IMS
MATHS
BOOK-10

* STATICS *

Virtual Work

§ 1. Displacement. Suppose a particle moves from a position P to any other position Q by whatever path. Then the vector $\vec{PQ}$ is called the displacement of the particle with regard to P. If $\mathbf{r}$ and $\mathbf{r}'$ be the position vectors of the points P and Q referred to some origin O, then the displacement of the particle from P to Q is the vector

$$\vec{PQ}=\mathbf{r}'-\mathbf{r}.$$

§ 2. A rigid body. A rigid body is a collection of particles such that for any displacement of the body the distance between any two particles of the body remains the same in magnitude. Thus in the case of the rigid body, referred to some origin O, if $\mathbf{r}_1, \mathbf{r}_2$ are respectively the position vectors of the two particles before displacement and $\mathbf{r}_1', \mathbf{r}_2'$ are their respective position vectors after displacement, then the condition of rigidity of the body requires that their mutual distance must remain the same before and after the displacement *i.e.*,

$$|\mathbf{r}_2-\mathbf{r}_1|=|\mathbf{r}_2'-\mathbf{r}_1'|, \text{ or } (\mathbf{r}_2-\mathbf{r}_1)^2=(\mathbf{r}_2'-\mathbf{r}_1')^2.$$

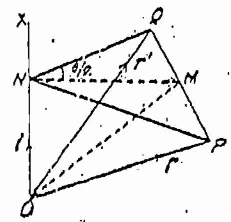
§ 3. Kinds of displacement of a rigid body. (*Translation, Rotation and General*).

One way of displacing a particle of a rigid body from one position to any other position is what we call *pure translation*. In this case the displacement is brought about without rotating the body. Thus if $\mathbf{r}$ be the position vector of a particle P referred to some origin O and if the particle is displaced from P to Q by giving a displacement $\mathbf{u}$ in the direction of OP only, then this displacement is called *translation* and we say that the displacement $\vec{PQ}=\mathbf{u}$ is a translation.

The other way of displacement of a particle is called *pure rotation*. In this case the displacement of the particle is brought about only by rotating the body about a fixed point, say O, so that the distance of the particle from the fixed point O does not change in the two positions P and Q before and after the displacement. Thus in the case of pure rotation, we have $OP=OQ$ in length but their directions are generally different.

If both the displacements translation and rotation take place simultaneously we call it a *general displacement* of the particle or of the body.

§ 4. Rotation of a rigid body about a point.



Suppose a rigid body rotates about a fixed point O. On account of this rotation suppose a particle is displaced from P to Q where

$$\vec{OP}=\mathbf{r} \text{ and } \vec{OQ}=\mathbf{r}'.$$

Since the displacement of the particle is that of rotation only about O, therefore

the length $OP=$ the length OQ.

Let M be the middle point of PQ, so that

$$\vec{OM}=\tfrac{1}{2}(\mathbf{r}+\mathbf{r}').$$

Draw a line OX, through O, perpendicular to PQ and let $\mathbf{i}$ be a unit vector in the direction OX. Also let N be the foot of the perpendicular from M to OX.

Since $OP=OQ$ and M is the middle point of PQ, therefore from the $\triangle OPQ$ we observe that PQ is perpendicular to OM.

Thus PQ, being perpendicular to OM and ON both, is perpendicular to the plane OMN. Consequently PQ is perpendicular to MN because MN lies in the plane OMN. Thus NM is the perpendicular bisector of PQ and so we have

$$NP=NQ.$$

If $\angle PNQ=\theta$, then $\angle PNM=\angle MNQ=\tfrac{1}{2}\theta$.

The vector $\vec{PQ}$ is perpendicular to the vectors $\mathbf{i}$ and $\vec{OM}$ which implies that $\vec{PQ}$ is parallel to the vector $\mathbf{i}\times\vec{OM}$.

Also $|\vec{PQ}|=PQ=2PM=2NM\tan\tfrac{1}{2}\theta$

$$=2(OM\sin\angle MON)\tan\tfrac{1}{2}\theta=2\,|\mathbf{i}\times\vec{OM}|\tan\tfrac{1}{2}\theta.$$

$$[\because\ |\mathbf{i}\times\vec{OM}|=1.OM.\sin\angle MON]$$

Thus $\vec{PQ}$ is parallel to the vector $\mathbf{i}\times\vec{OM}$ and

$$|\vec{PQ}|=(2\tan\tfrac{1}{2}\theta)\,|\,\mathbf{i}\times\vec{OM}\,|.$$

Therefore by the definition of the multiplication of a vector by a scalar, we have

$$\vec{PQ}=(2\tan\tfrac{1}{2}\theta)\,\mathbf{i}\times\vec{OM}=(2\tan\tfrac{1}{2}\theta\,\mathbf{i})\times\vec{OM}.$$

Thus if $\mathbf{q}$ is the displacement of the particle from P to Q due to this rotation of the rigid body, we have

$$\mathbf{q}=\vec{PQ}=\mathbf{e}\times\mathbf{h}, \text{ where}$$

$$\mathbf{e}=(2\tan\tfrac{1}{2}\theta)\,\mathbf{i} \text{ and } \mathbf{h}=\vec{OM}=\tfrac{1}{2}(\mathbf{r}+\mathbf{r}').$$

The vector $\mathbf{e}$ is called the finite rotation about O which brings the particle from $\mathbf{r}$ to $\mathbf{r}'$, the direction of the vector $\mathbf{e}$ is called the axis of rotation and θ is called the angle of rotation.

When the rotation is small, Q tends to P *i.e.*, $\mathbf{r}'$ tends to $\mathbf{r}$, and then we have

$$\mathbf{h}=\tfrac{1}{2}(\mathbf{r}+\mathbf{r}')\to\tfrac{1}{2}(\mathbf{r}+\mathbf{r})=\mathbf{r}$$

which tends to

$$\mathbf{q}=\mathbf{e}\times\mathbf{r}.$$

Remark. It can be easily seen that the displacement about a point is always a rotation. Also it can be easily shown that any displacement of a rigid body can be reduced to a translation together with a rotation.

§ 5. Position vector of a point after a general displacement.

Let $\mathbf{r}$ be the position vector of a point P referred to some origin O. If the particle is displaced from P to Q by giving only a displacement $\mathbf{u}$ in the direction of OP (*i.e.*, translation), then

$$\vec{PQ}=\mathbf{u}.$$

Also if P is displaced to Q by giving only a rotation $\mathbf{e}$ about O, then

$$\vec{PQ}=\mathbf{e}\times\tfrac{1}{2}(\mathbf{r}+\mathbf{r}')$$

where $\mathbf{r}$ and $\mathbf{r}'$ are the position vectors of P and Q respectively.

Now if the particle is displaced from P to Q by giving both the displacements translation $\mathbf{u}$ and rotation $\mathbf{e}$ simultaneously, then it is called a general displacement of the point. In this case, combining the above results, we have

$$\vec{PQ} = \mathbf{u} + \mathbf{e} \times \tfrac{1}{2}(\mathbf{r} + \mathbf{r}'). \qquad \ldots(1)$$

If this displacement is small, then writing $\mathbf{r}' = \mathbf{r} + d\mathbf{r}$ in the above result (1), we have

$$\vec{PQ} = \mathbf{u} + \mathbf{e} \times \tfrac{1}{2}(\mathbf{r} + \mathbf{r} + d\mathbf{r})$$
$$= \mathbf{u} + \mathbf{e} \times \mathbf{r} + \tfrac{1}{2}\mathbf{e} \times d\mathbf{r}$$
$$= \mathbf{u} + \mathbf{e} \times \mathbf{r}, \text{ because both the vectors } \mathbf{e} \text{ and } d\mathbf{r} \text{ are small.}$$

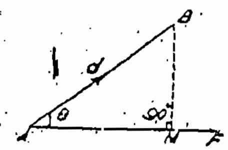
But $\vec{PQ} = \vec{OQ} - \vec{OP} = (\mathbf{r} + d\mathbf{r}) - \mathbf{r} = d\mathbf{r}$.

Therefore if $\mathbf{r}$ is the position vector of a point P and if it is given a small displacement $d\mathbf{r}$ consisting of a small translation $\mathbf{u}$ and a small rotation $\mathbf{e}$, then we have

$$d\mathbf{r} = \mathbf{u} + \mathbf{e} \times \mathbf{r}. \qquad \text{[Remember]}$$

§ 6. Work done by a force. [Meerut 74]

Suppose a force represented by the vector $\mathbf{F}$ acts at the point A. Let the point A be displaced to the point B where $\vec{AB} = \mathbf{d}$.



Then the work W done by the force $\mathbf{F}$ during the displacement $\mathbf{d}$ of its point of application is defined as

$$W = \mathbf{F} \cdot \mathbf{d}, \qquad \ldots(1)$$

where $\mathbf{F} \cdot \mathbf{d}$ is the scalar product of the vectors $\mathbf{F}$ and $\mathbf{d}$.

Let θ be the angle between the vectors $\mathbf{F}$ and $\mathbf{d}$. If $F = |\mathbf{F}|$ and $d = |\mathbf{d}| = AB$, then using the definition of the scalar product of two vectors, the equation (1) defining the work may be written as

$$W = F d \cos \theta. \qquad \ldots(2)$$

Obviously $d \cos \theta$ is the displacement of the point of application of the force $\mathbf{F}$ in the direction of the force. *Hence the work done by a force is equal to the magnitude of the force multiplied by the displacement of the point of application of the force in the direction of the force.*

From the equation (2), we make the following observations.

(i) If $\theta = \frac{1}{2}\pi$ i.e., if the displacement of the point of application of the force is perpendicular to the direction of the force, then $W = 0$.

(ii) If $0 \leqslant \theta < \frac{1}{2}\pi$ *i.e.*, if the displacement of the point of application of the force parallel to the line of action of the force is in the direction of the force, then W is positive.

(iii) If $\frac{1}{2}\pi < \theta \leqslant \pi$ *i.e.*, if the displacement of the point of application of the force parallel to the line of action of the force is opposite to the direction of the force, then W is negative.

Remark. The work done by a force $\mathbf{F}$ acting at the point $\mathbf{r}$ during a small displacement $d\mathbf{r}$ of its point of application is

$$= \mathbf{F} \cdot d\mathbf{r}.$$

§ 7. Work done by a system of concurrent forces.

Theorem. *The work done by the resultant of a number of concurrent forces is equal to the sum of the works done by the separate forces.*

Proof. Let there be n forces represented by the vectors $\mathbf{F}_1$, $\mathbf{F}_2, \ldots, \mathbf{F}_n$ acting at a point P. Then during any displacement of P represented by the vector $\mathbf{d}$, the works done by the separate forces are respectively equal to

$$\mathbf{F}_1 \cdot \mathbf{d}, \mathbf{F}_2 \cdot \mathbf{d}, \ldots, \mathbf{F}_n \cdot \mathbf{d}.$$

The total work done is therefore

$$= \mathbf{F}_1 \cdot \mathbf{d} + \mathbf{F}_2 \cdot \mathbf{d} + \ldots + \mathbf{F}_n \cdot \mathbf{d}$$
$$= (\mathbf{F}_1 + \mathbf{F}_2 + \ldots + \mathbf{F}_n) \cdot \mathbf{d}$$

[$\because$ scalar product is distributive]

$= \mathbf{R} \cdot \mathbf{d}$, where $\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \ldots + \mathbf{F}_n$ is the vector representing the resultant of these n concurrent forces.

But $\mathbf{R} \cdot \mathbf{d}$ is the work done by the resultant $\mathbf{R}$ during the displacement $\mathbf{d}$ of the point P.

Hence we have the result.

Example. *A particle acted on by constant forces* $4\mathbf{i} + \mathbf{j} - 3\mathbf{k}$ *and* $3\mathbf{i} + \mathbf{j} - \mathbf{k}$ *is displaced from the point* $\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$ *to the point* $5\mathbf{i} + 4\mathbf{j} + \mathbf{k}$. *Find the total work done by the forces.* [Kanpur 76]

Solution. Let $\mathbf{R}$ be the resultant of the two concurrent forces and $\mathbf{d}$ be the displacement of their point of application. Then, we have

$$\mathbf{R} = (4\mathbf{i} + \mathbf{j} - 3\mathbf{k}) + (3\mathbf{i} + \mathbf{j} - \mathbf{k}) = 7\mathbf{i} + 2\mathbf{j} - 4\mathbf{k}$$

and

$$\mathbf{d} = (5\mathbf{i} + 4\mathbf{j} + \mathbf{k}) - (\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) = 4\mathbf{i} + 2\mathbf{j} - 2\mathbf{k},$$

$\therefore$ the total work done $= \mathbf{R}\cdot\mathbf{d}$

$$= (7\mathbf{i}+2\mathbf{j}-4\mathbf{k})\cdot(4\mathbf{i}+2\mathbf{j}-2\mathbf{k})$$
$$= 28+4+8 = 40 \text{ units of work.}$$

§ 8. **Work done by a couple during a small displacement.**

Let the two forces $\mathbf{F}$ and $-\mathbf{F}$ acting on a rigid body at the points whose position vectors are $\mathbf{r}_1$ and $\mathbf{r}_2$ be equivalent to a couple of moment $\mathbf{G}$, then

$$\mathbf{G} = \mathbf{r}_1\times\mathbf{F}+\mathbf{r}_2\times(-\mathbf{F}) = (\mathbf{r}_1-\mathbf{r}_2)\times\mathbf{F}.$$

Suppose the body undergoes a small displacement consisting of a uniform translation $\mathbf{u}$ and a small rotation $\mathbf{e}$. Then

$$d\mathbf{r}_1 = \mathbf{u}+\mathbf{e}\times\mathbf{r}_1$$

and $$d\mathbf{r}_2 = \mathbf{u}+\mathbf{e}\times\mathbf{r}_2. \quad \text{[Refer § 5]}$$

$\therefore$ the work done by the couple

$$= \mathbf{F}\cdot d\mathbf{r}_1 + (-\mathbf{F})\cdot d\mathbf{r}_2$$
$$= \mathbf{F}\cdot(\mathbf{u}+\mathbf{e}\times\mathbf{r}_1)+(-\mathbf{F})\cdot(\mathbf{u}+\mathbf{e}\times\mathbf{r}_2)$$
$$= \mathbf{F}\cdot(\mathbf{e}\times\mathbf{r}_1)-\mathbf{F}\cdot(\mathbf{e}\times\mathbf{r}_2)$$
$$= \mathbf{e}\cdot(\mathbf{r}_1\times\mathbf{F})-\mathbf{e}\cdot(\mathbf{r}_2\times\mathbf{F}), \text{ by a property of scalar triple product}$$
$$= \mathbf{e}\cdot(\mathbf{r}_1-\mathbf{r}_2)\times\mathbf{F} = \mathbf{e}\cdot\mathbf{G},$$

which is independent of the translation and depends upon rotation only.

§ 9. **Work done by a system of forces during a small displacement.**

Let a system of forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ act at the points of a rigid body whose position vectors with respect to some origin O are $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_n$ respectively. Suppose this system of forces is equivalent to a single force $\mathbf{R}$ acting at O, together with a couple of moment $\mathbf{G}$. Then

$$\mathbf{R} = \sum_{p=1}^{n} \mathbf{F}_p \text{ and } \mathbf{G} = \sum_{p=1}^{n} \mathbf{r}_p\times\mathbf{F}_p. \quad \ldots(1)$$

If the body undergoes a small displacement consisting of a uniform translation $\mathbf{u}$ and a small rotation $\mathbf{e}$ about O, then for a typical particle displaced from $\mathbf{r}_p$ to $\mathbf{r}_p+d\mathbf{r}_p$, the general displacement $d\mathbf{r}_p$ is given by

$$d\mathbf{r}_p = \mathbf{u}+\mathbf{e}\times\mathbf{r}_p. \quad \ldots(2)$$

$\therefore$ the work done by the system of forces during this small displacement

$$= \sum_{p=1}^{n} \mathbf{F}_p\cdot d\mathbf{r}_p$$

$$= \sum_{p=1}^{n} \mathbf{F}_p\cdot(\mathbf{u}+\mathbf{e}\times\mathbf{r}_p) \quad \text{[by (2)]}$$
$$= \mathbf{u}\cdot\sum_{p=1}^{n} \mathbf{F}_p+\mathbf{e}\cdot\sum_{p=1}^{n} \mathbf{r}_p\times\mathbf{F}_p, \text{ as } \mathbf{u} \text{ and } \mathbf{e} \text{ are constant vectors}$$
$$= \mathbf{u}\cdot\mathbf{R}+\mathbf{e}\cdot\mathbf{G}, \text{ by (1).}$$

§ 10. **Virtual displacement and Virtual Work.** [Meerut 73, 74]

If a number of forces act on a body and displace it, these forces do some work actually. But if the forces are in equilibrium, then they do not displace their points of application and so there is actually no work done by these forces. However, if we imagine that the forces in equilibrium undergo some small displacement and find out the work done by the forces during that displacement, then such a displacement is called **virtual displacement** and such a work is called **virtual work.**

§ 11. **The principle of virtual work.**

The necessary and sufficient condition that a particle or a rigid body acted upon by a system of coplanar forces be in equilibrium is that the algebraic sum of the virtual works done by the forces during any small displacement consistent with the geometrical conditions of the system is zero to the first degree of approximation.

[Meerut 80, 81, 82, 83, 84, 85P, 85S, 89, 89P, 90, 90P; Lucknow 75, 76; Allahabad 75, 78; Kanpur 76, 78, 79, 80, 81, 82, 83, 87, 88; Jiwaji 89; Gorakhpur 78, 82; Rohilkhand 79, 80, 83, 86, 89]

Proof. Let a system of forces $\mathbf{F}_1, \ldots, \mathbf{F}_n$ act at the points of a rigid body whose position vectors with respect to some origin O are $\mathbf{r}_1, \ldots, \mathbf{r}_n$. Suppose this system of forces is equivalent to a single force $\mathbf{R} = \Sigma\,\mathbf{F}_1$ acting at O, together with a couple of moment $\mathbf{G} = \Sigma\mathbf{r}_1\times\mathbf{F}_1$. Then during any small displacement of the body consisting of a uniform translation $\mathbf{u}$ and a small rotation $\mathbf{e}$ about O, the sum of the works done by these forces

$$= \Sigma\,\mathbf{F}_1\cdot d\mathbf{r}_1 = \Sigma\,\mathbf{F}_1\cdot(\mathbf{u}+\mathbf{e}\times\mathbf{r}_1)$$
$$= \mathbf{u}\cdot\Sigma\,\mathbf{F}_1+\mathbf{e}\cdot\Sigma\,\mathbf{r}_1\times\mathbf{F}_1$$
$$= \mathbf{u}\cdot\mathbf{R}+\mathbf{e}\cdot\mathbf{G}. \quad \ldots(1)$$

The condition is necessary. Suppose the given system of forces is in equilibrium. Then $\mathbf{R}=0$ and $\mathbf{G}=0$. Therefore, from (1), the sum of the works done by the forces is zero. Hence the condition is necessary.

include those which are required in the final result. After giving the displacement we must note the points and the lengths that change and that do not change during the displacement. If any length or angle etc. is to change during the displacement, we should first find its value in terms of some variable symbol and then after solving the problem we should put its value in the position of equilibrium.

In many cases we are required to find the tension of an inextensible string or the thrust or tension of an inextensible rod. In order to find such a tension or thrust we must give the system a displacement in which the length of the string or the rod changes because otherwise the tension or thrust will not come in the equation of virtual work. But according to the geometrical conditions of the system we cannot give such a displacement to the body. So to get over this difficulty we replace the string or the rod by two equal and opposite forces T which are equivalent to the tension or the thirst in it. By doing so evidently the equation of virtual work is not affected while we become free to give the system a displacement in which the length of the string or the rod changes and consequently T will occur in the equation of virtual work and will thus be determined.

In any problem the virtual work done by the tension T of an extensible string of length l is $-T\delta l$ and the virtual work done by the thrust T of an extensible rod of length l is $+T\delta l$. In order to find the virtual work done by a force other than a tension or a thrust we first mark a fixed point or a fixed straight line. Then we measure the distance of the point of application of the force from this fixed point or line while moving along the line of action of the force. If this distance is x and the force is P, then the virtual work done by the force P during a small displacement is $P\delta x$ in magnitude. If the distance x is measured in the direction of the force P, the virtual work done by P is taken with positive sign and if the distance x is measured in the direction opposite to that of the force P, the virtual work done by P is taken with negative sign.

Equating to zero the total sum of the virtual works done by the forces, we get the equation of the virtual work. Solving this equation we get the value of the required thing to be determined.

If $f(x)$ is a function of x, then during a small displacement in which x changes to $x+\delta x$, we have

$$\delta f(x)=f(x+\delta x)-f(x)$$
$$=f(x)+\frac{\delta x}{1!}f'(x)+\ldots-f(x),$$

expanding $f(x+\delta x)$ by Taylor's theorem

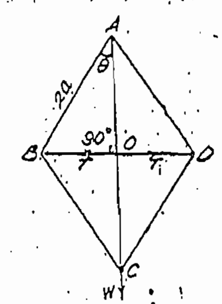
$$=f'(x)\,\delta x, \text{ to the first order of small quantities.}$$

In many cases, the only forces that remain in the equation of virtual work are those due to gravity. In such cases if W is the total weight and z the height or depth of its point of application (*i.e.*, the centre of gravity of the system) above or below a fixed horizontal level, then by the principle of virtual work for the equilibrium of the body we must have $W\delta z=0$ *i.e.*, $dz=0$, which shows that z is a *maximum or minimum* in the position of equilibrium.

Illustrative Examples

Ex. 1. *Five weightless rods of equal length are joined together so as to form a rhombus ABCD with one diagonal BD. If a weight W be attached to C and the system be suspended from A, show that there is a thrust in BD equal to* $W/\sqrt{3}$.

[Lucknow 76, 79; Kanpur 86]



Sol. Five equal weightless rods AB, BC, CD, DA and BD form the rhombus $ABCD$ and the diagonal BD. The system is suspended from A and a weight W is attached to C. Since the force of reaction at the point of suspension A balances the weight W at C, therefore the line AC must be vertical and so BD is horizontal. [Note that the diagonals of a rhombus bisect each other at right angles].

The rod BD prevents the points B and D from moving towards O and so it pushes them outwards, showing that there is a thrust in BD.

Let T be the thrust in the rod BD. Let $2a$ be the length of each of the rods AB, BC, CD and DA and let $\angle BAC=\theta$. In the position of equilibrium BD is also equal to $2a$ and so in the position of equilibrium ABD is an equilateral triangle and $\theta=\pi/6$.

To find the thrust T in the rod BD we shall have to give the system a displacement in which BD must change. So we replace the rod BD by two equal and opposite forces T as shown in the figure and then the distance BD can be changed.

Now we give the system a small symmetrical virtual displacement in which θ changes to $\theta+\delta\theta$. The point A remains fixed and so the distances will be measured from A. The points B, C and D change. The lengths of the rods AB, BC, CD and DA do not change while the length BD changes. The angle BOA will remain $90°$ because even after the displacement the figure remains a rhombus.

We have $\quad BD=2BO=2AB\sin\theta=4a\sin\theta,$

and $\quad AC=2AO=2.2a\cos\theta=4a\cos\theta.$

By the principle of virtual work, we have

$$T\delta(4a\sin\theta)+W\delta(4a\cos\theta)=0$$

or $\quad 4aT\cos\theta\,\delta\theta-4aW\sin\theta\,\delta\theta=0$

or $\quad 4a(T\cos\theta-W\sin\theta)\,\delta\theta=0$

or $\quad T\cos\theta-W\sin\theta=0 \qquad [\because \delta\theta\neq 0]$

or $\quad T\cos\theta=W\sin\theta$

or $\quad T=W\tan\theta.$

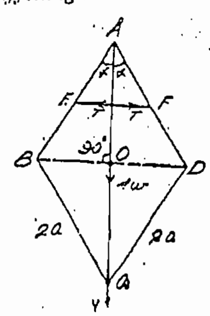
But in the position of equilibrium,

$$BD=2a \text{ and } \theta=\pi/6.$$

Therefore $\quad T=W\tan\tfrac{1}{6}\pi=W/\sqrt{3}.$

Ex. 2. *Four rods of equal weights w form a rhombus $ABCD$, with smooth hinges at the joints. The frame is suspended by the point A, and a weight W is attached to C. A stiffening rod of negligible weight joins the middle points of AB and AD, keeping these inclined at α to AC. Show that the thrust in this stiffening rod is*

$$(2W+4w)\tan\alpha.$$



Sol. $ABCD$ is a framework formed of four equal rods each of weight w and say of length $2a$. It is suspended by the point A and a weight W is attached to C. To keep the system in the form of a rhombus a light rod EF joins the middle points E and F of AB and AD respectively. Obviously the line AC must be vertical and so BD is horizontal.

We have $\quad \angle BAC=\angle DAC=\alpha.$

Let T be the thrust in the rod EF. The total weight $4w$ of all the four rods can be taken acting at the point of intersection O of the diagonals AC and BD.

Replace the rod EF by two equal and opposite forces T as shown in the figure.

Give the system a small symmetrical displacement about the vertical AC in which α changes to $\alpha+\delta\alpha$. The point A remains fixed and so the distances of the points of application of the weights $4w$ and W will be measured from A. The lengths of the rods AB, BC, CD and DA do not change, the length EF changes, the $\angle AOB$ remains $90°$ and the points O and C change.

We have

$$EF=2.AE\sin\alpha=2a\sin\alpha,$$

$AO=$ depth of O below the fixed point A

$\quad =AB\cos\alpha=2a\cos\alpha$ and $AC=2AO=4a\cos\alpha.$

By the principle of virtual work, we have

$$T\delta(2a\sin\alpha)+4w\delta(2a\cos\alpha)+W\delta(4a\cos\alpha)=0$$

or $\quad 2aT\cos\alpha\,\delta\alpha-8aw\sin\alpha\,\delta\alpha-4aW\sin\alpha\,\delta\alpha=0$

or $\quad 2a[T\cos\alpha-4w\sin\alpha-2W\sin\alpha]\,\delta\alpha=0$

or $\quad T\cos\alpha-4w\sin\alpha-2W\sin\alpha=0 \qquad [\because \delta\alpha\neq 0]$

or $\quad T\cos\alpha=(4w+2W)\sin\alpha$

or $\quad T=(2W+4w)\tan\alpha.$

Ex. 3. *Four equal heavy uniform rods are freely jointed so as to form a rhombus which is freely suspended by one angular point, and the middle points of the two upper rods are connected by a light rod so that the rhombus cannot collapse. Prove that the thrust of this light rod is*

$$4W\tan\alpha,$$

where W is the weight of each rod and 2α is the angle of the rhombus at the point of suspension. [Meerut 76; Kanpur 81]

Sol. Proceed as in Ex. 2. Here a weight W is not attached at C. The equation of virtual work is

$$T\delta(2a\sin\alpha)+4W\delta(2a\cos\alpha)=0,$$

giving $\quad T=4W\tan\alpha.$

Ex. 4. *Four equal uniform rods, each of weight w, are freely jointed to form a rhombus $ABCD$. The frame work is suspended freely from A and a weight W is attached to each of the joints B, C, D. If two horizontal forces each of magnitude P acting at B and D*

The condition is sufficient. Suppose the sum of the works done by the forces during any small displacement is zero. Then to prove that the forces are in equilibrium. We have, from (1)

$$\mathbf{u}\cdot\mathbf{R}+\mathbf{e}\cdot\mathbf{G}=0, \qquad \ldots(2)$$

for any small displacement consisting of a uniform translation u and a small rotation e about O.

Since the result (2) holds for any small displacement, therefore taking $\mathbf{e}=0$ and $\mathbf{u}\neq 0$, we get from (2)

$$\mathbf{u}\cdot\mathbf{R}=0. \qquad \ldots(3)$$

Again taking u not perpendicular to R, we get from (3)

$$\mathbf{R}=0.$$

Now taking $\mathbf{e}\neq 0$ and $\mathbf{u}=0$, we get from (2)

$$\mathbf{e}\cdot\mathbf{G}=0. \qquad \ldots(4)$$

Again taking e not perpendicular to G, we get from (4)

$$\mathbf{G}=0.$$

Thus if the result (2) holds for any small displacement u and e, we must have $\mathbf{R}=0$ and $\mathbf{G}=0$. Hence the forces are in equilibrium and this proves that the condition is sufficient.

Remark 1. The equation (2) formed by equating to zero the sum of the virtual works done by the forces is called the equation of virtual work.

Remark 2. The above principle of virtual work and its proof equally holds whether the forces are coplanar or not and whether the forces act upon a particle or upon a rigid body.

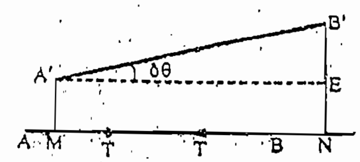
§ 12. Forces which are omitted in forming the equation of virtual work.

[Meerut 76, 85; Kanpur 79, 82, 87; Lucknow 75, 77; Allahabad 77; Gorakhpur 78, 80; Agra 75, 76; Jiwaji 81; Rohilkhand 78, 81]

The principle of virtual work gives us a very powerful method of attacking problems on equilibrium of forces. The mechanical advantage of this principle over other methods is that there are certain forces which are omitted in forming the equation of virtual work and consequently the solution of the problem becomes easy by this method. We now mention with proof the forces which are omitted in forming the equation of virtual work.

(i) *The work done by the tension of an inextensible string is zero during a small displacement.*

[Meerut 90P; Rohilkhand 77; Kanpur 83]



Let AB be an inextensible string of length l joining two points A and B of a rigid body. Let T be the tension in the string AB. After a small displacement let $A'B'$ be the position of the string and $\delta\theta$ be the small angle between AB and $A'B'$. Since the string is inextensible, therefore $A'B'=AB=l$. Draw $A'M$ and $B'N$ perpendiculars to AB. Also draw $A'E$ perpendicular to $B'N$.

On account of the tension in the string AB, there are two forces each equal to T acting on A and B in opposite directions as shown in the figure. After displacement A moves to A' and B moves to B'. The work done by the tension of the string AB during this displacement

$=T.AM-T.BN$ [Note that the displacement of B is in a direction opposite to that of the force T]

$=T.(AB-MB)-T.(MN-MB)$

$=T.(AB-MN)$

$=T.(AB-A'E)=T.(AB-A'B'\cos\delta\theta)$

$=T.(l-l\cos\delta\theta)$ $\quad [\because\ AB=A'B'=l]$

$=T.l\,(1-\cos\delta\theta)$

$=T.l\left[1-\left\{1-\dfrac{(\delta\theta)^2}{2!}+\ldots\ldots\right\}\right]$, expanding $\cos\delta\theta$ in powers of $\delta\theta$

$=T.l.0$, to the first order of small quantities

$=0.$

Alternative Method.

Let T be the tension in an inextensible string connecting two points A and B whose position vectors are $\mathbf{r}_1$ and $\mathbf{r}_2$. Then a force $\mathbf{T}$ acts at $\mathbf{r}_1$ and $-\mathbf{T}$ acts at $\mathbf{r}_2$. Since the string is inextensible, therefore for any displacement of A and B, we have

$$(\mathbf{r}_1-\mathbf{r}_2)^2=\text{constant}.$$

Differentiating,

$$2(\mathbf{r}_1-\mathbf{r}_2)\cdot(d\mathbf{r}_1-d\mathbf{r}_2)=0$$

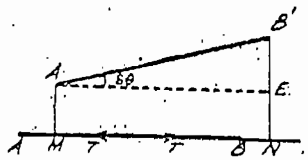
i.e., $\quad 2\mathbf{T}\cdot(d\mathbf{r}_1-d\mathbf{r}_2)=0 \quad [\because\ \mathbf{T}$ is parallel to $\mathbf{r}_1-\mathbf{r}_2]$

or $\quad \mathbf{T}\cdot d\mathbf{r}_1+(-\mathbf{T})\cdot d\mathbf{r}_2=0,$

showing that the total work done by $\mathbf{T}$ at $\mathbf{r}_1$ and $-\mathbf{T}$ at $\mathbf{r}_2$ during a small displacement is zero. Hence the work done by the tension of an inextensible string is zero during a small displacement.

(ii) *The work done by the thrust of an inextensible rod is zero during a small displacement.*

Let T be the thrust in an inextensible rod AB joining two points A and B of a rigid body. Proceed as in part (i). Here the work done by the thrust in the rod AB during a small displacement



$$= -T.AM+T.BN=0.$$

Remark. The forces of tension act inward and the forces of thrust act outwards. A common name for tension and thrust is stress. From (i) and (ii) we conclude that *if the distance between two particles of a system is invariable, the work done by the mutual action and reaction between the two particles is zero.*

(iii) *The reaction R of any smooth surface with which the body is in contact does no work.* For, if the surface is smooth, the reaction R on the point of contact A is along the normal to the surface. If A moves to a neighbouring point B, then the displacement AB is right angles to the direction of the force and so the work done by R is zero. If however, the surface is rough, the work done by the frictional force F *i.e.*, $F.(-AB)$ will come into the equation of virtual work.

(iv) *If a body rolls without sliding on any fixed surface, the work done in a small displacement by the reaction of the surface on the rolling body is zero.* For, the point of contact of the body is for the moment at rest, and so the normal reaction and the force of friction at the point of contact have zero displacements.

(v) *The work done by the mutual reaction between two bodies of a system is zero in any virtual displacement of the system.* For action and reaction are equal and opposite and so the work done by the action balances that done by the reaction.

(vi) *If a body is constrained to turn about a fixed point or a fixed axis, the virtual work of the reaction at the point or on the axis is zero.* For in this case the displacement of the point of application of the force is zero.

§ 13. (i) *To show that the work done by the tension T of an extensible string of length l during a small displacement is $-T\delta l$.*

[Allahabad 78; Jiwaji 82]

Refer figure on page 9.

Let T be the tension in an extensible string AB of length l joining two points A and B of a rigid body. After a small displacement let $A'B'$ be the position of the string and $\delta\theta$ be the small angle between AB and $A'B'$. Since the string is extensible, therefore let $A'B'=l+\delta l$. Draw $A'M$ and $B'N$ perpendiculars to AB. Also draw $A'E$ perpendicular to $B'N$.

On account of the tension in the string AB, there are two forces each equal to T acting on A and B in the opposite directions AB and BA respectively. After displacement A moves to A' and B moves to B'. The work done by the tension of the string AB during the displacement

$$=T.AM-T.BN$$
$$=T.(AB-MB)-T.(MN-MB)$$
$$=T.(AB-MN)=T.(AB-A'E)$$
$$=T.(AB-A'B'\cos\delta\theta)$$
$$=T.[l-(l+\delta l)\cos\delta\theta]$$
$$=T.\left[l-(l+\delta l)\left\{1-\frac{(\delta\theta)^2}{2!}+\ldots\right\}\right], \text{ expanding } \cos\delta\theta \text{ in powers of } \delta\theta$$
$$=T.[l-l-\delta l], \text{ to the first order of small quantities}$$
$$=-T.\delta l.$$

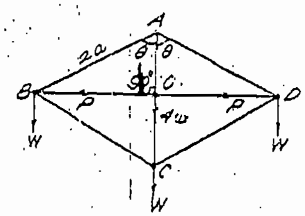
(ii) Similarly it can be shown that *the work done by the thrust T of an extensible rod of length l during a small displacement is $T\delta l$.*

[Allahabad 79]

§ 14. Application of the principle of virtual work. While applying the principle of virtual work we can give any small displacement to the system provided it is consistent with the geometrical conditions of the system. This displacement should be such as to exclude the forces which are not required and to *keep the angle BAD equal to 120°, prove that*

$$P=(W+w)\,2\sqrt{3}.$$

Sol. $ABCD$ is a framework formed of four equal rods each of weight w and say of length $2a$. It is suspended from the point A and a weight W is attached to each of the points B, C and D. To save the system from collapsing two horizontal forces each of magnitude P act at B and D and in equilibrium $\angle BAD=120°$. Obviously the nature of the forces P is like that of thrust.



The total weight $4w$ of all the four rods AB, BC, CD and DA can be taken acting at the point of intersection O of the diagonals AC and BD. Obviously the line AC must be vertical and so BD is horizontal.

To find P we have to give the system a displacement in which the length BD must change and consequently the angle BAD will change so let us assume that $\angle BAC=\theta=\angle DAC$.

Now give the system a small symmetrical displacement about AC in which θ changes to $\theta+\delta\theta$. The point A remains fixed and we shall measure the distances of the points of application of various forces from the point A. The points B, C, D and O change. The lengths of the rods AB, BC, CD and DA do not change while the length BD changes. The angle AOB will remain 90°.

We have

$$BD=2BO=2AB\sin\theta=4a\sin\theta,$$

the depth of B or D or O below A

$$=AO=2a\cos\theta,$$

and the depth of C below A

$$=AC=2AO=4a\cos\theta.$$

By the principle of virtual work, we have

$$P\delta\,(4a\sin\theta)+4w\delta\,(2a\cos\theta)+2W\delta\,(2a\cos\theta)+W\delta\,(4a\cos\theta)=0$$

or $\quad 4a\,P\cos\theta\,\delta\theta-8aw\sin\theta\,\delta\theta-4aW\sin\theta\,\delta\theta-4aW\sin\theta\,\delta\theta=0$

or $\quad 4a\,[P\cos\theta-2w\sin\theta-W\sin\theta-W\sin\theta]\,\delta\theta=0$

or $\quad P\cos\theta-2\,(W+w)\sin\theta=0$

or $\quad P=2\,(W+w)\tan\theta.$

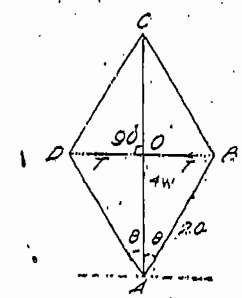
But in the position of equilibrium, $\theta=60°$,

$\therefore\quad P=2\,(W+w)\tan 60°=2\,(W+w)\sqrt{3}=(W+w)\,2\sqrt{3}.$

Ex. 5. *Four equal uniform rods, each of weight W, are jointed to form a rhombus ABCD, which is placed in a vertical plane with AC vertical and A resting on a horizontal plane. The rhombus is kept in the position in which $\angle BAC=\theta$ by a light string joining B and D. Find the tension of the string.*

(Jiwaji 79; Kanpur 80, 88; P.C.S. 74)

Sol. $ABCD$ is a framework formed of four equal rods each of weight W and say of length $2a$. It is placed in a vertical plane with AC vertical and A resting on a horizontal plane. To keep the system in the form of a rhombus a light string joins B and D and prevents the points B and D from moving in the directions OB and OD respectively. Let T be the tension in the string BD. The total weight $4W$ of all the four rods may be taken acting at the point of intersection O of the diagonals AC and BD.



Let $\angle DAC=\theta=\angle BAC$.

Give the system a small symmetrical displacement about AC in which θ changes to $\theta+\delta\theta$. The point A resting on the horizontal plane remains fixed. The points B, C, D and O will change. The lengths of the rods AB, BC, CD and DA will remain fixed while the length BD will change. The angle DOC will remain 90°.

We have $\quad BD=2BO=2AB\sin\theta=4a\sin\theta,$

and the height of O above the fixed point A

$$=AO=2a\cos\theta.$$

By the principle of virtual work, we have

$$-T\delta\,(4a\sin\theta)-4W\delta\,(2a\cos\theta)=0. \qquad \ldots(1)$$

[Note that in the equation (1) the work done by the weight $4W$ has been taken with negative sign because the distance AO of its point of application O from the fixed point A is in a direction opposite to the direction of $4W$.]

From the equation (1), we have

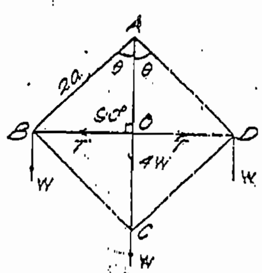
$$-4aT\cos\theta\,\delta\theta+8aW\sin\theta\,\delta\theta=0$$

or $\quad 4a\,[-T\cos\theta+2W\sin\theta]\,\delta\theta=0$

or $\quad -T\cos\theta+2W\sin\theta=0 \qquad [\because\ \delta\theta\neq 0]$

or $\quad T=2W\tan\theta.$

Ex. 6. *A square framework, formed of uniform heavy rods of equal weight W, jointed together, is hung up by one corner. A weight W is suspended from each of the three lower corners and the shape of the square is preserved by a light rod along the horizontal diagonal. Find the thrust of the light rod.*



Sol. $ABCD$ is a square framework formed of four rods each of weight W and say of length $2a$. It is suspended from the point A and a weight W is suspended from each of the three lower corners B, C and D. A light rod along the horizontal diagonal BD prevents the system from collapsing. Let T be the thrust in the rod BD. The total weight $4W$ of the rods AB, BC, CD and DA can be taken as acting at O.

To find T we shall have to give the system a displacement in which BD must change. So replace the rod BD by two equal and opposite forces T as shown in the figure and assume that $\angle BAC=\theta=\angle CAD$. [Note that the angle BAC will change during a displacement in which BD is to change.]

Now give the system a small symmetrical displacement about AC in which θ changes to $\theta+\delta\theta$. The point A remains fixed and the points B, O, D and C change. The lengths of the rods AB, BC, CD and DA do not change while the length BD changes.

We have $BD=2BO=2AB\sin\theta=4a\sin\theta$,

the depth of each of the points B, C and D below the fixed point A

$$=AO=2a\cos\theta,$$

and the depth of C below $A=2AO=4a\cos\theta$.

$\therefore$ By the principle of virtual work, we have

$$T\,\delta(4a\sin\theta)+4W\,\delta(2a\cos\theta)+2W\,\delta(2a\cos\theta)$$
$$+W\,\delta(4a\cos\theta)=0$$

or $4aT\cos\theta\,\delta\theta-8aW\sin\theta\,\delta\theta-4aW\sin\theta\,\delta\theta-4aW\sin\theta\,\delta\theta=0$

or $4a(T\cos\theta-4W\sin\theta)\,\delta\theta=0$

or $T\cos\theta-4W\sin\theta=0$ $\quad[\because\ \delta\theta\neq 0]$

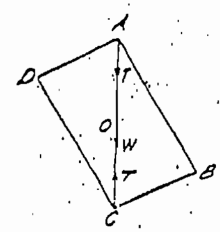
or $T=4W\tan\theta$.

But in the position of equilibrium $\theta=45°$.

$\therefore$ $T=4W\tan 45°=4W=$ the total weight of the four rods.

Ex. 7. *Four uniform rods are freely jointed at their extremities and form a parallelogram ABCD, which is suspended by the joint A, and is kept in shape by a string AC. Prove that the tension of the string is equal to half the weight of all the four rods.*

[Rohilkhand 88; Meerut 80, 88]



Sol. $ABCD$ is a framework in the shape of a parallelogram formed of four uniform rods. It is suspended from the point A and is kept in shape by a string AC. Let T be the tension in the string AC. The total weight W of all the four rods AB, BC, CD and DA can be taken as acting at O, the middle point of AC. Since the force of reaction at the point of suspension A balances the weight W at O, therefore the line AO must be vertical. Let $AC=2x$.

Give the system a small displacement in which x changes to $x+\delta x$ and AC remains vertical. The point A remains fixed, the point O changes and the length AC changes. We have, $AO=x$.

By the principle of virtual work, we have

$$-T\,\delta(AC)+W\,\delta(AO)=0$$

or $\quad -T\,\delta(2x)+W\,\delta(x)=0$

or $\quad -2T\,\delta x+W\,\delta x=0$

or $\quad (-2T+W)\,\delta x=0$

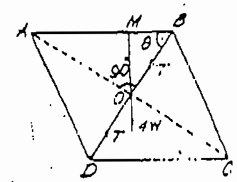
or $\quad -2T+W=0$ $\quad[\because\ \delta x\neq 0]$

or $\quad T=\frac{1}{2}W=\frac{1}{2}$ (total weight of all the four rods).

Ex. 8. *A string, of length a, forms the shorter diagonal of a rhombus formed of four uniform rods, each of length b and weight W, which are hinged together. If one of the rods be supported in a horizontal position, prove that the tension of the string is*

$$= \frac{2W(2b^2-a^2)}{b\sqrt{(4b^2-a^2)}}.$$

[Meerut 86, 88S, 90; Lucknow 77, 78, 79, 80; Rohilkhand 81, 85, 87; Kanpur, 86, P. C. S. 76]



Sol. $ABCD$ is a framework in the shape of a rhombus formed of four equal uniform rods each of length b and weight W. The rod AB is fixed in a horizontal position and B and D are joined by a string of length a forming the shorter diagonal of the rhombus.

Let T be the tension in the string BD. The total weight $4W$ of the rods AB, BC, CD and DA can be taken as acting at the point of intersection O of the diagonals AC and BD. We have $\angle AOB = 90°$.

Let $\angle ABO = \theta$. Draw OM perpendicular to AB.

Give the system a small symmetrical displacement in which θ changes to $\theta + \delta\theta$. The line AB remains fixed. The points O, C and D change. The lengths of the rods AB, AC, CD and DA do not change while the length BD changes. The $\angle AOB$ will remain $90°$.

We have $BD = 2BO = 2AB \cos\theta = 2b \cos\theta$.

[Note that in the position of equilibrium $BD = a$. But during the displacement BD changes and so we have found BD in terms of θ.]

The depth of O below the fixed line $AB = MO$,

$= BO \sin\theta = (AB \cos\theta) \sin\theta = b \sin\theta \cos\theta$.

By the principle of virtual work, we have

$$-T\delta(2b\cos\theta) + 4W\delta(b\sin\theta\cos\theta) = 0$$

or $\quad 2bT \sin\theta\,\delta\theta + 4bW(\cos^2\theta - \sin^2\theta)\,\delta\theta = 0$

or $\quad 2b\,[T\sin\theta - 2W(\sin^2\theta - \cos^2\theta)]\,\delta\theta = 0$

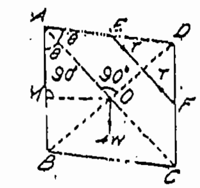
or $\quad T\sin\theta - 2W(\sin^2\theta - \cos^2\theta) = 0 \qquad [\because \delta\theta \neq 0]$

or $\quad T = \dfrac{2W(\sin^2\theta - \cos^2\theta)}{\sin\theta} = \dfrac{2W(1-2\cos^2\theta)}{\sqrt{(1-\cos^2\theta)}}$.

In the position of equilibrium, $BD = a$ or $BO = \frac{1}{2}a$. So in the position of equilibrium, $\cos\theta = \dfrac{BO}{AB} = \dfrac{\frac{1}{2}a}{b} = \dfrac{a}{2b}$.

$$\therefore \quad T = \frac{2W\{1-2(a^2/4b^2)\}}{\sqrt{\{1-(a^2/4b^2)\}}} = \frac{2W(2b^2-a^2)}{b\sqrt{(4b^2-a^2)}}.$$

Ex. 9. *Four equal uniform rods, each of weight W, are smoothly joined so as to form a square $ABCD$; the side AB is fixed (clamped) in a vertical position with A uppermost and the figure is kept in shape by a string joining the middle points of AD and DC. Find the tension of the string.*



Sol. $ABCD$ is a framework formed of four equal uniform rods each of weight W and say of length $2a$. The side AB is fixed in a vertical position with A uppermost. A string joins the middle points E and F of AD and DC respectively and in equilibrium $ABCD$ is a square.

Let T be the tension in the string EF. The total weight $4W$ of all the rods AB, BC, CD and DA acts at O, the point of intersection of the diagonals AC and BD. We have, $\angle AOD = 90°$. Let $\angle BAC = \theta = \angle DAC$. Draw OM perpendicular to AB.

[Note that we have drawn $ABCD$ as a rhombus and not as a square because in a displacement in which EF is to change the figure will not remain a square. After finding the value of the tension T we shall use the fact that in the position of equilibrium the figure is a square].

Give the system a small symmetrical displacement in which θ changes to $\theta + \delta\theta$. The line AB will remain fixed and so A is a fixed point. The points C, D and O will change. The lengths of the rods AB, BC, CD and DA do not change while the length EF changes. The $\angle AOD$ remains $90°$.

We have $EF = \frac{1}{2}AC = AO = AD\cos\theta = 2a\cos\theta$. Also the depth of O below the fixed point A i.e., the distance of O from the fixed point A in the direction of the force $4W$

$$= AM = AO\cos\theta = (2a\cos\theta)\cos\theta = 2a\cos^2\theta.$$

By the principle of virtual work, we have

$$-T\delta(2a\cos\theta) + 4W\delta(2a\cos^2\theta) = 0$$

or $\quad 2aT\sin\theta\,\delta\theta - 16aW\cos\theta\sin\theta\,\delta\theta = 0$

or $\quad 2a\sin\theta\,(T - 8W\cos\theta)\,\delta\theta = 0$

or $\quad T - 8W\cos\theta = 0 \qquad [\because \delta\theta \neq 0$ and $\sin\theta \neq 0]$

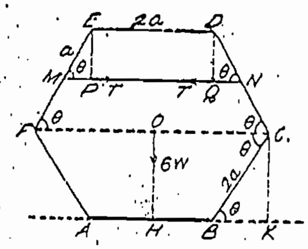
or $\quad T = 8W\cos\theta$.

But in the position of equilibrium, $\theta = 45°$.

$$\therefore \quad T = 8W\cos 45° = 8W(1/\sqrt{2}) = 4W\sqrt{2}.$$

Ex. 10. *Six equal heavy beams are freely jointed at their ends to form a hexagon, and are placed in a vertical plane with one beam resting on a horizontal plane; the middle points of the two upper slant beams, which are inclined at an angle θ to the horizon, are connected by a light cord. Find its tension in terms of W and θ, where W is the weight of each beam.* [Meerut 82]

Sol. $ABCDEF$ is a hexagon formed of six equal heavy beams each of weight W and say of length $2a$. The frame is placed in a vertical plane with the beam AB resting on a horizontal plane. To save the system from collapsing the middle points M and N of the beams FE and CD are connected by a light cord. Let T be the tension in the cord MN.



The line FC is horizontal. We have $\angle EFC = \theta = \angle DCF$.

Draw EP and DQ perpendiculars to MN.

The total weight $6W$ of all the six rods can be taken, acting at O, the middle point of FC. Draw OH and CK perpendiculars to AB. We have $\angle CBK = \theta$.

Give the system a small symmetrical displacement about the vertical line OH in which θ changes to $\theta + \delta\theta$. The line AB on the horizontal plane remains fixed, and the distance of the point of application O of the weight $6W$ will be measured from AB. The lengths of the rods AB, BC etc. remain fixed while the length MN changes. The point O also changes.

We have $\quad MN = MP + PQ + QN$

$$= a \cos \theta + 2a + a \cos \theta = 2a + 2a \cos \theta.$$

[Note that $PQ = ED = 2a$, because ED remains fixed].

Also the height of O above the fixed line AB

$$= HO = KC = 2a \sin \theta.$$

By the principle of virtual work, we have

$$-T\delta\,(2a + 2a \cos \theta) - 6W\,\delta\,(2a \sin \theta) = 0$$

[The work done by $6W$ is taken with $-$ive sign because the direction of HO is opposite to that of $6W$]

or $\quad 2aT \sin \theta\, \delta\theta - 12a\, W \cos \theta\, \delta\theta = 0$

or $\quad 2a\,(T \sin \theta - 6W \cos \theta)\, \delta\theta = 0$

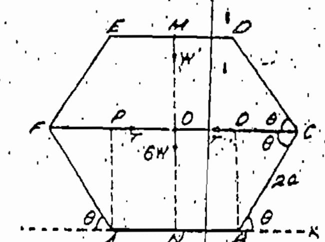
or $\quad T \sin \theta - 6W \cos \theta = 0 \qquad [\because \delta\theta \neq 0]$

or $\quad T = 6W \cot \theta.$

Ex. 11. *A regular hexagon $ABCDEF$ consists of six equal rods which are each of weight W and are freely joined together. The two opposite angles C and F are connected by a string, which is horizontal, AB being in contact with a horizontal plane. A weight W' is placed at the middle point of DE. Show that the tension of the string is $(3W + W')/\sqrt{3}$.*

[Meerut 79, 83, 86P, 80P; Rohilkhand 80; Kanpur 77]

Sol. $ABCDEF$ is a hexagon formed of six equal rods each of weight W and say of length $2a$. The hexagon rests in a vertical plane with AB in contact with a horizontal plane. The opposite points C and F are connected by a string and a weight W' is placed at the middle point M of DE. Let T be the tension in the string FC. The total weight $6W$ of the six rods AB, BC etc. can be taken acting at O, the middle point of FC.



Suppose BC and AF are inclined at an angle θ to the horizontal. Then in the position of equilibrium $\theta = \pi/3$ because in the position of equilibrium the hexagon is given to be a regular one and so $\angle CBK = 60°$.

Give the system a small symmetrical displacement about the vertical line MON in which θ changes to $\theta + \delta\theta$. The line AB on the horizontal plane remains fixed. The lengths of the rods AB, BC etc. do not change, the length FC changes and the points O and M also change.

We have

$$FC = FP + PQ + QC = 2a \cos \theta + 2a + 2a \cos \theta$$
$$= 2a + 4a \cos \theta,$$

the height of O above AB

$$= NO = BQ = 2a \sin \theta,$$

and the height of M above AB

$$= NM = 2NO = 4a \sin \theta.$$

By the principle of virtual work, we have

$$-T\delta\,(2a+4a\cos\theta)-6W\delta\,(2a\sin\theta)-W'\delta\,(4a\sin\theta)=0$$

or $\quad 4aT\sin\theta\,\delta\theta-12aW\cos\theta\,\delta\theta-4aW'\cos\theta\,\delta\theta=0$

or $\quad 4a\,[T\sin\theta-3W\cos\theta-W'\cos\theta]\,\delta\theta=0$

or $\quad T\sin\theta-(3W+W')\cos\theta=0 \qquad [\because\ \delta\theta\neq 0]$

or $\quad T=(3W+W')\cot\theta.$

But in the position of equilibrium, $\theta=60°$. Therefore

$$T=(3W+W')\cot 60°=(3W+W')/\sqrt{3}.$$

Ex. 12. *A regular hexagon ABCDEF consists of six equal rods which are each of weight W and are freely jointed together. The hexagon rests in a vertical plane and AB is in contact with a horizontal table. If C and F be connected by a light string, prove that its tension is $W\sqrt{3}$.* [Agra 76; Kanpur 79]

Sol. Proceed as in Ex. 11. Here a weight W' is not placed at the middle point M of DE otherwise the question is the same.

The equation of virtual work is

$$-T\delta\,(2a+4a\cos\theta)-6W\delta\,(2a\sin\theta)=0,$$

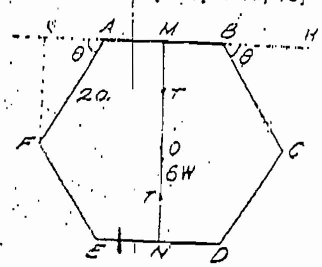
giving $\quad T=3W\cot\theta.$

But in the position of equilibrium $\theta=60°$. Therefore

$$T=3W\cot 60°=3W/\sqrt{3}=W\sqrt{3}.$$

Ex. 13. *Six equal rods AB, BC, CD, DE, EF and FA are each of weight W and are freely jointed at their extremities so as to form a hexagon; the rod AB is fixed in a horizontal position and the middle points of AB and DE are jointed by a string; prove that its tension is 3W.* [Kanpur 76, 78]

Sol. $ABCDEF$ is a hexagon formed of six equal rods each of weight W and say of length $2a$. The rod AB is fixed in a horizontal position and the middle points M and N of AB and DE are jointed by a string. Let T be the tension in the string MN. The total weight $6W$ of all the six rods AB, BC etc. can be taken acting at O, the middle point of MN. Let



$$\angle FAK=\theta=\angle CBH.$$

Give the system a small symmetrical displacement about the

vertical line MN in which θ changes to $\theta+\delta\theta$. The line AB remains fixed. The lengths of the rods AB, BC etc. remain fixed, the length MN changes and the point O also changes.

We have

$$MN=2MO=2KF=2AF\sin\theta=4a\sin\theta.$$

Also the depth of O below the fixed line AB

$$=MO=2a\sin\theta.$$

By the principle of virtual work, we have

$$-T\delta\,(4a\sin\theta)+6W\delta\,(2a\sin\theta)=0$$

or $\quad -4aT\cos\theta\,\delta\theta+12aW\cos\theta\,\delta\theta=0$

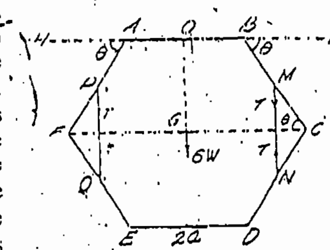
or $\quad 4a\,[-T+3W]\cos\theta\,\delta\theta=0$

or $\quad -T+3W=0 \qquad [\because\ \delta\theta\neq 0$ and $\cos\theta\neq 0]$

or $\quad T=3W.$

Ex. 14. *Six equal bars are freely jointed at their extremities forming a regular hexagon ABCDEF which is kept in shape by vertical strings joining the middle points of BC, CD and AF, FE respectively, the side AB being held horizontal and uppermost. Prove that the tension of each string is three times the weight of a bar.*

Sol. $ABCDEF$ is a hexagon formed of six equal bars say each of weight W and length $2a$. The rod AB is held horizontal and uppermost. The middle points M and N of BC and CD are joined by a string and the middle points P and Q of AF and FE are also joined by a string. Let T be the tension in each of the strings PQ and MN. The total weight $6W$ of all the six rods AB, BC etc. can be taken acting at G, the middle point of FC.



Let $\quad \angle HAF=\theta=\angle KBC.$

Give the system a small symmetrical displacement about the vertical line OG in which θ changes to $\theta+\delta\theta$. The line AB remains fixed. The lengths of the rods AB, BC etc. remain fixed, the lengths MN and PQ change and the point G also changes.

We have

$$PQ=MN=2MC\sin\theta=2a\sin\theta.$$

Also the depth of G below AB

$$=OG=BC\sin\theta=2a\sin\theta.$$

By the principle of virtual work, we have

$$-2T\delta\,(2a \sin \theta)+6W\delta\,(2a \sin \theta)=0$$

or $$-4aT \cos \theta\,\delta\theta+12aW \cos \theta\,\delta\theta=0$$

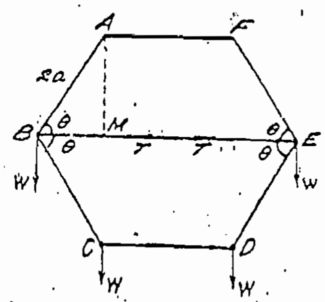
or $$4a \cos \theta\,(-T+3W)\,\delta\theta=0$$

or $$-T+3W=0 \qquad [\because\ \delta\theta\neq 0 \text{ and } \cos \theta\neq 0]$$

or $T=3W$ i.e., the tension of each string is three times the weight of a bar.

Ex. 15. *Six equal light rods are joined to form a hexagon ABCDEF which is suspended at A and F so that AF is horizontal. A rod BE, also light, keeps the figure from collapsing and is of such a length that the rods ending in the points B, E are inclined at an angle of 45° to the vertical. Equal weights are suspended from B, C, D, E. Find the stress in BE.*

Sol. $ABCDEF$ is a hexagon formed of six equal light rods say each of length $2a$. It is suspended at A and F so that AF is horizontal. Equal weights W are suspended from each of the points B, C, D and E. A light rod joining B and E saves the system from collapsing. Let T be the stress in the rod BE. Since the rod BE prevents the points B and E from moving inwards, therefore the stress in the rod BE is a thrust.



Let $\angle ABE=\theta=\angle FEB=\angle CBE=\angle DEB$.

Replace the rod BE by two equal and opposite forces T as shown in the figure. Give the system a small symmetrical displacement in which θ changes to $\theta+\delta\theta$. The line AF remains fixed. The points B, C, D and E change. The lengths of the rods AB, BC etc. do not change while the length BE changes.

We have

$$BE=AF+2BM=2a+2.2a \cos \theta=2a+4a \cos \theta,$$

the depth of each of the points B and E below AF

$$=AM=2a \sin \theta,$$

and the depth of each of the points C and D below AF

$$=2AM=4a \sin \theta.$$

By the principle of virtual work, we have

$$T\delta\,(2a+4a \cos \theta)+2W\delta\,(2a \sin \theta)+2W\delta\,(4a \sin \theta)=0$$

or $$-4aT \sin \theta\,\delta\theta+4aW \cos \theta\,\delta\theta+8aW \cos \theta\,\delta\theta=0$$

or $$4a\,(-T \sin \theta+W \cos \theta+2W \cos \theta)\,\delta\theta=0$$

or $$-T \sin \theta+3W \cos \theta=0 \qquad [\because\ \delta\theta\neq 0].$$

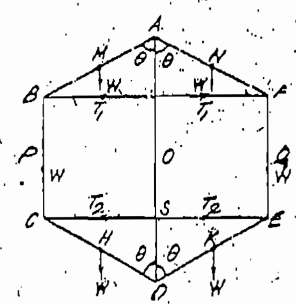
or $$T=3W \cot \theta.$$

But in the position of equilibrium each of the rods AB, BC, EF and ED makes an angle 45° with the vertical and so also with the horizontal BE. Therefore in the position of equilibrium, $\theta=45°$.

and $$T=3W \cot 45°=3W.$$

Ex. 16. *Six equal heavy rods, freely hinged at the ends, form a regular hexagon ABCDEF, which when hung up by the point A is kept from altering its shape by two light rods BF and CE. Prove that the thrusts of these rods are $(5\sqrt{3}/2)\,W$ and $(\sqrt{3}/2)\,W$, where W is the weight of each rod.* (Meerut 81, 84R, 85S)

Sol. Let the length of each of the rods AB, BC etc. be $2a$ and let θ be the angle which each of the slant rods AB, AF, DC and DE makes with the vertical AD.



Let T_1 and T_2 be the thrusts in the rods BF and CE respectively. Here A is the fixed point. The weights of the rods AB, BC etc. act at their respective middle points as shown in the figure.

Let us first find the thrust T_1.

Replace the rod BF by two equal and opposite forces T_1 as shown in the figure and keep the rod CE intact so that during any displacement the length CE does not change. Now give the system a small symmetrical displacement about the vertical line AD in which θ at the end A changes to $\theta+\delta\theta$ while θ at the end D does not change. The portion $BCDEF$ moves as it is. The length BF changes while the length CE does not change so that during this small displacement the work done by the thrust T_2 of the rod CE is zero. The centres of gravity of all the six rods AB, BC etc. are slightly displaced.

We have $$BF=4a \sin \theta.$$

In this case we cannot take the total weight of the rods AB, BC etc. act at the middle point O of AD. The depth of each of the points M and N below A is $a \cos \theta$, the depth of each of the points P and Q below A is $2a \cos \theta+a$, and the depth of each of the points H and K below A is $2a\cos\theta+2a+\frac{1}{2}SD$ where in this case SD is fixed.

By the principle of virtual work, we have

$$T_1\delta(4a\sin\theta)+2W\delta(a\cos\theta)+2W\delta(2a\cos\theta+a)$$
$$+2W\delta(2a\cos\theta+2a+\tfrac{1}{2}SD)=0$$

or $\quad 4aT_1\cos\theta\,\delta\theta-10aW\sin\theta\,\delta\theta=0$

or $\quad 2a(2T_1\cos\theta-5W\sin\theta)\,\delta\theta=0$

or $\quad 2T_1\cos\theta-5W\sin\theta=0 \qquad [\because \delta\theta\neq 0]$

or $\quad T_1=\frac{5}{2}W\tan\theta.$

But in the position of equilibrium, the hexagon is a regular one and so $\theta=\pi/3$.

Therefore $T_1=\frac{5}{2}W\tan\frac{1}{3}\pi=\frac{5}{2}W\sqrt{3}.$

Now let us proceed to find the thrust T_2.

Replace the rod BF by two equal and opposite forces T_1 as shown in the figure and so replace the rod CE by two equal and opposite forces T_2 as shown in the figure. Give the system a small symmetrical displacement about the vertical line AD in which θ at both the ends A and D changes to $\theta+\delta\theta$ so that both the lengths BF and CE change. In this case the total weight $6W$ of all the six rods AB, BC etc. can be taken acting at the middle point O of AD.

We have

$$BF=4a\sin\theta,\ CE=4a\sin\theta \text{ and } AO=2a\cos\theta+a.$$

By the principle of virtual work, we have

$$T_1\delta(4a\sin\theta)+T_2\delta(4a\sin\theta)+6W\delta(2a\cos\theta+a)=0$$

or $\quad 4aT_1\cos\theta\,\delta\theta+4aT_2\cos\theta\,\delta\theta-12aW\sin\theta\,\delta\theta=0$

or $\quad 4a[(T_1+T_2)\cos\theta-3W\sin\theta]\,\delta\theta=0$

or $\quad (T_1+T_2)\cos\theta-3W\sin\theta=0 \qquad [\because \delta\theta\neq 0]$

or $\quad T_1+T_2=3W\tan\theta.$

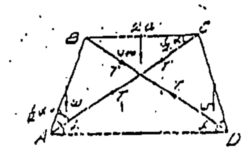
But in the position of equilibrium $\theta=\pi/3$.

$\therefore \quad T_1+T_2=3W\tan\frac{1}{3}\pi=3W\sqrt{3}.$

$\therefore \quad T_2=3W\sqrt{3}-T_1=3W\sqrt{3}-\dfrac{5W\sqrt{3}}{2}=\dfrac{W\sqrt{3}}{2}.$

Ex. 17. *Three equal uniform rods AB, BC, CD each of weight w, are freely jointed together at B and C, and rest in a vertical plane, A and D being in contact with a smooth horizontal table. Two equal light strings AC and BD help to support the framework, so that AB and CD are each inclined at an angle α to the horizontal. Show that if a mass of weight W be placed on BC at its middle point, then tension of each string will be of magnitude*

$$(w+\tfrac{1}{2}W)\cos\alpha\operatorname{cosec}\tfrac{1}{2}\alpha.$$



Sol. AB, BC and CD are three equal uniform rods each of weight w and say of length $2a$. The rods are freely jointed together at B and C and the frame rests in a vertical plane with the points A and D in contact with a smooth horizontal table. AC and BD are two equal light strings and $\angle BAD=\angle CDA=\alpha$. Obviously BC is horizontal. Let T be the tension in each of the strings AC and BD.

A mass of weight W is placed on BC at its middle point. The weights of the rods may be taken as acting at their middle points so that the total weight on the middle point of BC is $w+W$.

Since $AB=BC$, therefore $\angle BAC=\angle BCA=\angle CAD=\frac{1}{2}\alpha$.

Here the fixed horizontal level is AD. We have,

the height of the middle point of the rod AB or DC above AD

$$=a\sin\alpha,$$

and the height of the middle point of the rod BC above AD

$$=2a\sin\alpha.$$

The length of the string AC or $BD=4a\cos\frac{1}{2}\alpha$.

Give the system a small symmetrical displacement in which α changes to $\alpha+\delta\alpha$. The level of the line AD lying on the table remains fixed and the points A and D move on this line. The lengths of the rods AB, BC and CD do not change while the lengths of the strings AC and BD change. The middle points of the rods AB, BC and CD are slightly displaced.

The equation of virtual work is

$$-2T\delta(4a\cos\tfrac{1}{2}\alpha)-2w\delta(a\sin\alpha)-(w+W)\delta(2a\sin\alpha)=0$$

or $\quad -2T.4a.(-\sin\frac{1}{2}\alpha).\frac{1}{2}\delta\alpha-2aw\cos\alpha\,\delta\alpha$
$$-2a(w+W)\cos\alpha\,\delta\alpha=0$$

or $\quad 2a[2T\sin\frac{1}{2}\alpha-w\cos\alpha-(w+W)\cos\alpha]\,\delta\alpha=0$

or $\quad 2T\sin\frac{1}{2}\alpha-(2w+W)\cos\alpha=0 \qquad [\because \delta\alpha\neq 0]$

or $\quad 2T\sin\frac{1}{2}\alpha=(2w+W)\cos\alpha$

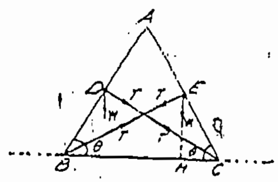
or $\quad T=(w+\frac{1}{2}W)\cos\alpha\operatorname{cosec}\frac{1}{2}\alpha.$

Ex. 18. *Two equal beams AC and AB, each of weight W, are connected by a hinge at A and are placed in a vertical plane with their extremities B and C resting on a smooth horizontal plane. They are prevented from falling by strings connecting B and C with the middle points of the opposite beams. Show that the tension of each string is*

$$\tfrac{1}{4}W\sqrt{(1+9\cot^2\theta)},$$

where θ is the inclination of each beam to the horizon.

Sol. AB and AC are equal beams each of weight W and say of length $2a$. They are hinged at A and are placed in a vertical plane with their ends B and C resting on a smooth horizontal plane. BE and CD are strings where D and E are the middle points of AB and AC respectively. Let T be the tension in each of the strings BE and CD. We have



$$\angle ABC = \angle ACB = \theta.$$

The weights of the rods AB and AC act at their middle points. Draw EH perpendicular to BC.

Here the fixed level is the horizontal line BC.

The height of the point D or E above BC

$$= EH = EC \sin\theta = a \sin\theta.$$

We have

$$BH = \tfrac{3}{4} BC = \tfrac{3}{4} \cdot 2 \cdot 2a \cos\theta = 3a \cos\theta.$$

$\therefore$ the length of the string $BE = \sqrt{(BH^2 + EH^2)}$

$$= \sqrt{(9a^2 \cos^2\theta + a^2 \sin^2\theta)} = a\sqrt{(1+8\cos^2\theta)}$$

$=$ the length of the string CD.

Give the system a small symmetrical displacement in which θ changes to $\theta + \delta\theta$. The level of the line BC lying on the horizontal plane remains fixed and the points B and C move on the line. The lengths of the rods AB and AC do not change while the lengths of the strings BE and CD change. The points D and E are slightly displaced.

The equation of virtual work is

$$-2T\delta\,[a\sqrt{(1+8\cos^2\theta)}] - 2W\delta\,(a \sin\theta) = 0$$

or $$-2aT\,\tfrac{1}{2}(1+8\cos^2\theta)^{-1/2}(-16\cos\theta \sin\theta)\,\delta\theta - aW \cos\theta\,\delta\theta = 0$$

or $$2a\,[8 \sin\theta\,(1+8\cos^2\theta)^{-1/2}\,T - W] \cos\theta\,\delta\theta = 0$$

or $$8 \sin\theta\,(1+8\cos^2\theta)^{-1/2}\,T - W = 0$$

$$[\because \delta\theta \neq 0 \text{ and } \cos\theta \neq 0]$$

or $$T = \frac{W\sqrt{(1+8\cos^2\theta)}}{8 \sin\theta} = \frac{W\sqrt{(\operatorname{cosec}^2\theta + 8\cot^2\theta)}}{8}$$

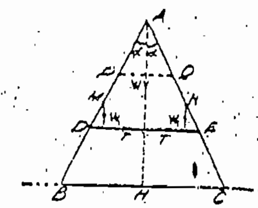
$$= \tfrac{1}{8} W\sqrt{(1+9\cot^2\theta)}.$$

Ex. 19. *A step ladder has a pair of legs which are jointed by a hinge at the top, and are connected by a cord attached at one-third of the distance from the lower end to the top. If the weight of each leg be W_1 and acts at their middle points and if a man of weight W is two-thirds the way up the ladder, show by the principle of virtual work, that the tension in the cord is*

$$\tfrac{1}{2}\,(W + \tfrac{3}{2}W_1) \tan\alpha,$$

α being the inclination of each leg to the vertical.

Sol. Let AB and AC be the two legs each of weight W_1 and say of length $2a$. The points B and C rest on a horizontal plane. The points D and E are connected by a cord where $BD = CE = \tfrac{1}{3}AB$.



Let T be the tension in the cord DE. The weight W_1 of each leg acts at its middle point M or N. A man of weight W is on the level PQ where

$$BP = CQ = \tfrac{2}{3} AB.$$

The line AH is vertical and

$$\angle BAH = \angle CAH = \alpha.$$

Here the fixed level is the horizontal line BC.

The height of M or N above BC

$$= \tfrac{1}{2} AH = \tfrac{1}{2} \cdot AB \cos\alpha = a \cos\alpha,$$

the height of the man (who is on PQ) above BC

$$= \tfrac{2}{3} AH = \tfrac{2}{3} \cdot 2a \cos\alpha = \tfrac{4}{3} a \cos\alpha.$$

The length of the cord (i.e., string) DE

$$= 2 \cdot AD \sin\alpha$$

$$= 2 \cdot \tfrac{2}{3} AB \sin\alpha = 2 \cdot \tfrac{2}{3} 2a \sin\alpha = \tfrac{8}{3} a \sin\alpha.$$

Give the system a small symmetrical displacement in which α changes to $\alpha + \delta\alpha$. The level of the line BC lying on the horizontal plane remains fixed, the lengths of the legs AB and AC do not change and the length DE changes.

The equation of virtual work is

$$-T\delta\,(\tfrac{8}{3} a \sin\alpha) - 2W_1\delta\,(a \cos\alpha) - W\delta\,(\tfrac{4}{3} a \cos\alpha) = 0$$

or $$-\tfrac{8}{3} aT \cos\alpha\,\delta\alpha + 2aW_1 \sin\alpha\,\delta\alpha + \tfrac{4}{3} aW \sin\alpha\,\delta\alpha = 0$$

or $$2a\,[-\tfrac{4}{3} T \cos\alpha + W_1 \sin\alpha + \tfrac{2}{3} W \sin\alpha]\,\delta\alpha = 0$$

or $$-\tfrac{4}{3} T \cos\alpha + W_1 \sin\alpha + \tfrac{2}{3} W \sin\alpha = 0 \qquad [\because \delta\alpha \neq 0]$$

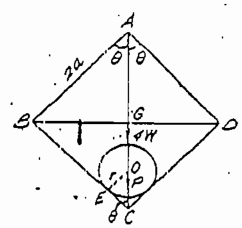
or $$\tfrac{4}{3} T \cos\alpha = (W_1 + \tfrac{2}{3} W) \sin\alpha$$

or $$T = (\tfrac{3}{4} W_1 + \tfrac{1}{2} W) \tan\alpha = \tfrac{1}{2}\,(W + \tfrac{3}{2} W_1) \tan\alpha.$$

Ex. 20. *Four equal uniform bars, each of weight W, are jointed together so as to form a rhombus. This is suspended vertically from* *one of the joints, and a sphere of weight P is balanced inside the rhombus so as to keep it from collapsing. Show that if 2θ be the angle at the fixed joint in the figure of equilibrium, then*

$$\frac{\sin^3 \theta}{\cos \theta}=\frac{Pr}{4(P+2W)a},$$

where r is the radius of the sphere and $2a$ is the length of each bar.



Sol. A rhombus $ABCD$ formed of four rods each of weight W and length $2a$ is suspended from the point A. A sphere of weight P and radius r is balanced inside the rhombus so as to keep it from collapsing. The diagonal AC of the rhombus must be vertical and the centre O of the sphere lies on it. The diagonal BD is horizontal.

The total weight $4W$ of all the four rods can be taken acting at G, the middle point of the diagonal AC and the weight P of the sphere acts at O. If the sphere touches the rod BC at E, then $\angle OEC=90°$, being the angle between the radius OE and the tangent BC. Also $\angle AGB=90°$.

We have

$$\angle BAC=\angle DAC=\theta=\angle BCA.$$

Give the system a small symmetrical displacement about AC in which θ changes to $\theta+\delta\theta$. The point A remains fixed and the points G and O slightly displace. The angles AGB and OEC remain $90°$. We have

the depth of G below $A=AG=2a \cos \theta$,

and the depth of O below A

$$=AO=AC-OC=4a \cos \theta-r \operatorname{cosec} \theta.$$

[Note that from the right angled triangle OEC,

$$OC=OE \operatorname{cosec} \theta]$$

The equation of virtual work is

$$4W\delta(2a \cos \theta)+P\delta(4a \cos \theta-r \operatorname{cosec} \theta)=0$$

or $\quad -8aW \sin \theta\, \delta\theta-4aP \sin \theta\, \delta\theta+Pr \operatorname{cosec} \theta \cot \theta\, \delta\theta=0$

or $\quad [-8aW \sin \theta-4aP \sin \theta+Pr \operatorname{cosec} \theta \cot \theta]\, \delta\theta=0$

or $\quad -8aW \sin \theta-4aP \sin \theta+Pr \operatorname{cosec} \theta \cot \theta=0 \quad [\because \delta\theta \neq 0]$

or $\quad 4a \sin \theta\,(2W+P)=Pr \operatorname{cosec} \theta \cot \theta$

or $\quad \dfrac{Pr}{4a(P+2W)}=\dfrac{\sin \theta}{\operatorname{cosec} \theta \cot \theta}=\dfrac{\sin^3 \theta}{\cos \theta}.$

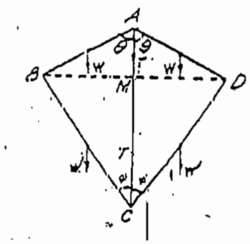
Problems involving two parameters.

Now we shall give the method to solve the problems involving two parameters.

Ex. 21. *A quadrilateral $ABCD$, formed of four uniform rods freely jointed to each other at their ends, the rods AB, AD being equal and also the rods BC, CD, is freely suspended from the joint A. A string joins A to C and is such that ABC is a right angle. Apply the principle of virtual work to show that the tension of the string is*

$$(W+W') \sin^2 \theta+W',$$

where W is the weight of an upper rod and W' of a lower rod and 2θ is equal to the angle BAD.



Sol. The quadrilateral is suspended from the point A.

Let $\quad AB=AD=2a$

and $\quad BC=DC=2b$.

The diagonal AC must be vertical and BD horizontal. Let T be the tension in the string AC. The weights of the rods AB, BC, CD and DA act at their respective middle points. We have

$$\angle BAC=\theta=\angle DAC.$$

Let $\quad \angle BCA=\phi=\angle DCA.$

Since in the position of equilibrium $\angle ABC=90°$, therefore in the position of equilibrium

$$\theta+\phi=90° \quad \text{or} \quad \phi=90°-\theta.$$

For initial calculation we cannot take $\angle ABC$ equal to $90°$ because during a displacement in which AC is to change this angle will not remain $90°$.

Now give the system a small symmetrical displacement about AC in which θ changes to $\theta+\delta\theta$ and ϕ changes to $\phi+\delta\phi$. The point A remains fixed. The lengths of the rods AB, BC, CD and DA do not change while the length AC changes. The $\angle AMB$ remains $90°$. We have

$$AC=AM+MC=AB \cos \theta+BC \cos \phi$$
$$=2a \cos \theta+2b \cos \phi,$$

the depth of the middle point of AB or AD below A

$$=a \cos \theta,$$

and the depth of the middle point of CB or CD below A

$$=2a \cos \theta + b \cos \phi.$$

The equation of virtual work is

$$-T\delta\,(2a \cos \theta + 2b \cos \phi) + 2W\delta\,(a \cos \theta) + 2W'\delta\,(2a \cos \theta + b \cos \phi) = 0$$

or $2aT \sin \theta\, \delta\theta + 2b\,T \sin \phi\, \delta\phi - 2aW \sin \theta\, \delta\theta - 4aW' \sin \theta\, \delta\theta - 2bW' \sin \phi\, \delta\phi = 0$

or $2a \sin \theta\,(T - W - 2W')\,\delta\theta = 2b \sin \phi\,(W' - T)\,\delta\phi.$ ...(1)

But the parameters θ and ϕ are not independent of each other. From the figure, we can find a relation between θ and ϕ. From the $\triangle AMB$, we have $BM = 2a \sin \theta$ and from the $\triangle CMB$, we have

$$BM = 2b \sin \phi.$$

Equating the two values of BM, we have

$$2a \sin \theta = 2b \sin \phi.$$

$\therefore$ $\delta\,(2a \sin \theta) = \delta\,(2b \sin \phi)$

or $2a \cos \theta\, \delta\theta = 2b \cos \phi\, \delta\phi.$ ...(2)

Dividing (1) by (2), we get

$$\frac{\sin \theta\,(T - W - 2W')}{\cos \theta} = \frac{\sin \phi\,(W' - T)}{\cos \phi} \quad ...(3)$$

[Note that $\delta\theta$ and $\delta\phi$ cancel because $\delta\theta \neq 0$ and $\delta\phi \neq 0$]

or $\sin \theta \cos \phi\,(T - W - 2W') = \cos \theta \sin \phi\,(W' - T)$

or $T\,(\sin \theta \cos \phi + \cos \theta \sin \phi) = W \sin \theta \cos \phi + W' \sin \theta \cos \phi + W'\,(\sin \theta \cos \phi + \cos \theta \sin \phi)$

or $T \sin\,(\theta + \phi) = (W + W') \sin \theta \cos \phi + W' \sin\,(\theta + \phi).$

But in the position of equilibrium

$$\theta + \phi = 90° \quad \text{or} \quad \phi = 90° - \theta.$$

$\therefore$ $T \sin 90° = (W + W') \sin \theta \cos\,(90° - \theta) + W' \sin 90°$

or $T = (W + W') \sin^2 \theta + W'.$

Ex. 22. *ABCD is a quadrilateral formed of four uniform freely jointed rods, of which $AB = AD$ and each of weight W, and $BC = CD$ each of weight W'. A string joins A to C. It is freely suspended from A. If $\angle BAD = 2\theta$ and $\angle BCD = 2\phi$, show that the tension in the string is*

$$\frac{W \tan \theta + W'\,(2 \tan \theta + \tan \phi)}{\tan \theta + \tan \phi}.$$ [Meerut 86S]

Sol. Proceed as in Ex. 21.

From the result (3) of Ex. 21, we have

$$\tan \theta\,(T - W - 2W') = \tan \phi\,(W' - T)$$

or $T\,(\tan \theta + \tan \phi) = W \tan \theta + W'\,(2 \tan \theta + \tan \phi)$

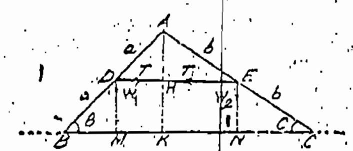
or $$T = \frac{W \tan \theta + W'\,(2 \tan \theta + \tan \phi)}{\tan \theta + \tan \phi}$$

Ex. 23. *Two uniform rods AB and AC smoothly jointed at A are in equilibrium in a vertical plane, B and C rest on a smooth horizontal plane and the middle points of AB and AC are connected by a string. Show that the tension of the string is*

$$\frac{W}{\tan B + \tan C}$$

where W is the total weight of the rods AB and AC.

[Gorakhpur 79; Jiwaji 78]



Sol. AB and AC are two uniform rods smoothly jointed at A. They rest in a vertical plane with the ends B and C placed on a smooth horizontal plane. Let T be the tension in the string connecting the middle points D and E of AB and AC respectively. Let

$$AB = 2a \text{ and } AC = 2b.$$

The weight W_1 of the rod AB acts at its middle point D and the weight W_2 of the rod AC acts at its middle point E. Therefore the total weight $W = W_1 + W_2$ of the two rods AB and AC acts at some point of the line DE which is parallel to BC.

Give the system a small displacement in which the angle B changes to $B + \delta B$ and C changes to $C + \delta C$. The level of the line BC lying on the horizontal plane remains fixed and the points B and C move on this line. The lengths of the rods AB and AC do not change, the length DE changes and the points D and E move. We have

$$DE = DH + HE = a \cos B + b \cos C,$$

the height of any point of the line DE above BC

$$= DM = a \sin B.$$

The equation of virtual work is

$$-T\delta\,(a \cos B + b \cos C) - W\delta\,(a \sin B) = 0$$

or $aT \sin B\, \delta B + bT \sin C\, \delta C - aW \cos B\, \delta B = 0$

or $a\,(W \cos B - T \sin B)\,\delta B = bT \sin C\, \delta C.$ ...(1)

From the figure,

$$DM = a \sin B \text{ and } EN = b \sin C.$$

Since $DM = EN$, therefore $a \sin B = b \sin C$.

$\therefore$ $\delta\,(a \sin B) = \delta\,(b \sin C)$

or $a \cos B\, \delta B = b \cos C\, \delta C.$ ...(2)

Dividing (1) by (2), we have

$$\frac{W\cos B - T\sin B}{\cos B}=\frac{T\sin C}{\cos C}$$

or $\quad W - T\tan B = T\tan C$

or $\quad T(\tan B+\tan C)=W$

or $\quad T=\dfrac{W}{\tan B+\tan C}$.

Ex. 24. *Two uniform rods AB, BC of weights W and W' are smoothly jointed at B and their middle points are joined across by a cord. The rods are tightly held in a vertical plane with their ends A, C resting on a smooth horizontal plane. Show by the principle of virtual work that the tension in the cord is*

$$(W+W')\cos A\cos C/\sin B.$$

Find the additional tension in the cord caused by suspending a weight W'' from B.

Sol. Draw figure and proceed as in Ex. 23.

In the first case, we shall get

$$T=\frac{(W+W')}{\tan A+\tan C}=\frac{(W+W')\cos A\cos C}{\sin A\cos C+\cos A\sin C}$$

$$=\frac{(W+W')\cos A\cos C}{\sin(A+C)}=\frac{(W+W')\cos A\cos C}{\sin(180^\circ-B)}$$

$$=\frac{(W+W')\cos A\cos C}{\sin B}.$$

In the second case when a weight W'' is also suspended from B, let T' be the tension in the cord. Write the new equation of virtual work and find T'.

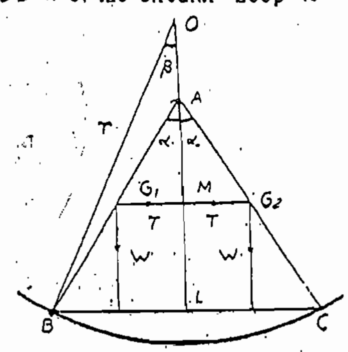
The required additional tension in the cord

$$=T'-T=(2W''\cos A\cos C)/\sin B.$$

Ex. 25. *Two equal uniform rods AB, AC each of weight W are freely jointed at A and rest with the extremities B and C on the inside of a smooth circular hoop, whose radius is greater than the length of either rod, the whole being in a vertical plane and the middle points of the rods being jointed by a light string. Show that, if the string is stretched, its tension is W (tan α − 2 tan β), where 2α is the angle between the rods, and β the angle either rod subtends at the centre.*

Sol. AB and AC are two uniform rods freely jointed at A and resting with their extremeties B and C on the inside of a

smooth circular hoop. The radius $OB=r$ of the circular hoop is greater than the length $2a$ of either rod. Let T be the tension in the string connecting the middle points G_1 and G_2 of the rods. The weights W and W of the two rods act at their middle points G_1 and G_2 and the total weight $W+W=2W$ will act at the middle point M of G_1G_2. Given that $\angle BAL=\angle CAL=\alpha$ and $\angle BOL=\beta$.



Give the system a small displacement in which the angle α changes to $\alpha+\delta\alpha$ and β changes to $\beta+\delta\beta$. The smooth circular hoop remains fixed and hence its centre O can be taken as the fixed point. The lengths of the rods AB and AC do not change while the length of the string G_1G_2 changes.

The equation of virtual work is

$$-T\delta(G_1G_2)+2W\delta(OM)=0$$

or $\quad -T\delta(2a\sin\alpha)+2W\delta(r\cos\beta-a\cos\alpha)=0$

or $\quad a(-T\cos\alpha+W\sin\alpha)\,\delta\alpha=Wr\sin\beta\,\delta\beta. \quad ...(1)$

In triangle OBL, $BL=r\sin\beta$,

and in triangle ABL, $BL=2a\sin\alpha$.

$\therefore \quad 2a\sin\alpha=r\sin\beta$

$\therefore \quad \delta(2a\sin\alpha)=\delta(r\sin\beta)$

or $\quad 2a\cos\alpha\,\delta\alpha=r\cos\beta\,\delta\beta. \quad ...(2)$

Dividing (1) by (2), we get

$$\frac{a(-T\cos\alpha+W\sin\alpha)}{2a\cos\alpha}=\frac{Wr\sin\beta}{r\cos\beta}$$

or $\quad -T+W\tan\alpha=2W\tan\beta$

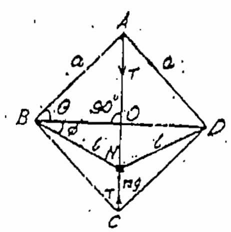
or $\quad T=W(\tan\alpha-2\tan\beta)$.

Ex. 26. *A frame, formed of four light rods, each of length a, freely jointed at A, B, C, D suspended at A. A mass m is suspended from B and D by two strings of length l (l > a/√2). The frame is kept in the form of a square by a string AC. Apply the method of virtual work to find the tension T in AC and show that when*

l = a√5, T = 2mg/3. [Ravishankar 86; Meerut 75]

Sol. The framework is suspended from A and so A is a fixed point from which the distances are to be measured. A mass m is suspended from B and D by means of two strings BN and DN each of length l. Thus a weight mg acts at N. Let T be the tension in the string AC. In the position of equilibrium the figure is a square.

Let $\angle ABD=\theta$ and $\angle NBO=\phi$.



Give the system a small symmetrical displacement about the vertical AC in which θ changes to $\theta+\delta\theta$ and ϕ changes to $\phi+\delta\phi$. The point A remains fixed. The lengths of the rods AB, BC, CD and DA remain fixed and the length AC changes. The lengths of the strings BN and DN remain fixed so that the work done by their tensions is zero. The point N is slightly displaced.

We have

$$AC=2AO=2a \sin \theta,$$

and the depth of N below A

$$=AN=AO+ON=a \sin \theta+l \sin \phi.$$

The equation of virtual work is

$$-T\delta\,(2a \sin \theta)+mg\,\delta\,(a \sin \theta+l \sin \phi)=0$$

or $\quad -2aT \cos \theta\,\delta\theta+a\,mg \cos \theta\,\delta\theta+l\,mg \cos \phi\,\delta\phi=0$

or $\quad a \cos \theta\,(2T-mg)\,\delta\theta=l\,mg \cos \phi\,\delta\phi. \qquad ...(1)$

Now from the $\triangle AOB$, $BO=a \cos \theta$

and from the $\triangle BON$, $\quad BO=l \cos \phi$.

$$\therefore \quad a \cos \theta=l \cos \phi$$

so that $\quad -a \sin \theta\,\delta\theta=-l \sin \phi\,\delta\phi$

or $\quad a \sin \theta\,\delta\theta=l \sin \phi\,\delta\phi. \qquad ...(2)$

Dividing (1) by (2), we have

$$\frac{\cos \theta\,(2T-mg)}{\sin \theta}=\frac{mg \cos \phi}{\sin \phi}$$

or $\quad \cot \theta\,(2T-mg)=mg \cot \phi$

or $\quad 2T-mg=mg \tan \theta \cot \phi$

or $\quad T=\tfrac{1}{2}\,mg\,(1+\tan \theta \cot \phi)$.

In the position of equilibrium $\theta=45°$,

$$BO=a \cos 45°=a/\sqrt{2},$$
$$ON=\sqrt{(BN^2-BO^2)}=\sqrt{\{l^2-(a^2/2)\}}$$
$$=\{\sqrt{(2l^2-a^2)}\}/\sqrt{2},$$

so that $\quad \cot \phi=\dfrac{BO}{ON}=\dfrac{a/\sqrt{2}}{\{\sqrt{(2l^2-a^2)}\}/\sqrt{2}}$

$$=\frac{a}{\sqrt{(2l^2-a^2)}}$$

$$\therefore \quad T=\tfrac{1}{2}\,mg\left\{1+\tan 45°\cdot\frac{a}{\sqrt{(2l^2-a^2)}}\right\}$$
$$=\tfrac{1}{2}\,mg\left\{1+\frac{a}{\sqrt{(2l^2-a^2)}}\right\}.$$

When $\quad l=a\sqrt{5}$, the tension T

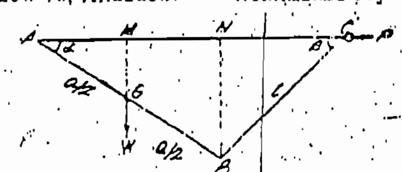
$$=\tfrac{1}{2}\,mg\left\{1+\frac{a}{\sqrt{(2a^2 \cdot 5-a^2)}}\right\}$$
$$=\tfrac{1}{2}\,mg\,(1+\tfrac{1}{3})=\tfrac{2}{3}\,mg.$$

Ex. 27. *A rod is movable about a point A, and to B is attached a string whose other end is tied to a ring. The ring slides along a smooth horizontal wire passing through A. Prove by the principle of virtual work that the horizontal force necessary to keep the ring at rest is*

$$\frac{W \cos \alpha \cos \beta}{2 \sin (\alpha+\beta)},$$

where W is the weight of the rod, and α, β the inclinations of the rod and the string to the horizontal.

[Lucknow 76; Allahabad Rohilkhand 87]



Sol. The rod AB is hinged at A. Let the length of the rod AB be a and the length of the string BC be l. At C there is a ring which can slide on a smooth horizontal wire AC.

Let P be the horizontal force applied at the ring C to keep it at rest. The weight W of the rod AB acts at its middle point G.

Let $\quad \angle BAC=\alpha$ and $\angle BCA=\beta$.

Give the system a small displacement in which α changes to $\alpha+\delta\alpha$ and β changes to $\beta+\delta\beta$. The point A remains fixed. The length of the rod AB remains fixed and the length of the string BC also remains fixed so that the work done by its tension is zero. The points G and C are slightly displaced. We have

the depth of G below $A=MG$

$$=AG \sin \alpha=\tfrac{1}{2}a \sin \alpha,$$

and the horizontal distance of C from $A=AC$

$$=AN+NC=a \cos \alpha+l \cos \beta.$$

The equation of virtual work is

$$W\delta\,(\tfrac{1}{2}a \sin \alpha)+P\delta\,(a \cos \alpha+l \cos \beta)=0$$

or $\quad \tfrac{1}{2}a\,W \cos \alpha\,\delta\alpha-aP \sin \alpha\,\delta\alpha-lP \sin \beta\,\delta\beta=0$

or $\quad a\,(\tfrac{1}{2}W \cos \alpha-P \sin \alpha)\,\delta\alpha=lP \sin \beta\,\delta\beta. \qquad ...(1)$

From the figure, equating the values of BN found from the triangles ANB and CNB, we get

$$a \sin \alpha = l \sin \beta,$$

so that $$a \cos \alpha \, \delta\alpha = l \cos \beta \, \delta\beta. \quad \ldots(2)$$

Dividing (1) by (2), we get

$$\frac{\frac{1}{2} W \cos \alpha - P \sin \alpha}{\cos \alpha} = \frac{P \sin \beta}{\cos \beta}$$

or $\frac{1}{2} W \cos \alpha \cos \beta - P \sin \alpha \cos \beta = P \cos \alpha \sin \beta$

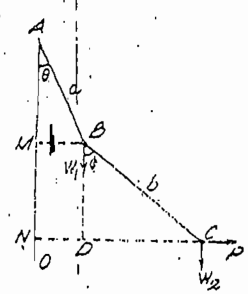
or $P (\sin \beta \cos \alpha + \cos \beta \sin \alpha) = \frac{1}{2} W \cos \alpha \cos \beta$

or $$P = \frac{W \cos \alpha \cos \beta}{2 \sin (\alpha + \beta)}.$$

Ex. 28. *Weights W_1, W_2 are fastened to a light inextensible string ABC at the points B, C the end A being fixed. Prove that, if a horizontal force P is applied at C and in equilibrium AB, BC are inclined at angles θ, ϕ to the vertical, then*

$$P = (W_1 + W_2) \tan \theta = W_2 \tan \phi. \quad \text{[Meerut 81, 82, 85]}$$

Sol. Let the length of the portion AB of the string be a and that of BC be b. The point A is fixed and the vertical line AO through A is a fixed line.



From the fixed point A, the depth of B

$= AM = a \cos \theta,$

and the depth of C

$= AN = AM + MN$

$= AM + BD = a \cos \theta + b \cos \phi.$

Also the horizontal distance of the point C from the fixed line $AO = NC$

$= ND + DC = MB + DC = a \sin \theta + b \sin \phi.$

Now give the system a small displacement in which θ changes to $\theta + \delta\theta$, ϕ changes to $\phi + \delta\phi$, the point A remains fixed, the length of the string remains unaltered and the points B and C are slightly displaced. The equation of virtual work is

$$W_1 \delta (a \cos \theta) + W_2 \delta (a \cos \theta + b \cos \phi) + P \delta (a \sin \theta + b \sin \phi) = 0$$

or $$-a W_1 \sin \theta \, \delta\theta - a W_2 \sin \theta \, \delta\theta - b W_2 \sin \phi \, \delta\phi + aP \cos \theta \, \delta\theta + bP \cos \phi \, \delta\phi = 0$$

or $$a [P \cos \theta - (W_1 + W_2) \sin \theta] \, \delta\theta = b [W_2 \sin \phi - P \cos \phi] \, \delta\phi, \quad \ldots(1)$$

where θ and ϕ are independent of each other.

Now consider a displacement when only θ changes and ϕ does not change so that $\delta\phi = 0$. Then putting $\delta\phi = 0$ in (1), we have

$$a [P \cos \theta - (W_1 + W_2) \sin \theta] \, \delta\theta = 0$$

or $P \cos \theta - (W_1 + W_2) \sin \theta = 0 \quad [\because \delta\theta \neq 0]$

or $P = (W_1 + W_2) \tan \theta. \quad \ldots(2)$

Again consider a displacement when only ϕ changes and θ does not change so that $\delta\theta = 0$. Thus putting $\delta\theta = 0$ in (1), we have

$$b [W_2 \sin \phi - P \cos \phi] \, \delta\phi = 0$$

or $W_2 \sin \phi - P \cos \phi = 0 \quad [\because \delta\phi \neq 0]$

or $P = W_2 \tan \phi. \quad \ldots(3)$

From (2) and (3), we have

$$P = (W_1 + W_2) \tan \theta = W_2 \tan \phi.$$

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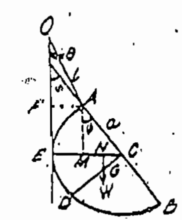
Ex. 29. *A solid hemisphere is supported by a string fixed to a point on its rim and to a point on a smooth vertical wall with which the curved surface of the hemisphere is in contact. If θ, ϕ are the inclinations of the string and the plane base of the hemisphere to the vertical, prove that*

$$\tan \phi = \tfrac{3}{8} + \tan \theta.$$

Sol. O is a fixed point in the wall to which one end of the string has been attached. Let l be the length of the string AO and a be the radius of the hemisphere the centre of whose base is C. The weight W of the hemisphere acts at its centre of gravity G which lies on the symmetrical radius CD and is such that $CG = \frac{3}{8} a.$



The hemisphere touches the wall at E. We have $\angle OEC = 90°$ so that EC is horizontal.

The string AO makes an angle θ with the wall and the base BA of the hemisphere makes an angle ϕ with the wall.

The depth of G below $O = OF + AM + NG$

$= l \cos \theta + a \cos \phi + \frac{3}{8} a \sin \phi.$

[Note that $\angle NCG = 90° - \angle ACM = 90° - (90° - \phi) = \phi$].

Give the system a small displacement in which θ changes to $\theta + \delta\theta$, ϕ changes to $\phi + \delta\phi$, the point O remains fixed, the length of the string AO does not change so that the work done by its tension is zero and the point G is slightly displaced. The $\angle OEC$ remains 90°.

The only force that contributes to the equation of virtual work is the weight W of the hemisphere acting at G whose depth below the fixed point O has been found above. The equation of virtual work is
$W\delta\,(l \cos\theta + a\cos\phi + \frac{3}{8}a\sin\phi) = 0$

or $-l\sin\theta\,\delta\theta - a\sin\phi\,\delta\phi + \frac{3}{8}a\cos\phi\,\delta\phi = 0$

or $l\sin\theta\,\delta\theta = a\,(\frac{3}{8}\cos\phi - \sin\phi)\,\delta\phi$. ...(1)

From the figure, $EC = a$.

Also $EC = EM + MC = FA + MC = l\sin\theta + a\sin\phi$.

$\therefore \quad a = l\sin\theta + a\sin\phi$.

Differentiating, $0 = l\cos\theta\,\delta\theta + a\cos\phi\,\delta\phi$

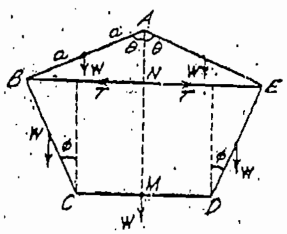
or $-l\cos\theta\,\delta\theta = a\cos\phi\,\delta\phi$. ...(2)

Dividing (1) by (2), we get

$-\tan\theta = \frac{3}{8} - \tan\phi$ or $\tan\phi = \frac{3}{8} + \tan\theta$.

Ex. 30. (a) *Five equal uniform rods, freely jointed at their ends, form a regular pentagon ABCDE and BE is joined by a weightless bar. The system is suspended from A in a vertical plane. Prove that the thrust in BE is W cot $\frac{1}{10}\pi$, where W is the weight of the rod.*

Sol. $ABCDE$ is a pentagon formed of five equal rods each of weight W and say of length $2a$. It is suspended from A and BF is jointed by a weightless bar. Let T be the thrust in the bar BE. The line AM joining A to the middle point M of CD is vertical and the line BE is horizontal. The weights of the rods AB, BC, CD, DE and EA act at their respective middle points. In the position of equilibrium the pentagon is a regular one so that each of its interior angles is $180° - 72°$ *i.e.*, $108°$ or $\frac{3}{5}\pi$ radians.



Let θ be the angle which the two upper slant rods AB and AE make with the vertical and ϕ be the angle which the two lower slant rods CB and DE make with the vertical.

Replace the rod BE by two equal and opposite forces T as shown in the figure.

Give the system a small symmetrical displacement about the vertical AM in which θ changes to $\theta + \delta\theta$ and ϕ changes to $\phi + \delta\phi$.

The point A remains fixed. The lengths of the rods AB, BC etc. remain fixed, the length BE changes and the middle points of the rods AB, BC etc. are slightly displaced. The $\angle ANB$ remains $90°$.

We have

$BE = 2BN = 2.2a\sin\theta = 4a\sin\theta$,

the depth of the middle point of AB or AE below $A = a\cos\theta$,

the depth of the middle point of BC or ED below A

$= 2a\cos\theta + a\cos\phi$,

and the depth of the middle point M of CD below A

$= 2a\cos\theta + 2a\cos\phi$.

The equation of virtual work is

$T\delta\,(4a\sin\theta) + 2W\delta\,(a\cos\theta) + 2W\delta\,(2a\cos\theta + a\cos\phi)$
$+ W\delta\,(2a\cos\theta + 2a\cos\phi) = 0$

or $4aT\cos\theta\,\delta\theta - 2aW\sin\delta\theta - 4aW\sin\theta\,\delta\theta - 2aW\sin\phi\,\delta\phi$
$-2aW\sin\theta\,\delta\theta - 2aW\sin\phi\,\delta\phi = 0$

or $4a\,(T\cos\theta - 2W\sin\theta)\,\delta\theta = 4aW\sin\phi\,\delta\phi$

or $(T\cos\theta - 2W\sin\theta)\,\delta\theta = W\sin\phi\,\delta\phi$. ...(1)

From the figure, finding the length BE in two ways *i.e.*, from the upper portion ABE and from the lower portion $BCDE$, we have

$4a\sin\theta = 2a + 4a\sin\phi$.

Differentiating, we get $4a\cos\theta\,\delta\theta = 4a\cos\phi\,\delta\phi$.

or $\cos\theta\,\delta\theta = \cos\phi\,\delta\phi$. ...(2)

Dividing (1) by (2), we get

$$\frac{T\cos\theta - 2W\sin\theta}{\cos\theta} = \frac{W\sin\phi}{\cos\phi}$$

or $T - 2W\tan\theta = 2W\tan\phi$

or $T = W(\tan\phi + 2\tan\theta)$.

But in the position of equilibrium,

$\theta = \frac{1}{2}\cdot\frac{3}{5}\pi = \frac{3}{10}\pi$, $\phi = \frac{3}{5}\pi - \frac{1}{2}\pi = \frac{1}{10}\pi$.

$$\therefore \quad T = W\left(\tan\frac{1}{10}\pi + 2\tan\frac{3}{10}\pi\right) = W\left[\tan\frac{1}{10}\pi + 2\cot\frac{2}{10}\pi\right]$$

$$\left[\because \tan\frac{3}{10}\pi = \cot\left(\frac{1}{2}\pi - \frac{3}{10}\pi\right) = \cot\frac{2}{10}\pi\right]$$

$$= W\left[\tan\frac{1}{10}\pi + 2\cdot\frac{1-\tan^2(\pi/10)}{2\tan(\pi/10)}\right]$$

$$\left[\because \cot 2\alpha = \frac{1}{\tan 2\alpha} = \frac{1-\tan^2\alpha}{2\tan\alpha}\right]$$

$= W\cot(\pi/10)$.

Ex. 30. (b) *A regular pentagon ABCDE formed of equal uniform rods each of weight W, is suspended from the point A and is maintained in shape by a light rod joining the middle points of BC and DE. Prove that the stress in the light rod is 2W cot (π/10).*

(Rohilkhand 83)

Sol. Proceed as in part (a).

Ex. 31. *A freely jointed framework is formed of five equal uniform rods each of weight W. The framework is suspended from one corner which is also joined to the middle point of the opposite side by an inextensible string; if the two upper and the two lower rods make angles θ and φ respectively with the vertical, prove that the tension of the string is to the weight of the rod as*

$$(4 \tan\theta + 2 \tan\phi) : (\tan\theta + \tan\phi).$$ [Lucknow 80]

Sol. Draw figure as in Ex. 30 (a). This question differs from the preceding one in having the string AM instead of the rod BE.

Let T be the tension in the string AM. The string AM is given to be inextensible, therefore before giving the displacement replace the string by two equal and opposite forces T so that the length AM may be changed.

Here $AM = 2a \cos\theta + 2a \cos\phi$.

The equation of virtual work is

$$-T\delta(2a\cos\theta + 2a\cos\phi) + 2W\delta(a\cos\theta) + 2W\delta(2a\cos\theta + a\cos\phi)$$
$$+ W\delta(2a\cos\theta + 2a\cos\phi) = 0$$

or $$2aT\sin\theta\,\delta\theta + 2aT\sin\phi\,\delta\phi + 2aW\sin\theta\,\delta\theta - 4aW\sin\theta\,\delta\theta$$
$$-2aW\sin\phi\,\delta\phi - 2aW\sin\theta\,\delta\theta - 2aW\sin\phi\,\delta\phi = 0$$

or $$2a\sin\theta\,(T-4W)\,\delta\theta = 2a\sin\phi\,(2W-T)\,\delta\phi$$

or $$\sin\theta\,(T-4W)\,\delta\theta = \sin\phi\,(2W-T)\,\delta\phi. \quad ...(1)$$

Also from the figure, we have

$$4a\sin\theta = 2a + 4a\sin\phi,$$

so that $$4a\cos\theta\,\delta\theta = 4a\cos\phi\,\delta\phi$$

or $$\cos\theta\,\delta\theta = \cos\phi\,\delta\phi. \quad ...(2)$$

Dividing (1) by (2), we get

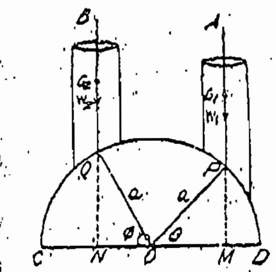
$$\tan\theta\,(T-4W) = \tan\phi\,(2W-T)$$

or $$T(\tan\theta + \tan\phi) = W(2\tan\phi + 4\tan\theta)$$

or $$\frac{T}{W} = \frac{4\tan\theta + 2\tan\phi}{\tan\theta + \tan\phi},$$ which proves the required result.

Ex. 32. *A flat semi-circular board with its plane vertical and curved edge upwards rests on a smooth horizontal plane and is pressed at two given points of its circumference by two beams which slide in smooth vertical tubes. If the board is in equilibrium, find the ratio of the weights of the beams.*

Sol. Let W_1 and W_2 be the weights of the beams AP and BQ



whose lengths are say $2l_1$ and $2l_2$ respectively. Let θ and ϕ be the angles which the radii OP and OQ make with the horizontal diameter COD of the board. Let a be the radius of the board.

Here COD is a fixed horizontal line. The weight W_1 of the beam AP acts at its centre of gravity G_1 whose height above

$$CD = MG_1 = l_1 + a\sin\theta.$$

The weight W_2 of the beam BQ acts at G_2 whose height above CD is $l_2 + a\sin\phi$.

Let the beams be imagined to undergo a small displacement in which θ changes to $\theta+\delta\theta$ and ϕ changes to $\phi+\delta\phi$. The equation of virtual work is $-W_1\delta(l_1 + a\sin\theta) - W_2\delta(l_2 + a\sin\phi) = 0$

or $$-aW_1\cos\theta\,\delta\theta - a\,W_2\cos\phi\,\delta\phi = 0$$

or $$-W_1\cos\theta\,\delta\theta = W_2\cos\phi\,\delta\phi. \quad ...(1)$$

If h be the distance between the tubes in which the beams slide, then from the figure

$$a\cos\theta + a\cos\phi = h = \text{constant}$$

so that, $$-a\sin\theta\,\delta\theta - a\sin\phi\,\delta\phi = 0$$

or $$-\sin\theta\,\delta\theta = \sin\phi\,\delta\phi. \quad ...(2)$$

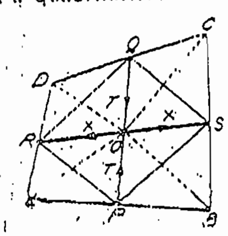
Dividing (1) by (2), we have

$$W_1\cot\theta = W_2\cot\phi$$

or $$\frac{W_1}{W_2} = \frac{\cot\phi}{\cot\theta} = \frac{\tan\theta}{\tan\phi},$$ which gives the required ratio.

Ex. 33. *A smoothly jointed framework of light rods forms a quadrilateral ABCD. The middle points P, Q of an opposite pair of rods are connected by a string in a state of tension T, and the middle points R, S of the other pair by a light rod in a state of thrust X: show, by the method of virtual work, that*

$$T/PQ = X/RS.$$



Sol. $ABCD$ is a framework in the form of a quadrilateral formed of four light rods. The middle points P and Q of the rods AB and DC are joined by a string in a state of tension T and the middle points R and S of the rods AD and BC are joined by a light rod in a state of thrust X. [The framework is to be taken as placed on some smooth horizontal plane].

Since P, S, Q, R are the middle points of the sides of the quadrilateral $ABCD$, therefore $PSQR$ is a parallelogram. Consequently the diagonals PQ and RS of this parallelogram bisect each other at O.

Replace the string PQ by two equal and opposite forces T as shown in the figure and replace the rod RS by two equal and opposite forces X as shown in the figure. Now give the system a small displacement in which PQ changes to $PQ+\delta(PQ)$ and RS changes to $RS+\delta(RS)$. The lengths of the rods AB, BC, CD, DA do not change. The equation of virtual work is

$$-T\delta(PQ)+X\delta(RS)=0$$

or $$T\delta(PQ)=X\delta(RS)$$

or $$\frac{\delta(PQ)}{\delta(RS)}=\frac{X}{T}. \quad \ldots(1)$$

Now let us find a relation between the parameters PQ and RS from the figure. Since OP is a median of the $\triangle OAB$, therefore

$$OA^2+OB^2=2OP^2+2AP^2=2(\tfrac{1}{2}PQ)^2+2(\tfrac{1}{2}AB)^2$$

$$=\tfrac{1}{2}(PQ^2+AB^2). \quad \ldots(2)$$

Similarly from $\triangle OCD$, we have

$$OC^2+OD^2=\tfrac{1}{2}(PQ^2+CD^2). \quad \ldots(3)$$

Adding (2) and (3), we get

$$OA^2+OB^2+OC^2+OD^2=\tfrac{1}{2}(2PQ^2+AB^2+CD^2) \quad \ldots(4)$$

Doing the same thing with $\triangle OAD$ and $\triangle OBC$, we get

$$OA^2+OB^2+OC^2+OD^2=\tfrac{1}{2}(2RS^2+BC^2+DA^2). \quad \ldots(5)$$

From (4) and (5), we get

$$\tfrac{1}{2}(2PQ^2+AB^2+CD^2)=\tfrac{1}{2}(2RS^2+BC^2+DA^2)$$

or $$2(PQ^2-RS^2)=BC^2+DA^2-AB^2-CD^2$$

or $$PQ^2-RS^2=\text{constant}, \quad \ldots(6)$$

since AB, BC, CD, DA are all of fixed lengths.

Differentiating (6), we get

$$2PQ\,\delta(PQ)-2RS\,\delta(RS)=0$$

or $$\frac{\delta(PQ)}{\delta(RS)}=\frac{RS}{PQ}. \quad \ldots(7)$$

Equating the values of $\dfrac{\delta(PQ)}{\delta(RS)}$ from (1) and (7), we get

$$\frac{X}{T}=\frac{RS}{PQ} \text{ or } \frac{X}{RS}=\frac{T}{PQ}.$$

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Ex. 34. *The middle points of the opposite sides of a jointed quadrilateral are connected by light rods of lengths l, l'. If T, T' be the tensions in these rods, prove that*

$$\frac{T}{l}+\frac{T'}{l'}=0.$$

[Rohilkhand 79; Kanpur 78, 82]

Sol. Proceed as in Ex. 33. Here the rod RS is also in a state of tension T'. The equation of virtual work is

$$-T\delta(PQ)-T'\delta(RS)=0$$

or $$\frac{\delta(PQ)}{\delta(RS)}=-\frac{T'}{T}.$$

Also $$\frac{\delta(PQ)}{\delta(RS)}=\frac{RS}{PQ}, \text{ as found in Ex. 33.}$$

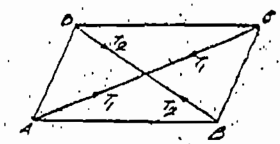
$$\therefore \quad -\frac{T'}{T}=\frac{RS}{PQ}$$

or $$\frac{T}{PQ}+\frac{T'}{RS}=0$$

or $$\frac{T}{l}+\frac{T'}{l'}=0. \quad [\because \text{ in the position of equilibrium } PQ=l \text{ and } RS=l'].$$

Ex. 35. *Four rods are jointed together to form a parallelogram, the opposite joints are joined by strings forming the diagonals and the whole system is placed on a smooth horizontal table. Show that their tensions are in the same ratio as their lengths.* [Rohilkhand 82]

Sol. A framework $ABCD$ is in the form of a parallelogram and is placed on a smooth horizontal table. Let T_1 and T_2 be the tensions in the strings AC and BD respectively.



Give the system a small displacement in the plane of the table in which AC changes to $AC+\delta(AC)$ and BD changes to $BD+\delta(BD)$. The lengths of the rods AB, BC, CD, DA do not change. During this displacement the weights of the rods do no work because the displacement of their points of application in the vertical direction is zero. The equation of virtual work is

$$-T_1\,\delta(AC)-T_2\,\delta(BD)=0$$

or $$\frac{\delta(AC)}{\delta(BD)}=-\frac{T_2}{T_1}. \quad \ldots(1)$$

Now let us find a relation between the parameters AC and BD from the figure. Since in a parallelogram the sum of the squares of the diagonals is equal to the sum of the squares of its sides, therefore

$$AC^2+BD^2=AB^2+BC^2+CD^2+DA^2=\text{constant}. \quad \ldots(2)$$

Differentiating (2), we get

$$2AC\,\delta(AC)+2BD\,\delta(BD)=0$$

or
$$\frac{\delta(AC)}{\delta(BD)}=-\frac{BD}{AC} \quad \ldots(3)$$

From (1) and (3), we get
$$-\frac{T_2}{T_1}=-\frac{BD}{AC}$$

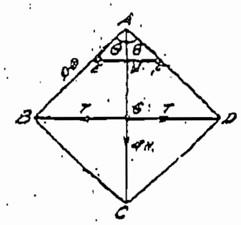
or
$$\frac{T_1}{AC}=\frac{T_2}{BD},$$

i.e., tensions are in the ratio of the lengths of the strings.

Problems relating to bodies or frameworks resting on pegs or on inclined planes.

Ex. 36. *Four equal rods, each of length 2a and weight W, are freely jointed to form a square ABCD which is kept in shape by a light rod BD and is supported in a vertical plane with BD horizontal, A above C and AB, AD in contact with two fixed smooth pegs which are at a distance 2b apart on the same level. Find the stress in the rod BD.*

Sol. The rods AB and AD of the framework rest on two fixed smooth pegs E and F which are at the same level and $EF=2b$. Let $2a$ be the length of each of the rods AB, BC, CD and DA. The total weight $4W$ of all the rods AB, BC, CD and DA can be taken acting at G, the middle point of AC.



Let T be the thrust in the rod BD and let $\angle BAC=\theta=\angle CAD$.

Replace the rod BD by two equal and opposite forces T as shown in the figure. Give the system a small symmetrical displacement in which θ changes to $\theta+\delta\theta$. The line EF joining the pegs remains fixed and the distance will be measured from this line. The lengths of the rods AB, BC, CD, DA do not change and the length BD changes. The $\angle AGB$ remains $90°$.

The forces contributing to the sum of virtual works are: (i) the thrust T in the rod BD, and (ii) the weight $4W$ acting at G. The reactions at the pegs do no work.

We have
$$BD=2BO=2.2a\sin\theta=4a\sin\theta.$$

Also the depth of G below the fixed line EF
$$=MG=AG-AM=AB\cos\theta-EM\cot\theta$$
$$=2a\cos\theta-b\cot\theta.$$

The equation of virtual work is
$$T\delta(4a\sin\theta)+4W\delta(2a\cos\theta-b\cot\theta)=0$$

or $\quad 4aT\cos\theta\,\delta\theta-8aW\sin\theta\,\delta\theta+4bW\operatorname{cosec}^2\theta\,\delta\theta=0$

or $\quad 4(aT\cos\theta-2aW\sin\theta+bW\operatorname{cosec}^2\theta)\,\delta\theta=0$

or $\quad aT\cos\theta-2aW\sin\theta+bW\operatorname{cosec}^2\theta=0 \quad [\because \delta\theta\neq 0$

or $\quad aT\cos\theta=2aW\sin\theta-bW\operatorname{cosec}^2\theta$

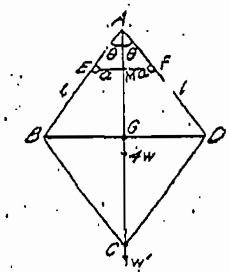
or
$$T=\frac{W}{a\cos\theta}(2a\sin\theta-b\operatorname{cosec}^2\theta)$$
$$=\frac{W}{a}\tan\theta\,(2a-b\operatorname{cosec}^3\theta).$$

But in the position of equilibrium, $\theta=45°$.
$$\therefore \quad T=\frac{W}{a}\tan 45°\,(2a-b\operatorname{cosec}^3 45°)$$
$$=\frac{W}{a}[2a-b(\sqrt{2})^3]=\frac{2W}{a}(a-b\sqrt{2}).$$

Remark. The pegs E and F may also be taken below the middle points of the rods AB and AD.

Ex. 37. *A rhombus is formed of rods each of weight W and length l with smooth joints. It rests symmetrically with its two upper sides in contact with two smooth pegs at the same level and at a distance 2a apart. A weight W′ is hung at the lowest point. If the sides of the rhombus make an angle θ with the vertical, prove that*
$$\sin^3\theta=\frac{a(4W+W')}{l(4W+2W')}.$$

Sol. The rods AB and AD of the framework rest on two smooth pages E and F which are at the same level and $EF=2a$. The length of each rod of the rhombus is l. The total weight $4W$ of all the rods AB, BC, CD and DA can be taken acting at G, the point of intersection of the diagonals AC and BD. A weight W' is hung at the lowest point C. The diagonal AC is vertical and BD is horizontal. We have
$$\angle BAC=\theta=\angle CAD.$$



Give the system a small symmetrical displacement in which θ changes to $\theta+\delta\theta$. The line EF joining the pegs remains fixed and the distances will be measured from this line. The $\angle AGB$ remains $90°$.

$\therefore$ We have, the depth of G below EF
$$=MG=AG-AM=l\cos\theta-a\cot\theta,$$
and the depth of C below EF
$$=MC=AC-AM=2l\cos\theta-a\cot\theta.$$

The equation of virtual work is

$$4W\,\delta\,(l\cos\theta - a\cot\theta) + W'\delta\,(2l\cos\theta - a\cot\theta) = 0$$

or $\quad -4lW\sin\theta\,\delta\theta + 4a\,W\operatorname{cosec}^2\theta\,\delta\theta - 2l\,W'\sin\theta\,\delta\theta + aW'\operatorname{cosec}^2\theta\,\delta\theta = 0.$

or $\quad [a\,(4W+W')\operatorname{cosec}^2\theta - l\,(4W+2W')\sin\theta]\,\delta\theta = 0.$

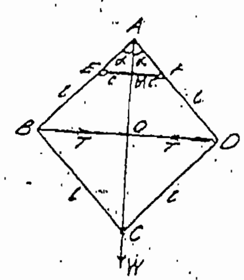
or $\quad a\,(4W+W')\operatorname{cosec}^2\theta - l\,(4W+2W')\sin\theta = 0 \qquad [\because\ \delta\theta \neq 0]$

or $\quad a\,(4W+W')\operatorname{cosec}^2\theta = l\,(4W+2W')\sin\theta$

or $\quad \sin^3\theta = \dfrac{a\,(4W+W')}{l\,(4W+2W')}.$

Ex. 38. *ABCD is a rhombus with four rods each of length l and negligible weight joined by smooth hinges. A weight W is attached to the lowest hinge C, and the frame rests on two smooth pegs in a horizontal line in contact with the rods AB and AD, B and D are in a horizontal line and are joined by a string. If the distance of the pegs apart is 2c and the angle at A is 2α, show that the tension in the string is*

$$W\tan\alpha\left(\frac{c}{2l}\operatorname{cosec}^3\alpha - 1\right)$$



Sol. The rods AB and AD of the frame rest on two smooth pegs E and F which are in the same horizontal line and $EF = 2c$. The length of each rod of the rhombus is l and the rods forming the rhombus are weightless. A weight W is attached to the lowest point C. Let T be the tension in the string BD. We have

$$\angle BAC = \alpha = \angle CAD.$$

The diagonal AC is vertical and BD is horizontal.

Give the system a small symmetrical displacement in which α changes to $\alpha + \delta\alpha$. The line EF joining the pegs remains fixed and the distances will be measured from this line. The $\angle AOB$ remains $90°$. We have

$$BD = 2BO = 2AB\sin\alpha = 2l\sin\alpha.$$

Also the depth of the point C below EF

$$= MC = AC - AM = 2AO - AM$$
$$= 2AB\cos\alpha - EM\cot\alpha = 2l\cos\alpha - c\cot\alpha.$$

The equation of virtual work is

$$-T\delta\,(2l\sin\alpha) + W\delta\,(2l\cos\alpha - c\cot\alpha) = 0$$

or $\quad -2lT\cos\alpha\,\delta\alpha - 2lW\sin\alpha\,\delta\alpha + Wc\operatorname{cosec}^2\alpha\,\delta\alpha = 0$

or $\quad (-2lT\cos\alpha - 2lW\sin\alpha + Wc\operatorname{cosec}^2\alpha)\,\delta\alpha = 0$

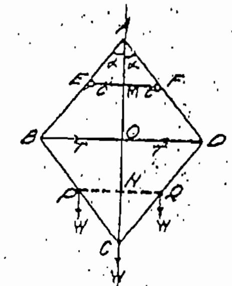
or $\quad -2lT\cos\alpha - 2lW\sin\alpha + Wc\operatorname{cosec}^2\alpha = 0 \quad [\because\ \delta\alpha \neq 0]$

or $\quad 2lT\cos\alpha = Wc\operatorname{cosec}^2\alpha - 2lW\sin\alpha$

or $\quad T = \dfrac{1}{2l\cos\alpha}\,[Wc\operatorname{cosec}^2\alpha - 2lW\sin\alpha]$

$$= W\tan\alpha\left[\frac{c}{2l}\operatorname{cosec}^3\alpha - 1\right]$$

Ex. 39. *A rhombus ABCD formed of four weightless rods each of length a freely jointed at the extremities, rests in a vertical plane on two smooth pegs, which are in a horizontal line distant 2c apart and in contact with AB and AD. Weights each equal to W are hung from the lowest corner C and from the middle points of two lower sides, while B and D are connected by a light inextensible string. If 2α be the angle of the rhombus at A, apply the principle of virtual work to find the tension of the string.*



Sol. The rods AB and AD are in contact with two smooth pegs E and F which are in a horizontal line and

$$EF = 2c.$$

The length of each rod of the rhombus is a and the rods forming the rhombus are light. Weights each equal to W are hung from the lowest corner C and from the middle points P and Q of the lower sides BC and CD. The diagonal AC is vertical and BD is horizontal. Let T be the tension in the inextensible string joining B and D. We have

$$\angle BAC = \alpha = \angle DAC.$$

Replace the string BD by two equal and opposite forces T as shown in the figure so that the distance BD can be changed. Give the system a small symmetrical displacement in which α changes to $\alpha + \delta\alpha$. The line EF joining the pegs remains fixed and the distances will be measured from this line. The $\angle AOB$ remains $90°$.

We have

$$BD = 2BO = 2AB\sin\alpha = 2a\sin\alpha.$$

The depth of C below $EF = MC = AC - AM$

$$= 2AO - AM = 2AB\cos\alpha - EM\cot\alpha$$
$$= 2a\cos\alpha - c\cot\alpha,$$

and the depth of P or Q below EF

$$= AN - AM = \tfrac{3}{2}AO - AM$$
$$= \tfrac{3}{2}a\cos\alpha - c\cot\alpha.$$

The equation of virtual work is

$$-T\delta\,(2a\sin\alpha) + W\delta\,(2a\cos\alpha - c\cot\alpha) + 2W\delta\,(\tfrac{3}{2}a\cos\alpha - c\cot\alpha) = 0$$

or $\quad (-2aT\cos\alpha - 2aW\sin\alpha + Wc\operatorname{cosec}^2\alpha - 3aW\sin\alpha + 2cW\operatorname{cosec}^2\alpha)\,\delta\alpha = 0$

or $$-2aT\cos\alpha-5aW\sin\alpha+3Wc\operatorname{cosec}^2\alpha=0 \qquad [\because\ \delta\alpha\neq 0]$$

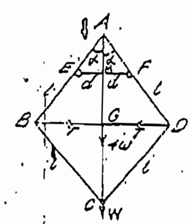
or $$2aT\cos\alpha=3Wc\operatorname{cosec}^2\alpha-5aW\sin\alpha$$

or $$T=\frac{W(3c\operatorname{cosec}^2\alpha-5a\sin\alpha)}{2a\cos\alpha}.$$

Ex. 40. *ABCD is a rhombus formed with four rods each of length l and of weight w joined by smooth hinges. A weight W is attached to the lowest hinge C and the frame rests on two smooth pegs in a horizontal line and B and D are joined by a string. If the distance of the pegs apart is 2d and the angle at A is 2α, show that the tension in the string is*

$$\tan\alpha\left[\frac{d}{2l}(W+4w)\operatorname{cosec}^3\alpha-(W+2w)\right].$$

Sol. The rods AB and AD are in contact with two smooth pegs E and F which are in a horizontal line and $EF=2d$. The length of each rod forming the rhombus is l. The total weight $4w$ of the rods forming the rhombus can be taken acting at G, the point of intersection of the diagonals AC and BD. A weight W is attached to the lowest point C. The diagonal AC is vertical and BD is horizontal. Let T be the tension in the string BD. We have



$$\angle BAC=\alpha=\angle DAC.$$

Give the system a small symmetrical displacement in which α changes to $\alpha+\delta\alpha$. The line EF joining the pegs remain fixed and the distances will be measured from this line. The $\angle AGB$ remains $90°$

We have the length of the string BD

$$=2BG=2AB\sin\alpha=2l\sin\alpha.$$

The depth of G below EF

$$=MG=AG-AM=l\cos\alpha-d\cot\alpha,$$

and the depth of C below EF

$$=MC=AC-AM=2l\cos\alpha-d\cot\alpha.$$

The equation of virtual work is

$$-T\delta(2l\sin\alpha)+4w\delta(l\cos\alpha-d\cot\alpha)+W\delta(2l\cos\alpha-d\cot\alpha)=0$$

or $$-2lT\cos\alpha\,\delta\alpha-4lw\sin\alpha\,\delta\alpha+4dw\operatorname{cosec}^2\alpha\,\delta\alpha-2lW\sin\alpha\,\delta\alpha+dW\operatorname{cosec}^2\alpha\,\delta\alpha=0$$

or $$[-2lT\cos\alpha-2l\sin\alpha\,(2w+W)+d\operatorname{cosec}^2\alpha\,(4w+W)]\,\delta\alpha=0$$

or $$-2lT\cos\alpha-2l\sin\alpha\,(2w+W)+d\operatorname{cosec}^2\alpha\,(4w+W)=0 \qquad [\because\ \delta\alpha\neq 0]$$

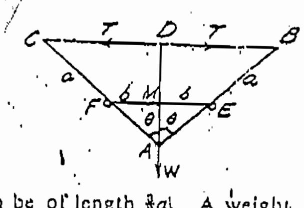
or $$2lT\cos\alpha=d\,(W+4w)\operatorname{cosec}^2\alpha-2l\sin\alpha\,(W+2w)$$

or $$T=\frac{1}{2l\cos\alpha}[d\,(W+4w)\operatorname{cosec}^2\alpha-2l\sin\alpha\,(W+2w)]$$

or $$T=\tan\alpha\left[\frac{d}{2l}(W+4w)\operatorname{cosec}^3\alpha-(W+2w)\right].$$

Ex. 41. *A frame ABC consists of three light rods, of which AB, AC are each of length a, BC of length $\frac{3}{2}a$, freely jointed together. It rests with BC horizontal, A below BC and the rods AB, AC over two smooth pegs E and F, in the same horizontal line, distant 2b apart. A weight W is suspended from A; find the thrust in the rod BC.*

Sol. ABC is a framework consisting of three light rods AB, AC and BC. The rods AB and AC rest on two smooth pegs E and F which are in the same horizontal line and $EF=2b$. Each of the rods AB and AC is of length a. Let T be the thrust in the rod BC which is given to be of length $\frac{3}{2}a$. A weight W is suspended from A. The line AD joining A to the middle point D of BC is vertical. Let



$$\angle BAD=\theta=\angle CAD.$$

Replace the rod BC by two equal and opposite forces T as shown in the figure. Now give the system a small symmetrical displacement in which θ changes to $\theta+\delta\theta$. The line EF joining the pegs remains fixed, the lengths of the rods AB and AC do not change and the length BC changes.

The forces contributing to the sum of virtual works are: (i) the thrust T in the rod BC, and (ii) the weight W acting at A.

We have,

$$BC=2BD=2AB\sin\theta=2a\sin\theta.$$

Also the depth of the point of application A of the weight W below the fixed line EF

$$=MA=ME\cot\theta=b\cot\theta.$$

The equation of virtual work is

$$T\delta(2a\sin\theta)+W\delta(b\cot\theta)=0$$

or $$2aT\cos\theta\,\delta\theta-bW\operatorname{cosec}^2\theta\,\delta\theta=0$$

or $$(2aT\cos\theta-bW\operatorname{cosec}^2\theta)\,\delta\theta=0$$

or $$2aT\cos\theta-bW\operatorname{cosec}^2\theta=0 \qquad [\because\ \delta\theta\neq 0]$$

or $$2aT\cos\theta=bW\operatorname{cosec}^2\theta$$

or $$T=\frac{Wb}{2a}\operatorname{cosec}^2\theta\sec\theta.$$

But in the position of equilibrium,

$$BC=\tfrac{3}{2}a \text{ and so } BD=\tfrac{3}{4}a.$$

Therefore $$\sin\theta=\frac{BD}{AB}=\frac{\frac{3}{4}a}{a}=\frac{3}{4}$$

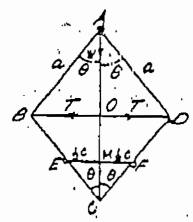
and $\cos\theta = \sqrt{(1-\sin^2\theta)} = \sqrt{(1-\tfrac{9}{16})} = \tfrac{1}{4}\sqrt{7}$.

$\therefore \quad T = \dfrac{Wb}{2a}\cdot\dfrac{16}{9}\cdot\dfrac{4}{\sqrt{7}} = \dfrac{32}{9\sqrt{7}}\dfrac{b}{a}W.$

Ex. 42. *A rhomboidal framework ABCD is formed of four equal light rods of length a smoothly jointed together. It rests in a vertical plane with the diagonal AC vertical, and the rods BC, CD in contact with smooth pegs in the same horizontal line at a distance c apart, the joints B, D being kept apart by a light rod of length b. Show that a weight W, being placed on the highest joint A, will produce in BD a thrust of magnitude*

$$W(2a^2c-b^3)/b^2(4a^2-b^2)^{1/2}.$$

Sol. The rods BC and CD of a rhomboidal framework $ABCD$ are in contact with two smooth pegs E and F which are in the same horizontal line and $EF=c$. The rods forming the rhombus are light and the length of each rod forming the rhombus is a. Let T be the thrust in the light rod BD joining B and D. A weight W is placed at the highest joint A. In the position of equilibrium, $BD=b$. The diagonal AC is vertical and BD is horizontal. Let



$$\angle BAC = \theta = \angle CAD.$$

Replace the rod BD by two equal and opposite forces T as shown in the figure. Give the system a small symmetrical displacement about the vertical line AC in which θ changes to $\theta+\delta\theta$. The line EF joining the pegs remains fixed. The lengths of the rods AB, BC, CD, DA do not change and the length BD changes. The only forces contributing to the sum of virtual works are: (i) the weight W placed at A, and (ii) the thrust T in the rod BD. The reactions of the pegs E and F do not work. We have

$$BD = 2BO = 2AB\sin\theta = 2a\sin\theta.$$

Also the height of A above the fixed line EF

$$= MA = CA - CM$$
$$= 2OA - CM = 2a\cos\theta - \tfrac{1}{2}c\cot\theta.$$

The equation of virtual work is

$$T\delta(2a\sin\theta) - W\delta(2a\cos\theta - \tfrac{1}{2}c\cot\theta) = 0$$

or $\quad 2aT\cos\theta\,\delta\theta + 2aW\sin\theta\,\delta\theta - \tfrac{1}{2}c\,W\operatorname{cosec}^2\theta\,\delta\theta = 0$

or $\quad (2aT\cos\theta + 2aW\sin\theta - \tfrac{1}{2}c\,W\operatorname{cosec}^2\theta)\,\delta\theta = 0$

or $\quad 2aT\cos\theta + 2aW\sin\theta - \tfrac{1}{2}c\,W\operatorname{cosec}^2\theta = 0 \qquad [\because \delta\theta \neq 0]$

or $\quad 2aT\cos\theta = \tfrac{1}{2}c\,W\operatorname{cosec}^2\theta - 2aW\sin\theta$

or $\quad T = W.\dfrac{\tfrac{1}{2}c\operatorname{cosec}^2\theta - 2a\sin\theta}{2a\cos\theta}. \qquad \ldots(1)$

But in the position of equilibrium, we have

$$BD = b \text{ so that } BO = \tfrac{1}{2}b.$$

$\therefore$ from $\triangle AOB$, we have

$$\sin\theta = \frac{BO}{AB} = \frac{\tfrac{1}{2}b}{a} = \frac{b}{2a}.$$

$\therefore \quad \operatorname{cosec}\theta = \dfrac{2a}{b}$ and $\cos\theta = \sqrt{(1-\sin^2\theta)}$

$$= \sqrt{\left(1-\frac{b^2}{4a^2}\right)} = \frac{\sqrt{(4a^2-b^2)}}{2a}.$$

Substituting in (1), we have

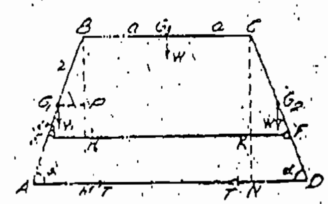
$$T = W\frac{\tfrac{1}{2}c.(4a^2/b^2) - 2a.(b/2a)}{2a.\{\sqrt{(4a^2-b^2)}/2a\}} = W\frac{2a^2c-b^3}{b^2(4a^2-b^2)^{1/2}}.$$

Ex. 43. *Three rigid rods AB, BC, CD each of length 2a, are smoothly jointed at B and C. The system is placed in a vertical plane so that rods AB, CD are in contact with two smooth pegs distant 2c apart in the same horizontal line, the rods AB, CD make equal angle α with the horizon. Prove that the tension of the string AD which will maintain this configuration is*

$$\tfrac{1}{4}W\operatorname{cosec}\alpha.\sec^2\alpha\,\{(3c/a)-(3+2\cos^3\alpha)\},$$

where W is the weight of either rod.

Sol. Three rigid rods AB, BC, CD, each of length $2a$ and weight W are smoothly jointed at B and C. The rods AB and CD are in contact with two smooth pegs E and F which are in the same horizontal line and



$$EF = 2c.$$

Let T be the tension in the string AD joining A and D. The weights W of the rods AB, BC and CD act at their respective middle points.

We have $\quad \angle BAD = \alpha = \angle CDA.$

Give the system a small symmetrical displacement in which α changes to $\alpha+\delta\alpha$. The line EF joining the pegs remains fixed. The lengths of the rods AB, BC, CD do not change and the length AD changes.

We have,

$$AD = AM + MN + ND$$
$$= 2a\cos\alpha + 2a + 2a\cos\alpha$$
$$= 4a\cos\alpha + 2a.$$

The height of G_1 or G_2 above the fixed line EF

$$= HP = HB - PB = EH\tan\alpha - BG_1\sin\alpha$$
$$= \tfrac{1}{2}(2c-2a)\tan\alpha - a\sin\alpha$$
$$= (c-a)\tan\alpha - a\sin\alpha,$$

and the height of G_3 above EF

$$= HB = (c-a)\tan\alpha.$$

The equation of virtual work is

$$-T\delta(4a\cos\alpha+2a)-2W\delta[(c-a)\tan\alpha-a\sin\alpha]$$
$$-W\delta[(c-a)\tan\alpha]=0$$

or $\quad 4aT\sin\alpha\,\delta\alpha-2(c-a)W\sec^2\alpha\,\delta\alpha$
$$+2aW\cos\alpha\,\delta\alpha-(c-a)W\sec^2\alpha\,\delta\alpha=0$$

or $\quad [4aT\sin\alpha-3(c-a)W\sec^2\alpha+2aW\cos\alpha]\,\delta\alpha=0$

or $\quad 4aT\sin\alpha-3(c-a)W\sec^2\alpha+2aW\cos\alpha=0 \qquad [\because\ \delta\alpha\neq 0]$

or $\quad 4aT\sin\alpha=3(c-a)W\sec^2\alpha-2aW\cos\alpha$

or $\quad T=\dfrac{1}{4a\sin\alpha}\,W[3c\sec^2\alpha-3a\sec^2\alpha-2a\cos\alpha]$

$$=\tfrac{1}{4}W\operatorname{cosec}\alpha\sec^2\alpha\,[(3c/a)-(3+2\cos^3\alpha)].$$

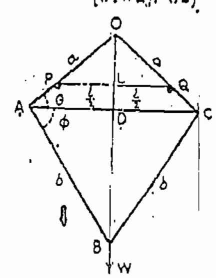
Ex. 44. *Four light rods are jointed together to form a quadrilateral OABC. The lengths are such that*

$$OA=OC=a,\ AB=CB=b.$$

The framework hangs in a vertical plane OA and OC resting in contact with two smooth pegs distant l apart and on the same horizontal level. A weight hangs at B. If θ, ϕ are the inclinations of OA, AB to the horizontal, prove that these values are given by the equations

$$a\cos\theta=b\cos\phi$$

and $\quad \tfrac{1}{2}l\sec^2\theta\sin\phi=a\sin(\theta+\phi).$ [Jiwaji 82]



Sol. $OABC$ is a framework formed of four light rods such that

$$OA=OC=a \text{ and } AB=CB=b.$$

The rods OA and OC are in contact with two smooth pegs P and Q which are in the same horizontal line and $PQ=l$. A weight W hangs at B. We have

$$\angle OAC=\theta \text{ and } \angle BAC=\phi.$$

Give the system a small displacement in which θ changes to $\theta+\delta\theta$ and ϕ changes to $\phi+\delta\phi$. The line PQ joining the pegs remains fixed. The only force contributing to the sum of virtual works is the weight W acting at B.

We have, the depth of B below $PQ=LB=OB-OL$
$$=OD+DB-OL=a\sin\theta+b\sin\phi-\tfrac{1}{2}l\tan\theta.$$

The equation of virtual work is

$$W\delta(a\sin\theta+b\sin\phi-\tfrac{1}{2}l\tan\theta)=0$$

or $\quad a\cos\theta\,\delta\theta+b\cos\phi\,\delta\phi-\tfrac{1}{2}l\sec^2\theta\,\delta\theta=0$

or $\quad (\tfrac{1}{2}l\sec^2\theta-a\cos\theta)\,\delta\theta=b\cos\phi\,\delta\phi. \qquad \ldots(1)$

Now let us find a relation between the parameters θ and ϕ from the figure. From the $\triangle OAD$, we have $AD=a\cos\theta$ and from the $\triangle BAD$, we have $AD=b\cos\phi$.

$$\therefore\quad a\cos\theta=b\cos\phi. \qquad \ldots(2)$$

Differentiating (2),

$$-a\sin\theta\,\delta\theta=-b\sin\phi\,\delta\phi$$

or $\quad a\sin\theta\,\delta\theta=b\sin\phi\,\delta\phi. \qquad \ldots(3)$

Dividing (1) by (3), we get

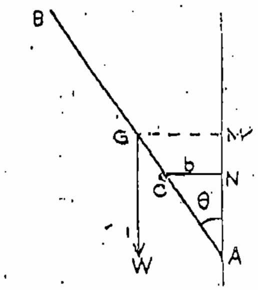
$$\frac{\tfrac{1}{2}l\sec^2\theta-a\cos\theta}{a\sin\theta}=\frac{b\cos\phi}{b\sin\phi}$$

or $\quad \tfrac{1}{2}l\sec^2\theta\sin\phi=a(\sin\theta\cos\phi+\cos\theta\sin\phi)$

or $\quad \tfrac{1}{2}l\sec^2\theta\sin\phi=a\sin(\theta+\phi). \qquad \ldots(4)$

Thus θ and ϕ are given by the equations (2) and (4).

Ex. 45. *A uniform beam of length 2a, rests in equilibrium against a smooth vertical wall and upon a smooth peg at a distance b from the wall. Show that in the position of equilibrium the beam is inclined to the wall at an angle $\sin^{-1}(b/a)^{1/3}$.* [Gorakhpur 80, 82]



Sol. A uniform beam AB of length $2a$ rests in equilibrium against a smooth vertical wall and upon a smooth peg C whose distance CN from the wall is b. Suppose the rod makes an angle θ with the wall i.e., $\angle BAM=\theta$. The weight W of the rod acts at its middle point G.

Give the rod a small displacement in which θ changes to $\theta+\delta\theta$. The peg C remains fixed. The only force that contributes to the sum of virtual works is the weight of the rod acting at G. The reactions at A and C do not work.

We have, the height of G above the fixed point C
$$=NM=AM-AN=AG\cos\theta-CN\cot\theta$$
$$=a\cos\theta-b\cot\theta.$$

The equation of virtual work is

$$-W\delta(a\cos\theta-b\cot\theta)=0,$$

or $\quad \delta(a\cos\theta-b\cot\theta)=0$

or $\quad -a\sin\theta\,\delta\theta+b\operatorname{cosec}^2\theta\,\delta\theta=0$

or $\quad (-a\sin\theta+b\operatorname{cosec}^2\theta)\,\delta\theta=0$

or $\quad -a\sin\theta+b\operatorname{cosec}^2\theta=0 \qquad [\because\ \delta\theta\neq 0]$

or $\quad a\sin\theta=b\operatorname{cosec}^2\theta$ or $\sin^3\theta=b/a$

or $\quad \theta=\sin^{-1}(b/a)^{1/3},$

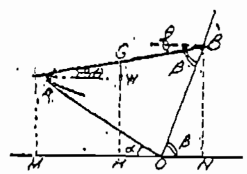
giving the inclination of the rod to the vertical in the position of equilibrium.

Ex. 46. *A heavy uniform rod, of length 2a, rests with its ends in contact with two smooth inclined planes, of inclination α and β to the horizon. If θ be the inclination of the rod to the horizon, prove, by the principle of virtual work, that*

$$\tan\theta=\tfrac{1}{2}(\cot\alpha-\cot\beta).$$

[Meerut 83, 83P, 84S, 87; Jiwaji 81]

Sol. Let AB be the rod of length $2a$ and G its middle point. Let AM, BN, and GH be the perpendiculars from A, B and G on the horizontal line through O, the point of intersection of the inclined planes OA and OB. The weight W of the rod acts at G and in equilibrium the rod makes an angle θ with the horizontal.



Give the rod a small displacement in which θ changes to $\theta+\delta\theta$. The horizontal line MON through O is the fixed line from which the distances will be measured. The angles α and β remain fixed. The only force that contributes to the sum of virtual works is the weight of the rod acting at G. The reactions at A and B do no work. We have

the height of G above the fixed line MON

$$=HG=\tfrac{1}{2}(AM+BN)=\tfrac{1}{2}(OA\sin\alpha+OB\sin\beta).$$

From the $\triangle AOB$ by the sine theorem of trigonometry, we have

$$\frac{OA}{\sin(\beta-\theta)}=\frac{OB}{\sin(\theta+\alpha)}=\frac{AB}{\sin\{\pi-(\alpha+\beta)\}}=\frac{2a}{\sin(\alpha+\beta)}.$$

$$\therefore \quad OA=2a\frac{\sin(\beta-\theta)}{\sin(\alpha+\beta)},\ OB=\frac{2a\sin(\theta+\alpha)}{\sin(\alpha+\beta)}.$$

$$\therefore \quad HG=\tfrac{1}{2}\cdot\frac{2a}{\sin(\alpha+\beta)}\{\sin(\beta-\theta)\sin\alpha+\sin(\theta+\alpha)\sin\beta\}.$$

The equation of virtual work is

$$-W\delta(HG)=0, \text{ or } \delta(HG)=0$$

or $$\delta\left[\frac{a}{\sin(\alpha+\beta)}\{\sin(\beta-\theta)\sin\alpha+\sin(\theta+\alpha)\sin\beta\}\right]=0$$

or $$\frac{a}{\sin(\alpha+\beta)}[-\cos(\beta-\theta)\sin\alpha+\cos(\theta+\alpha)\sin\beta]\,\delta\theta=0$$

or $$-\cos(\beta-\theta)\sin\alpha+\cos(\theta+\alpha)\sin\beta=0 \qquad [\because \delta\theta\neq 0]$$

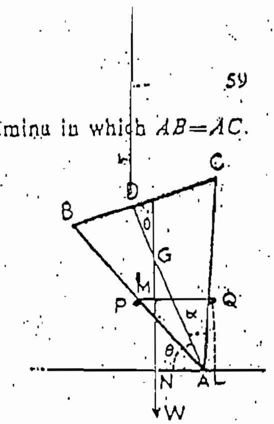
or $$-(\cos\beta\cos\theta+\sin\beta\sin\theta)\sin\alpha+(\cos\theta\cos\alpha-\sin\theta\sin\alpha)\sin\beta=0$$

or $$2\sin\alpha\sin\beta\sin\theta=\cos\theta(\cos\alpha\sin\beta-\cos\beta\sin\alpha)$$

or $$\tan\theta=\tfrac{1}{2}(\cot\alpha-\cot\beta),$$ giving the inclination of the rod to the horizontal in the position of equilibrium.

Ex. 47. *An isosceles triangular lamina, with its plane vertical rests with its vertex downwards, between two smooth pegs in the same horizontal line. Show that there will be equilibrium if the base makes an angle* $\sin^{-1}(\cos^2\alpha)$ *with the vertical,* 2α *being the vertical angle of the lamina and the length of the base being three times the distance between the pegs.* [Meerut 81, 84P]

Sol. ABC is an isosceles triangular lamina in which $AB=AC$. The sides AB and AC rest on two smooth pegs P and Q which are in the same horizontal line.



Let $PQ=a$ so that $BC=3a$.

If D is the middle point of BC, then the centre of gravity G of the lamina lies on the median AD and is such that

$$AG=\tfrac{2}{3}AD.$$

The weight W of the lamina acts vertically downwards at G. We have

$$\angle BAD=\angle CAD=\alpha.$$

Suppose in equilibrium the base BC of the lamina makes an angle θ with the vertical. Since the angle between two lines is equal to the angle between their perpendicular lines, therefore $\angle DAN=\theta$. [Note that DA is perpendicular to BC and AN is perpendicular to the vertical line NMG].

Now $\angle QPA=\angle PAN=\theta-\alpha$,

and $\angle QAL=\pi-(\theta+\alpha)$.

Give the lamina a small displacement in which θ changes to $\theta+\delta\theta$. The line PQ joining the pegs remains fixed and the distances will be measured from this line. The angle α remains fixed. The only force contributing to the sum of virtual works is the weight W of the lamina acting at G. We have, the height of G above the fixed line PQ

$$=MG=NG-NM=NG-LQ$$
$$=AG\sin\theta-AQ\sin\{\pi-(\theta+\alpha)\}$$
$$=\tfrac{2}{3}AD\sin\theta-AQ\sin(\theta+\alpha).$$

Now $AD=CD\cot\alpha=\tfrac{3}{2}a\cot\alpha$. Also from the $\triangle AQP$, by the sine theorem of trigonometry, we have

$$\frac{AQ}{\sin APQ}=\frac{PQ}{\sin PAQ}\ \text{i.e.,}\ \frac{AQ}{\sin(\theta-\alpha)}=\frac{a}{\sin 2\alpha}.$$

$$\therefore \quad AQ=\frac{a}{\sin 2\alpha}\sin(\theta-\alpha).$$

$$\therefore \quad MG=\tfrac{2}{3}\cdot\tfrac{3}{2}a\cot\alpha\sin\theta-\frac{a}{\sin 2\alpha}\sin(\theta-\alpha)\sin(\theta+\alpha)$$

$$=a\cot\alpha\sin\theta-\frac{a}{2\sin 2\alpha}\cdot 2\sin(\theta-\alpha)\sin(\theta+\alpha)$$

$$=a\cot\alpha\sin\theta-\frac{a}{4\sin\alpha\cos\alpha}(\cos 2\alpha-\cos 2\theta)$$

$$=a\cot\alpha\sin\theta-\frac{a\cos 2\alpha}{4\sin\alpha\cos\alpha}+\frac{a\cos 2\theta}{4\sin\alpha\cos\alpha}.$$

The equation of virtual work is

$$-W\delta\,(MG)=0, \quad \text{or} \quad \delta\,(MG)=0$$

or $\delta\left[a\cot\alpha\sin\theta-\dfrac{a\cos 2\alpha}{4\sin\alpha\cos\alpha}+\dfrac{a\cos 2\theta}{4\sin\alpha\cos\alpha}\right]=0$

or $\left[a\cot\alpha\cos\theta-\dfrac{2a\sin 2\theta}{4\sin\alpha\cos\alpha}\right]\delta\theta=0$

or $a\cot\alpha\cos\theta-\dfrac{4a\sin\theta\cos\theta}{4\sin\alpha\cos\alpha}=0 \qquad [\because \ \delta\theta\neq 0]$

or $a\cos\theta\left(\cot\alpha-\dfrac{\sin\theta}{\sin\alpha\cos\alpha}\right)=0.$

$\therefore$ $\cos\theta=0$ *i.e.*, $\theta=\dfrac{\pi}{2}$, giving one position of equilibrium in which the lamina rests symmetrically on the pegs

or $\cot\alpha-\dfrac{\sin\theta}{\sin\alpha\cos\alpha}=0$ *i.e.*, $\sin\theta=\cos^2\alpha$ *i.e.*, $\theta=\sin^{-1}(\cos^2\alpha)$,

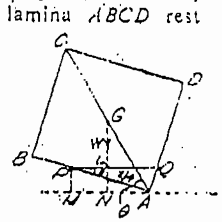
giving the other position of equilibrium.

Ex. 48. (a) *A square of side 2a is placed with its plane vertical between two smooth pegs which are in the same horizontal line at a distance c apart; show that it will be in equilibrium when the inclination of one of its edges to the horizon is either*

$$\frac{\pi}{4} \ \text{or} \ \frac{1}{2}\sin^{-1}\left(\frac{a^2-c^2}{c^2}\right).$$

[Meerut 80, 88P; Gorakhpur 81; Jiwaji 82; P.C.S. 81]

Sol. The sides AB and AD of the square lamina $ABCD$ rest on two smooth pegs P and Q which are in the same horizontal line. It is given that $PQ=c$ and $AB=2a$.



The weight W of the lamina acts at G, the middle point of the diagonal AC. Suppose in the position of equilibrium the side AB of the lamina makes an angle θ with the horizontal so that

$$\angle PAM=\theta=\angle QPA.$$

We have $\angle BAC=\frac{1}{4}\pi=$ constant.

Give the lamina a small displacement in which θ changes to $\theta+\delta\theta$. The line PQ joining the pegs remains fixed. The only force contributing to the sum of virtual works is the weight W of the lamina acting at G. We have, the height of G above the fixed line PQ

$$=LG=NG-NL=NG-MP$$
$$=AG\sin(\tfrac{1}{4}\pi+\theta)-AP\sin\theta$$

$$=a\sqrt{2}\sin(\tfrac{1}{4}\pi+\theta)-PQ\cos\theta\sin\theta$$

$[\because \ AG=\frac{1}{2}AC=\frac{1}{2}\,2a\sqrt{2}=a\sqrt{2},$ and $AP=PQ\cos\theta]$

$$=a\sqrt{2}\,(\sin\tfrac{1}{4}\pi\cos\theta+\cos\tfrac{1}{4}\pi\sin\theta)-c\cos\theta\sin\theta$$
$$=a\,(\cos\theta+\sin\theta)-c\cos\theta\sin\theta.$$

The equation of virtual work is

$$-W\delta\,(LG)=0, \quad \text{or} \quad \delta\,(LG)=0$$

or $\delta\,[a(\cos\theta+\sin\theta)-c\cos\theta\sin\theta]=0$

or $[a(-\sin\theta+\cos\theta)-c(\cos^2\theta-\sin^2\theta)]\,\delta\theta=0$

or $a(\cos\theta-\sin\theta)-c(\cos^2\theta-\sin^2\theta)=0 \qquad [\because \ \delta\theta\neq 0]$

or $(\cos\theta-\sin\theta)\,[a-c(\cos\theta+\sin\theta)]=0.$

$\therefore$ either $\cos\theta-\sin\theta=0$

i.e., $\sin\theta=\cos\theta$ *i.e.*, $\tan\theta=1$ *i.e.*, $\theta=\frac{1}{4}\pi$,

giving one position of equilibrium in which the lamina rests symmetrically on the pegs

or $a-c(\cos\theta+\sin\theta)=0$

i.e., $c^2(\cos\theta+\sin\theta)^2=a^2$

i.e., $c^2(1+\sin 2\theta)=a^2$

i.e., $\sin 2\theta=\dfrac{a^2}{c^2}-1=\dfrac{a^2-c^2}{c^2}$

i.e., $\theta=\frac{1}{2}\sin^{-1}\left(\dfrac{a^2-c^2}{c^2}\right)$,

giving the other position of equilibrium.

Ex. 48 (b). *A uniform rectangular board rests vertically in equilibrium with its sides a and b on two smooth pegs in the same horizontal line at a distance c apart. Prove by the principle of virtual work that the side of length a makes with the vertical an angle θ given by* $2c\cos 2\theta=b\cos\theta-a\sin\theta$. [Meerut 83, 87P]

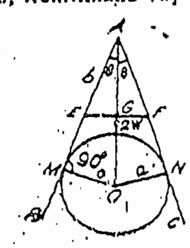
Sol. Proceed as in part (a).

Ex. 49. *Two equal rods, AB and AC, each of length 2b, are freely jointed at A and rest on a smooth vertical circle of radius a. Show that if 2θ be the angle between them, then*

$$b\sin^3\theta=a\cos\theta.$$

[Meerut 92; Lucknow 79; Kanpur 83, 87, 88; Jiwaji 80; Rohilkhand 78]

Sol. Let O be the centre of the given fixed circle and W be the weight of each of the rods AB and AC. If E and F are the middle points of AB and AC, then the total weight $2W$ of the two rods can be taken acting at G, the middle point of EF. The line AO is vertical. We have



$$\angle BAO=\angle CAO=\theta.$$

Also $AB=2b$, $AE=b$. If the rod AB touches the circle at M, then $\angle OMA=90°$ and OM = the radius of the circle $=a$.

Give the rods a small symmetrical displacement in which θ changes to $\theta+\delta\theta$. The point O remains fixed and the point G is slightly displaced.

The $\angle AMO$ remains 90°. We have,
the height of G above the fixed point O

$$=OG=OA-GA=OM \text{ cosec } \theta-AE\cos\theta$$
$$=a \text{ cosec } \theta-b\cos\theta.$$

The equation of virtual work is

$$-2W\delta(OG)=0, \quad \text{or} \quad \delta(OG)=0$$

or $\delta(a \text{ cosec } \theta-b\cos\theta)=0$

or $(-a \text{ cosec } \theta\cot\theta+b\sin\theta)\,\delta\theta=0$

or $-a \text{ cosec } \theta\cot\theta+b\sin\theta=0$ $[\because \delta\theta\neq 0]$

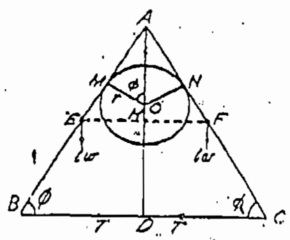
or $a \text{ cosec } \theta\cot\theta=b\sin\theta$

or $a\cos\theta=b\sin^3\theta.$

Ex. 50. *Two equal rods, each of weight wl and length l, are hinged together and placed astride a smooth horizontal cylindrical peg of radius r. Then the lower ends are tied together by a string and the rods are left at the same inclination ϕ to the horizontal. Find the tension in the string and if the string is slack, show that ϕ satisfies the equation*

$$\tan^3\phi+\tan\phi=l/2r.$$ [Gorakhpur 81; P.C.S. 78]

Sol. Let AB and AC be the equal rods which are placed on a fixed horizontal cylindrical peg. In the figure we have shown a cross-section of the cylinder by a vertical plane passing through the points of contact of the rods. This cross section is a circle whose centre is O and radius is r. We have $AB=AC=l$. The line AO is vertical and meets EC at its middle point D. The weights lw and lw of the rods AB and AC act at their respective middle points E and F. Let T be the tension in the string BC. We have



$$\angle ABC=\angle ACB=\phi.$$

If the rod AB touches the circle at the point M, then

$$\angle OMA=90°, \angle MOA=90°-\angle BAD$$
$$=90°-(90°-\phi)=\phi,$$

$OM=r$. Also $AE=\frac{1}{2}l$.

Give the rods a small symmetrical displacement in which ϕ changes to $\phi+\delta\phi$. The point O remains fixed, the length BC changes and the points E and F are slightly displaced. We have

$$BC=2BD=2AB\cos\phi=2l\cos\phi.$$

Also the depth of E or F below the fixed point O

$$=OH=AH-AO=AE\sin\phi-OM\sec\phi=\tfrac{1}{2}l\sin\phi-r\sec\phi.$$

The equation of virtual work is

$$-T\delta(2l\cos\phi)+2lw\delta(\tfrac{1}{2}l\sin\phi-r\sec\phi)=0$$

or $2lT\sin\phi\,\delta\phi+2lw.\tfrac{1}{2}l\cos\phi\,\delta\phi-2lwr\sec\phi\tan\phi\,\delta\phi=0$

or $(2lT\sin\phi+l^2w\cos\phi-2lrw\sec\phi\tan\phi)\delta\phi=0$

or $2lT\sin\phi+l^2w\cos\phi-2lrw\sec\phi\tan\phi=0$ $[\because \delta\phi\neq 0]$

or $2lT\sin\phi=2lrw\sec\phi\tan\phi-l^2w\cos\phi$

or $T=\dfrac{1}{2l\sin\phi}(2lrw\sec\phi\tan\phi-l^2w\cos\phi)$

$$=w(r\sec^2\phi-\tfrac{1}{2}l\cot\phi).$$

If the string is slack, $T=0$. So putting $T=0$ in (1), we get

$$0=w(r\sec^2\phi-\tfrac{1}{2}l\cot\phi)$$

or $r\sec^2\phi-\tfrac{1}{2}l\cot\phi=0$

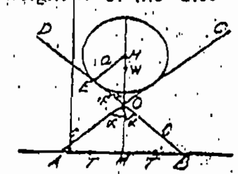
or $r(1+\tan^2\phi)=\tfrac{1}{2}l\cot\phi$

or $\dfrac{l}{2r}=\dfrac{1+\tan^2\phi}{\cot\phi}=\tan\phi+\tan^3\phi.$

Ex. 51. *Two light rods AOC, BOD are smoothly hinged at O, a point at a distance c from each of the ends A, B which are connected by a string of length 2c sin α. The rods rest in a vertical plane with the ends A and B on a smooth horizontal table. A smooth circular disc of radius a and weight W is placed on the rods above O with its plane vertical so that rods are tangents to the disc. Prove that the tension of the string is*

$$\tfrac{1}{2}W\{(a/c)\text{ cosec}^2\alpha+\tan\alpha\}.$$

Sol. If H is the centre of disc, then the weight W of the disc acts at H. The ends A and B of the rods AOC and BOD hinged at O are placed on a smooth horizontal table. Let T be the tension in the string AB. The line HO is vertical and meets AB at its middle point M. We have



$$AO=BO=c \text{ and } AB=2c\sin\alpha.$$

Therefore $\angle HOM=\angle BOM=\alpha=\angle HOD.$

If the rod BOD touches the disc at E, then $\angle OEH=90°$ and HE = the radius of the disc $=a$.

Give the system a small symmetrical displacement about the vertical line HOM in which α changes to $\alpha+\delta\alpha$. The level of the line AB lying on the table remains fixed. The point H is slightly displaced, the length AB changes and the lengths of the rods do not change.

We have $AB=2c\sin\alpha$.

Also the height of H above $AB=MH=MO+OH$

$$=AO\cos\alpha+HE \text{ cosec } \alpha=c\cos\alpha+a \text{ cosec } \alpha.$$

The equation of virtual work is

$$-T\delta(2c\sin\alpha)-W\delta(c\cos\alpha+a \text{ cosec } \alpha)=0$$

or $\quad -2cT \cos\alpha\, \delta\alpha + Wc \sin\alpha\, \delta\alpha + aW \operatorname{cosec}\alpha \cot\alpha\, \delta\alpha = 0$

or $\quad (-2cT \cos\alpha + Wc \sin\alpha + aW \operatorname{cosec}\alpha \cot\alpha)\, \delta\alpha = 0$

or $\quad -2cT \cos\alpha + Wc \sin\alpha + aW \operatorname{cosec}\alpha \cot\alpha = 0 \qquad [\because \delta\alpha \neq 0]$

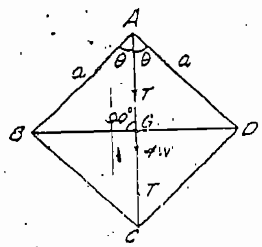
or $\quad 2cT \cos\alpha = W (c \sin\alpha + a \operatorname{cosec}\alpha \cot\alpha)$

or $\quad T = \dfrac{W}{2c \cos\alpha} (c \sin\alpha + a \operatorname{cosec}\alpha \cot\alpha)$

$\quad = \frac{1}{2} W (\tan\alpha + (a/c) \operatorname{cosec}^2 \alpha).$

Problems involving elastic strings

Ex. 52. *Four equal jointed rods, each of length a are hung from an angular point, which is connected by an elastic string with the opposite point. If the rods hang in the form of a square, and if the modulus of elasticity of the string be equal to the weight of a rod, show that the unstretched length of the string is $a\sqrt{2}/3$.* [Meerut 82, 85P]



Sol. $ABCD$ is a framework formed of four equal rods each of length a and say of weight W. It is suspended from the point A. A and C are connected by an elastic string and in equilibrium $ABCD$ is square. The diagonal AC is vertical and so BD is horizontal. Let T be the tension in the string AC. The total weight $4W$ of all the rods AB, BC, CD and DA can be taken acting at G, the point of intersection of the diagonals AC and BD. Let $\angle BAC = \angle DAC$.

Give the system a small symmetrical displacement about the vertical line AC in which θ changes to $\theta + \delta\theta$. The point A remains fixed, the length AC changes, the point G is slightly displaced, the lengths of the rods AB, BC, CD, DA do not change, and the $\angle BGA$ remains 90°. We have $\quad AC = 2AG = 2a \cos\theta$.

Also the depth of G below $A = AG = a \cos\theta$.

The equation of virtual work is

$$-T\delta (2a \cos\theta) + 4W\, \delta (a \cos\theta) = 0$$

or $\quad 2aT \sin\theta\ \delta\theta - 4aW \sin\theta\ \delta\theta = 0$

or $\quad 2a \sin\theta\, (T - 2W)\, \delta\theta = 0$

or $\quad T - 2W = 0 \qquad [\because \delta\theta \neq 0 \text{ and } \sin\theta \neq 0]$

or $\quad T = 2W.$

Let l be the natural length of the elastic string AC. In the position of equilibrium, $\angle BAC = 45°$ and so the extended length AC of the elastic string $= 2AG = 2a \cos 45° = 2a/\sqrt{2} = a\sqrt{2}$.

By Hooke's law, the tension T in the elastic string AC is given by $T = \lambda \dfrac{AC - l}{l}$, where λ is the modulus of elasticity of the string

$$= W \frac{a\sqrt{2} - l}{l} \qquad [\because \lambda = W].$$

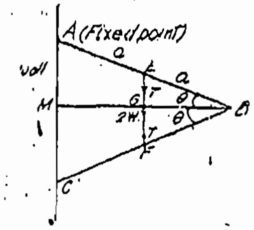
Equating the two values of T, we get

$$2W = W \frac{a\sqrt{2} - l}{l},$$

or $\quad 2l = a\sqrt{2} - l$, or $3l = a\sqrt{2}$

or $\quad l = a\sqrt{2}/3.$

Ex. 53. *One end of a uniform rod AB, of length $2a$ and weight W, is attached by a frictionless joint to a smooth vertical wall, and the other end B is smoothly jointed to an equal rod BC. The middle points of the rods are joined by an elastic string, of natural length a and modulus of elasticity $4W$. Prove that the system can rest in equilibrium in a vertical plane with C in contact with the wall below A, and the angle between the rods is $2 \sin^{-1} (3/4)$.*



Sol. AB and BC are two rods each of length $2a$ and weight W, smoothly joined together at B. The end A of the rod AB is attached to a smooth vertical wall and the end C of the rod BC is in contact with the wall. The middle points E and F of the rods AB and BC are connected by an elastic string of natural length a.

Let T be the tension in the string EF.

The total weight $2W$ of the two rods can be taken acting at the middle point of EF. The line BG is horizontal and meets AC at its middle-point M. Let $\quad \angle ABM = \theta = \angle CBM$.

Give the system a small symmetrical displacement about BM in which θ changes to $\theta + \delta\theta$. The point A remains fixed, the point G is slightly displaced, the length EF changes, the lengths of the rods AB and BC do not change.

We have $EF = 2EG = 2EB \sin\theta = 2a \sin\theta$.

Also the depth of G below the fixed point A

$$= AM = AB \sin\theta = 2a \sin\theta.$$

The equation of virtual work is

$$-T\delta (2a \sin\theta) + 2W\delta (2a \sin\theta) = 0$$

or $\quad (-2aT \cos\theta + 4aW \cos\theta)\, \delta\theta = 0$

or $\quad 2a \cos\theta\, (-T + 2W)\, \delta\theta = 0$

or $\quad -T + 2W = 0 \qquad [\because \delta\theta \neq 0 \text{ and } \cos\theta \neq 0]$

or $\quad T = 2W.$

Also by Hooke's law the tension T in the elastic string EF is given by

$$T = \lambda \frac{2a \sin\theta - a}{a},$$

where λ is the modulus of elasticity of the string

$$= 4W (2 \sin\theta - 1). \qquad [\because \lambda = 4W]$$

Equating the two values of T, we have

$$2W=4W(2\sin\theta-1)$$

or $\quad 1=2(2\sin\theta-1)$, or $1=4\sin\theta-2$

or $\quad 4\sin\theta=3$, or $\sin\theta=3/4$, or $\theta=\sin^{-1}(3/4)$.

$\therefore$ in equilibrium the whole angle between AB and BC

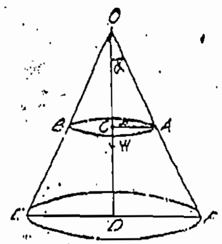
$$=2\theta=2\sin^{-1}(3/4).$$

Ex. 54. *A heavy elastic string, whose natural length is $2\pi a$, is placed round a smooth cone whose axis is vertical and whose semi-vertical angle is α. If W be the weight and λ the modulus of elasticity of the string, prove that it will be in equilibrium when in the form of a circle whose radius is*

$$a\left(1+\frac{W}{2\lambda\pi}\cot\alpha\right).$$

(Meerut 77; Lucknow 79; Gorakhpur 80; Kanpur 88; P.C.S. 79)

Sol. OEF is a smooth fixed cone of semi-vertical angle α, the axis OD of the cone being vertical. A heavy elastic string of natural length $2\pi a$ is placed round this cone and suppose it rests in the form of a circle whose centre is C and whose radius CA is x. The weight W of the string acts at its centre of gravity C. Let T be the tension in this string.



Give the string a small displacement in which x changes to $x+\delta x$. The point O remains fixed, the point C is slightly displaced, $\angle\alpha$ is fixed and the length of the string slightly changes.

We have the length of the string AB in the form of a circle of radius $x=2\pi x$ and so the work done by the tension T of this string is $-T\delta(2\pi x)$.

Also the depth of the point of application C of the weight W below the fixed point O

$$=OC=AC\cot\alpha=x\cot\alpha.$$

and so the work done by the weight W during this small displacement

$$=W\delta(x\cot\alpha).$$

Since the reactions at the various points of contact do no work, we have, by the principle of virtual work

$$-T\delta(2\pi x)+W\delta(x\cot\alpha)=0$$

or $\quad -2\pi T\,\delta x+W\cot\alpha\,\delta x=0 \quad$ or $\quad (-2\pi T+W\cot\alpha)\delta x=0$

or $\quad -2\pi T+W\cot\alpha=0 \qquad [\because \delta x\neq 0]$

or $\quad T=(W\cot\alpha)/2\pi.$

By Hooke's law the tension T in the elastic string AB is given by

$$T=\lambda\frac{2\pi x-2\pi a}{2\pi a}=\lambda\frac{x-a}{a}.$$

Equating the two values of T, we get

$$\frac{W\cot\alpha}{2\pi}=\lambda\frac{x-a}{a}$$

or $$x-a=\frac{a}{2\pi\lambda}W\cot\alpha$$

or $$x=a\left(1+\frac{W}{2\pi\lambda}\cot\alpha\right),$$

which gives the radius of the string in equilibrium.

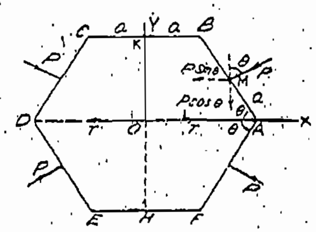
Ex. 55. *An endless chain of weight W rests in the form of a circular band round a smooth vertical cone which has its vertex upwards. Find the tension in the chain due to its weight, assuming the vertical angle of the cone to be 2α.*

Sol. Proceed as in Ex. 54. Here in place of a heavy elastic string of weight W we have a heavy endless chain of weight W. If T is the tension in this chain, then proceeding as in Ex. 54, we get $\quad T=(W\cot\alpha)/2\pi.$

Problems in which the nature of stress is to be found out.

Ex. 56. *$ABCDEF$ is a regular hexagon formed of light rods smoothly jointed at their ends with a diagonal rod AD. Four equal forces P act inwards at the middle points of AB, CD, DE, FA and at right angles to the respective sides. Find the stress in the diagonal AD and state whether it is a tension or thrust.*

Sol. Let $2a$ be the length of each side of the hexagon. The four forces, each equal to P, act inwards at the middle points of AB, CD, DE, FA and at right angles to the respective sides. Let M be the middle point of AB where the force P acts.



Let us assume that the stress in the rod AD is thrust and let it be T.

In the beginning we should not assume the hexagon to be a regular one, for we would give a displacement which will alter the length of AD, without altering the lengths of the sides of the hexagon. Let $\angle BAD=\theta=\angle FAD$.

Replace the rod AD by two equal and opposite forces T as shown in the figure. Take the centre O of the hexagon as origin, the line OA as the axis of x and the perpendicular line OK joining O to the middle point K of BC as the axis of y. Now give the hexagon a small symmetrical displacement about the centre O in which the centre O remains fixed and the lines OX and OY also remain fixed, θ changes to $\theta+\delta\theta$, the points A and D move on the axis of x and the points K and H move on the axis of y. The length AD changes and the middle points of AB, CD, DE and FA are slightly displaced.

We have $AD=2a+4a\cos\theta$. Therefore the work done by the thrust T in the rod AD during this small displacement

$$=T\delta(2a+4a\cos\theta)=-4a\,T\sin\theta\,\delta\theta.$$

By symmetry the forces P acting at the middle points of AB, CD, DE and FA contribute equal works, so that the sum of the works done by all of them is four times the work done by P acting at M.

The components (X, Y) of the force P acting at M along the fixed coordinate axes OX and OY are given by

$$X=-P\sin\theta,\ Y=-P\cos\theta.$$

Also the coordinates of the point M are $(a+a\cos\theta, a\sin\theta)$.

$\therefore$ the virtual work of the force P acting at M during this small displacement

$$=X\delta(a+a\cos\theta)+Y\delta(a\sin\theta)$$
$$=-P\sin\theta\,\delta(a+a\cos\theta)-P\cos\theta\,\delta(a\sin\theta)$$
$$=aP\sin^2\theta\,\delta\theta-aP\cos^2\theta\,\delta\theta=-aP(\cos^2\theta-\sin^2\theta)\,\delta\theta$$
$$=-aP\cos 2\theta\,\delta\theta.$$

Hence the total virtual work done by all the forces P

$$=-4aP\cos 2\theta\,\delta\theta.$$

Now the equation of virtual work is

$$-4a\,T\sin\theta\,\delta\theta-4a\,P\cos 2\theta\,\delta\theta=0$$

or $\quad 4a(T\sin\theta+P\cos 2\theta)\,\delta\theta=0$

or $\quad T\sin\theta+P\cos 2\theta=0 \qquad [\because\ \delta\theta\neq 0]$

or $\quad T=-\dfrac{P\cos 2\theta}{\sin\theta}.$

But in the position of equilibrium the hexagon is regular, so that

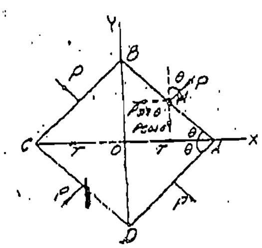
$$\theta=60°.$$

$$\therefore\ T=-\frac{P\cos 120°}{\sin 60°}=-\frac{P(-\frac{1}{2})}{(\sqrt{3}/2)}=\frac{P}{\sqrt{3}}.$$

The positive value of T means that our assumption that there is thrust in AD is correct. Hence there is a thrust $P/\sqrt{3}$ in the rod AD.

Ex. 57. *A frame consists of five bars forming the sides of a rhombus ABCD with diagonal AC. If four equal forces P act inwards at the middle points of the sides, and at right angles to the respective sides, prove that the tension in AC is $(P\cos 2\theta/\sin\theta)$ where θ denotes the angle BAC.*

Sol. Let $2a$ be the length of each side of the rhombus $ABCD$ which we shall assume as placed on a smooth horizontal table. The four forces, each equal to P, act inwards at the middle points of AB, BC, CD, DA and at right angles to the respective sides. Let M be the middle point of AB where the force P acts.



Let us assume that the stress in the rod AC is tension and let it be T. Let $\angle BAC=\theta=\angle CAD$.

Replace the rod AC by two equal and opposite forces T as shown in the figure. Take the centre O of the rhombus as origin, the line OA as the axis of x and the perpendicular line OB as the axis of y. Now give the rhombus a small symmetrical displacement about the centre O in which the centre O and the lines OX and OY remain fixed, θ changes to $\theta+\delta\theta$, the points A and C move on the axis of x and the points B and D move on the axis of y. The length AC changes, the lengths of the rods AB, BC, CD, DA do not change but their middle points are slightly displaced.

We have $AC=4a\cos\theta$ so that the work done by the tension T in the rod AC during this small displacement $=-T\delta(4a\cos\theta)$ $=4aT\sin\theta\,\delta\theta.$

By symmetry the forces P acting at the middle points of AB, BC, CD, DA contribute equal works, so that the sum of the works done by all of them is four times the work done by P acting at M.

The components (X, Y) of the force P acting at M along the fixed co-ordinate axes OX and OY are given by

$$X=-P\sin\theta,\ Y=-P\cos\theta.$$

Also the co-ordinates of the point M are $(a\cos\theta, a\sin\theta)$.

$\therefore$ the virtual work of the force P acting at M during this small displacement

$$=X\delta(a\cos\theta)+Y\delta(a\sin\theta)$$
$$=-P\sin\theta\,\delta(a\cos\theta)-P\cos\theta\,\delta(a\sin\theta)$$
$$=aP\sin^2\theta\,\delta\theta-aP\cos^2\theta\,\delta\theta=-aP\cos 2\theta\,\delta\theta.$$

Hence the total virtual work done by all the four forces P

$$=-4aP\cos 2\theta\,\delta\theta.$$

Now the equation of virtual work is

$$4aT\sin\theta\,\delta\theta-4aP\cos 2\theta\,\delta\theta=0$$

or $\quad 4a(T\sin\theta-P\cos 2\theta)\,\delta\theta=0$

or $\quad T\sin\theta-P\cos 2\theta=0 \qquad [\because\ \delta\theta\neq 0]$

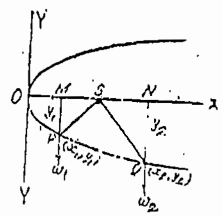
or $\quad T=(P\cos 2\theta/\sin\theta).$

Note. It may be seen that if 2θ is acute, then $\cos 2\theta$ is positive and so the value of T is positive which means that there is tension in the rod AC as we have assumed while solving the problem. But if 2θ is obtuse, then $\cos 2\theta$ is negative and so the value of T is negative which means that there is not tension but thrust in the rod AC.

Problems involving curves

Ex. 58. *Two heavy rings slide on a smooth parabolic wire, whose axis is horizontal and plane vertical, and are connected by a* *string passing round a smooth peg at the focus. Prove that in the position of equilibrium their weights are proportional to their vertical depths below the axis.* [Kanpur 82]

Sol. Let the equation of the parabola be $y^2=4ax$. For the sake of convenience the vertically downwards direction has been taken as the positive direction of y-axis.



There is a smooth peg at the focus S of the parabola. Let PSQ be the string of length l which passes over the peg at S. To the ends of the string two rings P and Q are attached which can slide on the parabolic wire. Let w_1 and w_2 be the weights of the rings P and Q respectively. Then a weight w_1 acts at P and a weight w_2 acts at Q. Let (x_1, y_1) and (x_2, y_2) be the co-ordinates of P and Q respectively.

Give the rings a small displacement in which y_1 changes to $y_1+\delta y_1$ and y_2 changes to $y_2+\delta y_2$. The line OX remains fixed and the length of the string PSQ remains unaltered so that the work done by the tension in this string during this small displacement is zero. The equation of virtual work is

$$w_1\,\delta(y_1)+w_2\,\delta(y_2)=0$$

or $$w_1\,\delta y_1=-w_2\,\delta y_2. \quad \ldots(1)$$

Now let us find a relation between the parameters y_1 and y_2 from the figure.

We have

the focal distance $SP=a+x_1$;

and the focal distance $SQ=a+x_2$.

$$\therefore \quad SP+SQ=2a+x_1+x_2$$

or $$l=2a+\frac{y_1^2}{4a}+\frac{y_2^2}{4a}. \quad \ldots(2)$$

Differentiating (2), we get

$$0=\frac{2y_1}{4a}\,\delta y_1+\frac{2y_2}{4a}\,\delta y_2$$

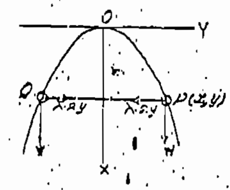
or $$y_1\delta y_1=-y_2\,\delta y_2. \quad \ldots(3)$$

Dividing (1) by (3), we get

$\dfrac{w_1}{y_1}=\dfrac{w_2}{y_2}$ *i.e.*, weights are proportional to their depths.

Ex. 59. *Two small rings of equal weights, slide on a smooth wire in the shape of a parabola whose axis is vertical and vertex upwards, and attract one another with a force which varies as the distance. If they can rest in any symmetrical position on the curve show that they will rest in all symmetrical positions.*

Sol. Here the vertex O is fixed. Taking the vertical line OX as the axis of x, let the equation of the parabola be $y^2=4ax$.



Suppose the two rings each of weight W are in equilibrium in any one symmetrical position P and Q. If (x, y) are the coordinates of P, then $PQ=2y$.

$\therefore$ force of attraction on each ring (P or Q) $=\lambda.2y$ and this force acts at P in the direction PQ and at Q in the direction QP.

A weight W acts at P and a weight W acts at Q. Give the rings a small displacement in which y changes to $y+\delta y$ and x changes to $x+\delta x$. The line OY remains fixed.

The depth of P or Q below OY is x and $PQ=2y$. The equation of virtual work is $\quad -\lambda.2y\,\delta(2y)+2W\delta x=0$.

or $$2\lambda y\,\delta y=W\delta x. \quad \ldots(1)$$

But $\quad y^2=4ax$. Therefore $2y\,\delta y=4a\,\delta x$

or $$y\,\delta y=2a\,\delta x. \quad \ldots(2)$$

Dividing (1) by (2), we get

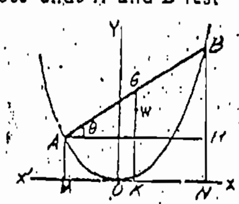
$$2\lambda=W/2a \quad \text{or} \quad \lambda=W/2a, \quad \ldots(3)$$

as the condition for the equilibrium of the rings. The condition (3) is independent of the position of P *i.e.*, of (x, y). Hence if the rings rest in any one symmetrical position, they will rest in all symmetrical positions.

Ex. 60. *A smooth parabolic wire is fixed with its axis vertical and vertex downwards, and in it is placed a uniform rod of length $2l$ with its ends resting on the wire. Show that, for equilibrium, the rod is either horizontal, or makes with the horizontal an angle θ given by $\cos^2\theta=2a/l$, $4a$ being the latus rectum of the parabola.*

[Rohilkhand 89; Lucknow 74; Jiwaji 79; Kanpur 77]

Sol. Let AB be the rod of length $2l$ whose ends A and B rest on a smooth parabolic wire whose axis is vertical and vertex downwards. Referred to OX and OY as the coordinate axes, let the equation of the parabola be $x^2=4ay$.



Let the coordinates of the point A be $(2at, at^2)$ and let $\angle BAH=\theta$. Then the coordinates of B are

$$(2at-2l\cos\theta,\ at^2+2l\sin\theta).$$

Since the point B lies on the parabola, therefore

$$(2at+2l\cos\theta)^2=4a\,(at^2+2l\sin\theta)$$

or $\quad 4a^2t^2+8alt\cos\theta+4l^2\cos^2\theta=4a^2t^2+8al\sin\theta$

or $\quad 8alt\cos\theta=8al\sin\theta-4l^2\cos^2\theta$

or $\quad t=\tan\theta-\dfrac{l}{2a}\cos\theta.$

If z be the height of the centre of gravity G of the rod above the fixed line OX, then

$$z=KG=\tfrac{1}{2}[MA+NB]=\tfrac{1}{2}[at^2+(at^2+2l\sin\theta)]$$

$$=al^2+l\sin\theta=a\left(\tan\theta-\frac{l}{2a}\cos\theta\right)^2+l\sin\theta$$

$$=a\tan^2\theta+\frac{l^2}{4a}\cos^2\theta.$$

Now give the rod a small displacement in which θ changes to $\theta+\delta\theta$, the ends of the rod remaining in contact with the wire. If W be the weight of the rod, then the equation of virtual work is

$$-W\delta z=0,\quad\text{or}\quad \delta z=0.$$

or $\quad \delta\left(a\tan^2\theta+\frac{l^2}{4a}\cos^2\theta\right)=0.$

or $\quad \left(2a\tan\theta\sec^2\theta-\frac{l^2}{2a}\cos\theta\sin\theta\right)\delta\theta=0$

or $\quad 2a\tan\theta\sec^2\theta-\frac{l^2}{2a}\cos\theta\sin\theta=0 \qquad [\because\ \delta\theta\neq 0]$

or $\quad \sin\theta\left(2a\sec^3\theta-\frac{l^2}{2a}\cos\theta\right)=0.$

$\therefore$ either $\sin\theta=0$ *i.e.*, the rod is horizontal

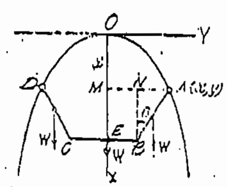
or $\quad 2a\sec^3\theta-\frac{l^2}{2a}\cos\theta=0$ *i.e.*, $2a\sec^3\theta=\frac{l^2}{2a}\cos\theta$

i.e., $\cos^4\theta=4a^2/l^2$ *i.e.*, $\cos^2\theta=2a/l$, giving the inclined position of the rod.

Ex. 61. *Three equal and similar rods AB, BC, CD freely jointed at B and C have small weightless rings attached to them at A and D. The rings slide on a smooth parabolic wire, whose axis is vertical and vertex upwards and whose latus rectum is half the sum of the lengths of the three rods. Prove that in the position of equilibrium, the inclination θ of AB or CD to the vertical is given by*

$$\cos\theta-\sin\theta+\sin 2\theta=0.$$ [Lucknow 80]

Sol. Let $AB=BC=CD=2a$, so that the sum of their lengths $=6a$.



Then the latus rectum of the parabola $=3a$.

Hence the equation of the parabola is $y^2=3ax$.

In the position of equilibrium let θ be the inclination of AB or CD to the vertical. The weight W of each of the rods AB, BC and CD acts at their respective middle points.

Let the coordinates of the point A be (x, y). Then $x=OM$, and $y=MA=MN+NA=EB+NA=a+2a\sin\theta$. Since the point (x, y) lies on the parabola $y^2=3ax$, therefore

$$(a+2a\sin\theta)^2=3ax. \qquad \ldots(1)$$

Differentiating (1), we get

$$2(a+2a\sin\theta)\,2a\cos\theta\,\delta\theta=3a\,\delta x$$

or $\quad \delta x=\frac{4}{3}a(1+2\sin\theta)\cos\theta\,\delta\theta. \qquad \ldots(2)$

Here OY is the fixed line. The depth of the middle point of AB or CD below $OY=x+a\cos\theta$ and the depth of the middle point of BC below $OY=x+2a\cos\theta$.

Let the rods be given a small symmetrical displacement about the axis OX in which θ changes to $\theta+\delta\theta$. Then the equation of virtual work is

$$2W\delta(x+a\cos\theta)+W\delta(x+2a\cos\theta)=0$$

or $\quad 2W\delta x-2aW\sin\theta\,\delta\theta+W\delta x-2aW\sin\theta\,\delta\theta=0$

or $\quad 3W\delta x-4aW\sin\theta\,\delta\theta=0$

or $\quad 3W\cdot\frac{4}{3}a(1+2\sin\theta)\cos\theta\,\delta\theta-4aW\sin\theta\,\delta\theta=0,$ substituting for δx from (2)

or $\quad 4aW[\cos\theta+2\sin\theta\cos\theta-\sin\theta]\,\delta\theta=0$

or $\quad \cos\theta+\sin 2\theta-\sin\theta=0, \qquad [\because\ W\neq 0$ and $\delta\theta\neq 0]$

which gives the required result.

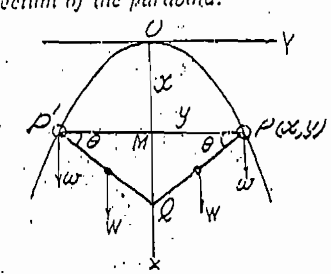
Ex. 62. *Two uniform straight rods PQ, $P'Q$, in all respects alike are smoothly jointed at Q and at P, P' carry small rings which slide on a smooth fixed parabolic wire whose axis is vertical and vertex upwards. Prove that in the symmetrical position of equilibrium the angle either rod makes with the horizontal is*

$$\sin^{-1}\left\{\frac{aW}{l(W+w)}\right\},$$

where W is the weight of either rod, w weight of either ring, l the length of either rod and $4a$ the latus rectum of the parabola.

Sol. Let the equation of the parabola be $y^2=4ax$. We have

$$PQ=P'Q=l.$$



The weights W of the rods PQ and $P'Q$ act at their respective middle points. The weight w of the ring P acts at P and the weight w of the ring P' acts at P'.

We have $\angle QPP'=\theta=\angle QP'P$.

Let the co-ordinates of the point P be (x, y). Then $OM=x$ and $MP=y$.

Here OY is a fixed line. The depth of P or P' below OY $=OM=x$,

and the depth of the middle point of PQ or $P'Q$ below OY $=x+\frac{1}{2}l\sin\theta.$

Give the system (*i.e.*, rods and rings) a small symmetrical displacement about the axis OX of the parabola such that θ changes to $\theta+\delta\theta$, x changes to $x+\delta x$ and the lengths of the rods do not change. The equation of virtual work is

$$2w\,\delta(x)+2W\,\delta(x+\tfrac{1}{2}l\sin\theta)=0$$

or $\quad 2w\,\delta x+2W\,\delta x+Wl\cos\theta\,\delta\theta=0$

or $\quad 2(w+W)\,\delta x+Wl\cos\theta\,\delta\theta=0$

or $\quad -Wl\cos\theta\,\delta\theta=2(w+W)\,\delta x.$ ...(1)

Now we should find a relation between the two parameters x and θ.

From $\triangle QMP$, we have $MP=PQ\cos\theta=l\cos\theta$, giving the y-coordinate y of the point P of the parabola $y^2=4ax$.

Putting $y=l\cos\theta$ in the equation $y^2=4ax$, we have

$$l^2\cos^2\theta=4ax. \quad ...(2)$$

Differentiating (2), we have

$$-2l^2\cos\theta\sin\theta\,\delta\theta=4a\,\delta x$$

or $$-l^2\cos\theta\sin\theta\,\delta\theta=2a\,\delta x. \quad ...(3)$$

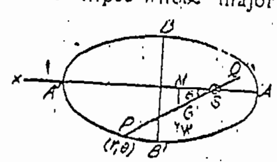
Dividing (3) by (1), we have

$$\frac{l\sin\theta}{W}=\frac{a}{w+W} \text{ or } \sin\theta=\frac{aW}{l(w+W)}$$

or $$\theta=\sin^{-1}\left\{\frac{aW}{l(w+W)}\right\}.$$

Ex. 63. *A smooth rod passes through a smooth ring at the focus of an ellipse whose major axis is horizontal, and rests with its lower end on the quadrant of the curve which is farthest removed from the focus. Find its position of equilibrium and show that its length must at least be $\frac{3}{4}a+\frac{1}{4}a\sqrt{(1+8e^2)}$ where $2a$ is the major axis and e is the eccentricity.*

Sol. There is a ring at the focus S of the ellipse whose major axis AA' is horizontal. The rod PQ, say of length $2c$, passes through the ring at S and the end P of the rod rests on the quadrant $A'B'$ of the ellipse which is farthest from the focus S. The weight W of the rod PQ acts at its middle point G.



Referred to the focus S as pole and SA' as the initial line, the polar equation of the ellipse is

$$l/r=1-e\cos\theta. \quad ...(1)$$

Let the coordinates of P be (r, θ) so that $SP=r$ and $\angle PSX=\theta$.

Here the point S is fixed. The depth of G below the fixed line $SX=MG=SG\sin\theta$

$$=(SP-PG)\sin\theta=(r-c)\sin\theta$$

$$=\left(\frac{l}{1-e\cos\theta}-c\right)\sin\theta. \quad [\text{Substituting for } r \text{ from (1)}]$$

Give the rod a small displacement in which the end P of the rod remains on the ellipse and θ changes to $\theta+\delta\theta$. The equation of virtual work is $\quad W\delta(MG)=0$, or $\delta(MG)=0$

or $$\delta\left[\left(\frac{l}{1-e\cos\theta}-c\right)\sin\theta\right]=0$$

or $$\left[\left(\frac{l}{1-e\cos\theta}-c\right)\cos\theta-\frac{le\sin\theta}{(1-e\cos\theta)^2}\sin\theta\right]\delta\theta=0$$

or $$\left[\left(\frac{l}{1-e\cos\theta}-c\right)\cos\theta-\frac{le\sin^2\theta}{(1-e\cos\theta)^2}\right]=0 \quad [\because \delta\theta\neq 0] \quad ...(2)$$

From (1), $e\cos\theta=1-\frac{l}{r}=\frac{r-l}{r}$

$$\therefore \quad \cos\theta=\frac{r-l}{er}. \quad ...(3)$$

With the help of (1) and (3), the equation (2) becomes

$$(r-c)\frac{r-l}{er}-le\left\{1-\frac{(r-l)^2}{r^2e^2}\right\}\cdot\frac{r^2}{l^2}=0$$

or $$(r-c)\frac{r-l}{er}-\frac{1}{le}\{r^2e^2-(r-l)^2\}=0. \quad ...(4)$$

The equation (2) or (4) gives the position of equilibrium. When the length of rod is least, its upper end Q is just at the focus S so that

$$r=2c.$$

Putting $r=2c$ in (4), we have

$$(2c-c)\frac{2c-l}{2ce}-\frac{1}{le}\{4c^2e^2-(2c-l)^2\}=0$$

or $$l(2c-l)-2\{4c^2e^2-(4c^2-4cl+l^2)\}=0$$

or $$8(1-e^2)c^2-6cl+l^2=0.$$

Solving it, we get

$$c=\frac{6l\pm\sqrt{[36l^2-32l^2(1-e^2)]}}{16(1-e^2)}=\frac{3l\pm l\sqrt{(1+8e^2)}}{8(1-e^2)}.$$

But we know that semi-latus rectum $l=b^2/a=a(1-e^2)$.

Hence $$c=\frac{3a\pm a\sqrt{(1+8e^2)}}{8}$$

or $$2c=\frac{3}{4}a\pm\frac{1}{4}a\sqrt{(1+8e^2)}.$$

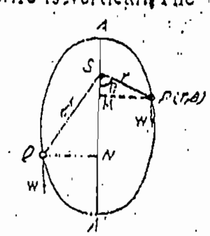
But $2c$ is definitely greater than SB' i.e. $2c>a$. Therefore neglecting the $-$ive sign, we get

$$2c=\frac{3}{4}a+\frac{1}{4}a\sqrt{(1+8e^2)},$$

giving the least length of the rod.

Ex. 64. *Two small smooth rings of equal weight slide on a fixed elliptical wire, whose major axis is vertical and they are connected by a string which passes over a small smooth peg at the upper focus; show that the weights will be in equilibrium wherever they are placed.*

Sol. The major axis AA' of the elliptical wire is vertical. The two rings are P and Q connected by the string PSQ which passes over a smooth peg at the focus S. Let c be the length of the string PSQ.



Referred to the focus S as pole and SA' as the initial line, the polar equation of the ellipse is $l/r=1-e\cos\theta$. ...(1)

Let the coordinates of the point P be (r, θ) so that

$SP = r$ and $\angle PSA' = \theta$.

Since $SP + SQ = c$, therefore $SQ = c - SP = c - r$.

The weight W of the ring P acts at P and the weight W of the ring Q acts at Q.

Here S is the fixed point. The depth of P below S

$$= SM = SP \cos\theta = r\cos\theta = \frac{r-l}{e}, \quad \text{from (1).}$$

Since the radius vector of the point Q is $c-r$, therefore the depth of Q below $S = \dfrac{c-r-l}{e}$.

Let the system be imagined to undergo a small displacement in which the length of the string remains unchanged and only the rings P and Q slightly slide on the wire. The sum of the virtual works done by the forces during this small displacement

$$= W\delta\,(SM) + W\delta\,(SN)$$

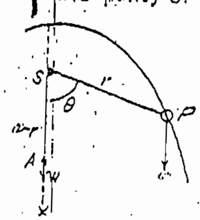
$$= W\delta\left(\frac{r-l}{e}\right) + W\delta\left(\frac{c-r-l}{e}\right)$$

$$= \frac{W}{e}\,\delta r - \frac{W}{e}\,\delta r = 0 \text{ (always).}$$

Thus the sum of virtual works is zero and is independent of r. Therefore the rings will be in equilibrium wherever they are placed.

Ex. 65. *A small heavy ring slides on a smooth wire whose plane is vertical, and is connected by a string passing over a small pulley in the plane of the curve with a weight which hangs freely. If the ring is in equilibrium in any position on the wire, show that the form of latter is a conic section whose focus is at the pulley.*

Sol. Let a string ASP of length a pass over a fixed pulley S. Let w be the weight of the ring P which slides on a smooth wire whose plane is vertical and W the weight attached to the other end A of the string.



Take the fixed point S as pole and the vertical line SAX as the initial line. Let the polar coordinates of the point P be (r, θ) so that $SP = r$ and $\angle PSX = \theta$.

We have $SA = a - r$.

Give the system a small displacement in which r changes to $r + \delta r$, θ changes to $\theta + \delta\theta$ and the length of the string remains unaltered so that the work done by its tension during this small displacement is zero. The only two forces contributing to the sum of virtual works are:

(i) the weight w of the ring acting at P whose depth below the fixed point S is $r\cos\theta$;

and (ii) the weight W acting at A whose depth below S is $a-r$.

Since the ring is in equilibrium in any position on the wire, therefore by the principle of virtual work, we have

$$w\delta\,(r\cos\theta) + W\delta\,(a-r) = 0 \text{ or } \delta\,[wr\cos\theta + W(a-r)] = 0.$$

Integrating it, we get

$$wr\cos\theta + W(a-r) = c, \text{ where } c \text{ is an arbitrary constant}$$

or $\quad Wa - c = Wr - wr\cos\theta$

or $\quad Wa - c = Wr\,\{1 - (w/W)\cos\theta\}$

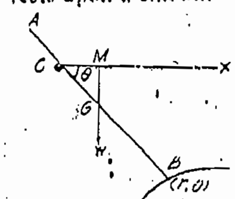
or $\quad \dfrac{a-(c/W)}{r} = 1 - \dfrac{w}{W}\cos\theta$

which is of the form $l/r = 1 - e\cos\theta$.

This is the polar equation of a conic section whose focus is at S.

Ex. 66. *A heavy rod, of length $2l$, rests upon a fixed smooth peg at C and with its end B upon a smooth curve. If it rests in all positions, show that the curve is a conchoid whose polar equation, with C as origin, is,* $\quad r = l + (a/\sin\theta)$.

Sol. The end B of the rod AB of length $2l$ rests upon a smooth curve and the rod rests against a smooth peg at C. Take the fixed point C as pole and the fixed horizontal line CX as the initial line. The weight W of the rod acts at its middle point G. Let the coordinates of B be (r, θ) so that $CB = r$ and $\angle BCX = \theta$. We have



$$CG = CB - GB = r - l.$$

It is given that the rod is in equilibrium in all positions. Give the rod a small displacement in which r changes to $r + \delta r$ and θ changes to $\theta + \delta\theta$. The length of the rod does not change and the line CX remains fixed. The only force that does virtual work is the weight W of the rod acting at G whose depth below the fixed line CX is

$$= MG = CG\sin\theta = (r-l)\sin\theta.$$

The equation of virtual work is

$$W\delta\,[(r-l)\sin\theta] = 0, \quad \text{or} \quad \delta\,[(r-l)\sin\theta] = 0.$$

Integrating it, we get $\quad (r-l)\sin\theta = a,$

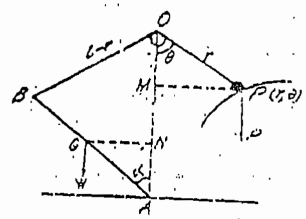
where a is an arbitrary constant

or $\quad r - l = \dfrac{a}{\sin\theta}, \quad$ or $\quad r = l + \dfrac{a}{\sin\theta}$.

This is the locus of B i.e., the equation of the curve on which the end B of the rod rests.

Ex. 67. *AB is a heavy beam which can turn about a horizontal axis at A. A cord fastened to B passes over a smooth pulley, at O, vertically above A, and is tied at the other end to a given weight P which moves on a smooth curve. Find the form of the curve if there is equilibrium in all positions.*

Sol. The beam AB of length $2a$ can turn about a horizontal axis at A. There is a smooth pulley at O which is vertically above A. One end of a cord of length l is fastened to B and the cord passes over the pulley at O and to the other end of the cord a weight P is tied which moves on a smooth curve. The beam is in equilibrium in all positions. The weight W of the beam acts at its middle point G.



Since both the points A and O are fixed, therefore the distance $AO=\text{constant}=c$ (say).

Take the fixed point O as pole and the fixed straight line OA as the initial line. Let the coordinates of P be (r, θ) so that $OP=r$ and $\angle POA=\theta$. We have $BO=l-r$. If $\angle BAO=\alpha$, then from $\triangle BAO$, by using the cosine formula, we have

$$\cos\alpha=\frac{4a^2+c^2-(l-r)^2}{4ac}. \qquad ...(1)$$

The depth of P below the fixed point $O=OM=r\cos\theta$, and the depth of G below $O=ON=OA-AN=c-a\cos\alpha$.

Give the system a small displacement in which the length of the string does not change. The equation of virtual work is

$$P\delta(OM)+W\delta(ON)=0$$

or $\quad P\delta(r\cos\theta)+W\delta(c-a\cos\alpha)=0$

or $\quad \delta[Pr\cos\theta+W(c-a\cos\alpha)]=0.$

Integrating it, we get

$Pr\cos\theta+W(c-a\cos\alpha)=k$, where k is an arbitrary constant

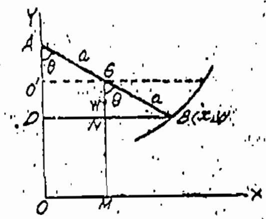
or $\quad Pr\cos\theta+W\left\{c-\dfrac{4a^2+c^2-(l-r)^2}{4c}\right\}=k$, substituting for $\cos\alpha$ from (1)

or $\quad 4cPr\cos\theta+W\{3c^2-4a^2+(l-r)^2\}=4ck.$

This is the equation of the curve on which the weight P rests.

Ex. 68. *One end of a beam rests against a smooth vertical wall and the other end on a smooth curve in a vertical plane perpendicular to the wall; if the beam rests in all positions, show that the curve is an ellipse whose major axis lies along the horizontal line described by the centre of gravity of the beam.* **[Kanpur 79]**

Sol. Let AB be the beam of weight W and length $2a$ which rests with its one end A against a smooth vertical wall OY and the other end B on a smooth curve which lies in a vertical plane perpendicular to the wall. The weight W of the beam acts at its middle point G. Let θ be the inclination of the beam to the vertical and z be the height of G above the fixed horizontal plane OX i.e. $MG=z$.



The beam rests in all positions. Give the beam a small displacement in which θ changes to $\theta+\delta\theta$; the end B remains on the curve and the end A remains on the wall. The only force which does the virtual work is the weight W of the beam. Hence the equation of virtual work is

$$-W\delta(MG)=0, \quad \text{or} \quad \delta(z)=0.$$

$\therefore \quad z=\text{constant}=h$ (say).

Thus, in all positions of rest, the centre of gravity of the beam remains at a constant height h from the fixed horizontal plane OX and so it describes a horizontal line at a height h from the fixed horizontal plane OX.

Referred to OX and OY as the coordinate axes, let (x, y) be the coordinates of the end B of the beam. Then

$$x=DB=AB\sin\theta=2a\sin\theta \qquad ...(1)$$

and $\quad y=OD=MN=MG-NG=h-a\cos\theta. \qquad ...(2)$

From (2), $y-h=-a\cos\theta$

or $\quad 2(y-h)=-2a\cos\theta. \qquad ...(3)$

Squaring and adding (1) and (3), we get

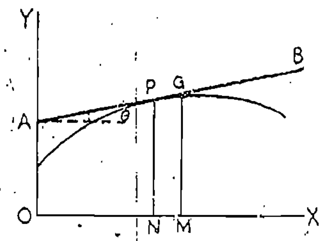
$$x^2+4(y-h)^2=4a^2$$

or $$\frac{x^2}{4a^2}+\frac{(y-h)^2}{a^2}=1. \qquad ...(4)$$

The equation (4) is the locus of the point B i.e., the equation of the curve on which the end B of the beam lies. We see that (4) is the equation of an ellipse whose centre is the point $(0, h)$ i.e., the point O' where the line described by G meets the wall and whose major axis is the line $y-h=0$ i.e., $y=h$ i.e., the horizontal line $O'G$ described by the centre of gravity of the beam.

Ex. 60. *A uniform beam rests tangentially upon a smooth curve in a vertical plane and one end of the beam rests against a smooth vertical wall; if the beam is in equilibrium in any position, find the equation of the curve.* [Kanpur 80]

Sol. Let AB be the beam of length $2a$ touching the curve at P and resting with its end A in contact with the vertical wall OY. Take the wall OY as the y-axis and a fixed horizontal line OX as the x-axis. The weight W of the beam acts at its middle point G. Let z be the height of G above the fixed horizontal line OX i.e., $MG=z$. Suppose the beam makes an angle θ with the horizontal. The beam is in equilibrium in all positions. If we give the beam a small displacement in which θ changes to $\theta+\delta\theta$, then the equation of virtual work is



$$-W\delta(MG)=0, \text{ i.e., } \delta(z)=0$$

for the reactions of the wall and the curve do not work.

$\therefore$ $z=\text{constant}=h$ (say).

Hence the coordinates of G are $(a\cos\theta, h)$.

Now the straight line AB passes through the point $G(a\cos\theta, h)$ and makes an angle θ with the x-axis. Therefore the equation of AB is $y-h=\tan\theta\,(x-a\cos\theta)$

i.e., $x\tan\theta-y=a\sin\theta-h$, ...(1)

where θ is the parameter.

Since AB touches the curve, therefore the curve is the envelope of AB for varying values of θ.

Differentiating (1) partially with respect to θ, we have

$$x\sec^2\theta=a\cos\theta, \text{ i.e., } x=a\cos^3\theta. \quad \text{...(2)}$$

If we now eliminate θ between (1) and (2), we get the envelope of (1) i.e., the curve upon which the beam rests.

Putting $x=a\cos^3\theta$ in (1), we get

$$y-h=a\cos^3\theta\tan\theta-a\sin\theta=-a\sin\theta\,(1-\cos^2\theta)=-a\sin^3\theta \quad \text{...(3)}$$

From (2) and (3), $x^{2/3}+(y-h)^{2/3}=(a\cos^3\theta)^{2/3}+(-a\sin^3\theta)^{2/3}$
$=a^{2/3}(\cos^2\theta+\sin^2\theta)=a^{2/3}$.

Hence the equation of the curve is $x^{2/3}+(y-a)^{2/3}=a^{2/3}$, which is an astroid.

Strings In Two Dimensions

(Catenary)

§ 1. Introduction.

In the present chapter we shall consider the equilibrium of perfectly flexible strings. All those strings which offer no resistance on bending at any point are called **flexible strings**. In such cases, the resultant action across any section of the string consists of a single force whose line of action is along the tangent to the curve formed by the string. The normal section of the string is taken to be so small that it may be regarded as a curved line. A chain with short and perfectly smooth links approximates a flexible string. [Kanpur 76]

§ 2. The Catenary.

When a uniform string or chain hangs freely under gravity between two points not in the same vertical line, the curve in which it hangs, is called a **catenary.**

A Uniform or Common Catenary. [Raj. T.D.C. 80]. If the weight per unit length of the suspended flexible string or chain is constant, then the catenary is called the **uniform or common catenary.**

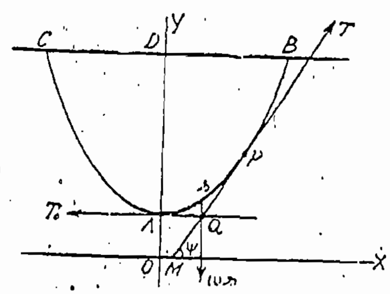
Note. Here we shall frequently use the word catenary for the common catenary i.e. the word catenary will always mean the common catenary in this chapter.

§ 3. Intrinsic equation of the common catenary.

[Raj. T.D.C. 78, 80; Kanpur 80; Agra 80; Rohilkhand 86, 88; Luck. 77, 79]

Let the uniform flexible string BAC hang in the form of a uniform catenary with A as its lowest point. Let P be any point on the portion AB of the string and s the distance of P from A measured along the arc length of the string. If w is the weight per unit length of the string, then the weight of the portion AP will be ws and will act vertically downwards through the centre of gravity of AP.

The portion AP of the string is in equilibrium under the action of the following three forces :



(Fig. 1)

(i) The weight ws of the string AP acting vertically downwards through its centre of gravity,

(ii) The tension T_0 at the lowest point A acting along the tangent to the curve at A which is horizontal,

and (iii) The tension T at P acting along the tangent to the curve at P inclined at an angle ψ to the horizontal.

Since the string AP is in equilibrium under the action of the three forces acting in the same vertical plane therefore the line of action of the weight ws, must pass through the point Q which is the point of intersection of the lines of action of the tensions T_0 and T.

Resolving the forces acting on AP horizontally and vertically, we have

$$T \cos \psi = T_0, \quad \ldots(1)$$

and

$$T \sin \psi = ws. \quad \ldots(2)$$

Dividing (2) by (1), we have

$$\tan \psi = \frac{ws}{T_0} \quad \ldots(3)$$

Now let

$$T_0 = wc, \quad \ldots(4)$$

i.e., let the tension at the lowest point be equal to the weight of the length c, of the string, then from (3) we have

$$\tan \psi = \frac{s}{c}$$

or

$$s = c \tan \psi, \quad \ldots(5)$$

which is the *intrinsic equation of the common catenary.*

Remark 1. From the equation (1) it is clear that the horizontal component of the tension at every point of the catenary is the same and is equal to T_0, the tension at the lowest point.

Remark 2. From the equation (2) we conclude that the vertical component of the tension at any point of the string is equal to the weight of the string between the vertex and that point.

Remark 3. From the relation (4) it follows that the tension at the lowest point is equal to the weight of the string of length c.

§ 4. Cartesian equation of the common catenary.

[Allahabad 78, 79; Rohilkhand 79, 81, 83, 85, 86; Lucknow 81; Agra 84, 86; Kanpur 83, 86, 87; Meerut 90, 90P]

The intrinsic equation of the common catenary is [see § 3]

$$s = c \tan \psi, \quad \ldots(1)$$

where ψ is the angle which the tangent at any point P of the catenary makes with some horizontal line to be taken as the axis of x and s is the arc length of the catenary measured from the vertex A to the point P.

We know that $dy/dx = \tan \psi$.

$\therefore$ from (1), we have

$$s = c \frac{dy}{dx}.$$

Differentiating both sides with respect to x, we have

$$\frac{ds}{dx} = c \frac{d^2y}{dx^2}$$

or

$$\sqrt{\left[1 + \left(\frac{dy}{dx}\right)^2\right]} = c \frac{d^2y}{dx^2}$$

Putting $\frac{dy}{dx} = p$, so that $\frac{d^2y}{dx^2} = \frac{dp}{dx}$, we get

$$\sqrt{(1 + p^2)} = c \frac{dp}{dx}$$

or

$$\frac{dx}{c} = \frac{dp}{\sqrt{(1 + p^2)}}.$$

Integrating, we have

$$\frac{x}{c} + A = \sinh^{-1}(p) = \sinh^{-1}\left(\frac{dy}{dx}\right), \quad \ldots(2)$$

where A is the constant of integration.

Now if we choose the vertical line throught the lowest point A of the catenary as the axis of y, then at the point A, we have

$$x = 0 \text{ and } dy/dx = 0$$

because the tangent at the point A is horizontal *i.e.*, parallel to the axis of x.

$\therefore$ from (2), we have $A=0$.

$$\therefore \quad \frac{x}{c}=\sinh^{-1}\left(\frac{dy}{dx}\right)$$

or
$$\frac{dy}{dx}=\sinh\left(\frac{x}{c}\right).$$

Integrating both sides with respect to x, we have

$$y=c\cosh\left(\frac{x}{c}\right)+B,$$

where B is a constant of integration.

If we take the origin O at a depth c below the lowest point A of the catenary, then at A, we have

$$x=0 \text{ and } y=c.$$

$\therefore$ from (3), we have $B=0$;

$$\therefore \quad y=c\cosh\left(\frac{x}{c}\right), \qquad \ldots(4)$$

which is the *cartesian equation of the common catenary.*

§ 5. Definitions.

1. **Axis of the catenary.** Since $\cosh(x/c)$ is an even function of x, therefore the curve is symmetrical about the axis of y which is along the vertical through the lowest point of catenary. *This vertical line of symmetry is called the* **axis of catenary.**

2. **Vertex of the catenary.** *The lowest point A of the common catenary at which the tangent is horizontal is called the* vertex of the catenary.

3. **Parameter of the catenary.** *The quantity c occuring in the cartesian equation $y=c\cosh(x/c)$ of the catenary is called the* **parameter of the catenary.**

4. **Directrix of the catenary.** *The horizontal line at a depth c below the lowest point i.e., the axis of x, is called the* directrix of the catenary.

5. **Span and Sag.** Let the string be suspended from the two points B and C in the same horizontal line. *Then the distance BC is called the* span of a catenary *and the depth DA (see fig. of § 3) of the lowest point below BC is called the* sag of the catenary.

§ 6. Some important relations for the common catenary.

1. **Relation between x and s.**

[Rohilkhand 85; Raj. T.D.C. 81; Kanpur 85, 87; Agra 88]

For the catenary, we have

$$s=c\tan\psi=c\frac{dy}{dx} \qquad \left[\because \frac{dy}{dx}=\tan\psi\right]$$

$$\therefore \quad \frac{dy}{dx}=\frac{s}{c}. \qquad \ldots(1)$$

Also $\quad y=c\cosh(x/c)$.

Differentiating, we have

$$\frac{dy}{dx}=\sinh\left(\frac{x}{c}\right) \qquad \ldots(2)$$

From (1) and (2), we have

$$\frac{s}{c}=\sinh(x/c),$$

or
$$s=c\sinh(x/c),$$

which is the relation between x and s.

2. **Relation between y and s.**

[Raj. T.D.C. 81; Kanpur 84; Rohilkhand 85, 88]

For a catenary, we have

$$y=c\cosh(x/c).$$

Also $\quad s=c\sinh(x/c). \qquad$ [see relation (3)]

$\therefore$ squaring and subtracting, we have

$$y^2-s^2=c^2[\cosh^2(x/c)-\sinh^2(x/c)]=c^2$$

or
$$y^2=c^2+s^2, \qquad \ldots(4)$$

which is the relation between y and s.

3. **Relation between y and ψ.**

[Gorakhpur 77; Garhwal 76; Agra 80, 87; Kanpur 81, 87; Raj. T.D.C. 79; Rohilkhand 82]

For any curve, we have

$$dy/ds=\sin\psi.$$

$$\therefore \quad \frac{dy}{d\psi}=\frac{dy}{ds}\cdot\frac{ds}{d\psi}=\sin\psi\cdot\frac{d}{d\psi}(c\tan\psi) \qquad [\because s=c\tan\psi]$$

$$=c\sin\psi\cdot\sec^2\psi=c\sec\psi\cdot\tan\psi.$$

Thus
$$\frac{dy}{d\psi}=c\sec\psi\tan\psi.$$

Integrating, we get $y=c\sec\psi+A$, where A is a constant of integration.

But when $\quad y=c,\ \psi=0;\ \therefore\ A=0$.

Hence $\quad y=c\sec\psi, \qquad \ldots(5)$

which is the relation between y and ψ.

Aliter. From relation (4), we have $y^2=c^2+s^2$.

Also $\quad s=c\tan\psi$.

$$\therefore \quad y^2=c^2+s^2=c^2+c^2\tan^2\psi=c^2(1+\tan^2\psi)=c^2\sec^2\psi$$

or $\quad y=c\sec\psi$.

4. **Relation between x and ψ.** [Garhwal 76; Alld. 79; Agra 84, 87; Gorakhpur 72, 75, 80; Kanpur 84]

For any curve, we have $dx/ds = \cos\psi$.

$$\therefore \quad \frac{dx}{d\psi} = \frac{dx}{ds}\cdot\frac{ds}{d\psi} = \cos\psi \frac{d}{d\psi}(c\tan\psi) \qquad [\because s = c\tan\psi]$$

$$= \cos\psi\,(c\sec^2\psi)$$

or $$\frac{dx}{d\psi} = c\sec\psi.$$

Integrating, we get

$$x = c\log(\sec\psi + \tan\psi) + B,$$

where B is a constant of integration.

But when $x=0,\ \psi=0;\ \therefore\ B=0.$

Hence $x = c\log(\sec\psi+\tan\psi)$, ...(6)

which is the relation between x and ψ.

Note. The equations (5) and (6) together form the parametric equations of the catenary, ψ being the parameter.

5. Relation between tension and ordinate.

[Rohilkhand 82; Allad. 78; Agra 85; Raj. T.D.C. 79(S); Gorakhpur 81; Kanpur 77]

From § 3, we have

$$T\cos\psi = T_0 \quad\text{and}\quad T_0 = wc$$

$$\therefore \quad T = T_0\sec\psi = wc\sec\psi.$$

But $y = c\sec\psi$.

Hence $T = wy$, ...(7)

which is the relation between T and y.

The relation (7) shows that the tension at any point of a catenary varies as the height of the point above the directrix.

6. Radius of curvature at any point of a catenary.

[Rohilkhand 82]

For a catenary, we have $s = c\tan\psi$.

$$\therefore \quad \rho = \frac{ds}{d\psi} = c\sec^2\psi. \qquad ...(8)$$

Illustrative Examples

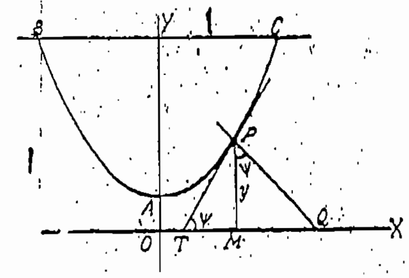
Ex. 1. *If the normal at any point P of a catenary meets the directrix at Q, show that $PQ=\rho$.*

Sol. If the tangent at the point P makes an angle ψ with the axis of x, then the normal PQ at P will make an angle ψ with the vertical through P.

Let PM be the ordinate of the point P. Then from the right angled triangle PMQ, we have

$$PQ = PM\sec\psi = y\sec\psi$$

$$= c\sec^2\psi \qquad [\because y = c\sec\psi]$$



(Fig. 2)

$=\rho$, because for the catenary $s = c\tan\psi$,

$$\rho = ds/d\psi = c\sec^2\psi.$$

Ex. 2. *If T be the tension at any point P of a catenary, and T_0 that at the lowest point A, prove that*

$$T^2 - T_0^2 = W^2,$$

W being the weight of the arc AP of the catenary.

[Rohilkhand 81,83; Agra 85]

Sol. Let arc $AP = s$ and ψ be the inclination of the tangent at P to the horizontal. If w is the weight per unit length of the string, then W = weight of the arc $AP = ws$.

If T is the tension at the point P of the catenary and T_0 that at the lowest point A, then we have

$$T\cos\psi = T_0 \quad\text{and}\quad T\sin\psi = ws = W.$$

Squaring and adding, we have

$$T^2 = T_0^2 + W^2 \quad\text{or}\quad T^2 - T_0^2 = W^2.$$

Ex. 3. *Prove that if a uniform inextensible chain hangs freely under gravity, the difference of the tensions at two points varies as the difference of their weights.*

Sol. Let T_1 and T_2 be the tensions at the two points $P(x_1, y_1)$ and $Q(x_2, y_2)$ respectively of the chain.

Then from $T = wy$, we have

$$T_1 = wy_1 \quad\text{and}\quad T_2 = wy_2.$$

$$\therefore \quad T_2 - T_1 = w(y_2 - y_1)$$

or $$T_2 - T_1 \propto (y_2 - y_1). \qquad [\because w \text{ is constant}]$$

Hence the difference of the tensions at P and Q varies as the difference of their heights y_1 and y_2.

Ex. 4. *Show that for a common catenary*

$$x=c \log\left(\frac{y+s}{c}\right).$$

[Agra 85; Gorakhpur 76; Rohilkhand 77]

Sol. The parametric equations of a catenary are

$$x=c \log (\sec \psi+\tan \psi), \qquad \ldots(1)$$

and
$$y=c \sec \psi. \qquad \ldots(2)$$

Also for a catenary,

$$s=c \tan \psi. \qquad \ldots(3)$$

From (2) and (3), we have

$$\sec \psi=y/c, \text{ and } \tan \psi=s/c.$$

Substituting in (1), we get

$$x=c \log\left(\frac{y}{c}+\frac{s}{c}\right), \text{ or } x=c \log\left(\frac{y+s}{c}\right).$$

Ex. 5. *If $(\bar{x}, \bar{y})$ be the coordinates of the centre of gravity of the arc measured from the vertex upto the point $P(x, y)$, prove that*

$$\bar{x}=x-c \tan(\psi/2), \quad \bar{y}=\tfrac{1}{2}(c/\cos \psi+x \cot \psi).$$

Sol. The parametric equations of a catenary are

$$x=c \log (\sec \psi+\tan \psi) \text{ and } y=c \sec \psi.$$

Also for a catenary $s=c \tan \psi$, $\therefore \; ds/d\psi=c \sec^2 \psi$.

We have

$$\bar{x}=\frac{\int x\, ds}{\int ds}=\frac{\int_0^\psi x \frac{ds}{d\psi}\, d\psi}{\int_0^\psi \frac{ds}{d\psi}\, d\psi} \qquad [\because \text{ At the vertex, } \psi=0]$$

$$=\frac{\int_0^\psi c \log (\sec \psi+\tan \psi)\, c \sec^2 \psi\, d\psi}{\int_0^\psi c \sec^2 \psi\, d\psi}$$

$$=\frac{c^2\left[\left\{\log (\sec \psi+\tan \psi) . \tan \psi\right\}_0^\psi-\int_0^\psi \frac{1}{(\sec \psi+\tan \psi)}(\sec \psi \tan \psi+\sec^2 \psi) \tan \psi\, d\psi\right]}{c\left[\tan \psi\right]_0^\psi}$$

[Integrating the numerator by parts taking $\sec^2 \psi$ as 2nd function]

$$=\frac{c\left[\tan \psi \log (\sec \psi+\tan \psi)-\int_0^\psi \sec \psi \tan \psi\, d\psi\right]}{\tan \psi}$$

$$=\frac{c\left[\tan \psi \log (\sec \psi+\tan \psi)-\left\{\sec \psi\right\}_0^\psi\right]}{\tan \psi}=\frac{x \tan \psi-c(\sec \psi-1)}{\tan \psi}$$

$$=x-\frac{c(\sec \psi-1)}{\tan \psi}=x-\frac{c(1-\cos \psi)}{\sin \psi}=x-c \frac{2 \sin^2 \frac{1}{2}\psi}{2 \sin \frac{1}{2}\psi \cos \frac{1}{2}\psi}$$

$$=x-c \tan \tfrac{1}{2}\psi,$$

and
$$\bar{y}=\frac{\int y\, ds}{\int ds}=\frac{\int_0^\psi y \frac{ds}{d\psi}\, d\psi}{\int_0^\psi \frac{ds}{d\psi}\, d\psi}=\frac{\int_0^\psi c \sec \psi . c \sec^2 \psi\, d\psi}{\int_0^\psi c \sec^2 \psi\, d\psi}$$

$$=\frac{c^2\left[\left\{\frac{1}{2} \sec \psi \tan \psi\right\}_0^\psi+\frac{1}{2}\int_0^\psi \sec \psi\, d\psi\right]}{c\left[\tan \psi\right]_0^\psi}$$

$$\left[\because \int \sec^n \theta\, d\theta=\frac{\sec^{n-2} \theta \tan \theta}{n-1}+\frac{n-2}{n-1}\int \sec^{n-2} \theta\, d\theta\right]$$

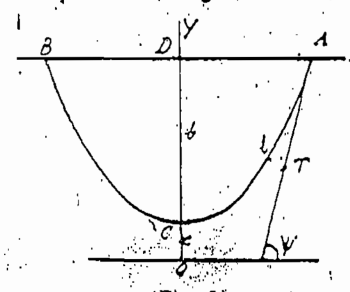
$$=\frac{c\left[\frac{1}{2} \sec \psi . \tan \psi+\frac{1}{2}\left\{\log (\sec \psi+\tan \psi)\right\}_0^\psi\right]}{\tan \psi}$$

$$=\frac{1}{2 \tan \psi}\left[c \sec \psi \tan \psi+c . \log (\sec \psi+\tan \psi)\right]$$

$$=\frac{1}{2 \tan \psi}\left[c \sec \psi \tan \psi+x\right]=\tfrac{1}{2}\left[\frac{c}{\cos \psi}+x \cot \psi\right]$$

Ex. 6. *A rope of length $2l$ feet is suspended between two points at the same level, and the lowest point of the rope is b feet below the points of suspension. Show that the horizontal component of the tension is $w(l^2-b^2)/2b$, w being the weight of the rope per foot of its length.* [G.N.U. 77]

Sol. Let the rope ACB of length $2l$ be suspended between two



(Fig. 3)

points A and B and the same level and let C be its lowest point. It is given that the sag $CD=b$.

Let $OC=c$ be the parameter of the catenary.

For the point A of the catenary, we have $s=l$ and $y=c+b$.

Substituting these values in the formula $y^2=c^2+s^2$, we have

$$(c+b)^2=c^2+l^2$$

or $$c^2+2cb+b^2=c^2+l^2$$

or $$2cb=l^2-b^2, \quad \text{or} \quad c=(l^2-b^2)/2b.$$

Now the horizontal component of the tension at any of the catenary is constant and is equal to wc, where w is the weight per unit length of the string.

$\therefore$ here the horizontal component of the tension $=\dfrac{w(l^2-b^2)}{2b}$.

Ex. 7. *A uniform chain, of length l, is to be suspended from two points A and B, in the same horizontal line so that either terminal tension is n times that at the lowest point. Show that the span AB must be*

$$\frac{l}{\sqrt{(n^2-1)}} \log \{n+\sqrt{(n^2-1)}\}.$$

[Agra 85; Gorkhpur 79, 82]

Sol. Draw figure as in Ex. 6 on page 9.

Let the uniform chain ACB of length l be suspended from two points A and B in the same horizontal line.

Let (x_A, y_A) be the coordinates of the point A and ϕ_A be the angle which the tangent at A makes with the x-axis.

If T is the tension at the terminal point A and T_0 that at the lowest point, then given that $T=nT_0$.

But $$T=wy_A \text{ and } T_0=wc.$$

$$\therefore \quad wy_A=nwc$$

or $$wc\sec\psi_A=nwc. \qquad [\because\ y=c\sec\psi_A]$$

$$\therefore \sec\psi_A=n. \qquad \ldots(1)$$

Now at the point A, $s=\text{arc } CA=\tfrac{1}{2}l$.

$\therefore$ from $s=c\tan\psi$, we have

$$\tfrac{1}{2}l=c\tan\psi_A$$

or $$c=\frac{l}{2\tan\psi_A}=\frac{l}{2\sqrt{(\sec^2\psi_A-1)}}=\frac{l}{2\sqrt{(n^2-1)}}. \qquad \ldots(2)$$

Hence the span $AB=2DA=2x_A$

$$=2c\log(\sec\psi_A+\tan\psi_A)$$

$$=2c\log\{\sec\psi_A+\sqrt{(\sec^2\phi_A-1)}\}$$

$$=\frac{l}{\sqrt{(n^2-1)}}\log\{n+\sqrt{(n^2-1)}\}.$$

[Substituting for $\sec\psi_A$ from (1) and for c from (2)]

Ex. 8. *A uniform chain of length l, is suspended from two points A, B in the same horizontal line. If the tension at A is twice that at the lowest point show that the span AB is*

$$\frac{l}{\sqrt{3}}\log(2+\sqrt{3}).$$

[Kanpur 80]

Sol. Proceed exactly as in the preceding example 7.

Here $n=2$.

Ex. 9. *A uniform chain of length l, which can just bear a tension of n times its weight, is stretched between two points on the same horizontal line. Show that the least possible sag in the middle is $l\{n-\sqrt{n^2-\tfrac{1}{4}}\}$.* [Raj T.D.C. 79, 81; Kanpur 84; Rohilkhand 86]

Sol. (Refer figure of Ex. 6 on Page 9).

Let the uniform chain of given length l be stretched between two points A and B in the same horizontal line. Obviously the sag in the middle is least when the span AB is maximum. Since the chain can just bear a tension of n times its weight, therefore in the case of maximum span AB the terminal tension at the end A should be equal to nlw, where w is the weight per unit length of the chain. So, for the least possible sag, if T is the tension at the end A, we have

$$\therefore \quad T=nwl. \qquad \ldots(1)$$

If (x_A, y_A) are the coordinates of the point A, then

$$T=wy_A. \qquad \ldots(2)$$

$\therefore$ from (1) and (2), we have

$nwl=wy_A$, or $y_A=nl$. Thus for the point A, $y=y_A=nl$ $\qquad \ldots(3)$

Now arc $CA=\tfrac{1}{2}l$, i.e., for the point A, $s=\tfrac{1}{2}l$.

Using the formula $y^2=c^2+s^2$ for the point A, we have

$$n^2l^2=c^2+\tfrac{1}{4}l^2,$$

or $$c^2=l^2(n^2-\tfrac{1}{4}), \quad \text{or} \quad c=l\sqrt{(n^2-\tfrac{1}{4})} \qquad \ldots(4)$$

$\therefore$ least possible sag in the middle

$$=CD=OD-OC=y_A-c$$

$$=nl-l\sqrt{(n^2-\tfrac{1}{4})}$$ [Substituting for y_A from (3) and for c from (4)]

$$=l\{n-\sqrt{(n^2-\tfrac{1}{4})}\}.$$

Ex. 10. *A given length, 2s, of a uniform chain has to be hung between two points at the same level and the tension has not to exceed the weight of a length b of the chain. Show that the greatest span is* $\sqrt{(b^2-s^2)}\cdot\log\left(\frac{b+s}{b-s}\right)$. [Raj. T. D. C. 81; Agra 79, 86]

Sol. [Refer figure of Ex. 6 on page. 9].

Let a uniform chain of given length $2s$ be hung between two points A and B at the same level. Let w be the weight per unit length of the chain. Then bw is the weight of a length b of the chain. From the formula $T=wy$, we observe that the tension in the chain is maximum at the terminal point A. Since the tension in the chain is not to exceed bw, therefore in the case of maximum span AB if T is the tension at A, we must have $T=bw$. ...(1)

If (x_A, y_A) are the coordinates of the point A, then $T=wy_A$. ...(2)

From (1) and (2), we have

$$wy_A=bw, \text{ or } y_A=b. \quad ...(3)$$

Now using the formula $y^2=c^2+s^2$ for the point A, we have

$$b^2=c^2+s^2, \text{ or } c^2=b^2-s^2,$$

$$\therefore\ c=\sqrt{(b^2-s^2)}. \quad ...(4)$$

Now the greatest span $=AB=2x_A$

$=2c\log(\sec\psi_A+\tan\psi_A)$, ψ_A being the inclination of the tangent at A to the x-axis

$=2c\log\left(\dfrac{c\sec\psi_A+c\tan\psi_A}{c}\right)$

$=2c\log\left(\dfrac{y_A+s}{c}\right)$ $[\because\ y_A=c\sec\psi_A$ and $s=c\tan\psi_A]$

$=2\sqrt{(b^2-s^2)}\log\left[\dfrac{b+s}{\sqrt{(b^2-s^2)}}\right]$, putting for y_A from (3) and for c from (4)

$=2\sqrt{(b^2-s^2)}\log\left[\dfrac{(b+s)}{\sqrt{\{(b+s)(b-s)\}}}\right]$

$=\sqrt{(b^2-s^2)}\log\left(\dfrac{b+s}{b-s}\right)^{1/2}$

$=\sqrt{(b^2-s^2)}\log\left(\dfrac{b+s}{b-s}\right)$.

Ex. 11. *If α, β be the inclinations to the horizon of the tangents at the extremities of a portion of a common catenary, and l the*

length of the portion, show that the height of one extremity above the other is $\dfrac{l\sin\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)}$, *the two extremities being on one side of the vertex of the catenary.*

[Rohilkhand 88, 89; Gorakhpur 81; Agra 87; Luck. 80]

Sol. Let α, β be the inclinations to the horizontal of the tangents at the extremities P and Q (P lying above Q) of a portion PQ of a common catenary, the points P and Q being on the same side of the vertex of the catenary. Draw the figure by taking an arc PQ of a catenary lying only on one side of the vertex.

If y_P and y_Q are the ordinates of the points P and Q respectively, then from $y=c\sec\psi$, we have

$$y_P=c\sec\alpha \text{ and } y_Q=c\sec\beta$$

$[\because\ \psi=\alpha$ at P and $\psi=\beta$ at $Q]$

$\therefore$ Height of the extremity P above the other extremity Q

$$=y_P-y_Q=c(\sec\alpha-\sec\beta). \quad ...(1)$$

If C is the lowest point (*i.e.*, the vertex) of the catenary of which PQ is an arc, then from the formula $s=c\tan\psi$, we have

arc $CP=c\tan\alpha$ and arc $CQ=c\tan\beta$.

$\therefore$ l = length of the arc PQ = arc CP − arc CQ

$=c(\tan\alpha-\tan\beta)$

or $$c=\frac{l}{(\tan\alpha-\tan\beta)}.$$

$\therefore$ from (1), the required height

$=\dfrac{l(\sec\alpha-\sec\beta)}{(\tan\alpha-\tan\beta)}=\dfrac{l(\cos\beta-\cos\alpha)}{(\sin\alpha\cos\beta-\cos\beta\sin\alpha)}$

$=\dfrac{2l\sin\frac{1}{2}(\beta+\alpha)\sin\frac{1}{2}(\alpha-\beta)}{\sin(\alpha-\beta)}$

$=\dfrac{2l\sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)}{2\sin\frac{1}{2}(\alpha-\beta)\cos\frac{1}{2}(\alpha-\beta)}=\dfrac{l\sin\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)}$.

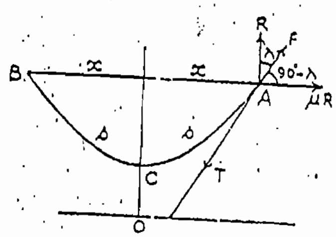
Ex. 12. *The end links of a uniform chain slide along a fixed rough horizontal rod. Prove that the ratio of the maximum span to the length of the chain is*

$$\mu\log\left\{\frac{1+\sqrt{(1+\mu^2)}}{\mu}\right\}$$

where μ is the coefficient of friction.

[Kanpur 85; Raj. T. D. C. 78; Meerut 90; Rohilkhand 88]

Sol. Let the end links A and B of a uniform chain slide along a fixed rough horizontal rod. If AB is the maximum span, then A and B are in the state of limiting equilibrium. Let R be the reaction of the rod at A acting perpendicular to the rod. Then the frictional force μR will act at A along the rod in the outward direction BA as shown in the figure. The resultant F of the forces R and μR at A will make an angle λ (where $\tan\lambda=\mu$) with the direction of R. For the equilibrium of A the resultant F of R and μR at A will be equal and opposite to the tension T at A.



(Fig. 4)

Since the tension at A acts along the tangent to the chain at A, therefore the tangent to the catenary at A makes an angle $\psi_A=\frac{1}{2}\pi-\lambda$ to the horizontal.

Thus for the point A of the catenary, we have $\psi=\psi_A=\frac{1}{2}\pi-\lambda$.

$\therefore$ the length of the chain

$$=2s=2c\tan\psi_A=2c\tan\left(\tfrac{1}{2}\pi-\lambda\right)$$

$$=2c\cot\lambda=\frac{2c}{\mu}. \quad [\because \tan\lambda=\mu].$$

If (x_A, y_A) are the coordinates of the point A, then the maximum span $AB=2x_A$

$$=2c\log(\tan\psi_A+\sec\psi_A)$$
$$=2c\log\{\tan\psi_A+\sqrt{(1+\tan^2\psi_A)}\}$$
$$=2c\log\{\cot\lambda+\sqrt{(1+\cot^2\lambda)}\} \quad [\because \psi_A=\tfrac{1}{2}\pi-\lambda]$$
$$=2c\log\left\{\frac{1}{\mu}+\sqrt{\left(1+\frac{1}{\mu^2}\right)}\right\}=2c\log\left\{\frac{1+\sqrt{(1+\mu^2)}}{\mu}\right\}.$$

Hence the required ratio

$$=\frac{2x}{2s}=\frac{2c\log\left\{\dfrac{1+\sqrt{(1+\mu^2)}}{\mu}\right\}}{(2c/\mu)}$$

$$=\mu\log\left\{\frac{1+\sqrt{(1+\mu^2)}}{\mu}\right\}.$$

Ex. 13. *If the ends of a uniform inextensible string of length l hanging freely under gravity slide on a fixed rough horizontal rod*

whose coefficient of friction is μ, show that at most they can rest at a distance $\mu l\log\left\{\dfrac{1+\sqrt{(1+\mu^2)}}{\mu}\right\}$. [Luck. 75]

Sol. [Refer figure of Ex. 12 on page 13]

Proceed as in the last example 12.

At A, $\psi=\frac{1}{2}\pi-\lambda$.

Also at the point A, $s=$ arc $CA=\frac{1}{2}l$.

Using the formula $s=c\tan\psi$ for the point A, we have

$$\tfrac{1}{2}l=c\tan\left(\tfrac{1}{2}\pi-\lambda\right)=c\cot\lambda=c/\mu.$$

$$\therefore \quad c=\mu l/2.$$

$\therefore$ the required maximum span AB

$$=2x=2c\log(\tan\psi+\sec\psi)$$
$$=2c\log[\tan\psi+\sqrt{(2+\tan^2\psi)}]$$
$$=2c\log[\cot\lambda+\sqrt{(1+\cot^2\lambda)}] \quad [\because \text{for } A,\ \psi=\tfrac{1}{2}\pi-\lambda]$$
$$=2c\log\left[\frac{1}{\mu}+\sqrt{\left(1+\frac{1}{\mu^2}\right)}\right] \quad [\because \cot\lambda=1/\tan\lambda=1/\mu]$$
$$=\mu l\log\left\{\frac{1+\sqrt{(1+\mu^2)}}{\mu}\right\} \quad [\because c=\mu l/2].$$

Ex. 14. *The extremities of a heavy string of length $2l$ and weight $2lw$, are attached to two small rings which can slide on a fixed wire. Each of these rings is acted on by a horizontal force equal to lw. Show that the distance apart of the rings is*

$$2l\log(1+\sqrt{2}).$$

Sol. [Refer figure of Ex. 12 on page 13]

Here the length of the string $ACB=2l$ and its weight $=2lw$.

$\therefore$ the weight per unit length of the string $=w$.

Here it is given that the small ring at A is acted on by a horizontal force lw. For the equilibrium of the small ring at A, the horizontal force lw must balance the horizontal component of the tension at A. But the horizontal component of the tension at any point of the string is equal to wc. So we must have

$$lw=wc, \quad \text{or} \quad c=l.$$

Since arc $CA=l$, therefore for the point A, $s=l$. So using the formula $s=c\tan\psi$ for the point A, we have

$$l=c\tan\psi, \quad \text{or} \quad \tan\psi=l/c=l/l=1.$$

$$\therefore \quad \text{for } A,\ \psi=\psi_A=45°.$$

If (x_A, y_A) are the corrdinates of the point A, then the distance betwen the rings $= AB = 2x_A$

$$= 2c \log (\tan \psi_A + \sec \psi_A)$$
$$= 2l \log (\tan 45° + \sec 45°) = 2l \log (1+\sqrt{2}).$$

Ex. 15. *A uniform chain of length $2l$ is suspended by its ends which are on the same horizontal level. The distance apart $2a$ of the ends is such that the lowest point of the chain is at a distance a vertically below the ends. Prove that if c be the distance of the lowest point from the directrix of the catenary, then*

$$\frac{2a^2}{l^2-a^2} = \log\left(\frac{l+a}{l-a}\right) \text{ and } \tanh \frac{a}{c} = \frac{2al}{l^2+a^2}. \quad \text{[Gorakhpur 76]}$$

Sol. [Refer figure of Ex. 6 on page 9]

Here arc $CA = l$, $AB = 2a$, $CD = a$ and $OC = c$.

If (x_A, y_A) are the coordinates of the point A, then

$$x_A = DA = \tfrac{1}{2} AB = a \text{ and } y_A = OC + CD = c + a.$$

At A, $s = \text{arc } CA = l$

$\therefore$ Using the formula $y^2 = c^2 + s^2$ for the point A, we have

$$(c+a)^2 = c^2 + l^2, \text{ or } c^2 + 2ac + a^2 = c^2 + l^2.$$

$$\therefore \quad c = \frac{l^2-a^2}{2a} \qquad \ldots(1)$$

Also using the formula $s = c \tan \psi$ for the point A, we have

$$l = c \tan \psi.$$

$$\therefore \text{ at } A, \tan \psi = \frac{l}{c} = \frac{2al}{l^2-a^2}.$$

Now at A, we have $x = a$. So using the formula

$$x = c \log (\tan \psi + \sec \psi),$$

for the point A, we have

$$a = c \log \{\tan \psi + \sqrt{(1+\tan^2 \psi)}\}$$

or $$a = \frac{l^2-a^2}{2a} \log \left[\frac{2al}{l^2-a^2} + \sqrt{\left\{1 + \frac{4a^2l^2}{(l^2-a^2)^2}\right\}}\right]$$

or $$\frac{2a^2}{l^2-a^2} = \log \left\{\frac{2al}{l^2-a^2} + \frac{l^2+a^2}{l^2-a^2}\right\} = \log \left(\frac{2al + l^2 + a^2}{l^2-a^2}\right)$$

$$= \log \frac{(l+a)^2}{(l-a)(l+a)}$$

$$\therefore \frac{2a^2}{l^2-a^2} = \log \left(\frac{l+a}{l-a}\right).$$

Again for a catenary $s = c \sinh (x/c)$. Since at A, $s = l$ and $x = x_A = a$, therefore we have

$$l = c \sinh \left(\frac{a}{c}\right), \text{ or } \sinh \frac{a}{c} = \frac{l}{c} = \frac{2al}{l^2-a^2}.$$

$$\therefore \tanh \frac{a}{c} = \frac{\sinh \frac{a}{c}}{\cosh \frac{a}{c}} = \frac{\sinh \frac{a}{c}}{\sqrt{\left(\sinh^2 \frac{a}{c} + 1\right)}}$$

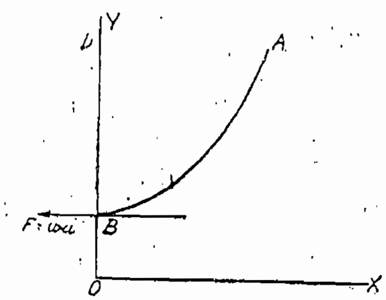
$$= \frac{\frac{2al}{l^2-a^2}}{\sqrt{\left\{\frac{4a^2l^2}{(l^2-a^2)^2} + 1\right\}}} = \frac{2al}{l^2+a^2}.$$

Ex. 16. *A heavy uniform string, of length l, is suspended from a fixed point A, and its other end B is pulled horizontally by a force equal to the weight of a length a of the string. Show that the horizontal and vertical distances between A and B are*

$$a \sinh^{-1} (l/a) \text{ and } \sqrt{(l^2+a^2)} - a,$$

respectively.

Sol. In the equilibrium position the arc AB will represent half of the arc of the complete catenary with B as its lowest point.



(Fig. 5)

The horizontal force $F = aw$ by which the end B is pulled is equal to the tension T_0 at the lowest point B.

$\therefore$ $aw = T_0 = wc$, so that $c = a$.

Let (x_A, y_A) be the coordinates of the point A.

We have arc $BA = l$ i.e., at A, $s = \text{arc } BA = l$.

$\therefore$ from $s=c\sinh(x/c)$, we have

$l=a\sinh(x_A/a)$ $[\because c=a]$

$\therefore$ $x_A=a\sinh^{-1}(l/a)=$ the horizontal distance between B and A.

Again from $y^2=s^2+c^2$, we have

$y^2{}_A=l^2+a^2$

But $y_A=OD=OB+BD=c+BD=a+BD$.

$\therefore$ $(a+BD)^2=l^2+a^2$, or $a+BD=\sqrt{(l^2+a^2)}$.

or $BD=\sqrt{(l^2+a^2)}-a=$ the vertical distance between B and A.

Hence the horizontal and vertical distances between A and B are $a\sinh^{-1}(l/a)$ and $\sqrt{(l^2+a^2)}-a$ respectively.

Ex. 17. *A box kite is flying at a height h with a length l of wire paid out, and with the vertex of the catenary on the ground. Show that at the kite the inclination of the wire to the ground is*

$$2\tan^{-1}(h/l),$$

and that its tensions there and at the ground are

$$w\frac{l^2+h^2}{2h} \text{ and } w\frac{l^2-h^2}{2h},$$

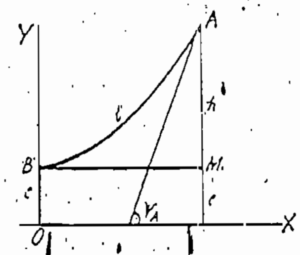
where w is the weight of the wire per unit of length.

[Luck. 80; Kanpur 87]

Sol. Let AB be the wire, B the vertex of the catenary on the ground and A the position of the kite.

The height of A, above B is h i.e., $AM=h$.

We have the ordinate of the point $A=y_A=c+h$ and at A, $s=$ arc $BA=l$.



(Fig. 6)

$\therefore$ from $y^2=c^2+s^2$,

we have $y_A{}^2=c^2+l^2$,

or $(c+h)^2=c^2+l^2$, or $c^2+ch+h^2=c^2+l^2$.

$$\therefore \quad c=\frac{l^2-h^2}{2h} \text{ and } y_A=c+h=\frac{l^2-h^2}{2h}+h=\frac{l^2+h^2}{2h}.$$

If the tangent at A is inclined at an angle ψ_A to the ground, then from $s=c\tan\psi$, we have

$$l=c\tan\psi_A$$

or $$\tan\psi_A=\frac{l}{c}=\frac{2hl}{l^2-h^2}=\frac{2\,(h/l)}{1-(h/l)^2}$$

$$\therefore \quad \psi_A=\tan^{-1}\left\{\frac{2\,(h/l)}{1-(h/l)^2}\right\}=2\tan^{-1}(h/l)$$

$$\left[\because 2\tan^{-1}x=\tan^{-1}\frac{2x}{1-x^2}\right]$$

$\because$ for a catenary $T=wy$,

$\therefore$ the tension at the point $A=T_A=wy_A=w\,\dfrac{l^2+h^2}{2h}$

and the tension at the point $B=T_B=wy_B=wc=w\,\dfrac{l^2-h^2}{2h}$

Ex. 18. *A is the lowest point of a uniform thread hanging from two fixed points, B and C. Let a, b be the heights of A and B above the directrix of the catenary formed by the thread. Show that the length of the thread between A and B equals $\sqrt{(b^2-a^2)}$.*

Sol. The lowest point A of the thread is the vertex of the catenary in which the thread hangs.

Let the length of the thread between the lowest point A and the point $B=l$.

Then for the point B, we have $s=s_B=l$.

The height of the lowest point A above the directrix OX i.e., the x-axis $=c=a$ (given)

and the height of the point B above the directrix OX

$=$ the ordinate of the point $B=y_B=b$ (given)

$\therefore$ applying the formula $y^2=c^2+s^2$ for the point B, we have

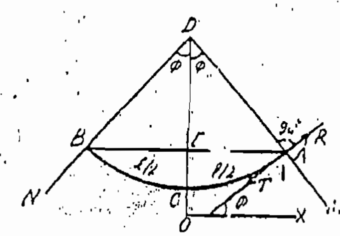
$y_B{}^2=a^2+s_B{}^2$ i.e., $b^2=a^2+l^2$.

$\therefore$ $l=\sqrt{(b^2-a^2)}$.

Ex. 19. *The end links of a uniform chain of length l can slide on two smooth rods in the same vertical plane which are inclined in opposite directions at equal angles φ to the vertical. Prove that the sag in the middle is $\frac{1}{2}l\tan\frac{1}{2}\phi$.*

[Gorakhpur 73]

Sol. The end links A and B of a uniform chain ACB slide on two smooth rods DM and DN which are inclined in opposite directions at equal angles ϕ to the vertical as shown in the figure. The point C is the vertex of the catenary in which the chain hangs.



(Fig. 7)

Let us consider the equilibrium of the link at A. Two forces are acting upon it : (i) the reaction R of the rod DM acting perpendicular to the rod, and (ii) the tension T of the chain acting tangentially to the chain at A. For the equilibrium of the link at A these two forces must be equal and opposite and their lines of action must coincide. Therefore the tangent at A to the catenary makes an angle ϕ with the horizontal or say with the directrix OX of the catenary.

Thus at A, we have $\psi=\psi_A=\phi$.

Also at A, we have $s=s_A=\text{arc } CA=\frac{1}{2}l$.

Using the formula $s=c\tan\psi$ for the point A, we have,

$s_A=c\tan\psi_A$, or $\frac{1}{2}l=c\tan\phi$, or $c=\frac{1}{2}l\cot\phi$.

Now the sag in the middle $=EC=OE-OC=y_A-c$

$$=c\sec\psi_A-c \qquad [\because\ y=c\sec\psi]$$

$$=c(\sec\phi-1)=\tfrac{1}{2}l\cot\phi(\sec\phi-1)=\tfrac{1}{2}l\,\frac{1-\cos\phi}{\sin\phi} \qquad [\because\ \psi_A=\phi]$$

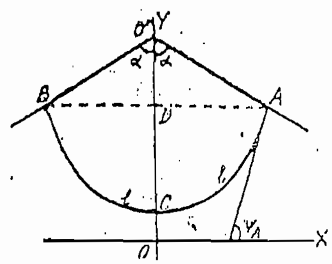
$$=\tfrac{1}{2}l\,\frac{2\sin^2\frac{1}{2}\phi}{2\sin\frac{1}{2}\phi\cos\frac{1}{2}\phi}=\tfrac{1}{2}l\tan\tfrac{1}{2}\phi.$$

Ex. 20. *A uniform heavy chain is fastened at its extremities to two rings of equal weight, which slide on smooth rods intersecting in a vertical plane, and inclined at the same angle α to the vertical, find the condition that the tension at the lowest point may be equal to half the weight of the chain, and in that case, show that the vertical distance of the rings from the point of intersection of the rods is*

$$l\cot\alpha\log(\sqrt{2}+1),$$

where $2l$ is the length of the chain.

Sol. Let the rods inclined at the same angle α to the vertical intersect at the point O'. Since the rods are inclined at the same angles, rings are of the same weight and the chain is uniform, hence in equilibrium the positions of the rings will be symmetrical with respect to the vertical line through the point O'. Let A and B be the positions of the rings in equilibrium. Clearly AB is horizontal. Let $O'D$ be perpendicular from O' on AB, OO' the axis of the catenary and OX the directrix. Let $OC=c$.



(Fig. 8)

If w is the weight per unit length of the chain, then its weight $=2wl$.

Then according to the question, if the tension at the lowest point is equal to half the weight of the chain, *i.e.*, if $T_0=wc=wl$, we have $c=l$.

For the point A of the catenary, we have $s=s_A=\text{arc } CA=l$. If the tangent at A is inclined at an angle ψ_A to the horizontal, then from $s=c\tan\psi$, we have

$s_A=c\tan\psi_A$, or $l=c\tan\psi_A$

or $l=l\tan\psi_A$ $\quad[\because\ c=l]$

or $\tan\psi_A=1$, or $\psi_A=\frac{1}{4}\pi$.

Hence the condition that the tension at the lowest point may be equal to half the weight of the chain is that the tangents at the ends A and B of the chain will make an angle $\frac{1}{4}\pi$ to the horizontal.

Now using the formula $x=c\log(\tan\psi+\sec\psi)$, for the point A, we have

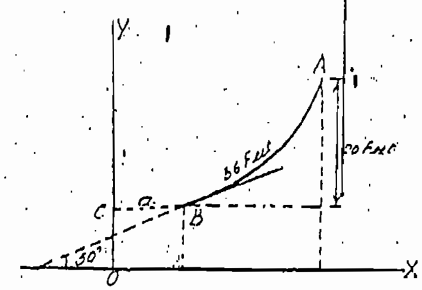
$$x_A=DA=c\log(\tan\tfrac{1}{4}\pi+\sec\tfrac{1}{4}\pi)=l\log(1+\sqrt{2}).$$

Hence the vertical distance of the rings from the point of intersection O' of the rods

$$=O'D=DA\cot\alpha=l\cot\alpha\log(1+\sqrt{2}).$$

Ex. 21. *A boat is towed by means of a rope attached to a ship and the lower end of the rope makes an angle of 30° to the horizontal. If the length of the rope is 36 feet, and the upper end is 20 feet higher than the lower end, find the resistance of the water to the motion of the boat, the weight of each foot of the rope being ten ounces.*

Sol. Let AB be the rope of length 36 feet with the lower end



(Fig. 9)

B making an angle of 30° to the horizontal. Let C be the lowest point *i.e.*, the vertex, OX the directrix and c the parameter of the catenary for which AB is an arc.

If arc $CB=a$ feet, then $a=c\tan 30°=c/\sqrt{3}$.

Let y_A and y_B be the ordinates of the ends A and B respectively.

Then $\qquad y_A-y_B=20$ ft. $\qquad$ ...(1)

But from $y=c\sec\psi$, we have

$$y_B=c\sec 30°=2c/\sqrt{3} \text{ feet.}$$

$$\therefore \quad y_A=y_B+20=\left(\frac{2c}{\sqrt{3}}+20\right) \text{ feet.}$$

Also $s_A=\text{arc } CA=\text{arc } CB+\text{arc } BA=a+36=\left(\frac{c}{\sqrt{3}}+36\right)$ feet.

Now using the formula $y^2=c^2+s^2$ for the point A, we have

$$y_A^2=c^2+\left(\frac{c}{\sqrt{3}}+36\right)^2$$

or $$\left(\frac{2c}{\sqrt{3}}+20\right)^2=c^2+\left(\frac{c}{\sqrt{3}}+36\right)^2$$

or $$\frac{4}{3}c^2+\frac{80}{\sqrt{3}}c+400=c^2+\frac{1}{3}c^2+\frac{72}{\sqrt{3}}c+1296$$

or $$\frac{8}{\sqrt{3}}c=1296-400. \quad \therefore \quad c=112\sqrt{3} \text{ feet.}$$

Now, $w=$ weight of one foot of the rope $=\frac{10}{16}=\frac{5}{8}$ lbs.

The resistance due to the water will act horizontally and therefore will be equal to

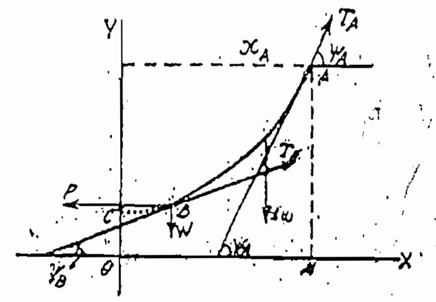
$$T_0=wc=\tfrac{5}{8}\times 112\sqrt{3}=70\times 1{\cdot}732=121{\cdot}2 \text{ lbs. wt.}$$

Ex. 22. *A weight W is suspended from a fixed point by a uniform string of length l and weight w per unit length. It is drawn aside by a horizontal force P. Show that in the position of equilibrium, the distance of W from the vertical through the fixed point is*

$$\frac{P}{w}\left\{\sinh^{-1}\left(\frac{W+lw}{P}\right)-\sinh^{-1}\left(\frac{W}{P}\right)\right\}.$$ (Luck. 78)

Sol. The end A of the string AB of length l is attached to the fixed point A and a weight W hanging at the other end B of the string is drawn aside by a horizontal force P.

Let us first consider the equilibrium of the end B of the string. There are three forces acting on it : (i) the horizontal force P applied at B, (ii) the weight W suspended at B and acting vertically downwards, and (iii) the tension T_B of the string BA acting along the tangent to the string at B which makes an angle ψ_B with OX.



(Fig. 10)

Resolving these forces horizontally and vertically, we have

$$T_B\cos\psi_B=P \qquad ...(1)$$

and $$T_B\sin\psi_B=W. \qquad ...(2)$$

Dividing (2) by (1), we have

$$\tan\psi_B=W/P. \qquad ...(3)$$

Since the horizontal component of the tension at any point of the string is constant and is equal to wc, therefore

$$T_B\cos\psi_B=wc=P, \text{ so that}$$
$$c=P/w. \qquad ...(4)$$

Now let us consider the equilibrium of the whole string AB. The forces acting on it are : (i) the horizontal force P applied at B, (ii) the weight W suspended at B, (iii) the weight lw of the string AB acting vertically downwards through the centre of gravity of the string AB, and (iv) the tension T_A at the end A acting along the tangent to the string at A which makes an angle ψ_A with OX. Resolving these forces horizontally and vertically, we have

$$T_A\cos\psi_A=P \qquad ...(5)$$

and $$T_A\sin\psi_A=W+lw. \qquad ...(6)$$

Dividing (6) by (5), we have

$$\tan\psi_A=(W+lw)/P. \qquad ...(7)$$

Now the distance of the weight W from the vertical AM through the fixed point A

$=$(the x-coordinate of A) $-$ (the x-coordinate of B)

$=x_A-x_B$.

But for a catenary, we have

$$s=c\sinh(x/c).$$

$\therefore\quad x=c\sinh^{-1}\left(\dfrac{s}{c}\right)=c\sinh^{-1}\left(\dfrac{c\tan\psi}{c}\right)$ [∵ $s=c\tan\psi$]

$\quad=c\sinh^{-1}(\tan\psi)$.

$\therefore\quad x_A-x_B=c\sinh^{-1}(\tan\psi_A)-c\sinh^{-1}(\tan\psi_B)$

$$=\frac{P}{w}\left\{\sinh^{-1}\left(\frac{W+lw}{P}\right)-\sinh^{-1}\left(\frac{W}{P}\right)\right\},$$

substituting for c from (4), for $\tan\psi_A$ from (7) and for $\tan\psi_B$ from (3).

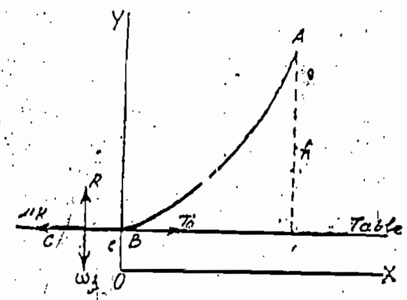
This gives the required distance of W from the vertical through the fixed point A.

Ex. 23. *A length l of a uniform chain has one end fixed at a height h above a rough table, and rests in a vertical plane so that a portion of it lies in a straight line on the table. Prove that if the chain is on the point of slipping, the length of the table is*

$$l+\mu h-\sqrt{\{(\mu^2+1)h^2+2\mu lh\}},$$

where μ is the coefficient of friction. **[Rohilkhand 87]**

Sol. Let one end of a uniform chain ABC of length l be fixed at A at a height h above the rough table and let the portion BC of the chain rest on the table.



(Fig. 11)

Let $BC=z$.

The chain rests in limiting equilibrium with the portion AB in the form of an arc of a catenary with B as its vertex i.e., the lowest point. Let us consider the limiting equilibrium of the portion BC of the chain lying on the table.

The forces acting on the portion BC of the chain are : (i) its weight wz acting vertically downwards, (ii) the normal reaction R of the table acting perpendicular to the table, (iii) the force of limiting friction μR along the table in the direction BC which is opposite to the direction CB in which the chain BC will begin to slip, and (iv) the tension T_0 in the chain at B.

For the equilibrium of the chain BC, resolving these forces horizontally and vertically, we have

$$R=wz\quad\text{and}\quad \mu R=T_0=wc.$$

$\therefore\quad \mu wz=wc,\quad$ or $\quad c=\mu z.$

Length of the arc $AB=s_A=l-z$;

and the ordinate of the point $A=y_A=c+h$.

$\therefore$ using the formula $y^2=c^2+s^2$ for the point A, we have

$$y_A^2=c^2+s_A^2$$

or $\quad (c+h)^2=c^2+(l-z)^2$

or $\quad z^2-2lz+(l^2-h^2)-2ch=0$

or $\quad z^2-2(l+\mu h)z+(l^2-h^2)=0.\quad$ [∵ $c=\mu z$]

$$\therefore\quad z=\frac{2(l+\mu h)\pm\sqrt{[4(l+\mu h)^2-4(l^2-h^2)]}}{2.1}$$

or $\quad z=(l+\mu h)\pm\sqrt{[(l+\mu h)^2-(l^2-h^2)]}.$

The plus sign will give $z>l$, which is impossible. Therefore, neglecting the plus sign, the required length z of the chain on the table is given by

$$z=(l+\mu h)-\sqrt{[(l+\mu h)^2-(l^2-h^2)]}$$
$$=(l+\mu h)-\sqrt{[(\mu^2+1)h^2+2\mu lh]}.$$

Ex. 24. *One extremity of a uniform string is attached to a fixed point and the string rests partly on a smooth inclined plane ; prove that the directrix of the catenary determined by the portion which is not in contact with the plane is the horizontal line drawn through the extremity which rests on the plane.*

If α is the inclination of the plane, β the inclination of the tangent at the fixed extremity, and l the whole length of the string, prove that the length of the portion in contact with the plane is

$$\frac{l\cos\beta}{\cos\alpha\cos(\beta-\alpha)}.$$

Sol. Let ABC be a string of length l of which the portion BC is on the plane inclined at an angle α to the horizontal. Let the length of the portion BC of the string be a. Then the length of the portion AB of the string hanging in the form of an arc of a catenary is $l-a$.

Let us consider the equilibrium of the portion BC of the string lying on the inclined plane. There are three forces acting on it: (i) its weight aw acting vertically downward through its centre of gravity, (ii) the normal reaction R of the inclined plane acting perpendicular to the plane, and (iii) the tension T_B of the string at B acting along the tangent at B to the string which is along the line CB lying in the inclined plane. Resolving these forces along the inclined plane, we have

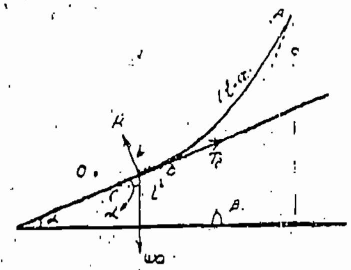


Fig. 12

$$T_B = wa \sin \alpha. \qquad \ldots(1)$$

Let BL be the perpendicular from B on the horizontal line through C. Then $BL = BC \sin \alpha = a \sin \alpha$.

$\therefore$ from (1), we have $T_B = w.BL$.

But for a catenary, from $T = wy$, we have

$$T_B = wy_B,$$

where y_B is the vertical distance of the point B from the directrix of the catenary.

Thus we have $\qquad wBL = wy_B$,

so that $\qquad y_B = BL$.

Hence the directrix of the catenary AB is the horizontal line CL through the exremity C of the portion of the string which rests on the inclined plane.

Second Part. Let O be the lowest point *i.e.*, the vertex of the catenary of which AB is a part. The inclinations to the horizontal of the tangents at B and A to the string are α and β, respectively.

Then from $s = c \tan \psi$, we have

arc $OB = c \tan \alpha$ and arc $OA = c \tan \beta$;

$\therefore$ arc $AB =$ arc $OA -$ arc $OB = c (\tan \beta - \tan \alpha)$

or $\qquad l - a = c (\tan \beta - \tan \alpha)$. $\qquad \ldots(2)$

From (1), we have

$$wa \sin \alpha = T_B$$

$$= wy_B = wc \sec \alpha \qquad [\because \text{ from } y = c \sec \psi, \text{ we have } y_B = c \sec \alpha]$$

$\therefore \qquad a = c \sec \alpha \operatorname{cosec} \alpha. \qquad \ldots(3)$

Diving (2) by (3), we have

$$\frac{l-a}{a} = \frac{c (\tan \beta - \tan \alpha)}{a \sec \alpha \operatorname{cosec} \alpha}$$

or $$\frac{l-a}{a} = \left(\frac{\sin \beta}{\cos \beta} - \frac{\sin \alpha}{\cos \alpha}\right) \sin \alpha \cos \alpha = \frac{\sin (\beta - \alpha) \sin \alpha}{\cos \beta}$$

or $\quad (l-a) \cos \beta = a \sin (\beta - \alpha) \sin \alpha$

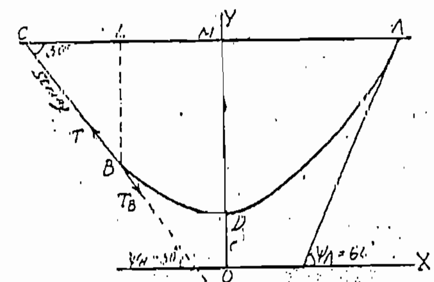
or
$$\begin{aligned} l \cos \beta &= a [\cos \beta + \sin (\beta - \alpha) \sin \alpha] \\ &= c [\cos \beta + (\sin \beta \cos \alpha - \cos \beta \sin \alpha) \sin \alpha] \\ &= a [\sin \beta \cos \alpha \sin \alpha + (1 - \sin^2 \alpha) \cos \beta] \\ &= a [\sin \beta \cos \alpha \sin \alpha + \cos^2 \alpha \cos \beta] \\ &= a \cos \alpha [\sin \alpha \sin \beta + \cos \alpha \cos \beta] \\ &= a \cos \alpha \cos (\beta - \alpha). \end{aligned}$$

$$\therefore \qquad a = \frac{l \cos \beta}{\cos \alpha \cos (\beta - \alpha)}.$$

Ex. 25. *A heavy uniform chain AB hangs freely under gravity, with the end A fixed and the other end B attached by a light string BC to a fixed point C at the same level as A. The lengths of the string and chain are such that the ends of the chain at A and B make angles 60° and 30° respectively with the horizontal. Prove that the ratio of these lengths is* $(\sqrt{3} - 1) : 1$.

[Gorakhpur '82; Kanpur '81; Agra 80, 86]

Sol. Let the lengths of the heavy uniform chain AB and the light string BC be l and a respectively.



(Fig. 13)

The chain AB being heavy will hang in the form of an arc of a catenary while the string BC being light will hang in the form of a straight line. Since the tension T_B of the chain at B will be balanced by the tension T in the string, therefore the string BC will be along the tangent at the point B of the chain. Let D be the lowest point *i.e.*, the vertex of the catenary and OX the directrix such that $OD=c$. If the tangents at A and B are inclined at angles ψ_A and ψ_B to the horizontal, then given that

$$\psi_A=60^\circ \quad \text{and} \quad \psi_B=30^\circ.$$

Let y_A and y_B be the ordinates of the points A and B respectively. Then from $y=c \sec\psi$, we have

$$y_B=c\sec\psi_B=c\sec 30^\circ=2c/\sqrt{3}$$

and

$$y_A=c\sec\psi_A=c\sec 60^\circ=2c.$$

Let BL be the perpendicular from B on AC.

Then $\quad BL=BC\sin 30^\circ=\frac{1}{2}a. \qquad [\because \ BC=a]$

$$\therefore \ a=2BL=2(y_A-y_B)=2\left(2c-\frac{2c}{\sqrt{3}}\right)=\frac{4c}{\sqrt{3}}(\sqrt{3}-1).$$

If the length of the arc DA be s_1 and that of the arc DB be s_2, then from $s=c\tan\psi$, we have

$$s_1=c\tan 60^\circ=c\sqrt{3}, \text{ and } s_2=c\tan 30^\circ=c/\sqrt{3}.$$

$\therefore \quad l=$ the length of the chain ADB

$$=s_1+s_2=c\sqrt{3}+c/\sqrt{3}=4c/\sqrt{3}.$$

Hence the ratio of the lengths of the string and the chain

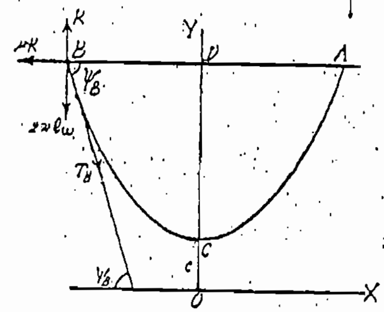
$$=\frac{a}{l}=\frac{(4c/\sqrt{3})(\sqrt{3}-1)}{(4c/\sqrt{3})}=\frac{\sqrt{3}-1}{1}=(\sqrt{3}-1):1.$$

Ex. 26. *A heavy chain, of length $2l$, has one end tied at A and the other is attached to a small heavy ring which can slide on a rough horizontal rod which passes through A. If the weight of the ring be n times the weight of the chain, show that its greatest possible distance from A is $\frac{2l}{\lambda}\log\{\lambda+\sqrt{(1+\lambda^2)}\}$, where $1/\lambda=\mu(2n+1)$ and μ is the coefficient of friction.*

[Agra 85; Kanpur 78, 81]

Sol. Let one end of a heavy chain of length $2l$ be fixed at A and the other end be attached to a small heavy ring which can slide on a rough horizontal rod ADB through A. Let B be the position of limiting equilibrium of the ring when it is at greatest possible distance from A.

In this position of limiting equilibrium the forces acting on the ring are : (i) the weight $2nlw$ of the ring acting vertically downwards, (ii) the normal reaction R of the rod, (iii) the force of limiting friction μR of the rod acting in the direction AB, and (iv) the tension T_B in the string at B acting along the tangent to the string at B.



(Fig. 14)

For the equilibrium of the ring at B, resolving the forces acting on it horizontally and vertically, we have

$$\mu R=T_B\cos\psi_B \qquad \ldots(1)$$

and

$$R=2nlw+T_B\sin\psi_B, \qquad \ldots(2)$$

where ψ_B is the angle of inclination of the tangent at B to the horizontal.

Let C be the lowest point of the catenary formed by the chain, OX be the directrix and $OC=c$ be the parameter. We have arc $CB=s_B=l$. By the formula $T\cos\psi=wc$, we have $T_B\cos\psi_B=wc$. Also by the formula $T\sin\psi=ws$, we have

$$T_B\sin\psi_B=ws_B=wl.$$

Putting these values in (1) and (2), we have

$$\mu R=wc \text{ and } R=2nlw+wl=(2n+1)wl.$$

$$\therefore \quad \mu(2n+1)wl=wc \quad \text{or} \quad \mu(2n+1)l=c.$$

But it is given that $\mu(2n+1)=1/\lambda$.

$$\therefore \quad l/\lambda=c. \qquad \ldots(3)$$

Using the formula $s=c\tan\psi$ for the point B, we have

$$l=c\tan\psi_B;$$

$$\therefore \quad \tan\psi_B=l/c=\lambda. \qquad \ldots(4)$$

Now the required greatest possible distance of the ring from A

$=AB=2DB=2x_B$

$=2c\log(\sec\psi_B+\tan\psi_B) \qquad [\because \ x=c\log(\sec\psi+\tan\psi)]$

$=2c\log[\tan\psi_B+\sqrt{(1+\tan^2\psi_B)}]$

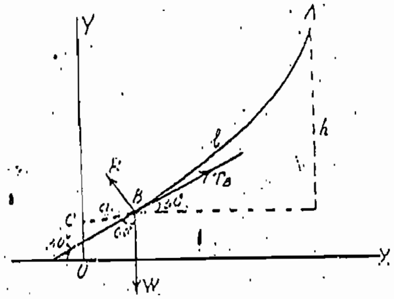
$= \frac{2l}{\lambda} \log [\lambda + \sqrt{(1+\lambda^2)}]$

[$\because$ from (3), $c = l/\lambda$ and from (4), $\tan \psi_B = \lambda$]

Ex. 27. *A uniform inextensible string of length l and weight wl, carries at one end B, a particle of weight W which is placed on a smooth plane inclined at $30°$ to the horizontal. The other end of the string is attached to a point A, situated at a height h above the horizontal through B and in the vertical plane through the line of greatest slope through B. Prove that the particle will rest in equilibrium with the tangent at B to the catenary lying in the inclined plane if*

$$\frac{W}{w} = \frac{(l-h)(l+h)}{(h-\frac{1}{2}l)}.$$

Sol. Let AB be the string of weight w and length l, carrying a particle of weight W at the end B which is placed on the plane inclined at an angle $30°$ to the horizontal. The other end of the string is attached to the fixed point A, at a height h above the



(Fig. 15)

horizontal through B. Let the particle rest in equilibrium with the tangent at B to the catenary lying in the inclined plane. And let C be the lowest point and OX the directrix of the catenary of which AB is a part and let the arc $CB = a$.

Let T_B be the tension at B and y_B be the ordinate of the point B.

The tension T_B at B is inclined at an angle $\psi_B = 30°$ to the horizontal.

The particle of weight W at P is in equilibrium under the action of three forces:

(i) the weight W acting vertically downwards,

(ii) the normal reaction R of the inclined plane and

(iii) the tension T_B of the string at B acting along the tangent at B.

Resolving these forces along the inclined plane, we have

T_B = the component of weight W along the inclined plane $= W \cos 60°$.

$$\therefore \quad T_B = \tfrac{1}{2} W.$$

Now $\quad y_B = c \sec \psi_B = c \sec 30° = 2c/\sqrt{3}$

and $\quad T_B = w y_B = 2wc/\sqrt{3}$.

$$\therefore \quad \frac{W}{2} = \frac{2wc}{\sqrt{3}}, \quad \text{or} \quad c = \frac{\sqrt{3}W}{4w}.$$

Also from $s = c \tan \psi$, at B, we have $a = c \tan \psi_B$

or $\quad a = c \tan 30° = \frac{\sqrt{3}W}{4w} \cdot \frac{1}{\sqrt{3}} = \frac{W}{4w}$.

Now at B, $s = a$, $y = y_B$

and at A, $s = a + l$, $y = y_B + h$.

$\therefore$ from $y^2 = c^2 + s^2$, we have

$y_B^2 = c^2 + a^2$ and $(y_B + h)^2 = c^2 + (a + l)^2$.

Subtracting, we have

$$h^2 + 2hy_B = l^2 + 2al$$

or $\quad h^2 + 2h \cdot \frac{2c}{\sqrt{3}} = l^2 + 2 \cdot \frac{W}{4w} l \quad [\because y_B = 2c/\sqrt{3}, \text{ and } a = W/4w]$

or $\quad h^2 + \frac{4h}{\sqrt{3}} \cdot \frac{\sqrt{3}W}{4w} = l^2 + \frac{Wl}{2w} \quad \left[\because c = \frac{\sqrt{3}W}{4w}\right]$

or $\quad \frac{W}{w} \left(h - \frac{l}{2}\right) = l^2 - h^2$

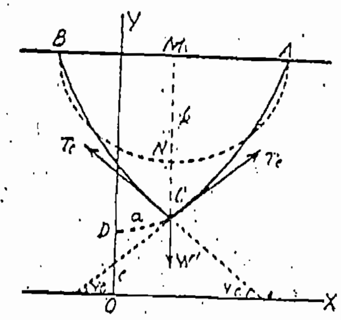
or $\quad \frac{W}{w} = \frac{(l^2 - h^2)}{(h - \frac{1}{2}l)} = \frac{(l-h)(l+h)}{(h-\frac{1}{2}l)}$.

Ex. 28. *A uniform chain, of length l and weight W, hangs between two fixed points at the same level, and a weight W' is attached at the middle point. If k be the sag in the middle, prove that the pull on either point of support is*

$$\frac{k}{2l} W + \frac{l}{4k} W' + \frac{l}{8k} W.$$

[Raj. T.D.C. 79 (S); Gorakhpur 81; Rohilkhand 80; Allhd. 73]

Sol. Let a string of length l and weight W suspended from two points A and B at the same level hang freely under gravity in the form of the catenary ANB.



(Fig. 16)

When a weight W' is attached at the middle point N of the string then it will descend downwards to C, and the two portions AC and BC of the string each of length $\frac{1}{2}l$ will be the parts of two equal catenaries. Let D be the lowest point, *i.e.*, the vertex and OX the directrix of the catenary of which AC is an arc.

The weight per unit length of the chain $= w = W/l$.

If T_C, T_C are the tensions at the point C in the string CA and CB acting along the tangents at C, then resolving vertically the forces acting at C, we have

$$2T_C \sin \psi_C = W', \qquad \ldots(1)$$

where the tangents at C to the arcs AC and BC are inclined at an angle ψ_C to the horizontal.

But from $T \sin \psi = ws$, we have

$$T_C \sin \psi_C = wa, \text{ where arc } DC = a.$$

$\therefore$ from (1), we have

$$2wa = W', \text{ or } a = \frac{W'}{2w} = \frac{W'l}{2W}. \qquad \ldots(2)$$

Let y_A and y_C be the ordinates of the points A and C respectively.

Now at A, $s = s_A = \text{arc } DA = \text{arc } DC + \text{arc } CA = a + \frac{1}{2}l$

and $y = y_A$;

at C, $s = s_C = \text{arc } DC = a$ and $y = y_C$.

Since sag in the middle $= CM = k$,

$\therefore$ $y_C + k = y_A$ or $y_C = y_A - k$.

$\therefore$ from $y^2 = c^2 + s^2$, we have

$$y_A^2 = c^2 + (a + \tfrac{1}{2}l)^2 \text{ and } y_C^2 = c^2 + a^2.$$

Subtracting, we have

$$y_A^2 - y_C^2 = al + \tfrac{1}{4}l^2$$

or $\quad y_A^2 - (y_A - k)^2 = al + \frac{1}{4}l^2 \qquad [\because \ y_C = y_A - k]$

or $\quad 2ky_A - k^2 = al + \frac{1}{4}l^2 = \frac{W'l^2}{2W} + \frac{l^2}{4}$

$[\because \ a = W'l/2W \text{ from (2)}]$

or $\quad y_A = \frac{k}{2} + \frac{l^2 W'}{4kW} + \frac{l^2}{8k}$.

Hence the pull (*i.e.*, the tension) at either point of support A or B

$$= T_A = wy_A = \frac{W}{l}\left(\frac{k}{2} + \frac{l^2 W'}{4kW} + \frac{l^2}{8k}\right) \qquad \left[\because \ w = \frac{W}{l}\right]$$

$$= \frac{k}{2l} W + \frac{l}{4k} W' + \frac{l}{8k} W.$$

Ex. 29 (a). *A uniform chain of length l and weight W hangs between two fixed points at the same level and a weight is suspended from its middle point so that the total sag in the middle is h. Show that if P is the pull on either point of support, the weight suspended is*

$$\frac{4h}{l} P - \left(\frac{1}{2} + \frac{2h^2}{l^2}\right) W. \qquad \text{[Rohilkhand 85]}$$

Sol. Proceed exactly as in the last Ex. 28.

If W' is the weight suspended at the middle point of the chain, then the pull P at either point of support is given by

$$P = \frac{h}{2l} W + \frac{l}{4h} W' + \frac{l}{8h} W.$$

$$\therefore \frac{l}{4h} W' = P - \left(\frac{h}{2l} + \frac{l}{8h}\right) W.$$

or $\quad W' = \frac{4h}{l} P - \left(\frac{2h^2}{l^2} + \frac{1}{2}\right) W.$

Ex. 29. (b). *A uniform chain of length $2l$ and weight W, is suspended from two points A and B, in the same horizontal line. A load P is now suspended from the middle point D of the chain and the depth of this point below AB is found to be h. Show that each terminal tension is* $\frac{1}{2}\left[P \frac{l}{h} + W \frac{h^2 + l^2}{2hl}\right]$.

[I.A.S. 79; Allad. 76; Kanpur 83]

Sol. Proceed as in Ex. 28.

Ex. 30. *A uniform string of weight W is suspended from two points at the same level and a weight W' is attached to its lowest point. If α and β are now the inclinations to the horizontal of the tangents at the highest and lowest points, prove that*

$$\frac{\tan \alpha}{\tan \beta} = 1 + \frac{W}{W'}.$$

[Raj. T.D.C. 80; Luck. 81; Kanpur 86; Agra 88]

Sol. [Refer figure of Ex. 28, on page 32. In this question the inclinations of the tangents at A and C to the horizontal are α and β respectively].

If l is the length of the string and w the weight per unit length, then $w = W/l$.

Let T_C, T_C be the tensions at the point C in the strings CA and CB acting along the tangents at C to the arcs CA and CB.

Resolving vertically the forces acting at C, we have

$$2T_C \sin \beta = W'.$$

[$\because$ the tangent at C is inclined at an angle β to the horizontal]

Since the horizontal component of the tension at each point of a catenary is equal to wc, where c is the parameter of the catenary, therefore

$$T_C \cos \beta = wc. \qquad \ldots(2)$$

Dividing (1) by (2), we have

$$2 \tan \beta = \frac{W'}{wc}. \qquad \ldots(3)$$

Let the tension at each point of support A and B be T acting along the tangents at these points which are inclined at an angle α to the horizontal.

Considering the equilibrium of the whole string and resolving the forces vertically, we have

$2T \sin \alpha$ = the total weight of the system acting vertically downwards,

or $$2T \sin \alpha = W + W'. \qquad \ldots(4)$$

Also the horizontal component of tension T at A,

$$T \cos \alpha = wc. \qquad \ldots(5)$$

Dividing (4) by (5) we have

$$2 \tan \alpha = \frac{W + W'}{wc}. \qquad \ldots(6)$$

Now dividing (6) by (3), we have

$$\frac{\tan \alpha}{\tan \beta} = \frac{W + W'}{W'}$$

or $$\frac{\tan \alpha}{\tan \beta} = 1 + \frac{W}{W'}.$$

Ex. 31. *A uniform chain, of length $2l$ and weight $2w$, is suspended from two points in the same horizontal line. A load w is now suspended from the middle point of the chain and the depth of this point below the horizontal line is h. Show that the terminal tension is*

$$\frac{1}{2} w \cdot \frac{h^2 + 2l^2}{hl}.$$

[Rohilkhand 83, 86, 89; Luck. 79, 86, 89]

Sol. [Refer figure of Ex. 28, page 32].

Let a weight w be suspended at the middle point C of the spring suspended from two points A and B in the same horizontal line.

Let $MC = h$ (Depth of the middle point C of the chain below the horizontal line AB).

If w' is the weight per unit length of the chain, then

$$w' = \frac{2w}{2l} = \frac{w}{l}.$$

Let T_C, T_C be the tensions at the point C in the strings CA and CB acting along the tangents at C.

Let the tangents at the point C be inclined at an angle ψ_C to the horizontal.

Resolving vertically the forces acting at C, we have

$$2T_C \sin \psi_C = w. \qquad \ldots(1)$$

Let D be the vertex of the catenary of which AC is an arc. If arc $DC = a$, then from $T \sin \psi = w's$, we have

$$T_C \sin \psi_C = w'a. \qquad \ldots(2)$$

$\therefore$ from (1) and (2), we have

$$2w'a = w, \quad \text{or} \quad a = \frac{w}{2w'} = \frac{l}{2}. \qquad \left[\because \ w' = \frac{w}{l}\right]$$

Let y_A and y_C be the ordinates of the points A and C respectively.

Then at C, $y = y_C$, s = arc $DC = a = l/2$

and at A, $y = y_A$, s = arc DA = arc DC + arc $CA = \frac{1}{2}l + l = \frac{3}{2}l$.

$\therefore$ from $y^2 = s^2 + c^2$, we have

$$y_C^2 = \tfrac{1}{4}l^2 + c^2 \quad \text{and} \quad y_A^2 = \tfrac{9}{4}l^2 + c^2.$$

Subtracting, we have

$$y_A^2 - y_C^2 = 2l^2. \qquad \ldots(3)$$

But $$y_C = y_A - CM = y_A - h.$$

$\therefore$ from (3), we have

$$y_A^2 - (y_A - h)^2 = 2l^2$$

or $$2hy_A = 2l^2 + h^2$$

or $$y_A = \frac{h^2 + 2l^2}{2h}.$$

The terminal tension T_A at A

$$=w'y_A=\frac{w}{l}\cdot\frac{h^2+2l^2}{2h}=\tfrac{1}{2}w\frac{(h^2+2l^2)}{hl}.$$

Ex. 32. *A heavy string of uniform density and thickness is suspended from two given points in the same horizontal plane. A weight, n times that of the string, is attached to its lowest point; show that if θ, φ be the inclinations to the vertical of the tangents at the highest and lowest points of the string*

$$\tan\phi=(1+n)\tan\theta.$$ [Gorakhpur 77]

Sol. [Refer figure of Ex. 28, Page 32].

Let W be the weight and $2l$ the length of the string suspended from two points A and B in the same horizontal line. Then the weight attached at middle point C of the string is W/n. The weight per unit length w of the string is given by

$$w=W/2l.$$

If ψ_C and ψ_A are the angles of inclination of the tangents at C and A to the horizontal respectively, then

$$\psi_C=\tfrac{1}{2}\pi-\phi,\ \psi_A=\tfrac{1}{2}\pi-\theta.$$

For the equilibrium of the point C, resolving the forces acting on it vertically, we have

$$2T_C\sin\psi_C=W/n, \quad \ldots(1)$$

where T_C is the tension at C in each of the strings AC and BC.

Also we have $T_A\cos\psi_C=wc$ where c is the parameter of the catenary of which AC is an arc.

$$\therefore\ T_C\cos\psi_C=\frac{W}{2l}c. \quad \ldots(2)$$
$[\because\ w=W/2l]$

Dividing (1) by (2), we have

$$2\tan\psi_C=\frac{2l}{nc},\ \text{or}\ \tan(\tfrac{1}{2}\pi-\phi)=\frac{l}{nc},\ \text{or}\ \cot\phi=\frac{l}{nc}.$$

$$\therefore\ c=(l/n)\tan\phi.$$

Now from $s=c\tan\psi$, we have

$$\text{arc } DC=c\tan\psi_C \text{ and arc } DA=c\tan\psi_A,$$

where D is the vertex of the catenary of which AC is an arc.

Subtracting, we have

$$\text{arc } DA-\text{arc } DC=c(\tan\psi_A-\tan\psi_C)$$

or $$\text{arc } CA=c[\tan(\tfrac{1}{2}\pi-\theta)-\tan(\tfrac{1}{2}\pi-\phi)]$$

or $$l=(l/n)\tan\phi.[\cot\theta-\cot\phi]$$

or $$n=\tan\phi\cot\theta-1,$$

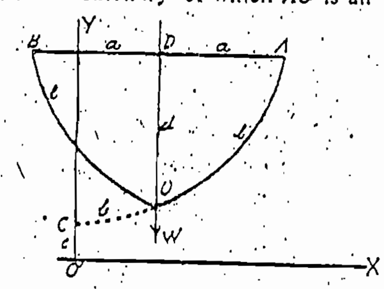
or $$1+n=\tan\phi\cot\theta$$

or $$\tan\phi=(1+n)\tan\theta.$$

Ex. 33 (a). *A and E are two points in the same horizontal line distant 2a apart; AO, OB are two equal heavy strings tied together at O and carrying a weight at O. If l is the length of each string, d the depth of O below AB, show that the parameter c of the catenary in which either string hangs is given by*

$$l^2-d^2=2c^2[\cosh(a/c)-1].$$

Sol. Let C be the vertex of the catenary of which AO is an arc and c be its parameter. Let $O'X$ be the directrix and $O'Y$ the axis of this catenary.



(Fig. 17)

Referred to $O'X$ and $O'Y$ as the coordinate axes, let (x_1, y_1) and (x_2, y_2) be the coordinates of the points O and A respectively and let arc $CO=b$. Then

$$\text{arc } CA=\text{arc } CO+\text{arc } OA$$
$$=b+l.$$

Given that

$OD=d$ and $AB=2a$,

so that $AD=a$.

We hav $y_2=y_1+d$ and $x_2=x_1+DA=x_1+a$.

$$b=c\sinh\frac{x_1}{c},$$ [for the point O]

and $$b+l=c\sinh\frac{x_2}{c}$$ [for the point A]

Subtracting, we have

$$l=c\left(\sinh\frac{x_1+a}{c}-\sinh\frac{x_1}{c}\right).$$ $[\because\ x_2=x_1+a]$

Also from $y=c\cosh\frac{x}{c}$, we have

$$y_1=c\cosh\frac{x_1}{c} \text{ and } y_2=c\cosh\frac{x_2}{c}.$$

Subtracting, we have

$$y_2-y_1=c\left(\cosh\frac{x_2}{c}-\cosh\frac{x_1}{c}\right),$$

or $$d=c\left(\cosh\frac{x_1+a}{c}-\cosh\frac{x_1}{c}\right).$$

$$\therefore\ l^2-a^2=c^2\left[\left(\sinh\frac{x_1+a}{c}-\sinh\frac{x_1}{c}\right)^2-\left(\cosh\frac{x_1+a}{c}-\cosh\frac{x_1}{c}\right)^2\right]$$

$$=c^2\left[-\left(\cosh^2\frac{x_1+a}{c}-\sinh^2\frac{x_1+a}{c}\right)-\left(\cosh^2\frac{x_1}{c}-\sinh^2\frac{x_1}{c}\right)\right.$$

$$\left.+2\left\{\cosh\frac{x_1+a}{c}\cdot\cosh\frac{x_1}{c}-\sinh\frac{x_1+a}{c}\cdot\sinh\frac{x_1}{c}\right\}\right]$$

$$=c^2\left[-1-1+2\cosh\left(\frac{x_1+a}{c}-\frac{x_1}{c}\right)\right]$$

$$=2c^2\,[\cosh\,(a/c)-1].$$

Ex. 33 (b). *A uniform chain of length l hangs between two points A and B which are at a horizontal distance a from one another, with B at a vertical distance b above A. Prove that the parameter of the catenary is given by*

$$2c\sinh\,(a/2c)=\sqrt{(l^2-b^2)}.$$

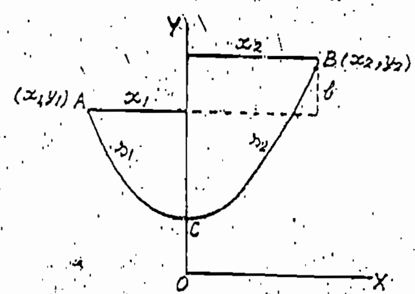
Prove also that, if the tensions at A and B are T_1 and T_2 respectively,

$$T_1+T_2=W\sqrt{\left(1+\frac{4c^2}{l^2-b^2}\right)}$$

and $$T_2-T_1=Wb/l,$$

where W is the weight of the chain. [Rohilkhand 82]

Sol. A uniform chain of length l and weight W hangs between two points A and B. Let C be the vertex, OX the directrix, OY the axis and c the parameter of the catenary in which the chain



(Fig. 18)

hangs. Let (x_1, y_1) and (x_2, y_2) be the coordinates of the points A and B respectively and let arc $CA=s_1$ and arc $CB=s_2$.

We have $$s_1+s_2=l.$$

Since the horizontal distance between A and B is a, therefore $$x_1+x_2=a.$$

Again since the vertical distance of B above A is b, therefore $$y_2-y_1=b.$$

Let w be the weight per unit length of the chain. Then $$W=lw,\quad\text{or}\quad w=W/l.$$

By the formula $s=c\sinh\,(x/c)$, we have

$$s_1=c\sinh\,(x_1/c)\ \text{and}\ s_2=c\sinh\,(x_2/c)]$$

$$\therefore\ l=s_1+s_2=c\,[\sinh x_1/c)+\sinh\,(x_2/c)]. \quad ...(1)$$

Again by the formula $y=c\cosh\,(x/c)$, we have

$$y_1=c\cosh\,(x_1/c)\ \text{and}\ y_2=c\cosh\,(x_2/c).$$

$$\therefore\ b=y_2-y_1=c\,[\cosh\,(x_2/c)-\cosh\,(x_1/c)] \quad ...(2)$$

Squaring and subtracting (1) and (2), we have

$$l^2-b^2=c^2\,[-\{\cosh^2\,(x_1/c)-\sinh^2\,(x_1/c)\}-\{\cosh^2\,(x_2/c)-\sinh^2\,(x_2/c)\}$$

$$+2\,\{\cosh\,(x_1/c)\cosh\,(x_2/c)+\sinh\,(x_1/c)\sinh\,(x_2/c)\}]$$

$$=c^2\,[-1-1+2\cosh\,(x_1/c+x_2/c)]$$

$$=c^2\,[-2+2\cosh\,((x_1+x_2)/c)]$$

$$=2c^2\left\{\cosh\frac{a}{c}-1\right\}=c^2\left\{1+2\sinh^2\frac{a}{2c}-1\right\}$$

$$=4c^2\sinh^2\frac{a}{2c} \quad ...(3)$$

$\therefore$ c is given by

$$2c\sinh\,(a/2c)=\sqrt{(l^2-b^2)}.$$

[Remember that

$$\cosh\,(\alpha+\beta)=\cosh\alpha\cosh\beta+\sinh\alpha\sinh\beta$$

and $$\cosh 2\alpha=1+2\sinh^2\alpha.]$$

Now let T_1 and T_2 be the tensions at the points A and B respectively. Then by the formula $T=wy$, we have

$$T_1=wy_1,\ T_2=wy_2.$$

$$\therefore\ T_2-T_1=w\,(y_2-y_1)=wb=(W/l)\,b=Wb/l.$$

Also $$T_1+T_2=w\,(y_1+y_2)=\frac{W}{l}(y_1+y_2)=W\frac{y_1+y_2}{s_1+s_2}$$

$$=W\frac{c\cosh\,(x_1/c)+c\cosh\,(x_2/c)}{c\sinh\,(x_1/c)+c\sinh\,(x_2/c)}.$$

$$= W \frac{\cosh (x_1/c) + \cosh (x_2/c)}{\sinh (x_1/c) + \sinh (x_2/c)}$$

$$= W \frac{2 \cosh \frac{1}{2} (x_1/c + x_2/c) \cosh \frac{1}{2} (x_1/c - x_2/c)}{2 \sinh \frac{1}{2} (x_1/c + x_2/c) \cosh \frac{1}{2} (x_1/c - x_2/c)}$$

$$= W \coth \left(\frac{x_1 + x_2}{2c}\right) = W \coth \frac{a}{2c}$$

$$= W \sqrt{\left[1 + \operatorname{cosech}^2 \frac{a}{2c}\right]} \qquad [\because \coth^2 \alpha = 1 + \operatorname{cosech}^2 \alpha]$$

$$= W \sqrt{\left[1 + \frac{4c^2}{l^2 - b^2}\right]}$$

substituting for $\operatorname{cosech}^2 (a/2c)$ from (3).

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Ex. 34. *Show that the length of an endless chain which will hang over a circular pulley of radius a so as to be in contact with two-thirds of the circumference of the pulley is*

$$a \left\{\frac{3}{\log (2 + \sqrt{3})} + \frac{4\pi}{3}\right\}.$$

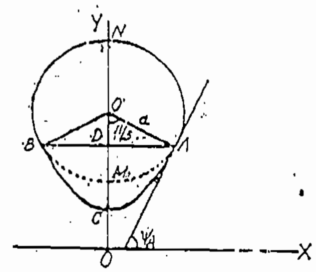
[**Gorakhpur 80; Agra 84; Luck. 79; Kanpur 87, 88; Meerut 90 P**]

Sol. Let $ANBMA$ be the circular pulley of radius a and $ANBCA$ the endless chain hanging over it.

Since the chain is in contact with the two-thirds of the circumference of the pulley, hence the length of this portion ANB of the chain

$= \frac{2}{3}$ (circumference of the pulley)

$= \frac{2}{3} (2\pi a) = \frac{4}{3} \pi a.$



(Fig. 20)

Let the remaining portion of the chain hang in the form of the catenary ACB, with AB horizontal. C is the lowest point *i.e.*, the vertex, $CO'N$ the axis and OX the directrix of this catenary.

Let $OC = c$ = the parameter of the catenary.

The tangent at A will be perpendicular to the radius $O'A$

$\therefore$ If the tangent at A is inclined at an angle ψ_A to the horizontal, then

$$\psi_A = \angle AO'D = \tfrac{1}{2} (\angle AO'B) = \tfrac{1}{2} (\tfrac{1}{3} . 2\pi) = \tfrac{1}{3}\pi.$$

From the triangle $AO'D$, we have

$$DA = O'A \sin \tfrac{1}{3}\pi = a\sqrt{3}/2.$$

$\therefore$ from $x = c \log (\tan \psi + \sec \psi)$, for the point A, we have

$$x = DA = c \log (\tan \psi_A + \sec \psi_A)$$

or $$\frac{a\sqrt{3}}{2} = c \log \left(\tan \frac{\pi}{3} + \sec \frac{\pi}{3}\right) = c \log (\sqrt{3} + 2).$$

$$\therefore c = \frac{a\sqrt{3}}{2 \log (2 + \sqrt{3})}.$$

From $s = c \tan \psi$ applied for the point A, we have

$$\text{arc } CA = c \tan \psi_A = c \tan \tfrac{1}{3}\pi = c\sqrt{3} = \frac{3a}{2 \log (2 + \sqrt{3})}.$$

Hence the total length of the chain

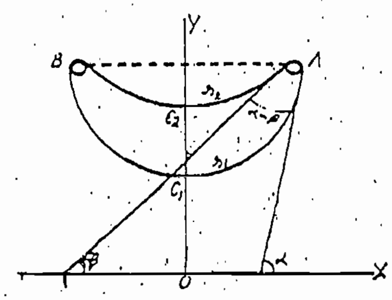
= arc ABC + length of the chain in contact with the pulley

$$= 2 . (\text{arc } CA) + \frac{4}{3} \pi a$$

$$= 2 \cdot \frac{3a}{2 \log (2 + \sqrt{3})} + \frac{4}{3} \pi a = a\left\{\frac{3}{\log (2 + \sqrt{3})} + \frac{4\pi}{3}\right\}.$$

Ex. 35. *An endless uniform chain is hung over two smooth pegs in the same horizontal line. Show that, when it is in a position of equilibrium, the ratio of the distance between the vertices of the two catenaries to half the length of the chain is the tangent of half the angle of inclination of the portions near the pegs.*

Sol. Let an endless uniform chain hang over two smooth pegs A and B in the same horizontal line. The two portions of



(Fig. 19)

the chain will hang in the form of two catenaries AC_1B and AC_2B with C_1, C_2 as their lowest point (vertices). Let c_1 and c_2 be the parameters of the two catenaries.

Also let $2s_1$ and $2s_2$ be the lengths of the portions AC_1B and AC_2B of the chain.

Let α and β be the inclinations to the horizontal of the tangents of the two catenaries AC_1B and AC_2B at A. Since the tension of a chain does not change while passing over a smooth peg, therefore the tension at the point A in each of the strings AC_1B and AC_2B is the same. Let y_1 and y_2 be the heights of the point A of the two catenaries AC_1B and AC_2B above their corresponding directrices. Then using the formula $T=wy$ for each of these catenaries for the point A, we have

$$wy_1=wy_2, \text{ so that } y_1=y_2.$$

Therefore the two catenaries AC_1B and AC_2B have the same directrix. Let it be OX. The common axis of the two catenaries is the line OY passing through their vertices C_1 and C_2.

Now using the formula $y=c \sec\psi$ for the two catenaries for the point A, we have $c_1 \sec\alpha=c_2 \sec\beta$, so that

$$c_2=\frac{c_1 \sec\alpha}{\sec\beta}=\frac{c_1 \cos\beta}{\cos\alpha}.$$

The distance between the vertices C_1 and C_2 of the two catenaries $=OC_2-OC_1$

$$=c_2-c_1=\frac{c_1 \cos\beta}{\cos\alpha}-c_1=c_1\left(\frac{\cos\beta-\cos\alpha}{\cos\alpha}\right).$$

Again using the formula $s=c\tan\psi$ for the two catenaries for the point A, we have

$$s_1=c_1 \tan\alpha \text{ and } s_2=c_2 \tan\beta.$$

$\therefore$ half the length of the chain $=s_1+s_2$

$$=c_1 \tan\alpha+c_2 \tan\beta=c_1 \tan\alpha+\frac{c_1 \cos\beta}{\cos\alpha}\cdot\tan\beta=c_1\left(\frac{\sin\alpha+\sin\beta}{\cos\alpha}\right).$$

Hence the required ratio

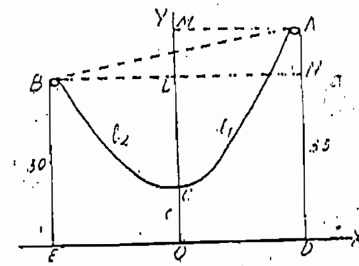
$$=\frac{c_2-c_1}{s_1+s_2}=\frac{\cos\beta-\cos\alpha}{\sin\alpha+\sin\beta}=\frac{2\sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)}{2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)}$$

$$=\tan\tfrac{1}{2}(\alpha-\beta).$$

Ex. 36. (a) *A heavy uniform string, 90 inches long, hangs over two smooth pegs at different heights. The parts which hang are of lengths 30 and 33 inches. Prove that the vertex*

nary divides the whole string in the ratio 4 : 5, and find the distance between the pegs. [Raj. T.D.C. 78; Gorakhpur 74]

Sol. Let the heavy string $DACBE$ of length 90 inches hang



(Fig. 21)

over two smooth pegs A and B at different heights. Let the portion ACB hang in the form of a catenary with C as its lowest point i.e. vertex. Let OX be the directrix and c the parameter of this catenary. The portions AD and BE of the string hang vertically and are of lengths 33 and 30 inches respectively. Now the tension of a string remains unaltered while passing over a smooth peg. Therefore the tension at the point A in the strings AD and AC is the same. But the tension at the point A due to the strings AD is $w.AD$, where w is the weight per unit length of the string. Also by the formula $T=wy$, the tension at the point A in the string AC is wy_A, where y_A is the height of the point A above the directrix of the catenary ACB. So we have $w.AD=w.y_A$, so that $y_A=AD$. Therefore the free end D of the string lies on the directrix OX of the catenary ACB. Similarly the free end E of the string also lies on the directrix of the catenary ACB. Thus for the catenary ACB, we have

$$y_A=AD=33 \text{ inches, and } y_B=BE=30 \text{ inches.}$$

Now let the lengths of the strings CA and CB be l_1 and l_2 inches respectively.

Then $AD+BE+l_1+l_2=90$

or $33+30+l_1+l_2=90$

or $l_1+l_2=27$. ...(1)

From $y^2=c^2+s^2$, we have

$$y_A^2=c^2+l_1^2 \text{ and } y_B^2=c^2+l_2^2$$

or $33^2=c^2+l_1^2$ and $30^2=c^2+l_2^2$.

Subtracting, we have

$$l_1^2-l_2^2=33^2-30^2=(33-30)(33+30)$$

or $$(l_1-l_2)(l_1+l_2)=3\times 63$$

$$\therefore\quad l_1-l_2=\frac{3\times 63}{l_1+l_2}=\frac{3\times 63}{27},$$

or $$l_1-l_2=7. \qquad \ldots(2)$$

Solving (1) and (2), we have

$$l_1=17 \text{ inches and } l_2=10 \text{ inches.}$$

$\therefore$ the lengths of the string on the two sides of the vertex C are

$$CB+BE=l_2+BE=10+30=40 \text{ inches,}$$

and $$CA+AD=l_1+AD=17+33=50 \text{ inches.}$$

Hence the required ratio $=40:50=4:5$.

Now from $y_A^2=c^2+l_1^2$, we have $33^2=c^2+17^2$

or $$c^2=33^2-17^2=(33-17)(33+17)=16\times 50.$$

$$\therefore\quad c=20\sqrt{2}.$$

If (x_A, y_A) and (x_B, y_B) are the coordinates of the points A and B respectively, then from $y=c\cosh(x/c)$, we have

$$y_A=c\cosh\frac{x_A}{c} \text{ and } y_B=c\cosh\frac{x_B}{c}.$$

$$\therefore\quad x_A=c\cosh^{-1}\frac{y_A}{c} \text{ and } x_B=c\cosh^{-1}\frac{y_B}{c}.$$

Hence the horizontal distance between the pegs

$$=x_A+x_B=c\left[\cosh^{-1}\frac{y_A}{c}+\cosh^{-1}\frac{y_B}{c}\right]$$

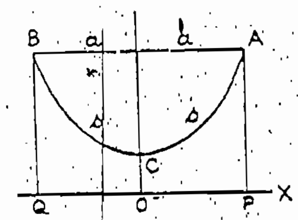
$$=20\sqrt{2}\cdot\left[\cosh^{-1}\left(\frac{33}{20\sqrt{2}}\right)+\cosh^{-1}\left(\frac{30}{20\sqrt{2}}\right)\right]$$

$$=20\sqrt{2}\cdot\left[\cosh^{-1}\left(\frac{33}{40}\sqrt{2}\right)+\cosh^{-1}\left(\frac{3}{4}\sqrt{2}\right)\right]$$

Ex. 36 (b). *A string hangs over two smooth pegs which are at the same level. Its free ends hanging vertically. Prove that when the string is of shortest possible length, the parameter of the catenary is equal to half the distance between the pegs, and find the whole length of the string.*

Sol. Suppose A and B are two smooth pegs at the same level and at a distance $2a$ apart *i.e.*, $AB=2a$. A string hangs over the

pegs A and B. The portions AP and BQ of the string hang vertically and the portion ACB hangs in the form of a catenary whose vertex is C and directrix is OX. Let w be the weight per unit length of the string.



(Fig. 22)

Now the tension of the string remains unaltered while passing over the smooth peg at A. Therefore the tension at the point A due to the string AP is equal to the tension at the point A due to the string ACB hanging in the form of a catenary. But the tension at the point A due to the string AP is equal to the weight $w\cdot AP$ of the string AP and by the formula $T=wy$, the tension at A in the catenary ACB is equal to $w\,y_A$, where y_A is the height of the point A above the directrix OX of the catenary ACB. So we have $w\,y_A=w\cdot AP$ *i.e.*, $y_A=AP$. Therefore the free end P of the string lies on the directrix OX of the catenary ACB. Similarly the other free end Q of the string also lies on the directrix OX.

Let c be the parameter of the catenary ACB *i.e.*, let $OC=c$. For the point A of the catenary ACB, we have

$$y=y_A=AP \text{ and } x=x_A=a.$$

By the formula $s=c\sinh(x/c)$, for the point A, we have

$$s=\text{arc } CA=c\sinh(a/c).$$

Also by the formula $y=c\cosh(x/c)$, for the point A, we have

$$y=y_A=AP=c\cosh(a/c).$$

Hence the total length of the string, say l, is given by

$$l=2(\text{arc } CA+y_A)=2\{c\sinh(a/c)+c\cosh(a/c)\}$$

$$=2c\{\tfrac{1}{2}(e^{a/c}-e^{-a/c})+\tfrac{1}{2}(e^{a/c}+e^{-a/c})\}=2c\,e^{a/c}. \qquad \ldots(1)$$

Now l is a function of c. For a maximum or a minimum of l, we must have $$dl/dc=0.$$

From (1), we have

$$\frac{dl}{dc}=2e^{a/c}+2c\,e^{a/c}\cdot\left(-\frac{a}{c^2}\right)=2e^{a/c}\left(1-\frac{a}{c}\right)$$

Putting $dl/dc=0$, we get $2e^{a/c}\{1-(a/c)\}=0$,

or $$1-(a/c)=0 \qquad [\because\ e^{a/c}\neq 0]$$

or $$c=a.$$

Now $$\frac{d^2l}{dc^2}=2e^{a/c}\cdot\frac{a}{c^2}+\left(1-\frac{a}{c}\right)\cdot 2e^{a/c}\left(-\frac{a}{c^2}\right)$$

$$=2e/a, \text{ when } c/a=+\text{ive}.$$

$\therefore$ l is minimum when $c=a$.

Thus when the string is of shortest possible length, we have $c=a=\frac{1}{4}(2a)=$ half the distance between the pegs.

Now putting $c=a$ in (1), the length l of the string in this case is given by $\quad l=2a\ e^{a/a}=2ae.$

Ex. 36. (c) *A heavy string hangs over two fixed small smooth pegs. The two ends of the string are free and the central portion hangs in a catenary. Show that the free ends of the string are on the directrix of the catenary. If the two pegs are on the same level and distant 2a apart, show that equilibrium is impossible unless the string is equal to or greater than 2ae.* [Rohilkhand 87]

Sol. For the first part of the question draw figure as in Ex. 36 (a) and proceed in the same way.

For the second part of the question draw figure as in Ex. 36 (b) and proceed in the same way.

The least possible length for the string to be in equilibrium comes out to be $2ae$. Therefore the equilibrium is impossible unless the string is equal to or greater than $2ae$.

§ 7. Approximations to the common catenary. [Kanpur 75]

1. The cartesian equation of the common catenary is

$$y=c\cosh(x/c)=\tfrac{1}{2}c\,(e^{x/c}+e^{-x/c})$$

$$=\frac{c}{2}\left[\left\{1+\left(\frac{x}{c}\right)+\frac{1}{2!}\left(\frac{x}{c}\right)^2+\frac{1}{3!}\left(\frac{x}{c}\right)^3+\frac{1}{4!}\left(\frac{x}{c}\right)^4+\ldots\right\}\right.$$
$$\left.+\left\{1-\left(\frac{x}{c}\right)+\frac{1}{2!}\left(\frac{x}{c}\right)^2-\frac{1}{3!}\left(\frac{x}{c}\right)^3+\frac{1}{4!}\left(\frac{x}{c}\right)^4\ldots\ldots\right\}\right]$$

$$=c\left[1+\frac{1}{2!}\left(\frac{x}{c}\right)^2+\frac{1}{4!}\left(\frac{x}{c}\right)^4+\ldots\right] \qquad (1)$$

Now if x/c is small, then neglecting the powers of x/c higher than two, the equation (1) reduces to

$$y=c\left[1+\frac{1}{2!}\left(\frac{x}{c}\right)^2\right],$$

or $\quad x^2=2c\,(y-c),$

which is the equation of a parabola of latus rectum $2c$ or $2T_0/w$.

Thus if x is small compared to c, the common catenary coincides very nearly with a parabola of latus rectum 2c or $2T_0/w$ and vertex at the point (0, c).

Examples of such a case are the electric transmission wires and the telegraphic wires tightly streched between the poles. Besides such cases of tightly stretched strings, even in the case of a common catenary not tightly stretched if we consider the portion of the curve near the vertex, x is small compared to c.

2. When x is large *i.e.* at points far removed from the lowest point, x/c is large and so $e^{-x/c}$ becomes very small, hence

$$y=c\cosh(x/c)=\tfrac{1}{2}c\,(e^{x/c}+e^{-x/c})$$

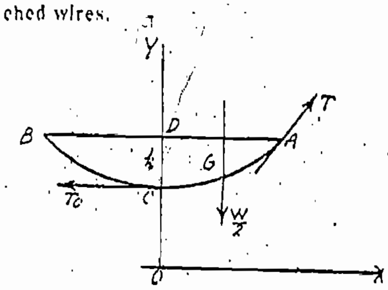
behaves as $y=\frac{1}{2}c\,e^{x/c}$,

which is an exponential curve.

Hence at points far removed from the lowest point, a common catenary behaves as an exponential curve.

§ 8. Sag of tightly stretched wires.

Consider a tightly stretched wire which appears nearly a straight line, as for example a telegraphic wire stretched tightly between the poles.



(Fig. 23)

Let A and B be two points in a horizontal line between which a wire is stretched tightly. Let C be the lowest point of the catenary formed by the wire. Let W be the weight and l the length of the wire ACB. Also let T_0 be the horizontal tension at the lowest point C. The portion CA of the wire is in equilibrium under the action of the following forces :

(i) the tension T_0 acting horizontally at the point C.

(ii) the tension T at A acting along the tangent at A,

and (iii) the weight $\frac{1}{2}W$ of the wire CA acting vertically downwards through its centre of gravity G.

Since the wire is tightly stretched, the distance of the centre of gravity G of the wire AC from the vertical line through A will be approximately equal to $\frac{1}{2}AC$ *i.e.* $\frac{1}{4}l$. Let k be the sag CD and a the span AB of the catenary.

Taking moments of the forces acting on the portion CA, about A, we have $\quad T_0 . k=\frac{1}{2}W.\frac{1}{4}l$

$$T_0=\frac{lW}{8k}. \qquad \ldots(1)$$

Now we calculate *the increase in the length of the wire on account of the sag.*

For a catenary, we have

$$s=c\sinh(x/c)=\tfrac{1}{2}c\,(e^{x/c}-e^{-x/c})$$

$$=\frac{c}{2}\left[\left\{1+\left(\frac{x}{c}\right)+\frac{1}{2!}\left(\frac{x}{c}\right)^2+\frac{1}{3!}\left(\frac{x}{c}\right)^3+\frac{1}{4!}\left(\frac{x}{c}\right)^4+\ldots\ldots\right\}\right.$$
$$\left.-\left\{1-\left(\frac{x}{c}\right)+\frac{1}{2!}\left(\frac{x}{c}\right)^2-\frac{1}{3!}\left(\frac{x}{c}\right)^3+\frac{1}{4!}\left(\frac{x}{c}\right)^4\ldots\ldots\right\}\right]$$
$$=c\left[\frac{x}{c}+\frac{1}{3!}\left(\frac{x}{c}\right)^3+\frac{1}{5!}\left(\frac{x}{c}\right)^5+\ldots\ldots\right]. \qquad (2)$$

The radius of curvature ρ of the catenary is given by

$$\rho=c\sec^2\psi.$$

At the vertex, $\psi=0$ and so $\rho=c$ at the vertex. This shows that if the curve is flat near its vertex C so that ρ is large at the vertex, then c will be very large and x will be small as compared to c. Thus for a tightly stretched wire x/c is very small.

Thus retaining only the first two terms in (2), we have

$$s=c\left[\frac{x}{c}+\frac{1}{3!}\left(\frac{x}{c}\right)^3\right]=x+\frac{x^3}{6c^2}$$

or $\quad s-x=\dfrac{x^3}{6c^2}.$

But $\quad c=T_0/w. \qquad [\because\ T_0=wc]$

$$\therefore\quad s-x=\frac{w^2x^3}{6T_0^2}.$$

Now putting $x=\frac{1}{2}a$, where a is the span AB, we have

$$s-\frac{a}{2}=\text{arc } CA-DA=\frac{a^3w^2}{48T_0^2}.$$

$\therefore$ *total increase in the length of the wire due to sagging*

$$=\text{arc } ACB-\text{span } AB=2s-a=\frac{a^3w^2}{24T_0^2}.$$

Illustrative Examples

Ex. 37. *Show that the maximum tension in a wire which weighs 15 lb. per yard and hangs with a sag of 1 foot in a horizontal span of 100 feet is about $62\frac{1}{2}$ lbs. wt.*

Sol. Refer figure of § 8 on page 47.

From the formula $T=wy$ it is obvious that the maximum tension in the wire will be at the extremities A or B.

Here $w=15/3=\cdot05$ lb. per foot, and span $AB=100$ feet.

The sag $CD=k=1$ foot.

If T is the maximum tension in the wire at A, then

$$T=wy_A, \qquad \ldots(1)$$

where y_A is the ordinate of the point A.

For a catenary, we have

$$y=c\cosh(x/c)=c\left[1+\frac{1}{2!}\left(\frac{x}{c}\right)^2+\frac{1}{4!}\left(\frac{x}{c}\right)^4+\ldots\ldots\right]. \qquad \ldots(2)$$

Since sag 1 foot is very small compared to the span 100 feet, hence the wire is tightly stretched and x/c is very small.

$\therefore$ neglecting higher powers of x/c in (2), we have

$$y=c\left[1+\frac{1}{2!}\left(\frac{x}{c}\right)^2\right]=c+\frac{x^2}{2c} \qquad \ldots(3)$$

For the point A, $x=DA=\frac{1}{2}AB=50$ feet. Therefore from (3), we have $\quad y_A=c+\dfrac{(50)^2}{2c}.$

But $\quad y_A=c+k=c+1.$

$$\therefore\quad c+\frac{(50)^2}{2c}=c+1 \quad\text{or}\quad c=1250.$$

$$\therefore\quad y_A=c+1=1250+1=1251\text{ feet.}$$

Hence from (3), the required maximum tension

$$T=wy_A=(\cdot05)\times(1251)=62\cdot55\text{ lbs. wt.}$$
$$=62\tfrac{1}{2}\text{ lbs. wt. nearly.}$$

Ex. 38. *A telegraph is constructed of No. 8 iron wire which weighs 7·3 lbs. per 100 feet; the distance between the posts is 150 feet and the wire sags 1 foot in the middle. Show that it is screwed up to a tension of about 205 lbs. wt.*

Sol. Here the sag $k=1$ foot is small as compared to the span 150 feet, hence the wire is tightly stretched between the posts.

Here $\quad w=\dfrac{7\cdot3}{100}=\cdot073$ lbs. per foot.

$\because$ wire is tightly stretched, $\quad\therefore\ x/c$ is very small.

$\therefore\quad y=c\cosh(x/c)$

$$=c\left[1+\frac{1}{2!}\left\{\frac{x}{c}\right\}^2+\frac{1}{4!}\left\{\frac{x}{c}\right\}^4+\ldots\ldots\right]$$
$$=c\left[1+\frac{1}{2!}\left(\frac{x}{c}\right)^2\right], \text{ neglecting higher powers of } x/c$$
$$=c+\frac{x^2}{2c}. \qquad \ldots(1)$$

If (x_A, y_A) are the coordinates of the extremity A of the wire, then $\quad x_A=\frac{150}{2}=75$ feet and $y_A=c+k=(c+1)$ feet. $\qquad \ldots(2)$

Also from (1), $\quad y_A=c+\dfrac{x_A^2}{2c} \quad$ or $\quad c+1=c+\dfrac{x_A^2}{2c},$

or $\quad 1=\dfrac{x_A^2}{2c}.$

$$\therefore \quad c = \tfrac{1}{2}x_A^2 = \tfrac{1}{2}(75)^2 = 2812 \cdot 5.$$

Hence the required tension at the extremity

$$T = wy_A = (\cdot 073) \times (c+1)$$
$$= (\cdot 073) \times (2812 \cdot 5 + 1)$$
$$= \cdot 073 \times 2813 \cdot 5 = 205 \text{ lbs. wt. nearly.}$$

Ex. 39. *The ends of a tightly stretched cable weighing $\frac{1}{2}$ lb. per yard are fixed to two points on a level 80 yards apart and the cable has a sag, at the middle, of 1 foot 4 inches. Find the tension in the wire at the lowest point.*

Sol. Here the cable is tightly stretched.

$$\therefore \quad y = c \cosh (x/c) = c + x^2/2c, \qquad \ldots(1)$$

neglecting higher powers of x/c which is small.

Also here $w = \frac{1}{6}$ lb. per foot; span $= 80 \times 3 = 240$ feet

and sag $k = \frac{4}{3}$ feet.

At the extremity of the wire,

$x = 120$ feet and $y = c + k = (c + \frac{4}{3})$ feet.

$\therefore$ from (1), we have $\quad c + \dfrac{4}{3} = c + \dfrac{(120)^2}{2c}$

or $\quad \dfrac{4}{3} = \dfrac{(120)^2}{2c} \quad \therefore \quad c = \dfrac{(120)^2 \times 3}{8} = 45 \times 120.$

Hence the tension in the wire at the lowest point

$$T_0 = wc = \tfrac{1}{6} \times 45 \times 120 = 900 \text{ lbs. wt.}$$

Ex. 40. *A heavy uniform string 155 ft. long, is suspended from two points A and B, 150 ft. apart on the same horizontal plane. Show that the tension at the lowest point is approximately equal to 1·08 times the weight of the string.*

Sol. If W is the weight of the string and w is the weight per foot of the string, then $\quad w = (W/155)$.

$\therefore$ tension at the lowest point,

$$T_0 = wc = (W/155)\, c. \qquad \ldots(1)$$

Now from $s = c \sinh (x/c)$, we have

$$s = c\left[\frac{x}{c} + \frac{1}{3!}\left(\frac{x}{c}\right)^3\right],$$

neglecting higher powers of x/c which is small because the string is tightly stretched. $\quad \therefore \quad s - x = (x^3/6c^2)$

or
$$2s - 2x = \frac{x^3}{3c^2}. \qquad \ldots(2)$$

But at the extremity of the string, we have

$2s = 155$ ft., $2x = 150$ ft., *i.e.*, $x = 75$ ft.

$\therefore$ from (2), we have

$$155 - 150 = \frac{(75)^3}{3c^2}$$

or
$$c^2 = \frac{(75)^3}{3 \times 5} = 5 \times (75)^2; \quad \therefore \quad c = 75\sqrt{5}.$$

Hence from (1)

$$T_0 = (W/155) \times 75\sqrt{5} = 1 \cdot 08W \text{ approximately.}$$

Ex. 41. *A uniform measuring chain of length l is tightly stretched over a river, the middle point just touching the surface of the water, while each of the extremities has an elevation k above the surface. Show that the difference between the length of the measuring chain and the breadth of the river is nearly $(8k^2/3l)$.*

Sol. Let l be the length of the chain, $2a$ the breadth of the river, and k the sag. It is required to show that

$$l - 2a = 8k^2/3l \text{ nearly.}$$

Since the chain is tightly stretched, therefore x/c is small. So neglecting higher powers of x/c, the equation $y = c \cosh (x/c)$ of the catenary approximates to $y = c + x^2/2c$.

At the extremity of the chain, $x = a$, and $y = c + k$.

$$\therefore \quad c + k = c + (a^2/2c), \text{ giving } c = (a^2/2k).$$

Now we have $s = c \sinh \dfrac{x}{c} = c\left[\dfrac{x}{c} + \dfrac{1}{3!}\dfrac{x^3}{c^3}\right]$ nearly.

At an extremity of the chain, $s = \frac{1}{2}l$ and $x = a$.

$$\therefore \quad \frac{l}{2} = c\left(\frac{a}{c} + \frac{a^3}{6c^3}\right), \text{ or } \frac{l}{2} - a = \frac{a^3}{6c^2}$$

or
$$l - 2a = \frac{a^3}{3c^2} = \frac{a^3}{3} \cdot \frac{4k^2}{a^4} \qquad \left[\because \quad c = \frac{a^2}{2k}\right]$$
$$= 4k^2/3a.$$

Since the string is tightly stretched,

$$\therefore \quad a = l/2 \text{ approximately.}$$

Hence
$$l - 2a = \frac{4k^2}{3} \cdot \frac{2}{l} = \frac{8k^2}{3l}.$$

Ex. 42. *A chain is suspended in a vertical plane from two fixed supports A and B, which lie in a horizontal line 462 feet apart. If the tangent to the chain at A be inclined at an angle $\tan^{-1}(3/4)$ to the horizon, find the length of the chain (take $\log_e 2 = \cdot 693$).*

Sol. Here the angle of inclination of the tangent at A,

$$\psi_A = \tan^{-1} \tfrac{3}{4}$$

or $\tan \psi_A = \frac{3}{4}$. $\therefore$ $\sec \psi_A = \sqrt{(1 + \tan^2 \psi_A)} = \sqrt{(1 + 9/16)} = 5/4.$

If the x-coordinate of the extremity A is x_A, then

$$x_A = \tfrac{1}{2} \times 462 = 231.$$

$\therefore$ from $x=c\log(\tan\psi+\sec\psi)$, we have

$$x_A=c\log(\tan\psi_A+\sec\psi_A),$$

or $$231=c\log\left(\tfrac{4}{3}+\tfrac{5}{3}\right)=c\log 2=\cdot 693c.$$

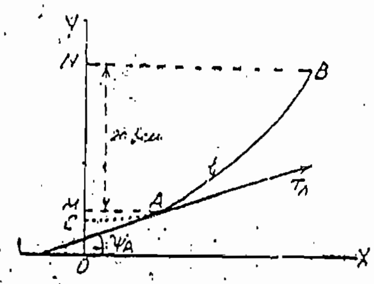
$$\therefore\quad c=\frac{331}{\cdot 693}=\frac{1000}{3}.$$

Hence the length of the chain

$$=2s_A=2c\tan\psi_A=2\times\frac{1000}{3}\times\frac{3}{4}=500 \text{ ft.}$$

Ex. 43. *A uniform chain has its ends fixed at A and B where B is 20 ft. above the level of A, and no part of the chain is below A. At A the chain is inclined at* $\sec^{-1}(5/3)$ *to the horizontal, and the tension there is equal to the wt. of 100 ft. of chain. Prove that the length of the chain is 23 ft. 11 inch. to the nearest inch.*

Sol. Let AB be the chain of length l feet with the end B at a height 20 ft. above the end A. The tension T_A at the point A acting along the tangent at A is inclined at an angle $\psi_A=\sec^{-1}(5/3)$ to the horizontal.



We have, $\sec\psi_A=5/3$

and $\tan\psi_A=\sqrt{(\sec^2\psi_A-1)}$

$$=4/3.$$

(Fig. 24)

If w is the weight of one foot length of the chain, then according to the question, $T_A=100w$.

Let C be the vertex and OX the directrix of the catenary of which AB is an arc.

Let y_A and y_B be the ordinates of the point A and B respectively and let arc $CA=a$ feet.

From $T=wy$, we have $T_A=wy_A$.

But $T_A=100w$.

$\therefore$ $wy_A=100w$ or $y_A=100$ feet.

Also we have $y_A=c\sec\psi_A$

$\therefore$ $c=y_A\cos\psi_A=100.\tfrac{3}{5}=60$ feet.

$\therefore$ from $s=c\tan\psi$, for the point A, we have

arc $CA=a=c\tan\psi_A=60.\tfrac{4}{3}=80.$

Now $y_B=y_A+MN=100+20=120$

and arc $CB=a+l=80+l.$

$\therefore$ from $y^2=s^2+c^2$, we have

$$y_B^2=(l+80)^2+60^2$$

or $$120^2=(l+80)^2+60^2$$

or $$(l+80)^2=120^2-60^2=(120-60)(120+60)=60\times 180$$

$$\therefore\quad l+80=\pm 60\sqrt{3}$$

or $l=-80+60\sqrt{3}$, neglecting the negative sign otherwise l will be negative

or $l=(60\times 1\cdot 732-80)$ ft. $=23\cdot 92$ ft. $=23$ ft. 11 in. nearly.

Ex. 44. *A kite is flown with 600 ft. of string from the hand to the kite, and a spring balance held in hand shows a pull equal to the weight of 100 ft. of the string, inclined at 30° to the horizontal. Find the vertical height of the kite above the hand.* **[Agra 78]**

Sol. [Refer figure 24 of Ex. 43 on page 52].

Let AB be the string of length 600 ft. with the kite at B. If w is the weight of one ft. length of the string, then the hand at A experiences a pull of 100 w *i.e.*, the tension at A, $T_A=100w$.

Let C be the vertex and OX the directrix of the catenary of which AB is an arc. If the tangent at A is inclined at an angle ψ_A to the horizontal, then according to the question $\psi_A=30°$.

Let y_A and y_B be the ordinates of the points A and B respectively.

From $T=wy$, we have $T_A=wy_A$.

But $T_A=100w$.

$\therefore$ $100w=wy_A$ or $y_A=100$ ft.

Also we have $y_A=c\sec\psi_A$.

$\therefore$ $100=c\sec 30°$ or $c=100\cos 30°=50\sqrt{3}$ ft.

From $s=c\tan\psi$, for the point A, we have,

arc $CA=a$ (say) $=c\tan\psi_A=50\sqrt{3}.\tan 30°=50$ ft.

Now $y_B=y_A+MN=100+MN$

and arc $CB=$ arc $CA+$ arc $AB=50+600=650$ ft.

$\therefore$ from $y^2=s^2+c^2$, we have

$$y_B^2=650^2+(50\sqrt{3})^2$$

or $$(100+MN)^2=50^2\times(13^2+3)=50^2.172=50^2\times 2^2\times 43.$$

$$\therefore\quad 100+MN=\pm 100\sqrt{(43)}$$

or $MN=100[\sqrt{(43)}-1]$, neglecting the negative sign, otherwise MN is negative

$=555\cdot 7$ ft. nearly.

Ex. 45. *A telegraph wire is made of a given material, and such a length l is stretched between two posts, distant d apart and of the same height, as will produce the least possible tension at the posts.*

Show that $l=(d/\lambda)\sinh\lambda$,

where λ *is given by the equation* $\lambda\tanh\lambda=1$.

[Rohilkhand 78; Kanpur 83; Luck. 77, 79; Agra 87]

Sol. Let T be the tension at either of the posts. Then

$$T=wy=wc\cosh(x/c)=wc\cosh(d/2c).$$

[Since d is the distance between the posts, $\therefore$ at either of the posts $x=d/2$.]

The tension T is a function of the parameter c of the catenary. We have

$$\frac{dT}{dc}=w\left(\cosh\frac{d}{2c}-\frac{d}{2c}\sinh\frac{d}{2c}\right)$$

and $$\frac{d^2T}{dc^2}=\frac{wd^2}{4c^3}\cosh\frac{d}{2c}, \text{ after simplification.}$$

For maximum or minimum of T,

$$(dT/dc)=0,$$

i.e., $$w\left(\cosh\frac{d}{2c}-\frac{d}{2c}\sinh\frac{d}{2c}\right)=0$$

or $$\frac{d}{2c}\tanh\frac{d}{2c}=1$$

or $$\lambda\tanh\lambda=1,\quad \text{where } \lambda=d/2c.$$

Since d^2T/dc^2 is positive, therefore T is minimum for $c=d/2\lambda$, λ being given by the equation $\lambda\tanh\lambda=1$.

Now for either of the posts, $x=\frac{1}{2}d$ and $s=\frac{1}{2}l$.

$\therefore$ from $s=c\sinh(x/c)$, we have

$$\tfrac{1}{2}l=c\sinh(d/2c)$$

or $$l=(2.d/2\lambda)\sinh\lambda=(d/\lambda)\sinh\lambda,$$

for producing the least possible tension at the posts.

Ex. 46. *If the length of a uniform chain suspended between posts at the same level is adjusted so that the tension at the posts of support is a minimum for that particular span 2d, show that the equation to determine c is* $\coth(d/c)=(d/c)$. [Kanpur 82]

Sol. Hint. Proceed as in Ex. 45.

Ex. 47. *A uniform chain is hung up from two points at the same level and distant 2a apart. If z is the sag at the middle, show that:* $z=c\{\cosh(a/c)-1\}$.

If z is small compared with a, show that

$2cz=a^2$ *nearly.* [Kanpur 78]

Sol. [Refer figure of § 8 on page 47].

Let ACB be the uniform chain, C the lowest point i.e., the

vertex of the catenary formed by the chain and OX its directrix.

We have, $y=c\cosh(x/c)$.

At A, $x=\frac{1}{2}AB=a$.

Therefore if $y=y_A$ for the point A, then

$$y_A=c\cosh(a/c).$$

$\therefore$ sag at the middle,

$$z=OD-OC=y_A-c=c\cosh(a/c)-c=c\,[\cosh(a/c)-1].$$

Now expanding $\cosh(a/c)$ in powers of a/c, we have,

$$z=c\left[1+\frac{1}{2!}\left(\frac{a}{c}\right)^2+\frac{1}{4!}\left(\frac{a}{c}\right)^4+\ldots\ldots-1\right]$$

$$=c\left[\frac{1}{2!}\left(\frac{a}{c}\right)^2+\frac{1}{4!}\left(\frac{a}{c}\right)^4+\ldots\ldots\right]$$

$$=\frac{1}{2!}\frac{a^2}{c}+\frac{1}{4!}\frac{a^4}{c^3}+\ldots\ldots$$

If z is small compared to a, then c must be large. Therefore neglecting the higher powers of $1/c$ in the above expansion, we have

$$z=\frac{1}{2!}\cdot\frac{a^2}{c}$$

or $$2cz=a^2.$$

Ex. 48. *A telegraph wire is supported by two poles distant 40 yards apart. If the sag be one foot and the weight of the wire half an ounce per foot, show that the horizontal pull on each pole is* $\frac{1}{4}$ *cwt. nearly.*

Sol. Proceed as in the last Ex. 47. The sag z at the middle is given by

$$2cz=a^2 \text{ (nearly).}$$

Here $z=1$ ft. and $a=\frac{1}{2}(40\times 3)=60$ ft.

$$\therefore\quad c=\frac{a^2}{2z}=\frac{60^2}{2.1}=1800 \text{ ft.}$$

Now the required horizontal pull at A

$$=T_0=wc=\frac{1}{32}\cdot 1800 \text{ lbs.}\qquad\left[\because\ w=\frac{1}{32}\text{ lbs}\right]$$

$$=\frac{1800}{32\times 112}\text{ cwt. nearly}=\tfrac{1}{2}\text{ cwt. nearly.}$$

Ex. 49. *A uniform chain, of length 2l, has its ends attached to two points in the same horizontal line at a distance 2a apart. If l is only a little greater than a, show that the tension of the chain is approximately equal to the weight of a length*

$$\sqrt{\left[\frac{a^3}{6(l-a)}\right]}$$

of the chain, and that the sag or depression of the lowest point of *the chain below its ends is nearly* $\frac{1}{2}\sqrt{\{6a\,(l-a)\}}$.

[Kanpur 74; Luck. 78, 80; Agra 86]

Sol. [Refer figure of § 8 on page 47]

Since the chain is tightly stretched, the tension at any point shall be the same nearly.

$\therefore$ the tension in the chain $=T=T_0$ (tension at the lowest point)

or $$T=wc. \quad \ldots(1)$$

Here span $=2a$ and the length of the chain $=2l$.

$\therefore$ at either of the supports, $x=a$ and $s=l$.

$\therefore$ from $s=c\sinh(x/c)$, we have

$$l=c\sinh\frac{a}{c}=c\left[\frac{a}{c}+\frac{1}{3\,!}\left(\frac{a}{c}\right)^3+\ldots\right] \quad \ldots(2)$$

Since the chain is tightly stretched, hence c is large *i.e.*, a/c is very small.

$\therefore$ neglecting higher powers of a/c in (2), we have

$$l=c\left[\frac{a}{c}+\frac{1}{3\,!}\left(\frac{a}{c}\right)^3\right]=a+\frac{a^3}{6c^2}$$

or $$\frac{a^3}{6c^2}=l-a, \text{ or } c=\sqrt{\left\{\frac{a^3}{6\,(l-a)}\right\}}.$$

$\therefore$ from (1), we have $T=w\sqrt{\left\{\dfrac{a^3}{6\,(l-a)}\right\}}$.

Hence the tension of the chain is approximately equal to the weight of a length $\sqrt{\left\{\dfrac{a^3}{6\,(l-a)}\right\}}$ of the chain.

At the support, $x=a$. $\therefore$ from $y=c\cosh(x/c)$, at the support, $y=y_A=c\cosh(a/c)$.

$\therefore$ sag in the middle $=y_A-c=c\cosh(a/c)-c$

$$=c\left[1+\frac{1}{2\,!}\left(\frac{a}{c}\right)^2+\frac{1}{4\,!}\left(\frac{a}{c}\right)^4+\ldots-1\right]$$

$=c\,[a^2/2c^2]$ nearly, neglecting higher powers of a/c

$=(a^2/2c)$

$$=\frac{a^2}{2}\sqrt{\left\{\frac{6\,(l-a)}{a^3}\right\}}=\frac{1}{2}\sqrt{\{6a\,(l-a)\}}.$$

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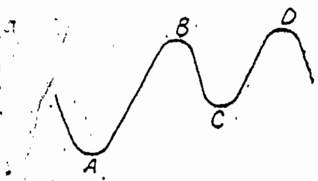
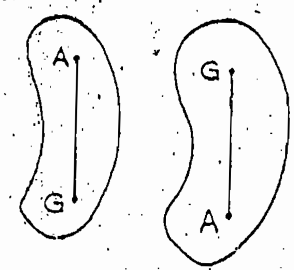
Stable and Unstable Equilibrium

§ 1. Introduction. Consider the motion of a body on a smooth curve in a vertical plane as shown in the figure. Obviously the body can rest at points A, B, C and D which are points of maxima or minima of the curve. If the body be slightly displaced from its position of rest at A or C (*i.e.*, the points of minima), it will tend to return to its original position of rest, while if displaced from its position of rest at B or D (*i.e.*, the points of maxima), it will tend to move still further away from its original position of rest. In the first case the equilibrium of the body is said to be *stable* and in the second case it is said to be *unstable*.



Take one more illustration. Consider the equilibrium of a rigid body fixed at one point say A. For the equilibrium of the body the centre of gravity G of the body must lie on the vertical line through the point of support A. There arise three cases.



Case 1. Suppose that the centre of gravity G lies below the point of support A. In this case if the body be slightly displaced from its position of equilibrium its centre of gravity will be raised. If the body be then let free, the force of gravity will bring the body back to its original position of equilibrium. In this case the body is said to be in *stable equilibrium*.

Case 2. Next suppose that the centre of gravity G lies above the point of support A. In this case if the body be slightly displaced from its position of equilibrium, its centre of gravity will be lowered. If the body be then let free, the force of gravity will still further move away the body from its original position of equilibrium. In this case the body is said to be *in unstable equilibrium.*

Case 3. If the centre of gravity G is at the point of support A, the body will still be in equilibrium when displaced. In this case we say that the body is in a state of *neutral equilibrium.*

Remark. It can be seen that among all the possible positions of the body, in the case 1 the height of the centre of gravity of the body above some fixed plane is minimum and in the case 2 it is maximum.

§ 2. Definitions. [Meerut 77, 90, 90 P]

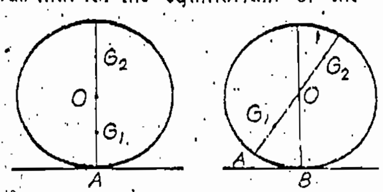
(I) **Stable equilibrium.** *A body is said to be in stable equilibrium if when slightly displaced from its position of equilibrium, the forces acting on the body tend to make it return towards its position of equilibrium.*

(II) **Unstable equilibrium.** *The equilibrium of a body is said to be unstable if when slightly displaced from its position of equilibrium, the forces acting on the body tend to move the body further away from its position of equilibrium.*

(III) **Neutral Equilibrium.** *A body is said to be in neutral equilibrium if the forces acting on it are such that they keep the body in equilibrium in any slightly displaced position.*

Further examples of stable, unstable and neutral equilibrium.

Consider the case of a heavy sphere resting on a horizontal plane. Suppose the centre of gravity of the sphere is not at its geometric centre O. It is obvious that for the equilibrium of the sphere its point of contact A with the plane, its geometric centre O and its centre of gravity must be in the same vertical line. One position of equilibrium is in which the centre of gravity G_1 is below the geometric centre O. In this case if the sphere be slightly displaced it would tend to come back to its original position of equilibrium. This is the position of stable equilibrium. The other position of equilibrium is in which the centre of gravity G_2 is above the geometric centre O. In this case if the sphere be slightly displaced it would not come back to its original position of equilibrium but would go further away from that position. This is the position of **unstable equilibrium.**



If, however, the centre of gravity of the sphere is at its geometric centre O, the sphere will still be in equilibrium when displaced. In this case the equilibrium is **neutral.**

If a right circular cone rests on a horizontal plane with its base in contact with the plane and its axis vertical, its equilibrium is stable. But if it rests with its vertex in contact with the plane and its axis vertical, its equilibrium is unstable. Again if it rests along a generator, it is in neutral equilibrium.

The equilibrium of a pendulum is stable, when it is displaced from its vertical position of equilibrium, for it returns towards the vertical position again. Any top heavy thing or a stick placed vertically on a finger is an example of unstable equilibrium.

§ 3. The Work Function. Suppose a material system is acted upon by a system of forces X, Y, Z parallel to the axes of coordinates. If during a small displacement whose projections on the coordinate axes are dx, dy, dz the work done by these forces is dW, then
$$dW = X\,dx + Y\,dy + Z\,dz.$$

The forces X, Y, Z generally depend upon the position of the particle. If we confine ourselves to the class of forces which are single-valued and are functions of x, y, z (and not of time t), then integrating the above equation from some standard position (x_0, y_0, z_0) to any position (x, y, z), we have
$$W = \int_{(x_0, y_0, z_0)}^{(x, y, z)} (X\,dx + Y\,dy + Z\,dz).$$

Such a function W is called the *work function.* It is work done by the forces in displacing the body from standard position to any position.

If W_A and W_B are the values of the work function at two positions A and B, then $W_B - W_A$ gives the work done by the forces in displacing the body from A to B.

If $X\,dx + Y\,dy + Z\,dz$ is an exact differential, the forces are called *conservative forces.*

§ 4. Work function test for the nature of stability of equilibrium.

Let A be the position of equilibrium of a rigid body under the action of a given system of forces and let W be the work function of the system in this position A. Suppose the body undergoes a small displacement and takes a position B near to the position of equilibrium A, then the value of the work function in the position B will be $W+dW$. Therefore the work done by the forces in displacing the body from the equilibrium position A to the nearby position B is dW. Since the body is in equilibrium in the position A, therefore by the principle of virtual work, we have $dW=0$. *Hence the work function W is stationary (maximum or minimum) in the position of equilibrium.*

First suppose that W is maximum at the equilibrium position A. Imagine that the body is slightly displaced to a position B and let W' be the work function there. Since W is maximum at A, therefore, $W' < W$, so that $W'-W$ is negative. It means that in displacing the body from A to B the work done by the forces is negative *i.e.*, the work is done against the forces and hence the forces will have a tendency to bring the body back to the original position of equilibrium A. Hence the equilibrium at A is stable.

Next suppose that W is minimum at the equilibrium position A. If W' is the value of the work function in a slightly displaced position B of the body, then in this case $W' > W$ so that $W'-W$ is positive. It means that in displacing the body from A to B the work done by the forces is positive *i.e.*, the work has been done by the forces and so the forces will have a tendency to move the body further away from the position of equilibrium. Hence in this case the equilibrium at A is unstable.

Thus in the positions of equilibrium of the body the work function W is either maximum or minimum. If it is maximum, the equilibrium is stable and if it is minimum, the equilibrium is unstable.

§ 5. Potential energy test for the nature of stability of equilibrium. [Lucknow 79; Meerut 83, 83P, 87P, 87S, 88, 88P, 89P]

Potential energy of a body. *The potential energy of a body, acted upon by a conservative system of forces, is defined as its capacity to do work by virtue of the position it has acquired. It is measured by the amount of work it can do in passing from the present position to some standard position.* If W be the work function of the body in any position referred to some standard position, and V be the potential energy of the body in that position referred to the same standard position, then $V=-W$. If V_A and V_B are the values of the potential energy at the two positions A and C, then V_A-V_B is the work done by the forces in displacing the body from A to B.

Let A be the position of equilibrium of a rigid body under the action of a given system of forces and V be the potential energy of

the body in this position A. Suppose the body undergoes a small displacement and takes a position B near to the position of equilibrium A, then the potential energy of the body in the position B will be $V+dV$. Therefore the work done by the forces in displacing the body from the equilibrium position A to the nearby position B is $V-(V+dV)$ *i.e.*, $-dV$. Since the body is in equilibrium in the position A, therefore by the principle of virtual work, we have $-dV=0 \Rightarrow dV=0$. *Hence the potential energy V is stationary (maximum or minimum) in the position of equilibrium.*

First suppose that V is minimum at the equilibrium position A. Imagine that the body is slightly displaced to a position B and let V' be the potential energy there. Since V is minimum at A, therefore $V' > V$, so that $V-V'$ is negative. It means that in displacing the body from A to B the work done by the forces acting on the body is negative *i.e.*, the work is done against the forces and so the forces will have a tendency to bring the body back to the original position of equilibrium A. Hence the equilibrium at A is stable.

Thus we see that *in the position of stable equilibrium, the potential energy of the body is minimum.* [Meerut 87, 87P, 88, 88P]

Next suppose that V is maximum at the equilibrium position A. If V' is the value of the potential energy in a slightly displaced position B of the body, then in this case $V' < V$, so that $V-V'$ is positive. It means that in displacing the body from A to B the work done by the forces is positive *i.e.*, the work is done by the forces and so the forces will tend to move the body further away from the position of equilibrium. Hence the equilibrium at A is unstable.

Thus in the positions of equilibrium of the body the potential energy V is either maximum or minimum. If it is minimum, the equilibrium is stable, and if it is maximum, the equilibrium is unstable.

For example, whenever gravitational energy is the only form of potential energy involved, the height of the centre of gravity of the body above a fixed horizontal plane must be a minimum for stable equilibrium and maximum for unstable equilibrium.

§ 6. z-test for the nature of stability.

Suppose a body is in equilibrium under its weight only *i.e.*, the force of gravity is the only external force acting on the body. Let z be the height of the centre of gravity of the body above a fixed horizontal plane. Express z as a function of some variable θ: i.e., let $z=f(\theta)$. By the principle of virtual work, for the equilibrium of the body, we must have

$-W\,\delta z=0$, where W is the weight of the body

$$\Rightarrow \delta z=0 \Rightarrow \frac{dz}{d\theta}\,\delta\theta=0 \Rightarrow \frac{dz}{d\theta}=0.$$

Thus the equilibrium positions of the body are given by the equation $dz/d\theta=0$. So in the position of equilibrium the height of the centre of gravity of the body above a fixed level must be either maximum or minimum.

Suppose the equation $dz/d\theta=0$ on solving gives $\theta=\alpha, \beta, \gamma$ etc. as the positions of equilibrium.

To test the nature of equilibrium at the position $\theta=\alpha$. We find $d^2z/d\theta^2$ for $\theta=\alpha$. If it is positive, then z is minimum for $\theta=\alpha$. So if we give a slight displacement to the body, the height of its centre of gravity will be raised and then on being set free the body will tend to come back to its original position of equilibrium. Therefore in this case the equilibrium is stable.

Again if $d^2z/d\theta^2$ for $\theta=\alpha$ is negative, then z is maximum for $\theta=\alpha$. So if we give a slight displacement to the body, the height of its centre of gravity will be lowered and then on being set free the force of gravity will still displace the body further away from its original position of equilibrium. Therefore in this case the equilibrium is unstable.

Thus the equilibrium positions of the body are given by the equation $dz/d\theta=0$. If for a root $\theta=\alpha$ of this equation, $d^2z/d\theta^2$ is positive, then z is minimum and the equilibrium is stable. But if for $\theta=\alpha$, $d^2z/d\theta^2$ is negative, then z is maximum and the equilibrium is unstable.

If however $d^2z/d\theta^2=0$ for $\theta=\alpha$, then we consider $d^3z/d\theta^3$ and $d^4z/d\theta^4$. Then for the position of equilibrium $\theta=\alpha$, we must have $d^3z/d\theta^3=0$, and the equilibrium is stable or unstable according as for this position $d^4z/d\theta^4$ is positive or negative.

Similar tests apply for the other positions of equilibrium $\theta=\beta, \gamma$ etc.

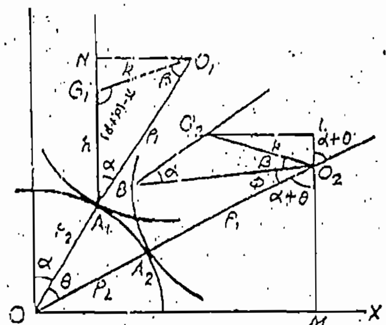
Remark. If $z=f(\theta)$ represents the depth of the centre of gravity of the body below some fixed horizontal plane, then the conditions for the stability and unstability of the equilibrium are reversed. In this case for equilibrium position we must have $dz/d\theta=0$. If for a root $\theta=\alpha$ of this equation $d^2z/d\theta^2$ is positive, then z is minimum and the equilibrium is unstable. But if for $\theta=\alpha$, $d^2z/d\theta^2$ is negative, then z is maximum and the equilibrium is stable. If however $d^2z/d\theta^2=0$ for $\theta=\alpha$, then we consider higher differential coefficients of z and conclude similarly.

§ 7. Stability of a body resting on a fixed rough surface.

Theorem. *A body rests in equilibrium upon another fixed body, the portions of the two bodies in contact have radii of curvatures ρ_1 and ρ_2 respectively. The centre of gravity of the first body is at a height h above the point of contact and the common normal makes an angle α with the vertical; it is required to prove that the equilibrium is stable or unstable according as $h <$ or $> \dfrac{\rho_1\rho_2}{\rho_1+\rho_2}\cos\alpha$.*

[Meerut 81]

Let O and O_1 be the centres of curvature of the lower and upper bodies in the position of rest and A_1 be their point of contact. In this position of equilibrium the common normal OA_1O_1 makes an angle α with the vertical OY. If G_1 is the centre of gravity of the upper body, then for equilibrium the line A_1G_1 must be vertical. It is given that $A_1G_1=h$.



Let $O_1G_1=k$

and $\angle CO_1G_1=\beta$.

Suppose the upper body is slightly displaced by pure rolling over the lower body which is fixed. Let A_2 be the new point of contact. O_2 is the new position of O_1 and the point A_1 of the upper body rolls up to the position B so that O_2B is the new position of the original normal O_1A_1. Also G_2 is the new position of G_1 so that $O_2G_2=O_1G_1=k$.

Suppose the common normal at A_2 makes angles θ and ϕ with the original normals OA_1 and O_2B.

We have $O_1A_1=\rho_1$ and $OA_1=\rho_2$. Also $O_2A_2=\rho_1$ and $OA_2=\rho_2$. Since the upper body rolls on the lower body without slipping therefore arc A_1A_2 = arc A_2B i.e., $\rho_2\theta=\rho_1\phi$.

$$\therefore \frac{d\phi}{d\theta}=\frac{\rho_2}{\rho_1}. \qquad \ldots(1)$$

Let z be the height of G_2 above the fixed horizontal line OX.

Then

$$z = LM = LO_2 + O_2M$$
$$= O_2G_2 \cos \angle G_2O_2L + OO_2 \cos(\alpha+\theta)$$
$$= k \cos[\pi - (\alpha+\theta+\phi+\beta)] + (\rho_1+\rho_2)\cos(\alpha+\theta)$$
$$[\because \quad O_2G_2 = O_1G_1 = k]$$
$$= (\rho_1+\rho_2)\cos(\alpha+\theta) - k\cos(\alpha+\theta+\phi+\beta).$$

$$\therefore \frac{dz}{d\theta} = -(\rho_1+\rho_2)\sin(\alpha+\theta) + k\sin(\alpha+\theta+\phi+\beta)\left(1+\frac{d\phi}{d\theta}\right)$$

[$\because$ α, β are constants and θ, ϕ are the only variables]

$$= -(\rho_1+\rho_2)\sin(\alpha+\theta) + k\sin(\alpha+\theta+\phi+\beta)\left(1+\frac{\rho_2}{\rho_1}\right) \quad \text{[from (1)]}$$

$$= \frac{\rho_1+\rho_2}{\rho_1}[-\rho_1 \sin(\alpha+\theta) + k\sin(\alpha+\theta+\phi+\beta)]$$

and

$$\frac{d^2z}{d\theta^2} = \frac{\rho_1+\rho_2}{\rho_1}\left[-\rho_1\cos(\alpha+\theta) + k\cos(\alpha+\theta+\phi+\beta)\left(1+\frac{d\phi}{d\theta}\right)\right]$$
$$= \frac{\rho_1+\rho_2}{\rho_1}\left[-\rho_1\cos(\alpha+\theta) + k\cos(\alpha+\theta+\phi+\beta)\left(1+\frac{\rho_2}{\rho_1}\right)\right]$$
$$= \frac{\rho_1+\rho_2}{\rho_1^2}[-\rho_1^2\cos(\alpha+\theta) + k(\rho_1+\rho_2)\cos(\alpha+\theta+\phi+\beta)].$$

In the position of equilibrium $\theta=0$ and $\phi=0$.

Thus the equilibrium is stable or unstable according as $d^2z/d\theta^2$ is positive or negative for $\theta=\phi=0$,

i.e., according as $k(\rho_1+\rho_2)\cos(\alpha+\beta) >$ or $< \rho_1^2 \cos\alpha$.

But from the $\triangle A_1G_1O_1$, we have

$$h = A_1G_1 = A_1N - G_1N = A_1O_1\cos\alpha - O_1G_1 \cos\angle O_1G_1N$$
$$= \rho_1 \cos\alpha - k\cos(\alpha+\beta).$$
$$\therefore \quad k\cos(\alpha+\beta) = \rho_1\cos\alpha - h.$$

Hence the equilibrium is stable or unstable according as

$$(\rho_1+\rho_2)(\rho_1\cos\alpha - h) > \text{ or } < \rho_1^2\cos^2\alpha$$

i.e., $(\rho_1+\rho_2)\rho_1\cos\alpha - (\rho_1+\rho_2)h >$ or $< \rho_1^2\cos^2\alpha$

i.e., $(\rho_1+\rho_2)h <$ or $> (\rho_1+\rho_2)\rho_1\cos\alpha - \rho_1^2\cos^2\alpha$

i.e., $(\rho_1+\rho_2)h <$ or $> \rho_1\rho_2\cos\alpha$

i.e., $$h < \text{ or } > \frac{\rho_1\rho_2}{\rho_1+\rho_2}\cos\alpha.$$

Cor. If $\alpha=0$, the above conditions give that the equilibrium is stable or unstable according as

$$h < \text{ or } > \frac{\rho_1\rho_2}{\rho_1+\rho_2} \quad i.e., \quad \frac{1}{h} > \text{ or } < \frac{\rho_1+\rho_2}{\rho_1\rho_2}$$

or

$$\frac{1}{h} > \text{ or } < \frac{1}{\rho_1} + \frac{1}{\rho_2}.$$

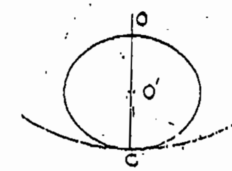
Thus suppose that a body rests in equilibrium upon another body which is fixed and the portions of the two bodies in contact have radii of curvatures ρ_1 and ρ_2 respectively. The C.G. of the first body is at a height h above the point of contact and the common normal coincides with the vertical. Then the equilibrium is stable or unstable according as

$$\frac{1}{h} > \text{ or } < \frac{1}{\rho_1} + \frac{1}{\rho_2}.$$

If the portions of the bodies in contact are spheres of radii r_1 and r_2, then in the above condition we put $\rho_1 = r_1$ and $\rho_2 = r_2$. Thus the equilibrium is stable or unstable according as

$$\frac{1}{h} > \text{ or } < \frac{1}{r_1} + \frac{1}{r_2}.$$

If the surface of the upper body at the point of contact is plane, then $\rho_1 = \infty$ and if the surface of the lower body at the point of contact is plane, then $\rho_2 = \infty$.

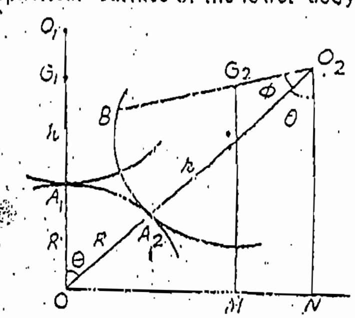


If the surface of the lower body at the point of contact instead of being convex is concave, then ρ_2 is to be taken with negative sign.

On account of its importance we shall now give an independent proof in case the surfaces in contact are spherical.

§ 8. *A body rests in equilibrium upon another fixed body, the portions of the two bodies in contact being spheres of radii r and R respectively and the straight line joining the centres of the spheres being vertical; if the first body be slightly displaced, to find whether the equilibrium is stable or unstable, the bodies being rough enough to prevent any sliding.* [Lucknow 76]

Let O be the centre of the spherical surface of the lower body which is fixed and O_1 that of the upper body which rests on the lower body, A_1 being their point of contact and the line OO_1 being vertical. If G_1 is the centre of gravity of the upper body, then for the equilibrium of the upper body, the line A_1G_1 must be vertical; let A_1G_1 be h. The figure is a section of the bodies by a vertical plane through G_1.



Suppose the upper body is slightly displaced by pure rolling over the lower body. Let A_2 be the new point of contact, O_2 is the new position of O_1 and the point A_1 of the upper body rolls up to the position B so that O_2B is the new position of O_1A_1. Also G_2 is the new position of G_1 so that $BG_2=A_1G_1=h$.

Let $\angle A_1OA_2=\theta$ and $\angle BO_2A_2=\phi$;

so that $\angle G_2O_2N=\theta+\phi$.

We have $O_1A_1=r$ and $OA_1=R$. Also $O_2A_2=O_2B=r$ and $OA_2=R$. Since the upper body rolls on the lower body without slipping, therefore

arc $A_1A_2=$ arc A_2B i.e., $R\theta=r\phi$ i.e., $\phi=(R/r)\,\theta$.

Now in order to find the nature of equilibrium, we should find the height z of the centre of gravity G_2 in the new position above the fixed horizontal line OX. We have

$$z=G_2M=O_2N-O_2G_2\cos(\theta+\phi)$$
$$=OO_2\cos\theta-(O_2B-BG_2)\cos(\theta+\phi)$$
$$=(R+r)\cos\theta-(r-h)\cos(\theta+\phi)$$
$$=(R+r)\cos\theta-(r-h)\cos\{\theta+(R/r)\,\theta\}\qquad[\because\ \phi=(R/r)\,\theta]$$
$$=(R+r)\cos\theta-(r-h)\cos\left\{\frac{\theta\,(r+R)}{r}\right\}.$$

For equilibrium, we have $dz/d\theta=0$

i.e., $$-(R+r)\sin\theta+(r-h)\sin\left\{\frac{\theta\,(r+R)}{r}\right\}\frac{r+R}{r}=0.$$

This is satisfied by $\theta=0$.

Now $$\frac{d^2z}{d\theta^2}=-(R+r)\cos\theta+(r-h)\cos\left\{\frac{\theta\,(r+R)}{r}\right\}\cdot\left(\frac{r+R}{r}\right)^2.$$

$$\therefore\quad\left(\frac{d^2z}{d\theta^2}\right)_{\theta=0}=-(R+r)+(r-h)\left(\frac{r+R}{r}\right)^2$$
$$=\left(\frac{r+R}{r}\right)^2\left\{(r-h)-\frac{r^2}{R+r}\right\}=\left(\frac{r+R}{r}\right)^2\left\{r-\frac{r^2}{R+r}-h\right\}$$
$$=\left(\frac{r+R}{r}\right)^2\left\{\frac{rR}{R+r}-h\right\}.$$

This will be positive if

$$\frac{rR}{R+r}>h\ \text{i.e.,}\ \frac{1}{h}>\frac{R+r}{rR}\ \text{i.e.,}\ \frac{1}{h}>\frac{1}{r}+\frac{1}{R}$$

and negative, if $\dfrac{rR}{R+r}<h$ i.e., $\dfrac{1}{h}<\dfrac{1}{r}+\dfrac{1}{R}$.

Hence the equilibrium is stable or unstable according as

$$\frac{1}{h}>\frac{1}{r}+\frac{1}{R}\quad\text{or}\quad\frac{1}{h}<\frac{1}{r}+\frac{1}{R}.$$

Here R is the radius of the lower body and r that of the upper body and h is the height of the C.G. of the upper body above the point of contact.

Now it remains to discuss the case when

$$1/h=1/r+1/R\quad\text{i.e.,}\quad h=rR/(R+r).$$

In this case $d^2z/d\theta^2=0$. Hence we find $d^3z/d\theta^3$ and $d^4z/d\theta^4$. We have

$$\frac{d^3z}{d\theta^3}=(R+r)\sin\theta-(r-h)\sin\left\{\frac{\theta\,(r+R)}{r}\right\}\cdot\left(\frac{r+R}{r}\right)^3,$$

and $$\frac{d^4z}{d\theta^4}=(R+r)\cos\theta-(r-h)\cos\left\{\frac{\theta\,(r+R)}{r}\right\}\cdot\left(\frac{r+R}{r}\right)^4.$$

Obviously $\left(\dfrac{d^3z}{d\theta^3}\right)_{\theta=0}=0.$

Also $$\left(\frac{d^4z}{d\theta^4}\right)_{\theta=0}=(R+r)-(r-h)\left(\frac{r+R}{r}\right)^4$$
$$=(R+r)\left\{1-\frac{r-h}{r}\left(\frac{r+R}{r}\right)^3\right\}$$
$$=(R+r)\left\{1-\frac{r-h}{r}\cdot\frac{R+r}{r}\cdot\left(\frac{R+r}{r}\right)^2\right\}$$
$$=(R+r)\left\{1-\left(r-\frac{rR}{R+r}\right)\cdot\frac{R+r}{r^2}\cdot\left(\frac{R+r}{r}\right)^2\right\}\qquad\left[\because\ h=\frac{rR}{r+R}\right]$$
$$=(R+r)\left\{1-\frac{r^2}{R+r}\cdot\frac{R+r}{r^2}\cdot\left(\frac{R+r}{r}\right)^2\right\}$$
$$=(R+r)\left\{1-\left(\frac{R+r}{r}\right)^2\right\}$$
$$=(R+r)\left\{1-\left(1+\frac{R}{r}\right)^2\right\},$$

which is negative.

This shows that z is maximum and so in this case the equilibrium is unstable.

Hence if $\dfrac{1}{h}>\dfrac{1}{r}+\dfrac{1}{R}$, then equilibrium is stable

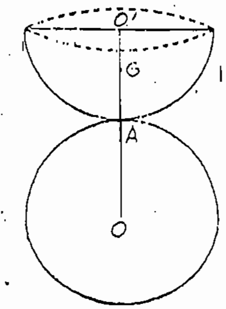
and if $\dfrac{1}{h}\leqslant\dfrac{1}{r}+\dfrac{1}{R}$, the equilibrium is unstable.

Remark. If the upper body has a plane face in contact with the lower body of radius R, then obviously $r=\infty$. And if the lower body be plane, then $R=\infty$.

Illustrative Examples

Ex. 1. *A hemisphere rests in equilibrium on a sphere of equal radius; show that the equilibrium is unstable when the curved, and*

stable when the flat surface of the hemisphere rests on the sphere.
[Meerut 79, 82, 83S]



Sol. (i) When the curved surface of the hemisphere rests on the sphere. A hemisphere of centre O' rests on a sphere of centre O with its curved surface in contact with the sphere. The point of contact is A and $OA = O'A = a$ (say). Also the line OAO' is vertical.

If G is the centre of gravity of the hemisphere, then G lies on $O'A$ and $O'G = \frac{3}{8}a$.

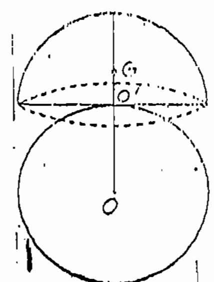
Here ρ_1 = the radius of curvature of the upper body at the point of contact = the radius of the hemisphere $= a$,
and ρ_2 = the radius of curvature of the lower body at the point of contact $= a$.

Also h = the height of the centre of gravity of the upper body above the point of contact A
$$= AG = O'A - O'G = a - \tfrac{3}{8}a = \tfrac{5}{8}a.$$

We have $\dfrac{1}{h} = \dfrac{1}{5a/8} = \dfrac{8}{5a}$,

and $\dfrac{1}{\rho_1} + \dfrac{1}{\rho_2} = \dfrac{1}{a} + \dfrac{1}{a} = \dfrac{2}{a} = \dfrac{10}{5a}$.

Thus $\dfrac{1}{h} < \dfrac{1}{\rho_1} + \dfrac{1}{\rho_2}$. Hence the equilibrium is unstable in this case.



(ii) When the flat surface of the hemisphere rests on the sphere. In this case a hemisphere of centre O' rests on a sphere of centre O and equal radius a with its flat surface (*i.e.*, the plane base) in contact with the sphere. The point of contact is O' and G is the C.G. of the hemisphere.

Here ρ_1 = the radius of curvature of the upper body at the point of contact $= \infty$,
[Note that the base of the hemisphere touches the sphere along a straight line]
and ρ_2 = the radius of curvature of the lower body at the point of contact = the radius of the sphere $= a$.

Also h = the height of the C.G. of the hemisphere above the point of contact $O' = O'G = \frac{3}{8}a$.

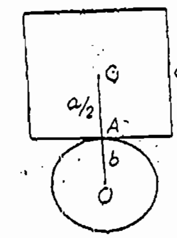
We have $\dfrac{1}{h} = \dfrac{1}{3a/8} = \dfrac{8}{3a}$,

and $\dfrac{1}{\rho_1} + \dfrac{1}{\rho_2} = \dfrac{1}{\infty} + \dfrac{1}{a} = 0 + \dfrac{1}{a} = \dfrac{1}{a} = \dfrac{3}{3a}$.

Obviously $\dfrac{1}{h} > \dfrac{1}{\rho_1} + \dfrac{1}{\rho_2}$. Hence in this case the equilibrium is stable.

Remark. Remember that for a straight line the radius of curvature at any point is infinity, and for a circle the radius of curvature at any point is equal to the radius of the circle.

Ex. 2. *A uniform cubical box of edge a is placed on the top of a fixed sphere, the centre of the face of the cube being in contact with the highest point of the sphere. What is the least radius of the sphere for which the equilibrium will be stable?* **[Meerut 72, 84S]**



Sol. A uniform cubical box of edge a is placed on the top of a fixed sphere of centre O. The point of contact is A. If G is the C.G. of the box, then for equilibrium the line OAG must be vertical. Let the radius of the sphere be b.

The figure shows the vertical section of the bodies through the point of contact A.

Here ρ_1 = the radius of curvature of the upper body at the point of contact $= \infty$,
and ρ_2 = the radius of curvature of the lower body at the point of contact $= b$.

Also h = the height of the C.G. of the box above the point of contact A = half the edge of the box $= \frac{1}{2}a$.

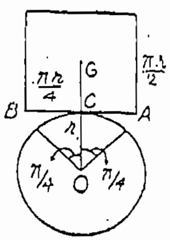
The equilibrium will be stable, if

$\dfrac{1}{h} > \dfrac{1}{\rho_1} + \dfrac{1}{\rho_2}$ *i.e.*, $\dfrac{1}{\frac{1}{2}a} > \dfrac{1}{\infty} + \dfrac{1}{b}$ *i.e.*, $\dfrac{2}{a} > \dfrac{1}{b}$ *i.e.*, $b > \frac{1}{2}a$.

Hence the least value of b for the equilibrium to be stable is $\frac{1}{2}a$.

Ex. 3. *A heavy uniform cube balances on the highest point of a sphere whose radius is r. If the sphere is rough enough to prevent sliding and if the side of the cube be $\pi r/2$, show that the cube can rock through a right angle without falling.* [Meerut 82(S)]

Sol. A heavy uniform cube balances on the highest point C of a sphere whose centre is O and radius r. The length of a side of the cube is $\pi r/2$. If G is the C.G. of the cube, then for equilibrium the line OCG must be vertical. In the figure we have shown a cross section of the bodies by a vertical plane through the point of contact C.



First we shall show that the equilibrium of the cube is stable.

Here ρ_1 = the radius of curvature of the upper body at the point of contact $C = \infty$,

and ρ_2 = the radius of curvature of the lower body at the point of contact $= r$.

Also h = the height of the centre of gravity G of the upper body above the point of contact C = half the edge of the cube $= \pi r/4$.

The equilibrium will be stable if

$$\frac{1}{h} > \frac{1}{\rho_1} + \frac{1}{\rho_2} \text{ i.e., } \frac{1}{\pi r/4} > \frac{1}{\infty} + \frac{1}{r}$$

i.e., $$\frac{4}{\pi r} > \frac{1}{r} \text{ i.e., } \frac{4}{\pi} > 1 \text{ i.e., } 4 > \pi$$

which is so because the value of π lies between 3 and 4.

Hence the equilibrium is stable. So if the cube is slightly displaced, it will tend to come back to its original position of equilibrium. During a swing to the right, the cube will not fall down till the right hand corner A of the lowest edge comes in contact with the sphere.

If θ is the angle through which the cube turns when the right hand corner A of the lowest edge comes in contact with the sphere, we have

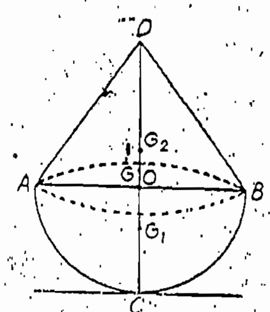
$$r\theta = \text{half the edge of the cube} = \pi r/4,$$

so that $$\theta = \pi/4.$$

Similarly the cube can turn through an angle $\pi/4$ to the left side on the sphere. Hence the total angle through which the cube can swing (or rock) without falling is $2 \cdot \frac{1}{4}\pi$ i.e., $\frac{1}{2}\pi$.

Ex. 4. *A body, consisting of a cone and a hemisphere on the same base, rests on a rough horizontal table the hemisphere being in contact with the table; show that the greatest height of the cone so that the equilibrium may be stable, is $\sqrt{3}$ times the radius of the hemisphere.* [Meerut 81]

Sol. AB is the common base of the hemisphere and the cone and COD is their common axis which must be vertical for equilibrium. The hemisphere touches the table at C.



Let H be the height OD of the cone and r be the radius OA or OC of the hemisphere. Let G_1 and G_2 be the centres of gravity of the hemisphere and the cone respectively. Then

$OG_1 = 3r/8$ and $OG_2 = H/4$.

If h be the height of the centre of gravity of the combined body composed of the hemisphere and the cone above the point of contact C, then using the formula $x = \frac{w_1x_1 + w_2x_2}{w_1 + w_2}$, we have

$$h = \frac{\frac{1}{3}\pi r^2 H \cdot CG_2 + \frac{2}{3}\pi r^3 \cdot CG_1}{\frac{1}{3}\pi r^2 H + \frac{2}{3}\pi r^3} = \frac{\frac{1}{3}\pi r^2 H(r + \frac{1}{4}H) + \frac{2}{3}\pi r^3 \cdot \frac{5}{8}r}{\frac{1}{3}\pi r^2 H + \frac{2}{3}\pi r^3}$$

$$= \frac{H(r + \frac{1}{4}H) + \frac{5}{4}r^2}{H + 2r}.$$

Here ρ_1 = the radius of curvature at the point of contact C of the upper body which is spherical $= r$,

and ρ_2 = the radius of curvature of the lower body at the point of contact $= \infty$.

$\therefore$ the equilibrium will be stable if

$$\frac{1}{h} > \frac{1}{\rho_1} + \frac{1}{\rho_2} \text{ i.e., } \frac{1}{h} > \frac{1}{r} + \frac{1}{\infty} \text{ i.e., } \frac{1}{h} > \frac{1}{r}$$

i.e., $$h < r$$

i.e., $$\frac{H(r + \frac{1}{4}H) + \frac{5}{4}r^2}{H + 2r} < r \text{ i.e., } Hr + \frac{1}{4}H^2 + \frac{5}{4}r^2 < Hr + 2r^2$$

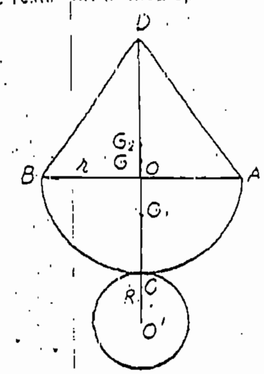
i.e., $$\frac{1}{4}H^2 < \frac{3}{4}r^2 \text{ i.e., } H^2 < 3r^2 \text{ i.e., } H < r\sqrt{3}.$$

Hence the greatest height of the cone consistent with the stable equilibrium of the body is $\sqrt{3}$ times the radius of the hemisphere.

Ex. 5. *A solid homogeneous hemisphere of radius r has a solid right circular cone of the same substance constructed on the base; the hemisphere rests on the convex side of the fixed sphere of radius R. Show that the length of the axis of the cone consistent with stability for a small rolling displacement is*

$$\frac{r}{R+r}\left[\sqrt{\{(3R+r)(R-r)\}} - 2r\right].$$

Sol. Let O be the centre of the common base AB of the hemisphere and the cone. The hemisphere rests on a fixed sphere of radius R and centre O', their point of contact being C. For equilibrium the line $O'COD$ must be vertical. Let H be the length of the axis OD of the cone. It is given that $OB=OC=r=$ the radius of the hemisphere.



If G_1 and G_2 are the centres of gravity of the hemisphere and the cone respectively, then

$OG_1=3r/8$ and $OG_2=H/4$.

Let G be the centre of gravity of the combined body composed of the hemisphere and the cone. If h be the height of G above the point of contact C, then

$$h=\frac{\frac{2}{3}\pi r^3 \cdot \frac{5}{8}r+\frac{1}{3}\pi r^2 H\cdot(r+\frac{1}{4}H)}{\frac{2}{3}\pi r^3+\frac{1}{3}\pi r^2 H}=\frac{H(r+\frac{1}{4}H)+\frac{5}{4}r^2}{H+2r}$$

Here $\rho_1=$ the radius of curvature at the point of contact C of the upper body $=r$.

and $\rho_2=$ the radius of curvature at C of the lower body $=R$.

The equilibrium will be stable if

$$\frac{1}{h}>\frac{1}{\rho_1}+\frac{1}{\rho_2}\quad i.e.,\ \frac{1}{h}>\frac{1}{r}+\frac{1}{R}$$

i.e., $$\frac{H+2r}{H(r+\frac{1}{4}H)+\frac{5}{4}r^2}>\frac{R+r}{rR}$$

i.e., $(R+r)\{Hr+\frac{1}{4}H^2+\frac{5}{4}r^2\}-rR(H+2r)<0$

i.e., $\frac{1}{4}H^2(R+r)+H\{(R+r)r-rR\}+\frac{5}{4}r^2(R+r)-2r^2R<0$

i.e., $H^2(R+r)+4r^2H+5r^3-3r^2R<0$

i.e., $H^2(R+r)+4r^2H-r^2(3R-5r)<0$

i.e., $$H^2+\frac{4r^2}{R+r}H-\frac{r^2(3R-5r)}{R+r}<0$$

i.e., $$\left(H+\frac{2r^2}{R+r}\right)^2-\frac{4r^4}{(R+r)^2}-\frac{r^2(3R-5r)}{R+r}<0$$

i.e., $$\left(H+\frac{2r^2}{R+r}\right)^2-\frac{4r^4+r^2(3R-5r)(R+r)}{(R+r)^2}<0$$

i.e., $$\left(H+\frac{2r^2}{R+r}\right)^2-\frac{r^2[4r^2+3R^2-2rR-5r^2]}{(R+r)^2}<0$$

i.e., $$\left(H+\frac{2r^2}{R+r}\right)^2-\frac{r^2(3R^2-2rR-r^2)}{(R+r)^2}<0$$

i.e., $$\left(H+\frac{2r^2}{R+r}\right)^2<\frac{r^2(3R+r)(R-r)}{(R+r)^2}$$

i.e., $$H+\frac{2r^2}{R+r}<\frac{r}{R+r}\sqrt{\{(3R+r)(R-r)\}}$$

i.e., $$H<\frac{r}{R+r}\sqrt{\{(3R+r)(R-r)\}}-\frac{2r^2}{R+r}$$

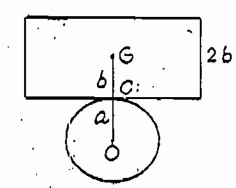
i.e., $$H<\frac{r}{R+r}[\sqrt{\{(3R+r)(R-r)\}}-2r].$$

Therefore the greatest value of H consistent with the stability of equilibrium is

$$\frac{r}{R+r}[\sqrt{\{(3R+r)(R-r)\}}-2r].$$

Ex. 6. *A uniform beam, of thickness 2b, rests symmetrically on a perfectly rough horizontal cylinder of radius a; show that the equilibrium of the beam will be stable or unstable according as b is less or greater than a.*

Sol. C is the point of contact of the beam and the cylinder and G is the centre of gravity of the beam. The figure shows the cross section of the bodies by a vertical plane through C. For equilibrium the line OCG is vertical.



Here $\rho_1=$ radius of curvature of the upper body at the point of contact $C=\infty$,

$\rho_2=$ radius of curvature of the lower body at $C=a$.

Also $h=$ the height of C.G. of the beam above the point of contact $C=\frac{1}{2}$ (thickness of the beam) $=\frac{1}{2}.2b=b$.

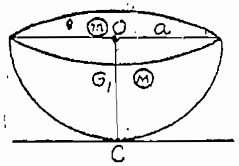
The equilibrium is stable or unstable according as

$$\frac{1}{h}>\text{ or }<\frac{1}{\rho_1}+\frac{1}{\rho_2}\quad i.e.,\ \frac{1}{h}>\text{ or }<\frac{1}{\infty}+\frac{1}{a}$$

i.e., $$\frac{1}{h}>\text{ or }<\frac{1}{a}$$

i.e., $h<$ or $>a$ *i.e.,* $b<$ or $>a$.

Ex. 7. (a). *A uniform solid hemisphere rests in equilibrium upon a rough horizontal plane with its curved surface in contact with the plane and a particle of mass m is fixed at the centre of the plane face. Show that for any value of m, the equilibrium is stable.*



Sol. C is the point of contact of the hemisphere and the plane and O is the centre of the base of the hemisphere. Let M be the mass of the hemisphere and a be its radius. A particle of mass m is placed at O. The mass M of the hemisphere acts at G_1 where $OG_1 = 3a/8$.

If h be the height of the centre of gravity of the combined body consisting of the hemisphere and the mass m above the point of contact C, then

$$h = \frac{M.\frac{5}{8}a + m.a}{M+m}$$

Here ρ_1 = the radius of curvature of the upper body at the point of contact $C = a$,

and ρ_2 = the radius of curvature of the lower body at the point of contact $C = \infty$.

The equilibrium will be stable if

$$\frac{1}{h} > \frac{1}{\rho_1} + \frac{1}{\rho_2} \text{ i.e., } \frac{1}{h} > \frac{1}{a} + \frac{1}{\infty} \text{ i.e., } \frac{1}{h} > \frac{1}{a} \text{ i.e., } h < a$$

i.e., $\dfrac{\frac{5}{8}aM + am}{M+m} < a$ i.e., $\frac{5}{8}aM + am < aM + am$

i.e., $\frac{5}{8}aM < aM$

i.e., $\frac{5}{8}a < a$, which is so whatever may be the value of m.

Hence for any value of m, the equilibrium is stable.

Ex. 7 (b). *A uniform hemisphere rests in equilibrium with its base upwards on the top of a sphere of double its radius. Show that the greatest weight which can be placed at the centre of the plane face without rendering the equilibrium unstable is one-eighth of the weight of the hemisphere.*

Sol. Draw figure yourself. Here a hemisphere rests on the top of a sphere. The base of the hemisphere is upwards. Let $2r$ be the radius of the sphere and r that of the hemisphere.

If W be the weight of the hemisphere and w be the weight placed at the centre of the base of the hemisphere, then

$$h = \frac{W.\frac{5}{8}r + w.r}{W+w}$$

Here $\rho_1 = r$ and $\rho_2 = 2r$. The equilibrium will be stable if

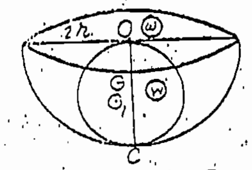
$$\frac{1}{h} > \frac{1}{r} + \frac{1}{2r} \text{ i.e., } \frac{1}{h} > \frac{3}{2r} \text{ i.e., } \frac{W+w}{\frac{5}{8}rW + wr} > \frac{3}{2r}$$

i.e., $2W + 2w > \frac{15}{8}W + 3w$ i.e., $\frac{1}{8}W > w$

i.e., $w < \frac{1}{8}W$,

which proves the required result.

Ex. 8 (a). *A solid sphere rests inside a fixed rough hemispherical bowl of twice its radius. Show that, however large a weight is attached to the highest point of the sphere, the equilibrium is stable.*



Sol. Let r be the radius of the solid sphere which rests inside a fixed rough hemispherical bowl of radius $2r$. Their point of contact is C and O is the highest point of the sphere so that $OC = 2r$. Let W and w be weights of the sphere and the weight attached to the highest point of the sphere. The weight W of the sphere acts at the middle point G_1 of its diameter OC.

If h is the height of the centre of gravity of the combined body consisting of the sphere and the weight w attached to O, then

$$h = \frac{W.r + w.2r}{W+w}$$

Here ρ_1 = the radius of curvature of the upper body at the point of contact C = the radius of the sphere $= r$,

and ρ_2 = the radius of curvature of the lower body at the point of contact $C = -2r$, the negative sign is taken because the surface of the lower fixed body i.e., the bowl at C is concave.

The equilibrium will be stable if

$$\frac{1}{h} > \frac{1}{\rho_1} + \frac{1}{\rho_2} \text{ i.e., } \frac{1}{h} > \frac{1}{r} - \frac{1}{2r} \text{ i.e., } \frac{1}{h} > \frac{1}{2r} \text{ i.e., } h < 2r$$

i.e., $\dfrac{Wr + 2wr}{W+w} < 2r$

i.e., $Wr + 2wr < 2Wr + 2wr$ i.e., $Wr < 2Wr$,

which is so whatever be the value of w.

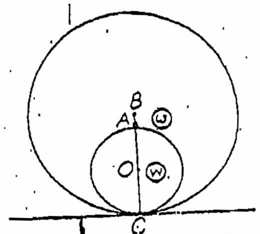
Hence, however large a weight is attached to the highest point of the sphere, the equilibrium is stable.

Ex. 8 (b). *A solid sphere rests inside a fixed rough hemispherical bowl of thrice its radius. Find the conditions and nature of equilibrium if a large weight is attached to the highest point of the sphere.* [Meerut 84, 85S]

Sol. Proceed exactly as in part (a). Equilibrium will be stable if weight of the sphere > weight attached.

Ex. 9. *A sphere of weight W and radius a lies within a fixed spherical shell of radius b, and a particle of weight w is fixed to the* *upper end of the vertical diameter. Prove that the equilibrium is stable if* $\dfrac{W}{w} > \dfrac{b-2a}{a}$.

Sol. C is the point of contact of the sphere and the spherical shell; O is the centre of the sphere, CA is the vertical diameter of the sphere, and B is the centre of the spherical shell. We have $OC=a$ and $BC=b$.



The weight W of the sphere acts at O and a particle of weight w is attached to A. If h be the height of the centre of gravity of the combined body consisting of the sphere and the weight w attached at A, then

$$h=\frac{W.a+w.2a}{W+w}=\frac{W+2w}{W+w}a.$$

Here $\rho_1=a$ and $\rho_2=-b$.

The equilibrium will be stable if

$$\frac{1}{h} > \frac{1}{\rho_1}+\frac{1}{\rho_2} \text{ i.e., } \frac{1}{h} > \frac{1}{a}-\frac{1}{b} \text{ i.e., } \frac{W+w}{a(W+2w)} > \frac{b-a}{ab}$$

i.e., $(W+w)\,ab > a\,(b-a)(W+2w)$

i.e., $(W+w)\,b > (b-a)(W+2w)$

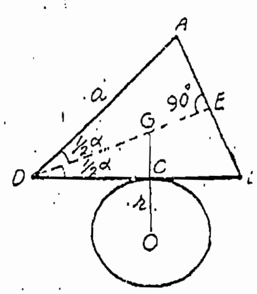
i.e., $W\{b-(b-a)\} > w\{2(b-a)-b\}$ i.e., $Wa > w\,(b-2a)$

i.e., $\dfrac{W}{w} > \dfrac{b-2a}{a}$.

Ex. 10. *A lamina in the form of an isosceles triangle, whose vertical angle is α, is placed on a sphere, of radius r, so that its plane is vertical and one of its equal sides is in contact with the sphere; show that, if the triangle be slightly displaced in its own plane, the equilibrium is stable if $\sin\alpha < 3r/a$, where a is one of the equal sides of the triangle.*

Sol. DAB is an isosceles triangular lamina in which

$DA=DB=a$ and $\angle ADB=\alpha$.



The centre of gravity G of the lamina lies on its median DE which is perpendicular to AB and also bisects the angle ADB. We have

$DG=\frac{2}{3}DE=\frac{2}{3}a\cos\frac{1}{2}\alpha$.

The lamina rests on a fixed sphere whose centre is O and radius r. Their point of contact is C. For equilibrium the line OCG must be vertical.

If h be the height of the C.G. of the lamina above the point of contact C, then

$$h=GC=DG\sin\tfrac{1}{2}\alpha=\tfrac{2}{3}a\cos\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\alpha=\tfrac{1}{3}a\sin\alpha.$$

Here $\rho_1=$ the radius of curvature of the upper body at the point of contact $C=\infty$,

and $\rho_2=$ the radius of curvature of the lower fixed body at the point $C=r$.

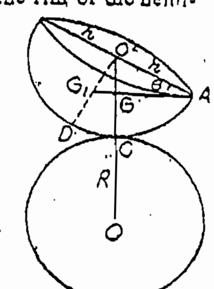
The equilibrium will be stable if

$$\frac{1}{h} > \frac{1}{\rho_1}+\frac{1}{\rho_2} \text{ i.e., } \frac{1}{h} > \frac{1}{\infty}+\frac{1}{r} \text{ i.e., } \frac{1}{h} > \frac{1}{r}$$

i.e., $h < r$ i.e., $\frac{1}{3}a\sin\alpha < r$ i.e., $\sin\alpha < 3r/a$.

Ex. 11. *A heavy hemispherical shell of radius r has a particle attached to a point on the rim, and rests with the curved surface in contact with a rough sphere of radius R at the highest point. Prove that if $R/r > \sqrt{5}-1$, the equilibrium is stable, whatever be the weight of the particle.* [Meerut 90P]

Sol. Let O' be the centre of the base of the hemispherical shell of radius r. Let a weight be attached to the rim of the hemispherical shell at A. The centre of gravity G_1 of the hemispherical shell is on its symmetrical radius $O'D$ and $O'G_1=\frac{1}{2}O'D=\frac{1}{2}r$.



Let G be the centre of gravity of the combined body consisting of the hemispherical shell and the weight at A. Then G lies on the line AG_1.

The hemispherical shell rests with its curved surface in contact with a rough sphere of radius R and centre O at the highest point C. For equilibrium the line $OCGO'$ must be vertical but AG_1 need not be horizontal.

Let $CG=h$. Also here $\rho_1=r$ and $\rho_2=R$.

The equilibrium will be stable if

$$\frac{1}{h} > \frac{1}{\rho_1}+\frac{1}{\rho_2} \text{ i.e., } \frac{1}{h} > \frac{1}{r}+\frac{1}{R} \text{ i.e., } \frac{1}{h} > \frac{R+r}{rR}$$

i.e., $$h < \frac{rR}{R+r}. \qquad \ldots(1)$$

The value of h depends on the weight of the particle attached at A. So the equilibrium will be stable, whatever be the weight of the particle attached at A, if the relation (1) holds even for the maximum value of h.

Now h will be maximum if $O'G$ is minimum *i.e.*, if $O'G$ is perpendicular to AG_1 or if $\triangle AO'G$ is right angled.

Let $\angle O'AG = \theta$. Then from right angled $\triangle AO'G_1$,

$$\tan\theta = \frac{O'G_1}{O'A} = \frac{\frac{1}{2}r}{r} = \frac{1}{2}. \quad \therefore \quad \sin\theta = \frac{1}{\sqrt{5}}.$$

$\therefore$ the minimum value of $O'G$

$$= O'A \sin\theta = r\,(1/\sqrt{5}) = r/\sqrt{5}.$$

$\therefore$ the maximum value of $h = r -$ the minimum value of $O'G$

$$= r - \frac{r}{\sqrt{5}} = \frac{r(\sqrt{5}-1)}{\sqrt{5}}.$$

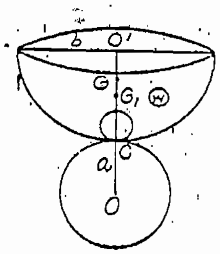
Hence the equilibrium will be stable, whatever be the weight of the particle at A, if

$$\frac{r(\sqrt{5}-1)}{\sqrt{5}} < \frac{rR}{R+r} \quad \textit{i.e., if} \quad \frac{\sqrt{5}-1}{\sqrt{5}} < \frac{R}{R+r}$$

i.e., if $(\sqrt{5}-1)\,R + (\sqrt{5}-1)\,r < R\sqrt{5}$

i.e., if $(\sqrt{5}-1)\,r < R$ *i.e.*, if $R/r > \sqrt{5}-1$.

Ex. 12. *A thin hemispherical bowl, of radius b and weight W rests in equilibrium on the highest point of a fixed sphere, of radius a, which is rough enough to prevent any sliding. Inside the bowl is placed a small smooth sphere of weight w; show that the equilibrium is not stable unless* $w < W\dfrac{a-b}{2b}$.



Sol. O is the centre, a the radius and C the highest point of the fixed sphere. A hemispherical bowl of radius b and weight W rests on the highest point C of this sphere and inside the bowl is placed a small smooth sphere of weight w. The weight W of the bowl acts at G_1 where $O'G_1 = \frac{1}{2}O'C$.

First we want to find out the height of the C.G. of the combined body consisting of the hemispherical bowl of weight W and sphere of weight w above the point of contact C. If the upper bowl be slightly displaced, the small smooth sphere placed inside it moves in such a way that the line of action of its weight w always passes through O', the centre of the base of the bowl. Hence so far as the question of the stability of the bowl is concerned the weight w of the small sphere may be taken to act at the centre O' of the bowl. If h be the height of the centre of gravity G of the combined body (*i.e.*, hemispherical shell of weight W and sphere of weight w) above the point of contact C, then

$$h = \frac{W \cdot \frac{1}{2}b + w \cdot b}{W+w} = \frac{(W+2w)\,b}{2\,(W+w)}.$$

Here $\rho_1 = b$ and $\rho_2 = a$. Hence the equilibrium will be stable if

$$\frac{1}{h} > \frac{1}{\rho_1} + \frac{1}{\rho_2} \;\textit{i.e.,}\; \frac{1}{h} > \frac{1}{b} + \frac{1}{a} \;\textit{i.e.,}\; \frac{1}{h} > \frac{a+b}{ab} \;\textit{i.e.,}\; h < \frac{ab}{a+b}$$

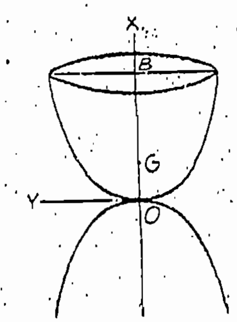
i.e., $\dfrac{(W+2w)\,b}{2\,(W+w)} < \dfrac{ab}{a+b}$ *i.e.*, $\dfrac{W+2w}{2\,(W+w)} < \dfrac{a}{a+b}$

i.e., $(a+b)\,(W+2w) < 2a\,(W+w)$

i.e., $w\,(2a+2b-2a) < W\,(2a-a-b)$

i.e., $2wb < W(a-b)$ *i.e.*, $w < \dfrac{W(a-b)}{2b}$.

Ex. 13. *A solid frustum of a paraboloid of revolution of height h and latus rectum $4a$, rests with its vertex on the vertex of a paraboloid of revolution, whose latus rectum is $4b$; show that the equilibrium is stable if* $h < \dfrac{3ab}{a+b}$.



Sol. The point of contact of the two bodies is O and $OB = h$.

Let the equation of the generating parabola of the upper paraboloid be

$$y^2 = 4ax.$$

The parabola $y^2 = 4ax$ passes through the origin and the y-axis is tangent at the origin. If ρ be the radius of curvature of this parabola at the origin, then by Newton's formula for the radius of curvature at the origin, we have

$$\rho = \lim_{x\to 0} \frac{y^2}{2x} = \lim_{x\to 0} \frac{4ax}{2x} = \lim_{x\to 0} 2a = 2a.$$

$\therefore$ the radius of curvature of the parabola $y^2 = 4ax$ at the vertex (*i.e.*, at the origin) is $2a$.

So here, $\rho_1 =$ the radius of curvature of the lower body at the point of contact $= 2a$,

and $\rho_2 =$ the radius of curvature of the upper body at the point of contact $= 2b$.

If H be the height of the centre of gravity G of the upper body above the point of contact O, then

$$H = OG = \bar{x} = \frac{\int x\,dm}{\int dm} = \frac{\int_0^h x\,\pi y^2\,dx}{\int_0^h \pi y^2\,dx}$$

$$=\frac{\int_0^h x.4ax\,dx}{\int_0^h 4ax\,dx}=\frac{\int_0^h x^2\,dx}{\int_0^h x\,dx}=\frac{\left[\frac{x^3}{3}\right]_0^h}{\left[\frac{x^2}{2}\right]_0^h}=\frac{\frac{h^3}{3}}{\frac{h^2}{2}}=\tfrac{2}{3}h.$$

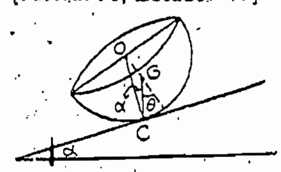
Now the equilibrium will be stable if

$$\frac{1}{H}>\frac{1}{\rho_1}+\frac{1}{\rho_2}\ \ i.e.,\ \frac{3}{2h}>\frac{1}{2a}+\frac{1}{2b}$$

i.e., $$\frac{3}{h}>\frac{a+b}{ab}\ i.e.,\ \frac{h}{3}<\frac{ab}{a+b}\ i.e.,\ h<\frac{3ab}{a+b}.$$

Ex. 14. *A solid hemisphere rests on a plane inclined to the horizon at an angle $\alpha < \sin^{-1}\frac{3}{8}$, and the plane is rough enough to prevent any sliding. Find the position of equilibrium and show that it is stable.* [Meerut 90; Lucknow 79]

Sol: Let O be the centre of the base of the hemisphere and r be its radius. If C is the point of contact of the hemisphere and the inclined plane, then $OC=r$. Let G be the centre of gravity of the hemisphere.



Then $OG=3r/8$. In the position of equilibrium the line CG must be vertical.

Since OC is perpendicular to the inclined plane and CG is perpendicular to the horizontal, therefore $\angle OCG=\alpha$. Suppose in equilibrium the axis of the hemisphere makes an angle θ with the vertical. From $\triangle OGC$, we have

$$\frac{OG}{\sin\alpha}=\frac{OC}{\sin\theta}\ i.e.,\ \frac{3r/8}{\sin\alpha}=\frac{r}{\sin\theta}.$$

$\therefore\ \sin\theta=\frac{8}{3}\sin\alpha$, or $\theta=\sin^{-1}(\frac{8}{3}\sin\alpha)$,

giving the position of equilibrium of the hemisphere.

Since $\sin\theta<1$, therefore $\frac{8}{3}\sin\alpha<1$

i.e., $\sin\alpha<\frac{3}{8}$ *i.e.*, $\alpha<\sin^{-1}\frac{3}{8}$.

Thus for the equilibrium to exist, we must have

$$\alpha<\sin^{-1}\tfrac{3}{8}.$$

Now let $CG=h$. Then

$$\frac{h}{\sin(\theta-\alpha)}=\frac{3r/8}{\sin\alpha},\quad \text{so that } h=\frac{3r\sin(\theta-\alpha)}{8\sin\alpha}.$$

Here $\rho_1=r$ and $\rho_2=\infty$.

The equilibrium will be stable if

$$h<\frac{\rho_1\rho_2\cos\alpha}{\rho_1+\rho_2}$$ [See § 7]

i.e., $\frac{1}{h}>\frac{\rho_1+\rho_2}{\rho_1\rho_2}\sec\alpha$ *i.e.*, $\frac{1}{h}>\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)\sec\alpha$

i.e., $\frac{1}{h}>\frac{1}{r}\sec\alpha$ $[\because\ \rho_1=r,\ \rho_2=\infty]$

i.e., $h<r\cos\alpha$

i.e., $\frac{3r\sin(\theta-\alpha)}{8\sin\alpha}<r\cos\alpha$ [substituting for h]

or, $3\sin(\theta-\alpha)<8\sin\alpha\cos\alpha$

or, $3\sin\theta\cos\alpha-3\cos\theta\sin\alpha<8\sin\alpha\cos\alpha$

or, $8\sin\alpha\cos\alpha-3\sin\alpha\sqrt{(1-\frac{64}{9}\sin^2\alpha)}<8\sin\alpha\cos\alpha$ $[\because\ \sin\theta=\frac{8}{3}\sin\alpha]$

or, $-\sin\alpha\sqrt{(9-64\sin^2\alpha)}<0$

or, $\sin\alpha\sqrt{(9-64\sin^2\alpha)}>0$. ...(2)

But from (1),

$\sin\alpha<\frac{3}{8}$ *i.e.*, $64\sin^2\alpha<9$ *i.e.*, $\sqrt{(9-64\sin^2\alpha)}$

is a positive real number. Therefore the relation (2) is true. Hence the equilibrium is stable.

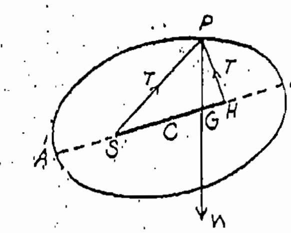
Ex. 15. *A rod SH, of length 2c and whose centre of gravity G is at a distance d from its centre, has a string, of length 2c sec α, tied to its two ends and the string is then slung over a small smooth peg P; find the position of equilibrium and show that the position which is not vertical is stable.*

Sol. We have

$SP+PH=$ the length of the string

$=2c\sec\alpha$,

as is given. The middle point of the rod SH is C and its centre of gravity is G such that $CG=d$.



Since in an ellipse the sum of the focal distances of any point on it is constant and is equal to the length $2a$ of its major axis, therefore the peg P must lie on an ellipse whose foci are S and H and for which the length of the major axis $2a=2c\sec\alpha$, so that

$$a=c\sec\alpha.$$

Now $SH=2c$ (given) and so $CH=c$. But $CH=ae$, where e is the eccentricity of this ellipse.

$\therefore\ ae=c$.

If b be the length of the semi minor axis of this ellipse, then

$$b^2=a^2(1-e^2)=a^2-a^2e^2=c^2\sec^2\alpha-c^2=c^2\tan^2\alpha,$$

Hence the equation of this ellipse with C as origin and CH as x-axis is

$$\frac{x^2}{c^2 \sec^2 \alpha} + \frac{y^2}{c^2 \tan^2 \alpha} = 1,$$

or $$x^2 \sin^2 \alpha + y^2 = c^2 \tan^2 \alpha.$$

Shifting the origin to the point G $(d, 0)$, it becomes

$$(x+d)^2 \sin^2 \alpha + y^2 = c^2 \tan^2 \alpha.$$

Changing to polar coordinates, it becomes

$$(r \cos \theta + d)^2 \sin^2 \alpha + r^2 \sin^2 \theta = c^2 \tan^2 \alpha \quad \ldots(1)$$

where G is the pole and GH is the initial line so that for the point P, $GP = r$ and $\angle PGH = \theta$.

If we find the value of θ for which r is maximum or minimum and regard the corresponding point P of the ellipse for the position of the peg and make PA vertical, we shall find the inclined position of equilibrium.

From (1),

$$r^2 \cos^2 \theta \sin^2 \alpha + 2rd \cos \theta \sin^2 \alpha + d^2 \sin^2 \alpha + r^2 - r^2 \cos^2 \theta = c^2 \tan^2 \alpha$$

or $r^2 \cos^2 \theta \cos^2 \alpha - 2rd \cos \theta \sin^2 \alpha + (c^2 \tan^2 \alpha - r^2 - d^2 \sin^2 \alpha) = 0.$

This is a quadratic in $\cos \theta$. Therefore

$$\cos \theta = \frac{2rd \sin^2 \alpha \pm \sqrt{[4r^2 d^2 \sin^4 \alpha - 4r^2 \cos^2 \alpha (c^2 \tan^2 \alpha - r^2 - d^2 \sin^2 \alpha)]}}{2r^2 \cos^2 \alpha}$$

$$= \frac{d \sin^2 \alpha \pm \sqrt{[d^2 \sin^4 \alpha - c^2 \sin^2 \alpha + r^2 \cos^2 \alpha + d^2 \sin^2 \alpha \cos^2 \alpha]}}{r \cos^2 \alpha}$$

$$= \frac{d \sin^2 \alpha \pm \sqrt{[r^2 \cos^2 \alpha - (c^2 - d^2) \sin^2 \alpha]}}{r \cos^2 \alpha}$$

For real values of $\cos \theta$, we must have

$$r^2 \cos^2 \alpha \geq (c^2 - d^2) \sin^2 \alpha \text{ i.e., } r^2 \geq (c^2 - d^2) \tan^2 \alpha.$$

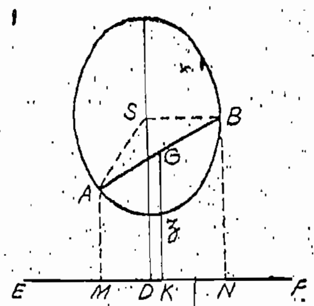
Therefore the least value of r is $\sqrt{(c^2 - d^2)} \tan \alpha$ and in that case

$$\cos \theta = \frac{d \sin^2 \alpha}{r \cos^2 \alpha} = \frac{d \sin^2 \alpha}{\sqrt{(c^2 - d^2)} \tan \alpha \cos^2 \alpha} = \frac{d \tan \alpha}{\sqrt{(c^2 - d^2)}}.$$

This gives the position of equilibrium in which the rod is not vertical. Since in this case r, the depth of the C.G. of the rod below the peg, is minimum, therefore the equilibrium is unstable.

The other two positions of equilibrium are when P is at A or A' *i.e.*, when the rod is vertical.

Ex. 16. *A smooth ellipse is fixed with its axis vertical and in it is placed a beam with its ends resting on the arc of the ellipse; if the length of the beam be not less than the latus rectum of the ellipse, show that when it is in stable equilibrium, it will pass through the focus.*



Sol. Let S be a focus and EF be the corresponding directrix of the ellipse. Referred to S as pole and the perpendicular SD from the focus to the directrix as the initial line, the polar equation of the ellipse is

$$l/r = 1 + e \cos \theta. \quad \ldots(1)$$

Let AB be the beam and G its middle point *i.e.*, its centre of gravity. Let z be the height of G above the fixed line EF. Then

$$z = GK = \tfrac{1}{2}(AM + BN).$$

But by the definition of the ellipse,

$$\frac{AS}{AM} = e \text{ and } \frac{BS}{BN} = e, \text{ so that } AM = \frac{1}{e} AS \text{ and } BN = \frac{1}{e} BS.$$

$$\therefore \quad z = \tfrac{1}{2}\left[\frac{AS}{e} + \frac{BS}{e}\right] = \frac{1}{2e}(AS + BS). \quad \ldots(2)$$

Now z will be minimum if $AS + BS$ is minimum *i.e.*, if A, S and B lie on the same straight line *i.e.*, if the beam AB passes through the focus S. But z is minimum implies that the equilibrium of the beam is stable. Hence the equilibrium of the beam is stable when it passes through the focus S.

In this case when the beam passes through the focus S, we have

$$AB = AS + BS$$

$$= \frac{l}{1 + e \cos \theta} + \frac{l}{1 + e \cos (\pi + \theta)}, \quad \text{by (1)}.$$

[Note that if the vectorial angle of B is θ then that of A is $\pi + \theta$]

$$= \frac{l}{1 + e \cos \theta} + \frac{l}{1 - e \cos \theta} = \frac{2l}{1 - e^2 \cos^2 \theta}.$$

$\therefore$ the length of the beam AB will be least when $1 - e^2 \cos^2 \theta$ is greatest *i.e.*, when $\cos \theta = 0$ or $\theta = \frac{1}{2}\pi$.

Then $AB = 2l =$ length of the latus rectum of the ellipse.

Therefore the least length of the beam is equal to the length of the latus rectum of the ellipse.

Problems based upon z-test

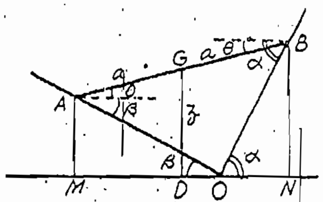
2007—15M

Ex. 17. *A uniform beam of length $2a$ rests with its ends on two smooth planes which intersect in a horizontal line. If the inclinations of the planes to the horizontal are α and β $(\alpha > \beta)$, show that the* *inclination θ of the beam to the horizontal in one of the equilibrium positions is given by*

$$\tan\theta = \tfrac{1}{2}(\cot\beta - \cot\alpha)$$

and show that the beam is unstable in this position. [Kanpur 80]

Sol. Let AB be a uniform beam of length $2a$ resting with its ends A and B on two smooth inclined planes OA and OB. Suppose the beam makes an angle θ with the horizontal. We have $\angle AOM = \beta$ and $\angle BON = \alpha$.



The centre of gravity of the beam AB is its middle point G. Let z be the height of G above the fixed horizontal line MN. We shall express z as a function of θ.

We have, $z = GD = \tfrac{1}{2}(AM + BN)$

$$= \tfrac{1}{2}(OA \sin\beta + OB \sin\alpha). \quad \ldots(1)$$

Now in the triangle OAB, $\angle OAB = \beta + \theta$, $\angle OBA = \alpha - \theta$ and $\angle AOB = \pi - (\alpha + \beta)$. Applying the sine theorem for the $\triangle OAB$, we have

$$\frac{OA}{\sin(\alpha-\theta)} = \frac{OB}{\sin(\beta+\theta)} = \frac{AB}{\sin[\pi-(\alpha+\beta)]} = \frac{2a}{\sin(\alpha+\beta)}$$

$$\therefore \quad OA = \frac{2a\sin(\alpha-\theta)}{\sin(\alpha+\beta)}, \quad OB = \frac{2a\sin(\beta+\theta)}{\sin(\alpha+\beta)}.$$

Substituting for OA and OB in (1), we have

$$z = \tfrac{1}{2}\left[\frac{2a\sin(\alpha-\theta)}{\sin(\alpha+\beta)}\sin\beta + \frac{2a\sin(\beta+\theta)}{\sin(\alpha+\beta)}\sin\alpha\right]$$

$$= \frac{a}{\sin(\alpha+\beta)}\left[\sin(\alpha-\theta)\sin\beta + \sin(\beta+\theta)\sin\alpha\right]$$

$$= \frac{a}{\sin(\alpha+\beta)}\left[(\sin\alpha\cos\theta - \cos\alpha\sin\theta)\sin\beta + (\sin\beta\cos\theta + \cos\beta\sin\theta)\sin\alpha\right]$$

$$= \frac{a}{\sin(\alpha+\beta)}\left[\sin\theta(\sin\alpha\cos\beta - \cos\alpha\sin\beta) + 2\cos\theta\sin\alpha\sin\beta\right].$$

$$\therefore \quad \frac{dz}{d\theta} = \frac{a}{\sin(\alpha+\beta)}\left[\cos\theta(\sin\alpha\cos\beta - \cos\alpha\sin\beta) - 2\sin\theta\sin\alpha\sin\beta\right] \quad \ldots(2)$$

For equilibrium of the beam, we have $\dfrac{dz}{d\theta} = 0$

i.e., $\cos\theta(\sin\alpha\cos\beta - \cos\alpha\sin\beta) - 2\sin\theta\sin\alpha\sin\beta = 0$

i.e., $2\sin\theta\sin\alpha\sin\beta = \cos\theta(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$

or $\dfrac{\sin\theta}{\cos\theta} = \tfrac{1}{2}\left(\dfrac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\sin\alpha\sin\beta}\right)$

or $\tan\theta = \tfrac{1}{2}(\cot\beta - \cot\alpha)$. $\quad \ldots(3)$

This gives the required position of equilibrium of the beam.

Differentiating (2), we have

$$\frac{d^2z}{d\theta^2} = \frac{a}{\sin(\alpha+\beta)}\left[-\sin\theta(\sin\alpha\cos\beta - \cos\alpha\sin\beta) - 2\cos\theta\sin\alpha\sin\beta\right]$$

$$= \frac{-2a\sin\alpha\sin\beta\cos\theta}{\sin(\alpha+\beta)}\left[\tfrac{1}{2}\tan\theta(\cot\beta - \cot\alpha) + 1\right]$$

$$= \frac{-2a\sin\alpha\sin\beta\cos\theta}{\sin(\alpha+\beta)}\left[\tan^2\theta + 1\right] \quad \text{[by (3)]}$$

= a negative quantity because θ, α and β are all acute angles and $\alpha + \beta < \pi$.

Thus in the position of equilibrium $d^2z/d\theta^2$ is negative *i.e.*, z is maximum. Hence the equilibrium is unstable.

Ex. 18. *A uniform heavy beam rests between two smooth planes, each inclined at an angle $\tfrac{1}{4}\pi$ to the horizontal, so that the beam is in a vertical plane perpendicular to the line of action of the planes. Show that the equilibrium is unstable when the beam is horizontal.*

Sol. Draw figure as in Ex. 17, taking $\alpha = \beta = \tfrac{1}{4}\pi$. If the beam makes an angle θ with the horizontal and z be the height of the C.G. of the beam above the fixed horizontal line MN, then proceeding as in Ex. 17, we have

$$z = \frac{a}{\sin\frac{1}{2}\pi}\left[\sin(\tfrac{1}{4}\pi - \theta)\sin\tfrac{1}{4}\pi + \sin(\tfrac{1}{4}\pi + \theta)\sin\tfrac{1}{4}\pi\right]$$

$$= a\left[\left(\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta\right)\frac{1}{\sqrt{2}} + \left(\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta\right)\frac{1}{\sqrt{2}}\right]$$

$= a\cos\theta.$

$\therefore \quad dz/d\theta = -a\sin\theta.$

For equilibrium of the beam, we have $dz/d\theta = 0$ *i.e.*, $\sin\theta = 0$ *i.e.*, $\theta = 0$.

$\therefore$ the beam rests in a horizontal position.

Now $d^2z/d\theta^2 = -a\cos\theta$.

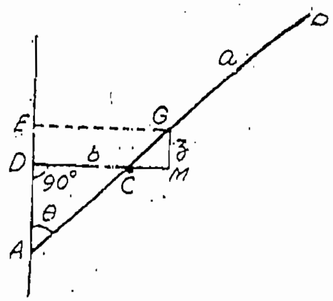
When $\theta = 0$, $d^2z/d\theta^2 = -a\cos 0 = -a$, which is negative.

Thus in the position of equilibrium $d^2z/d\theta^2$ is negative *i.e.*, z is maximum. Hence the equilibrium is unstable.

Ex. 19. *A heavy uniform rod rests with one end against a smooth vertical wall and with a point in its length resting on a smooth peg; find the position of equilibrium and show that it is unstable.*

[Lucknow 81; Meerut 80, 82, 84P, 85, 85P, 86S, 87S, 92]

Sol. Let AB be a uniform rod of length $2a$. The end A of the rod rests against a smooth vertical wall and the rod rests on a smooth peg C whose distance from the wall is say b i.e., $CD=b$.



Suppose the rod makes an angle θ with the wall. The centre of gravity of the rod is at its middle point G. Let z be the height of G above the fixed peg C i.e., $GM=z$. We shall express z in terms of θ. We have

$z=GM=ED=AE-AD$
$=AG\cos\theta-CD\cot\theta$
$=a\cos\theta-b\cot\theta$.

$\therefore\quad dz/d\theta=-a\sin\theta+b\operatorname{cosec}^2\theta$

and $\quad d^2z/d\theta^2=-a\cos\theta-2b\operatorname{cosec}^2\theta\cot\theta$.

For equilibrium of the rod, we have $dz/d\theta=0$

i.e., $\quad -a\sin\theta+b\operatorname{cosec}^2\theta=0$

or $\quad a\sin\theta=b\operatorname{cosec}^2\theta, \quad$ or $\quad \sin^3\theta=b/a$

or $\quad \sin\theta=(b/a)^{1/3}, \quad$ or $\quad \theta=\sin^{-1}(b/a)^{1/3}$.

This gives the position of equilibrium of the rod.

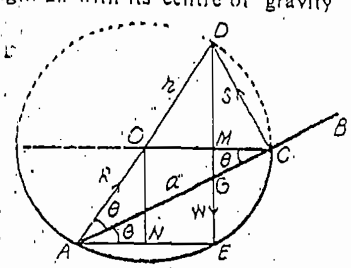
Again $d^2z/d\theta^2=-(a\cos\theta+2b\operatorname{cosec}^2\theta\cot\theta)$
$=-$negative for all acute values of θ.

Thus $d^2z/d\theta^2$ is negative in the position of equilibrium and so z is maximum. Hence the equilibrium is unstable.

Ex. 20. *A heavy uniform rod, length 2a, rests partly within and partly without a fixed smooth hemispherical bowl of radius r, the rim of the bowl is horizontal, and one point of the rod is in contact with the rim; if θ be the inclination of the rod to the horizon, show that* $2r\cos 2\theta=a\cos\theta$.

Show also that the equilibrium of the rod is stable.

Sol. Let AB be the rod of length $2a$ with its centre of gravity at G. A point C of its length is in contact with the rim of the bowl of radius r and centre O.



The rod is in equilibrium under the action of three forces. The reaction R of the bowl at A is along the normal AO and the reaction S of the rim at C is perpendicular to the rod. Let these reactions meet in a point D. Since the line AOD passes through the centre O of the bowl and $\angle ACD$ is a right angle, therefore AOD is a diameter of the sphere of which the bowl is a part.

The third force on the rod is its weight W acting vertically downwards through its middle point G. Since the three forces must be concurrent, therefore the line DG is vertical.

Suppose the line DG meets the surface of the bowl at the point E. Join AE; then AE is horizontal because $\angle AED=90°$, being the angle in a semi-circle.

We have $\angle BAE=\theta=\angle ACO \qquad [\because AE$ is parallel to $OC]$
$\qquad=\angle OAC. \qquad [\because OA=OC]$.

$\therefore\quad \angle DAE=2\theta$.

Suppose z is the depth of the centre of gravity G of the rod below the fixed horizontal line OC. Then

$$z=MG=ME-GE=ON-GE$$
$$=OA\sin 2\theta-AG\sin\theta=r\sin 2\theta-a\sin\theta.$$

$\therefore\quad dz/d\theta=2r\cos 2\theta-a\cos\theta$.

For the equilibrium of the rod, we must have $dz/d\theta=0$

i.e., $\quad 2r\cos 2\theta-a\cos\theta=0\quad$ i.e., $2r\cos 2\theta=a\cos\theta$.

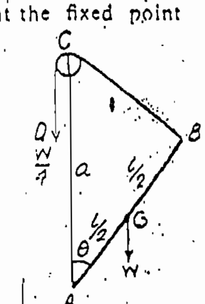
This gives the position of equilibrium of the rod.

Again $\quad d^2z/d\theta^2=-4r\sin 2\theta+a\sin\theta$
$=-2(2r\sin 2\theta)+a\sin\theta$
$=-2.DE+GE$, which is negative because $DE>GE$.

Thus the depth z of the C.G. of the rod below a fixed horizontal line is maximum. Hence the equilibrium is stable.

Ex. 21. *One end A of a uniform rod AE of weight W and length l is smoothly hinged at a fixed point, while B is tied to a light string which passes over a small smooth pulley at a distance a vertically above A and carries a weight W/4. If* $l<a<2l$, *show that the system is in stable equilibrium when AB is vertically upwards, and that there is also a configuration of equilibrium in which the rod is at a certain angle to the vertical.*

Sol. Let AB be the rod of length l hinged at the fixed point A. The weight W of the rod acts through its middle point G. Let h be the length of the string BCD which is attached to E and passes over a smooth pulley at C, AC being vertical and equal to a. The string carries a weight $W/4$ at its other end D. Let $\angle BAC=\theta$.



From $\triangle BAC$,

$BC=\sqrt{(AB^2+AC^2-2AB.AC.\cos\theta)}$
$=\sqrt{(l^2+a^2-2la\cos\theta)}$.

$\therefore$ the length of the portion CD of the string hanging vertically

$=b-BC=b-\sqrt{(l^2+a^2-2la\cos\theta)}.$

The weight W acts at the point G whose height above the fixed point A is $AG\cos\theta$ i.e., $\frac{1}{2}l\cos\theta$. The weight $W/4$ acts at D whose height above A is $a-b+\sqrt{(l^2+a^2-2la\cos\theta)}$.

Hence if z be the height, above the fixed point A, of the centre of gravity of the system consisting of the weight W and $W/4$, then

$$(W+\tfrac{1}{4}W)\,z=W.\tfrac{1}{2}l\cos\theta+\tfrac{1}{4}W(a-b+\sqrt{(l^2+a^2-2la\cos\theta)})$$

i.e., $5z=2l\cos\theta+a-b+\sqrt{(l^2+a^2-2la\cos\theta)}$.

$$\therefore\quad 5\frac{dz}{d\theta}=-2l\sin\theta+\frac{al\sin\theta}{\sqrt{(l^2+a^2-2la\cos\theta)}}$$

and $$5\frac{d^2z}{d\theta^2}=-2l\cos\theta+\frac{al\cos\theta}{\sqrt{(l^2+a^2-2la\cos\theta)}}-\frac{a^2l^2\sin^2\theta}{(l^2+a^2-2la\cos\theta)^{3/2}}$$

For the equilibrium of the system we must have $dz/d\theta=0$. Obviously $dz/d\theta$ vanishes when $\sin\theta=0$ i.e., $\theta=0$ i.e., the rod AB is vertically upwards. Thus the system is in equilibrium when the rod AB is vertically upwards.

For $\theta=0$, we have $$5\frac{d^2z}{d\theta^2}=-2l+\frac{al}{\sqrt{(l^2+a^2-2la)}}$$

$$=-2l+\frac{al}{a-l},\text{ if } a>l$$

$$=-l\frac{(a-2l)}{a-l},$$

which is positive if $l<a<2l$.

Thus if $l<a<2l$, then for $\theta=0$, $d^2z/d\theta^2$ is positive i.e., z is minimum. Hence this is a stable position of equilibrium.

Again $dz/d\theta$ also vanishes when

$$-2+\frac{a}{\sqrt{(l^2+a^2-2l\,a\cos\theta)}}=0\quad\text{or}\quad 4=\frac{a^2}{(l^2+a^2-2l\,a\cos\theta)}$$

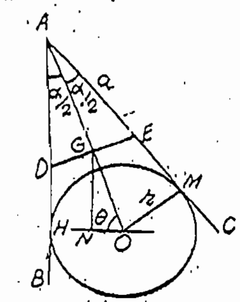
or, $\quad 4l^2+4a^2-8la\cos\theta=a^2$

or $\quad \cos\theta=\dfrac{3a^2+4l^2}{8la}$, which gives a real value of θ when $l<a<2l$.

So there is also a configuration of equilibrium in which the rod is inclined to the vertical.

Ex. 22. *Two equal uniform rods are firmly jointed at one end so that the angle between them is α, and they rest in a vertical plane on a smooth sphere of radius r. Show that they are in a stable or unstable equilibrium according as the length of the rod is $>$ or $<$ $4r$ cosec α.* [Lucknow 80]

Sol. Let AB and AC be two rods jointed at A and placed in a vertical plane on a smooth sphere of centre O and radius r. We have $\angle BAC=\alpha$. Since the rods are tangential to the sphere, therefore $\angle BAO=\angle CAO=\frac{1}{2}\alpha$.



Suppose $AB=AC=2a$.

If D and E are the middle points of the rods AB and AC, then the combined C.G. of the rods is at the middle point G of ED which must be on AO. Suppose the rod AC touches the sphere at M. We have,

$$OM=r,\ AE=a;\ \angle AMO=90^\circ,\ \angle AGE=90^\circ.$$

Suppose AO makes an angle θ with the horizontal line OH through the fixed point O. Let z be the height of the C.G. of the system above the horizontal through O. Then

$$z=GN=OG\sin\theta=(AO-AG)\sin\theta$$
$$=(r\operatorname{cosec}\tfrac{1}{2}\alpha-a\cos\tfrac{1}{2}\alpha)\sin\theta.$$

$$\therefore\ dz/d\theta=(r\operatorname{cosec}\tfrac{1}{2}\alpha-a\cos\tfrac{1}{2}\alpha)\cos\theta.$$

For the equilibrium of the rods, we must have $dz/d\theta=0$ i.e., $(r\operatorname{cosec}\frac{1}{2}\alpha-a\cos\frac{1}{2}\alpha)\cos\theta=0$ i.e., $\cos\theta=0$ i.e., $\theta=\frac{1}{2}\pi$.

Thus in the position of equilibrium of rods, the line AO must be vertical.

Also $$d^2z/d\theta^2=-(r\operatorname{cosec}\tfrac{1}{2}\alpha-a\cos\tfrac{1}{2}\alpha)\sin\theta$$
$$=-r\operatorname{cosec}\tfrac{1}{2}\alpha+a\cos\tfrac{1}{2}\alpha,\text{ for }\theta=\tfrac{1}{2}\pi.$$

The equilibrium will be stable or unstable according as the height z of the C.G. of the system is minimum or maximum in the position of equilibrium,

i.e., according as $d^2z/d\theta^2$ is positive or negative at $\theta=\frac{1}{2}\pi$

i.e., according as $a\cos\frac{1}{2}\alpha >$ or $< r\operatorname{cosec}\frac{1}{2}\alpha$.

i.e., according as $2a >$ or $< \dfrac{2r}{\cos\frac{1}{2}\alpha\sin\frac{1}{2}\alpha}$

i.e., according as $2a >$ or $< \dfrac{4r}{\sin\alpha}$

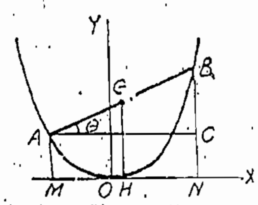
i.e., according as $2a >$ or $< 4r$ cosec α.

Ex. 23. *A uniform rod, of length $2l$, is attached by smooth rings at both ends of a parabolic wire, fixed with its axis vertical and vertex downwards, and of latus rectum $4a$. Show that the angle θ which the rod makes with the horizontal in a slanting position of* *equilibrium is given by* $\cos^2\theta=2a/l$, *and that, if these positions exist they are stable.*

Show also that the positions in which the rod is horizontal are stable or unstable according as the rod is below or above the focus. [Meerut 80]

Sol. Let AB be the rod of length $2l$. Take OX and OY as coordinate axes, so that the equation of the parabola be written as

$$x^2=4ay.$$



Let the coordinates of the point A be $(2at, at^2)$ and let the rod AB make an angle θ with the horizontal AC. Then the coordinates of B are $(2at+2l\cos\theta,\ at^2+2l\sin\theta)$. Since B lies on the parabola $x^2=4ay$, therefore

$$(2at+2l\cos\theta)^2=4a\,(at^2+2l\sin\theta)$$

or $\quad 8atl\cos\theta+4l^2\cos^2\theta=8al\sin\theta$

or $\quad (2al\cos\theta)\,t=2al\sin\theta-l^2\cos^2\theta$

or $\quad t=\tan\theta-(l/2a)\cos\theta.$ ...(1)

The centre of gravity of the rod AB is at its middle point G. If z be the height of G above the fixed horizontal line OX, then

$$z=GH=\tfrac{1}{2}\,(AM+BN)$$
$$=\tfrac{1}{2}\,[at^2+(at^2+2l\sin\theta)]=at^2+l\sin\theta$$
$$=a\,[\tan\theta-(l/2a)\cos\theta]^2+l\sin\theta \qquad \text{[from (1)]}$$
$$=(l^2/4a)\cos^2\theta+a\tan^2\theta=(1/4a)\,[l^2\cos^2\theta+4a^2\tan^2\theta].$$

$\therefore\quad dz/d\theta=(1/4a)\,[-2l^2\cos\theta\sin\theta+8a^2\tan\theta\sec^2\theta]$

$=(1/2a)\sin\theta\,[-l^2\cos\theta+4a^2\sec^3\theta].$

For the equilibrium of the rod, we must have $dz/d\theta=0$

i.e. $\quad (1/2a)\sin\theta\,(-l^2\cos\theta+4a^2\sec^3\theta)=0.$

$\therefore\quad$ either $\sin\theta=0\ $ *i.e.*, $\ \theta=0$, which gives the horizontal position of rest of the rod

or $\quad -l^2\cos\theta+4a^2\sec^3\theta=0$ *i.e.* $l^2\cos\theta=4a^2/\cos^3\theta$

i.e., $\quad \cos^4\theta=4a^2/l^2$ *i.e.*, $\cos^2\theta=2a/l$, which gives the inclined position of rest of the rod.

Now, $\quad d^2z/d\theta^2=(1/2a)\cos\theta\,[-l^2\cos\theta+4a^2\sec^3\theta]$
$\qquad +(1/2a)\sin\theta\,[l^2\sin\theta+12a^2\sec^3\theta\tan\theta].$ ...(2)

When $\cos^2\theta=2a/l$ *i.e.*, when $-l^2\cos\theta+4a^2\sec^3\theta=0$, we have

$$d^2z/d\theta^2=(1/2a)\sin\theta\,[l^2\sin\theta+12a^2\sec^3\theta\tan\theta],$$
$$=(1/2a)\sin^2\theta\,[l^2+12a^2\sec^4\theta], \text{ which is } >0.$$

Hence in the inclined position of rest of the rod, z is minimum and so the equilibrium is stable.

Again when the rod is horizontal *i.e.*, $\theta=0$, we have, from (2)

$$\frac{d^2z}{d\theta^2}=\frac{8a^2-2l^2}{4a}=\frac{4a^2-l^2}{2a}.$$

The equilibrium in this case is stable or unstable according as $d^2z/d\theta^2$ is positive or negative

i.e., $\quad$ according as $4a^2-l^2>$ or <0

i.e., $\quad$ according as $2a>$ or $<l$

i.e., $\quad$ according as $2l<$ or $>4a$

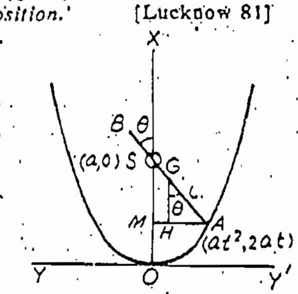
i.e., $\quad$ according as the rod is below or above the focus.

Ex. 24. *A uniform smooth rod passes through a ring at the focus of a fixed parabola whose axis is vertical and vertex below the focus, and rests with one end on the parabola. Prove that the rod will be in equilibrium if it makes with the vertical an angle* θ *given by the equation*

$$\cos^4\tfrac{1}{2}\theta=a/2c$$

where 4a is the latus rectum and 2c the length of the rod. Investigate also the stability of equilibrium in this position. [Lucknow 81]

Sol. Let the equation of the parabola be $\quad y^2=4ax.$



Let AB be the rod of length $2c$ with its end A on the parabola and passing through a ring at the focus S. Let the coordinates of A be $(at^2, 2at)$ and the coordinates of the focus S are $(a, 0)$. If the rod AB makes an angle θ with the vertical OX, then

$\tan\theta=$ the gradient of the line AB

$$=\frac{2at-0}{at^2-a}=\frac{2t}{t^2-1}=\frac{-2t}{1-t^2}.$$

$$\therefore\quad \frac{2\tan\frac{1}{2}\theta}{1-\tan^2\frac{1}{2}\theta}=\frac{2\,(-t)}{1-(-t)^2},\ \text{ or }\ \tan\tfrac{1}{2}\theta=-t.$$

Let z be the height of the centre of gravity G of the rod AB above the fixed horizontal line YOY'. Then

$$z=OM+HG=OM+AG\cos\theta$$
$$=at^2+c\cos\theta$$
$$[\because\ OM=x\text{-coordinate of } A \text{ and } AG=\tfrac{1}{2}AB]$$
$$=a\tan^2\tfrac{1}{2}\theta+c\cos\theta.$$

$$\therefore\quad dz/d\theta=2a\,(\tan\tfrac{1}{2}\theta\sec^2\tfrac{1}{2}\theta)\cdot\tfrac{1}{2}-c\sin\theta$$

$$=a \tan \tfrac{1}{2}\theta \sec^2 \tfrac{1}{2}\theta - c.2 \sin \tfrac{1}{2}\theta \cos \tfrac{1}{2}\theta$$
$$=\sin \tfrac{1}{2}\theta\,[a \sec^3 \tfrac{1}{2}\theta - 2c \cos \tfrac{1}{2}\theta].$$

For the equilibrium of the rod, we must have $dz/d\theta=0$

i.e., $\sin \tfrac{1}{2}\theta\,(a \sec^3 \tfrac{1}{2}\theta - 2c \cos \tfrac{1}{2}\theta)=0$.

$\therefore$ either $\sin \tfrac{1}{2}\theta=0$ *i.e.*, $\theta=0$,

which gives the vertical position of equilibrium,

or $a \sec^3 \tfrac{1}{2}\theta - 2c \cos \tfrac{1}{2}\theta=0$ *i.e.*, $a \sec^3 \tfrac{1}{2}\theta = 2c \cos \tfrac{1}{2}\theta$

i.e., $\cos^4 \tfrac{1}{2}\theta = a/2c$, which gives the inclined position of rest of the rod.

Now

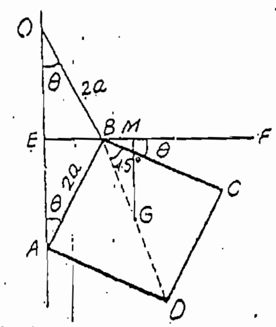
$$\frac{d^2z}{d\theta^2}=\tfrac{1}{2}\cos \tfrac{1}{2}\theta\,[a \sec^3 \tfrac{1}{2}\theta - 2c \cos \tfrac{1}{2}\theta]$$
$$+\sin \tfrac{1}{2}\theta\left[\frac{3a}{2}\sec^3 \tfrac{1}{2}\theta \tan \tfrac{1}{2}\theta + c \sin \tfrac{1}{2}\theta\right]$$
$$=\tfrac{1}{2}\cos \tfrac{1}{2}\theta\,[a \sec^3 \tfrac{1}{2}\theta - 2c \cos \tfrac{1}{2}\theta]+\sin^2 \tfrac{1}{2}\theta\,[\tfrac{3}{2}a \sec^4 \tfrac{1}{2}\theta + c],$$

which is >0 when $\cos^4 \tfrac{1}{2}\theta = a/2c$

i.e., when $a \sec^3 \tfrac{1}{2}\theta - 2c \cos \tfrac{1}{2}\theta = 0$.

Thus in the inclined position of equilibrium of the rod, $d^2z/d\theta^2$ is positive *i.e.*, z is minimum. Hence the equilibrium is stable in the inclined position of rest of the rod.

Ex. 25. *A square lamina rests with its plane perpendicular to a smooth wall one corner being attached to a point in the wall by a fine string of length equal to the side of the square. Find the position of equilibrium and show that it is stable.*



Sol. $ABCD$ is a square lamina of side $2a$. It is suspended from the point O in the wall by a fine string OB of length $2a$. The corner A of the lamina touches the wall and the plane of the lamina is perpendicular to the wall.

Let $\angle BAO=\theta$.

Then $\angle AOB=\angle BAO=\theta$.

$[\because AB=OB]$

Since BC is perpendicular to AB and the horizontal line EF is perpendicular to AO, therefore $\angle FBC=\theta$.

The centre of gravity of the lamina is the middle point G of the diagonal BD. We have

$$BG=\tfrac{1}{2}BD=\tfrac{1}{2}.2a\sqrt{2}=a\sqrt{2},$$

$\therefore \angle CBD=45°$ and $\angle FBG=45°+\theta$.

If z be the depth of G below the fixed point O, then

$$z=OE+MG=2a \cos\theta + BG \sin(45°+\theta)$$
$$=2a\cos\theta + a\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos\theta+\frac{1}{\sqrt{2}}\sin\theta\right)$$
$$=3a\cos\theta + a\sin\theta.$$

$\therefore$ $dz/d\theta=-3a\sin\theta + a\cos\theta$.

For equilibrium, $dz/d\theta=0$

i.e., $-3a\sin\theta+a\cos\theta=0$, *i.e.*, $\tan\theta=\tfrac{1}{3}$.

This gives the position of equilibrium *i.e.*, in equilibrium the side AB of the lamina makes an angle $\tan^{-1}\tfrac{1}{3}$ with the wall.

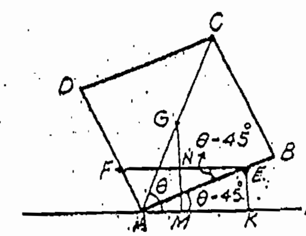
Now $d^2z/d\theta^2=-3a\cos\theta - a\sin\theta$

$$=-a\left(3\times\frac{3}{\sqrt{10}}-\frac{1}{\sqrt{10}}\right), \text{ when } \tan\theta=\tfrac{1}{3}$$

$=$ a negative number.

Thus in the position of equilibrium the depth z of the C.G. of the lamina below the fixed point O is maximum. Hence the equilibrium is stable.

Ex. 26. *A square lamina rests in a vertical plane on two smooth pegs which are in the same horizontal line. Show that there is only one position of equilibrium unless the distance between the pegs is greater than one-quarter of the diagonal of the square, but that if this condition is satisfied, there may be three positions of equilibrium and that the symmetrical position will be stable, but the other two positions of equilibrium will be unstable.*



Sol. $ABCD$ is a square lamina resting on the pegs E and F which are in the same horizontal line. Let $EF=c$ and $AC=2d$. Suppose the diagonal AC makes an angle θ with the horizontal AH. Then

$\angle EAK=\theta-\angle CAB=\theta-45°$.

The C.G. of the lamina is the middle point G of the diagonal AC.

Let z be the height of G above the fixed line EF.

Then $z=GN=GM-NM=GM-EK$

$=AG\sin\theta - AE\sin(\theta-45°)$

$=d\sin\theta - EF\cos(\theta-45°)\sin(\theta-45°)$

$=d\sin\theta-\tfrac{1}{2}c\sin 2(\theta-45°)$

$=d\sin\theta-\tfrac{1}{2}c\sin(2\theta-90°)$

$=d \sin\theta+\frac{1}{2}c \sin(90°-2\theta)=d \sin\theta+\frac{1}{2}c \cos 2\theta.$

$\therefore \quad dz/d\theta=d\cos\theta-c \sin 2\theta.$

For equilibrium,

$dz/d\theta=0$ *i.e.*, $d\cos\theta-c\sin 2\theta=0$

i.e., $d\cos\theta-2c\sin\theta\cos\theta=0$ *i.e.*, $\cos\theta\,(d-2c\sin\theta)=0.$

$\therefore \quad \cos\theta=0$ *i.e.*, $\theta=\frac{1}{2}\pi,$

or $d-2c\sin\theta=0$ *i.e.*, $\sin\theta=d/2c$ *i.e.*, $\theta=\sin^{-1}(d/2c).$

In the position of equilibrium given by $\theta=\frac{1}{2}\pi$, the diagonal AC is vertical and the square rests symmetrically on the pegs.

In the position of equilibrium given by $\theta=\sin^{-1}(d/2c)$, if $d/2c<1$, the diagonal AC is not vertical but is inclined at some angle to the vertical. So it gives inclined position of equilibrium.

But we know that $\sin\theta=\sin(\pi-\theta)$.

Hence we shall have two inclined positions of equilibrium given by

$\theta=\sin^{-1}(d/2c)$ and $\theta=\pi-\sin^{-1}(d/2c).$

The inclined position of equilibrium is possible only when

$d/2c<1$ [$\because \sin\theta<1$ for inclined position]

i.e., when $d<2c$, *i.e.*, when $c>\frac{1}{2}d$ *i.e.*, when $c>\frac{1}{4}.(2d)$

i.e., when the distance between the pegs $>\frac{1}{4}$ (length of the diagonal).

Thus there is only one position of equilibrium (*i.e.*, the symmetrical position) unless the distance between the pegs is greater than one-quarter of the diagonal of the square. Also if $2c>d$, there are three positions of equilibrium.

To determine the nature of equilibrium when $2c>d$.

We have,

$d^2z/d\theta^2=-d\sin\theta-2c\cos 2\theta$

$=-d\sin\theta-2c\,(1-2\sin^2\theta)=-d\sin\theta-2c+4c\sin^2\theta.$

For the symmetrical position of equilibrium $\theta=\frac{1}{2}\pi$,

$d^2z/d\theta^2=-d-2c+4c=2c-d>0,$ because $2c>d$.

$\therefore \quad d^2z/d\theta^2$ is positive when $\theta=\frac{1}{2}\pi$ and so z is minimum for $\theta=\frac{1}{2}\pi$. Hence the symmetrical position of equilibrium given by $\theta=\frac{1}{2}\pi$ is stable.

For the inclined position of equilibrium given by $\sin\theta=d/2c$, we have

$$\frac{d^2z}{d\theta^2}=-d.\frac{d}{2c}-2c+4c.\frac{d^2}{4c^2}=-\frac{d^2}{2c}+\frac{d^2}{c}-2c$$

$$=\frac{d^2-4c^2}{2c}<0, \text{ because } 2c>d.$$

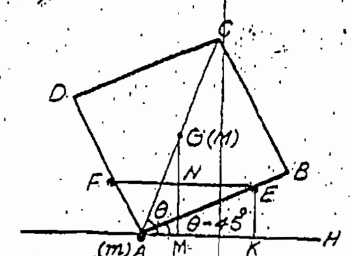
$\therefore \quad d^2z/d\theta^2$ is negative when $\sin\theta=d/2c$ and so z is maximum for the inclined positions of equilibrium. Hence the inclined positions of equilibrium are unstable.

Remark. When $2c<d$, there is only one position of equilibrium *i.e.*, the symmetrical position of equilibrium. For this position of equilibrium, $d^2z/d\theta^2=2c-d$ which is <0, because $2c<d$. Hence z is maximum and the equilibrium is unstable.

Ex. 27. *A uniform square board of mass M is supported in a vertical plane on two smooth pegs on the same horizontal level. The distance between the pegs is a and the diagonal of the square is D, where $D>4a$. If one diagonal is vertical and a mass m is attached to its lower end, prove that the equilibrium is stable, if*

$$4am>M.(D-4a).$$

Sol. $ABCD$ is a square board resting on the pegs E and F which are in the same horizontal line.



We have

$EF=a$ and $AC=D$.

The mass M of the lamina acts at the middle point G of AC and there is a mass m attached at A. Suppose the diagonal AC makes an angle θ with the horizontal AH. Then

$$\angle EAK=\theta-45°=\angle FEA.$$

The height of G (*i.e.*, the point where M acts) above EF

$=GN=GM-NM=GM-EK=AG\sin\theta-AE\sin(\theta-45°)$

$=\frac{1}{2}D\sin\theta-EF\cos(\theta-45°).\sin(\theta-45°)$

$=\frac{1}{2}D\sin\theta-\frac{1}{2}a\sin 2(\theta-45°)=\frac{1}{2}D\sin\theta-\frac{1}{2}a\sin(2\theta-90°)$

$=\frac{1}{2}D\sin\theta+\frac{1}{2}a\sin(90°-2\theta)=\frac{1}{2}D\sin\theta+\frac{1}{2}a\cos 2\theta.$

Also the depth of A (*i.e.*, the point where m acts) below EF

$=EK=AE\sin(\theta-45°)=EF\cos(\theta-45°).\sin(\theta-45°)$

$=\frac{1}{2}a\sin(2\theta-90°)=-\frac{1}{2}a\cos 2\theta.$

Let z be the height of C.G. of the system consisting of the masses M and m above the fixed line EF. Then

$$z=\frac{M\,(\frac{1}{2}D\sin\theta+\frac{1}{2}a\cos 2\theta)+m\,[-(-\frac{1}{2}a\cos 2\theta)]}{M+m}.$$

$$=\frac{\frac{1}{2}MD\sin\theta+(M+m)\cdot\frac{1}{2}a\cos 2\theta}{M+m}.$$

$$\therefore\quad \frac{dz}{d\theta}=\frac{1}{M+m}\left\{\tfrac{1}{2}MD\cos\theta-a(M+m)\sin 2\theta\right\}.$$

For equilibrium, $dz/d\theta=0$,

i.e., $\quad \frac{1}{2}MD\cos\theta-2a(M+m)\sin\theta\cos\theta=0$

i.e., $\quad \cos\theta\,[\frac{1}{2}MD-2a(M+m)\sin\theta]=0.$

$\therefore$ either $\cos\theta=0$ *i.e.*, $\theta=\frac{1}{2}\pi$,

or $\quad \frac{1}{2}MD-2a(M+m)\sin\theta=0$

i.e. $\quad \sin\theta=MD/\{4a(M+m)\}.$

Now $\theta=\frac{1}{2}\pi$ means the diagonal AC is vertical.

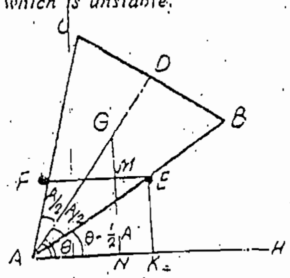
We have $\quad \dfrac{d^2z}{d\theta^2}=\dfrac{1}{M+m}\{-\frac{1}{2}MD\sin\theta-2a(M+m)\cos 2\theta\}$

$$=\frac{1}{M+m}\{-\tfrac{1}{2}MD+2a(M+m)\},\ \text{for }\theta=\tfrac{1}{2}\pi.$$

The equilibrium is stable at $\theta=\frac{1}{2}\pi$ if z is minimum at $\theta=\frac{1}{2}\pi$ *i.e.*, if $d^2z/d\theta^2$ is positive at $\theta=\frac{1}{2}\pi$ *i.e.*, if $-\frac{1}{2}MD-2a(M+m)>0$ or, $4am>MD-4aM$ or, $4am>M(D-4a)$.

Ex. 28 (a). *A uniform isosceles triangular lamina ABC rests in equilibrium with its equal sides AB and AC in contact with two smooth pegs in the same horizontal line at a distance c apart. If the perpendicular AD upon BC is h, show that there are three positions of equilibrium, of which the one with AD vertical is stable and the other two are unstable, if $h<3c$ cosec A; whilst if $h\geqslant 3c$ cosec A, there is only one position of equilibrium, which is unstable.*

Sol. ABC is an isosceles triangular lamina resting on two smooth pegs E and F which are in the same horizontal line and $EF=c$. The perpendicular AD from A upon BC is of length h. We have



$$\angle BAD=\angle CAD=\tfrac{1}{2}A.$$

The weight of the lamina acts at its centre of gravity G, where

$$AG=\tfrac{2}{3}AD=\tfrac{2}{3}h.$$

Suppose AD makes an angle θ with the horizontal AH, so that

$$\angle BAH=\theta-\tfrac{1}{2}A.$$

Let z be the height of G above the fixed horizontal line EF. Then

$$z=GM=GN-MN=GN-EK=AG\sin\theta-AE\sin(\theta-\tfrac{1}{2}A).$$

$$z=\tfrac{2}{3}h\sin\theta-AE\sin(\theta-\tfrac{1}{2}A). \qquad \ldots(1)$$

Since EF is parallel to AK, therefore

$$\angle FEA=\angle EAK=\theta-\tfrac{1}{2}A.$$

Now in the $\triangle AEF$, we have

$$\angle EFA=\pi-(A+(\theta-\tfrac{1}{2}A))=\pi-(\theta+\tfrac{1}{2}A).$$

Applying the sine theorem of trigonometry for the $\triangle AEF$, we have

$$\frac{AE}{\sin\angle EFA}=\frac{EF}{\sin\angle FAE}$$

i.e., $$\frac{AE}{\sin\{\pi-(\theta+\frac{1}{2}A)\}}=\frac{c}{\sin A}.$$

$$\therefore\quad AE=\frac{c}{\sin A}\sin(\theta+\tfrac{1}{2}A).$$

Substituting this value of AE in (1), we have

$$z=\tfrac{2}{3}h\sin\theta-\frac{c}{\sin A}\sin(\theta+\tfrac{1}{2}A)\sin(\theta-\tfrac{1}{2}A)$$

$$=\tfrac{2}{3}h\sin\theta-\frac{c}{2\sin A}[\cos A-\cos 2\theta]$$

$$=\tfrac{2}{3}h\sin\theta-\frac{c}{2}\cot A+\frac{c}{2\sin A}\cos 2\theta.$$

$$\therefore\quad \frac{dz}{d\theta}=\tfrac{2}{3}h\cos\theta-\frac{c}{\sin A}\sin 2\theta. \qquad \ldots(2)$$

For equilibrium, $\quad dz/d\theta=0$

i.e., $\quad \frac{2}{3}h\cos\theta-\dfrac{2c}{\sin A}\sin\theta\cos\theta=0$

i.e., $\quad 2\cos\theta\left[\frac{1}{3}h-\dfrac{c\sin\theta}{\sin A}\right]=0.$

$\therefore$ either $\cos\theta=0$ *i.e.* $\theta=\frac{1}{2}\pi$

or $\quad \frac{1}{3}h-\dfrac{c\sin\theta}{\sin A}=0$ *i.e.* $\sin\theta=\dfrac{h\sin A}{3c}=\dfrac{h}{3c\operatorname{cosec}A}.$

Now $\theta=\frac{1}{2}\pi$ gives the position of equilibrium in which AD is vertical and the triangle rests symmetrically on the pegs. The values of θ given by $\quad \sin\theta=h/(3c\operatorname{cosec}A)$

are real and not equal to $\frac{1}{2}\pi$ if $h<3c$ cosec A. Since

$$\sin(\pi-\theta)=\sin\theta,$$

therefore if $h<3c$ cosec A, the equation $\sin\theta=h/(3c\operatorname{cosec}A)$ gives two inclined positions of equilibrium, one θ and the other $\pi-\theta$. Thus if $h<3c$ cosec A, there are three positions of equilibrium, one symmetrical and the other two inclined.

If $h \geqslant 3c \operatorname{cosec} A$, then the equation $\sin\theta = h/(3c \operatorname{cosec} A)$ either gives no real value of θ or the value of θ given by it is also equal to $\frac{1}{2}\pi$. Thus in this case the symmetrical position of equilibrium, $\theta=\frac{1}{2}\pi$, is the only position of equilibrium.

Nature of equilibrium.

From (2), $$\frac{d^2z}{d\theta^2} = -\tfrac{2}{3}h \sin\theta - \frac{2c}{\sin A}\cos 2\theta. \quad \ldots(3)$$

For $\theta=\frac{1}{2}\pi$, $$\frac{d^2z}{d\theta^2} = -\tfrac{2}{3}h + \frac{2c}{\sin A} = \tfrac{2}{3}(-h+3c \operatorname{cosec} A),$$

which is positive or negative according as

$$h < \text{or} > 3c \operatorname{cosec} A.$$

Thus for $\theta=\frac{1}{2}\pi$, z is minimum or maximum according as

$$h < \text{or} > 3c \operatorname{cosec} A.$$

Hence for $\theta=\frac{1}{2}\pi$, the equilibrium is stable or unstable according as

$$h < \text{or} > 3c \operatorname{cosec} A.$$

For $\theta=\frac{1}{2}\pi$, $d^2z/d\theta^2=0$ when $h=3c \operatorname{cosec} A$. In this case we can see that $d^3z/d\theta^3=0$ and $d^4z/d\theta^4=-6c \operatorname{cosec} A$, which is negative. So in this case z is maximum and the equilibrium is unstable. Thus the symmetrical position of equilibrium is stable or unstable according as

$$h < \text{or} \geqslant 3c \operatorname{cosec} A.$$

Now we consider the inclined positions of equilibrium. From (3), we can write

$$\frac{d^2z}{d\theta^2} = -\tfrac{2}{3}h \sin\theta - \frac{2c}{\sin A}(1-2\sin^2\theta). \quad \ldots(4)$$

For the inclined positions of equilibrium, $\sin\theta = (h \sin A)/3c$. Putting $\sin\theta = (h\sin A)/3c$ in (4), we get

$$\frac{d^2z}{d\theta^2} = -\frac{2h}{3}\cdot\frac{h\sin A}{3c} - \frac{2c}{\sin A} + \frac{4c}{\sin A}\cdot\frac{h^2\sin^2 A}{9c^2}$$
$$= \frac{2h^2}{9c}\sin A - \frac{2c}{\sin A} = \frac{2}{9c}\sin A\,(h^2 - 9c^2 \operatorname{cosec}^2 A),$$

which is negative since for inclined positions of equilibrium

$$h < 3c \operatorname{cosec} A.$$

Thus for the inclined positions of equilibrium, z is maximum and so they are positions of unstable equilibrium.

Remark. For inclined positions of equilibrium to exist, we must have $h<3c \operatorname{cosec} A$. For these positions of equilibrium, θ is given by $\sin\theta = (h\sin A)/3c$.

Now $\frac{1}{2}A < \theta \Rightarrow \sin\frac{1}{2}A < \sin\theta \Rightarrow \sin\frac{1}{2}A < (h\sin A)/3c$

$$\Rightarrow \sin\tfrac{1}{2}A < \frac{2h\sin\frac{1}{2}A\cos\frac{1}{2}A}{3c} \Rightarrow h > \tfrac{3}{2}c\sec\tfrac{1}{2}A.$$

Thus for inclined positions of equilibrium, we must have

$$\tfrac{3}{2}c\sec\tfrac{1}{2}A < h < 3c \operatorname{cosec} A.$$

Ex. 28. (b) *An isosceles triangular lamina of an angle 2α and height h rests between two smooth pegs at the same level, distant $2c$ apart; prove that if*

$$3c\sec\alpha < h < 6c \operatorname{cosec} 2\alpha,$$

the oblique positions of equilibrium exist, which are unstable. Discuss the stability of the vertical position.

Sol. Proceed as in Ex. 28 (a). The complete question has been solved there.

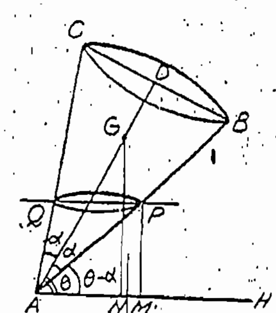
Ex. 29 (a). *A smooth solid right circular cone, of height h and vertical angle 2α, is at rest with its axis vertical in a horizontal circular hole of radius a. Show that if $16a > 3h\sin 2\alpha$, the equilibrium is stable, and there are two other positions of unstable equilibrium; and that if $16a<3h\sin 2\alpha$, the equilibrium is unstable, and the position in which the axis is vertical is the only position of equilibrium.*

Sol. ABC is a solid right circular cone whose height AD is h and vertical angle BAC is 2α. It rests in a horizontal circular hole PQ of radius a, so that $PQ=2a$. We have

$$\angle BAD = \angle CAD = \alpha.$$

The weight of the cone acts at its centre of gravity G, where

$$AG = \tfrac{3}{4}AD = \tfrac{3}{4}h.$$



Suppose AD makes an angle θ with the horizontal AH, so that

$$\angle BAH = \theta - \alpha.$$

Let z be the height of G above the fixed horizontal line PQ. Then

$$z = GN - PM$$
$$= AG\sin\theta - AP\sin(\theta-\alpha)$$
$$= \tfrac{3}{4}h\sin\theta - AP\sin(\theta-\alpha). \quad \ldots(1)$$

Since PQ is parallel to AM, therefore

$$\angle QPA = \angle PAM = \theta - \alpha.$$

Now in the $\triangle APQ$, we have

$$\angle PQA = \pi - \{2\alpha + (\theta-\alpha)\} = \pi - (\theta+\alpha).$$

Applying the sine theorem of trigonometry for the ΔAPQ, we have

$$\frac{AP}{\sin\{\pi-(\theta+\alpha)\}}=\frac{PQ}{\sin 2\alpha}.$$

$\therefore\ AP=\dfrac{2a}{\sin 2\alpha}\sin(\theta+\alpha)$, because $PQ=2a$.

Putting the value of AP in (1), we have

$$z=\tfrac{3}{4}h\sin\theta-\frac{2a\sin(\theta+\alpha)}{\sin 2\alpha}\sin(\theta-\alpha)$$

$$=\tfrac{3}{4}h\sin\theta-\frac{a}{\sin 2\alpha}[\cos 2\alpha-\cos 2\theta]$$

$$=\tfrac{3}{4}h\sin\theta-a\cot 2\alpha+\frac{a}{\sin 2\alpha}\cos 2\theta.$$

$$\therefore\ \frac{dz}{d\theta}=\tfrac{3}{4}h\cos\theta-\frac{2a}{\sin 2\alpha}\sin 2\theta. \qquad \ldots(2)$$

For equilibrium, $dz/d\theta=0$

i.e., $\tfrac{3}{4}h\cos\theta-\dfrac{4a}{\sin 2\alpha}\sin\theta\cos\theta=0$

i.e., $\cos\theta\left[\tfrac{3}{4}h-\dfrac{4a\sin\theta}{\sin 2\alpha}\right]=0$.

$\therefore$ either $\cos\theta=0$ *i.e.,* $\theta=\tfrac{1}{2}\pi$,

or $\tfrac{3}{4}h-\dfrac{4a\sin\theta}{\sin 2\alpha}=0$ *i.e.,* $\sin\theta=\dfrac{3h\sin 2\alpha}{16a}$.

Now $\theta=\tfrac{1}{2}\pi$ gives the position of equilibrium in which the axis AD of the cone is vertical. The values of θ given by

$$\sin\theta=(3h\sin 2\alpha)/16a$$

are real and not equal to $\tfrac{1}{2}\pi$ if $\sin\theta<1$ *i.e.,* if $16a>3h\sin 2\alpha$. Since $\sin(\pi-\theta)=\sin\theta$, therefore if $16a>3h\sin 2\alpha$, the equation

$$\sin\theta=(3h\sin 2\alpha)/16a$$

gives two oblique positions of equilibrium one θ and the other $\pi-\theta$. Thus if $16a>3h\sin 2\alpha$, there are three positions of equilibrium, one in which the axis AD is vertical and the other two inclined.

If $16a<3h\sin 2\alpha$, the equation

$$\sin\theta=(3h\sin 2\alpha)/16a$$

gives no real value of θ. Thus in this case the only position of equilibrium is that in which the axis of the cone is vertical.

Nature of equilibrium

From (2), $\dfrac{d^2z}{d\theta^2}=-\tfrac{3}{4}h\sin\theta-\dfrac{4a}{\sin 2\alpha}\cos 2\theta. \qquad \ldots(3)$

For $\theta=\tfrac{1}{2}\pi$;

$$\frac{d^2z}{d\theta^2}=-\tfrac{3}{4}h+\frac{4a}{\sin 2\alpha}=\frac{1}{4\sin 2\alpha}[-3h\sin 2\alpha+16a],$$

which is positive or negative according as

$$16a>\text{ or }<3h\sin 2\alpha.$$

Thus for $\theta=\tfrac{1}{2}\pi$, z is minimum or maximum according as

$$16a>\text{ or }<3h\sin 2\alpha.$$

Hence the vertical position of equilibrium is stable or unstable according as $16a>$ or $<3h\sin 2\alpha$.

Now we consider the inclined positions of equilibrium given by $\qquad \sin\theta=(3h\sin 2\alpha)/16a$.

These exist only if $16a>3h\sin 2\alpha$. From (3), we can write

$$\frac{d^2z}{d\theta^2}=-\tfrac{3}{4}h\sin\theta-\frac{4a}{\sin 2\alpha}(1-2\sin^2\theta).$$

Putting $\sin\theta=(3h\sin 2\alpha)/16a$ in it, we get

$$\frac{d^2z}{d\theta^2}=-\tfrac{3}{4}h.\frac{3h\sin 2\alpha}{16a}-\frac{4a}{\sin 2\alpha}+\frac{8a}{\sin 2\alpha}.\frac{9h^2\sin^2 2\alpha}{256a^2}$$

$$=\frac{9h^2}{64a}\sin 2\alpha-\frac{4a}{\sin 2\alpha}=\frac{9h^2\sin^2 2\alpha-256a^2}{64a\sin 2\alpha}=\frac{(3h\sin 2\alpha)^2-(16a)^2}{64a\sin 2\alpha}$$

which is negative since for inclined positions of equilibrium

$$16a>3h\sin 2\alpha.$$

Thus for the inclined positions of equilibrium, z is maximum and so they are positions of unstable equilibrium.

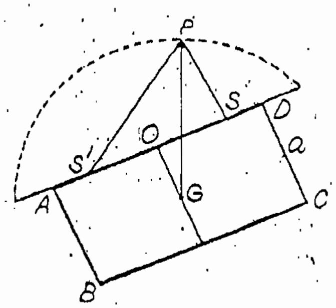
Ex. 29. (b) *A smooth cone is placed with vertex downwards in a circular horizontal hole. Prove that the position of equilibrium with the axis vertical is unstable or stable according as it is, or, is not, the only possible position of equilibrium.*

Sol. Proceed as in Ex. 29 (a). Also take help from Ex. 28.

Ex. 30. (a) *A rectangular picture hangs in a vertical position by means of a string, of length l, which after passing over a smooth nail has its ends attached to two points symmetrically situated in the upper edge of the picture at a distance c apart. If the height of the picture is a, show that there is no position of equilibrium in which a side of the picture is inclined to the horizon if $la>c\sqrt{(c^2+a^2)}$, whilst if $\qquad la<c\sqrt{(c^2+a^2)}$, there are two such positions which are both stable.*

Show also that in the latter case the position in which the side is vertical is stable for some and unstable for other displacements.

Sol. $ABCD$ is a rectangular picture which hangs by means of a string of length l passing over the peg P, the ends of the string being attached to two points S and S' symmetrically situated in the upper edge AD of the picture such that $SS'=c$. If O is the middle point of AD, then O is also the middle point of SS' because S and S' are symmetrically situated in AD. Therefore



$$OS=OS'=\tfrac{1}{2}c.$$

If G be the centre of gravity of the picture, then $OG=\frac{1}{2}a$, as height CD of the picture is given to be a.

We have, $\quad SP+S'P=l.$ ...(1)

From the relation (1), it is obvious that P lies on an ellipse whose foci are S and S' and the length say 2α, of whose major axis is l, so that $\alpha=\frac{1}{2}l$.

We have $OS=\alpha e$, where e is the eccentricity of the ellipse.

$$\therefore \quad \alpha e=\tfrac{1}{2}c.$$

If β be the semi major axis of the ellipse, then

$\beta^2=\alpha^2-\alpha^2e^2=\frac{1}{4}l^2-\frac{1}{4}c^2=\frac{1}{4}(l^2-c^2)$, so that $\beta=\frac{1}{2}\sqrt{(l^2-c^2)}$.

The centre of the ellipse is the middle point O of SS'. Take O as origin, OS as x-axis and a line perpendicular to OS through O as y-axis. Then the coordinates of G are $(0, -\frac{1}{2}a)$. Let the coordinates of P be $(\alpha\cos\theta, \beta\sin\theta)$.

Since the line PG is vertical, therefore if z be the depth of G below the fixed point P, then $z=PG$.

Now z is maximum or minimum according as z^2 or PG^2 is maximum or minimum.

Let $\quad u=PG^2=(\alpha\cos\theta-0)^2+(\beta\sin\theta+\frac{1}{2}a)^2$

$$=\alpha^2\cos^2\theta+\beta^2\sin^2\theta+a\beta\sin\theta+\tfrac{1}{4}a^2.$$

$$\therefore \quad \frac{du}{d\theta}=2(\beta^2-\alpha^2)\sin\theta\cos\theta+a\beta\cos\theta.$$

For equilibrium,

$$dz/d\theta=0 \text{ i.e., } du/d\theta=0,$$

i.e., $\quad \cos\theta\,[2(\beta^2-\alpha^2)\sin\theta+a\beta]=0.$

$\therefore$ either $\cos\theta=0$ i.e., $\theta=\frac{1}{2}\pi$,

or $\quad \sin\theta=\dfrac{a\beta}{2(\alpha^2-\beta^2)}=\dfrac{a\cdot\frac{1}{2}\sqrt{(l^2-c^2)}}{2[\frac{1}{4}l^2-\frac{1}{4}(l^2-c^2)]}=\dfrac{a\sqrt{(l^2-c^2)}}{c^2}$, ...(2)

after substituting the values of α and β.

Here, $\theta=\frac{1}{2}\pi$ gives the position of equilibrium, symmetrical about the peg P, in which the sides AB and CD of the picture hang vertically.

There is no inclined position of equilibrium if the value of $\sin\theta$ given by (2) is >1,

i.e., if $\quad a\sqrt{(l^2-c^2)}>c^2$, i.e., if $a^2l^2-a^2c^2>c^4$

i.e., if $\quad a^2l^2>c^2(c^2+a^2)$ i.e., if $al>c\sqrt{(a^2+c^2)}$.

Thus if $al>c\sqrt{(a^2+c^2)}$, there is no position of equilibrium in which a side of the picture is inclined to the horizon. In this case the symmetrical position $\theta=\frac{1}{2}\pi$ is the only position of equilibrium.

But if the value of $\sin\theta$ given by (2) is <1,

i.e., $\quad a\sqrt{(l^2-c^2)}<c^2$, or $al<c\sqrt{(a^2+c^2)}$,

then (2) gives real values of θ. Since $\sin\theta=\sin(\pi-\theta)$, therefore when $al<c\sqrt{(a^2+c^2)}$, we have two inclined positions of equilibrium given by (2). In these positions the side CD may be inclined towards either side of the vertical. In this case there are in all three positions of equilibrium, one symmetrical, given by $\theta=\frac{1}{2}\pi$, and the other two, which are inclined, given by (2).

Nature of the positions of equilibrium.

We have,

$$d^2u/d\theta^2=2(\beta^2-\alpha^2)(\cos^2\theta-\sin^2\theta)-a\beta\sin\theta$$

$$=2(\beta^2-\alpha^2)(1-2\sin^2\theta)-a\beta\sin\theta. \qquad ...(3)$$

For the symmetrical position of equilibrium given by $\theta=\frac{1}{2}\pi$,

$$\frac{d^2u}{d\theta^2}=-2(\beta^2-\alpha^2)-a\beta$$

$$=-2[\tfrac{1}{4}(l^2-c^2)-\tfrac{1}{4}l^2]-a\cdot\tfrac{1}{2}\sqrt{(l^2-c^2)}$$

$$=\tfrac{1}{2}c^2-\tfrac{1}{2}a\sqrt{(l^2-c^2)}=\tfrac{1}{2}[c^2-a\sqrt{(l^2-c^2)}],$$

which is positive or negative according as $a\sqrt{(l^2-c^2)}<$ or $>c^2$

i.e., according as $\quad al<$ or $>c\sqrt{(a^2+c^2)}$.

Thus if $al<c\sqrt{(c^2+a^2)}$, then u and so also z is minimum. Since z is the depth of G below the fixed point P, therefore the equilibrium is unstable in this case. Again if $al>c\sqrt{(c^2+a^2)}$, then u and so also z is maximum, and the equilibrium is stable. Hence the symmetrical equilibrium position $\theta=\frac{1}{2}\pi$ is unstable if

$$al<c\sqrt{(c^2+a^2)}$$

and stable if $al>c\sqrt{(c^2+a^2)}$.

Now consider the inclined positions of equilibrium given by

$$\sin\theta = \{a\sqrt{(l^2-c^2)}\}/c^2,$$

which give real values of θ only if

$$a\sqrt{(l^2-c^2)} < c^2, \text{ or, } al < c\sqrt{(c^2+a^2)}.$$

In this case putting $\sin\theta = \{a\sqrt{(l^2-c^2)}\}/c^2$ in (3), we get

$$\frac{d^2u}{d\theta^2} = 2\,[\tfrac{1}{2}(l^2-c^2)-\tfrac{1}{4}l^2]\left[1-2.\frac{a^2(l^2-c^2)}{c^4}\right]$$

$$-a.\tfrac{1}{2}\sqrt{(l^2-c^2)}.\frac{a}{c^2}\sqrt{(l^2-c^2)}$$

$$=-\frac{c^2}{2}+\frac{a^2(l^2-c^2)}{c^2}-\frac{a^2(l^2-c^2)}{2c^2}$$

$$=-\frac{c^2}{2}+\frac{a^2(l^2-c^2)}{2c^2}=\frac{1}{2c^2}[a^2(l^2-c^2)-c^4],$$

which is negative because $a\sqrt{(l^2-c^2)} < c^2$.

Thus in this case u and so also z is maximum and the equilibrium is stable. Hence if $al < c\sqrt{(c^2+a^2)}$, there are two inclined positions of equilibrium and they are both stable.

Ex. 30 (b). *A rectangular picture-frame hangs from a small perfectly smooth pulley by a string of length 2a attached symmetrically to two points on the upper edge at a distance 2c apart. Prove that if the depth of the picture is less than*

$$2c^2/\sqrt{(a^2-c^2)},$$

there are three positions of equilibrium of which the symmetrical one is unstable. If the depth exceeds the above value the symmetrical position of equilibrium is the only one and is stable.

Sol. Proceed as in Ex. 30 (a).

4. ALGORITHMS

4.0 Definition and origin of an algorithm

The term algorithm may be defined as sequence of instructions designed in such a way that if the instructions are executed in the specified sequence the desired results can be obtained. Further, the instructions should be unambiguous and the result should be obtained after a finite number of executed steps i.e., an algorithm must terminate and should not repeat one or more instructions infinitely. In other words, the algorithm represents the logic of the processing to be performed. The word algorithm originates from the word algorism, which means the process of doing arithmetic with Arabic numerals. Later, the word algorism combined with the word arithmetic to become algorithm.

4.1 Development of an algorithm

We first construct an algorithm that gives a very general manner in which computer could produce the solution to a given problem. Such an algorithm is known as general algorithm. In addition, we add details in the general algorithm, in a step-by-step manner, so that the algorithm becomes more detailed, is called refinements of the general algorithm.

The following example shows how an algorithm is written for a given task.

Consider the problem of calculating the average marks of a student in three different subjects.

An algorithm for this task involves the following steps:

Step 1: Read a set of three marks

Step 2: Find the average by summing them and dividing by three

Step 3: Stop

Successive refinements of this general algorithm add details until complete algorithm is achieved as per the user specifications/requirements. For example, the set of three marks may be obtained from an input device and placed in the variables SUB 1, SUB 2 and SUB 3. Compute the sum and place in the variable TOTMARK. This requires a refinement of step 1 and 2 which is done as follows:

Step 1.1: Obtain three marks through an input device and place them in the variables SUB 1, SUB 2 and SUB 3.

Step 2.1: Compute the total marks and place it in a variable TOTMARK

Step 2.2: Compute the average mark dividing TOTMARK by 3.

It is important to note that hierarchical number system has been used to indicate that these are refinement of steps 1 and 2 of the general algorithm.

If you decide to calculate average marks for a class of 100 students, then the algorithm needs further refinement. The refined algorithm defines a new variable COUNTER which ensures that all the students are taken care for processing.

Step 1: Initialise the counter, say COUNT = 1

Step 2: Read the student register number and his marks from an input device and place them in the variables REGNO, SUB 1, SUB 2 and SUB 3 respectively.

Step 3: Compute the average marks for each student by summing of the corresponding marks and dividing by 3.

Step 4: Display the register number and average marks of each student.

Step 5: Start increment COUNT BY 1.

Step 6: If COUNT < = 100, repeat steps 1 through 6 otherwise go to

Step 7: Stop

4.2 Essential steps to be followed for development

From this, it can be stated that the following steps are very essential to develop an algorithm for any task:

(1) Input step

(2) Assignment step

(3) Decision step

(4) Repetitive step and

(5) Output step

Remark: The input step takes the three marks along with register number for each student. Assignment step assigns three marks and register number to SUB1, SUB2, SUB3 and REGNO. The decision step checks whether or all the students have been considered. The repetitive steps concentrate on calculation of average marks for all the 100 students. The output step provides the result.

4.3 Illustrative examples of algorithms using Numerical methods

(1) Bisection method

Step 1: Read initial values say x_0 and x_1

Step 2: Read the allowed relative error say e (epsilon)

Step 3: $y_0 \longleftarrow f(x_0)$

Step 4: $y_1 \longleftarrow f(x_1)$

Step 5: $i \longleftarrow 0$

Step 6: If $y_0 y_1 > 0$ follow steps 14 to 16; otherwise follow steps 7 to 14

Step 7: When $|(x_1 - x_0)/x_1| > e$, follow steps 8 to 11

Step 8: $x_2 \longleftarrow (x_0 + x_1)/2$

Step 9: $y_2 \longleftarrow f(x_2)$

Step 10: $i \longleftarrow i + 1$

Step 11: If $y_0 y_2 > 0$; assign $x_0 = x_2$, otherwise $x_1 = x_2$

Step 12: Print "Solution converges to a root"

Step 13: Print i, x_2, y_2; goto step 16

Step 14: Print "Initial values unsuitable"

Step 15: Write x_0, x_1, y_0, y_1

Step 16: Stop

(2) Regula – Flasi method

Step 1: Read initial values say x_0 and x_1

Step 2: Read the prescribed error e and the maximum number of iterations say n

Step 3: $f_0 \longleftarrow f(x_0)$

Step 4: $f_1 \longleftarrow f(x_1)$

Step 5: for i = 1 to n

Step 6: $x_2 \longleftarrow (x_0 f_1 - x_1 f_0)/(f_1 - f_0)$

Step 7: $f_2 \longleftarrow f(x_2)$

Step 8: When $|f_2| <= e$ then print the message "Solution is convergent and print x_2, f_2 follow step 13; otherwise follow steps 9 to 12

Step 9: When $f_2 f_0 < 0$, assign $x_1 = x_2$ and $f_1 = f_2$ otherwise assign $x_0 = x_2$ and $f_0 = f_2$

Step 10: Next i

Step 11: Print "Does not converge in n iterations"

Step 12: Print x_2, f_2

Step 13: Stop

(3) Newton – Raphson method

Step 1: Assign an initial value to x (say x_0)

Step 2: Evaluate f(x), $f'(x_0)$ and $f(x_0)$

Step 3: Find the improved estimate of x_0

$$x_1 = \frac{x_0 - f(x_0)}{f'(x_0)}$$

Step 4: Check for accuracy of the latest estimate. It may be done by comparing the difference between the values of the latest estimate and the previous estimate to a predefined value, say E. If $|(x_1 - x_0)| < E$. Follow Step 6; otherwise continue.

Step 5: Replace x_0 by x_1 and repeat steps 3 and 4

Step 6: Stop

(4) Gauss-Elimination method

Step 1: Read n

Step 2: Read a_{ij}, i = 1 to n and j = 1, n + 1

Step 3: The equations are arranged such that $a_{11} \neq 0$

Step 4: Eliminate x_1 from all except the first equation. For this, normalise the first equation by dividing in by a_{11}.

Step 5: Subtract from the second equation 21 times the normalised first equation. Similarly continue the procedure for other equations till the nth equation is finished.

Step 6: Eliminate x_2 from the third to the last equation in the new set. Assume that $c_{22} \neq 0$. If $c_{22} = 0$, reorder the equations. Follow the subtraction procedure as stated in Step 5 for x_2. Continue this process till the last equation contains only one unknown say x_n.

Step 7: Obtain solution by back substitution. This back substitution can be continued till we get the solution for x_1.

Step 8: Print the results

Step 9: Stop

(5) Gauss-Jordan method

Step 1: Read n

Step 2: Read the coefficients of the equations say e_{ij}, i = 1 to n and j = 1, n + 1.

Step 3: Check for the pivotal values, and if they are zero, reorder the equations or rows.

Step 4: Perform normalisation of rows

Step 5: Eliminate the unknowns (i.e., in kth column)

Step 6: Print the results.

(6) Gauss – Seidel iterative method

Step 1: Read n

Step 2: Read the coefficients of the n equations say a_{ij} (i = 1 to n, j = 1 to n)

Step 3: Read initial values of x_i, i = 1 to n

Step 4: Read b_i (i = 1 to n)

Step 5: Check the values of aii and if they are zero, print a message "a_{ii} is zero" and follow Step 10, otherwise follow Step 6.

Step 6: Start the iterations, to calculate

$$x_i = \frac{b_i - S}{a_{ii}}, i = 1 \text{ to } n$$

where $S = \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij} x_j$

Step 7: For each iteration, the absolute difference between the successive values of x_i are computed and the largest difference is stored in E, If E is less than the desired value, follow step 9, otherwise follow step 8.

Step 8: When E is large, repeat the iterations. If the process does not converge after 100 iterations then follow step 10.

Step 9: Print the result.

Step 10: Stop.

(7) Newton's – Forward formula

Step 1: Read n

Step 2: Read x_i and y_i, i = 1 to n

Step 3: Read the value of x (say a)

Step 4: Construct the difference table

Step 5: Compute the value of $u = \frac{a - a_0}{h}$

Step 6: Expand the Newton-forward formula and substitute the values to get the function value for the given argument (say a)

Step 7: Print the function value of the given argument (say a)

Step 8: If you want to continue, follow Steps 3 to 7, otherwise follow Step 9

Step 9: Stop

(8) Newton's – Backward formula

Step 1: Read n

Step 2: Read x_i and y_i, i=1 to n

Step 3: Read the value of x (say a)

Step 4: Construct the difference table

Step 5: Compute the value of $u = \frac{a - a_0}{h}$

Step 6: Expand the Newton-forward formula and substitute the values, to get the function value for the given argument (say a)

Step 7: Print the function value of the given argument (say a)

Step 8: If you want to continue, follow Steps 3 to 7, otherwise follow Step 9

Step 9: Stop

(9) Lagrange's method

Step 1: Read the number of values (say n)

Step 2: Read x_i, i = 1 to n

Step 3: Read y_i, i = 1 to n

Step 4: Read the value of x (say a)

Step 5: Expand the lagrangian interpolation formula as per 'n'

Step 6: Substitute the values of a, x_i and y_i and evaluate.

Step 7: Print the results (i.e. x and y)

Step 8: If you want to try for another value follow Steps 4 to 8, otherwise follow Step 9

Step 9: Stop

(10) Simpson's One-third rule

Step 1: Read an initial value, say a

Step 2: Read the final value, say b

Step 3: Read number of divisions, say n

Step 4: Find the width of the interval

i.e. $h = \frac{b - a}{n}$

Step 5: Assign m=n/2 (to get the area under the curve for n intervals)

Step 6: Assign S = 0 and x = a

Step 7: Expand the Simpson's rule and is as follows

$$\frac{h}{3}(y_1 + 4y_2 + 2y_3 + 4y_4 + 2y_5$$

$$+ \ldots + 2y_{n-1} + 4y_n + y_{n+1})$$ and compute the values of h and y_i and evaluate.

Step 8: Print the results.

Step 9: Stop

(11) Trapezoidal rule

Step 1: Read two end points of the interval say x_1, x_2

Step 2: Read the allowed error in the integral say e

Step 3: $h \longleftarrow x_2 - x_1$

Step 4: $S_1 \longleftarrow (f(x_1) + f(x_2))/2$

Step 5: $I_1 \longleftarrow h \times S_1$

Step 6: $S_0 \longleftarrow 0$

Step 7: $I_0 \longleftarrow 0$

Step 8: $i \longleftarrow 1$

Step 9: When $|(I_1 - I_0)/I_1| > e$ then follow Step 10 to 16, otherwise follow Step 17

Step 10: $S_0 \longleftarrow S_1$

Step 11: $x \longleftarrow x_1 + h/2$

Step 12: for j = 1 to i

$S_1 \longleftarrow S_1 + f(x)$

$x \longleftarrow x + h$

next j

Step 13: $i \longleftarrow 2 \times i$

Step 14: $h \longleftarrow h/2$

Step 15: $I_0 \longleftarrow I_1$

Step 16: $I_1 \longleftarrow h \times S_1$

Step 17: Print the results I_1, h

Step 18: Stop

(12) Gaussian Quadrature formula

Step 1: First transform the independent variable

Step 2: Choose value of n

Step 3: Expand the Gauss formula

Step 4: Substitute the values for the parameters A_i and Z_i (use the following Table) and evaluate the formula

Parameters for Gaussian integration

n	i	A_i	Z_i
2	1	1.00000	-0.57735
	2	1.00000	0.57735
3	1	0.55556	-0.77460
	2	0.88889	0.00000
	3	0.55556	-0.77460
4	1	0.34785	-0.86114
	2	0.65215	-0.33998
	3	0.65215	0.33998
	4	0.34785	0.86114
5	1	0.23693	-0.90618
	2	0.47863	-0.53847
	3	0.56889	0.00000
	4	0.47863	0.53847
	5	0.23693	0.90618
6	1	0.17132	-0.93247
	2	0.36076	-0.66121
	3	0.46791	-0.23862
	4	0.46791	0.23862
	5	0.36076	0.66121
	6	0.17132	0.93247

(13) Euler's method

Step 1: Read initial values say a, b, y

Step 2: Read number of points of x (say n)

Step 3: Find the width of the interval say

$$h = \frac{b-a}{n-1}$$

Step 4: Assign a to x

Step 5: Assign I = 1

Step 6: If x > = (b + h), follow Step 9, otherwise follow Step 7 and 8

Step 7: Evaluate the Eulers formula $y_i + 1 = y_i + hf(x_i, y_i)$ and print the results

Step 8: Increment I, I = I+1, follow step 6.

Step 9: Stop

(14) Runge – Kutta method

Step 1: Read initial values say a, b, y

Step 2: Read number of points of x (say n)

Step 3: Find the width of the interval (say h)

Step 4: Assign the initial value 'a' to 'x'

Step 5: Compute the value of the parameters viz., k_1, k_2, k_3, k_4 and y and substitute in the Runge–Kutta fourth order formula as follows:

$$y = y + 1/6(k_1 + 2k_2 + 2k_3 + k_4)$$

Step 6: Increment x, say x = x + h

Step 7: Continue steps 5 and 6 till we get number of points equal to n.

Step 8: Stop

5. FLOW CHARTING

5.0 Introduction

A flow chart is a pictorial representation of an algorithm in which boxes of different shapes are used to denote different types of operation.

The actual operation are stated within the boxes. The boxes are connected by directed solid lines indicating the flow of operations. Usually a flow chart is drawn before writing the programs and the flow chart is expressed in the programming language to prepare a program. The main advantage of this two-tier system in program writing is that while drawing a flow chart one is not concerned with the details of the elements of a programming language, they concentrate only on logic of the procedure. Further, the flow chart shows the logic pictorially, any problem can be identified and eliminated immediately.

It is important to note that, for a beginner, it is recommended that a flow chart be drawn first in order to reduce the number of errors and omissions in the program. It is a good practice to have a flow chart which may help during the testing of the program as well as while incorporating further modifications in the program.

5.1 Classification of flow charts

Flow charts can be divided into two broad categories:

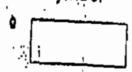
(1) Program flow charts and

(2) System flow charts

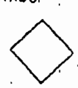
Program flow chart is the pictorial representation of a sequence of instructions for solving a problem. System flow chart indicates the flow of data throughout a data processing system, as well as the flow into and out of the system.

5.2 Flow chart symbols

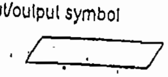
(1) Assignment symbol



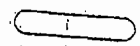
(2) Decision symbol



(3) Input/output symbol

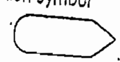


(4) Terminal symbol

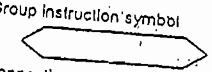


(5) Off-page connector symbol

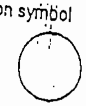
(6) Modification symbol



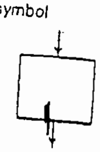
(7) Group instruction symbol



(8) Connection symbol



(9) Module symbol



5.3 Guidelines to draw efficient flow charts

(1) Analyse the nature of the problem completely and ensure that all the information relating to the problem is available

(2) Decide the sequence of actions to be taken so that solution can be obtained in a finite number of steps.

(3) Prepare the general algorithm

(4) Include refinements, wherever necessary

(5) Draw flow chart on the basis of the algorithm

(6) Draw flow chart symbols from top to bottom or left to right across the page to ensure the readability.

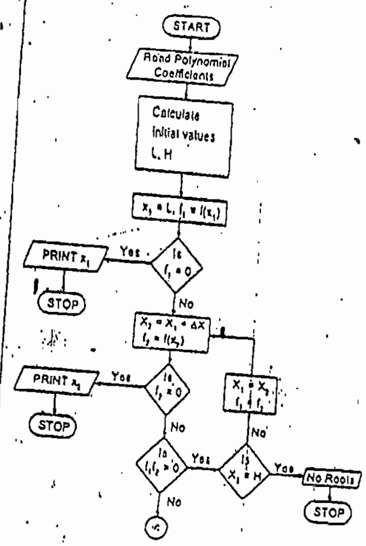
(7) Avoid intersecting directed lines. Use connector symbol wherever required.

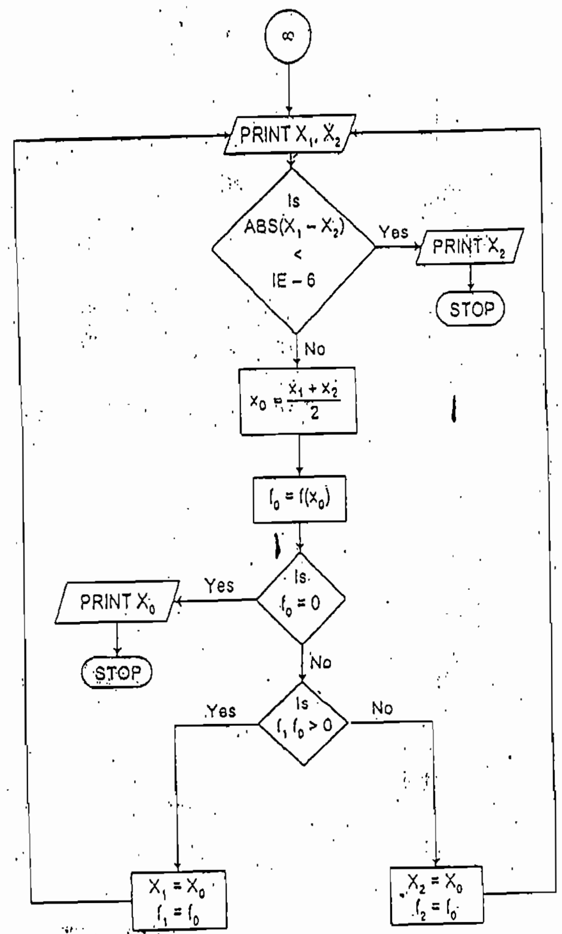
(8) Always use pencil to draw flow chart so that erasing and redrawing are easier. To draw flow chart, use template.

(9) Use visible arrow head on directed lines whenever flow is not from top to bottom or from left to right.

5.4 Flow Charts for Solving Numerical Analysis Problems

(1) Flow chart for Bisection method





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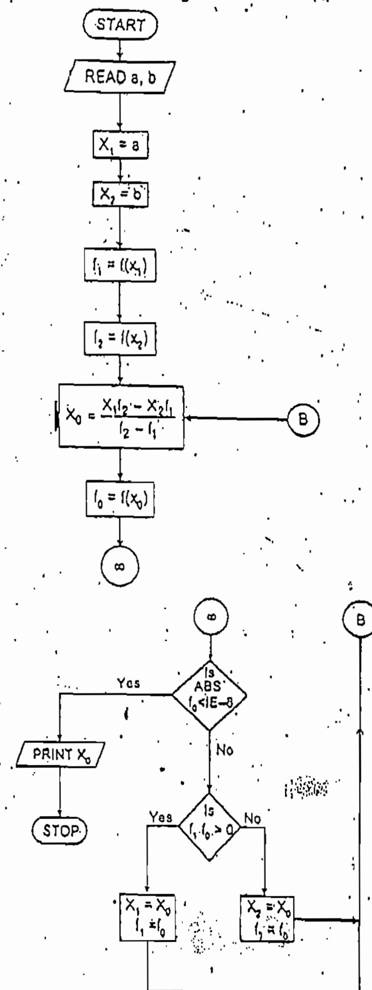
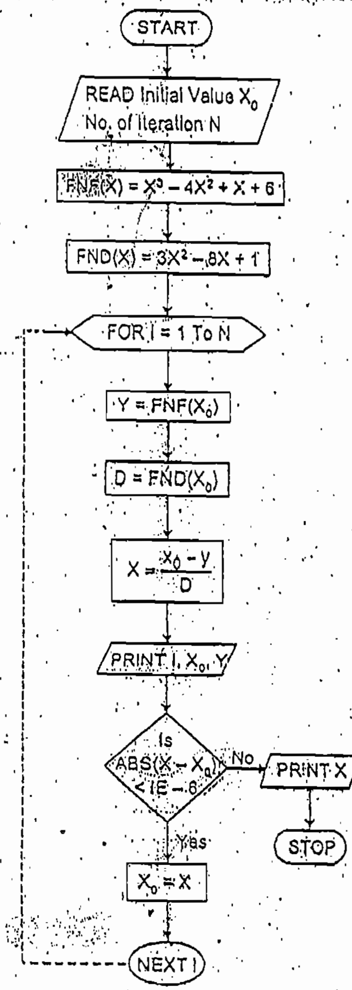
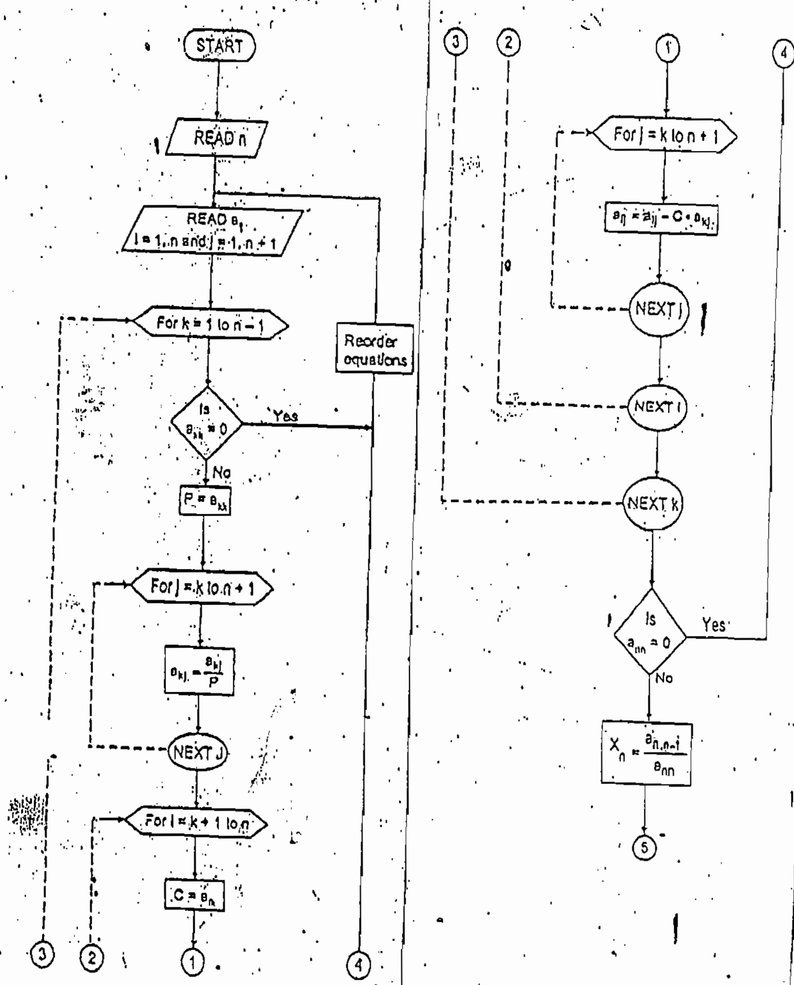
(2) Flow chart for Regula-Falsi method

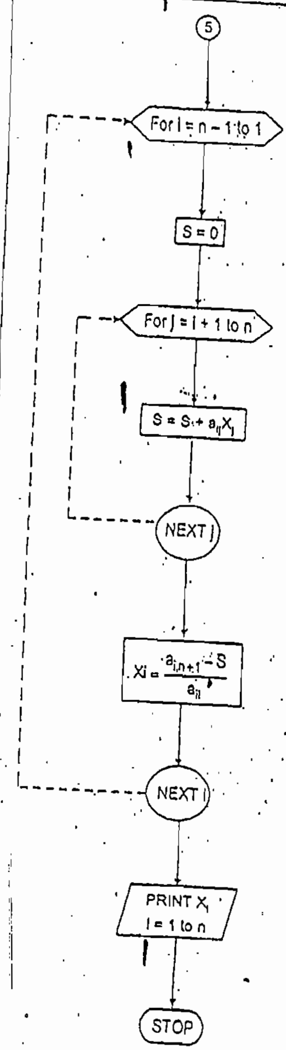


(3) Flow chart for Newton-Raphson Method

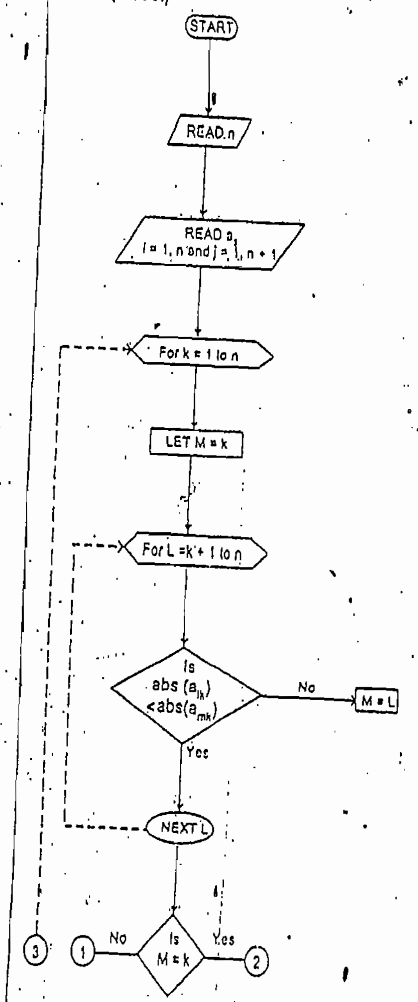


(4) Flow chart for Gauss elimination method

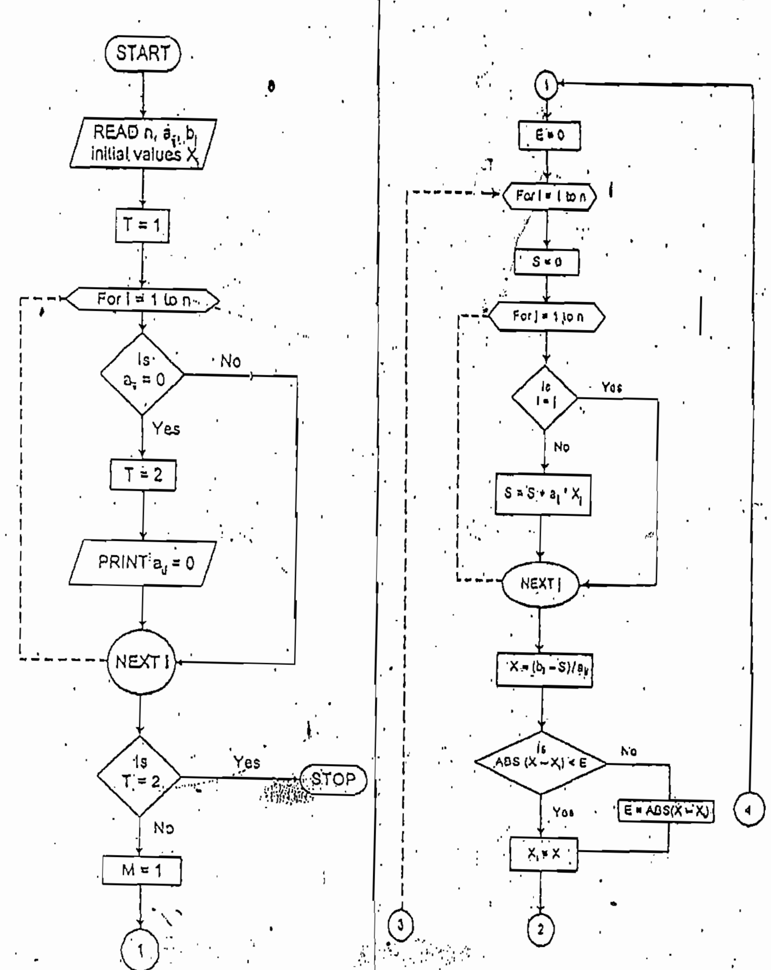


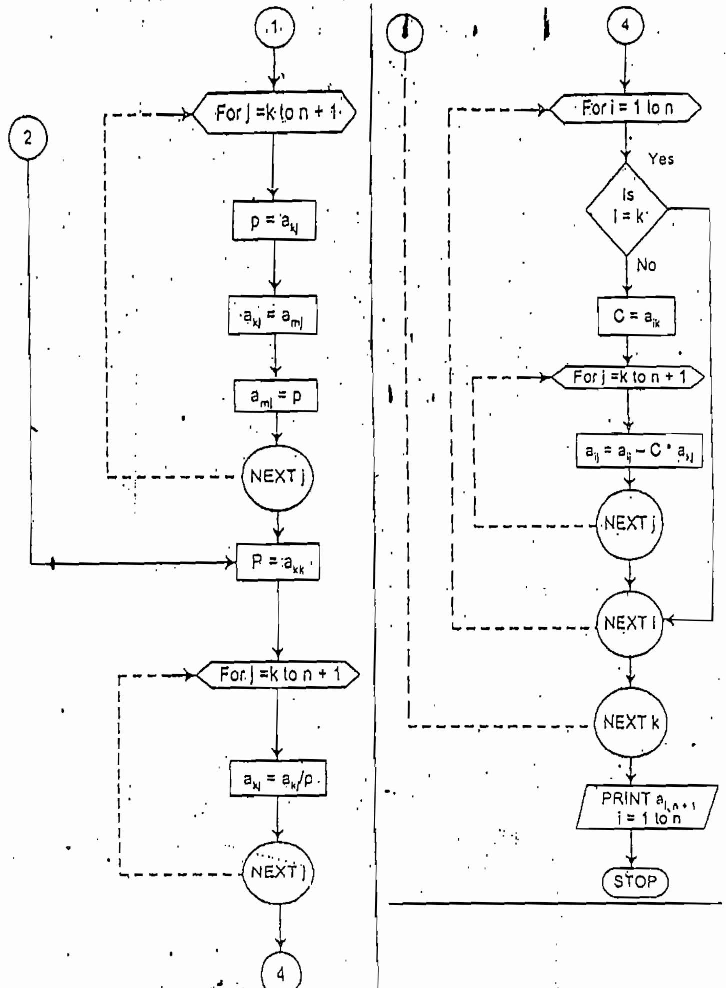


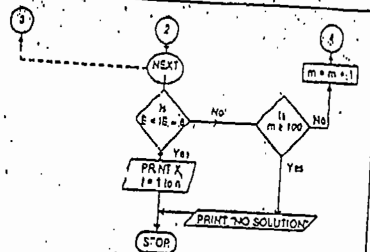
(5) Flow chart for Gauss-Jordan method (Direct)



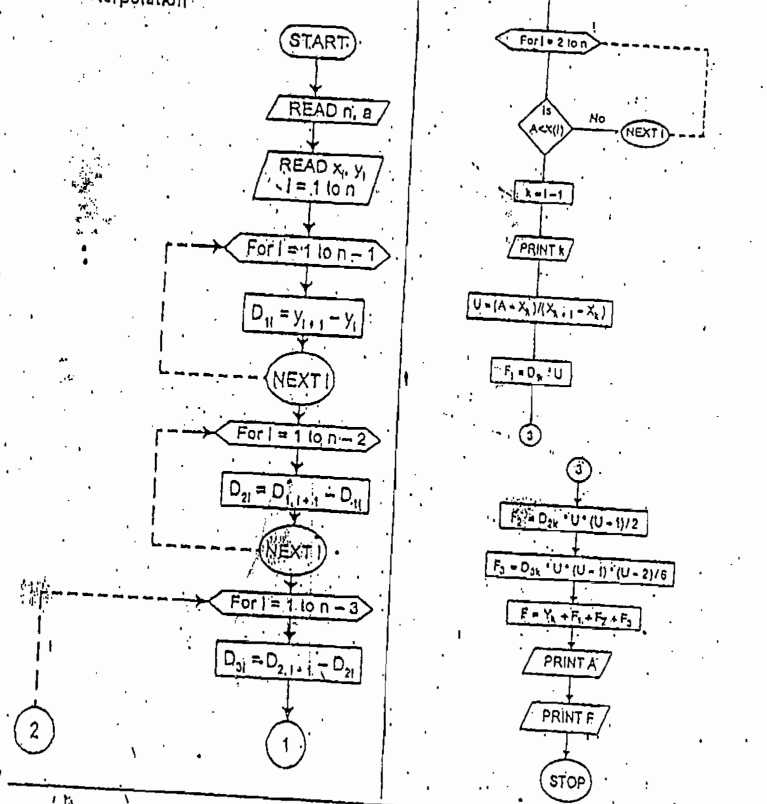
(6) Flow chart for Gauss-Seidel (iterative) method



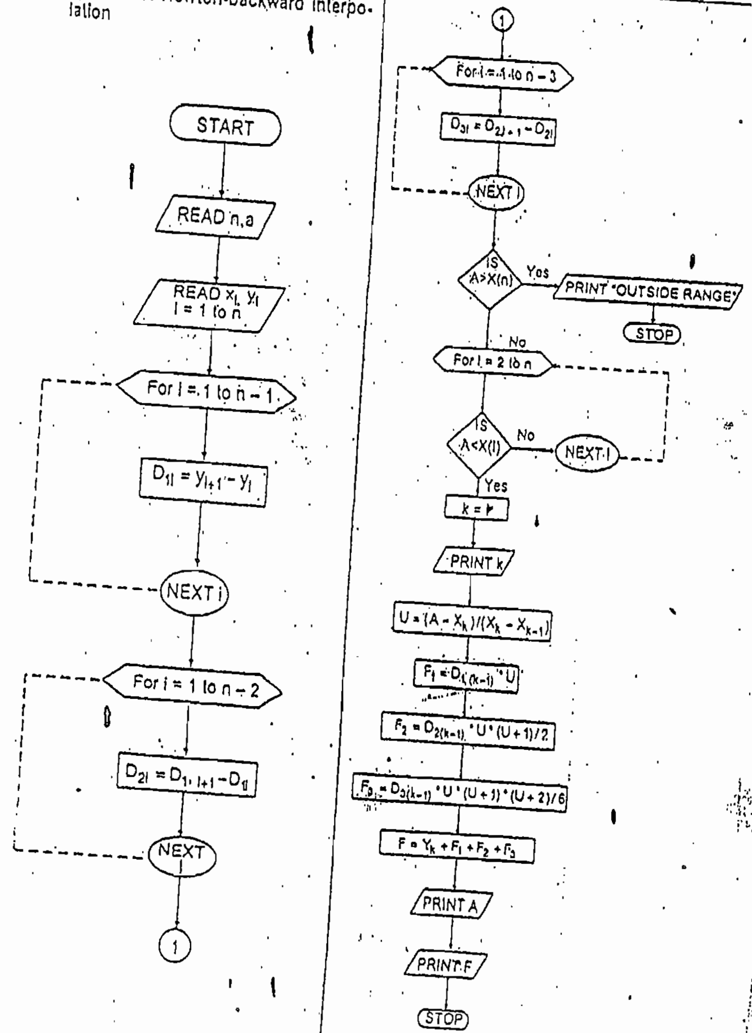




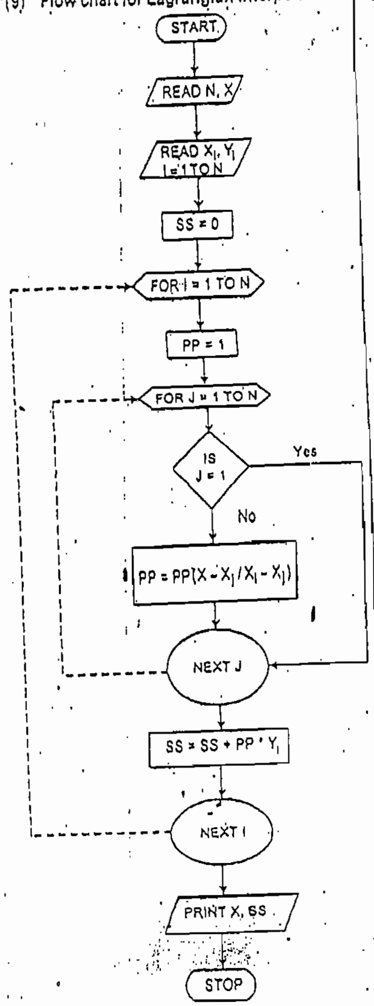
(7) Flow Chart for Newton-forward Interpolation



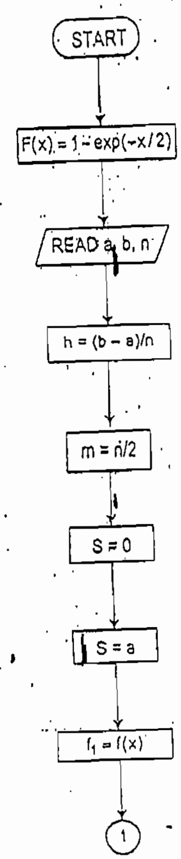
(8) Flow chart Newton-backward Interpolation

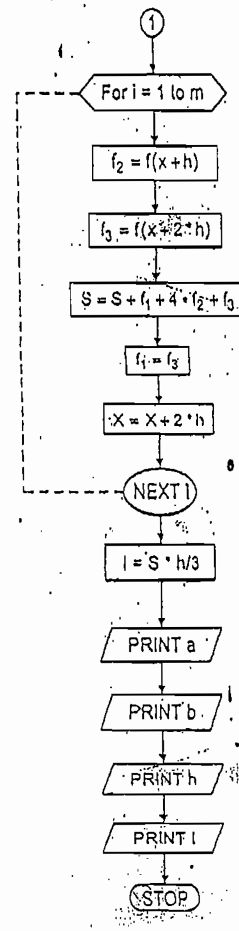


(9) Flow chart for Lagrangian Interpolation

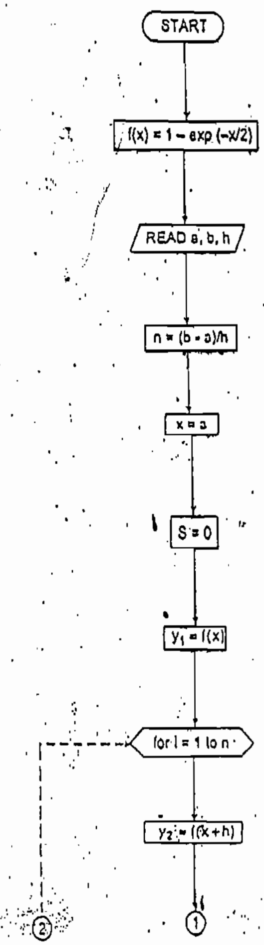


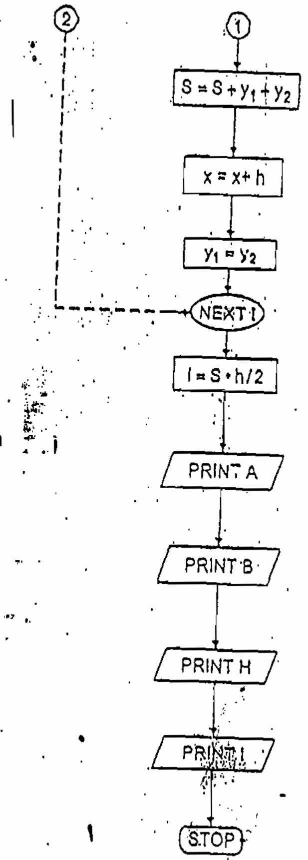
(10) Flow chart for Simpson's one-third rule





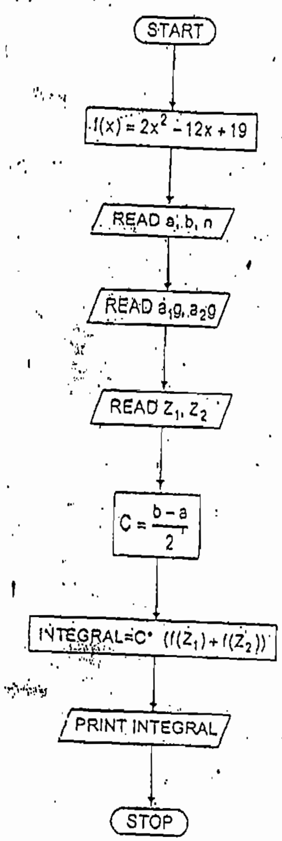
(11) Flow chart for Trapezoidal rule



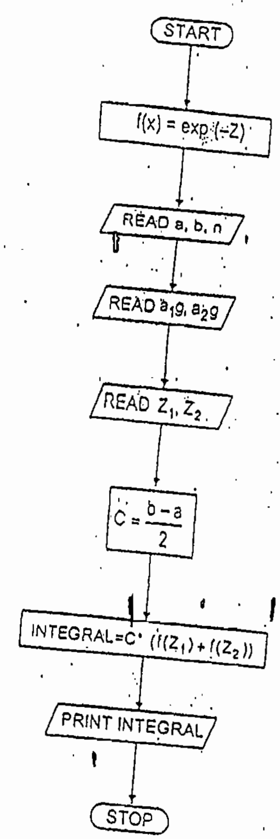


(12) Flow chart for Gaussian Quadrature formula

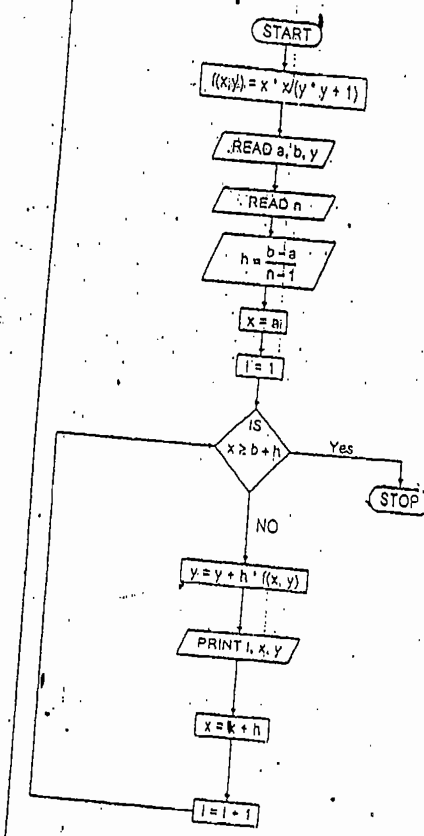
(a)



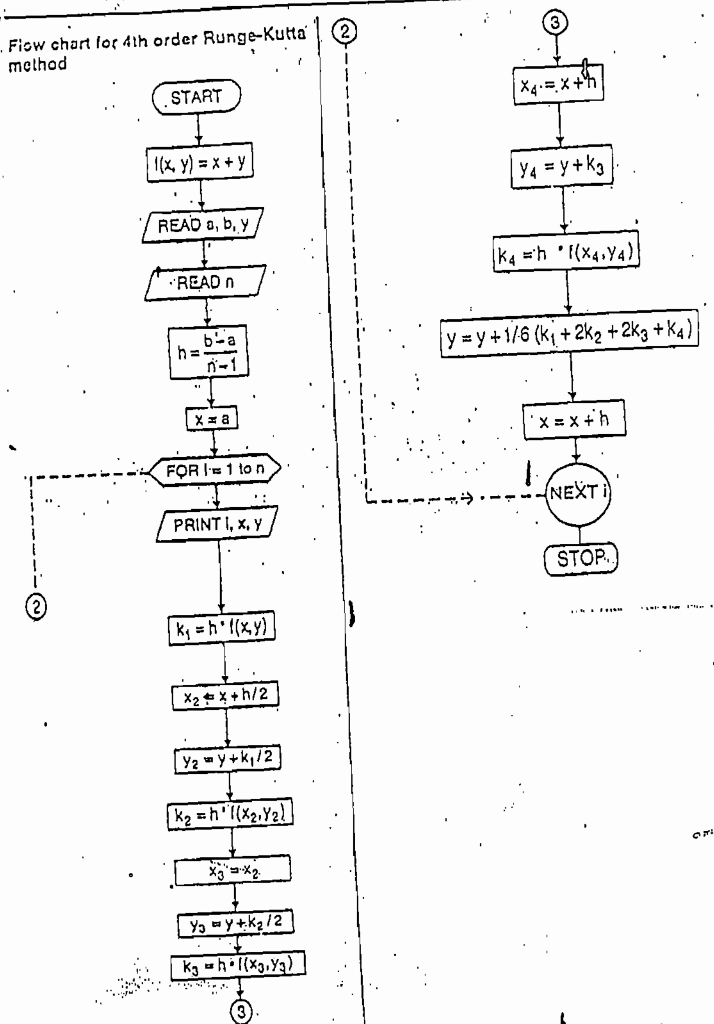
(b)



(13) Flow chart for Euler's method



(14) . Flow chart for 4th order Runge-Kutta method



(5)

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for PT.

Chapter 1

Number System and Codes

1.1 INTRODUCTION

The term *digital* in digital circuits is derived from the way circuits perform operations by counting digits. A digital circuit operates with binary numbers, i.e., only in two states. The output of the circuit is either low (0) or high (1) in a *positive logic system*. In general, 0 represents zero volts and 1 represents five volts. If the situation is reverse, it is known as a *negative logic system*.

In digital systems, as explained above, the data is usually in binary states (0 and 1) and is processed and stored electronically to prevent errors due to noise and interfering signals. At present, digital technology has progressed remarkably from vacuum-tube circuits to integrated circuits, microprocessors and microcontrollers. Digital circuits find applications in computers, telephony, data processing, radar navigation, military systems, medical instruments and consumer products. The general properties of number systems, methods of conversion from one to another, arithmetic operations, weighted codes, non-weighted codes, error detecting and correcting codes are discussed in this chapter.

1.2 NUMBER SYSTEM

The decimal number system (0, 1, 2, ..., 9) is commonly used eventhough there are many other number systems like binary, octal, hexadecimal, etc. It is possible to express a number in any base or radix 'X'. In the binary system, the base is 2. In general, any number with radix X, having m digits to the left and n digits to the right of the decimal point, can be expressed as:

$$a_m(X)^{m-1} + a_{m-1}(X)^{m-2} + \dots + a_2(X)^1 + a_1(X)^0 + b_1(X)^{-1} + b_2(X)^{-2} + \dots + b_n(X)^{-n}$$

where a_m is the digit in mth position. The coefficient a_m is termed as the Most Significant Digit (MSD) and b_n is termed as the Least Significant Digit (LSD).

1.2.1 Binary Numbers

The Binary number system is simple because it consists of only two digits, i.e. 0 and 1. Just as the decimal system with its ten digits is a base-ten system, the binary system with its two digits is a base-two system. The position of 0 or 1 in a binary number indicates its "weight" within the number. In a binary number, the weight of each successively higher position to the left is an increasing *power* of two.

For example, in the decimal system,

$$(198)_{10} = 1 \times 10^2 + 9 \times 10^1 + 8 \times 10^0$$

Hundreds Tens Units

Positional Weights

Similarly, binary numbers are also represented by positional weights.

For example,

$$(198)_{10} = (11000110)_2$$
$$= 1\times2^7 + 1\times2^6 + 0\times2^5 + 0\times2^4 - 0\times2^3 + 1\times2^1 + 0\times2^0$$
$$= 128 + 64 + 0 + 0 + 0 + 4 - 2 + 0 = 198$$

In the digital system, each of the binary digits is called a bit and a group of 4 and 8 bits are called a nibble and a byte respectively. The highest decimal number that can be represented by n-bits binary number is $2^n - 1$ (beginning with zero). Thus, with an 8-bit binary number, the maximum decimal number that can be represented is $2^8 - 1 = 255$.

1.2.2 Decimal–Binary Conversion

An easy method of converting a decimal number into a binary number is by dividing the decimal number by 2 progressively, until the quotient of zero is obtained. The binary number is obtained by taking the remainder after each division in the reverse order. This method is popularly known as the *double-dabble* method. The procedure for decimal to binary conversion is described in the following example.

Example 1.1 Convert the decimal number 53·625 into an equivalent binary number.

Solution

Step 1 The integer is 53. The fraction is 0·625.

Step 2 Integer conversion:

Division	Generated remainder
2) 53	
2) 26	→ 1
2) 13	→ 0
2) 6	→ 1
2) 3	→ 0
2) 1	→ 1
2) 0	→ 1 → MSB

Reading the remainders from bottom to top gives the binary equivalent. Thus, $(53)_{10} = (110101)_2$.

Step 3 Fractional conversion: If the decimal number is a fraction, its binary equivalent is obtained by multiplying the number continuously by 2, recording a *carry* in the integer position each time. The *carries* in the forward order give the required binary number.

Multiplication	Generated integer
0·625 × 2 = 1·25 →	1 → MSB
0·250 × 2 = 0·50 →	0
0·500 × 2 = 1·00 →	1
0·000 × 2 = 0·00 →	0

Further multiplication by two is not possible since the product is zero. The binary equivalent is obtained by reading the carry terms from top to bottom. Thus, $(0·625)_{10}$ is $(0·101)_2$. The combined number will give the binary equivalent as $(53·625)_{10} = (110101·101)_2$.

Example 1.2 Convert the binary number $(101111·1101)_2$ into its decimal equivalent.

Solution A binary number can be converted into a decimal number by multiplying the binary numbers 1 or 0 by their weight and adding the products.

1 0 1 1 1 1

$1 \times 2^0 = 1$
$1 \times 2^1 = 2$
$1 \times 2^2 = 4$
$1 \times 2^3 = 8$
$0 \times 2^4 = 0$
$1 \times 2^5 = 32$
$\overline{47}$

Therefore, $(101111)_2$ can be written as $(47)_{10}$.

Conversion of $(0·1101)_2$ is done in a similar manner.

0 · 1 1 0 1

$1 \times 2^{-4} = 0·0625$
$0 \times 2^{-3} = 0·0000$
$1 \times 2^{-2} = 0·2500$
$1 \times 2^{-1} = 0·5000$
$\overline{0·8125}$

Thus, $(0·1101)_2$ is equal to $(0·8125)_{10}$.

Therefore, $(101111·1101)_2$ is equal to $(47·8125)_{10}$

1.2.3 Octal Numbers

The Octal number system uses the digits 0, 1, 2, 3, 4, 5, 6 and 7. The *base* or *radix* of this system is eight. Each significant position in an octal number has a positional weight. The least significant position has a weight of 8^0, i.e. 1; the higher significant positions are given weights in the ascending powers of eight, i.e. $8^1, 8^2, 8^3$, etc. respectively. The octal equivalent of a decimal number can be obtained by dividing a given decimal number by 8 repeatedly, until a quotient of 0 is obtained. The procedure is exactly the same as the *double-dabble* method explained earlier. The *decimal* to *octal* conversion method is explained in the following example.

Example 1.3 Convert $(444 \cdot 456)_{10}$ to an octal number.

Solution Integer conversion:

Division		Generated remainder
8) 444		
8) 55	→	4
8) 6	→	7
8) 0	→	6

Reading the remainders from bottom to top, the decimal number $(444)_{10}$ is equivalent to octal $(674)_8$.

Fractional conversion:

Multiplication	Generated integer
$0 \cdot 456 \times 8 = 3 \cdot 648 \rightarrow$	3
$0 \cdot 648 \times 8 = 5 \cdot 184 \rightarrow$	5
$0 \cdot 184 \times 8 = 1 \cdot 472 \rightarrow$	1
$0 \cdot 472 \times 8 = 3 \cdot 776 \rightarrow$	3
$0 \cdot 776 \times 8 = 6 \cdot 208 \rightarrow$	6

The process is terminated when significant digits are obtained.

Thus, the octal equivalent of $(444.456)_{10}$ is $(674.35136)_8$.

The conversion from an octal to decimal number can be done by multiplying each significant digit of the octal number by its respective weight and adding the products. The following example illustrates the conversion from octal to decimal.

Example 1.4 Convert the octal numbers (a) $(237)_8$ and (b) $(120)_8$ to decimals.

Solution (a)

$$\begin{aligned}(237)_8 &= 2\times8^2 + 3\times8^1 + 7\times8^0\\ &= 2\times64 + 3\times8 + 7\times1\\ &= 128 + 24 + 7\\ &= (159)_{10}\end{aligned}$$

(b)

$$\begin{aligned}(120)_8 &= 1\times8^2 + 2\times8^1 + 0\times8^0\\ &= 1\times64 + 2\times8 + 0\times1\\ &= 64 + 16 + 0\\ &= (80)_{10}\end{aligned}$$

1.2.4 Octal–Binary Conversion

Conversion from octal to binary and vice versa can be easily carried out. For obtaining the binary equivalent of an octal number, each significant digit in the given number is replaced by its 3-bit binary equivalent. For example,

$$\begin{aligned}(376)_8 &= 3 \quad 7 \quad 6\\ &= 011 \quad 111 \quad 110\end{aligned}$$

Thus, $(376)_8 = (011111110)_2$. For converting a binary number to an octal, the reverse procedure is used, i.e. starting from the least significant bit, each group of 3 bits is replaced by its decimal equivalents. For example,

$$\begin{aligned}(10011010101)_2 &= 010 \quad 011 \quad 010 \quad 101\\ &= 2 \quad 3 \quad 2 \quad 5\end{aligned}$$

Thus, $(10011010101)_2 = (2325)_8$

1.2.5 Hexadecimal Numbers

The Hexadecimal number system has a radix of 16 and uses 16 symbols, namely 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, *A*, *B*, *C*, *D*, *E* and *F*. The symbols *A*, *B*, *C*, *D*, *E* and *F* represent the decimals 10, 11, 12, 13, 14 and 15 respectively. Each significant position in an hexadecimal number has a positional weight. The least significant position has a weight of 16^0, i.e. 1; the higher significant positions are given weights in the ascending powers of sixteen, i.e. $16^1, 16^2, 16^3$, etc. respectively. The hexadecimal equivalent of a decimal number can be obtained by dividing the given decimal number by 16 repeatedly, until a quotient of 0 is obtained. The following example illustrates how the hexadecimal equivalent of a given decimal is obtained.

Example 1.5 Convert (a) $(115)_{10}$ and (b) $(235)_{10}$ to hexadecimal numbers.

Solution

(a)

Division	Remainder
16) 115	
16) 7	3
16) 0	7

Reading the remainders from bottom to top, the decimal number $(115)_{10}$ is equivalent to the hexadecimal $(73)_{16}$. The hexadecimal $(73)_{16}$ can also be represented as 73 H.

(b)

Division	Remainder
16) 235	
16) 14	11 → B
16) 0	14 → E

Reading the remainders from bottom to top, the decimal number $(235)_{10}$ is equivalent to hexadecimal $(EB)_{16}$.

The conversion from an hexadecimal to a decimal number can be carried out by multiplying each significant digit of the hexadecimal by its respective weight and adding the products. This is illustrated in the following example.

Example 1.6 Convert the following hexadecimal numbers into decimal numbers. (a) A3BH and (b) 2F3H

Solution

(a)
$$\begin{aligned} A3BH = (A3B)_{16} &= A\times16^2+3\times16^1+B\times16^0 \\ &= 10\times16^2+3\times16^1+11\times16^0 \\ &= 10\times256+3\times16+11\times1 \\ &= 2560+48+11 \\ &= (2619)_{10} \end{aligned}$$

(b)
$$\begin{aligned} 2F3H = (2F3)_{16} &= 2\times16^2+F\times16^1+3\times16^0 \\ &= 2\times256+15\times16+3\times1 \\ &= 512+240+3 \\ &= (755)_{10} \end{aligned}$$

1.2.6 Hexadecimal–Binary Conversion

Conversion from hexadecimal to binary and vice versa can be easily carried out. For arriving at the binary equivalent of a hexadecimal number, each significant digit in the given number is replaced by its 4-bit binary equivalent.

For example,

$$\begin{aligned} (2D5)_{16} &= \;2 \qquad D \qquad 5 \\ &= 0010 \quad 1101 \quad 0101 \end{aligned}$$

Thus, $(2D5)_{16} = (0010\,1101\,0101)_2$. The reverse procedure is used for converting a binary number to an hexadecimal, i.e. starting from the least significant bit, each group of 4 bits is replaced by its decimal equivalents.

For example,

$$\begin{aligned} (11110110101)_2 &= 111 \quad 1011 \quad 0101 \\ &= \;\;7 \qquad B \qquad 5 \end{aligned}$$

Thus,

$$(11110110101)_2 = (7B5)_{16}$$

1.2.7 Hexadecimal–Octal Conversion

Conversion from hexadecimal to octal and vice versa is sometimes required. To convert a hexadecimal number to octal, the following steps can be applied

(i) Convert the given hexadecimal number to its binary equivalent.
(ii) Form groups of 3 bits, starting from the LSB (least significant digit).
(iii) Write the equivalent octal number for each group of 3 bits.

For example,

$$\begin{aligned} (47)_{16} &= (0100 \quad 0111)_2 \\ &= (01\,000\,111)_2 \\ &= (107)_8 \end{aligned}$$

Thus, 47 in hexadecimal is equivalent to 107 in the octal number system.

To convert an octal number to hexadecimal, the steps are as follows:

(i) Convert the given octal number to its binary equivalent.
(ii) Form groups of 4 bits, starting from the LSB.
(iii) Write the equivalent hexadecimal number for each group of 4 bits.

For example,

$$\begin{aligned} (32)_8 &= (011 \quad 010)_2 \\ &= (01 \quad 1010)_2 \\ &= (1A)_{16} \end{aligned}$$

Thus, 32 in octal is equivalent to 1A in the hexadecimal number system.

1.3 FLOATING POINT REPRESENTATION OF NUMBERS

In the decimal system, very large and very small numbers are expressed in scientific notation as follows: $4\cdot69\times10^{23}$ and $1\cdot601\times10^{-19}$. Binary numbers can also be expressed by the floating point representation. The floating point representation of a number consists of two parts: the first part represents a signed, fixed point number called the *mantissa* (*m*); the second part designates the position of the decimal (or binary) point and is called the *exponent* (*e*). The fixed point mantissa may be a fraction or an integer. The number of bits required to express the exponent and mantissa is determined by the accuracy desired from the computing system as well as its capability to handle such numbers. For example, the decimal number + 6132.789 is represented in floating point as follows:

$$\overset{\text{sign}}{0}\;\underbrace{0\cdot6132789}_{\text{mantissa}}\qquad\underbrace{\overset{\text{sign}}{0}\;\;04}_{\text{exponent}}$$

The mantissa has a 0 in the leftmost position to denote a plus. Here, the mantissa is considered to be a fixed point fraction. This representation is equivalent to the number expressed as a fraction 10 times by an exponent, that is $0.6132789\times10^{+04}$. Because of this analogy, the mantissa is sometimes called the *fraction part*.

Consider, for example, a computer that assumes integer representation for the mantissa and radix 8 for the numbers. The octal number $+\,36.754 = 36754\times8^{-3}$ in its floating point representation will look like this:

$$\underbrace{\overset{\text{sign}}{0}\;\;36754}_{\text{mantissa}}\qquad\underbrace{\overset{\text{sign}}{1}\;\;03}_{\text{exponent}}$$

When this number is represented in a register in its binary-coded form, the actual value of the register becomes 0 011 110 111 101 100 and 1 000 011.

Most computers and all electronic calculators have a built-in capacity to perform floating-point arithmetic operations.

Example 1.7 Determine the number of bits required to represent in floating point notation the exponent for decimal numbers in the range of $10^{\pm 86}$.

Solution Let n be the required number of bits to represent the number $10^{\pm 86}$.

$$2^n = 10^{86}$$
$$n \log 2 = 86$$
$$n = \frac{86}{\log 2} = \frac{86}{0.3010} = 285.7$$

Therefore, $10^{\pm 86} = 2^{\pm 285.7}$

The exponent ±285 can be represented by a 10-bit binary word. It has a range of exponents (+511 to −512).

1.4 ARITHMETIC OPERATION

Arithmetic operations in a computer are done using binary numbers and not decimal numbers and these take place in its *arithmetic unit*. The electronic circuit of a binary adder with suitable shift register can perform all arithmetic operations.

1.4.1 Binary Arithmetic

The arithmetic rules for Addition, Subtraction, Multiplication and Division of binary numbers are given below:

	Addition	Substraction	Multiplication	Division
(i)	$0 + 0 = 0$	$0 - 0 = 0$	$0 \times 0 = 0$	$0 \times 1 = 0$
(ii)	$0 + 1 = 1$	$1 - 0 = 1$	$0 \times 1 = 0$	$1 \times 1 = 1$
(iii)	$1 + 0 = 1$	$1 - 1 = 0$	$1 \times 0 = 0$	0×0 = not allowed
(iv)	$1 + 1 = 10$	$10 - 1 = 1$	$1 \times 1 = 1$	1×0 = not allowed

Binary addition Two binary numbers can be added in the same way as two decimal numbers are added. The addition is carried out from the least significant bits and it proceeds to higher significant bits, adding the carry resulting from the previous addition each time. Consider the addition of the binary numbers 1010 and 1111.

MSB LSB	Decimal
1 1 1 1	15
1 0 1 0	10
1 1 0 0 1	25

The addition carried out above can be explained as follows:

Step 1 The least significant bits are added, i.e. $0 + 1 = 1$ with a carry 0.

Step 2 The carry in the previous step is added to the next higher significant bits, i.e. $1 + 1 + 0 = 0$ with a carry 1.

Step 3 The carry in the above step is added to the next higher significant bits; i.e. $0 + 1 + 1 = 0$ with a carry 1.

Step 4 The preceding carry is added to the most significant bits, i.e. $1 + 1 + 1 = 1$ with a carry 1.

Thus, the sum is 11001. The addition is also shown in the decimal number system in order to compare the results.

Binary subtraction Binary subtraction is also carried out in the same way as decimal numbers are subtracted. The subtraction is carried out from the least significant bits and proceeds to the higher significant bits. When 1 is subtracted from 0, a 1 is borrowed from the immediate higher significant bit. The following problem explains the steps involved. Suppose that 1001 is subtracted from 1101.

Case 1:

MSB LSB	Decimal
1 1 0 1	13
1 0 0 1	9
0 1 0 0	4

The steps are described below.

Step 1 The LSB in the first column are 1 and 1. Hence, the difference is $1 - 1 = 0$.

Step 2 In the second column, the subtraction is performed as $0 - 0 = 0$.

Step 3 In the third column, the difference is given by $1 - 0 = 1$.

Step 4 In the fourth column (MSB), the difference is given by $1 - 1 = 0$.

Thus, the difference between the two binary numbers is 0100.

Case 2:

	Decimal
1 10	
1 0 0 1	9
0 1 1 1	7
0 0 1 0	2

The steps are described below.

Step 1 The least significant bits in the first column are 1 and 1. Hence, the difference is $1 - 1 = 0$.

Step 2 In the second column, it is not possible to subtract the 1 from 0. So, a 1 has to be borrowed from the next MSB (3rd bit). But since the 3rd bit is also 0, borrowing has to be done from the MSB (4th bit). The borrowing of 1 from the 4th bit (with weight 8) results in 1 and 10 with weight 4 in the 3rd column and 0 in 4th column as shown above. Now, the subtraction is performed as $10 - 1 = 1$.

Step 3 In the third column, the difference is given by $1 - 1 = 0$.

Step 4 In the fourth column (MSB), the difference is given by 0 – 0 = 0.

Thus, the difference between the two binary numbers is 0010.

Binary multiplication Binary multiplication is much simpler than decimal multiplication. The procedure is same as that of decimal multiplication. The binary multiplication procedure is as follows.

Step 1 The least significant bit of the multiplier is taken. If the multiplier bit is 1, the multiplicant is copied as such and, if the multiplier bit is 0, a 0 is placed in all the bit positions.

Step 2 The next higher significant bit of the multiplier is taken and the partial product is written with a shift to the left, as in step 1.

Step 3 Step 2 is repeated for all other higher significant bits and each time a left shift is given.

Step 4 When all the bits in the multiplier have been taken into account, the partial product terms are added, which give the actual product of the multiplier and the multiplicant. The following examples illustrate the multiplication procedure.

Example 1.8 Multiply the following binary numbers: (a) 1011 and 1101, (b) 100110 and 1001 and (c) 1·01 and 10·1.

Solution

(a) 1011 × 1101

```
          1 0 1 1
        × 1 1 0 1
        ---------
          1 0 1 1
        0 0 0 0
      1 0 1 1
    1 0 1 1
  ---------------
  1 0 0 0 1 1 1 1
```

(b) 100110 × 1001

```
          1 0 0 1 1 0
        ×     1 0 0 1
        -------------
          1 0 0 1 1 0
        0 0 0 0 0 0
      0 0 0 0 0 0
    1 0 0 1 1 0
  -------------------
    1 0 1 0 1 0 1 1 0
```

(c) 1.01 × 10.1

```
      1· 0 1
    × 1  0· 1
    ---------
      1  0  1
    0 0  0
  1 0 1
  ---------
  1 1· 0 0 1
```


Binary division Division in binary follows the same procedure as division in decimal. Division by 0 is meaningless. An example is given below.

Example 1.9 Divide the following: (a) 11001 ÷ 101 (b) 11101 ÷ 1100

Solution

(a) 11001 ÷ 101

```
       101
    ------
101)11001
    101
    ----
     0101
      101
     ----
     0000
```

(b) 11101 ÷ 1100

```
          10.01101
     -------------
1100)11101.00000
     1100
     -----
      10100
       1100
      -----
      10000
       1100
     ------
      010000
        1100
     ------
        0100
```

1.5 1's AND 2's COMPLEMENTS

Subtraction of a number from another can be accomplished by adding the complement of the subtrahend to the minuend. The exact difference can be obtained with minor manipulations.

1.5.1 1's Complement Subtraction

Subtraction of binary numbers using the 1's complement method allows subtraction only by addition. The 1's complement of a binary number can be obtained by changing all 1s to 0s and all 0s to 1s. To subtract a smaller number from a larger number, the 1's complement method is as follows:

(i) Determine the 1's complement of the smaller number.

(ii) Add this to the larger number.

(iii) Remove the carry and add it to the result. This carry is called *end-around-carry*.

Example 1.10 Subtract $(1010)_2$ from $(1111)_2$ using the 1's complement method. Also subtract using direct method and compare.

Solution

```
Direct subtraction                    1's Complement method
  1 1 1 1                                  1 1 1 1 (+)
 -1 0 1 0            1's Complement →      0 1 0 1
  0 1 0 1            Carry          →    1 0 1 0 0
                     Add Carry      →            1
                                           0 1 0 1
```

Subtraction of a larger number from a smaller one by the 1's complement method involves the following steps:

(i) Determine the 1's complement of the larger number.
(ii) Add this to the smaller number.
(iii) The answer is the 1's complement of the true result and is opposite in sign. There is no carry.

Example 1.11 Subtract $(1010)_2$ from $(1000)_2$ using the 1's complement method. Also subtract by direct method and compare.

Solution

```
Direct subtraction                    1's Complement method
  1 0 0 0                                1 0 0 0 (+)
 -1 0 1 0            1's Complement →    0 1 0 1
 -0 0 1 0                                1 1 0 1
```

No carry is obtained. The answer is the 1's complement of 1101 and is opposite in sign, i.e. –0010.

The 1's complement method is particularly useful in arithmetic logic circuits because subtraction can be accomplished with the help of an adder.

1.5.2 2's Complement Subtraction

The 2's complement of a binary number can be obtained by adding 1 to its 1's complement. Subtraction of a smaller number from a larger one by the 2's complement method involves following steps:

(i) Determine the 2's complement of the smaller number.
(ii) Add this to the larger number.
(iii) Omit the carry (there is always a carry in this case).

Example 1.12 Subtract $(1010)_2$ from $(1111)_2$ using the 2's complement method. Subtract by direct method also and compare.

Solution

```
Direct subtraction                    2's Complement method
  1 1 1 1                                  1 1 1 1 (+)
 -1 0 1 0            2's Complement →      0 1 1 0
  0 1 0 1            Carry          →    1 0 1 0 1
```


The carry is discarded. Thus, the answer is $(0101)_2$.

The 2's complement method for subtraction of a larger number from a smaller one is as follows:

(i) Determine the 2's complement of the larger number.
(ii) Add the 2's complement to the smaller number.
(iii) There is no carry. The result is in 2's complement form and is negative.
(iv) To get an answer in true form, take the 2's complement and change the sign.

Example 1.13 Subtract $(1010)_2$ from $(1000)_2$ using 2's complement method. Subtract by direct method also and compare.

Solution

```
Direct subtraction                    2's Complement method
  1 0 0 0                                1 0 0 0 (+)
 -1 0 1 0            2's Complement →    0 1 1 0
  0 0 1 0            No carry       →    1 1 1 0
```

No carry is obtained. Thus, the difference is negative and the true answer is the 2's complement of $(1110)_2$, i.e. $(0010)_2$.

Though both 1's and 2's complement methods of subtraction seem complex compared to the direct method of subtraction, both have distinct advantages when applied using logic circuits, because they allow subtraction to be done using only addition. The 1's and 2's complements of a binary number can be easily arrived at using logic circuits; the advantage in 2's complement method is that the *end-around-carry* operation present in the 1's complement method is not involved here.

1.5.3 Signed Binary Number Representation

Binary numbers are represented with a separate sign bit along with the magnitude, as shown below. For example, in an 8-bit binary number, the MSB is the sign bit and the remaining 7 bits correspond to magnitude. The magnitude part contains true binary equivalent of the number for positive numbers, while 2's complement form of the number for negative numbers. For example, +13, 0, –46 are represented as follows:

	Sign	Magnitude
+13	0	000 1101
0	0	000 0000
–46	1	010 1110

It is important to note that the number zero is assigned with the sign bit '0'. Therefore, the range of numbers that can be represented using 8-bit binary number is –128 to +127. In general, the range of numbers that can be represented by an n-bit number is (-2^{n-1}) to $(+2^{n-1}-1)$.

1.5.4 Addition in the 2's Complement System

Addition can be explained with four possible cases: (i) when both the numbers are positive, (ii) when augend is a positive and addend is a negative number, (iii) when augend is a negative and addend is a positive number, or (iv) when both the numbers are negative.

Case 1 Two Positive Numbers

Consider the addition of +29 and +19.

+29	→	0	001	1101	(augend)
+19	→	0	001	0011	(addend)
		0	011	0000	(sum = 48)
		↑ Sign bit			

The sign bits of both *augend* and *addend* are zero and the sign bit of the sum is 0, indicating that when the sum is positive they have the same number of bits. This must always be done in the 2's complement system.

Case 2 Positive augend Number and Negative addend Number

Consider the addition of +39 and −22. Remember that −22 will be in its 2's complement form. Therefore, +22 [00010110] must be converted to −22 [11101010].

	+39	→	0	010	0111	(augend)
	−22	→	1	110	1010	(addend)
1			0	001	0001	(result = 17)
↑ Carry			↑ Sign bit			

This carry is omitted and hence the result is 0001 0001.

In this case, the sign bit of addend is 1. Sign bits also participate in the process of addition. In fact, a carry is generated in the last position of addition. This carry is always omitted. Therefore, the final sum is 0001 0001, which is equivalent to +17.

Case 3 Positive addend Number and Negative augend Number

Consider the addition of −47 and +29.

−47	→	1	101	0001	(augend)
+29	→	0	001	1101	(addend)
		1	110	1110	(result = −18)
		↑ Sign bit			

The result has a sign bit of 1, indicating a negative number. It is in the 2's complement form. The last seven bits 110 1110 actually represent the 2's complement of the sum. The true magnitude of the sum can be found by taking the 2's complement of 110 1110; the result is 10010 (+18). Thus, 1 110 1110 represents −18.

Case 4 Two Negative Numbers

Consider the addition of −32 and −44.

	−32	→	1	110	0000	(augend)
	−44	→	1	101	0100	(addend)
1			1	011	0100	(result = −76)
↑ Carry			↑ Sign bit			

The carry is discarded and hence the result is 1011.0100

The true magnitude of the sum is the 2's complement of 011 0100, i.e. 1 1[illegible]01100 (−76). Thus, the 2's complement addition works in every case. This assumes that the decimal sum is within −128 to +127 range. Otherwise, we get an overflow.

1.5.5 Subtraction in the 2's Complement System

As in the case of addition, subtraction can also be carried out in four possible cases. Subtraction by the 2's complement system involves addition.

Case 1 Both the Numbers are Positive

Consider the case where +19 is to be subtracted from +28.

+28 → 0001 1100
+19 → 0001 0011

To subtract +19 from +28, the computer will send the +19 to a 2's complement circuit to produce

−19 → 1110 1101

The system will then add +28 and −19 as follows:

+28	→	0001	1100
−19	→	1110	1101
(Sum = 9)		10000	1001
		↑ Omit this carry	

Case 2 Positive Number and Smaller Negative Number

Consider that the minuend is +39 and the subtrahend is −21. In the 2's complement system, they appear as

+39 → 0010 0111
−21 → 1110 1011

The computer sends −21 to a 2's complement circuit to produce

+21 → 0001 0101

It then adds +39 and +21 as follows:

+39	→	0010	0111
+21	→	0001	0101
(Sum = 60)		0011	1100

Case 3 Positive Number and Larger Negative Number

Consider that the minuend is +19 and the subtrahend is −43. In the 2's complement system, they appear as

+19	→	0001 0011
−43	→	1101 0101

The computer sends the 2's complement of −43, i.e.

+43	→	0010 1011

It then adds +19 and +43 as shown below:

+19	→	0001 0011
+43	→	0010 1011
(Sum = 62)		0011 1110

Case 4 Both the Numbers are Negative

Consider the subtraction of −33 from −57. In the 2's complement representation,

−57	→	1100 0111
−33	→	1101 1111

Taking the 2's complement of −33;

+33	→	0010 0001

Then add +33 to −57. We have

−57	→	1100 0111
+33	→	0010 0001
(−24)		1110 1000

1.5.6 Arithmetic Overflow

When the number of bits in the *sum* exceeds the number of bits in each of the numbers added, *overflow* results. This appears in the ninth significant place, and is also called the excess-one. Overflow causes a sign change.

Assume that both the input numbers are in the range of −128 to +127. The problem arises only when the arithmetic circuit adds two positive numbers or two negative numbers. In such a case, it is possible for the sum to be outside the range of −128 to +127.

Case 1 Two Positive Numbers

Consider the addition of +120 and +65. As the decimal sum of +120 and +65 is +185, an overflow occurs into the MSD position. This overflow forces the sign bit of the answer to change.

+120	→	0111 1000
+65	→	+0100 0001
(185)		1011 1001

As the sign bit is 1, i.e. negative, the answer is not correct.

Case 2 Two Negative Numbers

Consider the addition of −77 and −122.

−77	→	1011 0011		
+(−122)	→	+1000 0110		
−199		1 0011 1001	→	0011 1001

The 8-bit answer is 0011 1001. Here the sign bit is positive. As the right answer has to contain a negative sign bit, the answer is not correct.

An overflow is a software problem and not a hardware problem. In digital computers, an overflow occurs when an operation results in a quantity beyond the capacity of the storage register. Therefore, a programmer must check the overflow after each addition or subtraction by looking for a change in the sign bit. Logic circuitry is used in each case to detect overflow.

1.5.7 Comparison Between 1's and 2's Complements

(i) The 1's complement can be easily obtained using an inverter. The 2's complement has to be arrived at by first obtaining the 1's complement and then adding one (1) to it.

(ii) The advantage in the 2's complement system is that only one arithmetic operation is required; the 1's complement requires two operations.

(iii) While the 1's complement is often used in logical manipulations for inversion operation, the 2's complement is used only for arithmetic applications.

1.6 9's COMPLEMENT

The 9's complement of a decimal number can be found by subtracting each digit in the number from 9. The 9's complement of decimal digits 0 to 9 is shown below:

Decimal digit	9's complement
0	9
1	8
2	7
3	6
4	5
5	4
6	3
7	2
8	1
9	0

Example 1.14 Find the 9's complement of each of the following decimal numbers:

(a) 19 (b) 146 (c) 469 and (d) 4397

Solution: Subtract each digit in the number from 9 to get the 9's complement.

(a)

$$\begin{array}{r} 99 \\ -19 \\ \hline 80 \end{array} \rightarrow \text{9's Complement of 19}$$

(b)

$$\begin{array}{r} 999 \\ -146 \\ \hline 853 \end{array} \rightarrow \text{9's Complement of 146}$$

(c)

$$\begin{array}{r} 999 \\ -469 \\ \hline 530 \end{array} \rightarrow \text{9's Complement of 469}$$

(d)

$$\begin{array}{r} 9999 \\ -4397 \\ \hline 5602 \end{array} \rightarrow \text{9's Complement of 4397}$$

1.6.1 9's Complement Subtraction

Subtraction of a smaller decimal number from a larger one in the 9's complement system is done by the addition of the 9's complement of the subtrahend to the minuend and then adding the carry to the result. Subtraction of a larger number from a smaller one does not produce a carry, and the result is a negative in the 9's complement form. This procedure has a distinct advantage in certain types of arithmetic logic.

Example 1.15 Perform the following subtractions by using the 9's complement method: (a) 18–06, (b) 39–23, (c) 34–49 and (d) 49–84.

Solution

	Regular subtraction	9's Complement subtraction	
(a)	18	18	
	−06	+93	← 9's Complement of 6
	12	(1) 11	
		+1	Add carry to result
		12	
(b)	39	39	
	−23	+76	← 9's Complement of 23
	16	(1) 15	
		+1	Add carry to result
		16	
(c)	34	34	
	−49	+50	← 9's Complement of 49
	−15	84	
		↓	
		−15	← 9's Complement of 84
(d)	49	49	
	−84	+15	← 9's Complement of 84
	−35	64	
		↓	
		−35	← 9's Complement of 64

1.7 10's COMPLEMENT

The 10's complement of a decimal number is equal to its 9's complement +1.

Example 1.16 Convert the following decimal numbers into its 10's complement form: (a) 9, (b) 46 and (c) 739.

Solution

(a)

$$\begin{array}{r} 9 \\ -9 \\ \hline 0 \\ +1 \\ \hline 1 \end{array} \begin{array}{l} \\ \\ \leftarrow \text{9's Complement of 9} \\ \\ \leftarrow \text{10's Complement of 9} \end{array}$$

(b)

$$\begin{array}{r} 99 \\ -46 \\ \hline 53 \\ +1 \\ \hline 54 \end{array} \begin{array}{l} \\ \\ \leftarrow \text{9's Complement of 46} \\ \\ \leftarrow \text{10's Complement of 46} \end{array}$$

(c)

$$\begin{array}{r} 999 \\ -739 \\ \hline 260 \\ +1 \\ \hline 261 \end{array} \begin{array}{l} \\ \\ \leftarrow \text{9's Complement of 739} \\ \\ \leftarrow \text{10's Complement of 739} \end{array}$$

1.7.1 10's Complement Subtraction

In the 10's complement method of subtraction, the minuend is added to the 10's complement of the subtrahend and the carry is dropped.

Example 1.17 Subtract the following decimal numbers using the 10's complement method:

(a) 9–4, (b) 24–09, (c) 69–32 and (d) 347–265.

Solution

	Regular subtraction	10's Complement subtraction	
(a)	9	9	
	−4	+6	← 10's Complement of 4
	5	(1) 5	Drop carry
(b)	24	24	
	−09	+91	← 10's Complement of 9
	15	(1) 15	Drop carry
(c)	69	69	
	−32	+68	← 10's Complement of 32
	37	(1) 37	Drop carry
(d)	347	347	
	−265	+735	← 10's Complement of 265
	82	(1) 082	Drop carry

1.8 BINARY CODED DECIMAL (BCD)

The Binary Coded Decimal (BCD) is a combination of four binary digits that represent decimal numbers. For example, the 8421 code is a type of binary coded decimal. It has four bits and represents the decimal digits 0 to 9. The numbers 8421 indicate the binary weights of the four bits. The ease of conversion between the 8421 code numbers and the familiar decimal numbers is the main advantage of this code. To express any decimal number in BCD, each decimal digit should be replaced by the appropriate four-bit code. Table 1.1 gives the binary and BCD codes for the decimal numbers 0 to 15.

Table 1.1 Decimal numbers, binary equivalents and BCD

Decimal number	Binary number	Binary coded decimal (BCD)
0	0000	0000
1	0001	0001
2	0010	0010
3	0011	0011
4	0100	0100
5	0101	0101
6	0110	0110
7	0111	0111
8	1000	1000
9	1001	1001
10	1010	0001 0000
11	1011	0001 0001
12	1100	0001 0010
13	1101	0001 0011
14	1110	0001 0100
15	1111	0001 0101

1.8.1 BCD Addition

BCD is a numerical code. Many applications require arithmetic operations. Addition is the most important of these because the other three operations, namely subtraction, multiplication and division, can be done using addition. The rule for addition of two BCD numbers is given below.

(i) Add the two numbers using the rules for binary addition.
(ii) If a four-bit sum is equal to or less than 9, it is a valid BCD number.
(iii) If a four-bit sum is greater than 9, or if a carry-out of the group is generated, it is an invalid result. Add 6 (0110_2) to the four-bit sum in order to skip the six invalid states and return the code to BCD. If a carry results when 6 is added, add the carry to the next four-bit group.

Example 1.18 Add the following BCD numbers: (a) 1001 and 0100 and (b) 00011001 and 00010100.

Solution

(a)

```
          1 0 0 1
        + 0 1 0 0
          1 1 0 1  → Invalid BCD number          9
        + 0 1 1 0  → Add 6                      +4
0 0 0 1   0 0 1 1  → Valid BCD number           13₁₀
```

(b)

```
  0 0 0 1   1 0 0 1
+ 0 0 0 1   0 1 0 0
  0 0 1 0   1 1 0 1  → Right group is invalid     19
          + 0 1 1 0  → Add 6                     +14
  0 0 1 1   0 0 1 1  → Valid BCD number           33₁₀
     3         3
```

1.8.2 BCD Subtraction

Method I Table 1.2 shows an algorithm for BCD subtraction. The 1's complement of the BCD subtrahend is entered into adder 1, and the complement (true) of the result is transferred to adder 2, where either a 1010 or 0000 is added, depending on the sign of the total result. Examples of a positive and negative total result are given in Table 1.2. Arrows indicate EAC (end-around-carry) or carry to the next decade.

Table 1.2 Algorithm for BCD subtraction

Decade result	Sign of total result	
	(+) EAC = 1	(−) EAC = 0
$C_n = 1$ $C_n = 0$	Transfer true results of adder 1 0000 added in adder 2 1010 added in adder 2	Transfer 1's complement of result of adder 1 1010 added in adder 2 0000 added in adder 2

Total Result Positive:

```
          10²      10¹    10⁰     EAC indicates a (+) total result
  835     1000     0011   0101
− 274     1101     1000   1011  ← 1's Complement of 0010 0111 0100
+ 561   1 0101     1011   0000
                      1      1
                   1100   0001    Transfer true output of adder 1
          0101     1100   0001
          0000     1010   0000
          0101   1 0110   0001
            5         6      1    Ignore this carry 1
```

Total Result Negative:

	10^2	10^1	10^0	
429	0100	0010	1001	No EAC Indicates (–) total result
–476	1011	1000	1001	← 1's Complement of 0100 0111 0110
–47	1111	1010	0010	
		1		
		1011		
	0000	0100	1101	
	0000	0000	1010	Transfer 1's Complement of adder 1 output
	0000	0100	1 0111	Ignore this carry

Method II Another method in BCD subtraction is the addition of the 9's complement of the subtrahend to the minuend.

Example 1.19 Subtract 748 from 983 using 9's complement method.

Solution

9's complement of 748 = 999 – 748 = 251

Direct method

983	983		
–748	+251	← 9's Complement of 748	
235	1 234		
	1	← EAC	
	235		

Example 1.20 Subtract the following using 9's complement method:

(a) 649–387 and (b) 891–786

Solution

(a)

649	649	
–387	+612	← 9's Complement of 387
262	1 261	Ignore carry
	1	← EAC
	262	

(b)

891	891	
–786	+213	← 9's Complement of 786
105	1 104	
	1	← EAC
	105	

Example 1.21 Subtract the following using the 10's complement method.

(a) 786 – 427, (b) 473 – 438 and (c) 357 – 294

Solution

(a)

786	789	
–427	+573	← 10's Complement of 427
359	1 359	Ignore carry

(b)

473	473	
–438	+562	← 10's Complement of 438
35	1 035	Ignore carry

(c)

357	357	
–294	+706	← 10's Complement of 294
63	1 063	Ignore carry

Example 1.22 Carry out BCD subtraction for (68) – (61) using 10's complement method.

Solution

68	68			1	1
–61	+39	← 10's Complement of 61		0110	1000
07	1 07	Ignore carry		0011	1001
				1010	0001
			Add 6	0110	0110
			Ignore carry	1 0000	0111

1.9 CODES

Code is a symbolic representation of discrete information, which may be present in the form of numbers, letters or physical quantities. The symbols used are the binary digits 0 and 1 which are arranged according to the rules of codes. These codes are used to communicate information to a digital computer and to retrieve messages from it. A code is used to enable an operator to feed data into a computer directly, in the form of decimal numbers, alphabets and special characters. The computer converts these data into binary codes and after computation, transforms the data into its original format (decimal numbers, alphabets and special characters).

When numbers, letters, or words are represented by a special group of symbols, this is called *encoding*, and the group of symbols is called a *code*. In Morse code, a series of dots and dashes represent alphabet, numerals and special characters.

Codes are broadly classified into five groups, *viz.*, (i) Weighted Binary Codes, (ii) Non-weighted Codes, (iii) Error-detecting Codes, (iv) Error-correcting Codes, and (v) Alphanumeric Codes.

1.9.1 Weighted Binary Codes

Weighted binary codes obey their positional weighting principles. Each position of a number represents a specific weight. In a weighted binary code, the bits are multiplied by the weights indicated; the sum of these weighted bits gives the equivalent decimal digit.

Straight binary coding is a method of representing a decimal number by its binary equivalent. The codes 8421, 2421, 5421 and 5211 are weighted binary codes. Each decimal digit is represented by a four-bit binary word, the three digits for the left being weighted. Table 1.3 consists of a few weighted 4-bit binary codes with their decimal numbers and complements.

Table 1.3 Some weighted 4-bit binary codes

Decimal number	8421	5421	2421	9's complement of 2421 code
0	0000	0000	0000	1111
1	0001	0001	0001	1110
2	0010	0010	0010	1101
3	0011	0011	0011	1100
4	0100	0100	0100	1011
5	0101	1000	1011	0100
6	0110	1001	1100	0011
7	0111	1010	1101	0010
8	1000	1011	1110	0001
9	1001	1100	1111	0000

BCD (or) 8421 code The Binary-coded Decimal (BCD) uses the binary number system to specify the decimal numbers 0 to 9. It has four bits. The weights are assigned according to the positions occupied by these digits. The weights of the first (right-most) position is 2^0(1), the second 2^1 (2), the third 2^2(4), and the fourth 2^3 (8). Reading from left to right, the weights are 8-4-2-1, and hence it is called 8421 code.

The binary equivalent of 7 is $[111]_2$, but the same number is represented in BCD in 4-bit form as $[0111]_2$. Also, the numbers from 0 to 9 are represented in the same way as in the binary system, but after 9 the representations in BCD are different. For example, the decimal number 12 in the binary system is $[1100]_2$ but the same number is represented as [0001 0010] in BCD.

Example 1.23 Give the BCD Code for the decimal number 874.

Solution
Decimal number → 874
BCD code → 1000 0111 0100
Hence, $(874)_{10} = (1000\ 0111\ 0100)_{BCD}$

Example 1.24 Give the BCD code equivalent for the decimal number 96.42.

Solution
Decimal number → 9 6 . 4 2
BCD Code → 1001 0110 . 0100 0010
Hence, $[96.42]_{10} = [1001\ 0110 . 0100\ 0010]_{BCD}$

2421 code This is a weighted code; its weights are 2, 4, 2 and 1. A decimal number is represented in 4-bit form and the total weight of the four bits = 2 + 4 + 2 + 1 = 9. Hence the 2421 code represents decimal numbers from 0 to 9. Upto 4, the 2421 code is the same as that in BCD; however, it varies for digits from 5 to 9. This code is also a self-complementing code i.e. the 9's complement of a number 'N' is obtained by complementing the 0s and 1s in the code word 'N'. For example, the 2421 code for 3 is 0011 and its natural complement 1100 gives 6 which is the 9's complement of 3. Table 1.3 gives the 2421 code of the decimal numbers and its complement. The bit combination 1101, when weighted by the reflective digits 2421, gives the decimal equivalent of $2 \times 1 + 4 \times 1 + 2 \times 0 + 1 \times 1 = 2 + 4 + 0 + 1 = 7$.

Reflective codes A code is said to be *reflective* when the code for 9 is the complement of the code for 0, 8 for 1, 7 for 2, 6 for 3, and 5 for 4. While the 2421, 5211 and Excess-3 codes are reflective codes, the 8421 code is not. While finding the 9's complement, such as in 9's complement subtraction, reflectivity is desirable in a code.

Sequential codes A code can be said to be sequential when each succeeding code is one binary number greater than its preceding code. This greatly helps mathematical manipulation of data. While the 8421 and Excess-3 codes are sequential, the 2421 and 5421 codes are not.

1.9.2 Non-weighted Codes

Non-weighted codes are codes that are not positionally weighted. This means that each position within a binary number is not assigned a fixed value. Excess-3 codes and Gray codes are examples of non-weighted codes.

Excess-3 code As the name indicates, the *excess-3* represents a decimal number, in binary form, as a number greater than 3. An excess-3 code is obtained by adding 3 to a decimal number. For example, to encode the decimal number 6 into an excess-3 code, we must first add 3 in order to obtain 9. The 9 is then encoded in its equivalent 4-bit binary code 1001. The excess-3 code is a self-complementing code, and this helps in performing subtraction operations in digital computers, especially in the earlier models. The excess-3 code is also a reflective code.

Example 1.25 Convert $[643]_{10}$ into its Excess-3 code.

Solution

Decimal number	6	4	3
Add 3 to each bit.	+3	+3	+3
Sum →	9	7	6

Converting the above sum into its BCD code, we have

$$\text{Sum} \rightarrow 9 \quad 7 \quad 6$$
$$\downarrow \quad \downarrow \quad \downarrow$$
$$\text{BCD} \rightarrow 1001\ 0111\ 0110$$

Hence, the Excess-3 code for $[643]_{10}$ is 1001 0111 0110.

Table 1.4 lists the BCD, Excess-3 code and 9's complement representations for decimal digits. Note that both codes use only 10 of the 16 possible 4-bit code groups. The excess-3 code, however, does not use the same code groups. Its invalid code groups are 0000, 0001, 0010, 1101, 1110, and 1111.

Table 1.4 Excess-3 codes

Decimal	8421(BCD) code	Excess-3 code	9's complement
0	0000	0011	1100
1	0001	0100	1011
2	0010	0101	1010
3	0011	0110	1001
4	0100	0111	1000
5	0101	1000	0111
6	0110	1001	0110
7	0111	1010	0101
8	1000	1011	0100
9	1001	1100	0011

Gray codes The *Gray code* belongs to a class of codes called *minimum-change codes*, in which only one bit in the code group changes when moving from one step to the next. The Gray code is a *non-weighted code*. Therefore, it is not suitable for arithmetic operations but finds applications in input/output devices and in some types of analog-to-digital converters. The Gray code is a reflective digital code which has a special property of containing two adjacent code numbers that differ by only one bit. Therefore, it is also called a *unit-distance code*.

Table 1.5 shows the Gray code representation for the decimal numbers 0 to 15, together with the straight binary code.

Table 1.5 Gray code

Decimal numbers	Binary code	Gray code
0	0000	0000
1	0001	0001
2	0010	0011
3	0011	0010
4	0100	0110
5	0101	0111
6	0110	0101

(Table 1.5 Contd.)

Decimal numbers	Binary code	Gray code
7	0111	0100
8	1000	1100
9	1001	1101
10	1010	1111
11	1011	1110
12	1100	1010
13	1101	1011
14	1110	1001
15	1111	1000

Conversion of a binary number to gray code A binary number can be converted to its Gray code when

(i) the first bit (MSB) of the Gray code is the same as the first bit of the binary number;

(ii) the second bit of the Gray code equals the exclusive-OR of the first and second bits of the binary number, i.e. it will be 1 if these binary code bits are different and 0 if they are the same;

(iii) the third Gray code bit equals the exclusive-OR of the second and third bits of the binary number, and so on.

Example 1.26 Convert $[10110]_2$ to Gray code.

Solution

Step 1 The first bit MSB of the Gray code is the same as the first bit of the binary number.

$$1 \quad 0 \quad 1 \quad 1 \quad 0 \quad \text{Binary}$$
$$\downarrow$$
$$1 \qquad\qquad\qquad \text{Gray}$$

Step 2 Add the first bit of the binary digit to the second bit of the binary. The addition of 1 and 0 is 1. The result is the second bit of the Gray code.

$$\boxed{1 \oplus 0} \quad 1 \quad 1 \quad 0 \quad \text{Binary}$$
$$\downarrow$$
$$1 \quad 1 \qquad\qquad \text{Gray}$$

Step 3 Add the second bit 0 to the third bit 1 of the binary.

$$1 \quad \boxed{0 \oplus 1} \quad 1 \quad 0 \quad \text{Binary}$$
$$\downarrow$$
$$1 \quad 1 \quad 1 \qquad \text{Gray}$$

Step 4 Add the third bit 1 to the fourth bit 1 and omit the carry. The exclusive OR addition of 1 and 1 is 0. This is the fourth bit of the Gray code.

$1 \quad 0 \quad \boxed{1 \oplus 1} \quad 0$ Binary

$\qquad\qquad\downarrow$

$1 \quad 1 \quad 1 \quad 0$ Gray

Step 5 Add the fourth bit 1 to the last bit of the binary.

$1 \quad 0 \quad 1 \quad \boxed{1 \oplus 0}$ Binary

$\qquad\qquad\qquad\downarrow$

$1 \quad 1 \quad 1 \quad 0 \quad 1$ Gray

As the conversion is complete, the Gray code of the binary 10110 is 11101.

Example 1.27 Convert the binary $[10101101]_2$ to its Gray code.

Solution

$$\begin{array}{ccccccccccccccccl} 1 & \oplus & 0 & \oplus & 1 & \oplus & 0 & \oplus & 1 & \oplus & 1 & \oplus & 0 & \oplus & 1 & & \text{Binary} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ 1 & & 1 & & 1 & & 1 & & 1 & & 0 & & 1 & & 1 & & \text{Gray code} \end{array}$$

Conversion from gray code to binary Conversion of a Gray code into its binary form involves the reverse of the procedure given above:

(i) The first binary bit (MSB) is the same as that of the first Gray code bit.

(ii) If the second Gray bit is 0, the second binary bit is the same as that of the first binary; if the second Gray bit is 1, the second binary bit is the inverse of its first binary bit.

(iii) Step 2 is repeated for each successive bit.

Example 1.28 Convert the Gray code 110101 to binary form.

Solution

Step 1 Write the first binary bit 1 which is the MSB of the Gray code.

$1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1$ Gray

$\downarrow$

1 Binary

Step 2 The second bit of the Gray code is 1 and therefore the second bit of the binary is 0, i.e. inverse of the first binary bit '1'.

$1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1$ Gray

$\qquad\downarrow$

$1 \rightarrow 0$ Binary

Step 3 The third bit of the Gray code is 0 and therefore the third bit of the binary is same as that of the second binary bit, i.e. 0.

$1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1$ Gray

$\qquad\qquad\downarrow$

$1 \quad 0 \rightarrow 0$ Binary

Step 4

$1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1$ Gray

$\qquad\qquad\qquad\downarrow$

$1 \quad 0 \quad 0 \rightarrow 1$ Binary

Step 5

$1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1$ Gray

$\qquad\qquad\qquad\qquad\downarrow$

$1 \quad 0 \quad 0 \quad 1 \rightarrow 1$ Binary

Step 6

$1 \quad 1 \quad 0 \quad 1 \quad 0 \quad 1$ Gray

$\qquad\qquad\qquad\qquad\qquad\downarrow$

$1 \quad 0 \quad 0 \quad 1 \quad 1 \rightarrow 0$ Binary

Therefore, $[110101]_G = [100110]_2$.

Example 1.29 Convert $[1010111]_G$ to binary.

Solution

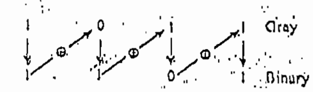
$$\begin{array}{cccccccccccccl} 1 & & 0 & & 1 & & 0 & & 1 & & 1 & & 1 & \text{Gray} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ 1 & \rightarrow & 1 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 1 & \rightarrow & 0 & \rightarrow & 1 & \text{Binary} \end{array}$$

Therefore, $[1010111]_G = [1100101]_2$

This process can be seen in another way. Each binary bit (except the first) can be obtained by taking the exclusive-OR of the corresponding Gray code bit and the previous binary bit. The reader should verify that this process gives the proper result.

Example 1.30 Convert $[1011]_G$ to binary.

Solution



Therefore, $[1011]_G = [1101]_2$

1.8.3 Error Detecting Codes

During the process of binary data transmission, errors may occur. In order to detect and correct such errors, two types of codes, namely (i) *error-detecting codes* and (ii) *error-correcting codes* may be used.

If a single error transforms a valid code word into an invalid one, it is said to be a single error-detecting code. The most simple and commonly used error detecting method is the *parity check*, in which an extra parity bit is included with the binary message, to make the total number of 1s either odd or even, resulting in two methods, viz. (i) *Even-parity* method and (ii) *Odd-parity* method. In the *even-parity* method, the total number of 1s in the code group (including the parity bit) must be an even number. Similarly, in the *odd-parity* method, the total number of 1s (including the parity bit) must be an odd number. The parity bit can be placed at either end of the code word, such that the receiver should be able to understand the parity bit and the actual data. Table 1.6 shows a message of three bits and its corresponding even and odd parity bits.

Table 1.6 Parity-bit generation

Message xyz	Even-parity code xyz	p	Odd-parity code xyz	p
000	000	0	000	1
001	001	1	001	0
010	010	1	010	0
011	011	0	011	1
100	100	1	100	0
101	101	0	101	1
110	110	0	110	1
111	111	1	111	0

If a single error occurs, it transforms the valid code into an invalid one. This helps in the detection of single bit errors. Though the parity code is meant for single error detection, it can detect any odd number of errors. However, in both the cases, the original code word can not be found. If an even number of errors occur, then the parity check is satisfied, giving an erroneous result.

Check sums The parity method can detect only a *single* error within a word and not *double errors*. Since the double error will not change the parity of the bits, the parity checker will not indicate any error. The check sum method is used to detect double errors and pinpoint erroneous bits. The working of this method is explained in the following lines.

Initially word A 10110111 is transmitted, next the word B 00100010 is transmitted. The binary digits in the two words are added and the sum obtained is retained in the transmitter. Then, a word C is transmitted and added to the previous sum and the new sum is retained. In the same manner, each word is added to the previous sum; after transmission of all the words, the final sum called the check sum is also transmitted. The same operation is done at the receiving end and the final sum obtained here is checked against the transmitted check sum. If the two sums are equal, then there is no error.

1.9.4 Error Correcting Codes

Hamming code R. W. Hamming developed a system that provides a methodical way to add one or more parity bits to a data character in order to detect and correct errors. The *Hamming distance* between two code words is defined as the number of bits changed from one code word to another.

Consider C_i and C_j to be any two code words in a particular block code. The Hamming distance d_{ij} between the two vectors C_i and C_j is defined by the number of

components in which they differ. Assuming that d_{ij} is determined for each pair of code words; the minimum value of the d_{ij} can be called the Hamming distance, d_{min}. For linear block codes, minimum weight is equal to minimum distance.

For example,

$$\begin{array}{l} C_i = 1\ 0\ 0\ 0\ 1\ 1\ 1 \\ \qquad\ \downarrow \qquad\ \ \downarrow\ \downarrow \\ C_j = 0\ 0\ 0\ 1\ 0\ 1\ 1 \end{array}$$

Here, these code words differ in the leftmost bit position and in the fourth and fifth bit positions from the left. Accordingly, $d_{ij} = 3$.

From Hamming's analysis of code distances, the following important properties have been derived:

(i) A minimum distance of at least two is required for single error detection.

(ii) Since the number of errors, $E \le [(d_{min} - 1)/2]$, a minimum distance of three is required for single error correction.

(iii) Greater distances will provide detection and/or correction of more number of errors.

The 7-bit Hamming (7, 4) code word $h_1\ h_2\ h_3\ h_4\ h_5\ h_6\ h_7$ associated with a 4-bit binary number $b_3\ b_2\ b_1\ b_0$ is:

$$\begin{array}{ll} h_1 = b_3 \oplus b_2 \oplus b_0 & h_3 = b_3 \\ h_2 = b_3 \oplus b_1 \oplus b_0 & h_5 = b_2 \\ h_4 = b_2 \oplus b_1 \oplus b_0 & h_6 = b_1 \\ & h_7 = b_0 \end{array}$$

where $\oplus$ denotes the Exclusive-OR operation. Note that bits h_1, h_2 and h_4 are even parity bits for the bit fields $b_3\ b_2\ b_0$, $b_3\ b_1\ b_0$ and $b_2\ b_1\ b_0$, respectively. In general, the parity bits (h_1, h_2, h_4, h_8, ...) are located in the positions corresponding to ascending powers of two (i.e., $2^0, 2^1, 2^2, 2^3, ... = 1, 2, 4, 8, ...$).

The h_1 parity bit has a 1 in the LSB of its binary representation. Therefore, it checks all bit positions, including it, that have 1's in the same location (i.e., LSB) in the binary representation (i.e., h_1, h_3, h_5 and h_7). The binary representation of h_2 has a 1 in the middle bit. Therefore, it checks all bit positions, including it, that have 1's in the same location (i.e., middle bit) in the binary representation (i.e., h_2, h_3, h_6 and h_7). The binary representation of h_4 has a 1 in the MSB. Therefore, it checks all bit positions, including it, that have 1's in the same location (i.e., MSB) in the binary representation (i.e., h_4, h_5, h_6 and h_7).

To decode a Hamming code, one must check for odd parity over the bit fields in which even parity was previously established. For example, a single bit error is indicated by a non-zero parity word $c_4\ c_2\ c_1$, where

$$\begin{array}{l} c_1 = h_1 \oplus h_3 \oplus h_5 \oplus h_7 \\ c_2 = h_2 \oplus h_3 \oplus h_6 \oplus h_7 \\ c_4 = h_4 \oplus h_5 \oplus h_6 \oplus h_7 \end{array}$$

If $c_4\, c_2\, c_1 = 0\,0\,0$, then there is no error in the Hamming code. If it has a non-zero value, it indicates the bit position in error. For example, if $c_4\, c_2\, c_1 = 1\,0\,1$, then bit 5 is in error. To correct this error, bit 5 has to be complemented.

Example 1.31 Encode data bits 0101 into a 7-bit even parity Hamming code.

Solution Given $b_3\, b_2\, b_1\, b_0 = 0101$.

Therefore,

$$h_1 = b_3 \oplus b_2 \oplus b_0 = 0 \oplus 1 \oplus 1 = 0 \quad h_3 = b_3 = 0$$
$$h_2 = b_3 \oplus b_1 \oplus b_0 = 0 \oplus 0 \oplus 1 = 1 \quad h_5 = b_2 = 1$$
$$h_4 = b_2 \oplus b_1 \oplus b_0 = 1 \oplus 0 \oplus 1 = 0 \quad h_6 = b_1 = 0$$
$$h_7 = b_0 = 1$$

h_1	h_2	h_3	h_4	h_5	h_6	h_7
0	1	0	0	1	0	1

Example 1.32 A 7-bit Hamming code is received as 0101101. What is its correct code?

Solution

h_1	h_2	h_3	h_4	h_5	h_6	h_7
0	1	0	1	1	0	1

Now, to find the error,

$$c_1 = h_1 \oplus h_3 \oplus h_5 \oplus h_7 = 0 \oplus 0 \oplus 1 \oplus 1 = 0$$
$$c_2 = h_2 \oplus h_3 \oplus h_6 \oplus h_7 = 1 \oplus 0 \oplus 0 \oplus 1 = 0$$
$$c_4 = h_4 \oplus h_5 \oplus h_6 \oplus h_7 = 1 \oplus 1 \oplus 0 \oplus 1 = 1$$

Thus, $c_4\, c_2\, c_1 = 1\,0\,0$. Therefore, bit 4 is in error and the corrected code word can be obtained by complementing the fourth bit in the received code word as 0100 101.

1.9.5 Alphanumeric Codes

If a computer is to be useful, it must be capable of handling non-numerical information. Similarly, printers and other similar devices must be able to recognize codes that represent numbers (0 to 9), letters, and special symbols. The codes that represent numbers, alphabetic letters and special symbols are called *alphanumeric codes*.

A complete and adequate set of necessary characters includes (i) 26 lower case letters, (ii) 26 upper case letters, (iii) 10 numeric digits, and (iv) about 25 special characters, such as +, /, #, %, etc. These total up to about 87 symbols. The representation of 87 symbols with some type of binary code would require at least 7 bits. Each character is represented by a 7-bit code; usually an 8-bit is inserted for parity. With 7 bits, there are $2^7 = 128$ possible binary numbers; 87 of these arrangements of 0 and 1 bits serve as code groups representing 87 different characters. Hence, this code consists of 128 symbols; 95 (= 87 + 8) characters represent *graphic symbols* that include upper and lower case letters, numerals 0 to 9, punctuation marks and special symbols. 23 characters represent *format effectors* which are functional characters for controlling the layout of printing or display devices such as carriage return, line feed, horizontal tabulation and back space; the other 10 characters are used to direct the data communication flow and report its status.

ASCII code A standardised code that has been widely accepted by the industry, the ASCII (pronounced "as-kee") code—*American Standards Code for Information Interchange*, used in most microcomputers by its manufacturers. The ASCII code represents a character with seven bits, which can be stored as one byte with one bit unused. The extra bit is often used to extend the ASCII code to represent an additional 128 characters. Table 1.7 partially lists the 7-bit ASCII code for each character and its octal and hexadecimal equivalents. The format of the ASCII code is $X_6X_5X_4X_3X_2X_1X_0$, where each bit is a '0' or a '1'. For example, the letter A is coded as 100 0001.

EBCDIC code Another alphanumeric code used in IBM equipment is the EBCDIC or *Extended Binary Coded Decimal Information Code*. It differs from ASCII only in its code grouping for the different alphanumeric characters. It uses eight bits for each character and a ninth bit for parity.

Hollerith code The Hollerith code is used in punched cards. A punched card consists of 80 columns and 12 rows. Each column represents an alphanumeric character of 12 bits by punching holes in the appropriate rows. The presence of a hole represents a 1, its absence represents a 0. The 12 rows are marked starting from the top, as 12, 11, 0, 1, 2, ...9. The first three rows are called the *zone* punch and the last nine are called the *numeric* punch. Decimal digits are represented by a single hole in a numeric punch. The alphabets are represented by two holes in a column; one in the zone punch and the other in the numeric punch. Special characters are represented by one, two, or three holes in a column; while the zone is always used; the other two holes, if used, are in a numeric punch with the 8-th punch being commonly used. The 12-bit card code is inefficient with respect to the number of bits used. Most computers convert the input 12-bit card code into an internal 6-bit code to conserve bits of memory. The Hollerith code is BCD. Thus, the transmission from EBCDIC is simple. Since large computers use punched cards, the Hollerith code is used for their card readers, and punches and EBCDIC are used within the computer.

Table 1.7 Partial listing of ASCII code

Character	7-bit ASCII	Octal	Hexa
A	100 0001	101	41
B	100 0010	102	42
C	100 0011	103	43
D	100 0100	104	44
E	100 0101	105	45
F	100 0110	106	46
G	100 0111	107	47
H	100 1000	110	48
I	100 1001	111	49
J	100 1010	112	4A
K	100 1011	113	4B
L	100 1100	114	4C
M	100 1101	115	4D

Character	7-bit ASCII	Octal	Hexa
N	100 1110	116	4E
O	100 1111	117	4F
P	101 0000	120	50
Q	101 0001	121	51
R	101 0010	122	52
S	101 0011	123	53
T	101 0100	124	54
U	101 0101	125	55
V	101 0110	126	56
W	101 0111	127	57
X	101 1000	130	58
Y	101 1001	131	59
Z	101 1010	132	5A
0	011 0000	060	30
1	011 0001	061	31
2	011 0010	062	32
3	011 0011	063	33
4	011 0100	064	34
5	011 0101	065	35
6	011 0110	066	36
7	011 0111	067	37
8	011 1000	070	38
9	011 1001	071	39
Blank	010 0000	040	20
.	010 1110	056	2E
(	010 1000	050	28
)	010 1001	051	29
+	010 1011	053	2B
$	010 0100	044	24
*	010 1010	052	2A
-	010 1101	055	2D
/	010 1111	057	2F
,	010 1100	054	2C
=	011 1101	075	3D
!	010 0001	041	21
"	010 0010	042	22
%	010 0101	045	25
&	010 0110	046	26
	010 0111		
:	011 1010	072	3A
;	011 1011	073	3B
<	011 1100	074	3C
>	011 1110	076	3E
?	011 1111	077	3F

REVIEW QUESTIONS

1. What are the common features between the different number systems?
2. How is a number expressed in a general number system?

3. What is the base of the decimal number system?
4. How are binary digits used to express the integer and fractional parts of a number?
5. Explain how multiplication and addition can be performed in digital systems.
6. How are subtraction and division performed in digital systems?
7. List the salient features of the BCD, Excess-3 and Gray codes.
8. What is a BCD Code? What are its advantages and disadvantages?
9. What are meant by the 1's and 2's complements of a binary number?
10. Explain the rules for binary subtraction using the 1's and 2's complement methods.
11. Explain how BCD addition is carried out.
12. What is meant by overflow? Is it a software problem or a hardware problem?
13. Distinguish between 1's and 2's complements.
14. Distinguish between 9's and 10's complements.
15. How do you add two decimal numbers in the BCD form if the sum is greater than 9?
16. Define odd and even parity check codes.
17. Write a short note on "weighted and non-weighted codes".
18. What is a Gray code? Why is it important?
19. What is a Hamming code and how is it used?

PROBLEMS

1. Convert the following decimal numbers to their binary equivalents:
(a) 37 (b) 14 (c) 167 (d) 72·45 (e) 0·4475 (f) 52 (g) 4097·188 (h) 2048·0625
Ans: (a) 100101 (b) 1110 (c) 10100111 (d) 1001000·01110 (e) 0·0111001 (f) 110100 (g) 1000000000001·001 (h) 100000000000·0001
2. Convert the following binary numbers to their decimal equivalents:
(a) 10110 (b) 10001101 (c) 10111·1011 (d) 0·011011 (e) 1101111·101
Ans: (a) 22 (b) 141 (c) 23·6875 (d) 0·421875 (e) 111·625
3. Convert the following numbers from octal to decimal.
(a) 743_8 (b) $36 \cdot 4_8$ (c) 2376_8 (d) 64_8 (e) 557_8 (f) 1024_8 (g) 465_8 (h) 7765_8
Ans: (a) 483 (b) 30·5 (c) 1278 (d) 52 (e) 367 (f) 532 (g) 309 (h) 4085.
4. Convert the following binary digits to octal numbers:
(a) 101101 (b) 101101110 (c) 10110111 (d) 110110·011 (e) 011·101101
Ans: (a) 55_8 (b) 556_8 (c) 267_8 (d) 66·3 (e) 3·55.
5. Convert the following decimal numbers to octal numbers and then to binary:
(a) 59 (b) 0·58 (c) 64·2 (d) 199·3
Ans: (a) 111011 (b) 0·100101000111 (c) 001000000·001100 (d) 011000111·010011001
6. Convert each of the following octal numbers to binary:
(a) 15_8 (b) 24_8 (c) 167_8 (d) 234_8 (e) 173_8 (f) 157_8 (g) 4653_8 (h) 1723_8 (i) 2645_8
Ans: (a) 001·101 (b) 010 100 (c) 001 110 111 (d) 010 011 100 (e) 001 111 011 (f) 001 101 111 (g) 100 110 101 011 (h) 001 111 010 011 (i) 010 110 100 101

7. Convert the following hexadecimal numbers to binary:
(a) 49_{16} (b) 324_{16} (c) 649_{16} (d) ABC_{16} (e) $5C8_{16}$ (f) $FB17_{16}$ (g) $4A \cdot 67_{16}$ (h) $8109 \cdot 4A_{16}$ (i) $EFF2 \cdot F_{16}$

Ans: (a) 0100 1001 (b) 0011 0010 0100 (c) 0110 0100 1001 (d) 1010 1011 1100 (e) 0101 1100 1000 (f) 1111 1011 0001 0111 (g) 0100 1010 · 0110 0111 (h) 1000 0001 0000 1001 · 0100 1010 (i) 1110 1111 1111 0010 · 1111

8. Convert the following binary numbers to octal and then to hexadecimal:
(a) 101100110011 (b) 1011101.1011.

Ans: (a) 5463, B33 (b) 135·54; 5D · B

9. Convert the following hexadecimal numbers to their decimal equivalents:
(a) 49_{16} (b) 632_{16} (c) 54_{16} (d) $AB0_{16}$ (e) $BC2_{16}$ (f) FFF_{16} (g) 649_{16} (h) $54A_{16}$ (i) 1622_{16}.

Ans: (a) 73_{10} (b) 1586_{10} (c) 84_{10} (d) 2736_{10} (e) 3010_{10} (f) 4095_{10} (g) 1609_{10} (h) 1354_{10} (i) 5666_{10}

10. Convert the following hexadecimal numbers to octal:
(a) $381B_{16}$ (b) $641A_{16}$ (c) $AB2_{16}$ (d) 1289_{16} (e) 2647_{16} (f) ABC_{16} (g) $1AF_{16}$ (h) $2A0_{16}$

Ans: (a) 34033_8 (b) 62032_8 (c) 5262_8 (d) 11211_8 (e) 23107_8 (f) 5274_8 (g) 657_8 (h) 1240_8

11. Convert the following octal numbers to hexadecimal:
(a) 137_8 (b) 4163_8 (c) 775_8 (d) 673_8 (e) 1275_8 (f) 3643_8 (g) 4555_8 (h) 4447_8

Ans: (a) $5F_{16}$ (b) 873_{16} (c) $1FD_{16}$ (d) $1BB_{16}$ (e) $2BD_{16}$ (f) $7A3_{16}$ (g) $96D_{16}$ (h) 927_{16}

12. What are the decimal numbers represented by each BCD code?
(a) 101000111 (b) 1000001001 (c) 10011000110011 (d) 1000100100101000.00100111 (e) 100100010111.0011

Ans: (a) 147_{10} (b) 209_{10} (c) 2633_{10} (d) $8928 \cdot 27_{10}$ (e) $917 \cdot 3_{10}$

13. Express the following decimal numbers in 2421 and 5421 codes:
(a) 169 (b) 264 (c) 6734 (d) 1993 (e) 9021

Ans: (a) 0001 1100 1111, 0001 1001 1100 (b) 0010 1100 0100, 0010 1001 0100 (c) 1100 1101 0011 0100, 1001 1010 0011 0100 (d) 0001 1111 1111 0011, 0001 1100 1100 0011 (e) 1111 0000 0010 0001, 1100 0000 0010 0001

14. Express the following 2421 code numbers in decimal form:
(a) 1111 1011 0100 (b) 0010 1110 0001 (c) 1011 0100 1111 (d) 0010 1100 1110 (e) 1100 1111 0011 0010 (f) 1101 1100 1011

Ans: (a) 954 (b) 281 (c) 549 (d) 268 (e) 6932 (f) 765

15. Express the following decimal numbers in Excess-3 code form:
(a) 426 (b) 739 (c) 1234 (d) 5678 (e) 984 (f) 3421

Ans: (a) 0111 0101 1001 (b) 1010 0110 1100 (c) 0100 0101 0110 0111 (d) 1000 1001 1010 1011 (e) 1100 1011 0111 (f) 0110 0111 0101 0100

16. Express the following Excess-3 codes as decimals:
(a) 0110 1011 1100 0111 (b) 0011 0101 1010 0100 (c) 0100 1000 1001 (d) 1001 0111 1100 (e) 1100 1010 0011

Ans: (a) 3894 (b) 0271 (c) 156 (d) 649 (e) 970

17. Convert the following binary numbers to Gray codes:
(a) 10110 (b) 1110111 (c) 101010001 (d) 101011101 (e) 110110011 (f) 1000111010 (g) 1010111000

Ans: (a) 11101 (b) 1001100 (c) 111111001 (d) 111110011 (e) 101101010 (f) 1100100111 (g) 1111100100

18. Express the following decimals in Gray code form:
(a) 4 (b) 7 (c) 82 (d) 324 (e) 15 (f) 457

Ans: (a) 0110 (b) 0100 (c) 01111011 (d) 000111100110 (e) 1000 (f) 000100101101

19. Convert the following Gray codes into binary digits:
(a) 10111 (b) 110011 (c) 1110011 (d) 1001101101 (e) 111011 (f) 101101 (g) 111011001 (h) 1010101101

Ans: (a) 11010 (b) 100010 (c) 1011101 (d) 1110110110 (e) 101101 (f) 110110 (g) 101101110 (h) 1100110110

20. Encode the following binary digits into 7-bit even-parity Hamming codes:
(a) 1000 (b) 0101 (c) 1011

Ans: (a) 1110000 (b) 0100101 (c) 0110011

21. Add the following binary numbers using the binary addition method:
(a) 1010 + 1011 (b) 1011.0111 + 1101.101 (c) 1110.1011 + 1001.1110

Ans: (a) 10101 (b) 11001.0001 (c) 11000.1001

22. Subtract the following groups of binary numbers:
(a) 11110 − 11011 (b) 10110.1 − 1100.01 (c) 101.101 − 100.1

Ans: (a) 0011 (b) 1010.01 (c) 1.001

23. Multiply the following binary numbers:
(a) 1011 × 111 (b) 1101.1 × 1101 (c) 11111 × 101

Ans: (a) 1001 101 (b) 10101111.1 (c) 10011011

24. Perform the following divisions:
(a) 111111 ÷ 1001 (b) 10111 ÷ 100 (c) 10110.1101 ÷ 11.1

Ans: (a) 111 (b) 101.11 (c) 111.00011

25. Convert the following numbers into floating point decimal notation:
(a) 81200 (b) 434 · 45 (c) 89 · 46 (d) 0 · 00379

Ans: (a) 812×10^2 (b) 43445×10^{-2} (c) 8946×10^{-2} (d) 379×10^{-5}

26. Give the 1's complement of the following numbers:
(a) 1011011 (b) 1101101 (c) 1000111101 (d) 111001101 (e) 11001100 (f) 01010111 (g) 010011011 (h) 101011001

Ans: (a) 0100100 (b) 0010010 (c) 0111000010 (d) 000110010 (e) 00110011 (f) 10101000 (g) 101100100 (h) 010100110

27. Give the 2's complement of the following numbers:
(a) 1010111 (b) 11001101

Ans: (a) 0101001 (b) 00110011

28. Subtract the following binary numbers using the 1's complement method:
(a) 101 − 10 (b) 111 − 11 (c) 1010 − 111 (d) 1101 − 1001 (e) 110110 − 11001 (f) 1000010 − 100101 (g) 110111 − 11011

Ans: (a) 011 (b) 100 (c) 011 (d) 100 (e) 11101 (f) 11101 (g) 11100

29. Perform the following additions using the 2's complement method:
(a) 1101 + 1110 (b) +39 and +19 (c) +40 and +26 (d) +63 and +37.

Ans: (a) 0001 1011 (b) 00111010 (c) 0100 0010 (d) 0110 0100

30. Add the following numbers using the 2's complement method:
(a) +38 and −22 (b) +64 and −29 (c) +49 and −37

Ans: (a) 0001 0000 (b) 0010 0011 (c) 0000 1000

31. Add the following numbers using the 2's complement method:
(a) −48 and +31 (b) −64 and +46 (c) −99 and +62
Ans: (a) 1110 1111 (b) 1110 1110 (c) 1101 1011

32. Add the following numbers using the 2's complement method:
(a) −32 and −16 (b) −42 and −31 (c) −64 and −13
Ans: (a) 1101 0000 (b) 1011 0111 (c) 1011 0001

33. Subtract the following numbers using the 2's complement method:
(a) +39 − (+16) (b) +49 − (+32) (c) +62 − (+29)
Ans: (a) 0001 0111 (b) 0001 0001 (c) 0010 0001

34. Give the 9's complement of the following decimal numbers:
(a) 49 (b) 245 (c) 7865
Ans: (a) 50 (b) 754 (c) 2134

35. Subtract the following decimal numbers using the 9's complement method:
(a) 19 − 12 (b) 39 − 15 (c) 349 − 436
Ans: (a) 7 (b) 24 (c) − 87

36. Give the 10's complement of the following decimal numbers:
(a) 12 (b) 345 (c) 6789
Ans: (a) 88 (b) 655 (c) 3211

37. Convert the following decimal numbers to BCD:
(a) 26_{10} (b) 379_{10} (c) 2019_{10}
Ans: (a) 0010 0110 (b) 0011 0111 1001 (c) 0010 0000 0001 1001

Chapter

2

Boolean Algebra and Minimization Techniques

2.1 INTRODUCTION

Binary logic deals with variables that take two discrete values—1 for TRUE and 0 for FALSE. A simple switching circuit containing active elements such as diode and the transistor can demonstrate the binary logic, which can either be ON (Switch closed) or OFF (Switch open). Electrical signals such as voltage or current exist throughout the digital system in 'either one of the two recognisable values', except during transition.

The Switching functions can be expressed with Boolean equations, Truth tables, Logic diagrams or Karnaugh maps. Truth tables offer an easy means of representing any switching function of n variables. The transformation of equations to truth tables or Karnaugh maps is little difficult. However, the reverse process of finding equations from tables and maps requires Minterms, Maxterms and Simplification methods.

Boolean algebra can be used to simplify the design of logic circuits. However, this method involves lengthy mathematical operations. An alternative method called the Karnaugh map can be used for the simplification of Boolean equations with up to four input variables. The use of Karnaugh map would become difficult if there are more than five input variables. Hence, it is better to employ the Quine-McCluskey method, which is a tabular method of minimization. These minimization techniques reduce the requirement of hardware.

2.2 DEVELOPMENT OF BOOLEAN ALGEBRA

Mathematician George Boole invented a new kind of algebra—the algebra of logic in the year 1854—popularly known as Boolean Algebra or Switching Algebra. He stated that symbols can be used to represent the structure of logical thought. Boolean algebra differs significantly from conventional algebra. This algebra deals with the rules by which the logical operations are carried out. Here, a digital circuit is represented by a set of input and output symbols and the circuit function expressed as a set of Boolean relationships between the symbols.

Boolean expressions are basically defined by stating that (i) a constant is a Boolean expression and (ii) a variable is a Boolean expression. For example, if A is a

Boolean expression, so is $\overline{A}$. The combination of variables such as $\overline{A}B$ and $\overline{A}B+C$ are also Boolean expressions. However, $A-B$ is not a Boolean expression.

2.3 BOOLEAN LOGIC OPERATIONS

A Boolean function is an algebraic expression formed using binary constants, binary variables and basic logical operation symbols. Basic logical operations include the AND function (logical multiplication), the OR function (logical addition) and the NOT function (logical complementation). A Boolean function can be converted into a logic diagram composed of the AND, OR and NOT (inverter) gates.

2.3.1 Logical AND Operation

The logical AND operation of two Boolean variables A and B, given as $Y = A \cdot B$, is represented by Table 2.1. The common symbol for this operation is the multiplication sign (.). The Table shows that the result of the AND operation on the variables A and B is logical 0 for all cases, except when both A and B are logical 1. Usually, the dot denoting the AND function is omitted and $A \cdot B$ is written as AB.

Table 2.1 Logical AND operation

Inputs		Output
A	B	$Y = A \cdot B$
0	0	0
0	1	0
1	0	0
1	1	1

2.3.2 Logical OR Operation

The logical OR operation between two Boolean variables A and B, given as $Y = A + B$, is represented by Table 2.2. This table shows that the result of the OR operation on the variables A and B is logical 1 when A or B (or both) are logical 1. The common symbol used for this logical addition operation is the plus sign (+).

Table 2.2 Logical OR operation

Inputs		Output
A	B	$Y = A + B$
0	0	0
0	1	1
1	0	1
1	1	1

2.3.3 Logical Complementation (Inversion)

The logical inverse operation converts the logical 1 to the logical 0 and vice versa. This method is also called the NOT operation. The symbol used for this operation is a bar over the function or the variable. Several notations, such as adding an asterisk, a star, prime, etc. over the variable, are used to indicate the NOT operation. "NOT A" or the complement of A is represented by $\overline{A}$.

2.4 BASIC LAWS OF BOOLEAN ALGEBRA

Logical operations can be expressed and minimized mathematically using the rules, laws, and theorems of Boolean algebra. It is a convenient and systematic method of expressing and analyzing the operation of digital circuits and systems. Boolean algebra uses binary arithmetic variables which have two distinct symbols 0 and 1. These are called levels or states of logic. For example, a binary 1 represents a High level and a binary 0 represents a Low level.

2.4.1 Boolean Addition

Addition by the Boolean method involves variables having values of either a binary 1 or a 0. The basic rules of Boolean addition are given below:

$$0 + 0 = 0$$
$$0 + 1 = 1$$
$$1 + 0 = 1$$
$$1 + 1 = 1$$

Boolean addition is same as the logical OR operation.

2.4.2 Boolean Multiplication

The basic rules of the Boolean multiplication method are as follows:

$$0 \cdot 0 = 0$$
$$0 \cdot 1 = 0$$
$$1 \cdot 0 = 0$$
$$1 \cdot 1 = 1$$

Boolean multiplication is same as the logical AND operation.

2.4.3 Properties of Boolean Algebra

Boolean Algebra is a mathematical system consisting of a set of two or more distinct elements, two binary operators denoted by the symbols (+) and (.) and one unary operator denoted by the symbol either bar (–) or prime ('). They satisfy the commutative, associative, distributive, absorption, consensus and idempotency properties of the Boolean Algebra.

Commutative property Boolean addition is commutative, given by

$$A + B = B + A \qquad (1a)$$

According to this property, the order of the OR operation conducted on the variables makes no difference. Boolean algebra is also commutative over multiplication, given by

$$A \cdot B = B \cdot A \qquad (1b)$$

This means that the order of the AND operation conducted on the variables makes no difference.

Associative property The associative property of addition is given by

$$A + (B + C) = (A + B) + C \qquad (2a)$$

The OR operation of several variables results in the same, regardless of the grouping of the variables. The associative law of multiplication is given by

$$A \cdot (B \cdot C) = (A \cdot B) \cdot C \qquad (2b)$$

According to this law, it makes no difference in what order the variables are grouped during the AND operation of several variables.

Distributive property (i) The Boolean addition is distributive over Boolean multiplication, given by

$$A + BC = (A + B)(A + C) \qquad (3a)$$

This property states that the AND operation (multiplication) of several variables and then the OR operation (addition) of the result with a single variable is equivalent to the OR operation of the single variable with each of the several variables and then the AND operation of the sums.

Proof

$$\begin{aligned}
A + BC &= A \cdot 1 + BC && (\because A \cdot 1 = A)\\
&= A(1 + B) + BC && (\because 1 + B = 1)\\
&= A \cdot 1 + AB + BC && (\because A(B + C) = AB + BC)\\
&= A \cdot (1 + C) + AB + BC && (\because 1 + C = 1)\\
&= A \cdot 1 + AC + AB + BC\\
&= A \cdot A + AC + AB + BC && (\because A \cdot A = A)\\
&= A(A + C) + B(A + C)\\
&= (A + B)(A + C)
\end{aligned}$$

(ii) Boolean multiplication is also distributive over Boolean addition given by

$$A \cdot (B + C) = A \cdot B + A \cdot C \qquad (3b)$$

According to this property, the OR operation of several variables and then the AND operation of the result with a single variable is equivalent to the AND operation of the single variable with each of the several variables and then the OR operation of the products.

Absorption laws

(i) $\boxed{A + AB = A}$ (4a)

Proof

$$\begin{aligned}
A + AB &= A \cdot 1 + AB\\
&= A(1 + B)\\
&= A \cdot 1 = A
\end{aligned}$$

(ii) $\boxed{A \cdot (A + B) = A}$ (4b)

Proof

$$\begin{aligned}
A(A + B) &= A \cdot A + A \cdot B\\
&= A + AB\\
&= A(1 + B)\\
&= A \cdot 1\\
&= A
\end{aligned}$$

(iii) $\boxed{A + \overline{A}B = A + B}$ (5a)

Proof

$$\begin{aligned}
A + \overline{A}B &= (A + \overline{A})(A + B) && [\because A + BC = (A + B)(A + C)]\\
&= 1 \cdot (A + B) && (\because A + \overline{A} = 1)\\
&= A + B
\end{aligned}$$

(iv) $\boxed{A \cdot (\overline{A} + B) = AB}$ (5b)

Proof

$$\begin{aligned}
A \cdot (\overline{A} + B) &= A \cdot \overline{A} + AB\\
&= AB (\because A\overline{A} = 0)
\end{aligned}$$

Consensus laws

(i) $\boxed{AB + \overline{A}C + BC = AB + \overline{A}C}$ (6a)

Proof

$$\begin{aligned}
AB + \overline{A}C + BC &= AB + \overline{A}C + BC \cdot 1\\
&= AB + \overline{A}C + BC(A + \overline{A}) && (\because A + \overline{A} = 1)\\
&= AB + \overline{A}C + ABC + \overline{A}BC\\
&= AB(1 + C) + \overline{A}C(1 + B) && (\because 1 + B = 1 = 1 + C)\\
&= AB + \overline{A}C
\end{aligned}$$

(ii) $\boxed{(A + B)(\overline{A} + C)(B + C) = (A + B)(\overline{A} + C)}$ (6b)

Proof

$$\begin{aligned}
(A + B)(\overline{A} + C)(B + C) &= (A + B)(\overline{A} + C)(B + C + 0)\\
&= (A + B)(\overline{A} + C)(B + C + A\overline{A})\\
&= (A + B)(\overline{A} + C)(B + C + A)(B + C + \overline{A})\\
&\quad [\because A + BC = (A + B)(A + C)]\\
&= (A + B)(A + B + C)(\overline{A} + C)(\overline{A} + C + B)\\
&= (A + B)(\overline{A} + C)\\
&\quad [\because A(A + B) = A]
\end{aligned}$$

The other basic laws (theorems) of Boolean algebra are given in Table 2.3. These theorems can be proved easily by adopting the truth table method or by using algebraic manipulation.

Table 2.3 Other laws of boolean algebra

No.	Boolean laws		
7	(a) $A + 0 = A$	(b) $A \cdot 1 = A$	—
8	(a) $A + 1 = 1$	(b) $A \cdot 0 = 0$	—
9	(a) $A + A = A$	(b) $A \cdot A = A$	Idempotency
10	(a) $A + \bar{A} = 1$	(b) $A \cdot \bar{A} = 0$	Full set, null set
11		$\bar{\bar{A}} = A$	Double inversion or involution

2.4.4 Principle of Duality

From the above properties and laws of Boolean algebra, it is evident that they are grouped in pairs as (a) and (b). One expression can be obtained from the other in each pair by replacing every 0 with 1, every 1 with 0, every (+) with (.) and every (.) with (+). Any pair of expression satisfying this property is called *dual expression*. This characteristic of Boolean algebra is called the *principle of duality*.

2.5 DEMORGAN'S THEOREMS

Two theorems that are an important part of Boolean algebra were proposed by DeMorgan. The first theorem states that the complement of a product is equal to the sum of the complements. That is, if the variables are A and B, then,

$$\overline{AB} = \bar{A} + \bar{B}$$

The second theorem states that the complement of a sum is equal to the product of the complements. In equation form, this can be written as

$$\overline{A+B} = \bar{A} \cdot \bar{B}$$

The complement of a Boolean logic function or a logic expression may be expanded or simplified by following the steps of DeMorgan's theorem.

(i) Replace the symbol (+) with symbol (.); the symbol (.) with symbol (+) given in the expression.

(ii) Complement each of the terms or variables in the given expression.

DeMorgan's theorems can be proved for any number of variables; proof of these two theorems for 2-input variables can be found in Table 2.4.

Table 2.4 Proof for DeMorgan's theorem by perfect induction method

1 A	2 B	3 $\bar{A}$	4 $\bar{B}$	5 $A+B$	6 $A \cdot B$	7 $\overline{A+B}$	8 $\bar{A} \cdot \bar{B}$	9 $\overline{A \cdot B}$	10 $\bar{A}+\bar{B}$
0	0	1	1	0	0	1	1	1	1
0	1	1	0	1	0	0	0	1	1
1	0	0	1	1	0	0	0	1	1
1	1	0	0	1	1	0	0	0	0

A study of Table 2.4 makes clear that columns 7 and 8 are equal. Therefore,

$$\overline{A+B} = \bar{A} \cdot \bar{B}$$

Similarly, columns 9 and 10 are equal. Therefore,

$$\overline{A \cdot B} = \bar{A} + \bar{B}$$

Also, De-Morgan's theorem can be proved by algebraic method as follows:

According to the first theorem, $(\bar{A} + \bar{B})$ is the complement of AB. From the Table 2.3, the Boolean Laws 10 (a) and 10 (b) are given as,

$$A + \bar{A} = 1 \quad \text{and} \quad A\bar{A} = 0.$$

Substituting AB for A and $(\bar{A} + \bar{B})$ for $\bar{A}$ in the above expressions,

$$\begin{aligned} AB + \bar{A} + \bar{B} &= 1 & \text{and} \quad AB(\bar{A} + \bar{B}) &= 0 \\ \bar{A} + B + \bar{B} &= 1 & \text{and} \quad AB\bar{A} + AB\bar{B} &= 0 \\ \bar{A} + 1 &= 1 & \text{and} \quad 0 + 0 &= 0 \\ 1 &= 1 & \text{and} \quad 0 &= 0 \end{aligned}$$

Thus De-Morgan's first theorem is proved algebraically.

Similarly, [illegible] De-Morgan's second theorem, $(\bar{A} \cdot \bar{B})$ is the complement of $(A + B)$ [illegible] $(A + B)$ for A and $(\bar{A} \cdot \bar{B})$ for $\bar{A}$ in 10 (a) and 10 (b).

$$\begin{aligned} [illegible]\ \bar{A} \cdot \bar{B} &= 1 & \text{and} \quad (A+B) \cdot (\bar{A} \cdot \bar{B}) &= 0 \\ A + B + \bar{B} &= 1 & \text{and} \quad A\bar{A}\bar{B} + B\bar{A}\bar{B} &= 0 \\ A + 1 &= 1 & \text{and} \quad 0 + 0 &= 0 \\ 1 &= 1 & \text{and} \quad 0 &= 0 \end{aligned}$$

Thus De-Morgan's second theorem is proved algebraically.

Minimization (Simplification) of Boolean expressions using algebraic method The switching or Boolean expressions can be simplified by applying properties, laws and theorems of Boolean Algebra. The simplification of different Boolean expressions are demonstrated in the following examples.

Example 2.1 Prove that $AB + BC + \bar{B}C = AB + C$.

Solution
$$\begin{aligned} AB + BC + \bar{B}C &= AB + C(B + \bar{B}) \\ &= AB + C \cdot 1 \\ &= AB + C \end{aligned}$$

Example 2.2 Simplify the expression $\bar{A} \cdot B + A \cdot B + \bar{A} \cdot \bar{B}$.

Solution
$$\begin{aligned} \bar{A} \cdot B + A \cdot B + \bar{A} \cdot \bar{B} &= (\bar{A} + A) \cdot B + \bar{A} \cdot \bar{B} \\ &= 1 \cdot B + \bar{A} \cdot \bar{B} \\ &= B + \bar{A} \cdot \bar{B} \\ &= B + \bar{A} \quad (\because A + \bar{A} \cdot B = A + B) \end{aligned}$$

Example 2.3 Simplify the given expression $A + A \cdot \bar{B} + \bar{A} \cdot B$.

Solution
$$\begin{aligned} A + A \cdot \bar{B} + \bar{A} \cdot B &= A(1 + \bar{B}) + \bar{A} \cdot B \\ &= A \cdot 1 + \bar{A} \cdot B \\ &= A + B \end{aligned}$$

Example 2.12 Prove the following Boolean expression:

$(A+B)(\bar{A}\bar{C}+C)(\overline{\bar{B}+AC}) = \bar{A}B.$

Solution $(A+B)(\bar{A}\bar{C}+C)(\overline{\bar{B}+AC}) = (A+B)+(\bar{A}\bar{C}+C)(\bar{\bar{B}}\cdot\overline{AC})$

$= (A+B)(\bar{A}\bar{C}+C)(B\cdot\overline{AC})$

$= [A\bar{A}\bar{C}+AC+\bar{A}\bar{C}B+BC][B(\bar{A}+\bar{C})]$

$= (AC+\bar{A}\bar{C}B+BC)\cdot(B\bar{A}+B\bar{C})$

$= AC\cdot B\bar{A}+AC\cdot B\bar{C}+\bar{A}\bar{C}B\cdot B\bar{A}$

$+\bar{A}\bar{C}B\cdot B\bar{C}+BC\cdot B\bar{A}+BC\cdot B\bar{C}$

$= 0+0+\bar{A}B\bar{C}+\bar{A}\bar{C}B+BC\bar{A}+0$

$= \bar{A}B(\bar{C}+\bar{C}+C)$

$= \bar{A}B$

Example 2.13 Prove that $\bar{A}\bar{B}\bar{C}+\bar{A}\bar{B}C+\bar{A}B\bar{C}+\bar{A}BC+A\bar{B}\bar{C} = \bar{A}+\overline{B+C}.$

Solution

$\bar{A}\bar{B}\bar{C}+\bar{A}\bar{B}C+\bar{A}B\bar{C}+\bar{A}BC+A\bar{B}\bar{C} = \bar{A}\bar{B}(\bar{C}+C)+\bar{A}B(\bar{C}+C)+A\bar{B}\bar{C}$

$= \bar{A}\bar{B}+\bar{A}B+A\bar{B}\bar{C}$

$= \bar{A}(\bar{B}+B)+A\bar{B}\bar{C}$

$= \bar{A}+A\bar{B}\bar{C}$

$= \bar{A}+\bar{B}\bar{C}$ [$\because A+\bar{A}B = A+B$]

$= A+\overline{B+C}$

Example 2.14 Find the complement of the expression $Y = ABC+AB\bar{C}+A\bar{B}C+\bar{A}BC.$

Solution $\bar{Y} = \overline{ABC+AB\bar{C}+A\bar{B}C+\bar{A}BC}$

$= (\overline{ABC})(\overline{AB\bar{C}})(\overline{A\bar{B}C})(\overline{\bar{A}BC})$

$= (\bar{A}+\bar{B}+\bar{C})(\bar{A}+\bar{B}+C)(\bar{A}+B+\bar{C})(A+\bar{B}+\bar{C})$

$= (\bar{A}+\bar{B}+C\bar{C})(\bar{A}+\bar{C}+B\bar{B})$ [$\because (A+B)(A+C) = A+BC$]

$= (\bar{A}+\bar{B})(\bar{A}+\bar{C})$

Example 2.15 Prove that $BCD+A\bar{C}\bar{D}+ABD = BCD+A\bar{C}\bar{D}+AB\bar{C}.$

Solution $BCD+A\bar{C}\bar{D}+ABD = BCD+A\bar{C}\bar{D}+(ABD)\cdot 1$

$= BCD+A\bar{C}\bar{D}+(ABD)(C+\bar{C})$

$= BCD+A\bar{C}\bar{D}+ABCD+AB\bar{C}D$

$= BCD(1+A)+A\bar{C}(\bar{D}+DB)$

$= BCD+A\bar{C}(\bar{D}+B)$

$= BCD+A\bar{C}\bar{D}+AB\bar{C}$

Hence, L.H.S = R.H.S

Example 2.16 Simplify the given expression $Y = \bar{A}B+ABD+A\bar{B}C\bar{D}+BC.$

Solution

$\bar{A}B+ABD+A\bar{B}C\bar{D}+BC = B(\bar{A}+AD)+C(B+\bar{B}A\bar{D})$

$= B(\bar{A}+D)+C(B+A\bar{D})$ [$\because A+\bar{A}B = A+B$]

$= \bar{A}B+BD+BC+AC\bar{D}$

$= \bar{A}B+BD+BC(A+\bar{A})+AC\bar{D}$ [$\because A+\bar{A} = 1$]

$= \bar{A}B+BD+ABC+\bar{A}BC+AC\bar{D}$

$= \bar{A}B(1+C)+BD+ABC+AC\bar{D}$

$= \bar{A}B+BD+AC\bar{D}$ [$\because AB+BC+\bar{A}C = AB+\bar{A}C$]

[Here $A = D$; $B = B$; $C = AC$]

Example 2.17 If $\bar{A}B+C\bar{D} = 0$, then prove that

$AB+\bar{C}(\bar{A}+\bar{D}) = AB+BD+\bar{B}\bar{D}+\bar{A}\bar{C}D.$

Solution

L.H.S $= AB+\bar{C}(\bar{A}+\bar{D})+0$

$= AB+\bar{C}(\bar{A}+\bar{D})+\bar{A}B+C\bar{D}$ (given that $\bar{A}B+C\bar{D} = 0$)

$= AB+\bar{A}\bar{C}+\bar{C}\bar{D}+\bar{A}B+C\bar{D}$

$= B(A+\bar{A})+\bar{D}(C+\bar{C})+\bar{A}\bar{C}$

$= B+\bar{D}+\bar{A}\bar{C}$

R.H.S $= AB+BD+\bar{B}\bar{D}+\bar{A}\bar{C}D+0$

$= AB+BD+\bar{B}\bar{D}+\bar{A}\bar{C}D+\bar{A}B+C\bar{D}$ (given that $\bar{A}B+C\bar{D} = 0$)

$= B(A+\bar{A})+BD+\bar{B}\bar{D}+\bar{A}\bar{C}D+C\bar{D}$

$= B(1+D)+\bar{B}\bar{D}+\bar{A}\bar{C}D+C\bar{D}$

$= B+\bar{B}\bar{D}+\bar{A}\bar{C}D+C\bar{D}$

$= B+\bar{D}+\bar{A}\bar{C}D+C\bar{D}$ ($\because A+\bar{A}B = A+B$)

$= B+\bar{D}(1+C)+\bar{A}\bar{C}D$

$= B+\bar{D}+D\bar{A}\bar{C}$

$= B+\bar{D}+\bar{A}\bar{C}$ ($\because A+\bar{A}B = A+B$)

Hence, L.H.S = R.H.S

Example 2.18 Simplify $Y = A\bar{B}+(\bar{A}+B)C$

Solution $Y = A\bar{B}+(\bar{A}+B)C$

$= A\bar{B}+(\overline{A\bar{B}})C$ ($\because \bar{A}+B = \overline{A\bar{B}}$)

$= A\bar{B}+C$ ($\because X+\bar{X}Y = X+Y$)

Example 2.19 Simplify $Y = A + \bar{A}B + \bar{A}\bar{B}C + \bar{A}\bar{B}\bar{C}D$

Solution

$$
\begin{aligned}
A + \bar{A}B + \bar{A}\bar{B}C + \bar{A}\bar{B}\bar{C}D &= A + B + \bar{A}\bar{B}C + \bar{A}\bar{B}\bar{C}D \quad [\because A + \bar{A}B = A + B]\\
&= A + B + \bar{B}C + \bar{A}\bar{B}\bar{C}D\\
&= A + B + C + \bar{A}\bar{B}\bar{C}D\\
&= A + B + C + \bar{B}\bar{C}D\\
&= A + B + C + \bar{C}D\\
&= A + B + C + D
\end{aligned}
$$

Example 2.20 If $A\bar{B} + \bar{A}B = C$, show that $A\bar{C} + \bar{A}C = B$.

Solution

$$
\begin{aligned}
A\bar{C} + \bar{A}C &= A(\overline{A\bar{B} + \bar{A}B}) + \bar{A}(A\bar{B} + \bar{A}B) \qquad (\text{given that } C = A\bar{B} + \bar{A}B)\\
&= A(\bar{A} + B)(A + \bar{B}) + \bar{A}A\bar{B} + \bar{A}\bar{A}B\\
&= (A\bar{A} + AB)(A + \bar{B}) + \bar{A}B\\
&= AB + AB\bar{B} + \bar{A}B\\
&= AB + \bar{A}B\\
&= B(A + \bar{A})\\
&= B
\end{aligned}
$$

2.6 SUM OF PRODUCTS AND PRODUCT OF SUMS

Logical functions are generally expressed in terms of logical variables. Values taken on by the logical functions and logical variables are in the binary form. An arbitrary logic function can be expressed in the following forms:

(i) Sum of Products (SOP)

(ii) Product of Sums (POS)

Product term The AND function is referred to as a *product*. In Boolean algebra, the word "product" loses its original meaning but serves to indicate an AND function. The logical product of several variables on which a function depends is considered to be a product term. The variables in a product term can appear either in complemented or uncomplemented form. $AB\bar{C}$, for example, is a product term.

Sum term An OR function (+ sign) is generally used to refer a *sum*. The logical sum of several variables on which a function depends is considered to be a *sum term*. Variables in a sum term can appear either in complemented or uncomplemented form. $A + \bar{B} + C$, for example, is a sum term.

Sum Of Products (SOP) The logical sum of two or more logical product terms, is called a *Sum of Products* expression. It is basically an OR operation of AND operated variables such as :

(i) $Y = AB + BC + AC$

(ii) $Y = \bar{A}B + \bar{A}C + B\bar{C}$

Product Of Sums (POS) A product of sums expression is a logical product of two or more logical sum terms. It is basically an AND operation of OR operated variables such as:

(i) $Y = (A + B)(B + C)(C + \bar{A})$

(ii) $Y = (A + B + C)(A + \bar{C})$

2.6.1 Minterm

A product term containing all the K variables of the function in either complemented or uncomplemented form is called a *Minterm*. A 2-variable function has four possible combinations, viz. $\bar{A}\bar{B}$, $\bar{A}B$, $A\bar{B}$, and AB. These product terms are called minterms or standard products or fundamental products. For a 3-binary input variable function, there are 8 minterms as shown in Table 2.5. Each minterm can be obtained by the AND operation of all the variables of the function. In the minterm, a variable appears either in uncomplemented form, if it possesses a value of 1 in the corresponding combination, or in complemented form, if it contains the value 0. The minterms of a 3-variable function can be represented by m_0, m_1, m_2, m_3, m_4, m_5, m_6 and m_7; the suffix indicates the decimal code corresponding to the minterm combination.

Table 2.5 The minterm table

A	B	C	Minterm
0	0	0	$\bar{A}\bar{B}\bar{C}$
0	0	1	$\bar{A}\bar{B}C$
0	1	0	$\bar{A}B\bar{C}$
0	1	1	$\bar{A}BC$
1	0	0	$A\bar{B}\bar{C}$
1	0	1	$A\bar{B}C$
1	1	0	$AB\bar{C}$
1	1	1	ABC

The main property of a minterm is that it possesses the value 1 for only one combination of K input variables; i.e., for a K variable function of the 2^K minterms, only one minterm will have the value 1, while the remaining 2^K-1 minterms will possess the value 0 for an arbitrary input combination. For example, as shown in Table 2.5, for input combination 010, i.e., for $A = 0$, $B = 1$ and $C = 0$, only the minterm $\bar{A}B\bar{C}$ will have the value 1, while the remaining seven minterms will have the value 0.

Canonical sum of product expression It is defined as the logical sum of all the minterms derived from the rows of a truth table, for which the value of the function is 1. It is also called a *minterm canonical* form. The *canonical sum of product expression* can be given in a compact form by listing the decimal codes in correspondence with the minterm containing a function value of 1. For example, if the canonical sum of product form of a 3-variable logic function Y has three minterms $\bar{A}B\bar{C}$, $\bar{A}BC$ and $AB\bar{C}$, this can be expressed as the sum of the decimal codes corresponding to these minterms as stated below:

$$Y = \Sigma_m(0,5,6)$$
$$= m_0 + m_5 + m_6$$
$$= \overline{A}\,\overline{B}\,\overline{C} + A\overline{B}C + AB\overline{C}$$

where $\Sigma_m(0,5,6)$ represents the summation of minterms corresponding to the decimal codes 0, 5 and 6.

Using the following procedure, the canonical sum of product form of a logic function can be obtained:

1. Examine each term in the given logic function. Retain it if it is a minterm; continue to examine the next term in the same manner.
2. Check for variables that are missing in each product which is not a minterm. Multiply the product by $(X+\overline{X})$, for each variable X that is missing.
3. Multiply all the products and omit the redundant terms.

The above procedures can be explained with the following examples.

Example 2.21 Obtain the canonical sum of product form of the function

$$Y(A,B) = A+B.$$

Solution The given function containing the two variables A and B has the variable B missing in the first term and the variable A missing in the second. Therefore, the first term has to be multiplied by $(B+\overline{B})$, the second term by $(A+\overline{A})$ as given below:

$$A+B = A\cdot 1 + B\cdot 1$$
$$= A\cdot(B+\overline{B}) + B\cdot(A+\overline{A})$$
$$= AB + A\overline{B} + BA + B\overline{A}$$
$$= AB + A\overline{B} + \overline{A}B \quad (\because AB + AB = AB)$$
$$Y(A,B) = A+B = AB + A\overline{B} + \overline{A}B$$

Example 2.22 Obtain the canonical sum of product form of the function

$$Y(A,B,C) = A+BC$$

Solution Here, neither the first term nor the second term is a minterm. The variables B and C in the first term and the varible A in the second term are missing. Therefore, the first term has to be multiplied by $B+\overline{B}$ and $C+\overline{C}$, the second term by $(A+\overline{A})$ as shown below:

$$A+BC = A(B+\overline{B})(C+\overline{C}) + BC(A+\overline{A})$$
$$= (AB + A\overline{B})(C+\overline{C}) + BCA + BC\overline{A}$$
$$= ABC + AB\overline{C} + A\overline{B}C + A\overline{B}\,\overline{C} + BCA + BC\overline{A}$$
$$= ABC + AB\overline{C} + A\overline{B}C + A\overline{B}\,\overline{C} + \overline{A}BC \quad (\because ABC + ABC = ABC)$$

Therefore, the canonical sum of product form of $Y = A+BC$ is given by

$$A+BC = ABC + AB\overline{C} + A\overline{B}C + A\overline{B}\,\overline{C} + \overline{A}BC$$

Example 2.23 Obtain the canonical sum of product form of the function

$$Y = AB + ACD$$

Solution

$$Y = AB + ACD$$
$$= AB(C+\overline{C})(D+\overline{D}) + ACD(B+\overline{B})$$
$$= (ABC + AB\overline{C})(D+\overline{D}) + ABCD + A\overline{B}CD$$
$$= ABCD + ABC\overline{D} + AB\overline{C}D + AB\overline{C}\,\overline{D} + ABCD + A\overline{B}CD$$
$$= ABCD + ABC\overline{D} + AB\overline{C}D + AB\overline{C}\,\overline{D} + A\overline{B}CD \quad (\because ABCD + ABCD = ABCD)$$

2.6.2 Maxterm

A sum term containing all the K variables of the function in either complemented or uncomplemented form is called a *Maxterm*. A 2-variable function has four possible combinations, viz. $A+B$, $A+\overline{B}$, $\overline{A}+B$ and $\overline{A}+\overline{B}$. These sum terms are called maxterms. So also, a 3-binary input variable function has 8 maxterms as shown in Table 2.6. Each maxterm can be obtained by the OR operation of all the variables of the function. In a maxterm, a variable appears either in uncomplemented form if it possesses the value 0 in the corresponding combination or in complemented form if it contains the value 1. The maxterms of a 3-variable function can be represented by M_0, M_1, M_2, M_3, M_4, M_5, M_6 and M_7; the suffix indicates the decimal code corresponding to the maxterm combination.

Table 2.6 The maxterm table

A	B	C	Maxterm
0	0	0	$A+B+C$
0	0	1	$A+B+\overline{C}$
0	1	0	$A+\overline{B}+C$
0	1	1	$A+\overline{B}+\overline{C}$
1	0	0	$\overline{A}+B+C$
1	0	1	$\overline{A}+B+\overline{C}$
1	1	0	$\overline{A}+\overline{B}+C$
1	1	1	$\overline{A}+\overline{B}+\overline{C}$

The most important property of a maxterm is that it possesses the value 0 for only one combination of K input variables; i.e., for a K variable function of the 2^K maxterms, only one maxterm will have the value 0, while all the remaining 2^K-1 maxterms will have the value 1 for an arbitrary input combination. For example, for input combination 101, i.e., for $A=1$, $B=0$ and $C=1$, only the maxterm $(\overline{A}+B+\overline{C})$ will have the value 0, while the remaining seven maxterms will have the value 1. This can be studied in Table 2.6.

From Tables 2.6 and 2.7, it is found that each maxterm is the complement of the corresponding minterm. For example, if the maxterm is $(A+B+C)$, then its complement (i.e., $\overline{A+B+C}$) $\overline{A}\,\overline{B}\,\overline{C}$ is its corresponding minterm.

Canonical product of sum expression This is defined as the logical product of all the maxterms derived from the rows of truth table, for which the value of function is 0. It is also known as the maxterm canonical form. The *canonical product of sum expression* can be given in a compact form by listing the decimal codes corresponding to the maxterms containing a function value of 0. For example, if the canonical product of sum form of a 3-variable logic function Y has four maxterms $(A+B+C)$, $(A+\overline{B}+C)$, $(\overline{A}+B+C)$ and $(\overline{A}+\overline{B}+\overline{C})$, then it can be expressed as the product of decimal codes as given below:

$$\begin{aligned} Y &= \Pi(0,2,4,7) \\ &= M_0 \cdot M_2 \cdot M_4 \cdot M_7 \\ &= (A+B+C)(A+\overline{B}+C)(\overline{A}+B+C)(\overline{A}+\overline{B}+\overline{C}) \end{aligned}$$

The following procedure can be used to obtain the canonical product of the sum form of a logic function:

1. Examine each term in the given logic function. Retain it if it is a maxterm; continue to examine the next term in the same manner.
2. Check for variables that are missing in each sum, which is not a maxterm. Add $(X\overline{X})$ to the sum term, for each variable X that is missing.
3. Expand the expression using the distributive property and eliminate the redundant terms.

The above procedures can be explained with the following examples.

Example 2.24 Obtain the canonical product of the sum expression of $Y(ABC) = (A+\overline{B})(B+C)(A+\overline{C})$.

Solution In the given expression, the variable C is missing in the first term, the variable A in the second term and the variable B in the third term. Therefore, $C\overline{C}$, $A\overline{A}$ and $B\overline{B}$ have to be added with the first, second and third terms respectively as shown below:

$$\begin{aligned} Y(ABC) &= (A+\overline{B})(B+C)(A+\overline{C}) \\ &= (A+\overline{B}+0)(B+C+0)(A+\overline{C}+0) \\ &= (A+\overline{B}+C\overline{C})(B+C+A\overline{A})(A+\overline{C}+B\overline{B}) \end{aligned}$$

Now, using the distributive property, each sum term can be expanded as

$$Y = (A+\overline{B}+C)(A+\overline{B}+\overline{C})(A+B+C)(\overline{A}+B+C)(A+B+\overline{C})(A+\overline{B}+\overline{C})$$

$$Y = (A+\overline{B}+C)(A+\overline{B}+\overline{C})(A+B+C)(\overline{A}+B+C)(A+B+\overline{C})$$

$$[\because (A+\overline{B}+\overline{C})(A+\overline{B}+\overline{C}) = (A+\overline{B}+\overline{C})]$$

This is called the maxterm canonical form or the canonical product of sum expression.

Example 2.25 Express the function $Y = A + \overline{B}C$ in (a) canonical SOP and (b) canonical POS form.

Solution

(a) Canonical sum of products form

$$\begin{aligned} Y &= A + \overline{B}C \\ &= A(B+\overline{B})(C+\overline{C}) + \overline{B}C(A+\overline{A}) \\ &= (AB + A\overline{B})(C+\overline{C}) + A\overline{B}C + \overline{A}\overline{B}C \\ &= ABC + AB\overline{C} + A\overline{B}C + A\overline{B}\overline{C} + A\overline{B}C + \overline{A}\overline{B}C \\ &= ABC + AB\overline{C} + A\overline{B}C + A\overline{B}\overline{C} + \overline{A}\overline{B}C \quad [\because A\overline{B}C + A\overline{B}C = A\overline{B}C] \end{aligned}$$

$$Y = m_7 + m_6 + m_5 + m_4 + m_1$$

Therefore, $Y = \Sigma(1, 4, 5, 6, 7)$

(b) Canonical product of sum form

$$\begin{aligned} Y &= A + \overline{B}C \\ &= (A+\overline{B})(A+C) \quad [\because A + B.C = (A+B)(A+C)] \\ &= (A+\overline{B}+C\overline{C})(A+C+B\overline{B}) \\ &= (A+\overline{B}+C)(A+\overline{B}+\overline{C})(A+B+C)(A+\overline{B}+C) \quad [\because (A+B)(A+\overline{B}) = A] \end{aligned}$$

$$Y = (A+\overline{B}+C)(A+\overline{B}+\overline{C})(A+B+C)$$

$$Y = M_2M_3M_0 \quad \text{or} \quad Y = M_0M_2M_3$$

Therefore, $Y = \Pi(0, 2, 3)$

2.6.3 Deriving Sum of Product (SOP) Expression from a Truth Table

The Sum of Product (SOP) expression for a Boolean function can be derived from its truth table by summing (OR operation) the product terms that correspond to the combinations containing a function value 1. In the product term, the input variable appears either in uncomplemented form if it possesses the value 1, or in complemented form if it contains the value 0.

Now, consider the truth table, shown in Table 2.7, for a 3-input function Y. Here, the Y value is 1 for the input combinations 010, 011, 101 and 111 and their corresponding product terms are $\overline{A}B\overline{C}$, $\overline{A}BC$, $A\overline{B}C$ and ABC respectively.

Table 2.7 Truth table

Inputs			Output	Product terms	Sum terms
A	B	C	Y		
0	0	0	0		$(A+B+C)$
0	0	1	0		$(A+B+\overline{C})$
0	1	0	1	$\overline{A}B\overline{C}$	
0	1	1	1	$\overline{A}BC$	
1	0	0	0		$(\overline{A}+B+C)$
1	0	1	1	$A\overline{B}C$	
1	1	0	0		$(\overline{A}+\overline{B}+C)$
1	1	1	1	ABC	

Now, the final SOP expression for the output Y is obtained by summing (OR operation of) the four product terms as follows:

$$Y = \overline{A}B\overline{C} + \overline{A}BC + A\overline{B}C + ABC$$

The procedure for obtaining the output expression in SOP form from a truth table can be summarised, in general, as follows:

1. Give a product term for each input combination in the table, containing an output value of 1.
2. Each product term contains its input variables in either complemented or uncomplemented form. If an input variable is 0, it appears in complemented form; if the input variable is 1, it appears in uncomplemented form.
3. All the product terms are OR operated together in order to produce the final SOP expression of the output.

2.6.4 Deriving Product of Sum (POS) Expression from a Truth Table

The Product of Sum (POS) expression for a Boolean (switching) function can also be obtained from a truth table by the AND operation of the sum terms corresponding to the combinations for which the function assumes the value 0. In the sum term, the input variable appears in an uncomplemented form if it has the value 0 in the corresponding combination and in the complemented form if it has the value 1.

Studying the truth table shown in Table 2.7, for a 3-input function Y, we find that the Y value is 0 for the input combinations 000, 001, 100 and 110 and that their corresponding sum terms are $(A+B+C)$, $(A+B+\overline{C})$, $(\overline{A}+B+C)$ and $(\overline{A}+\overline{B}+C)$ respectively.

Now the final POS expression for the output Y is obtained by the AND operation of the four sum terms as follows:

$$Y = (A+B+C)(A+B+\overline{C})(\overline{A}+B+C)(\overline{A}+\overline{B}+C)$$

The procedure for obtaining the output expression in POS form from a truth table can be summarised, in general, as follows:

1. Give a sum term for each input combination in the table, which has an output value of 0.
2. Each sum term contains all its input variables in complemented or uncomplemented form. If the input variable is 0, then it appears in an uncomplemented form; if the input variable is 1, it appears in the complemented form.
3. All the sum terms are AND operated together to obtain the final POS expression of the output.

The POS expression for a Boolean (switching) function can also be obtained from its SOP expression using $\overline{Y} = Y$ as given in the following example.

Consider a function,

$$X = \overline{A}B\overline{C} + \overline{A}BC + A\overline{B}C + ABC$$

$$Y = \overline{\overline{Y}} = \overline{\overline{A}B\overline{C} + \overline{A}BC + A\overline{B}C + ABC}$$

The complement $\overline{Y}$ can be obtained by the OR operation of the minterms which are not available in Y. Therefore,

$$\overline{Y} = \overline{A}\,\overline{B}\,\overline{C} + \overline{A}\,\overline{B}C + A\overline{B}\,\overline{C} + AB\overline{C}$$

$$Y = \overline{\overline{A}\,\overline{B}\,\overline{C} + \overline{A}\,\overline{B}C + A\overline{B}\,\overline{C} + AB\overline{C}}$$

$$= (\overline{\overline{A}\,\overline{B}\,\overline{C}})(\overline{\overline{A}\,\overline{B}C})(\overline{A\overline{B}\,\overline{C}})(\overline{AB\overline{C}})$$

$$= (A+B+C)(A+B+\overline{C})(\overline{A}+B+C)(\overline{A}+\overline{B}+C)$$

2.7 KARNAUGH MAP

The simplification of the switching functions using Boolean laws and theorems becomes complex with the increase in the number of variables and terms. The Karnaugh map technique provides a systematic method for simplifying and manipulating switching expressions. In this technique, the information contained in a truth table or available in the POS or SOP form is represented on the Karnaugh map (K-map). The K-map is actually a modified form of a truth table. Here, the combinations are conveniently arranged to aid the simplification process by applying the rule $Ax + Ax' = A$. In an n-variable K-map, there are 2^n cells. Each cell corresponds to one combination of n variables. Therefore, for each row of the truth table, i.e. for each minterm and for each maxterm, there is one specific cell in the K-map. The K-maps for 2, 3 and 4 variables are shown in Fig. 2.1. The decimal codes corresponding to the combination of variables are given inside the cells. The variables have been marked as A, B, C, and D, and the binary numbers formed by them are taken as AB, ABC, and $ABCD$ for 2, 3 and 4 variables respectively.

B \ A	0	1
0	0	2
1	1	3

(a) two-variable

C \ AB	00	01	11	10
0	0	2	6	4
1	1	3	7	5

(b) three-variable

CD \ AB	00	01	11	10
00	0	4	12	8
01	1	5	13	9
11	3	7	15	11
10	2	6	14	10

(c) four-variable

Fig. 2.1 Karnaugh maps

The 3 and 4 variable K-maps show that the column and row headings, used in representing the cells, are cyclic or unit distance code which result in adjacent cells, differing in just one variable. This helps the grouping of the adjacent cells and in their simplification by the application of the rule $Ax + Ax' = A$. In addition, the left and right-most cells of the 3-variable K-map are adjacent. For example, the cells 0 and 4 are adjacent, and the cells 1 and 5 are adjacent. This is because each pair differs in just a single variable. In the 4-variable K-map, the cells to the extreme left and right as well as those at the top and bottom-most position are adjacent.

A collection or group of 2^m cells, each adjacent to m cells, is called a group. This group can be expressed by a product containing $n-m$ variables, where n is the number of variables in the K-map. For example, in a 4-variable K-map (i.e. $n = 4$), if a group of 4 (i.e., $2^m = 4$; $m = 2$) 1s is formed, then this group can be expressed by $4 - 2 = 2$ variables. Similarly, if a group of eight 1s are combined, then this group can be expressed by $4 - 3 = 1$ variable. This can be better understood by the examples given later.

The entries in a truth table can be represented in a K-map given below. Consider the truth table shown in Table 2.8.

Table 2.8 Truth table of a digital system

Inputs			Output
A	B	C	Y
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

Here, the output Y can be written as

$$Y = \bar{A}\bar{B}C + \bar{A}B\bar{C} + A\bar{B}\bar{C} + ABC$$

$$Y(A, B, C) = m_1 + m_2 + m_4 + m_7$$

The K-map for the above three-variable expression is shown in Fig. 2.2.

Variables		$\bar{A}\bar{B}$ 00	$\bar{A}B$ 01	AB 11	$A\bar{B}$ 10
$\bar{C}$	0	0	1	0	1
C	1	1	0	1	0

Fig. 2.2

The value of the output variable Y (0 or 1) for each row of the truth table is entered in the corresponding cells of the K-map.

Simplification is based on the principle of combining the terms present in adjacent cells. The 1s in the adjacent cells can be grouped by drawing a loop around those cells following the given rules:

1. Construct the K-map and enter the 1s in those cells corresponding to the combinations for which function value is 1, then enter the 0s in the other cells.

2. Examine the map for 1s that cannot be combined with any other 1 cells and form groups with such single 1.
3. Next, look for those 1s which are adjacent to only one other 1 and form groups containing only 2 cells and which are not part of any group of 4 or 8 cells. A group of 2 cells is called a *pair*.
4. Group the 1s which results in groups of 4 cells but are not part of an 8-cells group. A group of 4 cells is called a *quad*.
5. Group the 1s which results in groups of 8 cells. A group of 8 cells is called an *octet*.
6. Form more pairs, quads and octets to include those 1s that have not yet been grouped, and use only a minimum number of groups. There can be overlapping of groups if they include common 1s.
7. Omit any redundant group.
8. Form the logical sum of all the terms generated by each group.

When one or more than one variable appear in both complemented and uncomplemented form within a group, then that variable(s) is eliminated from the term corresponding to that group. Variables that are the same for all the cells of the group must appear in the term corresponding to that group.

A larger group of 1s eliminates more variables. To be precise, a group of two eliminates one variable, a group of four eliminates two variables, similarly a group of eight eliminates three variables.

Example 2.26 Simplify the following expression using the Karnaugh map for the 4-variables A, B, C and D.

$$Y = m_1 + m_3 + m_5 + m_7 + m_8 + m_9 + m_{12} + m_{13}$$

Solution The K-map of the given equation is shown in Fig. E2.26. The expression is minimized using the following steps.

Step 1 Construct the K-map and enter 1 in the cells corresponding to the minterms present in the expression and 0 in the other cells.

Step 2 There are no 1s which are not adjacent to other 1s.

Step 3 There are no pairs which are not part of any larger groups.

Step 4 There are 2 quads. Cells 1, 3, 5 and 7 are grouped to form one quad and the second quad is made up of cells 12, 13, 8 and 9. The combinations corresponding to the cells in the first quad are $\bar{A}\bar{B}\bar{C}D$, $\bar{A}\bar{B}CD$, $\bar{A}B\bar{C}D$ and $\bar{A}BCD$. In the above group of four combinations, the variables $\bar{A}D$ are common in all the cells while B and C appear both in complemented and uncomplemented forms. From the preceding section, it is clear that only the variables that are the same in all the cells of the group must appear in the term corresponding to that group. Therefore, the minimized term for the first quad is $\bar{A}D$; and that of the second quad is $A\bar{C}$.

Step 5 There are no octets.

Step 6 All the 1s have already been grouped.

Step 2: The terms generated by the two groups are OR operated together to obtain the expression for Y as follows:

CD \ AB	00	01	11	10
00	0	0	1	1
01	1	1	1	1
11	1	1	0	0
10	0	0	0	0

$Y = A\bar{C} + \bar{A}D$

Fig. E2.26

Note: In the above *K*-map, if a third quad is formed as shown by the dotted lines, it results in a redundant expression because the 1s to be covered by the third quad are already covered by quads 1 and 2.

Example 2.27 Plot the logical expression $ABCD + A\bar{B}\bar{C}\bar{D} + A\bar{B}C + AB$ on a 4-variable *K*-map; obtain the simplified expression from the map.

Solution To enter into a *K*-map, a logic expression must be either in the canonical SOP form or in the canonical POS form. The canonical SOP form of the given expression can be obtained as follows:

$$
\begin{aligned}
Y &= ABCD + A\bar{B}\bar{C}\bar{D} + A\bar{B}C + AB \\
&= ABCD + A\bar{B}\bar{C}\bar{D} + A\bar{B}C(D + \bar{D}) + AB(C + \bar{C})(D + \bar{D}) \\
&= ABCD + A\bar{B}\bar{C}\bar{D} + A\bar{B}CD + A\bar{B}C\bar{D} + (ABC + AB\bar{C})(D + \bar{D}) \\
&= ABCD + A\bar{B}\bar{C}\bar{D} + A\bar{B}CD + A\bar{B}C\bar{D} + ABCD + ABC\bar{D} + AB\bar{C}D + AB\bar{C}\bar{D} \\
&= ABCD + A\bar{B}\bar{C}\bar{D} + A\bar{B}CD + A\bar{B}C\bar{D} + ABC\bar{D} + AB\bar{C}D + AB\bar{C}\bar{D} \\
&= m_{15} + m_8 + m_{11} + m_{10} + m_{14} + m_{13} + m_{12} \\
&= \Sigma_m(8, 10, 11, 12, 13, 14, 15)
\end{aligned}
$$

The *K*-map for the above expression is shown in Fig. E2.27.

CD \ AB	00	01	11	10
00	0	0	1	1
01	0	0	1	0
11	0	0	1	1
10	0	0	1	1

Fig. E2.27

In the *K*-map in Fig. E2.27, there are three quads; the minimised terms for them are AB, AC and $A\bar{D}$ and the simplified expression is:

$$Y = AB + AC + A\bar{D}$$

Example 2.28 Simplify the expression $Y = \Sigma_m(7, 9, 10, 11, 12, 13, 14, 15)$, using the *K*-map method.

Solution The *K*-map for the above function is shown in Fig. E2.28.

CD \ AB	00	01	11	10
00	0	0	1	0
01	0	0	1	1
11	0	1	1	1
10	0	0	1	1

Fig. E2.28

In the given *K*-map, there are three quads and one pair; the corresponding simplified terms are AB, AD, AC and BCD.

Now, the simplified expression is

$$Y = AB + AD + AC + BCD$$

Since the quads and pair formed in the above *K*-map overlap, the expression can be further simplified using the Boolean algebra as follows:

$$
\begin{aligned}
Y &= AB + AD + AC + BCD \\
&= A(B + D + C) + BCD
\end{aligned}
$$

Example 2.29 Simplify the expression $Y = m_1 + m_5 + m_{10} + m_{11} + m_{12} + m_{13} + m_{15}$, using the *K*-map method.

Solution The *K*-map for the above expression is shown in Fig. E2.29(a).

CD \ AB	00	01	11	10
00	0	0	1	0
01	1	1	1	0
11	0	0	1	1
10	0	0	0	1

Fig. E2.29(a)

As shown in Fig. E2.29(a), the K-map contains four pairs but no quads or octets; the corresponding simplified expression is given by

$$Y = \overline{A}\,\overline{C}D + AB\overline{C} + ABD + A\overline{B}C \qquad (1)$$

It is important to note that the simplified expression obtained from the K-map is not unique. This can be explained by grouping the pairs in a different manner as shown in Fig. E2.29(b).

From the K-map shown in Fig. E2.29(b), the simplified expression can be written as:

$$Y = \overline{A}\,\overline{C}D + AB\overline{C} + ACD + A\overline{B}C \qquad (2)$$

In equations (1) and (2), the third term is different, due to the different groupings done in Fig. E2.29(b). Though the simplified expression for any given function is not unique, both the above expressions are logically equivalent. Two expressions are said to be logically equivalent if and only if both the expressions have the same value for every combination of input variables.

CD \ AB	00	01	11	10
00	0	0	1	0
01	1	1	1	0
11	0	0	1	1
10	0	0	0	1

Fig. E2.29(b)

Example 2.30 Simplify the expression $Y = \Sigma_m(3, 4, 5, 7, 9, 13, 14, 15)$, using the K-map method.

Solution The K-map for the above function is shown in Fig. E2.30.

CD \ AB	00	01	11	10
00	0	1	0	0
01	0	1	1	1
11	1	1	1	0
10	0	0	1	0

Fig. E2.30

In the above K-map, the cells 5, 7, 13 and 15 can be grouped to form a quad as indicated by the dotted lines. In order to group the remaining 1s, four pairs have to be formed as shown in Fig. E2.30. However, all the four 1s covered by the quad are also covered by the pairs. So, the quad in the above K-map is redundant. Therefore, the simplified expression will be

$$Y = \overline{A}CD + ABC + A\overline{C}D + \overline{A}B\overline{C}$$

Example 2.31 Simplify the expression $Y = \Pi(0, 1, 4, 5, 6, 8, 9, 12, 13, 14)$, using the K-map method.

Solution The given function is in the POS form. This can also be written as

$$Y = (A+B+C+D)(A+B+C+\overline{D})(A+\overline{B}+C+D)(A+\overline{B}+C+\overline{D})(A+\overline{B}+\overline{C}+D)$$
$$(\overline{A}+B+C+D)(\overline{A}+B+C+\overline{D})(\overline{A}+\overline{B}+C+D)(\overline{A}+\overline{B}+C+\overline{D})(\overline{A}+\overline{B}+\overline{C}+D)$$

To simplify a POS expression, for each maxterm in the expression, a 0 has to be entered in the corresponding cells and groups must be formed with 0 cells instead of 1 cells to get the minimal expression. The simplified term corresponding to each group can be obtained by the OR operation of the variables that are same for all cells of that group. Here, a variable corresponding to 0 has to be represented in an uncomplemented form and that corresponding to 1 in the complemented form.

The K-map for the function is shown in Fig. E2.31.

CD \ AB	00	01	11	10
00	0	0	0	0
01	0	0	0	0
11	1	1	1	1
10	1	0	0	1

Fig. E2.31

In the above K-map, one octet and one quad are produced by combining 0 cells. The simplified sum term corresponding to the octet is C whereas for the quad is $(\overline{B}+D)$. Hence, the simplified POS expression for the given function is

$$Y = C(\overline{B}+D)$$

Example 2.32 Obtain (a) minimal sum of product and (b) minimal product of sum expressions for the function given below:

$$F(A, B, C, D) = \Sigma_m(0, 1, 2, 5, 8, 9, 10)$$

Solution Here, cells with 1 are grouped to obtain the minimal sum of product; cells with 0 are grouped to obtain minimal product of sum, as shown in Fig. E2.32.

CD \ AB	00	01	11	10
00	1	0	0	1
01	1	1	0	1
11	0	0	0	0
10	1	0	0	1

Fig. E2.32

(a) To obtain minimal sum of products: A quad with four corner 1s and two pairs can be formed as shown in Fig. E2.32. Hence, the minimal SOP expression is:

$$Y = \bar{B}\bar{D} + \bar{A}\bar{C}D + A\bar{B}\bar{C}$$

(b) To obtain minimal product of sum: Three quads can be formed as shown in Fig. E2.32 with the corresponding sum terms $(\bar{A}+\bar{B})$, $(\bar{C}+\bar{D})$ and $(\bar{B}+D)$. Thus, the minimal product of sum expression for the given function is:

$$Y = (\bar{A}+\bar{B})(\bar{C}+\bar{D})(\bar{B}+D)$$

2.7.1 Five-variable K-map

A 5-variable *K*-map contains 32 (2^5) cells which is used to simplify any 5-variable logic expression. A 5-variable *K*-map is shown in Fig. 2.3, with entries in each cell representing the decimal code of that cell. Three variables are used to mark the column headings and two variables are used to mark the row headings. The first four columns can be marked with headings in the same way as the 4-variable *K*-map, after which the remaining four columns can be marked with headings in the reverse order. In other words, the two least significant bits of headings in the last four columns are the mirror image of the corresponding bits in the first four columns. Add 0s to the first four column headings and 1s to the remaining four columns. The simplification using the 5-variable *K*-map can be done in the same way as 4-variable *K*-map.

DE \ ABC	000	001	011	010	110	111	101	100
00	0	4	12	8	24	28	20	16
01	1	5	13	9	25	29	21	17
11	3	7	15	11	27	31	23	19
10	2	6	14	10	26	30	22	18

Fig. 2.3 5-variable K-map

Example 2.33 Simplify $Y = \Sigma_m(3,6,7,8,10,12,14,17,19,20,21,24,25,27,28)$ using the *K*-map method.

Solution In the 5-variable *K*-map shown in Fig. E2.33, there are three quads and three pairs; the simplified expression is given by

$$Y = B\bar{D}\bar{E} + A\bar{C}E + \bar{A}B\bar{E} + A\bar{B}C\bar{D} + \bar{A}\bar{B}CD + \bar{A}\bar{B}DE$$

DE \ ABC	000	001	011	010	110	111	101	100
00	0	0	1	1	1	1	1	0
01	0	0	0	0	1	0	1	1
11	1	1	0	0	1	0	0	1
10	0	1	1	1	0	0	0	0

Fig. E2.33 5-variable K-map

Don't care combinations In certain digital systems, some input combinations never occur during the process of a normal operation because those input conditions are guaranteed never to occur. Such input combinations are *don't care combinations*. We don't care what the function output is for such combinations. These combinations can be plotted on a map to provide further simplification of the function.

The functions considered so far in the various examples, for simplification using the *K*-map method, are completely specified, i.e., it assumes the value 1 for some input combinations and the value 0 for others. Also, there are functions which assume the value 1 for some combinations, the value 0 for some other, and either 0 or 1 for the remaining combinations. Such functions are called *incompletely specified* functions, and the combinations for which the value of the function is not specified are called *don't care combinations*. The don't care combinations are represented by *d* or *x* or ϕ.

When an incompletely specified function, i.e., a function with don't care combinations, is simplified to obtain minimal SOP expression, the value 1 can be assigned to selected don't care combinations. This is done in order to increase the number of 1s in the selected groups, wherever further simplification is possible. Also, a don't care combination need not be used in grouping if it does not cover a large number of 1s. In each case, the choice depends only on the simplification that has to be achieved. Similarly, when a function is simplified to obtain a minimal POS expression, the value 0 can be assigned to selected don't care combinations in order to increase the number of 0s in the selected groups which results in further simplification.

Example 2.34 Simplify the Boolean function

$$F(A,B,C,D) = \Sigma_m(1,3,7,11,15) + \Sigma_d(0,2,5)$$

Solution The *K*-map for the given function is shown in Fig. E2.34 with entries *d* in cells corresponding to combinations 0, 2 and 5.

As discussed in the previous section, the 1s and d's are combined in order to enclose the maximum number of adjacent cells with 1. As shown in the K-map in Fig.E2.34, by combining the 1s and d's, two quads can be obtained. The d in cell 5 is left free since it does not contribute in increasing the size of any group. Now, the simplified expression in SOP form is

$$Y = \bar{A}\bar{B} + CD$$

CD \ AB	00	01	11	10
00	d	0	0	0
01	1	d	0	0
11	1	1	1	1
10	d	0	0	d

Fig. E2.34

Example 2.35 Using the K-map method, simplify the following Boolean function and obtain (i) minimal SOP and (ii) minimal POS expressions:

$$Y = \Sigma_m (0, 2, 3, 6, 7) + \Sigma_d (8, 10, 11, 15)$$

Solution The K-map for the above function is shown in Fig.E2.35.

Minimal SOP form By combining the 1s and d's as shown in the K-map, there are two quads; the simplified SOP expression is given by

$$Y = \bar{A}C + \bar{B}\bar{D}$$

CD \ AB	00	01	11	10
00	1	0	0	d
01	0	0	0	0
11	1	1	d	d
10	1	1	0	d

Fig. E2.35

Minimal POS form One octet and two quads can be obtained by combining the 0s and d's as shown in the K-map; the simplified POS expression is given by

$$Y = \bar{A}(C + \bar{D})(\bar{B} + C)$$

Example 2.36 Obtain the minimal SOP expression for the function

$$Y = \Sigma_m (1, 5, 7, 13, 14, 15, 17, 18, 21, 22, 25, 29) + \Sigma_d (6, 9, 19, 23, 30)$$

Solution The K-map for the given function is shown in Fig.E2.36.

DE \ ABC	000	001	011	010	110	111	101	100
00	0	0	0	0	0	0	0	0
01	1	1	1	d	1	1	1	1
11	0	1	1	0	0	0	d	d
10	0	d	1	0	0	d	1	1

Fig. E2.36

By combining the 1s and d's, one octet and two quads can be obtained as shown in Fig.E2.36. The simplified expression is

$$Y = \bar{D}E + \bar{A}CD + A\bar{E}D$$

2.8 QUINE–McCLUSKEY OR TABULAR METHOD OF MINIMIZATION OF LOGIC FUNCTIONS

The K-map is a very effective tool for minimization of logic functions with 4 or less variables. For logic expressions with more than 4 variables, the visualization of adjacent cells and the drawing of the K-map become more difficult. The Quine–McCluskey method, also known as the tabular method, can be employed in such cases to minimize switching functions. This method employs a systematic, step-by-step procedure to produce a simplified standard form of expression for a function with any number of variables. The steps to be followed in the Quine–McCluskey method are:

Step 1 A set of all prime implicants of the function must be obtained.

Step 2 From the set of all prime implicants, a set of essential implicants must be determined by preparing a prime implicant chart.

Step 3 The minterms which are not covered by the essential implicants are taken into consideration and a minimum cover is obtained from the remaining prime implicants.

Selecting prime implicants The procedure for selecting prime implicants is given below:

(i) Each minterm should be expressed by its binary representation.

(ii) The minterms should be arranged according to increasing index (index can be defined as the number of 1s in a minterm). Separate each set of minterms possessing the same index by lines.

(iii) Each term of index 'n' should be compared with each term of index $(n+1)$. For each pair of terms that can combine, the newly formed terms should be stated. If two minterms differ in only one variable, that variable should be removed and a dash placed at that position; thus a new term with one less literal is found. After all the pairs of terms with indices n and $(n+1)$ have been considered, a line should be drawn under the last term.

(iv) When the above process has been repeated for all the groups of terms, one stage of elimination will have been completed.

(v) The next stage of elimination or matching process should be repeated for the new terms. According to this stage, two terms can be combined only when they have dashes in the same positions.

(vi) The cycles have to be continued until no new list can be found (i.e. no further elimination of literals).

(vii) All terms which remain unchecked (do not match) during the process are considered to be prime implicants.

Prime implicant chart

(i) The prime implicants should be represented in rows and each minterm of the function in a column.

(ii) Crosses should be placed in each row to show the composition of minterms that makes the prime implicants.

(iii) A completed prime implicants table should be inspected for columns containing only a single cross. Prime implicants that cover minterms with a single cross in their column are called essential prime implicants.

Example 2.37 Find the minimal sum of products for the Boolean expression, $f = \Sigma(1, 2, 3, 7, 8, 9, 10, 11, 14, 15)$, using the Quine–McCluskey method.

Solution Firstly, these minterms are represented in the binary form as shown in Table E2.37(a). The above binary representation are grouped into a number of sections in terms of the number of 1s as shown in Table E2.37(b).

Table E2.37(a) Binary representation of minterms

Minterms	Variables			
	A	B	C	D
1	0	0	0	1
2	0	0	1	0
3	0	0	1	1
7	0	1	1	1
8	1	0	0	0
9	1	0	0	1
10	1	0	1	0
11	1	0	1	1
14	1	1	1	0
15	1	1	1	1

Table E2.37(b). Group of minterms for different number of 1s

Number of 1s	Minterms	Variables			
		A	B	C	D
1	1 ✓	0	0	0	1
	2 ✓	0	0	1	0
	8 ✓	1	0	0	0
2	3 ✓	0	0	1	1
	9 ✓	1	0	0	1
	10 ✓	1	0	1	0
3	7 ✓	0	1	1	1
	11 ✓	1	0	1	1
	14 ✓	1	1	1	0
4	15 ✓	1	1	1	1

Any two numbers in these groups which differ from each other by only one variable can be chosen and combined, to get two-cell combinations, as shown in Table E2.37(c).

Table E2.37(c). 2-cell combinations

Combination	A	B	C	D
(1,3) ✓	0	0	–	1
(1,9) ✓	–	0	0	1
(2,3) ✓	0	0	1	–
(2,10) ✓	–	0	1	0
(8,9) ✓	1	0	0	–
(8,10) ✓	1	0	–	0
(3,7) ✓	0	–	1	1
(3,11) ✓	–	0	1	1
(9,11) ✓	1	0	–	1
(10,11) ✓	1	0	1	–
(10,14) ✓	1	–	1	0
(7,15) ✓	–	1	1	1
(11,15) ✓	1	–	1	1
(14,15) ✓	1	1	1	–

From the two-cell combinations, one variable and a dash in the same position can be combined to form 4-cell combinations as shown in Table E2.37(d).

Table E2.37(d) 4-cell combinations

Combination	A	B	C	D
(1, 3, 9, 11)	–	0	–	1
(2, 3, 10, 11)	–	0	1	–
(8, 9, 10, 11)	1	0	–	–
(3, 7, 11, 15)	–	–	1	1
(10, 11, 14, 15)	1	–	1	–

Note that the cells (1,3) and (9,11) form the same 4-cell combination as the cells (1,9) and (3,11). The order in which the cells are placed in a combination does not have any effect. Thus, the (1,3,9,11) combination may be given as (1,9,3,11). Using Table E2.37(d), the *prime implicants table* can be plotted as shown in Table E2.37(e).

The columns having only one cross mark correspond to essential prime implicants. A tick mark is put against every column which has only one cross mark. A star mark is placed against every essential primary implicant. The sum of the prime implicants gives the function in its minimal SOP form, since all prime implicants are essential prime implicants.

Therefore, $f = \overline{B} \cdot D + \overline{B} \cdot C + A \cdot \overline{B} + C \cdot D + A \cdot C$

Table E2.37(e) Prime implicants table

Prime implicants	Minterms									
	1	2	3	7	8	9	10	11	14	15
(1, 3, 9, 11) *	X		X			X		X		
(2, 3, 10, 11) *		X	X				X	X		
(8, 9, 10, 11) *					X	X	X	X		
(3, 7, 11, 15) *			X	X				X		X
(10, 11, 14, 15) *							X	X	X	X
	✓	✓		✓	✓				✓	

Example 2.38 Find the minimal sum of products for the Boolean expression, $f(w,x,y,z) = \Sigma(1,3,4,5,9,10,11) + \Sigma\phi(6,8)$, using the Quine–McCluskey method.

Solution These minterms are firstly represented in binary form as shown in Table E2.38(a). The above binary representation is grouped into a number of sections in terms of a number of 1s as shown in Table E2.38(b).

Table E2.38(a) Binary representation of minterms

Minterms	Variables			
	w	x	y	z
1	0	0	0	1
3	0	0	1	1
4	0	1	0	0
5	0	1	0	1
6	0	1	1	0
8	1	0	0	0
9	1	0	0	1
10	1	0	1	0
11	1	0	1	1

Table E2.38(b) Group of minterms for different number of 1s

Number of 1s	Minterms	Variables			
		w	x	y	z
1	1 ✓	0	0	0	1
	4 ✓	0	1	0	0
	8 ✓	1	0	0	0
2	3 ✓	0	0	1	1
	5 ✓	0	1	0	1
	6 ✓	0	1	1	0
	9 ✓	1	0	0	1
	10 ✓	1	0	1	0
3	11 ✓	1	0	1	1

Any two numbers in these groups which differ from each other by only one variable can be chosen and combined to get two-cell combinations, as shown in Table E2.38(c).

Table E2.38(c) 2-cell combinations

Combination	w	x	y	z
(1,3) ✓	0	0	–	1
(1,5)	0	–	0	1
(1,9) ✓	–	0	0	1
(4,5)	0	1	0	–
(4,6)	0	1	–	0
(8,9) ✓	1	0	0	–
(8,10) ✓	1	0	–	0
(3,11) ✓	–	0	1	1
(9,11) ✓	1	0	–	1
(10,11) ✓	1	0	1	–

From the two-cell combinations, one variable and a dash in the same position can be combined to form 4-cell combinations as shown in Table E2.38(d).

Table E2.38(d) 4-cell combinations

Combination	w	x	y	z
(1, 3, 9, 11)	–	0	–	1
(8, 9, 10, 11)	1	0	–	–

Note that the cells (1,3) and (9,11) form the same 4-cell combination as the cells (1,9) and (3,11). The order in which the cells are placed in a combination has no effect. Therefore, the (1,3,9,11) combination may be written as (1, 9, 3, 11). Using table E2.38(d), the prime implicants table can be plotted as shown in Table E2.38(e).

Don't care minterms cannot be listed as column headings in the chart because they do not have to be covered by a minimal expression.

Table E2.38(e) Prime implicants table

Prime implicants	Minterms						
	1	3	4	5	9	10	11
(1,5)	X			X			
(4,5)			X	X			
(4,6)			X				
(1,3,9,11) *	X	X			X		X
(8,9,10,11) *					X	X	X
		✓				✓	

The columns having only one cross mark correspond to the essential prime implicants. A tick mark is put against every column which has only one cross mark. A star mark is put against every essential primary implicant. The prime implicant which covers the minterm (1,3,9,11) is the essential prime implicant. Therefore, in order to cover the remaining minterms, the reduced prime implicant chart is formed as shown in Table E2.38(f).

Table E2.38 (f) Reduced prime implicants table.

Prime Implicants	Minterms 1	3	4	5	9	10	11
(1, 5)	X			X			
(4, 5)			X	X			
(4, 6)			X				

To cover the minterms (4,5), the prime implicants (4,5) can be selected, in addition to the essential prime implicants, for obtaining the minimal expression of the given function.

Therefore, $f(w, x, y, z) = \bar{x}z + \bar{w}x\bar{y} + w\bar{x}$

REVIEW QUESTIONS

1. State the methods used to simplify the Boolean equations.
2. State and explain the basic Boolean logic operations.
3. What are the applications of Boolean algebra?
4. Define the truth table.
5. How is the AND multiplication different from ordinary multiplication?
6. How does OR addition differ from the ordinary addition method?
7. What are the basic laws of Boolean algebra?
8. State and prove Absorption and Simplification theorems.
9. State and prove Associative and Distributive theorems.
10. What is meant by duality in Boolean algebra?
11. State DeMorgan's theorem.
12. State and explain the DeMorgan's theorems which convert a sum into a product form and vice versa.
13. Explain the terms: (a) prime implicant, (b) input variable, (c) minterm and (d) maxterm.
14. Prove DeMorgan's theorem for a 4-variable function.
15. Many cars produced in Japan have an interlock system that allows the engine to start only if both the front seat occupants have their seat-belts on. Construct a truth table to indicate whether the car may be started based upon whether a passenger is present and whether both the passenger and the driver have buckled their seat-belts.
16. Draw a truth function table for a person crossing a river based upon whether (i) the river is frozen over, (ii) the boat leaks, and (iii) the person can swim.

PROBLEMS

1. Indicate whether Y is a 0 or a 1 in the equation $Y = A\bar{B}\bar{C} + AB$, under the following conditions: (a) $A = 1, B = 0, C = 1$; (b) $A = 0, B = 1, C = 1$; (c) $A = 0, B = 0, C = 0$

   Ans: (a) 0, (b) 0, (c) 0

2. Simplify the following expressions:
   (a) $A\bar{B}\bar{C} + \bar{A}\bar{B}\bar{C} + \bar{A}B\bar{C} + \bar{A}\bar{B}C$
   (b) $ABC + \bar{A}BC + A\bar{B}C + AB\bar{C} + A\bar{B}\bar{C} + \bar{A}B\bar{C} + \bar{A}\bar{B}\bar{C}$
   (c) $A(A+B+C)\cdot(\bar{A}+B+C)\cdot(A+\bar{B}+C)\cdot(A+B+\bar{C})$
   (d) $(A+B+C)\cdot(A+\bar{B}+\bar{C})\cdot(A+B+\bar{C})\cdot(A+\bar{B}+C)$

   Ans: (a) $\bar{B}\bar{C} + \bar{A}\bar{C} + \bar{A}\bar{B}$, (b) $A + B + \bar{C}$, (c) $A(B + C)$, (d) A is a minimal expression.

3. Draw a truth table for the equations given below:
   (a) $Y = AC + AB$, (b) $Y = A(\bar{B} + \bar{C})$
4. Draw a truth table for the equations given below:
   (a) $Y = AB(B + C + \bar{D})$ (b) $Y = (A + B + C)\bar{A}B$
   (c) $Y = A\bar{B}C(C + \bar{D})$ (d) $Y = AB + BA + C(A + B)$
5. Reduce the following Boolean expressions:
   (a) $ABC(ABC + 1)$ (b) $A + \bar{A} + B + C$
   (c) $AAB + ABB + BCC$ (d) $AB + B + A + C$

   Ans: (a) ABC, (b) 1, (c) $AB + BC$, (d) $A + B + C$
6. Reduce the following Boolean expressions:
   (a) $AB + BB + C + \bar{B}$ (b) $A(\bar{A} + B)$ (c) $AB(B + C)$ (d) $ABB(ABC + BC)$
   (e) $(AB + C)(AB + D)$

   Ans: (a) 1, (b) AB, (c) AB, (d) ABC, (e) $AB + CD$
7. Reduce the given Boolean expressions:
   (a) $A\bar{B}C + \bar{A}\bar{B}C$ (b) $A\bar{B}C + \bar{A}BC + ABC$ (c) $(\bar{A}B)(AB) + AB$
   (d) $(1 + B)(ABC)$ (e) $(A\bar{B} + A\bar{C})(BC + B\bar{C})(ABC)$

   Ans: (a) $\bar{B}C$ (b) $AC + BC$ (c) AB (d) ABC (e) 0
8. Reduce the given Boolean expressions:
   (a) $A + \bar{B}C(A + \bar{B}C)$ (b) $A(\overline{\bar{A}BC} + A\bar{B}C)$
   (c) $\overline{ABC + AB + BC}$ (d) $A[B + C\overline{(AB + AC)}]$

   Ans: (a) $A + \bar{B}C$ (b) $A(\bar{B} + \bar{C})$ (c) $\bar{B} + \bar{C}$ (d) AB
9. Derive the complements of the expressions given below:
   (a) $(A + BC + AB)$ (b) $(A + B)(B + C)(A + C)$
   (c) $AB + BC + CD$ (d) $AB(\bar{C}D + \bar{B}C)$ (e) $A(B + C)(\bar{C} + \bar{D})$

   Ans: (a) $\bar{A}(\bar{B} + \bar{C})$ (b) $\bar{A}\bar{B} + \bar{B}\bar{C} + \bar{A}\bar{C}$ (c) $\bar{A}\bar{C} + \bar{B}\bar{C} + \bar{B}\bar{D}$
   (d) $\overline{ABD} + C$ (e) $\bar{A} + \bar{B}\bar{C} + CD$
10. Simplify each of the following expressions:
    (a) $Y = RST + RS + (\bar{T} + V)$ (b) $Y = (M + N)(\bar{M} + P)(\bar{N} + P)$

    Ans: (a) $RS + \bar{T} + V$ (b) $P(M + N)$
11. Using Boolean techniques, simplify the given expressions:
    (a) $AB + A(B + C) + B(B + C)$ (b) $AB(C + B\bar{D}) + \overline{AB}$

    Ans: (a) $B + AC$, (b) $\overline{AB\bar{C}D}$
12. Simplify the following expressions:
    (a) $A + AB + A\bar{B}C$ (b) $(\bar{A} + B)C + ABC$ (c) $A\bar{B}C(BD + CDE) + A\bar{C}$

    Ans: (a) A, (b) $C(\bar{A} + B)$ (c) $A(\bar{C} + \bar{B}DE)$
13. Apply DeMorgan's theorem to each of the given expressions:
    (a) $\overline{A(B + C)}$ (b) $\overline{AB} + \overline{CD}$ (c) $\overline{AB + CD}$

    Ans: (a) $\bar{A} + \bar{B}\cdot\bar{C}$ (b) $\bar{A} + \bar{B} + \bar{C} + \bar{D}$ (c) $(\bar{A} + \bar{B})(\bar{C} + \bar{D})$ (d) $(A + \bar{B})(\bar{C} + D)$ (d) $\bar{A}B + \overline{CD}$
14. Apply DeMorgan's theorem to the given expressions:
    (a) $\overline{AB + (C + D)}$ (b) $\overline{AB(CD + EF)}$ (c) $\overline{(A + B + C + D)} + \overline{ABCD}$
    (d) $\overline{(\bar{A} + B + C + D) + AB\bar{C}D}$ (e) $\overline{AB(CD + EF)(AB + CD)}$

    Ans: (a) $(\bar{A} + B)\bar{C}D$ (b) $(\bar{A} + \bar{B}) + (\bar{C} + \bar{D})\cdot(\bar{E} + \bar{F})$
    (c) $\overline{ABCD}$ (d) $A\bar{B}\bar{C}D$ (e) $AB + (E + \bar{F}) + (\bar{C} + \bar{D})$.

15. Simplify the given expressions using the Boolean Algebra method.
(a) $BD + B(D+E) + \bar{D}(D+F)$ (b) $\bar{A}\bar{B}C + \overline{(A+B+\bar{C})} + \bar{A}\bar{B}\bar{C}D$
(c) $(B+BC)(B+\bar{B}C)(B+D)$ (d) $ABCD + AB(\overline{CD}) + (\overline{AB})CD$
(e) $ABC\left[AB + \bar{C}(BC + AC)\right]$

Ans: (a) $BD + BE + \bar{D}F$ (b) $\bar{A}\bar{B}(C+D)$ (c) B (d) $AB + CD$ (e) ABC

16. Give a Boolean expression for the following statements:
(a) Y is a 1 only if A is a 1 and B is a 1 or if A is a 0 and B is a 0.
(b) Y is a 1 only if A, B and C are all 1s or if only one of the variables is a 0.

Ans: (a) $Y = AB + \bar{A}\bar{B}$ (b) $Y = ABC + \bar{A}BC + A\bar{B}C + AB\bar{C}$

17. Simplify the following logic expressions:
(a) $(\bar{A} + B + \bar{C})(\bar{A} + B + D + \bar{E})(C + D)$ (b) $F = \bar{A}\bar{B}C + \bar{B}C + \bar{A}C$

Ans: (a) $(\bar{A} + B)(C + D) + \bar{C}D$ (b) C

18. Simplify the following expressions using the simplification theorem.
(a) $A + \bar{A}B + \overline{(A+B)}C + \overline{(A+B+C+D)}$ (b) $A\bar{B} + A\bar{C} + BC\bar{D} + \bar{D}$

Ans: (a) $A + B + C + \bar{D}$ (b) $A\bar{B} + BC + \bar{D}$

19. Using the absorption theorem, simplify the following expressions:
(a) $A + \bar{A}\bar{B} + BC\bar{D} + B\bar{D}$ (b) $A\bar{B}C + (\bar{B} + \bar{C})(\bar{B} + \bar{D}) + \overline{A + C + D}$

Ans: (a) $A + \bar{B} + \bar{D}$ (b) $\bar{B} + \overline{CD}$

20. Prove the following using Boolean theorems:
(a) $(A+C)(A+D)(B+C)(B+D) = AB + CD$
(b) $(\bar{A} + \bar{B} + \bar{D})(\bar{A} + B + \bar{D})(B + C + D)(A + \bar{C})(A + \bar{C} + D) = \bar{A}\bar{C}D + AC\bar{D} + B\bar{C}\bar{D}$

21. (a) Convert $f = ABCD + \bar{A}BC + \bar{B}\bar{C}$ into a sum of minterms by algebraic method.
(b) Convert $f = AB + \bar{B}CD$ into a product of maxterms by algebraic method.

Ans: (a) $f = ABCD + \bar{A}BCD + A\bar{B}\bar{C}D + \bar{A}BC\bar{D} + \bar{A}\bar{B}\bar{C}D + A\bar{B}\bar{C}\bar{D} + \bar{A}\bar{B}\bar{C}\bar{D}$

(b) $f = (A+B+C+D)(A+B+C+\bar{D})(A+B+\bar{C}+D)(A+\bar{B}+C+D)$
$(A+\bar{B}+C+\bar{D})(A+\bar{B}+\bar{C}+D)(A+\bar{B}+\bar{C}+\bar{D})(\bar{A}+B+C+D)$
$(\bar{A}+B+C+\bar{D})(\bar{A}+B+\bar{C}+D)$

22. Obtain the canonical sum of products and product of sums of the following expression:
$f = x_1x_2x_3 + x_1x_3x_4 + x_1x_2x_4$

Ans: $f = x_1x_2x_3x_4 + x_1x_2x_3\bar{x}_4 + x_1x_2\bar{x}_3x_4 + x_1\bar{x}_2x_3x_4$ [Canonical SOP]

$f = (x_1 + x_2 + x_3 + x_4)(x_1 + x_2 + x_3 + \bar{x}_4)(x_1 + x_2 + \bar{x}_3 + x_4)(x_1 + x_2 + \bar{x}_3 + \bar{x}_4)$
$(x_1 + \bar{x}_2 + x_3 + x_4)(x_1 + \bar{x}_2 + x_3 + \bar{x}_4)(x_1 + \bar{x}_2 + \bar{x}_3 + x_4)(x_1 + \bar{x}_2 + \bar{x}_3 + \bar{x}_4)$
$(\bar{x}_1 + x_2 + x_3 + x_4)(\bar{x}_1 + x_2 + x_3 + \bar{x}_4)(\bar{x}_1 + x_2 + \bar{x}_3 + x_4)(\bar{x}_1 + \bar{x}_2 + x_3 + x_4)$ [Canonical POS]

23. Using the K-map method, simplify the following function, obtain their (i) minimum sum of products, and (ii) minimum product of sums form.
$f(w, x, y, z) = \Sigma(1, 3, 4, 5, 6, 7, 9, 12, 13)$
$f(w, x, y, z) = \Sigma(1, 5, 6, 7, 11, 12, 13, 15)$

Ans: (a) (i) $f = \bar{w}z + \bar{w}x + \bar{y}z + x\bar{y}$ (ii) $f = (\bar{w} + \bar{y})(x + z)$
(b) (i) $f = x(w \oplus y) + z(\overline{w \oplus y})$
(ii) $f = (w + y + z)(\bar{w} + x + y)(w + x + \bar{y})(\bar{w} + \bar{y} + z)$

24. Using the K-map method, obtain the minimal sum of product expression of the following function.
$Y = \Sigma(0, 2, 3, 6, 7, 8, 10, 11, 12, 15)$

Ans: $Y = \bar{A}C + CD + \bar{B}\bar{D} + A\bar{C}\bar{D}$

25. Determine the don't care condition in the following Boolean expression $BE + \bar{B}D\bar{E}$ which is a simplified version of the expression $\bar{A}BE + BCDE + B\bar{C}\bar{D}E + \bar{A}\bar{B}D\bar{E} + \bar{B}\bar{C}D\bar{E}$

Ans: don't care combinations: $A\bar{B}CD\bar{E} + ABC\bar{D}E + \bar{A}\bar{B}CD\bar{E}$

26. Using the K-map method, simplify the following functions into minimal sum of products:
$f(u, w, x, y, z) = \Sigma(0, 2, 5, 7, 9, 11, 13, 15, 16, 18, 21, 23, 25, 27, 29, 31)$

Ans: $f = wz + xz + \overline{w}\overline{x}\overline{z}$

27. A corporation having 100 shares entitles the owner of each share to cast one vote at the share-holders' meeting. Assume that A has 40 shares, B has 30 shares, C has 20 shares and D has 10 shares. A two-third majority is required to pass a resolution in a share-holders' meeting. Each of these four men has a switch which he closes to vote YES and opens to vote NO for his percentage of shares. When the resolution is passed the output, LED must be ON. Derive a truth table for the output function and give the sum of product equation for it.

Ans: $f = AB\bar{C}\bar{D} + A\bar{B}CD + ABC\bar{D} + AB\bar{C}D + ABCD = AB + ACD$

28. (a) Express the following function as a product of maxterms $f = \Sigma(1, 3, 5, 7)$
(b) Express the complement of the function as a sum of minterms.
(c) Express the complement of the function as a product of maxterms.

Ans: (a) $(A+B+C)(A+\bar{B}+C)(\bar{A}+B+C)(\bar{A}+\bar{B}+C)$
(b) $\bar{A}\bar{B}\bar{C} + \bar{A}B\bar{C} + A\bar{B}\bar{C} + AB\bar{C}$
(c) $(A+B+\bar{C})(A+\bar{B}+\bar{C})(\bar{A}+B+\bar{C})(\bar{A}+\bar{B}+\bar{C})$

29. Prepare Karnaugh maps for the following functions:
(a) $f = ABC + \bar{A}BC + \bar{B}\bar{C}$ (b) $f = A + B + \bar{C}$ (c) $f = AB + \bar{B}CD$

Chapter

3

Logic Gates

3.1 INTRODUCTION

Boolean algebra is used in describing and simplifying logic circuits. Simplification of Boolean logic expressions is very important because it reduces the hardware required to design a specific system. The Boolean expression corresponding to a given gate network can be derived by systematically progressing from the input to the output of gates. The gating or logic network can be formed by interconnecting the OR, AND and NOT gates.

Combinational circuit. A combinational circuit consists of input variables, logic gates and output variables. The design of combinational circuits starts from the verbal outline of the problem or from a set of Boolean functions, and ends in a logic circuit diagram. The steps involved in the design of combinational circuits are as follows:

(i) State the problem in words.

(ii) Find the number of input and output variables.

(iii) Assign letter symbols to the input and output variables.

(iv) Obtain the truth table using the word statement.

(v) Obtain Boolean expressions for each output from the truth table.

(vi) Simplify the Boolean expressions to minimise the number of variables by using laws of Boolean algebra or Karnaugh map method or McCluskey's method.

(vii) Draw the logic circuit diagram corresponding to the simplified Boolean expression.

3.2 POSITIVE AND NEGATIVE LOGIC DESIGNATION

Logics 1 and 0 are generally represented by voltage levels. In a *Positive logic* system, the most positive voltage level (HIGH) represents the logical 1 state, and the most negative voltage level (LOW) represents the logical 0 state. In a *Negative logic* system, the most positive (HIGH) voltage level represents logical 0 state and the most negative (LOW) voltage level represents logical 1 state. For example, if the voltage levels are –0.1V and –5V, then in a positive logic system the –5V level represents a

0 state and the –0.1 V level represents a 1 state; in a negative logic system, –0.1 V level represents a 0 state and –5V represents a 1 state. Conversely, if the voltage levels are 0.1 V and 5V, then in a positive logic system the 5V level represents a 1 state and the 0.1 V represents a 0 state; in a negative logic system, the 0.1 V represents a 1 state and the 5 V level represents a 0 state.

The effect of changing from one logic designation to the other is equivalent to complementing the logic function. The simple method of converting the logic designation (i.e. from positive to negative logic or vice versa) is that all 0s are replaced with 1s and all 1s with 0s in the Truth Table. The resulting logic function is determined accordingly. For example, when 0s and 1s are interchanged in the truth table, the positive logic AND gate becomes negative logic OR gate, and positive logic NAND gate becomes negative logic NOR gate. As there is no real advantage to either designation, the choice of positive or negative logic is made by the individual logic designer.

3.3 LOGIC GATES

A logic gate is an electronic circuit which makes logical decisions. To arrive at these decisions, the most common logic gates used are OR, AND, NOT, NAND and NOR gates. The NAND and NOR gates are called as the *Universal gates*. The exclusive-OR gate is another logic gate which can be constructed using *basic gates* such as AND, OR and NOT gates.

Logic gates have two or more inputs and only one output except for the NOT gate, which has only one input. The output signal appears only for certain combinations of the input signals. The manipulation of binary information is done by the gates. The logic gates are the building blocks of hardware which are available in the form of various IC families. Each gate has a distinct logic symbol and its operation can be described by means of an algebraic function. The relationship between input and output variables of each gate can be represented in a tabular form called a *truth table*.

3.3.1 OR Gate

The OR gate performs logical addition, commonly known as OR function. The OR gate has two or more inputs and only one output. The operation of OR gate is such that a HIGH (1) on the output is produced when any of the inputs is HIGH(1). The output is LOW(0) only when all the inputs are LOW(0).

If A and B are the input variables of an OR gate and Y is its output, then

$$Y = A + B$$

Similarly, for more than two variables, the OR function can be expressed as

$$Y = A + B + C + D + \dots$$

An OR gate using diodes is shown in Fig. 3.1(a) in which A and B represent the inputs and Y the output. The resistance R_L is the load resistance.

If $A = 0$ & $B = 0$, both the diodes will not conduct, and hence the output $Y = 0$.

If $A = 1$ & $B = 0$, diode D_1 conducts, then $V_0 \approx 5V$ and so $Y = 1$.

If $A = 0$ & $B = 1$, diode D_2 conducts and hence $Y = 1$.

If $A = 1$ & $B = 1$, both the diodes conduct and hence $Y = 1$.

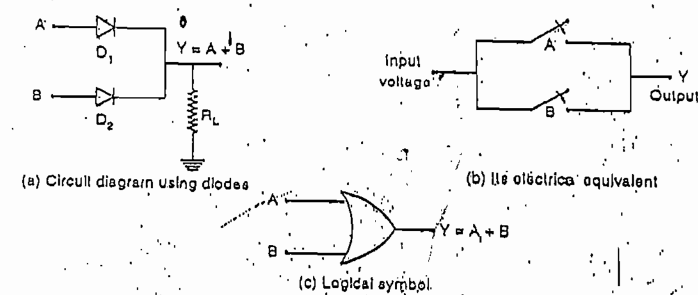


Fig. 3.1 2-input OR gate

The electrical equivalent circuit of an OR gate is shown in Fig. 3.1(b) where switches A and B are connected in parallel. If either A or B is closed or if both are closed, then the output is high. The logic symbol for a 2-input OR gate is shown in Fig. 3.1(c). The logical operation of the two-input OR gate is described in the truth table shown in Table 3.1.

Table 3.1 Truth table of OR gate

Inputs		Output
A	B	$Y = A + B$
0	0	0
0	1	1
1	0	1
1	1	1

The same idea can be extended to an OR gate with more than two inputs. Fig. 3.2 shows a 3-input OR gate. Table 3.2 gives the truth table of a 3-input OR gate.

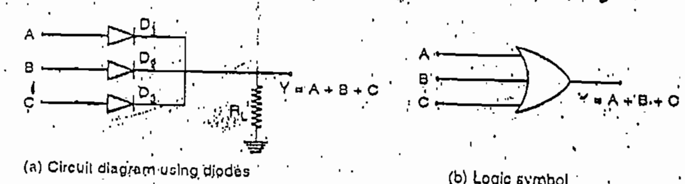


Fig. 3.2 3-input OR gate

Table 3.2 Truth table of 3-input OR gate

Inputs			Output
A	B	C	$Y = A + B + C$
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
1	0	1	1
1	1	0	1
1	1	1	1

In general, if n is the number of input variables, then there will be 2^n possible combinations, since each variable can take on either of two values.

3.3.2 AND Gate

The AND gate performs logical multiplication, commonly known as AND function. The AND gate has two or more inputs and a single output. The output of an AND gate is HIGH only when all the inputs are HIGH. Even if any one of the inputs is LOW, the output will be LOW.

If A and B are the input variables of an AND gate and Y is its output, then

$$Y = A \cdot B$$

where the dot (.) denotes the AND operation. Moreover, one typically deletes the dot and writes as $Y = AB$.

A 2-input AND gate using diodes is shown in Fig. 3.3(a) in which A and B represent the inputs and Y the output.

If $A = 0$ & $B = 0$, both the diodes conduct as they are forward biased, and hence the output is $Y = 0$.

If $A = 0$ & $B = 1$, the diode D_1 conducts and D_2 does not conduct, and hence the output is $Y = 0$.

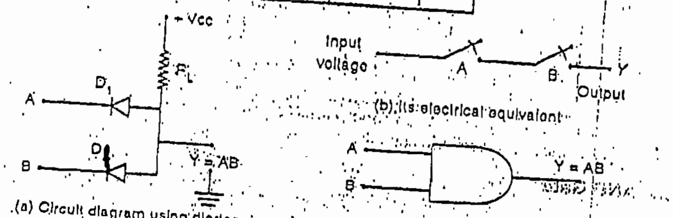
If $A = 1$ & $B = 0$, the diode D_1 does not conduct and D_2 conducts, and hence the output is $Y = 0$.

If $A = 1$ & $B = 1$, both the diodes do not conduct as they are reverse biased, and hence the output is $Y = 1$.

The electrical equivalent circuit of an AND gate is shown in Fig. 3.3(b) where two switches A and B are connected in series. If both A and B are closed, then the output is high. Logic symbol of the 2-input AND gate is shown in Fig. 3.3(c). The logical operation of the two input AND gate and the three-input AND gate are described in the truth tables shown in Tables 3.3. and 3.4.

Table 3.3 Truth table of a 2-input AND gate

Inputs		Output
A	B	$Y = A \cdot B$
0	0	0
0	1	0
1	0	0
1	1	1



(a) Circuit diagram using diodes (b) Its electrical equivalent (c) Logic symbol

Fig. 3.3 2-input AND gate

From Table 3.4, it is seen that the AND gate has a HIGH output only when A, B and C are HIGH. When there are more inputs, all inputs must be HIGH for a HIGH output. For this reason, the AND gate is also called an ALL Gate.

Table 3.4 Truth table of a 3-input AND gate

Inputs			Output
A	B	C	$Y = A \cdot B \cdot C$
0	0	0	0
0	0	1	0
0	1	0	0
0	1	1	0
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	1

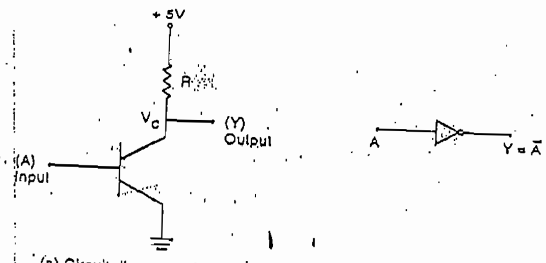
3.3.3 NOT Gate (Inverter)

The NOT gate performs the basic logical function called *inversion* or *complementation*. The purpose of this gate is to convert one logic level into the opposite logic level. It has one input and one output. When a HIGH level is applied to an inverter, a LOW level appears at its output and vice versa.

A NOT gate using a transistor is shown in Fig. 3.4(a) in which A represents the input and Y represents the output, i.e. $Y = \overline{A}$. When the input is HIGH, the transistor is in the ON state and the output ($V_c = V_{c(sat)}$) is LOW. If the input is LOW, the transistor is in the OFF state and the output $V_c = V_{cc}$ is HIGH. The symbol for the inverter is shown in Fig. 3.4(b). The truth table of a NOT gate is given in Table 3.5.

Table 3.5 Truth table of an inverter

Inputs	Output
A	$Y = \overline{A}$
0	1
1	0

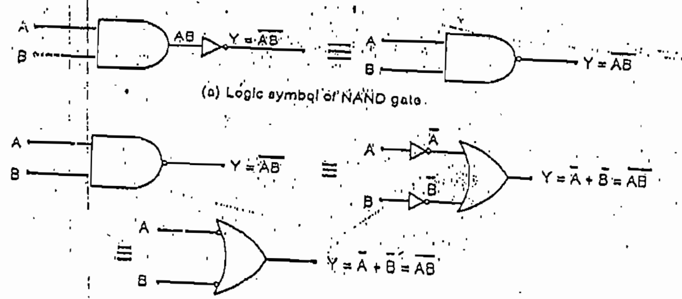


(a) Circuit diagram using transistor (b) Logic symbol

Fig. 3.4 NOT gate

3.3.4 NAND Gate

NAND is a contraction of the NOT–AND gates. It has two or more inputs and only one output, i.e. $Y = \overline{A \cdot B}$. When all the inputs are HIGH, the output is LOW. If any one or both the inputs are LOW, then the output is HIGH. The logic symbol for the NAND gate is shown in Fig. 3.5(a). The small circle or bubble represents the operation of inversion.



(a) Logic symbol of NAND gate

(b) NAND gate = Bubbled OR gate

Fig. 3.5 NAND gate

The truth table for the NAND gate is shown in Table 3.6.

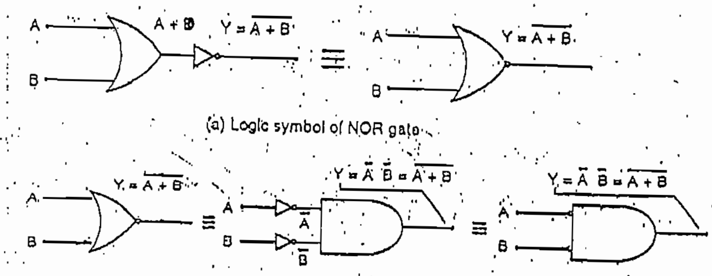
Table 3.6 Truth table of a 2-input NAND gate

Inputs		Output
A	B	$Y = \overline{A \cdot B}$
0	0	1
0	1	1
1	0	1
1	1	0

The NAND gate is equivalent to an OR gate with a bubble at its inputs which is shown in Fig. 3.5(b).

3.3.5 NOR Gate

NOR is a contraction of NOT–OR gates. It has two or more inputs and only one output, i.e. $Y = \overline{A + B}$. The output is HIGH only when all the inputs are LOW. If any one or both the inputs are HIGH, then the output is LOW. The logic symbol for the NOR gate is shown in Fig. 3.6(a). The small circle or bubble represents the operation of inversion.



(a) Logic symbol of NOR gate

(b) NOR gate = Bubbled AND gate

Fig. 3.6 NOR gate

The truth table of a two input NOR gate is shown in Table 3.7.

Table 3.7 Truth table of a 2-input NOR gate

Inputs		Output
A	B	$Y = \overline{A + B}$
0	0	1
0	1	0
1	0	0
1	1	0

The NOR gate is equivalent to an AND gate with a bubble at its inputs. This is shown in Fig. 3.6(b).

3.3.6 Universal Gates / Universal Building Blocks

NAND and NOR gates are called *Universal gates* or universal building blocks because both can be used to implement any gate like AND, OR and NOT gates or any combination of these basic gates. Fig. 3.7 shows how a NAND gate can be used to realise various logic gates while Fig. 3.8 shows how a NOR gate can be used for the same.

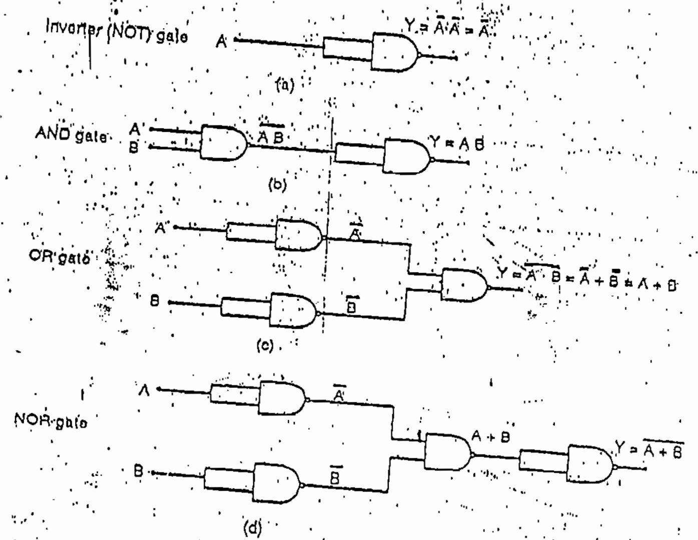


Fig. 3.7 Realisation of
(a) NOT, (b) AND, (c) OR and (d) NOR gates using NAND gates

Realisation of logic function using NAND gates Any logic function can be implemented using NAND gates. To achieve this, first the logic function has to be written in SOP (Sum of Products) form. Then, this can be easily realised using NAND gates. In other words, a logic gate circuit with AND gates in the first level and OR gates in the second level can be converted into a NAND–NAND gate circuit. To understand this concept, consider the following SOP expression.

$$Y = ABC + B\overline{C}D + A\overline{C}D$$

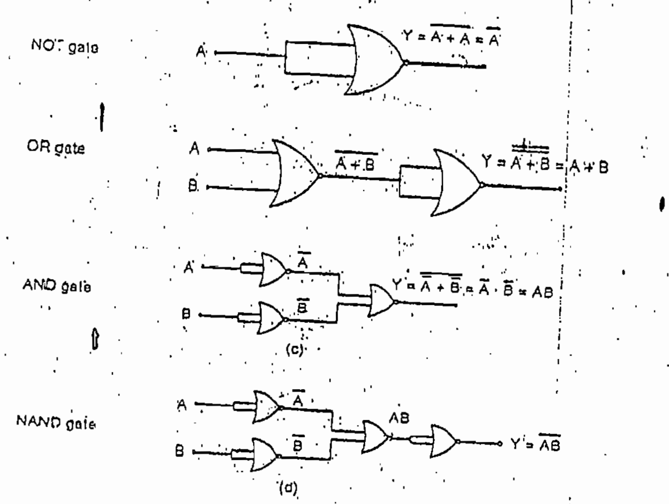


Fig. 3.8 Realisation of
(a) NOT, (b) OR, (c) AND and (d) NAND gates using NOR gates

The above expression can be implemented using three AND gates in the first level and one OR gate in the second level as shown in Fig. 3.9(a). If bubbles are introduced at the output of the AND gates and the inputs of OR gate, the above circuit will be modified as shown in Fig. 3.9(b). But, it has already been explained in the previous section that an OR gate with bubbles at its inputs is equivalent to a NAND gate. Therefore, the OR gate with bubbled inputs in Fig. 3.9(b) can be replaced by a NAND gate as shown in Fig. 3.9(c). Now, the above SOP expression is implemented using only NAND gates.

Realisation of logic functions using NOR gates Any logic function can also be implemented using NOR gates. To achieve this, first the logic function has to be written in POS (Product of Sums) form. Then, this can be easily realised using only NOR gates. In other words, a logic gate circuit with OR gates in the first level and AND gates in the second level can be converted into a NOR–NOR gate circuit. To understand this concept, consider the following POS expression.

$$Y = (A + B + C)(B + C + D)(A + B + D)$$

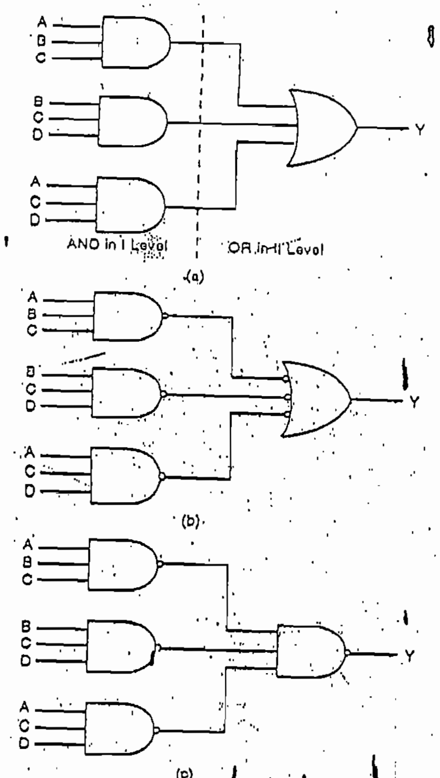


Fig. 3.9 Realisation of a logic expression (SOP form) using NAND gates

The POS expression can be implemented using three OR gates in the first level and one AND gate in the second level as shown in Fig. 3.10(a). If bubbles are introduced at the output of the OR-gates and the inputs of the AND gate, the above circuit will be modified as shown in Fig. 3.10(b). But, an AND gate with bubble at its inputs is equivalent to a NOR gate. Therefore, the AND gate with bubbled inputs in Fig. 3.10(b) can be replaced by a NOR gate as shown in Fig. 3.10(c). Now, the above POS expression is implemented using only NOR gates.

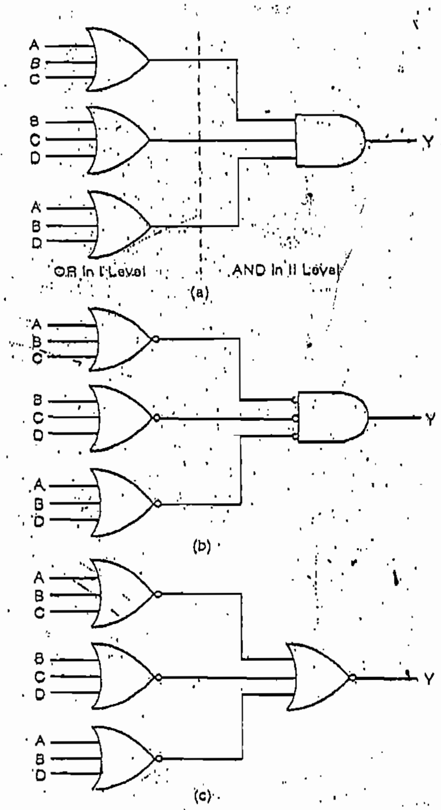


Fig. 3.10 Realisation of a logic expression (POS form) using NOR gates

3.3.7 Exclusive-OR (Ex-OR) Gate

An Exclusive-OR gate is a gate with two or more inputs and one output. The output of a two-input Ex-OR gate assumes a HIGH state if one and only one input assumes a HIGH state. This is equivalent to saying that the output is HIGH if either input A or input B is HIGH exclusively, and low when both are 1 or 0 simultaneously.

The logic symbol for the Ex-OR gate is shown in Fig. 3.11(a) and the truth table for the Ex-OR operation is given in Table 3.8.

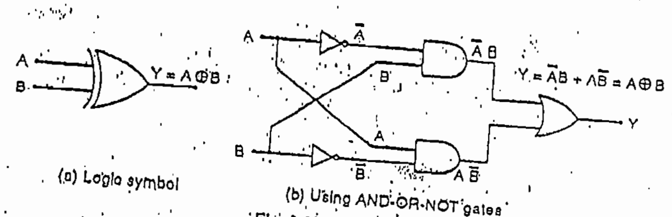


Fig. 3.11 Ex-OR gate

Table 3.8 Truth table of a 2-input Ex-OR gate

Inputs A	Inputs B	Output $Y = A \oplus B$
0	0	0
0	1	1
1	0	1
1	1	0

The truth table of the Ex-OR gate shows that the output is HIGH when any one, but not all, of the inputs is at 1. This exclusive feature eliminates a similarity to the OR gate. The Ex-OR gate responds with a HIGH output only when an odd number of inputs is HIGH. When there is an even number of HIGH inputs, such as two or four, the output will always be LOW. From the truth table of a 2-input Ex-OR gate, the Ex-OR function can be written as $Y = \overline{A}B + A\overline{B} = A \oplus B$.

The above expression can be read as Y equals A Ex-OR B. Using the above expression, a 2-input Ex-OR gate can be implemented using basic gates like AND, OR and NOT gates as shown in Fig. 3.11(b).

The 2-input Ex-OR gate can also be implemented using NAND gates as shown in Fig 3.12.

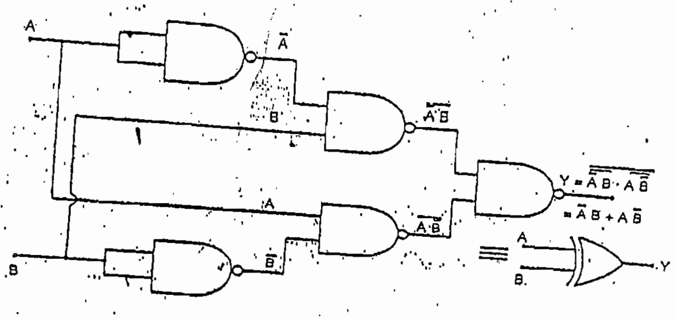


Fig. 3.12 Ex-OR gates using NAND gates

The main characteristic property of an Ex-OR gate is that it can perform modulo-2 addition. It should be noted that the same Ex-OR truth table applies when adding two *binary digits* (bits). A 2-input Ex-OR circuit is, therefore, sometimes called a *modulo-2 adder* or a *half-adder* without carry output. The name half-adder refers to the fact that possible carry-bit, resulting from an addition of two preceding bits, has not been taken into account. A full addition is performed by a second Ex-OR circuit with the output signal of the first circuit and the carry as input signals, as shown in Fig. 3.13.

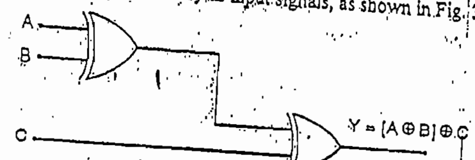


Fig. 3.13 Cascading of two Ex-OR circuits

The configuration of Fig. 3.13 is a cascading of two Ex-OR circuits resulting in an Ex-OR operation of three variables A, B, and C. Consequently, the sum output of a full adder for two bits is an Ex-OR operation of the 2 bits to be added and the carry of the preceding adding stage. The logic expression of the Ex-OR operation of three variables A, B, and C is given by

$$A \oplus B \oplus C = (A\overline{B} + \overline{A}B)\overline{C} + \overline{(A\overline{B} + \overline{A}B)}C$$
$$= (A\overline{B} + \overline{A}B)\overline{C} + (\overline{A}\overline{B} + AB)C$$
$$A \oplus B \oplus C = A\overline{B}\overline{C} + \overline{A}B\overline{C} + \overline{A}\overline{B}C + ABC$$

In general, Ex-OR operation of n variables results in a logical 1 output if an *odd* number of the input variables are 1s. An Ex-OR operation of n variables can be obtained by cascading 2-input Ex-OR gates.

Another important property of an Ex-OR gate is that it can be used as a *controlled inverter*, i.e., by using an Ex-OR gate, a logic variable can be complemented or allowed to pass through it unchanged. This is done by using one Ex-OR input as a control input and the other as the logic variable input as shown in Fig. 3.14. When the control input is HIGH, the output $Y = \overline{A}$ and when the control input is LOW, the output $Y = A$.

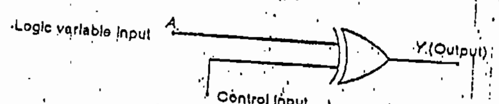


Fig. 3.14 Ex-OR gate as a controlled inverter

3.3.8 Exclusive-NOR (Ex-NOR) Gate

The exclusive-NOR gate, abbreviated Ex-NOR, is an Ex-OR gate, followed by an inverter. An exclusive-NOR gate has two or more inputs and one output. The output of a two-input Ex-NOR gate assumes a HIGH state if both the inputs assume the same logic state or have an even number of 1s, and its output is LOW when the inputs assume different logic states or have an odd number of 1s. The logic symbol of Ex-NOR gate is shown in Fig. 3.15 and its truth table is given in Table 3.9. From the truth table, it is clear that the Ex-NOR output is the complement of the Ex-OR gate. The Boolean expression for the Ex-NOR gate is

$$Y = \overline{A \oplus B}$$

Read the above expression as "Y equals A Ex-NOR B." According to DeMorgan's theorem,

$$\begin{aligned}\overline{A \oplus B} &= \overline{\bar{A}B + A\bar{B}}\\ &= \overline{\bar{A}B} \cdot \overline{A\bar{B}}\\ &= (A + \bar{B})(\bar{A} + B)\\ &= AB + \bar{A}\bar{B}\end{aligned}$$

Table 3.9. Truth table of 2-input Ex-NOR gate

Inputs		Output
A	B	$Y = \overline{A \oplus B}$
0	0	1
0	1	0
1	0	0
1	1	1

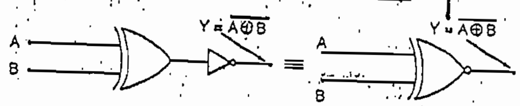


Fig. 3.15 Logic symbol of 2-input Ex-NOR gate

An important property of the Ex-NOR gate is that it can be used for bit comparison. The output of an Ex-NOR gate is 1 if both the inputs are similar, i.e., both are 0 or 1; otherwise, its output is 0. Hence, it can be used as a one-bit comparator. It is also called a *coincidence circuit*.

Another property of the Ex-NOR gate is that it can be used as an *even-parity checker*. The output of the Ex-NOR gate is 1 if the number of 1s in its inputs is even; if the number of 1s is odd, the output is 0. Hence, it can be used as an *even/odd parity checker*. Hence, the 2-input Ex-NOR gate is immensely useful for bit comparison and parity checking.

Example 3.1 Realise the logic expression $Y = \bar{B}\bar{C} + \bar{A}\bar{C} + \bar{A}\bar{B}$ using basic gates.

Solution In the given expression, there are 3 product terms each with two variables which can be implemented using three 2-input AND gates, and the product terms can be OR operated together using a 3-input OR gate. The complemented form of individual variable can be obtained by 3 NOT gates. Thus, the circuit for the given expression is realised as shown in Fig. E3.1.

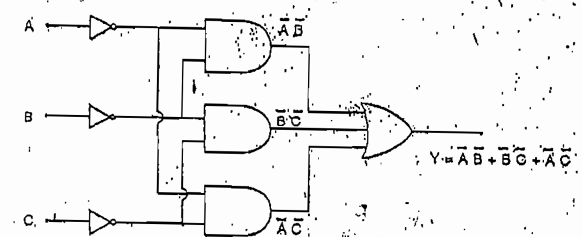


Fig. E3.1

Example 3.2 Realise the logic expression $Y = (A + B)(\bar{A} + C)(B + D)$ using basic gates.

Solution In the given expression, there are 3 sum terms which can be implemented using three 2-input OR gates and their outputs are AND operated together by a 3-input AND gate. A NOT gate can be used to obtain the inverse of A. Now, the realised circuit is shown in Fig. E3.2.

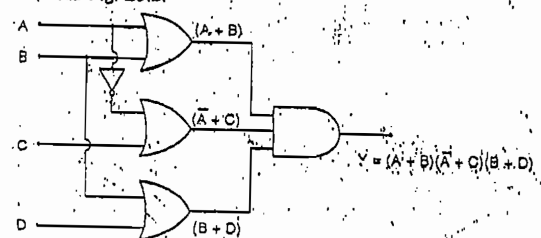


Fig. E3.2

Example 3.3 Implement $Y = \overline{AB} + A + \overline{(B + C)}$ using NAND gates only.

Solution The implementation of the given function is shown in Fig. E3.3.

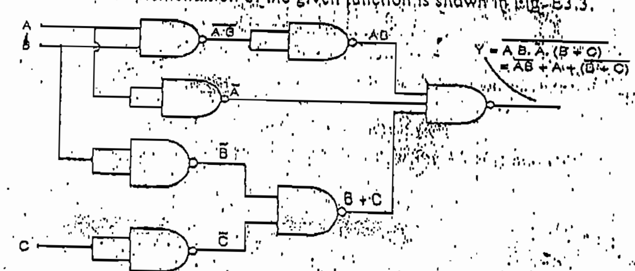


Fig. E3.3

... shown in Fig. E3.4(a).

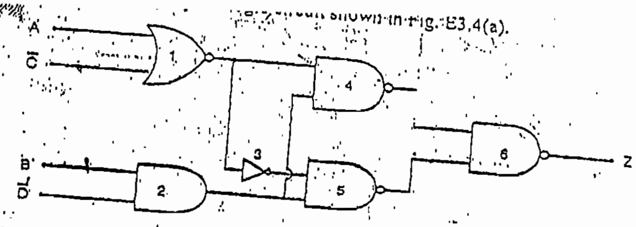


Fig. E3.4 (a)

Solution From the given logic circuit, the expression for Z can be written as

$$Z = \overline{\overline{(A+\bar C)} \cdot \overline{B\bar D} \cdot \overline{\overline{(A+\bar C)} \cdot B\bar D}}$$

$$= \overline{[\overline{(A+\bar C)} + \overline{B\bar D}]} \cdot \overline{[\overline{\overline{(A+\bar C)}} + \overline{B\bar D}]}$$

$$= \overline{[(A+\bar C) + B\bar D]} \cdot \overline{[\overline{(A+\bar C)} + \overline{B\bar D}]}$$

$$= B\bar D + (A+\bar C)\overline{(A+\bar C)} \qquad [\because (A+B)(\bar A+C) = A + BC]$$

$$= \overline{\overline{B\bar D}}$$

$$= B\bar D \qquad [\because A\bar A = 0]$$

Therefore, the above logic circuit can be simplified as shown in Fig. E3.4(b).

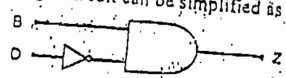


Fig. E3.4 (b)

Instead of using Boolean algebra, the logic circuit can be simplified directly as shown below. In the given logic circuit shown in Fig. E3.4 (a), the NAND gate (6) can be replaced by an OR gate with a bubble at its inputs as shown in Fig. E3.4(c).

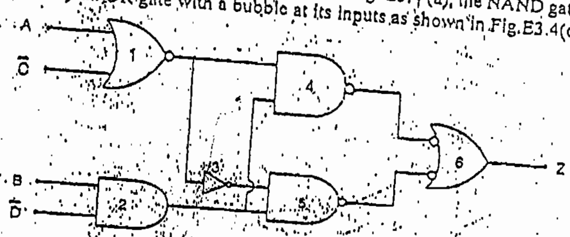


Fig. E3.4 (c)

Now, using $\bar{\bar A} = A$, the bubble at the outputs of gates 4 and 5 get cancelled with the bubble at the inputs of gate 6 as shown in Fig. E3.4(d).

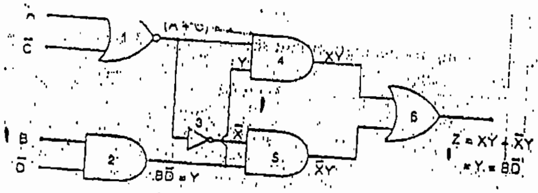


Fig. E3.4(d)

In the above figure, if we assume the output of gate 1 $(\overline{A+\bar C}) = X$ and the output of gate 2 $B\bar D = Y$, then the output of gate 4 is XY and the output of gate 5 is $\bar X Y$. If XY and $\bar X Y$ are OR operated in gate 6, then $XY + \bar X Y = Y(X + \bar X) = Y = B\bar D$. Therefore, the above circuit can be simplified as shown in Fig. E3.4(e).

Fig. E3.4(e)

Example 3.5 Realise (a) $Y = A + BC\bar D$ using NAND gates and (b) $Y = (A+C)(A+\bar D)(A+B+\bar C)$ using NOR gates.

Solution Using NAND gates

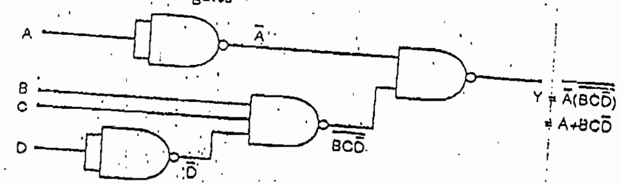


Fig. E3.5(a)

(b) Using NOR gates

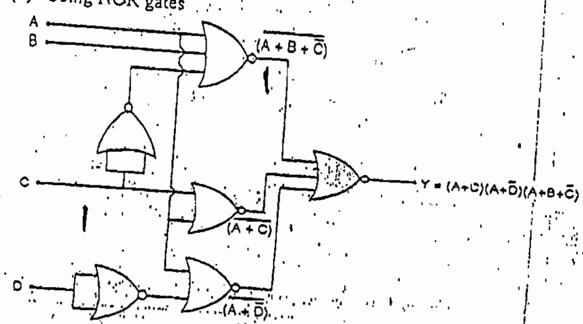


Fig. E3.5(b)

Example 3.6 A company has five directors, namely A, B, C, D and E, and their corresponding percentage of shares in the company are 30, 25, 20, 15 and 10 respectively. The directors are eligible to vote according to their percentage of shares (e.g. A can cast 30 votes; B can cast 25 votes and so on) in the board of directors meeting and two-third majority is required to pass any resolution. Design a combinational circuit to indicate whether a resolution is passed or not.

Solution. From the word description of the problem, the truth table for the output Y, i.e. whether a resolution is passed or not, can be written as follows.

Table E3.6 Truth table

Inputs					Output
A (30%)	B (25%)	C (20%)	D (15%)	E (10%)	Y
0	0	0	0	0	0
0	0	0	0	1	0
0	0	0	1	0	0
0	0	0	1	1	0
0	0	1	0	0	0
0	0	1	0	1	0
0	0	1	1	0	0
0	0	1	1	1	0
0	1	0	0	0	0
0	1	0	0	1	0
0	1	0	1	0	0
0	1	0	1	1	0
0	1	1	0	0	0
0	1	1	0	1	0
0	1	1	1	0	0
0	1	1	1	1	1
1	0	0	0	0	0
1	0	0	0	1	0
1	0	0	1	0	0
1	0	0	1	1	0
1	0	1	0	0	0
1	0	1	0	1	0
1	0	1	1	0	0
1	0	1	1	1	1
1	1	0	0	0	0
1	1	0	0	1	0
1	1	0	1	0	1
1	1	0	1	1	1
1	1	1	0	0	1
1	1	1	0	1	1
1	1	1	1	0	1
1	1	1	1	1	1

From the above truth table, the expression for Y can be written as

$$Y = \Sigma_m(15, 23, 26, 27, 28, 29, 30, 31)$$

This five variable functions can be simplified using Karnaugh map method as shown in Fig. E3.6(a).

DE \ ABC	000	001	011	010	110	111	101	100
00	0	0	0	0	0	1	0	0
01	0	0	0	0	0	1	0	0
11	0	0	1	0	1	1	1	0
10	0	0	0	0	1	1	0	0

Fig. E3.6(a) Karnaugh simplification

From the above K-map, the simplified expression for Y is given by

$$Y = ABC + ABD + ACDE + BCDE$$

Now, the above expression can be implemented as shown in Fig. E3.6(b).

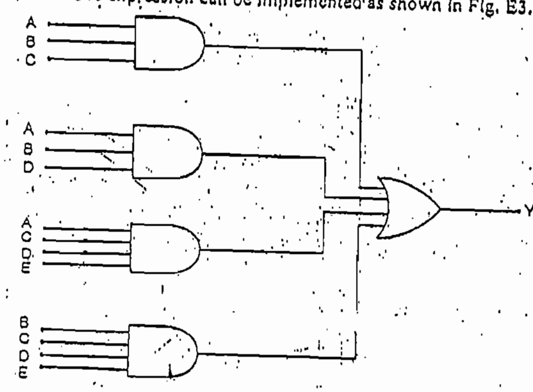


Fig. E3.6(b)

3.4 MIXED LOGIC

In a positive logic, a '1' (i.e., TRUE) is assigned to +5V and a '0' (i.e., FALSE) is assigned to 0V and in a negative logic, a '1' is assigned to 0V and a '0' is assigned to +5V. But, in mixed logic, the assignment of logical values to voltage values is not fixed, and is left to the discretion of designers. The notation of mixed logic provides a simplified mechanism for the analysis and design of digital circuits. Correct use of mixed logic notation provides logic expressions and logic diagrams that are analogue to each other. Additionally, a mixed logic diagram provides clear documentation as to the operation of a circuit. The positive logic interpretation of the "NAND" gate is that the output will be a "low" logic level for two high inputs. That is, if A is high "AND" B is high, the output will be low; if A is low "OR" B is low, the output will be high. By viewing both the "AND" and "OR" functionality associated with each logic gate, it becomes easy to design and analyze digital circuits.

3.4.1 Basic Mixed Logic Operators

In positive logic, the basic building blocks are "AND", "NAND", "NOR", "OR" and the "Inverter." The "Exclusive OR" can be obtained from a combination of "AND", "OR", and "Inverter". The truth tables associated with the "AND" and "NAND" positive logic operations are given in Tables 3.10 and 3.11 to show the difference in the positive logic and mixed logic "one" and "zero" notation.

Table 3.10 Positive Logic AND Truth Table

A	B	C
0	0	0
0	1	0
1	0	0
1	1	1

Table 3.11 Positive Logic NAND Truth Table

A	B	C
0	0	1
0	1	1
1	0	1
1	1	0

Mixed logic does not restrict a "1" to always represent a high and a "0" to always represent a low. The truth tables associated with the mixed logic "AND" and "NAND" are shown in Tables 3.12 and 3.13.

Table 3.12 Mixed Logic AND Truth Table

A	B	G
0	0	0
0	1	0
1	0	0
1	1	1

Table 3.13 Mixed Logic NAND Truth Table

A	B	C
0	0	0
0	1	0
1	0	0
1	1	1

The generality of the mixed logic approach allows all "AND" operations, regard-

3.4.2 Mixed Logic Symbols / Alternate GATE Symbols

The correct use of mixed logic requires the placement of the "AND" or "OR" symbol in the logic diagram anytime an "AND" or "OR" operator is called for in the Boolean Expression. The voltage requirements are met by the assignment of assertion levels.

Alternate gate symbols have been developed to clearly denote "AND" and "OR" operations in a logic diagram. The basic positive logic symbols for the AND, NAND, OR, NOR and NOT gates and their corresponding alternate gate symbols are shown in Fig.3.16.

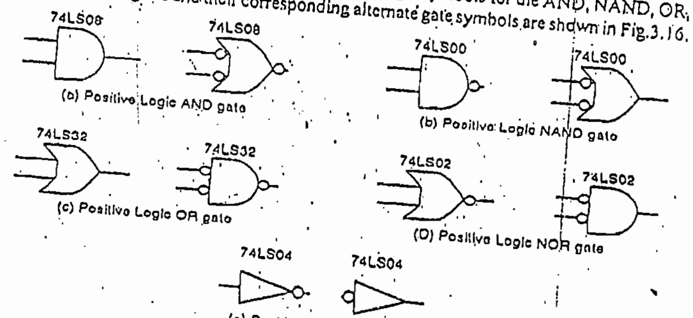
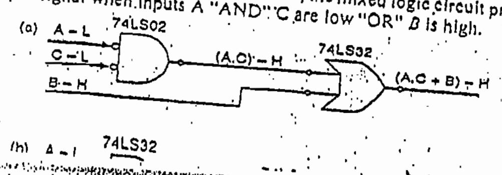


Fig.3.16 Mixed Logic Symbol Pairs

The alternate logic symbol is obtained by changing the operation, "AND" to "OR" or "OR" to "AND" and complementing assertion levels. The circle, or "bubble" on the output of the "AND" gate shown in Fig.3.16(b) means that the output will be low when asserted. The absence of bubbles on the inputs means that both of the gate inputs require a logic high to be asserted. The mixed logic interpretation for the logical "AND" circuit shown in Fig.3.16(b) is given by the expression $F = (A.B)_{-L}$. This expression is much clearer than the positive logic expression $F = \overline{A \cdot B}$ since $\overline{A \cdot B} = \bar{A} + \bar{B}$.

By conversion of "AND" and "OR" symbols, it is easy to realise any logic circuits and the operation of the circuit is always clearly communicated by the logic diagram. Two circuits using mixed logic notations that realise the expression $(A \cdot C + B)_{-H}$ are shown in Fig.3.17. Here, the mixed logic circuit produces a logic high output signal when inputs A "AND" C are low "OR" B is high.



The form of the "OR" operation depends on the choice made for the "AND" realisation. Regardless of the final realisation, the desired logic operation is clearly denoted by the logic diagram. Here, a positive logic "AND" gate is not required to implement the circuit.

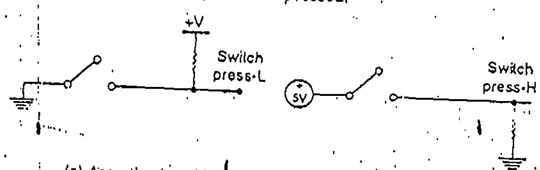
3.4.3 Assertion Levels and Polarity Indication

The mixed logic design approach allows the freedom to produce circuits that generate and respond to either high or low logic levels.

In terms of the logic diagram, a bubble on the output means that a "low" will be obtained when the gate conditions are met. Here, the logic expression at the gate output should contain $-L$, i.e., $F = (A.B)_{-L}$ at the end of the expression. Likewise, in a logic "high" output, the output expression should be labeled with a $-H$ i.e., $F = (A.B)_{-H}$. These notations are shown in Fig.3.17.

In mixed logic, the requirements of input and output voltage levels are expressed in terms of their assertion levels. For example, a positive logic NAND gate is expressed as being an "AND" function with an "asserted low" output and "asserted high" inputs. Assertion means "the affirmative position of an action related to a Boolean variable." This concept is demonstrated with the circuits shown in Fig.3.18.

Let us assume that the switches, shown in Fig.3.18, are spring loaded so that in order for contact to be made, they must be pressed.



(a) Assertion Level-Low (b) Assertion Level-High

Fig.3.18. Correct mixed logic assertion level labeling

In both cases, the output signal of both circuits is named "switch-pressed". If the circuits are examined, the signal generated by one circuit will be the opposite logic level of the signal generated by the other circuit. In one case when the switch is pressed, the output will go to a logic "low" and in the other case the output of switch circuit will be a logic "high". To clearly differentiate these signals, a $(-L)$ or $(-H)$ is added to denote asserted low or asserted high signals respectively. The mixed logic approach of denoting voltage levels using either $-L$ or $-H$ rather than an over bar (positive logic) is consistent with the expected Boolean expression.

In general, the assertion levels i.e., the polarity at an input to a gate or output of a gate can be indicated in different ways. If the assertion level is "low", it is indicated either by a symbol "L" or "bubble" or "half-way arrow mark" or "inverted triangle". If the assertion level is "high", it is indicated either by a symbol "H" or "upward triangle" or the absence of the "bubble" or the absence of "half-way arrow mark". For example, the logic expression $Y = \overline{A}\,\overline{B}$ is represented as shown in Fig. 3.19.

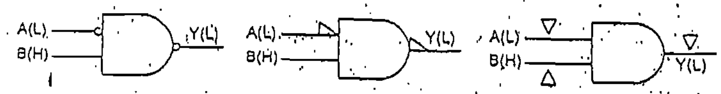


Fig. 3.19 Representation of $Y = \overline{AB}$

Easy interpretation of a circuits operation is possible by maintaining the consistent use of assertion levels. An example is provided using the circuit shown in Fig.3.20.

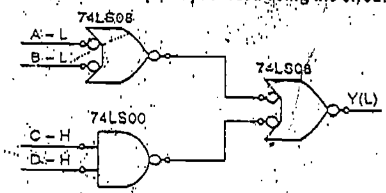


Fig. 3.20 Gate circuit demonstrating an asserted low output signal.

The output expression of the circuit is $Y = [(A + B) + (C . D)]_{-L}$. Here, the output will be a logic "low" when the input conditions are met. The input conditions are that either A "OR" B must be asserted "OR" C and D must be asserted in order that the output will be asserted low. Inspection of the input assertion levels indicates that in order for A to be asserted it must be low, likewise with B. The assertion of the C and D signals require logic highs. In contrast, a positive logic approach to the analysis of this circuit would begin by "barring" of all the variables which assert low as shown in Fig.3.21.

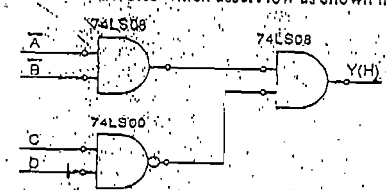


Fig. 3.21 Gate circuit demonstrating an asserted high output signal.

The "barring" of the terms in the positive logic approach obscures the functionality of the circuit. The diagram shown in Fig.3.21 implies that no "OR" operations occur in the circuit. Positive logic can obscure circuit understanding.

Mixed logic is a useful tool for logic design and analysis, and circuit documentation. The mixed logic designer provides design documentation with greater clarity, accuracy and comprehension of the digital logic design to the end users.

3.5 MULTILEVEL GATING NETWORKS

The maximum number of gates cascaded in series between an input and output is called *level of gates*. For example, a Sum of Product (SOP) expression can be implemented using a two level gate network i.e., AND gates in the first level and a OR gate

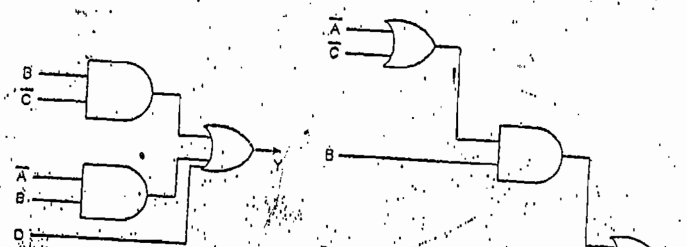
In the second level as shown in Fig.3.9(a) in section 3.3.6. Similarly, a Product of Sum (POS) expression can be implemented using a two level gate network i.e., OR gates in the first level and AND gate in the second level as shown in Fig.3.10(a) in section 3.3.7. It is important to note that the inverter gates are not considered to decide the level of the gate network.

The number of levels can be increased by factoring the Sum of Products (SOP) expression for AND-OR network or by multiplying out some terms in the Product of Sum (POS) expression for OR-AND network. If a switching expression is implemented using gates in more than two levels, then it is called Multilevel gate network. Generally, the propagation delay through a multi-level gate network is proportional to the number of levels in it. However, sometimes by increasing the number of levels of logic, the number of gates and number of gate inputs are decreased.

3.5.1 Implementation of Multilevel Gate Network

The implementation of the multilevel gate network can be explained with the following two cases.

Case 1: Consider the switching function, $Y = B\overline{C} + \overline{A}B + D$. This expression can be implemented using a two level AND-OR gate network as shown in Fig.3.22(a). It requires two 2-input AND gates and one 3-input OR gate with a total number of five literals.



(a) Two level AND-OR network (b) Multilevel AND-OR network

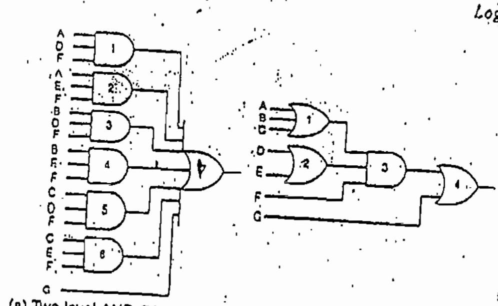
Fig. 3.22

If the above switching expression is factored into a different form as $Y = B(\overline{A} + \overline{C}) + D$, then it can be implemented using a three level gate network as shown in Fig.3.22(b). Now, this implementation requires two 2-input OR gates and one 2-input AND gate with a total number of four literals. Thus, it reduces the number of gate inputs by one.

Case 2: Next consider the following function

$$f(A,B,C,D,E,F,G) = ADF + AEF + BDF + BEF + CDF + CEF + G$$

It is already in its minimal sum of products form. Its implementation as a two



(a) Two level AND-OR network (b) Multilevel AND-OR network

Fig. 3.23

The above two-level expression can be replaced with a so-called factored form by factoring out common literals from the product terms whenever possible. By recursively factoring out the common literals, the expression can be written as:

$$= (AD + AE + BD + BE + CD + CE)\,F + G$$
$$= [(A + B + C)\,D + (A + B + C)\,E]\,F + G$$
$$= (A + B + C)(D + E)\,F + G$$

Now, the function requires one 3-input OR gate, two 2-input OR gates, and a 3-input AND gate, a total of four gates and nine literals. The implementation of the factored form is shown in Fig. 3.23(b). This implementation significantly reduces the number of wires and gates needed to implement the function, but it probably has worse delay because of the increased levels of logic. Compared to two level implementations, the multi-level implementations require less number of gates, thereby increasing the propagation delay.

3.5.2 Conversion to NAND-NAND and NOR-NOR Gate Networks

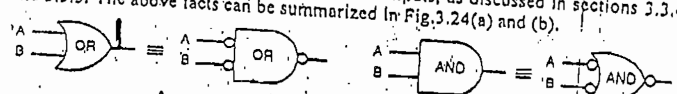
From De-Morgan's theorem, it is known that

$$\overline{A + B} = \overline{A}\,\overline{B} \text{ and } \overline{AB} = \overline{A} + \overline{B}$$

These expressions can also be written as

$$A + B = \overline{\overline{A}\,\overline{B}} \text{ and } AB = \overline{\overline{A} + \overline{B}}$$

It means that an OR gate is equivalent to a NAND gate with bubbles at its inputs and an AND gate is equivalent to a NOR gate with bubbles at its inputs. Also, a NAND gate is equivalent to an OR gate with bubbles at its inputs and a NOR gate is equivalent to an AND gate with bubbles at its inputs, as discussed in sections 3.3.4 and 3.3.5. The above facts can be summarized in Fig.3.24(a) and (b).



The schematic symbols on either side of the Fig.3.24 (a) and (b) can be freely exchanged without changing the truth table or logical value of the function.

AND-OR conversion to NAND-NAND gate networks

A two level AND-OR gate network can be easily converted to a NAND-NAND gate network as discussed in section 3.3.6.

AND-OR conversion to NOR-NOR gate networks

A two level AND-OR gate network, shown in Fig. 3.25(a) can be converted into NOR-NOR gate network by replacing the first level AND gate with NOR gates (i.e., AND with bubbles at its inputs) and the second level OR gate with a NOR gate. But, this is not logically equivalent. This can be corrected by introducing additional inverters at the inputs and the output, as shown in Fig. 3.25(b). Then the circuit shown in Fig. 3.25(b) can be modified as shown in Fig. 3.25(c).

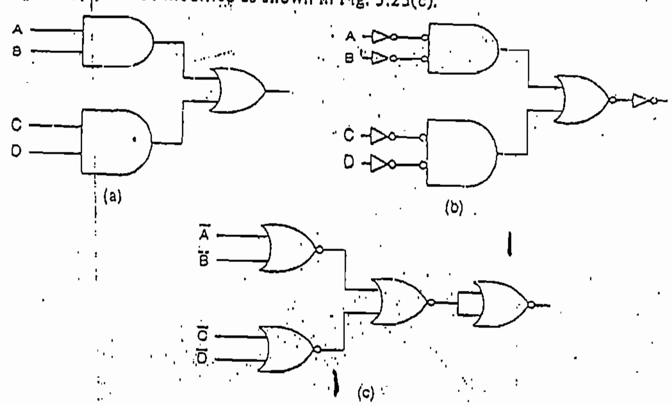


Fig. 3.25

OR-AND conversion to NOR-NOR gate networks

A two level OR-AND gate network can be easily converted to a NOR-NOR gate network as discussed in section 3.3.6.

OR-AND conversion to NAND-NAND gate networks

A two level OR-AND gate network, shown in Fig. 3.26(a) can be converted into NAND-NAND gate network by replacing the first level OR gate with NAND gates (i.e., OR with bubbles at its inputs) and the second level AND gate with a NAND gate. But this is not logically equivalent. This can be corrected by introducing additional inverters at the inputs and output as shown in Fig. 3.26(b). Then the circuit shown in Fig. 3.26(b) can be modified as shown in Fig. 3.26(c).

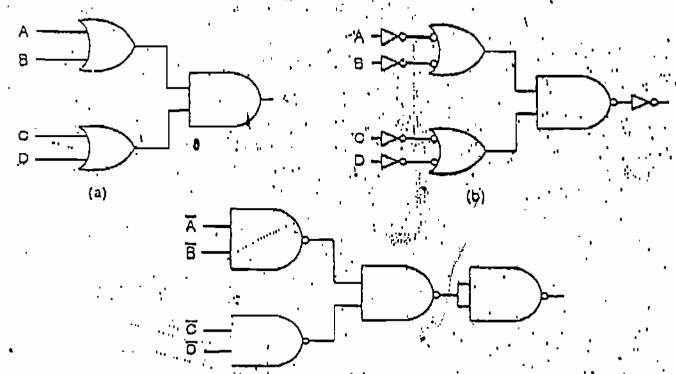


Fig. 3.26

Multilevel AND-OR conversion to NAND-NAND gate networks.

A multilevel AND-OR gate network can be easily converted into a gate network with NAND-NAND gates. Whenever a gate has a complemented switching variable at its input, it is logically equivalent to a gate with the un-complemented switching variable and a bubble at the input of that gate to which the complemented switching variable is connected. Also, two bubbles can be introduced at both ends of a line connecting two gates because their logical effect is nil. Using the above facts, the circuit shown in Fig. 3.27(a) can be modified as shown in Fig. 3.27(b).

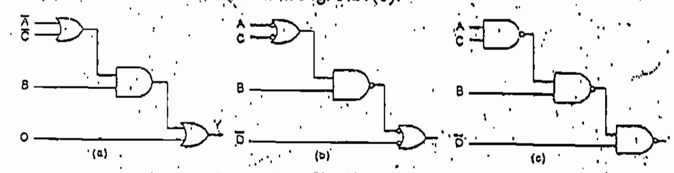


Fig. 3.27

It is known that an OR gate with bubbles at its inputs is equivalent to NAND gate. Now, the circuit can be drawn as shown in Fig. 3.27(c).

Example 3.7 Realise the following function as (i) multilevel NAND-NAND gate network and (ii) multilevel NOR-NOR network.

$$f = B(A + CD) + A\bar{C}$$

Solution

The given expression can be implemented as a four level AND-OR gate network as shown in Fig. E3.7(a).

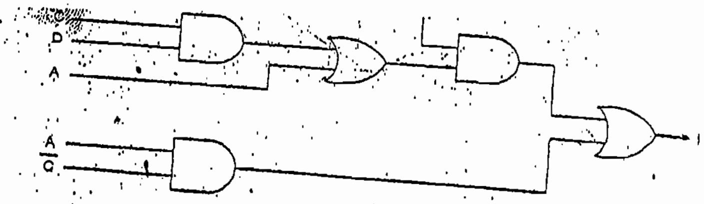


Fig. E3.7(a)

(i) Multilevel NAND-NAND Implementation

For NAND-NAND implementation, (i) each AND gate is replaced by a NAND gate followed by an inverter and (ii) each OR gate is replaced by OR gate with bubbles at its inputs. In addition, the corresponding input variables to bubbled OR gate is complemented so that the logical equivalence of the function is maintained as shown in Fig. E3.7(b).

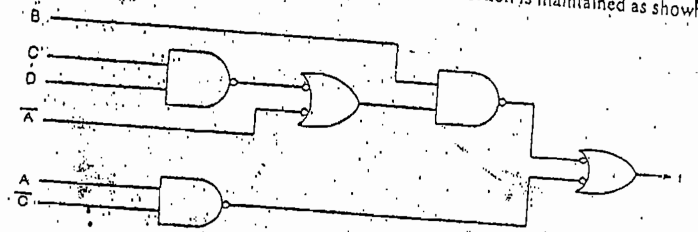


Fig. E3.7(b)

Now, the circuit shown in Fig. E3.7(b) can be modified as shown in Fig. E3.7(c).

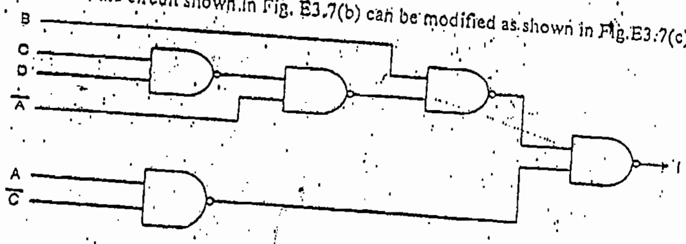


Fig. E3.7(c)

(ii) Multilevel NOR-NOR Implementation

For NOR-NOR implementation, (i) each OR gate is replaced by a NOR gate followed by an inverter and (ii) each AND gate is replaced by AND gate with bubbles at its inputs. In addition, the corresponding input variables to bubbled AND gate is complemented so that the logical equivalence of the function is maintained, as shown in Fig. E3.7(d).

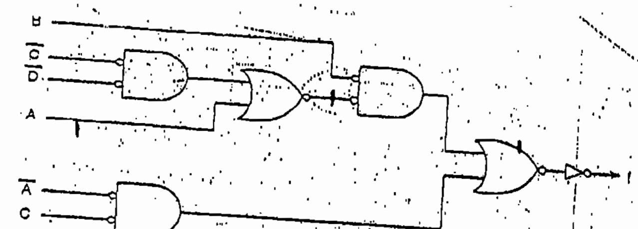


Fig. E3.7(d)

Now, the circuit shown in Fig. E3.7(d) can be modified as shown in Fig. E3.7(e).

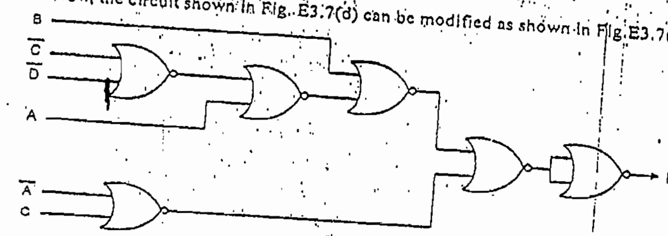


Fig. E3.7(e)

Example 3.8 Realize the following switching function using a multilevel gate network. Also, obtain the logically equivalent multilevel NAND-NAND gate circuit.

$$Y = A + (B + \overline{C})\,(\overline{D}E + F)$$

Solution

The given expression can be implemented as a four level gate network using basic AND and OR gates as shown in Fig. E3.8(a).

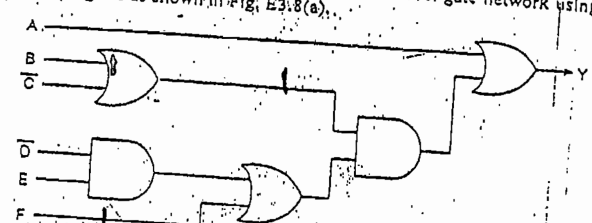


Fig. E3.8(a)

The four level gate network shown in Fig. E3.8(a) can be modified as shown in Fig. E3.8(b), so that it can be realised using universal gates.

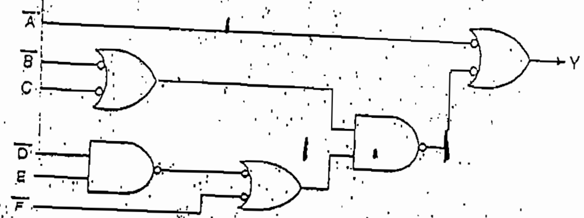


Fig. E3.8(b)

Now, the above circuit can be implemented using NAND gates as shown in Fig. E38(c).

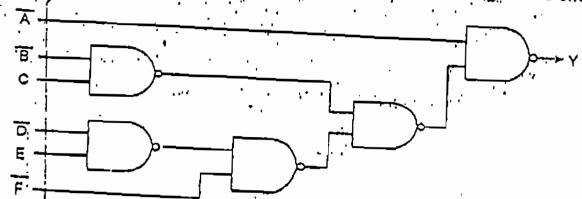


Fig. E3.8(c)

Example 3.9 : Convert the gate network, shown in Fig. E3.9(a) (i) all NAND gates network (ii) all NOR gates network by adding bubbles and inverters wherever necessary.

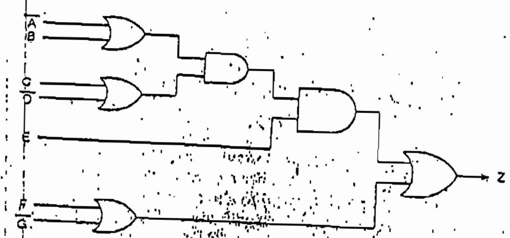


Fig. E3.9(a)

Solution

i) All NAND gate implementation

For all NAND gates implementation, (i) each AND gate is replaced by a NAND gate followed by an inverter and (ii) each OR gate is replaced by OR gate with bubble at its inputs and the corresponding input variables to bubbled OR gate is complemented. Using this procedure, the multilevel gate network, shown in Fig. E3.9(a) is modified as shown in Fig. E3.9(b).

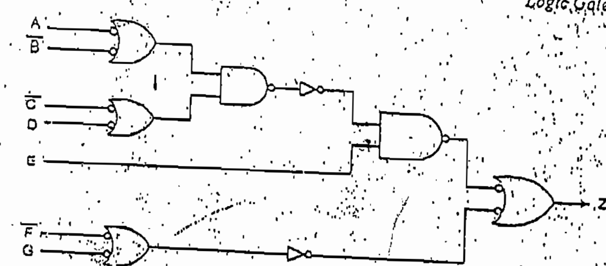


Fig. E3.9(b)

It is known that a bubbled OR gate is equivalent to a NAND gate. Also, a NOT gate can be implemented using a NAND gate. Therefore, the circuit shown in Fig. E3.9(b) can be modified as shown in Fig. E3.9(c).

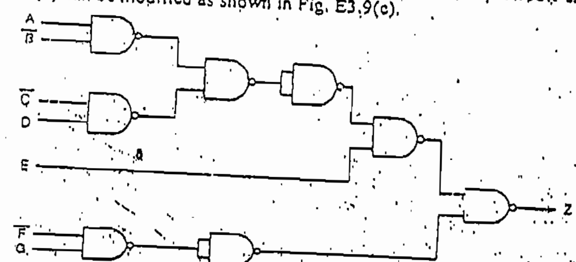


Fig. E3.9(c)

ii) All NOR gates implementation

For all NOR gates implementation, (i) each OR gate is replaced by a NOR gate followed by an inverter and (ii) each AND gate is replaced by AND gate with bubbles at its inputs and the corresponding input variables to bubbled AND gate is complemented. Using this procedure, the multilevel gate network shown in Fig. E3.9(a) can be modified as shown in Fig. E3.9(d).

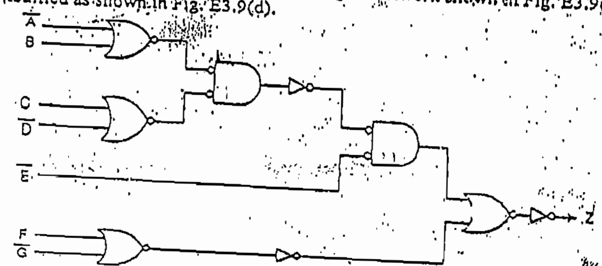


Fig. E3.9(d)

It's known that a bubbled-AND gate is equivalent to a NOR gate and a NOT gate can be implemented using NOR gate. Therefore, the circuit shown in Fig. E3.9(d) can be modified as shown in Fig. E3.9(c).

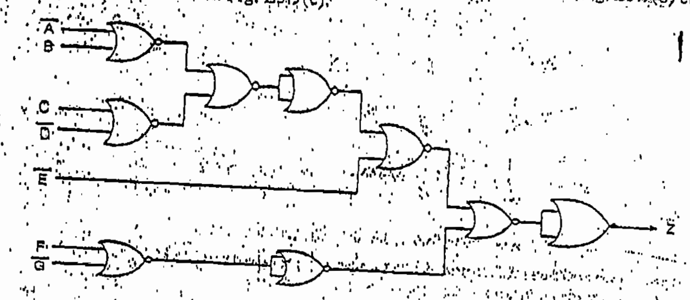


Fig. E3.9(e)

3.6 MULTIPLE OUTPUT GATE NETWORKS

A switching function with more than one output is called *Multioutput function* and the corresponding logic circuit is called *Multiple output gate network*. A multi-output function can be completely described or specified by a truth table in which for all possible combinations of inputs, the multi-outputs are specified. Code converters such as BCD to Excess-3 code converter, BCD to seven segment decoder, Binary to Gray code converter, Gray to Binary converter and arithmetic circuits such as Half and Full adder/subtractor are examples for multi-output functions. The design of BCD to seven segment decoder, Binary to Gray code converter and Gray to Binary converter are discussed in section 6.5.9, 6.10.2 and 6.10.3 respectively. The design of BCD to Excess-3 code converter is discussed below.

3.6.1 BCD to Excess-3 Code Conversion

The availability of a large variety of codes for the same discrete information results in the use of different codes by different digital systems. It is sometimes necessary to use the output of one system as the input to another. A conversion circuit must be inserted between two such systems. Thus, a code converter is a circuit that makes the two systems compatible eventhough each uses a different binary code.

The following example illustrates the conversion of BCD-to-excess-3 code. Table 3.14 shows the input in BCD and the output in Excess-3. The four input combinations from 10 to 15 never occur and therefore they are don't care combinations.

Table 3.14 BCD-to-excess-3 code conversion

Decimal number	Inputs				Outputs			
	A	*B*	*C*	*D*	*W*	*X*	*Y*	*Z*
0	0	0	0	0	0	0	1	1
1	0	0	0	1	0	1	0	0
2	0	0	1	0	0	1	0	1
3	0	0	1	1	0	1	1	0
4	0	1	0	0	0	1	1	1
5	0	1	0	1	1	0	0	0
6	0	1	1	0	1	0	0	1
7	0	1	1	1	1	0	1	0
8	1	0	0	0	1	0	1	1
9	1	0	0	1	1	1	0	0
10	1	0	1	0	d	d	d	d
11	1	0	1	1	d	d	d	d
12	1	1	0	0	d	d	d	d
13	1	1	0	1	d	d	d	d
14	1	1	1	0	d	d	d	d
15	1	1	1	1	d	d	d	d

From the truth table, one can write the logic expressions for Excess-3 code outputs *W*, *X*, *Y* and *Z* as follows.

$$W = \Sigma_m(5, 6, 7, 8, 9) + \Sigma_d(10, 11, 12, 13, 14, 15)$$
$$X = \Sigma_m(1, 2, 3, 4, 9) + \Sigma_d(10, 11, 12, 13, 14, 15)$$
$$Y = \Sigma_m(0, 3, 4, 7, 8) + \Sigma_d(10, 11, 12, 13, 14, 15)$$
$$Z = \Sigma_m(0, 2, 4, 6, 8) + \Sigma_d(10, 11, 12, 13, 14, 15)$$

where Σ_d represents the summation of don't care combinations.

Now, the above expressions for Excess-3 code outputs can be simplified using *K*-map method.

CD \ AB	00	01	11	10
00	0	0	d	1
01	0	1	d	1
11	0	1	d	d
10	0	1	d	d

(a) for W

$W = A + BD + BC$

CD \ AB	00	01	11	10
00	0	1	d	0
01	1	0	d	1
11	1	0	d	d
10	1	0	d	d

(b) for X

$X = \overline{B}D + \overline{B}C + B\overline{C}\overline{D}$
$= \overline{B}(C + D) + B(\overline{C + D}) = (B \oplus (C + D))$
$(\because \overline{A}B + A\overline{B} = A \oplus B)$

CD \ AB	00	01	11	10
00	1	1	d	1
01	0	0	d	0
11	1	1	d	d
10	0	0	d	d

(c) for Y

$Y = \bar{C}\bar{D} + CD$
$= \overline{C \oplus D}$

CD \ AB	00	01	11	10
00	1	1	d	1
01	0	0	d	0
11	0	0	d	d
10	1	1	d	d

(d) for Z

$Z = \bar{D}$

Fig. 3.28 K-map simplification for BCD-to-excess-3 code conversion

Using the above simplified expressions, the circuit diagram for BCD-to-excess-3 code converter can be drawn as shown in Fig. 3.29.

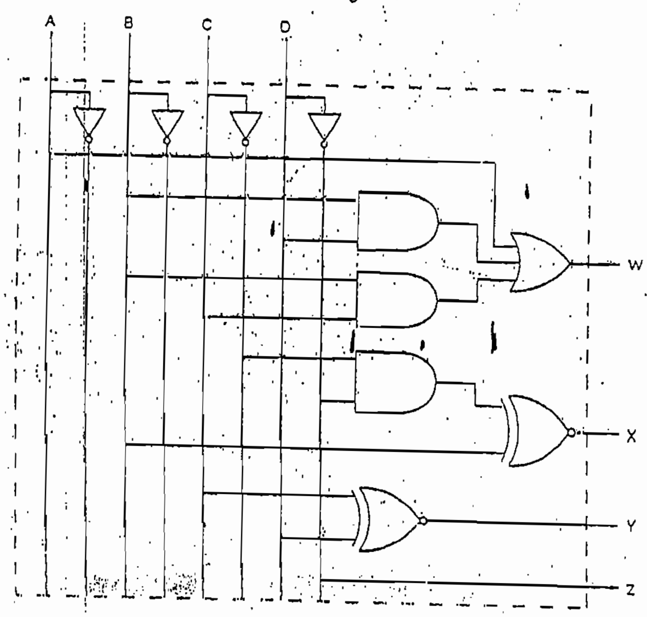


Fig. 3.29 Circuit diagram for BCD-to-excess-3 code converter

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Set - IV

3

Second Order Partial Differential Equations with Variable Coefficients

§ 1. A partial differential equation is said to be of second order if it contains at least one of the second order partial differential coefficients r, s and t but none of higher order. The differential coefficients p and q may also appear in the equation. Thus the general form of a second order partial differential equation is :

$$F(x, y, z, p, q, r, s, t)=0.$$

The complete solutions of these equations will contain two arbitrary functions.

Below we give some examples of equations that are readily solvable. It is to be noted that x and y, being independent, are constant with respect to each other in differentiation and integration.

Solved Examples

Ex. 1. *Solve* $s=\frac{x}{y}+a$.

Sol. The given equation can be written as

$$\frac{\partial^2 z}{\partial x\, \partial y}=\frac{x}{y}+a.$$

Integrating w.r.t. 'x', we get

$$\frac{\partial z}{\partial y}=\frac{1}{y}\cdot\frac{x^2}{2}+ax+f(y).$$

Now integrating w.r.t. 'y', we get

$$z=\frac{x^2}{2}\log y+axy+\int f(y)\, dy+\psi(x)$$

or $$z=\tfrac{1}{2}x^2\log y+axy+\phi(y)+\psi(x).$$

Ex. 2. *Solve* $xys=1$.

Sol. The given equation can be written as

$$\frac{\partial^2 z}{\partial x\, \partial y}=\frac{1}{xy}.$$

Integrating w.r.t. 'y', we get

$$\frac{\partial z}{\partial x}=\frac{1}{x}\log y+f(x).$$

Now integrating w.r.t. 'x', we get

$$z=\log x\log y+\int f(x)\, dx+\psi(y)$$

or $$z=\log x\log y+\phi(x)+\psi(y).$$

Ex. 3. *Solve* $s=2x+2y$.

Sol. The given equation can be written as

$$\frac{\partial^2 z}{\partial x\, \partial y}=2x+2y.$$

Integrating w.r.t. 'x', we get

$$\frac{\partial z}{\partial y}=x^2+2xy+f(y).$$

Now integrating w.r.t. 'y', we get

$$z=x^2y+xy^2+\int f(y)\, dy+\psi(x)$$

or $$z=x^2y+xy^2+\phi(y)+\psi(x).$$

Ex. 4. *Solve* $t=\sin xy$.

Sol. The given equation can be written as

$$\frac{\partial^2 z}{\partial y^2}=\sin xy.$$

Integrating w.r.t. 'y', we get

$$\frac{\partial z}{\partial y}=-\frac{1}{x}\cos xy+f(x).$$

Again integrating w.r.t. 'y', we get

$$z=-\frac{1}{x^2}\sin xy+y\, f(x)+\psi(x).$$

Ex. 5. *Solve* $ys+p=\cos(x+y)-y\sin(x+y)$.

Sol. The given equation can be written as

$$y\frac{\partial q}{\partial x}+\frac{\partial z}{\partial x}=\cos(x+y)-y\sin(x+y).$$

Integrating w.r.t. 'x', we get

$$yq+z=\sin(x+y)+y\cos(x+y)+f(y)$$

or $$y\frac{\partial z}{\partial y}+z=\sin(x+y)+y\cos(x+y)+f(y).$$

Now integrating w.r.t. 'y', we get

$$yz=y\sin(x+y)+\int f(y)\,dy+\psi(x)$$

or $$yz=y\sin(x+y)+\phi(y)+\psi(x).$$

Ex. 6. *Solve* $t-xq=x^2$.

Sol. The given equation can be written as

$\frac{\partial q}{\partial y}-xq=x^2$, which is linear in q regarding x as constant.

$\text{I.F.}=e^{-\int x\,dy}=e^{-xy}$.

$\therefore$ solution is $qe^{-xy}=\int x^2e^{-xy}\,dy+f(x)$

$$=-xe^{-xy}+f(x)$$

or $$q=\frac{\partial z}{\partial y}=-x+e^{xy}f(x).$$

Integrating, $z=-xy+f(x)\int e^{xy}\,dy+\psi(x)$

or $$z=-xy+\frac{1}{x}f(x)\,e^{xy}+\psi(x)$$

or $$z=-xy+\phi(x)\,e^{xy}+\psi(x).$$

Ex. 7. *Solve* $yt-q=xy$.

Sol. The given equation can be written as

$\frac{\partial q}{\partial y}-\frac{1}{y}q=x$, which is linear in q regarding x as constant.

$\text{I.F.}=e^{\int(-1/y)\,dy}=e^{-\log y}=1/y$.

$$\therefore\ q\,\frac{1}{y}=\int x.\frac{1}{y}\,dy+f(x)=x\log y+f(x).$$

or $$q=\frac{\partial z}{\partial y}=xy\log y+y\,f(x).$$

Integrating, $z=x\int y\log y\,dy+f(x)\int y\,dy+\psi(x)$

or $$z=x\left[\tfrac{1}{2}y^2\log y-\int\frac{1}{y}.\frac{y^2}{2}\,dy\right]+\frac{y^2}{2}f(x)+\psi(x)$$

or $$z=\tfrac{1}{2}xy^2\log y-\tfrac{1}{4}xy^2+\tfrac{1}{2}y^2f(x)+\psi(x).$$

Ex. 8. *Solve* $rx=(n-1)p$.

Sol. The given equation can be written as

$$\frac{\partial^2z/\partial x^2}{\partial z/\partial x}=\frac{n-1}{x}.$$

Integrating, $\log\frac{\partial z}{\partial x}=(n-1)\log x+\log f(y)$

or $$\frac{\partial z}{\partial x}=x^{n-1}f(y).$$

Again integrating w.r.t. 'x', we get

$$z=\frac{x^n}{n}f(y)+\psi(y)$$

or $$z=x^n\phi(y)+\psi(y).$$

Ex. 9. *Solve* $p+r+s=1$. [Kanpur 81]

Sol. The given equation can be written as

$$\frac{\partial z}{\partial x}+\frac{\partial p}{\partial x}+\frac{\partial q}{\partial x}=1.$$

Integrating w.r.t. 'x', we get

$$z+p+q=x+f(y)$$

or $$p+q=x+f(y)-z.$$

Lagrange's auxiliary equations for this are

$$\frac{dx}{1}=\frac{dy}{1}=\frac{dz}{x+f(y)-z}.$$

The first two members give $x-y=a$.

From the last two members, we get

$$\frac{dz}{dy}+z=x+f(y)$$

or $$\frac{dz}{dy}+z=a+y+f(y),$$ which is linear in z.

$\text{I.F.}=e^{\int dy}=e^y$.

$\therefore$ $ze^y=\int\{a+y+f(y)\}\,e^y\,dy+b$

$$=ae^y+\int\{y+f(y)\}\,e^y\,dy+b$$

$$=ae^y+\phi(y)+b$$

or $$z=a+e^{-y}\phi(y)+be^{-y}$$

or $$z=x-y+e^{-y}\phi(y)+e^{-y}\psi(x-y).$$

Ex. 10. *Solve* $s-t=\frac{x}{y^2}$.

Sol. The given equation can be written as

$$\frac{\partial p}{\partial y}-\frac{\partial q}{\partial y}=\frac{x}{y^2}.$$

Integrating w.r.t. 'y', we get

$$p-q=-\frac{x}{y}+f(x).$$

Lagrange's auxiliary equations for this are

$$\frac{dx}{1}=\frac{dy}{-1}=\frac{dz}{-(x/y)+f(x)}.$$

The first two members give, $x+y=a$.

Taking the first and the last members, we get

$$dz=-\frac{x}{y}\,dx+f(x)\,dx$$

or
$$dz=-\frac{x}{a-x}\,dx+f(x)\,dx$$

or
$$dz=\left(1-\frac{a}{a-x}\right)dx+f(x)\,dx.$$

Integrating, $z=x+a\log(a-x)+\phi(x)+b$

or $z=a\log(a-x)+\psi(x)+b$

or $z=(x+y)\log y+\psi(x)+F(x+y)$.

Ex. 11. *Solve* $xr+p=9x^2y^3$.

Sol. The given equation can be written as

$$\frac{\partial p}{\partial x}+\frac{1}{x}p=9xy^3,$$ which is linear in p.

$$\text{I.F.}=e^{\int(1/x)\,dx}=e^{\log x}=x.$$

$$px=\int 9x^2y^3\,dx+f(y)$$

or
$$px=3x^3y^3+f(y)$$

or
$$p=\frac{\partial z}{\partial x}=3x^2y^3+\frac{1}{x}f(y).$$

Integrating w.r.t. 'x', we get

$$z=x^3y^3+f(y)\log x+\phi(y).$$

Ex. 12. *Solve* $r=2y^2$.

Sol. The given equation can be written as

$$\frac{\partial p}{\partial x}=2y^2.$$

Integrating w.r.t. 'x', we get

$$p=\frac{\partial z}{\partial x}=2xy^2+f(y).$$

Again integrating w.r.t. 'x', we get

$$z=x^2y^2+xf(y)+\phi(y).$$

Ex. 13. *Solve* $\log s=x+y$.

Sol. The given equation can be written as

$$\frac{\partial^2 z}{\partial x\,\partial y}=e^{x+y}.$$

Integrating w.r.t. 'x', we get

$$\frac{\partial z}{\partial y}=e^{x+y}+f(y).$$

Again integrating w.r.t. 'y', we get

$$z=e^{x+y}+\int f(y)\,dy+\psi(x)$$

or
$$z=e^{x+y}+\phi(y)+\psi(x).$$

Ex. 14. *Solve* $r=6x$.

Sol. The given equation can be written as

$$\frac{\partial^2 z}{\partial x^2}=6x.$$

Integrating w.r.t. 'x', we get

$$\frac{\partial z}{\partial x}=3x^2+f(y).$$

Again integrating w.r.t. 'x', we get

$$z=x^3+xf(y)+\phi(y).$$

Ex. 15. *Solve* $s=0$.

Sol. The given equation can be written as

$$\frac{\partial^2 z}{\partial x\,\partial y}=0.$$

Integrating w.r.t. 'x', we get

$$\frac{\partial z}{\partial y}=f(y).$$

Again integrating w.r.t. 'y', we get

$$z=\int f(y)\,dy+\psi(x)$$

or
$$z=\phi(y)+\psi(x).$$

Ex. 16. *Solve* $xr+2p=0$.

Sol. The given equation can be written as

$$x\frac{\partial p}{\partial x}+2p=0$$

or
$$x^2\frac{\partial p}{\partial x}+2xp=0.$$

Integrating w.r.t. 'x', we get

$$x^2p=f(y)$$

or
$$p=\frac{\partial z}{\partial x}=\frac{1}{x^2}f(y).$$

Again integrating w.r.t. 'x', we get

$$z=-\frac{1}{x}f(y)+\phi(y).$$

Ex. 17. *Solve* $xr=p$.

Sol. The given equation can be written as

$$x\frac{\partial^2 z}{\partial x^2}=\frac{\partial z}{\partial x}$$

or

$$\frac{\partial^2 z/\partial x^2}{\partial z/\partial x}=\frac{1}{x}.$$

Integrating w.r.t. 'x', we get

$$\log\frac{\partial z}{\partial x}=\log x+\log f(y)$$

or

$$\frac{\partial z}{\partial x}=xf(y).$$

Again integrating w.r.t. 'x', we get

$$z=\tfrac{1}{2}x^2 f(y)+\phi(y).$$

Ex. 18. *Solve* $ar=xy$.

Sol. The given equation can be written as

$$a\frac{\partial^2 z}{\partial x^2}=xy.$$

Integrating w.r.t. 'x', $a\dfrac{\partial z}{\partial x}=\dfrac{x^2}{2}y+f(y).$

Again integrating w.r.t. 'x', we get

$$az=\frac{1}{6}x^3y+xf(y)+\phi(y).$$

Ex. 19. *Solve* $xs+q=4x+2y+2$.

Sol. The given equation can be written as

$$x\frac{\partial p}{\partial y}+\frac{\partial z}{\partial y}=4x+2y+2.$$

Integrating w.r.t. 'y', we get

$$xp+z=4xy+y^2+2y+f(x)$$

or

$$x\frac{\partial z}{\partial x}+z=4xy+y^2+2y+f(x).$$

Again integrating w.r.t. 'x', we get

$$zx=2x^2y+xy^2+2xy+\phi(x)+\psi(y).$$

Ex. 20. *Solve* $t+s+q=0$.

Sol. The given equation can be written as

$$\frac{\partial q}{\partial y}+\frac{\partial p}{\partial y}+\frac{\partial z}{\partial y}=0.$$

Integrating w.r.t. 'y', we get

$$q+p+z=f(x)$$

or

$$p+q=f(x)-z.$$

Lagrange's auxiliary equations are

$$\frac{dx}{1}=\frac{dy}{1}=\frac{dz}{f(x)-z}.$$

From the first two members, we get $x-y=a$.

Taking the first and the last members, we get

$$\frac{dz}{dx}+z=f(x),\ \text{which is linear in } z.$$

I.F. $=e^{\int dx}=e^x$.

$\therefore$ $ze^x=\int e^x f(x)\,dx+b=\phi(x)+b$

or $ze^x=\phi(x)+\psi(x-y)$.

Ex. 21. *Solve* $2yq+y^2t=1$.

Sol. The given equation can be written as

$$2y\frac{\partial z}{\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=1.$$

Integrating w.r.t. 'y', we get

$$y^2\frac{\partial z}{\partial y}=y+f(x)$$

or

$$\frac{\partial z}{\partial y}=\frac{1}{y}+\frac{1}{y^2}f(x).$$

Again integrating w.r.t. 'y', we get

$$z=\log y-\frac{1}{y}f(x)+\phi(x)$$

or

$$yz=y\log y-f(x)+y\phi(x).$$

Solutions of equations under given conditions.

After finding the general solutions by the usual methods, the geometrical conditions given in the problem are used to find the arbitrary functions.

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Ex. 22. *Find the surface passing through the parabolas* $z=0,\ y^2=4ax$ *and* $z=1,\ y^2=-4ax$ *and satisfying the equation*

$xr+2p=0$. [Meerut 89 ; Rohilkhand 76 ; Agra 73, 86]

Sol. The given equation can be written as

$$x\frac{\partial p}{\partial x}+2p=0$$

or

$$x^2\frac{\partial p}{\partial x}+2px=0.$$

Integrating w.r.t. 'x', we get

$$px^2=f(y)$$

or $$p=\frac{\partial z}{\partial x}=\frac{1}{x^2}f(y).$$

Again integrating w.r.t. 'x', we get
$$z=-\frac{1}{x}f(y)+\phi(y). \qquad \ldots(1)$$

Now using the geometrical conditions of the problem we are to determine the values of $f(y)$ and $\phi(y)$.

Since the required surface is to pass through the parabola $z=0$, $y^2=4ax$ hence putting $z=0$ and $x=y^2/4a$ in (1), we get
$$0=-\frac{4a}{y^2}f(y)+\phi(y). \qquad \ldots(2)$$

Again, putting $z=1$ and $x=-y^2/4a$ in (1), we get
$$1=\frac{4a}{y^2}f(y)+\phi(y). \qquad \ldots(3)$$

Solving (2) and (3) for $f(y)$ and $\phi(y)$, we have
$$\phi(y)=\tfrac{1}{2} \text{ and } f(y)=y^2/8a.$$

Thus we have determined the arbitrary functions.

Putting the values of $f(y)$ and $\phi(y)$ in (1), the required surface is
$$z=-\frac{1}{x}\cdot\frac{y^2}{8a}+\frac{1}{2}$$
or $$8axz-4ax+y^2=0.$$

Ex. 23. *Find a surface passing through the two lines $z=x=0$, $z-1=x-y=0$, satisfying $r-4s+4t=0$.*

[Meerut 90 ; Rohilkhand 80 ; Agra 81 ; I.A.S. 77]

Sol. The given equation can be written as
$$(D^2-4DD'+4D'^2)z=0. \qquad \ldots(1)$$

A.E. is $m^2-4m+4=0$

or $(m-2)^2=0$. $\therefore$ $m=2, 2$.

Hence the general solution of (1) is
$$z=\phi_1(y+2x)+x\,\phi_2(y+2x). \qquad \ldots(2)$$

If the surface (2) passes through the lines
$$z=x=0 \text{ and } z-1=x-y=0,$$
then we have
$$0=\phi_1(y+2x), \qquad \ldots(3)$$
and $$1=\phi_1(y+2x)+x\,\phi_2(y+2x). \qquad \ldots(4)$$

From (3) and (4), we get
$$\phi_2(y+2x)=\frac{1}{x}=\frac{3}{3x}=\frac{3}{2x+y}. \quad [\because\ y-x=0]$$

Putting the values of ϕ_1 and ϕ_2 in (2), the required surface is
$$z=x\cdot\frac{3}{2x+y}, \quad \text{or } z(2x+y)=3x.$$

Ex. 24. *Find a surface satisfying $t=6x^3y$ containing the two lines* $y=0=z$, $y=1=z$.

(Meerut 84, 88 ; Agra 70, 85 ; Kanpur 86)

Sol. The given equation can be written as
$$\frac{\partial q}{\partial y}=6x^3y.$$

Integrating w.r.t. 'y', we get
$$q=\frac{\partial z}{\partial y}=3x^3y^2+f(x).$$

Again integrating w.r.t. 'y', we get
$$z=x^3y^3+y\,f(x)+\phi(x). \qquad \ldots(1)$$

Putting $y=0$, $z=0$ in (1), we get
$$0=\phi(x). \qquad \ldots(2)$$

Again putting $y=1$, $z=1$ in (1), we get
$$1=x^3+f(x)+\phi(x). \qquad \ldots(3)$$

From (2) and (3), $\phi(x)=0$, $f(x)=1-x^3$.

Putting the values of $f(x)$ and $\phi(x)$ in (1), the required surface is
$$z=x^3y^3+y(1-x^3).$$

Ex. 25. *A surface is drawn satisfying $r+t=0$ and touching $x^2+z^2=1$ along its section by $y=0$. Find its equation in the form*
$$z^2(x^2+z^2-1)=y^2(x^2+z^2).$$
(Agra 83)

Sol. The given equation can be written as
$$\frac{\partial^2 z}{\partial x^2}+\frac{\partial^2 z}{\partial y^2}=0 \quad \text{or} \quad (D^2+D'^2)z=0. \qquad \ldots(1)$$

A.E. is $m^2+1=0$. $\therefore$ $m=\pm i$.

Hence the general solution of the equation (1) is
$$z=\phi_1(y+ix)+\phi_2(y-ix). \qquad \ldots(2)$$

From (2), $$p=\frac{\partial z}{\partial x}=i\,\phi_1'(y+ix)-i\,\phi_2'(y-ix)$$
and $$q=\frac{\partial z}{\partial y}=\phi_1'(y+ix)+\phi_2'(y-ix).$$

Also from $x^2+z^2=1$, $\qquad \ldots(3)$
$$p=\frac{\partial z}{\partial x}=\frac{-x}{\sqrt{(1-x^2)}}, \quad q=\frac{\partial z}{\partial y}=0.$$

If the surface (2) touches the surface (3) along its section by the plane $y=0$ then the values of p and q from (2) and (3) must be equal at $y=0$.

Thus $l\phi_1'(lx)-l\phi_2'(-lx)=-\frac{x}{\sqrt{(1-x^2)}}$ ...(4)

and $\phi_1'(lx)+\phi_2'(-lx)=0.$ ...(5)

Solving (4) and (5), we get

$$\phi_1'(lx)=\frac{xl}{2\sqrt{(1+l^2x^2)}},\quad \phi_2'(-lx)=\frac{-xl}{2\sqrt{(1+l^2x^2)}}.$$

$\therefore$ $\phi_1'(u)=\frac{u}{2\sqrt{(1+u^2)}}$, so that $\phi_1(u)=\frac{1}{2}\sqrt{(1+u^2)}+c_1$

and $\phi_2'(v)=\frac{v}{2\sqrt{(1+v^2)}}$, so that $\phi_2(v)=\frac{1}{2}\sqrt{(1+v^2)}+c_2$.

These give $\phi_1(y+lx)=\frac{1}{2}\sqrt{\{1+(y+lx)^2\}}+c_1$

and $\phi_2(y-lx)=\frac{1}{2}\sqrt{\{1+(y-lx)^2\}}+c_2$.

$\therefore$ $z=\frac{1}{2}\sqrt{\{1+(y+lx)^2\}}+\frac{1}{2}\sqrt{\{1+(y-lx)^2\}}+c_1+c_2$

or $z=\frac{1}{2}[\sqrt{\{1+(y+lx)^2\}}+\sqrt{\{1+(y-lx)^2\}}]+c$ ...(6)

where $c=c_1+c_2$.

Equating the two values of z from (3) and (6) when $y=0$, we have

$$\sqrt{(1-x^2)}=\sqrt{(1-x^2)}+c. \quad \therefore \quad c=0.$$

Hence, from (6), we get

$$2z=\sqrt{\{1+(y+lx)^2\}}+\sqrt{\{1+(y-lx)^2\}}$$

or $2z-\sqrt{\{1+(y+lx)^2\}}=\sqrt{\{1+(y-lx)^2\}}$.

Squaring both sides, we get

$$4z^2+1+(y+lx)^2-4z\sqrt{\{1+(y+lx)^2\}}=1+(y-lx)^2$$

or $z^2+lxy=z\sqrt{\{1+(y+lx)^2\}}$.

Again squaring both sides, we get

$$z^4+2lxyz^2+l^2x^2y^2=z^2\{1+(y+lx)^2\}$$

or $z^4-x^2y^2=z^2+z^2y^2-z^2x^2$

or $z^2(z^2+x^2-1)=y^2(z^2+x^2)$.

Ex. 26. *Find a surface satisfying*

$$2x^2r-5xys+2y^2t+2(px+qy)=0$$

and touching the hyperbolic paraboloid $z=x^2-y^2$ along its section by the plane $y=1$.

Sol. The given equation can be written as

$$2x^2\frac{\partial^2 z}{\partial x^2}-5xy\frac{\partial^2 z}{\partial x\,\partial y}+2y^2\frac{\partial^2 z}{\partial y^2}+2x\frac{\partial z}{\partial x}+2y\frac{\partial z}{\partial y}=0.$$

Putting $x=e^X, y=e^Y$ and denoting $\frac{\partial}{\partial X}$ and $\frac{\partial}{\partial Y}$ by D and D' respectively, the given equation transforms to

$$[2D(D-1)-5DD'+2D'(D'-1)+2D+2D']z=0$$

or $(2D-D')(D-2D')z=0.$

$\therefore$ $z=\phi_1(2Y+X)+\phi_2(Y+2X)$

$=\phi_1(2\log y+\log x)+\phi_2(\log y+2\log x)$

$=\phi_1(\log xy^2)+\phi_2(\log x^2y)$

or $z=f_1(xy^2)+f_2(x^2y)$. ...(1)

From (1), $p=\frac{\partial z}{\partial x}=y^2f_1'(xy^2)+2xyf_2'(x^2y)$

and $q=\frac{\partial z}{\partial y}=2xyf_1'(xy^2)+x^2f_2'(x^2y)$.

Also from $z=x^2-y^2$,

$$p=\frac{\partial z}{\partial x}=2x,\ q=\frac{\partial z}{\partial y}=-2y. \quad ...(2)$$

If the surface (2) touches the surface (1) along its section by the plane $y=1$ then the values of p and q from (1) and (2) must be equal at $y=1$.

$f_1'(x)+2xf_2'(x^2)=2x$ ...(3)

and $2xf_1'(x)+x^2f_2'(x^2)=-2.$ ...(4)

Solving (3) and (4), we get

$$f_1'(x)=-\frac{4}{3x}-\frac{2}{3}x$$

and $f_2'(x^2)=\frac{4}{3}+\frac{2}{3x^2}$ or $f_2'(u)=\frac{4}{3}+\frac{2}{3u}$, where $u=x^2$.

$\therefore$ $f_1(x)=-\frac{4}{3}\log x-\frac{1}{3}x^2+c_1$

and $f_2(u)=\frac{4}{3}u+\frac{2}{3}\log u+c_2$.

These give

$$f_1(xy^2)=-\frac{4}{3}\log(xy^2)-\frac{1}{3}(xy^2)^2+c_1$$

and $f_2(x^2y)=\frac{4}{3}x^2y+\frac{2}{3}\log(x^2y)+c_2$.

Putting the values of $f_1(xy^2), f_2(x^2y)$ in (1), we get

$$z=-\frac{4}{3}\log(xy^2)-\frac{1}{3}x^2y^4+\frac{4}{3}x^2y+\frac{2}{3}\log(x^2y)+c_1+c_2$$

$$= -\frac{4}{3}\log(xy^4) + \frac{2}{3}\log(x^2y) - \frac{1}{3}x^2y^4 + \frac{4}{3}x^2y + c \qquad \ldots(5)$$

where $c = c_1 + c_2$.

Equating the two values of z from (2) and (5) when $y = 1$, we have

$$(x^2 - 1) = -\frac{4}{3}\log x + \frac{2}{3}\log x^2 - \frac{1}{3}x^2 + \frac{4}{3}x^2 + c \Rightarrow c = -1.$$

Putting $c = -1$ in (5), the required surface is

$$z = \frac{4}{3}x^2y - \frac{1}{3}x^2y^4 - 2\log y - 1$$

or $$3z = 4x^2y - x^2y^4 - 6\log y - 3.$$

Ex. 27. *Find a surface satisfying $r + s = 0$ and touching the elliptic paraboloid $z = 4x^2 + y^2$ along its section by the plane $y = 2x + 1$.* [Meerut 71, 77, 82, 87]

Sol. The given equation can be written as

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x\,\partial y} = 0$$

or $$(D^2 + DD')z = 0 \text{ or } D(D + D')z = 0.$$

$\therefore$ $$z = \phi_1(y) + \phi_2(y - x). \qquad \ldots(1)$$

From (1), $p = \dfrac{\partial z}{\partial x} = -\phi_2'(y - x)$

and $$q = \frac{\partial z}{\partial y} = \phi_1'(y) + \phi_2'(y - x).$$

Also from, $z = 4x^2 + y^2$, $\qquad \ldots(2)$

$$p = \frac{\partial z}{\partial x} = 8x \text{ and } q = \frac{\partial z}{\partial y} = 2y.$$

If the surface (1) touches the surface (2) along its section by the plane $y = 2x + 1$ then the values of p and q from (1) and (2) must be equal when $y = 2x + 1$.

$\therefore$ $$-\phi_2'(y - x) = 8x, \qquad \ldots(3)$$
$$\phi_1'(y) + \phi_2'(y - x) = 2y, \qquad \ldots(4)$$
and $$y = 2x + 1. \qquad \ldots(5)$$

From (3) and (5), we get

$$-\phi_2'(y - x) = 8(y - x - 1).$$

Integrating, $-\phi_2(y - x) = 8\left[\frac{1}{2}(y - x)^2 - (y - x)\right] + c_1$.

$\therefore$ $$\phi_2(y - x) = -4(y - x)^2 + 8(y - x) + c_1.$$

Again from (3) and (4), we get

$$\phi_1'(y) = 8x + 2y$$

or $$\phi_1'(y) = \frac{8}{2}(y - 1) + 2y = 6y - 4.$$

Integrating, $\phi_1(y) = 3y^2 - 4y + c_2$.

Substituting the values of $\phi_1(y)$ and $\phi_2(y - x)$ in (1), we get

$$z = 3y^2 - 4y - 4(y - x)^2 + 8(y - x) + c_1 + c_2$$

or $$z = -4x^2 - y^2 + 8xy + 4y - 8x + c, \qquad \ldots(6)$$

where $$c = c_1 + c_2.$$

Equating the two values of z from (2) and (6), when $y = 2x + 1$, we have

$$-4x^2 - (2x + 1)^2 + 4(2x + 1)^0 - 8x + c = 4x^2 + (2x + 1)^2.$$

$\therefore$ $$c = -2.$$

Putting the value of c in (6) the required surface is

$$z + 4x^2 + y^2 - 8xy - 4y + 8x + 2 = 0.$$

Ex. 28. *Find a surface satisfying $r = 6x + 2$ and touching $z = x^3 + y^3$ along its section by the plane $x + y + 1 = 0$.* [Agra 78, 82]

Sol. The given equation can be written as

$$\frac{\partial^2 z}{\partial x^2} = 6x + 2.$$

Integrating, $\dfrac{\partial z}{\partial x} = 3x^2 + 2x + \phi_1(y)$.

Again, integrating, $z = x^3 + x^2 + x\phi_1(y) + \phi_2(y) \qquad \ldots(1)$

From (1), $p = \dfrac{\partial z}{\partial x} = 3x^2 + 2x + \phi_1(y)$,

$$q = \frac{\partial z}{\partial y} = x\phi_1'(y) + \phi_2'(y).$$

Also from $z = x^3 + y^3$, $\qquad \ldots(2)$

$$p = \frac{\partial z}{\partial x} = 3x^2 \text{ and } q = \frac{\partial z}{\partial y} = 3y^2.$$

If the surface (1) touches the surface (2) along its section by the plane $x + y + 1 = 0$ then the values of p and q from (1) and (2) must be equal when $x + y + 1 = 0$.

$\therefore$ $$3x^2 + 2x + \phi_1(y) = 3x^2, \qquad \ldots(3)$$
$$x\phi_1'(y) + \phi_2'(y) = 3y^2, \qquad \ldots(4)$$
and $$x + y + 1 = 0. \qquad \ldots(5)$$

From (3) and (5), we get

$$\phi_1(y) = -2x = 2(y + 1) \Rightarrow \phi_1'(y) = 2.$$

Substituting the value of $\phi_1'(y)$ in (4), we have

$$\phi_2'(y) = 3y^2 - 2x = 3y^2 + 2(y + 1).$$

Integrating, $\phi_2(y) = y^3 + y^2 + 2y + k$.

Putting the values of $\phi_1(y)$ and $\phi_2(y)$ in (1), we get

$$z = x^3 + x^2 + 2x(y + 1) + y^3 + y^2 + 2y + k. \qquad \ldots(6)$$

Equating the two values of z from (2) and (6) when $y=-(x+1)$, we get

$$x^3-(x+1)^3=x^3+x^3+2x(-x)-(x+1)^3+(x+1)^3-2(x+1)+k,$$

$$\therefore \quad k=1.$$

Putting the value of k in (6), the required surface is

$$z=x^3+x^3+2x(y+1)+x^3+y^3+2y+1$$

or $$z=x^3+y^3+(x+y+1)^3$$

Ex. 29. *Show that the surface satisfying $r-2s+t=6$ and touching the hyperbolic paraboloid $z=xy$ along its section the plane $y=x$ is $z=x^3-xy+y^3$.*

Sol. The given equation can be written as

$$(D^2-2DD'+D'^2)\,z=0$$

or $$(D-D')^2\,z=6. \qquad \ldots(1)$$

$\therefore$ C.F. $=\phi_1(y+x)+x\,\phi_2(y+x)$.

Now P.I. $=\dfrac{1}{(D-D')^2}6=\dfrac{1}{D^2}\left(1-\dfrac{D'}{D}\right)^{-2}6$

$$=\frac{1}{D^2}\left\{1+\frac{2D'}{D}+\ldots\right\}6=\frac{1}{D^2}6=3x^2.$$

Hence the general solution of (1) is

$$z=\text{C.F.}+\text{P.I.}=\phi_1(y+x)+x\phi_2(y+x)+3x^2. \qquad \ldots(2)$$

From (2),

$$p=\frac{\partial z}{\partial x}=\phi_1'(y+x)+x\phi_2'(y+x)+\phi_2(y+x)+6x$$

and $$q=\frac{\partial z}{\partial y}=\phi_1'(y+x)+x\phi_2'(y+x).$$

Also from $z=xy$, $\qquad \ldots(3)$

$$p=\frac{\partial z}{\partial x}=y, \qquad q=\frac{\partial z}{\partial y}=x.$$

If the surface (3) touches the surface (2) along its section by the plane $y=x$ then the values of p and q from (2) and (3) must be equal at $y=x$.

$$\therefore \quad \phi_1'(2x)+x\phi_2'(2x)+\phi_2(2x)+6x=x \qquad \ldots(4)$$

and $$\phi_1'(2x)+x\phi_2'(2x)=x \qquad \ldots(5)$$

Solving (4) and (5), we get $\phi_2(2x)=-6x$

so that $\quad \phi_2(u)=-3u$

or $\quad \phi_2(y+x)=-3(y+x)$.

Putting $\phi_2(2x)=-6x$ and $\phi_2'(2x)=-3$ in (4), we get

$$\phi_1'(2x)-3x-6x+6x=x$$

or $$\phi_1'(2x)=4x \text{ so that } \phi_1'(u)=2u.$$

$$\therefore \quad \phi_1(u)=u^2+c$$

or $$\phi_1(y+x)=(y+x)^2+c.$$

Putting the values of $\phi_1(y+x)$ and $\phi_2(y+x)$ in (2), we get

$$z=(y+x)^2-3x(y+x)+3x^2+c=x^2-xy+y^2+c \qquad \ldots(6)$$

Equating the two values of z from (3) and (6) when $y=x$, we get

$$x^2-x^2+x^2+c=x^2 \quad \text{or} \quad c=0.$$

Putting the value of c in (6) the required surface is

$$z=x^2-xy+y^2.$$

Ex. 30. *Solve the equation $r+t=2s$ and determine the arbitrary functions by the conditions that $bz=y^2$ when $x=0$ and $az=x^2$ when $y=0$.*

Sol. The given equation can be written as

$$r-2s+t=0$$

or $$(D^2-2DD'+D'^2)\,z=0$$

or $$(D-D')^2\,z=0. \qquad \ldots(1)$$

Hence the general solution of (1) is

$$z=\phi_1(y+x)+x\phi_2(y+x) \qquad \ldots(2)$$

Putting $x=0$, $z=y^2/b$ in (2), we get

$$\frac{y^2}{b}=\phi_1(y) \text{ so that } \phi_1(y+x)=\frac{(y+x)^2}{b}.$$

Again putting $y=0$, $z=x^2/a$ in (2), we get

$$\frac{x^2}{a}=\phi_1(x)+x\phi_2(x)=\frac{x^2}{b}+x\phi_2(x).$$

$$\therefore \quad \phi_2(x)=\frac{b-a}{ab}x \text{ so that } \phi_2(y+x)=\frac{b-a}{ab}(y+x).$$

Putting the values of $\phi_1(y+x)$ and $\phi_2(y+x)$ in (2) the required solution is

$$z=\frac{(y+x)^2}{b}+x\,\frac{b-a}{ab}(y+x)=(y+x)\left(\frac{x}{a}+\frac{y}{b}\right).$$

§ 2. Canonical Forms (Method of Transformations).

Now we shall consider the equation of the type

$$Rr+Ss+Tt+f(x,y,z,p,q)=0, \qquad \ldots(1)$$

where R, S, T are continuous functions of x and y possessing continuous partial derivatives of as high an order as necessary.

We shall show that any equation of the type (1) can be reduced to one of the three *canonical forms* by a suitable change of the independent variables. Suppose we change the independent variables from x, y to u, v, where

$$u=u(x, y),\ v=v(x, y). \qquad \ldots(2)$$

Then, we have

$$p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x},\quad q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

$$\therefore\ \frac{\partial}{\partial x}=\frac{\partial u}{\partial x}\cdot\frac{\partial}{\partial u}+\frac{\partial v}{\partial x}\cdot\frac{\partial}{\partial v},\quad \frac{\partial}{\partial y}=\frac{\partial u}{\partial y}\cdot\frac{\partial}{\partial u}+\frac{\partial v}{\partial y}\cdot\frac{\partial}{\partial v}$$

Now
$$r=\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\left(\frac{\partial u}{\partial x}\frac{\partial}{\partial u}+\frac{\partial v}{\partial x}\frac{\partial}{\partial v}\right)\left(\frac{\partial u}{\partial x}\frac{\partial z}{\partial u}+\frac{\partial v}{\partial x}\frac{\partial z}{\partial v}\right)$$
$$=\frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2+2\frac{\partial^2 z}{\partial u\,\partial v}\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2+\frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x^2}+\frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial x^2}$$

$$\left(\frac{\partial u}{\partial x}\frac{\partial}{\partial u}+\frac{\partial v}{\partial x}\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\right)=\frac{\partial u}{\partial x}\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\right)+\frac{\partial v}{\partial x}\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}\right)$$

$$s=\frac{\partial^2 z}{\partial x\,\partial y}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\left(\frac{\partial u}{\partial x}\frac{\partial}{\partial u}+\frac{\partial v}{\partial x}\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\right)$$
$$=\frac{\partial^2 z}{\partial u^2}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+\frac{\partial^2 z}{\partial u\,\partial v}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)+\frac{\partial^2 z}{\partial v^2}\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}+\frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial y\,\partial x}+\frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial y\,\partial x}$$

and
$$t=\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\left(\frac{\partial u}{\partial y}\frac{\partial}{\partial u}+\frac{\partial v}{\partial y}\frac{\partial}{\partial v}\right)\left(\frac{\partial u}{\partial y}\frac{\partial z}{\partial u}+\frac{\partial v}{\partial y}\frac{\partial z}{\partial v}\right)$$
$$=\frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial y}\right)^2+2\frac{\partial^2 z}{\partial u\,\partial v}\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}+\frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial y}\right)^2+\frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial y^2}+\frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial y^2}$$

Substituting these values of p, q, r, s and t in (1), it takes the form

$$A\frac{\partial^2 z}{\partial u^2}+2B\frac{\partial^2 z}{\partial u\,\partial v}+C\frac{\partial^2 z}{\partial v^2}+F\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right)=0 \qquad \ldots(3)$$

where
$$A=R\left(\frac{\partial u}{\partial x}\right)^2+S\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}+T\left(\frac{\partial u}{\partial y}\right)^2, \qquad \ldots(4)$$

$$B=R\frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial x}+\frac{1}{2}S\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)+T\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}, \qquad \ldots(5)$$

$$C=R\left(\frac{\partial v}{\partial x}\right)^2+S\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}+T\left(\frac{\partial v}{\partial y}\right)^2 \qquad \ldots(6)$$

and the function F is the transformed form of the function f.

Now the problem is to determine u and v so that the equation (3) takes the simplest possible form. The procedure is simple when the discriminant S^2-4RT of the quadratic equation

$$R\lambda^2+S\lambda+T=0 \qquad \ldots(7)$$

is everywhere either positive, negative or zero, and we shall discuss these three cases separately.

Case I. $S^2-4RT>0$. If this condition is satisfied then the roots λ_1, λ_2 of the equation (7) are real and distinct. The coefficients of $\frac{\partial^2 z}{\partial u^2}$ and $\frac{\partial^2 z}{\partial v^2}$ in the equation (3) will vanish if we choose u and v such that

$$\frac{\partial u}{\partial x}=\lambda_1\frac{\partial u}{\partial y}, \qquad \ldots(8)$$

and
$$\frac{\partial v}{\partial x}=\lambda_2\frac{\partial v}{\partial y}. \qquad \ldots(9)$$

The differential equations (8) and (9) will determine the form of u and v as functions of x and y.

For this, from (8), Lagrange's auxiliary equations are

$$\frac{dx}{1}=\frac{dy}{-\lambda_1}=\frac{du}{0}.$$

The last member gives $du=0$ *i.e.*, $u=$constant.

The first two members give

$$\frac{dy}{dx}+\lambda_1=0. \qquad \ldots(10)$$

Let $f_1(x, y)=$constant be the solution of the equation (10). Then the solution of the equation (8) can be taken as

$$u=f_1(x, y). \qquad \ldots(11)$$

Similarly, if $f_2(x, y)=$constant is a solution of

$$\frac{dy}{dx}+\lambda_2=0,$$

then the solution of the equation (9) can be taken as

$$v=f_2(x, y). \qquad \ldots(12)$$

Also it can be easily seen that, in general,

$$AC-B^2=(4RT-S^2)\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)^2,$$

so that when A and C are zero

$$B^2=(S^2-4RT)\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)^2, \qquad \ldots(13)$$

It follows that $B^2 > 0$ since $S^2 - 4RT > 0$ and hence we can divide both sides of the equation by it.

Thus making the substitutions defined by the equations (11) and (12), the equation (1) transforms to the form

$$\frac{\partial^2 z}{\partial u\, \partial v} = \phi\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right), \qquad \ldots(14)$$

which is the canonical form in this case.

Case II. $S^2 - 4RT = 0$. In this case the roots of the equation (7) are equal. We define the function u as in Case I and take v to be any function of x and y, which is independent of u. Then, we have, as before, $A = 0$.

Since $S^2 - 4RT = 0$, hence from (13), $B^2 = 0$ i.e., $B = 0$.

On the other hand, in this case, $C \neq 0$, otherwise v would be a function of u.

Putting $A = 0$, $B = 0$ and dividing by C, we see that in this case the canonical form of the equation (1) is,

$$\frac{\partial^2 z}{\partial v^2} = \phi\left(u, v, z, \frac{\partial z}{\partial u}, \frac{\partial z}{\partial v}\right). \qquad \ldots(15)$$

Case III. $S^2 - 4RT < 0$. Formally it is the same as Case I except that now the roots of the equation (7) are complex.

Proceeding as in Case I, we find that the equation (1) reduces to the form (14) but that the variables u, v are not real but are in fact complex conjugates.

To find a real canonical form let $u = \alpha + i\beta$, $v = \alpha - i\beta$ so that

$$\alpha = \tfrac{1}{2}(u + v),\ \beta = \tfrac{1}{2}i\,(v - u).$$

Now $\dfrac{\partial z}{\partial u} = \dfrac{\partial z}{\partial \alpha}\dfrac{\partial \alpha}{\partial u} + \dfrac{\partial z}{\partial \beta}\dfrac{\partial \beta}{\partial u} = \tfrac{1}{2}\left(\dfrac{\partial z}{\partial \alpha} - i\dfrac{\partial z}{\partial \beta}\right)$.

Similarly $\dfrac{\partial z}{\partial v} = \tfrac{1}{2}\left(\dfrac{\partial z}{\partial \alpha} + i\dfrac{\partial z}{\partial \beta}\right)$.

$$\therefore \quad \frac{\partial^2 z}{\partial u\, \partial v} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right) = \tfrac{1}{4}\left(\frac{\partial}{\partial \alpha} - i\frac{\partial}{\partial \beta}\right)\left(\frac{\partial z}{\partial \alpha} - i\frac{\partial z}{\partial \beta}\right)$$

$$= \tfrac{1}{4}\left(\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2}\right).$$

Thus, transforming the independent variables u, v to α, β the desired canonical form is

$$\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = \psi\left(\alpha, \beta, z, \frac{\partial z}{\partial \alpha}, \frac{\partial z}{\partial \beta}\right). \qquad \ldots(16)$$

Second order partial differential equations of the type (1) are classified by their canonical forms; we say that an equation of this type is

(i) Hyperbolic if $S^2 - 4RT > 0$,
(ii) Parabolic if $S^2 - 4RT = 0$,
(iii) Elliptic if $S^2 - 4RT < 0$.

Solved Examples

Ex. 1. *Reduce the equation*

$$(y-1)r - (y^2-1)s + y(y-1)t + p - q = 2ye^{2x}(1-y)^3 \qquad \ldots(1)$$

to canonical form and hence solve it. [Rohilkhand 82]

Sol. Comparing the equation (1) with

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0,$$ we have,

$$R = (y-1),\ S = -(y^2-1),\ T = y(y-1).$$

The quadratic equation $R\lambda^2 + S\lambda + T = 0$ therefore becomes

$$(y-1)\lambda^2 - (y^2-1)\lambda + y(y-1) = 0$$

or $$\lambda^2 - (y+1)\lambda + y = 0$$

or $$(\lambda - 1)(\lambda - y) = 0 \Rightarrow \lambda = 1, y \text{ (real and distinct roots).}$$

The equations $\dfrac{dy}{dx} + \lambda_1 = 0$ and $\dfrac{dy}{dx} + \lambda_2 = 0$ become

$$\frac{dy}{dx} + 1 = 0 \text{ and } \frac{dy}{dx} + y = 0.$$

These on integration give

$$x + y = \text{constant and } ye^x = \text{constant},$$

so that to change the independent variables from x, y to u, v, we take

$$u = x + y \text{ and } v = ye^x.$$

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} + ye^x\frac{\partial z}{\partial v} = \frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v},$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + e^x\frac{\partial z}{\partial v},$$

$$r = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial}{\partial u} + v\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial^2 z}{\partial u^2} + 2v\frac{\partial^2 z}{\partial u\, \partial v} + v^2\frac{\partial^2 z}{\partial v^2} + v\frac{\partial z}{\partial v},$$

$$s = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u} + e^x\frac{\partial z}{\partial v}\right)$$

$$=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)+e^x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)+e^x\frac{\partial z}{\partial v}$$

$$=\left(\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u}\right)+e^x\left(\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial v}\right)+e^x\frac{\partial z}{\partial v}$$

$$=\frac{\partial^2 z}{\partial u^2}+(e^x+v)\frac{\partial^2 z}{\partial u\,\partial v}+ve^x\frac{\partial^2 z}{\partial v^2}+e^x\frac{\partial z}{\partial v}$$

and $$t=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}+e^x\frac{\partial z}{\partial v}\right)=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)+e^x\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)$$

$$=\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}+e^x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$=\frac{\partial^2 z}{\partial u^2}+2e^x\frac{\partial^2 z}{\partial u\,\partial v}+e^{2x}\frac{\partial^2 z}{\partial v^2}.$$

Substituting these values in (1), it reduces to :

$$(1-y)^3 e^x\frac{\partial^2 z}{\partial u\,\partial v}=2y\,e^{2x}(1-y)^3$$

or $$\frac{\partial^2 z}{\partial u\,\partial v}=2y\,e^x \quad\text{or}\quad \frac{\partial^2 z}{\partial u\,\partial v}=2v, \qquad \ldots(2)$$

which is the canonical form of the equation (1).

Integrating (2) w.r.t. v, we get

$$\frac{\partial z}{\partial u}=v^2+\phi_1(u), \qquad \ldots(3)$$

where $\phi_1(u)$ is an arbitrary function of u.

Again integrating (3) w.r.t. u, we get

$$z=uv^2+\psi_1(u)+\psi_2(v),$$

where ψ_1 is an integral of ϕ_1 and ψ_2 is an arbitrary function

or $$z=(x+y)\,y^2e^{2x}+\psi_1(x+y)+\psi_2(ye^x).$$

Ex. 2. *Reduce the equation*

$$\frac{\partial^2 z}{\partial x^2}=x^2\frac{\partial^2 z}{\partial y^2}\ \textit{to canonical form.}$$ [Kanpur 82]

Sol. The given equation can be written as

$$r-x^2t=0. \qquad \ldots(1)$$

Comparing the equation (1) with

$$Rr+Ss+Tt+f(x,y,z,p,q)=0, \text{ we have}$$
$$R=1,\ S=0,\ T=-x^2.$$

The quadratic equation in $R\lambda^2+S\lambda+T=0$ therefore becomes

$$\lambda^2-x^2=0 \Rightarrow \lambda=x,\,-x \text{ (real and distinct roots).}$$

The equations $\frac{dy}{dx}+\lambda_1=0$ and $\frac{dy}{dx}+\lambda_2=0$ become

$$\frac{dy}{dx}+x=0 \text{ and } \frac{dy}{dx}-x=0.$$

These on integration give

$$y+\tfrac{1}{2}x^2=\text{constant and } y-\tfrac{1}{2}x^2=\text{constant},$$

so that to change the independent variables from x, y to u, v, we take

$$u=y+\tfrac{1}{2}x^2 \text{ and } v=y-\tfrac{1}{2}x^2.$$

$$\therefore\ p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=x\frac{\partial z}{\partial u}-x\frac{\partial z}{\partial v}=x\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right),$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v},$$

$$r=\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left\{x\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)\right\}$$

$$=x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)+1.\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)$$

$$=x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]+\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}$$

$$=x^2\left(\frac{\partial^2 z}{\partial u^2}-2\frac{\partial^2 z}{\partial u\,\partial v}+\frac{\partial^2 z}{\partial v^2}\right)+\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}$$

and $$t=\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\left(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)$$

$$=\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\,\partial v}+\frac{\partial^2 z}{\partial v^2}.$$

Substituting these values in (1), it reduces to

$$\frac{\partial^2 z}{\partial u\,\partial v}=\frac{1}{4x^2}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)$$

or $$\frac{\partial^2 z}{\partial u\,\partial v}=\frac{1}{4(u-v)}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right),$$

which is the required canonical form of the given equation.

Ex. 3. *Reduce the equation*

$$\frac{\partial^2 z}{\partial x^2}+2\frac{\partial^2 z}{\partial x\,\partial y}+\frac{\partial^2 z}{\partial y^2}=0$$

to canonical form and hence solve it. [G.N.D.U. 87 ; Raj. 83]

Sol. The given equation can be written as

$$r+2s+t=0. \qquad \ldots(1)$$

Comparing the equation (1) with

$$Rr+Ss+Tt+f(x, y, z, p, q)=0, \text{ we have}$$

$$R=1,\ S=2,\ T=1.$$

The quadratic equation $R\lambda^2+S\lambda+T=0$ is therefore given by

$$\lambda^2+2\lambda+1=0, \text{ or } (\lambda+1)^2=0.$$

$\therefore$ $\lambda=-1, -1.$ (equal roots).

The equation $\frac{dy}{dx}+\lambda=0$ becomes $\frac{dy}{dx}-1=0,$

which on integration gives $x-y=$ constant.

To change the independent variables x, y to u, v we take

$$u=x-y.$$

We have to take v as some function of x and y independent of u.

Let $v=x+y$.

Then $p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}=\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v},$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}=-\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v},$$

$$r=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\left(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)$$

$$=\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2},$$

$$s=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\left(\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right)\left(-\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)=-\frac{\partial^2 z}{\partial u^2}+\frac{\partial^2 z}{\partial v^2}$$

and

$$t=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\left(-\frac{\partial}{\partial u}+\frac{\partial}{\partial v}\right)\left(-\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)$$

$$=\frac{\partial^2 z}{\partial u^2}-2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}.$$

Substituting these values in (1), it reduces to

$$\frac{\partial^2 z}{\partial v^2}=0 \text{ which is the required canonical form.}$$

Integrating it w.r.t. v, we get

$$\frac{\partial z}{\partial v}=\phi_1(u).$$

Again integrating w.r.t. v, we get

$$z=v\,\phi_1(u)+\phi_2(u),$$

where ϕ_1 and ϕ_2 are arbitrary functions of u.

Hence the solution is

$$z=(x+y)\,\phi_1(x-y)+\phi_2(x-y).$$

Ex. 4. Reduce the equation

$$\frac{\partial^2 z}{\partial x^2}+x^2\frac{\partial^2 z}{\partial y^2}=0,$$

to canonical form. [Kanpur 82]

Sol. The given equation can be written as

$$r+x^2t=0. \qquad \dots(1)$$

Comparing the equation (1) with

$$Rr+Ss+Tt+f(x, y, z, p, q)=0, \text{ we have}$$

$$R=1,\ S=0,\ T=x^2.$$

The quadratic equation $R\lambda^2+S\lambda+T=0$ is therefore given by

$$\lambda^2+x^2=0 \Rightarrow \lambda=ix, -ix. \text{ (Complex roots)}$$

The equations $\frac{dy}{dx}+\lambda_1=0$ and $\frac{dy}{dx}+\lambda_2=0$ become

$$\frac{dy}{dx}+ix=0 \text{ and } \frac{dy}{dx}-ix=0.$$

These on integration give

$$y+\tfrac{1}{2}ix^2=\text{constant and } y-\tfrac{1}{2}ix^2=\text{constant},$$

so that to change the independent variables from x, y to u, v, we take

$$u=y+\tfrac{1}{2}ix^2=\alpha+i\beta \text{ (say)}$$

and

$$v=y-\tfrac{1}{2}ix^2=\alpha-i\beta.$$

Then $\alpha=y,\quad \beta=\tfrac{1}{2}x^2.$

Now we shall transform the independent variables x and y to α and β. We have

$$p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial x}+\frac{\partial z}{\partial \beta}\cdot\frac{\partial \beta}{\partial x}=x\frac{\partial z}{\partial \beta},$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial \alpha}\cdot\frac{\partial \alpha}{\partial y}+\frac{\partial z}{\partial \beta}\cdot\frac{\partial \beta}{\partial y}=\frac{\partial z}{\partial \alpha},$$

$$r=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial \beta}\right)=1\cdot\frac{\partial z}{\partial \beta}+x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \beta}\right)$$

$$=\frac{\partial z}{\partial \beta}+x\left[\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \alpha}{\partial x}+\frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right)\cdot\frac{\partial \beta}{\partial x}\right]=\frac{\partial z}{\partial \beta}+x^2\frac{\partial^2 z}{\partial \beta^2}$$

and

$$t=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial \alpha}\right)=\frac{\partial^2 z}{\partial \alpha^2}.$$

Substituting these values in (1), it reduces to

$$\frac{\partial^2 z}{\partial \alpha^2}+\frac{\partial^2 z}{\partial \beta^2}=-\frac{1}{2\alpha}\frac{\partial z}{\partial \alpha},$$

which is the required canonical form of (1).

Ex. 5. *Reduce the equation*

$$(n-1)^2 \frac{\partial^2 z}{\partial x^2} - y^{2n} \frac{\partial^2 z}{\partial y^2} = n y^{2n-1} \frac{\partial z}{\partial y},$$

to canonical form and find its general solution. [Meerut 84; Kanpur 83; Raj. 82]

Sol. The given equation can be written as

$$(n-1)^2 r - y^{2n} t - n y^{2n-1} q = 0. \quad \ldots(1)$$

Comparing the equation (1) with

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0, \text{ we have}$$

$$R = (n-1)^2, \quad S = 0, \quad T = -y^{2n}.$$

The quadratic equation $R\lambda^2 + S\lambda + T = 0$ is therefore given by

$$(n-1)^2 \lambda^2 - y^{2n} = 0 \text{ or } \lambda^2 = \frac{1}{(n-1)^2} y^{2n}.$$

$$\therefore \lambda = \frac{1}{(n-1)} y^n, -\frac{1}{(n-1)} y^n. \quad \text{(Real and distinct roots).}$$

The equations $\frac{dy}{dx} + \lambda_1 = 0$ and $\frac{dy}{dx} + \lambda_2 = 0$ become

$$\frac{dy}{dx} + \frac{1}{(n-1)} y^n = 0 \text{ and } \frac{dy}{dx} - \frac{1}{(n-1)} y^n = 0$$

or $\quad (n-1) y^{-n} dy + dx = 0$ and $(n-1) y^{-n} dy - dx = 0.$

These on integration give

$$x - y^{-n+1} = \text{constant and } x + y^{-n+1} = \text{constant},$$

so that to change the independent variables from x, y to u, v, we take

$$u = x - y^{-n+1} \text{ and } v = x + y^{-n+1}.$$

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \frac{\partial v}{\partial y} = (n-1) y^{-n} \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right),$$

$$r = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$$

and

$$t = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left[(n-1) y^{-n} \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\right]$$

$$= -n(n-1) y^{-n-1} \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) + (n-1) y^{-n} \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

$$= -n(n-1) y^{-n-1} \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

$$+ (n-1) y^{-n} \left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$= -n(n-1) y^{-n-1} \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) + (n-1)^2 y^{-2n} \left[\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right].$$

Substituting these values in (1), it reduces to

$$\frac{\partial^2 z}{\partial u \partial v} = 0, \quad \text{which is the required canonical form.}$$

Integrating it, w.r.t. v, we get

$$\frac{\partial z}{\partial u} = \phi_1(u), \quad \text{where } \phi_1 \text{ is an arbitrary function.}$$

Again integrating w.r.t. u, we get

$$z = \psi_1(u) + \psi_2(v),$$

where ψ_1 and ψ_2 are arbitrary functions.

Hence the required general solution is

$$z = \psi_1(x - y^{-n+1}) + \psi_2(x + y^{-n+1}).$$

Ex. 6. *Reduce the equation*

$$y^2 \frac{\partial^2 z}{\partial x^2} - 2xy \frac{\partial^2 z}{\partial x \partial y} + x^2 \frac{\partial^2 z}{\partial y^2} = \frac{y^2}{x} \frac{\partial z}{\partial x} + \frac{x^2}{y} \frac{\partial z}{\partial y}$$

to canonical form and hence solve it. [I.A.S. 82, 85]

Sol. The given equation can be written as

$$y^2 r - 2xys + x^2 t - \frac{y^2}{x} p - \frac{x^2}{y} q = 0. \quad \ldots(1)$$

Comparing the equation (1) with

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0, \text{ we have}$$

$$R = y^2, \quad S = -2xy, \quad T = x^2.$$

The quadratic equation $R\lambda^2 + S\lambda + T = 0$ is therefore given by

$$y^2 \lambda^2 - 2xy\lambda + x^2 = 0 \quad \text{or} \quad (y\lambda - x)^2 = 0.$$

$$\therefore \quad \lambda = \frac{x}{y}, \frac{x}{y} \text{ (equal roots).}$$

The equation $\frac{dy}{dx} + \lambda = 0$ becomes $\frac{dy}{dx} + \frac{x}{y} = 0$

or $\quad y\,dy + x\,dx = 0.$

Integrating, we get $x^2 + y^2 = \text{constant}.$

To change the independent variables x, y to u, v, we take

$$u = x^2 + y^2.$$

We have to take v as some function of x and y independent of u, let $v = x^2 - y^2$.

Then $p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}=2x\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)$,

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial y}=2y\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right),$$

$$r=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left\{2x\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)\right\}$$

$$=2\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)+2x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)$$

$$=2\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)+2x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)\cdot\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)\cdot\frac{\partial v}{\partial x}\right]$$

$$=2\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)+4x^2\left[\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}\right]$$

$$s=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial x}\left[2y\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)\right]=2y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)$$

$$=2y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)\cdot\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)\cdot\frac{\partial v}{\partial x}\right]$$

$$=4xy\left[\frac{\partial^2 z}{\partial u^2}-\frac{\partial^2 z}{\partial v^2}\right]$$

and $t=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial y}\left[2y\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)\right]$

$$=2\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)+2y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)$$

$$=2\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)+2y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)\cdot\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)\cdot\frac{\partial v}{\partial y}\right]$$

$$=2\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)+4y^2\left[\frac{\partial^2 z}{\partial u^2}-2\frac{\partial^2 z}{\partial u\,\partial v}+\frac{\partial^2 z}{\partial v^2}\right].$$

Substituting these values in (1), it reduces to

$\frac{\partial^2 z}{\partial v^2}=0$, which is the required canonical form,

Integrating it w.r.t. v, we get

$$\frac{\partial z}{\partial v}=\phi_1(u).$$

Again integrating w.r.t. v, we get

$$z=v\,\phi_1(u)+\phi_2(u),$$

where ϕ_1 and ϕ_2 are arbitrary functions of u.

Hence the solution is

$$z=(x^2-y^2)\,\phi_1(x^2+y^2)+\phi_2(x^2+y^2).$$

Ex. 7. *Reduce the equation*

$$xyr-(x^2-y^2)s-xyt+py-qx=2(x^2-y^2)$$

to canonical form and hence solve it.

Sol. The given equation can be written as

$$xyr-(x^2-y^2)s-xyt+py-qx-2(x^2-y^2)=0. \quad \ldots(1)$$

Comparing the equation (1) with

$Rr+Ss+Tt+f(x, y, z, p, q)=0$, we have

$R=xy$, $S=-(x^2-y^2)$, $T=-xy$.

The quadratic equation $R\lambda^2+S\lambda+T=0$ therefore becomes

$$xy\lambda^2-(x^2-y^2)\lambda-xy=0 \text{ or } (y\lambda-x)(x\lambda+y)=0$$

$$\Rightarrow \quad \lambda=\frac{x}{y},\ -\frac{y}{x} \text{ (distinct roots).}$$

The equations

$\frac{dy}{dx}+\lambda_1=0$ and $\frac{dy}{dx}+\lambda_2=0$ become

$$\frac{dy}{dx}+\frac{x}{y}=0 \text{ and } \frac{dy}{dx}-\frac{y}{x}=0.$$

These on integration give

$$x^2+y^2=\text{constant and } y/x=\text{constant},$$

so that to change the independent variables from x, y to u, v, we take

$$u=x^2+y^2 \text{ and } v=y/x.$$

$$\therefore \quad p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=2x\frac{\partial z}{\partial u}-\frac{y}{x^2}\frac{\partial z}{\partial v},$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=2y\frac{\partial z}{\partial u}+\frac{1}{x}\frac{\partial z}{\partial v},$$

$$r=\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(2x\frac{\partial z}{\partial u}-\frac{y}{x^2}\frac{\partial z}{\partial v}\right)$$

$$=2\frac{\partial z}{\partial u}+2x\left\{\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)\right\}+\frac{2y}{x^3}\frac{\partial z}{\partial v}-\frac{y}{x^2}\left\{\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)\right\}$$

$$=2\frac{\partial z}{\partial u}+2x\left\{\frac{\partial^2 z}{\partial u^2}\frac{\partial u}{\partial x}+\frac{\partial^2 z}{\partial u\,\partial v}\cdot\frac{\partial v}{\partial x}\right\}+\frac{2y}{x^3}\frac{\partial z}{\partial v}-\frac{y}{x^2}\left\{\frac{\partial^2 z}{\partial u\,\partial v}\frac{\partial u}{\partial x}+\frac{\partial^2 z}{\partial v^2}\frac{\partial v}{\partial x}\right\}$$

$$=4x^2\frac{\partial^2 z}{\partial u^2}-4\frac{y}{x}\frac{\partial^2 z}{\partial u\,\partial v}+\frac{y^2}{x^4}\frac{\partial^2 z}{\partial v^2}+2\frac{\partial z}{\partial u}+\frac{2y}{x^3}\frac{\partial z}{\partial v},$$

$$s=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial x}\left(2y\frac{\partial z}{\partial u}+\frac{1}{x}\frac{\partial z}{\partial v}\right)$$

$$=4xy\frac{\partial^2 z}{\partial u^2}+\left(2-\frac{2y^2}{x^2}\right)\frac{\partial^2 z}{\partial u\,\partial v}-\frac{y}{x^3}\frac{\partial^2 z}{\partial v^2}-\frac{1}{x^2}\frac{\partial z}{\partial v}$$

and
$$t=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial y}\left(2y\frac{\partial z}{\partial u}+\frac{1}{x}\frac{\partial z}{\partial v}\right)$$

$$=4y^2\frac{\partial^2 z}{\partial u^2}+\frac{4y}{x}\frac{\partial^2 z}{\partial u\,\partial v}+\frac{1}{x^2}\frac{\partial^2 z}{\partial v^2}+2\frac{\partial z}{\partial u}$$

Substituting these values in (1), it reduces to

$$\frac{\partial^2 z}{\partial u\,\partial v}=\frac{v^2-1}{(v^2+1)^2}$$

which is the required canonical form.

Integrating it w.r.t. 'v', we get

$$\frac{\partial z}{\partial u}=\int\frac{v^2-1}{(v^2+1)^2}\,dv+\phi_1(u).$$

Now $\displaystyle\int\frac{v^2-1}{(v^2+1)^2}\,dv=\int\frac{v^2+1-2}{(v^2+1)^2}\,dv=\int\frac{dv}{v^2+1}-\int\frac{2dv}{(v^2+1)^2}$.

Let $\displaystyle I=\int\frac{dv}{v^2+1}$.

Integrating by parts taking unity as the second function

$$\int\frac{dv}{v^2+1}=\frac{v}{v^2+1}+\int\frac{2v^2}{(v^2+1)^2}\,dv$$

$$=\frac{v}{v^2+1}+2\int\frac{dv}{v^2+1}-2\int\frac{dv}{(v^2+1)^2}.$$

$$\therefore\quad\int\frac{dv}{v^2+1}-2\int\frac{dv}{(v^2+1)^2}=-\frac{v}{v^2+1}.$$

$$\therefore\quad\frac{\partial z}{\partial u}=-\frac{v}{v^2+1}+\phi_1(u).$$

Now integrating it w.r.t. 'u', we get

$$z=-\frac{uv}{v^2+1}+\psi_1(u)+\psi_2(v),$$

where ψ_1 and ψ_2 are arbitrary functions.

Hence the required solution is

$$z=-xy+\psi_1(x^2+y^2)+\psi_2(y/x).$$

Ex. 8. *Reduce the equation*

$$x^2r-2xys+y^2t-xp+3yq=8y/x$$

to canonical form and hence solve it.

Sol. The given equation can be written as

$$x^2r-2xys+y^2t-xp+3yq-(8y/x)=0 \quad ...(1)$$

Comparing the equation (1) with

$$Rr+Ss+Tt+f(x,y,z,p,q)=0,$$

we have

$$R=x^2,\quad S=-2xy,\quad T=y^2.$$

The quadratic equation $R\lambda^2+S\lambda+T=0$ is, therefore given by

$$x^2\lambda^2-2xy\lambda+y^2=0\quad\text{or}\quad(x\lambda-y)^2=0,$$

$$\therefore\quad\lambda=\frac{y}{x},\frac{y}{x}\ \text{(equal roots)}.$$

The equation $\dfrac{dy}{dx}+\lambda=0$ becomes

$$\frac{dy}{dx}+\frac{y}{x}=0$$

or
$$ydx+xdy=0\ \Rightarrow\ xy=\text{constant}.$$

To change the independent variables x, y to u, v, we take

$$u=xy.$$

We have to take v as some function of x and y independent of u, let $v=y/x$.

Then
$$p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=y\frac{\partial z}{\partial u}-\frac{y}{x^2}\frac{\partial z}{\partial v},$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=x\frac{\partial z}{\partial u}+\frac{1}{x}\frac{\partial z}{\partial v},$$

$$r=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(y\frac{\partial z}{\partial u}-\frac{y}{x^2}\frac{\partial z}{\partial v}\right)$$

$$=y^2\frac{\partial^2 z}{\partial u^2}-\frac{2y^2}{x^2}\frac{\partial^2 z}{\partial u\,\partial v}+\frac{y^2}{x^4}\frac{\partial^2 z}{\partial v^2}+\frac{2y}{x^3}\frac{\partial z}{\partial v},$$

$$s=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial u}+\frac{1}{x}\frac{\partial z}{\partial v}\right)$$

$$=xy\frac{\partial^2 z}{\partial u^2}-\frac{y}{x^3}\frac{\partial^2 z}{\partial v^2}-\frac{1}{x^2}\frac{\partial z}{\partial v}+\frac{\partial z}{\partial u}$$

and
$$t=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial y}\left(x\frac{\partial z}{\partial u}+\frac{1}{x}\frac{\partial z}{\partial v}\right)$$

$$=x^2\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\,\partial v}+\frac{1}{x^2}\frac{\partial^2 z}{\partial v^2}.$$

Substituting these values in (1), it reduces to

$$v\frac{\partial^2 z}{\partial v^2}+2\frac{\partial z}{\partial v}=2, \qquad \ldots(2)$$

which is the required canonical form.

Let $\dfrac{\partial z}{\partial v}=Z$. Then (2) becomes

$$\frac{\partial Z}{\partial v}+\frac{2}{v}Z=\frac{2}{v}, \text{ which is linear;}$$

$$\text{I.F.}=e^{\int(2/v)\,dv}=e^{2\log v}=v^2,$$

$$\therefore \quad v^2 Z=\int\frac{2}{v}\,v^2\,dv+\phi_1(u)=v^2+\phi_1(u).$$

$$\therefore \quad Z=\frac{\partial z}{\partial v}=1+\frac{1}{v^2}\phi_1(u).$$

Integrating this again w.r.t. 'v', we get

$$z=v-\frac{1}{v}\phi_1(u)+\phi_2(u),$$

where ϕ_1 and ϕ_2 are arbitrary functions of u.

Hence the solution is

$$z=\frac{y}{x}-\frac{x}{y}\phi_1(xy)+\phi_2(xy)$$

or
$$z=\frac{y}{x}-\frac{x^2}{xy}\phi_1(xy)+\phi_2(xy)$$

or
$$z=\frac{y}{x}+x^2\psi_1(xy)+\phi_2(xy),$$

$$\text{where } \psi_1(xy)=-\frac{1}{xy}\phi_1(xy).$$

Ex. 9. *Reduce the equation*

$$x^2(y-1)r-x(y^2-1)s+y(y-1)t+xyp-q=0 \qquad \ldots(1)$$

to canonical form, and hence solve it.

Sol. Comparing the equation (1) with

$$Rr+Ss+Tt+f(x, y, z, p, q)=0, \text{ we have}$$

$$R=x^2(y-1),\ S=-x(y^2-1),\ T=y(y-1).$$

The quadratic equation $R\lambda^2+S\lambda+T=0$ therefore becomes

$$x^2(y-1)\lambda^2-x(y^2-1)\lambda+y(y-1)=0$$

or
$$x^2\lambda^2-x(y+1)\lambda+y=0$$

or
$$(x\lambda-y)(x\lambda-1)=0.$$

$$\therefore \quad \lambda=\frac{y}{x}, \frac{1}{x}.$$

The equations $\dfrac{dy}{dx}+\lambda_1=0$ and $\dfrac{dy}{dx}+\lambda_2=0$ become

$$\frac{dy}{dx}+\frac{y}{x}=0 \text{ and } \frac{dy}{dx}+\frac{1}{x}=0.$$

These on integration give

$$xy=\text{constant and } xe^y=\text{constant},$$

so that to change the independent variables from x, y to u, v, we take

$$u=xy \text{ and } v=xe^y,$$

$$\therefore \quad p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=y\frac{\partial z}{\partial u}+e^y\frac{\partial z}{\partial v},$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=x\frac{\partial z}{\partial u}+xe^y\frac{\partial z}{\partial v},$$

$$r=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(y\frac{\partial z}{\partial u}+e^y\frac{\partial z}{\partial v}\right)$$

$$=y^2\frac{\partial^2 z}{\partial u^2}+2ye^y\frac{\partial^2 z}{\partial u\,\partial v}+e^{2y}\frac{\partial^2 z}{\partial v^2},$$

$$s=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial u}+xe^y\frac{\partial z}{\partial v}\right)$$

$$=xy\frac{\partial^2 z}{\partial u^2}+xe^y(1+y)\frac{\partial^2 z}{\partial u\,\partial v}+xe^{2y}\frac{\partial^2 z}{\partial v^2}+e^y\frac{\partial z}{\partial v}+\frac{\partial z}{\partial u}$$

and
$$t=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial y}\left(x\frac{\partial z}{\partial u}+xe^y\frac{\partial z}{\partial v}\right)$$

$$=x^2\frac{\partial^2 z}{\partial u^2}+2x^2e^y\frac{\partial^2 z}{\partial u\,\partial v}+x^2e^{2y}\frac{\partial^2 z}{\partial v^2}+xe^y\frac{\partial z}{\partial v}.$$

Substituting these values in (1), it reduces to

$$\frac{\partial^2 z}{\partial u\,\partial v}=0,$$

which is the required canonical form.

Integrating it w.r.t. 'v', we get

$$\frac{\partial z}{\partial u}=\phi(u).$$

Now integrating w.r.t. 'u', we get

$$z=\phi_1(u)+\phi_2(v),$$

where ϕ_1 and ϕ_2 are arbitrary functions.

Hence the required solution is

$$z=\phi_1(xy)+\phi_2(xe^y).$$

Ex. 10. *Reduce the equation*

$$x(xy-1)r-(x^2y^2-1)s+y(xy-1)t+(x-1)p+(y-1)q=0 \qquad \ldots(1)$$

to canonical form and hence solve it.

Sol. Comparing the equation (1) with

$$Rr+Ss+Tt+f(x, y, z, p, q)=0, \text{ we have}$$

$$R=x(xy-1),\ S=-(x^2y^2-1),\ T=y(xy-1).$$

The quadratic equation $R\lambda^2+S\lambda+T=0$ therefore becomes

$$x(xy-1)\lambda^2-(x^2y^2-1)\lambda+y(xy-1)=0$$

or $$x\lambda^2-(xy+1)\lambda+y=0 \quad \text{or} \quad (x\lambda-1)(\lambda-y)=0.$$

$$\therefore \quad \lambda=\frac{1}{x},\ y.$$

The equations $\dfrac{dy}{dx}+\lambda_1=0$ and $\dfrac{dy}{dx}+\lambda_2=0$ become

$$\frac{dy}{dx}+\frac{1}{x}=0 \quad \text{and} \quad \frac{dy}{dx}+y=0.$$

These on integration give

$$xe^y=\text{constant and } ye^x=\text{constant},$$

so that to change the independent variables from x, y to u, v, we have

$$u=xe^y \quad \text{and} \quad v=ye^x.$$

$$p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=e^y\frac{\partial z}{\partial u}+ye^x\frac{\partial z}{\partial v},$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=xe^y\frac{\partial z}{\partial u}+e^x\frac{\partial z}{\partial v},$$

$$r=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(e^y\frac{\partial z}{\partial u}+ye^x\frac{\partial z}{\partial v}\right)$$

$$=e^{2y}\frac{\partial^2 z}{\partial u^2}+2ye^{x+y}\frac{\partial^2 z}{\partial u\,\partial v}+y^2e^{2x}\frac{\partial^2 z}{\partial v^2}+ye^x\frac{\partial z}{\partial v},$$

$$s=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial x}\left(xe^y\frac{\partial z}{\partial u}+e^x\frac{\partial z}{\partial v}\right)$$

$$=xe^{2y}\frac{\partial^2 z}{\partial u^2}+(xy+1)e^{x+y}\frac{\partial^2 z}{\partial u\,\partial v}+ye^{2x}\frac{\partial^2 z}{\partial v^2}$$

$$+e^y\frac{\partial z}{\partial u}+e^x\frac{\partial z}{\partial v},$$

and $$t=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial y}\left(xe^y\frac{\partial z}{\partial u}+e^x\frac{\partial z}{\partial v}\right)$$

$$=x^2e^{2y}\frac{\partial^2 z}{\partial u^2}+2xe^{x+y}\frac{\partial^2 z}{\partial u\,\partial v}+e^{2x}\frac{\partial^2 z}{\partial v^2}+xe^y\frac{\partial z}{\partial u}.$$

Substituting these values in (1), it reduces to

$$\frac{\partial^2 z}{\partial u\,\partial v}=0,$$

which is the required canonical form.

Integrating it w.r.t. 'v', we get

$$\frac{\partial z}{\partial u}=\phi(u).$$

Now integrating w.r.t. 'u', we get

$$z=\phi_1(u)+\phi_2(v);$$

where ϕ_1 and ϕ_2 are arbitrary functions.

Hence the solution is

$$z=\phi_1(xe^y)+\phi_2(ye^x).$$

Sol. Monge's equations are $y^2 dy^2 + 2y\,dy\,dx + dx^2 = 0$ ...(i)

and $y^2 dp\,dy + dq\,dx - (p + 6y)\,dy\,dx = 0$ ...(ii)

(i) gives $(y\,dy + dx)^2 = 0$ i.e. $y^2 + 2x = c$. ...(iii)

Putting $y\,dy = -dx$ in (ii), we get

$$y\,dp - dq + (p + 6y)dy = 0 \quad \text{or} \quad (y\,dp + p\,dy) - dq + 6y\,dy = 0$$

whose integral is $py - q + 3y^2 = a$

Combining this with (iii), we get the integral $py - q + 3y^2 = \phi(y^2 + 2x)$

The subsidiary equations for this Lagrange's linear equation are

$$\frac{dx}{y} = \frac{dy}{-1} = \frac{dz}{\phi(y^2 + 2x) - 3y^2}$$

From the first two equations, we have $y^2 + 2x = c$

Using this, we have $dz + [\phi(c) - 3y^2]\,dy = 0$

whose solution is $z + y\phi(c) - y^3 = b$.

Hence the required solution is $z = y^3 - y\phi(y^2 + 2x) + \psi(y^2 + 2x)$.

Problems 17.11

Solve :

1. $(q + 1)s = (p + 1)t$.
2. $r - t\cos^2 x + p\tan x = 0$. (Osmania, 1999)
3. $2x^2r - 5xys + 2y^2t + 2(px + qy) = 0$.
4. $xy(t - r) + (x^2 - y^2)(s - 2) = py - qx$.
5. $q^2r - 2pqs + p^2t = pq^2$.
6. $(1 + q)^2r - 2(1 + p + q + pq)s + (1 + p)^2t = 0$.

Set - 2

18

Applications of Partial Differential Equations

18.1 INTRODUCTION

In physical problems, we always seek a solution of the differential equation which satisfies some specified conditions known as the boundary conditions. The differential equation together with these boundary conditions, constitute a *boundary value problem*.

In problems involving ordinary differential equations, we may first find the general solution and then determine the arbitrary constants from the initial values. But the same process is not applicable to problems involving partial differential equations for the general solution of a partial differential equation contains arbitrary functions which are difficult to adjust so as to satisfy the given boundary conditions. Most of the boundary value problems involving linear partial differential equations can be solved by the following method.

18.2 METHOD OF SEPARATION OF VARIABLES

It involves a solution which breaks up into a product of functions each of which contains only one of the variables. The following example explains this method :

Example 18.1. *Solve (by the method of separation of variables) :*

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0. \qquad \textit{(Mysore, 1997 ; Osmania, 1995)}$$

Sol. Assume the trial solution $z = X(x)Y(y)$ ...(i)

where X is a function of x alone and Y that of y alone.

Substituting this value of z in the given equation, we have

$$X''Y - 2X'Y + XY' = 0, \quad \text{where } X' = \frac{dX}{dx},\ Y' = \frac{dY}{dy} \text{ etc.}$$

Separating the variables, we get $\dfrac{X'' - 2X'}{X} = -\dfrac{Y'}{Y}$ ...(ii)

Since x and y are independent variables, therefore, (ii) can only be true if each side is equal to the same constant, a (say).

$$\therefore \quad \frac{X'' - 2X'}{X} = a, \text{ i.e. } X'' - 2X' - aX = 0 \qquad ...(iii)$$

and $\quad -Y'/Y = a$, i.e. $Y' + aY = 0$ ...(iv)

To solve the ordinary linear equation (iii), the auxiliary equation is

$$m^2 - 2m - a = 0, \text{ whence } m = 1 \pm \sqrt{(1 + a)}.$$

∴ The solution of (iii) is $X = c_1 e^{(1 + \sqrt{(1+a)})x} + c_2 e^{(1 - \sqrt{(1+a)})x}$

and the solution of (iv) is $Y = c_3 e^{ay}$

Substituting these values of X and Y in (i), we get

[illegible]

i.e. [illegible]

which is the required solution.

Obs. In practical problems, [illegible] conditions [illegible] are to be determined from the given boundary [illegible]

Example 18.2. *Using the method of separation of variables, solve* $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ *where* $u(x, 0) = 6e^{-3x}$. (V.T.U., 2001; S. Patel, 1997)

Sol. Assume the solution $u(x, t) = X(x)\, T(t)$

Substituting in the given equation, we have

$$X'T = 2XT' + XT \quad \text{or} \quad (X' - X)\, T = 2XT'$$

or

$$\frac{X' - X}{2X} = \frac{T'}{T} = k \quad \text{(say)}$$

∴ $X' - X - 2kX = 0$ or $\frac{X'}{X} = 1 + 2k$ ...(i)

and $\frac{T'}{T} = k$ ...(ii)

Solving (i), $\log X = (1 + 2k)x + \log c$ or $X = ce^{(1 + 2k)x}$

From (ii), $\log T = kt + \log c'$ or $T = c'e^{kt}$

Thus $u(x, t) = XT = cc' e^{(1 + 2k)x} e^{kt}$ ...(iii)

Now $6e^{-3x} = u(x, 0) = cc' e^{(1 + 2k)x}$

∴ $cc' = 6$ and $1 + 2k = -3$ or $k = -2$

Substituting these values in (iii), we get

$u = 6e^{-3x} e^{-2t}$ i.e., $u = 6e^{-(3x + 2t)}$, which is the required solution.

PROBLEMS 18.1

Solve the following equations by the method of separation of variables:

1. $py^3 + qx^2 = 0$ (Madurai, 1997)
2. $x^2 \frac{\partial u}{\partial x} + y^2 \frac{\partial u}{\partial y} = 0$.
3. $4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u$, given $u = 3e^{-y} - e^{-5y}$ when $x = 0$. (V.T.U., 2003 S)
4. $3\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$, $u(x, 0) = 4e^{-x}$. (Osmania, 1999)
5. Find a solution of the equation $\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial y} + 2u$ in the form $u = f(x)\, g(y)$. Solve the equation subject to the conditions $u = 0$ and $\partial u/\partial x = 1 + e^{-3y}$, when $x = 0$ for all values of y. (Andhra, 2000; Nagpur, 1997)

18.3 PARTIAL DIFFERENTIAL EQUATIONS OF ENGINEERING [illegible]

A number of problems in engineering give rise to the following well-known partial differential equations:

(i) *Wave equation*: $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$.

(ii) *One dimensional heat flow equation*: $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$.

(iii) *Two dimensional heat flow equation* which in steady state becomes the two dimensional *Laplace's equation*: $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$.

(iv) *Transmission line equations.*

(v) *Vibrating membrane.* Two dimensional wave equation.

(vi) *Laplace's equation in three dimensions.*

Besides these, the partial differential equations frequently occur in the theory of Elasticity and Hydraulics.

Starting with the method of separation of variables, we find their solutions subject to specific boundary conditions and the combination of such solutions gives the desired solution. Quite often a certain condition is not applicable. In such cases, the most general solution is written as the sum of the particular solutions already found and the constants are determined using Fourier series so as to satisfy the remaining conditions.

18.4 VIBRATIONS OF A STRETCHED STRING — WAVE EQUATION

Consider a tightly stretched elastic string of length l and fixed ends A and B and subjected to constant tension T (Fig. 18.1). The tension T will be considered to be large as compared to the weight of the string so that the effects of gravity are negligible.

Let the string be released from rest and allowed to vibrate. We shall study the subsequent motion of the string, with no external forces acting on it, assuming that each point of the string makes small vibrations at right angles to the equilibrium position AB, of the string entirely in one plane.

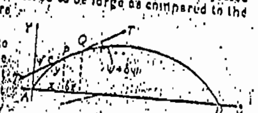

Fig. 18.1

Taking the end A as the origin, AB as the x-axis and AY perpendicular to it as the y-axis; so that the motion takes place entirely in the xy-plane. Fig. 18.1 shows the string in the position APB at time t. Consider the motion of the element PQ of the string between its points $P(x, y)$ and $Q(x + \delta x, y + \delta y)$, where the tangents make angles ψ and $\psi + \delta\psi$ with the x-axis. Clearly the element is moving upwards with the acceleration $\partial^2 y/\partial t^2$. Also the vertical component of the force acting on this element

$$= T \sin(\psi + \delta\psi) - T \sin\psi = T[\sin(\psi + \delta\psi) - \sin\psi]$$

$$= T[\tan(\psi + \delta\psi) - \tan\psi], \text{ since } \psi \text{ is small} = T\left[\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right]$$

If m be the mass per unit length of the string, then by Newton's second law of motion, we have

$$m\,\delta x \cdot \frac{\partial^2 y}{\partial t^2} = T\left[\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_x\right]$$

i.e.,
$$\frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\left[\frac{\left(\frac{\partial y}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial y}{\partial x}\right)_x}{\delta x}\right]$$

Taking limits as $Q \to P$, i.e. $\delta x \to 0$, we have $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$, where $c^2 = \frac{T}{m}$ ...(1)

This is the partial differential equation giving the transverse vibrations of the string. It is also called the one dimensional *wave equation*.

(2) Solution of the wave equation. Assume that a solution of (1) is of the form $y = X(x)T(t)$ where X is a function of x and T is a function of t only.

Then $\frac{\partial^2 y}{\partial t^2} = X \cdot T''$ and $\frac{\partial^2 y}{\partial x^2} = X'' \cdot T$

Substituting these in (1), we get $XT'' = c^2 X''T$ i.e., $\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T}$ ...(2)

Clearly the left side of (2) is a function of x only and the right side is a function of t only. Since x and t are independent variables, (2) can hold good if each side is equal to a constant k (say). Then (2) leads to the ordinary differential equations:

$$\frac{d^2X}{dx^2} - kX = 0 \quad ...(3)$$

and
$$\frac{d^2T}{dt^2} - kc^2T = 0 \quad ...(4)$$

Solving (3) and (4), we get

(i) *When k is positive and $= p^2$, say* $X = c_1e^{px} + c_2e^{-px}$; $T = c_3e^{cpt} + c_4e^{-cpt}$.

(ii) *When k is negative and $= -p^2$ say.* $X = c_5 \cos px + c_6 \sin px$; $T = c_7 \cos cpt + c_8 \sin cpt$.

(iii) *When k is zero.* $X = c_9x + c_{10}$; $T = c_{11}t + c_{12}$.

Thus the various possible solutions of wave-equation (1) are

$$y = (c_1e^{px} + c_2e^{-px})(c_3e^{cpt} + c_4e^{-cpt}) \quad ...(5)$$
$$y = (c_5 \cos px + c_6 \sin px)(c_7 \cos cpt + c_8 \sin cpt) \quad ...(6)$$
$$y = (c_9x + c_{10})(c_{11}t + c_{12}) \quad ...(7)$$

Of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem. As we will be dealing with problems on vibrations, y must be a periodic function of x and t. Hence their solution must involve trigonometric terms. Accordingly the solution given by (6), i.e., of the form $y = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt)$ ...(8)

is the only suitable solution of the wave equation.

Example 18.3. *A string is stretched and fastened to two points l apart. Motion is started by displacing the string in the form $y = a \sin (\pi x/l)$ from which it is released at time $t = 0$. Show that the displacement of any point at a distance x from one end at time t is given by*

$$y(x, t) = a \sin (\pi x/l) \cos (\pi ct/l).$$

(Assam, 1999)

(Andhra, 1998 ; Mysore, 1995)

Sol. The vibration of the string is given by $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ ...(i)

As the end points of the string are fixed, for all time,

$$y(0, t) = 0 \quad ...(ii)$$

and
$$y(l, t) = 0 \quad ...(iii)$$

Since the initial transverse velocity of any point of the string is zero,

therefore,
$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \quad ...(iv)$$

Also
$$y(x, 0) = a \sin (\pi x/l) \quad ...(v)$$

Now we have to solve (i) subject to the *boundary conditions* (ii) and (iii) and *initial conditions* (iv) and (v). Since the vibration of the string is periodic, therefore, the solution of (i) is of the form

$$y(x, t) = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt) \quad ...(vi)$$

By (ii), $y(0, t) = C_1(C_3 \cos cpt + C_4 \sin cpt) = 0$

For this to be true for all time, $C_1 = 0$.

Hence, $y(x, t) = C_2 \sin px(C_3 \cos cpt + C_4 \sin cpt)$ ...(vii)

and, $\frac{\partial y}{\partial t} = C_2 \sin px\,[C_3(-cp \cdot \sin cpt) + C_4 (cp \cdot \cos cpt)]$

$\therefore$ By (iv), $\left(\frac{\partial y}{\partial t}\right)_{t=0} = C_2 \sin px \cdot (C_4 cp) = 0$, whence $C_2C_4cp = 0$.

If $C_2 = 0$, (vii) will lead to the trivial solution $y(x, t) = 0$.

$\therefore$ the only possibility is that $C_4 = 0$.

Thus (vii) becomes $y(x, t) = C_2C_3 \sin px \cos cpt$ ...(viii)

$\therefore$ By (iii), $y(l, t) = C_2C_3 \sin pl \cos cpt = 0$ for all t.

Since C_2 and $C_3 \neq 0$, we have $\sin pl = 0$. $\therefore pl = n\pi$, i.e. $p = n\pi/l$, where n is an integer.

Hence (i) reduces to $y(x, t) = C_2C_3 \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}$.

[These are the solutions of (i) satisfying the boundary conditions. These functions are called the eigen functions corresponding to the eigen-values $\lambda_n = cn\pi/l$ of the vibrating string. The set of values $\lambda_1, \lambda_2, \lambda_3, \ldots$ is called its spectrum.]

Finally, imposing the last condition (v), we have $y(x, 0) = C_2C_3 \sin \frac{n\pi x}{l} = a \sin \frac{\pi x}{l}$

which will be satisfied by taking $C_2C_3 = a$ and $n = 1$.

Hence the required solution is $y(x, t) = a \sin \frac{\pi x}{l} \cos \frac{\pi ct}{l}$ ...(ix)

Obs. We have from (ix) $\frac{\partial^2 y}{\partial t^2} = -a\left(\frac{\pi c}{l}\right)^2 \sin \frac{\pi x}{l} \cos \frac{\pi ct}{l} = -\left(\frac{\pi c}{l}\right)^2 y$.

This shows that the motion of each point $y(x, t)$ of the string is simple harmonic with period $= 2\pi/(\pi c/l)$, i.e., $2l/c$.

Thus we can look upon (ix) as a sine wave $y = y_0 \sin (\pi x/l)$ of wave length l, wave-velocity c and amplitude $y_0 = a \cos (\pi ct/l)$ which varies harmonically with time t. Whatever t may be, $y = 0$ when $x = 0, l, 2l, 3l$ etc. and these points called nodes remain undisturbed during wave motion. Thus (ix) represents a *stationary sine wave* of varying amplitude whose frequency is $c/2l$. Such waves often occur in electrical and mechanical vibratory systems.

Example 18.4. A tightly stretched string of length l with fixed ends is initially in equilibrium position. It is set vibrating by giving each point a velocity $v_0 \sin^3 \pi x/l$. Find the displacement $y(x, t)$. (Madras, 1995)

Sol. The equation of the vibrating string is $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ ...(i)

The boundary conditions are $y(0, t) = 0$, $y(l, t) = 0$ ...(ii)

Also the initial conditions are $y(x, 0) = 0$ ...(iii)

and $\left(\frac{\partial y}{\partial t}\right)_{t=0} = v_0 \sin^3 \frac{\pi x}{l}$ ...(iv)

Since the vibration of the string is periodic, therefore, the solution of (i) is of the form

$$y(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$$

By (ii), $y(0, t) = c_1(c_3 \cos cpt + c_4 \sin cpt) = 0$.

For this to be true for all time $c_1 = 0$.

$\therefore$ $y(x, t) = c_2 \sin px (c_3 \cos cpt + c_4 \sin cpt)$

Also $y(l, t) = c_2 \sin pl (c_3 \cos cpt + c_4 \sin cpt) = 0$ for all t.

This gives $pl = n\pi$ or $p = \frac{n\pi}{l}$, n being an integer.

Thus $y(x, t) = c_2 \sin \frac{n\pi x}{l}\left(c_3 \cos \frac{cn\pi}{l} t + c_4 \sin \frac{cn\pi}{l} t\right)$

By (iii), $0 = c_2 c_3 \sin \frac{n\pi x}{l}$ for all x i.e. $c_2 c_3 = 0$.

$\therefore$ $y(x, t) = b_n \sin \frac{n\pi x}{l} \sin \frac{cn\pi t}{l}$ where $b_n = c_2 c_4$

Adding all such solutions, the general solution of (i) is

$$y(x, t) = \Sigma b_n \sin \frac{n\pi x}{l} \sin \frac{cn\pi t}{l}$$

Now $\frac{\partial y}{\partial t} = \Sigma b_n \sin \frac{n\pi x}{l} \cdot \frac{cn\pi}{l} \cos \frac{cn\pi t}{l}$ ...(v)

By (iv), $v_0 \sin^3 \frac{\pi x}{l} = \left(\frac{\partial y}{\partial t}\right)_{t=0} = \Sigma \frac{cn\pi}{l} b_n \sin \frac{n\pi x}{l}$

or $\frac{v_0}{4}\left(3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l}\right) = \Sigma \frac{cn\pi}{l} b_n \sin \frac{n\pi x}{l}$ $[\because \sin 3\theta = 3 \sin \theta - 4 \sin^3 \theta]$

$$= \frac{c\pi}{l} b_1 \sin \frac{\pi x}{l} + \frac{2c\pi}{l} b_2 \sin \frac{2\pi x}{l} + \frac{3c\pi}{l} b_3 \sin \frac{3\pi x}{l} + \ldots$$

Equating coefficients from both sides, we get

$$\frac{3v_0}{4} = \frac{c\pi}{l} b_1, \quad 0 = \frac{2c\pi}{l} b_2, \quad -\frac{v_0}{4} = \frac{3c\pi}{l} b_3, \ldots$$

$$b_1 = \frac{3lv_0}{4c\pi}, \quad b_3 = -\frac{lv_0}{12c\pi}, \quad b_2 = b_4 = b_5 = \ldots = 0$$

Substituting in (v), the desired solution is

$$y = \frac{lv_0}{12c\pi}\left(9 \sin \frac{\pi x}{l} \sin \frac{c\pi t}{l} - \sin \frac{3\pi x}{l} \sin \frac{3c\pi t}{l}\right)$$

Example 18.5. The points of trisection of a string are pulled aside through the same distance on opposite sides of the position of equilibrium and the string is released from rest. Derive an expression for the displacement of the string at subsequent time and show that the mid-point of the string always remains at rest.

Sol. Let B and C be the points of the trisection of the string $OA (= l)$. Initially the string is held in the form $OB'C'A$, where $BB' = CC' = a$ (say).

The displacement $y(x, t)$ of any point of the string is given by

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \ldots(i)$$

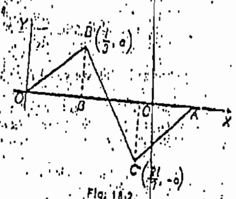

Fig. 18.2.

and the boundary conditions are

$y(0, t) = 0$ ...(ii)

$y(l, t) = 0$ ...(iii)

$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$ ...(iv)

The remaining condition is that at $t = 0$, the string rests in the form of the broken line $OB'C'A$. The equation of OB' is $y = (3a/l)x$;

the equation of $B'C'$ is $y - a = \frac{-2a}{(l/3)}\left(x - \frac{l}{3}\right)$, i.e. $y = \frac{3a}{l}(l - 2x)$

and the equation of $C'A$ is $y = \frac{3a}{l}(x - l)$

Hence the fourth boundary condition is $y(x, 0) = \frac{3a}{l}x$, $0 \le x \le \frac{l}{3}$

$$= \frac{3a}{l}(l - 2x), \quad \frac{l}{3} \le x \le \frac{2l}{3}$$

$$= \frac{3a}{l}(x - l), \quad \frac{2l}{3} \le x \le l \quad \ldots(v)$$

As in example 18.4, the solution of (i) satisfying the boundary conditions (ii), (iii) and (iv) is

$$y(x, t) = b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}$$

Adding all such solutions, the most general solution of (i) is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \quad \ldots(vi)$$

Putting $t = 0$, we have $y(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$ ...(vii)

In order that the condition (v) may be satisfied, (v) and (vii) must be same. This requires the expansion of $y(x, 0)$ into a Fourier half-range sine series in the interval $(0, l)$.

$\therefore$ By (1) of § 10.7,

$$b_n = \frac{2}{l}\left[\int_0^{l/3} \frac{3ax}{l} \sin \frac{n\pi x}{l} dx + \int_{l/3}^{2l/3} \frac{3a}{l}(l - 2x) \sin \frac{n\pi x}{l} dx + \int_{2l/3}^{l} \frac{3a}{l}(x - l) \sin \frac{n\pi x}{l} dx\right]$$

$$= \frac{6a}{l^2}\left[\left|x\left\{-\frac{\cos(n\pi x/l)}{(n\pi/l)}\right\} - 1\left\{-\frac{\sin(n\pi x/l)}{(n\pi/l)^2}\right\}\right|_0^{l/3}\right.$$

$$+ \left|(l-2x)\left\{-\frac{\cos(n\pi x/l)}{(n\pi/l)}\right\} - (-2)\left\{\frac{\sin(n\pi x/l)}{(n\pi/l)^2}\right\}\right|_{l/3}^{2l/3}$$

$$\left. + \left|(x-l)\left\{-\frac{\cos(n\pi x/l)}{(n\pi/l)}\right\} - (1)\left\{-\frac{\sin(n\pi x/l)}{(n\pi/l)^2}\right\}\right|_{2l/3}^{l}\right]$$

$$= \frac{6a}{l^2}\left[\left(-\frac{l^2}{3n\pi}\cos\frac{n\pi}{3} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{3}\right) + \frac{l^2}{3n\pi}\cos\frac{2n\pi}{3} - \frac{2l^2}{n^2\pi^2}\sin\frac{2n\pi}{3} + \frac{l^2}{3n\pi}\cos\frac{n\pi}{3}\right.$$

$$\left. + \frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{3} - \left(\frac{l^2}{3n\pi}\cos\frac{2n\pi}{3} + \frac{l^2}{n^2\pi^2}\sin\frac{2n\pi}{3}\right)\right]$$

$$= \frac{6a}{l^2}\cdot\frac{3l^2}{n^2\pi^2}\left(\sin\frac{n\pi}{3} - \sin\frac{2n\pi}{3}\right)$$

$$= \frac{18a}{n^2\pi^2}\sin\frac{n\pi}{3}\left[1 + (-1)^n\right] \qquad \left[\because \sin\frac{2n\pi}{3} = \sin\left(n\pi - \frac{n\pi}{3}\right) = -(-1)^n\sin\frac{n\pi}{3}\right]$$

Thus $b_n = 0$, when n is odd.

$$= \frac{36a}{n^2\pi^2}\sin\frac{n\pi}{3}, \text{ when } n \text{ is even.}$$

Hence (vi) gives

$$y(x, t) = \sum_{n=2,4,\ldots}^{\infty} \frac{36a}{n^2\pi^2}\sin\frac{n\pi}{3}\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \qquad [\text{Take } n = 2m]$$

$$= \frac{9a}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\sin\frac{2m\pi}{3}\sin\frac{2m\pi x}{l}\cos\frac{2m\pi ct}{l} \qquad \ldots(vii)$$

Putting $x = l/2$ in (vii), we find that the displacement of the midpoint of the string, i.e. $y(l/2, t) = 0$, because $\sin m\pi = 0$ for all integral values of m.

This shows that the midpoint of the string is always at rest.

(3) D'Alembert's solution of the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \qquad \ldots(1)$$

Let us introduce the new independent variables $u = x + ct$, $v = x - ct$ so that y becomes a function of u and v. Then $\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}$

and
$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right) = \frac{\partial}{\partial u}\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right) = \frac{\partial^2 y}{\partial u^2} + 2\frac{\partial^2 y}{\partial u\partial v} + \frac{\partial^2 y}{\partial v^2}$$

Similarly
$$\frac{\partial^2 y}{\partial t^2} = c^2\left(\frac{\partial^2 y}{\partial u^2} - 2\frac{\partial^2 y}{\partial u\partial v} + \frac{\partial^2 y}{\partial v^2}\right)$$

Substituting in (1), we get $\frac{\partial^2 y}{\partial u\partial v} = 0$ $\qquad \ldots(2)$

Integrating (2) w.r.t. v, we get $\frac{\partial y}{\partial u} = f(u)$ $\qquad \ldots(3)$

where $f(u)$ is an arbitrary function of u. Now integrating (3) w.r.t. u, we obtain

$$y = \int f(u)\,du + \psi(v)$$

where $\psi(v)$ is an arbitrary function of v. Since the integral is a function of u alone, we may denote it by $\phi(u)$. Thus

$$y = \phi(u) + \psi(v)$$

i.e.
$$y(x, t) = \phi(x + ct) + \psi(x - ct) \qquad \ldots(4)$$

This is the general solution of the wave equation (1).

Now to determine ϕ and ψ, suppose initially $y(x, 0) = f(x)$ and $\partial y(x, 0)/\partial t = 0$.

Differentiating (4) w.r.t. t, we get $\frac{\partial y}{\partial t} = c\phi'(x + ct) - c\psi'(x - ct)$

At $t = 0$, $\qquad \phi'(x) = \psi'(x)$

and $\qquad y(x, 0) = \phi(x) + \psi(x) = f(x)$ $\qquad \ldots(5)$

(5) gives, $\qquad \phi(x) = \psi(x) + k$ $\qquad \ldots(6)$

$\therefore$ (6) becomes $2\psi(x) + k = f(x)$

or
$$\psi(x) = \frac{1}{2}[f(x) - k] \text{ and } \phi(x) = \frac{1}{2}[f(x) + k].$$

Hence the solution of (4) takes the form

$$y(x, t) = \frac{1}{2}[f(x + ct) + k] + \frac{1}{2}[f(x - ct) - k] = f(x + ct) + f(x - ct) \qquad \ldots(7)$$

which is the *d'Alembert's solution** *of the wave equation* (1).

(Gulbarga, 1996 ; Calicut, 1991)

> **Obs.** The above solution gives a very useful method of solving partial differential equations by change of variables.

Example 18.6. *Find the deflection of a vibrating string of unit length having fixed ends with initial velocity zero and initial deflection $f(x) = k(\sin x - \sin 2x)$.* (Delhi, 1997)

Sol. By d'Alembert's method, the solution is

$$y(x, t) = \frac{1}{2}[f(x + ct) + f(x - ct)]$$

$$= \frac{1}{2}[k\{\sin(x + ct) - \sin 2(x + ct)\} + k\{\sin(x - ct) - \sin 2(x - ct)\}]$$

$$= k\{\sin x\cos ct - \sin 2x\cos 2ct\}$$

Also $\qquad y(x, 0) = k(\sin x - \sin 2x) = f(x)$

and $\qquad \partial y(x, 0)/\partial t = k(-c\sin x\sin ct + 2c\sin 2x\sin 2ct)_{t=0} = 0$

i.e. the given boundary conditions are satisfied.

PROBLEMS 18.2

1. Solve completely the equation $\partial^2 y/\partial t^2 = c^2\partial^2 y/\partial x^2$, representing the vibrations of a string of length l, fixed at both ends, given that $y(0, t) = 0$, $y(l, t) = 0$, $y(x, 0) = f(x)$ and $\partial y(x, 0)/\partial t = 0$, $0 < x < l$. (Osmania, 1990)

* See footnote of p. 351.

2. A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by $y = y_0 \sin^3 (\pi x/l)$. If it is released from rest from this position, find the displacement $y(x, t)$.
(V.T.U., 2003; J.N.T.U., 2002; Kottayam, 1999)

3. Find the solution of the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$, corresponding to the triangular initial deflection

$$f(x) = \frac{2k}{l} x \text{ when } 0 < x < \frac{l}{2}; = \frac{2k}{l}(l - x) \text{ when } \frac{l}{2} < x < l,$$

and initial velocity zero.

4. A tightly stretched string of length l has its ends fastened at $x = 0$, $x = l$. The mid-point of the string is then taken to height h and then released from rest in that position. Find the lateral displacement of a point of the string at time t from the instant of release.
(Madras, 2000)

5. A tightly stretched string with fixed end points at $x = 0$ and $x = 1$, is initially in a position given by

$$f(x) = \begin{cases} x, & 0 \le x \le \frac{1}{2} \\ 1 - x, & \frac{1}{2} \le x \le 1 \end{cases}$$

If it is released from this position with velocity c, perpendicular to the x-axis, show that the displacement $u(x, t)$ at any point x of the string at any time $t > 0$, is given by

$$u(x, t) = \frac{4\sqrt{2}}{\pi^2} \left[\sum_{n=1}^{\infty} \left\{ \frac{\sin [(4n-3)\pi x] \cos [(4n-3)\pi ct + \pi/4]}{(4n-3)^2} - \frac{\sin[(4n-1)\pi x] \cos [(4n-1)\pi ct + \pi/4]}{(4n-1)^2} \right\} \right]$$

6. A tightly stretched flexible string has its ends fixed at $x = 0$ and $x = l$. At time $t = 0$, the string is given a shape defined by $F(x) = \mu x(l - x)$, where μ is a constant, and then released. Find the displacement of any point x of the string at any time $t > 0$.
(Madras, 2003 S; Bhopal, 2002; Nagpur, 1997)

7. A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially at rest in its equilibrium position. If it is vibrating by giving to each of its points a velocity $\lambda x(l - x)$, find the displacement of the string at any distance x from one end at any time t.

8. A taut string of length 20 cms. fastened at both ends is displaced from its position of equilibrium, by imparting to each of its points an initial velocity given by

$$v = x \text{ in } 0 \le x \le 10$$
$$= 20 - x \text{ in } 10 \le x \le 20,$$

x being the distance from one end. Determine the displacement at any subsequent time.

9. Show that the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$, under the conditions

$$u(0, t) = 0, \; u(l, t) = 0 \text{ for all } t, \quad u(x, 0) = f(x), \; \left(\frac{\partial u}{\partial t}\right)_{t=0} = g(x),$$

has the solution of the form: $u(x, t) = \sum_{n=1}^{\infty} (B_n \cos \lambda_n t + C_n \sin \lambda_n t) \sin \frac{n\pi}{l} x$

where $\quad B_n = \frac{2}{l} \int_0^l f(x) \sin \frac{n\pi x}{l} dx, \; C_n = \frac{2}{cn\pi} \int_0^l g(x) \sin \frac{n\pi x}{l} dx.$

10. Using d' Alembert's method, find the deflection of a vibrating string of unit length having fixed ends, with initial velocity zero and initial deflection:

(i) $f(x) = a(x - x^2)$ (ii) $f(x) = a \sin^2 \pi x$.
(Gulbarga, 1996)

18.5 (1) ONE-DIMENSIONAL HEAT FLOW

Consider a homogeneous bar of uniform cross-section α (cm^2). Suppose that the sides are covered with a material impervious to heat so that the stream-lines of heat-flow are all parallel and perpendicular to the area α. Take one end of the bar as the origin and the direction of flow as the positive

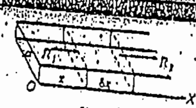

Fig. 18.3.

x-axis (Fig. 18.3). Let ρ be the density (gr/cm^3), s the specific heat (cal./gr. deg.) and k the thermal conductivity (cal./cm. deg. sec.).

Let $u(x, t)$ be the temperature at a distance x from O. If δu be the temperature change in a slab of thickness δx of the bar, then by § 12.7 (i) (p. 431), the quantity of heat in this slab $= s\rho\alpha\,\delta x\,\delta u$. Hence the rate of increase of heat in this slab, i.e. $s\rho\alpha\delta x \frac{\partial u}{\partial t} = R_1 - R_2$, where R_1 and R_2 are respectively the rate (cal./sec.) of inflow and outflow of heat.

Now by (A) of p. 431, $R_1 = -k\alpha \left(\frac{\partial u}{\partial x}\right)_x$ and $R_2 = -k\alpha \left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$

the negative sign appearing as a result of (i) on p. 438.

Hence $s\rho\alpha\delta x \frac{\partial u}{\partial t} = -k\alpha\left(\frac{\partial u}{\partial x}\right)_x + k\alpha\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$ i.e., $\frac{\partial u}{\partial t} = \frac{k}{s\rho}\left\{\frac{(\partial u/\partial x)_{x+\delta x} - (\partial u/\partial x)_x}{\delta x}\right\}$

Writing $k/s\rho = c^2$, called the *diffusivity* of the substance (cm^2/sec.), and taking the limit as $\delta x \to 0$, we get

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad ...(1)$$

This is the *one-dimensional heat-flow equation*.

(2) Solution of the heat equation. Assume that a solution of (1) is of the form

$$u(x, t) = X(x) \cdot T(t)$$

where X is a function of x alone and T is a function of t only.

Substituting this in (1), we get

$$XT' = c^2 X''T, \text{ i.e., } X''/X = T'/c^2 T. \qquad ...(2)$$

Clearly the left side of (2) is a function of x only and the right side is a function of t alone. Since x and t are independent variables, (2) can hold good if each side is equal to a constant k (say). Then (2) leads to the ordinary differential equations

$$\frac{d^2 X}{dx^2} - kX = 0 \quad ...(3) \qquad \text{and} \qquad \frac{dT}{dt} - kc^2 T = 0 \qquad ...(4)$$

Solving (3) and (4), we get

(i) *When k is positive and $= p^2$, say:*

$$X = c_1 e^{px} + c_2 e^{-px}, \; T = c_3 e^{c^2 p^2 t}$$

(ii) *When k is negative and $= -p^2$, say:*

$$X = c_4 \cos px + c_5 \sin px, \; T = c_6 e^{-c^2 p^2 t}$$

(iii) *When k is zero:*

$$X = c_7 x + c_8, \; T = c_9.$$

Thus the various possible solutions of the heat-equation (1) are

$$u = (c_1 e^{px} + c_2 e^{-px})\, c_3 e^{c^2 p^2 t} \qquad ...(5)$$
$$u = (c_4 \cos px + c_5 \sin px)\, c_6 e^{-c^2 p^2 t} \qquad ...(6)$$
$$u = (c_7 x + c_8) c_9 \qquad ...(7)$$

Of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem. As we are dealing with problems on heat conduction, it must be a transient solution, *i.e.*, u is to decrease with the increase of time t. Accordingly, the solution given by (6), *i.e.* of the form

$$u = (C_1 \cos px + C_2 \sin px)\, e^{-c^2p^2t} \qquad ...(8)$$

is the only suitable solution of the heat equation.

Example 18·7. *Solve the differential equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ for the conduction of heat along a rod without radiation, subject to the following conditions :*

(i) u is not infinite for $t \to \infty$, (ii) $\frac{\partial u}{\partial x} = 0$ for $x = 0$ and $x = l$,

(iii) $u = lx - x^2$ for $t = 0$, between $x = 0$ and $x = l$. (Assam, 1999)

Sol. Substituting $u = X(x)\,T(t)$ in the given equation, we get

$$XT' = \alpha^2 X''T \quad \text{i.e. } X''/X = \frac{T'}{\alpha^2 T} = -k^2 \quad \text{(say)}$$

$$\therefore \quad \frac{d^2X}{dx^2} + k^2 X = 0 \quad \text{and} \quad \frac{dT}{dt} + k^2\alpha^2 T = 0 \qquad ...(1)$$

Their solutions are

$$X = c_1 \cos kx + c_2 \sin kx, \quad T = c_3 e^{-k^2\alpha^2 t} \qquad ...(2)$$

If k^2 is changed to $-k^2$, the solutions are

$$X = c_4 e^{kx} + c_5 e^{-kx}, \quad T = c_6 e^{k^2\alpha^2 t} \qquad ...(3)$$

If $k^2 = 0$, the solutions are

$$X = c_7 x + c_8, \quad T = c_9 \qquad ...(4)$$

In (3), $T \to \infty$ for $t \to \infty$ therefore, u also $\to \infty$ *i.e.* the given condition (*i*) is not satisfied. So we reject the solutions (3) while (2) and (4) satisfy this condition.

Applying the condition (*ii*) to (4), we get $c_7 = 0$.

$$\therefore \quad u = XT = c_8 c_9 = a_0 \text{ (say)} \qquad ...(5)$$

From (2), $\quad \frac{\partial u}{\partial x} = (-c_1 \sin kx + c_2 \cos kx)\, k c_3 e^{-k^2\alpha^2 t}$

Applying the condition (*ii*), we get $c_2 = 0$ and $-c_1 \sin kl + c_2 \cos kl = 0$

i.e. $\quad c_2 = 0 \quad \text{and} \quad kl = n\pi \quad (n \text{ an integer})$

$$\therefore \quad u = c_1 \cos kx \cdot c_3 e^{-k^2\alpha^2 t} = a_n \cos\left(\frac{n\pi x}{l}\right) e^{-n^2\pi^2\alpha^2 t/l^2} \qquad ...(6)$$

Thus the general solution being the sum of (5) and (6), is

$$u = a_0 + \Sigma a_n \cos(n\pi x/l)\, e^{-n^2\pi^2\alpha^2 t/l^2} \qquad ...(7)$$

Now using the condition (*iii*), we get

$$lx - x^2 = a_0 + \Sigma a_n \cos(n\pi x/l)$$

This being the expansion of $lx - x^2$ as a half-range cosine series in $(0, l)$, we get

$$a_0 = \frac{1}{l}\int_0^l (lx - x^2)\,dx = \frac{1}{l}\left|\frac{lx^2}{2} - \frac{x^3}{3}\right|_0^l = \frac{l^2}{6}$$

and

$$a_n = \frac{2}{l}\int_0^l (lx - x^2)\cos\frac{n\pi x}{l}\,dx = \frac{2}{l}\left|(lx - x^2)\left(\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right) - (l - 2x)\left(-\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right) + (-2)\left(-\frac{l^3}{n^3\pi^3}\sin\frac{n\pi x}{l}\right)\right|_0^l$$

$$= \frac{2}{l}\left|0 - \frac{l^3}{n^2\pi^2}(\cos n\pi + 1) + 0\right| = -\frac{4l^2}{n^2\pi^2} \text{ when } n \text{ is even, otherwise } 0.$$

Hence taking $n = 2m$, the required solution is

$$u = \frac{l^2}{6} - \frac{l^2}{\pi^2}\sum_{m=1}^{\infty} \frac{1}{m^2}\cos\left(\frac{2m\pi x}{l}\right) e^{-4m^2\pi^2\alpha^2 t/l^2}.$$

Example 18·8. *(a) An insulated rod of length l has its ends A and B maintained at 0°C and 100°C respectively until steady state conditions prevail. If B is suddenly reduced to 0°C and maintained at 0°C, find the temperature at a distance x from A at time t.* (Andhra, 1998)

(b) Solve the above problem if the change consists of raising the temperature of A to 20°C and reducing that of B to 80°C. (Madras, 2000S)

Sol. (a) Let the equation for the conduction of heat be

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad ...(i)$$

Prior to the temperature change at the end B, when $t = 0$, the heat flow was independent of time (*steady state condition*). When u depends only on x, (*i*) reduces to $\partial^2 u/\partial x^2 = 0$ ...(*ii*)

Its general solution is $u = ax + b$

Since $u = 0$ for $x = 0$ and $u = 100$ for $x = l$, therefore, (2) gives $b = 0$ and $a = 100/l$.

Thus the *initial condition* is expressed by $u(x, 0) = \frac{100}{l}x$ ...(*iii*)

Also the *boundary conditions* for the subsequent flow are

$$u(0, t) = 0 \text{ for all values of } t \qquad ...(iv)$$

and

$$u(l, t) = 0 \text{ for all values of } t. \qquad ...(v)$$

Thus we have to find a temperature function $u(x, t)$ satisfying the differential equation (*i*) subject to the initial condition (*iii*) and the boundary conditions (*iv*) and (*v*).

Now the solution of (*i*) is of the form

$$u(x, t) = (C_1 \cos px + C_2 \sin px)\, e^{-c^2p^2t} \qquad ...(vi)$$

By (*iv*), $u(0, t) = C_1 e^{-c^2p^2t} = 0$, for all values of t.

Hence $C_1 = 0$ and (*vi*) reduces to $u(x, t) = C_2 \sin px \cdot e^{-c^2p^2t}$ ...(*vii*)

Applying (*v*), (*vii*) gives $u(l, t) = C_2 \sin pl \cdot e^{-c^2p^2t} = 0$, for all values of t.

This requires $\sin pl = 0$ i.e. $pl = n\pi$ as $C_2 \neq 0$ $\therefore$ $p = n\pi/l$, where n is any integer.

Hence (vii) reduces to $u(x, t) = b_n \sin \frac{n\pi x}{l} \cdot e^{-c^2 n^2 \pi^2 t/l^2}$, where $b_n = C_2$.

[These are the solutions of (i) satisfying the boundary conditions (iv) and (v). These are the eigen functions corresponding to the eigen values $\lambda_n = cn\pi/l$ of the problem.]

Adding all such solutions, the most general solution of (i) satisfying the boundary conditions (iv) and (v) is

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cdot e^{-c^2 n^2 \pi^2 t/l^2} \qquad \ldots(viii)$$

Putting $t = 0$, $\quad u(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$ $\qquad \ldots(ix)$

In order that the condition (iii) may be satisfied, (iii) and (ix) must be same. This requires the expansion of $100x/l$ as a half-range Fourier sine series in $(0, l)$. Thus

$$\frac{100x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \text{ where } b_n = \frac{2}{l}\int_0^l \frac{100x}{l} \cdot \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{200}{l^2}\left[x\left\{-\frac{\cos(n\pi x/l)}{(n\pi/l)}\right\} - (1)\left\{-\frac{\sin(n\pi x/l)}{(n\pi/l)^2}\right\}\right]_0^l = \frac{200}{l^2}\left(-\frac{l^2}{n\pi}\cos n\pi\right) = \frac{200}{n\pi}(-1)^{n+1}$$

Hence (viii) gives $u(x, t) = \frac{200}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin \frac{n\pi x}{l} \cdot e^{-(cn\pi/l)^2 t}$

(b) Here the initial condition remains the same as (iii) above, and the boundary conditions are

$$u(0, t) = 20 \text{ for all values of } t \qquad \ldots(x)$$

$$u(l, t) = 80 \text{ for all values of } t \qquad \ldots(xi)$$

In part (a), the boundary values (i.e. the temperature at the ends) being zero, we were able to find the desired solution easily. Now the boundary values being non-zero, we have to modify the procedure.

We split up the temperature function $u(x, t)$ into two parts as

$$u(x, t) = u_s(x) + u_t(x, t) \qquad \ldots(xii)$$

where $u_s(x)$ is a solution of (i) involving x only and satisfying the boundary conditions (x) and (xi); $u_t(x, t)$ is then a function defined by (xii). Thus $u_s(x)$ is a steady state solution of the form (ii) and $u_t(x, t)$ may be regarded as a transient part of the solution which decreases with increase of t.

Since $u_s(0) = 20$ and $u_s(l) = 80$, therefore, using (ii) we get

$$u_s(x) = 20 + (60/l)x \qquad \ldots(xiii)$$

Putting $x = 0$ in (xii), we have by (x),

$$u_t(0, t) = u(0, t) - u_s(0) = 20 - 20 = 0 \qquad \ldots(xiv)$$

Putting $x = l$ in (xii), we have by (xi),

$$u_t(l, t) = u(l, t) - u_s(l) = 80 - 80 = 0 \qquad \ldots(xv)$$

Also

$$u_t(x, 0) = u(x, 0) - u_s(x) = \frac{100x}{l} - \left(\frac{60x}{l} + 20\right) \quad \text{[by (iii) and (xiii)]}$$

$$= \frac{40x}{l} - 20 \qquad \ldots(xvi)$$

Hence (xiv) and (xv) give the boundary conditions and (xvi) gives the initial condition relative to the transient solution. Since the boundary values given by (xiv) and (xv) are both zero, therefore, as in part (a), we have $u_t(x, t) = (C_1 \cos px + C_2 \sin px)e^{-c^2p^2t}$

By (xiv),

$$u_t(0, t) = C_1 e^{-c^2p^2t} = 0 \text{ for all values of } t.$$

Hence $C_1 = 0$ and $\quad u_t(x, t) = C_2 \sin px \cdot e^{-c^2p^2t}$ $\qquad \ldots(xvii)$

Applying (xv), it gives $\quad u_t(l, t) = C_2 \sin pl \cdot e^{-c^2p^2t} = 0$ for all values of t.

This requires $\quad \sin pl = 0$, i.e. $pl = n\pi$ as $C_2 \neq 0$. $\quad p = n\pi/l$, when n is any integer.

Hence (xvii) reduces to $u_t(x, t) = b_n \sin \frac{n\pi x}{l} e^{-c^2n^2\pi^2t/l^2}$ where $b_n = C_2$.

Adding all such solutions, the most general solution of (xvii) satisfying the boundary conditions (xiv) and (xv) is $u_t(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{-c^2n^2\pi^2t/l^2}$

Putting $t = 0$, we have $u_t(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$ $\qquad \ldots(xviii)$

In order that the condition (xvi) may be satisfied, (xvi) and (xix) must be same. This requires the expansion of $(40/l)x - 20$ as a half-range Fourier sine series in $(0, l)$. Thus $\qquad \ldots(xix)$

$$\frac{40x}{l} - 20 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad \text{where } b_n = \frac{2}{l}\int_0^l \left(\frac{40x}{l} - 20\right)\sin \frac{n\pi x}{l}\, dx = -\frac{40}{n\pi}(1 + \cos n\pi)$$

i.e. $b_n = 0$, when n is odd; $= -80/n\pi$, when n is even.

Hence (xviii) becomes $u_t(x, t) = \sum_{n=2,4,\ldots}^{\infty} \left(\frac{-80}{n\pi}\right) \sin \frac{n\pi x}{l} \cdot e^{-c^2n^2\pi^2t/l^2}$

$$= -\frac{40}{\pi}\sum_{m=1}^{\infty} \frac{1}{m} \sin \frac{2m\pi x}{l} \cdot e^{-4c^2m^2\pi^2t/l^2} \quad \text{[taking } n = 2m\text{]} \qquad \ldots(xx)$$

Finally combining (xiii) and (xx), the required solution is

$$u(x, t) = \frac{40x}{l} + 20 - \frac{40}{\pi}\sum_{m=1}^{\infty} \frac{1}{m} \sin \frac{2m\pi x}{l} \cdot e^{-4c^2m^2\pi^2t/l^2}$$

Example 18.9. Bar with insulated ends. *A bar 100 cm long, with insulated sides, has its ends kept at 0°C and 100°C until steady state conditions prevail. The two ends are then suddenly insulated and kept so. Find the temperature distribution.*

Sol. The temperature $u(x, t)$ along the bar satisfies the equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \ldots(i)$$

By law of heat conduction, the rate of heat flow is proportional to the gradient of the temperature. Thus, if the ends $x = 0$ and $x = l$ ($= 100$ cm) of the bar are insulated (Fig. 18.4) so that no heat can flow through the ends, the boundary conditions are

$$\frac{\partial u(0,t)}{\partial x}=0,\ \frac{\partial u(l,t)}{\partial x}=0 \text{ for all } t \quad ...(ii)$$

Initially, under steady state conditions, $\frac{\partial^2 u}{\partial x^2}=0$. Its solution is $u = ax + b$.

Since $u = 0$ for $x = 0$ and $u = 100$ for $x = l$ $\therefore$ $b = 0$ and $a = 1$.

Thus the initial condition is $u(x, 0) = x \qquad 0 < x < l$. ...(iii)

Now the solution of (i) is of the form $u(x, t) = (c_1 \cos px + c_2 \sin px)\, e^{-c^2p^2t}$ ...(iv)

Differentiating partially w.r.t. x, we get

$$\frac{\partial u}{\partial x} = (-c_1 p \sin px + c_2 p \cos px)\, e^{-c^2p^2t} \quad ...(v)$$

Putting $x = 0$, $\left(\frac{\partial u}{\partial x}\right)_0 = c_2 p\, e^{-c^2p^2t} = 0$ for all t. [by (ii)]

$\therefore \quad c_2 = 0$

Putting $x = l$ in (v), $\left(\frac{\partial u}{\partial x}\right)_l = -c_1 p \sin pl\, e^{-c^2p^2t}$ for all t. [by (ii)]

$\therefore$ $c_1 p \sin pl = 0$ i.e., p being $\neq 0$, either $c_1 = 0$ or $\sin pl = 0$.

When $c_1 = 0$, (iv) gives $u(x, t) = 0$ which is a trivial solution, therefore $\sin pl = 0$.

or $\quad pl = n\pi \quad$ or $\quad p = n\pi/l,\ n = 0, 1, 2, ...$

$\therefore$ Hence (iv) becomes $u(x, t) = c_1 \cos \frac{n\pi x}{l}\, e^{-c^2n^2\pi^2t/l^2}$

$\therefore$ The most general solution of (i) satisfying the boundary conditions (ii) is

$$u(x, t) = \sum_{n=0}^{\infty} A_n \cos\frac{n\pi x}{l}\, e^{-c^2n^2\pi^2t/l^2} = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l}\, e^{-c^2n^2\pi^2t/l^2} \quad (\text{where } A_n = c_1) \ ...(vi)$$

Putting $t = 0$, $u(x, 0) = A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{l} = x$ [by (iii)]

This requires the expansion of x into a half range cosine series in $(0, l)$.

Thus $\qquad x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x/l \qquad$ where $a_0 = \frac{2}{l}\int_0^l x\,dx = l$

and $\qquad a_n = \frac{2}{l}\int_0^l x \cos\frac{n\pi x}{l}\,dx = \frac{2l}{n^2\pi^2}(\cos n\pi - 1)$

$= 0$, when n is even; $= -4l/n^2\pi^2$, when n is odd.

$\therefore \quad A_0 = \frac{a_0}{2} = l/2$, and $A_n = a_n = 0$ for n even; $= -4l/n^2\pi^2$ for n odd.

Hence (vi) takes the form

$$u(x, t) = \frac{l}{2} - \sum_{n=1,3,...} \frac{4l}{n^2\pi^2} \cos\frac{n\pi x}{l}\, e^{-c^2n^2\pi^2t/l^2}$$

$$= \frac{l}{2} - \frac{4l}{\pi^2}\sum_{1}^{\infty} \frac{1}{(2n-1)^2} \cos\frac{(2n-1)\pi x}{l}\, e^{-c^2(2n-1)^2\pi^2t/l^2} \quad ...(vii)$$

This is the required temperature at a point P_1 distant x from end A at any time t.

Obs. *The sum of the temperatures at any two points equidistant from the centre is always 100°C, a constant.*

Let P_1, P_2 be two points equidistant from the centre C of the bar so that $CP_1 = CP_2$ (Fig. 18·4).

If $AP_1 = BP_2 = x$ (say), then $AP_2 = l - x$.

$\therefore$ Replacing x by $l - x$ in (vii), we get the temperature at P_2 as

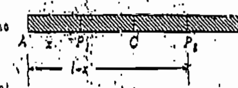

Fig. 18·4.

$$u(l-x, t) = \frac{l}{2} - \frac{4l}{\pi^2}\sum_{1}^{\infty} \frac{1}{(2n-1)^2} \cos\frac{(2n-1)\pi(l-x)}{l}\, e^{-c^2(2n-1)^2\pi^2t/l^2}$$

$$= \frac{l}{2} + \frac{4l}{\pi^2}\sum_{1}^{\infty} \frac{1}{(2n-1)^2} \cos\frac{(2n-1)\pi x}{l}\, e^{-c^2(2n-1)^2\pi^2t/l^2} \quad ...(viii)$$

$$\left[\because \cos\frac{(2n-1)\pi(l-x)}{l} = \cos\left[2n\pi - \pi - \frac{(2n-1)\pi x}{l}\right] = -\cos\frac{(2n-1)\pi x}{l}\right]$$

Adding (vii) and (viii), we get $u(x, t) + u(l-x, t) = l = 100°C$.

PROBLEMS 18.3

1. A homogeneous rod of conducting material of length 100 cm has its ends kept at zero temperature and the temperature initially is
$u(x, 0) = x, \qquad 0 \le x \le 50$
$\qquad = 100 - x, \quad 50 \le x \le 100.$
Find the temperature $u(x, t)$ at any time. *(Mangalore, 1997)*
2. Find the temperature $u(x, t)$ in a bar which is perfectly insulated laterally, whose ends are kept at temperature 0°C and whose initial temperature in (°C) is $f(x) = x(10 - x)$ given that its length is 10 cm, constant cross-section of area 1 cm², density 10.6 gm/cm³, thermal conductivity 1.04 cal/cm deg. sec., and specific heat 0.056 cal/gm deg. *(A.M.I.E., 1997)*
3. The ends A and B of a rod 20 cm. long have the temperature at 30°C and 80°C until steady state prevails. The temperature of the ends are changed to 40°C and 60°C respectively. Find the temperature distribution in the rod at time t. *(Kerala, 1995)*
4. A bar of 10 cm long, with insulated sides has its ends A and B maintained at temperatures 50°C and 100°C respectively, until steady-state conditions prevail. The temperature A is suddenly raised to 90°C and at the same time that at B is lowered to 60°C. Find the temperature distribution in the bar at time t. *(Mysore, 1997 ; Warangal, 1996)*
Show that the temperature at the middle point of the bar remains unaltered for all time, regardless of the material of the bar.
5. Solve the following boundary value problem :
$$\frac{\partial u}{\partial t} = a^2\frac{\partial^2 u}{\partial x^2},\ 0 < x < l, \quad \frac{\partial u(0,t)}{\partial x} = 0,\ \frac{\partial u(l,t)}{\partial x} = 0,\ u(x, 0) = x.$$
6. The temperatures at one end of a bar, 50 cm long with insulated sides, is kept at 0°C and that the other end is kept at 100°C until steady-state conditions prevail. The two ends are then suddenly insulated, so that the temperature gradient is zero at each end thereafter. Find the temperature distribution. *(Madras, 1998)*
7. Find the solution of $\frac{\partial \theta}{\partial t} = k\frac{\partial^2 \theta}{\partial x^2}$, such that
(i) θ is not infinite when $t \to +\infty$;
(ii) $\frac{\partial \theta}{\partial x} = 0$ when $x = 0$ $\}$ for all values of t;
$\theta = 0$, when $x = l$

(iii) $\theta = \theta_0$, when $t = 0$ for all values of x between 0 and l.

8. Solve the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ with boundary conditions $u(x, 0) = 3 \sin n\pi x$, $u(0, t) = 0$, $u(1, t) = 0$, where $0 < x < 1$, $t > 0$.

9. Find the solution of $\frac{\partial V}{\partial t} = h^2 \frac{\partial^2 V}{\partial x^2}$ having given that $V = V_0 \sin nt$ when $x = 0$ for all values of t and $V = 0$ when x is very large. (Osmania, 1995)

18.6 TWO-DIMENSIONAL HEAT FLOW

Consider the flow of heat in a metal plate of uniform thickness α (cm), density ρ (gr/cm³), specific heat s (cal/gr deg) and thermal conductivity k (cal/cm sec deg). Let XOY plane be taken in one face of the plate (Fig. 18.5). If the temperature at any point is independent of the z-coordinate and depends only on x, y and time t, then the flow is said to be two-dimensional. In this case, the heat flow is in the XY-plane only and is zero along the normal to the XY-plane.

Consider a rectangular element $ABCD$ of the plane with sides δx and δy. By (A) on p. 431, the amount of heat entering the element in 1 sec. from the side AB

$$= -k\alpha\delta x \left(\frac{\partial u}{\partial y}\right)_y$$

and the amount of heat entering the element in 1 second from the side $AD = -k\alpha\delta y \left(\frac{\partial u}{\partial x}\right)_x$

The quantity of heat flowing out through the side CD per sec. $= -k\alpha\delta x \left(\frac{\partial u}{\partial y}\right)_{y+\delta y}$

and the quantity of heat flowing out through the side BC per second $= -k\alpha\delta y \left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$

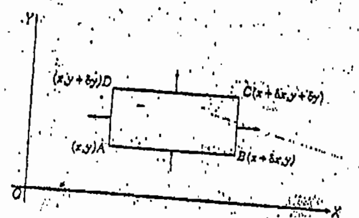

Fig. 18.5.

Hence the total gain of heat by the rectangular element $ABCD$ per second

$$= -k\alpha\delta x\left(\frac{\partial u}{\partial y}\right)_y - k\alpha\delta y\left(\frac{\partial u}{\partial x}\right)_x + k\alpha\delta x\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} + k\alpha\delta y\left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$$

$$= k\alpha\delta x\left[\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} - \left(\frac{\partial u}{\partial y}\right)_y\right] + k\alpha\delta y\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right]$$

$$= k\alpha\delta x\delta y\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x} + \frac{\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} - \left(\frac{\partial u}{\partial y}\right)_y}{\delta y}\right] \quad ...(1)$$

Also the rate of gain of heat by the element

$$= \rho\delta x\delta y\alpha s\frac{\partial u}{\partial t} \quad ...(2)$$

Thus equating (1) and (2),

$$k\alpha\delta x\delta y\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x} + \frac{\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} - \left(\frac{\partial u}{\partial y}\right)_y}{\delta y}\right] = \rho\delta x\delta y\alpha s\frac{\partial u}{\partial t}$$

Dividing both sides by $\alpha\delta x\delta y$ and taking limits as $\delta x \to 0$, $\delta y \to 0$, we get

$$k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \rho s\frac{\partial u}{\partial t}$$

i.e. $\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$ where $c^2 = k/\rho s$ is the *diffusivity*. ...(3)

Hence the equation (3) gives the temperature distribution of the plane in the *transient state*.

> **Cor.** In the *steady state*, u is independent of t, so that $\partial u/\partial t = 0$ and the above equation reduces to,
>
> $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$
>
> which is the well known Laplace's equation *in two dimensions*.
>
> **Obs.** When the stream lines are curves in space, *i.e.* the heat flow is three dimensional, we shall similarly arrive at the equation
>
> $$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$
>
> In a *steady state*, it reduces to $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$
>
> which is the *three dimensional Laplace's equation*.

18.7 SOLUTION OF LAPLACE'S EQUATION

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad ...(1)$$

Let $u = X(x)Y(y)$ be a solution of (1).

Substituting it in (1), we get $\frac{d^2X}{dx^2}Y + X\frac{d^2Y}{dy^2} = 0$

or separating the variables, $\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2}$ ...(2)

Since x and y are independent variables, (2) can hold good only if each side of (2) is equal to a constant k (say). Then (2) leads to the ordinary differential equations

$$\frac{d^2X}{dx^2} - kX = 0 \text{ and } \frac{d^2Y}{dy^2} + kY = 0.$$

Solving these equations, we get

(i) *When k is positive and is equal to* p^2, say

$$X = c_1e^{px} + c_2e^{-px},\ Y = c_3\cos py + c_4\sin py$$

(ii) *When k is negative, and is equal to* $-p^2$, say

$$X = c_5\cos px + c_6\sin px,\ Y = c_7e^{py} + c_8e^{-py}$$

(iii) *When k is zero*; $X = c_9x + c_{10},\ Y = c_{11}y + c_{12}$.

Thus the various possible solutions of (1) are

$$u = (c_1e^{px} + c_2e^{-px})(c_3\cos py + c_4\sin py) \quad ...(3)$$

$$u = (c_5\cos px + c_6\sin px)(c_7e^{py} + c_8e^{-py}) \quad ...(4)$$

$$u = (c_9x + c_{10})(c_{11}y + c_{12}) \quad ...(5)$$

Of these we take that solution which is consistent with the given boundary conditions.

Example 18.10. *An infinitely long, plane, uniform plate is bounded by two parallel edges and an end at right angles to them. The breadth is π; this end is maintained at a temperature u_0 at all points and other edges are at zero temperature. Determine the temperature at any point of the plate in the steady-state.* (J.N.T.U., 2002 S)

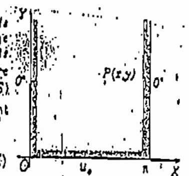

Fig. 18.6.

Sol. In the steady state, the temperature $u(x, y)$ at any point $P(x, y)$ satisfies the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad ...(i)$$

The boundary conditions are $u(0, y) = 0$ for all values of y ...(ii)

$$u(\pi, y) = 0 \text{ for all values of } y \quad ...(iii)$$

$$u(x, \infty) = 0 \text{ in } 0 < x < \pi \quad ...(iv)$$

$$u(x, 0) = u_0 \text{ in } 0 < x < \pi \quad ...(v)$$

Now the three possible solutions of (i) are

$$u = (c_1e^{px} + c_2e^{-px})(c_3\cos py + c_4\sin py) \quad ...(vi)$$

$$u = (c_5\cos px + c_6\sin px)(c_7e^{py} + c_8e^{-py}) \quad ...(vii)$$

$$u = (c_9x + c_{10})(c_{11}y + c_{12}) \quad ...(viii)$$

Of these, we have to choose that solution which is consistent with the physical nature of the problem. The solution (vi) cannot satisfy the condition (ii) for $u = 0$ for $x = 0$, for all values of y. The solution (viii) cannot satisfy the condition (iv). Thus the only possible solution is (vii), *i.e.*, of the form

$$u(x, y) = (C_1\cos px + C_2\sin px)(C_3e^{py} + C_4e^{-py}) \quad ...(ix)$$

By (ii), $u(0, y) = C_1(C_3e^{py} + C_4e^{-py}) = 0$ for all y.

Hence $C_1 = 0$ and (ix) reduces to

$$u(x, y) = C_2\sin px(C_3e^{py} + C_4e^{-py}) \quad ...(x)$$

By (iii), $u(\pi, y) = C_2\sin p\pi(C_3e^{py} + C_4e^{-py}) = 0$, for all y.

This requires $\sin p\pi = 0$, *i.e.*, $p\pi = n\pi$ as $C_2 \neq 0$. ∴ $p = n$, an integer.

Also to satisfy the condition (iv), *i.e.*, $u = 0$ as $y \to \infty$, $C_3 = 0$.

Hence (x) takes the form $u(x, y) = b_n \sin nx \cdot e^{-ny}$, where $b_n = C_2C_4$.

∴ The most general solution satisfying (ii), (iii) and (iv) is of the form

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin nx \cdot e^{-ny} \quad ...(xi)$$

Putting $y = 0$, $\quad u(x, 0) = \sum_{n=1}^{\infty} b_n \sin nx \quad ...(xii)$

In order that the condition (v) may be satisfied, (v) and (xii) must be same. This requires the expansion of u_0 as a half-range Fourier sine series in $(0, \pi)$. Thus

$$u_0 = \sum_{n=1}^{\infty} b_n \sin nx \text{ where } b_n = \frac{2}{\pi}\int_0^{\pi} u_0 \sin nx\, dx = \frac{2u_0}{n\pi}(1 - (-1)^n)$$

i.e., $\quad b_n = 0$, if n is even; $= 4u_0/n\pi$, if n is odd.

Hence (xi) becomes $u(x, y) = \frac{4u_0}{\pi}\left[e^{-y}\sin x + \frac{1}{3}e^{-3y}\sin 3x + \frac{1}{5}e^{-5y}\sin 5x + \ldots\right]$.

PROBLEMS 18.4

1. A long rectangular plate of width a cm with insulated surface has its temperature v equal to zero on both the long sides and one of the short sides so that $v(0, y) = 0$, $v(a, y) = 0$, $v(x, \infty) = 0$, $v(x, 0) = kx$. Show that the steady-state temperature within the plate is

$$v(x, y) = \frac{2ak}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\, e^{-n\pi y/a}\sin\frac{n\pi x}{a}.$$

2. A rectangular plate with insulated surface is 8 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature along one short edge $y = 0$ is given by

$$u(x, 0) = 100\sin(\pi x/8), \quad 0 < x < 8;$$

while the two long edges $x = 0$ and $x = 8$ as well as the other short edge are kept at 0°C, show that the steady-state temperature at any point of the plane is given by

$$u(x, y) = 100e^{-\pi y/8}\sin(\pi x/8).$$

3. A rectangular plate with insulated surface is 10 cm wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature of the short edge $y = 0$ is given by

$$u = 20x \quad \text{for } 0 \leq x \leq 5$$

and $\quad u = 20(10 - x) \quad \text{for } 5 \leq x \leq 10$

and the two long edges $x = 0$, $x = 10$ as well as the other short edge are kept at 0°C, prove that the temperature u at any point (x, y) is given by

$$u = \frac{800}{\pi^2}\sum_{1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\sin\frac{(2n-1)\pi x}{10}\cdot e^{-(2n-1)\pi y/10} \quad \text{(Madras, 1997 S)}$$

4. Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, subject to the conditions

$$u(0, y) = u(l, y) = u(x, 0) = 0$$

and $\quad u(x, a) = \sin\frac{n\pi}{l}x$. (Mangalore, 1997; S.V.T.U., 1997)

5. Solve $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ for $0 < x < \pi$, $0 < y < \pi$,

with conditions given: $u(0, y) = u(\pi, y) = u(x, \pi) = 0$, $u(x, 0) = \sin^2 x$.

6. A square plate is bounded by the lines $x=0$, $y=0$, $x=20$ and $y=20$. Its faces are insulated. The temperature along the upper horizontal edge is given by
$u(x, 20) = x(20-x)$ when $0 < x < 20$,
while other three edges are kept at 0°C. Find the steady state temperature in the plate.

7. The temperature u is maintained at 0° along three edges of a square plate of length 100 cm and the fourth edge is maintained at 100° until steady state conditions prevail. Find an expression for the temperature u at any point (x, y). Hence show that the temperature at the centre of the plate
$$= \frac{200}{\pi}\left[\frac{1}{\cosh(\pi/2)} - \frac{1}{3\cosh 3\pi/2} + \frac{1}{5\cosh 5\pi/2} - \ldots\right]$$
(Madras, 2003)

8. The function $v(x, y)$ satisfies the Laplace's equation in rectangular co-ordinates (x, y) and for the points within the rectangle $x=0$, $x=a$, $y=0$, $y=b$, it satisfies the conditions
$v(0, y) = v(a, y) = v(x, b) = 0$,
and $v(x, 0) = x(a-x)$, $0 < x < a$.
Show that $v(x, y)$ is given by
$$v(x, y) = \frac{8a^2}{\pi^3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}\sin\frac{(2n+1)\pi x}{a}\,\frac{\sinh\frac{(2n+1)\pi}{a}(b-y)}{\sinh\frac{(2n+1)\pi b}{a}}.$$
(Madras, 2003)

9. A square thin metal plate of side a, is bounded by the lines $x=0$, $x=a$, $y=0$, $y=a$. The edges $x=0$, $y=a$ are kept at zero temperature, the edge $y=0$ is insulated and the edge $x=a$ is kept at constant temperature T_0. Show that in the steady state conditions, the temperature $u(x, y)$ at the point (x, y) is given by
$$u(x, y) = \frac{4T_0}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\sinh\frac{(2n-1)\pi x}{2a}\cos\frac{(2n-1)\pi y}{2a}}{(2n-1)\sinh\frac{(2n-1)\pi}{2}}$$

10. A rectangular plate has sides a and b. Taking the side of length a as OX and that of length b as OY and other sides to be $x=a$ and $y=b$, the sides $x=0$, $x=a$, $y=b$ are insulated and the edge $y=0$ is kept at temperature $u_0\cos\frac{\pi x}{a}$. Find the temperature $u(x, y)$ in the steady state.

18.8 (1) LAPLACE'S EQUATION IN POLAR CO-ORDINATES

In the study of steady state temperature distribution in a rectangular plate, it is usually convenient to employ Cartesian co-ordinates as hitherto done. Sometimes Polar co-ordinates (r, θ) are found to be more useful and the Cartesian form of Laplace's equation is replaced by its polar form:
$$r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0$$
(See Ex. 6.10, p. 200)

(2) Solution of Laplace's equation
$$r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0 \quad \ldots(1)$$
Assume that a solution of (1) is of the form $u = R(r)\cdot\phi(\theta)$ where R is a function of r alone and ϕ is a function of θ only.

Substituting it in (1), we get $r^2R''\phi + rR'\phi + R\phi'' = 0$ or $\phi(r^2R'' + rR') + R\phi'' = 0$.

Separating the variables $\frac{r^2R'' + rR'}{R} = -\frac{\phi''}{\phi}$ ...(2)

Clearly the left side of (2) is a function of r only and the right side is a function of θ alone. Since r and θ are independent variables, (2) can hold good only if each side is equal to a constant k (say). Then (2) leads to the ordinary differential equations
$$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - kR = 0 \quad \ldots(3)$$
and
$$\frac{d^2\phi}{d\theta^2} + k\phi = 0 \quad \ldots(4)$$
Putting $r = e^t$, (3) reduces to $\frac{d^2R}{dt^2} - kR = 0$ ...(5)

Solving (5) and (4), we get

(i) When k is positive and $= p^2$, say
$$R = c_1e^{pt} + c_2e^{-pt} = c_1r^p + c_2r^{-p},\ \phi = c_3\cos p\theta + c_4\sin p\theta$$
(ii) When k is negative and $= -p^2$, say
$$R = c_5\cos pt + c_6\sin pt = c_5\cos(p\log r) + c_6\sin(p\log r),\ \phi = c_7e^{p\theta} + c_8e^{-p\theta}$$
(iii) When k is zero:
$$R = c_9t + c_{10} = c_9\log r + c_{10},\ \phi = c_{11}\theta + c_{12}$$
Thus the three possible solutions of (1) are
$$u = (c_1r^p + c_2r^{-p})(c_3\cos p\theta + c_4\sin p\theta) \quad \ldots(6)$$
$$u = [c_5\cos(p\log r) + c_6\sin(p\log r)](c_7e^{p\theta} + c_8e^{-p\theta}) \quad \ldots(7)$$
$$u = (c_9\log r + c_{10})(c_{11}\theta + c_{12}) \quad \ldots(8)$$
Of these solutions, we have to take that solution which is consistent with the physical nature of the problem. The general solution will consist of a sum of terms of type (6), (7) or (8).

Example 18.11. *The diameter of a semi-circular plate of radius a is kept at 0°C and the temperature at the semi-circular boundary is T°C. Show that the steady state temperature in the plate is given by*
$$u(r, \theta) = \frac{4T}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\left(\frac{r}{a}\right)^{2n-1}\sin(2n-1)\theta.$$
Sol. Take the centre of the circle as the pole and bounding diameter as the initial line as in Fig. 18.7. Let the steady state temperature at any point $P(r, \theta)$ be $u(r, \theta)$, so that u satisfies the equation
$$r^2\frac{\partial^2 u}{\partial r^2} + r\frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0 \quad \ldots(i)$$


Fig. 18.7.

The boundary conditions are:
$$u(r, 0) = 0 \quad \text{in } 0 \le r \le a \quad \ldots(ii)$$
$$u(r, \pi) = 0 \quad \text{in } 0 \le r \le a \quad \ldots(iii)$$
and
$$u(a, \theta) = T \quad \ldots(iv)$$
The three possible solutions of (i) are
$$u = (c_1r^p + c_2r^{-p})(c_3\cos p\theta + c_4\sin p\theta) \quad \ldots(v)$$

$$u = [c_5 \cos (p \log r) + c_6 \sin (p \log r)](c_7 e^{p\theta} + c_8 e^{-p\theta}) \quad ...(vi)$$

$$u = (c_9 \log r + c_{10})(c_{11}\theta + c_{12}) \quad ...(vii)$$

From (ii) and (iii), $u = 0$ when $r = 0$ i.e. u must be finite at the origin. Thus the solutions (vi) and (vii) are to be rejected. Hence the only suitable solution is (v).

By (ii), $u(r, 0) = (c_1 r^p + c_2 r^{-p}) c_3 = 0$

Hence $c_3 = 0$ and (v) becomes $u(r, \theta) = (c_1 r^p + c_2 r^{-p}) c_4 \sin p\theta$ ...(viii)

By (iii), $u(r, \pi) = (c_1 r^p + c_2 r^{-p}) c_4 \sin p\pi = 0$.

As $c_4 \neq 0$, $\sin p\pi = 0$, i.e., $p = n$, where n is any integer.

Hence (viii) reduces to $u(r, \theta) = (c_1 r^n + c_2 r^{-n}) c_4 \sin n\theta$ ...(ix)

Since $u = 0$, when $r = 0$, ∴ $c_2 = 0$ and (ix) becomes

$u(r, \theta) = b_n r^n \sin n\theta$, where $b_n = c_1 c_4$.

∴ The most general solution of (i) is of the form

$$u(r, \theta) = \sum_{n=1}^{\infty} b_n r^n \sin n\theta \quad ...(x)$$

Putting $r = a$, $u(a, \theta) = \sum_{n=1}^{\infty} b_n a^n \sin n\theta$ ...(xi)

In order that (iv) may be satisfied, (iv) and (xi) must be same. This requires the expansion of T as a half-range Fourier sine series in $(0, \pi)$. Thus

$$T = \sum_{n=1}^{\infty} B_n \sin n\theta \quad \text{where} \quad B_n = \frac{2}{\pi}\int_0^{\pi} T \sin n\theta \, d\theta = \frac{2T}{n\pi}(1 - \cos n\pi) \text{ and } B_n = b_n a^n$$

$$b_n = \frac{B_n}{a^n} = \frac{2T}{n\pi a^n}(1 - \cos n\pi)$$

i.e. $b_n = 0$, if n is even

$= \dfrac{4T}{n\pi a^n}$, if n is odd.

Hence (x) gives $u(r, \theta) = \dfrac{4T}{\pi}\left\{\dfrac{(r/a)}{1}\sin\theta + \dfrac{(r/a)^3}{3}\sin 3\theta + \dfrac{(r/a)^5}{5}\sin 5\theta + \ldots\right\}$

PROBLEMS 18.3

1. A semi-circular plate of radius a has its circumference kept at temperature $u(a, \theta) = k\theta(\pi - \theta)$ while the boundary diameter is kept at zero temperature. Find the steady state temperature distribution $u(r, \theta)$ of the plate assuming the lateral surfaces of the plate to be insulated. (Madras, 1997 S)
2. A semi-circular plate of radius 10 cm has insulated faces and heat flows in plane curves. The bounding diameter is kept at 0°C and on the circumference, the temperature distribution maintained is $u(10, \theta) = (400/\pi)(\pi\theta - \theta^2)$, $0 \le \theta \le \pi$. Determine the temperature distribution $u(r, \theta)$ at any point on the plate.
3. The bounding diameter of a semi-circular plate of radius a cm is kept at 0°C and the temperature along the semicircular boundary is given by $u(a, \theta) = \begin{cases} 50\theta \text{ when } 0 < \theta \le \pi/2 \\ 50(\pi - \theta) \text{ when } \pi/2 < \theta < \pi \end{cases}$ Find the steady-state temperature function $u(r, \theta)$. (Madras, 2003)
4. Find the harmonic (temperature distribution) function $F(r, \theta)$ inside the circle $|z| = 1$ taking the values

$$F(\theta) = \begin{cases} T, & 0 \le \theta \le \pi \\ -T, & \pi \le \theta \le 2\pi \end{cases} \text{ on its circumference.}$$

(Assume the lateral surface of the plate to be insulated.)

5. A plate in the shape of truncated quadrant of a circle, is bounded by $r = a$, $r = b$ and $\theta = 0$, $\theta = \pi/2$. It has its faces insulated and heat flows in plane curves. It is kept at temperature 0°C along three of the edges while along the edge $r = a$, it is kept at temperature $\theta(\pi/2 - \theta)$. Determine the temperature distribution.
6. Determine the steady state temperature at the points on the sector $0 \le \theta \le \pi/4$, $0 \le r \le a$ of a circular plate, if the temperature is maintained at 0°C along the side edges and at a constant temperature k°C along the curved edges.
7. Find the steady state temperature in a circular plate of radius a, which has one-half of its circumference at 0°C and the other half at 60°C. (Warangal, 1995)
8. If the radii of the inner and outer boundaries of a circular annulus are 10 cm and 20 cm and $u(10, \theta) = 15 \cos\theta$, $u(20, \theta) = 30 \sin\theta$, find the value of $u(r, \theta)$ in the annulus. [$u(r, \theta)$ satisfies Laplace equation in the interior of the annulus.]
9. A plate in the form of a ring is bounded by the lines $r = 2$ and $r = 4$. Its surfaces are insulated and the temperature along the boundaries are $u(2, \theta) = 10 \sin\theta + 6 \cos\theta$, $u(4, \theta) = 17 \sin\theta + 15 \cos\theta$. Find the steady state temperature $u(r, \theta)$ in the ring.

18.9 (1) VIBRATING MEMBRANE—TWO DIMENSIONAL WAVE EQUATION

We shall now derive the equation for the vibrations of a tightly stretched membrane, such as the membrane of a drum. We shall assume that the membrane is uniform and the tension T in it per unit length is the same in all directions at every point.

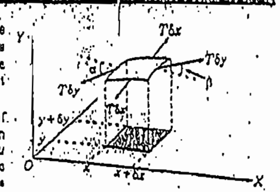

Fig. 18.8.

Consider the forces acting on an element $\delta x \delta y$ of the membrane (Fig. 18.8). Forces $T\delta x$ and $T\delta y$ act on the edges along the tangent to the membrane. Let u be its small displacement perpendicular to the xy-plane, so that the forces $T\delta y$ on its opposite edges of length δy make angles α and β to the horizontal. So their vertical component

$= T\delta y \sin\beta - T\delta y \sin\alpha$

$= T\delta y (\tan\beta - \tan\alpha)$ approximately, since α and β are small

$= T\delta y \left\{\left(\dfrac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\dfrac{\partial u}{\partial x}\right)_x\right\} = T\delta y \dfrac{\partial^2 u}{\partial x^2}\delta x$, up to a first order of approximation.

Similarly, the vertical component of the force $T\delta x$ acting on the edges of length δx

$= T\delta x \dfrac{\partial^2 u}{\partial y^2}\delta y$

If m be the mass per unit area of the membrane, then the equation of motion of the element is

$$m\,\delta x \delta y \frac{\partial^2 u}{\partial t^2} = T\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]\delta x \delta y$$

or $$\frac{\partial^2 u}{\partial t^2} = c^2\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right], \text{ where } c^2 = T/m. \quad ...(1)$$

This is the wave equation in two dimensions.

(2) Solution of the two dimensional wave equation—Rectangular membrane. Assume that a solution of (1) is of the form $u = X(x)\,Y(y)\,T(t)$.

Substituting this in (1) and dividing by XYT, we get

$$\frac{1}{c^2 T}\frac{d^2T}{dt^2} = \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2}$$

This can hold good if each member is a constant. Choosing the constants suitably, we have

$$\frac{d^2X}{dx^2} + k^2X = 0, \quad \frac{d^2Y}{dy^2} + l^2Y = 0 \quad \text{and} \quad \frac{d^2T}{dt^2} + (k^2 + l^2)c^2T = 0.$$

Hence a solution of (1) is

$$u = (c_1 \cos kx + c_2 \sin kx)(c_3 \cos ly + c_4 \sin ly) \times [c_5 \cos \sqrt{(k^2 + l^2)}ct + c_6 \sin \sqrt{(k^2 + l^2)}ct] \qquad ...(2)$$

Now suppose the membrane is rectangular and is stretched between the lines $x = 0$, $x = a$; $y = 0$, $y = b$. Then the condition $u = 0$ when $x = 0$ gives

$$0 = c_1(c_3 \cos ly + c_4 \sin ly)[c_5 \cos \sqrt{(k^2 + l^2)}ct + c_6 \sin \sqrt{(k^2 + l^2)}ct] \quad i.e., \quad c_1 = 0.$$

Then putting $c_1 = 0$ in (2) and applying the condition $u = 0$ when $x = a$, we get $\sin ka = 0$ or $k = m\pi/a$. (m being an integer)

Similarly, applying the conditions $u = 0$, when $y = 0$ and $y = b$, we obtain

$$c_3 = 0 \text{ and } l = n\pi/b \qquad (n \text{ being an integer})$$

Thus the solution (2) becomes

$$u(x, y, t) = c_2c_4 \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}(c_5 \cos p_{mn}t + c_6 \sin p_{mn}t)$$

where $\quad p_{mn} = \pi c\sqrt{[(m/a)^2 + (n/b)^2]} \qquad ...(3)$

[These are the solutions of the wave equation (1) which are zero on the boundary of the rectangular membrane. These functions are called eigen functions and the numbers p_{mn} are the eigen values of the vibrating membrane.]

Choosing the constants c_2 and c_4 so that $c_2c_4 = 1$, we can write the general solution of the equation (1) as

$$u = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}(A_{mn}\cos pt + B_{mn}\sin pt) \qquad ...(4)$$

If the membrane starts from rest from the initial position $u = f(x, y)$, i.e., $\frac{\partial u}{\partial t} = 0$ when $t = 0$, then (3) gives $B_{mn} = 0$.

Also using the condition $u = f(x, y)$ when $t = 0$, we get

$$f(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}.$$

This is *double Fourier series*. Multiplying both sides by $\sin(m\pi x/a)\sin(n\pi y/b)$ and integrating from $x = 0$ to $x = a$ and $y = 0$ to $y = b$, every term on the right except one, becomes zero. Hence we obtain

$$\int_0^a\int_0^b f(x, y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dy\,dx = \frac{ab}{4}A_{mn} \qquad ...(5)$$

which gives the coefficients in the solution and is called the generalised Euler's formula.

Example 18.12. *Find the deflection $u(x, y, t)$ of the square membrane with $a = b = 1$ and $c = 1$, if the initial velocity is zero and the initial deflection is $f(x, y) = A \sin \pi x \sin 2\pi y$.*

Sol. Taking $a = b = 1$ and $f(x, y) = A \sin \pi x \sin 2\pi y$ in (5), we get

$$A_{mn} = 4\int_0^1\int_0^1 A \sin\pi x \sin 2\pi y \sin m\pi x \sin n\pi y\,dy\,dx$$

$$= 4A\int_0^1 \sin\pi x \sin m\pi x\,dx\left(\int_0^1 \sin 2\pi y \sin n\pi y\,dy\right) = 0, \text{ for } m \neq 1$$

$$= 4A\left(\frac{1}{2}\right)\int_0^1 \sin 2\pi y \sin n\pi y\,dy, \text{ for } m = 1 \qquad \left[\because \int_0^1 \sin \pi x \sin \pi x\,dx = \frac{1}{2}\right]$$

i.e. $\quad A_{1n} = 2A\int_0^1 \sin 2\pi y \sin n\pi y\,dy = 0, \text{ for } n \neq 2$

$$= 2A\left(\frac{1}{2}\right) \text{ for } n = 2.$$

$\therefore A_{12} = A$. Also from (3), $p_{mn} = \pi\sqrt{(m^2 + n^2)}$ $\qquad [\because a = b = 1 = c]$

$\therefore \quad p_{12} = \pi\sqrt{(1^2 + 2^2)} = \sqrt{5}\pi.$

Hence from (4), the required solution is $u(x, y, t) = A \sin \pi x \sin 2\pi y \cos(\sqrt{5}\pi t)$.

Example 18.13. Circular membrane*. *A circular membrane of unit radius fixed along its boundary starts vibrating from rest and has initial deflection $u(r, 0) = f(r)$. Show that the deflection $u(r, t)$ of the membrane at any instant is given by*

$$u(r, t) = \sum_{m=1}^{\infty} A_m \cos(c\alpha_m t)\cdot J_0(\alpha_m r) \text{ where } A_m = \frac{2}{J_1^2(\alpha_m)}\int_0^1 r f(r) J_0(\alpha_m r)\,dr,$$

and α_m ($m = 1, 2, \ldots$) are the positive roots of the Bessel function $J_0(\lambda) = 0$.

Sol. The vibrations of a plane circular membrane are governed by 2-dimensional wave equation in polar coordinates i.e.

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2}\right)$$

For a radially symmetric membrane (in which u does not depend on θ), the above equation reduces to

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) \qquad ...(i)$$

For the given membrane fixed along its boundary, the boundary condition is

$$u(1, t) = 0 \quad \text{for all } t \geq 0 \qquad ...(ii)$$

For solutions not depending on θ,

initial deflection $u(r, 0) = f(r)$ $\qquad ...(iii)$

and initial velocity $\left(\frac{\partial u}{\partial t}\right)_{t=0} = 0$ $\qquad ...(iv)$

which are the initial conditions. We find the solutions $u(r, t) = R(r)\,T(t)$ satisfying the boundary condition (ii). $\qquad ...(v)$

* Drums, telephones and microphones provide examples of circular membrane and as such are quite useful in engineering.

Differentiating and substituting (v) in (i), we get

$$\frac{\partial^2 T/\partial t^2}{c^2 T} = \frac{1}{R}\left(\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r}\right) = -k^2 \text{ (say)}$$

This leads to $\frac{\partial^2 T}{\partial t^2} + p^2 T = 0$ where $p = ck$ ...(vi)

and $\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r} + k^2 R = 0$ ...(vii)

Now putting $s = kr$, (vii) transforms to $\frac{d^2R}{ds^2} + \frac{1}{s}\frac{dR}{ds} + R = 0$ which is Bessel's equation. Its general solution $R = a J_0(s) + b Y_0(s)$ where J_0 and Y_0 are Bessel's functions of the first and second kind of order zero.

Since the deflection of the membrane is always finite, we must have $b = 0$. Then taking $a = 1$, we get

$$R(r) = J_0(s) = J_0(kr)$$

On the boundary of the circular membrane, we must have $J_0(k) = 0$ which is satisfied for $k = \alpha_m$, $m = 1, 2, \ldots$

Thus the solutions of (vii) are $R(r) = J_0(\alpha_m r)$, $m = 1, 2, \ldots$ and the corresponding solutions of (vi) are $T(t) = A_m \cos p_m t + B_m \sin p_m t$, where $p_m = ck_m = c\alpha_m$.

Hence the general solution of (i) satisfying (ii) are

$$u(r, t) = (A_m \cos p_m t + B_m \sin p_m t) J_0(\alpha_m r)$$

which are the *eigen functions* of the problem and the corresponding *eigen values* are p_m.

To find that solution which also satisfies the initial conditions (iii) and (iv), consider the series

$$u(r, t) = \sum_{m=1}^{\infty} (A_m \cos p_m t + B_m \sin p_m t) J_0(\alpha_m r)$$

Putting $t = 0$ and using (iii), we get $u(r, 0) = \sum_{m=1}^{\infty} A_m J_0(\alpha_m r) = f(r)$

Here the constants A_m must be the coefficients of Fourier-Bessel series [(8) page 624] with $n = 0$, i.e.,

$$A_m = \frac{2}{J_1^2(\alpha_m)} \int_0^1 r f(r) J_0(\alpha_m r)\, dr$$

Using (iv), we get $B_m = 0$. Hence the result.

PROBLEMS 18.4

1. A tightly stretched unit square membrane starts vibrating from rest and its initial displacement is $k \sin 2\pi x \sin \pi y$. Show that the deflection at any instant is $k \sin 2\pi x \sin \pi y \cos(\sqrt{5}\pi ct)$.
2. Find the vibrations $u(x, y, t)$ of a rectangular membrane $(0 < x < a, 0 < y < b)$ whose boundary is fixed, given that it starts from rest and $u(x, y, 0) = kxy(a - x)(b - y)$.
3. Find the deflection $u(r, t)$ of the circular membrane of unit radius if $c = 1$, the initial velocity is zero and the initial deflection is $0.25(1 - r^2)$.

18.10. TRANSMISSION LINES

Consider a cable l km in length, carrying an electric current with resistance R ohms/km, inductance L henries/km, capacitance C farads/km and leakance G mhos/km (Fig. 18.9).

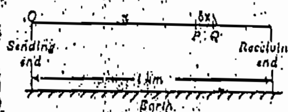

Fig. 18.9.

Let the instantaneous voltage and current at any point P, distant x km from the sending end O, and at time t sec be $v(x, t)$ and $i(x, t)$ respectively. Consider a small length $PQ (= \delta x)$ of the cable.

Now since the voltage drop across the segment δx

= voltage drop due to resistance + voltage drop due to inductance

$$\therefore \quad -\delta v = iR\delta x + L\delta x \cdot \frac{\partial i}{\partial t}$$

and dividing by δx and taking limits as $\delta x \to 0$, we get

$$-\frac{\partial v}{\partial x} = Ri + L\frac{\partial i}{\partial t} \quad \ldots(1)$$

Similarly the current lost between P and Q

= current lost due to capacitance and leakance

$$\therefore \quad -\delta i = C\frac{\partial v}{\partial t}\delta x + Gv\delta x \text{ from which as before, we get}$$

$$-\frac{\partial i}{\partial x} = C\frac{\partial v}{\partial t} + Gv \quad \ldots(2)$$

Rewriting the simultaneous partial differential equations (1) and (2) as

$$\left(R + L\frac{\partial}{\partial t}\right)i + \frac{\partial v}{\partial x} = 0 \quad \ldots(3)$$

and

$$\frac{\partial i}{\partial x} + \left(C\frac{\partial}{\partial t} + G\right)v = 0, \quad \ldots(4)$$

we shall eliminate i and v in turn.

$\therefore$ Operating (3) by $\frac{\partial}{\partial x}$ and (4) by $\left(R + L\frac{\partial}{\partial t}\right)$ and subtracting,

$$\frac{\partial^2 v}{\partial x^2} - \left(R + L\frac{\partial}{\partial t}\right)\left(C\frac{\partial}{\partial t} + G\right)v = 0$$

or

$$\frac{\partial^2 v}{\partial x^2} = LC\frac{\partial^2 v}{\partial t^2} + (LG + RC)\frac{\partial v}{\partial t} + RGv \quad \ldots(5)$$

Again operating (3) by $\left(C\frac{\partial}{\partial t} + G\right)$ and (4) by $\frac{\partial}{\partial x}$ and subtracting

$$\left(C\frac{\partial}{\partial t} + G\right)\left(R + L\frac{\partial}{\partial t}\right)i - \frac{\partial^2 i}{\partial x^2} = 0$$

or

$$\frac{\partial^2 i}{\partial x^2} = LC\frac{\partial^2 i}{\partial t^2} + (LG + RC)\frac{\partial i}{\partial t} + RGi \quad \ldots(6)$$

which is (5) with v replaced by i. The equations (5) and (6) are called the *telephone equations*.

Cor. 1. *If $L = G = 0$*, the equations (5) and (6) become

$$\frac{\partial^2 v}{\partial x^2} = RC\frac{\partial v}{\partial t} \quad ...(7) \qquad \frac{\partial^2 i}{\partial x^2} = RC\frac{\partial i}{\partial t} \quad ...(8)$$

which are known as the *telegraph equations*.

Rewriting (7) as $\frac{\partial v}{\partial t} = \frac{1}{RC}\frac{\partial^2 v}{\partial x^2}$, we observe that it is similar to the heat equation [(1) p. 565].

Cor. 2. *If $R = G = 0$*, the equations (5) and (6) become

$$\frac{\partial^2 v}{\partial x^2} = LC\frac{\partial^2 v}{\partial t^2} \quad ...(9) \qquad \frac{\partial^2 i}{\partial x^2} = LC\frac{\partial^2 i}{\partial t^2} \quad ...(10)$$

which are called the *radio equations*.

Rewriting (9) as $\frac{\partial^2 v}{\partial t^2} = \lambda^2\frac{\partial^2 v}{\partial x^2}$ where $\lambda^2 = \frac{1}{LC}$,

its general solution is $v(x, t) = f_1(x + \lambda t) + f_2(x - \lambda t)$.

Similarly from (10), $i(x, t) = F_1(x + \lambda t) + F_2(x - \lambda t)$. [See (4) p. 563]

Thus the voltage $v(x, t)$ [or the current $i(x, t)$] at any point along the lossless transmission line can be obtained by the superposition of a progressive wave and a receding wave travelling with equal velocities λ. This is the case of oscillations of $v(x, t)$ and $i(x, t)$ at high frequencies.

Cor. 3. *If $L = G = 0$*, e.g. in the case of a submarine cable, then (6) gives

$$\frac{d^2 v}{dx^2} = GRv, \text{ i.e. } (D^2 - GR)v = 0$$

$$\therefore v(x) = A\cosh(\sqrt{GR}\cdot x) + B\sinh(\sqrt{GR}\cdot x)$$

Since by (1), $Ri = -\frac{\partial v}{\partial x} = -\sqrt{GR}\,[A\sinh(\sqrt{GR}\cdot x) + B\cosh(\sqrt{GR}\cdot x)]$ ...(11)

$$\therefore i(x) = -\sqrt{(G/R)}\,[A\sinh(\sqrt{GR}\cdot x) + B\cosh(\sqrt{GR}\cdot x)]$$

If $v(0) = v_0$ and $i(0) = i_0$, then $v_0 = A$ and $i_0 = -\sqrt{(G/R)}\,B$. ...(12)

Hence writing $\sqrt{(GR)} = \gamma$ and $\sqrt{(R/G)} = z_0$, (11) and (12) give

$$v(x) = v_0\cosh\gamma x - i_0 z_0\sinh\gamma x$$

$$i(x) = i_0\cosh\gamma x - \frac{v_0}{z_0}\sinh\gamma x \quad ...(13)$$

Obs. Steady-state solutions. We have so far considered the transient state solutions only. The steady-state solutions of transmission lines are, however, obtained by assuming $v = Ve^{i\omega t}$ and $i = Ie^{i\omega t}$ where V and I are complex functions of x only. Substituting these in (5) and (6), we get two ordinary linear differential equations of the second order which can be solved at once.

Example 18.14. *Neglecting R and G, find the e.m.f. $v(x, t)$ in a line of length l, t seconds after the ends were suddenly grounded, given that $i(x, 0) = i_0$ and $v(x, 0) = e_1\sin\frac{\pi x}{l} + e_5\sin\frac{5\pi x}{l}$.*

Sol. Since R and G are negligible, we use the *Radio equation* $\frac{\partial^2 v}{\partial x^2} = LC\frac{\partial^2 v}{\partial t^2}$ ...(i)

Since the ends are suddenly grounded, we have the boundary conditions

$$v(0, t) = 0,\ v(l, t) = 0. \quad ...(ii)$$

Also the initial conditions are $i(x, 0) = i_0$

and
$$v(x, 0) = e_1\sin\pi x/l + e_5\sin 5\pi x/l \quad ...(iii)$$

$$\frac{\partial i}{\partial x} = -C\frac{\partial v}{\partial t} \text{ gives } \frac{\partial v}{\partial t}(x, 0) = 0 \quad ...(iv)$$

Let $v = X(x)\,T(t)$ be the solution of (i).

$\therefore$ (i) gives $X''T = LC\,XT''$

$$\frac{X''}{X} = LC\frac{T''}{T} = -k^2 \text{ (say)}$$

$$X'' + k^2X = 0 \text{ and } T'' + (k^2/LC)T = 0.$$

Solving these equations, we get

$$v = (c_1\cos kx + c_2\sin kx)\left(c_3\cos\frac{k}{\sqrt{LC}}t + c_4\sin\frac{k}{\sqrt{LC}}t\right)$$

Using the boundary conditions (ii), we get

$$c_1 = 0 \text{ and } k = n\pi/l.$$

$$\therefore v = \sin\frac{n\pi x}{l}\left(a_n\cos\frac{n\pi}{l\sqrt{LC}}t + b_n\sin\frac{n\pi}{l\sqrt{LC}}t\right)$$

Using the initial condition (iv), we get $b_n = 0$

$$\therefore v = a_n\sin\frac{n\pi x}{l}\cos\frac{n\pi}{l\sqrt{LC}}t$$

Thus the most general solution of (i) is

$$v = \sum_{n=1}^{\infty} a_n\sin\frac{n\pi x}{l}\cos\frac{n\pi t}{l\sqrt{LC}}$$

Finally by the initial condition (iii), we have

$$e_1\sin\frac{\pi x}{l} + e_5\sin\frac{5\pi x}{l} = \Sigma a_n\sin\frac{n\pi x}{l}$$

$$\therefore a_1 = e_1 \text{ and } a_5 = e_5, \text{ while all other } a\text{'s are zero.}$$

Hence
$$v = e_1\sin\frac{\pi x}{l}\cos\frac{\pi t}{l\sqrt{LC}} + e_5\sin\frac{5\pi x}{l}\cos\frac{5\pi t}{l\sqrt{LC}}$$

which is the required solution.

Example 18.15. *A telephone line 3000 km long has a resistance of 4 ohms/km and a capacitance of 5×10^{-7} farad/km. Initially both the ends are grounded so that the line is uncharged. At time $t = 0$, a constant e.m.f. E is applied to one end, while the other end is left grounded. Assuming the inductance and leakance to be negligible, show that the steady state current at the grounded end at the end of 1 sec. is 5.3%.*

Sol. Since $L = 0$, $G = 0$, we use the telegraph equation

$$\frac{\partial^2 v}{\partial x^2} = RC\frac{\partial v}{\partial t}.$$

Let $v = X(x)\,T(t)$ be its solution so that

$$TX'' = RC\,XT' \quad \text{or} \quad \frac{X''}{X} = RC\frac{T'}{T} = -k^2 \text{ (say)}$$

$$\therefore X'' + k^2X = 0 \text{ and } T' + (k^2/RC)T = 0.$$

Solving these equations, we get

$$X = c_1\cos kx + c_2\sin kx,\ T = c_3e^{-k^2t/RC}$$

giving
$$v = (c_1\cos kx + c_2\sin kx)\,c_3e^{-k^2t/RC} \quad ...(i)$$

When $t = 0$; $v = 0$ at $x = 0$ and $v = 0$ at $x = l$

$$\therefore 0 = c_1c_3;\ 0 = (c_1\cos kl + c_2\sin kl)\,c_3$$

i.e.
$$c_1c_3 = 0 \text{ and } kl = n\pi \text{ (n an integer)}$$

Putting these in (i) and adding a linear term, we have

$$v = a_0 x + b_0 + \sum_{n=1}^{\infty} \sin \frac{n\pi x}{l} e^{-n^2\pi^2 t/RCl^2} \qquad ...(ii)$$

The end conditions of the problem are

$$v = 0 \text{ at } x = 0 \quad \text{and} \quad v = E \text{ at } x = l$$

Using these, (ii) gives $b_0 = 0$ and $a_0 = E/l$

Then (ii) becomes $v = \frac{E}{l} x + \Sigma b_n \sin \frac{n\pi x}{l} e^{-n^2\pi^2 t/RCl^2}$

Also $v = 0$ when $t = 0$, we get $-Ex/l = \Sigma b_n \sin n\pi x/l$

This requires the expansion of $(-Ex/l)$ as a half-range sine series in $(0, l)$.

$$b_n = \frac{2}{l}\int_0^l \left(\frac{-Ex}{l}\right) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{l}\left[\left(\frac{-Ex}{l}\right)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - \left(\frac{-E}{l}\right)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right]_0^l$$

$$= \frac{2}{l}\left(\frac{El}{n\pi}\cos n\pi\right) = \frac{2E}{n\pi}(-1)^n.$$

Thus $$v = \frac{Ex}{l} + \frac{2E}{\pi}\sum_{n=1}^{\infty} \sin\frac{n\pi x}{l} e^{-n^2\pi^2 t/RCl^2} \qquad ...(iii)$$

Also when $L = 0$, $-\frac{\partial v}{\partial x} = Ri$

i.e. $$i = -\frac{1}{R}\frac{\partial v}{\partial x} = -\frac{E}{lR} - \frac{2E}{lR}\sum_{n=1}^{\infty}(-1)^n \cos\frac{n\pi x}{l} e^{-n^2\pi^2 t/RCl^2}$$

At the grounded end $(x = 0)$, the current is

$$i = -\frac{E}{lR} - \frac{2E}{lR}\sum_{n=1}^{\infty}(-1)^n e^{-n^2\pi^2 t/RCl^2}$$

When $t = 1$ sec, $$i = -\frac{E}{lR}\left(1 - 2e^{-\pi^2/RCl^2} + 2e^{-4\pi^2/RCl^2} - \ldots\right) \qquad ...(iv)$$

Since $$\frac{\pi^2}{RCl^2} = \frac{(3.14)^2}{4(5\times 10^{-7})(1000)^2} = 0.548$$

$$e^{-\pi^2/RCl^2} = e^{-0.548} = 0.578$$

When $t \to \infty$, $i \to -E/lR$

Hence from (iv), we have

$$i = -\frac{E}{lR}\left(1 - 2(0.578) + 2(0.578)^4 - 2(0.578)^9 + \ldots\right)$$

$$= -\frac{E}{lR}\left(1 - 1.156 + 0.223 - 0.014 + \ldots\right)$$

$$= i_\infty (0.053) = 5.3\% \text{ of } i_\infty.$$

Example 18.10. *A transmission line 1000 kilometres long is initially under steady state conditions with potential 1300 volts at the sending end (x = 0) and 1200 volts at the receiving end (x = 1000). The terminal end of the line is suddenly grounded, but the potential at the source is kept at 1300 volts. Assuming the inductance and leakance to be negligible, find the potential v(x, t).* (Andhra, 2000)

Sol. The equation of the telegraph line is

$$\frac{\partial^2 v}{\partial x^2} = RC\frac{\partial v}{\partial t} \quad \text{or} \quad \frac{\partial v}{\partial t} = \frac{1}{RC}\frac{\partial^2 v}{\partial x^2} \qquad ...(i)$$

v_1 = initial steady voltage satisfying $\frac{\partial^2 v}{\partial x^2} = 0 = 1300 - x/10 = v(x, 0)$ ...(ii)

v_s' = steady voltage (after grounding the terminal end) when steady conditions are ultimately reached $= 1300 - 1.3x$

$\therefore$ $v(x, t) = v_s' + v_t(x, t)$ where $v_t(x, t)$ is the transient part

$$= 1300 - 1.3x + \sum_{n=1}^{\infty} b_n e^{-(n^2\pi^2 t)/(l^2 RC)} \sin\frac{n\pi x}{l} \quad \text{[By (viii), p. 608]} \qquad ...(iii)$$

where $l = 1000$ kilometres.

Putting $t = 0$, we have from (ii) and (iii)

$$1300 - 0.1x = v(x, 0) = 1300 - 1.3x + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

i.e. $$1.2x = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \quad \text{where} \quad b_n = \frac{2}{l}\int_0^l 1.2 \sin\frac{n\pi x}{l}dx = \frac{2400}{\pi}\cdot\frac{(-1)^{n+1}}{n}$$

Hence $$v(x, t) = 1300 - 1.3x + \frac{2400}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} e^{-(n^2\pi^2 t)/(l^2 RC)} \sin\frac{n\pi x}{1000}$$

PROBLEMS 18.2

1. Find the current i and voltage e in a line of length l, t seconds after the ends are suddenly grounded, given that $i(x, 0) = i_0$, $e(x, 0) = e_0 \sin(\pi x/l)$. Also R and G are negligible.
2. Show that a transmission line with negligible resistance and leakage propagates waves of current and potential with a velocity equal to $1/\sqrt{(LC)}$, where L is the self-inductance and C is the capacitance.
3. A steady voltage distribution of 20 volts at the sending end and 12 volts at the receiving end is maintained in a telephone wire of length l. At time $t = 0$, the receiving end is grounded. Find the voltage and current t sec later. Neglect leakance and inductance.
4. Obtain the solution of the radio equation
$$\frac{\partial^2 v}{\partial x^2} = LC\frac{\partial^2 v}{\partial t^2}$$
appropriate to the case when a periodic e.m.f. $V_0 \cos pt$ is applied at the end $x = 0$ of the line. (Gorakhpur, 1991)

18.11 LAPLACE'S EQUATION IN THREE DIMENSIONS

We have seen that the three dimensional heat flow equation in steady state reduces to

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad ...(1)$$

which is the *Laplace's equation in three dimensions*.

Laplace's equation also arises in the study of gravitational potential at (x, y, z) of a particle of mass m situated at (ξ, η, ζ) given by

$$\frac{Gm}{r} \text{ where } r = \sqrt{(x-\xi)^2 + (y-\eta)^2 + (z-\zeta)^2}.$$

This function is called the *potential of the gravitational field* and satisfies the Laplace's equation.

If a mass of density ρ at (ξ, η, ζ) is distributed throughout a region R, then the gravitational potential u at an external point (x, y, z) is given by

$$u(x, y, z) = G \iiint_R \frac{\rho}{r} \, d\xi \, d\eta \, d\zeta \qquad \ldots(2)$$

Since $\nabla^2(1/r) = 0$ and ρ is independent of x, y, and z, we get

$$\nabla^2 u = \iiint_R \rho \nabla^2 (1/r) \, d\xi \, d\eta \, d\zeta = 0.$$

This shows that the gravitational potential defined by (2) also obeys Laplace's equation.

Thus Laplace's equation (1) is one of the most important equations arising in connection with numerous problems of physics and engineering. *The theory of its solutions is called the potential theory and its solutions are called the harmonic functions.*

In most of the problems leading to Laplace's equation, it is required to solve the equation subject to certain boundary conditions. A proper choice of co-ordinate system makes the solution of the problem simpler. Now we proceed to take up the solutions of (1) in its other forms.

18.12. SOLUTIONS OF THREE DIMENSIONAL LAPLACE'S EQUATION

(1) Cartesian form of $\nabla^2 u = 0$ is $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$ ...(1)

Let $u = X(x)\, Y(y)\, Z(z)$ ...(2)

be a solution of (1). Substituting (2) in (1) and dividing by XYZ, we obtain

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0 \qquad \ldots(3)$$

which is of the form $F_1(x) + F_2(y) + F_3(z) = 0$.

As x, y, z are independent, this will hold good only if F_1, F_2, F_3 are constants. Assuming these constants to be $k^2, l^2, -(k^2 + l^2)$ respectively, (3) leads to the equations

$$\frac{d^2X}{dx^2} - k^2X = 0, \quad \frac{d^2Y}{dy^2} - l^2Y = 0, \quad \frac{d^2Z}{dz^2} + (k^2 + l^2) Z = 0$$

Their solutions are $X = c_1 e^{kx} + c_2 e^{-kx}$, $Y = c_3 e^{ly} + c_4 e^{-ly}$,

$$Z = c_5 \cos \sqrt{(k^2 + l^2)}\, z + c_6 \sin \sqrt{(k^2 + l^2)}\, z$$

Thus a possible solution of (1) is

$$u = (c_1 e^{kx} + c_2 e^{-kx})(c_3 e^{ly} + c_4 e^{-ly})(c_5 \cos \sqrt{(k^2 + l^2)}\, z + c_6 \sin \sqrt{(k^2 + l^2)}\, z).$$

Since the three constants could have been taken as $-k^2, -l^2$ and $k^2 + l^2$, an alternative solution of (1) will be

$$u = (c_1 \cos kx + c_2 \sin kx)(c_3 \cos ly + c_4 \sin ly)\,(c_5 e^{\sqrt{(k^2+l^2)}\,z} + c_6 e^{-\sqrt{(k^2+l^2)}\,z}).$$

(2) Cylindrical form of $\nabla^2 u = 0$ is $\dfrac{\partial^2 u}{\partial \rho^2} + \dfrac{1}{\rho}\dfrac{\partial u}{\partial \rho} + \dfrac{1}{\rho^2}\dfrac{\partial^2 u}{\partial \phi^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$ ...(1)

Let $u = R(\rho)\, H(\phi)\, Z(z)$ [(ii) p. 347]

be a solution of (1). Substituting it in (1), and dividing by RHZ, we get

$$\frac{1}{R}\left(\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho}\right) + \frac{1}{\rho^2 H}\frac{d^2H}{d\phi^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0 \qquad \ldots(2)$$

Assuming that $\dfrac{d^2H}{d\phi^2} = -n^2H$ and $\dfrac{d^2Z}{dz^2} = k^2Z$, ...(3)

(2) reduces to

$$\frac{1}{R}\left(\frac{d^2R}{d\rho^2} + \frac{1}{\rho}\frac{dR}{d\rho}\right) - \frac{n^2}{\rho^2} + k^2 = 0 \quad \text{or} \quad \rho^2\frac{d^2R}{d\rho^2} + \rho\frac{dR}{d\rho} + (k^2\rho^2 - n^2)R = 0.$$

This is Bessel's equation [(§ 16.10)(1)] and its solution is $R = c_1 J_n(k\rho) + c_2 Y_n(k\rho)$.

Also the solutions of equations (3) are

$$H = c_3 \cos n\phi + c_4 \sin n\phi;\ Z = c_5 e^{kz} + c_6 e^{-kz}.$$

Thus a solution of (1) is

$$u = [c_1 J_n(k\rho) + c_2 Y_n(k\rho)][c_3 \cos n\phi + c_4 \sin n\phi][c_5 e^{kz} + c_6 e^{-kz}]$$

which is known as a *cylindrical harmonic*.

(3) Spherical form of $\nabla^2 u = 0$ is (Assam, 1999)

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 u}{\partial \phi^2} = 0 \qquad \ldots(1)$$

Let $u = R(r)\, G(\theta)\, H(\phi)$ be a solution of (1). [(iv) p. 342]

Then $\dfrac{1}{R}\left(r^2\dfrac{d^2R}{dr^2} + 2r\dfrac{dR}{dr}\right) + \dfrac{1}{G}\left(\dfrac{d^2G}{d\theta^2} + \cot\theta\dfrac{dG}{d\theta}\right) + \dfrac{1}{H\sin^2\theta}\dfrac{d^2H}{d\phi^2} = 0$

Putting $\dfrac{1}{R}\left(r^2\dfrac{d^2R}{dr^2} + 2r\dfrac{dR}{dr}\right) = n(n+1)$ ...(2)

and $\dfrac{1}{H}\dfrac{d^2H}{d\phi^2} = -m^2$, ...(3)

the above equation takes the form

$$\frac{d^2G}{d\theta^2} + \cot\theta\frac{dG}{d\theta} + [n(n+1) - m^2 \operatorname{cosec}^2\theta]G = 0 \qquad \ldots(4)$$

Now differentiating the *Legendre's equation* (§ 16.13)

$$(1 - x^2)y'' - 2xy' + n(n+1)y = 0,$$

m times with respect to x and writing $u = d^m y/dx^m$, we get

$$(1 - x^2)u'' - 2(m+1)xu' + (n-m)(n+m+1)u = 0 \qquad \ldots(5)$$

Now putting $G = (1 - x^2)^{m/2} u$ in (5), we get

$$(1 - x^2)\frac{d^2G}{dx^2} - 2x\frac{dG}{dx} + \left[n(n+1) - \frac{m^2}{1 - x^2}\right]G = 0 \qquad ...(6)$$

Now putting $x = \cos\theta$ in (6), it reduces to (4) and its solution is

$$G = c_1 P_n^m(\cos\theta) + c_2 Q_n^m(\cos\theta)$$

The solution of (3) is $H = c_3 \cos m\phi + c_4 \sin m\phi$

To solve (2), write $R = r^k$, so that $k(k-1) + 2k = n(n+1)$ which gives $k = n$ or $-(n+1)$

Thus $\qquad R = c_5 r^n + c_6 r^{-n-1}$

Hence the general solution of (1) is

$$u = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}[c_1 P_n^m(\cos\theta) + c_2 Q_n^m(\cos\theta)](c_3 \cos m\phi + c_4 \sin m\phi) \times (c_5 r^n + c_6 r^{-n-1})$$

Any solution of (1) is known as a *spherical harmonic*.

Example 18.17. *Find the potential in the interior of a sphere of unit radius when the potential on the surface is $f(\theta) = \cos^2\theta$.*

Sol. Take the origin at the centre of the given sphere S. Since the potential is independent of ϕ on S, so also is the potential at any point. Therefore, the Laplace's equation in spherical co-ordinates reduces to

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial\theta} = 0 \qquad ...(i)$$

Putting $u(r, \theta) = R(r).\Theta(\theta)$ in (i) and proceeding as in § 18.12 (3), we obtain the equations

$$\frac{d^2\Theta}{d\theta^2} + \cot\theta\frac{d\Theta}{d\theta} + n(n+1)\Theta = 0 \qquad ...(ii)$$

and

$$\frac{1}{R}\left(r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr}\right) = n(n+1) \qquad ...(iii)$$

Putting $\cos\theta = v$, (ii) takes the form

$$(1 - v^2)\frac{d^2\Theta}{dv^2} - 2v\frac{d\Theta}{dv} + n(n+1)\Theta = 0$$

which is Legendre's equation. Its solutions are

$$\Theta = P_n(v) = P_n(\cos\theta) \text{ for } n = 0, 1, 2, ...$$

The solutions of (iii) are $R_n(r) = r^n$, $\bar{R}_n(r) = 1/r^{n+1}$.

Hence the equation (i) has the following two sets of solutions

$u_n(r, \theta) = c_n r^n P_n(\cos\theta)$ and $\bar{u}_n(r, \theta) = c_n P_n(\cos\theta)/r^{n+1}$, where $n = 0, 1, 2, ...$

For points inside S, we have the general equation $u(r, \theta) = \sum_{n=0}^{\infty} c_n r^n P_n(\cos\theta) \qquad ...(iv)$

On the boundary of S, $u(1, \theta) = f(\theta)$ $\therefore$ $f(\theta) = \sum_{n=0}^{\infty} c_n P_n(\cos\theta)$

which is Fourier-Legendre expansion of $f(\theta)$. Hence by (5) p. 525, we have

$$c_n = \left(n + \frac{1}{2}\right)\int_{-1}^{1} f(\theta) P_n(x)\,dx \text{ where } x = \cos\theta.$$

$$= \left(n + \frac{1}{2}\right)\int_{-1}^{1} x^2 P_n(x)\,dx \qquad [\because f(\theta) = \cos^2\theta]$$

$$= \left(n + \frac{1}{2}\right)\int_{-1}^{1}\left[\frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)\right]P_n(x)\,dx \qquad [\because P_2(x) = \tfrac{1}{2}(3x^2 - 1)]$$

Using the orthogonality of Legendre polynomials, we get

$c_n = 0$, except for $n = 0, 2$. Hence

$$c_0 = \frac{1}{2}\cdot\frac{1}{3}\int_{-1}^{1} P_0^2(x)\,dx = \frac{1}{3},\quad c_2 = \frac{5}{2}\cdot\frac{2}{3}\int_{-1}^{1} P_2^2(x)\,dx = \frac{2}{3}.$$

Substituting in (iv), we get $u(r, \theta) = \frac{1}{3} + \frac{2}{3}r^2 P_2(\cos\theta)$ or $u(r, \theta) = \frac{1}{3} + r^2\left(\cos^2\theta - \frac{1}{3}\right)$.

Problems 18.[illegible]

1. Show that a solution of Laplace's equation in cylindrical co-ordinates, which remains finite at $r = 0$, may be expressed in the form

$$u = \sum_{n=0}^{\infty} J_n(kr)[e^{kz}(A_n \cos n\theta + B_n \sin n\theta) + e^{-kz}(C_n \cos n\theta + D_n \sin n\theta)].$$

2. The potential on the surface of a unit sphere is $f(\theta) = \cos 2\theta$. Show that the potential at all points of space is given by

$$u(r, \theta) = 2r^2(\cos^2\theta - 1/3) - \frac{1}{3} \text{ for } r < 1,$$

and $\qquad u(r, \theta) = 2r^{-3}(\cos^2\theta - 1/3) - r^{-1}/3$ for $r > 1$.

3. Show that in spherical polar co-ordinates (r, θ, ϕ), Laplace's equation possesses solutions of the form

$$(Ar^n + B/r^{n+1})P_n^m(\mu)e^{\pm im\phi},$$

where $\mu = \cos\theta$, A, B, m, n are constants and $P_n^m(\mu)$ satisfies Legendre's equation

$$(1 - \mu^2)\frac{d^2P_n}{d\mu^2} - 2\mu\frac{dP_n}{d\mu} + \left\{n(n+1) - \frac{m^2}{1 - \mu^2}\right\}P_n = 0.$$