Set – V

* Linear Equations *

→ Now we shall discuss the nature of solutions of a system of non-homogeneous linear Equations.

Now consider the system of m non homogeneous linear equations in 'n' unknowns $x_1, x_2, x_3, \ldots x_n$

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n = b_2 \\ \cdots \\ \cdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n = b_m \end{array}\right\} \text{—(1)}$$

The system (1) can be expressed as the matrix equation $AX = B$ —(2)

where $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \cdots & & & & \\ \cdots & & & & \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$

$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$, $B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$

→ The matrix A is called the coefficient matrix.

→ Any set of values of $x_1, x_2, \ldots x_n$ which simultaneously satisfy all these equations, is called a solution of the system (1) when the system of equations has one or more solutions, the equations are said to be consistent.

Otherwise they are said to be inconsistent.

The matrix

$$[A \vdots B] = \left[\begin{array}{cccc:c} a_{11} & a_{12} & a_{13} \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & a_{23} \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \\ a_{m1} & a_{m2} & a_{m3} & a_{mn} & b_m \end{array}\right]_{m\times(n+1)}$$

is called the augmented matrix of the given system of equations (1)

Imp ** → **Condition for Consistency:**

Theorem :– The equation $AX = B$ is consistent i e possesses solution iff the two matrices A & $[A|B]$ are of the same rank.

Proof – we write

$A = [C_1 \; C_2 \; C_3 \cdots C_r \; C_{r+1} \; C_{r+2} \cdots C_n]_{m\times n}$

where $C_1, C_2, \cdots C_n$ are matrices each of order $m \times 1$.

the system $AX = B$ is equivalent to

$$[C_1 \ C_2 \ C_3 \cdots C_n]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = B$$

$$\Rightarrow x_1C_1 + x_2C_2 + x_3C_3 + \cdots + x_nC_n = B \quad \text{--- (1)}$$

Let $\rho(A) = r$ then A has r linear Independent columns, without loss of generality.

let the first r columns of A be LI i.e the first r columns $C_1\ C_2 \cdots C_r$ form L.I set

$\therefore$ Each of the remaining $n-r$ columns is a linear combination of the first r columns $C_1, C_2 \cdots C_r$

Necessary Condition :-

Let $AX = B$ be Consistent then $\exists$ n Scalars (real numbers) $k_1, k_2, k_3 \cdots k_n$ such that

$$k_1C_1 + k_2C_2 + \cdots + k_nC_n = B \quad \text{--- (2)}$$

Let $\rho(A) = r$

Since each of the $(n-r)$ columns $C_{r+1}, C_{r+2}, C_{r+3} \cdots C_n$ is a linear Combination of the first r Columns $C_1, C_2, \cdots C_r$

$\therefore$ from (2) we have

B is also a linear Combination of $C_1, C_2 \cdots C_r$

$\therefore$ The maximum number of linearly independent columns of $[A \ B]$ is also r.

$$\therefore \rho(A|B) = r$$

$$\text{i.e } \rho(A) = \rho(A|B).$$

Sufficient Condition :-

Let $\rho(A) = \rho(A|B) = r$. then the maximum number of linearly independent columns of $[A|B]$ is 'r' and its columns $C_1, C_2 \cdots C_r$.

$\therefore$ B is a linear combination of $C_1, C_2 \cdots C_r$

Now $\exists$ Scalars $P_1, P_2, \cdots P_r$ Such that $P_1C_1 + P_2C_2 + \cdots + P_rC_r = B$

i.e. $P_1C_1 + P_2C_2 + \cdots + P_rC_r + 0C_{r+1}$

$$+ 0C_{r+2} + \cdots + 0C_n = B \quad \text{--- (3)}$$

Now comparing (1) & (3) we get

$x_1 = P_1, x_2 = P_2, \cdots x_r = P_r, x_{r+1} = 0,$

$$x_{r+2} = 0 \cdots x_n = 0.$$

$\therefore$ The system $AX = B$ has a solution

$\therefore$ The System is consistent.

Note:- Let $AX = B$ and $CX = D$ be two linear systems, each of m equations in 'n' unknowns.

If the augmented matrices $[A|B]$ & $[C|D]$ of these systems are row equivalent then both linear systems have exactly the same solutions.

If one system has no solution then the other system has no solution.

2000*

→ Theorem:-

If A is a non-singular matrix of order 'n' then the linear system $AX = B$ in 'n' unknowns has a unique solution

Proof:- Since A is a non-singular matrix of order 'n'

$\therefore |A| \neq 0$

$\Rightarrow \rho(A) = n$ & $\rho(A|B) = n$

$\therefore \rho(A) = \rho(A|B)$

$\Rightarrow AX = B$ is consistent and it has a solution.

Also A^{-1} exists. $(\because |A| \neq 0)$

$\therefore A^{-1}(AX) = A^{-1}B$

$\Rightarrow IX = A^{-1}B$

$\Rightarrow \boxed{X = A^{-1}B}$ is a solution of $AX = B$

If possible let X_1 & X_2 be two solutions of $AX = B$.

$\therefore AX_1 = B$ & $AX_2 = B$

$\Rightarrow AX_1 = AX_2$

$\Rightarrow A^{-1}(AX_1) = A^{-1}(AX_2)$

$\Rightarrow IX_1 = IX_2$

$\Rightarrow \boxed{X_1 = X_2}$

$\therefore$ The solution $X = A^{-1}B$ of $AX = B$ is unique.

* Working rule for finding the solution of equation $AX = B$:

Suppose the coefficient matrix A is of the type $m \times n$ ie we have m equations in 'n' unknowns.

– write the augmented matrix $[A|B]$ and reduce it to an echelon form by applying only Elementary row operations on it this echelon form will enable us to know the ranks of the augmented matrix $[A|B]$ and the coefficient matrix A then the following cases will arise.

Case(i): If $\rho(A) = \rho(A|B) =$ number of unknowns.

Then the given system of equations is consistent and has unique solution.

Case(ii): If $\rho(A) = \rho(A|B) <$ the number of unknowns then the given system of equations is consistent and has infinite solutions

Case(iii): If $\rho(A) \neq \rho(A|B)$ then the given system is not consistent and has no solution.

Problems

→ show that the equations $x+y+z=6$, $x+2y+3z=14$, $x+4y+7z=30$ are consistent and solve them.

Sol'n :- The matrix form of the given system is

$$AX = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \\ 30 \end{bmatrix} = B$$

The augmented matrix

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 1 & 4 & 7 & 30 \end{array}\right]$$

$R_2 \to R_2 - R_1,\ R_3 \to R_3 - R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 3 & 6 & 24 \end{array}\right]$$

$R_3 \to R_3 - 3R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

∴ which is in echelon form

∴ $\rho(A|B) = 2$ & $\rho(A) = 2$.

∴ $\rho(A|B) = \rho(A) = 2 <$ the number of three unknown variables x, y, z.

∴ The given system of equations consistent.

and has infinite number of solutions.

Now write the matrix equation with echelon-form.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x+y+z \\ y+2z \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \\ 0 \end{bmatrix}$$

$\therefore x+y+z=6$ —① $\quad y+2z=8$ —②

Now taking $z=t$ in ② where t is arbitrary constant

$$\boxed{y = 8-2t}$$

$\therefore$ ① $\equiv x = 6 - t - 8 + 2t$

$\Rightarrow \boxed{x = t-2}$

$\therefore x = t-2,\ y = 8-2t$ and $z = t$

where t is arbitrary constant constitute the general solution of the given system

→ Apply the test of rank to examine of the following equations are consistent:

$2x - y + 3z = 8$

$-x + 2y + z = 4$; $3x + y - 4z = 0$ and if consistent, find the complete solution.

Sol'n:- Now write the single matrix equation of the given system is

$$Ax = \begin{bmatrix} 2 & -1 & 3 \\ -1 & 2 & 1 \\ 3 & 1 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ 0 \end{bmatrix} = B$$

The augmented matrix

$$[A|B] = \left[\begin{array}{ccc|c} 2 & -1 & 3 & 8 \\ -1 & 2 & 1 & 4 \\ 3 & 1 & -4 & 0 \end{array}\right]$$

$R_1 \to R_2$

$$\sim \left[\begin{array}{ccc|c} -1 & 2 & 1 & 4 \\ 2 & -1 & 3 & 8 \\ 3 & 1 & -4 & 0 \end{array}\right]$$

$R_3 \to R_3 + 3R_1$

$R_2 \to R_2 + 2R_1$

$$\sim \left[\begin{array}{ccc|c} -1 & 2 & 1 & 4 \\ 0 & 3 & 5 & 16 \\ 0 & 7 & -1 & 12 \end{array}\right]$$

$R_3 \to 3R_3$

$$\sim \left[\begin{array}{ccc|c} -1 & 2 & 1 & 4 \\ 0 & 3 & 5 & 16 \\ 0 & 21 & -3 & 36 \end{array}\right]$$

$R_3 \to R_3 - 7R_2$

$$\sim \left[\begin{array}{ccc|c} -1 & 2 & 1 & 4 \\ 0 & 3 & 5 & 16 \\ 0 & 0 & -38 & -76 \end{array}\right]$$

$R_3 \to -\frac{1}{38}R_3$

$$\sim \left[\begin{array}{ccc|c} -1 & 2 & 1 & 4 \\ 0 & 3 & 5 & 16 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

∴ which is in echelon form

∴ $\rho(A|B) = 3$ & $\rho(A) = 3$

∴ $\rho(A|B) = \rho(A) = 3$

= the number of unknown variables.

∴ The given equations are consistent. and have a unique solution.

Again re write the matrix equation with echelon form is

$$\begin{bmatrix} -1 & 2 & 1 \\ 0 & 3 & 5 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 16 \\ 2 \end{bmatrix}$$

$\Rightarrow -x + 2y + z = 4$

$3y + 5z = 16$

$\boxed{z = 2}$

∴ $3y = 16 - 5(2)$

$= 6$

$\Rightarrow \boxed{y = 2}$ and $\boxed{x = 2}$

∴ $x = 2,\ y = 2,\ z = 2$

2002 → Solve the following system of linear equations.

$$x_1 - 2x_2 - 3x_3 + 4x_4 = -1$$
$$-x_1 + 3x_2 + 5x_3 - 5x_4 - 2x_5 = 0$$
$$2x_1 + x_2 - 2x_3 + 3x_4 - 4x_5 = 17.$$

2004 → verify whether the following system of equations are Consistent

$$x + 3z = 5$$
$$-2x + 5y - z = 0$$
$$-x + 4y + z = 4$$

→ show that the equations $x + y + z = -3$, $3x + y - 2z = -2$, $2x + 4y + 7z = 7$ are not consistent

Sol^n :- we write the single matrix equation of the system is

$$AX = \begin{bmatrix} 1 & 1 & 1 \\ 3 & 1 & -2 \\ 2 & 4 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ -2 \\ 7 \end{bmatrix} = B$$

The augmented matrix

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & -3 \\ 3 & 1 & -2 & -2 \\ 2 & 4 & 7 & 7 \end{array}\right]$$

$R_2 \to R_2 - 3R_1$
$R_3 \to R_3 - 2R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & -3 \\ 0 & -2 & -5 & 7 \\ 0 & 2 & 5 & 13 \end{array}\right]$$

$R_3 \to R_3 + R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & -3 \\ 0 & -2 & -5 & 7 \\ 0 & 0 & 0 & 20 \end{array}\right]$$

∴ which is in echelon form.

$\therefore \rho(A|B) = 3$ & $\rho(A) = 2$

$\therefore \rho(A|B) \neq \rho(A)$

∴ The given system of equations is not consistent.

→ show that the equations

$$x + 2y - z = 3$$
$$3x - y + 2z = 1$$
$$2x - 2y + 3z = 2$$
$$x - y + z = -1$$ are consistent

and solve them.

IFoS 201x → Investigate for what values of λ and μ the equations.

$x + y + z = 6$, $x + 2y + 3z = 10$

$x + 2y + \lambda z = \mu$ have —

(i) no solution (ii) a unique solution

(iii) infinitely many solutions.

Sol'n :- write the matrix equation of the given system.

$$AX = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 2 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \\ \mu \end{bmatrix} = B$$

The augmented matrix

$$[A|B] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & \lambda-1 & \mu-6 \end{array}\right]$$ $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$

$R_3 \longrightarrow$

$R_3 \to R_3 - R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right]$$

— If $\lambda = 3$ & $\mu \neq 10$ then

$\rho(A|B) = 3$ & $\rho(A) = 2$

$\therefore \rho(A|B) \neq \rho(A)$

$\therefore$ The given equations have no solutions.

— If $\lambda \neq 3$ and μ = any value then $\rho(A|B) = \rho(A) = 3$ = the number of unknown variables.

$\therefore$ The equations are consistent and have unique solution.

— If $\lambda = 3$ and $\mu = 10$ then $\rho(A|B) = \rho(A) = 2 <$ the number of unknown variables.

$\therefore$ the given equations are consistent and have infinite solutions.

Imp $\to$ For what values of the parameter λ will the following equations fail to have unique solutions. $3x - y + \lambda z = 1$, $2x + y + z = 2$, $x + 2y - \lambda z = -1$.

For the equations have any solution for these values of λ ?

write the matrix equation of the given system is

$$AX = \begin{bmatrix} 3 & -1 & \lambda \\ 2 & 1 & 1 \\ 1 & 2 & -\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = B$$

The augmented matrix

$$[A|B] = \left[\begin{array}{ccc|c} 3 & -1 & \lambda & 1 \\ 2 & 1 & 1 & 2 \\ 1 & 2 & -\lambda & -1 \end{array}\right]$$

$R_1 \leftrightarrow R_3$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -\lambda & -1 \\ 2 & 1 & 1 & 2 \\ 3 & -1 & \lambda & 1 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 3R_1$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -\lambda & -1 \\ 0 & -3 & 1+2\lambda & 4 \\ 0 & -7 & 4\lambda & 4 \end{array}\right]$$

$R_3 \to -3R_3$

$$\left[\begin{array}{ccc|c} 1 & 2 & -\lambda & -1 \\ 0 & -3 & 1+2\lambda & 4 \\ 0 & 21 & -12\lambda & -12 \end{array}\right]$$

$R_3 \to R_3 + 7R_2$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & -\lambda & -1 \\ 0 & -3 & 1+2\lambda & 4 \\ 0 & 0 & 2\lambda+7 & 16 \end{array}\right]$$

clearly which is in echelon form

If $2\lambda + 7 \neq 0$ then $\lambda \neq -7/2$

$\therefore \rho(A|B) = \rho(A) = 3$
= The number of unknown variables

$\therefore$ The given system is consistent and has unique solution.

If $\lambda = -7/2$ then

$\rho(A|B) = 3$ & $\rho(A) = 2$

$\therefore \rho(A|B) \neq \rho(A)$

$\therefore$ the given is inconsistent and has no solution.

Imp. For what values of λ the equations
$x+y+z=1$
$x+2y+4z=\lambda$
$x+4y+10z=\lambda^2$
have a solution and solve them completely in each case

$\longrightarrow$ Discuss for all values of λ, the system of equations $x+y+4z=6$,
$x+2y-2z=6$
$\lambda x+y+z=6$, as regards existence and nature of solutions.

Sol'n :- Now write the matrix equation of the system is

$$AX = \begin{bmatrix} 1 & 1 & 4 \\ 1 & 2 & -2 \\ \lambda & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \\ 6 \end{bmatrix} = B$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - \lambda R_1$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & -6 \\ 0 & 1-\lambda & 1-4\lambda \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 6-6\lambda \end{bmatrix} \quad (1)$$

The given system of equations will have unique solution iff coefficient matrix is non-singular matrix.

i.e $\begin{vmatrix} 1 & 1 & 4 \\ 0 & 1 & -6 \\ 0 & 1-\lambda & 1-4\lambda \end{vmatrix} \neq 0$

$\Rightarrow 1-4\lambda+6-6\lambda \neq 0$

$\Rightarrow \lambda \neq 7/10$.

If $\lambda = 7/10$ then the equation (1) becomes.

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & -6 \\ 0 & \frac{3}{10} & \frac{-18}{10} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 18/10 \end{bmatrix}$$

$R_3 \to R_3 - 3/10 R_2$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 4 \\ 0 & 1 & -6 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 18/10 \end{bmatrix}$$

showing that the equations are inconsistent

* Cramer's Rule of solving a system of 'n' non-homogeneous linear equations in 'n' unknowns:

Let the given system be

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ \cdots\cdots\cdots & \\ \cdots\cdots\cdots & \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 \cdots + a_{nn}x_n &= b_n \end{aligned}\right\}$$

Let

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \neq 0.$$

Let $A_{11}, A_{12}, A_{13}, \ldots$ etc denote the cofactors of $a_{11}, a_{12}, a_{13} \ldots$ in Δ then multiplying by the given equations respectively by $a_{11}, a_{21}, a_{31}, a_{41}, \cdots A_{n1}$ and adding.

we obtain,

$$x_1(a_{11}A_{11} + a_{21}A_{21} + a_{31}A_{31} + \cdots + a_{n1}A_{n1}) + x_2(0) + x_3(0) + \cdots + x_n(0) = b_1A_{11} + b_2A_{21} + b_3A_{31} + \cdots + b_nA_{n1}$$

$$\Rightarrow x_1\Delta = \Delta_1$$

where Δ_1 is the determinant obtained by replacing the elements in the first column of Δ by the elements $b_1, b_2, \ldots b_n$.

Similarly $x_2\Delta = \Delta_2$

$x_3\Delta = \Delta_3$

$\cdots\cdots$

$\cdots\cdots$

$x_n\Delta = \Delta_n$

where Δ_i is the determinant obtained by replacing the ith column in Δ by the elements $b_1, b_2 \cdots b_n$.

If $\Delta \neq 0$, then

$$x_1 = \frac{\Delta_1}{\Delta},\ x_2 = \frac{\Delta_2}{\Delta} \cdots\cdots x_n = \frac{\Delta_n}{\Delta}.$$

This method of solving n non-homogeneous linear equations in n' unknowns is called Cramer's Rule.

Note: If $\Delta = 0$, Cramer's rule of solving is not applicable

Problems:

→ solve the equations

$$x + y + z = 6,$$
$$x - y + z = 2$$
$$2x - y + 3z = 9$$

Sol'n:- ① By Cramer's Rule

The given system of 3 non homogeneous linear equations in 3 unknowns is

$$x+y+z=6$$
$$x-y+z=2$$
$$2x-y+3z=9$$

Let
$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & -1 & 3 \end{vmatrix} = -2 \neq 0$$

$\therefore$ By Cramer's Rule

$$x = \frac{\Delta_1}{\Delta}, \quad y = \frac{\Delta_2}{\Delta}, \quad z = \frac{\Delta_3}{\Delta}$$

where $\Delta_1 = \begin{vmatrix} 6 & 1 & 1 \\ 2 & -1 & 1 \\ 9 & -1 & 3 \end{vmatrix} = -2,$

$$\Delta_2 = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 2 & 1 \\ 2 & 9 & 3 \end{vmatrix} = -4$$

$$\Delta_3 = \begin{vmatrix} 1 & 1 & 6 \\ 1 & -1 & 2 \\ 2 & -1 & 9 \end{vmatrix} = -6$$

$\therefore x = \frac{-2}{2}, \quad y = \frac{-4}{-2}$ and $z = \frac{-6}{-2}$

$\Rightarrow x = -1, \quad y = 2$ and $z = 3$

(2) By Inversion Method :-

The given system can be expressed as $AX = B$.

where $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & -1 & 3 \end{bmatrix}$

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \qquad B = \begin{bmatrix} 6 \\ 2 \\ 9 \end{bmatrix}$$

Now $|A| = -2 \neq 0$

$\therefore A^{-1}$ exists

$$A^{-1} = \frac{1}{|A|}\operatorname{adj} A$$
$$= \frac{1}{-2}\begin{bmatrix} -2 & -4 & 2 \\ -1 & 1 & 0 \\ 1 & 3 & -2 \end{bmatrix}$$

Now $AX = B \Rightarrow X = A^{-1}B$

$$X = \frac{-1}{2}\begin{bmatrix} -2 & -4 & 2 \\ -1 & 1 & 0 \\ 1 & 3 & -2 \end{bmatrix}\begin{bmatrix} 6 \\ 2 \\ 9 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\Rightarrow x = 1, \; y = 2, \; z = 3.$

(3)

The given system can be expressed as

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & -1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 2 \\ 9 \end{bmatrix}$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - 2R_1$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & -3 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ -3 \end{bmatrix}$$

$R_3 \to R_3 - \frac{3}{2}R_2$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ -4 \\ 3 \end{bmatrix}$$

$$\Rightarrow \left.\begin{aligned} x+y+z &= 6 \\ -2y &= -4 \\ z &= 3 \end{aligned}\right\} \text{——(i)}$$

from (i) $\boxed{z=3}$, $\boxed{y=2}$, $\boxed{x=1}$

Imp. Solve the equations

$\lambda x + 2y - 2z = 1$

$4x + 2\lambda y - z = 2$

$6x + 6y + \lambda z = 3$ Considering specially the case when $\lambda = 2$

Sol'n :- write the matrix equation of the given system.

$$AX = \begin{bmatrix} \lambda & 2 & -2 \\ 4 & 2\lambda & -1 \\ 6 & 6 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = B \quad \text{——(1)}$$

The given system of equations will have a unique solution iff the coefficient matrix is non-singular.

$$\text{i.e} \begin{vmatrix} \lambda & 2 & -2 \\ 4 & 2\lambda & -1 \\ 6 & 6 & \lambda \end{vmatrix} \neq 0$$

$$\Rightarrow \lambda^3 + 11\lambda - 30 \neq 0$$

$$\Rightarrow (\lambda-2)(\lambda^2+2\lambda+15) \neq 0$$

Now the only the real root of the equation $(\lambda-2)(\lambda^2+2\lambda+15)=0$ is $\lambda = 2$

$\therefore$ If $\lambda \neq 2$ then the given system of equations will have a unique solution given by

$$x = \frac{\begin{vmatrix} 1 & 2 & -2 \\ 2 & 2\lambda & -1 \\ 3 & 6 & \lambda \end{vmatrix}}{\begin{vmatrix} \lambda & 2 & -2 \\ 4 & 2\lambda & -1 \\ 6 & 6 & \lambda \end{vmatrix}} \qquad y = \frac{\begin{vmatrix} \lambda & 1 & -2 \\ 4 & 2 & -1 \\ 6 & 3 & \lambda \end{vmatrix}}{\begin{vmatrix} \lambda & 2 & -2 \\ 4 & 2\lambda & -1 \\ 6 & 6 & \lambda \end{vmatrix}}$$

$$z = \frac{\begin{vmatrix} \lambda & 2 & 1 \\ 4 & 2\lambda & 2 \\ 6 & 6 & 3 \end{vmatrix}}{\begin{vmatrix} \lambda & 2 & -2 \\ 4 & 2\lambda & -1 \\ 6 & 6 & \lambda \end{vmatrix}}$$

In Case $\lambda = 2$

$$(1) \equiv \begin{bmatrix} 2 & 2 & -2 \\ 4 & 4 & -1 \\ 6 & 6 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - 3R_1$

$$\Rightarrow \begin{bmatrix} 2 & 2 & -2 \\ 0 & 0 & 3 \\ 0 & 0 & 8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{aligned} 2x+2y-2z &= 1 \\ 3z &= 0 \\ 8z &= 0 \end{aligned}$$

$\Rightarrow$ $z=0$, $2x+2y=1$

taking $y=c$ in the above where c is arbitrary

$\therefore$ $2x = 1-2c$

$$\boxed{x = \frac{1-2c}{2}}$$

$\therefore$ $x=\frac{1-2c}{2}$, $y=c$, $z=0$ constitute the general solution of the given system.

Imp $\rightarrow$ Investigate for what values of a, b the equations.

$x+2y+3z=4$,

$x+3y+4z=5$,

$x+3y+az=b$ have

(i) no solution (ii) a unique solution (iii) an infinite number of solutions.

$\longrightarrow$ solve completely the equations

$2x+3y+z=9$,

$x+2y+3z=6$,

$3x+y+2z=8$

* Homogeneous Linear Equations:—

Let
$$\left.\begin{aligned} a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n&=0\\ a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n&=0\\ &\vdots\\ a_{m1}x_1+a_{m2}x_2+\cdots+a_{mn}x_n&=0 \end{aligned}\right\} \rightarrow (1)$$

be a system of m homogeneous equations in 'n' unknowns $x_1, x_2 \cdots x_n$

then the system (1) can be expressed as the matrix equation $AX=O$ —— (2)

where $A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & & & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{bmatrix}_{m\times n}$

$X=\begin{bmatrix} x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix}_{n\times 1}$ $\quad O=\begin{bmatrix}0\\0\\ \vdots\\ 0\end{bmatrix}_{m\times 1}$

$\rightarrow$ The matrix A is called the Coefficient matrix

Now $x_1=0$, $x_2=0 \cdots x_n=0$

i.e $X=O$ is a solution of (1)

— This solution is called the trivial solution $AX=O$.

— Also the trivial solution is called the zero solution. and any other solution is called non-trivial solution. (i.e. non-zero solution)

$\therefore Ax=0$ is always consistent

Imp. If $AX=0$ is a homogeneous system and x_1, x_2 are two solutions of ① then the solution set of ① is a subspace of $V_n(F)$, vectorspace of n-tupples over F

Sol'n :- Given that x_1 & x_2 are two solutions of $AX=0$ —— ①

$\therefore AX_1=0$ & $AX_2=0$ —— ②

Now for $K_1, K_2 \in F$

we have $K_1(AX_1)=0$ &
$K_2(AX_2)=0$

$\Rightarrow A(K_1x_1)=0$ and
$A(K_2x_2)=0$

$\Rightarrow A[K_1x_1+K_2x_2]=0$

$\therefore K_1x_1+K_2X_2$ is also a solution of $AX=0$.

$\therefore$ The solution set of $AX=0$ is a subspace of $V_n(F)$

Imp $\rightarrow$ The number of L.I solutions of the linear system $Ax=0$ is $n-r$, r being the rank of the matrix $A_{m\times n}$.

Proof :- Let

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \text{ and } X=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$$

Since the rank of the coefficient matrix A is r

It has r L.I columns without loss of generality we can suppose that the first r columns from the left of the matrix A are L.I

Let $A=[C_1\ C_2\ C_3 \cdots C_r\ C_{r+1} \cdots C_n]_{1\times n}$

where $C_1, C_2, \cdots C_n$ are the columns vectors of the matrix A each of them being an m-vectors

$\therefore$ The equation $Ax=0$ becomes

$$[C_1\ C_2\ C_3 \cdots C_r\ C_{r+1} \cdots C_n]\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}=0$$

$$\Rightarrow x_1C_1+x_2C_2+x_3C_3+\cdots+x_rC_r+x_{r+1}C_{r+1}+\cdots+x_nC_n=0 \quad \text{—— ①}$$

Since each of the vectors $C_{r+1}, C_{r+2}, \cdots C_n$ is a linear combination of the vectors $C_1, C_2 \cdots C_r$ we have

$$\left.\begin{aligned} C_{r+1} &= K_{11}C_1+K_{12}C_2+\cdots+K_{1r}C_r \\ C_{r+2} &= K_{21}C_1+K_{22}C_2+\cdots+K_{2r}C_r \\ &\cdots\cdots\cdots \\ C_n &= K_{p1}C_1+K_{p2}C_2+\cdots+K_{pr}C_r \end{aligned}\right\} \rightarrow ②$$

where K's $\in F$ and $p=n-r$.

i.e

$$\left.\begin{aligned} &K_{11}C_1+K_{12}C_2+\cdots+K_{1r}C_r+(-1)C_{r+1}+0C_{r+2}+\cdots+0C_n=0 \\ &K_{21}C_1+K_{22}C_2+\cdots+K_{2r}C_r+0C_{r+1}+(-1)C_{r+2}+\cdots+0C_n=0 \\ &\cdots\cdots\cdots \\ &K_{p1}C_1+K_{p2}C_2+\cdots+K_{pr}C_r+0C_{r+1}+0C_{r+2}+\cdots+(-1)C_n=0 \end{aligned}\right\} \text{—— ③}$$

comparing ① & ③, we get

$$x_1 = \begin{bmatrix} k_{11} \\ k_{12} \\ \vdots \\ k_{1r} \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad x_2 = \begin{bmatrix} k_{21} \\ k_{22} \\ \vdots \\ k_{2r} \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \cdots x_p = x_{n-r} = \begin{bmatrix} k_{p_1} \\ k_{p_2} \\ \vdots \\ k_{p_r} \\ 0 \\ \vdots \\ -1 \end{bmatrix}$$

as $n-r$ solutions of $Ax=0$.

Now we show that these $(n-r)$ solutions form a L.I system.

Let $k_1x_1 + k_2x_2 + \cdots + k_{n-r}x_{n-r} = 0$ for some k's $\in F$ —— ④

then equating $(r+1)^{th}, (r+2)^{th}, (r+3)^{th}, \cdots n^{th}$ entries of columns on the two sides of ④, we get,

$-k_1 = 0, \ -k_2 = 0, \cdots -k_{n-r} = 0$

$\Rightarrow k_1 = k_2 = \cdots = k_{n-r} = 0$

It follows that $x_1, x_2 \cdots x_{n-r}$ form a $(n-r)$ linearly independent solutions.

Further we show that every solution of $AX=0$ is some suitable linear combination of these $(n-r)$ solutions.

Let x, with components $x_1, x_2 \cdots x_n$ be a solution of ①

Consider the vector

$$x + x_{r+1}x_1 + x_{r+2}x_2 + \cdots + x_{r+(n-r)}x_{n-r} \longrightarrow ⑤$$

Since ⑤ is a linear combination of solutions,

⑤ is also a solution of $AX=0$

It is quite obvious that the last $(n-r)$ components of ⑤ are all zero's. i.e.

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \\ \vdots \\ x_n \end{bmatrix} + x_{r+1}\begin{bmatrix} k_{11} \\ k_{12} \\ \vdots \\ k_{1r} \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_{r+2}\begin{bmatrix} k_{21} \\ k_{22} \\ \vdots \\ k_{2r} \\ 0 \\ -1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n\begin{bmatrix} k_{p_1} \\ k_{p_2} \\ \vdots \\ k_{p_r} \\ 0 \\ \vdots \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + x_{r+1}k_{11} + x_{r+2}k_{21} + \cdots + x_nk_{p_1} \\ x_2 + x_{r+1}k_{12} + x_{r+2}k_{22} + \cdots + x_nk_{p_2} \\ - - - - - - - - - \\ - - - - - - - - - \\ x_r + x_{r+1}k_{1r} + \cdots + x_nk_{p_r} \\ x_{r+1} + x_{r+1}(-1) + \cdots + x_n(0) \\ - - - - - - - - - \\ - - - - - - - - - \\ x_n + x_{r+1}(0) + x_{r+2}(0) + \cdots x_n(-1) \end{bmatrix}$$

$$= \begin{bmatrix} x_1 + x_{r+1}k_{11} + x_{r+2}k_{21} + \cdots + x_nk_{p_1} \\ x_2 + x_{r+1}k_{12} + x_{r+2}k_{22} + \cdots + x_nk_{p_2} \\ - - - - - - - - - \\ - - - - - - - - - \\ x_r + x_{r+1}k_{1r} + \cdots + x_nk_{p_r} \\ - - - - - - - - - \\ - - - - - - - - - \\ 0 + 0 + 0 + \cdots + 0 \\ 0 + 0 + 0 + \cdots + 0 \end{bmatrix}$$

Let the first r components (entries) be $y_1, y_2, y_3 \cdots y_r$

such a vector with components (entries) $y_1, y_2, y_3 \cdots y_r, 0, 0, 0, \cdots 0$ is also solution of $AX=0$.

$\therefore$ from ①,

$$y_1 C_1 + y_2 C_2 + \cdots + y_r C_r + 0C_{r+1} + 0C_{r+2} + \cdots + 0C_n = 0$$

$$\Rightarrow y_1 C_1 + y_2 C_2 + \cdots + y_r C_r = 0$$

But $C_1, C_2, \cdots C_r$ are L.I

$\therefore y_1 = 0, \ y_2 = 0, \cdots y_r = 0.$

$\therefore$ from ⑤

$$X + x_{r+1} X_1 + x_{r+2} X_2 + \cdots + x_n X_{n-r} = 0$$

$$\Rightarrow X = (-x_{r+1}) X_1 + (-x_{r+2}) X_2 + \cdots + (-x_n) X_{n-r}.$$

$\therefore$ Every solution X is a linear combination of the $(n-r)$ linearly independent solutions $X_1, X_2 \cdots X_{n-r}$.

where r is the rank of A.

$\therefore$ The set of solutions $\{X_1, X_2, \cdots X_{n-r}\}$ forms a basis for the vectorspace of all solutions of the system of equations $AX=0$

* Working rule for finding the solution of the equation $AX=0$:—

Let $AX=0$ be a given system of m homogeneous equations in 'n' variables then the coefficient matrix A is of the type $m \times n$.

— Reduce the coefficient matrix A to echelon form by applying elementry row transformations only.

This echelon form will help us to know the rank of the matrix A.

— Let $\rho(A) = r$ then the system $AX=0$ has $n-r$ L.I solutions.

and $\rho(A) \leq \min\{m,n\}$ then the following cases will arise

Case(i): If $\rho(A) = r = n$ (unknowns) then the system has $n-r = n-n = 0$ L.I solution.

i.e. it has no L.I solution and the only solution of $AX=0$ is the trivial solution $x_1 = x_2 = \cdots = x_n = 0$ (i.e. the zero solution).

Note: A set containing zero vector is always L.D.

Case(ii):- If $\rho(A) = r < n \leq m$ (or) $r < m \leq n$ then the system $AX=0$

has $(n-r)$ L.I solutions.

In the process of reducing the matrix A to echelon form, $(m-r)$ equations will be eliminated.

Therefore the given system of m equations will be replaced by an equivalent system of 'r' equations in 'n' unknown variables.

Express the values of some 'r' unknowns in terms of remaining $(n-r)$ unknown variables. These $(n-r)$ unknowns can be given any aribtrarily chosen values.

In this case the system has infinitely many solutions which form a vector space of dimension $(n-r)$.

Problems

→ Solve the system of equations

$$x+y-3z+2w=0$$
$$2x-y+2z-3w=0$$
$$3x-2y+z-4w=0$$
$$-4x+y-3z+w=0$$

Sol'n :- Now write the matrix equation of given system is

$$Ax=\begin{bmatrix}1&1&-3&2\\2&-1&2&-3\\3&-2&1&-4\\4&1&-3&1\end{bmatrix}\begin{bmatrix}x\\y\\z\\w\end{bmatrix}=\begin{bmatrix}0\\0\\0\\0\end{bmatrix}=0 \quad \text{—(1)}$$

The coefficient matrix

$$A=\begin{bmatrix}1&1&-3&2\\2&-1&2&-3\\3&-2&1&-4\\-4&1&-3&1\end{bmatrix}$$

$R_2\to R_2-2R_1$
$R_3\to R_3-3R_1$
$R_4\to R_4+4R_1$

$$\sim\begin{bmatrix}1&1&-3&2\\0&-3&8&-7\\0&-5&10&-10\\0&5&-15&9\end{bmatrix}$$

$R_2\longleftrightarrow R_3$

$$\begin{bmatrix}1&1&-3&2\\0&-5&10&-10\\0&-3&8&-7\\0&5&-15&9\end{bmatrix}$$

$R_2\to -\frac{1}{5}R_2$

$$\sim\begin{bmatrix}1&1&-3&2\\0&1&-2&2\\0&-3&8&-7\\0&5&-15&9\end{bmatrix}$$

$R_4\to R_4-5R_2$
$R_3\to R_3+3R_2$

$$\sim\begin{bmatrix}1&1&-3&2\\0&1&-2&2\\0&0&2&-1\\0&0&-5&-1\end{bmatrix}$$

$R_4\to -\frac{2}{5}R_4$

$$\sim\begin{bmatrix}1&1&-3&2\\0&1&-2&2\\0&0&2&-1\\0&0&2&2/5\end{bmatrix}$$

R.

$$\begin{bmatrix} 1 & 1 & -3 & 2 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 2 & -1 \\ 0 & 0 & 0 & 7/5 \end{bmatrix} \quad R_4 \to R_4 - R_3$$

∴ clearly which is in echelon form

∴ $\rho(A) = 4$

= the number of four unknowns.

∴ the given system of equations has the zero solution.

i.e. $x = y = z = w = 0$.

→ solve completeley the system of equations $x + 3y - 2z = 0$,

$2x - y + 4z = 0$,

$x - 11y + 14z = 0$.

sol^n :- write the matrix equation $AX = O$ where

$$A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}, \; X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \; O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{Now } A = \begin{bmatrix} 1 & 3 & -2 \\ 2 & -1 & 4 \\ 1 & -11 & 14 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$R_3 \to R_3 - R_1$

$$\sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & -14 & 16 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix} \quad R_3 \to R_3 - 2R_2$$

Clearly which is in echelon form.

∴ $\rho(A) = 2 <$ the number of three unknown variables

∴ The given system of equations has non-zero solution.

∴ The given system of equations will have $n - r = 3 - 2 = 1$ L.I solution.

Now we take the arbitrary value to $n - r = 3 - 2 = 1$ variable.

and the remaining 2 variables will be expressed interms of these

Again we write matrix equation in echelon form is

$$\begin{bmatrix} 1 & 3 & -2 \\ 0 & -7 & 8 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{rcases} x + 3y - 2z = 0 \\ -7y + 8z = 0 \end{rcases} \begin{matrix} (1) \\ (2) \end{matrix}$$

Let $\boxed{z = k_1}$ then (2) ≡ $\boxed{y = 8/7\, k_1}$

where k_1 is arbitrary constant and (1) ≡ $\boxed{x = -10/7\, k_1}$

∴ The general solution of (1) & (2) is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -10/7\, k_1 \\ 8/7\, k_1 \\ k_1 \end{bmatrix}$$

$$= K_1 \begin{bmatrix} -10/7 \\ 8/7 \\ 1 \end{bmatrix}$$

$\therefore$ the general solution of $AX = O$ is given by

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = K_1 \begin{bmatrix} -10/7 \\ 8/7 \\ 1 \end{bmatrix}$$ where K_1 is arbitrary constant.

therefore the solution $\begin{bmatrix} -10/7 \\ 8/7 \\ 1 \end{bmatrix}$ is L.I.

$\therefore \left\{\left(-10/7, 8/7, 1\right)^T\right\}$ is a basis of the solution space.

→ Find a basis and the dimension of the solution space S of the linear equations

$$x + 2y - 2z + 2s - t = 0$$
$$x + 2y - z + 3s - 2t = 0$$
$$2x + 4y - 7z + s + t = 0$$

Sol^n: write the matrix equation of the given system is $AX = O$

where $A = \begin{bmatrix} 1 & 2 & -2 & 2 & -1 \\ 1 & 2 & -1 & 3 & -1 \\ 2 & 4 & -7 & 1 & 1 \end{bmatrix}$

$$X = \begin{bmatrix} x \\ y \\ z \\ s \\ t \end{bmatrix}, \quad O = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Now

$$A = \begin{bmatrix} 1 & 2 & -2 & 2 & -1 \\ 1 & 2 & -1 & 3 & -1 \\ 2 & 4 & -7 & 1 & 1 \end{bmatrix}$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 + 3R_2$

$$\sim \begin{bmatrix} 1 & 2 & -2 & 2 & -1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & -3 & -3 & 3 \end{bmatrix}$$

$R_3 \to R_3 + 3R_2$

$$\begin{bmatrix} 1 & 2 & -2 & 2 & -1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\therefore$ which is in echelon form.

$\therefore \rho(A) = 2 <$ then number of five unknown variables.

$\therefore$ the given system has non-zero solution.

$\therefore$ The given system will have $n - r = 5 - 2 = 3$ L.I solutions.

and the dimension of the solution space $S = 3$

Now we take the arbitrary values to $n - r = 5 - 2 \Rightarrow 3$ Variables. and the remaining '2' variables will be expressed interms these

Now we write the matrix equation in echelon form is

$$\begin{bmatrix} 1 & 2 & -2 & 2 & -1 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x+2y-2z+2w-t=0$ —— ①

$z+w-t=0$ —— ②

Let $t=k_1$, $w=k_2$

where k_1, k_2 are arbitrary constants

then ② $\equiv$ $\boxed{z=k_1-k_2}$

Let $y=k_3$ then ① $\equiv x+2k_3-2(k_1-k_2)+2k_2-k_1=0$

where k_3 arbitrary constant

$\Rightarrow x+2k_3-3k_1+4k_2=0$

$\Rightarrow \boxed{x=3k_1-4k_2-2k_3}$

$\therefore$ The general solution of ① & ② is

$$\begin{bmatrix}x\\y\\z\\s\\t\end{bmatrix}=\begin{bmatrix}3k_1-4k_2-2k_3\\k_3\\k_1-k_2\\k_2\\k_1\end{bmatrix}=\begin{bmatrix}3k_1-4k_2-2k_3\\0k_1+0k_2+k_3\\k_1-k_2+0k_3\\0k_1+k_2+0k_3\\k_1+0k_2+0k_3\end{bmatrix}$$

$$=k_1\begin{bmatrix}3\\0\\1\\0\\1\end{bmatrix}+k_2\begin{bmatrix}-4\\0\\-1\\1\\0\end{bmatrix}+k_3\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix}$$

where k_1, k_2, k_3 are arbitrary constants.

$\therefore$ The solutions

$$\begin{bmatrix}3\\0\\1\\0\\1\end{bmatrix} \quad \begin{bmatrix}-4\\0\\-1\\1\\0\end{bmatrix} \quad \begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix}$$ are L.I.

$\therefore$ The set of solutions

$\{(3,0,1,0,1)^T, (-4,0,-1,1,0)^T, (-2,1,0,0,0)^T\}$ form a basis of the solution space 'S' of the given system of equations.

→ H.W solve completely the system of equations $x+y+z=0$, $2x-y-3z=0$, $3x-5y+4z=0$, $x+17y+4z=0$

→ Find a basis of the solution space W of the system of equations

$x+2y-2z+2r+s=0$; $2x+4y-6z+5r=$

$2x+4y-2z+3r+4s=0$

$3x+6y-8z+7r+s=0$.

→ Prove that a necessary and sufficient condition that values, not all zero may be assigned to 'n' variables $x_1, x_2, x_3, x_4, \ldots x_n$, so that n homogeneous equations

$a_{i1}x_1+a_{i2}x_2+a_{i3}x_3+\cdots+a_{in}x_n=0$; $(i=1,2\ldots n)$ hold simultaneously, is that the determinant of the coefficient matrix vanishes.

Sol'n:- the coefficient matrix is $A=[a_{ij}]_{n\times n}$ and the given system is $AX=0$.

Now the given system has non-zero solution iff

$n-r>0$ where $r=\rho(A)$

i.e iff $r<n$

i.e. iff $\rho(A)<n$

i.e. iff A is singular

i.e. iff $|A|=0$

→ Show that the only real value of λ for which the following system of equations has a non-zero solution is 6.

$x+2y+3z=\lambda x$, $3x+y+2z=\lambda y$,

$2x+3y+z=\lambda z$.

Sol'n :- Write the matrix equation of the system is $AX=0$ —— ①

where

$$A=\begin{bmatrix}1-\lambda & 2 & 3\\ 3 & 1-\lambda & 2\\ 2 & 3 & 1-\lambda\end{bmatrix} \quad X=\begin{bmatrix}x\\y\\z\end{bmatrix} \quad O=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Here the coefficient matrix A is a square matrix of order 3×3 and the number of unknowns is also 3

Now the given system $AX=0$ has non-zero solution

iff $3-r>0$ where $r=\rho(A)$

i.e iff $3>0$

i.e iff $r<3$

i.e iff $|A|=0$

i.e. iff $\begin{vmatrix}1-\lambda & 2 & 3\\ 3 & 1-\lambda & 2\\ 2 & 3 & 1-\lambda\end{vmatrix}=0$

$C_1\to C_1+C_2+C_3$

i.e iff $\begin{vmatrix}6-\lambda & 2 & 3\\ 6-\lambda & 1-\lambda & 2\\ 6-\lambda & 3 & 1-\lambda\end{vmatrix}=0$

i.e. iff

$(6-\lambda)\begin{vmatrix}1 & 2 & 3\\ 1 & 1-\lambda & 2\\ 1 & 3 & 1-\lambda\end{vmatrix}=0$

$R_2\to R_2-R_1$
$R_3\to R_3-R_1$

i.e iff $(6-\lambda)\begin{vmatrix}1 & 2 & 3\\ 0 & -1-\lambda & -1\\ 0 & 1 & -2-\lambda\end{vmatrix}=0$

i.e iff $(6-\lambda)\left[(1+\lambda)(2+\lambda)+1\right]=0$

i.e. iff $(6-\lambda)(\lambda^2+3\lambda+3)=0$

i.e. iff $\lambda=6$ $\lambda=\frac{-3\pm\sqrt{9-12}}{2}$

i.e. iff $\lambda=6$ $\lambda=\frac{-3\pm\sqrt{-3}}{2}$

∴ The only real value of λ for which the given system has a non trivial solution is 6