ANALYTIC GEOMETRY

: CSE - 2013 :

1(d): Find the equation of the plane which passes through the points (0,1,1) and (2,0,-1) and is parallel to the line joining the points (-1,1,-2), (3,-2,4). Find also the distance between the line and the plane.

→ Any plane through (0,1,1) is $Ax + B(y-1) + C(z-1) = 0$ — (1)

It passes through (2,0,-1) ≡ $2A - B - 2C = 0$ — (2)

DRs of line joining (-1,1,-2) and (3,-2,4) are $3+1, -2-1, 4+2$
$\Rightarrow 4, -3, 6$

Since the plane is parallel to the line joining these points, normal to the plane which has drs A, B, C are ⊥ar to the join of these points. Therefore,

$4A - 3B + 6C = 0$ — (3)

from (2) & (3): $\frac{A}{6} = \frac{B}{10} = \frac{C}{1}$ — (4)

∴ Eliminating A, B C between (1) & (4), we get

the reqd eqn of plane $6x + 10(y-1) + 1(z-1) = 0$
$\Rightarrow \boxed{6x + 10y + z = 11}$ — (5)

The line joining (-1,1,-2) & (3,-2,4) is parallel to the plane (5). Then, the distance b/w the line and the plane is equal to the ⊥ar distance from any point on the line to the plane.

∴ distance reqd is the distance b/w the point (-1,1,-2) and the plane $6x+10y+z=11$ measured perpendicularly

$$\therefore d = \frac{|6\cdot(-1) + 10(1) + 1\cdot(-2) - 11|}{\sqrt{6^2+10^2+1^2}} = \frac{|-9|}{\sqrt{137}} = \frac{9}{\sqrt{137}} \text{ units}$$

1(e). A sphere S has points (0,1,0), (3,-5,2) at opposite ends of diameter. Find the equation of the sphere having the intersection of the sphere S with the plane $5x-2y+4z+7=0$ as a great circle.

→ Sphere S has equation:

$$(x-0)(x-3)+(y-1)(y+5)+(z-0)(z-2)=0 \quad \text{[Diameter form]}$$

$$\Rightarrow x^2-3x+y^2+4y-5+z^2-2z=0$$

$$\Rightarrow x^2+y^2+z^2-3x+4y-2z-5=0 \quad \text{— (1)}$$

Any sphere through the intersection of sphere S & the given plane $5x-2y+4z+7=0$ is

$$x^2+y^2+z^2-3x+4y-2z-5+\lambda(5x-2y+4z+7)=0$$

$$\Rightarrow x^2+y^2+z^2+(-3+5\lambda)x+(4-2\lambda)y+(-2+4\lambda)z+(-5+7\lambda)=0 \quad \text{— (2)}$$

Since the required sphere has the intersection of the ~~great~~ given plane and the sphere S as a great circle, the centre of the reqd. sphere lies on the plane $5x-2y+4z+7=0$

centre of sphere (2) is $\left(-\frac{1}{2}(-3+5\lambda), -(2-\lambda), -(-1+2\lambda)\right)$

It lies on the given plane. Therefore:

$$-5\cdot\frac{1}{2}(-3+5\lambda)+2(2-\lambda)-4(-1+2\lambda)+7=0$$

$$-15+25\lambda-8+4\lambda+16\lambda-8-14=0$$

$$\Rightarrow 45\lambda = 45 \Rightarrow \lambda=1$$

$\therefore$ (2) $\equiv$ $x^2+y^2+z^2+2x+2y+2z+2=0$ which is the equation of the required sphere.

4(a) Show that three mutually perpendicular tangent lines can be drawn to the sphere $x^2+y^2+z^2=r^2$ from any point on the sphere $2(x^2+y^2+z^2)=3r^2$.

→ Let $S \equiv x^2+y^2+z^2-r^2$.

Let (α,β,γ) be any point from where three mutually $\perp$ar tangent lines can be drawn to sphere S.

Let $S_1 \equiv \alpha^2+\beta^2+\gamma^2-r^2$

Tangent plane to sphere S at (α,β,γ) is $T \equiv \alpha x+\beta y+\gamma z-r^2$.

Then, the enveloping cone to the sphere S where the vertex of this cone is (α,β,γ) is given by

$T^2 = SS_1 \Rightarrow (\alpha x+\beta y+\gamma z-r^2)^2 = (x^2+y^2+z^2-r^2)(\alpha^2+\beta^2+\gamma^2-r^2)$ — ①

Since the sphere S has 3 mutually $\perp$ar tangent lines drawn through (α,β,γ), these tangent lines are the 3 mutually $\perp$ar generators of the enveloping cone ①

The condition for three mutually $\perp$ar generators for the cone is that the sum of coeff of x^2, y^2 & z^2 is zero.. Now, in the cone ①,

Coeff of $x^2 \equiv \beta^2+\gamma^2-r^2$, coeff of $y^2 = \alpha^2+\gamma^2-r^2$

and coeff of $z^2 \equiv \alpha^2+\beta^2-r^2$

The sum of these coeff is zero.

$\therefore \beta^2+\gamma^2-r^2+\alpha^2+\gamma^2-r^2+\alpha^2+\beta^2-r^2=0$

$\Rightarrow 2(\alpha^2+\beta^2+\gamma^2) = 3r^2$.

The locus of (α,β,γ) is $\boxed{2(x^2+y^2+z^2)=3r^2}$ — (S_1)

Hence, three mutually $\perp$ar tangent lines can be drawn to the sphere S from any point on the sphere S_1

4(b) A cone has a guiding curve the circle $x^2+y^2+2ax+2by=0$, $z=0$, and passes through a fixed point $(0,0,c)$. If the section of the cone by the plane $y=0$ is a rectangular hyperbola, prove that the vertex lies on the fixed circle $x^2+y^2+z^2+2ax+2by=0$,

$2ax+2by+cz=0$.

$\rightarrow$ Let (α, β, γ) be the vertex of the cone. Then, any generator of the cone has eqn $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ —— ①

It meets plane $z=0$: $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{-\gamma}{n}$.

$\therefore$ It passes through the point $\left(\alpha - \frac{l\gamma}{n}, \beta - \frac{m\gamma}{n}, 0\right)$

This point lies on the given conic $x^2+y^2+2ax+2by=0$

$\Rightarrow \left(\alpha - \frac{l\gamma}{n}\right)^2 + \left(\beta - \frac{m\gamma}{n}\right)^2 + 2a\left(\alpha - \frac{l\gamma}{n}\right) + 2b\left(\beta - \frac{m\gamma}{n}\right) = 0$

Putting $\frac{l}{n} = \frac{x-\alpha}{z-\gamma}$ & $\frac{m}{n} = \frac{y-\beta}{z-\gamma}$ from ①, we have

$\left(\alpha - \frac{x-\alpha}{z-\gamma}\gamma\right)^2 + \left(\beta - \frac{y-\beta}{z-\gamma}\gamma\right)^2 + 2a\left(\alpha - \frac{x-\alpha}{z-\gamma}\gamma\right) + 2b\left(\beta - \frac{y-\beta}{z-\gamma}\gamma\right) = 0$

$\Rightarrow (\alpha z - \gamma x)^2 + (\beta z - \gamma y)^2 + 2a(\alpha z - \gamma x)(z-\gamma) + 2b(\beta z - \gamma y)(z-\gamma) = 0.$ —— ②

It passes through $(0,0,c)$. Therefore

$(\alpha c)^2 + (\beta c)^2 + 2a(\alpha c)(c-\gamma) + 2b(\beta c)(c-\gamma) = 0$ —— ③

The section of cone ② by $y=0$ is

$(\alpha z - \gamma x)^2 + (\beta z)^2 + 2a(\alpha z - \gamma x)(z-\gamma) + 2b(\beta z)(z-\gamma) = 0$

It this section is a rectangular hyperbola in xz-plane, then the sum of coeff of x^2 & z^2 is zero.

$\therefore$ coeff of $x^2 = \gamma^2$

coeff of $z^2 = \alpha^2 + \beta^2 + 2a\alpha + 2b\beta$.

$\therefore \alpha^2 + \beta^2 + \gamma^2 + 2a\alpha + 2b\beta = 0$ —— ④

The locus of (α, β, γ) from ③ & ④ is given by

$(x^2+y^2)c + 2ax(c-z) + 2by(c-z) = 0$ —— ⑤

and $x^2+y^2+z^2+2ax+2by=0$ —— ⑥

⑥ $\times c$ − ⑤ $\Rightarrow cz^2 + 2axz + 2byz = 0$

$\Rightarrow 2ax + 2by + cz = 0$ —— ⑦

Therefore, the vertex lies on the fixed circle

$x^2+y^2+z^2+2ax+2by=0, \quad 2ax+2by+cz=0$

4 (c) A variable generator meets two generators of the system through the extremities B & B' of the minor axis of the principal elliptic section of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ in P and P'. Prove that $BP \cdot B'P' = a^2+c^2$.

→ Generating lines of a hyperboloid in the standard form is given by

$$\frac{x-a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{-b\cos\theta}=\frac{z}{c} \quad \text{--- (1)}$$

and

$$\frac{x-a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{-b\cos\theta}=\frac{z}{-c} \quad \text{--- (2)}$$

From (1): we can obtain the equations of two generators passing through minor axis by putting $\theta=90^\circ$ & -90°.

$$\therefore \underbrace{\frac{x}{a}=\frac{y-b}{0}=\frac{z}{c}}_{(3)} \quad \& \quad \frac{x}{-a}=\frac{y+b}{0}=\frac{z}{c} \quad \text{--- (4)}$$

Now: we have $B(0,b,0)$ and $C(0,-b,0)$

P is the intersection of lines (2) & (3)

Any point on (3) is (ar, b, cr). Putting in (3), we get

$$\frac{ar-a\cos\theta}{a\sin\theta}=\frac{br-b\sin\theta}{-b\cos\theta}=\frac{cr}{-c}$$

$$\Rightarrow \frac{r-\cos\theta}{\sin\theta}=\frac{r-\sin\theta}{-\cos\theta}=-r \quad \Rightarrow \quad r-\cos\theta=-r\sin\theta$$

$$\Rightarrow \quad r=\frac{\cos\theta}{1+\sin\theta}$$

$\therefore$ Point of intersection is

$$P\left(\frac{a\cos\theta}{1+\sin\theta}, b, \frac{c\cos\theta}{1+\sin\theta}\right)$$

Similarly, P' is the intersection of lines (2) and (4).

$$\therefore P'\left(\frac{-a\cos\theta}{\sin\theta-1}, -b, \frac{c\cos\theta}{(\sin\theta-1)}\right)$$

$$BP = \sqrt{\left(\frac{a\cos\theta}{1+\sin\theta}\right)^2 + (b-b)^2 + \left(\frac{c\cos\theta}{1+\sin\theta}\right)^2} = \sqrt{a^2+c^2}\,\frac{\cos\theta}{1+\sin\theta}$$

$$B'P' = \sqrt{\left(\frac{-a\cos\theta}{(\sin\theta-1)}\right)^2 + (b-b)^2 + \left(\frac{c\cos\theta}{\sin\theta-1}\right)^2} = \sqrt{a^2+c^2}\,\frac{\cos\theta}{1-\sin\theta}$$

$$\therefore BP.\ B'P' = (a^2+c^2)\frac{\cos^2\theta}{1-\sin^2\theta} = a^2+c^2$$

$$\boxed{\therefore BP.\ B'P' = a^2+c^2}$$