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Rolle's Theorem, Mean value Theorems, Taylor's and Maclaurin's Theorems

§ 1. Rolle's Theorem.

(Meerut 1985, 91; Agra 82, 80, 77, 73; Indore 70; Gorakhpur 78)

If a function $f(x)$ is such that

(i) $f(x)$ is continuous in the closed interval $a \le x \le b$,

(ii) $f'(x)$ exists for every point in the open interval $a < x < b$,

(iii) $f(a) = f(b)$, then there exists at least one value of x, say c, where $a < c < b$, such that $f'(c) = 0$.

Proof. Since $f(a) = f(b)$, unless the function $f(x)$ is a constant in which case the theorem is at once established, $f(x)$ should either increase or decrease when x takes values greater than a. Suppose it increases; then since it again takes a value $f(b) = f(a)$, it must cease to increase and begin to dercrease at some point c, such that $a < c < b$.

At this point c the function $f(x)$ has a maximum value and so $f(c+h) - f(c)$ and $f(c-h) - f(c)$ are both negative, h being small and positive.

$$\therefore \quad \frac{f(c+h) - f(c)}{h} < 0 \quad \text{and} \quad \frac{f(c-h) - f(c)}{-h} > 0.$$

Obviously as $h \to 0$, the above expressions tend to being $-$ive and $+$ive respectively unless each of them has the limit zero.

If they have different limits, then $Rf'(c) \ne Lf'(c)$ and therefore $f'(c)$ does not exist, contradicting the hypothesis.

Hence each of the above limits must be zero,

i.e., $\quad f'(c) = 0$ where $a < c < b$.

Note 1. There may be more than one point like c at which $f'(x)$ vanishes.

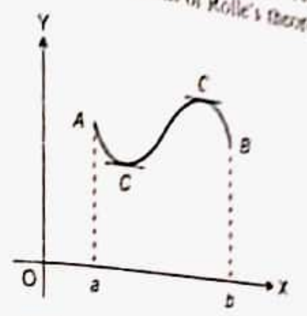
Note 2. Rolle's therorem will not hold good

(i) if $f(x)$ is discontinuous at some point in the interval $a \le x \le b$,

or **(ii)** if $f'(x)$ does not exist at some point in the interval $a < x < b$,

or **(iii)** if $f(a) \ne f(b)$.

Geometrical interpretation of Rolle's Theorem. Suppose the function $f(x)$ satisfies the conditions of Rolle's theorem in the interval



$[a, b]$. Then its geometrical interpretation is that on the curve $y = f(x)$ there is at least one point lying in the open interval (a, b) the tangent at which is parallel to the axis of x.

Solved Examples

Ex. 1 (a). *Discuss the applicability of Rolle's theorem for $f(x) = 2 + (x-1)^{2/3}$ in the interval $[0, 2]$.* **(G.N.E. 1977)**

Sol. Given $f(x) = 2 + (x-1)^{2/3}$. Obviously $f(0) = 3 = f(2)$, showing that the third condition of Rolle's therorem is satisfied.

The function $f(x)$, being an algebraic function of x, is continuous in the closed interval $[0, 2]$. Thus the first condition of Rolle's theorem is satisfied.

Now $f'(x) = \frac{2}{3}.[1/(x-1)^{1/3}]$. We observe that at $x = 1$, $f'(x)$ is not finite while $x = 1$ is a point of the open interval $0 < x < 2$. Thus the second condition for Rolle's theorem is not satisfied.

Hence the Rolle's theorem is not applicable for the function $2 + (x-1)^{2/3}$ in the given interval $[0, 2]$.

Ex. 1 (b). *Discuss the applicability of Rolle's theorem in the interval $[-1, 1]$ to the function $f(x) = |x|$.* **(Meerut 1974)**

Sol. Here $f(-1) = |-1| = 1$ and $f(1) = |1| = 1$, so that $f(-1) = f(1)$.

Also the function $f(x)$ is continuous throughout the closed interval $[-1, 1]$ but it is not differentiable at $x = 0$ which is a point of the open interval $(-1, 1)$. Therefore the second condition for Rolle's theorem is not satisfied, *i.e.*, the Rolle's theorem is not applicable here.

Ex. 2. *Are the conditions of Rolle's theorem satisfied in the case of the following functions ?*

(i) $f(x) = x^2$ *in* $2 \le x \le 3$,

(ii) $f(x) = \cos(1/x)$ *in* $-1 \le x \le 1$.

(iii) $f(x) = \tan x$ *in* $0 \le x \le \pi$.

Sol. (i) The function $f(x) = x^2$ is continuous and differentiable in the interval $[2, 3]$. Thus the first two conditions of Rolle's theorem are satisfied.

Also $f(2) = 4$ and $f(3) = 9$, so that $f(2) \ne f(3)$. Hence the third condition is not satisfied.

(ii) Here $f(-1) = \cos(-1) = \cos 1$ and $f(1) = \cos 1$. Thus $f(-1) = f(1)$ *i.e.*, the third condition is satisfied.

But the first two conditions of Rolle's theorem are not satisfied as the function is discontinuous at $x = 0$ and consequently is not differentiable there.

(iii) Here $f(0) = \tan 0 = 0$ and $f(\pi) = \tan \pi = 0$. Thus $f(0) = f(\pi)$ *i.e.*, the third condition is satisfied.

But the first two conditions of Rolle's theorem are not satisfied here as the function is not continuous at $x = \pi/2$ and consequently is non-differentiable there.

Ex. 3. *Discuss the applicability of Rolle's theorem to* $f(x) = \log\left[\dfrac{x^2 + ab}{(a+b)x}\right]$, *in the interval* $[a, b]$. **(Meerut 1990)**

Sol. We have $f(a) = \log\left[\dfrac{a^2 + ab}{(a+b)a}\right] = \log 1 = 0$,

and $f(b) = \log\left[\dfrac{b^2 + ab}{(a+b)b}\right] = \log 1 = 0.$

Hence $f(a) = f(b) = 0$.

Also $Rf'(x) = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$$= \lim_{h \to 0} \frac{1}{h}\left[\log\left\{\frac{(x+h)^2 + ab}{(a+b)(x+h)}\right\} - \log\left\{\frac{x^2 + ab}{(a+b)x}\right\}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\log \frac{(x^2 + 2xh + h^2 + ab)(a+b)x}{(a+b)(x+h)(x^2+ab)}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\log\left\{\frac{(x^2 + 2xh + h^2 + ab)}{(x^2+ab)} \times \frac{x}{x+h}\right\}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\log\left\{1 + \frac{xh + h^2}{x^2 + ab}\right\} - \log\left\{1 + \frac{h}{x}\right\}\right]$$

$$= \lim_{h \to 0} \frac{1}{h}\left[\frac{2hx + h^2}{x^2 + ab} - \frac{h}{x} + \dots\dots\right] \qquad \dots(1)$$

$[\because \log(1+y) = y - \tfrac{1}{2}y^2 + \dots\dots]$

$$= \frac{2x}{x^2 + ab} - \frac{1}{x}.$$

Again $Lf'(x) = \lim_{h \to 0} \left[\dfrac{f(x-h) - f(x)}{-h}\right]$

$$= \lim_{h \to 0} \frac{1}{(-h)}\left[\frac{-2hx + h^2}{x^2 + ab} - \frac{(-h)}{x} + \dots\right].$$

$$= \frac{2x}{x^2 + ab} - \frac{1}{x}. \qquad \text{[replacing } h \text{ by } -h \text{ in (1)]}$$

Thus $Rf'(x) = Lf'(x)$, showing that $f(x)$ is differentiable for all values of x in $[a, b]$. Consequently $f(x)$ is also continuous for all values of x in $[a, b]$. Hence all the three conditions of Rolle's theorem are staisfied.

$\therefore$ $f'(x) = 0$ for at least one value of x in the open interval $a < x < b$.

Now $f'(x) = 0$ where $\dfrac{2x}{x^2 + ab} - \dfrac{1}{x} = 0$ or $2x^2 - (x^2 + ab) = 0$ or $x^2 = ab$ or $x = \sqrt{(ab)}$, which being the geometric mean of a and b lies in the open interval (a, b). Hence Rolle's theorem is verified.

Ex. 4. *Verify Rolle's theorem in the case of the functions*

(i) $f(x) = 2x^3 + x^2 - 4x - 2$, **(Agra 1982, 80)**

(ii) $f(x) = \sin x$ *in* $[0, \pi]$,

(iii) $f(x) = (x - a)^m (x - b)^n$, *where m and n are +ve integers, and x lies in the interval* $[a, b]$. **(Agra 1981)**

Sol. (i) Here $f(x)$ is a rational integral function of x. So it is continuous and differentiable for all real values of x. Thus the first two conditions of Rolle's theorem are satisfied in any interval.

Now let $f(x) = 0$. Then $2x^3 + x^2 - 4x - 2 = 0$

or $(x^2 - 2)(2x + 1) = 0$ *i.e.*, $x = \pm\sqrt{2}, -\tfrac{1}{2}$.

Thus $f(\sqrt{2}) = f(-\sqrt{2}) = f(-\tfrac{1}{2}) = 0$.

Let us consider the interval $[-\sqrt{2}, \sqrt{2}]$. In this interval all the conditions of Rolle's theorem are satisfied. Therefore there is at least one value of x in the open interval $(-\sqrt{2}, \sqrt{2})$ where $f'(x) = 0$.

Now $f'(x) = 0$ where $6x^2 + 2x - 4 = 0$

or $3x^2 + x - 2 = 0$ or $(3x - 2)(x + 1) = 0$ or $x = -1, 2/3$.

Thus $f'(-1) = f'(2/3) = 0$.

Since both the points $x = -1$ and $x = 2/3$ lie in the open interval $(-\sqrt{2}, \sqrt{2})$, Rolle's theorem is verified.

(ii) Here $f(0) = \sin 0 = 0$ and $f(\pi) = \sin \pi = 0$. Thus $f(0) = f(\pi) = 0$.

Further $\sin x$ is continuous and differentiable in $[0, \pi]$. Hence all the three conditions of Rolle's theorem are satisfied. Therefore $f'(x) = 0$ for at least one value of x in the open interval $(0, \pi)$.

Now $f'(x) = 0$ gives $\cos x = 0$ or $x = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \pm \frac{5\pi}{2}, \cdots$ Since $x = \pi/2$ lies in the open interval $(0, \pi)$, the Rolle's theorem is verified.

(iii) Here $f(x) = (x-a)^m (x-b)^n$.

As m and n are positive integers, $(x-a)^m$ and $(x-b)^n$ are polynomials in x on being expanded by binomial theorem. Hence $f(x)$ is also a polynomial in x. Consequently $f(x)$ is continuous and differentiable in the closed interval $[a, b]$. Also $f(a) = f(b) = 0$. Thus all the three conditions of Rolle's theorem are satisfied. So $f'(x) = 0$ for at least one value of x lying in the open interval (a, b).

Now $f'(x) = (x-a)^m . n(x-b)^{n-1} + m(x-a)^{m-1}(x-b)^n$.

The equation $f'(x) = 0$, on being solved, gives

$$x = a, b, \frac{na + mb}{m+n}.$$

Out of these values the value $\frac{na+mb}{m+n}$ is a point lying in the open interval (a, b) as it divides the interval (a, b) internally in the ratio $m : n$. Thus the Rolle's theorem is verified.

Ex. 5. *Verify Rolle's theorem for*

(i) $f(x) = x^3 - 4x$ *in* $[-2, 2]$. **(G.N.U. 1975)**

(ii) $f(x) = x(x+3)e^{-x/2}$ *in* $[-3, 0]$. **(Gorakhpur 1970)**

(iii) $f(x) = e^x(\sin x - \cos x)$ *in* $[\pi/4, 5\pi/4]$.

Sol. (i) Here $f(x) = x^3 - 4x$. Since $f(x)$ is a polynomial in x, therefore it is continuous and differentiable for every real value of x; Also $f(-2) = 0 = f(2)$.

$\therefore$ $f(x)$ satisfies all the three conditions of Rolle's theorem.

$\therefore$ there must exist at least one number, say c, in the open interval $(-2, 2)$ for which $f'(c) = 0$.

Now $f'(x) = 0$ gives $3x^2 - 4 = 0$ or

$$x = \pm \frac{2}{\sqrt{3}} = \pm 1.155 \text{ (approx)}.$$

Both these values lie in the open interval $(-2, 2)$. Thus the theorem is verified.

(ii) Here $f(x) = x(x+3)e^{-x/2} = (x^2 + 3x)e^{-x/2}$.

We have $f'(x) = (2x+3)e^{-x/2} + (x^2+3x).e^{-x/2}.(-\frac{1}{2})$

$$= e^{-x/2}[2x + 3 - \tfrac{1}{2}(x^2 + 3x)] = -\tfrac{1}{2}(x^2 - x - 6)e^{-x/2},$$

which exists for every value of x in the interval $[-3, 0]$. Therefore $f(x)$ is differentiable and also continuous in the interval $[-3, 0]$.

Also $f(-3) = 0 = f(0)$. Therefore all the three conditions of Rolle's theorem are satisfied.

$\therefore$ there must exist at least one number, say c, in the open interval $(-3, 0)$ for which $f'(c) = 0$ i.e., $-\frac{1}{2}(c^2 - c - 6)e^{-c/2} = 0$ or $c^2 - c - 6 = 0$ or $(c-3)(c+2) = 0$ or $c = 3, -2$.

The value $c = -2$ lies in the open interval $(-3, 0)$. Hence the theorem is verified.

(iii) Here $f(x) = e^x(\sin x - \cos x)$.

We have $f(\pi/4) = e^{\pi/4}(\sin(\pi/4) - \cos(\pi/4))$

$$= e^{\pi/4}[(1/\sqrt{2}) - (1/\sqrt{2})] = 0$$

and $$f\left(\frac{5\pi}{4}\right) = e^{5\pi/4}\left[\sin\frac{5\pi}{4} - \cos\frac{5\pi}{4}\right] = e^{5\pi/4}\left[-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right] = 0.$$

$\therefore$ $f(\pi/4) = f(5\pi/4) = 0$.

Further the function $f(x)$ is continuous and differentiable in $[\pi/4, 5\pi/4]$. Therefore all the three conditions of Rolle's theorem are satisfied.

$\therefore$ there must exist at least one number, say c, in the open interval $(\pi/4, 5\pi/4)$ for which $f'(c) = 0$.

Now $f'(x) = e^x(\cos x + \sin x) + e^x(\sin x - \cos x) = 2e^x \sin x$.

From $f'(x) = 0$ we get $2e^x \sin x = 0$

or $\sin x = 0$, $[\because e^x \neq 0]$

or $x = 0, \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$

Out of these values $x = \pi$ lies in the open interval $(\pi/4, 5\pi/4)$. Thus the Rolle's theorem is verified.

Ex. 6. *If* $f(x), \phi(x), \psi(x)$ *have derivatives when* $a \le x \le b$, *show that there is a value* c *of* x *lying between* a *and* b *such that*

$$\begin{vmatrix} f(a) & \phi(a) & \psi(a) \\ f(b) & \phi(b) & \psi(b) \\ f'(c) & \phi'(c) & \psi'(c) \end{vmatrix} = 0.$$ **(Agra 1973)**

Sol. Consider the following function

$$F(x) = \begin{vmatrix} f(a) & \phi(a) & \psi(a) \\ f(b) & \phi(b) & \psi(b) \\ f(x) & \phi(x) & \psi(x) \end{vmatrix}$$

On expanding the determinant, we observe that the function $F(x)$ is of the form $Af(x) + B\phi(x) + C\psi(x)$, where A, B, C are some real numbers.

Since the functions $f(x)$, $\phi(x)$ and $\psi(x)$ have derivatives when $a \le x \le b$, therefore the function $F(x)$ also possesses derivatives when

$a \le x \le b$. Consequently $F(x)$ is also continuous when $a \le x \le b$. Further $F(a) = F(b) = 0$ because then the two rows of the determinant become identical. Thus $F(x)$ satisfies all the three conditions of Rolle's theorem. Hence $F'(x) = 0$ for at least one value of x, say $x = c$, lying between a and b. Thus there is a value c of x lying between a and b such that

$$\begin{vmatrix} f(a) & \phi(a) & \psi(a) \\ f(b) & \phi(b) & \psi(b) \\ f'(c) & \phi'(c) & \psi'(c) \end{vmatrix} = 0.$$

••§ 2. Lagrange's mean value theorem or First mean value theorem. **(Lucknow 1983, 81; Gorakhpur 77; Meerut 81, 84P, 86, 91; Delhi 76, Agra 78; Alld. 81)**

If a function $f(x)$ is

(*i*) *continuous in the closed interval $a \le x \le b$,*

and (*ii*) *differentiable in the open interval (a, b) i.e., $a < x < b$, then there exists at least one value 'c' of x lying in the open interval $a < x < b$ such that*

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Proof. Consider the function $\phi(x)$ defined by

$$\phi(x) = f(x) + Ax, \qquad \text{...(1)}$$

where A is a constant to be determined such that $\phi(a) = \phi(b)$ *i.e.,*

$$f(a) + Aa = f(b) + Ab$$

or
$$-A = \frac{f(b) - f(a)}{b - a} \qquad \text{...(2)}$$

Now $f(x)$ is given to be continuous in $a \le x \le b$ and differentiable in $a < x < b$.

Again, A being a constant, Ax is also continuous in $a \le x \le b$ and differentiable in $a < x < b$.

$\therefore$ $\phi(x) = f(x) + Ax$ is continuous in $a \le x \le b$ and differentiable in $a < x < b$. Also by our choice of A, we have $\phi(a) = \phi(b)$. Thus $\phi(x)$ satisfies all the conditions of Rolle's theorem in the interval $[a, b]$. Hence there exists at least one point, say $x = c$, of the open interval $a < x < b$, such that $\phi'(c) = 0$.

But $\phi'(x) = f'(x) + A$, from (1).

$\therefore$ $\phi'(c) = 0$ gives $f'(c) + A = 0$

or
$$f'(c) = -A = \frac{f(b) - f(a)}{b - a}, \text{ from (2).}$$

This proves the theorem.

Another form of Lagrange's mean value theorem.

If a function $f(x)$ is

(*i*) *continuous in the closed interval $[a, a+h]$,*

and (*ii*) *differentiable in the open interval $(a, a+h)$, then there exists at least one number θ lying between 0 and 1 such that*

$$f(a+h) = f(a) + hf'(a+\theta h).$$

(K.U. 1974)

Proof. Let $a + h = b$. Then $b - a = h$ = the length of the interval.

Now give the complete proof of Lagrange's mean value theorem.

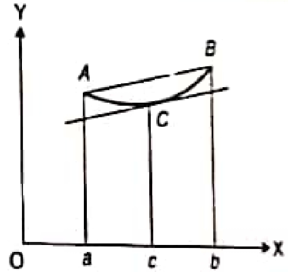
Since c lies between a and $a + h$, therefore it is greater than a by a fraction of h and may be written as $c = a + \theta h$, where $0 < \theta < 1$.

Hence the result of Lagrange's mean value theorem can be written as

$$f(a+h) - f(a) = hf'(a+\theta h). \qquad [0 < \theta < 1].$$

Geometrical interpretation of the mean value theorem. **(Meerut 1977, 78, 85 P; Lucknow 80)**

In the figure let ACB be the graph of $f(x)$ in (a, b) and let the chord AB make an angle α with the x-axis so that



$$\tan\alpha = \frac{f(b) - f(a)}{b - a}$$
$$= f'(c), \text{ by the Mean Value Theorem}$$

where $a < c < b$.

Thus there is some point c within (a, b) such that the tangent to the curve at the point $[c, f(c)]$ is parallel to the chord AB.

§ 3. Some important deductions from mean value theorem.

Theorem 1. *If a function $f(x)$ be such that $f'(x)$ is zero throughout the interval (a, b), then $f(x)$ must be constant throughout the interval.*

Proof. Let x_1, x_2 be any two points in the interval (a, b) such that $x_2 > x_1$. Since $f'(x)$ exists through out the interval (a, b), therefore $f(x)$ satisfies the conditions of Lagrange's mean value theorem in the interval $[x_1, x_2]$. So applying this theorem for $f(x)$ in the interval $[x_1, x_2]$, we get

$$\frac{f(x_2)-f(x_1)}{x_2-x_1}=f'(c), \text{ where } x_1<c<x_2.$$

But by hypothesis $f'(x)=0$ throughout the interval (a,b).

$\therefore \quad f'(c)=0$ or $f(x_2)-f(x_1)=0$ or $f(x_2)=f(x_1)$.

Thus the values of $f(x)$ at every two points of (a,b) are equal. Hence $f(x)$ must be constant throughout (a,b).

Theorem 2. *If $f(x)$ and $\phi(x)$ be two functions such that $f'(x)=\phi'(x)$ throughout the interval (a,b), then $f(x)$ and $\phi(x)$ differ only by a constant.*

Proof. Consider the function $F(x)=f(x)-\phi(x)$.

Throughout the interval (a,b), we have

$$F'(x)=f'(x)-\phi'(x)=0, \qquad [\because f'(x)=\phi'(x)].$$

Therefore, from theorem 1, we have

$$F(x)=\text{const}, \text{ or } f(x)-\phi(x)=\text{const}.$$

Theorem 3. *If $f(x)$ is*

(i) *continuous in the closed interval $[a,b]$,*

(ii) *differentiable in the open interval (a,b)*

and (iii) *$f'(x)$ is –ive in $a<x<b$, then*

$f(x)$ is a monotonically decreasing function in the closed interval $[a,b]$.

Proof. Let x_1, x_2 be any two points belonging to the closed interval $[a,b]$ such that $x_2>x_1$.

Applying Lagrange's mean value theorem to $f(x)$ in the interval $[x_1,x_2]$, we have

$$f(x_2)-f(x_1)=(x_2-x_1)f'(c), \text{ where } x_1<c<x_2. \qquad ...(1)$$

Now $x_2-x_1>0$. Since by hypothesis $f'(x)$ is negative for every x in (a,b), therefore $f'(c)<0$. Hence from (1), we have

$$f(x_2)-f(x_1)<0 \text{ i.e., } f(x_2)<f(x_1).$$

Thus $f(x)$ is a decreasing function of x in $[a,b]$.

Similarly we can prove that **a function having a positive derivative for every value of x in an interval is a monotonically increasing function in that interval.** **(Mysore 1971)**

Corollary. *The function $f(x)$ is strictly decreasing or increasing in $[a,b]$ if $f'(x)<0$ or $(f'(x)>0)$ for every x in (a,b) except for a finite number of points where the derivative is zero.*

§ 4. Cauchy's mean value theorem or second mean value theorem.

(Meerut 1991; Gorakhpur 82; Allahabad 82; Agra 79; Luck. 82)

If two functions $f(x)$ and $g(x)$ are

(i) *continuous in the closed interval $[a,b]$,*

(ii) *differentiable in the open interval (a,b),*

and (iii) *$g'(x)\neq 0$ for any point of the open interval (a,b), then there exists at least one value c of x in the open interval (a,b), such that*

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}, \; a<c<b.$$

Proof. First we note that $g(b)-g(a)\neq 0$. For if $g(b)-g(a)=0$ i.e., $g(b)=g(a)$, then the function $g(x)$ satisfies the conditions of Rolle's theorem and so its derivative $g'(x)$ should vanish for at least one value of x lying in the open interval (a,b). But this is contrary to our hypothesis.

Now consider the function $F(x)$ defined by

$$F(x)=f(x)+Ag(x), \qquad ...(1)$$

where A is a constant to be determined such that $F(a)=F(b)$ i.e.,

$$f(a)+Ag(a)=f(b)+Ag(b)$$

or
$$-A=\frac{f(b)-f(a)}{g(b)-g(a)}. \qquad ...(2)$$

Since $g(b)-g(a)\neq 0$, therefore A is a definite real number.

Now the function $F(x)$ obviously satisfies the conditions of Rolle's theorem in the interval $[a,b]$. Therefore there exists, at least one value, say c, of x in the open interval (a,b) such that $F'(c)=0$.

But $F'(x)=f'(x)+Ag'(x)$, from (1).

$\therefore \quad F'(c)=0$ gives $f'(c)+Ag'(c)=0$

or
$$-A=\frac{f'(c)}{g'(c)}. \qquad ...(3)$$

From (2) and (3), we get

$$\frac{f(b)-f(a)}{g(b)-g(a)}=\frac{f'(c)}{g'(c)}.$$

Another form. Let $b=a+h$. Then $a+\theta h=a$ when $\theta=0$ and $a+\theta h=b$ when $\theta=1$. Therefore $a+\theta h$, where $0<\theta<1$, means some value between a and b. So putting $b=a+h$ and $c=a+\theta h$, the result of the above theorem can be written as

$$\frac{f(a+h)-f(a)}{g(a+h)-g(a)}=\frac{f'(a+\theta h)}{g'(a+\theta h)}, \; 0<\theta<1.$$

Note. Lagrange's mean value theorem is a particular case of Cauchy's mean value theorem.

Let us set $g(x)=x$ in Cauchy's mean value theorem which is justified because $g(x)=x$ satisfies all the conditions of Cauchy's mean value theorem. But $g(x)=x$ means $g(b)=b$, $g(a)=a$, $g'(x)=1$ and so $g'(c)=1$. Putting these values in Cauchy's mean value theorem, we get

$$\frac{f(b)-f(a)}{b-a}=f'(c), \; (a<c<b)$$

which is nothing but the result of Lagrange's mean value theorem.

Solved Examples

Ex. 7 (a). *If* $f(x) = (x-1)(x-2)(x-3)$ *and* $a = 0, b = 4$, *find 'c' using Lagrange's mean value theorem.*

Sol. We have

$f(x) = (x-1)(x-2)(x-3) = x^3 - 6x^2 + 11x - 6.$

$\therefore \quad f(a) = f(0) = -6$, and $f(b) = f(4) = 6.$

$$\therefore \quad \frac{f(b)-f(a)}{b-a} = \frac{6-(-6)}{4-0} = \frac{12}{4} = 3.$$

Also $f'(x) = 3x^2 - 12x + 11$, so that $f'(c) = 3c^2 - 12c + 11$.

Substituting these values in Lagrange's mean value theorem

$\dfrac{f(b)-f(a)}{b-a} = f'(c), (a < c < b)$, we have

$3 = 3c^2 - 12c + 11$ or $3c^2 - 12c + 8 = 0$

or $$c = \frac{12 \pm \sqrt{(144-96)}}{6} = 2 \pm \frac{2\sqrt{3}}{3}.$$

Both of these values of c lie in the open interval $(0, 4)$. Hence both of these are the required values of c.

Ex. 7 (b). *Find 'c' of the mean value theorem, if*

$f(x) = x(x-1)(x-2); a = 0, b = \frac{1}{2}.$

Sol. Here $f(a) = f(0) = 0$ and

$f(b) = f(\frac{1}{2}) = \frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2) = \frac{3}{8}.$

$$\therefore \quad \frac{f(b)-f(a)}{b-a} = \frac{\frac{3}{8}-0}{\frac{1}{2}-0} = \frac{3}{4}.$$

Now $f(x) = x^3 - 3x^2 + 2x$.

$\therefore \quad f'(x) = 3x^2 - 6x + 2$, so that $f'(c) = 3c^2 - 6c + 2$.

Substituting these values in Lagrange's mean value theorem

$\dfrac{f(b)-f(a)}{b-a} = f'(c), (a < c < b)$, we have

$\frac{3}{4} = 3c^2 - 6c + 2$ or $12c^2 - 24c + 5 = 0$.

$$\therefore \quad c = \frac{24 \pm \sqrt{(24 \times 24 - 4 \times 12 \times 5)}}{24} = \frac{24 \pm 4\sqrt{(36-15)}}{24} = 1 \pm \frac{\sqrt{21}}{6}.$$

Out of these two values of c only $1 - \dfrac{\sqrt{21}}{6}$ lies in the open interval $(0, \frac{1}{2})$ which is therefore the required value of c.

Ex. 8. *Find 'c' so that*

$$f'(c) = \frac{f(b)-f(a)}{b-a} \text{ in the following cases :}$$

(i) $f(x) = x^2 - 3x - 1$; $a = -11/7, b = 13/7$.

(ii) $f(x) = e^x$; $a = 0, b = 1$.

Sol. **(i)** Here $f(a) = f\left(-\dfrac{11}{7}\right) = \dfrac{121}{49} + \dfrac{33}{7} - 1 = \dfrac{303}{49}$

and $$f(b) = f\left(\frac{13}{7}\right) = \frac{169}{49} - \frac{39}{7} - 1 = -\frac{153}{49}.$$

$$\therefore \quad \frac{f(b)-f(a)}{b-a} = \frac{-456/49}{24/7} = -\frac{19}{7}.$$

Now $f'(x) = 2x - 3$; $\therefore \quad f'(c) = 2c - 3$.

From Lagrange's mean value theorem, we have

$2c - 3 = -19/7$ or $c = 1/7$.

(ii) Here $f(a) = f(0) = e^0 = 1$, and $f(b) = f(1) = e^1 = e$. Also $f'(x) = e^x$, so that $f'(c) = e^c$

$\therefore$ using Lagrange's mean value theorem, we have

$\dfrac{e-1}{1-0} = e^c$ or $e^c = e - 1$ or $c = \log_e(e-1)$.

Ex. 9. *Compute the value of* θ *in the first mean value theorem*

$$f(x+h) = f(x) + hf'(x + \theta h),$$

if $f(x) = ax^2 + bx + c$.

Sol. We have $f(x) = ax^2 + bx + c$.

$\therefore \quad f(x+h) = a(x+h)^2 + b(x+h) + c,$

$f'(x) = 2ax + b, f'(x + \theta h) = 2a(x + \theta h) + b.$

Putting all these values in the Lagrange's mean value theorem, we have

$$a(x+h)^2 + b(x+h) + c = ax^2 + bx + c + h[2a(x+\theta h) + b] \quad \text{...(1)}$$

The relation (1) is identically true for all values of x. So when $x \to 0$, we get

$ah^2 + bh + c = c + h[2a\theta h + b]$

or $ah^2 = 2a\theta h^2$ or $\theta = 1/2$.

Ex. 10. *A function* $f(x)$ *is continuous in the closed interval* $0 \le x \le 1$ *and differentiable in the open interval* $0 < x < 1$, *prove that* $f'(x_1) = f(1) - f(0)$, *where* $0 < x_1 < 1$.

Sol. Here $a = 0, b = 1$. Therefore

$$\frac{f(b)-f(a)}{b-a} = \frac{f(1)-f(0)}{1-0} = f(1) - f(0).$$

If we take $c = x_1$ and substitute these values in the result of Lagrange's mean value theorem, we get

$f(1) - f(0) = f'(x_1)$ where $0 < x_1 < 1$.

Ex. 11. *Separate the intervals in which the polynomial* $2x^3 - 15x^2 + 36x + 1$ *is increasing or decreasing.*

Sol. Let $f(x) = 2x^3 - 15x^2 + 36x + 1$.

Then $f'(x) = 6x^2 - 30x + 36 = 6(x-2)(x-3)$.

Now $f'(x) > 0$ for $x < 2$; $f'(x) < 0$ for $2 < x < 3$; $f'(x) > 0$ for $x > 3$; $f'(x) = 0$ for $x = 2$ and 3.

Thus $f(x)$ is positive in the intervals $(-\infty, 2)$ and $(3, \infty)$ and negative in the interval $(2, 3)$.

Hence $f(x)$ is monotonically increasing in the intervals $(-\infty, 2]$, $[3, \infty)$ and monotonically decreasing in the interval $[2, 3]$.

Ex. 12. *Show that $x^3 - 3x^2 + 3x + 2$ is monotonically increasing in every interval.*

Sol. Let $f(x) = x^3 - 3x^2 + 3x + 2$.

Then $f'(x) = 3x^2 - 6x + 3 = 3(x-1)^2$.

We see that $f'(x) > 0$ for every real value of x except 1 where its value is zero. Hence $f(x)$ is monotonically increasing in every interval.

Ex. 13 (a). *Show that*

$$\frac{x}{1+x} < \log(1+x) < x \text{ for } x > 0.$$

(Delhi 1973)

Sol. Let $f(x) = \log(1+x) - \dfrac{x}{1+x}$.

Then

$$f'(x) = \frac{1}{1+x} - \frac{1.(1+x) - x.1}{(1+x)^2} = \frac{1}{1+x} - \frac{1}{(1+x)^2} = \frac{x}{(1+x)^2}.$$

We see that $f'(x) > 0$ for $x > 0$. Therefore $f(x)$ is monotonically increasing in the interval $[0, \infty)$. But $f(0) = 0$. Therefore $f(x) > f(0) = 0$ for $x > 0$ i.e., $\left[\log(1+x) - \dfrac{x}{1+x}\right] > 0$ for $x > 0$.

Hence $\log(1+x) > \dfrac{x}{1+x}$ for $x > 0$. ...(1)

Again let $\phi(x) = x - \log(1+x)$; then

$$\phi'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x}.$$

We see that $\phi'(x) > 0$ for $x > 0$. Therefore $\phi(x)$ is monotonically increasing in the interval $[0, \infty)$. But $\phi(0) = 0$. Therefore $\phi(x) > \phi(0) = 0$ for $x > 0$ i.e., $[x - \log(1+x)] > 0$ for $x > 0$.

Hence $x > \log(1+x)$ for $x > 0$. ...(2)

From (1) and (2), we have

$$\frac{x}{1+x} < \log(1+x) < x \text{ when } x > 0.$$

Ex. 13 (b). *Prove that for every $x > 0$, $\dfrac{x}{1+x^2} < \tan^{-1} x < x$.*

(Lucknow 1982)

Sol. Proceed exactly in the same way as in Ex. 13 (a). First take

$$f(x) = \tan^{-1} x - \frac{x}{1+x^2}.$$

Then $$f'(x) = \frac{2x^2}{(1+x^2)^2}.$$

Again take $\phi(x) = x - \tan^{-1} x$.

Then $\phi'(x) = \dfrac{x^2}{1+x^2}$.

Ex. 14. *State the conditions for the validity for the formula*

$$f(x+h) = f(x) + hf'(x + \theta h)$$

and investigate how far these conditions are satisfied and whether the result is true, when $f(x) = x \sin(1/x)$ (being defined to be zero at $x = 0$) and $x < 0 < x + h$.

Sol. The conditions for the validity of the given formula are :

(i) The function $f(x)$ must be continuous in the closed interval $[x, x+h]$.

(ii) The function $f(x)$ must be differentiable in the open interval $(x, x+h)$.

(iii) θ is a real number such that $0 < \theta < 1$.

Now consider the function $f(x)$ defined as :

$$f(x) = x \sin(1/x) \text{ for } x \neq 0, f(0) = 0.$$

The first condition is satisfied because $f(x)$ is continuous in the closed interval $[x, x+h]$ for $x < 0 < x + h$. [The students should show here that $f(x)$ is continuous at $x = 0$].

But the second condition is not satisfied because $f(x)$ is not differentiable at $x = 0$ which is a point lying in the open interval $(x, x+h)$ for $x < 0 < x + h$. [Show here that $f(x)$ is not differentiable at $x = 0$].

Hence the result of the given formula is not true for this function $f(x)$.

Ex. 15. *Verify Cauchy's mean value theorem for the functions x^2 and x^3 in the interval $[1, 2]$.*

Sol. Let $f(x) = x^2$ and $g(x) = x^3$. Both $f(x)$ and $g(x)$ are continuous in the closed interval $[1, 2]$ and differentiable in the open interval $(1, 2)$. Also $g'(x) = 3x^2 \neq 0$ for any point in the open interval $(1, 2)$. Therefore by Cauchy's mean value theorem there exists at least one real number c in the open interval $(1, 2)$, such that

$$\frac{f(2) - f(1)}{g(2) - g(1)} = \frac{f'(c)}{g'(c)} \quad \text{...(1)}$$

Now $\dfrac{f(2) - f(1)}{g(2) - g(1)} = \dfrac{4-1}{8-1} = \dfrac{3}{7}$. Also $f'(x) = 2x$, $g'(x) = 3x^2$.

Therefore $\dfrac{f'(c)}{g'(c)} = \dfrac{2c}{3c^2} = \dfrac{2}{3c}$.

Substituting these values in (1), we get $\frac{3}{7} = \frac{2}{3c}$ or $c = \frac{14}{9}$ which lies in the open interval $(1, 2)$. This verifies the theorem.

Ex. 16. *If in the Cauchy's mean value theorem, we write $f(x) = e^x$ and $g(x) = e^{-x}$, show that 'c' is the arithmetic mean between a and b.*

Sol. Here $\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{e^b - e^a}{e^{-b} - e^{-a}} = -e^a e^b = -e^{a+b}$.

Also $\frac{f'(x)}{g'(x)} = \frac{e^x}{-e^{-x}}$, so that $\frac{f'(c)}{g'(c)} = \frac{e^c}{-e^{-c}} = e^{2c}$.

Substituting these values in Cauchy's mean value theorem, we get $-e^{a+b} = -e^{2c}$ or $2c = a + b$ or $c = \frac{1}{2}(a + b)$.

Hence c is the arithmetic mean between a and b.

Ex. 17. *If, in the Cauchy's mean value theorem, we write*

(i) *$f(x) = \sqrt{x}$ and $g(x) = 1/\sqrt{x}$, then c is the geometric mean between a and b, and if*

(ii) *$f(x) = 1/x^2$ and $g(x) = 1/x$, then c is the harmonic mean between a and b.*

Sol. (i) Here $\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{\sqrt{b} - \sqrt{a}}{(1/\sqrt{b}) - (1/\sqrt{a})} = -\sqrt{(ab)}$.

Also $\frac{f'(x)}{g'(x)} = \frac{\frac{1}{2}x^{-1/2}}{-\frac{1}{2}x^{-3/2}}$, so that $\frac{f'(c)}{g'(c)} = -\frac{c^{-1/2}}{c^{-3/2}} = -c$.

Substituting these values in Cauchy's mean value theorem, we get $-\sqrt{(ab)} = -c$ or $c = \sqrt{(ab)}$ *i.e.*, c is the geometric mean between a and b.

(ii) From the Cauchy's mean value theorem, we have

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Putting $f(x) = 1/x^2$ and $g(x) = 1/x$, we get

$$\frac{(1/b^2) - (1/a^2)}{(1/b) - (1/a)} = \frac{-2c^{-3}}{-c^{-2}} \text{ or } \frac{a+b}{ab} = \frac{2}{c} \text{ or } c = \frac{2ab}{a+b}$$

i.e., c is the harmonic mean between a and b.

§ 5. Taylor's theorem with Lagrange's form of remainder after n terms.

(Delhi 1971; K.U. 73; Meerut 90)

If $f(x)$ is a single valued function of x such that

(i) *all the derivatives of $f(x)$ upto $(n-1)^{th}$ are continuous in $a \le x \le a + h$,*

and (ii) *$f^{(n)}(x)$ exists in $a < x < a + h$, then*

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots$$
$$+ \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n}{n!}f^{(n)}(a + \theta h), \text{ where } 0 < \theta < 1.$$

Proof. Consider the function $\phi(x)$ defined by

$$\phi(x) = f(x) + (a + h - x)f'(x) + \frac{(a + h - x)^2}{2!}f''(x) + \dots$$
$$+ \frac{(a + h - x)^{n-1}}{(n-1)!}f^{(n-1)}(x) + \frac{A}{n!}(a + h - x)^n,$$

where A is a constant to be determined such that

$$\phi(a) = \phi(a + h).$$

Now

$$\phi(a) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots$$
$$+ \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{A}{n!}h^n,$$

and $\phi(a + h) = f(a + h)$.

Therefore A is given by

$$f(a + h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \dots\dots$$
$$+ \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n}{n!}A. \quad \dots(1)$$

Now, by hypothesis, all the functions

$$f(x), f'(x), f''(x), \dots, f^{(n-1)}(x)$$

are continuous in the closed interval $[a, a + h]$ and differentiable in the open interval $(a, a + h)$.

Also $(a + h - x)$, $(a + h - x)^2/2!, \dots, (a + h - x)^n/n!$, all being polynomials, are continuous in the closed interval $[a, a + h]$ and differentiable in the open interval $(a, a + h)$. Further A is a constant.

$\therefore$ $\phi(x)$ is continuous in the closed interval $[a, a + h]$ and differentiable in the open interval $(a, a + h)$. Also by our choice of A, $\phi(a) = \phi(a + h)$. Thus $\phi(x)$ satisfies all the conditions of Rolle's theorem.

$\therefore$ $\phi'(a + \theta h) = 0$, where $0 < \theta < 1$.

Now

$$\phi'(x) = f'(x) - f'(x) + (a + h - x)f''(x) - (a + h - x)f''(x)$$
$$+ \dots + \frac{(a + h - x)^{n-1}}{(n-1)!}f^{(n)}(x) - \frac{A}{(n-1)!}(a + h - x)^{n-1}$$
$$= \frac{(a + h - x)^{n-1}}{(n-1)!}[f^{(n)}(x) - A], \text{ since other terms cancel in pairs.}$$

$\therefore\ \phi'(a+\theta h) = 0$ gives

$$\frac{[a+h-(a+\theta h)]^{n-1}}{(n-1)!}[f^{(n)}(a+\theta h) - A] = 0$$

or
$$\frac{h^{n-1}(1-\theta)^{n-1}}{(n-1)!}[f^{(n)}(a+\theta h) - A] = 0.$$

Now $h \neq 0$. Also $(1-\theta) \neq 0$ because $0 < \theta < 1$.

$\therefore\ f^{(n)}(a+\theta h) - A = 0$ or $A = f^{(n)}(a+\theta h)$.

Substituting this value of A in (1), we get

$$\mathbf{f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots\ldots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n}{n!}f^{(n)}(a+\theta h)}.$$

This is **Taylor's development** of $f(a+h)$ in ascending integral powers of h. The $(n+1)^{th}$ term $\frac{\mathbf{h}^n}{\mathbf{n}!}\mathbf{f}^{(n)}(\mathbf{a}+\theta\mathbf{h})$ is called **Lagrange's form of remainder** after n terms in Taylor's expansion of $f(a+h)$.

Note. If we take $n = 1$, we observe that Lagrange's mean value theorem is a particular case of Taylor's theorem.

Corollary. (Maclaurin's development). Instead of considering the interval $[a, a+h]$, let us take the interval $[0, x]$. Then changing a to 0 and h to x in Taylor's theorem, we get

$$\mathbf{f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots\ldots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + \frac{x^n}{n!}f^{(n)}(\theta x)},$$

which is known as **Maclaurin's theorem** or **Maclaurin's development** of $f(x)$ in the interval $[0, x]$ with **Lagrange's form of remainder** $\frac{\mathbf{x}^n}{\mathbf{n}!}\mathbf{f}^{(n)}(\theta\mathbf{x})$.

§ 6. Taylor's theorem with Cauchy's form of remainder.

(G.N.U. 1975; Meerut 91)

If $f(x)$ is a single valued function of x such that

(i) *all the derivatives of $f(x)$ upto $(n-1)^{th}$ are continuous in $a \le x \le a+h$,*

and (ii) *$f^{(n)}(x)$ exists in $a < x < a+h$, then*

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots\ldots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n}{(n-1)!}(1-\theta)^{n-1}f^{(n)}(a+\theta h),$$ *where* $0 < \theta < 1$.

Proof. Consider the function $\phi(x)$ defined by

$$\phi(x) = f(x) + (a+h-x)f'(x) + \frac{(a+h-x)^2}{2!}f''(x) + \ldots\ldots + \frac{(a+h-x)^{n-1}}{(n-1)!}f^{(n-1)}(x) + (a+h-x)A,$$

where A is a constant to be determined such that $\phi(a) = \phi(a+h)$.

Now $\phi(a) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + hA,$

and $\phi(a+h) = f(a+h)$.

Therefore A is given by

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + hA. \qquad \ldots(1)$$

As shown in § 5, we can easily show that $\phi(x)$ satisfies all the conditions of Rolle's theorem.

$\therefore\ \phi'(a+\theta h) = 0$, where $0 < \theta < 1$.

Now $\phi'(x) = \frac{(a+h-x)^{n-1}}{(n-1)!}f^{(n)}(x) - A$, since other terms cancel in pairs. $\therefore\ \phi'(a+\theta h) = 0$ gives

$$\frac{[a+h-(a+\theta h)]^{n-1}}{(n-1)!}f^{(n)}(a+\theta h) - A = 0$$

or
$$A = \frac{h^{n-1}}{(n-1)!}(1-\theta)^{n-1}f^{(n)}(a+\theta h).$$

Substituting this value of A in (1), we get

$$\mathbf{f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n}{(n-1)!}(1-\theta)^{n-1}f^{(n)}(a+\theta h)}.$$

The $(n+1)^{th}$ term $\frac{\mathbf{h}^n}{(\mathbf{n}-1)!}(1-\theta)^{n-1}\mathbf{f}^{(n)}(\mathbf{a}+\theta\mathbf{h})$ is called **Cauchy's form of remainder** after n terms in Taylor's expansion of $f(a+h)$ in ascending integral powers of h.

Corollary. (Maclaurin's development with Cauchy's form of remainder). Changing a to 0 and h to x in the above result, we get

$$\mathbf{f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots + \frac{x^{n-1}}{(n-1)!}f^{(n-1)}(0) + \frac{x^n}{(n-1)!}(1-\theta)^{n-1}f^{(n)}(\theta x)},$$

which is known as **Maclaurin's development** of $f(x)$ in the interval $[0, x]$ with **Cauchy's form** of remainder after n terms.

Solved Examples

Ex. 18. *Expand the following by Maclaurin's theorem with Lagrange's form of remainder after n terms*

(i) a^x, (ii) e^x.

Sol. (i) Here $f(x) = a^x$ ...(1)

$\therefore \quad f^{(n)}(x) = a^x (\log a)^n$ ...(2)

Putting $x = 0$ in (1) and (2), we get

$$f(0) = a^0 = 1,\ f^{(n)}(0) = a^0 (\log a)^n = (\log a)^n.$$

$$\therefore \quad f'(0) = \log a,\ f''(0) = (\log a)^2, \ldots f^{(n-1)}(0) = (\log a)^{n-1}.$$

Also changing x to θx in (2), we get

$$f^{(n)}(\theta x) = a^{\theta x} (\log a)^n.$$

Now by Maclaurin's theorem with Lagrange's form of remainder after n terms, we have

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0)$$

$$+ \frac{x^n}{n!} f^{(n)}(\theta x), \text{ where } 0 < \theta < 1. \quad \ldots(3)$$

Substituting the values found above in (3), we get

$$a^x = 1 + x \log a + \frac{x^2}{2!} (\log a)^2 + \ldots + \frac{x^{n-1}}{(n-1)!} (\log a)^{n-1}$$

$$+ \frac{x^n}{n!} a^{\theta x} (\log a)^n.$$

Here Lagrange's form of remainder after n terms

$$= \frac{x^n}{n!} a^{\theta x} (\log a)^n, \text{ where } 0 < \theta < 1.$$

(ii) Here $f(x) = e^x$. Therefore $f^{(n)}(x) = e^x$.

Putting $x = 0$ in these, we get

$$f(0) = e^0 = 1,\ f^{(n)}(0) = e^0 = 1. \text{ Also } f^{(n)}(\theta x) = e^{\theta x}.$$

Substituting these values in Maclaurin's theorem with Lagrange's form of remainder after n terms, we get

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^{n-1}}{(n-1)!} + \frac{x^n}{n!} e^{\theta x}.$$

Ex. 19. *Show that*

(i) $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \ldots + (-1)^{n-1} \dfrac{x^{2n-1}}{(2n-1)!}$

$$+ (-1)^n \frac{x^{2n}}{(2n)!} \sin \theta x.$$

(K.U. 1973)

for every real value of x.

(ii) $\log(1 + x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \ldots + (-1)^{n-2} \dfrac{x^{n-1}}{n-1}$

$$+ (-1)^{n-1} \frac{x^n}{n(1 + \theta x)^n} \text{ for } x > -1.$$

(Panjab University 1975)

Sol. (i) Here $f(x) = \sin x$ ...(1)

We know that $\sin x$ possesses derivatives of every order for every real number x and

$$f^{(n)}(x) = \sin (x + \tfrac{1}{2} n\pi). \quad \ldots(2)$$

Putting $x = 0$ in (1) and (2), we get

$$f(0) = \sin 0 = 0,\ f^{(n)}(0) = \sin (\tfrac{1}{2} n\pi).$$

$$\therefore \quad f'(0) = \sin (\tfrac{1}{2}\pi) = 1,\ f''(0) = \sin \pi = 0.$$

$$f'''(0) = \sin (3\pi/2) = -1,$$

$$f^{iv}(0) = \sin 2\pi = 0,$$

$$f^{v}(0) = \sin (5\pi/2) = \sin (2\pi + \tfrac{1}{2}\pi) = \sin \tfrac{1}{2}\pi = 1, \ldots$$

$$f^{(2n-2)}(0) = \sin (n-1)\pi = 0,$$

$$f^{(2n-1)}(0) = \sin \{\tfrac{1}{2}(2n-1)\pi\} = \sin (n\pi - \tfrac{1}{2}\pi)$$

$$= (-1)^n \sin (-\pi/2),$$

$$[\because \sin (n\pi + \theta) = (-1)^n \sin \theta]$$

$$= (-1)^n (-1) = (-1)^{n+1} = (-1)^{n-1}.$$

Also changing n to $2n$ and x to θx in (2), we get

$$f^{(2n)}(\theta x) = \sin (\theta x + n\pi) = (-1)^n \sin \theta x.$$

Substituting these values in Maclaurin's theorem with Lagrange's form of remainder after $2n$ terms *i.e.*,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \ldots$$

$$+ \frac{x^{2n-1}}{(2n-1)!} f^{(2n-1)}(0) + \frac{x^{2n}}{(2n)!} f^{(2n)}(\theta x),$$

we $\quad \sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dots$

$$+ (-1)^{n-1} \frac{x^{2n-1}}{(2n-1)!} + (-1)^n \frac{x^{2n}}{(2n)!} \sin \theta x.$$

(ii) Here $f(x) = \log(1 + x)$. ...(1)

We know that $\log(1 + x)$ possesses derivatives of every order when $(1 + x) > 0$ *i.e.*, $x > -1$.

Also, $\quad f^{(n)}(x) = (-1)^{n-1}(n-1)!(1+x)^{-n}$ ...(2)

Putting $x = 0$ in (1) and (2), we get

$$f(0) = \log 1 = 0, f^{(n)}(0) = (-1)^{n-1}(n-1)!.$$

Also changing x to θx in (2), we get

$$f^{(n)}(\theta x) = (-1)^{n-1}(n-1)!(1+\theta x)^{-n}.$$

Substituting these values in Maclaurin's theorem with Lagrange's form of remainder after n terms *i.e.*,

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \dots$$

$$+ \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) + \frac{x^n}{n!} f^{(n)}(\theta x),$$

we get

$$\log(1 + x) = 0 + \frac{x}{1!} \cdot 1 + \frac{x^2}{2!} \cdot (-1) \cdot 1! + \frac{x^3}{3!}(-1)^2 \cdot 2! + \dots$$

$$+ \frac{x^{n-1}}{(n-1)!}(-1)^{n-2}(n-2)!$$

$$+ \frac{x^n}{n!}(-1)^{n-1}(n-1)!(1+\theta x)^{-n}$$

$$= x - \frac{x^2}{2} + \frac{x^3}{3} - \dots + (-1)^{n-2}\frac{x^{n-1}}{n-1} + (-1)^{n-1}\frac{x^n}{n(1+\theta x)^n}.$$

Ex. 20. *Find* θ, *if*

$$f(x + h) = f(x) + hf'(x) + \frac{h^2}{2!} f''(x + \theta h),\ 0 < \theta < 1, \text{ and}$$

(*i*) $f(x) = ax^3 + bx^2 + cx + d$, (Lucknow 1981)

(*ii*) $f(x) = x^3$. (Lucknow 1983, 80)

Sol. (i) Here $f(x) = ax^3 + bx^2 + cx + d$.

$\therefore \quad f(x + h) = a(x + h)^3 + b(x + h)^2 + c(x + h) + d,$

$f'(x) = 3ax^2 + 2bx + c,\ f''(x) = 6ax + 2b,$

and so $\quad f''(x + \theta h) = 6a(x + \theta h) + 2b.$

Putting these values in the given relation

$f(x + h) = f(x) + hf'(x) + (h^2/2!) f''(x + \theta h)$, we

$$a(x + h)^3 + b(x + h)^2 + c(x + h) + d$$

$$= ax^3 + bx^2 + cx + d + h(3ax^2 + 2bx + c)$$

$$+ (h^2/2!)\{6a(x + \theta h) + 2b\} \quad ...(1)$$

The relation (1) is an identity in x. Letting $x \to 0$ on both sides of (1), we have

$$ah^3 + bh^2 + ch + d = d + ch + (h^2/2)(6a\theta h + 2b)$$

or $\quad ah^3 + bh^2 + ch + d = d + ch + 3a\theta h^3 + bh^2$

or $\quad ah^3 = 3a\theta h^3 \quad$ or $\quad \theta = 1/3. \quad [\because ah^3 \neq 0]$

(ii) Proceed as in part (i) of this question. The required value of θ is 1/3. $\square$