FLUID DYNAMICS

Introduction and Some Basic Concepts:

Fluid Dynamics is the study of motion of fluids (liqui and gases) or that of bodies in contact with fluids.

Isotropy: A fluid is said to be isotropic with respec to some property (pressure, density etc.) if that property is the same in all directions at a point. otherwise the fluid is anisotropic w.r.t a property.

Properties of fluid:

(i) Density:

$$\rho = \lim_{\delta v \to 0} \frac{\delta m}{\delta v} \quad \text{at a point (mass per unit volume)}$$

Specific weight

$$W = \rho g \quad \text{(weight per unit volume)}$$

Specific volume

$$\frac{1}{\rho} \quad \text{(volume per unit mass)}$$

(ii) Pressure:

At a point, $$P = \lim_{\delta s \to 0} \frac{\delta F}{\delta s} \quad \text{(Force per unit area)}$$

(iii) Temperature: T

(iv) Thermal conductivity:

Fourier's heat conduction law:

$$q_n \propto \frac{-\partial T}{\partial x} \quad \text{or} \quad q_n = -k\frac{\partial T}{\partial x}$$

q_n = heat flux (heat flow per unit area)

$\frac{\partial T}{\partial x}$ – temperature decrease per unit distance in a direction normal to the area.

(v) Specific heat:

$$C_v = \left(\frac{\partial Q}{\partial T}\right)_v \quad ; \quad C_p = \left(\frac{\partial Q}{\partial T}\right)_p$$

$$\gamma = \frac{C_p}{C_v}$$

(vi) Incompressible and Compressible fluids:

Gases - compressible

Liquids - can be treated as incompressible for most of the cases.

Compressibility and bulk modulus:

bulk modulus $K = \dfrac{dp}{-\left(\frac{dv}{v}\right)} = \dfrac{dp}{+\left(\frac{d\rho}{\rho}\right)} = \rho \dfrac{dp}{d\rho}$

$v \propto \frac{1}{\rho}$

$v\rho = \text{constant}$

$\rho\, dv + v\, d\rho = 0$

$\Rightarrow -\frac{dv}{v} = \frac{d\rho}{\rho}$.

K - ratio of volumetric stress to volumetric strain

co-efficient of compressibility, $\beta = \frac{1}{K}$.

Viscous (real) and inviscid (ideal or non-viscous, frictionless, perfect):

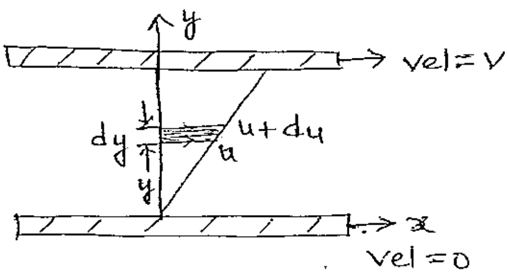
Viscous - Normal force and shear stress exists.

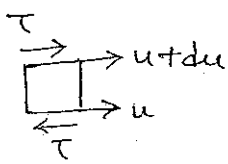
shear stress ↓ tangential force per unit area.

Inviscid - shear stress doesn't exist

(water, air are near to inviscid).

Viscosity / Internal friction:





$\tau = \mu\left(\frac{du}{dy}\right)$, μ - co-efficient of viscosity

Newton's law of viscosity

Fluids which obey Newton's law of viscosity are called Newtonian's fluids.

- Important types of flows:

(i) Laminar (streamline) and Turbulent flows:

curves traced out by
two different fluid
particles donot intersect

(ii) Steady and unsteady flows:

$$\frac{\partial P}{\partial t} = 0 \qquad\qquad \frac{\partial P}{\partial t} \neq 0$$

P – velocity, density, pressure, temperature etc. at a point.

(iii) Uniform and non-uniform flows:

→ Fluid particles possess equal velocities at each section of the channel or pipe.

(iv) Rotational and irrotational flows:

→ Fluid particles go on rotating about their own axis.

(v) Barotropic flow:

pressure $= f(\rho)$ – function of density.

Vector Calculus:

- $$\frac{d}{dt}(a \times b) = a \times \frac{db}{dt} + \frac{da}{dt} \times b$$

  $$\frac{d}{dt}(a \cdot b) = a \cdot \frac{db}{dt} + \frac{da}{dt} \cdot b$$

- Vector operator ∇ (del)

$$\nabla \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$$

Let $\phi(x,y,z)$ and $\psi(x,y,z)$ be scalar point functions and let $\bar{a}(x,y,z)$ and $\bar{b}(x,y,z)$ be vector point functions

$$\text{grad}\,\phi = \nabla\phi = \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z}.$$

$$\text{div } \bar{a} = \nabla \cdot \bar{a} = \frac{\partial a_1}{\partial x} + \frac{\partial a_2}{\partial y} + \frac{\partial a_3}{\partial z}$$

$$\text{Curl } \bar{a} = \nabla \times \bar{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_1 & a_2 & a_3 \end{vmatrix}$$

divergence of grad ϕ is known as Laplacian of ϕ.

$$\text{div grad } \phi = \nabla \cdot (\nabla \phi) = \nabla^2 \phi$$
$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

Laplacian operator, ∇^2 is defined as

$$\nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}.$$

Important vector Identities:

$$\text{grad } (\phi \psi) = \phi \text{ grad } \psi + \psi \text{ grad } \phi$$

$$\text{grad } (\bar{a} \cdot \bar{b}) = (\bar{a} \cdot \nabla) \bar{b} + (\bar{b} \cdot \nabla) \bar{a} + (\bar{a} \times \text{curl } \bar{b}) + (\bar{b} \times \text{curl } \bar{a})$$

$$\text{div } (\phi \bar{a}) = \phi \text{ div } \bar{a} + \text{grad } \phi \cdot \bar{a}$$

$$\text{div } (\bar{a} \times \bar{b}) = \bar{b} \cdot \text{curl } \bar{a} - \bar{a} \cdot \text{curl } \bar{b}$$

$$\text{Curl } (\phi \bar{a}) = \phi \text{ curl } \bar{a} + \text{grad } \phi \times \bar{a}$$

$$\text{Curl } (\bar{a} \times \bar{b}) = \bar{a} \text{ div } \bar{b} - \bar{b} \text{ div } \bar{a} + (\bar{b} \cdot \nabla) \bar{a} - (\bar{a} \cdot \nabla) \bar{b}$$

$$\text{Curl Curl } \bar{a} = \text{grad div } \bar{a} - \nabla^2 \bar{a}$$

$$\text{Curl grad } \phi = 0$$

$$\text{div Curl } \bar{a} = 0$$

$$(\bar{a} \cdot \nabla) = a_1 \left(\frac{\partial}{\partial x}\right) + a_2 \left(\frac{\partial}{\partial y}\right) + a_3 \left(\frac{\partial}{\partial z}\right)$$

Important Integral Theorems:

- The Divergence theorem (Gauss's Theorem):

Let S denote a surface boundary bounding a volume V and $\hat{n}$ the unit vector normal to the surface. Then,

$$\int_S \bar{a}\cdot\hat{n}\, ds = \int_V \operatorname{div} \bar{a}\, dV .$$

- Stoke's theorem:

Let S be a surface bounded by a closed curve C and $\hat{n}$ the unit vector normal to the surface. Then,

$$\int_S \operatorname{Curl} \bar{a}\cdot\hat{n}\, ds = \int_C \bar{a}\cdot d\bar{r}$$

- Green's theorem:

Let ϕ and ψ be two scalar point functions and S be a surface bounding a volume V. Then,

$$\int_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\, dV = \int_S \left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right) ds$$

If ϕ and ψ are harmonic so that

$$\nabla^2\phi = 0 = \nabla^2\psi ,\ \text{then}$$

$$\int_S \phi\frac{\partial\psi}{\partial n}\, ds = \int_S \psi\frac{\partial\phi}{\partial n}\, ds .$$

Orthogonal co-ordinate systems:

✓ Cartesian coordinates (x, y, z) w.r.t $(\hat{i}, \hat{j}, \hat{k})$

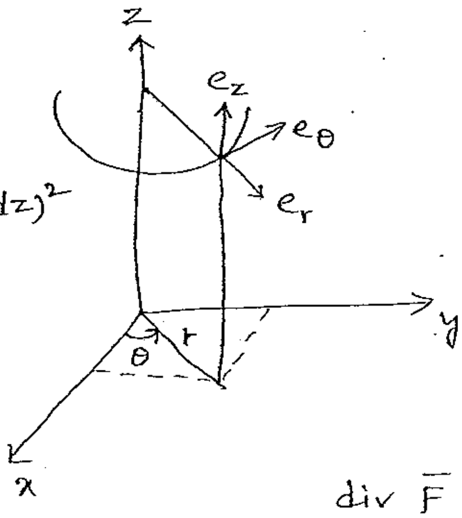
$$\operatorname{grad}\psi = \nabla(\psi) = \frac{\partial\psi}{\partial x}\hat{i} + \frac{\partial\psi}{\partial y}\hat{j} + \frac{\partial\psi}{\partial z}\hat{k}$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} .$$

$$\operatorname{div}\bar{F} = \nabla\cdot\bar{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

$$\operatorname{Curl}\bar{F} = \nabla\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

- Cylindrical coordinates (r,θ,z) w.r.t e_r, e_θ, e_z



$$(ds)^2 = (dr)^2 + (r\,d\theta)^2 + (dz)^2$$

$$\text{grad}\,\psi = \nabla\psi$$

$$= \frac{\partial\psi}{\partial r} e_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta} e_\theta + \frac{\partial\psi}{\partial z} e_z$$

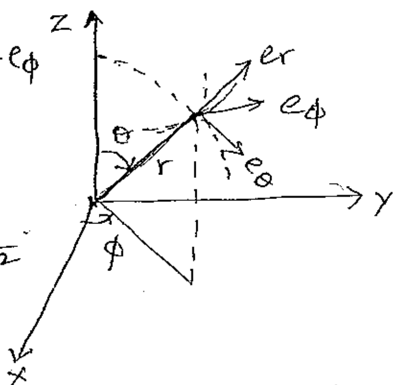
$$\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2\psi}{\partial z^2}$$

$$\text{div}\,\bar{F} = \nabla\cdot\bar{F} = \frac{1}{r}\frac{\partial(rF_r)}{\partial r} + \frac{1}{r}\frac{\partial F_\theta}{\partial\theta} + \frac{\partial F_z}{\partial z}$$

$$\text{curl}\,\bar{F} = \nabla\times\bar{F} = \frac{1}{r}\begin{vmatrix} e_r & e_\theta & e_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial z} \\ F_r & rF_\theta & F_z \end{vmatrix}$$

- Spherical co-ordinates (r,θ,ϕ) w.r.t e_r, e_θ, e_ϕ

$$\text{grad}\,\psi = \frac{\partial\psi}{\partial r} e_r + \frac{1}{r}\frac{\partial\psi}{\partial\theta} e_\theta + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi} e_\phi$$



$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

$$\text{div}\,\bar{F} = \nabla\cdot\bar{F} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, F_\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}$$

$$\text{curl}\,\bar{F} = \nabla\times\bar{F} = \frac{1}{r^2\sin\theta}\begin{vmatrix} e_r & re_\theta & r\sin\theta\, e_\phi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\phi} \\ F_r & rF_\theta & r\sin\theta\, F_\phi \end{vmatrix}$$

$$(ds)^2 = (dr)^2 + (r\,d\theta)^2 + (r\sin\theta\, d\phi)^2.$$

Kinematics of Fluids in motion:

Two methods for studying fluid motion mathematically:

- Lagrangian method → Individual time rate of change
- Eulerian (Flux) method → Local time rate of change

Lagrangian Method:

In this method we study the history of each fluid particle.

Initially let the fluid particle be at (x_0, y_0, z_0) at time, $t=0$. At later time $t=t$ let the coordinates of the same particle be (x, y, z)

$$x = f_1(x_0, y_0, z_0, t), \quad y = f_2(x_0, y_0, z_0, t), \quad z = f_3(x_0, y_0, z_0, t)$$

Let u, v, w and a_x, a_y, a_z be the components of velocity and acceleration respectively. Then,

$$u = \frac{\partial x}{\partial t}, \quad v = \frac{\partial y}{\partial t}, \quad w = \frac{\partial z}{\partial t}$$

$$a_x = \frac{\partial^2 x}{\partial t^2}, \quad a_y = \frac{\partial^2 y}{\partial t^2}, \quad a_z = \frac{\partial^2 z}{\partial t^2}.$$

These fundamental equations in Lagrangian method are non-linear and are difficult to solve.

This method will have an advantage only in some one-dimensional problems.

Eulerian Method:

In this method we select any point fixed in space occupied by the fluid and study the changes which take place in velocity, pressure and density as the fluid passes through this point.

Let u, v, w be the components of velocity at the point (x, y, z) at time t

$$u = F_1(x, y, z, t), \quad v = F_2(x, y, z, t), \quad w = F_3(x, y, z, t)$$

The point under consideration being fixed, x, y, z & t are independent variables and hence $\frac{dx}{dt}, \frac{d^2x}{dt^2}$, etc. have no meaning in this method.

Relationship b/w the Lagrangian and Eulerian methods:

(b/w particle parameters and space parameters)

(i) Lagrangian to Eulerian:

Let $\phi(x_0, y_0, z_0, t)$ be some physical quantity involving Lagrangian description

$$\phi = \phi(x_0, y_0, z_0, t)$$

$$x = f_1(x_0, y_0, z_0, t), \quad y = f_2(x_0, y_0, z_0, t), \quad z = f_3(x_0, y_0, z_0, t)$$

solving for x_0, y_0, z_0 we have

$$x_0 = g_1(x, y, z, t), \quad y_0 = g_2(x, y, z, t), \quad z_0 = g_3(x, y, z, t)$$

$$\Rightarrow \phi = \phi\left[g_1(x, y, z, t), g_2(x, y, z, t), g_3(x, y, z, t), t\right]$$

which expresses ϕ in terms of Eulerian description.

(ii) Eulerian to Lagrangian:

Let $\psi(x, y, z, t)$ be some physical quantity involving Eulerian description

$$\psi = \psi(x, y, z, t)$$

$$\Rightarrow u = F_1(x, y, z, t), \quad v = F_2(x, y, z, t), \quad w = F_3(x, y, z, t)$$

$$\Rightarrow \frac{dx}{dt} = F_1(x, y, z, t), \quad \frac{dy}{dt} = F_2(x, y, z, t), \quad \frac{dz}{dt} = F_3(x, y, z, t)$$

Integrating gives (x_0, y_0, z_0 – constants of integration)

$$x = f_1(x_0, y_0, z_0, t), \quad y = f_2(x_0, y_0, z_0, t), \quad z = f_3(x_0, y_0, z_0, t)$$

$$\Rightarrow \psi = \psi\left[f_1(x_0, y_0, z_0, t), f_2(x_0, y_0, z_0, t), f_3(x_0, y_0, z_0, t), t\right]$$

which expresses ψ in terms of Lagrangian description

Q1: For a two-dimensional flow the velocities at a point in a fluid may be expressed in the Eulerian coordinates by $u = x + y + 2t$ and $v = 2y + t$. Determine the Lagrange coordinates as functions of the initial positions x_0 and y_0 and the time t.

Sol: Given $u = x+y+2t$ and $v = 2y+t$

In terms of displacements x and y, we have

$$u = \frac{dx}{dt} \quad \text{and} \quad v = \frac{dy}{dt}$$

$$\Rightarrow \frac{dx}{dt} = x+y+2t \quad \text{and} \quad \frac{dy}{dt} = 2y+t$$

Considering $\frac{dy}{dt} - 2y = t$

$$I.F = e^{\int(-2)\,dt} = e^{-2t}$$

Then the solution is

$$y e^{-2t} = \int t(e^{-2t})\,dt + c_1$$

$$y e^{-2t} = c_1 + t\left(-\tfrac{1}{2}e^{-2t}\right) - \int 1\cdot\left(-\tfrac{1}{2}e^{-2t}\right)dt$$

$$= c_1 - \tfrac{1}{2}te^{-2t} - \tfrac{1}{4}e^{-2t} = c_1 - \tfrac{1}{4}(2t+1)e^{-2t}$$

$$\Rightarrow y = c_1 e^{2t} - (2t+1)/4$$

Substituting in $\frac{dx}{dt} = x+y+2t$

$$\frac{dx}{dt} - x = c_1 e^{2t} + \tfrac{1}{4}(6t-1)$$

$$I.F = e^{\int(-1)\,dt} = e^{-t}$$

$$\Rightarrow x e^{-t} = c_2 + c_1 e^{t} + \int \frac{6t-1}{4} e^{-t}\,dt$$

$$x e^{-t} = c_2 + c_1 e^{t} + \frac{6t-1}{4}(e^{-t}) - \int\left(\tfrac{6}{4}\right)(-e^{-t})\,dt$$

$$\Rightarrow x = c_2 e^{t} + c_1 e^{2t} - (6t+5)/4$$

Using initial conditions, $x = x_0$ and $y = y_0$ when $t = t_0 = 0$

$$\Rightarrow y_0 = c_1 - 1/4 \quad \text{and} \quad x_0 = c_2 + c_1 - (5/4)$$

$$\Rightarrow c_1 = y_0 + \tfrac{1}{4} \quad \text{and} \quad c_2 = x_0 - y_0 + 1$$

$$\Rightarrow x = (x_0 - y_0 + 1)e^{t} + (y_0 + \tfrac{1}{4})e^{2t} - (6t+5)/4$$

$$y = (y_0 + \tfrac{1}{4})e^{2t} - (2t+1)/4$$

These equations gives the desired displacements x and y in the Lagrangian system involving the initial positions x_0, y_0 and the time t.

Velocity of a fluid particle:

Let a fluid particle be at P at any time t and let it be at Q at time $t+\delta t$ such that

$$\vec{OP} = \vec{r} \quad \text{and} \quad \vec{OQ} = \vec{r} + \delta\vec{r}$$

$$\vec{PQ} = \delta\vec{r}.$$

$$\text{velocity, } \vec{q} = \lim_{\delta t \to 0} \left(\frac{\delta \vec{r}}{\delta t}\right) = \frac{d\bar{r}}{dt}.$$

The limit exists uniquely if we take fluid as continuous.

Clearly $\bar{q}$ is a function of r and t

$$\Rightarrow \bar{q} = f(\bar{r}, t)$$

If u, v, w are components of velocity along the axes $\bar{q} = u\hat{i} + v\hat{j} + w\hat{k}$

Calculating the rate of change of a physical quantity ϕ or $\vec{\alpha}$:

Let a fluid particle moves from $P(x,y,z)$ at time t to $Q(x+\delta x, y+\delta y, z+\delta z)$ at time $t+\delta t$.

Let $\phi(x,y,z,t)$ be a scalar function associated with some property (pressure, density, etc).

Let change of ϕ due to the movement of the fluid particle from P to Q be $\delta\phi$. Then, we have

$$\delta\phi = \frac{\partial\phi}{\partial x}\delta x + \frac{\partial\phi}{\partial y}\delta y + \frac{\partial\phi}{\partial z}\delta z + \frac{\partial\phi}{\partial t}\delta t$$

$$\frac{\delta\phi}{\delta t} = \left(\frac{\partial\phi}{\partial x}\right)\frac{\delta x}{\delta t} + \left(\frac{\partial\phi}{\partial y}\right)\frac{\delta y}{\delta t} + \left(\frac{\partial\phi}{\partial z}\right)\frac{\delta z}{\delta t} + \frac{\partial\phi}{\partial t}$$

as $\displaystyle\lim_{\delta t\to 0}\frac{\delta\phi}{\delta t} = \frac{d\phi}{dt}$, $\displaystyle\lim_{\delta t\to 0}\frac{\delta x}{\delta t} = u$, $\displaystyle\lim_{\delta t\to 0}\frac{\delta y}{\delta t} = v$, $\displaystyle\lim_{\delta t\to 0}\frac{\delta z}{\delta t} = w$

where $\bar{q} = u\hat{i} + v\hat{j} + w\hat{k}$

$$\Rightarrow \frac{d\phi}{dt} = \frac{\partial\phi}{\partial t} + u\frac{\partial\phi}{\partial x} + v\frac{\partial\phi}{\partial y} + w\frac{\partial\phi}{\partial z}$$

$$\nabla \equiv \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$$

$$\bar{q}\cdot\bar{\nabla} = \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)$$

$$\Rightarrow \quad \frac{d\phi}{dt} = \frac{\partial \phi}{\partial t} + (\bar{q}\cdot\nabla)\phi .$$

Similarly let $\bar{F} = F(x,y,z,t)$ be a vector function associated with some property of the fluid

$$\Rightarrow \quad \frac{d\bar{F}}{dt} = \frac{\partial \bar{F}}{\partial t} + (\bar{q}\cdot\nabla)\bar{F} .$$

For both scalar and vector functions

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\bar{q}\cdot\nabla)$$

- $\frac{d}{dt}$: Material (or particle) derivative → Differentiation following the motion of the fluid
- $\frac{\partial}{\partial t}$: Local derivative → Time variation at a fixed point
- $(\bar{q}\cdot\nabla)$: Convective derivative → change of physical quantity due to the motion of the fluid particle.

· Acceleration of a fluid particle:

$$\bar{a} = \frac{d\bar{q}}{dt} = \frac{\partial \bar{q}}{\partial t} + (\bar{q}\cdot\nabla)\bar{q}$$

$\bar{a}$ can be expressed as the material derivative of the velocity vector $\bar{q}$.

$$\bar{a} = \frac{\partial \bar{q}}{\partial t} + u\frac{\partial \bar{q}}{\partial x} + v\frac{\partial \bar{q}}{\partial y} + w\frac{\partial \bar{q}}{\partial z} .$$

· Components of acceleration in cartesian coordinates (x,y,z)

$$\bar{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k}$$

$$a_x\hat{i} + a_y\hat{j} + a_z\hat{k} = u\frac{\partial}{\partial x}(u\hat{i} + v\hat{j} + w\hat{k}) + v\frac{\partial}{\partial y}(\bar{q}) + w\frac{\partial}{\partial z}(\bar{q}) + \frac{\partial}{\partial t}(\bar{q})$$

$$\Rightarrow \quad a_x = u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} + \frac{\partial u}{\partial t},$$

$$a_y = u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} + \frac{\partial v}{\partial t},$$

$$a_z = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} + \frac{\partial w}{\partial t}.$$

- Components of acceleration in cylindrical co-ordinates (r,θ,z) with velocity components (v_r, v_θ, v_z):

$$a_r = \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}.$$

$$a_\theta = \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}$$

$$a_z = \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z}.$$

- Components of acceleration in spherical polar coordinates (r,θ,ϕ) with velocity components (v_r, v_θ, v_ϕ):

$$a_r = \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{v_\theta^2+v_\phi^2}{r}$$

$$a_\theta = \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_r v_\theta}{r} - \frac{v_\phi^2 \cot\theta}{r}$$

$$a_\phi = \frac{\partial v_\phi}{\partial t} + v_r \frac{\partial v_\phi}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta} + \frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi} + \frac{v_r v_\phi}{r} + \frac{v_\theta v_\phi \cot\theta}{r}$$

-Q : The velocity components of a flow in cylindrical polar coordinates are $(r^2 z\cos\theta,\ rz\sin\theta,\ z^2 t)$. Determine the components of the acceleration of a fluid particle.

Sol : Let v_r, v_θ, v_z be the components of velocity in cylindrical polar coordinates (r,θ,z).

Then we have

$$v_r = r^2 z\cos\theta, \quad v_\theta = rz\sin\theta, \quad v_z = z^2 t$$

Let a_r, a_θ, a_z be the components of acceleration,

Then

$$a_r = \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} + v_z \frac{\partial v_r}{\partial z} - \frac{v_\theta^2}{r}$$

$$= 0 + (r^2 z\cos\theta)(2rz\cos\theta) + \frac{rz\sin\theta}{r}(-2rz\sin\theta) + z^2 t(r^2\cos\theta) - \frac{(rz\sin\theta)^2}{r}$$

$$= rz^2(2r^2\cos^2\theta - 3\sin^2\theta + rt\cos\theta).$$

$$a_\theta = \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} + v_z \frac{\partial v_\theta}{\partial z} + \frac{v_r v_\theta}{r}$$

$$= (r^2 z\cos\theta)(z\sin\theta) + \frac{rz\sin\theta}{r}(rz\cos\theta) + z^2 t(r\sin\theta) + \frac{1}{r}(r^2 z\cos\theta)(rz\sin\theta)$$

$$= z^2 r\sin\theta\,(3r\cos\theta + t)$$

$$a_z = \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z}$$

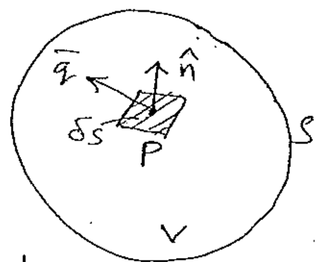
$$= z^2 + (r^2 z\cos\theta)\times 0 + \frac{rz\sin\theta}{r}\times 0 + (z^2 t)(2zt)$$

$$= z^2(1 + 2t^2 z).$$

Equation of Continuity : (by Euler's method)

Also called as equation of conservation of mass.

Let S be an arbitrary small closed surface drawn in the compressible fluid enclosing a volume V and let S be taken fixed in space.



Let $P(x,y,z)$ be any point of S and $\rho(x,y,z,t)$ be fluid density at P at any time t.

Let δs denote an element of the surface S enclosing P

Let $\hat{n}$ be the unit vector drawn normal at δs and $\bar{q}$ be the fluid velocity at P. Then the normal component of $\bar{q}$ measured outward from V is $\hat{n}\cdot\bar{q}$

$\therefore$ Rate of mass flow across $\delta s = \rho(\hat{n}\cdot\bar{q})\,\delta s$

$\therefore$ Total rate of mass flow across S

$$= \int_S \rho(\hat{n}\cdot\bar{q})\,ds = \int_V \nabla\cdot(\rho\bar{q})\,dV$$

by Gauss divergence theorem.

$\therefore$ Total rate of mass flow into $V = -\int_V \nabla\cdot(\rho\bar{q})\,dV$

Again, the mass of the fluid within S at time t

$$= \int_V \rho\,dV$$

$\therefore$ Total rate of mass increase within $S = \frac{\partial}{\partial t}\int_V \rho\,dV = \int_V \frac{\partial \rho}{\partial t}\,dV$

Suppose that the region V of the fluid contains neither sources nor sinks. Then by the law of conservation of the fluid mass, the rate of increase of the mass of the fluid within V must be equal to the total rate of mass flowing into V

$$\Rightarrow \int_V \frac{\partial \rho}{\partial t}\, dv = -\int_V \nabla\cdot(\rho\bar{q})\, dv$$

$$\Rightarrow \int_V \left[\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\bar{q})\right] dv = 0$$

which holds for arbitrary small volumes V,

if $\left[\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\bar{q}) = 0\right]$

This is called equation of continuity and it holds at all points of fluid free from sources and sinks.

Note :

1. Since $\nabla\cdot(\rho\bar{q}) = \rho\,\nabla\cdot\bar{q} + \nabla\rho\cdot\bar{q}$

[Other forms of equation of continuity are

$$\frac{\partial \rho}{\partial t} + \rho\nabla\cdot\bar{q} + \nabla\rho\cdot\bar{q} = 0$$

$$\frac{d\rho}{dt} + \rho\,\nabla\cdot\bar{q} = 0$$

$$\frac{d(\log \rho)}{dt} + \nabla\cdot\bar{q} = 0 \;]$$

2. For an incompressible and heterogeneous fluid the density of any fluid particle is invariable with time, so that $\frac{d\rho}{dt} = 0$

$$\Rightarrow \rho\,\nabla\cdot\bar{q} = 0 \Rightarrow \nabla\cdot\bar{q} = 0 \Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

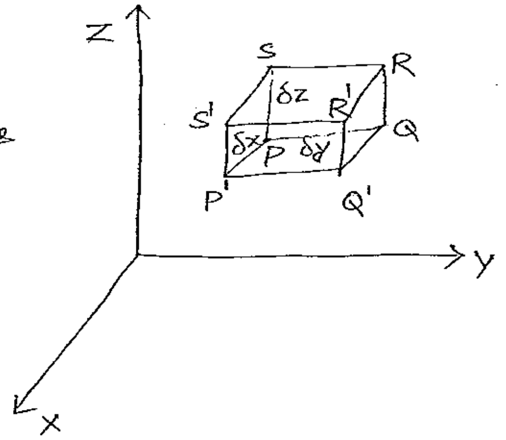
3. For an incompressible and homogeneous fluid, ρ is constant and hence $\frac{\partial \rho}{\partial t} = 0$

$$\Rightarrow \nabla\cdot(\rho\bar{q}) = 0 \Rightarrow \nabla\cdot\bar{q} = 0 \text{ or}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \text{ as } \rho \text{ is constant.}$$

. The equation of continuity in cartesian coordinates:

Let there be a fluid particle at $P(x,y,z)$ and $\rho(x,y,z,t)$ be the density of fluid at P at any time t and u,v,w be the velocity components at P parallel to the rectangular coordinate axes. Then by constructing a small parallelopiped with edges of length $\delta x, \delta y, \delta z$ parallel to their respective coordinate axes as shown in the figure



Mass of the fluid that passes through the face PQRS

$$= (\rho\, \delta y\, \delta z) u \text{ per unit time}$$
$$= f(x,y,z) \text{ (say)}$$

$\therefore$ Mass of the fluid that passes out through the opposite face P'Q'R'S'

$$= f(x+\delta x, y, z) \text{ per unit time}$$
$$= f(x,y,z) + \delta x \frac{\partial}{\partial x} f(x,y,z) + \ldots \text{ (by Taylor's theorem)}$$

$\therefore$ The net gain in mass per unit time within the element (rectangular parallelopiped) due to the flow through the faces PQRS and P'Q'R'S'

$$= f(x,y,z) - \left[f(x,y,z) + \delta x \frac{\partial}{\partial x} f(x,y,z) + \ldots\right]$$
$$= -\delta x \frac{\partial}{\partial x} f(x,y,z), \text{ to the first order of approximation}$$
$$= -\delta x \frac{\partial}{\partial x} (\rho\, \delta y\, \delta z\, u)$$
$$= -\delta x\, \delta y\, \delta z \frac{\partial(\rho u)}{\partial x},$$

Similarly, Net gain in mass per unit time within the element due to the flow through the faces PSS'P' and QRR'Q'

$$= -\delta x\, \delta y\, \delta z \frac{\partial(\rho v)}{\partial y},$$

Net gain in mass per unit time within the element due to the flow through the faces $PP'Q'Q$ and $SS'R'R$ $= -\delta x\, \delta y\, \delta z \dfrac{\partial(\rho w)}{\partial z}$.

$\therefore$ Total rate of mass flow into the elementary parallelopiped $= -\delta x\, \delta y\, \delta z \left[\dfrac{\partial(\rho u)}{\partial x} + \dfrac{\partial(\rho v)}{\partial y} + \dfrac{\partial(\rho w)}{\partial z} \right]$.

Again, the mass of the fluid within the chosen element at time $t = \rho\, \delta x\, \delta y\, \delta z$

$\therefore$ Total rate of mass increase within the element $= \dfrac{\partial}{\partial t}(\rho\, \delta x\, \delta y\, \delta z) = \delta x\, \delta y\, \delta z \dfrac{\partial \rho}{\partial t}$

Suppose the choosen region of the fluid contains neither sources nor sinks. Then by the law of conservation of the fluid mass, the rate of increase of the mass of the fluid within the element must be equal to the rate of mass flowing into the element.

$$\Rightarrow \delta x\, \delta y\, \delta z \frac{\partial \rho}{\partial t} = -\delta x\, \delta y\, \delta z \left[\frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} \right]$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

$$\Rightarrow \left(\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + v \frac{\partial \rho}{\partial y} + w \frac{\partial \rho}{\partial z} \right) + \rho \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0$$

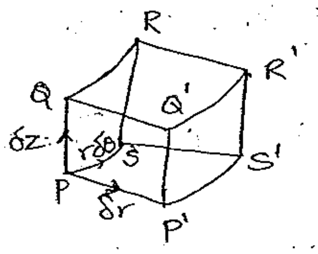
$$\Rightarrow \frac{d\rho}{dt} + \rho \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0$$

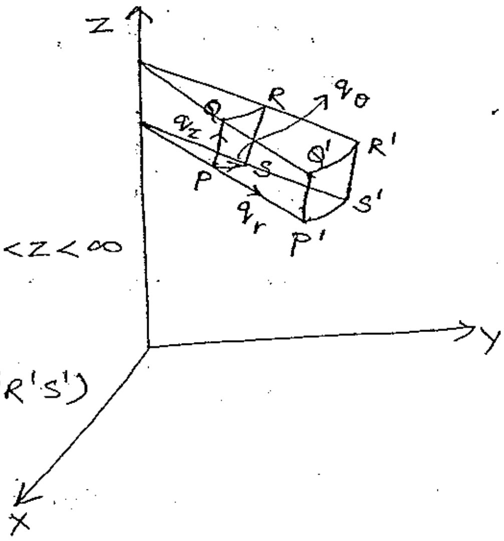
$$\Rightarrow \frac{d\rho}{dt} + \rho (\nabla \cdot \bar{q}) = 0, \quad \bar{q} = u\hat{i} + v\hat{j} + w\hat{k}.$$

Note:

✓ [If the fluid is incompressible and homogeneous or heterogeneous, ρ is constant or $\dfrac{d\rho}{dt} = 0$, then $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0$.]

The equation of continuity in cylindrical coordinates:





(r,θ,z) where $r \geq 0$, $0 \leq \theta \leq 2\pi$, $-\infty < z < \infty$

$\rho\,(r,\theta,z,t)$

Curvilinear parallelopiped (PQRS, P'Q'R'S')

q_r, q_θ, q_z velocity components in the direction of elements PP', arc PS and PQ respectively.

Net gain in mass per unit time within the chosen elementary parallelopiped due to flow through the faces PQRS and P'Q'R'S':

$$= f(r,\theta,z) - \left[f(r,\theta,z) + \delta r \cdot \frac{\partial}{\partial r} f(r,\theta,z) + \cdots\right]$$

$$= -\delta r \frac{\partial}{\partial r}\left(\rho r \delta\theta \delta z \cancel{\delta}\, q_r\right)$$

$$= -\delta r\, \delta\theta\, \delta z \;\frac{\partial(\rho r q_r)}{\partial r}.$$

Similarly, through PP'Q'Q and SS'R'R

$$= -\delta\theta \frac{\partial}{\partial \theta}\left(\rho\, \delta r\, \delta z\, q_\theta\right)$$

$$= -\delta r\, \delta\theta\, \delta z \;\frac{\partial(\rho q_\theta)}{\partial \theta}.$$

Through PP'S'S and QQ'R'R

$$= -\delta z \frac{\partial}{\partial z}\left(\rho\, r\delta\theta\, \delta r\, q_z\right)$$

$$= -r\,\delta r\, \delta\theta\, \delta z \;\frac{\partial(\rho q_z)}{\partial z}.$$

$\therefore$ Total rate of mass flow into the choosen element

$$= -\delta r\, \delta\theta\, \delta z\left[\frac{\partial}{\partial r}(\rho r q_r) + \frac{\partial}{\partial \theta}(\rho q_\theta) + r\frac{\partial}{\partial z}(\rho q_z)\right]$$

Again, mass of the fluid within the element at time, $t = \rho\, r\, \delta r\, \delta\theta\, \delta z$

$\therefore$ Total rate of increase of mass within the element

$$= \frac{\partial}{\partial t}\left(\rho\, r\, \delta r \delta\theta \delta z\right) = r\, \delta r \delta\theta \delta z\; \partial\rho/\partial t.$$

By law of conservation of fluid mass,

$$r\,\delta r\,\delta\theta\,\delta z\,\frac{\partial \rho}{\partial t} = -\,\delta r\,\delta\theta\,\delta z\left[\frac{\partial}{\partial r}(\rho r q_r)+\frac{\partial}{\partial \theta}(\rho q_\theta)+r\frac{\partial}{\partial z}(\rho q_z)\right]$$

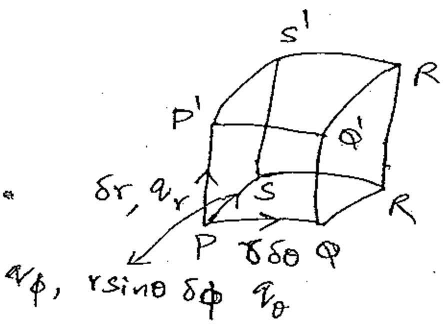
$$\Rightarrow r\frac{\partial \rho}{\partial t}+\left[\frac{\partial}{\partial r}(\rho r q_r)+\frac{\partial}{\partial \theta}(\rho q_\theta)+r\frac{\partial}{\partial z}(\rho q_z)\right]=0$$

$$\Rightarrow \frac{\partial \rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(\rho r q_r)+\frac{1}{r}\frac{\partial}{\partial \theta}(\rho q_\theta)+\frac{\partial}{\partial z}(\rho q_z)=0$$

Note: For incompressible fluid

$$\frac{1}{r}\frac{\partial}{\partial r}(r q_r)+\frac{1}{r}\frac{\partial}{\partial \theta}(q_\theta)+\frac{\partial}{\partial z}(q_z)=0$$

The equation of continuity in spherical polar coordinates:



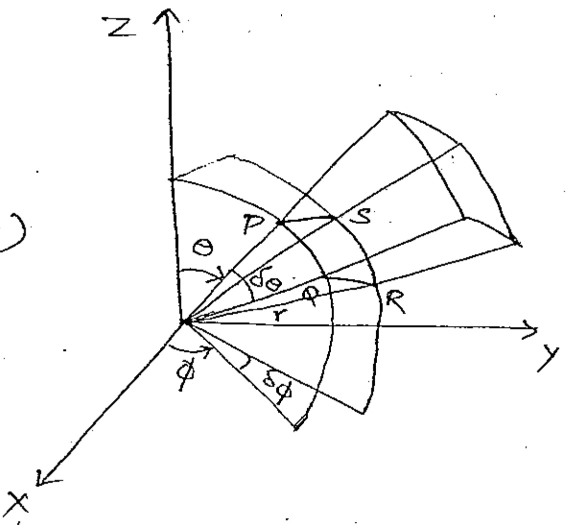
(r,θ,ϕ)

$r\geq 0$

$0\leq\phi\leq 2\pi$

$0\leq\theta\leq\pi$

$\rho\,(r,\theta,\phi,t)$



$$f(r,\theta,\phi)=\rho\, r\delta\theta\, r\sin\theta\,\delta\phi\cdot q_r$$

$$f(r+\delta r,\theta,\phi)=f(r,\theta,\phi)+\delta r\frac{\partial}{\partial r}f(r,\theta,\phi)+\cdots$$

$$\begin{aligned}f(r,\theta,\phi)-f(r+\delta r,\theta,\phi) &= -\delta r\frac{\partial}{\partial r}f(r,\theta,\phi)\\ &= -\delta r\frac{\partial}{\partial r}(\rho r^2\sin\theta\, q_r\,\delta\theta\,\delta\phi)\\ &= -\delta r\,\delta\theta\,\delta\phi\,\sin\theta\frac{\partial}{\partial r}(\rho r^2 q_r).\end{aligned}$$

$\|^{ly}$

$$\begin{aligned}g(r,\theta,\phi)-g(r,\theta+\delta\theta,\phi) &= -\delta\theta\frac{\partial}{\partial\theta}(q_\theta\,\rho\, r\sin\theta\,\delta\phi\,\delta r)\\ &= -\delta r\,\delta\theta\,\delta\phi\, r\frac{\partial}{\partial\theta}(\rho\sin\theta\, q_\theta)\end{aligned}$$

$$\begin{aligned}h(r,\theta,\phi)-h(r,\theta,\phi+\delta\phi) &= -\delta\phi\frac{\partial}{\partial\phi}(q_\phi\cdot r\,\delta\theta\,\delta r\cdot\rho)\\ &= -r\,\delta r\,\delta\theta\,\delta\phi\frac{\partial}{\partial\phi}(\rho q_\phi).\end{aligned}$$

$$\Rightarrow \frac{\partial}{\partial t}\left(\rho r^2 \sin\theta\, \delta r\, \delta\theta\, \delta\phi\right) = -\delta r\,\delta\theta\,\delta\phi\left[\sin\theta \frac{\partial}{\partial r}(\rho r^2 q_r) + r\frac{\partial}{\partial\theta}(\rho \sin\theta\, q_\theta) + r\frac{\partial}{\partial\phi}(\rho q_\phi)\right]$$

$$\Rightarrow r^2\sin\theta \frac{\partial\rho}{\partial t} + \sin\theta\frac{\partial}{\partial r}(\rho r^2 q_r) + r\frac{\partial}{\partial\theta}(\rho\sin\theta\, q_\theta) + r\frac{\partial}{\partial\phi}(\rho q_\phi) = 0$$

$$\Rightarrow \frac{\partial\rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 q_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\rho\sin\theta\, q_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(\rho q_\phi) = 0$$

Holds at all points of the fluid free from sources and sinks.

Equation of continuity in Lagrangian form:

$$[\rho J = \rho_0]$$

where $J = \dfrac{\partial(x,y,z)}{\partial(a,b,c)} = \begin{vmatrix} \frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} & \frac{\partial x}{\partial c} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} & \frac{\partial y}{\partial c} \\ \frac{\partial z}{\partial a} & \frac{\partial z}{\partial b} & \frac{\partial z}{\partial c} \end{vmatrix}$

$$\frac{dJ}{dt} = J\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

Relation b/w Euler and Lagrangian forms:

(i) $\rho J = \rho_0$

$$J\frac{d\rho}{dt} + \rho\frac{dJ}{dt} = \frac{d}{dt}(\rho_0) = 0$$

$$\Rightarrow \frac{d\rho}{dt}J + \rho J\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

$$\Rightarrow \frac{d\rho}{dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

(ii) $$\frac{d\rho}{dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

$$\frac{d\rho}{dt} + \rho\left(\frac{1}{J}\frac{dJ}{dt}\right) = 0$$

$$\Rightarrow J\frac{d\rho}{dt} + \rho\frac{dJ}{dt} = 0$$

$$\Rightarrow \frac{d}{dt}(\rho J) = 0$$

$$\Rightarrow \rho J = \rho_0$$

Some symmetrical forms:

(i) Cylindrical symmetry

$\frac{\partial}{\partial \theta} = 0$ and $\frac{\partial}{\partial z} = 0$; $q_r(r,t)$ – velocity $\perp^{ar}$ to axis.

$\rho(r,t)$ – density of fluid

$$\Rightarrow \frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r q_r) = 0$$

If ρ is constant, $\frac{\partial \rho}{\partial t} = 0$

$$\Rightarrow \frac{\partial}{\partial r}(\rho r q_r) = 0$$

Integrating w.r.t 'r', we have

$$r\rho q_r = \rho\, g(t) \Rightarrow r q_r = g(t)$$

If the flow is steady, $g(t)$ reduces to an absolute constant. Thus for a steady flow

$r q_r = c$, where c is a constant.

(ii) Spherical symmetry

$\frac{\partial}{\partial \theta} = 0$ and $\frac{\partial}{\partial \phi} = 0$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 q_r) = 0$$

If ρ is constant, $\frac{\partial \rho}{\partial t} = 0$

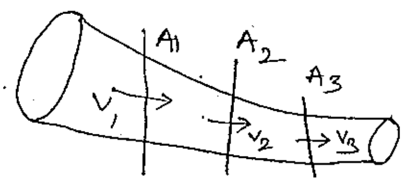
$$\Rightarrow \rho r^2 q_r = \rho\, g(t) \Rightarrow r^2 q_r = g(t)$$

If the flow is steady

$r^2 q_r = c$, where c is a constant.

Equation of continuity of a liquid flow through a channel or pipe:

Let an incompressible fluid flow through a channel or pipe.



Q – Quantity of fluid flowing across the section

Q – discharge

$Q_1 = A_1 V_1$

$Q_2 = A_2 V_2$

$Q_3 = A_3 V_3$

From law of conservation of mass

$Q_1 = Q_2 = Q_3 = \dots$ so on

Thus $A_1 V_1 = A_2 V_2 = A_3 V_3 = \dots$ is the equation of continuity

Working rule of writing the equation of continuity:

Let P be position of any fluid particle and let ρ be density at P.

With P as one corner construct a parallelopiped whose edges are $\lambda\delta\alpha$, $\mu\delta\beta$, $\nu\delta\gamma$

Lengths of elements : $\lambda\delta\alpha$, $\mu\delta\beta$, $\nu\delta\gamma$

Components of velocity : u , v , w

Calculate the rate of excess flow-in over flow-out along all the lengths.

$$-\delta\alpha\frac{\partial}{\partial\alpha}(\rho\mu\,\delta\beta\,\nu\,\delta\gamma\,u)-\delta\beta\frac{\partial}{\partial\beta}(\rho\,\lambda\delta\alpha\,\nu\,\delta\gamma\,v)-\delta\gamma\frac{\partial}{\partial\gamma}(\rho\,\delta\alpha\cdot\lambda\cdot\mu\delta\beta\,w)$$

The total mass of fluid in the element
$$=\rho\,\lambda\,\delta\alpha\,\mu\,\delta\beta\,\nu\,\delta\gamma.$$

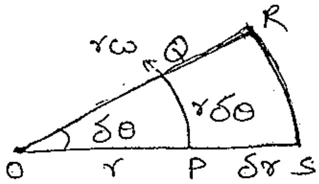
Rate of increase in mass of the element
$$=\frac{\partial}{\partial t}(\rho\,\lambda\,\delta\alpha\,\mu\,\delta\beta\,\nu\,\delta\gamma)$$

For equation of continuity

$$\frac{\partial}{\partial t}(\rho\,\lambda\,\delta\alpha\,\mu\,\delta\beta\,\nu\,\delta\gamma)=-\delta\alpha\frac{\partial}{\partial\alpha}(\rho\mu\,\delta\beta\,\nu\,\delta\gamma\,u)-\delta\beta\frac{\partial}{\partial\beta}(\rho v\,\lambda\delta\alpha\,\nu\,\delta\gamma)$$
$$-\delta\gamma\frac{\partial}{\partial\gamma}(\rho w\,\lambda\,\delta\alpha\,\mu\,\delta\beta)$$

3Q : A mass of fluid moves in such a way that each particle describes a circle in one plane about a fixed axis; show that the equation of continuity is $\frac{\partial\rho}{\partial t}+\frac{\partial(\rho\omega)}{\partial\theta}=0$, where ω is the angular velocity of a particle whose azimuthal angle is θ at time t.

Sol : Considering the motion in cylindrical polar co-ordinates (r,θ,z) with fixed axis as z-axis.



As the motion is confined to a plane and circular. The fluid particles have only v_θ component of velocity. $v_r=0=v_z$.

The rate of ~~excess~~ flow-in over flow-out $=-r\,\delta\theta\frac{\partial}{r\,\partial\theta}(\rho\cdot v_\theta\,\delta r\,\delta z)$
$$=-\delta\theta\frac{\partial}{\partial\theta}(\rho r\omega\,\delta r\,\delta z)=-\delta r\,\delta\theta\,\delta z\frac{\partial}{\partial\theta}(\rho r\omega).$$

The rate of increase in mass of the element

$$= \frac{\partial}{\partial t}(\rho\, \delta r\, r\delta\theta\, \delta z)$$

$$= r\,\delta r\,\delta\theta\,\delta z\, \frac{\partial \rho}{\partial t}$$

Equating for equation of continuity

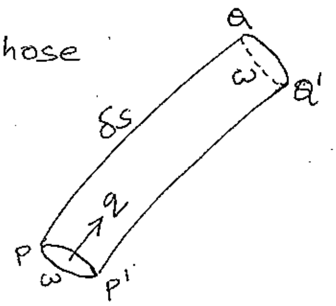
$$r\,\delta r\,\delta\theta\,\delta z\,\frac{\partial \rho}{\partial t} = -\delta r\,\delta\theta\,\delta z\,\frac{\partial}{\partial \theta}(\rho r\omega)$$

$$\Rightarrow \quad \frac{\partial \rho}{\partial t} + \frac{1}{r}\cdot r\,\frac{\partial}{\partial \theta}(\rho\omega) = 0$$

$$\Rightarrow \quad \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \theta}(\rho\omega) = 0.$$

4Q: If ω is the area of cross-section of a stream filament prove that the equation of continuity is $\frac{\partial}{\partial t}(\rho\omega) + \frac{\partial}{\partial s}(\rho\omega q) = 0$, where δs is an element of arc of the filament in the direction of flow and q is the speed.

Sol: Let $PP'Q'Q$ be stream filament whose area of cross-section is ω and arc $PQ = \delta s$.



The rate of the excess flow-in over the flow-out along PQ

$$= -\delta s\,\frac{\partial}{\partial s}(\rho q\omega)$$

Again, the total mass of the fluid within the stream filament is $\rho\omega\delta s$.

$\therefore$ The rate of increase in mass of the fluid in the stream filament $= \frac{\partial}{\partial t}(\rho\omega\delta s)$.

Hence the equation of continuity is given by

$$\frac{\partial}{\partial t}(\rho\omega\delta s) = -\delta s\,\frac{\partial}{\partial s}(\rho q\omega)$$

$$\Rightarrow \quad \frac{\partial}{\partial t}(\rho\omega) + \frac{\partial}{\partial s}(\rho q\omega) = 0$$

5Q: A pulse travelling along a fine straight uniform tube filled with gas causes the density at time t and distance x from the origin where the velocity is u_0 to become $\rho_0\,\phi(vt-x)$. Prove that the velocity u (at time t and distance x from the origin) is given by

$$v + \frac{(u_0-v)\,\phi(vt)}{\phi(vt-x)}$$

Sol: Let ρ be the density and u be the velocity at time t and at a distance x.

Then given that,

$$\rho = \rho_0\,\phi(vt-x) \quad -①$$

The equation of continuity is

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \rho\frac{\partial u}{\partial x} + u\frac{\partial \rho}{\partial x} = 0$$

From ①

$$\frac{\partial \rho}{\partial t} = \rho_0 v\,\phi'(vt-x) \quad \text{and} \quad \frac{\partial \rho}{\partial x} = -\rho_0\,\phi'(vt-x)$$

$$\Rightarrow \rho_0 v\,\phi'(vt-x) + \rho_0\,\phi(vt-x)\frac{\partial u}{\partial x} - u\rho_0\,\phi'(vt-x) = 0$$

$$\Rightarrow (v-u)\,\phi'(vt-x) + \phi(vt-x)\frac{\partial u}{\partial x} = 0$$

$$\Rightarrow \frac{du}{dx}\,\phi(vt-x) + (v-u)\,\phi'(vt-x) = 0$$

$$\Rightarrow \frac{du}{(v-u)} + \frac{\phi'(vt-x)}{\phi(vt-x)}\,dx = 0$$

Integrating, $-\log(v-u) - \log\phi(vt-x) = -\log c$

$$\Rightarrow (v-u)\,\phi(vt-x) = c$$

Given $u=u_0$ when $x=0$

$$\Rightarrow (v-u_0)\,\phi(vt) = c$$

$$\therefore (v-u)\,\phi(vt-x) = (v-u_0)\,\phi(vt)$$

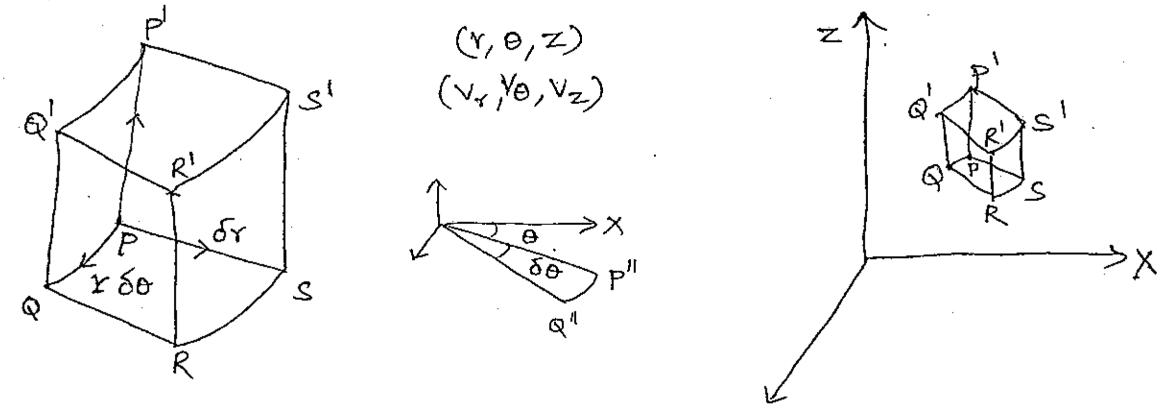
$$\Rightarrow u = v + \frac{(u_0-v)\,\phi(vt)}{\phi(vt-x)}.$$

6Q: A mass of fluid is in motion so that the lines of motion lie on the surface of co-axial cylinders. Show that the equation of continuity is

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho u) + \frac{\partial}{\partial z}(\rho v) = 0$$

where u, v are the velocity components perpendicular to z.

Sol:



Since lines of motion lie on the surface of co-axial cylinders

$\delta r = 0$, $V_r = 0$

Given $V_\theta = u$ and $V_z = v$.

Rate of excess of flow-in over the flow-out

along $PS = 0$

$$\left.\begin{aligned} PQ &= -\delta\theta \frac{\partial}{\partial\theta}(\rho u\, \delta r\, \delta z) \\ PP' &= -\delta z \frac{\partial}{\partial z}(\rho v\, r\, \delta\theta\, \delta r) \end{aligned}\right\} \quad -①$$

Again, the rate of increase in mass of the element $= \frac{\partial}{\partial t}(\rho r\, \delta\theta\, \delta r\, \delta z) \quad -②$

Equating ① and ② by law of conservation of mass,

$$\frac{\partial}{\partial t}(\rho r \delta\theta\, \delta r\, \delta z) + (\delta\theta\, \delta r\, \delta z)\left[\frac{\partial}{\partial\theta}(\rho u) + \frac{\partial}{\partial z}(\rho v r)\right] = 0$$

$$\Rightarrow r\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial\theta}(\rho u) + r\frac{\partial}{\partial z}(\rho v) = 0$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial\theta}(\rho u) + \frac{\partial}{\partial z}(\rho v) = 0$$

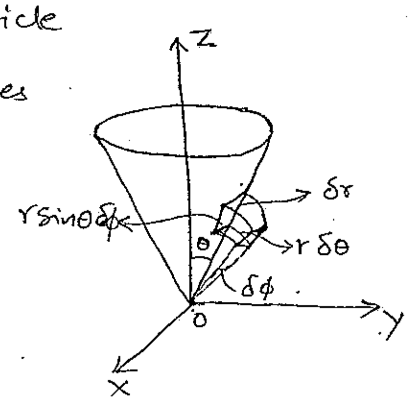
7Q: If the lines of motion are curves on the surface of cones having their vertices at the origin and the axis of z for common surface, prove that the equation of continuity is

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r}(\rho u) + \frac{2\rho u}{r} + \frac{\operatorname{Cosec}\theta}{r}\frac{\partial}{\partial\phi}(\rho w) = 0$$

where u and w are the velocity components in the directions in which r and ϕ increase.

Sol: let O, the vertex of cones, be the origin and let OZ, their common axis, be the axis z. let the semi-vertical angle be θ.

let us consider a fluid particle P whose spherical coordinates are (r, θ, ϕ).



According to question

$$\frac{\partial}{\partial \phi \theta} = 0$$

$$\Rightarrow \frac{\partial}{\partial t}(\rho\, \delta r \cdot r\delta\theta \cdot r\sin\theta\, \delta\phi)$$

$$= -\delta r \frac{\partial}{\partial r}(\rho u\, r\delta\theta \cdot r\sin\theta\, \delta\phi) - \cancel{r}\,\delta\phi \frac{\partial}{\partial \phi}(\rho w \cdot \delta r \cdot r\delta\theta)$$

$$\Rightarrow r^2\, \delta r\, \delta\theta\, \delta\phi\, \sin\theta \frac{\partial \rho}{\partial t} + \delta r\, \delta\theta\, \delta\phi \overset{\sin\theta}{\times} \frac{\partial}{\partial r}(\rho u r^2) + \delta r\, \delta\theta\, \delta\phi \cdot r \frac{\partial}{\partial \phi}(\rho w) = 0$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \frac{\sin\theta}{r^2 \sin\theta} \frac{\partial}{\partial r}(\rho r^2 u) + \frac{r}{r^2 \sin\theta} \frac{\partial}{\partial \phi}(\rho w) = 0$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \frac{1}{r^2}\left[r^2 \frac{\partial}{\partial r}(\rho u) + 2r\rho u\right] + \frac{1}{r\sin\theta} \frac{\partial}{\partial \phi}(\rho w) = 0$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r}(\rho u) + \frac{2\rho u}{r} + \frac{\operatorname{cosec}\theta}{r} \frac{\partial(\rho w)}{\partial \phi} = 0.$$

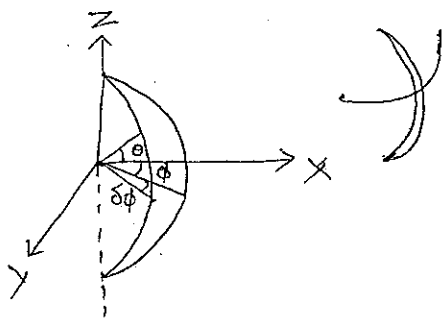
8 Q : If every particle moves on the surface of a sphere. Prove that the equation of continuity is

$$\frac{\partial \rho}{\partial t} \cancel{\cdot} \cos\theta + \frac{\partial}{\partial \theta}(\rho \omega \cos\theta) + \frac{\partial}{\partial \phi}(\rho \omega' \cos\theta) = 0,$$

ρ being the density, θ, ϕ the latitute and longitude of any element, ω and ω' the angular velocities of the element in latitude and longitude respectively.

Sol :

$$\frac{\partial}{\partial r} = 0$$



$$\delta r,\ r\delta\theta,\ r\cos\theta\, \delta\phi$$
$$0,\ r\omega,\ r\cos\theta\, \omega'$$

$$\frac{\partial}{\partial t}(\rho \cdot \delta r \cdot r\delta\theta \cdot r\cos\theta\, \delta\phi)$$

$$= -\delta\theta \frac{\partial}{\partial \theta}(\rho \cdot r\omega \cdot \delta r \cdot r\cos\theta\, \delta\phi) - \delta\phi \frac{\partial}{\partial \phi}(\rho \cdot r\cos\theta\, \omega' \cdot \delta r \cdot r\delta\theta)$$

$$\Rightarrow \left(\frac{\partial \rho}{\partial t}\right) \cdot r^2 \cos\theta \cdot \delta r\, \delta\theta\, \delta\phi + r^2 \delta r\, \delta\theta\, \delta\phi \left[\frac{\partial}{\partial \theta}(\rho \omega \cos\theta) + \frac{\partial}{\partial \phi}(\rho \omega' \cos\theta)\right]$$

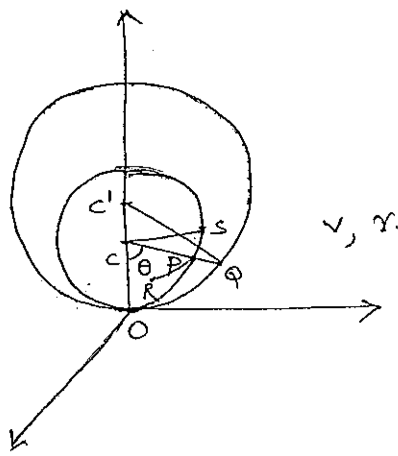
$$\Rightarrow \frac{\partial \rho}{\partial t} \cos\theta + \frac{\partial}{\partial \theta}(\rho \omega \cos\theta) + \frac{\partial}{\partial \phi}(\rho \omega' \cos\theta) = 0$$

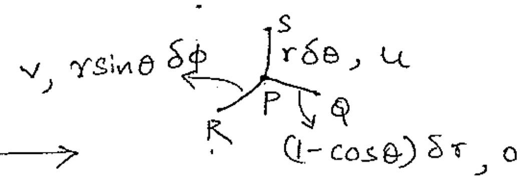
9 Q: If the lines of motion are curves on the surfaces of spheres, all touching the plane of xy at the origin O, the equation of continuity is

$$r\sin\theta \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial \phi}(\rho v) + \sin\theta \frac{\partial}{\partial \theta}(\rho u) + \rho u(1+2\cos\theta) = 0.$$

where r is the radius CP of one of the spheres, θ the angle PCO, u the velocity in the plane PCO, v the perpendicular velocity and ϕ the inclination of the plane PCO to a fixed plane through the axis of z.

Sol:





$CC' = \delta r$

$PS = r\delta\theta$

$PR = r\sin\theta\,\delta\phi$

$C'Q = r + \delta r$

$CP = r$

$PQ = CQ - CP$

$C'Q^2 = CC'^2 + CQ^2 - 2CC'.CQ\cos(180-\theta)$

$\Rightarrow (r+\delta r)^2 = \delta r^2 + (r+PQ)^2 - 2\delta r(r+PQ)\cos(\pi-\theta)$

$\Rightarrow 2r\delta r = 2rPQ + PQ^2 + 2r\delta r\cos\theta + 2\delta r\,PQ\cos\theta$

Since PQ and δr are very small, to first order of approximation

$$2r\delta r(1-\cos\theta) = 2rPQ$$

$$\Rightarrow PQ = (1-\cos\theta)\delta r.$$

$\therefore$ Rate of flow-in over flow-out along

$PS = -\delta\theta \frac{\partial}{\partial\theta}[\rho u.(1-\cos\theta)\delta r.\, r\sin\theta\,\delta\phi]$

$PQ = 0$

$PR = -\delta\phi \frac{\partial}{\partial\phi}[\rho v(1-\cos\theta)\delta r.\, r\,\delta\theta]$

$$\Rightarrow \frac{\partial}{\partial t}[\rho(1-\cos\theta)\delta r.\, r\delta\theta.\, r\sin\theta\,\delta\phi] = -\delta r\,\delta\theta\,\delta\phi.r.\frac{\partial}{\partial\theta}(\rho u(1-\cos\theta)\sin\theta) - \delta r\,\delta\theta\,\delta\phi.r.\frac{\partial}{\partial\phi}[\rho v(1-\cos\theta)].$$

$$\Rightarrow r\sin\theta\frac{\partial\rho}{\partial t} + \sin\theta\frac{\partial}{\partial\theta}(\rho u) + \frac{\partial}{\partial\phi}(\rho v) + \rho u(1+2\cos\theta) = 0.$$

10 Q : In a three dimensional incompressible flow, the velocity components in y and z directions are $v = ax^3 - by^2 + cz^2$, $w = bx^3 - cy^2 + az^2x$. Determine the missing component of velocity distribution such that continuity equation is satisfied.

Sol: $v = ax^3 - by^2 + cz^2$ $\qquad$ $w = bx^3 - cy^2 + az^2x$

$\frac{\partial v}{\partial y} = -2by$ $\qquad$ $\frac{\partial w}{\partial z} = 2azx$

Continuity equation for an incompressible fluid flow is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\Rightarrow \frac{\partial u}{\partial x} - 2by + 2azx = 0 \quad - (1)$$

$$[\partial u = (2by)\partial x + (-2azx)\partial x]$$

Integrating (1) w.r.t 'x'

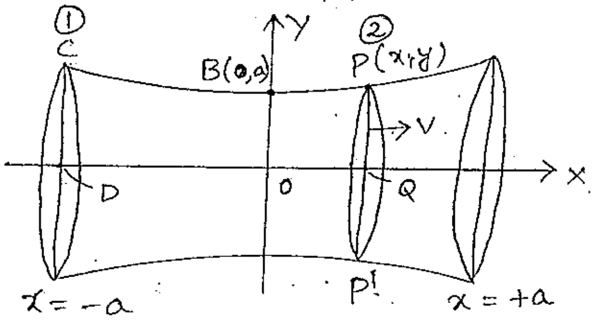
$$u = 2byx - 2az\frac{x^2}{2} + f(y,z)$$

or $\quad u = 2byx - azx^2 + f(y,z)$,

where $f(y,z)$ is an arbitrary function independent of x.

11 Q : Liquid flows through a pipe whose surface is the surface of revolution of the curve $y = a + \frac{kx^2}{a}$ about the x-axis $(-a \le x \le a)$. If the liquid enters at the end $x = -a$ of the pipe with velocity V, show that the time taken by a liquid particle to traverse the entire length of the pipe from $x = -a$ to $x = a$ is $\{2a/V(1+k^2)\}\{1 + \frac{2k}{3} + \frac{k^2}{5}\}$. Assume that k is so small that flow remains appreciably one dimensional throughout

Sol: $y - a = \frac{kx^2}{a}$ or $(x-0)^2 = \frac{a}{k}(y-a)$ $\qquad$ $v = \frac{dx}{dt}$



let $P(x,y)$ be any point on the curve

$PQ = y \quad \Rightarrow \quad y = a + \frac{k}{a}x^2$

Cross-sectional area at ①

$$S_1 = \pi CD^2 \qquad CD = \frac{K(-a)^2}{a} + a$$

$$S_1 = \pi a^2(1+K)^2 \qquad = a + Ka = a(1+K)$$

Cross-sectional area at ②

$$S_2 = \pi PQ^2$$

$$= \pi \left(a + \frac{Kx^2}{a}\right)^2$$

Velocity at section ① is V and at section ② can be given by $\frac{dx}{dt}$.

Since the motion is regarded as one-dimensional, by equation of continuity :

$$S_1V_1 = S_2 V_2$$

$$\Rightarrow \pi a^2(1+K)^2 V = \pi \left(a + \frac{Kx^2}{a}\right)^2 \frac{dx}{dt}$$

$$\Rightarrow dt = \left[(a^2 + Kx^2)^2 / a^4 (1+K)^2 V\right] \cdot dx$$

$$\Rightarrow dt = \frac{1}{V(1+K)^2} \left(1 + \frac{Kx^2}{a^2}\right)^2 dx$$

Let time of travelling from $x = -a$ to $x = +a$ be T

then

$$\int_0^T dt = \frac{1}{V(1+K)^2} \int_{-a}^{a} \left(1 + \frac{Kx^2}{a^2}\right)^2 dx$$

$$\Rightarrow T = \frac{1}{V(1+K)^2} \cdot 2 \int_0^a \left(1 + \frac{Kx^2}{a^2}\right)^2 dx$$

$$\therefore T = \frac{2}{V(1+K)^2} \int_0^a \left[1 + \frac{K^2x^4}{a^4} + \frac{2Kx^2}{a^2}\right] dx$$

$$T = \frac{2}{V(1+K)^2} \left[x + \frac{2Kx^3}{3a^2} + \frac{K^2x^5}{5a^4}\right]_0^a$$

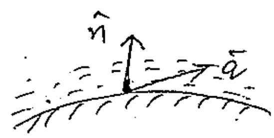
$$\Rightarrow T = \left(\frac{2a}{V}\right) \frac{1}{(1+K)^2} \left[1 + \frac{2K}{3} + \frac{K^2}{5}\right].$$

Boundary Surface conditions:

Let $F(r,t)=0$ or $F(x,y,z,t)=0$ may be a boundary surface. $\bar{q}$ - velocity of the fluid particle in contact with boundary surface.

$\bar{u}$ - velocity of the boundary surface.

The fluid and the surface with which contact is preserved must have the same velocity normal to the surface.



$$\bar{q}\cdot\hat{n} = \bar{u}\cdot\hat{n} \quad \text{or} \quad (\bar{q}-\bar{u})\cdot\hat{n} = 0$$

Direction ratios of $\hat{n}$ are $\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}$

$$\nabla F = \frac{\partial F}{\partial x}\hat{i} + \frac{\partial F}{\partial y}\hat{j} + \frac{\partial F}{\partial z}\hat{k}$$

$$\hat{n} = \frac{\nabla F}{|\nabla F|}$$

$$(\bar{q}-\bar{u})\cdot\hat{n} = 0$$

$$\Rightarrow (\bar{q}-\bar{u})\cdot\nabla F = 0$$

If $P(r,t)$ move to a point $Q(r+\delta r, t+\delta t)$ in time δt then Q must satisfy $F(r,t)=0$

$$\Rightarrow F(r+\delta r, t+\delta t) = 0$$

$$\Rightarrow F(r,t) + \delta\bar{r}\cdot\nabla F + \delta t\frac{\partial F}{\partial t} = 0 \quad \text{by Taylor's theorem.}$$

$$\Rightarrow \frac{\partial F}{\partial t} + \frac{\delta\bar{r}}{\delta t}\cdot\nabla F = 0$$

As $\delta\bar{r}\to 0$ $\delta t\to 0$ $\quad \underset{\delta t\to 0}{Lt}\frac{\delta\bar{r}}{\delta t} = \frac{d\bar{r}}{dt} = \bar{u}$

$$\Rightarrow \frac{\partial F}{\partial t} + \bar{u}\cdot\nabla F = 0 \Rightarrow \frac{\partial F}{\partial t} + \bar{q}\cdot\nabla F = 0$$

$$\Rightarrow \frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} = 0 \quad \text{or} \quad \frac{dF}{dt} = 0$$

Note:

1. When boundary surface is at rest $\frac{\partial F}{\partial t} = 0$

$$\Rightarrow u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} = 0$$

2. Normal velocity of the boundary $= \bar{u}\cdot\hat{n} = \bar{u}\cdot\frac{\nabla F}{|\nabla F|}$

$$= \frac{-\partial F/\partial t}{|\nabla F|}$$

12Q : Show that $\left(\frac{x^2}{a^2}\right)\tan^2 t + \left(\frac{y^2}{b^2}\right)\cot^2 t = 1$ is a possible form for the bounding surface of a liquid, and find an expression for the normal velocity.

Sol: For the present 2-D motion $\left(\frac{\partial F}{\partial z} = 0 \text{ and } \frac{\partial w}{\partial z} = 0\right)$, the surface

$$F(x,y,t) = \frac{x^2}{a^2}\tan^2 t + \frac{y^2}{b^2}\cot^2 t - 1 = 0$$

can be a possible boundary surface of a liquid, if it satisfies the boundary condition

$$\frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} = 0$$

and the same values of u and v satisfy the equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial F}{\partial t} = \frac{x^2}{a^2}\cdot 2\tan t\sec^2 t - \frac{y^2}{b}2\cot t\,\text{cosec}^2 t,$$

$$\frac{\partial F}{\partial x} = \frac{2x}{a^2}\tan^2 t, \quad \frac{\partial F}{\partial y} = \frac{2y}{b^2}\cot^2 t$$

$$\Rightarrow \frac{x\tan t}{a^2}(x\sec^2 t + u\tan t) + \frac{y\cot t}{b^2}(-y\,\text{cosec}^2 t + v\cot t) = 0,$$

which is identically satisfied if we take

$$x\sec^2 t + u\tan t = 0 \quad \text{and} \quad -y\,\text{cosec}^2 t + v\cot t = 0$$

$$u = -\frac{x}{\sin t\cos t} \quad \text{and} \quad v = \frac{y}{\sin t\cos t}$$

$$\frac{\partial u}{\partial x} = \frac{-1}{\sin t\cos t} \quad \text{and} \quad \frac{\partial v}{\partial y} = \frac{1}{\sin t\cos t}$$

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Normal velocity of the boundary

$$= \frac{u(\partial F/\partial x) + v(\partial F/\partial y)}{\sqrt{(\partial F/\partial x)^2 + (\partial F/\partial y)^2}} = \frac{\frac{1}{\sin t\cos t}\left[\frac{-2x^2\tan^2 t}{a^2} + \frac{2y^2\cot^2 t}{b^2}\right]}{\left[\left(\frac{2x\tan^2 t}{a^2}\right)^2 + \left(\frac{2y\cot^2 t}{b^2}\right)^2\right]^{1/2}}$$

$$= \frac{a^2y^2\cot t\,\text{cosec}^2 t - b^2x^2\tan t\sec^2 t}{\sqrt{(x^2b^4\tan^4 t + y^2a^4\cot^4 t)}}$$

Streamline or line of flow:

A streamline is a curve drawn in the fluid so that its tangent at each point is the direction of motion (i.e fluid velocity) at that point.

Let $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector of a point P on a stream line and $\bar{q} = u\hat{i} + v\hat{j} + w\hat{k}$ be the fluid velocity at P. Then $\bar{q}$ is parallel to $d\bar{r}$ at P on the streamline. Thus, the equation of streamlines is given by

$$\bar{q} \times d\bar{r} = 0$$

$$(u\hat{i} + v\hat{j} + w\hat{k}) \times (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}) = 0$$

$$\Rightarrow \quad v\,dz - w\,dy = 0 \; ; \quad w\,dx - u\,dz = 0 \; ; \quad u\,dy - v\,dx = 0$$

$$\Rightarrow \quad \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \quad \text{— (1)}$$

The equation (1) have a double infinite set of solutions. Through each point of the flow field where $u(x,y,z,t)$, $v(x,y,z,t)$ and $w(x,y,z,t)$ donot all vanish, there passes one and only one streamline at a given instant. (This fact can be verified by employing existence theorem for system of equations (1))

Note: If the velocity vanishes at a given point, various singularities occur at that point. Such a point is called as 'Critical point' or 'stagnation point'.

Pathline or path of a particle:

A path line is a curve or trajectory along which a particular fluid particle travels during its motion.

The differential equations of path lines are

$$\frac{d\bar{r}}{dt} = \bar{q} \quad \text{so that} \quad \frac{dx}{dt} = u, \; \frac{dy}{dt} = v \; \text{and} \; \frac{dz}{dt} = w$$

where $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\bar{q} = u\hat{i} + v\hat{j} + w\hat{k}$

Note: Let a fluid particle of fixed identity be at (x_0, y_0, z_0) when $t = t_0$, then path line is determined from equations

$\frac{dx}{dt} = u(x,y,z,t)$; $dy/dt = v(x,y,z,t)$; $dz/dt = w(x,y,z,t)$

Streak lines or filament lines:

A streak line is a line on which lie all those fluid particles that at some earlier instant passed through a certain point of space.

Thus a streak line presents the instantaneous picture of the position of all fluid particles, which have passed through a given point at some previous time

When a die is injected into a moving fluid at some fixed point, the coloured lines produced in the fluid are streak lines which have passed through the injected point.

The equation of streakline at time t can be derived by Lagrangian method.

Let a fluid particle (x_0, y_0, z_0) passes a fixed point (x_1, y_1, z_1)

$$x_1 = f_1(x_0, y_0, z_0, t), \quad y_1 = f_2(x_0, y_0, z_0, t), \quad z_1 = f_3(x_0, y_0, z_0, t)$$

Solving

$$x_0 = g_1(x_1, y_1, z_1, t), \quad y_0 = g_2(x_1, y_1, z_1, t), \quad z_0 = g_3(x_1, y_1, z_1, t)$$

Now, Streak line is the locus of the positions (x, y, z) of the particles which have passed through the fixed point (x_1, y_1, z_1). Hence the equation of the streak line at time t is given by

$$x = h_1(x_0, y_0, z_0, t), \quad y = h_2(x_0, y_0, z_0, t), \quad z = h_3(x_0, y_0, z_0, t).$$

Substituting x_0, y_0, z_0 the desired equation of streak line passing through (x, y, z) is given by

$$x = h_1(g_1, g_2, g_3, t), \quad y = h_2(g_1, g_2, g_3, t)$$

$$\text{and} \quad z = h_3(g_1, g_2, g_3, t)$$

Difference b/w the streamlines and path lines:

Streamlines are not, in general, the same as the path lines.

- Streamlines show how each particle is moving at a given instant of time while the path lines present the motion of the particle at each instant.
- Except in the case of steady motion, u, v, w are always functions of the time and hence the streamlines go on changing with the time and the actual path of any fluid particle will not in general coincide with a streamline.
- In the case of steady motion the streamlines remain unchanged as time progresses and hence they are also the path lines

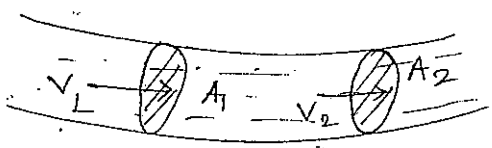
Stream tube (or) tube of flow:

A tube obtained by drawing streamlines from each point of a closed curve in a fluid is called the stream tube.

A stream tube of infinitesimal cross-section is called as stream filament.

Note:

Since there will be no movement of fluid across a streamline, no fluid can enter or leave the stream tube except at the ends.



For a steady motion of incompressible fluid

$$A_1 V_1 = A_2 V_2$$

Theorem:

The product of the speed and cross-sectional area is constant along a stream filament of a liquid in steady motion.

13 Q : The velocity components in a three-dimensional flow field for an incompressible fluid are $(2x, -y, -z)$. Is it a possible field? Determine the equations of the streamline passing through the point $(1,1,1)$. Sketch the streamlines.

Sol: Here, $u = 2x$, $v = -y$, $w = -z$

streamlines are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

i.e $$\frac{dx}{2x} = \frac{dy}{-y} = \frac{dz}{-z}$$

Considering $\frac{dx}{2x} = \frac{dy}{-y}$ and $\frac{dx}{2x} = \frac{dz}{-z}$

$$\Rightarrow \log xy^2 = \log c_1 \quad \text{and} \quad \log xz^2 = \log c_2$$

$$\Rightarrow xy^2 = c_1 \quad \text{and} \quad xz^2 = c_2$$

Here c_1 and c_2 are arbitrary constants. The streamlines are given by the curves of intersection of $xy^2 = c_1$ and $xz^2 = c_2$.

The required streamline passes through $(1,1,1)$

$\Rightarrow c_1 = 1$ and $c_2 = 1$.

$\therefore$ The desired streamline is given by the intersection of $xy^2 = 1$ and $xz^2 = 1$.

we also have $\frac{\partial u}{\partial x} = 2$, $\frac{\partial v}{\partial y} = -1$, $\frac{\partial w}{\partial z} = -1$

$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$ showing that the equation of continuity is satisfied for the given flow field for an incompressible fluid. Hence the given velocity components correspond to a possible field.

14 Q : Find the streamlines and paths of the particles when. $u = x/(1+t)$, $v = y/(1+t)$; $w = z/(1+t)$ (2000)

Sol: Streamlines are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}.$$

$$\Rightarrow \quad \frac{dx}{x/(1+t)} = \frac{dy}{y/(1+t)} = \frac{dz}{z/(1+t)}$$

$$\Rightarrow \quad \frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

Taking first two and last two we get

$$x/y = c_1 \quad \text{and} \quad y/z = c_2$$

The desired streamlines are given by the intersection of $x/y = c_1$ and $y/z = c_2$.

The paths of the particle are given by

$$\frac{dx}{dt} = u, \qquad \frac{dy}{dt} = v, \qquad \frac{dz}{dt} = w$$

$$\Rightarrow \frac{dx}{dt} = x/(1+t), \quad \frac{dy}{dt} = y/(1+t), \quad \frac{dz}{dt} = z/(1+t)$$

$$\Rightarrow \quad \frac{dx}{x} = \frac{dt}{1+t}, \quad \frac{dy}{y} = \frac{dt}{1+t}, \quad \frac{dz}{z} = \frac{dt}{1+t}$$

Integrating,

$$x = c_1'(1+t), \quad y = c_2'(1+t), \quad z = c_3'(1+t)$$

c_1', c_2', c_3' - arbitrary constants.

which give the desired paths of the particle.

15 Q: For an incompressible homogeneous fluid at the point (x,y,z) the velocity distribution is given by $u = -(c^2 y/r^2)$, $v = c^2 x/r^2$, $w = 0$, where r denotes the distance from the z-axis. Show that it is a possible motion and determine the surface which is orthogonal to streamlines. (2003)

Sol: Since r is the distance of the point (x,y,z) from the z-axis, we have $r = (x^2+y^2)^{1/2}$. Hence given velocity distribution becomes

$$u = -c^2 y/(x^2+y^2), \quad v = c^2 x/(x^2+y^2), \quad w = 0$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = \frac{2c^2 yx}{(x^2+y^2)^2}, \quad \frac{\partial v}{\partial y} = \frac{-2c^2 xy}{(x^2+y^2)^2}, \quad \frac{\partial w}{\partial z} = 0$$

$\Rightarrow \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0$, showing that the equation of continuity is satisfied and so the motion specified is possible.

The surfaces which are orthogonal to streamlines are given by

$$u\,dx + v\,dy + w\,dz = 0$$

or $$\frac{-c^2 y}{(x^2+y^2)}\,dx + \frac{c^2 x}{(x^2+y^2)}\,dy = 0$$

$$\Rightarrow \quad -y\,dx + x\,dy = 0$$

or $$\frac{1}{y}\,dy = \frac{1}{x}\,dx$$

$$\Rightarrow \quad \log y = \log k + \log x \quad \text{or} \quad y = kx, \ K\text{ - arbitrary constant.}$$

16Q : Determine the streamlines and the path lines of the particle when the components of the velocity field are given by $u = \dfrac{x}{1+t}$, $v = \dfrac{y}{2+t}$, $w = \dfrac{z}{3+t}$. Also state the condition for which the streamlines are identical with path lines. (2000)

Sol : Streamlines : $$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

$$(1+t)\frac{1}{x}\,dx = (2+t)\frac{1}{y}\,dy = (3+t)\frac{1}{z}\,dz$$

$$\Rightarrow \quad x^{(1+t)} = y^{(2+t)}.c_1 \quad \text{and} \quad y^{2+t} = c_2 z^{3+t}$$

$$\Rightarrow \quad \frac{x}{y^2} = c_1 (y/x)^t \quad \text{and} \quad \frac{y^2}{z^3} = c_2 (z/y)^t . \quad -(1)$$

The desired streamlines at a given instant $t = t_0$ are given by the intersection of the surfaces (1) by substituting t_0 for t.

Again, the path lines are given by

$$\frac{dx}{dt} = u, \quad \frac{dy}{dt} = v, \quad \frac{dz}{dt} = w$$

$$\Rightarrow \quad \frac{dx}{x} = \frac{dt}{1+t}, \quad \frac{dy}{y} = \frac{dt}{2+t}, \quad \frac{dz}{z} = \frac{dt}{3+t}$$

$$\Rightarrow x = c_3(1+t), \quad y = c_4(2+t), \quad z = c_5(3+t)$$

Desired path lines in terms of the parameter t.

- Condition : In case of steady motion streamlines remain unchanged as time progresses and hence they are identical with the path lines.

The velocity potential or velocity function:

Let fluid velocity at time t is $\bar{q} = (u, v, w)$ and there exists a scalar function $\phi(x, y, z, t)$ uniform throughout the entire field of flow and such that

$$-d\phi = u\,dx + v\,dy + w\,dz$$

i.e. $$-\left(\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz\right) = u\,dx + v\,dy + w\,dz$$

Then $u\,dx + v\,dy + w\,dz$ is an exact differential and we have

$$u = -\frac{\partial \phi}{\partial x}, \quad v = -\frac{\partial \phi}{\partial y}, \quad w = -\frac{\partial \phi}{\partial z}$$

$$\therefore \bar{q} = -\nabla \phi = -\text{grad}\,\phi \quad — ①$$

ϕ is called the velocity potential. The negative sign ensures that the flow takes place from the higher to lower potentials.

The necessary and sufficient condition for ① to hold is $\nabla \times \bar{q} = 0$ or $\text{curl}\,\bar{q} = 0$.

$$\Rightarrow \hat{i}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + \hat{j}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \hat{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

Note:

1. The surfaces $\phi(x, y, z, t) = \text{constant}$ are called equipotentials. The streamlines $\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$ are cut at right angles by the surfaces given by the differential equation

$$u\,dx + v\,dy + w\,dz = 0$$

and such orthogonal surfaces exists if

$$\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z}; \quad \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x} \quad \text{and} \quad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}.$$

At all points of field of flow the equipotentials are cut orthogonally by the streamlines.

2. If $\nabla \times \bar{q} = 0$ or $\text{curl}\,\bar{q} = 0$, then the flow is known as the 'potential kind'. It is also known as 'irrotational'. For such flow the field $\bar{q}$ is 'conservative'.

3. The equation of continuity of an incompressible fluid is $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

Suppose that the fluid move irrotationally. Then the velocity potential ϕ exists such that

$$u = -\frac{\partial \phi}{\partial x}, \quad v = -\frac{\partial \phi}{\partial y}, \quad w = -\frac{\partial \phi}{\partial z}$$

$$\Rightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0$$

shows that ϕ is a harmonic function satisfying the laplace equation $\nabla^2 \phi = 0$.

The Vorticity Vector:

Let $\bar{q} = u\hat{i} + v\hat{j} + w\hat{k}$ be the fluid velocity such that curl $\bar{q} \neq 0$. Then vorticity vector, $\Omega = \text{curl}\, \bar{q}$.

Let $\Omega = \Omega_x \hat{i} + \Omega_y \hat{j} + \Omega_z \hat{k}$

Then,

$$\Omega_x \hat{i} + \Omega_y \hat{j} + \Omega_z \hat{k} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}.$$

Some books/authors define

$$\Omega = \xi \hat{i} + \eta \hat{j} + \zeta \hat{k} = \frac{1}{2} \text{curl}\, \bar{q}$$

$$\Rightarrow \xi = \frac{1}{2}\Omega_x, \quad \eta = \frac{1}{2}\Omega_y, \quad \zeta = \frac{1}{2}\Omega_z$$

and $\Omega_x = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)$, $\Omega_y = \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)$, $\Omega_z = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$.

Note:

1. In two dimensional cartesian coordinates

Vorticity is given by $\Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$.

2. In two dimensional polar coordinates

$$\Omega_z = \frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}.$$

3. In 3-D cylindrical polar co-ordinates (r, θ, z)

$$\Omega_r = \frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}, \quad \Omega_\theta = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}, \quad \Omega_z = \frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}$$

4. In 3-D spherical polar coordinates (r, θ, ϕ)

$$\Omega_r = \frac{1}{r}\frac{\partial v_\phi}{\partial \theta} - \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_\phi}{r}\cot\theta$$

$$\Omega_\theta = \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{r}, \quad \Omega_\phi = \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}.$$

Vortex line:

A vortex line is a curve drawn in the fluid such that the tangent to it at every point is in the direction of the vorticity vector Ω.

$\Omega = \Omega_x \hat{i} + \Omega_y \hat{j} + \Omega_z \hat{k}$; $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$ be the position vector of a point P on a vortex line.

Then Ω is parallel to $d\bar{r}$ at P on the vortex line. Hence equation of vortex line is given by

$$\Omega \times d\bar{r} = 0$$

$$\Rightarrow (\Omega_x \hat{i} + \Omega_y \hat{j} + \Omega_z \hat{k}) \times (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}) = 0$$

$$\Rightarrow \frac{dx}{\Omega_x} = \frac{dy}{\Omega_y} = \frac{dz}{\Omega_z}$$ gives the desired equations of vortex lines.

Vortex tube:

A tube obtained by drawing the vortex lines from each point of a closed curve in the fluid is called the vortex tube.

A vortex tube of infinitesimal cross-section is known as vortex filament or simply a vortex.

Rotational and irrotational motion:

Irrotational	Rotational
• Ω, vorticity vector of every fluid particle is zero.	• Vorticity vector Ω is different from zero.
• Curl $\bar{q} = 0$ $\frac{\partial w}{\partial y} = \frac{\partial v}{\partial z}$, $\frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}$, $\frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$.	• $\Omega = \text{curl } \bar{q}$.
• No vortex lines.	• Also said to be vortex motion.

Note:

When curl $\bar{q} = 0$ then $\bar{q}$ must be of the form $(-\text{grad}\,\phi)$ for some scalar point function ϕ (say) because curl grad $\phi = 0$. Thus velocity potential exists whenever the fluid motion is irrotational. Again, when velocity potential exists the motion is irrotational because $\bar{q} = -\text{grad}\,\phi \Rightarrow \text{curl}\,\bar{q} = -\text{curl grad}\,\phi = 0$.

$\therefore$ The fluid motion is irrotational if and only if the velocity potential exists.

The angular velocity vector:

$$\bar{\omega} = \frac{1}{2}(\text{curl } \bar{q}) \quad , \quad 2\bar{\omega} = \text{Curl } \bar{q}$$

$$\Rightarrow \Omega = 2\bar{\omega}$$

$$\bar{\omega} = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{i} + \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{j} + \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}$$

$\bar{\omega}$ is also called rotation.

17 Q : Test whether the motion specified by $\bar{q} = \frac{k^2(x\hat{j} - y\hat{i})}{x^2+y^2}$, ($k$ = constant), is a possible motion for an incompressible fluid. If so, determine the equation of the streamlines. Also test whether motion is of the potential kind and if so determine the velocity potential.

Sol : Let $\bar{q} = u\hat{i} + v\hat{j} + w\hat{k}$

Then $u = -\frac{k^2 y}{x^2+y^2}$, $v = \frac{k^2 x}{x^2+y^2}$, $w = 0$ — ①

Equation of continuity for an incompressible fluid is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad \text{— ②}$$

From ① $\frac{\partial u}{\partial x} = \frac{2k^2 xy}{x^2+y^2}$, $\frac{\partial v}{\partial y} = \frac{-2k^2 xy}{x^2+y^2}$, $\frac{\partial w}{\partial z} = 0$

Since ② is satisfied and so the motion specified by given $\bar{q}$ is possible.

Equation of the streamlines:

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

$$\Rightarrow \frac{dx}{-k^2 y/x^2+y^2} = \frac{dy}{k^2 x/x^2+y^2} = \frac{dz}{0}$$

Considering, $dz = 0 \Rightarrow z = c_1$ — ③

considering first two fractions

$$\frac{dx}{(-y)} = \frac{dy}{x}$$

$$\Rightarrow x\,dx + y\,dy = 0$$

$$2x\,dx + 2y\,dy = 0$$

$$\Rightarrow x^2 + y^2 = c_2 \qquad \text{— ④}$$

③ and ④ together give the streamlines. The streamlines are circles whose centres are on the z-axis, and their planes $\perp^{ar}$ to z-axis.

Again, curl $\bar{q}$ $= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{-k^2 y}{x^2+y^2} & \frac{k^2 x}{x^2+y^2} & 0 \end{vmatrix} = k^2\left[\frac{y^2-x^2+x^2-y^2}{(x^2+y^2)^2}\right]\hat{k} = 0$

Hence the flow is of the potential kind and we can find velocity potential $\phi(x,y,z)$ such that $\bar{q} = -\nabla\phi$. Thus, we have

$$\frac{\partial\phi}{\partial x} = -u = \frac{k^2 y}{x^2+y^2}$$

$$\frac{\partial\phi}{\partial y} = -v = -\frac{k^2 x}{x^2+y^2}$$

$$\frac{\partial\phi}{\partial z} = -w = 0$$

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = -(u\,dx + v\,dy + w\,dz).$$

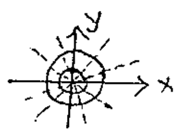
$$\Rightarrow \quad d\phi = k^2\left(\frac{y\,dx - x\,dy}{x^2+y^2}\right)$$

$$d\phi = k^2 \; d\left[\tan^{-1} x/y\right]$$

Integrating $\quad \phi = k^2 \tan^{-1} x/y + \text{Constant}$

Since the constant can be omitted while writing velocity potential, the required velocity potential is $\phi = k^2 \tan^{-1}\frac{x}{y}$.

The equipotentials are given by



$$\phi = c \Rightarrow \tan^{-1}\frac{x}{y} = \text{constant}$$

$$x = c'y$$

which are planes through z-axis.

18Q: (a) Show that $u = \frac{-2xyz}{(x^2+y^2)^2}$, $v = \frac{(x^2-y^2)z}{(x^2+y^2)^2}$, $w = \frac{y}{x^2+y^2}$

are the velocity components of a possible liquid motion. Is this motion irrotational. (2000)(2002)

(b) Show that a fluid of constant density can have a velocity $\bar{q}$ given by $\bar{q} = \left[\frac{-2xyz}{(x^2+y^2)^2}, \frac{(x^2-y^2)z}{(x^2+y^2)^2}, \frac{y}{(x^2+y^2)}\right]$

Find the vorticity vector.

Sol : (a) $\dfrac{\partial u}{\partial x} = -2yz \dfrac{y^2-3x^2}{(x^2+y^2)^3}$

$\dfrac{\partial v}{\partial y} = -2yz \dfrac{3x^2-y^2}{(x^2+y^2)^3}$; $\dfrac{\partial w}{\partial z} = 0$

Hence equation of continuity $\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} + \dfrac{\partial w}{\partial z} = 0$ is satisfied and so the fluid motion is possible.

$$\Omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = \frac{x^2-y^2}{(x^2+y^2)^2} - \frac{x^2-y^2}{(x^2+y^2)^2} = 0$$

$$\Omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = \frac{-2xy}{(x^2+y^2)^2} + \frac{2xy}{(x^2+y^2)^2} = 0$$

and $$\Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{2xz(3y^2-x^2)}{(x^2+y^2)^3} - \frac{2xz(3y^2-x^2)}{(x^2+y^2)^3} = 0$$

$$\therefore \frac{\partial w}{\partial y} = \frac{\partial v}{\partial z}, \quad \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x}, \quad \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$$

and hence the motion is irrotational.

(b) Let $q = (u, v, w)$

$$\Omega = \Omega_x \hat{i} + \Omega_y \hat{j} + \Omega_z \hat{k} = 0.$$

19/Q : If velocity distribution of an incompressible fluid at point (x, y, z) is given by $\left(\dfrac{3xz}{r^5}, \dfrac{3yz}{r^5}, \dfrac{kz^2-r^2}{r^5}\right)$, determine the parameter k such that it is a possible motion. Hence find its velocity potential. (2001)

Sol: $u = \dfrac{3xz}{r^5}$, $v = \dfrac{3yz}{r^5}$, $w = \dfrac{kz^2-r^2}{r^5}$.

where $r^2 = x^2+y^2+z^2$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\frac{\partial u}{\partial x} = \frac{3z}{r^5} - \frac{3x^2z}{r^7}\times 5 = \frac{3z}{r^5} - \frac{15x^2z}{r^7}$$

$$\frac{\partial v}{\partial y} = \frac{3z}{r^5} - \frac{15y^2z}{r^7}; \quad \frac{\partial w}{\partial z} = \frac{2kz}{r^5} - \frac{5kz^2}{r^6}\cdot\frac{z}{r} + \frac{3z}{r^5}.$$

$$= \frac{(2k+3)z}{r^5} - \frac{15z^3}{r^7}.$$

For possible liquid motion, the equation of continuity must be satisfied.

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

$$\Rightarrow \frac{(2K+9)z}{r^5} - \frac{15z}{r^7}(x^2+y^2+z^2) = 0$$

$$\Rightarrow \frac{(2K+9)z}{r^5} - \frac{15z}{r^5} = 0$$

$$\Rightarrow \frac{(2K-6)z}{r^5} = 0 \Rightarrow K = 3.$$

$$\therefore (u,v,w) = \left(\frac{3xz}{r^5}, \frac{3yz}{r^5}, \frac{3z^2-r^2}{r^5}\right)$$

$$d\phi = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = -(u\,dx + v\,dy + w\,dz),$$

by definition of ϕ.

$$\Rightarrow d\phi = -\left[\frac{3xz}{r^5}dx + \frac{3yz}{r^5}dy + \frac{3z^2-r^2}{r^5}dz\right]$$

$$d\phi = \frac{r^2dz - 3z(x\,dx + y\,dy + z\,dz)}{r^5}$$

$$d\phi = \frac{r^3dz - 3rz \cdot r\,dr}{r^6}$$

$$d\phi = \frac{r^3dz - 3r^2z\,dr}{(r^3)^2} = d\left(\frac{z}{r^3}\right)$$

Integrating, $\phi = z/r^3$ (omitting constant of integration, for it has no significance in ϕ).

$$\phi = \frac{z}{r^3}$$ $z = r\cos\theta$ in polar coordinates (spherical)

$$\Rightarrow \phi = \frac{\cos\theta}{r^2} \text{ or } \frac{z}{r^3}.$$

Note: For streamlines:

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \Rightarrow \frac{dx}{3xz} = \frac{dy}{3yz} = \frac{dz}{3z^2-r^2}$$

$$\frac{dx}{3xz} = \frac{dy}{3yz} \qquad = \frac{x\cdot dx + y\cdot dy + z\cdot dz}{3x^2z + 3y^2z + 3z^3 - r^2z}$$

$$\Rightarrow \frac{dx}{x} = \frac{dy}{y} \qquad = \frac{x\,dx + y\,dy + z\,dz}{3z(x^2+y^2+z^2) - r^2z}$$

$$\Rightarrow x/y = c_1 \quad -① \qquad = \frac{x\,dx + y\,dy + z\,dz}{2z\cdot(x^2+y^2+z^2)}$$

Intersection of ① and ② gives the streamlines.

$$\Rightarrow \frac{2}{3}\frac{dx}{x} = \frac{1}{2}\frac{d(x^2+y^2+z^2)}{x^2+y^2+z^2} \Rightarrow x^{2/3} = c_2(x^2+y^2+z^2)^{1/2} \quad -②$$

20 Q : If the velocity potential of a fluid is $\phi = \frac{z}{r^3}\tan^{-1} y/x$, where $r^2 = x^2+y^2+z^2$, then show that the lines of flow (streamlines) lie on the surfaces $x^2+y^2+z^2 = c\,(x^2+y^2)^{2/3}$, c being an arbitrary constant. (2008)

Sol:

$$\phi(x,y,z) = \frac{z}{r^3}\tan^{-1} y/x$$

$$r^2 = x^2+y^2+z^2$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$u = -\frac{\partial \phi}{\partial x} = \frac{z}{r^3}\cdot\frac{1}{(1+y^2/x^2)}\cdot\frac{y}{x^2} + \tan^{-1}\frac{y}{x}\cdot\frac{3z}{r^4}\cdot\frac{x}{r}$$

$$u = \frac{yz}{(x^2+y^2)r^3} + \frac{3zx\tan^{-1} y/x}{r^5}$$

$$v = -\frac{\partial \phi}{\partial y} = \frac{-xz}{(x^2+y^2)r^3} + \frac{3zy\tan^{-1} y/x}{r^5}$$

$$w = -\frac{\partial \phi}{\partial z} = \frac{3z^2\tan^{-1} y/x}{r^5} - \frac{1}{r^3}\tan^{-1} y/x .$$

Equation of flow lines:

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} = k$$

$$\Rightarrow k = \frac{x\,dx + y\,dy + z\,dz}{xu + yv + zw} = \frac{x\,dx + y\,dy}{xu + yv}$$

$$\Rightarrow \frac{x\,dx + y\,dy + z\,dz}{[(x^2+y^2+z^2)3z\cdot r^{-5} - zr^{-3}]\tan^{-1} y/x} = \frac{x\,dx + y\,dy}{(x^2+y^2)\,3z\cdot r^{-5}\cdot\tan^{-1} y/x}$$

$$\Rightarrow \frac{x\,dx + y\,dy + z\,dz}{2r^{-3}} = \frac{x\,dx + y\,dy}{3(x^2+y^2)\,r^{-5}}$$

$$\Rightarrow \frac{2(x\,dx + y\,dy + z\,dz)}{2r^2} = \frac{2(x\,dx + y\,dy)}{3(x^2+y^2)}$$

$$\log(x^2+y^2+z^2) = \frac{2}{3}\log(x^2+y^2) + \log c$$

$$\Rightarrow x^2+y^2+z^2 = c\,(x^2+y^2)^{2/3}.$$

21Q: Show that ~~$\phi = x f(x)$~~ the velocity potential given by $\phi = \frac{a}{2} \times (x^2+y^2-2z^2)$ satisfies the Laplace equation. Also determine the streamlines.

Sol:
$$\frac{\partial \phi}{\partial x} = ax, \quad \frac{\partial \phi}{\partial y} = ay, \quad \frac{\partial \phi}{\partial z} = -2az$$

$$\frac{\partial^2 \phi}{\partial x^2} = a, \quad \frac{\partial^2 \phi}{\partial y^2} = a, \quad \frac{\partial^2 \phi}{\partial z^2} = -2a$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0 \Rightarrow \nabla^2 \phi = 0$$

$\therefore$ ϕ satisfies the Laplace equation.

$$\bar{q} = -\nabla \phi = -(ax\,\hat{i} + ay\,\hat{j} - 2az\,\hat{k})$$

$$\Rightarrow u = -ax, \quad v = -ay, \quad w = -2az$$

streamlines:
$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

$$\Rightarrow \frac{dx}{x} = \frac{dy}{y} = \frac{dz}{-2z}$$

$$\Rightarrow \frac{dx}{x} = \frac{dy}{y} \quad \text{and} \quad \frac{dx}{x} = \frac{dz}{-2z}$$

$$x = c_1 y \quad \text{and} \quad x^2 z = c_2$$

$x = c_1 y$ & $x^2 z = c_2$ together give the equations of streamlines.

22Q: Show that all necessary conditions can be satisfied by a velocity potential of the form $\phi = \alpha x^2 + \beta y^2 + \gamma z^2$, and a bounding surface of the form $F = ax^4 + by^4 + cz^4 - \chi(t) = 0$, where $\chi(t)$ is a given function of the time and $\alpha, \beta, \gamma, a, b, c$ are suitable functions of the time.

Sol: Given expressions
$$\phi(x,y,z) = \alpha x^2 + \beta y^2 + \gamma z^2$$

and
$$F(x,y,z,t) = ax^4 + by^4 + cz^4 - \chi(t)$$

the following conditions must be satisfied:

(i) ϕ satisfies the Laplace equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0$$

(ii) F satisfies the condition for boundary surface

$$\frac{\partial F}{\partial t}+u\frac{\partial F}{\partial x}+v\frac{\partial F}{\partial y}+w\frac{\partial F}{\partial z}=0$$

$$\frac{\partial^2\phi}{\partial x^2}=2\alpha,\quad \frac{\partial^2\phi}{\partial y^2}=2\beta,\quad \frac{\partial^2\phi}{\partial z^2}=2\gamma$$

$$\Rightarrow \nabla^2\phi = 2\alpha+2\beta+2\gamma=0$$

$\Rightarrow \alpha+\beta+\gamma=0$ for which α,β,γ must be some suitable functions of time.

$$\frac{\partial F}{\partial t}=x^4\dot{a}+y^4\dot{b}+z^4\dot{c}-\dot{\chi}$$

$$\frac{\partial F}{\partial x}=4ax^3,\quad \frac{\partial F}{\partial y}=4by^3,\quad \frac{\partial F}{\partial z}=4cz^3$$

$$u=-\frac{\partial\phi}{\partial x}=-2\alpha x,\quad v=-2\beta y,\quad w=-2\gamma z$$

$$\therefore\ x^4(\dot{a}-8a\alpha)+y^4(\dot{b}-8b\beta)+z^4(\dot{c}-8c\gamma)-\dot{\chi}=0$$

This equation should be satisfied by all the points on $ax^4+by^4+cz^4-\chi(t)=0$

$$\Rightarrow \frac{\dot{a}-8a\alpha}{a}=\frac{\dot{b}-8b\beta}{b}=\frac{\dot{c}-8c\gamma}{c}=\frac{\dot{\chi}}{\chi}$$

Considering $\dfrac{\dot{a}-8a\alpha}{a}=\dfrac{\dot{\chi}}{\chi}$

$$\Rightarrow \frac{\dot{a}}{a}=8\alpha+\frac{\dot{\chi}}{\chi}\Rightarrow \log a=8\int\alpha\,dt+\log\chi$$

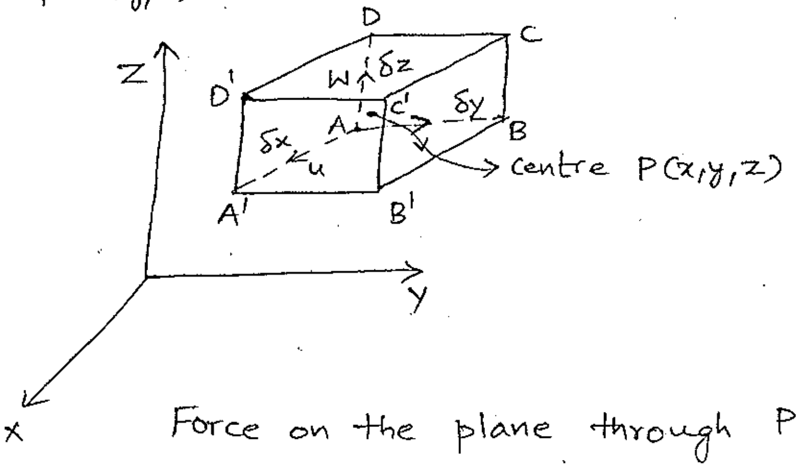
||^{ly} $\log b=8\int\beta\,dt+\log\chi,\quad \log c=8\int\gamma\,dt+\log\chi$

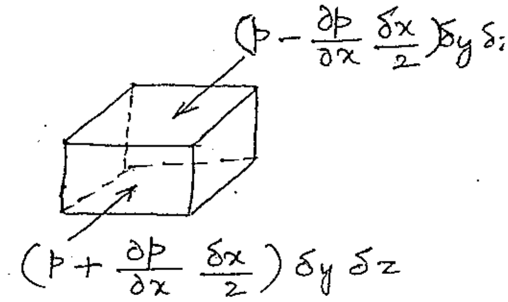
$\Rightarrow$ a, b and c as functions of t

Thus, velocity potential ϕ and boundary surface $F=0$ satisfy the necessary conditions if a, b, c, α, β and γ are some suitable functions of time.

Euler's equation of motion for inviscid flow:

Let p be the pressure and ρ be density at a point $P(x, y, z)$ in an inviscid (perfect) fluid. Consider an elementary parallelopiped with edges of lengths $\delta x, \delta y, \delta z$ parallel to their respective co-ordinate axes having P at its centre. Let (u, v, w) be components of velocity and (F_x, F_y, F_z) be the components of external force per unit mass at time 't' at P. Then if pressure, $p = f(x, y, z)$ we have





Force on the plane through P parallel to ABCD $= p\,\delta y\,\delta z = f(x, y, z)\cdot\delta y\,\delta z$

$\therefore$ pressure on face ABCD $= f(x - \frac{\delta x}{2}, y, z)\cdot\delta y\,\delta z$

$= (f - \frac{1}{2}\delta x\frac{\partial f}{\partial x} + \cdots)\,\delta y\cdot\delta z$, by Taylor's theorem

and pressure on face A'B'C'D'

$= f(x + \frac{\delta x}{2}, y, z)\cdot\delta y\cdot\delta z$

$= (f + \frac{1}{2}\delta x\frac{\partial f}{\partial x} + \cdots)\,\delta y\,\delta z$

$\therefore$ Net force in x-direction due to forces on ABCD and A'B'C'D' $= -(\frac{\partial f}{\partial x})\,\delta x\,\delta y\,\delta z$, to first order of approximation

$$= -\frac{\partial p}{\partial x}\cdot\delta x\,\delta y\cdot\delta z$$

The mass of the element is $\rho\,\delta x\,\delta y\,\delta z$. Hence the external force on the element in x-direction is $F_x\cdot\rho\,\delta x\,\delta y\,\delta z$. Also, we know that $\frac{du}{dt}$ is the total acceleration of the element in x-direction.

Mass × acceleration = Sum of external forces (Newton's II Law)

Along x-axis:

$$(\rho\,\delta x\,\delta y\,\delta z)\frac{du}{dt} = F_x\,\rho\,\delta x\,\delta y\,\delta z - \frac{\partial p}{\partial x}\,\delta x\,\delta y\,\delta z$$

$$\Rightarrow \quad \frac{du}{dt} = F_x - \frac{1}{\rho}\frac{\partial p}{\partial x} \quad -①$$

Similarly in y and z directions

$$\frac{dv}{dt} = F_y - \frac{1}{\rho}\frac{\partial p}{\partial y} \quad -② \; ; \quad \frac{dw}{dt} = F_z - \frac{1}{\rho}\frac{\partial p}{\partial z} \quad -③$$

①, ②, ③ - Euler's dynamical equations.

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = F_x - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = F_y - \frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = F_z - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

Vector Form:

Let $\bar{q} = u\hat{i} + v\hat{j} + w\hat{k}$ and $\bar{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$

Since, $\frac{\partial p}{\partial x}\hat{i} + \frac{\partial p}{\partial y}\hat{j} + \frac{\partial p}{\partial z}\hat{k} = \nabla p$.

①, ②, ③ can be combined as

$$\frac{d\bar{q}}{dt} = \bar{F} - \frac{1}{\rho}\nabla p \quad - ④$$

This is called Euler's equation of motion.

and also $\quad \frac{d\bar{q}}{dt} = \frac{\partial \bar{q}}{\partial t} + (\bar{q}\cdot\nabla)\bar{q}$

$$\Rightarrow \frac{\partial \bar{q}}{\partial t} + (\bar{q}\cdot\nabla)\bar{q} = \bar{F} - \frac{1}{\rho}\nabla p \quad - ⑤$$

Since $\quad \nabla(\bar{q}\cdot\bar{q}) = 2\left[\bar{q}\times \text{curl}\,\bar{q} + (\bar{q}\cdot\nabla)\bar{q}\right]$

$$\Rightarrow \quad (\bar{q}\cdot\nabla)\bar{q} = \frac{1}{2}\nabla q^2 - \bar{q}\times \text{curl}\,\bar{q}$$

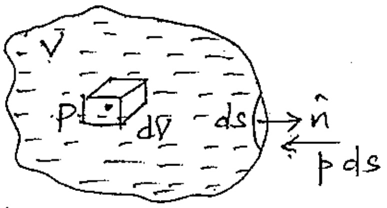
substituting in ⑤

$$\frac{\partial \bar{q}}{\partial t} + \nabla\left(\frac{1}{2}\bar{q}^2\right) - \bar{q}\times \text{curl}\,\bar{q} = \bar{F} - \frac{1}{\rho}\nabla p$$

$$\Rightarrow \frac{\partial \bar{q}}{\partial t} - \bar{q}\times \text{curl}\,\bar{q} = F - \frac{1}{\rho}\nabla p - \frac{1}{2}\nabla\bar{q}^2$$

The equation of motion of an inviscid fluid:

(vector/ Flux method)



Consider any arbitrary closed surface S drawn in the region occupied by the incompressible fluid and moving with it, so that it contains the same fluid particles at every instant.

By Newton's second law,

Total force acting on this mass of fluid

= The rate of change of linear momentum.

The fluid mass under consideration is subjected to the following ~~two~~ two forces

(i) The normal forces/ pressure thrusts on the boundary

(ii) The external force $\bar{F}$ per unit mass.

Let ρ be the density of the fluid particle P with in the closed surface and let dv be the volume enclosing P. The mass of element $\rho\, dv$ will always remain constant.

Let $\bar{q}$ be the velocity of the fluid particle P. Then the momentum M of the volume V is given by $M = \int_V \bar{q}\, \rho\, dv$

Time rate of change of linear momentum,

$$\frac{dM}{dt} = \frac{d}{dt}\int_V \bar{q}\,\rho\, dv = \int_V \frac{d\bar{q}}{dt}\rho\, dv + \int_V \bar{q}\,\underbrace{\frac{d}{dt}(\rho\, dv)}_{0}$$

$$\Rightarrow \frac{dM}{dt} = \int_V \frac{d\bar{q}}{dt}\rho\, dv.$$

$\frac{d}{dt}$ is the material derivative given by,

$$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + \bar{q}\cdot\nabla.$$

Total force on volume V due to external force

$$= \int_V \bar{F}\rho\, dv.$$

Total normal pressure thrust on the surface S

$$= \int_S p(-\hat{n})\, ds \quad (\because \text{ surface force acts inwards})$$

$$= -\int_V \nabla p\, dv \quad \text{by Gauss Theorem.}$$

$\therefore$ The total force acting on the volume V

$$= \int_V \bar{F}\rho\, dv - \int_V \nabla p\, dv = \int_V (\bar{F}\rho - \nabla p)\, dv$$

$$\Rightarrow \frac{dM}{dt} = \int_V \frac{d\bar{q}}{dt}\rho\, dv = \int_V (\bar{F}\rho - \nabla p)\, dv$$

$$\Rightarrow \int_V \left(\rho \frac{d\bar{q}}{dt} - \rho \bar{F} + \nabla p\right) dv = 0 \quad - (1)$$

Since the volume V enclosed by surface S is arbitrary, eqn. (1) holds if

$$\rho \frac{d\bar{q}}{dt} - \rho \bar{F} + \nabla p = 0$$

$$\Rightarrow \frac{d\bar{q}}{dt} = \bar{F} - \frac{1}{\rho}\nabla p$$, which is known as Euler's equation of motion.

Note: Euler's equation of motion in vector form

$$\frac{d\bar{q}}{dt} = \bar{F} - \frac{1}{\rho}\nabla p$$

$$\Rightarrow \frac{\partial \bar{q}}{\partial t} - \bar{q} \times \operatorname{curl} \bar{q} = \bar{F} - \frac{1}{\rho}\nabla p - \frac{1}{2}\nabla \bar{q}^2$$

$$\frac{\partial \bar{q}}{\partial t} + (\operatorname{curl} \bar{q}) \times \bar{q} = \bar{F} - \frac{1}{\rho}\nabla p - \frac{1}{2}\nabla \bar{q}^2$$

$$\Rightarrow \frac{\partial \bar{q}}{\partial t} + \Omega \times \bar{q} = \bar{F} - \frac{1}{\rho}\nabla p - \frac{1}{2}\nabla \bar{q}^2$$

$$\Rightarrow \frac{\partial \bar{q}}{\partial t} + \nabla\left(\frac{\bar{q}^2}{2}\right) + \Omega \times \bar{q} = \bar{F} - \frac{1}{\rho}\nabla p$$

Lamb's hydrodynamical equation.

Conservative field of force:

Workdone by the force F is independent of the path of motion.

If $F = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$, then a scalar point function $V(x,y,z)$ exists such that

$$F_x\, dx + F_y\, dy + F_z\, dz = -dV$$

$$\text{or} \quad \bar{F} = -\nabla V$$

so that, $F_x = -\frac{\partial V}{\partial x}$, $F_y = -\frac{\partial V}{\partial y}$, $F_z = -\frac{\partial V}{\partial z}$.

V is said to be 'force potential' and it measures the potential energy of the field.

Euler's equation of motion in cylindrical co-ordinates:

$$\frac{d\bar{q}}{dt} = \bar{F} - \frac{1}{\rho}\nabla p$$

Let (q_r, q_θ, q_z) be the velocity components and (F_r, F_θ, F_z) be the external force components in (r, θ, z) directions.

Then, $$\frac{d\bar{q}}{dt} = \left(\frac{dq_r}{dt} - \frac{q_\theta^2}{r}, \frac{dq_\theta}{dt} + \frac{q_r q_\theta}{r}, \frac{dq_z}{dt}\right)$$

$$F = (F_r, F_\theta, F_z), \quad \nabla p = \left(\frac{\partial p}{\partial r}, \frac{1}{r}\frac{\partial p}{\partial \theta}, \frac{\partial p}{\partial z}\right)$$

Substituting and equating the coefficients of $\hat{i}, \hat{j}, \hat{k}$

$$\frac{dq_r}{dt} - \frac{q_\theta^2}{r} = F_r - \frac{1}{\rho}\frac{\partial p}{\partial r}$$

$$\frac{dq_\theta}{dt} + \frac{q_r q_\theta}{r} = F_\theta - \frac{1}{\rho r}\frac{\partial p}{\partial \theta}$$

$$\frac{\partial q_z}{\partial t} = F_z - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

where $$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + q_r\frac{\partial}{\partial r} + \frac{q_\theta}{r}\frac{\partial}{\partial \theta} + q_z\frac{\partial}{\partial z}.$$

Euler's equation of motion in spherical co-ordinates:

$$\frac{d\bar{q}}{dt} = \bar{F} - \frac{1}{\rho}\nabla p$$

(q_r, q_θ, q_ϕ) ; (F_r, F_θ, F_ϕ) ; (r, θ, ϕ)

$$\frac{d\bar{q}}{dt} = \left[\frac{dq_r}{dt} - \frac{q_\theta^2 + q_\phi^2}{r}, \frac{dq_\theta}{dt} - \frac{q_\phi^2 \cot\theta}{r} + \frac{q_r q_\theta}{r}, \frac{dq_\phi}{dt} + \frac{q_\theta q_\phi \cot\theta}{r}\right];$$

$$F = (F_r, F_\theta, F_\phi) \ ; \ \nabla p = \left[\frac{\partial p}{\partial r}, \frac{1}{r}\frac{\partial p}{\partial \theta}, \frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi}\right]$$

$$\Rightarrow \frac{dq_r}{dt} - \frac{q_\theta^2 + q_\phi^2}{r} = F_r - \frac{1}{\rho}\frac{\partial p}{\partial r}$$

$$\frac{dq_\theta}{dt} - \frac{q_\phi^2 \cot\theta}{r} + \frac{q_r q_\theta}{r} = F_\theta - \frac{1}{\rho r}\frac{\partial p}{\partial \theta}$$

$$\frac{dq_\phi}{dt} + \frac{q_\theta q_\phi \cot\theta}{r} = F_z - \frac{1}{\rho r\sin\theta}\frac{\partial p}{\partial \phi}$$

where, $$\frac{d}{dt} \equiv \frac{\partial}{\partial t} + q_r\frac{\partial}{\partial r} + \frac{q_\theta}{r}\frac{\partial}{\partial \theta} + \frac{q_\phi}{r\sin\theta}\frac{\partial}{\partial \phi}$$

Note :

1. If the motion of an ideal fluid, for which density is a function of pressure p only, is steady and the external forces are conservative, then there exists a family of surfaces which contain the

streamlines and vortex lines

(or)

For steady motion of an inviscid isotropic fluid $p = f(\rho)$, $\int \frac{dp}{\rho} + \frac{1}{2}q^2 + \Omega$ = constant over a surface containing the streamlines and vortex lines.

Proof: $\frac{\partial \bar{q}}{\partial t} - \bar{q} \times \text{curl}\, \bar{q} = \bar{F} - \frac{1}{\rho}\nabla p - \frac{1}{2}\nabla \bar{q}^2$

Euler's equation in vector form

For ~~s~~ steady flow, $\frac{\partial \bar{q}}{\partial t} = 0$

Since external forces are conservative, $F = -\nabla V$

V - Force potential.

Given $p = f(\rho)$

Let $\nabla p' = \frac{1}{\rho}\nabla p$, p' is some function.

Then, $\nabla (V + p' + \bar{q}^2/2) = \bar{q} \times \text{curl}\, \bar{q} = \bar{q} \times \Omega$

Let $\bar{a} = \nabla (V + p' + \bar{q}^2/2)$

then $\bar{a} = \bar{q} \times \Omega$

$\bar{a} \cdot \bar{q} = (\bar{q} \times \Omega) \cdot \bar{q} = 0$

$\bar{a} \cdot \Omega = (\bar{q} \times \Omega) \cdot \Omega = 0$

$\Rightarrow$ $\bar{a}$ is $\perp^{ar}$ to both $\bar{q}$ and Ω.

Since $\nabla \phi$ is $\perp^{ar}$ everywhere to the surface ϕ = constant

shows that $\bar{a} = \nabla (V + p' + \frac{\bar{q}^2}{2})$ is $\perp^{ar}$ to the family of surfaces $V + p' + \bar{q}^2/2$ = constant

Thus $\bar{q}$ and Ω are both tangential to the surfaces $V + p' + \bar{q}^2/2 = c$. Hence the surface contains the streamlines and vortex lines.

2. If the given fluid is at constant temperature, then $p = K\rho$, K is a constant.

When the change is adiabatic

$p = K\rho^{\gamma}$

Homogeneous fluid

$\frac{1}{\rho}\nabla p = \nabla\left(\frac{p}{\rho}\right)$

3. Equations/relations for solving problems

a. Equation of motion

b. Equation of continuity

c. Relation between p and ρ

d. Initial and boundary conditions.

4. Bernoulli's equation / Pressure equation

$$-\frac{\partial \phi}{\partial t} + \frac{1}{2}q^2 + V + \int \frac{dp}{\rho} = F(t)$$ – arbitrary function of t.

ϕ - Velocity potential

For steady incompressible fluid

$$\frac{\bar{q}^2}{2} + V + \frac{p}{\rho} = \text{constant}$$

23 Q: A sphere of radius R, whose centre is at rest, vibrates radially in an infinite incompressible fluid of density ρ, which is at rest at infinity. If the pressure at infinity is Π, show that the pressure at the surface of the sphere at time t is

$$\Pi + \frac{1}{2}\rho\left\{\frac{d^2R^2}{dt^2} + \left(\frac{dR}{dt}\right)^2\right\}$$

Sol: Here the motion of the fluid will take place in such a manner so that each element of the fluid moves towards or away from the center. Hence the free surface is spherical.

Let P be the pressure on the surface of the sphere of radius R and V be the velocity there. Let p be the pressure at a distance r' in the fluid and v' be the velocity there.

Then the equation of continuity is

$$r'^2 v' = R^2 V$$

v' will be a function of r' and time t only

$$\therefore r'^2 v' = R^2 V = F(t)$$

$$\Rightarrow \frac{\partial v'}{\partial t} = \frac{F'(t)}{r'^2}$$

Equation of motion: $$\frac{\partial v'}{\partial t} + v'\frac{\partial v'}{\partial r'} = -\frac{1}{\rho}\frac{\partial p}{\partial r'}$$

$$\Rightarrow \frac{F'(t)}{r'^2} + \frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right) = -\frac{1}{\rho}\frac{\partial p}{\partial r'}$$

Integrating w.r.t r',

$$-\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = -\frac{p}{\rho} + c$$

when $r' = \infty$, then $v' = 0$ and $p = \Pi$

$$\Rightarrow \; c = \frac{\Pi}{\rho}$$

$$\therefore \; -\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = \frac{\Pi - p}{\rho}$$

$$\Rightarrow p = \Pi + \frac{1}{2}\rho\left[2\frac{F'(t)}{r'} - v'^2\right]$$

When $r' = R$, $v' = V$ and $p = P$

$$\Rightarrow \; P = \Pi + \frac{1}{2}\rho\left[\frac{2\{F'(t)\}_{r'=R}}{R} - V^2\right]$$

Also, $V = \frac{dR}{dt}$

$$[F'(t)]_{r'=R} = \frac{d}{dt}(R^2V) = \frac{d}{dt}\left(R^2\frac{dR}{dt}\right)$$

$$= R^2\frac{d^2R}{dt^2} + 2R\left(\frac{dR}{dt}\right)^2$$

(OR) $$[F'(t)]_{r'=R} = \frac{d}{dt}\left(R^2\frac{dR}{dt}\right) = \frac{d}{dt}\left[\frac{R}{2}\,2R\frac{dR}{dt}\right]$$

$$= \frac{d}{dt}\left[\frac{R}{2}\cdot\frac{dR^2}{dt}\right] = \frac{R}{2}\frac{d^2R^2}{dt^2} + R\left(\frac{dR}{dt}\right)^2$$

$$\Rightarrow \; P = \Pi + \frac{1}{2}\rho\left[\frac{d^2R^2}{dt^2} + 2\left(\frac{dR}{dt}\right)^2 - \left(\frac{dR}{dt}\right)^2\right]$$

$$\Rightarrow \; P = \Pi + \frac{1}{2}\rho\left[\frac{d^2R^2}{dt^2} + \left(\frac{dR}{dt}\right)^2\right]$$

24 Q : Liquid is contained between two parallel plates planes, the free surface is a circular cylinder of radius a whose axis is perpendicular to the planes. All the liquid within a concentric circular cylinder of radius b is ~~b~~ suddenly annihilated ; prove that if Π be the pressure at the outer surface, the

initial pressure at any point on the liquid distant r from the centre is

$$\Pi \frac{\log r - \log b}{\log a - \log b}. \qquad (2006)$$

Sol: Here the motion of the liquid will take place in such a manner so that each element of the liquid moves towards the axis of the cylinder $|z| = b$. Hence the free surface would be cylindrical.

Let v be the velocity of liquid at a distance r ($\alpha \ngtr r \ngtr \beta$) in the fluid from the axis of the cylinder $|z| = b$. Since the velocity is radial, it is a function of r and time t only. Let p be the pressure at a distance r.

Then the equation of continuity is

$$rv = F(t)$$

$$\frac{\partial v}{\partial t} = \frac{F'(t)}{r}$$

The equation of motion is

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

$$\frac{F'(t)}{r} + \frac{\partial}{\partial r}\left(\frac{1}{2}v^2\right) = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

Integrating,

$$F'(t)\log r + \frac{1}{2}v^2 = -\frac{p}{\rho} + C$$

Initially when $t=0$, $v=0$, $p=P$.

$$F'(0)\log r = -\frac{P}{\rho} + C$$

Again, $P = \Pi$ when $r = a$ and

$P = 0$ when $r = b$

$$\therefore\ F'(0)\log a = -\left(\frac{\Pi}{\rho}\right) + C \quad \text{and} \quad F'(0)\log b = C$$

$$\Rightarrow\ C = -\log b\,\frac{\Pi}{\rho\log\frac{a}{b}}, \qquad F'(0) = -\frac{\Pi}{\rho\log\frac{a}{b}}.$$

$$F'(0)\log r = -\frac{P}{\rho} + C$$

$$\Rightarrow\ \frac{P}{\rho} = \frac{\Pi\log r}{\rho\log a/b} - \frac{\Pi\log b}{\rho\log a/b}$$

$$\Rightarrow\ P = \Pi\frac{\log r - \log b}{\log a - \log b} \rightarrow \text{initial pressure}$$

25Q : A steady inviscid incompressible fluid flow has a velocity field $u = fx,\ v = -fy,\ w = 0$, where f is a constant. Derive an expression for the pressure field $p(x,y,z)$ if the pressure $p(0,0,0) = p_0$ and $\bar{F} = -g\,\hat{i}_z$.

(2006)

Sol : Given $u = +fx$, $v = -fy$, $w = 0$

$p = p_0$ when $x = 0, y = 0, z = 0$

$\bar{F} = -g\,\hat{i}_z \Rightarrow F_x = 0,\ F_y = 0$ and $F_z = -g$

Equations of motion for steady motion $\frac{\partial}{\partial t} = 0$ of an incompressible fluid flow are given by

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = F_x - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} = F_y - \frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = F_z - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

$$f^2 x = -\frac{1}{\rho}\frac{\partial p}{\partial x},\quad -f^2 y = -\frac{1}{\rho}\frac{\partial p}{\partial y} \quad \text{and}$$

$$0 = -g - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

$$\Rightarrow \frac{\partial p}{\partial x} = -f^2\rho x,\quad \frac{\partial p}{\partial y} = -f^2\rho y \quad \text{and} \quad \frac{\partial p}{\partial z} = -\rho g$$

Now $dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy + \frac{\partial p}{\partial z}dz$

$$dp = (-f^2\rho x)\,dx - (f^2\rho y)\,dy + (-\rho g)\,dz$$

Integrating,

$$p = -f^2\rho\frac{x^2}{2} - f^2\rho\frac{y^2}{2} - \rho g z + c$$

$p(0,0,0) = p_0 = c$

$$\Rightarrow p(x,y,z) = p_0 - \frac{\rho}{2}\left(f^2x^2 + f^2y^2 + 2gz\right)$$

26Q : For a steady motion of inviscid incompressible fluid of uniform density under conservative forces, show that the vorticity $\bar{w}$ and velocity $\bar{q}$ satisfies

$$(\bar{q}\cdot\nabla)\bar{w} = (\bar{w}\cdot\nabla)\bar{q}$$

Sol : Vector equation of motion for inviscid incompressible fluid is $\frac{\partial \bar{q}}{\partial t} + \nabla\left(\frac{\bar{q}^2}{2}\right) - \bar{q}\times\operatorname{curl}\bar{q} = \bar{F} - \frac{1}{\rho}\nabla p$

Since the motion is steady,

$$\frac{\partial \bar{q}}{\partial t} = 0$$

Since ρ is uniform,

$$\frac{1}{\rho} \nabla p = \nabla\left(\frac{p}{\rho}\right)$$

Since $\bar{F}$ is conservative, $\bar{F} = -\nabla\psi$, where ψ is some scalar function.

Again, by definition

vorticity vector $= \bar{w} = \text{curl } \bar{q}$

$$\Rightarrow \nabla \frac{\bar{q}^2}{2} - \bar{q}\times\bar{w} = -\nabla\psi - \nabla p/\rho$$

$$\Rightarrow \bar{q}\times\bar{w} = \nabla\left(\frac{q^2}{2} + \psi + \frac{p}{\rho}\right)$$

Taking the curl on both sides of the above equation and using the vector identity

Curl grad $\phi = 0$, we have

$$\text{Curl } (\bar{q}\times\bar{w}) = 0.$$

$$\Rightarrow (\nabla\cdot\bar{w})\bar{q} - (\bar{q}\cdot\nabla)\bar{w} + (\bar{w}\cdot\nabla)\bar{q} - (\nabla\cdot\bar{q})\bar{w} = 0$$

But $\nabla\cdot\bar{w} = \nabla\cdot\nabla\times\bar{q} = 0$ and $\nabla\cdot\bar{q} = 0$ (continuity equation)

$$\Rightarrow -(\bar{q}\cdot\nabla)\bar{w} + (\bar{w}\cdot\nabla)\bar{q} = 0$$

$$\Rightarrow (\bar{q}\cdot\nabla)\bar{w} = (\bar{w}\cdot\nabla)\bar{q} \longrightarrow \text{The desired result.}$$

27/Q : An infinite mass of fluid is acted on by a force $\mu/r^{3/2}$ per unit mass directed to the origin. If initially the fluid is at rest and there is a cavity in the form of the sphere $r = c$ in it, show that the cavity will be filled up after an interval of time $\left(\frac{2}{5\mu}\right)^{1/2} c^{5/4}$. (2003) (2009)

Sol : Method I: Here the motion of the fluid will take place in such a manner so that each element of the fluid moves towards the centre. Hence the free surface would be spherical. Thus the fluid velocity v' will be radial and hence v' will be function of r' (the radial distance from the centre of the sphere which is taken as origin) and time t. Also, let v be the velocity at a distance r.

Then the equation of continuity is

$$r'^2 v' = F(t) = r^2 v$$

$$\frac{\partial v'}{\partial t} = \frac{F'(t)}{r'^2}$$

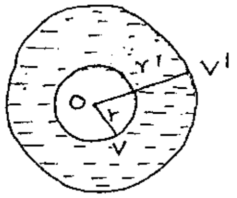
The equation of motion is

$$\frac{\partial v'}{\partial t} + v' \frac{\partial v'}{\partial r'} = \frac{-\mu}{r'^{3/2}} - \frac{1}{\rho}\frac{\partial p}{\partial r'}$$

$$\frac{F'(t)}{r'^2} + \frac{\partial}{\partial r'}\left(\frac{1}{2}v'^2\right) = \frac{-\mu}{r'^{3/2}} - \frac{1}{\rho}\frac{\partial p}{\partial r'}$$

Integrating w.r.t r', we have

$$-\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = \frac{2\mu}{r'^{1/2}} - \frac{p}{\rho} + c$$



When $r' = \infty$, $v' = 0$, $p = 0$

$$\Rightarrow c = 0$$

$$\therefore \frac{-F'(t)}{r'} + \frac{1}{2}v'^2 = \frac{2\mu}{r'^{1/2}} - \frac{p}{\rho}$$

Now when, $r' = r$, $v' = v$ and $p = 0$

$$\frac{-F'(t)}{r} + \frac{1}{2}v^2 = \frac{2\mu}{r^{1/2}}$$

$$F(t) = r^2 v \Rightarrow F'(t) = 2rv\frac{dr}{dt} + r^2\frac{dv}{dt}$$

$$= 2rv^2 + r^2 v\frac{dv}{dr}$$

$$\Rightarrow -\frac{\left[2rv^2 + r^2 v\frac{dv}{dr}\right]}{r} + \frac{v^2}{2} = \frac{2\mu}{r^{1/2}}$$

$$\text{or} \quad rv\frac{dv}{dr} + \frac{3}{2}v^2 = \frac{-2\mu}{r^{1/2}}$$

$$r^2(2rv\,dv + 3v^2 dr = -4\mu r^{-1/2} dr)$$

$$2r^3 v\,dv + 3v^2 r^2 dr = -4\mu r^{3/2} dr$$

$$d(r^3 v^2) = -4\mu r^{3/2} dr$$

$$r^3 v^2 = \frac{-8\mu}{5} r^{5/2} + D$$

When $r = c$, $v = 0$ $\Rightarrow D = \frac{8\mu}{5}c^{5/2}$

$$\Rightarrow r^3 v^2 = \frac{8\mu}{5}\left(c^{5/2} - r^{5/2}\right)$$

$$v = \frac{dr}{dt} = -\left(\frac{8\mu}{5}\right)^{1/2}\left[\frac{c^{5/2} - r^{5/2}}{r^3}\right]^{1/2}$$

Taking negative sign for $\frac{dr}{dt}$ since velocity increases as r decreases.

Let T be the time of filling up the cavity, then

$$T = -\left(\frac{5}{8\mu}\right)^{1/2} \int_c^0 \frac{r^{3/2}}{(c^{5/2}-r^{5/2})^{1/2}}\, dr$$

Let $r^{5/2} = c^{5/2}\sin^2\theta$

$$\Rightarrow \frac{5}{2} r^{3/2} dr = 2c^{5/2}\sin\theta\cos\theta\, d\theta$$

$$\Rightarrow T = \frac{4}{5}\left(\frac{5}{8\mu}\right)^{1/2} \int_0^{\pi/2} \frac{c^{5/2}\sin\theta\cos\theta}{c^{5/4}\cos\theta}\, d\theta$$

$$= \frac{4c^{5/4}}{5}\left(\frac{5}{8\mu}\right)^{1/2} \int_0^{\pi/2} \sin\theta\, d\theta$$

$$\therefore T = (2/5\mu)^{1/2} \times c^{5/4}.$$

Method II :

At any time t, let v' be the velocity at distance r' from the centre. Again, let r be the radius of the cavity and v its velocity. Then the equation of continuity yields

$$r'^2 v' = r^2 v$$

When the radius of the cavity is r, then

$$K.E = \int_r^\infty \frac{1}{2}(4\pi r'^2 \rho\, dr')\, v'^2$$

$$= 2\pi\rho r^4 v^2 \int_r^\infty \frac{dr'}{r'^2}$$

$$= 2\pi\rho r^3 v^2$$

Initial Kinetic energy $= 0$

Let V be the work function (or force potential) due to external forces. Then, we have

$$-\frac{\partial V}{\partial r'} = \frac{\mu}{r'^{3/2}} \Rightarrow V = \frac{2\mu}{r'^{1/2}}.$$

$\therefore$ work done $= \int_r^c V\,dm$ or $-\int_c^r V\,dm$, dm being the elementary mass

$$= \int_r^c \frac{2\mu}{r'^{1/2}}\, 4\pi r'^2 dr' \rho$$

$$= 8\pi\mu\rho \int_r^c r'^{3/2} dr' = \frac{16}{5}\pi\rho\mu(c^{5/2} - r^{5/2})$$

Using energy equation namely,

Increase in K.E = work done

$$\Rightarrow 2\pi\rho r^3 v^2 - 0 = \frac{16}{5}\pi\rho\mu\left(c^{5/2} - r^{5/2}\right)$$

$$\Rightarrow v = \frac{dr}{dt} = -\left(\frac{8\mu}{5}\right)^{1/2}\frac{\left(c^{5/2} - r^{5/2}\right)^{1/2}}{r^{3/2}}$$

$$T = \int_0^t dt = \left(2/5\mu\right)^{1/2} c^{5/4}$$

28 Q : A mass of perfect incompressible fluid of density ρ is bounded by Concentric spherical surfaces. The outer surface is contained by a flexible envelope which exerts continuously uniform pressure Π and contracts from radius R_1 to radius R_2. The hallow is filled with a gas obeying Boyle's law, its radius contracts from c_1 to c_2 and the pressure of the gas initially is p_1. Initially whole mass is at rest. Prove that, neglecting the mass of the gas, the velocity v of the inner surface when the configuration (R_2, c_2) is reached is given by

$$\frac{1}{2}v^2 = \frac{c_1^3}{c_2^3}\left\{\frac{1}{3}\left[1 - \frac{c_2^3}{c_1^3}\right]\frac{\Pi}{\rho} - \frac{p_1}{\rho}\log\frac{c_1}{c_2}\right\} \Big/ \left(1 - \frac{c_2}{R_2}\right)$$

Sol: Let p_2 be the pressure of the gas when the internal radius is c_2. Then, by Boyle's law,

$$\frac{4}{3}\times\pi c_1^3 p_1 = \frac{4}{3}\times\pi c_2^3 p_2$$

$$p_2 = \frac{c_1^3}{c_2^3}p_1$$

Equation of continuity is

$$v' = c_2^2 v / r'^2$$

Now, the initial Kinetic energy (K.E) = 0

$$\text{final K.E} = \int_{c_2}^{R_2}\frac{1}{2}\left(4\pi r'^2 dr'\rho\right)v'^2$$

$$= 2\pi\rho c_2^4 v^2\int_{c_2}^{R_2}\frac{dr'}{r'^2}$$

$$= 2\pi\rho c_2^4 v^2\left(\frac{1}{c_2} - \frac{1}{R_2}\right)$$

$$= 2\pi\rho c_2^3 v^2\left(1 - \frac{c_2}{R_2}\right)$$

Now, work done W by the external pressure Π and the internal pressure p_2 is given by

$$W = \int_{R_1}^{R_2} 4\pi R_2^2 \Pi (-dR_2) + \int_{c_1}^{c_2} 4\pi c_2^2 p_2 \, dc_2$$

$$W = -\frac{4}{3}\pi \Pi \left[\frac{R_2^3}{3} \right]_{R_1}^{R_2} + 4\pi \int_{c_1}^{c_2} c_2^2 \frac{c_1^3}{c_2^3} p_1 \, dc_2$$

$$W = \frac{4}{3} \times \pi \Pi (R_1^3 - R_2^3) + 4\pi p_1 c_1^3 \left[\log c_2\right]_{c_1}^{c_2}$$

Since the mass of the fluid remains constant, we have

$$\frac{4}{3}\pi (R_2^3 - c_2^3) = \frac{4}{3}\pi (R_1^3 - c_1^3)$$

$$\Rightarrow R_1^3 - R_2^3 = c_1^3 - c_2^3 .$$

$$\therefore W = \frac{4}{3}\pi \Pi (c_1^3 - c_2^3) + 4\pi p_1 c_1^3 \log (c_2/c_1) .$$

Increase in K.E = Total work done

$$\Rightarrow 2\pi \rho c_2^3 v^2 \left(1 - \frac{c_2}{R_2}\right) = \frac{4}{3}\pi \Pi (c_1^3 - c_2^3) + 4\pi p_1 c_1^3 \log\left(\frac{c_2}{c_1}\right)$$

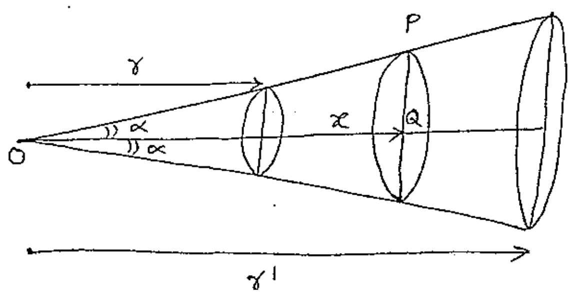
$$\frac{1}{2} v^2 c_2^3 \left(1 - \frac{c_2}{R_2}\right) = c_1^3 \left[\frac{1}{3}\left(1 - \frac{c_2^3}{c_1^3}\right)\frac{\Pi}{\rho} - \frac{p_1}{\rho} \log \frac{c_1}{c_2} \right]$$

$$\Rightarrow \frac{1}{2} v^2 = \frac{c_1^3}{c_2^3} \left\{ \frac{1}{3}\left(1 - \frac{c_2^3}{c_1^3}\right)\frac{\Pi}{\rho} - \frac{p_1}{\rho} \log \frac{c_1}{c_2} \right\} \Big/ \left(1 - \frac{c_2}{R_2}\right)$$

29 Q : A given quantity of liquid moves, under no forces, in a smooth conical tube having a small vertical angle and the distances of its nearer and farther extremities from the vertex at the time t are r and r', show that

$$2r\frac{d^2 r}{dt^2} + \left(\frac{dr}{dt}\right)^2 \left[3 - \frac{r}{r'} - \frac{r^2}{r'^2} - \frac{r^3}{r'^3}\right] = 0$$

Sol:



Let at any time t, let p' be the pressure at a distance x from the vertex and v' be the velocity there.

Let α be the semi-vertical angle of the conical tube. Then the equation of continuity is given by

$$v'(PQ)^2 = f(t) \Rightarrow v'(x\tan\alpha)^2 = f(t)$$

or $\quad v'x^2 = F(t)$, where $F(t) = \cot^2\alpha \, f(t)$.

The equation of motion is

$$\frac{\partial v'}{\partial t} + v'\frac{\partial v'}{\partial x} = -\frac{1}{\rho}\frac{\partial p'}{\partial x}$$

$$\frac{\partial v'}{\partial t} = \frac{1}{x^2} F'(t)$$

$$\Rightarrow \quad \frac{F'(t)}{x^2} + \frac{\partial}{\partial x}\left(\frac{1}{2}v'^2\right) = -\frac{1}{\rho}\frac{\partial p'}{\partial x}$$

Integrating w.r.t x, we have

$$-\frac{F'(t)}{x} + \frac{1}{2}v'^2 = C - \frac{p'}{\rho}$$

Let v and v' be the velocities when $x=r$ and $x=r'$ respectively and p_0 be the pressure there.

Then, $\quad -\frac{F'(t)}{r} + \frac{1}{2}v^2 = C - \frac{p}{\rho} \quad$ and

$$-\frac{F'(t)}{r'} + \frac{1}{2}v'^2 = C - \frac{p}{\rho}$$

$$\Rightarrow -F'(t)\left(\frac{1}{r} - \frac{1}{r'}\right) + \frac{1}{2}(v^2 - v'^2) = 0$$

From equation of continuity

$$F(t) = r^2 v = r'^2 v' \qquad F'(t) = \frac{d}{dt}(r^2 v)$$

$$v = \frac{dr}{dt} \quad \text{and} \quad v' = \frac{dr'}{dt}$$

$$v' = \frac{r^2 v}{r'^2}$$

$$\Rightarrow \quad -F'(t)\left(\frac{1}{r} - \frac{1}{r'}\right) + \frac{1}{2}\left(v^2 - \frac{r^4 v^2}{r'^4}\right) = 0$$

$$\Rightarrow \quad -F'(t)\frac{(r'-r)}{(rr')} + \frac{1}{2}v^2\left(\frac{r'^4 - r^4}{r'^4}\right) = 0$$

or $\quad F'(t)\left(\frac{r'-r}{rr'}\right) - \frac{v^2}{2}\frac{(r'-r)(r'+r)(r'^2+r^2)}{(r')^4} = 0.$

$$\Rightarrow \frac{2F'(t)}{r} - v^2 \frac{(r'^3 + r'^2 r + r' r^2 + r^3)}{r'^3} = 0$$

$$\Rightarrow 2\frac{d}{dt}(r^2 v) - v^2\, r\, \frac{(r'^3 + r'^2 r + r' r^2 + r^3)}{r'^3} = 0$$

$$\Rightarrow 2\left[2r\left(\frac{dr}{dt}\right)^2 + r^2\frac{d^2r}{dt^2}\right] - \left(\frac{dr}{dt}\right)^2 r \cdot \frac{(r'^3 + r'^2 r + r' r^2 + r^3)}{r'^3} = 0$$

$$\Rightarrow 2r^2\frac{d^2r}{dt^2} + \left(\frac{dr}{dt}\right)^2\left[4r - r - \frac{r \cdot r}{r'} - \frac{r^2 \cdot r}{r'^2} - \frac{r^3 \cdot r}{r'^3}\right] = 0$$

$$\Rightarrow 2r\frac{d^2r}{dt^2} + \left(\frac{dr}{dt}\right)^2\left[3 - \frac{r}{r'} - \frac{r^2}{r'^2} - \frac{r^3}{r'^3}\right] = 0$$

Motion in two dimensions:

If (x, y, z) are coordinates of any point in the fluid, then all physical quantities (velocity, density, pressure, etc.) associated with the fluid are independent of z or y or x.

Stream function / Current function:

Let u and v be the components of velocity in two-dimensional motion. Then the differential equations of lines of flow or streamlines is

$$\frac{dx}{u} = \frac{dy}{v} \quad \text{or} \quad v\,dx - u\,dy = 0 \qquad — (1)$$

and the equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad \text{or} \quad \frac{\partial v}{\partial y} = \frac{\partial(-u)}{\partial x} \qquad - (2)$$

(2) shows that L.H.S of (1) must be an exact differential, $d\psi$ (say). Thus, we have

$$v\,dx - u\,dy = d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy$$

$$\Rightarrow u = -\frac{\partial \psi}{\partial y} \quad \text{and} \quad v = \frac{\partial \psi}{\partial x}.$$

The function ψ is called 'stream function'

Streamlines are given by $d\psi = 0$ i.e $\psi = \text{constant}$.

Stream/current function exists by virtue of the equation of continuity and incompressibility of the fluid.

Hence stream function exists in all types of two dimensional motion whether rotational or irrotational.

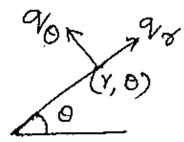
Physical Significance of Stream function:

Property of a stream function is that the difference of its values at two points represents the flow across any line joining the points.

Note:

In plane-polar co-ordinates (r, θ), velocity components in terms of ψ are given by

$$q_r = -\frac{1}{r}\frac{\partial \psi}{\partial \theta}$$



$$q_\theta = \frac{\partial \psi}{\partial r}.$$

Irrotational motion in two-dimensions:

Let there be an irrotational motion so that the velocity potential ϕ exists such that

$$u = -\frac{\partial \phi}{\partial x} \quad \text{and} \quad v = \frac{-\partial \phi}{\partial y} \quad - (1)$$

In two dimensional flow the stream function ψ always exists such that

$$u = -\frac{\partial \psi}{\partial y} \quad \text{and} \quad v = \frac{\partial \psi}{\partial x} \quad - (2)$$

From (1) and (2)

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \text{and} \quad \frac{\partial \phi}{\partial y} = \frac{-\partial \psi}{\partial x}$$

which are well known Cauchy-Riemann's equations.

Hence $\phi + i\psi$ is an analytic function of $z = x + iy$.

Moreover ϕ and ψ are known as conjugate functions.

And, $\frac{\partial \phi}{\partial x} \cdot \frac{\partial \psi}{\partial x} + \frac{\partial \phi}{\partial y} \cdot \frac{\partial \psi}{\partial y} = 0$

$\Rightarrow$ ϕ = constant and ψ = constant intersect orthogonally.

** The curves of equi-velocity potential and the stream lines intersect orthogonally.

And, $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ and $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0$

$\Rightarrow \phi$ and ψ satisfy Laplace's equation.

** when a two-dimensional irrotational motion is considered ϕ and ψ satisfy Laplace's equation.

Complex potential :

Let $w = \phi + i\psi$ be taken as a function of $z = x+iy$

Then, $w = f(z)$ or $\phi + i\psi = f(x+iy)$

$$\frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = f'(x+iy)$$

$$\frac{\partial \phi}{\partial y} + i\frac{\partial \psi}{\partial y} = i f'(x+iy)$$

$$\Rightarrow \frac{\partial \phi}{\partial y} + i\frac{\partial \psi}{\partial y} = i\left(\frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}\right)$$

Equating real and imaginary parts, we get

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \quad \text{and} \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

Cauchy - Riemann equations

Then w is an analytic function of z and w is known as 'complex potential'.

Note :

1. If w is an analytic function of z, then its real part is velocity potential (ϕ) and its imaginary part is the stream function (ψ) of an irrotational 2-D motion.

2. $w = \phi + i\psi$; $z = x+iy$

$$\frac{dw}{dz} \cdot \frac{\partial z}{\partial x} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} , \quad \frac{\partial z}{\partial x} = 1$$

$$\Rightarrow \frac{dw}{dz} = \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y} = -u + iv$$

$$\Rightarrow q = \left|\frac{dw}{dz}\right|$$

$$\because q = \sqrt{u^2+v^2} = \left|\frac{dw}{dz}\right|$$

Cauchy - Riemann equations in polar form:

Let $\phi + i\psi = f(z) = f(re^{i\theta})$

Differentiating w.r.t r and θ, we get

$$\frac{\partial \phi}{\partial r} + i\frac{\partial \psi}{\partial r} = f'(re^{i\theta}) \cdot e^{i\theta}$$

$$\frac{\partial \phi}{\partial \theta} + i\frac{\partial \psi}{\partial \theta} = f'(re^{i\theta}) \cdot r i e^{i\theta}$$

$$\Rightarrow \frac{\partial \phi}{\partial \theta} + i\frac{\partial \psi}{\partial \theta} = ri\left(\frac{\partial \phi}{\partial r} + i\frac{\partial \psi}{\partial r}\right)$$

$$\Rightarrow \frac{\partial \phi}{\partial \theta} = -r\frac{\partial \psi}{\partial r} \quad \text{and} \quad \frac{\partial \psi}{\partial \theta} = r\frac{\partial \phi}{\partial r}$$

$$\Rightarrow \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} \quad \text{and} \quad \frac{1}{r}\frac{\partial \phi}{\partial \theta} = -\frac{\partial \psi}{\partial r}.$$

30 Q : For a two-dimensional flow the velocity function is given by the expression, $\phi = x^2 - y^2$. Then

(i) Determine velocity components in x and y directions

(ii) show that the velocity components satisfy the conditions of flow continuity and irrotationality

(iii) Determine stream function and flow rate between the stream lines $(2,0)$ and $(2,2)$

(iv) Show that the streamlines and potential lines intersect orthogonally at the point $(2,2)$.

Sol : (i) The velocity components in x and y directions are

$$u = \frac{-\partial \phi}{\partial x} = -2x \quad \text{and} \quad v = \frac{-\partial \phi}{\partial y} = 2y$$

(ii) $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -2+2 = 0$$

$$\text{Curl } \bar{q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -2x & 2y & 0 \end{vmatrix}$$

Curl $\bar{q} = 0 \Rightarrow$ flow is irrotational

(iii) $$u = \frac{-\partial \psi}{\partial y} \quad \text{and} \quad v = \frac{\partial \psi}{\partial x}$$

$$\Rightarrow \frac{\partial \psi}{\partial y} = 2x \quad \text{and} \quad \frac{\partial \psi}{\partial x} = 2y$$

$$\Rightarrow \quad d\psi = \frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy$$

$$= 2\,(y\,dx + x\,dy)$$

$$\Rightarrow \quad \psi = 2xy + c$$

The required flow between the streamlines through $(2,0)$ and $(2,2)$ $= \psi(2,2) - \psi(2,0)$

$$= 8 - 0 = 8 \ m^3/sec$$

(iv) $\phi = x^2 - y^2$ and $\psi = 2xy + c$

Let m_1 = The slope of tangent at (x,y) to potential lines $\phi = c_1$

$$m_1 = \frac{-\partial\phi/\partial x}{\partial\phi/\partial y} = -\frac{2x}{-2y} = \frac{x}{y}$$

Let m_2 = The slope of tangent at (x,y) to $\psi = c_2$

$$m_2 = \frac{-\partial\psi/\partial x}{\partial\psi/\partial y} = -\frac{2y}{2x} = -\frac{y}{x}$$

$$m_1(2,2) = 1 \qquad m_2(2,2) = -1$$

$$m_1 m_2 \text{ at } (2,2) = 1 \times -1 = -1$$

$m_1 m_2 = -1$ shows that the streamlines and potential lines intersect orthogonally.

Sources and Sinks :

If the motion of a fluid consists of symmetrical radial flow in all directions proceeding from a point, the point is known as a 'simple source'. If the flow is directed radially inwards to a point then the point is known as 'simple sink'.

A source implies the creation of fluid at a point whereas a sink implies the annihilation of a fluid at a point.

Consider a source at a point. Then the mass of the fluid coming out from the point in a unit time is known as the strength of the source. Similarly the amount of fluid going into the sink in a unit time is called the strength of the sink.

Note:

Sources and sinks are examples of singularities of a flow field because infinitely many stream lines meet at such points. (velocity vector at the point is not unique).

If the velocity vector is unique at a point, usually no two streamlines intersect each other.

Sources and Sinks in two dimensions:

In two dimensions a source of strength m is such that the flow across any small curve surrounding is $2\pi m$. Sink is regarded as a source of strength $-m$.

Consider a circle of radius r with source at its centre. Then radial velocity q_r is given by

$$q_r = \frac{-1}{r}\frac{\partial \psi}{\partial \theta} \quad \text{and} \quad q_r = \frac{-\partial \phi}{\partial r}$$

$$\Rightarrow \frac{\partial \phi}{\partial r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta}.$$

But the flow across the circle is $2\pi r q_r$

$$\Rightarrow 2\pi r q_r = 2\pi m$$

$$\Rightarrow r q_r = m$$

$$\therefore \; r\left(\frac{-1}{r}\frac{\partial \psi}{\partial \theta}\right) = m$$

Integrating and omitting constant of integration, we get $\psi = -m\theta \quad -①$

And, $\frac{\partial \phi}{\partial r} = \frac{1}{r}(-m)$

$$\Rightarrow \phi = -m\log r \quad -②$$

Equation ① shows that streamlines are θ ~~ϕ~~ = constant i.e straight lines radiating from the source.

Equation ② shows that the curves of equivelocity potential are r = constant i.e concentric circles with centre at the source.

Complex potential due to a source:

Source strength $= m$

$$w = \phi + i\psi = -m\log r - im\theta = -m(\log r + \log e^{i\theta})$$
$$= -m\log(re^{i\theta}) = -m\log z.$$

If source is at z'

complex potential, $w = -m\log(z - z')$

Note: Let $m_1, m_2, m_3, ----$ sources of given strengths situated at points $z_1, z_2, z_3, ----$

Then, $w = -m_1\log(z - z_1) - m_2\log(z - z_2) - m_3\log(z - z_3) ---$

$$\phi = -m_1\log r_1 - m_2\log r_2 ----$$
$$\psi = -m_1\theta_1 - m_2\theta_2 - m_3\theta_3 ----$$

where, $r_n = |z - z_n|$ and $\theta_n = \arg(z - z_n)$, $n = 1, 2, 3, ---$

Doublet (or dipole) in two dimensions:

Combination of a source of strength m and sink of strength $-m$ at a small distance δs apart, where in the limit m is taken infinitely large and δs infinitely small but the product $m\delta s$ remain finite and equal to μ is called a doublet of strength μ.

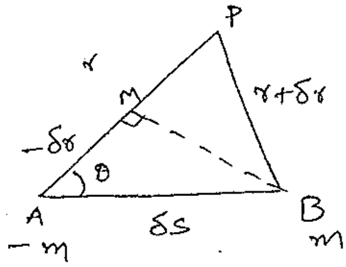
Line from $-m$ to $+m$ is taken as the axis of the doublet.

Complex potential due to a doublet in two-dimensions:

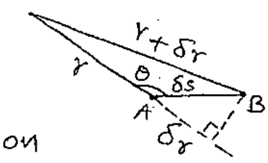
A - ~~Source~~ Sink   B - ~~Sink~~ Source

$AP = r$ $BP = r + \delta r$

$AM = -\delta r$ $AB = \delta s$



(or)



$$\phi = m\log r - m\log(r + \delta r)$$
$$= -m\log\frac{r + \delta r}{r}$$
$$= -m\log\left(1 + \frac{\delta r}{r}\right)$$

$\phi = -m\frac{\delta r}{r}$, to first order of approximation

$\cos(180 - \theta) = \frac{\delta r}{\delta s}$

$AM = AP - MP = r - (r + \delta r) = -\delta r$

$$\cos\theta = \frac{AM}{AB} = -\frac{\delta r}{\delta s}$$

$$\Rightarrow \quad \delta r = -\delta s \cos\theta .$$

$$\Rightarrow \phi = -m\frac{\delta r}{r} = m\delta s \frac{\cos\theta}{r}$$

$$= \frac{\mu \cos\theta}{r}.$$

$\mu = m\delta s$ - strength of the doublet.

$$\frac{\partial\phi}{\partial r} = \frac{-\mu\cos\theta}{r^2}$$

$$\frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{-\mu\cos\theta}{r^2} \Rightarrow \frac{\partial\psi}{\partial\theta} = \frac{-\mu\cos\theta}{r}$$

Integrating w.r.t θ, we get

$$\psi = \frac{-\mu\sin\theta}{r} + f(r)$$

$$\frac{1}{r}\frac{\partial\phi}{\partial\theta} = \frac{-\partial\psi}{\partial r}$$

$$\frac{1}{r}\left(\frac{-\mu\sin\theta}{r}\right) = \frac{-\mu\sin\theta}{r^2} - f'(r)$$

$$\Rightarrow f'(r) = 0 \quad \Rightarrow f(r) = \text{constant}$$

$$\therefore \psi = \frac{-\mu\sin\theta}{r}$$

complex potential due to the doublet is given by,

$$w = \phi + i\psi = \frac{\mu}{r}(\cos\theta - i\sin\theta)$$

$$w = \frac{\mu}{r}e^{-i\theta} = \frac{\mu}{re^{i\theta}} = \frac{\mu}{z}.$$

Note:

1. Equi potential curves $\phi = \text{constant}$

$$\frac{\mu\cos\theta}{r} = \text{constant} \Rightarrow \frac{\cos\theta}{r} = \text{constant}$$

$$\Rightarrow r\cos\theta = cr^2 \text{ (or) } x = c(x^2+y^2)$$

represents circles touching y-axis at origin.

2. Stream lines are given by $\psi = \text{constant}$ i.e

$$\frac{-\mu\sin\theta}{r} = \text{constant} \Rightarrow \frac{\sin\theta}{r} = c'$$

$\Rightarrow r\sin\theta = c' r^2$ or $y = c'(x^2+y^2)$

represents circles touching the x-axis at origin.

3. If doublet makes α angle with x-axis

$$w = \frac{\mu}{re^{i(\theta-\alpha)}} = \frac{\mu}{z} e^{i\alpha}.$$

If doublet is at point z'

Then, $w = \dfrac{\mu e^{i\alpha}}{z-z'}$.

4. For multiple doublets

$$w = \frac{\mu_1 e^{i\alpha_1}}{z-z_1} + \frac{\mu_2 e^{i\alpha_2}}{z-z_2} + \cdots.$$

31 Q : What arrangement of sources and sinks will give rise to the function $w = \log(z - a^2/z)$. Draw a rough sketch of the streamlines. Prove that two of the streamlines subdivide into the circle $r=a$ and axis of y.

Sol: Given, $w = \log(z - a^2/z)$

$$= \log\left[\frac{(z-a)(z+a)}{z}\right]$$

$$w = \log(z-a) + \log(z+a) - \log z.$$

which shows that there are two sinks of units strength at the points $z=a$ and $z=-a$ and a source of unit strength at origin.

Since, $w = \phi + i\psi$ and $z = x+iy$, we obtain

$$\phi + i\psi = \log(x+iy-a) + \log(x+iy+a) - \log(x+iy)$$

$$\therefore \phi + i\psi = \log[(x-a)+iy] + \log[(x+a)+iy] - \log(x+iy)$$

Equating imaginary parts on both sides, we have

$$\psi = \tan^{-1}\frac{y}{(x-a)} + \tan^{-1}\frac{y}{(x+a)} - \tan^{-1}\frac{y}{x},$$ Since $\log(\alpha+i\beta) = \frac{1}{2}\log(\alpha^2+\beta^2) + i\tan^{-1}\frac{\beta}{\alpha}$.

$$\psi = \tan^{-1}\frac{2xy}{x^2-y^2-a^2} - \tan^{-1}\frac{y}{x}$$

$$\psi = \tan^{-1}\frac{y(x^2+y^2+a^2)}{x(x^2+y^2-a^2)}$$

The desired streamlines are given by $\psi = \text{constant} = \tan^{-1}(c)$ (say) 1

$$\Rightarrow \frac{y(x^2+y^2+a^2)}{} = c.$$

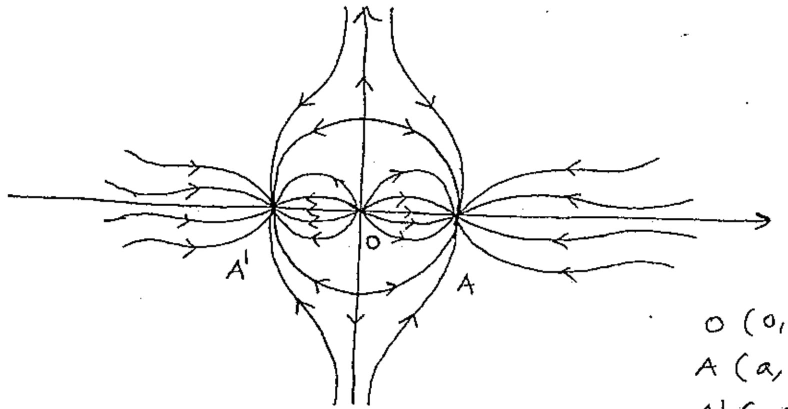
When $C=0$

$y=0 \Rightarrow$ x-axis is a streamline.

When $C \to \infty$

$x(x^2+y^2-a^2)=0$

$\Rightarrow x=0$ and $x^2+y^2=a^2$ $(r=a)$ are streamlines.



O (0,0)
A (a,0)
A' (−a,0)

32 Q : There is a source of strength m at $(0,0)$ and equal sinks at $(1,0)$ and $(-1,0)$. Discuss two-dimensional motion

Sol :

$$w = m\log(z-1) + m\log(z+1) - \cancel{2}\, m\log(z-0)$$

$$\phi + i\psi = m\ \log(x+iy-1) + \log(x+iy+1) - \log(x+iy)$$

$$\psi = m\left[\tan^{-1}\frac{y}{x-1} + \tan^{-1}\frac{y}{x+1} - \tan^{-1}\frac{y}{x}\right]$$

$$\frac{\psi}{m} = \tan^{-1}\frac{y(x^2+y^2+1)}{x(x^2+y^2-1)}$$

The desired streamlines are given by

$$\psi/m = \text{constant} = \tan^{-1} C$$

$$\Rightarrow \frac{y(x^2+y^2+1)}{x(x^2+y^2-1)} = C.$$

When $C=0$

$y=0 \Rightarrow$ x-axis is a streamline

When $C \to \infty$

$x=0$ and $x^2+y^2=1$ $(r=1)$ are streamlines.

(sketch of streamlines same as above).

33Q: Two sources, each of strength m are placed at the points $(-a,0)$, $(a,0)$ and a sink of strength $2m$ at the origin. Show that the streamlines are the curves $(x^2+y^2)^2 = a^2(x^2-y^2+\lambda xy)$ where λ is a variable parameter.

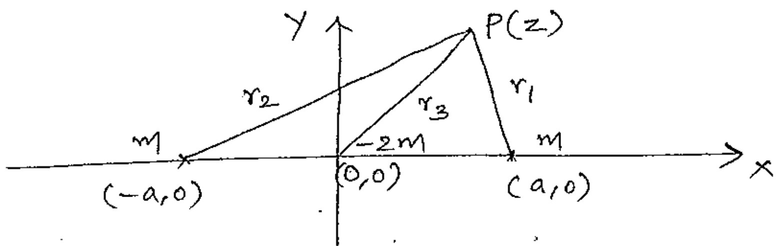
Show also that the fluid speed at any point is $2ma^2/r_1r_2r_3$ where r_1, r_2, r_3 are the distances of the points from the sources and the sink. (2003)

Sol: The complex potential at any point $P(z)$ is given by

$$w = -m\log(z-a) - m\log(z+a) + 2m\log z$$

or $$w = m[\log z^2 - \log(z^2-a^2)]$$

$$\Rightarrow \phi + i\psi = m[\log(x^2-y^2+2ixy) - \log(x^2-y^2-a^2+2ixy)]$$



Equating the imaginary parts, we have

$$\Rightarrow \psi = m\left[\tan^{-1}\frac{2xy}{x^2-y^2} - \tan^{-1}\frac{2xy}{x^2-y^2-a^2}\right]$$

$$\therefore \psi = m\tan^{-1}\left[\frac{-2a^2xy}{(x^2+y^2)^2 - a^2(x^2-y^2)}\right]$$

The desired streamlines are given by ψ = constant

let $\psi = m\tan^{-1}(-2/\lambda)$

$$\Rightarrow -2/\lambda = -2a^2xy/[(x^2+y^2)^2 - a^2(x^2-y^2)]$$

$$\Rightarrow (x^2+y^2)^2 - a^2(x^2-y^2) = a^2\lambda xy$$

$$\Rightarrow (x^2+y^2)^2 = a^2(x^2-y^2+\lambda xy).$$

$$\frac{dw}{dz} = \frac{-m}{z-a} - \frac{m}{z+a} + \frac{2m}{z} = -\frac{2a^2m}{z(z-a)(z+a)}$$

$$q = \left|\frac{dw}{dz}\right| = \frac{2a^2m}{|z||z-a||z+a|} = \frac{2a^2m}{r_1r_2r_3}$$

$$\therefore |\bar{q}| = \frac{2a^2m}{r_1r_2r_3}$$

34 Q : Find the stream function of the two-dimensional motion due to two equal sources and an equal sink situated midway between them.

Sol : Let there be two sources of strength m at the points $z=a$ and $z=-a$ and a sink of same strength at $z=0$ (origin).

Complex potential,

$$w = -m\log(z-a) - m\log(z+a) + m\log(z-0)$$

$$\phi + i\psi = m\log(x+iy) - m\log[(x-a)+iy] - m\log[(x+a)+iy]$$

Equating imaginary parts,

$$\psi = m\tan^{-1}\frac{y}{x} - m\left(\tan^{-1}\frac{y}{x-a} + \tan^{-1}\frac{y}{x+a}\right)$$

$$\Rightarrow \frac{\psi}{m} = \tan^{-1}\frac{y}{x} - \tan^{-1}\frac{2xy}{x^2-y^2-a^2}.$$

$$\Rightarrow \frac{\psi}{m} = \tan^{-1}\frac{y(x^2+y^2+a^2)}{x(a^2-x^2-y^2)}$$

$$\Rightarrow \psi = m\tan^{-1}\frac{y(x^2+y^2+a^2)}{x(a^2-x^2-y^2)}.$$

35 Q : In a two dimensional liquid motion ϕ and ψ are the velocity and current functions, show that a second fluid motion exists in which ψ is the velocity potential and $-\phi$ the current function; and prove that if the first motion be due to sources and sinks, the second motion can be built up by replacing a ~~sa~~ source and an equal sink by a line of doublets uniformly distributed along any curve joining them.

Sol: Since ϕ and ψ are the velocity potential and stream function respectively for the two-dimensional motion, we have

$$\frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} \quad \text{and} \quad \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}. \quad \text{— (1)}$$

Again if ψ and $-\phi$ be the velocity ~~an~~ potential and stream function respectively for another fluid motion in two-dimensions, then the conditions of the ① must be satisfied by ψ and $-\phi$ i.e, we must have

$$\frac{\partial \psi}{\partial x} = \frac{\partial(-\phi)}{\partial y} \quad \text{and} \quad \frac{\partial \psi}{\partial y} = -\frac{\partial(-\phi)}{\partial x}$$

$$\text{i.e} \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x} \quad \text{and} \quad \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$$

which is true by virtue of ①

It follows that if $w = \phi + i\psi$ exists then

$w' = \psi - i\phi = -i(\phi + i\psi) = -iw$, also exists

Consider a source of strength m at $A(a,0)$ and a sink of strength $-m$ at $B(-a,0)$. Then the complex potential function w due to them is given by $w = -m\log(z-a) + m\log(z+a)$

$$w = m\log\frac{z+a}{z-a}$$

Join A, B by an arbitrary curve. Then the axis of the doublet on this curve is normal to AB. If w' be the complex potential due to this line of doublets then,

$$w' = \int_A^B \frac{me^{i\pi/2}}{z-\alpha}\,d\alpha = me^{i\pi/2}\log\frac{z-a}{z+a}$$

$$= mi\log\frac{z-a}{z+a} = -iw$$

$$w' = iw.$$

Images:

If in a liquid a surface S can be drawn across which there is no flow then any system of sources, sinks and doublets on opposite sides of this surface is known as the image of the system with regard to the surface.

Moreover, if the surface S is treated as a rigid boundary and the liquid removed from one side of it the motion on the other side will remain unchanged.

As there is no flow across the surface, it must be a streamline. Thus the fluid flows tangentially to the surface and hence the normal velocity of the fluid at any point of the surface is zero.

The method of images is used to determine the complex potential due to sources, sinks and doublets in the presence of rigid boundaries.

Image of a source w.r.t a line:

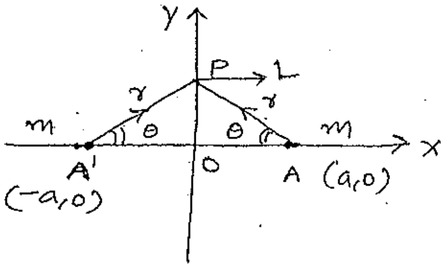


Image of the source m at $A(a,0)$ on x-axis w.r.t y-axis is at $A'(-a,0)$ with a source of strength m.

Resultant velocity at P due to sources at A and A' along PL $= \frac{m\cos\theta}{r} - \frac{m\cos\theta}{r} = 0$

showing that there will be no flow across oy.

Note: 1. Image of a sink w.r.t a line in two-dimensions is an equal sink equidistant from the line opposite to the sink.

2. The above results hold good if a line is replaced by a plane.

3. Image of a doublet w.r.t a line

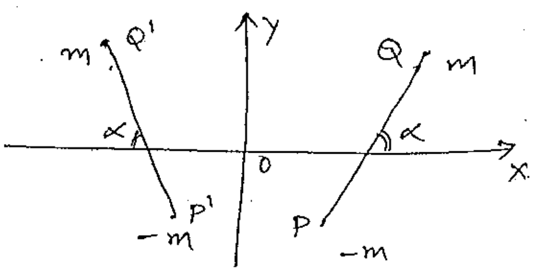
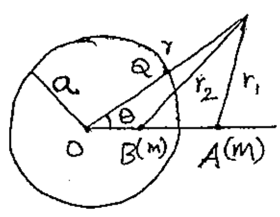


Image of a source with regard to a circle:

Let $OA = f$

Let B be inverse point of A with respective to the circle.

Let there be a source of strength m at B.



Complex potential $w = -m\log(z-f) - m\log(z - a^2/f)$

$(\because OA\cdot OB = a^2 \Rightarrow OB = a^2/f)$

$$\phi + i\psi = -\frac{m}{2}\left[\log\{(r\cos\theta - f)^2 + (r\sin\theta)^2\} + \log\{(r\cos\theta - a^2/f)^2 + (r\sin\theta)^2\}\right]$$

$$- i m\left[\tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta - f}\right) + \tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta - a^2/f}\right)\right]$$

$$\Rightarrow \phi = -\frac{m}{2}\left\{\log[(r\cos\theta - f)^2 + (r\sin\theta)^2] + \log[(r\cos\theta - a^2/f)^2 + (r\sin\theta)^2]\right\}$$

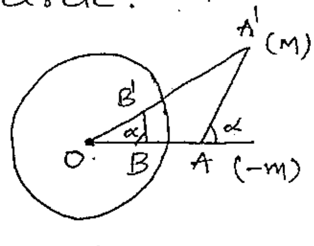
$$\frac{\partial\phi}{\partial r} = -\frac{m}{2}\left[\frac{2(r - f\cos\theta)}{r^2 + f^2 - 2fr\cos\theta} + \frac{2(r - \frac{a^2}{f}\cos\theta)}{r^2 + a^4/f^2 - 2r\frac{a^2}{f}\cos\theta}\right]$$

Hence normal velocity at any point Q on the circle $= -\left(\frac{\partial\phi}{\partial r}\right)_{r=a} = \frac{m}{a}$

If we place a source of strength $-m$ at O, the normal velocity due to it at Q will be $-\frac{m}{a}$.

$\therefore$ "The image system for a source outside a circle consists of an equal source at the inverse point and an equal sink at the centre of the circle".

Image of a doublet with regard to a circle:



$OA\cdot OB = OA'\cdot OB' = a^2$

Let μ be the strength of the doublet AA'.

Then $\mu' = \frac{\mu a^2}{f^2}$ $\quad (OA = OA' = f)$

$\therefore$ "The image of a two dimensional doublet at A with regard to a circle is another doublet at the inverse point B, the axes of the doublets making supplementary angles with the radius OBA."

strength at B is $-m$

strength at B' is m

strength at O $= m - m$

$= 0$

Vortex motion:

All possible motions of an ideal liquid can be sub-divided into:

- Vortex free irrotational or potential flows whose characteristics can be derived from a velocity potential $\phi(x, y, z, t)$.
- Vortex or rotational motions.

Vorticity, Vorticity components (or components of spin):

$\bar{q}$ - velocity vector of a fluid particle.

Then vector quantity, $\Omega = \text{curl}\, \bar{q}$ - vorticity vector/ Vorticity.

Ω is the measure of the angular velocity of an infinitesimal element.

Let $\Omega = \Omega_x \hat{i} + \Omega_y \hat{j} + \Omega_z \hat{k}$

so that $(\Omega_x, \Omega_y, \Omega_z)$ are the vorticity components.

If $\bar{q} = u\hat{i} + v\hat{j} + w\hat{k}$

$$\Omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}; \quad \Omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}; \quad \Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

If $\Omega_x = \Omega_y = \Omega_z = 0$, then the motion is irrotational and the velocity function ϕ exists.

and if $\Omega_x, \Omega_y, \Omega_z$ are not all zero, the motion is rotational.

In the case of two-dimensional motion, $w = 0$ and u and v are functions of x and y only, then

$$\Omega_x = 0 \text{ and } \Omega_y = 0 \text{ and } \Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}.$$

Ω_z or spin axis is $\perp^{ar}$ to the plane of the motion.

Vortex line:

A vortex line is a curve drawn in fluid such that the tangent to it at every point is in the direction of the vorticity vector. The differential equations of the vortex lines are

$$\frac{dx}{\Omega_x} = \frac{dy}{\Omega_y} = \frac{dz}{\Omega_z}.$$

In 2-D motion, since the axis of ~~motion~~ rotation at every point is $\perp^{ar}$ to the plane of motion and hence the vortex lines must be all parallel lines.

Vortex tube and vortex filament (or vortex):

The tube formed by the vortex lines drawn through each point of a closed curve enclosing a tubular space in the fluid is called the 'vortex tube'.

A vortex tube of infinitesimal cross section is known as a 'vortex filament' or 'vortex'.

Properties of vortex tube (Helmholtz's vorticity theorems):

1. The product of the cross section and vorticity (or angular velocity) at any point on a vortex filament is constant along the filament and for all time when the body forces are conservative and the pressure is a single-valued function of density only.

Ω - vorticity vector $\qquad$ ω - angular velocity

$$\Omega = 2\omega = \text{curl } \bar{q}$$

$$\Omega_1 \delta s_1 = \Omega_2 \delta s_2 \quad \text{or} \quad \omega_1 \delta s_1 = \omega_2 \delta s_2 .$$

$\Omega \delta s$ or $2\omega \delta s$ is constant over every section δ of the vortex tube. It is also called strength of the vortex tube.

A vortex tube of strength unity is called a unit vortex tube.

2. Vortex lines and tubes cannot originate or terminate at internal points in a fluid.

$$\int_S \Omega \cdot d\bar{s} = \int_S \hat{n} \cdot \Omega \, ds = \int_V \nabla \cdot \Omega \, dV = 0$$

$\Rightarrow$ The total strength of vortex tubes emerging from S must be equal to that entering S. They must either form closed curves or have their extremities on the boundary of the liquid.

36 Q : If $u = (ax-by)/(x^2+y^2)$; $v = (ay+bx)/(x^2+y^2)$; $w=0$, investigate the nature of motion of the liquid. Also show that

(i) the velocity potential is $-\{(\frac{a}{2}) \times \log(x^2+y^2) + b\tan^{-1}(\frac{y}{x})\}$,

(ii) the pressure at any point (x,y) is given by

$$\frac{p}{\rho} = \text{const.} - \frac{1}{2}\,\frac{a^2+b^2}{x^2+y^2}$$

Sol : Given $u = (ax-by)/(x^2+y^2)$; $v = (ay+bx)/(x^2+y^2)$; $w=0$ — ①

$$\frac{\partial u}{\partial x} = \frac{a(x^2+y^2) - 2x(ax-by)}{(x^2+y^2)^2} = \frac{ay^2 - ax^2 + 2bxy}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial y} = \frac{a(x^2+y^2) - 2y(ay+bx)}{(x^2+y^2)^2} = \frac{ax^2 - ay^2 - 2bxy}{(x^2+y^2)^2}$$

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$, hence the equation of continuity is satisfied by ①. Therefore ① represents a possible motion.

Moreover ① represents a 2-D motion and hence vorticity components are given by

$$\Omega_x = 0, \quad \Omega_y = 0, \quad \Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \quad \text{— ②}$$

$$\frac{\partial u}{\partial y} = \frac{-b(x^2+y^2) - 2y(ax-by)}{(x^2+y^2)^2}$$

$$= \frac{-bx^2 + by^2 - 2ayx}{(x^2+y^2)^2}$$

$$\frac{\partial v}{\partial \cancel{y}x} = \frac{b(x^2+y^2) - 2x(ay+bx)}{(x^2+y^2)^2}$$

$$= \frac{-bx^2 + by^2 - 2axy}{(x^2+y^2)^2}$$

$\Rightarrow \Omega_z = 0$, Thus, $\Omega_x = \Omega_y = \Omega_z = 0$, showing that the motion is irrotational

(i) $d\phi = (\frac{\partial \phi}{\partial x})dx + (\frac{\partial \phi}{\partial y})dy = -u\,dx - v\,dy$

$$d\phi = -\left[\frac{ax-by}{x^2+y^2}\,dx + \frac{ay+bx}{x^2+y^2}\,dy\right]$$

$$= -\left[\frac{a}{2}\cdot\frac{2x\,dx+2y\,dy}{x^2+y^2} + b\,\frac{x\,dy-y\,dy}{x^2+y^2}\right]$$

$$d\phi = -\frac{a}{2}\,d\{\log(x^2+y^2)\} - b\,d\left\{\tan^{-1}\frac{y}{x}\right\}$$

$$\phi = -\left(\frac{a}{2}\right)\times\log(x^2+y^2) - b\tan^{-1}\left(\frac{y}{x}\right)$$

(ii) Let q be the fluid velocity. Then, pressure is given by $\frac{p}{\rho}+\frac{q^2}{2} = \text{constant}$

$$\frac{p}{\rho} = \text{const.} - \frac{q^2}{2}$$

$$q^2 = u^2+v^2 = \frac{(ax-by)^2+(ay+bx)^2}{(x^2+y^2)^2}$$

$$q^2 = \frac{a^2(x^2+y^2)+b^2(x^2+y^2)}{(x^2+y^2)^2} = \frac{a^2+b^2}{x^2+y^2}$$

$$\Rightarrow \frac{p}{\rho} = \text{Const.} - \frac{1}{2}\cdot\frac{a^2+b^2}{x^2+y^2}.$$

37 Q : Prove that the necessary and sufficient condition that the vortex lines may be at right angles to the stream lines are

$u, v, w = \mu\,(\partial\phi/\partial x, \partial\phi/\partial y, \partial\phi/\partial z)$, where μ, ϕ are functions of x, y, z, t. (2005)

Sol : Streamlines are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \quad — (1)$$

and vortex lines are given by

$$\frac{dx}{\Omega_x} = \frac{dy}{\Omega_y} = \frac{dz}{\Omega_z} \quad — (2)$$

(1) and (2) will be at right angles if

$$u\Omega_x + v\Omega_y + w\Omega_z = 0$$

But $\Omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}$, $\Omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}$, $\Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$

$$\Rightarrow u\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + v\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + w\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

which is the necessary and sufficient condition in order that $u\,dx + v\,dy + w\,dz$ may be a perfect differential. So we may write

$$u\,dx + v\,dy + w\,dz = \mu\, d\phi = \mu\left(\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz\right)$$

$$\therefore\ u = \mu\left(\frac{\partial \phi}{\partial x}\right),\quad v = \mu\left(\frac{\partial \phi}{\partial y}\right),\quad w = \mu\left(\frac{\partial \phi}{\partial z}\right).$$

38Q: In an incompressible fluid the vorticity at every point is constant in magnitude and direction; show that the components of velocity u, v, w are solutions of Laplace's equation. (2004) (2010)

Sol: Let $\Omega_x, \Omega_y, \Omega_z$ be the components of vorticity Ω so that $\Omega = (\Omega_x^2 + \Omega_y^2 + \Omega_z^2)^{1/2}$ and direction cosines of its direction are

$$\frac{\Omega_x}{\Omega},\ \frac{\Omega_y}{\Omega},\ \frac{\Omega_z}{\Omega}$$

Given $\Omega_x, \Omega_y, \Omega_z$ are all constants.

$$\text{Also, } \Omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z},\quad \Omega_y = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x},\quad \Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

$$\frac{\partial}{\partial z}\Omega_y = 0 = \frac{\partial^2 u}{\partial z^2} - \frac{\partial w}{\partial x \partial z};\quad \frac{\partial}{\partial y}\Omega_z = 0 = \frac{\partial v}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial}{\partial z}\Omega_y - \frac{\partial}{\partial y}\Omega_z = 0 = \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

Equation of continuity,

$$\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = -\frac{\partial u}{\partial x}$$

$$\Rightarrow \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2} = 0$$

Shows that u satisfies Laplace's equation. Similarly we can show that v and w also satisfy Laplace's equation.

39Q: If $u\,dx + v\,dy + w\,dz = d\theta + \lambda\, d\chi$, where θ, λ, χ are functions of x, y, z, t, prove that the vortex lines at any time are the lines of intersection of the surfaces $\lambda = \text{constant}$ and $\chi = \text{constant}$.

Sol : Given $u\,dx + v\,dy + w\,dz = d\theta + \lambda\, d\chi$

$$\therefore\ u\,dx + v\,dy + w\,dz = \frac{\partial\theta}{\partial x}dx + \frac{\partial\theta}{\partial y}dy + \frac{\partial\theta}{\partial z}dz + \frac{\partial\theta}{\partial t}dt$$
$$+\lambda\left(\frac{\partial\chi}{\partial x}dx + \frac{\partial\chi}{\partial y}dy + \frac{\partial\chi}{\partial z}dz + \frac{\partial\chi}{\partial t}dt\right)$$

$$\Rightarrow\quad u = \frac{\partial\theta}{\partial x} + \lambda\frac{\partial\chi}{\partial x}\ ,\quad v = \frac{\partial\theta}{\partial y} + \lambda\frac{\partial\chi}{\partial y}\ ,$$
$$w = \frac{\partial\theta}{\partial z} + \lambda\frac{\partial\chi}{\partial z}\ ,\quad 0 = \frac{\partial\theta}{\partial t} + \lambda\frac{\partial\chi}{\partial t}$$

Hence components of vorticity Ω are given by

$$\Omega_x = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = \frac{\partial}{\partial y}\left(\frac{\partial\theta}{\partial z} + \lambda\frac{\partial\chi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial\theta}{\partial y} + \lambda\frac{\partial\chi}{\partial y}\right)$$

$$\Rightarrow\ \Omega_x = \frac{\partial^2\theta}{\partial y\,\partial z} + \frac{\partial\lambda}{\partial y}\frac{\partial\chi}{\partial z} + \lambda\frac{\partial^2\chi}{\partial y\,\partial z} - \frac{\partial^2\theta}{\partial z\,\partial y} - \frac{\partial\lambda}{\partial z}\frac{\partial\chi}{\partial y}$$
$$-\lambda\frac{\partial^2\chi}{\partial z\,\partial y}$$

$$\Rightarrow\ \Omega_x = \frac{\partial\lambda}{\partial y}\frac{\partial\chi}{\partial z} - \frac{\partial\lambda}{\partial z}\frac{\partial\chi}{\partial y}$$

$$\Rightarrow\ \Omega_x = \begin{vmatrix} \partial\lambda/\partial y & \partial\lambda/\partial z \\ \partial\chi/\partial y & \partial\chi/\partial z \end{vmatrix}$$

Similarly, $\Omega_y = \begin{vmatrix} \partial\lambda/\partial z & \partial\lambda/\partial x \\ \partial\chi/\partial z & \partial\chi/\partial x \end{vmatrix}$ and $\Omega_z = \begin{vmatrix} \partial\lambda/\partial x & \partial\lambda/\partial y \\ \partial\chi/\partial x & \partial\chi/\partial y \end{vmatrix}$

$$\therefore\ \Omega_x\frac{\partial\lambda}{\partial x} + \Omega_y\frac{\partial\lambda}{\partial y} + \Omega_z\frac{\partial\lambda}{\partial z} = \begin{vmatrix} \frac{\partial\lambda}{\partial x} & \frac{\partial\lambda}{\partial y} & \frac{\partial\lambda}{\partial z} \\ \frac{\partial\lambda}{\partial x} & \frac{\partial\lambda}{\partial y} & \frac{\partial\lambda}{\partial z} \\ \frac{\partial\chi}{\partial x} & \frac{\partial\chi}{\partial y} & \frac{\partial\chi}{\partial z} \end{vmatrix} = 0$$

$$\therefore\ \Omega_x\frac{\partial\lambda}{\partial x} + \Omega_y\frac{\partial\lambda}{\partial y} + \Omega_z\frac{\partial\lambda}{\partial z} = 0 \qquad -\ (1)$$

Similarly $\Omega_x\frac{\partial\chi}{\partial x} + \Omega_y\frac{\partial\chi}{\partial y} + \Omega_z\frac{\partial\chi}{\partial z} = 0 \qquad -\ (2)$

Equations (1) and (2) show that the vortex lines at any time are the lines of intersection of the surfaces given by λ = constant and χ = constant

Rectilinear Vortices:

Vortex lines being straight and parallel, all vortex tubes are cylindrical, with generators $\perp^{ar}$ to the plane of motion. Such vortices are known as rectilinear vortices.

Consider a rectilinear vortex filament with its axis parallel to the axis of z. The motion being similar in all planes parallel to xy-plane, $w=0$.

Moreover u and v are independent of z i.e

$$\frac{\partial u}{\partial z}=0 \; ; \; \frac{\partial v}{\partial z}=0 \quad -(1)$$

If $\Omega_x, \Omega_y, \Omega_z$ are vorticity components.

$$\Omega_x=0; \; \Omega_y=0 \text{ and } \Omega_z=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y} \quad -(2)$$

Now the equations of lines of flow are

$$\frac{dx}{u}=\frac{dy}{v} \; ; \; v\,dx-u\,dy=0 \quad -(3)$$

The equation of continuity is $\frac{\partial v}{\partial y}=\frac{\partial(-u)}{\partial x}$ $-(4)$

Equation (4) shows that $v\,dx-u\,dy$ must be a perfect differential, $d\psi$ (say). Thus we have

$$v\,dx-u\,dy = d\psi = \frac{\partial \psi}{\partial x}dx+\frac{\partial \psi}{\partial y}dy$$

$$\Rightarrow u=\frac{-\partial\psi}{\partial y} \text{ and } v=\frac{\partial\psi}{\partial x} \quad -(5)$$

Then the lines of flow are given by $d\psi=0$

i.e $\psi=$ constant. Hence ψ is the stream function.

$$\Omega_z=\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2} \quad -(6)$$

Thus stream function ψ satisfies

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2} = \Omega_z \text{, on the vortex filament}$$
$$= 0 \text{, outside the filament}$$

Let $P(r,\theta)$ be any point outside the vortex filament. Since the motion outside the vortex is irrotational, the velocity potential ϕ exists such that

$$\frac{\partial\psi}{\partial r}=\frac{-1}{r}\frac{\partial\phi}{\partial\theta} \quad -(7)$$

writing $\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=0$ in polar coordinates outside the vortex filament

$$\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}=0 \quad -(8)$$

Considering symmetry about the origin, ψ will be independent of θ.

Equation (8) reduces to

$$\frac{d^2\psi}{dr^2}+\frac{1}{r}\frac{d\psi}{dr}=0 \quad \text{or} \quad \frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right)=0$$

Integrating $\quad r\left(\frac{d\psi}{dr}\right)=c$

$$\frac{c}{r}=\frac{d\psi}{dr} \quad -(9)$$

Integrating $\quad \psi=c\log r \quad -(10)$

Since ψ is independent of θ equation (7) gives

$$\frac{d\psi}{dr}=-\frac{1}{r}\left(\frac{\partial\phi}{\partial\theta}\right) \quad -(11)$$

From (9) and (11) $\quad \frac{c}{r}=-\frac{1}{r}\frac{\partial\phi}{\partial\theta}$

$$\Rightarrow \phi=-c\theta \quad -(12)$$

If $w=\phi+i\psi$ be the complex potential outside the filament, then we have

$$w=-c\theta+ic\log r=ic(\log r+i\theta)$$
$$=ic\log re^{i\theta}=ic\log z$$

Let K be the circulation in the circuit embracing the vortex. ~~thus~~ Then we have

$$K=\int_0^{2\pi}\left(-\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)r\,d\theta=c\int_0^{2\pi}d\theta=2\pi c$$

$$\Rightarrow c=\frac{k}{2\pi}$$

$$\therefore \phi=\frac{-k}{2\pi}\theta, \quad \psi=\frac{k}{2\pi}\log r$$

and $\quad w=\frac{ik}{2\pi}\log z \quad -(13)$

(OR) $\frac{k}{2\pi}=K$, Then $w=iK\log z=\frac{ik}{2\pi}\log z \quad -(14)$

k or K is called the strength of the vortex.

If there be a rectilinear vortex of strength k at $z' = x' + iy'$, then

$$w = \frac{ik}{2\pi} \log(z - z') \quad - (15)$$

$$(r')^2 = (x - x')^2 + (y - y')^2$$ * r' - distance b/w z and z' (not from origin)

$$\psi = \frac{k}{2\pi} \log r'$$

$$u = \frac{-\partial \psi}{\partial y} = \frac{-\partial \psi}{\partial r'} \cdot \frac{\partial r'}{\partial y} = \frac{-k}{2\pi r'} \cdot \frac{y - y'}{r'}$$

$$= \frac{-k(y - y')}{2\pi (r')^2} = \frac{-K(y - y')}{(r')^2}.$$

$$v = \frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial r'} \cdot \frac{\partial r'}{\partial x} = \frac{k}{2\pi r'} \cdot \frac{x - x'}{r'}$$

$$= \frac{k(x - x')}{2\pi (r')^2} = \frac{K(x - x')}{(r')^2}$$

$$q = \sqrt{u^2 + v^2} = \frac{k}{2\pi r'^2} \cdot \sqrt{(x - x')^2 + (y - y')^2}$$

$$= \frac{k}{2\pi r'} \quad \text{or} \quad \frac{K}{r'}.$$

Note:

In case of several rectilinear vortices.

Let there be a number of vortices of strength $k_1, k_2, k_3 \ldots$ situated at $z_1, z_2, z_3 \ldots$. Then complex potential is given by

$$w = \frac{ik_1}{2\pi} \log(z - z_1) + \frac{ik_2}{2\pi} \log(z - z_2) + \frac{ik_3}{2\pi} \log(z - z_3) + \ldots$$

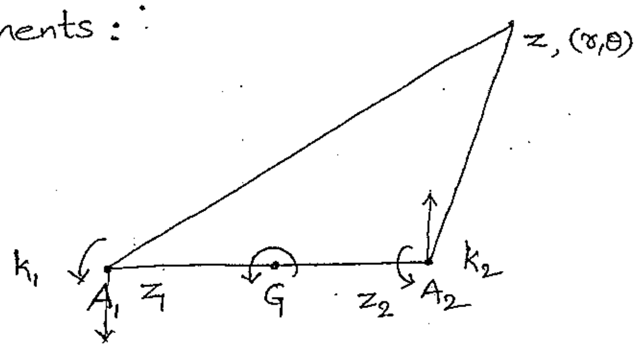
$$= \frac{i}{2\pi} \sum k_n \log(z - z_n)$$

$$\phi = \frac{-k_1}{2\pi} \theta_1 - \frac{k_2}{2\pi} \theta_2 - \frac{k_3}{2\pi} \theta_3 - \ldots$$

$$\psi = \frac{k_1}{2\pi} \log r_1 + \frac{k_2}{2\pi} \log r_2 + \frac{k_3}{2\pi} \log r_3 + \ldots$$

where, $r_n = |z - z_n|$ and $\theta_n = \arg(z - z_n)$, $n = 1, 2, 3, \ldots$

Two vortex filaments :



$$w = \frac{ik_1}{2\pi}\log(z-z_1) + \frac{ik_2}{2\pi}\log(z-z_2)$$

Due to presence of each other they start moving. Let u_1, v_1 be the components of the velocity q_1 of A_1 which is due to A_2 alone then,

$$u_1 - iv_1 = \left[\frac{1}{2\pi}\frac{ik_1}{z-z_1} + \left(-\frac{dw}{dz}\right)\right]_{z=z_1}$$

$$= \frac{-ik_2}{2\pi}\frac{1}{(z_1-z_2)}$$

$$q_1 = |u_1 - iv_1| = \frac{+k_2}{2\pi(A_1A_2)}$$

Similarly $\quad q_2 = |u_2 - iv_2| = \dfrac{k_1}{2\pi(A_1A_2)}$

And $\quad \dfrac{u_1 - iv_1}{k_2} = -\dfrac{(u_2 - iv_2)}{k_1}$

$$\Rightarrow (k_1u_1 + k_2u_2) - i(k_1v_1 + k_2v_2) = 0$$

$$\Rightarrow k_1u_1 + k_2u_2 = 0 \quad \text{and} \quad k_1v_1 + k_2v_2 = 0$$

$$\Rightarrow k_1 \cdot A_1G = k_2 \cdot A_2G \Rightarrow \frac{A_1G}{k_2} = \frac{A_2G}{k_1} = \frac{A_1A_2}{k_1+k_2}$$

$$\Rightarrow A_1G = \frac{k_2}{k_1+k_2}A_1A_2 \; ; \; A_2G = \frac{k_1}{k_1+k_2}A_1A_2$$

$q_1 = A_1G\,\omega \quad$ and $\quad q_1 = \dfrac{k_2}{2\pi(A_1A_2)}$

$$= \frac{k_2(A_1A_2)}{(k_1+k_2)} \cdot \frac{(k_1+k_2)}{(A_1A_2)^2}$$

$$= A_1G \cdot \omega$$

$$\Rightarrow \omega = \frac{k_1+k_2}{2\pi(A_1A_2)^2}$$

Note:

If $k_1 = k_2 = k$ and $A_1A_2 = 2a$

Then $q_1 = \frac{k}{4\pi a} = q_2$ and $\omega = \frac{k}{4\pi a^2}$

stream function

$$\psi = \frac{k}{2\pi}\log r_1 + \frac{k}{2\pi}\log r_2$$

$$\psi = \frac{k}{2\pi}\log (r_1 r_2).$$

The streamlines are given by $\psi =$ constant

i.e $r_1 r_2 =$ constant.

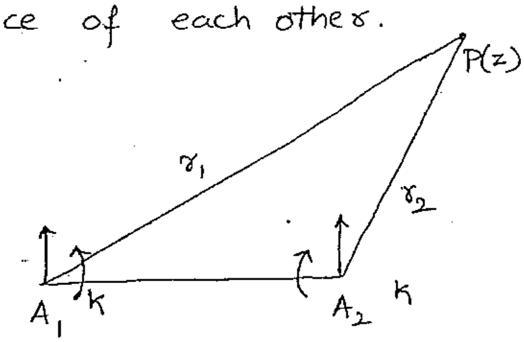
Vortex pair:

Two vortex filaments of strengths k and $-k$ form a vortex pair.

Let us consider rectilinear vortices of strengths k and $-k$ at $A_1\,(z=z_1)$ and $A_2\,(z=z_2)$. Then complex potential at any point $P(x,y)$ or $P(z)$ due to stationary system is

$$W = \frac{ik}{2\pi}\log(z-z_1) - \frac{ik}{2\pi}\log(z-z_2)$$

However the vortices situated at A_1 and A_2 would start moving due to presence of each other.



Let u_1, v_1 be the components of velocity q_1 of A_1 which is due to A_2 alone.

Then,
$$u_1 - iv_1 = \left[\frac{1}{2\pi}\frac{ik}{z-z_1} + \left(-\frac{dW}{dz}\right)\right]_{z=z_1}$$

$$= \frac{ik}{2\pi}\frac{1}{(z_1-z_2)}$$

$$q_1 = |u_1 - iv_1| = \frac{k}{2\pi|z_1-z_2|} = \frac{k}{2\pi(A_1A_2)}.$$

Similarly
$$u_2 - iv_2 = \frac{ik}{2\pi}\frac{1}{(z_2-z_1)}$$

$$q_2 = \frac{k}{2\pi(A_1A_2)}$$

$$\Rightarrow q_1 = q_2 = q \quad (\text{say})$$

The vortices situated at A_1 and A_2 move in the same direction perpendicular to A_1A_2 with uniform velocity q.

Let $w = \phi + i\psi$; $z = (x, y)$; $z_1 = (x_1, y_1)$; $z_2 = (x_2, y_2)$

$$\phi + i\psi = w = \frac{ik}{2\pi}\log\left[(x-x_1) + i(y-y_1)\right]$$

$$- \frac{ik}{2\pi}\log\left[(x-x_2) + i(y-y_2)\right]$$

Equating imaginary parts

$$\psi = \frac{k}{4\pi}\log r_1^2 - \frac{k}{4\pi}\log r_2^2$$

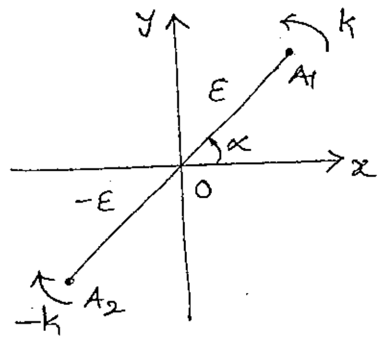
$$= \frac{k}{4\pi}\log\left(\frac{r_1}{r_2}\right)^2 = \frac{k}{2\pi}\log\frac{r_1}{r_2}$$

The stream lines are given by ψ = constant

i.e $\frac{r_1}{r_2}$ = constant.

$\frac{r_1}{r_2} = c$ form a system of co-axial circles having A_1 and A_2 as their limiting points.

Vortex doublet or dipole :



k and $-k$ δs apart

where $\delta s \to 0$ and $k \to \infty$

such that $\mu = \delta s \frac{k}{2\pi}$ is finite.

[Some may define $\mu = \delta s K$ or

$\mu = \delta s k$]

μ - strength of the vortex doublet.

Consider two vortex filaments of strengths k and $-k$ at A_1 $(z = \varepsilon e^{i\alpha})$ and $A_2 (z = -\varepsilon e^{i\alpha})$ so that $A_1A_2 = \delta s = 2\varepsilon$.

[A_1A_2 is the axis of the doublet]

Axis of the doublet is inclined at an angle α to the x-axis.

The complex potential is given by

$$w = \frac{ik}{2\pi}\left[\log(z - \varepsilon e^{i\alpha}) - \log(z + \varepsilon e^{i\alpha})\right]$$

$$w = \frac{ik}{2\pi}\left[\log\left(1 - \frac{\varepsilon e^{i\alpha}}{z}\right) - \log\left(1 + \frac{\varepsilon e^{i\alpha}}{z}\right)\right]$$

$$W = \frac{ik}{2\pi}\left[-\frac{\varepsilon e^{i\alpha}}{z} - \frac{\varepsilon^2 e^{2i\alpha}}{z^2} - \frac{\varepsilon^3 e^{3i\alpha}}{z^3} - \cdots - \left(\frac{\varepsilon e^{i\alpha}}{z} - \frac{\varepsilon^2 e^{2i\alpha}}{z^2} + \cdots\right)\right]$$

$$= \frac{-ik}{2\pi}\left(\frac{2\varepsilon e^{i\alpha}}{z}\right)$$ (to first order of approximation of small quantity ε)

$$W = \frac{-\varepsilon i k e^{i\alpha}}{\pi r e^{i\theta}}$$

Thus $$w = \frac{-\mu i}{r} e^{i(\alpha-\theta)}$$

$$\phi + i\psi = \frac{-\mu i}{r}\left[\cos(\alpha-\theta) + i\sin(\alpha-\theta)\right]$$

$$\therefore \phi = \frac{\mu}{r}\sin(\alpha-\theta) \quad \text{and} \quad \psi = \frac{-\mu}{r}\cos(\alpha-\theta).$$

Note:

If $\mu = Ua^2$ then $\psi = -\frac{Ua^2}{r}\sin\theta$, which is the stream function for a circular cylinder of radius a moving with velocity U along the x-axis and same as due to suitable vortex doublet placed at a centre with axis $\perp^{ar}$ to the direction of motion.

40.Q: Verify that the stream function ψ and velocity potential ϕ of a two-dimensional vortex flow satisfies the Laplace's equation.

Sol: We know that the stream function ψ and velocity potential ϕ of a 2-D vortex flow are given by

$$\phi = -\left(\frac{k\theta}{2\pi}\right)$$

$$\psi = \frac{k}{2\pi}\log r$$

$$\frac{\partial\phi}{\partial r} = 0, \quad \frac{\partial^2\phi}{\partial r^2} = 0, \quad \frac{\partial^2\phi}{\partial\theta^2} = 0$$

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0.$$

$$\frac{\partial\psi}{\partial r} = \frac{k}{2\pi r}, \quad \frac{\partial^2\psi}{\partial r^2} = \frac{-k}{2\pi r^2}, \quad \frac{\partial^2\psi}{\partial\theta^2} = 0$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{1}{r}\frac{\partial\psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} = 0.$$

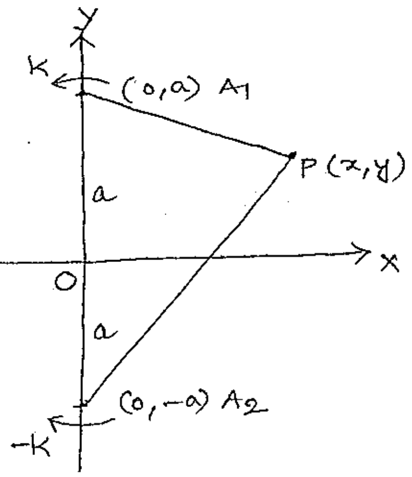
$\therefore \phi$ and ψ are satisfying the Laplace's equation.

41 Q: When an infinite liquid contains two parallel, equal and opposite rectilinear vortices at a distance $2a$, prove that the streamlines (paths of the fluid particles) relative to the vortex are given by the equation

$$\log \frac{x^2+(y-a)^2}{x^2+(y+a)^2} + \frac{y}{a} = c,$$

the origin being the middle point of the join, which is taken for axis of y.

Sol: Let $A_1\,(z = 0+ia)$ and $A_2\,(z = -ia)$ be the two rectilinear vortices of strengths k and $-k$ respectively.



$AA_1 = 2a$

Here we have a vortex pair and hence the vortex pair will move with a uniform velocity $\frac{k}{2\pi(A_1A_2)}$ or $\frac{k}{4\pi a}$ along the x-axis.

To determine the streamlines relative to the vortices, we must impose a velocity on the given system equal and opposite to the velocity $\frac{k}{4\pi a}$ of motion of the vortex pair. Accordingly, we add a term $\frac{kz}{4\pi a}$ to the complex potential of the vortex pair.

Note that $\quad -\frac{d}{dz}\left(\frac{kz}{4\pi a}\right) = \frac{-k}{4\pi a}$

and hence the term added is justified. So, for the case under consideration, the complex potential is given by

$$w = \frac{ik}{2\pi}\log(z-ia) - \frac{ik}{2\pi}\log(z+ia) + \frac{kz}{4\pi a}$$

Equating the imaginary parts, we have

$$\psi = \frac{k}{4\pi}\log\left[x^2+(y-a)^2\right] - \frac{k}{4\pi}\log\left[x^2+(y+a)^2\right] + \frac{ky}{4\pi a}$$

$$\Rightarrow \psi = \frac{k}{4\pi}\left[\log\left(\frac{x^2+(y-a)^2}{x^2+(y+a)^2}\right) + \frac{y}{a}\right]$$

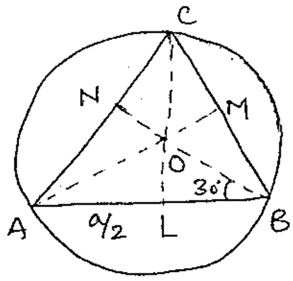
Hence, the required streamlines are given by ψ = constant. i.e $\log\frac{x^2+(y-a)^2}{x^2+(y+a)^2} + \frac{y}{a} = c$.

Note:

Stagnation points are given by $\frac{dW}{dz} = 0$.

42 Q: Three parallel rectilinear vortices of the same strength K and in the same sense meet any plane perpendicular to them in an equilateral triangle of side a. Show that all the vortices move round the same cylinder with uniform speed in time $2\pi a^2/3K$.

Sol: Let r be the radius of the circumcircle of the equilateral triangle ABC. Let O be the circumcentre.



From figure

$$r = OB = \frac{a}{2}\sec 30^\circ = \frac{a}{\sqrt{3}}.$$

There are three vortices of strength $(K = \frac{k}{2\pi})$ at A, B, C which are situated at the points

$$z_m = re^{2m\pi i/3}, \quad m = 1, 2, 3.$$

Then the complex potential of the vortices at A, B, C is given by

$$w = \frac{ik}{2\pi}\left[\log(z - re^{2\pi i/3}) + \log(z - re^{4\pi i/3}) + \log(z - re^{6\pi i/3})\right]$$

$$= \frac{ik}{2\pi}\log(z^3 - r^3).$$

Then the velocity induced at $z = re^{6\pi i/3} = r$, by others is given by

$$u_3 - iv_3 = \frac{-d}{dz}\left[\frac{ik}{2\pi}\log(z^3 - r^3) - \frac{ik}{2\pi}(z - r)\right]$$

$$= \frac{-ik}{2\pi}\,\frac{2z + r}{z^2 + zr + r^2}$$

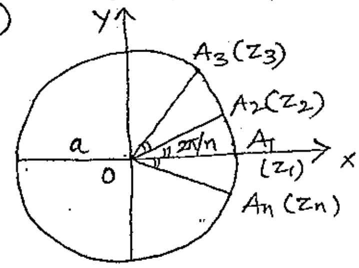
$$[u_3 - iv_3]_{z=r} = \frac{-ik}{2\pi}\,\frac{1}{r}$$

$$q_3 = |u_3 - iv_3| = \frac{k}{2\pi r} = \frac{K}{r}.$$

The required time $= \dfrac{\text{Circumference of the circumcircle}}{\text{velocity at } z = r}$

$$\Rightarrow T = \frac{2\pi a/\sqrt{3}}{K/r} = \frac{2\pi a/\sqrt{3}}{(K/a)\sqrt{3}} = \frac{2\pi a^2}{3K}. \quad \left(\text{as } r = \frac{a}{\sqrt{3}}\right).$$

43Q: If n rectilinear vortices of the same strength k are symmetrically arranged as generators of a circular cylinder of radius a in an infinite liquid, prove that the vortices will move round the cylinder uniformly in time $8\pi^2a^2/(n-1)k$, and find the velocity of any part of the liquid. (2013)



Sol: Let $A_1, A_2, \ldots A_n$ are n rectilinear vortices each of strength k be situated at points $z_m = ae^{2\pi i m/n}$, $m = 0, 1, 2, \ldots (n-1)$ on the circle of radius 'a' and centre at the origin 'O'.

Then the complex potential due to thes n vortices is given by

$$w = \frac{ik}{2\pi} \sum_{m=0}^{n-1} \log\left(z - ae^{2\pi i m/n}\right)$$

$$w = \frac{ik}{2\pi} \log \prod_{m=0}^{n-1}\left(z - ae^{2\pi i m/n}\right) = \frac{ik}{2\pi}\log(z^n - a^n)$$

Now the fluid velocity q at any point out of all the n vortices is given by

$$q = \left|-\frac{dw}{dz}\right| = \left|\frac{ik}{2\pi}\frac{nz^{n-1}}{z^n - a^n}\right| = \frac{kn}{2\pi}\left|\frac{z^{n-1}}{z^n - a^n}\right|$$

Again the velocity induced at $A_1 (z = a)$, by others is given by the complex potential

$$w' = \frac{ik}{2\pi}\log(z^n - a^n) - \frac{ik}{2\pi}\log(z-a) = \frac{ik}{2\pi}\log\frac{z^n - a^n}{z-a}.$$

$$\therefore\ w' = \frac{ik}{2\pi}\log\left(z^{n-1} + z^{n-2}a + \cdots + za^{n-2} + a^{n-1}\right)$$

$$\frac{dw'}{dz} = \frac{ik}{2\pi}\,\frac{(n-1)z^{n-2} + (n-2)z^{n-3}a + \cdots + a^{n-2}}{z^{n-1} + z^{n-2}a + \cdots + a^{n-1}}$$

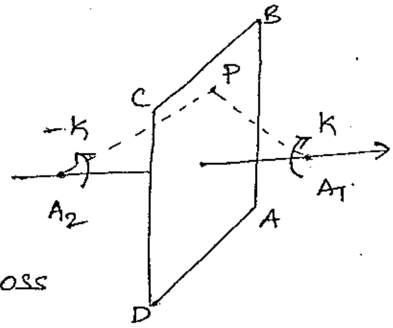
$$\Rightarrow \left(\frac{dw'}{dz}\right)_{z=a} = \frac{ik}{2\pi}\,\frac{(n-1)+(n-2)+\cdots+2+1}{na} = \frac{ik(n-1)}{4\pi a}$$

$$u_1 - iv_1 = \left[-\frac{dw'}{dz}\right]_{z=a} = \frac{-ik(n-1)}{4\pi a}$$

$\Rightarrow u_1 = 0$ and $v_1 = \dfrac{k(n-1)}{4\pi a}$. If q_r and q_θ be the radial and transverse components of the velocity at $z = a$. Then $q_r = 0$ and $q_\theta = k(n-1)/4\pi a$. Due to symmetry of the problem, it follows that each vortex moves with same transverse velocity.

$$\therefore \text{Required time} = \frac{2\pi a}{k(n-1)/4\pi a} = \frac{8\pi^2a^2}{(n-1)k}.$$

Image of a vortex filament in a plane:



$$\psi = \frac{ik}{2\pi} \log \frac{r_1}{r_2}$$

At P, $r_1 = r_2$

$\Rightarrow \psi = 0$

$\Rightarrow$ There would be no flow across the plane ABCD.

Hence the motion would remain unchanged if the plane were made a rigid barrier.

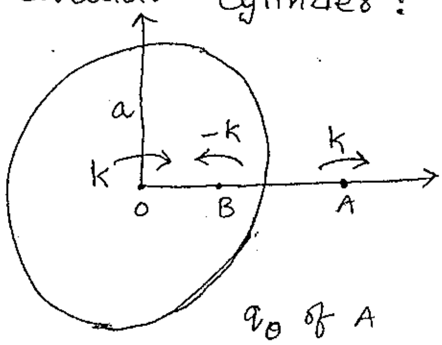
$\therefore$ "The image of a vortex filament in a plane to which it is parallel is an equal and opposite vortex filament at its optical image in the plane".

* Let $A_1A_2 = 2a$.

Then the uniform velocity of vortex filament A_1 parallel to the plane is given by $\frac{k}{4\pi a}$.

Moreover the velocity midway A_1 and A_2 due to both the vortices is $\frac{k}{\pi a}$. Thus the vortex moves parallel to the plane with $1/4$ th of the velocity of the liquid at the boundary.

Image of a vortex outside a circular cylinder:



$OA = c$

$OB = a^2/c$

The image system of a vortex k outside a circular cylinder consists of a vortex of strength $-k$ at the inverse point and a vortex of strength k at the centre.

q_θ of A

$$= \frac{k}{2\pi c} \frac{a^2}{c^2 - a^2}.$$

A moves round the cylinder.

(or)

The image system of a vortex k outside the circular cylinder consists of a vortex of strength $-k$ at the inverse point and a circulation of strength k round the cylinder.

Image of a vortex inside a circular cylinder:

The image of a vortex inside a circular cylinder would be an equal and opposite vortex at the inverse point.

$$OA \cdot OB = a^2$$

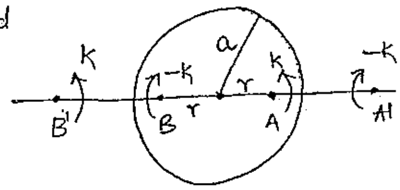
$$OB = a^2/c$$

$$q_\theta \text{ of } A = \frac{k}{2\pi} \frac{c}{(a^2-c^2)}$$

$$\omega = \frac{q_\theta}{OA} = \frac{q}{c} = \frac{k}{2\pi(a^2-c^2)}$$

44 Q: A vortex pair is situated within a cylinder. show that it will remain at rest if the distance of either from the centre is given by $(\sqrt{5}-2)^{1/2}a$, where a is the radius of the cylinder.

sol: Let a vortex pair be situated at A, B where $OA = OB = r$. Let A' and B' be the inverse points of A and B respectively with regard to the circular cylinder so that $OA' = a^2/r = OB'$.



The vortex will remain at rest if its velocity due to other three vortices be zero i-e

$$\frac{k}{2\pi}\left[\frac{1}{AA'} - \frac{1}{BA} + \frac{1}{B'A}\right] = 0$$

$$\Rightarrow \frac{1}{(a^2/r)-r} - \frac{1}{2r} + \frac{1}{(a^2/r)+r} = 0$$

$$\Rightarrow \frac{r}{a^2-r^2} + \frac{r}{a^2+r^2} - \frac{1}{2r} = 0$$

$$\Rightarrow r^4 + 4a^2r^2 - a^4 = 0$$

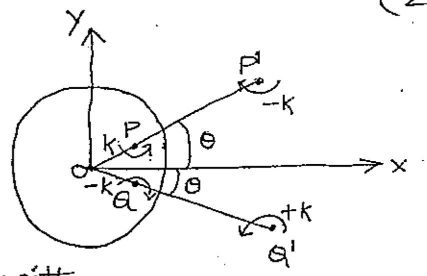
$$\Rightarrow (r^2/a^2)^2 + 4(r^2/a^2) - 1 = 0$$

$$\Rightarrow r^2/a^2 = \sqrt{5}-2$$

$$\therefore r = (\sqrt{5}-2)^{1/2}a.$$

45 Q : When a pair of equal and opposite rectilinear vortices are situated in a long circular cylinder at equal distances from its axis, show that the path of each vortex is given by the equation $(r^2\sin^2\theta - b^2)(r^2-a^2)^2 = 4a^2b^2r^2\sin^2\theta$, θ being measured from the line through the centre perpendicular to the joint of the vortices.
(2010)

Sol: Let k be the strength of the vortex at $P(r,\theta)$ and $-k$ at $Q(r,-\theta)$.



Let P' and Q' be the inverse points of P and Q respectively with regard to the circular cylinder $|z|=a$ so that $OP' = a^2/r = OQ'$.

Then the image of vortex k at P is a vortex $-k$ at P' and the image of vortex $-k$ at Q is a vortex k at Q'.

Hence the complex potential of the system of four vortices is given by

$$w = \frac{ik}{2\pi}\left[\log(z-re^{i\theta}) - \log\left(z-\frac{a^2}{r}e^{i\theta}\right) - \log(z-re^{-i\theta}) + \log\left(z-\frac{a^2}{r}e^{-i\theta}\right)\right]$$

$$W = \left(\frac{ik}{2\pi}\right)\log(z-re^{i\theta}) + w'$$

Since the motion of vortex P is solely due to other vortices, the complex potential of the vortex at P is given by the value of w' at $z=re^{i\theta}$.

$$\therefore [w']_{z=re^{i\theta}} = \frac{ik}{2\pi}\left[-\log\left[z-\frac{a^2}{r}e^{i\theta}\right] - \log(z-re^{-i\theta}) + \log\left[z-\frac{a^2}{r}e^{-i\theta}\right]\right]_{z=re^{i\theta}}$$

$$\therefore \phi + i\psi = \frac{-ik}{2\pi}\left[\log\left(re^{i\theta}-\frac{a^2}{r}e^{i\theta}\right) - \log(re^{i\theta}-re^{-i\theta}) + \log\left(re^{i\theta}-\frac{a^2}{r}e^{-i\theta}\right)\right]$$

$$\therefore \psi = \frac{-k}{2\pi}\left[\log\left(r-\frac{a^2}{r}\right) + \log(2r\sin\theta) - \frac{1}{2}\log\left\{r^2+\frac{a^4}{r^2}-2r\cdot\frac{a^2}{r}\cos 2\theta\right\}\right]$$

$$\therefore \quad \psi = \frac{-k}{4\pi} \log \frac{(r - a^2/r)^2 (2r\sin\theta)^2}{r^2 + \frac{a^4}{r^2} - 2a^2\cos 2\theta}.$$

So the required stream lines are given by

$$\psi = \text{constant, i.e,} \quad \frac{(r^2 - a^2)^2 r^2 \sin^2\theta}{r^4 + a^4 - 2a^2r^2\cos 2\theta} = b^2 \text{ (say)}$$

$$\Rightarrow b^2(r^4 + a^4 - 2a^2r^2\cos 2\theta) = r^2(r^2 - a^2)^2 \sin^2\theta.$$

$$\Rightarrow b^2\left[(r^2 - a^2)^2 + 2a^2r^2(1 - \cos 2\theta)\right] = r^2(r^2 - a^2)^2 \sin^2\theta.$$

$$\Rightarrow 2a^2b^2r^2(1 - \cos 2\theta) = (r^2 - a^2)^2 (r^2\sin^2\theta - b^2)$$

$$\Rightarrow 4a^2b^2r^2\sin^2\theta = (r^2\sin^2\theta - b^2)(r^2 - a^2)^2.$$

Infinite number of parallel vortices of the same strength in one row:

The motion due to a set of line vortices of strength k at points $z = \pm na$ $(n = 0, 1, 2, 3, \ldots)$ is given by the relation

$$w = \frac{ik}{2\pi} \log \sin\left(\frac{\pi z}{a}\right)$$

Let there be $(2n+1)$ vortices of strength k each situated at the points $(0,0)$ $(\pm a, 0)$, $(\pm 2a, 0), \ldots, (\pm na, 0)$. The complex potential of these $(2n+1)$ vortices at any point z is given by

$$w_{2n+1} = \frac{ik}{2\pi}\left[\log z + \log(z-a) + \log(z+a) + \log(z-2a) + \log(z+2a) + \cdots + \log(z-na) + \log(z+na)\right]$$

$$w_{2n+1} = \frac{ik}{2\pi} \log\left[z(z^2 - a^2)(z^2 - 2^2a^2) \cdots (z^2 - n^2a^2)\right]$$

$$w_{2n+1} = \frac{ik}{2\pi} \log\left[\frac{\pi z}{a}\left(1 - \frac{z^2}{a^2}\right)\left(1 - \frac{z^2}{2^2a^2}\right) \cdots \left(1 - \frac{z^2}{n^2a^2}\right)\right] + \frac{ik}{2\pi}\log\left[(-1)^n \frac{a}{\pi} a^2 \cdot 2^2a^2 \cdots n^2a^2\right] \quad \text{—①}$$

Neglecting the second term on R.H.S of ① being constant for the purpose of complex potential. Hence the complex potential given by ① may be also written as

$$w_{2n+1} = \frac{ik}{2\pi} \log\left[\frac{\pi z}{a}\left(1 - \frac{z^2}{a^2}\right)\left(1 - \frac{z^2}{2^2a^2}\right) \cdots \left(1 - \frac{z^2}{n^2a^2}\right)\right] \quad \text{—②}$$

Making $n \to \infty$ in ②, the complex potential w of the entire system of vortices at points $z = \pm na$ $(n = 0, 1, 2, 3, \ldots)$ is given by

$$w = \frac{ik}{2\pi} \log\left[\frac{\pi z}{a}\left(1 - \frac{z^2}{a^2}\right)\left(1 - \frac{z^2}{2^2a^2}\right) \cdots\right] \quad \text{③}$$

But $\sin\theta = \theta\left(1 - \theta^2/\pi^2\right)\left(1 - \theta^2/2^2\pi^2\right) \cdots$

Let $\theta = \frac{\pi z}{a}$ i.e $\frac{z}{a} = \theta/\pi$

$$\Rightarrow \sin\left(\frac{\pi z}{a}\right) = \frac{\pi z}{a}\left(1 - \frac{z^2}{a^2}\right)\left(1 - z^2/2^2a^2\right) \cdots \quad \text{④}$$

From ③ and ④

$$w = \frac{ik}{2\pi} \log\left[\sin\left(\frac{\pi z}{a}\right)\right]$$

Let u and v be the velocity components of any point of the fluid not occupied by any vortex filament. Then we have

$$u - iv = -\frac{dw}{dz} = \frac{-ik}{2a} \cot\frac{\pi z}{a}.$$

$$u - iv = \frac{-ik}{2a} \frac{\cos \pi(x+iy)/a}{\sin \frac{\pi(x+iy)}{a}} \times \frac{\sin \pi(x \mp iy)/a}{\sin \frac{\pi(x \mp iy)}{a}}.$$

$$= \frac{ik}{2a} \frac{\sin(2\pi x/a) - i \sinh(2\pi y/a)}{\cosh\left(\frac{2\pi y}{a}\right) - \cos\left(\frac{2\pi x}{a}\right)}$$

$\sin(i\theta) = i\sinh\theta$

$\cos i\theta = \cosh\theta.$

$$\therefore u = \frac{-k}{2a} \frac{\sinh(2\pi y/a)}{\cosh\left(\frac{2\pi y}{a}\right) - \cos\left(\frac{2\pi x}{a}\right)}$$

$$v = \frac{-k}{2a} \frac{\sin(2\pi x/a)}{\cosh\left(\frac{2\pi y}{a}\right) - \cos\left(\frac{2\pi x}{a}\right)}$$

Let q_0 be the velocity of vortex at the origin

$$q_0 = -\left\{\frac{d}{dz}\left[\frac{ik}{2\pi}\log\sin\frac{\pi z}{a} - \frac{ik}{2\pi}\log z\right]\right\}_{z=0}$$

$$q_0 = \frac{-ik}{2\pi}\left[\frac{\pi}{a}\cot\frac{\pi z}{a} - \frac{1}{z}\right]_{z=0}$$

$$q_0 = \frac{-ik}{2\pi} \lim_{z \to 0} \left[\frac{\pi \cos(\pi z/a)}{a \sin(\pi z/a)} - \frac{1}{z}\right]$$

$$q_0 = \frac{-ik}{2\pi a} \lim_{z \to 0} \left[\frac{\pi z \cos(\pi z/a) - a\sin(\pi z/a)}{z \sin(\pi z/a)} \right] \quad \frac{0}{0} \text{ for.}$$

using L' Hospital's rule

$$q_0 = \frac{-ik}{2\pi a} \times 0 = 0$$

Hence the vortex at origin is at rest. Similarly, it can be shown that the remaining vortices are also at rest. Thus we find that the vortex row induces no velocity on itself.

For streamlines

$$\phi + i\psi = \frac{ik}{2\pi} \log\left[\sin\left(\frac{\pi z}{a}\right)\right] \quad \text{—ⓐ}$$

$$\phi + i\psi = \frac{ik}{2\pi} \log \sin\left\{\frac{\pi}{a}(x+iy)\right\}$$

$$\therefore \phi - i\psi = \frac{-ik}{2\pi} \log \sin\left\{\frac{\pi}{a}(x-iy)\right\} \quad \text{—ⓑ}$$

$$ⓐ - ⓑ \Rightarrow 2i\psi = \frac{ik}{2\pi}\left[\log \sin\frac{\pi z}{a} + \log \sin\frac{\pi \bar{z}}{a}\right]$$

$$\psi = \frac{k}{4\pi} \log\left[\sin\frac{\pi}{a}(x+iy) \cdot \sin\frac{\pi}{a}(x-iy)\right]$$

$$\Rightarrow \psi = \frac{k}{4\pi} \log\left[\frac{1}{2}\left(\cosh\frac{2\pi y}{a} - \cos\frac{2\pi x}{a}\right)\right]$$

$$\therefore \psi = \frac{k}{4\pi} \log\left[\cosh\frac{2\pi y}{a} - \cos\frac{2\pi x}{a}\right] \quad \text{(omitting the irrelevent constant)}$$

The required streamlines are given by $\psi = \text{const.}$

i.e, $$\cosh\frac{2\pi y}{a} - \cos\frac{2\pi x}{a} = \text{constant}$$

when y is very large, second term can be omitted

$\Rightarrow$ The resultant streamlines are given by

$$\cosh\frac{2\pi y}{a} = \text{constant},$$

$$\Rightarrow y = \text{Constant}.$$

Therefore: At a great distance from the row of vortices, the streamlines are parallel to the row.

46 Q: An infinite row of equidistant rectilinear vortices are at a distance a apart. The vortices are of the same numerical strength k but they are alternately of opposite signs. Find the complex function that determines the velocity potential and the stream function. (2011)

Sol: Let the row of vortices be taken along the x-axis. Let there be vortices of strength k ~~at~~ each at the points $(0,0), (\pm 2a, 0), (\pm 4a, 0), ----$ and those of strength $-k$ each at the points $(\pm a, 0), (\pm 3a, 0), (\pm 5a, 0), ----$

The complex potential of the entire system is given by

$$W = \frac{ik}{2\pi}\left[\log z + \log(z-2a) + \log(z+2a) + \log(z-4a) + ----\right]$$
$$- \frac{ik}{2\pi}\left[\log(z-a) + \log(z+a) + \log(z-3a) + \log(z+3a) + ---\right]$$

$$W = \frac{ik}{2\pi}\log \frac{z(z^2-2^2a^2)(z^2-4^2a^2)---}{(z^2-a^2)(z^2-3^2a^2)----}$$

$$W = \frac{ik}{2\pi}\log \frac{2\left[\frac{\pi z}{2a}\left[1-\left(\frac{z}{2a}\right)^2\right]\left[1-\left(\frac{z}{4a}\right)^2\right]---\right]^2}{\frac{\pi z}{2a}\left[1-\left(\frac{z}{a}\right)^2\right]\left[1-\left(\frac{z}{2a}\right)^2\right]\left[1-\left(\frac{z}{3a}\right)^2\right]---} + \text{a constant}$$

$$\Rightarrow W = \frac{ik}{2\pi}\log\frac{2\sin^2\left(\frac{\pi z}{2a}\right)}{\sin\frac{\pi z}{a}} = \frac{ik}{2\pi}\log\frac{2\sin^2\frac{\pi z}{2a}}{2\sin\frac{\pi z}{2a}\cos\frac{\pi z}{2a}}$$

$$\Rightarrow W = \frac{ik}{2\pi}\log\frac{\sin\frac{\pi z}{2a}}{\cos\frac{\pi z}{2a}}$$

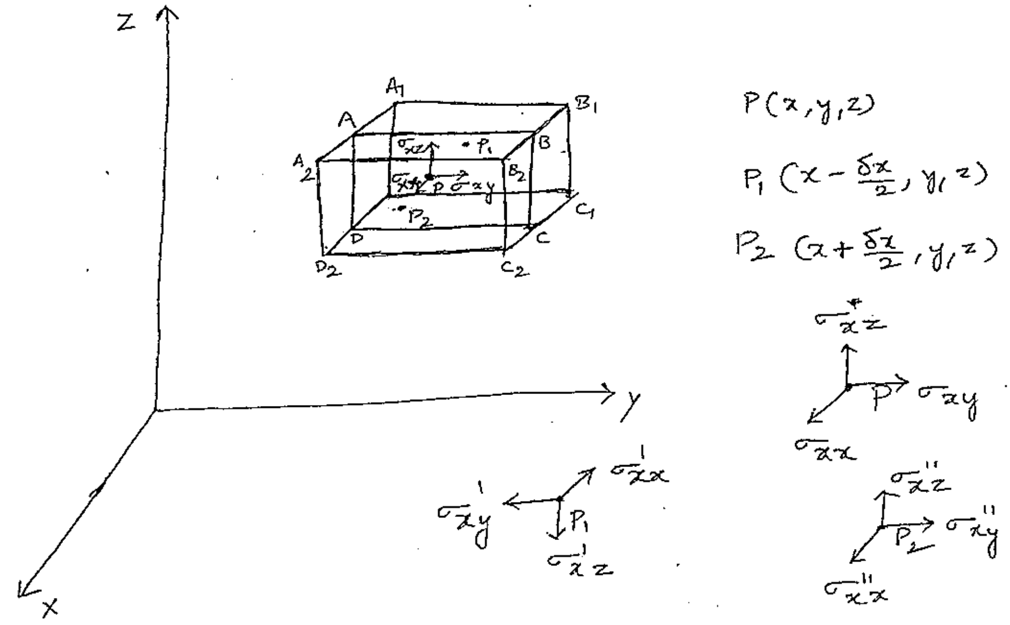
$$\Rightarrow w = \frac{ik}{2\pi}\log\tan\left(\frac{\pi z}{2a}\right).$$

$$\phi + i\psi = \frac{ik}{2\pi}\log\tan\frac{\pi}{2a}(x+iy)$$

$$\therefore \phi - i\psi = \frac{-ik}{2\pi}\log\tan\frac{\pi}{2a}(x-iy)$$

$$\Rightarrow \psi = \frac{k}{4\pi}\log\frac{\cosh\frac{\pi y}{a} - \cos\frac{\pi x}{a}}{\cosh\frac{\pi y}{a} + \cos\frac{\pi x}{a}}.$$

Navier - Stokes equations for a viscous fluid :



Let $P(x,y,z)$ be the centre and edges of lengths $\delta x, \delta y, \delta z$ parallel to fixed co-ordinate axes of an elementary parallelopiped.

Let the mass of element $= (\rho\, \delta x\, \delta y\, \delta z)$ be constant.

Let $P_1\left(x - \delta x/2, y, z\right)$ and $P_2\left(x + \frac{\delta x}{2}, y, z\right)$ be the points which are centers of the planes perpendicular to x-axis and surfaces of the parallelopiped.

At P, the force components parallel to OX, OY, OZ on the rectangular surface ABCD of area $\delta y\, \delta z$ through P are

$$[\sigma_{xx}\, \delta y\, \delta z,\ \sigma_{xy}\, \delta y\, \delta z,\ \sigma_{xz}\, \delta y\, \delta z]$$

($\hat{i}$ is unit normal of ABCD)

Then force components at P_2 are

$$\left[\left(\sigma_{xx} + \frac{\delta x}{2}\frac{\partial \sigma_{xx}}{\partial x}\right)\delta y\, \delta z,\ \left(\sigma_{xy} + \frac{\delta x}{2}\frac{\partial \sigma_{xy}}{\partial x}\right)\delta y\, \delta z,\right.$$

$$\left.\left(\sigma_{xz} + \frac{\delta x}{2}\frac{\partial \sigma_{xz}}{\partial x}\right)\delta y\, \delta z\right]$$

($\hat{i}$ is the unit normal measured outwards from the fluid corresponding to the surface $A_2B_2C_2D_2$)

At P_1, since $-\hat{i}$ is the unit normal measured outwards from the fluid, the force components at P_1 are

$$\left[-\left(\sigma_{xx} - \frac{\delta x}{2}\frac{\partial \sigma_{xx}}{\partial x}\right)\delta y\, \delta z,\ -\left(\sigma_{xy} - \frac{\delta x}{2}\frac{\partial \sigma_{xy}}{\partial x}\right)\delta y\, \delta z,\right.$$

$$\left.-\left(\sigma_{xz} - \frac{\delta x}{2}\frac{\partial \sigma_{xz}}{\partial x}\right)\delta y\, \delta z\right]$$

Hence the forces on the parallel planes $A_2B_2C_2D_2$ and $A_1B_1C_1D_1$ passing through P_1 and P_2 are equivalent to a single force at P with components

$$\left[\frac{\partial\sigma_{xx}}{\partial x}\delta x\,\delta y\,\delta z,\ \frac{\partial\sigma_{xy}}{\partial x}\delta x\,\delta y\,\delta z,\ \frac{\partial\sigma_{xz}}{\partial x}\delta x\,\delta y\,\delta z\right]$$

together with couples whose moments (to the third order of smallness) are

$-\sigma_{xz}\,\delta x\,\delta y\,\delta z$ about oy and $\sigma_{xy}\,\delta x\,\delta y\,\delta z$ about oz.

Similarly,

the forces on the parallel planes $\perp^{ar}$ to y-axis are equivalent to a single force at P with components

$$\left[\frac{\partial\sigma_{yx}}{\partial y}\delta x\,\delta y\,\delta z,\ \frac{\partial\sigma_{yy}}{\partial y}\delta x\,\delta y\,\delta z,\ \frac{\partial\sigma_{yz}}{\partial y}\delta x\,\delta y\,\delta z\right]$$

together with couples whose moments are

$-\sigma_{yx}\,\delta x\,\delta y\,\delta z$ about oz and $\sigma_{yz}\,\delta x\,\delta y\,\delta z$ about ox.

Again, the forces on the parallel planes $\perp^{ar}$ to z-axis are equivalent to a single force at P with components

$$\left[\frac{\partial\sigma_{zx}}{\partial z}\delta x\,\delta y\,\delta z,\ \frac{\partial\sigma_{zy}}{\partial z}\delta x\,\delta y\,\delta z,\ \frac{\partial\sigma_{zz}}{\partial z}\delta x\,\delta y\,\delta z\right]$$

together with couples whose moments are

$-\sigma_{zy}\,\delta x\,\delta y\,\delta z$ about ox and $\sigma_{zx}\,\delta x\,\delta y\,\delta z$ about oy.

Thus, the surface forces on all six faces of the rectangular parallelopiped are equivalent to a single force at P whose components are

$$\left[\left(\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\sigma_{yx}}{\partial y}+\frac{\partial\sigma_{zx}}{\partial z}\right)\delta v,\ \left(\frac{\partial\sigma_{xy}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\sigma_{zy}}{\partial z}\right)\delta v,\ \left(\frac{\partial\sigma_{xz}}{\partial x}+\frac{\partial\sigma_{yz}}{\partial y}+\frac{\partial\sigma_{zz}}{\partial z}\right)\delta v\right]$$

together with a vector couple having components

$$\left[(\sigma_{yz}-\sigma_{zy})\,\delta v,\ (\sigma_{zx}-\sigma_{xz})\,\delta v,\ (\sigma_{xy}-\sigma_{yx})\,\delta v\right],$$

where $\delta v = \delta x\,\delta y\,\delta z$.

3

Let $\bar{q} = u\hat{i} + v\hat{j} + w\hat{k}$ and $B = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$ be the velocity of the fluid at $P(x,y,z)$ at any time t and external body force at P per unit mass respectively.

The body force on the element has components

$$[B_x \rho \,\delta v,\ B_y \rho\, \delta v,\ B_z \rho\, \delta v].$$

Taking account of surface forces and body forces,
2. we find the total force component in the $\hat{i}$-direction on the element of the fluid is

$$\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z}\right)\delta v + B_x \rho\, \delta v$$

Since the mass $\rho\,\delta v$ of the element is treated to be constant, the equation of motion of the element in the $\hat{i}$-direction (i.e along x-axis) is

$$(\rho \delta v)\frac{Du}{Dt} \text{ or } \rho\,\delta v \frac{du}{dt} = \left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z}\right)\delta v + B_x \rho\,\delta v$$

$$\Rightarrow \quad \rho \frac{du}{dt} = \rho B_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z}$$

Thus by cyclic permutation we obtain three equations of motion in the i, j, k directions (i.e ox, oy, oz) :

$$\rho \frac{du}{dt} = \rho B_x + \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z}$$

$$\rho \cdot \frac{dv}{dt} = \rho B_y + \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z}$$

$$\rho \frac{dw}{dt} = \rho B_z + \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z}$$

The constitutive equations for a Newtonian (viscous) compressible fluied are given by

$$\sigma_{xx} = 2\mu\left(\frac{\partial u}{\partial x}\right) - (2\mu/3) \times \nabla\cdot\bar{q} - p$$

$$\sigma_{yy} = 2\mu\left(\frac{\partial v}{\partial y}\right) - \left(\frac{2\mu}{3}\right)\nabla\cdot\bar{q} - p$$

$$\sigma_{zz} = 2\mu\left(\frac{\partial w}{\partial z}\right) - \left(\frac{2\mu}{3}\right)\nabla\cdot\bar{q} - p$$

$$\sigma_{xy} = \sigma_{yx} = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

$$\sigma_{yz} = \sigma_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)$$

$$\Rightarrow \rho \frac{du}{dt} = \rho B_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[\mu\left\{2\frac{\partial u}{\partial x} - \frac{2}{3}(\nabla\cdot\bar{q})\right\}\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right]$$

$$\rho \frac{dv}{dt} = \rho B_y - \frac{\partial p}{\partial y} + \frac{\partial}{\partial y}\left[\mu\left\{2\frac{\partial v}{\partial y} - \frac{2}{3}(\nabla\cdot\bar{q})\right\}\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right]$$

$$\rho \frac{dw}{dt} = \rho B_z - \frac{\partial p}{\partial z} + \frac{\partial}{\partial z}\left[\mu\left\{2\frac{\partial w}{\partial z} - \frac{2}{3}(\nabla\cdot\bar{q})\right\}\right] + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right]$$

The above three equations are called the Navier-stokes equations of motion for a viscous compressible fluid in cartesian coordinates.

→ Incompressible viscous fluids :

$\rho = \text{constant} \Rightarrow \nabla\cdot\bar{q} = 0$ (from eqn. of continuity)

Considering temperature variation very small, viscosity may be taken to be constant.

$$\Rightarrow \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho B_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\Rightarrow \frac{du}{dt} = B_x - \frac{1}{\rho}\left(\frac{\partial}{\partial x}p\right) + \nu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

where $\nu = \mu/\rho$ - Kinematic viscosity.

μ - Dynamic viscosity or co-efficient of viscosity.

$||^{ly}$ $$\frac{dv}{dt} = B_y - \frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$\frac{dw}{dt} = B_z - \frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

→ Finally :

(i) Viscous incompressible fluid with constant velocity

$$\frac{\partial \bar{q}}{\partial t} + (\bar{q}\cdot\nabla)\bar{q} = \bar{B} - \frac{1}{\rho}\nabla p + \nu\nabla^2\bar{q}$$ (vector form)

or $$\frac{\partial \bar{q}}{\partial t} + (\bar{q}\cdot\nabla)\bar{q} = \bar{B} - \frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2\bar{q}$$

(ii) Viscous compressible fluid with constant ~~vel~~ viscosity

$$\rho\left[\frac{\partial \bar{q}}{\partial t} + (\bar{q}\cdot\nabla)\bar{q}\right] = \rho\bar{B} - \nabla p + \mu\nabla^2\bar{q} + \frac{\mu}{3}\cdot\nabla(\nabla\cdot\bar{q}).$$

or

$$\rho\left[\frac{\partial \bar{q}}{\partial t} + \nabla\left(\tfrac{1}{2}\bar{q}^2\right) - \bar{q}\times(\nabla\times\bar{q})\right] = \rho\bar{B} - \nabla p + \frac{4}{3}\mu\,\nabla(\nabla\cdot\bar{q})$$
$$- \mu\,\nabla\times(\nabla\times\bar{q}).$$

(iii) Non-viscous incompressible fluid

$$\frac{\partial \bar{q}}{\partial t} + (\bar{q}\cdot\nabla)\bar{q} = \bar{B} - \frac{1}{\rho}\nabla p.$$

well known Euler's equation.

(iv) Plane two-dimensional flow of incompressible viscous fluid.

we have $w=0$ and $\frac{\partial}{\partial z}=0$

$$\Rightarrow \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right) = \rho B_x - \frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = \rho B_y - \frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right)$$

Summary of basic equations governing the flow of viscous fluid :

A. Cartesian co-ordinates (x,y,z)

(i) Compressible fluid

Equation of continuity :

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$$

Navier - stokes equation :

$$\rho\frac{du}{dt} = \rho B_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[\mu\left\{2\frac{\partial u}{\partial x} - \frac{2}{3}(\nabla\cdot\bar{q})\right\}\right]$$
$$+ \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial v}{\partial z}\right)\right].$$

Similarly two equations of $\rho\frac{dv}{dt}$ and $\rho\frac{dw}{dt}$.

(ii) Incompressible fluid (ρ, μ - constant)

Equation of continuity :

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Navier - stokes equation : (vector form)

$$\rho\left(\frac{\partial \bar{q}}{\partial t} + (\bar{q}\cdot\nabla)\bar{q}\right) = \rho\bar{B} - \nabla p + \mu\nabla^2\bar{q}.$$

B. Cylindrical coordinates (r, θ, z) :

Equation of continuity :

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho r q_r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho q_\theta) + \frac{\partial}{\partial z}(\rho q_z) = 0$$

Navier – stokes equation :

$$\rho\left(\frac{dq_r}{dt} - \frac{q_\theta^2}{r}\right) = \rho B_r + \frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\partial \sigma_{zr}}{\partial z} + \frac{\sigma_{rr} - \sigma_{\theta\theta}}{r}$$

$$\rho\left(\frac{dq_\theta}{dt} + \frac{q_r q_\theta}{r}\right) = \rho B_\theta + \frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{\theta z}}{\partial z} + \frac{2\sigma_{r\theta}}{r}$$

$$\rho\left(\frac{dq_z}{dt}\right) = \rho B_z + \frac{\partial \sigma_{zr}}{\partial r} + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} + \frac{\sigma_{zr}}{r}.$$

$$\text{where, } \frac{d}{dt} \equiv \frac{\partial}{\partial t} + q_r\frac{\partial}{\partial r} + \frac{q_\theta}{r}\frac{\partial}{\partial \theta} + q_z\frac{\partial}{\partial z}.$$

Constitutive equations given by

$$\sigma_{rr} = 2\mu\, \epsilon_{rr} - \frac{2}{3}\mu(\nabla\cdot\bar{q}) - p$$

$$\sigma_{\theta\theta} = 2\mu\, \epsilon_{\theta\theta} - \frac{2}{3}\mu(\nabla\cdot\bar{q}) - p$$

$$\sigma_{zz} = 2\mu\, \epsilon_{zz} - \frac{2}{3}\mu(\nabla\cdot\bar{q}) - p$$

$$\sigma_{r\theta} = \mu\gamma_{r\theta}, \quad \sigma_{\theta z} = \mu\gamma_{\theta z}, \quad \sigma_{zr} = \mu\gamma_{zr}$$

$$\epsilon_{rr} = \frac{\partial q_\theta}{\partial r}, \quad \epsilon_{\theta\theta} = \frac{1}{r}\frac{\partial q_\theta}{\partial \theta} + \frac{q_r}{r}, \quad \epsilon_{zz} = \frac{\partial q_z}{\partial z}$$

$$\gamma_{r\theta} = \frac{\partial q_\theta}{\partial r} - \frac{q_\theta}{r} + \frac{1}{r}\frac{\partial q_r}{\partial \theta}$$

$$\gamma_{\theta z} = \frac{1}{r}\frac{\partial q_z}{\partial \theta} + \frac{\partial q_\theta}{\partial z}$$

$$\gamma_{zr} = \frac{\partial q_r}{\partial z} + \frac{\partial q_z}{\partial r}.$$

$$\nabla\cdot\bar{q} = \frac{\partial q_r}{\partial r} + \frac{1}{r}\frac{\partial q_\theta}{\partial \theta} + \frac{\partial q_z}{\partial z} + \frac{q_r}{r}.$$

Note:

For axi-symmetric flow

$$\frac{\partial}{\partial \theta} = 0.$$

C. Spherical coordinates (r, θ, ϕ) :

Equation of continuity :

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho r^2 q_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\rho \sin\theta\, q_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}(\rho q_\phi) = 0$$

Navier - stokes equation :

$$\rho\left(\frac{dq_r}{dt} - \frac{q_\theta^2 + q_\phi^2}{r}\right) = \rho B_r + \frac{1}{r^2}\frac{\partial(r^2\sigma_{rr})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sigma_{r\theta}\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\sigma_{\phi r}}{\partial\phi} - \left(\frac{\sigma_{\theta\theta}+\sigma_{\phi\phi}}{r}\right)$$

$$\rho\left(\frac{dq_\theta}{dt} + \frac{q_r q_\theta}{r} - \frac{q_\phi^2\cot\theta}{r}\right) = \rho B_\theta + \frac{1}{r^2}\frac{\partial(r^2\sigma_{r\theta})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sigma_{\theta\theta}\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\sigma_{\theta\phi}}{\partial\phi} + \frac{\sigma_{r\theta}}{r} - \frac{\sigma_{\phi\phi}\cot\theta}{r}$$

$$\rho\left(\frac{dq_\phi}{dt} + \frac{q_\phi q_r}{r} - \frac{q_\phi q_\theta\cot\theta}{r}\right) = \rho B_z + \frac{1}{r^2}\frac{\partial(r^2\sigma_{\phi r})}{\partial r} + \frac{1}{r\sin\theta}\frac{\partial(\sigma_{\theta\phi}\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\sigma_{\phi\phi}}{\partial\phi} + \frac{\sigma_{\phi r}}{r} - \frac{\sigma_{\theta\phi}\cot\theta}{r}$$

where, $\frac{d}{dt} \equiv \frac{\partial}{\partial t} + q_r\frac{\partial}{\partial r} + \frac{q_\theta}{r}\frac{\partial}{\partial\theta} + \frac{q_\phi}{r\sin\theta}\frac{\partial}{\partial\phi}$.

Constitutive equations given by

$$\sigma_{rr} = 2\mu\epsilon_{rr} - \frac{2\mu}{3}\nabla\cdot\bar{q} - p \qquad \sigma_{r\theta} = \mu\gamma_{r\theta}$$

$$\sigma_{\theta\theta} = 2\mu\epsilon_{\theta\theta} - \frac{2\mu}{3}\nabla\cdot\bar{q} - p \qquad \sigma_{\theta\phi} = \mu\gamma_{\theta\phi}$$

$$\sigma_{\phi\phi} = 2\mu\epsilon_{\phi\phi} - \frac{2\mu}{3}\nabla\cdot\bar{q} - p \qquad \sigma_{\phi r} = \mu\gamma_{\phi r}$$

$$\epsilon_{rr} = \frac{\partial q_r}{\partial r}, \quad \epsilon_{\theta\theta} = \frac{1}{r}\frac{\partial q_\theta}{\partial\theta} + \frac{q_r}{r}, \quad \epsilon_{\phi\phi} = \frac{1}{r\sin\theta}\frac{\partial q_\phi}{\partial\phi} + \frac{q_r}{r} + \frac{q_\theta\cot\theta}{r}$$

$$\gamma_{r\theta} = r\frac{\partial}{\partial r}\left(\frac{q_\theta}{r}\right) + \frac{1}{r}\frac{\partial q_r}{\partial\theta},$$

$$\gamma_{\theta\phi} = \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{q_\theta}{\sin\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial q_\theta}{\partial\phi}$$

$$\gamma_{\phi r} = \frac{1}{r\sin\theta}\frac{\partial q_r}{\partial\phi} + r\frac{\partial}{\partial r}\left(\frac{q_\phi}{r}\right).$$

$$\nabla\cdot\bar{q} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 q_r) + \frac{1}{r\sin\theta}\frac{\partial(q_\theta\sin\theta)}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial q_\phi}{\partial\phi}.$$