Set - V

Numerical Integration

Consider a function of single variable $y = f(x)$. If the function is known and simple, we can easily evaluate its definite integral. However, if we do not know the function as such or the function is complicated such as $f(x) = e$, $f(x) = \frac{\sin x}{x}$ which have no anti-derivatives expressible interms of elementary functions and is given in a tabular form at a set of points $x_0, x_1, \ldots x_n$, we use only numerical methods for integration of the function.

To evaluate the integral, we fit up a suitable interpolation polynomial to the given set of values of $f(x)$ and then integrate it with in the desired limits. Here we integrate an approximate interpolation formula instead of $f(x)$. When this technique is applied on a function of single variable, the process is called Quadrature.

We have studied several interpolation formulae which fits the given data (x_k, f_k) $k = 0, 1, 2 \ldots n$. So, the different integration formulae can be obtained depending upon the

Now we derive a general quadrature formula for numerical integration using Newton's forward difference formula.

Newton-cotes formula (A general quadrature formula for equidistant ordinates):-

Consider the integral $I = \int_a^b f(x)\,dx$. ——(1)

where $f(x)$ takes the values

$f(x_0) = y_0$, $f(x_0+h) = y_1$, $f(x_0+2h) = y_2$, ---- $f(x_0+nh) = y_n$ when $x = x_0$, $x = x_0+h$, $x = x_0+2h$, ---- $x = x_0+nh$ respectively.

To evaluate I, we replace $f(x)$ by a suitable interpolation formula.

Let the interval $[a,b]$ be divided into n sub-intervals of width h so that $x_0 = a$, $x_1 = x_0+h$, $x_2 = x_0+2h$, ---
---------- $x_n = x_0+nh = b$.

Approximating $f(x)$ by Newton's forward interpolation formula we can write the integral (1) as

$$I = \int_{x_0}^{x_n} f(x)\,dx.$$

$$= \int^{x_0+nh} f(x)\,dx$$

(2)

$$= \int_{x_0}^{x_0+nh} \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots\cdots \right] dx \quad \text{---} (2)$$

Since $p = \frac{x - x_0}{h}$

$\Rightarrow x = x_0 + ph$

$\Rightarrow dx = h\,dp$

when $x = x_0$, $p = 0$

and when $x = x_0 + nh$, $p = n$.

Equation (2) can be written as

$$I = h\int_0^n \left[y_0 + p\Delta y_0 + \frac{p^2 - p}{2!}\Delta^2 y_0 + \frac{p^3 - 3p^2 + 2p}{3!}\Delta^3 y_0 + \cdots\cdots \right] dp$$

$$= h\left[p y_0 + \frac{p^2}{2}\Delta y_0 + \frac{\frac{p^3}{3} - \frac{p^2}{2}}{2!}\Delta^2 y_0 + \frac{1}{6}\left(\frac{p^4}{4} - p^3 + p^2\right)\Delta^3 y_0 + \cdots\cdots \right]_0^n$$

$$= h\left[n y_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{n^4}{4} - n^3 + n^2\right)\Delta^3 y_0 + \cdots\cdots \right]$$

This is known as Newton-cotes quadrature formula. From this general formula, we can obtain different integration formulae by putting $n = 1, 2, 3, \ldots$ etc.

Trapezoidal Rule :-

putting $n=1$, in the quadrature formula, and taking the curve through (x_0, y_0) and (x_1, y_1) as a straightline i.e, a polynomial of first order so that differences of order higher than first become zero.

we get
$$\int_{x_0}^{x_0+h} f(x)\,dx = h\left[y_0 + \tfrac{1}{2}\Delta y_0\right]$$
$$= h\left[y_0 + \tfrac{1}{2}(y_1 - y_0)\right]$$
$$= \frac{h}{2}(y_0 + y_1)$$

For the next interval $[x_1, x_2]$, we deduce

Similarly
$$\int_{x_1}^{x_2} f(x)\,dx = \int_{x_0+h}^{x_0+2h} f(x)\,dx$$
$$= \frac{h}{2}\left(y_1 + \tfrac{1}{2}\Delta y_1\right)$$
$$= \frac{h}{2}(y_1 + y_2)$$

and so on.

for the last interval $[x_{n-1}, x_n]$,

we have
$$\int_{x_{n-1}}^{x_n} f(x)\,dx = \int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx$$
$$= \frac{h}{2}(y_{n-1} + y_n)$$

Adding these n integrals, we obtain

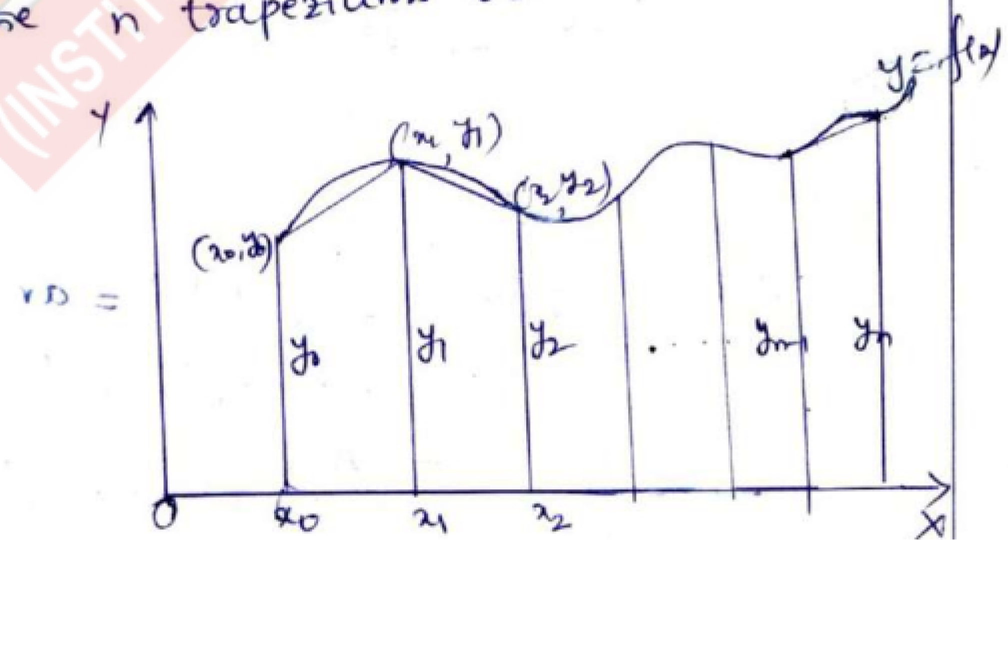
$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+\cdots y_{n-1})\right]$$

which is known as the trapezoidal rule.

Geometrical interpretation:

The geometrical significance of this rule is that the curve $y=f(x)$ is replaced by 'n' straightlines joining the points (x_0, y_0), and (x_1, y_1); (x_1, y_1) and (x_2, y_2); ......; (x_{n-1}, y_{n-1}) and (x_n, y_n).

The area bounded by the curve $y=f(x)$, the ordinates $x=x_0$ and $x=x_n$, and the x-axis is then approximately equivalent to the sum of the areas of the n trapeziums obtained.



The error of the trapezoidal formula can be obtained in the following way:

Let $f(x)$ be continuous, well-behaved and possess continuous derivatives in $[x_0, x_n]$.

Expanding y in a Taylor's series around $x = x_0$,

$$y(x) = y_0 + \frac{(x-x_0)}{1!} y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \cdots \quad \text{——} (1)$$

where $y_0' = [y'(x)]_{x=x_0}$.

$$\int_{x_0}^{x_1} y\,dx = \int_{x_0}^{x_1} \left[y_0 + \frac{(x-x_0)}{1!} y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \cdots \right] dx$$

$$= \left[y_0 x + \frac{(x-x_0)^2}{2!} y_0' + \frac{(x-x_0)^3}{3!} y_0'' + \cdots \right]_{x_0}^{x_1}$$

$$= y_0 (x_1 - x_0) + \frac{(x_1-x_0)^2}{2!} y_0' + \frac{(x_1-x_0)^3}{3!} y_0'' + \cdots$$

$$= h y_0 + \frac{h^2}{2!} y_0' + \frac{h^3}{3!} y_0'' + \cdots \quad \text{——} (2)$$

where h is the equal interval length.

Also $\int_{x_0}^{x_1} y\,dx = \frac{h}{2}(y_0 + y_1)$.

= area of the first trapezium

$= A_0$. —— (3)

Putting $x = x_1$ in (1), we get

$$y(x_1) = y_1 = y_0 + \frac{(x_1 - x_0)}{1!} y_0' + \frac{(x_1-x_0)^2}{2!} y_0'' + \cdots$$

$$\Rightarrow y_1 = y_0 + \frac{h}{\ldots} y_0' + \frac{h^2}{\ldots} y_0'' + \cdots \quad \text{——} (4)$$

$\therefore$ from ③ & ④, we have

$$A_0 \simeq \frac{h}{2}\left[y_0 + y_0 + \frac{h}{1!}y_0' + \frac{h^2}{2!}y_0'' + \cdots\right]$$

$$= hy_0 + \frac{h^2}{2}y_0' + \frac{h^3}{2\times 2!}y_0'' + \cdots$$

Subtracting A_0 value from ②

$$\int_{x_0}^{x_1} y\,dx - A_0 = h^3 y_0''\left[\frac{1}{3!} - \frac{1}{2\times 2!}\right] + \cdots$$

$$= -\frac{1}{12}h^3 y_0'' + \cdots$$

which is the error in the first interval $[x_0, x_1]$.

(neglecting the higher powers of h).

proceeding in a similar manner we obtain the errors in the remaining subintervals, viz., $[x_1, x_2]$, $[x_2, x_3]$, ... and $[x_{n-1}, x_n]$.

Thus we have

$$E = -\frac{1}{12}h^3\left[y_0'' + y_1'' + \cdots + y_{n-1}''\right]$$

where E is the ~~total~~ error.

Assuming that $y''(\bar{x})$ is the largest value of the n quantities y_0'', y_1'', ... y_{n-1}''.

we obtain
$$E = -\frac{1}{12}h^3 n y''(\bar{x})$$

$$= -\frac{(b-a)}{12}h^2 y''(\bar{x})$$

$(\because nh = b-a)$

Hence, the error in the trapezoidal rule is of the order h^2.

Simpson's one-third Rule:-

putting $n=2$ in the quadrature formula and taking the curve through (x_0, y_0), (x_1, y_1) and (x_2, y_2) as a parabola. i.e, a polynomial of second order so that differences of order higher than second vanish, we get

$$\int_{x_0}^{x_2} f(x)\,dx = \int_{x_0}^{x_0+2h} f(x)\,dx$$

$$= h\left[2y_0 + \frac{4}{2}\Delta y_0 + \frac{1}{2}\left(\frac{8}{3}-\frac{4}{2}\right)\Delta^2 y_0\right]$$

$$= h\left[2y_0 + 2(y_1 - y_0) + \frac{1}{3}\Delta^2 y_0\right]$$

$$= h\left[2y_1 + \frac{1}{3}(y_2 - 2y_1 + y_0)\right]$$

$$= \frac{h}{3}\left[y_0 + 4y_1 + y_2\right] \qquad \left(\because \Delta^2 y_0 = \Delta y_1 - \Delta y_0 = y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_1 + y_0.\right)$$

Similarly,

$$\int_{x_2}^{x_4} f(x)\,dx = \frac{h}{3}\left[y_2 + 4y_3 + y_4\right]$$

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$$\int_{x_{n-2}}^{x_n} f(x)\,dx = \frac{h}{3}\left[y_{n-2} + 4y_{n-1} + y_n\right], \text{ n being even.}$$

Adding all these integrals, if n is even +ve integer. i.e, the number of ordinates $y_0, y_1, y_2, \ldots y_n$ is odd,

we have $\displaystyle\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_0+nh} f(x)\,dx$

$$= \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \cdots\cdots + \int_{x_{n-2}}^{x_n} f(x)\,dx.$$

$$= \frac{h}{3}\left[(y_0+4y_1+y_2)+(y_2+4y_3+y_4)+(y_4+4y_5+y_6)\right.$$
$$\left.+\cdots\cdots+(y_{n-2}+4y_{n-1}+y_n)\right]$$

$$= \frac{h}{3}\left[(y_0+y_n)+2(y_2+y_4+\cdots y_{n-2})+4(y_1+y_3+\cdots+y_{n-1})\right].$$

$= \frac{h}{3}$ [sum of the first and last ordinates + 2 (sum of even ordinates) + 4 (sum of remaining odd ordinates)]

which is known as Simpson's $\frac{1}{3}$ rule. Simply Simpson's rule.

Note:-

1. Though y_2 has suffix even, it is the third ordinate (odd).

2. This rule requires the division of the whole range into an even number of subintervals of width 'h'.

Error in Simpson's formula:

Following the method outlined as in case of Trapezoidal rule, it can be shown that the error in Simpson's rule is given by

$$\boxed{E = \frac{-nh^5}{90}\, y^{iv}(\bar{x})}$$

where $y^{iv}(\bar{x})$ is the largest value of the fourth derivatives.

$$\boxed{E = -\frac{(b-a)}{180}\, h^4\, y^{iv}(\bar{x})}$$

$\left(\because (2n)h = b-a \Rightarrow nh = \frac{b-a}{2}\right)$

Hence, the error in Simpson's one third rule is of the order h^4.

Simpson's 3/8-Rule:-

putting $n=3$ in the quadrature formula and taking the curve through (x_i, y_i) ; $i=0,1,2,3$. as a polynomial of third order so that differences above the third order vanish.

we get

$$\int_{x_0}^{x_3} f(x)\,dx = \int_{x_0}^{x_0+3h} f(x)\,dx$$

$$= 3h\left(y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right)$$

$$= \frac{3h}{8}\left(y_0 + 3y_1 + 3y_2 + y_3\right).$$

Similarly

$$\int_{x_0+3h}^{x_0+6h} f(x)\,dx = \frac{3h}{8}\left(y_3 + 3y_4 + 3y_5 + y_6\right)$$

and so on.

Adding all these integrals from x_0 to x_0+nh where 'n' is a multiple of 3.

we obtain

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \int_{x_0}^{x_0+3h} f(x)\,dx + \int_{x_0+3h}^{x_0+6h} f(x)\,dx + \cdots\cdots + \int_{x_0+(n-3)h}^{x_0+nh} f(x)\,dx.$$

$$= \frac{3h}{8}\Big[\left(y_0 + 3y_1 + 3y_2 + y_3\right) + \left(y_3 + 3y_4 + 3y_5 + y_6\right) + \cdots\cdots + \left(y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n\right)\Big]$$

$$= \frac{3h}{8}\Big[(y_0+y_n)+3(y_1+y_2+y_4+y_5+\cdots\cdots+y_{n-2}+y_{n-1}) + 2(y_3+y_6+\cdots\cdots+y_{n-3})\Big]$$

which is known as simpson's 3/8 - rule.

Note: [1]. while applying Simpson's 3/8 - rule, the number of sub-intervals should be taken as multiple of 3.

→ The error in Simpson's 3/8 - rule is given by

$$\boxed{E = -\frac{3h^5}{80} y^{iv}(\bar{x})}$$

where $y^{iv}(\bar{x})$ is the largest value of the fourth order derivative

Note: It may be noted that the errors in Simpson's 1/3 and 3/8 - rules are of the same order. However, if we consider the magnitudes of the error terms, Simpson's 1/3 rule is superior to Simpson's 3/8 rule.

Hence Simpson's 3/8 - rule, is not so accurate as Simpson's rule, the dominant term in the error of this formula being $-\frac{3}{80} h^5 y^{iv}(\bar{x})$.

Example 1 : Calculate the value $\int_0^1 \frac{x}{1+x}dx$ correct upto three significant figures taking six intervals by Trapezoidal rule.

Solution : Here we have

$$f(x) = \frac{x}{1+x},$$

$$a = 0,\ b = 1 \text{ and } n = 6.$$

$$\therefore\ h = \frac{b-a}{n} = \frac{1-0}{6} = \frac{1}{6}.$$

x	0	1/6	2/6	3/6	4/6	5/6	6/6=1
$y=f(x)$	0.00000	0.14286	0.25000	0.33333	0.40000	0.45454	0.50000
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

The Trapezoidal rule can be written as

$$I = \frac{h}{2}\left[(y_0 + y_6) + 2(y_1 + y_2 + y_3 + y_4 + y_5)\right]$$

$$= \frac{1}{12}\left[(0.00000 + 0.50000) + 2(0.14286 + 0.25000 + 0.3333 + 0.40000 + 0.45454)\right]$$

$$= 0.30512.$$

$\therefore\ I = 0.0305$, correct to three significant figures.

IAS-2019

Example 2 : Find the value of $\int_0^1 \frac{dx}{1+x^2}$, taking 5 subinterval by Trapezoidal rule, correct to five significant figures. Also compare it with its exact value.

Solution : Here

$$f(x) = \frac{1}{1+x^2},$$

$$a = 0,\ b = 1, \text{ and } n = 5.$$

$$\therefore\ h = \frac{1-0}{5} = \frac{1}{5} = 0.2.$$

x	0.0	0.2	0.4	0.6	0.8	1
$y=f(x)$	1.000000	0.961538	0.832069	0.735294	0.609756	0.500000
	y_0	y_1	y_2	y_3	y_4	y_5

Using trapezoidal rule we get

$$I = \int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0 + y_5) + 2(y_1 + y_2 + y_3 + y_4)\right]$$

$$= \frac{0.2}{2}[(1.000000 + .500000) + 2(0.961538 + 0.862069 + 0.735294 + 0.609756)]$$

$= 0.7837314,$

$\therefore$ $I = 0.78373$, correct to five significant figures.

The exact value

$$= \int_0^1 \frac{1}{1+x^2} dx = \left[\tan^{-1} x\right]_0^1$$

$$= \tan^{-1} 1 - \tan^{-1} 0$$

$$= \frac{\pi}{4} = 0.7853981$$

$$\int_0^1 \frac{1}{1+x^2} dx = 0.78540,$$

correct to five significant figures.

$\therefore$ The error is $= 0.78540 - 0.78373$

$= 0.00167$

$\therefore$ Absolute error $= 0.00167$.

Example 3 : Find the value of $\int_1^5 \log_{10} x \, dx$, taking 8 subintervals correct to four decimal places by Trapezoidal rule.

Solution : Here

$$f(x) = \log_{10} x,$$

$$a = 1, b = 5, n = 8,$$

$$\therefore \; h = \frac{b-a}{n} = \frac{5-1}{8} = 0.5 .$$

x	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
$f(x)$	0.00000	0.17609	0.30103	0.39794	0.47712	0.54407	0.60206	0.65321	0.69897
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

Using Trapezoidal rule we can write

$$I = \frac{h}{2}\left[(y_0 + y_8) + 2(y_1 + y_2 + y_3 + y_4 + y_5 + y_6 + y_7)\right]$$

$$= \frac{0.5}{2}[(0.00000 + 0.69897) + 2(0.17609 + 0.30103 + 0.39794)]$$

$$+ \frac{0.5}{2}[2(0.47712 + 0.54407 + 0.60206 + 0.65321)]$$

$= 1.7505025$

$\therefore I = \int \log_{10} x \, dx = 1.75050$

Note:

Applications of Simpson's rule: If the various ordinates in (A) (i.e. Simpson's $\frac{1}{3}$ rule) represent equispaced cross-sectional areas, then Simpson's rule gives the volume of the solid. As such Simpson's rule is very useful for civil engineers for calculating the amount of earth that must be moved to fill a depression or make a dam. Similarly if the ordinates denote velocities at equal intervals of time, the Simpson's rule gives the distance travelled.

→ The velocity v (km/min) of a moped which starts from rest, is given at fixed intervals of time t (min) as follows.

t	2	4	6	8	10	12	14	16	18	20
v	10	18	25	29	32	20	11	5	2	0

Estimate approximately the distance covered in 20 minutes.

Sol: If s (km) be the distance covered in time t (min), then

$$\frac{ds}{dt} = v$$

$$\Rightarrow \quad s = \int_0^{20} v\,dt$$

Here $h=2$, $v_0=0$, $v_1=10$, $v_2=18$, $v_3=25$
$v_4=29$, $v_5=32$, $v_6=20$, $v_7=11$, $v_8=5$
$v_9=2$, $v_{10}=0$

By Simpson's $\frac{1}{3}$rd rule

$$s = \frac{h}{3}\left[(y_0+y_{10}) + 4(v_1+v_3+v_5+v_7+v_9) + 2(v_2+v_4+v_6+v_8)\right]$$

$= \frac{2}{3}(0 + 4\times 80 + 2\times 72) = 309.33$ km

which is the required distance

IAS-2011 A solid of revolution is formed by rotating about the x-axis, the area between the x-axis, the lines $x=0$ and $x=1$ and a curve through the points with the following co-ordinates.

x	0	0.25	0.50	0.75	1
y	1	0.9896	0.9589	0.9089	0.8415

Estimate the volume of the solid formed using Simpson's rule.

Sol: Here $h=0.25$, $y_0=1$, $y_1=0.9896$, $y_2=0.9589$, $y_3=0.9089$, $y_4=0.8415$

If V is the volume of the solid formed, then we know that $V=\pi\int_0^1 y^2\,dx$

The values of y^2 are tabulated below:

x	0	0.25	0.50	0.75	1
y^2	1	0.9793	0.9195	0.8261	0.7081

By Simpson's $\frac{1}{3}$rd rule

$$V = \frac{\pi(0.25)}{3}\left[1+4(0.9793+0.8261) + 2(0.9195+0.7081)\right]$$

$= 2.8192$

IFoS-2009 Use Simpson's $\frac{1}{3}$rd rule to find $\int_0^{0.6} e^{-x^2}dx$ by taking seven ordinates. (Ans: 0.5351)

(8)

Example 4 : Find the value $\int_0^{0.6} e^x dx$, taking $n = 6$, correct to five significant figures by Simpson's one–third rule.

Solution : We have

$$f(x) = e^x,$$

$$a = 0,\ b = 0.6,\ n = 6.$$

$$\therefore h = \frac{b-a}{n} = \frac{0.6-0}{6} = 0.1.$$

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6
$y=f(x)$	1.0000	1.10517	1.22140	1.34986	1.49182	1.64872	1.82212
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

The Simpson's rule is

$$I = \frac{h}{3}\left[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right]$$

$$= \frac{0.1}{3}\left[(1.00000 + 1.82212) + 4(1.10517 + 1.34986 + 1.64872) + 2(1.22140 + 1.49182)\right]$$

$$= \frac{0.1}{3}\left[(2.82212) + 4(4.10375) + 2(2.71322)\right]$$

$$= 0.8221186 \approx 0.82212$$

$$\therefore I = 0.82212.$$

Example 5 : The velocity of a train which starts from rest is given by the following table, the time being reckoned in minutes from the start and the speed in km/hour.

t (minutes)	2	4	6	8	10	12	14	16	18	20
v (km/hr)	16	28.8	40	46.4	51.2	32.0	17.6	8	3.2	0

Solution :

$$v = \frac{ds}{dt} \Rightarrow ds = v.dt$$

$$\Rightarrow \int ds = \int v.dt$$

$$s = \int_0^{20} v \cdot dt.$$

The train starts from rest, $\therefore$ the velocity $v = 0$ when $t = 0$.

The given table of velocities can be written

t	0	2	4	6	8	10	12	14	16	18	20
v	0	16	28.8	40	46.4	51.2	32.0	17.6	8	3.2	0
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}

$$h = \frac{2}{60} \text{ hrs} = \frac{1}{30} \text{ hrs}.$$

The Simpson's rule is

$$s = \int_0^{20} v \cdot dt = \frac{h}{3}\left[(y_0 + y_{10}) + 4(y_1 + y_3 + y_5 + y_7 + y_9) + 2(y_2 + y_4 + y_6 + y_8)\right]$$

$$= \frac{1}{30 \times 3}\left[(0+0) + 4(16 + 40 + 51.2 + 17.6 + 3.2) + 2(28.8 + 46.4 + 32.0 + 8)\right]$$

$$= \frac{1}{90}[0 + 4 \times 128 + 2 \times 115.2]$$

$$= 8.25 \text{ km.}$$

$\therefore$ The distance run by the train in 20 minutes = 8.25 kms.

Example 6 : A tank in discharging water through an orifice at a depth of x meter below the surface of the water whose area is Am^2. The following are the values of x for the corresponding values of A.

A	1.257	1.39	1.52	1.65	1.809	1.962	2.123	2.295	2.462	2.650	2.827
x	1.50	1.65	1.80	1.95	2.10	2.25	2.40	2.55	2.70	2.85	3.00

Using the formula $(0.018)T = \int_{1.5}^{3.0} \frac{A}{\sqrt{x}} dx$, calculate T the time in seconds for the level of the water to drop from 3.0 m to 1·5 m above the orifice.

Solution : We have $h = 0.15$

the table of values of x and the corresponding values of $\frac{A}{\sqrt{x}}$ is

x	1.50	1.65	1.80	1.95	2.10	2.25	2.40	2.55	2.70	2.85	3.00
$y=\frac{A}{\sqrt{x}}$	1.025	1.081	1.132	1.182	1.249	1.308	1.375	1.438	1.498	1.571	1.632

Using Simpson's rule, we get

$$\int_{1.5}^{3.0}\frac{A}{\sqrt{x}}dx=\frac{h}{3}\left[(y_0+y_{10})+4(y_1+y_3+y_5+y_7+y_9)+2(y_2+y_4+y_6+y_8)\right]$$

$$=\frac{0.15}{3}\left[(1.025+1.632)+4(1.081+1.182+1.308+1.438+1.571)\right]$$

$$+\frac{0.15}{3}\left[2(1.132+1.249+1.375+1.498)\right]$$

$$=1.9743$$

$$\therefore\ \int_{1.5}^{3}\frac{A}{\sqrt{x}}dx=1.9743\,.$$

Using the formula

$$(0.018)T=\int_{1.5}^{3}\frac{A}{\sqrt{x}}dx\,,$$

we get

$$(0.018)T=1.9743$$

$$\Rightarrow T=\frac{1.9743}{0.018}=110\text{ sec (approximately)}$$

$$\therefore\ T=110\text{ sec}\,.$$

Example 7 : Evaluate $\int_0^1\frac{1}{1+x^2}dx$, **by taking seven ordinates.**

Solution : We have

$$n\ +1=7\Rightarrow n>6$$

The points of division are

$$0\,,\ \frac{1}{6}\,,\ \frac{2}{6}\,,\ \frac{3}{6}\,,\ \frac{4}{6}\,,\ \frac{5}{6}\,,\ 1\,.$$

The table of values is

x	0	1/6	2/6	3/6	4/6	5/6	1
$y=\dfrac{1}{1+x^2}$	1.0000000	0.9729730	0.9000000	0.8000000	0.6923077	0.5901639	0.5000000

Here $h=\frac{1}{6}$,

the Simpson's three–eighth rule is

$$I=\frac{3h}{8}\left[\left(y_0+y_6\right)+3\left(y_1+y_2+y_4+y_5\right)+2\left(y_3\right)\right]$$

$$=\frac{3}{6\times 8}\left[(1+0.5000000)+3(0.9729730+0.9000000)\right]$$

$$+\frac{3}{6\times 8}\left[3(0.6923077+0.5901639)+2(0.800000)\right]$$

$$=\frac{1}{16}\left[1.5000000+9.4663338+1.6000000\right]$$

$$=0.7853959\,.$$

Simpson's 3/8 Rule

Example 8 : Calculate $\int_0^{\frac{\pi}{2}} e^{\sin x}\,dx$, correct to four decimal places.

Solution : We divide the range in three equal points with the division points

$$x_0=0,\ x_1=\frac{\pi}{6},\ x_2=\frac{\pi}{3},\ x_3=\frac{\pi}{2}$$

where $h=\frac{\pi}{6}$.

The table of values of the function is

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$y=e^{\sin x}$	1	1.64872	2.36320	2.71828
	y_0	y_1	y_2	y_3

By Simpson's three–eighth rule we get

$$I=\int_0^{\frac{\pi}{2}} e^{\sin x}\,dx=\frac{3h}{8}\left[\left(y_0+y_3\right)+3\left(y_1+y_2\right)\right.$$

(10)

$$=\frac{3}{8}\frac{\pi}{6}[(1+2.71828)+3(1.64872+2.36320)]$$

$$=\frac{\pi}{16}[(3.71828+12.03576)]$$

$$=0.091111$$

$$I=\int_0^{\frac{\pi}{2}} e^{\sin x}\,dx=0.091111\,.$$

Example 9 : Compute the integral $\int_0^{\frac{\pi}{2}}\sqrt{1-0.162\sin^2\phi}\,d\phi$ by Weddle's rule.

Solution : Here we have

$$y=f(\phi)=\sqrt{1-0.162\sin^2\phi}\,,$$

$$a=0,\ b=\frac{\pi}{2},$$

taking $n = 12$ we get

$$h=\frac{b-a}{n}=\frac{\frac{\pi}{2}-0}{12}=\frac{\pi}{24}\,.$$

f	$y=f(f)$		f	$y=f(f)$	
0	1.000000	y_0	$\frac{6\pi}{24}$	0.958645	y_6
$\frac{\pi}{24}$	0.998619	y_1	$\frac{7\pi}{24}$	0.947647	y_7
$\frac{2\pi}{24}$	0.994559	y_2	$\frac{8\pi}{24}$	0.937283	y_8
$\frac{3\pi}{24}$	0.988067	y_3	$\frac{9\pi}{24}$	0.928291	y_9
$\frac{4\pi}{24}$	0.979541	y_4	$\frac{10\pi}{24}$	0.921332	y_{10}
$\frac{5\pi}{24}$	0.969518	y_5	$\frac{11\pi}{24}$	0.916930	y_{11}
			$\frac{12\pi}{24}=\frac{\pi}{2}$	0.915423	y_{12}

By Weddle's rule we have

$$I = \int_0^{\frac{\pi}{2}} \sqrt{.-0.162\sin^2\phi}\, d\phi$$

$$= \frac{3h}{10}\left[\left(y_0 + y_{12}\right) + 5\left(y_1 + y_5 + y_7 + y_{11}\right)\right]$$

$$+ \frac{3h}{10}\left[\left(y_2 + y_4 + y_8 + y_{10}\right) + 6\left(y_3 + y_9\right) + 2y_6\right]$$

$$= \frac{3\pi}{240}\left[(11.000000 + 0.915423) + 5(0.998619 + 0.969518 + 0.947647 + 0.916930)\right]$$

$$+ \frac{3\pi}{240}\left[(0.994559 + 0.979541 + 0.937283 + 0.9213322)\right]$$

$$+ \frac{3\pi}{240}\left[6(0.988067 + 0.928291) + 2(0.958645)\right]$$

$\therefore\ I = 1.505103504$.

Example 10 : Find the value of $\int_4^{5.2} \log_e x\, dx$ by Weddle's rule.

Solution : Here $f(x) = \log_e n$, $a = x_0 = 4$, $b = x_n = 5.2$ taking $n = 6$ (a multiple of six) we have

$$h = \frac{5.2 - 4}{6} = 0.2,$$

x	4.0	4.2	4.4	4.6	4.8	5.0	5.2
$y=f(x)$	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094	1.6457

Weddle's rule is

$$I = \int_4^{5.2} \log_e x\, dx = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\right]$$

$$= \frac{3\times(0.2)}{10}\left[1.3863 + 7.1755 + 1.4816 + 9.1566 + 1.5686 + 8.0470 + 1.6487\right]$$

$$= 0.06\left[30.4643\right]$$

$$= 1.827858$$

$$\int^{5.2} \log_e x = 1.827858.$$

(11)

Exercise 9.1

1. Evaluate $\int_0^1 x^3 dx$ by Trapezoidal rule.

2. Evaluate $\int_0^1 (4x-3x^2)dx$ taking 10 intervals by Trapezoidal rule.

3. Given that $e^0 = 1$, $e^1 = 2.72$, $e^2 = 7.39$, $e^3 = 20.09$, $e^4 = 54.60$, find an approximation value of $\int_0^4 e^x dx$ by Trapezoidal rule.

4. Evaluate $\int_0^1 \sqrt{1-x^3}\, dx$ by (i) Simpson's rule and (ii) Trapezoidal rule, taking six interval correct to two decimal places.

5. Evaluate $\int_0^{\frac{\pi}{2}} \sqrt{\sin x}\, dx$ taking $x = 6$, correct to four significant figures by (i) Simpson's one–third rule and (ii) Trapezoidal rule.

6. Evaluate $\int_1^2 \frac{dx}{x}$ taking 4 subintervals, correct to five decimal places (i) Simpson's one third rule (ii) Trapezoidal rule.

7. Compute by Simpson's one–third rule, the integral $\int_0^1 x^2(1-x)dx$ correct to three places of decimal, taking step length equal to 0.1.

8. Evaluate $\int_0^1 \sin x^2\, dx$ by (i) Trapezoidal rule and (ii) Simpson's one–third rule, correct to four decimals taking $x = 10$.

9. Calculate approximate value of $\int_{-3}^3 \sin x^4\, dx$ by using (i) Trapezoidal rule and (ii) Simpson's rule, taking $n = 6$.

10. Find the value of $\int_0^{\frac{\pi}{2}} \sqrt{\cos x}\, dx$ by (ii) Trapezoidal rule and (ii) Simpson's one–third rule taking $x = 6$.

11. Compute $\int_1^{1.5} e^x\, dx$ by (i) Trapezoidal rule and (ii) Simpson's one–third rule taking $x = 10$.

12. Evaluate $\int_0^{0.5} \frac{x}{\cos x} dx$ taking $n = 10$, by (i) Trapezoidal rule and (ii) Simpson's one–third rule.

13. Evaluate $\int_0^{0.4} \cos x\, dx$ taking four–equal intervals by (i) Trapezoidal rule and (ii) Simpson's one–third rule.

14. Evaluate $\int_0^{\frac{\pi}{2}} \sqrt{\cos x}\, dx$ by Weddle's rule taking $n = 6$.

15. Evaluate $\int_0^{1} \frac{x^2+2}{x^2+1} dx$ by Weddle's rule, correct to four decimals taking $n = 12$.

16. Evaluate $\int_0^{2} \frac{1}{1+x^2} dx$ by using Weddle's rule taking twelve–intervals.

17. Evaluate $\int_{0.4}^{1.6} \frac{x}{\sinh x} dx$ taking thirteen–ordinates by Weddle's rule correct to five decimals.

18. Using Simpson's rule evaluate $\int_0^{\frac{\pi}{2}} \sqrt{2+\sin x}\, dx$ with seven ordinates.

19. Using Simpson's rule evaluate $\int^{2} \sqrt{x-1/x}\, dx$ with five ordinates.

20. Using Simpson's rule evaluate $\int_2^6 \frac{1}{\log_e x} dx$ taking $n = 4$.

21. A river is 80 unit wide. The depth at a distance x unit from one bank d is given by the following table

x	0	10	20	30	40	50	60	70	80
d	0	4	7	9	12	15	14	8	3

22. Find the approximate value of $\int_0^{\frac{\pi}{2}} \sqrt{\cos\theta}\, d\theta$ using Simpson's rule with six intervals.

Answers

1. 0.260 **2.** 0.995 **3.** 58.00 **4.** 0.83 **5.** 1.187, 1.170 **6.** 0.69326, 0.69702 **7.** 0.083 **8.** 0.3112, 0.3103 **9.** 115, 98 **10.** 1.170, 1.187 **11.** 1.764, 1.763 **12.** 0.133494, 0.133400 **13.** 0.3891, 0.3894 **14.** 1.18916 **15.** 1.7854 **16.** 1.1071 **17.** 1.1020 **18.** 2.545 **19.** 1.007 **20.** 3.1832 **21.** 710 Sq units **22.** 1.1872

9.7 Newton - Cotes formula

Consider Lagrange's interpolation formula

$$f(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)} f(x_0)$$

$$+\frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)} f(x_1)$$

$$+\ldots+\frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})} f(x_n),$$

Integrating between the limits x_0 and x_0+nh we get

$$\int_{x_0}^{x_0+h} f(x)dx = H_0 f(x_0) + H_1 f(x_1) + \ldots + H_r f(x_r) + \ldots + H_n f(x_n). \tag{5}$$

Expression (5) is known as Newton – Cotes formula. Taking

$$x_{r+1} - x_r = h$$

for all r such that

* Gaussian Integration:

So far the formulae derived for evaluation of $\int_a^b f(x)dx$, required the values of the function at equally spaced points of the interval.

But the Gauss derived a formulae which uses the same number of functions with different spacing and gives better accuracy.

— Gaussian formula imposes a restriction on the limits of integration to be from -1 to 1.

In general, the limits of the integral $\int_a^b f(x)dx$ are changed to -1 to 1 by means of the transformation: $x = \frac{1}{2}(b-a)u + \frac{1}{2}(b+a)$.

Let us consider the Gauss's formula in the form:

$$\int_{-1}^{1} f(x)dx = w_1 f(x_1) + w_2 f(x_2) + \cdots + w_n f(x_n)$$

$$= \sum_{i=1}^{n} w_i f(x_i). \quad \text{--- (1)}$$

where w_i and x_i are called the weights and abscissae respectively.

An advantage of this formula is that the abscissae and weights are symmetrical with respect to the middle point of the interval.

In the equation (1), there are altogether $2n$ arbitrary constants (i.e n weights & n abscissae) and therefore the weights and abscissae can be determined such that the formula is exact when $f(x)$ is a polynomial of degree not exceeding $2n-1$.

Hence, we consider,

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_{2n-1} x^{2n-1}. \quad \text{——— (2)}$$

$$\therefore (1) \equiv \int_{-1}^{1} f(x)\,dx = \int_{-1}^{1} [c_0 + c_1 x + c_2 x^2 + \cdots + c_{2n-1} x^{2n-1}]\,dx.$$

$$= 2c_0 + \tfrac{2}{3} c_2 + \frac{2}{5} c_4 + \cdots \quad \text{——— (3)}$$

by setting $x = x_i$ in (2), we obtain

$$f(x_i) = c_0 + c_1 x_i + c_2 x_i^2 + \cdots + c_{2n-1} x_i^{2n-1}.$$

Substituting these values on the right-hand side of (1), we obtain

$$\int_{-1}^{1} f(x)\,dx = \sum_{i=1}^{n} w_i f(x_i)$$

$$= w_1 f(x_1) + w_2 f(x_2) + \cdots + w_n f(x_n)$$

$$= w_1\left[c_0 + c_1 x_1 + c_2 x_1^2 + c_3 x_1^3 + \cdots\cdots + c_{2n-1} x_1^{2n-1}\right]$$
$$+ w_2\left[c_0 + c_1 x_2 + c_2 x_2^2 + c_3 x_2^3 + \cdots\cdots + c_{2n-1} x_2^{2n-1}\right]$$
$$+ \cdots\cdots\cdots$$
$$+ \cdots\cdots\cdots$$
$$+ w_n\left[c_0 + c_1 x_n + c_2 x_n^2 + c_3 x_n^3 + \cdots\cdots + c_{2n-1} x_n^{2n-1}\right].$$

$$= c_0\left[w_1 + w_2 + w_3 + \cdots + w_n\right] + c_1\left[w_1 x_1 + w_2 x_2 + \cdots + w_n x_n\right]$$
$$+ c_2\left[w_1 x_1^2 + w_2 x_2^2 + w_3 x_3^2 + \cdots + w_n x_n^2\right] +$$
$$\cdots\cdots\cdots$$
$$+ \cdots\cdots\cdots$$
$$+ c_{2n-1}\left[w_1 x_1^{2n-1} + w_2 x_2^{2n-1} + w_3 x_3^{2n-1} + \cdots + w_n x_n^{2n-1}\right]. \quad \text{—— (4)}$$

But the equations (3) & (4) are identical for all values of c_i, hence comparing coefficients of c_i, we obtain $2n$ equations in $2n$ unknowns w_i and x_i $(i = 1, 2, 3 \cdots n)$

$$\left.\begin{array}{l} w_1 + w_2 + w_3 + \cdots\cdots + w_n = 2 \\ w_1 x_1 + w_2 x_2 + w_3 x_3 + \cdots + w_n x_n = 0 \\ w_1 x_1^2 + w_2 x_2^2 + w_3 x_3^2 + \cdots + w_n x_n^2 = 2/3 \\ \vdots \\ w_1 x_1^{2n-1} + w_2 x_2^{2n-1} + w_3 x_3^{2n-1} + \cdots + w_n x_n^{2n-1} = 0 \end{array}\right\} \text{—— (5)}.$$

* One point formula:

Gauss formula for $n=1$ is

$$\int_{-1}^{1} f(x)\,dx = w_1 f(x_1). \quad \text{---} (6)$$

∴ The method has two unknowns w_1 & x_1

∴ from (5), $w_1 = 2$

&

$w_1 x_1 = 0.$

$\Rightarrow x_1 = 0 \quad (\because w_1 = 2)$.

$$\therefore (6) \equiv \boxed{\int_{-1}^{1} f(x)\,dx = 2f(0)} \quad \text{---} (7)$$

* Two-point formula:

Gauss formula for $n=2$ is

$$\int_{-1}^{1} f(x)\,dx \simeq \sum_{i=1}^{2} w_i f(x_i)$$

$$= w_1 f(x_1) + w_2 f(x_2) \quad \text{---} (8)$$

∴ The method has four unknowns w_1, w_2 and x_1, x_2

∴ from (5),

$$\left.\begin{aligned} w_1 + w_2 &= 2 \\ w_1 x_1 + w_2 x_2 &= 0 \\ w_1 x_1^2 + w_2 x_2^2 &= 2/3. \\ w_1 x_1^3 + w_2 x_2^3 &= 0. \end{aligned}\right\} \quad \text{---} (9)$$

solving these equations, we obtain

$w_1 = w_2 = 1$; $x_1 = \frac{-1}{\sqrt{3}}$, $x_2 = \frac{1}{\sqrt{3}}$.

$$\therefore (8) \equiv \boxed{\int_{-1}^{1} f(x)\,dx = 1\cdot f\left(\frac{-1}{\sqrt{3}}\right) + 1\cdot f\left(\frac{1}{\sqrt{3}}\right)} \quad (10)$$

which gives the correct value of the integral of f(x) in the range (-1, 1) for function upto 3rd order.

Equation (10) some times is known as Gauss–Legendre formula.

we can also, easily find for three-point formula;

Gauss formula formula for n = 3 is

$$\int_{-1}^{1} f(x)\,dx = \sum_{i=1}^{3} w_i f(x_i).$$

$$= w_1 f(x_1) + w_2 f(x_2) + w_3 f(x_3).$$

$$= \frac{5}{9} f\left(-\sqrt{\tfrac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\tfrac{3}{5}}\right).$$

which is exact for polynomial upto degree 5.

Note (1): This method, when applied to the general system (5) above, will be extremely complicated and difficult i.e theoretically, we can obtain $2n$ unknowns w_i's and x_i's by solving the above of non-linear system of equations (5) by usual algebraic methods but practically it is very difficult to solve even for small values of 'n'. In case, we know $x_1, x_2, \ldots x_n$, the system of equations (5) can be reduced to a system of linear equations in $w_1, w_2, \ldots w_n$.

An alternative must be chosen to solve the non-linear system (5):

It can be shown that the x_i's are the zero's (roots) of the $(n+1)^{th}$ Legendre polynomial $P_{n+1}(x)$ which can be generated using the recurrence relation

$$(n+1)P_{n+1}(x) = (2n+1)\,x\,P_n(x) - n\,P_{n-1}(x) \quad \text{(I)}$$

where $P_0(x) = 1$ and $P_1(x) = x$

(16)

$\therefore$ The first five Legendre polynomials are given by

$$\left.\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \frac{1}{2}(3x^2-1) \\ P_3(x) &= \frac{1}{2}(5x^3-3x) \\ P_4(x) &= \frac{1}{8}(35x^4-30x^2+3) \end{aligned}\right\} \longrightarrow \text{(II)}$$

etc

and the corresponding weights w_i are given by

$$w_i = \int_{-1}^{1} \prod_{\substack{j=0 \\ j\neq i}}^{n} \left(\frac{x-x_j}{x_i-x_j}\right) dx \longleftarrow \text{(III)}$$

where the x_i are the abscissae.

As an example, when $n=1$. (two point formula)

setting, $P_2(x)=0$

i.e $\frac{1}{2}(3x^2-1)=0$

$\Rightarrow 3x^2=1$

$\Rightarrow x = \pm\sqrt{\frac{1}{3}}$.

$$\Rightarrow \left.\begin{aligned} x_0 &= -\frac{1}{\sqrt{3}} \\ x_1 &= \frac{1}{\sqrt{3}} \end{aligned}\right\} \text{say.}$$

$\therefore$ The corresponding weights are given by

$$w_0 = \int_{-1}^{1} \left(\frac{x - x_1}{x_0 - x_1} \right) dx = 1.$$

and $$w_1 = \int_{-1}^{1} \left(\frac{x - x_0}{x_1 - x_0} \right) dx = 1.$$

Similarly $n = 2$; setting $P_3(x) = 0$.
(Three point formula)

$$\Rightarrow \frac{1}{2}(5x^3 - 3x) = 0.$$

$$\Rightarrow x(5x^2 - 3) = 0.$$

$$\Rightarrow x = 0;\ x = \pm\sqrt{3/5}$$

$$\Rightarrow x_0 = 0;\ x_1 = -\sqrt{3/5};\ x_2 = \sqrt{3/5}\ \text{say}$$

The corr. weights,

$$w_0 = \int_{-1}^{1} \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} dx = 8/9.$$

$$w_1 = \int_{-1}^{1} \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} dx = 5/9.$$

and $$w_2 = \int_{-1}^{1} \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)} dx = 5/9.$$

etc.

(17)

① Evaluate $\int_{-1}^{1} \frac{dx}{1+x^2}$ using Gauss formula for $n=2$ and $n=3$.

Sol^n:

Gauss formula for $n=2$ is i.e, Two point formula.

$$I = \int_{-1}^{1} \frac{dx}{1+x^2} = 1f\left(-\frac{1}{\sqrt{3}}\right) + 1f\left(\frac{1}{\sqrt{3}}\right)$$

where $f(x) = \frac{1}{1+x^2}$

$$I = 1 \cdot \frac{1}{1+\left(-1/\sqrt{3}\right)^2} + 1 \cdot \frac{1}{1+\left(1/\sqrt{3}\right)^2} = \frac{3}{4} + \frac{3}{4} = 1.5$$

Gauss formula for $n=3$ is i.e, Three-point formula

$$I = \frac{8}{9} f(0) + \frac{5}{9}\left[f\left(-\sqrt{3/5}\right) + f\left(\sqrt{3/5}\right)\right]$$

where $f(x) = \frac{1}{1+x^2}$

$$= \frac{8}{9}(1) + \frac{5}{9}\left(\frac{5}{8} + \frac{5}{8}\right) = \frac{8}{9} + \frac{50}{72} = 1.5833.$$

② Using three point Gaussian quadrature formula, evaluate $\int_0^1 \frac{dx}{1+x}$.

Sol: we first change the limits $(0,1)$ to $(-1,1)$ by using $x = \frac{1}{2}(b-a)u + \frac{1}{2}(b+a)$

i.e, $x = \frac{1}{2}(1-0)u + \frac{1}{2}(1+0) = \frac{1}{2}(u+1)$

(or)

put $u = ax+b$

$-1 = a(0)+b \Rightarrow b=-1$

& $1 = a(1)+b$

$\Rightarrow 1 = a+b$

$\Rightarrow a = 2$

$\therefore u = 2x-1$

$\Rightarrow x = \frac{u+1}{2}$

$$\therefore I = \int_0^1 \frac{dx}{1+x} = \int_{-1}^{1} \frac{\frac{1}{2}du}{1+\frac{1}{2}(u+1)}$$

$$= \int_{-1}^{1} \frac{du}{u+3}$$

Three point Gauss quadrature formula is

$$I = \frac{8}{9} f(0) + \frac{5}{9}\left\{ f\left(-\sqrt{3/5}\right) + f\left(\sqrt{3/5}\right)\right\}$$

where $f(u) = \frac{1}{u+3}$

$$\therefore I = \frac{8}{9}\left(\frac{1}{3}\right) + \frac{5}{9}\left\{\frac{1}{-\sqrt{3/5}+3} + \frac{1}{\sqrt{3/5}+3}\right\}$$

$$= 0.6931$$

---

(3) Evaluate $\int_0^2 \frac{x^2+2x+1}{1+(x+1)^4}\,dx$ by Gaussian 3-point formula.

[Hint: changing the limits of integration 0 to 2 to -1 to 1 by $x = \frac{1}{2}(b-a)u + \frac{1}{2}(b+a) = \frac{2-0}{2}u + \frac{2+0}{2} = u+1$]

$\boxed{\text{Ans: } 0.5365}$

---

(4) Using Gaussian two-point formula compute $\int_{-2}^{2} e^{-x/2}\,dx$

(5) Evaluate $\int_0^{\pi} \sin x\,dx$ by using Gauss-Legendre two point formula.

(6) Using three point Gaussian quadrature formula, evaluate

(i) $\int_1^5 \frac{1}{x}\,dx$ (ii) $\int_2^4 (1+x^4)\,dx$ (iii) $\int_{0.2}^{1.5} e^{-x^2}\,dx$

(7) Using four point Gauss formula, compute $\int_0^1 x\,dx$ correct to four decimal places.

(8) IAS-2008 Evaluate $\int_0^1 e^{-x^2}\,dx$; employing three point Gaussian quadrature formula, finding the required weights and residues. Use five decimal places for computation.