4.1. Suppose u and v belong to a vector space V. Simplify each of the following expressions:

(a) $E_1 = 3(2u - 4v) + 5u + 7v$, (c) $E_3 = 2uv + 3(2u + 4v)$

(b) $E_2 = 3u - 6(3u - 5v) + 7u$, (d) $E_4 = 5u - \dfrac{3}{v} + 5u$

Multiply out and collect terms:

(a) $E_1 = 6u - 12v + 5u + 7v = 11u - 5v$

(b) $E_2 = 3u - 18u + 30v + 7u = -8u + 30v$

(c) E_3 is not defined because the product uv of vectors is not defined.

(d) E_4 is not defined because division by a vector is not defined.

4.3. Show that (a) $k(u - v) = ku - kv$, (b) $u + u = 2u$.

(a) Using the definition of subtraction, that $u - v = u + (-v)$, and Theorem 4.1(iv), that $k(-v) = -kv$, we have

$$k(u - v) = k[u + (-v)] = ku + k(-v) = ku + (-kv) = ku - kv$$

(b) Using Axiom [M_4] and then Axiom [M_2], we have

$$u + u = 1u + 1u = (1 + 1)u = 2u$$

4.4. Express $v = (1, -2, 5)$ in $\mathbf{R}^3$ as a linear combination of the vectors

$$u_1 = (1, 1, 1), \qquad u_2 = (1, 2, 3), \qquad u_3 = (2, -1, 1)$$

We seek scalars x, y, z, as yet unknown, such that $v = xu_1 + yu_2 + zu_3$. Thus, we require

$$\begin{bmatrix} 1 \\ -2 \\ 5 \end{bmatrix} = x\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + y\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + z\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \qquad \text{or} \qquad \begin{aligned} x + y + 2z &= 1 \\ x + 2y - z &= -2 \\ x + 3y + z &= 5 \end{aligned}$$

(For notational convenience, we write the vectors in $\mathbf{R}^3$ as columns, because it is then easier to find the equivalent system of linear equations.) Reducing the system to echelon form yields the triangular system

$$x + y + 2z = 1, \qquad y - 3z = -3, \qquad 5z = 10$$

The system is consistent and has a solution. Solving by back-substitution yields the solution $x = -6$, $y = 3$, $z = 2$. Thus, $v = -6u_1 + 3u_2 + 2u_3$.

Alternatively, write down the augmented matrix M of the equivalent system of linear equations, where u_1, u_2, u_3 are the first three columns of M and v is the last column, and then reduce M to echelon form:

$$M = \begin{bmatrix} 1 & 1 & 2 & 1 \\ 1 & 2 & -1 & -2 \\ 1 & 3 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 2 & -1 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 5 & 10 \end{bmatrix}$$

The last matrix corresponds to a triangular system, which has a solution. Solving the triangular system by back-substitution yields the solution $x = -6$, $y = 3$, $z = 2$. Thus, $v = -6u_1 + 3u_2 + 2u_3$.

4.5. Express $v = (2, -5, 3)$ in $\mathbf{R}^3$ as a linear combination of the vectors

$$u_1 = (1, -3, 2), \quad u_2 = (2, -4, -1), \quad u_3 = (1, -5, 7)$$

We seek scalars x, y, z, as yet unknown, such that $v = xu_1 + yu_2 + zu_3$. Thus, we require

$$\begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix} = x\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} + y\begin{bmatrix} 2 \\ -4 \\ -1 \end{bmatrix} + z\begin{bmatrix} 1 \\ -5 \\ 7 \end{bmatrix} \qquad \text{or} \qquad \begin{aligned} x + 2y + z &= 2 \\ -3x - 4y - 5z &= -5 \\ 2x - y + 7z &= 3 \end{aligned}$$

Reducing the system to echelon form yields the system

$$x + 2y + z = 2, \qquad 2y - 2z = 1, \qquad 0 = 3$$

The system is inconsistent and so has no solution. Thus, v cannot be written as a linear combination of u_1, u_2, u_3.

4.6. Express the polynomial $v = t^2 + 4t - 3$ in $\mathbf{P}(t)$ as a linear combination of the polynomials

$$p_1 = t^2 - 2t + 5, \qquad p_2 = 2t^2 - 3t, \qquad p_3 = t + 1$$

Set v as a linear combination of p_1, p_2, p_3 using unknowns x, y, z to obtain

$$t^2 + 4t - 3 = x(t^2 - 2t + 5) + y(2t^2 - 3t) + z(t + 1) \qquad (*)$$

We can proceed in two ways.

Method 1. Expand the right side of (*) and express it in terms of powers of t as follows:

$$\begin{aligned} t^2 + 4t - 3 &= xt^2 - 2xt + 5x + 2yt^2 - 3yt + zt + z \\ &= (x + 2y)t^2 + (-2x - 3y + z)t + (5x + 3z) \end{aligned}$$

Set coefficients of the same powers of t equal to each other, and reduce the system to echelon form. This yields

$$\begin{aligned} x + 2y &= 1 \\ -2x - 3y + z &= 4 \\ 5x + 3z &= -3 \end{aligned} \qquad \text{or} \qquad \begin{aligned} x + 2y &= 1 \\ y + z &= 6 \\ -10y + 3z &= -8 \end{aligned} \qquad \text{or} \qquad \begin{aligned} x + 2y &= 1 \\ y + z &= 6 \\ 13z &= 52 \end{aligned}$$

The system is consistent and has a solution. Solving by back-substitution yields the solution $x = -3$, $y = 2$, $z = 4$. Thus, $v = -3p_1 + 2p_2 + 4p_2$.

Method 2. The equation (*) is an identity in t; that is, the equation holds for any value of t. Thus, we can set t equal to any numbers to obtain equations in the unknowns.

(a) Set $t = 0$ in (*) to obtain the equation $-3 = 5x + z$.

(b) Set $t = 1$ in (*) to obtain the equation $2 = 4x - y + 2z$.

(c) Set $t = -1$ in (*) to obtain the equation $-6 = 8x + 5y$.

Solve the system of the three equations to again obtain the solution $x = -3$, $y = 2$, $z = 4$. Thus, $v = -3p_1 + 2p_2 + 4p_3$.

4.7. Express M as a linear combination of the matrices A, B, C, where

$$M = \begin{bmatrix} 4 & 7 \\ 7 & 9 \end{bmatrix}, \quad \text{and} \quad A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}$$

Set M as a linear combination of A, B, C using unknown scalars x, y, z; that is, set $M = xA + yB + zC$. This yields

$$\begin{bmatrix} 4 & 7 \\ 7 & 9 \end{bmatrix} = x\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + y\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + z\begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} x + y + z & x + 2y + z \\ x + 3y + 4z & x + 4y + 5z \end{bmatrix}$$

Form the equivalent system of equations by setting corresponding entries equal to each other:

$$x + y + z = 4, \qquad x + 2y + z = 7, \qquad x + 3y + 4z = 7, \qquad x + 4y + 5z = 9$$

Reducing the system to echelon form yields

$$x + y + z = 4, \qquad y = 3, \qquad 3z = -3, \qquad 4z = -4$$

The last equation drops out. Solving the system by back-substitution yields $z = -1$, $y = 3$, $x = 2$. Thus, $M = 2A + 3B - C$.

4.10. Let $V = \mathbf{P}(t)$, the vector space of real polynomials. Determine whether or not W is a subspace of V, where

(a) W consists of all polynomials with integral coefficients.

(b) W consists of all polynomials with degree ≥ 6 and the zero polynomial.

(c) W consists of all polynomials with only even powers of t.

(a) No, because scalar multiples of polynomials in W do not always belong to W. For example,

$$f(t) = 3 + 6t + 7t^2 \in W \qquad \text{but} \qquad \tfrac{1}{2}f(t) = \tfrac{3}{2} + 3t + \tfrac{7}{2}t^2 \notin W$$

(b and c) Yes. In each case, W contains the zero polynomial, and sums and scalar multiples of polynomials in W belong to W.

4.11. Let V be the vector space of functions $f : \mathbf{R} \to \mathbf{R}$. Show that W is a subspace of V, where

(a) $W = \{f(x) : f(1) = 0\}$, all functions whose value at 1 is 0.

(b) $W = \{f(x) : f(3) = f(1)\}$, all functions assigning the same value to 3 and 1.

(c) $W = \{f(t) : f(-x) = -f(x)\}$, all *odd functions*.

Let $\hat{0}$ denote the zero function, so $\hat{0}(x) = 0$ for every value of x.

(a) $\hat{0} \in W$, because $\hat{0}(1) = 0$. Suppose $f, g \in W$. Then $f(1) = 0$ and $g(1) = 0$. Also, for scalars a and b, we have

$$(af + bg)(1) = af(1) + bg(1) = a0 + b0 = 0$$

Thus, $af + bg \in W$, and hence W is a subspace.

(b) $\hat{0} \in W$, because $\hat{0}(3) = 0 = \hat{0}(1)$. Suppose $f, g \in W$. Then $f(3) = f(1)$ and $g(3) = g(1)$. Thus, for any scalars a and b, we have

$$(af + bg)(3) = af(3) + bg(3) = af(1) + bg(1) = (af + bg)(1)$$

Thus, $af + bg \in W$, and hence W is a subspace.

(c) $\hat{0} \in W$, because $\hat{0}(-x) = 0 = -0 = -\hat{0}(x)$. Suppose $f, g \in W$. Then $f(-x) = -f(x)$ and $g(-x) = -g(x)$. Also, for scalars a and b,

$$(af + bg)(-x) = af(-x) + bg(-x) = -af(x) - bg(x) = -(af + bg)(x)$$

Thus, $ab + gf \in W$, and hence W is a subspace of V.

4.17. Determine whether or not u and v are linearly dependent, where

(a) $u = (1,2)$, $v = (3,-5)$, (c) $u = (1,2,-3)$, $v = (4,5,-6)$

(b) $u = (1,-3)$, $v = (-2,6)$, (d) $u = (2,4,-8)$, $v = (3,6,-12)$

Two vectors u and v are linearly dependent if and only if one is a multiple of the other.

(a) No. (b) Yes; for $v = -2u$. (c) No. (d) Yes, for $v = \frac{3}{2}u$.

4.18. Determine whether or not u and v are linearly dependent, where

(a) $u = 2t^2 + 4t - 3$, $v = 4t^2 + 8t - 6$, (b) $u = 2t^2 - 3t + 4$, $v = 4t^2 - 3t + 2$,

(c) $u = \begin{bmatrix} 1 & 3 & -4 \\ 5 & 0 & -1 \end{bmatrix}, v = \begin{bmatrix} -4 & -12 & 16 \\ -20 & 0 & 4 \end{bmatrix}$, (d) $u = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix}, v = \begin{bmatrix} 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}$

Two vectors u and v are linearly dependent if and only if one is a multiple of the other.

(a) Yes; for $v = 2u$. (b) No. (c) Yes, for $v = -4u$. (d) No.

4.19. Determine whether or not the vectors $u = (1,1,2)$, $v = (2,3,1)$, $w = (4,5,5)$ in $\mathbf{R}^3$ are linearly dependent.

Method 1. Set a linear combination of u, v, w equal to the zero vector using unknowns x, y, z to obtain the equivalent homogeneous system of linear equations and then reduce the system to echelon form. This yields

$$x\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + y\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + z\begin{bmatrix} 4 \\ 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{or} \quad \begin{aligned} x + 2y + 4z &= 0 \\ x + 3y + 5z &= 0 \\ 2x + y + 5z &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} x + 2y + 4z &= 0 \\ y + z &= 0 \end{aligned}$$

The echelon system has only two nonzero equations in three unknowns; hence, it has a free variable and a nonzero solution. Thus, u, v, w are linearly dependent.

Method 2. Form the matrix A whose columns are u, v, w and reduce to echelon form:

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 5 \\ 2 & 1 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & -3 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

The third column does not have a pivot; hence, the third vector w is a linear combination of the first two vectors u and v. Thus, the vectors are linearly dependent. (Observe that the matrix A is also the coefficient matrix in Method 1. In other words, this method is essentially the same as the first method.)

Method 3. Form the matrix B whose rows are u, v, w, and reduce to echelon form:

$$B = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 3 & 1 \\ 4 & 5 & 5 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 2 \\ 0 & 1 & -3 \\ 0 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$$

Because the echelon matrix has only two nonzero rows, the three vectors are linearly dependent. (The three given vectors span a space of dimension 2.)

4.20. Determine whether or not each of the following lists of vectors in $\mathbf{R}^3$ is linearly dependent:

(a) $u_1 = (1,2,5)$, $u_2 = (1,3,1)$, $u_3 = (2,5,7)$, $u_4 = (3,1,4)$,

(b) $u = (1,2,5)$, $v = (2,5,1)$, $w = (1,5,2)$,

(c) $u = (1,2,3)$, $v = (0,0,0)$, $w = (1,5,6)$.

(a) Yes, because any four vectors in $\mathbf{R}^3$ are linearly dependent.

(b) Use Method 2 above; that is, form the matrix A whose columns are the given vectors, and reduce the matrix to echelon form:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 5 & 5 \\ 5 & 1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & -9 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 24 \end{bmatrix}$$

Every column has a pivot entry; hence, no vector is a linear combination of the previous vectors. Thus, the vectors are linearly independent.

(c) Because $0 = (0,0,0)$ is one of the vectors, the vectors are linearly dependent.

4.21. Show that the functions $f(t) = \sin t$, $g(t) \cos t$, $h(t) = t$ from $\mathbf{R}$ into $\mathbf{R}$ are linearly independent.

Set a linear combination of the functions equal to the zero function $\mathbf{0}$ using unknown scalars x, y, z; that is, set $xf + yg + zh = \mathbf{0}$. Then show $x = 0, y = 0, z = 0$. We emphasize that $xf + yg + zh = \mathbf{0}$ means that, for every value of t, we have $xf(t) + yg(t) + zh(t) = 0$.

Thus, in the equation $x \sin t + y \cos t + zt = 0$:

(i)	Set $t = 0$	to obtain	$x(0) + y(1) + z(0) = 0$	or	$y = 0$.
(ii)	Set $t = \pi/2$	to obtain	$x(1) + y(0) + z\pi/2 = 0$	or	$x + \pi z/2 = 0$.
(iii)	Set $t = \pi$	to obtain	$x(0) + y(-1) + z(\pi) = 0$	or	$-y + \pi z = 0$.

The three equations have only the zero solution; that is, $x = 0$, $y = 0$, $z = 0$. Thus, f, g, h are linearly independent.

4.22. Suppose the vectors u, v, w are linearly independent. Show that the vectors $u + v$, $u - v$, $u - 2v + w$ are also linearly independent.

Suppose $x(u + v) + y(u - v) + z(u - 2v + w) = 0$. Then

$$xu + xv + yu - yv + zu - 2zv + zw = 0$$

or

$$(x + y + z)u + (x - y - 2z)v + zw = 0$$

Because u, v, w are linearly independent, the coefficients in the above equation are each 0; hence,

$$x + y + z = 0, \qquad x - y - 2z = 0, \qquad z = 0$$

The only solution to the above homogeneous system is $x = 0$, $y = 0$, $z = 0$. Thus, $u + v$, $u - v$, $u - 2v + w$ are linearly independent.

4.23. Show that the vectors $u = (1 + i, \ 2i)$ and $w = (1, \ 1 + i)$ in $\mathbf{C}^2$ are linearly dependent over the complex field $\mathbf{C}$ but linearly independent over the real field $\mathbf{R}$.

Recall that two vectors are linearly dependent (over a field K) if and only if one of them is a multiple of the other (by an element in K). Because

$$(1 + i)w = (1 + i)(1, \ 1 + i) = (1 + i, \ 2i) = u$$

u and w are linearly dependent over $\mathbf{C}$. On the other hand, u and w are linearly independent over $\mathbf{R}$, as no real multiple of w can equal u. Specifically, when k is real, the first component of $kw = (k, \ k + ki)$ must be real, and it can never equal the first component $1 + i$ of u, which is complex.

Basis and Dimension

4.24. Determine whether or not each of the following form a basis of $\mathbf{R}^3$:

(a) $(1, 1, 1), (1, 0, 1)$; (c) $(1, 1, 1), (1, 2, 3), (2, -1, 1)$;

(b) $(1, 2, 3), (1, 3, 5), (1, 0, 1), (2, 3, 0)$; (d) $(1, 1, 2), (1, 2, 5), (5, 3, 4)$.

(a and b) No, because a basis of $\mathbf{R}^3$ must contain exactly three elements because $\dim \mathbf{R}^3 = 3$.

(c) The three vectors form a basis if and only if they are linearly independent. Thus, form the matrix whose rows are the given vectors, and row reduce the matrix to echelon form:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & -3 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 5 \end{bmatrix}$$

The echelon matrix has no zero rows; hence, the three vectors are linearly independent, and so they do form a basis of $\mathbf{R}^3$.

4.29. Find a basis and dimension of the subspace W of $\mathbf{R}^3$ where

(a) $W = \{(a, b, c) : a + b + c = 0\}$, (b) $W = \{(a, b, c) : (a = b = c)\}$

(a) Note that $W \neq \mathbf{R}^3$, because, for example, $(1, 2, 3) \notin W$. Thus, $\dim W < 3$. Note that $u_1 = (1, 0, -1)$ and $u_2 = (0, 1, -1)$ are two independent vectors in W. Thus, $\dim W = 2$, and so u_1 and u_2 form a basis of W.

(b) The vector $u = (1, 1, 1) \in W$. Any vector $w \in W$ has the form $w = (k, k, k)$. Hence, $w = ku$. Thus, u spans W and $\dim W = 1$.

4.30. Let W be the subspace of $\mathbf{R}^4$ spanned by the vectors

$$u_1 = (1, -2, 5, -3), \qquad u_2 = (2, 3, 1, -4), \qquad u_3 = (3, 8, -3, -5)$$

(a) Find a basis and dimension of W. (b) Extend the basis of W to a basis of $\mathbf{R}^4$.

(a) Apply Algorithm 4.1, the row space algorithm. Form the matrix whose rows are the given vectors, and reduce it to echelon form:

$$A = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 2 & 3 & 1 & -4 \\ 3 & 8 & -3 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 14 & -18 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The nonzero rows $(1, -2, 5, -3)$ and $(0, 7, -9, 2)$ of the echelon matrix form a basis of the row space of A and hence of W. Thus, in particular, $\dim W = 2$.

(b) We seek four linearly independent vectors, which include the above two vectors. The four vectors $(1, -2, 5, -3)$, $(0, 7, -9, 2)$, $(0, 0, 1, 0)$, and $(0, 0, 0, 1)$ are linearly independent (because they form an echelon matrix), and so they form a basis of $\mathbf{R}^4$, which is an extension of the basis of W.

4.31. Let W be the subspace of $\mathbf{R}^5$ spanned by $u_1 = (1, 2, -1, 3, 4)$, $u_2 = (2, 4, -2, 6, 8)$, $u_3 = (1, 3, 2, 2, 6)$, $u_4 = (1, 4, 5, 1, 8)$, $u_5 = (2, 7, 3, 3, 9)$. Find a subset of the vectors that form a basis of W.

Here we use Algorithm 4.2, the casting-out algorithm. Form the matrix M whose columns (not rows) are the given vectors, and reduce it to echelon form:

$$M = \begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 2 & 4 & 3 & 4 & 7 \\ -1 & -2 & 2 & 5 & 3 \\ 3 & 6 & 2 & 1 & 3 \\ 4 & 8 & 6 & 8 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 6 & 5 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & 2 & 4 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivot positions are in columns C_1, C_3, C_5. Hence, the corresponding vectors u_1, u_3, u_5 form a basis of W, and $\dim W = 3$.

4.41. Find the rank and basis of the row space of each of the following matrices:

$$\text{(a)}\quad A = \begin{bmatrix} 1 & 2 & 0 & -1 \\ 2 & 6 & -3 & -3 \\ 3 & 10 & -6 & -5 \end{bmatrix}, \qquad \text{(b)}\quad B = \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{bmatrix}.$$

(a) Row reduce A to echelon form:

$$A \sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 2 & -3 & -1 \\ 0 & 4 & -6 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -1 \\ 0 & 2 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The two nonzero rows $(1,2,0,-1)$ and $(0,2,-3,-1)$ of the echelon form of A form a basis for rowsp(A). In particular, $\operatorname{rank}(A) = 2$.

(b) Row reduce B to echelon form:

$$B \sim \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & -3 & -6 & -3 & 3 \\ 0 & -1 & -2 & -1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 1 & -2 & -3 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The two nonzero rows $(1,3,1,-2,-3)$ and $(0,1,2,1,-1)$ of the echelon form of B form a basis for rowsp(B). In particular, $\operatorname{rank}(B) = 2$.

4.42. Show that $U = W$, where U and W are the following subspaces of $\mathbf{R}^3$:

$$U = \operatorname{span}(u_1, u_2, u_3) = \operatorname{span}(1,1,-1),\ (2,3,-1),\ (3,1,-5)\}$$
$$W = \operatorname{span}(w_1, w_2, w_3) = \operatorname{span}(1,-1,-3),\ (3,-2,-8),\ (2,1,-3)\}$$

Form the matrix A whose rows are the u_i, and row reduce A to row canonical form:

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 3 & -1 \\ 3 & 1 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Next form the matrix B whose rows are the w_j, and row reduce B to row canonical form:

$$B = \begin{bmatrix} 1 & -1 & -3 \\ 3 & -2 & -8 \\ 2 & 1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -3 \\ 0 & 1 & 1 \\ 0 & 3 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Because A and B have the same row canonical form, the row spaces of A and B are equal, and so $U = W$.

4.50. Find the dimension and a basis of the solution space W of each homogeneous system:

$$\begin{aligned} x+2y+2z-s+3t&=0\\ x+2y+3z+s+\ t&=0\\ 3x+6y+8z+s+5t&=0 \end{aligned} \qquad \begin{aligned} x+2y+\ z-2t&=0\\ 2x+4y+4z-3t&=0\\ 3x+6y+7z-4t&=0 \end{aligned} \qquad \begin{aligned} x+\ y+2z&=0\\ 2x+3y+3z&=0\\ x+3y+5z&=0 \end{aligned}$$

(a) (b) (c)

(a) Reduce the system to echelon form:

$$\begin{aligned} x+2y+2z-\ s+3t&=0\\ z+2s-2t&=0\\ 2z+4s-4t&=0 \end{aligned} \qquad \text{or} \qquad \begin{aligned} x+2y+2z-\ s+3t&=0\\ z+2s-2t&=0 \end{aligned}$$

The system in echelon form has two (nonzero) equations in five unknowns. Hence, the system has $5-2=3$ free variables, which are y, s, t. Thus, $\dim W=3$. We obtain a basis for W:

(1) Set $y=1, s=0, t=0$ to obtain the solution $v_1=(-2,1,0,0,0)$.
(2) Set $y=0, s=1, t=0$ to obtain the solution $v_2=(5,0,-2,1,0)$.
(3) Set $y=0, s=0, t=1$ to obtain the solution $v_3=(-7,0,2,0,1)$.

The set $\{v_1, v_2, v_3\}$ is a basis of the solution space W.

(b) (Here we use the matrix format of our homogeneous system.) Reduce the coefficient matrix A to echelon form:

$$A=\begin{bmatrix} 1 & 2 & 1 & -2\\ 2 & 4 & 4 & -3\\ 3 & 6 & 7 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & -2\\ 0 & 0 & 2 & 1\\ 0 & 0 & 4 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & -2\\ 0 & 0 & 2 & 1\\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This corresponds to the system

$$\begin{aligned} x+2y+2z-2t&=0\\ 2z+\ t&=0 \end{aligned}$$

The free variables are y and t, and $\dim W=2$.

(i) Set $y=1, z=0$ to obtain the solution $u_1=(-2,1,0,0)$.
(ii) Set $y=0, z=2$ to obtain the solution $u_2=(6,0,-1,2)$.

Then $\{u_1, u_2\}$ is a basis of W.

(c) Reduce the coefficient matrix A to echelon form:

$$A=\begin{bmatrix} 1 & 1 & 2\\ 2 & 3 & 3\\ 1 & 3 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2\\ 0 & 1 & -1\\ 0 & 2 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2\\ 0 & 1 & -1\\ 0 & 0 & 5 \end{bmatrix}$$

This corresponds to a triangular system with no free variables. Thus, 0 is the only solution; that is, $W=\{0\}$. Hence, $\dim W=0$.

5.10. Suppose $F: \mathbf{R}^3 \to \mathbf{R}^2$ is defined by $F(x, y, z) = (x + y + z,\ 2x - 3y + 4z)$. Show that F is linear.

We argue via matrices. Writing vectors as columns, the mapping F may be written in the form $F(v) = Av$, where $v = [x, y, z]^T$ and

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & -3 & 4 \end{bmatrix}$$

Then, using properties of matrices, we have

$$F(v + w) = A(v + w) = Av + Aw = F(v) + F(w)$$

and
$$F(kv) = A(kv) = k(Av) = kF(v)$$

Thus, F is linear.

5.11. Show that the following mappings are not linear:

(a) $F: \mathbf{R}^2 \to \mathbf{R}^2$ defined by $F(x, y) = (xy, x)$

(b) $F: \mathbf{R}^2 \to \mathbf{R}^3$ defined by $F(x, y) = (x + 3,\ 2y,\ x + y)$

(c) $F: \mathbf{R}^3 \to \mathbf{R}^2$ defined by $F(x, y, z) = (|x|,\ y + z)$

(a) Let $v = (1, 2)$ and $w = (3, 4)$; then $v + w = (4, 6)$. Also,

$$F(v) = (1(2), 1) = (2, 1) \quad \text{and} \quad F(w) = (3(4), 3) = (12, 3)$$

Hence,

$$F(v + w) = (4(6), 4) = (24, 6) \neq F(v) + F(w)$$

(b) Because $F(0, 0) = (3, 0, 0) \neq (0, 0, 0)$, F cannot be linear.

(c) Let $v = (1, 2, 3)$ and $k = -3$. Then $kv = (-3, -6, -9)$. We have

$$F(v) = (1, 5) \quad \text{and} \quad kF(v) = -3(1, 5) = (-3, -15).$$

Thus,

$$F(kv) = F(-3, -6, -9) = (3, -15) \neq kF(v)$$

Accordingly, F is not linear.

5.12. Let V be the vector space of n-square real matrices. Let M be an arbitrary but fixed matrix in V. Let $F: V \to V$ be defined by $F(A) = AM + MA$, where A is any matrix in V. Show that F is linear.

For any matrices A and B in V and any scalar k, we have

$$\begin{aligned} F(A + B) &= (A + B)M + M(A + B) = AM + BM + MA + MB \\ &= (AM + MA) = (BM + MB) = F(A) + F(B) \end{aligned}$$

and

$$F(kA) = (kA)M + M(kA) = k(AM) + k(MA) = k(AM + MA) = kF(A)$$

Thus, F is linear.

5.16. Let $F: \mathbf{R}^4 \to \mathbf{R}^3$ be the linear mapping defined by

$$F(x, y, z, t) = (x - y + z + t, \quad x + 2z - t, \quad x + y + 3z - 3t)$$

Find a basis and the dimension of (a) the image of F, (b) the kernel of F.

(a) Find the images of the usual basis of $\mathbf{R}^4$:

$$F(1,0,0,0) = (1,1,1), \qquad F(0,0,1,0) = (1,2,3)$$
$$F(0,1,0,0) = (-1,0,1), \qquad F(0,0,0,1) = (1,-1,-3)$$

By Proposition 5.4, the image vectors span Im F. Hence, form the matrix whose rows are these image vectors, and row reduce to echelon form:

$$\begin{bmatrix} 1 & 1 & 1 \\ -1 & 0 & 1 \\ 1 & 2 & 3 \\ 1 & -1 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \\ 0 & -2 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, $(1,1,1)$ and $(0,1,2)$ form a basis for Im F; hence, $\dim(\text{Im } F) = 2$.

(b) Set $F(v) = 0$, where $v = (x, y, z, t)$; that is, set

$$F(x, y, z, t) = (x - y + z + t, \; x + 2z - t, \; x + y + 3z - 3t) = (0, 0, 0)$$

Set corresponding entries equal to each other to form the following homogeneous system whose solution space is Ker F:

$$\begin{aligned} x - y + z + t &= 0 \\ x \qquad + 2z - t &= 0 \\ x + y + 3z - 3t &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} x - y + z + t &= 0 \\ y + z - 2t &= 0 \\ 2y + 2z - 4t &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} x - y + z + t &= 0 \\ y + z - 2t &= 0 \end{aligned}$$

The free variables are z and t. Hence, $\dim(\text{Ker } F) = 2$.

(i) Set $z = -1$, $t = 0$ to obtain the solution $(2, 1, -1, 0)$.

(ii) Set $z = 0$, $t = 1$ to obtain the solution $(1, 2, 0, 1)$.

Thus, $(2, 1, -1, 0)$ and $(1, 2, 0, 1)$ form a basis of Ker F.
[As expected, $\dim(\text{Im } F) + \dim(\text{Ker } F) = 2 + 2 = 4 = \dim \mathbf{R}^4$, the domain of F.]

5.17. Let $G: \mathbf{R}^3 \to \mathbf{R}^3$ be the linear mapping defined by

$$G(x, y, z) = (x + 2y - z, \quad y + z, \quad x + y - 2z)$$

Find a basis and the dimension of (a) the image of G, (b) the kernel of G.

(a) Find the images of the usual basis of $\mathbf{R}^3$:

$$G(1,0,0) = (1,0,1), \qquad G(0,1,0) = (2,1,1), \qquad G(0,0,1) = (-1,1,-2)$$

By Proposition 5.4, the image vectors span Im G. Hence, form the matrix M whose rows are these image vectors, and row reduce to echelon form:

$$M = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus, $(1,0,1)$ and $(0,1,-1)$ form a basis for Im G; hence, $\dim(\text{Im } G) = 2$.

(b) Set $G(v) = 0$, where $v = (x, y, z)$; that is,

$$G(x, y, z) = (x + 2y - z, \quad y + z, \quad x + y - 2z) = (0, 0, 0)$$

Set corresponding entries equal to each other to form the following homogeneous system whose solution space is Ker G:

$$\begin{aligned} x + 2y - z &= 0 \\ y + z &= 0 \\ x + y - 2z &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} x + 2y - z &= 0 \\ y + z &= 0 \\ -y - z &= 0 \end{aligned} \quad \text{or} \quad \begin{aligned} x + 2y - z &= 0 \\ y + z &= 0 \end{aligned}$$

The only free variable is z; hence, $\dim(\text{Ker } G) = 1$. Set $z = 1$; then $y = -1$ and $x = 3$. Thus, $(3, -1, 1)$ forms a basis of Ker G. [As expected, $\dim(\text{Im } G) + \dim(\text{Ker } G) = 2 + 1 = 3 = \dim \mathbf{R}^3$, the domain of G.]

5.24. Determine whether or not each of the following linear maps is nonsingular. If not, find a nonzero vector v whose image is 0.

(a) $F: \mathbf{R}^2 \to \mathbf{R}^2$ defined by $F(x,y) = (x - y, \; x - 2y)$.
(b) $G: \mathbf{R}^2 \to \mathbf{R}^2$ defined by $G(x,y) = (2x - 4y, \; 3x - 6y)$.

(a) Find Ker F by setting $F(v) = 0$, where $v = (x, y)$.

$$(x - y, \; x - 2y) = (0, 0) \qquad \text{or} \qquad \begin{aligned} x - y &= 0 \\ x - 2y &= 0 \end{aligned} \qquad \text{or} \qquad \begin{aligned} x - y &= 0 \\ -y &= 0 \end{aligned}$$

The only solution is $x = 0$, $y = 0$. Hence, F is nonsingular.

(b) Set $G(x, y) = (0, 0)$ to find Ker G:

$$(2x - 4y, \; 3x - 6y) = (0, 0) \qquad \text{or} \qquad \begin{aligned} 2x - 4y &= 0 \\ 3x - 6y &= 0 \end{aligned} \qquad \text{or} \qquad x - 2y = 0$$

The system has nonzero solutions, because y is a free variable. Hence, G is singular. Let $y = 1$ to obtain the solution $v = (2, 1)$, which is a nonzero vector, such that $G(v) = 0$.

6.1. Consider the linear mapping $F: \mathbf{R}^2 \to \mathbf{R}^2$ defined by $F(x, y) = (3x + 4y, \ \ 2x - 5y)$ and the following bases of $\mathbf{R}^2$:

$$E = \{e_1, e_2\} = \{(1, 0), \ (0, 1)\} \qquad \text{and} \qquad S = \{u_1, u_2\} = \{(1, 2), \ (2, 3)\}$$

(a) Find the matrix A representing F relative to the basis E.

(b) Find the matrix B representing F relative to the basis S.

(a) Because E is the usual basis, the rows of A are simply the coefficients in the components of $F(x, y)$; that is, using $(a, b) = ae_1 + be_2$, we have

$$\begin{aligned} F(e_1) &= F(1, 0) = (3, 2) \ \ = 3e_1 + 2e_2 \\ F(e_2) &= F(0, 1) = (4, -5) = 4e_1 - 5e_2 \end{aligned} \qquad \text{and so} \qquad A = \begin{bmatrix} 3 & 4 \\ 2 & -5 \end{bmatrix}$$

Note that the coefficients of the basis vectors are written as columns in the matrix representation.

(b) First find $F(u_1)$ and write it as a linear combination of the basis vectors u_1 and u_2. We have

$$F(u_1) = F(1, 2) = (11, -8) = x(1, 2) + y(2, 3), \qquad \text{and so} \qquad \begin{aligned} x + 2y &= 11 \\ 2x + 3y &= -8 \end{aligned}$$

Solve the system to obtain $x = -49$, $y = 30$. Therefore,

$$F(u_1) = -49u_1 + 30u_2$$

Next find $F(u_2)$ and write it as a linear combination of the basis vectors u_1 and u_2. We have

$$F(u_2) = F(2, 3) = (18, -11) = x(1, 2) + y(2, 3), \qquad \text{and so} \qquad \begin{aligned} x + 2y &= 18 \\ 2x + 3y &= -11 \end{aligned}$$

Solve for x and y to obtain $x = -76$, $y = 47$. Hence,

$$F(u_2) = -76u_1 + 47u_2$$

Write the coefficients of u_1 and u_2 as columns to obtain $B = \begin{bmatrix} -49 & -76 \\ 30 & 47 \end{bmatrix}$

(b′) Alternatively, one can first find the coordinates of an arbitrary vector (a, b) in $\mathbf{R}^2$ relative to the basis S. We have

$$(a, b) = x(1, 2) + y(2, 3) = (x + 2y, \ \ 2x + 3y), \qquad \text{and so} \qquad \begin{aligned} x + 2y &= a \\ 2x + 3y &= b \end{aligned}$$

Solve for x and y in terms of a and b to get $x = -3a + 2b$, $y = 2a - b$. Thus,

$$(a, b) = (-3a + 2b)u_1 + (2a - b)u_2$$

Then use the formula for (a, b) to find the coordinates of $F(u_1)$ and $F(u_2)$ relative to S:

$$\begin{aligned} F(u_1) &= F(1, 2) = (11, -8) \ \ = -49u_1 + 30u_2 \\ F(u_2) &= F(2, 3) = (18, -11) = -76u_1 + 47u_2 \end{aligned} \qquad \text{and so} \qquad B = \begin{bmatrix} -49 & -76 \\ 30 & 47 \end{bmatrix}$$

6.2. Consider the following linear operator G on $\mathbf{R}^2$ and basis S:

$$G(x, y) = (2x - 7y, \ 4x + 3y) \qquad \text{and} \qquad S = \{u_1, u_2\} = \{(1, 3), \ (2, 5)\}$$

(a) Find the matrix representation $[G]_S$ of G relative to S.

(b) Verify $[G]_S[v]_S = [G(v)]_S$ for the vector $v = (4, -3)$ in $\mathbf{R}^2$.

First find the coordinates of an arbitrary vector $v = (a, b)$ in $\mathbf{R}^2$ relative to the basis S. We have

$$\begin{bmatrix} a \\ b \end{bmatrix} = x\begin{bmatrix} 1 \\ 3 \end{bmatrix} + y\begin{bmatrix} 2 \\ 5 \end{bmatrix}, \qquad \text{and so} \qquad \begin{aligned} x + 2y &= a \\ 3x + 5y &= b \end{aligned}$$

Solve for x and y in terms of a and b to get $x = -5a + 2b$, $y = 3a - b$. Thus,

$$(a, b) = (-5a + 2b)u_1 + (3a - b)u_2, \qquad \text{and so} \qquad [v] = [-5a + 2b, \ \ 3a - b]^T$$

(a) Using the formula for (a, b) and $G(x, y) = (2x - 7y, \ 4x + 3y)$, we have

$$\begin{aligned} G(u_1) &= G(1, 3) = (-19, 13) = 121u_1 - 70u_2 \\ G(u_2) &= G(2, 5) = (-31, 23) = 201u_1 - 116u_2 \end{aligned} \qquad \text{and so} \qquad [G]_S = \begin{bmatrix} 121 & 201 \\ -70 & -116 \end{bmatrix}$$

(We emphasize that the coefficients of u_1 and u_2 are written as columns, not rows, in the matrix representation.)

(b) Use the formula $(a, b) = (-5a + 2b)u_1 + (3a - b)u_2$ to get

$$\begin{aligned} v &= (4, -3) = -26u_1 + 15u_2 \\ G(v) &= G(4, -3) = (20, 7) = -131u_1 + 80u_2 \end{aligned}$$

Then $$[v]_S = [-26, 15]^T \qquad \text{and} \qquad [G(v)]_S = [-131, 80]^T$$

(a) Find eigenvalues of the matrix A.

To find the eigenvalues of A, we calculate the characteristic polynomial $p(t)$ as follows.
We have

$$\begin{aligned} p(t) = \det(A - tI) &= \begin{vmatrix} 1-t & 2 \\ 4 & 3-t \end{vmatrix} \\ &= (1-t)(3-t) - 8 = t^2 - 4t - 5 = (t+1)(t-5). \end{aligned}$$

The eigenvalues of A are roots of its characteristic polynomial $p(t)$.
Hence the eigenvalues of A are -1 and 5.

(b) Find eigenvectors for each eigenvalue of A.

We first determine the eigenvectors of the eigenvalue -1 by solving the system $(A + I)\mathbf{x} = \mathbf{0}$.
We have

$$A + I = \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix} \xrightarrow[\text{then } \frac{1}{2} R_1]{R_2 - 2R_1} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

This yields that the eigenvectors corresponding to -1 are

$$a\begin{bmatrix}1\\-1\end{bmatrix}$$

for any nonzero scalar a.

Next, we find the eigenvectors corresponding to the eigenvalue 5 by solving $(A-5I)\mathbf{x}=\mathbf{0}$. We have

$$A-5I=\begin{bmatrix}-4 & 2\\4 & -2\end{bmatrix}\xrightarrow[\text{then }\frac{-1}{4}R_1]{R_2+R_1}\begin{bmatrix}1 & -1/2\\0 & 0\end{bmatrix}.$$

It follows that the eigenvectors corresponding to 5 are

$$a\begin{bmatrix}1\\2\end{bmatrix}$$

for any nonzero scalar a.

(c) Diagonalize the matrix A.

From part (a) and part (b), we have seen that A has eigenvalues -1 and 5 with corresponding eigenvectors

$$\mathbf{u} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ and } \mathbf{v} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

(Here we chose the scalars a to be 1 but you could use any nonzero values for the scalars a.)

Let

$$S = [\mathbf{u} \quad \mathbf{v}] = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}.$$

Then the general procedure of the diagonalization yields that the matrix S is invertible and

$$S^{-1}AS = D,$$

where D is the diagonal matrix given by

$$D = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}.$$

(d) Diagonalize the matrix $A^3 - 5A^2 + 3A + I$.

In part (c), we obtained

$$S^{-1}AS = D,$$

where

$$S = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \text{ and } D = \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}.$$

Note that we have $A = SDS^{-1}$ and

$$\begin{aligned} A^2 &= AA = SDS^{-1} \cdot SDS^{-1} = SD^2S^{-1} \\ A^3 &= A^2A = SD^2S^{-1} \cdot SDS^{-1} = SD^3S^{-1}. \end{aligned}$$

These relations gives

$$\begin{aligned} A^3 - 5A^2 + 3A + I &= SD^3S^{-1} - 5SD^2S^{-1} + 3SDS^{-1} + I \\ &= S(D^3 - 5D^2 + 3D + I)S^{-1}. \end{aligned}$$

Hence we obtain

$$
\begin{aligned}
&S^{-1}(A^3-5A^2+3A+I)S \\
&= D^3-5D^2+3D+I \\
&= \begin{bmatrix} (-1)^3 & 0 \\ 0 & 5^3 \end{bmatrix} - 5\begin{bmatrix} (-1)^2 & 0 \\ 0 & 5^2 \end{bmatrix} + 3\begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} -8 & 0 \\ 0 & 16 \end{bmatrix}.
\end{aligned}
$$

This completes the diagonalization of the matrix A^3-5A^2+3A+I.

(e) Calculate A^{100}.

In part (d), we have seen that $A=SDS^{-1}$, $A^2=SD^2S^{-1}$, $A^3=SD^3S^{-1}$. Repeating the same argument (or using mathematical induction), we also have

$$A^{100}=SD^{100}S^{-1}.$$

Thus, we have

$$\begin{aligned}
A^{100} &= SD^{100}S^{-1} \\
&= \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 5 \end{bmatrix}^{100} \frac{1}{3} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \\
&= \frac{1}{3} \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} (-1)^{100} & 0 \\ 0 & 5^{100} \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \\
&= \frac{1}{3} \begin{bmatrix} 2+5^{100} & -1+5^{100} \\ -2+2\cdot 5^{100} & 1+2\cdot 5^{100} \end{bmatrix}.
\end{aligned}$$

(f) Calculate $(A^3 - 5A^2 + 3A + I)^{100}$.

Let

$$B := A^3 - 5A^2 + 3A + I.$$

In part (d), we obtained

$$S^{-1}BS = \begin{bmatrix} -8 & 0 \\ 0 & 16 \end{bmatrix}.$$

Hence we have $B = S \begin{bmatrix} -8 & 0 \\ 0 & 16 \end{bmatrix} S^{-1}$, and

Hence we have $B = S\begin{bmatrix} -8 & 0 \\ 0 & 16 \end{bmatrix}S^{-1}$, and

$$\begin{aligned} B^{100} &= S\begin{bmatrix} -8 & 0 \\ 0 & 16 \end{bmatrix}^{100} S^{-1} \\ &= S\begin{bmatrix} (-8)^{100} & 0 \\ 0 & 16^{100} \end{bmatrix} S^{-1} \\ &= S\begin{bmatrix} 2^{300} & 0 \\ 0 & 2^{400} \end{bmatrix} S^{-1} \\ &= S\begin{bmatrix} w^3 & 0 \\ 0 & w^4 \end{bmatrix} S^{-1}, \end{aligned}$$

where we put $w = 2^{100}$.

Hence we have

$$\begin{aligned} B^{100} &= \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} w^3 & 0 \\ 0 & w^4 \end{bmatrix}\frac{1}{3}\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{w^3}{3}\begin{bmatrix} 2+w & -1+w \\ -2+2w & 1-2w \end{bmatrix}. \end{aligned}$$

4.2 Vector Spaces

The following defines the notion of a vector space V where K is the field of scalars.

DEFINITION: Let V be a nonempty set with two operations:

(i) ***Vector Addition:*** This assigns to any $u, v \in V$ a *sum* $u + v$ in V.
(ii) ***Scalar Multiplication:*** This assigns to any $u \in V$, $k \in K$ a *product* $ku \in V$.

Then V is called a *vector space* (over the field K) if the following axioms hold for any vectors $u, v, w \in V$:

[A_1] $(u + v) + w = u + (v + w)$
[A_2] There is a vector in V, denoted by 0 and called the *zero vector*, such that, for any $u \in V$,
$$u + 0 = 0 + u = u$$
[A_3] For each $u \in V$, there is a vector in V, denoted by $-u$, and called the *negative* of u, such that
$$u + (-u) = (-u) + u = 0.$$
[A_4] $u + v = v + u$.
[M_1] $k(u + v) = ku + kv$, for any scalar $k \in K$.
[M_2] $(a + b)u = au + bu$, for any scalars $a, b \in K$.
[M_3] $(ab)u = a(bu)$, for any scalars $a, b \in K$.
[M_4] $1u = u$, for the unit scalar $1 \in K$.

The above axioms naturally split into two sets (as indicated by the labeling of the axioms). The first four are concerned only with the additive structure of V and can be summarized by saying V is a *commutative group* under addition. This means

(a) Any sum $v_1 + v_2 + \cdots + v_m$ of vectors requires no parentheses and does not depend on the order of the summands.

(b) The zero vector 0 is unique, and the negative $-u$ of a vector u is unique.

(c) (Cancellation Law) If $u + w = v + w$, then $u = v$.

Also, *subtraction* in V is defined by $u - v = u + (-v)$, where $-v$ is the unique negative of v.

On the other hand, the remaining four axioms are concerned with the "action" of the field K of scalars on the vector space V. Using these additional axioms, we prove (Problem 4.2) the following simple properties of a vector space.

THEOREM 4.1: Let V be a vector space over a field K.

(i) For any scalar $k \in K$ and $0 \in V$, $k0 = 0$.
(ii) For $0 \in K$ and any vector $u \in V$, $0u = 0$.
(iii) If $ku = 0$, where $k \in K$ and $u \in V$, then $k = 0$ or $u = 0$.
(iv) For any $k \in K$ and any $u \in V$, $(-k)u = k(-u) = -ku$.