Mains Test Series – 2018

Test-12, Paper-II, Answer Key

1(a) Let G be a group and $a, b \in G$, such that $ab = ba$ and $O(a)$ and $O(b)$ are relatively prime. Then prove that $O(ab) = O(a)O(b)$.

Sol^n: Given that G is an abelian group and $a, b \in G$ such that $O(a) = m$ & $O(b) = n$

Since $O(a) = m$ & $O(b) = n$

i.e., m is the least +ve integer such that $a^m = e$

and $O(b) = n$

i.e., n is the least +ve integer such that $a^n = e$

Also $a, b \in G \Rightarrow ab \in G$

Let $O(ab) = P$

Now $(ab)^{mn} = a^{mn} b^{mn}$ ($\because$ G is abelian)

$= (a^m)^n (b^n)^m$

$= e^n e^m = e$

$\therefore (ab)^{mn} = e$

$\Rightarrow O(ab) / mn$

i.e. P / mn —— (1)

Also $(ab)^{Pn} = [(ab)^P]^n$

$= e^n = e$

and $(ab)^{Pn} = a^{Pn} b^{Pn} = a^{Pn} (b^n)^P = a^{Pn} e = a^{Pn}$

$\therefore a^{Pn} = e$ ($\because (ab)^{Pn} = e$)

$\Rightarrow O(a) / Pn$

ie. m / pn

Since $(m,n) = 1$

$\Rightarrow m/p$ — ②.

Similarly we can Prove n/p — ③

from ② & ③ and $(m,n) = 1$

we have mn/p — ④

$\therefore$ from ① & ④

we have $mn = p$

$\therefore o(ab) = p = mn$

$= o(a) \cdot o(b)$

1(b) → Prove that a finite integral domain is a field. what happens if the integral domain is infinite?

Sol^n: Let F be the finite Integral domain.

Let $F = \{a_1, a_2 \cdots a_n\}$ and F Contains 'n' distinct elements.

To prove that F is a field.

For this we are enough to prove that the non-zero elements of F have multiplicative inverse.

Let $a(\neq 0) \in F$

$\therefore aa_1, aa_2, \ldots, aa_n \in F$ (by closure prop)

All these elements are distinct.

because; if possible

let $aa_i = aa_j$, $a_i, a_j \in F$

$\Rightarrow a(a_i - a_j) = 0$

$\Rightarrow (a_i - a_j) = 0$ ($\because a \neq 0$ & F is an ID i.e. F doesnot contain zero divisors)

$\Rightarrow a_i = a_j$

This is a contradiction to hypothesis that F contains n distinct elements.

$\therefore$ Our assumption that $aa_i = aa_j$ is wrong

$\therefore aa_1, aa_2, \ldots, aa_n$ are all distinct elements in F which has exactly 'n' elements.

By the pigeon-hole principle, one of these products must be equal to one ($\because$ F is an ID)

Let $aa_r = 1$ for some $a_r \in F$

$\therefore a^{-1} = a_r$

$\therefore$ Every non-zero element of F has multiplicative inverse.

F is a field

An infinite integral domain may not be a field.

For example, $\mathbb{Z}$ (set of all integers) is an infinite integral domain, which is not a field. ($\because$ inverse prop. is not satisfied w.r.t $\times$)

1(c) Test the convergence of the following series.

$$1^p + \left(\frac{1}{2}\right)^p + \left(\frac{1.3}{2.4}\right)^p + \left(\frac{1.3.5}{2.4.6}\right)^p + \cdots$$

Sol^n: Neglecting the first term, we have

$$u_n = \left[\frac{1.3.5.\cdots(2n-1)}{2.4.6.\cdots 2n}\right]^p$$

and $u_{n+1} = \left[\frac{1.3.5.\cdots(2n-1)(2n+1)}{2.4.6\cdots(2n)(2n+2)}\right]^p$

$$\therefore \frac{u_n}{u_{n+1}} = \left(\frac{2n+2}{2n+1}\right)^p = \frac{\left(1+\frac{1}{n}\right)^p}{\left(1+\frac{1}{2n}\right)^p}$$

$$\therefore \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = 1.$$

$\therefore$ Ratio test fails

Now $\frac{u_n}{u_{n+1}} = \left(1+\frac{1}{n}\right)^p\left(1+\frac{1}{2n}\right)^{-p}$

$$= \left[1+\frac{p}{n}+\frac{p(p-1)}{2!}\cdot\frac{1}{n^2}+\cdots\right]\left[1-\frac{p}{2n}+\frac{p(p+1)}{2!}\frac{1}{4n^2}+\cdots\right]$$

$$= 1+\frac{p}{n}-\frac{p}{2n}+O\left(\frac{1}{n^2}\right)$$

where $O\left(\frac{1}{n^2}\right)$ represents terms of the order $\frac{1}{n^2}$ and higher powers of $\frac{1}{n}$.

$$= 1+\frac{p/2}{n}+O\left(\frac{1}{n^2}\right)$$

Comparing this with $\frac{u_n}{u_{n+1}} = 1+\frac{\lambda}{n}+O\left(\frac{1}{n^2}\right)$.

we have $\lambda = p/2$.

$\therefore$ By Gauss test, the given series $\sum u_n$ converges if $\frac{p}{2}>1$ i.e, $p>2$ and diverges if $\frac{p}{2}\le 1$ i.e, $p\le 2$.

Hence the series converges if $p>2$ and diverges if $p\le 2$.

1(d) → Discuss the continuity of the following complex valued function at $z=0$,

$$f(z) = \begin{cases} \dfrac{1-\exp(-|z|^2)}{|z|^2} & ; f(z)\neq 0 \\ 1 & ; f(z)=0 \end{cases}$$

Sol'n: let's approach 'o' along any coordinate axes path:

$$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \frac{1-e^{-|x+iy|^2}}{|x+iy|^2}$$

Along x-axis path:

$$\lim_{x\to 0} f(x,0) = \lim_{x\to 0} \frac{1-e^{-|x|^2}}{|x|^2}$$

$$= \lim_{x\to 0} \frac{1-e^{-x^2}}{x^2}$$

$$= \lim_{x\to 0} \frac{1-\left[1-\frac{x^2}{1!}+\frac{x^4}{2!}+\cdots\right]}{x^2}$$

$$= 1 = f(0)$$

It is going to become a continuous function.
please check it by ϵ-δ definition

1(e) A pine-apple firm produces two products: canned pine-apple and canned juice. The specific amounts of material, labour and equipment required to produce each product and the availability of each of these resources are shown in the table given below:

	Canned Juice	Pine-apple	Available Resources
Labour (man hrs)	3	2.0	12.0
Equipment (m/c hrs)	1	2.3	6.9
Material (units)	1	1.4	4.9

Solution

Let the units produced of canned juice and pine apple are x, y respectively.

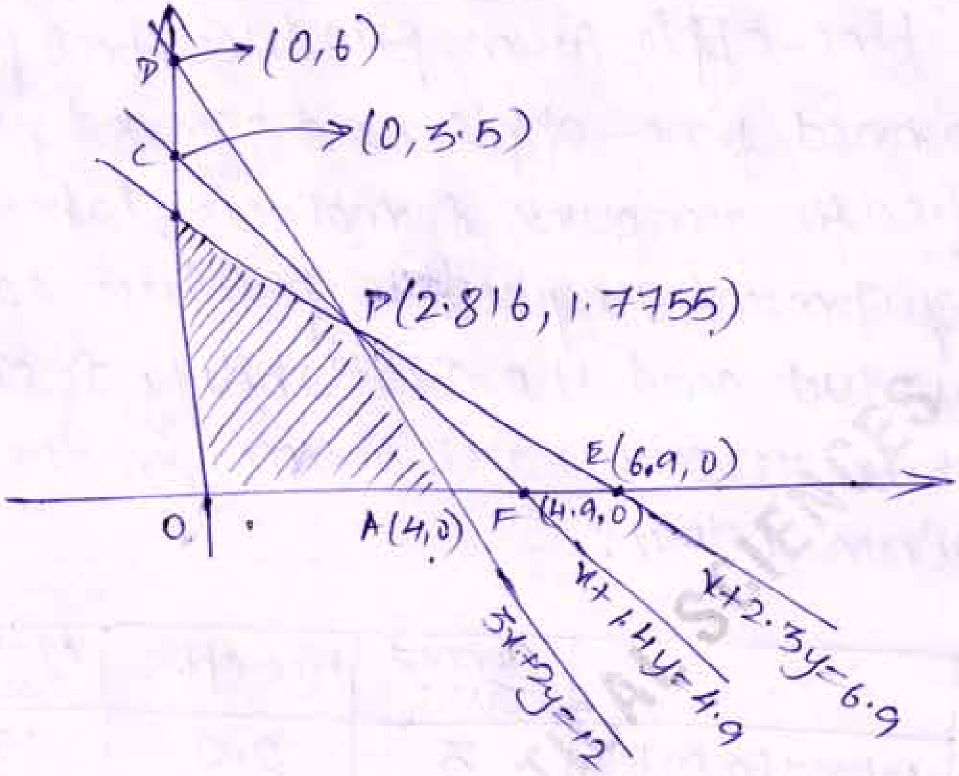
Then, L.P. problem is –

Maximize $z = 2x + y$

subject to

$$3x + 2y \leq 12$$
$$x + 2.3y \leq 6.9$$
$$x + 1.4y \leq 4.9$$

and $x, y \geq 0$.



hence, required region is OAPD

$z(A) = 2\times4+0 = 8$

$z(P) = 7.408$

$z(D) = 2\times0+3 = 3$

$\therefore\ z_{max} = 8$ at $x = 4, y = 0$

2(a) show that $Z[i] = \{m+ni / m, n \in Z, i = \sqrt{-1}\}$ is a Euclidean domain.

Sol^n: Given that $Z[i] = \{m+ni / m, n \in Z, i = \sqrt{-1}\}$ of Gaussian integers is an integral domain w.r.t $+^n$ and $\times^n$.

Let us define the mapping $d : Z[i] - \{0\} \to Z$ by $d(x+iy) = x^2+y^2 \; \forall \; x+iy \in Z[i]-\{0\}$

i.e. $d(x+iy) = (|x+iy|)^2$

$$= x^2+y^2 \; \forall \; x+iy \in Z[i]-\{0\} \quad \text{--- (1)}$$

we have $x \neq 0$ or $y \neq 0$ and hence $x^2+y^2 \geq 1$

$\therefore d(z) = d(x+iy) \geq 0 \; \forall \; z \in Z[i]-\{0\}$

Let $z_1, z_2 \in Z[i]-\{0\}$ then we have $z_1 = a+ib$, $z_2 = c+id$ where $a, b, c, d \in Z$ and $a \neq 0$ or $b = 0$; $c \neq 0$ or $d \neq 0$.

$\therefore z_1 z_2 = (ac-bd) + i(ad+bc)$

Now we have $d(z_1 z_2) = (ac-bd)^2 + (ad+bc)^2$

$$= (a^2+b^2)(c^2+d^2)$$
$$\geq a^2+b^2 = d(z_1)$$
$$(\because c^2+d^2 \geq 1)$$

$$\therefore \boxed{d(z_1) \leq d(z_1 \cdot z_2)}$$

Now we have

$$\frac{z_1}{z_2} = \frac{a+ib}{c+id}$$
$$= \frac{ac+bd}{c^2+d^2} + i\left[\frac{bc-ad}{c^2+d^2}\right]$$

$\frac{z_1}{z_2} = p + iq$ (say)

where $p = \frac{ac+bd}{c^2+d^2}$; $q = \frac{bc-ad}{c^2+d^2}$ are rational numbers.

Corresponding to the rational numbers p and q, we can find suitable integers p' and q' such that

$|p'-p| \le \frac{1}{2}$ and $|q'-q| \le \frac{1}{2}$

Let $t = p' + q'i$ then $t \in Z[i]$

and $\frac{z_1}{z_2} = \lambda$, where $\lambda = p + qi$

$$\Rightarrow z_1 = \lambda z_2$$
$$= (\lambda - t) z_2 + t z_2$$

$\boxed{z_1 = t z_2 + r}$ where $r = (\lambda - t) z_2$

Now $z_1, z_2, t \in Z[i]$

$$\Rightarrow z_1 - t z_2 \in Z[i]$$
$$\Rightarrow r \in Z[i]$$

$\therefore \exists\, t, r \in Z[i]$ such that $z_1 = t z_2 + r$ where $r = 0$ (or)

$$\begin{aligned} d(r) &= d[(\lambda - t) z_2] \\ &= d[(p+qi) - (p'+q'i)]\, d(z_2) \\ &= d[(p-p') + (q-q')i]\, d(z_2) \\ &= [(p-p')^2 + (q-q')^2]\, d(z_2) \\ &\le \left[\frac{1}{4} + \frac{1}{4}\right] d(z_2) \\ &= \frac{1}{2} d(z_2) \\ &< d(z_2) \end{aligned}$$

thus $\exists\, t, r \in Z[i]$ such that $z_1 = t z_2 + r$ where $r = 0$ (or) $d(r) < d(z_2)$.

2(b) The infinite product $\prod_{n=1}^{\infty}\left\{1+\frac{x^2}{(1+x^2)^n}\right\}$ is convergent for all values of x, but is not uniformly convergent on any interval I, which either contains zero as an interior point or has it as an end point.

sol: Let $f_n(x) = \frac{x^2}{(1+x^2)^n}$ $\forall x \in I$

Then it is sufficient to test $\sum f_n(x)$ on I.

Here $S_n(x) = \frac{x^2}{1+x^2} + \frac{x^2}{(1+x^2)^2} + \cdots + \frac{x^2}{(1+x^2)^n}$

$$= \frac{x^2}{1+x^2}\left[1+\frac{1}{(1+x^2)}+\frac{1}{(1+x^2)^2}+\cdots \frac{1}{(1+x^2)^{n-1}}\right]$$

$$= \frac{x^2}{1+x^2}\left[1-\frac{1}{(1+x^2)^n}\right]\left[1-\frac{1}{1+x^2}\right]^{-1}$$

$$= \cancel{\frac{x^2}{1+x^2}}\left[1-\frac{1}{(1+x^2)^n}\right]\cancel{\frac{x^2}{(1+x^2)}}$$

$$= 1-\frac{1}{(1+x^2)^n}$$

Therefore,

$$S(x) = \lim_{n\to\infty} S_n(x) = 1 \quad \text{when } x\neq 0$$
$$= 0 \quad \text{when } x=0$$

Hence, $\sum f_n(x)$ and so $\prod(1+f_n(x))$

converges for all values of $x \in I$

But $\prod (1+f_n(x))$ is not uniformly convergent on I, for $\sum f_n(x)$ is not.

As for $\epsilon > 0$, ~~then~~

$x \neq 0$, $|S_n(x) - S(x)| = \frac{1}{(1+x^2)^n} < \epsilon$

if $n > \frac{\log 1/\epsilon}{\log(1+x^2)}$

then $n \to \infty$ if $x^2 \to 0$

And so there is no ~~finite~~ m

for which we may have

$|S_n(x) - S(x)| < \epsilon \ \forall n \geq m$ and $x \in I$

2(C) show that $x^n - a = 0$ has atmost one real +ve root of n is a +ve integer

Sol^n: Let $f(x) = x^n - a$. The $f'(x) = nx^{n-1}$

Since $f'(x) > 0$ for $x > 0$

Hence f is increasing on $[0, \infty]$

Let $x_1, x_2 \in (0, \infty)$ and $0 < x_1 < r < x_2$

such that $f(r) = 0$

Then $f(x_1) < f(r) < f(x_2)$ (or) $f(x_1) < 0 < f(x_2)$

which shows that if $x \neq r$, $f(x) \neq 0$ on $(0, \infty)$

i.e. $x^n - a = 0$ has atmost one real positive root.

3(a) (i) Let $|G| = 33$. What are the possible orders for the elements of G? Show that G must ~~be~~ have an element of order 3.

(ii) prove that group $\frac{4Z}{12Z} \simeq Z_3$.

(iii) Give an example of an infinite integral domain that has characteristic 3.

Sol (i) The possible orders are: 1, 3, 11, 33.

let $|x| = 33$; $x \in G$.

then $|x^{11}| = 3$.

we may assume that there is no element of order 33.

Since we know that

No. of elements of order d in a finite group. In a finite group, the no. of elements of order d is a multiple of $\phi(d)$.

$\therefore$ The number of elements of order 11 is a multiple 10 so they account for 0, 10, 20 or 30 elements of the group.

The identity accounts for one more. So, at most we have accounted for 31 elements.

By Lagrange's theorem, the elements unaccounted for have order '3'.

(ii) Define $f: 4\mathbb{Z} \to \mathbb{Z}_3$ by $f(4n) = [n]$ $\forall\, n \in \mathbb{Z}$

Clearly f is onto. and

$$f(4n+4m) = f(4(n+m)) = [n+m]$$
$$= [n] + [m]$$
$$= f(4n) + f(4m).$$

$\therefore$ f is homomorphism and onto

$\therefore$ By the fundamental theorem of homomorphism,

$$\frac{4\mathbb{Z}}{\ker f} \cong \mathbb{Z}_3.$$

Now we have $\ker f = \{4n \in 4\mathbb{Z} / f(4n) = [0] \text{ in } \mathbb{Z}_3\}$

$$= \{4n \in 4\mathbb{Z} / [n] = [0] \text{ in } \mathbb{Z}_3\}$$
$$= \{4n \in 4\mathbb{Z} / n \text{ is a multiple of } 3\}$$
$$= \{4n \in 4\mathbb{Z} / n = 3k,\ k \in \mathbb{Z}\}$$
$$= 12\mathbb{Z}.$$

$\therefore \frac{4Z}{12Z} \cong Z_3.$

(iii). Let $Z_3[x]$ be the ring of all polynomials over the ring $(Z_3=\{0,1,2\}, +_3, \times_3)$

clearly it is an infinite integral domain.

$\because$ Char of $Z_3 = 3.$

$\therefore$ Char of $Z_3[x] = 3$

3(b) Prove that $\frac{x}{1+x} < \log(1+x) < x$ for all $x>0$.

Deduce that $\log\frac{2n+1}{n+1} < \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} < \log 2$, n being a +ve integer.

Sol'n: Let $f(t) = \log(1+t)\ \forall\ t\in[0,x]$ where $x>0$.

and $f'(t) = \frac{1}{1+t}\ \forall\ t\in(0,x)$

By Lagrange's Mean Value theorem $\exists\, c\in(0,x)$ such that

$$f'(c) = \frac{f(x)-f(0)}{x-0}$$

$$\Rightarrow \frac{1}{1+c} = \frac{\log(1+x)-\log 1}{x}$$

$$\Rightarrow \frac{1}{1+c} = \frac{\log(1+x)}{x} \quad \text{——} (1)$$

Since $c\in(0,x)$

$\Rightarrow 0<c<x$

$\Rightarrow 1<1+c<1+x$

$\Rightarrow 1 > \frac{1}{1+c} > \frac{1}{1+x} \Rightarrow 1 > \frac{\log(1+x)}{x} > \frac{1}{1+x}$ (by (1))

$\Rightarrow x > \log(1+x) > \frac{x}{1+x}$

i.e. $\frac{x}{1+x} < \log(1+x) < x$

Now, we have $\log(1+x) < x$

Let $x = \frac{1}{n+1}$

then $\log\left(1+\frac{1}{n+1}\right) < \frac{1}{n+1}$ i.e. $\log\left(\frac{n+2}{n+1}\right) < \frac{1}{n+1}$

$\log\left(1+\frac{1}{n+2}\right) < \frac{1}{n+2}$ i.e. $\log\left(\frac{n+3}{n+2}\right) < \frac{1}{n+2}$

Similarly $\log\left(\frac{n+4}{n+3}\right) < \frac{1}{n+3}$

$\log\left(\frac{n+5}{n+4}\right) < \frac{1}{n+4}$

$\vdots$

$\log\left(1+\frac{1}{n+n}\right) < \frac{1}{n+n}$ i.e. $\log\left(\frac{2n+1}{2n}\right) < \frac{1}{n+n}$

$\therefore$ Adding all the above, we get

$$\log\left(\frac{n+2}{n+1}\right) + \log\left(\frac{n+3}{n+2}\right) + \log\left(\frac{n+4}{n+3}\right) + \cdots + \log\left(\frac{2n+1}{2n}\right) < \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{n+n}$$

$$\Rightarrow \log\left(\frac{n+2}{n+1}\cdot\frac{n+3}{n+2}\cdot\frac{n+4}{n+3}\cdots\frac{2n}{2n-1}\cdot\frac{2n+1}{2n}\right) < \frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}$$

$$\Rightarrow \log\left(\frac{2n+1}{n+1}\right) < \frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{2n} \quad \text{—— (2)}$$

Also, we have $\frac{x}{1+x} < \log(1+x)$

let $x=\frac{1}{n}$, then $\frac{1/n}{1+1/n} < \log(1+\frac{1}{n}) \Rightarrow \frac{1}{n+1} < \log\left(\frac{n+1}{n}\right)$

$x=\frac{1}{n+1}$ then $\frac{\frac{1}{n+1}}{1+\frac{1}{n+1}} < \log\left(1+\frac{1}{n+1}\right) \Rightarrow \frac{1}{n+2} < \log\left(\frac{n+2}{n+1}\right)$

$\vdots$

$x=\frac{1}{2n-1}$ then $\frac{\frac{1}{2n-1}}{1+\frac{1}{2n-1}} < \log\left(1+\frac{1}{2n-1}\right) \Rightarrow \frac{1}{2n} < \log\frac{2n}{2n-1}$

Adding, we get

$$\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{n+n} < \log\left(\frac{n+1}{n}\right)+\log\left(\frac{n+2}{n+1}\right)+\cdots\log\frac{2n}{2n-1}$$

$$= \log \frac{n+1}{n}\cdot\frac{n+2}{n+1}\cdot\frac{n+3}{n+2}\cdots\frac{2n}{2n-1}$$

$$= \log\left(\frac{2n}{n}\right) = \log 2$$

$$\therefore \frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n} < \log 2 \quad \text{—— (3)}$$

$\therefore$ from (2) & (3)

$$\log\left(\frac{2n+1}{n+1}\right) < \frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n} < \log 2$$

3(C) A Product is produced by four factories F_1, F_2, F_3, F_4. The unit production costs in them are RS 1, RS 3, RS 1 and Rs. 5 respectively. Their production capacities are: F_1 – 50 units, F_2 – 70 units, F_3 – 30 units, F_4 – 50 units. These factories supply the product to four stores S_1, S_2, S_3 and S_4, demands of which are 25, 35, 105 and 20 units respectively. Unit transport cost in rupees from each factory to each store is given in the table below. Determine the extent of derivaties from each of the factories to each of the stores so that the total Production and transportation cost is minimum.

	S_1	S_2	S_3	S_4
F_1	2	4	6	11
F_2	10	8	7	5
F_3	13	3	9	12
F_4	4	6	8	3

Sol: First of all we shall form a new table of unit costs which consists of both production and transportation costs. The new cost matrix is given below

	S_1	S_2	S_3	S_4	a_i
F_1	2+2	4+2	6+2	11+2	50
F_2	10+3	8+3	7+3	5+3	70
F_3	13+1	3+1	9+1	12+1	30
F_4	4+5	6+5	8+5	3+5	50
Demand b_j	25	35	105	20	

Since $\sum a_i \neq \sum b_j$

it is an unbalanced transportation problem with surplus capacity = 15 units.

Therefore, we create a dummy store S_5 with associated cost coefficients which are taken as zero.

Therefore, the starting cost matrix becomes: —

	S_1	S_2	S_3	S_4	S_5	
F_1	4	6	8	13	0	50
F_2	13	11	10	8	0	70
F_3	14	4	10	13	0	30
F_4	9	11	13	8	0	50
	25	35	105	20	15	

Using the vogel's Approximation method, the initial basic feasible solution is given by

(25) 4	(5) 6	(20) 8	13	0
13	11	(70) 10	8	0
14	(30) 4	10	13	0
9	11	(15) 13	(20) 8	(15) 0

which is non-degenerate basic feasible solution.
since the no. of allocations $= m+n-1$
$= 4+5-1$
$= 8$ (basic variables)

Now finding the values of u_i and v_j:-

As the maximum no of basic cells exists in the first row and 3rd column. putting either $u_1=0$ or $v_3=0$

Let $u_1=0$ and the values of u_i's and v_j's and also the net evaluations $\Delta_{ij} = u_i+v_j-c_{ij}$

for all unoccupied cells are exhibited as shown below.

(25) 4	(5) 6	(20) 8	(−) 13	(−) 0	0
(−) 13	(−) 11	(70) 10	(−) 8	(−) 0	2
(−) 14	(30) 4	(−) 10	(−) 13	(−) 0	−2
9 (0)	11 (0)	(15) 13	(20) 8	(15) 0	5
4	6	8	3	−5	

Since all the net evaluations are ≤ 0 and atleast one $\Delta_{ij} = 0$, the current initial basic feasible solution is optimal but not unique.

There exists alternate optimal solution.

Therefore one of the optimal solution becomes

$x_{11} = 25, \quad x_{12} = 5, \quad x_{13} = 20$

$x_{23} = 70, \quad x_{32} = 30, \quad x_{43} = 15,$

$x_{44} = 20, \quad x_{45} = 15$

with optimum ~~transportation~~ plus product cost = 1465

4(a) Let $R = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} / a, b \in Z \right\}$ and let ϕ be the mapping that takes $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ to $a-b$.

(i) Show that ϕ is a homomorphism.

(ii) Determine the kernel of ϕ.

(iii) Show that $R/\ker\phi$ is isomorphic to Z.

Soln: Let $\phi : R \to Z$ such that

$$\phi \begin{bmatrix} a & b \\ b & a \end{bmatrix} = a-b$$

we have

$$\phi\left(\begin{bmatrix} a & b \\ b & a \end{bmatrix} + \begin{bmatrix} a_1 & b_1 \\ b_1 & a_1 \end{bmatrix}\right) = \phi \begin{bmatrix} a+a_1 & b+b_1 \\ b+b_1 & a+a_1 \end{bmatrix}$$

$$= (a+a_1) - (b+b_1)$$

$$= (a-b) + (a_1 - b_1)$$

$$= \phi\begin{pmatrix} a & b \\ b & a \end{pmatrix} + \phi\begin{pmatrix} a_1 & b_1 \\ b_1 & a_1 \end{pmatrix}$$

$\therefore \phi$ is homomorphism

Now $\ker\phi = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} / \phi\begin{pmatrix} a & b \\ b & a \end{pmatrix} = 0 \right\}$

since $\phi\begin{pmatrix} a & b \\ b & a \end{pmatrix} = a-b$

$\Rightarrow a-b=0$

$\Rightarrow \boxed{a=b}$

$\therefore \ker\phi = \left\{ \begin{pmatrix} a & a \\ a & a \end{pmatrix} / a \in Z \right\}$

for each $a \in Z$ $\exists$ $A \in R$ such that

$\phi(A) = a$

$\therefore$ ϕ is onto.

$\therefore$ By Fundamental theorem of Homomorphism.

$$\frac{R}{\ker\phi} \cong Z$$

4(b) Discuss the convergence of the sequence $\{x_n\}$

where $x_n = \dfrac{\sin\left(\frac{n\pi}{2}\right)}{8}$

Sol^n: let $x_n = \dfrac{\sin\left(\frac{n\pi}{2}\right)}{8}$ $\forall$ $n \in N$

Since $\lim\limits_{n\to\infty} x_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{1}{8} & \text{if } n = 4m+1 \\ -\frac{1}{8} & \text{if } n = 4m+3 \end{cases}$

does not exist

$\therefore$ (x_n) does not convergent.

4(c) Express $f(z) = \dfrac{1}{z(z+1)^2(z+2)^3}$ in a Laurent's series in the region $\frac{5}{4} \le |z| \le \frac{7}{4}$.

Let $f(z) = \dfrac{1}{z(z+1)^2(z+2)^3}$

$$= \frac{1}{z}\left[\frac{-3}{z+1} + \frac{1}{(z+1)^2} + \frac{3}{(z+2)} + \frac{3}{(z+2)^2} - \frac{1}{(z+2)^3}\right]$$

Obviously $f(z)$ is analytic everywhere except at $z = -1, -2, 0$

Hence $f(z)$ is analytic in the region $1 < |z| < 2$

The region $\frac{5}{4} \le |z| \le \frac{7}{4}$ is part of the region $1 < |z| < 2$.

i.e. the region $\frac{5}{4} \le |z| \le \frac{7}{4}$ is part of the annulus between ~~two~~ concentric circles $|z| = 1$ and $|z| = 2$ within which $f(z)$ is analytic.

Hence we can express it in Laurent's series at any z within the annulus

Now $f(z) = \dfrac{1}{z}\left[\dfrac{-(3z+2)}{(z+1)^2} + \dfrac{(3z^2+15z+17)}{(z+2)^3}\right]$

$$= \frac{1}{z}\left[\frac{-(3z+2)}{z^2}\left(1+\frac{1}{z}\right)^{-2} + (3z^2+15z+17)\cdot\frac{1}{8}\cdot\left(1+\frac{z}{2}\right)^{-3}\right.$$

$$= -\left(\frac{3}{z^2}+\frac{2}{z^3}\right)\sum_{n=0}^{\infty}(-1)^n(n+1)\frac{1}{z^n}$$

$$+\frac{1}{8}\left(3z+\frac{17}{z}+15\right)\sum_{n=0}^{\infty}\frac{(-1)^{n+1}(n+1)(n+2)}{2}\left(\frac{z}{2}\right)^n$$

$$= \left(\frac{3}{z^2}+\frac{2}{z^3}\right)\sum_{n=0}^{\infty}(-1)^{n+1}(n+1)\frac{1}{z^n}$$

$$+\frac{1}{16}\left(3z+\frac{17}{z}+15\right)\sum_{n=0}^{\infty}(-1)^{n+1}(n+1)(n+2)\left(\frac{z}{2}\right)^n$$

4(a) A department head has four tasks to be performed and three subordinates, the subordinates differ in efficiency. The estimates of the time, each subordinate would take to perform, is given below in the matrix. How should he allocate the tasks one to each man so as to maximize the total man-hours?

Solution

MEN

TASK	1	2	3
I	9	26	15
II	13	27	6
III	35	20	15
IV	18	30	20

Given problem is NOT balanced. To balance the problem, we add another column for man-4, having time = 0. The given problem then transforms to an ASSIGNMENT problem in which time is to be minimized.

	1	2	3	4 ← Dummy
I	9	26	15	0
II	13	27	6	0
III	35	20	15	0
IV	18	30	20	0

Now, applying Hungarian method (substracting the smallest element of a column from all the elements of that column), we obtain

0	6	9	0
4	7	0	0
26	0	9	0
9	10	14	0

Since, number of lines crossing all the zeroes is equal to the number of unknowns i.e. 4, hence, OPTIMALITY has been achieved.

∴ Assignment schedule is

	1	2	3	4
I	(0)			$\cancel{0}$
II			(0)	$\cancel{0}$
III		(0)		$\cancel{0}$
IV				(0)

(i.e.)
I → 1
II → 3
III → 2
IV → Dummy

thus, total man hours $= 9 + 20 + 6$
$= 35$

5(a) Find the equation of the surface satisfying $4yzp + q + 2y = 0$ and passing through $y^2+z^2=1$ and $x+z=2$.

Sol'n: Given equation can be written as

$$yzp + q = -2y \quad \text{--- (1)}$$

Lagrange's auxiliary equations are

$$\frac{dx}{4yz} = \frac{dy}{1} = \frac{dz}{-2y}$$

from first & 3rd fractions we have

$$dx + 2z\,dz = 0$$

Integrating, we get $x + z^2 = c_1$ --- (2)

Again from 2nd & 3rd fractions we have

$$2y\,dy + dz = 0$$

Integrating we get

$$y^2 + z = c_2 \quad \text{--- (3)}$$

The surface satisfying (1), are required to pass through $y^2+z^2=1$, $x+z=2$ --- (4)

Adding (2) & (3), we get $y^2+z^2+x+z = c_1+c_2$

$\Rightarrow 1+2 = c_1+c_2$ (by using 4)

$\Rightarrow \boxed{c_1+c_2=3}$

∴ The required surface is

$$\boxed{y^2+z^2+x+z=3}$$

5(b) Find a complete integral of $p^2x + q^2y = z$

Sol'n: Given equation is $f(x,y,z,p,q) = p^2x+q^2y-z=0$ --- (1)

Charpit's auxiliary equations are

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y+qf_z} = \frac{dz}{-pf_p-qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

$$\frac{dp}{-p+p^2} = \frac{dq}{-q+q^2} = \frac{dz}{-2(p^2x+q^2y)} = \frac{dx}{-2px} = \frac{dy}{-2qy} \text{ by } ① \quad —② $$

Now each fraction in ②

$$= \frac{2px\,dp + p^2dx}{2px(-p+p^2)+p^2(-2px)} = \frac{2qy\,dq + q^2dy}{2qy(-q+q^2)+q^2(-2qy)}$$

$$\Rightarrow \frac{d(p^2x)}{-2p^2x} = \frac{d(q^2y)}{-2q^2y}$$

$$\Rightarrow \frac{d(p^2x)}{p^2x} = \frac{d(q^2y)}{q^2y}$$

Integrating we get

$$\log(p^2x) = \log(q^2y) + \log a$$

$$\Rightarrow p^2x = aq^2y \quad —③$$

from ① & ③

$$aq^2y + q^2y = z$$

$$\Rightarrow q = \left[z/y(1+a)\right]^{1/2} \quad —④$$

from ③ & ④ $\quad p = q\left(\frac{ya}{x}\right)^{1/2} = \left[\frac{za}{(1+a)x}\right]^{1/2}$

Putting the above values of p and q in $dz = p\,dx + q\,dy$, we get

$$dz = \left[\frac{za}{(1+a)x}\right]^{1/2} dx + \left[\frac{z}{(1+a)y}\right]^{1/2} dy$$

$$\Rightarrow (1+a^2)^{1/2} z^{-1/2}\, dz = \sqrt{a}\, x^{-1/2} \cdot dx + y^{-1/2}\, dy$$

Integrating we get

$$(1+a)^{1/2}\sqrt{z} = \sqrt{a}\sqrt{x} + \sqrt{y} + b$$

a, b being arbitrary constants.

5(c) Use Regula-Falsi method to find a real root of the equation $\log x - \cos x = 0$, accurate to four decimal places after three successive approximations.

Solution:

Let $f(x) = \log x - \cos x$

$\because f(1) = -0.5403 < 0,\ f(2) = 0.71717683$

$\therefore$ there must lie a root between 1 & 2.

Now, according to Regula-falsi method

$$x_{n+1} = \frac{x_{n-1} f(x_n) - x_n f(x_{n-1})}{f(x_n) - f(x_{n-1})}$$

$\therefore$ taking $x_0 = 1,\ x_1 = 2$, we have

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)}$$

$$= \frac{1(0.71717683) - 2(-0.5403)}{0.71717683 - (-0.5403)}$$

$$= 1.42969945$$

again, $f(x_2) = 0.01461557470$

so, a root lies between x_0 & x_2

$\therefore\ x_3 = \dfrac{x_0 f(x_2) - x_2 f(x_0)}{f(x_2) - f(x_0)} = 1.41838$

again, $f(x_3) = -3.187469 \times 10^{-5} < 0$

$\therefore$ there must lie a root between x_3 and x_4

so, $x_4 = \dfrac{x_1 f(x_3) - x_3 f(x_1)}{f(x_3) - f(x_1)} = 1.418407$

$\therefore$ root of $f(x)$ correct to four decimal places is 1.4184.

5(d) A committee of three approves proposal by majority vote. Each member can vote for the proposal by pressing a button at the side of their chairs. These three buttons are connected to a light bulb. For a proposal, whenever the majority of votes takes place, a light bulb is turned on. Design a circuit as simple as possible so that the current passes and the light bulb is turned on only when the proposal is approved.

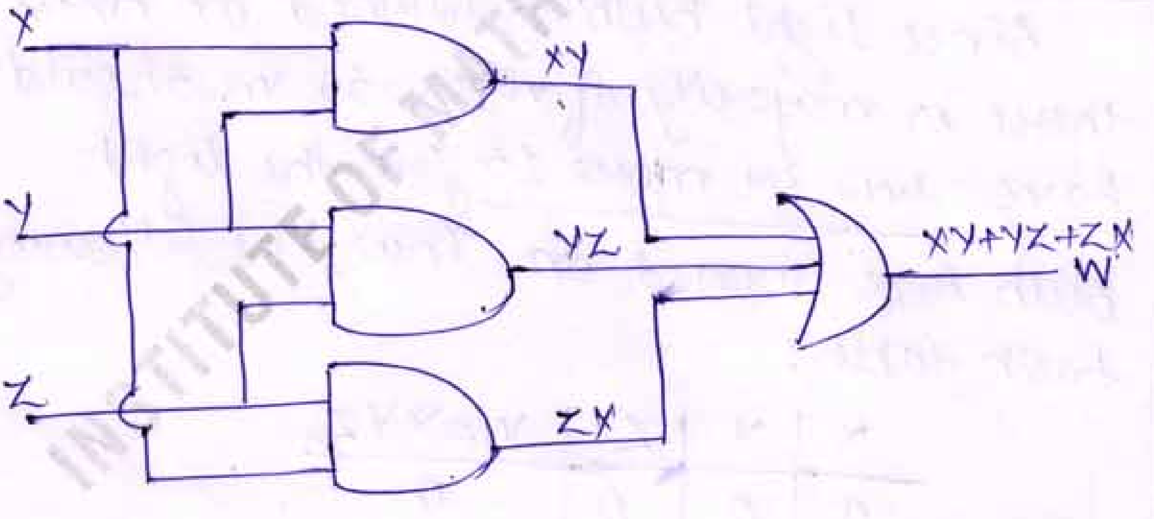
Solution:

Since light bulb is turned on when there is majority of votes, so we should have two or more 1s for the light bulb to be turned on. Thus, the following truth table:

x	y	z	W=XYZ
0	0	0	0
1	0	0	0
0	1	0	0
0	0	1	0
1	1	0	1
1	0	1	1
0	1	1	1
1	1	1	1

∴ Possible majority is–

$$W = XY\bar{Z} + X\bar{Y}Z + \bar{X}YZ + XYZ$$
$$= XY(Z+\bar{Z}) + X\bar{Y}Z + \bar{X}YZ$$
$$= XY + X\bar{Y}Z + \bar{X}YZ \quad (\because Z+\bar{Z}=1)$$
$$= X(Y+\bar{Y}Z) + \bar{X}YZ$$
$$= X\{(Y+\bar{Y})(Y+Z)\} + \bar{X}YZ$$
$$= X(Y+Z) + \bar{X}YZ$$
$$= XY + Z(X+\bar{X}Y)$$
$$= XY + Z\{(X+\bar{X})(X+Y)\}$$
$$= XY + YZ + ZX$$



5(e). Use Hamilton's equations to find the equations of motion of a projectile in space.

Sol'n: Let (x, y, z) be the coordinates of a projectile in space at time t, if K and V are the kinetic and potential energies, then

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) \text{ and } V = mgz$$

$$\therefore L = T - V = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - mgz$$

Here x, y, z are the generalised coordinates,

$$\therefore p_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}, \quad p_y = \frac{\partial L}{\partial \dot{y}} = m\dot{y}, \quad p_z = \frac{\partial L}{\partial \dot{z}} = m\dot{z} \quad \text{—— (1)}$$

Since L doesnot contain t explicity, therefore

$$H = T + V = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + mgz$$

$$\Rightarrow H = \frac{1}{2m}(p_x^2 + p_y^2 + p_z^2) + mgz, \qquad \text{(using relations (1))}$$

Hamilton's equations are

$$\dot{p}_x = -\frac{\partial H}{\partial x} = 0 \quad \text{—— } (H_1), \qquad \dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m} \quad \text{—— } (H_2)$$

$$\dot{p}_y = -\frac{\partial H}{\partial y} = 0 \quad \text{—— } (H_3), \qquad \dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m} \quad \text{—— } (H_4)$$

$$\dot{p}_z = -\frac{\partial H}{\partial z} = -mg \quad \text{—— } (H_5), \qquad \dot{z} = \frac{\partial H}{\partial p_z} = \frac{p_z}{m} \quad \text{—— } (H_6)$$

from (H_1) and (H_2), we have $\ddot{x} = (1/m)\dot{p}_x = 0$ —— (2)

from (H_3) and (H_4), we have $\ddot{y} = (1/m)\dot{p}_y = 0$ —— (3)

from (H_5) and (H_6) we have $\ddot{z} = (1/m)\dot{p}_z$ i.e. $\ddot{z} = -g$ —— (4)

Equations (2), (3), (4) are the equations of motion of projectile in space.

6(a) → Form a partial differential equation by eliminating the arbitrary constants a and b from

$\log(az-1) = x + ay + b$.

sol'n: Given $\log(az-1) = x+ay+b$ — ①

Differentiating ① partially w.r.t 'x'

we get $\frac{a}{az-1}\frac{\partial z}{\partial x} = 1$ — ②

Differentiating ① partially w.r.t 'y' we get

$$\frac{a}{az-1}\frac{\partial z}{\partial y} = 1 \quad \text{——} \quad ③$$

from ③ $az-1 = \frac{\partial z}{\partial y}$ so that $a = \frac{1+\partial z/\partial y}{z}$ — ④

putting the above values of $az-1$ and a in ②, we have

$$\frac{1+\partial z/\partial y}{z(\partial z/\partial y)}\frac{\partial z}{\partial x} = 1$$

$$\Rightarrow \left(1+\frac{\partial z}{\partial y}\right)\frac{\partial z}{\partial x} = z\frac{\partial z}{\partial y}$$

6(b) Reduce $r + 2xs + x^2 t = 0$ to canonical form

Sol'n: Comparing given equation with

$Rr + Ss + Tt + f(x, y, z, p, q) = 0$

$R = 1 \quad S = 2x \quad T = x^2$

$S^2 - 4RT = (2x)^2 - 4(x^2)(1) = 0$. ∴ parabolic

λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$

reduces to $\lambda^2 + 2x\lambda + x^2 = 0$

$(\lambda + x)^2 = 0$

$\lambda = -x, -x$

$dy/dx + (-x) = 0 \Rightarrow y - x^2/2 = c_1$

∴ $u = y - x^2/2 \qquad v = x$

$J(u,v) = \begin{vmatrix} u_x & v_x \\ u_y & v_y \end{vmatrix} = \begin{vmatrix} -x & 1 \\ 1 & 0 \end{vmatrix}$

$= -1 \neq 0.$

∴ u, v are linearly independent

$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = -x\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$

$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u}$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(-x\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$= -\frac{\partial z}{\partial u} - x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= -\frac{\partial z}{\partial u} - x\left[\frac{\partial^2 z}{\partial u^2}(-x) + \frac{\partial^2 z}{\partial u \partial v}\right] + \frac{\partial^2 z}{\partial u \partial v}(-x) + \frac{\partial^2 z}{\partial v^2}$$

$$r = -\frac{\partial z}{\partial u} - x^2\frac{\partial^2 z}{\partial u^2} - 2x\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \quad \text{--- (a)}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) = -x\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v} \quad \text{--- (b)}$$

$$T = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) = \frac{\partial^2 z}{\partial u^2} \quad \text{--- (c)}$$

putting (a), (b), (c) in given equation $r + 2xs + x^2 t = 0$

$$-\frac{\partial z}{\partial u} + \frac{\partial^2 z}{\partial v^2} = 0$$

$$\therefore \boxed{\frac{\partial^2 z}{\partial v^2} = \frac{\partial z}{\partial u}}$$ Required Canonical form

6(c) → The following table gives pressure of a steam at a given temperature. Using Newton's formula, compute the pressure for a temperature of 142°C.

Temp. °C	140	150	160	170	180
pressure/kgf/cm²	3.685	4.854	6.302	8.076	10.225.

Sol^n: The difference table

Temperature	pressure	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
140	3.685				
		1.169			
150	4.854		0.279		
		1.448		0.047	
160	6.302		0.326		0.02
		1.774		0.049	
170	8.076		0.375		
		2.149			
180	10.225				

$$P = \frac{x - x_0}{h} = \frac{142 - 140}{10} = 0.2$$

By Newtons forward formula.

$$y(42) = y_0 + P\Delta y_0 + \frac{P(P-1)}{2!}\Delta^2 y_0 + \frac{P(P-1)(P-2)}{3!}\Delta^3 y_0 + \frac{P(P-1)(P-2)(P-3)}{4!}\Delta^4 y_0$$

$$\therefore y = 3.685 + (0.2)(1.169) - \frac{(0.2)(0.8)}{2}(0.279) + \frac{(0.2)(0.8)(1.8)}{6}(0.047) - \frac{(0.2)(0.8)(1.8)(2.8)}{24}(0.002)$$

$$= 3.685 + 0.2338 + 0.002256 - 0.02232 - 0.0000672$$

$$= 3.8986688$$

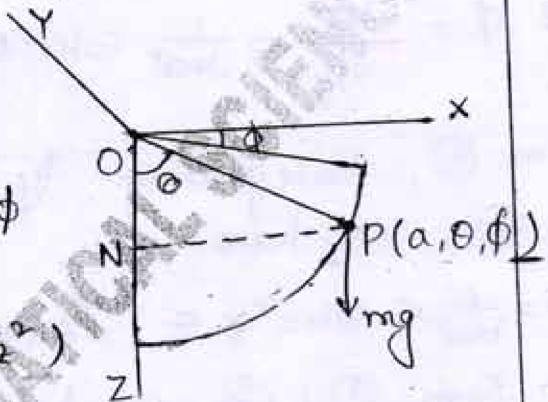
$\therefore$ Pressure at $142°C = 3.89866$ kgf/cm²

6(d) Determine the motion of spherical pendulum, by using Hamilton's equations.

Sol^n: Let m be the mass of the bob suspended by a light rod of length a. In a spherical pendulum of length a, the path of the motion of the bob is the surface of sphere of radius a and hence at the fixed point O.

At time t, let $P(a,\theta,\phi)$ be the position of the bob.

If (x,y,z) are the Cartesian coordinates of P then



$x = a\sin\theta\cos\phi,\quad y = a\sin\theta\sin\phi$

$z = a\cos\theta$

$\therefore$ K.E, $T = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2)$

$= \frac{1}{2}ma^2(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta)$

and potential $V = -mgz = -mga\cos\theta$

$\therefore L = T-V = \frac{1}{2}ma^2(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta)+mga\cos\theta$

Here θ and ϕ are the generalised coordinates.

$\therefore p_\theta = -\frac{\partial L}{\partial \theta} = ma^2\dot{\theta}$ and

$p_\phi = -\frac{\partial L}{\partial \phi} = ma^2\dot{\phi}\sin^2\theta$

Since L does not contain t explicity.

$\therefore H = T+V = \frac{1}{2}ma^2(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta) - mga\cos\theta$

Substituting the values of $\dot{\theta}$ and $\dot{\phi}$ from relations ①, we get

$H = \frac{1}{2ma^2}(p_\theta^2 + \operatorname{cosec}^2\theta\, p_\phi^2) - mga\cos\theta$

Hence the four Hamilton's equations are

$$\dot{p}_\theta = \frac{-\partial H}{\partial \theta} = \frac{1}{ma^2} \operatorname{cosec}^2\theta \cot\theta \, p_\phi^2 - mga\sin\theta \quad \text{—— (1)}$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{1}{ma^2} p_\theta \quad \text{—— (2)}$$

$$\dot{p}_\phi = -\frac{\partial H}{\partial \phi} = 0 \quad \text{—— (3)}$$

and $$\dot{\phi} = \frac{\partial H}{\partial p_\phi} = \frac{1}{ma^2} \operatorname{cosec}^2\theta \, p_\phi \quad \text{—— (4)}$$

from (3), Integrating, $p_\phi = C$ (Const).

$\therefore$ (4), we have

$$\dot{\phi} = \frac{1}{ma^2} C \operatorname{cosec}^2\theta = A/\sin^2\theta \quad \left[\text{where } A = \frac{C}{ma^2}\right] \quad \text{—— (5)}$$

Also from (1) & (2), we have

$$\ddot{\theta} = \frac{1}{ma^2} \dot{p}_\theta = \frac{1}{ma^2}\left[\frac{1}{ma^2} \frac{\cos\theta}{\sin^3\theta} p_\phi^2 - mga\sin\theta\right]$$

$$= \frac{1}{(ma^2)^2} C^2 \frac{\cos\theta}{\sin^3\theta} - \frac{g}{a}\sin\theta \quad \because p_\phi = C$$

$$= A^2 \frac{\cos\theta}{\sin^3\theta} - \frac{g}{a}\sin\theta \quad (\because A = C/ma^2)$$

Multiplying both sides by $2\dot{\theta}$ and integrating, we get

$$\dot{\theta}^2 = -\frac{A^2}{\sin^2\theta} + \frac{2g}{a}\cos\theta + B, \quad (B \text{ is a const}) \quad \text{—— (6)}$$

Equations (5) & (6) determine the required motion.

7(a) A square plate is bounded by the lines $x=0, y=0, x=10$ & $y=10$. Its faces are insulated. The temperature along the upper horizontal edge is given by $u(x,10) = x(10-x)$ while the other three faces are kept at $0°C$. Find the steady state temperature in the plate.

Sol^n: The steady state temperature $u(x,y)$ is the solution of Laplace equation $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ —— (1)

subject to boundary conditions

$u(0,y) = u(10,y) = 0, \quad 0 \le y \le 10$ — (2)

$u(x,0) = 0, \quad 0 \le x \le 10$ — (3a)

and $u(x,10) = 10x - x^2, \quad 0 \le x \le 10$ — (3b)

Suppose (1) has a solution of the form

$u(x,y) = X(x)\, Y(y)$ — (4)

Substituting this value of u in (1), we get

$X''Y + XY'' = 0 \Rightarrow \frac{X''}{X} = -\frac{Y''}{Y}$ — (5)

Since x and t are independent variables, each side of (5) must be equal to the constant, say μ.

Then (5) gives $X'' - \mu X = 0$ — (6)

and $Y'' + \mu Y = 0$ — (7)

Using (2), (4) gives $X(0)\,Y(y) = 0$ & $X(10)\,Y(y) = 0$

giving $X(0) = 0$ and $X(10) = 0$ — (8)

where $Y(y) \ne 0$, since otherwise $u \equiv 0$ which doesnot satisfy 3(b)

we now solve (6) under B.C. (8). Three cases arise.

case(1): Let $\mu = 0$. The solution of (6) is

$X(x) = Ax + B$ — (9)

Using B.C. (8), we get $A = B = 0$

so that $X(x) = 0$. This leads $u \equiv 0$ which doesnot satisfy 3(b). so we reject $\mu = 0$.

case(2): Let $\mu = \lambda^2, \lambda \ne 0$. Then solution of (6) is

$X(x) = A e^{x\lambda} + B e^{-\lambda x}$ — (10)

Using B.C (8), we get $A = B = 0$ so that $X(x) = 0$ and hence $u \equiv 0$.

So we reject $\mu = \lambda^2$.

Case (3): Let $\mu = -\lambda^2$, $\lambda \neq 0$. Then solution of (6) is

$X(x) = A\cos\lambda x + B\sin\lambda x$ — (11)

Using B.C. (8), (11) gives $0 = A$ and $0 = A\cos\lambda(10) + B\sin\lambda(10)$
$\Rightarrow \sin\lambda(10) = 0$, $B \neq 0$.

since otherwise $X(x) \equiv 0$ and hence $u \equiv 0$ which doesnot satisfy 3(5).

Now, $\sin\lambda a = 0 \Rightarrow \lambda(10) = n\pi$, $n = 1, 2, 3 \ldots$
$\Rightarrow \lambda = \frac{n\pi}{10}$; $n = 1, 2, \ldots$ — (12)

Hence non-zero solutions $X_n(x)$ of (6) are given by $X_n(x) = B_n \sin\left(\frac{n\pi x}{10}\right)$ — (13)

Using $\mu = -\lambda^2 = -\frac{n^2\pi^2}{(10)^2}$, (7) becomes $Y'' - \left(\frac{n^2\pi^2}{10^2}\right)Y = 0$ — (14)

whose general solution is
$Y_n(y) = C_n e^{n\pi y/10} + D_n e^{-n\pi y/10}$ — (15)

Using 3(a), (4) gives $0 = X(x)\,Y(0)$ so that $Y(0) = 0$, where we have taken $X(x) \neq 0$.
for otherwise we will get $u = 0$ which doesnot satisfy 3(5).

But $Y(0) = 0 \Rightarrow Y_n(y) = 0$ — (16)

putting $y = 0$ in (15) and using (16), we have
$0 = C_n + D_n \Rightarrow D_n = -C_n$. Then (15) reduces to
$Y_n(y) = C_n\left(e^{n\pi y/10} - e^{-n\pi y/10}\right)$ — (17)
$= 2\sinh(n\pi y/10)$

$\therefore u_n(x,y) = X_n(x)\,Y_n(y) = E_n \sin\left(\frac{n\pi x}{10}\right)\sinh\left(\frac{n\pi y}{10}\right)$ — (18)

are solutions of (1), satisfying (2) and 3(a).
Here $E_n = 2B_nC_n$.

In order to satisfy condition 3(5), now consider more general solution given by

$$u(x,y) = \sum_{n=1}^{\infty} u_n(x,y) = \sum_{n=1}^{\infty} E_n \sin\left(\frac{n\pi x}{10}\right) \sinh\left(\frac{n\pi y}{10}\right). \quad (18)$$

putting $y=10$ in (18) and using 3(b), we get

$$u(x,10) = \underset{=f(x)}{10x - x^2} = \sum_{n=1}^{\infty} E_n \sinh(n\pi) \sin\left(\frac{n\pi x}{10}\right).$$

which is the half range fourier sine series of $f(x)$ in $(0, 10)$.

Hence, we have

$$E_n \sinh(n\pi) = \frac{2}{10}\int_0^{10} (10x - x^2) \sin\left(\frac{n\pi x}{10}\right) dx.$$

$$= \frac{1}{5\sinh n\pi}\left[(10x - x^2)\left(-\frac{10}{n\pi}\cos\frac{n\pi x}{10}\right) - (10 - 2x)\left(\frac{-100}{n^2\pi^2}\right)\sin\frac{n\pi x}{10} + 2\left(\frac{1000}{n^3\pi^3}\right)\cos\frac{n\pi x}{10}\right]_0^{10}$$

$$= \frac{1}{5\sinh n\pi}\left[-\frac{2000(-1)^n}{n^3\pi^3} + \frac{2000}{n^3\pi^3}\right]$$

$$= \frac{400(1-(-1)^n)}{n^3\pi^3 \sinh n\pi}.$$

$$\therefore E_n = \begin{cases} 0, & \text{if } n = 2m \text{ and } m = 1,2,3,\ldots \\ \dfrac{800\,\mathrm{cosech}\,(2m-1)\pi}{(2m-1)^3\pi^3}, & \text{if } n = 2m-1,\ m = 1,2,\ldots \end{cases}$$

$\therefore$ from (18), we have

$$u(x,y) = \frac{800}{\pi^3}\sum_{m=1}^{\infty} \frac{\mathrm{cosech}(2m-1)\pi}{(2m-1)^3} \sin\frac{(2m-1)\pi x}{10} \sinh\frac{(2m-1)\pi y}{10}.$$

which is the required temperature

7(b). Find the solution, to three decimals of the system

$83x + 11y - 4z = 95$

$7x + 52y + 13z = 104$

$3x + 8y + 29z = 71$

using Gauss Seidel method.

Sol: By the Gauss Seidel method given system can be written as

$$x^{k+1} = \frac{1}{83}\left[95 - 11y^k + 4z^k\right]$$

$$y^{k+1} = \frac{1}{52}\left[104 - 7x^{k+1} - 13z^k\right]$$

$$z^{k+1} = \frac{1}{29}\left[71 - 3x^{k+1} - 8y^{k+1}\right]$$

where $k = 0, 1, 2, \ldots$

Now taking $x^{(0)} = 0$, we obtain the following iterations

Here $x^{(0)} = (0_x, 0_y, 0_z)$

first iteration:

$k=0$: $x^{(1)} = \frac{95}{83} = 1.1445$

$y^{(1)} = \frac{1}{52}\left[104 - 7(1.1445) - 13(0)\right] = 1.84592$

$z^{(1)} = \frac{1}{29}\left[71 - 3(1.1445) - 8(1.84592)\right] = 1.82065$

k=	x	y	z
1	0.98768	1.41188	1.95662
2	1.05176	1.36526	1.96175
3	1.0576	1.3672	1.9617
4	1.0579	1.3672	1.9617

∴ The solution is $x = 1.057$, $y = 1.3672$, $z = 1.9617$

7(C) If the velocity of an incompressible fluid at the point (x,y,z) is given by $\left(\frac{3xz}{r^5}, \frac{3yz}{r^5}, \frac{3z^2-r^2}{r^5}\right)$, $r^2=x^2+y^2+z^2$ then Prove that the liquid motion is possible and that the velocity potential is z/r^3. Also determine the streamlines.

Sol^n: Here $u=\frac{3xz}{r^5}$, $v=\frac{3yz}{r^5}$, $w=\frac{3z^2-r^2}{r^5}=\frac{3z^2}{r^5}-\frac{1}{r^3}$ —①

where $r^2=x^2+y^2+z^2$ —②

from ②, $\frac{\partial r}{\partial x}=x/r$, $\frac{\partial r}{\partial y}=y/r$, $\partial r/\partial z=z/r$ —③

from ①, ② and ③ we have

$$\frac{\partial u}{\partial x}=3z\left[\frac{1}{r^5}+(-5x)r^{-6}\frac{\partial r}{\partial x}\right]=\frac{3z}{r^5}-\frac{15x^2z}{r^7},$$

$$\frac{\partial v}{\partial y}=3z\left[\frac{1}{r^5}+(-5y)r^{-6}\frac{\partial r}{\partial y}\right]=\frac{3z}{r^5}-\frac{15y^2z}{r^7}$$

$$\frac{\partial w}{\partial z}=\frac{6z}{r^5}-15z^2r^{-6}\frac{\partial r}{\partial z}+3r^{-4}\frac{\partial r}{\partial z}=\frac{6z}{r^5}-\frac{15z^2}{6}\cdot\frac{z}{r}+\frac{3}{r^4}\cdot\frac{z}{r}=\frac{9z}{r^5}-\frac{15z^3}{r}$$

$$\therefore \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=\frac{15z}{r^7}(x^2+y^2+z^2)=\frac{15z}{r^5}-\frac{15z}{r^7}\times r^2=0$$

Since the equation of continuity is satisfied by the given values of u, v and w, the motion is possible. Let ϕ be the required velocity potential. Then

$$d\phi=\frac{\partial\phi}{\partial x}dx+\frac{\partial\phi}{\partial y}dy+\frac{\partial\phi}{\partial z}dz=-(u\,dx+v\,dy+w\,dz),\text{ by definition of }\phi$$

$$=-\left[\frac{3xz}{r^5}dx+\frac{3yz}{r^5}dy+\frac{3z^2-r^2}{r^5}dz\right]$$

$$=\frac{r^2dz-3z(x\,dx+y\,dy+z\,dz)}{r^5}=\frac{r^3dz-3r^2z\,dr}{(r^3)^2}=d\left(\frac{z}{r^3}\right),\text{ using ②}$$

Integrating, $\phi=z/r^3$

In spherical polar coordinates (r,θ,ϕ), we know that $z=r\cos\theta$. Hence the required potential is given by $\phi=(r\cos\theta)/r^3=\frac{\cos\theta}{r^2}$

we now obtain streamlines. The equations of streamlines are given by $\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w}$ i.e. $\frac{dx}{3xz/r^5}=\frac{dy}{3yz/r^5}=\frac{dz}{(3z^2-r^2)/r^5}$

$$\Rightarrow \frac{dx}{3xz}=\frac{dy}{3yz}=\frac{dz}{3z^2-r^2} \quad —③$$

Taking the first two members of ③ and simplifying we get

$dx/x-dy/y=0 \Rightarrow$ Integrating $\log x-\log y=c_1 \Rightarrow x/y=c_1$ (c_1 constant)

Now, each member in ③

$$= \frac{x\,dx + y\,dy + z\,dz}{3x^2z + 3y^2z + 3z^3 - r^2 z} = \frac{x\,dx + y\,dy + z\,dz}{3z(x^2+y^2+z^2) - (x^2+y^2+z^2)z}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{2z(x^2+y^2+z^2)}, \text{ by } ② \quad \text{—— } ⑤.$$

Taking the first member in ③ and ⑤, we get-

$$\frac{dx}{3xz} = \frac{x\,dx + y\,dy + z\,dz}{2z(x^2+y^2+z^2)} \Rightarrow \frac{2}{3}\frac{dx}{x} = \frac{1}{2}\frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2+y^2+z^2}$$

Integrating, $\frac{2}{3}\log x = \frac{1}{2}\log(x^2+y^2+z^2) + \log c_2$

$\Rightarrow x^{2/3} = c_2 (x^2+y^2+z^2)^{1/2}$, c_2 being arbitrary Const. ⑥

The required streamlines are the curves of intersection of ④ & ⑥

8(a) If the string of length l is initially at rest in equilibrium position and each of the points is given the velocity $v_0 \sin(3\pi x/l)\cdot\cos(2\pi x/l)$ where $0<x<l$ at $t=0$. Find the displacement function.

Sol Let the Required displacement function $y(x,t)$ is the the solution of wave equation

$$\frac{\partial^2 y}{\partial x^2} = (1/c^2)(\partial^2 y/\partial t^2) \quad \text{—— (1)}$$

Subject to boundary condition.

$$y(0,t) = y(l,t) = 0 \quad \text{for all } t \geq 0 \quad \text{— (2)}$$

Initial conditions

$$y(x,0) = 0 \quad \text{- equilibrium position}$$

$$\frac{\partial y}{\partial t} = v_0 \sin(3\pi x/l)\cos(2\pi x/l) \quad \text{— (3)}$$

- initial velocity

Suppose (1) has the solution of the form.

$$y(x,t) = X(x)\,T(t) \quad \text{— (4)}$$

Putting (4) in (1) $\quad X''T = \frac{1}{c^2}\cdot XT'' \Rightarrow \frac{X''}{X} = \frac{T''}{c^2 T} = \mu \quad$ -(5)

Now solve (7) $T'' + (n^2\pi^2c^2/l^2)T = 0$. Since $\mu = -\lambda^2 = \frac{n\pi}{l}$

$$T_n = C_n \cos\frac{n\pi c}{l}t + D_n \sin\frac{n\pi c}{l}t$$

$$y_n(x,t) = X_n(x)T_n(t) = \left(E_n\cos\left(\frac{n\pi ct}{l}\right) + F_n\sin\left(\frac{n\pi ct}{l}\right)\right)\sin\left(\frac{n\pi x}{l}\right)$$

$$y(x,t) = \sum_{n=1}^{\infty} y_n(x,t) = \sum_{n=1}^{\infty}\left(E_n\cos\left(\frac{n\pi ct}{l}\right) + F_n\sin\left(\frac{n\pi ct}{l}\right)\right)\sin\left(\frac{n\pi x}{l}\right)$$

from (8) equilibrium position

$$y(x,0) = 0.$$

$$E_n\sin\left(\frac{n\pi x}{l}\right) = 0 \Rightarrow E_n = 0.$$

$$\therefore y(x,t) = \sum_{n=1}^{\infty} F_n \sin\left(\frac{n\pi x}{l}\right)\sin\left(\frac{n\pi ct}{l}\right) \quad -(8)$$

differentiating (8) w.r.t t

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} F_n\left(\frac{n\pi c}{l}\right)\sin\left(\frac{n\pi x}{l}\right)\cdot\cos\left(\frac{n\pi ct}{l}\right).$$

$$\left.\frac{\partial y}{\partial t}\right|_{t=0} = \sum_{n=1}^{\infty} F_n\left(\frac{n\pi c}{l}\right)\sin\left(\frac{n\pi x}{l}\right) = V_0\sin\left(\frac{3\pi x}{l}\right)\cos\left(\frac{2\pi x}{l}\right)$$

$$(\pi cF_1/l)\sin\left(\frac{\pi x}{l}\right) + (2\pi cF_2/l)\sin\left(\frac{2\pi x}{l}\right)\ldots = \frac{V_0}{2}\left[\sin\left(\frac{5\pi x}{l}\right) + \sin\left(\frac{\pi x}{l}\right)\right]$$

$$\pi cF_1/l = \frac{V_0}{2} \qquad 5\pi cF_5/l = V_0/2$$

$$F_2 = F_3 = F_4 = F_6 = F_7 \ldots = 0.$$

$$F_1 = (lV_0/2\pi c) \qquad F_5 = (lV_0/10\pi c)$$

$$\therefore y(x,t) = (lV_0/2\pi c)\sin(\pi x/l)\cdot\sin(\pi ct/l) + \frac{lV_0}{10\pi c}\cdot\sin\left(\frac{5\pi x}{l}\right)\cdot\sin(5\pi ct/l)$$

from (5) $X'' - \mu X = 0$ —— (6)

$T'' - \mu c^2 T = 0$ —— (7)

from (2) $X(0)T(t) = 0$ and $X(a)T(t) = 0$.

Suppose $T(t) \neq 0$ for some t

$\Rightarrow \underline{X(0) = 0} \quad \underline{X(a) = 0}$

with these conditions we solve the (6)

$X'' - \mu X = 0$.

Case (i) $\mu = 0 \Rightarrow X(x) = Ax + B$

$X(0) = 0 \Rightarrow B = 0$

$X(a) = 0 \Rightarrow A = 0$.

$\therefore$ we reject $\mu = 0$

Case (ii) $\mu = \lambda^2 \Rightarrow X(x) = Ae^{\lambda x} + Be^{-\lambda x}$

$X(0) = 0 \Rightarrow A + B = 0$

$X(a) = 0 \Rightarrow Ae^{\lambda a} + Be^{-\lambda a} = 0$

$A(e^{\lambda a} - e^{-\lambda a}) = 0$

$\underline{A = 0} \Rightarrow B = 0$

$\therefore$ we ~~jeck~~ reject $\mu = \lambda^2$

Case (iii) $\mu = -\lambda^2 \Rightarrow X(x) = A\cos\lambda x + B\sin\lambda x$

$X(0) = 0 \quad A = 0$.

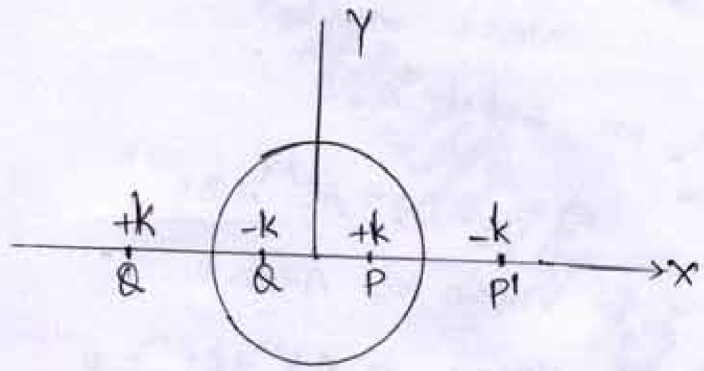
$X(a) = 0 \quad \lambda l = n\pi \Rightarrow \lambda = \frac{n\pi}{l}$

$\therefore X_n(x) = B_n \sin\left(\frac{n\pi x}{l}\right)$

8(c) → If a vortex pair is situated within a cylinder show that it will remain at rest if the distance of either from the centre is given by $a(\sqrt{5}-2)^{1/2}$, where a is the radius of the cylinder.

Sol^n: The vortex pair PQ consists of vortex $+k$ at P and vortex $-k$ at Q.

The image of vortex $+k$ at P is a vortex at $-k$ at P', the inverse point of P.



Similarly, the image of vortex $-k$ at Q is a vortex $+k$ at Q'.

Let $OP = OQ = r$.

Then $OP \cdot OP' = a^2 = OQ \cdot OQ'$

Hence $OP' = \frac{a^2}{r} = OQ'$.

Thus $z_p = r$, $z_Q = -r$

$z_Q' = -a^2/r$, $z_{p'} = a^2/r$

The complex potential for ~~this~~ motion ie

$$W = \frac{ik}{2\pi}\left[\log(z-z_p) - \log(z-z_p') - \log(z-z_Q) + \log(z-z_Q')\right]$$

The motion of P is due to other vortices

for the motion of P,

$$W_1 = W - \frac{ik}{2\pi}\log(z-z_p)$$

$$\frac{dW_1}{dz} = \frac{-ik}{2\pi}\left[\frac{1}{z-z_{p'}} + \frac{1}{z-z_Q} - \frac{1}{z-z_{Q'}}\right]$$

$$u_p - iv_p = \left(-\frac{dW_1}{dz}\right)_{z=z_p}$$

$$= \frac{ik}{2\pi}\left[\frac{1}{z_p-z_{p'}} + \frac{1}{z_p-z_Q} - \frac{1}{z_p-z_{Q'}}\right]$$

This implies $u_p = 0$, $v_p = \frac{-k}{2\pi}\left[\frac{r}{a^2-r^2} - \frac{1}{2r} + \frac{r}{r^2+a^2}\right]$

The vortex at P will be at rest if $q_p = 0$

i.e, $\sqrt{u_p^2+v_p^2}=0$

(or) $v_p=0$ (or) $\frac{r}{a^2-r^2}-\frac{1}{2r}+\frac{r}{r^2+a^2}=0$

$2r^2(r^2+a^2)-(a^2-r^2)(a^2+r^2)+2r^2(a^2-r^2)=0$

$\Rightarrow r^4+4r^2a^2-a^4=0$

$\Rightarrow \left(\frac{r^2}{a^2}\right)^2+4\left(\frac{r^2}{a^2}\right)-1=0$

$\Rightarrow \frac{r^2}{a^2}=\frac{-4\pm\sqrt{20}}{2}=-2\pm\sqrt{5}$

$r=a(-2\pm\sqrt{5})^{1/2}$

The value $(-2-\sqrt{5})^{1/2}a$ is not admissible because this root gives imaginary value of r.

Hence $r=a(-2+\sqrt{5})^{1/2}$