WORKSHEET - 8

Limits, Continuity and Differentiability of Functions of Several Variables

1. If $f(x,y)=x\sin 1/x+y\sin 1/y$ where $xy\neq 0$, $f(x,0)=x\sin 1/x$ when $x\neq 0$, $f(0,y)=y\sin 1/y$ when $y\neq 0$; $f(0,0)=0$. Show that f is continuous but not differentiable at (0, 0).

2. If $z=xyf(y/x)$, show that $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=2z$, and if z is a constant, then

$$\frac{f'(y/x)}{f(y/x)}=\frac{x\left(y+x\frac{dy}{dx}\right)}{y\left(y-x\frac{dy}{dx}\right)}$$

3. Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined as

$$f(x,y)=\frac{xy}{\sqrt{x^2+y^2}} \text{ if } (x,y)\neq 0 \text{ and}$$

$f(0,0)=0$.

Prove that f_x and f_y exist at (0, 0) but f is not differentiable at (0, 0).

4. Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined by setting

$$f(x,y)=\frac{xy}{\sqrt{x^2+y^2}}, \text{ where}$$

$(x,y)\neq(0,0)$, $f(0,0)=0$. Show that f_x and f_y exist at (0,0) but f is not differentiable at (0, 0). Also show that f is continuous at (0,0).

5. Discuss the continuity of the function

$$f(x,y)=\begin{cases}\dfrac{2xy^2}{x^2+3y^2} & ;(x,y)\neq(0,0)\\ 0 & ;(x,y)=(0,0)\end{cases}$$

6. Investigate the maxima and minima of $f(x,y)=x^2+3xy+y^2+x^3+y^3$.

7. Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined by setting

$f(x,0)=x^2\sin(1/x)$, when $x\neq 0$,

$f(0,y)=y^2\sin(1/y)$, when $y\neq 0$, $f(0,0)=0$.

Show that (i) f_x and f_y both exist at (0, 0) but are not continuous at (0, 0); (ii) f is differentiable at (0, 0).

8. If $f(x,y)=\frac{x^3y^3}{x^2+y^2}, (x,y)\neq(0,0)$ and $f(0,0)=0$, then show that f is continuous at (0, 0).

9. Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined as

$$f(x,y)=\frac{xy}{\sqrt{(x^2+y^2)}},(x,y)\neq(0,0)\quad f(0,0)=0$$

Prove that f_x and f_y exist at (0, 0), but f is not differentiable at (0, 0).

10. Show that the function given by

$$f(x,y)=\begin{cases}\frac{xy}{x^2+2y^2}; & (x,y)\neq(0,0)\\ 0 & ;\ (x,y)=(0,0)\end{cases}$$

is not continuous at (0, 0) but partial derivatives f_x & f_y exist at (0, 0)

11. (i) Discuss the convergence of

$$\int_0^\infty\left(\frac{1}{1+x}-e^{-x}\right)\frac{dx}{x}.$$

(ii) If $z=xf\left(\frac{y}{x}\right)+g\left(\frac{y}{x}\right)$

Show that $x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=0$

12. Define $f(0,0)$ in a way that extends $f(x,y)=xy\frac{x^2-y^2}{x^2+y^2}$ to be continuous at the origin.

13. If $W=f\left[\frac{xy}{x^2+y^2}\right]$ is a differentiable function of $u=\frac{xy}{x^2+y^2}$.

Show that $x\left(\frac{\partial w}{\partial x}\right)+y\left(\frac{\partial w}{\partial y}\right)=0$.

14. Compute $f_{xy}(0,0)$ and $f_{yx}(0,0)$ for the function

$$f(x,y)=\begin{cases}(x^2+y^2)\log(x^2+y^2), & (x,y)\neq(0,0)\\ 0 & ,(x,y)=(0,0)\end{cases}$$

Also discuss the continuity of f_{xy} and f_{yx} at (0,0)

15. Let ϕ be a function of two variables defined as

$\phi(x,y)=(x^3+y^3)/(x-y)$,

when $x\neq y$

$\phi(x,y)=0$, when $x=y$

Show that ϕ is discontinuous at the origin, but the first order partial derivatives exist at that point.

16. Let $z=f(t)$, $t=\frac{x+y}{xy}$. Show that

$$x^2\frac{\partial z}{\partial x}=y^2\frac{\partial z}{\partial y}$$

17. Show that the following function is discontinuous at (0, 0):

$$f(x,y)=\begin{cases}\frac{x^3+y^3}{x-y}, & x\neq y,\\ 0 & ,\ x=y\end{cases}$$

18. Discuss the continuity of the following function at (0, 0)

$$f(x,y)=\begin{cases}\frac{x^2y^2}{x^2+y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0).\end{cases}$$

Also find f_{xy} and f_{yx} at (0, 0).

19. (i) Find the maxima and minima of

$$f(x,y)=x^4+y^4-2(x-y)^2$$

20. (i) Show that the function given by

$$f(x,y)=\begin{cases}\frac{xy}{x^2+2y^2}; & (x,y)\neq(0,0)\\ 0 & ;\ (x,y)=(0,0)\end{cases}$$

is not continuous at (0, 0) but partial derivatives f_x & f_y exist at (0, 0)

(ii) If $z=xf\left(\frac{y}{x}\right)+g\left(\frac{y}{x}\right)$ Show that

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=0$$

21.

(ii) If $z = xyf(y/x)$, show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$, and if z is a constant,

then $\frac{f'(y/x)}{f(y/x)} = \frac{x\left(y + x\frac{dy}{dx}\right)}{y\left(y - x\frac{dy}{dx}\right)}$.

22. Let $f : R^2 \to R$ be defined by setting

$$f(x, y) = \begin{cases} \frac{x^2 y}{x^4 + y^2}, & if\,(x,y) \neq (0,0), \\ 0, & if\,(x,y) = (0,0), \end{cases}$$

Show that f possesses partial derivatives at (0,0) but is not differentiable at the point.

23. Let the function f (x,y) be defined by the relations $f(x,y) = \frac{\sin^2(x-y)}{|x|+|y|}$ for $|x| + |y| \neq 0$, $f(0,0) = 0$. Is f continuous at $x = 0, y = 0$?

24. If $z = f\left(\frac{x-y}{y}\right)$, show that

$$x\left(\frac{\partial z}{\partial x}\right) + y\left(\frac{\partial z}{\partial y}\right) = 0$$

25. If $f(x, y) = x\sin 1/x + y\sin 1/y$ where $xy \neq 0$, $f(x, 0) = x\sin 1/x$ when $x \neq 0$, $f(0,y) = y\sin 1/y$ when $y \neq 0$; y $f(0, 0) = 0$. Show that f is continuous but not differentiable at (0, 0).

26. If $z = xyf(y/x)$, show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2z$,

and if z is a constant,

then $\frac{f'(y/x)}{f(y/x)} = \frac{x\left(y + x\frac{dy}{dx}\right)}{y\left(y - x\frac{dy}{dx}\right)}$

27. If $z = x^m f\left(\frac{y}{x}\right) + x^n g\left(\frac{x}{y}\right)$, prove that

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} + mnz = (m + n - 1)\left(x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y}\right).$$

28. (i) Show that the function given by

$$f(x,y) = \begin{cases} \frac{xy}{x^2 + 2y^2}; & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

is not continuous at (0,0) but partial derivatives f_x & f_y exist at (1,0).

(ii) Show how the change of order in the integral $\int_0^\infty \int_0^\infty e^{-xy} \sin x\,dx\,dy$ leads to the evaluation of $\int_0^\infty \frac{\sin x}{x}dx$.

Hence evaluate it.

29. If $u = x\phi(x + y) + y\psi(x + y)$, Prove that

$$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0.$$

30. Given $f(x, y) = \frac{xy}{x^2 + y^2}$ when $(x, y) \neq (0,0)$ $f(0,0) = 0$.

Show that both the partial derivatives exist at (0. 0) but the function is continuous at (0, 0).

31. If $u = \frac{1}{\sqrt{x^2 + y^2 + z^2}}; x^2 + y^2 + z^2 \neq 0$

Show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial^2 y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$

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32. If $f(x,y) = \begin{cases} \frac{x^3 + y^3}{x - y} & ; x \neq y \\ 0 & ; x = y \end{cases}$ then show

that f is discontinuous at the origin but the partial derivatives exist at the origin.

33. If $z=(x+y)\phi(y/x)$, where ϕ is any arbitrary function. Prove that $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z$.

34. Suppose $f:\mathbb{R}^2\to\mathbb{R}$ is defined by

$$f(x,y)=\begin{cases}\frac{xy}{x^4+y^4} & ,(x,y)\neq(0,0)\\ 0 & ,(x,y)=(0,0)\end{cases}$$

Find the two partial derivatives at the points (0, 0), (a, 0), (0, b) and (a, b) where $a\neq 0, b\neq 0$.

35. If $u=f(x+2y)+g(x-2y)$.

Show that $4\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial y^2}$.

36. Show that

$$f(x,y)=\begin{cases}\frac{x^3-y^3}{x^2+y^2} ;(x,y)\neq(0,0)\\ 0 \quad ;(x,y)=(0,0)\end{cases}$$

is continuous and possesses partial derivatives but is not differentiable at the origin.

37. Prove that the function

$$f(x,y)=\frac{xy}{\sqrt{x^2+y^2}},\ (x,y)\neq(0,0)$$

$=0,(x,y)=(0,0)$ is continuous at the origin.

38. If $z=(x+y)\phi(y/x)$, where ϕ is any arbitrary function. Prove that $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=z$.

39. (i) Discuss the continuity of the function

$$f(x,y)=\begin{cases}\frac{2xy^2}{x^3+3y^3}; & (x,y)\neq(0,0)\\ 0 ; & (x,y)=(0,0)\end{cases}$$

(ii) If $z=f\left(\frac{ny-mz}{nx-lz}\right)$ prove that

$$(nx-lz)\frac{\partial z}{\partial x}+(ny-mz)\frac{\partial z}{\partial y}=0.$$

40. Find the maxima and minima of $x^2+y^2+z^2$ subject to the conditions $ax^2+by^2+cz^2=1$ and $lx+my+nz=0$.

41. If $u=f(x+2y)+g(x-2y)$

Show that $4\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial y^2}$

42. If $u=\sin^{-1}\left(\frac{x^3+y^3}{\sqrt{x}+\sqrt{y}}\right)$, Prove that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=\frac{5}{2}\tan u.$$

43. Can $f(0,0)$ be defined so that

$$f(x,y)=\frac{x^3 y}{x^2+y^2},(x,y)\neq(0,0)$$

and f is continuous at (0, 0) ?

44. Show that the function f is continuous at (0, 0) where $f(x,y)=e^{\frac{-|x-y|}{x^2-2xy+y^2}}$ when $(x,y)\neq(x,x)$ and $f(x,x)=0$.

45. If $u=\cot^{-1}\frac{x+y}{\sqrt{x}+\sqrt{y}}$, Show that

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+\frac{1}{4}\sin 24=0.$$

46. Show that for the function f of two real variable defined by

$$f(x,y)=\frac{x^2y^2}{x^2y^2+(x-y)^2}$$

the two repeated limits exist and are equal but the simultaneous limit does not exist when $(x,y)\to(0,0)$.

47. (i) If $Z=(x+y)\phi(y/x)$ where ϕ is any

arbitrary function. Prove that

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=Z.$$

(ii) Show that the function f, where

$$f(x,y)=\begin{cases} x\sin\frac{1}{x}+y\sin\frac{1}{y} & xy\neq 0\\ x\sin\frac{1}{x} & y=0, x\neq 0\\ y\sin\frac{1}{y} & x=0, y\neq 0\\ 0 & x=0=y\end{cases}$$

is continuous but not differentiable at the origin.

48. (i) Examine the continuity of the function $f(x,y)=\sqrt{|xy|}$ at the origin.

(ii) If $u=x\phi(x+y)+y\psi(x+y)$,

Prove that $\frac{\partial^2 u}{\partial x^2}-\frac{2\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}=0.$

49. Find the maxima and minima of the function $f(x,y)=x^3+y^3-3x-12y+20.$

50. (i) If

$$V=\log_e \sin\left\{\frac{\pi(2x^2+y^2+xz)^{1/2}}{2(x^2+xy+2yz+z^2)^{1/3}}\right\}.$$

Find the value $x\frac{\partial v}{\partial x}+y\frac{\partial v}{\partial y}+z\frac{\partial v}{\partial z}$

when $x=0, y=1, z=2$.

(ii) Prove that the maximum and minimum values of $f(x,y)=x^2+xy+y^2$ in the unit square $0\le x\le 1, 0\le y\le 1$ are 3 and 0 respectively.

51.(i) Let $f:\mathbb{R}^2\to\mathbb{R}$ be a function defined by

$$f(x,y)=\begin{cases}\frac{x^2}{x^2+y^2-x}, & \text{if } f(x,y)\neq(0,0)\\ 0, & \text{if } f(x,y)=(0,0)\end{cases}$$

Show that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

(ii) Let $f:\mathbb{R}^2\to\mathbb{R}$ be a function defined by

$$f(x,y)=\begin{cases}x^2\tan^{-1}(y/x)-y^2\tan^{-1}(x/y), & \text{if } x\neq 0, y\neq 0\\ 0 & \text{else where.}\end{cases}$$

Show that $f_{yx}(0,0)\neq f_{xy}(0,0)$.

52. If $z=x^m f\left(\frac{y}{x}\right)+x^n g\left(\frac{x}{y}\right)$, prove that

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}+mnz=(m+n-1)\left(x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right)$$

53. (i) Discuss the convergence of $\int_0^\infty\left(\frac{1}{1+x}-e^{-x}\right)\frac{dx}{x}.$

(ii) If $z=x f\left(\frac{y}{x}\right)+g\left(\frac{y}{x}\right)$

Show that

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=0$$

54. Examine the continuity of the function

$$f(x,y)=\begin{cases}\frac{x^2y}{x^4+y^2}, & \text{if } (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

55. Let $f(x,y)=\begin{cases}\frac{xy(x^2-y^2)}{x^2+y^2}, & \text{if } (x,y)\neq(0,0)\\ 0, & \text{if } (x,y)=(0,0)\end{cases}$

Show that:

(i) $f_{xy}(0,0)\neq f_{yx}(0,0)$

(ii) f is differentiable at (0, 0)

56. Show that the function defined by

$$f(x,y)=\begin{cases}\frac{x^3+y^3}{x-y}, & x\neq y\\ 0, & x=y\end{cases}$$

is discontinuous at the origin but possesses partial derivatives f_x and f_y thereat.

57. If $z = xf\left(\frac{y}{x}\right) + g\left(\frac{y}{x}\right)$ show that

$$x^2\frac{\partial^2 z}{\partial x^2} + 2xy\frac{\partial^2 z}{\partial x \partial y} + y^2\frac{\partial^2 z}{\partial y^2} = 0$$

58. Prove that if $z = \phi(y + ax) + \psi(y - ax)$ then $a^2\frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x^2} = 0$ for any twice differentiable ϕ and ψ; 'a' is a constant.

59. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$f(x, y) = \begin{cases} \dfrac{xy}{\sqrt{x^2 + y^2}}, & \text{if } (x, y) \neq (0,0) \\ 0, & \text{if } (x, y) = (0,0) \end{cases}$$

Is f continuous at (0, 0)? Compute partial derivatives of f at any point (x, y), if exist.

60. If $f(x, y)$ is a homogeneous function of degree n in x and y, and has continuous first - and second-order partial derivatives, then show that

(i) $x\frac{\partial f}{\partial x} + y\frac{\partial f}{\partial y} = nf$

(ii) $x^2\frac{\partial^2 f}{\partial x^2} + 2xy\frac{\partial^2 f}{\partial x \partial y} + y^2\frac{\partial^2 f}{\partial y^2} = n(n-1)f$

61. Find $\lim_{(x,y)\to(0,0)} \frac{x^3 y}{x^6 + y^2}$ if it exists.

62. For the function

$$f(x, y) = \begin{cases} \dfrac{x^2 - x\sqrt{y}}{x^2 + y}, & (x, y) \neq (0,0) \\ 0 & (x, y) = (0,0) \end{cases}$$

Examine the continuity and differentiability.

63. Show that the function given by

$$f(x, y) = \begin{cases} \dfrac{xy}{x^2 + 2y^2}, & (x, y) \neq (0,0) \\ 0, & (x, y) = (0,0) \end{cases}$$

is not continuous at (0, 0) but its partial derivatives f_x and f_y exist at (0, 0).

64. Let $f(x, y) = \begin{cases} \dfrac{(x + y)^2}{x^2 + y^2}, & \text{if } (x, y) \neq (0,0) \\ 1, & \text{if } (x, y) = (0,0). \end{cases}$

Show that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ exist at (0, 0) though $f(x, y)$ is not continuous at (0, 0).

65. A function f is defined as

$$f(x, y) = \frac{x^3 y^3}{x^2 + y^2}, \quad (x, y) \neq (0,0)$$
$$= 0, \text{ otherwise}$$

Prove that $f_{xy}(0,0) = f_{yx}(0,0)$ but neither f_{xy} nor f_{yx} is continuous at (0, 0).

66. If $f(x, y) = xy\frac{x^2 - y^2}{x^2 + y^2}$, when $(x, y) \neq (0,0)$ and $f(0,0) = 0$, show that at (0, 0) $\frac{\partial^2 f}{\partial x \partial y} \neq \frac{\partial^2 f}{\partial y \partial x}$

WORKSHEET - 9

Maxima, Minima and Lagrange Multipiers method

1. Find the volume common to cylinders $x^2+y^2=a^2$, $x^2+z^2=a^2$.

2. Prove that if $x+y+z=1$, $ayz+bzx+cxy$ has an extreme value equal to $\frac{abc}{2bc+2ca+2ab+a^2-b^2-c^2}$. Prove also that if a, b, c are all positive and c lies between $a+b-2\sqrt{ab}$ and $a+b+2\sqrt{ab}$ this value true maximum and that if a, b, c are all negative and c lies between $a+b\pm2\sqrt{ab}$, it is true minimum.

3. Let $f:\mathbb{R}^2\to\mathbb{R}$ be defined by $f(x,y)=x^3+y^3-12y+20$. Find the maxima and minima of this function.
   (0,2) – min, (0,−2) – max

4. Find the volume bounded by the plane $z=0$, surface $z=x^2+y^2+2$, and the cylinder $x^2+y^2=4$.

5. Find the maximum and minimum distances of the point (3, 4, 12) from the sphere $x^2+y^2+z^2=1$. Min → (3/13, 4/13, 12/13)
   Max → (−3/13, −4/13, −12/13)

6. Find the volume in the first octant bounded by the cylinder $x=4-y^2$, and the planes $z=y, x=0, z=0$.

7. Find the dimensions of the closed circular can of smallest surface area whose volume is $16\pi\,cm^3$. h = 4, r = 2, λ = −π

8. Find the maximum and minimum value of $f(x)=xy$ subject to the condition that $g = x^2+y^2+xy=a^2$
   max c→ x = −y when λ = 1
   min c→ x = y when λ = −1/3

9. Obtain the volume bounded by the elliptic paraboloids given by the equations $z=x^2+9y^2$ and $z=18-x^2-9y^2$

10. Find the maximum and minimum values of $f(x,y,z)=x-2y+5z$ on the sphere $x^2+y^2+z^2=30$.

11. Find the volume of the cylinder $x^2+y^2-2ax=0$, intercepted between the paraboloid $x^2+y^2=2az$ and the xy-plane.

12. A rectangular box, open at the top, is to have a volume of 32 cubic feet. What must be the dimensions so that the total surface is a minimum?

13. Find the maxima and minima of xy^2z^3 subject to the conditions $x+y+z=6$; $x>0, y>0, z>0$.
   (1,2,3) is only critical pt.

14. Find the volume enclosed between the two surfaces $z=x^2+3y^2$ and $z=8-x^2-y^2$.

15. Find the maxima and minima of the function $f(x,y)=x^3+y^3-3x-12y+20$
   4 points critical: (±1, ±2)
   Max @ (−1,−2)
   Min @ (1,2)

16. The sections of the enveloping cone of the surface $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ whose vertex is

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$P(x_1, y_1, z_1)$ by the plane $z = 0$ is a cirlce. Find the locus of the vertex P.

17. Find the volume cut off the sphere $x^2+y^2+z^2=a^2$ by the cone $x^2+y^2=z^2$.

18. Discuss the nature of the critical values of $f(x, y) = e^x \sin y$.

19. Find the maximum and minimum values of $f(x, y) = x^2 + 3y^2 + 2y$ on the unit disc $x^2 + y^2 \le 1$ Max @ (0,1) Min @ $(\pm\sqrt{3}/2, -1/2)$

20. Find the volume of the cylinder $x^2 + y^2 - 2ax = 0$ bounded by the planes $z = x \tan a$ and $z = x \tan b$ where $\beta > \alpha$.

21. Investigate the maxima and minima of $f(x,y)=x^2+3xy+y^2+x^3+y^3$

22. Find the maximum value of $f(x, y, z) = x^2 + 2y - z^2$ subject to the constraints $2x-y=0$ and $y + z = 0$

23. Find the points on the sphere $x^2+y^2+z^2=25$ where $f(x,y,z)=x+2y+3z$ has its maximum and minimum values.

Hm to Q.10

24. Using Lagrange multipliers, show that a rectangular box with lid of volume 1000 cubic units and of least surface area is a cube of side 10 units. Minimize $2xy+2yz+2zx$ → (1) $xyz=1000$ Substitute $z = 1000/xy$ in (1)

25. Find the extreme values of the function $f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$. (−2,−2) max only CRIT

26. A surface is defined by $z = x^3 + y^3 - 9xy + 27$. Prove that the only possible maxima and minima of z occur at (0, 0) or (3, 3). Prove that (0, 0) is neither a maximum nor a minimum. Determine whether (3, 3) is a maximum or a minimum.

27. Find the volume bounded above by the sphere $x^2 + y^2 + z^2 = 2a^2$ and below by the paraboloid $az = x^2 + y^2$.

28. Find the minimum distance from the origin to the surface $x^2 - z^2 - 1 = 0$

29. Compute $f_{xy}(0, 0)$ and $f_{yx}(0, 0)$ for the function

$$f(x, y) = \begin{cases} \dfrac{xy^3}{x+y^2}, & (x,y) \ne (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

Also, discuss the continuity of f_{xy} and f_{yx} at (0,0)

30. Use the method of Lagrange multipliers to find the dimensions of the rectangle of greatest area that can be inscried in the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ with sides parallel to the co-ordinate axes.

31. Find the volume common to the two cylinders $x^2 + y^2 = a^2$ and $x^2 + z^2 = a^2$

32. Using Lagrange's multiplier method, find the shortest distance between the line $y = 10 - 2x$ and the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$.

33. Find the volume of the cylinder $x^2 + y^2 - 2ax = 0$ bounded by the planes $z = x \tan \alpha$ and $z = x \tan \beta$ where $\beta > \alpha$.

Show that the function $f(x, y) = 2x^4 - 3x^2 y + y^2$ has neither a maximum nor a minimum at (0, 0).

34. Examine the following function for extreme values $f(x,y,z) = 2x^2+3x^2+4z^2-3xy+8z$.

35. Find the shortest distance from the origin to the hyperbola $x^2+8xy+7y^2=225, z=0$

36. Find the maximum value of

$$f(x,y,z)=(ax,by+cz)^{-\alpha^2x^2-\beta^2y^2-\gamma^2z^2}$$

37. Discuss the maxima and minima of the function $u=\sin x\sin y\sin z$, where x,y,z are the angles of a triangle.

38. Check whether the functiongiven by $f(x,y)=x^2-2x+\dfrac{y^2}{4}$ has maximum or minimum values.

39. Prove that the volume of the greatest rectangular parallelopiped, that can be inscribed in the ellipsoid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}=1$, is $\dfrac{8abc}{3\sqrt{3}}$.

40. Find the volume bounded by

$$y^2+z^2=4ax, y^2=ax, x=3a.$$

41. Find the volume under the plane $x+y+z=6$ and above the triangle in the xy plane bounded by $2x=3y, y=0, x=3$.

42. Find the volume and centroid of the region in the first octant bounded by $6x+3y+2z=6$.

43. Find the maximum and minimum distances of the point (3, 4, 12) from the sphere $x^2+y^2+z^2=1$.

44. Find the volume of the region lying below the paraboloid P with equation $z=4-x^2-y^2$

45. (i) Examine the continuity of the function $f(x,y)=\sqrt{|xy|}$ at the origin.

(ii) If $u=\phi(x+y)+y\psi(x+y)$,
Prove that

$$\frac{\partial^2 u}{\partial x^2}-\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}=0.$$

46. Find the extremum values of x^2+y^2 subject to the condition $3x^2+4xy+6y^2=140$.

47. Find the volume of the portion of the sphere $x^2+y^2+z^2=a^2$ lying inside the cylinder $x^2+y^2=ay$.

48. Prove that the function $f(x,y)=\sqrt{(xy)}$ is not differentiable at the point (0, 0), but that f_x and f_y both exist at the origin and have the value 0. Hence deduce that these two partial derivatives are continuous except at the origin.

49. Find the maximum and minimum values of

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4}, \text{ when } lx+my+nz=0$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

50. Find the vdeeme enclosed by the surfaces $x^2+y^2=cz,\ x^2+y^2=2ax, z=0.$

WORKSHEET - 10

Multiple Integrals

1. Evaluate the integral $\int_0^1 \int_{e^x}^{e} \frac{dx\,dy}{\log y}$ by changing the order of integration.

2. Show how the change of order in the integral $\int_0^\infty \int_0^\infty e^{-xy} \sin x\,dx\,dy$ leads to the evaluation of $\int_0^\infty \frac{\sin x}{x}\,dx$. Hence evaluate it.

3. Evaluate the integral $\int_0^\infty \int_0^x x e^{-x^2/y}\,dx\,dy$ by changing the order integration.

4. Find the volume bounded above by the sphere $x^2+y^2+z^2=2a^2$ and below by the paraboloid $az=x^2+y^2$.

5. Evaluate the integral $\int_0^1 \int_{\sqrt{y}}^1 e^{x/y}\,dx\,dy$ by changing the order of integration.

6. Evaluate the integral $\int_0^a \int_{\sqrt{ax}}^a \frac{y^2\,dx\,dy}{\sqrt{y^4-a^2x^2}}$ by changing the order of integration.

7. Evaluate the following integral $\int_0^{a/\sqrt{2}} \int_y^{\sqrt{(a^2-y^2)}} \log\left(x^2+y^2\right) dx\,dy\,(a>0)$ by changing the order of integration.

8. Evaluate the integral $\int_0^1 \int_{e^x}^{e} \frac{dx\,dy}{\log y}$ by changing the order of integration

9. (i) Evaluate $\iiint_V (2x+y)\,dx\,dy\,dz$, where V is the closed region bounded by the cylinder $z=4-x^2$ and the planes $x=0$, $y=0$, $y=2$ and $z=0$.

10. Evaluate $\int\int\int z\,dx\,dy\,dz$ over the volume enclosed between the cone $x^2+y^2=z^2$ and the sphere $x^2+y^2+z^2=1$ on the positive side of xy-plane.

11. Evaluate $\int_0^\infty \int_x^\infty \left(\frac{1}{y}\right) e^{-y/2}\,dy\,dx$ by changing the order of integration.

12. (i) Examine the continuity of the function

$$f(x,y)=\begin{cases} \frac{x^2y}{x^4+y^2}, & \text{if } (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$$

(ii) Evaluate the integral $\int_0^1 \int_{e^x}^{e} \frac{dx\,dy}{\log y}$ by changing the order of integration

13. By using the transformation x + y = u, y = uv, show that

$$\int_0^1 \int_0^{1-x} e^{y/(x+y)}\,dx\,dy = \frac{1}{2}(e-1).$$

14. Use the transformation $x = u+\left(\frac{1}{2}\right)v$, $y=v$ to evaluate the integral $\int_0^2 \int_{y/2}^{(y+4)/2} y^3(2x-y)e^{(2x-y)^2}\,dx\,dy$.

15. Evaluate $\iint_D xy\,dA$, where D is the region bounded by the line $y=x-1$ and the parabola $y^2=2x+6$.

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16. Evaluate $\iiint_V (2x+y)\,dx\,dy\,dz$, where V is the closed region bounded by the cylinder z = 4 – x^2 and the planes x = 0, y = 0, y = 2 and z = 0.

17. Change the order of integration in
$$\int_0^{a\cos\alpha}\int_{x\tan\alpha}^{\sqrt{a^2-x^2}} f(x,y)\,dx\,dy$$
and verify result when $f(x,y)=1$.

18. Change the order of integration in double integral $\int\int \frac{\phi'(y)\,dx\,dy}{\sqrt{(a-x)(x-y)}}$ and hence find its value.

19. Change the order of integration in
$$\int_0^{2a}\int_0^{\sqrt{(2ax-x^2)}} \frac{\phi'(y)(x^2+y^2)x\,dx\,dy}{\sqrt{\{4a^2x^2-(x^2+y^2)\}}}$$

20. Change the order of integration in $I=\int_0^1\int_{x^2}^{2-x} xy\,dx\,dy$ and hence evaluate the same.

21. Change the order of integration in the double integral $\int_0\int_x \frac{e^{-y}}{y}\,dx\,dy$ and hence find the vlaue.

22. Change the order of integration in double integral $\int_0^a\int_0^x \frac{\phi(y)\,dx\,dy}{\sqrt{(a-x)(x-y)}}$ and hence find its value.

23. Change the order of integration in $I=\int_0^1\int_{x^2}^{2-x} xy\,dx\,dy$ and hence evaluate the same.

24. Change the order of integration in the integral $\int_0^{4a}\int_{\frac{x^2}{4a}}^{2\sqrt{ax}} dy\,dx$ and evaluate it.

25. Evaluate the double integral
$$\int_0^2\int_0^{\sqrt{2x-x^2}} \frac{x\,dx\,dy}{\sqrt{x^2+y^2}}$$
by changing order of integration.

26. Evaluate $\iiint_V (2x+y)\,dx\,dy\,dz$, where V is the closed region bounded by the cylinder z = 4 – x^2 and the planes x = 0, y = 0, y = 2 and z = 0.

27. Change the order of integration and evaluate $\int\int^{2-y} dx\,dy$.

28. Evaluate the integral $\int_0\int_{e^x} \frac{dx\,dy}{\log y}$ by changing the order of integration

29. Evaluate the integral $\int_0^1\int_{\sqrt{x}}^1 e^{x/y}\,dx\,dy$ by changing the order of integration.

30. Evaluate $\iint_D x\sin(x+y)\,dx\,dy$, where D is the region bounded by $0\le x\le\pi$ and $0\le y\le\frac{\pi}{2}$

31. Evaluate $\iiint (x+y+z+1)^4\,dx\,dy\,dz$ over the region defined by $x\ge 0, y\ge 0, z\ge 0$ and $x+y+z\le 1$.

32. Evaluate $\iint_D (x+2y)\,dA$, where D is the region bounded by the parabolas y = $2x^2$ and y = 1 + x^2.

33. Change the order of integration in $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$ and hence evaluate it.

34. Evaluate the double integral

$$\int_0^a \int_y^a \frac{x\, dx dy}{x^2 + y^2}$$ by changing the order of integration.

35. Let D be the region determined by the inequalities $x > 0, y > 0, z < 8$ and $z > x^2 + y^2$, compute

$$\iiint_D 2x\, dx\, dy\, dz$$

36. Evaluate $\iiint (ax^2 + by^2 + cz^2)\, dx dy dz$ taken throughout the region $x^2 + y^2 + z^2 \le R^2$

37. Change the order of integration in

$$\int_0^{2a} \int_{\frac{x^2}{4a}}^{3a-x} \phi(x, y)\, dx\, dy$$

38. Evaluate $\iiint \sqrt{1-z}\, dx\, dy\, dz$ through the volume bounded by the surfaces $x = 0, y = 0, z = 0$ and $x + y + z = 1$.

39. Evaluate the double integral $\iint_R x^2\, dx\, dy$, where R is the region bounded by the line $y = x$ and the curve $y = x^2$.

40. Examine the following function for extrema:

$$f(x_1, x_2) = x_1^3 - 6x_1x_2 + 3x_2^2 - 24x_1 + 4$$

41. Evaluate

$$\iint xy(x + y)\, dx\, dy$$

over the area between $y = x^2$ and $y = x$.

42. Evaluate

$$\iint \sqrt{4x^2 - y^2}\, dx\, dy$$

over the triangle formed by the straight lines $y = 0$, $x = 1$, $y = x$.

43. Show that $\iint_D \frac{(x-y)}{(x+y)^3}\, dx\, dy$ does not exist, where

$$D = \{(x, y) \in R^2 / 0 \le x \le 1, 0 < y < 1\}$$

44. Show that $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx dy = \frac{\pi}{4}$.

45. Evaluate the integral $\int_0^a \int_{y^2/a}^{y} \frac{y\, dx dy}{(a-x)\sqrt{(ax-y^2)}}$

46. Evaluate the integral $\iint_R \frac{xe^{-y}}{y}\, dx\, dy$, where R is the triangular region in the first quadrant bounded by y = x and x = 0.

47. Evalicate $\int_0^{\pi/2} \int_0^{\pi/2} \sin x \sin^{-1}(\sin x \sin y)\, dx\, dy$