⇒ Functions of Several Variables :

* Lt / continuity using spherical or rectangular nbd
* Infinite approaches possible , limit , if exists , is independent of all

1) Limit of a Function : Let $f: X \to \mathbb{R}$, $X \subset \mathbb{R}^n$

$\lim_{x \to a} f(x) = l$ if $\forall \; \epsilon > 0$, $\exists$ a $\delta > 0$ st $|f(x) - l| < \epsilon$ whenever $0 < ||x - a|| < \delta$

2) Limits of 2 variable function

a) Repeated Limits (rl)

$$rl_1 = \lim_{x \to x_0} \lim_{y \to y_0} f(x,y) \qquad \text{or } rl_2 = \lim_{y \to y_0} \lim_{x \to x_0} f(x,y)$$

$$\Downarrow \qquad\qquad \Downarrow$$

if $\lim_{y \to y_0} f(x,y) = g(x)$ if $\lim_{x \to x_0} f(x,y) = h(y)$

then $rl_1 = \lim_{x \to x_0} g(x)$ then $rl_2 = \lim_{y \to y_0} h(y)$

b) Simultaneous Limit (sl) | Double Limit

$$sl = \lim_{(x,y) \to (x_0, y_0)} f(x,y)$$

if $\forall \; \epsilon > 0$, $\exists$ a δ (dependent on ϵ) > 0.

st $|f(x,y) - sl| < \epsilon$ whenever

$$\sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta \quad \text{or} \quad |x - x_0| < \delta, |y - y_0| < \delta$$

Relation :

a) SL may exist but rl may not exist, but if they exist $rl_1 = rl_2 = sl$

b) rls may exist but sl may not exist

c) if $rl_1 \neq rl_2 \Rightarrow$ sl doesn't exist

Examples :

i) $f(x,y) = y \sin\left(\frac{1}{x}\right)$ $\quad \lim_{(x,y)\to(0,0)} f(x,y) = 0$

but $rl_1 = \lim_{x\to 0} y_1 \sin\left(\frac{1}{x}\right)$ doesn't exist

[NOTE : rl_1 for $y = y_0$, rl_2 for $x = x_0$ does exist if sl exists]

ii) $f(x,y) = \begin{cases} y + x\sin\left(\frac{1}{y}\right) & \text{if } y \neq 0 \\ 0 & \text{if } y = 0 \end{cases}$

$\lim_{(x,y)\to(0,0)} f(x,y) = sl$ and $\lim_{y\to 0}\lim_{x\to 0} f(x,y) = rl_2$ exist

but $rl_1 = \lim_{x\to 0}\lim_{y\to 0} f(x,y)$ doesn't

Tip :

i) Homogenous functions $\Rightarrow$ use $(r\cos\theta, r\sin\theta) \to \delta = \sqrt{\epsilon}$ form or $y = mx$ forms

ii) $x\sin\left(\frac{1}{y}\right)$ forms, use δ, ϵ directly using $|\sin\theta| \leq 1$ and $\delta = \frac{\epsilon}{2}$ or something similar

3) Algebra of Limits $\lim_{(x,y)\to(a,b)} f(x,y) = l$ & $\lim_{(x,y)\to(a,b)} g(x,y) = m$

i) $\lim f \pm g = l \pm m$

ii) $\lim f \cdot g = l \cdot m$

iii) $\lim (f/g) = l/m$ provided $m \neq 0$

4) Continuity : $f(x,y)$ is cont. at (a,b) if
$\forall \ \epsilon > 0$, $\exists$ a $\delta > 0$ st $|f(x,y) - f(a,b)| < \epsilon$
whenever $|x-a| < \delta$ & $|y-b| < \delta$

5) If $f(x,y)$ is continuous at (a,b) then
$f(x,b)$ is cont. at $x=a$ and
$f(a,y)$ is cont. at $y=b$

But converse is not true.

Tip: cont. of $f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & x^2+y^2 \neq 0 \\ 0 & o/w \end{cases}$

$$|f(x,y) - 0| = \left|\frac{xy}{\sqrt{x^2+y^2}}\right| = \left|\frac{xy}{x^2+y^2}\right| \left|\sqrt{x^2+y^2}\right|$$

$\Downarrow$

$\leq \frac{1}{2}$ $\Leftarrow$ ☆

[$\because$ AM $\geq$ GM]

$$\leq \frac{\sqrt{x^2+y^2}}{2}$$

$$\leq \sqrt{x^2+y^2} < \epsilon$$

Take $\delta = \epsilon$.

or

directly use $x = r\cos\theta$
$y = r\sin\theta$

6) Partial Derivatives : $f_x(a,b) = \lim_{h \to 0} \frac{f(a+h, b) - f(a,b)}{h}$

$$f_y(a,b) = \lim_{k \to 0} \frac{f(a, b+k) - f(a,b)}{k}$$

* Go for direct differentiation in case of exp. or trig. fns.

Relation: Existence of PD doesn't imply continuity at a point and continuity doesn't imply that PD exists

TRICK: eg $f(x,y) = \begin{cases} \frac{x^3+y^3}{x-y} & x \neq y \\ 0 & x = y \end{cases}$

PD exist at (0,0) but discontinuous.

Put $\boxed{y = x - mx^3}$

RELATION:

7) Sufficient condition for continuity:

i) for a point : one of the PD exists & is bounded in a nbd of (a,b) & other PD exists at (a,b)

ii) in a closed region: Both the PD exist & are bounded throughout the region.

$\Downarrow$

Sufficient condition for Differentiability:

One of the PD is continuous at (a,b) & other PD exists at (a,b)

↳ not necessary though, it can be diff even when none of PD is cont.

But Not diff → PDs cannot be continuous at the pt.

8) Differentiability

$$f(a+h, b+k) - f(a,b) = Ah + Bk + h\phi(h,k) + k\psi(h,k)$$

(OR)

$$f(a+h, b+k) - f(a,b) = Ah + Bk + \sqrt{h^2+k^2}\,\phi(h,k)$$

st (i) A & B are constants independent of h and k but depend on f & (a, b)

(ii) ϕ & ψ are f (h,k) which $\to 0$ as $(h,k) \to 0$

TRICKS: (i) $f(x,y) = \dfrac{x}{y}$

$$f(a+h, b+k) - f(a,b) = \frac{a+h}{b+k} - \frac{a}{b} = \frac{-ak}{b(b+k)} + \frac{h}{b+k}$$

$$= \frac{-ak}{b^2}\left[1 - \frac{k}{b+k}\right] + \frac{h}{b}\left[1 - \frac{k}{b+k}\right]$$

$$A = \frac{1}{b} \quad B = \frac{-a}{b^2} \quad \phi = \frac{-k}{b(b+k)} \quad \psi = \frac{ak}{b^2(b+k)}$$

(ii) $f(x,y) = |x| + |y|$ is not diff'ble at (0,0)

Put in defn to get $|h| + |k| = Ah + Bk + h\phi + k\psi$

for $h=0,\ k>0 \Rightarrow B = 1$
for $h=0,\ k<0 \Rightarrow B = -1$ $\Rightarrow$ contradiction

RELATIONS:

(i) NECESSARY COND: If f is diff. at (a,b) $\rightarrow$ f is cont. at (a,b)
But converse not true. eg $f = \dfrac{xy}{\sqrt{x^2+y^2}}$

TIP: Use discontinuity at (a,b) to show non-diff. nature

(ii) Diff at (a,b) $\Rightarrow$ Both PD exist at (a,b)
Infact A & B in defn are f_x & f_y
Approximate $f(a+h, b+k) - f(a,b) = hf_x(a,b) + kf_y(a,b)$

9) $f(x,y)$ is a real valued function, diff at (a,b) then linear fn. $T : \mathbb{R}^2 \to \mathbb{R}$ defd by

$$T(h,k) = h f_x(a,b) + k f_y(a,b)$$ is called

the differential of f at $(a,b) = df(a,b)$

RELATION: f is cont. & PD exists $\not\Rightarrow$ f is diff. $\Rightarrow$ converse of (ii) is not true [of prev page]

eg $$f(x,y) = \begin{cases} \frac{x^3-y^3}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

eg. $$f(x,y) = \begin{cases} x^2 \sin\left(\frac{1}{x}\right) + y^2 \sin\left(\frac{1}{y}\right) & xy \neq 0 \\ x^2 \sin\left(\frac{1}{x}\right) & x \neq 0,\ y=0 \\ y^2 \sin\left(\frac{1}{y}\right) & y \neq 0 \quad x=0 \\ 0 & x=y=0 \end{cases}$$

f is diff at $(0,0)$ but neither PDs are continuous

10) Continuously Differentiable:

Both PDs exist in the nbd of (a,b) & are cont. at (a,b)

Cont. diff'ble $\Rightarrow$ diff'ble

11) Partial Differentials of Higher Order $f_{xx}, f_{xy}, f_{yx}, f_{yy}$

f_{xy} → applied 1st $= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)$

PD of a particular order doesn't imply existence of PD of higher order

Also not necessary that $f_{xy} = f_{yx}$

12) Condition for equality of f_{xy} & f_{yx}

i) f_{xy} & f_{yx} are cont. at $(a,b) \Rightarrow f_{xy}(a,b) = f_{yx}(a,b)$

ii) SCHWARTZ THEOREM :

a) f_x exits in a certain nbd of (a,b)

b) f_{xy} is cont. at (a,b)

$\Rightarrow$ then f_{yx} exists at (a,b) & $f_{xy} = f_{yx}$

iii) YOUNG'S THEOREM :

Both f_x & f_y are diff. at $(a,b) \Rightarrow f_{xy}(a,b) = f_{yx}(a,b)$

Both Schwartz & Young Theorem are sufficient, not necessary

eg. $f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

Here, f_{xy} is not cont., neither f_x & f_y are diff.
↳ use defn method

SUMMARY:

CONDITION at (a,b)	RESULT AT (a,b)
a) f_x & f_y exist + one PD is bdd	f is cont.
b) f_x & f_y exist + one PD is cont.	f is diff.
c) f_{xy} & f_{yx} are cont.	$f_{xy} = f_{yx}$
d) f_x exists in nbd (a,b) + f_{xy} cont.	$f_{xy} = f_{yx}$
e) Both f_x & f_y diff	$f_{xy} = f_{yx}$

13) Differentials of Higher Order :

a) $z = f(x, y)$ where x, y are indept variables

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$d^2z = d(dz) = \frac{\partial^2 z}{\partial x^2} dx^2 + \frac{\partial^2 z}{\partial y^2} dy^2 + 2\frac{\partial^2 z}{\partial x \partial y} dx\,dy$$

$$= \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy \right)^2$$

b) In general ; $d^n z = \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy \right)^n$

c) If $z = f(x, y)$ & $x = g(u, v)$ & $y = h(u, v)$
where u & v are IVs

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$d^2z = d(dz) = \left(\frac{\partial}{\partial x} dx + \frac{\partial}{\partial y} dy \right)^2 + \frac{\partial z}{\partial x} d^2x + \frac{\partial z}{\partial y} d^2y$$

If g & h are linear, d^2x & $d^2y = 0$
$d^n z$ reduces to (b) case

d) CHAIN RULE : $z = f(x, y)$ $x = \phi(t)$ $y = \psi(t)$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \frac{dx}{dt} + \frac{\partial z}{\partial y} \frac{dy}{dt}$$

If ϕ, ψ are linear $\Rightarrow$ $\frac{d^n z}{dt^n} = \left(h \frac{\partial}{\partial x} + k \frac{\partial}{\partial y} \right)^n z$

$x = a + ht$
$y = b + kt$

e) $z = f(x,y) \quad x = \phi(u,v) \quad y = \psi(u,v)$

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

f) IMPLICIT FUNCTION: $F(x,y) = 0$

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

g) EULER THEOREM ON HOMOGENOUS FUNCTIONS:

i) $z = f(x,y)$ is homogenous of degree 'n'

$$\boxed{x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz}$$

:→ $z = x^n g\left(\frac{y}{x}\right)$

$$\frac{\partial z}{\partial x} = nx^{n-1} g\left(\frac{y}{x}\right) - yx^{n-2} g'\left(\frac{y}{x}\right)$$

$$\frac{\partial z}{\partial y} = x^{n-1} g'\left(\frac{y}{x}\right)$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nx^n g\left(\frac{y}{x}\right) - \cancel{yx^{n-1} g'\left(\frac{y}{x}\right)} + \cancel{yx^{n-1} g'\left(\frac{y}{x}\right)}$$

$$= nz$$

ii) $$\boxed{x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = n(n-1)z}$$

⇒ Extreme values of a function:

i) $f(x,y) - f(a,b)$ should keep the same sign for (a,b) to be the pt with extrema

ii) NECESSARY CONDITION: If f_x & f_y exist at a point which has the extremum, then $f_x = f_y = 0$ at the pt.

So $f_x \neq 0$ or $f_y \neq 0$ ⇒ no extrema

But it is not a suff. condⁿ: $f_x = f_y = 0 \not\Rightarrow$ extrema

iii) f can have extrema even if f_x & f_y do not exist

eg $f(x,y) = |x| + |y|$ at $(0,0)$

iv) Stationary Point: $f_x(a,b) = f_y(a,b) = 0$

v) SUFFICIENT CONDITION: $f_x(a,b) = f_y(a,b) = 0$

a) $(f_{xx} f_{yy} - f_{xy}^2)(a,b) > 0$

then $f(a,b)$ is max. if f_{xx} (or f_{yy}) < 0

$f(a,b)$ is min. if f_{xx} (or f_{yy}) > 0

b) Further investigation if $f_{xx} f_{yy} - f_{xy}^2 = 0$ [check sign of $f(x,y) - f(a,b)$ in nbd (a,b)]

c) Neither max, nor minima if $f_{xx} f_{yy} - f_{xy}^2 < 0$

Such points are called Saddle Points

VI) n-variable function

Necessary : All PDs $f_{x_1}, f_{x_2} \ldots f_{x_n}$ vanish, if they exist

Sufficient : a) $df(a,b,c) = f_x dx + f_y dy + f_z dz = 0$

so that $f_x = f_y = f_z = 0$

b) $d^2f(a,b,c) = f_{xx}(dx)^2 + f_{yy}(dy)^2 + f_{zz}(dz)^2$

$+ 2f_{xy}\,dx\,dy + 2f_{yz}dy\,dz + 2f_{xz}dx\,dz$

f_x, f_y, f_z terms are already 0.

- if d^2f keeps same sign, maxima if $d^2f < 0$, minima if $d^2f > 0$
- if d^2f doesn't keep same sign, neither max. nor min
- if d^2f keeps sign but vanishes at some pt, further check

Condition to keep same sign : Matrix = $\begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix} = M$

a) d^2f is always +ve if $|f_{xx}|, \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix}, |M|$ are all +ve
$\Downarrow$
minima

b) d^2f is always -ve if D_1, D_2, D_3 are -ve, +ve, -ve alternatively
$\Downarrow$
maxima

Generalised :

$d_{ij} = \dfrac{\partial^2 f}{\partial x_i \partial x_j}$ $\quad D_1 = |d_{11}| \quad D_2 = \begin{vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{vmatrix}, D_3 = \begin{vmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{vmatrix}$

a) minima if $D_1, D_2, D_3 \ldots$ are all +ve

b) maxima if D_1, D_2, D_3 are alternatively -ve & +ve

⇒ Lagrange Multipliers

To Locate stationary points when the variables are subject to some additional conditions.

eg. Maximise $z = f(x,y)$ subject to $g(x,y) = 0$

We could have, theoretically, eliminated y to get z in terms of x & then find the maxima

$$\therefore \quad \frac{dz}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0$$

$$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} = 0 \quad \text{—} (1)$$

From $g(x,y)=0$, at extrema we have

$$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y}\frac{dy}{dx} = 0 \quad \text{—} (2)$$

$$(1) + \lambda \times (2) \Rightarrow \left(\frac{\partial f}{\partial x} + \lambda\frac{\partial g}{\partial x}\right) + \left(\frac{\partial f}{\partial y} + \lambda\frac{\partial g}{\partial y}\right)\frac{dy}{dx} = 0$$

We choose λ st $\frac{\partial f}{\partial y} + \lambda\frac{\partial g}{\partial y} = 0$ — (3)

we get $\frac{\partial f}{\partial x} + \lambda\frac{\partial g}{\partial x} = 0$ — (4)

Also $g(x,y) = 0$ — (5)

Using (3), (4), (5), we can get x, y, λ

STEPS: i) Write the auxillary function F

ii) Find the stationary pts using $dF = 0$

$$dF = \frac{\partial F}{\partial x_1} dx_1 + \frac{\partial F}{\partial x_2} dx_2 + \cdots + \frac{\partial F}{\partial \lambda_n} d\lambda_n = 0$$

$\vdots$

iii) check maxima or minima using $d^2F \lessgtr 0$ [little difficult]

TIP: (i) For eqn with single constraint like $\max f(x,y,z)$ subject to $g(x,y,z) = 0$

Compute f_x & f_y by treating x & y independent

Put $\frac{\partial z}{\partial x}$ & $\frac{\partial z}{\partial y}$ by using $d(g(x,y,z)) = 0$
$\Downarrow$
$g_x = 0$ & $g_y = 0$

compute f_{xx}, f_{yy}, f_{xy} and infer as per sign of $[f_{xx} f_{yy} - (f_{xy})^2]$

(ii) eg Max of $xy + yz + zx$ occurs at stationary pt. if $[2lm + 2mn + 2nl - \Sigma l^2]^{-1} > 0$ given that $\Sigma lx = 1$

We have $F = f + \lambda\phi$ $\quad f = \Sigma xy \quad \phi = (\Sigma lx) - 1$

To show maxima compute $F_{xx}, F_{yy}, F_{zz}, F_{xy}, F_{yz}, F_{zx}, \phi_x, \phi_y, \phi_z$

maxima if $d^2F < 0$

or
$$\begin{vmatrix} F_{xx} & F_{xy} & F_{xz} & \phi_x \\ F_{yx} & F_{yy} & F_{yz} & \phi_y \\ F_{zx} & F_{zy} & F_{zz} & \phi_z \\ \phi_x & \phi_y & \phi_z & 0 \end{vmatrix} < 0$$

⇒ Jacobian

i) $J\left(\frac{u_1, u_2, \ldots, u_n}{x_1 x_2 x_3 \ldots x_n}\right) = \frac{\partial(u_1, u_2, \ldots, u_n)}{\partial(x_1, x_2, \ldots x_n)} = \begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots & \frac{\partial u_2}{\partial x_n} \\ \vdots & & & \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$

ii) $J = \frac{\partial(u, v)}{\partial(x, y)}$ $\quad J' = \frac{\partial(x, y)}{\partial(u, v)}$ $\quad JJ' = 1$

iii) CHAIN RULE: $\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(u, v)}{\partial(r, s)} \cdot \frac{\partial(r, s)}{\partial(x, y)}$

iv)
$$u_1 = f_1(x_1)$$
$$u_2 = f_2(x_1, x_2)$$
$$u_3 = f_3(x_1, x_2, x_3)$$
$$\vdots$$
$$u_n = f_n(x_1, x_2, \ldots, x_n)$$

$$J = \frac{\partial(u_1, u_2, \ldots, u_n)}{\partial(x_1, x_2, \ldots, x_n)} = \frac{\partial u_1}{\partial x_1} \frac{\partial u_2}{\partial x_2} \frac{\partial u_3}{\partial x_3} \cdots \frac{\partial u_{n-1}}{\partial x_{n-1}} \frac{\partial u_n}{\partial x_n}$$

v) IMPLICIT FUNCTION:

n variables $y_1, \ldots, y_n$

n variables $x_1, \ldots, x_n$

n equations $f_1, f_2, \ldots, f_n$

$$\frac{\partial(f_1, f_2, \ldots, f_n)}{\partial(y_1, y_2, \ldots, y_n)} \cdot \frac{\partial(y_1, y_2, \ldots, y_n)}{\partial(x_1, x_2, \ldots, x_n)} = (-1)^n \frac{\partial(f_1, f_2, \ldots f_n)}{\partial(x_1, x_2, \ldots, x_n)}$$

eg. $u = \frac{x}{(1-r^2)^{1/2}}$ $v = \frac{y}{(1-r^2)^{1/2}}$ $w = \frac{z}{(1-r^2)^{1/2}}$ $r^2 = \Sigma x^2$

TRICK

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \frac{1}{(1-r^2)^{5/2}}$$

use
$$f_1 = x^2 - u^2(1-\Sigma x^2) = 0$$
$$f_2 = y^2 - v^2(1-\Sigma x^2) = 0$$
$$f_3 = z^2 - w^2(1-\Sigma x^2) = 0$$

} easier to evaluate then directly using J def^n.

vi)
$$f_1(x_1, x_2 \ldots, x_n, y_1) = 0$$
$$f_2(x_2, x_3 \ldots, x_n, y_1, y_2) = 0$$
$$\vdots$$
$$f_n(x_n, y_1, y_2 \ldots, y_n) = 0$$

$$\Rightarrow \frac{\partial(f_1 f_2 \ldots f_n)}{\partial(y_1, y_2 \ldots y_n)} = \frac{\partial f_1}{\partial y_1} \frac{\partial f_2}{\partial y_2} \ldots \frac{\partial f_n}{\partial y_n}.$$

$$\frac{\partial(f_i)}{\partial x_i} = \frac{\partial f_1}{\partial x_1} \frac{\partial f_2}{\partial x_2} \ldots \frac{\partial f_n}{\partial x_n}$$

$$\therefore \frac{\partial(y_1 y_2 \ldots y_n)}{\partial(x_1 x_2 \ldots x_n)} = (-1)^n \frac{\frac{\partial f_1}{\partial x_1} \frac{\partial f_2}{\partial x_2} \ldots \frac{\partial f_n}{\partial x_n}}{\frac{\partial f_1}{\partial y_1} \frac{\partial f_2}{\partial y_2} \ldots \frac{\partial f_3}{\partial y_3}}$$

vii) $\boxed{\text{Necessary \& Sufficient condition}}$ that $y_1, y_2 \ldots y_n$ - n functions of n independent variable $x_1 x_2 \ldots x_n$ are functionally related (i.e., $f(y_1, y_2, \ldots, y_n) = 0$) iff $J = \frac{\partial(y_1 y_2 \ldots y_n)}{\partial(x_1 x_2 \ldots x_n)} = 0$

$\Rightarrow$ $J = 0 \iff y_1, y_2 \ldots y_n$ are not independent. They have a relation.

eg
$$u = x + y - z$$
$$v = x - y + z$$
$$w = x^2 + y^2 + z^2 - 2yz$$

Here $J\frac{(u,v,w)}{(x,y,z)} = 0$.

⇒ Multiple Integral

i) **Important Integrations :**

a) $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$

b) $\int \frac{1}{\sqrt{1+x^2}}\,dx = \log\{x+\sqrt{1+x^2}\} + C$

c) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

d) $\int \sqrt{a^2-x^2}\,dx = \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + \frac{x\sqrt{a^2-x^2}}{2} + C$

e) $\int_0^{\pi/2} \sin^n\theta \,d\theta$ (n is even) $= \left(\frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{3}{4}\cdot\frac{1}{2}\right)\times\frac{\pi}{2}$

f) $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

g) $\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{if } f(x) = f(-x) \text{ f is even} \\ 0 & \text{if } -f(x) = f(-x) \text{ f is odd} \end{cases}$

h) $\int \sec\theta\,d\theta = \int \frac{\sec\theta(\sec\theta+\tan\theta)}{(\sec\theta+\tan\theta)}\,d\theta = \ln(\sec\theta+\tan\theta) + C$

i) $\int_0^\infty e^{-x^2} = \sqrt{\pi}/2$ j) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$

k) $\int \sqrt{a^2+x^2}\,dx = \frac{x\sqrt{a^2+x^2}}{2} + \frac{a^2}{2}\log\left(x+\sqrt{a^2+x^2}\right)$ l) $\int \frac{1}{\sqrt{a^2+x^2}}\,dx = \log\left(x+\sqrt{a^2+x^2}\right)$

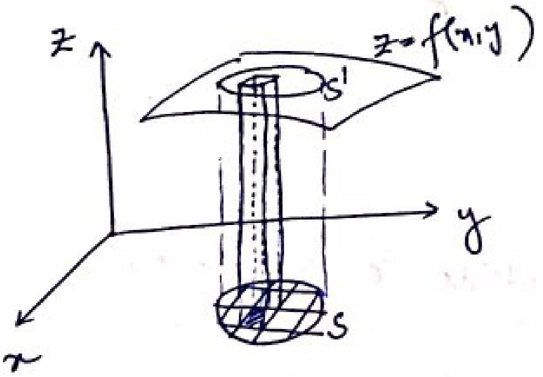
m) $\int \sqrt{x^2-a^2}\,dx = \frac{x\sqrt{x^2-a^2}}{2} - \frac{a^2}{2}\log\left(x+\sqrt{x^2-a^2}\right)$ n) $\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \log\left|x+\sqrt{x^2-a^2}\right|$

ii) $$\int_0^1 dx \int_0^1 \frac{x-y}{(x+y)^3}\, dy \neq \int_0^1 dy \int_0^1 \frac{x-y}{(x+y)^3}\, dx$$

Order is important but most of the integrations are well behaved, so don't think much

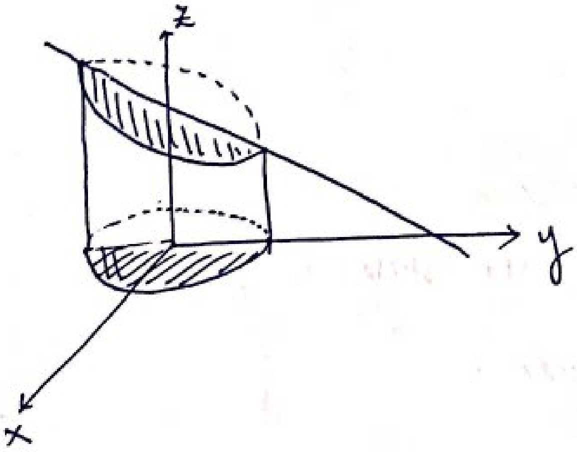
iii) Area by double integration $$A = \int_{x_1}^{x_2} \int_{f_1(x)}^{f_2(x)} dx\, dy$$

iv) Volume as double Integral : $$V = \iint_S f(x,y)\, dx\, dy$$



where S is the area projected on x-y plane

eg. Find the volume bounded by the cylinder $x^2+y^2=4$ & the plane $y+z=4$ and $z=0$



TRY TO DRAW GOOD FIGURES.

$$\text{vol} = 2\int_{-2}^{2} \int_0^{\sqrt{4-y^2}} (4-y)\, dx\, dy$$

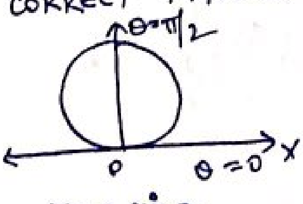
$$= \int_0^{2\pi} \int_0^{2} (4-r\sin\theta)\, r\, dr\, d\theta$$

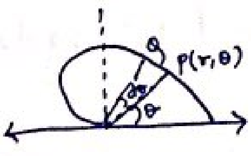
v) Double Integral in Polar Coordinates :

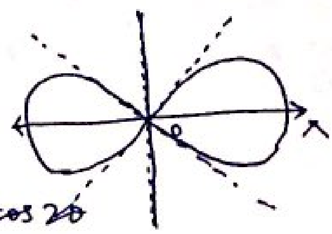
DRAW CORRECT FIGURES

FIGURES :

NOTE! $\boxed{r \geq 0}$

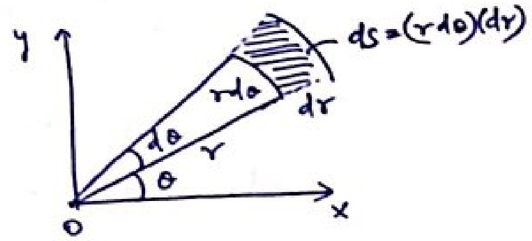






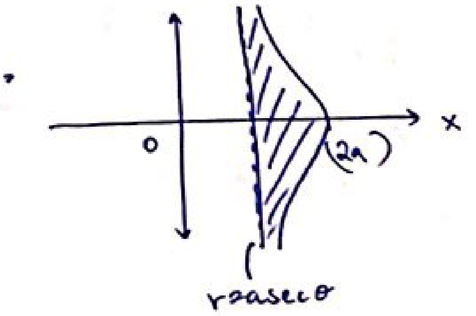
vi) Area enclosed by plane curves : Polar coordinates

$$\iint_A dx\,dy = \iint r\,dr\,d\theta$$



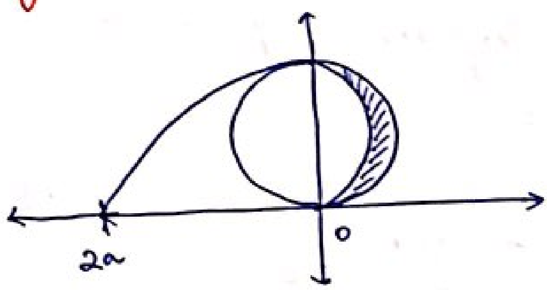
eg.1. Area included b/w $r = a(\sec\theta + \cos\theta)$ & asymptote

at $\theta = \frac{\pi}{2}$ $r \simeq a\sec\theta \Rightarrow$ asymptote



$$\text{Area} = \frac{5\pi a^2}{4}$$

eg.2 Area inside $r = a\sin\theta$ & outside $r = a(1-\cos\theta)$



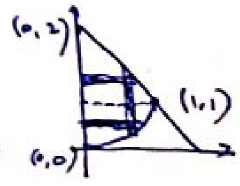
$$\text{Area} = a^2\left(1 - \frac{\pi}{4}\right)$$

vii) Change order of integration :

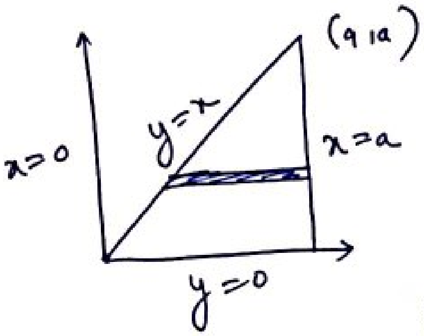
Draw diagram & interchange the strips & find the new limits for the given area.

Sometimes splitting of integral is also needed.

eg. $$\int_0^1 \int_{x^2}^{2-x} xy\,dx\,dy = \int_0^1 dy \int_0^{\sqrt{y}} xy\,dx + \int_1^2 dy \int_0^{2-y} xy\,dx$$



eg. 1: Change order in $\int_0^a \int_0^x \frac{\phi'(y)\, dx\, dy}{\sqrt{(a-x)(x-y)}}$



$\Rightarrow \int_0^a \int_y^a \frac{\phi'(y)\, dx\, dy}{\sqrt{(a-x)(x-y)}}$

Put $x = a\sin^2\theta + y\cos^2\theta$

$a-x = a(1-\sin^2\theta) - y\cos^2\theta = (a-y)\cos^2\theta$

$x-y = a\sin^2\theta + y\cos^2\theta - y = (a-y)\sin^2\theta$

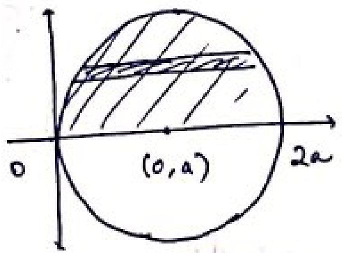
at $x=a$ $\quad a\cos^2\theta = y\cos^2\theta \Rightarrow \theta = \pi/2$

at $x=y$ $\quad y\sin^2\theta = a\sin^2\theta \Rightarrow \theta = 0$

$$I = \int_0^a \int_0^{\pi/2} \frac{\phi'(y)\, 2\cancel{(a-y)}\, \cancel{\sin\theta}\, \cancel{\cos\theta}\, d\theta\, dy}{\cancel{(a-y)}\, \cancel{\sin\theta}\, \cancel{\cos\theta}} = 2\int_0^a \int_0^{\pi/2} \phi'(y)\, d\theta\, dy$$

$$= \pi \int_0^a \phi'(y)\, dy = \pi\left(\phi(a) - \phi(0)\right)$$

eg. 2. $I = \int_0^{2a} \int_0^{\sqrt{2ax-x^2}} \frac{\phi'(y)(x^2+y^2)\, x\, dx\, dy}{\sqrt{4a^2x^2-(x^2+y^2)^2}}$



$\Rightarrow I = \int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} \frac{\phi'(y)(x^2+y^2)\, x\, dx\, dy}{\sqrt{4a^2x^2-(x^2+y^2)^2}}$

Put $x^2+y^2 = 2at$

$$I = \int_0^a \int_{a-\sqrt{a^2-y^2}}^{a+\sqrt{a^2-y^2}} \frac{\phi'(y)\, a\, t\, dt\, dy}{\sqrt{(a^2-y^2)-(t-a)^2}}$$

Put $(t-a) = \sqrt{a^2-y^2}\sin\alpha$

eg. 3:

$$I = \int_0^{\pi/2}\int_0^{\pi/2} \sin x \, \sin^{-1}(\sin x \sin y)\, dx\, dy$$

Put $\sin x \sin y = \sin\theta$

$\sin x \cos y \, dy = \cos\theta\, d\theta$

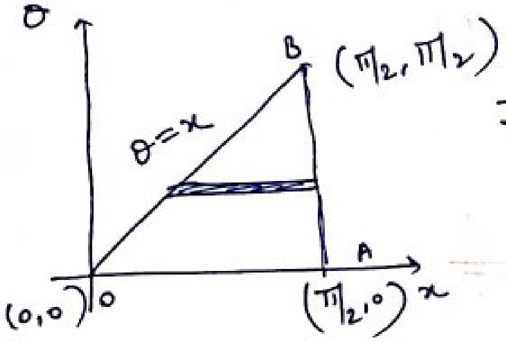
at $y = 0 \Rightarrow \theta = 0$

at $y = \pi/2 \quad \theta = x$

$$I = \int_0^{\pi/2}\int_0^{x} \cancel{\sin x} \cdot \theta \, dx \, \frac{\cos\theta\, d\theta}{\cancel{\sin x} \cos y}$$

$$= \int_0^{\pi/2}\int_0^{x} \frac{\theta\cos\theta\, d\theta\, dx}{\cos y} = \int_0^{\pi/2}\int_0^{x} \frac{\theta\cos\theta\, d\theta\, dx}{\sqrt{1-\frac{\sin^2\theta}{\sin^2 x}}}$$

$$= \int_0^{\pi/2}\int_0^{x} \frac{\theta \sin x \cos\theta\, d\theta\, dx}{\sqrt{\sin^2 x - \sin^2\theta}} = \int_0^{\pi/2}\int_0^{x} \frac{\theta \sin x \cos\theta\, d\theta\, dx}{\sqrt{\cos^2\theta - \cos^2 x}}$$



$$I = \int_{\theta=0}^{\pi/2} \theta\cos\theta\, d\theta \int_{\theta}^{\pi/2} \frac{\sin x\, dx}{\sqrt{\cos^2\theta - \cos^2 x}}$$

$\cos x = t$

$-\sin x\, dx = dt$

$$I = \int_{\theta=0}^{\pi/2} \theta\cos\theta\, d\theta \int_{\cos\theta}^{0} \frac{-dt}{\sqrt{\cos^2\theta - t^2}} = I = \int_{\theta=0}^{\pi/2} \theta\cos\theta \, \sin^{-1}\left(\frac{t}{\cos\theta}\right)\Bigg|_0^{\cos\theta} d\theta$$

$$= \int_{\theta=0}^{\pi/2} \theta\cos\theta \times \frac{\pi}{2}\, d\theta = \frac{\pi}{2}\int_0^{\pi/2} \theta\cos\theta\, d\theta$$

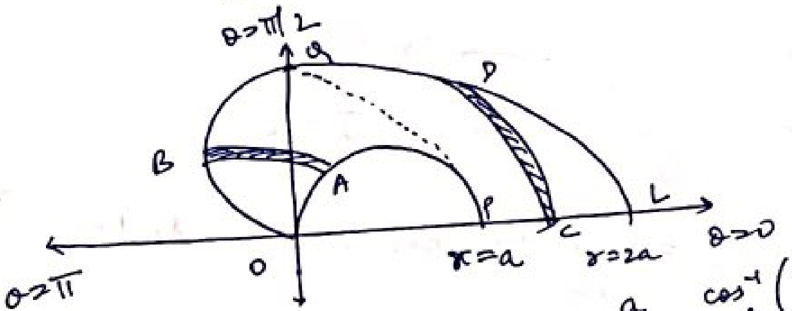
$$= \frac{\pi}{2}\left[\theta\sin\theta\Big|_0^{\pi/2} - \int_0^{\pi/2}\sin\theta\, d\theta\right] = \frac{\pi}{2}\left[\left(\frac{\pi}{2}-0\right) + (-1)\right] = \frac{\pi}{2}\left(\frac{\pi}{2}-1\right)$$

viii) Change order in polar coordinates

$$\int_{\theta=}\int_{r=} f(r,\theta)\,dr\,d\theta \Rightarrow \int_{r=}\int_{\theta=} f(r,\theta)\,d\theta\,dr$$

↳ for a given r identify the values θ can take.

eg. 1 $$\int_{0}^{\pi/2}\int_{a\cos\theta}^{a(1+\cos\theta)} f(r,\theta)\,r\,d\theta\,dr + \int_{\pi/2}^{\pi}\int_{a\cos\theta}^{a(1+\cos\theta)} f(r,\theta)\,r\,dr\,d\theta$$



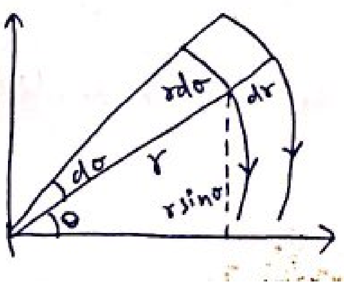
for OAPQBO : $I_1 = \int_{r=0}^{a}\int_{\cos^{-1}\left(\frac{r}{a}\right)}^{\cos^{-1}\left(\frac{r-a}{a}\right)} f(r,\theta)\,r\,dr\,d\theta$

for PCLDQP : $I_2 = \int_{a}^{2a}\int_{0}^{\cos^{-1}\left(\frac{r-a}{a}\right)} f(r,\theta)\,r\,dr\,d\theta$

ix) Volume of Solid of Revolution (About x-axis)

Volume (cartesian) $= \iint_A 2\pi y\,dx\,dy$

Volume (polar) $= \iint_A (2\pi r\sin\theta)(r\,d\theta)(dr)$



$= \iint_A 2\pi r^2 \sin\theta\,d\theta\,dr$

x) Change of variables

$$\iint_{R_{xy}} f(x,y)\,dx\,dy = \iint_{R'_{uv}} f[\phi(u,v),\psi(u,v)]\,|J|\,du\,dv$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} \neq 0$$

where transformation is $x = \phi(u,v)$ & $y = \psi(u,v)$

a) Cartesian to polar $(x,y) \longrightarrow (r,\theta)$

$x = r\cos\theta \quad y = r\sin\theta \quad J = r$

$$\therefore \iint_{R_{xy}} f(x,y)\,dx\,dy = \iint_{R'_{r\theta}} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$$

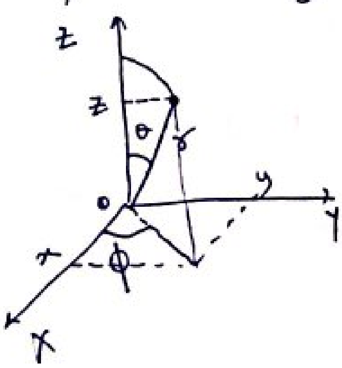
b) Rectangular to cylindrical $(x,y,z) \longrightarrow (r,\phi,z)$

$x = r\cos\phi \quad y = r\sin\phi \quad z = z$

$$J = \frac{\partial(x,y,z)}{\partial(r,\phi,z)} = r$$

$$\iiint_{R_{xyz}} f\,dx\,dy\,dz = \iiint_{R'_{r\phi z}} f\,r\,dr\,d\phi\,dz$$

c) Rectangular to polar $(x,y,z) \longrightarrow (r,\theta,\phi)$

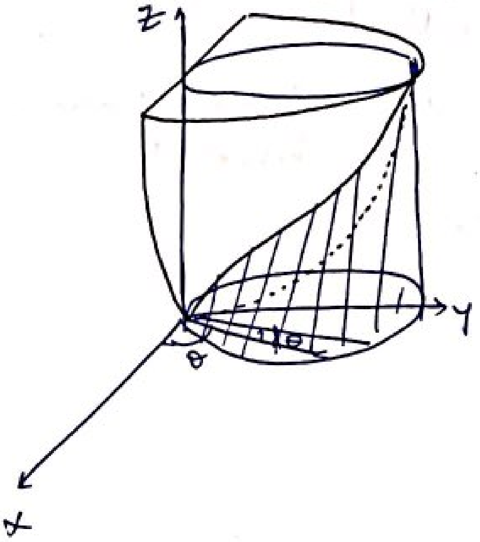


$x = r\sin\theta\cos\phi \quad y = r\sin\theta\sin\phi \quad z = r\cos\theta$

$J = r^2\sin\theta$

$$\iiint_{R_{xyz}} f\,dx\,dy\,dz = \iiint_{R'_{r\theta\phi}} f\,(r^2 dr)(r\sin\theta\,d\theta)(d\phi)$$

eg. 1 volume bounded by paraboloid $x^2+y^2=az$
cylinder $x^2+y^2=2ay$ & $z=0$

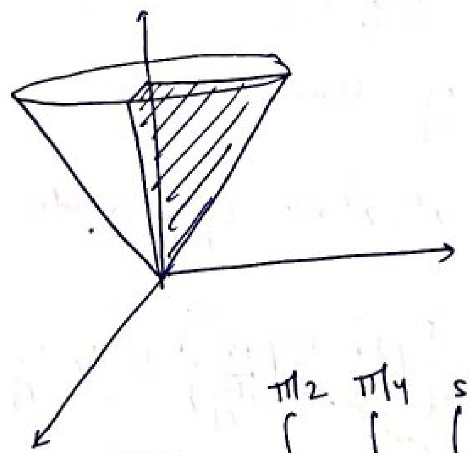


$$\Rightarrow \int_{\theta=0}^{\pi}\int_{r=0}^{2a\sin\theta} \frac{r^2}{a}\cdot r\,dr\,d\theta = \frac{1}{a}\int_0^{\pi} d\theta \int_0^{2a\sin\theta} r^3 dr$$

$$= \frac{1}{4a}\int_0^{\pi} 2^4a^4 \sin^4\theta\, d\theta = 8a^3\int_0^{\pi/2}\sin^4\theta\, d\theta$$

$$= 8a^3 \times \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{3\pi a^3}{2}$$

eg. 2. $$\int_0^1\int_0^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{1} \frac{dz\,dy\,dx}{\sqrt{x^2+y^2+z^2}}$$



$x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$

from $z=\sqrt{x^2+y^2}$
$r\cos\theta = r\sin\theta$
$\theta = \pi/4$

$\theta = 0 \rightarrow \pi/4$
$\phi = 0 \rightarrow \pi/2$
$r = 0 \rightarrow \sec\theta$

$$I = \int_0^{\pi/2}\int_0^{\pi/4}\int_0^{\sec\theta} \frac{r^{\cancel{2}}\sin\theta\,dr\,d\theta\,d\phi}{\cancel{r}}$$

$$= \int_0^{\pi/2}\int_0^{\pi/4} \sin\theta\,d\theta\,d\phi\cdot\frac{\sec^2\theta}{2} = \frac{1}{2}\int_0^{\pi/2}\int_0^{\pi/4}\sec\theta\tan\theta\,d\theta\,d\phi$$

$$= \frac{1}{2}\sec\theta\Big|_0^{\pi/4}\ \phi\Big|_0^{\pi/2}$$

$$= \frac{1}{2}\times(\sqrt{2}-1)\times\frac{\pi}{2} = \frac{\pi}{4}(\sqrt{2}-1)$$

xi) Area of Curved surface

Let projection of surface S on xy plane be A

$\delta A = \delta x \, \delta y = \delta S \cos\gamma$ where γ = angle b/w z-axis & normal at δS

dcs of Normal is given by directional derivative

$$= \left(\frac{-\partial z}{\partial x}, \frac{-\partial z}{\partial y}, 1\right)$$

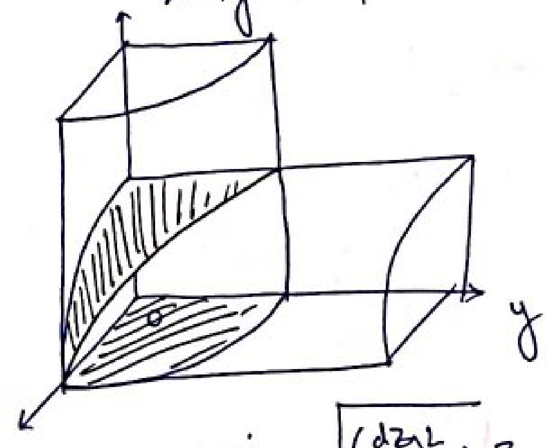
$$\therefore \cos\gamma = \frac{1}{\sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}}$$

$$\therefore \delta S = \frac{\delta x \, \delta y}{\cos\gamma} = \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1} \; \delta x \, \delta y$$

$$\therefore S = \underset{\delta S \to 0}{\text{Lt}} \sum \delta S = \iint_A \sqrt{\left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2 + 1} \; dx \, dy$$

↳ A is on xy plane

eg. 1. Area of portion of cylinder $x^2+z^2=4$ lying inside $x^2+y^2=4$



$$A = \iint_{A_{xy}} \sqrt{\left(\frac{dz}{dx}\right)^2 + \left(\frac{dz}{dy}\right)^2 + 1} \; dx \, dy$$

$$z = \sqrt{4-x^2}$$

$$\frac{dz}{dx} = \frac{-x}{\sqrt{4-x^2}} \qquad \frac{dz}{dy} = 0$$

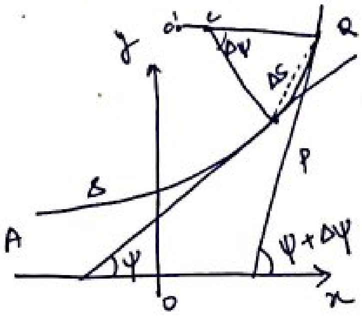
$$\therefore \sqrt{\left(\frac{dz}{dx}\right)^2 + 1} = \sqrt{\frac{x^2}{4-x^2} + 1} = \sqrt{\frac{4}{4-x^2}}$$

$$\therefore A = 8 \int_0^2 \int_0^{\sqrt{4-x^2}} \sqrt{\frac{4}{4-x^2}} \, dx \, dy = 8 \int_0^2 \sqrt{\frac{4}{\cancel{4-x^2}}} \times \cancel{\sqrt{4-x^2}} \, dx = 16 \int_0^2 dx$$

$$= 32 \text{ sq. unit}$$

⇒ Curvature

* Numerical measure of how much a curve is curved at a pt.



curvature $= \boxed{\dfrac{d\psi}{ds} = \dfrac{1}{r}}$

↳ radius of curvature

$$\left[\because \; s = r\psi \qquad \frac{d\psi}{ds} = \frac{1}{r}\right]$$

In $\Delta PQC'$: $\dfrac{PC'}{\sin(PQC')} = \dfrac{\text{chord } PQ}{\sin PC'Q} = \dfrac{QC'}{\sin C'PQ} = \dfrac{\text{chord } PQ}{\text{arc } PQ} \cdot \dfrac{\widehat{PQ}}{\sin(PC'Q)}$

$$\Rightarrow \left(\frac{\overline{PQ}}{\widehat{PQ}}\right) \frac{\Delta S}{\sin(\Delta\psi)} = \frac{\overline{PQ}}{\widehat{PQ}} \; \frac{\Delta S}{\Delta \psi} \; \frac{\Delta\psi}{\sin(\Delta\psi)}$$

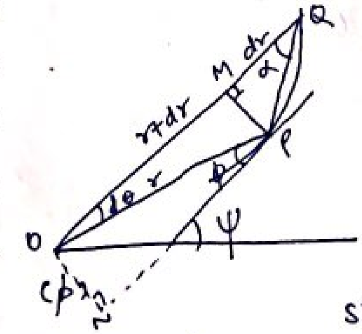
as $Q \to P$, $\angle PQC' \to \pi/2$, $\Delta\psi \to 0$, $C \to C'$

$$\lim_{Q\to P} \frac{\overline{PQ}}{\widehat{PQ}} = 1 \qquad \lim_{\Delta\psi \to 0} \frac{\Delta\psi}{\sin(\Delta\psi)} = 1 \qquad \lim_{\Delta\psi\to 0} \frac{\Delta S}{\Delta\psi} = \frac{ds}{d\psi}$$

$$\therefore \frac{PC'}{\sin(\pi/2)} = \frac{ds}{d\psi} \Rightarrow PC = r = \boxed{\frac{ds}{d\psi} = \underline{\rho}}$$

* Cartesian formula : $\boxed{\rho = \dfrac{\left[1+(y')^2\right]^{3/2}}{y''}}$

Polar coordinates : $\boxed{\rho = \dfrac{(r^2+r'^2)^{3/2}}{r^2+2r'^2-rr''}}$ $\quad r' = \dfrac{dr}{d\theta} \quad r'' = \dfrac{d^2r}{d\theta^2}$



as $Q \to P$ $\quad \overline{PQ} \to \widehat{PQ} = ds$

$MQ = dr$

$\alpha \to \phi$

$\sin\phi = \lim_{Q\to P} \sin\alpha = \boxed{\dfrac{r\,d\theta}{ds} = \sin\phi}$ $\quad \boxed{\cos\phi = \dfrac{dr}{ds}}$ $\quad \boxed{\tan\phi = \dfrac{r\,d\theta}{dr}}$

$$\frac{\overline{PQ}}{\sin(d\theta)} = \frac{OP}{\sin\alpha}$$

Length of $\perp$ from pole to tangent $= p = r \sin\phi$

'Pedal' curve $\boxed{\dfrac{ds}{d\psi} = r\dfrac{dr}{dp}}$

$$\boxed{\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2}$$

$$\frac{1}{p^2} = \frac{1}{r^2}\operatorname{cosec}^2\phi = \frac{1}{r^2}(1+\cot^2\phi) = \frac{1}{r^2}\left(1+\frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right)$$

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2$$

eg. Pedal eq^n helps in finding ρ with more ease

like curve $r^2 = a^2 \sin 2\theta$

$$2r\frac{dr}{d\theta} = a^2 \times 2\cos 2\theta$$

$$r\frac{dr}{d\theta} = a^2 \cos 2\theta$$

$$\therefore \frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\frac{a^4\cos^2 2\theta}{r^2} = \frac{r^4 + a^4\cos^2 2\theta}{r^6}$$

$$= \frac{a^4\sin^2 2\theta + a^4\cos^2 2\theta}{r^6} = \frac{a^4}{r^6}$$

Eliminate other variables & try to get p in terms of 'r' & constants

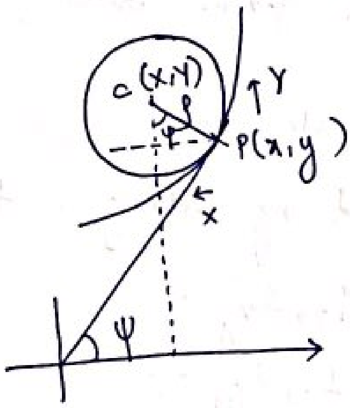
$$\therefore p = \frac{r^3}{a^2} \qquad \frac{dp}{dr} = \frac{3r^2}{a^2}$$

$$\therefore \rho = r\frac{dr}{dp} = r \cdot \frac{a^2}{3r^2} = \frac{a^2}{3r}$$

* Centre of curvature :

$$X = x - \frac{(1+y'^2)}{y''} y' \qquad Y = y + \frac{(1+y'^2)}{y''}$$

NOTE: Similar to ρ formula. Remember signs using fig



$X = x - \rho \sin\psi$

$Y = y + \rho \cos\psi$

$\tan\psi = y'$

$\sin\psi = \frac{y'}{(1+y')^{1/2}} \quad \cos\psi = \frac{1}{(1+y')^{1/2}}$

$\rho = \frac{(1+y')^{3/2}}{y''}$

Locus of centre of curvature is called Evolute of the curve

If C_1 is evolute of C_2, then C_2 is Involute of C_1

eg. Evolute of $2xy = a^2$ is $(x+y)^{2/3} - (x-y)^{2/3} = 2a^{2/3}$

$x = \frac{at}{2} \quad y = \frac{a}{t} \quad \frac{dy}{dt} = \frac{-a}{t^2} \quad \frac{dx}{dt} = \frac{a}{2} \quad y' = \frac{-2}{t^2}$

$y'' = \frac{8}{at^3}$

$X = \frac{3at}{4} + \frac{a}{t^3} \qquad Y = \frac{3a}{2t} + \frac{at^3}{8}$

$X + Y = a\left(\frac{t^2+2}{2t}\right)^3 \qquad X - Y = a\left(\frac{2-t^2}{2t}\right)^3$

$$(X+Y)^{2/3} - (X-Y)^{2/3} = a^{2/3}\left[\left(\frac{2+t^2}{2t}\right)^2 - \left(\frac{2-t^2}{2t}\right)^2\right]$$

$$= a^{2/3}\left[\frac{2+t^2+2-t^2}{2t} \cdot \frac{2+t^2-2+t^2}{2t}\right]$$

TRY TO USE PARAMETRIC FORMS

$$= a^{2/3} \cdot \frac{\cancel{4}}{\cancel{2t}} \cdot \frac{2t^2}{\cancel{2t}} = 2a^{2/3}$$

⇒ Beta and Gamma Functions :

1) $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx \qquad m, n > 0$

$= B(n,m)$

2) Properties :

a) If $m, n \in \mathbb{Z}^+$ $\quad B(m,n) = \dfrac{(m-1)!\,(n-1)!}{(m+n-1)!}$

TRICK: use $\int x^m (1-x)^{n-1} \Rightarrow \int x^m \underline{(1-x)}(1-x)^{n-2}$

$\Rightarrow \int x^m (1-x)^{n-2} - x^{m+1}(1-x)^{n-2}$ ⇐ [use Integration by parts]

b) $\dfrac{B(p, q+1)}{q} = \dfrac{B(p+1, q)}{p} = \dfrac{B(p,q)}{p+q}$

TIP: To express in $B(m,n)$ form, we need to substitute $f(x) = z$ to get $\int_0^1 z^{m-1}(1-z)^{n-1}\,dz$ form

eg $\int_0^1 \dfrac{x^2}{\sqrt{1-x^5}}\,dx \qquad$ Put $x^5 = z$

$\int_0^1 x^m (p^2 - x^2)^n\,dx \qquad$ Put $z = \left(\dfrac{x}{p}\right)^2$

$\int_0^1 \dfrac{x^{m-1}(1-x)^{n-1}}{(a+bx)^{m+n}}\,dx \qquad$ Put $z = \dfrac{(a+b)x}{(a+bx)}$

$\int_a^b (x-a)^{m-1}(b-x)^{n-1}\,dx \qquad$ Put $x = a + (b-a)z$

$\int_0^{\pi/2} \dfrac{\cos^{2m-1}\theta \sin^{2n-1}\theta}{(a\cos^2\theta + b\sin^2\theta)^{m+n}} = \int_0^{\pi/2} \dfrac{(\cos^2\theta)^{m-1}(\sin^2\theta)^{n-1}\cos\theta\sin\theta\,d\theta}{(a\cos^2\theta + b\sin^2\theta)}$

Put $\sin^2\theta = x$

3) $\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} \, dx \qquad n > 0$

4) Properties

a) $\Gamma(n) = (n-1)\,\Gamma(n-1)$

$= (n-1)! \quad \text{if } n \in \mathbb{Z}^+$

TRICK : $\gamma > \beta$ so $\Gamma(n)$ is $0 \rightarrow \infty$ & $B(m,n)$ is $0 \rightarrow 1$

Need e^{-x} to handle ∞ & all powers of x & $(1-x)$ are $m-1$, $n-1$ form

b) $B(m,n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} \qquad m > 0,\ n > 0$

c) $\Gamma(1/2) = \sqrt{\pi} \qquad \left[\because B\left(\frac{1}{2}, \frac{1}{2}\right) = \Gamma(1/2)\Gamma(1/2) \right.$

↳ use $x = \sin^2\theta$ to evaluate

$\Downarrow$

$\int_0^\infty e^{-x^2} dx = \dfrac{\sqrt{\pi}}{2} \qquad [\because \text{Put } x^2 = z]$

d) $\int_0^{\pi/2} \sin^p x \cos^q x \, dx = \dfrac{1}{2} \dfrac{\Gamma\left(\frac{p+1}{2}\right) \Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+1}{2} + \frac{q+1}{2}\right)}$

Don't forget

⇒ Singular Points

i) DOUBLE POINT : pt through which 2 tangents pass

CUSP : Both tangents coincide
NODE : 2 different tangents
CONJUGATE : Imaginary tangents

ii) Tangent at origin : equating to 0 the lowest degree terms of the given curve

iii) Position and Nature of double points :

Double points are obtained by solving . $f(x,y)=0$, $f_x=0$ $f_y=0$

Node if $(f_{xy})^2 - f_{xx}f_{yy} > 0$

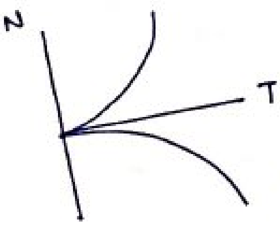
Cusp if $(f_{xy})^2 - f_{xx}f_{yy} = 0$

Isolated point if $(f_{xy})^2 - f_{xx}f_{yy} < 0$

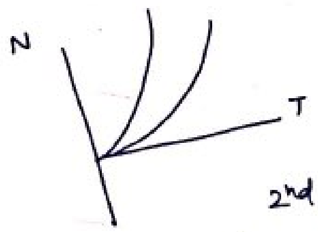
If all $f_{xx} = f_{yy} = f_{xy} = 0$ ⇒ multiple point of order > 2

iv) CUSPS are classified based on relative position of the branches to normal & tangent

SINGLE CUSP :

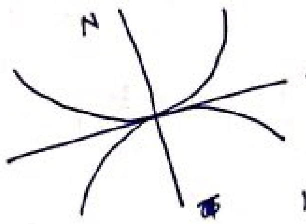


1st species

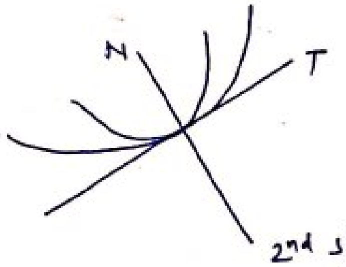


2nd species

DOUBLE CUSP

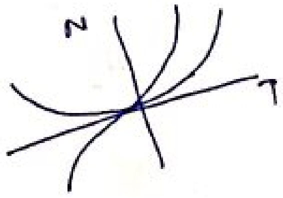


1st species



2nd species

OSCU - INFLEXION



Tangent → same side 2nd species
opp 1st species

Normal : Same side single
opp side double

* Concavity

Concave upwards : curve lies above tangent $f''(c) > 0$

concave downwards : curve lies below tangent $f''(c) < 0$

point of inflexion : $f''(c) = 0$ & $f'''(c) \neq 0$

[$f''(x)$ changes sign at $x=c$]

eg. 1 $(y - 4x^2)^2 = x^7$ $y - 4x^2 = \pm x^{7/2}$

$y = 4x^2 \pm x^{7/2}$

lowest degree

$\therefore$ tangent $= y = 0 \Rightarrow$ cusp

x can't be negative $\therefore x > 0 \Rightarrow$ same side of normal $\Rightarrow$ single cusp ~~species~~

Now $y_1 = 4x^2 + x^{7/2}$ is > 0 $\forall\ x > 0$

for $y_2 = 4x^2 - x^{7/2}$ if $4x^2 > x^{7/2}$

$\Rightarrow 4 > x^{3/2}$

$\Rightarrow x < 4^{2/3} \Rightarrow y > 0$

$\therefore$ for $x \in (0, 4^{2/3})$ $y_2 > 0$

$\therefore$ Near origin both branches are on same side of tangent $\Rightarrow$ second species

$\therefore$ single cusp of 2^{nd} species.

⇒ Asymptotes:

1) $\perp$ distance from a line to the curve tends to 0 at either or both x or y tend to ∞, then the line is asymptote

2) Asymptotes $\parallel$ to x-axis & y-axis to $y = f(x)$

a) $y = b$ is asymptote if $\lim_{x \to \infty} f(x) = b$ or $\lim_{x \to -\infty} f(x) = b$

b) $x = a$ is asymptote if $\lim_{x \to a^+} f(x) = \pm\infty$ or $\lim_{x \to a^-} f(x) = \pm\infty$

3) Working Rule for $\left[f(x) = x^n \phi_n\left(\frac{y}{x}\right) + x^{n-1} \phi_{n-1}\left(\frac{y}{x}\right) + \cdots = 0 \right]$

i) Put $x = 1$ & $y = m$ in highest degree terms of eq^n & get $\phi_n(m)$. Get roots of $\phi_n(m) = 0 \Rightarrow (m_1, m_2, \ldots, m_n)$

ii) Find $c_i = \dfrac{-\phi_{(n-1)}(m_i)}{\phi_n'(m_i)}$ for each m_i

Asymptote $y = m_i x + c_i$

* If $c_i = \frac{0}{0}$; solve $\dfrac{c^2}{2!} \phi_n''(m) + c\,\phi_{n-1}'(m) + \phi_{n-2}(m) = 0$

n asymptotes – real or img

if n is odd, at least one real asymptote

4) If x^n or y^n is absent, some asymptotes can be found directly ⇒ ASYMPTOTES PARALLEL TO AXES

If ϕ_n is not of degree 'n', you might miss asymptote

a) if y^n is absent $\Rightarrow$ one asymptote with $m = \infty$
Find coeff of y^{n-1} & equate to 0

b) if x^n is absent $\Rightarrow$ one asymptote with $m = 0$
Find coeff of x^{n-1} & equate to 0

eg. 1 Find asymptotes of $x^3 + 2x^2y + xy^2 - x^2 - xy + 2 = 0$

$\phi_3(m) = 1 + 2m + m^2$ ~~[illegible]~~ $= (m+1)^2 =$ roots $= -1, -1$

$\phi_2(m) = -1 - m$

$$c = \frac{-\phi_2(m)}{\phi_3'(m)} = \frac{-\phi_2(m)}{2m+2} = \frac{m+1}{2(m+1)} = \frac{0}{0}$$

$$\therefore \frac{c^2}{2!}\phi_3''(m) + c\phi_2'(m) + \phi_1(m) = 0$$

$$\frac{c^2}{2}(2) + c(-1) + 0 \Rightarrow c^2 - c = 0 \quad c \Rightarrow 0, 1$$

asymptotes $= y = -x, \; y = -x + 1$

3rd asymptote $=$ coeff of $y^2 = 0 \Rightarrow x = 0$

eg. 2 Find asymptotes $\parallel$ to axes for $x^2y^2 - a^2(x^2+y^2) = 0$

Both x^4 & y^4 are absent

so coeff of $x^2 = 0 \Rightarrow a^2 - y^2 = 0 \Rightarrow y = \pm a$
coeff of $y^2 = 0 \Rightarrow a^2 - x^2 = 0 \Rightarrow x = \pm a$

5) Another method

a) Rearrange Equation as $ax + by + c + \frac{F_{n-1}}{P_{n-1}} = 0$

$$\text{Asymptote} = ax + by + c + \lim_{y = -\frac{ax}{b} \to \infty}\left(\frac{F_{n-1}}{P_{n-1}}\right) = 0$$

b) If we can write as $(ax+by)^2 P_{n-2} + (ax+by) F_{n-2} + f_{n-2} = 0$

Asymptotes = $(ax+by)^2 + (ax+by) \lim \frac{F_{n-2}}{P_{n-2}} + \lim \frac{f_{n-2}}{P_{n-2}} = 0$

We get 2 asymptotes $ax+by = \alpha$ & $ax+by = \beta$

where α, β are roots of $t^2 + t \lim \frac{F_{n-2}}{P_{n-2}} + \lim \frac{f_{n-2}}{P_{n-2}} = 0$

* Try to use (a) before going for (b)

eg Asymptote of $(x+y)^2 (x+2y+2) = x+9y-2$

$$(x+y)^2 = \lim_{y=-x\to\infty} \frac{x+9y-2}{x+2y+2} = \lim_{x\to-\infty} \frac{-8x-2}{-x+2}$$

$$= 8$$

$$(x+y) = \pm 2\sqrt{2}$$

3rd : $$x+2y+2 = \lim_{y=-\frac{x}{2}\to\infty} \frac{x+9y-2}{(x+y)^2}$$

$$= \lim_{x\to-\infty} \frac{-\frac{7x}{2}-2}{\frac{x^2}{4}} = 0$$

$$x+2y+2 = 0$$

6) Position of curve wrt asymptote — use binomial expansion to get in this form

Asymp is $y = mx+c$

curve $y = mx + c + A/x + B/x^2 + C/x^3 + \cdots$

If $A \neq 0$ above if A & x have same sign, else below

$A=0$ above if $B>0$, below if $B<0$

7) If curve is $F_n + F_{n-2} = 0$, then asymptotes are $F_n = 0$

Total intersections are $n(n-2) = 0$

8) Asymptote in Polar Coordinates

a) Express eqn as $\frac{1}{r} = f(\theta)$ [$\because$ we need $\frac{1}{r} = 0$ at ∞]

b) Let α be root of $f(\theta) = 0$

c) $r \sin(\theta - \alpha) = \frac{1}{f'(\alpha)}$ $f'(\alpha) \neq 0$ is asymptote

[$r\sin\theta = c$ is a line in polar]

eg $r = \frac{1}{1-2\sin\theta}$ $\frac{1}{r} = 1 - 2\sin\theta$

$1 - 2\sin\theta = 0$ $\Rightarrow \sin\theta = \frac{1}{2}$ $\Rightarrow$ $\alpha = \pi/6$

Asymptote = $r\sin\left(\theta - \frac{\pi}{6}\right) = \frac{1}{-\sqrt{3}}$

9) Circular Asymptote

If $r = f(\theta)$ be the curve, & $\lim_{\theta \to \infty} f(\theta) = a$, then

$r = a$ is circular asymptote

⇒ Tracing of Curves :

1) Things to check for Cartesian coordinates :

a) symmetricity about any line
   i) about x-axis ⇒ if only even powers of y
   ii) about y-axis ⇒ if only even powers of x.
   iii) about both-axis
   iv) about $y = x$ if $f(x,y) = f(y,x)$
   v) about $y = -x$ if $f(x,y) = f(-y,-x)$
   vi) in opposite quadrants if $f(x,y) = f(-x,-y)$

b) If origin ~~passes~~ satisfies, then tangents are obtained by equating ~~to~~ lowest degree term to 0.

c) Find asymptotes
d) Find impossible regions (such as $y^2 = x$ & $x < 0$)

e) Find the nature of tangents (∥ to axis) & concavity

* Not All of the above are necessary

2) If $x = f(t)$ & $y = g(t)$, try to eliminate 't'

If can't eliminate, then

a) If x is odd & y is even function, symmetry abt y-axis
—"— even & y is odd —"— , —"— x-axis

b) Find limits / extremas of x & y

c) Find intersection with axes
d) Find $y' = 0$ pts
e) Plot few values to get an idea

3) Polar Curve

a) Find symmetry about initial line OX if $f(r,\theta) = f(r,-\theta)$

b) If $f(r,\theta) = f(r,\pi-\theta) \Rightarrow$ symmetry about OY

$f(r,\theta) = f(r,\frac{\pi}{2}-\theta) \Rightarrow$ about $\theta = \pi/4$

c) Find limits for r and θ

$f(r,\theta) = f(r,\frac{3\pi}{2}-\theta) \Rightarrow$ about $\theta = \frac{3\pi}{4}$

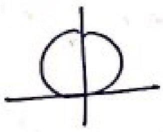
d) Find impossible regions

e) Asymptotes in direction of θ which make $r \longrightarrow \infty$

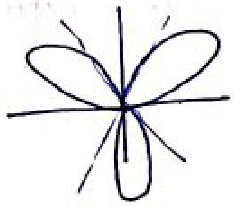
f) tangent = $\tan\phi = r\frac{d\theta}{dr} = 0$ or ∞ pts.

TRICK: If we have to trace $f(n\theta)$ & know for $f(\theta)$, then $f(n\theta)$ is compressed version of $f(\theta)$ in $\left(\frac{360}{k}\right)$ section repeated 'k' times $k = \begin{cases} n & \text{if } n \text{ is odd} \\ 2n & \text{if } n \text{ is even} \end{cases}$

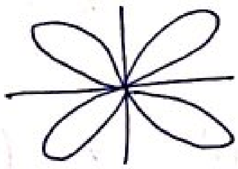
eg $r = a\sin\theta$



$r = a\sin(3\theta)$

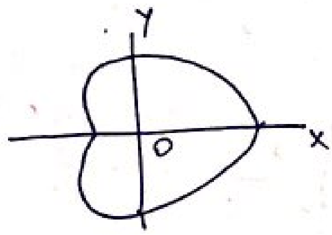


$r = a\sin(2\theta)$

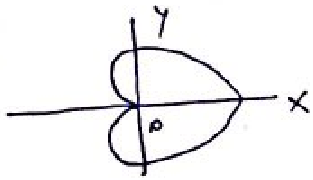


eg. $r = a + b\cos\theta$

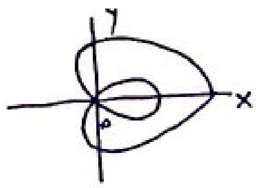
$a > b$



$a = b$



$a < b$



⇒ Taylor's theorem for multiple variables :

1) n^{th} Taylor polynomial of f at (x_0, y_0)

$$T_n(x,y) = \sum_{i,j=0}^{i+j \le n} \frac{1}{i!\,j!} \left[\frac{\partial^{i+j} f}{\partial x^i \partial y^j} (x_0, y_0) \right] (x-x_0)^i (y-y_0)^j$$

2) Taylor's theorem

for a given $(x,y) \neq (x_0, y_0)$ in some nbd of (x_0, y_0)

$\exists\ (c_1, c_2)$ st on line joining (x_0, y_0) & (x, y)

st. $f(x,y) = T_n(x,y) + R_{n+1}(x,y)$

where $T_n(x,y) = \displaystyle\sum_{i,j=0}^{i+j \le n} \frac{1}{i!\,j!} \left(\frac{\partial^{i+j} f}{\partial x^i \partial y^j} \right)_{(x_0, y_0)} (x-x_0)^i (y-y_0)^j$

and $R_{n+1}(x,y) = \displaystyle\sum_{i+j = n+1} \frac{1}{i!\,j!} \left(\frac{\partial^{i+j} f}{\partial x^i \partial y^j} \right)_{(c_1, c_2)} (x-x_0)^i (y-y_0)^j$

no '≤' sign

3) 2nd Taylor expansion

$$f(x,y) = f(x_0, y_0) + \left[(x-x_0) \frac{\partial}{\partial x} + (y-y_0) \frac{\partial}{\partial y} \right] f(x_0, y_0)$$
$$+ \frac{1}{2!} \left[(x-x_0) \frac{\partial}{\partial x} + (y-y_0) \frac{\partial}{\partial y} \right]^2 f(x_0, y_0) + R_3(x,y)$$