IAS MATHEMATICS (OPT.)-2017

PAPER - I : SOLUTIONS

1(a) Let $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$. Find a non-singular matrix P such that $P^{-1}AP$ is a diagonal matrix.

Sol^n: we have $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)(3-\lambda) - 2 = 0$$

$$\Rightarrow \lambda^2 - 5\lambda + 4 = 0$$

$$\Rightarrow \lambda = 1; 4.$$

which are the eigen values of A.

Let us find the eigen vector corresponding to $\lambda = 1$.

i.e. $(A - \lambda I)X = 0$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} X = 0$$

$$\Rightarrow x + 2y = 0$$

$\therefore X_1 = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ is a non-zero solution of the system and so is an eigen vector of A corresponding to $\lambda = 1$.

Let us find the eigen vector corresponding to $\lambda = 4$.

i.e. $(A - I\lambda)X = 0$

$$\Rightarrow \begin{bmatrix} -2 & -2 \\ 1 & -1 \end{bmatrix} X = 0$$

$$\Rightarrow x - y = 0$$

$\therefore X_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is a non-zero solution and so is an eigen vector of A corresponding to $\lambda = 4$.

Since A has two independent eigen vectors. A is diagonalizable.

Let $P = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$

Then $P^{-1}AP = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix}$

Q.1(b) Show that similar matrices have the same Characteristic polynomial.

Sol:- Let A and B be similar matrices

Then $\exists$ an invertible matrix P such that

$$B = P^{-1}AP$$

$$\therefore B - \lambda I = P^{-1}AP - \lambda I$$

$$= P^{-1}AP - \lambda P^{-1}P$$

$$= P^{-1}AP - P^{-1}\lambda P$$

$$= P^{-1}AP - P^{-1}\lambda(IP)$$

$$= P^{-1}AP - P^{-1}(\lambda I)P$$

$$= P^{-1}(A - \lambda I)P$$

$$\therefore |B - \lambda I| = |P^{-1}(A - \lambda I)P|$$

$$= |P^{-1}||A - \lambda I||P|$$

$$= |A - \lambda I||P^{-1}||P|$$

$$= |A - \lambda I||P^{-1}P|$$

$$= |A - \lambda I||I|$$

$$= |A - \lambda I|$$

$$\therefore |B - \lambda I| = |A - \lambda I|$$

$\therefore$ A and B have the same characteristics polynomial and hence same characteristic roots.

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1(c) Integrate the function $f(x,y) = xy(x^2+y^2)$ over the domain $R:\{-3 \le x^2-y^2 \le 3,\ 1 \le xy \le 4\}$. ? —(1)

Sol: To find $\iint_R f(x,y)\,dx\,dy$.

let $u = x^2-y^2$ $\therefore -3 \le u \le 3$

$v = xy$ $\quad 1 \le v \le 4.$

$$\iint_R f(x,y)\,dx\,dy = \iint f(u,v)\cdot J(u,v)\,du\,dv \quad \text{—(2)}$$

we know that

$$\frac{\partial(x,y)}{\partial(u,v)} \times \frac{\partial(u,v)}{\partial(x,y)} = (-1)^2 = 1.$$

$$\therefore \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 2x & -2y \\ y & x \end{vmatrix} = 2(x^2+y^2) \quad \text{—(3)}$$

$\Rightarrow$ from (2) & (3)

$$\Rightarrow (x^2+y^2)\,dx\,dy = \frac{1}{2}\,du\,dv$$

$$\therefore \iint_R f(x,y)\,dx\,dy = \frac{1}{2}\int_{v=1}^{4}\int_{u=-3}^{3} v\,du\,dv$$

$$= \frac{1}{2}\int_{v=1}^{4} v\,[u]_{-3}^{3}\,dv = \frac{1}{2}\int_{v=1}^{4} 6v\,dv$$

$$= \frac{3}{2}\left[v^2\right]_1^4 = \frac{3}{2}[16-1] = \frac{45}{2}.$$

$$\therefore \iint_R f(x,y)\,dx\,dy = \frac{45}{2}$$

1(d) Find the equation of tangent plane at point (1,1,1) to the conicoid $3x^2-y^2=2z$.

Sol:- Given, conicoid is $3x^2-y^2=2z$

Let equation of line passing through (1,1,1) be

$$\frac{x-1}{l}=\frac{y-1}{m}=\frac{z-1}{n}=r \text{ (say)} \quad \text{—— (1)}$$

Any point of this line is $(lr+1,\, mr+1,\, nr+1)$

The point of intersection of line and conicoid is given by.

$$3(lr+1)^2-(mr+1)^2=2(nr+1)$$

$$(3l^2-m^2)r^2+(6l-2m-2n)r=0$$

$$\text{or} \quad 6l-2m-2n=0 \quad \text{—— (2)}$$

Substituting value of l, m, n from (1), we get locus of tangent line (i.e. tangent plane)

$$6(x-1)-2(y-1)-2(z-1)=0$$

$$\text{or} \quad 6x-2y-2z-6+2+2=0$$

$$6x-2y-2z-2=0$$

$$\boxed{3x-y-z-1=0}$$

which is the required equation of tangent.

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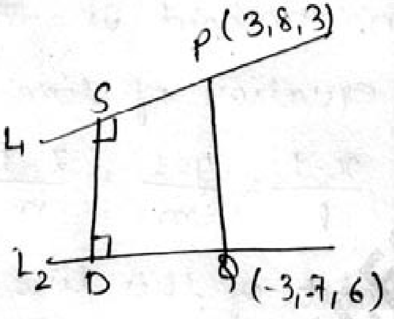
1(e) Find the shortest distance between skew-lines:

$$\frac{x-3}{3}=\frac{8-y}{1}=\frac{z-3}{1} \text{ and } \frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}.$$

Sol: Given lines are.



$$L_1 = \frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$$

$$L_2 = \frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$$

let point $P(3,8,3)$ on line L_1

and point $Q(-3,7,6)$ on line L_2.

The shortest distance between L_1 & L_2 is the projection of PQ over SD ——— (1)

Let, Direction ratios of SD be (l,m,n), which are perpendicular to L_1 & L_2

$$\Rightarrow \begin{matrix} 3l-m+n=0 \\ -3l+2m+4n=0 \end{matrix} \Rightarrow \frac{l}{-6}=\frac{m}{-15}=\frac{n}{3}$$

$$\text{or } \frac{l}{2}=\frac{m}{5}=\frac{n}{-1}$$

The direction cosines of SD = $\left[\frac{2}{\sqrt{30}}, \frac{5}{\sqrt{30}}, \frac{-1}{\sqrt{30}}\right]$

Shortest distance = SD = projection of PQ on SD

$$= [3-(-3)]\frac{2}{\sqrt{30}} + (8-(-7))\frac{5}{\sqrt{30}} + (3-6)\left(\frac{-1}{\sqrt{30}}\right)$$

$$= \frac{12+75+3}{\sqrt{30}} = \frac{90}{\sqrt{30}} = \underline{\underline{3\sqrt{30}}}$$

$\therefore$ SD between skew lines $= 3\sqrt{30}$

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Q.2(a) Find the volume of the solid above the x-y plane and directly below the portion of the elliptic paraboloid $x^2+\frac{y^2}{4}=z$; which is cutoff by the plane $z=9$.

Sol:-

$$V=\iiint dx\,dy\,dz = \iint (z_2-z_1)\,dx\,dy.$$

$$V=\iint\left(9-x^2-\frac{y^2}{4}\right)dx\,dy$$

for bounds of y; $y=0$ to $2\sqrt{9-x^2}$

for bounds of x; $x=0$ to 3

$$\therefore V=\int_0^3\int_0^{2\sqrt{9-x^2}}\left(9-x^2-\frac{y^2}{4}\right)dx\,dy$$

$$V=\int_0^3\left[9y-x^2y-\frac{y^3}{12}\right]_0^{2\sqrt{9-x^2}}dx$$

$$V=\int_0^3\left[18\sqrt{9-x^2}-2x^2\sqrt{9-x^2}-\frac{8(9-x^2)\sqrt{9-x^2}}{12}\right]dx$$

$$V=\int_0^3 18(9-x^2)^{1/2}-2x^2(9-x^2)^{1/2}-\frac{2}{3}(9-x^2)^{3/2}\Big]dx$$

$$V=\int_0^3\left[2(9-x^2)^{3/2}-\frac{2}{3}(9-x^2)^{3/2}\right]dx$$

$$V=\frac{4}{3}\int_0^3(9-x^2)^{3/2}dx \qquad \text{Put } x=3\sin\theta,\ dx=3\cos\theta\,d\theta.$$

$$V=\int_0^{\pi/2}\frac{4}{\cancel{3}}\cdot\cancel{27}^{9}\cos^3\theta\cdot 3\cos\theta\,d\theta$$

$$V=(27\times4)\int_0^{\pi/4}\cos^4\theta\,d\theta = 27\times\cancel{4}\times\frac{3\pi}{4\times\cancel{2}\times\cancel{2}}$$

$$\Rightarrow \boxed{V=\frac{81\pi}{4}}$$

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2(b) A plane passes through a fixed point (a, b, c) and cuts the axes at the points A, B, C respectively. Find the locus of the centre of sphere which passes through the origin O and A, B, C.

Sol :- Any sphere through point on $A(p,0,0)$ $B(0,q,0)$, $C(0,0,r)$ and $O(0,0,0)$ is of the form -

$$x^2+y^2+z^2-px-qy-rz=0 \quad \text{——— (1)}$$

The centre of this sphere is $(p/2, q/2, r/2) = (\alpha, \beta, \gamma)$ say — (A)

let, the equation of the plane be

$$\frac{x}{p}+\frac{y}{q}+\frac{z}{r}=1 \quad \text{——— (2)}$$

Since, it passes through (a, b, c); eq^n (2) becomes

$$(2) \equiv \frac{a}{p}+\frac{b}{q}+\frac{c}{r}=1.$$

$$\Rightarrow \frac{a}{2\alpha}+\frac{b}{2\beta}+\frac{c}{2\gamma}=1 \quad \text{(from (A)) — Putting coordinates of centre of sphere.}$$

Hence, the locus of centre of sphere becomes.

$$\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}=2$$

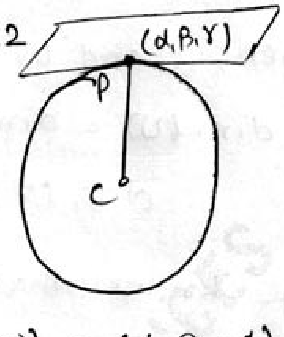
Put (α, β, γ) as (x, y, z) respectively.

locus of centre of sphere is

$$\boxed{\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2}$$

Q.2(c) Show that the plane $2x-2y+z=12$ touches the sphere $x^2+y^2+z^2-2x-4y+2z-3=0$. Find the point of contact?

Sol: Equation of sphere.



$x^2+y^2+z^2-2x-4y+2z-3=0$

By comparing to standard equation of sphere; $u=-1$, $v=-2$, $w=1$, $d=-3$

Hence centre of sphere $=(-u,-v,-w)=(1,2,-1)$

Hence; radius $=\sqrt{u^2+v^2+w^2-d}=\sqrt{1+4+1+3}=3.$

Now, Distance of 'C' from the given plane.

$$=\left|\frac{2(1)-2(2)+(-1)+12}{\sqrt{2^2+(-2)^2+1^2}}\right|=\frac{9}{3}=3=r.$$

Since; perpendicular distance of centre from given plane is equal to radius; hence, it's a Tangent plane.

For point of contact, it becomes the foot of perpendicular of point $(1,2,-1)$ on the plane.

Let it be $P(\alpha,\beta,\gamma)$;

Direction ratio's of PC $=(\alpha-1,\beta-2,\gamma+1)$

These are parallel to the normal of plane

$$\Rightarrow \frac{\alpha-1}{2}=\frac{\beta-2}{-2}=\frac{\gamma+1}{1}=\lambda \text{ (say)}$$

Now; $(\alpha,\beta,\gamma)=(2\lambda+1,-2\lambda+2,\lambda-1)$ lies on plane.

$$\Rightarrow 2(2\lambda+1)-2(-2\lambda+2)+(\lambda-1)+12=0$$

$$9\lambda=-9 \quad \Rightarrow \lambda=-1.$$

$\therefore$ point of contact $(\alpha,\beta,\gamma)=((2\times-1)+1,-2\times-1+2,-1,-1)$

$$(\alpha,\beta,\gamma)=(1,4,-2)$$

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2(d) Suppose U and W are distinct four dimentional subspaces of a vector space 'V', where dim V = 6. Find the possible dimension of subspace $U \cap W$?

Sol: Given; U and W are distinct Subspaces; such that

$$\left.\begin{array}{l}\dim(U) = \dim(W) = 4 \\ \dim(V) = 6\end{array}\right\} \text{ of a vector space } V. \quad \text{--- (1)}$$

we know that

$$U, W \subseteq U+W \subseteq V \quad \text{--- (2)}$$

Using formula.

$$\dim(U) + \dim(W) - \dim(U\cap W) = \dim(U+W)$$

$$\dim(U\cap W) = -\dim(U+W) + \dim(U) + \dim(W)$$

$$\dim(U\cap W) = -\dim(U+W) + 4 + 4$$

$$\dim(U\cap W) = 8 - \dim(U+W) \quad \text{--- (3)}$$

from (2)

$$\dim(U) = 4 \leq \dim(U+W) \leq 6 \; (=\dim(V)) \quad \text{--- (4)}$$

Using (3) and (4) we get

$$2 \leq \dim(U\cap W) \leq 4$$

Since; U, W are distinct

$$U\cap W \neq U \text{ or } W \text{ or } \dim(U\cap W) \neq 4$$

Hence; $\dim(U\cap W) = \underline{\underline{2 \text{ or } 3}}$

3(a) Consider the matrix mapping $A: R^4 \to R^3$; where $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & -2 \\ 3 & 8 & 13 & -3 \end{bmatrix}$. Find a basis and dimension of the image of 'A' and those of the Kernel 'A'.

Sol: Given Matrix Mapping : $A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & -2 \\ 3 & 8 & 13 & -3 \end{bmatrix}$

To find basis and dimension of.

(i) Image of A : converting given matrix to Echlon form

$$\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & -2 \\ 3 & 8 & 13 & -3 \end{bmatrix} \underset{R_3 \to R_3 - 3R_1}{\overset{R_2 \to R_2 - R_1}{\sim}} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -3 \\ 0 & 2 & 4 & -6 \end{bmatrix} \overset{R_3 \to R_3 - 2R_2}{\sim} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is clearly the echlon form.

Hence, dimension of Image of A = number of non-zero rows = 2

i.e $\rho(A) = 2$ and basis of $\rho(A) = \{(1,2,3,1), (0,1,2,-3)\}$

(ii) Kernel of A ($N(A)$)

Let $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 5 & -2 \\ 3 & 8 & 13 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = 0$

or $x + 2y + 3z + t = 0$ and $y + 2z - 3t = 0$

$\therefore y = 3t - 2z$; $x = z - 7t$

$$\begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} z - 7t \\ 3t - 2z \\ z \\ t \end{bmatrix} = z \begin{bmatrix} 1 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -7 \\ 3 \\ 0 \\ 1 \end{bmatrix}$$

Hence; $\dim(N(A)) = 2$

Basis = $\{(1,-2,1,0), (-7,3,0,1)\}$

Q.3(b) Prove that distinct non-zero eigen vectors of a matrix are linearly independent?

Correct one.→ The characteristic vectors corresponding to distinct characteristic roots of a matrix are linearly independent.

Sol. Let $X_1, X_2, X_3 \cdots X_n$ be a set of non-zero eigen vectors of matrix A and corresponding eigen values be $\lambda_1, \lambda_2, \ldots \lambda_n$, which are distinct.

$\therefore A_1X_1 = \lambda_1 X_1$; $A_2X_2 = \lambda_2 X_2 \ldots$ so on —(1)

Let us assume $X_1, X_2, X_3 \cdots X_r$ be linearly independent and $X_1, X_2, X_3 \cdots X_r, X_{r+1}$ be linearly dependent vector.

$\therefore a_1X_1 + a_2X_2 + a_3X_3 + \cdots + a_rX_r + a_{r+1}X_{r+1} = 0$

Pre-multiplying by A, we get

$a_1AX_1 + a_2AX_2 + \cdots + a_rAX_r + a_{r+1}AX_{r+1} = 0$

$a_1\lambda_1X_1 + a_2\lambda_2X_2 + \cdots + a_r\lambda_rX_r + a_{r+1}\lambda_{r+1}X_{r+1} = 0$ —(2)

Since, $X_1, X_2 \ldots X_r$ are linearly independent.

$\Rightarrow a_1AX_1 + a_2AX_2 + \cdots + a_rAX_r = 0$ —(3)

Using (2) and (3); $a_{r+1}A_{r+1}X_{r+1} = 0$

Since $\lambda_{r+1} \neq 0$; $X_{r+1} \neq 0$

Hence $\Rightarrow a_{r+1} = 0$.

Hence; $X_1, X_2, \ldots X_{r+1}$ are also linearly independent

Hence; our Assumption is wrong.

3(c) If $f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0). \end{cases}$

Calculate $\dfrac{\partial^2 f}{\partial x \partial y}$ and $\dfrac{\partial^2 f}{\partial y \partial x}$ at $(0,0)$.

Sol: $\dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial}{\partial y} f(x,y)\right) = \lim\limits_{h\to 0} \dfrac{f_y(x+h,y) - f_y(x,y)}{h}$ —(1)

$\dfrac{\partial^2 f}{\partial y \partial x} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial}{\partial x} f(x,y)\right) = \lim\limits_{k\to 0} \dfrac{f_x(x,y+k) - f_x(x,y)}{k}$ —(2)

Now at $(x,y) = (0,0)$

$f_y(0,0) = \lim\limits_{k\to 0} \dfrac{f(0,k) - f(0,0)}{k} = 0$

$f_y(h,0) = \lim\limits_{k\to 0} \dfrac{f(h,k) - f(0,0)}{k} = \lim\limits_{k\to 0} \dfrac{\dfrac{h\cancel{k}(h^2-k^2)}{h^2+k^2} - 0}{\cancel{k}}$

$f_y(h,0) = \dfrac{h^3}{h^2} = h.$

$\therefore \dfrac{\partial^2 f}{\partial x \partial y}(0,0) = \lim\limits_{h\to 0} \dfrac{f_y(h,0) - f_y(0,0)}{h} = \dfrac{h-0}{h} = \dfrac{h}{h} = 1$

Again. $f_x(0,0) = \lim\limits_{h\to 0} \dfrac{f(h,0) - f(0,0)}{h} = 0$

$f_x(0,k) = \lim\limits_{h\to 0} \dfrac{f(h,k) - f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{\dfrac{\cancel{h}k(h^2-k^2)}{h^2+k^2}}{\cancel{h}}$

$f_x(0,k) = -k.$

$\therefore \dfrac{\partial^2 f}{\partial y \partial x}(0,0) = \lim\limits_{k\to 0} \dfrac{f_x(0,k) - f_x(0,0)}{k} = \lim\limits_{k\to 0} \dfrac{-k-0}{k} = -1.$

$\dfrac{\partial^2 f}{\partial y \partial x} \neq \dfrac{\partial^2 f}{\partial x \partial y}$

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3(d). Find the locus of the point of intersection of 3 mutually perpendicular tangent planes to $ax^2+by^2+cz^2=1$.

Sol:- Let $lx+my+nz=p$ be the tangent to the conicoid $ax^2+by^2+cz^2=1$; then we know that $\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p^2$; i.e equation of tangent becomes

$$lx+my+nz=\pm\sqrt{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}} \quad \text{--- (1)}$$

Now; let a set of three mutually perpendicular tangents of conicoid are given by

$$l_1x+m_1y+n_1z=\sqrt{\frac{l_1^2}{a}+\frac{m_1^2}{b}+\frac{n_1^2}{c}} \quad \text{--- (2)}$$

$$l_2x+m_2y+n_2z=\sqrt{\frac{l_2^2}{a}+\frac{m_2^2}{b}+\frac{n_2^2}{c}} \quad \text{--- (3)}$$

$$l_3x+m_3y+n_3z=\sqrt{\frac{l_3^2}{a}+\frac{m_3^2}{b}+\frac{n_3^2}{c}} \quad \text{--- (4)}$$

Squaring & adding (2), (3) & (4), we get

$$(5) \equiv (l_1^2+l_2^2+l_3^2)x^2+2xy(l_1m_1+l_2m_2+l_3m_3) \quad 1/a(l_1^2+l_2^2+l_3^2)$$
$$(m_1^2+m_2^2+m_3^2)+2yz(m_1n_1+m_2n_2+m_3n_3)=+\frac{1}{b}(m_1^2+m_2^2+m_3^2)$$
$$(n_1^2+n_2^2+n_3^2)+2zx(n_1l_1+n_2l_2+n_3l_3) \quad +\frac{1}{c}(n_1^2+n_2^2+n_3^2) \quad \text{--- (5)}$$

and we know, that $l_1^2+l_2^2+l_3^2=1$

for set of $\perp^r$ planes $\Rightarrow l_1m_1+l_2m_2+l_3m_3=0$

$\therefore$ (5) becomes

$$x^2+y^2+z^2=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$; which is the required equation.

4(a) Reduce the following equation to the standard form and hence determine the nature of conicoid:

$$x^2+y^2+z^2-yz-zx-xy-3x-6y-9z+21=0.$$

Sol. Given, $f(x,y,z)=x^2+y^2+z^2-yz-zx-xy-3x-6y-9z+21$.

Comparing it with $ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$

Discriminating cube :

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = \begin{vmatrix} 1-\lambda & -1/2 & -1/2 \\ -1/2 & 1-\lambda & -1/2 \\ -1/2 & -1/2 & 1-\lambda \end{vmatrix} = 0$$

$$=(1-\lambda)\left[(1-\lambda)^2-1/4\right]+\frac{1}{2}\left[\frac{-1}{2}(1-\lambda)-\frac{1}{4}\right]-\frac{1}{2}\left[\frac{1}{4}+\frac{1}{2}(1-\lambda)\right]$$

$$=(1-\lambda)^3-\frac{1}{4}+\frac{\lambda}{4}+\frac{1}{2}\left[\frac{-3}{4}+\frac{\lambda}{2}\right]-\frac{1}{2}\left[\frac{3}{4}-\frac{\lambda}{2}\right]$$

$$=(1-\lambda)^3+\frac{3\lambda}{4}-1=0$$

$$-\lambda^3+3\lambda^2-\frac{9}{4}\lambda=0 \Rightarrow \lambda^3-3\lambda^2+\frac{9}{4}\lambda=0$$

$$\therefore \lambda=0,\frac{3}{2},\frac{3}{2}.$$

Hence, the given surface can be the form:

$A(x^2+y^2)+Bz=0$ or $A(x^2+y^2)+D=0$

calculating Direction ratios of axis for $\lambda=0$; (l_3,m_3,n_3)

$$\left.\begin{aligned} l_3-\frac{m_3}{2}-\frac{n_3}{2}&=0 \\ -\frac{l_3}{2}+m_3-\frac{n_3}{2}&=0 \\ -\frac{l_3}{2}-\frac{m_3}{2}+n_3&=0 \end{aligned}\right\} \Rightarrow \frac{l_3}{3/4}=\frac{m_3}{3/4}=\frac{n_3}{3/4}$$

$\Rightarrow (l_3,m_3,n_3)$ in DC form $\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$

Now; $k=ul_3+vm_3+wn_3 \Rightarrow \left(\frac{-3}{2}-3-\frac{9}{2}\right)\frac{1}{\sqrt{3}}=-3\sqrt{3}$

$\therefore$ given eq^n represents paraboloid of revolution.

$$\frac{3}{2}(x^2+y^2)+2(-3\sqrt{3})z=0 \quad \left\{\lambda_1x^2+\lambda_2y^2+2kz=0\right\}$$

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4(b). Consider the following system of equation in x, y, z:

$$x + 2y + 2z = 1$$
$$x + ay + 3z = 3$$
$$x + 11y + az = b$$

(i) For which values of 'a' does the system have a unique solution.

(ii) For which pair of values (a, b) does the system have more than one solution.

Sol:- The Augumented matrix $[A:B]$ is given by

$$\begin{bmatrix} 1 & 2 & 2 & : & 1 \\ 1 & a & 3 & : & 3 \\ 1 & 11 & a & : & b \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{array}$$

$$\begin{bmatrix} 1 & 2 & 2 & : & 1 \\ 0 & a-2 & 1 & : & 2 \\ 0 & 9 & a-2 & & b-1 \end{bmatrix} \quad R_3 \to R_3 - \frac{9}{(a-2)} R_2$$

$$\begin{bmatrix} 1 & 2 & 2 & : & 1 \\ 0 & a-2 & 1 & : & 2 \\ 0 & 0 & (a-2)-\frac{9}{a-2} & : & b-1-\frac{18}{a-2} \end{bmatrix}$$ which is the Echlon form.

(i) For unique solution $\rho(A:B) = \rho(A) =$ no. of unknown.

or $a-2 - \dfrac{9}{a-2} \neq 0$

$(a-2)^2 - 9 \neq 0 \Rightarrow (a-2)^2 = 9$

$(a-2) \neq \pm 3 \Rightarrow a \neq -1$ or 5

(ii) For more than one solution; i.e infinite solution.

$\rho(A) = \rho(A/B) <$ no. of un.known

$a-2 - \dfrac{9}{a-2} = 0$	$b-1-\dfrac{18}{a-2} = 0$
$(a-2)^2 = 9$	$b = \dfrac{a-2-18}{a-2} \Rightarrow b = -5, 7$
$a-2 = \pm 3$	
$a = -1, 5$	$\boxed{\therefore (a,b) = (-1,-5) \text{ or } (5,7)}$

4(c) Examine, if the improper integral $\int_0^3 \frac{2x}{(1-x^2)^{2/3}}$ exists?

Sol. let $I = \int_0^3 \frac{2x}{(1-x^2)^{2/3}} dx.$

The only point of discontinuity shall be '1', which belongs to (0,3)

$$I = \underset{\epsilon\to 0}{lt}\int_0^{1-\epsilon} \frac{2x}{(1-x^2)^{2/3}} dx + \underset{\epsilon\to 0^+}{lt}\int_{1+\epsilon}^3 \frac{2x}{(1-x^2)^{2/3}} dx.$$

$$= \underset{\epsilon\to 0}{lt}\left[-3(1-x^2)^{1/3}\right]_0^{1-\epsilon} + \underset{\epsilon\to 0^+}{lt}\left[-3(1-x^2)^{1/3}\right]_{1+\epsilon}^3$$

$$= \underset{\epsilon\to 0}{lt}\left[-3(1-(1-\epsilon^2))^{1/3} + (-3(1-3^2)^{1/3} + 3(1-(1+\epsilon)^2)^{1/3}\right)$$

$$= -3[1-1]^{1/3} + (-3(-8)^{1/3} + 3(1-1)^{1/3}$$

$$= -3[0] + (-3((-2)^{3\times 1/3}) + 3(0)$$

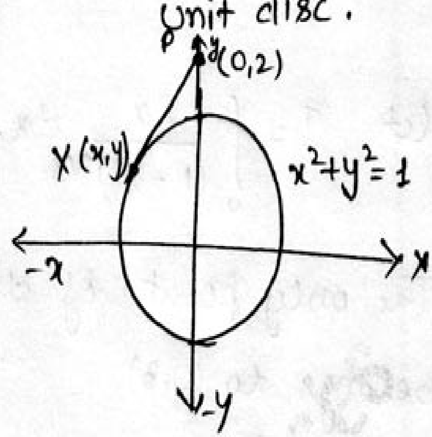
$$= 0 - 3\times -2 + 0$$

$$= 0 + 6 + 0 = 6.$$

which is finite.

Hence, I exists.

4(d) Prove that $\frac{\pi}{3} \leq \iint_D \frac{dx\,dy}{\sqrt{x^2+(y-2)^2}} \leq \pi$; where D is the unit disc.



Sol: let $f(x,y) = \sqrt{x^2+(y-2)^2}$ be a function; which given the distance between (0,2) and any point $X(x,y)$ on the unit circle : $x^2+y^2=1$.

Clearly; distance (max) or (min) of PX is such that

$$1 \leq PX \leq 3$$

or $$1 \leq \sqrt{x^2+(y-2)^2} \leq 3$$

$$1 \geq \frac{1}{\sqrt{x^2+(y-2)^2}} \geq \frac{1}{3}$$

$$\Rightarrow \iint_D \frac{1}{3}\,dx\,dy \leq \iint_D \frac{1\,dx\,dy}{(x^2+(y-2)^2)^{1/2}} \leq \int_D dx\,dy$$

$$= \frac{1}{3}\left[\pi(1)\right] \leq \iint_D \frac{dx\,dy}{\sqrt{x^2+(y-2)^2}} \leq \pi(1)^2.$$

$$\Rightarrow \frac{\pi}{3} \leq \iint_D \frac{dx\,dy}{\sqrt{x^2+(y-2)^2}} \leq \pi$$

which is the required result.

5(a). Find the differential equation representing all the circles of x-y plane.

Sol:- let the equation of circles in x-y plane be

$$x^2+y^2+2gx+2fy+c=0 \quad \text{—— (1)}$$

Differentiating w.r.t x, we get

$$2x+2y\frac{dy}{dx}+2g+2f\frac{dy}{dx}=0$$

Differentiating w.r.t x we get;

$$2+2y\frac{d^2y}{dx^2}+2\left(\frac{dy}{dx}\right)^2+2f\frac{d^2y}{dx^2}=0$$

Dividing by $\frac{d^2y}{dx^2}$ and differentiating w.r.t x

$$\frac{2}{\left(\frac{d^2y}{dx^2}\right)}+2y+\frac{2\left(\frac{dy}{dx}\right)^2}{\left(\frac{d^2y}{dx^2}\right)}+2f=0$$

$$0=\frac{-\cancel{2}\cdot -1}{\left(\frac{d^2y}{dx^2}\right)^2}\frac{d^3y}{dx^3}+\cancel{2}\frac{dy}{dx}+\cancel{2}\frac{\left[2\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2-2\left(\frac{dy}{dx}\right)^2\frac{d^3y}{dx^3}\right]}{\left(\frac{d^2y}{dx^2}\right)^3}$$

on Simplification

$$\left[1+\left(\frac{dy}{dx}\right)^2\right]\frac{d^3y}{dx^3}-3\frac{dy}{dx}\left(\frac{d^2y}{dx^2}\right)^2=0$$

which is the required Differential Equation.

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5(b) Suppose that the streamlines of the fluid flow are given by a family of curves $xy = c$. Find the equipotential lines, that is, the orthogonal trajectories of the family of curves representing the streamlines.

Sol: Given family of curve is $xy = c$ —(1)

differentiating (1) w.r.t x

$$y + x\frac{dy}{dx} = 0 \quad \text{—(2)}$$

An orthogonal trajectory of above curves will have slope $= -1/\frac{dy}{dx}$; therefore, differential equation of orthogonal curves of (1) is

$$\frac{x}{-\frac{dy}{dx}} + y = 0 \Rightarrow \frac{dy}{dx} = \frac{x}{y}$$

$$\Rightarrow y\,dy - x\,dx = 0 \quad \text{—(3)}$$

Integrating above eqn, we get

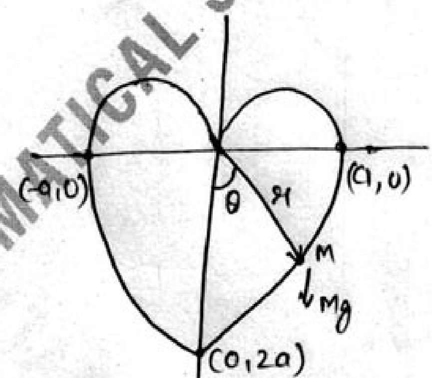
$$\boxed{y^2 - x^2 = c_2}$$

which is the required family of orthogonal trajectory c_2, being arbitrary constant.

5(c) A fixed wire is in the shape of Cardiod $r = a(1+\cos\theta)$ the initial line being the downward vertical. A small ring of mass 'm' can slide on the wire and is attached to the point $r=0$ of the cardiod by an elastic string of natural length 'a' and modulus of elasticity 4mg. The string is released from rest when the string is horizontal. Show by using laws of conservation of energy that

$a\dot\theta^2(1+\cos\theta) - g\cos\theta(1-\cos\theta) = 0$; g being the acceleration due to gravity.

Sol. Given, fixed wire is in shape of Cardiod.



$r = a(1+\cos\theta)$

$r' = -a\sin\theta$

loss in P.E of m

= gain in K.E of m + gain in PE of the string

$\downarrow mg$

$$mg(r\cos\theta) = \frac{1}{2}mv^2 + \int_0^{r-a} \frac{\lambda x}{a}\,dx$$

here; $x = r\sin\theta$; $y = r\cos\theta$

$$v^2 = \dot x^2 + \dot y^2 = \left(r\sin\theta + r\cos\theta\frac{d\theta}{dt}\right)^2 + \left(-r\sin\theta\frac{d\theta}{dt} + r\cos\theta\right)^2$$

$$= \dot r^2 + r^2\dot\theta^2 = (-a\sin\theta\cdot\dot\theta)^2 + (a(1+\cos\theta)\dot\theta)^2$$

$$\Rightarrow mgr\cos\theta = \frac{1}{2}m(a^2\dot\theta^2[2+2\cos\theta] + \frac{\lambda}{2a}(r-a)^2$$

$$\boxed{\lambda = 4mg}$$

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$= mg\,a\,(1+\cos\theta)\cdot\cos\theta$

$= m\,a^2\cdot\dot{\theta}^2\,[1+\cos\theta] + \frac{2mg}{a}\cdot a\cos^2\theta\cdot$

$\Rightarrow g\cos\theta\,(1+\cos\theta) = a\theta^2\,[1+\cos\theta] + 2g\cos^2\theta$

$\Rightarrow \boxed{a\dot{\theta}^2(1+\cos\theta) - g\cos\theta\,(1-\cos\theta) = 0}$

Q.5(d) For what values of the constant a, b and c the vector $\vec{V} = (x+y+az)\hat{i} + (bx+2y-z)\hat{j} + (-x+cy+2z)\hat{k}$ is irrotational. Find the divergence in cylindrical co-ordinates of this vector with these values?

Sol: Given; $\vec{V} = (x+y+az)\hat{i} + (bx+2y-z)\hat{j} + (-x+cy+2z)\hat{k}$

$$\vec{V} = V_1\hat{i} + V_2\hat{j} + V_3\hat{k}$$

For V to be irrotational

$$\nabla \times V = 0$$

i.e $\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ V_1 & V_2 & V_3 \end{vmatrix}$

$$= i(c+1) - j(-1-a) + \hat{k}(b-1) = \vec{0}$$

or $\boxed{a=-1,\ b=1,\ c=1}$

Now; $\nabla\cdot\vec{V} = \dfrac{\partial V_1}{\partial x} + \dfrac{\partial V_2}{\partial y} + \dfrac{\partial V_3}{\partial z}$

$$\nabla\cdot\vec{V} = 1+2+2 = 5;$$

which is same for cylindrical co-ordinates of given vector.

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5(e) The position vector of a moving point at time t is $\vec{r} = \sin t\,\hat{i} + \cos 2t\,\hat{j} + (t^2+2t)\hat{k}$. Find the components of acceleration $\vec{a}$ in the directions parallel to the velocity vector $\vec{v}$ and perpendicular to the plane of $\vec{r}$ and $\vec{v}$ at time $t=0$.

Sol:
$$r(t) = \sin t\,\hat{i} + \cos 2t\,\hat{j} + (t^2+2t)\hat{k} \quad \text{--- (1)}$$

$$v(t) = \frac{dr}{dt} = \cos t\,\hat{i} - 2\sin 2t\,\hat{j} + (2t+2)\hat{k} \quad \text{--- (2)}$$

$$a(t) = \frac{d^2r}{dt^2} = -\sin t\,\hat{i} - 4\sin 2t\,\hat{j} + 2\hat{k} \quad \text{--- (3)}$$

at $t=0$

$r(0) = 0\hat{i} + \hat{j} + 0\hat{k} = \hat{j}$

$v(0) = \hat{i} + 0\hat{j} + 2\hat{k} = i + 2\hat{k}$

Unit vector $\hat{v} = \dfrac{i+2\hat{k}}{\sqrt{5}}$

$a(0) = 0\hat{i} - 4\hat{j} + 2\hat{k} = -4\hat{j} + 2\hat{k}$

component of $\vec{a}$, in direction parallel to $\vec{v}$ is

$$= \frac{\vec{a}\cdot\vec{v}}{|\vec{v}|}(\hat{v}) = \frac{4}{5}(\hat{i}+2\hat{k})$$

Now, let w be the direction perpendicular to r and v at $t=0$; then

$$\vec{\omega} = \vec{r}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 2\hat{i} - \hat{k}$$

unit vector $\hat{\omega} = \dfrac{2\hat{i}-\hat{k}}{\sqrt{5}}$

Component of a in direction of $\vec{\omega}$, is

$$= \vec{a}\cdot\frac{\vec{\omega}}{|\vec{\omega}|}\cdot(\vec{\omega}) = -\frac{2}{5}(2\hat{i}-\hat{k}).$$

6(a)(i) Solve the following simultaneous linear differential equations:-
$(D+1)y = z+e^x$ and $(D+1)z = y+e^x$; where y and z are functions of independent variable x and $D = \frac{d}{dx}$.

Sol: Given that; $(D+1)y = z+e^x$ —— ①
$(D+1)z = y+e^x$ —— ②

$\{(D+1)\times ①\} + ②$

$(D+1)^2 y + \cancel{(D+1)}z = \cancel{(D+1)}z + (D+1)e^x + y + e^x$

$(D^2+2D+1)y = 2e^x + y + e^x$

$(D^2+2D)y + \cancel{y} = 3e^x + \cancel{y}$

$(D^2+2D)y = 3e^x$ —— ③

The general solution of ③ is given by

$y = y_c + y_p$ $\left\{ \begin{array}{l} y_c = \text{complimentary function} \\ y_p = \text{Particular Integral} \end{array} \right\}$

Auxillary equation because; m^2+2m
$m(m+2)=0$
$\Rightarrow m=0,-2$

$\therefore y_c = c_1 + c_2 e^{-2x}$; $y_p = \frac{1}{D^2+2D}(3e^x) = \frac{1}{1+2}\cdot 3\cdot e^x = e^x$

$\boxed{\therefore\ y = c_1 + c_2 e^{-2x} + e^x}$ —— ④

Putting this value of 'y' in ① ; we get

$(D+1)y = z+e^x$

$-2c_2 e^{-2x} + e^x + c_1 + c_2 e^{-2x} + \cancel{e^x} = z + \cancel{e^x}$

$\boxed{\therefore\ z = c_1 - c_2 e^{-2x} + e^x}$ —— Ans

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6(a)ii) If the growth rate of the population of bacteria at any time 't' is proportional to the amount present at that time and population doubles in one week, then how much bacterias can be expected after 4 weeks?

Sol:- Let population of bacteria be given by 'y' at time 't'

given $\frac{dy}{dt} \propto y$ (or) $\frac{dy}{dt} = ky$ (where k is constant)

$\Rightarrow \frac{dy}{y} = k \cdot dt \Rightarrow \int \frac{dy}{y} = \int k \cdot dt$

or $\log y = kt + c$ ——— (1)

let initial population be y_0; at $t=0$

then $c = \log y_0$

(1) becomes $\log y = kt + \log y_0$

$\Rightarrow$ Also; at $t=1$; $y = 2y_0$

$\Rightarrow \log 2y_0 = k(1) + \log y_0$

$k = \log\left(\frac{2y_0}{y_0}\right) \Rightarrow \boxed{k = \log 2}$

Now, at $t = 4$

$\log y = (\log 2) 4 + \log y_0$

$\log y = \log 2^4 + \log y_0$

$\Rightarrow \log y = \log(16 y_0)$

or $\boxed{y = 16 y_0}$

i.e. Population of bacterias becomes 16 times the initial population of bacteria (y_0).

6(b)(i) Consider the differentiable equation $xyp^2 - (x^2+y^2-1)p + xy = 0$; where $p = \frac{dy}{dx}$. Substituting $u = x^2$ and $v = y^2$ reduce the equation to Clairaut's form in terms of u, v and $p' = \frac{dv}{du}$. Hence, or otherwise solve the equation.

Sol:- $xyp^2 - (x^2+y^2-1)p + xy = 0$

Substituting $u = x^2$, $v = y^2$

$du = 2x\,dx$; $dv = 2y\,dy$

$$p' = \frac{dv}{du} = \frac{y}{x}\frac{dy}{dx} = \frac{y}{x}p \quad \text{or} \quad p = \frac{x}{y}p'$$

$$xy\left(\frac{x^2}{y^2}\right)p'^2 - (u+v-1)\frac{x}{y}p' + 1 = 0$$

$\cancel{\frac{u}{v}p'^2} - \cancel{\frac{(u+v-1)}{v}}p'$ Dividing by xy, we get

$$\frac{x^2}{y^2}\cdot p'^2 - (u+v-1)\frac{x}{x\cdot y\cdot y}p' + 1 = 0$$

$$\frac{u}{v}p'^2 - \frac{(u+v-1)}{v}\cdot p' + 1 = 0$$

$$up'^2 - (u+v-1)p' + v = 0$$

$$(1-p')v = up' - up'^2 - p'$$

$$v = \frac{up'(1-p')}{1-p'} - \frac{p'}{1-p'} = up' - \frac{p'}{1-p'}$$

which is clearly in Clairaut form,

hence, solution is

$$v = uc - \frac{c}{1-c} \Rightarrow y^2 = cx^2 + \frac{c}{c+1}$$

which is the required solution.

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6(b)(ii) Solve the following initial value differential Equations:

$$20y''+4y'+y=0 \quad ; \quad y(0)=3\cdot2 \text{ and } y'(0)=0.$$

Sol Given equation; $20y''+4y'+y=0$

$$\text{or} \quad \left(D^2+\frac{D}{5}+\frac{1}{20}\right)y=0 \quad ; \quad D=\frac{d}{dx}$$

which is clearly homogenous,

Auxillary Equation ; $m^2+\frac{m}{5}+\frac{1}{20}=0$

$$\Rightarrow m=\frac{-1\pm 2i}{10}$$

$$\therefore y=e^{-x/10}\left(c_1\cos\frac{2}{10}x+c_2\sin\frac{2}{10}x\right)$$

given, $y(0)=3\cdot2$

$$\Rightarrow 3\cdot2=e^{-0/10}(c_1+c_2(0)) \Rightarrow c_1=3\cdot2$$

Also; $y'(0)=0$

$$e^{-x/10}\left(-c_1\sin\frac{x}{5}+c_2\cos\frac{x}{5}-\frac{1}{10}\left[c_1\cos\frac{x}{5}+c_2\sin\frac{x}{5}\right]\right)=0$$

at $x=0$

$$\text{or} \quad e^0\left(0+c_2-\frac{1}{10}(c_1+0)\right]=0$$

$$\Rightarrow c_2=\frac{c_1}{10}=0\cdot32$$

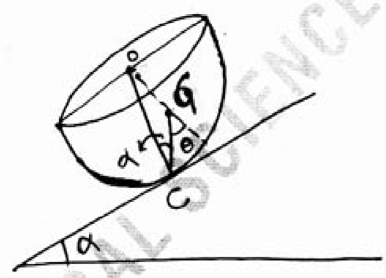
Hence, the general solution is given by

$$\boxed{y=e^{-x/10}\left[3\cdot2\cos\left(\frac{x}{5}\right)+0\cdot32\sin\left(\frac{x}{5}\right)\right]}$$

6(c). A uniform solid hemisphere rests on a rough plane inclined to the horizon at an angle ϕ with its curved surface touching the plane. Find the greatest admissible value of the inclination ϕ for equilibrium. If ϕ be less than this value, is the equilibrium stable?

Sol:- Let, O be the centre of the base of the hemisphere and 'r' be its radius.

If 'C' is the point of contact of the hemisphere and the inclined plane; the $OC = r$.



Let, G be the centre of gravity of the hemisphere.

Then $OG = 3r/8$.

In the position of equilibrium the line CG must be vertical.

Since, OC is perpendicular to the inclined plane and CG is perpendicular to the horizontal, therefore $\angle OCG = \alpha$. Suppose in equilibrium the axis of the hemisphere makes an angle 'θ' with the vertical.

From ΔOGC, we have

$$\frac{OG}{\sin\alpha} = \frac{OC}{\sin\theta} \quad \text{i.e} \quad \frac{3r/18}{\sin\alpha} = \frac{r}{\sin\theta}$$

$$\therefore \sin\theta = \frac{8}{3}\sin\alpha \quad (\text{or}) \quad \boxed{\theta = \sin^{-1}\left(\frac{8}{3}\sin\alpha\right)}$$

giving the position of equilibrium of the hemisphere

Since; $\sin\theta < 1$; $\therefore \frac{8}{3}\sin\alpha < 1$

i.e $\sin\alpha < \frac{3}{8}$ i.e $\alpha < \sin^{-1}\left(\frac{3}{8}\right)$

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Thus for the equilibrium to exist, we must have

$$\alpha < \sin^{-1}\left(\frac{3}{8}\right).$$

Now, let $CG = h$; Then

$$\frac{h}{\sin(\theta-\alpha)} = \frac{3r/8}{\sin\alpha} \text{ ; so that } h = \frac{3r\sin(\theta-\alpha)}{8\sin\alpha}$$

Here $\rho_1 = r$ and $\rho_2 = \infty$

The equilibrium will be stable if

$$h < \frac{\rho_1\rho_2\cos\alpha}{\rho_1+\rho_2} \quad \text{i.e.} \quad \frac{1}{h} > \frac{\rho_1+\rho_2}{\rho_1\rho_2}\sec\alpha$$

$$\text{i.e.} \quad \frac{1}{h} > \left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)\sec\alpha \Rightarrow \frac{1}{h} > \frac{1}{r}\sec\alpha$$

$$[\because \rho_1 = r, \rho_2 = \infty]$$

$$\therefore h < r\cos\alpha$$

$$\text{i.e.} \quad \frac{3r\sin(\theta-\alpha)}{8\sin\alpha} < r\cos\alpha.$$

$$\Rightarrow 3r\sin(\theta-\alpha) < 8r\sin\alpha\cos\alpha$$

$$\Rightarrow 3\sin\theta\cos\alpha - 3\cos\theta\sin\alpha < 8\sin\alpha\cos\alpha$$

$$\text{or} \quad 8\sin\alpha\cos\alpha - 3\sin\alpha\sqrt{\left(1-\frac{64}{9}\sin^2\alpha\right)} < 8\sin\alpha\cos\alpha$$

$$\left[\because \sin\theta = \frac{8}{3}\sin\alpha\right]$$

$$\text{or} \quad -\sin\alpha\sqrt{(9-64\sin^2\alpha)} < 0$$

$$\text{or} \quad \sin\alpha\sqrt{(9-64\sin^2\alpha)} > 0$$

But for (1)

$\sin\alpha < \frac{3}{8}$ i.e $64\sin^2\alpha < 9$ i.e $\sqrt{(9-64\sin^2\alpha)}$ is a positive real number. Therefore, the relation (2) is true. Hence, the equilibrium is stable.

7(a) Find the curvature vector and its magnitude at any point $\vec{r} = (\theta)$ of the curve $\vec{r} = (a\cos\theta, a\sin\theta, a\theta)$ show that the locus of the feet of the perpendicular from the origin to the tangent is a curve that completely lies on the hyperboloid $x^2 + y^2 - z^2 = a^2$.

Sol:-

$$r(\theta) = a\cos\theta\,\hat{i} + a\sin\theta\,\hat{j} + a\theta\,\hat{k}$$

$$\frac{ds}{d\theta} = \left|\frac{dr}{d\theta}\right| = \left|-a\sin\theta\,\hat{i} + a\cos\theta\,\hat{j} + a\hat{k}\right|$$

$$= \sqrt{2}\,a$$

$$\frac{dr}{ds} = \frac{dr}{d\theta}\cdot\frac{d\theta}{ds} = \frac{1}{\sqrt{2}a}\left(-a\sin\theta\,\hat{i} + a\cos\theta\,\hat{j} + a\hat{k}\right)$$

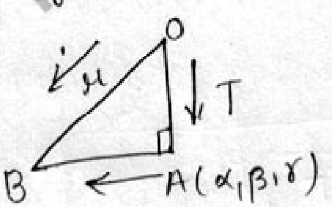
$$\text{or}\quad \frac{dr}{ds} = \frac{1}{\sqrt{2}}\left(-\sin\theta\,\hat{i} + \cos\theta\,\hat{j} + \hat{k}\right) = T \quad \text{——} (1)$$

Curvature vector;

$$\frac{dT}{ds} = \frac{dT}{d\theta}\cdot\frac{d\theta}{ds} = \frac{1}{\sqrt{2}}\left[-\cos\theta\,\hat{i} - \sin\theta\,\hat{j}\right)\cdot\frac{1}{\sqrt{2}a}$$

$$\frac{dT}{ds} = \frac{-1}{2a}\left[\cos\theta\,\hat{i} + \sin\theta\,\hat{j}\right]$$

Magnitude of Curvature vector; $= \left|\frac{dT}{ds}\right| = \frac{1}{2a}$



$$\overrightarrow{BA} = \vec{r} - \overrightarrow{OA}$$

$$\overrightarrow{BA} = \vec{r} - \frac{(\vec{r}\cdot\vec{T})}{|T|}\cdot\hat{T}$$

$$\boxed{\overrightarrow{BA} = \vec{r} - \frac{a\theta}{\sqrt{2}}(\hat{T})}$$

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$$(\alpha\hat{i}+\beta\hat{j}+\gamma\hat{j}) = \left(a\cos\theta + \frac{a\theta\sin\theta}{2}\right)\hat{i} + \left(a\sin\theta - \frac{a\theta\cos\theta}{2}\right)\hat{j} + \left(\frac{a\theta}{2}\right)\hat{k}$$

$$\therefore \alpha = a\cos\theta + \frac{a\theta}{2}\sin\theta$$

$$\beta = a\sin\theta - \frac{a\theta}{2}\cos\theta$$

$$\gamma = \frac{a\theta}{2}$$

$\therefore$ locus of feet of perpendicular;

$$\alpha^2+\beta^2-\gamma^2 = a^2$$

or $x^2+y^2-z^2=a^2$

7(b)(i) Solve the differential equation:

$$x\frac{d^2y}{dx^2} - \frac{dy}{dx} - 4x^3y = 8x^3\sin(x^2)$$

Sol: Given equation can be re-written as;

$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} - 4x^4y = 8(x^2)^2.\sin x^2 \quad \text{[multiplied equation with } x\text{]} \quad \text{——(1)}$$

let $x^2 = u$; $\frac{du}{dx} = 2x$

& $\frac{dy}{dx} = \frac{dy}{du}.\frac{du}{dx} = 2x.\frac{dy}{du}$

Similarly; $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(2x\frac{dy}{dx}\right) = 2\frac{dy}{du} + 2x\frac{d^2y}{dx^2}.\frac{du}{dx}$

$$\frac{d^2y}{dx^2} = 2\frac{dy}{du} + 4x^2\frac{d^2y}{dx^2}$$

Substituting these values in eqn (1)

$$u\left(2\frac{dy}{du} + 4u\frac{d^2y}{dx^2}\right) - 2u\frac{dy}{du} - 4u^2y = 8u^2\sin u \quad \text{——(2)}$$

$$\frac{d^2y}{dx^2} - y = \sin u$$

Auxillary Equation — $m^2 - 1 = 0 \Rightarrow m = \pm 1$

or $y_c = c_1e^u + c_2e^{-u}$ ——(3) [y_c = complimentary function

$y_p = \frac{1}{D^2-1}(2\sin u)$ { y_p = Particular Integral }

$y_p = \frac{+u}{-2D}(2\sin u)$

$\therefore$ general solution; $y = c_1e^u + c_2e^{-u} - u\cos u$

or $\boxed{y = c_1e^{x^2} + c_2e^{-x^2} - x^2\cos(x^2)}$

Where, c_1 & c_2 are arbitrary constants.

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7(b)(ii) Solve the following differential Equation using method of variation of parameters:

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 44 - 76x - 48x^2.$$

Sol:- Given; equation can be re-written as

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \text{; where} \quad \text{---} (1)$$

$P = -1$; $Q = -2$; $R = 44 - 76x - 48x^2$

Using the method of variation of parameters:

The homogenous equation becomes:

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0 \quad \text{(or)} \quad (D^2 - D - 2)y = 0 \quad \left[\because D = \frac{d}{dx}\right] \quad \text{---} (2)$$

Auxillary Equation ; $m^2 - m - 2 = 0 \Rightarrow m = -1, 2$

complementary function ; $y_c = c_1 e^{-x} + c_2 e^{2x}$ --- (3)

let $u = e^{-x}$ and $v = e^{x}$; hence, the wronksian of (u, v)

$$\omega(u,v) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{2x} \\ -e^{-x} & 2 \cdot e^{2x} \end{vmatrix} = 2e^{x} + e^{x} = 3e^{x}$$

Hence ; Particular Integral ; $y_p = Au + Bv$ --- (4)

where, $A = \int \frac{-vR}{\omega} dx = \int \frac{-e^{2x} e^{x}}{3e^{x}} (44 - 76x - 48x^2) dx.$

or $A = \frac{-1}{3} \int e^{x} (44 - 76x - 48x^2) dx = \frac{-1}{3} \int e^{x} [24 + (20 + 20x) - 96x - 48x^2] dx$

$= \frac{-1}{3} (24e^{x} + 20e^{x} \cdot x - 48e^{x} \cdot x^2) = e^{x} (-8 - \frac{20}{3} x + 16x^2)$.

Similarly ; $B = \int \frac{uR}{\omega} = \int \frac{e^{-x}}{3e^{x}} (44 - 76x - 48x^2) dx$

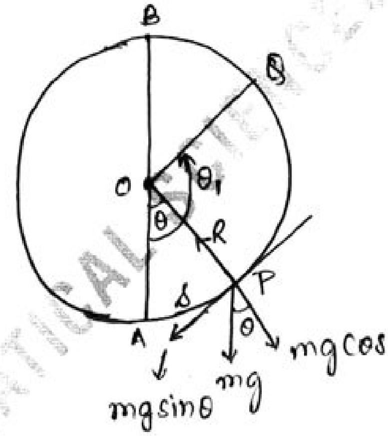
$B = e^{-2x} [3 + \frac{62}{3} x + 8x^2]$

Hence ; general solution be $y = y_c + y_p$

$y = c_1 e^{-x} + c_2 e^{x} + 24x^2 + 14x - 5$ A

Q.7(c) A particle is free to move on a smooth vertical circular wire of radius 'a'. At time $t=0$; it is projected along circle from its lowest point 'A' with velocity just sufficient to carry it to the highest point 'B'. Find the time 'T' at which the reaction between the particle and the wire is zero.

Sol: Let, a particle of mass 'm' be projected from the lowest point 'A' of a vertical circle of radius 'a' with velocity 'u' which is just sufficient to carry it to the highest point 'B'.



If 'P' is the position of the particle at any time 't' such that $\angle AOP = \theta$ and arc $AP = s$, then the equations of motion of the particle along the tangent and normal are

$$\frac{m d^2 s}{dt^2} = -mg\sin\theta \quad \text{——— (1)}$$

and $$m\frac{v^2}{a} = R - mg\cos\theta \quad \text{——— (2)}$$

Also $$s = a\theta \quad \text{——— (3)}$$

From (1) and (3); we have $a\dfrac{d^2\theta}{dt^2} = -g\sin\theta$

Multiplying both side by $2a\left(d\theta/dt\right)$ and integrating, we have

$$v^2 = \left(a\frac{d\theta}{dt}\right)^2 = 2ag\cos\theta + A.$$

But According to the question; $v=0$ at the highest point 'B'; where $\theta = \pi$

$$\therefore 0 = 2ag\cos\pi + A \quad \text{or} \quad A = 2ag$$

$$\therefore v^2 = \left(a\frac{d\theta}{dt}\right)^2 = 2ag\cos\theta + 2ag \quad \text{——— (4)}$$

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From (2) and (4), we have

$$R = \frac{m}{a}(v^2 + ag\cos\theta) = \frac{m}{a}(2ag + 3ag\cos\theta) \qquad \text{---}(5)$$

If the reaction $R=0$ at the point 'Q'; where $\theta = \theta_1$ then from (5), we have

$$0 = \frac{m}{a}(2ag + 3ag\cos\theta_1)$$

$$\boxed{\cos\theta_1 = -\frac{2}{3}} \qquad \text{---}(6)$$

from (4), we have

$$\left(a\frac{d\theta}{dt}\right)^2 = 2ag(\cos\theta + 1) = 2ag \cdot 2\cos^2\left(\tfrac{1}{2}\theta\right) = 4ag\cos^2\frac{\theta}{2}.$$

$\therefore \frac{d\theta}{dt} = 2\sqrt{(g/a)}\cos\theta/2$; the positive sign being taken before the radical sign because 'θ' increases as 't' increases.

or $dt = \frac{1}{2}\sqrt{(a/g)}\sec\theta/2 \, d\theta$

Integrating from $\theta = 0$ to $\theta = \theta_1$; the required time 't' is given by

$$t = \frac{1}{2}\sqrt{(a/g)}\int_{\theta=0}^{\theta=\theta_1} \sec\frac{\theta}{2}\,d\theta = \frac{2}{2}\sqrt{a/g}\left[\log(\sec\theta/2 + \tan\theta/2)\right]_0^{\theta_1}$$

$$\boxed{t = \sqrt{(a/g)}\,\log\left(\sec\theta_1/2 + \tan\frac{\theta_1}{2}\right)} \qquad \text{---}(7)$$

From (6), we have.

$$2\cos^2\theta_1/2 - 1 = -2/3 \Rightarrow 2\cos^2\theta_1/2 = 1 - 2/3$$

$$\cos^2\theta_1/2 = \frac{1}{6} \quad \Rightarrow \sec^2\theta_1/2 = 6$$

$$\therefore \quad \sec\theta_1/2 = \sqrt{6}$$

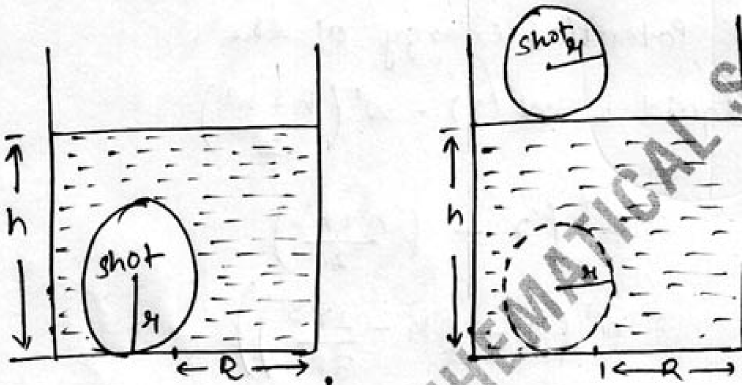
and $\tan\frac{\theta_1}{2} = \sqrt{(\sec^2\theta_1/2 - 1)} = \sqrt{6-1} = \sqrt{5}$

Substituting in (7), the required time is given by

$$\boxed{t = \sqrt{(a/g)}\,\log(\sqrt{6} + \sqrt{5})}$$

Q.8(a) A spherical shot of 'W' gm weight and radius 'r' cm, lies at the bottom of cylindrical bucket of radius 'R' cm. The bucket is filled with water up to a depth of 'h' cm ($h > 2r$); show that minimum amount of work done in lifting the shot just clear of water must be $\left[W\left(h-\frac{4r^3}{3R^2}\right)+W'\left(r-h+\frac{2r^3}{3R^2}\right)\right]$ cm gm. W' gm is the weight of water displaced by the spherical shot.

Sol:-



Assuming the spherical shot was lifted slowly, only one can ignore the drage losses and any changes in kinetic energy; so the problem is reduted to the changes in the potential energy of the spherical shot and the water displaced by the shot.

r – radius of spherical shot
R – radius of cylindrical bucket
h – height of filled water ($h > 2r$)

The depth of water 'h' will decrease to h', which can be found by equating volume of spherical shot to the volume of water displaced.

Thus – $\frac{4}{3}\pi r^3 = \pi R^2 h - \pi R^2 h'$

$$\Rightarrow \frac{4}{3} r^3 = R^2(h-h')$$

$$\Rightarrow \boxed{h' = h - \frac{4}{3}\frac{r^3}{R^2}}$$

change in P.E of the shot $= mg(h'+r) - mg(r)$

$$= w(h') = w\left(h - \frac{4r^3}{3R^2}\right).$$

Effectively the displaced liquid occupies the volume of the spherical shot.

$\Rightarrow$ change in Potential Energy of the displaced liquid $= w'(r) - w'\left(\frac{h+h'}{2}\right)$

$$= w'\left(r - \left(\frac{h+h'}{2}\right)\right)$$

$$= w'\left(r - \left(h - \frac{2r^3}{3R^2}\right)\right)$$

$$= w'\left(r - h + \frac{2r^3}{3R^2}\right)$$

Since, the least work done is the change in Potential Energy

$\therefore$ Work done = Change in P.E of shot + change in P.E of displaced liquid

$$\boxed{\text{Work done} = w\left(h - \frac{4r^3}{3R^2}\right) + w'\left(r - h + \frac{2r^3}{3R^2}\right)} \text{ cm gm}$$

8(b) Solve the following initial value problem using Laplace transform.

$$\frac{d^2y}{dx^2} + 9y = r(x) \; ; \; y(0) = 0 \; ; \; y'(0) = 4.$$

where; $r(x) = \begin{cases} 8\sin x & ; \text{if } 0 < x < \pi \\ 0 & ; \text{if } x \geq \pi \end{cases}$

Sol:- Given; $\frac{d^2y}{dx^2} + 9y = r(x)$; $y(0) = 0$; $y'(0) = 4$

we can rewrite the above equation

$$y'' + 9y = r(x) \quad \text{—(1)} \; ; \; y(0) = 0 \; ; \; y'(0) = 4$$

let $L(y) = p$

$$L(y'') = s^2 p - s(y(0)) - y'(0)$$

$$= s^2 p - s(0) - 4$$

$$L(y'') = s^2 p - 4. \quad \text{——(2)}$$

$$r(x) = \begin{cases} 8\sin x & ; \text{if } 0 < x < \pi \\ 0 & ; \text{if } x \geq \pi \end{cases}$$

$$L(r(x)) = \int_0^\infty e^{-st} \cdot r(t)\, dt$$

$$L(r(x)) = 8\int_0^\pi e^{-st} \cdot \sin t\, dt \quad \text{——(3)}$$

let $I = \int_0^\pi e^{-st} \cdot \sin t\, dt$

$$I = [-e^{-st} \cdot \cos t]_0^\pi - s\int_0^\pi e^{-st} \cos t\, dt$$

$$I = e^{-\pi s} + 1 - s\left[\left[e^{-st}\sin t\right]_0^{\pi} + s\int_0^{\pi} e^{-st}\sin t\, dt\right]$$

$$I = e^{-\pi s} + 1 - s^2\int_0^{\pi} e^{-st}\cdot \sin t\, dt$$

$$I = e^{-\pi s} + 1 - s^2 I$$

$$I(1+s^2) = e^{-\pi s} + 1$$

$$I = \frac{e^{-\pi s}+1}{1+s^2} = \frac{e^{-\pi s}}{1+s^2} + \frac{1}{1+s^2} \quad \text{—(4)}$$

Applying Laplace to (1) & using (2), (3) & (4)

$$s^2 p - 4 + 9p = \frac{8e^{-\pi s}}{1+s^2} + \frac{8}{1+s^2}$$

$$(s^2+9)p = \frac{8e^{-\pi s}}{1+s^2} + \frac{8}{1+s^2} + 4$$

$$p = \frac{8e^{-\pi s}}{(1+s^2)(s^2+9)} + \frac{8}{(1+s^2)(9+s^2)} + \frac{4}{s^2+9}$$

$$= \frac{e^{-\pi s}}{s^2+1} - \frac{e^{-\pi s}}{s^2+9} + \frac{1}{s^2+1} - \frac{1}{s^2+9} + \frac{4}{s^2+9}$$

$$p = \frac{e^{-\pi s}}{s^2+1} - \frac{e^{-\pi s}}{s^2+9} + \frac{1}{s^2+1} + \frac{3}{s^2+9}$$

$$L^{-1}(p) = u(t-\pi)\sin(t-\pi) - \frac{1}{3}u(t-\pi)\sin 3(t-\pi) + \sin t + \sin 3t$$

$$\therefore\ y(x) = u(x-\pi)\sin(x-\pi) - \frac{1}{3}u(x-\pi)\sin 3(x-\pi) + \sin x + \sin 3x.$$

$$\therefore\ y(x) = \begin{cases} \sin x + \sin 3x & ;\ 0 < x < \pi \\ \frac{4}{3}\sin 3x & ;\ x \geq \pi \end{cases}$$

8(c) (i) Evaluate the integral : $\iint_S \vec{F}\cdot\hat{n}\,ds$; where $\vec{F} = 3xy^2\hat{i} + (yx^2 - y^3)\hat{j} + 3zx^2\hat{k}$ and 'S' is a surface of cylinder $y^2+z^2 \le 4$; $-3 \le x \le 3$; using divergence theorem.

Sol: Using Gauss divergence theorem.

$$\iiint \nabla\cdot f\, d\delta = \iint_S \vec{F}\cdot\hat{n}\,ds$$

Now, $\vec{F} = 3xy^2\hat{i} + (yx^2 - y^3)\hat{j} + 3zx^2\hat{k}$

$$\nabla\cdot\vec{F} = \cancel{3y^2} + x^2 - \cancel{3y^2} + 3x^2 = x^2 + 3x^2 = 4x^2$$

$$\therefore \iint F\cdot\hat{n}\,ds = \iiint \nabla\cdot F\, d\delta$$

$$I = \iiint 4x^2\,dx\,dy\,dz$$

converting to cylinder co-ordinates.

$y = r\cos\theta$; $z = r\sin\theta$; $x = x$

$$I = \int_{\theta=0}^{2\pi}\int_{r=0}^{2}\int_{-3}^{3} 4x^2\,dx\,dr\,d\theta\,[r]$$

$$I = \int_{\theta=0}^{2\pi}\int_{r=0}^{2}\left[\frac{4x^3}{3}\right]_{-3}^{3} = 72\int_0^{2\pi}\int_0^{2} r\,dr\,d\theta$$

$$I = 72\int_0^{2\pi}\left[\frac{r^2}{2}\right]_0^2 d\theta = 144\int_0^{2\pi} d\theta$$

$$\boxed{I = 288\pi}$$

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8(c)(ii) Using Green's theorem; evaluate the $\int_C F(\vec{r}).d\vec{r}$ Counter-clockwise; where $F(\vec{r}) = (x^2+y^2)\hat{i} + (x^2-y^2)\hat{j}$ and $d\vec{r} = dx\,\hat{i} + dy\,\hat{j}$ and the curve 'C' is the boundary of the region $R = \{(x,y) \mid 1 \leq y \leq 2-x^2\}$.

Sol: $$\int \vec{F}\cdot d\vec{r} = \int_C \left[(x^2+y^2)\hat{i} + (x^2-y^2)\hat{j}\right]\left[dx\,\hat{i} + dy\,\hat{j}\right]$$

$$I = \int_C (x^2+y^2)\,dx + (x^2-y^2)\,dy$$

$$I = \int_C P\,dx + Q\,dy \quad \text{— say (1)}$$

Using Green's theorem.

$$\int_C P\,dx + \int_D Q\,dy = \iint_S \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

$$\therefore \int_C F\cdot d\vec{r} = \int_{x=-1}^{1}\int_{y=1}^{2-x^2} (2x-2y)\,dx\,dy$$

$$= \int_{-1}^{1}\left[2xy - \frac{\cancel{2}y^2}{\cancel{2}}\right]_0^{2-x^2} dx$$

$$= \int_{-1}^{1} 2x(2-x^2) - (2-x)^2 - 2x + 1\,dx$$

$$= \int_{-1}^{1} -x^4 - \cancel{2x^3} + 4x^2 + \cancel{2x^3} - 3\,dx$$

$$= 2\left[\frac{-x^5}{5} + \frac{4x^3}{3} - 3x\right]_0^1.$$

$$\int \vec{F}\cdot d\vec{r} = 2\left[\frac{-1}{5} + \frac{4}{3} - 3\right] = -\frac{56}{15}$$

$$\boxed{\therefore \int \vec{F}\cdot d\vec{r} = \frac{-56}{15}}$$

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