$\boxed{2010}$

(1) $f(z) = u + iv$

$$v = e^x(x\sin y + y\cos y)$$

$$\frac{\partial v}{\partial x} = e^x(x\sin y + y\cos y) + e^x(\sin y)$$

$$= e^x(x\sin y + y\cos y + \sin y)$$

$$\frac{\partial v}{\partial y} = e^x(x\cos y + \cos y - y\sin y)$$

Now, $w = f(z) = u + iv$

$$\frac{dw}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$\frac{dw}{dz} = +\frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x} \quad \text{(by CR eq}^{\underline{n}}\text{)}$$

Replacing x by z and y by 0

by Milne thompson, we have

$$\frac{dw}{dz} = +e^z(z+1) + i\,e^z(0)$$

$$w = \int e^z(z+1)\,dz$$

$$= (z+1)\cdot e^z - e^z + C$$

$$\boxed{w = f(z) = ze^z + C}$$

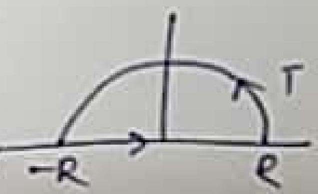
Scanned with CamScanner

② $\displaystyle\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)^2(x^2+2x+2)}$

consider $\displaystyle\int_C \frac{z^2\,dz}{(z^2+1)^2(z^2+2z+2)}$ where C is a contour consisting of large semicircle Γ of Radius R from $-R$ to R on real axis

by Cauchy Residue theorem

$$\int_C \frac{z^2\,dz}{(z^2+1)^2(z^2+2z+2)} = 2\pi i\,\Sigma\,\text{Residue}$$



$$= \int_{-R}^{R} \frac{x^2\,dx}{(x^2+1)^2(x^2+2x+2)} + \int_\Gamma \frac{z^2\,dz}{(z^2+1)^2(z^2+2z+2)} = 2\pi i\,\Sigma\,\text{Res}$$

by applying $R \to \infty$

we have $\displaystyle\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)^2(x^2+2x+2)} = 2\pi i\,\Sigma\,\text{Residue}.$

Now to calculate Residue, we have to take poles of $f(z) = \dfrac{z^2}{(z^2+1)^2(z^2+2z+2)}$

$$(z^2+1)^2(z^2+2z+2) = 0$$

$$z^2 = -1, -1 \qquad ((z+1)^2+1) = 0$$

$$z = \pm i, \pm i \qquad z = -1 \pm i$$

Residue at $z = i$

$$= \lim_{z\to i} \frac{d}{dz} \frac{z^2}{(z+i)^2(z^2+2z+2)}$$

$$= \lim_{z\to i} \frac{2(z^2+2z+2)(z+i)^2 z - z^2[(z+i)^2(2z+2) - 2(z+i)(z^2+2z+2)]}{(z+i)^4(z^2+2z+2)^2}$$

$$= \frac{2(i^2+2i+2)(2i)^2(1) - i^2\left[(2i)^2 2(i+1) - 2\times 2i\times((i^2+2i+2)\right]}{(2i)^4 \; (i^2+2i+2)^2}$$

$$= \frac{-8i(2i+1) + \left[-8(i+1) - 4i(1+2i)\right]}{16 \; (1+2i)^2}$$

$$= \frac{-8(i+1) - 12i(2i+1)}{16(4i-3)}$$

$$= \frac{16-20i}{16(4i-3)}$$

$$= \frac{1}{4} \frac{4-5i}{(4i-3)} = \frac{-(4-5i)(4i+3)}{4\times 25}$$

Residue at $z=-i$

$$\lim_{z\to -i} \frac{d}{dz} \frac{z^2}{(z-i)^2(z^2+2z+2)}$$

$$= \lim_{z\to -i} \frac{2(z-i)^2(z^2+2z+2)z - z^2\left[2(z-i)(z^2+2z+2) + 2(z-i)^2(2z)\right]}{(z-i)^4 \; (z^2+2z+2)^2}$$

$$= \lim_{z\to -i} \frac{2(-i)(-4)(1-2i) + \left[2(-2i)(1-2i) + 2(-4)(1-i)\right]}{16(1-2i)^2}$$

$$= \lim_{z\to -i} \frac{8i(1-2i) - 4i(1-2i) - 8(1-i)}{-16(4i+3)}$$

$$= \frac{-12i}{-16(4i+3)} = -\frac{3i}{4}\frac{(3-4i)}{25}$$

Residue at $z=-1+i$

$$\lim_{z\to(-1+i)} \frac{z^2}{(z^2+1)^2\left(z-(-1-i)\right)}$$

$$= \frac{(-1+i)^2}{\left[\left(\underbrace{(-1+i)^2}_{-2i}+1\right]^2 (2i)}$$

$$= \frac{-2i}{2i\,(1-2i)^2} = +\frac{1}{4i+3} = \frac{3-4i}{25}$$

Residue at $z=-1-i$

$$\lim_{z\to(-1-i)} \frac{z^2}{(z^2+1)^2\left(z-(-1+i)\right)}$$

$$= \frac{2i}{-2i\,(4i-3)} = -\frac{1}{4i-3} = \frac{(4i+3)}{25}$$

$$\Sigma\,\text{Residue} = -\frac{(12+20+i)}{100} \mp \frac{(12+9i)}{100} + \frac{3-4i}{25} + \frac{4i+3}{25}$$

$$= -\frac{44}{100} - \frac{2}{9} \cancel{-\frac{10i}{100}} = -\frac{20}{100} - \frac{10i}{100}$$

$$= -\frac{1}{10}(2+i)$$

$$= \cancel{-\frac{398}{900}} \cancel{-\frac{10i}{100}}$$

$$\int_{-\infty}^{\infty} \frac{x^2\,dx}{(x^2+1)^2(x^2+2x+2)} \cancel{= 2\pi i\left(-\frac{398}{900} - \frac{i}{10}\right)}$$

Ans.

$$= 2\pi i\,(\Sigma\,\text{Residue})$$

$$= -\frac{2\pi i}{10}(2+i) = \boxed{\frac{\pi}{5}(2-4i)}$$

Scanned with CamScanner

(3) $f(z) = \dfrac{1}{(z+1)(z+3)} = \dfrac{1}{2}\left[\dfrac{1}{(z+1)} - \dfrac{1}{(z+3)}\right]$

in $0 < |z+1| < 2$, let $z+1 = t$, then $z+3 = t+2$

$$g(t) = \frac{1}{2}\left[\frac{1}{t} - \frac{1}{t+2}\right]$$

$$= \frac{1}{2}\left[\frac{1}{t} - \frac{1}{2(1+t/2)}\right] \quad (\because t/2 < 1)$$

$$= \frac{1}{2}\left[\frac{1}{t} - \frac{1}{2}\left(1+\frac{t}{2}\right)^{-1}\right]$$

$$= \frac{1}{2}\left(\frac{1}{t} - \frac{1}{2}\left(1 - \frac{t}{2} + \frac{t^2}{4} - \frac{t^3}{8} \cdots\right)\right]$$

$$= \frac{1}{2t} - \frac{1}{4} + \frac{t}{8} - \frac{t^3}{16} \cdots$$

$$\boxed{f(z) = \frac{1}{2(z+1)} - \frac{1}{4} + \frac{z+1}{8} - \frac{(z+1)^3}{16} \cdots}$$

where $0 < |z+1| < 2$