Revision is the most crucial part of mathematics preparation in Civil Services Examination. For that purpose, I had prepared short notes on each topic of the syllabus which I am sharing here. I would suggest aspirants to use this as just a reference for quick revision and not as your primary source. This optional requires good understanding of the topic which cannot be derived from short notes.

It is the process of preparing such notes that gives a candidate confidence about his or her preparation and brings conceptual clarity. Hence I would encourage aspirants to do this exercise themselves even if it is time consuming.

All the best!

Regards,
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AIR 8 - CSE 2015

REAL ANALYSIS / CALCULUS

Real Number System

① A rational number can be shown either as terminating decimal or recurring decimal.

Bet^n any 2 rational (irrational) numbers, there lie ∞ many rational (irrational) numbers.

② An Ordered field satisfies 4 properties

O1 : for $a, b \in \mathbb{R}$, exactly one of following 3 holds
$a > b$, $a < b$ or $a = b$
This is law of trichotomy.

O2 : Law of transitivity
$a > b,\ b > c \Rightarrow a > c$

O3 : Monotone property for addition
$a > b \Rightarrow a + c > b + c$

O4 : Monotone property for multiplication
$a > b$ & $c > 0 \Rightarrow ac > bc$

e.g. $(\mathbb{R}, +, \cdot)$ & $(\mathbb{Q}, +, \cdot)$ are ordered fields.

③ Finite interval → $[a,b]$, (a,b), $(a,b]$, $[a,b)$
Infinite interval → $(-\infty, b]$, $[a, \infty)$, $(-\infty, \infty)$ etc.

Every interval is an infinite set, but every infinite set need not be an interval.

④ Extended Real Number System.
Includes $-\infty$ & $+\infty$. → $[-\infty, +\infty]$.
Basic properties ⇒ $x + \infty = \infty + x = -x + \infty = \infty - x = \infty$ | $\frac{x}{\infty} = 0$ etc.
$\infty - \infty$, $0 \times \infty$, $\frac{\infty}{\infty}$ etc. are meaningless.

Real Analysis Notes Continued after few Fluid Pages →

REAL ANALYSIS (contd.)

5. Greatest Lower Bound (glb) or infimum
If bounded below set 'S' is taken & set of its lower bounds has greatest number 't' then t is infimum.

If t is infimum, then $\forall \epsilon > 0, \exists x \in S$ s.t. $t < x < t+\epsilon$

Least Upper bound or Supremum:-
Similarly, least member of set of all upper bounds.

e.g. $S = \{2^n \mid n \in \mathbb{N}\}$ $\inf(S) = 2$, supremum doesn't exist.

6. If both upper & lower bounds exist $\Rightarrow$ bounded set.

Every finite set is bdd & has inf & sup.
Null set is bdd but inf & sup of ϕ doesn't exist.
Every real no. is upper/lower bound for null set.

7. Greatest member of a subset of $\mathbb{R}$.
If supremum belongs to set, then it is greatest number.
If subset attains its infimum, then it is lowest number.
Sup/inf need not belong to the set.

8. Completeness Property of $\mathbb{R}$.
Every non-empty subset which is bounded above has the supremum in $\mathbb{R}$.

$\mathbb{R}$ is completely ordered field. Bcoz it satisfies
(a) Field axioms (b) Order axioms (c) Completeness axiom.

$\mathbb{Q}$ is ordered field but is not complete. eg. $\left(1+\frac{1}{n}\right)^n \to e$

(9) The Archimedian Property
If $a \& b \in \mathbb{R}$ & $a > 0$, $\exists n \in \mathbb{N}$ s.t. $an > b$

(10) Some properties with absolute values
$|x+y| \le |x| + |y|$
$|x-y| \geq \big| |x| - |y| \big|$
$|xy| = |x|\,|y|$

(11) Neighbourhood of a point 'a'.
Open interval $(a-\delta, a+\delta) = N_\delta(a)$ or $N(\delta, a)$

★ A subset $S \subseteq \mathbb{R}$ is said to be neighbourhood of $a \in \mathbb{R}$ if $\exists\, \delta > 0$ s.t. $(a-\delta, a+\delta) \subset S$

So, an open interval is neighbourhood of all its points.
A closed interval is nbd for all except end points.
$\mathbb{Q}$ is not a nbd for any of its points.

Intersection of 2 nbd is also a nbd.
Union of 2 nbd is also a nbd.

(12) Interior point
Let $P \in S$, P is interior pt. if S is a nbd of P.

Every non-empty finite set has no interior point.

(13) Interior of a set.
Set of all interior points S is called interior of S & is denoted by S° or $\mathrm{int}(S)$.
e.g. $\mathbb{N}^\circ = \phi$, $\mathbb{Z}^\circ = \phi$

(14) Open Set
A subset of $\mathbb{R}$ where every point is an interior point.
i.e. S is open iff $S^\circ = S$

Union of 2 open set is an open set.
Intersection of finite no. of open sets is an open set.
[(Infinite need not be open.
Let $S_n = (-\frac{1}{n}, \frac{1}{n}) \rightarrow \bigcap S_n = \{0\} \Rightarrow$ not open]

Every open interval is an open set but not vice-versa.
$(1,2) \cup (3,4)$ is open set but not interval.

(15) Limit point of set S
P is limit point of S if every neighbourhood of P contains a point of S other than P.

Limit pt. = Cluster pt. = Condensation pt. = Accumulation pt.
It may or may not belong to S.

P be limit point of S, then $(P-\epsilon, P+\epsilon)$ contains ∞ points of S.
Every real no. is a limit point of $\mathbb{Q}$.

If supremum/inf. doesn't belong to the set, then it is a limit pt. of set.

(16) Isolated Point
$P \in S$ is called isolated point if it is not a limit point.

Discrete Set
S is discrete set if all its points are isolated points.

(17) Derived Set

Set of limit points of set S is derived set.

Denoted by $D(S)$ or S'.

This can go to $2^{nd}, 3^{rd}, \ldots n^{th}$ level.

$D^{(n)}(S)$ or $S^{(n)}$ shows derived set of n^{th} level.

(18) Set of 'first species'

→ It has finite no. of derived sets.

Set of 'Second Species'

→ infinite no. of derived sets. e.g. $\mathbb{R}$

A set whose n^{th} derived set is finite then its $(n+1)^{th}$ derived set is empty. Then it is called set of n^{th} order.

(19) Adherant Point

Point P is adherant point of S if

$\forall$ nbd N of P; $N \cap S \neq \phi$.

So, for P to be adherant point of S;

either $P \in S$ or $P \in D(S)$

(20) Closure of a set = Set of adherant points of S.

i.e. $\bar{S} = S \cup D(S)$

Dense Set

Set S is called dense if $\bar{S} = \mathbb{R}$

Dense in itself

S is called dense-in-itself if every point of S is a limit point of S.

Perfect set $\Rightarrow$ $S = D(S)$. i.e. $S = \bar{S}$

e.g. $D(Q) = \mathbb{R}$ $\therefore Q$ is dense-in-itself
But Q is not perfect $\because$ $D(Q) \neq Q$.

$D(\phi) = \phi$ $\therefore$ Dense-in-itself & perfect set.

(21) Bolzano-Weistrass Theorem
Existance of limit points of a set.
→ Every infinite bounded set has a limit point.

(22) Some results on derived sets
$A \subset B \Rightarrow D(A) \subset D(B)$
$D(A \cup B) = D(A) \cup D(B)$
$D(A \cap B) \subseteq D(A) \cap D(B)$
$D(D(A)) \subseteq D(A)$

Derived set of any infinite bounded set attains its bounds.
Bounds of derived set denoted by
$\underline{\lim}(S) \rightarrow$ lower limit of S
$\overline{\lim}(S) \rightarrow$ upper limit of S

(23) Closed set
S is closed if its compliment is open.
or S is closed if $D(S) \subseteq S$.
e.g. Let S be $\mathbb{N}$. Then $D(S) = \phi$ $\therefore D(S) \subseteq S$
$\therefore \mathbb{N}$ is closed.
Any non-empty finite set is also closed, similarly.

Now, $D(Q) = \mathbb{R}$ $\therefore D(Q) \not\subseteq Q$ $\therefore Q$ is not closed.
Not very intuitive. So remember def^n of closed, $D(S) \subseteq S$.

e.g. $S = \{ \frac{1}{n} / n \in \mathbb{N} \}$ $D(S) = \{0\}$ $D(S) \not\subseteq S$

$\therefore$ S is not closed.

(24) Intersection of arbitrary no. of closed sets is closed set.

Union of finite closed sets → closed set.

Union of infinite closed sets → need not be closed.

e.g. $S_n = [\frac{1}{n}, 1]$ → $\bigcup_{i=1}^{\infty} S_n = (0,1]$

(25) If A is closed & B open, then

A−B is closed.

B−A is open.

(26) Compact Set = Closed + Bounded.

e.g. S = $[1,2] \cup [3,4]$ → It is compact.

$\mathbb{R}$ or (a,b) are not compact.

Union of finite family of compact sets is compact.

Intersection of arbitrary compact sets, containing at least 1 common point, is compact.

(27) Cover of a set.

Let S be a set & $\{G_\alpha\}$ a family of sets.

G_α is called cover of S if $S \subseteq \bigcup_\alpha G_\alpha$

If all sets of G_α are open, it is called open cover.

e.g. $G = \{(-n, n) / n \in \mathbb{N}\}$ is an open cover of $\mathbb{R}$.

(28) Finite Subcover

If G is open cover of S.
Then E is finite subcover if
(a) E is contained in G
(b) E has finite no. of sets
(c) E is a cover of S.

Summary

		Definition
1	Completeness prop. of $\mathbb{R}$	Every non-empty bdd. above subset of $\mathbb{R}$ has supremum in $\mathbb{R}$. ($\mathbb{Q}$ is not complete)
2	Interior point	$P \in S$ is interior if $N_\delta(P) \subseteq S$ for some δ
3	Open set	$S^\circ = S$ i.e. set of interior point is the set itself
4	Isolated point	$P \in S$ isolated if P is not a limit point.
5	Discrete set	Every point of set is isolated.
6	Derived set $D(S)$	set of all limit points
7	Closure of set	$\bar{S} = S \cup D(S)$
8	Dense-in-itself set	$S \subseteq D(S)$
9	Dense set	$\bar{S} = R$ $\mathbb{Q}$ is Dense
10	Closed set	$D(S) \subseteq S$ or compliment is open
	Compact set	Closed + bounded

Sequences

Date
Page

(1) A sequence is a function from $\mathbb{N}$ to $\mathbb{R}$.
Set of all distinct terms in a sequence = range of sequence.

(2) Sum of sequence $X + Y = (x_n + y_n)$
Similarly subtraction, multiplication & division of sequences defined.

If range of sequence is bdd., seq. is called bounded.
i.e. seq. bdd iff $\exists\ M \in \mathbb{N}$ s.t. $|x_n| \leq M \quad \forall n \in \mathbb{N}$.

(3) Limit of a sequence
L is limit of (x_n) if $\forall \epsilon > 0$, $\exists\ k \in \mathbb{N}$ s.t.
$|x_n - L| < \epsilon \quad \forall n \geq k$

(4) If $\lim_{n\to\infty} x_n = x \in \mathbb{R}$, it is a convergent seq.

If $\lim_{n\to\infty} x_n = +\infty$ or $-\infty$, it is divergent seq.

Diverges to $+\infty$ if $\forall k \in \mathbb{R}$, $\exists\ m \in \mathbb{N}$ s.t.
$x_n > k \quad \forall n > m$.

Similar for divergence to $-\infty$.

If seq. doesn't converge to real no. or $+\infty, -\infty$, then it is called oscillatory seq.
If it is bounded, then finite oscillatory sequence.
If it is unbounded, then infinite oscillatory seq.

(5) Null sequences
(x_n) is called null sequence if it converges to 0.

(6) Thm:- Every convergent sequence has unique limit
i.e. it can't converge to ~~2~~ more than 1 limit.

(7) Thm :- Every convergent sequence is bounded.

Proof :- Given $\epsilon > 0$, $\exists$ $k \in \mathbb{N}$ s.t. $|x_n - x| < \epsilon$ $\forall$ $n \geq k$.

We have $|x_n| < |x| + \epsilon$ $\forall$ $n \geq k$

Let $M = \sup\{|x_1|, |x_2|, \dots, |x_{k-1}|, \epsilon + |x|\}$ & so on.

(8) Thm:-

Let (x_n), (a_n) be 2 seq. & $a_n \to 0$.

If for constant $c > 0$; $m \in \mathbb{N}$, $|x_n - x| \leq c a_n$ $\forall n \geq m$,

then $x_n \to x$.

(9) Bernoulli's Inequality

If $x > -1$, then $(1+x)^n \geq 1 + nx$ $\forall n \in \mathbb{N}$

(10) Thm

Let $(x_n) \to x$ & $(y_n) \to y$ then

$(x_n + y_n) \to x + y$, $(x_n - y_n) \to x - y$, $(x_n y_n) \to xy$, $(c x_n) \to cx$.

Also if $y_n \neq 0$ & $y \neq 0$ then $\left(\frac{x_n}{y_n}\right) \to \left(\frac{x}{y}\right)$

This will also apply to addition, product of finite convergent sequences.

Also, if seq. $(a_n) \to a$ & $k \in \mathbb{N}$ then $Lt.(a_n^k) = (Lt.(a_n))^k$

(11) Thm:- If $(x_n) \to x$ & $x_n \geq 0$ $\forall n \in \mathbb{N}$ then $x \geq 0$.

If $(x_n) \to x$ & $(y_n) \to y$ & $x_n \leq y_n$ $\forall n \in \mathbb{N}$ then $x \leq y$

If $a \leq x_n \leq b$ $\forall n \in \mathbb{N}$ then $a \leq Lt.\, x_n \leq b$ (given x_n converges)

(12) Squeeze theorem

Let $x_n \leq y_n \leq z_n$ $\forall n \in \mathbb{N}$ & $Lt.\, x_n = Lt\, z_n$

then y_n is convergent & $Lt\, x_n = Lt\, y_n = Lt\, z_n$

(13) If $Lt.\ x_n = x$ then $Lt.\ |x_n| = |x|$.

(14) Theorem

(x_n) be seq. of +ve real no.

Let $l = \lim \left(\frac{x_n}{x_{n+1}}\right)$ exist.

If $l > 1$ then (x_n) converges & $\lim\limits_{n\to\infty} x_n = 0$

(15) Cauchy's first theorem on limits

$\lim\limits_{n\to\infty} a_n = l \Rightarrow \lim\limits_{n\to\infty} \frac{a_1+a_2+\cdots+a_n}{n} = l$

Proof:- Let $b_n = a_n - l$ so $b_n \to 0$.

Then $x_n = \frac{a_1+\cdots+a_n}{n} = \frac{b_1+b_2+\cdots+b_n}{n} + l$

To prove $x_n \to l$, it is enough to show that

$\frac{b_1+b_2+\cdots+b_n}{n} \to 0$ (use that b_n is bdd.)

(16) Converse of Cauchy's first thm need not be true.

Let $\{a_n\} = (-1)^n \Rightarrow x_n = 0$ if n is even

$-\frac{1}{n}$ if n is odd

now, $\{x_n\} \to 0$ but a_n is ~~dis~~ not convergent.

(17) Theorem

Let a_n be +ve sequence & $Lt.\ a_n = l$ then

$Lt\ (a_1 \cdot a_2 \cdots a_n)^{1/n} = l$

Proof:- Let $b_n = \log a_n$ & use Cauchy's first thm on limits

18) Cauchy's Second Theorem

If (a_n) is sequence s.t. $a_n > 0 \ \forall n$ & $\underset{n\to\infty}{Lt}\left(\frac{a_{n+1}}{a_n}\right) = l$

then $\underset{n\to\infty}{Lt}(a_n)^{1/n} = l$

Proof :- Define $\{b_n\}$ as

$b_1 = a_1, \ b_2 = a_2/a_1, \ b_3 = a_3/a_2 \ \ldots \ b_n = a_n/a_{n-1} \ \ldots$

& then using earlier thm.

Converse is not true. e.g. $a_n = 2^{-n+(-1)^n}$

An example :- $Lt\,(n!)^{1/n} = ?$

$\Rightarrow$ let $a_n = n!$ then $\frac{a_{n+1}}{a_n} = n+1 \Rightarrow \underset{n\to\infty}{Lt}\frac{a_{n+1}}{a_n} = \infty$

$\therefore Lt\,(a_n)^{1/n} = \infty \quad \Rightarrow \quad Lt\,(n!)^{1/n} = \infty$

(In any question of limit sequence that involves $(x_n)^{1/n}$; try to use Cauchy's 2^{nd} thm. ~~Aas~~ As shown above)

e.g. $x_n = \frac{n!}{n^n}$ $\lim x_n = ?$

$\frac{x_n}{x_{n+1}} = \left(1+\frac{1}{n}\right)^n \quad \therefore \lim \frac{x_n}{x_{n+1}} = e \quad \Big/ \Rightarrow \lim \frac{x_{n+1}}{x_n} = 1/e < 1$

$\therefore \lim x_n = 0.$

Monotonic Sequences

(1) Monotonically increasing $x_n \le x_{n+1}$
" decreasing $x_n \ge x_{n+1}$
Strictly monotonically increasing $x_n < x_{n+1}$
" decreasing $x_n > x_{n+1}$
$\Bigg\} \forall n.$

A seq. is said to be monotonic if either mono. increasing or mono. decreasing.

A seq. is said to be strictly monotonic if it is either strictly increasing or strictly decreasing.

(2) Monotonically increasing seq. that is bounded above converges to its supremum.
Unbounded increasing sequence converges to $+\infty$.

Similar for decreasing.

(3) Monotone Convergence Theorem
A monotone seq. is convergent iff. it is bounded.

(4) Limit point of a sequence
l is a limit pt. of sequence $\{x_n\}$ if every nbd of l contains ∞ points of $\{x_n\}$
e.g. $\{x_n\} = (-1)^n$ has 1 & -1 as limit points.

Limit of seq. is limit pt. but converse need not be true.

If for $\epsilon > 0$, $x_n \in (l-\epsilon, l+\epsilon)$ for finitely many values then l is not a limit point of x_n.

(5) Bolzano - Weistrass Theorem for sequences.
→ Every bounded sequences has at least 1 limit point.

(6) Cauchy's general principle of convergence
Necessary & sufficient condition for convergence of a sequence $\{x_n\}$:-
$\forall \epsilon > 0, \exists m \in \mathbb{N}$ s.t. $|x_{n+p} - x_n| < \epsilon \quad \forall n \geq m \; \& \; p \geq 1$

→ Proof of sufficient condition :-
So, $\forall \epsilon > 0, \exists m \in \mathbb{N}$ s.t. $|x_{m+p} - x_n| < \epsilon \quad \forall n \geq m, \; p \geq 1$
$\therefore \{x_n\}$ is bounded.
By bolzano-weistrass thm, it has at least 1 limit pt. l.
We prove that x_n converges to l.

now, for a given ϵ_0, $\exists m \in \mathbb{N}$ s.t. $|x_{n+p} - x_n| < \frac{\epsilon_0}{3}, \forall n \geq m$

$\therefore |x_{m+p} - x_m| < \epsilon/3 \quad \forall p \geq 1$
Since, l is a limit point, $\exists m_1 > m$ s.t. $|x_{m_1} - l| < \epsilon/3$
& since $m_1 > m$, $|x_{m_1} - x_m| < \epsilon/3$

$\therefore |x_{m+p} - l| = |x_{m+p} - x_m + x_m - x_{m_1} + x_{m_1} - l|$
$\leq |x_{m+p} - x_m| + |x_{m_1} - x_m| + |x_{m_1} - l| = \epsilon.$

(7) Cauchy Sequence
$\{x_n\}$ is a cauchy or fundamental sequence if
$\forall \epsilon > 0, \exists m \in \mathbb{N}$ s.t. $|x_{n+p} - x_n| < \epsilon \quad \forall n \geq m \; \& \; p \geq 1$.

Of course, every cauchy sequence is bounded.

(8) Thm :- If $\{x_n\}$ is a convergent sequence, then it is a cauchy sequence.

(9) Example using Bernoulli's Inequality, $(1+x)^n \geq 1+nx$, $x>-1$

(a) $0<b<1$, Prove $\lim_{n\to\infty} b^n = 0$

$\rightarrow$ Take $b = \frac{1}{1+a}$, now $a>0$.

By bernoulli's inequality, $(1+a)^n \geq 1+an$

i.e. $b^n = \frac{1}{(1+a)^n} \leq \frac{1}{(1+an)}$ & so on.

(b) If $c>0$ then $\underset{n\to\infty}{Lt}(c^{1/n}) = 1$

$\rightarrow$ case (i) Let $c>1$ then let $c^{1/n} = 1+d_n$, $d_n>0$

$\therefore c = (1+d_n)^n \geq 1+nd_n$

$\therefore d_n \leq \frac{c-1}{n}$ $\therefore d_n \to 0$ & so on.

(10) Proving $\lim \left(1+\frac{1}{n}\right)^n = e$.

(a) Prove $\left(1+\frac{1}{n}\right)^n$ is monotonically increasing.

$$x_n = \left(1+\frac{1}{n}\right)^n = {}^nC_0 (1)^n + {}^nC_1 (1)^{n-1}\left(\frac{1}{n}\right) + \dots + {}^nC_n (1)^0 \left(\frac{1}{n}\right)^n$$

$$= 1+1+\frac{n(n-1)}{2!}\left(\frac{1}{n^2}\right) + \dots$$

$$= 1+1+\frac{1}{2!}\left(1-\frac{1}{n}\right) + \frac{1}{3!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) + \dots$$

Similarly;

$$x_{n+1} = \left(1+\frac{1}{n+1}\right)^{n+1} = 1+1+\frac{1}{2!}\left(1-\frac{1}{n+1}\right) + \frac{1}{3!}\left(1-\frac{1}{n+1}\right)\left(1-\frac{2}{n+1}\right) + \dots$$

Since, $\left(1-\frac{1}{n}\right) < \left(1-\frac{1}{n+1}\right)$ $\therefore x_{n+1} > x_n$.

Hence, monotonically increasing sequence.

(6) Proving $\left(1+\frac{1}{n}\right)^n$ is convergent.

→ $x_n < 1+1+\frac{1}{2!}+\frac{1}{3!}+\cdots+\frac{1}{n!}$

$< 1+1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots,+\frac{1}{2^{n-1}}$

$= 1+\frac{\left(1-\frac{1}{2^n}\right)}{\left(1-\frac{1}{2}\right)} = 3-\frac{1}{2^{n-1}}$ $\therefore x_n < 3 \;\forall n \in \mathbb{N}$

& since it is bounded monotone, it converges.

(11) Sequence defined by $x_{n+1} = \sqrt{3x_n}$, $x_1 = 1$, converges to 3

→

Use mathematical induction.

$x_2 = \sqrt{3}$ $\therefore x_2 > x_1$. Now, assume $x_{n+1} > x_n$

$\therefore 3x_{n+1} > 3x_n$ $\therefore \sqrt{3x_{n+1}} > \sqrt{3x_n}$

$\therefore x_{n+2} > x_{n+1}$

$\therefore x_n$ is monotonically increasing by induction.

We now find upper bound.

$x_1 < 3$ & $x_2 < 3$, let $x_n < 3 \Rightarrow \sqrt{3x_n} < \sqrt{3.3} = 3$

$\therefore x_{n+1} < 3$ by induction, 3 is upper bd.

$\therefore x_n$ converges.

Let l be the limit.

$\therefore \text{Lt}\, x_n = l$ & $\text{Lt}\, x_{n+1} = l$

$\therefore \text{Lt}\sqrt{3x_n} = \text{Lt}\, x_n$ $\therefore \sqrt{3l} = l \Rightarrow l = 3$ (l can't be 0.)

(12) Another example:-

$y_1 = \sqrt{P}$, $P > 0$, $y_{n+1} = \sqrt{P+y_n}$ Show that y_n converges.

→ Simple to show y_n is increasing.

Then, $y_n < y_{n+1} \Rightarrow y_n < \sqrt{P+y_n} \Rightarrow y_n^2 < P+y_n \Rightarrow y_n^2 - y_n - P < 0$

$\Rightarrow \left[y_n - \frac{1+\sqrt{1+4P}}{2}\right]\left[y_n - \frac{1-\sqrt{1+4P}}{2}\right] < 0$

obv. $\left[y_n - \frac{1+\sqrt{1+4P}}{2}\right] < 0$ $\forall n$ $\therefore y_n$ is bounded.

Infinite Series

① Series of positive terms - all terms are +ve.
Alternating series - terms are alternatively +ve & -ve.

Partial Sum
$\sum x_n = x_1 + x_2 + \dots + x_n + \dots$ is infinite series then
$S_n = x_1 + x_2 + \dots + x_n$ is n^{th} partial sum.
$\{S_n\}$ is sequence of partial sums corresponding to x_n series

② Convergent series ⇒ $\{S_n\}$ converges
Divergent series ⇒ $\{S_n\}$ → $+\infty$ or $-\infty$.
Oscillatory series ⇒ neither cgs or dgs.
Oscillating infinitely ⇒ series is unbounded & neither converges nor diverges.

Obv. a constant series is divergent unless it is 0 series.

③ Geometric Series $\sum_{n=0}^{\infty} r^n$ → oscillates finitely for $r=-1$
& oscillates infinitely for $r<-1$.

Geometric Series converges when common ratio is less than 1.
In an infinite series, following acts don't change its nature
→ ⓐ changing / adding / omitting finite terms
ⓑ division / multiplication by non-zero constant to whole series.

Different Tests for Convergence :-

ⓐ P - Test

$$\sum \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \dots$$

Converges for $p > 1$
Diverges for $p \leq 1$

(b) n^{th} - term test

If Σx_n converges then $\lim_{n\to\infty} x_n = 0$

- A positive term series either converges or diverges to $+\infty$
- A positive term series with $\lim x_n \neq 0$ diverges to $+\infty$.

(c) Comparison Test

Let $\{x_n\}$ & $\{y_n\}$ be non-negative sequences & for some $k \in \mathbb{N}$,
$0 \le x_n \le y_n$ $\forall\ n \geq k$ then

(a) Convergence of $\Sigma y_n \Rightarrow$ convergence of Σx_n
(b) Divergence of $\Sigma x_n \Rightarrow$ Divergence of Σy_n.

(d) Limit Comparison Test

If x_n & y_n are strictly positive sequences & let this limit exist $l = \lim_{n\to\infty}\left(\frac{x_n}{y_n}\right)$ then

(a) if $l \neq 0$ & finite, then x_n, y_n conv. or dvg together.
(b) if $l = 0$ & y_n cvg $\Rightarrow x_n$ cvg
(c) if $l = \infty$ & y_n dvg $\Rightarrow x_n$ dvg

(e) D'Alembert's Ratio Test

If Σu_n is positive term series s.t. $Lt \frac{u_n}{u_{n+1}} = l$ then
(a) Converges if $l > 1$
(b) diverges if $l < 1$
(c) may or may not cvg. if $l = 1$

Ratio test is generally applied when fractionals & combination of powers is involved.

(f) **Cauchy's root test**

If Σu_n is +ve term series s.t.

(a) $Lt. (u_n)^{1/n} = l$ then

$l < 1 \Rightarrow$ convergence, $l > 1 \Rightarrow$ divergence, $l = 1 \Rightarrow$ don't know.

(b) $Lt (u_n)^{1/n} = \infty$ then it is divergent.

Root Test is used when powers are involved. Root test is more general than ratio test because if $Lt (u_n)^{1/n}$ exists then $Lt\ u_{n+1}/u_n$ may or may not exist. But $Lt\ u_{n+1}/u_n$ existance means $Lt (u_n)^{1/n}$ exists.

(g) **Raabe's Test**

If $Lt\ n\left(\frac{u_n}{u_{n+1}} - 1\right) = l$ then

$l > 1 \Rightarrow$ convergence, $l < 1 \Rightarrow$ divergence, $l = 1 \Rightarrow$ don't know.

Obv. Raabe's test is stronger than ratio test.

Also, $l = \infty \Rightarrow$ convergence & $l = -\infty \Rightarrow$ divergence

(f) **Logarithmic Test**

$Lt\ n \log \frac{u_n}{u_{n+1}} = l$

then $l > 1 \Rightarrow$ convergence, $l < 1 \Rightarrow$ dvg, $l = 1 \Rightarrow$ don't know.

Obv. when e is involved, apply logarithmic test.

(9) Gauss Test

If ΣU_n is +ve term series such that

$$\frac{U_n}{U_{n+1}} = 1 + \frac{\lambda}{n} + \frac{\alpha_n}{n^{1+\delta_n}}$$ where $\delta_n > 0$ & α_n is bdd. sequence

then (i) Σu_n converges if $\lambda > 1$

(ii) Diverges if $\lambda \leq 1$

Gauss test never fails as $\lambda = 1$ diverges. It is used on failure of ratio test & when U_n/U_{n+1} can be expanded in powers of $(1/n)$ using binomial theorem etc.

Note that essentially Gauss is rearrangement of terms in Raabe's test $n\left(\frac{u_n}{u_{n+1}} - 1\right) = \lambda$

(10) De Morgan's & Bertrand's test

$$\text{Lt} \left[\left\{ n\left(\frac{U_n}{U_{n+1}} - 1\right) - 1 \right\} \log n \right] = l$$

$l > 1 \Rightarrow$ converges, $l < 1 \Rightarrow$ diverges

An alternative

$$\text{Lt} \left[\left(n \log \frac{U_n}{U_{n+1}} - 1 \right) \log n \right] = l$$

$l > 1 \Rightarrow$ cvg, $l < 1 \Rightarrow$ dvg

(l) Cauchy's Condensation Test

Let $\Sigma a(n)$ be s.t. $a(n)$ is decreasing seq. of +ve numbers

Then

$\sum_{n=1}^{\infty} (a(n))$ converges or diverges same as $\sum_{n=1}^{\infty} 2^n a(2^n)$

Helps when series terms contain logarithm.

Decreasing seq. cond^n necessary o/w you could have a dvt. seq. with just 2^n terms 0 & then call $\Sigma 2^n a(2^n) = 0$. ∴ note to seq.

(j) Comparison Test

Σu_n & Σw_n be 2 series of +ve terms
& let h, k be +ve no. s.t. $h w_n < u_n < k w_n \ \forall n$

Then

Σu_n & Σw_n converge & diverge together.

(k) Leibnitz's test on alternating Series

The alternating series

$\Sigma (-1)^{n+1} u_n = u_1 - u_2 + u_3 - u_4 + \dots$ $\quad u_n > 0 \ \forall n$

converges if

(i) $u_n \geq u_{n+1} \ \forall n$ &

(ii) $\lim_{n \to \infty} u_n = 0$

Essentially, u_n should be decreasing +ve seq. tending to 0.
Then alternating Series made of u_n converges.

(l) Abel's Test

If $\sum_{n=1}^{\infty} a_n$ is convergent &
sequence $\{b_n\}$ is monotonic & bounded
then $\Sigma a_n \cdot b_n$ converges.

(m) Dirichilet's Test

Let Σa_n be a series whose n^{th} partial sum is bounded
$\{b_n\}$ be a monotonic sequence converging to zero
then $\Sigma a_n \cdot b_n$ converges.

Note :- Leibnitz test is particular case of dirichilet
since $\Sigma (-1)^n$ has bounded partial sums.

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Series Convergence Tests

No.	Test	Important Term
1	P-test	$\Sigma \frac{1}{n^p}$ crg for $p>1$ dvg for $p \leq 1$
2	n^{th} term test	If Σx_n cvg $\Rightarrow \lim_{n\to\infty} x_n = 0$
3	Comparison Test	$0 \leq x_n \leq y_n \Rightarrow$ Conv $\Sigma y_n \Rightarrow$ Conv. Σx_n | dvg $\Sigma x_n \Rightarrow$ dvg Σy_n
4	Limit Comparison	$\lim\left(\frac{x_n}{y_n}\right) = l$, l finite $(\neq 0) \Rightarrow$ cvg, dvg together etc.
5	D'Almbert's Ratio	$\lim \left(\frac{u_n}{u_{n+1}}\right)$
6	Cauchy's Root	$\lim (u_n)^{1/n} = l$ (Note $l>1 \Rightarrow$ divergence)
7	Raabe's Test	$\lim n\left(\frac{u_n}{u_{n+1}} - 1\right)$
8	Logarithmic Test	$\lim n\left(\log \frac{u_n}{u_{n+1}}\right)$
9	Bertrand / Demorgan	$\lim\left\{\left\{n\left(\frac{u_n}{u_{n+1}} - 1\right) - 1\right\} \log n\right\}$
10	Alternative	$\lim\left\{\left(n \log \frac{u_n}{u_{n+1}} - 1\right) \log n\right\}$
11	Gauss Test	$\frac{u_n}{u_{n+1}} = 1 + \frac{\lambda}{n} + \frac{\alpha_n}{n^{1+\delta_n}}$, $\left\{\begin{array}{l}\alpha_n \text{ bdd seq } \& \ \delta_n > 0 \\ \lambda > 1 \Rightarrow \text{cvg}, \ \lambda \leq 1 \Rightarrow \text{dvg}\end{array}\right\}$
12	Cauchy Condensation	a_n is $\downarrow$ +ve seq. / $\Sigma a_n \Leftrightarrow \Sigma 2^n a_{2^n}$
13	Leibnitz test on alt.	u_n is $\downarrow$ +ve seq & $\lim u_n \to 0$ then $\Sigma (-1)^n u_n$ cvg.
14	Dirichilet Test	$\{b_n\}$ monotone & $\to 0$ / $\Sigma a_n \to$ part. sum bdd. $\Rightarrow \Sigma a_n \cdot b_n$ cvg.
15	Abel's Test	Σa_n cvg & $\{b_n\}$ monotone & bounded $\Rightarrow \Sigma a_n \cdot b_n$ cvg
16	Comparison test	$h w_n < u_n < k w_n$ $\forall n$, h & k const. $\Rightarrow \Sigma u_n$ & Σw_n cvg, dvg together

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(5) Examples on these tests

(a) Check convergence of Σx_n, $x_n = \frac{1}{n^2+n}$

$\rightarrow$ Let $y_n = \frac{1}{n^2}$ $\quad \underset{n\to\infty}{Lt} \frac{x_n}{y_n} = 1$ $\therefore$ By limit comparison, Σx_n cgs.

(b) $\Sigma\left(\sqrt{n^3+1} - \sqrt{n^3}\right)$

$\rightarrow$ Let $x_n = \sqrt{n^3+1} - \sqrt{n^3} = \frac{1}{n^{3/2}\left(1+\sqrt{1+1/n^3}\right)}$

let $y_n = 1/n^{3/2}$ $\quad \therefore Lt \frac{x_n}{y_n} = \frac{1}{2}$ $\quad \therefore$ Converges.

(c) $\Sigma \sqrt[3]{n+1} - \sqrt[3]{n}$

$\rightarrow$ Let $x_n = \sqrt[3]{n+1} - \sqrt[3]{n} = n^{1/3}\left(\left(1+\frac{1}{n}\right)^{1/3} - 1\right)$

$$= n^{1/3}\left(1+\frac{1}{3}\cdot\frac{1}{n} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\frac{1}{n^2} + \cdots - 1\right)$$

$$= \frac{1}{n^{2/3}}\left[\frac{1}{3} - \frac{1}{9n} + \cdots\right]$$

Let $y_n = 1/n^{2/3}$

then $Lt \frac{x_n}{y_n} = 1/3$ & as Σy_n dvg; Σx_n dvg.

Rationalisation is effective only when square roots are involved whereas above binomial expansion method is more general.

(d) Checking cgs. of $\Sigma \sin \frac{1}{n}$ or $\Sigma \frac{1}{\sqrt{n}} \tan\frac{1}{n}$ etc.

$\rightarrow$ Obv. use $y_n = \frac{1}{n}$ since $\lim_{n\to\infty} \frac{\sin\frac{1}{n}}{1/n} = 1$ & so on.

(e) $1 + \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \cdots\cdots$; $x > 0$

$\rightarrow$ leaving first term; let $u_n = \frac{x^{2n}}{2n}$ $\quad \therefore u_{n+1} = \frac{x^{2(n+1)}}{2(n+1)}$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{1}{x^2}\left(1+\frac{1}{n}\right) \quad \therefore Lt. \frac{u_n}{u_{n+1}} = \frac{1}{x^2}$$

$\therefore x < 1 \Rightarrow$ cgs, $x > 1 \Rightarrow$ dgs.

Need to separately consider $x=1$ case, as ratio test fails here,

then $u_n = \frac{1}{2n}$ $\therefore$ dgs.

(f)

(g)

(h)

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(f) $\frac{1^2}{2^2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} + \cdots\cdots$

→ Raabe's test etc. fails. So we apply Gauss.

$$\frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2} = \left(1+\frac{1}{n}\right)^2\left(1+\frac{1}{2n}\right)^{-2}$$

$$= \left(1+\frac{2}{n}+\frac{1}{n^2}\right)\left(1-\frac{2}{2n}+\frac{3}{4n^2} \cdots\right)$$

$$= \left(1-\frac{2}{2n}+\frac{3}{4n^2}-\cdots\right)+\left(\frac{2}{n}-\frac{4}{2n^2}+\frac{6}{4n^3}\right)+\left(\frac{1}{n^2}-\frac{2}{2n^3}+\cdots\right)$$

$$= 1+\frac{1}{n}+O\left(\frac{1}{n^2}\right)$$

$\therefore \lambda = 1$ By gauss test, divergent.

(g) $$\sum_{n=3}^{\infty} \frac{1}{n(\log n)(\log\log n)}$$

→ Let $a(n) = \frac{1}{n(\log n)(\log\log n)}$

Since, it is a monotone, we apply cauchy condensation test

$\therefore \sum 2^n a(2^n) = \sum \frac{1}{(n\log 2)\log(n\log 2)} = \sum x_n$ (say)

We can see $x_n > \frac{1}{\log 2} \cdot \frac{1}{n\log n}$ $(\because \log 2 < 1)$

let $\sum y_n = \sum \frac{1}{\log 2} \cdot \frac{1}{n\log n}$ again apply cond. test & so on

(h) Some manipulations involving logarithmic series:-

(i) $\sum \frac{1}{(\log n)^{\log n}}$ → we can find n large enough so that

$$\log(\log n) > 2$$

$$\therefore (\log n).[\log(\log n)] > 2\log n$$

$$\therefore (\log n)^{\log n} > n^2$$

$$\therefore \frac{1}{(\log n)^{\log n}} < \frac{1}{n^2}$$

& $\sum \frac{1}{n^2}$ converges $\therefore$ given series also converges.

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(ii) $\sum r^{\log n}$

$\rightarrow$ now $\quad \log r \cdot \log n = \log n \cdot \log r$

$\therefore \log\left(n^{\log r}\right) = \log\left(r^{\log n}\right)$

$\therefore n^{\log r} = r^{\log n} = \sum \frac{1}{n^{-\log r}}$

By P-test, this converges if $-\log r > 1$ i.e. if $r < \frac{1}{e}$

(i) Show that series $\sum_{n=2}^{\infty} \frac{(n^3+1)^{\frac{1}{3}} - n}{\log n}$ is convergent.

$\rightarrow$ let $a_n = (n^3+1)^{\frac{1}{3}} - n$ & $b_n = \frac{1}{\log n}$

Upon binomial expansion, we get $a_n = \frac{1}{n^2}\left[\frac{1}{3} - \frac{1}{9n^2} + \cdots\right]$

Obv. $\sum a_n$ converges by limit comparison.

Now $\sum b_n$ is monotonically decreasing & bounded below.

$\therefore$ By abel's test, $\sum a_n \cdot b_n$ converges.

(j) Discuss convergence of $\sum \frac{(-1)^{n-1}}{n^p}$ $(p > 0)$

Let $a(n) = (-1)^{n-1}$ & $b(n) = 1/n^p$

$\therefore \sum a(n)$ has bounded partial sums.

& b_n is decreasing seq. going to 0.

$\therefore$ By dirichlet's test, $\sum a_n \cdot b_n$ converges.

(6) Absolute & Conditional Convergence:-

Σu_n is absolutely cgt. if $\Sigma |u_n|$ converges.
If Σu_n converges but not absolutely, then it is called conditional convergence or semi-convergence.

e.g. $\Sigma \frac{(-1)^{n-1}}{n}$ converges by Leibnitz test but $\Sigma \frac{1}{n}$ doesn't.

Hence, it is conditional convergence.

(7) Rearrangement of Terms :-
Σb_n & Σa_n are said to rearrangement of each other if $\exists$ one to one correspondence in terms. (i.e. for each a_n, $\exists$ b_n & vice versa)

Rearrangement can change nature of series completely.

(8) Dirichilet's Theorem
(i) If Σa_n is absolutely cgt., then every ~~deran~~ derangement also absolutely converges to same no.
(ii) If Σa_n is +ve terms series & diverges then every derangement diverges. (so its not completely non-intuitive !!!)

(9) Riemann's Theorem
A conditionally convergent series can be made by derangement
(a) To converge to any real no.
(b) To diverge to $+\infty$ or $-\infty$
(c) To oscillate finitely or infinitely.

Example:- Check for convergence $1+\frac{1}{3^2}-\frac{1}{2^2}+\frac{1}{5^2}-\frac{1}{4^2}+\frac{1}{6^2}-\dots$
→ It is rearrangement of $1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}\dots$ which is absolutely convergent. Hence by dirichilet thm, it is convergent

(10) Pringsheim's Method

Rearrangement questions will have series $\sum (-1)^{n+1} \frac{1}{n}$ mostly. This is a conditionally convergent series. Its sum is $\log 2$.

Now, new arrangement if has α positive terms followed by β negative terms then new sum becomes

$$\log 2 + \frac{1}{2} \log \left(\frac{\alpha}{\beta}\right)$$

(We can see ~~at~~ how any value can be achieved.) (almost.)

Example :- find sum of series

$$1 - \tfrac{1}{2} - \tfrac{1}{4} + \tfrac{1}{3} - \tfrac{1}{6} - \tfrac{1}{8} + \tfrac{1}{5} \dots$$

$\rightarrow \alpha = 1$ & $\beta = 2$ $\therefore$ Sum $= \log 2 + \frac{1}{2} \log \frac{1}{2} = \log 2 - \frac{1}{2} \log 2$

$$= \frac{1}{2} \log 2$$

Example :- 2007 :→ Rearrange $\sum (-1)^{n+1} \frac{1}{n}$ to converge to 1

$\rightarrow \therefore \cancel{\tfrac{1}{2}} \log 2 + \frac{1}{2} \log\left(\frac{\alpha}{\beta}\right) = 1 \Rightarrow \frac{\alpha}{\beta} = 4e^2$

$\therefore$ Not possible by Pring Scheim's method.

⊳ So essentially remember the formula $\log 2 + \frac{1}{2} \log\left(\frac{\alpha}{\beta}\right)$

Product of Series

① Cauchy product of infinite series

$$\sum_{n=1}^{\infty} C_n = \left(\sum_{n=1}^{\infty} a_n\right)\left(\sum_{n=1}^{\infty} b_n\right)$$

$$= a_1b_1 + (a_1b_2 + a_2b_1) + (a_1b_3 + a_2b_2 + a_3b_1) + \dots$$

$$= C_1 + C_2 + C_3 + \dots$$

Slight difference when series starts from 0;

$$\sum_{n=0}^{\infty} C_n = \left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right)$$

$$= a_0b_0 + (a_1b_0 + a_0b_1) + (a_0b_2 + a_1b_1 + a_2b_0) + \dots$$

$$= C_0 + C_1 + C_2 + \dots$$

If 2 series converge, their product need not converge.

② ⓐ If Σa_n & Σb_n are non-negative & converge to A & B then $\Sigma C_n = AB$

ⓑ If Σa_n & Σb_n are absolutely cgt. then ΣC_n abs. cgs. to AB

ⓒ Merten's Thm :- Let Σa_n abs. cgt. & Σb_n be cgt. $\Rightarrow \Sigma C_n = AB$

③ Abel's test
Let Σa_n & Σb_n be convergent.
If ΣC_n converges, then $\Sigma C_n = AB$.

④ ex. S.T. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\sqrt{n}}$ Cauchy product with itself is not convergent

$\longrightarrow$ By leibnitz test, Σa_n & Σb_n are cgt. ($\Sigma a_n = \Sigma b_n = \Sigma \frac{(-1)^{n-1}}{\sqrt{n}}$

Now,

$$C_n = \frac{(-1)^0}{\sqrt{1}} \cdot \frac{(-1)^{n-1}}{\sqrt{n}} + \frac{(-1)^2}{\sqrt{2}} \cdot \frac{(-1)^{n-2}}{\sqrt{n-1}} + \dots + \frac{(-1)^{n-1}}{\sqrt{n}} \cdot \frac{(-1)^0}{\sqrt{1}}$$

$$= (-1)^{n-1}\left[\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{2.(n-1)}} + \cdots + \frac{1}{\sqrt{n.1}}\right]$$

$$\geq \left[\frac{1}{\sqrt{n.n}} + \frac{1}{\sqrt{n.n}} + \cdots + \frac{1}{\sqrt{n.n}}\right] = 1$$

$\therefore |C_n| \geq 1 \ \forall n \Rightarrow \lim_{n\to\infty} C_n \neq 0 \quad \therefore \sum C_n$ ~~diverges~~ doesn't converge.

This is a good idea.
Find general representation of C_n & show $C_n \not\to 0$.

(5) e.g. S.T. cauchy product of 2 divergent series is convergent

$\sum a_n = 2 + 2^1 + 2^2 + \cdots$

$\sum b_n = -1 + 1 + 1 + 1 \cdots$

$\to$ $C_n = a_1 b_n + a_2 b_{n-1} + \cdots + a_n b_1$

$$= 2.1 + 2.1 + 2^2.1 + \cdots + 2^{n-2}.1 + 2^{n-1}.(-1)$$

$$= 2 + 2\frac{(2^{(n-2)} - 1)}{2-1} - 2^{n-1} = 0 \ \forall n \geq 2$$

$C_1 = a_1 b_1 = 2(-1) = -2$

$\therefore \sum C_n = -2 + 0 + 0 \cdots = -2$

(6) Check ex. on page 21

Infinite Products

① Partial Product $P_n = a_1 \cdot a_2 \cdots a_n$
Product $\prod a_n$ converges in seq. of partial product cgs.

So in many questions we have to use this basic def^n & find P_n which tells us about convergence of product.

② Convergence -
ⓐ If infinitely many a_n are zero → diverge to 0.

ⓑ If finite a_n are 0 & their removal means product convergence
→ then we say product converges

ⓒ If finite a_n are negative then choose m s.t. $\forall n \geq m$; $a_n > 0$
& then consider $\prod_{n=m}^{\infty} a_n$ for deciding cgs or dgs.

ⓓ If P_n seq. oscillates → $\prod a_n$ oscillates.

③ For convenience we write product as $\prod(1+a_n)$
& assume $a_n > -1$ s.t. $\log(1+a_n)$ is defined.

④ ex. S.T. Product cgs.
$\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots$

→ Find P_n. simple job

⑤ See ex. on page 24 for proving cgs. using $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4}$ Series.

(6) Necessary condⁿ for convergence

If $\prod_{n=1}^{\infty}(1+a_n)$ cgs ~~when~~ then $\lim_{n\to\infty} a_n = 0$

(7) Important results

(a) If $a_n \geq 0$ then Σa_n & $\Pi(1+a_n)$ converge & diverge together. $(\Sigma a_n \to -\infty \Rightarrow \Pi(1+a_n) \to 0)$

(b) If Σa_n^2 is convergent, then $\Pi(1+a_n)$ & Σa_n converge or diverge together.
This is needed in tricky questions containing terms like $\Pi\left(1-\frac{(-1)^{n-1}}{\sqrt{n}}\right)$ etc.

Remember to check convergence of Σa_n^2 also in such questions containing $(-1)^n$.

(c) If Σa_n^2 diverges & Σa_n converges or oscillates finitely then $\Pi(1+a_n)$ diverges to 0.

(8) Absolute Convergence

Defⁿ → $\Pi(1+a_n)$ is called abs. cgt. if $\Pi(1+|a_n|)$ is cgt.

Test → $\Pi(1+a_n)$ is abs. cgt. iff Σa_n is abs. cgt.
iff $\Sigma \log(1+a_n)$ is abs. cgt.

The factors of absolutely convergent product can be rearranged in any order without affecting its convergence.

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(9) e.g. Check $\prod \left(1+\frac{(-1)^n}{n^\alpha}\right)$.

→ $\sum \frac{(-1)^n}{n^\alpha}$ is cgt. when $\alpha > 0$ as per Leibnitz.

& $\sum a_n^2 = \sum \frac{1}{n^{2\alpha}}$ converges if $\alpha > 1/2$.

∴ given product cgs when $\alpha > 1/2$

(10) example. Check $\prod \left[1+\left(\frac{nx}{n+1}\right)^n\right]$

→ let $a_n = \left(\frac{nx}{n+1}\right)^n$ ∴ $a_n^{1/n} = \frac{nx}{n+1}$ ∴ $\lim a_n^{1/n} = x$

∴ cgt. if $x<1$ & dgt. if $x>1$.

if $x=1$ $\sum a_n$ is dgt. as $\lim a_n = 1/e \neq 0$.

∴ dgt. if $x=1$.

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Quick Real Analysis Notes
Covering major points

Limits & Continuity

(1) Limits $\forall x$ s.t. $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \epsilon$ cont.
$|f(x) - l| < \epsilon$ limit

Sequential criterion: $\forall x_n \to x$; $f(x_n) \to l$
useful for proving discont.

(2) In any questions of ~~finding~~ (proving) limit; start with $|f(x) - l|$ & reach $|x - x_0|$ with some manipulations.

(3) Squeeze thm: obvious.
use $\Rightarrow$

$$-1 \leq \cos x \leq 1$$

integrating gives $-x \leq \sin x \leq x$
again int. $1 - \frac{x^2}{2} \leq \cos x \leq 1$

(4) Limits at ∞ or infinite limits: quite intuitive

(5) Removable discontinuity $\to$ function can be redefined & then cont.
Discon. of first kind = jump disc.
Discon. of 2^{nd} kind = both left & right limit doesn't exist
Infinite dis $\to$ obv.

(6) Dirichlet function = $f(x) = 1 \quad x \in \mathbb{Q}$
$f(x) = 0 \quad x \notin \mathbb{Q}$
Discont. can be proved using seq. criterion. One seq. limit 0
Other seq. limit 1 & so on

(7) f cont. & satisfies functional eq. $f(x+y) = f(x) + f(y)$
s.t. $f(x) = cx$.
$\to$ Easy to get $f(qx) = q f(x) \quad q \in \mathbb{Q}$ with simple manipul..
Then use seq. of rationals tending to irrational
& so on.

(8) Thm: If f is cont. on $[a,b] \Rightarrow$ bounded.
$\rightarrow$ Take ϵ; partition $[a,b]$ s.t. each subinterval has $|f(x_1)-f(x_2)| < \epsilon$
& so on.

(9) Thm: If f is cont. on $[a,b] \Rightarrow$ it attains its bounds.
$\rightarrow$ If possible let $f(x)$ not attain its bounds. Let $\sup = M$.
Consider $g(x) = \frac{1}{M-f(x)}$; this is defined everywhere & cont.
$\therefore g(x)$ has to be bounded. now use $\frac{1}{M-f(x)} \leq k \Rightarrow f(x) \leq M - \frac{1}{k}$
& so on.

Converse not true; attaining bound doesn't imply cont.
e.g. $\sin(1/x)$ in $[-1,1]$

(10) Sign preservation thm;
- If f cont. on $[a,b]$ & $a<c<b$ s.t. $f(c) \neq 0$
then sign is preserved in nbd. of c.

- If cont. on $[a,b]$ & $f(a).f(b) < 0 \Rightarrow \exists\, c$ s.t. $f(c) = 0$.

Intermediate value thm.
If cont. on $[a,b]$, then every value bet^n $f(a)$ & $f(b)$ attained atleast once.

(11) Uniform continuity

If given $\epsilon > 0$, $\exists\, \delta > 0$ that works for all x.
Uniform cont. is a global property & cont. is a local prop.

$F(x)$ is not uniform if $\exists$ some $\epsilon > 0$ for which no δ works.
$\forall \delta > 0$, $\exists\, x_1, x_2$ s.t. $|x_1 - x_2| < \delta$ & $|f(x_1) - f(x_2)| \geq \epsilon$

(12) If a function is continuous on $[a,b]$ then it is uniformly continuous.

Differentiability

① $f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c} = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}$

② For an even function, $f(x) = f(-x)$
& $f'(x) = -f'(-x)$

③ Interior extremum theorem
If extremum at c & $f'(c)$ exists then $f'(c) = 0$.
Converse not true. (x^3)

If $f'(c) = 0$, c is called stationary point.

④ Roll's theorem: ⓐ $f(a) = f(b) = 0$, ⓑ Derivative exists on (a,b)
ⓒ & cont. on $[a,b]$; $\exists\, c$ s.t. $f'(c) = 0$, $c \in (a,b)$.
→ Proof uses fact that $[a,b]$ attains its bounds & then extremum will have deriv. 0 & so on.

Roll's thm fails if not derivable on (a,b), not cont. on (a,b)
or $f(a) \neq f(b)$

⑤ Remember 3 conditions of Roll's theorem & show they are satisfied in exam before applying it.

⑤ Lagrange Mean Value Thm
ⓐ f cont. on $[a,b]$ ⓑ f derivable on (a,b)
then $\exists\, c$ s.t. $f'(c) = \frac{f(a) - f(b)}{a - b}$

Corollary ①: $f'(x) = 0 \ \forall x \in (a,b) \Rightarrow$ constant function.

Corollary ② increasing $\Leftrightarrow f'(x) \geq 0$
decreasing $\Leftrightarrow f'(x) \leq 0$

(6) In some inequality proving questions, brute force work is needed; just take difference & show $f'(x) \geq 0 \ \forall x$.

e.g. $x - \frac{x^2}{2} < \log(1+x) < x - \frac{x^2}{2(1+x)}$ here we use brute-force.

(7) Lagrange Mean Value useful in ineq.

→ Idea is to show $f'(c) = \frac{f(x) - f(0)}{x}$

& then start with $0 < c < x$ & with manipulations; arrive at term 1 $< f'(c) <$ term 2.

Here we replace $f'(c)$ by $\frac{f(x) - f(0)}{x}$.

e.g. P.T. $\frac{2}{7} < \log(1.4) < \frac{2}{5}$.

→ use that $\frac{\log(1+t) - \log(1)}{t} = f'(c)$ where $f(x) = \log(1+x)$

then $0 < c < x$

$\therefore 1 > \frac{1}{1+c} > \frac{1}{1+x}$

$\therefore 1 > \frac{\log(1+x)}{x} > \frac{1}{1+x}$ & putting $x = 2/5$.

(8) In tricky questions like

P.T. $f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a+\theta h)$ $\theta \in (0,1)$.

Try to find a function $\phi(x)$ s.t. $\phi(a) = \phi(b)$

& then apply Roll's theorem.

Some inspiration can be taken from final expected result while deciding $\phi(x)$.

Now, final result is expected to have some $f'(c)$ or $f''(c)$ etc. generally replace this c by x & see what function you get in terms of a & b.

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e.g. P.T. for good f & g; $\exists c$ $\dfrac{f(c)-f(a)}{g(b)-g(c)} = \dfrac{f'(c)}{g'(c)}$

$\longrightarrow$ now, replacing c by x & getting eq. we have

$$(f(x)g(x))' - (g'(x)f(a) + f'(x)g(b)) = 0$$

$\therefore$ we take $\phi(x)$ as integral of above

$$\phi(x) = g(x)f(x) - g(x)f(a) - f(x)g(b)$$

luckily $\phi(a) = \phi(b)$ & so on.

(9) Cauchy's Mean Value theorem $\rightarrow$ 2nd mean value thm.

Let f, g be well behaved on $[a,b]$, then $\exists\, c$ s.t.

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

(10) Generalised mean value theorem

$$\begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0$$

Good ex.☆ P.T. $\left(1+\frac{1}{x}\right)^x > \left(1+\frac{1}{y}\right)^y$ if $x > y > 0$

$\longrightarrow$ let $f(x) = \left(1+\frac{1}{x}\right)^x$

(take log etc.)

then we get $f'(x) = \left(1+\frac{1}{x}\right)^x \left[\log\left(1+\frac{1}{x}\right) - \dfrac{\frac{1}{x}}{1+\frac{1}{x}}\right]$

Now, we brilliantly replace $1/x$ by x & consider $\phi(x) = \log(1+x) - \frac{x}{1+x}$

& $\phi'(x) = \frac{x}{1+x^2} > 0 \;\forall x > 0$ & $\phi(0) = 0$ $\therefore$ $\phi(x)$ is strictly in

$\therefore \phi(x) > 0 \;\forall x > 0$

$\therefore \phi\left(\frac{1}{x}\right) > 0 \;\forall x > 0$

$\therefore f'(x) > 0$ & so on.

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(11) Taylor's theorem

Let f be defined on [a,b] s.t.
(i) $f^{(n)}$ derivative exists on (a,b)
(ii) $f^{(n-1)}$ derivative is cont. on [a,b].
Then $\exists\ c \in (a,b)$ s.t.

$$f(b) = f(a) + (b-a) f'(a) + \frac{(b-a)^2}{2!} f''(a) + \cdots$$

$$\cdots + \frac{(b-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{(b-a)^p (b-c)^{n-p}}{p\,(n-1)!} f^{(n)}(c)$$

P is a given integer.

Just mug up this remainder n^{th} term $\dfrac{(b-a)^p\,(b-c)^{n-p}}{p\,(n-1)!} f^{(n)}(c)$

→ Total power is obv. retained to n but (b-a) is accompanied by c & denominator has $p(n-1)!$ instead of $n!$.

This P form is called Roche's form of remainder.
P = 1 gives cauchy's form of remainder
P = n gives lagrange's form of remainder.

We like lagrange's simple remainder $\dfrac{(b-a)^n}{n!} f^{(n)}(c)$

(12) Taylor Series

$$f(a+h) = f(a) + h f'(a) + \cdots + \frac{h^n}{n!} f^{(n)}(a) + \cdots \quad \cancel{\theta \in (0,1)}$$

Maclaurin's series, even more elegant $x = h$ & $a = 0$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^n}{n!} f^{(n)}(0) + \cdots$$

(13) e.g. Using taylor theorem S.T. $1+x+\frac{x^2}{2} < e^x < 1+x+\frac{x^2}{2}e^x$ $x>0$

$\longrightarrow$ Taylor theorem gives $e^x = 1+x+\frac{x^2}{2}e^{\theta x}$ $\theta \in (0,1)$

$\therefore$ obv.

(14) In questions of finding taylor or maclaurin series, remember to show remainder tends to 0.

i.e.
$$\frac{(b-a)^n}{n!} f^{(n)}(a+b\theta) \to 0.$$

e.g. find series expansion of e^x.

$\to$ Remainder $R_n = \frac{x^n}{n!} e^{x\theta}$

$$\therefore \lim_{n\to\infty} R_n = e^{x\theta} \lim_{n\to\infty} \frac{x^n}{n!}$$

let $a_n = \frac{x^n}{n!}$ $\therefore \frac{a_n}{a_{n+1}} = \frac{n+1}{x}$

$$\therefore \lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \infty > 1 \quad \therefore \lim_{n\to\infty} a_n = 0$$

$$\therefore \lim_{n\to\infty} R_n = 0$$

& then find coefficients.

(15) Darboux's theorem

If f is diff. on $[a,b]$ & $f'(a)$ & $f'(b)$ have k in bet^n them, then $\exists\ c \in (a,b)$ s.t. $f'(c) = k$.

$\longrightarrow$ let $g(x) = kx - f(x)$.

g is diff. & bdd $\therefore$ attains bound & so on.

(16) General rule for finding maxima / minima

Let $f' = f'' = \dots = f^{(n-1)} = 0$

& $f^{(n)} \neq 0$

then (i) n even $\Rightarrow$ minima if $f^{(n)}(x) > 0$

maxima if $f^{(n)}(x) < 0$

(ii) n odd $\Rightarrow$ no minima or maxima.

(17) L'Hospital's Rule for indeterminate forms.

If $\frac{f(x)}{g(x)}$ has indeterminate form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ at lim $\to a$

then, $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$

In some $\frac{\infty}{\infty}$ problems, we have to convert it to $\frac{0}{0}$;

Otherwise process will never end.

(18) Manipulations for other forms like ~~0^∞~~ $0 \times \infty$ etc.

e.g. $\lim_{x\to 0} x^x$

$\rightarrow$ let lim $= l$ $\Rightarrow$ $\log l = \lim (x \log x) = \lim \frac{\log x}{1/x}$ & so on.

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Easy to
common

Riemann Integral

(1) Norm of partition
Maximum of the lengths of subinterval is known as norm of partition, $\|P\|$.

(2) Upper Darboux Sum corresponding to partition P
$= \sum_{r=1}^{n} M_r \delta_r$ where M_r is supremum of subinterval with length δ_r & so on.
Similarly lower Darboux sum $= \sum_{r=1}^{n} m_r \delta_r$

(3) Lower Riemann integral

Let $L(P,f)$ denote set of ~~all~~ lower darboux sums for all possible partitions.
lower riemann integral $= \sup \{ L(P,f) \} = \underline{\int_a^b} f(x)dx$

Similarly Upper Riemann integral $= \inf \{ U(P,f) \} = \overline{\int_a^b} f(x)dx$

(4) Riemann integral
A function is called Riemann integrable if
$\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx$ & this value is Riemann integral of f over $[a,b]$

The family of all bounded riemann integrable functions over $[a,b]$ is shown as $R[a,b]$.

(5) Every bdd. function need not be R integrable. → Dirichlet

(6) In questions asking to find integral $[0,1]$ from basic def^n of Riemann integral; consider ~~partitio~~ partition given by $\left[\frac{r}{n}, \frac{r+1}{n}\right]$. Generally works.

(7)

(8)

(9)

(10)

(1

Easy to remember & must remember.
common term $\frac{\sin\frac{n\alpha}{2}}{\sin\frac{\alpha}{2}}$ (n is no. of terms). Other term is average of first & last term.

(7) In questions involving Riemann int. of sin functions, remember [similarly $\cos\alpha + \cos(\alpha+\beta) + \cdots + \cos(\alpha+(n-1)\beta) = \frac{\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}} \cos\left(\alpha+\frac{n-1}{2}\beta\right)$

$\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \cdots + \sin(\alpha+(n-1)\beta)$

$$= \frac{\sin\left(\alpha+\left(\frac{n-1}{2}\right)\beta\right)\sin\frac{n\beta}{2}}{\sin\frac{\beta}{2}}$$

~~Comes from~~ $e^{i\alpha}\left(1+e^{i\beta}+e^{2i\beta}+\cdots+e^{(n-1)i\beta}\right) = e^{i\alpha}\left(\frac{1-e^{ni\beta}}{1-e^{i\beta}}\right)$

(no: use $2\sin a\cos B = \sin(a+B) - \sin(a-B)$ multiply series by $\sin\frac{\beta}{2}$

(8) ~~Darboux~~ Thm: A bounded function f on $[a,b]$ is integrable iff $\forall\, \epsilon>0\ \exists$ a partition P s.t. $U(P,f) - L(P,f) < \epsilon$.

(9) (a) If f is cont. on $[a,b]$
(b) If f is monotonous on $[a,b]$
(c) If $\exists$ finite discont. pt. of f
(d) If set of disc. points have finite limit points

} in each case f is integrable on $[a,b]$.

(10) Riemann sum $= S(P,f) = \sum_{r=1}^{n} f(\xi_r)\delta_r$ $\quad \xi_r$ is arb. point in r^{th} subinterval.

(11) 2^{nd} def^n of Riemann integral

f is Riemann int. if $\lim_{\|P\|\to 0} S(P,f)$ exists

& $\lim_{\|P\|\to 0} S(P,f) = \int_a^b f(x)\,dx$.

Corollary: If f is int. on $[a,b]$ then

$$\int_a^b f(x)\,dx = \lim_{n\to\infty}\sum_{r=1}^{n} h\, f(a+rh) \quad \text{where } h = \frac{b-a}{n}$$

$\underset{n\to\infty}{Lt} \frac{1}{n} f\left(\frac{r}{n}\right)$.

(12) In examples, you will have $\underset{n\to\infty}{}$ & then we get
Write $\frac{r}{n}$ as x & $\frac{1}{n}$ as dx
& $\underset{n\to\infty}{Lt}$ as $\int_0^1$

$$\underset{n\to\infty}{Lt} \frac{1}{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)\,dx$$

(13) Corollary 2 → Consider partition given by
$$\{a, ah, ah^2, \dots, ah^n = b\}$$
$$\therefore h = \left(\frac{b}{a}\right)^{1/n} \quad \& \quad \int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{r=1}^{n} a(h^r - ah^{r-1}) \cdot f(ah^r)$$

(14) Ex. $f(x) = \frac{1}{2^n}$ when $\frac{1}{2^{n+1}} < x \leq \frac{1}{2^n}$

S.t. f is int. on $[0,1]$ despite inf. pt. of dis & $\int_0^1 f(x) = 2/3$.

→ Discont. is at $1/2^n$ points & limit point is 0.
$\therefore$ f. is bdd., and finite limit pt. in set of disc. pts.
$\therefore$ integrable.

$$\& \int_0^1 f(x) = \int_{1/2}^1 f + \int_{1/4}^{1/2} f + \int_{1/8}^{1/4} + \dots$$

$$= \left(1 - \tfrac{1}{2}\right) + \tfrac{1}{2}\left(\tfrac{1}{2} - \tfrac{1}{2^2}\right) + \dots = 2/3 \text{ after some work.}$$

(15) Just remember that $\sum_{n=1}^{\infty} \frac{1}{n^2} = \pi^2/6$

(16) If $|f| \in R[a,b]$, f need not ⇒ dirichlet $\begin{cases} -1 & \text{for } \mathbb{Q} \\ 1 & \text{for } \mathbb{R}-\mathbb{Q} \text{ \& so on} \end{cases}$
(in all these tricky ex., use dirichlet)

(17) If $f, g \in R[a,b] \Rightarrow \alpha f + \beta g \in R[a,b]$
& $f^2 \in R[a,b]$

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21 b
nent

(18) If $|f(x)| \le k \Rightarrow \left|\int_a^b f(x)dx\right| \le (b-a)k$

(19) Integral function or indefinite integral $\phi(t) = \int_a^t f(x)dx$ $t \in [a,b]$.

Integral function is continuous on $[a,b]$

First fundamental theorem of integral calculus.
If $f \in R[a,b]$ & f is cont. at $c \in [a,b]$ then
$\phi(t)$ is derivable at c & $\phi'(c) = f(c)$

(20) Primitive ~~of~~ or antiderivative of f :
If $f \in R[a,b]$ & $\exists$ ϕ s.t. $\phi'(x) = f(x)$ $\forall x$ $[a,b]$ then
ϕ is primitive/antiderivative.

Of course, primitive is not unique.

(21 a) Cont. of function is not a necessary condition for existance of primitive.

21b on next page

e.g. $\phi(x) = \begin{cases} x^2 \sin(1/x) & x \neq 0 \\ 0 & x = 0 \end{cases}$

Generally we see functions where derive is also cont. This is good ex. of function with discont. derivative

Then $\phi'(x) = \begin{cases} 2x\sin(\frac{1}{x}) - \cos(\frac{1}{x}) & ; x \neq 0 \\ 0 & x = 0 \end{cases}$

& we know $\phi'(x)$ is not cont. at 0.

(22) Fundamental theorem of integral calculus.
If $f \in R[a,b]$ & ϕ be its antiderivative; then
$\int_a^b f(x) = \phi(b) - \phi(a)$

Proof :- Uses lagrange mean value thm.
Consider partion $P = \{a, x_1, x_2, \dots, x_{r-1}, x_r, \dots, x_n = b\}$
In interval $[x_{r-1}, x_r]$; $\phi'(\xi_r) = \dfrac{\phi(x_r) - \phi(x_{r-1})}{x_r - x_{r-1}}$

$\therefore \sum_{r=1}^{n} \left[\phi(x_r) - \phi(x_{r-1}) \right] = \sum_{r=1}^{n} \phi'(\xi_r)\, \delta_r$

& so on.

(23) Mean value theorem of integral calculus.

Let $f, g \in R[a,b]$

If (i) $g(x) \geq 0$ or $g(x) \leq 0 \quad \forall\, x \in [a,b]$

Then $\exists\, \mu \in \text{Range}(f)$ s.t.

$$\int_a^b f(x)\, g(x)\, dx = \mu \int_a^b g(x)\, dx$$

(24) Bonnet's Mean Value Theorem

Let $g \in R[a,b]$ & f be $\downarrow$ & ≥ 0 then

$$\int_a^b f(x)\, g(x) = f(a) \int_a^{\xi} g(x)\, dx \qquad \xi \in [a,b].$$

Let $g \in R[a,b]$ & f be $\uparrow$ & ≥ 0 then

$$\int_a^b f(x)\, g(x)\, dx = f(b) \int_{\xi}^{b} g(x)\, dx.$$

$\rightarrow$ e.g. S.T. $\left| \int_a^b \frac{\sin x}{x} \right| \leq \frac{2}{a}$ Use above theorem

(25) 2^{nd} Mean Value Theorem

Let $g \in R[a,b]$ & f be bounded & monotonic, then

$$\int_a^b f(x)\, g(x)\, dx = f(a) \int_a^{\xi} g + f(b) \int_{\xi}^{b} g$$

Remember that in all these properties involving $f(x)g(x)$, we always use biggest guy of the monotone & cut short integral of remaining guy.

(26)

(21 b)

(26) e.g. Find $\lim_{n\to\infty} \left\{\frac{n!}{n^n}\right\}^{1/n}$

$\longrightarrow$ Let $A = \lim_{n\to\infty} \left\{\frac{n!}{n^n}\right\}^{1/n}$ $\therefore \log A = \lim_{n\to\infty} \frac{1}{n}\log\left(\frac{1}{n}\cdot\frac{2}{n}\cdots\cdots\frac{n}{n}\right)$

$= \lim_{n\to\infty} \sum \frac{1}{n}\log\left(\frac{r}{n}\right) = \int_0^1 \log x\,dx$ &so on.

(21b) P.T. $f(x)$ admits a primitive $F(x)$ but $\int_{-1}^{1} f(x)\,dx$ doesn't exist.

$$f(x) = \begin{cases} x\sin\frac{1}{x^2} - \frac{1}{x}\cos\frac{1}{x^2} & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \end{cases}$$

$$F(x) = \begin{cases} \frac{1}{2}x^2\sin\frac{1}{x^2} & x\neq 0 \\ 0 & x = 0 \end{cases}$$

$\longrightarrow$ We can find $F'(x)$ for $x\neq 0$ using chain rule & $F'(0)$ using limit. They give $f(x)$.

But $\int_{-1}^{1} f(x)\,dx$ doesn't exist since $f(x)$ is not bounded on $[-1,1]$. (integral requires bounded function)

If possible let $g(x) = \frac{1}{x}\cos\frac{1}{x^2}$ & let $|g(x)| < M$ $\forall x$

By archimedes property $\exists N$ s.t. $N > M$.

Also, consider $M_2 > N^2 > 1$ $M_2 \in \mathbb{N}$.

let $x_0 = \frac{1}{\sqrt{2\pi M_2}}$ $\therefore g(x_0) = \sqrt{2\pi M_2}\cos 2\pi M_2 = \sqrt{2\pi M_2} > N$

$\therefore$ Contradiction. $\therefore g(x)$ is unbounded

$\therefore f(x) = x\sin\frac{1}{x^2} - g(x)$ is also unbounded.

since $x\sin\frac{1}{x^2}$ is bounded.

Remember $F(x) = x^2\sin\frac{1}{x^2}$ $x\neq 0$ & 0 when $x = 0$;

This is a good function to produce when asked about function admitting primitive but not ~~pri~~ integral over $[-1,1]$.

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Sequences & Series of function
Uniform Convergence

(1) Sequence of function f_n converges pointwise if $\forall x$ the seq. $\{f_1(x), f_2(x), \ldots, f_n(x) \ldots\}$ converges.
The limiting value for each x gives $\lim_{n\to\infty} f_n(x) = f(x)$

Pointwise convergence is simple to test & solve examples.

(2) Pointwise cgn. is not strong enough to transmit properties such as cont., diff or int. to limiting function.

e.g. $f_n(x) = \frac{\sin nx}{\sqrt{n}}$ $\lim f_n(x) = 0 = f(x)$ $\therefore f'(x) = 0$

but $f_n'(x) = \sqrt{n}\cos nx$ $\therefore \lim f_n'(x) = \infty$

$\therefore$ diff. not transmitted.

Uniform convergence solves this.

(3) Uniform cgn. of seq. of functions.
$\{f_n\}$ uni. cgt. if $\forall \epsilon > 0 \ \exists \ m \in \mathbb{N}$ s.t.
$$|f_k(x) - f(x)| < \epsilon \quad \forall k \geq m \ \& \ \forall x$$

Uniform $\Rightarrow$ pointwise convergence.

(4) Uniformly bounded seq. of function: $|f_n(x)| \leq k \ \forall n \ \& \ \forall x$.

(5) M_n Test for uniform convergence.
Let $M_n = \text{Sup}\{|f_n(x) - f(x)| \ / \ \forall x \in I\}$
Uniform convergence iff $\lim_{n\to\infty} M_n = 0$

So, if M_n doesn't tend to 0, no uniform convergence.
Imp. test to prove given seq. not uniformly convergent.

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Series of real valued functions

(6) Pointwise convergence : If partial sum $\sum_{r=1}^{n} f_r(x)$ converges pointwise.

(7) Cauchy's Criterion for uniform convergence of series of functions
$\forall \epsilon > 0 ; \exists m \in \mathbb{N}$ s.t. $|S_{n+p}(x) - S_n(x)| < \epsilon \quad \forall n \geq m, p > 0$
(not useful in exam).

(8) Useful Weierstrass's M-Test.

If there exist cgt. $\sum_{n=1}^{\infty} M_n$ s.t. $|f_n(x)| \leq M_n \quad \forall n \,\&\, x \in I$
& $M_n \geq 0$.

Then uniform cgt. series.

(9) e.g. $0 < r < 1$ s.t. $\sum r^n \cos nx$ is uni. cgt.
$\rightarrow f_n(x) = r^n \cos nx \quad |f_n(x)| \leq r^n = M_n$ & $\sum M_n$ cgs. ∴ uniform.

e.g. Test U.C. of $\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$

→ We can't use much tricks here. Need to find maxima of $f_n(x)$ by making $f_n'(x) = 0$ & $f_n''(x) \lessgtr 0$

(10) If $\{f_n\}$ is uniformly cgt. & f_n cont. ⇒ $f(x)$ cont.
If $\sum f_n$ — // ——— & f_n cont. ⇒ $f(x)$ cont.

Note that this is sufficient cond^n but uniform cgt. is not necessary cond^n for f to be cont.
This gives useful test for uniform cgs. If $f(x)$ of cont. $f_n(x)$ is discont. ; then we can claim it is not uniform cgt.

e.g. $f_n(x) = x^n \quad 0 \leq x \leq 1$ ∴ $\lim f_n = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1 \end{cases}$

∴ limit is discont. ∴ not uniformly cgt.

(11) If $f_n(x) \in R[a,b]$ & unif. cgt. $\Rightarrow$ $f(x)$ is integrable

If $f_n(x) \in R[a,b]$ & unif. cgt. $\Rightarrow$ $\sum f_n(x)$ is integrable

$$\& \quad \sum \int_a^b f_n(x)\,dx = \int_a^b \sum f_n(x)\,dx$$

i.e. series is term by term integrable.

$\therefore$ If $\sum \int_a^b f_n(x)\,dx \neq \int_a^b \sum f_n(x)\,dx$ $\Rightarrow$ no uniform convergence.

(12) e.g. S.T. the series $1-x+x^2-x^3+\dots \quad 0 \le x \le 1$ allows term by term int. though it is not uni. cgt.

$\longrightarrow$ $0 \le x < 1 \Rightarrow 1-x+x^2-x^3+\dots = \dfrac{1}{1+x}$

& for $x=1 \Rightarrow 1-1+1-1$ $\therefore$ series oscillates.

$\therefore$ Obv. no uniform convergence.

But now, $\displaystyle\int_0^1 \frac{1}{(1+x)} = \log(1+x)$ & term by term int. gives

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots = \log(1+x)$$

$\therefore$ Term by term allowed.

(13) Uniform cgs & diff.

Here we don't ask for uni. cgs. of ~~fn~~ f_n or $\sum f_n$ to f but we need uni. cgs of f_n' & $\sum f_n'$.

Thm: If $\{f_n\}$ is s.t. (i) f_n converges to f

(ii) f_n' is continuous

(iii) f_n' ~~is~~ conv. uniformly to g

Then $f' = g$.

Remember that in these questions, we need to check uniform convergence of $\Sigma f_n'$ & not Σf_n. Then only we can say $\Sigma f_n' = g$

Thm:- If Σf_n is s.t. (i) Σf_n converges to f
(ii) f_n' is continuous
(iii) $\Sigma f_n'$ converges uniformly to g.

Then $f' = g$.

(14) e.g. S.T. $f_n(x) = \dfrac{nx}{1+n^2x^2}$ $\quad 0 \le x \le 1$

can't be differentiated term by term at $x=0$.

$\rightarrow \lim f_n(x) = 0 \quad \therefore \quad f'(0) = 0$

& $f_n'(0) = \lim\limits_{h \to 0} \dfrac{\frac{nh}{1+n^2h^2} - 0}{h} = n \quad \therefore \quad f_n'(0) \to \infty$ as $n \to \infty$ $\therefore$ proved.

(15) Note that integrating term by term or differentiating term by term is not restricted to series of functions. We can say this for sequence of functions also. In this case we check if $\dfrac{d}{dx}\left(\lim\limits_{n\to\infty} f_n(x)\right) = \lim\limits_{n\to\infty}\left(\dfrac{d}{dx} f_n(x)\right)$

or $\displaystyle\int \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \int f_n(x)$

Be careful that sometimes in exam they might ask to check term by term integration of a series for which $f_n = \ldots$ It might means they are asking above seq. convergence thing.

Improper Integral

Improper integral of first kind $\rightarrow$ limits of int. $\infty \rightarrow \int_1^\infty \frac{dx}{x^2}$ etc

(1) Improper integral of second kind $\rightarrow$ $f(x)$ is unbounded on $[a,b]$ $\int_a^b f(x)\,dx \rightarrow \int_0^1 \frac{1}{x^2}\,dx$.

Improper integral of third kind $\rightarrow \int_0^\infty e^x\,dx$ both limit & $f(x)$ unbdd

(2) If $f(x)$ becomes ~~def~~ infinite at $x=b$; we define

$$\int_a^b f(x) = \underset{\epsilon\to 0}{Lt} \int_a^{b-\epsilon} f(x)\,dx \qquad 0<\epsilon<b-a.$$

Improper int. cgt. if Lt. exists.

$$\& \int_a^\infty f(x)\,dx = \underset{t\to\infty}{lt} \int_a^t f(x)\,dx$$

(3) e.g. $\int_1^\infty \frac{\tan^{-1}x}{x^2}\,dx = \underset{t\to\infty}{Lt} \int_1^t \frac{\tan^{-1}x}{x^2}\,dx$ Put $x=\tan\theta$ $\therefore dx = \sec^2\theta\,d\theta$

$\therefore \int \frac{\tan^{-1}x}{x^2}\,dx = \int \frac{\theta}{\tan^2\theta}\sec^2\theta\,d\theta = \int \theta\,\mathrm{cosec}^2\theta\,d\theta = -\theta\cot\theta + \log\sin\theta$

& so on.

So don't rush for comparison test once you see $\int_1^\infty$, sometimes basic def^n works.

Similarly $\int_0^{1/e} \frac{dx}{x(\log x)^2}$ can be easily solved with $\lim_{\epsilon\to 0}\int_\epsilon^{1/e} \frac{dx}{x(\log x)^2}\,dx$

4

5

6

7

(4) Comparison Test - I
$f(x) \le g(x) \ \forall x$ if $\int_a^b g(x)$ cgs. $\Rightarrow \int_a^b f(x)$ cgs.
$\int_a^b f(x)$ dgs. $\Rightarrow \int_a^b g(x)$ dgs.

(5) Comparison Test II
If a is pt. of ∞ discontinuity, & $\lim_{x\to a} \frac{f(x)}{g(x)} = l$ where l is finite & non zero;
then f & g converge diverge together.

(6) If a is ∞ disc. & $\lim_{x\to a} (x-a)^n f(x)$ exists & is non-zero finite,
then $\int_a^b f(x)$ converges iff $n<1$. (comes from $\int_a^b \frac{dx}{(x-a)^n}$ convergence)

(7) Some good examples using above rule.
(a) Check $\int_1^2 \frac{\sqrt{x}}{\log x}$

$\rightarrow$ at $x=1$; consider $\lim_{x\to 1} \frac{(x-1)^n\sqrt{x}}{\log x} = \lim \frac{n(x-1)^{n-1}\sqrt{x} + (x-1)^n \frac{1}{2\sqrt{x}}}{1/x} = 1$ if $n=1$

$\therefore$ diverges (note we needed nonzero finite lim. as per (6) rule)

(b) Check $\int_0^1 \frac{(x^p + x^{-p})\log(1+x)}{x} dx$

$\rightarrow$ case (i) $p>0$; let $g(x) = 1/x^p$
$\therefore \text{Lt} \frac{f(x)}{g(x)} = \text{Lt} \frac{(x^{2p}+1)\log(1+x)}{x} = \text{Lt} \frac{(x^{2p}+1)}{1} \cdot \text{Lt} \frac{\log(1+x)}{x}$
$=1$. Now $\int_0^1 g(x)$ cgs for $p<1$. $\therefore 0<p<1$ cgs

case (ii) $p=0$ gives $\text{Lt}_{x\to 0} f(x)$ exists. $\therefore$ integrable.

case (iii) $p<0$. consider $g(x) = \frac{1}{x^{-p}} \Rightarrow \text{Lt} \frac{f(x)}{g(x)} = \frac{\left(1+\frac{1}{x^{2p}}\right)\log(1+x)}{x}$

$= \text{Lt}\left(1+\frac{1}{x^{2p}}\right) \text{Lt} \frac{\log(1+x)}{x} = (1)(1) \ (\because p<0)$.

$g(x)$ cgs for $p>-1$ $\therefore$ cgs for $-1<p<1$.

in questions that involve $(x-a)^p$ already in question; try to use $(x-a)^p$ to multiply when $p>0$ & $(x-a)^{-p}$ when $p<0$ & so on.

Also check $\int_0^1 \left(\log \frac{1}{x}\right)^n dx$ question on page 11.

(8) Absolute convergent $\int_a^b |f(x)| dx$.

Absolute convergent $\Rightarrow$ convergent.

(9) For $\int_a^\infty f(x)$; similar comparison test I & II as earlier.

(10) $\int_a^\infty \frac{dx}{x^n}$ $(a>0)$ converges iff $n>1$.

Very useful in most problems.

(11) e.g. $\int_1^\infty x^n e^{-x} dx$ of ~~cou~~ course converges since $\frac{1}{e^x}$ rapidly goes to 0.

$\therefore \frac{1}{e^x} < \frac{1}{x^m}$ $\forall m$ & so on. using comp. I test.

(12) Abel Test

If $\int_a^\infty f(x)dx$ converges & $g(x)$ is bdd. monotone then $\int_a^\infty f(x)g(x)dx$ converges at ∞.

Dirichlet's test

If $\int_a^t f(x)dx$ is bounded $\forall t \geq a$ & $g(x)$ is bounded & monotone tending to 0 as $x \to \infty$ then $\int^\infty f(x)g(x)dx$ converges.

(13) Dirichlet is very useful in proving conv. of type $\int_1^{\infty} \frac{\sin x^m}{x^n}$

→ here $\int_1^t \sin x^m$ is bounded & other term becomes monotone & so on.
(need to put $x^m = t$ & so on)

Beta & Gamma Function

① $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

obv. $B(m,n) = B(n,m)$

② Properties of Beta function

ⓐ If m, n are integers

$$B(m,n) = \frac{(m-1)!\,(n-1)!}{(m+n-1)!}$$

ⓑ $\frac{B(p,q+1)}{q} = \frac{B(p+1,q)}{p} = \frac{B(p,q)}{p+q}$

easy, comes from int. by parts on basic def^n of Beta function.

③ In all examples involving 'find B(m,n) representing given integral,' we just have to find appropriate $z = f(x)$ s.t. given integral converts in $\int_0^1 z^{m-1}(1-z)^{n-1}$.

ⓐ $\int_0^1 \frac{x^2}{\sqrt{1-x^5}}\,dx$ obv. $z = x^5$

ⓑ $\int_0^p x^m (p^q - x^q)^n\,dx$ obv. $z = \left(\frac{x}{p}\right)^q$

ⓒ $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}\,dx$ you will realize $z = \frac{(a+b)x}{(a+bx)}$

④ Always remember that B(m,n) has $x^{(m-1)}$ & $(1-x)^{(n-1)}$.
Don't make mistake of going for x^m or $(1-x)^n$.

5) Gamma Function

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx \quad \forall n > 0$$

(easy to remember ; Beta is simple $\int_0^1$ & Gamma is bigger $\int_0^\infty$.
Obv. since Γ goes upto ∞, we need e^{-x} to make integration possible & for all power of x or $1-x$ we need $m-1$ or $n-1$. Here we have only n so we have x^{n-1}).

6) Properties of gamma function.

(a) $\Gamma(n) = (n-1)\Gamma(n-1)$

(b) $\Gamma(n) = (n-1)!$ $\quad n \in \mathbb{N}$

(c) $B(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$

7) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$ (Very useful in many questions, especially Laplace in ODE).

$\longrightarrow$ use $B\left(\frac{1}{2}, \frac{1}{2}\right) = \dfrac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2})}{\Gamma(1)}$ & in $B\left(\frac{1}{2}, \frac{1}{2}\right)$ use $x = \sin^2\theta$

Cor. $\displaystyle\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$ (put $x^2 = z$)

8) Some things to remember in gamma related questions.

(a) Express given integral first in terms of Beta function & then use 6 (c) property.

(b) In questions with $\sin\theta, \cos\theta$; putting $\sin^2\theta = z$ works

(c) Final answer should generally be in the form of $\Gamma(n)$ where $0 < n < 1$.
So $\Gamma\left(\frac{4}{3}\right)$ should be further simplified to $\frac{1}{3}\Gamma\left(\frac{1}{3}\right)$.

(9) e.g. S.T. $\int_0^\infty \frac{x^a}{a^x}dx = \frac{\Gamma(a+1)}{(\log a)^{a+1}}$ $(a>1)$

$\longrightarrow$ naturally put $a^x = e^z$ i.e. $z = x\log a$ & so on.

(10) Useful Duplication formula

$$\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin(\pi n)} \quad (\text{obv. } 0<n<1)$$

This gives $\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \sqrt{2}\,\pi$ etc. results, could help in tricky problems.

(1)

(2)

(3)

Functions of Several Variables

(1) Limit of a function :→ Let $f: X \to \mathbb{R}$, $X \subseteq \mathbb{R}^n$?

$\underset{x \to a}{Lt} f(x) = l$ if $\forall \epsilon > 0, \exists \delta > 0$ s.t. $|f(x) - l| < \epsilon$

$\forall x$ s.t. $0 < \|x - a\| < \delta$.

(2) Limits of 2 variable functions

- (a) Repeated limits

(a1) $\underset{x \to x_0}{Lt} \; \underset{y \to y_0}{Lt} f(x, y)$

If $\underset{y \to y_0}{Lt} f(x, y) = g(x)$ then above limit is $\underset{x \to x_0}{Lt} g(x)$.

(a2) $\underset{y \to y_0}{Lt} \; \underset{x \to x_0}{Lt} f(x, y)$

If $\underset{x \to x_0}{Lt} f(x, y) = h(y)$ then above limit is $\underset{y \to y_0}{Lt} h(y)$.

- (b) Simultaneous or double limit

$\underset{(x,y) \to (x_0, y_0)}{Lt} f(x, y) = l$ if $\forall \epsilon > 0 \; \exists \delta > 0$ s.t.

$|f(x, y) - l| < \epsilon \;\; \forall x$ s.t. $0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta$.

(3) Now, relation betn existance of repeated & simultaneous limit is not straight forward.

(a) Double limit may exist but repeated may not exist.
Repeated limit —"— but double may not exist.

(b) But if both exists ; repeated limit = double limit
If repeated limits are not equal , double limit won't exist.

(4) e.g. $\lim\limits_{(x,y) \to (0,0)} y \sin\left(\frac{1}{x}\right) = 0$ but $\lim\limits_{x \to 0} y' \sin\left(\frac{1}{x}\right)$ doesn't exist for $y' \neq 0$.

$\therefore$ Repeated limit won't exist.

e.g. $f(x,y) = y + x\sin(1/y)$ if $y \neq 0$
$\qquad\qquad = 0$ if $y = 0$.

Simultaneous limit exist & $\underset{y\to 0}{Lt}\ \underset{x\to 0}{Lt}$ exists but $\underset{x\to 0}{Lt}\ \underset{y\to 0}{Lt}$ won't exist.

e.g. $f(x,y) = \dfrac{y-x}{y+x} \cdot \dfrac{1+x}{1+y}$

$\underset{x\to 0;\, y\to 0}{Lt} f(x,y) = -1$ & $\underset{y\to 0}{Lt}\ \underset{x\to 0}{Lt} f(x,y) = 1$

∴ Repeated limits exist but are unequal. Hence simultaneous limit won't exist.

(5) In proving existance of simultaneous limits in questions involving $y\sin(1/x)$ etc; useful to use basic ϵ, δ definition.
e.g. $f(x,y) = x\sin(1/y) + y\sin(1/x)$ use $\delta = \epsilon/2$ gives lt at (0,0)

(6) If $f(x,y)$ is continuous at (a,b) then $f(x,b)$ is cont. at $x=a$ & $f(a,y)$ is cont. $y=b$.

But converse not true. Obv.

(7) In homogeneous functions asking limit at origin; use $r\cos\theta, r\sin\theta$ & go for $r\to 0$. Easy job.

(8) Partial Derivatives

$\dfrac{\partial f}{\partial x} = \lim\limits_{h\to 0} \dfrac{f(x+h,y) - f(x,y)}{h}$ Similarly $\dfrac{\partial f}{\partial y}$

Existance of partial derivatives at a point need not imply continuity at that point. & cont. doesn't imply P.D. exist.

e.g. $f(x,y) = \dfrac{x^3+y^3}{x-y}$ $x \neq y$ & 0 if $x = y$.
S.t. f is discontinuous at origin even though P.D. exist.
→
Use $y - x = mx^3$ (brilliant trick. Remember. Might not be able to construct in exam

(9) Sufficient condition for continuity : at (a, b)

→ Both P.D. exist at (a, b) & one P.D. is bounded in a nbd. of (a, b).

later we see suff. condⁿ for differentiability as

→ Both P.D. exist at (a, b) & one P.D. is continuous at (a, b)

∴ If P.D. exist & f is not diff. ⇒ P.D. can't be cont.

Differentiation in multiple variables

(1) Def^n:-
$f(a+h, b+k) - f(a,b) = Ah + Bk + h\phi(h,k) + k\psi(h,k)$

or $f(a+h, b+k) - f(a,b) = Ah + Bk + \sqrt{h^2+k^2}\,\phi(h,k)$

s.t. A & B are constants ind. of h, k but dep. on (a, b)
ϕ, ψ are functions tending to 0 as $(h,k) \to (0,0)$.

(2) e.g. s.t. $f(x,y) = x/y$ is diff. in its domain of def^n;

$\rightarrow f(a+h, b+k) - f(a,b) = \frac{a+h}{b+k} - \frac{a}{b} = \frac{-ak}{b(b+k)} + \frac{h}{(b+k)}$

$= \frac{-ak}{b^2}\left[1 - \frac{k}{b+k}\right] + \frac{h}{b}\left[1 - \frac{k}{b+k}\right]$

(nice trick).

& we get $A = 1/b$ $B = \frac{-a}{b^2}$ $\phi(h,k) = \frac{-k}{b(b+k)}$ & $\psi = \frac{ak}{b^2(b+k)}$

(3) e.g. S.t. $f(x,y) = |x| + |y|$ is not diff. at (0,0).
$\longrightarrow$ let $h=0$ & $k>0 \Rightarrow |k| = Bk + k\psi(0,k) \Rightarrow B = 1$
$h=0$ & $k<0 \Rightarrow B = -1$ $\therefore \#$.

(4) e.g. S.t. $f(x,y) = \begin{cases} xy/\sqrt{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$ is not diff. at origin

$\longrightarrow f(0+h, 0+k) - f(0,0) = 0.h + 0.k + \sqrt{h^2+k^2}\left(\frac{hk}{h^2+k^2}\right)$

we want $\frac{hk}{h^2+k^2} \to 0$ as $(h,k) \to (0,0)$. $\#$ as limit doesn't exist

(5) Discont. $\Rightarrow$ not diff.
$\therefore$ In questions asking to show given f^n is not diff.; see if it is dis cont. Much ea...

(6) If f is diff. ⇒ partial derivatives exist.
In defⁿ of diff; $A = f_x(a,b)$ & $B = f_y(a,b)$

(7) As seen earlier:
Both P.D. exist + one P.D. cont. ⇒ function is diff. at (a,b).

(8) A $f(x,y)$ is called continuously differentiable if both partial derivatives are continuous at (a,b).

(9) Partial Derivatives of higher order

$$f_{xx}(a,b) = \underset{h\to 0}{Lt} \frac{f_x(a+h,b) - f_x(a,b)}{h}$$

$$f_{xy}(a,b) = \underset{h\to 0}{Lt} \frac{f_y(a+h,b) - f_y(a,b)}{h}$$ & so on.

Existance of partial derivative of a particular order doesn't imply existance of other P.D. of same order.
i.e. f_{xy} exist $\not\Rightarrow$ f_{yx} will exist etc.

(10) Thm:- If both f_{xy} & f_{yx} are cont. ⇒ $f_{xy} = f_{yx}$

(11) Schwartz thm
If f_{xy} is cont. at (a,b) & f_x exists in a nbd. of (a,b) then f_{yx} exists & $f_{xy} = f_{yx}$.
This is sufficient but not necessary.

(12) Young's Thm
If f_x & f_y both are differentiable at (a,b) then
$f_{xy} = f_{yx}$ at (a,b)

(13) Summary of these imp. theorem

	Condition at (a,b)	Result at (a,b)
1	Both f_x & f_y exists + one P.D. bounded	f Continuous
2.	Both f_x & f_y exist + one P.D. cont.	f differentiable
3.	Young's thm f_x & f_y are differentiable	$f_{xy} = f_{yx}$
4.	f_{xy} & f_{yx} are continuous	$f_{xy} = f_{yx}$
5.	Schwartz thm f_{xy} cont. + f_x defined in nbd of (a,b)	$f_{xy} = f_{yx}$.

(14) S.T. $f(x,y) = \frac{x^2y^2}{x^2+y^2}$ $(x,y) \neq (0,0)$

0 $(x,y) = (0,0)$

S.T. $f_{xy}(0,0) = f_{yx}(0,0)$ even though Young thm condⁿ is unsatisfied.

→ If possible let f_x be diff.

$\therefore f_x(h,k) - f_x(0,0) = h f_{xx}(0,0) + k f_{yx}(0,0) + h\phi(h,k) + k\psi(h,k)$

we get $f_{xx} = f_{yx} = 0$ with some work.

& $f_x(h,k) = \frac{2hk^4}{(h^2+k^2)^2}$

$\therefore \frac{2hk^4}{(h^2+k^2)^2} = h\phi(h,k) + k\psi(h,k)$ this can be shown ~~#~~ using $r\cos\theta, r\sin\theta$.

we need $2\cos\theta\sin^4\theta$ ...

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Derivatives of Higher order

(1) Let $z = f(x, y)$

we know $dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$

In general $d^n z = \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy\right)^n z$

(useful in Lagrange multiplier questions if you want to provide COMPLETE solution)

(2) Derivative of composite functions : Chain Rule

Let $x = \phi(t)$ & $y = \psi(t)$ & $z = f(x, y)$ then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

ex. $z = xy f(y/x)$ find $\frac{f'(y/x)}{f(y/x)}$

→ if z is constant.

then we take

$\log z = \log x + \log y + \log f(y/x)$

& take derivative w.r.t. x

& so on. Much easier

(3) Derivative of implicit function.

If $f(x, y) = 0$

$\therefore f_x + f_y \frac{dy}{dx} = 0 \quad \therefore \frac{dy}{dx} = \frac{-f_x}{f_y}$

e.g. Find $\frac{\partial u}{\partial x}$ if $u = \sin(x^2 + y^2)$ & x, y satisfy $a^2x^2 + b^2y^2 = c^2$

→ $2a^2x + 2b^2yy' = 0 \quad \therefore y' = \frac{-a^2x}{b^2y}$ & $\frac{\partial u}{\partial x} = \cos(x^2+y^2)(2x + 2yy')$ & so on.

(4) Euler thm on homogeneous functions

→ $z(x, y)$ → homo of degree $n \Rightarrow x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz$

(straightfwd. assume $z = x^2 + y^2$ if ~~...~~ you can't remember formula.)

e.g. $u = \cot^{-1}\left(\frac{x+y}{\sqrt{x} + \sqrt{y}}\right)$ S.T. $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{4}\sin 2u = 0$.

→ $\cot u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$ homo.

Comes from basic chain rule also. So no worries if don't remember.

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Lagrange Multipliers

(1) Following ex. explains concept.
Find largest & smallest values of $f(x,y) = x+2y$ on circle $x^2+y^2=1$.

→ Auxilary function using lagrange multiplier becomes.
$F(x,y,\lambda) = (x+2y) + \lambda(x^2+y^2-1)$
Solve $\frac{\partial F}{\partial x} = 0$; $\frac{\partial F}{\partial y} = 0$ & $\frac{\partial F}{\partial \lambda} = 0$ together for answer.

(2) Technical process

(a) Write auxillary function F.

(b) Find stationary point $dF = 0$

i.e. $\frac{\partial F_1}{\partial x_1} dx_1 + \frac{\partial F_2}{\partial x_2} dx_2 + \cdots + \frac{\partial F_1}{\partial \lambda_1} d\lambda_1 + \cdots = 0$

i.e. $\frac{\partial F_1}{\partial x_1} = \frac{\partial F_2}{\partial x_2} = \cdots = \frac{\partial F_1}{\partial \lambda_1} = \cdots = 0$

(c) Then find d^2F & check its sign. $d^2F > 0 \Rightarrow$ minima
& $d^2F < 0 \Rightarrow$ maxima.

This last step can get complex & hence is optional.

(3) In question with max. $f(x,y,z)$ & one condition $g(x,y,z)=0$;
→ Finding extrema easier with lagrange multipliers.
But for showing it is maxima or minima; we can use technique of finding max. of $F(x,y)$ by $F_x = 0$ & $F_y = 0$
& $F_{xx}F_{yy} - F_{xy}^2 > 0$ & (then $F_{xx} < 0 \Rightarrow$ maxima, $F_{xx} > 0$, minima)

We find $f_x(x,y,z)$ & then use $g(x,y,z)=0$ condition to eliminate z.
& so on

Jacobian

(1) Jacobian of $u_1, u_2 \ldots, u_n$ w.r.t. $x_1, x_2, \ldots, x_n$ is

$$\frac{\partial(u_1, u_2 \ldots, u_n)}{\partial(x_1, x_2, \ldots, x_n)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \vdots & & & \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

(2) Properties - (Though 2 variable used, they are relevant everywhere)

Let $J = \frac{\partial(u,v)}{\partial(x,y)}$ & $J' = \frac{\partial(x,y)}{\partial(u,v)}$ $\Rightarrow JJ' = 1$

(3) Chain Rule

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \cdot \frac{\partial(r,s)}{\partial(x,y)}$$

(again applies to more than 2 variable)

(4) Particular case of Jacobian

Let $u_1 = f_1(x_1)$

$u_2 = f_2(x_1, x_2)$

$u_3 = f_3(x_1, x_2, x_3)$

$\vdots$

$u_n = f_n(x_1, x_2, \ldots, x_n)$

Then $$\frac{\partial(u_1, u_2 \ldots, u_n)}{\partial(x_1, x_2 \ldots x_n)} = \frac{\partial u_1}{\partial x_1} \cdot \frac{\partial u_2}{\partial x_2} \cdot \frac{\partial u_3}{\partial x_3} \cdots \frac{\partial u_n}{\partial x_n}$$

(5) Jacobian of implicit functions

If $y_1, y_2 \ldots, y_n$ & $x_1, x_2, \ldots, x_n$ are implicitely connected by n functions $f_1, f_2 \ldots f_n$ as

$f_1(x_1, x_2 \ldots x_n, y_1, y_2 \ldots y_n) = 0$

$\vdots$

$f_n(x_1, x_2, \ldots x_n, y_1, y_2 \ldots y_n) = 0$

Note we need
n variable in x
n variable in y
& n functions f_n
implicitely connecting

then

$$\frac{\partial(f_1, f_2 \ldots f_n)}{\partial(y_1, y_2 \ldots y_n)} \cdot \frac{\partial(y_1, y_2 \ldots y_n)}{\partial(x_1, x_2 \ldots x_n)} = (-1)^n \frac{\partial(f_1, f_2 \ldots f_n)}{\partial(x_1, x_2 \ldots x_n)}$$

(remember $(-1)^n$ as it is imp'; rest is intuitive chain rule)

(6) example. Roots of this eq. in λ are u, v, w.

$$(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0.$$

S.T. $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{-2(y-z)(z-x)(x-y)}{(v-w)(w-u)(u-v)}$$

$\longrightarrow$ eq. is $3\lambda^3 - 3\lambda^2(x+y+z) + 3\lambda(x^2+y^2+z^2) - (x^3+y^3+z^3) = 0$

Since u, v, w are roots;

simple but remember $\longrightarrow$

$u+v+w = x+y+z \quad \therefore f_1 = u+v+w-x-y-z = 0$

$uv+vw+wz = x^2+y^2+z^2 \quad f_2 = uv+vw+wz-x^2-y^2-z^2 = 0.$

$uvw = \dfrac{x^3+y^3+z^3}{3} \quad f_3 = uvw - \dfrac{(x^3+y^3+z^3)}{3} = 0$

& then use

$$\frac{\partial(f_1, f_2, f_3)}{\partial(u,v,w)} \cdot \frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\partial(f_1 f_2 f_3)}{\partial(x,y,z)}$$ & so on.

(7) e.g. If $u = \dfrac{x}{(1-r^2)^{1/2}} \quad v = \dfrac{y}{(1-r^2)^{1/2}} \quad w = \dfrac{z}{(1-r^2)^{1/2}}$

S.T. $$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{1}{(1-r^2)^{5/2}}$$

$\longrightarrow$ & $u = x/\sqrt{1-r^2} \quad \therefore x^2 = u^2(1-x^2-y^2-z^2)$

$\therefore f_1 = x^2 - u^2(1-x^2-y^2-z^2) = 0$

$f_2 = y^2 - v^2(1-x^2-y^2-z^2) = 0$ & so on.

$f_3 = z^2 - w^2(1-x^2-y^2-z^2) = 0$

You could use basic def^n of jacobian also but this becomes much less complex.

(8) Particular case of Jacobian of implicit functions

Let $f_1(x_1, x_2 \dots x_n, y_1) = 0$

$f_2(x_2, x_3 \dots, y_1, y_2) = 0$

$f_3(x_3, x_4 \dots y_1, y_2, y_3) = 0$

$\vdots$

$f_n(x_n, y_1, y_2 \dots y_n) = 0$

$$\therefore \frac{\partial(y_1, y_2 \dots y_n)}{\partial(x_1, x_2 \dots x_n)} = (-1)^n \frac{\frac{\partial f_1}{\partial x_1} \cdot \frac{\partial f_2}{\partial x_2} \cdots \frac{\partial f_n}{\partial x_n}}{\frac{\partial f_1}{\partial y_1} \cdot \frac{\partial f_2}{\partial y_2} \cdots \frac{\partial f_n}{\partial y_n}}$$

(note that x_i is in numerator)

(9) Necessary & sufficient condition that n functions $y_1, y_2 \dots y_n$ of ind. variables $x_1, x_2 \dots x_n$ are functionally related

$f(y_1, y_2 \dots y_n) = 0$

iff $\frac{\partial(y_1, y_2 \dots y_n)}{\partial(x_1, x_2 \dots x_n)} = 0$

i.e. Jacobian $= 0 \Leftrightarrow y_1, y_2 \dots y_n$ are not independent. You can find a relation between them

e.g. $u = x + y - z$; $v = x - y + z$; $w = x^2 + y^2 + z^2 - 2yz$.

S.T. they are not ind. & find a relation betn them.

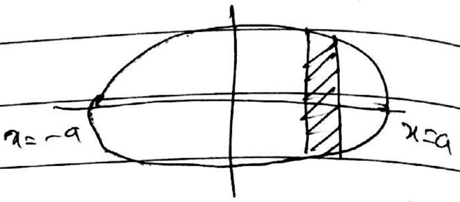
$\rightarrow \frac{\partial(u, v, w)}{\partial(x, y, z)}$ turns out to be 0. Relation comes with some manipulations.

Multiple Integral

(1) As definite integral can be interpreted as area, double integral can be interpreted as volume. $\iint f(x,y)\,dx\,dy$

(2) Essentially we divide given area into convenient strips.

e.g. Evaluate $\iint (x+y)^2\,dx\,dy$ over ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



$$\rightarrow I = \int_{-a}^{a}\int_{-b\sqrt{1-\frac{x^2}{a^2}}}^{b\sqrt{1-\frac{x^2}{a^2}}} (x+y)^2\,dx\,dy$$

$$= 4b\int_0^a \left\{x^2\sqrt{1-\frac{x^2}{a^2}} + \frac{b^2}{3}\left(1-\frac{x^2}{a^2}\right)^{3/2}\right\}dx$$

Putting $x = a\sin\theta$ & so on.

(3) $\int_0^1 dx \int_0^1 \frac{x-y}{(x+y)^3}dy \neq \int_0^1 dy \int_0^1 \frac{x-y}{(x+y)^3}dx$ $\Big\} \rightarrow$ which variable you choose first changes things.

(But most of the questions have ~~good~~ well behaving functions, so don't think much on this)

(4) Drawing proper figure & finding intersection points between 2 given curve is very important for correct solution.

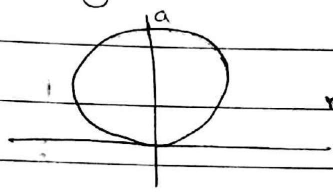
(5) Volume as double integral.
$z = f(x,y)$ forms top of the surface & limits of integration are taken on projection of surface on X-Y plane.

(6) Find volume bounded by cylinder $x^2+y^2=4$ & planes $y+z=4$ & $z=0$.

→ Drawing good figure should help in understanding as well as marks.
Given integral has to be taken over circle $x^2+y^2=4$ & $z=4-y$ $\therefore V = \int_0^{2\pi}\int_0^2 (4-r\sin\theta)\, r\,dr\,d\theta$ & so on.

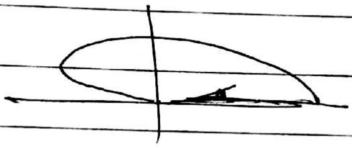
7) Double Integral in Polar Co-ordinates.

→ Here also drawing proper figure matters.
$r = a\sin\theta$ →



note that limits on integral for θ would be 0 to π & not 0 to 2π as π to 2π repeats same curve.

$r = a(1+\cos\theta)$
Cardioid.

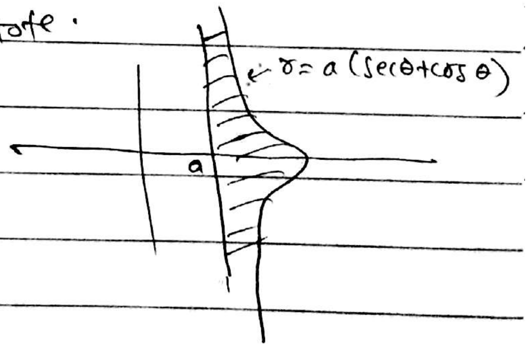


8) Area enclosed betn 2 curves : polar coordinates.

→ Area $= \iint r\,dr\,d\theta$ limits according to enclosed area.

e.g. Calculate area included between curve $r = a(\sec\theta+\cos\theta)$ & its asymptotes.
→ Notice that as $\theta \to \frac{\pi}{2}$; $r \simeq a\sec\theta$. And $r = a\sec\theta$ is a straight line $x=a$. This is our asymptote.

$\therefore$ Area becomes $= 2\int_0^{\pi/2}\int_{a\sec\theta}^{a(\sec\theta+\cos\theta)} r\,dr\,d\theta$



turns out to be $\frac{5\pi a^2}{4}$.

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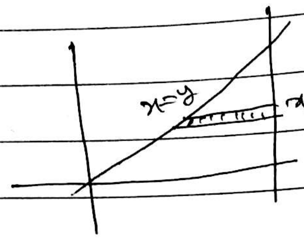
9. Change of order of integration (essentially interchange horizontal & vertical strips)

e.g. Change order of integration & solve.

$$\int_0^a \int_0^x \frac{\phi'(y)\, dx\, dy}{\sqrt{(a-x)(x-y)}}$$

⟶ 

new limits obv. become

$$\int_0^a \int_y^a (\quad)\, dy\, dx.$$

Then use brilliant substitution $x = a\cos^2\theta + y\sin^2\theta$.

This reduces integral to $\int_0^a \int_0^{\pi/2} 2\phi'(y)\, dy\, d\theta$ & so on.

10. Solve $\displaystyle\int_0^{\pi/2}\int_0^{\pi/2} \sin x \sin^{-1}(\sin x \sin y)\, dx\, dy$.

⟶ Nice concepts involved in this sol^n. Clarifies concepts.

Let $\sin x \sin y = \sin\theta$

$\sin x \cos y\, dy = \cos\theta\, d\theta$ — Since initially we are keeping x constant in double integral, hence obv. only replace y by θ.

New limits ⟹ $y = 0 \Rightarrow \sin\theta = 0 \quad \therefore \theta = 0$

$y = \pi/2 \Rightarrow \sin\theta = \sin x \quad \therefore \theta = x$ $\quad \therefore$ limits $\int_0^x$

$$\therefore I = \int_0^{\pi/2}\int_0^x \frac{\theta\cos\theta}{\cos y}\, dx\, d\theta = \int_0^{\pi/2}\int_0^x \frac{\theta \cos\theta \sin x\, dx\, d\theta}{\sqrt{\cos^2\theta - \cos^2 x}}$$

Note that we have $\cos x$ in denom. & $\sin x$ above, hence makes sense to change order of integration.

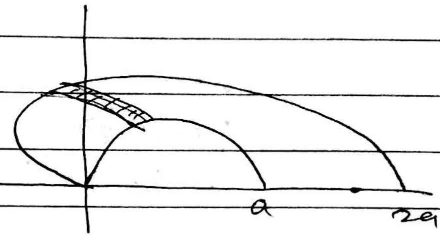
$$\therefore I = \int_0^{\pi/2}\int_\theta^{\pi/2} \frac{\theta\cos\theta\sin x}{\sqrt{\cos^2\theta - \cos^2 x}}\, d\theta\, dx \text{ \& so on.}$$

(simple int. using $\int \frac{1}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a}$)

(11) Changing order of integration in polar coordinates

e.g. Change order in $\int_{0}^{\pi/2}\int_{a\cos\theta}^{a(1+\cos\theta)} f(r,\theta)\,r\,dr\,d\theta + \int_{\pi/2}^{\pi}\int_{a\cos\theta}^{a(1+\cos\theta)} f(r,\theta)\,r\,d\theta\,dr$

⟶ The figure becomes as shown.

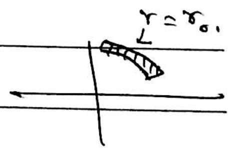


Now, when r is in range 0 to a;
θ ~~#~~ changes from $\cos^{-1}\left(\frac{r}{a}\right)$ to $\cos^{-1}\left(\frac{r-a}{a}\right)$

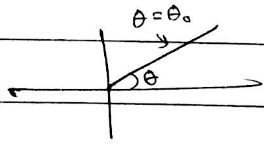
Now when r is from a to 2a; range of θ becomes 0 to $\cos^{-1}\left(\frac{r-a}{a}\right)$.

$$\therefore I = \int_{0}^{a}\int_{\cos^{-1}(r/a)}^{\cos^{-1}\left(\frac{r-a}{a}\right)} f(\,) + \int_{a}^{2a}\int_{0}^{\cos^{-1}\left(\frac{r-a}{a}\right)} f(\,).$$

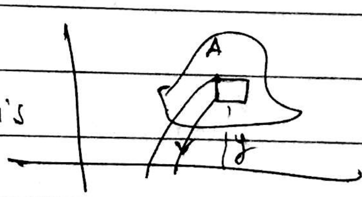
(12) In polar questions, note that when r is constant & θ changes, we are considering an arc of circle $r = r_0$



When θ_0 is constant; we are seeing r as it changes from 0 onwards.



(13) Volume of Solid of revolution



Vol^m formed by revolving area A about x-axis

$$= \iint_A 2\pi y\,dx\,dy = \iint_A 2\pi\, r\sin\theta \cdot r\,dr\,d\theta$$

Vol^m formed by revolving about Y-axis

$$= \iint_A 2\pi x\,dx\,dy$$

(14) Change of variables.

We use jacobian to find new dV or dS.

Note that we use $|J|$ not just J. modulus is taken.

Changing rectangular (x,y,z) to cylindrical (ρ,ϕ,z)

$$\frac{\partial(x,y,z)}{\partial(\rho,\phi,z)} = \rho \quad \therefore \iiint f(x,y,z)\,dV = \iiint f(\rho\cos\theta,\rho\sin\theta,z)\,\rho\,d\rho\,d\theta\,dz$$

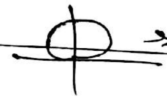
Changing rectangular to polar coordinates (r,θ,ϕ).

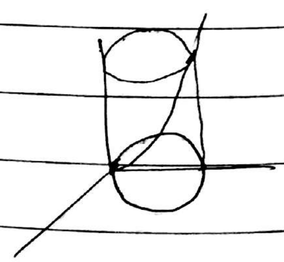
$$J = \frac{\partial(\cancel{r,\theta,\phi})\,x,y,z}{\partial(r,\theta,\phi)} = r^2\sin\theta \quad \therefore \iiint dV = \iiint f(\ )\,r^2\sin\theta\,dr\,d\theta\,d\phi.$$

(15) If you have questions of finding vol^m bound by a conicoid & cylinder; it's simple.

Cylinder base is circle over which integration has to be done. For other conicoid, express $z=f(x,y)$ & then integrate $\iint_C f(x,y)\,dx\,dy$ where C is base of cylinder.

(1). Find vol^m bounded by $x^2+y^2=az$ & cylinder $x^2+y^2=2ay$ & plane $z=0$.

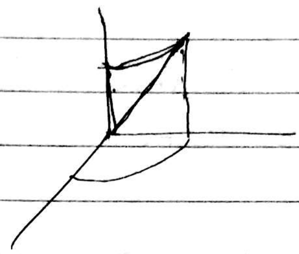
→ Cylinder base is  $\Rightarrow r = 2a\sin\theta$.



& conicoid has $z = \dfrac{x^2+y^2}{a}$

$$\therefore V = \int_0^{\pi}\int_0^{2a\sin\theta} z\cdot r\,dr\,d\theta = \int_0^{\pi}\int_0^{2a\sin\theta} \frac{r^3}{a}\,dr\,d\theta = \frac{3\pi a^3}{2}$$

(16) $$\int_0^1 \int_0^{\sqrt{1-x^2}} \int_{\sqrt{x^2+y^2}}^{1} \frac{dx\,dy\,dz}{\sqrt{x^2+y^2+z^2}}$$



$\longrightarrow$ The figure becomes a cone in positive quadrant.

$\therefore$ limits become in polar coordinates

$$\int_0^{\pi/2} \int_0^{\pi/4} \int_0^{\sec\theta} \frac{1}{r} \cdot r^2 \sin\theta \, dr\, d\theta\, d\phi = \frac{(\sqrt{2}-1)\pi}{4}$$

(17) Area of a curved surface

let $F(x,y) = z$ $\therefore$ directional derivative is given by gradient & it turns out to be $\frac{-\partial z}{\partial x}, \frac{-\partial z}{\partial y}, 1$.

$\therefore$ $\delta S = \frac{dx \cdot dy}{\cos\gamma}$ now $\cos\gamma = \frac{1}{\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}}$

$\therefore$ $\delta S = \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy$

$\therefore$ Area of surface $= \iint_A \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\, dx\, dy$

(A is projection of XY plane)

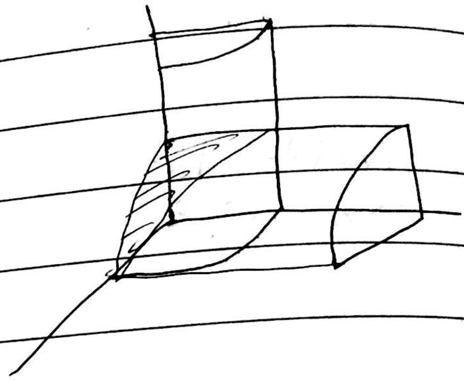
$\left\{\right.$ Remember length of curve formula for $(x,y) = (x(t), y(t))$ way $t \in [a,b]$

$$\int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\, dt \left.\right\}$$

(18) See examples on ~~page~~ last page of multiple integrals notes.

(19) Find area of $x^2+z^2=4$ lying inside $x^2+y^2=4$

→ In all these area examples, we split required area into 4 or 8 symmetric areas.



Here we have 8 areas.

For surface; $x^2+z^2=4$;

$$\frac{\partial z}{\partial x} = \frac{-x}{z} \; ; \; \frac{\partial z}{\partial y} = 0$$

$$\therefore \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2} = \sqrt{\frac{4}{4-x^2}}$$

$$\therefore \text{Area} = 8\int_0^2\int_0^{\sqrt{4-x^2}} \frac{2}{\sqrt{4-x^2}}\,dx\,dy = 16\times 2 = 32.$$

In these questions, need to get projection area right & what is surface of function above this projection.

Taylor's Theorem for multiple variables.

(1) N^{th} taylor function of $f(x,y)$

$$T_n(x,y) = \sum_{i,j=0}^{i+j\le n} \frac{1}{i!\,j!} \left[\frac{\partial^{i+j} f(x_0,y_0)}{\partial x^i \partial y^j}\right] (x-x_0)^i (y-y_0)^j$$

(2) Taylor's theorem

$\exists$ point (c_1, c_2) such that

$$f(x,y) = T_n(x,y) + R_{n+1}$$

where $$T_n = \sum_{i,j=0}^{i+j\le n} \frac{1}{i!\,j!} \left(\frac{\partial^{i+j} f(x_0,y_0)}{\partial x^i \partial y^j}\right) (x-x_0)^i (y-y_0)^j$$

$$R_{n+1} = \sum_{i+j=n+1} \frac{1}{i!\,j!} \left(\frac{\partial^{i+j} f(c_1,c_2)}{\partial x^i \partial y^j}\right) (x-x_0)^i (y-y_0)^j$$

(note here we want only n+1 ~~degree~~ order terms).

(3) e.g. find taylor polynomial $T_3(x,y)$ for function $\sin(x+y)$ at (0.

→ First find all partial deri. upto order 3.

f_x, f_y

f_{xx}, f_{yy}, f_{xy}

$f_{xxx}, f_{xxy}, f_{xyx}, f_{yyy}$ etc. & then put in formula.

Remaining Real Contd. After Lin. Algebra Notes.

Real Analysis continued

Extreme values.

(1) Stationary point $\Rightarrow f_x = f_y = 0$

If function has P.D. & f_x or $f_y \neq 0 \Rightarrow$ no extremum.

But there can be extremum at points where f_x, f_y don't exist.

$|x| + |y|$ has minima at $(0,0)$.

(2) Sufficient cond^n for $f(x,y)$ extremum at (a,b).

$f_x = f_y = 0$

& $(f_{xx}f_{yy} - f_{xy}^2) > 0$

maxima $\Leftrightarrow f_{xx}$ (or f_{yy}) negative } generally both

minima $\Leftrightarrow f_{xx}$ (or f_{yy}) positive } have same sign in questions.

Note (1) $f_{xx}f_{yy} - f_{xy}^2 = 0 \Rightarrow$ further investigation needed.

Note (2) $f_{xx}f_{yy} - f_{xy}^2 < 0 \Rightarrow$ no extrema.

(3) Test for higher variable functions.

Let f be function of $(x_1, x_2, x_3, \ldots)$

& let $d_{ij} = \dfrac{\partial^2 f}{\partial x_i \partial x_j}$ & let all first order P.D. be 0 at $(a_1, a_2 \ldots a_n)$.

let $D_1 = \begin{vmatrix} d_{11} \end{vmatrix}$ $\quad D_2 = \begin{vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{vmatrix}$ $\quad D_3 = \begin{vmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{vmatrix} \ldots$

Then $(a_1, a_2, a_3 \ldots)$ has

(i) minima if $D_1, D_2 \ldots D_n$ are all +ve.

(ii) maxima if $D_1, D_2 \ldots D_n$ are alternatively ~~+ve~~ -ve & +ve

- This will help in remembering conditions for $f(x,y)$ extrema al

CURVATURE

① Radius of curvature $\rho = \dfrac{(1+y'^2)^{3/2}}{y''}$

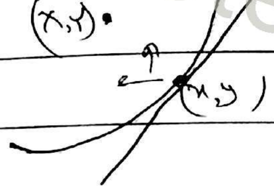
Curvature $= 1/\rho$

Radius of curvature in polar $\rho = \dfrac{(r^2+r'^2)^{3/2}}{r^2+2r'^2-rr''}$

$r' = \dfrac{dr}{d\theta} \quad r'' = \dfrac{d^2r}{d\theta^2}$

If you can't remember, ~~go~~ find y', y'' from parametric form & use cartesian formula.

② Centre of circle of curvature at (x,y)



$X = x - \dfrac{(1+y'^2)y'}{y''}$

$Y = y + \dfrac{(1+y'^2)}{y''}$

Picture above figure; obv. x needs subtraction & y needs addition.

Now, the part to be added or subtracted is similar to rad. of curv. formula. We just don't go for 3/2 in numerator & need y' for X.

So $-\dfrac{(1+y'^2)}{y''}y'$ in X & $\dfrac{(1+y'^2)}{y''}$ in Y.

Locus of such centres is called 'evolute' of the curve.

Original curve for whom we are finding evolute is called 'involute'.

Singular Points

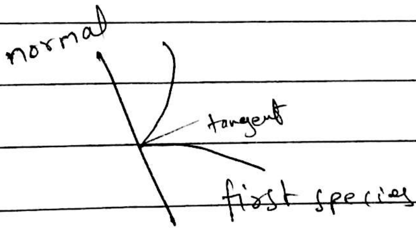
(1) Double point → Point through which 2 tangents pass.
→ Cusp = both tangents coincide
node = 2 different tangents. (node in electrical means ✓)
conjugate = imaginary tangent.

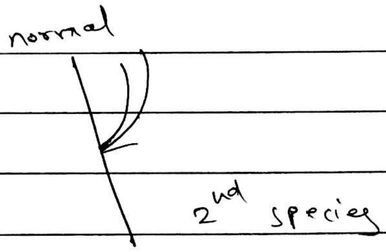
(2) Tangent at origin
→ It is given by equating to 0 the lowest degree terms in the equation of the curve.

(3) Obtaining double points of a curve $f(x,y)=0$.
→ Given by solving $f_x=0$ & $f_y=0$.
If $f_{xx}f_{yy}-f_{xy}^2=0$ → cusp
If $f_{xx}f_{yy}-f_{xy}^2<0$ → node

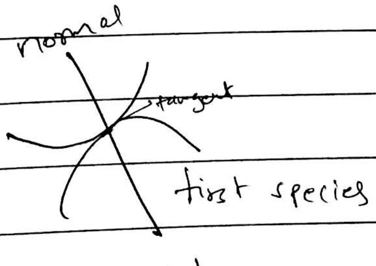
(4) A cusp has single tangent at double point. So we can see 2 curves going from cusp. Based on relative position of these 2 curves to normal; we say if it is single cusp or double cusp.

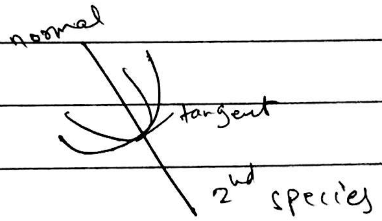
Single cusp



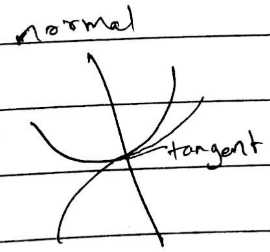


Double cusp





Point of oscu-inflexion

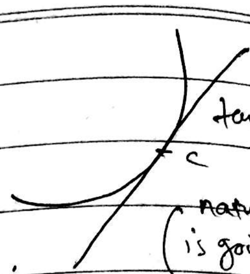


So, on same side of normal is single cusp & on both sides in double cusp.

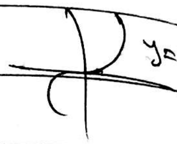
Species is given by whether 2 curves are on same side of tangent or not, different sides → 1st species; same side → 2nd species

(5) Concavity

Concave upward → curve lies above tangent ⇒ $f''(c) > 0$.



Concave downward → curve below tangent ⇒ $f''(c) < 0$



Point of inflexion → $f''(c) = 0$ & $f'''(c) \neq 0$
(f'' changes sign on either side of c.)

Asymptotes

① An asymptote of the curve is straight line which touches the curve at ∞.
(So the distance betn curve & line tends to 0 as x or $y \to \infty$)

② 2 types of asymptotes for curve $y = f(x)$ parallel to either axis.

ⓐ line $x = a$ is vertical asymptote if
$\lim_{x \to a^+} f = \pm\infty$ or $\lim_{x \to a^-} f = \pm\infty$ (left or right limit)

ⓑ line $y = b$ is horizontal asymptote if
$\lim_{x \to \infty} f = b$ or $\lim_{x \to -\infty} f = b$.

③ Working rule to find eq. of asymptotes

ⓘ Put $x = 1$ & $y = m$ in highest degree terms of equation & get $\phi_n(m)$. Get soln of $\phi_n(m) = 0 \to (m_1, m_2, \ldots, m_n)$.

ⓘⓘ We find $c_i = -\dfrac{\phi_{(n-1)}(m_i)}{\phi_n'(m_i)}$ for each m_i

& hence asymptotes are $y = m_i x + c_i$.

If $c = \frac{0}{0}$; we solve
$\frac{c^2}{2!}\phi_n''(m) + c\,\phi_{n-1}'(m) + \phi_{n-2}(m) = 0$
This is imp.
(similar to taylor)

Of course we have n asymptotes, real or imaginary.
If n is odd; we have at least 1 real asymptotes.

④ Now if y^n or x^n is absent in $\phi_n(x)$; we can find asymptote directly with some trick.
When you find ϕ_n, be especially careful if you are actually getting eq. in m with degree n; o/w you will miss an asymptote if y^n is missing.

(a) y^n is absent $\Rightarrow$ one asymptote with $m=\infty$.
just find coefficient of y^{n-1} & equate it to 0.

(b) x^n is absent $\Rightarrow$ one asymptote with $m=0$
just find coefficient of x^{n-1} & equate it to 0.

5 e.g. Find assymptotes of $x^3+2x^2y+xy^2-x^2-xy+2=0$.
$\rightarrow \phi_3(m) = m^2+2m+1 \Rightarrow$ roots $-1,-1$.

$C = \dfrac{-\phi_2(m)}{\phi_3'(m)} = 0/0$ $\therefore$ we solve $\dfrac{C^2}{2!}\phi_3''(m) + C\phi_2'(m)+\phi_1(m)=0$

$\therefore\; C^2 - C = 0 \Rightarrow C = 0, 1.$

$\therefore$ 2 asymptotes given by $y = -x+0$ & $y = -x+1$.

3rd asymptote obtained by equation coefficients of y^2 to 0 $\Rightarrow$ $x=0$

$\therefore$ 3 asymptotes ; $x=0$, $x+y=0$ & $x+y=1$.

e.g. Find asymptotes $\parallel$ to coordinate axes.
(i) $x^2y^2 - a^2(x^2+y^2) = 0$
$\longrightarrow$ Degree 4 but both x^4 & y^4 absent.
Coeff. of $x^2 \rightarrow$ $y^2 - a^2 = 0$ $\Rightarrow y = \pm a$.
Coeff. of $y^2 \rightarrow$ $x^2 - a^2 = 0$ $\Rightarrow x = \pm a$
4 asymptotes

6. Another method of finding asymptotes.

(a) → If possible we rearrange given equation as

$$ax+by+c+\frac{(F_{n-1})}{(P_{n-1})}=0$$

& then for asymptote, we go to ∞ by $ax+by=0$ direction i.e. asymptote becomes

$$ax+by+c+\lim_{y=-\frac{a}{b}x\to\infty}\left(\frac{f_{n-1}}{P_{n-1}}\right)=0$$

(b) Similarly we try to find a quadratic in $(ax+b)$; i.e.

$$(ax+by)^2+(ax+by)\lim_{y=-\frac{a}{b}x\to\infty}\frac{F_{n-2}}{P_{n-2}}+\lim_{y=-\frac{ax}{b}\to\infty}\frac{f_{n-2}}{P_{n-2}}=0$$

Then we get 2 asymptotes $ax+by=\alpha$ & $ax+by=\beta$ where α,β are roots of

$$t^2+t\lim_{\substack{y\to\infty\\=-\frac{ax}{b}}}\frac{F_{n-2}}{P_{n-2}}+\lim_{\substack{y\to\infty\\=-\frac{ax}{b}}}\frac{f_{n-2}}{P_{n-2}}=0$$

Before going for quadratic, we try to see if any $(a'x+b'y+c')$ can be taken out common first & use 6a rule.

7. e.g. Find asymptotes of $(x+y)^2(x+2y+2)=x+9y-2$

→ ∴ eq. $(x+2y+2)=\lim_{y=-\frac{x}{2}\to\infty}\frac{x+9y-2}{(x+y)^2}=\lim_{x\to\infty}\frac{-\frac{7x}{2}-2}{\left(\frac{x}{2}\right)^2}=0$

∴ 1 asymptote $x+2y+2=0$

then $(x+y)^2=\lim_{x=-y\to\infty}\frac{x+9y-2}{x+2y+2}=8$ ∴ asymptotes are

$x+y=2\sqrt{2}$

& $x+y=-2\sqrt{2}$.

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(8) Position of curve w.r.t. asymptote. (11)

→ Let curve be $y = mx + c + \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \dots$

Now if $A \neq 0$ & A and x have same sign; curve lies above asymptote.

if $A \neq 0$ & A & x have different sign, curve lies below

Naturally; $A = 0$ mean $B > 0 \to$ above & $B < 0 \to$ below.

Quite intuitive.

Just try to get eq. in the form $y = mx + c + \frac{A}{x} + \frac{B}{x^2} + \dots$ to find position of curve w.r.t. asymptote.

(9) Intersection of a curve with asymptotes.

If given curve is of type $F_n + F_{n-2} = 0$; we can see that asymptotes will be given by $F_n = 0$.

{ Since we take one degree out & $(ax+by+c) \times \lim \frac{F_{n-2}}{F_{n-1}} = 0$ & this limit will always be 0 & so on }

(9b) Now each asymptote meets the curve in 2 coincident points at ∞.

∴ ~~if~~ n asymptotes total meet in $n(n-2)$ points the curve.

(10) e.g. Find cubic which has same asymptotes as the cubic $x^3 - 6x^2y + 11xy^2 - 6y^3 + x + y + 1 = 0$ & touches y-axis at origin & (3,2) satisfies

→ This is $F_3 + F_1 = 0$

∴ asymptote is $F_3 = 0$.

∴ Req. eq. is of type $x^3 - 6x^2y + 11xy^2 - 6y^3 + ax + by + c = 0$

Since origin satisfies ⇒ $c = 0$. & as diff. gives $a + by' = 0$ at origin

∴ $b = \frac{-a}{y'}$ & y' is ∞ ∴ $b = 0$ & (3,2) putting gives

(11) Asymptote in polar coordinates

→ express eq. as $f(\theta) = \frac{1}{r}$ ($\because$ we want $\frac{1}{r} = 0$, i.e., at ∞)

Let α be root of $f(\theta) = 0$

Then asymptote is $r \sin(\theta - \alpha) = \frac{1}{f'(\alpha)}$

($r \sin(\theta - \alpha) = c$ is a line in polar)

Tracing of Curves

(1) While tracing a curve, checking following things can help

(a) Check if curve is symmetrical about X-axis or Y-axis based on whether it has even powers of x or y.

(b) If interchanging x & y keeps eq. same $\Rightarrow$ symmetrical about $x=y$

(c) If origin satisfies it; tangent can be found by equating lowest order term to 0.

(d) See what happen when $x \to \infty$ & $y \to \infty$. (asymptote essentially) ~~Also at which value~~

(e) Find impossible regions s.t. $y^2 = x$ $\therefore$ $x < 0$ impossible etc.

(f) See if y' is +ve or −ve always.
Also can see y'' for concave upward / downward stuff.

(2) Note that some of ~~the~~ above points need to be checked; not all.
Can check some ex. on page 5 of notes etc.

(3) If you are given parametric form $x = f_1(t)$ & $y = f_2(t)$, try to ~~fi~~ eliminate t & use above method.

(4) If you can't eliminate t; following things can help;
(i) If x is odd & y is even function of t, then curve is symmetrical about y-axis.

(ii) Find extrema of x & y so as to find strips parallel to axes within which the curve lies.

(iii) Find intersection with axes.

(iv) For series of values of θ, plot corresponding points; this gives some idea.

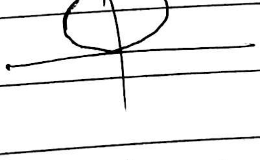
5) Tracing Polar Curves $r = f(\theta)$

(1) Find limits of r so that you see which circle have curve confined.
See if any θ makes $r = \infty$. It has asymptote in that θ direction.

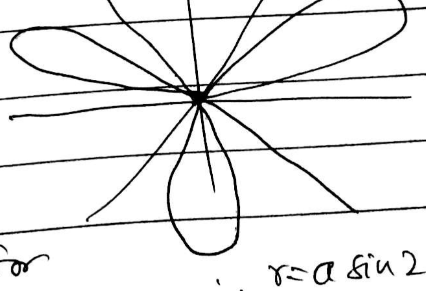
(2) If $f(\theta)$ contains only $\cos\theta$ terms; curve is symmetrical to x-axis. If only $\sin\theta$ terms are there; symmetric about Y-axis.

(3) If curve has only even powers of r, it is symmetric about the pole.

(4) One brilliant trick; if you have asked $f(n\theta)$ $(n = odd)$ & know curve for $f(\theta)$, then $f(n\theta)$ is compressed version of $f(\theta)$ in $\left(\frac{360}{n}\right)$ section repeated n times.

So $r = a\sin\theta =$ 

you get n leaves when n is odd.

then $r = a\sin 3\theta =$ 

360° curve compressed to 120° each repeated thrice.

you get $2n$ leaves for even n.

$\therefore r = a\sin 2\theta \rightarrow$ 

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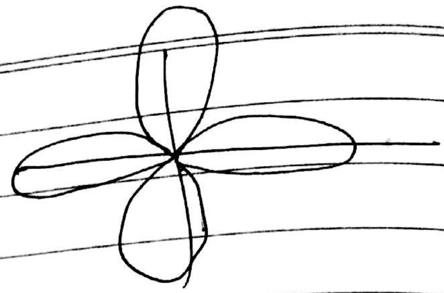
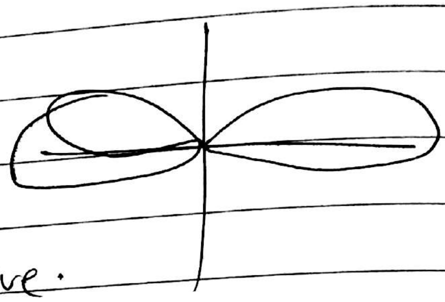
→ note $r^2 = a\cos 2\theta$ →



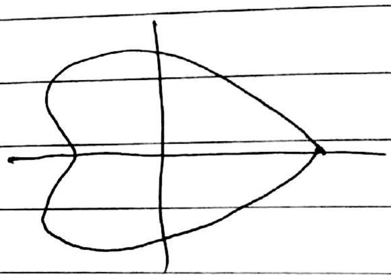
but $r^2 = a^2 \cos 2\theta$ →



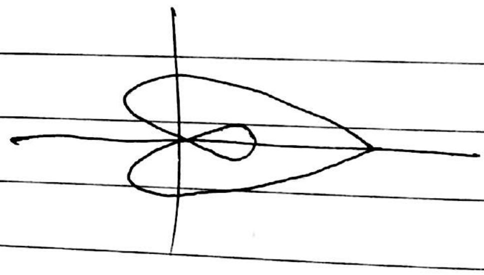
bcoz r^2 can't be negative.

→ $r = a + b\cos\theta$

case (i) $a > b$



Case (ii) $a < b$



Case (iii) $a = b$