2019

#5b) Newton Forward

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.61	1.840431				
		0.018497			
0.62	1.858928		0.000185		
		0.018682		0.000004	
0.63	1.877610		0.000189		0.000004
		0.018871		0.000000	
0.64	1.896481		0.000189		
		0.019060			
0.65	1.915541				

Here $x = 0.612$, $x_0 = 0.61$, $h = 0.01$

$$\therefore u = \frac{0.612 - 0.61}{0.01} = 0.2$$

Now, by the Newton Forward Formula,

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0$$

$$\therefore f(0.612) = 1.840431 + 0.2 \times 0.018497 + \frac{0.2 \times (-0.8)}{2} \times 0.000185 + \frac{0.2 \times (-0.8) \times (-1.8)}{6} \times 0.000004 + \frac{0.2 \times (-0.8) \times (-1.8) \times (-2.8)}{24} \times 0.000004$$

$$\simeq 1.840431 + 0.0036994 - 0.0000148 + 0.0000001 9$$

$$\simeq 1.84378633$$

⑤c Simpson 1/3

x	y	y_i $(i=0,6)$	y_i $(i=1,3,5)$	y_i $(i=2,4)$
$x_0=0.0$	0.00	0.00	—	—
$x_1=0.5$	0.75	—	0.75	—
$x_2=1.0$	1.00	—	—	1.00
$x_3=1.5$	0.75	—	0.75	—
$x_4=2.0$	0.00	—	—	0.00
$x_5=2.5$	-1.25	—	-1.25	—
$x_6=3.0$	-3.00	-3.00	—	—
		$\sum y_i=-3.00$	$\sum y_i=0.25$	$\sum y_i=1.00$

$$\therefore h=\frac{3-0}{6}=1/2$$

$\therefore$ Simpson 1/3 rule is,

$$I_s^c=\frac{h}{3}\left[y_0+y_6+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$$=\frac{1}{6}\left[-3.00+(4\times 0.25)+(2\times 1.00)\right]$$

$$=\frac{1}{6}\left[-3+1+2\right]=0$$

#6b) Gauss-Jordan

The augmented matrix is,

$$\left[\begin{array}{ccc:ccc} 1 & 1 & 1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 0 & 1 & 0 \\ 1 & -1 & -1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{c:ccc:cc} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & -2 & -2 & -1 & 0 & 1 \end{array}\right] \begin{array}{c} \\ -2 \\ -1 \end{array}$$

$a'_{21} = 3 - 2\times1 = 1$; $a'_{22} = 1 - 2\times1 = -1$; $a'_{23} = 0 - 2\times\frac{1}{1} = -2$

$a'_{31} = -1 - 1\times1 = -2$; $a'_{32} = -1 - 1\times1 = -2$; $a'_{33} = 0 - 1\times\frac{1}{1} = -1$

$$\sim \left[\begin{array}{cc:ccc:c} 1 & 0 & 2 & 3 & -1 & 0 \\ 0 & 1 & -1 & -2 & 1 & 0 \\ 0 & 0 & -4 & -5 & 2 & 1 \end{array}\right] \begin{array}{c} -1 \\ \\ +2 \end{array}$$

$a''_{11} = 1 - 1\times(-1) = +2$; $a''_{12} = 1 - 1\times(-2) = 3$; $a''_{13} = 0 - 1\times\frac{1}{1} = -1$

$a''_{31} = -2 - (-2)\times(-1) = -4$; $a''_{32} = -1 - (-2)\times(-2) = -5$; $a''_{33} = 0 - (-2)\times\frac{1}{1} = 2$

$$\sim \left[\begin{array}{ccc:ccc} 1 & 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 1 & 0 & -3/4 & 1/2 & -1/4 \\ 0 & 0 & 1 & +5/4 & -2/4 & -1/4 \end{array}\right] \begin{array}{c} -2 \\ +1 \\ \\ \end{array}$$

$a'''_{11} = 3 - 2\times\frac{5}{4} = 1/2$; $a'''_{12} = -1 - 2\times\frac{2}{-4} = 0$; $a'''_{13} = 0 - 2\times\frac{1}{-4} = 1/2$

$a'''_{21} = -2 - (-1)\times\frac{5}{4} = -3/4$; $a'''_{22} = 1 - (-1)\times\frac{2}{-4} = 1/2$; $a'''_{23} = 0 - (-1)\times -\frac{1}{4} = -1/4$

Thus $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 & 1/2 \\ -3/4 & 1/2 & -1/4 \\ 5/4 & -2/4 & -1/4 \end{bmatrix} \begin{bmatrix} 3 \\ 6 \\ -3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3/2 \\ 3/2 \end{bmatrix}$

$\therefore x_1 = 0, \ x_2 = 3/2, \ x_3 = 3/2$

7a Runge-Kutta

Here $x_0 = 0$, $y_0 = 1$, $f(x,y) = x^2 + y^2$, $h = 0.1$

Thus, $K_1 = hf(x_0, y_0) = 0.1 \times f(0,1) = 0.1$

$K_2 = hf(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}) = 0.1 \times f(0.05, 1.05) = 0.1105$

$K_3 = hf(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}) = 0.1 \times f(0.05, 1.05525) = 0.111605$

$K_4 = hf(x_0 + \frac{h}{2}, y_0 + K_3) = 0.1 \times f(0.1, 1.111605) = 0.124567$

$\therefore y(0.1) = y_0 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$= 1 + \frac{1}{6}(0.668777) = 1.111463$

Now for $y(0.2)$, $x_1 = 0.1$, $y_1 = 1.111463$

$K_1 = hf(x_1, y_1) = 0.1 \times f(0.1, 1.111463) = 0.124535$

$K_2 = hf(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}) = 0.1 \times f(0.15, 1.173730) = 0.140014$

$K_3 = hf(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}) = 0.1 \times f(0.15, 1.181470) = 0.141837$

$K_4 = hf(x_1 + h, y_1 + K_3) = 0.1 \times f(0.2, 1.2533) = 0.161076$

$\therefore y(0.2) = y_1 + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$= 1.111463 + \frac{1}{6}(0.849313)$

$= 1.253015$

(8c) Gauss quadrature

$x_1 = -0.23861919$; $\omega_1 = 0.46791393$

$x_2 = -0.66120935$; $\omega_2 = 0.36076157$

$x_3 = -0.93246951$; $\omega_3 = 0.7132449$

$x_4 = +0.23861919$; $\omega_4 = 0.46791393$

$x_5 = +0.66120935$; $\omega_5 = 0.36076157$

$x_6 = +0.93246951$; $\omega_6 = 0.7132449$

For doing Gauss quadrature formula, we have to convert the limits -1 to 1.

Here $f(x) = \dfrac{1}{1+x^2}$

we know that,

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\int_{-1}^{1} f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right)dx$$

$$\therefore \int_0^1 f(x)\,dx = \frac{1-0}{2}\int_{-1}^{1} f\left(\frac{1-0}{2}x + \frac{1+0}{2}\right)dx = \frac{1}{2}\int_{-1}^{1} f(0.5x+0.5)\,dx$$

Now for Gauss quadrature's 6 point formula,

$$\int_{-1}^{1} f(x) \simeq \omega_1 f(x_1) + \omega_2 f(x_2) + \omega_3 f(x_3) + \omega_4 f(x_4) + \omega_5 f(x_5) + \omega_6 f(x_6)$$

$$\therefore 0.5\int_{-1}^{1} f(0.5x+0.5)\,dx = 0.5\Big[0.46791393 \times f(0.5\times -0.23861919 + 0.5) + 0.17132449 \times f(0.5\times -0.93246951 + 0.5)$$
$$+ 0.36076157 \times f(0.5\times -0.66120939 + 0.5) + 0.46791393 \times f(0.5\times 0.23861919 + 0.5) + 0.36076157 \times f(0.5\times 0.66120939 + 0.5)$$
$$+ 0.17132449 \times f(0.5\times 0.93246951 + 0.5)\Big]$$

$$= 0.5\Big[0.46791393 \times f(0.380690405) + 0.17132449 \times f(0.033766524)$$
$$+ 0.36076157 \times f(0.169395305) + 0.36076157 \times f(0.830604695)$$
$$+ 0.46791393 \times f(0.619309595) + 0.17132449 \times f(0.966234755)\Big]$$

$$= 0.5\Big[0.46791393 \times 0.8734195156 + 0.36076157 \times 0.97210565 + 0.17132449 \times 0.9988611204 + 0.46791393 \times 0.7227812989$$
$$+ 0.36076157 \times 0.5917495347 + 0.17132449 \times 0.5171674774\Big]$$

$$= 0.5\Big[0.408685158 + 0.3506983614 + 0.171129372 + 0.3381994381 + 0.2134804912 + 0.5171674774\Big]$$

$$= 0.5 \times 1.999360298 \simeq 0.999680149$$