Previous Years' Papers (Solved)

IFS MATHEMATICS EXAM., 2011

PAPER-I

Instructions : Candidates should attempt Questions No. 1 and 5 which are compulsory, and any THREE of the remaining questions, selecting at least ONE question from each Section. All questions carry equal marks. Marks allotted to parts of a question are indicated against each. Answers must be written in ENGLISH only. Assume suitable data, if considered necessary, and indicate the same clearly. Unless indicated otherwise, symbols & notations carry their usual meaning.

Section-A

1. Answer any *four* of the following:

(a) Let V be the vector space of 2×2 matrices over the field of real numbers R. Let $W = \{A \in V \mid \text{Trace } A = 0\}$. Show that W is a subspace of V. Find a basis of W and dimension of W. 10

(b) Find the linear transformation from R^3 into R^3 which has its range the subspace spanned by $(1, 0, -1)$, $(1, 2, 2)$. 10

(c) Show that the function defined by

$$f(x,y) = \begin{cases} \dfrac{x^3 + y^3}{x - y}, & x \neq y \\ 0, & x = y \end{cases}$$

is discontinuous at the origin but possesses partial derivatives f_x and f_y thereat. 10

(d) Let the function f be defined by

$$f(t) = \begin{cases} 0, & \text{for } t < 0 \\ t, & \text{for } 0 \leq t \leq 1 \\ 4, & \text{for } t > 1 \end{cases}$$

(*i*) Determine the function

$$F(x) = \int_0^x f(t)\, dt.$$

(*ii*) Where is F non-differentiable? Justify your answer. 10

(e) A variable plane is at a constant distance p from the origin and meets the axes at A, B, C. Prove that the locus of the centroid of the tetrahedron OABC is

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{16}{p^2}.$$ 10

2. (a) Let $V = \{(x, y, z, u) \in R^4 : y + z + u = 0\}$, $W = \{(x, y, z, u) \in R^4 : x + y = 0, z = 2u\}$ be two subspaces of R^4. Find bases for V, W, V + W and $V \cap W$. 10

(b) Find the characteristic polynomial of the matrix

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$$

and hence compute A^{10}. 10

(c) Let $A = \begin{pmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{pmatrix}$.

Find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

(d) Find an orthogonal transformation to reduce the quadratic form $5x^2 + 2y^2 + 4xy$ to a canonical form. 10

3. (a) Show that the equation $3^x + 4^x = 5^x$ has exactly one root. 8

(b) Test for convergence the integral

$$\int_0^\infty \sqrt{x e^{-x}}\, dx.$$ 8

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(c) Show that the area of the surface of the sphere $x^2 + y^2 + z^2 = a^2$ cut off by $x^2 + y^2 = ax$ is $2(\pi - 2)a^2$. 12

(d) Show that the function defined by

$$f(x, y, z) = 3 \log (x^2 + y^2 + z^2) - 2x^2 - 2y^3 - 2z^3, (x, y, z) \neq (0, 0, 0)$$

has only one extreme value, $\log\left(\frac{3}{e^2}\right)$. 12

4. (a) Find the equation of the right circular cylinder of radius 2 whose axis is the line $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-3}{2}$. 10

(b) Find the tangent planes to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ which are parallel to the plane $lx + my + nz = 0$. 10

(c) Prove that the semi-latus rectum of any conic is a harmonic mean between the segments of any focal chord. 8

(d) Tangent planes at two points P and Q of a paraboloid meet in the line RS. Show that the plane through RS and middle point of PQ is parallel to the axis of the paraboloid. 12

Section-B

5. Answer any ***four*** of the following:

(a) Find the family of curves whose tangents form an angle $\pi/4$ with hyperbolas $xy = c$. 10

(b) Solve :

$$\frac{d^2y}{dx^2} - 2\tan x \frac{dy}{dx} + 5y = \sec x \cdot e^x.$$ 10

(c) The apses of a satellite of the Earth are at distances r_1 and r_2 from the centre of the Earth. Find the velocities at the apses in terms of r_1 and r_2. 10

(d) A cable of length 160 meters and weighing 2 kg per meter is suspended from two points in the same horizontal plane. The tension at the points of support is 200 kg. Show that the span of the cable is $120 \cosh^{-1}\left(\frac{5}{3}\right)$ and also find the sag. 10

(e) Evaluate the line integral $\oint_C (\sin x\, dx + y^2 dy - dz)$, where C is the circle $x^2 + y^2 = 16$, $z = 3$, by using Stokes' theorem. 10

6. (a) Solve :

$$p^2 + 2\, py \cot x = y^2,$$

where $p = \frac{dy}{dx}$. 10

(b) Solve :

$$\{x^4D^4 + 6x^3D^3 + 9x^2D^2 + 3xD + 1\}y = (1 + \log x)^2,$$

where $D \equiv \frac{d}{dx}$. 15

(c) Solve :

$$(D^4 + D^2 + 1)y = ax^2 + be^{-x} \sin 2x,$$

where $D \equiv \frac{d}{dx}$. 15

7. (a) One end of a uniform rod AB, of length $2a$ and weight W, is attached by a frictionless joint to a smooth wall and the other end B is smoothly hinged to an equal rod BC. The middle points of the rods are connected by an elastic cord of natural length a and modulus of elasticity 4W. Prove that the system can rest in equilibrium in a vertical plane C in contact with the wall below A, and the angle between the rod is $2\sin^{-1}\left(\frac{3}{4}\right)$. 13

(b) AB is a uniform rod, of length $8a$, which can turn freely about the end A, which is fixed. C is a smooth ring, whose weight is twice that of the rod, which can slide on the rod, and is attached by a string CD to a point D in the same horizontal plane as the point A. If AD and CD are each of length a, fix the position of the ring and the tension of the string when the system is in equilibrium.

Show also that the action on the rod at the fixed end A is a horizontal force equal to $\sqrt{3}$ W, where W is the weight of the rod. 14

(c) A stream is rushing from a boiler through a conical pipe, the diameter of the ends of which are D and d; if V and v be the corresponding velocities of the stream and if the motion is supposed to be that of the divergence from the vertex of cone, prove that

$$\frac{v}{V}=\frac{D^2}{d^2}e^{\left(v^2-V^2\right)/2K}$$

where K is the pressure divided by the density and supposed constant. 13

8. (a) Find the curvature, torsion and the relation between the arc length S and parameter u for the curve :

$$\vec{r}=\vec{r}(u)=2\log_e u\hat{i}+4u\hat{j}+(2u^2+1)\hat{k} \quad 10$$

(b) Prove the vector identity :

curl $(\vec{f}\times\vec{g})=\vec{f}\operatorname{div}\vec{g}-\vec{g}\operatorname{div}\vec{f}+(\vec{g}\cdot\nabla)\vec{f}-(\vec{f}\cdot\nabla)\vec{g}$

and verify it for the vectors $\vec{f}=x\hat{i}+z\hat{j}+y\hat{k}$ and $\vec{g}=y\hat{i}+z\hat{k}$. 10

(c) Verify Green's theorem in the plane for

$$\oint_C[(3x^2-8y^2)dx+(4y-6xy)dy],$$

where C is the boundary of the region enclosed by the curves $y=\sqrt{x}$ and $y=x^2$. 10

(d) The position vector $\vec{r}$ of a particle of mass 2 units at any time t, referred to fixed origin and axes, is

$$\vec{r}=(t^2-2t)\hat{i}+\left(\frac{1}{2}t^2+1\right)\hat{j}+\frac{1}{2}t^2\hat{k}.$$

At time $t=1$, find its kinetic energy, angular momentum, time rate of change of angular momentum and the moment of the resultant force, acting at the particle, about the origin. 10

PAPER-II

Instructions : *Candidates should attempt Question Nos. 1 and 5 which are compulsory, and any THREE of the remaining questions, selecting at least ONE question from each Section. All questions carry equal marks. The number of marks carried by each part of a question is indicated against each. Answers must be written in ENGLISH only. Assume suitable data, if considered necessary, and indicate the same clearly. Symbols and notations have their usual meanings, unless indicated otherwise.*

Section-A

1. Answer any ***four*** parts from the following:

(a) Let G be a group, and x and y be any two elements of G. If $y^5=e$ and $yxy^{-1}=x^2$, then show that $O(x)=31$, where e is the identity element of G and $x\neq e$. 10

(b) Let Q b

Show t

is a fie

multiplication.

(c) Determine whether

$$f(x)=2x\sin\frac{1}{x}-\cos\frac{1}{x}$$

is Riemann-integrable on [0, 1] and justify your answer. 10

(d) Expand the function

$$f(z)=\frac{2z^2+11z}{(z+1)(z+4)}$$

in a Laurent's series valid for $2<|z|<3$. 10

ANSWERS

PAPER-I

Section-A

1. (a): Given, v is a vector space of 2×2 matrices over R

i.e., $$V = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in R\right\}$$

And W is a subset of V such that Trace $(A) = 0$ when $A \in W$

Clearly $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in W.$

i.e., W is not empty.

Now, let $A_1, A_2 \in W$

then, Trace $(A_1) = 0$

and, Trace $(A_2) = 0$

then, $\operatorname{tr}(xA_1 + yA_2) = x\operatorname{Tr}(A_1) + y\operatorname{Tr}(A_2)$

$= x.0 + y.0 = 0$

i.e., $xA_1 + yA_2 = 0$

$\Rightarrow$ $xA_1 + yA_2 \in W$

i.e., W is a subspace of V.

Part II

if $\begin{bmatrix} x & y \\ z & w \end{bmatrix} \in W$

then $x + w = 0$

i.e., it can have at maximum three free variables.

Hence, dimension of $W = 4 - 1 = 3$

and the basis of W are $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

and $\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

i.e., $$W = \left\{\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right\}.$$

(b): Let T be the required linear transformation such that the range of it is spanned by $(1, 0, -1), (1, 2, 2)$.

As $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ are the standard basis of R_3.

Hence, we can assume

$T(1, 0, 0) = (1, 0, -1)$

$T(0, 1, 0) = (1, 2, 2)$

and $T(0, 0, 1) = (0, 0, 0)$

Also, $(x, y, z) = x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1)$

$\Rightarrow$ $T(x, y, z) = xT(1, 0, 0) + yT(0, 1, 0) + zT(0, 0, 1)$

$= x(1, 0, -1) + y(1, 2, 2) + z(0, 0, 0)$

$= (x + y, 2y, -x + 2y)$

i.e., $T(x, y, z) = (x + y, 2y, -x, + 2y)$

is the required transformation.

(c): The given function is

$$f(x, y) = \frac{x^3 + y^3}{x - y} \qquad x \neq y$$

$$= 0 \qquad x = y$$

The above function is continuous at origin if it is equal to $\lim\limits_{(x,y)\to(0,0)} f(x, y) = f(0, 0) = 0.$

Whatever the path taken by the function to approach origin.

Now $\lim\limits_{(x,y)\to(0,0)} f(x, y) = \lim\limits_{(x,y)\to(0,0)} \dfrac{x^3 + y^3}{x - y}$

let $y = x - mx^3$ be the path through which this (x, y) approaches to origin, then,

clearly $y \to 0$

when $x \to 0$

then

$$\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{x\to 0} \frac{x^3 + (x - mx^3)^3}{x - x + mx^3}$$

$$= \lim_{x\to 0} \frac{x^3 + x^3 - 3x^2 \cdot mx^3 + 3x \cdot m^2x^6 - m^3x^9}{mx^3}$$

$$= \lim_{x\to 0} \frac{2x^3 - 3mx^5 + 3m^2x^7 - m^3x^9}{mx^3}$$

$$= \frac{2}{m}$$

i.e., it approaches to different values depending on the value of m.

i.e., the function is discontinuous at the origin.

Again, $f(x, y) = \dfrac{x^3 + y^3}{x - y}$

$$f_x(0, 0) = \lim_{h\to 0} \frac{f(0+h, 0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{h^3/h - 0}{h} = 0$$

and $$f_y(0, 0) = \lim_{k\to 0} \frac{f(0, 0+k) - f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{\frac{k^3}{-k} - 0}{k} = 0$$

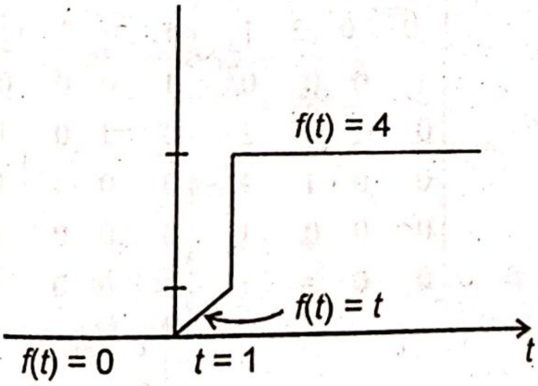
i.e., $f_x(0, 0)$ and $f_y(0, 0)$ exist at origin.

1.(d): The given function f is defined as

$f(t) = 0$, for $t < 0$

$= t$, for $0 \le t \le 1$

$= 4$, for $t > 1$



Now we have to calculate $F(x) = \displaystyle\int_0^x f(t)dt$

Case I When $0 < x \le 1$

then $$F(x) = \int_0^x f(t)dt = \int_0^x t\,dt = \frac{x^2}{2}$$

i.e., $$F(x) = \frac{x^2}{2} \quad 0 < x \le 1$$

Case II $x > 1$

then $$F(x) = \int_0^x f(t)dt$$

$$= \int_0^1 f(t)dt + \int_1^x f(t)dt$$

$$= \int_0^1 t\,dt + \int_1^x 4\,dt$$

$$= \frac{1}{2} + 4(x-1) = 4x - \frac{T}{2}$$

i.e., $$F(x) = 4x - \frac{T}{2} \quad x > 1$$

i.e., $$F(x) = \frac{x^2}{2} \qquad 0 < x \le 1$$

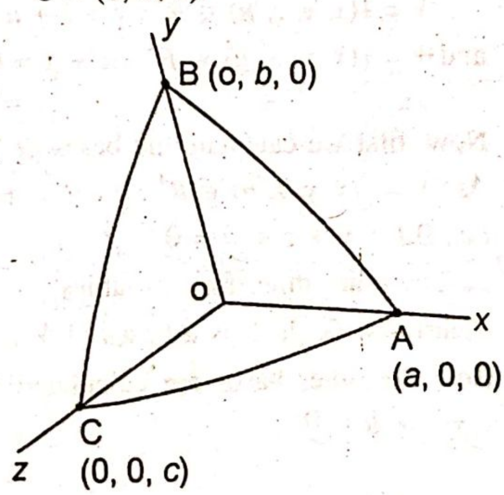
$$= 4x - \frac{T}{2} \qquad x > 1.$$

Clearly the function $F(x)$ is not differentiable at $x = 1$.

Infact it is not even continuous at this point so the point of differentiably does not arise at all.

This is due to the nature of integrand of $F(x)$, which itself is discontinuous at the point $t = 1$.

1.e) Let the required plane cut the axes at A, B, C such that $A = (a, 0, 0)$, $B = (0, b, 0)$ and $C = (0, 0, c)$.



Then the equation of this plane is given by

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1 \quad ...(1)$$

Now from question,
The length of perpendicular to this plane from origin is p

then, $$\frac{|0+0+0-1|}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}} = p$$

or, $$\frac{1}{p^2} = \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \quad ...(2)$$

Again, let (α, β, γ) be the centroid of the tetrahedron.

then, $$\alpha = \frac{0+a+0+0}{4},$$

$$\beta = \frac{0+0+b+0}{4},$$

$$\gamma = \frac{0+0+0+c}{4}$$

or, $a = 4\alpha, \quad b = 4\beta, \quad c = 4\gamma$

putting a, b, c is equation (2), we get

$$\frac{1}{p^2} = \frac{1}{16\alpha^2}+\frac{1}{16\beta^2}+\frac{1}{16\gamma^2}$$

or, $$\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2} = \frac{16}{p^2}$$

Hence, locus of (α, β, γ) is given by

$$\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2} = \frac{16}{p^2}$$ **Proved.**

2 (a): The given subspace of R^4 are

$V = \{(x, y, z, u) \in R^4 : y + z + u = 0\}$

and $W = \{(x, y, z, u) \in R^4 : x + y = 0, z = 2u\}$.

Now, first we calculate the bases of V.

As $V = \{(x, y, z, u) \in R^4 : y + z + u = 0\}$

i.e., $0.x + y + z + u = 0$

$\Rightarrow$ There are three free variables

Clearly $(1, 0, 0, 0)$ is a bases of V.

Now the other bases are calculated from $y + z + u = 0$ by independently choosing two free variables then compute the other variables satisfying the above condition.

let $z = 0, \quad u = 1$

then, $y = -1$

i.e., $(0, -1, 0, 1)$ is one of the bases.

let $z = 1, u = 0$

then, $y = -1$

i.e. $(0, -1, 1, 0)$ is one of the bases.

i.e., the bases of V are given by $\{(1, 0, 0, 0), (0, -1, 0, 1), (0, -1, 1, 0)\}$

To calculate the bases of W.

as $W = \{x, y, z, u) \in R^4 : x + y = 0, z = 2u\}$

i.e., there will be two free variables.

Choosing y and u be two free variables.

Now if $y = 1, \quad u = 0$

then, $x = -1, \quad z = 0$

and $y = 0, \quad u = 1$

then, $x = 0, \quad z = 2$

i.e., $W = \{(-1, 1, 0, 0), (0, 0, 2, 1)\}$

Now the bases of $V + W$ are given by the number of independent rows in the matrix formed by the bases of V and W in the form of row vector

i.e., $$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix} \sim$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

i.e., No. of independent rows in above matrix = 4.

i.e., dimension of $V + W = 4$.

i.e., the bases of $V + W$ are given by
$\{(1, 0, 0, 0), (0, -1, 0, 1), (0, 0, 1, -1), (0, 0, 0, 1)\}$

Now we calculate the bases of $V \cap W$.

Clearly it should satisfy $y + z + u = 0$, $x + y = 0$, $z = 2u$.

i.e., there is only one free variable in this subspace.

Choose $u = 1$ is the free variable

then, $z = 2$, $y = -3$, $x = 3$

i.e., $\{(3, -3, 2, 1)\}$ is the basis of $V \cap W$.

And the dimension of this subspace – 1.

(b): The given matrix A is $\begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{bmatrix}$.

Then, the characteristic equation of this polynomial is given by $|A - \lambda I| = 0$.

i.e., $\begin{vmatrix} 3-\lambda & 1 & 1 \\ 2 & 4-\lambda & 2 \\ -1 & -1 & 1-\lambda \end{vmatrix} = 0$

or $(3 - \lambda)\{(4 - \lambda)(1 - \lambda) + 2\} - 1\} 2(1 - \lambda) + 2\} + 1\{-2 + 4 - \lambda\} = 0$

or, $16 - 20\lambda + 8\lambda^2 - \lambda^3 = 0$

or, $\lambda^3 - 8\lambda^2 + 20\lambda - 16 = 0$

Hence, by Cayley-Hamilton Theorem

it should be satisfy the by matrix A.

i.e., $A^3 - 8A^2 + 20A - 16I = 0$

or, the characteristic polynomial is given by

$$A^3 - 8A^2 + 20A - 16I = 0$$

from the given expression, it is difficult to calculate A^{10}.

(c): The given matrix A is $\begin{bmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{bmatrix}$.

The characteristic equation of this matrix is given by

$$|A - \lambda| = 0$$

or, $\begin{vmatrix} 1-\lambda & -3 & 3 \\ 0 & -5-\lambda & 6 \\ 0 & -3 & 4-\lambda \end{vmatrix} = 0$

or, $(1 - \lambda)\{(\lambda + 5)(\lambda - 4) + 18\} = 0$

or, $(1 - \lambda)\{\lambda^2 + \lambda - 20 + 18\} = 0$

$\Rightarrow$ $(1 - \lambda)(\lambda^2 + \lambda - 2) = 0$

$(1 - \lambda)(\lambda^2 + 2\lambda - \lambda - 2) = 0$

$\Rightarrow$ $\lambda = 1, 1, -2$

i.e., $\lambda = 1, 1, -2$ are the eigen values of the matrix A.

Now, for $\lambda = 1$, the eigen vector is given by

$$[A - I][X] = 0$$

where $[X] = [x\ y\ z]^T$

or, $\begin{bmatrix} 0 & -3 & 3 \\ 0 & -6 & 6 \\ 0 & -3 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

or, $\begin{bmatrix} 0 & -3 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

i.e. $-3y + 3z = 0$

or, $-y + z = 0$.

Clearly, this will possess two eigen vectors as there are two free variables satisfying the above condition.

Hence, the eigen vectors corresponding to $\lambda = 1$ is given by,

$[1\ 0\ 0]^T$ and $[0\ 1\ 1]^T$

for $\lambda = -2$, the eigen vector is given by

$$[A + 2I][X] = 0$$

or, $\begin{bmatrix} 3 & -3 & 3 \\ 0 & -3 & 6 \\ 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\Rightarrow$ $x - y + z = 0$

$-y + 2z = 0$

i.e. It'll possesses only one free variable.

Choose $z = 1$ as the free variable then.

$y = 2$

and $x = 1$

i.e., $[1\ 2\ 1]^T$ is the required eigen vector.

Hence, the invertible matrix (P) is given by

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}^T$$

It will reduce the matrix A to a diagonal matrix by operation $P^{-1}AP = D$ where

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Verification

As $$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$\Rightarrow$ $|\det P| = -1$

Now $$P^{-1} = \frac{\operatorname{Adj} P}{|P|} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

Now $$P^{-1}AP = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -4 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Hence the result.

(d): The given quadratic form is

$5x^2 + 2y^2 + 4xy$

its associated matrix A can be written as

$$\begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Apply congruent operation

$R_2 \to R_2 - \frac{2}{5}R_1$ and $C_2 \to C_2 - \frac{2}{5}C_1$.

We get

$$\begin{bmatrix} 5 & 0 \\ 0 & \frac{6}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{2}{5} & 1 \end{bmatrix} A \begin{bmatrix} 1 & -\frac{2}{5} \\ 0 & 1 \end{bmatrix}$$

Now apply $R_1 \to R_1 \cdot \frac{1}{\sqrt{5}}$ and $C_1 = \frac{1}{\sqrt{5}} \cdot C_1$

we get

$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{6}{5} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} & 0 \\ -\frac{2}{5} & 1 \end{bmatrix} A \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ 0 & 1 \end{bmatrix}$$

Apply $R_2 \to \sqrt{\frac{5}{6}}\, R_2$ and $C_2 \to \sqrt{\frac{5}{6}}\, C_2$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} & 0 \\ -\sqrt{\frac{2}{15}} & \sqrt{\frac{5}{6}} \end{bmatrix} A \begin{bmatrix} \frac{1}{\sqrt{5}} & -\sqrt{\frac{2}{15}} \\ 0 & \sqrt{\frac{5}{6}} \end{bmatrix}$$

Hence the orthogonal transformation is

$$\begin{bmatrix} \frac{1}{\sqrt{5}} & -\sqrt{\frac{2}{15}} \\ 0 & \sqrt{\frac{5}{6}} \end{bmatrix}.$$

3. (a): The given equation is

$$3^x + 4^x = 5^x \quad \text{...(1)}$$

Dividing both the sides by 5^x, we get

$$\left(\frac{3}{5}\right)^x + \left(\frac{4}{5}\right)^x = 1 \quad \text{...(2)}$$

let $\sin\theta = \frac{3}{5}$ then $\cos\theta = \frac{4}{5}$

hence the equation (2) is reduced to

$(\sin\theta)^x + (\cos\theta)^x = 1$

which is true for $x = 2$.

i.e. $x = 2$ is the only root of equation (1) infact, this is a well-known theorem known as Fermas theorem.

It states that

$$a^n + b^n \neq c^n \quad \text{for} \quad n > 2$$

where $a, b, c \in z$ and $n \in N$.

(b): The given is $\int_0^\infty \sqrt{xe^{-x}}\,dx$

$$= \int_0^\infty \sqrt{x}e^{-x/2}\,dx$$

let $y = \frac{x}{2}$ then $x = 2y$ and $dx = 2dy$

$$= \int \sqrt{2}ye^{-y}2dy = 2\sqrt{2}\int_0^\infty \sqrt{y}e^{-y}dy$$

let $f(y) = \sqrt{y}e^{-y} = \frac{e^{-y}}{y^{-1/2}}$

Clearly the function has an infinite discontinuity at $y = 0$.

Hence we've to examine the convergence at 0 and ∞ both consider, ∞

$$\int_0^\infty \sqrt{y}e^{-y}dy = \int_0^1 \sqrt{y}e^{-y}dy + \int_0^\infty \sqrt{y}e^{-y}dy$$

We test the two integrals on the right for convergence at 0 and ∞ respectively.

Convergence at 0

let $\qquad g(y) = \sqrt{y}$

such that

$$\lim_{y\to 0}\frac{f(y)}{g(y)} = e^{-y} \to 1 \quad \text{as } x \to 0$$

However $\int_0^1 g(y)\,dy = \left.\frac{y^{3/2}}{32}\right|_0^1$ converges

$$\Rightarrow \int_0^1 \sqrt{y}e^{-y}dy \text{ coverges} \qquad ...(1)$$

Convergence at ∞

let $\qquad g(y) = \frac{1}{y^2}$

then $\lim_{y\to\infty}\frac{f(y)}{g(y)} = \lim_{y\to\infty}\frac{\sqrt{y}e^{-y}}{1/y^2}$

$$= \lim_{y\to\infty}\frac{y^{5/2}}{e^y} \to 0 \text{ as } y \to \infty$$

As $\int_1^\infty g(y)\,dy$ converges if $g(y) = \frac{1}{y^2}$

i.e., $\int_1^\infty \sqrt{y}e^{-y}dy$ converges $\qquad ...(2)$

from (1) and (2)

$\int_0^\infty \sqrt{y}e^{-y}dy$ converges

infact the given integral $2\sqrt{2}\int_0^\infty \sqrt{y}e^{-y}dy$

$$= 2\sqrt{2}\int_0^\infty y^{\frac{3}{2}-1}e^{-y}dy = 2\sqrt{2}\,\Gamma\!\left(\frac{3}{2}\right)$$

$$= 2\sqrt{2}\cdot\frac{1}{2}\cdot\sqrt{\pi} = \sqrt{2\pi}.$$

(c): The given sphere is $x^2 + y^2 + z^2 = a^2$

$$\therefore \quad \frac{\partial z}{\partial x} = -\frac{x}{z},$$

$$\frac{\partial z}{\partial y} = -\frac{y}{z}$$

or $\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2} = \frac{1}{z}\sqrt{x^2+y^2+z^2}$

$$= \frac{a}{\sqrt{a^2-x^2-y^2}}$$

Now the surface of area is

$$\iint ds = \iint\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dx\,dy$$

$$= 4\iint\frac{a}{\sqrt{a^2-x^2-y^2}}\,dx\,dy$$

over half the circle $x^2 + y^2 = ax$

let $\qquad x = r\cos\theta, \quad y = r\sin\theta$

then $x^2 + y^2 = ax$

becomes $\quad r = a\cos\theta$

and $\quad dxdy = rd\theta dr$

$$\therefore \quad S = 4a \int_0^{\pi/2} \int_0^{a\cos\theta} \frac{r d\theta dr}{a^2 - r^2}$$

$$= 4a \int_0^{\pi/2} \left[-\sqrt{a^2 - r^2}\right]^{a\cos\theta} d\theta$$

$$= 4a \int_0^{\pi/2} (1 - \sin\theta) d\theta$$

$$= 4a^2 [\theta + \cos\theta]_0^{\pi/2}$$

$$= 4a^2 \left[\left(\frac{\pi}{2} + 0\right) - (0 + 1)\right]$$

$$= 2a^2 \ (\pi - 2) \text{ units}$$

i.e., $S = 2(\pi - 2)a^2$ units **Proved.**

(d): The given function is

$$f(x, y, z) = 3\log\,(x^2 + y^2 + z^2) - 2x^3 - 2y^3 - 2z^3$$

$$(x, y, z) \neq (0, 0, 0)$$

for extremum value

$$f_x = f_y = f_z = 0$$

Now, $$f_x = 3 \cdot \frac{2x}{x^2 + y^2 + z^2} - 6x^2$$

$$= \frac{6x}{x^2 + y^2 + z^2} - 6x^2 = 0$$

$$\Rightarrow \frac{6x[1 - x(x^2 + y^2 + z^2)]}{(x^2 + y^2 + z^2)} = 0$$

as, $(x, y, z) \neq (0, 0, 0)$

$$\Rightarrow \quad 1 - x\,(x^2 + y^2 + z^2) = 0$$

$$x(x^2 + y^2 + z^2) = 1 \qquad \text{...(1)}$$

Similarly, $f_y = 0$

$$\Rightarrow \quad y(x^2 + y^2 + z^2) = 1 \qquad \text{...(2)}$$

and $f_z = 0$

$$\Rightarrow \quad z(x^2 + y^2 + z^2) = 1 \qquad \text{...(3)}$$

from equations (1), (2) and (3)

we get $x = y = z$

i.e., $x(x^2 + x^2 + x^2) = 1$

or, $3x^3 = 1$

$$\Rightarrow \quad x = \left(\frac{1}{3}\right)^{\frac{1}{3}} = \frac{1}{3^{1/3}}$$

i.e., $x = y = z = \dfrac{1}{3^{1/3}}$

Hence, the value of $f(x, y, z)$ at the point $\left(\frac{1}{3^{1/3}}, \frac{1}{3^{1/3}}, \frac{1}{3^{1/3}}\right)$ is given by

$$f\left(\frac{1}{3^{1/3}}, \frac{1}{3^{1/3}}, \frac{1}{3^{1/3}}\right)$$

$$= 3\log\left(\frac{1}{3^{2/3}} + \frac{1}{3^{2/3}} + \frac{1}{3^{2/3}}\right) - 2\left(\frac{1}{3} + \frac{1}{3} + \frac{1}{3}\right)$$

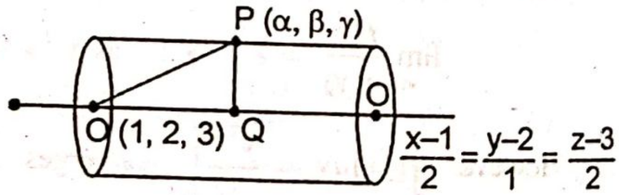
$$= 3\log\left(\frac{3}{3^{2/3}}\right) - 2$$

$$= 3\log 3^{1/3} - 2 = \frac{3}{3}\log 3 - 2$$

$$= \log 3 - 2 = \log\left(\frac{3}{e^2}\right)$$

i.e., the only extreme value of $f(x, y, z)$ is $\log\left(\dfrac{3}{e^2}\right)$.

4. (a): Let OO' be the axis of the right circular cylinder which has radius 2.



From question,

The equation of the line OO' is

$$\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-3}{2}$$

clearly it passes through (1, 2, 3)

let $O \equiv (1, 2, 3)$

Now, let P (α, β, γ) be a point which lies on the cylinder, then from the figure, the projection of OP on the line OO' is given by

$$(\alpha-1)\frac{2}{3}+(\beta-2).\frac{1}{3}+(\gamma-3).\frac{2}{3}$$

Now in right angled triangle OPQ.

We have, $OP^2 = PQ^2 + OQ^2$

$$\Rightarrow (\alpha-1)^2 + (\beta-2)^2 + (\gamma-3)^2 = 2^2 + \frac{1}{9}$$
$$[2(\alpha-1) + (\beta-2) + (\gamma-3)]^2$$
$$\Rightarrow 9[(\alpha-1)^2 + (\beta-2)^2 + (\gamma-3)^2] = 36 + [2\alpha + \beta + 2\gamma - 10]^2$$
$$\Rightarrow 9[\alpha^2 + \beta^2 + \gamma^2 + 14 - 2\alpha - 4\beta - 6\gamma]$$
$$= 36 + [4\alpha^2 + \beta^2 + 4\gamma^2 + 100 + 4\alpha\beta + 4\beta\gamma + 8\alpha\gamma - 40\alpha - 20\beta - 40\gamma]$$
$$\Rightarrow 5\alpha^2 + 8\beta^2 + 5\gamma^2 - 4\alpha\beta - 8\alpha\gamma + 2\alpha - 16\beta + 4\gamma - 10 = 0$$

Hence, equation of right circular cylinder is given by the locus (α, β, γ).

i.e., $5x^2 + 8y^2 + 5z^2 - 4xy - 4yz - 8xz + 22x - 16y + 4z - 10 = 0.$

(b): The equation of the given ellipsoid is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad ...(1)$$

The equation of the given plane is

$$lx + my + nz = 0 \qquad ...(2)$$

The equation of plane parallel to (2) can be written in the form of

$$lx + my + nz = p \qquad ...(3)$$

Now, if (3) represent a plane which is tangent to ellipsoid (1), then f

$$p^2 = a^2l^2 + b^2m^2 + c^2n^2$$

<using standard result>

Hence $\Rightarrow$ $p = \pm\sqrt{a^2l^2+b^2m^2+c^2n^2}$

Hence, the equation of the required plane is given by

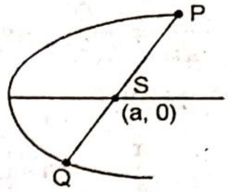
$$lx + my + nz = \pm\sqrt{a^2l^2+b^2m^2+c^2n^2}\,.$$

(c): Consider a parabola <conic> whose equation is

$$y^2 = 4ax \qquad ...(1)$$

then the length of semi-latus rectum = $2a$.

Let $S \equiv (a, 0)$ be the focus of this parabola and PSQ be any focal chord of this parabola,



let $\qquad P \equiv (at^2, 2at)$

then, $\qquad Q \equiv \left(\frac{a}{t^2}, \frac{-2a}{t}\right)$

Now, $\qquad SP^2 = a^2(1-t^2)^2 + 4a^2t^2$
$$= a^2(1+t^2)^2$$
$\Rightarrow \qquad SP = a(1+t^2)$

Similarly, $SQ = a\left(1+\frac{1}{t^2}\right)$

Now, the harmonic mean of SP and SQ is given by

$$\frac{2\cdot SP\cdot SQ}{SP+SQ} = \frac{2a^2(1+t^2)\left(1+\frac{1}{t^2}\right)}{a\left[1+t^2+1+\frac{1}{t^2}\right]}$$
$$= \frac{2a\left[1+t^2+\frac{1}{t^2}+1\right]}{\left[2+t^2+\frac{1}{t^2}\right]}$$
$$= \frac{2a\left[2+t^2+\frac{1}{t^2}\right]}{\left[2+t^2+\frac{1}{t^2}\right]} = 2a$$

which is equal to semi-latus rectum.

<Hence the result>

4(d) ?

Section-B

5. (a): Let the family of required curve is $y = f(x)$

The tangent at a general point $P(x, y)$ is

$$\frac{dy}{dx} = m_1(\text{say})$$

DATE

4(d) Tangent planes at two points P and Q of a paraboloid meet in the line RS. Show that the plane through RS and middle point of PQ is parallel to the axis of the paraboloid. (12)

Let standard eqn of parabola be

$$2cz = ax^2 + by^2 \quad —(*)$$

and Points $P(x_1, y_1, z_1)$ & $Q(x_2, y_2, z_2)$

Tangent Planes at P and Q

$$c(z+z_1) = ax_1x + by_1y \quad —(1)$$

$$c(z+z_2) = ax_2x + by_2y$$

Hence, Equation of plane passing through line of intersection of (1) & (2) is

$$\left[(ax_1)x + (by_1)y - cz - cz_1\right] + \lambda\left[(ax_2)x + (by_2)y - cz - cz_2\right] = 0 \quad —(3)$$

Middle point of PQ, $M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{z_1+z_2}{2}\right)$ lies on the above plane.

Hence, we obtain the value of λ as

$$\lambda = \frac{-\left[ax_1\left(\frac{x_1+x_2}{2}\right) + by_1\left(\frac{y_1+y_2}{2}\right) - c\left(\frac{z_1+z_2}{2}\right) - cz_1\right]}{\left[ax_2\left(\frac{x_1+x_2}{2}\right) + by_2\left(\frac{y_1+y_2}{2}\right) - c\left(\frac{z_1+z_2}{2}\right) - cz_2\right]}$$

$$= \frac{-\left[ax_1x_2 + by_1y_2 - c(z_1+z_2)\right]}{\left[ax_2x_1 + by_2y_1 - c(z_1+z_2)\right]} = -1$$

$\left[\because P \text{ \& } Q \text{ lies on Paraboloid}\right.$

$\therefore$

$$ax_1^2 + by_1^2 - 2cz_1 = 0$$

$$\left. ax_2^2 + by_2^2 - 2cz_2 = 0 \right]$$

Hence, eqn of plane (from (3))

$$a(x_1+x_2)x + b(y_1+y_2)y = c(z_1+z_2)$$

D.R. of Normal of this plane are

$$\langle a(x_1+x_2), b(y_1+y_2), 0 \rangle$$

Axis of the paraboloid (*) is z-axis

D.R. of z-axis are $\langle 0, 0, 1 \rangle$

$$\therefore a(x_1-x_2)\times 0 + b(y_1-y_2)\times 0 + 0\times 1 = 0$$

Therefore, the above plane is parallel to axis of paraboloid.

5.(a). $\left(\frac{dy}{dx}\right) = m_1$ (say)

Also the given curve is $xy = C$

$$\Rightarrow \quad y = \frac{C}{x}$$

$$\text{or} \quad \frac{dy}{dx} = \frac{-C}{x^2} = m_2(\text{say})$$

From the question,

$$\frac{\pi}{4} = \tan^{-1}\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right)$$

$$\text{or} \quad \tan\frac{\pi}{4} = \frac{\frac{dy}{dx} + \frac{C}{x^2}}{1 - \frac{dy}{dx}\cdot\frac{C}{x^2}}$$

$$\Rightarrow \quad 1 - \frac{dy}{dx}\cdot\frac{C}{x^2} = \frac{dy}{dx} + \frac{C}{x^2}$$

$$\Rightarrow \quad \frac{dy}{dx}\left[1 + \frac{C}{x^2}\right] = 1 - \frac{C}{x^2}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{x^2 - C}{x^2 + C}$$

$$= \frac{x^2 + C - 2C}{x^2 + C}$$

$$= 1 - \frac{2C}{x^2 + C}$$

$$\therefore \quad dy = \left(1 - \frac{2C}{x^2 + C}\right)dx$$

integrating both the sides, we get

$$y = x - \frac{2C}{\sqrt{C}}\tan^{-1}\frac{x}{\sqrt{C}} + C^1$$

(C^1 = integration constants)

$$\text{or } y = x - 2\sqrt{C}\tan^{-1}\frac{x}{\sqrt{C}} + C^1$$

is the required family of curves.

(b): $$\frac{d^2y}{dx^2} - 2\tan x\frac{dy}{dx} + 5y = \sec x.e^x \quad ...(1)$$

This is the general form of second order differential equation. It can be solved by reducing it into normal form (i.e., by removal first order derivative).

Comparing with the general form, 2nd order differential equation $\frac{d^2y}{dx^2} + \frac{Pdy}{dx} + Qy = R$.

We get $P = -2\tan x$, $Q = 5$ and $R = e^x\sec x$.

Now, the first order derivative can be removed if u is solution of C.F. of the equation (1), where u is given by

$$P + \frac{2}{u}\cdot\frac{dy}{dx} = 0$$

$$i.e. \quad -2\tan x + \frac{2}{u}\frac{du}{dx} = 0$$

$$\text{or} \quad \frac{du}{u} = \tan x\, dx$$

to integrating both sides, we get $u = \sec x$.

Now assume $y = uv$ be the complete solution of (1), then the equation (1) can be reduced to

$$\frac{d^2v}{dx^2} + Iv = S \quad ...(2)$$

$$\text{where} \quad I = Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2$$

$$\text{and} \quad S = \frac{R}{u}$$

$$\text{Hence,} \quad I = 5 - \frac{1}{2}(-2\sec^2 x) - \frac{1}{4}.4\tan^2 x$$

$$= 6$$

$$\text{and} \quad S = \frac{R}{u} = \frac{e^x \cdot \sec x}{\sec x} = e^x$$

$$i.e. \quad \frac{d^2v}{dx^2} + 6v = e^x \quad ...(3)$$

the complementary function of (3) is given by

$$\text{C.F.} = C_1\sin\sqrt{6}x + C_2\cos\sqrt{6}x$$

and the Particular Integral is given by,

$$v = \frac{1}{(D^2 + 6)}e^x = \frac{e^x}{1+6} = \frac{e^x}{7}$$

i.e. $v = \text{C.F.} + \text{P.I.}$

$$= C_1 \sin\sqrt{6}x + C_2\cos\sqrt{6}x + \frac{e^x}{7}$$

Hence the complete solution of (1) is given by.

$$y = uv$$

$$= \sec x\left[C_1 \sin\sqrt{6}x + C_2\cos\sqrt{6}x + \frac{e^x}{7}\right]$$

$$= C_1 \sec x.\sin\sqrt{6}x + C_2\sec x\cos\sqrt{6}x + \frac{e^x.\sec x}{7}.$$

(c): Let the satellite of the Earth moves under the inverse square law $= \frac{\mu}{r^2}$.

Clearly the satellite will move in elliptical orbit and the velocity at a distance r is given by

$$v^2 = \mu\left[\frac{2}{r} - \frac{1}{a}\right]$$

where $2a$ = major axis of elliptical orbit

Now at apse

$$r_1 = a + ae$$

and $r_2 = a - ae \quad \Rightarrow 2a = r_1 + r_2$

Now from question at $r = r_1$, $v = v_1$

hence $$v_1^2 = \mu\left[\frac{2}{r_1} - \frac{1}{a}\right]$$

$$= \mu\left[\frac{2}{r_1} - \frac{2}{r_1 + r_2}\right]$$

or $$v_1^2 = 2\mu\left[\frac{1}{r_1} - \frac{1}{r_1 + r_2}\right]$$

$$= 2\mu\left[\frac{r_1 + r_2 - r_1}{r_1(r_1 + r_2)}\right]$$

or $$v_1^2 = \frac{2\mu r_2}{r_1 + r_2}$$

$$\Rightarrow \quad v_1 = \sqrt{\frac{2\mu r_2}{r_1(r_1 + r_2)}}$$

5(d)? Similarly, $v_2 = \sqrt{\dfrac{2\mu r_1}{r_2(r_1 + r_2)}}$.

(e): The given line integral is

$$\int_C (\sin x\,dx + y^2 dy - dz)$$

where C is the circle $x^2 + y^2 = 16$, $z = 3$

$$\int_C (\sin x\,\hat{i} + y^2\hat{j} - \hat{k})\cdot(dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= \int_C \bar{F}\cdot d\bar{r}$$

where $\bar{F} = \sin x\hat{i} + y^2\hat{j} - \hat{k}$

Now from the Stokes' theorem

$$\int_C \bar{F}\cdot d\bar{r} = \oiint_S (\bar{\nabla}\times\bar{F})\cdot d\bar{s}$$

where S is the surface enclosed by the curves.

$$= \oiint_S (\bar{\nabla}\times\bar{F})\cdot k\,dx\,dy$$

$$x^2 + y^2 = 16$$

$$\text{Now } (\bar{\nabla}\times\bar{F})\cdot\bar{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin x & y^2 & -1 \end{vmatrix}\cdot K = 0$$

$$\Rightarrow \quad \oint \bar{F}\cdot d\bar{r} = 0$$

6. (a): Solve:

$$p^2 + 2py\cot x = y^2 \qquad \text{...(1)}$$

or, $p^2 + 2py\cot x - y^2 = 0$

Solving the above equation for the quadratic in p, we get

$$p = \frac{-2y\cot x \pm\sqrt{4y^2\cot^2 x + 4y^2}}{2}$$

or $p = -y\cot x \pm y\,\text{cosec}x$

Case I

when $p = -y\cot x + y\,\text{cosec}x$

then $\frac{dy}{dx} = y(-\cot x - \text{cosec}x)$

or, $\frac{dy}{y} = (-\cot x + \operatorname{cosec} x)dx$

or $\frac{dy}{y} = \left(-\frac{\cos x}{\sin x} + \frac{1}{\sin x}\right)dx$

$$= \left(\frac{2\sin^2 x/2}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right)dx$$

or $\frac{dy}{y} = \tan\frac{x}{2}dx$

integrating both the sides, we get

$$\log y = 2\log\sec\frac{x}{2} + \log C_1$$

<$\log C_1$ = integration constant>

or, $y = C_1 \sec^2\frac{x}{2}$

or, $y - C_1 \sec^2\frac{x}{2} = 0$

is one solution.

Case II

when $p = -y \cot x - y \operatorname{cosec} x$

then, $\frac{dy}{dx} = -y(\cot x - \operatorname{cosec} x)dx$

or, $\frac{dy}{y} = (-\cot x - \operatorname{cosec} x)dx$

or, $\frac{dy}{y} = -\left(\frac{1+\cos x}{\sin x}\right)dx = \frac{-\cos x/2}{\sin x/2}dx$

integrating both the sides we get

$$\log y = 2\log\operatorname{cosec}\frac{x}{2} + \log C_2$$

<$\log C_2$ = integration constant>

or, $y - C_2 \operatorname{cosec}^2\frac{x}{2} = 0$ is another solution.

Hence, the required solution is given by,

$$\left(y - C_1\sec^2\frac{x}{2}\right)\left(y - C_2\operatorname{cosec}^2\frac{x}{2}\right) = 0$$

where C_1 and C_2 are arbitrary constant.

(b): The given differential equation is

$$[x^4D^4 + 6x^3D^3 + 9x^2D^2 + 3xD + 1]y = \langle 1+\log x\rangle^2 \quad ...(1)$$

The above equation is also known as linear. Homogeneous form or Euler-Cauchy form of differential equation.

It is solved by putting $x = e^z$ then reducing the above equation in the form of y and z.

Since, $x = e^z$

then, $\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}$

or, $x\frac{dy}{dx} = \frac{dy}{dz}$

or, $x\frac{dy}{dx} = D_1 y$ (where $D_1 = \frac{d}{dz}$)

Again, $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right)$

$$= \frac{d}{dz}\left(\frac{dy}{dx}\right)\cdot\frac{dz}{dx}$$

$$= \frac{d}{dz}\left(\frac{1}{x}\cdot\frac{dy}{dz}\right)\cdot\frac{1}{x}$$

$$= \frac{1}{x}\left[\frac{1}{x}\cdot\frac{d^2y}{dz^2} - \frac{1}{x}\frac{dy}{dz}\right]$$

or, $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$

or, $x^2\frac{d^2y}{dx^2} = D_1(D_1 - 1)y$

Similarly, $x^3\frac{d^3y}{dx^3} = D_1(D_1 - 1)(D_1 - 2)y$

and $x^4\frac{d^4y}{dx^4} = D_1(D_1 - 1)(D_1 - 2)(D_1 - 3)y$

Putting these values in equation (1), we get

$$[D_1(D_1 - 1)(D_1 - 2)(D_1 - 3) + 6D_1(D_1 - 1)(D_1 - 2) + 9D_1(D_1 - 1) + 3D_1 + 1]y = (1 + e^z)^2$$

$$\Rightarrow [D_1(D_1^3 - 6D_1^2 + 11D_1 - 6) + 6D_1(D_1^2 - 3D_1 + 2) + 9D_1(D_1 - 1) + 3D_1 + 1]y = 1 + 2e^z + e^{2z}.$$

$(D_1^4 + 2D_1^2 + 1)y = 1 + 2e^z + e^{2z}$ ...(2)

The auxiliary equation is given by

$m^4 + 2m^2 + 1 = 0$

$\Rightarrow \quad (m^2 + 1)^2 = 0$

$\Rightarrow \quad m = \pm i, \pm i$

Hence, the complementary function is given by

$y = (C_1 + C_2z)\cos z + (C_3 + C_4z)\sin z$

where C_1, C_2, C_3 and C_4 are arbitrary constant

Now, the Particular Integral is given by

$$y = \frac{1}{(1+2D_1^2+D_1^4)}(1+2e^z+e^{2z})$$

$$= \frac{1}{(1+D_1^2)^2} + 2\cdot\frac{1}{(1+D_1^2)^2}e^z + \frac{e^{2z}}{(1+D_1^2)^2}$$

$$= 1 + \frac{2\cdot e^z}{(1+1)^2} + \frac{e^{2z}}{(1+4)^2} = 1 + \frac{e^z}{2} + \frac{e^{2z}}{25}$$

i.e., the General solution is given by

$$y = \text{C.F.} + \text{P.I.}$$

$$= (C_1 + C_2z)\cos z + (C_3 + C_4z)\sin z + 1 + \frac{e^z}{2} + \frac{e^{2z}}{25}$$

Putting the value of z, we get

$y = (C_1 + C_2\log x)\cos(\log x) + (C_3 + C_4\log x)\sin(\log x) + 1 + \frac{x}{2} + \frac{x^2}{25}$ is the required solution.

(c): Solve:

$(D^4 + D^2 + 1)y = ax^2 + be^{-x}\sin 2x$ ...(1)

The auxiliary equation is given by

$m^4 + m^2 + 1 = 0$

or $\quad m^4 + 2m^2 + 1 - m^2 = 0$

or $\quad (m^2 + 1)^2 - m^2 = 0$

$\Rightarrow (m^2 + m + 1)(m^2 - m + 1) = 0$

i.e. $\quad m = \frac{-1\pm\sqrt{1-4}}{2}, \frac{1+\sqrt{1-4}}{2}$

i.e. $\quad m = \frac{-1}{2}\pm\frac{\sqrt{3}i}{2}$ and $m = \frac{1}{2}\pm\frac{\sqrt{3}i}{2}$ are the roots of auxiliary equation.

Hence, the complementary function is given by

$$y = e^{-x/2}\left[C_1\cos\left(\frac{\sqrt{3}}{2}x\right) + C_2\sin\left(\frac{\sqrt{3}}{2}x\right) + e^{x/2}\right]$$

$$\left[C_3\cos\left(\frac{\sqrt{3}x}{2}\right)\right] + C_4\sin\left(\frac{\sqrt{3}x}{2}\right)$$

Now, the Particular Integral is given by

$$y = \frac{1}{(1+D^2+D^4)}\{ax^2 + be^{-x}\sin 2x\}$$

Consider

$$\frac{a}{(1+D^2+D^4)}x^2 = a\langle 1+D^2+D^4\rangle^{-1}x^2$$

$$= a\langle 1-(D^2+D^4)+(D^2+D^4)^2+\ldots\rangle x^2$$

$$= a[(x^2) - D^2(x)^2] = a[x^2 - 2]$$

Now consider

$$\frac{1}{(1+D^2+D^4)}e^{-x}\sin 2x$$

$$= e^{-x}\frac{1}{[1+(D-1)^2+(D-1)^4)}\sin 2x$$

$$= e^{-x}\frac{1}{[1+D^2-2D+1+D^4-4D^3+6D^2-4D+1]}\sin 2x$$

$$= e^{-x}\frac{1}{(D^4-4D^3+7D^2-6D+3)}\sin 2x$$

$$= e^{-x}\frac{1}{(D^2)^2-4D(D^2)+7\cdot D^2-6D+3}\sin 2x$$

$$= e^{-x}\frac{1}{(-2^2)^2-4D(-2^2)+7(-2^2)-6D+3}\sin 2x$$

$$= e^{-x}\frac{1}{16+16D-28-6D+3}\sin 2x$$

$$= e^{-x}\frac{1}{10D-9}\sin 2x$$

$$= e^{-x}\frac{(10D+9)}{(100D^2-81)}\sin 2x$$

$$= e^{-x}\frac{(10D+9)\sin 2x}{-400-81}$$

$$= \frac{-e^{-x}}{481}(20\cos 2x + 9\sin 2x)$$

$$\therefore\ y = \frac{1}{(1+D^2+D^4)}(ax^2 + be^{-x}\sin 2x)$$

$$= a(x^2-2) - \frac{be^{-x}}{481}(20\cos 2x + 9\sin 2x)$$

Hence the general solution is given by,

$$y = e^{\frac{-x}{2}}\left[C_1\cos\left(\frac{\sqrt{3}}{2}x\right)+C_2\sin\left(\frac{\sqrt{3}}{2}x\right)\right] + e^{\frac{x}{2}}\left[C_3\cos\left(\frac{\sqrt{3}}{2}x\right)+C_4\sin\left(\frac{\sqrt{3}}{2}x\right)\right] + a(x^2-2) - \frac{be^{-x}}{481}(20\cos 2x + 9\sin 2x).$$

8. (a): The parametric equation of the given curve is

$$\vec{r} = \vec{r}(u) = 2\log_e u\hat{i} + 4u\hat{j} + (2u^2+1)\hat{k}$$

where u is the parameter.

Now, $\dot{\vec{r}} = \frac{d\vec{r}}{du} = \frac{2.1}{u}\hat{i} + 4\hat{j} + (4u)\hat{k}$

and $\ddot{\vec{r}} = \frac{d^2r}{du^2} = \frac{-2}{u^2}\hat{i} + 0 + 4\hat{k}$

and $\dddot{\vec{r}} = \frac{d^3r}{du^2} = \frac{4}{u^3}\hat{i}$

Now, we know the curvature (k) of the curve is given by

$$K = \frac{|\dot{\vec{r}}\times\ddot{\vec{r}}|}{|\dot{\vec{r}}|^3}$$

Now $\dot{\vec{r}}\times\ddot{\vec{r}} = \begin{bmatrix} i & j & k \\ \frac{2}{u} & 4 & 4u \\ \frac{-2}{u^2} & 0 & 4 \end{bmatrix}$

$$= 16\hat{i} - \frac{16}{u}\hat{j} + \frac{8}{u^2}\hat{k}$$

$$\Rightarrow\ |\dot{\vec{r}}\times\ddot{\vec{r}}| = \sqrt{16^2 + \left(\frac{16}{u}\right)^2 + \left(\frac{8}{u^2}\right)^2}$$

$$= \frac{8(1+2u^2)}{u^2}$$

and $|\dot{\vec{r}}| = \left|\frac{2}{u}\hat{i} + 4\hat{j} + 4u\hat{k}\right|$

$$= 2\sqrt{\left(\frac{1}{u}\right)^2 + 2^2 + (2u)^2}$$

$$= \frac{2(1+2u^2)}{u}$$

$$\therefore\ K = \frac{|\dot{\vec{r}}\times\ddot{\vec{r}}|}{|\dot{\vec{r}}|^3}$$

$$= \frac{8(1+2u^2)}{u^2}\cdot\frac{u^3}{8(1+2u^2)^3}$$

$$= \frac{u}{(1+2u^2)^2}$$

$$\therefore\ K = \frac{u}{(1+2u^2)^2}$$

and the Torsion (T) is given by

$$T = \frac{[\dot{\vec{r}}\ \ddot{\vec{r}}\ \dddot{\vec{r}}]}{|\dot{\vec{r}}\times\ddot{\vec{r}}|^2}$$

Now, $[\dot{\vec{r}}\ \ddot{\vec{r}}\ \dddot{\vec{r}}] = \begin{bmatrix} \frac{2}{u} & 4 & 4u \\ \frac{-2}{u^2} & 0 & 4 \\ \frac{4}{u^3} & 0 & 0 \end{bmatrix} = \frac{64}{u^3}$

$$\therefore\ T = \frac{64}{u^3}\cdot\frac{u^4}{64(1+2u^2)^2}$$

$$= \frac{u}{(1+2u^2)^2}$$

i.e. Torsion $(T) = \frac{u}{(1+2u^2)^2}$

Now the arc length S is given by the

$$\int ds = \int \left|\frac{d\bar{r}}{du}\right| \cdot du$$

$\because$ $$\frac{d\bar{r}}{du} = \frac{2}{u}\hat{i} + 4\hat{j} + 4u\hat{k}$$

$\Rightarrow$ $$\left|\frac{d\bar{r}}{du}\right| = \sqrt{\frac{4}{u^2} + 16 + 16u^2}$$

$$= \frac{2(1+2u^2)}{u}$$

$\therefore$ $$\int ds = \int 2\left(\frac{1+2u^2}{u}\right)du$$

$$= 2\int \frac{1}{u}du + 4\int u du$$

$\therefore$ $S = 2 \log u + 2u^2$

$\Rightarrow$ $S = 2\,(u^2 + \log u)$

is the required relation between S and parameter u.

(b): The given vector identity is

$$\nabla \times (\bar{f} \times \bar{g}) = \bar{f}(\bar{\nabla} \cdot \bar{g}) - \bar{g}(\bar{\nabla} \cdot \bar{f}) + (\bar{g} \cdot \bar{\nabla})\bar{f} - (\bar{f} \cdot \bar{\nabla})\bar{g}$$

$$\text{L.H.S.} = \bar{\nabla} \times (\bar{f} \times \bar{g})$$

$$= \Sigma i \times \frac{\partial}{\partial x}(\bar{f} \times \bar{g})$$

$$= \Sigma \bar{i} \times \left[\left(\frac{\partial \bar{f}}{\partial x} \times \bar{g}\right) + \left(\bar{f} \times \frac{\partial \bar{g}}{\partial x}\right)\right]$$

$$= \Sigma \bar{i} \times \left(\frac{\partial \bar{f}}{\partial x} \times \bar{g}\right) + \Sigma \bar{i} \times \left(\bar{f} \times \frac{\partial \bar{g}}{\partial x}\right)$$

Now consider

$$\bar{i} \times \left(\frac{\partial \bar{f}}{\partial x} \times \bar{g}\right) = (\bar{g} \cdot \bar{i})\frac{\partial \bar{f}}{\partial x} - \left(\bar{i} \cdot \frac{\partial \bar{f}}{\partial x}\right)\bar{g}$$

[using relation $\bar{A} \times (\bar{B} \times \bar{C}) = (\bar{A} \cdot \bar{C})\bar{B} - (\bar{A} \cdot \bar{B})\bar{C}$

$\therefore$ $$\Sigma \bar{i} \times \left(\frac{\partial \bar{f}}{\partial x} \times \bar{g}\right)$$

$$= \Sigma(\bar{g}.\bar{i})\frac{\partial \bar{f}}{\partial x} - \Sigma\left(\bar{i} \cdot \frac{\partial \bar{f}}{\partial x}\right)\bar{g}$$

$$= \bar{g} \cdot \left(\Sigma i \frac{\partial}{\partial x}\right)\bar{f} - \Sigma\left(i \frac{\partial}{\partial x} \cdot \bar{f}\right)\bar{g}$$

$$= (\bar{g} \cdot \bar{\nabla})\bar{f} - (\bar{\nabla} \cdot \bar{f})\bar{g}$$

$$= (\bar{g} \cdot \bar{\nabla})\bar{f} - \bar{g}(\bar{\nabla} \cdot \bar{f})$$

Similarly for

$$\Sigma i \times \left(\bar{f} \times \frac{\partial \bar{g}}{\partial x}\right)$$

$$= \Sigma\left(\bar{i} \cdot \frac{\partial \bar{g}}{\partial x}\right) \cdot \bar{f} - \Sigma(\bar{i} \cdot \bar{f})\frac{\partial \bar{g}}{\partial x}$$

$$= \bar{f}(\bar{\nabla} \cdot \bar{g}) - (\bar{f} \cdot \bar{\nabla})\bar{g}$$

hence

$$\bar{\nabla}(\bar{f} \times \bar{g}) = f(\bar{\nabla} \cdot \bar{g}) - g(\bar{\nabla} \cdot \bar{f}) + (\bar{g} \cdot \bar{\nabla})\bar{f} - (\bar{f} \cdot \bar{\nabla})\bar{g}$$

which proves the identity.

Now if $\vec{f} = x\hat{i} + z\hat{j} + y\hat{k}$

and $\vec{g} = y\hat{i} + z\hat{k}$

then $$\bar{f} \times \bar{g} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & z & y \\ y & 0 & z \end{vmatrix}$$

$$= z^2\hat{i} + (y^2 - xz)\hat{j} - yz\hat{k}$$

Now $$\bar{\nabla} \times (\bar{f} \times \bar{g}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & y^2 - xz & -yz \end{vmatrix}$$

$$\hat{i}\left\{\frac{-\partial}{\partial y}(yz) - \frac{\partial}{\partial z}(y^2 - xz)\right\} - \hat{j}\left\{\frac{\partial}{\partial x}(-yz) - \frac{\partial}{\partial z}(z^2)\right\} + \hat{k}\left\{\frac{\partial}{\partial x}(y^2 - xz) - \frac{\partial z^2}{\partial y}\right\}$$

$$= \hat{i}(-z + x) + \hat{j}(2z) + \hat{k}(-z)$$

$$= (x - z)\hat{i} + 2z\,\hat{j} - z\,\hat{k}$$

$\because$ $\vec{g} = (y\hat{i} + z\hat{k})$

$$\Rightarrow \quad \bar{\nabla}\cdot\bar{g} = 1$$

and $\quad \bar{f} = \left(x\hat{i}+z\hat{j}+y\hat{k}\right)$

$$\Rightarrow \quad \bar{\nabla}\cdot\bar{f} = 1$$

Also $\quad (\bar{g}\cdot\bar{\nabla})\bar{f} = \left(\dfrac{y\partial}{\partial x}+\dfrac{z\partial}{\partial z}\right)(x\hat{i}+z\hat{j}+y\hat{k})$

$$= (y\hat{i}+z\hat{j})$$

and

$$(\bar{f}\cdot\bar{\nabla})\bar{g} = \left(x\frac{\partial}{\partial x}+z\frac{\partial}{\partial y}+y\frac{\partial}{\partial z}\right)(y\hat{i}+z\hat{k})$$

$$= (z\hat{i}+y\hat{k})$$

Hence

$$\bar{f}(\bar{\nabla}\cdot\bar{g})-\bar{g}(\bar{\nabla}\cdot\bar{f})+(\bar{g}\cdot\bar{\nabla})\bar{f}-(\bar{f}\cdot\bar{\nabla})\bar{g}$$

$$= \bar{f}-\bar{g}+\left(y\hat{i}+z\hat{j}\right)-\left(z\hat{i}+y\hat{k}\right)$$

$$= \left(x\hat{i}+z\hat{j}+y\hat{k}\right)-\left(y\hat{i}+z\hat{k}\right)+\left(y\hat{i}+z\hat{j}\right)-\left(z\hat{i}+y\hat{k}\right)$$

$$= (x-z)\hat{i}+2z\hat{j}-z\hat{k}.$$

i.e. The given identity is verified for the vector

$$\bar{f} = x\hat{i}+z\hat{j}+y\hat{k}$$

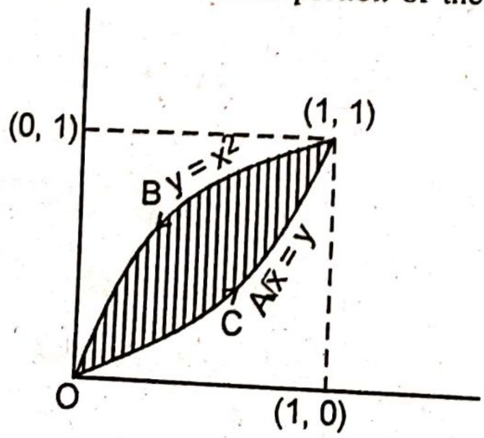
and $\quad \vec{g} = y\hat{i}+z\hat{k}$

(c): The Green's theorem in a plane is given by

$$\oint_C M\,dx+N\,dy = \iint_S\left(\frac{\partial M}{\partial y}-\frac{\partial N}{\partial x}\right)dx\,dy$$

where S is the area enclosed by the boundary of the curve C.

< shown as shaded portion of the figure>



Now from question, the given curve is $y = \sqrt{x}$ and $y = x^2$ and we have to verify Green's theorem for

$$\oint_C[(3x^2-8y^2)dx+(4y-6xy)dy]$$

$$= \oint_{OAC}[3x^2-8y^2)dx+(4y-6xy)dy+ \oint_{CBO}(3x^2-8y^2)dx+(4y-6xy)dy$$

Consider $\oint_{OAC}(3x^2-8y^2)dx+(4y-6xy)dy$

along OAC path $y = \sqrt{x}$ or $x = y^2$

$$\therefore \quad dx = 2y\,dy$$

$\therefore$ the integral can be changed to

$$\int_{y=0}^{1}(3y^4-8y^2)2y\,dy+(4y-6y^3)dy$$

$$= \int_{y=0}^{1}(6y^5+16y^2+4y-6y^3)dy$$

$$= \int_{y=0}^{1}(6y^5-22y^2+4y)\,dy$$

$$= \left[y^6-\frac{22}{4}y^4+\frac{4}{2}y^2\right]_0^1 = \left(1-\frac{22}{4}+2\right)$$

$$= 3-\frac{11}{2} = \frac{-5}{2}.$$

Consider

$$\int_{CBO}(3x^2-8y^2)dx+(4y-6xy)dy$$

along this path $y = x^2$

$$= \int_{x=1}^{0}(3x^2-8x^4)dx+(4x^2-6x^3)2x\,dx$$

$$= \int_{x=1}^{0}(3x^2-8x^4+8x^3-12x^4)dx$$

$$= \int_{1}^{0}(3x^2+8x^3-20x^4)dx$$

$$= 3\frac{x^3}{3}+8\frac{x^4}{4}-20\frac{x^5}{5}\Big|_1^0 = 1$$

$$\therefore \int_C (3x^2-8y^2)dx+(4y-6xy)dy$$

$$= \frac{-5}{2}+1 = \frac{-3}{2}$$

Now consider the integral

$$\oint\!\!\int_S \left(\frac{\partial N}{\partial y}-\frac{\partial N}{\partial x}\right)dx\,dy$$

$$= \iint\{-16y-(-6y)\}dx\,dy = -10\iint y\,dx\,dy$$

$$= -10\int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} y\,dx\,dy = -10\int_0^1 \frac{y^2}{2}\Big|_{x^2}^{\sqrt{x}} dx$$

$$= -5\int_0^1 (x-4^4)dx = -\frac{3}{2}$$

Hence the Green's theorem is verified.

(d): The position vector of the particle of mass 2 unit at time t is given by

$$\vec{r} = (t^2-2t)\hat{i}+\left(\frac{1}{2}t^2+1\right)\hat{j}+\frac{1}{2}t^2\hat{k}$$

Now we know that the kinetic energy is given by

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m(\vec{v}\cdot\vec{v})$$

$$\because \quad \vec{r} = (t^2-2t)\hat{i}+\left(\frac{1}{2}t^2+1\right)\hat{j}+\frac{t^2}{2}\vec{k}$$

Hence $\vec{v} = \dfrac{d\vec{r}}{dt} = (2t-2)\hat{i}+t\hat{j}+t\hat{k}$

$$\therefore \vec{v}\cdot\vec{v} = 4(t-1)^2+t^2+t^2$$
$$= [2(t-1)^2+t^2]$$

$$\therefore \quad K = \frac{1}{2}\cdot 2\cdot 2\left[2(t-1)^2+t^2\right]$$

$\therefore$ At $t = 1$, the K.E. is given by

$K = 2[2(1-1)^2+1^2] = 2$ **Units.**

$$\vec{v}\big|_{t=1} = \hat{j}+\hat{k}$$

$$\vec{r}\big|_{t=1} = -\hat{i}+\frac{3}{2}\hat{j}+\frac{1}{2}\hat{k}$$

$$\therefore \vec{L}\big|_{t=1} = \vec{r}\times m\vec{v} = \begin{bmatrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & \frac{3}{2} & \frac{1}{2}\\ 0 & 2 & 2\end{bmatrix}$$

$$\Rightarrow \quad \vec{L} = 2(\hat{i}+\hat{j}+\hat{k})$$

Since, $\vec{L} = \vec{r}\times m\vec{v}$

differentiating both sides with respect to t, we get

$$\frac{d\vec{L}}{dt} = \frac{d\vec{r}}{dt}\times m\vec{v}+\vec{r}\times m\frac{d\vec{v}}{dt} = \vec{r}\times m\frac{d\vec{v}}{dt}$$

$$\because \quad \vec{v} = (2t-2)\hat{i}+t\hat{j}+t\hat{k}$$

$$\Rightarrow \quad \frac{d\vec{v}}{dt} = 2\hat{i}+\hat{j}+\hat{k}$$

$$\therefore \frac{d\vec{L}}{dt}\Big|_{t=1} = 2\begin{bmatrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & \frac{3}{2} & \frac{1}{2}\\ 2 & 1 & 1\end{bmatrix} = 2\hat{i}+4\hat{j}-8\hat{k}$$

Finally, the moment of the resultant force is given by,

$$\vec{\tau} = \vec{r}\times\vec{F} = \vec{r}\times m\frac{d\vec{v}}{dt}$$

$$\Rightarrow \quad \vec{\tau}\big|_{t=1} = 2\begin{bmatrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & \frac{3}{2} & \frac{1}{2}\\ 2 & 1 & 1\end{bmatrix}$$

$$= 2\hat{i}+4\hat{j}-8\hat{k}$$

In sum, at $t = 1$,

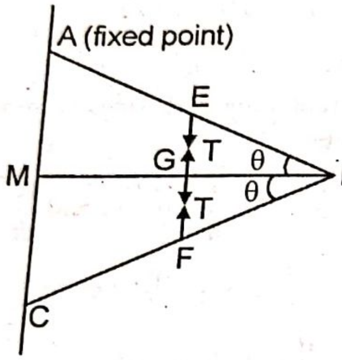
Kinetic energy = 2 units.

Angular momentum $(\vec{L}) = 2(\hat{i}+\hat{j}-\hat{k})$ units

Time Rate of change of angular momentum $= (2\hat{i}+4\hat{j}-8\hat{k})$ units

and moment of the resultant force $= (2\hat{i}+4\hat{j}-8\hat{k})$ untis.

7. (a): AB and BC are two rods each of length $2a$ and weight W smoothly jointed together at B. The end A of the rod AB is attached to a smooth vertical wall and the end C of the rod BC is in contact with the wall. The middle points E and F of rods AB and BC are connected by an elastic string of natural length a.



Let T be the tension in the string EF. The total weight 2W of the two rods can be taken acting at the middle point of EF. The line BG is horizontal and meets AC at its middle point M.

Let $\angle ABM = \theta = \angle CBM$.

Give the system a small symmetrical displacement about BM in which θ changes to $\theta + \delta\theta$. The point A remains fixed, the point G is slightly displaced, the length EF changes, the lengths of the rods AB and BC do not change.

We have $EF = 2EG = 2EB \sin\theta = 2a \sin\theta$

Also the depth of G below the fixed point $A = AM = AB \sin\theta = 2a \sin\theta$.

The equation of virtual work is

$$-T\delta(2a \sin\theta) + 2W\delta\,(2a \sin\theta) = 0$$

or, $(-2aT\cos\theta + 4aW\cos\theta)\delta\theta = 0$

or, $2a\cos\theta(-T + 2W)\delta\theta = 0$

or, $-T + 2W = 0$

$[\because\ \delta\theta \neq 0 \text{ and } \cos\theta \neq 0]$

or, $T = 2W$

Also, by Hooke's law the tension T in the elastic string EF is given by

$$T = \lambda\cdot\frac{2a\sin\theta - a}{a}$$

where λ is the modulus of elasticity of the string

$T = 4W\,(2\sin\theta - 1)$

Equating the two values of T, we have

$2W = 4W(2\sin\theta - 1)$

or, $1 = 2(2\sin\theta - 1)$

or, $4\sin\theta - 2 = 1$

or, $4\sin\theta = 3$

or, $\sin\theta = \frac{3}{4}$

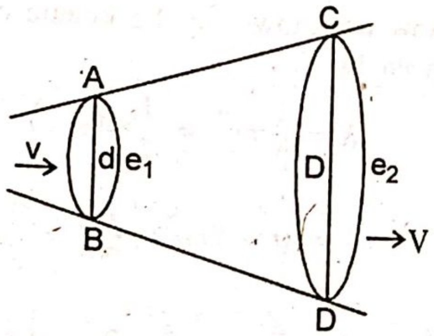
or, $= \sin^{-1}\left(\frac{3}{4}\right)$

∴ In equilibrium, the whole angle between AB and BC

7(b) = ?

$$= 2\theta = 2\sin^{-1}\left(\frac{3}{4}\right).$$

(c): Let e_1 and e_2 be the densities of steam at the ends of the conical pipe AB and CD. By the principle of conservation of mass, the mass of the steam that enters and leaves at the ends AB and CD are the same. Thus we have



$$\pi\left(\frac{1}{2}d\right)^2 ve_1 = \pi\left(\frac{1}{2}D\right)^2 Ve_2$$

or $$\frac{v}{V} = \frac{D^2 e_2}{d^2 e_1} \quad ...(1)$$

let p be the pressure, e the density and u the velocity at distance r from AB, then the equation of motion is given by

$$u\frac{\partial u}{\partial r} = -\frac{1}{e}\frac{\partial p}{\partial r},$$

$$p = Ke$$

or, $$u\frac{\partial u}{\partial r} = -\frac{K}{e}\frac{\partial e}{\partial r}$$

By integrating, we have

$$\frac{1}{2}u^2 = -K\log e + K\log E$$

where E is an arbitrary constant

or, $$\log\frac{e}{E} = -\frac{u^2}{2K}$$

or, $$e = E\exp\left(-\frac{u^2}{2K}\right)$$

Again $e = e_1$ when $u = v$

then $$e_1 = E\exp\left(-\frac{v^2}{2K}\right)$$

and $e = e_2$ when $u = v$,

then $$e_2 = E\exp\left(\frac{-V^2}{2K}\right)$$

or, $$\frac{e_1}{e_2} = \frac{\exp\left(-v^2/2K\right)}{\exp\left(-V^2/2K\right)} \quad ...(2)$$

from (1) and (2), we have

$$\frac{v}{V} = \frac{D^2}{d^2}\exp.\left(\frac{v^2 - V^2}{2K}\right)$$ **Proved.**

PAPER-II

Section-A

1. (a): It is given that $(G, 0)$ is a group and $\exists\ x, y, \in G$, such that $y^5 = e$ and $yxy^{-1} = x^2$.

Now, we have

$$(yxy^{-1})^2 = yxy^{-1}\ .\ yxy^{-1} = yx^2y^{-1}.$$
$$= y.yxy^{-1}y^{-1} = y^2xy^{-2}$$

$\therefore$ $$(yxy^{-1})^4 = (y^2xy^{-2})\ (y^2xy^{-2})$$
$$= y^2x^2y^{-2} = y^2.yxy^{-1}y^{-2}$$

i.e., $$(yxy^{-1})^4 = y^3xy^{-3}$$

Again $$(yxy^{-1})^8 = (y^3xy^{-3})\ .\ (y^3xy^{-3})$$
$$= y^3x^2y^{-3}$$
$$= y^3.yxy^{-1}y^{-3} = y^4xy^{-4}$$

or $$(yxy^{-1})^{16} = (y^4xy^{-4}).(y^4xy^{-4}) = y^4x^2y^{-4}$$

or $$(yxy^{-1})^{16} = y^4.yxy^{-1}y^{-4}$$
$$= y^5xy^{-5} = x \qquad [\because y^5 = e]$$

or, $(x^2)^{16} = x \Rightarrow x^{32} = x$

$\Rightarrow$ $x^{31} = e$

$\Rightarrow$ $0(x) = 31$ **Proved.**

(b): First we show $Q(\sqrt{2})$ is a ring under usual addition and multiplication.

i.e., We'll show that $(Q\sqrt{2}, +1)$ is a group under addition and closed under multiplication.

Let $x, y \in Q\sqrt{2}$ then $x = a + b\sqrt{2}$, $y = c + d\sqrt{2}$ where $a, b, c, d \in Q$.

Now $x + y = (a + b\sqrt{2}) + (c + d\sqrt{2}) = (a + c) + (b + d)\sqrt{2}$.

Since $a + c$ and $b + d$ are elements of Q, therefore $(a + c) + (b + d)\sqrt{2} \in Q\sqrt{2}$.

Thus $x + y \in Q\sqrt{2}\ \forall\ x, y \in Q\sqrt{2}$.

Therefore $Q\sqrt{2}$ is closed with respect to addition.

Also, the elements of $Q\sqrt{2}$ are all real numbers and the addition of real number is associative.

Also $O + O\sqrt{2} \in Q\sqrt{2}$ as $O \in Q$.

If $a + b\sqrt{2} \in Q\sqrt{2}$ then $(O + O\sqrt{2}) + (a + b\sqrt{2}) = (a + b\sqrt{2}) + (O + O\sqrt{2}) = (a + b\sqrt{2})$.