values of $f(0), f'(0), f''(0), \ldots, f^m(0)$, we get the finite expansion

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \ldots + x^m, \ \forall \ x$$

(b) When m is not a positive integer, $(1+x)^m$ possesses continuous derivatives of all orders provided $x \neq -1$.

Let $-1 < x < 1$.

Taking Cauchy's form of remainder, we have

$$R_n = \frac{x^n}{(n-1)!}(1-\theta)^{n-1} f^n(\theta x)$$

$$= \frac{x^n}{(n-1)!}(1-\theta)^{n-1} m(m-1) \ldots (m-n+1)(1+\theta x)^{m-n}$$

$$= \left(\frac{m(m-1)\ldots(m-n+1)x^n}{(n-1)!}\right)\left(\frac{1-\theta}{1+\theta x}\right)^{n-1}(1+\theta x)^{m-1}$$

We know for $|x| < 1$,

$$\frac{m(m-1)(m-2)\ldots(m-n+1)}{(n-1)!}x^n \to 0 \text{ as } n \to \infty,$$

and

$$\frac{1-\theta}{1+\theta x} < 1, \text{ so that } \left(\frac{1-\theta}{1+\theta x}\right)^{n-1} \to 0 \text{ as } n \to \infty.$$

Also

$$(1+\theta x)^{m-1} < \left(1+|x|\right)^{m-1}, m > 1, 0 < \theta < 1$$

and

$$(1+\theta x)^{m-1} = \frac{1}{(1+\theta x)^{1-m}} < \frac{1}{\left(1-|x|\right)^{1-m}}, \text{ when } m < 1$$

Thus $R_n \to 0$ when $n \to \infty$, for $|x| < 1$.

Hence, the conditions of Maclaurin's infinite expansion are satisfied.

Making the substitutions $f(0) = 1$, $f'(0) = m, \ldots, f^n(0) = m(m-1)\ldots(m-n+1)$, we get

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \ldots \text{ for } |x| < 1$$

Note: When m is not a positive integer the expansion is not possible if $|x| > 1$, for then as $n \to \infty$, $\frac{m(m-1)\ldots(m-n+1)x^n}{(n-1)!}$ and so R_n does not tend to zero.

EXERCISE

1. Expand, if possible, $\sin x$ in ascending powers of x.
2. Assuming the validity of expansion, show that

(i) $e^x \cos x = 1 + x - \dfrac{2x^3}{3!} - \dfrac{2^2 x^4}{4!} - \dfrac{2^2 x^5}{5!} + \ldots$

(ii) $\log \sec x = \dfrac{1}{2}x^2 + \dfrac{1}{12}x^4 + \ldots$

(iii) $\tan^{-1} x = \tan^{-1}\dfrac{\pi}{4} + \dfrac{x - \pi/4}{1 + \pi^2/16} - \dfrac{\pi(x - \pi/4)^2}{4(1 + \pi^2/16)^2} + \ldots$

(iv) $\sin\left(\dfrac{\pi}{4} + \theta\right) = \dfrac{1}{\sqrt{2}}\left(1 + \theta - \dfrac{\theta^2}{2!} - \dfrac{\theta^3}{3!} + \ldots\right)$

(v) $$f(x) = f(a) + 2\left[\frac{x-a}{2} f'\left(\frac{x+a}{2}\right) + \frac{(x-a)^3}{8.(3)!} f'''\left(\frac{x+a}{2}\right) + \frac{(x-a)^5}{32(5)!} f^{\text{v}}\left(\frac{x+a}{2}\right) + \ldots\right]$$

3. Use Taylor's theorem to show that

(i) $\cos x \geq 1 - \dfrac{x^2}{2}$, for all real x.

(ii) $x - \dfrac{x^3}{6} < \sin x < x$, for $x > 0$

(iii) $x - \dfrac{x^3}{6} < \sin x < x - \dfrac{x^3}{6} + \dfrac{x^5}{120}$, $\forall\, x > 0$

(iv) $1 + x + \dfrac{x^2}{2} < e^x < 1 + x + \dfrac{x^2}{2} e^x$, $x > 0$

4. If $0 < x \leq 2$, then prove that

$$\log x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \ldots$$

5. If $f(x) = \exp(-1/x^2)$, for $x \neq 0$ and $f(0) = 0$, then show that

(i) $f^n(0) = 0$, for all $n = 0, 1, 2, \ldots$, and

(ii) The Taylor's series for f about 0 agrees with $f(x)$ only at $x = 0$.

[*Hint*: First, prove by induction that, for any $x \neq 0$,

$$f^n(x) = \exp(-1/x^2)\, P_n(1/x),$$

where P_n is a polynomial of degree $3n$. Second, using $e^x > x^n/n!\ (x > 0)$, show that

$$\lim_{x \to 0} \exp(-1/x^2)\, P(1/x) = 0,$$

where P is any polynomial. Then apply induction to prove (*i*).]