IFos-2016

Q 1 Obtain the curve passing through (1,2) & slope $-\frac{2xy}{x^2+1}$.
obtain one asymptote to the curve.

sol. Given $\frac{dy}{dx} = \frac{-2xy}{x^2+1}$

$\Rightarrow \int \frac{dy}{y} = -\int \frac{2x\,dx}{x^2+1}$

$\Rightarrow \ln y = -\ln(x^2+1) + \ln C$

$\Rightarrow \ln y + \ln(x^2+1) = \ln C$

$\Rightarrow \ln(y(x^2+1)) = \ln C$

$\Rightarrow y(x^2+1) = C$

As the curve passes through (1,2)

$\therefore \quad 2(1^2+1) = C$

$\Rightarrow \quad C = 4$

$\therefore$ Curve is $y(x^2+1) = 4$

or $\boxed{y = \frac{4}{x^2+1}}$

Asymptote

Highest ~~cofficient~~ order of x is x^2 & its coefficient is y.

$\therefore$ $y = 0$ is horizontal asymptote of the curve.

Q 2. Solve DE to get particular integral of $\frac{d^4y}{dx^2} + 2\frac{d^2y}{dx^2} + y = x^2 \cos x$.

sol. $P.I. = \frac{1}{D^4+2D^2+1} x^2 \cos x$

$= \text{Real part of } \frac{1}{D^4+2D^2+1} x^2 e^{ix}$

$= \text{Real part of } \frac{1}{(D^2+1)^2} x^2 e^{ix}$

$= \text{Real part of } \left[e^{ix} \frac{1}{((D+i)^2+1)^2} x^2 \right]$

$$= \text{Real part of } \left[e^{ix} \frac{1}{(D^2+2iD-1+1)^2} x^2 \right]$$

$$= \text{Real part of } \left[e^{ix} \frac{1}{(D^2+2iD)^2} x^2 \right]$$

$$= \text{Real part of } \left[e^{ix} \frac{1}{(2iD)^2 \left(1+\frac{D}{2i}\right)^2} x^2 \right]$$

$$= \text{Real part of } \left[e^{ix} \frac{1}{-4D^2} \left(1+\frac{D}{2i}\right)^{-2} x^2 \right]$$

$$= \text{Real part of } \left[e^{ix} \frac{1}{-4D^2} \left[1-2\left(\frac{D}{2i}\right)+3\left(\frac{D}{2i}\right)^2+\dots\right] x^2 \right]$$

$$= \text{Real part of } \left[e^{ix} \frac{1}{-4D^2} \left[x^2 - \frac{D(x^2)}{i} - \frac{3}{4}D^2(x^2) + 0\right] \right]$$

$$= \text{Real part of } \left[e^{ix} \frac{1}{-4D^2} \left(x^2 - \frac{2x}{i} - \frac{3}{4}(2)\right) \right]$$

$$= \cancel{\text{Real part of } \left[\frac{e^{ix}}{-4} \iint \left(x^2+2ix-\frac{3}{2}\right) dx\,dx \right]}$$

$$= \text{Real part of } \left[\frac{e^{ix}}{-4} \frac{1}{D} \left(\frac{x^3}{3} - \frac{2x^2}{2i} - \frac{3}{2}x\right) \right]$$

$$= \text{Real part of } \left[\frac{e^{ix}}{-4} \left(\frac{x^4}{12} - \frac{x^3}{3i} - \frac{3x^2}{4}\right) \right]$$

$$= \text{Real part of } \left[e^{ix} \left(\frac{x^4}{-48} + \frac{ix^3}{-12} + \frac{3x^2}{16}\right) \right]$$

$$= \text{Real part of } \left[(\cos x + i\sin x) \left[\left(\frac{3x^2}{16} - \frac{x^4}{48}\right) - i\frac{x^3}{12}\right] \right]$$

$$\cancel{= \text{Real part of } \left[\cos x \left(\frac{3x^2}{16} - \frac{x^4}{48}\right) \right.}$$

$$= \left(\frac{3x^2}{16} - \frac{x^4}{48}\right)\cos x + \frac{x^3}{12}\sin x$$

$$= \frac{x^2}{4}\left[\left(\frac{3}{4} - \frac{x^2}{12}\right)\cos x + \frac{x}{3}\sin x\right]$$

$\therefore$ Particular integral of the equation is

$$\frac{x^2}{4}\left[\left(\frac{3}{4} - \frac{x^2}{12}\right)\cos x + \frac{x}{3}\sin x\right] \quad \text{Ans.}$$

Q3 Solve by variation of parameters $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^2e^x$

sol: Homogeneous part is $(x^2D^2 + xD - 1)y = 0$ $\quad (D = \frac{d}{dx})$

Let $x = e^z$ then $xD = \theta$

$x^2D = \theta(\theta-1)$ where $\theta = \frac{d}{dz}$

Hence we get, $(\theta(\theta-1) + \theta - 1)y = 0$

$\Rightarrow (\theta^2 - 1)y = 0$

$\therefore$ Auxiliary eq^n is $m^2 - 1 = 0$

$m = \pm 1$

$$C.F. = C_1e^{-z} + C_2e^z = \frac{C_1}{x} + C_2x$$

$\therefore u = \frac{1}{x}$ and $v = x$ are solutions of homogeneous part.

$$W = \begin{vmatrix} u & u' \\ v & v' \end{vmatrix} = \begin{vmatrix} \frac{1}{x} & -\frac{1}{x^2} \\ x & 1 \end{vmatrix} = \frac{1}{x} + \frac{1}{x} = \frac{2}{x} \neq 0$$

$\therefore u$ & v are independent, standard form of eq^n is

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{y}{x^2} = e^x$$

Variation of Parameters

Let $y_p = Au + Bv$ where A & B are parameters in x.

then $A = -\int \frac{vR}{W}$ & $B = \int \frac{uR}{W}$ where $R = e^x$

$$\therefore A = -\int \frac{x\,e^x}{(2/x)}dx = -\int \frac{x^2}{2}e^x = -\left(\frac{x^2}{2}e^x - \frac{2x}{2}e^x + \frac{2}{2}e^x\right)$$

$$= -\left(\frac{x^2}{2} - x + 1\right)e^x$$

$$\therefore A = -e^x\left(\frac{x^2}{2} - x + 1\right)$$

$$B = \int \frac{uR}{W} = \int \frac{\left(\frac{1}{x}\right)e^x}{(2/x)} = \int \frac{e^x}{2}dx = \frac{e^x}{2}$$

$$\therefore y_p = \frac{A}{x} + Bx = \frac{-e^x\left(\frac{x^2}{2} - x + 1\right)}{x} + \frac{e^x}{2}(x)$$

$$= -e^x\left(\frac{x}{2} - 1 + \frac{1}{x}\right) + \left(e^x\frac{x}{2}\right) = e^x\left(-\frac{x}{2} + 1 - \frac{1}{x} + \frac{x}{2}\right)$$

$= e^x\left(1-\frac{1}{x}\right) = \frac{e^x(x-1)}{x}$

$\therefore\ y = C.F. + y_p$

$$\boxed{y = \frac{c_1}{x} + c_2 x + \frac{e^x(x-1)}{x}}$$

Q4 Obtain singular solution of DE $y^2 - 2pxy + p^2(x^2-1) = m^2$

sol. $y^2 - 2pxy + p^2x^2 - p^2 = m^2$

$\Rightarrow (y-px)^2 - p^2 = m^2$

$\Rightarrow (y-px)^2 = p^2 + m^2$

$\Rightarrow y - px = \pm\sqrt{p^2+m^2}$

$\Rightarrow \boxed{y = px \pm \sqrt{p^2+m^2}}$ Clairaut's form

$\therefore$ Solution of $y = cx \pm \sqrt{c^2+m^2}$

or $(y-cx)^2 = c^2+m^2$

$\Rightarrow y^2 - 2cxy + c^2x^2 = c^2 + m^2$

$\Rightarrow (x^2-1)c^2 - 2cxy + y^2 - m^2 = 0$

C- discriminant (B^2-4AC)

$A = x^2-1 \quad B = -2xy \quad C = y^2-m^2$

$\therefore\ B^2 - 4AC = 0$ is the singular sol.

$\Rightarrow (-2xy)^2 - 4(x^2-1)(y^2-m^2) = 0$

$\Rightarrow 4x^2y^2 - 4(x^2y^2 - m^2x^2 - y^2 + m^2) = 0$

$\Rightarrow x^2y^2 - x^2y^2 + m^2x^2 + y^2 - m^2 = 0$

$\Rightarrow \boxed{y^2 + m^2(x^2-1) = 0}$ is the required singular solution of the given differential equation.

Q5 Solve the DE: $\frac{dy}{dx} - y = y^2(\sin x + \cos x)$

sol. Dividing the eq^n by $-y^2$

$$-\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{y} = \sin x + \cos x \quad \text{(Bernoulli's equation)}$$

Let $\frac{1}{y} = z \Rightarrow -\frac{1}{y^2}\,dy = dz$

$$\therefore -\frac{1}{y^2}\frac{dy}{dx} = \frac{dz}{dx}$$

$$\Rightarrow \frac{dz}{dx} + z = \sin x + \cos x \quad \text{(Linear form)}$$

$$\Rightarrow \text{I.F.} = e^{\int 1\,dx} = e^x$$

solution:

$$z(e^x) = \int e^x(\sin x + \cos x)\,dx$$

$$z e^x = \int e^x \sin x\,dx + \int e^x \cos x\,dx$$

$$= e^x(-\cos x) \quad \int e^x$$

$$z e^x = \sin x \int e^x dx - \int\left(\frac{d}{dx}(\sin x)\int e^x dx\right)dx + \int e^x \cos x\,dx$$

$$z e^x = e^x \sin x - \int e^x \cos x\,dx + \int e^x \cos x\,dx$$

$$z e^x = e^x \sin x + c$$

$$\therefore z = \sin x + ce^{-x}$$

$$\Rightarrow \frac{1}{y} = \sin x + ce^{-x}$$

$\Rightarrow \boxed{y(\sin x + ce^{-x}) = 1}$ is the required solution.