Set-iv

Second Order Partial Differential Equations with Variable Coefficients

§ 1. A partial differential equation is said to be of second order if it contains at least one of the second order partial differential coefficients r, s and t but none of higher order. The differential coefficients p and q may also appear in the equation. Thus the general form of a second order partial differential equation is

$$F(x, y, z, p, q, r, s, t)=0.$$

The complete solutions of these equations will contain two arbitrary functions.

Below we give some examples of equations that are readily solvable. It is to be noted that x and y, being independent, are constant with respect to each other in differentiation and integration.

Solved Examples

Ex. 1. *Solve* $s=\frac{x}{y}+a.$

Sol. The given equation can be written as

$$\frac{\partial^2 z}{\partial x\,\partial y}=\frac{x}{y}+a.$$

Integrating w.r.t. 'x', we get

$$\frac{\partial z}{\partial y}=\frac{1}{y}\cdot\frac{x^2}{2}+ax+f(y).$$

Now integrating w.r.t. 'y', we get

$$z=\frac{x^2}{2}\log y+axy+\int f(y)\,dy+\psi(x)$$

or $$z=\frac{1}{2}x^2 \log y+axy+\phi(y)+\psi(x).$$

Ex. 2. *Solve* $xys=1$.

Sol. The given equation can be written as

$$\frac{\partial^2 z}{\partial x\,\partial y}=\frac{1}{xy}.$$

Integrating w.r.t. 'y', we get

$$\frac{\partial z}{\partial x}=\frac{1}{x}\log y+f(x).$$

Now integrating w.r.t. 'x', we get

$$z=\log x \log y+\int f(x)\,dx+\psi(y)$$

or $$z=\log x \log y+\phi(x)+\psi(y).$$

Ex. 3. *Solve* $s=2x+2y$.

Sol. The given equation can be written as

$$\frac{\partial^2 z}{\partial x\,\partial y}=2x+2y.$$

Integrating w.r.t. 'x', we get

$$\frac{\partial z}{\partial y}=x^2+2xy+f(y).$$

Now integrating w.r.t. 'y', we get

$$z=x^2y+xy^2+\int(y)\,dy+\psi(x)$$

or $$z=x^2y+xy^2+\phi(y)+\psi(x).$$

Ex. 4. *Solve* $t=\sin xy$.

Sol. The given equation can be written as

$$\frac{\partial^2 z}{\partial y^2}=\sin xy.$$

Integrating w.r.t. 'y', we get

$$\frac{\partial z}{\partial y}=-\frac{1}{x}\cos xy+f(x).$$

Again integrating w.r.t. 'y', we get

$$z=-\frac{1}{x^2}\sin xy+y\,f(x)+\psi(x).$$

Ex. 5. *Solve* $ys+p=\cos(x+y)-y\sin(x+y)$.

Sol. The given equation can be written as

$$y\frac{\partial q}{\partial x}+\frac{\partial z}{\partial x}=\cos(x+y)-y\sin(x+y).$$

Integrating w.r.t. 'x', we get

$$yq+z=\sin(x+y)+y\cos(x+y)+f(y)$$

$\frac{\partial}{\partial y}(yz)$

or $$y\frac{\partial z}{\partial y}+z=\sin(x+y)+y\cos(x+y)+f(y).$$

Now integrating w.r.t. 'y', we get
$$yz=y\sin(x+y)+\int f(y)\,dy+\psi(x)$$
or $$yz=y\sin(x+y)+\phi(y)+\psi(x).$$

Ex. 6. *Solve* $t-xq=x^2$.

Sol. The given equation can be written as

$\frac{\partial q}{\partial y}-xq=x^2$, which is linear in q regarding x as constant.

$\text{I.F.}=e^{-\int x\,dy}=e^{-xy}$.

$\therefore$ solution is $qe^{-xy}=\int x^2e^{-xy}\,dy+f(x)$
$$=-xe^{-xy}+f(x)$$
or $$q=\frac{\partial z}{\partial y}=-x+e^{xy}f(x).$$

Integrating, $z=-xy+f(x)\int e^{xy}\,dy+\psi(x)$
or $$z=-xy+\frac{1}{x}f(x)\,e^{xy}+\psi(x)$$
or $$z=-xy+\phi(x)\,e^{xy}+\psi(x).$$

Ex. 7. *Solve* $yt-q=xy$.

Sol. The given equation can be written as

$\frac{\partial q}{\partial y}-\frac{1}{y}q=x$, which is linear in q regarding x as constant.

$\text{I.F.}=e^{\int(-1/y)\,dy}=e^{-\log y}=1/y$.

$\therefore\; q\,\frac{1}{y}=\int x.\frac{1}{y}\,dy+f(x)=x\log y+f(x)$

or $$q=\frac{\partial z}{\partial y}=xy\log y+y\,f(x).$$

Integrating, $z=x\int y\log y\,dy+f(x)\int y\,dy+\psi(x)$
or $$z=x\left[\tfrac{1}{2}y^2\log y-\int\frac{1}{y}.\frac{y^2}{2}dy\right]+\frac{y^2}{2}f(x)+\psi(x)$$
or $$z=\tfrac{1}{2}xy^2\log y-\tfrac{1}{4}xy^2+\tfrac{1}{2}y^2f(x)+\psi(x).$$

Ex. 8. *Solve* $rx=(n-1)\,p$.

Sol. The given equation can be written as
$$\frac{\partial^2z/\partial x^2}{\partial z/\partial x}=\frac{n-1}{x}.$$

Integrating, $\log\frac{\partial z}{\partial x}=(n-1)\log x+\log f(y)$

or
$$\frac{\partial z}{\partial x}=x^{n-1} f(y).$$

Again integrating w.r.t. 'x', we get
$$z=\frac{x^n}{n} f(y)+\psi(y)$$
or
$$z=x^n \phi(y)+\psi(y).$$

Ex. 9. *Solve* $p+r+s=1$. [Kanpur 81]

Sol. The given equation can be written as
$$\frac{\partial z}{\partial x}+\frac{\partial p}{\partial x}+\frac{\partial q}{\partial x}=1.$$

Integrating w.r.t. 'x', we get
$$z+p+q=x+f(y)$$
or
$$p+q=x+f(y)-z.$$

Legrange's auxiliary equations for this are
$$\frac{dx}{1}=\frac{dy}{1}=\frac{dz}{x+f(y)-z}.$$

The first two members give $x-y=a$.

From the last two members, we get
$$\frac{dz}{dy}+z=x+f(y)$$
or
$$\frac{dz}{dy}+z=a+y+f(y),$$ which is linear in z.

$$\text{I.F.}=e^{\int dy}=e^y.$$
$$\therefore \quad ze^y=\int\{a+y+f(y)\}\, e^y\, dy+b$$
$$=ae^y+\int\{y+f(y)\}\, e^y\, dy+b$$
$$=ae^y+\phi(y)+b$$
or
$$z=a+e^{-y}\phi(y)+be^{-y}$$
or
$$z=x-y+e^{-y}\phi(y)+e^{-y}\psi(x-y).$$

Ex. 10. *Solve* $s-t=\dfrac{x}{y^2}$.

Sol. The given equation can be written as
$$\frac{\partial p}{\partial y}-\frac{\partial q}{\partial y}=\frac{x}{y^2}.$$

Integrating w.r.t. 'y', we get
$$p-q=-\frac{x}{y}+f(x).$$

Lagrange's auxiliary equations for this are
$$\frac{dx}{1}=\frac{dy}{-1}=\frac{dz}{-(x/y)+f(x)}.$$

The first two members give, $x+y=a$.

Taking the first and the last members, we get

$$dz=-\frac{x}{y}\,dx+f(x)\,dx$$

or $$dz=-\frac{x}{a-x}\,dx+f(x)\,dx$$

or $$dz=\left(1-\frac{a}{a-x}\right)dx+f(x)\,dx.$$

Integrating, $$z=x+a\log(a-x)+\phi(x)+b$$

or $$z=a\log(a-x)+\psi(x)+b$$

or $$r=(x+y)\log y+\psi(x)+F(x+y).$$

Ex. 11. *Solve* $xr+p=9x^2y^3$.

Sol. The given equation can be written as

$$\frac{\partial p}{\partial x}+\frac{1}{x}p=9xy^3, \text{ which is linear in } p.$$

$$\text{I.F.}=e^{\int(1/x)\,dx}=e^{\log x}=x.$$

$\therefore$ $$px=\int 9x^2y^3\,dx+f(y)$$

or $$px=3x^3y^3+f(y)$$

or $$p=\frac{\partial z}{\partial x}=3x^2y^3+\frac{1}{x}f(y).$$

Integrating w.r.t. 'x', we get

$$z=x^3y^3+f(y)\log x+\phi(y).$$

Ex. 12. *Solve* $r=2y^2$.

Sol. The given equation can be written as

$$\frac{\partial p}{\partial x}=2y^2.$$

Integrating w.r.t. 'x', we get

$$p=\frac{\partial z}{\partial x}=2xy^2+f(y).$$

Again integrating w.r.t. 'x', we get

$$z=x^2y^2+x\,f(y)+\phi(y).$$

Ex. 13. *Solve* $\log s=x+y$.

Sol. The given equation can be written as

$$\frac{\partial^2 z}{\partial x\,\partial y}=e^{x+y}.$$

Integrating w.r.t. 'x', we get

$$\frac{\partial z}{\partial y}=e^{x+y}+f(y).$$

Again integrating w.r.t. 'y', we get

$$z = e^{x+y} + \int f(y)\, dy + \psi(x)$$

or

$$z = e^{x+y} + \phi(y) + \psi(x).$$

Ex. 14. *Solve* $r = 6x$.

Sol. The given equation can be written as

$$\frac{\partial^2 z}{\partial x^2} = 6x.$$

Integrating w.r.t. 'x', we get

$$\frac{\partial z}{\partial x} = 3x^2 + f(y).$$

Again integrating w.r.t. 'x', we get

$$z = x^3 + x\, f(y) + \phi(y).$$

Ex. 15. *Solve* $s = 0$.

Sol. The given equation can be written as

$$\frac{\partial^2 z}{\partial x\, \partial y} = 0.$$

Integrating w.r.t. 'x', we get

$$\frac{\partial z}{\partial y} = f(y).$$

Again integrating w.r.t. 'y', we get

$$z = \int f(y)\, dy + \psi(x)$$

or

$$z = \phi(y) + \psi(x).$$

Ex. 16. *Solve* $xr + 2p = 0$.

Sol. The given equation can be written as

$$x \frac{\partial p}{\partial x} + 2p = 0$$

or

$$x^2 \frac{\partial p}{\partial x} + 2xp = 0.$$

Integrating w.r.t. 'x', we get

$$x^2 p = f(y)$$

or

$$p = \frac{\partial z}{\partial x} = \frac{1}{x^2} f(y).$$

Again integrating w.r.t. 'x', we get

$$z = -\frac{1}{x} f(y) + \phi(y).$$

Ex. 17. *Solve* $xr = p$.

Sol. The given equation can be written as

$$x\frac{\partial^2 z}{\partial x^2}=\frac{\partial z}{\partial x}$$

or $$\frac{\partial^2 z/\partial x^2}{\partial z/\partial x}=\frac{1}{x}.$$

Integrating w.r.t. 'x', we get

$$\log\frac{\partial z}{\partial x}=\log x+\log f(y)$$

or $$\frac{\partial z}{\partial x}=x\,f(y).$$

Again integrating w.r.t. 'x', we get

$$z=\tfrac{1}{2}x^2 f(y)+\phi(y).$$

Ex. 18. *Solve* $ar=xy$.

Sol. The given equation can be written as

$$a\frac{\partial^2 z}{\partial x^2}=xy.$$

Integrating w.r.t. 'x', $a\frac{\partial z}{\partial x}=\frac{x^2}{2}y+f(y).$

Again integrating w.r.t. 'x', we get

$$az=\frac{1}{6}x^3y+x\,f(y)+\phi(y).$$

Ex. 19. *Solve* $xs+q=4x+2y+2$.

Sol. The given equation can be written as

$$x\frac{\partial p}{\partial y}+\frac{\partial z}{\partial y}=4x+2y+2.$$

Integrating w.r.t. 'y', we get

$$xp+z=4xy+y^2+2y+f(x)$$

or $$x\frac{\partial z}{\partial x}+z=4xy+y^2+2y+f(x).$$

Again integrating w.r.t. 'x', we get

$$zx=2x^2y+xy^2+2xy+\phi(x)+\psi(y).$$

Ex. 20. *Solve* $t+s+q=0$.

Sol. The given equation can be written as

$$\frac{\partial q}{\partial y}+\frac{\partial p}{\partial y}+\frac{\partial z}{\partial y}=0.$$

Integrating w.r.t. 'y', we get

$$q+p+z=f(x)$$

or $$p+q=f(x)-z.$$

Lagrange's auxiliary equations are

$$\frac{dx}{1}=\frac{dy}{1}=\frac{dz}{f(x)-z}.$$

From the first two members, we get $x-y=a$.

Taking the first and the last members, we get

$$\frac{dz}{dx}+z=f(x), \text{ which is linear in } z.$$

$$\text{I.F.}=e^{\int dx}=e^x.$$

$$\therefore \qquad ze^x=\int e^x f(x)\, dx+b=\phi(x)+b$$

or $$ze^x=\phi(x)+\psi(x-y).$$

Ex. 21. *Solve* $2yq+y^2t=1$.

Sol. The given equation can be written as

$$2y\frac{\partial z}{\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=1.$$

Integrating w.r.t. 'y', we get

$$y^2\frac{\partial z}{\partial y}=y+f(x)$$

or $$\frac{\partial z}{\partial y}=\frac{1}{y}+\frac{1}{y^2}f(x).$$

Again integrating w.r.t. 'y', we get

$$z=\log y-\frac{1}{y}f(x)+\phi(x)$$

or $$yz=y\log y-f(x)+y\phi(x).$$

Solutions of equations under given conditions.

After finding the general solutions by the usual methods, the geometrical conditions given in the problem are used to find the arbitrary functions.

Ex. 22. *Find the surface passing through the parabolas* $z=0,\ y^2=4ax$ *and* $z=1,\ y^2=-4ax$ *and satisfying the equation*

$xr+2p=0.$ **[Meerut 89 ; Rohilkhand 76 ; Agra 73, 86]**

Sol. The given equation can be written as

$$x\frac{\partial p}{\partial x}+2p=0$$

or $$x^2\frac{\partial p}{\partial x}+2px=0.$$

Integrating w.r.t. 'x', we get

$$px^2=f(y)$$

or $$p=\frac{\partial z}{\partial x}=\frac{1}{x^2}\ f(y).$$

Again integrating w.r.t. 'x', we get

$$z=-\frac{1}{x}f(y)+\phi\ (y). \qquad \ldots(1)$$

Now using the geometrical conditions of the problem we are to determine the values of $f(y)$ and $\phi\ (y)$.

Since the required surface is to pass through the parabola $z=0$, $y^2=4ax$ hence putting $z=0$ and $x=y^2/4a$ in (1), we get

$$0=-\frac{4a}{y^2}f(y)+\phi\ (y). \qquad \ldots(2)$$

Again, putting $z=1$ and $x=-y^2/4a$ in (1), we get

$$1=\frac{4a}{y^2}f(y)+\phi\ (y). \qquad \ldots(3)$$

Solving (2) and (3) for $f(y)$ and $\phi\ (y)$, we have

$$\phi(y)=\tfrac{1}{2} \text{ and } f(y)=y^2/8a.$$

Thus we have determined the arbitrary functions.

Putting the values of $f(y)$ and $\phi\ (y)$ in (1), the required surface is

$$z=-\frac{1}{x}\cdot\frac{y^2}{8a}+\tfrac{1}{2}$$

or $$8axz-4ax+y^2=0.$$

Ex. 23. *Find a surface passing through the two lines $z=x=0$, $z-1=x-y=0$, satisfying $r-4s+4t=0$.*

[**Meerut 90 ; Rohilkhand 80 ; Agra 81 ; I.A.S. 77**]

Sol. The given equation can be written as

$$(D^2-4DD'+4D'^2)\ z=0. \qquad \ldots(1)$$

A.E. is $$m^2-4m+4=0$$

or $$(m-2)^2=0. \qquad \therefore \quad m=2,2.$$

Hence the general solution of (1) is

$$z=\phi_1\ (y+2x)+x\ \phi_2\ (y+2x). \qquad \ldots(2)$$

If the surface (2) passes through the lines

$$z=x=0 \text{ and } z-1=x-y=0,$$

then we have

$$0=\phi_1\ (y+2x), \qquad \ldots(3)$$

and $$1=\phi_1\ (y+2x)+x\ \phi_2\ (y+2x). \qquad \ldots(4)$$

From (3) and (4), we get

$$\phi_2\ (y+2x)=\frac{1}{x}=\frac{3}{3x}=\frac{3}{2x+y}. \qquad [\because \quad y-x=0]$$

Putting the values of ϕ_1 and ϕ_2 in (2), the required surface is

$$z=x.\frac{3.}{2x+y}, \quad \text{or } z\,(2x+y)=3x.$$

Ex. 24. *Find a surface satisfying $t=6x^3y$ containing the two lines* $y=0=z$, $y=1=z$.

(Meerut 84, 88 ; Agra 70, 85 ; Kanpur 86]

Sol. The given equation can be written as

$$\frac{\partial q}{\partial y}=6x^3y.$$

Integrating w.r.t. 'y, we get

$$q=\frac{\partial z}{\partial y}=3x^3y^2+f(x).$$

Again integrating w.r.t. 'y', we get

$$z=x^3y^3+y\,f(x)+\phi\,(x). \qquad \ldots(1)$$

Putting $y=0$, $z=0$ in (1), we get

$$0=\phi\,(x). \qquad \ldots(2)$$

Again putting $y=1$, $z=1$ in (1), we get

$$1=x^3+f(x)+\phi\,(x). \qquad \ldots(3)$$

From (2) and (3), $\phi(x)=0$, $f(x)=1-x^3$.

Putting the values of $f(x)$ and $\phi(x)$ in (1), the required surface is

$$z=x^3y^3+y\,(1-x^3).$$

Ex. 25. *A surface is drawn satisfying $r+t=0$ and touching $x^2+z^2=1$ along its section by $y=0$. Find its equation in the form*

$$z^2\,(x^2+z^2-1)=y^2\,(x^2+z^2).$$ **[Agra 83]**

Sol. The given equation can be written as

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0 \quad \text{or} \quad (D^2+D'^2)\,z=0. \qquad \ldots(1)$$

A.E. is $m^2+1=0$. $\therefore$ $m=\pm i$.

Hence the general solution of the equation (1) is

$$z=\phi_1\,(y+ix)+\phi_2\,(y-ix). \qquad \ldots(2)$$

From (2), $p=\dfrac{\partial z}{\partial x}=i\phi_1'\,(y+ix)-i\,\phi_2'\,(y-ix)$.

and $$q=\frac{\partial z}{\partial y}=\phi_1'\,(y+ix)+\phi_2'(y-ix).$$

Also from $x^2+z^2=1$, $\qquad \ldots(3)$

$$p=\frac{\partial z}{\partial x}=\frac{-x}{\sqrt{(1-x^2)}}, \quad q=\frac{\partial z}{\partial y}=0.$$

If the surface (2) touches the surface (3) along its section by the plane $y=0$ then the values of p and q from (2) and (3) must be equal at $y=0$.

Thus $$i\phi_1'(ix)-i\phi_2'(-ix)=-\frac{x}{\sqrt{(1-x^2)}} \qquad ...(4)$$

and $$\phi_1'(ix)+\phi_2'(-ix)=0. \qquad ...(5)$$

Solving (4) and (5), we get

$$\phi_1'(ix)=\frac{xi}{2\sqrt{(1+i^2x^2)}}, \quad \phi_2'(-ix)=\frac{-xi}{2\sqrt{(1+i^2x^2)}}.$$

$$\therefore \quad \phi_1'(u)=\frac{u}{2\sqrt{(1+u^2)}}, \text{ so that } \phi_1(u)=\tfrac{1}{2}\sqrt{(1+u^2)}+c_1$$

and $$\phi_2'(v)=\frac{v}{2\sqrt{(1+v^2)}}, \text{ so that } \phi_2(v)=\tfrac{1}{2}\sqrt{(1+v^2)}+c_2.$$

These give $\phi_1(y+ix)=\frac{1}{2}\sqrt{\{1+(y+ix)^2\}}+c_1$

and $\phi_2(y-ix)=\frac{1}{2}\sqrt{\{1+(y-ix)^2\}}+c_2$.

$$\therefore \quad z=\tfrac{1}{2}\sqrt{\{1+(y+ix)^2\}}+\tfrac{1}{2}\sqrt{\{1+(y-ix)^2\}}+c_1+c_2$$

or $$z=\tfrac{1}{2}\left[\sqrt{\{1+(y+ix)^2\}}+\sqrt{\{1+(y-ix)^2\}}\right]+c, \qquad ...(6)$$

where $c=c_1+c_2$.

Equating the two values of z from (3) and (6) when $y=0$, we have

$$\sqrt{(1-x^2)}=\sqrt{(1-x^2)}+c. \qquad \therefore \quad c=0.$$

Hence, from (6), we get

$$2z=\sqrt{\{1+(y+ix)^2\}}+\sqrt{\{1+(y-ix)^2\}}$$

or $$2z-\sqrt{\{1+(y+ix)^2\}}=\sqrt{\{1+(y-ix)^2\}}.$$

Squaring both sides, we get

$$4z^2+1+(y+ix)^2-4z\sqrt{\{1+(y+ix)^2\}}=1+(y-ix)^2$$

or $$z^2+ixy=z\sqrt{\{1+(y+ix)^2\}}.$$

Again squaring both sides, we get

$$z^4+2ixyz^2+i^2x^2y^2=z^2\{1+(y+ix)^2\}$$

or $$z^4-x^2y^2=z^2+z^2y^2-z^2x^2$$

or $$z^2(z^2+x^2-1)=y^2(z^2+x^2).$$

Ex. 26. *Find a surface satisfying*

$$2x^2r-5xys+2y^2t+2(px+qy)=0$$

and touching the hyperbolic paraboloid $z=x^2-y^2$ along its section by the plane $y=1$.

Sol. The given equation can be written as

$$2x^2\frac{\partial^2 z}{\partial x^2}-5xy\frac{\partial^2 z}{\partial x\,\partial y}+2y^2\frac{\partial^2 z}{\partial y^2}+2x\frac{\partial z}{\partial x}+2y\frac{\partial z}{\partial y}=0.$$

Putting $x=e^X, y=e^Y$ and denoting $\frac{\partial}{\partial X}$ and $\frac{\partial}{\partial Y}$ by D and D' respectively, the given equation transforms to

$$[2D(D-1)-5DD'+2D'(D'-1)+2D+2D']\,z=0$$

or
$$(2D-D')(D-2D')\,z=0.$$

$$\therefore \quad z=\phi_1(2Y+X)+\phi_2(Y+2X)$$
$$=\phi_1(2\log y+\log x)+\phi_2(\log y+2\log x)$$
$$=\phi_1(\log xy^2)+\phi_2(\log x^2y)$$

or
$$z=f_1(xy^2)+f_2(x^2y). \quad \ldots(1)$$

From (1), $p=\dfrac{\partial z}{\partial x}=y^2f_1'(xy^2)+2xyf_2'(x^2y)$

and
$$q=\frac{\partial z}{\partial y}=2xyf_1'(xy^2)+x^2f_2'(x^2y).$$

Also from $z=x^2-y^2$, $\quad\ldots(2)$

$$p=\frac{\partial z}{\partial x}=2x,\ q=\frac{\partial z}{\partial y}=-2y.$$

If the surface (2) touches the surface (1) along its section by the plane $y=1$ then the values of p and q from (1) and (2) must be equal at $y=1$.

$$\therefore \quad f_1'(x)+2xf_2'(x^2)=2x \quad \ldots(3)$$

and
$$2xf_1'(x)+x^2f_2'(x^2)=-2. \quad \ldots(4)$$

Solving (3) and (4), we get

$$f_1'(x)=-\frac{4}{3x}-\frac{2}{3}x$$

and $f_2'(x^2)=\dfrac{4}{3}+\dfrac{2}{3x^2}$ or $f_2'(u)=\dfrac{4}{3}+\dfrac{2}{3u}$, where $u=x^2$.

$$\therefore \quad f_1(x)=-\frac{4}{3}\log x-\frac{1}{3}x^2+c_1$$

and
$$f_2(u)=\frac{4}{3}u+\frac{2}{3}\log u+c_2.$$

These give

$$f_1(xy^2)=-\frac{4}{3}\log(xy^2)-\frac{1}{3}(xy^2)^2+c_1$$

and
$$f_2(x^2y)=\frac{4}{3}x^2y+\frac{2}{3}\log(x^2y)+c_2.$$

Putting the values of $f_1(xy^2)$, $f_2(x^2y)$ in (1), we get

$$z=-\frac{4}{3}\log(xy^2)-\frac{1}{2}x^2y^4+\frac{4}{3}x^2y+\frac{2}{3}\log(x^2y)+c_1+c_2$$

$$=-\frac{4}{3}\log(xy^2)+\frac{2}{3}\log(x^2y)-\frac{1}{3}x^2y^4+\frac{4}{3}x^2y+c \qquad ...(5)$$

where $c=c_1+c_2$.

Equating the two values of z from (2) and (5) when $y=1$, we have

$$(x^2-1)=-\frac{4}{3}\log x+\frac{2}{3}\log x^2-\frac{1}{3}x^2+\frac{4}{3}x^2+c \Rightarrow c=-1.$$

Putting $c=-1$ in (5), the required surface is

$$z=\frac{4}{3}x^2y-\frac{1}{3}x^2y^4-2\log y-1$$

or $$3z=4x^2y-x^2y^4-6\log y-3.$$

Ex. 27. *Find a surface satisfying $r+s=0$ and touching the elliptic paraboloid $z=4x^2+y^2$ along its section by the plane $y=2x+1$.* [Meerut 71, 77, 82, 87]

Sol. The given equation can be written as

$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial x\,\partial y}=0$$

or $$(D^2+DD')z=0 \text{ or } D(D+D')z=0.$$

$$\therefore \quad z=\phi_1(y)+\phi_2(y-x). \qquad ...(1)$$

From (1), $p=\frac{\partial z}{\partial x}=-\phi_2'(y-x)$

and $$q=\frac{\partial z}{\partial y}=\phi_1'(y)+\phi_2'(y-x).$$

Also from, $z=4x^2+y^2$, ...(2)

$$p=\frac{\partial z}{\partial x}=8x \text{ and } q=\frac{\partial z}{\partial y}=2y.$$

If the surface (1) touches the surface (2) along its section by the plane $y=2x+1$ then the values of p and q from (1) and (2) must be equal when $y=2x+1$.

$$\therefore \quad -\phi_2'(y-x)=8x, \qquad ...(3)$$

for any (x, y) pair

$$\phi_1'(y)+\phi_2'(y-x)=2y, \qquad ...(4)$$

and $$y=2x+1. \Rightarrow y-x=x+1 \Rightarrow x=y-x-1 \qquad ...(5)$$

From (3) and (5), we get

$$-\phi_2'(y-x)=8(y-x-1).$$

Integrating, $-\phi_2(y-x)=8\left[\frac{1}{2}(y-x)^2-(y-x)\right]+a.$

$$\therefore \quad \phi_2(y-x)=-4(y-x)^2+8(y-x)+c_1.$$

Again from (3) and (4), we get

$$\phi_1'(y)=8x+2y$$

or $$\phi_1'(y)=\frac{8}{2}(y-1)+2y=6y-4.$$

Integrating, $\phi_1(y)=3y^2-4y+c_2$.

Substituting the values of $\phi_1(y)$ and $\phi_2(y-x)$ in (1), we get

$$z=3y^2-4y-4(y-x)^2+8(y-x)+c_1+c_2$$

or $$z=-4x^2-y^2+8xy+4y-8x+c, \quad \ldots(6)$$

where $$c=c_1+c_2.$$

Equating the two values of z from (2) and (6), when $y=2x+1$, we have

$$-4x^2-(2x+1)^2+4(2x+1)^2-8x+c=4x^2+(2x+1)^2.$$

$$\therefore \quad c=-2.$$

Putting the value of c in (6) the required surface is

$$z+4x^2+y^2-8xy-4y+8x+2=0.$$

Ex. 28. *Find a surface satisfying $r=6x+2$ and touching $z=x^3+y^3$ along its section by the plane $x+y+1=0$.* **[Agra 78, 82]**

Sol. The given equation can be written as

$$\frac{\partial^2 z}{\partial x^2}=6x+2.$$

Integrating, $$\frac{\partial z}{\partial x}=3x^2+2x+\phi_1(y).$$

Again, integrating, $$z=x^3+x^2+x\phi_1(y)+\phi_2(y) \quad \ldots(1)$$

From (1), $$p=\frac{\partial z}{\partial x}=3x^2+2x+\phi_1(y),$$

$$q=\frac{\partial z}{\partial y}=x\phi_1'(y)+\phi_2'(y).$$

Also from $$z=x^3+y^3, \quad \ldots(2)$$

$$p=\frac{\partial z}{\partial x}=3x^2 \text{ and } q=\frac{\partial z}{\partial y}=3y^2.$$

If the surface (1) touches the surface (2) along its section by the plane $x+y+1=0$ then the values of p and q from (1) and (2) must be equal when $x+y+1=0$.

$$\therefore \quad 3x^2+2x+\phi_1(y)=3x^2, \quad \ldots(3)$$

$$x\phi_1'(y)+\phi_2'(y)=3y^2, \quad \ldots(4)$$

and $$x+y+1=0. \quad \ldots(5)$$

From (3) and (5), we get

$$\phi_1(y)=-2x=2(y+1) \Rightarrow \phi_1'(y)=2.$$

Substituting the value of $\phi_1'(y)$ in (4), we have

$$\phi_2'(y)=3y^2-2x=3y^2+2(y+1).$$

Integrating, $\phi_2(y)=y^3+y^2+2y+k$.

Putting the values of $\phi_1(y)$ and $\phi_2(y)$ in (1), we get

$$z=x^3+x^2+2x(y+1)+y^3+y^2+2y+k. \quad \ldots(6)$$

Equating the two values of z from (2) and (6), when $y=-(x+1)$, we get

$$x^3-(x+1)^3=x^3+x^2+2x(-x)-(x+1)^3+(x+1)^2-2(x+1)+k.$$

$$\therefore \qquad k=1.$$

Putting the value of k in (6), the required surface is

$$z=x^3+x^2+2x(y+1)+y^3+y^2+2y+1$$

or $$z=x^3+y^3+(x+y+1)^2$$

Ex. 29. *Show that the surface satisfying $r-2s+t=6$ and touching the hyperbolic paraboloid $z=xy$ along its section the plane $y=x$ is $z=x^2-xy+y^2$.*

Sol. The given equation can be written as

$$(D^2-2DD'+D'^2)z=0 \qquad \ldots(1)$$

or $$(D-D')^2 z=6.$$

$$\therefore \quad \text{C.F.}=\phi_1(y+x)+x\phi_2(y+x).$$

$$\text{Now P.I.}=\frac{1}{(D-D')^2}6=\frac{1}{D^2}\left(1-\frac{D'}{D}\right)^{-2}6$$

$$=\frac{1}{D^2}\left\{1+\frac{2D'}{D}+\ldots\right\}6=\frac{1}{D^2}6=3x^2.$$

Hence the general solution of (1) is

$$z=\text{C.F.}+\text{P.I.}=\phi_1(y+x)+x\phi_2(y+x)+3x^2. \qquad \ldots(2)$$

From (2),

$$p=\frac{\partial z}{\partial x}=\phi_1'(y+x)+x\phi_2'(y+x)+\phi_2(y+x)+6x$$

and $$q=\frac{\partial z}{\partial y}=\phi_1'(y+x)+x\phi_2'(y+x).$$

Also from $z=xy$, $\qquad \ldots(3)$

$$p=\frac{\partial z}{\partial x}=y, \qquad q=\frac{\partial z}{\partial y}=x.$$

If the surface (3) touches the surface (2) along its section by the plane $y=x$ then the values of p and q from (2) and (3) must be equal at $y=x$.

$$\therefore \quad \phi_1'(2x)+x\phi_2'(2x)+\phi_2(2x)+6x=x \qquad \ldots(4)$$

and $$\phi_1'(2x)+x\phi_2'(2x)=x \qquad \ldots(5)$$

Solving (4) and (5), we get $\phi_2(2x)=-6x$

so that $\qquad \phi_2(u)=-3u$

or $\qquad \phi_2(y+x)=-3(y+x).$

Putting $\phi_2(2x)=-6x$ and $\phi_2'(2x)=-3$ in (4), we get

$$\phi_1'(2x)-3x-6x+6x=x$$

or $$\phi_1'(2x)=4x \text{ so that } \phi_1'(u)=2u.$$

$\therefore$ $$\phi_1(u)=u^2+c$$

or $$\phi_1(y+x)=(y+x)^2+c.$$

Putting the values of $\phi_1(y+x)$ and $\phi_2(y+x)$ in (2), we get

$$z=(y+x)^2-3x(y+x)+3x^2+c=x^2-xy+y^2+c \quad ...(6)$$

Equating the two values of z from (3) and (6) when $y=x$, we get

$$x^2-x^2+x^2+c=x^2 \quad \text{or} \quad c=0.$$

Putting the value of c in (6) the required surface is

$$z=x^2-xy+y^2.$$

Ex. 30. *Solve the equation $r+t=2s$ and determine the arbitrary functions by the conditions that $bz=y^2$ when $x=0$ and $az=x^2$ when $y=0$.*

Sol. The given equation can be written as

$$r-2s+t=0$$

or $$(D^2-2DD'+D'^2)z=0$$

or $$(D-D')^2 z=0. \quad ...(1)$$

Hence the general solution of (1) is

$$z=\phi_1(y+x)+x\phi_2(y+x). \quad ...(2)$$

Putting $x=0$, $z=y^2/b$ in (2), we get

$$\frac{y^2}{b}=\phi_1(y) \text{ so that } \phi_1(y+x)=\frac{(y+x)^2}{b}.$$

Again putting $y=0$, $z=x^2/a$ in (2), we get

$$\frac{x^2}{a}=\phi_1(x)+x\phi_2(x)=\frac{x^2}{b}+x\,\phi_2(x).$$

$$\therefore \quad \phi_2(x)=\frac{b-a}{ab}x \text{ so that } \phi_2(y+x)=\frac{b-a}{ab}(y+x).$$

Putting the values of $\phi_1(y+x)$ and $\phi_2(y+x)$ in (2) the required solution is

$$z=\frac{(y+x)^2}{b}+x\frac{b-a}{ab}(y+x)=(y+x)\left(\frac{x}{a}+\frac{y}{b}\right).$$

§2. Canonical Forms (Method of Transformations).

Now we shall consider the equation of the type

$$Rr+Ss+Tt+f(x,y,z,p,q)=0, \quad ...(1)$$

where R, S, T are continuous functions of x and y possessing continuous partial derivatives of as high an order as necessary.

We shall show that any equation of the type (1) can be reduced to one of the three canonical forms by a suitable change of the independent variables. Suppose we change the independent variables from x, y to u, v, where

$$u=u(x, y),\ v=v\ (x, y). \qquad \ldots(2)$$

Then, we have

$$p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x},\quad q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}.$$

$$\therefore\quad \frac{\partial}{\partial x}\equiv\frac{\partial u}{\partial x}\cdot\frac{\partial}{\partial u}+\frac{\partial v}{\partial x}\cdot\frac{\partial}{\partial v},\ \frac{\partial}{\partial y}\equiv\frac{\partial u}{\partial y}\cdot\frac{\partial}{\partial u}+\frac{\partial v}{\partial y}\cdot\frac{\partial}{\partial v}.$$

Now $$r=\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\left(\frac{\partial u}{\partial x}\frac{\partial}{\partial u}+\frac{\partial v}{\partial x}\frac{\partial}{\partial v}\right)\left(\frac{\partial u}{\partial x}\frac{\partial z}{\partial u}+\frac{\partial v}{\partial x}\frac{\partial z}{\partial v}\right)$$

$$\left(\frac{\partial u}{\partial x}\frac{\partial}{\partial u}+\frac{\partial v}{\partial x}\frac{\partial}{\partial v}\right)\left(\frac{\partial u}{\partial x}\frac{\partial z}{\partial u}+\frac{\partial v}{\partial x}\frac{\partial z}{\partial v}\right)=\frac{\partial u}{\partial x}\frac{\partial}{\partial u}\left(\frac{\partial u}{\partial x}\frac{\partial z}{\partial u}+\frac{\partial v}{\partial x}\frac{\partial z}{\partial v}\right)+\frac{\partial v}{\partial x}\frac{\partial}{\partial v}\left(\frac{\partial u}{\partial x}\frac{\partial z}{\partial u}+\frac{\partial v}{\partial x}\frac{\partial z}{\partial v}\right)$$

$$=\frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2+2\frac{\partial^2 z}{\partial u\,\partial v}\cdot\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2+\frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x^2}+\frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial x^2},$$

$$s=\frac{\partial^2 z}{\partial x\,\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\left(\frac{\partial u}{\partial x}\frac{\partial}{\partial u}+\frac{\partial v}{\partial x}\cdot\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\right)$$

$$=\frac{\partial^2 z}{\partial u^2}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial^2 z}{\partial u\,\partial v}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)+\frac{\partial^2 z}{\partial v^2}\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}+\frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial y\,\partial x}+\frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial y\,\partial x}$$

and $$t=\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\left(\frac{\partial u}{\partial y}\frac{\partial}{\partial u}+\frac{\partial v}{\partial y}\frac{\partial}{\partial v}\right)\left(\frac{\partial u}{\partial y}\frac{\partial z}{\partial u}+\frac{\partial v}{\partial y}\frac{\partial z}{\partial v}\right)$$

$$=\frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial y}\right)^2+2\frac{\partial^2 z}{\partial u\,\partial v}\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial y}+\frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial y}\right)^2+\frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial y^2}+\frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial y^2}.$$

Substituting these values of p, q, r, s and t in (1), it takes the form

$$A\frac{\partial^2 z}{\partial u^2}+2B\frac{\partial^2 z}{\partial u\,\partial v}+C\frac{\partial^2 z}{\partial v^2}+F\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right)=0 \qquad \ldots(3)$$

where $$A=R\left(\frac{\partial u}{\partial x}\right)^2+S\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+T\left(\frac{\partial u}{\partial y}\right)^2, \qquad \ldots(4)$$

$$B=R\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x}+\tfrac{1}{2}S\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)+T\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}. \qquad \ldots(5)$$

$$C=R\left(\frac{\partial v}{\partial x}\right)^2+S\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}+T\left(\frac{\partial v}{\partial y}\right)^2 \qquad \ldots(6)$$

and the function F is the transformed form of the function f.

Now the problem is to determine u and v so that the equation (3) takes the simplest possible form. The procedure is simple when the discriminant S^2-4RT of the quadratic equation

$$R\lambda^2+S\lambda+T=0 \qquad \ldots(7)$$

is everywhere either positive, negative or zero, and we shall discuss these three cases separately.

Case I. $S^2-4RT>0$. If this condition is satisfied then the roots λ_1, λ_2 of the equation (7) are real and distinct. The coefficients of $\dfrac{\partial^2 z}{\partial u^2}$ and $\dfrac{\partial^2 z}{\partial v^2}$ in the equation (3) will vanish if we choose u and v such that

$$\frac{\partial u}{\partial x}=\lambda_1\frac{\partial u}{\partial y}. \qquad \ldots(8)$$

and

$$\frac{\partial v}{\partial x}=\lambda_2\frac{\partial v}{\partial y}. \qquad \ldots(9)$$

The differential equations (8) and (9) will determine the form of u and v as functions of x and y.

For this, from (8), Lagrange's auxiliary equations are

$$\frac{dx}{1}=\frac{dy}{-\lambda_1}=\frac{du}{0}.$$

The last member gives $du=0$ *i.e.*, $u=$constant.

The first two members give

$$\frac{dy}{dx}+\lambda_1=0. \qquad \ldots(10)$$

Let $f_1(x, y)=$constant be the solution of the equation (10).

Then the solution of the equation (8) can be taken as

$$u=f_1(x, y). \qquad \ldots(11)$$

Similarly, if $f_2(x, y)=$constant is a solution of

$$\frac{dy}{dx}+\lambda_2=0,$$

then the solution of the equation (9) can be taken as

$$v=f_2(x, y). \qquad \ldots(12)$$

Also it can be easily seen that, in general,

$$AC-B^2=(4RT-S^2)\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)^2,$$

so that when A and C are zero

$$B^2=(S^2-4RT)\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)^2, \qquad \ldots(13)$$

It follows that $B^2 > 0$ since $S^2-4RT > 0$ and hence we can divide both sides of the equation by it. (1) by B^2.

Thus making the substitutions defined by the equations (11) and (12), the equation (1) transforms to the form

$$\frac{\partial^2 z}{\partial u\,\partial v}=\phi\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right). \qquad ...(14)$$

which is the canonical form in this case.

Case II. $S^2-4RT=0$. In this case the roots of the equation (7) are equal. We define the function u as in Case I and take v to be any function of x and y, which is independent of u. Then, we have, as before, $A=0$.

Since $S^2-4RT=0$, hence from (13), $B^2=0$ *i.e.*, $B=0$.

On the other hand, in this case, $C\neq 0$, otherwise v would be a function of u.

Putting $A=0, B=0$ and dividing by C, we see that in this case the canonical form of the equation (1) is,

$$\frac{\partial^2 z}{\partial v^2}=\phi\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right). \qquad ...(15)$$

Case III. $S^2-4RT<0$. Formally it is the same as Case I except that now the roots of the equation (7) are complex.

Proceeding as in Case I, we find that the equation (1) reduces to the form (14) but that the variables u, v are not real but are in fact complex conjugates.

To find a real cononical form let $u=\alpha+i\beta$, $v=\alpha-i\beta$

so that $\qquad \alpha=\frac{1}{2}(u+v),\ \beta=\frac{1}{2}i\,(v-u).$

Now $\dfrac{\partial z}{\partial u}=\dfrac{\partial z}{\partial \alpha}\dfrac{\partial \alpha}{\partial u}+\dfrac{\partial z}{\partial \beta}\cdot\dfrac{\partial \beta}{\partial u}=\frac{1}{2}\left(\dfrac{\partial z}{\partial \alpha}-i\dfrac{\partial z}{\partial \beta}\right).$

Similarly $\dfrac{\partial z}{\partial v}=\frac{1}{2}\left(\dfrac{\partial z}{\partial \alpha}+i\dfrac{\partial z}{\partial \beta}\right).$

$$\therefore \quad \frac{\partial^2 z}{\partial u\,\partial v}=\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)=\frac{1}{4}\left(\frac{\partial}{\partial \alpha}-i\frac{\partial}{\partial \beta}\right)\left(\frac{\partial z}{\partial \alpha}-i\frac{\partial z}{\partial \beta}\right)$$

$$=\frac{1}{4}\left(\frac{\partial^2 z}{\partial \alpha^2}+\frac{\partial^2 z}{\partial \beta^2}\right).$$

Thus, transforming the independent variables u, v to α, β the desired canonical form is

$$\frac{\partial^2 z}{\partial \alpha^2}+\frac{\partial^2 z}{\partial \beta^2}=\psi\left(\alpha, \beta, z, \frac{\partial z}{\partial \alpha}, \frac{\partial z}{\partial \beta}\right). \qquad ...(16)$$

Note:-

Second order partial differential equations of the type (1) are classified by their canonical forms ; we say that an equation of this type is (i.e an eqn is said to be)

(i) Hyperbolic if $S^2-4RT>0$,

(ii) Parabolic if $S^2-4RT=0$,

(iii) Elliptic if $S^2-4RT<0$.

Solved Examples

Ex. 1. *Reduce the equation*

$$(y-1)\,r-(y^2-1)\,s+y\,(y-1)\,t+p-q=2ye^{2x}\,(1-y)^3 \qquad \ldots(1)$$

to canonical form and hence solve it. **[Rohilkhand 82]**

Sol. Comparing the equation (1) with

$$Rr+Ss+Tt+f\,(x, y, z, p, q)=0, \text{ we have}$$

$$R=(y-1),\ S=-(y^2-1),\ T=y\,(y-1).$$

The quadratic equation $R\lambda^2+S\lambda+T=0$ therefore becomes

$$(y-1)\,\lambda^2-(y^2-1)\,\lambda+y\,(y-1)=0$$

or $$\lambda^2-(y+1)\,\lambda+y=0$$

or $$(\lambda-1)\,(\lambda-y)=0 \Rightarrow \lambda=1,\ y \text{ (real and distinct roots).}$$

The equations $\dfrac{dy}{dx}+\lambda_1=0$ and $\dfrac{dy}{dx}+\lambda_2=0$ become

$$\frac{dy}{dx}+1=0 \text{ and } \frac{dy}{dx}+y=0.$$

These on integration give

$$x+y=\text{constant and } ye^x=\text{constant},$$

so that to change the independent variables from x, y to u, v, we take

$$u=x+y \quad \text{and} \quad v=ye^x.$$

$$\therefore\quad p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}=\frac{\partial z}{\partial u}+ye^x\,\frac{\partial z}{\partial v}=\frac{\partial z}{\partial u}+v\,\frac{\partial z}{\partial v}.$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}=\frac{\partial z}{\partial u}+e^x\frac{\partial z}{\partial v}.$$

$$r=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\left(\frac{\partial}{\partial u}+v\,\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u}+v\,\frac{\partial z}{\partial v}\right)$$

$$=\frac{\partial^2 z}{\partial u^2}+2v\,\frac{\partial^2 z}{\partial u\,\partial v}+v^2\,\frac{\partial^2 z}{\partial v^2}+v\,\frac{\partial z}{\partial v}.$$

$$s=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}+e^x\,\frac{\partial z}{\partial v}\right)$$

$$=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)+e^x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)+e^x\frac{\partial z}{\partial v}$$

$$=\left(\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u}\right)+e^x\left(\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial v}\right)+e^x\frac{\partial z}{\partial v}$$

$$=\frac{\partial^2 z}{\partial u^2}+(e^x+v)\frac{\partial^2 z}{\partial u\,\partial v}+ve^x\frac{\partial^2 z}{\partial v^2}+e^x\frac{\partial z}{\partial v}$$

and $$t=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}+e^x\frac{\partial z}{\partial v}\right)=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)+e^x\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)$$

$$=\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}$$
$$+e^x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$=\frac{\partial^2 z}{\partial u^2}+2e^x\frac{\partial^2 z}{\partial u\,\partial v}+e^{2x}\frac{\partial^2 z}{\partial v^2}$$

Substituting these values in (1), it reduces to

$$(1-y)^3\,e^x\frac{\partial^2 z}{\partial u\,\partial v}=2y\,e^{2x}\,(1-y)^3$$

or $$\frac{\partial^2 z}{\partial u\,\partial v}=2y\,e^x \quad \text{or} \quad \frac{\partial^2 z}{\partial u\,\partial v}=2v, \qquad \ldots(2)$$

which is the canonical form of the equation (1).

Integrating (2) w.r.t. v, we get

$$\frac{\partial z}{\partial u}=v^2+\phi_1'(u), \qquad \ldots(3)$$

where $\phi_1(u)$ is an arbitrary function of u.

Again integrating (3) w.r.t. u, we get

$$z=uv^2+\psi_1(u)+\psi_2(v),$$

where ψ_1 is an integral of ϕ_1 and ψ_2 is an arbitrary function

or $$z=(x+y)\,y^2\,e^{2x}+\psi_1(x+y)+\psi_2(ye^x).$$

Ex. 2. *Reduce the equation*

$$\frac{\partial^2 z}{\partial x^2}=x^2\frac{\partial^2 z}{\partial y^2}, \text{ to canonical form.}$$ [Kanpur 82]

Sol. The given equation can be written as

$$r-x^2t=0. \qquad \ldots(1)$$

Comparing the equation (1) with

$$Rr+Ss+Tt+f(x,y,z,p,q)=0, \text{ we have}$$
$$R=1,\ S=0,\ T=-x^2.$$

The quadratic equation in $R\lambda^2+S\lambda+T=0$ therefore becomes

$$\lambda^2-x^2=0 \Rightarrow \lambda=x,\ -x \text{ (real and distinct roots).}$$

The equations $\dfrac{dy}{dx}+\lambda_1=0$ and $\dfrac{dy}{dx}+\lambda_2=0$ become

$$\frac{dy}{dx}+x=0 \text{ and } \frac{dy}{dx}-x=0.$$

These on integration give

$$y+\tfrac{1}{2}x^2=\text{constant and } y-\tfrac{1}{2}x^2=\text{constant},$$

so that to change the independent variables from x, y to u, v, we take

$$u=y+\tfrac{1}{2}x^2 \text{ and } v=y-\tfrac{1}{2}x^2.$$

$$\therefore\ p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}=x\frac{\partial z}{\partial u}-x\frac{\partial z}{\partial v}=x\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right).$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}=\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}.$$

$$r=\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left\{x\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)\right\}$$

$$=x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)+1.\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)$$

$$=x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)\cdot\frac{\partial v}{\partial x}\right]+\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}$$

$$=x^2\left(\frac{\partial^2 z}{\partial u^2}-2\frac{\partial^2 z}{\partial u\,\partial v}+\frac{\partial^2 z}{\partial v^2}\right)+\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}$$

and
$$t=\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\left(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)$$

$$=\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\,\partial v}+\frac{\partial^2 z}{\partial v^2}.$$

Substituting these values in (1), it reduces to

$$\frac{\partial^2 z}{\partial u\,\partial v}=\frac{1}{4x^2}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)$$

or
$$\frac{\partial^2 z}{\partial u\,\partial v}=\frac{1}{4(u-v)}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right),$$

which is the required canonical form of the given equation.

Ex. 3. *Reduce the equation*

$$\frac{\partial^2 z}{\partial x^2}+2\frac{\partial^2 z}{\partial x\,\partial y}+\frac{\partial^2 z}{\partial y^2}=0$$

to canonical form and hence solve it. **[G.N.D.U. 87 ; Raj. 83]**

Sol. The given equation can be written as

$$r+2s+t=0. \qquad \ldots(1)$$

Comparing the equation (1) with

$Rr+Ss+Tt+f(x, y, z, p, q)=0$, we have

$R=1,\ S=2,\ T=1.$

The quadratic equation $R\lambda^2+S\lambda+T=0$ is therefore given by

$\lambda^2+2\lambda+1=0$, or $(\lambda+1)^2=0$.

$\therefore\quad \lambda=-1, -1.$ (equal roots).

The equation $\dfrac{dy}{dx}+\lambda=0$ becomes $\dfrac{dy}{dx}-1=0$,

which on integration gives $x-y=$ constant.

To change the independent variables x, y to u, v we take

$u=x-y$.

We have to take v as some function of x and y independent of u. Let $v=x+y$.

Then $p=\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial x}+\dfrac{\partial z}{\partial v}\cdot\dfrac{\partial v}{\partial x}=\dfrac{\partial z}{\partial u}+\dfrac{\partial z}{\partial v}$,

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}=-\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v},$$

$$r=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\left(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)$$

$$=\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2},$$

$$s=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\left(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right)\left(-\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)=-\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}$$

and $$t=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\left(-\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right)\left(-\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)$$

$$=\frac{\partial^2 z}{\partial u^2}-2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}.$$

Substituting these values in (1), it reduces to

$\dfrac{\partial^2 z}{\partial v^2}=0$ which is the required canonical form.

Integrating it w.r.t. v, we get

$$\frac{\partial z}{\partial v}=\phi_1(u).$$

Again integrating w.r.t. v, we get

$$z=v\,\phi_1(u)+\phi_2(u),$$

where ϕ_1 and ϕ_2 are arbitrary functions of u.

Hence the solution is

$$z=(x+y)\,\phi_1(x-y)+\phi_2(x-y).$$

Ex. 4. *Reduce the equation*

$$\frac{\partial^2 z}{\partial x^2}+x^2\frac{\partial^2 z}{\partial y^2}=0,$$

to canonical form. **[Kanpur 82]**

Sol. The given equation can be written as

$$r+x^2 t=0. \quad ...(1)$$

Comparing the equation (1) with

$Rr+Ss+Tt+f(x, y, z, p, q)=0$, we have

$R=1,\ S=0,\ T=x^2.$

The quadratic equation $R\lambda^2+S\lambda+T=0$ is therefore given by

$\lambda^2+x^2=0 \Rightarrow \lambda=ix,\ -ix.$ (Complex roots)

The equations $\dfrac{dy}{dx}+\lambda_1=0$ and $\dfrac{dy}{dx}+\lambda_2=0$ become

$$\frac{dy}{dx}+ix=0 \text{ and } \frac{dy}{dx}-ix=0.$$

These on integration give

$y+\frac{1}{2}ix^2=\text{constant}$ and $y-\frac{1}{2}ix^2=\text{constant}$,

so that to change the independent variables from x, y to u, v, we take

$u=y+\frac{1}{2}ix^2=\alpha+i\beta$ (say)

and $v=y-\frac{1}{2}ix^2=\alpha-i\beta.$

Then $\alpha=y,\quad \beta=\frac{1}{2}x^2.$

Now we shall transform the independent variables x and y to α and β. We have

$$p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial x}+\frac{\partial z}{\partial \beta}\cdot\frac{\partial \beta}{\partial x}=x\frac{\partial z}{\partial \beta},$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial y}+\frac{\partial z}{\partial \beta}\cdot\frac{\partial \beta}{\partial y}=\frac{\partial z}{\partial \alpha},$$

$$r=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial \beta}\right)=1.\frac{\partial z}{\partial \beta}+x.\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \beta}\right)$$

$$=\frac{\partial z}{\partial \beta}+x\left[\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \alpha}{\partial x}+\frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right)\cdot\frac{\partial \beta}{\partial x}\right]=\frac{\partial z}{\partial \beta}+x^2\frac{\partial^2 z}{\partial \beta^2}$$

and $$t=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial \alpha}\right)=\frac{\partial^2 z}{\partial \alpha^2}.$$

Substituting these values in (1), it reduces to

$$\frac{\partial^2 z}{\partial \alpha^2}+\frac{\partial^2 z}{\partial \beta^2}=-\frac{1}{2\alpha}\frac{\partial z}{\partial \alpha}, \quad -\frac{1}{2\beta}\frac{\partial z}{\partial \beta}$$

which is the required canonical form of (1)

Ex. 5. *Reduce the equation*

$$(n-1)^2 \frac{\partial^2 z}{\partial x^2} - y^{2n} \frac{\partial^2 z}{\partial y^2} = ny^{2n-1} \frac{\partial z}{\partial y},$$

to canonical form and find its general solution.

[Meerut 84; Kanpur 83; Raj. 82]

Sol. The given equation can be written as

$$(n-1)^2\, r - y^{2n}\, t - ny^{2n-1}\, q = 0. \qquad \ldots(1)$$

Comparing the equation (1) with

$Rr+Ss+Tt+f(x, y, z, p, q)=0$, we have

$$R=(n-1)^2, \quad S=0, \quad T=-y^{2n}.$$

The quadratic equation $R\lambda^2+S\lambda+T=0$ is therefore given by

$$(n-1)^2 \lambda^2 - y^{2n} = 0 \text{ or } \lambda^2 = \frac{1}{(n-1)^2} y^{2n}.$$

$$\therefore \quad \lambda = \frac{1}{(n-1)} y^n, -\frac{1}{(n-1)} y^n. \qquad \text{(Real and distinct roots).}$$

The equations $\dfrac{dy}{dx}+\lambda_1=0$ and $\dfrac{dy}{dx}+\lambda_2=0$ become

$$\frac{dy}{dx}+\frac{1}{(n-1)} y^n = 0 \text{ and } \frac{dy}{dx}-\frac{1}{(n-1)} y^n = 0$$

or $$(n-1)\, y^{-n}\, dy + dx = 0 \text{ and } (n-1)\, y^{-n}\, dy - dx = 0.$$

These on integration give

$$x - y^{-n+1} = \text{constant and } x + y^{-n+1} = \text{constant},$$

so that to change the independent variables from x, y to u, v, we take

$$u = x - y^{-n+1} \text{ and } v = x + y^{-n+1}.$$

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v},$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} = (n-1)\, y^{-n} \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right),$$

$$r = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u\, \partial v} + \frac{\partial^2 z}{\partial v^2}$$

and $$t = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left[(n-1)\, y^{-n}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\right]$$

$$= -n\,(n-1)\, y^{-n-1}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) + (n-1)\, y^{-n} \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

$$= -n\,(n-1)\, y^{-n-1}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

$$+(n-1)\,y^{-n}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$=-n\,(n-1)\,y^{-n-1}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)$$

$$+(n-1)^2\,y^{-2n}\left[\frac{\partial^2 z}{\partial u^2}-2\,\frac{\partial^2 z}{\partial u\,\partial v}+\frac{\partial^2 z}{\partial v^2}\right].$$

Substituting these values in (1), it reduces to

$$\frac{\partial^2 z}{\partial u\,\partial v}=0,$$ which is the required canonical form.

Integrating it, w.r.t. v, we get

$$\frac{\partial z}{\partial u}=\phi_1(u),$$ where ϕ_1 is an arbitrary function.

Again integrating w.r.t. u, we get

$$z=\psi_1(u)+\psi_2(v),$$

where ψ_1 and ψ_2 are arbitrary functions.

Hence the required general solution is

$$z=\psi_1(x-y^{-n+1})+\psi_2(x+y^{-n+1}).$$

Ex. 6. *Reduce the equation*

$$y^2\frac{\partial^2 z}{\partial x^2}-2xy\frac{\partial^2 z}{\partial x\partial y}+x^2\frac{\partial^2 z}{\partial y^2}=\frac{y^2}{x}\frac{\partial z}{\partial x}+\frac{x^2}{y}\frac{\partial z}{\partial y}$$

to canonical form and hence solve it. [I.A.S. 82, 85]

Sol. The given equation can be written as

$$y^2 r-2xys+x^2 t-\frac{y^2}{x}p-\frac{x^2}{y}q=0. \qquad \ldots(1)$$

Comparing the equation (1) with

$Rr+Ss+Tt+f(x,y,z,p,q)=0$, we have

$R=y^2,\ S=-2xy,\ T=x^2.$

The quadratic equation $R\lambda^2+S\lambda+T=0$ is therefore given by

$$y^2\lambda^2-2xy\,\lambda+x^2=0 \quad\text{or}\quad (y\lambda-x)^2=0.$$

$\therefore\quad \lambda=\dfrac{x}{y},\dfrac{x}{y}$ (equal roots).

The equation $\dfrac{dy}{dx}+\lambda=0$ becomes $\dfrac{dy}{dx}+\dfrac{x}{y}=0$

or $\qquad ydy+xdx=0.$

Integrating, we get $x^2+y^2=$ constant.

To change the independent variables x, y to u, v, we take

$$u=x^2+y^2.$$

We have to take v as some function of x and y independent of u, let $v=x^2-y^2$.

Then $p=\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial u}\cdot\dfrac{\partial u}{\partial x}+\dfrac{\partial z}{\partial v}\cdot\dfrac{\partial v}{\partial x}=2x\left(\dfrac{\partial z}{\partial u}+\dfrac{\partial z}{\partial v}\right)$,

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}=2y\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right),$$

$$\begin{aligned}r&=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left\{2x\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)\right\}\\&=2\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)+2x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)\\&=2\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)+2x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)\cdot\frac{\partial u}{\partial x}\right.\\&\qquad\left.+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)\cdot\frac{\partial v}{\partial x}\right]\\&=2\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)+4x^2\left[\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}\right],\end{aligned}$$

$$\begin{aligned}s&=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial x}\left[2y\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)\right]=2y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)\\&=2y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)\cdot\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)\cdot\frac{\partial v}{\partial x}\right]\\&=4xy\left[\frac{\partial^2 z}{\partial u^2}-\frac{\partial^2 z}{\partial v^2}\right]\end{aligned}$$

and
$$\begin{aligned}t&=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial y}\left[2y\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)\right]\\&=2\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)+2y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)\\&=2\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)+2y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)\cdot\frac{\partial u}{\partial y}\right.\\&\qquad\left.+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)\cdot\frac{\partial v}{\partial y}\right]\\&=2\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)+4y^2\left[\frac{\partial^2 z}{\partial u^2}-2\frac{\partial^2 z}{\partial u\,\partial v}+\frac{\partial^2 z}{\partial v^2}\right].\end{aligned}$$

Substituting these values in (1), it reduces to

$\dfrac{\partial^2 z}{\partial v^2}=0$, which is the required canonical form.

Integrating it w.r.t. v, we get

$$\frac{\partial z}{\partial v}=\phi_1(u).$$

Again integrating w.r.t. v, we get

$$z=v\,\phi_1(u)+\phi_2(u),$$

where ϕ_1 and ϕ_2 are arbitrary functions of u.

Hence the solution is

$$z=(x^2-y^2)\,\phi_1\,(x^2+y^2)+\phi_2\,(x^2+y^2).$$

Ex. 7. *Reduce the equation*

$$xyr-(x^2-y^2)\,s-xyt+py-qx=2\,(x^2-y^2)$$

to canonical form and hence solve it.

Sol. The given equation can be written as

$$xyr-(x^2-y^2)\,s-xyt+py-qx-2\,(x^2-y^2)=0. \qquad \ldots(1)$$

Comparing the equation (1) with

$Rr+Ss+Tt+f\,(x, y, z, p, q)=0$, we have

$R=xy,\ S=-(x^2-y^2),\ T=-xy.$

The quadratic equation $R\lambda^2+S\lambda+T=0$ therefore becomes

$$xy\lambda^2-(x^2-y^2)\,\lambda-xy=0 \text{ or } (y\lambda-x)\,(x\lambda+y)=0$$

$$\Rightarrow \qquad \lambda=\frac{x}{y},\ -\frac{y}{x} \text{ (distinct roots).}$$

The equations

$$\frac{dy}{dx}+\lambda_1=0 \text{ and } \frac{dy}{dx}+\lambda_2=0 \text{ become}$$

$$\frac{dy}{dx}+\frac{x}{y}=0 \quad\text{and}\quad \frac{dy}{dx}-\frac{y}{x}=0.$$

These on integration give

$$x^2+y^2=\text{constant and } y/x=\text{constant},$$

so that to change the independent variables from x, y to u, v, we take

$$u=x^2+y^2 \quad\text{and}\quad v=y/x.$$

$$\therefore\quad p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=2x\frac{\partial z}{\partial u}-\frac{y}{x^2}\frac{\partial z}{\partial v},$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=2y\frac{\partial z}{\partial u}+\frac{1}{x}\frac{\partial z}{\partial v},$$

$$r=\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(2x\frac{\partial z}{\partial u}-\frac{y}{x^2}\frac{\partial z}{\partial v}\right)$$

$$=2\frac{\partial z}{\partial u}+2x\left\{\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)\right\}+\frac{2y}{x^3}\frac{\partial z}{\partial v}-\frac{y}{x^2}\left\{\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)\right\}$$

$$=2\frac{\partial z}{\partial u}+2x\left\{\frac{\partial^2 z}{\partial u^2}\frac{\partial u}{\partial x}+\frac{\partial^2 z}{\partial u\,\partial v}\cdot\frac{\partial v}{\partial x}\right\}$$

$$+\frac{2y}{x^3}\frac{\partial z}{\partial v}-\frac{y}{x^2}\left\{\frac{\partial^2 z}{\partial u\,\partial v}\frac{\partial u}{\partial x}+\frac{\partial^2 z}{\partial v^2}\frac{\partial v}{\partial x}\right\}$$

$$=4x^2\frac{\partial^2 z}{\partial u^2}-4\frac{y}{x}\frac{\partial^2 z}{\partial u\,\partial v}+\frac{y^2}{x^4}\frac{\partial^2 z}{\partial v^2}+2\frac{\partial z}{\partial u}+\frac{2y}{x^3}\frac{\partial z}{\partial v},$$

$$s=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial x}\left(2y\frac{\partial z}{\partial u}+\frac{1}{x}\frac{\partial z}{\partial v}\right)$$
$$=4xy\ \frac{\partial^2 z}{\partial u^2}+\left(2-\frac{2y^2}{x^2}\right)\frac{\partial^2 z}{\partial u\,\partial v}-\frac{y}{x^3}\frac{\partial^2 z}{\partial v^2}-\frac{1}{x^2}\frac{\partial z}{\partial v}$$

and
$$t=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial y}\left(2y\frac{\partial z}{\partial u}+\frac{1}{x}\frac{\partial z}{\partial v}\right)$$
$$=4y^2\frac{\partial^2 z}{\partial u^2}+\frac{4y}{x}\frac{\partial^2 z}{\partial u\,\partial v}+\frac{1}{x^2}\frac{\partial^2 z}{\partial v^2}+2\frac{\partial z}{\partial u},$$

Substituting these values in (1), it reduces to

$$\frac{\partial^2 z}{\partial u\,\partial v}=\frac{v^2-1}{(v^2+1)^2},$$

which is the required canonical form.

Integrating it w.r.t. 'v', we get

$$\frac{\partial z}{\partial u}=\int\frac{v^2-1}{(v^2+1)^2}\ dv+\phi_1(u).$$

Now $\displaystyle\int\frac{v^2-1}{(v^2+1)^2}\ dv=\int\frac{v^2+1-2}{(v^2+1)^2}\ dv=\int\frac{dv}{v^2+1}-\int\frac{2dv}{(v^2+1)^2}.$

Let $\displaystyle I=\int\frac{dv}{v^2+1}.$

Integrating by parts taking unity as the second function

$$\int\frac{dv}{v^2+1}=\frac{v}{v^2+1}+\int\frac{2v^2}{(v^2+1)^2}\,dv$$
$$=\frac{v}{v^2+1}+2\int\frac{dv}{v^2+1}-2\int\frac{dv}{(v^2+1)^2}.$$

$$\therefore\quad \int\frac{dv}{v^2+1}-2\int\frac{dv}{(v^2+1)^2}=-\frac{v}{v^2+1}.$$

$$\therefore\quad \frac{\partial z}{\partial u}=-\frac{v}{v^2+1}+\phi_1(u).$$

Now integrating it w.r.t. 'u', we get

$$z=-\frac{uv}{v^2+1}+\psi_1(u)+\psi_2(v),$$

where ψ_1 and ψ_2 are arbitrary functions.

Hence the required solution is

$$z=-xy+\psi_1(x^2+y^2)+\psi_2(y/x).$$

Ex. 8. *Reduce the equation*

$$x^2r-2xys+y^2t-xp+3yq=8y/x$$

to canonical form and hence solve it.

Sol. The given equation can be written as

$$x^2r-2xys+y^2t-xp+3yq-(8y/x)=0 \qquad \ldots(1)$$

Comparing the equation (1) with

$$Rr+Ss+Tt+f(x,y,z,p,q)=0,$$

we have

$$R=x^2, \quad S=-2xy, \quad T=y^2.$$

The quadratic equation $R\lambda^2+S\lambda+T=0$ is therefore given by

$$x^2\lambda^2-2xy\lambda+y^2=0 \quad \text{or} \quad (x\lambda-y)^2=0.$$

$$\therefore \qquad \lambda=\frac{y}{x},\ \frac{y}{x} \text{ (equal roots).}$$

The equation $\dfrac{dy}{dx}+\lambda=0$ becomes

$$\frac{dy}{dx}+\frac{y}{x}=0$$

or
$$ydx+xdy=0 \Rightarrow xy=\text{constant}.$$

To change the independent variables x, y to u, v, we take

$$u=xy.$$

We have to take v as some function of x and y independent of u, let $v=y/x$.

Then $p=\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial x}+\dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial x}=y\dfrac{\partial z}{\partial u}-\dfrac{y}{x^2}\dfrac{\partial z}{\partial v}$,

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=x\frac{\partial z}{\partial u}+\frac{1}{x}\frac{\partial z}{\partial v},$$

$$r=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(y\frac{\partial z}{\partial u}-\frac{y}{x^2}\frac{\partial z}{\partial v}\right)$$

$$=y^2\frac{\partial^2 z}{\partial u^2}-\frac{2y^2}{x^2}\frac{\partial^2 z}{\partial u\,\partial v}+\frac{y^2}{x^4}\frac{\partial^2 z}{\partial v^2}+\frac{2y}{x^3}\frac{\partial z}{\partial v},$$

$$s=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial u}+\frac{1}{x}\frac{\partial z}{\partial v}\right)$$

$$=xy\frac{\partial^2 z}{\partial u^2}-\frac{y}{x^3}\frac{\partial^2 z}{\partial v^2}-\frac{1}{x^2}\frac{\partial z}{\partial v}+\frac{\partial z}{\partial u}$$

and
$$t=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial y}\left(x\frac{\partial z}{\partial u}+\frac{1}{x}\frac{\partial z}{\partial v}\right)$$

$$=x^2\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\,\partial v}+\frac{1}{x^2}\frac{\partial^2 z}{\partial v^2}.$$

Substituting these values in (1), it reduces to

$$v\frac{\partial^2 z}{\partial v^2}+2\frac{\partial z}{\partial v}=2, \qquad ...(2)$$

which is the required canonical form.

Let $\dfrac{\partial z}{\partial v}=Z$. Then (2) becomes

$$\frac{\partial Z}{\partial v}+\frac{2}{v}Z=\frac{2}{v}, \text{ which is linear.}$$

$$\text{I.F.}=e^{\int(2/v)\,dv}=e^{2\log v}=v^2.$$

$$\therefore \quad v^2 Z=\int\frac{2}{v}.v^2\,dv+\phi_1(u)=v^2+\phi_1(u).$$

$$\therefore \quad Z=\frac{\partial z}{\partial v}=1+\frac{1}{v^2}\phi_1(u).$$

Integrating this again w.r.t. 'v', we get

$$z=v-\frac{1}{v}\phi_1(u)+\phi_2(u),$$

where ϕ_1 and ϕ_2 are arbitrary functions of u.

Hence the solution is

$$z=\frac{y}{x}-\frac{x}{y}\phi_1(xy)+\phi_2(xy)$$

or
$$z=\frac{y}{x}-\frac{x^2}{xy}\phi_1(xy)+\phi_2(xy)$$

or
$$z=\frac{y}{x}+x^2\psi_1(xy)+\phi_2(xy),$$

where $\psi_1(xy)=-\dfrac{1}{xy}\phi_1(xy)$.

Ex. 9. *Reduce the equation*

$$x^2(y-1)r-x(y^2-1)s+y(y-1)t+xyp-q=0 \qquad ...(1)$$

to canonical form and hence solve it.

Sol. Comparing the equation (1) with

$Rr+Ss+Tt+f(x,y,z,p,q)=0$, we have

$R=x^2(y-1)$, $S=-x(y^2-1)$, $T=y(y-1)$.

The quadratic equation $R\lambda^2+S\lambda+T=0$ therefore becomes

$$x^2(y-1)\lambda^2-x(y^2-1)\lambda+y(y-1)=0$$

or
$$x^2\lambda^2-x(y+1)\lambda+y=0$$

or
$$(x\lambda-y)(x\lambda-1)=0.$$

$$\therefore \quad \lambda=\frac{y}{x},\frac{1}{x}.$$

The equations $\dfrac{dy}{dx}+\lambda_1=0$ and $\dfrac{dy}{dx}+\lambda_2=0$ become

$$\frac{dy}{dx}+\frac{y}{x}=0 \quad \text{and} \quad \frac{dy}{dx}+\frac{1}{x}=0.$$

These on integration give

$$xy=\text{constant and } xe^y=\text{constant},$$

so that to change the independent variables from x, y to u, v, we take

$$u=xy \quad \text{and} \quad v=xe^y.$$

$$\therefore \quad p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=y\frac{\partial z}{\partial u}+e^y\frac{\partial z}{\partial v},$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=x\frac{\partial z}{\partial u}+xe^y\frac{\partial z}{\partial v},$$

$$r=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(y\frac{\partial z}{\partial u}+e^y\frac{\partial z}{\partial v}\right)$$

$$=y^2\frac{\partial^2 z}{\partial u^2}+2ye^y\frac{\partial^2 z}{\partial u\,\partial v}+e^{2y}\frac{\partial^2 z}{\partial v^2},$$

$$s=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial u}+xe^y\frac{\partial z}{\partial v}\right)$$

$$=xy\frac{\partial^2 z}{\partial u^2}+xe^y(1+y)\frac{\partial^2 z}{\partial u\,\partial v}+xe^{2y}\frac{\partial^2 z}{\partial v^2}+e^y\frac{\partial z}{\partial v}+\frac{\partial z}{\partial u}$$

and
$$t=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial y}\left(x\frac{\partial z}{\partial u}+xe^y\frac{\partial z}{\partial v}\right)$$

$$=x^2\frac{\partial^2 z}{\partial u^2}+2x^2e^y\frac{\partial^2 z}{\partial u\,\partial v}+x^2e^{2y}\frac{\partial^2 z}{\partial v^2}+xe^y\frac{\partial z}{\partial v}.$$

Substituting these values in (1), it reduces to

$$\frac{\partial^2 z}{\partial u\,\partial v}=0,$$

which is the required canonical form.

Integrating it w.r.t. 'v', we get

$$\frac{\partial z}{\partial u}=\phi(u).$$

Now integrating w.r.t. 'u', we get

$$z=\phi_1(u)+\phi_2(v),$$

where ϕ_1 and ϕ_2 are arbitrary functions.

Hence the required solution is

$$z=\phi_1(xy)+\phi_2(xe^y).$$

Ex. 10. *Reduce the equation*

$$x(xy-1)r-(x^2y^2-1)s+y(xy-1)t+(x-1)p+(y-1)q=0 \quad \ldots(1)$$

to canonical form and hence solve it.

Sol. Comparing the equation (1) with

$Rr+Ss+Tt+f(x, y, z, p, q)=0$, we have

$$R=x(xy-1),\ S=-(x^2y^2-1),\ T=y(xy-1).$$

The quadratic equation $R\lambda^2+S\lambda+T=0$ therefore becomes

$$x(xy-1)\lambda^2-(x^2y^2-1)\lambda+y(xy-1)=0$$

or $$x\lambda^2-(xy+1)\lambda+y=0 \quad \text{or} \quad (x\lambda-1)(\lambda-y)=0.$$

$$\therefore \qquad \lambda=\frac{1}{x},\ y.$$

The equations $\dfrac{dy}{dx}+\lambda_1=0$ and $\dfrac{dy}{dx}+\lambda_2=0$ become

$$\frac{dy}{dx}+\frac{1}{x}=0 \quad \text{and} \quad \frac{dy}{dx}+y=0.$$

These on integration give

$$xe^y=\text{constant and } ye^x=\text{constant},$$

so that to change the independent variables from x, y to u, v, we have

$$u=xe^y \quad \text{and} \quad v=ye^x.$$

$$\therefore \quad p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=e^y\frac{\partial z}{\partial u}+ye^x\frac{\partial z}{\partial v},$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=xe^y\frac{\partial z}{\partial u}+e^x\frac{\partial z}{\partial v},$$

$$r=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(e^y\frac{\partial z}{\partial u}+ye^x\frac{\partial z}{\partial v}\right)$$

$$=e^{2y}\frac{\partial^2 z}{\partial u^2}+2ye^{x+y}\frac{\partial^2 z}{\partial u\,\partial v}+y^2e^{2x}\frac{\partial^2 z}{\partial v^2}+ye^x\frac{\partial z}{\partial v},$$

$$s=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial x}\left(xe^y\frac{\partial z}{\partial u}+e^x\frac{\partial z}{\partial v}\right)$$

$$=xe^{2y}\frac{\partial^2 z}{\partial u^2}+(xy+1)e^{x+y}\frac{\partial^2 z}{\partial u\,\partial v}+ye^{2x}\frac{\partial^2 z}{\partial v^2}+e^y\frac{\partial z}{\partial u}+e^x\frac{\partial z}{\partial v}$$

and $$t=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial y}\left(xe^y\frac{\partial z}{\partial u}+e^x\frac{\partial z}{\partial v}\right)$$

$$=x^2e^{2y}\frac{\partial^2 z}{\partial u^2}+2xe^{x+y}\frac{\partial^2 z}{\partial u\,\partial v}+e^{2x}\frac{\partial^2 z}{\partial v^2}+xe^y\frac{\partial z}{\partial u}.$$

Substituting these values in (1), it reduces to

$$\frac{\partial^2 z}{\partial u \partial v}=0,$$

which is the required canonical form.

Integrating it w.r.t. 'v', we get

$$\frac{\partial z}{\partial u}=\phi(u).$$

Now integrating w.r.t. 'u', we get

$$z=\phi_1(u)+\phi_2(v),$$

where ϕ_1 and ϕ_2 are arbitrary functions.

Hence the solution is

$$z=\phi_1(xe^y)+\phi_2(ye^x).$$