WORKSHEET - 4

Real Number System and Sequence & Series of Real Numbers

1. Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series of real numbers.

   Suppose $\sum_{n=1}^{\infty} a_{2n} = \frac{9}{8}$ and $\sum_{n=0}^{\infty} a_{2n+1} = \frac{-3}{8}$.

   What is $\sum_{n=1}^{\infty} a_n$? $\rightarrow 3/4$

   Justify your answer. (Majority of marks is for the correct justification).

2. (I). Let $S = \left\{ m + \frac{1}{n} : m \in \mathbb{N}, n \in \mathbb{N} \right\}$. Find the derived set of S.

   (II). $S = \left\{ (-1)^n \left(1 + \frac{1}{n} \right) : n \in \mathbb{N} \right\}$.

   (i) Show that -1 and 1 are limit points of S.
   (ii) Find the derived set of S. $\{-1, 1\}$

3. Show that the series $\sum \frac{1}{n} \log \frac{n+1}{n}$ is convergent.

4. Prove that the sequence $\{a_n\}$ recursively defined by $a_1 = \sqrt{5}, a_{n+1} = \sqrt{5 + a_n}, n \geq 1$ converges to the positive root of the equation $x^2 - x - 5 = 0$.

5. Show that the series $\sum (-1)^n \left[\sqrt{n^2+1} - n \right]$ is conditionally convergent.

6. Find how the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \ldots\ldots$ should be deranged so that the sum is doubled.

   $K = 4$ $\Rightarrow \boxed{K = 4\beta}$

7. Discuss the convergence of the series

   $$1 + \frac{3}{7}x + \frac{3.6}{7.10}x^2 + \frac{3.6.9}{7.10.13}x^3 + \ldots\ldots, x > 0.$$

   cgt for $x \leq 1$, dgt for $x > 1$

8. Is the intersection of an arbitrary collection of open sets open? Justify your answer by a proof or by a counter example.

9. Discuss the convergence of the series

   $$x + \frac{2^2 x^2}{2!} + \frac{3^3 x^3}{3!} + \frac{4^4 x^4}{4!} + \frac{5^5 x^5}{5!} + \ldots$$ $\longrightarrow \frac{1}{e} \Rightarrow$ dgt.

10. Test for convergence of the following series:

    $$\sum \frac{1^2.3^2.5^2\ldots\ldots(2n-1)^2}{2^2.4^2\ldots\ldots(2n)^2} x^{n-1}, x > 0$$

    cgt $x < 1$, dgt $x \geq 1$

11. (i) Is every finite set open? Justify your answer by giving an example.
    (ii) Is the union of an arbitrary collection of closed sets closed? Justify your answer by an example.
    (iii) Give an example of a family $\{ I_n : n \in \mathbb{N} \}$ of non-empty closed intervals such that

    $$I_1 \supset I_2 \supset I_3 \supset \ldots\ldots \text{ and } \bigcap_{n=1}^{\infty} I_n = \phi.$$

    (iv) Let

    $$S = \left\{ \frac{1}{n} / \ n \in N \right\} \bigcup \left\{ 1 + \frac{3}{2n} / \ n \in N \right\} \bigcup \left\{ 6 - \frac{1}{3n} / \ n \in N \right\}.$$

    Find derived set S ′ of S. Also find supremum of S and greatest number of S.

12. What derangement of the series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots\ldots$ will reduce its sum to $\frac{1}{2} \log 2$?

    $K = \frac{1}{2}$ $\boxed{2\alpha = \beta}$

IMS
H.O. 25/8 Old Rajender Nagaer Market, Delhi-60
B.O.: 105-106, Top Floor, Mukherjee Tower, Dr. Mukherjee Nagar, delhi-9
Ph : 011-45629987, 09999197625 || Email : ims4ims2010@gmail.com, www.ims4maths. com

13. Test the convergence of the following series.

cgt. (i) $1^p + \left(\frac{1}{2}\right)^p + \left(\frac{1.3}{2.4}\right)^p + \left(\frac{1.3.5}{2.4.6}\right)^p + \ldots\ldots$

(ii) Discuss the convergence of the sequence

$\{X_n\}$ where $X_n = \dfrac{\sin\left(\frac{\pi x}{2}\right)}{8}$

14. Test the convergence and absolute convergence of the series $\sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{n}{n^2+1}$.

- Not abs cgt
- leibnitz (conditional cgt)

$\frac{n}{n^2+n} < \frac{n}{n^2+1} \Rightarrow \frac{1}{n+1} < \frac{1}{n^2+1}$
$\Rightarrow dgt.$

15. If (a_n) is a sequence such that $a_n > 0 \, \forall n$ and $\underset{n\to\infty}{Lt} \dfrac{a_{n+1}}{a_n} = 1$ then $\underset{n\to\infty}{Lt} (a_n)^{\frac{1}{n}} = 1$.

Is a converse true? Justify your answer.

16. Define $\{x_n\}$ by $x_1 = 5$ and $x_{n+1} = \sqrt{4 + x_n}$ for $n > 1$. Show that the sequence converges to $\dfrac{1+\sqrt{17}}{2}$

17. Show that the set

$$S = \left\{1 + \frac{(-1)^n}{2^n} : n \text{ is a positive integer}\right\}$$

is bounded. Show that 1 is a limit point of S. Are there any other limit points of S ?

18. Discuss the convergence of the infinite product

$$\prod_{n=1}^{\infty}\left\{1 + \left(\frac{nx}{n+1}\right)^n\right\}$$

19. Test for convergence the series

$$1 + \frac{1!}{2}x + \frac{2!}{3^2}x^2 + \frac{3!}{4^3}x^3 + \frac{4!}{5^4}x^4 + \ldots$$

$a_n = \dfrac{n!\,x^n}{(n+1)^n}$

D.A.n Test, then log test
cgt for $x < e$
dgt for $x \geq e$

20. Prove that the sequence $\{S_n\}$ defined bythe recursion formula

$S_{n+1} = \sqrt{7 + S_n}, S_1 = \sqrt{7}$, converges to the positive root of $x^2 - x - 7 = 0$.

21. If (a_n) is a sequence such that $a_n > 0 \, \forall n$ and

$\underset{n\to\infty}{Lt} \dfrac{a_{n+1}}{a_n} = 1$ then $\underset{n\to\infty}{Lt} (a_n)^{1/n} = 1$.

Is a converse true? Justify your answer.

22. Prove that the sequence whose n^{th} term is $\dfrac{3n+4}{2n+1}$

(i) is monotonically decreasing.

(ii) is bounded and

(iii) converges to $\dfrac{3}{2}$

23. Discuss the convergence of the sequence $\{x_n\}$ where

$x_n = \dfrac{\sin\left(\frac{n\pi}{2}\right)}{8}$.

24. Discuss the convergence of the series

cgt for $x < 1$
dgt for $x \geq 1$

$1 + \frac{1}{2}x + \frac{1.3}{2.4}x^2 + \frac{1.3.5}{2.4.6}x^3 + \ldots\ldots (x > 0)$.

25. Show that a bounded infinite subset of $\mathbb{R}$ must have a limit point.

26. Prove that the sets A= [0, 1], B = (0, 1) are equivalent sets.

27. Show that the series:

$$\left(\frac{1}{3}\right)^2 + \left(\frac{1\cdot4}{3\cdot6}\right)^2 + \ldots\ldots +$$

$$\left(\frac{1\cdot4\cdot7\ldots\ldots(3n-2)}{3\cdot6\cdot9\ldots3n}\right)^2 + \ldots\ldots \text{ converges.}$$

Rabbe test : $\frac{4}{3} > 1 \Rightarrow$ cgt.

28. Show that:

$$\underset{x\to 1}{Lt}\sum_{n=1}^{\infty}\frac{n^3x^2}{n^4+x^4}=\sum_{n=1}^{\infty}\frac{n^3}{n^4+1}$$

Justify all steps of your answer by quoting the theorems you are using.

29. Examine the convergence of the series

use Cauchy condensation test.

$$\frac{1}{(\log 2)^p}+\frac{1}{(\log 3)^p}+\frac{1}{(\log 4)^p}+\ldots..$$

30. Find how the series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\ldots\ldots\ldots$$ Q-6

should be deranged so that the sum is doubled.

31. Show that the sequence (x_n) defined by $x_{n+1}=\sqrt{3x_n}, x_1=1$ converges to 3.

32. Investigate what derangement of the series

$k = \frac{1}{4}$

$\Rightarrow \boxed{\alpha = \frac{\beta}{4}}$

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\ldots\ldots$$ will reduce its sum to zero.

33. Show that the series $\sum\frac{1}{n(n+1)}$ is equivalent to $\frac{1}{2}\prod_{2}^{\infty}\left(1+\frac{1}{n^2-1}\right)$

34. Show that the cauchy product of $\sum_{n=0}^{\infty}(-1)^n\big/\sqrt{n+1}$ with itself diverges.

Leibnitz test.

35. Define a sequence S_n of real numbers by

$$s_n=\sum_{i=1}^{n}\frac{(\log(n+i)-\log n)^2}{n+i}$$

Does $\lim_{n\to\infty} s_n$ exist? If so, compute the value of this limit and justify your answer.

36. Given that the terms of a sequence $\{a_n\}$ are such that each a_k, $k\leq 3$ is the arithmetic mean of its two immediately proceeding terms. Show that the sequence convergence. Also find the limit of the sequence.

37. Determine the values of x for which the infinite product $\prod_{n=0}^{\infty}\left(1+\frac{1}{x^{2^n}}\right)$ converges absolutely. Find its value whenever it converges.

38. Find all the positive values of a for which the series $\sum_{n=1}^{\infty}\frac{(a_n)^n}{n!}$ converges.

39. Let 'a' be a positive real number and $\{x_n\}$ a sequence of rational numbers such that $\underset{n\to\infty}{Lt}\, x_n=0$

Show that $\underset{n\to\infty}{Lt}\, a^{x_n}=1$

40. Rearrange the series $\sum_{n=1}^{\infty}(-1)^{n+1}\cdot\frac{1}{n}$ to converge to 1

$k = e^2/4$

$\alpha = \beta\cdot e^2/n$

41. Let

$$T=\left\{\frac{1}{n},n\in N\right\}\cup\left\{1+\frac{3}{2n},n\in N\right\}\cup\left\{6-\frac{1}{3n},n\in N\right\}.$$ $\{0,1,6\}=T'$

Find derived set T′ of T. Also find supremum of T and greatest number of T.

Sup = 6

greatest no. =

42. Discuss the convergence of the series

$$\frac{x}{2}+\frac{1\cdot 3}{2\cdot 4}x^2+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^3+\ldots..,x>0$$ Q. 24

43. Find the derived set of each of the following:

(i) $(1,\infty)$

(ii) $(-\infty,-1)$

(iii) $\left\{\frac{1+(-1)^n}{n};n\in\mathrm{N}\right\}$ $\to 0$

WORKSHEET - 5

Improper Integrals

1. Examine the convergence of $\int_0^1 x^{n-1}\operatorname{Log}x\,dx$

2. Evaluate $\int_0^1 (x\ell nx)^3\,dx$. not improper.

3. Show that the integral $\int_0^1 \frac{dx}{x^{1/3}(1+x^2)}$ is convergent.

4. Discuss the convergence of $\int_0^{\pi/2} \frac{\sin x}{x^p}\,dx$.
cgt for $p<2$

5. Show that $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ converges if $m>0$ & $n>0$

6. Examine the convergence of the integerals
(i) $\int_1^\infty \frac{\tan^{-1}x}{x^2}\,dx$ (ii) $\int_0^4 \frac{dx}{x(4-x)}$ dvg

7. Show that $\int_0^\infty x^{n-1}e^{-x}\,dx$ converges iff $n>0$.

8. Examine the convergence of the integral $\int_2^3 \frac{dx}{(x-2)^{\frac{1}{4}}(3-x)^{\frac{1}{2}}}$

9. Show that $\int_0^{\frac{\pi}{2}} \log(\sin x)\,dx$ isconvergent and hence evaluate it.

10. Examine the convergence of $\int_0^\infty \left(\frac{1}{1+x}-e^{-x}\right)\frac{dx}{x}$

11. Show that $\int_0^\infty \log\left(x+\frac{1}{x}\right)\frac{dx}{1+x^2} = \pi\log 2$

12. Examine the convergence of the integrals
(i) $\int_0^1 \frac{dx}{\sqrt{x(1-x)}}$ (ii) $\int_1^\infty \frac{\log x}{x^2}\,dx$

13. (i) Evaluate $\int_0^\infty \frac{\log\left(1+\frac{1}{x^2}\right)}{1+x^2}\,dx$
(ii) Find the volume of the largest possible right circular cylinder that can be inscribed in a sphere of radius a.

14. Evaluate the integral
$$I=\int_0^\infty 2^{-ax^2}\,dx$$
using Gamma function.

15. Test the convergence of the integral $\int_0^\infty \frac{x^{2m}}{1+x^{2n}}\,dx, m,n>0$

16. Show that $\int_0^1 x^{m-1}(1-x)^{n-1}$ exists if and only if m and n are both positive.

17. Examine the convergence of the integral $\int_0^\infty \frac{\sin x}{x}\,dx$

IMS
H.O. 25/8 Old Rajender Nagaer Market, Delhi-60
B.O.: 105-106, Top Floor, Mukherjee Tower, Dr. Mukherjee Nagar, delhi-9
Ph : 011-45629987, 09999197625 || Email : ims4ims2010@gmail.com, www.ims4maths. com

18. Examine the convergence of the following

(i) $\int_0^1 \frac{x^n \log x}{(1+x)^2} dx$

(ii) $\int_1^\infty \frac{x^3}{(1+x)^5} dx$

19. Prove that $\int_0^\infty e^{-x^2} dx = \frac{1}{2}\sqrt{\pi}$.

20. Determine the value of

$$\left[\int_0^1 \frac{x^2}{(1-x^4)^{1/2}} dx\right]\left[\int_0^1 \frac{dx}{(1+x^4)^{1/2}}\right]$$

21. Test the convergence of the following

(i) $\int_1^\infty \frac{dx}{\sqrt{x^3+1}}$ (ii) $\int_0^1 x^{n-1} e^{-x} dx$. see Q.7

22. Determine the value of

$$\left(\int_0^1 \frac{x^2}{(1-x^4)^{1/2}} dx\right)\left(\int_0^1 \frac{dx}{(1+x^4)^{1/2}}\right)$$

23. Examine the convergence of the integral

$\int_0^1 x^{n-1} \log x \, dx$

24. Show that $\int_0^\infty \sqrt{x}\, e^{-x^2} dx \times \int_0^\infty \frac{e^{-x^2}}{\sqrt{x}} dx = \frac{\pi}{2\sqrt{2}}$

25. Evaluate $\int_0^1 x^m \left(\ln \frac{1}{x}\right)^n dx,\ m, n > -1$

26. Discuss the convergence ~~of~~ $\cancel{\int_0^1 x^{n-1} \log x \, dx}$.

27. Examine the convergence of the integral

$\int_0^1 \left(x^p + x^{-p}\right) \log(1+x) \frac{dx}{x}$.

28. If $B(p,q)$ be the Beta function, show that $pB(n,q) = (q-1)B(p+1, q-1)$, where p, q are real p>0, q>1. Hence or otherwise find B(p,n) when 'n' is an integer (>0).

29. Prove the relation between beta and gamma functions $\beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

30. Examine the convergence of the integral

$\int_0^\infty \frac{\sin x^m}{x^n} dx$

31. Discuss the existence of the improper integral $\int_{-\infty}^{\infty} e^{-x^2/2} dx$

32. Discuss the convergence of $\cancel{\int_0^{\pi/2} \log \sin x \, dx}$ and evaluate it, if it is convergent.

33. Prove that $\int_0^\infty \frac{\sin x}{x} dx$ converges and conditionally converges.

34. Test the convergence of the integrals:

(i) $\cancel{\int_0^1 \frac{dx}{x^{1/2}(1+x^2)}}$

(ii) $\int_0^\infty \frac{\sin^2 x}{x^2} dx$

35. Express $\int_0^1 x^m (1-x^n)^p dx$ interms of Gamma function and hence evaluate the integral.

$\int_0^1 x^6 \sqrt{(1-x^2)}\, dx$.

36. Show that $e^{-x} x^n$ is bounded on $[0, \infty)$ for all positive integral values of n. Using this result show that $\int_0^\infty e^{-x} x^n dx$ exists.

37. Find all the real values of p and q so that the integral $\int_0^1 x^p \left(\log \frac{1}{x}\right)^q dx$ converges.

38. Show that $\int_0^{\pi/2} \frac{x^n}{\sin^m x}\,dx$ exists iff $m < n+1$

39. Examine the convergence of $\int_0^1 \frac{dx}{x^{1/3}(1-x)^{1/3}}$

40. Test the convegence of the improper integral $\int^{\infty} \frac{dx}{x^2(1+e^{-x})}$.

41. Test the convergence of the integral $\int_0^1 \frac{\sin\frac{1}{x}}{\sqrt{x}}\,dx$ cgt.

∵ $\sin \frac{1}{x}$ is defined everywhere between $(0,1)$

$\left|\sin \frac{1}{x}\right| < 1$

$\therefore |f(x)| < \frac{1}{\sqrt{x}} = g(x)$

∵ $g(x)$ is cgt.

WORKSHEET - 6

SEQUENCE & SERIES OF FUNCTIONS
(Uniform Convergence)

1. Show that the series for which $S_n(x) = nx(1-x)^n$ can be integrated term by term on [0, 1], though it is not uniformly convergent on [0,1].

2. Examine for uniform convergence and continuity of the limit function of the sequnce $<f_n>$, where

$$f_n(x) = \frac{nx}{1+n^2x^2}, 0 \le x \le 1.$$

Not UC

3. Show that the function represented by $\sum_{n=1}^{\infty} \frac{\sin nx}{n^3}$ is differentiable for every x and its derivative is $\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$

4. Show that the sequence $\{x^{n-1}(1-x)\}$ is uniformly convergent on [0, 1].

5. Examine for term by term integration the series the sum of whose first n terms is $n^2x(1-x)^n$, $(0 \le x \le 1)$.

6. Examine the convergence of the integrals

(i) $\int_1^2 \frac{dx}{(1+x)\sqrt{2-x}}$ (ii) $\int_0^{\infty} \frac{x^2}{\sqrt{x^5+1}} dx$

7. Show that the function represented by $\sum_{n=1}^{\infty} \frac{\sin nx}{n^3}$ is differetiable for every x and its derivative is $\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$.

8. Let $f_n(x) = \frac{x}{1+nx^2}$ for all real x. Show that f_n converges uniformly to a function f. What is f? Show that for $x \ne 0$, $f'_n(x) \to f'(x)$ but $f'_n(0)$ does not converge to $f'(0)$. Show that the maximum value $|f_n(x)|$ can take is $\frac{1}{2\sqrt{n}}$.

9. Show that the sequence $\langle x^{n-1}(1-x) \rangle$ is uniformly convergent on [0, 1].

10. Test the uniform convergence and continuity of $\{f_n\}$ where $f_n(x) = \frac{x}{1+nx}, 0 \le x \le 1$.

Not UC

11. Show that the sequence of functions $\{f_n\}$ where $f_n(x) = \frac{x}{1+nx^2}, x \in \mathbb{R}$ converges uniformly on any closed interval [a, b].

12. Show that the sequence of functions $\{f_n\}$, where $f_n(x) = nx(1-x)^n$ is not uniformly convergent on [0, 1].

13. Show that the series $\sum_{n=1}^{\infty} \frac{\sin nx}{n^p}$ is uniformly and absolutely convergent for all real values of x and $p > 1$.

14. Express the following in terms of Gamma function.

$$\int_0^a x^{p-1}(a-x)^{q-1} dx \text{ where } p > 0, q > 0$$

H.O. 25/8 Old Rajender Nagaer Market, Delhi-60
B.O.: 105-106, Top Floor, Mukherjee Tower, Dr. Mukherjee Nagar, delhi-9
Ph : 011-45629987, 09999197625 || Email : ims4ims2010@gmail.com, www.ims4maths.com

15. Test for uniform convergence and continuity of the sum function of the series for which

(i) $f_n(x) = \frac{1}{1+nx} (0 \le x \le 1)$

(ii) $f_n(x) = nx(1-x)^n (0 \le x \le 1)$.

16. Show that the series $\sum_1^\infty \frac{nx^2}{n^3+x^3}$ is uniformly convergent on [0, k] for any k > 0.

17. Show that the sequence of faunctions $\{f_n\}$, where $f_n(x) = nx(1-x)^n$ is not uniformly convergent on [0, 1]

18. Examine for term by term integration the series the sum of whose first n terms is $n^2x(1-x)^n, 0 \le x \le 1$. NOT TBTI

19. Show that the sequence $\langle x^{n-1}(1-x) \rangle$ is uniformly convergent on [0, 1].

20. Show that the function represented by $\sum_{n=1}^\infty \frac{\sin nx}{n^3}$ is differentiable for every x and its derivative is $\sum_{n=1}^\infty \frac{\cos nx}{n^2}$.

21. Show that the sequence $\langle f_n \rangle$, where $f_n(x) = nxe^{-nx^2}, x \ge 0$ is not uniformly convergent on $[0, k], k > 0$.

22. Test the uniform convergence of the series $\sum_{n=1}^\infty \frac{x}{n(1+nx^2)}$ Take $f_n = \frac{x}{n(1+nx^2)}$ Maximise f_n to use WMT.

23. Test for uniform convergence of the sequence $\{S_n(x)\}$ where $S_n(x) = nx(1-x)^n$ when $0 \le x \le 1$ NOT UC

24. Find the values of x for which the series $\sum_{n=1}^\infty \frac{x^\alpha}{1+n^2x^2}, x \ge 0, \alpha \ge 0$, converges uniformly on (i) [0, 1] and (ii) $[0, \infty)$

25. The functions f_n in on [0, 1] are given by

$f_n(x) = \frac{nx}{1+n^2x^p}, (p > 0)$ p<2

For what values of p does the sequence $\{f_n\}$ converge uniformly to its limit f? Examine whether $\int_0^1 f_n \to \int_0^1 f$ for p = 2 and p = 4.

26. Test the series $\sum_{n=1}^\infty \frac{x}{(n+x^2)^2}$ for uniform convergence.

27. Test uniform convergence of the series $\sum_{n=1}^\infty \frac{\sin nx}{n^p}$, where $P > 0$. UC for p>1

28. Let $f_n(x) = nx(1-x)^n, x \in [0, 1]$. Examine the uniform convergence of $\{f_n(x)\}$ on [0, 1].

29. Show that the series for which the sum of first n terms

$f_n(x) = \frac{nx}{1+n^2x^2}, 0 \le x \le 1$

cannot be differentiated term - by - term at x = 0.

What happens at $x \ne 0$?

30. Show that if $S(x) = \sum_{n=1}^\infty \frac{1}{n^3+n^4x^2}$, then its derivative

$S'(x) = -2x\sum_{n=1}^\infty \frac{1}{n^2(1+nx^2)^2}$ for all x.

31. Let $f_n(x)=\begin{cases} 0, & \text{if } x<\dfrac{1}{n+1}, \\ \sin\dfrac{\pi}{x}, & \text{if } \dfrac{1}{n+1}\le x\le\dfrac{1}{n}, \\ 0, & \text{if } x>\dfrac{1}{n} \end{cases}$

Show that $f_n(x)$ converges to a continuous function but not uniformly.

32. Show that the series $\sum_{n=1}^{\infty}\left(\frac{\pi}{\pi+1}\right)^n n^6$ is convergent.

33. Test the series of functions $\sum_{n=1}^{\infty}\frac{nx}{(1+n^2x^2)}$ for uniform convergence.

34. Test the convergence of the integral

$$\int_0^1 \frac{\sin\frac{1}{x}}{\sqrt{x}}\,dx$$

35. State the weierstrass M-test for uniform convergence of an infinite series of functions. Prove that the series $\sum_{n=1}^{\infty}\frac{x}{n^{\alpha}(1+nx^2)}$ with $\alpha<0$ is uniformly convergent on $(-\infty,\infty)$

36. Q. 21 Show that the sequence $\{f_n\}$, where $f_n(x)=nxe^{-nx^2}$ is pointwise, but not uniformly convergent in $[0,\infty)$.

37. Let $f_n(x)=\frac{x}{1+nx^2}$ for all real x. Show that f_n converges uniformly to a function f. What is f? Show that for $x\neq 0$, $f_n'(x)\to f'(x)$ but $f_n'(0)$ does not converge to $f'(0)$. Show that the maximum value $|f_n(x)|$ can take is $\frac{1}{2\sqrt{n}}$.

38. Show that the series for which $S_n(x)=nx(1-x)^n$ can be integrated term by term on $[0,1]$, though it is not uniformly convergent on $[0,1]$.

39. Examine the convergence of $\int_0^{\infty}\frac{x^{n-1}}{1+x}$.

40. Show that the sequence of functions $\{f_n\}$, where $f_n(x)=nx(1-x)^n$ is not uniformly convergent on $[0,1]$.

WORKSHEET - 7

COMBINED CHAPTERS:

Limits & Continuity, Differentiability, Reimann Integration, Real Number System, Sequence & Series of Real Numbers, Improper Integrals, Sequence & Series of Real Valued Functions

1. Given the series $\sum_{n=1}^{\infty} f_n$ for which $S_n(x) = \frac{1}{2n^2}\log\left(1+n^4x^2\right), 0 \le x \le 1$. Show that the series $\sum_{n=1}^{\infty} f_n'$ does not converge uniformly, but the given series can be differentiated term by term.

2. For what value of a does $\frac{\sin 2x + a\sin x}{x^3}$ tend to a finite limit l as $x \to 0$? When a has this value, what is the value of l ?

3. If $|x| < 1$, show that
$$\frac{1}{1-x}\log\frac{1}{1-x} = \sum_{n=1}^{\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}\right)x^n.$$

4. Prove that $\prod_{n=1}^{\infty}\left(1-\frac{1}{n^{2/3}}\right)e^{[illegible]}$ is absolutely convergent.

5. Evaluate $\lim_{x\to 0}\left(\frac{e^{ax}-e^{bx}+\tan x}{x}\right)$

6. Examine the derivability of the function f defined by
$$f(x) = \begin{cases} x^m \sin\frac{1}{x}, & \text{if } x \ne 0 \\ 0, & \text{if } x = 0. \end{cases}$$
determine m when f' is continuous at x = 0.

7. Determine the values of A and B for which $\underset{x\to 0}{Lt}\ \frac{\sin 3x + A\sin 2x + B\sin x}{x^5}$ exists and find the limit

8. (i) Prove that $\left(1+\frac{1}{x}\right)^x > \left(1+\frac{1}{y}\right)^y$ if $x, y \in \mathbb{R}$ and $x > y > 0$.

   (ii) If $z = xyf(y/x)$, show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$, and if z is a constant, then $\frac{f'(y/x)}{f(y/x)} = \frac{x\left(y+x\frac{dy}{dx}\right)}{y\left(y-x\frac{dy}{dx}\right)}$.

9. Discuss the convergence of the integral
$$\int_0^1 \frac{x^n \log x}{(1+x^2)}dx$$

10. If 0 < a < b, prove that $\left|\int_a^b \frac{\sin x}{x}dx\right| < \frac{4}{a}$.

11. Prove that $\prod_{n=1}^{\infty}\left(1+\frac{x}{n}\right)e^{-x/n}$ is absolutely convergent for any real x.

12. (i) Find the values of a and b in order that $\underset{x\to 0}{Lt}\ \frac{x(1-a\cos x)+b\sin x}{x^3}$ may be equal to $\frac{1}{3}$.

    (ii) Prove that the sequence $\{a_n\}$ recursively defined by $a_1 = \sqrt{5}, a_{n+1} = \sqrt{5+a_n}, n \ge 1$

converges to the positive root of the equation $x^2 - x - 5 = 0$.

13. Show that if a > 1, $\int_0^\infty \frac{x^a}{a^x} dx = \frac{\Gamma(a+1)}{(\log a)^{a+1}}$.

14. Show that $\prod_{n=0}^{\infty}(1+x^{2^n})$ converges to $\frac{1}{1-x}$ if $|x| < 1$.

15. Examine the convergence $\int_0^1 \frac{x^n}{1-x} dx$.

16. By applying the mean value theorem of integral calculus, Show that $e^{1/4} < \frac{4}{3} < e^{1/3}$ by considering $\int_3^4 \frac{1}{x} dx$.

17. Show that

$$\int_0^p x^m \left(p^q - x^q\right)^n dx = \frac{p^{m+nq+1}}{q} B\left(n+1, \frac{n+1}{q}\right)$$

if $p > 0, q > 0, m > -1, n > -1$.

18. Show that $B(m,n) = \frac{\gamma(m)\gamma(n)}{\gamma(m+n)}$ where m > 0, n > 0.

19. Show that $\lim \{I_n\}$, where $I_n = \int_0^\pi \frac{\sin nx}{x} dx, n \in I$ exist and that the limit is equal to $\frac{\pi}{2}$.

20. (i) Use Lagrange's mean value theorem to prove that $1 + x < e^x < 1 + xe^x \ \forall \ x > 0$.

(ii) Show that $\int_0^1 \frac{\operatorname{cosec} x}{x^n} dx$ is divergent if $n \geq 1$.

21. Show that the sequence (x_n), where $x_1 = 1$ and $x_n = \sqrt{2 + x_{n-1}}\ \forall n \geq 2$ is convergent and converges to 2.

22. Prove tht between amfong two real roots of $e^x \sin x = x$, there is atleast one root of $\cos x + \sin x = e^{-x}$;

23 Let $f(x) = \begin{cases} -1, -2 \leq x \leq 0 \\ x-1, 0 < x \leq 2 \end{cases}$, and

$g(x) = f(|x|) + |f(x)|$.

Test the differentiability of g(x) in (–2, 2).

24. For what value of a does $\frac{\sin 2x + a \sin x}{x^3}$ tend to a finite limit l as $x \to 0$? When a has this value, what is the value of l?

(i) $\underset{x \to 0}{Lt} \sin x \log x$

(ii) Show that

$$\int_0^\infty \sqrt{x} e^{-x^2} dx \times \int_0^\infty \frac{e^{-x^2}}{\sqrt{x}} dx = \frac{\pi}{2\sqrt{2}}.$$

25. Prove that the sequence $\{x_n\}$ defined by $x_1 = \sqrt{7}, x_{n+1} = \sqrt{4 + x_n}$ converges to positive root of the equation $x^2 - x - 7 = 0$.

26. Show that the function defined by

$$f(x) = x\left\{1 + \frac{1}{3}\sin \log(x)^2\right\}, f(0) = 0.$$

is everywhere continuous and monotone but has no differential coefficient at the origin.

27. Show that for any fixed value of x, the series $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ is convergent.

28. A function $f: \mathbb{R} \to \mathbb{R}$ is continuous on $\mathbb{R}$ and $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$ for all 125.

$x, y \in \mathbb{R}$. Prove that $f(x) = ax + b, (a, b \in \mathbb{R})$ for all $x \in \mathbb{R}$.

29. If $f(x) = e^{-1/x^2}\sin^{(1/x)}$, for $x \neq 0$ and $f(0) = 0$,

Show that

(i) The function f has at every point a differential coefficient and this is continuous at x=0.

(ii) The differential coefficient vanishes at $x = 0$ and at an infinite number of points in every neighbourhood of x = 0.

30. Show that f defined on (0, 2) by setting

$$f(x) = \begin{cases} x^2 & \text{for rational } x \in (0,2) \\ 2x-1 & \text{for irrational } x \in (0,2) \end{cases}$$

is differentiable only at x =1 and that $f'(1) \neq 0$

Is the inverse function differentiable at $1 = y = f(1)$?

31. (i) Examine the function f defined on R by setting

$$f(x) = \frac{e^{\frac{1}{x}}\sin(1/x)}{1+e^{\frac{1}{x}}}, \text{ if } x \neq 0,$$

$f(0) = 0$, for points of discontinuity, if any.

(ii) Show that $\log(1+x) - \dfrac{2x}{2+x}$ is increasing when x > 0.

32. Prove that $\prod_{n=1}^{\infty}\left(1+\frac{x}{n}\right)e^{-x/n}$ is absolutely convergent for any real x.

33. Prove that $\dfrac{\pi^2}{9} \leq \displaystyle\int_{\pi/6}^{\pi/3}\frac{x}{\sin x}dx \leq \frac{2\pi^2}{9}$

34. Show that $\prod_{n=0}^{\infty}(1+x^{2^n})$ converges to $\dfrac{1}{1-x}$ if $|x|<1$.

35. Prove that $\dfrac{1}{\pi} \leq \displaystyle\int_0^1 \frac{\sin \pi x}{1+x^2}dx \leq \frac{2}{\pi}$.

Consider the series $\displaystyle\sum_{n=0}^{\infty}\frac{x^2}{(1+x^2)^n}$.

Find the values of x for which it is convergent and also the sum function. Is the convergence uniform? Justify your answer.

36. Does the integral $\displaystyle\int_{-1}^{1}\sqrt{\frac{1+x}{1-x}}\,dx$ exist?

If so, find its value.

37. Evaluate

(i) $\lim_{x\to 2} f(x)$,

where $f(x) = \begin{cases} \dfrac{x^2-4}{x-2}, & x \neq 2 \\ \pi, & x = 2 \end{cases}$

(ii) $\displaystyle\int_0^1 \ell n x\,dx$.

38. Evaluate

$$\int_0^1 \frac{\log_e(1+x)}{1+x^2}dx.$$

39. Evaluate the following integral:

$$\int_{\pi/6}^{\pi/3}\frac{\sqrt[3]{\sin x}}{\sqrt[3]{\sin x}+\sqrt[3]{\cos x}}dx.$$

40. Show that $\int_0^\infty \frac{dx}{1+x^2\sin^2 x}$ is divergent.

47. Let f be a continuous function on [0, 1]. Using first mean value theorem on integration prove that

$$\underset{n\to\infty}{Lt}\int_0^1 \frac{n\,f(x)}{1+n^2x^2}\,dx=\frac{\pi}{2}f(0)$$

42. Consider the series $\sum_{n=0}^{\infty}\frac{x^2}{\left(1+x^2\right)^n}$

Find the values of x for which it is convergent and also the sum function.
Is the convergence uniform? Justify your answer.

43. Let $f_n(x)=x^n$ on $-1<x\le 1$ for $n=1,2,\ldots$. Find the limit function. Is the convergence uniform? Justify your answer.

44. Let $f(x)$ be differentiable on [0, 1] such that

$f(1)=f(0)=0$ and $\int_0^1 f^2(x)dx=1$. Prove that

$$\int_0^1 xf(x)f'(x)dx=-\frac{1}{2}$$

46. Investigate the continuity of the function $f(x)=\frac{|x|}{x}$ for $x\neq 0$ and $f(0)=-1$

47. Let (a, b) be any open interval, f a function defined and differentiable on (a, b) such that its derivative is bounded on (a, b). Show that f is uniformly continuous on (a, b).

48. If f is a continuous function on [a, b] and if $\int_a^b f^2(x)dx=0$ then show that $f(x)=0$ for all x in [a, b]. Is this true if f is not continuous?

49. Show that the function f defined by $f(x)=\frac{1}{2}$, $x\in[1,\infty)$ is uniformly continuous on $[1,\infty)$

50. Show that the series

$x^4+\frac{x^4}{1+x^4}+\frac{x^4}{(1+x^4)^2}+\cdots$ is not uniformly convergent on [0, 1].

51. If f is continuous on [a, b] and $\int_a^b fg\,dx=0$ for any continuous function g on [a, b], then show that f = 0 for all $x\in[a,b]$

52. If f(x) is monotonic in the interval $0<x<a$, and the integral $\int_0^a x^n f(x)dx$ exists, then show that $\lim_{x\to 0^+} x^{n+1}f(x)=0$.

53. Show that

$$\int_0^\infty \frac{\tan^{-1}\alpha x\tan^{-1}\beta x}{x^2}dx=\frac{\pi}{2}\log\left[\frac{(\alpha+\beta)^{(\alpha+\beta)}}{\alpha^\alpha+\beta^\beta}\right],$$

$\alpha,\beta>0$.

54. Discuss the convergence of the integral

$$\int_0^\infty \frac{dx}{1+x^4\sin^2 x}$$

55. Let the function f be defined by

$$f(t)=\begin{cases}0, \text{ for} & t<0\\ t, \text{ for} & 0\le t\le 1\\ 4, \text{ for} & t>1\end{cases}$$

(i) Determine the function $F(x)=\int_0^x f(t)dt$.

(ii) Where is F non-differentiable? Justify your answer.

56. Test for convergence the integral $\int_0^\infty \sqrt{xe^{-x}}\,dx$

57. Evaluate the following in terms of Gamma function:

$$\int_0^a \sqrt{\left(\frac{x^3}{a^3-x^3}\right)}dx.$$

58. Let $f(x)$ be a real-valued function defined on the interval $(-5,5)$ such that $e^{-x}f(x) = 2+\int_0^x \sqrt{t^4+1}\,dt$ for all $x \in (-5,5)$.

Let $f^{-1}(x)$ be the inverse function of $f(x)$. Find $(f^{-1})'(2)$.

59. For $x > 0$, let $f(x) = \int_1^x \frac{\ln t}{1+t}dt$.

Evaluate $f(e)+f\left(\frac{1}{e}\right)$.

60. Evaluate $\lim_{x\to 0}\left(\frac{2+\cos x}{x^3 \sin x}-\frac{3}{x^4}\right)$.

61. Examine whether the series for which $S_n(x) = \frac{1}{n+n^3x^2}$ is differentiable term by term.