7 Applications of Taylor's Theorem

MAXIMA AND MINIMA, INDETERMINATE FORMS

In this chapter we shall discuss applications of Taylor's theorem to two types of problems: (*a*) extreme values of a function, and (*b*) the evaluation of certain limits, popularly known as Indeterminate Forms.

1. EXTREME VALUES (Definitions)

Let c be an interior point of the domain $[a, b]$ of a function f.

Definition 1. The point c is said to be the *stationary point* and $f(c)$, the *stationary value* of the function f if $f'(c) = 0$.

Definition 2. The function f is said to have a *maximum value* (a *maxima* or a *maximum*) at c if $f(c)$ is the greatest value of the function in a small neighbourhood $]c - \delta, c + \delta[, \delta > 0$ of c.

Thus for all $x \in]c - \delta, c + \delta], x \neq c$, we have

$$f(c) > f(x)$$

$\Rightarrow$ $f(x) - f(c)$ is negative, for all values of x in $]c - \delta, c + \delta[$ other than c.

Definition 3. $f(c)$ is said to be the *minimum value*, a *minimum* or a *minima* if $f(c)$ is the least value of the function in a small neighbourhood $]c - \delta, c + \delta[$ of c. Thus for all $x \in]c - \delta, c + \delta[, x \neq c$,

$$f(c) < f(x)$$

$\Rightarrow$ $f(x) - f(c)$ is positive, for all values of x in a deleted neighbourhood $]c - \delta, c + \delta[$ of c.

Definition 4. The function f is said to have an *extreme value* at c if $f(c)$ is either a maximum or a minimum value. Thus at an extreme point c, $f(x) - f(c)$ keeps the same sign, for all values of x in a deleted neighbourhood $]c - \delta, c + \delta[$ of c.

1.1 A Necessary Condition for Extreme Values

Theorem 1. *If $f(c)$ is an extreme value of a function f then $f'(c)$, in case it exists, is zero.*

For the sake of definiteness, let us assume that $f(c)$ is a maximum value.

Hence $\exists$ a $\delta > 0$ such that

$$f(x) < f(c), \quad \forall\, x \in]c - \delta, c + \delta[, x \neq c$$

In case $f'(c)$ exists, there are three possibilities:

$$f'(c) > 0, \; f'(c) < 0, \; f'(c) = 0$$

If $f'(c) > 0$, then $\exists$ an interval $]c, c + \delta_1[, 0 < \delta_1 < \delta$ for every point of which $f(x) > f(c)$, which contradicts the fact that $f(c)$ is a maximum value.

Again if $f'(c) < 0$, then $\exists$ an interval $]c - \delta_2, c[, 0 < \delta_2 < \delta$ for every point of which $f(x) > f(c)$ which again is a contradiction.

Hence, the only possibility, $f'(c) = 0$.

Remarks:

1. The above theorem states that if the derivative exists, it must vanish at the extreme value. A function may however have an extreme value at a point without being derivable there at. Consider, for example, the function $f(x) = |x|$, which has a minimum at the origin even though $f'(0)$ does not exist.
2. The vanishing of the derivative at a point is only a necessary condition for the existence of an extreme value, it is not sufficient. Functions exist for which the derivative vanishes at a point but do not have an extreme value there at, *e.g.*, $f(x) = x^3$ at $x = 0$, so that the stationary points are not necessarily points of extreme values.

1.2 Investigation of the Points of Maximum and Minimum Values

At a point of extreme value the derivative of the function either does not exist or in case it exists, it must vanish.

Let c be an extreme point of a function f with domain $[a, b]$. If c is a *point of maximum value*, then $\exists$ a neighbourhood $]c - \delta, c + \delta[$ of c such that

$$f(x) < f(c), \quad \forall\, x \in]c - \delta, c + \delta[, x \neq c$$

$$\therefore \qquad f(x) < f(c), \quad \forall\, x \in]c - \delta, c[$$

$\Rightarrow$ f is increasing (to $f(c)$) in a small interval to the left of c.

Again

$$f(x) < f(c), \quad \forall\, x \in]c, c + \delta[$$

$\Rightarrow$ f is decreasing (from $f(c)$) in a small interval to the right of c.

We may therefore state:

c is a point of maximum value if the function changes from an increasing to a decreasing function as x passes through c. Therefore, in case f is derivable, the derivative changes sign from positive to negative as x passes through c.

It can similarly be seen that c is a point of *minimum value* if f changes from a decreasing to an increasing function and in case f is derivable, the derivative f' changes sign from negative to positive as x passes through c.

Notes:

1. It appears that f' is a decreasing (increasing) function in a neighbourhood of the point of maxima (minima) and therefore the second derivative f'' in case it exists, would be negative (positive) at such a point.
2. The above conditions are sufficient but not necessary.

1.3 Before discussing the subject further, let us investigate the extreme values in a few cases.

Example 1. Examine the function $(x - 3)^5 (x + 1)^4$ for extreme values.

- Let $f(x) = (x - 3)^5 (x + 1)^4$.

$$\therefore \qquad f'(x) = (x - 3)^4 (x + 1)^3 (9x - 7)$$

The function is derivable for all $x \in \mathbf{R}$ and the derivative f' vanishes for $x = -1, 3, \frac{7}{9}$ which may now be tested for extreme values.

(*a*) $x = -1$

f' is positive for a value of x slightly less than -1, and negative for slightly greater than -1. Thus f' changes sign from + to − as x passes through -1.
Hence, -1 is a point of maximum value.

(*b*) $x = 3$

f' remains positive as x passes through 3.

Hence, $x = 3$ is neither a maxima nor a minima.

(*c*) $x = \frac{7}{9}$

Since f' changes from − to + as x passes through $\frac{7}{9}$, therefore, f has a minimum value at $x = \frac{7}{9}$.

Example 2. Prove that a conical tent of a given capacity will require the least amount of canvas when the height is $\sqrt{2}$ times the radius of the base.

- Let the tent be a cone of semi-vertical angle α and radius of the base r. The volume V is fixed, while the surface area S has to be the minimum.

Now

$$V = \frac{1}{3}\pi r^3 \cot \alpha \qquad ...(1)$$

and

$$S = \pi r^2 \operatorname{cosec} \alpha \qquad ...(2)$$

Differentiating (1), we get

$$\frac{\pi}{3}\left[3r^2 \cot \alpha \frac{dr}{d\alpha} - r^3 \operatorname{cosec}^2 \alpha\right] = 0$$

or

$$\frac{dr}{d\alpha} = \frac{r \operatorname{cosec}^2 \alpha}{3 \cot \alpha}$$

Again from (2),

$$\frac{dS}{d\alpha} = \pi\left[2r \operatorname{cosec} \alpha \frac{dr}{d\alpha} - r^2 \cot \alpha \operatorname{cosec} \alpha\right]$$

$$= \frac{\pi r^2 \operatorname{cosec} \alpha}{3 \cot \alpha}\left[2 \operatorname{cosec}^2\alpha - 3 \cot^2\alpha\right]$$

$$= \frac{\pi r^2}{3 \cos \alpha}\left[2 - \cot^2\alpha\right]$$

$$\therefore \qquad \frac{dS}{d\alpha} = 0, \text{ when } \cot\alpha = \sqrt{2}$$

$$\Rightarrow \qquad \alpha = \cot^{-1}\sqrt{2}.$$

Also $dS/d\alpha$ changes sign from negative to positive as α passes through the value $\cot^{-1}\sqrt{2}$.

Therefore S has a minimum value at $\alpha = \cot^{-1}\sqrt{2}$.

Hence, the height of the tent $= r\cot\alpha = r\sqrt{2} = \sqrt{2}$ times the radius of the base.

1.4 In most cases the conclusion of § 1.2 suffices to determine the points of extreme values but to simplify the matters, recourse may be had to higher derivatives. In that context the following theorem will prove very useful.

Theorem 2. *If c is an interior point of the domain of a function f and $f'(c) = 0$, then the function has a maxima or a minima at c according as $f''(c)$ is negative or positive.*

The existence of $f''(c)$ implies that f and f' exist and are continuous at c. Continuity at c implies the existence of f and f' in a certain neighbourhood $]c-\delta, c+\delta[, \delta > 0$ of c, the neighbourhood being itself contained in the domain of f.

Let $f''(c) > 0$.

This implies that $f'(x)$ is an increasing function of c.

$$\Rightarrow \qquad f'(x) > f'(c) = 0, \ \forall\, x \in\,]c, c+\delta_1[, \delta_1 < \delta$$

and

$$f'(x) < f'(c) = 0 \ \ \forall\, x \in\,]c-\delta_1, c[$$

Thus first implies that $f'(x)$ is positive and hence f is an increasing function in $]c, c+\delta_1[$, *i.e.*, $f(x) > f(c)$ in $]c, c+\delta_1[$.

Similarly $f(x) > f(c)$ in $]c-\delta_1, c[$.

The last two results imply that

$$f(x) > f(c), \ \forall\, x \in\,]c-\delta_1, c+\delta_1[, x \neq c.$$

$\Rightarrow \qquad f$ has a minima at c.

Similarly $f''(c) < 0 \Rightarrow f(x)$ has a maxima at c.

Example 3. Examine the function $\sin x + \cos x$ for extreme values.

- Let

$$f(x) = \sin x + \cos x$$
$$f'(x) = \cos x - \sin x$$
$$f''(x) = -\sin x - \cos x$$
$$f'(x) = 0 \text{ when } \tan x = 1, \text{ so that}$$
$$x = n\pi + \frac{\pi}{4}$$

where n is zero or any integer

$$f''\left(n\pi + \frac{1}{4}\pi\right) = -\left\{\sin\left(n\pi + \frac{1}{4}\pi\right) + \cos\left(n\pi + \frac{1}{4}\pi\right)\right\}$$

$$= (-1)^{n+1}\left(\sin\frac{\pi}{4} + \cos\frac{\pi}{4}\right) = (-1)^{n+1}\sqrt{2}$$

Also
$$f\left(n\pi + \frac{1}{4}\pi\right) = \sin\left(n\pi + \frac{1}{4}\pi\right) + \cos\left(n\pi + \frac{1}{4}\pi\right) = (-1)^n\sqrt{2}$$

When n is zero or an even integer, $f''(n\pi + \pi/4)$ is negative and therefore $x = n\pi + \frac{1}{4}\pi$ makes $f(x)$ a maxima with the maximum value $\sqrt{2}$.

When n is an odd integer, $f''(n\pi + \frac{1}{4}\pi)$ is positive and therefore $x = n\pi + \frac{1}{4}\pi$ makes $f(x)$ a minima with the minimum value $\sqrt{2}$.

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1.5* In cases where the second derivative vanishes, the above theorem fails to give any result. In those cases, we make use of still higher derivatives, and the following theorem proves very helpful.

General Theorem. *If c is an interior point of the domain $[a, b]$ of a function f and is such that*

(*i*) $f'(c) = f''(c) = f'''(c) = \ldots = f^{n-1}(c) = 0$, *and*

(*ii*) $f^n(c)$ *exists and is not zero*,

then for n odd, $f(c)$ is not an extreme value, while for n even $f(c)$ is a maximum or minimum value according as $f^n(c)$ is negative or positive.

Condition (*ii*) of the existence of $f^n(c)$ implies that $f, f', f'', \ldots, f^{n-1}$ all exist and are continuous at c. Also continuity at c implies the existence of $f, f', \ldots, f^{n-1}$ in a certain neighbourhood $]c - \delta_1, c + \delta_1[$ of c, $\delta_1 > 0$.

As $f^n(c) \neq 0$ there exists a neighbourhood $]c - \delta, c + \delta[$, $0 < \delta < \delta_1$ such that for $f^n(c) > 0$,

and
$$\left.\begin{aligned} f^{n-1}(x) < f^{n-1}(c) = 0,\ x \in\ ]c - \delta, c[\\ f^{n-1}(x) > f^{n-1}(c) = 0,\ x \in\ ]c, c + \delta[\end{aligned}\right\} \quad \ldots(1)$$

and for $f^n(c) < 0$,

and
$$\left.\begin{aligned} f^{n-1}(x) > f^{n-1}(c) = 0,\ x \in\ ]c - \delta, c[\\ f^{n-1}(x) < f^{n-1}(c) = 0,\ x \in\ ]c, c + \delta[\end{aligned}\right\} \quad \ldots(2)$$

Again for any real number h, where $|h| < \delta$, we have by Taylor's Theorem

$$f(c + h) - f(c) + hf'(c) + \frac{h^2}{2!}f''(c) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{n-1}(c + \theta h),\ 0 < \theta < 1$$

or
$$f(c + h) - f(c) = \frac{h^{n-1}}{(n-1)!}f^{n-1}(c + \theta h) \quad \ldots(3)$$

where $c + \theta h \in\ ]c - \delta, c + \delta[$.

For n odd: Clearly $h^{n-1} > 0$ for any real number h, and further:

(*i*) When $f^n(c) > 0$, we deduce from (1) that for h negative, $(c + \theta h)$ is in $[c - \delta, c[$, and so $f^{n-1}(c + \theta h) < 0$, and for h positive, $f^{n-1}(c + \theta h) > 0$.

Thus from (3)

$$f(c + h) < f(c), \text{ when } c - \delta < c + h < c$$

and

$$f(c + h) > f(c), \text{ when } c < c + h < c + \delta$$

Thus $f(c)$ is not an extreme value.

(*ii*) When $f^n(c) < 0$, it may similarly be shown that $f(c)$ is not an extreme value.

For n even: As h^{n-1} is positive or negative according as h is positive or negative, we deduce as before from (1) and (3) that if $f^n(c) > 0$ then for every point $x = c + h \in]c - \delta, c + \delta[$ except c,

$$f(c + h) > f(c)$$

i.e., $f(c)$ is a minimum value.

It may similarly be deduced from (1) and (3) that $f(c)$ is a maximum value if $f^n(c) < 0$.

Notes:

1. As a consequence of the above theorem, if f' vanishes at c, then c is a point of maxima if $f''(c) < 0$ and a minima if $f''(c) > 0$.
2. Extreme value exists only if the first non-zero derivative is of even order.

Example 4. Examine the function $(x - 3)^5 (x + 1)^4$ for extreme values.

- Clearly f is derivable.

Let $f(x) = (x - 3)^5 (x + 1)^4$

$$\therefore \quad f'(x) = (x - 3)^4 (x + 1)^3 [9x - 7]$$

$$f''(x) = 8(x - 3)^3 (x + 1)^2 (9x^2 + 14x + 1)$$

$$f'''(x) = 24(x - 3)^2 (x + 1)(21x^3 - 49x^2 + 7x + 13)$$

$$f^{iv}(x) = 24(x - 3)(3x - 1)(21x^3 - 49x^2 + 7x + 13) + 168(x + 3)^2 (x + 1)(9x^2 - 14x + 1)$$

$$f^{v}(x) = 48(3x - 5)(21x^3 - 49x^2 + 7x + 13) + 336(x - 3)(9x^2 - 14x + 1)(3x - 1) + 336(x - 3)^2 (x + 1)(9x - 7)$$

Now f' vanishes for $x = -1, 3, \dfrac{7}{9}$.

Let us now test these for extreme values.

At $x = -1$, f^{iv} is the first non-vanishing derivative and this is negative. Thus $x = -1$ is a point of maxima.

At $x = 3$ the first non-vanishing derivative is the fifth which is of odd order. Therefore, the function has neither maximum nor minimum at $x = 3$. Such a point is called the *point of inflexion* of the function.

At $x = \frac{7}{9}$, f'' is the first non-vanishing derivative and is positive and therefore it is a point of minima.

EXERCISE

1. Show that the maximum value of the function $(x-1)(x-2)(x-3)$ is $\frac{2\sqrt{3}}{9}$ at $x = 2 - \frac{1}{\sqrt{3}}$.
2. Show that $x^5 - 5x^4 + 5x^3 - 1$ has a maxima at $x = 1$ and a minima at $x = 3$ and neither at $x = 0$.
3. Find the maximum and the minimum as well as the greatest and the least value of $x^3 - 12x^2 + 45x$ in the interval $[0, 7]$.
4. Find the maximum or minimum of
$$\frac{x^4}{(x-1)(x-3)^3}.$$
5. Show that the maximum value of $(1/x)^x$ is $e^{1/e}$.
6. Show that the maximum value of $(\log x)/x$ in $0 < x < \infty$ is $1/e$.
7. Show that $\sin x(1 + \cos x)$ is maximum at $x = \pi/3$.
8. If $(x-a)^{2n}(x-b)^{2m+1}$, where m and n are positive integers, is the derivative of a function f then show that $x = b$ gives a minimum but $x = a$ gives neither a maximum nor a minimum.
9. Show that the semi-vertical angle of a cone of maximum volume and of given slant height is $\tan^{-1}\sqrt{2}$.
10. Show that the volume of the greatest cylinder which can be inscribed in a cone of height h and semi-vertical angle α is $(4/27)\pi h^3 \tan^2 \alpha$.
11. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius a is $2a/\sqrt{3}$.

ANSWERS

3. Max 54, min 50, greatest 70, least 0.
4. Max at $\frac{6}{5}$, min at $x = 0$.

2. INDETERMINATE FORMS

We shall now discuss the evaluation of limits of functions generally known as *Indeterminate forms*. They are not indeterminate but have acquired this name by usage of the word.

In general, the limit of $\phi(x)/\psi(x)$ when $x \to a$, in case the limits of both the functions exist, is equal to the limit of the numerator divided by the limit of the denominator. But what happens when both these limits are zero? The division (0/0) then becomes meaningless. A case like this is known as Indeterminate form. Other such forms are ∞/∞, $0 \times \infty$, $\infty - \infty$, 0^0, 1^∞, and ∞^0. Ordinary methods of evaluating the limits are of little help. Particular methods are required to evaluate these peculiar limits. We shall now discuss these particular methods, generally called *L' Hospital rule*, due to the French mathematician L' Hospital (also called L' Hospital).

It should, however, be clearly understood, that in what follows, we do not find the value of 0/0 or of any of the other indeterminate forms. We only find the limits of combinations of functions which assume these forms when the limits of functions are taken separately.

2.1 It will be of help to remember the following points concerning the limits and continuity:

(*i*) $\lim_{x\to a} f(x) = l$

i.e., the function f is defined for every point ξ of the deleted neighbourhood $]a-\delta, a+\delta[$ of a and $f(\xi) \to l$ as $\xi \to a$.

(*ii*) Continuity of $f(x)$ at $x = a \Rightarrow \lim_{x\to a} f(x) = f(a)$,

i.e., the function f is defined for every point ξ of a neighbourhood $]a-\delta, a+\delta[$ of a and $f(\xi) \to f(a)$ as $\xi \to a$.

(*iii*) $\lim_{x\to a} \dfrac{f(x)}{g(x)} = l \ (\neq 0 \text{ or } \infty)$, then

(*a*) $\lim_{x\to a} g(x) \neq 0 \ (\text{or } \infty) \Rightarrow \lim_{x\to a} f(x)$ exists finitely

(*b*) $\lim_{x\to a} g(x) = 0 \ (\text{or } \infty) \Rightarrow \lim_{x\to a} f(x) = 0 \ (\text{or } \infty)$

for

$$\lim f(x) = \lim\left(\frac{f(x)}{g(x)} \cdot g(x)\right) = \lim\left(\frac{f(x)}{g(x)}\right) \lim g(x) = l \cdot (\lim g(x))$$

2.2 Indeterminate Form, 0/0

We shall now discuss some theorems concerning the indeterminate form 0/0. The reader will do well to note the differences in the hypothesis and the line of proof of the theorems.

Theorem 3. *If f, g be two functions such that*

(*i*) $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ *and*

(*ii*) $f'(a), g'(a)$ *exist and* $g'(a) \neq 0$ *then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

Since the functions f and g are derivable at $x = a$, therefore, they are continuous there at, *i.e.*,

$$\lim_{x\to a} f(x) = f(a) \text{ and } \lim_{x\to a} g(x) = g(a)$$

Thus from condition (*i*), $f(a) = 0 = g(a)$.

Also

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{f(x)}{x - a}$$

and

$$g'(a) = \lim_{x \to a} \frac{g(x) - g(a)}{x - a} = \lim_{x \to a} \frac{g(x)}{x - a}$$

$$\therefore \qquad \frac{f'(a)}{g'(a)} = \lim_{x \to a} \frac{f(x)}{g(x)}$$

Note: Condition (*i*) can be replaced by $f(a) = g(a) = 0$.

Theorem 4. *L' Hospital's Rule for 0/0 form. If f, g are two functions such that*

(*i*) $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$,

(*ii*) $f'(x), g'(x)$ *exist and* $g'(x) \neq 0, \forall x \in]a - \delta, a + \delta[, \delta > 0$ *except possibly at a, and*

(*iii*) $\lim_{x \to a} \frac{f'(x)}{g'(x)}$ *exists*,

then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Let us define two functions F and G such that

$$F(x) = \begin{cases} f(x), \forall x \in]a - \delta, a + \delta[\text{ except } a \\ \lim_{x \to a} f(x), \text{ at } x = a \end{cases}$$

$$G(x) = \begin{cases} g(x), \forall x \in]a - \delta, a + \delta[\text{ except } a \\ \lim_{x \to a} g(x), \text{ at } x = a \end{cases}$$

Clearly F and G are continuous and derivable on $]a - \delta, a + \delta[$ except possibly at a. Also, since

$$\lim_{x \to a} F(x) = \lim_{x \to a} f(x) = F(a)$$

and

$$\lim_{x \to a} G(x) = \lim_{x \to a} g(x) = G(a)$$

hence F and G are continuous at a as well.

Let x be a point of $]a - \delta, a + \delta[$ other than a.

For $x > a$, F and G satisfy the conditions of Cauchy's Mean Value Theorem in $[a, x]$, so that

$$\frac{F(x)-F(a)}{G(x)-G(a)} = \frac{F'(\xi)}{G'(\xi)}, \text{ where } a < \xi < x$$

But

$$F(a) = 0 = G(a)$$

$$\therefore \quad \frac{F(x)}{G(x)} = \frac{F'(\xi)}{G'(\xi)} \qquad ...(1)$$

Proceeding to limits,

$$\lim_{x\to a+0} \frac{F(x)}{G(x)} = \lim_{\xi\to a+0} \frac{F'(\xi)}{G'(\xi)} = \lim_{x\to a+0} \frac{F'(x)}{G'(x)}$$

$$\Rightarrow \quad \lim_{x\to a+0} \frac{f(x)}{g(x)} = \lim_{x\to a+0} \frac{f'(x)}{g'(x)}$$

For $x < a$, we can similarly prove that

$$\lim_{x\to a-0} \frac{f(x)}{g(x)} = \lim_{x\to a-0} \frac{f'(x)}{g'(x)}$$

Hence

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

It may be noted in the above theorem that if $g'(x) = 0$ at any point of the interval $]a-\delta, a+\delta[$, then $f'(x)$ also vanishes there at so that $f'(x)/g'(x)$ then takes up the indeterminate form 0/0. In such situations we have to proceed to the next two theorems which may be considered as generalisations of these theorems and may be proved by repeated applications of the above theorems.

Theorem 5. *Generalised L'Hospital's Rule for 0/0 form. If f, g be two functions such that*

(*i*) $f^n(x)$, $g^n(x)$ *exist, and* $g'(x) \neq 0$ ($r = 0, 1, 2, ..., n$) *for any* x *in* $]a-\delta, a+\delta[$ *except possibly at* $x = a$,

(*ii*) *when* $x \to a$, $\begin{cases} \lim f(x) = \lim f'(x) = ... = \lim f^{n-1}(x) = 0 \\ \lim g(x) = \lim g'(x) = ... = \lim g^{n-1}(x) = 0 \end{cases}$

and (*iii*) $\lim_{x\to a} \dfrac{f^n(x)}{g^n(x)}$ *exists, then*

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f^n(x)}{g^n(x)}$$

Theorem 6. *If f, g be two functions such that*

(*i*) *when* $x \to a$, $\begin{cases} \lim f(x) = \lim f'(x) = ... = \lim f^{n-1}(x) = 0 \\ \lim g(x) = \lim g'(x) = ... = \lim g^{n-1}(x) = 0 \end{cases}$

and (ii) $f^n(a)$, $g^n(a)$ exist, and $g^n(a) \neq 0$ then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f^n(a)}{g^n(a)}$$

[*Hint:* Using Theorem (3) it can be easily seen that

$$\lim_{x \to a} \frac{f^{n-1}(x)}{g^{n-1}(x)} = \frac{f^n(a)}{g^n(a)}$$

Also repeated applications of theorem (4) gives

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f^{n-1}(x)}{g^{n-1}(x)}]$$

Remark: The rule in the form of theorem 4 or 5 is more useful than the other forms because we may cancel out common factors or affect any other simplification in the quotient $f'(x)/g'(x)$ before proceeding to the limits. Also repeated application of the rule is possible in this form.

L'Hospital's rule holds even in the case when each $f(x)$ and $g(x)$ tends to ∞ when $x \to a$ or when $x \to \infty$. Theorems 7 and 8 will show how these cases can be handled in an exactly similar fashion.

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Example 5. If f' exists in the *nbd* of $x = a$ and $f''(a)$ exists. Show that

$$\lim_{h \to 0} \frac{f(a+2h) - 2f(a+h) + f(a)}{h^2}$$

exists and is equal to $f''(a)$. Give an example to show that the limit may exist even though $f'(a)$ does not exist.

■ $$\lim_{h \to 0} \frac{f'(a+2h) - f'(a)}{2h} = \lim_{h \to 0} \frac{f'(a+h) - f'(a)}{h} = f''(a)$$

Hence

$$\lim_{h \to 0} \frac{2f'(a+2h) - 2f'(a+h)}{2h} = \lim_{h \to 0}\left(2\left(\frac{f'(a+2h) - f'(a)}{2h}\right) - \left(\frac{f'(a+h) - f'(a)}{h}\right)\right)$$

$$= f''(a)$$

Then by theorem 5,

$$\lim_{h \to 0}\left(\frac{f(a+2h) - 2f(a+h) + f(a)}{h^2}\right)$$

exists and is equal to $f''(a)$.

Theorem 7. *L' Hospital's rule for infinite limits. If f, g be two functions such that*

(i) $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = 0$.

(ii) $f'(x)$, $g'(x)$ *exist, and* $g'(x) \neq 0$, $\forall\, x > 0$ *except possibly at* ∞, *and*

(iii) $\lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ *exists, then*

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

Let us put $z = 1/x$ so that $z \to 0+0$ as $x \to \infty$, and define two functions F and G where $F(z) = f(1/z)$ and $G(z) = g(1/z)$.

We see that the functions F and G satisfy the conditions of Theorem 4, viz.,

(i) $\lim_{z\to 0+0} F(z) = \lim_{z\to 0+0} G(z) = 0$,

(ii) $F'(z)$, $G'(z)$ exist and $G'(z) \neq 0$, $\forall\, z$ in $]-\delta, \delta[$ except possibly at $z = 0$, and

(iii) $\lim_{z\to 0+0} \frac{F'(z)}{G'(z)}$ exists.

Consequently,

$$\lim_{z\to 0+0} \frac{F(z)}{G(z)} = \lim_{z\to 0+0} \frac{F'(z)}{G'(z)}$$

$$\Rightarrow \qquad \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

2.3 Indeterminate Form, ∞/∞

If $f(x)$ and $g(x)$ both tend to ∞ as $x \to a$ then $\lim_{x\to a} \frac{f(x)}{g(x)}$ takes ∞/∞ form. We shall prove L' Hospital rule for this indeterminate form when x tends to a finite limit a. The rule, however, holds good even for infinite limits, $x \to \infty$ and may be deduced from the following theorem by the procedure indicated in Theorem 7.

Theorem 8*. *L' Hospital's rule for ∞/∞ form. If f, g be two functions such that*

(i) $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$,

(ii) $f'(x)$, $g'(x)$ *exist and* $g'(x) \neq 0$, $\forall\, x$ *in* $]a-\delta, a+\delta[$, $\delta > 0$ *except possibly at* a, *and*

(iii) $\lim_{x\to a} \frac{f'(x)}{g'(x)} = l$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = l$$

Consider the function

$$\phi(x) = f(x) - l g(x) \qquad \text{...(1)}$$

[Division by $g(x)$ would indicate that $f(x)/g(x) \to l$, if $\phi(x)/g(x) \to 0$. With this in mind we proceed to prove that the latter limit is actually equal to zero.]

Clearly, in view of condition (*ii*), $\phi(x)$ is derivable and continuous in $]a - \delta, a + \delta[$, except possibly at $x = a$. Also $g'(x) \neq 0$ there at.

$$\therefore \qquad \frac{\phi'(x)}{g'(x)} = \frac{f'(x)}{g'(x)} - l$$

But condition (*iii*) implies that $\exists$ a positive $\delta_1 < \delta$ and $\varepsilon > 0$ such that

$$\left| \frac{f'(x)}{g'(x)} - l \right| < \frac{\varepsilon}{2}, \text{ when } |x - a| < \delta_1$$

$$\therefore \qquad \left| \frac{\phi'(x)}{g'(x)} \right| < \frac{\varepsilon}{2}, \text{ for } |x - a| > \delta_1 \qquad \text{...(2)}$$

Let $x \neq a$ be a point of $]a - \delta_1, a + \delta_1[$.

For $x > a$, $\phi(x)$ and $g(x)$ satisfy all the conditions of Cauchy's Mean Value Theorem in $[x, a + \delta_1]$, so that

$$\frac{\phi(x) - \phi(a + \delta_1)}{g(x) - g(a + \delta_1)} = \frac{\phi'(\xi)}{g'(\xi)}, \text{ when } x < \xi < a + \delta_1 \qquad \text{...(3)}$$

Now in view of (2),

$$\left| \frac{\phi'(\xi)}{g'(\xi)} \right| < \frac{\varepsilon}{2}$$

Also since $g(x) \to \infty$ as $x \to a$, we have

(*i*) $\exists$ a positive $\delta_2 < \delta_1$ such that

and
$$\left.\begin{aligned} & g(x) > 0 \\ & g(x) > (2/\varepsilon)\,|\phi(a + \delta_1)| \end{aligned}\right\} \text{ when } (x - a) < \delta_2$$

(*ii*) $\exists$ a positive $\delta_3 < \delta_1$ such that

$$g(x) > g(a + \delta_1), \text{ when } (x - a) < \delta_3$$

Thus for $\delta_4 = \min(\delta_2, \delta_3)$, we have

and
$$\left.\begin{aligned} & g(x) > 0 \\ & g(x) - g(a + \delta_1) < g(x) \end{aligned}\right\} \text{ when } (x - a) < \delta_4$$

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Accordingly (3) gives

$$\frac{|\phi(x)-\phi(a+\delta_1)|}{g(x)}<\frac{\varepsilon}{2}$$

Now,

$$\left|\frac{\phi(x)}{g(x)}\right| \le \left|\frac{\phi(x)-\phi(a+\delta_1)}{g(x)}\right| + \left|\frac{\phi(a+\delta_1)}{g(x)}\right|$$

$$<\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon,\ \text{for } (x-a)<\delta_4$$

Thus, $\lim\limits_{x\to a+0}\frac{\phi(x)}{g(x)}=0$

For $x<a$, we can similarly prove that

$$\lim_{x\to a-0}\frac{\phi(x)}{g(x)}=0$$

$\therefore$

$$\lim_{x\to a}\frac{\phi(x)}{g(x)}=0$$

But since $\phi(x)=f(x)-lg(x)$, and $g(x)\neq 0$ for any $x\in\,]a-\delta_4, a+\delta_4[$ by (α), therefore dividing by $g(x)$ and taking limits,

$$\lim_{x\to a}\left[\frac{f(x)}{g(x)}-l\right]=\lim_{x\to a}\frac{\phi(x)}{g(x)}=0$$

so that $\lim\limits_{x\to a}\frac{f(x)}{g(x)}=l$

Particular case. $l=0$, *i.e.*, $\lim\limits_{x\to a}\frac{f'(x)}{g'(x)}=0$.

The auxiliary function $\phi(x)$ reduces to $f(x)$, so that we have to proceed without introducing any such function. And we shall have the simplification, $\lim\limits_{x\to a}\frac{f'(x)}{g'(x)}=0$, in place of $\lim\limits_{x\to a}\frac{\phi'(x)}{g'(x)}=0$.

So proceeding in a similar way, we may prove the theorem:

If f, g be two functions such that

(*i*) $\lim\limits_{x\to a}f(x)=\lim\limits_{x\to a}g(x)=\infty$.

(*ii*) $f'(x), g'(x)$ *exist and* $g'(x)\neq 0$, *in a deleted neighbourhood of a, and*

(*iii*) $\lim\limits_{x\to a}\frac{f'(x)}{g'(x)}=0$

then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = 0$$

Remarks:

1. $\lim_{x \to a} \frac{f(x)}{g(x)}$ when $f(x)$ and $g(x)$ both tend to infinity, can be dealt with in the same way as 0/0 form.
2. The above theorem holds good in the case of infinite limits $x \to \infty$ as well.
3. Sometimes repeated applications of L'Hospital's rule may be necessary to evaluate a limit. We must then ensure at each step that the expression to which the rule is applied, is actually an indeterminate form.
4. The forms 0/0 and ∞/∞ can be interchanged and so care should be taken to select the form which would enable us to evaluate the limit most quickly.

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2.4 (*a*) **Form $0 \times \infty$**

When $f(x) \to 0$ and $g(x) \to \infty$ as $x \to a$, $f(x) \cdot g(x)$ takes $0 \times \infty$ form.

However $f(x) \cdot g(x)$ may be expressed as

$$\frac{f(x)}{1/g(x)} \text{ or } \frac{g(x)}{1/f(x)}$$

which has respectively 0/0 and ∞/∞ forms.

(*b*) **Form $\infty - \infty$**

This can be reduced to the form 0/0 or ∞/∞. For

$$f(x) - g(x) = \frac{\frac{1}{g(x)} - \frac{1}{f(x)}}{\frac{1}{f(x)\,g(x)}} \qquad \left(\frac{0}{0} \text{ form}\right)$$

(*c*) **Form $0^0, 1^\infty, \infty^0$**

These forms can be made to depend upon one of the previous forms by putting $k = \{f(x)\}^{g(x)}$, so that

$$\log k = g(x) \cdot \log f(x)$$

$\therefore \lim \log k = \lim \{g(x) \log f(x)\}$

Also $\lim k = \lim e^{\log k} = e^{\lim \log k}$

Thus the limit may be evaluated by one of the previous methods.

2.5 Let us now evaluate some limits which take up these forms, we shall not hesitate to make use of certain known limits, such as $\lim_{x \to 0} \frac{\sin x}{x}$, $\lim_{x \to 0} \frac{\tan x}{x}$, $\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$ etc. or expansions of functions such as $\log(1 + x)$, $\sin x$, etc. either in the beginning or at some intermediate stage because it simplifies and shortens the process of evaluation of a limit to a considerable extent.

Example 6. Evaluate $\lim_{x \to 0} \frac{x - \tan x}{x^3}$.

- Let $\frac{x - \tan x}{x^3} = \frac{f(x)}{g(x)}$, where $f(x) = x - \tan x$ and $g(x) = x^3$.

Now

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} (x - \tan x) = 0$$

$$\lim_{x \to 0} g(x) = \lim_{x \to 0} x^3 = 0$$

Hence, $\lim_{x \to 0} \frac{f(x)}{g(x)}$ is of $\frac{0}{0}$ form, so that L'Hospital's rule is applicable, *i.e.*,

$$\lim_{x \to 0} \frac{f(x)}{g(x)} = \lim_{x \to 0} \frac{f'(x)}{g'(x)}$$

$$\therefore \quad \lim_{x \to 0} \frac{x - \tan x}{x^3} = \lim_{x \to 0} \frac{1 - \sec^2 x}{3x^2}$$

$$= -\frac{1}{3} \lim_{x \to 0} \left(\frac{\tan x}{x} \right)^2 = -\frac{1}{3}$$

Example 7. Evaluate $\lim_{x \to 0} \frac{xe^x - \log(1 + x)}{x^2}$.

- It is a 0/0 form and therefore

$$\lim_{x \to 0} \frac{xe^x - \log(1 + x)}{x^2} = \lim_{x \to 0} \frac{xe^x + e^x - \frac{1}{1 + x}}{2x} \qquad \left(\frac{0}{0} \text{ form} \right)$$

$$= \lim_{x \to 0} \frac{xe^x + 2e^x + \frac{1}{(1 + x)^2}}{2} = \frac{3}{2}$$

Example 8. Find $\lim_{x \to 0} \frac{(1 + x)^{1/x} - e}{x}$.

- Since $\lim_{x \to 0} (1 + x)^{1/x} = e$, therefore it is a 0/0 form

$$\therefore \quad \lim_{x \to 0} \frac{(1 + x)^{1/x} - e}{x} = \lim_{x \to 0} \frac{\frac{d}{dx}(1 + x)^{1/x}}{1}$$

$$= \lim_{x\to 0} \frac{(1+x)^{1/x}\{x-(1+x)\log(1+x)\}}{x^2(1+x)}$$

$$= e\cdot\lim_{x\to 0} \frac{x-(1+x)\log(1+x)}{x^2(1+x)} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= e\cdot\lim_{x\to 0} \frac{-\log(1+x)}{2x+3x^2} \qquad \left(\frac{0}{0}\text{ form}\right)$$

$$= e\cdot\lim_{x\to 0} \frac{-1}{(2+6x)(1+x)} = -\frac{e}{2}$$

Aliter. Let us first find an algebraic expansion for $(1+x)^{1/x}$.

Let $y=(1+x)^{1/x}$

$$\log y = \frac{1}{x}\log(1+x)$$

$$= \frac{1}{x}\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\right), \text{ if } |x|<1$$

$$= 1-\frac{x}{2}+\frac{x^2}{3}-\ldots$$

$$\therefore \qquad y = \exp\left(1-\frac{x}{2}+\frac{x^2}{3}-\ldots\right)$$

$$= e\cdot\exp\left(-\frac{x}{2}+\frac{x^2}{3}-\ldots\right)$$

$$= e\left[1+\left(-\frac{x}{2}+\frac{x^2}{3}-\ldots\right)+\frac{1}{2!}\left(-\frac{x}{2}+\frac{x^2}{3}-\ldots\right)^2+\ldots\right]$$

$$= e\left[1-\frac{x}{2}+\frac{11}{24}x^2-\ldots\right]$$

$$\therefore \qquad \lim \frac{(1+x)^{1/x}-e}{x} = \lim_{x\to 0}\frac{e\left(1-\frac{x}{2}+\frac{11}{24}x^2-\ldots\right)-e}{x} = -\frac{e}{2}$$

Example 9. Find $\lim\limits_{x\to 1-0} \dfrac{\log(1-x)}{\cot(\pi x)}$.

- It is a ∞/∞ form and therefore

$$\lim_{x\to 1-0}\frac{\log(1-x)}{\cot(\pi x)}=\lim_{x\to 1-0}\frac{\frac{-1}{1-x}}{-\pi\operatorname{cosec}^2(\pi x)}$$

$$=\lim_{x\to 1-0}\frac{\sin^2(\pi x)}{\pi(1-x)}\qquad\left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to 1-0}\frac{\pi\sin(2\pi x)}{-\pi}=0$$

Example 10. Evaluate $\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{\sin^2 x}\right)$.

■ It is a $(\infty-\infty)$ form, we therefore write as

$$\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{\sin^2 x}\right)=\lim_{x\to 0}\frac{\sin^2 x-x^2}{x^2\sin^2 x}\qquad\left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to 0}\frac{\sin 2x-2x}{2x\sin^2 x+x^2\sin 2x}\qquad\left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to 0}\frac{\cos 2x-1}{\left(\sin^2 x+2x\sin 2x+x^2\cos 2x\right)}\qquad\left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to 0}\frac{-2\sin 2x}{3\sin 2x+6x\cos 2x-2x^2\sin 2x}\qquad\left(\frac{0}{0}\text{ form}\right)$$

$$=\lim_{x\to 0}\frac{-\cos 2x}{3\cos 2x-4x\sin 4x-x^2\cos 2x}=-\frac{1}{3}$$

Aliter.

$$\lim_{x\to 0}\left(\frac{1}{x^2}-\frac{1}{\sin^2 x}\right)=\lim_{x\to 0}\frac{\sin^2 x-x^2}{x^2\sin^2 x}$$

$$=\lim_{x\to 0}\frac{\sin^2 x-x^2}{x^4}\left(\frac{x}{\sin x}\right)^2$$

$$\left[\text{Using }\lim_{x\to 0}\left(\frac{x}{\sin x}\right)=1\right]$$

$$= \lim_{x\to 0} \frac{\sin^2 x - x^2}{x^4} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{\sin 2x - 2x}{4x^3} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{\cos 2x - 1}{6x^2}$$

$$= \lim_{x\to 0} \left[-\frac{1}{3}\left(\frac{\sin x}{x}\right)^2\right] = -\frac{1}{3}$$

Note: It should be noted that when a given function can be put as a product of two or more factors, the limit of each of which can be easily found, then limit of the entire function can be determined by evaluating the limit of each factor separately, provided that the product of these limits is not an indeterminate form.

Example 11. Evaluate $\lim\limits_{x\to 0+0} (\sin x \log x)$.

- It is a $(0 \times \infty)$ form. Let us write

$$\lim_{x\to 0+0} (\sin x \log x) = \lim_{x\to 0+0} \frac{\log x}{\operatorname{cosec} x} \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$$= -\lim_{x\to 0+0} \frac{1/x}{\operatorname{cosec} x \cot x} = -\lim_{x\to 0+0} \left(\frac{\sin x}{x}\right) \tan x = 0$$

Example 12. Evaluate $\lim\limits_{x\to 0} \left(\frac{\tan x}{x}\right)^{1/x^2}$

- It is a (1^∞) form. Therefore, let $K = \left(\frac{\tan x}{x}\right)^{1/x^2}$, so that

$$\log K = \frac{1}{x^2} \log\left(\frac{\tan x}{x}\right)$$

$$\therefore \quad \lim_{x\to 0} \log K = \lim_{x\to 0} \frac{\log\left(\frac{\tan x}{x}\right)}{x^2} = \lim_{x\to 0} \frac{\frac{\sec^2 x}{\tan x} - \frac{1}{x}}{2x} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x\to 0} \frac{x \sec^2 x - \tan x}{2x^2 \tan x} = \lim_{x\to 0} \frac{\sec^2 x \tan x}{2\tan x + x\sec^2 x} \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to 0} \frac{\tan x}{\sin 2x + x} = \lim_{x \to 0} \frac{\sec^2 x}{2\cos 2x + 1} = \frac{1}{3} \qquad \left(\frac{0}{0} \text{ form}\right)$$

i.e.,
$$\lim_{x \to 0} \log K = \frac{1}{3} \Rightarrow \lim_{x \to 0} K = e^{1/3}$$

$\therefore$
$$\lim_{x \to 0} K = \lim_{x \to 0} \left(\frac{\tan x}{x}\right)^{1/x^2} = e^{1/3}$$

Example 13. Evaluate $\lim_{x \to 1-0} (1 - x^2)^{1/(\log(1-x))}$.

- It is a (0^0) form. Therefore, let $K = (1 - x^2)^{1/(\log(1-x))}$, so that

$$\log K = \frac{\log\left(1 - x^2\right)}{\log(1 - x)}$$

$\therefore$
$$\lim_{x \to 1-0} \log K = \lim_{x \to 1-0} \frac{\log\left(1 - x^2\right)}{\log(1 - x)} = \lim_{x \to 1-0} \frac{2x(1 - x)}{1 - x^2} = \lim_{x \to 1-0} \frac{2x}{1 + x} = 1 \qquad \left(\frac{\infty}{\infty} \text{ form}\right)$$

$\Rightarrow$
$$\lim_{x \to 1-0} \log K = 1 \Rightarrow \lim_{x \to 1-0} K = e.$$

EXERCISE

Evaluate the following limits:

1. $\lim_{x \to 0} \frac{x - \sin x}{x^3}$
2. $\lim_{x \to 0} \frac{x - \log(1 + x)}{1 - \cos x}$
3. $\lim_{x \to 0} \frac{\log\left(1 + x^3\right)}{\sin^3 x}$
4. $\lim_{x \to \infty} \frac{x^3}{e^x}$
5. $\lim_{x \to \pi/2+0} \frac{\log\left(x - \frac{\pi}{2}\right)}{\tan x}$
6. $\lim_{x \to \infty} x \tan \frac{1}{x}$
7. $\lim_{x \to 0} \left(\cot^2 x - \frac{1}{x^2}\right)$
8. $\lim_{x \to \pi/2} (\sec x - \tan x)$
9. $\lim_{x \to 0+0} (\cot x)^{\sin x}$
10. $\lim_{x \to \pi/2} (\sin x)^{\tan x}$
11. $\lim_{x \to 0} \frac{(1 + x)^{1/x} - e + \frac{1}{2}ex}{x^2}$

    [*Hint*: Use algebraic expansion of $(1 + x)^{1/x}$.]
12. $\lim_{x \to 0} \left(\frac{\sin x}{x}\right)^{1/x}$

13. $\lim\limits_{x \to 0} \dfrac{e^x - 2\cos x + e^{-x}}{x \sin x}$

14. $\lim\limits_{x \to 0} \left(\dfrac{1}{x} - \dfrac{1}{\sin x} \right)$

15. $\lim\limits_{x \to 0} (1 - x)^{1/x}$

16. $\lim\limits_{x \to \infty} (1 + a/x)^x,\ a \neq 0$

17. $\lim\limits_{x \to a} \left(2 - \dfrac{x}{a} \right)^{\tan(\pi x/2a)}$

18. $\lim\limits_{x \to 1} x^{1/(x-1)}$

19. Find the values of a and b in order that $\lim\limits_{x \to 0} \dfrac{x(1 + a\cos x) - b\sin x}{x^3}$ may be equal to 1.

20. Find the values of a and the limit in order that $\lim\limits_{x \to 0} \dfrac{\sin 2x + a \sin x}{x^3}$ be finite.

21. If $f''(x)$ exists and is continuous in a neighbourhood of $x = a$, then show that

$$\lim_{h \to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2} = f''(a).$$

CHAPTER 7

ANSWERS

1. $\frac{1}{6}$	2. 1	3. 1	4. 0	5. 0	6. 1
7. $-\frac{2}{3}$	8. 0	9. 1	10. 1	11. $\frac{11}{24}e$	12. 1
13. 2	14. 0	15. e^{-1}	16. e^a	17. $e^{2/\pi}$	18. e

19. $a = -5/2,\ b = -3/2$

20. $a = -2$, limit $= -1$.