin\alpha}$$ Acceleration of sphere rolling down.

4. Given a sphere of radius R

Density of fluid $\rightarrow \rho$

Pressure at infinity $\rightarrow \pi$

Given that sphere is water radially about centre, fluid particle move symmetrically about centre 'O'.

So continuity equation is $r^2 q_r = \text{Constant}$

Now $q_r \rightarrow f(r, t)$

Let at radius R, velocity of fluid be V, and for same r', velocity $\rightarrow v'$

$$\therefore \; r'^2 v' = R^2 V = f(t) \quad \text{— (1)}$$

$$\text{Now} \quad \frac{\partial v'}{\partial t} = \frac{f'(t)}{r'^2} \quad \text{— (2)}$$

Equation of motion is

$$\frac{\partial v'}{\partial t} + \frac{1}{2}\frac{\partial}{\partial r'}(v'^2) = -\frac{1}{\rho}\frac{\partial}{\partial r'}(p)$$

(As motion is about centre only)

Using (2)

$$\frac{f'(t)}{r'^2} + \frac{1}{2}\frac{\partial}{\partial r'}(v'^2) = -\frac{1}{\rho}\frac{\partial}{\partial r'}(p)$$

$$-\frac{f'(t)}{r'} + \frac{1}{2}v'^2 = -\frac{1}{\rho}p + C. \qquad C \rightarrow \text{Constant}$$

At $r' \rightarrow \infty$, $v' \rightarrow 0$, $p \rightarrow \pi$

$\therefore \quad C = \pi/\rho$

$$\therefore \quad p = \pi + \rho\left[\frac{F'(t)}{r'} - \frac{1}{2}v'^2\right].$$

Now at $r' = R$, $v' = V$.

$$p = \pi + \rho\left[\frac{[F'(t)]_{r=R}}{R} - \frac{1}{2}V^2\right].$$

From $R^2 V = [F(t)]_{r=R}$ $\qquad V = \frac{dR}{dt}$.

$$\frac{d}{dt}(R^2V) = [F'(t)]_{r=R}$$

$$\frac{dR^2}{dt}\cdot\frac{dR}{dt} + R^2\frac{d^2R}{dt^2} = [F'(t)]_{r=R}$$

$$[F'(t)]_{r=R} = 2R\left(\frac{dR}{dt}\right)^2 + R^2\left(\frac{d^2R}{dt^2}\right).$$

Put this value in p

$$p = \pi + \rho\left[\frac{2}{R}\left(\frac{dR}{dt}\right)^2 + R\left(\frac{d^2R}{dt^2}\right) - \frac{1}{2}\left(\frac{dR}{dt}\right)^2\right]$$

Now $\frac{d^2R^2}{dt^2} = \frac{d}{dt}\left(\frac{d}{dt}(R^2)\right) = \frac{d}{dt}\left(2R\frac{dR}{dt}\right)$

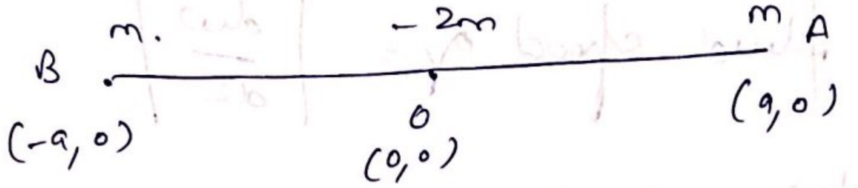
$$\frac{d^2R^2}{dt^2} = 2R\frac{d^2R}{dt^2} + 2\left(\frac{dR}{dt}\right)^2$$

$$\therefore \quad \boxed{p = \pi + \frac{\rho}{2}\left[\frac{d^2R^2}{dt^2} + \left(\frac{dR}{dt}\right)^2\right].}$$

5.

$m \rightarrow$ source

$-2m \rightarrow$ sink.



$$w = -m\log(z-a) - m\log(z+a) + 2m\log z$$

$$w = m\log\left(\frac{z^2}{z^2-a^2}\right)$$

$$w = m\log\left(x^2-y^2+2ixy\right) - m\log\left(x^2-a^2-y^2+2ixy\right)$$

$$\psi = m\left[\tan^{-1}\left(\frac{2xy}{x^2-y^2}\right) - \tan^{-1}\left(\frac{2xy}{x^2-y^2-a^2}\right)\right].$$

$$= m\left[\tan^{-1}\left(\frac{\frac{2xy}{x^2-y^2} - \frac{2xy}{x^2-y^2-a^2}}{1+\frac{4x^2y^2}{(x^2-y^2)(x^2-y^2-a^2)}}\right)\right].$$

$$\psi = m\left[\tan^{-1}\left[\frac{-2xya^2}{(x^2-y^2)^2-a^2(x^2-y^2)+4x^2y^2}\right]\right].$$

Now stream lines are ψ = Constant

$\therefore$ Put $\dfrac{-2xya^2}{(x^2+y^2)^2 - a^2(x^2-y^2)} = -\dfrac{2}{\lambda}$. $\lambda \rightarrow$ arbitrary Constant

$$\boxed{(x^2+y^2)^2 = a^2\left(x^2-y^2+\lambda xy\right)}$$

Now fluid speed $q = \left|\frac{dw}{dz}\right|$

$$\frac{dw}{dz} = \frac{m}{z^2} \times 2\cancel{z} - \frac{m}{(z^2-a^2)} \times 2z$$

$$\frac{dw}{dz} = \frac{-2m\cancel{z}a^2}{z^{\cancel{2}}(z^2-a^2)}$$

$$\frac{dw}{dz} = \frac{2ma^2}{z(z-a)(z+a)}$$

$r_1 \to |z|$, $r_2 \to |z-a|$, $r_3 \to |z+a|$

$$\therefore q = \left|\frac{dw}{dz}\right| = \frac{2ma^2}{|z|\,|z-a|\,|z+a|}$$

$$\boxed{q = \frac{2ma^2}{r_1 r_2 r_3}}$$