G-20 (Maths).

IFS P-2 2020

classmate

Date

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3(a) Find the extreme values of $f(x,y,z) = 2x+3y+z$ such that $x^2+y^2 = 5$ and $x+z=1$. (10)

Sol: $f(x,y,z) = 2x+3y+z$; $x^2+y^2-5=0$; $x+z-1=0$

Let $F = 2x+3y+z+\lambda(x^2+y^2-5)+\mu(x+z-1)$ —①

Then,
$$\left.\begin{aligned} \frac{\partial F}{\partial x} &= 2+2\lambda x+\mu \\ \frac{\partial F}{\partial y} &= 3+2\lambda y \\ \frac{\partial F}{\partial z} &= 1+\mu \end{aligned}\right\} \quad \text{—②}$$

For extreme values, $\partial F = 0$

i.e. $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$

$$\left.\begin{aligned} 2+2\lambda x+\mu &= 0 \\ 3+2\lambda y &= 0 \\ 1+\mu &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} &\mu = -1 \\ &x = \frac{-1}{2\lambda}\ ;\ y = \frac{-3}{2\lambda} \end{aligned}$$

Put in $x^2+y^2 = 5$

$$\Rightarrow \frac{1}{4\lambda^2}+\frac{9}{4\lambda^2} = 5 \Rightarrow \frac{\cancel{10}^{\,2}}{\cancel{4}_{2}\lambda^2} = \cancel{5} \Rightarrow \lambda = \pm\frac{1}{\sqrt{2}}$$

If $\lambda = 1/\sqrt{2}$, $(x,y,z) = \left(\frac{-1}{\sqrt{2}}, \frac{-3}{\sqrt{2}}, 1+\frac{1}{\sqrt{2}}\right)$ (Say A)

$\lambda = -1/\sqrt{2}$, $(x,y,z) = \left(\frac{1}{\sqrt{2}}, \frac{3}{\sqrt{2}}, 1-\frac{1}{\sqrt{2}}\right)$ (Say B)

Which are the extreme points

Value of $f(x,y,z) = 2x+3y+z$ at A is $1+5\sqrt{2}$
at B is $1-5\sqrt{2}$

which are max & min.

3(b) Let G be a finite group and let p be a prime. If p^m divides order of G, then show that G has a subgroup of order p^m, where m is a positive integer. (15)

Sol: (Sylow's First Theorem)
We prove it by induction on $o(G)$
It is vacuously true when $o(G) = 1$.

Assume it to be true for all groups with order less than $o(G)$.

Let $p^m \mid o(G)$.
If K is a subgroup of G s.t. $K \neq G$ and $p^m \mid o(K)$, then by induction $\exists\ H \leq K$ s.t. $o(H) = p^m$.

Hence, $H \leq K \Rightarrow H \leq G$.

So, the result holds in this case.

Assume p^m does not divide order of any proper subgroup of G.
Consider class equation of G.

$$o(G) = o(Z(G)) + \sum_{a \notin Z(G)} \frac{o(G)}{o(N(a))}$$

$$a \notin Z(G) \Rightarrow N(a) \neq G \Rightarrow p^m \nmid o(N(a))$$

But, $p^m \mid o(G) \Rightarrow p^m \mid \frac{o(G)}{o(N(a))} \cdot o(N(a))$

$\Rightarrow p \mid \frac{o(G)}{o(N(a))}$ for all $a \notin Z(G)$ as $p^m \nmid o(N(a))$

$$\Rightarrow p \mid \sum_{a \notin Z(G)} \frac{o(G)}{o(N(a))}$$

$$\Rightarrow p \mid o(G) - \sum_{a \notin Z(G)} \frac{o(G)}{o(N(a))} = o(Z(G))$$

$$\Rightarrow \exists\ x \in Z(G) \text{ s.t. } o(x) = p$$

Let $K = \langle x \rangle \subseteq Z(G) \Rightarrow K$ is normal in G.

Now, $o(G/K) < o(G)$ and $p^m \mid o(G) = \cancel{o(G/K)}$

i.e. $p^m \mid o(G/K) \cdot o(K)$, $p^m \nmid o(G)$

and thus $p^{m-1} \mid p^m \mid o(G/K)$.

(Note in case $m=1$, the result is true by Cauchy's theorem).

By induction hypothesis $\exists$ a subgroup H/K of G/K s.t. $o(H/K) = p^{m-1}$

$$\therefore o(H) = p^m, \quad \frac{H}{K} \leq \frac{G}{K} \Rightarrow H \leq G.$$

Thus result is true in this case also.
Hence, by induction the theorem follows.

3(c) Solve the following LPP by Simplex method:

$$\text{Max } Z = 2x_1 + x_2$$

Sub to
$$2x_1 - 2x_2 \le 1$$
$$2x_1 - 4x_2 \le 3$$
$$2x_1 + x_2 \le 2$$
$$x_1, x_2 \ge 0$$

Does there exist an alternate optimal solution? If yes, give one and hence find all the optimal solutions. (15)

Sol: Adding slack-variables, $s_1, s_2, s_3 \ge 0$

$$\text{Max } Z = 2x_1 + x_2 + 0 \cdot S_1 + 0 \cdot S_2 + 0 \cdot S_3$$

Sub to
$$2x_1 - 2x_2 + S_1 = 1$$
$$2x_1 - 4x_2 + S_2 = 3$$
$$2x_1 + x_2 + S_3 = 2$$
$$x_1, x_2, S_1, S_2, S_3 \ge 0$$

	C_j	2	1	0	0	0	0	
C_B	B.V.	x_1	x_2	S_1	S_2	S_3	b	Ratio
0	S_1	$\boxed{2}$	−2	1	0	0	1	$1/2$ →
0	S_2	2	−4	0	1	0	3	$3/2$
0	S_3	2	1	0	0	1	2	$2/2$
$\Delta_j =$	$Z_j - C_j$	−2 ↑	−1	0	0	0		
2	x_1	1	−1	$1/2$	0	0	$1/2$	—
0	S_2	0	−2	−1	1	0	2	—
0	S_3	0	$\boxed{3}$	−1	0	1	1	$1/3$ →
$\Delta_j = Z_j - C_j$		0	−3 ↑	1	0	0		

	c_j	2	1	0	0	0		
C_B	B.V.	x_1	x_2	s_1	s_2	s_3	b	Ratio
2	x_1	1	0	$\boxed{1/6}$	0	$1/3$	$5/6$	5 →
0	s_2	0	0	$-5/3$	1	$2/3$	$8/3$	
1	x_2	0	$-1/3$	0	$1/3$	$1/3$	$1/3$	
$\Delta_j =$	$z_j - c_j$	0	0	0 ↑	0	1		
0	s_1	6	0	1	0	2	5	
0	s_2	10	0	0	1	4	11	
1	x_2	2	1	0	0	1	2	
$\Delta_j =$	$z_j - c_j$	$\boxed{0}$	0	0	0	1		

Yes, there is alternate optimal solution as value of Δ_j is zero for one non-basic variable.

Two such solutions are

$A\left(\frac{5}{6}, \frac{1}{3}\right)$ and B: $x_2 = 2$, $s_1 = 5$

$\therefore\ 2x_1 - 2x_2 + s_1 = 1 \Rightarrow 2x_1 - 4 + 5 = 1 \Rightarrow x_1 = 0$

i.e. $B(0, 2)$.

Optimal value (Max) $z = 2$

Again, All optimal solutions are given by the points lying on the line-segment joining the points A and B.

$$x_2 - 2 = \frac{2 - 1/3}{0 - 5/6}(x_1 - 0) \quad \text{i.e. } 2x_1 + x_2 = 2.$$

$\therefore$ All optimal solutions : $(t, 2-2t)$ with $0 \le t \le 5/6$