MAINS TEST SERIES - 2020
Test - 05 - Paper - II (Answer key)
Full length Test.

1(a), Find a basis for a subspace U of V in the following

(i) $U = \left\{ (x_1, x_2, x_3, x_4, x_5) \in V_5 \,/\, \begin{matrix} x_1 + x_2 + x_3 = 0 \\ 3x_1 - x_4 + 7x_5 = 0 \end{matrix} \right\}$

(ii) $U = \{ p \in P_4 \,/\, p(x_0) = 0 \}$, $V = P_4$, where P_4 is the set of all polynomials of degree ≤ 4.

(i) Sol^n: Given that

$U = \left\{ (x_1, x_2, x_3, x_4, x_5) \in V_5 \,/\, \begin{matrix} x_1 + x_2 + x_3 = 0 \\ 3x_1 - x_4 + 7x_5 = 0 \end{matrix} \right\}$

Let $\alpha = (x_1, x_2, x_3, x_4, x_5) \in V_5$

such that $x_1 + x_2 + x_3 = 0$ — (1)

and $3x_1 - x_4 + 7x_5 = 0$ — (2)

from (1) $x_1 = -x_2 - x_3$

from (2) $x_4 = 3x_1 + 7x_5 = 3(-x_2 - x_3) + 7x_5$

i.e., $x_4 = -3x_2 - 3x_3 + 7x_5$

$\therefore \alpha = (x_2 - x_3, x_2, x_3, -3x_2 - 3x_3 + 7x_5, x_5)$

$= x_2(1, 1, 0, -3, 0) + x_3(-1, 0, 1, -3, 0) + x_5(0, 0, 0, 7, 1)$.

$\therefore$ Basis of $U = \{(1, 1, 0, -3, 0), (-1, 0, 1, -3, 0), (0, 0, 0, 7, 1)\}$

(ii) Given that

$U = \{ p \in P_4 \mid p(x_0) = 0 \}$

Let $P_4 = ax^4 + bx^3 + cx^2 + dx + e$

since $p(x_0) = 0$,

$\therefore$ x_0 is the factor of P_4.

$\therefore$ $P_4 = (x - x_0) P_3$. The most obvious for P_3 would be $\{x^3, x^2, x, 1\}$

$\therefore$ $P_4 = (x - x_0)(ax^3 + bx^2 + cx + d)$

$= ax^4 + bx^3 + cx^2 + dx - ax_0x^3 - bx_0x^2 - cx_0x - dx_0.$

$= a(x^4 - x_0x^3) + b(x^3 - x_0x^2) + c(x^2 - x_0x) + d(x - x_0).$

$\therefore$ Basis of U

$= \{ x^4 - x_0x^3, x^3 - x_0x^2, x^2 - x_0x, x - x_0 \}$

1(b) If $A = \begin{bmatrix} 1 & 0 & 0 \\ i & \frac{-1+i\sqrt{3}}{2} & 0 \\ 0 & 1+2i & \frac{-1-\sqrt{3}i}{2} \end{bmatrix}$ then find trace of A^{102}.

Sol'n: Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ i & \frac{-1+i\sqrt{3}}{2} & 0 \\ 0 & 1+2i & \frac{-1-\sqrt{3}i}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ i & \omega & 0 \\ 0 & 1+2i & \omega^2 \end{bmatrix} \text{ say}$$

where $1, \omega, \omega^2$ are cube roots of units.

If $A = \begin{bmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{bmatrix}$ then trace of $A^n = a^n + c^n + f^n$

$$\therefore tr(A^{102}) = a^{102} + c^{102} + f^{102}$$

$$\Rightarrow tr(A^{102}) = 1^{(102)} + \omega^{(102)} + (\omega^2)^{102}$$

$$\Rightarrow tr(A^{102}) = 1 + (\omega^3)^{34} + (\omega^3)^{68}$$

$$= 1 + (1)^{34} + (1)^{68}$$

$$= 1 + 1 + 1$$

$$= 3$$

$$\therefore tr(A^{102}) = 3.$$

1(C). Show that $\int_0^{\pi/2} \log(\alpha^2\cos^2\theta+\beta^2\sin^2\theta)\,d\theta = \pi\log\frac{\alpha+\beta}{2}$.

Sol'n: Let $u = \int_0^{\pi/2} \log(\alpha^2\cos^2\theta+\beta^2\sin^2\theta)\,d\theta$

$$\frac{du}{d\alpha} = \int_0^{\pi/2} \frac{2\alpha\cos^2\theta}{\alpha^2\cos^2\theta+\beta^2\sin^2\theta}\,d\theta$$

$$= \int_0^{\pi/2} \frac{2\alpha\cos^2\theta}{(\alpha^2-\beta^2)\cos^2\theta+\beta^2}\,d\theta$$

$$= \frac{2\alpha}{(\alpha^2-\beta^2)}\int_0^{\pi/2} \frac{(\alpha^2-\beta^2)\cos^2\theta+\beta^2-\beta^2}{(\alpha^2-\beta^2)\cos^2\theta+\beta^2}\,d\theta$$

$$= \frac{2\alpha}{(\alpha^2-\beta^2)}\int_0^{\pi/2} \left[1-\frac{\beta^2}{(\alpha^2-\beta^2)\cos^2\theta+\beta^2}\right]d\theta$$

$$= \frac{2\alpha}{(\alpha^2-\beta^2)}\int_0^{\pi/2} \left[1-\frac{\beta^2\sec^2\theta}{(\alpha^2-\beta^2)+\beta^2\sec^2\theta}\right]d\theta$$

$$= \frac{2\alpha}{(\alpha^2-\beta^2)}\int_0^{\pi/2} \left[1-\frac{\beta^2\sec^2\theta}{\alpha^2+\beta^2\tan^2\theta}\right]d\theta$$

$$= \frac{2\alpha}{\alpha^2-\beta^2}\left[\theta-\frac{\beta^2}{\alpha\beta}\tan^{-1}\frac{\beta\tan\theta}{\alpha}\right]_0^{\pi/2}$$

$$= \frac{2\alpha}{\alpha^2-\beta^2}\left[\frac{\pi}{2}-\frac{\beta}{\alpha}\cdot\frac{\pi}{2}\right] = \frac{\pi}{\alpha+\beta}$$

Integrating, we get $u = \pi\log(\alpha+\beta)+C$ —— (1)

when $\alpha=\beta$, $u = \int_0^{\pi/2} \log\alpha^2(\cos^2\theta+\sin^2\theta)\,d\theta = \int_0^{\pi/2}\log\alpha^2\,d\theta$

$= \frac{\pi}{2}\log\alpha^2 = \pi\log\alpha$

Substituting in (1), we get
$\pi\log\alpha = \pi\log 2\alpha + C \Rightarrow C = \pi\log\frac{1}{2}$

Hence (1) gives $u = \pi\log(\alpha+\beta)+\pi\log\frac{1}{2} = \pi\log\frac{\alpha+\beta}{2}$.

1(d) Prove that a conical tent of a given capacity will require the least amount of canvas when the height is $\sqrt{2}$ times the radius of the base.

Sol'n: Let the tent be a cone of semi-vertical angle α and radius of the base r.

The volume V is fixed, while the surface area S has to be the minimum.

$$\text{Now} \quad V = \frac{1}{3}\pi r^3 \cot\alpha \quad \text{——} ①$$

$$\text{and} \quad S = \pi r^2 \operatorname{cosec}\alpha \quad \text{——} ②$$

Differentiating ①, we get

$$\frac{\pi}{3}\left[3r^2\cot\alpha\frac{dr}{d\alpha} - r^3\operatorname{cosec}^2\alpha\right] = 0$$

$$\Rightarrow \frac{dr}{d\alpha} = \frac{r\operatorname{cosec}^2\alpha}{3\cot\alpha}$$

Again from ②,

$$\frac{dS}{d\alpha} = \pi\left[2r\operatorname{cosec}\alpha\frac{dr}{d\alpha} - r^2\cot\alpha\operatorname{cosec}\alpha\right]$$

$$= \frac{\pi r^2\operatorname{cosec}\alpha}{3r\cot\alpha}\left[2\operatorname{cosec}^2\alpha - 3\cot^2\alpha\right]$$

$$= \frac{\pi r^2}{3\cot\alpha}\left[2 - \cot^2\alpha\right]$$

$\therefore \dfrac{dS}{d\alpha} = 0$, when $\cot\alpha = \sqrt{2}$

$\Rightarrow \alpha = \cot^{-1}\sqrt{2}$.

Also $\dfrac{dS}{d\alpha}$ changes sign from negative to positive as α passes through the value $\cot^{-1}\sqrt{2}$.

$\therefore$ S has a minimum value at $\alpha = \cot^{-1}\sqrt{2}$.

Hence the height of the tent $= r\cot\alpha = r\sqrt{2}$

$= \sqrt{2}$ times the radius of the base.

1(e) If d be the distance between the centres of two spheres of radii r_1 and r_2, Prove that the angle between them is $\cos^{-1}\left[(r_1^2+r_2^2-d^2)/2r_1r_2\right]$. Hence find the angle of intersection of the sphere $x^2+y^2+z^2-2x-4y-6z+10=0$ with the sphere, the extremities of whose diameter are $(1,2,-3)$ and $(5,0,1)$.

Sol'n: Let C_1, C_2 be the centres of the spheres and P be their point of intersection. Then the angle between the spheres is the angle between their radii C_1P and C_2P.

$\therefore$ In ΔC_1PC_2; $C_1P = r_1$, $C_2P = r_2$ and $C_1C_2 = d$.

$\therefore$ If θ be the required angle, then

$$\cos\theta = \cos\angle C_1PC_2 = \frac{C_1P^2+C_2P^2-C_1C_2^2}{2C_1P.C_2P} = \frac{r_1^2+r_2^2-d^2}{2r_1r_2}$$

Now for the second part, the given spheres are

$$x^2+y^2+z^2-2x-4y-6z+10=0 \quad \text{——(i)}$$

and $(x-1)(x-5)+(y-2)(y-0)+(z+3)(z-1)=0$

$$\Rightarrow x^2+y^2+z^2-6x-2y+2z+2=0 \quad \text{——(ii)}$$

Centre and radius of (i) are $(1,2,3)$ and 2

Centre and radius of (ii) are $(3,1,-1)$ and 3.

$\therefore$ Here $r_1=2$, $r_2=3$,

$$d^2 = \left[(3-1)^2+(1-2)^2+(-1-3)^2\right] = 21$$

The required angle $\cos^{-1}\left[\frac{r_1^2+r_2^2-d^2}{2r_1r_2}\right]$

$$= \cos^{-1}\left[\frac{4+9-21}{2.2.3}\right]$$

$$= \cos^{-1}(-2/3).$$

2(a) → (i) Determine whether or not $V = (3,9,-4,-2)$ in R^4 is a linear combination of $u_1 = (1,-2,0,3)$, $u_2 = (2,3,0,-1)$ and $u_3 = (2,-1,2,1)$, that is, whether or not $V \in \text{span}(u_1, u_2, u_3)$.

(ii) Find conditions on a, b and c so that $(a,b,c) \in R^3$ belongs to the space spanned by $u = (2,1,0)$, $v = (1,-1,2)$ and $w = (0,3,-4)$.

Sol^n. (i) Given $V = (3,9,-4,-2) \in \mathbb{R}^4$

$$\left.\begin{array}{l} u_1 = (1,-2,0,3),\ u_2 = (2,3,0,-1),\ u_3 = (2,-1,2,1) \end{array}\right\} \text{ --- (1)}$$

For V to be linear combination of u_1, u_2 & u_3.

Let $\exists$ scalars a, b, c such that

$$V = au_1 + bu_2 + cu_3$$

$$(3,9,-4,-2) = a(1,-2,0,3) + b(2,3,0,-1) + c(2,-1,2,1)$$

$$\Rightarrow a + 2b + 2c = 3 \text{ --- (2)}$$

$$-2a + 3b - c = 9 \text{ --- (3)}$$

$$2c = -4 \text{ --- (4)}$$

$$3a - b + c = -2 \text{ --- (5)}$$

from (4) $c = -2$

putting $c = -2$ in (2) & (3)

$$a + 2b = 7$$

$$-2a + 3b = 7$$

solving we get $b = 3$, $a = 1$

$\therefore \exists$ scalars $a = 1$, $b = 3$, $c = -2$

such that $V = u_1 + 3u_2 - 2u_3$

$$\therefore V \in \text{span}(u_1, u_2, u_3)$$

(ii) Given $u = (2,1,0)$, $v = (1,-1,2)$, $w = (0,3,-4)$ --- (1)

Now for $(a,b,c) \in \mathbb{R}^3$ to belong to span of (u, v, w)

we must have scalars x, y, z such that

$$(a, b, c) = xu + yv + zw \quad \text{--- (2)}$$

$$\Rightarrow (a, b, c) = x(2, 1, 0) + y(1, -1, 2) + z(0, 3, -4)$$

$$\Rightarrow 2x + y = a \quad \text{--- (3)}$$

$$x - y + 3z = b \quad \text{--- (4)}$$

$$2y - 4z = c \quad \text{--- (5)}$$

from (3) & (4)

$$3(x + z) = a + b \Rightarrow x + z = \frac{a+b}{3} \quad \text{--- (6)}$$

Also $2 \times (4) + (5)$ gives

$$2x + 2z = 2b + c \Rightarrow x + z = \frac{2b+c}{2} \quad \text{--- (7)}$$

from (6) & (7)

$$\frac{a+b}{3} = \frac{2b+c}{2}$$

$$\Rightarrow 2a + 2b = 6b + 3c$$

$$\Rightarrow 2a - 4b - 3c = 0$$

which is the condition of a, b & c for (a, b, c) to belong to span of (u, v, w).

2(b) → Determine whether the following matrices have the same row space:

$$A=\begin{bmatrix}1&1&5\\2&3&13\end{bmatrix},\quad B=\begin{bmatrix}1&-1&-2\\3&-2&-3\end{bmatrix},\quad C=\begin{bmatrix}1&-1&-1\\4&-3&-1\\3&-1&3\end{bmatrix}$$

Sol^n: Matrices have the row space if and only if their row canonical forms have the same non-zero rows; hence row reduce each matrix to row canonical form:

$$A=\begin{bmatrix}1&1&5\\2&3&13\end{bmatrix}\ R_2\to R_2-2R_1$$

$$=\begin{bmatrix}1&1&5\\0&1&3\end{bmatrix}\ R_1\to R_1-R_2$$

$$=\begin{bmatrix}1&0&2\\0&1&3\end{bmatrix}$$

$$B=\begin{bmatrix}1&-1&-2\\3&-2&-3\end{bmatrix}$$

$$=\begin{bmatrix}1&-1&-2\\0&1&3\end{bmatrix}\ R_2\to R_2-3R_1 \quad =\begin{bmatrix}1&0&1\\0&1&3\end{bmatrix}\ R_1\to R_1+R_2$$

$$C=\begin{bmatrix}1&-1&-1\\4&-3&-1\\3&-1&3\end{bmatrix}=\begin{bmatrix}1&-1&-1\\0&1&3\\0&2&6\end{bmatrix}\ \begin{matrix}R_2\to R_2-4R_1\\R_3\to R_3-3R_1\end{matrix}$$

$$=\begin{bmatrix}1&-1&-1\\0&1&3\\0&0&0\end{bmatrix}\ R_3-2R_2 \quad =\begin{bmatrix}1&0&2\\0&1&3\\0&0&0\end{bmatrix}\ R_1\to R_1+R_2$$

Since the non-zero rows of the reduced form of A and of the reduced form of C are the same, A and C have the same row space. On the other hand, the non-zero rows of the reduced form of B are not the same as the others and so B has a different row space.

2(c) Find the maximum and minimum values of $x^2+y^2+z^2$ subject to the conditions $\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}=1$ & $x+y-z=0$.

Sol'n: Let $f = x^2+y^2+z^2$

$$\phi_1 = \frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}-1=0$$

and $\phi_2 = x+y-z=0$

Consider a function F of independent variables x, y, z

where $F = x^2+y^2+z^2+\lambda_1\left(\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}-1\right)+\lambda_2(x+y-z)$

$$\therefore dF = \left(2x+\frac{x}{2}\lambda_1+\lambda_2\right)dx+\left(2y+\frac{2y}{5}\lambda_1+\lambda_2\right)dy$$
$$+\left(2z+\frac{2z}{25}\lambda_1-\lambda_2\right)dz.$$

$[dF = F_x dx + F_y dy + F_z dz]$

As x, y, z are independent variables, we get

$\frac{\partial F}{\partial x}=0 \Rightarrow 2x+\frac{x}{2}\lambda_1+\lambda_2=0$

$\frac{\partial F}{\partial y}=0 \Rightarrow 2y+\frac{2y}{5}\lambda_1+\lambda_2=0$

$\frac{\partial F}{\partial z}=0 \Rightarrow 2z+\frac{2y}{25}\lambda_1-\lambda_2=0$

$\therefore x = \frac{-2\lambda_2}{\lambda_1+4}, \quad y=\frac{-5\lambda_2}{2\lambda_1+10}, \quad z=\frac{25\lambda_2}{2\lambda_2+50}$ ——— (1)

Substituting in $x+y=z$, we get

$$\frac{-2\lambda_2}{\lambda_1+4}+\frac{(-5\lambda_2)}{5\lambda_1+10}-\frac{25\lambda_2}{2\lambda_2+50}=0$$

$$\lambda_2\left[\frac{2}{\lambda_1+4}+\frac{5}{5\lambda_1+10}+\frac{25}{2\lambda_2+50}\right]=0 \quad \text{——— (2)}$$

if $\lambda_2=0$ then from (1)

$x=0, y=0, z=0$, but $(x,y,z)=(0,0,0)$ doesnot satisfy the other condition of the constant.

$\therefore$ from (2),

$$17\lambda_1^2+245\lambda_1+750=0$$
$$\Rightarrow \lambda_1=-10, \quad \lambda_1=-75/17.$$

For $\lambda_1 = -10$ from ①

$$x = \frac{1}{3}\lambda_2,\ y = \frac{1}{2}\lambda_2,\ z = \frac{5}{6}\lambda_2 \quad \text{---③}$$

Now substituting ③ in $\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} = 1$

we get $\lambda_2^2\left[\frac{1}{36} + \frac{1}{20} + \frac{1}{36}\right] = 1$

$$\frac{19\lambda_2^2}{180} = 1 \Rightarrow \lambda_2^2 = \frac{180}{19} \text{ (or) } \lambda_2 = \pm 6\sqrt{5/19}$$

The corresponding stationary points are

$\left(2\sqrt{5/19},\ 3\sqrt{5/19},\ 5\sqrt{5/19}\right)$ and $\left(-2\sqrt{5/19},\ -3\sqrt{5/19},\ -5\sqrt{5/19}\right)$

The value of $x^2+y^2+z^2$ corresponding to these points is 10.

For $\lambda_1 = -75/17$

from ① $x = \frac{34}{7}\lambda_2,\ y = -\frac{17}{4}\lambda_2,\ z = \frac{17}{28}\lambda_2$ ---②

which on substitution in $\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} = 1$ give

$$\lambda_2 = \pm\frac{140}{17\sqrt{646}}$$

Substituting $\lambda_2 = \pm\frac{140}{17\sqrt{646}}$ in ②

Then the corresponding stationary points are

$\left(\frac{40}{\sqrt{646}},\ \frac{-35}{\sqrt{646}},\ \frac{5}{\sqrt{646}}\right)$ and $\left(\frac{-40}{\sqrt{646}},\ \frac{35}{\sqrt{646}},\ \frac{-5}{\sqrt{646}}\right)$

The value of $x^2+y^2+z^2$ corresponding to these points is $75/17$.

$\therefore$ The maximum value is 10 and the minimum value is $75/17$.

2(d) (i) A variable plane is at a constant distance p from the origin and meets the axes in A, B and C. Show that the locus of the centroid of the tetrahedron OABC is
$x^{-2}+y^{-2}+z^{-2}=16p^{-2}$.

(ii) If $x/1 = y/2 = z/3$ represent one of a set of three mutually llar generators of the cone $5yz-8zx-3xy=0$, find the equations of the other two.

Sol'n: (i) Let the equation of the variable plane be

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \quad \text{——} \quad (1)$$

It is given that this plane is at a distance p from $(0,0,0)$.

$$p=\frac{1}{\sqrt{\left(\frac{1}{a}\right)^2+\left(\frac{1}{b}\right)^2+\left(\frac{1}{c}\right)^2}}$$

$$\Rightarrow \frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \quad \text{——} \quad (2)$$

Also the plane (1) meets the axes in A, B and C. So the coordinates of O, A, B and C are $(0,0,0)$, $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$ respectively.

Let (x,y,z) be the centroid of the tetrahedron OABC, then

$$x=\tfrac{1}{4}(0+a+0+0)=a/4$$

Similarly $y=b/4$ and $z=c/4$.

$$\Rightarrow a=4x,\quad b=4y,\quad c=4z$$

Substituting these values of a, b and c in (2), we get

$$\frac{1}{p^2}=\frac{1}{16x^2}+\frac{1}{16y^2}+\frac{1}{16z^2}$$

$$\Rightarrow x^{-2}+y^{-2}+z^{-2}=16p^{-2}$$

which is the required locus.

2(d)(ii) Sol^n: If $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ is one of the three mutually $\perp$lar generators, then it is normal to the plane through the vertex cutting the cone in two $\perp$lar generators and therefore the equation of the plane is

$$x+2y+3z=0 \quad \text{——} \ (1).$$

Now we are to find the lines of intersection of this plane and the given cone and let one of the lines of intersection be

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$$

Then we have $l+2m+3n=0$ and $5mn-8nl-3lm=0$ —— (2)

Eliminating l between these we get

$$5mn-(8n+3m)[-(2m+3n)]=0$$

$$\Rightarrow 24n^2+30mn+6m^2=0$$

$$\Rightarrow m^2+5mn+4n^2=0$$

$$\Rightarrow (m+n)(m+4n)=0$$

when $m=-n$, from (2) we get $l+n=0 \Rightarrow l=-n$

$$\therefore \frac{l}{1}=\frac{m}{1}=\frac{n}{(-1)} \quad \text{——} \ (3)$$

when $m=-4n$ from (2) we get $l-5n=0 \Rightarrow l=5n$

$$\therefore \frac{l}{5}=\frac{m}{(-4)}=\frac{n}{1} \quad \text{——} \ (4)$$

Hence from (3) and (4) the other two generators are $\frac{x}{1}=\frac{y}{1}=\frac{z}{(-1)}$ and $\frac{x}{5}=\frac{y}{-4}=\frac{z}{1}$

and evidently these are $\perp$lar as $1.5+1(-4)+(-1)1=0$ and also each one of them is $\perp$lar to the given generator.

3(a) (i) Let P_n denote the vectorspace of all real polynomials of degree atmost n and $T: P_2 \to P_3$ be a linear transformation given by $T(p(x)) = \int_0^x p(t)\,dt$, $p(x) \in P_2$. Find the matrix of T with respect to the bases $\{1, x, x^2\}$ and $\{1, x, 1+x^2, 1+x^3\}$ of P_2 and P_3 respectively. Also, find the nullspace of T.

(ii) Let V be an n-dimensional vectorspace and $T: V \to V$ be an invertible linear operator. If $\beta = \{x_1, x_2, \ldots, x_n\}$ is a basis of V, show that $B' = \{Tx_1, Tx_2, \ldots, Tx_n\}$ is also a basis of V.

Sol^n: (i) Given $T(p(x)) = \int_0^x p(t)\,dt$; $p(x) \in P_2$.

basis for P_2 is $\{1, x, x^2\}$ and

basis for P_3 is $\{1, x, 1+x^2, 1+x^3\}$

Now

$$T(1) = \int_0^x 1\,dt = x = 0\cdot 1 + 1\cdot x + 0(1+x^2) + 0(1+x^3)$$

$$T(x) = \int_0^x t\,dt = \frac{x^2}{2} = -\tfrac{1}{2}\cdot 1 + 0\cdot x + \tfrac{1}{2}(1+x^2) + 0(1+x^3)$$

$$T(x^2) = \int_0^x t^2\,dt = \frac{x^3}{3} = -\tfrac{1}{3}\cdot 1 + 0\cdot x + 0\cdot(1+x^2) + \tfrac{1}{3}(1+x^3)$$

$\therefore$ Matrix of T w.r.t bases β_1 and β_2 is

$$[T : \beta_1 \beta_2] = \begin{bmatrix} 0 & -\tfrac{1}{2} & -\tfrac{1}{3} \\ 1 & 0 & 0 \\ 0 & \tfrac{1}{2} & 0 \\ 0 & 0 & \tfrac{1}{3} \end{bmatrix}$$

Null space of T will be given by

$\int_0^x p(t)\,dt = 0$ i.e. if $p(x) = a_0 + a_1 x + a_2 x^2$

$$\begin{bmatrix} 0 & -\frac{1}{2} & -\frac{1}{3} \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow a_0 = 0;\ a_1 = 0;\ a_2 = 0.$

$\therefore a_0 = a_1 = a_2 = 0$

$\therefore P(x) = 0$

$\therefore$ Null space of T contains only '0' single element $\{0\}$.

(ii) Since it is given that $T : V \to V$ is invertible. so, 'T' must also be one-one and onto. Also T is linear.

So $T(\alpha) = 0 \Leftrightarrow \alpha = 0$. —— (1)

Again V is a vectorspace of dimension 'n', so any Linearly independent set of dimension 'n' can form its basis. —— (2)

Consider set $B' = \{TX_1, TX_2, \ldots TX_n\}$

Let $\alpha_1, \alpha_2 \ldots \alpha_n$ be n scalars such that

$$\alpha_1 TX_1 + \alpha_2 TX_2 + \cdots + \alpha_n TX_n = 0 \quad \text{—— (3)}$$

by property of linear transformation (3) becomes

$$T(\alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_n X_n) = 0$$

from (1) $\alpha_1 X_1 + \alpha_2 X_2 + \cdots + \alpha_n X_n = 0$

$B(X_1, X_2 \ldots X_n)$ forms a basis of V. So must be LI.

$$\alpha_1 = \alpha_2 = \cdots = \alpha_n = 0$$

$TX_1, TX_2 \ldots TX_n$ are LI.

So, from (2) set B' forms a basis for V.

3(b) (i) Determine $\lim \left(2-\frac{x}{a}\right)^{\tan\frac{\pi x}{2a}}$ as $x \to a$.

Sol^n: Let $y = \left(2-\frac{x}{a}\right)^{\tan\frac{\pi x}{2a}}$

$$\log y = \tan\frac{\pi x}{2a} \log\left(2-\frac{x}{a}\right)$$

$$\Rightarrow \lim_{x\to a} \log y = \lim_{x\to a} \tan\frac{\pi x}{2a} \log\left(2-\frac{x}{a}\right)$$

$$= \lim_{x\to a} \frac{\log\left(2-\frac{x}{a}\right)}{\cot\frac{\pi x}{2a}}$$

$$= \lim_{x\to a} \frac{-\frac{1}{a}\cdot\frac{1}{2-\frac{x}{a}}}{-\frac{\pi}{2a}\operatorname{cosec}^2\frac{\pi x}{2a}}$$

$$= \frac{2}{\pi}$$

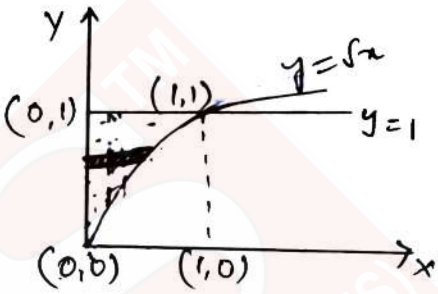
$$\Rightarrow \log \lim_{x\to a} y = \frac{2}{\pi}$$

$$\Rightarrow \lim_{x\to a} y = e^{2/\pi}$$

3(b)(ii) Evaluate the integral $\int_0^1 \int_{\sqrt{x}}^1 e^{x/y}\,dx\,dy$, by changing the order of integration.

sol'n: Given Curves are $y=\sqrt{x}$; $y=1$

$\Rightarrow y^2 = x$; $y=1$



Taking the limits of

y from 0 to 1

x from 0 to y^2

$$\therefore \int_{y=0}^{1}\int_{x=0}^{y^2} e^{x/y}\,dx\,dy = \int_{y=0}^{1}\left[y e^{x/y}\right]_{x=0}^{y^2} dy$$

$$= \int_{y=0}^{1} y(e^{y}-1)\,dy$$

$$= \int_0^1 (ye^{y} - y)\,dy$$

$$= \left[ye^{y} - e^{y} - \frac{y^2}{2}\right]_0^1$$

$$= \frac{1}{2}$$

3(C). The normal at a variable point P of the ellipsoid $\Sigma (x^2/a^2)=1$ meets the xy-plane in G_3 and G_3Q is drawn parallel to z-axis and equal to G_3P. Prove that the locus of Q is given by $\frac{x^2}{(a^2-c^2)}+\frac{y^2}{(b^2-c^2)}+\frac{z^2}{c^2}=1$.

Find the locus of R, if OR is drawn from the centre equal and parallel to G_3P.

Sol'n: Let P be (α,β,γ), then the equations of the normal to the given ellipsoid at $P(\alpha,\beta,\gamma)$ are

$$\frac{x-\alpha}{(p\alpha/a^2)}=\frac{y-\beta}{(p\beta/b^2)}=\frac{z-\gamma}{(p\gamma/c^2)}=r\ (\text{say}) \quad \text{--- (1)}$$

where $$\frac{1}{p^2}=\frac{\alpha^2}{a^4}+\frac{\beta^2}{b^4}+\frac{\gamma^2}{c^4} \quad \text{--- (2)}.$$

$\therefore$ The coordinates of any point Q on the normal (1) are

$$\left(\alpha+\frac{p\alpha}{a^2}r,\ \beta+\frac{p\beta}{b^2}r,\ \gamma+\frac{p\gamma}{c^2}r\right),$$

where r is the distance of Q from P.

We know that $PG = -c^2/p$, so substituting $-c^2/p$ for r, the coordinates of G_3 are

$$\left(\alpha-\frac{c^2\alpha}{a^2},\ \beta-\frac{c^2\beta}{b^2},\ 0\right)$$

[Let the normal at $P(\alpha,\beta,\gamma)$ meet the coordinate planes viz yz, zx & xy planes at G_1, G_2 & G_3, then putting $x=0$, $y=0$ & $z=0$ in succession in the above equation (1) of the normal we have respectively $PG_1=-a^2/p$, $PG_2=-b^2/p$ & $PG_3=-c^2/p$]

$\therefore$ The equations of the line G_3Q, which passes through G_3 and is parallel to z-axis are.

$$\frac{x-\{\alpha-(c^2\alpha/a^2)\}}{0}=\frac{y-\{\beta-(c^2\beta/b^2)\}}{0}=\frac{z-0}{1}=r_1\ (\text{say})$$

where r_1 denotes the distance of any point on this line G_3Q from G_3.

$\therefore G_3Q = G_3P$, so putting $r_1 = G_3P = -c^2/p$ we have the coordinates of Q as $\left[\alpha - \frac{c^2\alpha}{a^2}, \beta - \frac{c^2\beta}{b^2}, -\frac{c^2}{p}\right]$

$\therefore$ If Q is (x_1, y_1, z_1), then $x_1 = \alpha - \frac{c^2\alpha}{a^2}$, $y_1 = \beta - \frac{c^2\beta}{b^2}$, $z_1 = \frac{-c^2}{p}$

$$\Rightarrow \alpha = \frac{a^2 x_1}{a^2 - c^2}, \quad \beta = \frac{b^2 y_1}{b^2 - c^2}, \quad \frac{z_1}{c} = -\frac{c_1}{p} \quad \text{——— (3)}$$

Also as $P(\alpha, \beta, \gamma)$ lies on the given ellipsoid, so we have

$$\alpha^2/a^2 + \beta^2/b^2 + \gamma^2/c^2 = 1 \quad \text{——— (4)}$$

The required locus of Q is obtained by eliminating α, β, γ between the relations given in (3) & (4) and generalising x_1, y_1, z_1.

Now from (3), we get

$$\frac{z_1^2}{c^2} = \frac{c^2}{p} = c^2\left[\frac{\alpha^2}{a^4} + \frac{\beta^2}{b^4} + \frac{\gamma^2}{c^4}\right], \text{ from (4)}$$

$$\Rightarrow \frac{z_1^2}{c^2} = \frac{c^2\alpha^2}{a^4} + \frac{c^2\beta^2}{b^4} + \frac{c^2\gamma^2}{c^2} \Rightarrow \frac{\gamma^2}{c^2} = \frac{z_1^2}{c^2} - \frac{c^2\alpha^2}{a^4} - \frac{c^2\beta^2}{b^4} \text{ — (5)}$$

Substituting the values of α, β and γ from (3) and (5) in (4), we get

$$\frac{a^2x_1^2}{(a^2-c^2)^2} + \frac{b^2y_1^2}{(b^2-c^2)^2} + \left(\frac{z_1^2}{c^2} - \frac{c^2\alpha^2}{a^4} - \frac{c^2\beta^2}{b^4}\right) = 1$$

$$\Rightarrow \frac{a^2x_1^2}{(a^2-c^2)^2} + \frac{b^2y_1^2}{(b^2-c^2)^2} + \frac{z_1^2}{c^2} - \frac{c^2x_1^2}{(a^2-c^2)^2} - \frac{c^2y_1^2}{(b^2-c^2)^2} = 1, \text{ from (3)}$$

$$\Rightarrow \frac{x_1^2}{(a^2-c^2)^2}(a^2-c^2) + \frac{y_1^2}{(b^2-c^2)^2}(b^2-c^2) + \frac{z_1^2}{c^2} = 1$$

$$\Rightarrow \frac{x_1^2}{(a^2-c^2)} + \frac{y_1^2}{(b^2-c^2)} + \frac{z_1^2}{c^2} = 1$$

$\therefore$ The locus of $Q(x_1, y_1, z_1)$ is $\frac{x^2}{(a^2-c^2)} + \frac{y^2}{(b^2-c^2)} + \frac{z^2}{c^2} = 1$.

Hence proved.

Again the equations of the line OR, drawn through the centre O(0,0,0) of the given ellipsoid and parallel to the normal through P and equal to $PG_3 = -c^2/p$

are $\dfrac{x-0}{(p\alpha/a^2)} = \dfrac{y-0}{(p\beta/b^2)} = \dfrac{z-0}{(p\gamma/c^2)} = \dfrac{-c^2}{p}$.

$\therefore$ The coordinates of R are $\left(-\dfrac{c^2\alpha}{a^2}, -\dfrac{c^2\beta}{b^2}, -\gamma\right)$.

If R is (x_2, y_2, z_2) then $x_2 = -c^2\alpha/a^2$, $y_2 = -c^2\beta/b^2$, $z_2 = -\gamma$

$\Rightarrow \alpha = -a^2x_2/c^2$, $\beta = -b^2y_2/c^2$, $\gamma = -z_2$

Substituting these values in (4) and generalising (x_2, y_2, z_2), we get

$$\frac{a^2x^2}{c^4} + \frac{b^2y^2}{c^4} + \frac{z^2}{c^2} = 1$$

$$\Rightarrow a^2x^2 + b^2y^2 + c^2z^2 = c^4,$$

which is the required locus of R.

4(a)(i) (A) Let $A = \begin{bmatrix} 2 & 2 \\ 1 & 3 \end{bmatrix}$. Is a diagonalizable? If Yes, find P such that $P^{-1}AP$ is diagonal.

(B) If interchanging the eigenvectors of P, does P still diagonalize A?

Sol^n: we have $|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)(3-\lambda) - 2 = 0$$

$$\Rightarrow \lambda^2 - 5\lambda + 4 = 0$$

$$\Rightarrow \lambda = 1, 4$$

which are the eigenvalues of A.

Let us find the eigen vector corresponding to $\lambda = 1$.

i.e, $(A-\lambda I)X = 0$

$$\Rightarrow \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} X = 0$$

$$\Rightarrow x + 2y = 0$$

$\therefore X_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ is a non-zero solution of the system and so is an eigenvector of A corresponding to $\lambda = 1$.

Let us find the eigen vector corresponding to $\lambda = 4$.

i.e, $(A-\lambda I)X = 0$

$$\Rightarrow \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix} X = 0$$

$$\Rightarrow x - y = 0$$

$\therefore X_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is a non-zero solution and so is an eigenvector of A corresponding to $\lambda = 4$.

Since A has two independent eigen vectors.

A is digonalizable.

Let $P = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$

Then $P^{-1}AP = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$

(B) If interchanging the eigenvectors of P i.e. $P = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$, then P still diagonalize A.

However, now $P^{-1}AP = \begin{bmatrix} 4 & 0 \\ 0 & 1 \end{bmatrix}$.

In otherwords, the order of the eigen values in $P^{-1}AP$ corresponds to the eigenvectors in P.

4(a)(ii) Show that no skew-symmetric matrix can be of rank 1.

Sol'n: Let $A = \begin{bmatrix} 0 & h & g & l \\ -h & 0 & f & m \\ -g & -f & 0 & n \\ -l & -m & -n & 0 \end{bmatrix}$ be an 4x4 skew-symmetric matrix.

If h, g, l, m, n are all equal to zero, the matrix A will be of rank zero. If at least one of these elements, say g is not equal to zero, then atleast on 2-rowed minor of the matrix A, i.e. the minor $\begin{vmatrix} 0 & g \\ -g & 0 \end{vmatrix}$ is not equal to zero as its value is g^2 which is not equal to zero.

$\therefore$ the rank of the matrix A is ≥ 2.

Thus in either case the rank of the matrix A is not equal to one.

Note: The method of proof can be given in the case of a skew symmetric matrix of any order.

4(b). If $z = xyf(y/x)$, show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$; and if z is a constant, then $\frac{f'(y/x)}{f(y/x)} = \frac{x\left(y + x\frac{dy}{dx}\right)}{y\left(y - x\frac{dy}{dx}\right)}$.

Sol'n: Given that $z = xyf(y/x)$ —— (1)

Differentiating equation (1) partially w.r.t x and y respectively, we get

$$\frac{\partial z}{\partial x} = yf(y/x) + xyf'(y/x)(-y/x^2)$$

$$= yf(y/x) - \frac{y^2}{x}f'(y/x) \quad \text{—— (2)}$$

& $$\frac{\partial z}{\partial y} = xf(y/x) + xyf'(y/x)\left(\frac{1}{x}\right)$$

$$= xf(y/x) + yf'(y/x) \quad \text{—— (3)}$$

Multiply (2) by x & (3) by y, we have

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = xyf(y/x) - \cancel{y^2f'(y/x)} + xyf(y/x) + \cancel{y^2f'(y/x)}$$

$$= 2xyf(y/x)$$

$$= 2z$$

$$\therefore x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z.$$

If z is constant:

$$z = xyf(y/x)$$

Taking logarithm on bothsides, we have

$$\log xyf(y/x) = \log z$$

$$\Rightarrow \log x + \log y + \log f(y/x) = \log z$$

Differentiating w.r.t x, we get

$$\frac{1}{x}+\frac{1}{y}\frac{dy}{dx}+\frac{1}{f(y/x)}f'(y/x)\left(-\frac{y}{x^2}+\frac{1}{x}\frac{dy}{dx}\right)=0$$

$$\Rightarrow \frac{f'(y/x)}{f(y/x)}\left[\frac{-y+x\frac{dy}{dx}}{x^2}\right]+\left[\frac{y+x\frac{dy}{dx}}{xy}\right]=0$$

$$\Rightarrow \frac{f'(y/x)}{f(y/x)}=\frac{-\left[y+x\frac{dy}{dx}\right]/xy}{\left[-y+x\frac{dy}{dx}\right]/x^2}$$

$$=\frac{x\left[y+x\frac{dy}{dx}\right]}{\left[y-x\frac{dy}{dx}\right]y}$$

$$=\frac{x\left[y+x\frac{dy}{dx}\right]}{y\left[y-x\frac{dy}{dx}\right]}$$

4(c) → Show that the function

$$f(x,y)=\begin{cases} x^2y/(x^2+y^2), & \text{when } x^2+y^2\neq 0 \\ 0, & \text{when } x^2+y^2=0 \end{cases}$$

is continuous but not differentiable at (0,0)

Sol'n Given that $f(x,y)=\begin{cases} x^2y/(x^2+y^2), & \text{when } x^2+y^2\neq 0 \\ 0, & \text{when } x^2+y^2=0 \end{cases}$

putting $x=r\cos\theta$, $y=r\sin\theta$; we get

$$|f(x,y)-f(0,0)|=\left|\frac{r^2\cos^2\theta\cdot r\sin\theta}{r^2}-0\right|$$

$$=r|\cos\theta||\cos\theta||\sin\theta|$$

$$\leq r=\sqrt{x^2+y^2}$$

Let $\epsilon>0$ be given, choose $\delta=\epsilon$. Then

$$|f(x,y)-f(0,0)|<\epsilon \text{ if } \sqrt{x^2+y^2}<\delta$$

Hence f is continuous at the origin.

Now $f_x(0,0)=\lim\limits_{h\to0}[f(h,0)-f(0,0)]/h=\lim\limits_{h\to0}\frac{0-0}{h}=0$.

Similarly $f_y(0,0)=0$

Let, if possible, f be differentiable at (0,0). Then

$$f(h,k)-f(0,0)=Ah+Bk+\sqrt{h^2+k^2}\,g(h,k)$$

where $A=f_x(0,0)$, $B=f_y(0,0)$ and $g(h,k)\to0$ as $(h,k)\to(0,0)$ —— (1)

$$\therefore \frac{h^2k}{h^2+k^2}=\sqrt{h^2+k^2}\,g(h,k)\Rightarrow g(h,k)=\frac{h^2k}{(h^2+k^2)^{3/2}}$$

Now $\lim\limits_{h\to0} g(h,mh)=\frac{m}{(1+m^2)^{3/2}}$ $(k=mh)$

$$\therefore \lim_{(h,k)\to(0,0)} g(h,k)=\frac{m}{(1+m^2)^{3/2}}$$

which depends on m and so the limit does not exist. This contradicts (1). Hence f is not differentiable at (0,0).

4(d) Show that the feet of the normals from the point (α, β, γ) on the paraboloid $x^2+y^2=2az$ lie on a sphere.

Sol^n: Let (x_1, y_1, z_1) be any point on the given paraboloid, then $x_1^2+y_1^2 = 2az_1$ ——— ①.

The tangent plane to this paraboloid at (x_1, y_1, z_1) is

$$xx_1+yy_1 = a(z+z_1) \Rightarrow xx_1+yy_1-az = az_1$$

$\therefore$ The equation of the normal to the given paraboloid at (x_1, y_1, z_1) i.e, the line through (x_1, y_1, z_1) at right angles to the above tangent plane is

$$\frac{x-x_1}{x_1} = \frac{y-y_1}{y_1} = \frac{z-z_1}{-a}$$

If this normal passes through the fixed point (α, β, γ) then we have

$$\frac{\alpha-x_1}{x_1} = \frac{\beta-y_1}{y_1} = \frac{\gamma-z_1}{-a} \quad \text{——— ②}$$

$$\Rightarrow \frac{\alpha-x_1}{x_1} = \frac{\beta-y_1}{y_1} = \frac{\gamma-z_1}{-a} = \frac{x_1(\alpha-x_1)+y_1(\beta-y_1)}{x(x_1)+y_1(y_1)}$$

$$= \frac{z_1(\gamma-z_1)}{z_1(-a)}$$

from the last two fractions we have $\dfrac{\alpha x_1-x_1^2+\beta y_1-y_1^2}{x_1^2+y_1^2} = \dfrac{\gamma z_1-z_1^2}{-az_1}$

$$\Rightarrow \frac{(\alpha x_1+\beta y_1)-(x_1^2+y_1^2)}{2az_1} = \frac{2(\gamma z_1-z_1^2)}{-2az_1} \quad \text{from ①}$$

$$\Rightarrow x_1^2+y_1^2-(\alpha x_1+\beta y_1)-2\gamma z_1+2z_1^2 = 0$$

$$\Rightarrow x_1^2+y_1^2+2z_1^2-2\gamma z_1 = \alpha x_1+\beta y_1 \quad \text{——— ③}$$

Also from ② we have, $\dfrac{\alpha}{x_1}-1 = \dfrac{\beta}{y_1}-1$

$$\Rightarrow \alpha y_1 = \beta x_1 \quad \text{——— ④}$$

Now from ③ we have $x_1^2+y_1^2+2z_1^2-2\gamma z_1 = (\alpha\beta x_1+\beta^2 y_1)/\beta$

$= (\alpha^2 y_1+\beta^2 y_1)/\beta$ from ④

$\Rightarrow x_1^2+y_1^2+2z_1^2-2\gamma z_1 = (\alpha^2+\beta^2)y_1/\beta$ —— ⑤

adding ① and ⑤ we get

$2x_1^2+2y_1^2+2z_1^2-2\gamma z_1 = 2az_1+[\{2(\alpha^2+\beta^2)y_1\}/2\beta]$

$\Rightarrow x_1^2+y_1^2+z_1^2-(\gamma+a)z - \{(\alpha^2+\beta^2)y_1/(2\beta)\} = 0$

$\therefore$ The locus of the foot (x_1, y_1, z_1) of the normal form (α, β, γ) is

$x^2+y^2+z^2-(\gamma+a)z-\{(\alpha^2+\beta^2)/2\beta\}y = 0.$

Hence proved.

5(a) → Find the differential equation of the family of circles $x^2+y^2+2cx+2c^2-1=0$ (c arbitrary constant). Determine singular solution of the differential equation.

Sol^n: Given that $x^2+y^2+2cx+2c^2-1=0$ —①

Differentiating ① w.r.t x, we get

$2x+2yy'+2c=0 \Rightarrow c=-x-yy'$ —②

∴ from ① & ②, we have

$x^2+y^2+2(-x-yy')x+2(-x-yy')^2-1=0$

$\Rightarrow x^2+y^2-2x^2-2xyy'+2x^2+2y^2y'^2+4xyy'-1=0$

$\Rightarrow x^2+y^2+2xyy'+2y^2y'^2-1=0$

$\Rightarrow x^2+y^2+2xyp+2y^2p^2-1=0$ (where $y'=\frac{dy}{dx}=p$)

i.e. $2y^2p^2+2xyp+x^2+y^2-1=0$

Here p-discriminant relation is

$(2xy)^2-4(2y^2)(x^2+y^2-1)=0$

$\Rightarrow x^2y^2-2y^2(x^2+y^2-1)=0$

$\Rightarrow y^2[x^2-2(x^2+y^2-1)]=0$

$\Rightarrow y^2[x^2+2y^2-2]=0$ —③

from the eq^n ①, c-discriminant relation is

$(2x)^2-4(2)(x^2+y^2-1)=0$

$\Rightarrow x^2-2(x^2+y^2-1)=0$

$\Rightarrow x^2+2y^2-2=0$ —④

∴ from ③ & ④, we conclude that $x^2+2y^2-2=0$ is the only singular solution since it occurs once both in p and c-discriminant relations and also satisfies ①

5(b) Solve $(D^2+1)y = x^2\sin 2x$

Sol'n: Here the auxiliary equation is $D^2+1=0$ so that $D=\pm i$.

$\therefore$ C.F $= c_1\cos x + c_2\sin x$; c_1, c_2 being arbitrary constants.

$$P.I = \frac{1}{D^2+1}x^2\sin 2x$$

$$= I.P \text{ of } \frac{1}{D^2+1}x^2e^{2ix}, \text{ where I.P stands for imaginary part.}$$

$$= I.P \text{ of } e^{2ix}\frac{1}{(D+2i)^2+1}x^2 = IP \text{ of } e^{2ix}\frac{1}{D^2+4iD-3}x^2, \text{ as } i^2=-1.$$

$$= I.P \text{ of } \frac{e^{2ix}}{-3}\frac{1}{\{1-(4iD+D^2)/3\}}x^2$$

$$= IP \text{ of } \frac{e^{2ix}}{-3}\left[1-\left(\frac{4iD}{3}+\frac{D^2}{3}\right)\right]^{-1}x^2$$

$$= I.P \text{ of } \frac{e^{2ix}}{-3}\left[1+\left(\frac{4iD}{3}+\frac{D^2}{3}\right)+\left(\frac{4iD}{3}+\frac{D^2}{3}\right)^2+\cdots\right]x^2$$

$$= IP \text{ of } \frac{e^{2ix}}{-3}\left[1+\frac{4iD}{3}+\frac{D^2}{3}-\frac{16D^2}{9}+\cdots\right]x^2$$

$$= I.P \text{ of } \frac{e^{2ix}}{-3}\left[1+\frac{4iD}{3}-\frac{13D^2}{9}+\cdots\right]x^2$$

$$= I.P \text{ of } \frac{e^{2ix}}{-3}\left[x^2+\left(\frac{4i}{3}\times 2x\right)-\left(\frac{13}{9}\times 2\right)\right]$$

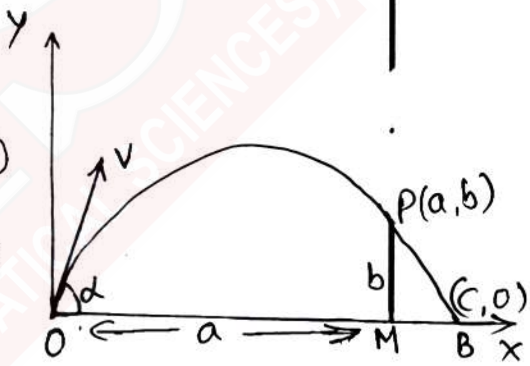
$$= I.P \text{ of } \left(-\frac{1}{3}\right)(\cos 2x+i\sin 2x)\left\{x^2+\left(\frac{8}{3}\right)ix-\frac{26}{9}\right\}$$

$$= \left(-\frac{1}{3}\right)\left[\left(x^2-\frac{26}{9}\right)\sin 2x+\frac{8}{3}x\cos 2x\right]$$

$\therefore$ solution is $y = c_1\cos x + c_2\sin x - \frac{1}{3}\left[\left(x^2-\frac{26}{9}\right)\sin 2x+\frac{8}{3}x\cos 2x\right]$.

5(c) A particle just clears a wall of height b at a distance a and strikes the ground at a distance c from the point of projection. Prove that the angle of projection is $\tan^{-1}\left\{\frac{bc}{a(c-a)}\right\}$, and the velocity of projection V is given by $\frac{2V^2}{g} = \frac{a^2(c-a)^2+b^2c^2}{ab(c-a)}$.

Solⁿ: Let the particle be projected from O with a velocity V at an angle α to the horizontal. Take the horizontal and vertical lines OX and OY in the plane of projection as the coordinate axes.



The equation of the trajectory is

$$y = x\tan\alpha - \frac{1}{2}g\frac{x^2}{V^2\cos^2\alpha} \quad \text{—(1)}$$

The particle just clears the wall PM of height b at a distance a from O and strikes the ground at the point B at a distance c from O. Thus both the points (a,b) and $(c,0)$ lie on the curve (1).

Therefore $$b = a\tan\alpha - \frac{1}{2}g\frac{a^2}{V^2\cos^2\alpha} \quad \text{—(2)}$$

and $$0 = c\tan\alpha - \frac{1}{2}g\frac{c^2}{V^2\cos^2\alpha} \quad \text{—(3)}$$

To eliminate V^2, we multiply (2) by c^2 and (3) by a^2 and subtract. Thus we get

$$bc^2 = ac^2\tan\alpha - a^2c\tan\alpha \Rightarrow bc^2 = ac\tan\alpha(c-a)$$

$$\Rightarrow \tan\alpha = \frac{bc}{a(c-a)} \quad \text{—(4)}$$

Now from (3), $$\frac{2V^2}{g} = \frac{c\sec^2\alpha}{\tan\alpha} = \frac{c(1+\tan^2\alpha)}{\tan\alpha}$$

Substituting the value of $\tan\alpha$ from (4), we have

$$\frac{2V^2}{g} = \frac{c\left[1+\{b^2c^2/a^2(c-a)^2\}\right]}{bc/\{a(c-a)\}} = \frac{a^2(c-a)^2+b^2c^2}{ab(c-a)}.$$

5(d). Find the directional derivative of $f = x^2yz^3$ along $x = e^{-t}$, $y = 1+2\sin t$, $z = t-\cos t$ at $t=0$.

Sol^n: Given that $f = x^2yz^3$

then $\text{grad} f = \frac{\partial f}{\partial x} i + \frac{\partial f}{\partial y} j + \frac{\partial f}{\partial z} k$

$= 2xyz^3 i + x^2z^3 j + 3x^2yz^2 k$ —(1)

At the point $t=0$

$(x, y, z) = (e^{-t}, 1+2\sin t, t-\cos t) = (1, 1, -1)$

$\therefore$ (1) $\Rightarrow$ $\text{grad} f\Big|_{\text{at } (1,1,-1)} = -2i - j + 3k.$

and $\vec{a} = xi + yj + zk.$

$= e^{-t} i + (1+2\sin t) j + (t-\cos t) k.$

$= i + j - k$, at $t=0$

$\therefore \hat{a} = \frac{\vec{a}}{|\vec{a}|} = \frac{i+j-k}{\sqrt{3}}$ which is unit vector in the direction of $\vec{a}$.

$\therefore$ Directional derivative of f

$= \nabla f \cdot \hat{a}$

$= (-2i - j + 3k) \cdot \left(\frac{i+j-k}{\sqrt{3}}\right)$

$= -\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}} - \frac{3}{\sqrt{3}}$

$= \frac{-6}{\sqrt{3}}$

$= -2\sqrt{3}$

5(e) → Apply Green's theorem in the plane to evaluate $\int_C [(2x^2-y^2)dx + (x^2+y^2)dy]$, where C is the boundary of the surface enclosed by the x-axis & the semicircle $y=(1-x^2)^{1/2}$.

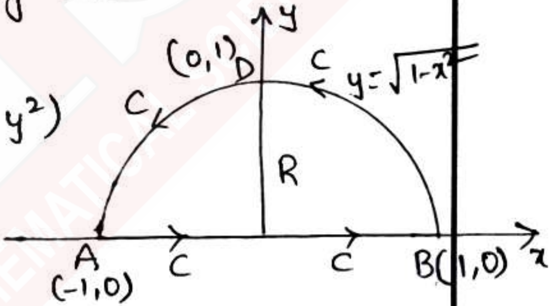
Sol'n: Here C is the closed curve traversed in the +ve direction by the straight line AOB and the semi-circle BDA. Also R is the region bounded by this curve C.

we have $\int_C (2x^2-y^2)dx + (x^2+y^2)dy$

$$= \int_C M\,dx + N\,dy$$

where $M = 2x^2-y^2$ & $N = x^2+y^2$

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy, \text{ by Green's theorem.}$$



$$= \iint_R \frac{\partial}{\partial x}(x^2+y^2) - \frac{\partial}{\partial y}(2x^2-y^2)$$

$$= \iint_R (2x+2y)\,dx\,dy$$

$$= \int_{x=-1}^{1}\int_{y=0}^{\sqrt{1-x^2}} 2(x+y)\,dx\,dy,$$

Since for the region R, y varies from 0 to $\sqrt{1-x^2}$ and x varies from -1 to 1.

$$= 2\int_{x=-1}^{1} \left[xy + \frac{y^2}{2}\right]_{y=0}^{\sqrt{1-x^2}} dx$$

$$= 2\int_{-1}^{1} \left[x\sqrt{1-x^2} + \frac{1}{2}(1-x^2)\right]dx$$

$$= 2\int_0^1 (1-x^2)\,dx \qquad \left[\because \int_{-1}^{1} x\sqrt{1-x^2}\,dx = 0\right)$$

$$= 2\left[x - \frac{x^3}{3}\right]_0^1$$

$$= 2\left(1-\frac{1}{3}\right) = 4/3.$$

6(a) (i) obtain Laplace Inverse transform of

$$\left\{\ln\left(1+\frac{1}{s^2}\right)+\frac{s}{s^2+25}\right\}e^{-\pi s}$$

(ii) Solve $(1+y^2)+\left(x-e^{-\tan^{-1}y}\right)\frac{dy}{dx}=0$

Sol'n: (i) Let $f(s)=\frac{s}{s^2+25}$

$f(t)=L^{-1}(f(s))=L^{-1}\left\{\frac{s}{s^2+25}\right\}=\cos 5t.$

$\therefore L^{-1}\left\{e^{-s\pi}f(s)\right\}=\begin{cases}f(t-\pi) & t>\pi\\ 0 & t<0\end{cases}$ by 2nd shifting theorem

$L^{-1}\left\{e^{-s\pi}\frac{s}{s^2+25}\right\}=f(t-\pi)H(t-\pi)$

$=\cos 5(t-\pi)H(t-\pi)=-H(t-\pi)\cos 5t.$

$L^{-1}(f(s))=L^{-1}\left[\log(s^2+1)-2\log s\right]=f(t)$

$\Rightarrow L(f(t))=\log(1+s^2)-2\log s$

$L[tf(t)]=\frac{-d}{ds}\left[\log(1+s^2)-2\log s\right]$

$=\frac{-2s}{1+s^2}+\frac{2}{s}$

$t.f(t)=L^{-1}\left[\frac{-2s}{s^2+1}+\frac{2}{s}\right]$

$=-2\cos t+2$

$\Rightarrow f(t)=\frac{-2\cos t}{t}+\frac{2}{t}$

$\therefore L^{-1}\left[\log\left(1+\frac{1}{s^2}\right)e^{-s\pi}\right\}=\begin{cases}f(t-\pi), & t>\pi\\ 0 & t<\pi\end{cases}$

$=\begin{cases}\frac{2-2\cos(t-\pi)}{t-\pi} & t>\pi\\ 0 & t<\pi\end{cases}$

$=\frac{2+2\cos t}{t-\pi}H(t-\pi)$

where $H(t-\pi)$ is the Heaviside unit step function.

$\therefore L^{-1}\left\{\left(\log\left(1+\frac{1}{s^2}\right)+\frac{s}{s^2+25}\right)e^{-\pi s}\right\}=\left[-\cos 5t+\frac{2+2\cos t}{t-\pi}\right]H(t-\pi)$

6(a)(ii) Sol'n

Given equation is $(1+y^2) + (x - e^{-\tan^{-1}y})\frac{dy}{dx} = 0$

This equation can be written as

$$\frac{dx}{dy} + \frac{x - e^{-\tan^{-1}y}}{1+y^2} = 0$$

$$\Rightarrow \frac{dx}{dy} + \frac{x}{1+y^2} = \frac{e^{-\tan^{-1}y}}{1+y^2}$$

which is in the form of $\frac{dx}{dy} + P(y)x = Q(y)$

Here $P(y) = \frac{1}{1+y^2}$; $Q(y) = \frac{e^{-\tan^{-1}y}}{1+y^2}$

Now $e^{\int P\,dy} = e^{\int \frac{1}{1+y^2}dy}$

$$= e^{\tan^{-1}y}$$

$\therefore$ Integrating factor (I.F) $= e^{\int P\,dy} = e^{\tan^{-1}y}$

solution of the given differential equation is

$$x(\text{I.F}) = \int \frac{e^{-\tan^{-1}y}}{1+y^2} e^{\tan^{-1}y}\,dy + C$$

$$= \int \frac{1}{1+y^2}\,dy + C$$

$$x e^{\tan^{-1}y} = \tan^{-1}y + C$$

$$\therefore \boxed{x e^{\tan^{-1}y} = \tan^{-1}y + C}$$

6(b). Find the orthogonal trajectories of the system of circles touching a given straight line at a given point.

Sol'n: Let the given point be O (0,0) and the given straight line be y-axis. Now if a be the radius, then equation family of given circles is

$$(x-a)^2+(y-0)^2=a^2 \Rightarrow x^2+y^2=2ax \quad \text{---} (1)$$ where 'a' is a parameter

Differentiating (1) w.r.t x

$$2x+2y\,(dy/dx)=2a$$

$$\Rightarrow x+y\left(\frac{dy}{dx}\right)=a \quad \text{---} (2)$$

Eliminating a from (1) & (2), we get

$$x^2+y^2=2x\left(x+y\frac{dy}{dx}\right)$$

$$\Rightarrow 2xy\left(\frac{dy}{dx}\right)=y^2-x^2$$

which is the differential equation of the given family of circles (1). Replacing dy/dx by $-dx/dy$, the differential equation of the required orthogonal trajectories is

$$-2xy\frac{dx}{dy}=y^2-x^2$$

$$\Rightarrow \frac{dy}{dx}=\frac{2xy}{x^2-y^2}=\frac{2(y/x)}{1-(y/x)^2} \quad \text{---} (3)$$

which is a homogeneous differential equation.

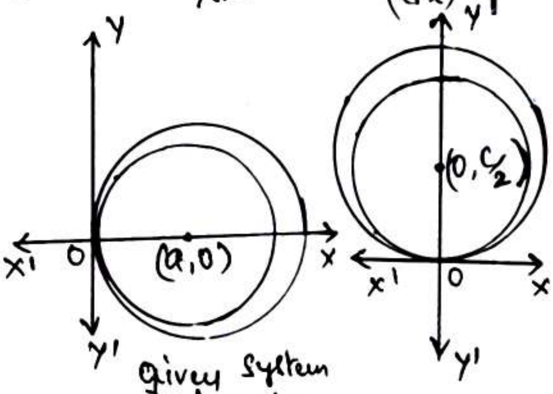
put $y/x=v \Rightarrow y=xv$ so that $dy/dx=v+x\left(\frac{dv}{dx}\right)$

$\therefore$ (3) gives $\quad v+x\frac{dv}{dx}=\frac{2v}{1-v^2}$

$$\Rightarrow x\frac{dv}{dx}=\frac{2v}{1-v^2}-v$$

$$\Rightarrow x\frac{dv}{dx}=\frac{v+v^3}{1-v^2}$$

$$\Rightarrow \frac{dx}{x}=\frac{1-v^2}{v(1+v^2)}\,dv$$



Given system of circles

$\Rightarrow \frac{dx}{x} = \left(\frac{1}{v} - \frac{2v}{1+v^2}\right)dv$, on resolving into partial fractions.

Integrating,

$$\log x = \log v - \log(1+v^2) + \log c$$

$$\Rightarrow x = \frac{cv}{(1+v^2)}$$

$$\Rightarrow x(1+v^2) = cv$$

$$\Rightarrow x(1+ y^2/x^2) = c(y/x) \quad \text{as } v = y/x$$

$$\therefore x^2+y^2 = cy, \quad c \text{ being parameter.} \text{ — (4)}$$

Note: Here the orthogonal trajectories (4) again represents a family of circles touching x-axis at $O(0,0)$ and having variable radius $(c/2)$.

6(c) → Apply the method of variation of parameters to solve
$x^2y_2 + xy_1 - y = x^2\log x,\ x>0.$

Soln: The given equation is

$$x^2y_2 + xy_1 - y = x^2\log x. \quad \text{--- (1)}$$

Re-writing the given equation, we have

$$y_2 + \frac{y_1}{x} - \frac{y}{x^2} = \log x \quad \text{--- (2)}$$

Comparing (2) with $y_2 + Py_1 + Qy = R$.

$$P = \frac{1}{x},\ Q = -\frac{1}{x^2},\ R = \log x$$

Consider $y_2 + \frac{1}{x}y_1 - \frac{1}{x^2}y = 0$.

$$\Rightarrow (x^2D^2 + xD - 1)y = 0 \quad \text{--- (3)}$$

Let $x = e^z \Rightarrow \log x = z$ and $D_1 = d/dz$

$$\therefore (3) \equiv [D_1(D_1-1) + D_1 - 1]y = 0$$

$$\Rightarrow (D_1^2 - 1)y = 0$$

$$D_1 = \pm 1$$

$\therefore$ C.F. of (3) is $y_c = c_1e^z + c_2e^{-z}$
$= c_1x + c_2x^{-1}$, c_1 and c_2 being arbitrary constants.

Let $u = x,\ v = x^{-1}$.

Also here $R = \log x$.

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} x & x^{-1} \\ 1 & -x^{-2} \end{vmatrix} = -x^{-1} - x^{-1} = -2x^{-1} \neq 0.$$

P.I. of (1) $= y_p = uf(x) + vg(x)$.

$$\therefore A = \int -\frac{vR}{W}\,dx = -\int \frac{x^{-1}\log x}{-2x^{-1}}\,dx = -\frac{1}{2}\int \log x\, dx$$

$$= -\frac{1}{2}\left[x\log x - \int \frac{1}{x}\cdot x\, dx\right]$$

$$= -\frac{1}{2}x[\log x - 1].$$

$$B = \int \frac{uR}{W}\,dx = \int \frac{x\log x}{2x^{-1}}$$

$$= \frac{1}{2}\int x^2 \log x\,dx.$$

$$= \frac{1}{2}\left[\log x \cdot \frac{x^3}{3} - \int \frac{1}{x}\cdot\frac{x^3}{3}\,dx\right]$$

$$= \frac{1}{2}\left[\frac{x^3}{3}\log x - \frac{x^3}{9}\right]$$

P.I. of (1) $y_p = x\left[-\frac{x}{2}(\log x - 1)\right] + \frac{x^{-1}}{2}\left[\frac{x^3}{3}\log x - \frac{x^3}{9}\right]$

$$y_p = -\frac{x^2}{2}\log x + \frac{x^2}{2} + \frac{x^2}{6}\log x - \frac{x^2}{18}$$

$$= -\frac{x^2}{3}\log x + \frac{4}{9}x^2.$$

$\therefore\ y = y_c + y_p$

$$y = c_1 x + c_2 x^{-1} - \frac{x^2}{3}\log x + \frac{4}{9}x^2$$

which is the required solution

6(d) Solve the following initial value problem using Laplace transform:

$$\frac{d^2y}{dx^2}+9y=r(x),\ y(0)=0,\ y'(0)=4$$

where $r(x)=\begin{cases} 8\sin x & \text{if } 0<x<\pi \\ 0 & \text{if } x\geq\pi \end{cases}$.

Sol^n: Given $\frac{d^2y}{dx^2}+9y=r(x)$; $y(0)=0$; $y'(0)=4$

we can rewrite the above equation.

$$y''+9y=r(x) \quad \text{---} (1) ;\ y(0)=0;\ y'(0)=4$$

Let $L(y)=p$

$$L(y'')=s^2p-s(y(0))-y'(0)$$

$$=s^2p-s(0)-4$$

$$L(y'')=s^2p-4 \quad \text{---} (2)$$

$$r(x)=\begin{cases} 8\sin x & ;\ \text{if } 0<x<\pi \\ 0 & ;\ \text{if } x\geq\pi \end{cases}$$

$$L(r(x))=\int_0^\infty e^{-st}\cdot r(t)\,dt$$

$$L(r(x))=8\int_0^\pi e^{-st}\cdot\sin t\,dt \quad \text{---} (3)$$

Let $I=\int_0^\pi e^{-st}\cdot\sin t\,dt$

$$I=\left[-e^{-st}\cdot\cos t\right]_0^\pi-s\int_0^\pi e^{-st}\cos t\,dt$$

$$I=e^{-\pi s}+1-s\left[\left[e^{-st}\sin t\right]_0^\pi+s\int_0^\pi e^{-st}\sin t\,dt\right]$$

$$I=e^{-\pi s}+1-s^2\int_0^\pi e^{-st}\cdot\sin t\,dt$$

$$I=e^{-\pi s}+1-s^2 I$$

$$I(1+s^2) = e^{-\pi s} + 1$$

$$I = \frac{e^{-\pi s}+1}{1+s^2} = \frac{e^{-\pi s}}{1+s^2} + \frac{1}{1+s^2} \quad \text{———} \quad (4)$$

Applying Laplace to (1) & using (2), (3) & (4).

$$s^2 p - 4 + 9p = \frac{8e^{-\pi s}}{1+s^2} + \frac{8}{1+s^2}$$

$$(s^2+9)p = \frac{8e^{-\pi s}}{1+s^2} + \frac{8}{1+s^2} + 4$$

$$p = \frac{8e^{-\pi s}}{(1+s^2)(s^2+9)} + \frac{8}{(1+s^2)(9+s^2)} + \frac{4}{s^2+9}$$

$$= \frac{e^{-\pi s}}{s^2+1} - \frac{e^{-\pi s}}{s^2+9} + \frac{1}{s^2+1} - \frac{1}{s^2+9} + \frac{4}{s^2+9}$$

$$p = \frac{e^{-\pi s}}{s^2+1} - \frac{e^{-\pi s}}{s^2+9} + \frac{1}{s^2+1} + \frac{3}{s^2+9}$$

$$L^{-1}(p) = u(t-\pi)\sin(t-\pi) - \tfrac{1}{3}u(t-\pi)\sin 3(t-\pi) + \sin t + \sin 3t$$

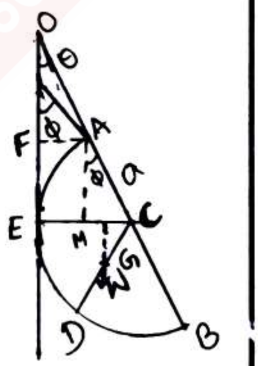
$$\therefore y(x) = u(x-\pi)\sin(x-\pi) - \tfrac{1}{3}u(x-\pi)\sin 3(x-\pi) + \sin x + \sin 3x$$

$$\therefore y(x) = \begin{cases} \sin x + \sin 3x \; ; & 0 < x < \pi \\ \tfrac{4}{3}\sin 3x \; ; & x \geq \pi \end{cases}$$

7(a). A solid hemisphere is supported by a string fixed to a point on its rim and to a point on a smooth vertical wall with which the curved surface of the hemisphere is in contact. If θ, ϕ are the inclinations of the string and the plane base of the hemisphere to the vertical, prove that

$\tan\phi = \frac{3}{8} + \tan\theta.$

Sol'n: 'O' is a fixed point in the wall to which one end of the string has been attached. Let l be the length of the string AO and a be the radius of the hemisphere the centre of whose base is C. The weight W of the hemisphere acts at its centre of gravity G which lies on the symmetrical radius CD and is such that $CG = \frac{3}{8}a$.



The hemisphere touches the wall at E.

we have $\angle OEC = 90^\circ$ so that EC is horizontal.

The string AO makes an angle θ with the wall and the base BA of the hemisphere makes an angle ϕ with the wall.

The depth of G below O = OF + AM + NG

$$= l\cos\theta + a\cos\phi + \frac{3}{8}a\sin\phi$$

[Note that $\angle NCG = 90^\circ - \angle ACM = 90^\circ - (90^\circ - \phi) = \phi$]

Give the system a small displacement in which θ changes to $\theta + \delta\theta$, ϕ changes to $\phi + \delta\phi$, the point O remains fixed, the length of the string AO does not change so that the workdone by its tension is zero & the point G is slightly displaced. The $\angle OEC$ remains 90°.

The only force that contributes to the equation of virtual work is the weight W of the hemisphere acting at G whose depth below the fixed point O has been found above. The equation of virtual work is

$$W\delta\left(l\cos\theta + a\cos\phi + \frac{3}{8}a\sin\phi\right) = 0$$

$$\Rightarrow -l\sin\theta\,\delta\theta - a\sin\phi\,\delta\phi + \frac{3}{8}a\cos\phi\,\delta\phi = 0$$

$$\Rightarrow l\sin\theta\,\delta\theta = a\left(\frac{3}{8}\cos\phi - \sin\phi\right)\delta\phi \quad \text{--- (1)}$$

from the fig, $EC = a$

Also $EC = EM + MC = FA + MC$

$$= l\sin\theta + a\sin\phi$$

$$\therefore a = l\sin\theta + a\sin\phi$$

Differentiating, $0 = l\cos\theta\,\delta\theta + a\cos\phi\,\delta\phi$

$$\Rightarrow -l\cos\theta\,\delta\theta = a\cos\phi\,\delta\phi \quad \text{--- (2)}$$

Dividing (1) by (2), we get

$$-\tan\theta = \frac{3}{8} - \tan\phi$$

$$\tan\phi = \frac{3}{8} + \tan\theta.$$

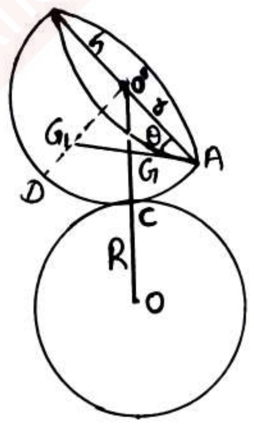
7(b) A heavy hemispherical shell of radius r has a particle attached to a point on the rim, and rests with the curved surface in contact with a rough sphere of radius R at the highest point. Prove that if $R/r > \sqrt{5}-1$, the equilibrium is stable, whatever be the weight of the particle.

Sol'n: Let O' be the centre of the base of the hemispherical shell of radius r. Let a weight be attached to the rim of the hemispherical shell at A. The centre of gravity G_1 of the hemispherical shell is on its symmetrical radius $O'D$ &

$O'G_1 = \frac{1}{2}O'D = \frac{1}{2}r$.

Let G be the centre of gravity of the combined body consisting of the hemispherical shell and the weight at A. Then G lies on the line AG_1.

The hemispherical shell rests with its curved surface in contact with a rough sphere of radius R and centre O at the highest point C.



For equilibrium the line $OCGO'$ must be vertical but AG_1 need not be horizontal.

Let $CG = h$. Also here $\rho_1 = r$ and $\rho_2 = R$.

The equilibrium will be stable if

$$\frac{1}{h} > \frac{1}{\rho_1} + \frac{1}{\rho_2} \text{ i.e, } \frac{1}{h} > \frac{1}{r} + \frac{1}{R}$$

$$\text{i.e, } \frac{1}{h} > \frac{R+r}{rR}$$

$$\text{i.e; } h < \frac{rR}{R+r} \qquad \text{——— (1)}$$

The value of h depends on the weight of the particle attached at A. So the equilibrium will be stable, whatever be the weight of the particle attached at A, if the relation ① holds even for the maximum value of h.

Now h will be maximum if O'G is minimum i.e, if O'G is ⊥lar to AG_1 or if $\Delta AO'G$ is right angled.

Let $\angle O'AG = \theta$. Then from right angled $\Delta AO'G_1$,

$$\tan\theta = \frac{O'G_1}{O'A}$$

$$= \frac{\frac{1}{2}r}{r} = \frac{1}{2}$$

$$\therefore \sin\theta = \frac{1}{\sqrt{5}}$$

$\therefore$ the minimum value of $O'G$

$$= O'A \sin\theta = r\left(\frac{1}{\sqrt{5}}\right) = \frac{r}{\sqrt{5}}$$

$\therefore$ the maximum value of $h = r -$ the minimum value of $O'G$

$$= r - \frac{r}{\sqrt{5}} = \frac{r(\sqrt{5}-1)}{\sqrt{5}}$$

Hence the equilibrium will be stable, whatever be the weight of the particle at A, if

$$\frac{r(\sqrt{5}-1)}{\sqrt{5}} < \frac{rR}{R+r} \quad \text{i.e if} \quad \frac{\sqrt{5}-1}{\sqrt{5}} < \frac{R}{R+r}$$

i.e, if $(\sqrt{5}-1)R + (\sqrt{5}-1)r < R\sqrt{5}$

i.e, if $(\sqrt{5}-1)r < R$ i.e, if $\frac{R}{r} > \sqrt{5}-1$

7(c) → A particle moves in a plane under a central force which varies inversely as the square of the distance from the fixed point, find the orbit.

Sol'n: we know that referred to the centre of force as pole the differential equation of a central orbit in pedal form is

$$\frac{h^2}{p^3}\frac{dp}{dr} = P, \quad \text{———} \quad (1)$$

where P is the central acceleration assumed to be attractive.

Here $P = \mu/r^2$. Putting $P = \mu/r^2$ in (1), we get

$$\frac{h^2}{p^3}\frac{dp}{dr} = \frac{\mu}{r^2}$$

$$\Rightarrow \frac{h^2}{p^3}dp = \frac{\mu}{r^2}dr$$

$$\Rightarrow -2\frac{h^2}{p^3}dp = -\frac{2\mu}{r^2}dr$$

Integrating bothsides, we get

$$v^2 = \frac{h^2}{p^2} = \frac{2\mu}{r} + C \quad \text{———} \quad (2)$$

Let $v = v_0$ when $r = r_0$.

Then $v_0^2 = \frac{2\mu}{r_0} + C \Rightarrow C = v_0^2 - \frac{2\mu}{r_0}$.

Putting this value of C in (2), the pedal equation of the central orbit is

$$\frac{h^2}{p^2} = \frac{2\mu}{r} + v_0^2 - \frac{2\mu}{r_0} \quad \text{———} \quad (3)$$

Case 1: Let $v_0^2 = \frac{2\mu}{r_0}$. Then the equation (3) becomes

$$\frac{h^2}{p^2} = \frac{2\mu}{r}$$ which is of the form $p^2 = ar$.

This is the pedal equation of a parabola referred to focus as pole. Hence in this case the orbit is a parabola with centre of force at the focus.

Case 2: Let $v_0^2 < \frac{2\mu}{r_0}$. In this case the equation (3) reduces to the form $\frac{b^2}{p^2} = \frac{2a}{r} - 1$.

This is the pedal equation of an ellipse referred to a focus as pole. Hence in this case the orbit is an ellipse with centre of force at its focus.

Case 3: Let $v_0^2 > \frac{2\mu}{r_0}$. In this case the equation ③ reduces to the form

$$\frac{b^2}{p^2} = \frac{2a}{r} + 1.$$

This is the pedal equation of a hyperbola referred to a focus as pole. It represents that branch of the hyperbola which is nearer to the focus taken as pole.

Hence we conclude that under inverse square law the central orbit is always a conic with centre of force at the focus.

8(a) (i) The position vector of a moving point at time t is $\vec{r} = \sin t\,\hat{i} + \cos 2t\,\hat{j} + (t^2+2t)\hat{k}$. Find the components of acceleration $\bar{a}$ in the directions parallel to the velocity vector $\bar{v}$ and ⊥ar to the plane of $\bar{r}$ and $\bar{v}$ at time $t=0$.

(ii) Prove that vector $f(r)\vec{r}$ is irrotational.

(iii) Prove that $\text{Curl}(\psi\nabla\phi) = \nabla\psi \times \nabla\phi = -\text{Curl}(\phi\nabla\psi)$

sol'n:
$$r(t) = \sin t\,\hat{i} + \cos 2t\,\hat{j} + (t^2+2t)\hat{k} \quad \text{——} \ (1)$$
$$v(t) = \frac{dr}{dt} = \cos t\,\hat{i} - 2\sin 2t\,\hat{j} + (2t+2)\hat{k} \quad \text{——} \ (2).$$
$$a(t) = \frac{d^2r}{dt^2} = -\sin t\,\hat{i} - 4\cos 2t\,\hat{j} + 2\hat{k} \quad \text{——} \ (3)$$

at $t=0$

$$r(0) = 0\hat{i} + \hat{j} + 0\hat{k} = \hat{j}$$
$$v(0) = \hat{i} + 0\hat{j} + 2\hat{k} = \hat{i} + 2\hat{k}$$
$$\text{unit vector } \hat{v} = \frac{\hat{i}+2\hat{k}}{\sqrt{5}}$$
$$a(0) = 0\hat{i} - 4\hat{j} + 2\hat{k} = -4\hat{j} + 2\hat{k}$$

Component of $\vec{a}$, in direction parallel to $\vec{v}$ is
$$= \frac{\vec{a}\cdot\vec{v}}{|\vec{v}|}(\hat{v}) = \frac{4}{5}(\hat{i}+2\hat{k})$$

Now, let w be the direction ⊥ar to r and v at $t=0$, then
$$\vec{\omega} = \vec{r}\times\vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{vmatrix} = 2\hat{i} - \hat{k}.$$

$$\text{unit vector } \hat{\omega} = \frac{2\hat{i}-\hat{k}}{\sqrt{5}}$$

Component of $\vec{a}$ in direction of $\vec{\omega}$ is
$$= \vec{a}\cdot\frac{\vec{\omega}}{|\vec{\omega}|}\cdot(\hat{\omega}) = -\frac{2}{5}(2\hat{i}-\hat{k}).$$

(ii) The vector $f(r)\vec{r}$ will be irrotational if

$$\text{Curl}[f(r)\vec{r}] = 0$$

we know that $\text{Curl}(\phi A) = (\text{grad}\,\phi)\times A + \phi\,\text{curl}\,A$.

putting $\phi = f(r)$ and $A = r$ in this identity, we get

$$\text{Curl}[f(r)\vec{r}] = [\text{grad}\, f(r)]\times\vec{r} + f(r)\,\text{Curl}\,\vec{r}$$

$$= [f'(r)\,\text{grad}\, r]\times\vec{r} + f(r)\,0 \quad [\because \text{curl}\, r = 0]$$

$$= \left[f'(r)\frac{1}{r}\vec{r}\right]\times\vec{r} = f'(r)\frac{1}{r}(\vec{r}\times\vec{r}) = 0$$

Since $\vec{r}\times\vec{r} = 0$.

$\therefore$ The vector $f(r)\vec{r}$ is irrotational.

(iii) we know that $\text{curl}(\phi A) = (\nabla\phi)\times A + \phi\,\text{curl}\,A$

In the above formula replacing ϕ by ψ and A by $\nabla\phi$,

we have. $\text{Curl}(\psi\nabla\phi) = (\nabla\psi)\times\nabla\phi + \psi\,\text{Curl}\,\nabla\phi$

$$= \nabla\psi\times\nabla\phi + 0$$

$$= \nabla\psi\times\nabla\phi \quad \text{———} (1) \quad [\because \text{Curl}\,\nabla\phi = \text{curl}\,\text{grad}\,\phi = 0]$$

Similarly $\text{curl}(\phi\nabla\psi) = (\nabla\phi)\times\nabla\psi + \phi\,\text{Curl}\,\nabla\psi$

$$= \nabla\phi\times\nabla\psi + 0$$

$$= \nabla\phi\times\nabla\psi$$

$$= -\nabla\psi\times\nabla\phi \quad \text{———} (2)$$

from (1) and (2) we have

$$\text{Curl}(\psi\nabla\phi) = \nabla\psi\times\nabla\phi = -\,\text{Curl}(\phi\nabla\psi).$$

8(b) show that $F = (\sin y + z)\hat{i} + (x\cos y - z)\hat{j} + (x-y)\hat{k}$ is a conservative vector field and find a function ϕ such that $F = \nabla\phi$.

Sol^n: we have $\operatorname{curl} F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin y + z & x\cos y - z & x-y \end{vmatrix}$

$$= \hat{i}\left[\frac{\partial}{\partial y}(x-y) - \frac{\partial}{\partial z}(x\cos y - z)\right] + \hat{j}\left[\frac{\partial}{\partial z}(\sin y + z) - \frac{\partial}{\partial x}(x-y)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}(x\cos y - z) - \frac{\partial}{\partial y}(\sin y + z)\right]$$

$$= (-1+1)\hat{i} + (1-1)\hat{j} + (\cos y - \cos y)\hat{k}$$

$$= 0\hat{i} + 0\hat{j} + 0\hat{k} = 0.$$

$\therefore$ The vector field F is conservative.

i.e, $(\sin y + z)\hat{i} + (x\cos y - z)\hat{j} + (x-y)\hat{k} = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$

then $\frac{\partial\phi}{\partial x} = \sin y + z$ hence $\phi = x\sin y + xz + f_1(y,z)$ —— (1)

$\frac{\partial\phi}{\partial y} = x\cos y - z$ hence $\phi = x\sin y - yz + f_2(z,x)$ —— (2)

$\frac{\partial\phi}{\partial z} = x - y$ hence $\phi = xz - yz + f_3(x,y)$ —— (3)

(1), (2), (3) each represents ϕ. These agree if we choose

$f_1(y,z) = -yz$, $f_2(z,x) = xz$, $f_3(x,y) = x\sin y$

$\therefore \phi = x\sin y + xz - yz$

$\therefore \phi = x\sin y + xz - yz$ to which may be added any constant.

Hence $\phi = x\sin y + xz - yz + C$.

8(C) By using divergence theorem evaluate

$$\iint_S (a^2x^2+b^2y^2+c^2z^2)^{1/2}\,ds$$ over the ellipsoid

$$ax^2+by^2+cz^2=1.$$

<u>sol'n</u>: Let us first put the integral

$$\iint_S (a^2x^2+b^2y^2+c^2z^2)^{1/2}\,ds \text{ in the form } \iint_S F\cdot n\,ds$$

where n is a unit normal vector to the closed surface S whose equation is $ax^2+by^2+cz^2=1$.

The normal vector to $\phi(x,y,z)\equiv ax^2+by^2+cz^2-1=0$ is

$$\nabla\phi = 2ax\hat{i}+2by\hat{j}+2cz\hat{k}.$$

$$\therefore n=\frac{\nabla\phi}{|\nabla\phi|}=\frac{2ax\hat{i}+2by\hat{j}+2cz\hat{k}}{\sqrt{(4a^2x^2+4b^2y^2+4c^2z^2)}}=\frac{ax\hat{i}+by\hat{j}+cz\hat{k}}{\sqrt{(a^2x^2+b^2y^2+c^2z^2)}}$$

Now we are to choose F such that

$$F\cdot n=\sqrt{(a^2x^2+b^2y^2+c^2z^2)}$$

Obviously $F=ax\hat{i}+by\hat{j}+cz\hat{k}$

Now $\iint_S (a^2x^2+b^2y^2+c^2z^2)^{1/2}\,ds$

$= \iint_S F\cdot n\,ds$, where $F=ax\hat{i}+by\hat{j}+cz\hat{k}$

$= \iiint_V \operatorname{div} F\,dV$, by divergence theorem; V is the volume enclosed by the closed surface S.

$= \iiint_V \left[\frac{\partial}{\partial x}(ax)+\frac{\partial}{\partial y}(by)+\frac{\partial}{\partial z}(cz)\right]dV$

$=(a+b+c)\iiint_V dV=(a+b+c)V$

$=(a+b+c)\cdot\frac{4}{3}\pi\left(\frac{1}{\sqrt{a}}\cdot\frac{1}{\sqrt{b}}\cdot\frac{1}{\sqrt{c}}\right)=\frac{4\pi}{3}\frac{(a+b+c)}{\sqrt{(abc)}}.$

Note that the equation of the ellipsoid $ax^2+by^2+cz^2=1$ can be written as $\frac{x^2}{1/a}+\frac{y^2}{1/b}+\frac{z^2}{1/c}=1$ and the volume of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is $\frac{4}{3}\pi abc$.

8(d) Verify Stoke's theorem for $F = (x^2+y-4)\hat{i} + 3xy\hat{j} + (2xz+z^2)\hat{k}$ where S is the upper half of the sphere $x^2+y^2+z^2=16$ and C is its boundary.

Sol^n: The boundary C of S is the circle $x^2+y^2=16$, $z=0$ lying in the xy-plane. Suppose $x=4\cos t$, $y=4\sin t$, $z=0$, $0 \le t < 2\pi$ are parametric equations of C. Then

$$\oint_C F\cdot dr = \oint_C \left[(x^2+y-4)\hat{i} + 3xy\hat{j} + (2xz+z^2)\hat{k}\right]\cdot(dx\hat{i}+dy\hat{j}+dz\hat{k})$$

$$= \oint_C \left[(x^2+y-4)\,dx + 3xy\,dy + (2xz+z^2)\,dz\right]$$

$$= \oint_C (x^2+y-4)\,dx + 3xy\,dy \quad (\because \text{ on } C\ z=0 \Rightarrow dz=0)$$

$$= \int_0^{2\pi}\left[(x^2+y-4)\frac{dx}{dt} + 3xy\frac{dy}{dt}\right]dt$$

$$= \int_0^{2\pi}\left[(16\cos^2 t + 4\sin t - 4)(-4\sin t) + 3.16\sin t\cos t.4\cos t\right]dt$$

$$= 128\int_0^{2\pi}\cos^2 t\sin t\,dt - 16\int_0^{2\pi}\sin^2 t\,dt + \int_0^{2\pi}\sin t\,dt$$

$$= 128(0) - 16(4)\int_0^{\pi/2}\sin^2 t\,dt + 16(0)$$

$$= -64\left(\tfrac{1}{2}\right)\left(\tfrac{\pi}{2}\right) = -16\pi \quad \text{—— (1)}$$

Now let us evaluate $\iint_S \text{Curl}\,F\cdot\hat{n}\,ds$

$$\text{Curl}\,F = \nabla\times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2+y-4 & 3xy & 2xz+z^2 \end{vmatrix}$$

$$= -zj + (3y-1)k$$

If S_1 is the plane region bounded by the circle C, then by an application of Gauss divergence theorem, we have

$$\iint_S \text{Curl}\, F \cdot \hat{n}\, ds = \iiint_V \text{div}(\text{Curl}\, F)\, dV = 0$$

[Here s' is the surface constant of S and S_1. The s' is closed surface and let V be the volume bounded by s']

$$\therefore \iint_S \text{Curl}\, F \cdot \hat{n}\, ds + \iint_{S_1} (\text{Curl}\, F \cdot \hat{n})\, ds = 0$$

[$\because$ div(curl F) = 0]

$$\Rightarrow \iint_S \text{Curl}\, F \cdot \hat{n}\, ds = \iint_{S_1} \text{Curl}\, F \cdot k\, ds \quad (\because \text{ on } S_1,\ \hat{n} = -k)$$

$$= \iint_{S_1} (3y - 1)\, ds$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{4} (3r\sin\theta - 1)\, r\, d\theta\, dr$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{4} 3r^2 \sin\theta\, d\theta - \int_0^{2\pi} \int_0^{4} r\, d\theta\, dr$$

$$= 0 - \int_{\theta=0}^{2\pi} \left(\frac{r^2}{2}\right)_0^4 d\theta \quad \left(\because \int_0^{2\pi} \sin\theta = 0\right)$$

$$= -8[\theta]_0^{2\pi}$$

$$= -8(2\pi)$$

$$= -16\pi$$