IAS MATHEMATICS (OPT.)-2009

PAPER - I : SOLUTIONS

1(a) Find a Hermitian and a skew-Hermitian matrix each whose sum is the matrix $\begin{bmatrix} 2i & 3 & -1 \\ 1 & 2+3i & 2 \\ -i+1 & 4 & 5i \end{bmatrix}$

Soln: Let $A = \begin{bmatrix} 2i & 3 & -1 \\ 1 & 2+3i & 2 \\ -i+1 & 4 & 5i \end{bmatrix}$

then $A+A^{\theta}$ is Hermitian and

$A-A^{\theta}$ is skew-Hermitian.

$\therefore \frac{1}{2}(A+A^{\theta})$ is Hermitian and

$\frac{1}{2}(A-A^{\theta})$ is skew-Hermitian.

Given that $A = \frac{1}{2}(A+A^{\theta}) + \frac{1}{2}(A-A^{\theta})$

$= P+Q$ (say)

where P is Hermitian and

Q is skew-Hermitian.

TO find P and Q:

Now $A^{\theta} = (\bar{A})^T$

$$= \begin{bmatrix} -2i & 3 & -1 \\ 1 & 2-3i & 2 \\ +i+1 & 4 & -5i \end{bmatrix}^T$$

$$= \begin{bmatrix} -2i & 1 & i+1 \\ 3 & 2-3i & 4 \\ -1 & 2 & -5i \end{bmatrix}$$

we have

$$P = \frac{1}{2}(A + A^{\theta})$$

$$= \frac{1}{2}\left(\begin{bmatrix} 2i & 3 & -1 \\ 1 & 2+3i & 2 \\ -i+1 & 4 & 5i \end{bmatrix} + \begin{bmatrix} -2i & 1 & i+1 \\ 3 & 2-3i & 4 \\ -1 & 2 & -5i \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 0 & 4 & i \\ 4 & 4 & 6 \\ -i & 6 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 & i/2 \\ 2 & 2 & 3 \\ -i/2 & 3 & 0 \end{bmatrix}$$

now we have

$$Q = \frac{1}{2}\left(\begin{bmatrix} 2i & 3 & -1 \\ 1 & 2+3i & 2 \\ -i+1 & 4 & 5i \end{bmatrix} - \begin{bmatrix} -2i & 1 & i+1 \\ 3 & 2-3i & 4 \\ -1 & 2 & -5i \end{bmatrix}\right)$$

$$= \frac{1}{2}\begin{bmatrix} 4i & 2 & -i-2 \\ -2 & 6i & -2 \\ -i+2 & 2 & 10i \end{bmatrix}$$

$$= \begin{bmatrix} 2i & 1 & \frac{-i}{2}-1 \\ -1 & 3i & -1 \\ \frac{-i}{2}+1 & 1 & 5i \end{bmatrix}$$

$\therefore$ The required Hermitian and Skew-Hermitian matrices are $\begin{bmatrix} 0 & 2 & i/2 \\ 2 & 2 & 3 \\ -i/2 & 3 & 0 \end{bmatrix}$ and

$\begin{bmatrix} 2i & 1 & \frac{-i}{2}-1 \\ -1 & 3i & -1 \\ \frac{-i}{2}+1 & 1 & 5i \end{bmatrix}$ respectively.

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1(b)

2. Prove that the set V of the vectors (x_1, x_2, x_3, x_4) in $\mathbb{R}^4$ which satisfy the equations $x_1+x_2+2x_3+x_4=0$ and $2x_1+3x_2-x_3+x_4=0$, is a subspace of $\mathbb{R}^4$. What is the dimension of this subspace? Find one of its bases.

Sol. Let $\mathbb{R}^4 = \{(x_1, x_2, x_3, x_4) / x_1, x_2, x_3, x_4 \in \mathbb{R}\}$ be the given vectorspace.

Let $V = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \middle/ \begin{array}{l} x_1+x_2+2x_3+x_4=0 \\ 2x_1+3x_2-x_3+x_4=0; \\ x_1, x_2, x_3, x_4 \in \mathbb{R} \end{array} \right\} \subseteq \mathbb{R}^4.$

Since $(0,0,0,0) \in \mathbb{R}^4$; $0+0+2(0)+0=0$ and $2(0)+3(0)-0+0=0$

$\therefore (0,0,0,0) \in V$.

$\therefore V$ is non-empty subset of $\mathbb{R}^4$.

Let $\alpha = (x_1, x_2, x_3, x_4)$

$\beta = (y_1, y_2, y_3, y_4) \in V$ then $x_1+x_2+2x_3+x_4=0$

$2x_1+3x_2-x_3+x_4=0$;

$y_1+y_2+2y_3+y_4=0$

$2y_1+3y_2-y_3+y_4=0$.

Let $a, b \in \mathbb{R}$ then we have

$a\alpha+b\beta = (ax_1+by_1, ax_2+by_2, ax_3+by_3, ax_4+by_4)$. — (1)

Since $(ax_1+by_1)+(ax_2+by_2)+2(ax_3+by_3)+(ax_4+by_4)$

$= a(x_1+x_2+2x_3+x_4)+b(y_1+y_2+2y_3+y_4)$

$= a(0)+b(0)$

$= 0$.

and $2(ax_1+by_1)+3(ax_2+by_2)-(ax_3+by_3)+(ax_4+by_4)$

$= a(2x_1+3x_2-x_3+x_4)+b(2y_1+3y_2-y_3+y_4)$

$= a(0)+b(0) = 0$.

Since the number of elements in a basis 'S' is 2.

$\therefore \boxed{\dim V = 2}$.

3-Dimensional Geometry, Paper-I, IAS

1(e) [2009] 2009 1(e). A line drawn through a variable point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, z=0$ to meet two fixed lines $y=mx, z=c$ and $y=-mx, z=-c$. Find the locus of the line.

Sol^n: The given lines are

$$y-mx=0,\quad z-c=0 \quad \text{---} (1)$$
$$y+mx=0,\quad z+c=0 \quad \text{---} (2)$$

and the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \;;\; z=0 \quad \text{---} (3)$$

Any line intersecting (1) & (2) is

$$\left.\begin{array}{l} y-mx+k_1(z-c)=0 \\ y+mx+k_2(z+c)=0 \end{array}\right\} \text{---} (4)$$

If it meets the ellipse (3), we have to eliminate x, y, z from (3) & (4).

(3) ⟹ putting $z=0$ in (4), we get

$$y-mx+k_1(-c)=0$$
$$y+mx+k_2(c)=0$$

Solving:

$$\frac{y}{-mk_2c+mk_1c}=\frac{x}{-ck_1-ck_2}=\frac{1}{m+m}$$

$$\Rightarrow x=\frac{-(k_1+k_2)}{2m}\;;\; y=\frac{c(k_1-k_2)}{2}$$

putting these values of x, y in (3) we get

$$\frac{c^2(k_1+k_2)^2}{4a^2m^2}+\frac{c^2(k_1-k_2)^2}{4b^2}=1.$$

$$\Rightarrow b^2c^2(k_1+k_2)^2+a^2c^2(k_1-k_2)^2=4a^2b^2m^2$$

$$\Rightarrow b^2c^2\left[\frac{mx-y}{z-c}+\frac{(-mx-y)}{z+c}\right]^2+c^2m^2a^2\left[\frac{mx-y}{z-c}+\frac{mx+y}{z+c}\right]^2=4a^2b^2m^2$$

$$\Rightarrow b^2c^2\left[\frac{(z+c)(mx-y)-(mx+y)(z-c)}{z^2-c^2}\right]^2+c^2a^2m^2\left[\frac{(mx-y)(z+c)+(mx+y)(z-c)}{z^2-c^2}\right]^2=4a^2b^2m^2$$

$$\Rightarrow \frac{b^2c^2}{(z^2-c^2)^2}\left[\cancel{mxz}-yz+cmx-\cancel{yc}-\cancel{mxz}+mxc-yz+\cancel{yc}\right]^2$$
$$+\frac{c^2a^2m^2}{(z^2-c^2)^2}\left[mxz-\cancel{yz}+\cancel{mxc}-\cancel{yc}+mxz-\cancel{mxc}+\cancel{yz}-yc\right]^2=4a^2b^2m^2$$

$$\Rightarrow b^2c^2(2mxc-2yz)^2+c^2a^2m^2[2mxz-2yc]^2=4a^2b^2m^2(z^2-c^2)^2$$

$$\Rightarrow 4b^2c^2(cmx-yz)^2+4c^2a^2m^2(mxz-yc)^2=4a^2b^2m^2(z^2-c^2)^2$$

$$\Rightarrow b^2c^2(cmx-yz)^2+c^2a^2m^2(mxz-yc)^2=a^2b^2m^2(z^2-c^2)^2$$

which is the required locus.

$\to$ 1(f) Find the equation of the sphere having its centre on the plane $4x-5y-z=3$, and passing through the circle $x^2+y^2+z^2-12x-3y+4z+8=0$, $3x+4y-5z+3=0$.

→ 1(f) Find the equation of the sphere having its centre on the plane $4x-5y-z=3$, and passing through the circle $x^2+y^2+z^2-12x-3y+4z+8=0$
$3x+4y-5z+3=0$.

Sol^n: The given circle is

$$x^2+y^2+z^2-12x-3y+4z+8=0$$
$$3x+4y-5z+3=0.$$

Any sphere through the circle is

$$x^2+y^2+z^2-12x-3y+4z+8+\lambda(3x+4y-5z+3)=0 \quad \text{(1)}$$

$$\Rightarrow x^2+y^2+z^2+x(-12+3\lambda)+y(-3+4\lambda)+z(4-5\lambda)+8+3\lambda=0$$

Its centre is $\left(\frac{12-3\lambda}{2}, \frac{3-4\lambda}{2}, \frac{5\lambda-4}{2}\right)$

Since it lies on the plane $4x-5y-z=3$

$$\Rightarrow 4\left(\frac{12-3\lambda}{2}\right)-5\left(\frac{3-4\lambda}{2}\right)-\left(\frac{5\lambda-4}{2}\right)=3$$

$$\Rightarrow 48-12\lambda-15+20\lambda-5\lambda+4=6$$

$$\Rightarrow 3\lambda=-31$$

$$\Rightarrow \boxed{\lambda=-\frac{31}{3}}$$

∴ putting the value of λ in (1), we get

$$x^2+y^2+z^2-12x-3y+4z+8-\frac{31}{3}(3x+4y-5z+3)=0$$

$$\Rightarrow 3(x^2+y^2+z^2)-36x-9y+12z+24-93x-124y+155z-93=0$$

$$\Rightarrow 3(x^2+y^2+z^2)-129x-133y+167z-69=0.$$

which is the required equation of the sphere.

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2(a)

(3) Let $\beta = \{(1,1,0), (1,0,1), (0,1,1)\}$ and $\beta' = \{(2,1,1), (1,2,1), (-1,1,1)\}$ be the two ordered bases of R^3. Then find a matrix representing the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ which transforms β into β'. Use this matrix representation to find $T(\bar{x})$, where $\bar{x} = (2,3,1)$.

Sol Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be the given linear transformation.

Let $\beta = \{(1,1,0), (1,0,1), (0,1,1)\}$ and

$\beta' = \{(2,1,1), (1,2,1), (-1,1,1)\}$ be two ordered bases of $\mathbb{R}^3$.

then we have

$$\left.\begin{aligned} T(1,1,0) &= (2,1,1) \\ T(1,0,1) &= (1,2,1) \\ T(0,1,1) &= (-1,1,1). \end{aligned}\right\} \text{—— (A)}.$$

Since β' is the basis of $\mathbb{R}^3$

let $\alpha = (x,y,z) \in \mathbb{R}^3$ then

$$(x,y,z) = a(2,1,1) + b(1,2,1) + c(-1,1,1). \text{ —— (B)}$$

$$\Rightarrow \quad 2a + b - c = x \text{ —— (i)}$$
$$a + 2b + c = y \text{ —— (ii)}$$
$$a + b + c = z \text{ —— (iii)}$$

from (ii) & (iii),

$$\boxed{b = y - z}$$

from (i) & (ii), we have

$$3a + 3b = x + y$$

$$\Rightarrow a = \frac{x+y}{3} - (y-z)$$

$$\boxed{a = \frac{x - 2y + 3z}{3}}$$

from (iii)

$$a + b + c = 0$$

$$\Rightarrow c = -(a+b)$$

$$= -\left[\frac{x - 2y + 3z}{3} + y - z\right]$$

$$\boxed{c = -\left[\frac{x+y}{3}\right]}$$

$\therefore$ from (B),

$$(x, y, z) = \left(\frac{x - 2y + 3z}{3}\right)(2,1,1) + (y-z)(1,2,1) + \left(\frac{x+y}{3}\right)(-1,1,1) \quad \text{(C)}$$

$\therefore$ from (A)

$$T(1,1,0) = \frac{2-2+3}{3}(2,1,1) + 0(1,2,1) + \left(\frac{2+1}{3}\right)(-1,1,1)$$

$$= 1(2,1,1) + 0(1,2,1) + 1(-1,1,1) \quad \text{(i)}$$

$$T(1,0,1) = \left(\frac{1-4+3}{3}\right)(2,1,1) + (2-1)(1,2,1) + \left(\frac{1+3}{3}\right)(-1,1,1)$$

$$= 0(2,1,1) + 1(1,2,1) + \frac{4}{3}(-1,1,1) \quad \text{(ii)}$$

$$T(0,1,1) = \frac{-1-2+3}{3}(2,1,1) + (1-1)(1,2,1) + \frac{-1+1}{3}(-1,1,1)$$

$$= 0(2,1,1) + 0(1,2,1) + 0(-1,1,1) \quad \text{(iii)}$$

Now the matrix of linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is

$$[T : B, B'] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 4/3 & 0 \end{bmatrix}.$$

To find linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ explicitly by using this matrix:

~~Let~~ since S is the basis of $\mathbb{R}^3$

let $\alpha = (p,q,r) \in \mathbb{R}^3 : p,q,r \in \mathbb{R}$

then we have

$$(p,q,r) = x(1,1,0) + y(1,0,1) + z(0,1,1) \quad (1).$$

$$\Rightarrow x+y = p,$$
$$x+z = q$$
$$y+z = r$$

After solving these equations, we get

$$x = \frac{q-r+p}{2}, \quad y = \frac{p-q+r}{2}, \quad z = \frac{q+r-p}{2}.$$

$\therefore$ from (1),

$$(p,q,r) = \frac{p+q-r}{2}(1,1,0) + \frac{p-q-r}{2}(1,0,1) + \frac{-p+q+r}{2}(0,1,1).$$

$$\Rightarrow T(p,q,r) = \frac{p+q-r}{2}T(1,1,0) + \frac{p-q-r}{2}T(1,0,1) + \frac{-p+q+r}{2}T(0,1,1) \quad (\because T \text{ is LT}).$$

$$= \frac{p+q-r}{2}(2,1,1) + \frac{p-q-r}{2}(1,2,1) + \frac{-p+q+r}{2}(-1,1,1)$$

$$\boxed{T(p,q,r) = \left(\frac{4p-2r}{2}, \frac{2p+2r}{2}, \frac{p+q+r}{2}\right)}$$

To find $T(\bar{x})$: where $\bar{x} = (2,3,1)$.

$$\therefore T(2,3,1) = \left(\frac{4(2)-2(1)}{2}, \frac{2(2)+2(1)}{2}, \frac{2+3+1}{2}\right)$$
$$= (1,2,3).$$

2017(4) 2009 **3(a)** Let $L: \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation defined by

$L(x_1, x_2, x_3, x_4) = (x_3+x_4-x_1-x_2,\ x_3-x_2,\ x_4-x_1)$.

Then find the rank and nullity of L. Also, determine null space and range space of L.

Sol^n: Given that $L: \mathbb{R}^4 \to \mathbb{R}^3$ is a linear transformation such that

$$L(x_1,x_2,x_3,x_4) = (x_3+x_4-x_1-x_2,\ x_3-x_2,\ x_4-x_1). \quad \text{--- (1)}$$

Range space of $L = \{\beta \in \mathbb{R}^3 / T(\alpha) = \beta \text{ for } \alpha \in \mathbb{R}^4\}$

$\therefore$ The range space consists of all vectors of the type $(x_3+x_4-x_1-x_2,\ x_3-x_2,\ x_4-x_1)$ for all $(x_1,x_2,x_3,x_4) \in \mathbb{R}^4$.

$$\therefore \boxed{R(L) = \{(x_3+x_4-x_1-x_2,\ x_3-x_2,\ x_4-x_1) / x_1,x_2,x_3,x_4 \in \mathbb{R}\}.} \quad \text{--- (A)}$$

Let $\beta = (x_3+x_4-x_1-x_2,\ x_3-x_2,\ x_4-x_1) \in R(L)$

then $\beta = x_3(1,1,0)+x_4(1,0,1)+x_1(-1,0,-1) + x_2(-1,-1,0)$.

$\in L'(S)$ ($\because$ linear span of S).

where $S = \{(1,1,0),(1,0,1),(-1,0,-1),(-1,-1,0)\} \subseteq R(L)$.

$\therefore \beta \in R(L) \Rightarrow \beta \in L'(S)$.

$\therefore R(L) \subseteq L'(S)$. --- (2)

Since $S \subseteq R(L)$

$\Rightarrow L'(S) \subseteq R(L)$ --- (3).

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$\therefore$ from (2) and (3), we have

$$L'(S) = R(L).$$

i.e S spans $R(L)$.

Now we construct a matrix, whose rows are vectors of the subset S' of $R(L)$, and convert into echelon form by using E-row transformations.

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}_{4\times 3} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \quad \begin{aligned} R_2 &\to R_2 - R_1 \\ R_3 &\to R_3 + R_1 \\ R_4 &\to R_4 + R_1 \end{aligned}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 + R_2$$

clearly which is in echelon form and the number of non-zero rows of echelon form is 2.

$\therefore$ the set $S_1 = \{(1,1,0),(1,0,1)\}$ forms a basis of $R(L)$. and the number of elements of $S' = 2$.

$$\therefore \dim(R(L)) = 2.$$

$$\text{rank of } L = \rho(L)$$
$$= 2.$$

We know that rank of L + nullity of $L = \dim R^4$

$$\Rightarrow \quad 2 + \text{nullity of } L = 4$$

$$\Rightarrow \boxed{\text{nullity of } L = 2.}$$

Now we find nullspace of L:

Null space of $L = N(L)$

$= \{\alpha \in \mathbb{R}^4 \mid T(\alpha) = (0,0,0) \text{ in } \mathbb{R}^3\} \subseteq \mathbb{R}^4.$ — (2)

Let $\alpha \in N(L)$

i.e $(x_1, x_2, x_3, x_4) \in N(L)$

$\Rightarrow L(x_1, x_2, x_3, x_4) = (0,0,0)$

$\Rightarrow (x_3 + x_4 - x_1 - x_2,\ x_3 - x_2,\ x_4 - x_1) = (0,0,0)$

$\Rightarrow \left.\begin{array}{l} x_3 + x_4 - x_1 - x_2 = 0 \text{ — (i)} \\ x_3 - x_2 = 0 \text{ — (ii)} \\ x_4 - x_1 = 0 \text{ — (iii)} \end{array}\right\}$

$\Rightarrow \boxed{x_3 = x_2}$
$\boxed{x_4 = x_1}$

$\therefore\ N(L) = \{(x_1, x_2, x_2, x_1) \mid x_1, x_2 \in \mathbb{R}\} \subseteq \mathbb{R}^4.$

Clearly which is the required nullspace of L.

20M 2009-I

3(b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0). \end{cases}$$

Is f continuous at $(0,0)$? Compute partial derivatives of f at any point (x,y), if exist.

Sol: Given that

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Let $\epsilon > 0$ be given

Now we have

$$|f(x,y) - f(0,0)| = \left|\frac{xy}{\sqrt{x^2+y^2}} - 0\right|$$

$$= \left|\frac{xy}{\sqrt{x^2+y^2}}\right|$$

$$= |xy| \left|\frac{1}{\sqrt{x^2+y^2}}\right|$$

$$= |xy| \frac{1}{\sqrt{x^2+y^2}}$$

$$= \left|\frac{xy}{x^2+y^2}\right| \sqrt{x^2+y^2}$$

$$\leq \frac{1}{2}\sqrt{x^2+y^2} \quad \left(\because 2|xy| \leq x^2+y^2 \Rightarrow \left|\frac{xy}{x^2+y^2}\right| \leq \frac{1}{2} \text{ if } (x,y) \neq (0,0)\right)$$

$$< \sqrt{x^2+y^2}$$

$$< \epsilon \quad \text{whenever } x^2+y^2 < \epsilon^2 = \delta \text{ (choosing)}$$

$\therefore |f(x,y) - f(0,0)| < \epsilon$ whenever $x^2+y^2 < \delta$

$\therefore f(x,y)$ is continuous at $(0,0)$.

NOW

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{h(0)}{\sqrt{h^2+0}} - 0}{h}$$

$$= 0$$

and $$f_y(0,0) = \lim_{k\to 0} \frac{f(0,0+k) - f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{\frac{0(k)}{\sqrt{0+k^2}} - 0}{k}$$

$$= 0$$

$\therefore f$ possesses partial derivatives at $(0,0)$.

2009 4(c) Prove that the normals from the point (α, β, γ) to the paraboloid $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2z$ lie on the cone.

$$\frac{\alpha}{x-\alpha} + \frac{\beta}{y-\beta} + \frac{a^2-b^2}{z-\gamma} = 0.$$

Sol'n:- The given paraboloid is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2z$ —— ①

Let any line through (α, β, γ) be

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \quad \text{—— ②}$$

be the normal at (x_1, y_1, z_1) to (1).

The equation of the tangent plane at (x_1, y_1, z_1) to (1) is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - (z+z_1) = 0 \quad \text{—— (3)}$$

Since (2) is normal to (3) ∴ it is ∥ lel to the normal to (3)

$$\therefore \frac{l}{\frac{x_1}{a^2}} = \frac{m}{\frac{y_1}{b^2}} = \frac{n}{-1} = k \text{ (say)} \quad \text{—— (4)}$$

Again if the normal at (x_1, y_1, z_1) to (1) passes through (α, β, γ) then $x_1 = \frac{a^2\alpha}{a^2+\lambda}$, $y_1 = \frac{b^2\beta}{b^2+\lambda}$, $z_1 = \gamma+\lambda$ —— (5)

from (4), $l = k\frac{x_1}{a^2} = \frac{k}{a^2} \cdot \frac{a^2\alpha}{a^2+\lambda} = \frac{k\alpha}{a^2+\lambda}$ [Using (5)]

or $a^2+\lambda = \frac{k\alpha}{l}$ —— (6)

$m = k\frac{y_1}{b^2} = \frac{k}{b^2} \cdot \frac{b^2\beta}{b^2+\lambda} = \frac{k\beta}{b^2+\lambda}$ (or) $b^2+\lambda = \frac{k\beta}{m}$ —— (7)

$n = -k$ —— (8)

Subtracting (7) from (6), we get

$$a^2 - b^2 = k\left(\frac{\alpha}{l} - \frac{\beta}{m}\right)$$

$$= -n\left(\frac{\alpha}{l} - \frac{\beta}{m}\right) \quad \text{—— (9)}$$

(using 8)

To find the locus, we have to eliminate l, m, n from (2) and (9). Putting the value of l, m, n from (2) in (9), we have

$$a^2 - b^2 = -(z-\gamma)\left(\frac{\alpha}{x-\alpha} - \frac{\beta}{y-\beta}\right)$$

(or) $$\frac{a^2-b^2}{z-\gamma} = -\frac{\alpha}{x-\alpha} + \frac{\beta}{y-\beta}$$

(or) $$\frac{\alpha}{x-\alpha} - \frac{\beta}{y-\beta} + \frac{a^2-b^2}{z-\gamma} = 0$$

which is the required result.

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