IAS - 2012 (1)

1) b) Q Let p, q be +ve real numbers s.t $1/p + 1/q = 1$
s.t for real numbers $a, b \geq 0$; $a^p/p + \frac{b^q}{q} \geq ab$

Sol Given that

$$p+q = pq \quad (1)$$

also let us assume that $\frac{a^p}{p} + \frac{b^q}{q} < ab$

$$\Rightarrow \quad (a^p)q + (b^q)p < ab(p+q)$$

This doesn't hold good because p, q are +ve real numbers

$$\therefore \quad (a^p)q + (b^q)p \geq ab(p+q)$$

Hence proved.

(or)

we know a inequality if $0 < t < 1$ then

$$x^t \leq 1 - t + xt$$

$$\forall x \geq 0$$

now put $x = a^p b^{-q}$, $t = 1/p$

$$\Rightarrow (a^p b^{-q})^{1/p} \leq (1 - 1/p) + \frac{(a^p b^{-q})}{p} \quad [\because \text{ multiplying } b^q]$$

$$\Rightarrow (a^p b^{-q})^{1/p} \cdot b^q \leq \frac{b^q}{q} + \frac{a^p}{p} \quad \left[\because 1/p = 1 - 1/q\right.$$

& vice versa]

$\Rightarrow a\, b^{q}(1-1/p) \le \frac{a^p}{p} + \frac{b^q}{q} \Rightarrow ab \le \frac{a^p}{p} + \frac{b^q}{q}$

Hence proved.

(10) find local extremum & saddle points of f

Sol Given $f(x,y) = x^3 + y^3 - 63(x+y) + 12xy$ — (1)

$f_x = 3x^2 - 63 + 12y$, $f_y = 3y^2 - 63 + 12x$

$f_{xx} = 12$ $\quad$ $f_{yy} = 6y$

put $f_x = 0 = f_y$

$3x^2 - 63 + 12y = 0$, $3y^2 - 63 + 12x = 0$

$x^2 + 4y - 21 = 0$ — (2) ; $y^2 + 4x - 21 = 0$ — (3)

solving (2) and (3) by (2) − (3)

$\Rightarrow x^2 - y^2 = 4(x-y)$

$\Rightarrow (x+y)\cancel{(x-y)} = 4\cancel{(x-y)} \longrightarrow x = y$ (4)

$x + y = 4$ (5)

put (4) in (2) we get $(-7,-7)$ & $(3,3)$ (6)

put (5) in (2) $x^2 + 4(4-x) - 21 = 0$

$\Rightarrow (-1,5)\ (5,-1)$. (7)

(6), (7) give four stationary points.

at $(-7,-7)$ $\quad rt - s^2 > 0$ & $r < 0 \Rightarrow$ max at $(-7,-7)$

at $(3,3)$ $\quad r > 0$ & $rt - s^2 > 0 \Rightarrow$ min at $(3,3)$

also $rt - s^2 = g = 36xy - 144$ ③

at $(-1, 5) \Rightarrow g < 0$; at $(5, -1)$ $g < 0$

$\therefore$ saddle points are $(-1, 5)$ $(5, -1)$.

3(b) Define a sequence S_n of reals by $S_n = \sum_{i=1}^{n} \frac{[\log(n+i) - \log n]^2}{n+i}$

does $\lim_{n\to\infty} S_n$ exist and compute it

Sol Given, $\lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\left[\log\left(\frac{n+i}{n}\right)\right]^2}{n+i}$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \frac{[\log(1+i/n)]^2}{n(1+i/n)} \quad — ①$$

[$\because$ Reimann sum] $\lim_{n\to\infty} \sum_{i=0}^{n} \to \int_0^1$ ② $\frac{i}{n} \to x$ ③ $\frac{1}{n} \to dx$ — ②

put ② in ①

$$\Rightarrow \lim_{n\to\infty} S_n = \int_0^1 \frac{[\log(1+x)]^2}{1+x}\,dx$$

put $\log(1+x) = t$

$\frac{dx}{1+x} = dt$

$$= \int_0^{\log 2} t^2\,dt$$

$$= \frac{(\log 2)^3}{3}$$

$\therefore \lim_{n\to\infty} S_n$ exists

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* This is a corrected question, given question has denominator of $1+n$, which problem can't be proceeded.

⑩ find the real values of 'p' and 'q' so that $\int_0^1 x^p (\log 1/x)^q$ converges

Sol $\log 1/x = t \Rightarrow x = e^{-t}$

$\Rightarrow \int_0^\infty t^q e^{-(p+1)t} dt = I$

$\Rightarrow \int_0^\infty t^q e^{-(p+1)t} dt$

$= \int_0^1 t^q e^{-(p+1)t} dt + \int_1^\infty t^q e^{-(p+1)t} dt$

[$\because$ 0 — infinite discontinuity point]

at 0 : $f(t) = \dfrac{e^{-(p+1)t}}{t^{-q}}$ ①

① Converges only if $-q < 1$

$\Rightarrow q > -1 \ \forall p$

at ∞

$f(t) = \dfrac{t^q}{e^{(p+1)t}}$

let $g(t) = 1/t^2$

$\dfrac{f(t)}{g(t)} = \dfrac{t^{q+2}}{e^{(p+1)t}} \to 0$ as $t \to \infty$

$\forall q$ and $p+1 > 0$

also $\int_1^\infty g(t)$ converges

hence $f(t)$ converges $\forall q$ iff $p+1 > 0$

$\therefore$ ~~for~~ $p > -1$, $q > -1$.