Q128. Find the range of $p(>0)$ for which the series

$\frac{1}{(1+a)^p} - \frac{1}{(2+a)^p} + \frac{1}{(3+a)^p} - \cdots, a > 0$ is

(i) Is absolutely convergent

(ii) Conditionally convergent

**(Year 2018)**

**(20 Marks)**

Q129. Prove the inequality: $\frac{\pi^2}{9} < \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{x}{\sin x} dx < \frac{2\pi^2}{9}$

**(Year 2018)**

**(10 Marks)**

Q130. Show by applying residue theorem that $\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}, a > 0.$

**(Year 2018)**

**(15 Marks)**

Q131. Show that of a function $f$ defined on an open interval (a, b) of $\mathbb{R}$ is convex, then $f$ is a continuous. Show , by example, if the condition of open interval is dropped then the convex function need not to be continous.

**(Year 2018)**

**(15 Marks)**

Q132. Suppose $\mathbb{R}$ be the set of all real numbers and $f: \mathbb{R} \to \mathbb{R}$ is a function such that the following equations hold for all $x, y \in \mathbb{R}$:

(i) $f(x+y) = f(x) + f(y)$

(ii) $f(xy) = f(x)f(y)$

Show that $\forall\, x \in \mathbb{R}$ either $f(x) = 0$ or $f(x) = x$.

**(Year 2018)**

**(20 Marks)**

Q133. Show that the function

$$f(x,y)=\begin{cases}\dfrac{x^2-y^2}{x-y}, & (x,y)\neq(1,-1),(1,1)\\ 0, & (x,y)\neq(1,-1),(1,1)\end{cases}$$

Is continous and differntiable at (-1, 1).

**(Year 2019)**

**(10 Marks)**

Q134. Evaluate

$$\int_0^\infty \frac{\tan^{-1}(ax)}{x(1+x^2)}dx, a>0, a\neq 1$$

**(Year 2019)**

**(10 Marks)**

Q135. Discuss the uniform convergence of

$$f_n(x)=\frac{nx}{1+n^2x^2}, \forall\, x\in\mathbb{R}\,(-\infty,\infty), n=1,2,3,\ldots$$

**(Year 2019)**

**(15 Marks)**

Q136. Find the maximum value of $f(x,y,z)=x^2y^2z^2$ subject to the subsidiary condition

$x^2+y^2+z^2=c^2, (x,y,z>0)$.

**(Year 2019)**

**(15 Marks)**

Q137. Discuss the convergence of $\int_1^2 \frac{\sqrt{x}}{\ln x}dx$

**(Year 2019)**

**(15 Marks)**

find the range of $(p>0)$ for which the series:

$$\frac{1}{(1+a)^p} - \frac{1}{(2+a)^p} + \frac{1}{(3+a)^p} - \cdots, \quad a>0 \text{ is}$$

(i) absolutely convergent and

(ii) conditionally convergent.

Let $\Sigma u_n$ be the given series and $v_n = |u_n|$

Then $\Sigma v_n$ is a series of positive real numbers

and $v_n = \frac{1}{(n+a)^p}$.

Let $w_n = \frac{1}{n^p}$. Then $\lim \frac{v_n}{w_n} = 1$

By comparision test $\Sigma v_n$ is convergent if $p>1$

$\Sigma v_n$ is divergent if $0<p\leq 1$.

Case-1. $p>1$

In this case $\Sigma u_n$ is an alternating series and $\Sigma |u_n|$ is convergent.

Therefore $\Sigma u_n$ is absolutely convergent.

Case-2 $0<p\leq 1$

In this case $\{v_n\}$ is a monotone decreasing sequence of positive real numbers and $\lim v_n = 0$

By Leibnitz's test, $\Sigma (-1)^{n+1} v_n$

i.e. $\Sigma u_n$ is convergent.







$\Sigma |u_n|$ is divergent, $\Sigma u_n$ is conditionally ergent.

Prove the inequality : $\frac{\pi^2}{9} < \int_{\pi/6}^{\pi/2} \frac{x}{\sin x} dx < \frac{2\pi^2}{9}$.

$$1 \le \frac{1}{\sin x} \le 2 \text{ for all } x \in \left[\frac{\pi}{6}, \frac{\pi}{2}\right].$$

Therefore $x \le \frac{x}{\sin x} \le 2x$ for all $x \in [\pi/6, \pi/2]$

Let $f(x) = \frac{x}{\sin x}$

$\phi(x) = x$

$\psi(x) = 2x$, $x \in [\pi/6, \pi/2]$.

$f$ and $\phi$ are both bounded and integrable on $[\pi/6, \pi/2]$ and $f(x) \ge \phi(x)$ for all $x \in [\pi/6, \pi/2]$

Also $f$ and $\phi$ are both continuous at $\pi/3$ and $f(\pi/3) > \phi(\pi/3)$.

Hence, $\int_{\pi/6}^{\pi/2} f(x)\,dx > \int_{\pi/6}^{\pi/2} \phi(x)\,dx$

$$= \int_{\pi/6}^{\pi/2} x\,dx$$

$$= \frac{\pi^2}{9}.$$

$f$ and $\psi$ are both bounded and integrable on $[\pi/6, \pi/2]$ and $f(x) \le \psi(x)$ for all $x \in [\pi/6, \pi/2]$

Also $f$ and $\psi$ are both continuous at $\pi/3$ and $f(\pi/3) < \psi(\pi/3)$.

Hence $\int_{\pi/6}^{\pi/2} f(x)\,dx < \int_{\pi/6}^{\pi/2} \psi(x)\,dx$

$$= 2\int_{\pi/6}^{\pi/2} x\,dx$$

$$= \frac{2\pi^2}{9}$$

Consequently, $\frac{\pi^2}{9} < \int_{\pi/6}^{\pi/2} \frac{x}{\sin x}\,dx < \frac{2\pi^2}{9}$.

Show that if a function $f$ defined on an open interval $(a, b)$ of $\mathbb{R}$ is convex, then $f$ is continuous. Show by example if the condition of open interval is dropped, then the ~~continuous~~ convex function need not be cts.

Suppose $f$ is convex on $(a,b)$ and let $[c,d] \subseteq (a,b)$. Choose $c_1$ and $d_1$ such that

$$a < c_1 < c < d < d_1 < b$$

If $x, y \in [c, d]$ with $x < y$. Then, we know that,

$$\frac{f(y)-f(x)}{y-x} \geq \frac{f(x)-f(c)}{x-c} \geq \frac{f(c)-f(c_1)}{c-c_1}$$

and,
$$\frac{f(y)-f(x)}{y-x} \leq \frac{f(d)-f(y)}{d-y} \leq \frac{f(d_1)-f(d)}{d_1-d}$$

showing the set

$$\left\{ \left| \frac{f(y)-f(x)}{y-x} \right| : c \leq x < y \leq d \right\}$$

is bounded by $M > 0$. It follows that

$$|f(y)-f(x)| \leq M\,|y-x|$$

and therefore $f$ is uniformly continuous on $[c, d]$. Recalling that uniformly cts $\Rightarrow$ continuity.

$\Rightarrow f$ is cts on $[c, d]$.
Since the interval $[c, d]$ was arbitrary, $f$ is continuous on $(a, b)$.

Example: Suppose $a > 0$ and define $f: [a, b] \to \mathbb{R}$ by

$$f(x) = \begin{cases} x^2 & \text{for } a \leq x \leq c \\ (x-c)^2 + c^2 & \text{for } c \leq x \leq b \end{cases}$$

Clearly $f$ is cts at $x=c$ as

$$f(c) = c^2$$

$$\lim_{h \to 0} f(x$$

$$\lim_{x \to c^-} f(x+0) = \lim_{x \to c^-} f(x) = \lim_{x \to c^-} c^2 = c^2$$

$$\lim_{x \to c^+} f(x) = \lim_{x \to c^+} (x-c)^2 + c^2 = c^2$$

$\Rightarrow f$ is cts on $[a, b]$.

but $f$ is not convex on $[a, b]$

as, $f^-(x) = 2x\Big]_{x=c} = 2c$

$f^+(x) = 2(x-c)\Big]_{x=c} = 0$

Now, $f^-(x) \not\leq f^+(x)$

$\forall x \in (a, b)$

$\therefore$ $f$ is not convex on $[a, b]$ as left hand derivative of $f(x)$ is not less than or equal to $f^+(x)$.

Suppose $\mathbb{R}$ be the set of all real numbers and $f : \mathbb{R} \to \mathbb{R}$ is a function such that the following equation hold for all $x, y \in \mathbb{R}$.

(i) $f(x+y) = f(x) + f(y)$

(ii) $f(x.y) = f(x) f(y)$

show that $\forall\ x \in \mathbb{R}$ either $f(x) = 0$ or $f(x) = x$.

Let us consider that $f(x) \neq 0$, then we shall show that $f(x) = x$

As $f(x+x) = f(x) + f(x) = 2f(x)$ —— ①

$$f(2x) = 2f(x) \quad \forall\ x, y \in \mathbb{R}.$$

from ① —

$$f(2) f(x) = 2f(x)$$

$$\Rightarrow f(2) f(x) - 2f(x) = 0$$

$$\therefore f(x)(f(2) - 2) = 0$$

$\because f(x) \neq 0 \Rightarrow f(2) = 2$

Similarly $\forall\ k \in \mathbb{R}$, $f(kx) = f(x + \cdots + x)$

$$= f(x) + \cdots + f(x) = k f(x)$$

$$\Rightarrow f(k) f(x) = k f(x) \text{ —— ①}$$

$$(\because f(x) \neq 0)$$

$$\therefore f(k) = k \quad \forall\ k \in \mathbb{R}.$$

$$\Rightarrow f(x) = x \quad \forall\ x \in \mathbb{R}$$

Now let us assume that $f(x) \neq x$, for some $x$, then need to show that $f(x) = 0$

As, $f(k)f(x) = k f(x)$ (from ①)

$\Rightarrow f(x)[f(k) - k] = 0$

Since, $f(x) = 0$ or $f(k) = k$

but $f(k) \neq k \Rightarrow f(x) = 0$

Hence we are done.