RING THEORY

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1. RINGS AND FIELDS

1. 4b 2020 IFoS

Let K be a finite field. Show that the number of elements in K is p^n, where p is a prime, which is characteristic of K and $n \geq 1$ is an integer. Also, prove that $\frac{\mathbb{Z}_3[X]}{(X^2+1)}$ is a field. How many elements does this field have? 15

2. 1a 2019 IFoS

Let R be an integral domain. Then prove that ch R (characteristic of R) is 0 or a prime. 8

3. 3a 2018

Find all the proper subgroups of the multiplicative group of the field $(\mathbb{Z}_{13}, +_{13}, \times_{13})$, where $+_{13}$ and $\times_{13}$ represent addition modulo 13 and multiplication modulo 13 respectively. 20

4. 1b 2015

Give an example of a ring having identity but a subring of this having a different identity. 10

5. 4a 2015

Do the following sets form integral domains with respect to ordinary addition and multiplication? If so, state if they are fields : 5+6+4=15

(i) $b\sqrt{2}$ के रूप की संख्याओं का समुच्चय, जहाँ b परिमेय संख्या है

The set of numbers of the form $b\sqrt{2}$ with b rational

(ii) सम पूर्णांकों का समुच्चय

The set of even integers

(iii) धनात्मक पूर्णांकों का समुच्चय

The set of positive integers

6. 2a 2015 IFoS

2. (a) If p is a prime number and e a positive integer, what are the elements 'a' in the ring $\mathbb{Z}_{p^e}$ of integers modulo p^e such that $a^2 = a$? Hence (or otherwise) determine the elements in $\mathbb{Z}_{35}$ such that $a^2 = a$. *14*

7. 2a 2014

Show that $\mathbb{Z}_7$ is a field. Then find $([5]+[6])^{-1}$ and $(-[4])^{-1}$ in $\mathbb{Z}_7$. 15

8. 3a 2014

Show that the set $\{a + b\omega : \omega^3 = 1\}$, where a and b are real numbers, is a field with respect to usual addition and multiplication. 15

9. 4a 2014

Prove that the set $Q(\sqrt{5}) = \{a + b\sqrt{5} : a, b \in Q\}$ is a commutative ring with identity. 15

10. 2a IFoS 2014

2. *(a)* Let J_n be the set of integers mod n. Then prove that J_n is a ring under the operations of addition and multiplication mod n. Under what conditions on n, J_n is a field? Justify your answer. 10

11. 2a 2013 IFoS

(a) Show that any finite integral domain is a field.

12. 2b 2013 IFoS

(b) Every field is an integral domain — Prove it.

13. 1b 2012 IFoS

(b) Show that every field is without zero divisor. 10

14. 1b 2011 IFoS

(b) Let Q be the set of all rational numbers. Show that

$$Q(\sqrt{2}) = \{a + b\sqrt{2} : a, b \in Q\}$$

is a field under the usual addition and multiplication. 10

15. 2b 2010

(b) Let $C = \{ f : I = [0, 1] \to \mathbb{R} \mid f \text{ is continuous} \}$.

Show C is a commutative ring with 1 under pointwise addition and multiplication.

Determine whether C is an integral domain. Explain. 15

16. 1b 2010 IFoS

(b) Let F be a field of order 32. Show that the only subfields of F are F itself and $\{0, 1\}$. 10

17. 4b 2009 IFoS

(b) Show that a field is an integral domain and a non-zero finite integral domain is a field. 13

18. 2c 2009 IFoS

(c) Find the multiplicative inverse of the element

$$\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$$

of the ring, M, of all matrices of order two over the integers. 13

2. IDEALS AND QUOTIENT RINGS

1. 1b 2020

Let R be a principal ideal domain. Show that every ideal of a quotient ring of R is principal ideal and R/P is a principal ideal domain for a prime ideal P of R. 10

2. 2a 2020 IFoS

Let R be a non-zero commutative ring with unity. Show that M is a maximal ideal in a ring R if and only if R/M is a field. 10

3. 3b 2018 IFoS

(b) Show by an example that in a finite commutative ring, every maximal ideal need not be prime. 10

4. 2c 2017 IFoS

2.(c) Let A be an ideal of a commutative ring R and $B = \{x \in R : x^n \in A$ for some positive integer $n\}$. Is B an ideal of R ? Justify your answer. 10

5. 3b 2013

(b) Let R^C = ring of all real valued continuous functions on [0, 1], under the operations

$$(f + g)\, x = f(x) + g(x)$$

$$(fg)\, x = f(x)\, g(x).$$

Let $M = \left\{ f \in R^C \,\middle|\, f\left(\frac{1}{2}\right) = 0 \right\}$.

Is M a maximal ideal of R ? Justify your answer. 15

6. 3b 2013 IFoS

(b) Prove that :

(i) the intersection of two ideals is an ideal.

(ii) a field has no proper ideals.

7. 3a 2012

3. (a) Is the ideal generated by 2 and X in the polynomial ring $\mathbb{Z}[X]$ of polynomials in a single variable X with coefficients in the ring of integers $\mathbb{Z}$, a principal ideal ? Justify your answer. 15

8. 4a 2012

4. (a) Describe the maximal ideals in the ring of Gaussian integers $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$. 20

9. 3b 2009

(b) How many elements does the quotient ring $\frac{\mathbb{Z}_5[X]}{(X^2+1)}$ have ? Is it an integral domain ? Justify yours answers. 15

10. 2a 2009

2. (a) How many proper, non-zero ideals does the ring Z_{12} have ? Justify your answer. How many ideals does the ring $Z_{12} \oplus \mathbb{Z}_{12}$ have ? Why ? 2+3+4+6=15

3. HOMOMORPHISM OF RINGS

1. 3a 2020

Let R be a finite field of characteristic $p(>0)$. Show that the mapping $f: R \to R$ defined by $f(a) = a^p$, $\forall a \in R$ is an isomorphism. 15

2. 2a 2019 IFoS

Let I and J be ideals in a ring R. Then prove that the quotient ring $(I + J)/J$ is isomorphic to the quotient ring $I/(I \cap J)$. *10*

3. 2d 2018 IFoS

(d) Let R be a commutative ring with unity. Prove that an ideal P of R is prime if and only if the quotient ring R/P is an integral domain. 10

4. 2a 2015

If R is a ring with unit element 1 and ϕ is a homomorphism of R onto R', prove that $\phi(1)$ is the unit element of R'. 15

5. 1a 2013

Show that the set of matrices $S = \left\{ \begin{pmatrix} a & -b \\ b & a \end{pmatrix} \middle| \ a, b \in \mathbb{R} \right\}$ is a field under the usual binary operations of matrix addition and matrix multiplication. What are the additive and multiplicative identities and what is the inverse of $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$? Consider the map $f: \mathbb{C} \to S$ defined by $f(a + ib) = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. Show that f is an isomorphism. (Here $\mathbb{R}$ is the set of real numbers and $\mathbb{C}$ is the set of complex numbers.) *10*

6. 3b 2010

(b) Show that the quotient ring $\mathbb{Z}[i]/(1+3i)$ is isomorphic to the ring $\mathbb{Z}/10\mathbb{Z}$ where $\mathbb{Z}[i]$ denotes the ring of Gaussian integers. 15

4. EUCLIDEAN RINGS, PID

1. 3d 2019

Let a be an irreducible element of the Euclidean ring R, then prove that $R/(a)$ is a field. 10

2. 2d 2017 IFoS

2.(d) Prove that the ring

$\mathbb{Z}[i] = \{a + ib : a, b \in \mathbb{Z},\ i = \sqrt{-1}\}$ of Gaussian integers is a Euclidean domain. 10

3. 2c 2016 IFoS

2.(c) Show that in the ring $R = \{a + b\sqrt{-5} \mid a, b \text{ are integers}\}$, the elements $\alpha = 3$ and $\beta = 1 + 2\sqrt{-5}$ are relatively prime, but $\alpha\gamma$ and $\beta\gamma$ have no g.c.d in R, where $\gamma = 7(1 + 2\sqrt{-5})$. 10

4. 3a 2013

Let $J = \{a + bi \mid a, b \in \mathbb{Z}\}$ be the ring of Gaussian integers (subring of $\mathbb{C}$). Which of the following is J : Euclidean domain, principal ideal domain, unique factorization domain ? Justify your answer. *15*

5. 4a 2010 IFoS

4. *(a)* Let R be a Euclidean domain with Euclidean valuation d. Let n be an integer such that $d(1) + n \geq 0$. Show that the function $d_n : R - \{0\} \to S$, where S is the set of all negative integers defined by $d_n(a) = d(a) + n$ for all $a \in R - \{0\}$ is a Euclidean valuation. 13

6. 4a 2009 IFoS

4. *(a)* Show that $d(a) < d(ab)$, where a, b be two non-zero elements of a Euclidean domain R and b is not a unit in R. 13

5. POLYNOMIAL RINGS, UFD

1. 1a 2018

Let R be an integral domain with unit element. Show that any unit in $R[x]$ is a unit in R. 10

2. 2c 2017

Let F be a field and F[X] denote the ring of polynomials over F in a single variable X. For $f(X), g(X) \in F[X]$ with $g(X) \neq 0$, show that there exist $q(X), r(X) \in F[X]$ such that degree $(r(X)) <$ degree $(g(X))$ and

$$f(X) = q(X) \cdot g(X) + r(X).$$ 20

3. 1a 2016

Let $\mathbb{K}$ be a field and $\mathbb{K}[X]$ be the ring of polynomials over $\mathbb{K}$ in a single variable X. For a polynomial $f \in \mathbb{K}[X]$, let (f) denote the ideal in $\mathbb{K}[X]$ generated by f. Show that (f) is a maximal ideal in $\mathbb{K}[X]$ if and only if f is an irreducible polynomial over $\mathbb{K}$. 10

4. 4a 2016

Show that every algebraically closed field is infinite. 15

5. 3a 2014 IFoS

3. (a) Let R be an integral domain with unity. Prove that the units of R and $R[x]$ are same. 10

6. 3c 2012 IFoS

(c) If R is an integral domain, show that the polynomial ring R[x] is also an integral domain. 14

7. 3a 2011

3. (a) Let F be the set of all real valued continuous functions defined on the closed interval [0, 1]. Prove that (F, +, ·) is a Commutative Ring with unity with respect to addition and multiplication of functions defined pointwise as below :

$$\left.\begin{aligned}(f + g)(x) &= f(x) + g(x)\\ \text{and}\quad (f \,.\, g)(x) &= f(x) \,.\, g(x)\end{aligned}\right\}\ x \in [0, 1]$$

where f, g ∈ F. 15

8. 3a 2010

3. (a) Consider the polynomial ring $Q[x]$. Show $p(x) = x^3 - 2$ is irreducible over Q. Let I be the ideal in $Q[x]$ generated by $p(x)$. Then show that $Q[x]/I$ is a field and that each element of it is of the form $a_0 + a_1 t + a_2 t^2$ with a_0, a_1, a_2 in Q and $t = x + I$. 15

9. 3a 2009

Show that $\mathbb{Z}[X]$ is a unique factorization domain that is not a principal ideal domain ($\mathbb{Z}$ is the ring of integers). Is it possible to give an example of principal ideal domain that is not a unique factorization domain ? ($\mathbb{Z}[X]$ is the ring of polynomials in the variable X with integer.) 15

*MISCELLANEOUS (EXTENSION FIELD)

1. 3a 2016

Let K be an extension of a field F. Prove that the elements of K, which are algebraic over F, form a subfield of K. Further, if $F \subset K \subset L$ are fields, L is algebraic over K and K is algebraic over F, then prove that L is algebraic over F. *20*