(JACOBIAN)

16 Implicit Functions

1. DEFINITION

It is generally assumed that a functional equation $f(x, y) = 0$ determines y as a function of x, but such an equation may not define any such function or it may define one or more than one such function. For example, the equation

$$x^2 + y^2 - 5 = 0$$

determines two functions

$$y = \sqrt{5 - x^2}, \text{ and } y = -\sqrt{5 - x^2}, \text{ for } x^2 \leq 5,$$

whereas the equation

$$x^2 + y^2 + 5 = 0$$

determines no such function.

Definition. Let $f(x, y)$ be a function of two variables, and $y = \phi(x)$ be a function of x such that, for every value of x, for which $\phi(x)$ is defined, $f(x, \phi(x))$ vanishes identically, *i.e.*, $y = \phi(x)$ is a root of the functional equation $f(x, y) = 0$, then $y = \phi(x)$ is an *implicit function* defined by the functional equation $f(x, y) = 0$.

It is only in elementary cases, such as those given above, that it may be possible to express y as a function of x (*i.e.*, determine the implicit function). For more complicated functional equations no such determination of implicit function is possible. The difficulty of actual determination of an analytical expression does not rule out the possibility of the *existence* of the implicit function or functions, defined by a functional equation; the actual determination may demand new processes or may be, from a practical standpoint, too laborious. We now consider an Existence theorem, known as *Implicit function theorem*, that specifies conditions which guarantee that a functional equation does define an implicit function even though actual determination may not be possible. For many purposes, however, it is the fact that an equation does define a function, rather than an expression for the implicit function thus defined, that is of real importance; hence the value of Existence theorem.

1.1 Existence Theorem (*Case of two variables*)

Let $f(x, y)$ be a function of two variables x and y, and (a, b) be a point of its domain of definition such that

(*i*) $f(a, b) = 0$,

(*ii*) *the partial derivatives f_x and f_y exist, and are continuous in a certain neighbourhood of (a, b), and* (*iii*) $f_y(a, b) \neq 0$, *then there exists a rectangle* $(a - h, a + h; b - k, b + k)$ *about* (a, b) *such that for every value of* x *in the interval* $[a - h, a + h]$, *the equation* $f(x, y) = 0$ *determines one and only one value* $y = \phi(x)$, *lying in the interval* $[b - k, b + k]$, *with the following properties:*

(1) $b = \phi(a)$,

(2) $f[x, \phi(x)] = 0$, *for every* x *in* $[a - h, a + h]$, *and*

(3) $\phi(x)$ *is derivable, and both* $\phi(x)$ *and* $\phi'(x)$ *are continuous in* $[a - h, a + h]$.

Existence. Let f_x, f_y be continuous in a neighbourhood

$$R_1 : (a - h_1, a + h_1; b - k_1, b + k_1), \text{ of } (a, b)$$

Since f_x, f_y exist and are continuous in R_1, therefore f is differentiable and hence continuous in R_1.

Again, since f_y is continuous, and $f_y(a, b) \neq 0$, there exists a rectangle

$$R_2 : (a - h_2, a + h_2; b - k_2, b + k_2), h_2 < h_1, k_2 < k_1$$

(R_2 contained in R_1) such that for every point of this rectangle, $f_y \neq 0$.

Since $f = 0$ and $f_y \neq 0$ (it is therefore either positive or negative) at the point (a, b), a positive number $k < k_2$ can be found such that

$$f(a, b - k), f(a, b + k)$$

are of opposite signs, for, f is either an increasing or a decreasing function of y, when $y = b$.

Again, since f is continuous, a positive number $h < h_2$ can be found such that for all x in $[a - h, a + h]$,

$$f(x, b - k), f(x, b + k),$$

respectively, may be as near as we please to $f(a, b - k), f(a, b + k)$ and therefore have opposite signs.

Thus, for all x in $[a - h, a + h]$, f is a continuous function of y and changes sign as y changes from $b - k$ to $b + k$. Therefore it vanishes for some y in $[b - k, b + k]$.

Thus, for each x in $[a - h, a + h]$, there is a y in $[b - k, b + k]$ for which $f(x, y) = 0$; this y is a function of x, say $\phi(x)$ such that properties (1) and (2) are true.

Uniqueness. We, now, show that $y = \phi(x)$ is a unique solution of $f(x, y) = 0$ in $R_3 : (a - h, a + h; b - k, b + k)$, that is, $f(x, y)$ cannot be zero for more than one value of y in $[b - k, b + k]$.

Let, if possible, there be two such values y_1, y_2 in $[b - k, b + k]$ so that $f(x, y_1) = 0, f(x, y_2) = 0$. Also $f(x, y)$ considered as a function of a single variable y is derivable in $[b - k, b + k]$, so that by Rolle's theorem, $f_y = 0$ for a value of y between y_1 and y_2, which contradicts the fact that $f_y \neq 0$ in $R_2 \supset R_3$. Hence our supposition is wrong and there cannot be more than one such y.

Derivability. Let (x, y), $(x + \delta x, y + \delta y)$ be two points in $R_3 : (a - h, a + h; b - k, b + k)$ such that

$$y = \phi(x), \; y + \delta y = \phi(x + \delta x)$$

and

$$f(x, y) = 0, \; f(x + \delta x, y + \delta y) = 0$$

Since f is differentiable in R_1 and consequently in R_3 (contained in R_1),

$$\therefore \quad 0 = f(x + \delta x, y + \delta y) - f(x, y)$$
$$= \delta x f_x + \delta y f_y + \delta x \psi_1 + \delta y \psi_2$$

where ψ_1, ψ_2 are functions of δx and δy, and tend to 0 as

$$(\delta x, \delta y) \to (0, 0)$$

or

$$\frac{\delta y}{\delta x} = -\frac{f_x}{f_y} - \frac{\psi_1}{f_y} - \frac{\delta y}{\delta x}\frac{\psi_2}{f_y} \qquad (f_y \neq 0 \text{ in } R_3)$$

Proceeding to limits as $(\delta x, \delta y) \to (0, 0)$, we get

$$\phi'(x) = \frac{dy}{dx} = -\frac{f_x}{f_y}$$

Thus, $\phi(x)$ is derivable and hence continuous in R_3. Also $\phi'(x)$, being a quotient of two continuous functions, is itself continuous in R_3.

Note: It should be clearly understood that the theorem is essentially of a *local character*. That is, the implicit function $y = \phi(x)$ is a unique solution of $f(x, y) = 0$ in a certain neighbourhood $(a - h, a + h; b - k, b + k)$ of (a, b).

It may have a different solution $y = \psi(x)$ if a different neighbourhood of (a, b) is considered.

Example 1. Let $f(x, y) = x^2 + y^2 - 1$, and a point $(0, 1)$ so that

$$f(0, 1) = 0 \text{ and } f_y(0, 1) \neq 0$$

Of the two possible solutions $y = \pm\sqrt{1 - x^2}$.

(*i*) $y = +\sqrt{1 - x^2}$ is the implicit function in a *nbd* of $(0, 1)$, where $|x| < 1, y > 0$.

(*ii*) $y = -\sqrt{1 - x^2}$ is the implicit function in a *nbd* of $(0, -1)$, where $|x| < 1, y < 0$.

Example 2. Let $f(x, y) = y^2 - yx^2 - 2x^5$, and a point $(1, -1)$ so that

$$f(1, -1) = 0, \text{ and } f_y(1, -1) \neq 0.$$

It can be easily verified that the partial derivatives

$$f_x(x, y) = -2xy - 10x^4, \text{ and } f_y(x, y) = 2y - x^2$$

are continuous in a *nbd* of $(1, -1)$.

Of the two possible solutions

$$y = \frac{x^2}{2}(1 \pm \sqrt{1 + 8x}),\ x > -1/8,$$

$$y = \frac{x^2}{2}(1 - \sqrt{1 + 8x}),\ x > -1/8,$$

is the unique solution of $f(x, y) = 0$ in a *nbd* of $(1, -1)$, since, $-1 = y(1)$.

1.2 General Case

Let $f(x_1, x_2, ..., x_n, y)$ be a function of $(n + 1)$ variables, $x_1, x_2, ..., x_n, y$ and $(a_1, a_2, ..., a_n, b)$ be a point of its domain of definition such that *(i)* $f(a_1, a_2, ..., a_n, b) = 0$

(ii) the partial derivatives with respect to all the $(n+1)$ *variables exist and are continuous in a certain neighbourhood of* $(a_1, a_2, ..., a_n, b)$, *and*

(iii) $f_y(a_1, a_2, ..., a_n, b) \neq 0$,

then there exists a neighbourhood

$$(a_1 - h_1, a_1 + h_1; a_2 - h_2, a_2 + h_2; ...; a_n - h_n, a_n + h_n; b - k, b + k)$$

of $(a_1, a_2, ..., a_n, b)$ *such that for every point* $(x_1, x_2, ..., x_n)$ *of the neighbourhood*

$$R : (a_1 - h_1, a_1 + h_1; a_2 - h_2, a_2 + h_2; ...; a_n - h_n, a_n + h_n)$$

the equation $f(x_1, x_2, ..., x_n, y) = 0$ *determines one and only one value* $y = \phi(x_1, x_2, ..., x_n)$ *lying in* $[b - k, b + k]$ *with the following properties*:

(1) $b = \phi(a_1, a_2, ..., a_n)$.

(2) $f(x_1, x_2, ..., x_n, \phi) = 0$ *for every point* $(x_1, x_2, ..., x_n)$ *in R*,

(3) ϕ *is continuous and possesses continuous partial derivatives of the first order with respect to* $x_1, x_2, ..., x_n$ *in R*.

The proof is essentially a repetition of that given for the preceding theorem and offers no fresh difficulties.

Ex. Examine the following equations for the existence of unique implicit function near the points indicated and verify by direct calculation. Also find the first derivatives of the solutions whenever these exist.

1. $y^2 + 2x^2y + x^5 = 0, (1, -1)$
2. $y^2 + yx^3 + x^2 = 0, (0, 0)$
3. $y^3 + x^3 - 3xy + y = 0\ (0, 0)$
4. $2xy - \log xy = 2, (1, 1)$
5. $y^4 + y^2x^2 - 2x^5 = 0, (1, 1)$
6. $y^2 - yx^2 - 2x^5 = 0, (0, 0)$
7. $x^2 + xy + y^2 - 1 = 0, (1, 0)$
8. $x_1x_2 - y \log x_2 + \exp(x_1, y) - 1 = 0, (0, 1, 1)$
9. $y^3 \cos x + y^2 \sin^2 x - 7 = 0, (\pi/3, 2)$
10. $xy \sin x + \cos y = 0, (0, \pi/2)$
11. $ax + by + Ax^2 + Hxy + By^2 = 0, (0, 0) (b \neq 0)$

1.3 Derivative of Implicit Functions

When the equation $f(x, y) = 0$ defines y as a function of x that has a derivative dy/dx, that derivative may be obtained simply by differentiating the equation with respect to x, on the understanding that y is a function $\phi(x)$ of x. Thus

$$f_x + f_y \frac{dy}{dx} = 0 \qquad ...(1)$$

If the higher partial derivatives of $f(x, y)$ are continuous, we obtain the higher derivatives of y or $\phi(x)$ by successive differentiation of (1), provided always that f_y is not zero; thus

$$f_{xx} + f_{xy}\frac{dy}{dx} + \left(f_{xy} + f_{yy}\frac{dy}{dx}\right)\frac{dy}{dx} + f_y\frac{d^2y}{dx^2} = 0$$

or

$$f_{xx} + 2f_{xy}\frac{dy}{dx} + f_{yy}\left(\frac{dy}{dx}\right)^2 + f_y\frac{d^2y}{dx^2} = 0 \qquad \text{...(2)}$$

Provided f_y is not zero, this equation determines the second derivative, and in a similar way the third and higher derivatives may be found.

Note: If $f(x, y) = 0$, and y is a function of x, we have from (1)

$$\frac{dy}{dx} = \frac{-p}{q}$$

and also from (2),

$$\frac{d^2y}{dx^2} = -\frac{r + 2s(-p/q) + t(-p/q)^2}{q} = \frac{-(rq^2 - 2spq + tp^2)}{q^3}$$

2. JACOBIANS

For further development of the subject, acquaintance with the notion of Jacobians is necessary. We shall now define a Jacobian and also prove some of its important properties.

If $u_1, u_2, ..., u_n$ be n differentiable functions of n variables $x_1, x_2, ..., x_n$, then the determinant

$$\begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

is called the *Jacobian* or the *Functional Determinant* of the functions $u_1, u_2, ..., u_n$ with respect to $x_1, x_2, ..., x_n$ and is denoted by

$$\frac{\partial(u_1, u_2, ..., u_n)}{\partial(x_1, x_2, ..., x_n)} \text{ or } J\left(\frac{u_1, u_2, ..., u_n}{x_1, x_2, ..., x_n}\right)$$

2.1 Some Properties

Jacobians have the remarkable property of behaving like the derivatives of functions of one variable. A few of the important relations are given here and the proofs depend upon the algebra of determinants.

For $n = 1$, the determinant is simply $\frac{\partial y_1}{\partial x_1}$ or $\frac{dy_1}{dx_1}$, the derivative of y_1 with respect to x_1; the first of the notations for a Jacobian is suggested by a certain analogy between the properties of the Jacobian and the derivative.

Theorem 1. *If $u_1, u_2, ..., u_n$ are functions of $y_1, y_2, ..., y_n$ and $y_1, y_2, ..., y_n$ are themselves functions of $x_1, x_2, ..., x_n$, then*

$$\frac{\partial(u_1, u_2, ..., u_n)}{\partial(x_1, x_2, ..., x_n)} = \frac{\partial(u_1, u_2, ..., u_n)}{\partial(y_1, y_2, ..., y_n)} \frac{\partial(y_1, y_2, ..., y_n)}{\partial(x_1, x_2, ..., x_n)} \quad ...(1)$$

For $n = 1$, the theorem reduces to the usual notation

$$\frac{du_1}{dx_1} = \frac{du_1}{dy_1}\frac{dy_1}{dx_1}$$

The proof of the theorem depends on the "row by column" rule of multiplication of determinants combined with the rule for the derivative of a function of a function.

Thus for determinants on the right hand side of (1), rth row of the first is $\frac{\partial u_r}{\partial y_1}, \frac{\partial u_r}{\partial y_2}, ..., \frac{\partial u_r}{\partial y_n}$, sth column of the second is $\frac{\partial y_1}{\partial x_s}, \frac{\partial y_2}{\partial x_s}, ..., \frac{\partial y_n}{\partial x_s}$, so that the element in the rth row and the sth column of the product is

$$\frac{\partial u_r}{\partial y_1}\frac{\partial y_1}{\partial x_s} + \frac{\partial u_r}{\partial y_2}\frac{\partial y_2}{\partial x_s} + ... + \frac{\partial u_r}{\partial y_n}\frac{\partial y_n}{\partial x_s}$$

and this is equal to $\frac{\partial u_r}{\partial x_s}$, which is the element in the rth row and the sth column of the Jacobian on the left hand side. Hence the theorem.

Corollary. If $x_r = u_r$, $r = 1, 2, ..., n$ and assuming the existence of inverse functions $x_1, x_2, ..., x_n$ (that is, assuming that the equations which define $y_1, y_2, ..., y_n$ as functions of $x_1, x_2, ..., x_n$ determine $x_1, x_2, ..., x_n$ as functions of $y_1, y_2, ..., y_n$) we find

$$\frac{\partial(x_1, x_2, ..., x_n)}{\partial(y_1, y_2, ..., y_n)} \cdot \frac{\partial(y_1, y_2, ..., y_n)}{\partial(x_1, x_2, ..., x_n)} = \frac{\partial(x_1, x_2, ..., x_n)}{\partial(x_1, x_2, ..., x_n)} = 1 \quad ...(2)$$

since $\frac{\partial x_i}{\partial x_j} = 0$, for $i \neq j = 1$, for $i = j$

Theorem 2. *If $y_1, y_2, ..., y_n$ are determined as functions of $x_1, x_2, ..., x_n$ by the equations*

$$\phi_r(x_1, x_2, ..., x_n, y_1, y_2, ..., y_n) = 0, \ r = 1, 2, ..., n$$

then

$$\frac{\partial(\phi_1, \phi_2, \ldots, \phi_n)}{\partial(x_1, x_2, \ldots, x_n)} = (-1)^n \frac{\partial(\phi_1, \phi_2, \ldots, \phi_n)}{\partial(y_1, y_2, \ldots, y_n)} \frac{\partial(y_1, y_2, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_n)} \quad \ldots(3)$$

[Theorem 1 is a particular form of this theorem.]

Differentiating the equations $\phi_r = 0$ with respect to x_s, we get

$$\frac{\partial \phi_r}{\partial x_s} + \frac{\partial \phi_r}{\partial y_1}\frac{\partial y_1}{\partial x_s} + \frac{\partial \phi_r}{\partial y_2}\frac{\partial y_2}{\partial x_s} + \ldots + \frac{\partial \phi_r}{\partial y_n}\frac{\partial y_n}{\partial x_s} = 0$$

or

$$\frac{\partial \phi_r}{\partial y_1}\cdot\frac{\partial y_1}{\partial x_s} + \frac{\partial \phi_r}{\partial y_2}\frac{\partial y_2}{\partial x_s} + \ldots + \frac{\partial \phi_r}{\partial y_n}\frac{\partial y_n}{\partial x_s} = -\frac{\partial \phi_r}{\partial x_s}$$

so that the element in the rth row and the sth column of the determinant which is the product of the two determinants on the right of (3) is $-\dfrac{\partial \phi_r}{\partial x_s}$, from which the result follows.

Theorem 3. *(i) If $y_{m+1}, y_{m+2}, \ldots, y_n$ are constant with respect to $x_1, x_2, \ldots, x_m$, or (ii) if $y_1, y_2, \ldots, y_m$ are constant with respect to $x_{m+1}, x_{m+2}, \ldots, x_n$, then*

$$\frac{\partial(y_1, y_2, \ldots, y_m, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_m, \ldots, x_n)} = \frac{\partial(y_1, y_2, \ldots, y_m)}{\partial(x_1, x_2, \ldots, x_m)}\cdot\frac{\partial(y_{m+1}, y_{m+2}, \ldots, y_n)}{\partial(x_{m+1}, x_{m+2}, \ldots, x_n)} \quad \ldots(4)$$

(*i*) $\dfrac{\partial y_r}{\partial x_s} = 0$, when $r = m+1, m+2, \ldots, n$; $s = 1, 2, \ldots, m$.

$$\frac{\partial(y_1, y_2, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_n)} = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_m} & \frac{\partial y_1}{\partial x_{m+1}} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_m} & \frac{\partial y_2}{\partial x_{m+1}} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \vdots & \vdots & \vdots\;\vdots & & \vdots & \vdots\;\vdots & \\ \frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_m} & \frac{\partial y_m}{\partial x_{m+1}} & \cdots & \frac{\partial y_m}{\partial x_n} \\ \frac{\partial y_{m+1}}{\partial x_1} & \frac{\partial y_{m+1}}{\partial x_2} & \cdots & \frac{\partial y_{m+1}}{\partial x_m} & \frac{\partial y_{m+1}}{\partial x_{m+1}} & \cdots & \frac{\partial y_{m+1}}{\partial x_n} \\ \vdots & \vdots & \vdots\;\vdots & & \vdots & \vdots\;\vdots & \\ \frac{\partial y_n}{\partial x_1} & \frac{\partial y_n}{\partial x_2} & \cdots & \frac{\partial y_n}{\partial x_m} & \frac{\partial y_n}{\partial x_{m+1}} & \cdots & \frac{\partial y_n}{\partial x_n} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_m} & \frac{\partial y_1}{\partial x_{m+1}} & \cdots & \frac{\partial y_m}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_m} & \frac{\partial y_m}{\partial x_{m+1}} & \cdots & \frac{\partial y_m}{\partial x_n} \\ 0 & 0 & \cdots & 0 & \frac{\partial y_{m+1}}{\partial x_{m+1}} & \cdots & \frac{\partial y_{m+1}}{\partial x_n} \\ 0 & 0 & \cdots & 0 & \frac{\partial y_{m+2}}{\partial x_{m+1}} & \cdots & \frac{\partial y_{m+2}}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & \frac{\partial y_n}{\partial x_{m+1}} & \cdots & \frac{\partial y_n}{\partial x_n} \end{vmatrix}$$

$$= \frac{\partial(y_1, y_2, \ldots, y_m)}{\partial(x_1, x_2, \ldots, x_n)} \cdot \frac{\partial(y_{m+1}, y_{m+2}, \ldots, y_n)}{\partial(x_{m+1}, x_{m+2}, \ldots, x_n)}$$

(*ii*) may also be proved similarly.

Corollary. *In particular,*

$$\frac{\partial(y_1, \ldots, y_m, x_{m+1}, \ldots, x_n)}{\partial(x_1, \ldots, x_m, x_{m+1}, \ldots, x_n)} = \frac{\partial(y_1, \ldots, y_m)}{\partial(x_1, \ldots, x_m)} \qquad \ldots(5)$$

Theorem 4. *If u, v are functions of ξ, η, ζ, and the variables ξ, η, ζ, are themselves functions of the independent variables x and y, then*

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(u, v)}{\partial(\xi, \eta)} \cdot \frac{\partial(\xi, \eta)}{\partial(x, y)} + \frac{\partial(u, v)}{\partial(\eta, \xi)} \cdot \frac{\partial(\eta, \xi)}{\partial(x, y)} + \frac{\partial(u, v)}{\partial(\zeta, \xi)} \cdot \frac{\partial(\zeta, \xi)}{\partial(x, y)} \qquad \ldots(6)$$

We have

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial u}{\partial \zeta}\frac{\partial \zeta}{\partial x} \qquad \ldots(7)$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial u}{\partial \zeta}\frac{\partial \zeta}{\partial y} \qquad \ldots(8)$$

and if we substitute these values in the Jacobian $\frac{\partial(u, v)}{\partial(x, y)}$, we get

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial u}{\partial \xi}\frac{\partial(\xi, v)}{\partial(x, y)} + \frac{\partial u}{\partial \eta}\frac{\partial(\eta, v)}{\partial(x, y)} + \frac{\partial u}{\partial \zeta}\frac{\partial(\zeta, v)}{\partial(x, y)} \qquad \ldots(9)$$

which is a linear expression of the Jacobians of (ξ, v), (η, v) and (ζ, v) with respect to x and y.

Now in each Jacobian on the right of equation (9), substitute the expressions for $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ which are similar to (7) and (8). Each of these Jacobians will be given as a linear expression of the Jacobians of (ξ, η), (η, ζ) and (ζ, ξ) since those of (ξ, ξ), (η, η) and (ζ, ζ) have two identical parallel lines and so vanish. Thus we see that the terms which involve the Jacobian of (ξ, η) are

$$\frac{\partial u}{\partial \xi}\frac{\partial v}{\partial \eta}\frac{\partial(\xi, \eta)}{\partial(x, y)}+\frac{\partial u}{\partial \eta}\frac{\partial v}{\partial \xi}\frac{\partial(\eta, \xi)}{\partial(x, y)}$$

which is equal to $\frac{\partial(u, v)}{\partial(\xi, \eta)}\frac{\partial(\xi, \eta)}{\partial(x, y)}$, the first terms on the right of equation (6).

Similarly we obtain the remaining two terms and the formula is established.

Example 3. If $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, then show that

$$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = r^2\sin\theta$$

■ $$\frac{\partial(x, y, z)}{\partial(r, \theta, \phi)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta \begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\theta \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix}$$

Adding $(\cos\phi)\, R_1$ to $(\sin\phi)\, R_2$,

$$= \frac{r^2\sin\theta}{\sin\phi} \begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta & \cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \end{vmatrix} = r^2\sin\theta$$

Example 4. If $y_1 + y_2 + ... + y_n = x_1$, $y_2 + y_3 + ... + y_n = x_1x_2$, ..., $y_r + y_{r+1} + ... + y_n = x_1x_2 ... x_r$, ..., $y_n = x_1x_2 ... x_n$, then show that

$$\frac{\partial(y_1, y_2, ..., y_n)}{\partial(x_1, x_2, ..., x_n)} = x_1^{n-1}x_2^{n-2} ... x_{n-2}^2 x_{n-1}.$$

■ Solving for $y_1, y_2, ..., y_n$, we get

$$y_1 = x_1 - x_1x_2 = x_1(1 - x_2)$$
$$y_2 = x_1x_2 - x_1x_2x_3 = x_1x_2(1 - x_3)$$
$$\vdots$$
$$y_{n-1} = x_1x_2 ... x_{n-1}(1 - x_n)$$
$$y_n = x_1x_2 ... x_n$$

$$\therefore \quad \frac{\partial(y_1, y_2, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_n)} =$$

$$\begin{vmatrix} 1-x_2 & -x_1 & 0 & \cdots & 0 \\ x_2(1-x_3) & x_1(1-x_3) & -x_1x_2 & \cdots & 0 \\ x_2x_3(1-x_4) & x_1x_3(1-x_4) & x_1x_2(1-x_4) & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ x_2x_3\ldots x_{n-1}(1-x_n) & x_1x_3\ldots x_{n-1}(1-x_n) & x_1x_2x_4\ldots x_{n-1}(1-x_n) & \cdots & x_1x_2\ldots x_{n-1} \\ x_3x_3\ldots x_n & x_1x_2x_4\ldots x_n & x_1x_3\ldots x_n & \cdots & x_1x_2\ldots x_{n-1} \end{vmatrix}$$

Adding R_n to R_{n-1}, then R_{n-1} to R_{n-2}, ..., then R_2 to R_1 and expanding by last column

$$= (x_1x_2\ldots x_{n-1})\,(x_1x_2\ldots x_{n-2})\ldots(x_1x_2)(x_1)$$

$$= x_1^{n-1}x_2^{n-2}\ldots x_{n-2}^2x_{n-1}$$

Example 5. The roots of the equation in λ

$$(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$$

are u, v, w. Prove that

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = -2\frac{(y-z)(z-x)(x-y)}{(v-w)(w-u)(u-v)}$$

Here u, v, w are roots of the equation

$$\lambda^3 - (x+y+z)\lambda^2 + (x^2+y^2+z^2)\lambda - \frac{1}{3}(x^3+y^3+z^3) = 0$$

Let $$x+y+z=\xi,\ x^2+y^2+z^2=\eta,\ \frac{1}{3}(x^3+y^3+z^3)=\zeta \qquad \text{...(1)}$$

and

$$u+v+w=\xi,\ vw+wu+uv=\eta,\ uvw=\zeta \qquad \text{...(2)}$$

Hence from (1),

$$\frac{\partial(\xi,\eta,\zeta)}{\partial(x, y, z)} = \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= 2(y-z)(z-x)(x-y) \qquad \text{...(3)}$$

and from (2),

$$\frac{\partial(\xi,\eta,\zeta)}{\partial(u, v, w)} = \begin{vmatrix} 1 & 1 & 1 \\ v+w & w+u & u+v \\ vw & wu & uv \end{vmatrix}$$

$$= -(v-w)(w-u)(u-v) \qquad \text{...(4)}$$

Hence from (3) and (4) and using theorem 1, we get

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \frac{\partial(u, v, w)}{\partial(\xi, \eta, \zeta)} \cdot \frac{\partial(\xi, \eta, \zeta)}{\partial(x, y, z)} = -2\frac{(y-z)(z-x)(x-y)}{(v-w)(w-u)(u-v)}$$

Example 6. If $y_r = \frac{u_r}{u}$, $r = 1, 2, \ldots, n$, and if u and u_r are functions of the n independent variables x_1, $x_2, \ldots, x_n$, prove that

$$\frac{\partial(y_1, y_2, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_n)} = \frac{1}{u^{n+1}} \begin{vmatrix} u & \frac{\partial u}{\partial x_1} & \frac{\partial u}{\partial x_2} & \cdots & \frac{\partial u}{\partial x_n} \\ u_1 & \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n} \\ u_2 & \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u_n & \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

■ Now

$$\frac{\partial y_r}{\partial x_s} = \frac{1}{u}\frac{\partial u_r}{\partial x_s} - \frac{u_r}{u^2}\frac{\partial u}{\partial x_s}$$

$$\therefore \quad \frac{\partial(y_1, y_2, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_n)} = \begin{vmatrix} \frac{1}{u}\frac{\partial u_1}{\partial x_1} - \frac{u_1}{u^2}\frac{\partial u}{\partial x_1} & \cdots & \frac{1}{u}\frac{\partial u_1}{\partial x_n} - \frac{u_1}{u^2}\frac{\partial u}{\partial x_n} \\ \frac{1}{u}\frac{\partial u_2}{\partial x_1} - \frac{u_2}{u^2}\frac{\partial u}{\partial x_1} & \cdots & \frac{1}{u}\frac{\partial u_2}{\partial x_n} - \frac{u_2}{u^2}\frac{\partial u}{\partial x_n} \\ \vdots & \vdots & \vdots \\ \frac{1}{u}\frac{\partial u_n}{\partial x_1} - \frac{u_n}{u^2}\frac{\partial u}{\partial x_1} & \cdots & \frac{1}{u}\frac{\partial u_n}{\partial x_n} - \frac{u_n}{u^2}\frac{\partial u}{\partial x_n} \end{vmatrix}$$

Taking out $\frac{1}{u}$ from each column and bordering the determinant, we get

$$\frac{\partial(y_1, y_2, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_n)} = \frac{1}{u^n} \begin{vmatrix} 1 & 0 & \cdots & 0 \\ u_1 & \frac{\partial u_1}{\partial x_1} - \frac{u_1}{u}\frac{\partial u}{\partial x_1} & \cdots & \frac{\partial u_1}{\partial x_n} - \frac{u_1}{u}\frac{\partial u}{\partial x_n} \\ u_2 & \frac{\partial u_2}{\partial x_1} - \frac{u_2}{u}\frac{\partial u}{\partial x_1} & \cdots & \frac{\partial u_2}{\partial x_n} - \frac{u_2}{u}\frac{\partial u}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ u_n & \frac{\partial u_n}{\partial x_1} - \frac{u_n}{u}\frac{\partial u}{\partial x_1} & \cdots & \frac{\partial u_n}{\partial x_n} - \frac{u_n}{u}\frac{\partial u}{\partial x_n} \end{vmatrix}$$

Operating $C_2 + \frac{1}{u}\frac{\partial u}{\partial x_1}C_1,\ C_3 + \frac{1}{u}\frac{\partial u}{\partial x_2}C_1, \dots, C_{n+1} + \frac{1}{u}\frac{\partial u}{\partial x_n}C_1$

$$= \frac{1}{u^n}\begin{vmatrix} 1 & \frac{1}{u}\frac{\partial u}{\partial x_1} & \cdots & \frac{1}{u}\frac{\partial u}{\partial x_n} \\ u_1 & \frac{\partial u_1}{\partial x_1} & \cdots & \frac{\partial u_1}{\partial x_n} \\ u_2 & \frac{\partial u_2}{\partial x_1} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ u_n & \frac{\partial u_n}{\partial x_1} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

$$= \frac{1}{u^{n-1}}\begin{vmatrix} u & \frac{\partial u}{\partial x_1} & \frac{\partial u}{\partial x_2} & \cdots & \frac{\partial u}{\partial x_n} \\ u_1 & \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n} \\ u_2 & \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ u_n & \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

Example 7. If $u = \frac{x^2+y^2+z^2}{x}$, $v = \frac{x^2+y^2+z^2}{y}$, and $w = \frac{x^2+y^2+z^2}{z}$ find $\frac{\partial(x, y, z)}{\partial(u, v, w)}$.

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} 1 - \frac{y^2+z^2}{x^2} & \frac{2y}{x} & \frac{2z}{x} \\ \frac{2x}{y} & 1 - \frac{x^2+z^2}{y^2} & \frac{2z}{y} \\ \frac{2x}{z} & \frac{2y}{z} & 1 - \frac{x^2+y^2}{z^2} \end{vmatrix}$$

- Applying $C_1 \to C_1 + \frac{y}{x}C_2 + \frac{z}{x}C_3$

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{x^2+y^2+z^2}{x^2} & \frac{2y}{x} & \frac{2z}{x} \\ \frac{x^2+y^2+z^2}{xy} & 1 - \frac{x^2+z^2}{y^2} & \frac{2z}{y} \\ \frac{x^2+y^2+z^2}{xz} & \frac{2z}{z} & 1 - \frac{x^2+y^2}{z^2} \end{vmatrix}$$

$$= \frac{(x^2+y^2+z^2)}{x^2 \cdot xy \cdot xz} \begin{vmatrix} 1 & 2xy & 2xz \\ 1 & xy - \frac{x}{y}(x^2+z^2) & 2xz \\ 1 & 2yz & xz - \frac{x}{z}(x^2+y^2) \end{vmatrix}$$

$$= \frac{(x^2+y^2+z^2)}{x^4 yz} \begin{vmatrix} 1 & 2xy & 2xz \\ 0 & -\frac{x(x^2+y^2+z^2)}{y} & 0 \\ 0 & 0 & -\frac{x}{z}(x^2+y^2+z^2) \end{vmatrix}$$

$$= \frac{(x^2+y^2+z^2)^3}{x^2y^2z^2}$$

$$\therefore \quad \frac{\partial(x, y, z)}{\partial(u, v, w)} = \frac{x^2y^2z^2}{(x^2+y^2+z^2)^3}$$

Ex. 1. If $u = \cos x$, $v = \sin x \cos y$, $w = \sin x \sin y \cos z$, then show that

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = (-1)^2 \sin^3 x \sin^2 y \sin z.$$

Ex. 2. If $u = a \cosh x \cos y$, $v = a \sinh x \sin y$, then show that

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{1}{2} a^2 (\cosh 2x - \cos 2y).$$

Ex. 3. If $x + y + z = u$, $y + z = uv$, $z = uvw$, then show that

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = u^2 v.$$

Ex. 4. If α, β, γ are the roots of the equation in t, such that

$$\frac{u}{a+t} + \frac{v}{b+t} + \frac{w}{c+t} = 1,$$

then prove that

$$\frac{\partial(u, v, w)}{\partial(\alpha, \beta, \gamma)} = -\frac{(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)}{(b-c)(c-a)(a-b)}.$$

Ex. 5. If $u = x/(1-r^2)^{1/2}$, $v = y/(1-r^2)^{1/2}$, $w = z/(1-r^2)^{1/2}$ where $r^2 = x^2 + y^2 + z^2$, then show that

$$J\begin{pmatrix} u, v, w \\ x, y, z \end{pmatrix} = \frac{1}{(1-r^2)^{5/2}}.$$

3. STATIONARY VALUES UNDER SUBSIDIARY CONDITIONS

To find the stationary values of the function

$$f(x_1, x_2, ..., x_n, u_1, u_2, ..., u_m) \qquad ...(1)$$

of $(n+m)$ variables which are connected by m differentiable equations

$$\phi_r(x_1, x_2, ..., x_n, u_1, u_2, ..., u_m) = 0, \; r = 1, 2, ..., m \qquad ...(2)$$

If the m variables $u_1, u_2, ..., u_m$ are determinate as functions of $x_1, x_2, ..., x_n$ from the system of m equations of (2), then f can be regarded as a function of n independent variables $x_1, x_2, ..., x_n$.

At a stationary point of f, $df = 0$.

Hence at a stationary point (by § 11.1, Ch. 15).

$$0 = df = f_{x_1}dx_1 + f_{x_2}dx_2 + ... + f_{x_n}dx_n + f_{u_1}du_1 + ... + f_{u_m}du_m \qquad ...(3)$$

Again differentiating the equation (2), we get

$$\left.\begin{aligned} \frac{\partial \phi_1}{\partial x_1}dx_1 + ... + \frac{\partial \phi_1}{\partial x_n}dx_n + \frac{\partial \phi_1}{\partial u_1}du_1 + ... + \frac{\partial \phi_1}{\partial u_m}du_m &= 0 \\ \frac{\partial \phi_2}{\partial x_1}dx_1 + ... + \frac{\partial \phi_2}{\partial x_n}dx_n + \frac{\partial \phi_2}{\partial u_1}du_1 + ... + \frac{\partial \phi_2}{\partial u_m}du_m &= 0 \\ &\vdots \\ \frac{\partial \phi_m}{\partial x_1}dx_1 + ... + \frac{\partial \phi_m}{\partial x_n}dx_n + \frac{\partial \phi_m}{\partial u_1}du_1 + ... + \frac{\partial \phi_m}{\partial u_m}du_m &= 0 \end{aligned}\right\} \qquad ...(4)$$

From these m equations of (4), the differentials $du_1, du_2, ..., du_m$ of the m dependent variables may be found in terms of the n differentials $dx_1, dx_2, ..., dx_n$ and substituted in (3). This way df has been expressed in terms of the differentials of the independent variables, and since the differentials of the independent variables are arbitrary and $df = 0$, the coefficients of each of these n differentials may be equated to zero. These n equations together with the m equations of (2) constitute a system of $(n+m)$ equations to determine the $(n+m)$ coordinates of the stationary points of f.

Example 8. $F(x, y, z)$ is a function subject to the constraint condition $G(x, y, z) = 0$. Show that at a stationary point

$$F_xG_y - F_yG_x = 0$$

■ We may consider z as a function of the independent variables x, y.

As a stationary point, $dF = 0$

$$\therefore \qquad 0 = dF = F_x dx + F_y dy + F_z dz \qquad ...(1)$$

Differentiating the relation $G(x, y, z) = 0$, we get

$$G_x\,dx + G_y\,dy + G_z\,dz = 0 \qquad \text{...(2)}$$

Putting the value of dz from (2) into (1), or what is same thing, eliminating dz from (1) and (2), we get

$$(F_x G_z - G_x F_z)\,dx + (F_y G_z - G_y F_z)\,dy = 0 \qquad \text{...(3)}$$

Since dx, dy (being differentials of independent variables) are arbitrary, therefore

$$F_x G_z - G_x F_z = 0$$
$$F_y G_z - G_y F_z = 0$$

which give

$$F_x G_y - F_y G_x = 0.$$

Ex. 1. Find the stationary points of the function xy^2z^2 subject to the conditions

$$x + y + z = 6, x > 0, y > 0, z > 0.$$

Ex. 2. Find the stationary points of $x^2 + y^2$ subject to the condition

$$3x^2 + 4xy + 6y^2 = 140.$$

Ex. 3. Find the stationary points of the function $x^2y^2z^2$ subject to the condition

$$x^2 + y^2 + z^2 = a^2.$$

Ex. 4. Find the stationary points of the function xyz, when x, y, z are connected by the equation

$$x^2/9 + y^2/16 + z^2/36 = 1.$$

3.1 Lagrange's Undetermined Multipliers

In this section we shall discuss the determination of stationary points from a modified point of view. The process consists in the introduction of undetermined multipliers, a method due to Lagrange.

Before we discuss the method proper, let us notice that in the above section (§ 3) the differentials $du_1, du_2, ..., du_m$ of the m dependent variables were found from equations of (4) and substituted in (3), so as to express df in terms of the differentials of the independent variables. All this amounts to elimination of the differentials of the dependent variables from (3) and (4). This process of elimination is effected due to Lagrange by the introduction of multipliers. The process is as follows:

Multiply each of the equations of (4) by $\lambda_1, \lambda_2, ..., \lambda_m$, which are to be specified later, and add to (3). Now the m multipliers are so chosen that the coefficients of the m differentials of dependent variables all vanish. This gives m equations to determine λ's. Thus in equation (3) there remain only the differentials of n independent variables, the coefficient of each one of them may be equated to zero. The process thus gives $n + m$ equations which along with m constraint conditions form $n + 2m$ equations, to determine the m multipliers $\lambda_1, \lambda_2, ..., \lambda_m$, and $m + n$ variables $x_1, x_2, ..., x_n, u_1, u_2, ..., u_m$, for which the function has a stationary value.

3.2 Lagrange's Method of Multipliers

To find the stationary points of the function

$$f(x_1, x_2, ..., x_n, u_1, u_2, ..., u_m) \qquad \text{...(1)}$$

of $n + m$ variables which are connected by the equations

$$\phi_r(x_1, x_2, ..., x_n, u_1, u_2, ..., u_m) = 0, r = 1, 2, ..., m \qquad \text{...(2)}$$

As in § 3, if the m variables $u_1, u_2, ..., u_m$ from equations of (2) are expressed in terms of variables $x_1, x_2, ..., x_n$ the function f may be considered as a function of n independent variables

$x_1, x_2, \ldots, x_n$

For stationary values, $df = 0$

$$\therefore \quad 0 = df = \frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \ldots + \frac{\partial f}{\partial x_n}dx_n + \frac{\partial f}{\partial u_1}du_1 + \ldots + \frac{\partial f}{\partial u_m}du_m \qquad \ldots(3)$$

Differentiating equations of (2), we get

$$\left.\begin{aligned}
&\frac{\partial \phi_1}{\partial x_1}dx_1 + \ldots + \frac{\partial \phi_1}{\partial x_n}dx_n + \frac{\partial \phi_1}{\partial u_1}du_1 + \ldots + \frac{\partial \phi_1}{\partial u_m}du_m = 0\\
&\frac{\partial \phi_2}{\partial x_1}dx_1 + \ldots + \frac{\partial \phi_2}{\partial x_n}dx_n + \frac{\partial \phi_2}{\partial u_1}du_1 + \ldots + \frac{\partial \phi_2}{\partial u_m}du_m = 0\\
&\vdots\\
&\frac{\partial \phi_m}{\partial x_1}dx_1 + \ldots + \frac{\partial \phi_m}{\partial x_n}dx_n + \frac{\partial \phi_m}{\partial u_1}du_1 + \ldots + \frac{\partial \phi_m}{\partial u_m}du_m = 0
\end{aligned}\right\} \qquad \ldots(4)$$

Multiplying the equations of (4) by $\lambda_1, \lambda_2, \ldots, \lambda_m$ respectively and adding to the equation (3), we get

$$0 = df = \left(\frac{\partial f}{\partial x_1} + \Sigma\lambda_r \frac{\partial \phi_r}{\partial x_1}\right)dx_1 + \ldots + \left(\frac{\partial f}{\partial x_n} + \Sigma\lambda_r \frac{\partial \phi_r}{\partial x_n}\right)dx_n$$
$$+ \left(\frac{\partial f}{\partial u_1} + \Sigma\lambda_r \frac{\partial \phi_r}{\partial u_1}\right)du_1 + \ldots + \left(\frac{\partial f}{\partial u_m} + \Sigma\lambda_r \frac{\partial \phi_r}{\partial u_m}\right)du_m \qquad \ldots(5)$$

Let the m multipliers $\lambda_1, \lambda_2, \ldots, \lambda_m$ be so chosen that the coefficients of the m differentials $du_1, du_2, \ldots, du_m$ all vanish, *i.e.*,

$$\frac{\partial f}{\partial u_1} + \Sigma\lambda_r \frac{\partial \phi_r}{\partial u_1} = 0, \ldots, \frac{\partial f}{\partial u_m} + \Sigma\lambda_r \frac{\partial \phi_r}{\partial u_m} = 0 \qquad \ldots(6)$$

Then (5) becomes

$$0 = df = \left(\frac{\partial f}{\partial x_1} + \Sigma\lambda_r \frac{\partial \phi_r}{\partial x_1}\right)dx_1 + \ldots + \left(\frac{\partial f}{\partial x_n} + \Sigma\lambda_r \frac{\partial \phi_r}{\partial x_n}\right)dx_n$$

so that the differential df is expressed in terms of the differentials of independent variables only. Hence

$$\frac{\partial f}{\partial x_1} + \Sigma\lambda_r \frac{\partial \phi_r}{\partial x_1} = 0, \ldots, \frac{\partial f}{\partial x_n} + \Sigma\lambda_r \frac{\partial \phi_r}{\partial x_n} = 0 \qquad \ldots(7)$$

Equations (2), (6) and (7) form a system of $n + 2m$ equations which may be simultaneously solved to determine the m multipliers $\lambda_1, \lambda_2, \ldots, \lambda_m$ and the $n + m$ coordinates $x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m$ of the stationary points of f.

An Important Rule. For practical purposes, the process of obtaining equations (6) and (7) of the above section, may be put in a precise form as follows:

Define a function

$$F = f + \lambda_1\phi_1 + \lambda_2\phi_2 + \ldots + \lambda_m\phi_m$$

and consider all the variables $x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m$ as independent.

At a stationary point of F, $dF = 0$. Therefore,

$$0 = dF = \frac{\partial F}{\partial x_1}dx_1 + \frac{\partial F}{\partial x_2}dx_2 + \ldots + \frac{\partial F}{\partial x_n}dx_n + \frac{\partial F}{\partial u_1}du_1 + \ldots + \frac{\partial F}{\partial u_m}du_m$$

$$\therefore \quad \frac{\partial F}{\partial x_1} = 0, \ldots, \frac{\partial F}{\partial x_n} = 0, \frac{\partial F}{\partial u_1} = 0, \ldots, \frac{\partial F}{\partial u_m} = 0$$

which are same as equations (7) and (6).

Thus, the stationary points of f may be found by determining the stationary points of the function F, where

$$F = f + \lambda_1\phi_1 + \lambda_2\phi_2 + \ldots + \lambda_m\phi_m$$

and considering all the variables as independent variables.

A stationary point will be an extreme point of f if d^2F keeps the same sign, and will be a maxima or minima according as d^2F is negative or positive.

Note: It may be easier to deal with d^2F by expressing it in terms of two variables only or by expressing $dx\,dy$, $dy\,dz$, $dz\,dx$ in terms of dx^2, dy^2, dz^2, with the help of constraint conditions. Solved examples will illustrate the procedure.

Example 9. Find the shortest distance from the origin to the hyperbola

$$x^2 + 8xy + 7y^2 = 225,\ z = 0.$$

■ We have to find the minimum value of $x^2 + y^2$ (the square of the distance from the origin to any point in the xy plane) subject to the constraint

$$x^2 + 8xy + 7y^2 = 225$$

Consider the function

$$F = x^2 + y^2 + \lambda(x^2 + 8xy + 7y^2 - 225)$$

where x, y are independent variables and λ, a constant.

$$dF = (2x + 2x\lambda + 8y\lambda)dx + (2y + 8x\lambda + 14y\lambda)dy$$

$$\therefore \quad \left.\begin{aligned}(1+\lambda)x + 4\lambda y = 0\\ 4\lambda x + (1+7\lambda)y = 0\end{aligned}\right\} \quad \therefore\ \lambda = 1, -\frac{1}{9}$$

For $\lambda = 1$, $x = -2y$ and substitution in $x^2 + 8xy + 7y^2 = 225$, gives $y^2 = -45$, for which no real solution exists.

For $\lambda = -\frac{1}{9}$, $y = 2x$ and substitution in $x^2 + 8xy + 7y^2 = 225$, gives $x^2 = 5$, $y^2 = 20$, and so $x^2 + y^2 = 25$.

$$d^2F = 2(1+\lambda)dx^2 + 16\lambda\,dx\,dy + 2(1+7\lambda)\,dy^2$$

$$= \frac{16}{9}dx^2 - \frac{16}{9}dx\,dy + \frac{4}{9}dy^2, \text{ at } \lambda = -\frac{1}{9}$$

$$= \frac{4}{9}(2dx - dy)^2$$

> 0, and cannot vanish because $(dx, dy) \neq (0, 0)$.

Hence, the function $x^2 + y^2$ has a minimum value 25.

Note: Here F is a function of two variables and so its maximum or minimum values can be verified by the method of functions of two variables, $AC - B^2 > 0$ also.

Example 10. Find the maximum and minimum values of $x^2 + y^2 + z^2$ subject to the conditions $\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} = 1$, and $z = x + y$.

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Let us consider a function F of independent variables x, y, z where

$$F = x^2 + y^2 + z^2 + \lambda_1\left(\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} - 1\right) + \lambda_2(x + y - z)$$

$$\therefore \quad dF = \left(2x + \frac{x}{2}\lambda_1 + \lambda_2\right)dx + \left(2y + \frac{2y}{5}\lambda_1 + \lambda_2\right)dy + \left(2z + \frac{2z}{25}\lambda_1 - \lambda_2\right)dz$$

As x, y, z are independent variables, we get

$F_x = 0 \Rightarrow$ $$2x + \frac{x}{2}\lambda_1 + \lambda_2 = 0$$

$f_y = 0 \Rightarrow$ $$2y + \frac{2y}{5}\lambda_1 + \lambda_2 = 0$$

$f_z = 0 \Rightarrow$ $$2z + \frac{2z}{25}\lambda_1 - \lambda_2 = 0$$

$$\therefore \quad x = \frac{-2\lambda_2}{\lambda_1 + 4}, \quad y = \frac{-5\lambda_2}{2\lambda_1 + 10}, \quad z = \frac{25\lambda_2}{2\lambda_2 + 50}$$

Substituting in $x + y = z$, we get

$$\frac{2}{\lambda_1 + 4} + \frac{5}{2\lambda_1 + 10} + \frac{25}{2\lambda_1 + 50} = 0, \ \lambda_2 \neq 0 \qquad ...(1)$$

for if, $\lambda_2 = 0$, $x = y = z = 0$, but $(0, 0, 0)$ does not satisfy the other condition of constraint.

Hence from (1), $17\lambda_1^2 + 245\lambda_1 + 750 = 0$, so that $\lambda_1 = -10, -75/17$.

For $\lambda_1 = -10$,

$$x = \frac{1}{3}\lambda_2, \ y = \frac{1}{2}\lambda_2, \ z = \frac{5}{6}\lambda_2$$

Substituting in $\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} = 1$, we get

$$\lambda_2^2 = 180/19 \text{ or } \lambda_2 = \pm 6\sqrt{5/19}$$

The corresponding stationary points are

$$(2\sqrt{5/19}, 3\sqrt{5/19}, 5\sqrt{5/19}), (-2\sqrt{5/19}, -3\sqrt{5/19}, -5\sqrt{5/19})$$

The value of $x^2 + y^2 + z^2$ corresponding to these point is 10.

For $\lambda_1 = -75/17$,

$$x = \frac{34}{7}\lambda_2, \ y = -\frac{17}{4}\lambda_2, \ z = \frac{17}{28}\lambda_2,$$

which on substitution in $\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} = 1$ give

$$\lambda_2 = \pm 140/(17\sqrt{646})$$

The corresponding stationary points are

$$(40/\sqrt{646}, -35/\sqrt{646}, 5/\sqrt{646}), (-40/\sqrt{646}, 35/\sqrt{646}, -5/\sqrt{646})$$

The value of $x^2 + y^2 + z^2$ corresponding to these points is 75/17.

Thus, the maximum value is 10 and the minimum 75/17.

Notes:

1. We have not theoretically established the existence of maximum or minimum value. We have simply shown that of all the possible values, 10 is the maximum and 75/17 the minimum.
2. Using constraint conditions, $dz = dx + dy; \frac{x}{4}dx + \frac{y}{5}dy + \frac{z}{25}dz = 0$, so that dz, dy and consequently d^2F may be expressed in terms of dx (or dx^2) alone. It can, then, be easily verified that 10 is a maximum value and 75/17 the minimum.

Example 11. Prove that the volume of the greatest rectangular parallelopiped, that can be inscribed in the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, is $\frac{8abc}{3\sqrt{3}}$.

■ We have to find the greatest value of $8xyz$ subject to the conditions

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \ x > 0, \ y > 0, \ z > 0 \qquad \text{...(1)}$$

Let us consider a function F of three independent variables x, y, z, where

$$F = 8xyz + \lambda\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right)$$

$$\therefore \quad dF = \left(8yz + \frac{2x\lambda}{a^2}\right)dx + \left(8zx + \frac{2y\lambda}{b^2}\right)dy + \left(8xy + \frac{2z\lambda}{c^2}\right)dz$$

At stationary points,

$$8yz + \frac{2x\lambda}{a^2} = 0,\ 8zx + \frac{2y\lambda}{b^2} = 0,\ 8xy + \frac{2z\lambda}{c^2} = 0 \qquad ...(2)$$

Multiplying by x, y, z respectively and adding,

$$24xyz + 2\lambda = 0 \text{ or } \lambda = -12xyz \qquad \text{[using (1)]}$$

Hence from (2), $x = a/\sqrt{3},\ y = b/\sqrt{3},\ z = c/\sqrt{3}$, and so

$$\lambda = -4abc/\sqrt{3}$$

Again

$$d^2F = 2\lambda\left(\frac{dx^2}{a^2} + \frac{dy^2}{b^2} + \frac{dz^2}{c^2}\right) + 16z\,dx\,dy + 16x\,dy\,dz + 16y\,dz\,dx$$

$$= -\frac{8abc}{\sqrt{3}}\Sigma\frac{1}{a^2}dx^2 + \frac{16}{\sqrt{3}}\Sigma c\,dx\,dy \qquad ...(3)$$

Now from equations. (1), we have

$$x\frac{dx}{a^2} + y\frac{dy}{b^2} + z\frac{dz}{c^2} = 0 \text{ or } \frac{dx}{a} + \frac{dy}{b} + \frac{dz}{c} = 0 \qquad ...(4)$$

Hence squaring,

$$\Sigma\frac{dx^2}{a^2} + 2\Sigma\frac{dx\,dy}{ab} = 0$$

or

$$abc\,\Sigma\frac{dx^2}{a^2} = -2\Sigma c\,dx\,dy$$

$$\therefore \qquad d^2F = -\frac{16}{\sqrt{3}}abc\,\Sigma\frac{dx^2}{a^2}$$

which is always negative.

Hence $\left(\frac{a}{\sqrt{3}}, \frac{b}{\sqrt{3}}, \frac{c}{\sqrt{3}}\right)$ is a point of maxima and the maximum value of $8xyz$ is $\frac{8abc}{3\sqrt{3}}$.

Note: The sign of d^2F can also be decided by expressing it in terms of dx and dy alone, by putting into (3) the value of dz from (4).

Example 12. Show that the length of the axes of the section of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ by the plane $lx + my + nz = 0$ are the roots of the quadratic in r^2,

$$\frac{l^2a^2}{r^2 - a^2} + \frac{m^2b^2}{r^2 - b^2} + \frac{n^2c^2}{r^2 - c^2} = 0.$$

- We have to find the stationary values of the function r^2 where $r^2 = x^2 + y^2 + z^2$, subject to the two equations of condition

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad ...(1)$$

$$lx + my + nz = 0 \quad ...(2)$$

Let us consider a function F of independent variables x, y, z,

$$F = x^2 + y^2 + z^2 + \lambda_1\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1\right) + 2\lambda_2(lx + my + nz)$$

$$\therefore \quad dF = 2\left(x + \frac{x\lambda_1}{a^2} + \lambda_2 l\right)dx + 2\left(y + \frac{y\lambda_1}{b^2} + \lambda_2 m\right)dy + 2\left(z + \frac{z\lambda_1}{c^2} + \lambda_2 n\right)dz$$

At stationary points,

$$x + \frac{x}{a^2}\lambda_1 + l\lambda_2 = 0,\ y + \frac{y}{b^2}\lambda_1 + m\lambda_2 = 0,\ z + \frac{z}{c^2}\lambda_1 + n\lambda_2 = 0 \quad ...(3)$$

Multiplying by x, y, z, respectively and adding, we get

$$\lambda_1 = -(x^2 + y^2 + z^2) = -r^2$$

$$\therefore \quad x = \frac{a^2 l\lambda_2}{r^2 - a^2},\ y = \frac{b^2 m\lambda_2}{r^2 - b^2},\ z = \frac{c^2 n\lambda_2}{r^2 - c^2}$$

But $$0 = lx + my + nz = \lambda_2\left\{\frac{a^2l^2}{r^2 - a^2} + \frac{b^2m^2}{r^2 - b^2} + \frac{c^2n^2}{r^2 - c^2}\right\}$$

and since $\lambda_2 \neq 0$, we get the quadratic in r^2 giving the stationary values:

$$\frac{a^2l^2}{r^2 - a^2} + \frac{b^2m^2}{r^2 - b^2} + \frac{c^2n^2}{r^2 - c^2} = 0$$

Example 13. If the variables x, y, z satisfy the equation

X

$$\phi(x)\phi(y)\phi(z) = k^3 \quad ...(1)$$

and $\phi(a) = k \neq 0,\ \phi'(a) \neq 0$, show that the function

$$f(x) + f(y) + f(z) \quad ...(2)$$

has a maximum, when $x = y = z = a$, provided that

$$f'(a)\left\{\frac{\phi''(a)}{\phi'(a)} - \frac{\phi'(a)}{\phi(a)}\right\} > f''(a)$$

- Let us consider a function

$$F = f(x) + f(y) + f(z) + \lambda\{\phi(x)\phi(y)\phi(z) - k^3\}$$

$$\therefore \quad dF = \Sigma\{f'(x) + \lambda\phi'(x)\phi(y)\phi(z)\}dx$$

At stationary points,
$$\left.\begin{aligned} f'(x) + \lambda\phi'(x)\phi(y)\phi(z) = 0 \\ f'(y) + \lambda\phi'(y)\phi(z)\phi(x) = 0 \\ f'(z) + \lambda\phi'(z)\phi(x)\phi(y) = 0 \end{aligned}\right\} \quad \ldots(3)$$

If the function has a maximum at (a, a, a), we must have
$$f'(a) + \lambda\phi'(a)\phi(a)\phi(a) = 0$$

or
$$\lambda = -\frac{f'(a)}{\phi'(a)\phi^2(a)} = -\frac{f'(a)}{k^2\phi'(a)};\ \phi'(a) \neq 0,\ \phi(a) \neq 0$$

Again
$$d^2F = \Sigma\{f''(x) + \lambda\phi''(x)\phi(y)\phi(z)\}dx^2 + 2\lambda\Sigma\phi'(x)\phi'(y)\phi(z)dx\,dy$$

At the stationary point (a, a, a),
$$d^2F = \{f''(a) + \lambda k^2\phi''(a)\}\,\Sigma dx^2 + 2\lambda k[\phi'(a)]^2\Sigma\, dx\,dy$$

From the given condition (1), we have
$$\Sigma\phi'(x)\phi(y)\phi(z)\,dx = 0$$

$\therefore$
$$k^2\phi'(a)(dx + dy + dz) = 0, \text{ at } (a, a, a)$$

or
$$dx + dy + dz = 0$$

$\therefore$
$$2\Sigma\, dx\,dy = -\Sigma\, dx^2$$

$\therefore$
$$d^2F = \{f''(a) + \lambda k^2\phi''(a)\}\,\Sigma\, dx^2 - \lambda k[\phi'(a)]^2\,\Sigma dx^2$$
$$= \left\{f''(a) - f'(a)\frac{\phi''(a)}{\phi'(a)} + f'(a)\frac{\phi'(a)}{\phi(a)}\right\}\Sigma\, dx^2$$

For a maximum value at (a, a, a), d^2F is to be negative, *i.e.*,
$$f'(a)\left\{\frac{\phi''(a)}{\phi'(a)} - \frac{\phi'(a)}{\phi(a)}\right\} > f''(a).$$

Example 14. If $f(x, y, z) = (a^2x^2 + b^2y^2 + c^2z^2)/x^2y^2z^2$, where $ax^2 + by^2 + cz^2 = 1$, and a, b, c are positive, show that the minimum value of $f(x, y, z)$ is given by
$$x^2 = \frac{u}{2a(u+a)},\quad y^2 = \frac{u}{2b(u+b)},\quad z^2 = \frac{u}{2c(u+c)},$$

where u is the positive root of the equation
$$u^3 - (bc + ca + ab)\,u - 2abc = 0 \ (\textit{Schlömilch})$$

- Consider a function F of independent variables x, y, z, where
$$F = (a^2x^2 + b^2y^2 + c^2z^2)/x^2y^2z^2 + \lambda\,(ax^2 + by^2 + cz^2 - 1)$$

$$\therefore \qquad dF = \Sigma\left(2ax\lambda - \frac{2(b^2y^2 + c^2z^2)}{x^3y^2z^2}\right)dx$$

At stationary points

$$\left.\begin{aligned} 2ax\lambda - \frac{2(b^2y^2+c^2z^2)}{x^3y^2z^2} &= 0 \\ 2by\lambda - \frac{2(c^2z^2+a^2x^2)}{x^2y^3z^2} &= 0 \\ 2cz\lambda - \frac{2(a^2x^2+b^2y^2)}{x^2y^2z^3} &= 0 \end{aligned}\right\}$$

or

$$ax^2\lambda = \frac{b^2y^2+c^2z^2}{x^2y^2z^2},\ by^2\lambda = \frac{c^2z^2+a^2x^2}{x^2y^2z^2},\ cz^2\lambda = \frac{a^2x^2+b^2y^2}{x^2y^2z^2} \qquad \text{...(1)}$$

Adding,
$$\lambda = \frac{2\Sigma a^2x^2}{x^2y^2z^2} \qquad (\because\ \Sigma ax^2 = 1)$$

Clearly λ is positive,

Again from (1),

$$\frac{b^2y^2+c^2z^2}{2ax^2} = \frac{c^2z^2+a^2x^2}{2by^2} = \frac{a^2x^2+b^2y^2}{2cz^2} = \frac{\Sigma a^2x^2}{1}$$

or
$$\frac{a^2x^2}{1-2ax^2} = \frac{b^2y^2}{1-2by^2} = \frac{c^2z^2}{1-2cz^2} = \frac{\Sigma a^2x^2}{1} = \frac{u}{2},\ \text{say}$$

Clearly $u > 0$ and the coordinates of the stationary point are given by

$$\left.\begin{aligned} &2a^2x^2 = u(1-2ax^2) \Rightarrow x^2 = \frac{u}{2a(u+a)} \\ &\text{Similarly,}\quad y^2 = \frac{u}{2b(u+b)}, \qquad z^2 = \frac{u}{2c(u+c)} \end{aligned}\right\} \qquad \text{...(2)}$$

Again substituting these values in the constraint condition, $\Sigma ax^2 = 1$, we get

$$1 = \frac{u}{2(u+a)} + \frac{u}{2(u+b)} + \frac{u}{2(u+c)}$$

which on simplification shows that u is a positive root of the equation,

$$u^3 - (bc + ca + ab)\,u - 2abc = 0 \qquad \text{...(3)}$$

Again
$$d^2F = \Sigma 2\left(a\lambda + \frac{3(b^2y^2+c^2z^2)}{x^4y^2z^2}\right)dx^2 + 2\Sigma\frac{4c^2}{x^3y^3}dx\,dy$$

Clearly
$$F_{xx} = 2\left(a\lambda + \frac{3(b^2y^2 + c^2z^2)}{x^4y^2z^2}\right) > 0$$

Also
$$\begin{vmatrix} F_{xx} & F_{xy} \\ F_{xy} & F_{yy} \end{vmatrix} = F_{xx}F_{yy} - (F_{xy})^2 = 4\left(a\lambda + \frac{3(b^2y^2 + c^2z^2)}{x^4y^2z^2}\right)\left(b\lambda + \frac{3(c^2z^2 + z^2x^2)}{x^2y^4z^2}\right) - \left(\frac{4c^2}{x^3y^3}\right)^2$$

Comparison of the term containing $\frac{1}{x^3y^3}$ shows that this expression is positive.

It may be similarly shown that

$$\begin{vmatrix} F_{xx} & F_{xy} & F_{zx} \\ F_{xy} & F_{yy} & F_{yz} \\ F_{zx} & F_{yz} & F_{zz} \end{vmatrix} \text{ is also positive}$$

So all the three principal minors are positive.

Thus, d^2F is positive and so the function has a minimum value at the stationary point given by (2).

EXERCISE

1. Show that

   (*i*) if $2x + 3y + 4z = a$, the maximum value of $x^2y^3z^4$ is $\left(\frac{a}{9}\right)^9$.

   (*ii*) if $a^2x^2 + 2by^3 + z^4 = c^4$, the maximum value of x^4yz^2 is given by
   $$17a^2x^2 = 12c^4,\ 17by^3 = c^4,\ 17z^4 = 3c^4$$

2. If $xyz = abc$, the minimum value of $bcx + cay + abz$ is $3abc$.
3. If $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, the maximum value of xyz is $abc/3\sqrt{3}$.
4. If $xyz = a^2(x + y + z)$, the minimum value of $yz + zx + xy$ is $9a^2$.
5. If $x^2 + y^2 = 1$, the minimum value of $(ax^2 + by^2)/(a^2x^2 + b^2y^2)^{1/2}$ is $2(ab)^{1/2}/(a + b)$.
6. If $xyz = k^3$, the product $(x + a)(y + b)(z + c)$ is a minimum, when $\frac{x}{a} = \frac{y}{b} = \frac{z}{c} = \frac{k}{(abc)^{1/3}}$; a, b, c are positive.
7. Show that the points on the ellipse $5x^2 - 6xy + 5y^2 = 4$ for which the tangent is at the greatest distance from the origin are $(1, 1)$ and $(-1, -1)$.
8. Show that the point on the sphere $x^2 + y^2 + z^2 = 1$ which is farther from $(2, 1, 3)$ is $(-2/\sqrt{14}, -1/\sqrt{14}, -3/\sqrt{14})$.
9. Show that the shortest distance from the origin to the curve of intersection of the surfaces $xyz = a$ and $y = bx$, where $a > 0$, $b > 0$, is $\sqrt[3]{a(b^2 + 1)/2b}$.
10. If $ax^2 + by^2 = ab$, show that the maximum and minimum values of $x^2 + xy + y^2$ will be the values of λ, given by the equation
$$4(\lambda - a)(\lambda - b) - ab = 0$$

11. If $1/x + 1/y + 1/z = 1$, show that a stationary value of $a^3x^2 + b^3y^2 + c^3z^2$ is given by $ax = by = cz$, and this gives an extreme value if $abc(a + b + c)$ is positive. (To see the sign of d^2F, change it to two variables only with the help of the condition $\Sigma 1/x = 1$).

12. Find the shortest distance between the points P and Q, when P moves on the plane $x + y + z = 2a$, and Q on the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

13. Find the maximum and the minimum distance from the origin to the curve of intersection of surface $(x^2+y^2+z^2)^2 = a^2x^2 + b^2y^2 + c^2z^2$ and the plane $lx + my + nz = 0$.

14. If $lx + my + nz = 1$, l, m, n are positive constants, show that the stationary value of $xy + yz + zx$ is

$$(2lm + 2mn + 2nl - l^2 - m^2 - n^2)^{-1}$$

and that this value is a maximum when it is positive.

15. Show that the function $xy + yz + zx$ has no extreme value, but has a maximum value when x, y, z are constrained by the condition $ax + by + cz = 1$, where a, b, c are positive constants satisfying the condition

$$2(ab + bc + ca) > (a^2 + b^2 + c^2)$$

16. If $\phi(x, y) = 0$, show that the determinant

$$\begin{vmatrix} f_{xx} + \lambda\phi_{xx} & f_{xy} + \lambda\phi_{xy} & \phi_x \\ f_{xy} + \lambda\phi_{xy} & f_{yy} + \lambda\phi_{yy} & \phi_y \\ \phi_x & \phi_y & 0 \end{vmatrix}$$

where λ is Lagrange's multiplier, is positive, in case the function $f(x, y)$ attains a maximum.

17. If $ax^2 + by^2 + cz^2 + 2fyz + 2gxz + 2hxy$ is equal to a constant k, and $lx + my + nz = 0$, find the maximum and minimum values of $x^2 + y^2 + z^2$.

18. If $ax + by + cz = 1$, show that in general $x^3 + y^3 + z^3 - 3xyz$ has the two stationary values 0 and $(a^3 + b^3 + c^3 - 3abc)^{-1}$, of which the first is a maximum or a minimum according as $a + b + c \gtrless 0$, but the second is not an extreme value. Discuss in particular, the cases when (*i*) $a + b + c = 0$, (*ii*) $a = b = c$.

19. If $x + y + z = 1$, find the stationary values of

$$x^3 + y^3 + z^3 + 3mxyz, \ (m \neq 2)$$

and show that the symmetrical stationary value is a maximum or minimum according as $m \lessgtr 2$, but the other stationary values are not extreme values.

Also show that $x^3 + y^3 + z^3 + 6xyz$ has only one stationary value and no extreme value.