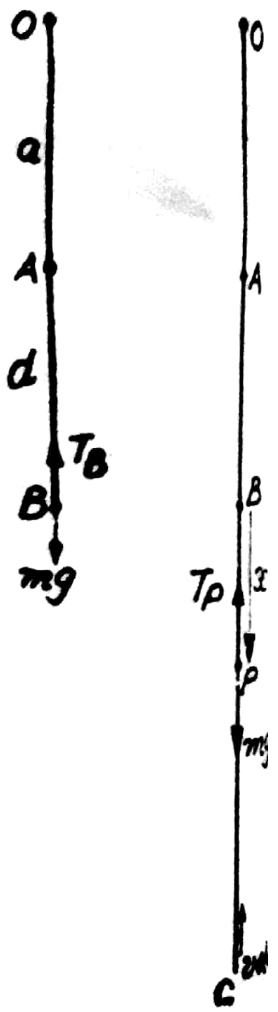
(b) A heavy particle is attached to one end of an elastic string, the other end of which is fixed. The modulus of elasticity of the string is equal to the weight of the particle. The string is drawn vertically down till it is four times its natural length a and then let go. Find the time taken by the particle to return to the starting point. **8**

Ex. 59. *A heavy particle is attached to one end of an elastic string, the other end of which is fixed. The modulus of elasticity of the string is equal to the weight of the particle. The string is drawn vertically down till it is four times its natural length and then let go. Show that the particle will return to this point in time* $\sqrt{\left(\frac{a}{g}\right)}\left[\frac{4\pi}{3}+2\sqrt{3}\right]$, *where a is the natural length of the string.*

[Lucknow 1976; Kanpur 83; Agra 80; Meerut 88]



Sol. Let $OA=a$ be the natural length of an elastic string whose one end is fixed at O. Let B be the position of equilibrium of a particle of mass m attached to the other end of the string and $AB=d$. If T_B is the tension in the string OB, then by Hooke's law,

$$T_B=\lambda\frac{OB-OA}{OA}=\lambda\frac{d}{a},$$

where λ is the modulus of elasticity of the string. Considering the equilibrium of the particle at B, we have

$$mg=T_B=\lambda\frac{d}{a}=mg\frac{d}{a}, \quad \left[\because \quad \lambda=mg, \text{ as given}\right]$$

$$\therefore \quad d=a.$$

Now the particle is pulled down to a point C such that $OC=4a$ and then let go. It starts moving towards B with velocity zero at C. Let P be the position of the particle at time t, where $BP=x$. [**Note** that we have taken the position of equilibrium B as origin. The direction BP is that of x increasing and the direction PB is that of x decreasing.]

When the particle is at P, there are two forces acting upon it.

(i) The tension $T_P=\lambda\frac{a+x}{a}=\frac{mg}{a}(a+x)$ in the string OP acting in the direction PO *i.e.*, in the direction of x decreasing.

(ii) The weight mg of the particle acting vertically downwards *i.e.*, in the direction of x increasing.

Hence by Newton's second law of motion ($P=mf$), the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2}=mg-\frac{mg}{a}(a+x)=-\frac{mgx}{a}.$$

Thus $$\frac{d^2x}{dt^2}=-\frac{g}{a}x, \qquad \ldots(1)$$

which is the equation of a S.H.M. with centre at the origin B and the amplitude $BC=2a$ which is greater than $AB=a$.

Multiplying both sides of (1) by $2(dx/dt)$ and integrating w.r.t. t, we have

$$\left(\frac{dx}{dt}\right)^2=-\frac{g}{a}x^2+k, \text{ where } k \text{ is a constant.}$$

At the point C, $x=BC=2a$, and the velocity $dx/dt=0$; —

$$\therefore \quad k=\frac{g}{a}\cdot 4a^2.$$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2=\frac{g}{a}(4a^2-x^2). \qquad \ldots(2)$$

Taking square root of (2), we have

$$\frac{dx}{dt}=-\sqrt{\left(\frac{g}{a}\right)}\sqrt{(4a^2-x^2)},$$

the —ive sign has been taken because the particle is moving in the direction of x decreasing.

Separating the variabes, we have

$$dt=-\sqrt{\left(\frac{a}{g}\right)}\frac{dx}{\sqrt{(4a^2-x^2)}}. \qquad \ldots(3)$$

If t_1 be the time from C to A, then integrating (3) from C to A, we get

$$\int_0^{t_1} dt=-\sqrt{\left(\frac{a}{g}\right)}\int_{2a}^{-a}\frac{dx}{\sqrt{(4a^2-x^2)}}$$

or $$t_1=\sqrt{\left(\frac{a}{g}\right)}\left[\cos^{-1}\frac{x}{2a}\right]_{2a}^{-a}$$

$$=\sqrt{\left(\frac{a}{g}\right)}[\cos^{-1}(-\tfrac{1}{2})-\cos^{-1}(1)]=\sqrt{\left(\frac{a}{g}\right)}\cdot\frac{2\pi}{3}.$$

Let v_1 be the velocity of the particle at A. Then at A

$$x=-a \text{ and } (dx/dt)^2=v_1^2.$$

So from (2), we have $v_1^2=(g/a)(4a^2-a^2)$

or $v_1=\sqrt{(3ag)}$, the direction of v_1 being vertically upwards.

Thus the velocity at A is $\sqrt{(3ag)}$ and is in the upwards direction so that the particle rises above A. Since the tension of the string vanishes at A, therefore at A the simple harmonic motion ceases and the particle when rising above A moves freely under gravity. Thus the particle rising from A with velocity $\sqrt{(3ag)}$ moves upwards till this velocity is destroyed. The time t_2 for this motion is given by

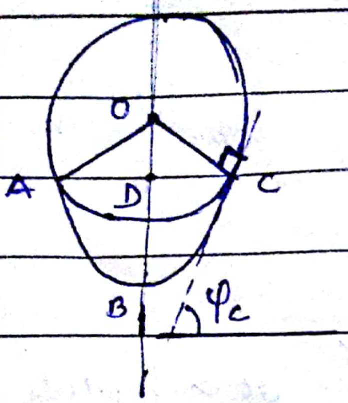
$$0=\sqrt{(3ag)}-gt_2, \quad \text{so that } t_2=\sqrt{\left(\frac{3a}{g}\right)}.$$

Conditions being the same, the equal time t_2 is taken by the particle in falling freely back to A. From A to C the particle will take the same time t_1 as it takes from C to A. Thus the whole time taken by the particle to return to $C=2\,(t_1+t_2)$

$$=2\left[\sqrt{\left(\frac{a}{g}\right)}\cdot\frac{2\pi}{3}+\sqrt{\left(\frac{3a}{g}\right)}\right]=\sqrt{\left(\frac{a}{g}\right)}\left[\frac{4\pi}{3}+2\sqrt{3}\right].$$

7\. *(a)* Determine the length of an endless chain which will hang over a circular pulley of radius a so as to be in contact with two-thirds of the circumference of the pulley. 15

$ABC \rightarrow$ catenary



$\varphi_c = \angle COD = \frac{1}{2}\angle COA = \frac{1}{2}\cdot\frac{1}{3}\cdot 2\pi = \pi/3$

$AD = DC = OC\sin\pi/3 = a\cdot\sqrt{3}/2$

$x = c\log_e(\tan\varphi + \sec\varphi)$

$C: \sqrt{3}/2\, a = c\log_e\left(\tan\frac{\pi}{3} + \sec\frac{\pi}{3}\right) = c\log_e(\sqrt{3}+$

$\Rightarrow c = \dfrac{a\sqrt{3}}{2\log_e(2+\sqrt{3})}$

$s = c\tan\varphi$

$\Rightarrow \text{arc } BC = \dfrac{a\sqrt{3}}{2\log_e(2+\sqrt{3})} \times \sqrt{3} = \dfrac{3a}{2\log_e(2+\sqrt{3})}$

Length of chain $= \frac{2}{3}\times 2\pi a + 2\times \text{arc } BC$

$= \frac{4}{3}\pi a + 2\times\dfrac{3a}{2\log_e(2+\sqrt{3})} = \dfrac{4\pi a}{3} + \dfrac{3a}{\log_e(2+\sqrt{3})}$

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(c) A particle moves with a central acceleration which varies inversely as the cube of the distance. If it be projected from an apse at a distance a from the origin with a velocity which is $\sqrt{2}$ times the velocity for a circle of radius a, determine the equation to its path. 15

Ex. 28. *A particle moves with a central acceleration which varies inversely as the cube of the distance. If it be projected from an apse at a distance a from the origin with a velocity which is $\sqrt{2}$ times the velocity for a circle of radius a, show that the equation to its path is $r \cos(\theta/\sqrt{2}) = a$.* **[Rohilkhand 77, 81; Allahabad 78; Meerut 78; Agra 86]**

Sol. Here the central acceleration varies inversely as the cube of the distance *i.e.*, $P = \mu/r^3 = \mu u^3$, where μ is a constant.
If V is the velocity for a circle of radius a, then

$$\frac{V^2}{a} = \left[P\right]_{r=a} = \frac{\mu}{a^3}$$

or
$$V = \sqrt{(\mu/a^2)}.$$

$\therefore$ the velocity of projection $v_1 = \sqrt{2}V = \sqrt{(2\mu/a^2)}$.

The differential equation of the path is

$$h^2\left[u+\frac{d^2u}{d\theta^2}\right]=\frac{P}{u^2}=\frac{\mu u^3}{u^2}=\mu u.$$

Multiplying both sides by 2 $(du/d\theta)$ and integrating, we have

$$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu u^2+A, \qquad ...(1)$$

where A is a constant.

But initially when $r=a$ *i.e.*, $u=1/a$, $du/d\theta=0$ (at an apse), and $v=v_1=\sqrt{(2\mu/a^2)}$.

$\therefore$ from (1), we have

$$\frac{2\mu}{a^2}=h^2\left[\frac{1}{a^2}\right]=\frac{\mu}{a^2}+A.$$

$\therefore$ $h^2=2\mu$ and $A=\mu/a^2$.

Substituting the values of h^2 and A in (1), we have

$$2\mu\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu u^2+\frac{\mu}{a^2}$$

or $$2\left(\frac{du}{d\theta}\right)^2=\frac{1}{a^2}+u^2-2u^2=\frac{1-a^2u^2}{a^2}$$

or $$\sqrt{2}\,a\frac{du}{d\theta}=\sqrt{(1-a^2u^2)} \text{ or } \frac{d\theta}{\sqrt{2}}=\frac{a\,du}{\sqrt{(1-a^2u^2)}}.$$

Integrating, $(\theta/\sqrt{2})+B=\sin^{-1}(au)$, where B is a constant.

But initially, when $u=1/a, \theta=0$. $\therefore$ $B=\sin^{-1}1=\frac{1}{2}\pi$.

$\therefore$ $(\theta/\sqrt{2})+\frac{1}{2}\pi=\frac{1}{2}\sin^{-1}(au)$ or $au=a/r=\sin\{\frac{1}{2}\pi+(\theta/\sqrt{2})\}$

or $a=r\cos(\theta/\sqrt{2})$, which is the required equation of the path.