ANALYTIC GEOMETRY

: IFoS - 2017 :

①(e) Find the equations of the plane parallel to the plane $3x-2y+6z+8=0$ and at a distance 2 from it

→ Any plane parallel to the given plane is $3x-2y+6z+k=0$ ——①

Distance between the two planes is 2.

$$\therefore \left|\frac{k-8}{\sqrt{9+4+36}}\right| = 2 \quad \Rightarrow (k-8)^2 = 4\times 49.$$

$$\Rightarrow k-8 = \pm 14$$

$$\Rightarrow k = 22, -6.$$

$\therefore$ Reqd planes are $3x-2y+6z+22=0$ &

$3x-2y+6z-6=0$

②(d) Show that the angles between the planes given by the equation $2x^2-y^2+3z^2-xy+7zx+2yz=0$ is $\tan^{-1}\frac{\sqrt{50}}{4}$

→ Let the eqn of the two planes be

$2x+b_1y+cz=0$ & $x+b_2y+c_2z=0$

Then, the pair of planes is given by :

$(2x+b_1y+c_1z)(x+b_2y+c_2z)=0$

$$2x^2+(2b_2+b_1)xy+(2c_2+c_1)zx+b_1b_2y^2+(b_1c_2+b_2c_1)yz + c_1c_2z^2=0 \quad ⑥$$

$2x^2 + b_1b_2 y^2 + c_1c_2 z^2 + (b_1+2b_2)xy + (c_1+2c_2)zx + (b_1c_2+b_2c_1)yz = 0$

Comparing coeff of respective terms between this plane & the given plane, we get

$b_1b_2 = -1$, $c_1c_2 = 3$, $b_1+2b_2 = -1$, $c_1+2c_2 = 7$, $b_1c_2 + b_2c_1 = 2$
$\quad$ ① $\quad$ ② $\quad$ ③ $\quad$ ④ $\quad$ ⑤

③ ≡ $b_1 = -1-2b_2$. ① ≡ $b_1b_2 = -1 \Rightarrow (1+2b_2)b_2 = 1$

$\Rightarrow 2b_2^2 + b_2 - 1 = 0$

$b_1 = -1-2b_2$ $\qquad \Rightarrow 2b_2^2 + 2b_2 - b_2 - 1 = 0$

$= -1 - 2\cdot\frac{1}{2}$ & $-1-2\cdot(-1)$ $\qquad \Rightarrow (2b_2-1)(b_2+1) = 0$

$= -2$ & 1 $\qquad b_2 = \frac{1}{2}$, $b_2 = -1$

$\therefore b_2 = \frac{1}{2}$, $b_1 = -2$ & $b_2 = -1$, $b_1 = 1$

$c_1 + 2c_2 = 7 \Rightarrow c_1 = 7 - 2c_2$. Now $c_1c_2 = 3$

$c_2(7-2c_2) = 3$

$2c_2^2 - 7c_2 + 3 = 0$

$c_1 = 7 - 2c_2 = 7 - 2\cdot\frac{1}{2}$ or $7-2\cdot3$ $\qquad 2c_2^2 - 6c_2 - c_2 + 3 = 0$

$c_1 = 6$ or 1 $\qquad (2c_2-1)(c_2-3) = 0$

$c_2 = \frac{1}{2}$, $c_2 = 3$

$\therefore c_1 = 6, c_2 = \frac{1}{2}$ or $c_1 = 1, c_2 = 3$

Taking $b_1 = -2$, $b_2 = \frac{1}{2}$, $c_1 = 6$, $c_2 = \frac{1}{2}$, then

$b_1c_2 + b_2c_1 = -1 + 3 = 2 \longrightarrow$ satisfies ⑤

Taking $b_1 = -2$, $b_2 = \frac{1}{2}$, $c_1 = 1$, $c_2 = 3$

$b_1c_2 + b_2c_1 = -2 + \frac{3}{2} \neq 2 \longrightarrow$ does not satisfy ⑤

Taking $b_2 = -1$, $b_1 = 1$, $c_1 = 1$, $c_2 = 3$.

$b_1c_2 + b_2c_1 = 3 - 1 = 2 \longrightarrow$ satisfies ⑤.

$\therefore$ our planes are $2x + (-2)y + 6z = 0$ & $x + \frac{1}{2}y + \frac{1}{2}z = 0$

$\rightarrow x - y + 3z = 0$ & $2x + y + z = 0$

If θ is the angle between the planes, then

$\cos\theta = \dfrac{1\cdot2 - 1\cdot1 + 3\cdot1}{\sqrt{1+1+9}\sqrt{4+1+1}} = \dfrac{4}{\sqrt{11}\sqrt{6}}$

$\cos^2\theta = \frac{16 \cdot ?}{1 \cdot 63} = \frac{8}{33} \Rightarrow \sec^2\theta = \frac{33}{8}$

$\Rightarrow 1 + \tan^2\theta = \frac{33}{8} \Rightarrow \tan^2\theta = \frac{33}{8} - 1 = \frac{25}{8} = \frac{50}{16}$

$\Rightarrow \tan\theta = \underline{\underline{\frac{\sqrt{50}}{4}}} \Rightarrow \theta = \tan^{-1}\underline{\underline{\frac{\sqrt{50}}{4}}}$

[illegible] Find the angle between the lines [illegible]

$\longrightarrow$

$l + m + n = 0 \Rightarrow n = -(l+m)$.

Putting in $2lm + 2nl - mn = 0$, we get,

$2lm - 2l(l+m) + m(l+m) = 0$

$2lm - 2l^2 - 2lm + ml + m^2 = 0$

$\Rightarrow 2\left(\frac{l}{m}\right)^2 - \frac{l}{m} - 1 = 0 \Rightarrow 2\left(\frac{l}{m}\right)^2 - 2\frac{l}{m} + \frac{l}{m} - 1 = 0$

$\Rightarrow \left(\frac{2l}{m} + 1\right)\left(\frac{l}{m} - 1\right) = 0 \Rightarrow \frac{2l}{m} = -1$ & $\frac{l}{m} = 1$

$\frac{l}{-1} = \frac{m}{2}$ & $\frac{l}{1} = \frac{m}{1}$

Now. $n = -(l+m)$.

(i) $\underline{m = -2l}$: $n = -(l - 2l) = l \Rightarrow \frac{l}{-1} = \frac{n}{-1}$

(ii) $\underline{m = l}$ $n = -(l + l) = -2l \Rightarrow \frac{l}{1} = \frac{n}{-2}$

$\therefore \frac{l}{-1} = \frac{m}{2} = \frac{n}{-1}$ & $\frac{l}{1} = \frac{m}{1} = \frac{n}{-2}$

Angle between the lines is

$\cos\theta = \frac{-1 \cdot 1 + 2 \cdot 1 - 1 \cdot (-2)}{\sqrt{1+4+1}\,\sqrt{1+1+4}}$

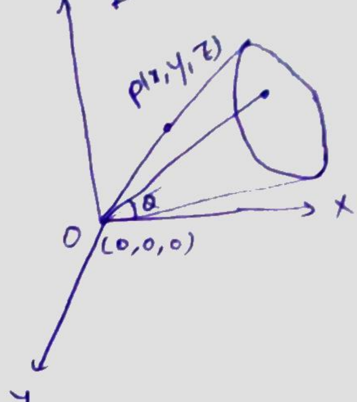
$= \frac{3}{6} = \frac{1}{2}$

$\boxed{\theta = \frac{\pi}{3}}$ if acute angle is taken & $\boxed{\theta = \frac{2\pi}{3}}$ if obtuse angle is taken.

4 (c) Find the equation of right circular cone with vertex at the origin & whose axes make equal angles with coordinate axes & the generator is a line passing through the origin with drs 1, -2, 2.

→ Eq^n of axis: $\frac{x}{1} = \frac{y}{1} = \frac{z}{1}$

DRs. of generators : 1, -2, 2



If θ is the semivertical angle, then

$$\cos\theta = \frac{1.1 + 1.(-2) + 1.2}{\sqrt{1+1+1}\ \sqrt{1+4+4}}$$

$$\cos\theta = \frac{1}{3\sqrt{3}} \quad —— ①$$

Let P(x, y, z) be any point on the right circular cone. Then OP is a generator whose drs are x, y, z. Then,

$$\cos\theta = \frac{x.1 + y.1 + z.1}{\sqrt{x^2+y^2+z^2}\ \sqrt{3}} \quad — ②$$

$$\therefore\ ① \ \& \ ② : \quad \frac{x+y+z}{\sqrt{x^2+y^2+z^2}\ \sqrt{3}} = \frac{1}{3\sqrt{3}}$$

$$\rightarrow (x^2+y^2+z^2) = 9(x+y+z)^2$$

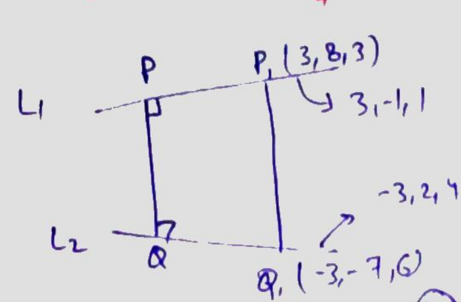
$$\rightarrow 8x^2+8y^2+8z^2 + 18xy + 18yz + 18xz = 0$$

$$\rightarrow 4(x^2+y^2+z^2) + 9(yz+zx+xy) = 0$$

5 (d) Find the shortest distance & the eqn of S.D. line between the lines $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$ & $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$

→ L_1: $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$

L_2: $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$



Let the dcs of S.D. line PQ be l, m, n. Then since it is ⊥ to L_1 & L_2, the condition for perpendicularity is

$l l_1 + m m_1 + n n_1 = 0$. Therefore, we have

$3l - m + n = 0$

$-3l + 2m + 4n = 0 \Rightarrow \frac{l}{-6} = \frac{m}{-15} = \frac{n}{3} \Rightarrow \frac{l}{2} = \frac{m}{5} = \frac{n}{-1} = \frac{1}{\sqrt{30}}$

$\therefore l = \frac{2}{\sqrt{30}},\ m = \frac{5}{\sqrt{30}},\ n = \frac{-1}{\sqrt{30}}$. The shortest distance line is the projection of the join of $P_1(3, 8, 3)$ & $Q_1(-3, -7, 6)$ on line whose dcs are l, m, n.

The formula of projection is $PQ = SD = l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$

$\Rightarrow S.D = \frac{1}{\sqrt{30}}\left[2(3+3) + 5(8+7) - 1(3-6)\right] = \frac{90}{\sqrt{30}}$

$\Rightarrow \boxed{SD = 3\sqrt{30} \text{ units}}$

Now: Eqn of plane ~~beta~~ containing L_1 & PQ is given by

$$\begin{vmatrix} x-3 & y-8 & z-3 \\ 3 & -1 & 1 \\ 2 & 5 & -1 \end{vmatrix} = 0 \Rightarrow (x-3)(-4) + (y-8)5 + (z-3)17 = 0$$

$\Rightarrow 4x - 5y - 17z + 79 = 0$ —— ①

Eqn of plane containing L_2 & PQ is given by

$$\begin{vmatrix} x+3 & y+7 & z-6 \\ -3 & 2 & 4 \\ 2 & 5 & -1 \end{vmatrix} = 0 \Rightarrow (x+3)(-22) + (y+7)5 + (z-6)(-19) = 0$$

$\Rightarrow 22x - 5y + 19z - 83 = 0$ —— ②

Then, the required S.D. line is the line of intersection of the planes:

$4x - 5y - 17z + 79 = 0 = 22x - 5y + 19z - 83$