# 3

# Volumes and Surfaces of Solids of Revolution

**§ 1. Definitions :**

**Solid of revolution.** If a plane area is revolved about a fixed line in its own plane, then the body so generated by the revolution of the plane area is called a solid of revolution.

**Surface of revolution.** If a plane curve is revolved about a fixed line lying in its own plane, then the surface generated by the perimeter of the curve is called a surface of revolution.

**Axis of revolution.** The fixed straight line, say $AB$, about which the area revolves is called the axis of revolution or axis of rotation.

**§ 2. Volumes of Solids of Revolution.**

**(a) The axis of rotation being x-axis.**

*If a plane area bounded by the curve $y = f(x)$, the ordinates $x = a$, $x = b$ and the x-axis revolves about the **x-axis** then the volume of the solid thus generated is*

$$\int_a^b \pi y^2 \, dx = \int_a^b \pi \, [f(x)]^2 \, dx,$$

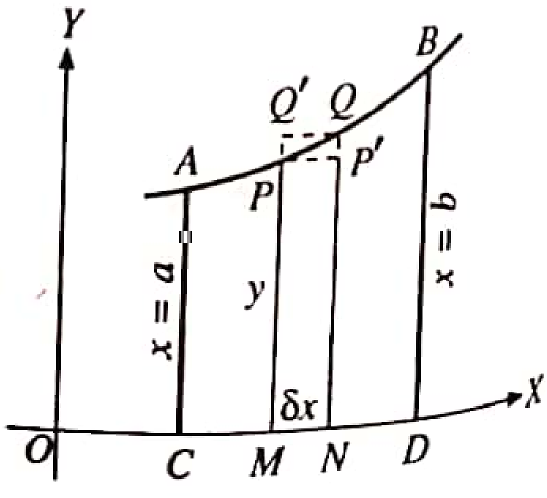
*where $y = f(x)$ is a finite, continuous and single valued function of $x$ in the interval $a \le x \le b$.*

**Or**

*The volume of the solid generated by the revolution of the area bounded by the curve $y = f(x)$, x-axis and the ordinates $x = a, x = b$ about the x-axis is $\int_a^b \pi y^2 \, dx$.*

(G.N.U. 1974)

**Proof.** Let $AB$ be the arc of the curve $y = f(x)$ included between the ordinates $x = a$ and $x = b$. It is being assumed that the curve does not cut the $x$-axis and $f(x)$ is a continuous function of $x$ in the interval $(a, b)$.




Let $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ be any two neighbouring points on the curve $y = f(x)$. Draw the ordinates $PM$ and $QN$. Also draw $PP'$ and $QQ'$ perpendiculars to these ordinates.

Let $V$ denote the volume of the solid generated by the revolution of the area $ACMP$ about the $x$-axis and let the volume of revolution obtained by revolving the area $ACNQ$ about $x$-axis be $V + \delta V$, so that volume of the solid generated by the revolution of the strip $PMNQ$ about the $x$-axis is $\delta V$.

Now $PM = y$, $QN = y + \delta y$ and $MN = (x + \delta x) - x = \delta x$. Then the volume of the solid generated by revolving the area $PMNP' = \pi y^2 \delta x$ and the volume of the solid generated by revolving the area $Q'MNQ = \pi (y + \delta y)^2 \delta x$.

Also the volume of the solid generated by the revolution of the area $PMNQP$ (*i.e.*, the volume $\delta V$) lies between the volumes of the right circular cylinders generated by the revolution of the areas $PMNP'P$ and $MNQQ'$ *i.e.*, $\delta V$ lies between

$$\pi y^2 \delta x \text{ and } \pi (y + \delta y)^2 \delta x$$

or $$(\delta V / \delta x) \text{ lies between } \pi y^2 \text{ and } \pi (y + \delta y)^2$$

*i.e.*, $$\pi y^2 < (\delta V / \delta x) < \pi (y + \delta y)^2.$$

In the limiting position $Q \to P$, $\delta x \to 0$ (and therefore $\delta y \to 0$), we have

$$dV/dx = \pi y^2 \quad \text{or} \quad dV = \pi y^2 \, dx.$$

Hence $\int_a^b \pi y^2 \, dx = \int_a^b dV = \Big[V\Big]_{x=a}^{x=b}$

= (value of $V$ for $x = b$) $-$ (value of $V$ for $x = a$)

= volume generated by the area $ACDB - 0$

= volume of the solid generated by the revolution of the given area $ACDB$ about the axis of $x$.

$\therefore$ the required volume $= \pi \int_a^b y^2 \, dx$.

**(b) The axis of rotation being y-axis.**

Similarly, it can be shown that the *volume of the solid generated by the revolution about y-axis of the area between the curve $x = f(y)$, the y-axis and the two abscissae $y = a$ and $y = b$ is given by*

$$\int_a^b \pi x^2 \, dy.$$

**(c) Volume of a solid of revolution when the equations of the generating curve are given in parametric form.**

**(i)** If the curve is given by the parametric equations, say $x = \phi(t)$, $y = \psi(t)$, then the volume of the solid generated by the revolution about **x-axis** of the area bounded by the curve, the axis of $x$ and the ordinates at the points where $t = a$ and $t = b$ is

$$= \int_a^b \pi y^2 \frac{dx}{dt} \, dt = \pi \int_a^b \{\psi(t)\}^2 \phi'(t) \, dt.$$

(ii) The volume of the solid generated by the revolution about $y$-axis of the area between the curve $x = \phi(t), y = \psi(t)$, the $y$-axis and the abscissae at the points where $t = a$, $t = b$ is

$$= \int_a^b \pi x^2 \frac{dy}{dt}\, dt = \pi \int_a^b \{\phi(t)\}^2 . \psi'(t)\, dt.$$

**(d) Volume of solid of revolution when the equation of the generating curve is given in polar co-ordinates.**

If the equation of the generating curve is given in polar co-ordinates, say $r = f(\theta)$, and the curve revolves about the axis of $x$, the volume generated

$$= \pi \int_{x=a}^{b} y^2\, dx = \pi \int_{\theta=\alpha}^{\beta} y^2 \frac{dx}{d\theta}\, d\theta,$$

where $\alpha$ and $\beta$ are the values of $\theta$ at the points where $x = a$ and $x = b$ respectively.

Now $x = r\cos\theta$ and $y = r\sin\theta$. Therefore the volume

$$= \pi \int_{\theta=\alpha}^{\beta} r^2 \sin^2\theta \frac{d}{d\theta}(r\cos\theta)\, d\theta,$$

in which the value of $r$ in terms of $\theta$ must be substituted from the equation of the given curve.

A similar procedure can be adopted in case the curve revolves about the axis of $y$.

**Alternative method in the case of polar curves.**

*The volume of the solid generated by the revolution of the area bounded by the curve $r = f(\theta)$ and radii vectors $\theta = \theta_1, \theta = \theta_2$*

(i) **about the initial line $\theta = 0$ (i.e., the x-axis) is**

$$\int_{\theta_1}^{\theta_2} \tfrac{2}{3}\pi r^3 \sin\theta\, d\theta,$$

(ii) **about the line $\theta = \pi/2$ (i.e., the y-axis) is**

$$\int_{\theta_1}^{\theta_2} \tfrac{2}{3}\pi r^3 \cos\theta\, d\theta,$$

(iii) **about any line $(\theta = \gamma)$ is**

$$\int_{\theta_1}^{\theta_2} \tfrac{2}{3}\pi r^3 \sin(\theta - \gamma)\, d\theta,$$

where in each of the above three formulae the value of $r$ in terms of $\theta$ must be substituted from the equation of the given curve.

**Ex. 1. *Prove that the volume of the solid formed by the rotation about the line $\theta = 0$ of the area bounded by the curve $r = f(\theta)$ and the lines $\theta = \theta_1, \theta = \theta_2$ is*

$$\tfrac{2}{3}\pi \int_{\theta_1}^{\theta_2} r^3 \sin\theta\, d\theta.$$



**Sol.** Let $OAB$ be the a… bounded by the curve $r = f(\theta)$ … the radii vectors $\theta = \theta_1$ … $\theta = \theta_2$. We have to find … volume formed by the revolu… of the area $OAB$ about the in… line $OX$.

Take any point $P(r, \theta)$ in… the area $OAB$ and take a s… element of the area $r\delta\theta\delta r$ at… point $P$. Drop $PM$ perpendic… from $P$ to the axis of rota… $OX$. We have

$PM = OP \sin$…

Now the volume of the … area $r\delta\theta\delta r$ about $OX = 2\pi r \sin$…

$\therefore$ the whole volume f… $OX$

$= \int_{\theta=\theta_1}^{\theta_2} \int_{r=0}^{f(\theta)} 2\pi r^2 \sin$…

$= \int_{\theta=\theta_1}^{\theta_2} 2\pi \sin\theta \left[\frac{r^3}{3}\right]_0^{f(\ldots}$

$= \frac{2}{3}\pi \int_{\theta=\theta_1}^{\theta_2} [f(\theta)]^3 \sin$…

$= \frac{2}{3}\pi \int_{\theta_1}^{\theta_2} r^3 \sin\theta\, d\theta$, w…

**Note.** Proceeding as a… the solid formed by the rot… the line $\theta = \pi/2$ is equal to…

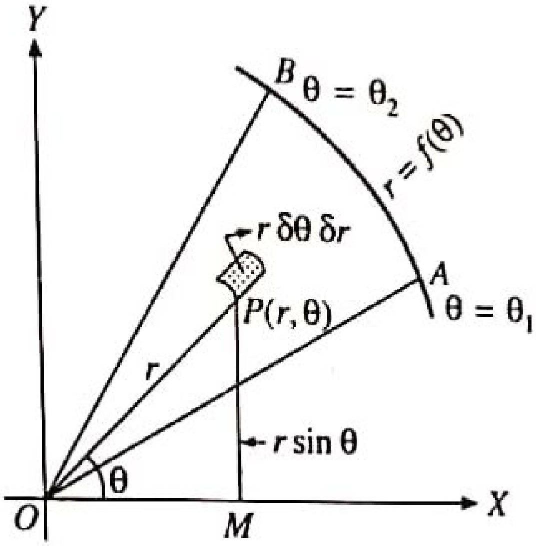
$\frac{2}{3}\pi \int_{\theta_1}^{\theta_2} r^3$…

****(e) The axis of rot…**

If, however, the axis o… any other line $CD$, then the v… about $CD$ of the area boun… perpendiculars $AC$, $BD$ on t…

where $PM$ … $\int^{OD}$ …

**Sol.** Let $OAB$ be the area bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \theta_1$ and $\theta = \theta_2$. We have to find the volume formed by the revolution of the area $OAB$ about the initial line $OX$.




Take any point $P(r, \theta)$ inside the area $OAB$ and take a small element of the area $r\delta\theta\delta r$ at the point $P$. Drop $PM$ perpendicular from $P$ to the axis of rotation $OX$. We have

$$PM = OP \sin\theta = r \sin\theta.$$

Now the volume of the ring formed by revolving the element of area $r\delta\theta\delta r$ about $OX = 2\pi r \sin\theta \,.\, r\delta\theta\delta r = 2\pi r^2 \sin\theta\, \delta\theta\delta r$.

$\therefore$ the whole volume formed by revolving the area $OAB$ about $OX$

$$= \int_{\theta=\theta_1}^{\theta_2} \int_{r=0}^{f(\theta)} 2\pi r^2 \sin\theta\, d\theta\, dr$$

$$= \int_{\theta=\theta_1}^{\theta_2} 2\pi \sin\theta \left[\frac{r^3}{3}\right]_0^{f(\theta)} d\theta$$

$$= \frac{2}{3}\pi \int_{\theta=\theta_1}^{\theta_2} [f(\theta)]^3 \sin\theta\, d\theta$$

$= \frac{2}{3}\pi \int_{\theta_1}^{\theta_2} r^3 \sin\theta\, d\theta$, where $r$ is to be replaced from the equation of the curve $r = f(\theta)$.

**Note.** Proceeding as above we can also show that the volume of the solid formed by the rotation of the above mentioned area about the line $\theta = \pi/2$ is equal to

$$\frac{2}{3}\pi \int_{\theta_1}^{\theta_2} r^3 \cos\theta\, d\theta.$$

**(e) The axis of rotation being any line.

If, however, the axis of rotation is neither $x$-axis nor $y$-axis, but is any other line $CD$, then *the volume of the solid generated by the revolution about $CD$ of the area bounded by the curve $AB$, the axis $CD$ and the perpendiculars $AC$, $BD$ on the aixs is*

$$\int_{OC}^{OD} \pi\, (PM)^2\, d\,(OM),$$

*where $PM$ is the perpendicular drawn from any point $P$ on the curve to the axis of rotation and $O$ is some fixed point on the axis of rotation.*

**Important Remarks.**

(i) If the given curve is symmetrical about $x$-axis and we have to find the volume generated by the revolution of the area about $x$-axis, then in such case we shall revolve only one of the two symmetrical areas and **shall not double it** as in the case of area or length. Obviously each of the two symmetrical parts will generate the same volume.

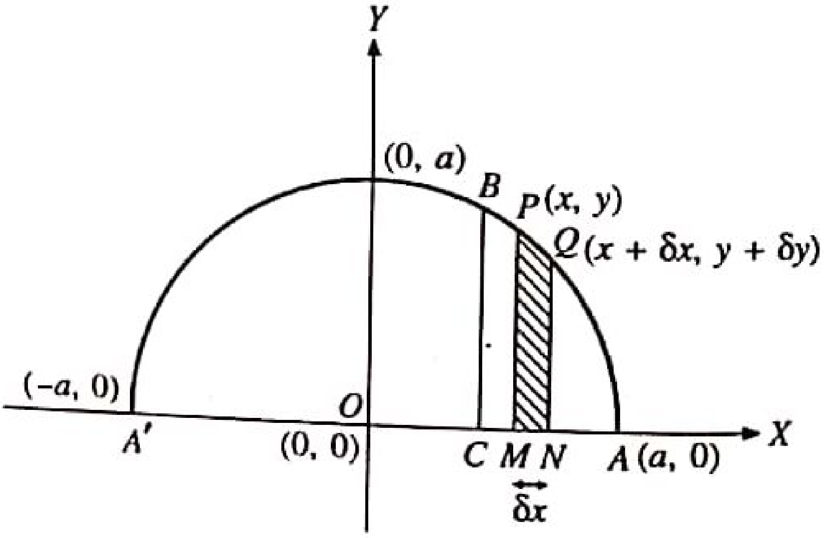
(ii) If the curve is symmetrical about $x$-axis and it is required to find the volume generated by the revolution of the area about $y$-axis, then the volume generated **will be twice** the volume generated by half of the symmetrical portion of the curve.

**Examples on volumes of solids of revolution** (*Cartesian equations*).

**Ex. 1 (a).** *Show that the volume of a sphere of radius a is* $\frac{4}{3}\pi a^3$.

**(Kanpur 1970; Meerut 95)**

**Sol.** The sphere is generated by the revolution of a semi-circular area about its bounding diameter. The equation of the generating circle of radius $a$ and centre as origin is $x^2 + y^2 = a^2$.




Let $AA'$ be the bounding diameter about which the semi-circle revolves.

Take an elementary strip $PMNQ$ where $P$ is the point $(x, y)$ and $Q$ is the point $(x + \delta x, y + \delta y)$. We have $PM = y$ and $MN = \delta x$.

Now volume of the elementary disc formed by revolving the strip $PMNQ$ about the diameter $AA'$ is

$$= \pi . PM^2 . MN = \pi y^2 \delta x = \pi (a^2 - x^2) \delta x.$$

Also the semi-circle is symmetrical about the $y$-axis and for the portion of the curve lying in the first quadrant $x$ varies from 0 to $a$.

$\therefore$ the required volume of the sphere

$$= 2 \int_0^a \pi (a^2 - x^2)\, dx = 2\pi \left[a^2 x - \tfrac{1}{3}x^3\right]_0^a$$

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$= 2\pi$

Ex. 1 (b). Find

Sol. Proceed exa

by the revolution of a

The limits for the volu

hemi-sphere is $\frac{2}{3}\pi a^3$.

Ex. 1 (c). Find

from a sphere of radiu

Sol. The limits

will be from $a - h$ to

volume

$= \int_{a-h}^{a} \pi y^2\, dx$

$= \pi [a^3 - \frac{1}{3}a^3 -$

$= \pi [a^3 - \frac{1}{3}a^3 -$

$= \pi [ah^2 - \frac{1}{3}h^3]$

Ex. 1 (d). A seg

at a dista... $\frac{1}{2}a$ from

5/32 of the volume o

Sol. Draw the

$x = \frac{1}{2}a$.

The segment o

ABCA of the circle

of the segment will

$\therefore$ the volume

$= \int_{a/2}^{a} \pi$

$= \pi \left[a^2 x\right.$

$= \pi [\frac{5}{24}a^3$

Also volume o

$\therefore$ Volume
Volume

i.e.

$= 2\pi [a^3 - \frac{1}{3}a^3] = \frac{4}{3}\pi a^3.$

Ex. 1 (b). *Find the volume of a hemisphere.* (Lucknow 1972)

Sol. Proceed exactly as in Ex. 1 (a). The hemi-sphere is generated by the revolution of a quadrant of the circle $x^2 + y^2 = a^2$ about $x$-axis. The limits for the volume will be from 0 to $a$. The required volume of hemi-sphere is $\frac{2}{3}\pi a^3$.

Ex. 1 (c). *Find the volume of a spherical cap of height h cut off from a sphere of radius a.*

Sol. The limits for the volume of the spherical cap of height $h$ will be from $a - h$ to $a$. Proceeding as in Ex. 1 (a), we get the required volume

$$= \int_{a-h}^{a} \pi y^2 \, dx = \pi \int_{a-h}^{a} (a^2 - x^2) \, dx = \pi \left[a^2 x - \frac{x^3}{3}\right]_{a-h}^{a}$$

$$= \pi [a^3 - \tfrac{1}{3}a^3 - a^2(a - h) + \tfrac{1}{3}(a - h)^3]$$

$$= \pi [a^3 - \tfrac{1}{3}a^3 - a^3 + a^2 h + \tfrac{1}{3}a^3 - a^2 h + ah^2 - \tfrac{1}{3}h^3]$$

$$= \pi [ah^2 - \tfrac{1}{3}h^3] = \pi h^2 [a - \tfrac{1}{3}h].$$

Ex. 1 (d). *A segment is cut off from a sphere of radius a by a plane at a distance $\frac{1}{2}a$ from the centre. Show that the volume of the segment is 5/32 of the volume of the sphere.*

Sol. Draw the figure as in Ex. 1 (a). Let $BC$ be the line $x = \frac{1}{2}a$.

The segment of the sphere is generated by revolving the area $ABCA$ of the circle about the $x$-axis. Hence the limits for the volume of the segment will be from $x = \frac{1}{2}a$ to $x = a$.

$\therefore$ the volume of the segment of the sphere

$$= \int_{a/2}^{a} \pi y^2 \, dx = \int_{a/2}^{a} \pi (a^2 - x^2) \, dx, \qquad [\because x^2 + y^2 = a^2]$$

$$= \pi \left[a^2 x - \frac{x^3}{3}\right]_{a/2}^{a} = \pi [a^3 - \tfrac{1}{3}a^3 - (\tfrac{1}{2}a^3 - \tfrac{1}{24}a^3)]$$

$$= \pi [\tfrac{5}{24}a^3] = \tfrac{5}{32}[\tfrac{4}{3}\pi a^3].$$

[See Ex. 1 (a)]

Also volume of the sphere $= \frac{4}{3}\pi a^3$.

$$\therefore \quad \frac{\text{Volume of the segment}}{\text{Volume of the sphere}} = \frac{\frac{5}{32} \cdot [\frac{4}{3}\pi a^3]}{\frac{4}{3}\pi a^3} = \frac{5}{32}$$

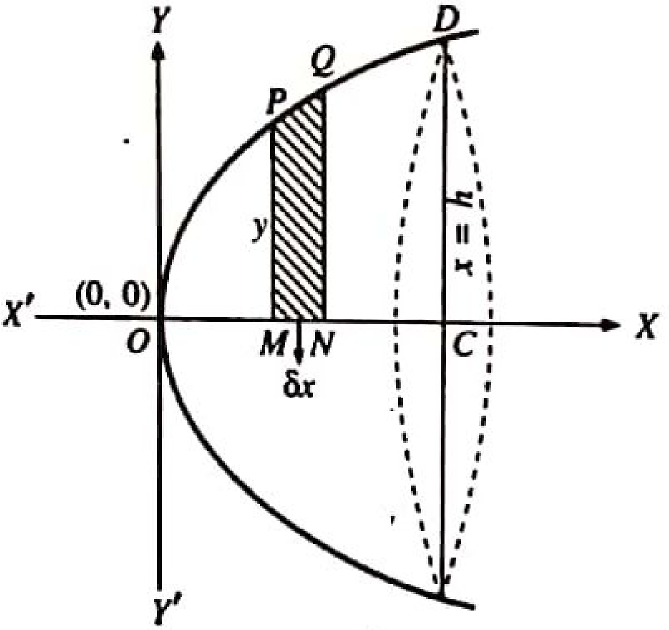
i.e., Volume of the segment $= \frac{5}{32}$ of the volume of the sphere.

**Ex. 2 (a).** *Find the volume of the paraboloid generated by the revolution about the x-axis of the parabola $y^2 = 4ax$ from $x = 0$ to $x = h$.* **(Bhopal 1983; Gorakhpur 74)**

**Sol.** The given parabola is $y^2 = 4ax$.

Take an elementary strip $PMNQ$, where $P$ is the point $(x, y)$ and $Q$ is the point $(x + \delta x, y + \delta y)$. Then $PM = y$ and $MN = ON - OM = (x + \delta x) - x = \delta x$. Now volume of the elementary disc formed by revolving the strip $PMNQ$ about the $x$-axis

$$= \pi . PM^2 . MN = \pi y^2 \delta x.$$




The paraboloid is formed by the revolution of the area $ODC$ about $x$-axis.

Also for the area $ODC$, $x$ varies from $x = 0$ to $x = h$.

$\therefore$ the required volume

$$= \int_0^h \pi y^2 \, dx = \int_0^h \pi (4ax) \, dx = 4\pi a \left[\frac{x^2}{2}\right]_0^h = 2\pi a h^2.$$

**Ex. 2 (b).** *The area of the parabola $y^2 = 4ax$ lying between the vertex and the latus rectum is revolved about the x-axis. Find the volume generated.*

**Sol.** At the vertex we have $x = 0$ and for the latus rectum we have $x = a$. Therefore the limits for the volume generated by revolving the area of the parabola $y^2 = 4ax$ between the vertex and the latus rectum are from $x = 0$ to $x = a$.

$\therefore$ the required volume

$$= \int_0^a \pi y^2 \, dx = \pi \int_0^a 4ax \, dx, \qquad [\because y^2 = 4ax]$$

$= 4a\pi\left[\frac{1}{2}x^2\right]_0^a = 4a\pi.\frac{1}{2}a^2 = 2\pi a^3.$

**Ex. 3.** *A paraboloid of revolution is generated by rotating the parabola $y^2 = 4ax$ about $OX$. Find the volume generated by that portion of the curve which lies between $x = 0$ and $x = L$. If $R$ is the area of the cross-section at $x = L$, show that the volume is half that of a cylinder of base area $R$ and length $L$.*

**Sol.** Proceeding as in Ex. 2 (a), the volume of the paraboloid of revolution $= 2\pi aL^2$. ...(1)

Also the radius of the cross-section at $x = L$ is

$$\sqrt{(4aL)}, \qquad (\because y^2 = 4ax)$$

$\therefore\ R =$ area of cross-section at $x = L$

$= \pi.(\text{radius})^2 = \pi\{\sqrt{(4aL)}\}^2 = 4a\pi L.$

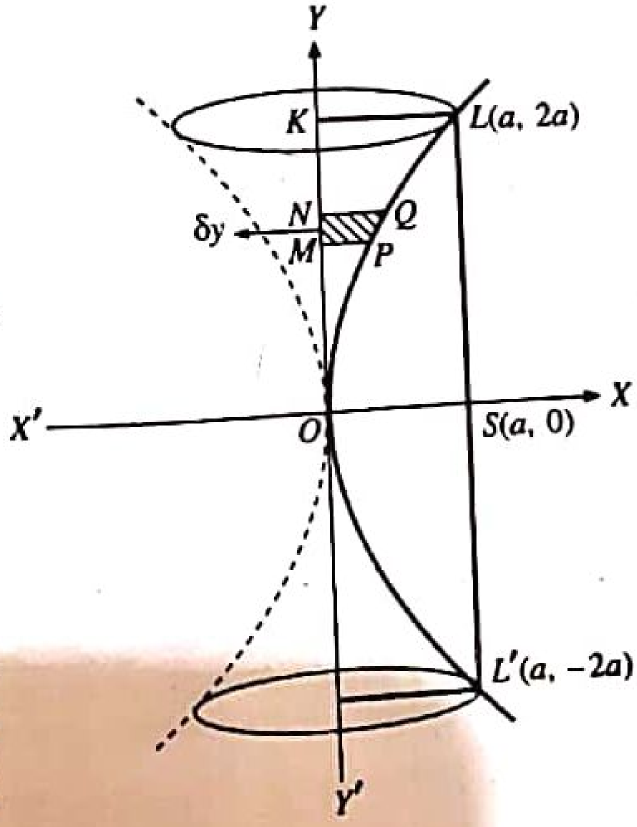
Now from (1), the volume of the paraboloid of revolution

$= 2\pi aL^2 = \frac{1}{2}\cdot(4\pi aL^2) = \frac{1}{2}[4a\pi L \times L] = \frac{1}{2}[R \times L]$

$= \frac{1}{2}$ volume of the cylinder of base area $R$ and length $L$.

***Ex. 4.** *The part of the parabola $y^2 = 4ax$ cut off by the latus rectum revolves about the tangent at the vertex. Find the volume of the reel thus generated.* **(Meerut 1984S, 85P; Allahabad 73)**

**Sol.** The given parabola is $y^2 = 4ax$. It is symmetrical about $x$-axis. The tangent at the vertex is $x = 0$ *i.e.*, $y$-axis. $LL'$ is the latus rectum.




A reel is formed by revolving about $y$-axis the area enclosed between the arc $L'OL$ of the parabola and the axis of $y$.

The volume of the reel generated by the revolution of the arc cut off by the latus rectum $LL'$ about $y$-axis $= 2 \times$ volume generated by revolving the area $OLK$ about $y$-axis.

Consider an elementary strip $PMNQ$ parallel to the axis of $x$, where $P$ is the point $(x, y)$ and $Q$ is the point $(x+\delta x, y+\delta y)$ on the parabola $y^2 = 4ax$. Then $PM = x$ and $NM = ON - OM = (y+\delta y) - y = \delta y$.

Now volume of the elementary disc formed by revolving the strip $PMNQ$ about $y$-axis $= \pi\,(PM)^2 . (NM) = \pi x^2\,\delta y$.

Also as the length of the semi-latus rectum $SL$ is $2a$, therefore $y$ varies from 0 to $2a$.

$\therefore$ the required volume

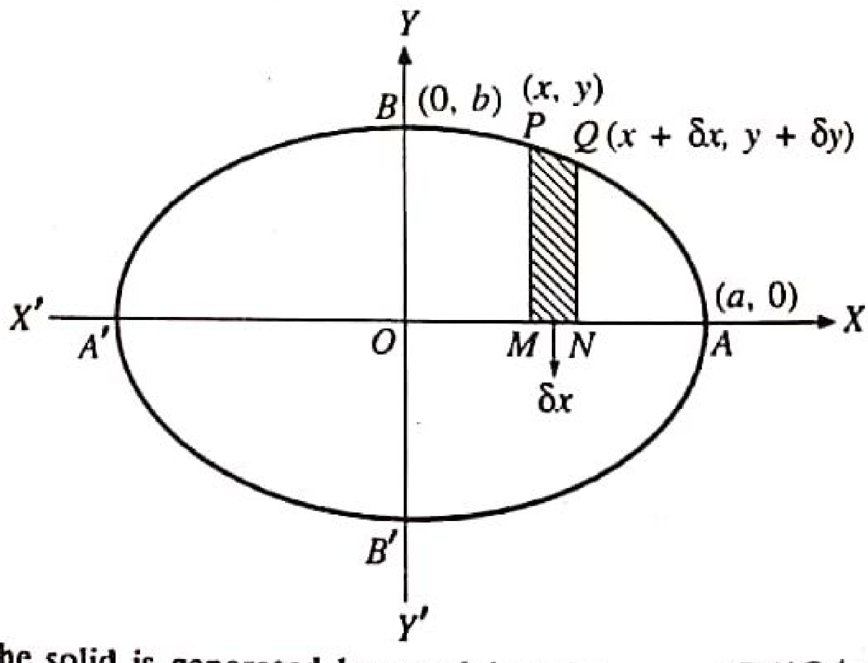
$$= 2\int_0^{2a} \pi x^2\,dy = 2\int_0^{2a} \pi\left[\frac{y^2}{4a}\right]^2 dy, \qquad (\because y^2 = 4ax)$$

$$= \frac{\pi}{8a^2}\int_0^{2a} y^4\,dy = \frac{\pi}{8a^2}\left[\frac{y^5}{5}\right]_0^{2a} = \frac{\pi}{40a^2}\cdot 32a^5 = \tfrac{4}{5}\pi a^3.$$

**Ex. 5.** *Find the volume of the solid generated by revolving the ellipse $x^2/a^2 + y^2/b^2 = 1$ about the x-axis.* **(Meerut 1984S, 85, 95 BP)**

**Sol.** The given equation of the ellipse is

$$x^2/a^2 + y^2/b^2 = 1 \qquad \text{...(1)}$$




The solid is generated by revolving the area $ABA'OA$ about the $x$-axis.

Take an elementary strip $PMNQ$ perpendicular to the axis of $x$. We have $PM = y$ and $MN = \delta x$. Now volume of the elementary disc formed by revolving the strip $PMNQ$ about the $x$-axis $= \pi . (PM)^2 . MN = \pi y^2\,\delta x$.

Also the ellipse is symmetrical about the $y$-axis and for the portion of the curve lying in the first quadrant $x$ varies from 0 to $a$. Therefore the required volume of the solid formed

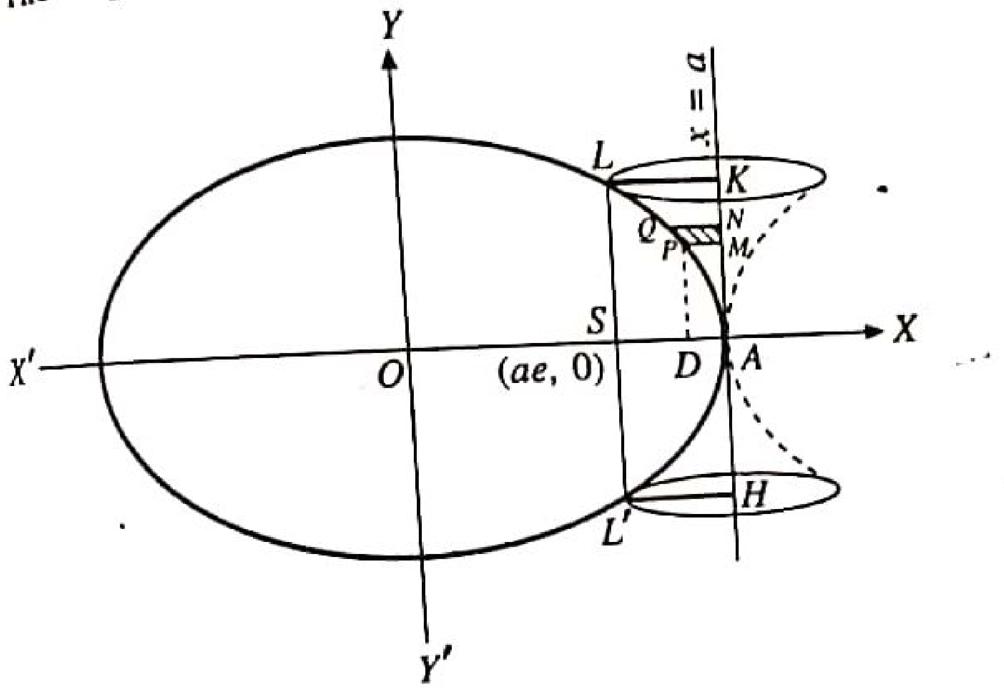
$$= 2\pi\int_0^a y^2\,dx = 2\pi\int_0^a \frac{b^2}{a^2}(a^2 - x^2)\,dx, \qquad \text{from (1)}$$

$$= 2\pi \frac{b^2}{a^2}\left[a^2x - \tfrac{1}{3}x^3\right]_0^a$$

$$= 2\pi \frac{b^2}{a^2}(a^3 - \tfrac{1}{3}a^3) = \frac{4}{3}\pi ab^2.$$

**Ex. 6.** *The part of the ellipse $x^2/a^2 + y^2/b^2 = 1$ cut off by a latus rectum revolves about the tangent at the nearer vertex. Find the volume of the reel thus generated.* **(Indore 1972)**

**Sol.** The given ellipse is $x^2/a^2 + y^2/b^2 = 1$. Let $LSL'$ be a latus rectum. The tangent at the nearer vertex $A$ is the line $x = a$.




Draw $LK$ and $L'H$ perpendicular to the line $x = a$.

We have to find the volume of the reel generated by the revolution of the area $LAL'HKL$ about the line $x = a$. Take an elementary strip $PMNQ$ perpendicular to the line $x = a$, where $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ are two neighbouring points on the arc $AL$ of the ellipse and $PM$ and $QN$ are perpendiculars from $P$ and $Q$ respectively to the tangent at $A$.

Then
$$PM = OA - OD = a - x$$
and
$$MN = AN - AM = y + \delta y - y = \delta y.$$

Now volume of the elementary disc formed by revolving the strip $PMNQ$ about the line $x = a$

$$= \pi (PM)^2 . MN = \pi (a - x)^2 \delta y.$$

Also as the length of semi-latus rectum $SL$ is $(b^2/a)$, therefore on the ellipse from $A$ to $L$, $y$ varies from $y = 0$ to $y = b^2/a$.

Now the required volume of the reel thus generated

$= 2 \times$ volume generated by revolving the area $LAKL$ about the tangent at the vertex $A$

$$= 2\int_0^{b^2/a} \pi (a-x)^2\, dy, \qquad \text{where } \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$= 2\pi \int_0^{b^2/a} (a^2 - 2ax + x^2)\, dy, \qquad \text{where } x^2 = \frac{a^2}{b^2}(b^2-y^2)$$

$$= 2\pi \int_0^{b^2/a} \left\{a^2 - 2a\cdot\frac{a}{b}\sqrt{(b^2-y^2)} + \frac{a^2}{b^2}(b^2-y^2)\right\} dy$$

$$= \frac{2\pi a^2}{b^2}\int_0^{b^2/a} \{2b^2 - 2b\sqrt{(b^2-y^2)} - y^2\}\, dy$$

$$= \frac{2\pi a^2}{b^2}\left[2b^2 y - 2b\left\{\tfrac{1}{2}y\sqrt{(b^2-y^2)} + \tfrac{1}{2}b^2 \sin^{-1}\left(\frac{y}{b}\right)\right\} - \frac{y^3}{3}\right]_0^{b^2/a}$$

$$= \frac{2\pi a^2}{b^2}\left[2b^2 . \frac{b^2}{a} - 2b\left\{\frac{1}{2}\frac{b^2}{a}\cdot\sqrt{\left(b^2 - \frac{b^4}{a^2}\right)} + \frac{1}{2}b^2 \sin^{-1}\frac{b}{a}\right\} - \frac{b^6}{3a^3}\right]$$

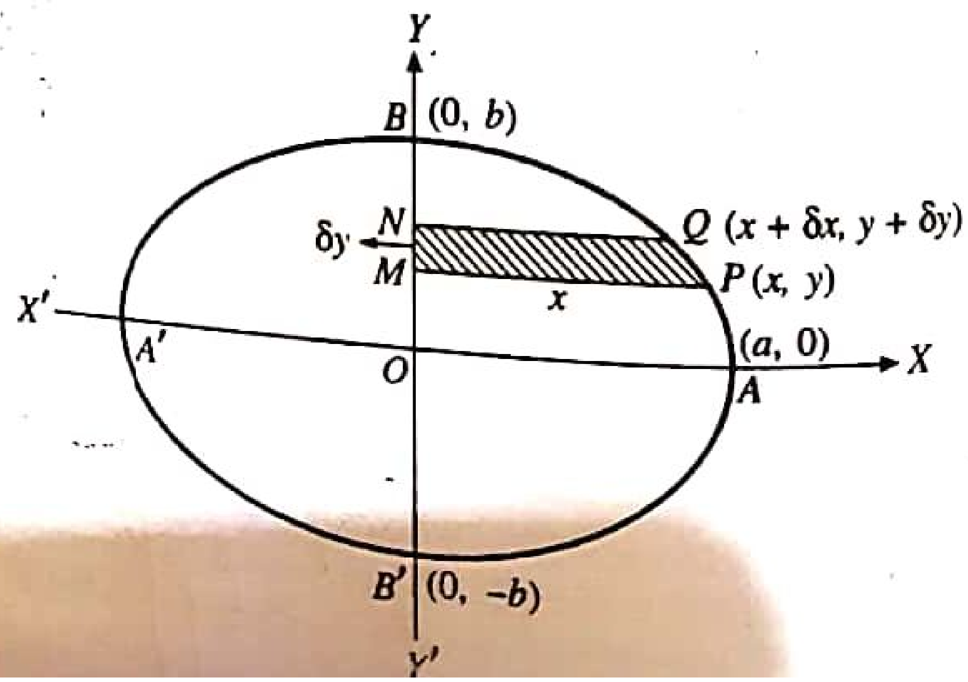
$$= \frac{2\pi a^2}{b^2}\cdot\left[2\frac{b^4}{a} - \frac{b^4}{a^2}\cdot\sqrt{(a^2-b^2)} - b^3\sin^{-1}\frac{b}{a} - \frac{b^6}{3a^3}\right]$$

$$= \frac{2\pi b}{3a}\left\{6a^2 b - 3ab\sqrt{(a^2-b^2)} - 3a^3 \sin^{-1}\frac{b}{a} - b^3\right\}.$$

**Ex. 7.** *Prove that the volume of the solid generated by the revolution of an ellipse round its minor axis is a mean proportional between those generated by the revolution of the ellipse and of the auxiliary circle about the major axis.* **(Rohilkhand 1980)**

**Sol.** Let the ellipse be $\quad x^2/a^2 + y^2/b^2 = 1.$ ...(1)
Also the equation of its auxiliary circle is $x^2 + y^2 = a^2$. ...(2)

Now the volume of the solid generated by the revolution of (1) about the major axis *i.e.*, the axis of $x$ is $\frac{4}{3}\pi ab^2 = V_1$, say. [See Ex. 5 page 132; prove it here].



Also the volume of the sphere formed by the revolution of (2) about $x$-axis is $\frac{4}{3}\pi a^3 = V_2$, say. [See Ex. 1 (a) page 128; prove it here].

Now we have to find the volume of the solid formed by the revolution of the ellipse about the minor axis *i.e.*, the axis of $y$. Consider an elementary strip $PMNQ$ perpendicular to $y$-axis. Then the volume of the elementary disc formed by revolving the strip $PMNQ$ about $y$-axis $= \pi x^2\, dy$.

The ellipse is symmetrical about $x$-axis and on the arc $AB$ of the ellipse $y$ varies from 0 to $b$.

$\therefore$ the volume of the solid generated by the revolution of the ellipse about $y$-axis $= 2\int_0^b \pi x^2\, dy$ **(Note)**

$$= 2\pi \frac{a^2}{b^2}\int_0^b (b^2 - y^2)\, dy, \quad \because \text{ from (1), } x^2 = \frac{a^2}{b^2}(b^2 - y^2)$$

$$= \frac{2\pi a^2}{b^2}\left[b^2 y - \frac{y^3}{3}\right]_0^b = \frac{2\pi a^2}{b^2}\left[b^3 - \frac{b^3}{3}\right] = \frac{4\pi a^2 b}{3} = V_3, \text{ say.}$$

Now mean proportional between $V_1$ and $V_2$

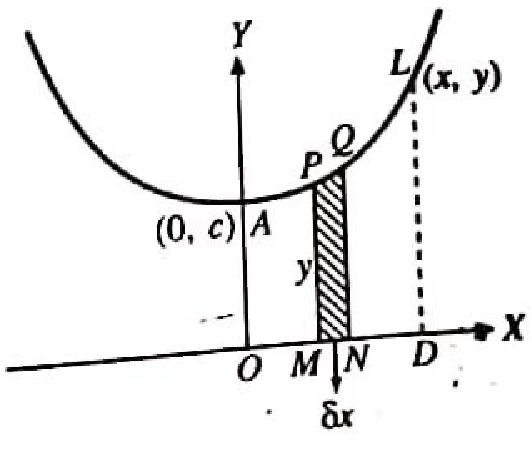
$= \sqrt{(V_1 V_2)} = \sqrt{(\frac{4}{3}\pi ab^2 \cdot \frac{4}{3}\pi a^3)} = \frac{4}{3}\pi a^2 b = V_2$

= volume generated when ellipse is revolved about minor axis.

**Ex. 8.** *Find the volume of the solid generated by the revolution of an arc of the catenary $y = c \cosh (x/c)$ about the x-axis.*

**(Rohilkhand 1979; Meerut 75, 81, 87S, 96, 97)**

**Sol.** The given equation of catenary is $y = c \cosh (x/c)$. Let $AL$ be an arc of this catenary where $L$ is the point $(x, y)$.




Take an elementary strip $PMNQ$ perpendicular to the axis of $x$, so that $PM = y$ and $MN = dx$.

Now volume of the elementary disc formed by revolving the strip $PMNQ$ about the axis of $x = \pi . PM^2 . MN = \pi y^2\, dx$.

$\therefore$ the required volume $= \int_0^x \pi y^2\, dx$

$$= \pi \int_0^x c^2 \cosh^2 \frac{x}{c}\, dx, \qquad [\because y = c \cosh (x/c)]$$

$$= \frac{\pi c^2}{2}\int_0^x \left(1 + \cosh \frac{2x}{c}\right) dx = \frac{\pi c^2}{2}\left[x + \frac{c}{2}\sinh \frac{2x}{c}\right]_0^x$$

$$= \frac{\pi c^2}{2}\left[x + \frac{c}{2}\sinh\frac{2x}{c}\right].$$

The solid of revolution formed by revolving a catenary about its directrix is called a **catenoid**.

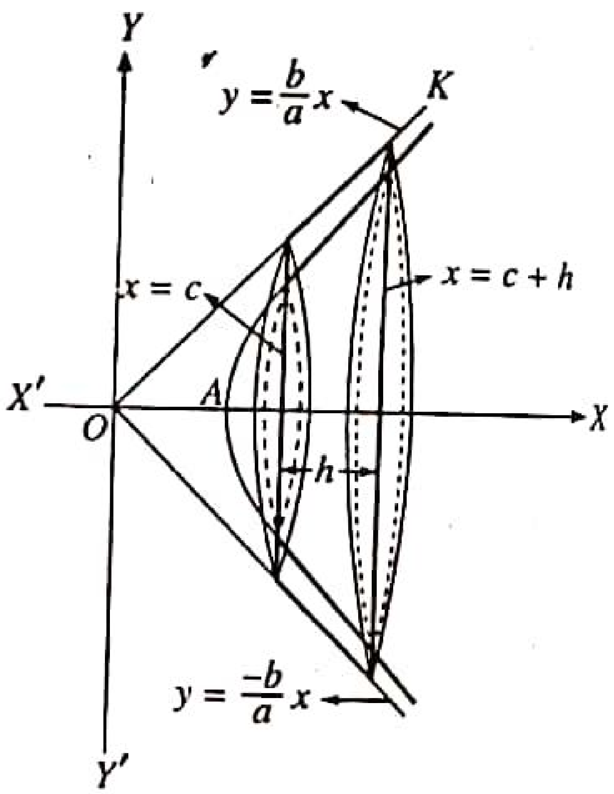
•**Ex. 9.** *If the hyperbola $x^2/a^2 - y^2/b^2 = 1$ revolves about the x-axis, show that the volume included between the surface thus generated, the cone generated by the asymptotes and two planes perpendicular to the axis of x, at a distance h apart, is equal to that of a circular cylinder of height h and radius b.*

**Sol.** The given hyperbola is $x^2/a^2 - y^2/b^2 = 1$, ...(1)

and the equation of its asymptotes is

$$x^2/a^2 - y^2/b^2 = 0$$

or $y = \pm (b/a)x$.




Let the two given planes be at distances $c$ and $(c + h)$ from the origin $O$. Then the volume of the portion of the cone generated by the asymptotes between $x = c$ and $x = c + h$ is

$$= \int_c^{c+h} \pi y^2\, dx,$$

where $y = \frac{b}{a}x$ **(Note)**

$$= \pi \int_c^{c+h} \frac{b^2}{a^2} x^2\, dx$$

$$= \frac{\pi b^2}{a^2}\left[\frac{x^3}{3}\right]_c^{c+h}$$

$$= \frac{\pi b^2}{3a^2}[(c+h)^3 - c^3]$$

$$= \frac{\pi b^2}{3a^2}[h^3 + 3ch^2 + 3c^2h] = V_1, \text{ say.}$$

Now the volume of the portion of the solid generated by the hyperbola between the two given planes (*i.e.*, between $x = c$ and $x = c + h$) is

$$= \int_c^{c+h} \pi y^2\, dx, \text{ where } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \textbf{(Note)}$$

$$= \frac{\pi b^2}{a^2}\int_c^{c+h} (x^2 - a^2)\, dx = \frac{\pi b^2}{a^2}\left[\frac{x^3}{3} - a^2 x\right]_c^{c+h}$$

$$=\frac{\pi b^2}{a^2}\left[\tfrac{1}{3}\{(c+h)^3-c^3\}-a^2\{(c+h)-c\}\right]$$

$$=\frac{\pi b^2}{3a^2}[h^3+3ch^2+3c^2h-3a^2h]=V_2, \text{ say.}$$

$\therefore$ the required volume $= V_1 - V_2$

$$=\frac{\pi b^2}{3a^2}[h^3+3ch^2+3c^2h]-\frac{\pi b^2}{3a^2}[h^3+3ch^2+3c^2h-3a^2h]$$

$$=\frac{\pi b^2}{3a^2}\cdot 3a^2h=\pi b^2 h = \text{volume of the cylinder of radius } b \text{ and height } h.$$

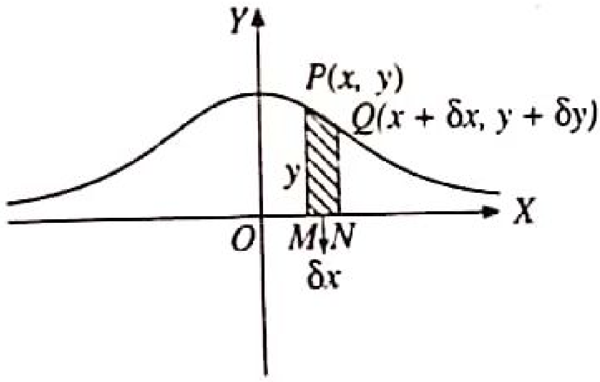
**Ex. 10.** *Find the volume of the solid generated by the revolution of the curve $y = a^3/(a^2+x^2)$ about its asymptote.*

**(Agra 1981, 77, 74; Meerut 74, 86, 86S, 88P, 93P; U.P. P.C.S. 96; Magadh 74; Jiwaji 72)**

**Sol.** The given curve is $y = a^3/(a^2+x^2)$

or $\qquad x^2y = a^2(a-y)$. ...(1)

Equating to zero, the coefficient of the highest power of $x$, the asymptote parallel to $x$-axis is $y = 0$ *i.e.*, $x$-axis. The shape of the curve is as shown in the figure. Take an elementary strip $PMNQ$ where $P(x, y)$ and $Q(x+\delta x, y+\delta y)$ are two neighbouring points on the curve. We have $PM = y$ and $MN = \delta x$.




Now volume of the elementary disc formed by revolving the strip $PMNQ$ about the axis of $x$ is $\pi PM^2 . MN = \pi y^2 \delta x$.

The curve is symmetrical about $y$-axis and for the portion of the curve in the positive quadrant $x$ varies from 0 to $\infty$.

$\therefore$ the required volume

$$=2\int_0^\infty \pi y^2\,dx = 2\pi\int_0^\infty \frac{a^6}{(x^2+a^2)^2}\,dx, \quad \text{from (1)}$$

$$=2\pi a^6\int_0^{\pi/2}\frac{a\sec^2\theta\,d\theta}{a^4(1+\tan^2\theta)^2}, \text{ putting } x = a\tan\theta$$

so that $dx = a\sec^2\theta\,d\theta$

$$=2\pi a^3\int_0^{\pi/2}\frac{\sec^2\theta\,d\theta}{\sec^4\theta}=2\pi a^3\int_0^{\pi/2}\cos^2\theta\,d\theta$$

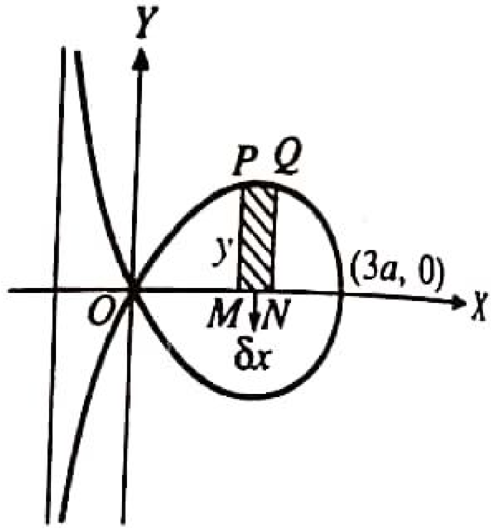
$= 2\pi a^3 \cdot \frac{1}{2} \cdot \frac{1}{2}\pi = \frac{1}{2}\pi^2 a^3.$

**Ex. 11. *The curve $y^2(a+x) = x^2(3a-x)$ revolves about the axis of $x$. Find the volume generated by the loop.***

**(Meerut 1984, 98; Kanpur 76; Gorakhpur 75)**

**Sol.** The given curve is $y^2(a+x) = x^2(3a-x)$. ...(1)

It is symmetrical about $x$-axis. Putting $y = 0$ in (1), we get $x = 0$ and $x = 3a$ *i.e.*, a loop is formed between $(0, 0)$ and $(3a, 0)$.




The volume generated by the revolution of the whole loop about $x$-axis is the same as the volume generated by the revolution of the upper half of the loop about $x$-axis.

Take an elementary strip $PMNQ$ where $P$ is the point $(x, y)$ and $Q$ is the point $(x + \delta x, y + \delta y)$. We have $PM = y$ and $MN = \delta x$.

Now volume of the elementary disc formed by revolving the strip $PMNQ$ about the axis of $x$ is $= \pi PM^2 . MN = \pi y^2 \delta x$.

$\therefore$ the required volume generated by the loop

$$= \int_0^{3a} \pi y^2 \, dx = \pi \int_0^{3a} \frac{x^2(3a-x)}{a+x} dx, \text{ from (1)}$$

$$= \pi \int_0^{3a} \left\{ -x^2 + 4ax - 4a^2 + \frac{4a^4}{x+a} \right\} dx,$$

dividing the Nr. by the Dr.

$$= \pi \left[ -\frac{x^3}{3} + \frac{4ax^2}{2} - 4a^2x + 4a^3 \log(x+a) \right]_0^{3a}$$

$$= \pi [-9a^3 + 18a^3 - 12a^3 + 4a^3 (\log 4a - \log a)]$$

$$= \pi [-3a^3 + 4a^3 \log 4] = \pi a^3 [8 \log 2 - 3].$$

**Ex. 12.** *Find the volume formed by the revolution of the loop of the curve $y^2(a+x) = x^2(a-x)$ about the axis of $x$.*

**(Kanpur 1974; Ranchi 74; Meerut 87P, 96P, 98)**

**Sol.** Proceed exactly as in Ex. 11. The required volume $= 2\pi a^3 (\log 2 - \frac{2}{3})$.

**Ex. 13.** *Find the volume of the solid generated by the revolution of the loop of the curve $y^2 = x^2(a-x)$ about the axis of $x$.*

**Sol.** The given equation of the curve is $y^2 = x^2(a-x)$ ...(1)

Putting $y = 0$ in (1) we get $x = 0, x = a$ *i.e.*, the loop is formed between $(0, 0)$ and $(a, 0)$.

$\therefore$ the required volume $= \int_0^a \pi y^2 \, dx$, (proceeding as in Ex. 11)

$= \pi \int_0^a x^2 (a - x) \, dx$, from (1)

$= \pi \int_0^a (x^2 a - x^3) \, dx = \pi \left[ a \dfrac{x^3}{3} - \dfrac{x^4}{4} \right]_0^a$

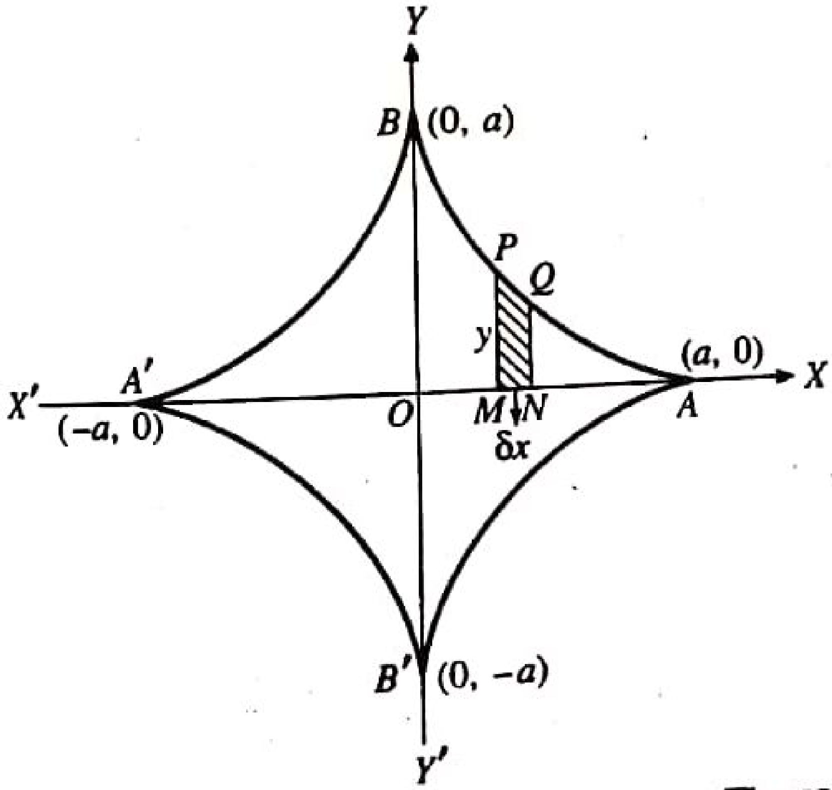
$= \pi a^4 \left[ \frac{1}{3} - \frac{1}{4} \right] = \frac{1}{12} \pi a^4$.

**Ex. 14.** *Show that the volume of the solid generated by the revolution of the upper half of the loop of the curve $y^2 = x^2 (2 - x)$ about x-axis is $\frac{4}{3}\pi$.*

**Sol.** Proceed exactly as in Ex. 13 or put $a = 2$ in Ex. 13.

**Ex. 15.** *Find the volume of the spindle shaped solid generated by revolving the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ about the x-axis.* **(Meerut 1986)**

**Sol.** The given curve is $x^{2/3} + y^{2/3} = a^{2/3}$. ...(1)




The curve is symmetrical about both the axes. The coordinates of $B$ are $(0, a)$ and those of $A$ are $(a, 0)$.

Take an elementary strip $PMNQ$ where $P$ is the point $(x, y)$ and $Q$ is the point $(x + \delta x, y + \delta y)$ on the curve. We have $PM = y$ and $MN = \delta x$.

Now volume of the elementary disc formed by revolving the strip $PMNQ$ about the axis of $x$ is $= \pi y^2 \delta x$.

$\therefore$ the required volume $= 2 \int_0^a \pi y^2 \, dx$, by symmetry

$$= 2\pi \int_0^a (a^{2/3} - x^{2/3})^3 \, dx,$$

$[\because$ from (1), $y^{2/3} = (a^{2/3} - x^{2/3})$ so that $y^2 = (a^{2/3} - x^{2/3})^3]$

$$= 2\pi \int_0^{\pi/2} a^2 \cos^6 \theta \,.\, 3a \sin^2 \theta \cos \theta \, d\theta,$$ putting $x = a \sin^3 \theta$ so that $dx = 3a \sin^2 \theta \cos \theta \, d\theta$

$$= 6\pi a^3 \int_0^{\pi/2} \sin^2 \theta \cos^7 \theta \, d\theta$$

$$= 6\pi a^3 \cdot \frac{1.6.4.2}{9.7.5.3.1} = \frac{32\pi a^3}{105}.$$

**Ex. 16.** *The area of the curve $x^{2/3} + y^{2/3} = a^{2/3}$ lying in the first quadrant revolves about x-axis. Find the volume of the solid generated.*

**Sol.** Proceed exactly as in Ex. 17. The required volume

$$= \tfrac{1}{2} \cdot \frac{32\pi a^3}{105} = \frac{16}{105} \pi a^3.$$

**Ex. 17.** *Find the volume of the solid obtained by revolving the loop of the curve $a^2y^2 = x^2 (2a - x)(x - a)$ about x-axis.*

**(Delhi 1981; Meerut 73, 86S)**

**Sol.** The given curve is $a^2y^2 = x^2 (2a - x)(x - a)$. ...(1)

The curve (1) is symmetrical about $x$-axis. It passes through the origin but the origin is a conjugate point. The curve cuts the $x$-axis at the points $(a, 0)$ and $(2a, 0)$ and so the loop of the curve is formed between $(a, 0)$ and $(2a, 0)$.

$\therefore$ the required volume $= \int_{x=a}^{2a} \pi y^2 \, dx$

$$= \pi \int_a^{2a} \frac{x^2 (2a - x)(x - a)}{a^2} dx, \text{ from (1)}$$

$$= \frac{\pi}{a^2} \int_a^{2a} (-x^4 + 3ax^3 - 2a^2x^2) \, dx$$

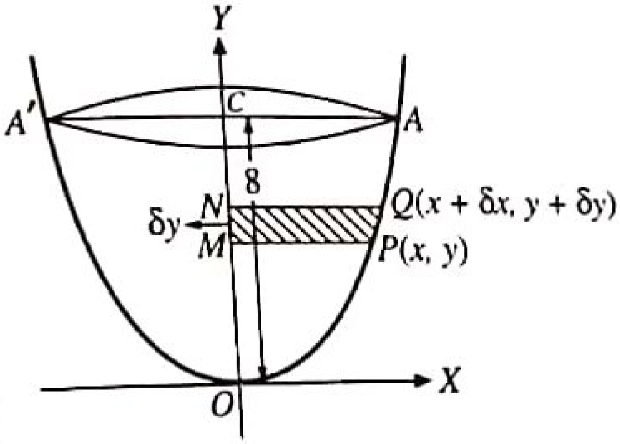
$$= \frac{\pi}{a^2} \left[ -\frac{x^5}{5} + \frac{3ax^4}{4} - \frac{2a^2x^3}{3} \right]_a^{2a}$$

$$= \frac{\pi}{a^2} \left[ \left( -\frac{32a^5}{5} + 12a^5 - \frac{16a^5}{3} \right) - \left( \frac{a^5}{5} + \frac{3a^5}{4} - \frac{2a^5}{3} \right) \right]$$

$$= \pi a^3 \left[ -\frac{32}{5} + 12 - \frac{16}{3} - \frac{1}{5} - \frac{3}{4} + \frac{2}{3} \right] = \frac{23}{60} \pi a^3.$$

•**Ex. 18.** *A basin is formed by the revolution of the curve $x^3 = 64y$, $(y > 0)$ about the axis of y. If the depth of the basin is 8 inches, how many cubic inches of water it will hold ?*

**Sol.** The given curve is $x^3 = 64y$. ...(1)




The curve (1) passes through the origin and the tangent there is the line $y=0$. When $y>0$, we have $x>0$ and so no portion of the curve lies in the second quadrant.

Take an elementary strip $PMNQ$ where $P(x,y)$ and $Q(x+\delta x, y+\delta y)$ are two neighbouring points on the curve and $PM$ and $QN$ are perpendiculars from the points $P$ and $Q$ respectively, on the $y$-axis.

We have $PM = x$ and $MN = \delta y$.

Now volume of the elementary disc formed by revolving the strip $PMNQ$ about $y$-axis is

$= \pi \, . \, PM^2 . \, MN = \pi x^2 \, \delta y.$

Clearly to form the required basin $y$ varies from 0 to 8.

$\therefore$ the required volume (*i.e.*, the capacity in cubic inches)

$$= \int_{y=0}^{8} \pi x^2 \, dy = \int_0^8 \pi \, (64y)^{2/3}, \text{ from (1)}$$

$$= 16\pi \int_0^8 y^{2/3} \, dy = 16\pi \, . \left[\frac{3}{5} y^{5/3}\right]_0^8 = \frac{48\pi}{5} \cdot 32$$

$$= \frac{1536\pi}{5} \text{ cubic inches.}$$

**Revolution about any axis :**

**Ex. 19.** *Find the volume of the solid generated by the revolution of the cissoid* $y^2 (2a - x) = x^3$ *about its asymptote.*

**(Garhwal 1983; Meerut 71)**

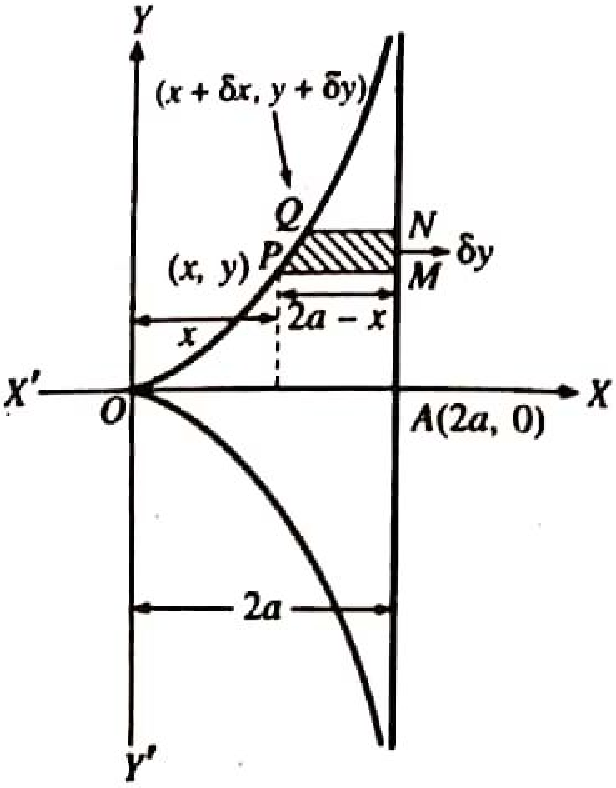
**Sol.** The given curve is $y^2 (2a - x) = x^3$. Its shape is as shown in the figure. Equating to zero the coefficient of highest power of $y$, the asymptote parallel to the axis of $y$ is $x = 2a$. Take an elementary strip $PMNQ$ perpendicular to the asymptote $x = 2a$ where $P$ is the point $(x, y)$ and $Q$ is the point $(x + \delta x, y + \delta y)$.

We have $\quad PM = 2a - x \quad$ and $\quad MN = \delta y$.

Now volume of the elementary disc formed by revolving the strip $PMNQ$ about the line $x = 2a$ is

$= \pi \, . \, PM^2 . \, MN = \pi \, (2a - x)^2 \, \delta y.$

The given curve is symmetrical about $x$-axis and for the portion of the curve above $x$-axis $y$ varies from 0 to $\infty$.




$\therefore$ the required volume $= 2\int_{y=0}^{\infty} \pi (2a - x)^2 \, dy.$ ...(1)

From the given equation of the curve $y^2 (2a - x) = x^3$ we observe that the value of $x$ cannot be easily found in terms of $y$. Hence for the sake of integration we change the independent variable from $y$ to $x$.

**(Note)**

The curve is $y^2 = \dfrac{x^3}{2a - x}$;

$$\therefore \quad 2y \frac{dy}{dx} = \frac{(2a - x) \cdot 3x^2 - x^3(-1)}{(2a - x)^2} = \frac{2(3a - x)x^2}{(2a - x)^2}$$

or $$dy = \frac{(3a - x)x^2}{(2a - x)^2} \cdot \frac{\sqrt{(2a - x)}}{x\sqrt{x}} dx = \frac{(3a - x)\sqrt{x}\sqrt{(2a - x)}}{(2a - x)^2} dx.$$

Also when $y = 0, x = 0$ and when $y \to \infty, x \to 2a$.

Hence from (1), the required volume

$$= 2\pi \int_{x=0}^{2a} (2a - x)^2 \left[\frac{(3a - x)\sqrt{x}\sqrt{(2a - x)}}{(2a - x)^2}\right] dx$$

$$= 2\pi \int_0^{2a} (3a - x)\sqrt{x}\sqrt{(2a - x)} \, dx.$$

Now put $x = 2a \sin^2\theta$ so that $dx = 4a \sin\theta \cos\theta \, d\theta$. When $x = 0, \theta = 0$ and when $x = 2a, \theta = \pi/2$. Therefore the required volume

$$= 2\pi \int_0^{\pi/2} (3a - 2a \sin^2\theta) \sqrt{(2a)} \sin\theta \sqrt{[2a(1 - \sin^2\theta)]}$$

$$\times 4a \sin\theta \cos\theta \, d\theta$$

$$= 16\pi a^3 \int_0^{\pi/2} (3 \sin^2\theta \cos^2\theta - 2\sin^4\theta\cos^2\theta)\, d\theta$$

$$= 16\pi a^3 \left[\frac{3\Gamma(\frac{3}{2})\Gamma(\frac{3}{2})}{2\Gamma(3)} - \frac{2\Gamma(\frac{5}{2})\Gamma(\frac{3}{2})}{2\Gamma(4)}\right]$$

$$= 16\pi a^3 \left[\frac{3\cdot\frac{1}{2}\cdot\sqrt{\pi}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{2.2.1} - \frac{2\cdot\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}\cdot\frac{1}{2}\cdot\sqrt{\pi}}{2.3.2.1}\right]$$

$$= 16\pi a^3 \left[\frac{3\pi}{16} - \frac{\pi}{16}\right] = 2\pi^2 a^3.$$

**Note.** If the given curve is $y^2(a-x) = x^3$, then the required volume can be obtained by putting $a$ for $2a$ in the above Exercise. The volume so obtained is $\frac{1}{2}\pi^2 a^3$.

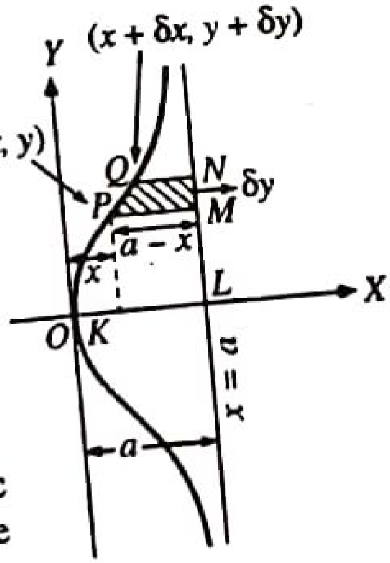
**Important Remark.** *When we are to revolve an area about a line which is neither the x-axis nor the y-axis we must take an elementary strip which is perpendicular to the line of revolution as explained in the above example.*

**Ex. 20.** *Show that the volume of the solid generated by the revolution of the curve $(a-x)y^2 = a^2x$, about its asymptote is $\frac{1}{2}\pi^2 a^3$.*

**(Meerut 1978; Gorakhpur 77; Agra 76; Magadh 73)**

**Sol.** The given curve is $(a-x)y^2 = a^2x$. ...(1)

Its shape is as shown in the figure. Equating to zero, the coefficient of highest power of $y$, the asymptote parallel to the axis of $y$ is $a - x = 0$ *i.e.*, $x = a$.




Take an elementary strip $PMNQ$, where $P(x, y)$ and $Q(x+\delta x, y+\delta y)$ are two neighbouring points on the curve and $PM$, $QN$ are perpendiculars to the asymptote from the points $P$ and $Q$ respectively. We have $PM = OL - OK = a - x$ and $MN = \delta y$.

Now volume of the elementary disc formed by revolving the strip $PMNQ$ about the line $x = a$ is

$$= \pi . PM^2 . MN = \pi (a-x)^2 \delta y.$$

The given curve is symmetrical about $x$-axis and for the portion of the curve above $x$-axis $y$ varies from 0 to $\infty$.

$\therefore$ the required volume

$$= 2\int_{y=0}^{\infty} \pi (a-x)^2\, dy$$

$$= 2\pi \int_0^\infty \left(a - \frac{ay^2}{y^2+a^2}\right)^2 dy, \qquad \left[\because \text{ from (1)}, x = \frac{ay^2}{y^2+a^2}\right]$$

$$= 2\pi a^6 \int_0^\infty \frac{dy}{(y^2+a^2)^2}.$$

Now put $y = a\tan\theta$ so that $dy = a\sec^2\theta\, d\theta$. When $y = 0, \theta = 0$ and when $y \to \infty, \theta \to \pi/2$. Therefore the required volume

$$= 2\pi a^6 \int_0^{\pi/2} \frac{a\sec^2\theta\, d\theta}{a^4\sec^4\theta} = 2\pi a^3 \int_0^{\pi/2} \cos^2\theta\, d\theta$$

$$= 2\pi a^3 \cdot \tfrac{1}{2} \cdot \tfrac{1}{2}\pi = \tfrac{1}{2}\pi^2 a^3.$$

**Ex. 21.** *The figure bounded by a quadrant of a circle of radius a and tangents at its extremities revolves about one of the tangents. Prove that the volume of the solid generated is*

$$\left(\tfrac{5}{3} - \tfrac{1}{2}\pi\right)\pi a^3.$$

**Sol.** Let the arc $AB$ be the quadrant of the circle $x^2 + y^2 = a^2$. The area bounded by the arc $AB$ and the tangents $AC$ and $BC$ (*i.e.*, the area $ABCA$) is revolved about the tangent $AC$, (say). Take an elementary strip $PMNQ$ where $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ are two neighbouring points on the arc $AB$ and $PM$, $QN$ are perpendiculars from $P$ and $Q$ on the tangent $AC$.

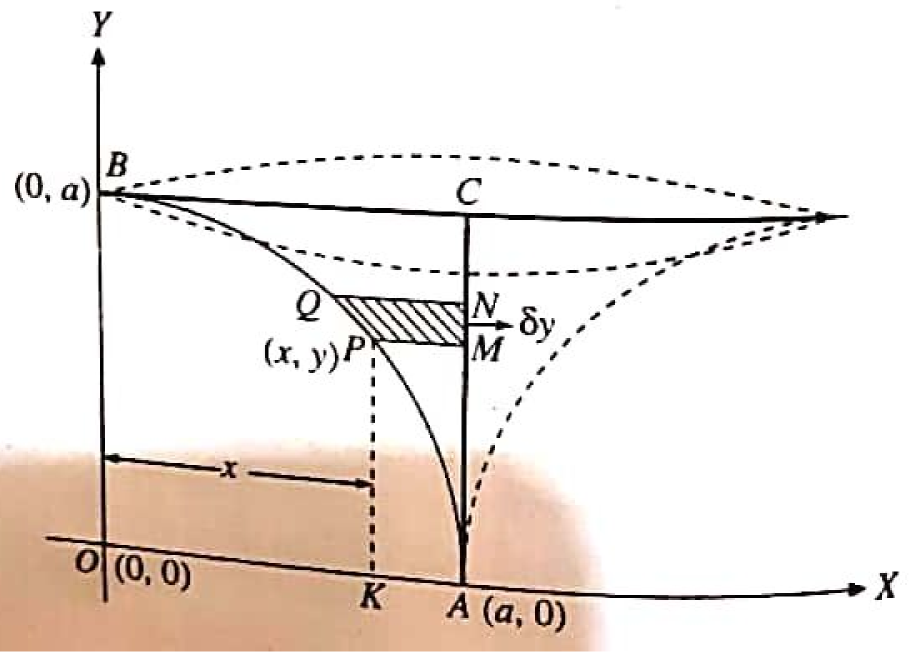
We have

$$PM = OA - OK = a - x$$

and $$MN = AN - AM = y + \delta y - y = \delta y.$$

Now volume of the elementary disc formed by revolving the strip $PMNQ$ about the tangent $AC$ (*i.e.*, the line $x = a$) is

$$= \pi . PM^2 . MN = \pi (a - x)^2 \delta y.$$



Also for the arc $AB$, $y$ varies from 0 to $a$.

$\therefore$ the required volume $= \int_{y=0}^{a} \pi (a-x)^2 \, dy$

$$= \pi \int_0^a (a^2 + x^2 - 2ax) \, dy$$

$$= \pi \int_0^a \{a^2 + (a^2 - y^2) - 2a\sqrt{(a^2 - y^2)}\} \, dy, \qquad [\because x^2 = a^2 - y^2]$$

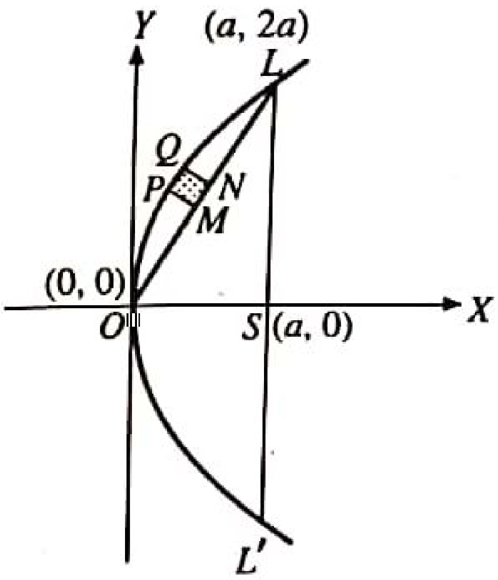
$$= \pi \left[2a^2 y - \frac{y^3}{3} - 2a \left\{\tfrac{1}{2} y \sqrt{(a^2 - y^2)} + \tfrac{1}{2} a^2 \sin^{-1}(y/a)\right\}\right]_0^a$$

$$= \pi \left[2a^3 - \tfrac{1}{3}a^3 - a^3 \sin^{-1}(1)\right] = \pi a^3 \left[\tfrac{5}{3} - \tfrac{1}{2}\pi\right].$$

**Ex. 22.** *The area cut off from the parabola $y^2 = 4ax$ by the chord joining the vertex to an end of the latus rectum is rotated through four right angles about the chord. Find the volume of the solid generated.*

**Sol.** The given parabola is

$$y^2 = 4ax.$$




Let $LL'$ be the latus rectum. The area bounded by the arc $OL$ and the chord $OL$ is revolved about the chord $OL$.

Let $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ be any two neighbouring points on the arc $OL$ and $PM$, $QN$ be the perpendiculars from $P$ and $Q$ respectively on the axis of revolution $OL$.

Now volume of the elementary disc formed by revolving the strip $PMNQ$ about the chord $OL$ is

$$= \pi . PM^2 . MN = \pi . PM^2 . d(OM). \qquad \textbf{(Note)}$$

Also equation of the chord $OL$ is

$$y - 0 = \frac{2a - 0}{a - 0}(x - 0) \text{ i.e., } 2x - y = 0 \qquad ...(1)$$

$\therefore$ $PM$ = the length of the perpendicular from $(x, y)$ to (1)

$$= \frac{2x - y}{\sqrt{(2^2 + 1^2)}} = \frac{2x - y}{\sqrt{5}},$$

and $$OM = \sqrt{(OP^2 - MP^2)} = \sqrt{\left[(x^2 + y^2) - \frac{(2x - y)^2}{5}\right]} = \frac{x + 2y}{\sqrt{5}}.$$

Now the required volume $= \int_{x=0}^{a} \pi (PM)^2 \, d(OM)$,

$[\because$ for the arc $OL$, $x$ varies from 0 to $a]$

$$= \int_{x=0}^{a} \pi \left(\frac{2x-y}{\sqrt{5}}\right)^2 d\left(\frac{x+2y}{\sqrt{5}}\right)$$

$$= \int_{x=0}^{a} \pi \left(\frac{2x-2\sqrt{(ax)}}{\sqrt{5}}\right)^2 \frac{d}{dx}\left(\frac{x+2.2\sqrt{(ax)}}{\sqrt{5}}\right) dx, \quad [\because y = 2\sqrt{(ax)}]$$

$$= \frac{\pi}{5\sqrt{5}} \int_0^a \{2x - 2\sqrt{(ax)}\}^2 \left(1 + 4\sqrt{a} \cdot \frac{1}{2\sqrt{x}}\right) dx, \qquad \text{(Note)}$$

$$= \frac{4\pi}{5\sqrt{5}} \int_0^a [x^2 - 2\sqrt{a}\,x^{3/2} + ax]\left[1 + 2\sqrt{\left(\frac{a}{x}\right)}\right] dx$$

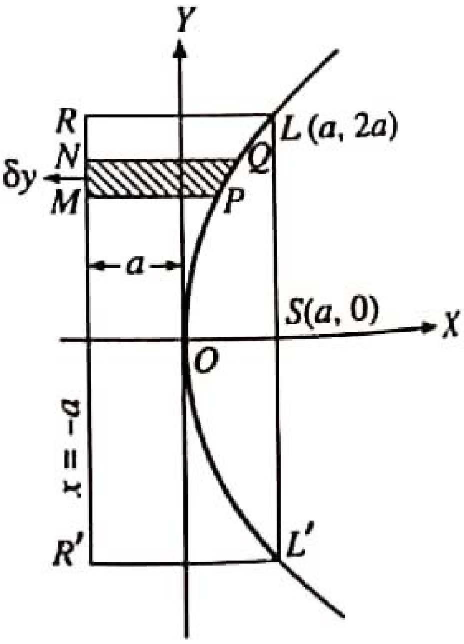
$$= \frac{4\pi}{5\sqrt{5}} \int_0^a [x^2 - 3ax + 2a^{3/2}\sqrt{x}]\, dx$$

$$= \frac{4\pi}{5\sqrt{5}} \left[\frac{x^3}{3} - \frac{3ax^2}{2} + \frac{2a^{3/2}x^{3/2}}{3/2}\right]_0^a$$

$$= \frac{4\pi}{5\sqrt{5}} \left[\frac{a^3}{3} - \frac{3a^3}{2} + \frac{4a^3}{3}\right] = \frac{2\pi a^3}{15\sqrt{5}} = \frac{2\sqrt{5}}{75}\pi a^2.$$

**Ex. 23.** *The area between a parabola and its latus rectum revolves about the directrix. Find the ratio of the volume of the ring thus obtained to the volume of the sphere whose diameter is the latus rectum.*

**Sol.** Let the parabola be $y^2 = 4ax$. Then the directrix is the line $x = -a$. Let $LL'$ be the latus rectum. The area $LOL'SL$ is revolved about the directrix. The volume of the ring thus obtained = the volume $V_1$ of the cylinder formed by the revolution of the rectangle $LL'R'R$ about the directrix − the volume $V_2$ of the reel formed by the revolution of the arc $LOL'$ about the directrix.




Now the volume $V_1$ of the cylinder

$$= \pi r^2 h = \pi (LR)^2 . LL'$$
$$= \pi (2a)^2 . 4a = 16\pi a^3.$$

To find the volume $V_2$ of the reel consider an elementary strip $PMNQ$ where $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ are two neighbouring points on the arc $OL$ and $PM, QN$ are perpendiculars from $P$ and $Q$ on the directrix.

We have $PM = a + x$ and $MN = \delta y$.

$\therefore$ the volume $V_2$ of the reel

$= 2\int_0^{2a} \pi (a + x)^2 \, dy$, [by symmetry about $x$-axis]

$= 2\int_0^{2a} \pi (a^2 + 2ax + x^2) \, dy = 2\pi \int_0^{2a} \left(a^2 + 2a \cdot \frac{y^2}{4a} + \frac{y^4}{16a^2}\right) dy,$

$[\because x = y^2/4a]$

$= 2\pi \left[a^2 y + \frac{1}{2}\frac{y^3}{3} + \frac{1}{16a^2} \cdot \frac{y^5}{5}\right]_0^{2a}$

$= 2\pi \left[2a^3 + \frac{4}{3}a^3 + \frac{2}{5}a^3\right] = 2\pi a^3 \cdot \frac{56}{15} = \frac{112\pi a^3}{15}.$

$\therefore$ Volume of the ring = volume of the cylinder – volume of the reel

$= V_1 - V_2 = 16\pi a^3 - \frac{112}{15}\pi a^3 = \frac{128}{15}\pi a^3.$

Volume of the sphere whose diameter is the latus rectum $4a$ *i.e.*, the radius is $2a = \frac{4}{3}\pi r^3 = \frac{4}{3}\pi (2a)^3 = \frac{32}{3}\pi a^3.$

$\therefore$ the rquired ratio $= \dfrac{128\pi a^3/15}{32\pi a^3/3} = \dfrac{4}{5}.$

### Examples on volumes of solids of revolution (Parametric equations)

****Ex. 24.** *Find the volume of the solid formed by revolving the cycloid*

$$x = a(\theta - \sin\theta), y = a(1 - \cos\theta)$$

**(Meerut 1985 S, 92)**
**(Kashmir 1974)**

(i) *about its base*

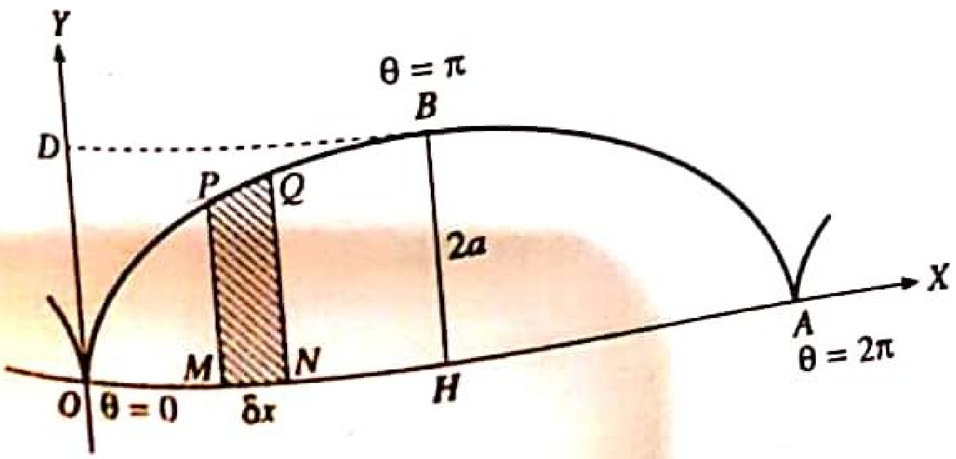
(ii) *about the y-axis.*

**Sol.** The given equations of the cycloid are

$$x = a(\theta - \sin\theta), y = a(1 - \cos\theta). \quad ...(1)$$

(i) The arc $OBA$ is revolved about the base *i.e.*, the $x$-axis. For the arc $OBA$, $\theta$ varies from 0 to $2\pi$ and at $B$, $\theta = \pi$.

Take an elementary strip $PMNQ$ where $P$ is the point $(x, y)$ and $Q$ is the point $(x + \delta x, y + \delta y)$. We have $PM = y$ and $MN = \delta x$.



Now the volume of the elementary disc formed by revolving the strip $PMNQ$ about the base (*i.e.*, the $x$-axis) is

$$\pi\, PM^2 . MN = \pi y^2\, \delta x .$$

Now the cycloid is symmetrical about the line $BH$.

$\therefore$ the required volume $= 2 \int \pi y^2\, dx$, the limits of integration being extended from $O$ to $B$

$$= 2\pi \int_{\theta=0}^{\pi} y^2 \frac{dx}{d\theta}\, d\theta$$

$$= 2\pi \int_0^{\pi} a^2 (1-\cos\theta)^2\, a\,(1-\cos\theta)\, d\theta, \text{ from (1)}$$

$$= 2\pi \int_0^{\pi} a^3 (1-\cos\theta)^3\, d\theta$$

$$= 2\pi a^3 \int_0^{\pi} \left(2\sin^2\frac{\theta}{2}\right)^3 d\theta = 16\pi a^3 \int_0^{\pi} \sin^6\frac{\theta}{2}\, d\theta$$

$$= 32\pi a^3 \int_0^{\pi/2} \sin^6\phi\, d\phi \text{ putting } \frac{\theta}{2} = \phi \text{ so that } d\theta = 2\,d\phi$$

$$= 32\pi a^3 \cdot \tfrac{5}{6} \cdot \tfrac{3}{4} \cdot \tfrac{1}{2} \cdot \tfrac{1}{2}\pi = 5\pi^2 a^3.$$

**(ii)** When the curve revolves about $y$-axis, the required volume of the solid generated = the volume generated by the revolution of the area $OABDO$ about $y$-axis $-$ the volume generated by the revolution of the area $OBDO$ about the $y$-axis. ...(2)

Also at $A, \theta = 2\pi$; at $B, \theta = \pi$ and at $O, \theta = 0$.

Now the area $OABD$ is bounded by the arc $AB$ of the cycloid and the axis of $y$. Therefore volume of the solid generated by the revolution of the area $OABDO$ about $y$-axis

$$= \int_{\theta=2\pi}^{\pi} \pi x^2\, dy = \int_{\theta=2\pi}^{\pi} \pi x^2 \frac{dy}{d\theta}\, d\theta$$

$$= \pi \int_{2\pi}^{\pi} a^2 (\theta - \sin\theta)^2\, a \sin\theta\, d\theta, \qquad \text{[from (1)]}$$

$$= \pi \int_{2\pi}^{\pi} a^2 (\theta^2 - 2\theta\sin\theta + \sin^2\theta)\, a\sin\theta\, d\theta$$

$$= \pi a^3 \int_{2\pi}^{\pi} (\theta^2\sin\theta - 2\theta\sin^2\theta + \sin^3\theta)\, d\theta$$

$$= \pi a^3 \int_{2\pi}^{\pi} [\theta^2\sin\theta - \theta(1-\cos 2\theta) + \tfrac{1}{4}(3\sin\theta - \sin 3\theta)]\, d\theta$$

(Note)

$$= \pi a^3 \Big[\theta^2 . (-\cos\theta) - 2\theta(-\sin\theta) + 2\cos\theta - \tfrac{1}{2}\theta^2 + \theta(\tfrac{1}{2}\sin 2\theta) - 1(-\tfrac{1}{4}\cos 2\theta) - \tfrac{3}{4}\cos\theta + \tfrac{1}{12}\cos 3\theta\Big]_{2\pi}^{\pi} .$$

the values of the integrals $\int \theta^2 \sin\theta\, d\theta$ and $\int \theta\cos 2\theta\, d\theta$ have been written after applying integration by parts

$$= \pi a^3 \left[\left(\pi^2 - 2 - \tfrac{1}{2}\pi^2 + \tfrac{1}{4} + \tfrac{3}{4} - \tfrac{1}{12}\right) - \left(-4\pi^2 + 2 - 2\pi^2 + \tfrac{1}{4} - \tfrac{3}{4} + \tfrac{1}{12}\right)\right]$$

$$= \pi a^3 \left[\tfrac{13}{2}\pi^2 - \tfrac{8}{3}\right] \qquad \text{...(3)}$$

Again volume of the solid generated by the revolution of the area $OBDO$ about $y$-axis

$$= \int_{\theta=0}^{\pi} \pi x^2\, dy = \int_{\theta=0}^{\pi} \pi x^2 \frac{dy}{d\theta}\, d\theta$$

$$= \pi \int_0^{\pi} a^2 (\theta - \sin\theta)^2 . a \sin\theta\, d\theta$$

$$= \pi a^3 \int_0^{\pi} (\theta^2 - 2\theta \sin\theta + \sin^2\theta) \sin\theta\, d\theta$$

$$= \pi a^3 \int_0^{\pi} (\theta^2 \sin\theta - 2\theta \sin^2\theta + \sin^3\theta)\, d\theta$$

$$= \pi a^3 \int_0^{\pi} [\theta^2 \sin\theta - \theta(1 - \cos 2\theta) + \tfrac{1}{4}(3\sin\theta - \sin 3\theta)]\, d\theta$$

$$= \pi a^3 \Big[\theta^2(-\cos\theta) - 2\theta(-\sin\theta) + 2\cos\theta - \tfrac{1}{2}\theta^2 + \theta\left(\tfrac{1}{2}\sin 2\theta\right) - 1\left(-\tfrac{1}{4}\cos 2\theta\right) - \tfrac{3}{4}\cos\theta + \tfrac{1}{12}\cos 3\theta\Big]_0^{\pi}$$

$$= \pi a^3 \left[\left(\pi^2 - 2 - \tfrac{1}{2}\pi^2 + \tfrac{1}{4} + \tfrac{3}{4} - \tfrac{1}{12}\right) - \left(2 + \tfrac{1}{4} - \tfrac{3}{4} + \tfrac{1}{12}\right)\right]$$

$$= \pi a^3 \left(\tfrac{1}{2}\pi^2 - \tfrac{8}{3}\right). \qquad \text{...(4)}$$

$\therefore$ from (2), the required volume $= (3) - (4)$

$$= \pi a^3 \left[\tfrac{13}{2}\pi^2 - \tfrac{8}{3}\right] - \pi a^3 \left[\tfrac{1}{2}\pi^2 - \tfrac{8}{3}\right] = \pi a^3 [6\pi^2] = 6\pi^3 a^3.$$

**Ex. 25.** *Find the volume of the solid generated by the revolution of the cycloid $x = a(\theta + \sin\theta)$, $y = a(1 - \cos\theta)$, $0 \le \theta \le \pi$,*

(i) *about the x-axis.* (ii) *about the base.*

**Sol.** The equations of the cycloid are

$$x = a(\theta + \sin\theta), y = a(1 - \cos\theta).$$

The cycloid is symmetrical about the $y$-axis. For half of the curve $\theta$ varies from 0 to $\pi$.

Now proceed exactly as in Ex. 24.

**Ex. 26.** *Show that the volume of the solid generated by the revolution of the cycloid*

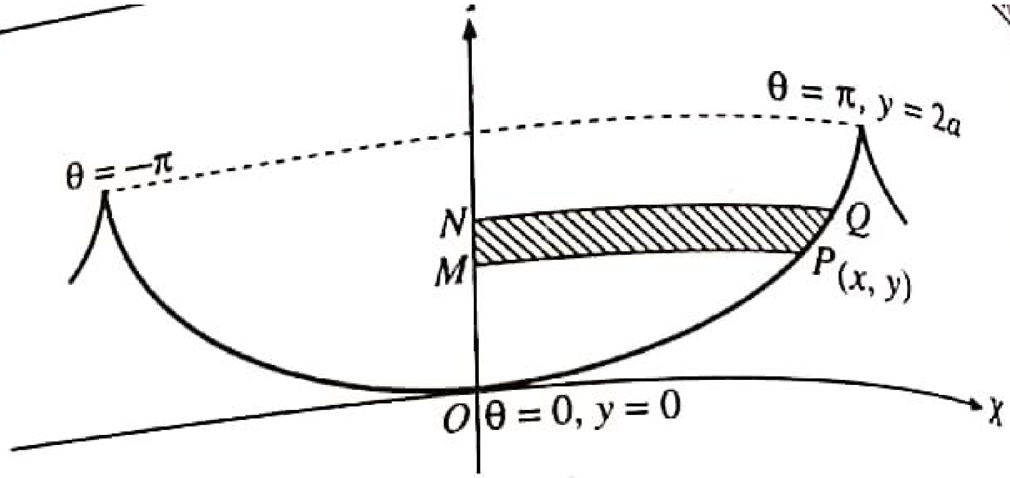
$$x = a(\theta + \sin\theta), y = a(1 - \cos\theta), 0 \le \theta \le \pi,$$

*about the y-axis is $\pi a^3 \left(\tfrac{3}{2}\pi^2 - \tfrac{8}{3}\right)$.* **(Meerut 1979S; Kanpur 79)**

**Sol.** The curve is as shown in the figure.

The required volume

$$= \int_{y=0}^{2a} \pi x^2\, dy = \pi \int_{\theta=0}^{\pi} x^2 \frac{dy}{d\theta}\, d\theta$$




$$= \pi \int_{\theta=0}^{\pi} a^2 (\theta + \sin\theta)^2 a \sin\theta \, d\theta$$

$$= \pi a^3 \int_{\theta=0}^{\pi} (\theta^2 + 2\theta \sin\theta + \sin^2\theta) \sin\theta \, d\theta$$

$$= \pi a^3 \int_0^{\pi} (\theta^2 \sin\theta + 2\theta \sin^2\theta + \sin^3\theta) \, d\theta = \pi a^3 (I_1 + 2I_2 + I_3),$$

where $\quad I_1 = \int_0^{\pi} \theta^2 \sin\theta \, d\theta = \left[-\theta^2 \cos\theta\right]_0^{\pi} + \int_0^{\pi} 2\theta \cos\theta \, d\theta$

$$= \pi^2 + \left[2\theta \sin\theta\right]_0^{\pi} - 2\int_0^{\pi} \sin\theta \, d\theta = \pi^2 + 2\left[\cos\theta\right]_0^{\pi}$$

$$= \pi^2 + 2(-1-1) = \pi^2 - 4,$$

$$I_2 = \int_0^{\pi} \theta \sin^2\theta \, d\theta = \int_0^{\pi} (\pi - \theta) \sin^2(\pi - \theta) \, d\theta,$$

$$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right]$$

$$= \int_0^{\pi} \pi \sin^2\theta \, d\theta - \int_0^{\pi} \theta \sin^2\theta \, d\theta = \pi \int_0^{\pi} \sin^2\theta \, d\theta - I_2$$

so that $\quad 2I_2 = \pi \int_0^{\pi} \sin^2\theta \, d\theta = 2\pi \int_0^{\pi/2} \sin^2\theta \, d\theta$

or $\quad I_2 = \pi \int_0^{\pi/2} \sin^2\theta \, d\theta = \pi \cdot \frac{1}{2} \cdot \frac{1}{2}\pi = \frac{1}{4}\pi^3,$

and $\quad I_3 = \int_0^{\pi} \sin^3\theta \, d\theta = 2\int_0^{\pi/2} \sin^3\theta \, d\theta = 2 \cdot \frac{2}{3.1} = \frac{4}{3}.$

$\therefore$ the required volume $= \pi a^3 (I_1 + 2I_2 + I_3)$

$$= \pi a^3 \left[(\pi^2 - 4) + 2 \cdot \tfrac{1}{4}\pi^2 + \tfrac{4}{3}\right] = \pi a^3 \left[\tfrac{3}{2}\pi^2 - \tfrac{8}{3}\right].$$

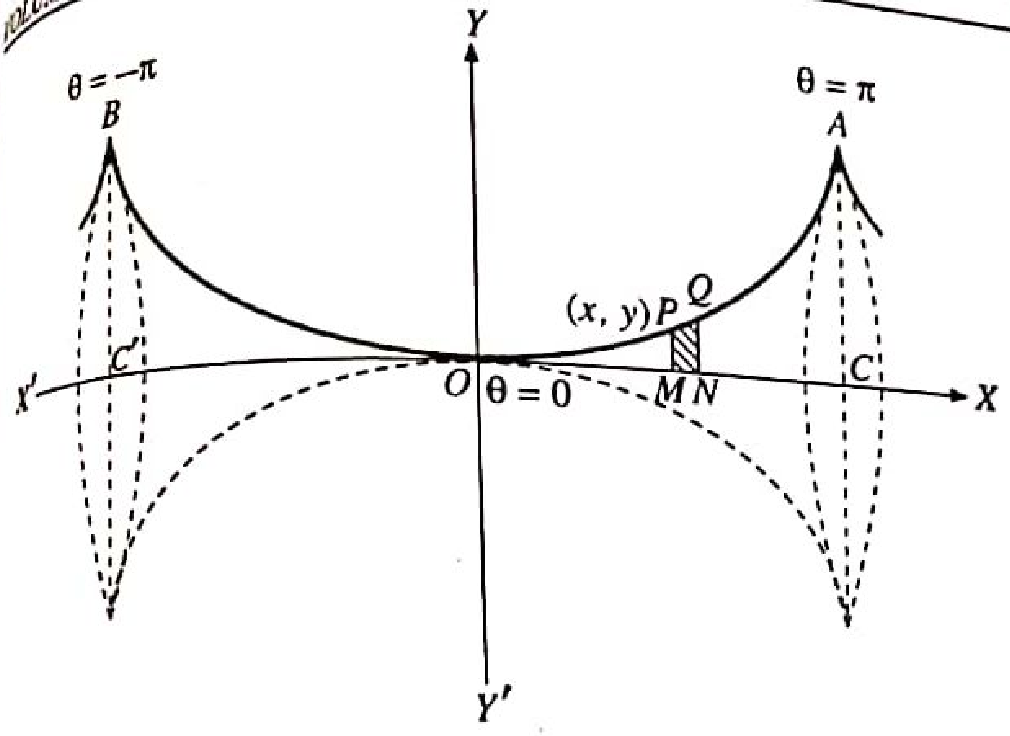
**Ex. 27. *Prove that the volume of the reel formed by the revolution of the cycloid***

$$x = a(\theta + \sin\theta), \; y = a(1 - \cos\theta)$$

*about the tangent at the vertex is* $\pi^2 a^3$.

**(Delhi 1983; Rohilkhand 82; Meerut 94)**

**Sol.** The given cycloid is symmetrical about the $y$-axis and the tangent at the vertex is $x$-axis. The reel is formed by the revolution




about $x$-axis of the area enclosed between the cycloid and the $x$-axis. For the arc $OA$ of the curve $\theta$ varies from 0 to $\pi$.

Take an elementary strip $PMNQ$ where $P$ is the point $(x, y)$ and $Q$ is the point $(x + \delta x, y + \delta y)$. We have $PM = y$ and $MN = \delta x$.

Now volume of the elementary disc formed by revolving the strip $PMNQ$ about the tangent at the vertex (*i.e.*, about $x$-axis) is

$$= \pi\, PM^2 . MN = \pi y^2\, \delta x.$$

$\therefore$ the required volume

$= 2 \int \pi y^2\, dx$, between the limits of integration from $O$ to $A$

$$= 2 \int_0^{\pi} \pi y^2 \frac{dx}{d\theta}\, d\theta$$

$= 2\pi \int_0^{\pi} a^2 (1 - \cos\theta)^2 . a (1 + \cos\theta)\, d\theta$, putting for $y$ and $\frac{dx}{d\theta}$

$$= 2\pi a^3 \int_0^{\pi} \left(2 \sin^2 \frac{\theta}{2}\right)^2 \cdot \left(2 \cos^2 \frac{\theta}{2}\right) d\theta$$

$= 2\pi a^3 \int_0^{\pi/2} 4 \sin^4 t . 2 \cos^2 t . 2\, dt$, putting $\frac{\theta}{2} = t$ so that $d\theta = 2\, dt$

$$= 32\pi a^3 \int_0^{\pi/2} \sin^4 t \cos^2 t\, dt = 32\, \pi a^3 \cdot \frac{3.1.1}{6.4.2} \cdot \frac{\pi}{2} = \pi^2 a^3.$$

***Ex. 28.** *Prove that the volume of the solid generated by the revolution about the x-axis of the loop of the curve*

$x = t^2, y = t - \frac{1}{3} t^3$ *is* $\frac{3}{4}\pi$. **(Kanpur 1980; G.N.U. 72)**

**Sol.** The given parametric equations of the curve are

$$x = t^2, y = t - \tfrac{1}{3} t^3. \qquad ...(1)$$

Eliminating $t$, we have

$$y^2 = t^2 (1 - \tfrac{1}{3} t^2)^2 = x (1 - \tfrac{1}{3} x)^2.$$

The curve is thus symmetrical about the $x$-axis. The curve cuts the $x$-axis at the points $(0, 0)$ and $(3, 0)$. Therefore the loop of the curve lies between these points. Putting $y = 0$ in (1), we get

$$t (1 - \tfrac{1}{3} t^2) = 0 \text{ giving } t = 0, \pm \sqrt{3}.$$

Therefore for the upper half of the loop $t$ varies from 0 to $\sqrt{3}$.

$\therefore$ the required volume

$$= \int_0^{\sqrt{3}} \pi y^2 \cdot \frac{dx}{dt} dt = \int_0^{\sqrt{3}} \pi (t - \tfrac{1}{3} t^3)^2 \, 2t \, dt, \text{ from (1)}$$

$$= 2\pi \int_0^{\sqrt{3}} t (t^2 + \tfrac{1}{9} t^6 - \tfrac{2}{3} t^4) \, dt = 2\pi \int_0^{\sqrt{3}} (t^3 + \tfrac{1}{9} t^7 - \tfrac{2}{3} t^5) \, dt$$

$$= 2\pi \left[\frac{t^4}{4} + \frac{1}{9} \cdot \frac{t^8}{8} - \frac{2}{3} \cdot \frac{t^6}{6}\right]_0^{\sqrt{3}} = 2\pi \left[\frac{9}{4} + \frac{1}{9} \cdot \frac{81}{8} - \frac{2}{3} \cdot \frac{27}{6}\right]$$

$$= 2\pi \left[\frac{9}{4} + \frac{9}{8} - 3\right] = \frac{3\pi}{4}.$$

**Ex. 29.** *Find the volume of the spindle shaped solid generated by revolving the astroid $x^{2/3} + y^{2/3} = a^{2/3}$ about the $x$-axis.* **(Delhi 1980)**

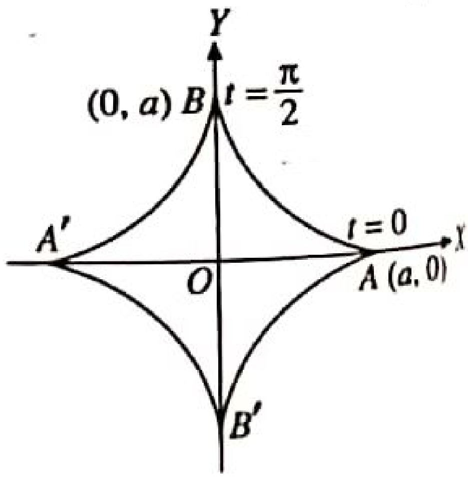
**Sol.** The parametric equations of the given curve

$$x^{2/3} + y^{2/3} = a^{2/3}$$

are $x = a \cos^3 t, y = a \sin^3 t$. ...(1)

The curve is symmetrical about both the axes.

At the point $B$, $x = 0$ and so $t = \frac{1}{2}\pi$. Again at the point $A$, $x = a$ and so $t = 0$.




Therefore for the portion of the curve lying in the first quadrant $t$ varies from $\frac{1}{2}\pi$ to 0.

$\therefore$ the required volume

= 2 × volume generated by revolving the area lying in the Ist quadrant

$$= 2 \int_{x=0}^{a} \pi y^2 \, dx$$

$$= 2 \int_{\pi/2}^{0} \pi y^2 \cdot \frac{dx}{dt} dt = 2\pi \int_{\pi/2}^{0} a^2 \sin^6 t \,.\, (-3a \cos^2 t \sin t \, dt),$$

from (1)

$$= 6\pi a^3 \int_0^{\pi/2} \sin^7 t \,.\, \cos^2 t \, dt = 6\pi a^3 \cdot \frac{6.4.2.1}{9.7.5.3.1} = \frac{32}{105} \pi a^3.$$

**Ex. 30. *Find the volume of the solid generated by the revolution of the tractrix $x = a \cos t + \frac{1}{2} a \log \tan^2 (t/2), y = a \sin t$ about its asymptote.***

**(Gorakhpur 1976; Kanpur 73; Agra 73; Allahabad 71; Vikram 71; Jiwaji 70)**

**Sol.** The given curve is

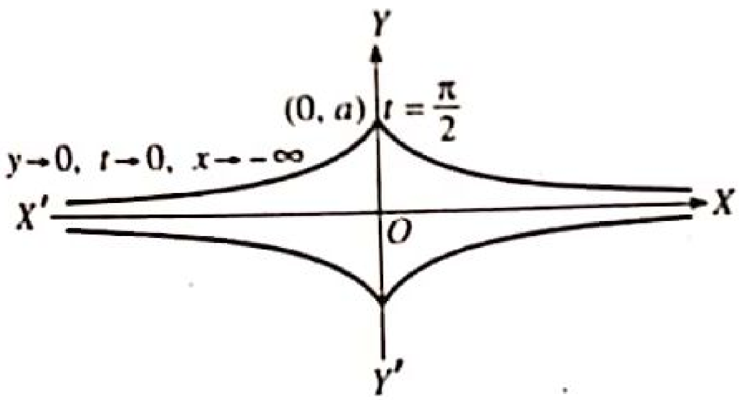
$$x = a \cos t + \tfrac{1}{2} a \log \tan^2 (t/2), y = a \sin t. \qquad ...(1)$$

$$\therefore \quad \frac{dx}{dt} = -a \sin t + \tfrac{1}{2} a \cdot \frac{1}{\tan^2 (t/2)} \cdot 2 \tan (t/2) \sec^2 (t/2) \cdot \tfrac{1}{2}$$

$$= -a \sin t + \frac{a}{2 \sin (t/2) \cos (t/2)} = -a \sin t + \frac{a}{\sin t}$$

$$= a \frac{(1 - \sin^2 t)}{\sin t} = a \frac{\cos^2 t}{\sin t} \qquad ...(2)$$

Now the given curve is symmetrical about both the axes and the asymptote is the line $y = 0$ *i.e.*, $x$-axis.




For the portion of the curve lying in the second quadrant $y$ varies from $a$ to 0, $t$ varies from $\pi/2$ to 0 and $x$ varies from 0 to $-\infty$.

$\therefore$ the required volume

$$= 2 \int_{-\infty}^{0} \pi y^2 \, dx = 2 \int_0^{\pi/2} \pi y^2 \frac{dx}{dt} \cdot dt$$

$$= 2\pi \int_0^{\pi/2} a^2 \sin^2 t \, . \frac{a \cos^2 t}{\sin t} dt, \qquad \text{from (1) and (2)}$$

$$= 2\pi a^3 \int_0^{\pi/2} \cos^2 t \sin t \, dt = 2\pi a^3 \frac{1}{3.1} = \frac{2}{3} \pi a^3.$$

**Ex. 31. *Prove that the volume of the solid generated by the revolution of the tractrix $x = a \cos t + \frac{1}{2} a \log \tan^2 \frac{1}{2} t, y = a \sin t$, about its asymptote equals half of a sphere of radius $a$.***

**Sol.** Proceeding exactly as in Ex. 30, we get the volume of the solid generated by the revolution of the given tractrix $= \frac{2}{3}\pi a^3 = \frac{1}{2} \cdot \frac{4}{3}\pi a^3 = \frac{1}{2} \cdot$ volume of the sphere of radius $a$.

**Ex. 32.** *Find the volume of the solid generated by the revolution of the cissoid $x = 2a \sin^2 t, y = 2a \sin^3 t/\cos t$ about its asymptote.*

**(Meerut 1982P, 83S, 85, 87; Bhopal 81; Kanpur**

**Sol.** The given parametric equations of the cissoid are

$$x = 2a \sin^2 t, y = 2a \sin^3 t / \cos t.$$ [Remember

Let us eliminate $t$ between these equations.

We have $\sin^2 t = x/2a$. ...(1)

Now $y^2 = \left[2a \dfrac{\sin^3 t}{\cos t}\right]^2 = 4a^2 \dfrac{\sin^6 t}{\cos^2 t} = 4a^2 \dfrac{(\sin^2 t)^3}{1 - \sin^2 t}$

$= \dfrac{\{4a^2 (x/2a)^3\}}{\{1 - (x/2a)\}}$, from (1)

$= \dfrac{x^3}{2a - x}$.

Thus $y^2 (2a - x) = x^3$ is the cartesian equation of the given cissoid and for the shape of the curve see Ex. 19 page 142.

Proceeding as in Ex. 19 the required volume

$$= 2\pi \int_{y=0}^{\infty} (2a - x)^2 \, dy = 2\pi \int_{t=0}^{\pi/2} (2a - x)^2 \frac{dy}{dt} \, dt$$

[$\because$ $t = 0$, when $y = 0$ and $t \to \frac{1}{2}\pi$ when $y \to \infty$]

$$= 2\pi \int_0^{\pi/2} (2a - 2a \sin^2 t)^2 \cdot 2a \frac{3 \sin^2 t \cos^2 t + \sin^4 t}{\cos^2 t} \, dt$$

$$= 16\pi a^3 \int_0^{\pi/2} \cos^2 t \, (3 \sin^2 t \cos^2 t + \sin^4 t) \, dt$$

$$= 16\pi a^3 \left[\int_0^{\pi/2} 3 \sin^2 t \cos^4 t \, dt + \int_0^{\pi/2} \sin^4 t \cos^2 t \, dt\right]$$

$$= 16\pi a^3 \left[3 \cdot \frac{1.3.1}{6.4.2} \cdot \frac{1}{2}\pi + \frac{3.1.1}{6.4.2} \cdot \frac{1}{2}\pi\right]$$

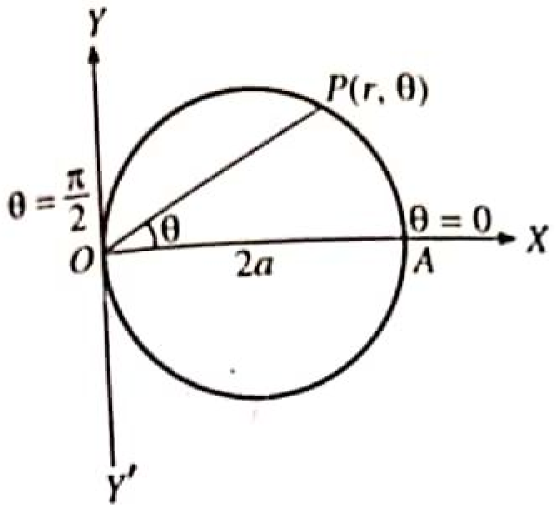
$$= 16\pi a^3 \cdot \frac{\pi}{32} \cdot (3 + 1) = 2\pi^2 a^3.$$

**Volumes of solids of revolution (polar equations)**

**Ex. 33.** *Find the volume of the solid generated by the revolution of $r = 2a \cos \theta$ about the initial line.* (Delhi 19..

**Sol.** The given curve $r = 2a \cos \theta$ is a circle passing through the pole. It is symmetrical about the initial line (*i.e.*, $x$-axis). We have $\theta = 0$ at the point $A$ and $\theta = \pi/2$ at the point $O$ where $r = 0$.

Thus for the upper half of the circle $\theta$ varies from 0 to $\frac{1}{2}\pi$.




$\therefore$ the required volume $= \dfrac{2}{3}\displaystyle\int_0^{\pi/2} \pi r^3 \sin\theta\, d\theta$

[Note. We have used the formula given on page 126].

$= \dfrac{2}{3}\pi \displaystyle\int_0^{\pi/2} (2a\cos\theta)^3 \sin\theta\, d\theta$ $\quad [\because r = 2a\cos\theta]$

$= \dfrac{16\pi a^3}{3}\displaystyle\int_0^{\pi/2} \cos^3\theta \sin\theta\, d\theta$

$= -\dfrac{16\pi a^3}{3}\displaystyle\int_0^{\pi/2} \cos^3\theta \,.\, (-\sin\theta)\, d\theta = -\dfrac{16\pi a^3}{3}\cdot\left[\dfrac{\cos^4\theta}{4}\right]_0^{\pi/2}$

$= -\frac{4}{3}\pi a^3 [0-1] = \frac{4}{3}\pi a^3.$

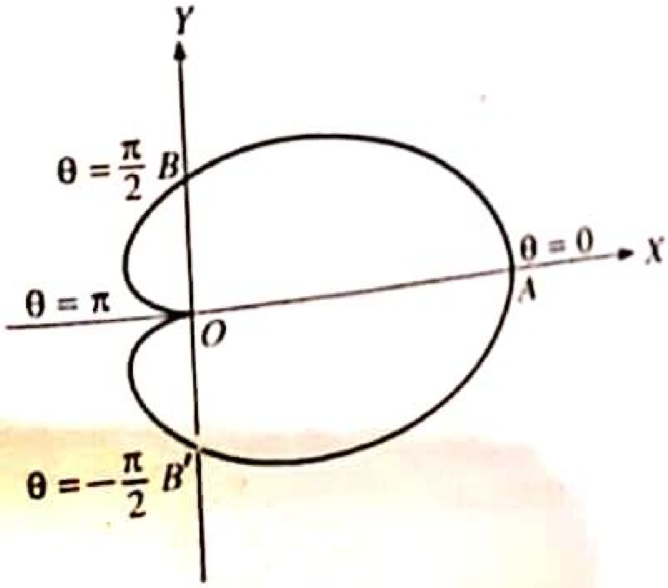
**Ex. 34. *The cardioid $r = a(1+\cos\theta)$ revolves about the initial line. Find the volume of the solid thus generated.*

**(Agra 1983, 79; Meerut 77, 87S, 88S, 93, 96 BP; Bhopal 82; Kanpur 77; U.P. P.C.S. 95; Rohilkhand 82; Ranchi 75)**

Sol. The given curve is $r = a(1+\cos\theta)$. ...(1)

It is symmetrical about the initial line. We have $r = 0$ when $\cos\theta = -1$ i.e., $\theta = \pi$.

Also $r$ is maximum when $\cos\theta = 1$ i.e., $\theta = 0$ and then $r = 2a$. As $\theta$ increases from 0 to $\pi$, $r$ decreases from $2a$ to 0. Hence the shape of the curve is as shown in the figure. For the upper half of the curve, $\theta$ varies from 0 to $\pi$.




$\therefore$ the required volume

$= \dfrac{2}{3}\displaystyle\int_0^{\pi} \pi r^3 \sin\theta\, d\theta$

$$= \frac{2\pi}{3} \int_0^{\pi} a^3 (1 + \cos\theta)^3 \sin\theta \, d\theta \quad \text{from (1)}$$

$$= -\tfrac{2}{3}\pi a^3 \int_0^{\pi} (1 + \cos\theta)^3 (-\sin\theta) \, d\theta \quad \text{(Note)}$$

$$= -\tfrac{2}{3}\pi a^3 \left[ \frac{(1 + \cos\theta)^4}{4} \right]_0^{\pi}, \text{ using power formula}$$

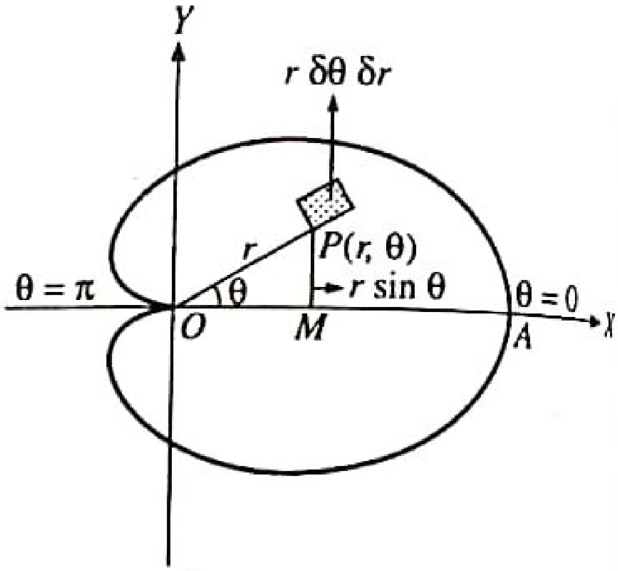
$$= -\tfrac{1}{6}\pi a^3 (0 - 2^4) = \tfrac{8}{3}\pi a^3.$$

**Aliter. (By double integration)**

Take a small element $r\delta\theta\, \delta r$ at any point $P(r, \theta)$ lying within the area of the upper half of the cardioid. Draw $PM$ perpendicular to $OX$. Then $PM = r \sin\theta$. The volume of the elementary ring formed by revolving the element $r\,\delta\theta\,\delta r$ about $OX$

$= 2\pi (r \sin\theta)\, r\, \delta\theta\, \delta r$

$= 2\pi r^2 \sin\theta\, \delta\theta\, \delta r$.




$\therefore$ the required volume formed by revolving the whole cardioid about the initial line

$$= \int_{\theta=0}^{\pi} \int_{r=0}^{a(1+\cos\theta)} 2\pi r^2 \sin\theta \, d\theta \, dr$$

$$= \int_0^{\pi} 2\pi \left[ \frac{r^3}{3} \right]_0^{a(1+\cos\theta)} \sin\theta \, d\theta = \frac{2\pi}{3} \int_0^{\pi} a^3 (1 + \cos\theta)^3 \sin\theta \, d\theta$$

$$= -\frac{2\pi a^3}{3} \int_0^{\pi} (1 + \cos\theta)^3 (-\sin\theta) \, d\theta = -\frac{2\pi a^3}{3} \left[ \frac{(1 + \cos\theta)^4}{4} \right]_0^{\pi}$$

$$= -\frac{2\pi a^3}{3} \cdot \frac{1}{4} [0 - 2^4] = \frac{2}{3} \cdot \pi a^3 \cdot \frac{1}{4} \cdot 16 = \frac{8}{3}\pi a^3.$$

**Ex. 35.** *Find the volume of the solid generated by the revolution of the cardioid $r = a(1 - \cos\theta)$ about the initial line.* **(Meerut 1994 P; Magadh 77)**

**Sol.** Proceed exactly as in Ex. 34. Required volume $= \frac{8}{3}\pi a^3$.

**Ex. 36.** *The arc of the cardioid $r = a(1 + \cos\theta)$, specified by $-\pi/2 \le \theta \le \pi/2$, is rotated about the line $\theta = 0$, prove that the volume generated is $\frac{5}{2}\pi a^3$.* **(Meerut 1978)**

**Sol.** See figure of Ex. 34. Here the portion $B'AB$ of the cardioid is rotated about the initial line (*i.e.*, $x$-axis). Obviously the volume generated is the same as is the volume generated by the revolution of the portion $AB$ about $x$-axis. For the portion $AB$, $\theta$ varies from 0 to $\frac{1}{2}\pi$.

$$\therefore \text{ the required volume } = \int_0^{\pi/2} \tfrac{2}{3}\pi r^3 \sin\theta\, d\theta$$

$$= \frac{2\pi a^3}{3}\int_0^{\pi/2} (1+\cos\theta)^3 \sin\theta\, d\theta = -\frac{2\pi a^3}{3}\left[\frac{(1+\cos\theta)^4}{4}\right]_0^{\pi/2}$$

$$= -\tfrac{1}{6}\pi a^3 [1-16] = \tfrac{15}{6}\pi a^3 = \tfrac{5}{2}\pi a^3.$$

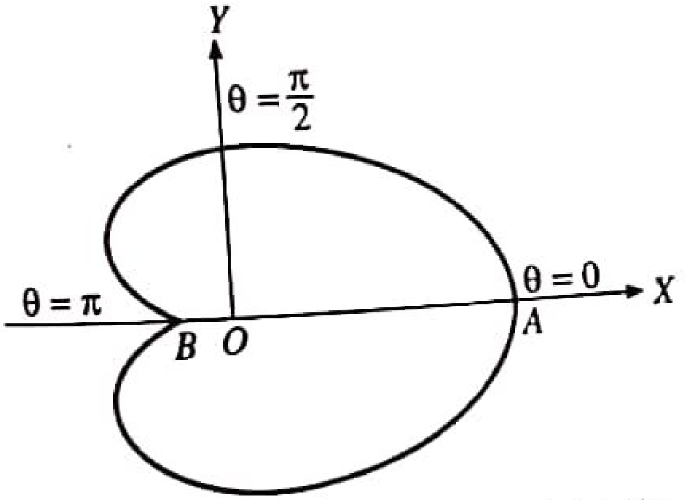
**Ex. 37.** *Show that the volume of the solid formed by the revolution of the curve $r = a + b\cos\theta$ $(a > b)$ about the initial line is*

$$\tfrac{4}{3}\pi a\,(a^2+b^2).$$

**(Meerut 1983; Delhi 82)**

**Sol.** The given equation of the curve is

$$r = a + b\cos\theta \ (a > b). \quad \text{...(1)}$$




It is symmetrical about the initial line and for the upper half of the curve $\theta$ varies from 0 to $\pi$.

$\therefore$ the required volume formed by revolving the whole curve about the initial line

$$= \int_0^{\pi} \frac{2}{3}\pi r^3 \sin\theta\, d\theta = \frac{2\pi}{3}\int_0^{\pi} (a+b\cos\theta)^3 \sin\theta\, d\theta, \quad \text{from (1)}$$

$$= -\frac{2\pi}{3b}\int_0^{\pi} (a+b\cos\theta)^3(-b\sin\theta)\, d\theta$$

$$= -\frac{2\pi}{3b}\left[\frac{(a+b\cos\theta)^4}{4}\right]_0^{\pi} = -\frac{2\pi}{3b}\left[\frac{(a-b)^4}{4} - \frac{(a+b)^4}{4}\right]$$

$$= \frac{\pi}{6b}\left[(a+b)^4 - (a-b)^4\right]$$

$= \frac{\pi}{6b} [(a+b)^2 + (a-b)^2] [(a+b)^2 - (a-b)^2]$

$= \frac{\pi}{6b} 2(a^2+b^2) \cdot 4ab = \frac{4\pi a}{3}(a^2+b^2).$

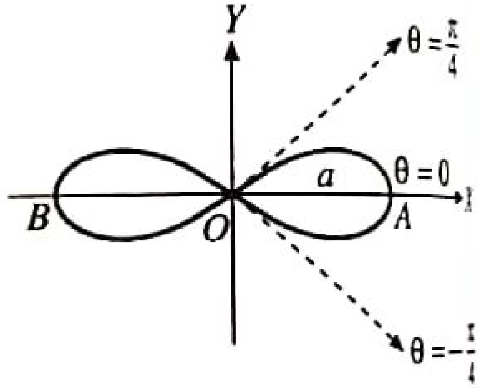
**Note.** If $b = a$, then the given curve becomes $r = a(1+\cos\theta)$ i.e., a cardioid and hence the volume of the solid generated by the revolution of the cardioid $r = a(1+\cos\theta)$ about the initial line $= \frac{4}{3}\pi a(a^2+a^2) = \frac{8}{3}\pi a^3.$

**•Ex. 38.** *Find the volume of the solid generated by revolving one loop of the lemniscate $r^2 = a^2 \cos 2\theta$ about the line $\theta = \frac{1}{2}\pi$.*

**(Lucknow 1971)**

**Sol.** The given curve is $r^2 = a^2 \cos 2\theta.$ ...(1)

It is symmetrical about the initial line. We have $r = 0$ when $\cos 2\theta = 0$ i.e., $2\theta = \pm\frac{1}{2}\pi$ or $\theta = \pm\frac{1}{4}\pi$. Thus for one loop $\theta$ varies from $-\pi/4$ to $\pi/4$. And for the upper half of one loop $\theta$ varies from 0 to $\frac{1}{4}\pi$.




Hence the required volume of the solid generated by revolving one loop about the line $\theta = \frac{1}{2}\pi$ (i.e., y-axis)

$= 2\int_0^{\pi/4} \frac{2}{3}\pi r^3 \cos\theta \, d\theta$

[**Note.** We have used the formula given on page 126]

$= \frac{4\pi}{3}\int_0^{\pi/4} a^3 (\cos 2\theta)^{3/2} \cos\theta \, d\theta,$ from (1)

$= \frac{4\pi a^3}{3}\int_0^{\pi/4} (1 - 2\sin^2\theta)^{3/2} \cos\theta \, d\theta.$

Now put $\sqrt{2}\sin\theta = \sin\phi$ so that $\sqrt{2}\cos\theta \, d\theta = \cos\phi \, d\phi$. [**Note the substitution**]

Also when $\theta = 0, \phi = 0$ and when $\theta = \pi/4, \phi = \pi/2$.

Then the required volume

$= \frac{4\pi a^3}{3}\int_0^{\pi/2} (1-\sin^2\phi)^{3/2} \cdot \frac{1}{\sqrt{2}} \cos\phi \, d\phi$

$= \frac{4\pi a^3}{3\sqrt{2}}\int_0^{\pi/2} \cos^4\phi \, d\phi = \frac{4\pi a^3}{3\sqrt{2}} \cdot \frac{3.1}{4.2} \cdot \frac{\pi}{2} = \frac{\pi^2 a^3}{4\sqrt{2}}.$

**••Ex. 39.** *Find the volume of the solid formed by revolving one loop of the curve $r^2 = a^2 \cos 2\theta$ about the initial line.*

**(Meerut 1981S; Agra 81)**

Sol. As shown in Ex. 38, for the upper half of the loop $\theta$ varies from 0 to $\pi/4$. Here the curve is revolving about the initial line (*i.e.*, ...)

∴ the required volume $= \frac{2}{3}\pi \int_0^{\pi/4} r^3 \sin\theta \, d\theta$

$= \frac{2\pi}{3} \int_0^{\pi/4} \{a\sqrt{(\cos 2\theta)}\}^3 \sin\theta \, d\theta$ $\quad [\because r^2 = a^2 \cos 2\theta]$

$= \frac{2\pi a^3}{3} \int_0^{\pi/4} (2\cos^2\theta - 1)^{3/2} \sin\theta \, d\theta.$ $\quad$ **(Note)**

Put $\sqrt{2}\cos\theta = \sec\phi$ so that $-\sqrt{2}\sin\theta \, d\theta = \sec\phi \tan\phi \, d\phi$.

When $\theta = 0, \phi = \pi/4$ and when $\theta = \pi/4, \phi = 0$.

∴ the required volume

$$= \frac{2\pi a^3}{3} \int_{\pi/4}^{0} (\sec^2\phi - 1)^{3/2} \frac{(-\sec\phi \tan\phi)}{\sqrt{2}} d\phi$$

$$= \frac{\sqrt{2}\pi a^3}{3} \int_0^{\pi/4} \tan^4\phi \sec\phi \, d\phi = \frac{\sqrt{2}\pi a^3}{3} \int_0^{\pi/4} (\sec^2\phi - 1)^2 \sec\phi \, d\phi$$

$$= \frac{\sqrt{2}\pi a^3}{3} \int_0^{\pi/4} (\sec^5\phi - 2\sec^3\phi + \sec\phi) \, d\phi. \quad \text{...(1)}$$

Also we know the reduction formula

$$\int \sec^n\phi \, d\phi = \frac{\sec^{n-2}\phi \tan\phi}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2}\phi \, d\phi.$$ **[Establish it here]**

$$\int_0^{\pi/4} \sec^5\phi \, d\phi = \left[\frac{\sec^3\phi \tan\phi}{4}\right]_0^{\pi/4} + \frac{3}{4} \int_0^{\pi/4} \sec^3\phi \, d\phi$$

$$= \frac{\sqrt{2}}{2} + \frac{3}{4}\left\{\left[\frac{\sec\phi \tan\phi}{2}\right]_0^{\pi/4} + \frac{1}{2}\int_0^{\pi/4} \sec\phi \, d\phi\right\}$$

$$= \frac{\sqrt{2}}{2} + \frac{3}{4}\left\{\frac{\sqrt{2}}{2} + \frac{1}{2}\left[\log(\sec\phi + \tan\phi)\right]_0^{\pi/4}\right\}$$

$$= \frac{\sqrt{2}}{2} + \frac{3\sqrt{2}}{8} + \frac{3}{8}\log(\sqrt{2}+1)$$

$$= \frac{7\sqrt{2}}{8} + \frac{3}{8}\log(\sqrt{2}+1),$$

$$\int_0^{\pi/4} \sec^3\phi \, d\phi = \left[\frac{\sec\phi \tan\phi}{2}\right]_0^{\pi/4} + \frac{1}{2}\int_0^{\pi/4} \sec\phi \, d\phi$$

$$= \frac{\sqrt{2}}{2} + \frac{1}{2}\log(\sqrt{2}+1)$$

and $\int_0^{\pi/4} \sec\phi \, d\phi = \log(\sqrt{2}+1)$.

Hence the required volume from (1) is

$$= \frac{\sqrt{2}\pi a^3}{3}\left[\frac{7\sqrt{2}}{8} + \frac{3}{8}\log(\sqrt{2}+1) - 2\left\{\frac{\sqrt{2}}{2} + \frac{1}{2}\log(\sqrt{2}+1)\right\} + \log(\sqrt{2}+1)\right]$$

$$= \frac{\sqrt{2}\pi a^3}{3}\left[\frac{3}{8}\log(\sqrt{2}+1) - \frac{\sqrt{2}}{8}\right] = \frac{\pi a^3\sqrt{2}}{24}[3\log(\sqrt{2}+1) - \sqrt{2}].$$

**Aliter :** The equation of the given curve is $r^2 = a^2\cos 2\theta$, or $r^4 = a^2 r^2(\cos^2\theta - \sin^2\theta)$.

Changing to cartesians, the equation becomes

$(x^2+y^2)^2 = a^2(x^2-y^2)$ or $y^4 + y^2(2x^2+a^2) + x^4 - a^2x^2 = 0$.

Solving for $y^2$, we have

$y^2 = [-(2x^2+a^2) \pm \sqrt{\{(2x^2+a^2)^2 - 4(x^4 - a^2x^2)\}}]/2.$

Neglecting the negative sign because $y^2$ cannot be –ive, we have

$$y^2 = \frac{-(2x^2+a^2) + \sqrt{(8a^2x^2+a^4)}}{2}$$

$$= \frac{-(2x^2+a^2) + 2\sqrt{2}a\sqrt{(x^2+\frac{1}{8}a^2)}}{2}.$$

Now for one loop of the given curve $x$ varies from 0 to $a$.

$\therefore$ the required volume $= \pi\int_0^a y^2\,dx$

$$= \frac{\pi}{2}\int_0^a [-2x^2 - a^2 + 2\sqrt{2}a\sqrt{(x^2+\tfrac{1}{8}a^2)}]\,dx$$

$$= \frac{\pi}{2}\left[-\frac{2}{3}x^3 - a^2x + 2\sqrt{2}a\cdot\frac{x}{2}\sqrt{(x^2+\tfrac{1}{8}a^2)} + 2\sqrt{2}a\cdot\tfrac{1}{16}a^2\log\{x+\sqrt{(x^2+\tfrac{1}{8}a^2)}\}\right]_0^a$$

$$= \frac{\pi}{2}\left[-\frac{2}{3}a^3 - a^3 + 2\sqrt{2}a\cdot\frac{a}{2}\cdot\frac{3a}{2\sqrt{2}} + \frac{1}{8}\sqrt{2}a^3\left\{\log\left(a+\frac{3a}{2\sqrt{2}}\right) - \log\frac{a}{2\sqrt{2}}\right\}\right]$$

$$= \frac{\pi}{2}\left[-\frac{5}{3}a^3 + \frac{3}{2}a^3 + \frac{1}{8}\sqrt{2}a^3\log\left\{\frac{a(2\sqrt{2}+3)}{2\sqrt{2}}\cdot\frac{2\sqrt{2}}{a}\right\}\right]$$

$$= \frac{\pi}{2}\left[-\frac{1}{6}a^3 + \frac{1}{8}\sqrt{2}a^3\log(2\sqrt{2}+3)\right]$$

$$= \frac{\pi}{2}\left[-\frac{1}{6}a^3 + \frac{1}{8}\sqrt{2}a^3\log(\sqrt{2}+1)^2\right]$$

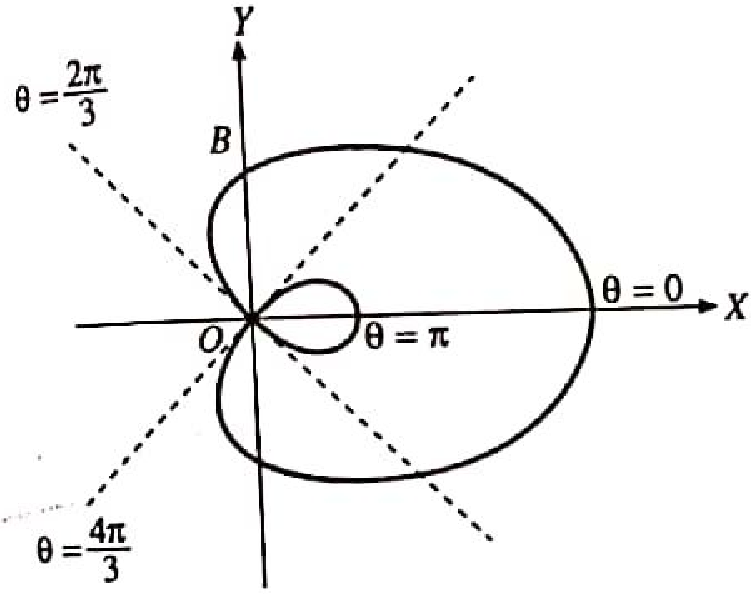
$$= \frac{\pi a^3}{2}\left[2\cdot\frac{1}{8}\sqrt{2}\log(\sqrt{2}+1) - \frac{1}{6}\right]$$

$$= \frac{\pi a^3}{2}\left[\frac{1}{4}\sqrt{2}\log(\sqrt{2}+1) - \frac{1}{6}\right]$$

$$= \frac{\pi a^3}{24}\left[3\sqrt{2}\log(\sqrt{2}+1) - 2\right]$$

$$= \frac{\pi a^3\sqrt{2}}{24}\left[3\log(\sqrt{2}+1) - \sqrt{2}\right].$$

**Ex. 40.** *The area of the inner loop of the curve $r = 1 + 2\cos\theta$ is rotated through two right angles about the initial line. Show that the volume of the solid so formed is $\pi/12$.*




**Sol.** The equation of the curve is $r = 1 + 2\cos\theta$. ...(1)

It is symmetrical about the initial line.

Some values of $\theta$ and $r$ are as given below :

| | | | | | |
|---|---|---|---|---|---|
| $\theta = 0$ | $\frac{\pi}{3}$ | $\frac{\pi}{2}$ | $\frac{2\pi}{3}$ | $\pi$ | $\frac{4\pi}{3}$ |
| $r = 3$ | 2 | 1 | 0 | $-1$ | 0 |

So the curve consists of an inner loop lying between $\theta = 2\pi/3$ and $\theta = 4\pi/3$ and for the upper half of this inner loop $\theta$ varies from $\pi$ to $4\pi/3$.

Now the volume generated by the revolution of the whole inner loop through two right angles about the initial line is the same as the volume generated by the revolution through four right angles of the upper half of the inner loop.

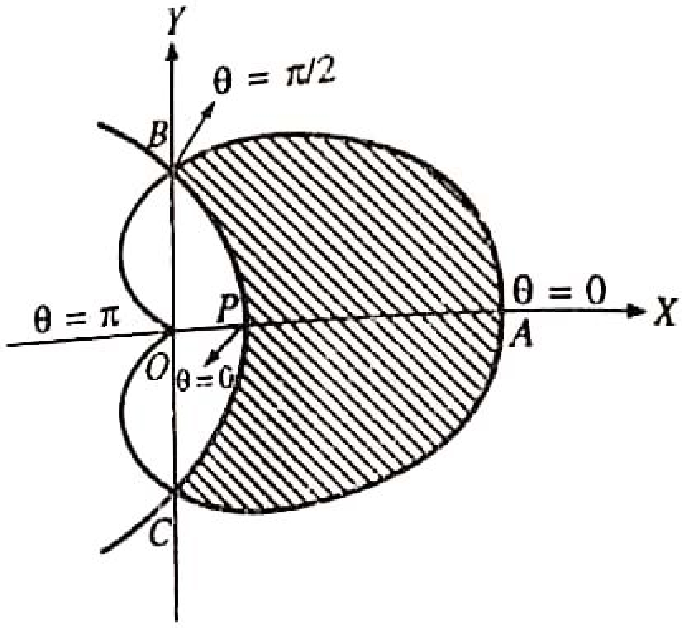
$\therefore$ the required volume $= \displaystyle\int_{\pi/3}^{4\pi/3} \frac{2}{3}\pi r^3 \sin\theta \, d\theta$

$= \dfrac{2\pi}{3}\displaystyle\int_{\pi/3}^{4\pi/3} (1 + 2\cos\theta)^3 \sin\theta \, d\theta$, substituting for $r$ from (1)

$$= \frac{2\pi}{3}\left[\frac{(1+2\cos\theta)^4}{-8}\right]_{\pi/3}^{4\pi/3} = -\frac{\pi}{12}[0-1] = \frac{\pi}{12}.$$

**Ex. 41. *Show that if the area lying within the cardioid $r = 2a(1 + \cos\theta)$ and without the parabola $r(1 + \cos\theta) = 2a$ revolves about the initial line, the volume generated is $18\pi a^3$.***

**(Bhopal 1980; Meerut 76, 8..)**




**Sol.** The equation of the cardioid is

$$r = 2a(1 + \cos\theta), \quad \text{...(1)}$$

and that of the parabola is $r = 2a/(1 + \cos\theta)$. ...(2)

Equating the values of $r$ from (1) and (2), we get

$$2a(1 + \cos\theta) = 2a/(1 + \cos\theta)$$

or $$(1 + \cos\theta)^2 = 1$$

or $$\cos\theta(\cos\theta + 2) = 0.$$

Now $\cos\theta \neq -2$. Therefore $\cos\theta = 0$

*i.e.,* $$\theta = \pi/2, -\pi/2.$$

Thus the curves (1) and (2) intersect where $\theta = \pi/2$ and $\theta = -\pi/2$.

Also both the curves are symmetrical about the initial line (i.e. $x$-axis). The required volume is generated by revolving the upper half of the shaded area about the initial line.

$\therefore$ the required volume = Volume generated by the revolution of the area $OABO$ of the cardioid − volume generated by the revolution of the area $OPBO$ of the parabola

$$= \frac{2\pi}{3}\int_0^{\pi/2} r^3 \sin\theta\, d\theta - \frac{2\pi}{3}\int_0^{\pi/2} r^3 \sin\theta\, d\theta$$

(for cardioid) (for parabola)

$$= \frac{2\pi}{3}\int_0^{\pi/2}\left[8a^3(1 + \cos\theta)^3 - \frac{8a^3}{\ldots}\right]\sin\theta\, d\theta$$

$$= \frac{-16\pi a^3}{3} \int_0^{\pi/2} [(1+\cos\theta)^3 - (1+\cos\theta)^{-3}](-\sin\theta)\, d\theta$$

**[Note]**

$$= \frac{-16\pi a^3}{3} \left[ \frac{(1+\cos\theta)^4}{4} - \frac{(1+\cos\theta)^{-2}}{-2} \right]_0^{\pi/2},$$

using power formula

$$= \frac{-16\pi a^3}{3} \left[ \frac{1}{4}(1-16) + \frac{1}{2}\left(1 - \frac{1}{4}\right) \right] = \frac{-16}{3}\pi a^3 \left[ -\frac{15}{4} + \frac{3}{8} \right]$$

$$= \left( -\frac{16}{3}\pi a^3 \right) \left( \frac{-27}{8} \right) = 18\pi a^3.$$

...3 **Surfaces of solids of revolution.**

(a) **Revolution about the axis of x.** *To prove that the curved surface of the solid generated by the revolution, about x-axis, of the area bounded by the curve $y = f(x)$, the ordinates $x = a, x = b$ and the x-axis is*

$$\int_{x=a}^{x=b} 2\pi y \, ds$$

*where s is the length of the arc measured from $x = a$ to any point $(x, y)$.*

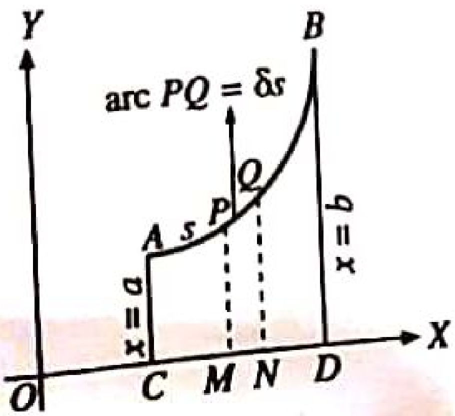
**Or**

*Show that the area of the surface of the solid obtained by revolving about x-axis the arc of the curve intercepted between the points whose abscissae are a and b is*

$$\int_a^b 2\pi y \frac{ds}{dx} dx.$$ **(G.N.U. 1974S)**

**Proof.** Let $AB$ be the arc of the curve $y = f(x)$ included between the ordinates $x = a$ and $x = b$. It is being assumed that the curve does not cut x-axis and $f(x)$ is a continuous function of $x$ in the interval $(a, b)$.

Let $P(x, y)$ and $Q(x + \delta x, y + \delta y)$ be any two neighbouring points on the curve $y = f(x)$.




Let the length of the arc $AP$ be $s$ and arc $AQ = s + \delta s$ so that arc $PQ = \delta s$.

Draw the ordinates $PM$ and $QN$. Let $S$ denote the curved surface of the solid generated by the revolution of the area $CMPA$ about the $x$-axis. Then the curved surface of the solid generated by the revolution of the area $MNQP = \delta S$.

We shall take it as an axiom that the curved surface of the solid generated by the revolution of the area $MNQP$ about the $x$-axis lies between the curved surfaces of the right circular cylinders whose radii are $PM$ and $NQ$ and which are of the same thickness (height) $\delta s$. There is no loss in assuming so because ultimately $Q$ is to tend to $P$.

Thus $\delta S$ lies between $2\pi y \delta s$ and $2\pi (y + \delta y) \delta s$

i.e., $$2\pi y \, \delta s < \delta S < 2\pi (y + \delta y) \, \delta s$$

or $$2\pi y < (\delta S / \delta s) < 2\pi (y + \delta y).$$

Now as $Q$ approaches $P$ *i.e.*, $\delta s \to 0$, $\delta y$ will also tend to zero. Hence by taking limits as $\delta s \to 0$, we have

$$\frac{dS}{ds} = 2\pi y \quad \text{or} \quad dS = 2\pi y \, ds.$$

$$\therefore \quad \int_{x=a}^{x=b} 2\pi y \, ds = \int_{x=a}^{x=b} dS = \Big[S\Big]_{x=a}^{x=b}$$

= (the value of $S$ when $x = b$) $-$ (the value of $S$ when $x = a$)

= surface of the solid generated by the revolution of the area $ACDB - 0$.

$\therefore$ the required curved surface $= \int_{x=a}^{x=b} 2\pi \mathbf{y} \, \mathbf{ds}$

$$= \int_{x=a}^{x=b} 2\pi y \frac{ds}{dx} dx, \text{ where } \frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}.$$

**(b) Axis of revolution as y-axis.** *Similarly the curved surface of the solid generated by the revolution about the y-axis, of the area bounded by the curve $x = f(y)$, the lines $y = a$, $y = b$ and the y-axis is*

$$2\pi \int_{y=a}^{y=b} x \, ds$$

or $$S = 2\pi \int_{y=a}^{b} x \frac{ds}{dy} dy, \text{ where } \frac{ds}{dy} = \sqrt{\left\{1 + \left(\frac{dx}{dy}\right)^2\right\}}.$$

**(c) Surface formula for parametric equations.** Suppose the equation of the curve is given in parametric form $x = f(t), y = \phi(t)$, $t$ being the variable parameter. Then the curved surface of the solid formed by the revolution about the $x$-axis

$$= \int 2\pi y \frac{ds}{dt} dt, \text{ between the suitable limits}$$

where $$\frac{ds}{dt} = \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}}.$$

**(d) Surface formula for Polar equations.** Suppose the equation of the curve is given in the polar form $r = f(\theta)$. Then the curved surface generated by the revolution about the initial line, of the arc intercepted between the radii vectors $\theta = \alpha$ and $\theta = \beta$ is

$$\int_{\theta=\alpha}^{\theta=\beta} 2\pi (r \sin \theta) \frac{ds}{d\theta} d\theta, \text{ where } \frac{ds}{d\theta} = \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}}.$$

$[\because y = r \sin \theta]$

**Note.** In some cases we may use the formula

$$S = \int 2\pi y \frac{ds}{dr} dr, \text{ where } \frac{ds}{dr} = \sqrt{\left\{1 + \left(r\frac{d\theta}{dr}\right)^2\right\}}.$$

(e) **Revolution about any axis.** If the given arc $AB$ is revolved about a line $CD$ other than the coordinate axes, then the curved surface thus generated is

$= 2\pi \int (PM)\, ds$, (between the proper limits of integration),

where $PM$ is the perpendicular drawn from any point $P$ on the arc $AB$ to the axis of revolution $CD$ and $ds$ is the length of an element of the arc $AB$ at the point $P$.

**Important Remark.** If an arc length revolves about $x$-axis, the basic formula for the surface of revolution in all cases is $\int 2\pi y\, ds$, between the suitable limits. If we want to integrate w.r.t. $x$, we shall change $ds$ as $(ds/dx)\, dx$ and adjust the limits accordingly.

A similar transformation can be made if we want to integrate w.r.t. $y$ or with respect to $\theta$ or w.r.t. some parameter, say $t$.

**Examples on Surfaces of Revolution.** (*Cartesian equations*)

**Revolution about x-axis**

**Ex. 1.** *Find the curved surface of a hemisphere of radius $a$.* **(Meerut 1996)**

**Sol.** A hemisphere is generated by the revolution of a quadrant of a circle about one of its bounding radii.

Let the equation of the circle be $x^2 + y^2 = a^2$. ...(1)

Let the hemisphere be formed by revolving about $x$-axis the arc of the circle (1) lying in the first quadrant.

Differentiating (1), w.r.t. $x$, we get

$$2x + 2y\,(dy/dx) = 0 \quad \text{or} \quad dy/dx = -x/y.$$

Therefore
$$\frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} = \sqrt{\left\{1 + \frac{x^2}{y^2}\right\}}$$

$$= \sqrt{\left\{\frac{y^2 + x^2}{y^2}\right\}} = \sqrt{\left(\frac{a^2}{y^2}\right)}, \text{ from (1)}$$

$$= a/y.$$

For the arc of the circle (1) lying in the first quadrant $x$ varies from 0 to $a$.

$\therefore$ the required surface

$$= 2\pi \int_{x=0}^{a} y\, ds = 2\pi \int_0^a y \frac{ds}{dx} \cdot dx$$

$$= 2\pi \int_0^a y \cdot \frac{a}{y} dx = 2\pi \int_0^a a\, dx = 2\pi a \left[x\right]_0^a$$

$= 2\pi a \cdot a = 2\pi a^2.$

•**Ex. 2.** *Find the surface of a sphere of radius a.* (Rohilkhand 1989; Meerut 90P, 97, 98)

**Sol.** Suppose the sphere is generated by the revolution of a semi-circle of radius $a$ about its bounding diameter (say x-axis).

Let the equation of the circle be $x^2 + y^2 = a^2$, ...(1)

the centre being the origin.

Then as in Ex. 1., $ds/dx = a/y$.

Also for the semi-circle, $x$ varies from $-a$ to $a$.

$\therefore$ the required surface

$$= 2\pi \int_{x=-a}^{a} y\, ds = 2\pi \int_{-a}^{a} y \frac{ds}{dx}\, dx = 2\pi \int_{-a}^{a} y \cdot \frac{a}{y}\, dx$$

$$= 2\pi \int_{-a}^{a} a\, dx = 2\pi a \left[x\right]_{-a}^{a}$$

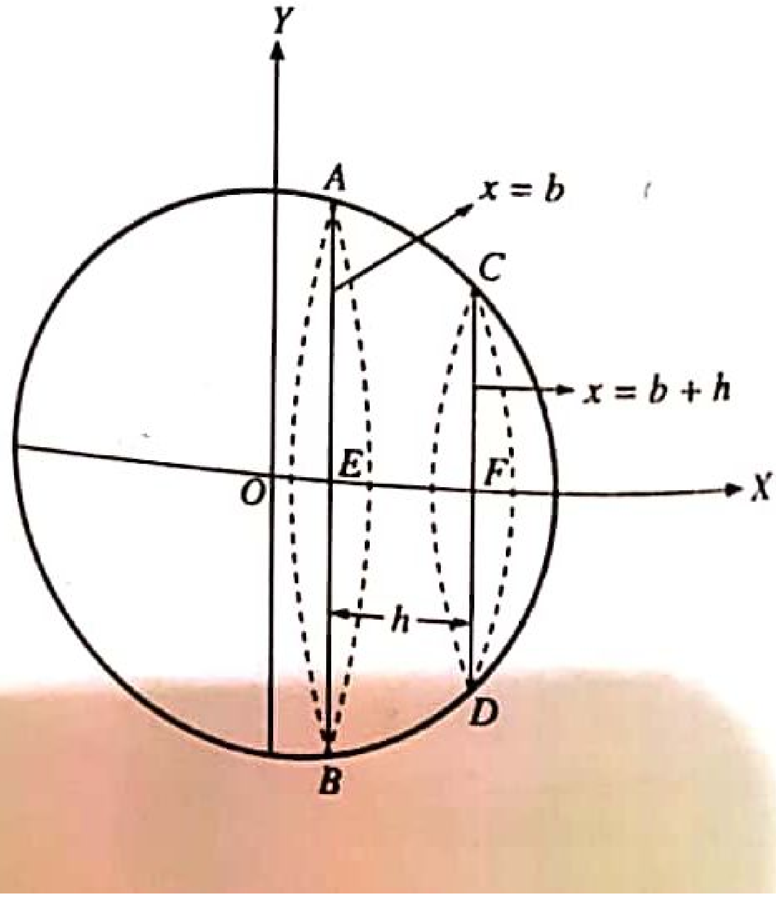
$$= 2\pi a\,(a + a) = 4\pi a^2.$$

**Ex. 3.** *Show that the surface of the spherical zone contained between two parallel planes is $2\pi ah$ where a is the radius of the sphere and h the distance between the planes.* (Kumaun 1983)

**Sol.** Let the sphere be generated by the revolution about the x-axis of the circle

$$x^2 + y^2 = a^2. \qquad \text{...(1)}$$

Let the two parallel planes bounding the spherical zone be formed by the revolution of the lines $x = b$ and $x = b + h$.



Then the required surface is generated by the revolution of the arc $AC$ about $x$-axis.

Proceeding as in Ex. 1, we get

$$\frac{ds}{dx} = \frac{a}{y}.$$

$\therefore$ the required surface

$$= \int_{b}^{b+h} 2\pi y \frac{ds}{dx} dx$$

$$= 2\pi \int_{b}^{b+h} y \cdot \frac{a}{y} dx = 2\pi a \int_{b}^{b+h} dx = 2\pi a \left[ x \right]_{b}^{b+h}$$

$$= 2\pi a (b + h - h) = 2\pi ab.$$

**Ex. 4.** *Find the area of the surface formed by the revolution of the parabola $y^2 = 4ax$ about the $x$-axis by the arc from the vertex to one end of the latus rectum.* **(Delhi 1978)**

**Sol.** The given parabola is $y^2 = 4ax$. ...(1)

Differentiating (1) w.r.t. $x$, we get

$$2y\,(dy/dx) = 4a \quad \text{or} \quad dy/dx = 2a/y.$$

$$\therefore \quad \frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} = \sqrt{\left(1 + \frac{4a^2}{y^2}\right)} = \sqrt{\left(\frac{y^2 + 4a^2}{y^2}\right)}$$

$$= \frac{\sqrt{(4ax + 4a^2)}}{y}, \text{ from (1)}$$

$$= \frac{2\sqrt{a}\,\sqrt{(x + a)}}{y}. \qquad ...(2)$$

For the given arc from the vertex $(0, 0)$ to one end of the latus rectum, say the end $(a, 2a)$, $x$ varies from 0 to $a$.

$\therefore$ the required surface

$$= \int_{x=0}^{a} 2\pi y \frac{ds}{dx} dx$$

$$= 2\pi \int_{0}^{a} y \cdot \frac{2\sqrt{a}\,\sqrt{(x + a)}}{y} dx, \text{ from (2)}$$

$$= 4\pi \sqrt{a} \int_{0}^{a} (x + a)^{1/2} dx = 4\pi \sqrt{a} \cdot \left[\frac{2}{3}(x + a)^{3/2}\right]_{0}^{a}$$

$$= \frac{8\pi \sqrt{a}}{3} [(2a)^{3/2} - a^{3/2}] = \frac{8\pi a^2}{3} [2\sqrt{2} - 1].$$

**Ex. 5.** *Find the curved surface of the solid generated by the revolution, about the $x$-axis of the area bounded by the $x$-axis the parabola $y^2 = 4ax$ and the ordinate $x = h$.*

**Sol.** Proceeding exactly as in Ex. 4, the required curved surface

$$= \int_{x=0}^{h} 2\pi y \frac{ds}{dx} dx = 2\pi \int_0^h y \cdot \frac{2\sqrt{a}\sqrt{(x+a)}}{y} dx$$

$$= 4\pi\sqrt{a} \int_0^h (x+a)^{1/2} dx = \frac{8\pi}{3}\sqrt{a}\,[(h+a)^{3/2} - a^{3/2}].$$

**Ex. 6. *Find the surface generated by the revolution of an arc of the catenary $y = c \cosh (x/c)$ about the axis of $x$.*

**(Rohilkhand 1979, 78; Kanpur 77; Agra 79, 75)**

**Sol.** The given curve is, $y = c \cosh (x/c)$. ...(1)

Differentiating (1) w.r.t $x$, we get

$$\frac{dy}{dx} = c \sinh \frac{x}{c} \cdot \frac{1}{c} = \sinh \frac{x}{c}.$$

$$\therefore \quad \frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} = \sqrt{\left\{1 + \sinh^2 \frac{x}{c}\right\}} = \cosh \frac{x}{c} \quad \text{...(2)}$$

If the arc be measured from the vertex ($x = 0$) to any point $(x, y)$, then the required surface formed by the revolution of this arc about $x$-axis

$$= \int_{x=0}^{x} 2\pi y \frac{ds}{dx} dx = 2\pi \int_0^x c \cosh \frac{x}{c} \cdot \cosh \frac{x}{c} dx, \text{ from (1) and (2)}$$

$$= \pi c \int_0^x 2 \cosh^2 \frac{x}{c} dx = \pi c \int_0^x \left[1 + \cosh \frac{2x}{c}\right] dx \quad \text{(Note)}$$

$$= \pi c \left[x + \frac{c}{2} \sinh \frac{2x}{c}\right]_0^x = \pi c \left[x + \frac{c}{2} \sinh \frac{2x}{c}\right]$$

$$= \pi c \left[x + c \sinh \frac{x}{c} \cosh \frac{x}{c}\right].$$

**Ex. 7.** *Find the surface generated by the revolution of the curve $y = c \cosh (x/c)$ about the $x$-axis, between the planes $x = a$ and $x = b$.*

**Sol.** Proceeding exactly as in Ex. 6, the required surface

$$= 2\pi \int_{x=a}^{b} y \frac{ds}{dx} dx = \pi c \left[x + \frac{c}{2} \sinh \left(\frac{2x}{c}\right)\right]_a^b$$

$$= \pi c \left[(b-a) + \frac{c}{2} \sinh \frac{2b}{c} - \frac{c}{2} \sinh \frac{2a}{c}\right].$$

***Ex. 8.** *For a catenary $y = a \cosh (x/a)$, prove that*

$$aS = 2V = \pi a\,(ax + sy),$$

*where $s$ is the length of the arc from the vertex, $S$ and $V$ are respectively the area of the curved surface and volume of the solid generated by the revolution of the arc about $x$-axis.*

**Sol.** The given equation of catenary is $y = a \cosh (x/a)$. ...(1)

The vertex of the catenary (1) is the point $(0, a)$.

Differentiating (1) w.r.t. $x$ we get $dy/dx = \sinh (x/a)$.

$$\therefore \frac{ds}{dx}=\sqrt{\left\{1+\left(\frac{dy}{dx}\right)^2\right\}}=\sqrt{\left\{1+\sinh^2\frac{x}{a}\right\}}=\cosh\frac{x}{a}$$

$ds=\cosh(x/a)\,dx$.

If $s$ is the length of the arc from the vertex ($x=0$) to any point $(x, y)$ on the catenary, then we have

$$s=\int_0^x \cosh\left(\frac{x}{a}\right)dx=\left[a\sinh\frac{x}{a}\right]_0^a=a\sinh\frac{x}{a}. \quad ...(2)$$

Now $S$ = area of the curved surface of the solid generated by the revolution of the arc about $x$-axis

$$=\int_0^x 2\pi y\frac{ds}{dx}dx=\pi a\left[x+\frac{a}{2}\sinh\frac{2x}{a}\right] \quad ...(3)$$

[As proved in Ex. 6 page 168. Prove it here]

Also $V$ = the volume generated by the revolution of the arc about axis

$$=\int_0^x \pi y^2\,dx=\pi\int_0^x a^2\cosh^2\frac{x}{a}dx=\frac{\pi a^2}{2}\int_0^x 2\cosh^2\frac{x}{a}dx$$

$$=\frac{\pi a^2}{2}\int_0^x\left[1+\cosh\frac{2x}{a}\right]dx=\frac{\pi a^2}{2}\left[x+\frac{a}{2}\sinh\frac{2x}{a}\right]_0^x$$

$$=\frac{\pi a^2}{2}\left[x+\frac{a}{2}\sinh\frac{2x}{a}\right]. \quad ...(4)$$

From (3), $aS=\pi a^2[x+\frac{1}{2}a\sinh(2x/a)]$. ...(5)

From (4), $2V=\pi a^2[x+\frac{1}{2}a\sinh(2x/a)]=aS$. ...(6)

From (5) and (6), we have $aS=2V$.

Also from (2), $\pi a(ax+sy)=\pi a[ax+a^2\sinh(x/a)\cosh(x/a)]$,

[$\because y=a\cosh(x/a)$]

$$=\pi a\left[ax+\frac{a^2}{2}\cdot 2\sinh\frac{x}{a}\cosh\frac{x}{a}\right]$$

$=\pi a^2[x+(a/2)\sinh(2x/a)]=aS$, from (5).

Hence $aS=2V=\pi a(ax+sy)$.

**Ex. 9.** *Prove that the surface of the prolate spheroid formed by the revolution of the ellipse of eccentricity e about its major axis is equal to* $2\times$ *area of the ellipse* $\times[\sqrt{(1-e^2)}+(1/e)\sin^{-1}e]$.

**(Meerut 1983, 87P; Kanpur 74)**

**Sol.** **[Note : Prolate spheroid is generated by the revolution of an ellipse about its major axis]**

Let the equation of the ellipse be $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, ...(1)

the $x$-axis being the major axis so that $a>b$.

The parametric equations of (1) are $x = a\cos t, y = b\sin t$.

$\therefore \quad dx/dt = -a\sin t$ and $dy/dt = b\cos t$.

We have $\dfrac{ds}{dt} = \sqrt{\left\{\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2\right\}} = \sqrt{(a^2\sin^2 t + b^2\cos^2 t)}$

$= \sqrt{\{a^2\sin^2 t + a^2(1-e^2)\cos^2 t\}}$,

$[\because$ for the ellipse $b^2 = a^2(1-e^2)]$

$= a\sqrt{(1 - e^2\cos^2 t)}$. ...(2)

Now the ellipse (1) is symmetrical about $y$-axis and for the arc of the ellipse lying in the first quadrant $t$ varies from 0 to $\pi/2$. At the point $(a, 0)$ we have $t = 0$ and at the point $(0, b)$ we have $t = \pi/2$.

Hence the required surface $S$ formed by the revolution of the ellipse (1) about the $x$-axis

$= 2\int 2\pi y\, ds$ between the suitable limits

$$= 4\pi\int_0^{\pi/2} y\frac{ds}{dt}dt = 4\pi\int_0^{\pi/2} b\sin t \,.\, a\sqrt{(1-e^2\cos^2 t)}\,dt,$$

$[\because y = b\sin t$ and $ds/dt = a\sqrt{(1-e^2\cos^2 t)}$, from (2)]

$$= 4\pi ab\int_0^{\pi/2}\sin t\sqrt{(1-e^2\cos^2 t)}\,dt.$$

Put $e\cos t = z$ so that $-e\sin t\,dt = dz$. When $t = 0, z = e$ and when $t = \frac{1}{2}\pi$, $z = 0$.

$$\therefore \quad S = -4\pi ab\int_e^0 \frac{1}{e}\sqrt{(1-z^2)}\,dz = \frac{4\pi ab}{e}\int_0^e\sqrt{(1-z^2)}\,dz$$

$$= \frac{4\pi ab}{e}\left[\frac{z}{2}\sqrt{(1-z^2)} + \tfrac{1}{2}\sin^{-1} z\right]_0^e$$

$$= \frac{4\pi ab}{e}\left[\frac{e}{2}\sqrt{(1-e^2)} + \tfrac{1}{2}\sin^{-1} e\right]$$

$$= 2\pi ab\,[\sqrt{(1-e^2)} + (1/e)\sin^{-1} e]$$

$= 2 \times$ area of the ellipse $\times [\sqrt{(1-e^2)} + (1/e)\sin^{-1} e]$.

**Remark.** The solid of revolution formed by revolving an ellipse about its minor axis is called **an oblate spheroid.**

***Ex. 10.** *Find the surface of the solid generated by the revolution of the ellipse $x^2 + 4y^2 = 16$ about its major axis.*

**(Meerut 1978, 85, 98; Gorakhpur 74; Vikram 72; Kanpur 79)**

**Sol.** The given ellipse is $x^2 + 4y^2 = 16$. ...(1)

The equation (1) of the ellipse can be written as

$$\frac{x^2}{16} + \frac{y^2}{4} = 1.$$

Comparing it with the standard form of the equation of the ellipse $x^2/a^2 + y^2/b^2 = 1$, we see that $a = 4, b = 2$ and the major axis is along the $x$-axis. So we have to revolve the curve (1) about $x$-axis.

Differentiating (1) w.r.t. $x$, we get

$$2x + 8y\frac{dy}{dx} = 0 \quad \text{or} \quad \frac{dy}{dx} = -\frac{x}{4y}.$$

$$\therefore \quad \frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} = \sqrt{\left\{1 + \frac{x^2}{16y^2}\right\}} = \sqrt{\left(\frac{16y^2 + x^2}{16y^2}\right)}$$

$$= \sqrt{\{(64 - 4x^2) + x^2\}}/4y = \sqrt{(64 - 3x^2)}/4y. \quad \text{...(2)}$$

Now the ellipse (1) is symmetrical about both axes and for the arc of the ellipse lying in the first quadrant $x$ varies from 0 to 4.

$\therefore$ the required surface $= 2 \times$ surface generated by the revolution of the arc in the first quadrant

$$= 2\int_{x=0}^{4} 2\pi y \frac{ds}{dx}\,dx = 4\pi \int_0^4 y \cdot \frac{\sqrt{(64 - 3x^2)}}{4y}\,dx, \text{ from (2)}$$

$$= \pi \int_0^4 \sqrt{(64 - 3x^2)}\,dx = \pi\sqrt{3}\int_0^4 \sqrt{\left\{\left(\frac{8}{\sqrt{3}}\right)^2 - x^2\right\}}\,dx$$

$$= \pi\sqrt{3}\left[\frac{x}{2}\sqrt{\left\{\left(\frac{8}{\sqrt{3}}\right)^2 - x^2\right\}} + \frac{1}{2}\cdot\frac{64}{3}\sin^{-1}\left(\frac{x\sqrt{3}}{8}\right)\right]_0^4$$

$$= \pi\sqrt{3}\left[2\sqrt{\left(\frac{64}{3} - 16\right)} + \frac{32}{3}\sin^{-1}\left(\frac{\sqrt{3}}{2}\right)\right]$$

$$= \pi\sqrt{3}\left[\frac{8}{\sqrt{3}} + \frac{32}{3}\cdot\frac{\pi}{3}\right], \qquad \left[\because \sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}\right]$$

$$= 8\pi\left[1 + \frac{4\pi}{3\sqrt{3}}\right].$$

**Examples on revolution about y-axis.**

**Ex. 11.** *Find the surface of the solid formed by the revolution, about the axis of y, of the part of the curve $ay^2 = x^3$ from $x = 0$ to $x = 4a$ which is above the x-axis.* **(Kanpur 1978; Meerut 84R)**

**Sol.** The given curve is $ay^2 = x^3$. ...(1)

Differentiating (1) w.r.t. $x$, we get

$$2ay\frac{dy}{dx} = 3x^2 \quad \text{or} \quad \frac{dy}{dx} = \frac{3x^2}{2ay}.$$

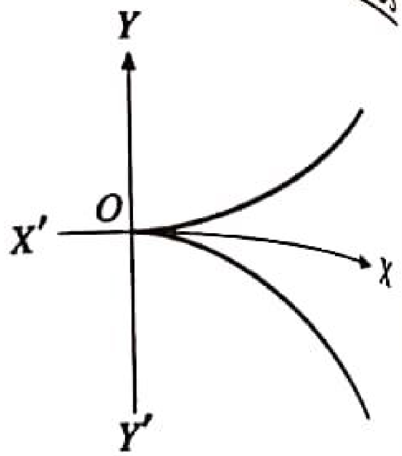
$$\therefore \quad \frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} = \sqrt{\left\{1 + \frac{9x^4}{4a^2y^2}\right\}}$$

$$= \sqrt{\left\{1 + \frac{9x^4}{4a \cdot x^3}\right\}}, \text{ from (1)}$$

$$= \sqrt{\left(1 + \frac{9x}{4a}\right)} = \frac{1}{2\sqrt{a}}\sqrt{(4a + 9x)}.$$

$\therefore$ the required surface

$$= \int_{x=0}^{4a} 2\pi x \, ds = \int_0^{4a} 2\pi x \frac{ds}{dx} dx \quad \textbf{(Note)}$$




$$= \int_0^{4a} 2\pi x \cdot \frac{1}{2\sqrt{a}} \sqrt{(4a + 9x)}\, dx$$

$$= \frac{\pi}{\sqrt{a}} \int_0^{a} x\sqrt{(4a + 9x)}\, dx.$$

Put $4a + 9x = t^2$ so that $9\,dx = 2t\,dt$. When $x = 0$, $t = \sqrt{(4a)}$ and when $x = 4a$, $t = \sqrt{(40a)}$.

$$\therefore \quad \text{the required surface} = \frac{\pi}{\sqrt{a}} \int_{\sqrt{(4a)}}^{\sqrt{(40a)}} \frac{t^2 - 4a}{9} \cdot t \cdot \frac{2t\,dt}{9}$$

$$= \frac{2\pi}{81\sqrt{a}} \int_{\sqrt{(4a)}}^{\sqrt{(40a)}} (t^4 - 4at^2)\, dt$$

$$= \frac{2\pi}{81\sqrt{a}} \left[\frac{1}{5}t^5 - \frac{4}{3}at^3\right]_{\sqrt{(4a)}}^{\sqrt{(40a)}}$$

$$= \frac{128}{1215}\pi a^2 [125\sqrt{(10)} + 1].$$

**Ex. 12.** *The curve $ay^2 = x^3$ revolves about the axis of $y$; find the surface area and the volume generated between the planes perpendicular to the axis of revolution at the origin and through the point where $27y = 8a$.*

**Sol.** The given curve is $ay^2 = x^3$. ...(1)

As proved in Ex. 11, $\dfrac{ds}{dx} = \sqrt{\left(1 + \dfrac{9x}{4a}\right)}$.

Let us find the value of $x$ at the point on the curve (1) where $27y = 8a$.

Putting $y = 8a/27$ in (1), we get

$$64a^3/27^2 = x^3 \quad \text{or} \quad x = 4a/9.$$

$\therefore$ the lines perpendicular to the axis of revolution, at the origin and through the point where $27y = 8a$ meet the curve (1) at the points where $x = 0$ and $x = 4a/9$.

$$\therefore \quad \text{the required surface} = \int_{x=0}^{4a/9} 2\pi x \frac{ds}{dx} dx$$

$$= 2\pi \int_0^{4a/9} x \cdot \sqrt{\left(\frac{4a + 9x}{4a}\right)} dx = \frac{\pi}{\sqrt{a}} \int_0^{4a/9} x\sqrt{(4a + 9x)}\, dx.$$

Now put $4a + 9x = t^2$ so that $9dx = 2t\,dt$.

Also when $x = 0, t = 2\sqrt{a}$ and when $x = 4a/9, t = 2\sqrt{2}.\sqrt{a}$.

$\therefore$ the required surface

$$= \frac{\pi}{\sqrt{a}} \int_{2\sqrt{a}}^{2\sqrt{2}\sqrt{a}} \frac{(t^2 - 4a)}{9} \cdot t \cdot \frac{2t\, dt}{9}$$

$$= \frac{2\pi}{81\sqrt{a}} \int_{2\sqrt{a}}^{2\sqrt{2}\sqrt{a}} (t^4 - 4at^2)\, dt = \frac{2\pi}{81\sqrt{a}} \left[\frac{t^5}{5} - \frac{4at^3}{3}\right]_{2\sqrt{a}}^{2\sqrt{2}\sqrt{a}}$$

$$= \frac{2\pi}{81\sqrt{a}} \left[\left\{\frac{(2\sqrt{2}\sqrt{a})^5}{5} - \frac{4a}{3}(2\sqrt{2}\sqrt{a})^3\right\} - \left\{\frac{(2\sqrt{a})^5}{5} - \frac{4a\,(2\sqrt{a})^3}{3}\right\}\right]$$

$$= \frac{2\pi}{81\sqrt{a}} \left[\frac{128\sqrt{2}.a^2\sqrt{a}}{5} - \frac{64\sqrt{2}a^2\sqrt{a}}{3} - \frac{32a^2\sqrt{a}}{5} + \frac{32a^2\sqrt{a}}{3}\right]$$

$$= \frac{2\pi a^2}{81 \times 15} [384\sqrt{2} - 320\sqrt{2} - 96 + 160]$$

$$= \frac{2\pi a^2}{1215} [64\sqrt{2} + 64] = \frac{128\pi a^2}{1215} [\sqrt{2} + 1].$$

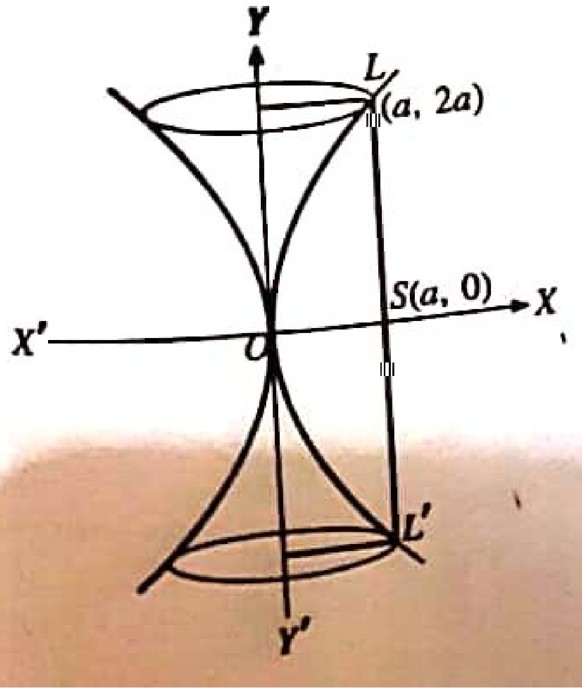
****Ex. 13.** *The part of the parabola $y^2 = 4ax$ cut off by the latus rectum revolves about the tangent at the vertex. Find the curved surface of the reel thus generated.* **(Allahabad 1973; Meerut 84S)**

**Sol.** The given parabola is $y^2 = 4ax$. ...(1)

Differentiating (1) w.r.t. $x$, we get $dy/dx = 2a/y$.

$$\therefore \quad \frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}} = \sqrt{\left\{1 + \frac{4a^2}{y^2}\right\}}$$

$$= \sqrt{\left\{1 + \frac{4a^2}{4ax}\right\}} = \sqrt{\left(\frac{x+a}{x}\right)}.$$



The required curved surface is generated by the revolution of the arc $LOL'$ ($LSL'$ is the latus rectum), about the tangent at the vertex *i.e.*, $y$-axis. The curve is symmetrical about $x$-axis and for the arc $OL$, $x$ varies from 0 to $a$.

$\therefore$ the required surface $= 2\int_{x=0}^{a} 2\pi x \dfrac{ds}{dx}\,dx$

$$= 4\pi \int_0^a x\sqrt{\left(\frac{x+a}{x}\right)}\,dx = 4\pi\int_0^a \sqrt{(x^2+ax)}\,dx$$

$$= 4\pi \int_0^a \sqrt{\left\{\left(x+\frac{a}{2}\right)^2 - \left(\frac{a}{2}\right)^2\right\}}\,dx \qquad \text{(Note)}$$

$$= 4\pi\left[\tfrac{1}{2}\left(x+\frac{a}{2}\right)\sqrt{(x^2+ax)} - \tfrac{1}{2}\cdot\frac{a^2}{4}\log\left\{\left(x+\frac{a}{2}\right)+\sqrt{(x^2+ax)}\right\}\right]_0^a$$

$$\left[\because \int \sqrt{(x^2-a^2)}\,dx = \tfrac{1}{2}x\sqrt{(x^2-a^2)} - \tfrac{1}{2}a^2\log\{x+\sqrt{(x^2-a^2)}\}\right]$$

$$= 4\pi\left[\tfrac{1}{2}\cdot\tfrac{3}{2}a\,a\sqrt{2} - \tfrac{1}{8}a^2\log\{\tfrac{3}{2}a + a\sqrt{2}\} + \tfrac{1}{8}a^2\log(\tfrac{1}{2}a)\right]$$

$$= 4\pi\left[\tfrac{3}{4}a^2\sqrt{2} - \tfrac{1}{8}a^2\log\{(\tfrac{3}{2}a + a\sqrt{2})/(\tfrac{1}{2}a)\}\right]$$

$$= \pi a^2\left[3\sqrt{2} - \tfrac{1}{2}\log(3+2\sqrt{2})\right]$$

$$= \pi a^2\left[3\sqrt{2} - \tfrac{1}{2}\log(\sqrt{2}+1)^2\right] \qquad \text{(Note)}$$

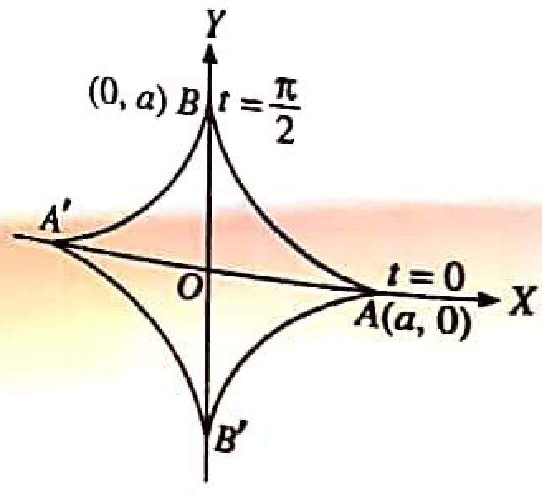
$$= \pi a^2\left[3\sqrt{2} - \log(\sqrt{2}+1)\right].$$

**Examples on Surfaces of Revolution (*parametric equations*)**

****Ex. 14.** *Find the surface of the solid generated by revolution of the astroid $x^{2/3}+y^{2/3}=a^{2/3}$ or $x = a\cos^3 t$, $y = a\sin^3 t$ about the $x$-axis.*

**(Garhwal 1983; Kumayun 83; Meerut 77, 82S, 84, 85S, 86, 88; Delhi 81; Lucknow 76; Vikram 76; Gorakhpur 73; Agra 73; Kanpur 72)**

**Sol.** The parametric equations of the the curve are
$x = a\cos^3 t$, $y = a\sin^3 t$.



$\therefore$ $\dfrac{dx}{dt} = -3a\cos^2 t\sin t$ and $\dfrac{dy}{dt} = 3a\sin^2 t\cos t$.

Hence $\dfrac{ds}{dt} = \sqrt{\left[\left(\dfrac{dx}{dt}\right)^2 + \left(\dfrac{dy}{dt}\right)^2\right]}$

$= \sqrt{[9a^2\cos^4 t\sin^2 t + 9a^2\sin^4 t\cos^2 t]}$

$= \sqrt{[9a^2\sin^2 t\cos^2 t(\cos^2 t + \sin^2 t)]} = 3a\sin t\cos t$.

Also the given curve (astroid) is symmetrical about both the axes and for the curve in the first quadrant, $t$ varies from 0 to $\pi/2$.

$\therefore$ the required surface $= 2\displaystyle\int_{t=0}^{\pi/2} 2\pi y\frac{ds}{dt}dt$

$= 4\pi\displaystyle\int_0^{\pi/2} a\sin^3 t \,.\, 3a\sin t\cos t\,dt = 12\pi a^2\int_0^{\pi/2}\sin^4 t\cos t\,dt$

$= 12\pi a^2\left[\dfrac{\sin^5 t}{5}\right]_0^{\pi/2} = 12\pi a^2\left[\frac{1}{5} - 0\right] = \dfrac{12\pi a^2}{5}$.

**Ex. 15. *Find the surface area of the solid generated by revolving the cycloid $x = a(\theta - \sin\theta), y = a(1 - \cos\theta)$ about the x-axis.*

**(G.N.U. 1973; Delhi 71; Meerut 85)**

Sol. The given parametric equations of the cycloid are

$$x = a(\theta - \sin\theta), y = a(1 - \cos\theta). \quad ...(1)$$

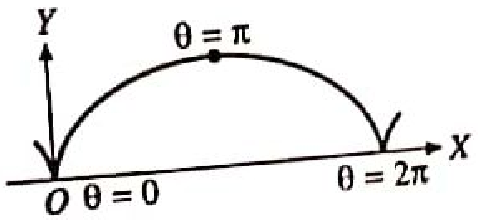
$\therefore$ $\dfrac{dx}{d\theta} = a(1 - \cos\theta)$ and $\dfrac{dy}{d\theta} = a\sin\theta$.

Hence $\dfrac{ds}{d\theta} = \sqrt{\left[\left(\dfrac{dx}{d\theta}\right)^2 + \left(\dfrac{dy}{d\theta}\right)^2\right]} = \sqrt{[a^2(1-\cos\theta)^2 + a^2\sin^2\theta]}$

$= a\sqrt{[1 - 2\cos\theta + \cos^2\theta + \sin^2\theta]} = a\sqrt{[2(1-\cos\theta)]}$

$= a\sqrt{[2\,.\,2\sin^2(\theta/2)]} = 2a\sin(\theta/2)$. ...(2)

We have $y = 0$ when $1 - \cos\theta = 0$ *i.e.*, $\cos\theta = 1$ giving $\theta = 0$ and $2\pi$. When $\theta = 0, x = 0$ and when $\theta = 2\pi, x = 2a\pi$. Also $y$ is maximum when $\cos\theta = -1$ *i.e.*, $\theta = \pi$ and then $y = 2a$ and $x = a\pi$. Thus for one arch of the given curve $\theta$ varies from 0 to $2\pi$ and this arch is symmetrical about the line $x = a\pi$ which meets the curve at the point $\theta = \pi$.




$\therefore$ the required surface $= 2\displaystyle\int_0^{\pi} 2\pi y\frac{ds}{d\theta}d\theta$

$= 4\pi\displaystyle\int_0^{\pi} a(1-\cos\theta)\,.\,2a\sin(\theta/2)\,d\theta$, from (1) and (2)

$= 8\pi a^2\displaystyle\int_0^{\pi} 2\sin^2\frac{\theta}{2}\cdot\sin\frac{\theta}{2}d\theta = 16\pi a^2\int_0^{\pi}\sin^3\frac{\theta}{2}d\theta$

$= 16\pi a^2 \int_0^{\pi/2} \sin^3 t \,.\, 2dt$, putting $\frac{\theta}{2} = t$ so that $d\theta = 2dt$

$= 32\pi a^2 \cdot \frac{2}{3.1} = \frac{64\pi a^2}{3}$.

**Ex. 16.** *Find the area of the surface generated by revolving an arch of the cycloid $x = a(\theta + \sin\theta), y = a(1 - \cos\theta)$ about the tangent at its vertex.*

**Sol.** The given parametric equations of the cycloid are

$$x = a(\theta + \sin\theta), y = a(1 - \cos\theta).$$

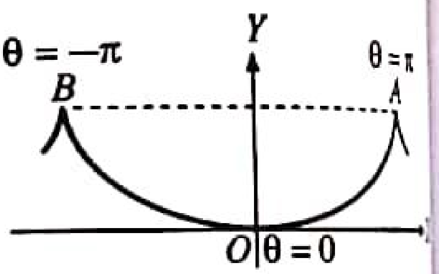
$\therefore \quad \frac{dx}{d\theta} = a(1 + \cos\theta)$ and $\frac{dy}{d\theta} = a\sin\theta$.

Hence $\frac{ds}{d\theta} = \sqrt{\left[\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right]} = \sqrt{[a^2(1+\cos\theta)^2 + a^2\sin^2\theta]}$

$= a\sqrt{[1 + 2\cos\theta + \cos^2\theta + \sin^2\theta]} = a \,.\, \sqrt{[2(1+\cos\theta)]}$

$= a \,.\, \sqrt{\{2 \,.\, 2\cos^2(\theta/2)\}} = 2a\cos(\theta/2)$.

Also for one arch of the given curve, $\theta$ varies from $-\pi$ to $\pi$ and this arch is symmetrical about the $y$-axis which meets the curve at the point $\theta = 0$.




$\therefore$ the required surface

$= 2\int_0^{\pi} 2\pi y \frac{ds}{d\theta} d\theta$

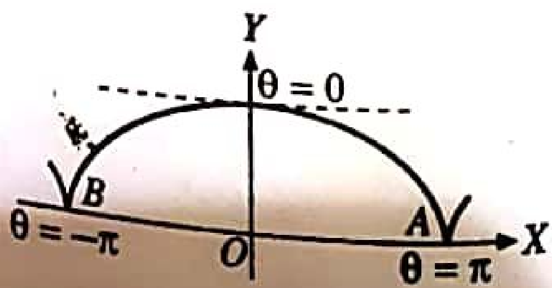
$= 4\pi \int_0^{\pi} a(1 - \cos\theta) \,.\, 2a\cos(\theta/2)\, d\theta$, from (1) and (2)

$= 8\pi a^2 \int_0^{\pi} 2\sin^2\frac{\theta}{2} \cdot \cos\frac{\theta}{2} d\theta = 16\pi a^2 \int_0^{\pi} \sin^2\frac{\theta}{2}\cos\frac{\theta}{2} d\theta$

$= 16\pi a^2 \int_0^{\pi/2} \sin^2 t \cos t \,.\, 2\,dt$, putting $\frac{\theta}{2} = t$ so that $d\theta = 2dt$

$= 32\pi a^2 \int_0^{\pi/2} \sin^2 t \cos t\, dt = 32\pi a^2 \cdot \frac{1}{3.1} = \frac{32\pi a^2}{3}$.

***Ex. 17.** *The portion between the consecutive cusps of the cycloid $x = a(\theta + \sin\theta), y = a(1 + \cos\theta)$ is revolved about the x-axis. Prove that the area of the surface so formed is to the area of the cycloid as 64 : 9*



**Sol.** The given parametric equations of the cycloid are
$$x = a(\theta + \sin\theta),\ y = a(1 + \cos\theta). \qquad ...(1)$$

$\therefore$ $\dfrac{dx}{d\theta} = a(1 + \cos\theta),\ \dfrac{dy}{d\theta} = -a\sin\theta.$

Hence $\dfrac{ds}{d\theta} = \sqrt{\left\{\left(\dfrac{dx}{d\theta}\right)^2 + \left(\dfrac{dy}{d\theta}\right)^2\right\}} = \sqrt{[a^2(1 + \cos\theta)^2 + a^2\sin^2\theta]}$

$= a\sqrt{[1 + 2\cos\theta + \cos^2\theta + \sin^2\theta]} = a\sqrt{[2(1 + \cos\theta)]}$

$= a\sqrt{[2 \cdot 2\cos^2(\theta/2)]} = 2a\cos(\theta/2).$ ...(2)

For one arch of the given curve (*i.e.*, for the portion between two successive cusps) $\theta$ varies from $-\pi$ to $\pi$. Also this arch is symmetrical about the $y$-axis which meets the arch at the point where $\theta = 0$. The base of the given cycloid is the axis of $x$.

$\therefore$ the surface $S$ generated by the revolution of the cycloid about the $x$-axis $= 2\displaystyle\int_0^{\pi} 2\pi y \frac{ds}{d\theta}\, d\theta$

$= 4\pi \int_0^{\pi} a(1 + \cos\theta) \cdot 2a\cos(\theta/2)\, d\theta$, from (1) and (2)

$= 8\pi a^2 \displaystyle\int_0^{\pi} 2\cos^2\frac{\theta}{2}\cos\frac{\theta}{2}\, d\theta = 16\pi a^2 \int_0^{\pi} \cos^3\frac{\theta}{2}\, d\theta$

$= 16\pi a^2 \int_0^{\pi/2} (\cos^3 t) \cdot 2\, dt$, putting $\theta/2 = t$ so that $d\theta = 2\, dt$

$= 32\pi a^2 \cdot \dfrac{2}{3.1} = \dfrac{64\pi a^2}{3}.$

Also the area $A$ of the given cycloid

$= 2\displaystyle\int_0^{\pi} y\frac{dx}{d\theta}\, d\theta = 2\int_0^{\pi} a(1 + \cos\theta) \cdot a(1 + \cos\theta)\, d\theta$

$= 2a^2 \int_0^{\pi} (1 + \cos\theta)^2\, d\theta = 2a^2 \displaystyle\int_0^{\pi} \left(2\cos^2\frac{\theta}{2}\right)^2 d\theta$

$= 8a^2 \displaystyle\int_0^{\pi} \cos^4\frac{\theta}{2}\, d\theta$

$= 8a^2 \int_0^{\pi/2} (\cos^4 t) \cdot 2\, dt$, putting $t = \theta/2$

$= 16a^2 \cdot \dfrac{3.1}{4.2} \cdot \dfrac{\pi}{2} = 3\pi a^2.$ ...(4)

From (3) and (4), we get

the required ratio $= \dfrac{S}{A} = \dfrac{\frac{64}{3}\pi a^2}{3\pi a^2} = \dfrac{64}{9}.$

***Ex. 18.** *Prove that the surface area of the solid generated by the revolution, about the x-axis of the loop of the curve*

$x = t^2, y = t - \frac{1}{3}t^3$ *is* $3\pi$.

**(Agra 1976; Kanpur 75, 80; Jiwaji 71; Meerut 87S)**

**Sol.** The given equations of the curve are

$$x = t^2, y = t - \tfrac{1}{3}t^3. \qquad \text{...(1)}$$

$\therefore \quad dx/dt = 2t$ and $dy/dt = 1 - t^2$.

Hence $$\frac{ds}{dt} = \sqrt{\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right]} = \sqrt{[(2t)^2 + (1-t^2)^2]}$$

$$= \sqrt{[4t^2 + 1 - 2t^2 + t^4]} = \sqrt{(1+t^2)^2} = (1+t^2). \qquad \text{...(2)}$$

Putting $y = 0$ in (1), we get $t - \frac{1}{3}t^3 = 0$ which gives $t = 0$ or $t = \pm\sqrt{3}$. For the upper half of the loop $y$ is positive and so for the upper half of the loop $t$ varies from 0 to $\sqrt{3}$.

$\therefore$ the required surface $= \displaystyle\int_0^{\sqrt{3}} 2\pi y \frac{ds}{dt}\, dt$

$$= 2\pi \int_0^{\sqrt{3}} (t - \tfrac{1}{3}t^3)(1+t^2)\, dt = 2\pi \int_0^{\sqrt{3}} (t + \tfrac{2}{3}t^3 - \tfrac{1}{3}t^5)\, dt$$

$$= 2\pi \left[\frac{t^2}{2} + \frac{2}{3}\cdot\frac{t^4}{4} - \frac{1}{3}\frac{t^6}{6}\right]_0^{\sqrt{3}} = 2\pi\left[\frac{t^2}{2} + \frac{t^4}{6} - \frac{t^6}{18}\right]_0^{\sqrt{3}}$$

$$= 2\pi\left[\frac{3}{2} + \frac{9}{6} - \frac{27}{18}\right] = 3\pi.$$

**Ex. 19.** *Prove that the surface of the solid generated by the revolution of the tractrix*

$$x = a\cos t + \tfrac{1}{2}a\log\tan^2\tfrac{1}{2}t,\ y = a\sin t$$

*about its asymptote is equal to the surface of a sphere of radius* $a$.

**(Meerut 1992; Rohilkhand 82; Agra 81, 74; Vikram 72)**

**Sol.** The given tractrix is

$$x = a\cos t + \tfrac{1}{2}a\log\tan^2\tfrac{1}{2}t,\ y = a\sin t.$$

$$\therefore \quad \frac{dx}{dt} = -a\sin t + a\frac{\sec^2\frac{1}{2}t}{\tan\frac{1}{2}t}\cdot\frac{1}{2} = a\left(-\sin t + \frac{1}{2\sin\frac{1}{2}t\cos\frac{1}{2}t}\right)$$

$$= a\left(-\sin t + \frac{1}{\sin t}\right) = a\frac{(-\sin^2 t + 1)}{\sin t} = \frac{a\cos^2 t}{\sin t}$$

and $\quad dy/dt = a\cos t$.

Hence $$\frac{ds}{dt} = \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}}$$

$$= \sqrt{\left\{\frac{a^2\cos^4 t}{\sin^2 t} + a^2\cos^2 t\right\}} = \frac{a\cos t}{\sin t}.$$

The given curve is symmetrical about both the axes and the asymptote is the line $y = 0$ *i.e.*, $x$-axis. For the arc of the curve lying in the second quadrant $t$ varies from 0 to $\frac{1}{2}\pi$.

$\therefore$ the required surface $= 2 . \int_0^{\pi/2} 2\pi y \frac{ds}{dt} dt$ **(Note)**

$= 4\pi \int_0^{\pi/2} a \sin t . \frac{a \cos t}{\sin t} dt = 4\pi a^2 \int_0^{\pi/2} \cos t \, dt$

$= 4\pi a^2 \left[\sin t\right]_0^{\pi/2} = 4\pi a^2$

= the surface of a sphere of radius $a$.

**Ex. 20.** *Prove that the surface of the oblate spheroid formed by the revolution of the ellipse of the semi-major axis a and eccentricity e is*

$$2\pi a^2 \left[1 + \frac{1 - e^2}{2e} \log \left(\frac{1 + e}{1 - e}\right)\right].$$

**Sol.** [Note : **Oblate spheroid** *is generated by the revolution of the ellipse about its minor axis.*]

Let the parametric equations of the ellipse be

$x = a \cos t, y = b \sin t$, where $b^2 = a^2 (1 - e^2)$. ...(1)

$\therefore$ $dx/dt = - a \sin t$ and $dy/dt = b \cos t$.

$\therefore$ $\frac{ds}{dt} = \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}} = \sqrt{\{a^2 \sin^2 t + b^2 \cos^2 t\}}$

$= \sqrt{\{a^2 \sin^2 t + a^2 (1 - e^2) \cos^2 t\}} = a \sqrt{(1 - e^2 \cos^2 t)}$ ...(2)

The ellipse is symmetrical about both the axes and for the arc of the ellipse lying in the first quadrant $t$ varies from 0 to $\pi/2$.

We have to revolve the ellipse about its minor axis which is the $y$-axis.

$\therefore$ the required surface $= 2 \int_0^{\pi/2} 2\pi x \frac{ds}{dt} dt$ **(Note)**

$= 4\pi \int_0^{\pi/2} a \cos t . a \sqrt{(1 - e^2 \cos^2 t)} \, dt$

$= 4\pi a^2 \int_0^{\pi/2} \sqrt{(1 - e^2 + e^2 \sin^2 t)} \cos t \, dt$

$= \frac{4\pi a^2}{e} \int_0^e \sqrt{[(1 - e^2) + z^2]} \, dz$, putting $e \sin t = z$ so that

$e \cos t \, dt = dz$

$= \frac{4\pi a^2}{e} \left[\frac{z}{2} \sqrt{\{(1 - e^2) + z^2\}} + \frac{1}{2} (1 - e^2) \log \{z + \sqrt{(1 - e^2 + z^2)}\}\right]_0^e$

$= \frac{2\pi a^2}{e} [e + (1 - e^2) \log (e + 1) - (1 - e^2) \log \sqrt{(1 - e^2)}]$

$$= \frac{2\pi a^2}{e}\left[e + (1 - e^2)\log\frac{1+e}{\sqrt{(1-e^2)}}\right]$$

$$= 2\pi a^2\left[1 + \frac{1-e^2}{e}\log\sqrt{\left(\frac{1+e}{1-e}\right)}\right]$$

$$= 2\pi a^2\left[1 + \frac{1-e^2}{2e}\log\frac{1+e}{1-e}\right].$$

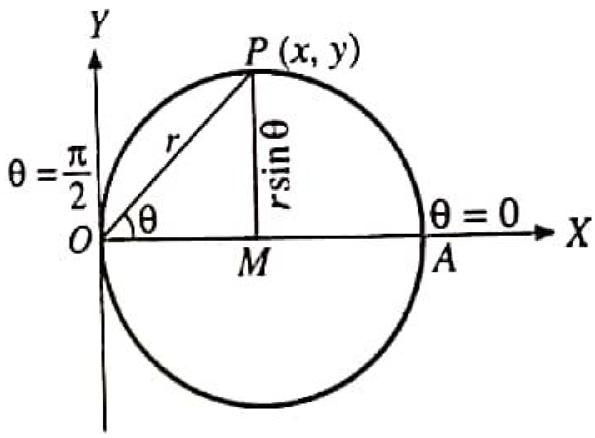
**Examples on surfaces of revolution (Polar equations).**

**Ex. 21.** *Find the area of the surface of revolution formed by revolving the curve $r = 2a\cos\theta$ about the initial line.*

(Meerut 199

**Sol.** The given curve is $r = 2a\cos\theta$, ...(1)

which is clearly a circle of radius $a$ passing through the pole and having diameter through the pole as initial line. At $O$, $r = 0$ and so (1) gives $\theta = \pi/2$ at $O$.




Differentiating (1) w.r.t $\theta$, we have $dr/d\theta = -2a\sin\theta$.

$$\therefore \quad \frac{ds}{d\theta} = \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}} = \sqrt{\{4a^2\cos^2\theta + 4a^2\sin^2\theta\}}$$

$$= 2a\sqrt{(\cos^2\theta + \sin^2\theta)} = 2a.$$

The given curve is revolved about the initial line (*i.e.*, the $x$-axis) and for the upper half of the curve, $\theta$ varies from 0 to $\pi/2$.

$\therefore$ the required surface $= \displaystyle\int_0^{\pi/2} 2\pi y \frac{ds}{d\theta}\, d\theta$, where $y = r\sin\theta$

$$= 2\pi\int_0^{\pi/2} r\sin\theta \,.\, 2a\, d\theta = 4a\pi\int_0^{\pi/2} 2a\cos\theta\sin\theta\, d\theta, \quad \text{from (1)}$$

$$= 8\pi a^2\int_0^{\pi/2}\sin\theta\cos\theta\, d\theta$$

$$= 8\pi a^2\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2} = 8\pi a^2\left(\tfrac{1}{2} - 0\right) = 4\pi a^2.$$

**Ex. 22.** *Find the surface of the solid generated by the revolution of lemniscate $r^2 = a^2 \cos 2\theta$ about the initial line.*

**(Meerut 1983S, 85P, 96 BP; Ranchi 73)**

**Sol.** The given curve is $r^2 = a^2 \cos 2\theta$. ...(1)

Differentiating (1) w.r.t. $\theta$, we get

$$2r\frac{dr}{d\theta} = -2a^2 \sin 2\theta \quad \text{or} \quad \frac{dr}{d\theta} = \frac{-a^2 \sin 2\theta}{r}.$$

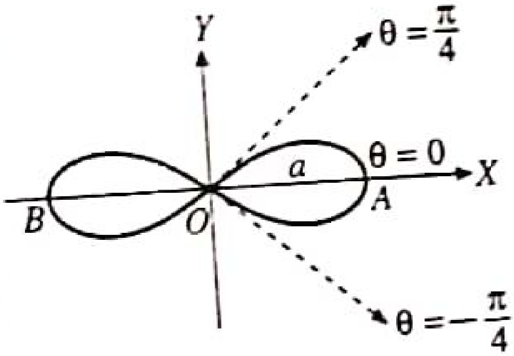
$$\therefore \frac{ds}{d\theta} = \sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}} = \sqrt{\left\{a^2 \cos 2\theta + \frac{a^4 \sin^2 2\theta}{r^2}\right\}}$$

$$= \frac{1}{r}\sqrt{\{r^2 . a^2 \cos 2\theta + a^4 \sin^2 2\theta\}}$$

$$= \frac{1}{r}\sqrt{\{a^4 \cos^2 2\theta + a^4 \sin^2 2\theta\}}, \qquad [\because r^2 = a^2 \cos 2\theta]$$

$$= a^2/r. \qquad ...(2)$$

The given curve is symmetrical about the initial line and about the pole.




Putting $r = 0$ in (1), we get $\cos 2\theta = 0$ giving $2\theta = \pm \frac{1}{2}\pi$ *i.e.*, $\theta = \pm \frac{1}{4}\pi$.

Therefore one loop of the curve lies between $\theta = -\frac{1}{4}\pi$ and $\theta = \frac{1}{4}\pi$.

There are two loops in the curve and for the upper half of one of these two loops $\theta$ varies from 0 to $\frac{1}{4}\pi$.

$\therefore$ the required surface $= 2 \times$ the surface generated by the revolution of one loop

$$= 2 . \int_0^{\pi/4} 2\pi y \frac{ds}{d\theta} d\theta, \text{ where } y = r \sin \theta$$

$$= 4\pi \int_0^{\pi/4} r \sin \theta \cdot \frac{a^2}{r} d\theta, \text{ from (2)}$$

$$= 4\pi a^2 \int_0^{\pi/4} \sin \theta \, d\theta = 4\pi a^2 \left[-\cos \theta\right]_0^{\pi/4}$$

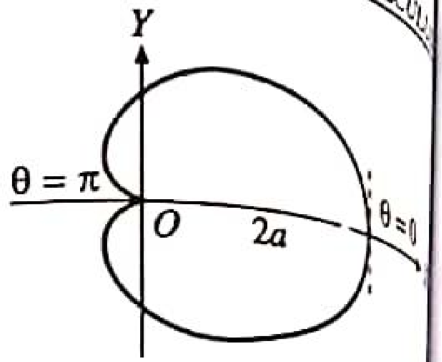
$$= 4\pi a^2 [-(1/\sqrt{2}) + 1] = 4\pi a^2 [1 - (1/\sqrt{2})].$$

****Ex. 23.** *Find the surface of the solid formed by the revolution of the cardioid $r = a(1 + \cos \theta)$ about the initial line.*

**(Meerut 1981; Agra 77; Gorakhpur 77, 75; Delhi 83, 80; Ranchi 75)**

**Sol.** The given curve is $r = a(1 + \cos \theta)$. ...(1)

It is symmetrical about the initial line and for the upper half of the curve, $\theta$ varies from 0 to $\pi$.




Differentiating (1) w.r.t. $\theta$, we get

$dr/d\theta = a\,(-\sin\theta) = -a\sin\theta.$

$$\therefore \quad \frac{ds}{d\theta} = \sqrt{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]}$$

$$= \sqrt{[a^2(1+\cos\theta)^2 + a^2\sin^2\theta]}$$

$$= a\sqrt{[2(1+\cos\theta)]} = 2a\cos\tfrac{1}{2}\theta.$$

$$\therefore \quad \text{the required surface} = \int_0^{\pi} 2\pi y\,\frac{ds}{d\theta}\,d\theta, \text{ where } y = r\sin\theta$$

$$= \int_0^{\pi} 2\pi\,.\,r\sin\theta\,.\,2a\cos\tfrac{1}{2}\theta\,d\theta$$

$$= 2\pi\int_0^{\pi} a\,(1+\cos\theta)\sin\theta\cos\tfrac{1}{2}\theta\,d\theta$$

$$= 4\pi a^2\int_0^{\pi} 2\cos^2\frac{\theta}{2}\cdot 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\cdot\cos\frac{\theta}{2}\,d\theta$$

$$= 16\pi a^2\int_0^{\pi}\cos^4\frac{\theta}{2}\sin\frac{\theta}{2}\,d\theta$$

$$= 16\pi a^2\int_0^{\pi/2}(\cos^4 t\sin t)\,.\,2\,dt, \text{ putting } \frac{\theta}{2} = t \text{ so that } d\theta = 2\,dt$$

$$= 32\pi a^2\int_0^{\pi/2}\cos^4 t\sin t\,dt = 32\pi a^2\cdot\frac{3.1.1}{5.3.1} = \frac{32\pi a^2}{5}.$$

**Ex. 24.** *Find the surface of the solid generated by the revolution of the curve $r = a\,(1-\cos\theta)$ about the initial line.* (Meerut 19..)

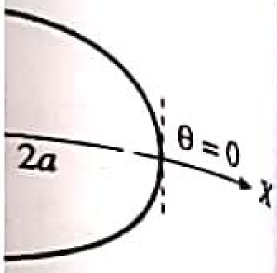
**Sol.** Proceed exactly as in Ex. 23. The required surface $= (32/5)\,\pi a^2$.

**Ex. 25.** *The arc of the cardioid $r = a\,(1+\cos\theta)$, specified by $-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi$ is rotated about the line $\theta = 0$, prove that the area of the surface generated is $\frac{4}{5}(8-\sqrt{2})\,\pi a^2$.*

**Sol.** Proceed as in Ex. 23. Obviously the surface generated by revolving the arc lying between $-\frac{1}{2}\pi \le \theta \le \frac{1}{2}\pi$ about the initial line is the same as that generated by revolving the arc lying between $\theta = 0$ and $\theta = \frac{1}{2}\pi$.

$$\therefore \quad \text{the required surface} = \int_0^{\pi/2} 2\pi y\,\frac{ds}{d\theta}\,d\theta$$

$$= 16\pi a^2\int_0^{\pi/2}\cos^4\frac{\theta}{2}\sin\frac{\theta}{2}\,d\theta, \text{ proceeding as in Ex. 23}$$




$$= 16\pi a^2 \left[\frac{-\cos^5 \frac{1}{2}\theta}{5 \cdot \frac{1}{2}}\right]_0^{\pi/2}$$

$$= -\frac{32}{5}\pi a^2 [(1/\sqrt{2})^5 - 1] = \frac{32\pi a^2}{5}\left(1 - \frac{1}{4\sqrt{2}}\right)$$

$$= \frac{4\pi a^2}{5}\left(8 - \frac{8}{4\sqrt{2}}\right) = \frac{4\pi a^2}{5}(8 - \sqrt{2}).$$

**Ex. 26.** *The arc of the cardioid $r = a(1 + \cos\theta)$, included between $-\frac{1}{2}\pi \le \theta \le \frac{1}{2}\pi$ is rotated about the line $\theta = \frac{1}{2}\pi$. Find the area of the surface generated.* **(Meerut 1988P)**

**Sol.** The given cardioid is $r = a(1 + \cos\theta)$, ...(1)

which is symmetrical about the initial line.

As proved in Ex. 23, $ds/d\theta = 2a \cos(\theta/2)$. (Prove it here).

The curve is rotated about the line $\theta = \pi/2$ *i.e.,* the $y$-axis and for the upper half of the cardioid lying in the first quadrant, $\theta$ varies from 0 to $\frac{1}{2}\pi$.

$$\therefore \text{ the required surface } = \int_0^{\pi/2} 2\pi x \frac{ds}{d\theta}\, d\theta, \qquad \textbf{(Note)}$$

$$= 4\pi \int_0^{\pi/2} r\cos\theta \cdot 2a\cos\frac{\theta}{2}\, d\theta, \qquad [\because x = r\cos\theta]$$

$$= 8\pi a \int_0^{\pi/2} a(1 + \cos\theta) \cdot \cos\theta \cdot \cos\frac{\theta}{2}\, d\theta, \qquad \text{from (1)}$$

$$= 8\pi a^2 \int_0^{\pi/2} 2\cos^2\frac{\theta}{2} \cdot \left(1 - 2\sin^2\frac{\theta}{2}\right)\cos\frac{\theta}{2}\, d\theta$$

$$= 16\pi a^2 \int_0^{\pi/2} \left(1 - \sin^2\frac{\theta}{2}\right)\left(1 - 2\sin^2\frac{\theta}{2}\right)\cos\frac{\theta}{2}\, d\theta \qquad \textbf{(Note)}$$

$$= 16\pi a^2 \int_0^{\pi/2} \left(1 - 3\sin^2\frac{\theta}{2} + 2\sin^4\frac{\theta}{2}\right)\cos\frac{\theta}{2}\, d\theta.$$

Put $\sin(\theta/2) = t$ so that $\frac{1}{2}\cos(\theta/2)\, d\theta = dt$.

Also when $\theta = 0, t = 0$ and when $\theta = \pi/2, t = 1/\sqrt{2}$.

$\therefore$ the required surface

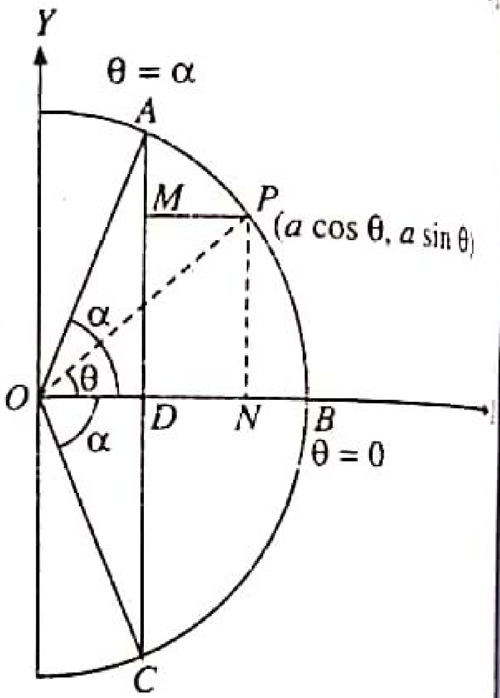
$$= 16\pi a^2 \int_0^{1/\sqrt{2}} (1 - 3t^2 + 2t^4) \cdot 2\, dt = 32\pi a^2 \left[t - t^3 + 2 \cdot \frac{t^5}{5}\right]_0^{1/\sqrt{2}}$$

$$= 32\pi a^2 \left[\frac{1}{\sqrt{2}} - \frac{1}{2\sqrt{2}} + \frac{1}{10\sqrt{2}}\right] = \frac{48\sqrt{2}}{5}\pi a^2.$$

**Examples on surfaces of revolution formed by rotation about any axis.**

**Ex. 27. *A circular arc revolves about its chord. Find the area of the surface generated, when $2\alpha$ is the angle subtended by the arc at the centre.*** (Me... 1983P, 85S, 86S, 87S, ...; Kanpur 76; Agra 72; Vikram ...)

**Sol.** Let the parametric equations of the circle be $x = a\cos\theta, y = a\sin\theta$, $\theta$ being the parameter.




Take any point $P(a\cos\theta, a\sin\theta)$ on the circular arc $ABC$ which is symmetrial about the $x$-axis and which subtends an angle $2\alpha$ at the centre $O$ so that $\angle AOB = \alpha$.

We have $OD = OA\cos\alpha = a\cos\alpha$.

Draw $PM$ perpendicular from $P$ to $AC$, the axis of rotation. Then $PM = ON - OD$

$= a\cos\theta - a\cos\alpha$. ...(2)

For the upper half of the arc to be rotated *i.e.*, for the arc $BA$, $\theta$ varies from 0 to $\alpha$.

Also $$\frac{ds}{d\theta} = \sqrt{\left\{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right\}}$$

$$= \sqrt{\{a^2\sin^2\theta + a^2\cos^2\theta\}} = a.$$

$\therefore$ the required surface $= 2 \times$ surface generated by the revolution of the arc $BA$ about the chord $AC$

$$= 2 \times \int_0^\alpha 2\pi (PM) \frac{ds}{d\theta} d\theta$$

$$= 4\pi \int_0^\alpha (a\cos\theta - a\cos\alpha) \,.\, a \,.\, d\theta,$$ from (2)

$$= 4\pi a^2 \left[\sin\theta - \theta\cos\alpha\right]_0^\alpha = 4\pi a^2 [\sin\alpha - \alpha\cos\alpha].$$

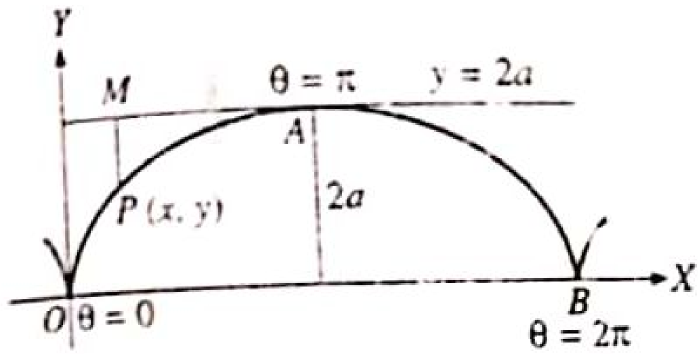
***Ex. 28.** *A quadrant of a circle of radius a revolves about its chord. Show that the surface of the spindle generated is*

$$2\pi a^2 \sqrt{2}\,(1 - \tfrac{1}{4}\pi).$$

**Sol.** Let us take the quadrant of the circle in such a way that it is placed symmetrially about the $x$-axis. The quadrant of the circle subtends an angle $\pi/2$ at the centre.

...w draw the figure as in Ex. 27 by taking $2a = \pi/2$ and then ...ed exactly in the same way as in Ex. 27 by taking $a = \pi/4$. Thus ...required result is obtained by putting $a = \pi/4$ in the result of Ex. ...

**Ex. 29.** *Find the area of the surface generated if an arch of the ... $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ revolves about the line ...* (Delhi 1982)




**Sol.** The given parametric equations of the cycloid are

$$x = a(\theta - \sin\theta),\ y = a(1 - \cos\theta). \qquad \text{...(1)}$$

Differentiating (1) w.r.t. $\theta$, we get

$dx/d\theta = a(1 - \cos\theta)$ and $dy/d\theta = a\sin\theta$.

$$\therefore \quad \frac{ds}{d\theta} = \sqrt{\left\{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right\}} = \sqrt{\{a^2(1 - \cos\theta)^2 + a^2\sin^2\theta\}}$$

$$= a\sqrt{\{2(1 - \cos\theta)\}} = a\sqrt{\{2 \cdot 2\sin^2(\theta/2)\}} = 2a\sin(\theta/2).$$

Take $P(x, y)$ as any point on the arc $OA$. Draw $PM$ perpendicular to the line $y = 2a$, which is tangent to the cycloid at the vertex $A$. Then $PM = 2a - y = 2a - a(1 - \cos\theta) = a(1 + \cos\theta)$.

Also the given cycloid is symmetrical about a line which is perpendicular to $x$-axis and which meets the curve at the point $A$ where $\theta = \pi$. For the arc $OA$, $\theta$ varies from 0 to $\pi$.

$\therefore$ the required surface

$= 2 \times$ the surface formed by the revolution of the arc $OA$ about the line $y = 2a$

$$= 2 \times \int_0^\pi 2\pi(PM)\frac{ds}{d\theta}\,d\theta = 4\pi\int_0^\pi a(1 + \cos\theta) \cdot 2a\sin\frac{\theta}{2}\,d\theta$$

$$= 8\pi a^2\int_0^\pi 2\cos^2\frac{\theta}{2} \cdot \sin\frac{\theta}{2}\,d\theta$$

$$= -32\pi a^2\int_0^\pi \cos^2\frac{\theta}{2}\left(-\frac{1}{2}\sin\frac{\theta}{2}\right)d\theta$$

making adjustment for the application of the power formula

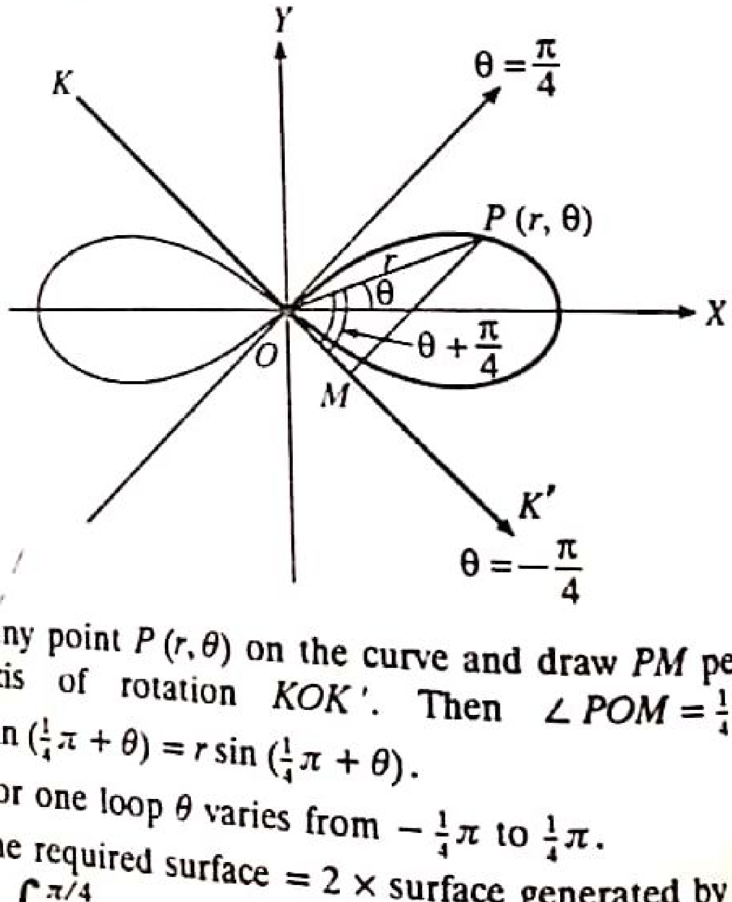
$$= -32\pi a^2 \left[\frac{\cos^3 \frac{1}{2}\theta}{3}\right]_0^{\pi} = -\frac{32}{3}\pi a^2 [0-1] = \frac{32}{3}\pi a^2.$$

**Ex. 30.** *The lemniscate $r^2 = a^2 \cos 2\theta$ revolves about a tangent at the pole. Show that the surface of the solid generated is $4\pi a^2$.* **(Meerut 1986, 87, 93)**

**Sol.** The given curve is $r^2 = a^2 \cos 2\theta$. ...(1)

Proceeding as in Ex. 22, we get $ds/d\theta = a^2/r$.

Putting $r = 0$ in (1), we get $\cos 2\theta = 0$ giving $2\theta = \pm\frac{1}{2}\pi$ i.e., $\theta = \pm\frac{1}{4}\pi$. Therefore one loop of the curve lies between $\theta = -\frac{1}{4}\pi$ and $\theta = \frac{1}{4}\pi$. The curve consists of two loops and both the lines $\theta = -\frac{1}{4}\pi$ and $\theta = \frac{1}{4}\pi$ are tangents at the pole. Let the curve be revolved about the line $KOK'$ which is a tangent at the pole.




Take any point $P(r, \theta)$ on the curve and draw $PM$ perpendicular to the axis of rotation $KOK'$. Then $\angle POM = \frac{1}{4}\pi + \theta$ and $PM = OP \sin(\frac{1}{4}\pi + \theta) = r \sin(\frac{1}{4}\pi + \theta)$.

Also for one loop $\theta$ varies from $-\frac{1}{4}\pi$ to $\frac{1}{4}\pi$.

$\therefore$ the required surface $= 2 \times$ surface generated by one loop

$$= 2 \times \int_{-\pi/4}^{\pi/4} 2\pi (PM) \frac{ds}{d\theta} d\theta$$

$$= 4\pi \int_{-\pi/4}^{\pi/4} r \sin(\tfrac{1}{4}\pi + \theta) \cdot \frac{a^2}{r} d\theta,$$

$$\left[\because \frac{ds}{d\theta} = \frac{a^2}{r}, PM = r \sin(\tfrac{1}{4}\pi + \theta\right.$$

$$= 4\pi a^2 \int_{-\pi/4}^{\pi/4} \sin(\tfrac{1}{4}\pi + \theta)\, d\theta = 4\pi a^2 \left[ -\cos(\tfrac{1}{4}\pi + \theta) \right]_{-\pi/4}^{\pi/4}$$

$$= 4\pi a^2 [0 + 1] = 4\pi a^2.$$

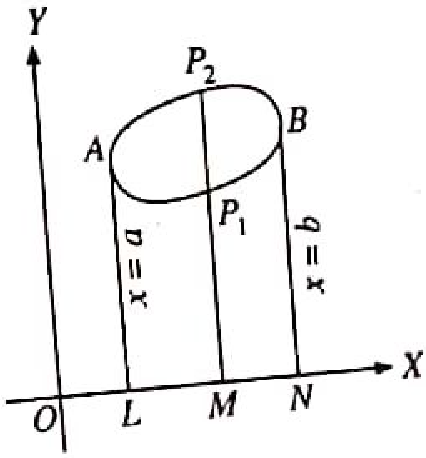
## § 4. Theorems of Pappus or Guldin

**Theorem 1. Volume of a Solid of Revolution :**

*If a closed plane curve revolves about a straight line in its plane which does not intersect it, the volume of the ring thus obtained is equal to the area of the region enclosed by the curve multiplied by the length of the path described by the centroid of the region.*

**(Kanpur 1980; Vikram 74; Utkal 73)**

**Proof.** Let $AP_1BP_2A$ be the closed plane curve and let it rotate about the axis of $x$.




Let $AL$ $(x = a)$ and $BN$ $(x = b)$ be the tangents to the curve parallel to the $y$-axis $(a < b)$. Also let any ordinate meet the curve at $P_1, P_2$ and let $MP_1 = y_1$, $MP_2 = y_2$ so that $y_1, y_2$ are functions of $x$.

Now volume of the ring generated by the revolution of the closed curve $AP_1BP_2A$ about the axis of $x$ = volume generated by the area $ALNBP_2A$ − volume generated by the area $ALNBP_1A$

$$= \pi \int_a^b y_2^2\, dx - \pi \int_a^b y_1^2\, dx = \pi \int_a^b (y_2^2 - y_1^2)\, dx. \qquad ...(1)$$

Also if $\bar{y}$ be the ordinate of the centroid of the area of the closed curve, then

$$\bar{y} = \frac{\int_a^b \frac{1}{2}(y_1 + y_2)(y_2 - y_1)\, dx}{A} = \frac{\frac{1}{2}\int_a^b (y_2^2 - y_1^2)\, dx}{A}, \qquad ...(2)$$

where $A$ is the area of the closed curve.

[See the chapter on centre of gravity.]

Hence from (1) and (2), the required volume

$= 2\pi A \bar{y} = A \times 2\pi \bar{y}$

= area of the closed curve × circumference of the circle of radius $\bar{y}$

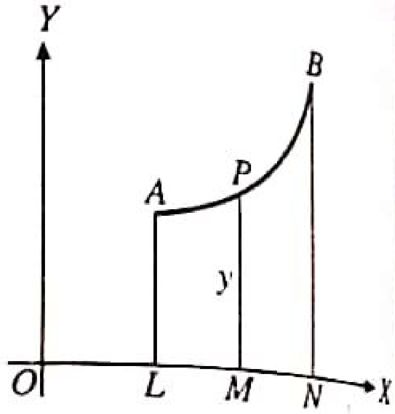
= area of the curve × length of the arc described by the centroid of the region bounded by the closed curve.

**Theorem 2. Surface of a solid of revolution :**

*If an arc of a plane curve revolves about a straight line in its plane, which does not intersect it, the surface of the solid thus obtained is equal* *to the arc multiplied by the length of the path described by the centroid of the arc.* **(Kanpur 1980; Indore 72)**

**Proof.** Let $l$ be the length of the arc $AB$ and let it revolve about $OX$.

Let the abscissae of the extremities $A$ and $B$ of the arc be $a$ and $b$.




Then the surface generated by the revolution of the arc $AB$ about $x$-axis is

$$= \int_{x=a}^{x=b} 2\pi y \, ds \qquad \text{...(1)}$$

Also we know that (see the chapter on centre of gravity) the ordinate $\bar{y}$, of the centroid of the arc from $x = a$ to $x = b$, of length $l$, is given by

$$\bar{y} = \frac{\int_{x=a}^{b} y \, ds}{l} \qquad \text{...(2)}$$

From (1) and (2), we get the required surface

$$= 2\pi\bar{y}l = l \times 2\pi\bar{y}$$

= length of the arc × length of the path described by the centroid of the arc

**Note 1.** The closed curve or arc in the above theorems must not cross the axis of revolution but may be terminated by it.

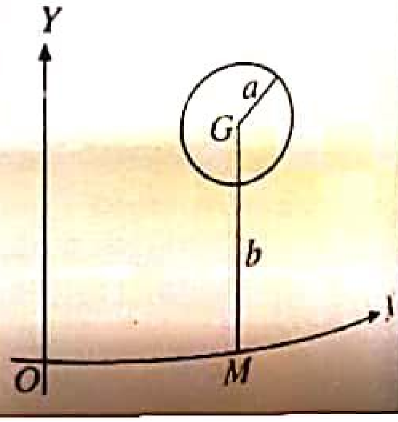
**Note 2.** When the volume or surface generated is known, the theorems may be applied to find the position of the centroid of the generating area or arc.

***Ex.** *State and prove the theorems of Pappus and Guldin.* **(Meerut 1983)**

**Exmaples solved by Pappus theorem.**

**Ex. 31.** *Find the volume and surface-area of the anchor-ring generated by the revolution of a circle of radius a about an axis in its own plane distant b from its centre* $(b > a)$. **(Utkal 1973)**

**Sol.** Here the given curve (circle) does not intersect the axis of rotation, so Pappus theorem can be applied. In this case




$A$ = area of the region of the closed curve

= area of the circle of radius $a$

$= \pi a^2$

and $l$ = length of the arc of the curve = circumference of the circle $= 2\pi a$.

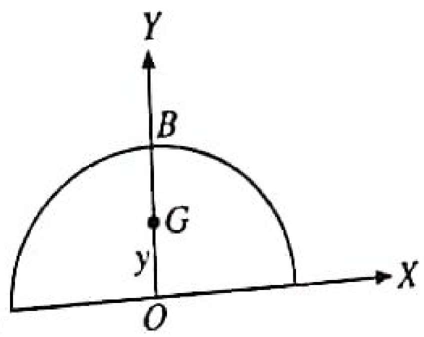
As the centroid of the area of a circle and also of its circumference lies at the centre, so $\bar{y} = b$ in both the cases and hence the length of the path described by the C.G. $= 2\pi b$.

Now by Pappus theorem, the required volume of the anchor-ring
= area of the circle × circumference of the circle generated by the centroid $= \pi a^2 . 2\pi b = 2\pi^2 a^2 b$.

And the surface area of the anchor-ring
= arc length of the circle × circumference of the circle generated by the centroid $= 2\pi a . 2\pi b = 4\pi^2 ab$.

**Ex. 32.** *Find the position of the centroid of a semi-circular area.* **(Kanpur 1970)**

**Sol.** Clearly the C.G. of the semi-circular area will lie somewhere on the radius which is perpendicular to the bounding diameter. Let the distance of the centroid from the centre $O$ be $y$. Also a sphere will be generated by the rotation of the semi-circular area about the bounding diameter.



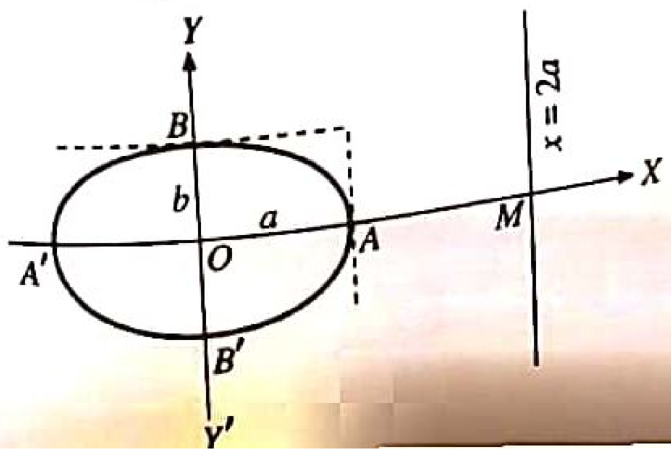

∴ by Pappus theorem, volume of the solid of revolution
= area of semi-circle × Circumference of the circle generated by the centroid of this area.

Hence $\frac{4}{3}\pi a^3 = \frac{1}{2}\pi a^2 . 2\pi y$, where $a$ is the radius of the semi-circle

or
$$y = \frac{1}{2\pi} \cdot \frac{\frac{4}{3}\pi a^3}{\frac{1}{2}\pi a^2} = \frac{4a}{3\pi}.$$

**Ex. 33.** *Show that the volume generated by the revolution of the ellipse* $x^2/a^2 + y^2/b^2 = 1$ *about the line* $x = 2a$ *is* $4\pi^2 a^2 b$. ...(1)

**Sol.** Area of the given ellipse is $\pi ab$.



The C.G. of the ellipse will describe a circle of radius $2a$ when revolved about the line $x = 2a$. Hence the length of the arc described by the C.G. $= 2\pi (2a) = 4\pi a$.

$\therefore$ by Pappus theorem the required volume
$=$ area of the ellipse $\times$ length of the arc described by its C.G.
$= \pi ab \,.\, 4\pi a = 4\pi^2 a^2 b$.

Ex. 34. *Show that the volume generated by the revolution of an ellipse having semi-axes $a$ and $b$ about a tangent at the vertex is $2\pi^2a^2b$ or $2\pi^2ab^2$.*

Sol. Area of the given ellipse $= \pi ab$. ...(1)

The C.G. of the ellipse will describe a circle of radius $a$ when revolved about the tangent at $A$ or a circle of radius $b$ when revolved about the tangent at $B$. [See fig. of Ex. 33].

Hence the length of the arc described by the C.G. will be
$2\pi \,.\, 2a$ *i.e.*, $4\pi a$ or $2\pi \,.\, 2b$ *i.e.*, $4\pi b$.

$\therefore$ by Pappus theorem, the required volume

(*i*) when revolved about the tangent at the vertex $A$
$$= \pi ab \,.\, 2\pi a = 2\pi^2 a^2 b,$$

(*ii*) when revolved about the tangent at the vertex $B$
$$= \pi ab \,.\, 2\pi b = 2\pi^2 ab^2.$$

Ex. 35. *Find the volume of the ring generated by the revolution of an ellipse of eccentricity $1/\sqrt{2}$ about a straight line parallel to the minor axis and situated at a distance from the centre equal to three times the major axis.*

Sol. Let $a$ be the semi-major axis of the ellipse. Then its semi-minor axis
$$b = a\sqrt{(1-e^2)} = a\sqrt{(1-\tfrac{1}{2})} = a/\sqrt{2}. \quad [\because e = 1/\sqrt{2}]$$

$\therefore$ area of the ellipse $= \pi ab = \pi a \,.\, (a/\sqrt{2}) = \pi a^2/\sqrt{2}$.

Distance of the C.G. of the ellipse from the axis of revolution is $3.2a = 6a$, (given).

As the ellipse revolves about the given line its C.G. will describe a circle of radius $6a$ whose perimeter will be
$$= 2\pi . 6a = 12\pi a.$$

Now by Pappus theorem, the required volume
$=$ area of the ellipse $\times$ length of the arc described by its C.G.
$= (\pi a^2/\sqrt{2}) \,.\, 12\pi a = 12\pi^2 a^3/\sqrt{2}$
$= 6\sqrt{2}\pi^2 a^3$, where $a$ is the semi-major axis.

Ex. 36. *The loop of the curve $2ay^2 = x(x-a)^2$ revolves about the straight line $y = a$. Find the volume of the solid generated.* (Meerut 1986 P)

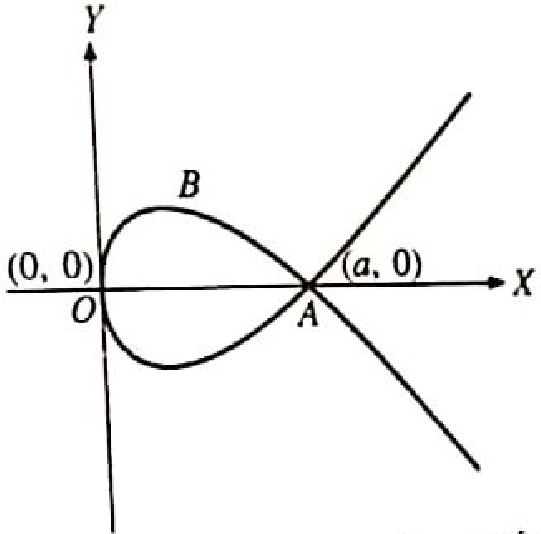
Sol. The given curve is $2ay^2 = x(x-a)^2$. ...(1)

The curve (1) is symmetrical about the $x$-axis and the loop lies between $x = 0$ and $x = a$.

Differentiating (1) w.r.t. $x$, we get

$4ay\,(dy/dx) = 2x\,(x-a) + (x-a)^2 = 3x^2 - 4ax + a^2.$

Now $(dy/dx) = 0$ when $3x^2 - 4ax + a^2 = 0$ or when $x = a/3$ which gives from (1), $y = (a\sqrt{2})/(3\sqrt{3})$ *i.e.*, $< a$ showing that the loop does not intersect the straight line $y = a$.




By symmetry the C.G. of the loop lies on $x$-axis *i.e.*, the distance of the C.G. from the axis of revolution ($y = a$) is $a$. When the loop is rotated about $y = a$, its C.G. will describe a circle of radius $a$ whose perimeter is $2\pi a$.

Also the area $A$ of the loop

$= 2\int_0^a y\,dx$

$$= 2\int_0^a \frac{(x-a)\sqrt{x}}{\sqrt{(2a)}}\,dx, \qquad \left[\because \text{ from (1) } y = \frac{(x-a)\sqrt{x}}{\sqrt{(2a)}}\right]$$

$$= \sqrt{\left(\frac{2}{a}\right)} \int_0^a (x^{3/2} - ax^{1/2})\,dx = \sqrt{\left(\frac{2}{a}\right)} \left[\frac{x^{5/2}}{5/2} - \frac{ax^{3/2}}{3/2}\right]_0^a$$

$$= \frac{4}{15}\sqrt{2}\,a^2.$$

$\therefore$ by Pappus theorem, the required volume

$$= 2\pi a \times A = 2\pi a \times \frac{4}{15}\sqrt{2}a^2 = \frac{8}{15}\sqrt{2}\pi a^3.$$

**Ex. 37.** *Find the volume of the ring generated by the revolution of the cardioid $r = a\,(1 + \cos\theta)$ about the line $r\cos\theta + a = 0$, given that the centroid of the cardioid is at a distance $5a/6$ from the origin.*

**Sol.** The given curve is

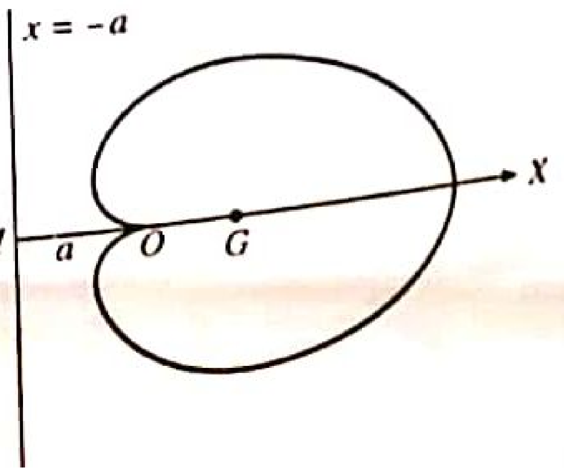
$r = a\,(1 + \cos\theta)$. ...(1)

And the given line of rotation is

$r\cos\theta + a = 0$

or $x + a = 0$, $(\because x = r\cos\theta)$

or $x = -a$.




By symmetry the centre of gravity $G$ of the cardioid lies on the initial line $OX$. If $G$ be the centroid of the area of the cardioid, then $OG = 5a/6$ (given).

Also $GM$ = the length of the perpendicular from G on the line of rotation

$$= GO + OM = (5a/6) + a = 11a/6.$$

Also the area $A$ of the cardioid $= 2 \times \int_0^{\pi} \frac{1}{2} r^2 \, d\theta$

$= \int_0^{\pi} a^2 (1 + \cos\theta)^2 \, d\theta,$ from (1)

$= a^2 \int_0^{\pi} (2 \cos^2 \frac{1}{2}\theta)^2 \, d\theta$

$= 8a^2 \int_0^{\pi/2} \cos^4 \phi \, d\phi$, putting $\theta/2 = \phi$ so that $d\theta = 2\, d\phi$

$= 8a^2 \cdot \dfrac{3}{4} \cdot \dfrac{1}{2} \cdot \dfrac{1}{2}\pi = \dfrac{3}{2}\pi a^2.$

$\therefore$ by Pappus theorem, the required volume

$= (2\pi \, . \, GM) \times A = 2\pi\,(11a/16) \times \dfrac{3}{2}\pi a^2 = \dfrac{11}{2}\pi^2 a^3.$