Vector [Extra Problems]

$$\frac{dF}{\partial x\,\partial y} = \frac{\partial}{\partial y}\left\{\frac{\partial f}{\partial x}\right\}$$

① Show $\nabla^2\left(\frac{x}{r^3}\right) = 0$.

$$\frac{\partial^2}{\partial x^2}\left(\frac{x}{r^3}\right) = \cancel{x}\,\frac{\partial}{\partial x}\left(\frac{\partial}{\partial x}\left(\frac{x}{r^3}\right)\right)$$

$$= \cancel{x}\,\frac{\partial}{\partial x}\left(\frac{1}{r^3} - \frac{3x}{r^4}\cdot\frac{x}{r}\right)$$

$$= \cancel{x}\left(\frac{(-3)x}{r^4\cdot r} - \frac{6x}{r^5} + \frac{5\cdot 3x^2}{r^6}\cdot\frac{x}{r}\right)$$

$$= \cancel{x} - \frac{9x}{r^5} + \frac{15x^3}{r^7}$$

$$\frac{\partial^2}{\partial y^2}\left(\frac{x}{r^3}\right) = \frac{\partial}{\partial y}\left(\frac{\partial}{\partial y}\,\frac{x}{r^3}\right)$$

$$= \frac{\partial}{\partial y}\left(-\frac{3x}{r^4}\cdot\frac{y}{r}\right)$$

$$= -\frac{3x}{r^5} - \frac{3xy\cdot(-5)}{r^6}\,\frac{y}{r} = -\frac{3x}{r^5} + \frac{15xy^2}{r^7}$$

$$\frac{\partial^2}{\partial z^2}\left(\frac{x}{r^3}\right) = -\frac{3x}{r^5} + \frac{15xz^2}{r^7}$$

Adding $= -\frac{15x}{r^5} + 15x\left(\frac{\cancel{r^2}}{r^5}\right) = 0$ Proved

② div $(A \times B) = B.\text{curl } A - A.\text{curl } B$

$$\sum i.\frac{\partial}{\partial x}(A\times B) = \sum i.\left(\frac{\partial A}{\partial x}\times B\right) + i.\left(A\times\frac{\partial B}{\partial x}\right)$$

$$= \left(\sum\left(i\times\frac{\partial A}{\partial x}\right).B - \left(i\times\frac{\partial B}{\partial x}\right).A\right)$$

$$= \text{Curl } A.B - \text{curl } B.A$$

③ grad $(A.B) = (B.\nabla)A + (A.\nabla)B + B\times\text{curl } A + A\times\text{curl } B.$

$$\sum i\frac{\partial}{\partial x}(A.B) = \sum\left\{\left(A.\frac{\partial B}{\partial x}\right)i\right\} + \sum\left(B.\frac{\partial A}{\partial x}\right)i$$

Now get curl $\Rightarrow$ $a\times(b\times c) = (a.c)b - (a.b)c.$

$$\left(A.\frac{\partial B}{\partial x}\right)i = \cancel{A\times\left(i\times\frac{\partial B}{\partial x}\right) = \left(A.\frac{\partial B}{\partial x}\right)i - (A.i)\frac{\partial B}{\partial x}}.$$

Same order as given ; (A-B-i)

$$= A\times\left(\frac{\partial B}{\partial x}\times i\right) = (A.i)\frac{\partial B}{\partial x} - \left(A.\frac{\partial B}{\partial x}\right)i$$

$$\rightarrow (A.i)\frac{\partial B}{\partial x} - A\times\left(\frac{\partial B}{\partial x}\times i\right)$$

$$\sum\left(A.\frac{\partial B}{\partial x}\right)i = (A.\nabla)B + A\times(\nabla\times B)$$

Add

Illy $$\sum\left(B.\frac{\partial A}{\partial x}\right)i = (B.\nabla)A + B\times(\nabla\times A)$$

$$\boxed{\text{grad}(A.A) = 2(A.\nabla)A + 2A\times(\nabla\times A)}$$

② $div(A \times B) = B \cdot curl\, A - A \cdot curl\, B$

$$\sum i \cdot \frac{\partial}{\partial x}(A \times B) = \sum i \cdot \left(\frac{\partial A}{\partial x} \times B\right) + i \cdot \left(A \times \frac{\partial B}{\partial x}\right)$$

$$= \left[\sum \left(i \times \frac{\partial A}{\partial x}\right) \cdot B - \left(i \times \frac{\partial B}{\partial x}\right) \cdot A\right.$$

$$= Curl\, A \cdot B - curl\, B \cdot A$$

③ $grad(A \cdot B) = (B \cdot \nabla)A + (A \cdot \nabla)B + B \times curl\, A + A \times curl\, B.$

$$\sum i \frac{\partial}{\partial x}(A \cdot B) = \sum \left\{\left(A \cdot \frac{\partial B}{\partial x}\right) i\right\} + \sum \left(B \cdot \frac{\partial A}{\partial x}\right) i$$

Now get curl $\Rightarrow \nabla \times (b \times c) = (a \cdot c)b - (a \cdot b)c.$

$$\left(A \cdot \frac{\partial B}{\partial x}\right) i \equiv \cancel{A \times \left(i \times \frac{\partial B}{\partial x}\right) = \left(A \cdot \frac{\partial B}{\partial x}\right) i - (A \cdot i)\frac{\partial B}{\partial x}}$$

Same order as given $\Rightarrow$ (A–B–i)

$$\equiv A \times \left(\frac{\partial B}{\partial x} \times i\right) = (A \cdot i)\frac{\partial B}{\partial x} - \left(A \cdot \frac{\partial B}{\partial x}\right) i$$

$$\to (A \cdot i)\frac{\partial B}{\partial x} - A \times \left(\frac{\partial B}{\partial x} \times i\right)$$

$$\sum \left(A \cdot \frac{\partial B}{\partial x}\right) i = (A \cdot \nabla)B + A \times (\nabla \times B)$$

Add

$|||^{ly}$ $$\sum \left(B \cdot \frac{\partial A}{\partial x}\right) i = (B \cdot \nabla)A + B \times (\nabla \times A)$$

$$\boxed{grad(A \cdot A) = 2(A \cdot \nabla)A + 2A \times (\nabla \times A)}$$

(4) $\operatorname{div}(r^n \bar{r}) = (n+3) r^n$.

$\operatorname{div}(\phi \bar{A}) = (\nabla \phi) \cdot \bar{A} + r^n \operatorname{div} \bar{r}$

$= (\nabla r^n) \cdot \bar{r} + r^n \times 3 = n r^{n-2} (\bar{r} \cdot \bar{r}) + 3 r^n$

$i \frac{\partial}{\partial x}(r^n) = n r^{n-1} \frac{x i}{r}$

$= \boxed{n r^n + 3 r^n}$

(5) Show $f(r)\bar{r}$ is irrotational.

$\operatorname{curl}(f(r)\bar{r}) = (\operatorname{grad} f(r)) \times \bar{r} + f(r)(\operatorname{curl} \bar{r})^{0}$

$= \left(\sum i \frac{\partial}{\partial x}(f(r))\right) \times \bar{r} = \sum f'(r) \frac{\partial r}{\partial x} i \times \bar{r}$

$= f'(r) [\operatorname{grad} r] \times \bar{r}$

$= f'(r) \left(\frac{\bar{r}}{r} \times \bar{r}\right) = \underline{\underline{0}}$

(6) Prove $\frac{1}{2} \nabla a^2 = (a \cdot \nabla) a + (a \times \operatorname{curl} a)$

Consider ~~$\nabla \times (a \times b)$ identity~~

$\nabla (a \cdot b)$ identity $\equiv$ where $b = a$.

$\nabla(a \cdot b) = \sum i \frac{\partial}{\partial x}(A \cdot B) = \sum i \left(A \cdot \frac{\partial B}{\partial x}\right) + \sum i \left(\frac{\partial A}{\partial x} \cdot B\right)$

$= \underbrace{\sum \left(A \cdot \frac{\partial B}{\partial x}\right) i}_{} + \sum \left(\frac{\partial A}{\partial x} \cdot B\right) i$

$A \times \left(\frac{\partial B}{\partial x} \times i\right) = (A \cdot i) \frac{\partial B}{\partial x} - \left(A \cdot \frac{\partial B}{\partial x}\right) i$ $\Downarrow$ $\Downarrow$ Illy.

$(A \cdot \nabla) B + A \times (\operatorname{curl} B)$.

$b = a \Rightarrow$

$\nabla(a \cdot a) = 2((A \cdot \nabla) A + A \times (\nabla \times A))$ Proved

$div(A \times B) = B.\, curl\, A - A.\, curl\, B$

(7) $div\,(\nabla\phi \times \nabla\psi) = 0.$

$$\nabla.(\nabla\phi \times \nabla\psi) = \nabla\psi.(\underbrace{\nabla \times \nabla\phi}_{0}) - \nabla\phi.(\underbrace{\nabla \times \nabla\psi}_{0})$$

$$= 0.$$

(8) $\nabla.\left\{ r\nabla\left(\frac{1}{r^3}\right)\right\} = \frac{3}{r^4}$

$$(\nabla . r) . \nabla\left(\frac{1}{r^3}\right) + r\,\nabla^2\left(\frac{1}{r^3}\right).$$

$$\left(\frac{\bar{r}}{r}\right) . \left(\frac{(-3)\,\bar{r}}{r^4 . r}\right) + r\,\nabla.\left(-\frac{3}{r^4 . r}\bar{r}\right)$$

$$-\frac{3}{r^4} + r(-3)\nabla.\frac{\bar{r}}{r^5} = \cancel{\sum i . \frac{\partial}{\partial x}\frac{\bar{r}}{r^5}}$$

$$\left(\nabla\frac{1}{r^5}\right) . \bar{r} + \frac{1}{r^5} \times (\nabla . \bar{r}).$$

$$\frac{-5\cancel{r^2}}{r^5} + \frac{3}{r^5} = -\frac{2}{r^5}.$$

$$-\frac{3}{r^4} + (+3)\cancel{r}.\frac{(+2)}{r^5} = \boxed{\frac{3}{r^4}}$$ Proved

$$\nabla.(A \times B) = B.\,curl\,A - A.\,curl\,B$$

(9) $\nabla \times (F \times \bar{r}) = 2F - (\nabla \cdot F) r + (r \cdot \nabla) F$

Using Identity $\Rightarrow$

$$F(\underbrace{\nabla \cdot \bar{r}}_{3}) - \bar{r}(\nabla \cdot \bar{F}) + (r \cdot \nabla)F - (F \cdot \nabla) r$$

$$3F - \bar{r}(\nabla \cdot \bar{F}) + (\bar{r} \cdot \nabla)F - (F \cdot \nabla)\bar{r}$$

Let F be $F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k}$

$$(F \cdot \nabla)\bar{r} = \left(F_1 \frac{\partial}{\partial x} + F_2 \frac{\partial}{\partial y} + F_3 \frac{\partial}{\partial z}\right)(x\hat{i} + y\hat{j} + z\hat{k}$$

$$= F_1 \hat{i} + F_2 \hat{j} + F_3 \hat{k} = F$$

$$\boxed{2F + (r \cdot \nabla)F - (\nabla \cdot \bar{F})\bar{r}} \quad \text{Ans}$$

(10) $\text{grad}\,[(r \times a) \cdot (r \times b)] = (b \times r) \times a + (a \times r) \times b$

Lagrange Identity $\Rightarrow$ $(r \times a) \cdot (r \times b) = (a \cdot b)(r \cdot r) - (r \cdot b)(r \cdot a)$

$$\text{grad}\,[(a \cdot b)(r \cdot r)] - \text{grad}\,[(r \cdot b)(r \cdot a)]$$

$$(a \cdot b)\,\text{grad}\underbrace{(r \cdot r)}_{2r} + (r \cdot r)\,\overset{0}{\text{grad}}\,(a \cdot b) - (r \cdot b)\,\text{grad}\underbrace{(r \cdot a)}_{a} - (r \cdot a)\,\text{grad}\underbrace{(r \cdot b)}_{b}$$

$$(a \cdot b)2r - (r \cdot b)a - (a \cdot r)b$$

$$[(a \cdot b)r - (r \cdot a)b] + [(a \cdot b)r - (r \cdot b)a]$$

$$(b \times r) \times a + (a \times r) \times b$$

$$(\bar{r} \times \bar{a}) \cdot (\bar{r} \times \bar{b}) = \begin{vmatrix} \bar{r} \cdot \bar{r} & \bar{r} \cdot \bar{a} \\ \bar{r} \cdot b & \bar{a} \cdot \bar{b} \end{vmatrix}$$

(11) $a.\{\nabla(v.a) - \nabla\times(v\times a)\} = \text{div } v$

~~$(\nabla v).a + (\nabla a).v$~~

$$grad(A.B) = (A.\nabla)B + (B.\nabla)A + A\times(\nabla\times B) + B\times(\nabla\times A)$$

$$\nabla(v.a) = \cancel{(v.\nabla)a}^{0} + (a.\nabla)v + \cancel{v\times(\nabla\times a)}^{0} + a\times(\nabla\times v)$$

$v = v_1\hat{i} + v_2\hat{j} + v_3\hat{k}$ — (1)

$$= (a.\nabla)v + (a\times(\nabla\times v))$$

$$\nabla\times(v\times a) = \cancel{(\nabla.a)}^{0}v - (\nabla.v)a + (a.\nabla)v - \cancel{(v.\nabla)}^{0}a$$

— (2)

$$(1) - (2) \Rightarrow a\times(\nabla\times v) + (\nabla.v)a$$

$$a.((1)-(2)) = a.[\cancel{a\times}^{0}(\nabla\times v)] + a.[(\nabla.v)a]$$

$= (\nabla.v)\,a^2 \Leftarrow$ a given unit vector

$= (\nabla.v)$ Ans [div v]

* $\nabla\phi . d\bar{r} = d\phi$ $\Rightarrow$ $\nabla\phi = \frac{\partial\phi}{\partial x} i + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$

$d\bar{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$

$\Rightarrow \nabla\phi . d\bar{r} = \frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz = d\phi$

* Directional derivative $\equiv$ $(\text{grad} f . \hat{a})$

Max rate of change $\equiv$ $|\text{grad} f|$.

* Prove $\nabla \times (\nabla \times A) = \nabla(\nabla . A) - \nabla^2 A$

$\nabla \times A = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_1 & A_2 & A_3 \end{vmatrix}$ .... and so on.

* $\nabla \times [r \times (a \times r)] = 3 r \times a$

$\nabla \times [(r.r)\bar{a} - (r.a)\bar{r}] = [\nabla(r^2) \times \bar{a}] + r^2[\cancel{\nabla \times \bar{a}}]^{0} - [\nabla(r.a) \times \bar{r}]$

$- [(\cancel{\nabla \times \bar{r}})^{0}(\bar{r}.\bar{a})]$

$= \left[2\frac{r}{r}\bar{r} \times \bar{a}\right] + [\bar{r} \times \bar{a}] = 3[\bar{r} \times \bar{a}]$ Ans

S = arc length

$\frac{d\bar{r}}{dS} = \hat{t}$ (unit tangent)

* $\bar{r}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$

* Surface Integral $\equiv \iint_S f(x,y,z)\, dS$

Flux $= \iint_S \bar{F}\cdot\hat{n}\, dS$ OR $\iint_S \bar{F}\cdot d\bar{S}$ ($d\bar{S}$ includes $\hat{n}$)

* $\int_c f\, ds = \int_c f \frac{ds}{dt}\cdot dt \qquad \left[\frac{ds}{dt} = \left|\frac{d\bar{r}}{dt}\right|\right]$

* Green's Theorem $\equiv$ R is region (closed, bdd), C is smooth boundary

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \oint_c M dx + N dy$$

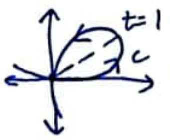
M, N continuous and $\frac{\partial M}{\partial y}, \frac{\partial N}{\partial x}$ also continuous in R.

C such that, R lies on left as we move on C.

ALITER $\equiv \boxed{\oint_c F\cdot d\bar{r} = \iint_R (\nabla \times F)\cdot k\, dS}$

- Area $= \boxed{\frac{1}{2}\oint x\,dy - y\,dx}$

* Folium $\equiv$



$x^3 + y^3 = 3axy$

$\boxed{\begin{aligned} y &= tx \\ x &= \frac{3at}{1+t^3} \end{aligned}}$

* $\iint_S n \times B\, dS = \iiint_V \nabla \times B\, dV$

Circulation of F around C.

① $\oint_C F.dr$ $\quad F = i\cos y - j\, x\sin y$

$C \equiv y = \sqrt{1-x^2}$ $\quad (1,0)$ to $(0,1)$

$$\int_C F.dr = \int_C (i\cos y - j\, x\sin y).(dx\,\hat{i} + dy\,\hat{j})$$

$$= \int_C \cos y\, dx - x\sin y\, dy = \int_C d(x\cos y) = x\cos y \Big]_{1,0}^{0,1}$$

$= -1$ Ans

②(✱) $\int_C x^{-1}(y+z)\, ds$ $\quad C \equiv x^2+y^2=4$ from $(2,0,0)$ to $(\sqrt{2},\sqrt{2},0)$

$x = 2\cos t,\ y = 2\sin t,\ z = 0$

$t \in [0, \pi/4]$

$$\int_C x^{-1}(y+z)\frac{ds}{dt}\, dt.$$

$$\frac{dr}{dt} = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} = -2\sin t\,\hat{i} + 2\cos t\,\hat{j}$$

$$\left(\frac{dr}{dt}\right)^2 = 4 \qquad \left(\frac{dr}{dt}\right)^2 = \left(\frac{dr}{ds}\frac{ds}{dt}\right)^2 = \left(\frac{ds}{dt}\right)^2 \hat{t}^2 = \left(\frac{ds}{dt}\right)^2$$

So $\boxed{\frac{ds}{dt} = 2}$

$$\int_0^{\pi/4} 2\tan t\, dt = \log 2 \text{ Ans}$$

(2.1) $\int \bar{F}.d\bar{r}$ along $x^2+y^2=1,\ z=1$ from $(0,1,1)$ to $(1,0,1)$

$F = (2x+yz)\hat{i} + xz\hat{j} + (xy+2z)\hat{k}$

Aliter
$x = \cos t,\ y = \sin t.$

$dr = dx\,\hat{i} + dy\,\hat{j}.$

$$\int \bar{F}.d\bar{r} = \int_0^1 2x+y\, dx + \int_1^0 x\, dy$$

$$= \int_0^1 2x + \sqrt{1-x^2}\, dx + \int_0^1 \sqrt{1-y^2}\, dy.$$

$= 1 + 0$ Ans

(✱) $\int_C t.dr = \int_C t.\frac{dr}{ds}\, ds = \int \frac{dr}{ds}.\frac{dr}{ds}\, ds$ $\quad$ t is unit tangent. C is unit circle

$$= \int_C dS = 2\pi$$

(3) $F = \dfrac{-y\hat{i} + x\hat{j}}{x^2+y^2}$. Evaluate $\int_C F \cdot dr$ around any closed path C in x-y plane

$r = x\hat{i} + y\hat{j} \rightarrow dr = dx\,\hat{i} + dy\,\hat{j}$

$$\int_C F \cdot dr = \int_C \frac{-y\,dx + x\,dy}{x^2+y^2}$$

Put $x = r\cos\theta,\ y = r\sin\theta$; $dx = -r\sin\theta\, d\theta + \cos\theta\, dr$

$dy = r\cos\theta\, d\theta + \sin\theta\, dr$

$$\int_C d\theta = \int_C F \cdot dr$$

Case I $\Rightarrow$ If origin lies inside C $\theta \in [0, 2\pi]$

Case II $\Rightarrow$ If origin is outside $\theta \in [\theta_0, \theta_0]$

$\int_C d\theta = 0$	$\int_C d\theta = 2\pi$

(4) $\int_C \{(2xy^3 - y^2\cos x)\,dx + (1 - 2y\sin x + 3x^2y^2)\,dy\}$ $C \equiv 2x = \pi y^2$ $(0,0)\ (\frac{\pi}{2}, 1)$

$M\,dx + N\,dy \rightarrow \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

$$\boxed{6xy^2 - 2y\cos x = -2y\cos x + 6xy^2}$$

$M = \dfrac{\partial \phi}{\partial x} \Rightarrow \phi = x^2y^3 - y^2\sin x + f_1(y)$

$\dfrac{\partial \phi}{\partial y} = 1 - 2y\sin x + 3x^2y^2 \Rightarrow \phi = y - y^2\sin x + x^2y^3 + f_2(x)$

$f_1(y) = y$, $f_2(x) = 0 \Rightarrow \phi = y - y^2\sin x + x^2y^3$

$$\Big[y - y^2\sin x + x^2y^3\Big]_{0,0}^{\frac{\pi}{2},1} = \frac{\pi^2}{4}\ \underline{\text{Ans}}$$

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* $\iint_S F.n \, dS$

① Find $\hat{n}$ vector of Surface S $= \frac{\nabla S}{|\nabla S|}$

② Perform $F.\hat{n}$ and integrate out.

③ $\iint_S F.n\,dS = \iint_R f.n \frac{dx\,dy}{n.k}$ [Done after projecting it on one of principal planes.

[If on XY → divide by $\hat{n}.\hat{k}$]

⑤ Evaluate $\iint_S F.n\,ds$ $F = z\hat{i} + x\hat{j} - 3y^2 z\hat{k}$ and S is surface of cylinder $x^2+y^2=16$ between $z=0$, $z=5$ in 1st octant.

$$\hat{n} = \frac{\nabla(x^2+y^2)}{|\nabla(x^2+y^2)|} = \frac{2x\hat{i}+2y\hat{j}}{\sqrt{4x^2+4y^2}} = \frac{x\hat{i}+y\hat{j}}{4}$$

$$\iint_S F.n\,dS = \iint_R F.n \frac{dx\,dz}{|n.\hat{j}|}$$

$$\iint_R \frac{xz+xy}{\cancel{4}} \frac{dx\,dz}{y/\cancel{4}}$$

limits ⇒ $y = \sqrt{16-x^2}$

$x = 0, 4$ $z = 0, 5$

$$\int_0^5 \int_0^4 \frac{xz}{\sqrt{16-x^2}} + x\,dx\,dz = \boxed{90} \text{ Ans}$$

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* Projection of S on X-Y plane only if line ⊥r to XY plane meets S in no more than one point

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limits = $x=0, y=0, z=0$ $4x+2y+z=8$

$$\int_0^2 \int_0^{4-2x} \int_0^{8-4x-2y}$$

(6) $\iint_S F.n\, dS$ $F = (x+y^2)\hat{i} - 2x\hat{j} + 2yz\hat{k}$

S = Surface of $2x+y+2z=6$ in 1st od.

$$\hat{n} = \frac{(\nabla S)}{|\nabla S|} = \cancel{\hat{i} + 2y\hat{k}} \quad \boxed{\frac{2\hat{i}+\hat{j}+2\hat{k}}{3.}}$$

$$\frac{1}{3}\iint_{S.} (\cancel{2x} + 2y^2 - \cancel{2x} + 4yz)\, dS.$$

On X-Y plane $|\hat{n}.\hat{k}| = 2/3$

$$\frac{1}{\cancel{3}}\iint \frac{(2y^2+4yz)}{2}.\cancel{3}\, dx\, dy.$$

$Z = 3 - x - y/2.$

$y = 0, 6-2x$

$x = 0, 3.$

$$\int_0^3 \int_0^{6-2x} y^2 + 2y\left(3-x-\frac{y}{2}\right) dx\, dy.$$

$$\int_0^3 \left[\frac{y^3}{2} + 3y^2 - xy^2 - \frac{y^3}{3}\right]_0^{6-2x} dx$$

$$\int_0^3 \frac{\cancel{(6-2x)^3}}{\cancel{2}} + 3(6-2x)^2 - x(6-2x)^2 - \frac{1}{2}(6-2x)^3\, dx$$

$$\int_0^3 \cancel{108} - \cancel{36x} - \cancel{72x} + 72x^2 + 12x^2 - \cancel{4x^3} - \cancel{108} + \cancel{108x} - 36x^2 + \cancel{4x^3}\, dx$$

$$\int_0^3 48x^3\, dx \Rightarrow 12x^4\Big]_0^3 \Rightarrow$$

X

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(7) $\iint F \cdot n\, dS$ $\quad S = x^2 + y^2 = 9 \quad z = 0, 4$

$F = z\hat{i} + x\hat{j} - yz\hat{k}$

$$\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{2x\hat{i} + 2y\hat{j}}{6}$$

$$\iint_S \frac{xz}{\cancel{3}} + \frac{xy}{\cancel{3}} \frac{dy\,dz}{(\cancel{x})} \qquad (\hat{n} \cdot \hat{i}) = \frac{x}{3}.$$

$$\int_0^4 \int_0^3 (z + y)\, dy\, dz.$$

$$\int_0^4 \left[zy + \frac{y^2}{2}\right]_0^3 dz = \int_0^4 3z + \frac{9}{2}\, dz = \frac{3z^2}{2} + \frac{9}{2}z\Big]_0^4$$

$= 24 + 18 = 42$ Ans

(8) Find area bounded by simple closed curve C to be

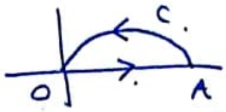
$\frac{1}{2} \oint_c x\,dy - y\,dx$ $\qquad M\,dx + N\,dy$

$$\text{Area} = \iint_S dS = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

Take $N = x$, $M = -y \Rightarrow 2 \underbrace{\iint dx\,dy}_{\text{Area}} = \oint_c -y\,dx + x\,dy.$

So, Area $= \frac{1}{2} \oint_c x\,dy - y\,dx$

⑨ Find area under of one arc $\equiv$ $x = a(\theta - \sin\theta)$; $y = a(1-\cos\theta)$



One arc $\equiv$ $\theta \in [0, 2\pi]$.

$$\text{Area} = \frac{1}{2}\oint_C x\,dy - y\,dx$$

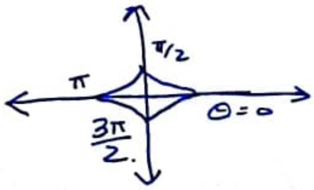
$$= \frac{1}{2}\Bigg[\int_{OA} x\,dy - y\,dx + \int_{ACO} x\,dy - y\,dx\Bigg]$$

OA: $y=0,\ dy=0$ $\downarrow$ 0

$$= \frac{1}{2}\int_{2\pi}^{0} \Big(a(\theta-\sin\theta)\,a\sin\theta - a(1-\cos\theta)[a - a\cos\theta]\Big)\,d\theta$$

$$= \boxed{3\pi a^2} \quad \text{Ans}$$

⑩ Area under $x^{2/3} + y^{2/3} = a^{2/3}$



$x = a\cos^3\theta \quad y = a\sin^3\theta$.

$$\text{Area} \equiv \frac{1}{2}\oint x\,dy - y\,dx$$

$$= \frac{3a^2}{2}\int_0^{2\pi} \cos^4\theta\sin^2\theta + \sin^4\theta\cos^2\theta\ d\theta$$

$$= \frac{3\pi a^2}{8} \quad \text{Ans}$$

Show Green's can be written as: $\iint_R \operatorname{div} A\,dx\,dy = \oint_C A\cdot n\,dS$.

Put $A = Ni - Mj \rightarrow \text{RHS} \equiv \iint \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\,dx\,dy = \oint_C M\,dx + N\,dy = \oint_C (Mi+Nj)\cdot d\vec{r}$

$\Rightarrow \oint_C ((Mi+Nj)\cdot t)\,ds = \oint_C (Mi+Nj)\cdot(k\times n)\,ds = \oint_C (Ni - Mj)\cdot n\,dS.$

* Gauss Divergence Theorem

$$\iiint_V \nabla \cdot F \, dV = \iint_S F \cdot n \, dS$$

Easier to Calculate

$F \leftarrow$ Continuously differentiable vector function

$S \leftarrow$ closed surface

$\hat{n}$ = unit outward.

* Green's Identities

① Put $F = \phi \nabla \psi$ in Gauss.

$$\iiint_V [\phi \nabla^2 \psi + (\nabla \phi) \cdot (\nabla \psi)] \, dV = \iint_S (\phi \nabla \psi) \cdot n \, dS$$

② Put $F = \psi \nabla \phi$ and subtract from ①.

$$\iiint_V (\phi \nabla^2 \psi - \psi \nabla^2 \phi) dV = \iint_S (\phi \nabla \psi - \psi \nabla \phi) \cdot n \, dS$$

③ $$\iiint_V \nabla \phi \, dV = \iint_S \phi \hat{n} \, dS$$

$F = \phi \vec{A}$

$\vec{A}$ is constant

④ $$\iiint_V \nabla \times B \, dV = \iint_S n \times B \, dS$$

$F = \vec{A} \times \vec{B}$

$\vec{A}$ is constant

Questions

① $\int_S \nabla\phi \times \nabla\psi \cdot dS$

$$\rightarrow = \int_S (\nabla\phi \times \nabla\psi)\cdot n\, dS = \int_V \nabla\cdot(\nabla\phi \times \nabla\psi)\, dV = 0$$

② $\int_V \nabla\phi \cdot \text{curl}\, F\, dV = \int_S (F \times \nabla\phi)\cdot dS$

$$\rightarrow \int_S (F \times \nabla\phi)\cdot n\, dS = \int_V \nabla\cdot(F \times \nabla\phi)\, dV$$

$$= \int_V (\nabla\phi\cdot \text{curl}\, F - F\cdot \cancelto{0}{\text{curl}\, \nabla\phi})\, dV$$

$$= \int_V \nabla\phi \cdot \text{curl}\, F\, dV$$

③ If ϕ is harmonic in V,

$$\iint_S \phi \frac{\partial\phi}{\partial n}\, dS = \iiint_V |\nabla\phi|^2\, dV.$$

$$\iint_S \left(\phi \frac{\partial\phi}{\partial n} n\right)\cdot n\, dS = \iint_S (\phi\nabla\phi)\cdot n\, dS$$

$$= \iiint_V \nabla\cdot(\phi\nabla\phi)\, dV \quad \text{(By Gauss divergence)}$$

$$= \iiint_V \left[(\nabla\phi)^2 + \phi\cancelto{0}{\nabla^2\phi}\right] dV$$

$$= \iiint_V |\nabla\phi|^2\, dV \quad \underline{\text{Ans}}$$

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$\boxed{\nabla\cdot(A\times B) = B\cdot \text{Curl } A - A\cdot \text{Curl } B}$

(4) Prove $\iint_S r\times n \, dS = 0$

$c\cdot \iint_S r\times n\, dS = 0 \Rightarrow \iint_S c\cdot(r\times n)\, dS$

$\Rightarrow \iint_S (c\times r)\cdot n\, dS = \iiint_V \nabla\cdot(c\times r)\, dV$

$= \iiint_V (r\cdot \text{curl}\, \vec{c} - c\cdot \text{curl}\, \vec{r})\, dV = 0$

(5) $\iint_S n\times(a\times r)\, dS = 2Va$.

$\boxed{\iint_S n\times a\, dS = \iiint_V \nabla\times a\, dV}$

$\iiint_V \nabla\times(a\times r)\, dV = \iiint_V (\nabla\cdot r)a - (\nabla\cdot a)r\, dV$

$\boxed{\text{Curl } A\times B = (\nabla\cdot B)A - (\nabla\cdot A)B + (A\cdot\nabla)B - (B\cdot\nabla)A}$ $= \iiint 2a\, dV = \boxed{2aV}$ Ans

Not LIKE THIS

- Check value of curl $(a\times r) = 3a$ or $2a$ – ???

(6) $\iint_S (4xz\, dy\, dz - y^2\, dz\, dx + yz\, dx\, dy)$ S is Cube $(1,1,1)$

$\iiint_V \frac{\partial}{\partial x}(4xz) - \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}(yz)\, dV$

$\iiint_V 4z - 2y + y\, dV = \int_0^1\int_0^1\int_0^1 4z - y = \int_0^1\int_0^1 4z - y\, dy\, dz$

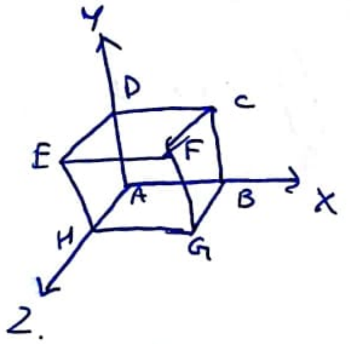
$= \int_0^1 4z - \frac{1}{2} \Rightarrow 2z^2 - \frac{z}{2}\Big]_0^1$

$= 2 - \frac{1}{2} = \boxed{\frac{3}{2}}$

$\iint x \underbrace{dy\, dz}_{n\, dS} = \iint F\cdot n\, dS$

$F = x$

⑦ Verify divergence theorem for $F = (2x-z)\hat{i} + x^2y\hat{j} - xz^2\hat{k}$ over region (0,0,0) to (1,1,1)



$$\iint_S F.n\, dS.$$

$$ABCD \equiv \iint (-xz^2)\hat{k}.(-\hat{k})\, dx\, dy = 0$$

$$EFGH \equiv \int_0^1\int_0^1 -x\,\hat{k}.\hat{k}\, dx\, dy$$

$$= \int_0^1 [-xy]_0^1\, dx \Rightarrow -\frac{x^2}{2}\Big]_0^1 = \boxed{-\frac{1}{2}}$$

$$ADEH \equiv \int_0^1\int_0^1 (+z)\hat{i}.(+i)\, dy\, dz. = \frac{1}{2}.$$

$$BCFG \equiv \int_0^1\int_0^1 (2-z)\, dy\, dz = \int_0^1 2 - z\, dz = 2z - \frac{z^2}{2}\Big]_0^1 = 2 - \frac{1}{2} = \frac{3}{2}$$

$$CDEF = \int_0^1\int_0^1 x^2\hat{j}.\hat{j}\, dx\, dz = \frac{1}{3}.$$

$$BAHG \equiv \int_0^1\int_0^1 0\,(-\hat{j}).\, dx\, dz. = 0.$$

$$\frac{3}{2} + \frac{1}{3} = \frac{9+2}{6} = \boxed{\frac{11}{6}}\ \text{An}$$

$$\iiint_V \nabla.F\, dV = \int_0^1\int_0^1\int_0^1 2 + x^2 - 2xz\ dx\, dy\, dz$$

$$= \boxed{\frac{11}{6}}\ \text{An}$$

(8) $\iint_S x^3\,dy\,dz + x^2 y\,dz\,dx + x^2 z\,dx\,dy$

where S is closed surface bounded by $z=0$, $z=b$ and $x^2+y^2=a^2$

$$\iiint_V \nabla\cdot F\,dV = \iiint_V 3x^2+x^2+x^2\,dx\,dy\,dz$$

$$= 4\times\int_0^b\int_0^a\int_0^{\sqrt{a^2-y^2}} 5x^2\,dx\,dy\,dz$$

$$= 20\int_0^b\int_0^a (a^2-y^2)^{3/2}\,dy\,dz$$

$$= \frac{20}{3}b\int_0^a (a^2-y^2)^{3/2}\,dy = \boxed{\frac{5}{4}\pi a^4 b}$$ Ans

(9) $\iint_S (y^2z^2\hat{i} + z^2x^2\hat{j} + z^2y^2\hat{k})\cdot n\,dS$

where S is part of sphere $x^2+y^2+z^2=1$ above xy-plane and bounded by this plane.

Gauss $\Rightarrow$ $\iiint_V 2zy^2\,dV$

Use spherical coords $\Rightarrow (r,\theta,\phi)$

$dV = (dr)(r\,d\theta)(r\sin\theta\,d\phi)$

$= r^2\sin\theta\,dr\,d\theta\,d\phi$

$$= 2\int_0^1\int_0^{\pi/2}\int_0^{2\pi} r^5\sin^3\theta\cos\theta\sin^2\phi\,dr\,d\theta\,d\phi$$

$z = r\cos\theta \quad y = r\sin\theta\sin\phi$

Limits $\equiv$ $r\in[0,1]$ $\quad\theta\in[0,\pi/2]$

$\phi\in[0,2\pi]$

$= \frac{\pi}{12}$

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(10) $\iint_S (x\hat{i}+y\hat{j}+z^2\hat{k}).n\,dS$.

S is closed surface $x^2+y^2=z^2$ and $z=1$.

$$\iiint_V \frac{\partial x}{\partial x}+\frac{\partial y}{\partial y}+\frac{\partial z^2}{\partial z}\,dV.$$

$$\int_0^1\int_{-z}^{z}\int_{-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} (2+2z\;\; dx\,dy\,dz.$$

$\equiv$ Limits Importan

$$\boxed{\int\sqrt{a^2-x^2} = \frac{x}{2}\sqrt{a^2-x^2}+\frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)}$$

$$\int_0^1\int_{-z}^{z} 2(1+z).2\sqrt{z^2-y^2}\;dy\,dz.$$

$$4\int_0^1 (1+z)\left[\frac{y}{2}\sqrt{z^2-y^2}+\frac{z^2}{2}\sin^{-1}\left(\frac{y}{z}\right)\right]_0^z dz$$

$$2\pi\int_0^1(z^2+z^3)\,dz = 2\pi.\frac{7}{126} = \boxed{\frac{7\pi}{6}}\ \text{Ans}$$

(11) $\iint_S (x^2\hat{i}+y^2\hat{j}+z^2\hat{k}).n\,dS$ $\qquad S\equiv \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c}=1$

$$2\iiint (x+y+z)\;dx\,dy\,dz.$$

$$4a\int_{-c}^{c}\int_{-b\sqrt{1-\frac{z^2}{c^2}}}^{b\sqrt{1-\frac{z^2}{c^2}}} (y+z)\sqrt{1-\frac{y^2}{b^2}-\frac{z^2}{c^2}}.\,dy\,dz.$$

$$8a\int_{-c}^{c}\int_0^{b\sqrt{1-\frac{z^2}{c^2}}} \frac{z\cancel{c}}{b\cancel{c}}\sqrt{\left(b^2\cancel{c}^2-\frac{b^2}{c^2}z^2\right)-\cancel{c}y^2}$$

$$\frac{8a}{b}\int_{-c}^{c} z\left(\frac{y}{2}\sqrt{b^2\left(1-\frac{z^2}{c^2}\right)-y^2}+\frac{b^2}{2}\left(1-\frac{z^2}{c^2}\right)\sin^{-1}\left(\frac{y\sqrt{c^2}}{\sqrt{b^2(c^2-z^2)}}\right)\right)_0^{b\sqrt{1-\frac{z^2}{c^2}}}$$

$$\frac{8a}{b}\int_{-c}^{c} z\left(\frac{b^2}{2}\left(1-\frac{z^2}{c^2}\right)\frac{\pi}{2}\right)dz = 0\ \text{Ans}$$

REM

* E

S

No

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REMEMBER: Gauss Div. Theorem only on closed surfaces.
So, if not closed integrate $\iint dS$ itself.

* Evaluating $(\nabla \times F) \cdot n$ over $\underline{\underline{S}}$

- By Gauss Theorem, $\iint_S (\nabla \times F) \cdot n \, dS = 0$ over closed S.

[Q] Evaluate $\iint_S (\nabla \times F) \cdot n \, dS$ where S is surface of sphere $x^2+y^2+z^2=a^2$ above xy-plane. $\boxed{F \equiv y\hat{i} + (x-2xz)\hat{j} - xy\hat{k}}$

S meets, $z=0$ in a circle $x^2+y^2=a^2, z=0 \equiv S_1$ be its surface

Now, by Gauss div application.

$$\iint_{S+S_1} (\nabla \times F) \cdot n \, dS = 0.$$

$$\iint_S (\nabla \times F) \cdot n \, dS = -\iint_{S_1} (\nabla \times F) \cdot n \, dS.$$

$$= + \iint (\nabla \times F) \cdot k \, dx\,dy.$$

$$= \iint_{S_1} 2z \, dx\,dy, \quad z=0 \equiv \underline{\underline{0}} \text{ Ans}$$

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x-2xz & -xy \end{vmatrix}$$

$-x+2x = \cancel{1}-2z-\cancel{1}$

$x. \quad x. \quad -2z.$

$$\int_{0}^{a}\int_{-a} \frac{z}{x} xy \frac{dx\,dz}{\frac{y}{x}}$$

0

* $\iint F.n\, dS.$

① When S is a curved surface of cylinder

$x = a\cos\theta,\ y = a\sin\theta,\ z$

$dS = a\, d\theta\, dz.$

* Volume ellipsoid $= \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \Rightarrow \cancel{\frac{4}{3}\pi\sqrt{abc}}$

$$\boxed{\frac{4}{3}\pi abc}$$

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* Gauss' Theorem ≡ Divergence Theorem

$$\iint_S F \cdot n \, dS = \iiint_V \nabla \cdot F \, dV.$$

* Stokes Theorem

$$\oint F.dr = \iint_S (\nabla \times F).n\,dS = \iint_S (\text{curl } F).n\,dS$$

→ Green's a special case of Stokes.

Q $\oint_c r.dr = 0.$

$$\oint_c r.dr = \iint_S (\text{curl } r).n\,dS = 0$$

Q Prove $\oint \phi \Delta\psi.dr = \iint_S [\nabla\phi \times \nabla\psi].n\,dS.$

$$\oint_C \phi\nabla\psi.dr = \iint_S [\nabla\times(\phi\nabla\psi)].n\,dS$$

$$= \iint_S [\nabla\phi\times\nabla\psi + \phi\,\underbrace{\text{curl grad }\psi}_{0}].n\,dS$$

$$= \iint_S (\nabla\phi\times\nabla\psi).n\,dS$$

Q Prove div curl $F=0$ by Stoke's

Let V be volume enclosed by surface (closed) S.

$$\iiint_V \nabla.(\text{curl } F)\,dV = \iint_S (\text{curl } F).n\,dS = \underbrace{\iint_{S_1} + \iint_{S_2}}$$

Divide S into 2 portions by curve C.

$$= \oint_c F.dr - \oint_c F.dr = 0$$

So div curl $F = \underline{\underline{0}}$.

C direction ≡ When walking C, head along outward normal of main surface/curve surface lies on left, then (+)

9 Verify Stokes theorem for

$F = (x^2 + y - 4)\hat{i} + 3xy\,\hat{j} + (2xz + z^2)\hat{k}$.

S is upper half of sphere $x^2+y^2+z^2 = 16$ and C its boundary

$C \equiv x^2+y^2 = 16, z=0 \quad x = 4\cos t \quad y = 4\sin t$

$$\oint F.dr = \int (x^2+y-4)dx + 3xy\,dy + (2x\overset{0\,(z=0)}{z} + z^2)dz$$

$$\int (16\cos^2 t + 4\sin t - 4).4(-\sin t)\,dt.$$

$$+ 48 \sin t \cancel{\cos t}.\,4\cos t.$$

$$\int_0^{2\pi} 128 \overset{0}{\cancel{\cos^2 t \sin t}} - 16\sin^2 t + 16\overset{0}{\cancel{\sin t}}\, dt$$

$\boxed{\sin(2\pi - t) = -\sin t}$

$$-16\int_0^{2\pi} \sin^2 t\, dt = \boxed{-16\pi} \text{ Ans}$$

(By Gauss Div Thm)

$$\iint_S (\nabla\times F).n\,dS + \iint_{S_1} (\nabla\times F).n\,dS_1 = 0$$

$$= \iint_{S_1} (\nabla\times F).k\,dS_1$$

$$\begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y-4 & 3xy & 2xz+z^2 \end{vmatrix}$$

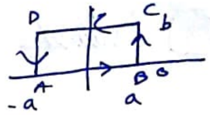
$3y - 1 = 2y.$

$$\iint (3y-1)\,dx\,dy \qquad x = r\cos t \quad y = r\sin t.$$

$\iint r \sin$

$$\int_0^{2\pi}\int_0^{4} (3r\sin\theta - 1)\,r\,dr\,d\theta = \boxed{-16\pi} \text{ Ans}$$

Q. Verify Stokes $F = (x^2+y^2)\hat{i} - 2xy\hat{j}$ around rectangle $x = \pm a,\ y=0,\ y=b$.



$\oint_C F.dr = \iint_S (\nabla\times F).n\,dS$

curl F
$\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \\ x^2+y^2 & -2xy & \end{vmatrix}$

$\nabla\times F = -2y-2y$
$= -4y$.

$\oint (x^2+y^2)\,dx - (2xy)\,dy$

$\int_0^{b}\int_{-a}^{a} -4y\;dx\,dy$.

AB $\Rightarrow \int_{-a}^{a} x^2\,dx = 2a^3/3$.

$-4.2a.\frac{b^2}{2} = \boxed{-4ab^2}$

BC $\Rightarrow \int_0^b -2ay\,dy = \frac{-2ab^2}{2} = -ab^2$

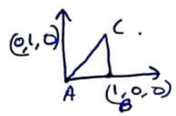
CD $\Rightarrow \int_a^{-a} x^2+b^2\,dx = \left.\frac{x^3}{3}\right]_a^{-a} + \left.b^2\right]_a^{-a} = -\frac{2a^3}{3} - 2ab^2$.

DA $\Rightarrow \int_b^0 2ay\,dy = \frac{-2a(+b)^2}{2} = -ab^2$

$\Downarrow$

$\boxed{-4ab^2}$ Hence Proved

Q. Evaluate $\oint_C F.dr$ $F = y^2\hat{i} + x^2\hat{j} - (x+z)\hat{k}$



$\frac{\partial}{\partial x}$ $\frac{\partial}{\partial y}$
y^2 x^2.
$(2x-2y)$

$\int_0^1\int_0^x (2x-2y)\;dx\,dy$.

$\int_0^1 \left.2xy-y^2\right]_0^x dx = \int_0^1 2x^2-x^2\,dx = \boxed{\frac{1}{3}}$ Ans

(9) Evaluate $\iint_S (\nabla \times A) . n \, dS$ $A \equiv (x-2)\hat{i} + (x^3+yz)\hat{j} - 3xy^2\hat{k}$

$S \equiv$ surface $z = 2 - \sqrt{x^2+y^2}$ above x-y plane.

Using Stoke's Theorem, this is equal to

$\oint_C A \cdot dr$ $\quad C \equiv x^2+y^2=4 \quad z=0.$

$x = \cancel{4}\,2\cos\theta \,, \; y = \cancel{4}\,2\sin\theta.$

$\int_C (x-2)dx + (x^3+yz)dy - (3xy^2)dz$

As $z=0$, $dz=0$.

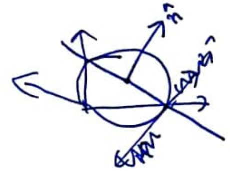
$\int x\,dx + x^3\,dy = \int_0^{2\pi} 2\cos\theta(-2)\sin\theta \, d\theta + 8\cos^3\theta . 2\cos\theta \, d\theta.$

$= (-2)\int_0^{2\pi} \sin 2\theta \, d\theta + 16 \int_0^{2\pi} \cos^4\theta \, d\theta$

$= 0 + \cancel{16}^{4} . \frac{3}{\cancel{4}}\pi = \boxed{12\pi}$ Ans

9 $\int_C (y\,dx + x\,dy + x\,dz) = -2\sqrt{2}\pi a^2$

C is $\equiv x^2+y^2+z^2-2ax-2ay=0,\ x+y=2a$ begins at $(2a,0,0)$ goes below z plane



$$\nabla\times F \equiv \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$$

$\Downarrow$

$(-1)\ (-1)\ (-1)$

$\hat{n} \Rightarrow +\frac{(i+j)}{\sqrt{2}}$

$\iint_S (\nabla\times F).\,n\,dS.$

$\iint_S -(i+j+k)\,\frac{(i+j)}{\sqrt{2}}\,dS.$

$(-\sqrt{2})\iint_S dS. \leftarrow$ area of circle. $= \boxed{(-\sqrt{2})\,2\pi a^2}$ Ans

Some results

- $\int_C F \cdot dr$ is independent of path $C \Leftrightarrow F = \text{grad}\,\phi$, some ϕ

Single-valued

Cont. 1st partial deriv.

$\Leftrightarrow \oint F \cdot dr = 0 \ \forall$ simple closed path

$\Leftrightarrow$ Irrotational $\boxed{\nabla \times F = 0}$

Q Show the differential form is exact and find ϕ s.t. it equals $d\phi$

- $(y^2z^3\cos x - 4x^3z)\,dx + 2z^3y\sin x\,dy + (3y^2z^2\sin x - x^4)\,dz$.

Also evaluate $\int_C$ over any path $(0,1,0)$ to $(1,1,1)$.

To show exact $\equiv$

Curl $F = 0$

$$\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2z^3\cos x - 4x^3z & 2z^3y\sin x & 3y^2z^2\sin x - x^4 \end{vmatrix}$$

$i\,(\cancel{6yz^2\sin x} - \cancel{6yz^2\sin x})$

$j\,(\cancel{3y^2z^2\cos x} - 4x^3 + 4x^3 \cancel{3y^2z^2})$

$k\,(\cancel{2z^3y\cos x} - \cancel{2yz^3\cos x})$

$= 0$.

So, exact.

To find ϕ

$\frac{\partial \phi}{\partial x} = y^2z^3\cos x - 4x^3z \Rightarrow \phi = y^2z^3\sin x - x^4z + f(y,z) \to 0$

$\frac{\partial \phi}{\partial y} = 2z^3y\sin x \Rightarrow \phi = z^3y^2\sin x + g(z,x) \to x^4z$.

$\frac{\partial \phi}{\partial z} = 3y^2z^2\sin x - x^4 \Rightarrow \phi = y^2z^3\sin x - x^4z + h(x,y) \to 0$

So, $(\phi = y^2z^3\sin x - x^4z + C)$ Satisfies all

$\int_C d(\phi) = \phi\Big]_{0,1,0}^{1,1,1} = \sin 1 - 1 - 0 = \boxed{\sin 1 - 1}$ Ans

9 Evaluate $\int_c yz\,dx + (xz+1)\,dy + xy\,dz$

where C is any path from 1,0,0 to 2,1,4

$$\Rightarrow \nabla \times F = \begin{vmatrix} i & j & k \\ \partial x & \partial y & \partial z \\ yz & xz+1 & xy \end{vmatrix} = i(x-x) + j(y-y) \; \hat{k}(z-z) = 0$$

So, exact differential.

$$\frac{\partial \phi}{\partial x} = yz \equiv xyz + f$$

$$\frac{\partial \phi}{\partial y} = xz+1 \equiv xyz + y + g$$

$$\frac{\partial \phi}{\partial z} \equiv xy \equiv xyz + h$$

$f = h = y$

$g = 0$

$$\phi = xyz + y$$

$$\int_c (d\phi) = \phi \Big]_{1,0,0}^{2,1,4} = 8 + 1 - 0 = \boxed{9} \text{ Ans}$$

(*) Prove $\oint_c \phi \nabla \psi . dr = - \oint_c \psi \nabla \phi . dr$

Consider $\oint_c \nabla(\phi\psi) . dr = \iint_S \text{curl grad}(\phi\psi) . n \, dS = 0.$

$$\oint_c (\phi \nabla \psi + \psi \nabla \phi) . dr = 0 \Rightarrow \text{Proved}$$

(*) Prove $\oint_c \phi \, dr = \iint_S dS \times \nabla \phi$

Let $\bar{F} = \phi \bar{A}$; Apply Stokes on $\bar{F}$

$$\oint_c \bar{F} . dr = \iint_S (\nabla \times (\phi A)) . n \, dS.$$

$$= \iint_S \left[(\nabla \phi \times A) + \phi \overset{0}{(\cancel{\nabla \times A})} \right] . n dS = \iint_S (\nabla \phi \times A) . dS$$

$$\oint_c (\phi A) . dr = \iint_S A . (dS \times \nabla \phi)$$

$\Rightarrow A . \oint_c \phi \, dr = \iint_S A . (dS \times \nabla \phi) \Rightarrow \boxed{\oint_c \phi \, dr = \iint_S (dS \times \nabla \phi)}$ Ans

* Vector Geometry

- parameter t to s.

$\gamma(t) = \{R\cos t, R\sin t, mt\}$; $\gamma'(t) = \{-R\sin t, R\cos t, m\}$

$$S(t) = \int_0^t \sqrt{\langle \gamma'(t) \cdot \gamma'(t) \rangle}\, dt = \int_0^t \sqrt{R^2+m^2}\, dt = \sqrt{R^2+m^2}\, t.$$

$$t = \frac{S}{\sqrt{R^2+m^2}}$$

So, $\gamma(s) = \left\{R\cos\frac{s}{\sqrt{R^2+m^2}}, R\sin\frac{s}{\sqrt{R^2+m^2}}, \frac{ms}{\sqrt{R^2+m^2}}\right\}$

So, $\gamma(t) \equiv \gamma(s)$

- Orthonormal triad.

① $\tau = \dfrac{\gamma'(s)}{\|\gamma(s)\|}$ (a unit ~~normal~~ tangent) $\quad \tau = \dfrac{d\bar{r}}{ds} = \dfrac{d\bar{r}/dt}{ds/dt}$

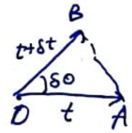
$\tau \cdot \tau = 1 \Rightarrow \tau' \cdot \tau + \tau \cdot \tau' = 0 \Rightarrow \tau' \perp \tau.$

② $N = \dfrac{\tau'}{\|\tau'\|}.$ (unit normal)

③ $B = \tau \times N$

* TS : $\left|\frac{dt}{ds}\right| = K.$

At 2 pts let tangents be t, $t+\delta t$



$Ar(\Delta OAB) = \frac{1}{2}\left(t \times (t+\delta t)\right) = \frac{1}{2} OA.OB \sin\delta\theta$

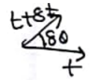
$= |t \times \delta t| = \sin\delta\theta$

$\left|t \times \frac{\delta t}{\delta s}\right| = \frac{\sin\delta\theta}{\delta\theta}\,\frac{\delta\theta}{\delta s} = 1.\,\frac{\delta\theta}{\delta s}\,(=K)$

As curvature (K) is rate of change of θ.

$\left|t \times \frac{\delta t}{\delta s}\right| = K. \Rightarrow \left|\frac{dt}{ds}\right| = K$ $\Leftarrow$ $\boxed{\text{As } t \text{ is of constant } l \text{ it is } \perp r \text{ to } dt/ds}$

$\Rightarrow$ $\boxed{t \cdot t = 1 \Rightarrow t \cdot t' = 0 \Rightarrow t \perp dt/ds}$



$\frac{1}{2}(t \times \delta t) = \frac{1}{2} OA.OB$

$\Rightarrow \left|\frac{t \times \delta t}{\delta s}\right| = \left|\frac{\sin\delta\theta}{\delta\theta}\,\frac{\delta\theta}{\delta s}\right|$

$\Rightarrow \left|\frac{dt}{ds}\right| = K.$

IIIly, replace t by b to get $\left|\frac{db}{ds}\right| = \tau$

- $K = \left|\frac{dt}{ds}\right| = \left|\frac{d^2 r}{ds^2}\right|$
- $\tau = \left|\frac{db}{ds}\right|$

* A curve is a plane curve iff. $\tau = 0$ at all points (*)

$\Rightarrow$ Plane curve : τ, N lie in plane of curve

$b = \tau \times n \Rightarrow b$ is always $\perp r$ to the plane and in a fixed direction (+) b is of constant magnitude

$\frac{db}{ds} = 0 \Rightarrow \tau = 0$

$\Leftarrow \frac{db}{ds} = 0.$

$\frac{d}{ds}(r.b) = \frac{dr}{ds}.b + \frac{db}{ds}.r = 0$

$r.b$ = constant

$\Rightarrow$ Projection of r on b is constant

$\Rightarrow$ Curve lies on plane

Serret Frenet ⊛

$$\begin{bmatrix} 0 & K & 0 \\ -K & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} \dot{T} \\ \dot{N} \\ \dot{B} \end{bmatrix}$$

$\frac{dt}{ds} = Kn. \longrightarrow \left|\frac{dt}{ds}\right| = K$ (already)

Unit tangent $\hat{T} = d\bar{r}/ds$; $|\hat{T}| = 1$.
$\hat{T} \cdot d\hat{T}/ds = 0$ ⊕ dT/ds lies in osc. plane

$t \cdot t = c \rightarrow t \cdot t' = 0 \Rightarrow t \perp \frac{dt}{ds}$.

$\Rightarrow \frac{d\hat{T}}{ds} \parallel \vec{N} \Rightarrow \boxed{\frac{d\bar{T}}{ds} = K\bar{N}}$

$\frac{db}{ds} = \cancel{+\tau n}$ $-\tau n$ $\longrightarrow$ $t \cdot b = 0 \rightarrow \frac{dt}{ds} \cdot b + \frac{db}{ds} \cdot t = 0$.

$\rightarrow K\cancel{n}\cdot\vec{B} + \frac{db}{ds} \cdot t = 0$

$\frac{db}{ds} \perp t$, Also b of const magnitude.

$\hookrightarrow b \perp \frac{db}{ds}$

$\hookrightarrow \frac{db}{ds} = \tau \hat{n}$ [+, − sign if $n, \frac{db}{ds}$ have same/opposite dir]

$\frac{dn}{ds} = -Kt \cancel{+} \tau b \longrightarrow n = b \times t$

$\frac{dn}{ds} = \frac{db}{ds} \times t + b \times \frac{dt}{ds}$.

$= -\tau n \times t \cancel{+} b \times Kn$

$= \cancel{+}\tau b - Kt$

• In terms of $r(t)$.

$K = \left|\frac{dr}{dt} \times \frac{d^2r}{dt^2}\right| \Big/ \left|\frac{dr}{dt}\right|^3$.

$K^2\tau = + \left[\frac{dr}{dt}\ \frac{d^2r}{dt^2}\ \frac{d^3r}{dt^3}\right] \Big/ \left|\frac{dr}{dt}\right|^6$

τ $\Downarrow$

$$\frac{\left[\frac{dr}{dt}\ \frac{d^2r}{dt^2}\ \frac{d^3r}{dt^3}\right]}{\left|\frac{dr}{dt} \times \frac{d^2r}{dt^2}\right|^2}$$

• $\rho = 1/K$.

$\sigma = 1/\tau$

1. Osculating Plane (T, N).

$$(R-r)\cdot\left(\frac{dr}{dt}\times\frac{d^2r}{dt^2}\right)=0.$$

2. Normal Plane (N, B)

$$(R-r)\cdot\frac{dr}{dt}=0$$

3. Rectifying Plane (T, B)

Curvature $\longrightarrow$	$\kappa = \left\lvert\frac{d\bar{r}}{ds}\times\frac{d^2\bar{r}}{ds^2}\right\rvert$	$\left\lvert\frac{d\bar{r}}{dt}\times\frac{d^2\bar{r}}{dt^2}\right\rvert \Big/ \left\lvert\frac{d\bar{r}}{dt}\right\rvert^3$
Torsion $\longrightarrow$	$\tau = \dfrac{\left[d\bar{r}/ds \;\; d^2\bar{r}/ds^2 \;\; d^3\bar{r}/ds^3\right]}{\left\lvert\frac{d\bar{r}}{ds}\times\frac{d^2\bar{r}}{ds^2}\right\rvert^2}$	$\dfrac{\left[\frac{d\bar{r}}{dt} \;\; \frac{d^2\bar{r}}{dt^2} \;\; \frac{d^3\bar{r}}{dt^3}\right]}{\left\lvert\frac{d\bar{r}}{dt}\times\frac{d^2\bar{r}}{dt^2}\right\rvert^2}$

Q. Find osculating plane, curvature, torsion

$$x = a\cos 2t,\quad y = a\sin 2t,\quad z = 2a\sin t$$

∴ Osculation Plane.

$$\frac{dr}{dt} = \langle -2a\sin 2t,\ 2a\cos 2t,\ 2a\cos t\rangle$$

$$\frac{d^2r}{dt^2} = \langle -4a\cos 2t,\ -4a\sin 2t,\ -2a\sin t\rangle$$

$- 4a^2\sin t\cos 2t + 8a^2\sin 2t\cos t$

$-8a^2\cos 2t\cos t$

$-4a^2\sin 2t\sin t$

$8a^2$ ~~sin 2t~~

$$\frac{d^3r}{dt^3} = \langle +8a\sin 2t,\ -8a\cos 2t,\ -2a\cos t\rangle.$$

$$\text{Plane} \equiv (R-r)\cdot\left(\frac{dr}{dt}\times\frac{d^2r}{dt^2}\right) = 0$$

WRONG ANSWER in BOOK

$$x = a(3t - t^3)\quad y = 3at^2\quad z = a(3t + t^3).$$

$$\frac{dr}{dt} \equiv \langle 3a - 3at^2,\ 6at,\ 3a + 3at^2\rangle$$

$$\frac{d^2r}{dt^2} \equiv \langle -6at,\ 6a,\ 6at\rangle$$

$$\frac{d^3r}{dt^3} \equiv \langle -6a,\ 0,\ 6a\rangle.$$

$$K = \frac{\left|\frac{dr}{dt}\times\frac{d^2r}{dt^2}\right|}{\left|\frac{dr}{dt}\right|^3} = \frac{\langle 18a^2t^2 - 18a^2,\ -36a^2t,\ 18a^2t^2 + 18a^2\rangle}{3a\sqrt{(1-t^2)^2 + 4t^2 + (1+t^2)^2}}$$

$$= \frac{1}{3a(1+t^2)^2}$$

$$\tau = \frac{\left(\frac{dr}{dt}\times\frac{d^2r}{dt^2}\right)\cdot\frac{d^3r}{dt^3}}{\left|\frac{dr}{dt}\times\frac{d^2r}{dt^2}\right|^2} = \frac{216a^3}{\{18\sqrt{2}\,a^2(1+t^2)\}^2} = \frac{1}{3a(1+t^2)^2}$$

??

$\boxed{8}$ Find T, N, B, K, τ. $[3\cos t, 3\sin t, 4t]$

$\Rightarrow$

$$T = \frac{d\vec{r}}{ds} = \frac{d\vec{r}/dt}{ds/dt} = \frac{d\bar{r}/dt}{|d\bar{r}/dt|} = \boxed{\frac{1}{5}(-3\sin t\,\hat{i} + 3\cos t\,\hat{j} + 4\hat{k})}\ \text{Ans}$$

$$K = \left|\frac{dT}{ds}\right| = \frac{dT/dt}{ds/dt} = \left|\frac{1}{25}\left[(-3\cos t)\hat{i} - (3\sin t)\hat{j}\right]\right| = \frac{3}{25}\ \text{Ans}$$

$$\frac{dT}{ds} = K\vec{N} \Rightarrow$$

$$\bar{N} = \frac{1}{K}\frac{dT}{ds} = \boxed{-\cos t\,\hat{i} - \sin t\,\hat{j}}$$

$$\bar{B} = \bar{T} \times \bar{N} = \boxed{\frac{1}{5}(4\sin t\,\hat{i} - 4\cos t\,\hat{j} + 3\hat{k})}$$

Q) Prove Serret Frenet formulae can be written as

$\frac{dt}{ds} = w \times t \quad \frac{dn}{ds} = w \times n \quad \frac{db}{ds} = w \times b$ Find w.

$$\frac{dt}{ds} = Kn = \tau(t \times t) + K\, b \times t$$

$$\frac{dn}{ds} = \tau b - Kt = \tau(t \times n) + K\, b \times n$$

$$\frac{db}{ds} = (\tau t + Kb) \times b = -\tau n.$$

So, $\boxed{w = \tau t + Kb}$

Q) Show $x'''^2 + y'''^2 + z'''^2 = \frac{1}{\rho^2 \sigma^2} + \frac{1+\rho'^2}{\rho^4}$

LHS $= \left|\frac{d^3 r}{ds^3}\right|^2$

$\left|\frac{d^3 r}{ds^3}\right|^2$

$\frac{dr}{ds} = t \Rightarrow \frac{d^2 r}{ds^2} = K\hat{n} = \frac{\hat{n}}{\rho}$

$$\frac{dr}{ds} = t.$$

$$\frac{d^2 r}{ds^2} = \frac{dt}{ds} = K\hat{n} = \frac{\hat{n}}{\rho}$$

$$\frac{d^3 r}{ds^3} = -\frac{\rho'}{\rho^2} n + \frac{1}{\rho}\frac{dn}{ds} = -\frac{\rho'}{\rho^2}\hat{n} + \frac{1}{\rho}\left[\tau\hat{b} - K\hat{t}\right].$$

$$\left|\frac{d^3 r}{ds^3}\right|^2 = \frac{\rho'^2}{\rho^4} + \frac{\cancel{\tau^2} 1}{\rho^2 \sigma^2} + \frac{\cancel{K^2} 1}{\rho^4} = \frac{1}{\rho^2 \sigma^2} + \frac{1+\rho'^2}{\rho^4}$$

Q Tangent, binomial at a pt on curve makes θ and ϕ with a fixed direction.

1992 Show $\frac{\sin\theta}{\sin\phi}\frac{d\theta}{d\phi} = -\frac{K}{\tau}$.

Let fixd dire $\to a$.

$$t.a = \cos\theta \Rightarrow \frac{dt}{ds}.a = -\sin\theta\frac{d\theta}{ds} = Kn.a$$

$$b.a = \cos\phi \Rightarrow \frac{db}{ds}.a = -\sin\phi\frac{d\phi}{ds} = -\tau n.a.$$

$$\boxed{\frac{\sin\theta \; d\theta}{\sin\phi \; d\phi} = -\frac{K}{\tau}}$$

Q If tangent to curve makes constant α with a fixed line Show $\sigma = \pm \rho \tan\alpha$.

Dir = e

$t.e = \cos\alpha \Rightarrow \frac{dt}{ds}.e = 0 \Rightarrow Kn.e = 0.$

b, t, e are coplanar

So, $b.e = \pm\sin\alpha$. (As b, t are $\perp$r)

$\frac{dn}{ds}.e = 0 \Rightarrow (K t + \tau b).e = 0 \Rightarrow K\cos\alpha \pm \tau\sin\alpha = 0$

$\Rightarrow \sigma = \pm\rho\tan\alpha.$

- If σ/ρ is constant, tangent makes a constant angle with fixed direction.

$\sigma = a\rho \Rightarrow \frac{dt}{ds} = \frac{1}{\rho}n \quad , \frac{db}{ds} = \frac{1}{\sigma}n.$

$\rho\frac{dt}{ds} = n = \sigma\frac{db}{ds} \Rightarrow \frac{dt}{ds} = \frac{\sigma}{\rho}\frac{db}{ds} = a\frac{db}{ds}$

Integrating $\Rightarrow \quad t = ab + c \quad = t.t = t.c = 1$

So, t makes constant angle with fixed vector c

* $\int_C t \cdot dr$ $\quad \hat{t} \equiv$ unit tangent
$C \equiv$ unit circle x-y plane.

$$\int_C t \cdot \frac{dr}{ds}\, ds = \int_C t \cdot t \, ds$$
$$= \int_C ds = \int_{s=0}^{2\pi} ds$$
$$= \boxed{2\pi} \text{ Ans}$$

* $\iint_S (y^2z^2 \hat{i} + z^2x^2 \hat{j} + x^2y^2 \hat{k}) \cdot \hat{n}\, dS$
where S is the surface of sphere $x^2+y^2+z^2 = 1$ above xy-plane.

$$-\hat{n} = \frac{\nabla(x^2+y^2+z^2)}{|\nabla S|} = (x\hat{i} + y\hat{j} + z\hat{k}) \quad ; \quad dS = \frac{dx\,dy}{|(\hat{n}\cdot\hat{k})|} = \frac{dx\,dy}{z}$$

$$\iint_S xy\cancel{z}\,(yz + zx + xy)\,\frac{dx\,dy}{\cancel{z}}$$ over area of circle $x^2+y^2=1$

$$\iint_S \left(xy + (x+y)\left(\sqrt{1-x^2-y^2}\right)\right) xy\, dx\,dy$$ Parametrize $x = r\cos\theta$, $y = r\sin\theta$.

$$\int_0^{2\pi}\int_0^1 \left\{ r^4\cos^2\theta\sin^2\theta + r^3(\sin^2\theta\cos\theta + \cos^2\theta\sin\theta)\sqrt{1-r^2} \right\} r\,dr\,d\theta$$
$0 \quad 0$

by definite integral property
$\int_0^{2\pi} \sin^2\theta\cos\theta\, d\theta = 0$.
$\cos^2\theta \sin\theta \, d\theta = 0$.

$$= \int_0^{2\pi}\int_0^1 r^5 \cos^2\theta \sin^2\theta \, d\theta\, dr$$

$$\Rightarrow \int_0^{2\pi} \frac{1}{6}\cos^2\theta\sin^2\theta\, d\theta \Rightarrow \frac{1}{6\cdot4}\int_0^{2\pi} \cancel{\sin 2\theta}\; 1 - \cos^2 2\theta \, d\theta$$

$$\Rightarrow \frac{1}{24}\int_0^{2\pi} 1 - \left(\frac{1+\cos 4\theta}{2}\right) d\theta.$$

$$= \frac{\pi}{24}$$

* Show $\iint_S n\,dS = 0$

Take any C vector.

$$C \cdot \iint_S n\,dS = \iint_S C \cdot n\,dS = \iiint_V (\nabla \cdot C)\,dV$$
$$= 0$$

* Stokes Theorem
  - Closed Surface not necessary

* $\left(\frac{\partial \psi}{\partial n} n\right) = \nabla \psi$

$$\frac{\partial \psi}{\partial n} dS = \left(\frac{\partial \psi}{\partial n} n\right) \cdot n\,dS$$

- 'Then use Gauss'

* For cylinder surface $\Rightarrow a\,d\theta\,dz$