IFS 2016 P-2

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4(a) Evaluate the integral $\int_0^\infty \frac{dx}{\sqrt{x}(1+x)}$

Put $\sqrt{x} = t$ i.e.

$x = t^2 \Rightarrow dx = 2t\,dt$

$$I = \int_0^\infty \frac{dx}{\sqrt{x}(1+x)}$$

x	0	∞
t	0	∞

$$= \int_0^\infty \frac{2\cancel{t}\,dt}{\cancel{t}(1+t^2)}$$

$$= \lim_{p\to\infty} 2\int_0^p \frac{dt}{1+t^2}$$

$$= 2 \lim_{p\to\infty} \left[\tan^{-1} t\right]_0^p$$

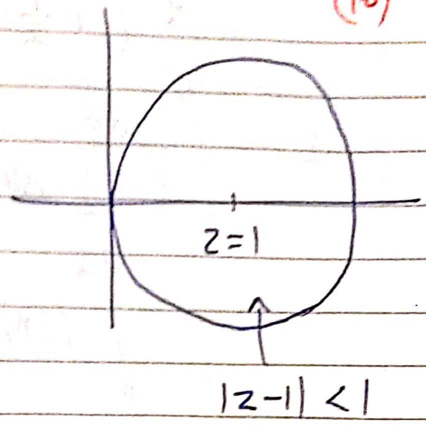
$$= 2\left[\lim_{p\to\infty} \tan^{-1} p - \tan^{-1} 0\right]$$

$$= 2\left(\frac{\pi}{2} - 0\right) = \pi.$$

4(b) Find the Laurent series for the function $f(z) = \frac{1}{1-z^2}$ with centre $z=1$. (10)

$$f(z) = \frac{1}{1-z^2}$$

$$= \frac{1}{2}\left[\frac{1}{z+1} - \frac{1}{z-1}\right]$$



Let $u = z-1$

$$f(z) = \frac{1}{2}\left[\frac{1}{u+2} - \frac{1}{u}\right] = \frac{1}{2}\left[\frac{1}{2(1+\frac{u}{2})} - \frac{1}{u}\right]$$

When $|z-1| = |u| < 2$ i.e. $|u|/2 < 1$

$$f(z) = \frac{1}{2}\left[\frac{1}{2}\left(1+\frac{u}{2}\right)^{-1} - \frac{1}{u}\right]$$

$$= \frac{1}{4}\left(1 - \frac{u}{2} + \frac{u^2}{4} - \dots\right) - \frac{1}{2u}$$

$$= \frac{-1}{2(z-1)} + \frac{1}{4}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z-1}{2}\right)^n$$

Now, we fined the Laurent series for $|z-1| = |u| > 2$ i.e. $2/|u| < \frac{2}{2} \leq 1$

$$f(z) = \frac{1}{2}\left[\frac{1}{u+2} - \frac{1}{u}\right] = \frac{1}{2}\left[\frac{1}{u(1+\frac{2}{u})} - \frac{1}{u}\right]$$

$$= \frac{1}{2u}\left[\left(\cancel{1} - \frac{2}{u} + \frac{4}{u^2} - \dots\right) - \cancel{1}\right]$$

$$= \frac{-1}{u^2} + \frac{2}{u^3} - \frac{4}{u^4} + \dots$$

$$= \frac{1}{2(z-1)}\sum_{n=0}^{\infty}(-1)^{n+1}\left(\frac{2}{(z-1)}\right)^{n+1}$$

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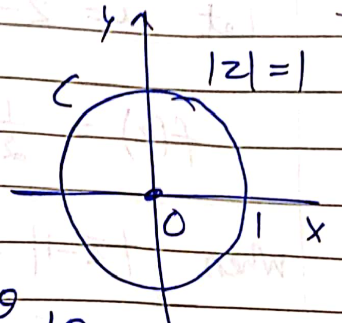
4(c) Evaluate by Contour Integration

$$\int_0^{\pi} \frac{d\theta}{\left(1+\frac{1}{2}\cos\theta\right)^2} \qquad (10)$$

$$I = \int_0^{\pi} \frac{d\theta}{\left(1+\frac{1}{2}\cos\theta\right)^2} = \frac{1}{2}\int_0^{2\pi} \frac{4\,d\theta}{(2+\cos\theta)^2}$$

$$\left[\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx \text{ if } f(2a-x)=f(x)\right]$$



Let contour 'C' be the unit circle $|z|=1$ with centre at the origin.

$$z = e^{i\theta} \Rightarrow dz = i e^{i\theta}\, d\theta \Rightarrow \boxed{d\theta = \frac{dz}{iz}}$$

$$I = \int_C \frac{2\,dz}{iz\left(2+\left(\frac{z+z^{-1}}{2}\right)\right)^2}$$

$$= \int_C \frac{8z\,dz}{i\left(z^2+4z+1\right)^2}$$

$$= \frac{8}{i}\int_C f(z)\,dz \quad ; \quad f(z) = \frac{z}{\left(z^2+4z+1\right)^2}$$

Poles of $f(z)$ are given by $\left(z^2+4z+1\right)^2 = 0$ i.e.

$$z = \frac{-4 \pm \sqrt{16-4}}{2} = -2 \pm \sqrt{3}$$

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Let $\alpha = -2+\sqrt{3}$, $\beta = -2-\sqrt{3}$

$|\beta| > 1$ & $|\alpha\beta| = 1 \Rightarrow |\alpha| < 1$

Pole inside C is at $z = \alpha$ of order 2.

Residue of $f(z)$ at $z = \alpha$

$$= \lim_{z \to \alpha} \frac{d}{dz}\left[(z-\alpha)^2 f(z)\right]$$

$$= \lim_{z \to \infty} \frac{d}{dz} (z-\alpha)^2 \cdot \frac{z}{(z^2+4z+1)^2}$$

$$= \lim_{z \to \alpha} \frac{d}{dz} \frac{z}{(z-\beta)^2} = \lim_{z \to \alpha} \frac{1\cdot(z-\beta)^2 - z\cdot 2(z-\beta)}{(z-\beta)^4}$$

$$= \frac{(\alpha-\beta)^2 - 2\alpha(\alpha-\beta)}{(\alpha-\beta)^4}$$

$$= \frac{(\alpha-\beta) - 2\alpha}{(\alpha-\beta)^3} = \frac{-(\alpha+\beta)}{(\alpha-\beta)^3}$$

$$= \frac{-(-4)}{(2\sqrt{3})^3} = \frac{1}{6\sqrt{3}} = \frac{\sqrt{3}}{18}$$

$$\therefore\ I = \frac{8}{i}\int_C f(z)\,dz$$

$$= \frac{8}{\cancel{i}} \times \cancel{2}\pi \cancel{i} \left(\frac{\sqrt{3}}{\cancel{18}_{9}}\right) = \frac{8\pi\sqrt{3}}{9}$$

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4.c) A company manufacturing air coolers . . .

	Mangalore	Bengalore	Delhi	Goa	supply
Bengaluru	90	90	100	100	200
Mumbai	50	70	130	85	100
Demand	75	100	100	25	300 / 300

(Profit Matrix)

First we notice that it ~~is~~ profit matrix is given, so it is maximization problem.

We convert it into minimization problem (loss matrix), by subtracting each cell value from maximum value of the table i.e. 130

					supply
	40	40	30	30	200
	80	60	0	45	100
Demand	75	100	100	25	300 / 300

Also, total Demand = 200+100 = 300

total supply = 75+100+100+25 = 300

Both values are equal, hence it is a balanced problem.

We use Voggle's Approximation method (VAM) to find initial basis feasible solution (IBFS)

					Penalities
40 (75)	40 (100)	30	30 (25)	200	0 / 10
80	60	0 (100)	45	~~100~~ 0	45 x
75	100	~~100~~ 0	25		
40	20	30	15		
40	40	X	30		

Penalities

To calculate Penalities, we take the difference between the lowest value and the next lowest value. Then we choose the row/column with largest penality and ~~of~~ make assignment in the cell with least cost.

Total number of assignment cells $= 4$

$\neq m+n-1 = 4+2-1 = 5$

Hence the Initial solution is non-basic. We assign a zero allocation to a cell with minimum cost, but this cell should not make many closed loop to satisfy the criteria of being linearly independent. We make assignment in cell (1,3) with cost 30.

So, the IBFS is

40 (75)	40 (100)	30 (0)	30 (25)
80	60	0 (100)	45

u-v method to check optimality

				u_i
40	40	30	30	0
10 (70)	10 (50)	0	0 (45)	-30
v_j: 40	40	30	30	

We calculate, $\Delta_{ij} = c_{ij} - (u_i + v_j)$ for all non-basic cells.

Since all $\Delta_{ij} \geq 0$, hence the optimal solution is obtained.
Optimal solution is degenerate.

	Mangalore	Bengaluru	Delhi	Goa	Supply
Bengaluru	90 $\boxed{75}$	90 $\boxed{100}$	100 $\boxed{0}$	100 $\boxed{25}$	200
Mumbai	50	70	130 $\boxed{100}$	85	100
Demand	75	100	100	25	

Hence, Maximum Profit is given by

$= 90 \times 75 + 90 \times 100 + 100 \times 0 + 100 \times 25 + 130 \times 100$

$= 31,250$ Rs.