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2017

Ques 2

Ques) for what values of the constant a, b & c the vector $\vec{V} = (x+y+za)\hat{i} + (bx+2y-z)\hat{j} + (-x+cy+2z)\hat{k}$ is irrotational.
find the divergence in cylindrical coordinates of this vector with these values

Ans) We know that, for a vector $\vec{V}$ to be irrotational,

$$\nabla \times \vec{V} = 0$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+y+az & bx+2y-z & -x+cy+2z \end{vmatrix} = 0$$

$$\hat{i}(c+1) - \hat{j}(-1-a) + \hat{k}(b-1) = 0$$

Equating & comparing,

$$a=-1, \quad b=1, \quad c=-1$$

$$\therefore \vec{V} = (x+y-z)\hat{i} + (x+2y-z)\hat{j} + (-x-y+2z)\hat{k}$$

Ques 2 The position vector of a moving point a time t is $\vec{r} = \sin t\, \hat{i} + \cos 2t\, \hat{j} + (t^2+2t)\hat{k}$. find the component of the acceleration $\vec{a}$ in the direction parallel to the velocity vector $\vec{v}$ & perpendicular to the plane of $\vec{r}$ & $\vec{v}$ at time $t=0$.

Ans 2

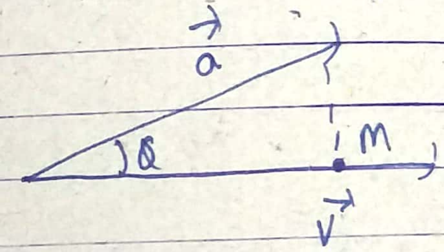
$\vec{r} = \sin t\, \hat{i} + \cos 2t\, \hat{j} + (t^2 + 2t)\hat{k}$

If: $\vec{a}$ compounents in dirn ∥ to velocity vector $\vec{v}$ & ⊥ to the plane of $\vec{r}$ & $\vec{v}$ at $t = 0$.

$\vec{v} = \frac{d\vec{r}}{dt} = \cos t\, \hat{i} - 2\sin(2t)\, \hat{j} + (2t+2)\hat{k}$

$\vec{a} = \frac{d^2 r}{dt^2} = -\sin t\, \hat{i} - 4\cos(2t)\, \hat{j} + 2\hat{k}$

i) Compount ∥ to velocity vector $\vec{v}$



OM is the compount of $\vec{a}$ along v

$\vec{a} \cdot \vec{v} = |\vec{a}|\, |\vec{v}| \cos\theta$

$\left(-\sin t \cos t + 8\sin(2t)\cos(2t) + 4t + 4\right) = |\vec{v}| \left(|\vec{a}| \cos\theta\right)$

$\left(\frac{\sin(2t)}{-2} + 4\sin(4t) + 4t + 4\right) / |\vec{v}| = \left(\vec{a} \cos\theta\right)$

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$$\vec{a}_v = \frac{1}{|\vec{v}|}\left(\frac{\sin(2t)}{-2} + 4\sin(4t) + 4t + 4\right)\frac{\vec{v}}{|\vec{v}|}$$

$$= \frac{1}{v^2}\left(\frac{-1}{2}\sin 2t + 4\sin 4t + 4t + 4\right)\left(\cos t\,\hat{i} - 2\sin 2t\,\hat{j} + (2t+2)\,\hat{k}\right)$$

At $t = 0$

$$= \cancel{\frac{1}{1+0+4+4}}\ \frac{1}{(1+4)}(0+0+0+4)(\hat{i} - 0 + 2\hat{k})$$

$$= \left(\frac{1}{5}\right)(4)(\hat{i} + 2\hat{k})$$

$$= \frac{4}{5}(\hat{i} + 2\hat{k})$$

~~(iii) Component $\perp$ to $\vec{x}$ & $\vec{v}$.~~

~~Vector lying in the plane of $\vec{u}$ & $\vec{v}$.~~

$$\cancel{= \hat{i}(\sin t - \cos t) + \hat{j}(\cos(2t) + 2\sin(2t)) + \hat{k}(t^2 - 2)}$$

(ii) Component $\perp$ to the plane of $\vec{u}$ & $\vec{v}$.

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \sin t & \cos(2t) & (t^2+2t) \\ \cos t & -2\sin(2t) & (2t+2) \end{vmatrix}$$

$$= \hat{i}\left(2t\cos(2t) + 2\cos(2t) + 2\sin(2t).t^2 + 4t\sin(2t)\right)$$

$$\cancel{-\hat{j}\left(2\sin(t)(2t)\right)}$$

$$-\hat{j}\left(2t\sin t + 2\sin t - t^2\cos t - 2t\cos t\right)$$

$$+ k\left(-2\sin(2t)\sin(t) - \cos t\cos(2t)\right)$$

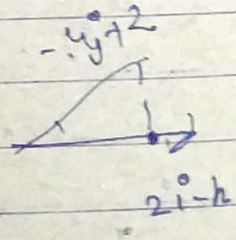
At $t = 0$

$$= \hat{i}(2) - \hat{j}(0) + k(-1)$$

$$= \underline{2\hat{i} - k}$$

Component of $\vec{a}$ along $2\hat{i} - k$,

$$\vec{a} \text{ at } t=0 = -4\hat{j} + 2k$$



$$\Rightarrow \frac{(-4\hat{j}+2k)\cdot(2\hat{i}-k)}{|2\hat{i}-k|} \cdot \left[\frac{(2\hat{i}-k)}{|2\hat{i}-k|}\right]$$

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$$= \frac{-2}{\sqrt{5}\cdot\sqrt{5}}(2\hat{i}-\hat{k}) = \frac{2}{5}(-2\hat{i}+\hat{k})$$

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Ques3 find the curvature vector & its magnitude at any point $\vec{r} = (\theta)$ of the curve $\vec{r} = (a\cos\theta, a\sin\theta, a\theta)$.
Show that the locus of the feet of the perpendicular from the origin to the tangent is a curve that completely lies on the hyperboloid $x^2 + y^2 + z^2 = a^2$.

We know,

$$K = \frac{dT}{ds}$$ where T is the unit tangential vector.

$$T = \frac{dr}{ds} = \frac{dr/d\theta}{ds/d\theta}$$

Now,

$$\left(\frac{dr}{d\theta}\right) = \left(\frac{dr}{ds}\right)\left(\frac{ds}{d\theta}\right)$$

$$\left|\frac{dr}{d\theta}\right| = \left|\frac{ds}{d\theta}\right| \qquad \left(\frac{dr}{ds} \text{ is a unit vector}\right)$$

$$\left|\frac{ds}{d\theta}\right| = a\sqrt{2}$$

$$T = \frac{dr/d\theta}{ds/d\theta}$$

$$= \frac{1}{a\sqrt{2}}\left(-a\sin\theta\, i + a\cos\theta\, j + ak\right)$$

$$= \frac{-1}{\sqrt{2}}\left(\sin\theta\, i - \cos\theta\, j - k\right)$$

$$K = \frac{dT}{ds} = \frac{dT/d\theta}{ds/d\theta}$$

$$= \frac{-1}{\sqrt{2}}\frac{\left(\cos\theta\, i + \sin\theta\, j\right)}{a\sqrt{2}}$$

$$= \left(\frac{-1}{2a}\right)\left(\cos\theta\, i + \sin\theta\, j\right)$$

$$|K| = \frac{1}{2a} \quad //.$$

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Ques 4 Evaluate the integral $\iint_S \vec{F} \cdot \hat{n} \, ds$

where $\vec{F} = 3xy^2 \hat{i} + (yx^2 - y^3) \hat{j} + 3zx^2 \hat{k}$

& S is a surface of the cylinder

$y^2 + z^2 \leq 4$, $-3 \leq x \leq 3$ using divergence

theorem.

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Ans 4

$$\iint_S \vec{F}\cdot\vec{n}\, ds$$

$$\vec{F} = 3xy^2\hat{i} + (yx^2 - y^3)\hat{j} + 3zx^2\hat{k}$$

S: cylinder surface
$y^2+z^2 \le 4$
$-3 \le x \le 3$

using GD.

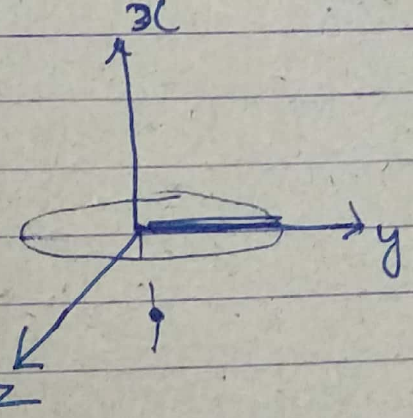
WKT using Gauss divergence theorem,

$$\iint_S \vec{F}\cdot\vec{n}\, ds = \cancel{\iint (\nabla\times F)\cdot\hat{n}\, ds}$$

$$= \iiint_V (\nabla\cdot F)\, dV$$

$$(\nabla\cdot F) = 3y^2 + x^2 - 3y^2 + 3x^2$$

$$= 4x^2$$



$$\Rightarrow \iiint_V (4x^2)\, dV$$

$$4\int_{x=-3}^{3}\int_{y=-2}^{2}\int_{z=-\sqrt{4-y^2}}^{\sqrt{4-y^2}} x^2\, dz\, dy\, dx$$

$$4\int_{x=-3}^{3}\int_{y=-2}^{2} 2\cdot x^2\sqrt{4-y^2}\, dy\, dx$$

$$8\int_{x=-3}^{3} x^2 dx \int_{y=-2}^{2} \sqrt{4-y^2}\, dy$$

$$8\int_{x=-3}^{3} x^2 \left[\frac{y}{2}\sqrt{4-y^2} + \frac{4}{2}\sin^{-1}\left(\frac{y}{2}\right) \right]_{-2}^{2} dx$$

$$8\int_{x=3}^{3} x^2 \left[0 + 2\cdot\frac{\pi}{2} - 0 + 2\cdot\frac{\pi}{2} \right]$$

$$8\int_{x=-3}^{3} 2\pi\, x^2\, dx$$

$$2\cdot 16\pi \int_{x=0}^{3} x^2\, dx \Rightarrow \frac{32\pi}{3}\cdot 27\ 9$$

$$\Rightarrow 288\pi \quad //$$

OR

$$\iiint_V 4x^2 \, dV$$

using cyclindual coordinates,

$x : -3$ to 3

$z : r\sin\theta$

$y : r\cos\theta$

$$4\int_{x=-3}^{3}\int_{r=0}^{2}\int_{\theta=0}^{2\pi} x^2 \, r \, dr \, d\theta \, dx$$

$$4\int_{x=-3}^{3}\int_{r=0}^{2} \left(x^2 r \cdot 2\pi\right) dr$$

$$\frac{8\pi}{\cancel{2}} \cdot 4 \cdot \cancel{2} \cdot \left(\frac{27}{3}\right) = 32\pi \times 9$$

$$= 288\pi \, . \; //$$

Ques 5 Using Green's theorem, evaluate the $\int F(\vec{r}) \cdot d\vec{r}$ counterclockwise where

$F(\vec{r}) = (x^2+y^2)\hat{i} + (x^2-y^2)\hat{j}$ & $d\vec{r} = dx\hat{i} + dy\hat{j}$ & the curve C is the boundary of the region

$R = \{(x, y) \mid 1 \le y \le 2-x^2\}$

Ans 5

$\cancel{f(u)}\; f = (x^2+y^2)\hat{i} + (x^2-y^2)\hat{j}$

$du = dx\,\hat{i} + dy\,\hat{j}$

C: $\oint$ Boundary of the region

$\{(x,y) \mid 1 \le y \le 2-x^2\}$

using GT.

WKT using green's theorem,

$$\int_C f_1 dx + f_2 dy = \iint_S \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx\,dy$$

$$\int f \cdot du = \int_C (x^2+y^2)\,dx + (x^2-y^2)\,dy$$

Comparing, $F_1 = x^2+y^2$

$f_2 = x^2-y^2$

$$= \iint_S \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right) dx\,dy = \iint_S (2x-2y)\,dx\,dy$$

Now,

$y \ge 1$

$y \le 2-x^2$

$x^2 \le 2-y$

$x^2 \le 1 \Rightarrow -1 \le x \le 1$

$\cancel{1 \le 2-x^2}$

$\cancel{x^2 \le 1}$

$$\iint_S (2x-2y)\, dx\,dy$$

$$= 2\int_{x=-1}^{1}\int_{y=1}^{2-x^2} (x-y)\, dy\, dx$$

$$= 2\int_{x=-1}^{1} \left[xy - \frac{y^2}{2}\right]_{y=1}^{2-x^2} dx$$

$$-\left(4+x^4 - 4x^2\right)/2$$

$$= 2\int_{x=-1}^{1} 2x - x^3 - \frac{(2-x^2)^2}{2} - x + \frac{1}{2}\, dx$$

$$= 2\left[x^2 - \frac{x^4}{4} - \frac{x^2}{2} + \frac{x}{2} - 2x - \frac{x^5}{10} + \frac{2x^3}{3}\right]_{x=-1}^{1}$$

$$= 2\left[\cancel{1} - \cancel{\frac{1}{4}} - \cancel{\frac{1}{2}} + \cancel{\frac{1}{2}} - 2 - \frac{1}{10} + \frac{2}{3} - \cancel{1} + \cancel{\frac{1}{4}} + \frac{1}{2} + \frac{1}{2} \right.$$

$$\left. -2 - \frac{1}{10} + \frac{2}{3}\right]$$

$$\frac{-3+4}{3}$$

$$-\frac{5}{3} - \frac{1}{5}$$

$$\frac{-25-3}{15}$$

$$= 2\left[\frac{-28}{15}\right) = -\frac{56}{15} \quad //.$$

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