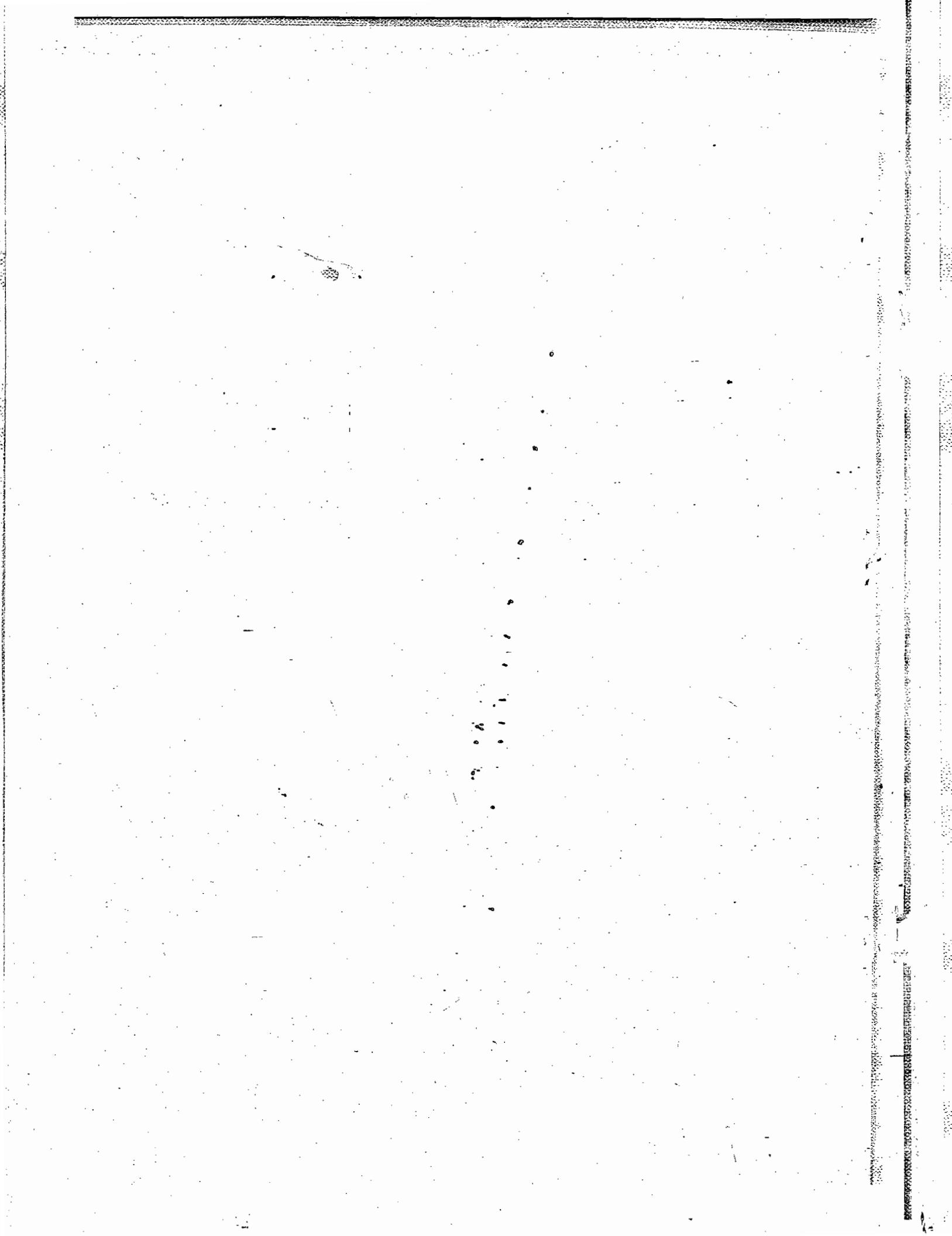


**IMS**  
**MATHS**  
**BOOK-03**



## \* REAL NUMBER SYSTEM \*

→ the set of natural numbers  
 $N = \{1, 2, 3, \dots\}$

→ the set of whole numbers  
 $W = \{0, 1, 2, 3, \dots\}$

→ the set of integers

$I = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$

→ the set of +ve integers

$I^+ = \{1, 2, 3, \dots\}$

→ the set of -ve integers

$I^- = \{\dots, -3, -2, -1\}$

→ the set of rational numbers

$Q = \left\{ \frac{P}{q} \mid P, q \in I, q \neq 0 \right\}$

→ the set of irrational numbers

$Q' =$  the numbers which cannot be expressed in the form of  $P/q$  ( $q \neq 0$ ) are known as irrational numbers.

Ex:  $\sqrt{2}, \sqrt{5}, e, \pi$  etc.

Note:

→ (1). the rational number can be expressed either as a terminating decimal (or) non-terminating recurring decimal.

→ (2). An irrational number can be expressed as non-terminating non-recurring decimal.

The set of real numbers  $R = Q \cup Q'$   
 i.e. the set of real numbers  $R$  which contains the set of rational and irrational numbers.

Note:

- (1). NCWCIC & CIR and  $Q' \subset R$ .
- (2). Between any two distinct consecutive integers, there exists no integer.
- (3). Between any two distinct rational numbers, there lie infinitely many rational numbers.
- (4). Between any two rational numbers there lie infinitely many irrational numbers.
- (5). Between any two irrational numbers there lie infinitely many irrational numbers as well as infinitely many rational numbers.
- (6). Between any two real numbers there lie infinitely many real numbers.

Note: The symbols  $\exists$  and  $\forall$  are known as Quantifiers and the symbols  $\Rightarrow$ ,  $\Leftrightarrow$  as connectives.

→ Some important Properties of real numbers in the form of Axioms. These axioms can be

divided into three types:

1. Field axioms
2. Order axioms
3. Completeness axiom.

#### (1) Field Axioms:

Let  $\mathbb{R}$  be the set of real numbers, then the algebraic structure  $(\mathbb{R}, +, \cdot)$  satisfies the following axioms.

(i)  $(\mathbb{R}, +)$  is an abelian group.

i.e. (i) Closure Property:

$$\forall a, b \in \mathbb{R} \Rightarrow a+b \in \mathbb{R}$$

(ii) Associative Property:

$$\forall a, b, c \in \mathbb{R} \Rightarrow a+(b+c) = (a+b)+c$$

(iii) Existence of identity:

$\forall a \in \mathbb{R}, \exists 0 \in \mathbb{R}$  such that

$$a+0 = 0+a = a$$

the real number '0' is called the additive identity of  $\mathbb{R}$ .

(iv) Existence of inverse:

$\forall a \in \mathbb{R}, \exists b \in \mathbb{R}$  such that

$$a+b = 0 = b+a$$

the real number 'b' is called the additive inverse of 'a'.

(v) Commutative Property:

$$\forall a, b \in \mathbb{R} \Rightarrow a+b = b+a$$

(II)  $(\mathbb{R}, \cdot)$  is an abelian group.

i.e. (i) Closure Property:

$$\forall a, b \in \mathbb{R} \Rightarrow a \cdot b \in \mathbb{R}$$

(ii) Associative Property:

$$\forall a, b, c \in \mathbb{R} \Rightarrow a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

(iii) Existence of Identity:

$\forall a \in \mathbb{R}, \exists 1 \in \mathbb{R}$  such that

$$a \cdot 1 = 1 \cdot a = a$$

the real number '1' is called the multiplicative identity of  $\mathbb{R}$ .

(iv) Existence of inverse:

$\forall a \in \mathbb{R}, a \neq 0, \exists b \in \mathbb{R}$  such that  $a \cdot b = b \cdot a = 1$

the real number 'b' is called the multiplicative inverse of 'a' and is denoted by  $a^{-1}$ .

(v) Commutative Property:

$$\forall a, b \in \mathbb{R} \Rightarrow a \cdot b = b \cdot a$$

### III. Distributivity :-

multiplication is distributive with respect to addition in  $\mathbb{R}$ .

i.e.  $\forall a, b, c \in \mathbb{R}$

$$\Rightarrow a \cdot (b+c) = a.b + a.c \text{ (L.D.L.)}$$

and

$$(b+c) \cdot a = b.a + c.a \text{ (R.D.L.)}$$

$\rightarrow$  A non-empty set  $S$  is said to be a field if it possesses the two compositions  $+^n$  &  $\times^n$  and satisfied all the above axioms.

Ex:-  $(\mathbb{Q}, +, \cdot)$  is a field but  $(\mathbb{Z}, +, \cdot)$  &  $(\mathbb{N}, +, \cdot)$  are not fields.

### 2. Order Axioms:

The order relation ' $>$ ' between pairs of real numbers  $\mathbb{R}$  satisfies the following axioms:

Let  $a, b, c \in \mathbb{R}$  then

O<sub>1</sub>: for  $a, b \in \mathbb{R}$ , exactly one of the following holds

- (i)  $a > b$  (ii)  $a = b$  and
- (iii)  $b > a$

which is known as the law of trichotomy.

O<sub>2</sub>: For  $a, b, c \in \mathbb{R}$ ;

$$a > b, b > c \Rightarrow a > c$$

which is known as the law of transitivity.

O<sub>3</sub>:  $\forall a, b, c \in \mathbb{R}$ ;

$$a > b \Rightarrow a + c > b + c$$

which is known as the monotone property for  $+^n$ .

O<sub>4</sub>:  $\forall a, b, c \in \mathbb{R}$ ;

$$a > b \text{ and } c > 0 \Rightarrow ac > bc$$

which is known as the monotone property for  $\times^n$ .

$\rightarrow$  A field satisfying the above properties is called an ordered field.

Hence  $(\mathbb{R}, +, \cdot)$  is an ordered field.

Note:  $(\mathbb{Q}, +, \cdot)$  is an ordered fi

\* Some more definitions :-

$\rightarrow$  Less than relation: For  $a, b \in \mathbb{R}$

$$a < b \Leftrightarrow b > a$$

$\rightarrow$  Two real numbers:

$a \in \mathbb{R}$  is said to be between if  $a$  and is denoted by  $I^+$ .

### → -ve real numbers :

$a \in \mathbb{R}$  is said to be '-ve' if  $a < 0$  and is denoted by  $\mathbb{I}\mathbb{R}^-$ .

$$\therefore \mathbb{I}\mathbb{R} = \mathbb{I}\mathbb{R}^- \cup \{0\} \cup \mathbb{I}\mathbb{R}^+$$

→ If  $a \in \mathbb{I}\mathbb{R}^+$  and  $b \in \mathbb{I}\mathbb{R}^-$  then  $a > b$ .

→ A real number ' $a$ ' is said to be greater than (or) equal to ' $b$ ' (i.e.  $a \geq b$ ) if either  $a > b$  (or)  $a = b$ .

→ A real number ' $a$ ' is said to be less than (or) equal to ' $b$ ' (i.e.  $a \leq b$ ) if either  $a < b$  (or)  $a = b$ .

### \* → Some Properties of order relation:

→  $a \in \mathbb{I}\mathbb{R}^+ \Leftrightarrow a > 0$  and  $a \in \mathbb{I}\mathbb{R}^- \Leftrightarrow a < 0$

→  $\forall a, b \in \mathbb{I}\mathbb{R}^+ \Rightarrow a+b \in \mathbb{I}\mathbb{R}^+$  and  
 $ab \in \mathbb{I}\mathbb{R}^+$  i.e.  $a > 0, b > 0$   
 $\therefore$  here  $\Rightarrow a+b > 0$  &  $ab > 0$ .

→  $\forall a, b \in \mathbb{I}\mathbb{R}^- \Rightarrow a+b \in \mathbb{I}\mathbb{R}^-$  and  
 $ab \in \mathbb{I}\mathbb{R}^+$ .  
i.e.  $a < 0, b < 0 \Rightarrow a+b < 0$  &  $ab > 0$ .

→  $a < b$  and  $b < c \Rightarrow a < c$   
(Law of transitivity)

→  $a < b \Leftrightarrow a+c < b+c$  &  $a < b$   
and  $c < 0 \Rightarrow ac > bc$ .

$$\rightarrow a < 0 \Leftrightarrow -a > 0$$

$$a > 0 \Leftrightarrow -a < 0$$

$$\rightarrow a > b \Leftrightarrow (a-b) > 0$$

$$a < b \Leftrightarrow (a-b) < 0$$

$$\rightarrow a > b \Leftrightarrow -a < -b$$

$$\rightarrow a > 0 \Leftrightarrow \frac{1}{a} > 0$$

$$\rightarrow a > b > 0 \Rightarrow \frac{1}{b} > \frac{1}{a} > 0$$

$$\rightarrow a \neq 0 \Rightarrow a^2 > 0$$

$$\rightarrow a \geq b > 0 \Rightarrow a^2 > b^2$$

$$a < b < 0 \Rightarrow a^2 > b^2$$

→ the relations  $\geq$  and  $\leq$  are known as the weak inequalities and the relations  $>$  and  $<$  are known as the strict inequalities.

## \* Intervals :-

Intervals are two types:

- ① finite intervals
- ② infinite intervals.

### 1. Finite Intervals:

Let  $a, b \in \mathbb{R}$  with  $a < b$  then

(i) the set  $\{x | x \in \mathbb{R}, a \leq x \leq b\}$

is called a closed interval  
and is denoted by  $[a, b]$ ,

$a$  and  $b$  are called the end  
points of the interval.

$a$  is called the left  
end point while  $b$  is called the  
right end point.

Here both the end points  $a$  &  $b$   
belong to the interval.

(ii), the set  $\{x | x \in \mathbb{R}; a < x < b\}$   
is called an open interval  
and is denoted by  $(a, b)$   
or  $]a, b[$ .

Here both the end points  
do not belong to the interval.

(iii), the set  $\{x | x \in \mathbb{R}; a \leq x < b\}$   
is called left-half closed  
interval (or right-half open  
interval). And is denoted

by  $[a, b)$  or  $[a, b[$ .

Here the left end point  $a$  belongs  
to the interval and right end point  
 $b$  does not belong to the interval.

(iv), the set  $\{x | x \in \mathbb{R}, a < x \leq b\}$  is  
called right-half closed interval

(or left-half open interval) and  
is denoted by  $(a, b]$  or  $]a, b]$ .

Note: — If  $a = b$

$(a, a) = \emptyset$  and  $[a, a] = \{a\}$ .

### 2. Infinite intervals:

Let  $a \in \mathbb{R}$  then

(i) the set  $\{x | x \in \mathbb{R}, x \geq a\}$   
i.e.  $x \geq a$

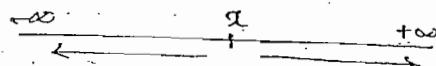
is called a closed right ray and  
is denoted by  $[a, \infty)$ .

(ii), the set  $\{x | x \in \mathbb{R}, x > a\}$  is  
called open right ray and is  
denoted by  $(a, \infty)$ .

(iii), the set  $\{x | x \in \mathbb{R}, x \leq a\}$  is called  
closed left-ray and is denoted  
 $(-\infty, a]$ .

(iv), the set  $\{x | x \in \mathbb{R}, x < a\}$  is  
called open left-ray and is  
denoted by  $(-\infty, a)$ .

(v) The set  $\{x | x \in \mathbb{R}\}$  is also called an interval and has no end points. It is denoted by  $(-\infty, \infty)$ .



→ Length of an interval :-

For each interval whose end points are any real numbers  $a & b$  such that  $a < b$ , the length of the interval is  $b-a$ .

Obviously the length of each of the intervals  $[a, b]$ ,  $]a, b[$ ,  $]a, b]$  and  $[a, b[$  is  $b-a$ . These intervals are called finite intervals because the length of each of them is finite.

The intervals  $[a, \infty[$ ,  $]a, \infty[$ ,  $]-\infty, a]$ ,  $]-\infty, a[$  and  $]-\infty, \infty[$

are called infinite intervals

because the length of each of them is infinite.

Note:- (i) Every interval is an infinite set but every infinite set need not be an interval.

- Ex:-
- ①  $\mathbb{N}$  is not an interval
  - ②  $\mathbb{Z}$  is not an interval
  - ③  $\mathbb{Q}$  is not an interval
  - ④  $\mathbb{R} - \mathbb{Q}$  is not an interval
  - ⑤  $\emptyset, \mathbb{R}$  sets are intervals.

2. A finite interval is also an infinite set because the word finite only signifies that the length of the interval is finite.
3. A ray is an infinite interval.

### One system (IR):

To extend the real number system by adjoining two "ideal points" denoted by  $+\infty$  and  $-\infty$ . The enlarged set is called the set of extended real numbers.

Note:  $\mathbb{R}$  is denoted by  $(-\infty, \infty)$  and  $\mathbb{R}^*$  by  $[-\infty, \infty]$ .

If  $x \in \mathbb{R}$  then  $-\infty < x < \infty$ ,

$$x + \infty = \infty + x = -x + \infty = \infty - x = \infty;$$

$$x - \infty = -\infty + x = -\infty - x = -x - \infty = -\infty;$$

$$\frac{x}{\infty} = 0;$$

$$\frac{\infty}{x} = \infty \times x = x \times \infty = \begin{cases} \infty & \text{if } x > 0 \\ -\infty & \text{if } x < 0 \end{cases}$$

$$\text{Further } \infty \times \infty = (-\infty) \times (-\infty) = \infty + \infty = \infty.$$

$$\infty \times (-\infty) = (-\infty) \times \infty = -\infty - \infty = -\infty.$$

The following combinations are meaningless.

$$\infty - \infty, -\infty + \infty, 0 \times \infty, \infty^{\infty}, \frac{\infty}{\infty}$$

### Bounds of Set:

#### Lower bound of a subset of $\mathbb{R}$ :

Let  $S$  be a non-empty subset of  $\mathbb{R}$ . If there exists a number  $a \in \mathbb{R}$  such that

$a \leq x$  for all  $x \in S$  then  $a$  is called a lower bound of  $S$ .

e.g. (i)  $N = \{1, 2, 3, \dots\} \subseteq \mathbb{R}$ .

$$1 \leq x \forall x \in N.$$

$\therefore 1$  is called the lower bound of  $N$ .

(ii) The set  $S = \{0, 1, 2, 3, \dots\} \subseteq \mathbb{R}$

$$0 \leq x \forall x \in S$$

$\therefore 0$  is the lower bound.

### Bounded below set:

A non-empty subset  $S$  of  $\mathbb{R}$  (i.e.,  $S \subseteq \mathbb{R}$ ) is said to be bounded below if it has lower bound.

e.g. (i)  $S = \{1, 2, 3, 4, \dots\} \subseteq \mathbb{R}$

is bounded below.

Since  $1$  is lower bound.

(ii)  $R^+ = \{x/x > 0\} = (0, \infty)$

is bounded below.

Since  $0$  is lower bound of

(iii)  $S = \{x/x \geq 0\} = [0, \infty)$  is bounded below.

Since  $0$  is lower bound of

note:  $0$  is lower bound of  $S$ .

every real number smaller than  $0$  is also a lower bound of  $S$ .

i.e. if a set  $S$  is bounded below then the set of all such

bounds of  $S$  is infinite.

### Greatest lower bound (glb) or infimum:

Let  $S$  be a non-empty subset of  $\mathbb{R}$ . If a set  $'S'$  is bounded below and if the set of all lower bounds of  $S$  has a greatest member, say  $t$  then  $t$  is called greatest lower bound or infimum of  $S$ .

(Or)

If  $t$  is a lower bound of  $S$  and any real number greater than  $t$  is not lower bound of  $S$  then  $t$  is called the greatest lower bound or infimum of  $S$ .

(Or)

If  $S$  is bounded below, then a number  $t$  is said to be greatest lower bound or infimum of  $S$  if it satisfies the conditions

1.  $t$  is lower bound of  $S$  and
2. if  $w$  is any lower bound of  $S$  then  $w \leq t$ .

$$0 - \{1, 2, 3\} \subset \mathbb{R}$$

Since  $-1 < x \forall x \in S$ .

$-1$  is lower bound of  $S$  but  $-1$  is not greatest lower bound of  $S$ .

Since  $0 < x \forall x \in S$ .

$0$  is a lower bound of  $S$  but  $0$  is not greatest lower bound of  $S$ .

Since  $0.9 < x \forall x \in S$

$0.9$  is a lower bound of  $S$  but  $0.9$  is not greatest lower bound of  $S$ .

$1 \leq x \forall x \in S$   
 $\therefore 1$  is a lower bound of  $S$  and is greatest lower bound of  $S$ .

because, the greatest of all lower bounds of  $S$  is  $1$ .

Note :- If  $t$  is infimum of  $S$  then for each  $\epsilon > 0$  (however small), the number  $t + \epsilon$  is not a lower bound of  $S$ , there exists atleast one member  $x \in S$  such that  $t \leq x < t + \epsilon$ .

Upper bound :- Let  $S$  be a non-empty subset of  $\mathbb{R}$ . If there exists a number  $M \in \mathbb{R}$  such that

$x \leq v \forall x \in S$  then  $v$  is called an upper bound of  $S$ .

Ex:-  $S = \{-3, -2, -1\} \subseteq \mathbb{R}$   
 $x \leq -1 \forall x \in S$

$\therefore -1$  is called the upper bound of  $S$ .

### Bounded above set

A non-empty subset  $S$  of  $\mathbb{R}$  (i.e.  $S \subseteq \mathbb{R}$ ) is said to be bounded above if it has an upper bound.

Ex:- (1)  $\mathbb{R} = \{x \in \mathbb{R} : x < 0\} = (-\infty, 0)$   
is bounded above

Since  $0$  is an upper bound and  $0 \notin \mathbb{R}$

(2).  $S = \{x \in \mathbb{R} : x \leq 0\} = (-\infty, 0]$  is bounded above.

Since  $0$  is an upper bound and  $0 \in S$ .

Note:- If  $v$  is an upper bound of a set  $S$  then every real number greater than  $v$  is also an upper bound of  $S$ . i.e. if a set  $S$  is bounded above then set of all such numbers that are upper bounds of  $S$  is infinite.

Least upper bound

### Supremum

Let  $S$  be a non-empty subset of  $\mathbb{R}$ . If a set  $S$  is bounded above and if the set of all upper bounds of  $S$  has a least member say  $t_1$  then  $t_1$  is called least upper bound (or) supremum of  $S$ .

(or)

If  $t_1$  is an upper bound of  $S$  & any real number less than  $t_1$  is not an upper bound of  $S$  then  $t_1$  is called least upper bound (or) supre of  $S$ .

(or)

If  $S$  is bounded above then a number  $t_1$  is said to be least upper bound supremum of  $S$  if it satisfies the following conditions.

- (1)  $t_1$  is an upper bound of  $S$  and
- (2) If  $w_1$  is any upper bound of  $S$ , then  $t_1 \leq w_1$ .

Ex:-  $S = \{ \dots, 48, 49, 50 \} \subseteq \mathbb{R}$

Since  $x < 51 \forall x \in S$

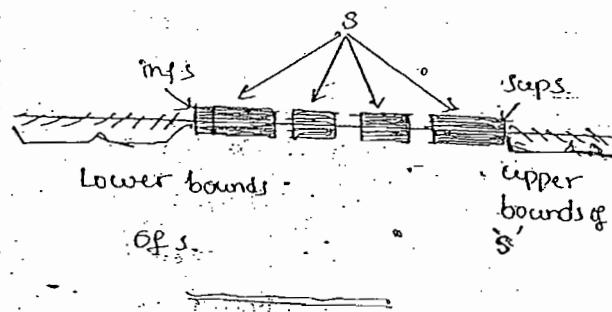
$\therefore 51$  is an upper bound of  $S$  but is not supremum of  $S$ .

Since  $x < 50.5 \forall x \in S$

$\therefore 50.5$  is an upper bound of  $S$  but is not supremum of  $S$ .

$x \leq 50 \forall x \in S$

Note:— If  $t_i$  is supremum of  $s$  then for each  $\epsilon > 0$  (however small), the number  $t_i - \epsilon$  is not an upper bound of  $s$ , there exists at least one member  $x \in s$  such that  $t_i - \epsilon < x \leq t_i$ .



→ Find the infimum & supremum of the following sets and also find whether they belong to set or not.

(1)  $S = \{3, 4, 7\} \subseteq \mathbb{R}$   
 $\inf S = 3 \in S$ ;  $\sup S = 7 \in S$

(2)  $S = \{\frac{n}{n+1} | n \in \mathbb{N}\} = \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\} \subseteq \mathbb{R}$

since  $n \in \mathbb{N}; n > 0$

$$\Rightarrow 0 < \frac{1}{n} \leq 1 \quad \forall n \in \mathbb{N}$$

$$\Rightarrow 0 < \frac{n}{n+1} \leq 1 \quad \forall n \in \mathbb{N}$$

$$\therefore \inf S = 0 \notin S$$

$$\sup S = 1 \in S$$

(3)  $S = \{-k_n | n \in \mathbb{N}\} = \{-1, -2, -3, -4, \dots\} \subseteq \mathbb{R}$   
 since  $n \in \mathbb{N}; n > 0$

$$\Rightarrow 0 < \frac{1}{n} \leq 1$$

$$\Rightarrow -1 \leq -k_n < 0$$

$$\therefore \inf = -1 \in S \text{ & } \sup = 0 \notin S$$

(4)  $S = \{\frac{1}{3^n} | n \in \mathbb{N}\} = \{\frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3}, \dots\}$

$$\sup = \frac{1}{3} \in S, \inf = 0 \notin S$$

(5)  $S = \{\frac{(-1)^n}{n} | n \in \mathbb{N}\}$

$$= \left\{ -1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \dots \right\}$$

$$= \left\{ -1, -\frac{1}{3}, -\frac{1}{5}, \dots, \frac{1}{2}, \frac{1}{4}, \dots \right\}$$

$$\inf = -1 \in S; \sup = \frac{1}{2} \in S$$

(6)  $S = \{a + \frac{1}{n} | n \in \mathbb{N}\} = \{a+1, a+\frac{1}{2}, \dots\}$

since  $n \in \mathbb{N}, n > 0$

$$\Rightarrow 0 < \frac{1}{n} \leq 1$$

$$\Rightarrow a < a + \frac{1}{n} \leq a+1$$

$$\therefore \inf = a \notin S; \sup = a+1 \in S$$

(7)  $S = \{1\}$

$$\inf = \sup = 1 \in S$$

(8)  $S = \{\frac{3n+2}{2n+1} | n \in \mathbb{N}\} \subseteq \mathbb{R}$

$$= \left\{ \frac{5}{3}, \frac{8}{5}, \dots \right\}$$

$$\sup = \frac{5}{3} \in S; \inf = 1 \in S$$

$$= \frac{3}{2} \notin S$$

(9)  $S = \{y_{5^n} | n \in \mathbb{Z}; n \neq 0\}$

$$= \left\{ \pm y_5, \pm y_{10}, \pm y_{15}, \dots \right\}$$

$$\inf = -y_5 \in S; \sup = y_5 \in S$$

$$⑩. S = \{2^n | n \in \mathbb{N}\} = \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \dots\}$$

$$\inf_{n \rightarrow \infty} S = 2 \in S; \sup_{n \rightarrow \infty} S = \lim_{n \rightarrow \infty} 2^n = \infty.$$

∴ Supremum does not exist.

$$⑪. S = \{1 - \frac{1}{n} | n \in \mathbb{N}\}$$

$$⑫. S = \{x | -5 < x < 3\}$$

$$⑬. S = \{x | x = (-1)^n; n \in \mathbb{N}\}$$

$$⑭. S = \{x | x = (-1)^n \cdot n; n \in \mathbb{N}\}$$

$$= \{-1, 2, -3, 4, -5, 6, -7, \dots\}$$

$$= \{-1, -3, -5, \dots; 2, 4, 6, \dots\}$$

∴ inf S = does not exist, &

sup S = does not exist.

### \* Bounded Subset S of real Numbers:

A subset 'S' of  $\mathbb{R}$  is said to be bdd if it is bdd below as well as bdd above.

i.e. A set 'S' is bdd iff there exist two real numbers  $u, v$  such that  $u \leq x \leq v \forall x \in S$ .

i.e.  $x \in [u, v] \forall x \in S$

i.e.  $S \subseteq [u, v]$

i.e.  $S$  is a subset of  $[u, v]$ .

Ex:- (1). Every finite set is bdd and it has inf & sup.

(2). we can say that in inf & sup of  $\emptyset$  does not exist

Because:

The null set  $\emptyset$  is bdd above if any real number then it is an upper bound for  $\emptyset$  obviously condition  $x \leq u$  for all  $x \in \emptyset$  is vacuously satisfied because  $\emptyset$  has no elements.

thus every real number is an upper bound for  $\emptyset$ . Since the set of all real numbers has no smallest member.

∴ Sup  $\emptyset$  does not exist

Similarly inf  $\emptyset$  does not exist.

(3).  $\mathbb{N} = \{1, 2, 3, \dots\}$  is bdd below but not bdd above.

(4).  $\mathbb{R}^+ = \{x \in \mathbb{R} | x > 0\}$  is bdd below but not bdd above.

(5).  $\mathbb{R}^- = \{x \in \mathbb{R} | x < 0\}$  is bdd above but not bdd below.

### \* Greatest & Least members of a subset of $\mathbb{R}$ :

If the supremum of a subset 'S' of  $\mathbb{R}$  is a member of 'S'

i.e. 'S' attains its supremum) then this supremum is called greatest

→ If the inf of a subset 's' of IR is a member of 's' (i.e. 's' attains its infimum) then this infimum is least member of 's'.

→ Ex :- (1)  $s = [2, 3]$

$$\text{i.e. } s = \{x / 2 \leq x \leq 3\}$$

$$\sup s = 3 \notin s \text{ & } \inf s = 2 \notin s.$$

∴ 3 is not greatest member & 2 is not least member.

(2)  $s = [2, 3]$

$$\text{i.e. } s = \{x / 2 \leq x \leq 3\}$$

(3).  $s = [1, 2)$  i.e.  $s = \{x / 1 \leq x < 2\}$

(4).  $s = (1, 2]$  i.e.  $s = \{x / 1 < x \leq 2\}$

(5). The unbounded intervals are

$[a, \infty)$ ,  $(a, \infty]$ ,

$[-\infty, a]$ ,  $(-\infty, a]$ ,  $[-\infty, \infty]$ .

Now suppose

$$s = [a, \infty) = \{x / x \geq a\}$$

$\inf s = a \in s$  &  $\sup s$  does not exist.

∴ least member of  $s = a$ .

\* Note (1). Every finite set has two bounds

(2) Every infinite set may or may not have bounds.

Every member of a set may or may not belong to the set.

(1). Supremum & infimum of a bounded set need not belong to the set.

(2). Every greatest member of a set 's' is the supremum of 's' but every sup. of 's' need not be the greatest member of 's'.

(3). Every least member of a set 's' is the infimum of 's' but every inf. of 's' need not be the least member of 's'.

### \* Completeness Property of IR

(Or Completeness Axiom):

Every non-empty set of real numbers has a least upper bound.

i.e. If 's' is any non-empty subset of IR which is bounded above, then the set of all upper bounds of 's' must have smallest member i.e.  $\$$  must possess the least upper bound which is a member of IR.

This property of real numbers (3) : Completeness axiom.

is known as completeness.

(This property is also called the Supremum property of IR).

(Or)

→ Every non-empty subset of real numbers which is bounded below has the infimum (or glb) in IR. This property of real numbers is known as completeness. This property is also called Infimum property of IR.

### \* Complete Ordered Field

An ordered field F is said to be a complete ordered field if every non-empty subset S of F (i.e.  $S \subset F$ ) which is bounded above has the supremum (or least upper bound) in F.

Ex:- The set IR of real numbers is complete ordered field.

Because IR satisfies

① field axioms

② order axioms and

Ex:- (2). The set Q of rational numbers is an ordered field but not completeness.

→ Now we shall show that the ordered field of rational numbers is not a complete ordered field.

for this we are enc to show that there exists a non-empty subset of Q which bounded above but which does not have a supremum in Q.

i.e. no rational number exists which can be the supremum.

Let us consider the set of all those irrational numbers whose squares are less than 2. i.e. let  $S = \{x : x \in \mathbb{Q} \text{ and } x^2 < 2\}$  and  $a \in S$ . Since  $1 \in S$   $\frac{a+1}{2} \in S$   $\therefore S \neq \{\}$

i.e. S is non-empty.

Clearly 2 is an upper bound of S.

$\therefore S$  is bounded above.

$\therefore S$  is a non-empty subset of F and is bounded above.

If possible suppose that in rational number  $k$  be its least upper bound.

Clearly  $k$  is +ve.

By law of trichotomy, which holds good in  $\mathbb{Q}$  one and only one of (i)  $k^2 < 2$  (ii)  $k^2 = 2$  (iii)  $k^2 > 2$  holds.

(i)  $k^2 < 2$

Let us consider the +ve rational number -  $y = \frac{4+3k}{3+2k}$

$$\text{then } k-y = k - \left( \frac{4+3k}{3+2k} \right)$$

$S = \{y \mid \text{new}\}$

$= \{x \mid \text{real, } 0 < x \leq 1\} \subset Q$

$\exists \text{ real } z < y \text{ s.t. } k < z$  (i.e.  $k^2 < z^2$ )

$\therefore k-y < 0$  (i.e.  $k^2 - z^2 < 0$ )

$$\text{Also } 2-y^2 = 2 - \left( \frac{4+3k}{3+2k} \right)^2$$

$$= \frac{2-k^2}{(3+2k)^2}$$

$> 0$  (i.e.  $k^2 < 2$  i.e.  $2-k^2 > 0$ )

$\therefore 2-y^2 > 0$

$$\Rightarrow y^2 < 2$$

$\Rightarrow \boxed{\text{yes}}$

$\therefore$  The member  $y$  of  $S$  is greater than  $k$  so that  $k$  cannot be an

upper bound.

(ii)  $k^2 = 2$ , we know that there exists no rational number whose square is equal to 2.

$\therefore$  This case is not possible.

(iii)  $k^2 > 2$

let us consider the +ve rational number:

$$y = \frac{4+3k}{3+2k} (> 0)$$

$$\text{then } k-y = k - \left( \frac{4+3k}{3+2k} \right)$$

$$= \frac{2(k^2-2)}{3+2k}$$

$$> 0 \quad (\because k^2 > 2)$$

$$\Rightarrow k^2-2 > 0$$

$$\therefore k-y > 0$$

$$\Rightarrow \boxed{k > y}$$

$$\text{Also } 2-y^2 = 2 - \left( \frac{4+3k}{3+2k} \right)^2$$

$$= \frac{2-k^2}{(3+2k)^2} < 0$$

$$\therefore 2-y^2 < 0 \quad (\because 2-k^2 < 0)$$

$$\Rightarrow 2 < y^2$$

$$\Rightarrow \boxed{y^2 > 2}$$

$$\therefore y < k \text{ & } y^2 > 2 \Rightarrow y^2 < k^2 \text{ & } y^2 > 2$$

$$\Rightarrow 2 < y^2 < k^2$$

If  $x$  is any member of  $S$  then

$$0 < x^2 < 2 < y^2 < k^2$$

$$\Rightarrow 0 < x^2 < y < k$$

which shows  $x$  &  $y$  are own upper bounds of  $S$ .

But  $y < k$

$\therefore k$  cannot be the supremum.

$\therefore$  since  $k$  is any rational number, we conclude that no rational number can be the supremum of  $S$ .

### \* The Archimedean Property:

If  $a \& b$  be any two real numbers and if  $a > 0$ , then there exists a +ve integer  $n$  such that  $na > b$ .

Proof :- Let  $a \& b$  be any two real numbers and  $a > 0$ .

Now if possible suppose that for all +ve integers  $n$ ,

i.e.  $na \leq b$ .

Let  $S = \{na \mid n \in \mathbb{N}\}$ : then  $S$  is bounded above by  $b$  (i.e.  $b$  is an upper bound of  $S$ ).

$\therefore$  By Completeness Property of the ordered field of real numbers, the set  $S$  must have a supremum  $M$  (say)

$\therefore a \leq M \& n \in \mathbb{N}$

$$\Rightarrow (M-a) \cdot n = m$$

$$\Rightarrow na \leq M-a \& n \in \mathbb{N}^+$$

$\therefore M-a$  is an upper bound of  $S$

the number  $M-a$  is less than

Supremum  $M$  (least upper bound is an upper bound of  $S$ ).

$\therefore$  which is a contradiction.

$\therefore$  Our supposition is wrong.  
Hence theorem.

### \* Absolute Value (modulus of a real number):

If  $x \in \mathbb{R}$  then the modulus

or absolute value or numerical

value of  $x$  is denoted by  $|x|$  and

defined as  $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

#### Properties:

Prove that (i)  $|x| = \max\{x, -x\}$

(ii)  $|x|^2 = x^2$  (iii)  $x \leq |x|$  and  $-x \leq |x|$

(iv)  $|x| = |-x|$ .

Proof :- (i) Since  $x \in \mathbb{R}$ , either  $x \geq 0$

or  $x < 0$ . If  $x \geq 0$  then  $|x| = x$  and  $x \geq -x$

and if  $x < 0$ , then  $|x| = -x$  and  $-x > x$ .

$\therefore |x|$  is greater of two numbers  $x$  &  $-x$ .

$$|x| = \max\{x, -x\}$$

(ii) Since  $|x| = x$  if  $x \geq 0$   
 $= -x$  if  $x < 0$

$$\therefore |x|^2 = x^2 \text{ (or)} (-x)^2 \\ = x^2$$

$$\boxed{|x|^2 = x^2}$$

(iii) Since  $|x| = \max\{x, -x\}$

$$\therefore |x| \geq x \text{ or } -x \\ \therefore x \leq |x| \text{ and } -x \leq |x|.$$

(iv) Since  $|x| = \max\{x, -x\}$

$$\text{and } |-x| = \max\{-x, x\} = \max\{x, -x\}$$

$$\therefore |x| = |-x|$$

Note:  $|x|^2 = x^2$

$$\therefore |x| = \pm \sqrt{x^2}$$

since  $|x| \geq 0$ .

∴ rejecting the -ve sign,

$$\text{we have } \boxed{|x| = \sqrt{x^2}}$$

$$(v) |x| = \sqrt{x^2}$$

$$\text{and } |-x| = \sqrt{(-x)^2}$$

$$= \sqrt{x^2}$$

$$\text{and } \sqrt{(-x)^2} = \sqrt{x^2}$$

$$\therefore |x| = |-x|$$

→ If  $x$  &  $y$  are any two real numbers

$$\text{then (a)} |x+y| \leq (|x|+|y|)$$

$$\text{For (b)} |x-y| \geq (|x|-|y|)$$

$$(c) |xy| = |x||y|$$

$$(d) \left| \frac{x}{y} \right| = \sqrt{\left( \frac{x}{y} \right)^2}$$

$$(a) |x+y| = \sqrt{(x+y)^2}$$

$$= \sqrt{x^2+y^2+2xy}$$

$$\leq \sqrt{x^2+y^2+2|x||y|}$$

$$(\because x \geq 0 \text{ & } y < 0)$$

$$= \sqrt{|x|^2+|y|^2+2|x||y|}$$

$$= \sqrt{[|x|+|y|]^2}$$

$$= |x| + |y| \quad (\because |x| = \sqrt{x^2})$$

$$= |x| + |y| \quad (\because |x|+|y| \geq 0)$$

i.e., lataa if  $x > 0$

$$\boxed{|x+y| \leq (|x|+|y|)}$$

$$(b) |x-y| = \sqrt{(x-y)^2}$$

$$= \sqrt{x^2+y^2-2xy}$$

$$\geq \sqrt{x^2+y^2-2|x||y|}$$

$$(\because x \leq |x| \text{ & } y \leq |y|)$$

$$\Rightarrow xy \leq |x||y|$$

$$\Rightarrow -xy \geq -|x||y|$$

$$= \sqrt{|x|^2+|y|^2-2|x||y|}$$

$$= \sqrt{(|x|-|y|)^2}$$

$$= |x| - |y| \quad (\because |x| = \sqrt{x^2})$$

$$\boxed{|x-y| \geq (|x|-|y|)}$$

$$(c) |xy| = \sqrt{(xy)^2}$$

$$= \sqrt{x^2 \cdot y^2}$$

$$= |x||y|.$$

$$(d) \left| \frac{x}{y} \right| = \sqrt{\left( \frac{x}{y} \right)^2}$$

$$= \frac{\sqrt{x^2}}{\sqrt{y^2}} = \frac{|x|}{|y|} \quad \text{provided } y \neq 0.$$

→ If  $\delta > 0$  then prove that

$$(a) |x| < \delta \Leftrightarrow -\delta < x < \delta$$

$$(b) |x-a| < \delta \Leftrightarrow a-\delta < x < a+\delta$$

$$\text{SOL: } (a) |x| < \delta \Leftrightarrow \max\{|x|, -x\} < \delta$$

$$\Leftrightarrow x < \delta \text{ and } -x < \delta$$

$$\Leftrightarrow x < \delta \text{ and } x > -\delta$$

$$\Leftrightarrow -\delta < x < \delta$$

$$\Leftrightarrow -\delta < x < \delta$$

$$(b) |x-a| < \delta \Leftrightarrow \max\{|x-a|, -(x-a)\} < \delta$$

$$\Leftrightarrow x-a < \delta \text{ and } -(x-a) < \delta$$

$$\Leftrightarrow x < a+\delta \text{ and } x > a-\delta$$

$$\Leftrightarrow a-\delta < x < a+\delta$$

$$\Leftrightarrow a-\delta < x < a+\delta$$

$$\Leftrightarrow a-\delta < x < a+\delta$$

→ Prove that (i)  $|x+y| \leq |x| + |y|$

$$(ii) a < x < b \Leftrightarrow |x - \left(\frac{a+b}{2}\right)| < \frac{b-a}{2}$$

Proof: (i)  $|x+y| = |x+(-y)|$

$$\leq |x| + |-y|$$

$$= |x| + |y|$$

$$(\because |-y| = |y|)$$

$$\therefore |x+y| \leq |x| + |y|.$$

(ii)  $a < x < b$ :

Adding throughout  $-\left(\frac{a+b}{2}\right)$ ,

we get

$$\Leftrightarrow a - \left(\frac{a+b}{2}\right) < x - \left(\frac{a+b}{2}\right) < b - \left(\frac{a+b}{2}\right)$$

$$\Leftrightarrow \frac{a-\delta}{2} < x - \left(\frac{a+b}{2}\right) < \frac{b-\delta}{2}$$

$$\Leftrightarrow -\left(\frac{b-a}{2}\right) < x - \left(\frac{a+b}{2}\right) < \left(\frac{b-a}{2}\right)$$

$$\Leftrightarrow \left|x - \left(\frac{a+b}{2}\right)\right| < \frac{b-a}{2}$$

$$(\because |x| < \delta \Leftrightarrow -\delta < x < \delta)$$

### \* Neighbourhood of a point:

If  $a$  is any real number and  $\delta > 0$  (however small), then the

open interval  $(a-\delta, a+\delta)$  is called a  $\delta$ -neighbourhood of  $a$

and is denoted by  $N_\delta(a)$  or

$N(\delta, a)$ , i.e.  $N_\delta(a) = (a-\delta, a+\delta)$

shortly written as  $\text{neighborhood of } a$

$\delta$  is called the radius of the neighborhood.

$x \in (a-\delta, a+\delta)$

→ If from the neighbourhood of a point, the point itself is excluded we get the deleted neighbourhood of that point, i.e.  $N_\delta(a) - \{a\}$

i.e.  $N_\delta(a) - \{a\}$  is a deleted neighbourhood of a point  $a$ .

and is denoted by  $N_{\delta d}(a)$

i.e.  $N_{\delta d}(a) = N_\delta(a) - \{a\}$

Ex: If  $a = 5$ ,  $\delta = 0.2 > 0$ , then

$(4.8, 5.2)$  is a neighbourhood of  $a$ .

Now  $x \in (4.8, 5.2) - \{5\} \Rightarrow x \in (4.8, 5)$

$x \neq 5$  is a deleted

Note:-  $x \in N_\delta(a)$

$$\Leftrightarrow x \in (a-\delta, a+\delta)$$

$$\Leftrightarrow a-\delta < x < a+\delta$$

$$\Leftrightarrow -\delta < x-a < \delta$$

$$\Leftrightarrow |x-a| < \delta$$

$$(\text{likewise} \Leftrightarrow -\delta < a-x < \delta)$$

and  $x \in N_\delta(d(a))$

$$\Leftrightarrow x \in (a-\delta, a+\delta) - \{a\}$$

$$\Leftrightarrow x \in (a-\delta, a+\delta); x \neq a$$

$$\Leftrightarrow a-\delta < x < a+\delta; x \neq a$$

$$\Leftrightarrow |x-a| < \delta; x \neq a$$

$$\Leftrightarrow 0 < |x-a| < \delta$$

### \* Neighbourhood of a set

→ A subset  $S$  of  $\mathbb{R}$  (i.e.  $S \subseteq \mathbb{R}$ ) is said to be neighbourhood of a point  $a \in S$  if there exists a  $\delta > 0$  (however small) such that

$$(a-\delta, a+\delta) \subseteq S$$

Note:- If  $S$  is a neighbourhood of a point  $a$ , then  $S - \{a\}$  is called deleted neighbourhood of  $a$ .

Ex:- (1) If  $a \in \mathbb{R} \subseteq \mathbb{R}$  then  $\mathbb{R}$  is a neighbourhood of  $a$  because  $a \in (a-\delta, a+\delta) \subseteq \mathbb{R}$ .

(2) If  $a \in Q \subseteq \mathbb{R}$  then  $Q$  is not neighbourhood of  $a$  because  $a \in (a-\delta, a+\delta) \notin Q$ .

Ex:-  $S = \{3\} \subseteq \mathbb{R}$

is not neighbourhood of 3.

neighbourhood of  $a'$

because  $a' \in (a-\delta, a+\delta) \notin S$

(3). If  $a \in N \subseteq \mathbb{R}$  then  $N$  is not

neighbourhood of  $a$  because

$a \in (a-\delta, a+\delta) \notin N$ .

### Problems

→ Any open interval is a

neighbourhood of each of its

points.

Sol'n :- Let  $S = (a,b)$

Let  $P$  be any point of  $(a,b)$

i.e.  $P \in (a,b)$

$$\Rightarrow a < P < b$$

$$\Rightarrow \epsilon = \min\{P-a, b-P\} > 0$$

$$\Rightarrow \epsilon \leq P-a; \epsilon \leq b-P$$

$$\Rightarrow a \leq P-\epsilon; b \geq P+\epsilon$$

$$\Rightarrow a \leq P-\epsilon < P < P+\epsilon \leq b$$

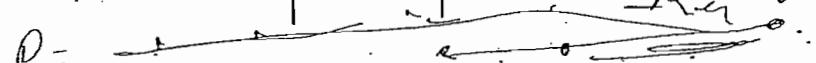
$$\Rightarrow P \in (P-\epsilon, P+\epsilon) \subset (a,b)$$

$(a,b)$  is a neighbourhood of  $P$ .

→ A closed interval  $[a,b]$  is a neighbourhood of each of its points except the two end points  $a$  &  $b$ .

Sol'n :- Let  $S = [a,b]$

Let  $P \in [a,b]$



$$\Rightarrow a \leq p \leq b$$

$$\Rightarrow (i) a < p < b$$

$$ii, p = a \&$$

$$iii, p = b$$

$$\text{Let } \epsilon = \min \{p-a, b-p\} > 0.$$

$$i) p \in (p-\epsilon, p+\epsilon) \subset (a, b) \subset [a, b]$$

$$\therefore p \in (p-\epsilon, p+\epsilon) \subset [a, b]$$

$\therefore [a, b]$  is a neighbourhood of  $p$ .

i.e.  $[a, b]$  is a neighbourhood of each  $p \in [a, b]$ .

$$ii, p = a$$

$$\Rightarrow (p-\epsilon, p+\epsilon) = (a-\epsilon, a+\epsilon)$$

$$\not\subset [a, b]$$

$[a, b]$  is not neighbourhood of  $a$ .

$$iii, b = p = b$$

$$\Rightarrow (p-\epsilon, p+\epsilon) = (b-\epsilon, b+\epsilon)$$

$$\not\subset [a, b]$$

$[a, b]$  is not neighbourhood of  $b$ .

$[a, b]$  is a neighbourhood of each of its points except  $a$ .

$\rightarrow [a, b]$  is a neighbourhood

of each of its points except

$\rightarrow$  A non-empty finite set can't be a neighbourhood any of its points.

Sol'n - Let  $s$  be any non-empty finite set.

Let  $p$  be any point of  $s$ .

Let  $\epsilon > 0$  (however small)

then  $(p-\epsilon, p+\epsilon)$  is an infinit

set.

$$(p-\epsilon, p+\epsilon) \not\subset s$$

$\therefore s$  is not a neighbourhood of  $p$ .

$\rightarrow$  Empty set  $\phi$  is a neighbourhood of each of its points.

Sol'n - The empty set  $\phi$  is a

neighbourhood of each of its points

because there is no point at all in  $\phi$ .

and so there is no point in  $\phi$  which it is not a neighbourhood.

$\rightarrow$  Show that the set 'N' of all natural is not a neighbourhood of any of its points.

Sol'n - Let  $p \in N$  and let  $\epsilon > 0$ .

then  $(P-\epsilon, P+\epsilon)$  contains infinitely many rational and irrational numbers.

$$\therefore (P-\epsilon, P+\epsilon) \notin N$$

$\therefore N$  is not a neighbourhood of any point  $\underline{P \in N}$ .

Similarly, the set ' $\omega$ ' of all whole numbers is not a neighbourhood of any of its points.

- And the set ' $I$ ' of integers is not a neighbourhood of any of its points.

$\rightarrow$  Show that the set  $Q$  of all rational numbers is not a neighbourhood of any of its points.

Sol: Let  $P \in Q$  and let  $\epsilon > 0$ .  
 (however small)

Then  $(P-\epsilon, P+\epsilon)$  contains infinitely many irrational numbers which are not members of  $Q$ .

$$\therefore (P-\epsilon, P+\epsilon) \notin Q$$

$\therefore Q$  is not a neighbourhood of any point  $\underline{P \in Q}$ .

Similarly, the set  $\Omega$  of all irrational numbers is not a neighbourhood of any of its points.

is a nbd of each of its points.

Sol: Let  $p \in R$  and let  $\epsilon > 0$ .

then  $(p-\epsilon, p+\epsilon)$  contains infinitely many real numbers

$$p \in (p-\epsilon, p+\epsilon) \subset R$$

$\therefore$  The set  $R$  of real numbers is a nbd of its points.

$\rightarrow$  Any set ' $S$ ' cannot be a nbd of any point of the set  $R-S$ .

Sol: Let  $p \in R-S$

then  $p \notin S$ .

let  $\epsilon > 0$ .

$$\Rightarrow (p-\epsilon, p+\epsilon) \notin S$$

$\therefore S$  is not a nbd of any point  $\underline{p \in R-S}$ .

$\rightarrow$  Every superset of nbd of a point ' $p$ ' is also nbd of  $p$ .

Sol: Let  $S$  be a nbd of  $P$ .

$$\Rightarrow (P-\epsilon, P+\epsilon) \subset S$$

If  $T$  is any superset of  $S$ , then  $S \subset T$ .

$$(P-\epsilon, P+\epsilon) \subset S \subset T$$

$$\Rightarrow (P-\epsilon, P+\epsilon) \subset T$$

$\therefore T$  is a nbd of  $P$ .

→ The intersection of two nbds of a point is also a nbd of that point.

Sol: Let  $M_1$  and  $M_2$  be two nbds of  $p$ .

∴  $\exists \epsilon_1, \epsilon_2 > 0$  (however small) such that

$$p \in (p - \epsilon_1, p + \epsilon_1) \subset M_1 \text{ and } p \in (p - \epsilon_2, p + \epsilon_2) \subset M_2$$

$$\text{let } \epsilon = \min\{\epsilon_1, \epsilon_2\}$$

$$(p - \epsilon, p + \epsilon) \subset (p - \epsilon_1, p + \epsilon_1) \subset M_1$$

$$\text{and } (p - \epsilon, p + \epsilon) \subset (p - \epsilon_2, p + \epsilon_2) \subset M_2$$

$$\therefore p \in (p - \epsilon, p + \epsilon) \subset M_1 \cap M_2$$

∴  $M_1 \cap M_2$  is also a nbd of  $p$ .

→ If  $M_1$  is a nbd of  $p$  or  $M_2$  is a nbd of  $p$

then  $M_1 \cup M_2$  is also a nbd of  $p$ .

### Interior point of a set

Let  $S \subset \mathbb{R}$ ,  $p \in S$  is called an interior point of a set  $S$  if  $\exists \epsilon > 0$  (however small) such that

$$(p - \epsilon, p + \epsilon) \subset S$$

Ex: (1) Every point of an open interval  $(a, b)$  is an interior point of the interval.

(2) Every point of a closed interval  $[a, b]$  is an interior point of the interval except the end points  $a$  and  $b$ .

(3) Every point of a semi-closed interval  $[a, b)$  is an interior point of the interval.

- except the left end point 'a'.
- Every point of a semi open interval  $(a, b]$  is an interior point of the interval except the right end point 'b'.
- Every point of the empty set is an interior point.
- Every non-empty finite set has no interior point.
- $\mathbb{N}$  has no interior point.
- Similarly  $\mathbb{W}, \mathbb{Z}, \mathbb{Q}, \mathbb{R} - \mathbb{Q}$ .
- Every point of a real number set is an interior point of  $\mathbb{R}$ .

### Interior of a set:

The set of all interior points of a set ' $S$ ' is called interior of a set ' $S$ ', and is denoted by  $S^\circ$  (or)  $\text{int}(S)$ .

Ex: If  $S = (a, b)$  then  $S^\circ = S$  because every point of  $S^\circ$  is an interior point of  $S$ .

(2) If  $S = [a, b]$  then  $S^\circ = (a, b)$  because every point of  $S^\circ$  is an interior point of  $S$  except the end points 'a' & 'b'.

(3) If  $S = [a, b)$  then  $S^\circ = (a, b)$ .

(4) If  $S = (a, b]$  then  $S^\circ = (a, b)$ .

(5)  $\mathbb{R}^\circ = \mathbb{R}$  because every point of  $\mathbb{R}$  is an interior point.

(6) If  $S$  is a non-empty finite set then  $S^\circ = \emptyset$ .

(7)  $\mathbb{N}^\circ = \emptyset, \mathbb{Z}^\circ = \emptyset, \mathbb{Q}^\circ = \emptyset, (\mathbb{R} - \mathbb{Q})^\circ = \emptyset$  &  $\mathbb{W}^\circ = \emptyset$ .

because  $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R} - \mathbb{Q}, \mathbb{W}$  are not nbd of any points and therefore, no point is an interior point of  $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R} - \mathbb{Q}$  or  $\mathbb{W}$ .

(8) If  $S = \emptyset$  then  $S^\circ = \emptyset$ .

→ Find the interior of the sets  
(i)  $\{1, 2, 3, 4, 5\}$  (ii)  $[0, 1]$  (iii)  $[0, 1] \cup [3, 5]$  (iv)  $\{\frac{1}{n} / n \in \mathbb{N}\}$

Sol: (i) Let  $A = \{1, 2, 3, 4, 5\}$ .

then  $A$  is a non-empty finite set.

⇒  $A$  is not nbhd of any point.

∴ no point is an interior point of  $A$ .

$$\Rightarrow A^0 = \emptyset.$$

(ii) Let  $A = [0, 1] \cup [3, 5]$ .

then  $A$  is nbhd of each point of  $[0, 1] \cup [3, 5]$ .

$$\therefore A^0 = (0, 1) \cup (3, 5).$$

(iv) Let  $A = \{\frac{1}{n} / n \in \mathbb{N}\}$

$$= \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$$

Let  $p$  be any point of  $A$ ,

i.e.,  $p \in A$ ,

then there is no  $\delta > 0$  such that

$(p - \delta, p + \delta) \subset A$ .

⇒  $p$  is not an interior point of  $A$ .

i.e.,  $A$  has no interior point

$$\therefore A^0 = \emptyset.$$

open set: A subset  $S$  of  $\mathbb{R}$  is said to be an open set if  $S$  is a nbd of each of its points i.e., if for each  $p \in S$  there is an  $\epsilon > 0$  such that

$$(p-\epsilon, p+\epsilon) \subset S$$

(or)  
if 'S' is a subset of  $\mathbb{R}$  is said to be open if every point of  $S$  is an interior point of  $S$ . i.e.,  $S$  is open  $\Leftrightarrow S^o = S$ .

Ex: (1) Every open interval is an open set.

$$\text{Sol: Let } S = (a, b)$$

$$\text{then } S^o = (a, b)$$

$$\therefore S^o = S$$

$\Rightarrow S$  is open set



(2)  $S = [a, b]$ , then  $S^o = (a, b)$ .

$\text{but } S \neq S^o$   
 $\Rightarrow S$  is not an open set.

Similarly  $[a, b]$ ,  $(a, b]$  are not open sets.

(3)  $S = \mathbb{N}$ , then  $S^o = \emptyset$ .

$$\therefore S \neq S^o$$

$\Rightarrow S$  is not open set.

Similarly  $\mathbb{Q}$ ,  $\mathbb{Z}$ ,  $\mathbb{W}$  and  $\mathbb{R} - \overline{\mathbb{Q}}$  are not open sets.

(4)  $S = \mathbb{R}$ , then  $S^o = \mathbb{R}$ .

$$\therefore S = S^o$$

$\Rightarrow S$  is open.

(5)  $S = \mathbb{R}^+ = (0, \infty)$  is an open set.

because let  $a \in S$

$\exists$  an  $\epsilon > 0$  (however small) such

$$\text{Suppose } a \in (x-\epsilon, x+\epsilon) \Rightarrow$$

$$\therefore S = S^o.$$

(6)  $S = \mathbb{R}^c = (-\infty, \infty)$  is an open set.

(7) Every non-empty-finite set is not an open set.

because for every nbhd of a point contains infinitely many points.

(8)  $S = \{q\}$  is an open set.

because  $S^o = \emptyset$

~~(9)~~  $S = \{1, 1/n | n \in \mathbb{N}\}$  is not an open set.

because  $S \neq S^o$ .

Union of two open sets is an open set.

Sol: Let  $S_1$  and  $S_2$  be two open sets.

$$\text{Let } S = S_1 \cup S_2.$$

Let  $x \in S \Rightarrow x \in S_1 \cup S_2$

$\Rightarrow x \in S_1 \text{ or } x \in S_2$

If  $x \in S_1$ , then  $\exists \epsilon > 0$  such that  $(x-\epsilon, x+\epsilon) \subset S_1$  ( $S_1$  is open)

$x \in (x-\epsilon, x+\epsilon) \subset S_1 \subset S_1 \cup S_2$  ( $S_1 \cup S_2$  is open)

If  $x \in S_2$ , then  $\exists \epsilon > 0$  such that

$x \in (x-\epsilon, x+\epsilon) \subset S_2 \subset S_1 \cup S_2$  ( $S_1 \cup S_2$  is open)

$x \in (x-\epsilon, x+\epsilon) \subset S_1 \cup S_2$

$\therefore x$  is an interior point of  $S_1 \cup S_2 = S$ .

$\therefore S_1 \cup S_2$  is an open set.

~~∴ Union of two sets is open~~

~~∴ Union of two sets is open~~

→ The union of an arbitrary family of open sets  
is an open set.

The intersection of two open sets is an open set.

Let  $S_1$  &  $S_2$  be two open sets.

To prove  $S_1 \cap S_2$  is also an open set.

Let  $s = S_1 \cap S_2$ .

Let  $x \in s \Rightarrow x \in S_1 \cap S_2$

$\Rightarrow x \in S_1$  and  $x \in S_2$

$\Rightarrow x \in (x - \epsilon_1, x + \epsilon_1) \subset S_1$  and

$x \in (x - \epsilon_2, x + \epsilon_2) \subset S_2$

( $\because S_1$  &  $S_2$  are two  
open sets)

Choosing  $\epsilon = \min\{\epsilon_1, \epsilon_2\} > 0$  (1)

$x \in (x - \epsilon, x + \epsilon) \subset (x - \epsilon_1, x + \epsilon_1) \subset S_1$  and

$x \in (x - \epsilon, x + \epsilon) \subset (x - \epsilon_2, x + \epsilon_2) \subset S_2$

$\Rightarrow x \in (x - \epsilon, x + \epsilon) \subset S_1 \cap S_2 = s$

$\therefore s = S_1 \cap S_2$  is an open set.

The intersection of a finite collection of  
open sets is an open set.

The intersection of an infinite collection  
of open sets need not be an open set.

Ex) Let  $s_n = (-\frac{1}{n}, \frac{1}{n})$

$$(1) \quad s_1 \cap s_2 \cap s_3 \cap \dots = (-\frac{1}{3}) \cap (-\frac{1}{2}, \frac{1}{2}) \cap \dots$$

$$\text{and } (2) \quad \dots = \{0\}$$

which is not an open set.

Because  $(0 \in s, 0 \notin \{0\})$

The intersection of an infinite collection  
of open sets is not an open set.

(ii)  $\text{int } S_n = \cup_{i=1}^n S_i$

$$\begin{aligned}\text{Then } S_1 \cap S_2 \cap \dots &= (0,1) \cap (0,2) \cap \dots \\ &= (0,1)\end{aligned}$$

which is an open set.

The intersection of an infinite collection  
of open sets need not be an open set.

Note: Every open interval is an open set.  
but every open set need not be an  
open interval.

for example, :

let  $S_1 = (1,2)$ ;  $S_2 = (3,4)$  are two  
open sets.

$S_1 \cup S_2 \in (1,2) \cup (3,4)$  is an open set

but  $(1,2) \cup (3,4)$  is not an open  
interval.

\* Limit point of a subset  $S$  of  $\mathbb{R}$

A point  $p \in \mathbb{R}$  is said to be a limit point  
of a subset  $S$  of  $\mathbb{R}$ , if every nbd of  $p$  has  
at least one point of  $S$  other than  $p$  itself.

(Or)

A point  $p \in \mathbb{R}$  is said to be a limit point  
of a subset  $S$  of  $\mathbb{R}$  if every nbd of  $p$   
has an infinite number of points of  $S$ .

(Or)

A point  $p \in \mathbb{R}$  is said to be a limit point of  
subset  $S$  of  $\mathbb{R}$  if every nbd of  $p$  contains

at least one point of  $S$  other than  $p$ .

i.e.,  $p$  is a limit point of  $S \Leftrightarrow$

$$(p-\epsilon, p+\epsilon) \cap S - \{p\} \neq \emptyset.$$

Notes: (1) limit point is also called cluster point (or)

condensation point (or) accumulation point

(2) A limit point of ' $S$ ' may or may not belong to the set ' $S$ '.

(3) A set may have no limit point, a unique limit point or a finite or infinite number of limit points.

(4)  $p \in \mathbb{R}$  is not a limit point of a subset

' $S$ ' of  $\mathbb{R}$  if there exists a nbhd of ' $p$ ' which does not contain any point of ' $S$ '

(5)  $p$  is not a limit point of ' $S$ ' if for some

$$\epsilon > 0, (p-\epsilon, p+\epsilon) \cap S = \emptyset \text{ (or)}$$

$$(p-\epsilon, p+\epsilon) \cap S = \{p\}.$$

Max infinite set has no limit point

Set  $A$  be a finite set

Suppose, If possible suppose that  $p$  lie a limit point of  $A$ . Then let  $\epsilon > 0$ .

Then  $(p-\epsilon, p+\epsilon)$  contains infinite number of points of ' $A$ '.

$\therefore A$  is infinite.

It is a contradiction.

$\therefore A$  has no limit points

For a finite set it has no limit points

### Closure of a set :-

The set of all adherent points of a set 'S' is called the closure of S and is denoted by  $\text{cls } S$  or  $\bar{S}$ .

$$\text{Thus } \bar{S} = S \cup D(S).$$

### Dense set:

A subset 'S' of  $\mathbb{R}$  is said to be dense (or dense in  $\mathbb{R}$  or everywhere dense) if every point of  $\mathbb{R}$  is a point of  $S$  or a limit point of  $S$  or both.

(or)

Let  $S \subseteq \mathbb{R}$  then 'S' is said to be dense if  $\bar{S} = \mathbb{R}$ .

### Dense in itself:

A set 'S' is said to be dense-in-itself if every point of  $S$  is a limit point of  $S$ .

(or)

A subset 'S' of  $\mathbb{R}$  is said to be dense-in-itself if  $S \subseteq D(S)$ .

(or)

A subset 'S' of  $\mathbb{R}$  is said to be dense-in-itself if it possesses no isolated points.

### perfect set:

A set 'S' is said to be perfect set if  $S = D(S)$ .

(or)

A set 'S' is said to be perfect set if it is dense-in-itself and if it contains all its limit points.

→ The set  $\mathbb{Q}$  of rational numbers.

Let  $S = \mathbb{Q} \subseteq \mathbb{R}$

Let  $x$  be any real number. Then for each  $\epsilon > 0$  (however small),

$(x - \epsilon, x + \epsilon)$  is a nbhd of  $x$

and it contains infinitely many rational numbers other than  $x$ .

i.e.  $(x - \epsilon, x + \epsilon) \cap \mathbb{Q} - \{x\} \neq \emptyset$

⇒  $x$  is a limit point of  $S = \mathbb{Q}$ .

⇒ every real number is a limit point of  $\mathbb{Q}$ .

Hence the set of the limit points of  $\mathbb{Q}$

is the set of all real numbers  $\mathbb{R}$

$$\therefore [D(\mathbb{Q}) = \mathbb{R}]$$

$$\text{also } S = S \cup D(S) -$$

$$= \mathbb{Q} \cup \mathbb{R}$$

$$\boxed{S = \mathbb{R}}$$

clearly 'S' is dense in  $\mathbb{R}$  and  $S \subseteq D(S)$

∴  $S = \mathbb{Q}$  is dense in itself.

Since  $S \neq D(S)$  -

i.e.  $\mathbb{Q} \neq D(\mathbb{Q})$

∴  $S = \mathbb{Q}$  is not a perfect set

→ The  $\mathbb{R} - \mathbb{Q}$  of irrational numbers.

Let  $S = \mathbb{R} - \mathbb{Q} \subseteq \mathbb{R}$  then

$$D(S) = \mathbb{R}$$

→ The set  $\mathbb{N}$  of natural numbers.

Let  $S = \mathbb{N} \subseteq \mathbb{R}$

Let  $x \in \mathbb{R}$

### Adherent point:

A real number ' $p$ ' is called an adherent point of a set  $S \subset \mathbb{R}$  if every nbhd of  $p$  contains a point of  $S$ .

i.e. point  $p \in \mathbb{R}$  is an adherent point of  $S \subset \mathbb{R}$   
 $\Leftrightarrow$  for each nbhd  $N$  of  $p$ ,  $N \cap S \neq \emptyset$

Note: Due to a close resemblance between the definitions of an adherent point of a set and a limit point of a set, the distinction between the two should be carefully noted.

for a point ' $p$ ' to be a limit of a set  $S$ , every nbhd  $N$  of ' $p$ ' must contain a point of  $S$  other than  $p$ .

$$\text{i.e. } N \cap S - \{p\} \neq \emptyset.$$

for a point ' $p$ ' to be an adherent point of a set  $S$ ; every nbhd of  $p$  must contain a point of  $S$  which can be ' $p$ ' itself.

$$\text{i.e. } N \cap S \neq \emptyset.$$

If  $p \in S$ , then ' $p$ ' is an adherent point of  $S$ , since every nbhd of  $p$  contains  $p$  which belongs to  $S$ .

If  $p \notin S$ , then  $p$  is a limit point of  $S$  and, therefore, every nbhd of ' $p$ ' contains a point of  $S$  other than  $p$ .

Thus  $p$  is also an adherent point of  $S$ .  
Clearly, a real number  $p$  is an adherent point  $\Leftrightarrow$  either  $p \in S$ , or  $p \in D(S)$ .

Every point of  $S$  is an adherent of  $S$ .

Every limit point of  $S$  is an adherent point.  
But an adherent point of  $S$  need not be a limit point of  $S$ .

### Derived Set:

The set of all limit points of a subset 'S' of  $\mathbb{R}$  is called the derived set of  $S$  and is denoted by  $S'$  or  $D(S)$ .

i.e.,  $D(S)$  or  $S' = \{x \in \mathbb{R} / x \text{ is a limit point of } S\}$

Again the derived set of  $D(S)$  is called the second derived set of  $S$  and is denoted by  $D''(S)$ , or  $S''$ .

In general, the  $n^{\text{th}}$  derived set of  $S$  denoted by  $D^{(n)}(S)$  or  $S^{(n)}$ .

→ A set is said to be of first species if it has only a finite number of derived sets.

It is said to be of second species if the number of derived sets is infinite.

Note: ① If the set  $S$  is finite, then ' $S$ ' has no limit point and consequently,  $D(S) = \emptyset$ .

② If a set ' $S$ ' is of first species, then its last derived set must be empty.

③ A set whose  $n^{\text{th}}$  derived set is a finite set

so that its  $(n+1)^{\text{th}}$  derived set is empty  
is called a set of  $n^{\text{th}}$  order.

- $N = \{1, 2, 3, \dots\} \subseteq \mathbb{R}$  has no limit points.
- $A = \{\dots, -3, -2, -1\}$  has no limit points.
- Every point of the set  $\mathbb{R}$  of all real numbers is a limit point of  $\mathbb{R}$ .
- Let  $S = \mathbb{R}$   
let  $p \in \mathbb{R}; \epsilon > 0$ .  
 $(p-\epsilon, p+\epsilon) \cap \mathbb{R} = \text{infinite number of real numbers}$
- Every real number is a limit point of the  $\mathbb{Q}$  of all rational numbers.  
Let  $S = \mathbb{Q}$   
let  $p \in \mathbb{Q}; \epsilon > 0$ .  
 $(p-\epsilon, p+\epsilon) \cap S = \text{infinite number of rational numbers}$
- Similarly ;  $S = \mathbb{Q}'$  or  $\mathbb{R} \setminus \mathbb{Q}$  has no limit points.
- The empty set  $\emptyset$  has no limit points.
- Let  $S = \emptyset$   
let  $p \in \mathbb{R}; \epsilon > 0$ .  
 $(p-\epsilon, p+\epsilon) \cap S = \emptyset$  not an infinite set.
- $S = (a, b)$ .  
Every point of  $S$  is a limit point of  $S$ .
- $S = [a, b]; (a, b]; [a, b)$ :  
Every point of  $S$  is a limit point of  $S$ .
- $S = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \dots \right\} \subseteq \mathbb{R}$   
Let  $o \in \mathbb{R}; \epsilon > 0$ .  
 $(o-\epsilon, o+\epsilon) \cap S = \text{infinite set}$ .  
 $\therefore o$  is a limit point of  $S$ .
- (Q)  
 $\lim_{n \rightarrow \infty} \frac{1}{n} = \frac{1}{\infty} = 0 \notin S$  (i.e.  $0$  is not a member of  $S$ )

$$\rightarrow S = \left\{ \frac{n}{n+1} \mid n \in \mathbb{N} \right\} = \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots \right\}$$

(or)

$$= \left\{ 1 - \frac{1}{n+1} \mid n \in \mathbb{N} \right\}.$$

Since  $\epsilon > 0$ ,  $\exists \delta > 0$  such that  $(1-\delta, 1+\delta)$  contains infinitely many points of  $S$ .

$\therefore 1$  is a limit point of  $S$  and  $1 \notin S$ .

$\lim_{n \rightarrow \infty} S = 1 \notin S$ .  $1$  is a limit point.

$$\rightarrow S = \left\{ 1 - \frac{1}{n} \mid n \in \mathbb{N} \right\}.$$

$\lim_{n \rightarrow \infty} S = 1 \notin S$   
 $1$  is a limit point of  $S$ .

$$\rightarrow S = \left\{ (-1)^n / n \mid n \in \mathbb{N} \right\} = \left\{ \frac{-1}{1}, \frac{1}{2}, \frac{-1}{3}, \frac{1}{4}, \dots \right\}$$

since  $\lim_{n \rightarrow \infty} S = \lim_{n \rightarrow \infty} (-1)^n = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$

$\therefore S$  has two limit points  $-1$  and  $1$ , which are members of  $S$ .

$$\rightarrow S = \left\{ (-1)^n n \mid n \in \mathbb{N} \right\} = \left\{ -1, +2, -3, +4, -5, \dots \right\}$$

has no limit points.

see  $\lim_{n \rightarrow \infty} S = \lim_{n \rightarrow \infty} (-1)^n n = \begin{cases} \pm \infty & \text{if } n \text{ is odd} \\ \pm \infty & \text{if } n \text{ is even} \end{cases}$

Note: (1) Every finite set has no limit points.

(2) Every infinite set may or may not have limit points.

(3) Every interior point is a limit point, but every limit point need not be an interior point.

ex: let  $S = (a, b)$ , then  $D(S) = [a, b]$  &  $S^0 = (a, b)$ .

$a$  &  $b$  are limit points but not interior points.

→ If the supremum of a set does not belong to the set, then it is a limiting point of the set.

Sol Let  $S$  be the non-empty subset of real number set  $\mathbb{R}$ .  
and has supremum but not belong to the set  $S$ .

Let it be ' $u$ '  
i.e.  $u = \text{supremum of } S$  but  $u \notin S$ .

Now we have to prove that ' $u$ ' is a limiting point of a set  $S$ .  
for this we have to prove that every  
if the point  $u$  contains a point of  $S$   
other than  $u$ .

Let  $(u-\epsilon, u+\epsilon)$  be any nbh of the point  $u$   
where  $\epsilon > 0$ .

Since  $u = \text{l.u.b (supremum) of } S$ .  
 $\therefore u$  is not an upper bound of  $S$ .

$\therefore \exists$  some  $a \in S$  s.t  $a > u - \epsilon$  (1)

Also  $a < u + \epsilon$  (2)  $(\because a \in S \text{ and } u \in S)$

from (1) and (2), we have -

$u - \epsilon < a < u + \epsilon$  where  $a \neq u$ .

$\Rightarrow (u-\epsilon, u+\epsilon)$  contains a  
point  $a \in S$  (3)

$\Rightarrow u$  is a limiting point of  
the set  $S$ .

~~def~~ If the supremum of a set does not belong to the set, then it is a limit point of a set.

for example:

①  $S = (-\infty, a) \subset \mathbb{R}$

$\therefore S$  is bdd above

and  $a \notin S$

$\therefore a$  is a limiting point of  $S$ .

②  $S = (a, \infty)$

$\therefore S$  is bdd below by  $a$

and  $a \notin S$

$\therefore a$  is a limiting point of  $S$ .

Isolated point:

A point  $p \in S$  is called an isolated point of  $S$  if  $p$  is not a limit point of  $S$ , i.e. if  $\exists$  a nbd of  $p$  which contains no points of  $S$  other than  $p$  at self.

A set ' $S$ ' is called a discrete set if all pts points are isolated points.

for example: Let  $S = \{1, 2, 3, \dots\}$

Since all the points of the set ' $S$ ' are its isolated points and so it is a discrete set.

If  $x$  is the inf of  $N$  (i.e.  $x = \inf N$ )  
contains no point of  $N$  other than  $x$   
 $\therefore x$  is not limit point of  $N$  of natural numbers.

If  $x \notin N$ , then the inf of  $x$  does not contain any point of  $S$   
 $\therefore x$  is not limit point of the set  $N$  of natural numbers.  
 $\therefore N$  has no limit points.

$$\therefore D(N) = \emptyset.$$

Since no point of  $N$  is a limit point of  $N$ .  
All the points of  $N$  are isolated points.  
Hence  $N$  is discrete set.

Also  $[N$  is of first species. ( $\because D(N) = \emptyset$ )

The set  $\mathbb{W}$  of all whole numbers.  
The set  $\mathbb{Z}$  of all integers.

Let  $S = \emptyset \subseteq \mathbb{R}$ .  
Let  $x \in \mathbb{R}$ , then for each  $\epsilon > 0$ , there is  
 $(x-\epsilon, x+\epsilon) \cap \emptyset = \emptyset$   
 $\therefore x$  is not a limit point of  $S = \emptyset$ .  
 $\therefore$  No real number is a limit of  $\emptyset$ .  
 $\therefore D(\emptyset) = \emptyset$ .

$$\begin{aligned} \bar{S} &= S \cup D(S) \\ &= \emptyset \cup D(\emptyset) \end{aligned}$$

$$\therefore \bar{S} = \emptyset$$

Since  $\emptyset \subseteq D(\emptyset)$   
 $\therefore S$  is dense set & self.

Also  $S = D(S)$ , i.e.  $\emptyset = D(\emptyset)$   
 $\therefore \emptyset$  is perfect set.

$\rightarrow$  One set, i.e.  $\mathbb{R}$

Let  $x \in \mathbb{R} \subseteq \mathbb{R}$

Then for each  $\epsilon > 0$ , however small  
the interval  $(x-\epsilon, x+\epsilon)$   
contains infinitely many real numbers  
other than  $x$ .

i.e.  $x$  is a limit point of  $\mathbb{R}$

$\Rightarrow$  every real number is a limit  
point of  $\mathbb{R}$ .

$$\therefore D(\mathbb{R}) = \mathbb{R}$$

$$\bar{\mathbb{R}} = \mathbb{R} \cup D(\mathbb{R})$$

$$[\bar{\mathbb{R}} = \mathbb{R}]$$

$\mathbb{R}$  is dense set

and it is dense-in-itself set

Also it is perfect set ( $\because \mathbb{R} = D(\mathbb{R})$ )

Note! we have  $D(\mathbb{R}) = \mathbb{R}$ ,  $D^2(\mathbb{R}) = D(\mathbb{R})$

$D^3(\mathbb{R}) = \mathbb{R}$  and so on.

$\therefore$  for every non-negative integer  $n$ ,  $D^n(\mathbb{R}) = \mathbb{R}$

$\therefore$  The number of derived sets of  $\mathbb{R}$   
is infinite

$\therefore \mathbb{R}$  is of the second species.

$$\rightarrow S = (a, b)$$

~~Ques.~~ If  $x \in [a, b]$

then  $x = a$  or  $x = b$  or  $x \in (a, b)$

If  $x = a$ , then for every  $\epsilon > 0$ ,

$(a-\epsilon, a+\epsilon) = (a-\epsilon, a+\epsilon)$  contains  
infinitely many points of  $(a, b)$  to the  
right of  $a$ .

If  $x = b$ , then for every  $\epsilon > 0$ ,

$(x-\epsilon, x+\epsilon) = (b-\epsilon, b+\epsilon)$  contains infinitely many points of  $(a, b)$  to the left of 'b'.

If  $x \in (a, b)$ , then for every  $\epsilon > 0$ ,

$(x-\epsilon, x+\epsilon)$  contains infinitely many points of  $(a, b)$ .

Thus, if  $x \in [a, b]$ , then for every  $\epsilon > 0$ ,

$(x-\epsilon, x+\epsilon)$  is a wbd of 'x' containing

infinitely many points of  $(a, b)$ .

$\Rightarrow$  every point of  $[a, b]$  is a limit point of  $(a, b)$ .

$$\therefore D([a, b]) = [a, b].$$

$$\text{since } S = S \cup D(S)$$

$$= [a, b] \cup [c, d]$$

$$\boxed{S = [a, b] \subseteq \mathbb{R}}$$

Since  $S \subseteq D(S)$  i.e.  $[a, b] \subseteq [a, b]$ .

$\therefore S$  is dense-in-itself.

Since  $S \neq D(S)$ .

$\therefore S$  is not a perfect set.

$$\rightarrow S = [a, b]$$

$$\rightarrow S = [a, b]$$

$$\rightarrow S = [a, b]$$

$$\rightarrow S = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\} \subseteq \mathbb{R} \quad \text{Here: } S \text{ is infinite, } S \subset (0, 1) \subseteq \mathbb{R}$$

Let  $p \in \mathbb{R}$ . Then for each  $\epsilon > 0$  (however small)  
If  $p = 0$ , then for each  $\epsilon > 0$  (however small)  
 $(0-\epsilon, 0+\epsilon)$  is a wbd of '0'.  
and it contains infinitely many  
points of  $S$  other than '0'

$\therefore 0$  is the limit point of  $S$

NOW we shall show that no other real number  $p$  other than '0' can be a limit point of  $S$ .  
The following cases arise:

case(i): If  $p < 0$ , then  $(-\infty, 0)$  is a nbd of  $p$  which contains no point of  $S$  i.e.,  $(-\infty, 0) \cap S = \emptyset$ .  
 $\therefore p$  is not a limit point of  $S$ .

case(ii): If  $p > 1$ , then  $(1, \infty)$  is a nbd of  $p$  which does not contain any point of  $S$ . i.e.,  $(1, \infty) \cap S = \emptyset$ .  
 $\therefore p$  is not a limit point of  $S$ .

Case(iii): If  $p=1$ , then  $(\frac{1}{2}, \infty)$  is a nbd of  $p$  which contains no point of  $S$  other than  $p$ . i.e.,  $(\frac{1}{2}, \infty) \cap S \cap \{p\} = \emptyset$ .  
 $\therefore p$  is not limit point of  $S$ .

Conclusion:

If  $0 < p < 1$ , then  $p \neq 0$ .

$\therefore \exists$  a unique natural number 'n' such that  $n \leq \frac{1}{p} < n+1$ .

$$\Rightarrow \frac{1}{n} \geq p > \frac{1}{n+1}$$

$$\Rightarrow \frac{1}{n+1} < p \leq \frac{1}{n} < \frac{1}{n}$$

$\Rightarrow$  The nbd  $\left(\frac{1}{n+1}, \frac{1}{n}\right)$  of  $p$  contains only one point  $p$  of  $S$ .

$\therefore p$  is not limit point of  $S$ .

Hence  $0$  is the only limit point of  $S$ .

$$D(S) = \{0\}$$

$$\text{Also } D(S) = D(U) = \emptyset.$$

The set  $S$  is of the first species and  
of first order.

$$\text{and } S = S \cup D(S)$$

$$= \left\{ \frac{1}{n} \mid n \in \mathbb{N}, n \neq 0 \right\}$$

$\rightarrow$  find  $S' \subset D(S)$   
where  $S' = \left\{ \frac{1}{n} \mid n \in \mathbb{Z}, n \neq 0 \right\}$ .

$$\text{Let } S = \left\{ \frac{1}{n} \mid n \in \mathbb{Z}, n \neq 0 \right\} \subseteq [-1, 1] \subset \mathbb{R}$$

Let  $p = 0 \in \mathbb{R}$  then the nbhd of '0' contains  
infinitely many numbers.  
 $\therefore 0$  is a limit of  $S$ .

now we shall show that no real number  
 $p$  other than 0 can be a limit point of  $S'$ .

The following cases will arise:

case(i) If  $p < -1$  then  $(-\infty, -1)$  is a nbhd of  $p$   
which contains no point of  $S$ .  
i.e.  $(-\infty, -1) \cap S = \emptyset$ .  
 $\therefore p$  is not a limit point of  $S$ .

case(ii) If  $p > 1$  then  $(1, \infty)$  is a nbhd of  $p$ .  
which contains no point of  $S$ .  
i.e.  $(1, \infty) \cap S = \emptyset$ .

case(iii): If  $p = 1$  then  $(1, \infty)$  is a nbhd of  $p$ .  
which does not contain any  
point of  $S$  other than  $p$ .  
i.e.  $(1, \infty) \cap S = \{1\} = \emptyset$ .  
 $\therefore p$  is not a limit point of  $S$ .

case(iv): If  $p = -1$  then  $(-\infty, -1)$  is a nbhd of  $p$ .  
which does not contain any  
point of  $S$  other than  $p$ .  
i.e.  $(-\infty, -1) \cap S = \{-1\} = \emptyset$ .

f S.

case (v) If  $0 < p < 1$ , then

$(\frac{1}{n+1}, \frac{1}{n})$  is a nbhd of p

which contains only one point

$\gamma_n$  of S

i.e. a finite number of points of S

$\therefore p$  is not a limit point of S

case (vi) If  $-1 < p < 0$ , so that  $0 < -p < 1$  and

$-\frac{1}{p} > 0$ ,  $\exists$  a unique n such that

$$n \leq -\frac{1}{p} < n+1$$

$$\Rightarrow -\frac{1}{n+1} < p < -\frac{1}{n}$$

$$\Rightarrow \gamma_{n+1} < \frac{1}{n+1} \leq p < -\frac{1}{n}$$

$\Rightarrow$  the nbhd  $(\frac{1}{n+1}, \frac{1}{n})$  of p  
contains only one point  $-\frac{1}{n}$   
of S:

$\therefore p$  is not a limit point of S.

Hence '0' is the only limit point of S.

$$\therefore D(S) = \{0\}$$

$$\text{and } \bar{S} = S \cup \{0\}$$

→ find the derived set of each of the following:

(i)  $(1, \infty)$  (ii)  $(-\infty, 1)$  (iii)  $\{\frac{n}{n+1} / n \in \mathbb{N}\}$ .

(iv)  $\{a + \frac{1}{n} / a \in \mathbb{R} \text{ and } n \in \mathbb{N}\}$

(v)  $\{\frac{1 + (-1)^n}{n} / n \in \mathbb{N}\}$ , (vi)  $\{\frac{1}{2^n} / n \in \mathbb{N}\}$  (vii)  $\{\frac{1}{3^n} / n \in \mathbb{N}\}$ .

SOL

(i) Let  $s = (1, \infty) = \dots$

Let  $x$  be any real number.

If  $x < 1$ , then for  $\epsilon < 1 - x$ ,

$$(x - \epsilon, x + \epsilon) \cap (1, \infty) = \emptyset.$$

$\Rightarrow$  any real number  $< 1$  is not a limit point of  $(1, \infty)$ .

If  $x \in [1, \infty)$ , then for every  $\epsilon > 0$ ,  $(x - \epsilon, x + \epsilon)$  contains infinitely many points of  $(1, \infty)$  to the right of  $1$ .  
 $\Rightarrow$  Every elt of  $[1, \infty)$  is a limit point of  $(1, \infty)$ .  
 $\therefore (1, \infty)^l = [1, \infty).$

(ii) Ans:  $(-\infty, -1)^l = (-\infty, -1]$ .

(iii) Let  $S = \left\{ \frac{1+(-1)^n}{n} \mid n \in \mathbb{N} \right\} \subseteq \mathbb{R}$ .

when 'n' is odd,

$$\frac{1+(-1)^n}{n} = \frac{1-1}{n} = 0.$$

when 'n' is even,

$$\frac{1+(-1)^n}{n} = \frac{1+1}{n} = \frac{2}{n}.$$

$$\therefore S = \{0\} \cup \left\{ \frac{2}{n} \mid n \in \mathbb{N} \text{ and } n \text{ is even} \right\}.$$

$$= \{0\} \cup \left\{ \frac{2}{2}, \frac{2}{4}, \frac{2}{6}, \dots \right\}.$$

$$= \{0\} \cup \left\{ 1, \frac{1}{2}, \frac{1}{3}, \dots \right\} \subseteq [0, 1] \subseteq \mathbb{R}.$$

Let  $p \in \mathbb{R}$

If  $p = 0$  then for each  $\epsilon > 0$  (however small),

$(0 - \epsilon, 0 + \epsilon)$  is a nbd of '0'  
and it contains infinitely many points of  $S$  other than '0'.

$\therefore 0$  is the limit point of  $S$ .

Now we shall show that no other real number  $p$  other than '0' can be a limit point of  $S$ :

The following cases arise:

case(i) If  $p < 0$  then  $(-\infty, 0)$  is a hbd of 'p' which contains no point of  $S$   
i.e.  $(-\infty, 0) \cap S = \emptyset$ .

$\therefore p$  is not a limit point of  $S$ .

case(ii) If  $p > 1$  then  $(1, \infty)$  is a hbd of 'p' which does not contain any point of  $S$   
i.e.  $(1, \infty) \cap S = \emptyset$

$\therefore p$  is not a limit point of  $S$ .

case(iii) If  $p=1$ , then  $(\frac{1}{2}, \infty)$  is a hbd of 'p' which contains no point of  $S$   
otherwise

$$\text{i.e. } (\frac{1}{2}, \infty) \cap S - \{1\} = \emptyset$$

$\therefore 'p'$  is not a limit point of  $S$ .

case(iv) If  $0 < p < 1$ , then  $\frac{1}{p} > 0$ .

$\therefore \exists$  a unique natural number 'n'  
such that  $n \leq \frac{1}{p} < n+1$

$$\Rightarrow \frac{1}{n+1} < p < \frac{1}{n}$$

$$\Rightarrow \frac{1}{n+1} < p \leq \frac{1}{n} < \frac{1}{n-1}$$

$$\Rightarrow \text{The hbd } \left( \frac{1}{n+1}, \frac{1}{n} \right) \text{ of } p$$

contains only one point  $\frac{1}{n}$  of  $S$ .

$\therefore p$  is not limit point of  $S$ .

Hence '0' is the only limit point of  $S$ .

$$\therefore D(S) = \{0\}$$

$$\rightarrow S = \left\{ \cos\left(\frac{n\pi}{2}\right) \mid n \in \mathbb{Z} \right\} = \dots$$

$$= \{\dots, -1, 0, 1, 0, -1, 0, 1, \dots\}$$

clearly  $-1, 0, 1$  are limit points of  $S$

$$\therefore D(S) = \{-1, 0, 1\}.$$

$$\rightarrow \text{Let } S = \left\{ \sin\left(\frac{n\pi}{2}\right) \mid n \in \mathbb{Z} \right\} \subseteq \mathbb{R}$$

$$\text{Then } D(S) = \{-1, 0, 1\}.$$

$$\rightarrow \text{Let } S = \left\{ \cos\left(\frac{n\pi}{3}\right) \mid n \in \mathbb{Z} \right\} \subseteq \mathbb{R}$$

$$\text{Then } D(S) = \left\{ -\frac{1}{2}, 1, \frac{1}{2}, \pm 1 \right\}.$$

$$\rightarrow \text{Let } S = \left\{ \sin\frac{n\pi}{3} \mid n \in \mathbb{Z} \right\} \subseteq \mathbb{R} \text{ Then}$$

$$D(S) = \left\{ -\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2} \right\}.$$

\* Existence of limit points of a set:

Bolzano-Weierstrass theorem:

We have seen that a finite set has no limit points. Also we have observed that an infinite set may or may not have a limit point.

for example!

The infinite set  $\mathbb{N}$  of natural numbers has no limit point whereas the infinite set  $S = \left\{ \frac{1}{n} \mid n \in \mathbb{Z} \right\}$  has '0' as its limit point.

We observe that the set  $S$  is bounded.

Now we shall give a theorem which gives us a set of sufficient conditions for a set to have a limit point. This theorem is known as Bolzano-Weierstrass theorem.

### Statement:

Every infinite bounded set of real numbers has a limit point.

Note: The converse of the above need not be true.  
i.e., An infinite set has a limit point, then the set is not bounded.

for example:

Ex-1  $[a, \infty)$  is an infinite set and has limit points but it is not bounded.

Ex-2  $S = \mathbb{Q}, \mathbb{R} - \mathbb{Q}, \mathbb{R}$

### Some results on Derived sets:

→ If A and B be any two subsets of  $\mathbb{R}$ ; then

$$(1) A \subset B \Rightarrow D(A) \subseteq D(B)$$

$$(2) D(A \cup B) = D(A) \cup D(B)$$

$$(3) D(A \cap B) \subseteq D(A) \cap D(B)$$

$$(4) D(D(A)) \subseteq D(A)$$

$$\rightarrow D(\bigcup_{i=1}^n A_i) = D(A_1) \cup D(A_2) \cup \dots$$

$$\rightarrow D(\bigcap_{i=1}^n A_i) \subseteq D(A_1) \cap D(A_2) \cap \dots$$

### Note:

(1) The derived set of any bounded set is bounded.

(2) Every infinite bounded set has the greatest and the smallest limit points.

i.e., the derived set of any infinite bounded set obtains its bounds.

(3). The smallest and the greatest members of the derived set  $D(S)$  of an infinite and bounded set  $S$  always exist.

They are usually denoted by  $\underline{\lim} S$  and  $\overline{\lim} S$  respectively and are called the inferior (or lower) limit of  $S$  and the superior (or upper) limit of  $S$ .

Also  $\underline{\lim} S \leq \overline{\lim} S$

(4) The supremum (or infimum) of a bounded set  $S$  is always members of  $\bar{S}$

(5) If  $S$  is bounded then  $\bar{S}$  is also bounded.

### Closed Set

A subset  $S$  of  $\mathbb{R}$  is said to be closed if its complement (i.e.,  $S^c = \mathbb{R} - S$ ) is an open set.

(or)

A set  $S \subset \mathbb{R}$  is said to be closed if every limit point of the set  $S$  is a member of the set  $S$ .

(or)

A subset  $S$  of  $\mathbb{R}$  is said to be closed if  $D(S) \subseteq S$

$\rightarrow$  (i)  $S$  is closed set  $\Leftrightarrow D(S) \subseteq S$

(ii)  $S$  is closed set  $\Leftrightarrow S^c$  is open.

(iii)  $S$  is open set  $\Leftrightarrow S^c$  is closed

(iv)  $S$  is closed set  $\Leftrightarrow \bar{S} = S$

If  $S$  is a closed set then every limit point of  $S$  is a member of  $S$ , but every point of  $S$  is not limit point.

(Ex)

$$\text{Q1) If } S = \{a\} \text{ then } S^c = \mathbb{R} - S \\ = \mathbb{R} - \{a\} \\ = (-\infty, a) \cup (a, \infty)$$

Since union of two open sets is again open

$\therefore S^c$  is open.

$\Rightarrow S$  is closed.

(Qv)

$$S = \{a\} \text{ then } D(S) = \emptyset \subseteq S \\ \text{i.e., } D(S) \subseteq S \\ \therefore S \text{ is closed.}$$

(Or)

$$S = \{a\} \text{ then } D(S) = \emptyset. \\ \therefore \overline{S} = S \cup D(S) \\ = \{a\} \cup \emptyset = \{a\} \\ = S.$$

$$\therefore \overline{S} = S$$

$\therefore S$  is closed.

(2) If  $S$  be a non-empty finite set.

$$\text{then } D(S) = \emptyset \subseteq S$$

$$\text{i.e., } D(S) \subseteq S$$

$\therefore S$  is closed.

(3) If  $S = \mathbb{N}$  then  $D(S) = \emptyset \subseteq S$ .

$$\Rightarrow D(S) \subseteq S$$

$\therefore S$  is closed

Similarly,  $S = \mathbb{W}, \mathbb{I}$ .

$$(4) S = \mathbb{Q}$$

then  $D(S) = \mathbb{R} \not\subseteq S$ .

$\Rightarrow S$  is not closed.

$$(5) S = \mathbb{R} - \mathbb{Q}$$

then  $D(S) = \mathbb{R} \not\subseteq S$

$\Rightarrow S$  is not closed.

$$(6) S = (a, b)$$

then  $D(S) = [a, b] \not\subseteq S$

$\therefore S$  is not closed.

$$(7) S = [a, b), [a, b]$$

then  $D(S) = [a, b] \not\subseteq S$

$\therefore S$  is not closed

$$(8) \bar{S} = [a, b]$$

then  $D(S) = [a, b] \subseteq S$

$\therefore S$  is closed.

$$(9) \bar{S} = \mathbb{R} \text{ then } D(S) = \mathbb{R}$$

$\therefore D(S) \subset \mathbb{R}$

$\therefore S$  is closed.

$$(10) S = \mathbb{R}^+$$

$$\text{i.e., } S = \{x / x > 0\}$$

$$= (0, \infty)$$

$$\Rightarrow D(S) = [0, \infty)$$

$\therefore D(S) \not\subseteq S$

$\therefore S$  is not closed.

$$(11) S = \mathbb{R}$$

$$= (0, 0) \Rightarrow D(S) = (-\infty, 0] \not\subseteq S$$

$\therefore S$  is not closed.

$$(12) S = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}$$

then  $D(S) = \{0\} \not\subseteq S$  ( $\because 0 \notin S$ )

$\therefore S$  is not closed.

$$(13) S = \{t_n \mid n \in \mathbb{Z}\} \text{ then } D(S) = \{0\} \not\subseteq S$$

$\therefore S$  is not closed.

Note: If a set has no limit point then  $\bar{S} = S$ .

The intersection of an arbitrary family of closed sets is a closed set.

Sol: Let  $s_1, s_2, s_3, \dots$  be closed sets.

then  $s_1^c, s_2^c, s_3^c, \dots$  be the open sets.

$$\text{Let } S = s_1 \cap s_2 \cap s_3 \cap \dots$$

$$\Rightarrow S^c = (s_1 \cap s_2 \cap s_3 \cap \dots)^c$$

$$= s_1^c \cup s_2^c \cup s_3^c \cup \dots$$

Since the union of arbitrary family of open sets is open.

$$\Rightarrow S^c \text{ is open}$$

$$\Rightarrow S \text{ is closed set}$$

→ The union of a finite collection of closed sets is a closed set.

Sol: Let  $s_1, s_2, \dots, s_n$  be closed sets.

then  $s_1^c, s_2^c, s_3^c, \dots, s_n^c$  be the open sets.

$$\text{Let } S = s_1 \cup s_2 \cup \dots \cup s_n$$

$$\Rightarrow S^c = (s_1 \cup s_2 \cup \dots \cup s_n)^c$$

$$= s_1^c \cap s_2^c \cap \dots \cap s_n^c$$

Since intersection of finite collection of open sets is open.

$$\therefore S^c \text{ is an open set}$$

$$\Rightarrow S \text{ is closed set}$$

→ The union of an infinite collection of closed sets need not be a closed set.

Sol<sup>3</sup> Let  $S_n = [\frac{1}{n}, 1]$  when

Then each  $S_n$  is a closed set.

Now  $\bigcup_{n=1}^{\infty} S_n = S_1 \cup S_2 \cup S_3 \cup \dots$

$= \{1\} \cup [\frac{1}{2}, 1] \cup [\frac{1}{3}, 1] \cup \dots$

$= (0, 1] = S$  (say)

which is not a closed set ( $\because D(S) \neq \emptyset$ )

The union of an infinite collection of closed sets need not be a closed set.

Let 'A' be a closed set and 'B' be an open set.

then (i)  $A-B$  is closed (ii)  $B-A$  is open.

Sol<sup>4</sup> Since A is closed  $\Rightarrow A^c$  is open

B is open  $\Rightarrow B^c$  is closed.

(i)  $B-A = B \cap A^c$ .

Since B and  $A^c$  are open,

$\Rightarrow B \cap A^c$  open.

$\therefore B-A$  is open.

(ii)  $A-B = A \cap B^c$ .

Since A and  $B^c$  are closed

$\Rightarrow A \cap B^c$  is closed

$\therefore A-B$  is closed.

### Compact sets

A non-empty subset of  $\mathbb{R}$  is said to be compact if it is closed and bounded.

(i)  $S = \emptyset$ .

$D(S) = \emptyset \subseteq S$

$\Rightarrow S$  is closed and bounded.

$\Rightarrow S$  is compact.

$$D(S) = [a, b] \subseteq S$$

$\therefore S$  is closed and bounded.  
 $\therefore S$  is compact.

(3)  $S = [-1, 1] \cup [2, 3]$

Since the union of two closed sets is closed  
and bounded.

$\therefore S$  is compact.

(4)  $S = N,$

$$D(S) = \emptyset \subset N$$

$\therefore S$  is closed but not bounded.  
 $\therefore S$  is not compact.

Similarly,  $S = W, Z$ .

(5)  $S = Q \Rightarrow D(S) = \mathbb{R} \not\subseteq S$

i.e.,  $D(S) \not\subseteq S$ .  
 $\therefore S$  is not closed and not bounded.  
 $\therefore S$  is not compact.

(6)  $S = \mathbb{R} - Q$ .

$$\Rightarrow D(S) = \mathbb{R} \not\subseteq \mathbb{R} - Q$$

i.e.,  $D(S) \not\subseteq S$ .  
 $\therefore S$  is not closed and bounded  
 $\therefore S$  is not compact.

(7)  $S = \mathbb{R}; D(S) = \mathbb{R}$ .

$\therefore D(S) \subseteq \mathbb{R}$ .  
 $\therefore S$  is closed but not bounded.  
 $\therefore S$  is not closed, compact.

(8)  $S = (a, b) \Rightarrow \overline{D(S)} = [a, b] \not\subseteq S$

i.e.,  $D(S) \not\subseteq S$   
 $\therefore S$  is not closed but  $S$  is bounded  
 $\therefore S$  is not compact.

Similarly  $S = [a, b), (a, b]$ .

$$(iv) S = \{x : a \leq x\}.$$

$$= [a, \infty).$$

$$\Rightarrow D(S) = [a, \infty) \subset S.$$

$$\therefore D(S) \subseteq S.$$

$\therefore S$  is closed but is not bounded.

$\therefore S$  is not compact.

(v)

$$S = \{1^r, 2^r, 3^r, \dots, (23)^r\}.$$

Since  $S$  is finite.

$$\Rightarrow D(S) = \emptyset \subseteq S$$

$$\Rightarrow D(S) \subseteq S$$

$\therefore S$  is closed and bounded

$\therefore S$  is compact.

$\Rightarrow$  The union of finite family of compact sets is compact.

soln: Let  $S_1, S_2, \dots, S_n$  be compact sets.

Then  $S_1, S_2, S_3, \dots, S_n$  are closed and bounded

$$\text{Let } S = \bigcup_{i=1}^n S_i$$

Since the union of finite collection -

of closed sets is a closed

$\therefore S$  is closed.

$\therefore S$  is closed and bounded.

Now we have to show that  $S$  is bounded.

Also  $S_i \subset [a_i, b_i]$ ,  $1 \leq i \leq n$ .

If  $a = \min\{a_1, a_2, \dots, a_n\}$

and  $b = \max\{b_1, b_2, \dots, b_n\}$

$$\text{then } S = \bigcup_{i=1}^n S_i \subset [a, b].$$

$\therefore S$  is bounded.

$\therefore S$  is closed and bounded.

$\therefore S$  is compact.

→ The intersection of an arbitrary family of compact sets, containing atleast one point in common, is compact.

Sol: Let  $S_1, S_2, \dots, S_n, \dots$  be arbitrary family of compact sets.

Then  $S_1, S_2, \dots, S_n, \dots$  be closed and bounded.

$$S = \bigcap_{i=1}^{\infty} S_i$$

Since the intersection of arbitrary family of closed sets is closed.

∴  $S$  is closed.

Also  $S \subseteq S_i$  for each  $i$ .

and each  $S_i$  bounded.

$S$  is bounded.

∴  $S$  is closed and bounded.

∴  $S$  is compact.

### Cover of a set

→ Let 'S' be a set and  $\{G_\alpha\}$  be a family of sets.

We say that  $\{G_\alpha\}$  is a cover of S, if the union of members of  $\{G_\alpha\}$  contains  $S$  as a subset i.e., if every point of  $S$  belongs to some member of the family  $\{G_\alpha\}$ .

→ we say that  $\{G_\alpha\}$  is an open cover if every member of  $\{G_\alpha\}$  is an open set.

(or)

Let  $S$  be a set and  $\{G_\alpha\}$  be a collection of some open subsets of  $\mathbb{R}$  such that  $S \cup G_\alpha$ . Then  $\{G_\alpha\}$  is called an open cover of S.

$\rightarrow$  S.T  $G = \{(-n, n) / n \in \mathbb{N}\}$

is an open cover of the set  $\mathbb{R}$ .

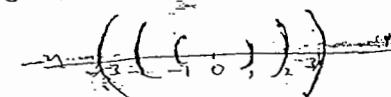
Sol:

Given  $G = \{(-n, n) / n \in \mathbb{N}\}$

$$= \left\{ (-1, 1), (-2, 2), (-3, 3), \dots \right\}$$

Since every  $x \in \mathbb{R}$  belongs to at least one of the open intervals in  $G$ .

$\therefore G_1$  is an open cover of  $\mathbb{R}$ .



Also  $\mathbb{R} = \bigcup_{n \in \mathbb{N}} G_n$ , where  $G_n = (-n, n)$

Similarly:

$$G_1 = \{(-2n, 2n) / n \in \mathbb{N}\}$$

$$G_2 = \{(-n, n+2) / n \in \mathbb{Z}\}$$

$$G_3 = \{(-n, n+1) / n \in \mathbb{Z}\}$$
 are open covers of  $\mathbb{R}$ .

$\rightarrow$  Show that  $G_1 = \left\{ \left(\frac{1}{4}, \frac{5}{4}\right), \left(\frac{3}{4}, \frac{7}{4}\right), \left(\frac{5}{4}, \frac{9}{4}\right) \right\}$  is open cover of

interval  $[1, 2]$ , whereas  $G_2 = \left\{ \left(\frac{1}{2}, \frac{5}{4}\right), \left(\frac{3}{2}, \frac{9}{4}\right) \right\}$  is not an open cover of the interval  $[1, 2]$ .

Sol:

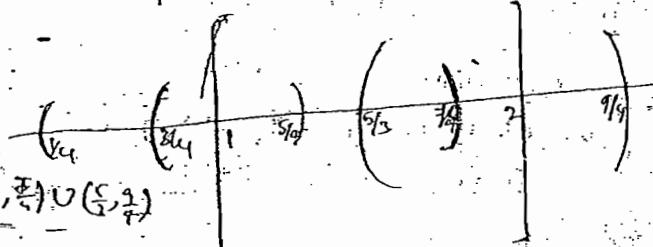
Given  $G_1 = \left\{ \left(\frac{1}{4}, \frac{5}{4}\right), \left(\frac{3}{4}, \frac{7}{4}\right), \left(\frac{5}{4}, \frac{9}{4}\right) \right\}$

is an open cover of the interval  $[1, 2]$

Since every element of the set  $S = [1, 2] = \{x\}$  belongs to at least one of the subsets of  $G_1$ .

and each of the subsets of  $G_1$  is an open set.

$\therefore G_1$  is an open cover of  $S = [1, 2]$

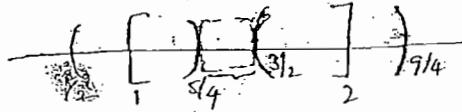


$$\therefore [1, 2] \subset \underline{\left( \frac{1}{4}, \frac{5}{4} \right)} \cup \underline{\left( \frac{3}{4}, \frac{7}{4} \right)} \cup \underline{\left( \frac{5}{4}, \frac{9}{4} \right)}$$

$G_2 = \left\{ \left( \frac{1}{2}, \frac{5}{4} \right), \left( \frac{5}{4}, \frac{9}{4} \right) \right\}$  is not an open cover  
of the interval  $S = [1, 2] = \{x / 1 \leq x \leq 2\}$ .

because none of points lies in the interval

$\left[ \frac{5}{4}, \frac{3}{2} \right]$  belongs to any of the subsets of  $G_2$ .  
i.e.,  $S = [1, 2]$  is not covered by union of open sets  $\left( \frac{1}{2}, \frac{5}{4} \right) \cup \left( \frac{5}{4}, \frac{9}{4} \right)$   
 $\therefore G_2$  is not an open cover of  $S$ .



### Subcover and finite subcover of a set?

Let  $G$  be an open cover of a set  $S$ . A subcollection  $E$  of  $G$  is called a subcover of  $S$  if  $E$  too is a cover of  $S$ .

further, if there are only a finite number of sets in  $E$ , then we say that  $E$  is a finite subcover of the open cover  $G$  of  $S$ .

If  $G$  is an open cover of a set  $S$ , then a collection  $E$  is a finite subcover of the open cover  $G$  provided the following three conditions hold:

- (i)  $E$  is contained in  $G$ .
- (ii)  $E$  is a finite collection.
- (iii)  $E$  is itself a cover of  $S$ .

### Heine-Borel property:

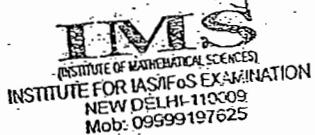
A subset  $S$  of  $\mathbb{R}$  is said to have the Heine-Borel property if every open cover of  $S$  has a finite sub-cover.

Sequence :- A function whose domain is the set  $\mathbb{N}$  of all natural numbers and the range is a subset of real numbers is called a sequence. (or) Real sequence.

The sequence is denoted by  $x: \mathbb{N} \rightarrow \mathbb{R}$ , (or)  $\{x_n\}_{n \in \mathbb{N}}$ .

A set of numbers which are in correspondence with natural numbers is called a sequence.

- NOTE
- The domain for a sequence is always natural numbers.
  - A sequence is specified by the values  $x(n)$  (or)  $x_n$ .
  - A sequence may be denoted by  $\{x_n : n \in \mathbb{N}\}$ .
  - (or)  $(x_n : n \in \mathbb{N})$  (or)  $x \in \{x_1, x_2, \dots, x_n, \dots\}$
- The values  $x_1, x_2, \dots, x_n, \dots$  are called first, second, third... terms of the sequence.
- The  $m^{\text{th}}$  &  $n^{\text{th}}$  terms  $x_m$  &  $x_n$  for  $m \neq n$  are treated as distinct terms even if  $x_m = x_n$ .
  - i.e., the terms of a sequence are arranged in a definite order as first, second, third, ... terms and the terms occurring at different positions are treated as distinct terms even if they have the same value.



Range of a sequence

The set of all distinct terms for a sequence is called its range.

- Note: In a sequence  $\{x_n : n \in \mathbb{N}\}$  and  $\mathbb{N}$  is infinite, the number of terms in a sequence is always infinite.
- The range of a sequence may be finite set.
  - Ex: If  $x_n \in \{-1, 0, 1\}; n \in \mathbb{N}$  then  $\{x_n\} = \{-1, 0, 1, -1, 0, \dots\}$

$\therefore$  There are only two distinct elements.  
 $\therefore$  The range of a sequence  $\{x_n\} = \{-1, +1\}$ .  
 which is finite.

Eg:  $\{x_n\} = \left\{ \frac{1}{n} \right\}_{n \in \mathbb{N}}$   
 $= \{1, \frac{1}{2}, \dots, \frac{1}{n}\}$

All the elements of the sequence are distinct.  
 $\therefore$  The range of a sequence is infinite.

Constant Sequence: - A sequence  $\{x_n\}$  is defined by  $x_n = c \forall n \in \mathbb{N}$  is called constant sequence.  
 i.e.,  $\{x_n\} = \{c, c, c, \dots, c\}$  is constant sequence  
 with a range  $\{c\}$   
 which is a singleton set.

Eg:  $\{x_n\}_{n \in \mathbb{N}} = \{1\}$

problem 5

The sequence  $\{x_n\}$  is defined by the following formulas for the  $n^{\text{th}}$  term. write the first five terms in each case.

(a)  $x_n = 1 + (-1)^n$  (b)  $x_n = \frac{(1)^n}{n}$  (c)  $x_n = \frac{1}{n(n+1)}$  (d)  $x_n = \frac{1}{n+2}$

The first few terms of a sequence  $(a_n)$  are given below. Assuming that the "natural pattern" indicated by these terms persists, give a formula for the  $n^{\text{th}}$  term  $a_n$ .

(a) 5, 7, 9, 11, ... (b)  $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$

(c)  $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$  (d) 1, 4, 9, 16, ...

Sol: Let  $\alpha = (5, 7, 9, 11, \dots) = (2n+3/n \in \mathbb{N})$ .

all two sequences in  $\mathbb{R}$  then  $x+y = (x_n+y_n)$  in  $\mathbb{R}$   
is called sum of two sequences.

→ Difference of sequences :- If  $x = (x_n)$  and  $y = (y_n)$   
are two sequences in  $\mathbb{R}$  then  $x-y = (x_n-y_n)$  in  $\mathbb{R}$   
is called difference of two sequences.

→ Product of sequences :- If  $x = (x_n)$  and  $y = (y_n)$   
are two sequences then  $x \cdot y = (x_n \cdot y_n)$  in  $\mathbb{R}$  is  
called product of two sequences.

→ Quotient : If  $x = (x_n)$ ,  $y = (y_n)$  are two sequences  
in  $\mathbb{R}$  then  $\frac{x}{y} = \left( \frac{x_n}{y_n} \right) (y_n \neq 0)$  is called  
quotient.

Bounds of a sequence :- If the range of a sequence

is bdd. below, then the sequence is said to be bdd

i.e., A sequence  $\{x_n\}$  is said to be bdd below if there is  $K \in \mathbb{R}$  s.t.  $x_n \geq K$   $\forall n \in \mathbb{N}$ .

→ If the range of a sequence is bdd above then the  
sequence is said to be bdd above.

i.e., A sequence  $\{x_n\}$  is said to be bdd above

if  $\exists K \in \mathbb{R}$  s.t.  $x_n \leq K$   $\forall n \in \mathbb{N}$ .

→ If the range of a sequence is bdd, the sequence

is said to be bdd.

i.e., A sequence  $\{x_n\}$  is bdd, if it has two real

numbers  $k, K$  s.t.  $k \leq x_n \leq K$   $\forall n \in \mathbb{N}$ .

→ A sequence is said to be unbdd if it is not bdd.

→ If  $K$  is a lowerbound of the sequence  $\{x_n\}$ , every  
real number less than  $K$  is also lower bound of  $\{x_n\}$ .  
The greatest of all lower bounds is called glb (or) inf of  $\{x_n\}$ .

→ If  $K$  is an upperbound of the seq.  $\{x_n\}$ , every real number greater  
than  $K$  is also an upperbound of a seq.  $\{x_n\}$ . The least of all such

$\rightarrow$  (1)  $x = (x_n)$  or  $\{x_n\}$  where  $x_n \in \mathbb{R}$  for  $n \in \mathbb{N}$

$\{x_n\} = \{q_{n/\sqrt{N}}\} = \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$  is not bdd sequence.

Since L.B = -1 ; U.B is not defined.

$\therefore$  Sequence  $\{x_n\}$  is bdd below.

$\rightarrow$  (2)  $x = \{x_n / n \in \mathbb{N}\}$ .

$\rightarrow$  (3)  $x = \{(-1)^n / n \in \mathbb{N}\} = \{-1, 1, -1, 1, \dots\} = \{-1, +1\}$  is bdd sequence.  
L.B = -1, U.B = +1

$\rightarrow$  (4)  $x = \{(-1)^{?n} / n \in \mathbb{N}\}$

$\rightarrow$  (5)  $x = \{\sqrt{n} / n \in \mathbb{N}\}$ .

Definition: A sequence  $\{x_n\}$  is bounded if  $\exists$  only if  
 $\exists$  the real number  $M$  (i.e.,  $M > 0$ ) s.t.  $\forall n \in \mathbb{N}$ ,

N.C:

Let  $\{x_n\}$  be a bdd seq.

$\Rightarrow$  By defn  $\exists$  two real numbers  $b, k$  s.t  
 $b \leq x_n \leq k \quad \forall n \in \mathbb{N}$ .  $\quad \text{(1)}$

Let  $M = \max\{|b|, |k|\}$

$\Rightarrow |b| \leq M, |k| \leq M$

$\Rightarrow -M \leq b \leq M, -M \leq k \leq M \quad \text{(2)}$

from (1), (2) & (3)

$-M \leq b \leq x_n \leq k \leq M \quad \forall n \in \mathbb{N}$

$\Rightarrow -M \leq x_n \leq M \quad \forall n \in \mathbb{N}$

$\Rightarrow |x_n| \leq M \quad \forall n \in \mathbb{N}$ .

S.C

$|x_n| \leq M \quad \forall n \in \mathbb{N}$

$\Rightarrow -M \leq x_n \leq M \quad \forall n \in \mathbb{N}$

$\therefore \{x_n\}$  is bdd.

\* Limit of a sequence: Let  $x = (x_n)$  be a sequence

and  $a \in \mathbb{R}$ , the real number  $a$  is said to be the limit of the sequence  $\{x_n\}$  if for each  $\epsilon > 0$  (however

$\exists N \in \mathbb{N}$  depending on  $\epsilon$ ; i.e.,  $N(\epsilon)$ )  $\forall n > N$   $|x_n - a| < \epsilon$ .

Converges  $\Rightarrow -\epsilon < x_n - x < \epsilon$  true  
 $\Rightarrow x - \epsilon < x_n < x + \epsilon$  true  
 $\Rightarrow x_n \in (x - \epsilon, x + \epsilon)$  true

\* Cgt of a sequence — Let  $(x_n)$  be a sequence  
 If  $x_n = x$  as  $n \rightarrow \infty$  then the sequence  $(x_n)$  is said  
 to be cgs to  $x$ .

If a sequence  $(x_n)$  has a limit then the sequence  
 $(x_n)$  is called cgt sequence.

(i) A sequence  $(x_n)$  is said to be cgs to  $x$ , if for  
 given  $\epsilon > 0$  (however small),  $\exists$  a  $\text{fix integer } k$   
 $\text{c } k \text{ depending on } \epsilon$ , i.e.,  $k(\epsilon)$  s.t.  $|x_n - x| < \epsilon$  for  
 all  $n \geq k$ . Here the real number  $x$  is limit of the sequence  $(x_n)$ .

\* Divergence of a sequences — Let  $(x_n)$  be a sequence,  
 If  $\lim_{n \rightarrow \infty} x_n = +\infty$  (i.e.)  $\forall$  then the sequence  $(x_n)$   
 is called dgt sequence.

(ii) A sequence  $(x_n)$  has no limit then the sequence  
 is called dgt sequence.

(iii) A sequence  $(x_n)$  is said to dgs to  $+\infty$ .  
 If given any fix real number  $k$  (however large)  
 $\exists$  a  $\text{fix integer } m$  (depending on  $k$ ) s.t.  
 $x_n > k \text{ for } n \geq m$ .

i.e., If  $x_n = +\infty$  (i.e.)  $x_n \rightarrow \infty$  as  $n \rightarrow \infty$ .

(iv) A sequence  $(x_n)$  is said to dgs to  $-\infty$ . If given any  
 fix real number  $k$  (however large)  $\exists$  a  $\text{fix integer}$   
 $m$  (fixing  $m$  — i.e.  $x_n < k$  for  $n \geq m$ ).

$n \rightarrow \infty$

i.e.,  $a_n \rightarrow \infty$  as  $n \rightarrow \infty$

Oscillatory sequence - if a sequence  $(a_n)$

neither goes to a finite number nor diverges to  $+\infty$  (or)  $-\infty$ , then the sequence  $(a_n)$  is called an oscillatory sequence.

→ If the oscillatory sequence is bdd then the sequence is called finite oscillatory sequence.

→ If the oscillatory sequence is unbdd then the sequence is called an infinite oscillatory seq.

Ex: (1)  $(a_n) = (\frac{1}{n}) = (1, \frac{1}{2}, \frac{1}{3}, \dots)$

$\therefore U.B = 1; L.B = 0$

$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{1}{n} = 0$

∴  $(a_n)$  is cgt.

(2)  $(a_n) = \frac{1}{n}$

(3)  $(a_n) = n^2$

(4)  $(a_n) = -n$

(5)  $(a_n) = (-1)^n = (-1, +1, -1, +1, \dots)$

$L.B = -1; U.B = +1$

$\lim_{n \rightarrow \infty} a_n = -1 \text{ if } n \text{ is odd}$

$= +1 \text{ if } n \text{ is even}$

$(a_n)$  is neither cgt nor dgt.

If it is oscillatory sequence and it is bdd see.

finite oscillatory sequence.

(6)  $(a_n) = ((-1)^n \cdot n) = (-1, +2, -3, +4, \dots)$

$U.B = \text{not defined}; L.B = \text{not defined}$ .

$\lim_{n \rightarrow \infty} a_n = +\infty \text{ if } n \text{ is even}$

$= -\infty \text{ if } n \text{ is odd}$

∴ It is neither cgt nor dgt.

i. If it is oscillatory sequence and it is unbdd.

ii. If it is finite oscillatory sequence.

Null sequence - a sequence  $(a_n)$  is said to be an null sequence

if either  $a_n = 0$  for all  $n$  or  $\lim_{n \rightarrow \infty} a_n = 0$  then  $(a_n)$  is called a null sequence.

i.e., a sequence cannot converge to more than one limit.

Proof : If possible let a sequence  $(a_n)$  converge to two distinct limits  $x^1 \neq x^2$ .

since  $x^1 \neq x^2 \Rightarrow |x^1 - x^2| > 0$ .  
let  $\epsilon = \frac{1}{2} |x^1 - x^2|$

since the sequence  $(a_n)$  cgs to  $x^1$ .

Given  $\epsilon > 0$ ,  $\exists$  a +ve integer  $k^1$  (depending on  $\epsilon$ )  
 $|a_n - x^1| < \epsilon/2 \forall n \geq k^1$ .

and also the sequence  $(a_n)$  cgs to  $x^2$ .

Given  $\epsilon > 0$ ,  $\exists$  a +ve integer  $k^2$  (depending on  $\epsilon$ )

$$|a_n - x^2| < \epsilon/2 \forall n \geq k^2$$

$$\text{Let } K = \max \{k^1, k^2\}$$

$$\text{then } |a_n - x^1| < \epsilon/2 \text{ & } |a_n - x^2| < \epsilon/2 \quad \forall n \geq K$$

$$\begin{aligned} |a^1 - a^2| &= |(a^1 - a_K) + (a_K - a_n) + (a_n - a^2)| \\ &\leq |a^1 - a_K| + |a_K - a_n| + |a_n - a^2| \\ &< \epsilon/2 + \epsilon/2 = \epsilon \end{aligned}$$

$$\therefore |a^1 - a^2| < \epsilon \quad \forall n \geq K$$

which is a contradiction to  $\epsilon = \frac{1}{2} |x^1 - x^2|$

our assumption that a sequence cgs to two distinct limits  $x^1, x^2$  is wrong.

$$\therefore x^1 = x^2$$

Theorem Every cgt sequence is bdd.

Pf: Let  $X = (a_n)$  be a cgt sequence.  
it cgs to  $x$  (say).

Given  $\epsilon > 0$ ,  $\exists$  a natural number  $K$  s.t.  $|a_n - x| < \epsilon$   $\forall n \geq K$

$$\leq |a_{n-1}| + |a_n|$$

$$\leq L + \epsilon.$$

$$\text{Let } M = \sup \{ |a_1|, |a_2|, \dots, |a_{k-1}|, \epsilon + |a_k| \}$$

$$\therefore |a_n| \leq M \quad \forall n \in N$$

$\therefore (a_n)$  is bdd.

Note: The converse of above theorem need not be true.

i.e., Every bdd sequence need not be cgt.

$$\text{Ex: } (a_n) = ((-1)^n) \text{ is bdd.}$$

but  $a_n = -1$  if  $n$  is odd

$= +1$  if  $n$  is even.

$\therefore$  It is an oscillatory sequence.

→ Use the defn. of the limit of a sequence to prove

$$\lim \left( \frac{1}{n} \right) = 0$$

Sol: Given  $\epsilon > 0$

$$\text{we have } |\frac{1}{n} - 0| = \frac{1}{n} \quad \text{--- (1)}$$

for given  $\epsilon > 0$ , by Archimedean Property  $\exists k \in \mathbb{N}$  such that

if  $k \in \mathbb{N}$ ,

$$\Rightarrow \frac{1}{k} < \epsilon \quad \text{--- (2)}$$

Now we have,  $\forall n \geq k \Rightarrow \frac{1}{n} \leq \frac{1}{k}$ .

$$\Rightarrow \frac{1}{n} \leq \frac{1}{k} < \epsilon \quad (\text{by (2)}) \quad \text{--- (3)}$$

$$\therefore (0 \leq |\frac{1}{n} - 0| = \frac{1}{n} \leq \epsilon \quad (\text{by (3)}) \quad \forall n \geq k)$$

$\therefore |\frac{1}{n} - 0| < \epsilon \quad \forall n \geq k$

i.e.,  $\frac{1}{n} \rightarrow 0$  as  $n \rightarrow \infty$ . i.e.,  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$

$$\begin{array}{lllll} \text{(1)} \lim \frac{1}{n+1} = 0 & \text{(2)} \lim \frac{2n}{n+1} = 2 & \text{(3)} \lim \frac{n-1}{n+3} = 1 & \text{(4)} \lim \frac{2n}{n+2} = 2 & \text{(5)} \lim \frac{(-1)^n n}{n+1} = 0 \\ \text{(6)} \lim \left( \frac{n}{n+1} \right) = 0 & \text{(7)} \lim \frac{3n+1}{2n+5} = \frac{3}{2} & \text{(8)} \lim \frac{1}{n+7} = 0 & \text{(9)} \lim \frac{5}{n+1} = 0 & \text{(10) for any b.c. } \lim \frac{b}{n+1} = 0 \\ \text{P.T. } \lim \frac{b}{n+1} = 0 & & & & \end{array}$$

and let  $x \in \mathbb{R}$ . If  $(a_n)$  is a sequence of real numbers with  $\lim_{n \rightarrow \infty} a_n = 0$  and if for some constant  $c > 0$  and some  $m \in \mathbb{N}$ , we have  $|x_n - x| \leq c a_n$  for all  $n \geq m$ , then it follows that  $\lim_{n \rightarrow \infty} x_n = x$ .

Proof: Since  $\lim_{n \rightarrow \infty} a_n = 0$

i.e.,  $a_n \rightarrow 0$  as  $n \rightarrow \infty$

Given  $\epsilon > 0$ ,  $\exists K \in \mathbb{N}$  s.t.  $\forall n \geq K$

$|a_n - 0| < \frac{\epsilon}{c}$

$|a_n| < \frac{\epsilon}{c}$

$|a_n| < \frac{\epsilon}{c} \quad \text{--- (1)}$

Since  $|x_n - x| \leq c a_n$

i.e.,  $\forall n \geq m \Rightarrow |x_n - x| \leq c a_n < c \left( \frac{\epsilon}{c} \right) \rightarrow$   
 $|x_n - x| < \epsilon \quad (\text{from (1)})$

∴ we have  $\forall n \geq m \Rightarrow |x_n - x| < \epsilon$ .

i.e.,  $x_n \rightarrow x$  as  $n \rightarrow \infty$

i.e.,  $a_n \rightarrow 0$  as  $n \rightarrow \infty$

Corresponding Bernoulli's Inequality  
 problem in pg 283  
 If  $n > 1$  then  $(1+x)^n \geq 1+nx \quad \forall n \in \mathbb{N}$

Theorem:

(a) Let  $x = (x_n)$  and  $y = (y_n)$  be sequences of real numbers that converge to  $x$  and  $y$  respectively and  $c \in \mathbb{R}$  then the sequences  $x+y$ ,  $xy$ ,  $x/y$  and  $cx$  converge to  $x+y$ ,  $xy$ ,  $x/y$  and  $cx$  respectively.

(b) If  $x = (x_n)$  converges to  $x$  and  $z = (z_n)$  is a sequence of non-zero real numbers that converges to  $z$  and if  $z \neq 0$  then the quotient sequence  $x/z$  converges to  $x/z$ .

$\rightarrow$  If  $a_1, a_2, \dots, a_n$  are cgt sequences

then  $a + b + \dots + z = (a_1 + b_1 + \dots + z_1) + \dots + (a_n + b_n + \dots + z_n)$  is also cgt  
and  $Lt(a_1 + b_1 + \dots + z_n) = Lt a_1 + Lt b_1 + \dots + Lt z_n$ .

$\rightarrow$  (2)  $A \cdot B = (a_1 b_1 \dots z_n)$  is cgt sequence  
and  $Lt(a_1 b_1 \dots z_n) = Lt a_1 \cdot Lt b_1 \dots Lt z_n$ .

$\rightarrow$  (3) If  $x = (x_n)$  is a cgt sequence then

$$Lt x = \underline{\underline{(Lt(x_n))}}$$

Theorem : If  $x = (x_n)$  is cgt to  $x$  and if  $x_n \geq 0$  then  
then  $x = Lt x \geq 0$  (i.e.,  $x \geq 0$ )

Proof : If possible suppose that  $x < 0$ .

Since the sequence  $(x_n)$  cgt to  $x$ .

$\therefore \exists k \in \mathbb{N}$  s.t.  $\forall n \geq k$ ,

$$\Rightarrow x - \epsilon < x_n < x + \epsilon \quad \forall n \geq k.$$

$$\text{Taking } \epsilon = -x/2 > 0 \quad (x < 0)$$

$$\therefore x + \frac{x}{2} < x_n < x - \frac{x}{2} \quad \forall n \geq k.$$

$$x_n < \frac{x}{2} < 0 \quad \forall n \geq k.$$

$\therefore$  But which is contradiction to the hypothesis

that  $x_n \geq 0 \quad \forall n$ .

$\therefore$  Our supposition that  $x < 0$  is wrong.

$$\therefore x \geq 0.$$

~~Theorem~~ If  $x = (x_n)$  and  $y = (y_n)$  are cgt and if

$x_n \leq y_n \quad \forall n$  then  $Lt x \leq Lt y$ .

Proof : Since  $(x_n)$  &  $(y_n)$  cgt sequences  
and converge to  $x$  &  $y$  (say).

$$\therefore Lt x = x ; Lt y = y$$

Let  $z_n = y_n - x_n$ ; then  $z_n \geq 0 \quad \forall n$  ( $\because y_n \geq x_n$ ).

NOW  $\frac{Lt z_n}{n} = Lt y_n - Lt x_n \Rightarrow y - x \geq 0 \Rightarrow y \geq x$   
 $\Rightarrow z_n \geq 0 \Rightarrow Lt x \leq Lt y$ .

Theorem :- If  $x = (x_n)$  is a convergent sequence and if  $a \leq x_n \leq b \forall n \in \mathbb{N}$ , then  $a \leq \lim x_n \leq b$ .

Proof :- Let  $y_n = b - x_n$  then

$$y_n \geq 0 \forall n (\because b \geq x_n)$$

$$\begin{aligned} \therefore \lim y_n &= \lim (b - x_n) \\ &= b - \lim x_n \\ \Rightarrow b - \lim x_n &\geq 0 \quad (\because y_n \geq 0) \\ \Rightarrow b &\geq \lim x_n \\ \Rightarrow \lim x_n &\leq b \end{aligned}$$

Similarly  $\lim x_n \geq a$  (Let  $y_n = x_n - a$ )

$$\therefore a \leq \lim x_n \leq b$$

### Squeeze theorem

Suppose that  $x = (x_n)$ ,  $y = (y_n)$  and  $z = (z_n)$  are sequences of real numbers such that  $x_n \leq y_n \leq z_n \forall n \in \mathbb{N}$  and that  $\lim z_n = \lim x_n = \omega$  then  $y = (y_n)$  is convergent and

$$\lim x_n = \lim y_n = \lim z_n = \omega.$$

Proof :- Let  $\lim x_n = \lim z_n = \omega$

i.e. the sequences  $(x_n)$  &  $(z_n)$  are convergent to  $\omega$ .

$\therefore$  Given  $\epsilon > 0$ ,  $\exists k \in \mathbb{N}$  such that

$$n \geq k \Rightarrow |x_n - \omega| < \epsilon; |z_n - \omega| < \epsilon$$

$$n \geq k \Rightarrow \omega - \epsilon < x_n < \omega + \epsilon$$

$$\text{and } \omega - \epsilon < z_n < \omega + \epsilon$$

Since  $x_n \leq y_n \leq z_n \forall n \in \mathbb{N}$ .

$\therefore$  we have  $\omega - \epsilon < x_n \leq y_n \leq z_n < \omega + \epsilon \forall n \geq k$

$$\Rightarrow \omega - \epsilon < y_n < \omega + \epsilon \quad \forall n \geq k$$

$$\Rightarrow |y_n - \omega| < \epsilon \quad \forall n \geq k$$

$$\therefore \lim y_n = \omega \quad n \rightarrow \infty$$

$\therefore (y_n)$  converges to  $\omega$ .

and also  $\lim x_n = \lim y_n = \lim z_n$ .

Theorem :- Let the sequence  $x = (x_n)$  converge to  $\omega$  then the sequence  $(|x_n|)$  of absolute values converges to  $|\omega|$ . i.e. if  $\lim x_n = \omega$  then  $\lim (|x_n|) = |\omega|$ .

Proof :- Since  $x = (x_n)$  is convergent to  $\omega$ .

$\therefore$  Given  $\epsilon > 0$ ,  $\exists k \in \mathbb{N}$  such that

$$|x_n - \omega| < \epsilon \quad \forall n \geq k$$

Now we have

$$|(|x_n| - |\omega|)| \leq |x_n - \omega| < \epsilon \quad \forall n \geq k$$

$$\therefore |x_n| - |\omega| < \epsilon \quad \forall n \geq k$$

$\therefore |x_n|$  converges to  $|\omega|$ .

### Converse part

$$\text{Ex} : (x_n) = (-1)^n \quad \forall n \in \mathbb{N}$$

Theorem :- Let  $(x_n)$  be a sequence of real numbers such that

$$\lim_{n \rightarrow \infty} \left( \frac{|x_n|}{x_n} \right) \text{ exists.}$$

If  $|L| < 1$  then  $(x_n)$  converges and

$$\lim_{n \rightarrow \infty} (x_n) = 0$$

Proof :- Since  $x_n > 0 \forall n$

$$\Rightarrow x_{n+1} < x_n$$

$$\therefore \frac{x_{n+1}}{x_n} > 0 \quad \forall n$$

$$\therefore \lim_{n \rightarrow \infty} \left( \frac{x_{n+1}}{x_n} \right) = L \geq 0$$

i.e.  $L \geq 0$ .

$$\text{Since } \lim_{n \rightarrow \infty} \frac{x_{n+1}}{x_n} = L$$

$\therefore$  Given  $\epsilon > 0$ ,  $\exists k \in \mathbb{N}$  such that

$$\left| \frac{x_{n+1}}{x_n} - L \right| < \epsilon \quad \forall n \geq k.$$

$$\Rightarrow -\epsilon < \frac{x_{n+1}}{x_n} - L < \epsilon \quad \forall n \geq k$$

$$\Rightarrow L - \epsilon < \frac{x_{n+1}}{x_n} < L + \epsilon \quad \forall n \geq k \quad \text{--- (1)}$$

Now replacing  $n$  by  $k, k+1, k+2, \dots, n-1$  in (1) we get

$$L - \epsilon < \frac{x_{k+1}}{x_k} < L + \epsilon$$

$$L - \epsilon < \frac{x_{k+2}}{x_{k+1}} < L + \epsilon$$

$$L - \epsilon < \frac{x_{k+3}}{x_{k+2}} < L + \epsilon$$

$$L - \epsilon < \frac{x_n}{x_{n-1}} < L + \epsilon$$

Now multiplying the above  $(n-k)$  inequalities, we have

$$(L - \epsilon)^{n-k} < \frac{x_n}{x_k} < (L + \epsilon)^{n-k} \quad \text{--- (2)}$$

Since  $L < 1$ :

$$\therefore 0 \leq L < L + \epsilon < 1$$

$$\therefore 0 < L + \epsilon < 1 \quad \text{--- (3)}$$

$\therefore$  From (2), we have

$$\frac{x_n}{x_k} < (L + \epsilon)^{n-k}$$

$$\Rightarrow x_n < x_k (L + \epsilon)^{n-k}$$

$$\Rightarrow x_n < x_k (L + \epsilon)^{n-1} \cdot \frac{1}{(L + \epsilon)^k}$$

since  $x_n > 0 \quad \forall n$

$$\therefore 0 < x_n < x_k (L + \epsilon)^{n-1} \cdot \frac{1}{(L + \epsilon)^k} \quad \text{--- (4)}$$

$$\text{Let } m = x_k \cdot \frac{1}{(L + \epsilon)^k} > 0$$

$$\therefore 0 < x_n < m(L + \epsilon)^{n-1} \quad (\text{by (4)}) \quad \text{--- (5)}$$

since  $0 < L + \epsilon < 1$  (by (3))

$$\therefore \lim_{n \rightarrow \infty} (L + \epsilon)^{n-1} = 0$$

since the equ (5) is of the form  $y_n < x_n < z_n \quad \forall n$

with  $\lim y_n = \lim z_n = 0$ .

By squeeze theorem,  $\lim x_n = 0$  and  $(x_n)$  is convergent to zero.

$$\left[ \frac{x_n}{x_k} = x_{n-k} \right]$$

$$70 \geq 50$$

$$n \geq k.$$

$$n = 50, 50+1, 50+2, \dots, 69 \\ k, k+1, k+2, \dots, n-1$$

$$69 - 50 = 20 \text{ terms}$$

$$n-k = 20$$

### Problems

→ Apply above theorem

(i.e. let  $(x_n)$  be a sequence of the real numbers such that

$$L = \lim_{n \rightarrow \infty} \left( \frac{x_{n+1}}{x_n} \right) \text{ exists. If } L < 1,$$

then  $(x_n)$  converges and

$\lim_{n \rightarrow \infty} (x_n) = 0$  to the following

sequences, where  $a, b$  satisfy  $0 < a < 1, -b > 1$ .

$$(a) \left( \frac{n}{b^n} \right), (b) \left( \frac{2^{3n}}{3^{2n}} \right), (c) \left( \frac{b^n}{2^n} \right)$$

$$\text{Sol'n:- } a) \left( \frac{n}{b^n} \right)$$

$$\text{Let } x_n = \frac{n}{b^n} \text{ then } x_{n+1} = \frac{n+1}{b^{n+1}}$$

$$\begin{aligned} \text{Now } \frac{x_{n+1}}{x_n} &= \frac{n+1}{b^{n+1}} \times \frac{b^n}{n} \\ &= \frac{n+1}{b^n} \\ &= \frac{1+b}{b} \end{aligned}$$

$$\lim \left( \frac{x_{n+1}}{x_n} \right) = \lim \left( \frac{1+b}{b} \right)$$

$$= \frac{1+0}{b} = \frac{1}{b} < 1$$

( $\because b > 1$ )

$$\therefore \lim \frac{x_{n+1}}{x_n} = \frac{1}{b} < 1. \text{ (i.e. } L < 1\text{)}$$

$\therefore (x_n)$  converges &  $\lim x_n = 0$ .

$$(b) \left( \frac{2^{3n}}{3^{2n}} \right)$$

$$\text{Let } x_n = \frac{2^{3n}}{3^{2n}} \text{ then } x_{n+1} = \frac{2^{3n+3}}{3^{2n+2}}$$

$$\text{Now } \lim_{n \rightarrow \infty} \left( \frac{x_{n+1}}{x_n} \right) = \lim_{n \rightarrow \infty} \left[ \frac{2^{3n+3}}{3^{2n+2}} \times \frac{3^{2n}}{2^{3n}} \right]$$

$$= \lim_{n \rightarrow \infty} \left( \frac{2^3}{3^2} \right) = \frac{8}{9} < 1$$

$\therefore (x_n)$  is convergent &  $\lim x_n = c$

$$(c) \left( \frac{b^n}{2^n} \right)$$

$$\text{Let } x_n = \frac{b^n}{2^n} \text{ then } x_{n+1} = \frac{b^{n+1}}{2^{n+1}}$$

$$\begin{aligned} \text{Now } \frac{x_{n+1}}{x_n} &= \frac{b^{n+1}}{2^{n+1}} \cdot \frac{2^n}{b^n} \\ &= b/2 \end{aligned}$$

$$\therefore \lim \left( \frac{x_{n+1}}{x_n} \right) = b/2$$

If  $1 < b < 2$  then  $\lim \left( \frac{x_{n+1}}{x_n} \right) = b/2 <$

$\therefore (x_n)$  is convergent &  $\lim x_n = 0$ .

If  $b > 2$  then  $\lim \left( \frac{x_{n+1}}{x_n} \right) = b/2 > 1$

$\therefore (x_n)$  is not convergent &  $\lim x_n \neq$

PROOF

\* Cauchy's First theorem

limits

If  $\{x_n\}$  converges to  $l$  then

converges  $\{x_n\}$  where  $x_n = a_n + l$

also converges to  $l$  (i.e.)

$\lim_{n \rightarrow \infty} (a_n + l) = l$  or  $\lim_{n \rightarrow \infty} a_n + \lim_{n \rightarrow \infty} l = l$

PROOF :- Let  $b_n = a_n - l$  then

$$\lim_{n \rightarrow \infty} b_n = \lim_{n \rightarrow \infty} a_n - l$$

$$= l - l \\ = 0$$

$$\therefore \lim_{n \rightarrow \infty} b_n = 0$$

$n \rightarrow \infty$

i.e.  $b_n \rightarrow 0$  as  $n \rightarrow \infty$  — (1)

Since  $a_n = b_n + l \forall n$

$$\therefore a_n = \frac{(b_1 + l) + (b_2 + l) + \dots + (b_n + l)}{n}$$

$$= \frac{(b_1 + b_2 + \dots + b_n) + nl}{n}$$

$$= \frac{b_1 + b_2 + \dots + b_n}{n} + l$$

In order to prove that  $a_n \rightarrow l$ ,

For this we are enough to show that

$$\frac{b_1 + b_2 + \dots + b_n}{n} \rightarrow 0. \quad (2)$$

From (1),  $\lim b_n = 0$  i.e.  $b_n \rightarrow 0$  as  $n \rightarrow \infty$ .

$\therefore$  Given  $\epsilon > 0, \exists m \in \mathbb{N}$  such that

$$|b_n - 0| < \epsilon/2 \quad \forall n \geq m.$$

$$\Rightarrow |b_n| < \epsilon/2 \quad \forall n \geq m. \quad (3)$$

Also the sequence  $\{b_n\}$  is convergent

$\therefore \{b_n\}$  is bounded.

$\therefore \exists M > 0$  such that  $|b_n| \leq M \forall n$ .

— (4)

Now let us prove (2).

We have

$$\begin{aligned} \left| \frac{b_1 + b_2 + \dots + b_n}{n} - 0 \right| &= \left| \frac{b_1 + b_2 + \dots + b_n}{n} \right| \\ &= \frac{|b_1 + b_2 + \dots + b_n|}{|n|} \end{aligned}$$

$$= \frac{1}{n} \left| b_1 + b_2 + \dots + b_m + b_{m+1} + b_{m+2} + \dots + b_n \right|$$

( $\because n \in \mathbb{N} \Rightarrow |n| = n$ )

$$\leq \frac{1}{n} \left[ (|b_1| + |b_2| + \dots + |b_m|) + (|b_{m+1}| + |b_{m+2}| + \dots + |b_n|) \right]$$

$$< \frac{1}{n} \left[ (M + M + \dots + M \text{ (m times)}) + (\epsilon/2 + \epsilon/2 + \dots + \epsilon/2 \text{ (n-m times)}) \right].$$

$\forall n \geq m$ .

(using (3) & (4))

$$\left| \frac{b_1 + b_2 + \dots + b_n}{n} - 0 \right| < \frac{1}{n} [mM + (n-m)\epsilon/2] \quad \forall n \geq m$$

$$\left| \frac{b_1 + b_2 + \dots + b_n}{n} - 0 \right| < \frac{mM}{n} + \frac{(n-m)\epsilon}{2n} \quad \forall n \geq m$$

$$< \frac{mM}{n} + \frac{\epsilon}{2}$$

$$\left( \because \frac{n-m}{n} = 1 - \frac{m}{n} < 1 \right) \forall n \geq m \quad (5)$$

$$\text{Now } \frac{mM}{n} < \frac{\epsilon}{2} \text{ if } \frac{n}{mM} > \frac{2}{\epsilon}$$

$$\therefore \Rightarrow \text{if } n > \frac{2mM}{\epsilon}$$

If  $P$  is a natural number  $> \frac{2mM}{\epsilon}$

then  $n \geq P$ .

$$\text{Let } q = \max \{P, m\}$$

$\therefore$  From (5),

$$\left| \frac{b_1 + b_2 + \dots + b_n}{n} - 0 \right| < \frac{\epsilon}{2} + \frac{\epsilon}{2} \quad \forall n \geq q \\ = \epsilon.$$

$$\left| \frac{b_1 + b_2 + \dots + b_n}{n} - 0 \right| < \epsilon \quad \forall n \geq q.$$

$$\frac{b_1 + b_2 + \dots + b_n}{n} \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$x_n \rightarrow l \text{ as } n \rightarrow \infty$$

$$\text{i.e. } \lim_{n \rightarrow \infty} x_n = l$$

Hence the theorem.

Note:— The converse of the above theorem need not be true.

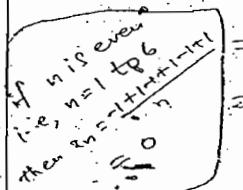
i.e. If the sequence  $\{x_n\}$  converges to  $l$  then the sequence  $\{a_n\}$  need not converge to  $l$ .

$$\text{where } x_n = \frac{a_1 + a_2 + \dots + a_n}{n}$$

$$\text{Ex:— Let } \{a_n\} = \{(-1)^n\}$$

$$a = \{-1, +1, -1, +1, \dots\}$$

$$\text{then } x_n = \frac{a_1 + a_2 + \dots + a_n}{n}$$



$$\therefore \lim_{n \rightarrow \infty} x_n = 0 \text{ i.e. the sequence } \{x_n\}$$

Convergent to 0.

But  $\{a_n\}$  is not convergent.

$$\text{because } \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} (-1)^n = +1 \text{ if } n \text{ is even.}$$

$$= -1 \text{ if } n \text{ is odd.}$$

$\therefore \{a_n\}$  is oscillatory sequence.

It is not convergent.

LEMMA 3:

$$\rightarrow \text{show that } \lim_{n \rightarrow \infty} \frac{1}{n} (1 + \frac{1}{2} + \dots + \frac{1}{n}) = 1$$

$$\text{Sol:— Let } a_n = \frac{1}{n} \text{ then } \lim_{n \rightarrow \infty} a_n = 0$$

∴ By Cauchy's first theorem on lim

$$\lim_{n \rightarrow \infty} \frac{a_1 + a_2 + \dots + a_n}{n} = 0$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{1 + \frac{1}{2} + \dots + \frac{1}{n}}{n} = 0$$

→ show that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \left( \frac{2}{1} + \frac{3}{2} + \dots + \frac{n+1}{n} \right) = 1$$

$$\text{Let } a_n = \frac{n+1}{n} \text{ then } \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} \frac{n+1}{n} = 1$$

∴ By Cauchy's first theorem on lim

$$\lim_{n \rightarrow \infty} \frac{a_1 + a_2 + \dots + a_n}{n} = 1$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{\frac{2}{1} + \frac{3}{2} + \dots + \frac{n+1}{n}}{n} = 1$$

→ show that

$$\lim_{n \rightarrow \infty} \left[ \frac{1}{\sqrt{1+n^2}} + \frac{1}{\sqrt{1+2/n^2}} + \dots + \frac{1}{\sqrt{1+(n-1)/n^2}} \right] = 1$$

$$\text{L.H.S. } \lim_{n \rightarrow \infty} \left[ \frac{1}{\sqrt{1+n^2}} + \frac{1}{\sqrt{1+2/n^2}} + \dots + \frac{1}{\sqrt{1+(n-1)/n^2}} \right]$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \left[ \frac{1}{\sqrt{1+1/n^2}} + \frac{1}{\sqrt{1+2/n^2}} + \dots + \frac{1}{\sqrt{1+(n-1)/n^2}} \right]$$

$$= \lim_{n \rightarrow \infty} \frac{1}{n} \left[ \frac{1}{\sqrt{1+1/n^2}} + \frac{1}{\sqrt{1+2/n^2}} + \dots + \frac{1}{\sqrt{1+1}} \right]$$

$$\lim_{n \rightarrow \infty} a_n = \frac{1}{\sqrt{1+\frac{1}{n}}} \quad \text{as } n \rightarrow \infty$$

∴ By Cauchy's first theorem on limits

$$\lim_{n \rightarrow \infty} \frac{a_1 + a_2 + \dots + a_n}{n} = l$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{\left[ \frac{1}{\sqrt{1+\frac{1}{1}}}, \frac{1}{\sqrt{1+\frac{1}{2}}}, \dots, \frac{1}{\sqrt{1+\frac{1}{n}}} \right]}{n} = l$$

$$\text{i.e. } \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n^2+1}} + \frac{1}{\sqrt{n^2+2}} + \dots + \frac{1}{\sqrt{n^2+n}} = l$$

→ show that

$$\lim_{n \rightarrow \infty} \left[ \frac{1}{\sqrt{n}}, \frac{1}{\sqrt{n+1}}, \dots, \frac{1}{\sqrt{2n}} \right] = \infty$$

$$\text{L.H.S. } \lim_{n \rightarrow \infty} \frac{1}{n} \left[ \frac{n}{\sqrt{n}}, \frac{n}{\sqrt{n+1}}, \dots, \frac{n}{\sqrt{2n}} \right] \quad (i)$$

$$\text{Let } a_n = \frac{n}{\sqrt{2n}} \text{ then } a_n = \frac{1}{\sqrt{2}} \sqrt{n}$$

$$\therefore \lim_{n \rightarrow \infty} a_n = \infty$$

∴ By Cauchy's first theorem on limits

$$\lim_{n \rightarrow \infty} \left[ \frac{n}{\sqrt{n}}, \frac{n}{\sqrt{n+1}}, \dots, \frac{n}{\sqrt{2n}} \right] = \infty$$

$$\text{i.e. } \lim_{n \rightarrow \infty} \left[ \frac{1}{\sqrt{n}}, \dots, \frac{1}{\sqrt{2n}} \right] = \infty$$

→ show that

$$\lim_{n \rightarrow \infty} \left[ \frac{1}{n^2} + \frac{1}{(n+1)^2} + \dots + \frac{1}{(2n)^2} \right] = 0$$

If  $\{a_n\}$  is a sequence of five terms for all  $n$  and if  $a_n = l$  then  $\lim_{n \rightarrow \infty} (a_1, a_2, \dots, a_n) = l$

$$\lim_{n \rightarrow \infty} (a_1, a_2, \dots, a_n) = l$$

Proof — Let  $b_n = \log a_n \forall n (\because a_n > 0)$

$$\text{Since } \lim_{n \rightarrow \infty} a_n = l$$

$$\therefore \lim_{n \rightarrow \infty} b_n = \log l (\because l > 0 \text{ because } a_n > 0 \forall n)$$

∴ By Cauchy's first theorem on limits.

$$\lim_{n \rightarrow \infty} \frac{b_1 + b_2 + \dots + b_n}{n} = \log l$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{\log a_1 + \log a_2 + \dots + \log a_n}{n} = \log l \quad (\text{by (i)})$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{\log [a_1 a_2 \dots a_n]}{n} = \log l$$

$$\Rightarrow \lim_{n \rightarrow \infty} \log (a_1 \dots a_n)^{1/n} = \log l$$

$$\Rightarrow \log \left[ \lim_{n \rightarrow \infty} (a_1 a_2 \dots a_n)^{1/n} \right] = \log l$$

$$\Rightarrow \lim_{n \rightarrow \infty} (a_1 a_2 \dots a_n)^{1/n} = l$$

Theorem — If  $\{a_n\}$  is a sequence such that

$a_n > 0 \forall n$  and  $\lim_{n \rightarrow \infty} a_n = l$

then  $\lim_{n \rightarrow \infty} (a_n)^{1/n} = l$ .

Proof — Let us define the

sequence  $\{b_n\}$  such that

$$b_1 = a_1, b_2 = \frac{a_2}{a_1}, b_3 = \frac{a_3}{a_2}, \dots, b_n = \frac{a_n}{a_{n-1}}$$

$$\text{Now } \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = l$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{a_n}{a_{n-1}} = l \Rightarrow \lim_{n \rightarrow \infty} b_n = l \quad \text{①}$$

Since  $a_n > 0 \forall n$

$$\therefore b_n > 0 \forall n$$

Now we have a sequence  $\{b_n\}$

such that  $b_n > 0 \forall n$  and  $\lim_{n \rightarrow \infty} b_n = l$

$$\therefore \lim_{n \rightarrow \infty} (b_1 \cdot b_2 \cdots b_n)^{1/n} = l \quad (\text{By previous theorem})$$

$$\Rightarrow \lim_{n \rightarrow \infty} (a_n)^{1/n} = l$$

Note :- the converse of above theorem need not be true.

Ex:- Let  $a_n = 2^{-n} + (-1)^n$

$$\text{then } a_n^{1/n} = 2^{-1} + (-1)^{n/2}$$

$$\therefore \lim_{n \rightarrow \infty} a_n^{1/n} = 2^{-1} + \lim_{n \rightarrow \infty} \frac{(-1)^{n/2}}{n} = 0$$

$$= 1/2$$

$$\text{But } \frac{a_{n+1}}{a_n} = \frac{2^{-(n+1)} + (-1)^{n+1}}{2^{-n} + (-1)^n}$$

$$= 2^{-1} + (-1)^{n+1} - (-1)^n$$

$$\text{Now } \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 2^{-1} - 1 = -1/2 \text{ if } n \text{ is even.}$$

$$= 2^{-1} + 1 = 1/2 \text{ if } n \text{ is odd}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 1/2$$

= 2 if n is odd.

∴  $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n}$  does not exist.

Note :- The above theorem known as Cauchy's second theorem on limit

Problems

→ Show that  $\{n^{1/n}\}$  converges to 1.

Sol'n: Let  $a_n = n$ , then

$$\frac{a_{n+1}}{a_n} = \frac{(n+1)^{1/(n+1)}}{n^{1/n}}$$

$$= (1 + 1/n)^{1/(n+1)}$$

$$\text{Now } \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = 1$$

By Cauchy's second theorem on limits.

$$\lim_{n \rightarrow \infty} a_n^{1/n} = 1$$

→ Show that

$$\frac{1}{n} (1 + 2^{1/2} + 3^{1/3} + \cdots + n^{1/n}) = 1$$

First theorem.

→ Find  $\lim_{n \rightarrow \infty} (n!)^{1/n}$

Let  $a_n = n!$  then  $\frac{a_{n+1}}{a_n} = (n+1)!$

$$\text{Now } \frac{a_{n+1}}{a_n} = n+1$$

$$\therefore \lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = \infty$$

By Cauchy's second theorem on limits.

$$\lim_{n \rightarrow \infty} (a_n)^{1/n} = \infty$$

$$\text{i.e. } \lim_{n \rightarrow \infty} (n!)^{1/n} = \infty$$

$$n^n \quad n \rightarrow \infty$$

Sol'n :- Now  $x_n = \frac{n!}{n^n}$

$$\Rightarrow x_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$$

Now  $\frac{x_{n+1}}{x_n} = \frac{n! \times (n+1)^{n+1}}{n^n \times (n+1)!}$

$$= \frac{(n+1)^n}{n^n}$$

$$= \left(1 + \frac{1}{n}\right)^n$$

$$\therefore \lim_{n \rightarrow \infty} \frac{x_{n+1}}{x_n} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = e$$

$$\Rightarrow \lim_{n \rightarrow \infty} \frac{x_{n+1}}{x_n} = \frac{1}{e} < 1$$

$$\lim x_n = 0$$

$$\Rightarrow \text{If } x_n = \left[ \left(\frac{2}{1}\right) \left(\frac{3}{2}\right)^2 \left(\frac{4}{3}\right)^3 \cdots \left(\frac{n+1}{n}\right)^n \right]$$

then show that  $\lim_{n \rightarrow \infty} x_n = e$ .

$$\text{Let } a_n = \left(\frac{2}{1}\right) \left(\frac{3}{2}\right)^2 \left(\frac{4}{3}\right)^3 \cdots \left(\frac{n+1}{n}\right)^n$$

$$\text{then } a_{n+1} = \left(\frac{2}{1}\right) \left(\frac{3}{2}\right) \left(\frac{4}{3}\right)^3 \cdots \left(\frac{n+2}{n+1}\right)^{n+1}$$

$$\text{Now } \frac{a_{n+1}}{a_n} = \left(\frac{n+2}{n+1}\right)^{n+1}$$

$$= \left(1 + \frac{1}{n+1}\right)^{n+1}$$

$$\therefore \lim \frac{a_{n+1}}{a_n} = e$$

Cauchy's second theorem on limits.

$$\lim_{n \rightarrow \infty} (a_n)^n = e$$

i.e.  $\lim x_n = e$

$$\xrightarrow{\text{H.W.}} \text{Prove that } \lim_{n \rightarrow \infty} \left(\frac{n^n}{n!}\right)^{1/n} = e$$

(Cauchy's second theorem)

$$\xrightarrow{\text{H.W.}} \text{Show that } \lim_{n \rightarrow \infty} \left(\frac{n!}{n^n}\right)^{1/n} = \frac{1}{e}$$

(Cauchy's second theorem)

$$\text{sol'n} : \lim_{n \rightarrow \infty} \left(\frac{n!}{n^n}\right)^{1/n}$$

Sequence  $(x_n)$  be convergent

Sequence.

Then we have to Prove that  
 $(x_n)$  is bounded.

### Sufficient Condition :-

Let the sequence  $(x_n)$  be monotone bounded sequence.

Then we have to Prove that the sequence  $(x_n)$  is convergent.

Since  $(x_n)$  is monotone bounded sequence.

$\therefore$  it is either  $M \uparrow$  sequence or  $M \downarrow$  sequence.

Also it is bounded above as well as bounded below.

i) Suppose that the sequence  $(x_n)$  is bounded  $M \uparrow$  sequence then  $(x_n)$  is bounded above.

ii) Suppose that the sequence  $(x_n)$  is bounded  $M \downarrow$  sequence then  $(x_n)$  is bounded below.

### \* Limit Points of a sequence

A real number  $l$  is said to be limit point of a sequence  $(x_n)$  if every neighbourhood of  $l$  contains infinitely many terms of the sequence.

i.e.  $l \in \mathbb{R}$  is limit point of the

sequence  $(x_n) \Leftrightarrow$  every neighbourhood of  $l$  contains infinitely many terms of the sequence.

$\Leftrightarrow \forall \epsilon > 0, x_n \in (l-\epsilon, l+\epsilon)$  for infinitely many values of  $n$ .

$\Leftrightarrow \forall \epsilon > 0, |x_n - l| < \epsilon$  for infinitely many values of  $n$ .

Ex:- (i)  $(x_n) = (-1)^n$

$$= (-1, +1, -1, +1, -1, \dots)$$

has two limit points  $-1$  &  $+1$ .

Let  $x_n = (-1)^n \forall n$

then  $x_n = -1$  if  $n$  is odd.

and  $x_n = +1$  if  $n$  is even.

$\therefore$  Every neighbourhood of  $-1$  contains all the odd terms of sequence  $(x_n)$ .

$\therefore -1$  is a limit point.

Similarly every neighbourhood of  $+1$  contains all even terms of the sequence  $(x_n)$ .

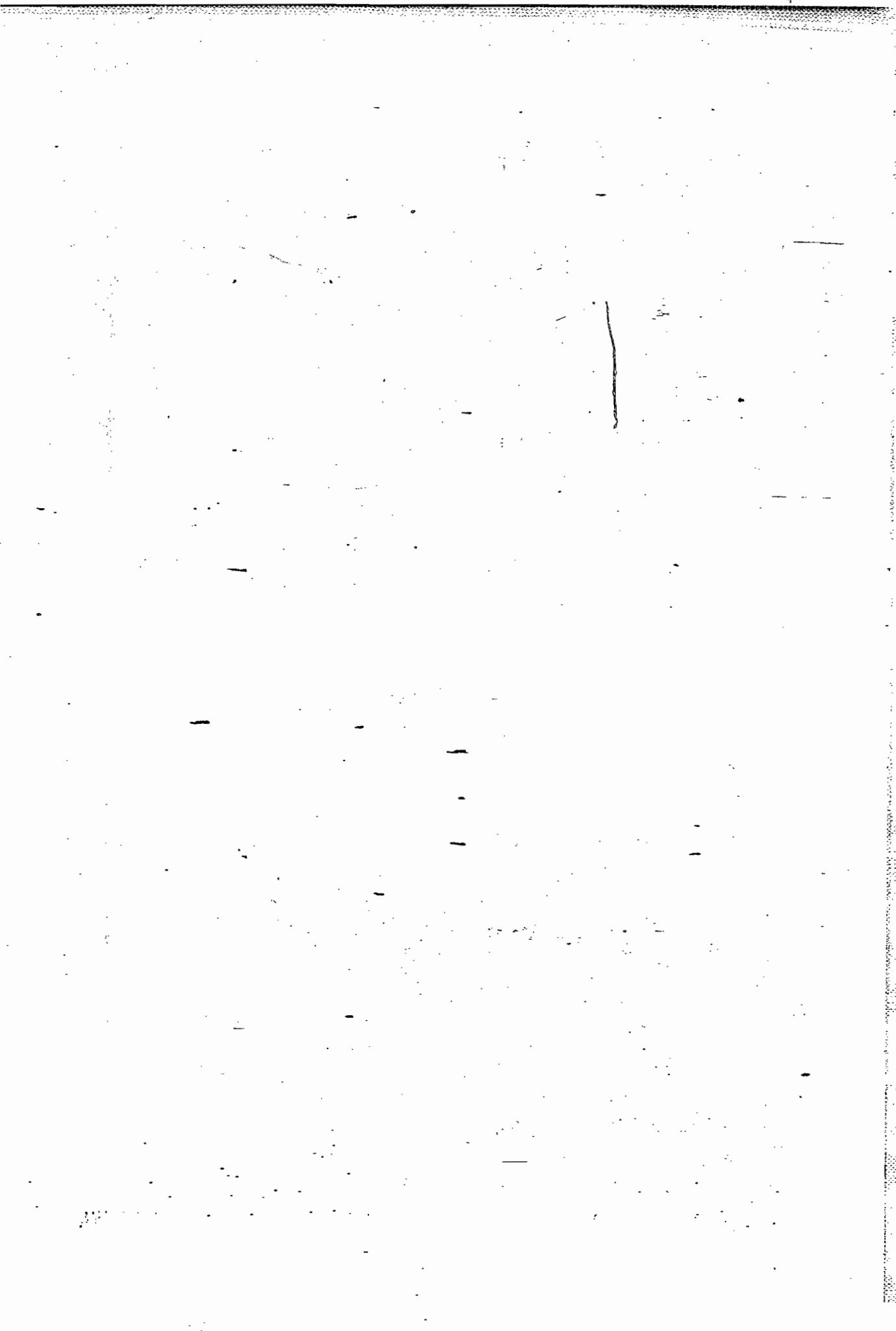
$\therefore +1$  is a limit point.

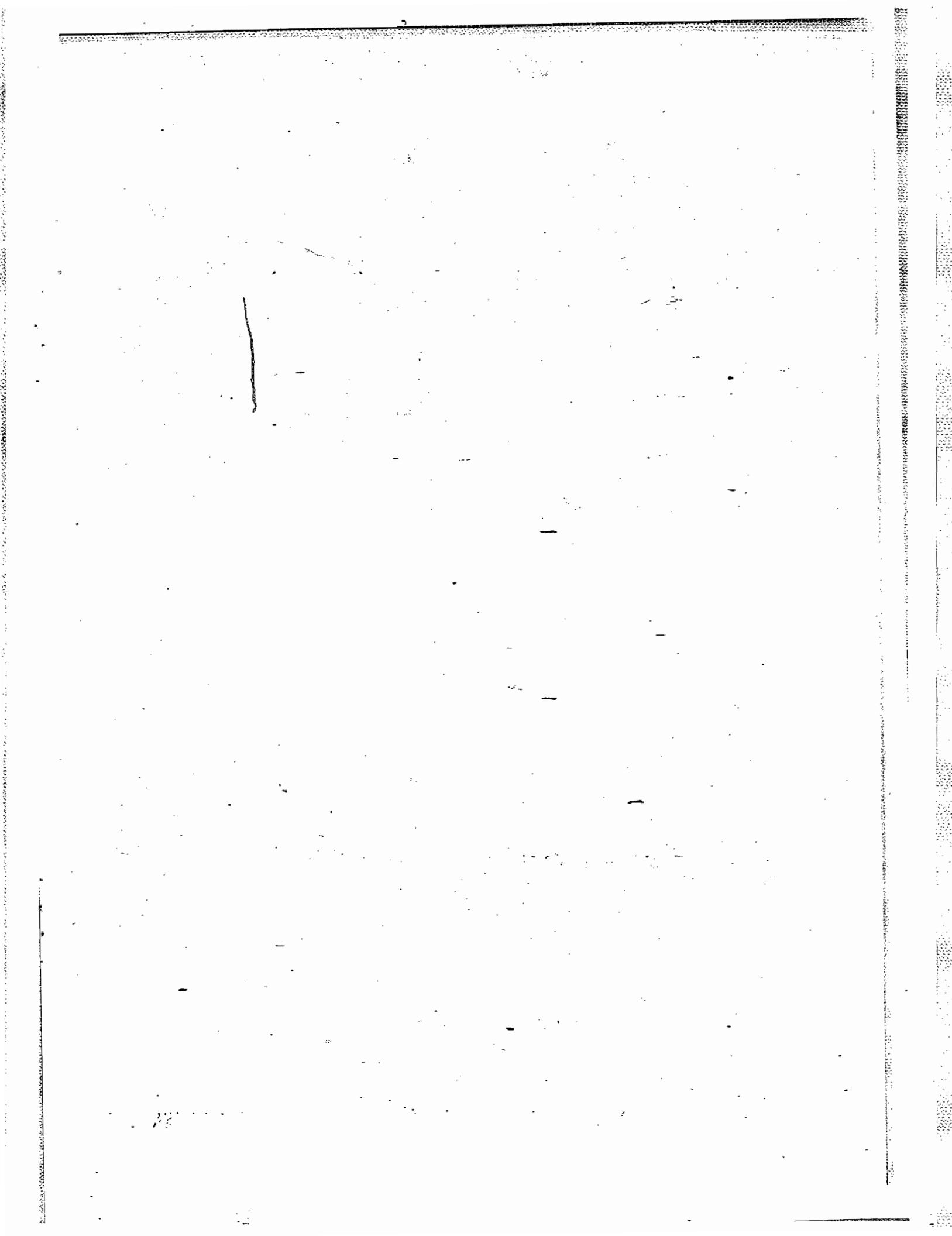
Ex:(ii)  $(x_n) = (\frac{1}{n})$

$$= (\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \dots)$$

a limit point '0'.

Because the neighbourhood of 0 contains infinitely many terms of the sequence.





where  $x_n = l \forall n \in \mathbb{N}$  has the only limit point  $l$ .

Note :- (1) A limit point of a sequence is also called cluster point (or) an accumulation point (or)

Condensation point of the sequence.

(2) Limit point of a sequence is different from limit of a sequence i.e. if  $l \in \mathbb{R}$  is the limit of a sequence  $(x_n)$  then for  $\epsilon > 0$ ,  $\exists m \in \mathbb{N}$  such that  $|x_n - l| < \epsilon \forall n \geq m$ .

$\Leftrightarrow x_n \in (l-\epsilon, l+\epsilon) \forall n \geq m$ .  
i.e. Every neighbourhood of  $l$  contains all except a finite number of terms of the sequence.

whereas if  $l \in \mathbb{R}$  is a limit point of the sequence  $(x_n)$  then every neighbourhood of  $l$  contains infinitely many terms of the sequence  $(x_n)$  does not exclude the possibility of an infinite number of terms of the sequence lying outside that neighbourhood.

Hence limit of a sequence is a limit point of the sequence, but a limit point of a sequence need not be the limit of the

(3) If  $x_n = l$  for infinitely many values of  $n$  then  $l$  is a limit point of  $(x_n)$ .

(4) If for  $\epsilon > 0$ ,  $x_n \in (l-\epsilon, l+\epsilon)$  for finitely many values of  $n$  then  $l$  is not a limit point of the sequence  $(x_n)$ .

(5) Limit point of a sequence need not be a term of the sequence.

Bolzano - Weierstrass Theorem for Sequences:

Every bounded sequence has at least one limit point.

Cauchy's General principle of Convergence:-

A necessary and sufficient condition for the convergence of a sequence  $(x_n)$  is that, for each  $\epsilon > 0$ ,  $\exists m \in \mathbb{N}$  such that

$$|x_{n+p} - x_n| < \epsilon, \forall n \geq m \text{ and } p \geq 1.$$

Necessary Condition:-

Let the sequence  $(x_n)$  be convergent and let it be convergent to  $l$ .

$$\therefore \lim_{n \rightarrow \infty} x_n = l \\ \text{i.e., } x_n \rightarrow l \text{ as } n \rightarrow \infty$$

Given  $\epsilon > 0$ ,  $\exists m \in \mathbb{N}$  such that

$$|x_n - l| < \frac{\epsilon}{2} \quad \forall n \geq m. \quad \text{--- (1)}$$

Since  $p \geq 1 \Rightarrow np \geq n+1 \geq m$ .

$$|x_{n+p} - l| < \frac{\epsilon}{2} \quad \forall n \geq m \text{ & } p \geq 1 \quad \text{--- (2)}$$

Now we have

$$\begin{aligned} |x_{n+p} - x_n| &= |x_{n+p} - l + l - x_n| \\ &\leq |x_{n+p} - l| + |x_n - l| \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} \quad \forall n \geq m \\ &\quad \text{and } p \geq 1. \end{aligned}$$

$$\Rightarrow |x_{n+p} - x_n| < \epsilon \quad \forall n \geq m \text{ and } p \geq 1.$$

### Sufficient condition:-

Given that for each  $\epsilon > 0$ ,

$\exists m \in \mathbb{N}$  such that  $|x_{n+p} - x_n| < \epsilon$   
 $\forall n \geq m$  and  $p \geq 1$

In particular  $n = m$ .

$$\therefore |x_{m+p} - x_m| < \epsilon \quad \forall p \geq 1.$$

$$\Rightarrow -\epsilon < x_{m+p} - x_m < \epsilon \quad \forall p \geq 1.$$

$$\sqrt{\{x_n\}} = \{x_m\} \Rightarrow x_m - \epsilon < x_{m+p} < x_m + \epsilon$$

$$x_m = \frac{1}{m}$$

$$x_{m+1} = \frac{1}{m+1}$$

$$x_{m+2} = \frac{1}{m+2}$$

$$x_{m+3} = \frac{1}{m+3}$$

$$x_{m+4} = \frac{1}{m+4}$$

$$x_{m+5} = \frac{1}{m+5}$$

$$x_{m+6} = \frac{1}{m+6}$$

$$x_{m+7} = \frac{1}{m+7}$$

$$x_{m+8} = \frac{1}{m+8}$$

$$x_{m+9} = \frac{1}{m+9}$$

$$x_{m+10} = \frac{1}{m+10}$$

$$x_{m+11} = \frac{1}{m+11}$$

$$x_{m+12} = \frac{1}{m+12}$$

$$x_{m+13} = \frac{1}{m+13}$$

$$x_{m+14} = \frac{1}{m+14}$$

$$x_{m+15} = \frac{1}{m+15}$$

$$x_{m+16} = \frac{1}{m+16}$$

$$x_{m+17} = \frac{1}{m+17}$$

$$x_{m+18} = \frac{1}{m+18}$$

$$x_{m+19} = \frac{1}{m+19}$$

$$x_{m+20} = \frac{1}{m+20}$$

$$x_{m+21} = \frac{1}{m+21}$$

$$x_{m+22} = \frac{1}{m+22}$$

$$x_{m+23} = \frac{1}{m+23}$$

$$x_{m+24} = \frac{1}{m+24}$$

$$x_{m+25} = \frac{1}{m+25}$$

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$$x_{m+97} = \frac{1}{m+97}$$

$$x_{m+98} = \frac{1}{m+98}$$

$$x_{m+99} = \frac{1}{m+99}$$

$$x_{m+100} = \frac{1}{m+100}$$

i.e. the sequence  $(x_n)$  has a limit point say  $l$ .

we shall show that the sequence  $(x_n)$  converges to  $l$ :  
i.e.  $\lim x_n = l \quad \text{--- (3)}$

Given that, for each  $\epsilon > 0$ ,  $\exists m$

such that  
 $|x_{n+p} - x_n| < \frac{\epsilon}{3} \quad \forall n \geq m \text{ & } p \geq 1 \quad \text{--- (4)}$

In particular,  $n = m$

$\therefore |x_{m+p} - x_m| < \frac{\epsilon}{3} \quad \forall p \geq 1 \quad \text{--- (5)}$

since  $l$  is a limit point:

$\exists m_1 > m$  such that  
 $|x_{m_1} - l| < \frac{\epsilon}{3} \quad \text{--- (6)}$

since  $m_1 > m$

$\therefore$  from (5),

$$|x_{m_1} - x_m| < \frac{\epsilon}{3} \quad \text{--- (7)}$$

Now we have

$$|x_{m+p} - l| = |x_{m+p} - x_m + x_m - x_{m_1} + x_{m_1} - l|$$

$$\leq |x_{m+p} - x_m| + |x_m - x_{m_1}|$$

$$+ |x_{m_1} - l|$$

$$< \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} \quad \forall p \geq 1$$

$$= \epsilon$$

$$\therefore |x_{m+p} - l| < \epsilon \quad \forall p \geq 1$$

$$\therefore |x_n - l| < \epsilon \quad \forall n \geq m.$$

$\therefore (x_n)$  is convergent to  $l$ .

$\therefore$  By Bolzano-Weierstrass theorem

every bounded sequence has at

$$x_{m+101} = \frac{1}{m+101}$$

$$x_{m+102} = \frac{1}{m+102}$$

$$x_{m+103} = \frac{1}{m+103}$$

$$x_{m+104} = \frac{1}{m+104}$$

$$x_{m+105} = \frac{1}{m+105}$$

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$$x_{m+111} = \frac{1}{m+111}$$

$$x_{m+112} = \frac{1}{m+112}$$

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$$x_{m+147} = \frac{1}{m+147}$$

$$x_{m+148} = \frac{1}{m+148}$$

$$x_{m+149} = \frac{1}{m+149}$$

$$x_{m+150} = \frac{1}{$$

Cauchy Sequence

A sequence  $(x_n)$  is said to be Cauchy sequence (or) fundamental sequence.

if for each  $\epsilon > 0$ ,  $\exists m \in \mathbb{N}^+$

such that

$$|x_{n+p} - x_n| < \epsilon \quad \forall n \geq m \text{ and } p \geq 1.$$

or)  $|s_p - s_q| < \epsilon \quad \forall p, q \geq m.$

Theorem Every Cauchy's sequence is bounded.

Theorem If  $x = (x_n)$  is a convergent sequence of real numbers then  $x$  is a Cauchy sequence.

Note: A sequence cannot converge if for each  $\epsilon > 0$ ,  $\exists m \in \mathbb{N}^+$  such that

$$|x_{n+p} - x_n| \neq \epsilon \quad \forall n \geq m \text{ and } p \geq 1.$$

converge which is bounded above  
2. the least upper bound  
sequence ( $x_n$ ).

$$\leq x \forall n.$$

is given then  $x - \epsilon$  is not  
bound of the sequence ( $x_n$ )

if one term of the sequence  
 $x_m$  in the interval  $(x - \epsilon, x]$ .

$$n \in (x - \epsilon, x]$$

$$-\epsilon < x_m \leq x < x + \epsilon \quad (1)$$

$x$  is monotonically increasing

$$x_n \leq x_{n+1} \forall n \in \mathbb{N}$$

$$x_m \leq x_{m+1} \leq x_{m+2} \leq \dots$$

$$x < x + \epsilon$$

$$\epsilon < x_n < x + \epsilon \quad \forall n \geq m$$

$$|x_n - x| < \epsilon \quad \forall n \geq m$$

$$|x_n - x| < \epsilon \quad \forall n \geq m.$$

$$\lim_{n \rightarrow \infty} x_n = x$$

the sequence  $(x_n)$  converges

Let the sequence  $(x_n)$  be  
increasing and let  $(x_n)$  be  
bounded above which is not bounded

Prove that it diverges to  $\infty$ .  
 $(x_n)$  is M.I.P. and which is

Now if at least one term of the  
sequence  $(x_n)$  is  $x_m$  such that  
 $x_m > K$ ,  $K > 0$  (however large) (1)  
since the sequence  $(x_n)$  is M.I.P  
sequence.

$$\therefore x_n \leq x_{n+1} \forall n$$

$$(1) \Rightarrow K < x_m \leq x_{m+1} \leq \dots$$

$$\Rightarrow K < x_n \forall n \geq m$$

$\therefore (x_n)$  diverges to  $\infty$

$$\text{i.e., } \lim x_n = \infty$$

Theorem : If  $y = (y_n)$  is a bounded  
decreasing sequence then

$$\lim (y_n) = \inf \{y_n : n \in \mathbb{N}\}$$

further if  $(y_n)$  is an unbounded  
decreasing then  $\lim y_n = -\infty$   
(OR)

Every monotonically decreasing  
sequence, which is bounded below  
converges to its greatest lower  
bound.

further, Every monotonically  
decreasing sequence which is not  
bounded below diverges to  $-\infty$ .

### Monotone Convergence Theorem

A monotone sequence of real  
numbers converges iff it is bounded.

Necessary Condition : Let the monotone

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\*monotonic Sequences :-

→ A sequence  $(x_n)$  is said to be monotonically increasing if

$$x_n \geq x_{n+1} \quad \forall n \in \mathbb{N}; \text{i.e. } x_n \leq x_{n+1} \quad \forall n$$

$$\text{i.e. } x_1 \leq x_2 \leq x_3 \leq \dots \leq x_n \leq x_{n+1}$$

→ A sequence  $(x_n)$  is said to be monotonically decreasing if

$$x_n \leq x_{n+1} \quad \forall n \in \mathbb{N}; \text{i.e. } x_n \geq x_{n+1} \quad \forall n$$

$$\text{i.e. } x_1 \geq x_2 \geq x_3 \geq \dots \geq x_n \geq x_{n+1}$$

→ A sequence  $(x_n)$  is said to be monotonic if it is either monotonically increasing or monotonically decreasing.

→ A sequence is said to be strictly monotonically increasing if  $x_n < x_{n+1} \quad \forall n$ .

→ A sequence  $(x_n)$  is said to be strictly monotonically decreasing if  $x_n > x_{n+1} \quad \forall n$ .

→ A sequence  $(x_n)$  is said to be strictly monotonic if it is either strictly increasing or strictly decreasing.

Ex-(1) :-

$$(1, 2, 3, 4, \dots, n, \dots), (1, 2, 2, 3, 3, 4, 4, 4, \dots)$$

$(a, a^2, a^3, \dots, a^n, \dots)$  if  $a > 1$  are increasing sequences

$$(1 - \frac{1}{n}; n \in \mathbb{N}) = (1, 1 - \frac{1}{2}, 1 - \frac{1}{3}, \dots)$$

$$(1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{n}), (1, \frac{1}{2}, \frac{1}{2}, \dots, \frac{1}{2^{n-1}})$$

$(b, b^2, b^3, \dots, b^n, \dots)$  if  $0 < b < 1$ .

$$(1 + \frac{1}{n}; n \in \mathbb{N}) = (1+1, 1+\frac{1}{2}, 1+\frac{1}{3}, \dots)$$
 are

decreasing sequences.

$$③ (1, -1, 1, -1, \dots (-1)^{n+1}, \dots);$$

$(-1, 2, -3, \dots, (-1)^n, n, \dots)$  are not monotonic sequences. Because which are neither increasing nor decreasing.

Theorem : If  $x = (x_n)$  is a bounded increasing sequence, then

$$\text{Lt } (x_n) = \sup_{n \rightarrow \infty} \{x_n : n \in \mathbb{N}\}. \text{ Furthermore}$$

if  $(x_n)$  is unbounded increasing sequence then  $\text{Lt}_{n \rightarrow \infty} x_n = \infty$ .

(OR) -

Every monotonically increasing sequence which is bounded above converge to its least upper bound.

Further Every monotonically increasing sequence which is not bounded above diverges to  $\infty$ .

Proof :- Case I :

Let  $x = (x_n)$  be a bounded increasing sequence.

and let  $x = (x_n)$  be a monotonic increasing which is bounded above

TO Prove that the  $(x_n)$  converge to its least upper bound.

Since  $(x_n)$  is monotonically increasing

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Problems

→ use the definition of the limit of a sequence to establish the following limits.

$$(1) \lim_{n \rightarrow \infty} \left( \frac{3n+1}{2n+5} \right) = \underline{\underline{3/2}}$$

Sol'n :- Let  $\epsilon > 0$  be given

$$\text{Now } \left| \frac{3n+1}{2n+5} - \frac{3}{2} \right| = \left| \frac{-13}{4n+10} \right| \\ = \frac{13}{4n+10} < \frac{13}{4n} < \epsilon$$

$$\text{if } \frac{13}{13} n > \frac{1}{\epsilon}$$

$$\text{i.e. } n > \frac{13}{4\epsilon}$$

Now if  $m$  is a +ve integer  $> \frac{13}{4\epsilon}$

$$\text{then } \left| \frac{3n+1}{2n+5} - \frac{3}{2} \right| < \epsilon \quad \forall n \geq m$$

$$\therefore \frac{3n+1}{2n+5} \rightarrow \underline{\underline{3/2}} \text{ as } n \rightarrow \infty$$

$$\text{i.e. } \lim_{n \rightarrow \infty} \frac{3n+1}{2n+5} = \underline{\underline{3/2}}$$

(or)

Now we have

$$\left| \frac{3n+1}{2n+5} - \frac{3}{2} \right| = \left| \frac{-13}{4n+10} \right|$$

$$= \frac{13}{4n+10} < \frac{13}{4n}$$

For  $\epsilon > 0$ , by Archimedean property

$\exists K \in \mathbb{Z}^+$  such that  $K \epsilon > \frac{13}{4}$

$$\Rightarrow \frac{1}{K} < \frac{4}{13} \quad \text{--- (2)}$$

Now we have

$$n \geq K \Rightarrow \frac{1}{n} \leq \frac{1}{K}$$

$$\Rightarrow \frac{1}{n} \leq \frac{1}{K} < \frac{4}{13} \quad \text{by (2)} \quad \text{--- (3)}$$

$$\therefore (1) \equiv \left| \frac{3n+1}{2n+5} - \frac{3}{2} \right| < \frac{13}{4} \left( \frac{4}{13} \right) \epsilon \quad \forall n \geq K \quad (\text{by (3)})$$

$$\therefore \left| \frac{3n+1}{2n+5} - \frac{3}{2} \right| < \epsilon \quad \forall n \geq K$$

$$\text{i.e. } \frac{3n+1}{2n+5} \rightarrow \underline{\underline{3/2}} \text{ as } n \rightarrow \infty$$

$$\text{i.e. } \lim_{n \rightarrow \infty} \frac{3n+1}{2n+5} = \underline{\underline{3/2}}$$

$$(2) \quad \lim_{n \rightarrow \infty} \frac{1}{\sqrt{n+7}} = 0$$

$$(3) \quad \lim_{n \rightarrow \infty} [(-1)^n \cdot \frac{n}{n^2+1}] = 0$$

$$(4) \quad \lim_{n \rightarrow \infty} \frac{1}{3^n} = 0$$

Sol'n :- Let  $\epsilon > 0$  be given

Now we have

$$\left| \frac{1}{3^n} - 0 \right| = \left| \frac{1}{3^n} \right|$$

$$= \frac{1}{3^n} < \epsilon \quad \text{if } 3^n > \frac{1}{\epsilon}$$

$$\therefore \text{if } n \log 3 > \log(\frac{1}{\epsilon})$$

$$\text{i.e. if } n > \frac{\log(\frac{1}{\epsilon})}{\log 3} \quad (\because \log$$

Now if  $m$  is +ve integer  $> \frac{\log(\frac{1}{\epsilon})}{\log 3}$

$$\text{then } \left| \frac{1}{3^n} - 0 \right| < \epsilon \quad \forall n \geq m$$

$$\text{i.e. } \frac{1}{3^n} \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$\text{i.e. } \lim_{n \rightarrow \infty} \frac{1}{3^n} = 0 \quad \underline{\underline{0}}$$

Ex If  $x_n = 1 + \frac{(-1)^n}{2^n}$ , find the value

least +ve integer  $m$  such that

$$|x_n - 1| < \frac{1}{10^3} \quad \forall n > m.$$

Sol'n: Now  $|x_n - 1| = \left| 1 + \frac{(-1)^n}{2^n} - 1 \right|$

$$= \left| \frac{(-1)^n}{2^n} \right|$$
$$= \frac{1}{2^n} \quad \text{--- (1)}$$

Since  $|x_n - 1| \leq \frac{1}{10^3}$

$$\Rightarrow \frac{1}{2^n} < \frac{1}{10^3} \quad (\text{by (1)})$$

$$\Rightarrow 2^n > 10^3$$

$$\Rightarrow n > 500$$

Taking  $m = 500$ ,

we have

$$|x_n - 1| < \frac{1}{10^3} \quad \forall n > m \text{ where } m = 500$$

H.W: If  $x_n = 2 + \frac{(-1)^n}{n^2}$ ,

find the least +ve integer  $m$  such

$$\text{that } |x_n - 2| < \frac{1}{10^4} \quad \forall n > m.$$

Theorem Let  $(x_n)$  be a sequence of real numbers and let  $x \in \mathbb{R}$ , if  $(a_n)$  is a sequence of +ve real numbers with  $\lim_{n \rightarrow \infty} a_n = 0$  and if for some

constant  $c > 0$  and some  $n \in \mathbb{N}$ .

we have  $|x_n - x| \leq c a_n \quad \forall n \geq n$ .

then it follows that

$$\lim_{n \rightarrow \infty} x_n = x.$$

If  $a \geq -1$  then  $(1+a)^n \geq 1+na$

Problems

Q1) If  $a > 0$  then  $\lim_{n \rightarrow \infty} \left( \frac{1}{1+na} \right) = 0$

Sol'n: Since  $a > 0$

$$\Rightarrow na > 0 \quad \forall n \in \mathbb{N}$$

$$\Rightarrow 0 < na < 1+na \quad \forall n \in \mathbb{N}$$

$$\Rightarrow 0 < \frac{1}{1+na} < \frac{1}{na} \quad \forall n \in \mathbb{N}$$

Now we have

$$\left| \frac{1}{1+na} - 0 \right| = \left| \frac{1}{1+na} \right|$$

$$= \frac{1}{1+na} < \frac{1}{na} \quad \forall n \in \mathbb{N}$$

$$= \left( \frac{1}{a} \right) \left( \frac{1}{n} \right) \quad \forall n \in \mathbb{N}.$$

$$\left| \frac{1}{1+na} - 0 \right| < \left( \frac{1}{a} \right) \left( \frac{1}{n} \right) \quad \forall n \in \mathbb{N} \quad \text{--- (1)}$$

Since  $\lim_{n \rightarrow \infty} \left( \frac{1}{n} \right) = 0$

and  $a > 0$

$$\Rightarrow \frac{1}{a} > 0$$

$$\lim_{n \rightarrow \infty} \left( \frac{1}{1+na} \right) = 0.$$

Q2) If  $0 < b < 1$  then  $\lim_{n \rightarrow \infty} (b^n) = 0$

Sol'n: Since  $0 < b < 1$

$$\text{Take } b = \frac{1}{1+a}$$

$$\text{where } a = \left( \frac{1}{b} \right) - 1$$

$$\Rightarrow a > 0$$

By Bernoulli's inequality,

$$\text{we have } (1+a)^n \geq 1+na \quad \forall n.$$

$$\Rightarrow \frac{1}{(1+a)^n} \leq \frac{1}{1+na} \quad \forall n \in \mathbb{N} \quad \text{--- (2)}$$

$$\begin{aligned}
 \text{Now, } 0 < b^n &= \frac{1}{(1+\alpha)^n} \\
 &\leq \frac{1}{1+n\alpha} \quad (\text{by (1)}) \\
 &< \frac{1}{n\alpha} \quad \forall n \in \mathbb{N} \quad \text{--- (2)}
 \end{aligned}$$

Now we have

$$\begin{aligned}
 |b^n - 0| &= \left| \frac{1}{(1+\alpha)^n} - 0 \right| \\
 &= \frac{1}{1+n\alpha} \\
 &< \frac{1}{n\alpha} \quad (\text{by (2)}), \forall n \in \mathbb{N} \\
 &= \left( \frac{1}{\alpha} \right) \left( \frac{1}{n} \right)
 \end{aligned}$$

$$\text{since } \lim_{n \rightarrow \infty} \left( \frac{1}{n} \right) = 0$$

and  $\alpha > 0$

$$\Rightarrow \frac{1}{\alpha} > 0.$$

$$\therefore \lim_{n \rightarrow \infty} b^n = 0.$$


---



Sol'n :- Case(i):

Let  $c=1$  then

$$(c^{v_n}) = (1, 1, 1, \dots)$$

$$\therefore \lim_{n \rightarrow \infty} (c^{v_n}) = 1$$

Case(ii): Let  $c > 1$

then  $c^{v_n} = 1 + d_n$  for some  $d_n > 0$

$$c^{v_n} - 1 = d_n$$

$$\text{and } c = (1 + d_n)^n \geq (1 + nd_n)^n$$

(By Bernoulli's

$$\Rightarrow c \geq 1 + nd_n \quad \forall n \quad \text{(inequality)}$$

$$\Rightarrow \frac{c-1}{n} \geq d_n \quad \forall n \quad \text{--- (1)}$$

Now we have,

$$\begin{aligned} |c^{v_n} - 1| &= d_n \\ &\leq \frac{c-1}{n} \quad \forall n \quad (\text{by (1)}) \\ &= (c-1) \left( \frac{1}{n} \right) \quad \text{--- (2)} \end{aligned}$$

$$\text{since } \lim_{n \rightarrow \infty} (v_n) = 0$$

$$\text{and } (c-1) > 0 \quad (\because c > 1)$$

$$\therefore \lim_{n \rightarrow \infty} (c^{v_n}) = 1$$

Case(iii):

Let  $0 < c < 1$

then  $c^{v_n} = \frac{1}{1+h_n}$  for some  $h_n > 0$

$$\Rightarrow c = \frac{1}{(1+h_n)^n} \quad \text{--- (3)}$$

By Bernoulli's inequality,

$$(1+h_n)^n \geq 1+nh_n \quad \forall n \in \mathbb{N}$$

Rough Idea

If  $c = 2$  then  $c^{v_n} = 2^{v_n}$

$$= 2^1, 2^{\frac{1}{2}}, 2^{\frac{1}{3}}, \dots$$

$$= 2, \sqrt{2}, \sqrt[3]{2}, \dots$$

$$= 2, 1.414, \dots$$

$$= 1+1, 1+0.414, \dots$$

$$= 1+d_n; d_n > 0$$

If  $c = 3$  then  $c^{v_n} = 3^{v_n}$

$$= 3^1, 3^{\frac{1}{2}}, 3^{\frac{1}{3}}, \dots$$

$$= 3, \sqrt{3}, \sqrt[3]{3}, \dots$$

$$= 3, 1.732, \dots$$

$$= 1+2, 1+0.432, \dots$$

$$= 1+d_n; d_n > 0.$$

Rough Idea

$$c = 0.5$$

$$= \frac{1}{2}$$

$$c^{v_n} = \left(\frac{1}{2}\right)^{v_n}$$

$$= \left(\frac{1}{2}\right)^1, \left(\frac{1}{2}\right)^{\frac{1}{2}}, \left(\frac{1}{2}\right)^{\frac{1}{3}}, \dots$$

$$= \frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt[3]{2}}, \dots$$

$$= \frac{1}{1+h_n}; h_n > 0.$$

$$\therefore \frac{1}{1+h_n} \leq \frac{1}{1+n h_n} \quad \forall n$$

from ③ & ④

$$c = \frac{1}{(1+h_n)^n} \leq \frac{1}{1+n h_n} \quad \forall n$$

$$\Rightarrow c \leq \frac{1}{1+n h_n} < \frac{1}{n h_n} \quad \forall n$$

now we have,

$$0 < c < \frac{1}{n h_n}$$

$$\Rightarrow 0 < h_n < \frac{1}{n c}$$

$$\Rightarrow 0 < h_n < \left(\frac{1}{c}\right) \left(\frac{1}{n}\right)$$

now we have, ⑥

$$|c^{n+1}| = \left| \frac{1}{1+h_n} - 1 \right|$$

$$= \left| \frac{-h_n}{1+h_n} \right|$$

$$= \frac{h_n}{1+h_n} < h_n \quad \forall n$$

$$< \left(\frac{1}{c}\right) \left(\frac{1}{n}\right)$$

(by ⑥)

Since  $\lim_{n \rightarrow \infty} \frac{1}{n} = 0$

$\therefore c > 0$ .

$$\Rightarrow \frac{1}{c} > 0$$

$$\therefore \lim_{n \rightarrow \infty} c^n = 1.$$

2003 Let 'a' be a +ve real number (ie  $a > 0$ ) and  $\{x_n\}$  a sequence of rational numbers such that  $\lim_{n \rightarrow \infty} x_n = \infty$

Show that

$$\lim_{n \rightarrow \infty} \frac{x_n}{a^n} = 1.$$

Soln: Given that  $\{x_n\}$  a sequence of rational numbers such that  $\lim_{n \rightarrow \infty} x_n = \infty$

Let the sequence

$$\{x_n\} = \{\frac{1}{n}\}$$

then we show that

$$\lim_{n \rightarrow \infty} \frac{x_n}{a^n} = 1.$$

(proceed as in the above problem.)

→ prove that the sequence whose  $n$ th terms are given below, are monotonic. find out whether they are increasing (or) decreasing

$$(i) \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^n}$$

Soln: Let  $x_n = \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^n}$

$$\text{and } x_{n+1} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^n} + \frac{1}{2^{n+1}}$$

$$\text{Now } x_{n+1} - x_n = \frac{1}{2^{n+1}} > 0 \quad \forall n$$

$$\Rightarrow x_{n+1} > x_n \quad \forall n$$

$$\Rightarrow x_n < x_{n+1} \quad \forall n$$

$\therefore (x_n)$  is an increasing sequence.

$\therefore (x_n)$  is monotonic sequence.

$$(i) \frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n-1}$$

$$(ii) \frac{3n+7}{4n+8} \quad (iv) \frac{2n+7}{3n+8}$$

$$(v) 1 + \frac{1}{1!} + \frac{1}{2!} + \dots + \frac{1}{(n-1)!}$$

$$(vi) \frac{1}{2n+5} \quad (vii) \frac{1}{2n+1}$$

$$(viii) 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \quad (ix) (1 + \frac{1}{n})^n$$

$$\text{Solt: Let } x_n = \left(1 + \frac{1}{n}\right)^n$$

$$= n c_0 \left(1\right)^n \left(\frac{1}{n}\right)^0 + n c_1 \left(1\right)^{n-1} \left(\frac{1}{n}\right)^1 + \\ + n c_2 \left(1\right)^{n-2} \left(\frac{1}{n}\right)^2 + \dots + \\ + n c_n \left(1\right)^{n-n} \left(\frac{1}{n}\right)^n.$$

$$= 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{2!} \left(\frac{1}{n^2}\right) + \\ + \frac{n(n-1)(n-2)}{3!} \left(\frac{1}{n^3}\right) + \dots \\ + \frac{n(n-1)(n-2)\dots(2 \cdot 1)}{n!} \cdot \frac{1}{n^n}$$

$$= 1 + 1 + \frac{1}{2!} \left(1 - \frac{1}{n}\right) + \frac{1}{3!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) + \\ + \dots + \frac{1}{n!} \left(1 - \frac{1}{n}\right) \left(1 - \frac{2}{n}\right) \dots \underbrace{\left[1 - \frac{(n-1)}{n}\right]}_{\text{Ans}}$$

$$\text{Now } x_{n+1} = 1 + 1 + \frac{1}{2!} \left(1 - \frac{1}{n+1}\right) +$$

$$+ \frac{1}{3!} \left(1 - \frac{1}{n+1}\right) \left(1 - \frac{2}{n+1}\right) +$$

$$+ \frac{1}{(n+1)!} \left(1 - \frac{1}{n+1}\right) \left(1 - \frac{2}{n+1}\right) \dots$$

$$\dots \underbrace{\left(1 - \frac{n}{n+1}\right) \left(1 - \frac{n-1}{n+1}\right)}_{\text{Ans}}$$

Now we have,

$$\frac{1}{n+1} < \frac{1}{n} \quad \forall n \in \mathbb{N}$$

$$\Rightarrow \frac{k}{n+1} < \frac{k}{n} \quad \forall n \in \mathbb{N}$$

$$\Rightarrow \frac{-k}{n+1} > \frac{-k}{n} \quad \forall n \in \mathbb{N}$$

$$\Rightarrow 1 - \frac{k}{n+1} > 1 - \frac{k}{n}; k = 3 \cdot 2^{-n}$$

From (i) & (ii) we have

$$x_{n+1} > x_n \quad \forall n$$

$\therefore (x_n)$  is  $M \uparrow$

$\therefore (x_n)$  is monotonic sequence.

$$(x) a_1 = 1 \text{ and } a_n = \sqrt{2+a_{n-1}}$$

Solt: Given that

$$a_1 = 1 \quad \& \quad a_n = \sqrt{2+a_{n-1}}$$

$$\therefore a_2 = \sqrt{2+a_1}$$

$$= \sqrt{2+1}$$

$$= \sqrt{3} > 1 = a_1$$

$$\therefore a_2 > a_1$$

Similarly  $a_3 > a_2$ :

NOW

Suppose,  $a_n > a_{n-1}$  for some  $n$ ,

$n \geq 1$

$$\Rightarrow 2+a_n > 2+a_{n-1}$$

$$\Rightarrow \sqrt{2+a_n} > \sqrt{2+a_{n-1}}$$

$$\therefore a_{n+1} > a_n$$

By mathematical induction

$$a_n < a_{n+1} \quad \forall n$$

$(a_n)$  is an increasing sequence.

$\therefore (a_n)$  is monotonic sequence

Show that the sequence  $(x_n)$  defined by  $x_n = (1 + \frac{1}{n})^n$  is

cgt.

Sol: Given that  $x_n = (1 + \frac{1}{n})^n$   $\forall n \in \mathbb{N}$

$$\Rightarrow x_n = 1 + 1 + \frac{1}{2!}(1 - \frac{1}{n}) +$$

$$+ \frac{1}{3!}(1 - \frac{1}{n})(1 - \frac{2}{n}) + \dots$$

$$+ \frac{1}{n!}(1 - \frac{1}{n})(1 - \frac{2}{n}) \dots (1 - \frac{n-1}{n})$$

$$\text{and } x_{n+1} = 1 + 1 + \frac{1}{2!}(1 - \frac{1}{n+1})$$

$$+ \frac{1}{3!}(1 - \frac{1}{n+1})(1 - \frac{2}{n+1})$$

$$+ \dots + \frac{1}{(n+1)!}(1 - \frac{1}{n+1})$$

$$(1 - \frac{2}{n+1}) \dots (1 - \frac{n}{n+1})$$

$$\therefore a_n \leq x_{n+1} \quad \forall n$$

$(x_n)$  is an increasing sequence.

$(x_n)$  is monotonic sequence

$$\text{Since } x_n = 1 + 1 + \frac{1}{2!}(1 - \frac{1}{n}) +$$

$$+ \frac{1}{3!}(1 - \frac{1}{n})(1 - \frac{2}{n}) + \dots$$

$$+ \frac{1}{n!}(1 - \frac{1}{n})(1 - \frac{2}{n}) \dots (1 - \frac{n-1}{n})$$

$$< 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \dots +$$

$$= 1 + 1 + \frac{1}{2!} + \frac{1}{3 \cdot 2 \cdot 1} + \dots$$

$$\frac{1}{n(n-1) \dots 3 \cdot 2 \cdot 1}$$

$$< 1 + 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots$$

$$= 1 + \frac{(1 - \frac{1}{2^n})}{1 - \frac{1}{2}} \quad (\text{by G. a.l.})$$

$$= 1 + 2(1 - \frac{1}{2^n})$$

$$= 3 - \frac{1}{2^{n-1}}$$

$$< 3 \quad \forall n$$

$$\therefore a_n < 3 \quad \forall n$$

$(a_n)$  is bdd above.

Since  $(a_n)$  is monotonically increasing & bdd above.

$\therefore (a_n)$  is cgt.

Note: Clearly,  $2 \leq x_n \quad \forall n$

$$\therefore 2 \leq x_n \leq 3 \quad \forall n$$

$$\Rightarrow 2 \leq x_n \leq 3$$

$$\Rightarrow 2 \leq \lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n \leq 3$$

$$\Rightarrow \lim_{n \rightarrow \infty} (1 + \frac{1}{n})^n = e$$

→ Discuss the convergence of the sequence  $(x_n)$

where

$$(i) \quad x_n = \frac{n+1}{n} \quad (ii) \quad x_n = \frac{n}{n+1}$$

$$(iii) \quad x_n = 1 + \frac{1}{3} + \frac{1}{3^2} + \dots + \frac{1}{3^n}$$

→ P.T the sequence  $\left\{\frac{2n+7}{3n+2}\right\}$

- (i) is monotonically increasing
- (ii) is bounded
- (iii) tends to the limit  $\frac{2}{3}$ .

Sol: Let  $x_n = \frac{2n+7}{3n+2} \forall n$

$$x_{n+1} < \frac{2n+5}{3n+5}$$

$$\therefore x_{n+1} - x_n = \frac{25}{(3n+5)(3n+2)} > 0 \forall n$$

$$\therefore x_{n+1} > x_n \forall n$$

∴  $(x_n)$  is monotonically increasing

(ii) The given sequence

$$\left\{-1, -\frac{3}{8}, -\frac{1}{11}, -\frac{1}{14}, -\frac{3}{17}, \dots\right\}_2 \text{ approach to } 1$$

Clearly  $x_n > -1 \forall n$

$$\text{and also } 1-x_n = 1 - \frac{2n+7}{3n+2} = \frac{n+9}{3n+2} > 0 \forall n$$

$$\therefore 1-x_n > 0 \forall n$$

$$\Rightarrow 1 > x_n \forall n$$

$$\Rightarrow x_n < 1 \forall n$$

$$\Rightarrow -1 < x_n < 1 \forall n$$

∴  $(x_n)$  is bounded.

(iii) Since  $(x_n)$  is M ↑ and bdd above.

∴ It cgs.

$$\text{Now if } x_n = \lim_{n \rightarrow \infty} \frac{2n+7}{3n+2}$$

$$\begin{aligned} &= \lim_{n \rightarrow \infty} \frac{2 + \frac{7}{n}}{3 + \frac{2}{n}} \\ &= \frac{2}{3}. \end{aligned}$$

∴  $(x_n)$  cgs to  $\frac{2}{3}$

→ P.T the sequence

whose nth term is  $\frac{3n+4}{2n+1}$

(i) is monotonically decreasing

(ii) is bdd and

(iii) cgs to  $\frac{3}{2}$ .

Ques: Show that the sequence

(i) defined by  $x_{n+1} = \sqrt{3x_n} \forall n$

$x_1 = 1$  cgs to  $\frac{3}{2}$ .

Sol: Given that  $x_1 = 1$ ,

$$x_{n+1} = \sqrt{3x_n} \forall n$$

$$\begin{aligned} \text{Now } x_2 &= \sqrt{3x_1} \\ &= \sqrt{3(1)} \\ &= \sqrt{3} > 1 = x_1 \\ \therefore x_2 &> x_1 \end{aligned}$$

Similarly  $x_3 > x_2$

Now suppose  $x_{n+1} > x_n$

$$\Rightarrow 3x_{n+1} > 3x_n$$

$$\Rightarrow \sqrt{3x_{n+1}} > \sqrt{3x_n}$$

$$\Rightarrow x_{n+2} > x_{n+1}$$

By mathematical induction

$$x_{n+1} > x_n \quad \forall n$$

$\therefore (x_n)$  is monotonically increasing.

$$\text{Now } x_1 = 1 < 3$$

$$x_2 = \sqrt{3x_1}$$

$$= \sqrt{3} < 3$$

$$x_3 = \sqrt{3x_2}$$

$$= \sqrt{3 \cdot \sqrt{3}} < 3.$$

Suppose  $x_n < 3$

$$\text{then } x_{n+1} = \sqrt{3x_n}$$

$$< \sqrt{3 \cdot 3} = 3$$

$$\therefore x_{n+1} < 3.$$

By mathematical induction

$$x_n < 3 \quad \forall n.$$

$\therefore (x_n)$  is bdd above by 3.  
Since  $(x_n)$  M.T. and bdd above.

$\therefore$  It is cgt.

Now let  $\lim x_n = l$

$$\underset{n \rightarrow \infty}{\text{then}}$$

$$\text{then } \lim x_{n+1} = l$$

$$\text{Now } x_{n+1} = \sqrt{3x_n} \quad \forall n$$

$$\Rightarrow \lim x_{n+1} = \sqrt{3 \lim x_n}$$

$$\Rightarrow l = \sqrt{3l}$$

$$\Rightarrow l^2 - 3l = 0$$

$$\Rightarrow l(l-3) = 0$$

$$\Rightarrow l=0, l=3.$$

But  $l \neq 0$ , since  $x_n > 1 \forall$

$\therefore \lim x_n = 3$

$$\underset{n \rightarrow \infty}{\text{l}}$$

Show that the sequence

$(x_n)$ , where  $x_1 = 1$  and

$$x_n = \sqrt{2+x_{n-1}} \quad \forall n \geq 2 \text{ is}$$

cgt and cgs to 2.

P.T the sequence  $\{x_n\}$  defined by  $x_1 = \sqrt{7}$ ,  $x_{n+1} = \sqrt{7+x_n}$  cgs to the true root of the equation  $x^2 - x - 7 =$

Soln: Given in  $x_1 = \sqrt{7}$ ,  $x_{n+1} = \sqrt{7+x_n}$

$$\begin{aligned}x_2 &= \sqrt{7+x_1} \\&= \sqrt{7+\sqrt{7}} > \sqrt{7} = x_1\end{aligned}$$

$$\therefore x_2 > x_1$$

$$\text{Similarly } x_3 > x_2$$

Suppose  $x_{n+1} > x_n$  for some  $n$ .

$$\Rightarrow 7+x_{n+1} > 7+x_n$$

$$\Rightarrow \sqrt{7+x_{n+1}} > \sqrt{7+x_n}$$

$$\Rightarrow x_{n+2} > x_{n+1}$$

$\therefore$  By mathematical induction

$$x_{n+1} > x_n \quad \forall n$$

$\therefore (x_n)$  is M↑

$$\text{Now } x_1 = \sqrt{7} < 7$$

$$x_2 = \sqrt{7+\sqrt{7}} \leq 7$$

$$\text{Similarly } x_3 < 7$$

$$\text{Suppose } x_n < 7$$

$$\Rightarrow 7+x_n < 14$$

$$\Rightarrow \sqrt{7+x_n} < \sqrt{14}$$

$$\Rightarrow x_{n+1} < \sqrt{14} < \sqrt{49} = 7$$

$$\Rightarrow x_{n+1} < 7$$

By mathematical induction

$$x_n < 7 \quad \forall n$$

$\therefore (x_n)$  is bdd above.

Since  $(x_n)$  is M↑ \*

bdd above.

$\therefore$  It is cgt.

$$\text{Let } 7+x_n = l \quad *$$

$\forall n$

$$\begin{aligned}\lim_{n \rightarrow \infty} 7+x_n &= l \\&\Rightarrow l = \sqrt{7+l}\end{aligned}$$

$$\text{Now } x_{n+1} = \sqrt{7+x_n}$$

$$\Rightarrow \lim_{n \rightarrow \infty} x_{n+1} = \sqrt{7+\lim_{n \rightarrow \infty} x_n}$$

$$\Rightarrow l = \sqrt{7+l}$$

$$\Rightarrow l^2 - l - 7 = 0$$

$$\Rightarrow l = \frac{1 \pm \sqrt{29}}{2}$$

$$\text{But } l = \frac{1 + \sqrt{29}}{2} < 0 \text{ where}$$

as  $x_n > 0$

$$\therefore l \neq \frac{1 + \sqrt{29}}{2} \dots$$

$$\therefore x_n \text{ cgs to } \frac{1 + \sqrt{29}}{2}$$

which is +ve root of  
the equation  $x^2 - x - 7 = 0$

H.W.P. of the sequence  $\{x_n\}$

defined by  $x_1 = \sqrt{7}$ ,  $x_{n+1} = \sqrt{7+x_n}$

cgs to the +ve root of

the equation

$$x^2 - x - 7 = 0$$

$\Rightarrow$  Let  $x_1 = 8$  and  $x_{n+1} = \frac{1}{2}x_n + 2$

Show that  $(x_n)$  is bdd and monotone, find the limit.

Sol: Given that

$$x_1 = 8, x_{n+1} = \frac{1}{2}x_n + 2$$

$$x_2 = \frac{1}{2}x_1 + 2$$

$$\therefore \frac{1}{2}(8) + 2 = 6 < 8 = x_1$$

$$\therefore x_2 < x_1$$

Similarly  $x_3 < x_2$ .

Now suppose  $x_{n+1} < x_n$  for some  $n$ .

$$\Rightarrow \frac{1}{2}x_{n+1} < \frac{1}{2}x_n$$

$$\Rightarrow \frac{1}{2}x_{n+1} + 2 < \frac{1}{2}x_n + 2$$

$$\Rightarrow x_{n+2} < x_{n+1}$$

$\therefore$  by mathematical induction

$$x_{n+1} < x_n \quad \forall n.$$

$\therefore (x_n)$  is monotonically decreasing.

But w.k.t every decreasing sequence is always bdd below.

Since  $x_n > x_{n+1} \quad \forall n \in \mathbb{N}$ .

$$\Rightarrow x_n > \frac{1}{2}x_n + 2 \quad \forall n$$

$$\Rightarrow 2x_n > x_n + 4 \quad \forall n$$

$$\Rightarrow x_n > 4 \quad \forall n$$

$\therefore (x_n)$  is bdd below

$\therefore (x_n)$  is bdd.

Since  $(x_n)$  is  $M \downarrow$  and bdd below.

$\therefore$  It is cgt.

$$\text{Let } H(x_n) \rightarrow l \quad \text{as } n \rightarrow \infty$$

$$\text{Since } x_{n+1} = \frac{1}{2}x_n + 2$$

$$\Rightarrow H(x_{n+1}) = \frac{1}{2}H(x_n) + 2$$

$$\Rightarrow l = \frac{1}{2}l + 2$$

$$\Rightarrow l = 4$$

$$\therefore H(x_n) = 4, \quad n \rightarrow \infty$$

H.D. Let  $y = (y_n)$  be defined inductively by  $y_1 = 1$ ;

$$y_{n+1} = \frac{1}{4}(2y_n + 3) \quad \forall n \geq 1$$

Show that  $\lim y_n = \frac{3}{2}$

H.W. Let  $z = (z_n)$  be the sequence of real numbers defined by

$$z_1 = 1, z_{n+1} = \sqrt{2z_n} \quad \text{for } n \in \mathbb{N}$$

then show that

$$\lim z_n = 2$$

$$\rightarrow \text{Let } y_1 = \sqrt{p}, \forall n \in \mathbb{N}, \text{ then } y_{n+1} = \sqrt{p+y_n} \text{ for } n \in \mathbb{N}$$

Show that  $(y_n)$  cgs and find limit.

$$\text{Sol: } y_1 = \sqrt{p}; p > 0 \quad \&$$

$$y_{n+1} = \sqrt{p+y_n} \quad \forall n$$

$$\text{Now } y_2 = \sqrt{py_1}$$

$$= \sqrt{p+p} > \sqrt{p} = y_1$$

$$\therefore y_2 > y_1$$

$$\text{Similarly } y_3 > y_2$$

$$\text{Now suppose, } y_{n+1} > y_n$$

$$\Rightarrow p+y_{n+1} > p+y_n; \quad p > 0$$

$$\Rightarrow \sqrt{p+y_{n+1}} > \sqrt{p+y_n}$$

$$\Rightarrow y_{n+2} > y_{n+1}$$

$\therefore$  By mathematical induction

$$y_{n+1} > y_n \quad \forall n$$

$$\therefore (y_n) \text{ is M.P.}$$

$$\text{Since } y_n \leq y_{n+1} \quad \forall n$$

$$\Rightarrow y_n \leq \sqrt{p+y_n} \quad \forall n$$

$$\Rightarrow y_n - y_n - p < 0 \quad \forall n$$

$$\Rightarrow \left[ y_n - \frac{1+\sqrt{1+4p}}{2} \right] \left[ y_n - \frac{1-\sqrt{1+4p}}{2} \right] < 0$$

$$\Rightarrow \left[ y_n - \frac{1+\sqrt{1+4p}}{2} \right] < 0 \quad (\because y_n - \frac{1-\sqrt{1+4p}}{2} > 0)$$

$$\Rightarrow y_n < \frac{1+\sqrt{1+4p}}{2} \quad \forall n$$

$$< \frac{1+1+\sqrt{1+4p}}{2}$$

$$< 1 + \frac{\sqrt{1+4p}}{2}$$

$$< 1 + \frac{\sqrt{4p}}{2}$$

$$< 1 + \sqrt{p}$$

$\therefore (y_n)$  is bdd.

Since  $(y_n)$  is M.P. & bdd above.

$\therefore$  It is cgt.

To find limit of  $(y_n)$

$$\text{Let } \lim_{n \rightarrow \infty} y_n = l \quad \& \quad \lim_{n \rightarrow \infty} y_{n+1} = l$$

$$\text{Now } y_{n+1} = \sqrt{p+y_n}; p > 0$$

$$\Rightarrow \lim_{n \rightarrow \infty} y_{n+1} = \lim_{n \rightarrow \infty} \sqrt{p+y_n}$$

$$\Rightarrow l = \sqrt{p+l}$$

$$\Rightarrow l^2 - l - p = 0$$

$$\Rightarrow l = \frac{1 \pm \sqrt{1+4p}}{2}$$

$$\therefore l = \frac{1+\sqrt{1+4p}}{2} \quad (\because l = 1 - \frac{\sqrt{1+4p}}{2} \text{ out } y_n > 0)$$

$$\therefore \text{If } y_n = \frac{l}{2} (1 + \sqrt{1+4p}).$$

Let  $y_n = \sqrt{n+1} - \sqrt{n}$  when

Show that  $(y_n)$  and  $(\sqrt{n}y_n)$  converge. find their limits.

$$\text{Soln: } (i) y_n = \sqrt{n+1} - \sqrt{n}$$

$$= \frac{(\sqrt{n+1} - \sqrt{n})(\sqrt{n+1} + \sqrt{n})}{\sqrt{n+1} + \sqrt{n}}$$

$$= \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

$$\leq \frac{1}{\sqrt{n+1}} \quad (\because \sqrt{n+1} > \sqrt{n})$$

$$\therefore y_n \leq \frac{1}{2\sqrt{n}} \quad (A)$$

Since  $0 < y_n < \infty$

$\therefore$  (B)

from (A) & (B),

we have

$$0 < y_n \leq \frac{1}{2\sqrt{n}}$$

which is in the form

$$\text{of } z_n < y_n \leq Z_n \quad (D)$$

$$\text{with } \lim_{n \rightarrow \infty} z_n = 0 = \lim_{n \rightarrow \infty} Z_n$$

$\therefore$  By squeeze theorem

$$\lim_{n \rightarrow \infty} y_n = 0$$

$\therefore (\sqrt{n}y_n)$  goes to zero.

$$(ii) \sqrt{n}y_n = \sqrt{n} \left( \frac{1}{\sqrt{n+1} + \sqrt{n}} \right)$$

$$\leq \sqrt{n} \left( \frac{1}{2\sqrt{n}} \right) \quad (\text{By (A)})$$

$$= \frac{1}{2}$$

$$\therefore \sqrt{n}y_n \leq \frac{1}{2} \quad (C)$$

Now since

$$\sqrt{n+1} + \sqrt{n} > \sqrt{n+1} + \sqrt{n}$$

$$\Rightarrow 2\sqrt{n+1} > \sqrt{n+1} + \sqrt{n}$$

$$\Rightarrow \frac{1}{2\sqrt{n+1}} < \frac{1}{\sqrt{n+1} + \sqrt{n}}$$

$$\Rightarrow \frac{\sqrt{n}}{2\sqrt{n+1}} < \frac{\sqrt{n}}{\sqrt{n+1} + \sqrt{n}}$$

$$\Rightarrow \frac{1}{2\sqrt{1+\frac{1}{n}}} < \sqrt{n}y_n. \quad (D)$$

from (C) & (D), we have

$$\frac{1}{2\sqrt{1+\frac{1}{n}}} < \sqrt{n}y_n \leq \frac{1}{2}$$

of the form

$$x_n < y_n \leq z_n$$

$$\text{with } \lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} z_n = \frac{1}{2}$$

$\therefore$  By squeeze theorem

$$\lim_{n \rightarrow \infty} (\sqrt{n}y_n) = \frac{1}{2}$$

$$\therefore \lim_{n \rightarrow \infty} \sqrt{n}y_n = \frac{1}{2}$$

$\therefore$  Establish the convergence  
of the sequence  $(y_n)$ ,

$$\text{where } y_n = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n}; \quad n \in \mathbb{N}.$$

$$\text{Sol: } y_{n+1} = \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{2n+2}$$

$$\text{Now } y_{n+2} - y_{n+1} = \left[ \frac{1}{n+3} + \frac{1}{n+4} + \dots + \frac{1}{2n+2} \right] - \left[ \frac{1}{n+2} + \frac{1}{n+3} + \dots + \frac{1}{2n+1} \right]$$

$$= \frac{1}{n+1} + \frac{1}{2n+2} - \frac{1}{n+1}$$

$$= \frac{1}{2(n+1)(2n+1)} > 0 \quad \forall n$$

$\therefore y_{n+1} > y_n \quad \forall n$

$\therefore (y_n)$  is M.I.

$$\text{Now } y_n = \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{n+n}$$

$$< \frac{1}{n+1} + \frac{1}{n+1} + \dots + \frac{1}{n+1}$$

$$< \frac{1}{n} + \frac{1}{n} + \dots + \frac{1}{n}$$

$$= \frac{n}{n}$$

$$= 1$$

$\therefore y_n < 1 \quad \forall n \in \mathbb{N}$

$\therefore (y_n)$  is bdd above.

$\therefore (y_n)$  is cgt.

$\rightarrow$  If  $(b_n)$  is bdd sequence  
and  $\lim(a_n) = 0$  then show  
 $\lim(a_n b_n) = 0$ .

Sol: Since  $(b_n)$  is bdd  
sequence.

$\exists M > 0$  such that

$$|b_n| \leq M \quad \forall n$$

and since  $\lim a_n = 0$   
 $\Rightarrow a_n \rightarrow 0$

i.e.,  $a_n \rightarrow 0 \Leftrightarrow n \rightarrow \infty$

$\therefore$  Given,  $\epsilon > 0, \exists K \in \mathbb{N}$

such that  $|a_n - 0| < \frac{\epsilon}{M}$ ,

$$(M > 0)$$

$$a_n > K$$

Now we have

$$|a_n b_n - 0| = |a_n b_n|$$

$$= |a_n| |b_n|$$

$$< \frac{\epsilon}{M} \cdot M$$

$$\text{Since } \frac{\epsilon}{M} < \epsilon$$

$\therefore |a_n b_n - 0| < \epsilon \quad \forall n \in \mathbb{N}$

$\therefore \lim a_n b_n = 0$

$n \rightarrow \infty$

## Set - III

Infinite Series

If  $\{x_n\}$  is a sequence of real numbers, then the expression  $x_1 + x_2 + x_3 + \dots + x_n + \dots$  (i.e., the sum of the terms of the sequence, which are infinite in number) is called an infinite series.

The infinite series  $x_1 + x_2 + x_3 + \dots + x_n + \dots$  is usually denoted by  $\sum_{n=1}^{\infty} x_n$  or  $\Sigma x_n$ .

The numbers  $x_1, x_2, x_3, \dots, x_n, \dots$  are called the first, second, third, ...,  $n^{\text{th}}$  term (or general term) of the series.

Series of positive terms: If all the terms of the series  $\Sigma x_n = x_1 + x_2 + x_3 + \dots + x_n + \dots$  are positive i.e., if  $x_n > 0$ , then the series  $\Sigma x_n$  is called a series of positive terms.

Alternating Series:-

A series in which the terms are alternatively +ve and -ve is called an alternating series.

$$\text{i.e., } \sum_{n=1}^{n-1} (-1)^{n-1} x_n = x_1 - x_2 + x_3 - x_4 + \dots + (-1)^{n-1} x_n$$

where  $x_n > 0 \forall n$  is an alternating series.

Partial Sums:-

If  $\Sigma x_n = x_1 + x_2 + \dots + x_n + \dots$  is an infinite series where the terms may be +ve or -ve then

$S_n = x_1 + x_2 + \dots + x_n$  is called the  $n^{\text{th}}$  partial sum of  $\Sigma x_n$ .

The  $n^{\text{th}}$  partial sum of an infinite series is

the sum of its first 'n' terms.

$s_1, s_2, s_3, \dots$  are the first, second, third, ... partial sums of the series.

Since next,  $\{s_n\}$  is a sequence of called the sequence of partial sums of the infinite series  $\sum x_n$ .

To every infinite series  $\sum x_n$ , there corresponds a sequence of  $\{s_n\}$  of its partial sums.

### Nature of infinite series:

(i) The series  $\sum x_n$  is said to be cgs if the sequence of its partial sums cgs.

i.e.,  $\sum x_n$  is cgt if  $\lim_{n \rightarrow \infty} s_n = l$  (finite)

(ii) If  $\lim_{n \rightarrow \infty} s_n = +\infty$  (or)  $-\infty$  then the series  $\sum x_n$  is called dgs.

(iii) If the series  $\sum x_n$  is neither cgt nor dgs, the series  $\sum x_n$  is called oscillatory series.

→ The series  $\sum x_n$  oscillates finitely if the sequence  $\{s_n\}$  of its partial sums oscillates finitely.

i.e.,  $\sum x_n$  oscillates finitely if  $\{s_n\}$  is bounded and neither cgt nor dgs.

→ The series  $\sum x_n$  oscillates infinitely if the sequence  $\{s_n\}$  of its partial sums oscillates infinitely.

i.e.,  $\sum x_n$  oscillates infinitely if  $\{s_n\}$  is unbounded and neither cgs nor dgs.

Discuss the cgs and dgs.

2

$$(1) \sum_{n=1}^{\infty} x_n = \sum_{n=1}^{\infty} n$$

$$= 1+2+3+\dots+n+\dots$$

$$\text{Let } S_n = 1+2+\dots+n$$

$$= \frac{n(n+1)}{2}$$

$$\lim S_n = \infty$$

$\therefore \{S_n\}$  is dgf to  $\infty$

$\therefore \{x_n\}$  is dgf to  $\infty$ .

$$(2) \sum x_n = \sum n^2$$

$$= 1^2+2^2+\dots+n^2+\dots$$

$$\text{Let } S_n = 1+2+\dots+n^2$$

$$= \frac{n(n+1)(2n+1)}{6}$$

$$\lim S_n = +\infty$$

$\therefore \{S_n\}$  is dgf.

$\therefore \{x_n\}$  is dgf.

$$(3) \sum_{n=0}^{\infty} \frac{1}{3^n} = \frac{1}{3^0} + \frac{1}{3^1} + \frac{1}{3^2} + \dots + \frac{1}{3^{n-1}} + \frac{1}{3^n} + \dots$$

$$S_n = \frac{1}{3^0} + \frac{1}{3^1} + \frac{1}{3^2} + \dots + \frac{1}{3^{n-1}}$$

$$= 1 + \frac{1}{3} + \frac{1}{3^2} + \dots + \frac{1}{3^{n-1}}$$

$$= \frac{1 - \left(\frac{1}{3}\right)^n}{1 - \frac{1}{3}}$$

$$= \frac{3}{2} \left(1 - \frac{1}{3^n}\right)$$

$$\lim S_n = 3/2$$

$\therefore \{S_n\}$  is Cgf.

$\therefore \{x_n\}$  is Cgf.

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

$$\sum_{n=1}^{\infty} k (\text{constant}) = k + k + \dots + k + \dots$$

$$S_n = k + k + \dots + k. \quad (n \text{ terms})$$

$$= nk.$$

$$\therefore \lim S_n = \infty$$

$\because \{S_n\}$  is dgt.

$\therefore$  The series  $\sum S_n$  is dgt.

Note: Every constant sequence is cgt but the constant series is dgt.

$$\sum \frac{1}{n(n+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{n(n+1)} + \dots$$

$$\begin{aligned} \text{Let } S_n &= \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \dots + \frac{1}{n(n+1)} \\ &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &\quad + \left(\frac{1}{n+1} - \frac{1}{n+2}\right) \end{aligned}$$

$$\therefore \lim S_n = 1 - 0$$

$$= 1$$

$\therefore \{S_n\}$  is cgt to 1

$\therefore \sum S_n$  is cgt to 1.

### Arithmetic Series:-

$$\sum S_n = a + (a+d) + (a+2d) + \dots + (a+(n-1)d) + \dots$$

$$\text{Let } S_n = a + (a+d) + (a+2d) + \dots + (a+(n-1)d)$$

$$S_n = \frac{n}{2}[2a + (n-1)d]$$

$$\therefore \lim S_n = \infty$$

$\because \{S_n\}$  is dgt.

$\therefore \sum S_n$  is dgt.

### Geometric series

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + \dots + r^{n-1} + \dots$$

(i) cgs if  $-1 < r < 1$  i.e.  $|r| < 1$

(ii) dgs if  $r > 1$

(iii) oscillates finitely if  $r = -1$

(iv) oscillates infinitely if  $r > 1$ .

so let  $s_n = 1 + r + r^2 + \dots + r^{n-1}$

then  $s_n = \frac{1-r^n}{1-r}$

(i) if  $-1 < r < 1$  i.e.  $|r| < 1$

$$\therefore r^n \rightarrow 0 \text{ as } n \rightarrow \infty$$

$$\lim_{n \rightarrow \infty} s_n = \lim_{n \rightarrow \infty} \frac{1-r^n}{1-r} = \frac{1}{1-r} \text{ (finite number)}$$

$\therefore \{s_n\}$  is cgt.

$\therefore \sum r^n$  is cgt.

(ii) If  $r > 1$

sub case (i): if  $r = 1$

$$\text{then } s_n = 1 + 1 + \dots + 1 \text{ (nd term)}$$

$$= n(1)$$

$$= n$$

$$\text{now } \lim_{n \rightarrow \infty} s_n = \infty$$

$\therefore \{s_n\}$  is dgt  $\Rightarrow \sum r^n$  is dgt.

sub case (ii): if  $r > 1$   
i.e.  $1 < r \rightarrow \infty$

$$\therefore r^n \rightarrow \infty \text{ as } n \rightarrow \infty$$

$$\begin{aligned} \lim_{n \rightarrow \infty} s_n &= \lim_{n \rightarrow \infty} \frac{(1-r^n)}{1-r} \text{ or } \lim_{n \rightarrow \infty} \frac{r^n}{1-r} \\ &= +\infty \text{ or } -\infty \end{aligned}$$

$\therefore \{s_n\}$  is dgt.

$\sum r^n$  is dgt.

(iii) If  $r \leq -1$

$$\text{then } s_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

$$\therefore L + s_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$$

$\therefore \{s_n\}$  is oscillatory seq.

$\therefore \sum s_n$  is oscillatory series.

This oscillatory series is finite.  
oscillatory series.

(iv) If  $r < -1$  then  $-r > 1$

$\Rightarrow r > 1$  where  $x = -r$ .

$\Rightarrow 1 < x^n \rightarrow \infty$  as  $n \rightarrow \infty$

i.e.  $(-r)^n \rightarrow \infty$  as  $n \rightarrow \infty$ .

$$\therefore s_n = \frac{(1 - (-r)^n)}{1 - (-r)} = \frac{1 - (-r)^n}{1 + r}$$

$$= \begin{cases} \frac{1 + r^n}{1 + r} & \text{if } n \text{ is odd} \\ \frac{1 - r^n}{1 + r} & \text{if } n \text{ is even} \end{cases}$$

$$\text{Now } L + s_n = \begin{cases} +\infty & \text{if } n \text{ is odd} \\ -\infty & \text{if } n \text{ is even.} \end{cases}$$

$\therefore$  the series  $\sum s_n$  is oscillating series.  
This oscillates infinitely.

Note! - (1) The g.s. egs only when common ratio is numerically less than 1.

(2) In an infinite series, if the terms are changed, a finite number of terms are added or omitted and each term of the series is multiplied and divided by the fixed numbers  $k$  then the nature of the series does not change.

### Problems

\* The geometric Series

$$\sum_{n=0}^{\infty} r^n = 1 + r + r^2 + \dots + r^{n-1} + \dots$$

- (i) Converges if  $|r| < 1$ , i.e.  $|r| < 1$
- (ii) Diverges if  $r \geq 1$
- (iii) Oscillates finitely if  $r = 1$
- (iv) Oscillates infinitely if  $r < -1$ .

\* P-test (or) P-Series:

$$\text{The series } \sum \frac{1}{n^p}$$

$$= \frac{1}{1^p} + \frac{1}{2^p} + \dots + \frac{1}{n^p}$$

- (i) Converges if  $p > 1$
- (ii) Diverges if  $p \leq 1$ .

\* The  $n^{\text{th}}$  term Test:

If the series  $\sum x_n$  converges

then  $\lim_{n \rightarrow \infty} (x_n) = 0$ .

Note!— (1)  $\sum x_n$  converges  $\Rightarrow \lim x_n = 0$

(2)  $\lim x_n = 0 \Rightarrow \sum x_n$  may (con) may not converge.

(3)  $\lim x_n \neq 0 \Rightarrow \sum x_n$  is not convergent.

(4) The series  $\sum_{n=1}^{\infty} \frac{1}{n}$  is called the harmonic series.

\* A positive term series either converges (or) diverges to  $\infty$ .

\* If  $x_n > 0 \forall n$  and  $\lim_{n \rightarrow \infty} x_n \neq 0$  then  $\sum x_n$  diverges to  $\infty$ .

\* Comparison Test: Let  $x = (x_n)$  and  $y = (y_n)$  be non-negative

Sequences of real numbers and suppose that for some  $K \in \mathbb{N}$ , we have

$$0 \leq x_n \leq y_n \text{ for } n \geq K.$$

Then

(a) The convergence of  $\sum y_n \Rightarrow$  the convergence of  $\sum x_n$ .

(b) The divergence of  $\sum x_n \Rightarrow$  the divergence of  $\sum y_n$ .

### Limit Comparison Test:

Suppose that  $x = (x_n)$  and  $y = (y_n)$  are strictly +ve sequences and suppose that the following limit exists in  $\mathbb{R}$ :

$$\gamma = \lim_{n \rightarrow \infty} \left( \frac{x_n}{y_n} \right)$$

(a) If  $\gamma \neq 0$  (finite) then  $\sum x_n$  is convergent (or divergent) iff  $\sum y_n$  is convergent (or divergent).

(b) If  $\gamma = 0$  and if  $\sum y_n$  is convergent then  $\sum x_n$  is convergent.

(c) If  $\gamma = \infty$  and if  $\sum y_n$  diverges then  $\sum x_n$  diverges.

### Problems

\* The series  $\sum_{n=1}^{\infty} \frac{1}{n^2 + n}$  converges.

Sol'n— Clearly the inequality

$$0 < \frac{1}{n^2 + n} < \frac{1}{n^2} \quad \forall n$$

which is in the form of  $0 < x_n < y_n$

$$\text{where } x_n = \frac{1}{n^2 + n} \quad \& \quad y_n = \frac{1}{n^2}$$

Now  $\sum y_n = \sum \frac{1}{n^2}$  is of the form

$$\sum \frac{1}{n^p} \text{ where } p = 2 > 1$$

$\therefore$  By P-Test.

$\sum y_n$  is convergent.

$\therefore$  By Comparison Test

$\sum x_n$  is convergent.

$\therefore$  (Or)

$$\text{Let } x_n = \frac{1}{n^2+n} = \frac{1}{n^2(1+\frac{1}{n})}$$

$$\text{Let } y_n = \frac{1}{n^2}$$

$$\text{then } \frac{x_n}{y_n} = \frac{1}{1+n}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{x_n}{y_n} = 1 \neq 0.$$

Since  $\sum y_n = \sum \frac{1}{n^2}$  is convergent

by P-Test.

$\therefore$  By Comparison test

$\sum x_n$  is convergent.

Now the series  $\sum_{n=1}^{\infty} \frac{1}{n^2-n+1}$  is

convergent.

→ The series  $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}}$  is divergent.

\* By using partial fractions,

Show that

$$@ \sum_{n=0}^{\infty} \frac{1}{(n+1)(n+2)} = 1$$

$$(b) \sum_{n=0}^{\infty} \frac{1}{(\alpha+n)(\alpha+n+1)} = \frac{1}{\alpha} > 0 \quad \text{if } \alpha > 0$$

$$③ \sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)} = \frac{1}{4}$$

$$\text{sol'n: (a) Let } x_n = \frac{1}{(n+1)(n+2)} \\ = \frac{1}{n+1} - \frac{1}{n+2}$$

$\therefore$  The  $n$ th partial sum

$$S_n = x_0 + x_1 + \dots + x_{n-1}$$

$$= (1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \dots$$

$$= 1 - \frac{1}{n+1}$$

$$\therefore \lim S_n = 1$$

$$\therefore \sum x_n = 1$$

$$④ \text{Let } x_n = \frac{1}{(\alpha+n)(\alpha+n+1)}$$

$$= \frac{1}{\alpha+n} - \frac{1}{\alpha+n+1}$$

$$\therefore S_n = x_0 + x_1 + \dots + x_{n-1}$$

$$= (\frac{1}{\alpha} - \frac{1}{\alpha+1}) + (\frac{1}{\alpha+1} - \frac{1}{\alpha+2}) + \dots$$

$$= \frac{1}{\alpha} - \frac{1}{\alpha+n}$$

$$\therefore \lim S_n = \frac{1}{\alpha} > 0 \text{ if } \alpha > 0$$

$$⑤ \text{Let } x_n = \frac{1}{n(n+1)(n+2)}$$

$$= \frac{1}{2n} - \frac{1}{n+1} + \frac{1}{2(n+2)}$$

$$\therefore S_n = x_1 + x_2 + \dots + x_{n-1} + x_n$$

→ Discuss the Convergence or divergence of the following series.

$$\textcircled{a} \quad \sqrt{\frac{1}{4}} + \sqrt{\frac{2}{6}} + \sqrt{\frac{3}{8}} + \dots + \sqrt{\frac{n}{2(n+1)}} + \dots$$

$$\textcircled{b} \quad \frac{1}{\sqrt{1 \cdot 2}} + \frac{1}{\sqrt{2 \cdot 3}} + \frac{1}{\sqrt{3 \cdot 4}} + \dots$$

$$\underline{\text{sol'n}}: \text{ Let } \sum x_n = \sum \sqrt{\frac{n}{2(n+1)}}$$

$$= \sqrt{\frac{1}{4}} + \sqrt{\frac{2}{6}} + \dots + \sqrt{\frac{n}{2(n+1)}}$$

$$\text{Here } x_n = \sqrt{\frac{n}{2(n+1)}} + \dots$$

$$= \sqrt{\frac{1}{2(1+k_n)}}$$

$$\therefore \lim x_n = \lim \frac{1}{\sqrt{2(1+k_n)}}$$

$$= \frac{1}{\sqrt{2}} \neq 0$$

$$\therefore \lim x_n \neq 0$$

since  $x_n > 0 \forall n$  and  $\lim x_n \neq 0$ .

$\therefore \sum x_n$  diverges to  $\infty$ .

\textcircled{b} Given that

$$\frac{1}{\sqrt{1 \cdot 2}} + \frac{1}{\sqrt{2 \cdot 3}} + \frac{1}{\sqrt{3 \cdot 4}} + \dots + \frac{1}{\sqrt{n(n+1)}}$$

$$\text{Let } x_n = \frac{1}{\sqrt{n(n+1)}} + \dots$$

$$= \frac{1}{n} \left( \frac{1}{\sqrt{1+k_n}} \right)$$

$$\text{Let } y_n = \frac{1}{n}$$

$$\text{then } \frac{x_n}{y_n} = \frac{1}{\sqrt{1+k_n}}$$

$$\therefore \lim \frac{x_n}{y_n} = 1 \neq 0$$

Since  $\sum y_n = \sum \frac{1}{n}$  is divergent.  
(by P-test)

∴ By Comparison test the series

$\sum x_n$  is divergent.

$$\textcircled{c} \quad \frac{2}{1^p} + \frac{3}{2^p} + \frac{4}{3^p} + \frac{5}{4^p} + \dots$$

$$\underline{\text{sol'n}}: \text{ Let } x_n = \frac{n+1}{n^p}$$

$$= \frac{1}{n^{p-1}} (1+k_n)$$

$$\text{Let } y_n = \frac{1}{n^{p-1}}$$

$$\text{then } \frac{x_n}{y_n} = 1+k_n.$$

$$\therefore \lim_{n \rightarrow \infty} \left( \frac{x_n}{y_n} \right) = 1 \neq 0.$$

Since  $\sum y_n = \sum \frac{1}{n^{p-1}}$  is convergent.

if  $p-1 > 1$

i.e.,  $p > 2$

∴ By Comparison test

$\sum x_n$  is convergent.

also  $\sum y_n = \sum \frac{1}{n^{p-1}}$  is divergent.

if  $p-1 \leq 1$

i.e.,  $p \leq 2$

∴ By Comparison test

$\sum x_n$  is divergent.

$$\text{H.W. } \sum (3(n+1))^{-1}$$

$$\rightarrow \sum (n+1)^{-2}$$

$$\rightarrow \frac{(n+1)^{-1}}{n^4+1}$$

$$\rightarrow \frac{1}{n^p(n+1)^{p+1}}$$

$$\sum (\sqrt{n^3+1} - \sqrt{n^3})$$

Sol'n: Let  $x_n = \sqrt{n^3+1} - \sqrt{n^3}$

$$= (\sqrt{n^3+1} - \sqrt{n^3}) \times \frac{\sqrt{n^3+1} + \sqrt{n^3}}{\sqrt{n^3+1} + \sqrt{n^3}}$$

$$\frac{n^3+1 - n^3}{\sqrt{n^3+1} + \sqrt{n^3}}$$

$$= \frac{1}{\sqrt{n^3+1} + \sqrt{n^3}}$$

$$= \frac{1}{n^{3/2}(1 + \sqrt{1 + \frac{1}{n^3}})}$$

$$\text{Let } y_n = \frac{1}{n^{3/2}}$$

$$\text{then } \frac{x_n}{y_n} = \frac{1}{1 + \sqrt{1 + \frac{1}{n^3}}}$$

$$\therefore \lim \frac{x_n}{y_n} = \frac{1}{2} \neq 0.$$

Since  $\sum y_n = \sum \frac{1}{n^{3/2}}$  is convergent

by P-test. Here  $P = \frac{3}{2} > 1$

$\therefore$  By comparison test

$\sum x_n$  is convergent.

$$\rightarrow \sum (\sqrt{n^2+1} - n)$$

$$\rightarrow \sum (\sqrt{n^4+1} - n^2)$$

$$\rightarrow \sum ((\sqrt[n]{n^4+1} - \sqrt[n]{n^2}))$$

Sol'n: Let  $x_n = \sqrt[n]{n^4+1} - \sqrt[n]{n^2}$

$$= ((n+1)^{1/n} - n^{1/n})$$

$$= n^{1/3} \left[ \left( 1 + \frac{1}{n^3} \right)^{1/n} - 1 \right]$$

$$= n^{1/3} \left[ \left( 1 + \frac{1}{n^3} \right)^{1/n} - 1 \right] + \dots$$

$$= n^{1/3} \left[ \left( 1 + \frac{1}{3n} + \frac{1}{9n^2} + \dots \right)^{1/n} - 1 \right]$$

$$= n^{1/3} \left[ \frac{1}{3n} - \frac{1}{9n^2} + \dots \right]$$

$$= \frac{1}{n^{2/3}} \left[ \frac{1}{3} - \frac{1}{9n} + \dots \right]$$

$$\text{Let } y_n = \frac{1}{n^{2/3}}$$

$$\text{then } \frac{x_n}{y_n} = \frac{1}{3} - \frac{1}{9n} + \dots$$

$$\therefore \lim \frac{x_n}{y_n} = \frac{1}{3} \neq 0.$$

since  $\sum y_n = \sum \frac{1}{n^{2/3}}$  is divergent

by P-test where  $P = \frac{2}{3} < 1$

∴ By comparison test

$\sum x_n$  is divergent.

$$\rightarrow \sum (\sqrt[3]{n^3+1} - n)$$

Note:

Rationalisation is effective only if square roots are involved where as this method of binomial expansion is general.

→ Test the convergence of the series.

$$(a) \sum \sin \frac{1}{n} \quad (b) \sum \frac{1}{n} \sin \frac{1}{n}$$

$$(c) \sum \frac{1}{\sqrt{n}} \sin \frac{1}{n} \quad (d) \sum \cos \frac{1}{n}$$

$$(e) \sum \tan^{-1} \frac{1}{n} \quad (f) \sum \cot^{-1} n$$

$$(g) \sum \frac{1}{\sqrt{n}} \tan \frac{1}{n}$$

Sol'n: (a)  $\sum \sin \frac{1}{n}$

$$\text{Let } a_n = \sin \frac{1}{n}$$

$$\text{and let } y_n = \frac{1}{n}$$

$$\text{then } \lim_{n \rightarrow \infty} \frac{x_n}{y_n} = \lim_{n \rightarrow \infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}}$$

$$= \lim_{n \rightarrow \infty} \frac{\sin \frac{1}{n}}{\frac{1}{n}} \quad (\because n \rightarrow \infty \Rightarrow \frac{1}{n} \rightarrow 0)$$

$\neq 0$

since  $\sum y_n = \sum \frac{1}{n}$  is divergent  
by P-Test where  $P=1$ .

∴ By Comparison Test  
 $\sum x_n$  is divergent.

(e). Let  $x_n = \frac{1}{n} \sin \frac{1}{n}$

$$\text{Let } y_n = \frac{1}{n^2}$$

$$\text{then } \lim_{n \rightarrow \infty} \frac{x_n}{y_n} = \lim_{n \rightarrow \infty} \frac{\sin \frac{1}{n}}{\frac{1}{n^2}}$$

$$= 1 \neq 0.$$

since  $\sum y_n = \sum \frac{1}{n^2}$  is convergent  
by P-Test where  $P=2 > 1$ .

∴ By Comparison test  $\sum x_n$  is  
convergent.

(f). Let  $x_n = \frac{1}{\sqrt{n}} \sin \frac{1}{n}$

$$\text{Let } y_n = \frac{1}{n^{3/2}}$$

(g). Let  $x_n = \cos \frac{1}{n}$

$$= 1 - \frac{(y_n)^2}{2!} + \frac{(y_n)^4}{4!} -$$

$$+ \frac{(y_n)^6}{6!} + \dots$$

$$= 1 - \frac{1}{n^2 \cdot 2!} + \frac{1}{n^4 \cdot 4!} -$$

$$\text{Let } y_n = \frac{1}{n}$$

$$\text{then } \frac{x_n}{y_n} = n - \frac{1}{n \cdot 2!} + \frac{1}{n^3 \cdot 4!} -$$

$$\therefore \lim_{n \rightarrow \infty} \frac{x_n}{y_n} = \infty$$

since  $\sum y_n = \sum \frac{1}{n}$  is divergent  
(by P-Test)

∴ By comparison test  $\sum x_n$  is divergent.  
(or)

$$\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} \cos \frac{1}{n}$$

$$= 1 \neq 0.$$

since  $\sum x_n$  is a series of +ve terms.  
i.e.,  $x_n > 0 \forall n$ .

and  $\lim x_n \neq 0$ .

∴  $\sum x_n$  is divergent.

(e). Let  $x_n = \tan^{-1} \frac{1}{n}$

$$= \frac{1}{n} - \frac{1}{2n^3} + \frac{1}{5n^5} - \frac{1}{7n^7} + \dots$$

(f). Let  $x_n = \cot^{-1} n^2$   $\left| \begin{array}{l} \because \cot^{-1} x = 0 \\ \Rightarrow x = \cot 0 \\ \Rightarrow x = \frac{1}{\tan 0} \\ \Rightarrow \tan 0 = \frac{1}{n} \\ \Rightarrow 0 = \tan^{-1} \left( \frac{1}{n} \right) \end{array} \right.$

(g). Let  $x_n = \frac{1}{\sqrt{n}} \tan^{-1} \left( \frac{1}{n} \right)$

$$\text{Let } y_n = \frac{1}{n^{3/2}}$$

$$\rightarrow \sum \frac{1}{n^{1/2} y_n}$$

Sol: Let  $x_n = \frac{1}{n \cdot n^{1/2}}$

$$\text{let } y_n = \frac{1}{n} \text{ then } \frac{x_n}{y_n} = \frac{1}{n^{1/2}}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{x_n}{y_n} = \lim_{n \rightarrow \infty} \frac{1}{n^{1/2}} = \frac{1}{1} = 1$$

since  $\sum y_n = \sum \frac{1}{n}$  is divergent by  
P-Test

∴ By comparison test  
 $\sum x_n$  is divergent.

### D'Alembert's Ratio Test:

If  $\sum u_n$  is a series of the terms such that (i) If  $\frac{u_n}{u_{n+1}} = l$  then

(i)  $\sum u_n$  converges if  $l > 1$ .

(ii)  $\sum u_n$  diverges if  $l < 1$ .

(iii)  $\sum u_n$  may converge or diverge if  $l = 1$ .

(ie) the test fails if  $l = 1$ .

Notice: In general the ratio test is applied when

Fractions & combination of powers involved.

Example: the convergence of the following series.

$$1 + \frac{2!}{2^2} + \frac{3!}{3^3} + \frac{4!}{4^4} + \dots$$

Sol'n: Let  $u_n = \frac{n!}{n^n}$  then

$$u_{n+1} = \frac{(n+1)!}{(n+1)^{n+1}}$$

$$\text{Now } \frac{u_n}{u_{n+1}} = \frac{n!}{n^n} \times \frac{(n+1)^{n+1}}{(n+1)!}$$

$$= \frac{n(1+k_n)^{n+1}}{(n+1)^{n+1}}$$

$$= (1+k_n)^n$$

$$\therefore \lim \frac{u_n}{u_{n+1}} = \lim (1+k_n)^n$$

$$= e > 1$$

∴ By D'Alembert's Ratio test.

∴  $\sum u_n$  is convergent.

$$\rightarrow \frac{1 \cdot 2^2}{1!} + \frac{2^2 \cdot 3^2}{2!} + \frac{3^2 \cdot 4^2}{3!} + \dots$$

Sol'n: Let  $u_n = \frac{n^2(n+1)^2}{n!}$

$$\text{then } u_{n+1} = \frac{(n+1)^2(n+2)^2}{(n+1)!}$$

$$\rightarrow (n+1)^4$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{n^2(n+1)^2}{n!} \times \frac{(n+1)!}{(n+1)^2(n+2)^2}$$

$$= \frac{(n+1)}{\left(1 + \frac{2}{n}\right)^2}$$

$$= n \cdot \frac{(1+k_n)}{(1+\frac{2}{n})^2}$$

$$\therefore \lim \left( \frac{u_n}{u_{n+1}} \right) = \infty > 1$$

∴ By D'Alembert's Ratio test

$\sum u_n$  is convergent.

$$\rightarrow \frac{2}{1^2+1} + \frac{2^2}{2^2+1} + \frac{2^3}{3^2+1} + \dots$$

Sol'n: Let  $u_n = \frac{2^n}{n^2+1}$

$$\text{then } u_{n+1} = \frac{2^{n+1}}{(n+1)^2+1}$$

$$\text{Now } \frac{u_n}{u_{n+1}} = \frac{2^n}{n^2+1} \times \frac{(n+1)^2+1}{2^{n+1}}$$

$$= \frac{1}{2} \frac{(1+k_n)^2 + k_n}{(1+k_{n+1})^2}$$

$$\therefore \lim \frac{u_n}{u_{n+1}} = \frac{1}{2} < 1$$

∴ By D'Alembert's ratio test

$\sum u_n$  is divergent.

$$\rightarrow \sum \frac{x^n}{3^n \cdot n^2}, x > 0.$$

Sol'n: Let  $u_n = \frac{x^n}{3^n \cdot n^2}$

$$\text{then } u_{n+1} = \frac{x^{n+1}}{3^{n+1} \cdot (n+1)^2}$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{x^n}{3^n \cdot n^2} \times \frac{3^{n+1} \cdot (n+1)^2}{x^{n+1}}$$

$$= \frac{3}{x} (1 + \ln)^2$$

$$\text{Now } \lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \frac{3}{x}$$

$\therefore$  By D'Alembert's test

$\sum u_n$  converges if  $3/x > 1$

$$\text{i.e. } x < 3$$

and diverges if  $3/x < 1$  i.e.  $x > 3$ .

if  $x = 3$  then  $\lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = 1$

$\therefore$  Ratio test fails.

Now if  $x = 3$ ,  $u_n = \frac{3^n}{3^n \cdot n^2} = \frac{1}{n^2}$

$\therefore \sum u_n = \sum \frac{1}{n^2}$  is convergent by P-test.

$\therefore \sum u_n$  is convergent if  $x \leq 3$

and divergent if  $x > 3$ .

$$\rightarrow \sum \frac{x^n}{n + \sqrt{n}} : x > 0$$

$$\rightarrow \sum \frac{n!}{n^{3/2}} \cdot x^n : x > 0$$

$$\rightarrow 1 + \frac{x^2}{2} + \frac{x^4}{4} + \frac{x^6}{6} + \dots : x > 0.$$

Sol'n Let  $u_n = \frac{x^n}{2^n}$  (leaving the first term)

$$\text{then } u_{n+1} = \frac{x^{n+1}}{2^{n+1}}$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{1}{x} (1 + \ln)$$

$$\therefore \lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \frac{1}{x^2}$$

$\therefore$  By D'Alembert's ratio test

$\sum u_n$  is convergent.

$$\text{if } \frac{1}{x^2} > 1$$

$$\text{i.e. } x^2 < 1$$

$$\text{i.e. if } x < 1$$

and the series diverges.

$$\text{if } \frac{1}{x^2} < 1$$

$$\text{i.e. } x^2 > 1$$

$$\text{i.e. } x > 1$$

If  $x^2 = 1$  i.e.  $x = 1$ , then the ratio test fails.

If  $x = 1$  then  $u_n = \frac{1}{2^n}$

Let  $v_n = \frac{1}{n}$  then

$$\frac{u_n}{v_n} = \frac{1}{2^n} \cdot n$$

$$\therefore \lim_{n \rightarrow \infty} \frac{u_n}{v_n} = \frac{1}{2} \neq 0.$$

Since  $\sum v_n = \sum \frac{1}{n}$  is divergent by P-test.

$\therefore$  By comparison test  $\sum u_n$  is divergent.

$\therefore \sum u_n$  is divergent.

$$\text{if } x \geq 1$$

and  $\sum u_n$  is convergent  
if  $x < 1$ .

\* Final note: Root test :-

$\rightarrow$  If  $\sum u_n$  is a series of positive terms

such that

(i)  $\lim_{n \rightarrow \infty} (u_n)^{1/n} = l$  then

(ii)  $\sum u_n$  converges if  $l < 1$ .

(iii)  $\sum u_n$  diverges if  $l > 1$ .

(iv)  $\sum u_n$  may converge or diverge

$$\text{if } l = 1.$$

(then the test fails if  $l = 1$ ).

(b) If  $\lim_{n \rightarrow \infty} (u_n)^{1/n} = \infty$  then  $\sum u_n$  is

divergent.

Note: The root test is used when powers are involved.

Problems:

\* Test the convergence of the following series:

$$\rightarrow \text{(a)} \sum_{n=1}^{\infty} \left( \frac{n}{n+1} \right)^{n^2} \quad \text{(b)} \sum_{n=1}^{\infty} x^n$$

$$\text{(c)} \sum_{n=2}^{\infty} \frac{1}{(\log n)^n}$$

$$\text{(d)} \sum_{n=1}^{\infty} n x^n, x > 0 \quad \text{(e)} \sum_{n=1}^{\infty} \left( \frac{n+1}{3n} \right)^n$$

$$\underline{\text{Sol'n}}: \text{(a)} \text{ Let } x_n = \left( \frac{n}{n+1} \right)^{n^2}$$

$$\text{then } (x_n)^{1/n} = \left[ \left( \frac{n}{n+1} \right)^{n^2} \right]^{1/n}$$

$$= \left( \frac{n}{n+1} \right)^n$$

$$= \left( \frac{n+1}{n} \right)^{-n}$$

$$= \left[ \left( 1 + \frac{1}{n} \right)^{+n} \right]^{-1}$$

$$\text{Now } \lim_{n \rightarrow \infty} (x_n)^{1/n} = e^{-1}$$

$$= \frac{1}{e} < 1$$

∴ By Cauchy's root test

$\sum x_n$  converges.

$$\text{(b)} \text{ Let } x_n = \frac{1}{(\log n)^n}$$

$$\text{then } x_n^{1/n} = \frac{1}{\log n}$$

$$\therefore \lim_{n \rightarrow \infty} x_n^{1/n} = \lim_{n \rightarrow \infty} \frac{1}{\log n} = 0 < 1.$$

∴ By root test.

$\sum x_n$  is convergent.

$$\rightarrow \sum 5^{-n} (-1)^n$$

$$\underline{\text{Sol'n}}: \text{ Let } x_n = 5^{-n} (-1)^n$$

$$\text{then } (x_n)^{1/n} = 5^{-1} \cdot \frac{(-1)^n}{n}$$

$$= 5^{-1 - \frac{1}{n}} \text{ if } n \text{ is even}$$

$$= 5^{-1 + \frac{1}{n}} \text{ if } n \text{ is odd.}$$

$$\therefore \lim_{n \rightarrow \infty} x_n^{1/n} = 5^{-1} = \frac{1}{5} < 1$$

∴ By root test

$\sum x_n$  is convergent.

$$\rightarrow \sum_{n=2}^{\infty} \frac{1}{[\log(\log n)]^n}$$

$$\rightarrow \sum_{n=1}^{\infty} 3^{-2n-5} (-1)^n$$

Note: Cauchy's root test is more general than D'Alembert's ratio test.

because:

$$(i) \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} \text{ exists} \Rightarrow \lim_{n \rightarrow \infty} u_n^{1/n} \text{ exists.}$$

$$\text{and } \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} u_n^{1/n}$$

(Cauchy's second theorem on limits)

∴ whenever ratio test is applicable, so is the root test.

$$(ii) \text{ If } \lim_{n \rightarrow \infty} u_n^{1/n} \text{ exists, then}$$

$$\lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} \text{ may not exist.}$$

∴ when the ratio test fails the root test succeeds.

$\therefore$  The root test is more general than the ratio test.

$\rightarrow$  show that Cauchy's root test establishes the convergence of the series  $\sum 3^{-n} - (-1)^n$

while D'Alembert's ratio test fails to do so.

Sol'n: Let  $u_n = 3^{-n} - (-1)^n$

$$\text{then } \lim_{n \rightarrow \infty} u_n^{1/n} = \lim_{n \rightarrow \infty} 3^{-1} - (-1)^{n/2}$$

$$= 3^{-1}$$

$$= \frac{1}{3} < 1$$

$\therefore$  By root test  $\sum u_n$  is convergent.

Now if  $n$  is odd (so that  $n+1$  is even)

$$u_n = 3^{-n+1}, u_{n+1} = 3^{-(n+1)-1}$$

$$= 3^{-n-2}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \lim_{n \rightarrow \infty} \frac{3^{-n+1}}{3^{-n-2}}$$

$$= \lim_{n \rightarrow \infty} \left( \frac{3^1}{3^{-2}} \right)$$

$$= 3^3$$

$$= 27 > 1$$

when  $n$  is even

(so that  $n+1$  is odd)

$$\therefore u_n = 3^{-n-1}, u_{n+1} = 3^{-n}$$

$$= 3^{-n}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = 3^1 = \frac{1}{3} < 1.$$

$\therefore \lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}}$  does not exist.

$\therefore$  D'Alembert's ratio test fails.

### \* Raabe's Test

If  $\sum u_n$  is a series of +ve terms

such that

$$\lim_{n \rightarrow \infty} n \left( \frac{u_n}{u_{n+1}} - 1 \right) = l \text{ then}$$

(i)  $\sum u_n$  converges if  $l > 1$ .

(ii)  $\sum u_n$  diverges if  $l < 1$ .

(iii) the test fails if  $l = 1$ .

Note! - Raabe's test is stronger than D'Alembert's ratio test and may succeed where the ratio test fails.

For example:

$$\sum \frac{1}{n^2}$$

$$\text{Let } u_n = \frac{1}{n^2}$$

$$u_{n+1} = \frac{1}{(n+1)^2}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \lim_{n \rightarrow \infty} \frac{(n+1)^2}{n^2}$$

$$= \lim_{n \rightarrow \infty} \left( 1 + \frac{1}{n} \right)^2$$

$$= 1$$

Here  $l = 1$ ,

$\therefore$  the ratio test fails.

$$\text{But } \lim_{n \rightarrow \infty} n \left( \frac{u_n}{u_{n+1}} - 1 \right) = \lim_{n \rightarrow \infty} n \left[ \left( 1 + \frac{1}{n} \right)^2 - 1 \right]$$

$$= \lim_{n \rightarrow \infty} n \left[ \frac{(n+1)^2 - 1}{n^2} \right]$$

$$= \lim_{n \rightarrow \infty} n \left[ \frac{x^n + 1 + 2n - x^n}{n^2} \right]$$

$$= \lim_{n \rightarrow \infty} \left( \frac{1}{n} + 2 \right) = 2 > 1$$

∴ By Raabe's test.

$\sum u_n$  is convergent.

Note:-

① If  $\lim_{n \rightarrow \infty} n \left( \frac{u_n}{u_{n+1}} - 1 \right) = \infty$  then

$\sum u_n$  is convergent.

② If  $\lim_{n \rightarrow \infty} n \left( \frac{u_n}{u_{n+1}} - 1 \right) = -\infty$  then

$\sum u_n$  is divergent.

③ In general Raabe's test is used when D'Alembert's ratio test fails and the ratio  $\frac{u_n}{u_{n+1}}$  does not involve the number 'e'.

→ when  $\frac{u_n}{u_{n+1}}$  involves 'e' we apply logarithmic test after the ratio test and not Raabe's test.

### \* Logarithmic Test:

If  $\sum u_n$  is a series of five terms such that  $\lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = l$

then (i)  $\sum u_n$  converges if  $l > 1$

(ii)  $\sum u_n$  diverges if  $l < 1$

(iii) the test fails if  $l = 1$ .

$$\text{Sol'n: Let } u_n = \frac{n^n \cdot x^n}{n!}$$

$$\text{then } u_{n+1} = \frac{(n+1)^{n+1} \cdot x^{n+1}}{(n+1)!}$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{n^n \cdot x^n}{n!} \times \frac{(n+1)!}{(n+1)^{n+1} \cdot x^{n+1}}$$

$$= \frac{(n+1) \cdot n^n}{(n+1)^{n+1} \cdot x}$$

$$= \frac{1}{(1+\frac{1}{n})^n} \cdot \frac{1}{x}$$

$$\text{Now } \lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \frac{1}{e^x}$$

∴ By D'Alembert's test, the series  $\sum u_n$  converges if  $\frac{1}{e^x} > 1$ .

i.e.  $e^x < 1$

i.e.  $x < \ln e$

and diverges if  $\frac{1}{e^x} < 1$

i.e.  $e^x > 1$

i.e.  $x > \ln e$

If  $x = \ln e$  then the ratio test fails.

Now if  $x = \ln e$  then  $\frac{u_n}{u_{n+1}} = \frac{1}{(1+\frac{1}{n})^n} e$

Since  $\frac{u_n}{u_{n+1}}$  involves the number 'e'

∴ we apply the logarithmic test.

$$\text{Now } \log \left( \frac{u_n}{u_{n+1}} \right) = \log \left[ \frac{1}{(1+\frac{1}{n})^n} e \right]$$

$$= \log e - n \log \left( 1 + \frac{1}{n} \right)$$

$$= 1 - n \left[ \frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \dots \right]$$

$$= \frac{1}{2n} - \frac{1}{3n^2} + \dots$$

$$\therefore \lim_{n \rightarrow \infty} n \log \frac{u_n}{u_{n+1}} = \dots$$

$$= \lim_{n \rightarrow \infty} n \left[ \frac{1}{2n} - \frac{1}{3n^2} + \dots \right]$$

$$= \frac{1}{2} < 1$$

$\therefore$  By logarithmic test  
the series  $\sum u_n$  is divergent.

$\therefore \sum u_n$  is divergent if  $x \geq \frac{1}{e}$  and  
converges if  $-x < \frac{1}{e}$ .

$\Rightarrow$  Discuss the convergence of  
the series.

$$1 + \frac{1}{2} nx + \frac{1 \cdot 3}{2 \cdot 4} x^2 + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} x^3 + \dots - (x > 0)$$

Sol<sup>n</sup> :- Neglecting the first term,

$$\text{let } u_n = \frac{1 \cdot 3 \cdot 5 \dots (2n-1)}{2 \cdot 4 \cdot 6 \dots (2n)} x^n$$

$$u_{n+1} = \frac{1 \cdot 3 \cdot 5 \dots (2n-1)(2n+1)}{2 \cdot 4 \cdot 6 \dots (2n)(2n+2)} x^{n+1}$$

$$\text{Now } \frac{u_n}{u_{n+1}} = \frac{2n+2}{2n+1} \cdot \frac{1}{x} = \frac{1+\frac{1}{2n}}{1+\frac{1}{2n}} \cdot \frac{1}{x}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \frac{1}{x}$$

$\therefore$  By Ratio test  $\sum u_n$  converges if  $\frac{1}{x} > 1$

i.e.  $x < 1$  & diverges if  $\frac{1}{x} < 1$  i.e.  $x > 1$

If  $x = 1$  then the ratio fails.

$$\text{but when } x = 1, \frac{u_n}{u_{n+1}} = \frac{2n+2}{2n+1}$$

Clearly which is not involving in e.  
so we apply the Raabe's test.

$$n \left( \frac{u_n}{u_{n+1}} - 1 \right) = \frac{n}{2n+1} = \frac{1}{2+\frac{1}{n}} \dots 9$$

$$\text{Now } \lim_{n \rightarrow \infty} n \left( \frac{u_n}{u_{n+1}} - 1 \right) = \frac{1}{2} < 1.$$

$\therefore$  By Raabe's Test,  $\sum u_n$  is divergent.

Hence  $\sum u_n$  is Convergent if  $x \leq 1$

and divergent if  $x \geq 1$ .

2008  $\Rightarrow$  Discuss the convergence of the

$$\text{series } \frac{x}{2} + \frac{x^3}{2 \cdot 4} + \frac{x^5}{2 \cdot 4 \cdot 6} + \dots + \frac{x^{2n-1}}{2 \cdot 4 \cdot 6 \dots (2n)} - (x > 0)$$

### Gauss' Test :-

If  $\sum u_n$  is a series of positive  
terms such that  $\frac{u_n}{u_{n+1}} = 1 + \frac{1}{n} + \frac{\alpha_n}{n^{1+\delta}}$

where  $\delta > 0$  and  $(\alpha_n)$  is a bounded  
sequence, then

(i)  $\sum u_n$  converges if  $\lambda > 1$

(ii)  $\sum u_n$  diverges if  $\lambda \leq 1$ .

### Note ! -

(1) the test never fails as we know  
that the series diverges for  $\lambda = 1$ .  
moreover, the test is applied after  
the failure of Ratio test and when  
it is possible to expand  $\frac{u_n}{u_{n+1}}$  in powers  
of  $\frac{1}{n}$  by Binomial Theorem (or) by  
any other method.

(2) Raabe's Test (or) Gauss' Test. If

$\frac{u_n}{u_{n+1}}$  does not involve the number e.

(i) If  $\frac{u_n}{u_{n+1}}$  involves the number  $e$ , apply logarithmic test.

(ii) For application of Gauss test,

expand  $\frac{u_n}{u_{n+1}}$  in powers of  $\frac{1}{n}$ :

$$\frac{u_n}{u_{n+1}} = 1 + \frac{\lambda}{n} + O\left(\frac{1}{n^2}\right) \text{ where}$$

$O\left(\frac{1}{n^2}\right)$  stands for terms of order  $\frac{1}{n^2}$  and higher powers of  $\frac{1}{n}$ .

### \* De Morgan's and Bertrand's Test

If  $\sum u_n$  is a series of positive terms such that

$$\lim_{n \rightarrow \infty} \left[ \left\{ n \left( \frac{u_n}{u_{n+1}} - 1 \right) - 1 \right\} \log n \right] = l$$

then (1)  $\sum u_n$  converges if  $l > 1$ .

(2)  $\sum u_n$  diverges if  $l < 1$ .

Note! - This test is to be applied when both D'Alembert's ratio test and Raabe's test fails.

### \* An alternative to Bertrand's Test:

If  $\sum u_n$  is a series of positive terms such that

$$\lim_{n \rightarrow \infty} \left[ \left( n \log \frac{u_n}{u_{n+1}} - 1 \right) \log n \right] = l$$

then (1)  $\sum u_n$  converges if  $l > 1$ .

(2)  $\sum u_n$  diverges if  $l < 1$ .

Note! - This test is to be applied when the logarithmic test fails.

Problems:-

$$\rightarrow \frac{1}{2} + \frac{1 \cdot 3}{2 \cdot 4} + \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} + \dots$$

$$\text{Let } u_n = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n}$$

$$\text{Then } u_{n+1} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)}{2 \cdot 4 \cdot 6 \cdots (2n)(2n+2)}$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2 \cdot 4 \cdot 6 \cdots 2n} \times \frac{2 \cdot 4 \cdot 6 \cdots (2n)(2n+2)}{1 \cdot 3 \cdot 5 \cdots (2n-1)(2n+1)} \\ = \frac{2n+2}{2n+1} = \frac{1 + \frac{1}{2n}}{1 + \frac{1}{2n}}$$

$$\text{Now if } \lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = 1$$

i.e. D'Alembert's ratio test fails.

Now we apply the Raabe's test

$$\therefore n \left[ \frac{u_n}{u_{n+1}} - 1 \right] = n \left[ \frac{2n+2}{2n+1} - 1 \right]$$

$$= \frac{n}{2n+1} = \frac{1}{2 + \frac{1}{n}}$$

$$\therefore \lim_{n \rightarrow \infty} n \left[ \frac{u_n}{u_{n+1}} - 1 \right] = \frac{1}{2} < 1$$

i.e. By Raabe's test,  $\sum u_n$  diverges.

$$\rightarrow \frac{1^2}{2^2} + \frac{1^2 \cdot 3^2}{2^2 \cdot 4^2} + \frac{1^2 \cdot 3^2 \cdot 5^2}{2^2 \cdot 4^2 \cdot 6^2} + \dots$$

$$\text{Sol'n: - Let } u_n = \frac{1^2 \cdot 3^2 \cdots (2n-1)^2}{2^2 \cdot 4^2 \cdots (2n)^2}$$

$$\text{then } u_{n+1} = \frac{1^2 \cdot 3^2 \cdots (2n-1)^2 (2n+1)^2}{2^2 \cdot 4^2 \cdots (2n)^2 (2n+2)^2}$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{(2n+2)^2}{(2n+1)^2} = \frac{\left(1 + \frac{1}{2n}\right)^2}{\left(1 + \frac{1}{2n}\right)^2}$$

$$\therefore \lim \frac{u_n}{u_{n+1}} = 1$$

$\therefore$  D'Alembert's Ratio test fails

Now apply Raabe's test

$$\begin{aligned} \therefore n \left[ \frac{u_n}{u_{n+1}} - 1 \right] &= n \left[ \frac{(2n+2)^2}{(2n+1)^2} - 1 \right] \\ &= n \left[ \frac{4n+3}{(2n+1)^2} \right] \\ &= \frac{4n^2 + 3n}{(2n+1)^2} \\ &= \frac{1 + \frac{3}{4n}}{\left(1 + \frac{1}{2n}\right)^2} \end{aligned}$$

$$\therefore \lim_{n \rightarrow \infty} n \left[ \frac{u_n}{u_{n+1}} - 1 \right] = 1$$

$\therefore$  Raabe's test fails.

Now we apply Gauss test

$$\begin{aligned} \frac{u_n}{u_{n+1}} &= \frac{(2n+2)^2}{(2n+1)^2} \\ &= \left(1 + \frac{1}{n}\right)^2 \left(1 + \frac{1}{2n}\right)^2 \end{aligned}$$

$$\begin{aligned} &= \left(1 + \frac{2}{n} + \frac{1}{n^2}\right) \left(1 - \frac{2}{2n} + \frac{3}{4n^2} - \dots\right) \\ &= \left(1 - \frac{2}{2n} + \frac{3}{4n^2} - \dots\right) + \left(\frac{2}{n} - \frac{4}{2n^2} + \frac{6}{4n^3} - \dots\right) \\ &\quad + \left(\frac{1}{n^2} - \frac{2}{2n^3} + \frac{3}{4n^4} - \dots\right) \\ &= 1 + \frac{1}{n} - \frac{1}{4n^2} + \dots \end{aligned}$$

$$= 1 + \frac{1}{n} + O\left(\frac{1}{n^2}\right)$$

Now comparing with

$$\frac{u_n}{u_{n+1}} = 1 + \frac{\lambda}{n} + O\left(\frac{1}{n^2}\right)$$

we have  $\lambda = 1$

$\therefore$  By Gauss Test

$\sum u_n$  is divergent.

Note: When D'Alembert's test fails then we may directly apply Gauss test

$$\rightarrow 1 + \frac{2}{3^2} + \frac{2^2 \cdot 4^2}{3^2 \cdot 5^2} + \frac{2^2 \cdot 4^2 \cdot 6^2}{3^2 \cdot 5^2 \cdot 7^2} + \dots$$

Sol'n :- Omitting the first term,

we have

$$u_n = \frac{2^2 \cdot 4^2 \cdot 6^2 \dots (2n)^2}{3^2 \cdot 5^2 \cdot 7^2 \dots (2n+1)^2}$$

$$\rightarrow 1 + \frac{3}{4} + \frac{3 \cdot 6}{4 \cdot 10} x^2 + \frac{3 \cdot 6 \cdot 9}{4 \cdot 10 \cdot 13} x^4 + \dots$$

Sol'n :- Leaving the first term,

we have

$$u_n = \frac{3 \cdot 6 \cdot 9 \dots (3n)}{7 \cdot 10 \cdot 13 \dots (3n+4)} x^n$$

$$\Rightarrow u_{n+1} = \frac{3 \cdot 6 \cdot 9 \dots (3n)(3n+3)}{7 \cdot 10 \cdot 13 \dots (3n+4)(3n+7)} x^{n+1}$$

$$\therefore \frac{u_n}{u_{n+1}} = \frac{3n+7}{3n+3} \cdot \frac{1}{x}$$

$$= \frac{1 + \frac{4}{3n}}{1 + \frac{1}{3n}} \cdot \frac{1}{x}$$

$$\therefore \lim \frac{u_n}{u_{n+1}} = \frac{1}{x}$$

$\therefore$  By D'Alembert's Ratio test

$\sum u_n$  converges if  $\frac{1}{x} > 1$   
i.e.  $x < 1$

and  $\sum u_n$  diverges if  $\frac{1}{x} < 1$   
i.e.  $x > 1$

If  $x=1$  then, the ratio test fails.

$$\text{When } x=1, \frac{u_n}{u_{n+1}} = \frac{3n+7}{3n+3}$$

Now we apply Gauss Test.

$$\begin{aligned}\therefore \frac{u_n}{u_{n+1}} &= \frac{3n+7}{3n+3} \\ &= \left(1 + \frac{7}{3n}\right) \left(1 + \frac{1}{n}\right)^{-1} \\ &= \left(1 + \frac{7}{3n}\right) \left(1 - \frac{1}{n} + \frac{1}{n^2} - \frac{1}{n^3} + \dots\right) \\ &= \left(1 - \frac{1}{n} + \frac{1}{n^2} - \dots\right) + \left(\frac{7}{3n} - \frac{7}{3n^2} + \dots\right) \\ &= 1 + \frac{4}{3n} - \frac{4}{3n^2} + \dots \\ &= 1 + \left(\frac{4}{3}\right) \frac{1}{n} + O\left(\frac{1}{n^2}\right)\end{aligned}$$

Comparing it with

$$\frac{u_n}{u_{n+1}} = 1 + \frac{\lambda}{n} + O\left(\frac{1}{n^2}\right)$$

where  $\lambda = \frac{4}{3} > 1$ .

∴ By Gauss Test,

$\sum u_n$  is Convergent.

∴ The given series converges if  $x \leq 1$   
and diverges if  $x > 1$ .

$$\begin{aligned}&\rightarrow \frac{x}{1} + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4} \cdot \frac{x^5}{5} + \dots \\ &\quad \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6} \cdot \frac{x^7}{7} + \dots\end{aligned}$$

Soln:— Neglecting the first term,

we have

$$u_n = \frac{1 \cdot 3 \cdot 5 \cdot 7 \dots (2n-1)}{2 \cdot 4 \cdot 6 \dots (2n)} \frac{x^{2n+1}}{(2n+1)}$$

### \* Cauchy's Condensation Test:

Let  $\sum_{n=1}^{\infty} a(n)$  be such that  $a(n)$

i.e. a decreasing sequence of strictly positive numbers.

$\sum_{n=1}^{\infty} a(n)$  converges (or diverges) iff

$\sum_{n=1}^{\infty} 2^n a(2^n)$  converges (or diverges).

Problems:-

$$(i) \sum_{n=2}^{\infty} \frac{1}{\ln n} \quad (ii) \sum_{n=2}^{\infty} \frac{1}{n \ln n}$$

$$(iii) \sum_{n=3}^{\infty} \frac{1}{n(\ln n)(\ln \ln n)}$$

$$(iv) \sum_{n=4}^{\infty} \frac{1}{n(\ln n)(\ln \ln n)(\ln \ln \ln n)}$$

Soln:— (i) Here given that

$$\sum_{n=2}^{\infty} \frac{1}{\ln n} = \sum_{n=2}^{\infty} \frac{1}{\log n}$$

$$\text{Put } \sum_{n=2}^{\infty} a(n) = \sum_{n=2}^{\infty} \frac{1}{\log n}$$

$$\text{Here } a(n) = \frac{1}{\log n}$$

since  $(\log n)$  is an increasing sequence.

$\therefore (a(n)) = \left(\frac{1}{\log n}\right)$  is a decreasing sequence.

$$\begin{aligned} \sum_{n=2}^{\infty} 2^n a(2^n) &= \sum_{n=2}^{\infty} 2^n \frac{1}{\log(2^n)} \\ &= \sum_{n=2}^{\infty} 2^n \cdot \frac{1}{n \log 2} \\ &= \frac{1}{\log 2} \sum_{n=2}^{\infty} 2^n \cdot \frac{1}{n} \quad \text{--- (1)} \end{aligned}$$

$$\text{Let } v_n = \frac{2^n}{n}$$

$$\text{Then } v_n v_n = \frac{2}{n} v_n$$

$$\begin{aligned} \therefore \lim_{n \rightarrow \infty} v_n v_n &= \frac{2}{1} \\ &= 2 > 1 \end{aligned}$$

$\therefore$  By Cauchy's root test,  
 $\sum v_n$  is divergent.

$\therefore \sum_{n=2}^{\infty} 2^n a(2^n)$  is divergent.

$\therefore$  By Cauchy's condensation test

$$\sum a(n) = \sum \frac{1}{\ln n}$$

is divergent.

(ii) Given that

$$\sum_{n=2}^{\infty} \frac{1}{n \ln n} = \sum \frac{1}{n \log n}$$

$$\text{put } \sum a(n) = \sum \frac{1}{n \log n}$$

$$\text{Here } a(n) = \frac{1}{n \log n}$$

since  $(n \log n)$  is an increasing

sequence.

$\therefore (a(n)) = \frac{1}{(n \log n)}$  is a decreasing sequence.

$$\begin{aligned} \sum_{n=2}^{\infty} 2^n a(2^n) &= \sum_{n=2}^{\infty} 2^n \frac{1}{2^n \log(2^n)} \\ &= \sum_{n=2}^{\infty} \frac{1}{n \log 2} \\ &= \frac{1}{\log 2} \sum_{n=2}^{\infty} \frac{1}{n} \end{aligned}$$

is divergent by P-Test where  $P=1$ .

$\therefore$  By Cauchy's condensation test.

$$\sum a(n) = \sum \frac{1}{n \ln n}$$

is divergent.

$$\begin{aligned} \text{(iii) Given } \sum_{n=3}^{\infty} \frac{1}{n(\ln n)(\ln \ln n)} &= \\ \sum_{n=3}^{\infty} \frac{1}{n(\log n)(\log \log n)} & \end{aligned}$$

$$\text{Put } \sum a(n) = \sum \frac{1}{n(\log n)(\log \log n)}$$

$$\text{Here } a(n) = \frac{1}{n(\log n)(\log \log n)}$$

Since  $(n(\log n)(\log \log n))$  is an increasing sequence.

$\therefore (a(n))$  is a decreasing sequence.

$$\begin{aligned} \sum 2^n a(2^n) &= \sum 2^n \frac{1}{2^n (\log 2)(\log \log 2)} \\ &= \sum \frac{1}{(\log 2) \log(\log 2)} \\ &= \sum x_n \text{ (say)} \quad \text{--- (A)} \end{aligned}$$

Since  $\log 2 < 1$

$$\Rightarrow n \log_2 < n$$

$$\Rightarrow \log(n \log_2) < \log n$$

$$\Rightarrow \frac{1}{\log(n \log_2)} > \frac{1}{\log n}$$

$$\Rightarrow \frac{1}{n \log_2} \cdot \frac{1}{\log(n \log_2)} > \frac{1}{n \log_2} \cdot \frac{1}{\log n}$$

$$\Rightarrow x_n > \frac{1}{\log_2} \cdot \frac{1}{n \log n}$$

$$= y_n \text{ (say)} \quad \textcircled{B}$$

$$\therefore x_n > y_n \forall n$$

$$\text{i.e. } y_n < x_n \forall n \quad \textcircled{C}$$

$$\text{Since } \sum y_n = \sum \frac{1}{\log_2} \cdot \frac{1}{n \log n}$$

$$= \frac{1}{\log_2} \sum \frac{1}{n \log n}$$

diverges (by (ii))

By comparison test,

$\sum x_n$  diverges.

By Cauchy's Condensation test,

$\sum a(n)$  diverges.

(iv) Given that

$$\sum_{n=4}^{\infty} \frac{1}{n(\ln n)(\ln \ln n)(\ln \ln \ln n)}$$

$$= \sum_{n=4}^{\infty} \frac{1}{n(\log n)(\log \log n)(\log \log \log n)}$$

$$\text{Put } \sum_{n=4}^{\infty} a(n) =$$

$$\sum_{n=4}^{\infty} \frac{1}{n(\log n)(\log \log n)(\log \log \log n)}$$

$$\text{Here } a(n) = \frac{1}{n(\log n)(\log \log n)(\log \log \log n)}$$

Since  $(n \log n)(\log \log n)(\log \log \log n)$  is an increasing sequence.

$\{a(n)\}$  is a decreasing sequence.

$$\therefore \sum 2^n a(2^n) =$$

$$\sum 2^n \frac{1}{2^n (\log 2^n)(\log \log 2^n)(\log \log \log 2^n)}$$

$$= \sum \frac{1}{(n \log_2)(\log(n \log_2))(\log \log(n \log_2))}$$

$$= \frac{1}{\log_2} \sum \frac{1}{n(\log n \log_2)(\log \log(n \log_2))}$$

$$= \sum x_n \text{ (say)}$$

Since  $\log_2 < 1$

$$\Rightarrow \log n \log_2 < \log n \quad \textcircled{A}$$

$$\Rightarrow \log \log n \log_2 < \log \log n$$

$$\Rightarrow \frac{1}{\log \log n \log_2} > \frac{1}{\log \log n} \quad \textcircled{B}$$

$$\textcircled{A} \equiv$$

$$\frac{1}{\log n \log_2} > \frac{1}{\log n} \quad \textcircled{C}$$

$$\text{But } \frac{1}{n} = \frac{1}{n} \quad \textcircled{D}$$

from ①, ②, ③ give,

$$\frac{1}{n(\log n \log 2) (\log \log n \log 2)} > \frac{1}{n(\log n) (\log \log n)}$$

$= y_n$  say

$$\therefore x_n > y_n \quad \forall n$$

$$\text{i.e. } y_n < x_n \quad \forall n$$

But by ③,

$$\sum y_n = \sum \frac{1}{n(\log n) (\log \log n)}$$

diverges (by (iii))

$\therefore$  By comparison test

$\sum x_n$  also diverges.

$\therefore$  By Cauchy's Condensation test,

$\sum a(n)$  diverges.

If  $c > 1$  then show that the following series are convergent.

$$\textcircled{a} \quad \sum \frac{1}{n(\log n)^c}$$

$$\textcircled{b} \quad \sum \frac{1}{n(\log n) (\log \log n)^c}$$

$$\underline{\text{Sol'n:}} \textcircled{a} \quad \sum a(n) = \sum \frac{1}{n(\log n)^c}$$

is decreasing for  $c > 1$ .

$$\therefore \sum 2^n a(2^n) = \sum 2^n \cdot \frac{1}{2^n (\log 2^n)^c}$$

$$= \sum \frac{1}{(n \log 2)^c}$$

$$= \frac{1}{(\log 2)^c} \sum \frac{1}{n^c}$$

Since  $\sum \frac{1}{n^c}$  is convergent for  $c > 1$ .

$\therefore \sum 2^n a(2^n)$  is convergent.

$\therefore$  By Cauchy's Condensation test,

$\sum a(n)$  is convergent.

$\therefore \sum \frac{1}{(\log n)^{\log n}}$  is convergent.

(ii) since  $\lim_{n \rightarrow \infty} \log(\log \log n) = \infty$

$\therefore$  we can find  $n$

so large that

$$\log(\log \log n) > 2$$

$$\Rightarrow \log n \cdot \log(\log \log n) > 2 \log n$$

$$\Rightarrow \log(\log \log n) \log n > \log n^2$$

$$\Rightarrow (\log \log n)^{\log n} > n^2$$

$$\Rightarrow \frac{1}{(\log \log n)^{\log n}} < \frac{1}{n^2}$$

since  $\sum \frac{1}{n^2}$  is convergent.

$\therefore$  By comparison test.

$\sum \frac{1}{(\log \log n)^{\log n}}$  is convergent.

(iii), since the multiplication of numbers is commutative.

$$\therefore \log n \log r = \log r \log n$$

$$\Rightarrow \log(r^{\log n}) = \log(n^{\log r})$$

$$\Rightarrow r^{\log n} = n^{\log r}$$

$$\therefore \sum r^{\log n} = \sum n^{\log r}$$

$$\therefore \sum \frac{1}{n^{\log r}}$$

By P-test it is convergent.

if  $\log r > 1$

i.e. if  $\log r < -1$

i.e. if  $\log r < -\log e$

Comparison test :-

Note! Let  $\sum u_n$  and  $\sum v_n$  be two series of positive terms and let  $k & k'$  be two real numbers such that  
 $k u_n < v_n < k' u_n \quad \forall n$ .

Then the series

$\sum u_n$  &  $\sum v_n$  converge (or)  
diverge together.

→ Examine the following series for convergence.

(i)  $\sum \frac{1}{(\log n)^{\log n}}$  (ii)  $\sum \frac{1}{(\log \log n)^{\log n}}$

(iii)  $\sum r^{\log n}$

Sol'n :- (i) since  $\lim_{n \rightarrow \infty} \log(\log n) = \infty$

$\therefore$  we can find  $n$

so large that  $\log(\log n) > 2$

$$\log[\log(\log n)] > 2 \log n$$

$$\log(\log n)^{\log n} > \log n^2$$

$$\Rightarrow (\log n)^{\log n} > n^2$$

$$\Rightarrow \frac{1}{(\log n)^{\log n}} < \frac{1}{n^2} \quad \text{--- (1)}$$

Since  $\sum \frac{1}{n^2}$  is convergent (by P-test)

i.e. if  $\log r < \log e^1$

i.e. if  $r < \frac{1}{e}$

$\therefore \sum r^n \log n$  converges iff  $r < \frac{1}{e}$ .

→ If  $\sum_{n=1}^{\infty} u_n$  converges but not absolutely.

i.e.  $\sum_{n=1}^{\infty} |u_n|$  diverges then the

### \* Alternating Series :-

A series with terms alternatively +ve and -ve is called an alternative series.

i.e.  $u_1 - u_2 + u_3 - u_4 + \dots + (-1)^{n-1} u_n$   
where  $u_n > 0 \forall n$  is alternating series and is shortly written as

$$\sum_{n=1}^{\infty} (-1)^{n-1} u_n.$$

### \* Leibnitz's Test on Alternating Series:

The alternating series

$$\sum (-1)^{n-1} u_n = u_1 - u_2 + u_3 - u_4 + \dots, \\ u_n > 0 \forall n$$

Converges if i)  $u_n \geq u_{n+1} \forall n$  and

$$\text{ii)} \lim_{n \rightarrow \infty} u_n = 0.$$

Note:- The alternating series will not be convergent if any one of the two conditions is not satisfied.

### \* Absolute and Conditional Convergence:

→ A series  $\sum_{n=1}^{\infty} u_n$  is said to be absolutely convergent if the series

$$\sum_{n=1}^{\infty} |u_n|$$
 is convergent.

Series  $\sum_{n=1}^{\infty} u_n$  is called conditional convergent (or) semi-convergent (or) non-absolutely convergent.

Note:- Every absolutely convergent series is convergent but convergent series need not be absolute convergent.

Ex:-

$$\sum \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

Sol'n:- Let  $u_n = \frac{1}{n}$

then  $u_n > 0 \forall n$

Since  $\frac{1}{n} > \frac{1}{n+1} \forall n$ ,

$$\Rightarrow u_n > u_{n+1} \forall n$$

$$\text{and } \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \frac{1}{n} = 0.$$

∴ By Leibnitz's test,

$$\sum \frac{(-1)^{n-1}}{n} \text{ convergent.}$$

But the series  $\sum \left| \frac{(-1)^{n-1}}{n} \right|$  -

$$= \sum \frac{1}{n} \text{ is divergent.}$$

(by P-Test)

### Problems:-

Test the convergence and absolute convergence of the series.

$$(i) 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

Sol'n: The given series is

$$\sum u_n = \sum \frac{(-1)^{n-1}}{2n-1}$$

$$= \sum_{n=1}^{\infty} (-1)^{n-1} v_n$$

It is an alternating series.

$$\text{Here } v_n = \frac{1}{2n-1} > 0 \quad \forall n$$

$$v_{n+1} = \frac{1}{2n+1} \quad \forall n$$

$$\text{Since } 2n-1 < 2n+1 \quad \forall n$$

$$\frac{1}{2n-1} > \frac{1}{2n+1} \quad \forall n$$

$$\Rightarrow v_n > v_{n+1} \quad \forall n$$

$$\text{and If } v_n = 0 \\ \lim_{n \rightarrow \infty} v_n = 0$$

$\therefore$  By Leibnitz's test, the series is convergent.

$$\text{Now } |u_n| = \frac{1}{2n-1}$$

$$\text{Since } \frac{1}{2n-1} > \frac{1}{2n} \quad \forall n$$

$$\therefore \sum \frac{1}{2n} = \frac{1}{2} \sum \frac{1}{n} \text{ is divergent} \\ (\text{by P-test}).$$

$\therefore$  By Comparison test,

$$\sum \frac{1}{2n-1} \text{ is divergent.}$$

$$\therefore \sum |u_n| \text{ is divergent.}$$

$\therefore$  the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1} \text{ is Conditional Convergent.}$$

$$\rightarrow \frac{1}{3} - \frac{1}{5} + \frac{1}{7} - \frac{1}{9} + \dots$$

$$\rightarrow \frac{1}{1 \cdot 3} - \frac{1}{2 \cdot 4} + \frac{1}{3 \cdot 5} - \frac{1}{4 \cdot 6} + \dots$$

Sol'n: The given series is

$$\sum v_n = \sum \frac{(-1)^{n-1}}{n(n+2)} = \sum (-1)^{n-1} v_n$$

It is an alternating series.

$$\rightarrow \frac{1}{\log 2} - \frac{1}{\log 3} + \frac{1}{\log 4} - \frac{1}{\log 5} + \dots$$

Sol'n: The given series is

$$\sum u_n = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{\log(n+1)}$$

$$= \sum (-1)^{n-1} v_n$$

It is alternating series.

$$\text{Here } v_n = \frac{1}{\log(n+1)}$$

$$\text{and } v_{n+1} = \frac{1}{\log(n+2)}$$

$$\text{Since } (n+1) < (n+2) \quad \forall n$$

$$\log(n+1) < \log(n+2) \quad \forall n$$

$$\Rightarrow \frac{1}{\log(n+1)} > \frac{1}{\log(n+2)} \quad \forall n$$

$$\Rightarrow v_n > v_{n+1} \quad \forall n$$

$$\text{and If } v_n = \lim_{n \rightarrow \infty} \frac{1}{\log(n+1)}$$

$$= 0$$

$\therefore$  By Leibnitz's test, the series is convergent.

$$\text{Now } |u_n| = \frac{1}{\log(n+1)}$$

$$\text{Since } \log(n+1) < (n+1) \quad \forall n$$

$$\Rightarrow \frac{1}{\log(n+1)} > \frac{1}{n+1} + u_n \quad (A)$$

Since  $\sum \frac{1}{n+1} = \sum u_n$  (say)

$$\text{let } a_n = \frac{1}{n+1} - \frac{1}{n(n+1)}$$

and  $y_n = b_n$

then  $\frac{x_n}{y_n} = \frac{1}{(1+b_n)}$

$\therefore \lim \frac{x_n}{y_n} = 1 \neq 0$ .

Since  $\sum y_n = \sum \frac{1}{n}$  is divergent

(by P-Test)

$\therefore$  By Comparison test

$\sum x_n = \sum \frac{1}{n+1}$  is divergent.

again by Comparison test

$\sum |u_n| = \sum \frac{1}{\log(n+1)}$  is divergent.

$\therefore \sum u_n = \sum \frac{(-1)^{n-1}}{n(n+1)}$

is conditional convergent.

→ show that  $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{n!}$  is

absolutely convergent.

Sol'n :-  $\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{2^n}{n!}$

$|u_n| = \frac{2^n}{n!}, |u_{n+1}| = \frac{2^{n+1}}{(n+1)!}$

$\therefore \lim \left| \frac{u_n}{u_{n+1}} \right| = \lim \left( \frac{n+1}{2} \right) = \infty > 1$

$\therefore$  By D'Alembert's ratio test,

$\sum |u_n|$  is convergent.

$\therefore$  The given alternating series is absolutely convergent.

Abel's Test :-

If  $\sum_{n=1}^{\infty} a_n$  is convergent and the sequence  $\{b_n\}$  is monotonic and bounded, then  $\sum_{n=1}^{\infty} a_n b_n$  is convergent.

Problem ① : Test the convergence of

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} (1+b_n)^n$$

Sol'n :- Let  $a_n = \frac{(-1)^{n-1}}{n}$  and

$$b_n = \left(1 + \frac{1}{n}\right)^n + n$$

Clearly  $\sum a_n$  is convergent

(by Leibnitz's Test) and the sequence  $\{b_n\}$  is monotone (increasing) and bounded.

Hence by Abel's Test,

the series  $\sum_{n=1}^{\infty} a_n b_n$  is convergent.

Problem ② : Test the convergence of

$$1 - \frac{1}{3 \cdot 2^2} + \frac{1}{5 \cdot 3^2} - \frac{1}{7 \cdot 4^2} + \dots$$

Sol'n :-  $1 - \frac{1}{3 \cdot 2^2} + \frac{1}{5 \cdot 3^2} - \frac{1}{7 \cdot 4^2} + \dots$

$$= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n(2n-1)n^2}$$

Let  $a_n = \frac{(-1)^{n-1}}{n^2}$  and  $b_n = \frac{1}{2n-1}$

Problem: Show that the series

$$\sum_{n=2}^{\infty} \frac{(n^3+1)^{1/3} - n}{\log n}$$
 is convergent.

Sol'n :- Let  $a_n = (n^3+1)^{1/3} - n$ ,  $b_n = \frac{1}{\log n}$

then the series can be written as

$$\sum_{n=1}^{\infty} a_n b_n.$$

$$\text{Now } a_n = (n^3+1)^{1/3} - n = n \left(1 + \frac{1}{n^3}\right)^{1/3} - n$$

$$= n \left[ 1 + \frac{1}{3} \cdot \frac{1}{n^3} + \frac{1/3 \cdot (1/3-1)}{2!} \cdot \frac{1}{n^6} + \dots \right]$$

$$= \frac{1}{n^2} \left[ \frac{1}{3} - \frac{1}{9n^3} + \dots \right]$$

Take  $c_n = \frac{1}{n^2}$  then

$$\frac{a_n}{c_n} = \frac{1}{3} - \frac{1}{9n^3} + \dots$$

If  $\frac{a_n}{c_n} = \frac{1}{3}$  which is finite and non-zero.

∴ By comparison test,  $\sum a_n$  and  $\sum c_n$  converge (or) diverge together.

But  $\sum c_n = \sum \frac{1}{n^2}$  is convergent.

∴  $\sum a_n$  is convergent.

Also  $\{b_n\}$  is a monotonically decreasing sequence of tve terms and bounded below.

∴ By Abel's test the series

$$\sum_{n=2}^{\infty} a_n b_n$$
 is convergent.

#### \* Dirichlet's Test

If  $\sum_{n=1}^{\infty} a_n$  is a series whose

n<sup>th</sup> Partial Sum  $\{s_n\}$  is bounded

and  $\{b_n\}$  is a monotonic sequence

Converging to zero then  $\sum_{n=1}^{\infty} a_n b_n$  is convergent.

→ Note: Leibnitz's Test as a particular

case of Dirichlet's Test!

The series  $\sum_{n=1}^{\infty} (-1)^{n-1}$  has bounded

partial sums,

$$\text{Since } s_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

$$\begin{aligned} s_1 &= 1 + (-1) \\ s_2 &= 1 + (-1) + 1 \\ s_3 &= 1 + (-1) + 1 + (-1) \end{aligned}$$

If  $\{a_n\}$  is a monotonically decreasing sequence of tve numbers convergent to '0' i.e. if (i)  $a_n > 0$  then

$$(ii) a_n \geq a_{n+1} \forall n$$

$$(iii) a_n \rightarrow 0 \text{ as } n \rightarrow \infty$$

then by Dirichlet's test,

$$\text{the series } \sum_{n=1}^{\infty} (-1)^{n-1} a_n$$

i.e. the alternating series

$$a_1 - a_2 + a_3 - a_4 + \dots$$
 is convergent.

Problem:- Discuss the convergence of

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}, (p > 0)$$

Sol'n :- Let  $a_n = (-1)^{n-1}$  and  $b_n = \frac{1}{n^p}$ ,  $(p > 0)$

then the series  $\sum a_n = \sum (-1)^{n-1}$

has bounded partial sums.

$$\text{Since } s_n = \begin{cases} 1 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$$

and the sequence  $\{b_n\} = \left\{ \frac{1}{n^p} \right\} (p > 0)$

is monotonically decreasing sequence of tve numbers convergent to '0'.

i.e. (i)  $b_n > 0 \forall n$

(ii)  $b_n \geq b_{n+1} \forall n$

(iii)  $b_n \rightarrow 0$  as  $n \rightarrow \infty$

Hence by Dirichlet's Test  $\sum_{n=1}^{\infty} a_n b_n$  is

Convergent.

#### \* Rearrangement of Terms:

A series  $\sum_{n=1}^{\infty} b_n$  is said to arise from a series  $\sum_{n=1}^{\infty} a_n$  by a rearrangement of terms if there exists a one-to-one correspondence between the terms of the two series so that every  $a_n$  is some  $b_n$  and conversely.

for example, the series.

$$1 - \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \dots$$

is a rearrangement of series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$$

$$\text{i.e. } 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$$

on rearranging the terms so that each positive term is followed by two negative terms,

the series

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \dots$$

If we add finitely many numbers, their sum has the same value, no matter how the terms of the sum are arranged. But this is not so when infinite series are involved. An

arrangement (or equally well derangements) or change in the order of the terms in an infinite series may not only alter the sum but may change its nature all together.

#### Dirichlet's Theorem (I):

- (i) If  $\sum_{n=1}^{\infty} a_n$  is a convergent series converging to 's', then any derangement  $\sum_{n=1}^{\infty} a_n$  also converges to 's'.
- (ii) If  $\sum_{n=1}^{\infty} a_n$  is divergent positive term series then so also is  $\sum_{n=1}^{\infty} b_n$ .

#### Dirichlet's Theorem (II):

If  $\sum_{n=1}^{\infty} a_n$  is an absolutely convergent series then every derangement  $\sum_{n=1}^{\infty} b_n$  also converges absolutely to the same sum as the original series.

#### \* Riemann's Theorem:

A conditionally convergent series can be made by derangement of terms. (i) to converge to any real number.

- (ii) to diverge to any  $+\infty$  or  $-\infty$ .
- (iii) to oscillate finitely or infinitely.

problem: ① Discuss the convergence of the series  $1 + \frac{1}{3^2} - \frac{1}{2^2} + \frac{1}{5^2} - \frac{1}{4^2} + \frac{1}{7^2} - \dots$

sol'n: - the given series is a rearrangement of the series

$$1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

which is absolutely convergent.

Hence by the Dirichlet's theorem, the given series is convergent.

Note: - Riemann's method is of theoretical importance only. For practical applications, the method given by Pringsheim's is useful. (Imp.)

### Pringsheim's Method:

Let  $f(n)$  be a tve fn decreasing to zero as  $n \rightarrow \infty$ . Then by

Leibnitz's test, the alternating series  $\sum_{n=1}^{\infty} (-1)^{n-1} f(n)$  is convergent.

Let the terms of the series

$\sum_{n=1}^{\infty} (-1)^{n-1} f(n)$  be rearranged by

taking alternatively  $\alpha$  positive terms and  $\beta$  negative terms.

If  $g = m f(m)$  and  $K = \alpha/\beta$ .

then the alternation in the sum due to this rearrangement is

$\log K$ .

In particular, if  $f(n) = \frac{1}{n}$ .

$$\text{then } \sum_{n=1}^{\infty} (-1)^{n-1} f(n) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} \\ = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$

then we know that the series is Conditionally convergent and its sum is  $\log 2$ .

$$\text{Also } g = m f(m)$$

$$= m \cdot \frac{1}{m} = 1$$

∴ If the terms are rearranged by taking alternately  $\alpha$  tve terms &  $\beta$  -ve terms,

then the sum of new series is

$$\log 2 + \frac{1}{2} g \log K = \log 2 + \frac{1}{2} \log K$$

Problems: ① Find the sum of the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$$

Sol'n: The given series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$$

is rearrangement of the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n}$$

and is Conditionally convergent and whose sum is  $\log 2$ .

Here the rearranged given series is formed by taking alternatively one tve and two -ve terms.

Let  $\alpha$  be the tve terms &  $\beta$  be the -ve terms.

$$\text{then } K = \alpha/\beta = 1/2$$

$$\text{and } g = m f(m) = m \cdot \frac{1}{m} = 1$$

∴ the sum of the rearranged given series is  $\log_2 + \frac{1}{2}g \log k$ .

$$\begin{aligned}&= \log_2 + \frac{1}{2} \log k \\&= \log_2 - \frac{1}{2} \log_2 \\&= \underline{\underline{\frac{1}{2} \log_2}}.\end{aligned}$$

→ find the sum of the series.

(i)  $1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} - \dots$

(ii)  $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{2} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{4} + \dots$

(iii)  $1 + \frac{1}{3} + \frac{1}{5} - \frac{1}{2} - \frac{1}{4} + \frac{1}{7} + \frac{1}{9} + \frac{1}{6} - \frac{1}{8} + \dots$

→ Investigate what derangement of the series  $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots$

will reduce its sum to zero.

Sol'n: The given series is

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \dots = \sum (-1)^{n+1} \frac{1}{n}$$

It is conditionally convergent with sum  $\log_2$ .

Let it be deranged by taking alternately  $\alpha$  +ve &  $\beta$ -ve terms

$$\text{so that } k = \alpha/\beta.$$

$$\text{and } g = m f(m)$$

$$= m \cdot \frac{1}{m} = 1$$

∴ The sum of the deranged.

$$= \log_2 + \frac{1}{2}g \log k.$$

$$= \log_2 + \frac{1}{2} \log k.$$

But the sum is given to be zero.

$$\therefore \log_2 + \frac{1}{2} \log k = 0$$

$$\Rightarrow \frac{1}{2} \log k = -\log_2$$

$$\Rightarrow \log k = -2 \log_2$$

$$\Rightarrow \log k = \log \frac{1}{4}$$

$$\Rightarrow k = \frac{1}{4}$$

$$\Rightarrow \alpha/\beta = \frac{1}{4}$$

$$\Rightarrow \alpha = 1 \text{ (one +ve term)}$$

$$\beta = 4 \text{ (four -ve terms)}$$

∴ To get the sum zero, one +ve term should be followed by four -ve terms.

The deranged series.

$$\begin{aligned}&+ \frac{1}{2} - \frac{1}{4} - \frac{1}{6} - \frac{1}{8} + \frac{1}{3} - \frac{1}{10} - \frac{1}{12} - \frac{1}{14} - \frac{1}{16} \\&\quad + \frac{1}{5} + \dots\end{aligned}$$

H.W. what derangement of the series  $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$

will reduce its sum to  $\frac{1}{2} \log_2$ .

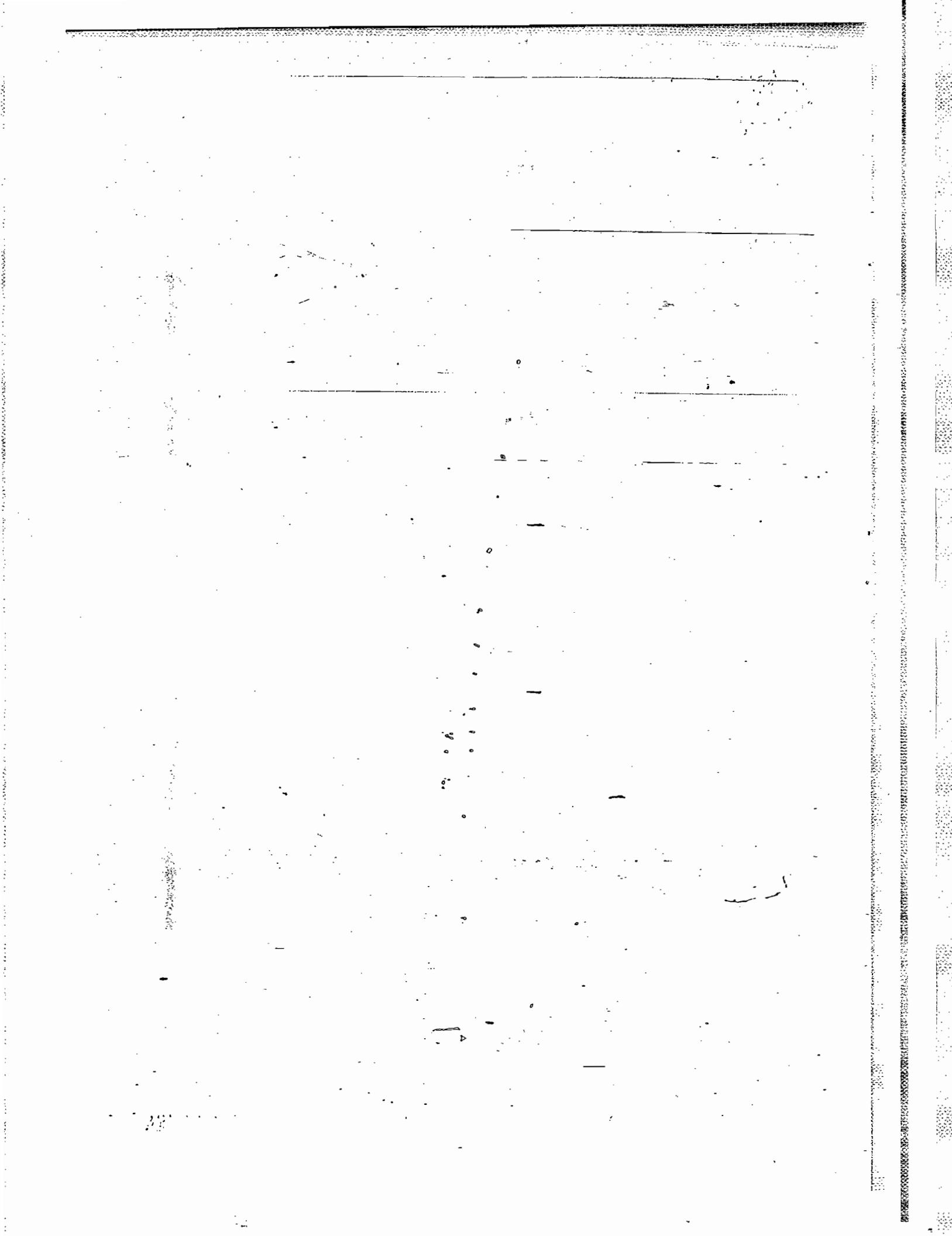
Ques. Rearrange the series.

H.W.  $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$  to converge to 1.

i.e. what derangement of the

series  $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$  will reduce its

sum to 1.



## Cauchy product of TWO infinite series !

If  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  are two infinite series, then their product, called the Cauchy product, is defined,

$$\sum_{n=1}^{\infty} c_n$$

where  $c_n = a_1 b_1 + a_2 b_2 + a_3 b_3 + \dots + a_n b_n$ ,

$$= \sum_{r=1}^n a_r b_{n-r} \text{ for each } n \in \mathbb{N}$$

$$\text{Thus } \sum_{n=1}^{\infty} c_n = \left( \sum_{n=1}^{\infty} a_n \right) \left( \sum_{n=1}^{\infty} b_n \right)$$

$$= (a_1 + a_2 + \dots) (b_1 + b_2 + \dots)$$

$$= a_1 b_1 + (a_1 b_2 + a_2 b_1) + (a_1 b_3 + a_2 b_2 + a_3 b_1) + \dots$$

The terms in the product are so arranged

that all the terms which have the same sum of suffices are bracketed together.

Note:- (1) The Cauchy product of  $\sum_{n=0}^{\infty} a_n$  and  $\sum_{n=0}^{\infty} b_n$  is defined as  $\sum_{n=0}^{\infty} c_n$ .

where  $c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \dots + a_n b_0$

$$= \sum_{r=0}^n a_r b_{n-r} \text{ for each } n \in \mathbb{N}$$

$$(2) c_n = \sum_{r=1}^n a_r b_{n-r+1} = \sum_{r=1}^n a_{n-r+1} b_r$$

$$\text{and } c_n = \sum_{r=0}^n a_r b_{n-r} = \sum_{r=0}^n a_n b_r$$

(3) If  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  converge then,

it is not necessary that  $\sum_{n=1}^{\infty} c_n = \left( \sum_{n=1}^{\infty} a_n \right) \left( \sum_{n=1}^{\infty} b_n \right)$

must converge

$\sum_{n=1}^{\infty} a_n$  converges if

(i)  $\sum_{n=1}^{\infty} |a_n|$  and  $\sum_{n=1}^{\infty} |b_n|$  are convergent series  
of non-negative terms

(ii)

(ii)  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  are absolutely convergent  
(or)

(iii)  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  are convergent and

one of them is absolutely convergent.

i.e.

(i) If  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  are two series of non-negative terms converging to A and B respectively then their Cauchy product  $\sum_{n=1}^{\infty} c_n$  converges to AB.

(ii) If  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  are two absolutely convergent series such that  $\sum_{n=1}^{\infty} a_n = A$  and  $\sum_{n=1}^{\infty} b_n = B$  Then their Cauchy product  $\sum_{n=1}^{\infty} c_n$  is also absolutely convergent and  $\sum_{n=1}^{\infty} c_n = AB$ .

(iii) Merten's Theorem

Let  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  be two convergent series and let  $\sum_{n=1}^{\infty} a_n$  converge absolutely.

If  $\sum_{n=1}^{\infty} a_n = A$  and  $\sum_{n=1}^{\infty} b_n = B$ ; then their product  $\sum_{n=1}^{\infty} c_n$  converges to AB.

→ Cesaro's Theorem:

If two sequences  $(a_n)$  and  $(b_n)$  converge to 'a' and 'b' respectively, then the sequence  $(x_n)$  where  $x_n = \frac{a_1 b_1 + a_2 b_2 + \dots + a_n b_n}{n}$  converge to ab.

→ Abel's Test:- Let  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  be two convergent series such that  $\sum_{n=1}^{\infty} a_n = A$  and  $\sum_{n=1}^{\infty} b_n = B$ . If their Cauchy product  $\sum_{n=1}^{\infty} c_n$  converges, then  $\sum_{n=1}^{\infty} c_n = AB$ .

problems

① S.T. the Cauchy product of the convergent series  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$  w.r.t itself is not convergent.

Sol. Given that  $\sum_{n=1}^{\infty} (-1)^{n+1}$

$$\text{Let } a_n = b_n = \frac{(-1)^{n+1}}{n}, \forall n$$

$\therefore$  By Leibnitz's test, the series  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  are both convergent (but not absolutely).

the Cauchy's product of the two series is

$$\sum_{n=1}^{\infty} c_n$$

$$\text{where } c_n = a_1 b_n + a_2 b_{n-1} + \dots + a_n b_1$$

$$= \frac{(-1)^0}{1} \cdot \frac{(-1)^{n+1}}{n} + \frac{(-1)^1}{2} \cdot \frac{(-1)^{n-1}}{n-1} + \dots + \frac{(-1)^{n-1}}{n} \cdot \frac{(-1)^0}{1}$$

$$= (-1)^{n+1} \left[ \frac{1}{1 \cdot n} + \frac{1}{2 \cdot (n-1)} + \frac{1}{3 \cdot (n-2)} + \dots + \frac{1}{n \cdot 1} \right]$$

$$\geq (-1)^{n+1} \left[ \frac{1}{n \cdot n} + \frac{1}{n \cdot n} + \dots + \frac{1}{n \cdot n} \right]$$

$$\left( \because r \leq n \Rightarrow \frac{1}{r} \geq \frac{1}{n} \right)$$

$$= (-1)^{n+1} \left[ \frac{1}{n \cdot n} \cdot (n \text{ times}) \right]$$

$$= (-1)^{n+1} \left[ \frac{n}{n \cdot n} \right]$$

$$= (-1)^{n+1} \left[ \frac{1}{n} \right]$$

$$\therefore c_n \geq (-1)^{n+1} \left( \frac{1}{n} \right) \quad \forall n$$

$$\Rightarrow |c_n| \geq \left| (-1)^{n+1} \frac{1}{n} \right| \quad \forall n$$

$$\Rightarrow \frac{1}{n} \leq |c_n| \quad \forall n$$

Now  $\sum \frac{1}{n}$  is divergent  $\Rightarrow \sum |c_n|$  is divergent.

$$\Rightarrow \lim_{n \rightarrow \infty} c_n \neq 0 \quad (\text{by comparison})$$

$$\begin{aligned} c_n &\geq \frac{(-1)^{n+1}}{n} \\ \frac{c_n}{n} &\leq c_n \\ \sum \frac{c_n}{n} &\leq \sum c_n \\ \text{if } \sum c_n \text{ is divergent} \end{aligned}$$

Hence  $\sum_{n=1}^{\infty} c_n$  can not converge.

Note:- The above example illustrates that the Cauchy product of two conditionally convergent series need not be necessarily convergent.

(Q) Show that the Cauchy product of the convergent series  $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$  with itself is not convergent.

Sol. Let  $c_n = b_n = \frac{(-1)^{n+1}}{\sqrt{n}}$ , now

By Leibnitz's test, the series  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  are both convergent (but not absolutely).

The Cauchy product of the two series

is  $\sum_{n=1}^{\infty} c_n$ ; where

$$c_n = a_1 b_n + a_2 b_{n-1} + \dots + a_n b_1$$

$$\begin{aligned} &= \frac{(-1)^0}{\sqrt{1}} \cdot \frac{(-1)^{n+1}}{\sqrt{n}} + \frac{(-1)^1}{\sqrt{2}} \cdot \frac{(-1)^{n+2}}{\sqrt{n-1}} + \dots \\ &\quad - \dots + \frac{(-1)^{n+1}}{\sqrt{n}} \cdot \frac{(-1)^0}{\sqrt{1}} \\ &= (-1)^{n+1} \left[ \frac{1}{\sqrt{1 \cdot n}} + \frac{1}{\sqrt{2 \cdot (n-1)}} + \dots + \frac{1}{\sqrt{n \cdot 1}} \right] \end{aligned}$$

$$\geq (-1)^{n+1} \left[ \frac{1}{\sqrt{1 \cdot n}} + \frac{1}{\sqrt{2 \cdot n}} + \dots + \frac{1}{\sqrt{n \cdot n}} \right]$$

$$= (-1)^{n+1} \left[ \frac{1}{n} + \frac{1}{n} + \frac{1}{n} + \dots + \frac{1}{n} \right]$$

$$= (-1)^{n+1} \left[ \frac{n}{n} \right]$$

$$= (-1)^{n+1}$$

$$\Rightarrow |c_n| \geq 1 \forall n \in \mathbb{N}$$

$$\Rightarrow \lim_{n \rightarrow \infty} c_n \neq 0.$$

Hence  $\sum_{n=1}^{\infty} c_n$  cannot converge.

① Show that the Cauchy product of the convergent series  $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+1}}$  with itself is not convergent. (19)

Sol Let  $a_n = b_n = \frac{(-1)^n}{\sqrt{n+1}}$ ,  $\forall n \in \mathbb{N}$ .

By Leibnitz's test, the series  $\sum_{n=1}^{\infty} a_n$  and  $\sum_{n=1}^{\infty} b_n$  are both convergent (but not absolutely).

The Cauchy product of the two series

is  $\sum_{n=1}^{\infty} c_n$ , where

$$\begin{aligned} c_n &= a_1 b_n + a_2 b_{n-1} + \dots + a_n b_1 \\ &= \frac{(-1)^1}{\sqrt{2}} \cdot \frac{(-1)^n}{\sqrt{n+1}} + \frac{(-1)^2}{\sqrt{3}} \cdot \frac{(-1)^{n-1}}{\sqrt{n}} + \dots + \frac{(-1)^n}{\sqrt{n+1}} \cdot \frac{(-1)^1}{\sqrt{2}} \\ &= (-1)^{n+1} \left[ \frac{1}{\sqrt{2(n+1)}} + \frac{1}{\sqrt{3(n+1)}} + \dots + \frac{1}{\sqrt{(n+1)(n+1)}} \right] \\ &\geq (-1)^{n+1} \left[ \frac{1}{\sqrt{(n+1)(n+1)}} + \frac{1}{\sqrt{(n+1)(n+1)}} + \dots + \frac{1}{\sqrt{(n+1)(n+1)}} \right] \\ &= (-1)^{n+1} \cdot \left( \frac{n}{n+1} \right). \end{aligned}$$

$$\therefore c_n \geq (-1)^{n+1} \left( \frac{n}{n+1} \right) = \text{r.m.s.}$$

$$|c_n| \geq \frac{n}{n+1}, \text{ when}$$

since  $\sum_{n=1}^{\infty} \frac{n}{n+1}$  is divergent (By comparison test)

$\Rightarrow \sum_{n=1}^{\infty} |c_n|$  is divergent.

$$\Rightarrow \lim_{n \rightarrow \infty} c_n \neq 0.$$

Hence  $\sum_{n=1}^{\infty} c_n$  cannot converge

④ Show that the Cauchy product of two divergent series  $\sum_{n=1}^{\infty} a_n = 2 + 2 + 2^2 + 2^2 + \dots$

and  $\sum_{n=1}^{\infty} b_n = 1 + 1 + 1 + 1 + 1 + \dots$  is convergent.

Sol for  $n \geq 2$ ,

$\sum a_n$  and  $\sum b_n$  are geometric series  
with common ratios 2 and 1 respectively.

Since the geometric series  $\sum r^n$  is divergent

for  $r \geq 1$ .

i. the series  $\sum a_n$  and  $\sum b_n$  are both divergent.

The Cauchy product of the two gives

series is  $\sum_{n=1}^{\infty} c_n$ , where

$$\begin{aligned}c_n &= a_1 b_n + a_2 b_{n-1} + a_3 b_{n-2} + \dots + a_n b_1 \\&= 2 \cdot 1 + 2 \cdot 1 + 2^2 \cdot 1 + 2^3 \cdot 1 + \dots \\&\quad \dots + 2^{n-2} \cdot 1 + 2^{n-1} (-1) \\&= 2 + (2 + 2^2 + 2^3 + \dots + 2^{n-2}) - 2^{n-1} \\&= 2 + \frac{2(2^{n-2} - 1)}{2-1} - 2^{n-1} \quad (\because r \geq 1) \\&= 2 + 2^{n-1} - 2 - 2^{n-1} \\&= 0\end{aligned}$$

and  $c_1 = a_1 b_1$   
 $= 2(-1)$   
 $= -2$

J.H.  $\Rightarrow -2$   
Thus  $\sum_{n=1}^{\infty} c_n = -2 + 0 + 0 + 0 + \dots$

clearly which goes to  $-2$ .

⑤ Show that the Cauchy product of two divergent

series  $\sum_{n=0}^{\infty} a_n = 1 - \frac{3}{2} - \left(\frac{3}{2}\right)^2 - \left(\frac{3}{2}\right)^3 - \dots$

and  $\sum_{n=0}^{\infty} b_n = 1 + \left(2 + \frac{1}{2^2}\right) + \frac{3}{2} \left(2^2 + \frac{1}{2^3}\right) + \left(\frac{3}{2}\right) \left(2^3 + \frac{1}{2^4}\right) + \dots$

is convergent.

Sol for  $\sum_{n=1}^{\infty} a_n$ ,  $\sum a_n$  is a geometric series with common ratio  $\frac{3}{2} (> 1)$ .

$\Rightarrow \sum a_n$  is divergent.

Also  $\sum b_n$  is a series of positive terms and  $b_n \neq 1$  for all

Since  $\lim_{n \rightarrow \infty} b_n \neq 0$

$\therefore \sum b_n$  is divergent.

The Cauchy product of the two given series

is  $\sum_{n=0}^{\infty} c_n$ , where

$$\begin{aligned}c_0 &= a_0 b_0 \\&= 1 \times 1 \\&= 1\end{aligned}$$

and for  $n \geq 1$ ,

$$c_n = a_0 b_n + a_1 b_{n-1} + a_2 b_{n-2} + \dots + a_{n-1} b_1 + a_n b_0$$

$$\begin{aligned}&= 1 \cdot \left(\frac{3}{2}\right)^{n-1} \left(2^n + \frac{1}{2^{n+1}}\right) - \left(\frac{3}{2}\right) \cdot \left(\frac{3}{2}\right)^{n-2} \left(2^{n-1} + \frac{1}{2^n}\right) \\&\quad - \left(\frac{3}{2}\right) \cdot \left(\frac{3}{2}\right)^{n-3} \left(2^{n-2} + \frac{1}{2^{n-1}}\right) - \dots \\&\quad - \left(\frac{3}{2}\right)^{n-2} \left(2^1 + \frac{1}{2^2}\right) - \left(\frac{3}{2}\right)^0\end{aligned}$$

$$\begin{aligned}&= \left(\frac{3}{2}\right)^{n-1} \left[ \left(2^n + \frac{1}{2^{n+1}}\right) - \left(2^{n-1} + 2^{n-2} + \dots + 2\right) \right. \\&\quad \left. - \left(\frac{1}{2^n} + \frac{1}{2^{n-1}} + \dots + \frac{1}{2^2}\right) \right] - \left(\frac{3}{2}\right)^0\end{aligned}$$

$$\begin{aligned}&= \left(\frac{3}{2}\right)^{n-1} \left[ 2^n + \frac{1}{2^{n+1}} - \frac{2(2^{n-1} - 1)}{2^1} - \frac{\frac{1}{2^n}(1 - \frac{1}{2^{n-1}})}{1 - \frac{1}{2}} \right] - \left(\frac{3}{2}\right)^0\end{aligned}$$

$$\begin{aligned}&= \left(\frac{3}{2}\right)^{n-1} \left[ 2^n + \frac{1}{2^{n+1}} - 2^n + 2 - \frac{1}{2} + \frac{1}{2^n} \right] - \left(\frac{3}{2}\right)^0\end{aligned}$$

$$\begin{aligned}&= \left(\frac{3}{2}\right)^{n-1} \left[ \frac{3}{2} + \frac{1}{2^{n+1}} + \frac{1}{2^n} \right] - \left(\frac{3}{2}\right)^0\end{aligned}$$

$$\begin{aligned}&= \left(\frac{3}{2}\right)^{n-1} \left[ \frac{3}{2} + \frac{3}{2^{n+1}} - \frac{3}{2} \right] = \frac{3^n}{2^{2n}} = \left(\frac{3}{4}\right)^n\end{aligned}$$

$$\Rightarrow \sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \left(\frac{3}{4}\right)^n$$

clearly which is a geometric series of positive terms with common ratio  $\frac{3}{4} < 1$ .  
is absolutely convergent.

(6)  $\rightarrow$  S.T. the Cauchy product of two divergent series

$$\sum_{n=1}^{\infty} a_n = 1 + \frac{3}{2} + \left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^3 + \dots$$

$$\text{and } \sum_{n=1}^{\infty} b_n = 1 + \left(2 + \frac{1}{2^n}\right) + \frac{3}{2} \left(2^2 + \frac{1}{2^{2n}}\right) + \dots$$

$$\left(\frac{3}{2}\right)^2 \left(2^2 + \frac{1}{2^4}\right) + \dots$$

convergent

(Note: In example (6)  $a_n$  is the  $(n+1)^{\text{th}}$  term of  $\sum_{n=0}^{\infty} a_n$  whereas in example (6),  $a_n$  is the  $n^{\text{th}}$  term of  $\sum_{n=0}^{\infty} a_n$ ).

$\rightarrow$  prove that the Cauchy product of the two series

$$3 + \sum_{n=1}^{\infty} 3^n \text{ and } -2 + \sum_{n=1}^{\infty} 2^n$$

is absolutely convergent, although both the series are divergent.

$$\text{Sol: Let } \sum_{n=0}^{\infty} a_n = 3 + 3 + 3^2 + 3^3 + \dots = 3 + \sum_{n=1}^{\infty} 3^n$$

$$\text{and } \sum_{n=0}^{\infty} b_n = -2 + 2 + 2^2 + 2^3 + \dots = -2 + \sum_{n=1}^{\infty} 2^n$$

$\rightarrow$  Show that

$$\left(1 - \frac{1}{2} + \frac{1}{3} - \dots\right)^2 = \sum_{n=1}^{\infty} (-1)^{n+1} \left[ \frac{1}{1 \cdot n} + \frac{1}{2(n-1)} + \dots + \frac{1}{n \cdot 1} \right]$$

$$\text{Sol: Let } 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \sum_{n=1}^{\infty} a_n,$$

then  $\sum a_n$  converges (conditionally).

By Abel's test, if the Cauchy product  $\sum_{n=1}^{\infty} c_n$  of  $\sum_{n=1}^{\infty} a_n$  with itself converges,

$$\text{then } \left( \sum_{n=1}^{\infty} a_n \right)^2 = \sum_{n=1}^{\infty} c_n \quad \text{--- (1)}$$

Now,

$$\begin{aligned} c_n &= 1 \cdot \frac{(-1)^{n+1}}{n} \left[ \frac{1}{2} + \frac{(-1)^{n-2}}{n-1} + \dots + \frac{(-1)^{n-2}}{2} \left( -\frac{1}{2} \right) + \frac{(-1)^n}{n} \cdot 1 \right] \\ &= (-1)^{n+1} \left[ \frac{1}{1 \cdot n} + \frac{1}{2(n-1)} + \dots + \frac{1}{(n-1) \cdot 2} + \frac{1}{n \cdot 1} \right] \quad \text{--- (2)} \\ &= \frac{(-1)^{n+1}}{n+1} \left[ \frac{n+1}{1 \cdot n} + \frac{n+1}{2(n-1)} + \dots + \frac{n+1}{(n-1) \cdot 2} + \frac{n+1}{n \cdot 1} \right] \\ &= \frac{(-1)^{n+1}}{n+1} \left[ \left( 1 + \frac{1}{n} \right) + \left( \frac{1}{2} + \frac{1}{n-1} \right) + \dots + \left( \frac{1}{n-1} + \frac{1}{2} \right) + \left( \frac{1}{n} + 1 \right) \right] \\ &= \frac{(-1)^{n+1}}{n+1} \left[ 2 + \frac{2}{2} + \dots + \frac{2}{n-1} + \frac{2}{n} \right] \\ &= (-1)^{n+1} \frac{2}{n+1} \left[ 1 + \frac{1}{2} + \dots + \frac{1}{n-1} + \frac{1}{n} \right] \\ \therefore |c_n| &= \frac{2}{n+1} \left[ 1 + \frac{1}{2} + \dots + \frac{1}{n} \right] \\ &= \frac{2n}{n+1} \left[ \frac{1 + \frac{1}{2} + \dots + \frac{1}{n}}{n} \right] \\ &= \frac{2}{1 + \frac{1}{n}} \left[ \frac{1 + \frac{1}{2} + \dots + \frac{1}{n}}{n} \right] \rightarrow 0 \text{ as } n \rightarrow \infty \end{aligned}$$

(by Cauchy's first theorem on limits)

$$\begin{aligned} \text{Also } |c_{n+1}| - |c_n| &= \frac{2}{n+2} \left( 1 + \frac{1}{2} + \dots + \frac{1}{n} + \frac{1}{n+1} \right) \\ &\quad - \frac{2}{n+1} \left( 1 + \frac{1}{2} + \dots + \frac{1}{n} \right) \\ &= \left( \frac{2}{n+2} - \frac{2}{n+1} \right) \left( 1 + \frac{1}{2} + \dots + \frac{1}{n} \right) + \frac{2}{(n+2)(n+1)} \\ &= \frac{-2}{(n+2)(n+1)} \left( 1 + \frac{1}{2} + \dots + \frac{1}{n} - 1 \right) \\ &= \frac{-2}{(n+2)(n+1)} \left( \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n} \right) < 0 \end{aligned}$$

$$\Rightarrow |c_n| > |c_{n+1}|$$

∴ By Leibnitz's test, the alternating series

$$\sum_{n=1}^{\infty} c_n = \sum_{n=1}^{\infty} (-1)^{n-1} \left[ \frac{1}{1 \cdot n} + \frac{1}{2(n-1)} + \dots + \frac{1}{n \cdot 1} \right] \quad (\text{by Q.})$$

Converges.  
Hence, from (1), we have

$$\left( 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \right) = \sum_{n=1}^{\infty} (-1)^{n-1} \left[ \frac{1}{1 \cdot n} + \frac{1}{2(n-1)} + \dots + \frac{1}{n \cdot 1} \right]$$

H.W. Show that

$$\left( 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots \right)^2 = 2 \left[ \frac{1}{2} - \frac{1}{3} (1 + \frac{1}{2}) + \frac{1}{4} (1 + \frac{1}{2} + \frac{1}{3}) - \frac{1}{5} (1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4}) + \dots \right]$$

H.W. Show that

$$\frac{1}{2} \left( 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots \right) = \frac{1}{2} - \frac{1}{4} (1 + \frac{1}{3}) + \frac{1}{6} (1 + \frac{1}{3} + \frac{1}{5}) - \dots$$

## \* Infinite products :-

If  $(a_n)$  is a sequence, then the product  $a_1 a_2 a_3 \dots a_n \dots$  is called an infinite product.

and is denoted by  $\prod_{n=1}^{\infty} a_n$  or simply by  $\prod a_n$   
i.e.  $\prod a_n = a_1 a_2 a_3 \dots a_n \dots$

$a_n$  is called the  $n^{\text{th}}$  factor of the product.

The product of first 'n' terms of the sequence  $(a_n)$  is called the  $n^{\text{th}}$  partial product and is denoted by  $P_n$ .

$$\therefore \text{this } P_n = a_1 a_2 a_3 \dots a_n \\ = \prod_{r=1}^n a_r$$

The sequence  $(P_n)$  is called the sequence of partial products of the sequence  $(a_n)$ .

## \* Convergence of infinite products :

Let  $P_n = \prod_{r=1}^n a_r$  be the  $n^{\text{th}}$  partial product of the infinite product  $\prod_{n=1}^{\infty} a_n$ .

(i) If no factor  $a_n$  is 'zero', then the product  $\prod_{n=1}^{\infty} a_n$  converges if the sequence  $(P_n)$  converges to a non-zero finite number  $P$  (say),

i.e. if  $\lim_{n \rightarrow \infty} P_n = P$ , then  $P$  is called the value of the product and we write  $\prod a_n = P$ .

If  $\lim_{n \rightarrow \infty} P_n = \infty$  then the product  $\prod_{n=1}^{\infty} a_n$  is

said to diverge to  $\infty$ .

If  $\lim_{n \rightarrow \infty} P_n = 0$ , then the product  $\prod_{n=1}^{\infty} a_n$  is said to diverge to '0'.

(92)

(i) If infinitely many factors  $a_n$  are zero, then the product  $\prod_{n=1}^{\infty} a_n$  is said to diverge to 0.

(ii) If finitely many factors  $a_n$  are zero, then the product  $\prod_{n=1}^{\infty} a_n$  is said to converge if it converges when the zero factors are removed.

(iii) If a finite number of factors are negative, then there exists a positive integer  $m$  such that  $a_m > 0 \forall n > m$  and the product  $\prod_{n=1}^{\infty} a_n$  is said to converge if the product  $\prod_{n=m+1}^{\infty} a_n$  converges.

(iv) If the sequence  $(P_n)$  oscillates, then the product  $\prod_{n=1}^{\infty} a_n$  is said to oscillate.

Note: (1) It is usually convenient to write the factors of the infinite product as  $1+a_n$  instead of  $a_n$ .

Thus an infinite product is usually written as  $\prod_{n=1}^{\infty} (1+a_n)$  and  $P_n = \prod_{k=1}^n (1+a_k)$ .

(2) we shall assume throughout our discussion that  $a_n > -1$  &  $a_n > 0 \forall n$  so that  $\log(1+a_n)$  is defined for all  $n$ .

(3) for  $a_n > -1$ , let  $P_n$  denote the  $n^{th}$  partial product of  $\prod_{n=1}^{\infty} (1+a_n)$ , then

$$P_n = (1+a_1)(1+a_2) \cdots \cdots (1+a_n)$$

$$\Rightarrow \log P_n = \log(1+a_1) + \log(1+a_2) + \cdots + \log(1+a_n)$$

where  $s_n = \sum_{r=1}^n \log(1+a_r)$  is the  $n^{\text{th}}$  partial sum of the series  $\sum_{n=1}^{\infty} \log(1+a_n)$

(27)

$$\Rightarrow P_n = e^{s_n}$$

If  $\lim_{n \rightarrow \infty} s_n = s$  then  $\lim_{n \rightarrow \infty} P_n = e^s$ .

Thus, to say that the product  $\prod_{n=1}^{\infty} (1+a_n)$  diverges to zero,

i.e.  $\lim_{n \rightarrow \infty} P_n = 0$  is equivalent to saying that the series  $\sum_{n=1}^{\infty} \log(1+a_n)$  diverges to  $-\infty$ .

i.e.  $\lim_{n \rightarrow \infty} s_n = -\infty$ . (i.e.  $\lim_{n \rightarrow \infty} P_n = e^{-\infty} = 0$ ).

### Problem

Show that the infinite product

$$(1 - \frac{1}{2^2})(1 - \frac{1}{3^2})(1 - \frac{1}{4^2}) \cdots$$

converges to  $\frac{1}{2}$ .

Sol. The given infinite product is

$$\begin{aligned} & (1 - \frac{1}{2^2})(1 - \frac{1}{3^2})(1 - \frac{1}{4^2}) \cdots = \prod_{n=1}^{\infty} \left(1 - \frac{1}{(n+1)^2}\right) \\ & = \prod_{n=1}^{\infty} \frac{(n+1)^2 - 1}{(n+1)^2} \\ & = \prod_{n=1}^{\infty} \frac{n(n+2)}{(n+1)^2} \\ & = \prod_{n=1}^{\infty} \left(\frac{n}{n+1} \cdot \frac{n+2}{n+1}\right) \end{aligned}$$

$$\therefore P_n = \left(\frac{1}{2} \cdot \frac{3}{2}\right) \left(\frac{2}{3} \cdot \frac{4}{3}\right) \left(\frac{3}{4} \cdot \frac{5}{4}\right) \cdots \left(\frac{n}{n+1} \cdot \frac{n+2}{n+1}\right)$$

$$= \left(\frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdots \frac{n}{n+1}\right) \left(\frac{3}{2} \cdot \frac{4}{3} \cdots \frac{n+2}{n+1}\right)$$

$$= \left(\frac{1}{n+1}\right) \left(\frac{n+2}{2}\right) = \frac{1}{2} \left(1 + \frac{1}{n+1}\right)$$

$$\Rightarrow \lim_{n \rightarrow \infty} P_n = \frac{1}{2}$$

Hence the given infinite product converges to  $\frac{1}{2}$ .

$$\text{i.e. } \prod_{n=1}^{\infty} \left(1 - \frac{1}{(n+1)^2}\right) = \frac{1}{2}.$$

QED show that the infinite product

$$(1 - \frac{2}{2 \cdot 3}) (1 - \frac{2}{4 \cdot 5}) (1 - \frac{2}{6 \cdot 7}) \dots$$

converges to  $\frac{1}{3}$ .

→ show that the infinite products

$$(i) \prod_{n=1}^{\infty} (1 + \frac{1}{n}) \text{ and } (ii) \prod_{n=2}^{\infty} (1 - \frac{1}{n}) \text{ are both divergent}$$

Sol: (i) Given infinite product is

$$\prod_{n=1}^{\infty} (1 + \frac{1}{n}) = \prod_{n=1}^{\infty} (\frac{n+1}{n})$$

$$= \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \frac{5}{4} \cdot \dots \cdot \frac{(n+1)}{n} \dots$$

$$\text{Let } P_n = \prod_{r=1}^n \frac{(r+1)}{r}.$$

$$= \frac{2}{1} \cdot \frac{3}{2} \cdot \frac{4}{3} \cdot \frac{5}{4} \cdot \dots \cdot \frac{(n+1)}{n}.$$

$$= n+1$$

$$\text{Now if } P_n = \infty \text{ as } n \rightarrow \infty$$

∴ The given infinite product  $\prod_{n=1}^{\infty} (1 + \frac{1}{n})$  is divergent and goes to  $\infty$ .

$$\text{i.e. } \prod_{n=1}^{\infty} (1 + \frac{1}{n}) = \infty.$$

$$(ii) \prod_{n=2}^{\infty} (1 - \frac{1}{n}) = \prod_{n=2}^{\infty} (\frac{n-1}{n})$$

$$\therefore P_n = \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{4} \cdot \dots \cdot \frac{n-1}{n} = \frac{1}{n+1}$$

$$\Rightarrow \lim_{n \rightarrow \infty} P_n = 0$$

$$\therefore \prod_{n=2}^{\infty} (1 - \frac{1}{n}) \text{ goes to } 0$$

→ show that the infinite product

$$(1 - \frac{1}{2}) (1 - \frac{1}{3}) (1 - \frac{1}{4}) \dots$$

is convergent.

*(Please see the notes for the proof)*  
Given infinite product is clearly convergent

$$\text{Let } P = \left(1 - \frac{1}{2^2}\right) \left(1 - \frac{1}{3^2}\right) \left(1 - \frac{1}{4^2}\right) \dots$$

Then  $\log P = \log \left(1 - \frac{1}{2^2}\right) + \log \left(1 - \frac{1}{3^2}\right) + \dots$

(24)

$$= \sum_{n=2}^{\infty} \log \left(1 - \frac{1}{n^2}\right)$$

$$= \sum_{n=2}^{\infty} a_n \text{ (say)} \quad \text{--- (1)}$$

$$\text{Now } a_n = \log \left(1 - \frac{1}{n^2}\right)$$

$$= \left[ \frac{1}{n^2} + \frac{1}{2n^4} + \frac{1}{3n^6} + \dots \right]$$

$$\therefore \log \left(1 - \frac{1}{n^2}\right) = -\left(\alpha + \frac{\alpha^2}{2} + \frac{\alpha^3}{3} + \dots\right)$$

$$= -\frac{1}{n^2} \left[ 1 + \frac{1}{2n^2} + \frac{1}{3n^4} + \dots \right].$$

$$\text{Let } b_n = \frac{1}{n^2} + \alpha n.$$

Then  $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \left[ 1 + \frac{1}{2n^2} + \dots \right]$

$$= 1 \neq 0$$

Here  $\sum b_n = \sum \frac{1}{n^2}$  is cgt (by p-test)

∴ by comparison test,

$$\sum a_n \text{ is cgt.}$$

and is convergent to finite number  $S$  (say)

∴ from (1),  $\log P = \text{a finite number's when } n \rightarrow \infty$   
if  $P = \text{a finite number's when } n \rightarrow \infty$

∴ The given product is cgt.

~~Now~~ in the infinite product  
 ~~$\left(1 - \frac{1}{2^2}\right) \left(1 - \frac{1}{3^2}\right) \left(1 - \frac{1}{4^2}\right) \dots$~~  is converges

QED Show that the infinite product

$$\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{5}{6} \cdot \frac{7}{6} \cdots \frac{(2n-1)}{2n} \cdot \frac{2n+1}{2n}$$

tends to a finite limit as  $n \rightarrow \infty$ .

S.T. the infinite product

$$(1 + \frac{1}{2})(1 - \frac{1}{3})(1 + \frac{1}{4})(1 - \frac{1}{5}) \cdots$$

converges to 1.

S.Q. Given infinite product is

$$(1 + \frac{1}{2})(1 - \frac{1}{3})(1 + \frac{1}{4})(1 - \frac{1}{5}) \cdots$$

$$\text{Let } P = (1 + \frac{1}{2})(1 - \frac{1}{3})(1 + \frac{1}{4})(1 - \frac{1}{5}) \cdots = (\frac{1 + \frac{1}{2}}{2})(\frac{1 - \frac{1}{3}}{3}) \cdots$$

$$\therefore \log P = \log \left[ (1 + \frac{1}{2})(1 + \frac{1}{4}) \right] + \log \left[ (1 - \frac{1}{3})(1 - \frac{1}{5}) \right]$$

$$+ \cdots + \log \left[ (1 + \frac{1}{m})(1 - \frac{1}{m+1}) \right] + \cdots$$

$$= \sum_{n=1}^{\infty} \log \left[ (1 + \frac{1}{2n})(1 - \frac{1}{2n+1}) \right]$$

$$= \sum_{n=1}^{\infty} \log \left[ 1 + \frac{1}{2n} - \left( 1 + \frac{1}{2n} \right) \left( \frac{1}{2n+1} \right) \right]$$

$$= \sum_{n=1}^{\infty} \log \left[ 1 + \left( \frac{1}{2n} - \frac{1}{2n+1} \right) - \frac{1}{2n(2n+1)} \right].$$

$$= \sum_{n=1}^{\infty} \log \left[ 1 + \frac{1}{2n(2n+1)} - \frac{1}{2n(2n+1)} \right]$$

$$= \sum_{n=1}^{\infty} \log [1] = 0.$$

$$\therefore \log P = 0 \Rightarrow P = e^0 = 1.$$

$\therefore$  The given infinite product is convergent and it converges to 1.

A necessary condition for convergence:

If the product  $\prod_{n=1}^{\infty} (1+a_n)$  is convergent, then

$$\lim_{n \rightarrow \infty} a_n = 0$$

proof Given that  $\prod_{n=1}^{\infty} (1+a_n)$  is cgt.  
and it cgts to  $P$  (say)

(25)

$\therefore P \neq 0; \lim_{n \rightarrow \infty} p_n = P$  and  $\lim_{n \rightarrow \infty} p_{n+1} = P$ .

$$\text{Now } \frac{p_n}{p_{n-1}} = \frac{(1+a_1)(1+a_2) \dots (1+a_{n-1})(1+a_n)}{(1+a_1)(1+a_2) \dots (1+a_{n-1})}$$

$$\frac{p_n}{p_{n-1}} = (1+a_n)$$

$$\therefore \lim_{n \rightarrow \infty} (1+a_n) = \lim_{n \rightarrow \infty} \frac{p_n}{p_{n-1}} = \lim_{n \rightarrow \infty} \frac{p_{n+1}}{p_n} = 1.$$

$$\lim_{n \rightarrow \infty} (1+a_n) = 1 \Rightarrow \lim_{n \rightarrow \infty} 1+a_n = 1 \Rightarrow \boxed{\lim_{n \rightarrow \infty} 1+a_n = 0}.$$

Note: - The converse of the above need not be true. i.e. if  $\lim_{n \rightarrow \infty} a_n = 0$  then  $\prod_{n=1}^{\infty} (1+a_n)$  need not be cgt.

for example

The infinite product is

$$\prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right) = \prod_{n=1}^{\infty} (1+a_n)$$

Here  $a_n = \frac{1}{n} \rightarrow 0$  as  $n \rightarrow \infty$

but the product is divergent. (already done)

General principle of convergence of an infinite product:

A necessary and sufficient condition for the convergence of the infinite product

for the convergence of the infinite product  $\prod_{n=1}^{\infty} (1+a_n)$  is that for every  $\epsilon > 0$ , there exists

a positive integer 'm' s.t.  $\left| \frac{p_{n+1}}{p_n} - 1 \right| < \epsilon \quad \forall n \geq m$ .

Note: In order to establish the convergence (or divergence) of infinite product, we now give the following statements:

- If  $a_n > 0$  then the series  $\sum_{n=1}^{\infty} a_n$  and  
the product  $\prod_{n=1}^{\infty} (1+a_n)$  converge or diverge  
together.
- If  $-1 < a_n \leq 0$ , then the series  $\sum_{n=1}^{\infty} a_n$  and  
the product  $\prod_{n=1}^{\infty} (1+a_n)$  converge or diverge  
together.
- If  $0 \leq b_n < 1$  then  $\prod_{n=1}^{\infty} (1+b_n)$  converges to  
non-zero finite limit, if  $\sum_{n=1}^{\infty} b_n$  converges and  
diverges to zero if  $\sum_{n=1}^{\infty} b_n$  diverges.
- If the series  $\sum_{n=1}^{\infty} a_n^2$  is convergent,  
then the product  $\prod_{n=1}^{\infty} (1+a_n)$  and series  $\sum_{n=1}^{\infty} a_n$   
converge or diverge together.
- If  $\sum_{n=1}^{\infty} a_n^2$  is convergent, then we have  $\sum_{n=1}^{\infty} a_n$  and  
 $\sum_{n=1}^{\infty} \log(1+a_n)$  converge or diverge together.  
Also  $\sum_{n=1}^{\infty} \log(1+a_n)$  and  $\sum_{n=1}^{\infty} (1+a_n)$  converge or diverge  
together.

- We have
- if  $\sum_{n=1}^{\infty} a_n$  converges, then  $\sum_{n=1}^{\infty} \log(1+a_n)$  converges  
and therefore  $\prod_{n=1}^{\infty} (1+a_n)$  converges.
- if  $\sum_{n=1}^{\infty} a_n$  diverges to  $\infty$ , then  $\sum_{n=1}^{\infty} \log(1+a_n)$  diverges  
to  $\infty$  and therefore  $\prod_{n=1}^{\infty} (1+a_n)$  diverges to  $\infty$ .
- if  $\sum_{n=1}^{\infty} a_n$  diverges to  $-\infty$ , then  $\sum_{n=1}^{\infty} \log(1+a_n)$  diverges  
to  $-\infty$  and therefore  $\prod_{n=1}^{\infty} (1+a_n)$  diverges to zero.

Also, if  $\sum_{n=1}^{\infty} a_n$  diverges and  $\sum_{n=1}^{\infty} |a_n|$  converges or oscillates finitely, then  $\prod_{n=1}^{\infty} (1+a_n)$  diverges to zero. (26)

→ Absolute convergence of infinite products

Def: The product  $\prod_{n=1}^{\infty} (1+a_n)$  is said to be absolutely convergent if the product  $\prod_{n=1}^{\infty} (1+|a_n|)$  is convergent.

→ The product  $\prod_{n=1}^{\infty} (1+a_n)$  is absolutely convergent iff the series  $\sum_{n=1}^{\infty} |a_n|$  is absolutely convergent.

→ The product  $\prod_{n=1}^{\infty} (1+a_n)$  is absolutely convergent iff the series  $\sum_{n=1}^{\infty} \log(1+a_n)$  is absolutely convergent.

→ Every absolutely convergent infinite product is convergent.  
i.e. If  $\prod_{n=1}^{\infty} (1+a_n)$  is an absolutely convergent  
(i.e.  $\prod_{n=1}^{\infty} (1+|a_n|)$  is cgt).

Then  $\prod_{n=1}^{\infty} (1+a_n)$  is cgt.

Note!:- The factors of an absolutely convergent infinite product may be rearranged in any order without affecting its convergence.

problems

→ Discuss the convergence of the infinite products:

$$(i) \prod_{n=1}^{\infty} \left(1 + \frac{1}{n^2}\right) \quad (ii) \prod_{n=1}^{\infty} \left(1 + \frac{1}{n^{3/2}}\right) \quad (iii) \prod_{n=1}^{\infty} \left(1 + \frac{1}{n^x}\right), x > 1$$

$$(iv) \prod_{n=1}^{\infty} \left(1 + \frac{1}{n^x}\right), 0 < x \leq 1 \quad (v) \prod_{n=1}^{\infty} \left(1 + \frac{1}{\sqrt[n]{n}}\right) \quad (vi) \prod_{n=1}^{\infty} \left(\frac{n+1}{n}\right)$$

Sol. (i) The given product is

$$\prod_{n=1}^{\infty} \left(1 + \frac{1}{n^2}\right) = \prod_{n=1}^{\infty} (1+a_n), \text{ where } a_n = \frac{1}{n^2} > 0$$

Since  $\sum_{n=1}^{\infty} \frac{1}{n^2}$  is cgt (by p-test)  
here  $p=2 > 1$

$$\therefore \text{the product } \prod_{n=1}^{\infty} \left(1 + \frac{1}{n^2}\right) = \prod_{n=1}^{\infty} \left(1 + \frac{1}{n^2}\right)$$

is cgt

→ Discuss the convergence of the infinite products:

$$(i) \prod_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right) \quad (ii) \prod_{n=2}^{\infty} \left(1 - \frac{1}{n}\right) \quad (iii) \prod_{n=2}^{\infty} \left(1 - \frac{1}{\sqrt{n}}\right)$$

$$(iv) \frac{\frac{3}{4} \cdot \frac{5}{7} \cdot \frac{7}{10} \cdots \frac{2n-1}{3n+1}}{=} \prod_{n=1}^{\infty} \left(1 - \frac{1}{3n+1}\right)$$

$$= \prod_{n=1}^{\infty} \left(1 - \frac{1}{2n+1}\right)$$

Sol (i) The given product is

$$= \prod_{n=2}^{\infty} \left(1 - \frac{1}{n^2}\right) = \prod_{n=2}^{\infty} \left(1 - b_n\right), \text{ where } b_n = \frac{1}{n^2}$$

and  $n \geq 2$

so that  $0 < b_n < 1$

∴ The product  $\prod_{n=2}^{\infty} \left(1 - b_n\right)$  and the series  $\sum_{n=2}^{\infty} b_n$  converge or diverge together.

But the series  $\sum_{n=2}^{\infty} b_n = \sum_{n=2}^{\infty} \frac{1}{n^2}$  is cgt  
(by p-test)

∴ the given product is cgt

(ii) The product is  $\prod_{n=2}^{\infty} \left(1 - \frac{1}{3n+1}\right) = \prod_{n=1}^{\infty} \left(1 - b_n\right)$

where  $b_n = \frac{1}{3n+1}$  and  $n \geq 1$

so that  $0 < b_n < 1$ .

∴ The product  $\prod_{n=1}^{\infty} \left(1 - b_n\right)$  and the series  $\sum_{n=1}^{\infty} b_n$  converge or diverge together.

But the series  $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{3n+1}$  does

∴ The product  $\prod_{n=1}^{\infty} \left(1 - b_n\right)$  does not go to zero.

→ Discuss the convergence of the infinite products:

$$(i) \prod_{n=1}^{\infty} \left(1 + \sin^2 \frac{\alpha}{n}\right) \quad (ii) \prod_{n=1}^{\infty} \left(1 + n \sin \frac{\alpha}{n^2}\right)$$

$$(iii) \prod_{n=1}^{\infty} \left(1 + \frac{a}{n^p}\right), \text{ where } a \text{ is a real number}$$

Sol The given product is  $\prod_{n=1}^{\infty} \left(1 + \sin^2 \frac{\alpha}{n}\right) = \prod_{n=1}^{\infty} (1 + a_n)$

$$\text{where } a_n = \sin^2 \frac{\alpha}{n} \geq 0 \text{ for all } n.$$

The product  $\prod_{n=1}^{\infty} (1 + a_n)$  and the

series  $\sum a_n$  converge or diverge

together.

$$\begin{aligned} \text{Now, } a_n &= \sin^2 \frac{\alpha}{n} = \left(\sin \frac{\alpha}{n}\right)^2 \\ &= \left(\frac{\alpha}{n} - \frac{1}{2!} \cdot \frac{\alpha^3}{n^3} + \frac{1}{5!} \cdot \frac{\alpha^5}{n^5} - \dots\right)^2 \\ &= \frac{\alpha^2}{n^2} - 2 \left(\frac{1}{2!} \cdot \frac{\alpha^4}{n^4}\right) + \dots \\ &= \frac{1}{n^2} \left[\alpha^2 - 2 \left(\frac{1}{2!} \cdot \frac{\alpha^4}{n^4}\right) + \dots\right] \end{aligned}$$

$$\text{Take } b_n = \frac{1}{n^2}$$

$$\therefore \lim_{n \rightarrow \infty} \frac{1 + a_n}{b_n} = \alpha^2$$

Since  $\sum b_n = \sum \frac{1}{n^2}$  is g.t.

∴ by comparison test  $\sum a_n$  is g.t.

Hence the given product is g.t.

$$(iii) \prod_{n=1}^{\infty} \left(1 + \frac{a}{n^p}\right) = \prod_{n=1}^{\infty} (1 + a_n)$$

$$\text{where } a_n = \frac{a}{n^p} \geq 0 \text{ for all } n.$$

∴ The product  $\prod_{n=1}^{\infty} (1 + a_n)$  and the series  $\sum a_n$  converge or diverge together.

$$\text{Now } \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{a}{n^p} = a \sum_{n=1}^{\infty} \frac{1}{n^p}$$

which is cgt if  $p > 1$

and dgt if  $p \leq 1$ .

Hence the given product is also convergent  
if  $p > 1$  and dgt if  $p \leq 1$ .

$$\rightarrow \prod_{n=1}^{\infty} (1+a) \left(1+\frac{a}{2}\right) \left(1+\frac{a}{3}\right) \dots$$

dgs to  $+\infty$  or to '0' according as  
 $a > 0$  or  $a < 0$ .

Sol The given product is

$$\prod_{n=1}^{\infty} \left(1+\frac{a}{n}\right) = \prod_{n=1}^{\infty} (1+a_n)$$

where  $a_n = \frac{a}{n}$ .

$$\text{Now } \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{a}{n} = a \sum_{n=1}^{\infty} \frac{1}{n}$$

which dgs to  $+\infty$  if  $a > 0$

dgs to  $-\infty$  if  $a < 0$ .

Hence the given product dgs to  $\infty$  if  $a > 0$   
dgs to  $-\infty$  if  $a < 0$ .

Discuss the convergence of the product:

$$(i) \prod_{n=2}^{\infty} \left(1 + \frac{(-1)^n}{n}\right)$$

$$(ii) \left(1 - \frac{1}{2}\right) \left(1 + \frac{1}{3}\right) \left(1 + \frac{1}{4}\right) \left(1 + \frac{1}{5}\right) \dots$$

$$(iii) \left(1 + \frac{1}{2}\right) \left(1 + \frac{1}{3}\right) \left(1 + \frac{1}{4}\right) \left(1 + \frac{1}{5}\right) \dots$$

$$(iv) \left(1 + \frac{1}{2}\right) \left(1 - \frac{1}{3}\right) \left(1 + \frac{1}{4}\right) \left(1 - \frac{1}{5}\right) \dots$$

$$(v) \prod_{n=1}^{\infty} \left(1 + \frac{(-1)^{n+1}}{\sqrt{n}}\right).$$

Sol (1) The given product is

$$\prod_{n=2}^{\infty} \left(1 + \frac{(-1)^n}{n}\right) = \prod_{n=2}^{\infty} (1+a_n) \text{ where } a_n = \frac{(-1)^n}{n}$$

now  $\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{(-1)^n}{n}$  is ~~cgt~~ by Leibnitz's test

and  $\sum_{n=2}^{\infty} a_n^2 = \sum_{n=2}^{\infty} \frac{1}{n^2}$  is ~~cgt~~ given

$\therefore$  the product is ~~cgt~~.

→ Discusses the convergence of the infinite product  $(1 - \frac{1}{2})(1 + \frac{1}{2})(1 - \frac{1}{2})(1 + \frac{1}{2})\dots$

Sol The given product is  $\prod_{n=1}^{\infty} \left(1 + (-1)^n \cdot \frac{1}{2}\right)$

$$= \prod_{n=1}^{\infty} (1+a_n) \text{ where } a_n = (-1)^n \cdot \frac{1}{2}$$

$$\text{Now } \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{2}$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} (-1)^n$$

$$= \frac{1}{2} (-1 + 1 - 1 + 1 - \dots)$$

which oscillates b/w  $-\frac{1}{2}$  and  $0$ .

$$\sum_{n=1}^{\infty} a_n^2 = \sum_{n=1}^{\infty} \frac{1}{4} = \frac{1}{4} + \frac{1}{4} + \dots$$

which is ~~cgt~~ ( $\rightarrow \infty$ )

Hence the given product ~~div~~ to zero.

H.W Discuss the convergence of  $\prod_{n=1}^{\infty} (1 + (-1)^n)$ .

→ Show that the infinite product

$\prod_{n=2}^{\infty} \left(1 + \frac{(-1)^n}{n^2}\right)$  is convergent if  $\alpha > \frac{1}{2}$ .

Sol The given product is

$$\prod_{n=2}^{\infty} \left(1 + \frac{(-1)^n}{n^2}\right) = \prod_{n=2}^{\infty} (1+a_n)$$

where  $a_n = \frac{(-1)^n}{n^2}$

Now  $\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2}$  cgs if  $\alpha > 0$  (by Leibniz's test)

Also  $\sum_{n=2}^{\infty} a_n = \sum_{n=2}^{\infty} \frac{1}{n^2}$  cgs if  $\alpha > 1$

Given  
Hence the product cgs if  $\alpha > 1$ .

Sol Discuss the convergence of the product

$$\prod_{n=1}^{\infty} \left[1 + \left(\frac{n\alpha}{n+1}\right)^n\right]$$

Sol Here  $a_n = \left(\frac{n\alpha}{n+1}\right)^n$

$$\therefore a_n^{1/n} = \frac{n\alpha}{n+1} = \frac{\alpha}{1 + \frac{1}{n}}$$

$$\therefore \lim_{n \rightarrow \infty} a_n^{1/n} = \alpha.$$

By Cauchy's root test, the series  $\sum_{n=1}^{\infty} a_n$  is  
cgt if  $\alpha > 1$  and  
dgt if  $\alpha < 1$ .

Hence the given product is cgt if  $\alpha < 1$

and dgt if  $\alpha > 1$ .

If  $\alpha = 1$  then  $a_n = \left(\frac{n}{n+1}\right)^n = \left(\frac{1}{1 + \frac{1}{n}}\right)^n$

$$\therefore \lim_{n \rightarrow \infty} a_n = \frac{1}{e} \neq 0 \text{ and } a_n > 0 \text{ for all } n.$$

$\therefore \sum a_n$  is dgt.

Hence  $\prod_{n=1}^{\infty} (1+a_n)$  is dgt.

Thus the given product is cgt if  $\alpha < 1$  and dgt if  $\alpha > 1$ .

→ Discusses absolute convergence of the following infinite products:

$$(i) \prod_{n=1}^{\infty} \cos \frac{\alpha}{n} \quad (ii) \prod_{n=1}^{\infty} \left[ \frac{\sin \frac{\alpha}{n}}{\frac{\alpha}{n}} \right].$$

Sol (i) Here  $1+a_n = \cos \frac{\alpha}{n}$ .

$$1 - \frac{1}{2!} \cdot \frac{\alpha^2}{n^2} + \frac{1}{4!} \cdot \frac{\alpha^4}{n^4} - \dots$$

$$\Rightarrow a_n = \frac{1}{2!} \cdot \frac{\alpha^2}{n^2} - \frac{1}{4!} \cdot \frac{\alpha^4}{n^4} - \dots$$

$$= \frac{1}{n^2} \left( -\frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \dots \right)$$

$$\text{Now } |a_n| = \frac{1}{n^2} \left| -\frac{\alpha^2}{2!} + \frac{\alpha^4}{4!} - \dots \right|.$$

$$\text{Let } b_n = \frac{1}{n^2} \text{ then } \lim_{n \rightarrow \infty} \frac{|a_n|}{b_n} = \frac{\alpha^2}{2},$$

(a finite quantity)

But  $\sum b_n = \sum \frac{1}{n^2}$  is  $\text{gt}$  (by p-test).

$\sum |a_n|$  is  $\text{gt}$  (by comparison test).

$\therefore \sum a_n$  is absolutely convergent.

The product  $\prod_{n=1}^{\infty} (1+a_n) \leq \prod_{n=1}^{\infty} \cos \frac{\alpha}{n}$  is

absolutely  $\text{gt}$ .

(ii) Here  $1+a_n = \frac{\sin \frac{\alpha}{n}}{\frac{\alpha}{n}}$

$$= \frac{1}{(\frac{\alpha}{n})} \left[ \frac{\alpha}{n} - \frac{1}{2!} \cdot \frac{\alpha^2}{n^2} + \frac{1}{4!} \cdot \frac{\alpha^4}{n^4} - \dots \right]$$

$$= 1 - \frac{1}{3!} \cdot \frac{\alpha^2}{n^2} + \frac{1}{5!} \cdot \frac{\alpha^4}{n^4} - \dots$$

$$\Rightarrow a_n = -\frac{1}{3!} \cdot \frac{\alpha^2}{n^2} + \frac{1}{5!} \cdot \frac{\alpha^4}{n^4} - \dots$$

$$\therefore = \frac{1}{n^n} \left( -\frac{x^2}{3!} + \frac{x^3}{5!} - \dots \right) \text{ proceed in this way.}$$

$\rightarrow$  prove that  $\prod_{n=1}^{\infty} \left(1 + \frac{x}{n}\right) e^{-x/n}$  is absolutely convergent for any real  $x$ .

$$\underline{\text{Sol}} \quad \text{Here } 1+a_n = \left(1 + \frac{x}{n}\right) e^{-x/n}$$

$$= \left(1 + \frac{x}{n}\right) \left(1 - \frac{x}{n} + \frac{x^2}{2!n^2} - \frac{x^3}{3!n^3} \dots\right)$$

$$= 1 - \frac{x^2}{n^2} + \frac{x^2}{2!n^2} + \frac{x^3}{2!n^3} - \frac{x^3}{6!n^6} \dots$$

$$\Rightarrow a_n = \frac{-x^2}{2!n^2} + \frac{x^3}{3!n^3} \dots$$

$$= \frac{1}{n^n} \left( -\frac{x^2}{2} + \frac{x^3}{2!} - \dots \right) \dots$$

proceed

in this way.

P.T  $\prod_{n=1}^{\infty} \left(1 + \frac{x}{n^n}\right) e^{-\frac{x}{n^n}}$  is absolutely cgt  
for all values of  $x$ .

H.W P.T  $\prod_{n=1}^{\infty} \left(1 + \frac{1}{n}\right) e^{-1/n}$  is absolutely cgt.

$\rightarrow$  Test the absolutely convergence of the infinite product  $\prod_{n=1}^{\infty} \frac{(x+x^{2n})}{1+x^{2n}}$ .

$$\underline{\text{Sol}} \quad \text{Here } 1+a_n = \frac{x+x^{2n}}{1+x^{2n}} = \frac{(1+x^{2n})+(x-1)}{1+x^{2n}}$$

$$\Rightarrow 1+a_n = 1 + \frac{x-1}{1+x^{2n}}$$

$$\Rightarrow \boxed{a_n = \frac{x-1}{1+x^{2n}}}$$

Now, when  $|x| > 1$ , we have

$$|a_n| = \left| \frac{n-1}{1+x^{2n}} \right| = \frac{|x-1|}{|1+x^{2n}|} = \frac{|x-1|}{1+x^{2n}} < \frac{|x-1|}{x^{2n}} \quad (1)$$

Now  $\sum_{n=1}^{\infty} \frac{1}{x^{2n}} = \sum u_n$  (say).

Here  $u_n = \frac{1}{x^{2n}}$ .

$$u_n \approx \frac{1}{x^2}$$

$$\therefore \text{L.R.} = \frac{1}{x^2} < 1 \quad (\because |x| > 1)$$

$\therefore$  by Cauchy's root test  $\sum \frac{1}{x^{2n}}$  is ge.

$\therefore$  by comparison test,

$\sum |a_n|$  is ge.

$\therefore \sum a_n$  is absolutely ge.

Hence  $\prod_{n=1}^{\infty} (1+a_n)$  is absolutely convergent.

Now, when  $|x| < 1$  (i.e.  $-1 < x < 1$ ),

we have

$$1+a_n = \frac{x+x^{2n}}{1+x^{2n}} = \frac{x(1+x^{2n-1})}{1+x^{2n}} \rightarrow x \quad (\because -1 < x < 1)$$

$\therefore 1+a_n \rightarrow x$  as  $n \rightarrow \infty$ .

$\Rightarrow a_n \rightarrow x-1$ . as  $n \rightarrow \infty$ .

$\Rightarrow a_n$  does not tend to 0 ( $\because -1 < x < 1$ )

i.e.  $x+a_n \neq 0$ .

$\therefore$  the product  $\prod_{n=1}^{\infty} (1+a_n)$  is divergent.

Now, when  $x = 1$ ,

every factor is unity

$$\text{i.e. } \prod_{n=1}^{\infty} \left( \frac{x+x^{2n}}{1+x^{2n}} \right) = \left( \frac{1+1}{1+1} \right) \left( \frac{1+1^4}{1+1^4} \right) \dots \dots \dots$$

$$= \frac{2}{2} \cdot \frac{2}{2} \cdot \dots \dots \dots$$

$$= 1 \cdot 1 \cdot \dots \dots \dots$$

Hence the product is convergent.

Now, when  $x = -1$ ,

every factor is zero.

Hence the product is divergent.

Discuss the convergence of the infinite product

$$\prod_{n=1}^{\infty} \left(1 + \frac{x^n}{x^{2n} + 1}\right).$$

Sol Here  $1 + a_n = 1 + \frac{x^n}{x^{2n} + 1}$  so that  $a_n = \frac{x^n}{x^{2n} + 1}$

Now  $a_{n+1} = \frac{x^{n+1}}{x^{2n+2} + 1}$

$$\begin{aligned} \left| \frac{a_n}{a_{n+1}} \right| &= \left| \frac{x^n}{x^{2n} + 1} \cdot \frac{x^{2n+2} + 1}{x^{n+1}} \right| \\ &= \left| \frac{x^{2n+2} + 1}{x(x^{2n} + 1)} \right| = \frac{|x^{2n+2} + 1|}{|x||x^{2n} + 1|} \end{aligned}$$

If  $|x| < 1$  (i.e.  $-1 < x < 1$ ),

$$\lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| = \frac{1}{|x|} > 1$$

By ratio test,  $\sum |a_n|$  cgs and hence  
 $\prod (1 + a_n)$  cgs absolutely.

If  $|x| > 1$ ,

$$\begin{aligned} \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| &= \lim_{n \rightarrow \infty} \left| \frac{x^{2n+2} + 1}{x^{2n+1} + x} \right| \\ &= \lim_{n \rightarrow \infty} \left| \frac{x + \frac{1}{x^{2n+1}}} {1 + \frac{1}{x^{2n}}} \right| = |x| > 1 \end{aligned}$$

By ratio test,  $\sum |a_n|$  cgs and hence  
 $\prod (1 + a_n)$  cgs absolutely.

$$\text{If } x=1, \quad a_n = \frac{1}{2} + n.$$

$$\lim_{n \rightarrow \infty} a_n \neq 0.$$

The product  $\prod_{n=1}^{\infty} (1+a_n)$  is dgt.  
(or)

$$\text{If } x \neq 1, \quad a_n = \frac{1}{2} + n.$$

$$\therefore \sum a_n = \sum_{n=1}^{\infty} \left( \frac{1}{2} + n \right) = \infty$$

$$\text{Let } S_n = \left( \frac{1}{2} + 1 + \dots + \frac{1}{2} + n \right) \text{ (n times)}$$

$$= \frac{n}{2}$$

$$\lim_{n \rightarrow \infty} S_n = \infty.$$

$\therefore \sum a_n$  is dgt.

Hence the product  $\prod_{n=1}^{\infty} (1+a_n)$  is dgt

If  $x=-1$ , the product  $\prod_{n=1}^{\infty} \left(1 + \frac{x^n}{n+1}\right)$  becomes

$$\left(1 - \frac{1}{2}\right) \left(1 + \frac{1}{2}\right) \left(1 - \frac{1}{2}\right) \left(1 + \frac{1}{2}\right) \dots$$

which dgs to '0' (already we have done.)

Show that  $\prod_{n=2}^{\infty} \left[1 - \left(1 - \frac{1}{n}\right)^{-n} x^n\right]$  dgs absolutely  
for  $|x| > 1$ .

sol Here if  $a_n = 1 - \left(1 - \frac{1}{n}\right)^{-n} x^n$  so that

$$a_n = -\left(1 - \frac{1}{n}\right)^{-n} x^n$$

$$\text{Now } a_{n+1} = -\left(1 - \frac{1}{n+1}\right)^{-n-1}$$

$$\therefore \frac{a_n}{a_{n+1}} = \frac{-\left(1 - \frac{1}{n}\right)^{-n} x^n}{-\left(1 - \frac{1}{n+1}\right)^{-n-1} x^{n+1}} = \frac{\left(1 - \frac{1}{n}\right)^{n+1}}{\left(1 - \frac{1}{n+1}\right)^n} \cdot x$$

$$\therefore \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right| = \frac{e}{e} |a_1|$$

$$= |a_1| > 1$$

∴ by ratio test,  $\sum a_n$  is ct.

Hence the infinite product is absolutely.

→ Show that  $\prod_{n=0}^{\infty} (1+a^{2^n})$  cgs if  $|a| < 1$ .

Sol Given infinite product is

$$\prod_{n=0}^{\infty} (1+a^{2^n}) = (1+a) (1+a^2) (1+a^4) \cdots (1+a^{2^{n-1}}) (1+a^{2^n}) \cdots \cdots$$

Let

$$P_n = \prod_{n=0}^{n-1} (1+a^{2^n}) = (1+a) (1+a^2) (1+a^4) \cdots (1+a^{2^{n-1}})$$

$$= \left( \frac{1}{1-a} \right) \left[ (1+a) \cdot (1+a^2) \cdot (1+a^4) \cdot (1+a^8) \cdot (1+a^{16}) \cdots (1+a^{2^{n-1}}) \right]$$

$$= \frac{1}{1-a} \left[ (1-a^2) (1+a^2) (1+a^4) \cdots (1+a^{2^{n-1}}) \right]$$

$$= \frac{1}{1-a} \left[ (1-(a^2)^2) (1+a^2)^2 (1+a^4)^2 \cdots (1+a^{2^{n-1}})^2 \right]$$

$$= \frac{1}{1-a} \left[ (1-a^{4 \cdot 2^{n-1}}) (1+a^{4 \cdot 2^{n-1}}) (1+a^{8 \cdot 2^{n-1}}) \cdots (1+a^{2^{n-1}}) \right]$$

$$= \frac{1}{1-a} \left[ ((1-a^4)^2) (1+a^4)^2 (1+a^{8 \cdot 2^{n-1}}) \cdots (1+a^{2^{n-1}}) \right]$$

$$= \frac{1}{1-a} \left[ ((1-a^8)^2) (1+a^8)^2 (1+a^{16 \cdot 2^{n-1}}) \cdots (1+a^{2^{n-1}}) \right]$$

$$= \frac{1}{1-a} \left[ ((1-a^{16})^2) (1+a^{16})^2 \cdots (1+a^{2^{n-1}}) \right]$$

$$= \frac{1}{1-a} \left[ ((1-a^{2^4})^2) (1+a^{2^4})^2 \cdots (1+a^{2^{n-1}}) \right]$$

$$= \frac{1}{1-a} \left[ ((1-a^2)^2) (1+a^2)^2 \cdots (1+a^{2^{n-1}}) \right]$$

$$= \frac{1}{1-a} (1-a^{2^n}).$$

Now if  $|z| < 1$  (i.e.  $-1 < z < 1$ )

~~Then  $z^n \rightarrow 0$  as  $n \rightarrow \infty$ .~~

$$P_n \rightarrow \frac{1}{1-z} \text{ as } n \rightarrow \infty$$

the infinite product  $\prod_{n=0}^{\infty} (1+z^{2^n})$  goes to

$$\frac{1}{1-z}$$

~~Since  $\sum_{n=0}^{\infty} \left(1 + \left(\frac{1}{2}\right)^n\right)$  goes to  $\infty$~~

$$(1+z) \text{ goes to } \infty$$

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## Set - IV

### Limits and Continuity

**IMSc** (1)

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### Real valued functions:

Constant function: A function  $f: \mathbb{R} \rightarrow \mathbb{R}$

defined by  $f(x) = k$ , ( $k \in \mathbb{R}$ ) is called a constant function.

Range of  $f = \{k\}$  a singleton set.

Identity function: A function  $f: \mathbb{R} \rightarrow \mathbb{R}$  defined

by  $f(x) = x$  is called the Identity function.

Range of  $f = \mathbb{R} = \text{Domain of } f$

Polynomial function: A function  $f: \mathbb{R} \rightarrow \mathbb{R}$  defined

by  $f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ , where  $a_0, a_1, a_2, \dots, a_n \in \mathbb{R}$ .

$n \in \mathbb{N}$  and  $a_n \neq 0$  is called a polynomial function of  $n^{\text{th}}$  degree.

If  $a_0 = a_1 = \dots = a_n = 0$  then  $f(x) = 0 \forall x \in \mathbb{R}$ .

In this case we say that  $f$  is a zero polynomial function.

Rational function: If  $f, g$  are two polynomials

functions and  $A = \{x/x \in \mathbb{R}, g(x) \neq 0\}$  then the

function  $h: A \rightarrow \mathbb{R}$  defined by  $h(x) = \frac{f(x)}{g(x)}$

is called a rational function.

Ex:  $h(x) = \frac{1}{x}$  is a rational function with domain  $\mathbb{R} - \{0\}$ .

Power function: A function  $f: \mathbb{R} \rightarrow \mathbb{R}$  defined

by  $f(x) = x^n$  where  $n \in \mathbb{N}$  is called power function.

if the square root function defined

→ Absolute value function (or) Mod function:

The function  $f: \mathbb{R} \rightarrow \mathbb{R}$  defined by

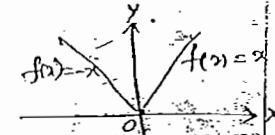
$$f(x) = x \text{ if } x \geq 0$$

$$= -x \text{ if } x < 0$$

is called mod function.

It is denoted by  $f(x) = |x|$ .

Range of  $f = [0, \infty)$ .



→ Signature function:

The function  $f: \mathbb{R} \rightarrow \mathbb{R}$  defined by

$$f(x) = 1 ; x > 0$$

$$= 0 ; x = 0$$

$$= -1 ; x < 0$$

is called signature function.

It is denoted by  $f(x) = \operatorname{sgn}(x)$ .

$$\text{i.e., } \operatorname{sgn}(x) = 1 \text{ if } x > 0$$

$$= 0 \text{ if } x = 0$$

$$= -1 \text{ if } x < 0$$

$$\text{i.e., } \operatorname{sgn}(x) = \begin{cases} \frac{|x|}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Range of  $\operatorname{sgn}(x) = \{-1, 0, 1\}$

→ Integral part function or Step function  
or greatest integer function

The function  $f: \mathbb{R} \rightarrow \mathbb{R}$  defined by

$f(x) = [x] = \text{integral part of } x$ , i.e. called

step function.

i.e.,  $f(x) = [x]$  is a greatest integer  $\leq x$ ,

is called the greatest integer function.

i.e., for every  $x \in \mathbb{R}$ ,  $\exists$  unique  $n \in \mathbb{Z}$  such that

$$n \leq x < n+1 \text{ and } [x] = n.$$

The range of the step function  $= \mathbb{Z}$ .

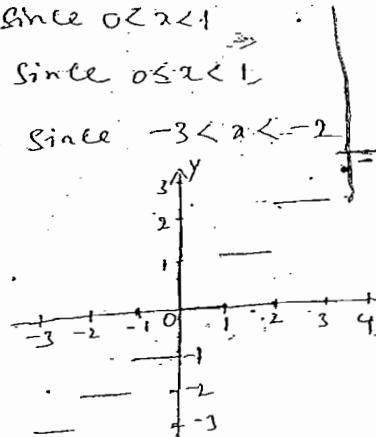
Ex:  $x = 2.5 ; [x] = 2$  since  $2 \leq x < 3$ .  
 $\therefore 2 \leq x < 2+1$

$$x = 0.1 ; [x] = 0 \text{ since } 0 \leq x < 1$$

$$x = 0 ; [x] = 0 \text{ since } 0 \leq x < 1$$

$$x = -2.5 ; [x] = -3 \text{ since } -3 \leq x < -2$$

$$x = 1.5 ; [x] = 1$$



### → Exponential function:

The function  $f: \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(x) = e^x$  is called exponential function.

The range of exponential function  $= \mathbb{R}^+$ .

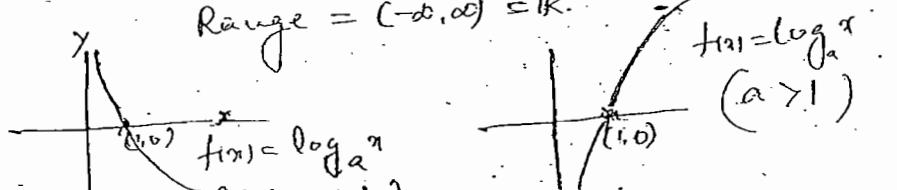
→ If  $a \in \mathbb{R} - \{1\}$  then  $f(x) = a^x$  from  $\mathbb{R} \rightarrow \mathbb{R}^+$  is also called exponential function.

### → Logarithmic function:

The exponential function  $f: \mathbb{R} \rightarrow \mathbb{R}^+$  defined by  $f(x) = e^x$  is both  $1-1$  and onto. The inverse function of this exponential function is called Logarithmic function.

$f: \mathbb{R}^+ \rightarrow \mathbb{R}$  defined by  $f(x) = \log x$  is the natural logarithmic function.

$$\text{Range} = (-\infty, \infty) \subset \mathbb{R}$$



$$f(x) = \log_a x \quad (a > 1)$$

## Trigonometric functions:

The function  $f: \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(x) = \sin x$   
is called sine function.

$$\text{Range } f = [-1, 1]$$

The function  $f: \mathbb{R} \rightarrow \mathbb{R}$  defined by  $f(x) = \cos x$   
is called cosine function.

$$\text{Range } f = [-1, 1]$$

If  $A = \{x \in \mathbb{R} / x = n\pi + \frac{\pi}{2}; n \in \mathbb{Z}\}$  then the  
(or)  $\{x \in \mathbb{R} / x = (2n+1)\frac{\pi}{2}; n \in \mathbb{Z}\}$   
function  $f: (\mathbb{R} - A) \rightarrow \mathbb{R}$  defined by

$$f(x) = \frac{\sin x}{\cos x} = \tan x$$

is called tangent function.

Domain  $f = \mathbb{R} - A$  (odd values)

$$\text{Range } f = \mathbb{R}$$

If  $A = \{x \in \mathbb{R} / x = n\pi; n \in \mathbb{Z}\}$  then the function  
 $f: \mathbb{R} - A \rightarrow \mathbb{R}$  defined by  $f(x) = \frac{\cos x}{\sin x} = \cot x$ ,  
is called cotangent function.

Domain  $f = \mathbb{R} - A$

$$\text{Range } f = \mathbb{R}$$

If  $A = \{x \in \mathbb{R} / x = (2n+1)\frac{\pi}{2}; n \in \mathbb{Z}\}$  then the  
function  $f: (\mathbb{R} - A) \rightarrow \mathbb{R}$  defined by  $f(x) = \frac{1}{\cos x}$

Domain  $f = \mathbb{R} - A$

$$\text{Range } f = \mathbb{R} - \{-1, 1\}$$

If  $A = \{x \in \mathbb{R} / x = n\pi; n \in \mathbb{Z}\}$  then the function  
 $f: (\mathbb{R} - A) \rightarrow \mathbb{R}$  defined by  $f(x) = \frac{1}{\sin x} = \operatorname{cosec} x$

is called cosecant function.

Domain  $f = \mathbb{R} - A$

$$\text{Range } f = \mathbb{R} - \{-1, 1\}$$

### Boundedness of a function:

A function  $f$  is said to be bounded if its range is bounded. Otherwise it is unbounded i.e., A function  $f$  is said to be bounded on a domain  $D$  if there exist two real numbers  $h, k$  such that  $h \leq f(x) \leq k \forall x \in D$  where  $h$  is called a lower bound of  $f$  &  $k$  is called an upperbound of  $f$ .

(or) A function  $f$  is said to be bounded on a domain  $D$  if there exist a real number  $M$

such that  $|f(x)| \leq M \forall x \in D$ .

i.e.,  $M > 0$  such that  $f(x) = \sin x, f(x) = \cos x$  are bounded

Ex:  $f(x) = \sin x, f(x) = \cos x$  are bounded functions on  $\mathbb{R}$

But  $f(x) = \tan x$  is not bounded on  $\mathbb{R}$ .

Cluster point of a set or limit point of a set:

Let  $A \subseteq \mathbb{R}$ . A point  $c \in \mathbb{R}$  is a cluster point of  $A$

if for every  $\delta > 0$  there exists atleast one point

$x \in A, x \neq c$  such that  $|x - c| < \delta$

i.e.,  $0 < |x - c| < \delta$ .

(or) Let  $A \subseteq \mathbb{R}$ , A point  $c \in \mathbb{R}$  is a cluster point of  $A$  if every  $\delta$ -nbd of  $c$  contains atleast one point of  $A$  other than  $c$ .

One point of  $A$  other than  $c$

i.e.,  $\delta > 0, (c-\delta, c+\delta)$  contains atleast one point of the set  $A$  other than  $c$ .

(or) A point  $c \in \mathbb{R}$  is a cluster point of  $A$  if every

nbd of  $C$  contains infinitely many points of  $A$ :  
 i.e.,  $\exists \delta > 0$ ,  $(C - \delta, C + \delta)$  contains infinitely many points of  $A$ .

Eg: (1) for the open interval  $A_1 = (0, 1)$ , every point of the closed interval  $[0, 1]$  is a cluster point of  $A_1$ .

The points  $0$  &  $1$  are cluster points of  $A_1$ ,

but do not belong to  $A_1$ .

All the points of  $A_1$  are cluster points of  $A_1$ .

(2) A finite set has no cluster points.

(3) The infinite set  $N$  has no cluster points.

(4) The set  $A_4 = \left\{ \frac{1}{n} \mid n \in \mathbb{N} \right\}$  has only the point '0' as a cluster point.

None of the points in  $A_4$  is a cluster point of  $A_4$ .

Note: A cluster point of the set  $A$  may (or) may not belong to the set  $A$ .

(A real no.  $L$  is said to be a limit of  $f$  at  $x_0$ )  
 (if given  $\epsilon > 0$  (however small),  $\exists \delta > 0$

### Limit of a function:

Let  $A \subseteq \mathbb{R}$  and let  $C$  be a cluster point of  $A$ . for a function  $f: A \rightarrow \mathbb{R}$ , [a Real number  $L$  is said to be a limit of  $f$  at  $C$ ; if given any  $\epsilon > 0$ , there exists a  $\delta > 0$  (depending on  $\epsilon$ , i.e.,  $\delta(\epsilon)$ ) such that if  $x \in A$  and  $0 < |x - C| < \delta$  then

$$|f(x) - L| < \epsilon.$$

i.e.,  $|f(x) - L| < \epsilon$  whenever  $0 < |x - C| < \delta$ .

$$\text{i.e., } \exists \delta(\epsilon) \in (L - \epsilon, L + \epsilon) \forall x \in (C - \delta, C + \delta); \\ x \neq C$$

*↳ Notes  
in brief*

Note: (1) If  $L$  is a limit of  $f$  at  $C$ , then we say  $f$  goes to  $L$  at  $C$ .

rewrite  $\lim_{x \rightarrow c} f(x) = L$  (or)  $\lim_{x \neq c} f = L$

(4)

we also say that  $f(x)$  approaches  $L$  as

$x$  approaches  $c$ .

i.e.  $f(x) \rightarrow L$  as  $x \rightarrow c$ .

(2) If the limit of  $f$  at  $c$  does not exist,  
we say that  $f$  diverges at  $c$ .

(3) If ' $c$ ' is not a cluster point of  $A$  then  
the limit of a function  $f$  does not discuss  
at ' $c$ '.

(4) The function  $f$  may (or) may not be  
defined at the limit point.

e.g. if  $A = (0, 1)$  and if  $f: A \rightarrow \mathbb{R}$  then  
 $1$  is a cluster point of  $A$   
but  $f$  is not defined at  $1$ .  
Similarly at ' $0$ '.

(5) In order to prove that  $\lim_{x \rightarrow c} f(x) \neq L$ , we have

to show that for any  $\epsilon > 0$ , and any  $\delta > 0$   
there is  $x \in A$ ,  $0 < |x - c| < \delta \Rightarrow |f(x) - L| \geq \epsilon$ .

(6) If  $f: A \rightarrow \mathbb{R}$  and if ' $c$ ' is a cluster point  
of  $A$  then  $f$  can have only one limit at ' $c$ '.

### Sequential Criterion

Let  $f: A \rightarrow \mathbb{R}$  and let ' $c$ ' be a cluster point of  
 $A$  then the following are equivalent.

(i)  $\lim_{x \rightarrow c} f(x) = L$

(ii) for every  $(x_n)$  in  $A$   $\rightarrow c$  such that,

$a_n \neq c$  &  $n \in \mathbb{N}$ , the sequence  $(f(a_n))$  goes to L.

→ Use the  $\epsilon$ - $\delta$  definition of limit, to show that

$$\lim_{x \rightarrow c} \frac{1}{x} = \frac{1}{c}, \quad c > 0.$$

Soln: Let  $f(x) = \frac{1}{x}; x > 0$

and let  $c > 0$ .

To show that  $\lim_{x \rightarrow c} f(x) = \frac{1}{c}$ .

for this we are enough to show that for any  $\epsilon > 0$ ;  $\exists \delta > 0$  (depends on  $\epsilon$ ) such that

$|f(x) - \frac{1}{c}| < \epsilon$  whenever  $0 < |x - c| < \delta$ .

$$\begin{aligned} \text{Now, we have } |f(x) - \frac{1}{c}| &= \left| \frac{1}{x} - \frac{1}{c} \right| \\ &= \left| \frac{c - x}{xc} \right| \\ &= \frac{|x - c|}{|xc|} \quad \text{--- (1)} \end{aligned}$$

for  $x \rightarrow c$ ,

by taking  $x$  sufficiently close to  $c$

we have  $0 < |x - c| < \frac{1}{2}c$ . ( $\because \delta \leq \frac{1}{2}c$ )

$\Rightarrow |x - c| > 0$  and  $|x - c| < \frac{1}{2}c$ .

$x \neq c$  and  $-\frac{1}{2}c < x - c < \frac{1}{2}c$ .

$x \neq c$  and  $\frac{c}{2} < x < \frac{3}{2}c$ .

$x \neq c$  and  $\frac{c}{2} < x$ .

$x \neq c$  and  $x < \frac{c^2}{2}$

and  $|xc| > \frac{c^2}{2}$

and  $\frac{|x - c|}{|xc|} < \frac{2}{c^2}$

$\therefore \text{Q.E. } |f(x) - \frac{1}{c}| < \frac{2}{c^2} |x - c|$

$< \epsilon$  whenever  $|x - c| < \frac{c^2}{2}\epsilon$

Choosing  $\delta = \min\left\{\frac{1}{2}c, \frac{c^2}{2}\epsilon\right\}$

(5)

$\therefore |f(x) - \frac{1}{c}| < \epsilon$  whenever  $0 < |x-c| < \delta$

$\therefore f(x) \rightarrow \frac{1}{c}$  as  $x \rightarrow c$ .

$$\lim_{x \rightarrow c} f(x) = \frac{1}{c}; c > 0.$$

→ Use either the  $\epsilon-\delta$  definition of limit ( $0^\circ$ ) or the sequential criterion for limits to establish the following limits.

$$(1) \lim_{x \rightarrow 2} \frac{1}{1-x} = -1 \quad (2) \lim_{x \rightarrow 1} \frac{x}{1+x} = \frac{1}{2}$$

$$(3) \lim_{x \rightarrow 0} \frac{x^2}{1+x} = 0 \quad (4) \lim_{x \rightarrow 1} \frac{x^2+x+1}{x+1} = \frac{1}{2}$$

Sol:

(i)  $\epsilon-\delta$  Method.

Let  $f(x) = \frac{1}{1-x}$ . Then we prove that

$$\lim_{x \rightarrow 2} f(x) = -1.$$

for this we are enough to prove that for each  $\epsilon > 0$   $\exists \delta > 0$  such that  $|f(x) - (-1)| < \epsilon$  whenever  $0 < |x-2| < \delta$ .

we have

$$\begin{aligned} |f(x) - (-1)| &= \left| \frac{1}{1-x} - (-1) \right| \\ &= \left| \frac{1}{1-x} + 1 \right| \\ &= \left| \frac{2-x}{1-x} \right| \\ &= \left| \frac{x-2}{x-1} \right| \end{aligned}$$

$$\therefore |f(x) - (-1)| = \left| \frac{x-2}{x-1} \right| \quad (6)$$

for  $x \rightarrow 2$ ,

by taking  $x$  sufficiently close to 2.

we have  $0 < |x-2| < 1$  ( $\because 0 < \delta \leq 1$ )

$$\Rightarrow |x-2| > 0 \text{ and } |x-2| < 1$$

$$\Rightarrow x \neq 2 \text{ and } -1 < x-2 < 1$$

$$\Rightarrow x \neq 2 \text{ and } 2-1 < x < 2+1$$

i.e.,  $1 < x < 3$

Since  $x > 1$

$$\Rightarrow x-1 > 0 \Rightarrow \frac{1}{x-1} > 0$$

$$\Rightarrow \left| \frac{1}{x-1} \right| > 0$$

$$\Rightarrow 0 < \left| \frac{1}{x-1} \right| \leq 1$$

$$\Rightarrow \frac{1}{|x-1|} \leq 1$$

$$\therefore ① = |f(x) - (-1)| \leq \frac{1}{|x-1|} < \epsilon \text{ whenever } |x-2| < \frac{\epsilon}{1}$$

choosing  $\delta = \min \left\{ 1, \frac{\epsilon}{1} \right\}$

$\therefore |f(x) - (-1)| < \epsilon \text{ whenever } 0 < |x-2| < \delta$

$\therefore f(x) \rightarrow -1 \text{ as } x \rightarrow 2$

$$\therefore \lim_{x \rightarrow 2} f(x) = -1$$

### Sequential Method:

$$\text{Let } f(x) = \frac{1}{1-x}; c=2$$

$$\text{Take } x_n = \frac{2n}{n+1} \rightarrow n \in \mathbb{N}$$

$$\therefore \lim x_n = 2$$

$n \rightarrow \infty$

$$f(x_n) = \frac{1}{1-x_n}$$

$$= \frac{1}{1-\frac{2n}{n+1}} = \frac{n+1}{n-1}$$

(6)

$$\lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} \left( \frac{1+n}{1-n} \right)$$

$$= \lim_{n \rightarrow \infty} \left( \frac{\frac{1}{n} + 1}{\frac{1}{n} - 1} \right)$$

$$= \frac{0+1}{0-1} = -1$$

$\therefore$  The sequence  $(f(x_n))$  goes to  $-1 = L$

$$\therefore \lim f(x) = L$$

$$\begin{aligned} & x \rightarrow 2 \\ & \Rightarrow \lim_{x \rightarrow 2} f(x) = -1 \end{aligned}$$

(3) By E-S Method:

$$\begin{aligned} \left| \frac{x^2}{|x|} - 0 \right| &= \frac{|x^2|}{|x|} \\ &= \frac{|x|^2}{|x|} \\ &= |x| < \epsilon \text{ (say)} \end{aligned}$$

$\therefore \left| \frac{x^2}{|x|} - 0 \right| < \epsilon$  whenever  $|x| < \delta = \epsilon$ .

$\forall \epsilon > 0$ ,  $\exists \delta = \epsilon > 0$  such that

$\left| \frac{x^2}{|x|} - 0 \right| < \epsilon$  whenever  $|x - 0| < \delta$ .

$$\begin{aligned} & \therefore \lim_{x \rightarrow 0} \frac{x^2}{|x|} = 0 \\ & x \rightarrow 0 \end{aligned}$$

### Divergence Criteria

A  $\subseteq \mathbb{R}$ , let  $f: A \rightarrow \mathbb{R}$  and let  $c \in \mathbb{R}$  be a

cluster point of A.

- (a) If  $L \in \mathbb{R}$  then  $f$  does not have limit  $L$  at  $c$  iff there exists a sequence  $(x_n)$  in  $A$  with  $x_n \neq c$  forever such that the sequence  $(x_n)$  goes to  $c$  but the sequence  $(f(x_n))$  does not converge to  $L$ .

b) The function  $f$  does not have a limit at  $c$  iff there exists a sequence  $(x_n)$  in  $A$  with  $x_n \neq c$   $\forall n \in \mathbb{N}$  such that the sequence  $(x_n)$  converges to  $c$  but the sequence  $(f(x_n))$  does not converge in  $\mathbb{R}$ .

Eg:  $\lim_{x \rightarrow 0} \frac{1}{x}$  does not exist in  $\mathbb{R}$ .

$$\text{Soln: Let } f(x) = \frac{1}{x}; c=0$$

$$\text{Let } x_n = \frac{1}{n} \quad \forall n$$

$$\text{then } \lim_{n \rightarrow \infty} x_n = 0 = c$$

$\therefore (x_n)$  converges to 0.

$$\text{Now } f(x_n) = \frac{1}{x_n} = \frac{1}{\frac{1}{n}} = n \quad \forall n$$

$$\therefore \lim_{n \rightarrow \infty} f(x_n) = \lim_{n \rightarrow \infty} n = +\infty$$

$\therefore (f(x_n))$  is not cpt in  $\mathbb{R}$ .

$\therefore \lim_{x \rightarrow 0} f(x)$  does not exist in  $\mathbb{R}$ .

→ Show that the following limits do not exist.

$$(a) \lim_{x \rightarrow 0} \frac{1}{x^2} \quad (b) \lim_{x \rightarrow 0} \frac{1}{\sqrt{x}}$$

$$(c) \lim_{x \rightarrow 0} \operatorname{sgn}(x) \quad (d) \lim_{x \rightarrow 0} (x + \operatorname{sgn}(x)) \quad (e) \lim_{x \rightarrow 0} \sin\left(\frac{1}{x}\right)$$

$$(f) \lim_{x \rightarrow 0} \sin\left(\frac{1}{x}\right)$$

$$\text{Soln: (c) } \lim_{x \rightarrow 0} \operatorname{sgn}(x)$$

$$\text{Let } f(x) = \operatorname{sgn}(x) = \begin{cases} +1 & \text{if } x > 0 \\ 0 & \text{if } x = 0 \\ -1 & \text{if } x < 0 \end{cases}$$

$$\text{Now, } \operatorname{sgn}(x) = \frac{|x|}{x} \text{ if } x \neq 0.$$

Now we have to show that  $\text{sgn}(x)$  does not have a limit at  $x=0$ . (7)

$$\text{Let } x_n = \frac{(-1)^n}{n} \quad \forall n \quad \text{then} \quad \lim_{n \rightarrow \infty} x_n = 0$$

$\therefore (x_n)$  goes to '0'.

$$\text{Now } \text{sgn}(x_n) = \frac{(-1)^n/n}{|(-1)^n/n|} = (-1)^n \quad \forall n$$

$$\therefore \lim_{n \rightarrow \infty} \text{sgn}(x_n) = \begin{cases} -1 & \text{if } n \text{ is odd} \\ 1 & \text{if } n \text{ is even} \end{cases}$$

$\therefore \text{sgn}(x_n)$  does not exist.

$\therefore \lim_{n \rightarrow 0} \text{sgn}(x)$  does not exist.

$$(e) \quad \lim_{x \rightarrow 0} \sin\left(\frac{1}{x}\right)$$

$$\text{Let } f(x) = \sin\left(\frac{1}{x}\right); \quad c=0$$

By introducing two sequences  $(x_n)$  &  $(y_n)$ .

$$\text{Let } x_n = \frac{1}{n\pi} \quad \forall n$$

$$\text{Then} \quad \lim_{n \rightarrow \infty} x_n = 0$$

$$\text{Now } f(x_n) = \sin(n\pi) \\ = 0 \quad \forall n$$

$$\therefore \lim_{n \rightarrow \infty} f(x_n) = 0$$

$$\text{and let } y_n = \frac{1}{\frac{1}{2}\pi + 2n\pi}$$

$$\text{Then} \quad \lim_{n \rightarrow \infty} y_n = 0$$

$$\text{Now } f(y_n) = \sin\left(\frac{1}{\frac{1}{2}\pi + 2n\pi}\right)$$

$$= \sin\left(\frac{1}{\pi} + \frac{4n}{\pi}\right)$$

$$= 1 \quad \forall n$$

$$\therefore \lim_{n \rightarrow \infty} f(y_n) = 1$$

$$\lim_{n \rightarrow \infty} f(x_n) \neq \lim_{n \rightarrow \infty} f(y_n)$$

$\therefore \lim_{x \rightarrow 0} f(x)$  does not exist.

$$(f) \quad \lim_{x \rightarrow 0} \sin\left(\frac{1}{x^2}\right)$$

$$\text{Let } f(x) = \sin\left(\frac{1}{x^2}\right); \quad c=0$$

$$\text{Let } x_n = \frac{1}{n}$$

## Algebra of limits:

Let  $A \subseteq \mathbb{R}$ . Let  $f$  &  $g$  be two functions on  $A$  to  $\mathbb{R}$ .

and  $c \in \mathbb{R}$  be a cluster point of  $A$ . further

let  $b \in \mathbb{R}$  if  $\lim_{x \rightarrow c} f = L$  and  $\lim_{x \rightarrow c} g = M$ .

$$\text{then (i) } \lim_{x \rightarrow c} (f \pm g) = L \pm M \quad \begin{aligned} \text{if } \lim_{x \rightarrow c} f(x) = L \\ \lim_{x \rightarrow c} g(x) = M \end{aligned}$$

$$\text{(ii) } \lim_{x \rightarrow c} (fg) = LM$$

$$\text{(iii) } \lim_{x \rightarrow c} (bf) = bL \quad \text{(iv) } \lim_{x \rightarrow c} \left( \frac{f}{g} \right) = \frac{L}{M} \text{ provided } M \neq 0.$$

Theorem Let  $A \subseteq \mathbb{R}$ , let  $f: A \rightarrow \mathbb{R}$  and let  $c \in \mathbb{R}$  be

a cluster point of  $A$ .

$$\text{If } a \leq f(x) \leq b \quad \forall x \in A; x \neq c$$

$$\text{and } \lim_{x \rightarrow c} f(x) \text{ exists.}$$

$$\text{then } a \leq \lim_{x \rightarrow c} f(x) \leq b.$$

## Squeeze theorem:

Let  $A \subseteq \mathbb{R}$ , let  $f, g, h: A \rightarrow \mathbb{R}$  and let  $c \in \mathbb{R}$  be a cluster point of  $A$ . if  $f(x) \leq g(x) \leq h(x) \quad \forall x \in A; x \neq c$ .

$$\text{and if } \lim_{x \rightarrow c} f(x) = L = \lim_{x \rightarrow c} h(x)$$

$$\text{then } \lim_{x \rightarrow c} g(x) = L$$

Note: if  $x \in \mathbb{R}; x \geq 0$  then

$$\text{we have (i) } -x \leq S(x) \leq x$$

$$(ii) 1 - \frac{1}{2}x^2 \leq C(x) \leq 1$$

$$(iii) x - \frac{1}{6}x^3 \leq S(x) \leq x$$

$$(iv) 1 - \frac{1}{2}x^2 \leq C(x) \leq 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$$

Here  $\sin x = S(x)$  &  $\cos x = C(x)$

Sol: Let  $-1 \leq C(t) \leq 1 \forall t \in \mathbb{R}$

(8)

If  $x > 0$  then

$$-\int_{t=0}^x dt \leq \int_0^x C(t) dt \leq \int_0^x dt$$

$$\Rightarrow -x \leq S(x) \leq x$$

Integrating, we get

$$-\frac{x^2}{2} \leq -\cos x + 1 \leq \frac{x^2}{2}$$

$$\Rightarrow -\frac{x^2}{2} \leq \cos x - 1 \leq \frac{x^2}{2}$$

$$\Rightarrow 1 - \frac{x^2}{2} \leq \cos x \leq 1 + \frac{x^2}{2}$$

$$\Rightarrow \boxed{1 - \frac{x^2}{2} \leq \cos x \leq 1} \quad (\text{range of cos } \leq 1)$$

and so on.

### Problems

$$\rightarrow \lim_{x \rightarrow 0} x^{3/2} = 0; (x > 0)$$

$x > 0$

Sol: Let  $g(x) = x^{3/2}; x > 0$

We have  $x < x^{3/2} \leq 1$  for  $0 < x \leq 1$

$$\Rightarrow x < x^{3/2} \leq x \text{ for } 0 < x \leq 1$$

is of the form  $f(x) \leq g(x) \leq h(x)$ .

where  $f(x) = x^2, g(x) = x^{3/2}, h(x) = x$ .

$$\therefore \lim_{x \rightarrow 0} f(x) = 0 = \lim_{x \rightarrow 0} h(x)$$

$x \rightarrow 0$

By squeeze theorem

$$\lim_{x \rightarrow 0} g(x) = 0$$

$x \rightarrow 0$

$$\rightarrow \lim_{x \rightarrow 0} \sin x = 0$$

Since  $-x \leq \sin x \leq x \forall x \geq 0$ .

is of the form  $f(x) = -x, g(x) = \sin x, h(x) = x$ .

$$\therefore \lim_{x \rightarrow 0} f(x) = 0 = \lim_{x \rightarrow 0} h(x)$$

$\therefore$  By squeeze theorem  $\lim_{x \rightarrow 0} g(x) = 0$ .

$$\Rightarrow \lim_{x \rightarrow 0} \cos x = 1.$$

Sol: Since  $1 - \frac{x^2}{2} \leq \cos x \leq 1$

$$\Rightarrow \lim_{x \rightarrow 0} \left( \frac{\cos x - 1}{x} \right) = 0$$

Sol: Since  $1 - \frac{x^2}{2} \leq \cos x \leq 1 \quad \forall x > 0$

$$\Rightarrow -\frac{x^2}{2} \leq \cos x - 1 \leq 0$$

$$\Rightarrow -\frac{x}{2} \leq \frac{\cos x - 1}{x} \leq 0.$$

is of the form  $f(x) \leq g(x) \leq h(x)$

$$\text{where } f(x) = -\frac{x}{2} \text{ ; } g(x) = \frac{\cos x - 1}{x}$$

and  $h(x) = 0$ .

$$\therefore \lim_{x \rightarrow 0} f(x) = 0 = \lim_{x \rightarrow 0} h(x)$$

$\therefore$  By squeeze theorem

$$\lim_{x \rightarrow 0} g(x) = 0$$

$$\Rightarrow \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

Sol: Since  $x - \frac{x^3}{6} \leq \sin x \leq x \quad \forall x > 0$

$$\Rightarrow 1 - \frac{x^2}{6} \leq \frac{\sin x}{x} \leq 1 \quad \forall x > 0$$

is of the form  $f(x) \leq g(x) \leq h(x)$

$$\text{where } f(x) = 1 - \frac{x^2}{6}; \quad g(x) = \frac{\sin x}{x}$$

and  $h(x) = 1$

$$\therefore \lim_{x \rightarrow 0} f(x) = 1 = \lim_{x \rightarrow 0} h(x)$$

$\therefore$  By squeeze theorem

$$\lim_{x \rightarrow 0} g(x) = 1$$

$$\therefore \lim_{x \rightarrow 0} \left( x \sin \frac{1}{x} \right) = 1$$

ans. Let  $f(x) = x \sin \frac{1}{x}$

Since  $-1 \leq \sin \frac{1}{x} \leq 1 ; x \neq 0$

$$\Rightarrow -x \leq x \sin \frac{1}{x} \leq x \quad x \neq 0$$

is of the form  $f(x) \leq g(x) \leq h(x)$

where  $f(x) = -x ; g(x) = x \sin \frac{1}{x}$  and  $h(x) = x$

$$\therefore \lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} h(x) = 0$$

∴ By squeeze theorem,

$$\lim_{x \rightarrow 0} g(x) = 0$$

$$\stackrel{H.W.}{\rightarrow} \lim_{x \rightarrow 0} x \sin \left( \frac{1}{x} \right) = ? ; x \neq 0$$

$$\rightarrow \lim_{x \rightarrow 0} \operatorname{sgn} \left( \sin \frac{1}{x} \right) = ?$$

$$\text{Soln: Let } f(x) = \operatorname{sgn} \left( \sin \frac{1}{x} \right) ; x \neq 0$$

$$= \frac{\sin \frac{1}{x}}{|\sin \frac{1}{x}|} ; x \neq 0$$

Since  $\lim_{x \rightarrow 0} \sin \frac{1}{x}$  does not exist.

$$x \rightarrow 0$$

∴  $\lim_{x \rightarrow 0} \operatorname{sgn} \left( \sin \frac{1}{x} \right)$  does not exist.

$$x \rightarrow 0$$

### One-Sided Limits:

Let  $A \subseteq \mathbb{R}$  and  $f: A \rightarrow \mathbb{R}$   
if  $c \in \mathbb{R}$  is a cluster point of the set

$$A \cap (c, \infty) = \{x \in A / x > c\}$$

then we say that  $L_R$  is a right-hand limit

of  $f$  at  $c$  if given  $\epsilon > 0$ ;  $\exists \delta > 0$  such that

$x \in A$  with  $0 < x - c < \delta$ , then  $|f(x) - L| < \epsilon$ .

i.e.,  $|f(x) - L| < \epsilon$  whenever  $0 < x - c < \delta$ .

The right-hand limit ( $RHL$ ) is denoted by

$$\lim_{x \rightarrow c^+} f(x) \quad (\text{or}) \quad \lim_{x \rightarrow c^+} f$$

(ii) If  $c \in \mathbb{R}$  is a cluster point of the set

$$A \cap (-\infty, c) = \{x \in A / x < c\},$$

then we say that  $L \in \mathbb{R}$  is a left-hand limit of  $f$  at  $c$ .

If given any  $\epsilon > 0$ , if a  $\delta > 0$  such that

for all  $x \in A$  with  $0 < c - x < \delta$ , then  $|f(x) - L| < \epsilon$

i.e.,  $|f(x) - L| < \epsilon$  whenever  $0 < c - x < \delta$ .

The left-hand limit (LHL) is denoted by

$$\underset{x \rightarrow c^-}{\text{Lt}} f(x) \quad (\text{or}) \quad \underset{x \rightarrow c^-}{\text{Lt}} f.$$

### Existence of a limit

$$\text{If } \underset{x \rightarrow c}{\text{f}(x)} = L \iff \underset{x \rightarrow c^+}{\text{Lt}} f(x) = L = \underset{x \rightarrow c^-}{\text{Lt}} f(x)$$

### Sequential criteria

Let  $A \subseteq \mathbb{R}$ , let  $f: A \rightarrow \mathbb{R}$  and let  $c \in \mathbb{R}$  be a cluster point of  $A \cap (c, \infty)$  then the following statements are equivalent.

(i)  $\underset{x \rightarrow c^+}{\text{Lt}} f(x) = L$

$x \rightarrow c^+$

(ii) for every sequence  $(x_n)$  that converges to  $c$  such that  $x_n \in A$  and  $x_n > c$  then, the sequence  $(f(x_n))$  converges to  $L$ .

In this way for left-hand limit

### Example:

$\rightarrow \underset{x \rightarrow 0^+}{\text{Lt}} \text{sgn}(x) = ?$

Sol: Let  $f(x) = \text{sgn}(x); x \neq 0$ .

$$= \frac{x}{|x|}, x \neq 0$$

$$= 1 \text{ if } x > 0$$

(16)

Now  $\lim_{x \rightarrow 0^+} f(x) = 1$  &  $\lim_{x \rightarrow 0^-} f(x) = -1$

$$\therefore \lim_{x \rightarrow 0^+} f(x) \neq \lim_{x \rightarrow 0^-} f(x)$$

$\therefore \lim_{x \rightarrow 0} f(x)$  does not exist.

$$\rightarrow \lim_{x \rightarrow 0} \sin\left(\frac{1}{x}\right) = ?$$

$$\text{Let } f(x) = \sin\left(\frac{1}{x}\right); x \neq 0 \quad (\text{i.e. } x < 0 \text{ or } x > 0)$$

LHL:

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \sin\frac{1}{x}$$

Now  $x \rightarrow 0^+$ ,  $\sin\frac{1}{x}$  is finite and oscillates between  $-1$  &  $1$ .

$\therefore$  It does not tend to any unique number.

$$\therefore \lim_{x \rightarrow 0^+} \sin\frac{1}{x} \text{ does not exist.}$$

Similarly,  $\lim_{x \rightarrow 0^-} \sin\frac{1}{x}$  does not exist.

$\therefore \lim_{x \rightarrow 0} \sin\frac{1}{x}$  does not exist.

$$\rightarrow \lim_{x \rightarrow 0} x \sin\frac{1}{x} = ?$$

$$\text{Soln: Let } f(x) = x \sin\frac{1}{x}; x \neq 0$$

$$\text{LHL: } \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} x \sin\frac{1}{x}$$

$$= 0 \times [\text{finite number between } -1 \text{ & } 1] \\ = 0$$

$$\text{RHL: } \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} x \sin\frac{1}{x}$$

$$= 0 \times (\text{finite number between } -1 \text{ & } 1) \\ = 0$$

$$\therefore \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} f(x) = 0$$

$$\therefore \lim_{x \rightarrow 0} f(x) = 0$$

## Limits at infinity and Infinite limits

(i)  $\lim_{x \rightarrow \infty} f(x) = L$ .

A function  $f(x)$  is said to tend to 'L' as  $x \rightarrow \infty$ , if given any  $\epsilon > 0$  (however small),  $\exists$  a +ve number  $K$  (depends on  $\epsilon$ ) such that  $x \geq K \Rightarrow |f(x) - L| < \epsilon$ .

(ii)  $\lim_{x \rightarrow -\infty} f(x) = L$ .

A function  $f(x)$  is said to tend to L as  $x \rightarrow -\infty$ , if given any  $\epsilon > 0$ ,  $\exists$  a +ve number  $K$  (depends on  $\epsilon$ ) such that  $x \leq -K \Rightarrow |f(x) - L| < \epsilon$ .

(iii)  $\lim_{x \rightarrow c} f(x) = +\infty$ .

A function  $f(x)$  is said to tend to  $\infty$  as  $x \rightarrow c$ , if given any  $K > 0$  (however large),  $\exists$  a +ve number  $\delta$  such that  $0 < |x - c| < \delta \Rightarrow f(x) > K$ .

(iv)  $\lim_{x \rightarrow c} f(x) = -\infty$

A function  $f(x)$  is said to tend to  $-\infty$  as  $x \rightarrow c$ , if given  $K > 0$  (however large),  $\exists$  a  $\delta > 0$  such that  $0 < |x - c| < \delta \Rightarrow f(x) < -K$ .

(v)  $\lim_{x \rightarrow \infty} f(x) = +\infty$

A function  $f(x)$  is said to tend to  $+\infty$  as  $x \rightarrow \infty$ , if given any  $K > 0$  (however large),  $\exists$  a number  $K' > 0$  such that  $x \geq K' \Rightarrow f(x) > K$ .

(vi)  $\lim_{x \rightarrow \infty} f(x) = -\infty$ .

A function  $f(x)$  is said to tend to  $-\infty$  as  $x \rightarrow \infty$ , if given any  $K > 0$  (however large),  $\exists$  a number  $K' > 0$  such that  $x \geq K' \Rightarrow f(x) < -K$ .

(vii)  $\lim_{x \rightarrow -\infty} f(x) = \infty$ .

A function  $f(x)$  is said to tend to  $\infty$  as  $x \rightarrow -\infty$ , if given any  $K > 0$  (however large),  $\exists$  a number  $K' > 0$  such that  $x \leq -K' \Rightarrow f(x) > K$ .

(viii)  $\lim_{x \rightarrow -\infty} f(x) = -\infty$ .

A function  $f(x)$  is said to tend to  $-\infty$  as  $x \rightarrow -\infty$ , if given any  $K > 0$  (however large),  $\exists$   $K' > 0$  (depends on  $K$ ) such that  $x \leq -K' \Rightarrow f(x) < -K$ .

### Continuous functions:

Let  $A \subseteq \mathbb{R}$ ,  $f: A \rightarrow \mathbb{R}$  and  $c \in A$  be a cluster point of  $A$  then we say that ' $f$ ' is continuous

at ' $c$ ' if  $\lim_{x \rightarrow c} f(x) = f(c)$

$$\text{(or) } \lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x) = f(c)$$

(or)

Let  $A \subseteq \mathbb{R}$ ,  $f: A \rightarrow \mathbb{R}$  and  $c \in A$  be a cluster point of  $A$  then we say that  $f$  is continuous at ' $c$ ' if given  $\epsilon > 0$ .  $\exists$  a  $\delta > 0$  (depending on  $\epsilon$ ) such that if  $x \in A$  satisfying  $|x - c| < \delta$

$$\text{then } |f(x) - f(c)| < \epsilon$$

i.e.,  $|f(x) - f(c)| < \epsilon$  whenever  $|x - c| < \delta$ .

$$\text{i.e., } f(x) \in (f(c) - \epsilon, f(c) + \epsilon)$$

$$\Rightarrow x \in (c - \delta, c + \delta)$$

### Continuous from the left at a point:

A function  $f$  is continuous from the left

(or left-continuous) at the point

$$x = c \quad \text{if } \lim_{x \rightarrow c^-} f(x) = f(c).$$

(or)

Let  $A \subseteq \mathbb{R}$ , let  $f: A \rightarrow \mathbb{R}$ ,  $c \in A$  if a cluster point of  $A \cap (-\infty, c]$  then we

say that  $f$  is left continuous at ' $c$ ',

if given any  $\epsilon > 0$  (however small),

$\exists \delta > 0$  (depends on  $\epsilon$ ) such that

$$c - \delta < x \leq c \Rightarrow |f(x) - f(c)| < \epsilon$$

### Continuity from the right at a point:

A function  $f$  is continuous from the right (or) right continuous at the point

$$\begin{aligned} x = c &\text{ if } \lim_{x \rightarrow c^+} f(x) = f(c) \\ &\text{(or)} \end{aligned}$$

Let  $A \subseteq \mathbb{R}$ , let  $f: A \rightarrow \mathbb{R}$ ,  $c \in A$  is a cluster point of  $A \cap [c, \infty) = \{x \in A / x \geq c\}$

then we say that  $f$  is right continuous at ' $c$ ', if given any  $\epsilon > 0$  (however small)

$\exists \delta > 0$  (depends on  $\epsilon$ ) such that

$$c \leq x < c + \delta \Rightarrow |f(x) - f(c)| < \epsilon$$

### Discontinuity:

If  $f$  is not continuous at ' $c$ '  
then  $f$  is said to be discontinuous at ' $c$ '

$$\text{i.e., } \lim_{x \rightarrow c} f(x) \neq f(c)$$

(or)

$A \subseteq \mathbb{R}$ , let  $f: A \rightarrow \mathbb{R}$ ;  
c is a cluster point  
of A then f is not continuous at c, if  $\epsilon > 0$ ,  $\forall \delta > 0$

s.t.  $x$  is any point of  
A satisfying  $|x - c| < \delta \Rightarrow$

$$|f(x) - f(c)| \geq \epsilon$$

NOTE:- (i) If c is a  
cluster point of A then  
the following three

conditions must hold for:

f to be continuous at c

(i) f should be defined at c  
(i.e.  $f(c)$  exists).

(ii).  $\lim_{x \rightarrow c} f(x)$  exists and

(iii)  $f(c) = \lim_{x \rightarrow c} f(x)$  are  
equal.

(2) f is discontinuous

at  $x=c$  because of  
any one of the following  
reason:

(i) f is not defined at c

(ii)  $\lim_{x \rightarrow c} f(x)$  does not exist

i.e.  $\lim_{x \rightarrow c^-} f(x) \neq \lim_{x \rightarrow c^+} f(x)$

(or)

one of the limit does  
not exist or

both of the limits do not exist.

(iii)  $\lim_{x \rightarrow c} f(x) \neq f(c)$  iff (12)  
but are not equal.

⇒ Sequential Criterion  
for continuity.

If function  $f: A \rightarrow \mathbb{R}$   
is continuous at the point  
 $c \in A$  iff for every sequence  
( $a_n$ ) in A that cgs to 'c', the  
sequence ( $f(a_n)$ ) cgs to  $f(c)$ .

⇒ Discontinuity criterion

Let  $A \subseteq \mathbb{R}$  and let  $f: A \rightarrow \mathbb{R}$ ,  
let  $c \in A$ . Then f is  
discontinuous at c iff  
there exists a sequence ( $a_n$ )  
in A that cgs to 'c',  
but the sequence ( $f(a_n)$ )  
does not converge to  $f(c)$ .

Let  $A \subseteq \mathbb{R}$  and let  $f: A \rightarrow \mathbb{R}$ .

if  $B \subseteq A$ , we say that  
f is continuous on the set B  
if f is continuous at every  
point of B.

### Continuity in an open interval :-

A function  $f$  is said to be continuous in an open interval  $(a, b)$ , if it is continuous at every point of  $(a, b)$ .

$$\text{i.e. } \lim_{x \rightarrow c} f(x) = f(c), \quad x \in (a, b)$$

### continuity in a closed interval

A function  $f$  is said to be continuous in a closed interval  $[a, b]$  if it is

- (i) right conti at ' $a$ '  
i.e.  $\lim_{x \rightarrow a^+} f(x) = f(a)$

(ii) left conti at ' $b$ '  
i.e.  $\lim_{x \rightarrow b^-} f(x) = f(b)$

(iii) conti in  $(a, b)$

$$\text{i.e. } \lim_{x \rightarrow c} f(x) = f(c), \quad c \in (a, b)$$

A function which is not continuous even at a single point of an interval is said to be discontinuous in that interval.

### The types of Discontinuity

#### 1. Removable discontinuity:-

If  $\lim_{x \rightarrow c} f(x)$  exists but is not equal to  $f(c)$  then  $f$  is

said to be removable discontinuity at ' $c$ '

$$\text{i.e. } \lim_{x \rightarrow c} f(x) \neq f(c)$$

Ex:- ①

$$f(x) = \begin{cases} \sin x & \text{if } x \neq 0 \\ 2 & \text{if } x=0 \end{cases}$$

Sol

$$\lim_{x \rightarrow 0} f(x) = \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$$

$$x \neq 0,$$

$$f(0) = 2$$

$$\therefore \lim_{x \rightarrow 0} f(x) \neq f(0)$$

Ex:- ②

$$f(x) = \begin{cases} x^2 - 2 & \text{if } x > 2 \\ 4 - x & \text{if } x \leq 2 \\ 1 & \text{if } x = 2 \end{cases}$$

Sol LHL

$$\lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (4 - x) = 4 - 2 = 2$$

RHL

$$\lim_{x \rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} x^2 - 2 = 4 - 2 = 2$$

at  $x = 2$

$$f(2) = 1$$

$$\therefore (\lim_{x \rightarrow 2} f(x) \neq f(2)) \neq f(2)$$

(2) Discontinuity of first kind (or)  
jump discontinuity (or)

Ordinary discontinuity:

If  $\lim_{x \rightarrow c^-} f(x)$  and  $\lim_{x \rightarrow c^+} f(x)$

both exist but are not equal and  $f(c)$  exists,  
it is equal to the either

(or) neither of  $\lim_{x \rightarrow c^-} f(x)$  (or)

$\lim_{x \rightarrow c^+} f(x)$  then  $f$  is called

discontinuity of first kind.

$\rightarrow f$  is said to be discontinuity of first kind from the left at  $c$  if  $\lim_{x \rightarrow c^-} f(x)$  exists but it is not equal to  $f(c)$ .

$\rightarrow f$  is said to be discontinuity of the first kind from right at  $c$  if  $\lim_{x \rightarrow c^+} f(x)$  exists but is not equal to  $f(c)$ .

$$\text{Ex:- } f(x) = \begin{cases} x & \text{if } x > 2 \\ 3-x & \text{if } x < 2 \\ 1 & \text{if } x = 2 \end{cases}$$

(3) Disconti. of second kind: (13)

$\rightarrow$  If  $\lim_{x \rightarrow c^-} f(x)$  &  $\lim_{x \rightarrow c^+} f(x)$

both do not exist they  
f is called discontinuity of second kind at  $c$ .

$\rightarrow f$  is said to be a discontinuity of the second kind from the left at  $c$  if  $\lim_{x \rightarrow c^-} f(x)$  does not exist.

$\rightarrow f$  is said to be a discontinuity of the second kind from the right at  $c$  if  $\lim_{x \rightarrow c^+} f(x)$  does not exist.

$$\text{Ex:- } f(x) = \begin{cases} \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

$\Rightarrow$  LHL

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \sin\left(\frac{1}{x}\right)$$

$$= l \quad (\because -1 \leq l \leq 1)$$

$\therefore l$  is finite number  
but it is not fixed because  
l rotates with  $-1$  to  $1$ .

$\therefore$   $\lim_{x \rightarrow 0^-} f(x)$  does not exist

∴ RHL does not exist

(4) Mixed discontinuity:-

If a function  $f$  has discontinuity of the second kind on one side of  $c$  and other side a discontinuity of first kind.

(or) may be continuous.  
 Then  $f$  is called a mixed discontinuity at  $c$ .  
 (or)

If one of the limits  $\lim_{x \rightarrow c^-} f(x)$  &  $\lim_{x \rightarrow c^+} f(x)$  exist but not the other then  $f$  is called mixed discontinuous at  $c$ .

i.e.  $\lim_{x \rightarrow c^-} f(x)$  does not exist and  $\lim_{x \rightarrow c^+} f(x)$  exists and may (or) may not equal to  $f(c)$ .

(or)  
 $\lim_{x \rightarrow c^+} f(x)$  does not exist and  $\lim_{x \rightarrow c^-} f(x)$  exists and may (or) may not equal to  $f(c)$ .

$$\text{Ex:- } f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x > 0 \\ 2-x & \text{if } x < 0 \\ 0 & \text{if } x=0 \end{cases}$$

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(5) Infinite discontinuity:-

If one ( $\infty$ ) both limits  $\lim_{x \rightarrow c^-} f(x)$  &  $\lim_{x \rightarrow c^+} f(x)$  are  $\infty$  then  $f$  is called infinite discontinuity at  $c$ .

$$\text{Ex:- } f(x) = \begin{cases} \frac{1}{x-2} & \text{if } x \neq 2 \\ \infty & \text{if } x=2 \end{cases}$$

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Algebra of continuous functions:-

If  $f(x)$  &  $g(x)$  are continuous functions at  $x=c$  then  $\lim_{x \rightarrow c} f(x) = f(c)$  &  $\lim_{x \rightarrow c} g(x) = g(c)$ .

$$(i) \lim_{x \rightarrow c} (f \pm g)(x) = \lim_{x \rightarrow c} (f(x) \pm g(x)) \\ = \lim_{x \rightarrow c} f(x) \pm \lim_{x \rightarrow c} g(x) \\ = f(c) \pm g(c).$$

$$(ii) \lim_{x \rightarrow c} (f \cdot g)(x) = \lim_{x \rightarrow c} (f(x) \cdot g(x)) \\ = \lim_{x \rightarrow c} f(x) \cdot \lim_{x \rightarrow c} g(x) \\ = f(c) \cdot g(c).$$

$$(iii) \lim_{x \rightarrow c} (f/g)(x) = \lim_{x \rightarrow c} [f(x)/g(x)] \\ = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)}$$

... if  $g(c) \neq 0$

$$\begin{aligned}
 \text{(iv)} \quad & \lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{\lim f(x)}{\lim g(x)} \\
 &= \frac{\lim f(x)}{\lim g(x)} \\
 &\stackrel{x \rightarrow c}{=} \frac{f(c)}{g(c)} \\
 &= \left(\frac{f}{g}\right)(c) \\
 &\text{provided } g \neq 0.
 \end{aligned}$$

problems

using  $\epsilon-\delta$  definition,  
prove that

(i)  $f(x) = 3x + 1$  is continuous  
at  $x=2$

$$\text{(ii)} \quad f(x) = \begin{cases} \frac{x-4}{x-2} & \text{if } x \neq 2 \\ 4 & \text{if } x=2 \end{cases}$$

$$\text{sol} \quad \text{(i)} \quad f(x) = 3x + 1;$$

$$\begin{aligned}
 &\text{at } x=2 \\
 &f(2) = 3(2) + 1 \\
 &= 7.
 \end{aligned}$$

Let  $\epsilon > 0$  be given.

we have

$$\begin{aligned}
 |f(x) - f(2)| &= |3x + 1 - 7| \\
 &= |3x - 6| \\
 &= 3|x - 2| \leq \epsilon \\
 &\text{whenever } |x - 2| \leq \frac{\epsilon}{3}
 \end{aligned}$$

If we choose  $\delta = \frac{\epsilon}{3}$ , then

$$|f(x) - f(2)| \leq \epsilon \text{ whenever } |x - 2| < \delta$$

$f(x)$  is conti at  $x=2$ .

$$\text{(ii)} \quad f(x) = \frac{x-4}{x-2} \quad \text{if } x \neq 2$$

$$\text{at } x=2; \quad f(2) = 4.$$

Let  $\epsilon > 0$  be given. (14)

Now we have

$$|f(x) - f(2)| = \left| \frac{x-4}{x-2} - 4 \right|$$

$$= \left| \frac{x^2 - 4x - 4x + 8}{x-2} \right|$$

$$= \left| \frac{x^2 - 4x + 4}{x-2} \right|$$

$$= \left| \frac{(x-2)^2}{x-2} \right|$$

$$= |x-2| \leq \epsilon \quad \text{whenever } |x-2| \leq \epsilon$$

Choosing  $\delta = \epsilon$ ,

$$|f(x) - f(2)| \leq \epsilon \text{ whenever } |x-2| \leq \epsilon$$

$f(x)$  is conti at  $x=2$ .

→ the constant function

$f(x) = b$  is conti on  $\mathbb{R}$ .

→  $g(x) = x$  is conti on  $\mathbb{R}$ .

→  $h(x) = x^2$  is conti on  $\mathbb{R}$ .

→  $\phi(x) = \frac{1}{x}$  is conti on  $\mathbb{R} \setminus \{x=0\}$

sol Let  $x = \text{const}$

$$\text{then } \phi(x) = \frac{1}{x}$$

$$\text{and } \lim_{x \rightarrow c} \phi(x) = \frac{1}{c}$$

$$\therefore \lim_{x \rightarrow c} \phi(x) = \phi(c).$$

$\phi(x)$  is conti at  $x=c$ .

$\phi(c) = \frac{1}{2}$  is not conti  
 $\text{at } x=0$

because

$\phi$  is not defined at  
 $x=0$  and

$\lim_{x \rightarrow 0} \phi$  does not exist

$\rightarrow$  the signum function  $\operatorname{sgn}$   
is not conti. at  $x=0$

sol

Let  $f(x) = \operatorname{sgn}(x)$

$$= \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ +1 & \text{if } x > 0 \end{cases}$$

$\lim_{x \rightarrow 0^-} f(x) = -1$  &  $\lim_{x \rightarrow 0^+} f(x) = 1$

$\therefore \lim_{x \rightarrow 0} f(x) \neq \lim_{x \rightarrow 0} f(x)$

$\therefore \lim_{x \rightarrow 0} f(x)$  does not exist.

$\therefore f(x)$  is not conti. at  $x=0$ .

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Let  $A = \mathbb{R}$  and

let  $f: A \rightarrow \mathbb{R}$  be defined

$$\text{by } f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$$

which is known as

- Dirichlet's function.

$\checkmark$  S.T. that the Dirichlet's function is not continuous at any point of  $\mathbb{R}$ .

sol Let  $f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational.} \end{cases}$

let  $c = c$  for they  
c is either rational  
or irrational number.  
If c is a rational  
number:

$$\therefore f(c) = 1.$$

Let  $(x_n)$  be sequence

of rational numbers  
that converges to 'c'

since  $f(x_n) = 0 \forall n$

( $\because x_n$  is irrational)

$$\therefore \lim_{n \rightarrow \infty} f(x_n) = 0 \neq f(c).$$

$\therefore (f(x_n))$  does not converge to  $f(c)$ .

$\therefore f(x)$  is not continuous  
at the rational numbers.

If c is an irrational

number:

$$\therefore f(c) = 0$$

Let  $(x_n)$  be a sequence  
of rational numbers that  
converges to 'c'

since  $f(x_n) = 1 \forall n$

$$\therefore \lim_{n \rightarrow \infty} f(x_n) = 1 \neq f(c)$$

$\therefore (f(x_n))$  does not converge to  $f(c)$ .

$\therefore f(x)$  is not conti. at the  
irrational number 'c'.

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$\rightarrow$  prove that the function  $f$  defined by

$$f(x) = \begin{cases} 2x & \text{when } x \text{ is rational} \\ x+3 & \text{when } x \text{ is irrational} \end{cases}$$

is nowhere continuous.

(1)  $\rightarrow$  prove that the function  $f$  defined by

$$f(x) = \begin{cases} \frac{1}{x}, & \text{if } x \text{ is rational} \\ \frac{1}{3}, & \text{if } x \text{ is irrational} \end{cases}$$

is discontinuous everywhere.

$\rightarrow$  Define  $g: \mathbb{R} \rightarrow \mathbb{R}$  by  

$$g(x) = \begin{cases} 2x & \text{for } x \text{ rational} \\ x+3 & \text{for } x \text{ irrational} \end{cases}$$

Find all points at which  $g$  is continuous.

sol Given that

$$g(x) = \begin{cases} 2x & \text{for } x \text{ rational} \\ x+3 & \text{for } x \text{ irrational} \end{cases}$$

Let  $x$  be any real number,  
 for each  $n \in \mathbb{N}$ ,  $\exists$  a rational number  $r_n$  and an irrational number  $i_n$  such that

$$x - r_n < c_n < x + i_n \text{ and}$$

$$|r_n - x| < \epsilon \text{ and } |i_n - x| < \epsilon$$

$$\Rightarrow |c_n - x| < \epsilon \text{ and } |x + i_n - x| < \epsilon \quad (1)$$

$$\Rightarrow \lim_{n \rightarrow \infty} c_n = x \text{ and } \lim_{n \rightarrow \infty} x + i_n = x \quad \text{by (1)}$$

$$\Rightarrow \lim_{n \rightarrow \infty} c_n = x = \lim_{n \rightarrow \infty} x \quad (1)$$

If  $g$  is continuous at  $x$ ,  
 then we must have

$$\lim_{n \rightarrow \infty} g(c_n) = g(x) = \lim_{n \rightarrow \infty} g(x)$$

$$\text{but } g(c_n) = 2c_n \neq x.$$

$$g(b_n) = b_n + 3.$$

$$\therefore \lim_{n \rightarrow \infty} 2c_n = g(x) = \lim_{n \rightarrow \infty} (b_n + 3)$$

$$\Rightarrow 2\lim_{n \rightarrow \infty} c_n = g(x) = \lim_{n \rightarrow \infty} b_n + 3$$

$$\Rightarrow 2x = g(x) = x + 3 \quad (\text{by (1)})$$

$$\Rightarrow x = 3$$

$\therefore 3$  is the only possible point of continuity and discontinuity.  
 Now we show  $g + f$  is continuous at  $x=3$ .

$$\text{Let } \epsilon > 0 \text{ be given,}$$

for a rational number  $x$ ,  
 we have

$$|g(x) - g(3)| = |2x - 6| \\ = 2|x - 3| \rightarrow (2)$$

for an irrational number  $x$ ,

$$\text{we have } |x+3 - 6| = |x-3| \rightarrow (3)$$

from ②,

$$|g(x) - g(3)| \leq 2|x-3| < \epsilon$$

whenever  $|x-3| < \frac{\epsilon}{2}$

choosing  $\delta = \frac{\epsilon}{2}$

$$\therefore |g(x) - g(3)| \leq \epsilon \text{ whenever } |x-3| < \delta$$

from ③,

$$|g(x) - g(3)| \leq 1 \cdot |x-3| < \epsilon$$

whenever  $|x-3| < \frac{\epsilon}{1}$

choosing  $\delta = \frac{\epsilon}{1}$

$$\therefore |g(x) - g(3)| \leq \epsilon \text{ whenever } |x-3| < \delta$$

$\therefore g(x)$  is continuous at  $x=3$ .

Q3  
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Let  $f$  be defined on  $\mathbb{R}$  by setting  $f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 1-x, & \text{if } x \text{ is irrational} \end{cases}$

Show that  $f$  is continuous

at  $x = \frac{1}{2}$  but discontinuous

nowhere else.

Q3  
Show that the function

$f$  defined by

$$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$$

is continuous only at  $x=0$ .

Q3  
Show that the function

$f(x) = \begin{cases} x, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$

\* examine the continuity of the following functions at the indicated point.

(i)  $f(x) = \begin{cases} e^x - 1, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

(ii)  $f(x) = \begin{cases} \frac{e^x - 1}{1+e^x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

(iii)  $f(x) = \begin{cases} \frac{e^{2x} - e^{-2x}}{e^{2x} + e^{-2x}}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$

(iv)  $f(x) = \begin{cases} \frac{e^{-x} - 1}{e^{-x} + 1}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

(v)  $f(x) = \begin{cases} \frac{x e^x}{1+e^x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$

Q3

Since  $x \rightarrow 0^- \Rightarrow \frac{1}{x} \rightarrow -\infty$

and  $x \rightarrow 0^+ \Rightarrow \frac{1}{x} \rightarrow +\infty$

LHL

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}$$

$$= \frac{e^{-\infty} - 1}{e^{-\infty} + 1} = \frac{0 - 1}{0 + 1} = -1.$$

RHL

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} \frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}$$

$$= \lim_{x \rightarrow 0^+} \left[ \frac{1-e^{-x}}{1+e^{-x}} \right]$$

$$= \frac{1-e^{-\infty}}{1+e^{\infty}}$$

$$= \frac{1-0}{1+0} = 1.$$

$$\lim_{x \rightarrow 0^+} f(x) \neq \lim_{x \rightarrow 0^+} f(x)$$

$\therefore \lim_{x \rightarrow 0^+} f(x)$  does not exist

(iv)

$$\text{Since } x \rightarrow \infty \Rightarrow (e-x) \rightarrow 0^-$$

$$\Rightarrow \frac{1}{e^{x-a}} \rightarrow \infty$$

$$\text{and } x \rightarrow \infty \Rightarrow (e-x) \rightarrow 0^+$$

$$\Rightarrow \frac{1}{e^{x-a}} \rightarrow +\infty.$$

LHL

$$\lim_{x \rightarrow a^-} f(x) = \lim_{x \rightarrow a^-} (x-a) \left[ \frac{e^{(x-a)} - 1}{e^{(x-a)} + 1} \right]$$

$$= 0 \times \left[ \frac{e^0 - 1}{e^0 + 1} \right]$$

$$= 0 \times \left[ \frac{0-1}{0+1} \right] = 0 \times (-1)$$

$$= 0.$$

RHL

$$\lim_{x \rightarrow a^+} f(x) = \lim_{x \rightarrow a^+} (x-a) \left[ \frac{e^{(x-a)} - 1}{e^{(x-a)} + 1} \right]$$

$$= \lim_{x \rightarrow a^+} (x-a) \left[ \frac{1 - e^{-\frac{1}{x-a}}}{1 + e^{-\frac{1}{x-a}}} \right]$$

$$= 0 \times \left[ \frac{1 - e^0}{1 + e^0} \right]$$

$$= 0 \times \left[ \frac{1-0}{1+0} \right]$$

$$= 0 \times 1 = 0$$

$$\lim_{x \rightarrow a} f(x),$$

$$f(a) = 0.$$

(16)

$$\therefore (\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow a} f(x)) \Rightarrow f(x) = f(x),$$

$$\Rightarrow \lim_{x \rightarrow a} f(x) = f(a).$$

$\therefore f(x)$  is continuous

Discuss the continuity of the following functions at  $x=a$

$$(i) f(x) = \begin{cases} 2^x & \text{if } x \neq 0 \\ -a & \text{if } x=0 \end{cases}$$

$$(ii) f(x) = \begin{cases} \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x=0 \end{cases}$$

$$(iii) f(x) = \begin{cases} x \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x=0 \end{cases}$$

$$(iv) f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x=0 \end{cases}$$

$$(v) f(x) = \begin{cases} \cos \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x=0 \end{cases}$$

$$(vi) f(x) = \begin{cases} x \cos \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x=0 \end{cases}$$

$$(vii) f(x) = \begin{cases} x^2 \cos \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x=0 \end{cases}$$

$$(viii) f(x) = \begin{cases} \log x & \text{if } x > 0 \\ \omega x & \text{if } x < 0 \end{cases}$$

$$\text{SOL } \textcircled{i} \quad \begin{aligned} \text{Since } x &\rightarrow 0^+ \\ &\Rightarrow \frac{1}{x} \rightarrow +\infty \\ x &\rightarrow 0^- \\ &\Rightarrow \frac{1}{x} \rightarrow -\infty. \end{aligned}$$

$$\text{LHL} \quad \begin{aligned} \text{Lt f(x)} &= \text{Lt}_{x \rightarrow 0^+} 2^{\frac{1}{x}} \\ &= 2^{\infty} = \infty \\ &= \frac{1}{2^0} = \frac{1}{1} \\ &= 1. \end{aligned}$$

$$\text{RHL} \quad \begin{aligned} \text{Lt f(x)} &= \text{Lt}_{x \rightarrow 0^+} 2^{\frac{1}{x}} \\ &= 2^{\infty} = \infty. \\ \therefore \text{Lt f(x)} &\text{ does not exist} \\ &\text{at } x=0. \\ \therefore f &\text{ is discontinuous at } x=0. \end{aligned}$$

$$\text{(ii) Since } x \rightarrow 0^+ \Rightarrow \frac{1}{x} \rightarrow +\infty \\ \Rightarrow x \rightarrow 0^+ \Rightarrow \frac{1}{x} \rightarrow -\infty.$$

$$\text{LHL} \quad \begin{aligned} \text{Lt f(x)} &= \text{Lt}_{x \rightarrow 0^-} 2^{\frac{1}{x}} \\ &= l \quad (\because -1 \leq l \leq 1) \end{aligned}$$

Since  $l$  is finite number  
but it is not fixed number  
because  $l$  rotates with  $-1$  to  $1$   
 $\therefore \text{Lt f(x)} \text{ does not exist}$

$\therefore$  RHL does not exist.

$\therefore f$  is discontinuous at  $x=0$ .

$$\text{(iii) Since } x \rightarrow 0^+ \Rightarrow \frac{1}{x} \rightarrow \infty \\ \Rightarrow x \rightarrow 0^+ \Rightarrow \frac{1}{x} \rightarrow -\infty.$$

$$\text{LHL} \quad \begin{aligned} \text{Lt f(x)} &= \text{Lt}_{x \rightarrow 0^+} 2^{\frac{1}{x}} \\ &= 2^{+\infty} \quad (\because -1 \leq l \leq 1) \\ &= \infty. \end{aligned}$$

$$\text{RHL} \quad \begin{aligned} \text{Lt f(x)} &= \text{Lt}_{x \rightarrow 0^+} 2^{\frac{1}{x}} \\ &= 2^{-\infty} \\ &= 0^-. \end{aligned}$$

at  $x=0$   
 $f(0) = 0.$

$$\therefore [\text{Lt f(x)} = \text{Lt}_{x \rightarrow 0^+} \text{f(x)}] = \text{f}(0)$$

$$\Rightarrow \text{Lt f(x)} = \text{f}(0).$$

$\therefore f$  is continuous at  $x=0$ .

$$\text{(v) Since } x \rightarrow 0 \Rightarrow \frac{1}{x} \rightarrow \infty \\ \Rightarrow x \rightarrow 0^+ \Rightarrow \frac{1}{x} \rightarrow +\infty.$$

$$\text{LHL} \quad \begin{aligned} \text{Lt f(x)} &= \text{Lt}_{x \rightarrow 0^-} 2^{\frac{1}{x}} \\ &= l \end{aligned}$$

Since  $l$  is finite number  
but it is not fixed  
because  $l$  rotates with  
 $-1$  to  $1$ .  
 $\therefore \text{Lt f(x)} \text{ does not exist.}$

$\therefore$  RHL does not exist.

$\therefore f$  is not continuous  
 $\Rightarrow x=0$ .

discusses the continuity  
of the following function  
at  $x=a$ .

$$\text{(i) } f(x) = \begin{cases} (x-a) \sin\left(\frac{1}{x-a}\right) & \text{if } x \neq a \\ 0 & \text{if } x=a \end{cases}$$

$$\text{(ii) } f(x) = \begin{cases} (x-a) \cos\left(\frac{1}{x-a}\right) & \text{if } x \neq a \\ 0 & \text{if } x=a \end{cases}$$

$$\text{(iii) } f(x) = \begin{cases} \frac{1}{x-a} \csc(x-a) & \text{if } x \neq a \\ 0 & \text{if } x=a \end{cases}$$

Examine the discontinuity of the following functions at the indicated point.

$$\text{(i) } f(x) = \begin{cases} \frac{|x|}{x}, \text{ when } x \neq 0 \\ 1 \text{ when } x=0 \end{cases}$$

$$\text{(ii) } f(x) = |x| + |x-1| \text{ at } x=0$$

$$\text{(iii) } f(x) = \begin{cases} \frac{x-1}{x}, \text{ if } x \neq 0 \\ 1 \text{ if } x=0 \end{cases}$$

$$\text{(iv) } f(x) = \begin{cases} \frac{|x-1|}{x-2} & \text{if } x \neq 2 \\ -1 & \text{if } x=2 \end{cases}$$

$$\text{(v) } f(x) = \begin{cases} \frac{x}{|x|+x} & \text{if } x \neq 0 \\ 0 & \text{if } x=0 \end{cases}$$

$$\text{(vi) } f(x) = \begin{cases} \frac{|x|}{x^2+2x} & \text{if } x \neq 0 \\ 0 & \text{if } x=0 \end{cases}$$

$$\text{(vii) } f(x) = \begin{cases} \frac{2|x|+x^2}{x}, \text{ if } x \neq 0 \\ -1 & \text{if } x=0 \end{cases}$$

SOL.

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$$|x| = x \text{ if } x > 0$$

$$\rightarrow 0 \text{ if } x < 0$$

LHL

$$\lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} \frac{|x|}{x} = \lim_{x \rightarrow 0^-} \frac{x}{x} = \lim_{x \rightarrow 0^-} 1 = 1$$

$$= LHL$$

RHL

$$\lim_{x \rightarrow 0^+} f(x) = ?$$

$$\text{at } x=0, f(0)=1$$

$$\therefore LHL \neq RHL$$

$$\therefore f(x) \text{ is not conti at } x=0$$

$$\text{(ii) } f(x) = |x| + |x-1|$$

$$\xrightarrow{-} \xrightarrow{+} \xrightarrow{+} \xrightarrow{+}$$

$$x < 0, 0 \leq x \leq 1,$$

$$\text{if } x < 0 \text{ then } |x| = -x \text{ & }$$

$$|x-1| = -(x-1) \\ = 1-x$$

$$\therefore f(x) = |x| + |x-1|$$

$$= -x + 1 - x$$

$$= 1 - 2x$$

If  $0 \leq x \leq 1$  then  $|x| = x$

and  
 $|x-1| = -(x-1)$   
 $= 1-x$

$$\therefore f(x) = |x| + |x-1|$$

$$= x + 1-x$$

$$= 1.$$

$\Rightarrow$   
If  $x > 1$  then  $|x| = x$  and

$$|x-1| = x-1.$$

$$\therefore f(x) = x + x-1$$

$$= 2x-1.$$

$$\Rightarrow$$

$$\therefore f(x) = 1-2x \text{ if } x < 0$$

$$1 \text{ if } 0 \leq x \leq 1.$$

$$2x-1 \text{ if } x > 1.$$

Continuity at  $x=0$ :

$$\text{at } x=0, f(0)=1.$$

$$\underline{\underline{\text{LHL}}} \quad \text{Lt } f(x) = \text{Lt}_{x \rightarrow 0^-} (1-2x)$$

$$= 1-2(0) = 1.$$

$$\underline{\underline{\text{RHL}}} \quad \text{Lt } f(x) = \text{Lt}_{x \rightarrow 0^+} (1)$$

$$= 1.$$

$$\therefore \text{Lt}_{x \rightarrow 0^-} f(x) = \text{Lt}_{x \rightarrow 0^+} f(x) = f(0).$$

$\therefore f$  is conti. at  $x=0$ .

Continuity at  $x=1$ :

$$\text{At } x=1, f(1)=1.$$

$$\underline{\underline{\text{LHL}}} \quad \text{Lt } f(x) = \text{Lt}_{x \rightarrow 1^-} (1) = 1$$

$$\underline{\underline{\text{RHL}}} \quad \text{Lt } f(x) = \text{Lt}_{x \rightarrow 1^+} (2x-1)$$

$$= 2(1)-1$$

$$= 1.$$

$$\therefore (\text{Lt } f(x) = \text{Lt}_{x \rightarrow 1^+} f(x)) = f(1)$$

$$\Rightarrow \text{Lt } f(x) = f(1)$$

$\therefore f$  is conti. at  $x=1$ .

$$\text{at } x=2 \quad f(x) = -1.$$

LHL

$$\text{Lt } f(x) = \text{Lt}_{x \rightarrow 2^-} \frac{1-x}{x-2}$$

$$= \text{Lt}_{x \rightarrow 2^-} \frac{-(x-1)}{x-2}$$

$$\left( \begin{array}{l} \therefore x \rightarrow 2^- \\ \Rightarrow x < 2 \\ \therefore (x-1) < 0 \end{array} \right)$$

$$= \text{Lt}_{x \rightarrow 2^-} (-1)$$

$$= -1.$$

RHL

$$\text{Lt } f(x) = \text{Lt}_{x \rightarrow 2^+} (-1)$$

$$\therefore \text{Lt } f(x) \neq \text{Lt } f(x)$$

$\therefore \text{Lt } f(x)$  does not exist

$\therefore f$  is not continuous

QED

Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be

such that

$$f(x) = \begin{cases} \sin(\cot x) + \tan x, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

otherwise the value is not defined.

$$\underline{\text{RHL}} \quad \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} f(0) = 0.$$

$$\therefore \text{LHL} = \text{RHL} = f(0)$$

$\therefore f$  is continuous at '0'.

$$\underline{\text{LHD}}: \lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x-0}$$

$$= \lim_{x \rightarrow 0^+} \frac{-x-0}{x}$$

$$= -1$$

$$\underline{\text{RHD}}: \lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x-0}$$

$$= \lim_{x \rightarrow 0^+} \frac{x-0}{x}$$

$$= +1$$

$\therefore \text{LHD} \neq \text{RHD}$

$\therefore f$  is not differentiable at '0'.

Note(2): If  $f$  is not continuous at any point, it can not be derivable at that point.

Note(3): Let  $I \subseteq \mathbb{R}$  be an interval, let  $c \in I$  and let  $f: I \rightarrow \mathbb{R}$  and  $g: I \rightarrow \mathbb{R}$  be functions that are differentiable at 'c'. Then

(i) If  $x \in \mathbb{R}$  then the function  $xf$  is differentiable at 'c' and  $(xf)'(c) = xf'(c)$

(ii) The function  $f+g$  is differentiable at 'c' and  $(f+g)'(c) = f'(c) + g'(c)$

(iii) The function  $fg$  is differentiable at 'c' and

$$(fg)'(c) = f(c)g'(c) + f'(c)g(c).$$

④ If  $g(c) \neq 0$  then the function  $fg$  is differentiable at 'c' and  $(fg)'(c) = \frac{f'(c)g(c) - g'(c)f(c)}{(g(c))^2}$

### Problems:

→ use the definition to find the derivative of each of the following functions.

$$(i) f(x) = x^3 \forall x \in \mathbb{R}$$

$$(ii) g(x) = \frac{1}{x} \forall x \in \mathbb{R}; x \neq 0$$

$$(iii) h(x) = \sqrt{x} \forall x > 0$$

$$(iv) k(x) = \frac{1}{\sqrt{x}} \text{ for } x > 0.$$

Sol'n (i) Let  $x = c > 0$

$$\text{then } h(c) = \sqrt{c}$$

$$\text{Now } h'(c) = \lim_{x \rightarrow c} \frac{h(x) - h(c)}{x-c}$$

$$= \lim_{x \rightarrow c} \frac{\sqrt{x} - \sqrt{c}}{x-c}$$

$$= \lim_{x \rightarrow c} \left( \frac{\sqrt{x} - \sqrt{c}}{x-c} \right) \times \left( \frac{\sqrt{x} + \sqrt{c}}{\sqrt{x} + \sqrt{c}} \right)$$

$$= \lim_{x \rightarrow c} \frac{(x-c)}{(x-c)(\sqrt{x} + \sqrt{c})}$$

$$= \lim_{x \rightarrow c} \frac{1}{\sqrt{x} + \sqrt{c}}$$

$$= \frac{1}{\sqrt{c} + \sqrt{c}} = \frac{1}{2\sqrt{c}} \text{ exists.}$$

$\therefore f$  is defined for all +ve values

$$\text{of } \mathbb{R} \text{ and } f'(x) = \frac{1}{2\sqrt{x}}$$

$\Rightarrow$  show that  $f(x) = \sqrt[3]{x}$ ;  $x \in \mathbb{R}$  is not differentiable at  $x=0$ .

Sol'n: at  $x=0$ ;  $f(0)=0$

$$\text{Now } f'(0) = \lim_{x \rightarrow 0} \frac{f(x)-f(0)}{x-0}$$

$$= \lim_{x \rightarrow 0} \frac{\sqrt[3]{x}-0}{x}$$

$$= \lim_{x \rightarrow 0} \left(\frac{1}{x}\right)^{1/3} \text{ does not exist.}$$

$\therefore f$  is not differentiable at  $x=0$ .

$\Rightarrow$  Let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be defined

by

$$f(x) = \begin{cases} x^2 & \text{for } x \text{ rational} \\ 0 & \text{for } x \text{ irrational.} \end{cases}$$

Show that  $f$  is differentiable at  $x=0$  and find  $f'(0)$ .

$$\text{Sol'n: Let } f'(0) = \lim_{x \rightarrow 0} \frac{f(x)-f(0)}{x-0}$$

$$= L \quad \text{--- (1)}$$

(i) when  $x$  is rational number

$$\text{then } f'(0) = \lim_{x \rightarrow 0} \frac{x^2-0}{x-0}$$

$$= \lim_{x \rightarrow 0} \frac{x^2}{x}$$

$$= \lim_{x \rightarrow 0} x = 0$$

(ii) when  $x$  is irrational number

$$\text{then } f'(0) = \lim_{x \rightarrow 0} \frac{0-0}{x-0}$$

$$= \lim_{x \rightarrow 0} 0 = 0$$

$$\therefore f'(0) = \lim_{x \rightarrow 0} \frac{f(x)-f(0)}{x-0} = 0$$

Now, we have

$$\left| \frac{f(x)-f(0)}{x-0} - L \right| = \left| \frac{f(x)-f(0)}{x-0} - 0 \right|$$

$$= \left| \frac{x^2-0}{x} - 0 \right|$$

( $\because x$  is rational)

$$= |x| < \epsilon \text{ whenever } |x| < \frac{\epsilon}{1}$$

$$\text{choosing } \delta = \frac{\epsilon}{1}$$

$$\therefore \left| \frac{f(x)-f(0)}{x-0} - L \right| < \epsilon \text{ whenever } 0 < |x-0| < \delta \quad @$$

Now we have

$$\left| \frac{f(x)-f(0)}{x-0} - L \right| = \left| \frac{f(x)-f(0)}{x-0} - 0 \right|$$

$$= \left| \frac{0-0}{x-0} - 0 \right|$$

$\therefore x$  is irrational

$$= 0 < \epsilon \text{ whenever } 0 < |x-0| < \delta$$

$$\therefore \left| \frac{f(x)-f(0)}{x-0} - L \right| < \epsilon \text{ whenever } 0 < |x-0| < \delta$$

$\therefore f$  is differentiable at  $x=0$ .

$$\text{and } f'(0) = 0.$$

P-I  
2006 Find  $a$  &  $b$  so that  $f'(0)$  exists

where

$$f(x) = \begin{cases} \frac{1}{|x|} & \text{if } |x| > 2 \\ a+bx^2 & \text{if } |x| \leq 2 \end{cases}$$

∴ f(x) is continuous at  $x=0$ .

Soln: at  $x=0$

$$f(0)=c$$

LHL

$$\begin{aligned} \lim_{x \rightarrow 0^-} f(x) &= \lim_{x \rightarrow 0^-} \frac{\sin((c+1)x) + \sin x}{x} \\ &= \lim_{x \rightarrow 0^-} \left[ \frac{\sin((c+1)x)}{x} + \frac{\sin x}{x} \right] \\ &= \lim_{x \rightarrow 0^-} \left[ (c+1) \frac{\sin((c+1)x)}{(c+1)x} + \lim_{x \rightarrow 0^-} \frac{\sin x}{x} \right] \\ &= (c+1) \lim_{x \rightarrow 0^-} \frac{\sin((c+1)x)}{(c+1)x} + \lim_{x \rightarrow 0^-} \frac{\sin x}{x} \\ &\quad (\because x \rightarrow 0^- \\ &\quad \Rightarrow c+1 \neq 0) \\ &= (c+1)(1) + 0 \quad (\because \lim_{x \rightarrow 0^-} \frac{\sin x}{x} = 1) \\ &= c+2 \end{aligned}$$

RHL

$$\begin{aligned} \lim_{x \rightarrow 0^+} f(x) &= \lim_{x \rightarrow 0^+} \frac{(x+bx)^{\frac{1}{bx}} - 1}{bx^{\frac{1}{bx}}} \\ &= \lim_{x \rightarrow 0^+} \frac{x^{\frac{1}{bx}} - (1+bx)^{\frac{1}{bx}} + 1}{bx^{\frac{1}{bx}}} \\ &= \lim_{x \rightarrow 0^+} \frac{(1+bx)^{\frac{1}{bx}} - 1}{bx} \\ &= \lim_{x \rightarrow 0^+} \frac{\left(\sqrt{1+bx}-1\right) \times \left(\sqrt{1+bx}+1\right)}{bx} \\ &= \lim_{x \rightarrow 0^+} \frac{\sqrt{1+bx}-1}{\sqrt{1+bx}+1} \\ &= \lim_{x \rightarrow 0^+} \frac{1}{\sqrt{1+bx}+1} \\ &= \frac{1}{\sqrt{1+b(0)}+1} \quad (\because b \neq 0) \\ &= \frac{1}{2} \end{aligned}$$

This is the P.E.

so that b can have any non-zero real value. (18)

Since f is conti at  $x=0$ , we have,

$$\lim_{x \rightarrow 0^+} f(x) = L + f(0) = f(0)$$

$$\Rightarrow a+2 = \frac{1}{2} \Rightarrow a = -\frac{3}{2}$$

$$\Rightarrow a+2 = \frac{1}{2} \Rightarrow a = -\frac{3}{2}$$

$$\Rightarrow a = -\frac{3}{2} - \infty \Rightarrow a = \infty$$

$$\therefore a = -\frac{3}{2}, a \neq 0 \text{ or } a = \infty$$

→ Discuss the continuity of the function  $f(x) = [x]$  at the points  $x = \frac{1}{k}$ , where  $[x]$  denotes the greatest integer  $\leq x$ .

$$\text{Sol: } f(x) = [x].$$

Conti at  $x = \frac{1}{k}$ :

$$\text{at } x = k, f(k) = [k] = 0$$

$$\begin{aligned} \text{LHL} \quad \lim_{x \rightarrow k^-} f(x) &= \lim_{x \rightarrow k^-} [x] \\ &= k \end{aligned}$$

$$= 0 \quad (\because x \rightarrow k^- \\ \Rightarrow x < k \\ \Rightarrow x \in \mathbb{R})$$

$$\text{RHL} \quad \lim_{x \rightarrow k^+} f(x) = [x] = ?$$

$$\therefore \left( \lim_{x \rightarrow 0^-} f(x) + \lim_{x \rightarrow 0^+} f(x) \right) = f(0)$$

$f(x)$  is conti at  $x=0$

conti at  $x=1$ !

$$f(1) = [1] = 1$$

$$\underline{\text{LHL}} \quad \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} [x]$$

$$= 0 \quad (\because x \rightarrow 1^- \\ \Rightarrow x < 1 \\ x = 0, 1, 2, 3, \dots)$$

$$\underline{\text{RHL}} \quad \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} [x]$$

$$= 1 \quad (\because x \rightarrow 1^+ \\ \Rightarrow x > 1 \\ x = 1, 2, 3, \dots)$$

$$\therefore \lim_{x \rightarrow 1^-} f(x) \neq \lim_{x \rightarrow 1^+} f(x)$$

$\therefore f(x)$  is not conti at  $x=1$

$\rightarrow$  Discuss the continuity of  $f$  at  $x=1$ , where

$$f(x) = [x] + [x-1].$$

$$\underline{\text{sol}} \quad f(1) = [1-1] + [1-1] \\ = [0]_1 + [0] \\ = 0.$$

$$\underline{\text{LHL}} \quad \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} [x-1] + [x]$$

$$= \lim_{x \rightarrow 1^-} [1-x] + \lim_{x \rightarrow 1^-} [x]$$

$$= 0 + (-1) = -1$$

$$\therefore \lim_{x \rightarrow 1^-} f(x) = -1 \\ \Rightarrow x = 0, 1, 2, 3, \dots$$

RHL

$$\lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} ([x-1] + [x])$$

$$= \lim_{x \rightarrow 1^+} [1-x] + \lim_{x \rightarrow 1^+} [x]$$

$$= -1 + 0$$

$$= -1$$

$$\therefore \left( \lim_{x \rightarrow 1^-} f(x) + \lim_{x \rightarrow 1^+} f(x) \right) \neq f(1)$$

$\therefore f$  is not conti at  $x=1$

$\rightarrow$  Show that the function  $f$  defined by

$$f(x) = \begin{cases} [x-1] + 1 & \text{if } x \neq 1 \\ 0 & \text{if } x=1 \end{cases}$$

is discontinuity at  $x=1$ .

$\rightarrow$  examine the continuity of  $f$  at  $x=2$ ,

where  $[x-2]$ , if  $x < 2$

$$f(x) = \begin{cases} [x-2] & \text{if } x < 2 \\ 1 & \text{if } x=2 \\ 3x-5 & \text{if } x > 2 \end{cases}$$

→ Determine the points of continuity of the following functions:

(i)  $f(x) = [x]$ , (ii)  $f(x) = x[x]$

(iii)  $k(x) = x - [x]$  ✓  $k(x) = \left\{ \frac{x}{n} \right\}$

Sol) (i)  $f(x) = [x]$ ;  $x \in \mathbb{R}$

(ii) Let  $x = c + z$  (i.e. non-integer value)  
then  $f(c) = [c] = c$

LHL  $\underline{\underline{L+ f(x)}} = \underline{\underline{L+ [x]}}$

putting  $x = c - h$  ( $h > 0$ )

$$\begin{aligned} \therefore L+ f(x) &= L+ [c-h] \\ &\xrightarrow{z \rightarrow c^-} \\ &= L+ (c-1) \quad (\because c < c-h < c) \\ &\xrightarrow{h \rightarrow 0} \\ &= c-1 \quad (\text{Ans}) \end{aligned}$$

RHL  $\underline{\underline{L+ f(x)}} = \underline{\underline{L+ [x]}}$

putting  $x = c+h$  ( $h > 0$ )

$$\begin{aligned} \therefore L+ f(x) &= L+ [c+h] \\ &\xrightarrow{z \rightarrow c^+} \\ &= L+ (c+1) \quad (\because c < c+h < c+1) \\ &\xrightarrow{h \rightarrow 0} \\ &= c+1 \end{aligned}$$

$\therefore L+ f(x) \neq L+ f(x)$

$\therefore f(x)$  is not continuous  
 $\Leftrightarrow x = c \in \mathbb{Z}$

(b) Let  $x = c + z$  (19)  
i.e.  $x$  is non-integer value  
If  $n$  is the greatest integer less than  $c$  then  $[c] = n$ .  
where  $n \in \mathbb{Z}_{\text{int}}$

Now  $f(x) = [c] = n$ .

LHL  $\underline{\underline{L+ f(x)}} = \underline{\underline{L+ [x]}}$

$\xrightarrow{z \rightarrow c^-}$   $\xrightarrow{x \rightarrow c^-}$   
 $= L+ [c-h]$  (put  $x = c-h$ )

$\xrightarrow{h \rightarrow 0} = L+ (n)$  ( $\because n < (c-h) < n+1$ )

$$\begin{cases} x \rightarrow c^- \\ x \rightarrow c-0.5 \\ x = 2.1 \text{ or } \dots \\ f(x) = 2^n \end{cases}$$

RHL

$\underline{\underline{L+ f(x)}} = \underline{\underline{L+ [x]}}$

$\xrightarrow{z \rightarrow c^+} \xrightarrow{x \rightarrow c^+}$   
 $= L+ [c+h]$  (put  $x = c+h$ )

$\xrightarrow{h \rightarrow 0} = L+ (n)$  ( $\because n < (c+h) < n+1$ )

$\xrightarrow{h \rightarrow 0}$

$\therefore (L+ f(x) = L+ f(x)) \Rightarrow f(x)$

$\therefore f(x)$  is cont. i.e.  $x = c \in \mathbb{Z}$

i.e.  $f(x)$  is continous at the non-integer values.

(iii)  $k(x) = \left\{ \frac{1}{x} \right\}; (x \neq 0) \rightarrow \text{aER}$

(iv) Let  $x = c$  be an integer (except 0 & ±1).

If  $n$  is the greatest integer less than  $\frac{1}{c}$  then  $[\frac{1}{c}] = n$ .

where

$$n < \frac{1}{c} < n+1$$

NOW

LHL

$$\lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^-} [\frac{1}{x}]$$

$$= \lim_{h \rightarrow 0} [\frac{1}{c-h}] \quad (\text{put } x=c-h; h>0)$$

$$= \lim_{h \rightarrow 0} (n) \quad (\because n < \frac{1}{c-h} < n+1)$$

$$= n.$$

RHL

$$\lim_{x \rightarrow c^+} f(x) = \lim_{x \rightarrow c^+} [\frac{1}{x}]$$

$$= \lim_{h \rightarrow 0} [\frac{1}{c+h}] \quad (\text{put } x=c+h; h>0)$$

$$= \lim_{h \rightarrow 0} (n) \quad (\because n < \frac{1}{c+h} < n+1)$$

$$= n.$$

$$\therefore \left( \lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^+} f(x) \right) = f(c).$$

$\therefore f(x)$  is conti at  $x=c$  except 0 &  $\pm\frac{1}{2}$ .

b) Let  $a = c \in \mathbb{R} - \mathbb{Z}$ :

then ①  $\frac{1}{a} = \frac{1}{c}$  is an

integer if  $a = c = \pm\frac{1}{2}, \pm\frac{1}{3}, \dots$

②  $\frac{1}{a} = \frac{1}{c}$  is not an

integer if  $a = c$

$$\pm\frac{1}{2}, \pm\frac{1}{3}, \dots$$

① if  $\frac{1}{a} = \frac{1}{c}$  is not an

integer then  $[\frac{1}{c}] = [\frac{1}{a}]$ .

LHL

$$\lim_{x \rightarrow c^-} f(x) = \lim_{x \rightarrow c^-} [\frac{1}{x}]$$

$$= \lim_{h \rightarrow 0} [\frac{1}{c-h}] \quad (\text{put } x=c-h; h>0)$$

$$= \lim_{h \rightarrow 0} \left( \frac{1}{c} - 1 \right) \quad (\because \frac{1}{c} < \frac{1}{c-h} < \frac{1}{c})$$

$$= \frac{1}{c} - 1$$

$$\begin{aligned} x &\rightarrow c^- \\ x &\rightarrow c-\frac{1}{2} \\ \frac{1}{x} &\rightarrow \frac{1}{c} = 2 \\ \frac{1}{x} &= 4 \cdot 6 \cdot 8 \dots 1 + 14 \\ \left[ \frac{1}{x} \right] &= 1 = 2^1 \\ \frac{1}{x} &\rightarrow \frac{1}{c}-1 \end{aligned}$$

RHL

$$\lim_{x \rightarrow c^+} f(x) = \lim_{x \rightarrow c^+} \frac{1}{x}$$

$$= \lim_{h \rightarrow 0} \left[ \frac{1}{c+h} \right] \quad (\text{put } x=c+h; h>0)$$

$$= \lim_{h \rightarrow 0} \left( \frac{1}{c} \right) \quad \left( \frac{1}{c} < \frac{1}{c+h} < \frac{1}{c} \right)$$

$$= \frac{1}{c}$$

$\therefore \lim_{x \rightarrow c^-} f(x) \neq \lim_{x \rightarrow c^+} f(x)$

$\therefore f$  is not conti at

$$x = \pm\frac{1}{2}, \pm\frac{1}{3}, \dots$$

(2) if  $\frac{1}{x} = \frac{1}{c}$  is not an integer if

$$x \neq \pm \frac{1}{2}, \pm \frac{1}{3}, \dots$$

Let  $n$  be the greatest integer less than  $\frac{1}{c}$ . Then  $\left[ \frac{1}{c} \right] = n$ . Then  $n < \frac{1}{c} < n+1$ .

$$\begin{aligned} LHL &= Lf(x) = Lf\left(\frac{1}{c+h}\right) \\ &= Lf\left(\frac{1}{cn}\right). \quad (\text{Put } h \rightarrow 0) \end{aligned}$$

$$= \lim_{h \rightarrow 0} f(n) \quad (\because n < \frac{1}{c+h} < n+1)$$

$$= \frac{n}{2}$$

$$RHL \quad Lf(x) = \lim_{x \rightarrow c^+} \left[ \frac{1}{x} \right]$$

$$= \lim_{h \rightarrow 0} \left[ \frac{1}{c+h} \right] \quad (\text{Put } h \rightarrow 0)$$

$$= \lim_{h \rightarrow 0} (n) \quad (\because n < \frac{1}{c+h} < n+1)$$

$$= n$$

$$\therefore Lf(x) = Rf(x) = f(c)$$

$\therefore f(x)$  is continuous at  $x \neq \pm \frac{1}{2}, \pm \frac{1}{3}, \dots$

Show that the absolute function  $f(x) = |x|$  is continuous at every point  $c \in \mathbb{R}$

Sol: Given that  $f(x) = |x|$  when  $x \in \mathbb{R}$  then for the limit

Now we will show that

$$Lf(x) = f(c)$$

$$\text{i.e. } f(x) \rightarrow f(c) \text{ as } x \rightarrow c.$$

Let  $\epsilon > 0$  be given. Now we have

$$|f(x) - f(c)| = ||x| - |c||$$

$$\leq |x - c|$$

$$\leq \epsilon \text{ whenever } |x - c| < \frac{\epsilon}{1}$$

choosing  $\delta = \frac{\epsilon}{1}$ .

$\therefore |f(x) - f(c)| < \epsilon$  whenever  $|x - c| < \delta$ .

i.e.  $f(x) \rightarrow f(c)$  as  $x \rightarrow c$ .

$$Lf(x) = f(c).$$

$\therefore f(x)$  is continuous at  $x = c$

① Given let  $K > 0$  and let  $f: \mathbb{R} \rightarrow \mathbb{R}$

satisfy the condition

$$|f(x) - f(y)| \leq K|x - y| \forall x, y \in \mathbb{R}$$

Show that  $f$  is continuous

at every point  $c \in \mathbb{R}$ .

② Given  $K > 0$ ,  $f: \mathbb{R} \rightarrow \mathbb{R}$

satisfies the condition

$$|f(x) - f(y)| \leq K|x - y| \quad \forall x, y \in \mathbb{R}$$

Now we shall show that

$$Lf(x) = f(c).$$

$$\text{i.e. } f(x) \rightarrow f(c)$$

as  $x \rightarrow c$ .

for this we are enough to show that,

given any  $\epsilon > 0$  (however small),

exists  $\delta > 0$  (depends on  $\epsilon$ ) s.t

$$|f(x) - f(c)| < \epsilon \text{ whenever } |x - c| < \delta$$

Now from ①,

we have

$$|f(x) - f(y)| \leq k|x-y|$$

$\forall x, y \in (c-\delta, c)$

Taking  $x=c$ ,  $y=c$ , we get

$$|f(c) - f(c)| \leq k|c-c|$$

$\Leftrightarrow$  whenever

$$|c-c| < \frac{\epsilon}{k}$$

choosing  $\delta = \frac{\epsilon}{k}$

$$\therefore |f(c) - f(c)| < \epsilon \text{ whenever } |c-c| < \delta$$

i.e.  $f(c) = f(c)$  as  $c \neq c$

$$\therefore L + f(c) = f(c)$$

$\therefore f(c)$  is continuous at  $x=c$ .

Let  $g$  be defined on  $\mathbb{R}$

$$\text{by } g(x) = \begin{cases} 0 & \text{for } x=1 \\ 2 & \text{for } x \neq 1 \end{cases}$$

and let  $f(x) = x+1 \forall x \in \mathbb{R}$ . if  $f(x+y) = f(x) + f(y)$  for all  $x, y$

$$\text{Show that } L + (g \circ f)(x) \neq (g \circ f)(0)$$

$$\text{sol} \quad g(x) = \begin{cases} 2 & \text{for } x \neq 1 \\ 0 & \text{for } x=1 \end{cases}$$

and  $f(x) = x+1 \forall x \in \mathbb{R}$ .

Now  $f(x) = x+1 \forall x \in \mathbb{R}$ .

$$\text{now } (g \circ f)(x) = g(f(x))$$

$$= g(x+1)$$

$$= \begin{cases} 2 & \text{for } x+1 \neq 1 \\ 0 & \text{for } x+1=1 \end{cases}$$

$$= \begin{cases} 2 & \text{for } x \neq 0 \\ 0 & \text{for } x=0 \end{cases}$$

$$= \begin{cases} 2 & \text{for } x \neq 0 \\ 0 & \text{for } x=0 \end{cases}$$

$$\text{Now } \lim_{x \rightarrow 0^-} (g \circ f)(x) = \lim_{x \rightarrow 0^-} (2) = 2$$

$\therefore$

$$\text{and } \lim_{x \rightarrow 0^+} (g \circ f)(x) = \lim_{x \rightarrow 0^+} (2) = 2$$

$\therefore$

$$\left( \lim_{x \rightarrow 0^-} (g \circ f)(x) = \lim_{x \rightarrow 0^+} (g \circ f)(x) \right) = 2$$

$$\Rightarrow \lim_{x \rightarrow 0} (g \circ f)(x) = 2$$

But  $\lim_{x \rightarrow 0} (g \circ f)(x) = 0$ ,

$$(g \circ f)(0) = 0.$$

from ① & ②

$$\lim_{x \rightarrow 0} (g \circ f)(x) \neq (g \circ f)(0).$$

def: A function  $f: \mathbb{R} \rightarrow \mathbb{R}$

is said to be additive

if  $f(x+y) = f(x) + f(y)$  for all  $x, y$

prove that if  $f$  is continuous at some  $x_0$ , then it is continuous at every point of  $\mathbb{R}$ .

sol

continuity at the point  $x_0$ :

$$\lim_{x \rightarrow x_0} f(x) = \lim_{h \rightarrow 0} f(x_0+h) \quad \begin{array}{l} \text{putting} \\ x = x_0 + h, \\ h \neq 0 \end{array}$$

$$= \lim_{h \rightarrow 0} [f(x_0) + f(h)] \quad \begin{array}{l} \because f(x+y) = \\ f(x)+f(y) \end{array}$$

$$= \lim_{h \rightarrow 0} f(x_0) + \lim_{h \rightarrow 0} f(h)$$

$$= f(x_0) + \lim_{h \rightarrow 0} f(h)$$

$\therefore \dots + f(h)$

$$\text{Sly } L + f(x) = L + f(x_0 + h)$$

$x \rightarrow x_0^+$   $h \rightarrow 0$

putting  
 $x = x_0 + h$   
 $h > 0$

$$= L + \lim_{h \rightarrow 0} [f(x_0) + f(h)]$$

$$\begin{aligned} &\therefore f(x+h) = \\ &f(x) + f(h) \end{aligned}$$

$$= L + f(x_0) + \lim_{h \rightarrow 0} f(h)$$

$$= f(x_0) + f(0)$$

$$\boxed{\begin{aligned} L + f(x) &= f(x_0) + f(0) \\ x &\rightarrow x_0^+ \end{aligned}}$$

Since  $f$  is continuous at  $x = x_0$

$$\therefore L + f(x) = L + f(x_0) = f(x_0)$$

$$\therefore L + f(x) = L + f(x_0) = f(x_0) + \lim_{h \rightarrow 0} f(h)$$

$$\therefore L + f(h) = f(x_0)$$

$$\therefore L + f(-h) = L + f(h) = 0$$

$\Rightarrow \text{Q.E.D.}$

Let  $c$  be any real number

$$\text{then } L + f(x) \underset{x \rightarrow c^-}{\text{def}} \lim_{h \rightarrow 0} f(c-h)$$

$$= L + \lim_{h \rightarrow 0} (f(c) + f(-h))$$

$\because f(x+h) = f(x) + f(h)$

$$= L + f(c) + \lim_{h \rightarrow 0} f(-h)$$

$$= f(c) + 0 \quad (\text{from Q})$$

$$\boxed{L + f(x) = f(c)}$$

also

$$L + f(x) = L + f(c+h)$$

$x \rightarrow c$

$$= L + \lim_{h \rightarrow 0} [f(c) + f(h)]$$

$$= L + f(c) + \lim_{h \rightarrow 0} f(h)$$

$$= f(c) + 0 \quad (\text{from Q})$$

$$\boxed{L + f(x) = f(c)}$$

$$\therefore L + f(x) = L + f(c) = f(c)$$

$\therefore f(x)$  is continuous at  $x = c$

Given If  $f$  a function of  $x$  is continuous at  $x = c$  then  $|f|$  is also continuous at  $x = c$ .

proof Since  $f$  is continuous

at  $x = c$

$\therefore$  Given  $\epsilon > 0$   $\exists \delta > 0$  s.t.

$|f(a) - f(c)| < \epsilon$  whenever  $|a - c| < \delta$   $\text{Q.E.D.}$

Now we have

$$|f(a) - f(c)| \leq |f(a) - f(c)| < \epsilon$$

whenever  $|a - c| < \delta$   $\text{Q.E.D.}$

$\therefore |f|$  is continuous at  $c$

Note:- The converse of the above theorem need not be true

i.e. if  $|f|$  is continuous at  $c$  then  $f$  need not

For example

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ -1 & \text{if } x < 0 \end{cases}$$

so

$$\lim_{x \rightarrow 0} |f(x)| = \lim_{x \rightarrow 0} |f(0)| = 1, \forall x \in \mathbb{R}.$$

$$\lim_{x \rightarrow 0^+} |f(x)| = \lim_{x \rightarrow 0^+} |f(0)| = 1$$

at  $x=0$ ,

$$|f(0)| = |f(0)|$$

$$= 1$$

$$\lim_{x \rightarrow 0^-} |f(x)| = \lim_{x \rightarrow 0^-} |f(0)|.$$

$\therefore |f|$  is continuous at  $x=0$

$$\begin{array}{l} \text{Now} \\ \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (-1) = -1. \quad \text{by} \\ \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (1) = 1. \end{array}$$

$$\lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} (1) = 1.$$

$$\therefore \lim_{x \rightarrow 0^+} f(x) \neq \lim_{x \rightarrow 0^-} f(x)$$

$\therefore f(x)$  is not continuous at  $x=0$ .

Theory  $\Rightarrow$  Let  $f, g: \mathbb{R} \rightarrow \mathbb{R}$

i) continuous at 'c' and

$$\text{let } h(x) = \max \{f(x), g(x)\}$$

i.e.,  $\sup \{f(x), g(x)\}$  for  $x \neq c$

$$\text{show } \lim_{x \rightarrow c} h(x) = \frac{1}{2} (f(c) + g(c)) + \frac{1}{2} |f(c) - g(c)|$$

$\forall \epsilon > 0$

use this to show that  $h$  is continuous at 'c'.

Proof: Since  $f: \mathbb{R} \rightarrow \mathbb{R}$  &  $g: \mathbb{R} \rightarrow \mathbb{R}$  are two continuous functions at 'c'

$$\lim_{x \rightarrow c} f(x) = f(c) \quad \text{&} \quad$$

$$\lim_{x \rightarrow c} g(x) = g(c)$$

and since

$$h(x) = \sup \{f(x), g(x)\}; \forall x \in \mathbb{R}$$

$$\text{i.e., } h(x) = \max \{f(x), g(x)\}; \forall x \in \mathbb{R}$$

$$(i) \quad h(x) = \begin{cases} f(x), & \text{if } f(x) \geq g(x) \\ g(x), & \text{if } f(x) \leq g(x) \end{cases}$$

now since,  $\forall x \in \mathbb{R}$ ,

$$\frac{1}{2} (f(x) + g(x)) + \frac{1}{2} |f(x) - g(x)|$$

$$= \begin{cases} \frac{1}{2} (f(x) + g(x)) + \frac{1}{2} [f(x) - g(x)], & \text{if } f(x) \geq g(x) \\ \frac{1}{2} (f(x) + g(x)) + \frac{1}{2} [-(f(x) - g(x))] & \text{if } f(x) \leq g(x) \end{cases}$$

$$= \begin{cases} f(x), & \text{if } f(x) \geq g(x) \\ g(x), & \text{if } f(x) \leq g(x). \end{cases}$$

$$= h(x) \quad (\text{by } (i)).$$

$\Rightarrow$

(ii) To show  $h$  is continuous at 'c'.

since  $f, g$  are continuous at 'c'

$\therefore f+g$  is also continuous at 'c'

$\Rightarrow \frac{1}{2} (f+g)$  is also continuous at 'c'

$\therefore \frac{1}{2} (f-g)$  is continuous at 'c'

$\Rightarrow |f-g|$  is also continuous at 'c'

$\Rightarrow \frac{1}{2} |f-g|$  is also continuous at 'c'

from (C) & (D)

$\frac{1}{2}(f+g) + \frac{1}{2}|f-g|$  is also continuous at 'c'.

$$\therefore h(x) = \frac{1}{2}(f(x) + g(x)) + \frac{1}{2}|f(x) - g(x)|$$

continuous at  $x=c$ .

$$\begin{aligned} \lim_{x \rightarrow c} h(x) &= \lim_{x \rightarrow c} \frac{1}{2}[f(x) + g(x)] + \frac{1}{2}|f(x) - g(x)| \\ &= \frac{1}{2}\left[\lim_{x \rightarrow c} f(x) + \lim_{x \rightarrow c} g(x)\right] + \frac{1}{2}\left[\lim_{x \rightarrow c} |f(x) - g(x)|\right] \\ &= \frac{1}{2}[f(c) + g(c)] + \frac{1}{2}|f(c) - g(c)| \\ &= h(c) \end{aligned}$$

$$\therefore \lim_{x \rightarrow c} h(x) = h(c).$$

$\therefore h(x)$  is continuous at 'c'.

H.W Let  $f, g: \mathbb{R} \rightarrow \mathbb{R}$  be continuous at 'c' and let

$$h(x) = \inf\{f(x), g(x)\}$$

$$\text{i.e., } h(x) = \min\{f(x), g(x)\} \text{ for } x \in \mathbb{R}$$

$$\text{Show that } h(x) = \frac{1}{2}(f(x) + g(x)) - \frac{1}{2}|f(x) - g(x)| \quad \forall x \in \mathbb{R}$$

Use this to show that  $h$  is continuous at 'c'.

Theorem A function  $f$  is continuous at 'c' iff for each  $\epsilon > 0$ ,  $\exists$  a  $\delta > 0$  such that  $|f(x_1) - f(x_2)| < \epsilon$  whenever  $x_1, x_2 \in (c-\delta, c+\delta)$ .

proof: (i) Let  $f$  be a continuous function at 'c'.

then for each  $\epsilon > 0$ ,  $\exists$  a  $\delta > 0$  such that

$$|f(x) - f(c)| < \frac{\epsilon}{2} \text{ whenever } |x - c| < \delta.$$

$$\Rightarrow |f(x) - f(c)| < \frac{\epsilon}{2}, \text{ whenever } -\delta < x - c < \delta.$$

$$\Rightarrow |f(x) - f(c)| < \frac{\epsilon}{2}, \text{ whenever } c - \delta < x < c + \delta.$$

$$\Rightarrow |f(x) - f(c)| < \frac{\epsilon}{2} \text{ whenever } x \in (c-\delta, c+\delta)$$

Now for  $x_1, x_2 \in (c-\delta, c+\delta)$

$$|f(x_1) - f(x_2)| < \frac{\epsilon}{2} \text{ & } |f(x_2) - f(c)| < \frac{\epsilon}{2} \quad (1)$$

Now we have

$$\begin{aligned}|f(x_1) - f(x_2)| &= |f(x_1) - f(c) + f(c) - f(x_2)| \\&\leq |f(x_1) - f(c)| + |f(x_2) - f(c)| \\&< \epsilon_1 + \epsilon_2 = \epsilon \quad (\text{by } (i))\end{aligned}$$

$\therefore |f(x_1) - f(x_2)| < \epsilon$  whenever  $x_1, x_2 \in (c-\delta, c+\delta)$

(ii) Conversely Suppose that for each  $\epsilon > 0$ ,  $\exists$  a  $\delta > 0$  such that  $|f(x_1) - f(x_2)| < \epsilon$  whenever  $x_1, x_2 \in (c-\delta, c+\delta)$ .

Taking  $x_1 = x$  &  $x_2 = c$

we have  $|f(x) - f(c)| < \epsilon$  whenever

$$x \in (c-\delta, c+\delta)$$
  
 $\therefore f \text{ is continuous at } x=c$

Theorem If a function  $f$  is continuous at 'c' then it is bounded in some nbd of 'c'.

Proof: Since  $f$  is continuous at 'c'

Given  $\epsilon > 0$ ,  $\exists$  a  $\delta > 0$  such that  $|x-c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon$  whenever  $x \in (c-\delta, c+\delta)$ .

$$\Rightarrow f(c) - \epsilon < f(x) < f(c) + \epsilon \quad \text{whenever } (c-\delta, c+\delta) \cap D_f$$

$$\text{Let } M = \max \{ |f(c) - \epsilon|, |f(c) + \epsilon| \}$$

then  $-M \leq f(x) \leq M$  whenever  $x \in (c-\delta, c+\delta) \cap D_f$

$$\Rightarrow |f(x)| \leq M \text{ whenever } x \in (c-\delta, c+\delta) \cap D_f$$

$\therefore f$  is bounded in some nbd of 'c'.

Ex:  $f(x) = \sin x$  is continuous for all  $x \in \mathbb{R}$

and the range of  $\sin x$  is  $[-1, 1]$

$$\therefore -1 \leq \sin x \leq 1 \quad \forall x \in \mathbb{R}$$

$$\inf = -1 \text{ & } \sup = 1$$

$\therefore f$  is bdd. (for each nbd of  $x$ )

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32/4

If  $f$  is a continuous function of  $x$  satisfying

(2a)

the functional equation

$$f(x+y) = f(x) + f(y)$$

Show that  $f(x) = ax$ , where ' $a$ ' is a constant.

$\forall x \in \mathbb{R}$ .

Sol:

Given that 'f' is continuous and  $f(x+y) = f(x) + f(y)$  — (1)

Taking  $x = 0 = y$  in (1)

$$\text{②} \Rightarrow f(0+0) = f(0) + f(0)$$

$$\Rightarrow f(0) = f(0) + f(0)$$

$$\Rightarrow f(0) \neq 0 = f(0) + f(0)$$

$$\Rightarrow f(0) = 0$$

Taking  $y = -x$ .

$$\text{③} \Rightarrow f(x+(-x)) = f(x) + f(-x)$$

$$\Rightarrow f(0) = f(x) + f(-x)$$

$$\Rightarrow -\text{②} = f(x) + f(-x)$$

$$\Rightarrow f(-x) = -f(x)$$

If  $x$  be a +ve integer,

we have

$$f(x) = f(1+1+1+\dots+1)$$

$$= f(1) + f(1) + f(1) + \dots + f(1)$$

$$= x f(1)$$

$$= ax, \text{ say}$$

where  $f(1) = a$ .

Let now let  $x$  be a -ve integer.

We write  $x = -y$  so that  $y$  is the integer.

we have

$$\begin{aligned}f(x) &= f(-y) \\&= -f(y) \quad [\because f(-y) = -f(y)] \\&= -ay \\&= a(-y) \\&= ax.\end{aligned}$$

Again, let  $x = \frac{p}{q}$  be a rational number;  
 $q$  being +ve.

we have

$$\begin{aligned}f(p) &= f\left(\frac{p}{q} \cdot q\right) \\&= f\left(\frac{p}{q} + \frac{p}{q} + \dots \text{ } q \text{ times}\right) \\&= f\left(\frac{p}{q}\right) + f\left(\frac{p}{q}\right) + \dots \text{ } q \text{ times} \\&= q \cdot f\left(\frac{p}{q}\right)\end{aligned}$$

$$\therefore f(p) = q \cdot f\left(\frac{p}{q}\right)$$

$$\Rightarrow ap = q \cdot f\left(\frac{p}{q}\right) \quad (\because f(p) = ap)$$

$$\Rightarrow f\left(\frac{p}{q}\right) = a \cdot \frac{p}{q}$$

$$\Rightarrow f(x) = ax. \quad (\because \frac{p}{q} = x)$$

Now,

Suppose that  $x$  is any real number.

Let  $\{x_n\}$  be a sequence of rational numbers such  
that  $\lim x_n = x$ .

We have,  $x_n$ , being rational,

$$f(x_n) = ax_n \quad \text{--- (2)}$$

Let  $n \rightarrow \infty$ .

As  $f$  is a continuous function,

we obtain from (2)

$$f(x) = ax \forall x.$$

Hence the result

Partition of a closed interval :-

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Let  $[a, b]$  be a closed

સુરત

If  $a = x_0 < x_1 < x_2 < \dots < x_{r-1} < x_r < x_{r+1}$

Then the finite set

$$P = \{x_0, x_1, x_2, \dots, x_{n-1}, x_n, \dots, x_b\} \quad (5)$$

called a partition of  $[a,b]$

→ the  $(n+1)$  points  $x_0, x_1, \dots, x_n$

- are called partition point

of the set  $P$ :

The closed intervals  $[x_0, x_1]$ ,  $[x_1, x_2]$ ,  $\dots$

$$\{x_1, x_n\}, \dots, \{x_{m-1}, x_n\} = \{x_{m+1}, x_n\}$$

are called the  $n$  subintervals

are placed in various  
of use.

$\text{subinterval } [x_{r-1}, x_r]$

is denoted by  $\sigma$ .

length  $x - x_1$  is indicated by 5.

It is denoted by  
 $S_x = m \omega_0$ .

$f$  is cont<sup>f</sup> on  $[a, b]$

$\Rightarrow$  If  $f$  is cont<sup>f</sup> on  $[a,b]$

~~thus~~ gives  $\epsilon \rightarrow 0$  (however small),

The clock period [a,b] can

divided into a finite

box of substances etc.

number of -  
- which the oscillations

each of which the base  
will be about 6

of his lesson -

$$\text{i.e. } |f(x_1) - f(x_2)| < \epsilon \text{ for}$$

any two points  $x_1$  &  $x_2$

any time  
in the same silent

卷之三

definition

If  $f$  is continuous  
in  $[a, b]$  then  $f$  is odd  
for that interval.

proof Since  $f$  is continuous  
in  $[a, b]$

Given  $\epsilon > 0$  (arbitrary),  
 $[a, b]$  can be divided into  
finite number of subintervals  
in each of which the  
oscillation of  $f$  is less than  
 $\epsilon$ .

i.e.  
Let  $[a = a_0, a_1], [a_1, a_2], \dots$   
 $[a_{n-1}, a_n]$

$$\text{S.T } |f(x_1) - f(x_2)| < \epsilon. \quad \textcircled{1}$$

for any two points  $x_1, x_2$   
belonging to the  
same subinterval.

Let  $x$  be any point in the  
first subinterval  $[a, a_1]$  then

by  $\textcircled{1}$ ,

$$|f(x) - f(x_1)| < \epsilon$$

$$\begin{aligned} |f(x)| &= |f(x_1) - f(x) + f(x_1)| \\ &\leq |f(x_1) - f(x)| + |f(x_1)| \end{aligned}$$

$$< \epsilon + |f(x_1)|.$$

For particular  $x = a_1$

$$|f(a_1)| < \epsilon + |f(x_1)|. \quad \textcircled{2}$$

Let  $x \in [a_1, a_2]$  then by  $\textcircled{1}$

$$|f(x) - f(x_1)| < \epsilon$$

$$\therefore |f(x)| = |f(x) - f(x_1) + f(x_1)|$$

$$\leq |f(x) - f(x_1)| + |f(x_1)|$$

$$\therefore |f(x_1)| < \epsilon$$

$$= 2\epsilon + |f(x_1)|$$

$$\therefore |f(x)| < 2\epsilon + |f(x_1)|.$$

For particular  $x = a_2$ ,

$$|f(x_2)| < 2\epsilon + |f(x_1)|. \quad \textcircled{3}$$

proceeding,

similarly, we have

$$|f(x)| < n\epsilon + |f(x_1)|.$$

This inequality is  
satisfied over the whole  
interval  $[a, b]$ .

$f$  is odd on  $[a, b]$ .

Note!— The converse of the  
above theorem need not be  
true.

i.e. If  $f$  is odd on  $[a, b]$   
the  $f$  need not be continuous  
on  $[a, b]$ .

E.g.:

$$f(x) = \begin{cases} \sin \frac{1}{x}, & \text{if } x \neq 0 \\ 0, & \text{if } x = 0 \end{cases}$$

$$\forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

$$\text{SOL. } f(x) = \sin \frac{1}{x}$$

$$\Rightarrow f\left(\frac{\pi}{2}\right) = \sin\left(\frac{2}{\pi}\right) = 1$$

$$\& f\left(-\frac{\pi}{2}\right) = \sin\left(\frac{-2}{\pi}\right) = -1.$$

$$\therefore -1 \leq f(x) \leq 1 \quad \forall x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right];$$

$$\therefore f \text{ is odd on } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].$$

but is not continuous at  $x=0$ .  $\rightarrow$  If  $f$  is continuous on  $[a,b]$

Because:

$$at x=0; f(0)=0.$$

Now since  $-1 \leq \sin \frac{1}{x} \leq 1$

then

$\rightarrow$  If  $f$  is continuous on  $[a,b]$

(26)

$[a,b]$  then  $f$  attains its bounds.

(08)

$\forall x \in [0, \frac{\pi}{2}],$  If  $f$  is conti on  $[a,b]$

$\exists c \neq 0$  such that  $f$  attains its supremum  
is of the form & minimum atleast once in  $[a,b]$

$$g(x) \leq f(x) \leq h(x)$$

then

$$g(x) = -1, \text{ for all } x \in$$

$$h(x) = 1$$

$$\text{with } L + g(x) \neq L + h(x)$$

$$x \rightarrow 0$$

by squeeze theorem

$\therefore$   $\lim_{x \rightarrow 0} f(x)$  does not exist.

$\therefore f(x)$  is not conti. at  $x=0$

$\therefore f(x)$  is not conti on  $[\frac{\pi}{2}, \frac{\pi}{2}]$

proof:

Let  $f$  be conti. on  $[a,b]$

then  $f$  is odd on  $[a,b]$ .

$\therefore$  sup  $f$  & inf  $f$  on  $[a,b]$  exist.

Let  $M = \sup f$  &

(sup)

$m = \inf f$  on  $[a,b]$

(inf)

$\therefore f(x) \leq M \text{ & } f(x) \geq m$

on  $[a,b]$

now we have to show that  
 $f$  attains its sup & inf atleast  
once in  $[a,b]$ .

i.e.  $\exists x_1, x_2 \in [a,b]$  such that

$$f(x_1) = M \text{ & } f(x_2) = m.$$

$f(x_1) = M$  &  $f(x_2) = m$ .

now if possible suppose

that  $f$  doesn't attain

$M$  or  $m$  on  $[a,b]$ .

$\therefore f(x) \neq M$  on  $[a,b]$ .

$M - f(x) \neq 0$  on  $[a,b]$ .

since  $M$  is constant

it is continuous for all  $x$

and  $f$  is continuous on  $[a,b]$ .

$\therefore M - f(x)$  is continuous on  $[a,b]$

$\Rightarrow \frac{1}{M-f(x)}$  is also conti on  $[a,b]$  ( $\because M-f(x) \neq 0$ )

$\Rightarrow 1$  is odd on  $[a,b]$ .

Because:  $\therefore x > 0$

$$\Rightarrow \frac{1}{x} > 0$$

$$\Rightarrow 0 < \frac{1}{x} < \infty$$

$\forall x \in (0, \infty)$

$$\Rightarrow 0 < \frac{1}{x} < \infty \in (0, \infty)$$

$$\Rightarrow 0 < f(x) < \infty \in (0, \infty)$$

$\therefore f$  is not odd.

$\exists$  real number  $k$   
(i.e.  $k > 0$ )

$$f + \frac{1}{1-f(x)} \leq k \text{ on } [a,b]$$

$$\Rightarrow M - f(x) \geq \frac{1}{k} \text{ on } [a,b]$$

$$\Rightarrow M - \frac{1}{k} \geq f(a) \text{ and } f(b)$$

$$\Rightarrow f(x) \leq M - \frac{1}{k} \text{ on } [a,b]$$

$$< M \text{ on } [a,b].$$

$\Rightarrow M - \frac{1}{k}$  is an upper bound  
of  $f$  on  $[a,b]$

and this upper bound less

than sup of  $f$  on  $[a,b]$ .

which is contradiction to  
the hypothesis that

$M$  is sup(lub) of  $f$  on  $[a,b]$

$\therefore \exists x \in [a,b] \text{ s.t. } f(x) = M$

$\therefore f$  attains its sup  
at least once on  $[a,b]$

Step 2:  $f$  attains its inf  
at least once on  $[a,b]$

Note: The above theorem

is not true

if the interval is not  
closed

Ex:  $f(x) = x \text{ if } x \in [0,1]$

$f$  is conti. on  $(0,1]$

and is bdd on  $(0,1]$

because  $f(0) = 0$      $f(1) = 1$   
 $\inf_{x \in (0,1]} f(x) < 1$      $\sup_{x \in (0,1]} f(x) < 1$

clearly  $f$  attains

sup. but not attains  
inf on  $(0,1]$

Step 3:  $f$  on  $[0,1]$  attains

the infimum but  
not the sup

Step 4:  $f$  on  $(0,1)$  does not  
attain inf & sup.

Note (2): The converse  
of above theorem  
need not be true

Ex:  $f(x) = \begin{cases} \sin(\frac{1}{x}) & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$   
 $x \in [\frac{-\pi}{2}, \frac{\pi}{2}]$

Theorem

If  $f$  is continuous  
on  $[a,b]$  then  $f$  is bdd  
and attains its bound  
at least once on  $[a,b]$

Proof: Above two theorems  
proofs combined

→ Sign preservation theorem:

If  $f$  is continuous on  $[a,b]$  and  $a < c < b$  such that  $f(c) \neq 0$ , then  $\exists \delta > 0$  such that  $f(x)$  has the same sign as  $f(c) \forall x \in (c-\delta, c+\delta)$ .

Then → If a function  $f$  is continuous on  $[a,b]$

and  $f(a) \& f(b)$  are of opposite signs then  $\exists$  atleast one point  $c \in (a,b)$  such that  $f(c)=0$

→ Intermediate value theorem:

If  $f$  is continuous on  $[a,b]$  and  $f(a) \neq f(b)$  then  $f$  assumes every value between  $f(a) \& f(b)$  atleast once.

Uniform Continuity-

w.r.t a function  $f$  is continuous at a point  $x_0$  of an interval  $I$ , if given  $\epsilon > 0$ ,  $\exists \delta > 0$  such that  $|f(x) - f(x_0)| \leq \epsilon$  whenever  $|x - x_0| < \delta$ .

Here  $\delta$  depends, in general, not only on  $\epsilon$  but also on the point  $x_0$  at which the continuity of ' $f$ ' is considered.

$$\therefore i.e., \delta = \delta(\epsilon, x_0)$$

for example:

$$f(x) = x^2 \quad \forall x \in \mathbb{R}$$

$$\text{Let } \epsilon = 1, \& x_0 = 0$$

$$\text{then } |f(x) - f(x_0)| = |x^2 - 0|$$

$$= |x^2|$$

$$= |x|^2 < \epsilon$$

$$\text{whenever } |x| < \frac{\sqrt{\epsilon}}{1}$$

Since  $\epsilon = \frac{1}{4}$

$$\therefore |f(x) - f(x_0)| < \frac{1}{4} \text{ whenever } |x| < \frac{1}{2}$$

Taking  $\delta = \frac{1}{2}$

$$|f(x) - f(x_0)| < \epsilon \text{ whenever } |x - x_0| < \delta.$$

$\therefore \delta = \frac{1}{2}$  works at  $x_0 = 0$ . Corresponding

$$\text{to } \epsilon = \frac{1}{4}$$

Now let  $\epsilon = \frac{1}{4}$  and  $x_0 = 1$ , then  $\delta = \frac{1}{2}$  does not work.

because: Let  $x = 1.4$ , then

$$|x - x_0| = |1.4 - 1| = 0.4 < \frac{1}{2}$$

$$\begin{aligned} \text{But } |f(x) - f(x_0)| &= |1.96 - 1| \\ &= 0.96 \neq \frac{1}{4} (\neq \epsilon) \end{aligned}$$

$\therefore \epsilon > 0$ , the same value of  $\delta$  does not work for different points of the interval.

$\therefore$  If a continuous function  $f$  is such that given  $\epsilon > 0$ , we can find a uniform  $\delta > 0$  which depends only on  $\epsilon$  and not on the point  $x_0$  at which the continuity is considered, then we say that  $f$  is uniformly continuous.

Defn: A function defined on an interval  $I$  is said to be uniformly continuous on  $I$ , if given  $\epsilon > 0$ ,  $\exists$  a  $\delta > 0$  (depends on  $\epsilon$  only) such that  $|f(x_1) - f(x_2)| < \epsilon$ , whenever  $|x_1 - x_2| < \delta$

where  $x_1, x_2 \in I$ .

Note: Uniform continuity of a function is a global property we talk of.

2. Continuity on the other hand is a local property.
3. A function  $f$  is not uniformly continuous on  $\mathbb{R}$  if  $\exists$  some  $\epsilon > 0$  for which no  $\delta > 0$  works such that i.e., for any  $\delta > 0$ ,  $\exists x_1, x_2 \in \mathbb{R}$  such that  $|x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| \geq \epsilon$ .

Problems

→ every constant function is uniformly continuous on  $\mathbb{R}$ .

Sol: Let  $f(x) = c (\forall x \in \mathbb{R})$  constant function.

Gives  $\epsilon > 0$ ,

now choosing  $\delta > 0$  such that

$$|x_1 - x_2| < \delta ; x_1, x_2 \in \mathbb{R}.$$

$$\Rightarrow |f(x_1) - f(x_2)| = |c - c| = 0 < \epsilon$$

$\therefore f(x) = c (\forall x \in \mathbb{R})$  is uniformly continuous on  $\mathbb{R}$ .

→ The identity function  $f(x) = x \forall x \in \mathbb{R}$  is uniformly continuous on  $\mathbb{R}$ .

Sol: Gives  $f(x) = x \forall x \in \mathbb{R}$

Let  $\epsilon > 0$  be given.

Let  $x_1, x_2 \in \mathbb{R}$  such that  $|x_1 - x_2| \leq \delta$ .

Now we have

$$|f(x_1) - f(x_2)| = |x_1 - x_2|$$

$< \epsilon$  whenever  $|x_1 - x_2| \leq \frac{\epsilon}{1}$

Choosing  $\delta = \frac{\epsilon}{1}$

$$\therefore |f(x_1) - f(x_2)| < \epsilon.$$

$\therefore f(x) = x$  is uniformly continuous on  $\mathbb{R}$ .

$\Rightarrow$   $f(x) = x^2$  is uniformly continuous on  $[-1, 1]$

Sol: Let  $\epsilon > 0$  be given and let  $x_1, x_2 \in [-1, 1]$   
 $\Rightarrow x_1 \in [-1, 1] \text{ & } x_2 \in [-1, 1]$ .

$$\Rightarrow -1 \leq x_1 \leq 1 \text{ & } -1 \leq x_2 \leq 1$$

$$\Rightarrow |x_1| \leq 1 \text{ & } |x_2| \leq 1.$$

Now we have

$$\begin{aligned} |f(x_1) - f(x_2)| &= |x_1^2 - x_2^2| \\ &= |(x_1 - x_2)(x_1 + x_2)| \\ &= |x_1 - x_2| |x_1 + x_2| \\ &\leq (|x_1| + |x_2|) (|x_1 - x_2|) \\ &\leq (1+1) |x_1 - x_2| \end{aligned}$$

$$= 2 |x_1 - x_2|$$

$\therefore$  whenever  $|x_1 - x_2| < \frac{\epsilon}{2}$

Choosing  $\delta = \frac{\epsilon}{2}$ .

$$\therefore |f(x_1) - f(x_2)| < \epsilon \text{ whenever } |x_1 - x_2| < \delta.$$

$\therefore f$  is uniformly continuous  
on  $[-1, 1]$

$\Rightarrow$   $f(x) = \frac{x}{x+1}$  is uniformly continuous  
on  $[0, 2]$

Sol: Let  $\epsilon > 0$  be given,

let  $x_1, x_2 \in [0, 2]$

$$\Rightarrow 0 \leq x_1 \leq 2 \text{ & } 0 \leq x_2 \leq 2 \quad \textcircled{1}$$

we have

$$|f(x_1) - f(x_2)| = \left| \frac{x_1}{x_1+1} - \frac{x_2}{x_2+1} \right|$$

$$= \left| \frac{x_1 - x_2}{(x_1+1)(x_2+1)} \right|$$

$$= \frac{|x_1 - x_2|}{(x_1+1)(x_2+1)} \quad \textcircled{2}$$

$$\begin{aligned} \textcircled{1} &\equiv |x_1 + 1| \leq 3 \quad \& \quad |x_2 + 1| \leq 3 \\ \Rightarrow & |x_1 + 1| \geq 1 \quad \& \quad |x_2 + 1| \geq 1 \\ \Rightarrow & \frac{1}{|x_1 + 1|} \leq 1 \quad \& \quad \frac{1}{|x_2 + 1|} \leq 1 \end{aligned}$$

$$\textcircled{2} \equiv |f(x_1) - f(x_2)| \leq M(1)(|x_1 - x_2|)$$

$\Leftarrow$  whenever  $|x_1 - x_2| < \frac{\epsilon}{M}$

Choosing  $\delta = \frac{\epsilon}{M}$ .

$$\therefore |f(x_1) - f(x_2)| < \epsilon \text{ whenever } |x_1 - x_2| < \delta$$

$\therefore f$  is uniformly continuous on  $[0, 2]$

$$\rightarrow \text{S.T } f(x) = \frac{2x}{2x+1} \text{ is uniformly continuous on } [1, \infty)$$

Sol: Let  $\epsilon > 0$ , be given.

Let  $x_1, x_2 \in [1, \infty)$

then  $x_1 \geq 1$  &  $x_2 \geq 1$   $\text{--- } \textcircled{1}$

we have

$$\begin{aligned} |f(x_1) - f(x_2)| &= \left| \frac{2x_1}{2x_1+1} - \frac{2x_2}{2x_2+1} \right| \\ &= \frac{2|x_1 - x_2|}{(2x_1+1)(2x_2+1)} \text{ --- } \textcircled{2} \end{aligned}$$

$$\textcircled{1} \equiv 2x_1+1 \geq 1 \quad \& \quad 2x_2+1 \geq 1$$

$$\Rightarrow |2x_1+1| \geq 1 \quad \& \quad |2x_2+1| \geq 1$$

$$\Rightarrow \frac{1}{|2x_1+1|} \leq 1 \quad \& \quad \frac{1}{|2x_2+1|} \leq 1.$$

$\textcircled{2} \equiv$

we have

$$|f(x_1) - f(x_2)| \leq M(1)(2)(|x_1 - x_2|)$$

$\Leftarrow$  whenever  $|x_1 - x_2| < \frac{\epsilon}{2M}$

Choosing  $\delta = \frac{\epsilon}{2M}$ .

$$\therefore |f(x_1) - f(x_2)| < \epsilon \text{ whenever } |x_1 - x_2| < \delta.$$

$\therefore f$  is uniformly continuous on  $[1, \infty)$

Note: Every uniformly continuous is always continuous but not converse.  
i.e., every continuous function need not be uniformly continuous.

for example:  
 $f(x) = x^2$  is continuous on  $\mathbb{R}$ , but not uniformly continuous on  $\mathbb{R}$ .

because:

Let  $\epsilon > 0$  be given.

Now we shall show that for each  $\delta > 0$ ,

$\exists x_1, x_2 \in \mathbb{R}$  such that  $|x_1 - x_2| < \delta$

$$\Rightarrow |f(x_1) - f(x_2)| > \epsilon.$$

Taking  $x_2 = x_1 + \frac{\delta}{2}$

$$\therefore |x_1 - x_2| = |x_1 - x_1 - \frac{\delta}{2}|$$

$$= \frac{\delta}{2} < \delta.$$

$$\text{Now } |f(x_1) - f(x_2)| = |x_1^2 - x_2^2|$$

$$= |x_1 - x_2| |x_1 + x_2|$$

$$= \frac{\delta}{2} |x_1 + x_1 + \frac{\delta}{2}|$$

$$= \left(\frac{\delta}{2}\right) \left(2x_1 + \frac{\delta}{2}\right)$$

$$= x_1 \delta + \frac{\delta^2}{4} \quad (\because x_1 > 0)$$

$$> \epsilon.$$

Since  $\frac{\delta^2}{4} > 0$  and  $x_1 \delta < \epsilon \Rightarrow x_1 < \frac{\epsilon}{\delta}$ ;  $x_1 > 0$ .

$\epsilon + \delta$  is impossible,  $\delta$  depends on  $\epsilon$  &  $x_1$

$\therefore$  The given function is not uniformly continuous on  $\mathbb{R}$ .

$\rightarrow$  If a function  $f$  is continuous on  $[a, b]$  then it is uniformly continuous on  $[a, b]$ .

$$\text{Ex: } f(x) = x^2 + 5x + 3. \quad \forall x \in [0, 4].$$

Differentiability

Geometrical Meaning of Derivative at a point:

Consider the curve  $y=f(x)$  defined in an open interval  $(a, b)$ .

Let  $x = c \in (a, b)$ .

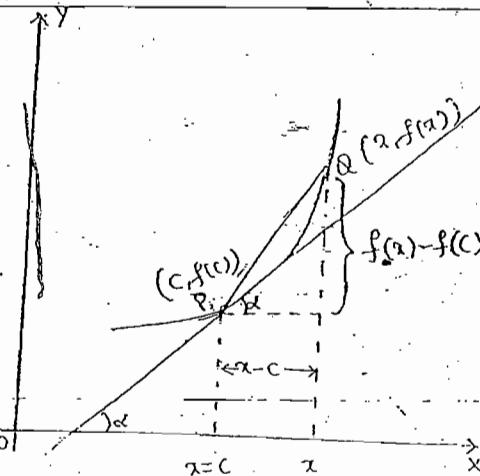
Let  $y = f(x)$  be differentiable at  $x = c$ .

Let  $P(c, f(c))$  be a point

on the curve  $y = f(x)$ .

and let  $Q(x, f(x))$  be a

neighbouring point on the curve.



$$\text{Now the slope of the chord } PQ = \frac{f(x) - f(c)}{x - c} \quad [\text{i.e. } \frac{y_2 - y_1}{x_2 - x_1}]$$

Taking limit as  $Q \rightarrow P$

i.e.  $x \rightarrow c$ , we get

$$\text{slope of the chord } PQ = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} \quad \text{①}$$

As  $Q \rightarrow P$ , chord  $PQ$  becomes tangent at  $P$ .

from ①, we have

slope of the tangent at  $P$ .

$$= \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} = \left[ \frac{d}{dx}(f(x)) \right]_{x=c} \quad \text{or} \quad f'(c)$$

i.e., the derivative of a function at a point  $x=c$  is the slope of the tangent to the curve  $y = f(x)$  at the point  $(c, f(c))$ .

- If a function is not differentiable at  $x=c$  only if the point  $(c, f(c))$  is a corner point of the curve  $y = f(x)$ . i.e., the curve suddenly changes its direction

at a point  $(c, f(c))$ .

→ Consider the function  $f(x)$  defined on  $(a, b)$ .

Let  $P(c, f(c))$  be a point on the curve  $y = f(x)$ .

Let  $Q(c-h, f(c-h))$  &  $R(c+h, f(c+h))$  be two neighbouring points on the left hand side (LHS) and RHS respectively of the point  $P$ .

Now slope of the chord  $PQ = \frac{f(c-h) - f(c)}{(c-h) - c}$

$$= \frac{f(c-h) - f(c)}{-h}$$

and slope of chord  $PR = \frac{f(c+h) - f(c)}{c+h - c}$

$$= \frac{f(c+h) - f(c)}{h}$$

Now taking limit as  $Q \rightarrow P$   
i.e.  $h \rightarrow 0$

$$\therefore \lim_{Q \rightarrow P} (\text{slope of chord } PQ) = \lim_{h \rightarrow 0} \frac{f(c-h) - f(c)}{-h} \quad \textcircled{1}$$

$$\text{similarly, } \lim_{R \rightarrow P} (\text{slope of chord } PR) = \lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h} \quad \textcircled{2}$$

As  $Q \rightarrow P$  &  $R \rightarrow P$ , the chords  $PQ$  &  $PR$  become tangent at  $P$ .

∴ from  $\textcircled{1}$  &  $\textcircled{2}$ , we have the slope of the tangent at  $P$ .

$$\lim_{h \rightarrow 0} \frac{f(c-h) - f(c)}{-h} = \lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h}$$

∴  $f(x)$  is differentiable at  $x=c$

$$\Leftrightarrow \lim_{h \rightarrow 0} \frac{f(c-h) - f(c)}{-h} = \lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h}$$

\* Derivative of a function at a point:

Let  $f: [a, b] \rightarrow \mathbb{R}$  be a function and  $c \in (a, b)$ , then  $f$  is said to be derivable (or differentiable) at  $c$ , if

$$\lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} \text{ (or) } \lim_{h \rightarrow 0} \frac{f(c+h) - f(c)}{h}$$

exists. The limit is called the derivative (or) the differential coefficient of the function  $f$  at  $x=c$  and is denoted by  $f'(c)$ .

$$\text{i.e. } f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$$

(or)

Let  $f: [a, b] \rightarrow \mathbb{R}$  and  $c \in (a, b)$ .

Then we say that a real number  $L$  is the derivative of  $f$  at  $c$  if given any  $\epsilon > 0$ ,  $\exists \delta(\epsilon) > 0$  such that if  $x \in I$  satisfies

$$0 < |x - c| < \delta$$

$$\text{then } \left| \frac{f(x) - f(c)}{x - c} - L \right| < \epsilon$$

In this case, we say that  $f$  is differentiable at  $c$  and we write  $f'(c)$  for  $L$ .

→ Left-hand Derivative:

Let  $f: [a, b] \rightarrow \mathbb{R}$  be a function and  $c \in (a, b)$  if

$$\lim_{x \rightarrow c^-} \frac{f(x) - f(c)}{x - c} \text{ (or) } \lim_{h \rightarrow 0^+} \frac{f(c-h) - f(c)}{-h}$$

exists.

then this limit is called the left-hand derivative of  $f$  at  $c$  and is denoted by  $f'(c-0)$  (or)  $Lf'(c)$ .

→ Right-hand Derivative:

Let  $f: [a, b] \rightarrow \mathbb{R}$  be a function and  $c \in (a, b)$ .

$$\text{If } \lim_{x \rightarrow c^+} \frac{f(x) - f(c)}{x - c} \text{ (or) } \lim_{h \rightarrow 0^+} \frac{f(c+h) - f(c)}{h}$$

exists, then this limit is called the right hand derivative of  $f$  at  $c$  and denoted by  $f'(c+)$  (or)  $Rf'(c)$  (or)  $f'(c+0)$ .

Note: - The derivative  $f'(c)$  exists

$$\Leftrightarrow Lf'(c) = Rf'(c).$$

→ Differentiability in an interval:

\* A function  $f: [a, b] \rightarrow \mathbb{R}$  is said to be derivable in the open interval  $(a, b)$  if  $f'(c)$  exists for each  $c \in (a, b)$ .

\* A function  $f: [a, b] \rightarrow \mathbb{R}$  is said to be derivable in  $[a, b]$  if

- (i)  $f'(c)$  exists at  $c \in (a, b)$
- (ii)  $Lf'(a)$  exists
- (iii)  $Rf'(b)$  exists.

\* A function  $f: I \rightarrow \mathbb{R}$  is said to be derivable on  $I$  if  $f$  is derivable at every point of  $I$ .

$$\underline{\text{Ex:}} \quad \underline{\text{Q}} \quad f(x) = x^2 \quad \forall x \in \mathbb{R}$$

$$\text{let } x = c \in \mathbb{R}$$

$$\text{then } f(c) = c^2$$

$$\text{Now, } f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$$

$$= \lim_{x \rightarrow c} \frac{x^2 - c^2}{x - c}$$

$$= \lim_{x \rightarrow c} (x + c)$$

$$= 2c \quad (\text{exists})$$

$\therefore f(x)$  is derivable function at  $x = c \in \mathbb{R}$

$\therefore f'(x)$  is defined on  $\mathbb{R}$  and  $f'(x) = 2x$   $\forall x \in \mathbb{R}$ .

$$\underline{\text{Ex:}} \quad \underline{\text{Q}} \quad f(x) = |x| \quad \forall x \in \mathbb{R}$$

$$\text{Sol'n: } f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$\text{at } x = 0, f(0) = 0.$$

$$\underline{\text{LHD}} \quad Lf'(0) = \lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \rightarrow 0^-} \frac{-x - 0}{x}$$

$$= \lim_{x \rightarrow 0^-} (-1) = -1$$

$$\underline{\text{RHD}} \quad Rf'(0) = \lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \rightarrow 0^+} \frac{x - 0}{x}$$

$$= \lim_{x \rightarrow 0^+} (1) = 1$$

$$\therefore Lf'(0) \neq Rf'(0)$$

Theorem :- If  $f: I \rightarrow \mathbb{R}$  has a derivative at  $c \in I$  then  $f$  is continuous at  $'c'$ .

Proof : Since  $f$  has a derivative at  $c \in I$ .

$$\therefore f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} \quad \forall c \in I \quad \underline{\text{Q}}$$

Now for  $x \in I; x \neq c$ ,

we have:

$$f(x) - f(c) = \left( \frac{f(x) - f(c)}{x - c} \right) * (x - c)$$

Now applying limit on both sides at  $x \rightarrow c$ , we get

$$\lim_{x \rightarrow c} (f(x) - f(c)) = \lim_{x \rightarrow c} \left( \frac{f(x) - f(c)}{x - c} \right) (x - c)$$

$$= f'(c) \lim_{x \rightarrow c} (x - c) \quad (\text{by Q})$$

$$= f'(c) \times 0$$

$$= 0$$

$$\therefore \lim_{x \rightarrow c} f(x) - \lim_{x \rightarrow c} f(c) = 0.$$

$$\Rightarrow \lim_{x \rightarrow c} f(x) = f(c)$$

$\therefore f(x)$  is continuous at  $x = c$ .

Note: (i) The converse of the above theorem need not be true.

$$\underline{\text{Ex:}} \quad f(x) = |x| \quad \forall x \in \mathbb{R}$$

$$= \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

$$\text{At } x = 0, f(0) = 0$$

$$\underline{\text{LHL}} \quad \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} (-x)$$

$$= 0$$

Soln: Since  $f$  is derivable at  $|x|=2$   
i.e. at  $x=2$

$\therefore f$  is continuous at  $|x|=2$

Now LHL

$$\lim_{|x|\rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} (ax + bx^2)$$

$$= \lim_{x \rightarrow 2^-} (a + b|x|^2)$$

$$= (a + 4b)$$

Now RHL

$$\lim_{|x|\rightarrow 2^+} f(x) = \lim_{x \rightarrow 2^+} \left( \frac{1}{|x|} \right)$$

$$= \frac{1}{2}$$

and at  $|x|=2$ , i.e.  $x=2$

$$f(2) = a + 4b$$

Since  $f$  is continuous at  $|x|=2$   
 $\therefore \lim_{|x|\rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^+} f(x) = f(2)$

$$\Rightarrow a + 4b = \frac{1}{2} = a + 4b$$

$$\Rightarrow a + 4b = \frac{1}{2} \quad \text{--- (1)}$$

Now LHD

$$\text{LH}(2) = \lim_{|x|\rightarrow 2^-} \frac{f(x) - f(2)}{x-2}$$

$$= \lim_{x \rightarrow 2^-} \frac{(ax + bx^2) - (a + 4b)}{x-2}$$

$$= \lim_{x \rightarrow 2^-} \frac{bx^2 - 4b}{x-2}$$

$$= \lim_{x \rightarrow 2^-} \frac{b(x^2 - 4)}{x-2}$$

$$= \lim_{x \rightarrow 2^-} b(x+2)$$

$$= b(2+2) = 4b.$$

Now RHD

$$\text{RH}(2) = \lim_{|x|\rightarrow 2^+} \frac{f(x) - f(2)}{x-2}$$

$$= \lim_{x \rightarrow 2^+} \left[ \frac{\frac{1}{|x|} - (a+4b)}{x-2} \right]$$

$$= \lim_{x \rightarrow 2^+} \left[ \frac{\frac{1}{|x|} - \frac{1}{2}}{x-2} \right] \text{ (using (1))}$$

$$= \lim_{x \rightarrow 2^+} \left[ \frac{2 - |x|}{2|x|(x-2)} \right]$$

$$\begin{aligned} &= \lim_{x \rightarrow 2^+} \left[ \frac{1x| - 2}{2|x|(x-2)} \right] \\ &= - \lim_{x \rightarrow 2^+} \left[ \frac{6x-12}{2x(x-2)} \right] \end{aligned}$$

$$= \lim_{x \rightarrow 2^+} \left[ \frac{1}{2x} \right]$$

$$= -\frac{1}{2(2)} = -\frac{1}{4}$$

Since  $f$  is derivable at  $|x|=2$ .

$$\therefore \text{LH}(2) = \text{RH}(2)$$

$$\Rightarrow 4b = -\frac{1}{4}$$

$$\Rightarrow b = -\frac{1}{16}$$

$$(1) \Rightarrow a + 4(-\frac{1}{16}) = \frac{1}{2}$$

$$\Rightarrow a - \frac{1}{4} = \frac{1}{2}$$

$$\Rightarrow a = \frac{1}{4}$$

$$\therefore a = \frac{1}{4} \text{ & } b = -\frac{1}{16}$$

Q3 For all real numbers  $a, f(x)$  is given as

$$f(x) = \begin{cases} e^x + a \sin x & ; x < 0 \\ b(x-1)^2 + x - 2 & ; x \geq 0 \end{cases}$$

Find the values of  $a$  &  $b$  for which  $f$  is differentiable at  $x=0$ .

Q4 The function  $f$  defined by

$$f(x) = \begin{cases} x^2 + 3x + a & \text{if } x \leq 1 \\ bx+2 & \text{if } x > 1 \end{cases}$$

is given to be derivable for every  $x$ . Find the values of  $a$  and  $b$  if  $x=1$ .

P.S. Since  $f$  is derivable for every  $x$

$f$  must be derivable at  $x=1$

and hence  $f$  must be continuous at  $x=1$ .

Q5 For what choice of  $a$  &  $b$ , if  $y$  will the function

$$f(x) = \begin{cases} ax+b & \text{if } x > 1 \\ bx^2 & \text{if } x \leq 1 \end{cases}$$

become differentiable at  $x=1$ ?

Q6 (i) Determine if  $f(x)$  has derivative at  $x=0$  when

$$f(x) = \begin{cases} x \sin \frac{1}{x} & ; x \neq 0 \\ 0 & ; x=0 \end{cases}$$

Examine the function

$$f(x) = \begin{cases} x^2 \cos \frac{1}{x} & ; x \neq 0 \\ 0 & ; x=0 \end{cases}$$

for the existence of derivative at  $x=0$ .

Discuss the continuity and differentiability of the following functions at  $x=a$ .

$$(i), f(x) = \begin{cases} (x-a) \cdot \sin\left(\frac{1}{x-a}\right) & ; x \neq a \\ 0 & ; x=a \end{cases}$$

$$(ii), f(x) = \begin{cases} (x-a)^2 \sin\left(\frac{1}{x-a}\right) & ; x \neq a \\ 0 & ; x=a \end{cases}$$

Soln (i): since  $x \rightarrow a^- \Rightarrow (x-a) \rightarrow 0^-$   
 $\Rightarrow \frac{1}{x-a} \rightarrow -\infty$

$$\begin{aligned} x \rightarrow a^+ &\Rightarrow (x-a) \rightarrow 0^+ \\ &\Rightarrow \frac{1}{x-a} \rightarrow +\infty \end{aligned}$$

Continuous at  $x=a$ :

$$\text{at } x=a \\ f(a) = 0$$

$$\begin{aligned} \text{LHL} & \quad \underset{x \rightarrow a^-}{\lim} f(x) = \underset{x \rightarrow a^-}{\lim} (x-a) \sin\left(\frac{1}{x-a}\right) \\ & = 0 \times l \quad (\because -1 \leq l \leq 1) \\ & = 0 \end{aligned}$$

$$\begin{aligned} \text{RHL} & \quad \underset{x \rightarrow a^+}{\lim} f(x) = \underset{x \rightarrow a^+}{\lim} (x-a) \sin\left(\frac{1}{x-a}\right) \\ & = 0 \times l \quad (\because -1 \leq l \leq 1) \\ & = 0 \end{aligned}$$

$$\therefore \text{LHL} = \text{RHL} = f(0)$$

$\therefore f$  is continuous at  $x=a$ .

Differentiable at  $x=a$ :

$$\text{LHD: } \underset{x \rightarrow a^-}{\lim} \frac{(f(x) - f(a))}{x-a}$$

$$= \underset{x \rightarrow a^-}{\lim} \frac{\left((x-a) \sin\left(\frac{1}{x-a}\right) - 0\right)}{x-a}$$

$$= \underset{x \rightarrow a^-}{\lim} \sin\left(\frac{1}{x-a}\right) = l \quad (\because -1 \leq l \leq 1)$$

# IMS

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$f$  is continuous and differentiable over each subinterval. The only doubtful points are the breaking points  $x=1$  and  $x=2$ .

At  $x=1$ :

$$f(1) = 1$$

$$\begin{aligned} \text{LHL} &= \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (3 - 2x) \\ &= 3 - 2(1) \\ &= 1 \end{aligned}$$

$$\begin{aligned} \text{RHL} &= \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} (1) = 1 \end{aligned}$$

$$\therefore \text{LHL} = \text{RHL} = f(1)$$

$\therefore f$  is continuous at  $x=1$ .

Now LHD:

$$\begin{aligned} Lf'(1) &= \lim_{x \rightarrow 1^-} \frac{f(x) - f(1)}{x - 1} \\ &= \lim_{x \rightarrow 1^-} \frac{3 - 2x - 1}{x - 1} \\ &= \lim_{x \rightarrow 1^-} \left( \frac{-2x + 2}{x - 1} \right) \\ &= -2 \lim_{x \rightarrow 1^-} \frac{(x-1)}{x-1} \\ &= -2(1) \\ &= -2 \end{aligned}$$

$$\begin{aligned} \text{RHD} : Rf'(1) &= \lim_{x \rightarrow 1^+} \frac{f(x) - f(1)}{x - 1} \\ &= \lim_{x \rightarrow 1^+} \frac{1 - 1}{x - 1} = 0 \end{aligned}$$

$$\therefore Lf'(1) \neq Rf'(1). \quad 6$$

$\therefore f$  is not differentiable at  $x=1$ .  
similarly we can easily show that  $f$  is continuous at  $x=2$  but not differentiable at  $x=2$ .

$\therefore f$  is continuous on  $[0, 3]$  also differentiable on  $[0, 3]$  except at  $x=1$  and  $x=2$ .

H.W. Discuss the continuity and differentiability of the function

$$f(x) = |x-2| + 2|x-3| \text{ in } [1, 4].$$

→ Determine where each of the following functions from  $\mathbb{R} \rightarrow \mathbb{R}$  is differentiable and find derivative.

$$(A) f(x) = |x| + |x+1|$$

$$(B) g(x) = 2x + |x|$$

$$(C) h(x) = x|x|.$$

Sol'n: (A)  $f(x) = |x| + |x+1|$  the value of  $f$  depends on

$$x < 0, x > 0;$$

$$x+1 > 0, x+1 < 0. \quad \begin{array}{c} x \\ \xleftarrow{-1} \xleftarrow{+1} \xleftarrow{+1} \\ -1 \quad 0 \end{array}$$

$$(Or) x+1 < 0, x+1 > 0 \Rightarrow x < 0, x > 0$$

$$\text{i.e. } x < -1, x > -1, x < 0, x > 0.$$

$$\text{i.e. } x < -1, -1 < x < 0; x > 0.$$

$$\text{If } x < -1; |x| = -x \text{ & } |x+1| = -(x+1)$$

$$\therefore f(x) = -2x - 1$$

$$\text{If } -1 < x < 0; |x| = -x \text{ & } |x+1| = x+1$$

$$\therefore f(x) = 1$$

If  $x > 0$ ;  $|x| = x$  &  $|2x+1| = x+1$

$$\therefore f(x) = 2x+1$$

$$f(x) = \begin{cases} -2x-1 & , x < -1 \\ 1 & , -1 < x < 0 \\ 2x+1 & , x > 0 \end{cases}$$

$$\begin{aligned} f'(x) /_{x < -1} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{-2(x+h)+1 - (-2x+1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{-2h}{h} \\ &= \lim_{h \rightarrow 0} (-2) \\ &= \underline{\underline{-2}} \end{aligned}$$

$$\begin{aligned} f'(x) /_{-1 < x < 0} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \left( \frac{1-1}{h} \right) = 0 \end{aligned}$$

$$\begin{aligned} f'(x) /_{x > 0} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{2(x+h)+1 - (2x+1)}{h} \\ &= \lim_{h \rightarrow 0} \frac{2h}{h} \\ &= \lim_{h \rightarrow 0} (2) \\ &= \underline{\underline{2}} \end{aligned}$$

$$\therefore f'(x) = \begin{cases} -2 & \text{if } x < -1 \\ 0 & \text{if } -1 < x < 0 \\ 2 & \text{if } x > 0 \end{cases}$$

$$\textcircled{O} \quad f(x) = x|x|$$

The value of  $f$  depends on  $x < 0$  and  $x > 0$ .

If  $x < 0$  then  $|x| = -x$

$$\therefore f(x) = -x^2$$

If  $x > 0$  then  $|x| = x$

$$\therefore f(x) = x^2$$

$$f(x) = \begin{cases} -x^2 & \text{if } x < 0 \\ x^2 & \text{if } x > 0 \end{cases}$$

$$\begin{aligned} f'(x) /_{x < 0} &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{-(x+h)^2 - x^2}{h} \\ &= \underline{\underline{-2x}} \end{aligned}$$

$$\text{and } f'(x) /_{x > 0} = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

$$\begin{aligned} &= \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} \\ &= \underline{\underline{2x}} \\ \therefore f'(x) &= \begin{cases} -2x & ; x < 0 \\ 2x & ; x > 0 \end{cases} \\ &= 2|x|. \end{aligned}$$

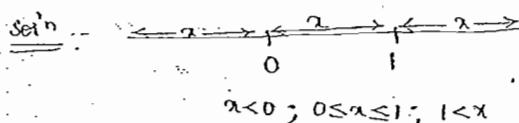
Prove that if  $f: \mathbb{R} \rightarrow \mathbb{R}$  is an even function (i.e.  $f(-x) = f(x)$   $\forall x \in \mathbb{R}$ )

and has a derivative at every point then the derivative  $f'$  is an odd function.

(i.e.  $f'(-x) = -f'(x) \forall x \in \mathbb{R}$ ). Also

Examine its continuity and derivability at  $x = \pi/2$ :

→ show that the function defined by  $f(x) = |x| + |x-1|$  is continuous but not derivable at  $x=0$  and  $x=1$ .



if  $x < 0$ :

$$\begin{aligned} |x| &= -x \\ |x-1| &= 1-x \end{aligned}$$

$$\therefore f(x) = 1-2x$$

if  $0 \leq x \leq 1$ ;  $|x| = x$ ,

$$|x-1| = 1-x$$

$$\therefore f(x) = 1$$

if  $x > 1$ :

$$|x| = x$$

$$|x-1| = x-1$$

$$\therefore f(x) = 2x-1$$

$$\therefore f(x) = 1-2x \text{ if } x < 0$$

$$1 \quad \text{if } 0 \leq x \leq 1$$

$$2x-1 \quad \text{if } x > 1$$

Continuity at  $x=0$ :

Derivability at  $x=0$ : not

Continuity at  $x=1$ :

Derivability at  $x=1$ : not

→ show that the function  $f(x)$

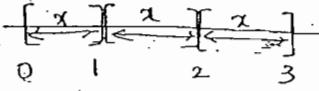
defined by  $f(x) = |x-1| + 2|x-2|$ .

is continuous but not derivable at 1 and 2.

→ Discuss the continuity and differentiability of the function.

$f(x) = |x-1| + 2|x-2|$  in the interval  $[0, 3]$

Sol'n:



$$0 \leq x \leq 1; 1 \leq x < 2; 2 \leq x < 3.$$

if  $0 \leq x \leq 1$

$$|x-1| = (1-x) &$$

$$|x-2| = 2-x$$

$$\therefore f(x) = 3-2x$$

if  $1 \leq x < 2$

$$|x-1| = x-1 &$$

$$|x-2| = 2-x$$

$$\therefore f(x) = 1$$

if  $2 \leq x < 3$

$$|x-1| = x-1 &$$

$$|x-2| = x-2$$

$$\therefore f(x) = 2x-3$$

$$\therefore f(x) = \begin{cases} 3-2x & \text{if } 0 \leq x \leq 1 \\ 1 & \text{if } 1 \leq x < 2 \\ 2x-3 & \text{if } 2 \leq x < 3 \end{cases}$$

Since  $f$  is a linear (Polynomial) function or Constant function over the various subintervals.

Here  $\ell$  is finite but not fixed because it rotates with  $-1$  to  $+1$ .

$\therefore$  LHD does not exist.

Similarly RHD does not exist.

$\therefore$   $f$  is not differentiable at  $x=0$

$$\text{Ques. } \rightarrow \text{Let } f(x) = \begin{cases} x^p \sin \frac{1}{x}; & x \neq 0 \\ 0; & x=0 \end{cases}$$

Obtain condition  $P$  so that

- (i)  $f$  is continuous at  $x=0$  and
- (ii)  $f$  is differentiable at  $x=0$ .

Sol'n: (i) At  $x=0$

$$f(0)=0$$

$$\text{LHL } \lim_{x \rightarrow 0^-} f(x) = \lim_{x \rightarrow 0^-} x^p (\sin \frac{1}{x}) \quad (1)$$

$$\text{RHL } \lim_{x \rightarrow 0^+} f(x) = \lim_{x \rightarrow 0^+} x^p (\sin \frac{1}{x}) \quad (2)$$

$f$  is continuous at  $x=0$

If the limits (1) & (2) both must be equal  
this is possible only if  $P > 0$

$\therefore$  the required condition for  $f$  at  $x=0$  is  $P > 0$ .

$$\text{iii, LHD } Lf'(0) = \lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x - 0} \\ = \lim_{x \rightarrow 0^-} \frac{x^p \sin \frac{1}{x} - 0}{x} \\ = \lim_{x \rightarrow 0^-} x^{(p-1)} \sin \frac{1}{x} \quad (3)$$

$$\text{RHD } Rf'(0) = \lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0} \\ = \lim_{x \rightarrow 0^+} \frac{x^p \sin \frac{1}{x} - 0}{x} \\ = \lim_{x \rightarrow 0^+} x^{(p-1)} \sin \frac{1}{x} \quad (4)$$

$f$  is differentiable at  $x=0$  if the limits (3) & (4) both must be zero.

This is possible only when  $(P-1) > 0$ . 5

$\therefore$  the required condition for differentiability of  $f$  at  $x=0$  is  $P > 1$ .

H.W:

$$\text{Let } f(x) = \begin{cases} x^m \sin \frac{1}{x}; & x \neq 0 \\ 0; & x=0 \end{cases}$$

What conditions should be imposed on  $m$  so that

- i)  $f$  may be continuous at  $x=0$ .
- ii)  $f$  may be differentiable at  $x=0$

H.W Show that the following function is continuous at  $x=1$ , for all values of  $p$ .

$$f(x) = \begin{cases} px+1 & \text{if } x \geq 1 \\ x^p + p & \text{if } x < 1 \end{cases}$$

Find the left-hand & right-hand derivatives of  $f(x)$  at  $x=1$ .

Hence find the condition for the existence of the derivative at that point.

$$\text{H.W. Let } f(x) = \begin{cases} \frac{e^{Vx} - e^{-Vx}}{2}; & x \neq 0 \\ 0; & x=0 \end{cases}$$

Show that  $f$  is continuous but not differentiable at  $x=0$ .

H.W A function  $f(x)$  is defined as follows.

$$f(x) = \begin{cases} 1 + \sin x & \text{for } 0 < x < \pi/2 \\ 2 + (x - \pi/2)^2 & \text{for } x \geq \pi/2 \end{cases}$$

Prove if  $g: \mathbb{R} \rightarrow \mathbb{R}$  is a differentiable odd function then  $g'$  is an even function.

Soln:- (i) Since  $f$  is even function

$$\therefore f(-x) = f(x) \quad \forall x \in \mathbb{R}$$

Let  $x=c \in \mathbb{R}$  then

$$f(-c) = f(c)$$

$$\text{Now } g'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} \quad (i)$$

$$\text{Now } f(-c) = \lim_{x \rightarrow c} \frac{f(-x) - f(-c)}{-x - (-c)}$$

$$= \lim_{x \rightarrow c} \frac{f(x) - f(c)}{-(x - c)}$$

( $\because f$  is even)

$$= - \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$$

$$= -f'(c) \quad (\text{from (i)})$$

$$\therefore f'(-c) = f'(c)$$

$\therefore f'$  is an odd function.

(ii) Since  $g$  is odd function

$$\therefore g(-x) = -g(x) \quad \forall x \in \mathbb{R}$$

Let  $x=c \in \mathbb{R}$  then  $g(-c) = -g(c)$

$$\text{Now } g'(c) = \lim_{x \rightarrow c} \frac{g(x) - g(c)}{x - c} \quad (i)$$

$$\text{Now } g'(-c) = \lim_{x \rightarrow c} \frac{g(-x) - g(-c)}{-x - (-c)}$$

$$= \lim_{x \rightarrow c} \frac{-g(x) + g(c)}{-(x - c)}$$

$$= \lim_{x \rightarrow c} \frac{g(x) - g(c)}{x - c}$$

$$= g'(c)$$

$$\therefore g'(-c) = g'(c)$$

$$\therefore g'(-x) = g'(x)$$

$\therefore g'$  is an even function.

P.R.  
Ques Let  $f(x)$  ( $x \in (-\pi, \pi)$ ) be defined

by  $f(x) = \sin|x|$ . Is  $f$  continuous  
on  $(-\pi, \pi)$ ? If it is continuous,  
then is it differentiable on  $(-\pi, \pi)$ ?

$$\text{Soln:- } f(x) = \sin|x| = \begin{cases} \sin x & x \geq 0 \\ \sin(-x) & x < 0 \end{cases} \quad \forall x \in (-\pi, \pi)$$

$$= \begin{cases} \sin x & x \geq 0 \\ -\sin x & x < 0 \end{cases} \quad \forall x \in (-\pi, \pi)$$

Let  $g: \mathbb{R} \rightarrow \mathbb{R}$  be defined by

$$g(x) = \begin{cases} x^2 \sin(1/x^2) & \text{for } x \neq 0 \\ 0 & \text{for } x=0 \end{cases}$$

Show that  $g$  is differentiable for all  $x \in \mathbb{R}$ .

Also show that the derivative  $g'$  is not bounded on the interval  $[-1, 1]$ .

$$\text{Soln:- } g(x) = x^2 \sin\left(\frac{1}{x^2}\right) + x^2 \left(\frac{-2}{x^3}\right) \cos\left(\frac{1}{x^2}\right)$$

$$= 2x \sin\left(\frac{1}{x^2}\right) - \frac{2}{x} \cos\left(\frac{1}{x^2}\right) \quad (i)$$

$\therefore g'(x)$  is well defined for  $x \neq 0$ .

Now at  $x=0$ :

$$g(0) = 0$$

$$\therefore g'(0) = \lim_{x \rightarrow 0} \frac{g(x) - g(0)}{x - 0}$$

$$= \lim_{x \rightarrow 0} \frac{x^r \sin\left(\frac{1}{x^2}\right) - 0}{x}$$

$$= \lim_{x \rightarrow 0} x^r \sin\left(\frac{1}{x^2}\right)$$

Now we have

$$-1 \leq \sin\left(\frac{1}{x^2}\right) \leq 1 \quad \forall x \in \mathbb{R}; x \neq 0.$$

$$\Rightarrow -x^r \leq x^r \sin\left(\frac{1}{x^2}\right) \leq x^r \quad \forall x > 0, \text{ is}$$

of the form

$$f(x) \leq g(x) \leq h(x)$$

$$\text{Here } f(x) = -x^r; \quad t(x) = x^r \sin\left(\frac{1}{x^2}\right)$$

$$h(x) = x^r$$

$$\therefore \lim_{x \rightarrow 0} f(x) = 0, \quad \lim_{x \rightarrow 0} h(x) = 0$$

By squeeze theorem

$$\lim_{x \rightarrow 0} x^r \sin\left(\frac{1}{x^2}\right) = 0.$$

$$\therefore g'(0) = 0$$

$\therefore g$  is differentiable at  $x=0$ .

(ii)  $\exists g'(x)$  is not bounded.

on  $[-1, 1]$  as  $x \in [-1, 1]$

If  $\theta > 0$  is a rational number

let  $f: \mathbb{R} \rightarrow \mathbb{R}$  be defined by

$$f(x) = \begin{cases} x^\theta \sin\left(\frac{1}{x^2}\right) & ; x \neq 0 \\ 0 & ; x=0 \end{cases}$$

Determine these values of  $r$  for which  $f'(0)$  exists.

Soln: At  $x=0; f(0)=0$

$$\text{Now } f'(0) = \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \rightarrow 0} \frac{x^\theta \sin\left(\frac{1}{x^2}\right)}{x}$$

$$= \lim_{x \rightarrow 0} x^{\theta-1} \sin\left(\frac{1}{x^2}\right)$$

$$= \lim_{x \rightarrow 0} x^{\theta-1} \lim_{x \rightarrow 0} \sin\left(\frac{1}{x^2}\right)$$

$$= 0 \times \lim_{x \rightarrow 0} \sin\left(\frac{1}{x^2}\right) \quad (\because \theta > 0) \\ \Rightarrow (\theta-1) > 0$$

$$= 0$$

$\therefore f'(0)$  exists for  $\theta > 1$ .

### Extreme Value (Definition):

→ A real number  $x'$  is called an interior point of a set  $A$  if  $A$  is neighbourhood of  $x'$ .

i.e.  $\exists \epsilon > 0$  such that  $(x-\epsilon, x+\epsilon) \subset A$ .

Ex:- (1) Every point of  $(a, b)$  is its interior point.

(2). Every point  $[a, b]$  is its interior point except  $a$  &  $b$ .

→ The function  $f: I \rightarrow \mathbb{R}$  is said to have a relative maximum (or) maximum value (or) maxima at  $c \in I$  if  $f(c)$  is the greatest value of the

function  $f$  in a small neighbourhood  $V = V_\delta(c)$  of  $c$ .

i.e. for all  $x \in (c-\delta, c+\delta); \delta > 0$  such that  $f(x) \leq f(c) \forall x \in V \cap I$ .

→ The function  $f: I \rightarrow \mathbb{R}$  is said to have a relative minimum (or) minimum value (or) minima at  $c \in I$  if  $f(c)$  is the least value of the function in a small neighbourhood  $V = V_\delta(c)$  [i.e.  $(c-\delta, c+\delta)$ ] of  $c$ .

i.e. for all  $x \in (c-\delta, c+\delta); \delta > 0$  such that  $f(x) \geq f(c) \forall x \in V \cap I$ .

→ The function  $f: I \rightarrow \mathbb{R}$  is said to

have relative extremum (or) extreme value at  $c \in I$ , iff  $f$  has either relative maximum (or) relative minimum at  $c$ .

### Interior Extremum Theorem:

Let  $c$  be an interior point of the interval  $I$  at which  $f: I \rightarrow \mathbb{R}$  has a relative extremum at  $c$ . If the derivative of  $f$  at  $c$  exists then  $f'(c) = 0$ .

Proof:- Since  $f$  has a relative extremum at  $c$ .

Suppose that  $f$  has a relative maximum at  $c$ .

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$$f(x) \leq f(c) \quad \forall x \in V_\delta(c).$$

If possible let  $f'(c) \neq 0$ .  
then  $f'(c) > 0$  or  $f'(c) < 0$ .

Case(i): If  $f'(c) > 0$

$$\text{i.e. } \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} > 0.$$

$$\therefore \frac{f(x) - f(c)}{x - c} > 0 \quad \forall x \in V_\delta(c); x \neq c$$

$$\text{Now if } x \in V_\delta(c) \text{ and } x > c \\ \text{then } f(x) - f(c) = \frac{(f(x) - f(c))}{x - c}(x - c) > 0$$

$$\Rightarrow f(x) - f(c) > 0$$

$$\Rightarrow f(x) > f(c) \quad \text{--- (2)}$$

But (1) & (2) are contradiction.

$$\therefore f'(c) \neq 0 \quad \text{--- (1)}$$

Case ii): If  $f'(c) < 0$  then

$$\text{Lt } \frac{f(x) - f(c)}{x-c} < 0$$

$$\therefore \frac{f(x) - f(c)}{x-c} < 0 \forall x \in V_s(c); \\ x \neq c$$

If  $x \in V_s(c)$  and  $x < c$  then

$$f(x) - f(c) = \left[ \frac{f(x) - f(c)}{(x-c)} \right] \times (x-c) \\ > 0$$

$$\therefore f(x) - f(c) > 0$$

$$\Rightarrow f(x) > f(c) \quad \text{--- (3)}$$

But (1) & (3) are contradiction.

$$\therefore f'(c) \neq 0. \quad \text{--- (4)}$$

from (1) & (4)

$$f'(c) = 0$$

Note! (1) If  $f$  has relative extremum at 'c' then  $f'(c)$  may not exist.  
if it exists then  $f'(c) = 0$ .

$$\text{Ex!- } f(x) = |x| \forall x \in [-1, 1]$$

Sol'n Let  $x=c=0 \in [-1, 1]$

$$\text{for } x = V_s(0) \rightarrow 0 \rightarrow$$

$$\Rightarrow x \in (-\delta, \delta)$$

$$(i) x \in (-\delta, 0)$$

$$\Rightarrow f(x) > f(0) = 0$$

$\therefore f(x)$  has minimum at  $x=0$   $\forall x \in (-\delta, \delta)$

$$(ii) x \in (0, \delta)$$

$$\Rightarrow f(x) > f(0) = 0$$

$\therefore f(x)$  has minimum at  $x=0$

$\therefore f$  has relative extremum at  $x=0$

$$f'(0) = \text{Lt } \frac{f(x) - f(0)}{x-0}$$

$$= \text{Lt } \frac{|x|-0}{x-0}$$

$$= \frac{|x|}{x}$$

$$\text{Now } \text{Lt } \frac{|x|}{x} = \text{Lt } \frac{-x}{x}$$

$$= \text{Lt } (-1) = -1$$

$$\text{Now } \text{Lt } \frac{|x|}{x} = \text{Lt } \frac{x}{x} = \text{Lt } (1) = 1$$

$\therefore f'(0)$  does not exist.

Note Q.: (a) The converse of above theorem need not be true.

If  $f'(c) = 0$  then

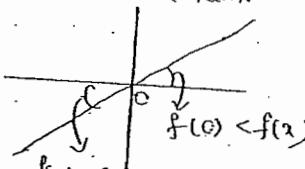
$f(c)$  may not be an extreme value.

$$\text{Ex!- } f(x) = x^3 \forall x \in \mathbb{R}$$

$$\therefore f'(x) = 3x^2$$

$$\text{At } x=0; f'(0)=0.$$

But  $f$  is strictly increasing in  $\mathbb{R}$  and has no local extremum.



Definition:-

The point 'c' is said to be stationary point and  $f(c)$  the stationary value of the function  $f$  if  $f'(c) = 0$ .

\*Rolle's Theorem :- [Only Problems]

Suppose that  $f$  is continuous on  $I = [a, b]$  that the derivative  $f'$

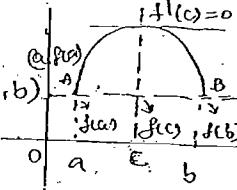
exists at every point of  $(a, b)$   
and  $f(a) = f(b) = 0$ . Then there  
exists at least one point  $c \in (a, b)$   
such that  $f'(c) = 0$ .

Proof: Case(i)

If  $f(x) = 0$  on  $I = [a, b]$  then

$f'(x) = 0 \forall x \in [a, b]$

$\therefore f'(c) = 0 \forall c \in (a, b)$



Case(ii)

If  $f(x) \neq 0 \forall x \in [a, b]$  then  $f(x) > 0$   
or  $f(x) < 0$ .

Suppose that  $f(x) > 0 \forall x \in [a, b]$

i.e.  $f$  assumes the five values  
in  $I = [a, b]$

Since  $f$  is continuous on  $I = [a, b]$

$\therefore f$  attains its supremum (lub)  
at least once in  $[a, b]$ .

i.e. let  $f$  attains its supremum  
at some point  $c \in [a, b]$ .

$$\therefore f(c) = \sup\{f(x) / x \in I = [a, b]\} > 0$$

at  $x = c \in I$

$$\Rightarrow x \in (c-s, c+s)$$

Since  $f$  takes some five values.

$$\therefore f(x) \leq f(c)$$

$$\forall x \in I \cap (c-s, c+s)$$

$\therefore f$  has relative maximum at  $c$ .

$$f(c) > 0.$$

$$\text{since } f(a) = f(b) = 0$$

$$\text{and then } c \neq a, c \neq b$$

$$\Rightarrow c \in (a, b)$$

Since  $f'$  exists at every point of  
 $(a, b)$ .

$$\therefore f'(c) \text{ exists.}$$

$\therefore f$  has relative maximum at  $c$   
and  $f$  has derivative at  $c$ .

$\therefore$  By interior extremum theorem

$$\therefore f'(c) = 0$$

for at least one point  $c \in (a, b)$ .  
Hence the theorem.

### \* Failure of Rolle's theorem :-

Rolle's theorem fails to hold good for a function which does not satisfy all three conditions of the theorem.

The theorem is not applicable if

their (i)  $f$  is not continuous in  $[a, b]$

(ii) (iii)  $f$  is not derivable in  $(a, b)$

(iv)  $f(a) \neq f(b)$ .

Note: The converse of Rolle's theorem is not true i.e.  $f'(c) = 0$  at  $c \in (a, b)$  without  $f(x)$  satisfying all the three conditions of Rolle's theorem.

Ex:-

$$f(x) = \begin{cases} 0 & \text{when } 0 \leq x \leq 1 \\ x+1 & \text{when } 1 < x \leq 2 \end{cases} \quad \forall x \in [0, 2].$$

Clearly  $f$  is not continuous and not

derivable at  $x=1$ .

$\therefore f$  is not continuous in  $[0, 2]$  and  
 $f$  is not derivable in  $(0, 2)$ .

Also  $f(0) \neq f(2)$ .

But  $f(x)=0 \forall x \in (0, 1) \subset (0, 2)$ .

i.e.  $f'(x)=0$  for at least one point  $x \in (0, 2)$ .

Note(2): Another form of Rolle's theorem.

If  $f$  is continuous on  $[a, a+h]$

derivable on  $(a, a+h)$  and

$f(a)=f(a+h)=0$ , then ~~that~~ at least

one real number  $\theta \in (0, 1)$  such that  $f'(a+\theta h)=0$ .

Here,  $b=a+h$ ;  $h>0$  and  $c=a+\theta h$

since  $c \in (a, b)$

$$\Rightarrow a < c < b$$

$$\Rightarrow a < a+\theta h < a+h$$

$$\Rightarrow 0 < \theta h < h$$

$$\Rightarrow 0 < \theta < 1 \quad (\because h>0)$$

$$\Rightarrow \theta \in (0, 1).$$

Problems:

Verify Rolle's theorem in the following cases:

i)  $f(x)=(x-a)^m(x-b)^n$

where  $m$  &  $n$  are the integers

in the interval  $[a, b]$ .

Sol'n :- we have

$$f(x)=(x-a)^m(x-b)^n$$

i) Since  $m$  &  $n$  are +ve integers.

$\therefore f(x)$  is polynomial in  $x$ .

(On expansion by binomial theorem).

Since every polynomial function is continuous function of  $x$  for all values of  $x$ .

$\therefore f(x)$  is continuous function for all values of  $x$ .

$\therefore$  It is continuous on  $[a, b]$ .

$$(ii) f'(x)=m(x-a)^{m-1}(x-b)^n+n(x-a)^m$$

$$(x-b)^{n-1}$$

$$=(x-a)^{m-1}(x-b)^{n-1}[m(x-b)+n(x-a)]$$

$$=(x-a)^{m-1}(x-b)^{n-1}[(m+n)x-(mb+na)]$$

exists in  $(a, b)$

$\therefore f(x)$  is derivable in  $(a, b)$ .

(iii)  $f(a)=f(b)=0$ .

$\therefore f(x)$  satisfies all the three conditions of Rolle's theorem.

$\therefore \exists$  at least one value  $x=c \in (a, b)$  such that  $f'(c)=0$

$$f'(c)=(c-a)^{m-1}(c-b)^{n-1}[c(m+n)-(mb+na)]$$

$$=0$$

$$\Rightarrow c(m+n)-(mb+na)=0$$

$$(\because c \neq a, c \neq b)$$

$$\Rightarrow c(m+n)=mb+na$$

$$\Rightarrow c = \frac{mb+na}{m+n} \in (a, b)$$

$\therefore$  Rolle's theorem is verified.

$$\text{Hence } f(x) = (x-a)^3(x-b)^4 \forall x \in [a,b]$$

$$\rightarrow f'(x) = 2(x-1)^{2/3} \forall x \in [0,2]$$

$$\text{Soln: Since } f'(x) = \frac{2}{3}(x-1)^{-1/3}$$

$$= \frac{2}{3(x-1)^{1/3}}$$

which does not exist in  $x=1 \in (0,2)$

$\therefore f'(x)$  does not exist in  $(0,2)$

$\therefore f$  is not derivable in  $(0,2)$ .

$\therefore$  Rolle's theorem is not applicable to  $f(x)$  in  $[0,2]$ .

$$\rightarrow f(x) = e^x \sin x \forall x \in [0,\pi]$$

Soln: (i) since  $e^x$  &  $\sin x$  are both continuous functions for values of  $x$ .

$\therefore e^x \sin x$  is also continuous for all values of  $x$ .

$\therefore f(x)$  is continuous in  $[0,\pi]$ .

$$(ii), f'(x) = e^x \cos x + e^x \sin x$$

which exists in  $(0,\pi)$ .

$\therefore f(x)$  is derivable in  $(0,\pi)$ .

$$(iii), f(0) = e^0 \sin(0)$$

$$= 0$$

$$- f(\pi) = e^\pi \sin(\pi)$$

$$= 0$$

$$\therefore f(0) = f(\pi) = 0$$

$\therefore$  the conditions of Rolle's theorem are satisfied.

$\therefore$  at least one value  $c \in (0,\pi)$  such that  $f'(c) = 0$

$$f'(c) = e^c (\cos c + \sin c) = 0$$

$$\Rightarrow \cos c + \sin c = 0 \quad (\because e^c \neq 0)$$

$$\Rightarrow \cos c = -\sin c$$

$$\Rightarrow 1 = -\tan c$$

$$\Rightarrow \tan c = -1$$

$$\Rightarrow \tan c = -\tan(\pi/4)$$

$$\Rightarrow c = \pi - \pi/4$$

$$\Rightarrow c = 3\pi/4 \in (0,\pi)$$

$\therefore$  Rolle's theorem is verified.

$$\text{Hence } f(x) = x(x+3)e^{-x/2} \forall x \in [-3,0]$$

$$\rightarrow f(x) = |x| \forall x \in [-1,1]$$

Soln: (i) since  $f(x) = |x|$  is continuous for all values of  $x$ .

(ii) since  $f(x)$  is not derivable at  $x=0 \in (-1,1)$

(iii)  $\therefore f$  is not derivable in  $(-1,1)$

$\therefore$  The Rolle's is not applicable to

$$f(x) = |x| \text{ in } [-1,1]$$

$$\rightarrow f(x) = \log \left[ \frac{x^2+ab}{x(a+b)} \right] \forall x \in [a,b]$$

$$0 \notin [a,b]$$

$$\text{Soln: (i), } f(x) = \log(x^2+ab) - \log(x(a+b))$$

$$= \log(x^2+ab) - \log x - \log(a+b)$$

$\therefore$  It is continuous in  $[a,b]$   $0 \notin [a,b]$

$$(ii), f'(x) = \frac{2x}{x^2+ab} - \frac{1}{x}$$

$$= \frac{x^2-ab}{x(x^2+ab)} \text{ exists in } (a,b)$$

$\therefore f(x)$  is derivable in  $(a, b)$ .

$$(ii) f(a) = \log \left[ \frac{a^x + ab}{a(a+b)} \right]$$

$$= \log \left( \frac{a^x + ab}{a^2 + ab} \right)$$

$$= \log(1) = 0$$

$$f(b) = \log \left[ \frac{b^x + ab}{b(a+b)} \right]$$

$$= \log(1) = 0$$

$$\therefore f(a) = f(b) = 0$$

The conditions of Rolle's theorem are satisfied.

$\exists$  atleast one point  $c \in (a, b)$  such that  $f'(c) = 0$ .

$$f'(c) = \frac{c^2 - ab}{c(c^2 + ab)} = 0$$

$$\Rightarrow c^2 - ab = 0$$

$$\Rightarrow c^2 = ab$$

$$\Rightarrow c = \pm \sqrt{ab}$$

$$\Rightarrow c = \pm \sqrt{ab} \in (a, b)$$

$\therefore$  Rolle's is verified. (neglecting  $\pm \sqrt{ab}$ )

H.W.

$$\rightarrow f(x) = \log \left( \frac{x^2 + 3}{4x} \right) \forall x \in [1, 3]$$

$$\rightarrow f(x) = x^2 - 6x + 8 \forall x \in [2, 4]$$

$$\rightarrow f(x) = 8x - x^2 \forall x \in [2, 6]$$

$$\rightarrow f(x) = \begin{cases} x^2 + 1 & \text{for } 0 \leq x \leq 1 \\ 3-x & \text{for } 1 \leq x \leq 2 \end{cases}$$

Sol': Here  $f(x)$  is defined in  $[0, 2]$

Since  $f(x) = x^2 + 1$  for  $0 \leq x \leq 1$

$\therefore$  it is polynomial.

$\therefore$  It is continuous & derivable.

Since  $f(x) = 3-x$  for  $1 \leq x \leq 2$  is a polynomial.

$\therefore$  It is continuous & derivable in  $[1, 2]$ .

Since the domain of function  $f(x)$  is  $[0, 2]$  which is partitioned at  $x=1$  we are not sure about the continuity and derivability of

$f(x)$  at  $x=1$

$$\text{Now LHL } \lim_{x \rightarrow 1^-} f(x) = \lim_{x \rightarrow 1^-} (x^2 + 1) = 2$$

$$\text{RHL } \lim_{x \rightarrow 1^+} f(x) = 2$$

$$\text{at } x=1 : f(1) = 2$$

$$f(1) = 2$$

$\therefore f$  is continuous at  $x=1$ .

$$\begin{aligned} \text{LHD} \quad f'(1) &= \lim_{x \rightarrow 1^-} \frac{f(x) - f(1)}{x - 1} \\ &= \lim_{x \rightarrow 1^-} \frac{x^2 + 1 - 2}{x - 1} \\ &= \lim_{x \rightarrow 1^-} \frac{(x-1)(x+1)}{x-1} \\ &= \lim_{x \rightarrow 1^-} (x+1) \\ &= 2 \end{aligned}$$

$$= \lim_{x \rightarrow 1^-} f(x).$$

$$= 2.$$

$$\begin{aligned} \text{RHD } 2f'(1) &= \lim_{x \rightarrow 1^+} \frac{f(x) - f(1)}{x - 1} \\ &= \lim_{x \rightarrow 1^+} \frac{3-x-2}{x-1} \\ &= \lim_{x \rightarrow 1^+} \frac{1-x}{x-1} \\ &= -1. \end{aligned}$$

$$\therefore \text{LHD} \neq \text{RHD}$$

$\therefore f$  is not derivable at  $x=1$

$\therefore f$  is not derivable in  $(0, 2)$

$\therefore$  Rolle's theorem is not applicable to  $f(x)$  in  $[0, 2]$ .

$$\begin{aligned} \text{Imp.} \quad \text{Let } & \frac{a_0}{n+1} + \frac{a_1}{n} + \frac{a_2}{n-1} + \dots + \\ & + \frac{a_{n-1}}{2} + a_n = 0 \end{aligned}$$

Show that the function  $a_0x^n + a_1x^{n-1} + \dots + a_n$  vanishes at least once in  $(0, 1)$ .

$$\begin{aligned} \text{Sol'n:} \quad \text{Let } f(x) = & a_0 \frac{x^{n+1}}{n+1} + a_1 \frac{x^n}{n} \\ & + a_2 \frac{x^{n-1}}{n-1} + \dots + \frac{a_{n-1}}{2} x^2 + a_n x \end{aligned}$$

$$\forall x \in [0, 1].$$

Since  $f(x)$  is a polynomial

which is continuous & derivable for all  $x$ .

$\therefore f$  is continuous in  $[0, 1]$  & derivable

in  $(0, 1)$ .

$$\text{Also } f(0) = 0.$$

$$\text{and } f(1) = \frac{a_0}{n+1} + \frac{a_1}{n} + \dots + \frac{a_{n-1}}{2} + a_n = 0 \text{ (given)}$$

$\therefore$  The conditions of Rolle's theorem are satisfied.

$\therefore \exists$  atleast one point  $x \in (0, 1)$  such that  $f'(x) = 0$ .

$$\Rightarrow f'(x) = a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n = 0.$$

H.W.: By Considering the function  $(x-4) \log x$ , show that the equation  $x \log x = 4-x$  is satisfied by at least one value of  $x \in (1, 4)$ .

$$\text{Sol'n: Let } f(x) = (x-4) \log x$$

$\rightarrow$  Show that between any two roots of  $e^x \cos x = 1$ ,  $\exists$  atleast one root of  $e^x \sin x - 1 = 0$ . i.e.  $\sin x - e^{-x} = 0$

Sol'n: Let  $\alpha = a$  &  $\beta = b$  be two distinct roots of the given equation  $e^x \cos x = 1$ .

$$\begin{aligned} \therefore e^a \cos a &= 1 \quad \& e^b \cos b = 1 \\ \Rightarrow \cos a &= e^{-a} \quad \& \cos b = e^{-b} \quad \text{--- (1)} \end{aligned}$$

$$\text{Let } f(x) = -\cos x + e^{-x} \forall x \in [a, b]$$

(i) Since  $\cos x$  &  $e^{-x}$  are continuous in  $[a, b]$ .

i)  $f(x)$  is continuous in  $[a, b]$

ii)  $f'(x) = \sin x - e^{-x}$

which exists for all  $x \in (a, b)$ .

$\therefore f$  is derivable in  $(a, b)$ .

iii)  $f(a) = -\cos a + e^{-a}$

$= 0 \quad (\text{by } \textcircled{1})$

&  $f(b) = -\cos b + e^{-b}$

$= 0 \quad (\text{by } \textcircled{1})$

$\therefore f(a) = f(b) = 0$

∴ The conditions of Rolle's

theorem are satisfied.

∴ ∃ at least one point  $c \in (a, b)$

such that  $f'(c) = 0$ .

$$\Rightarrow f'(c) = \sin c - e^{-c} = 0$$

$$\Rightarrow \sin c = e^{-c}$$

$$\Rightarrow e^c \sin c - 1 = 0$$

$\Rightarrow x = c \in (a, b)$  is a root of  
the equation  $e^x \sin x - 1 = 0$

$\therefore e^x \sin x - 1$  has at least one root b/w  
any two roots of the equation.

$$e^x \cos x = 1$$

Q.E.D. Prove that b/w any two  
roots of  $e^x \sin x = 1$ , ∃ at least  
one real root of

$$e^x \cos x + 1 = 0$$

## \* Lagrange's Mean Value

### Theorem: —

(First Mean Value theorem of Differential Calculus) —

Statement: Suppose that  $f$  is continuous on  $I = [a, b]$  and  $f$  has a derivative in  $(a, b)$ . Then there exists at least one point  $c \in (a, b)$  such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

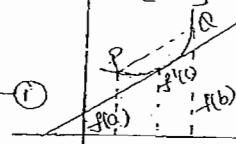
$$\text{i.e. } f(b) - f(a) = f'(c)(b-a)$$

Proof: Consider the function

$$\phi(x) = f(x) - f(a) - k(x-a) \quad \forall x \in [a, b]$$

where

$$k = \frac{f(b) - f(a)}{b-a} \quad \text{①}$$



Since  $f(x)$  is continuous on  $I = [a, b]$

since  $(x-a)$  is polynomial it continuous on  $I$  and  $f(a)$  &  $k$  are constants.

$\therefore \phi(x)$  is continuous on  $[a, b]$ .

Now  $\phi'(x) = f'(x) - k$  exists in  $(a, b)$  ②

$\because f'(x)$  exists in  $(a, b)$

Now  $\phi(a) = 0$

and  $\phi(b) = f(b) - f(a) - k(b-a)$

$$= (f(b) - f(a)) - \left( \frac{f(b) - f(a)}{b-a} \right) (b-a)$$

$$= 0$$

$$\therefore \phi(a) = \phi(b) = 0$$

$\therefore \phi(x)$  satisfies the conditions of Rolle's theorem.

$\therefore \exists$  at least one  $c \in (a, b)$  such that

$$\phi'(c) = 0.$$

$$\phi'(c) = f'(c) - k = 0$$

$$\Rightarrow f'(c) = k$$

$$\Rightarrow f'(c) = \frac{f(b) - f(a)}{b-a}.$$

Another statement: —

If a function  $f$  defined on  $[a, b]$  is

i) Continuous on  $[a, a+h]$

ii), derivable on  $(a, a+h)$  then  $\exists$  atleast one real number  $\theta \in (0, 1)$

such that  $f(a+h) = f(a) + h f'(a+\theta h)$

Here  $b = a+h$

$$\& c = a+h$$

## \* Deductions from Lagrange's

### Mean Value theorem: —

$\rightarrow$  If a function  $f$  is continuous on closed interval  $I = [a, b]$  and derivable on  $(a, b)$  and  $f'(x) = 0 \quad \forall x \in (a, b)$  then  $f$  is constant on  $I = [a, b]$

Sol'n: Let  $x_1, x_2$  (with  $x_1 < x_2$ ) be any two distinct points of  $[a, b]$  so that  $[x_1, x_2] \subset [a, b]$

then  $f$  satisfies both conditions of Lagrange's mean value theorem on  $[x_1, x_2]$ .

$\therefore \exists c \in (x_1, x_2)$  such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c) \quad (1)$$

But  $f'(x) = 0 \quad \forall x \in (a, b)$  and

$$x_1 < c < x_2$$

$$\therefore f'(c) = 0$$

$$\text{from (1), } \frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0 \quad \therefore$$

$$\Rightarrow f(x_2) - f(x_1) = 0$$

$$\Rightarrow f(x_1) = f(x_2)$$

Since  $x_1$  &  $x_2$  are any two distinct points of  $[a, b]$ .

it follows that  $f$  keeps the same value for every  $x \in [a, b]$ .

$f(x)$  is constant on  $[a, b]$ .

$\rightarrow$  If two functions  $f$  &  $g$  are continuous on  $[a, b]$ , differentiable on  $(a, b)$  and  $f'(x) = g'(x) \forall x \in (a, b)$  then  $f-g$  is a constant on  $[a, b]$

Qn: Let us consider  $\phi(x) = f(x) - g(x) \quad \forall x \in [a, b]$

Since  $f$  &  $g$  continuous on  $[a, b]$  and differentiable on  $(a, b)$ .

$\therefore \phi$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$ .

$\therefore \phi'(x) = f'(x) - g'(x)$  exists on  $(a, b)$ .

Since  $f'(x) = g'(x) \quad \forall x \in [a, b]$

$\therefore \phi'(x) = 0 \quad \forall x \in [a, b]$ .

Since  $\phi$  is continuous on  $[a, b]$ , differentiable on  $(a, b)$  and

$\phi'(x) = 0 \quad \forall x \in (a, b)$

$\therefore \phi$  is a constant function on  $[a, b]$ .

i.e.  $f-g$  is constant on  $[a, b]$ .

### \* Increasing and Decreasing Functions:

If in a part of the domain of the function  $f(x)$ ,

$\rightarrow x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$  then  $f(x)$  is called monotonically increasing function in that part.

$\rightarrow x_1 < x_2 \Rightarrow f(x_1) < f(x_2)$  then  $f(x)$  is called strictly monotonically increasing function in that part.

$\rightarrow x_1 < x_2 \Rightarrow f(x_1) \geq f(x_2)$

Monotonically decreasing.

$\rightarrow x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$

Strictly Monotonically decreasing.

Theorem:

Let  $f: I \rightarrow \mathbb{R}$  be differentiable on  $I$  then

(a)  $f$  is increasing on  $I$

$$\text{iff } f'(x) \geq 0 \quad \forall x \in I.$$

(b)  $f$  is decreasing on  $I$

$$\text{iff } f'(x) \leq 0 \quad \forall x \in I.$$

Proof - (a) Suppose that  $f'(x) \geq 0$   $\forall x \in I$ .

Let  $x_1, x_2 \in I$  with  $x_1 < x_2$ ,

$$\text{so that } [x_1, x_2] \subset I.$$

Since  $f$  is differentiable on  $I$

$\therefore$  If is differentiable on  $[x_1, x_2]$

and therefore it is continuous on  $[x_1, x_2]$ .

$\therefore$   $f$  satisfies both the conditions of Lagrange's mean value theorem on  $[x_1, x_2]$ .

$\therefore \exists c \in (x_1, x_2)$  such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c).$$

$$\Rightarrow f(x_2) - f(x_1) = (x_2 - x_1)f'(c) \quad \text{--- (1)}$$

Since  $x_1 < x_2 \Rightarrow x_2 - x_1 > 0$ ,

$$f'(x) \geq 0 \quad \forall x \in I \text{ and } x_1 < c < x_2$$

$$\Rightarrow f'(c) \geq 0.$$

$\therefore$  from (1),  $f(x_2) - f(x_1) \geq 0$

$$\Rightarrow f(x_1) \leq f(x_2).$$

$$\text{Since } x_1 < x_2 \Rightarrow f(x_1) \leq f(x_2)$$

$\therefore f$  is an increasing on  $I$ .

Conversely - Suppose that  $f$  is differentiable on  $I$  and  $f$  is an increasing on  $I$ .

Now for  $x \neq c \in I$  then  $x > c$  or  $x < c$

case i) if  $x > c$  (i.e.  $x - c > 0$ )  
then  $f(x) \geq f(c)$ . ( $\because f$  is increasing on  $I$ ).

$$\Rightarrow f(x) - f(c) \geq 0$$

$$\Rightarrow \frac{f(x) - f(c)}{x - c} \geq 0 \quad (\because x - c > 0)$$

$$\Rightarrow \lim_{x \rightarrow c^+} \frac{f(x) - f(c)}{x - c} \geq 0 \quad \text{--- (2)}$$

case ii) if  $x < c$  (i.e.  $x - c < 0$ )  
then  $f(x) \leq f(c)$  ( $\because f$  is increasing on  $I$ )

$$\Rightarrow f(x) - f(c) \leq 0$$

$$\Rightarrow \frac{f(x) - f(c)}{x - c} \geq 0 \quad (\because x - c < 0)$$

$$\Rightarrow \lim_{x \rightarrow c^-} \frac{f(x) - f(c)}{x - c} \geq 0 \quad \text{--- (3)}$$

Since  $f$  is differentiable on  $I$ .

Let  $f$  be differentiable at  $c \in I$ .

$$f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$$

$\therefore$  from (2) & (3)

we have

$$f'(c) \geq 0.$$

(b) The proof of part (b) is similar.

Problems:

\* Verify Lagrange's mean value theorem for the following functions in the specified intervals:

$$\rightarrow f(x) = x(x-1)(x-2) \forall x \in [0, \frac{1}{2}]$$

Sol'n:  $f(x) = x^3 - 3x^2 + 2x$  is a polynomial in  $x$ .

which is continuous in  $[0, \frac{1}{2}]$ .

$$f'(x) = 3x^2 - 6x + 2 \text{ exists in } (0, \frac{1}{2})$$

$\therefore f$  is differentiable in  $(0, \frac{1}{2})$ .

$\therefore f$  satisfies the conditions of Lagrange's Mean value theorem.

$\therefore \exists c \in (0, \frac{1}{2})$  such that

$$f'(c) = \frac{f(\frac{1}{2}) - f(0)}{\frac{1}{2} - 0}$$

$$\Rightarrow 3c^2 - 6c + 2 = \frac{3/8 - 0}{\frac{1}{2}}$$

$$\Rightarrow 3c^2 - 6c + 2 = \frac{3}{4}$$

$$\Rightarrow 12c^2 - 24c + 8 = 3$$

$$\Rightarrow 12c^2 - 24c + 5 = 0$$

$$\Rightarrow c = \frac{24 \pm \sqrt{576 - 240}}{24}$$

$$\Rightarrow c = \frac{24 \pm \sqrt{336}}{24}$$

$$\Rightarrow c = \frac{24 \pm 4\sqrt{21}}{24}$$

$$\Rightarrow c = \frac{6 \pm \sqrt{21}}{6}$$

Now the two values of  $c$  are

$$+ \frac{1}{6}\sqrt{21}, - \frac{1}{6}\sqrt{21}$$

In these, two values of  $c$  the second value  $- \frac{1}{6}\sqrt{21} \in (0, \frac{1}{2})$ .

$\therefore \exists$  at least one value of  $c$  in

$$c = - \frac{1}{6}\sqrt{21} \in (0, \frac{1}{2}) \text{ such that}$$

$$\frac{f(\frac{1}{2}) - f(0)}{\frac{1}{2} - 0} = f'(c)$$

$\therefore$  The Lagrange Mean value theorem is verified.

H.W.

$$\rightarrow f(x) = x^2 - 3x + 2 \quad \forall x \in [-2, 3]$$

$$\rightarrow f(x) = x^3 + x^2 - 6x \quad \forall x \in [-1, 4]$$

$$\rightarrow f(x) = e^x \text{ on } [0, 1]$$

$$\rightarrow f(x) = \log x \quad \forall x \in [1, e] \text{ where } e = 2.71828$$

Sol'n: Since  $f(x) = \log x$  is continuous for all +ve values of  $x$ .

$\therefore$  It is continuous on  $[1, e]$  and

$$f'(x) = \frac{1}{x} \text{ exists in } (1, e)$$

$\therefore f$  is derivable in  $(1, e)$

$\therefore f$  satisfies the conditions of

Lagrange's Mean value theorem.

$\therefore \exists$  at least one  $c \in (1, e)$  such that

$$- f'(c) = \frac{f(e) - f(1)}{e - 1}$$

$$\Rightarrow \frac{1}{c} = \frac{\log e - \log 1}{e - 1}$$

$$\Rightarrow \frac{1}{c} = \frac{1 - 0}{e - 1}$$

$$\Rightarrow c-1 = c$$

$$\Rightarrow c = e-1 \in (1, e)$$

$\therefore$  The Lagrange's Mean Value theorem is satisfied.

$$\text{Hence } f(x) = \sqrt{x^2-4} \quad \forall x \in [2, 4]$$

$$\Rightarrow f(x) = \begin{cases} 2 & \text{if } x=1 \\ x^2 & \text{if } 1 < x < 2 \\ 1 & \text{if } x=2 \end{cases}$$

Sol'n: Since  $f(x) = x^2$  is a polynomial function in  $1 < x < 2$  and every polynomial function is continuous for all values of  $x$ .

$\therefore$  It is continuous on  $(1, 2)$

Now at  $x=1$ :

$$f(1) = 2$$

$$\text{Now } \lim_{x \rightarrow 1^+} f(x) = \lim_{x \rightarrow 1^+} x^2 = 1$$

$$\therefore \lim_{x \rightarrow 1^+} f(x) \neq f(1).$$

At  $x=2$ :

$$f(2) = 4$$

$$\text{Now } \lim_{x \rightarrow 2^-} f(x) = \lim_{x \rightarrow 2^-} x^2 = 4$$

$$\therefore \lim_{x \rightarrow 2^-} f(x) \neq f(2)$$

$\therefore f(x)$  is not continuous at  $x=1$  &  $2$

$\therefore f(x)$  is continuous in  $(1, 2)$

but not in  $[1, 2]$

$\therefore f(x)$  does not satisfy the conditions of Lagrange's Mean Value theorem.

$\therefore$  Lagrange's Mean Value theorem is not applicable to  $f(x)$ .

$$\Rightarrow f(x) = |x| \quad \forall x \in [-1, 2]$$

Sol'n: It is continuous on  $[-1, 2]$  and it is differentiable at each point in  $(-1, 2)$  except at  $x=0$ .

$\therefore f(x)$  is not differentiable in  $(-1, 2)$

$\therefore f(x)$  does not satisfy the

conditions of Lagrange's Mean Value

Theorem.

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Lagrange's Mean Value theorem is not applicable to  $f(x)$ .

$\therefore$  If  $f(x) = (x-1)(x-2)(x-3)$ ;  $a=0, b=4$  find  $c$  of Lagrange mean value theorem.

$$\text{Sol'n: } f(x) = (x-1)(x-2)(x-3)$$

$$= x^3 - 6x^2 + 11x - 6$$

$$f(a) = f(0)$$

$$= -6$$

$$f(b) = f(4)$$

$$= (3)(2)(1) = 6$$

$$f'(x) = 3x^2 - 12x + 11$$

$$f'(c) = 3c^2 - 12c + 11$$

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$$3c^2 - 12c + 11 = \frac{6 - (-6)}{4 - 0}$$

$$\Rightarrow 3c^2 - 12c + 11 = 12/4$$

$$\Rightarrow 3c^2 - 12c + 11 = 3$$

$$\Rightarrow 3c^2 - 12c + 8 = 0$$

$$\Rightarrow c = \frac{12 \pm \sqrt{144 - 96}}{2 \times 3}$$

$$\Rightarrow c = \frac{12 \pm \sqrt{48}}{6}$$

$$\Rightarrow c = \frac{12 \pm 4\sqrt{3}}{6}$$

$$\Rightarrow c = 2 \pm \frac{2}{\sqrt{3}} \in (0, 4).$$

$$\rightarrow f(x) = \frac{1}{x} + x \in [-1, 1]$$

Sol'n:  $f(0)$  is not finite while  $x \in [-1, 1]$ .

LHL

$$\text{Lt } f(x) = -\infty \text{ &} \\ x \rightarrow 0^-$$

RHL

$$\text{Lt } f(x) = \infty \\ x \rightarrow 0^+$$

$\therefore f(x)$  is not continuous at  $x=0$ .

$\therefore f(x)$  is not continuous on  $[-1, 1]$ .

Lagrange's Mean Value theorem is not applicable to  $f(x)$ .

$$(q) f(x) = x^{2/3} \text{ in } [-1, 1].$$

$$\text{of'n}: f'(x) = \frac{1}{3}x^{-1/3} = \frac{1}{3x^{2/3}}$$

does not exist at  $x=0 \in (-1, 1)$

Lagrange's Mean Value theorem is

not applicable to  $f(x)$ .

$$\text{However, } \frac{f(1) - f(-1)}{1 - (-1)} = f'(c)$$

$$\Rightarrow \frac{1 - (-1)}{-2} = \frac{1}{3c^{2/3}}$$

$$\Rightarrow 3c^{2/3} = 1$$

$$\Rightarrow c^{2/3} = \frac{1}{3}$$

$$\Rightarrow c^{1/3} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow c = \frac{1}{3\sqrt{3}} \in (-1, 1)$$

$\therefore$  the hypothesis of Lagrange's Mean Value theorem is not valid.

i.e., the two conditions of Lagrange's Mean Value theorem are sufficient but not necessary.

$\rightarrow$  show that if  $x > 0$ ,  $\log(1+x) > \frac{x}{1+x}$

and hence prove that  $x^{-1} \log(\frac{1}{1+x})$  decreases monotonically as  $x$  increases from 0 to  $\infty$ .

$$\text{Sol'n: Let } f(x) = \log(1+x) - \frac{x}{1+x}$$

$$f'(x) = \frac{1}{1+x} - \left[ \frac{(1+x) \cdot 1 - x}{(1+x)^2} \right]$$

$$= \frac{1}{1+x} - \frac{1}{1+x} + \frac{x}{(1+x)^2}$$

$$= \frac{x}{(1+x)^2} > 0 \quad (\because x > 0)$$

$\therefore f'(x) > 0$  when  $x > 0$

i.e.  $f(x)$  is an increasing when

$$\therefore f(x) > f(0), \quad x > 0.$$

$$\text{Now } f(0) = \log(1+0) - \frac{0}{1+0} \\ = \log 1 - 0 \\ = 0$$

$$\therefore f'(x) > 0$$

$$\Rightarrow \log(1+x) - \frac{x}{1+x} > 0$$

$$\Rightarrow \log(1+x) > \frac{x}{1+x}$$

$$\text{Let } F(x) = x^{-1} \log(1+x)$$

$$= \frac{\log(1+x)}{x}$$

$$F'(x) = \frac{x \cdot \frac{1}{1+x} - \log(1+x) \cdot 1}{x^2}$$

$$= \frac{-\left[\log(1+x) - \frac{x}{1+x}\right]}{x^2}$$

$$= \frac{-f(x)}{x^2} < 0 \text{ for } x > 0$$

( $\because f(x) > 0$ )

$$\therefore F'(x) < 0 \text{ for } x > 0$$

$\therefore F(x)$  is an decreasing function

in  $(0, \infty)$ .

2004 P-I  
Show that

$$x - \frac{x^2}{2} < \log(1+x) < x - \frac{x^2}{2(1+x)} ; x > 0$$

$$\text{sol'n: Let } f(x) = x - \frac{x^2}{2} - \log(1+x)$$

$$\begin{aligned} f'(x) &= 1 - x - \frac{1}{1+x} \\ &= \frac{1-x^2-1}{1+x} \\ &= \frac{-x^2}{1+x} < 0 \text{ for } x > 0. \end{aligned}$$

$$\therefore f'(x) < 0 \text{ for } x > 0.$$

$\therefore f(x)$  is a decreasing function

for  $x > 0$ .

$$\therefore f(0) > f(x).$$

$$\begin{aligned} \text{Now } f(0) &= 0 - 0 - \log 1 \\ &= 0 \end{aligned}$$

$$\therefore f(x) < 0$$

$$\Rightarrow x - \frac{x^2}{2} - \log(1+x) < 0$$

$$\Rightarrow x - \frac{x^2}{2} < \log(1+x) \quad \text{--- (1)}$$

$$\text{Now let } g(x) = \log(1+x) - x + \frac{x^2}{2(1+x)},$$

$$\Rightarrow g'(x) = \frac{1}{1+x} - 1 + \frac{1}{2} \cdot \frac{(1+x)2x - 2x^2(1)}{(1+x)^2}$$

$$= \frac{1}{1+x} - 1 + \frac{1}{2} \cdot \frac{2x+x^2}{(1+x)^2}$$

$$= \frac{1}{1+x} - 1 + \frac{1}{2} \cdot \frac{2x+x^2}{(1+x)^2}$$

$$= \frac{2(1+x) - 2(1+x)^2 + 2x+x^2}{2(1+x)^2}$$

$$= \frac{-x^2}{2(1+x)^2} < 0 \text{ for } x > 0.$$

$$\therefore g'(x) < 0 \text{ for } x > 0.$$

$\therefore g(x)$  is a decreasing function.

for  $x > 0$ .

$$\therefore g(0) > g(x).$$

But  $g(0) = 0$

$\therefore g(x) < 0$

$$\Rightarrow \log(1+x) - x + \frac{x^2}{2(1+x)} < 0$$

$$\Rightarrow \log(1+x) < x - \frac{x^2}{2(1+x)} \quad \text{--- (2)}$$

Combining (1) & (2),

$$\frac{x-x^2}{2} < \log(1+x) < x - \frac{x^2}{2(1+x)}$$

2002 P.T. show that

$$\frac{b-a}{\sqrt{1-a^2}} \leq \sin^{-1} b - \sin^{-1} a \leq \frac{b-a}{\sqrt{1-b^2}}$$

for  $0 < a < b < 1$ .

Sol'n: Let  $f(x) = \sin^{-1} x \forall x \in [a, b]$   
where  $a > 0, b < 1$   
i.e.  $0 < a < b < 1$ .

$f(x)$  is continuous & derivable in  $[a, b]$  and  $f'(x) = \frac{1}{\sqrt{1-x^2}} \forall x \in (a, b)$

: By Lagrange's Mean Value

Theorem,  $\exists c \in (a, b)$  such that

$$f'(c) = \frac{f(b) - f(a)}{b-a}$$

$$\Rightarrow \frac{1}{\sqrt{1-c^2}} = \frac{\sin^{-1} b - \sin^{-1} a}{b-a} \quad \text{--- (1)}$$

Since  $c \in (a, b)$

$$\Rightarrow a < c < b$$

$$\Rightarrow a^2 < c^2 < b^2$$

$$\Rightarrow -a^2 > -c^2 > -b^2$$

$$\Rightarrow 1-a^2 > 1-c^2 > 1-b^2$$

$$\Rightarrow \sqrt{1-a^2} > \sqrt{1-c^2} > \sqrt{1-b^2}$$

$$\Rightarrow \frac{1}{\sqrt{1-a^2}} < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-b^2}}$$

$$\Rightarrow \frac{1}{\sqrt{1-a^2}} < \frac{\sin^{-1} b - \sin^{-1} a}{b-a} < \frac{1}{\sqrt{1-b^2}}$$

$$\Rightarrow \frac{b-a}{\sqrt{1-a^2}} < \sin^{-1} b - \sin^{-1} a < \frac{b-a}{\sqrt{1-b^2}} \quad \text{--- (1)}$$

$$\Rightarrow \text{Prove that } \frac{\pi}{6} + \frac{\sqrt{3}}{15} < \sin^{-1} 0.6 < \frac{\pi}{6} + \frac{1}{8}$$

Sol'n: putting  $b = 3/5, a = 1/2$  then  
from (1),

$$\frac{\frac{3}{5} - \frac{1}{2}}{\sqrt{1-\frac{1}{4}}} < \sin^{-1} \frac{3}{5} - \sin^{-1} \frac{1}{2} < \frac{\frac{3}{5} - \frac{1}{2}}{\sqrt{1-\frac{9}{25}}}$$

$$\Rightarrow \frac{1}{10} \times \frac{2}{\sqrt{3}} < \sin^{-1}(0.6) - \frac{\pi}{6} < \frac{1}{10} \times \frac{2}{\sqrt{3}}$$

$$\Rightarrow \frac{1}{5\sqrt{3}} < \sin^{-1}(0.6) - \frac{\pi}{6} < \frac{1}{8}$$

$$\Rightarrow \frac{\sqrt{3}}{15} + \frac{\pi}{6} < \sin^{-1}(0.6) < \frac{\pi}{6} + \frac{1}{8}$$

2008 (6) If  $x > 0$ , show that

$$\frac{x}{1+x} < \log(1+x) < x$$

Sol'n: Let  $f(t) = \log(1+t) \forall t \in [0, x]$

where  $x > 0$

$f(t)$  is continuous & differentiable  
in  $[0, x]$ .

and  $f'(t) = \frac{1}{1+t} \forall t \in (0, x)$

By Lagrange's Mean Value theorem

$\exists c \in (0, x)$  such that

$$\begin{aligned}f'(c) &= \frac{f(x) - f(0)}{x-0} \\ \Rightarrow \frac{1}{1+c} &= \frac{\log(1+x) - \log 1}{x} \\ \Rightarrow \frac{1}{1+c} &= \frac{\log(1+x) - 0}{x} \\ \Rightarrow \frac{1}{1+c} &= \frac{\log(1+x)}{x} \quad \text{①}\end{aligned}$$

Since  $c \in (0, x)$

$$\begin{aligned}\Rightarrow 0 < c < x \\ \Rightarrow 1 < 1+c < 1+x \\ \Rightarrow 1 > \frac{1}{1+c} > \frac{1}{1+x} \\ \Rightarrow 1 > \frac{\log(1+x)}{x} > \frac{1}{1+x} \quad \text{(by ①)} \\ \Rightarrow x > \log(1+x) > \frac{x}{1+x} \quad (\because x > 0) \\ \Rightarrow \frac{x}{1+x} < \log(1+x) < x.\end{aligned}$$

P.T. Use the Mean value theorem to prove that  $\frac{2}{7} < \log(1.4) < \frac{2}{5}$ .

Sol'n: Let  $f(t) = \log(1+t)$   $\forall t \in [0, x]$  where  $x > 0$ .

$f(t)$  is continuous & differentiable on  $[0, x]$ .

$$\text{and } f'(t) = \frac{1}{1+t} \quad \forall t \in (0, x)$$

By Lagrange's Mean Value theorem,  $\exists c \in (0, x)$  such that

$$f'(c) = \frac{f(x) - f(0)}{x-0}$$

$$\begin{aligned}\Rightarrow \frac{1}{1+c} &= \frac{\log(1+x) - \log 1}{x} \\ \Rightarrow \frac{1}{1+c} &= \frac{\log(1+x)}{x} \quad \text{①}\end{aligned}$$

since  $c \in (0, x)$

$$\begin{aligned}\Rightarrow 0 < c < x \\ \Rightarrow 1 < 1+c < 1+x \\ \Rightarrow 1 > \frac{1}{1+c} > \frac{1}{1+x} \\ \Rightarrow 1 > \frac{\log(1+x)}{x} > \frac{1}{1+x} \quad \text{(by ①)} \\ \Rightarrow \frac{1}{1+x} < \frac{\log(1+x)}{x} < 1 \\ \Rightarrow \frac{x}{1+x} < \log(1+x) < x \quad (\because x > 0) \\ \text{Putting } x = \frac{2}{7}, \text{ we get:} \\ \frac{2}{1+\frac{2}{7}} < \log\left(1+\frac{2}{7}\right) < \frac{2}{7} \\ \Rightarrow \frac{2}{5} < \log\left(\frac{9}{7}\right) < \frac{2}{7} \\ \Rightarrow \frac{2}{7} < \log(1.4) < \frac{2}{5}.\end{aligned}$$

H.W. show that

$$\frac{x^2}{2} > x - \log(1+x) > \frac{x^2}{2(1+x)} \quad \text{for } x > 0$$

H.W. Prove that

$$\begin{aligned}\frac{x^2}{2} + \frac{x^3}{3(1+x)} &< \log(1+x) < \\ x - \frac{x^2}{2} + \frac{x^3}{3} &\quad \text{for } x > 0.\end{aligned}$$

→ Apply Lagrange's Mean Value theorem to the function  $\log(1+x)$  to show that

$$0 < [\log(1+x)]' - x' < 1 \quad \forall x > 0.$$

Sol'n: Let  $f(t) = \log(1+t) \forall t \in [0, x]$  where  $x > 0$ .

(which is continuous & differentiable on  $[0, x]$ ).

$$\text{and } f'(t) = \frac{1}{1+t} \quad \forall t \in (0, x).$$

By Lagrange's mean value theorem

$$\exists c \in (0, x) \text{ such that } f'(c) = \frac{f(x) - f(0)}{x - 0}$$

$$\Rightarrow \frac{1}{1+c} = \frac{\log(1+x) - \log(1)}{x - 0}$$

$$\Rightarrow \frac{1}{1+c} = \frac{\log(1+x)}{x} \quad \text{--- (1)}$$

Since  $c \in (0, x) : x > 0$

$$\Rightarrow 0 < c < x$$

$$\Rightarrow 1 < 1+c < 1+x.$$

$$\Rightarrow 1 > \frac{1}{1+c} > \frac{1}{1+x}$$

$$\Rightarrow \frac{1}{1+x} < \frac{\log(1+x)}{x} < 1 \quad (\text{by (1)})$$

$$\Rightarrow (1+x) > \frac{x}{\log(1+x)} > 1$$

$$\Rightarrow \frac{1}{x} + 1 > \frac{1}{\log(1+x)} > \frac{1}{x}$$

$$\Rightarrow 1 > \frac{1}{\log(1+x)} - \frac{1}{x} > 0$$

$$\Rightarrow 0 < [\log(1+x)]^{-1} - \frac{1}{x} < 0$$

for  $x > 0$ .

$\therefore$  Use Lagrange's mean value

theorem to prove that  $1+x < e^x < 1+xe^x$

Let  $f(t) = e^t \quad \forall t \in [0, x]$  where  $x > 0$

$\therefore$  show that

$$\frac{e^x - e^0}{1+x^2} < \tan^{-1} x - \tan^{-1} 0 < \frac{x - 0}{1+x^2} \text{ if } 0 < x < \pi/2$$

deduce that

$$\frac{\pi}{4} + \frac{3}{8} < \tan^{-1} \frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$$

$$\text{Soln: Let } f(x) = \tan^{-1} x \quad \forall x \in [0, \pi/2]$$

$$\text{where } 0 < x < \pi/2$$

$\rightarrow$  use the Mean Value theorem

to prove that  $|\sin x - \sin y| \leq |x-y| \forall x, y \in \mathbb{R}$

Soln: If  $x=y$  then there is nothing to prove.

If  $x > y$  then consider the function

$$f(t) = \sin t \quad \forall t \in [y, x]$$

Clearly if  $f$  is continuous on  $[y, x]$

and  $f'(t) = \cos t$  exists on  $[y, x]$

$\therefore$  By mean value theorem

$\exists c \in (y, x)$  such that

$$f'(c) = \frac{f(x) - f(y)}{x - y}$$

$$\Rightarrow \cos c = \frac{\sin x - \sin y}{x - y}$$

$$\Rightarrow \left| \frac{\sin x - \sin y}{x - y} \right| = |\cos c|$$

$$\Rightarrow \frac{|\sin x - \sin y|}{|x - y|} = |\cos c| \leq 1$$

$$\Rightarrow |\sin x - \sin y| \leq |x - y|$$

$\therefore \forall x, y \in \mathbb{R}$

$$|\sin x - \sin y| \leq |x - y|$$

H.W. use the Mean value theorem to prove that  $\frac{x^2}{x} < \ln x < (x-1)$  for  $x > 1$

Let  $f(t) = \ln t \quad \forall t \in [1, x]$  where  $x > 1$

$$\Rightarrow f(t) = \log t$$

BOOK P-II

→ Using Lagrange's mean value theorem, show that  $|f(b) - f(a)| \leq |b-a|$

→ If a function  $f$  is such that its derivative  $f'$  is continuous on  $[a,b]$

and derivable on  $(a,b)$ , then show that

$$f(b) = f(a) + (b-a)f'(a) + \frac{1}{2}(b-a)^2 f''(c).$$

Sol: Let  $\phi(x) = f(x) + (b-x)f'(x) + (b-x)^2 K, \forall x \in [a,b]$

$$\text{where } K = \frac{f(b) - f(a) - (b-a)f'(a)}{(b-a)^2}$$

Since  $f'$  is continuous on  $[a,b]$

⇒  $f'$  exists on  $[a,b]$

⇒  $f$  is derivable on  $[a,b]$

⇒  $f$  is continuous on  $[a,b]$

∴ the functions  $f$  and  $f'$  are continuous function on  $[a,b]$  and derivable on  $(a,b)$ .

$(b-x), (b-x)^2$  and  $K$  are continuous on  $[a,b]$  and derivable on  $(a,b)$ .

∴  $\phi(x)$  is continuous on  $[a,b]$  and derivable on  $(a,b)$ .

Now

$$\phi(a) = f(a) + (b-a)f'(a) + (b-a)^2 K.$$

$$\Rightarrow \phi(a) = f(a) + (b-a)f'(a) + (b-a)^2 \left[ \frac{f(b) - f(a) - (b-a)f'(a)}{(b-a)^2} \right]$$

$$= f(b)$$

$$\text{and } \phi(b) = f(b)$$

$$\therefore \phi(a) = \phi(b)$$

(or) Let the function  $\phi$  on  $[a,b]$  defined by

$$\phi(x) = f(x) + (b-x)f'(x) + (b-x)^2 K$$

where  $K$  is a constant to be determined such that  $\phi(a) = \phi(b)$

$$\begin{aligned} & f(a) + (b-a)f'(a) + (b-a)^2 K = f(b) \\ & K = \frac{f(b) - f(a) - (b-a)f'(a)}{(b-a)^2} \end{aligned}$$

$\therefore \phi$  satisfies the conditions of Rolle's theorem.

$\exists c \in (a, b)$  such that  $\phi'(c) = 0$  ————— (1)

but

$$\begin{aligned}\phi'(x) &= f'(x) + (-1)f'(x) + (b-x)f''(x) \\ &\quad + 2(b-x)(-1)K\end{aligned}$$

$$\rightarrow \phi'(c) = (b-c)f''(c) +$$

$$2(b-c)(-1)K$$

$$\Rightarrow 0 = (b-c) [f''(c) - 2K] \quad (\text{by (1)})$$

$$\Rightarrow f''(c) - 2K = 0 \quad (\because b-c \neq 0) \\ \text{i.e. } c \in (a, b) \\ \Rightarrow a < c < b$$

$$\Rightarrow f''(c) = 2K$$

$$\Rightarrow K = \frac{1}{2}f''(c)$$

$$\Rightarrow \frac{f(b) - f(a) - (b-a)f'(a)}{(b-a)^2} = \frac{1}{2}f''(c)$$

$$\Rightarrow f(b) - f(a) - (b-a)f'(a) = - \frac{1}{2}(b-a)^2 f''(c)$$

$$\Rightarrow f(b) = f(a) + (b-a)^2 f'(a) \\ + \frac{1}{2}(b-a)^2 f''(c)$$

If a function  $f$  is twice differentiable on  $[a, a+h]$  then show that  $f(a+h) = f(a) + hf'(a) + \frac{h^2 f''(a+\theta h)}{2!}$

for some real number  $\theta$  where  $\theta \in (0, 1)$ .

Sol'n:- Since  $f$  is twice differentiable on  $[a, a+h]$ .

$\Rightarrow f', f''$  exist on  $[a, a+h]$

$\Rightarrow f, f'$  are differentiable on  $[a, a+h]$

$\Rightarrow f, f'$  are continuous on  $[a, a+h]$

Let  $\phi(x) =$

$$f(x) + (a+h-x)f'(x) + \frac{(a+h-x)^2}{2!} K$$

where  $K$  is a constant to be determined such that  $\phi(a) = \phi(a+h)$

$$f(a) + hf'(a) + \frac{h^2}{2!} K = f(a+h)$$

$$\Rightarrow K = f(a+h) - f(a) - \frac{hf'(a)}{\left(\frac{h^2}{2!}\right)} \quad (1)$$

Since  $f$  &  $f'$  are continuous on  $[a, a+h]$ ,  $[a+h-x]$  and  $\frac{(a+h-x)^2}{2!} K$  are

continuous functions on  $[a, a+h]$

$\Rightarrow \phi$  is continuous on  $[a, a+h]$

Since  $f$  &  $f'$  are derivable on  $(a, a+h)$  and  $(a+h-x)$ ,  $\frac{(a+h-x)^2}{2!} K$  are derivable on  $(a, a+h)$ .

$\Rightarrow \phi$  is derivable on  $(a, a+h)$ .

Also  $\phi(a) = \phi(b)$

$\phi$  satisfies the conditions of Rolle's theorem.

$\exists$  a real number  $\theta \in (0, 1)$  such that

$$\phi'(a+\theta h) = 0 \quad (2)$$

$$\begin{aligned}\text{But } \phi'(x) &= f'(x) - f'(x) + (a+h-x)\frac{f''(x)}{2!} \\ &\quad - (a+h-x)K \\ &= (a+h-x)[f''(x) - K]\end{aligned}$$

$$\Rightarrow f'(a+\theta h) = (a+h-a-\theta h) \cdot [f''(a+\theta h)-k]$$

$$\Rightarrow 0 = (h-\theta h) [f''(a+\theta h)-k] \quad (\text{by } @)$$

$$\Rightarrow f''(a+\theta h)-k=0 \quad (\because h-\theta h \neq 0)$$

$$\Rightarrow f''(a+\theta h)=k$$

$$\Rightarrow f''(a+\theta h) = \frac{f(a+h)-f(a)-hf'(a)}{\left(\frac{h^2}{2!}\right)}$$

$$\Rightarrow \frac{h^2}{2!} \cdot f''(a+\theta h) = f(a+h)-f(a)-hf'(a)$$

$$\Rightarrow f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a+\theta h)$$

P-II [2004], P-I  
2005 [2001], DM

A twice differentiable function  $f$  on  $[a, b]$  is such that  $f(a)=f(b)=0$

and  $-f'(c)>0$  for  $a < c < b$ .

Prove that there is at least

one value  $\xi$ ,  $a < \xi < b$  for which

$$f''(\xi) < 0.$$

Soln:  $f$  is twice differentiable on  $[a, b]$

$\Rightarrow f', f''$  exist on  $[a, b]$

$\Rightarrow f, f'$  are differentiable on  $[a, b]$

$f, f'$  are continuous functions on  $[a, b]$ .

Since  $a < c < b$ , applying

Lagrange's Mean Value theorem to on the intervals  $[a, c]$  and  $[c, b]$

we get

$$\frac{f(c)-f(a)}{c-a} = f'(z_1)$$

where  $a < z_1 < c$  and

$$\frac{f(b)-f(c)}{b-c} = f'(z_2) \text{ where } c < z_2 <$$

$$\text{But } f(a) = f(b) = 0$$

$$\therefore f'(z_1) = \frac{f(c)}{c-a} \text{ and}$$

$$f'(z_2) = \frac{f(c)}{b-c} \text{ where}$$

$$a < z_1 < c < z_2 < b.$$

Again  $f'$  is continuous and derivable on  $[z_1, z_2]$

By Lagrange's Mean Value theorem we have

$$\frac{f'(z_2) - f'(z_1)}{z_2 - z_1} = f''(\xi)$$

$$\text{where } z_1 < \xi < z_2$$

Substituting the values of  $f'(z_1)$  and  $f'(z_2)$ , we get

$$f''(\xi) = \frac{-f(c) - f(a)}{b-c - c-a} = \frac{f(c)}{z_2 - \xi}$$

$$= \frac{-f(c)}{z_2 - z_1} \left[ \frac{1}{b-c} - \frac{1}{c-a} \right]$$

$$= \frac{-f(c)}{z_2 - z_1} \left[ \frac{b-a}{b-c} \right]$$

Since  $a < z_1 < c < \xi < b$  and  $-f(c) > 0$

$\therefore f''(\xi) < 0$  where  $a < \xi < b$ .

### Cauchy's Mean Value

Theorem (Second Mean Value theorem)

Statement: Let  $f$  and  $g$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$  and assume that  $g'(x) \neq 0$ .

$\forall x \in (a, b)$  then  $\exists c \in (a, b)$

such that  $\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$

Proof: Let  $\phi(x) = f(x) - f(a) -$

$$\text{where } K = \frac{f(b) - f(a)}{g(b) - g(a)} \\ \therefore \phi(x) = f(x) - f(a) - K[g(x) - g(a)]$$

If possible, let  $g(a) = \lim_{x \rightarrow a} g(x) = \phi(b)$ .

Since  $g(x)$  is continuous on  $[a, b]$  and differentiable on  $(a, b)$

$\therefore g$  satisfies the conditions of Rolle's theorem.

$\exists c \in (a, b)$  such that  $g'(c) = 0$ .

which is contradiction to  $g'(x) \neq 0$ .

$$\forall x \in (a, b)$$

$\therefore g(a) \neq g(b)$

$\therefore \phi(x)$  is well defined.

since  $f(x)$  &  $g(x)$  are continuous functions on  $[a, b]$ .

and  $f(a), g(a)$  and  $K$  are constants.

These are continuous for all  $x$ .

$\therefore \phi(x)$  is continuous on  $[a, b]$ .

and  $\phi'(x) = f'(x) - kg'(x)$  exists on  $(a, b)$ .

because  $f$  &  $g$  are differentiable functions on  $(a, b)$ .

$\therefore \phi$  is differentiable function on  $(a, b)$ .

Now  $\phi(a) = 0$

$$\text{and } \phi(b) = f(b) - f(a) - K[g(b) - g(a)]$$

$$= [f(b) - f(a)] - \frac{f(b) - f(a)}{g(b) - g(a)} [g(b) - g(a)]$$

$\therefore 0$

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$$= \phi(b)$$

$\phi(x)$  satisfies the conditions of Rolle's theorem.

$\therefore \exists c \in (a, b)$  such that  $\phi'(c) = 0$

$$\text{But } \phi'(x) = f'(x) - kg'(x) \quad \forall x \in (a, b).$$

$$\Rightarrow \phi'(c) = f'(c) - kg'(c)$$

$$\Rightarrow 0 = f'(c) - kg'(c) \quad (\because \phi(c) = 0)$$

$$\Rightarrow f'(c) = kg'(c)$$

$$\Rightarrow K = \frac{f'(c)}{g'(c)}$$

$$\Rightarrow \frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

### \* Another form of. the statement

If two functions  $f$  and  $g$  defined on  $[a, a+b]$  are

- (i) Continuous on  $[a, a+b]$
  - (ii) differentiable on  $(a, a+b)$
  - (iii)  $g'(x) \neq 0$  for any  $x \in (a, a+b)$

ther)  $\exists$  at least one real number  $\theta \in (0, 1)$  such that

$$\frac{f'(a+\theta h)}{g'(a+\theta h)} = \frac{f(a+h)-f(a)}{g(a+h)-g(a)}$$

→ If  $f', g'$  are continuous and differentiable on  $[a, b]$  then show that for  $a < c < b$ .

$$\frac{f(b) - f(a) - (b-a)f'(a)}{g(b) - g(a) - (b-a)g'(a)} = \frac{f'(c)}{g'(c)}$$

Sol<sup>b</sup> :- Let us consider

$$f(x) = f(a) + (b-a)f'(x) + K \{g(x) + (b-x)g'(x)\}$$

$\forall x \in [a, b]$

where  $K$  is a constant to be determined such that  $\phi(a) = \phi(b)$ .

$$f(a) + (b-a)f'(a) + K[g(a) + (b-a)g'(a)] \\ \equiv f(b) + Kg(b).$$

$$\Rightarrow k = \frac{f(b) - f(a) - (b-a)f'(a)}{g(a) + (b-a)g'(a) - g(b)} \quad (1)$$

Since  $f, g$  are continuous and differentiable functions on  $[a, b]$ .

$\therefore \phi(t)$  is continuous and differentiable on  $[a, b]$

$\therefore f(x)$  satisfies the conditions  
Rolle's theorem on an interval  $[a, b]$

$\therefore \exists c \in (a, b) \text{ such that } \phi'(c) = 0.$

$$\begin{aligned} \text{But } \Phi'(x) &= f'(x) + (b-x)g''(x) - g'(x) \\ &\quad + k[g'(x) + (b-x)g''(x) - g'(x)] \\ &= (b-x)f''(x) + k(b-a)g''(x) \end{aligned}$$

$$\Rightarrow \Phi(c) = (b-c)f''(c) + k(b-c)g''(c)$$

$$\Rightarrow 0 = (b-c)f''(c) + k(b-c)g''(c)$$

(Since  $\Phi(c) = 0$ )

$$\Rightarrow K = -\frac{f''(c)}{g''(c)} \quad (\because b-c \neq 0)$$

$$\frac{f(b) - f(a) - (b-a)f'(a)}{g(a) + (b-a)g'(a) - g(b)} = \frac{-f''(c)}{g''(c)}$$

$$\Rightarrow \frac{f(b) - f(a) - (b-a)f'(a)}{g(b) - g(a) - (b-a)g'(a)} = \frac{f''(c)}{g''(c)}$$

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2005

If  $f'(x)$  and  $g'(x)$  exist for all  $x \in [a, b]$  and if  $g'(x)$  does not vanish anywhere on  $(a, b)$  then prove that for some  $c$  between  $a$  and  $b$ :

$$\frac{f(c) - f(a)}{g(b) - g(c)} = \frac{f'(c)}{g'(c)}$$

Sol'n: Let us consider

$$\phi(x) = f(x)g(x) - f(a)g(a) - g(b)f(x) \forall x \in [a, b]$$

Since  $f'$  and  $g'$  exists in  $[a, b]$ .

$\therefore f$  and  $g$  are derivable functions on  $[a, b]$ .

$\therefore f$  and  $g$  are continuous functions on  $[a, b]$ .

$\therefore \phi(x)$  is continuous and derivable on  $[a, b]$ .

and  $\phi(a) = -f(a)g(b)$ :

$$\phi(b) = -f(a)g(b)$$

$$\therefore \phi(a) = \phi(b)$$

$\therefore \phi(x)$  satisfies the conditions of Rolle's theorem on  $[a, b]$ .

$\therefore \exists$  at least one point  $c \in (a, b)$  such that  $\phi'(c) = 0$

$$\text{But } \phi'(x) = f'(x)g(x) + f(x)g'(x)$$

$$= f(a)g'(x) - g(b)f'(x)$$

$$\Rightarrow \phi'(c) = f'(c)g(c) + f(c)g'(c)$$

$$= f(a)g'(c) - g(b)f'(c)$$

$$\Rightarrow 0 = f'(c)g(c) + f(c)g'(c)$$

$$= f(a)g'(c) - g(b)f'(c)$$

$$(\because \phi(c) = 0)$$

$$g'(c)[f(c) - f(a)] + f'(c)[g(c) - g(b)] = 0$$

$$\Rightarrow g'(c)[f(c) - f(a)] = -f'(c)[g(b) - g(c)]$$

$$\Rightarrow \frac{f(c) - f(a)}{g(b) - g(c)} = \frac{f'(c)}{g'(c)}$$

$$[\because g'(x) \neq 0 \forall x \in (a, b)]$$

### Generalised Mean Value Theorem:

If three functions  $f, g$  and  $h$  defined on  $[a, b]$  are

i) Continuous on  $[a, b]$

ii) Differentiable on  $(a, b)$

there exists a real number  $c \in (a, b)$

such that  $\begin{vmatrix} f(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0$ .

Proof: Consider the function  $\phi$  on  $[a, b]$  defined by

$$\phi(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix}$$

$$= f(x) \begin{vmatrix} g(a) & h(a) \\ g(b) & h(b) \end{vmatrix} - g(x) \begin{vmatrix} f(a) & h(a) \\ f(b) & h(b) \end{vmatrix} + h(x) \begin{vmatrix} f(a) & g(a) \\ f(b) & g(b) \end{vmatrix}$$

$$= Af(x) + Bg(x) + Ch(x)$$

where  $A, B, C$  are constants.

Since  $f, g, h$  are continuous

functions on  $[a, b]$

$\therefore \phi(x)$  is continuous on  $[a, b]$  and

$f, g, h$  are differentiable on  $(a, b)$

$\therefore \phi'(x)$  is differentiable on  $(a, b)$  and  $\phi(a) = \phi(b) = 0$ .

$\therefore \phi$  satisfies the conditions of Rolle's theorem.

$\exists c \in (a, b)$  such that  $\phi'(c) = 0$ . (1)

But  $\phi'(x) = Af'(x) + Bg'(x) + Ch'(x)$  in  $(a, b)$

$$= \begin{vmatrix} f'(x) & g'(x) & h'(x) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix}$$

$$\Rightarrow \phi'(c) = \begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix}$$

$$\Rightarrow 0 = \begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix}$$

where  $c \in (a, b)$ .

$\rightarrow$  when  $h$  is a constant function the above theorem reduces to

Cauchy's mean value theorem.

Let  $h(x) = k$  (constant) then

$h(a) = h(b) = k$  and  $h'(c) = 0$ .

Substituting generalised mean value theorem, we get,

$$\begin{vmatrix} f(c) & g(c) & k \end{vmatrix}$$

$$\begin{vmatrix} -f(a) & g(a) & k \end{vmatrix} = 0$$

$$\begin{vmatrix} f(b) & g(b) & k \end{vmatrix}$$

$$\Rightarrow \frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

which is the Cauchy's Mean Value theorem.

$\rightarrow$  when  $g(x)=x$  and  $h(x)=k$  (constant)  
the above theorem (Generalised Mean Value) reduces to Lagrange's mean value theorem.

$$g(x)=x \text{ and } h(x)=k$$

$$\Rightarrow g'(x)=1 \text{ and } h'(x)=0$$

$$\Rightarrow g'(c)=1, h'(c)=0 \text{ and}$$

$$g(a)=a; g(b)=b; h(a)=h(b)=k$$

From generalised mean value theorem,

$$\begin{vmatrix} f'(c) & g'(c) & h'(c) \end{vmatrix}$$

$$\begin{vmatrix} f(a) & g(a) & h(a) \end{vmatrix} = 0$$

$$\begin{vmatrix} f(b) & g(b) & h(b) \end{vmatrix}$$

$$\begin{vmatrix} f'(c) & 1 & 0 \end{vmatrix}$$

$$\begin{vmatrix} -f(a) & a & k \end{vmatrix} = 0$$

$$\begin{vmatrix} f(b) & b & k \end{vmatrix}$$

$$\Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}$$

which is the Lagrange's Mean Value theorem.

Problems:-

$\rightarrow$  Verify Cauchy's Mean Value theorem for the following pairs of functions in the specified intervals.

$$f(x)=x^2 \text{ & } g(x)=x^3 \quad \forall x \in [1,2]$$

Sol'n: Since  $f$  &  $g$  are continuous

on  $[1,2]$  and differentiable on  $(1,2)$

Also  $g'(x)=3x^2 \neq 0$  for any  $x \in (1,2)$

$\therefore f$  &  $g$  satisfy the conditions of Cauchy's Mean Value theorem.

$\therefore \exists c \in (1,2)$  such that

$$\frac{f'(c)}{g'(c)} = \frac{f(2)-f(1)}{g(2)-g(1)} \quad \text{--- (1)}$$

$$\text{But } g'(x)=3x^2 \text{ & } f'(x)=2x$$

$$\therefore f'(c)=2c \text{ & } g'(c)=3c^2$$

$$(1) \equiv \frac{2c}{3c^2} = \frac{4-1}{8-1}$$

$$\Rightarrow \frac{2}{3c} = \frac{3}{7}$$

$$\Rightarrow 9c=14$$

$$\Rightarrow c = \frac{14}{9} \in (1,2)$$

Cauchy's Mean Value theorem is verified.

→ Find 'c' of Cauchy's Mean Value Theorem for the following pairs of functions.

i)  $f(x) = e^x$ ,  $g(x) = e^{-x}$   $\forall x \in [a, b]$

Soln:  $f(a) = e^a$ ;  $f(b) = e^b$

$g(a) = e^{-a}$ ;  $g(b) = e^{-b}$

$f'(x) = e^x \Rightarrow f'(c) = e^c$

$g'(x) = -e^{-x} \Rightarrow g'(c) = -e^{-c}$

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

$$\Rightarrow \frac{e^c}{-e^{-c}} = \frac{e^b - e^a}{e^{-b} - e^{-a}}$$

$$\begin{aligned} \Rightarrow -e^{2c} &= \frac{-e^b - e^a}{\frac{1}{e^b} - \frac{1}{e^a}} \\ &= -[-e^b + e^a] \\ &= \frac{e^a - e^b}{e^a \cdot e^b} \\ &= -e^{a-b} \\ &= -e^{(a+b)} \end{aligned}$$

$$\Rightarrow 2c = a+b$$

$$\Rightarrow c = \frac{a+b}{2} \in (a, b)$$

H.W.

ii)  $f(x) = x^2$ ,  $g(x) = x$   $\forall x \in [a, b]$

iii)  $f(x) = \sin x$ ,  $g(x) = \cos x$   $\forall x \in [-\pi/2, 0]$

→ Show that  $\frac{\sin \alpha - \sin \beta}{\cos \beta - \cos \alpha} = \cot \theta$

where  $0 < \alpha < \theta < \beta < \pi/2$ .

Soln: Let  $f(x) = \sin x$

$$g(x) = \cos x \forall x \in [\alpha, \beta]$$

Since  $f$  and  $g$  are both continuous on  $[\alpha, \beta]$  and differentiable on  $(\alpha, \beta)$ .

$$g(x) = -\sin x \neq 0 \text{ for any } x \in (\alpha, \beta)$$

∴ By Cauchy's Mean value theorem

$\exists \theta \in (\alpha, \beta)$  such that

$$\frac{f(\theta)}{g(\theta)} = \frac{f(\beta) - f(\alpha)}{g(\beta) - g(\alpha)} \quad \text{--- (1)}$$

$$\text{But } f'(\alpha) = \cos \alpha; g'(\alpha) = -\sin \alpha$$

$$\Rightarrow f'(\theta) = \cos \theta; g'(\theta) = -\sin \theta$$

$$\text{①} \equiv \frac{\sin \beta - \sin \alpha}{\cos \beta - \cos \alpha} = \frac{\cos \theta}{-\sin \theta}; \theta \in (\alpha, \beta)$$

$$\Rightarrow \frac{\sin \alpha - \sin \beta}{\cos \alpha - \cos \beta} = \cot \theta, \theta \in (\alpha, \beta)$$

## \* Miscellaneous Problems :-

→ Assuming  $f''$  to be continuous on  $[a,b]$ , show that

$$f(c) - f(a) \left( \frac{b-c}{b-a} \right) - \left( \frac{c-a}{b-a} \right) f(b) = \frac{1}{2} (c-a)(c-b) f''(\xi)$$

where  $c$  and  $\xi$  both lie in  $[a,b]$   
i.e.  $c, \xi \in [a,b]$ .

Sol: We have to show that

$$(b-a)f(c) - (b-c)f(a) - (c-a)f(b) = \frac{1}{2} (b-a)(c-a)(c-b) f''(\xi)$$

Let us consider the function for  $x \in [a,b]$  defined by

$$\phi(x) = (b-a)f(x) - (b-x)f(a) - (x-a)f(b) - (b-a)(x-a)(x-b)k$$

where  $k$  is a constant to be determined such that  $\phi(c) = 0$ .

$$0 = (b-a)f(c) - (b-c)f(a) - (c-a)f(b) - (b-a)(c-a)(c-b)k$$

$$= \frac{(b-a)f(c) - (b-c)f(a) - (c-a)f(b)}{(b-a)(c-a)(c-b)}$$

Clearly  $\phi(a) = \phi(b) = 0$  and  $\phi(x)$  is differentiable in  $[a,b]$ .

The function  $\phi$  satisfies all the

conditions of Rolle's theorem on each intervals  $[a,c]$  and  $[c,b]$ .

∴ ∃ two numbers  $\xi_1, \xi_2$  in  $(a,c)$  and  $(c,b)$  such that  $\phi'(\xi_1) = 0$  and  $\phi'(\xi_2) = 0$

$$\text{But } \phi'(x) = (b-a)f'(x) + f(a) - f(b) - (b-a)\{2x - (a+b)\}k.$$

which is continuous on  $[a,b]$  and derivable on  $(a,b)$ .

∴ Continuous and derivable on  $[\xi_1, \xi_2]$ .

$$\text{Also } \phi'(\xi_1) = \phi'(\xi_2) = 0$$

∴ By Rolle's theorem,

∃  $\xi \in (\xi_1, \xi_2)$  such that  $\phi''(\xi) = 0$

$$\text{But } \phi''(x) = (b-a)f''(x) - 2(b-a)k.$$

$$\therefore f''(\xi) - 2k = 0. \quad (\because b-a \neq 0 \& \\ \phi''(\xi) = 0)$$

$$\Rightarrow k = \frac{1}{2} f''(\xi) \text{ where}$$

$$a < \xi_1 < \xi < \xi_2 < b$$

(2)

from (1) & (2), we have

$$\frac{(b-a)f(c) - (b-c)f(a) - (c-a)f(b)}{(b-a)(c-a)(c-b)} = \frac{1}{2} f''(\xi)$$

$$\Rightarrow f(c) - \left( \frac{b-c}{b-a} \right) f(a) - \left( \frac{c-a}{b-a} \right) f(b) = \frac{1}{2} (c-a)(c-b) f''(\xi).$$

$$= \underline{\underline{\frac{1}{2} (c-a)(c-b) f''(\xi)}}.$$

Let  $R$  be the set of real numbers and  $f: R \rightarrow R$  such that for all  $x$  and  $y \in R$ ,  $|f(x) - f(y)| \leq |x-y|^3$ .

Prove that  $f(x)$  is a constant function.

Soln: Given  $|f(x) - f(y)| \leq |x-y|^3$

$$\forall x, y \in R \quad \text{--- (1)}$$

Let  $y \in R$  and  $x$  be chosen arbitrarily close to  $y$  but not equal to  $y$ .

$$\therefore \text{--- (1)} \equiv \left| \frac{f(x) - f(y)}{x-y} \right| \leq |x-y|^2$$

Taking limit when  $x \rightarrow y$  we get

$$\lim_{x \rightarrow y} \left| \frac{f(x) - f(y)}{x-y} \right| \leq \lim_{x \rightarrow y} |x-y|^2$$

$$\Rightarrow \left| \lim_{x \rightarrow y} \frac{f(x) - f(y)}{x-y} \right| \leq \left| \lim_{x \rightarrow y} (x-y) \right|^2$$

$$\Rightarrow |f'(y)| = 0 \left[ \because \lim_{x \rightarrow y} \frac{f(x) - f(y)}{x-y} = f'(y) \right. \\ \left. \text{and } f'(y) \geq 0 \right]$$

$$\Rightarrow f'(y) = 0$$

$\therefore f(x)$  is constant.

2004 Prove that an equation of the form  $x^n = a$  where  $n \in \mathbb{N}$  and  $a > 0$  is a real number, has a positive root.

(OR)

Show that  $x^n - a = 0$  has at most one real  $-ve$  root if  $n$  is a

$+ve$  integer.

Soln: Let  $f(x) = x^n - a$

$$\text{then } f'(x) = nx^{n-1}$$

Since  $f'(x) > 0$  for  $x > 0$ .

hence  $f(x)$  is increasing on  $(0, \infty)$ .

Let  $x_1, x_2 \in (0, \infty)$  and  $0 < x_1 < x_2$  such that  $f(x) \neq 0$ .

$$\text{then } f(x_1) \leq f(x) \leq f(x_2) \quad (\text{or})$$

$$f(x_1) \leq 0 \leq f(x_2)$$

$\therefore$  This shows that if  $x \neq a$ ,  $f(x) \neq 0$  on  $(0, \infty)$ .

i.e.  $x^n - a = 0$  has at most one  $real +ve$  root.

2008 Prove that  $\frac{\tan x}{x} > \frac{x}{\sin x}$

whenever  $0 < x < \pi/2$ .

$$\text{Soln: } \frac{\tan x}{x} - \frac{x}{\sin x} = \frac{\tan x - x \cos x}{x \sin x}$$

Since  $x \sin x > 0 \forall x \in (0, \pi/2)$

$\therefore$  we are enough to show that

$$\tan x \cdot \sin x - x^2 > 0 \quad \forall x \in (0, \pi/2)$$

$$\text{Let } f(x) = \tan x \cdot \sin x - x^2 \quad \forall x \in (0, \pi/2)$$

$$\Rightarrow f'(x) = \sec^2 x \sin x + \tan x \cos x - 2x \\ = \sin x (\sec^2 x + 1) - 2x$$

We cannot decide about the sign of  $f'(x)$  (because of the presence of  $2x$  term)

$$\text{Let } g(x) = f'(x) \quad \forall x \in (0, \pi/2)$$

$$\Rightarrow g(x) = \cos x (\sec^2 x + \tan x) - 2 \\ = \sec x + \cos x - 2 + 2\sin^2 x \sec^3 x \\ = (\sqrt{\sec x} - \sqrt{\cos x})^2 - 2\sin^2 x \sec^3 x.$$

Since  $g'(x) > 0 \quad \forall x \in (0, \pi/2)$ .

$\Rightarrow g(x)$  is an increasing function in  $(0, \pi/2)$ .

$$\Rightarrow g(0) < g(x) \text{ in } 0 < x < \pi/2$$

$$\text{since } g(0) = 0$$

$$\therefore g(x) > 0$$

$$\Rightarrow f'(x) > 0 \text{ whenever } 0 < x < \pi/2$$

$\therefore f$  is an increasing function in  $0 < x < \pi/2$ .

$$\Rightarrow f(0) < f(x)$$

$$\Rightarrow 0 < f(x)$$

$$\Rightarrow \tan x \sin x - x^2 > 0 \text{ in } (0, \pi/2).$$

$$\Rightarrow \frac{\tan x \sin x - x^2}{\sin x} > 0$$

$$\Rightarrow \frac{\tan x}{x} - \frac{x}{\sin x} > 0$$

$$\Rightarrow \frac{\tan x}{x} > \frac{x}{\sin x} \text{ whenever } 0 < x < \pi/2$$

$\rightarrow$  Prove that if  $f$  be defined for all real  $x$  such that

$|f(x) - f(y)| < (x-y)^2$  then  $f$  is constant.

Soln:- Here we have to show that  $f'(x) = 0 \quad \forall x \in \mathbb{R}$ .

Let  $a = c \in \mathbb{R}$ .

Now we have

$$\left| \frac{f(x) - f(c)}{x - c} \right| - 0 \quad \text{for } x \neq c.$$

$$= \left| \frac{f(x) - f(c)}{x - c} \right|$$

$$= \frac{|f(x) - f(c)|}{|x - c|} < \frac{(x-c)^2}{|x-c|} = \frac{|x-c|^2}{|x-c|}$$

(by hyp).

$$= |x-c| < \epsilon \text{ whenever } |x-c| < \epsilon$$

$$\therefore \left| \frac{f(x) - f(c)}{x - c} \right| - 0 < \epsilon \text{ whenever}$$

$|x-c| < \delta$  by choosing  $\delta = \frac{\epsilon}{1}$

$$\therefore \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c} = 0$$

$$\text{i.e. } f'(c) = 0 \quad \forall c \in \mathbb{R}.$$

$\Rightarrow f$  is constant function.

$\rightarrow$  Find the interval in which the function  $f(x) = \sin(\log_e x) - \cos(\log_e x)$  is strictly increasing.

Soln: Given that

$$f(x) = \sin(\log_e x) - \cos(\log_e x)$$

Here domain is  $x > 0$  as  $\log_e x$

exists when  $x > 0$ ,

$$f'(x) = \frac{\cos(\log_e x) + \sin(\log_e x)}{x}$$

$$= \frac{\sqrt{2} \left\{ \sin \frac{\pi}{4} \cos(\log_e x) + (\cos \frac{\pi}{4}) \sin(\log_e x) \right\}}{x}$$

$$= \frac{\sqrt{2} \sin \left( \frac{\pi}{4} + \log_e x \right)}{x}$$

Since  $f(x)$  is strictly increasing when  $f'(x) \geq 0$ .

$$\text{i.e. } \sin \left( \frac{\pi}{4} + \log_e x \right) \geq 0$$

$$\Rightarrow 2n\pi \leq \frac{\pi}{4} + \log_e x \leq (2n+1)\pi \quad \forall n \in \mathbb{Z}$$

$$\Rightarrow 2n\pi - \frac{\pi}{4} \leq \log_e x \leq 2n\pi + \pi - \frac{\pi}{4}$$

$$\Rightarrow e^{2n\pi - \frac{\pi}{4}} \leq x \leq e^{2n\pi + \frac{3\pi}{4}}$$

$\therefore f(x)$  is strictly increasing when

$$x \in \left[ e^{2n\pi - \frac{\pi}{4}}, e^{2n\pi + \frac{3\pi}{4}} \right]$$

$$\Rightarrow \text{Let } g(x) = f(x) + f(1-x)$$

$$\text{and } f''(x) > 0 \quad \forall x \in (0,1)$$

find the intervals of increase and decrease of  $g(x)$ .

Sol'n:- We have

$$g(x) = f(x) + f(1-x) \quad \text{--- (1)}$$

$$\text{then } g'(x) = f'(x) - f'(1-x) \quad \text{--- (2)}$$

$$\text{since } -f''(x) > 0 \quad \forall x \in (0,1)$$

$\therefore f'(x)$  is increasing on  $(0,1)$ .

Hence two cases:

Case(i):  $x > 1-x$  and  $f'(x)$  is increasing  $\Leftrightarrow x > \frac{1}{2}$  in  $(0,1)$

$$\Rightarrow f'(1-x) < f'(x) \quad \forall x > \frac{1}{2}$$

$$\Rightarrow f'(x) - f'(1-x) > 0 \quad \forall x > \frac{1}{2}$$

$$\therefore g'(x) > 0 \quad \forall x > \frac{1}{2} \text{ in } (0,1)$$

$$\text{i.e. } g'(x) > 0 \quad \forall x \in \left( \frac{1}{2}, 1 \right)$$

$\Rightarrow g(x)$  is increasing in  $\left( \frac{1}{2}, 1 \right)$ .

Case(ii)

$x < 1-x$  and  $f'(x)$  increasing for  $0 < x < \frac{1}{2}$  in  $(0,1)$ .

$$\Rightarrow f'(x) < f'(1-x) \text{ for } 0 < x < \frac{1}{2}$$

$$\Rightarrow f'(x) - f'(1-x) < 0 \text{ for } 0 < x < \frac{1}{2}$$

$$\Rightarrow g'(x) < 0 \quad \forall x \in (0, \frac{1}{2})$$

$\Rightarrow g(x)$  is decreasing function in  $(0, \frac{1}{2})$ .

→ show that

$$\frac{x}{\pi} < \frac{\sin x}{x} < 1, \quad 0 < x < \frac{\pi}{2}$$

Sol'n:- Let

$$f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x=0 \quad \forall x \in [0, \frac{\pi}{2}] \end{cases}$$

then  $f$  is continuous in  $[0, \pi/2]$

and derivable in  $(0, \pi/2)$

and  $f'(x) = \frac{x\cos x - \sin x}{x^2}$ ,

$$x \in (0, \pi/2) \quad \text{--- (1)}$$

Let  $F(x) = x\cos x - \sin x; x \in (0, \pi/2)$

$$F'(x) = \cos x - x\sin x - \cos x$$

$$= -x\sin x$$

$$< 0; x \in (0, \pi/2)$$

$\therefore F$  is decreasing in  $(0, \pi/2)$

$\therefore F(x) < F(0)$  for  $x > 0$  in  $(0, \pi/2)$

$\Rightarrow F(x) < 0$  for  $x \in (0, \pi/2)$

$$( \because F(0) = 0 )$$

$\Rightarrow f'(x) < 0; x \in (0, \pi/2)$

$\therefore f(x)$  is decreasing in  $(0, \pi/2)$

$\Rightarrow f(0) > f(x) > f(\pi/2)$  for

$$0 < x < \pi/2$$

$$\Rightarrow 1 > \frac{\sin x}{x} > \frac{2}{\pi}$$

$$\Rightarrow \frac{2}{\pi} < \frac{\sin x}{x} < 1 \quad \forall x \in (0, \pi/2)$$

## Taylor's Theorem:

### statement:

If a function  $f$  is defined on  $[a, b]$ , is such that  
 i) the  $(n-1)^{th}$  derivative  $f^{(n-1)}$  is continuous on  $[a, b]$   
 ii) the  $n^{th}$  derivative  $f^{(n)}$  exists on  $(a, b)$ .  
 then there exist at least one real number  $c \in (a, b)$   
 such that  $f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!}f''(a) + \dots + \frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{(b-a)^n}{n!}f^{(n)}(c)$   
 where  $p$  is a given +ve integer..

proof: Let  $G(x) = f(x) - \left(\frac{b-x}{b-a}\right)^p F(a)$

$$\text{where } F(x) = f(b) - f(x) - (b-x)f'(x) - \frac{(b-x)^2}{2!}f''(x) - \dots - \frac{(b-x)^{n-1}}{(n-1)!}f^{(n-1)}(x)$$

Since the  $(n-1)^{th}$  derivative  $f^{(n-1)}$  is continuous  
on  $[a, b]$ .

$\therefore f, f', f'', \dots, f^{(n-1)}$  are continuous on  $[a, b]$ .  
and  $(b-x)^x$ ,  $x = 1, 2, 3, \dots, n-1$   
is continuous for all  $x$ .

$\therefore F(x)$  is continuous on  $[a, b]$ .

$\therefore G(x)$  is continuous on  $[a, b]$ .

Since the  $n^{th}$  derivative  $f^{(n)}$  exists on  $(a, b)$ .

$\therefore f, f', f'', \dots, f^{(n-1)}$  are differentiable on  $(a, b)$ .  
and  $(b-x)^x$ ,  $x = 1, 2, 3, \dots, (n-1)$   
is differentiable for all  $x$ .

$\therefore f(x)$  is differentiable on  $(a, b)$ .

$\therefore G(x)$  is differentiable on  $(a, b)$ .

$$\text{Now } G(a) = f(a) - \frac{(b-a)^P}{(b-a)} f(a)$$

$$= 0$$

$$\text{and } G(b) = f(b) - 0$$

$$= f(b) - f(b) - (b-b) f'(b) - \dots - \frac{(b-b)^{n-1}}{(n-1)!} f^{(n-1)}(b)$$

$$= 0$$

$$\therefore G(a) = G(b) = 0$$

$\therefore G(x)$  satisfies the conditions of

Roll's theorem,

$\exists$  at least one real number  $c \in (a, b)$

such that  $G'(c) = 0$ .

$$\text{But } G'(x) = f'(x) + p \cdot \frac{(b-x)^{P-1}}{(b-a)^P} f(a)$$

$$\text{Now } G'(x) = 0 - f'(x) + f'(x) - (b-x) f''(x) + (b-x) f'''(x) \\ - \frac{(b-x)^2}{2!} f^{(4)}(x) + \dots + \frac{(n-1)(b-x)}{(n-1)!} f^{(n-1)}(x) \\ - \frac{(b-x)^{n-1}}{(n-1)!} f^{(n)}(x)$$

$$\therefore f'(x) = - \frac{(b-x)^{n-1}}{(n-1)!} f^{(n)}(x) \quad \text{--- (2)}$$

$$\text{Q.E.D. } G'(c) = f'(c) + p \frac{(b-c)^{P-1}}{(b-a)^P} f(a)$$

$$\Rightarrow 0 = f'(c) + p \frac{(b-c)^{P-1}}{(b-a)^P} f(a) \quad (\because G'(c) = 0)$$

$$\Rightarrow f'(c) = - \frac{p(b-c)^{P-1}}{(b-a)^P} f(a)$$

$$\Rightarrow - \frac{(b-c)^{n-1}}{(n-1)!} f^{(n)}(c) = - \frac{p(b-c)^{P-1}}{(b-a)^P} f(a) \quad (\text{from (2)})$$

$$\Rightarrow f(a) = \frac{(b-a)^P (b-c)^{n-P}}{p(n-1)!} f^{(n)}(c)$$

$$\Rightarrow f(b) - f(a) - \frac{(b-a)}{1!} f'(a) - \frac{(b-a)^2}{2!} f''(a) - \dots - \frac{(b-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) \\ = \frac{(b-a)^P (b-a)^{n-P}}{p(n-1)!} f^{(n)}(c)$$

$$\Rightarrow f(b) = f(a) + \frac{(b-a)}{1!} f'(a) + \frac{(b-a)^2}{2!} f''(a) + \dots$$

$$+ \frac{(b-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{(b-a)^p (b-c)^{n-p}}{P(n-1)!} f^{(n)}(c)$$

$$= P_n(a) + R_n(x)$$

Note: [ii] After  $n$  terms  $R_n(x) = T_{n+1}$

$$= \frac{(b-a)^p (b-c)^{n-p}}{P(n-1)!} f^{(n)}(c)$$

for some point  $c \in (a, b)$ .  
 This formula for  $R_n$  is referred to as the  
Rolle's form (or derivative form) of the  
 remainder.

[2]

(i) for  $p=1$ ,  
 $R_n = \frac{(b-a)(b-c)^{n-1}}{(n-1)!} f'(c)$ . Called Cauchy's  
 form of remainder.

(ii) for  $p=n$ ,  
 $R_n = \frac{(b-a)^n}{n!} f^{(n)}(c)$ . called Lagrange's  
 form of remainder.

[3] Another form of Taylor's Theorem:

If a function  $f$  defined on  $[a, a+h]$  is  
 such that  $f^{(n-1)}$  is continuous  
 (i) the  $(n-1)^{th}$  derivative  $f^{(n-1)}$   
 on  $[a, a+h]$   
 (ii) the  $n^{th}$  derivative  $f^{(n)}$  exists on  $(a, a+h)$   
 Then  $\exists c \in (0, 1)$  such that  

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a)$$

$$+ \frac{h^n (1-c)}{P(n-1)!} f^{(n)}(a+ch)$$
 where  $p$  is the integer.

#### 4. Maclaurin's theorem:

putting  $a=0, h=x$  in Taylor's theorem  
 i.e. If a function  $f$  defined on  $[0, x]$  is  
 such that (i) the  $(n-1)^{th}$  derivative  $f^{(n-1)}$  is  
 continuous on  $[0, x]$   
 (ii) the  $n^{th}$  derivative  $f^{(n)}$  exists on  $(0, x)$

then  $\exists \theta \in (0, 1)$  such that

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) + \frac{x^n (1-\theta)}{n!} f^{(n)}(\theta x).$$

#### Taylor's and Maclaurin's series:

Let a function  $f$  be continuous derivatives of every order in  $[a, a+h]$  then for all  $n \in \mathbb{N}$  we have by Taylor's theorem

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{n!} f^{(n)}(a+\theta h)$$

where  $\theta \in (0, 1)$

$$\text{Let } P_n = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a)$$

$$R_n = \frac{h^n}{n!} f^{(n)}(a+\theta h)$$

(which is Taylor's remainder after  $n$  terms)

$$\text{Then } f(a+h) = P_n + R_n$$

If  $R_n \rightarrow 0$  as  $n \rightarrow \infty$ ,

$$\text{we have } \lim_{n \rightarrow \infty} P_n = f(a+h)$$

⇒ The infinite series

$$f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \dots$$

converges to  $f(a+h)$

$f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \dots$  is called

Taylor's series which is eqs to  $f(a+h)$

$$\text{if } \lim_{n \rightarrow \infty} R_n = 0$$

Hence if  $f: [a, a+h] \rightarrow \mathbb{R}$  possesses continuous derivatives of every order in  $[a, a+h]$

and Taylor's remainder  $R_n \rightarrow 0$  as  $n \rightarrow \infty$

$$\text{then } f(a+h) = f(a) + h f'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \dots$$

→ If we put  $a=0, h=x$ ; we get

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^n}{n!} f^{(n)}(0)$$

This is called MacLaurin's series.

NOTE: This series is useful in finding the expansion of functions.

Problems:

$$\text{If } f(x+h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x+8h), \text{ find}$$

the value of  $\theta$  as  $x \rightarrow a$  if  $f(x) = (x-a)^{5/2}$

Sol: Given  $f(x) = (x-a)^{5/2}$

$$\Rightarrow f(x+h) = (x+h-a)^{5/2}$$

$$\text{and } f'(x) = \frac{5}{2} (x-a)^{3/2}$$

$$\Rightarrow f''(x) = \frac{15}{4} (x-a)^{1/2}$$

$$\Rightarrow f''(x+8h) = \frac{15}{4} (x+8h-a)^{1/2}$$

$$\therefore f(x+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a+0h)$$

$$\Rightarrow (x+a)^{\frac{5}{2}} = (a-a)^{\frac{5}{2}} + h\left(\frac{5}{2}\right)(a-a)^{\frac{3}{2}} + \frac{h^2}{2}\left(\frac{15}{4}\right)(x+0h-a)^{\frac{1}{2}}$$

when  $x \rightarrow a$ , we get

$$h^{\frac{5}{2}} = \frac{h^2}{2}\left(\frac{15}{4}\right)(0h)^{\frac{1}{2}}$$

$$\Rightarrow h^{\frac{5}{2}} = \frac{h^2}{2}\left(\frac{15}{4}\right) 0^{\frac{1}{2}}$$

$$\Rightarrow \frac{8}{15} = 0^{\frac{1}{2}}$$

$$\Rightarrow 0 = \frac{64}{225}$$

~~Q.S. 2009~~  
Hence if  $f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0x)$ ; find the value of  $\theta$  as  $x \rightarrow 1$  of  $f(x) = (1-x)^{\frac{5}{2}}$

Using Taylor's theorem, show that

$$(i) \cos x \geq 1 - \frac{x^2}{2} \quad \forall x \in \mathbb{R}$$

$$(ii) 1+x + \frac{x^2}{2} < e^x < 1+x + \frac{x^2}{2} e^x, x > 0$$

$$(iii) x - \frac{x^3}{3!} < \sin x < x, x > 0$$

$$(iv) x - \frac{x^3}{3!} \leq \sin x \leq x - \frac{x^3}{3!} + \frac{x^5}{5!}, x > 0$$

Soln (i)  $\cos x \geq 1 - \frac{x^2}{2} \quad \forall x \in \mathbb{R}$

Case (i) Let  $x \rightarrow 0$ ,

$$\text{then } \cos x = 1 \quad ; \quad 1 - \frac{x^2}{2} = 1$$

$$\therefore \cos x = 1 - \frac{x^2}{2}$$

Case (ii) Let  $x > 0$  and  $f(x) = \cos x$

$$\Rightarrow f'(x) = -\sin x, \quad f''(x) = -\cos x$$

$$\text{Since } f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0x)$$

where  $0 < \theta < 1$

$$\therefore \cos x = 1 - \frac{x^2}{2} \cos \theta x.$$

But  $\cos \theta x < 1$ ;  $\theta x < x$ ;  $x > 0$ .

$$\therefore 1 - \frac{x^2}{2} \cos \theta x > 1 - \frac{x^2}{2}$$

$$\Rightarrow \cos x > 1 - \frac{x^2}{2}.$$

Case(3):

let  $x < 0 \Rightarrow -x > 0$

put  $y = -x$ ;  $y > 0$

By Case(2),  $\cos y > 1 - \frac{y^2}{2}$

$$\Rightarrow \cos(-x) > 1 - \frac{(-x)^2}{2}$$

$$\Rightarrow \cos x > 1 - \frac{x^2}{2}$$

combining all cases,

$$\cos x \geq 1 - \frac{x^2}{2} \quad \forall x \in \mathbb{R}$$

$$(ii) \quad 1 + x + \frac{x^2}{2} \leq e^x \leq 1 + x + \frac{x^2}{2} e^{x^2}; \quad x > 0$$

Soln: let  $f(x) = e^x$ ;  $x > 0$

then  $f'(x) = e^x = f''(x)$

since  $f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0x)$   
where  $0 < 0x < 1$ .

$$f(x) = 1 + x + \frac{x^2}{2} e^{0x} \quad \text{--- (A)}$$

$0 < 0x < x; \quad x > 0$

$$e^0 < e^{0x} < e^x$$

$$1 < e^{0x} < e^x$$

$$\Rightarrow \frac{x^2}{2} < \frac{x^2}{2} e^{0x} \leq \frac{x^2}{2} e^x$$

$$\Rightarrow 1 + x + \frac{x^2}{2} < 1 + x + \frac{x^2}{2} e^{0x} < 1 + x + \frac{x^2}{2} e^x$$

$$\Rightarrow 1 + x + \frac{x^2}{2} \leq e^x \leq 1 + x + \frac{x^2}{2} e^x \quad (\text{by (A)})$$

$\rightarrow$  expand  $e^x$  as an infinite series

Soln: let  $f(x) = e^x$

$$f(x) = e^x \Rightarrow f(0) = 1$$

$$f'(x) = e^x \Rightarrow f'(0) = 1$$

$$f''(x) = e^x \Rightarrow f''(0) = e^0$$

Clearly  $f$  and its derivatives exist and are continuous for every value of  $x$ .

$$f_n(x) = \frac{x^n}{n!} e^{bx}$$
 (LFR)

$$\lim_{n \rightarrow \infty} R_n = e^{bx} + \lim_{n \rightarrow \infty} \frac{x^n}{n!} \quad \text{--- (1)}$$

$$\text{Now let } a_n = \frac{x^n}{n!} \text{ then}$$

$$\Rightarrow a_{n+1} = \frac{x^{n+1}}{(n+1)!} \text{ then}$$

$$\therefore \lim_{n \rightarrow \infty} \left( \frac{a_{n+1}}{a_n} \right) = \lim_{n \rightarrow \infty} \frac{x}{n+1} \quad -$$

$$= 0 < 1$$

$$\therefore \lim_{n \rightarrow \infty} a_n = 0 \quad (\because \lim_{n \rightarrow \infty} \left( \frac{a_{n+1}}{a_n} \right) = 1 < 1 \\ \text{then } \lim_{n \rightarrow \infty} a_n = 0)$$

$$\therefore \lim_{n \rightarrow \infty} R_n = 0$$

$\therefore$  the conditions of Maclaurin's series are satisfied.

$$\begin{aligned} e^x &= f(x) \\ &= f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots + \frac{x^n}{n!} f^{(n)}(0) + \cdots \\ &= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \end{aligned}$$

$\rightarrow$  Expand  $\sin x$  as infinite series

$$\text{Sol: Let } f(x) = \sin x.$$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \cdots$$

Here  $R_n(x)$  must be tend to '0'.

$$\text{Now } f(x) = \sin x; \quad f'(x) = \cos x; \quad f''(x) = -\sin x; \quad f'''(x) = -\cos x$$

$$\therefore f(0) = 0 \quad \therefore f'(0) = 1 \quad \therefore f''(0) = 0 \quad \therefore f'''(0) = -1$$

$$\begin{aligned} f^{(4)}(x) &= \sin x; \quad f^{(4)}(x) = +\cos x \\ \therefore f^{(4)}(0) &= 0 \quad \therefore f^{(4)}(0) = 1. \end{aligned}$$

$$\text{Generally } f^{(n)}(x) = \sin \left( x + \frac{n\pi}{2} \right)$$

$\therefore f$  and all its derivatives exist and continuous for every real value of  $x$ .

$$R_n(x) = \frac{x^n}{n!} \sin\left(\theta x + \frac{n\pi}{2}\right)$$

$$\text{Now } \lim_{n \rightarrow \infty} R_n = \lim_{n \rightarrow \infty} \frac{x^n}{n!} \times \lim_{n \rightarrow \infty} \sin\left(\theta x + \frac{n\pi}{2}\right)$$

$$= 0 \times l \quad (-1 \leq l \leq 1)$$

$$= 0$$

$$\therefore \lim_{n \rightarrow \infty} R_n = 0$$

$\therefore$  The conditions of Maclaurin's series is satisfied  $\forall x \in \mathbb{R}$ .

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\text{Similarly } \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots$$

$$\rightarrow f(x) = e^x$$

$$\rightarrow f(x) = x^a$$

$$\rightarrow f(x) = \log(1+x) \text{ etc.}$$



## \* Extreme Values of a function:

### Maxima and Minima:-

[Some definitions discussed in Pg No.(8)].

### \* Theorem (first derivative Test):

→ Let  $f$  be continuous on  $I = [a, b]$  and let  $c$  be an interior point on  $I$ . Assume that  $f$  is differentiable on  $(a, c)$  and  $(c, b)$ . Then -

(i) If there is a neighbourhood

$(c-\delta, c+\delta) \subseteq I$  such that  $f'(x) \geq 0$  for  $c-\delta < x < c$  and  $f'(x) \leq 0$  for  $c < x < c+\delta$  then  $f'$  has maximum at  $c$ .

(ii) If there is neighbourhood  $(c-\delta, c+\delta) \subseteq I$  such that  $f'(x) \leq 0$  for  $c-\delta < x < c$  and  $f'(x) \geq 0$  for  $c < x < c+\delta$  then  $f$  has a minimum at  $c$ .

### \* Theorem :-

Let  $I \subseteq \mathbb{R}$  be an interval, let  $f: I \rightarrow \mathbb{R}$  let  $c \in I$  and assume that  $f$  has a derivative at  $c$  then.

(i) If  $f'(c) > 0$  then  $\exists$  a  $\delta > 0$  such that  $f(x) > f(c)$  for  $x \in I$  such that  $c < x < c+\delta$

(ii) If  $f'(c) < 0$  then  $\exists$  a  $\delta > 0$  such that  $f(x) > f(c)$  for  $x \in I$  such that  $c-\delta < x < c$ .

## \* Darboux's Theorem:-

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If  $f$  is differentiable on  $I = [a, b]$  and if  $K$  is a number between  $f(a)$  &  $f'(b)$  then  $\exists c \in (a, b)$  such that  $f'(c) = K$ .

Proof :- Since  $K$  is number between  $f(a)$  &  $f'(b)$ .

Suppose that  $f'(a) < K < f'(b)$ .

Now we define

$$g(x) = Kx - f(x) \quad \forall x \in [a, b] \quad \textcircled{1}$$

Since  $f$  is differentiable on  $I$ .

$\therefore f$  is continuous on  $I$  and  $Kx$  is a polynomial which is continuous on  $I$ .

$\therefore g(x)$  is continuous on  $I$ .

$\therefore g(x)$  attains its supremum (infimum) atleast once on  $[a, b] = I$ .

$$\text{since } g'(x) = K - f'(x) \quad \forall x \in [a, b]$$

$$\Rightarrow g'(a) = (K - f'(a)) > 0 \\ (\because f'(a) < K < f'(b))$$

$$\Rightarrow g'(a) > 0.$$

We know that  $g$  has derivative at  $a$  and  $g'(a) > 0$ . then  $\exists$  a  $\delta > 0$

such that  $g(x) > g(a) \quad \forall x \in I$ .

such that  $a < x < a+\delta$ .

$\therefore g$  does not have the maximum at  $x=a$ .

Similarly  $g$  does not have the minimum at  $x=b$ .

$\therefore g$  has maximum at  $c \in (a, b)$

- Interior extremum theorem
- $f'(c) = 0 \forall c \in (a, b)$
- $f'(c) = k \forall c \in (a, b)$ .

### \* Generalised Test :-

Let  $I$  be an interval, let  $x_0 \in I$  and let  $n \geq 2$ .

Suppose that the derivatives  $f', f'', \dots, f^{(n)}$  exist and are continuous in a neighbourhood of  $x_0$  and that

$$f'(x_0) = f''(x_0) = \dots = f^{(n-1)}(x_0) = 0$$

but  $f^n(x_0) \neq 0$ .

i) If  $n$  is even and  $f^n(x_0) > 0$  then  $f$  has minimum at  $x_0$ .

ii) If  $n$  is even and  $f^n(x_0) < 0$  then  $f$  has maximum at  $x_0$ .

iii) If  $n$  is odd, then  $f$  has neither a minimum nor maximum at  $x_0$ .

### \* Working Rule for finding Maxima and Minima :-

#### Maxima and Minima :-

i) Denote the given function by  $f(x)$ .

ii) Find  $f'(x)$  and equate it to zero.

Let its roots be  $x_1, x_2, \dots$

iii) Find  $f''(x)$ , put  $x = x_1$ . If

$f''(x_1) < 0$ ,  $f(x)$  has a maximum at  $x = x_1$ .

If  $f''(x_1) > 0$ ,  $f(x)$  has a minimum at  $x = x_1$ .

(iv) If  $f''(x_1) = 0$ , find  $f'''(x_1)$ .

If  $f'''(x_1) \neq 0$ , there is neither

maximum nor minimum at  $x = x_1$ .

If  $f'''(x_1) = 0$ , find  $f^{(iv)}(x_1)$ .

If  $f^{(iv)}(x_1) < 0$ ,  $f(x)$  has a maximum at  $x = x_1$ .

If  $f^{(iv)}(x_1) > 0$ ,  $f(x)$  has a minimum at  $x = x_1$ , so on.

### \* Working Rule for finding Maxima and Minima :-

(Second Method by First Derivative Test)

(i) Denote the given function by  $f(x)$ .

(ii) Find  $f'(x)$  and equate it to zero.

Let its roots be  $x_1, x_2, x_3, \dots$

(iii) Test these values in succession.

Consider  $x = x_1$  (say).

If there is a neighbourhood

$x \in (x_1 - \delta, x_1 + \delta)$  such that

$f'(x) > 0$  for  $x_1 - \delta < x < x_1$  and  $f'(x) \leq 0$  for  $x_1 < x < x_1 + \delta$  then  $f$  has maximum at  $x_1$ .

$f'(x) \leq 0$  for  $x_1 - \delta < x < x_1$  and  $f'(x) \geq 0$

for  $x_1 < x < x_1 + \delta$  then  $f$  has minimum at  $x_1$ .

$f'(x) \leq 0$  ( $\geq 0$  only) for

$x_1 - \delta < x < x_1$  and  $x_1 < x < x_1 + \delta$  then

$f$  is neither maximum nor minimum at  $x_1$ .

(4) Similarly test all these values of  $x$  obtained in (3).

Problems :-

Examine the following function for extreme values.  $(x-3)^5 (x+1)^4$ .

Soln:- Let  $f(x) = (x-3)^5 (x+1)^4$

$$\begin{aligned} f'(x) &= (x-3)^5 + 4(x+1)^3 + (x+1)^4 \cdot 5(x-3)^4 \\ &= (x-3)^4 (x+1)^3 [(x-3) + 5(x+1)] \\ &= (x-3)^4 (x+1)^3 [4x - 12 + 5x + 5] \\ &= (x-3)^4 (x+1)^3 [9x - 7] \end{aligned}$$

for maximum or minimum  $f'(x)=0$

$$\Rightarrow (x-3)^4 (x+1)^3 (9x-7)=0$$

$$\Rightarrow [x=3, -1, \frac{7}{9}]$$

Second Method:-

Take  $x=3 \in (3-\delta, 3+\delta)$ ;  $\delta > 0$  for

$3-\delta < x < 3 \Rightarrow f'(x) > 0$  and for

$3 < x < 3+\delta \Rightarrow f'(x) > 0$ .

$\therefore f'(x) > 0$  for  $3-\delta < x < 3$  and  $3 < x < 3+\delta$ .

$\therefore f(x)$  is neither minimum nor maximum at  $x=3$ .

Take  $x=-1 \in (-1-\delta, -1+\delta)$ ,  $\delta > 0$

for  $-1-\delta < x < -1 \Rightarrow f'(x) > 0$  and for

$-1 < x < -1+\delta \Rightarrow f'(x) < 0$

$\therefore$  by first derivative test for

extrema,

$f(x)$  has maximum at  $x=-1$ .

$$\therefore f_{\max} = f(-1) = 0.$$

$$\text{Take } x=\frac{7}{9} \in \left(\frac{7}{9}-\delta, \frac{7}{9}+\delta\right)$$

for  $\frac{7}{9}-\delta < x < \frac{7}{9}$ :

$$\Rightarrow f'(x) < 0$$

for  $\frac{7}{9} < x < \frac{7}{9}+\delta \Rightarrow f'(x) > 0$ .

$\therefore f(x)$  has minimum at  $x=\frac{7}{9}$ .

(By first derivative test for extreme)-

$$\therefore f_{\min} = f\left(\frac{7}{9}\right) = \frac{-4 \cdot 3 \cdot 5^5}{3^{18}}.$$

H.W Examine for maxima and minima of the function defined by

$$f(x) = x^x (1-x)^3$$

show that  $\sin x (1+\cos x)$  is maximum when  $x=\pi/3$ .

First Method:-

Soln: Let  $f(x) = \sin x (1+\cos x)$

$$\text{then } f'(x) = \cos x (1+\cos x) + \sin x (-\sin x)$$

$$= \cos^2 x - \sin^2 x + \cos x$$

$$= \cos 2x + \cos x$$

$$\text{and } f''(x) = -2\sin 2x - \sin x$$

For maxima or minima  $f'(x)=0$

$$\Rightarrow \cos 2x + \cos x = 0$$

$$\Rightarrow 2\cos^2 \frac{x}{2} \cos \frac{x}{2} = 0$$

$$\Rightarrow \text{either } \frac{3x}{2} = \frac{\pi}{2} \text{ or } -\frac{x}{2} = \frac{\pi}{2}$$

$$\Rightarrow x = \frac{\pi}{3} \text{ (or) } x = \pi.$$

Here we consider only the point

$$x = \frac{\pi}{3}$$

$$\begin{aligned} f''\left(\frac{\pi}{3}\right) &= -2\sin\left(\frac{2\pi}{3}\right) - 3\sin\left(\frac{\pi}{3}\right) \\ &= -2\sin(120^\circ) - \sin 60^\circ \\ &= -2\sin(180^\circ - 60^\circ) - \sin 60^\circ \\ &= -2\sin 60^\circ - \sin 60^\circ \\ &= -3\sin 60^\circ \\ &= -3\sqrt{3}/2 < 0. \end{aligned}$$

$\therefore f(x)$  has a maximum at  $x = \frac{\pi}{3}$ .

$$\begin{aligned} \therefore f_{\max} &= f\left(\frac{\pi}{3}\right) = \sin\frac{\pi}{3} (1 + \cos\frac{\pi}{3}) \\ &= \frac{\sqrt{3}}{2} \left(1 + \frac{1}{2}\right) \\ &= \frac{\sqrt{3}}{2} \left(\frac{3}{2}\right) = \frac{3\sqrt{3}}{4}. \end{aligned}$$

Answer

→ Find the maximum and minimum values of any of the function

$$(1-x)^2 e^x.$$

$$\text{Sol'n: Let } f(x) = (1-x)^2 e^x$$

$$\begin{aligned} \text{then } f'(x) &= (1-x)^2 e^x - 2(1-x)e^x \\ &= [1+x^2-2x-2+2x]e^x \\ &= (x^2-1)e^x \end{aligned}$$

For maximum or minimum,

$$f'(x) = 0$$

$$\Rightarrow e^x(x^2-1) = 0$$

$$\Rightarrow x^2-1 = 0 \quad (e^x \neq 0)$$

$$\Rightarrow x = \pm 1$$

$$\begin{aligned} \text{when } x = 1 : \quad f''(x) &= e^x(x^2-1) + e^x(2x) \\ &= e^x[x^2+2x-1] \\ \therefore f''(1) &= e^1(1+2-1) \end{aligned}$$

$$= 2e > 0$$

$\therefore f$  is minimum at  $x = 1$ .

$$\therefore f_{\min} = f(1) = 0.$$

$$\text{When } x = -1 : \quad f''(x) = e^x(x^2+2x-1)$$

$$\therefore f''(-1) = e^{-1}(1-2-1)$$

$$= \frac{-2}{e} < 0$$

$\therefore f$  is maximum at  $x = -1$ .

$$\therefore f_{\max} = f(-1) = \frac{4}{e}.$$

→ Find the maximum value of  $\log x$ ,

$$0 < x < \infty.$$

$$\text{Sol'n: Let } f(x) = \frac{\log x}{x},$$

$$\begin{aligned} \text{then } f'(x) &= \frac{x\left(\frac{1}{x}\right) - \log x}{x^2} \\ &= \frac{1 - \log x}{x^2} \end{aligned}$$

$$\text{and } f''(x) = \frac{x^2(-\frac{1}{x^2}) - (1 - \log x) \cdot 2x}{x^4}$$

$$= \frac{-2 - 2x + 2x \log x}{x^4}$$

For maximum or minimum,

$$f'(x) = 0$$

$$\Rightarrow 1 - \log x = 0$$

$$\Rightarrow \log x = 1$$

$$\Rightarrow x = e$$

$$\text{when } x = e^{\frac{1}{2}}$$

$$-f''(e) = \frac{-e - 2e + 2e \log e}{e^4}$$

$$= \frac{-3e + 2e}{e^4}$$

$$= \frac{-e}{e^4}$$

$$= -\frac{1}{e^3} < 0.$$

$\therefore f$  is maximum at  $x=e$ .

$$\therefore f_{\max} = f(e) = \frac{\log e}{e} = \frac{1}{e}$$

→ Prove that the function  $(\frac{1}{x})^x$ ,  $x > 0$ , has a maximum at  $x=1/e$ .

Sol'n Let  $f(x) = (\frac{1}{x})^x$ ;  $x > 0$

$$\Rightarrow \log f(x) = x \log \frac{1}{x}$$

$$\Rightarrow \log f(x) = x[-\log x]$$

$$\Rightarrow \log f(x) = -x \log x$$

$$\Rightarrow \frac{1}{f(x)} f'(x) = -\left[\frac{x(1)}{x^2} + \log x\right]$$

$$\Rightarrow f'(x) = -f(x)[1 + \log x]$$

and  $f''(x) = -f(x)\frac{1}{x} - f'(x)[1 + \log x]$

For maximum (or) minimum  $f'(x)=0$

$$\Rightarrow -f(x)[\log x + 1] = 0$$

$$\Rightarrow 1 + \log x = 0 \quad (\because f(x) \neq 0)$$

$$\Rightarrow \log x = -1$$

$$\Rightarrow x = e^{-1}$$

Now when  $x = e^{-1}$ :

$$f''(e^{-1}) = -f(e^{-1})\frac{1}{e^{-1}} - f'(e^{-1})[1 + \log(e^{-1})]$$

$$= -(e)^{-1}\frac{1}{e^{-1}} - 0[1 + \log(e^{-1})]$$

$$= -e^{e^{-1}} \cdot e^1$$

$$= -(\bar{e})^{\bar{e}} \cdot e$$

$$< 0$$

$\therefore f$  is maximum at  $x=1/e$ .

$$\therefore f_{\max} = f\left(\frac{1}{e}\right) = \left(\frac{1}{e}\right)^{\frac{1}{e}} = (\bar{e})^{\bar{e}}$$

H.W. Prove that the function  $x^x$ ,  $x > 0$  has a minimum at  $x=1/e$ .

→ find the maximum and minimum values of the following functions:

$$\textcircled{1} \quad 2x^3 - 9x^2 + 24x - 20$$

$$\textcircled{2} \quad (x-1)(x-2)(x-3)$$

→ for each of the following functions on  $\mathbb{R} \rightarrow \mathbb{R}$ , find points of extrema, the intervals on which the function is increasing, and those on it is decreasing.

$$\text{i}, \quad f(x) = x^2 - 3x + 5$$

$$\text{ii}, \quad g(x) = 3x - 4x^2$$

$$\text{iii}, \quad h(x) = x^3 - 3x - 4$$

Sol'n i),  $f(x) = x^2 - 3x + 5$

$$f'(x) = 2x - 3$$

for maximum (or) minimum  $f'(x)=0$

$$x = 3/2$$

Now  $x = 3/2 \in (3/2 - \delta, 3/2 + \delta)$

For  $3/2 - \delta < x < 3/2 \Rightarrow f'(x) < 0$

and  $3/2 < x < 3/2 + \delta \Rightarrow f'(x) > 0$

∴ By first derivative test for extrema

$f(x)$  has minimum at  $x=3/2$ :

$$f_{\min} = f(3/2) = (3/2)^2 - 3(3/2) + 5$$

$$= 9/4 - 9/2 + 5$$

$$= \frac{9 - 18 + 20}{4}$$

$$= 11/4$$

$$\text{Now } f'(x) = 2x - 3 \quad \xrightarrow{x < \frac{3}{2}}$$

if  $x < \frac{3}{2} \Rightarrow f'(x) < 0.$

$\therefore f(x)$  is an decreasing in  $(-\infty, \frac{3}{2})$

if  $x > \frac{3}{2}$

$$\Rightarrow f'(x) > 0$$

$\therefore f(x)$  is an increasing in  $(\frac{3}{2}, \infty)$

$$h(x) = x^3 - 3x - 4$$

$$\text{Sofn: } h'(x) = 3x^2 - 3$$

for maximum or minimum  $h'(x) = 0$

$$\Rightarrow 3x^2 - 3 = 0$$

$$\Rightarrow x^2 - 1 = 0$$

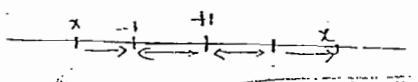
$$\Rightarrow x = \pm 1$$

At  $x=1$ :  $h(x)$  has minimum.

At  $x=-1$ :  $h(x)$  has maximum.

$$\text{Now } h'(x) = 3x^2 - 3$$

$$= 3(x-1)(x+1) \quad \text{--- (1)}$$



If  $x < -1$

$$\Rightarrow (x-1) < 0 ; (x+1) < 0$$

$$\therefore \text{--- (1)} \equiv h'(x) > 0$$

$\therefore h(x)$  is increasing in  $(-\infty, -1)$

If  $-1 < x < 1 \Rightarrow (x-1) < 0 ; (x+1) > 0$

$$\therefore \text{--- (1)} \equiv h'(x) < 0$$

$\therefore h(x)$  is decreasing in  $(-1, 1)$

If  $x > 1 \Rightarrow (x-1) > 0 ; (x+1) > 0$ .

$$\therefore \text{--- (1)} \equiv h'(x) > 0$$

$\therefore h(x)$  is increasing in  $(1, \infty)$ .

$\therefore$  In  $(-\infty, -1) \cup (1, \infty)$ ,

$h(x)$  is increasing.

and in  $(-1, 1)$ ,  $h(x)$  is decreasing.

## \* Hospital's Rule\*

### Ineterminate Forms:-

If  $A = \lim_{x \rightarrow c} f(x)$  and  $B = \lim_{x \rightarrow c} g(x)$ .

$$\text{and if } B \neq 0 \text{ then } \lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{A}{B}$$

However, if  $B=0$  then it has no conclusion.

If  $B=0$  and  $A \neq 0$  then the limit is infinite. (when it exists).

If  $A=0$  &  $B=0$  then  $\lim_{x \rightarrow c} \frac{f(x)}{g(x)}$  is

said to be indeterminate form.

Ex:- (1) If  $a$  is any real number

and if we define  $f(x) = ax$  and

$g(x) = x$  then

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow 0} \frac{ax}{x} \quad \left| \begin{array}{l} 0/0 \text{ form} \\ \end{array} \right. \\ &= \lim_{x \rightarrow 0} (a) \\ &= a \end{aligned}$$

Ex:- (2): If  $f(x) = x^2 - 1$  and  $g(x) = x - 1$

(with  $a=1$ )

then we have.

$$\begin{aligned} \lim_{x \rightarrow a} \frac{f(x)}{g(x)} &= \lim_{x \rightarrow 1} \frac{x^2 - 1}{x - 1} \quad \left| \begin{array}{l} 0/0 \text{ form} \\ \end{array} \right. \\ &= \lim_{x \rightarrow 1} (x+1) \\ &= 2. \end{aligned}$$

→ other indeterminate form are represented by the symbols  $\frac{\infty}{\infty}, 0 \cdot \infty, 0^0, 1^\infty, \infty^0$  and  $\infty - \infty$ .

→ our attention will be focused on the indeterminate forms  $\frac{0}{0}$  and  $\frac{\infty}{\infty}$ .

The other indeterminate cases are usually reduced to the form  $\frac{0}{0}$  or  $\frac{\infty}{\infty}$  by taking logarithms, exponentials, or algebraic manipulations.

\* We first establish an elementary result that is based simply on the definition of the derivative.

### Theorem:-

Let  $f$  and  $g$  be defined on  $[a, b]$  let  $f(a) = g(a) = 0$  and  $g'(x) \neq 0$  for  $x \in (a, b)$  (i.e.  $a < x < b$ ). If  $f$  and  $g$  are differentiable at  $a$  and if  $g'(a) \neq 0$ . then the limit of  $\frac{f}{g}$  at  $a$  exists and is equal to  $\frac{f'(a)}{g'(a)}$

$$\text{i.e. } \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

### \* Working Rule for finding the

Value of  $\lim_{x \rightarrow a} \frac{f(x)}{g(x)}$ :

where  $f(a) = g(a) = 0$ .

(1) Differentiate the numerator and

denominator separately.

(2) put  $x=a$  and remove the word limit.

3) If the indeterminate form  $\frac{0}{0}$  still persists, repeat the above process.

Problems :-

\* Evaluate the following limits:

$$\lim_{x \rightarrow 0} \frac{(1+x)^n - 1}{x}$$

soln:  $\lim_{x \rightarrow 0} \frac{(1+x)^n - 1}{x}$  |  $\frac{0}{0}$  form

$$= \lim_{x \rightarrow 0} \frac{n(1+x)^{n-1}}{1} \quad (\text{By differentiating numerator & denominator separately})$$

$$= n$$

Ques  $\lim_{x \rightarrow 0} \frac{x^2 - \log(1+x)}{x^2}$

Ques  $\lim_{x \rightarrow 0} \frac{x - \sin x}{x^3}$

Ques  $\lim_{x \rightarrow 0} \frac{\log(1-x^2)}{\log \cos x}$

Ques  $\lim_{x \rightarrow 1} \left( \frac{x^x - x}{1 - x + \log x} \right)$

Soln: Let  $u = x^x$  .  
then  $\log u = x \log x$ .

$$\Rightarrow \frac{1}{u} \frac{du}{dx} = x \left( \frac{1}{x} \right) + \log x$$

$$\Rightarrow \frac{du}{dx} = u \left( 1 + \log x \right) \\ = x^x (1 + \log x)$$

Now  $\lim_{x \rightarrow 1} \left( \frac{x^x - x}{1 - x + \log x} \right)$  | form  $\frac{0}{0}$

$$= \lim_{x \rightarrow 1} \frac{x^x (1 + \log x) - 1}{-1 + \frac{1}{x}} \quad \left| \begin{array}{l} \text{form } \frac{0}{0} \\ \text{put } x = 1 \end{array} \right.$$

$$= \lim_{x \rightarrow 1} \frac{x^x \left( \frac{1}{x} \right) + x^x (1 + \log x)(1 + \log x)}{-1} \\ = \frac{1' \left( \frac{1}{1} \right) + 1' (1 + \log_1)^2}{-1}$$

$$= \frac{1 + 1(1+0)}{-1} \\ = \frac{1+1}{-1} = -2$$

### \* L'Hospital's Rule - I :-

Let  $-\infty \leq a < b \leq \infty$  and let  $f, g$  be differentiable on  $(a, b)$  such that  $g'(x) \neq 0$   $\forall x \in (a, b)$

Suppose that

$$\lim_{x \rightarrow a^+} f(x) = 0 = \lim_{x \rightarrow a^+} g(x)$$

① If  $\lim_{x \rightarrow a^+} \frac{f'(x)}{g'(x)} = L \in \mathbb{R}$  then  $\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = L$

② If  $\lim_{x \rightarrow a^+} \frac{f'(x)}{g'(x)} = L \in \{-\infty, \infty\}$  then

$$\lim_{x \rightarrow a^+} \frac{f(x)}{g(x)} = L$$

Problems:

Evaluate the following limits.

(a)  $\lim_{x \rightarrow 0} \frac{e^x - 2\cos x + e^{-x}}{x \sin x}$

(b)  $\lim_{x \rightarrow 0} \frac{\tan x - x}{x^2 \tan x}$

(c)  $\lim_{x \rightarrow a} \frac{x^a - a^x}{x^2 - a^2}$

$$(d) \lim_{x \rightarrow 0} \frac{\cosh x - \cos x}{x \sin x}$$

$$\Rightarrow \frac{1}{u} \frac{du}{dx} = x^{\frac{1}{a}} + \log x$$

$$\Rightarrow \frac{du}{dx} = x^a (1 + \log x)$$

$$(e) \lim_{x \rightarrow 0} \frac{x \cos x - \log(1+x)}{x^2}$$

$$\lim_{x \rightarrow a} \left( \frac{x^a - a^a}{x^a - a^a} \right)$$

$$(f) \lim_{x \rightarrow 0} \frac{e^x \sin x - x - x^2}{x^2 + 2 \log(1-x)}$$

$$= \lim_{x \rightarrow a} \frac{a^{a-1} - a^a \log a}{x^a (1 + \log a) - 0}$$

$$\text{Sol'n (b): } \lim_{x \rightarrow 0} \frac{\tan x - x}{x^2 \tan x} \quad \left| \begin{array}{l} \frac{0}{0} \text{ form} \\ \end{array} \right.$$

$$= \frac{a^{a-1} - a^a \log a}{a^a (1 + \log a)}$$

$$= \lim_{x \rightarrow 0} \frac{\tan x - x}{x^3} \cdot \frac{dt}{x \rightarrow 0} \frac{x}{\tan x}$$

$$= \frac{a \cdot a^2 - \log a}{1 + \log a}$$

$$= \lim_{x \rightarrow 0} \frac{\tan x - x}{x^3} \quad (1)$$

$$= \frac{1 - \log a}{1 + \log a}$$

$$= \lim_{x \rightarrow 0} \frac{\tan x - x}{x^3} \quad \left| \begin{array}{l} \frac{0}{0} \text{ form} \\ \end{array} \right.$$

$$\text{→ what is wrong with the following application of L'Hospital's rule:}$$

$$\lim_{x \rightarrow 1} \frac{x^3 + 3x - 4}{2x^2 + x - 3} = \lim_{x \rightarrow 1} \frac{3x^2 + 3}{4x + 1} = \lim_{x \rightarrow 1} \frac{6x}{4} = 3/2$$

$$= \lim_{x \rightarrow 0} \frac{\sec^2 x - 1}{3x^2} \quad \left| \begin{array}{l} \frac{0}{0} \text{ form} \\ \end{array} \right.$$

$$\text{Sol'n: } \lim_{x \rightarrow 1} \frac{x^3 + 3x - 4}{2x^2 + x - 3} \quad \left| \begin{array}{l} \frac{0}{0} \text{ form} \\ \end{array} \right.$$

$$= \lim_{x \rightarrow 0} \frac{2 \sec x \sec x \tan x}{6x}$$

$$= \lim_{x \rightarrow 1} \frac{3x^2 + 3}{4x + 1}$$

$$= \lim_{x \rightarrow 0} \frac{2 \sec^2 x}{6} \cdot \lim_{x \rightarrow 0} \frac{\tan x}{x}$$

Now the expression  $\frac{3x^2 + 3}{4x + 1}$  is not of the form  $\frac{0}{0}$  as  $x \rightarrow 1$

$$= \frac{2(1)}{6} \cdot (1)$$

∴ It is not correct to apply

L'Hospital's Rule to evaluate  $\lim_{x \rightarrow 1} \frac{3x^2 + 3}{4x + 1}$

$$= \frac{1}{3}$$

$$\therefore \lim_{x \rightarrow 1} \frac{3x^2 + 3}{4x + 1} = \frac{3(1) + 3}{4(1) + 1} = 6/5$$

$$(g) \lim_{x \rightarrow a} \frac{x^a - a^a}{x^a - a^a}$$

→ what is wrong with the following use of L'Hospital's Rule:

$$\lim_{x \rightarrow 1} \frac{x^4 - 4x^3 + 3}{3x^2 - x - 2} = \lim_{x \rightarrow 1} \frac{4x^3 - 12x^2}{6x - 1}$$

$$= \lim_{x \rightarrow 0} \frac{12x^2 - 24x}{6}$$

$$= -2$$

$\rightarrow$  For what value of 'a' does  $\frac{\sin 2x + a \sin x}{x^2}$  tend to a finite

limit l as  $x \rightarrow 0$ ? When 'a' has this value, what is the value of l?

$$\text{Sol'n: } \lim_{x \rightarrow 0} \frac{\sin 2x + a \sin x}{x^3} \quad \left| \begin{array}{l} \text{form } \frac{0}{0} \\ \text{form } \frac{0}{0} \end{array} \right.$$

$$= \lim_{x \rightarrow 0} \frac{2\cos 2x + a \cos x}{3x^2} \quad \text{(i)}$$

The denominator (i)  $\rightarrow 0$  as  $x \rightarrow 0$  but (i)  $\rightarrow$  a finite limit l.

: The numerator  $(2\cos 2x + a \cos x)$  must be tend to zero as  $x \rightarrow 0$ .

$$\therefore 2\cos(0) + a\cos(0) = 0$$

$$\Rightarrow 2+a=0$$

$$\Rightarrow a = -2$$

With this value of a

$$(i) = \lim_{x \rightarrow 0} \frac{2\cos 2x - 2\cos x}{3x^2} \quad \left| \begin{array}{l} \text{form } \frac{0}{0} \\ \text{form } \frac{0}{0} \end{array} \right.$$

$$= \lim_{x \rightarrow 0} \frac{-4\sin 2x + 2\sin x}{6x} \quad \left| \begin{array}{l} \text{form } \frac{0}{0} \\ \text{form } \frac{0}{0} \end{array} \right.$$

$$= \lim_{x \rightarrow 0} \frac{-8\cos 2x + 2\cos x}{6} \quad \left| \begin{array}{l} \text{form } \frac{0}{0} \\ \text{form } \frac{0}{0} \end{array} \right.$$

$$= \frac{-8(1) + 2(1)}{6} = -6/6 = -1$$

$$\therefore l = -1$$

$\rightarrow$  Find the values of 'a' and 'b' in order that  $\lim_{x \rightarrow 0} \frac{x(1-a\cos x) + b\sin x}{x^3}$

may be equal to  $\frac{1}{3}$ :

$$\text{Sol'n: } \lim_{x \rightarrow 0} \frac{x(1-a\cos x) + b\sin x}{x^3} \quad \left| \begin{array}{l} \text{form } \frac{0}{0} \\ \text{form } \frac{0}{0} \end{array} \right.$$

$$= \lim_{x \rightarrow 0} \frac{x(\sin x) + (1-a\cos x) + b\cos x}{3x^2} \quad \text{(i)}$$

The denominators of (i)  $\rightarrow 0$  as  $x \rightarrow 0$  but (i)  $\rightarrow \frac{1}{3}$  as  $x \rightarrow 0$ .

: The numerator of (i).

$a(\sin x) + (1-a\cos x) + b\cos x$  tends to zero as  $x \rightarrow 0$ .

$$\Rightarrow 0(0) + (1-a(1)) + b(1) = 0$$

$$\Rightarrow 1-a+b=0 \quad \text{(2)}$$

If the relation (2) holds then from (i)

$$\lim_{x \rightarrow 0} \frac{a\sin x + (1-a\cos x) + b\cos x}{3x^2}$$

(is of the form  $\frac{0}{0}$ )

$$= \lim_{x \rightarrow 0} \frac{a\sin x + a\cos x + a\sin x - b\sin x}{6x} \quad \left| \begin{array}{l} \text{form } \frac{0}{0} \\ \text{form } \frac{0}{0} \end{array} \right.$$

$$= \lim_{x \rightarrow 0} \frac{a\cos x + a\cos x - a\sin x + a\sin x - b\cos x}{6}$$

$$= \frac{a(1) + a(1) - a(0) + a(1) - b(1)}{6}$$

$$= \frac{3a-b}{6}$$

but the limit of (i) equal to  $\frac{1}{3}$  (given)

$$\therefore \frac{3a-b}{6} = \frac{1}{3}$$

$$\Rightarrow 3a - b = 2 \quad \text{--- (3)}$$

From (2) & (3), we get

$$a = \frac{1}{2}, b = -\frac{1}{2}$$

H.W. Find the values of  $p$  and  $q$  for

which  $\lim_{x \rightarrow 0} \frac{x(1 + \cos x) - q \sin x}{x^3}$  exists

and equals 1.

2006 P-I find the values of  $a$  and  $b$

such that

$$\lim_{x \rightarrow 0} \frac{a \sin^2 x + b \log \cos x}{x^4} = \frac{1}{2}$$

$$\text{Sol'n: } \lim_{x \rightarrow 0} \frac{a \sin^2 x + b \log \cos x}{x^4} \quad | \text{ form } \frac{0}{0}$$

$$= \lim_{x \rightarrow 0} \frac{a(2 \sin x \cos x) + \frac{b}{\cos x}(-\sin x)}{4x^3}$$

$$= \lim_{x \rightarrow 0} \frac{a \sin 2x - b \tan x}{4x^3} \quad | \text{ form } \frac{0}{0}$$

$$= \lim_{x \rightarrow 0} \frac{2a \cos 2x - b \sec^2 x}{12x^2} \quad \text{--- (1)}$$

the denominator of (1)  $\rightarrow 0$  as  $x \rightarrow 0$ .

but (1)  $\rightarrow$  a finite limit value  $\frac{1}{2}$ .

$\therefore$  The numerator of (1) must be zero as  $x \rightarrow 0$ .

$$\therefore (1) \equiv 2a \cos(0) - b \sec^2(0) = 0$$

$$\Rightarrow 2a - b = 0 \quad \text{--- (2)}$$

with this form (1).

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$$\lim_{x \rightarrow 0} \frac{2a \cos 2x - b \sec^2 x}{12x^2} \quad | \text{ form } \frac{0}{0}$$

$$= \lim_{x \rightarrow 0} \frac{-4a \sin 2x - b[2 \sec^2 x \tan x]}{24x} \quad | \text{ form } \frac{0}{0}$$

$$= \lim_{x \rightarrow 0} \left[ \frac{-4a \sin x}{24x} - \frac{2b \sec^2 x \tan x}{24x} \right]$$

$$= \frac{-a}{3} \lim_{x \rightarrow 0} \frac{\sin x}{2x} + \frac{b}{12} \lim_{x \rightarrow 0} \sec^2 x \lim_{x \rightarrow 0} \frac{\tan x}{x}$$

$$= \frac{-a}{3}(1) + \frac{b}{12}(1)(1)$$

$$= \frac{-a}{3} + \frac{b}{12} = \frac{-4a+b}{12}$$

but limit of (1) is equal to  $\frac{1}{2}$

$$\therefore \frac{-4a+b}{12} = \frac{1}{2}$$

$$\Rightarrow -4a + b = 6 \quad \text{--- (3)}$$

$$(2) - (3) \Rightarrow 6a = 6$$

$$\Rightarrow [a = 1]$$

$$(2) \equiv 2(1) - b = 0$$

$$\Rightarrow 2 - b$$

$$\Rightarrow [b = 2]$$

$$\therefore a = 1, b = 2$$

### \* Hospital's Rule - 2:

Let  $-\infty \leq a < b \leq \infty$  and let

f.g be differentiable on  $(a, b)$ .

such that  $g'(x) \neq 0 \forall x \in (a, b)$

Suppose that  $\lim_{x \rightarrow a^+} g(x) = \pm \infty$ .

④ If  $\lim_{x \rightarrow a+} \frac{f(x)}{g(x)} = L \in \mathbb{R}$  then

$$\lim_{x \rightarrow a+} \frac{f(x)}{g(x)} = L$$

⑤ If  $\lim_{x \rightarrow a+} \frac{f(x)}{g(x)} = L \in \{-\infty, \infty\}$ , then

$$\lim_{x \rightarrow a+} \frac{f(x)}{g(x)} = L$$

Note! In most of the Problems of the form  $\frac{\infty}{\infty}$ , it is necessary to change it into the form  $\frac{0}{0}$  at the proper stage, otherwise the process will never end.

### Problems:

Evaluate the following limits:

①  $\lim_{x \rightarrow 0} \frac{\log x}{\cot x^2}$

Sol'n:

$$= \lim_{x \rightarrow 0} \frac{2 \log x}{\cot x^2} \quad \left| \begin{array}{l} \text{form } \frac{\infty}{\infty} \\ \because \log 0 = -\infty \\ \& \cot 0 = \infty \end{array} \right.$$

$$= \lim_{x \rightarrow 0} \frac{2 \cdot \frac{1}{x}}{(-\csc^2 x^2) (2x)}$$

$$= \lim_{x \rightarrow 0} \frac{-2}{2x^2 (\csc^2 x^2)}$$

$$= \lim_{x \rightarrow 0} \frac{\left( \frac{\sin^2 x^2}{x^2} \right)}{2} \quad \left| \begin{array}{l} \text{0/0 form} \\ \frac{\sin x^2}{x^2} \rightarrow 1 \end{array} \right.$$

$$= \lim_{x \rightarrow 0} \frac{-2 \sin^2 x^2 \cdot \cos x^2 \cdot (2x)}{2x}$$

$$= \lim_{x \rightarrow 0} -\sin(2x^2)$$

$$= 0$$

$$\lim_{x \rightarrow \pi/2} \frac{\log(\theta - \pi/2)}{\tan \theta}$$

$$\lim_{x \rightarrow 0} \frac{\cot x}{\log x}$$

$$\lim_{x \rightarrow \pi/2} \frac{\tan 5x}{\tan x}$$

$$\lim_{x \rightarrow 0+} \frac{\log(\tan x)}{\log x}$$

$$\text{Sol'n: } \lim_{x \rightarrow 0+} \frac{\log(\tan x)}{\log x} \quad \left| \begin{array}{l} \text{0/0 form} \\ \tan x \rightarrow 0 \end{array} \right.$$

$$= \lim_{x \rightarrow 0+} \left[ \frac{1}{\tan x} \sec^2 x \right]_{y_x}$$

$$= \lim_{x \rightarrow 0+} \frac{x}{\sin x \cos x}$$

$$= \lim_{x \rightarrow 0+} \frac{2x}{\sin 2x} \quad \left| \begin{array}{l} \text{0/0 form} \\ \sin 2x \rightarrow 0 \end{array} \right.$$

$$= \lim_{x \rightarrow 0+} \frac{2}{2 \cos 2x}$$

$$= \frac{2}{2(1)} = 1$$

### \* Other Indeterminate forms:

The indeterminate forms  $\infty - \infty$ ,

$0 \times \infty$ ,  $\infty^0$ ,  $0^0$ ,  $\infty^\infty$  can be

reduced to any one of the two indeterminate forms  $\frac{0}{0}$  or  $\frac{\infty}{\infty}$

by algebraic manipulations and the exponential functions.

This is illustrated by the

following examples.

Form  $\infty - \infty$

$$\rightarrow \lim_{x \rightarrow 0^+} \left( \frac{1}{x} - \frac{1}{\sin x} \right)$$

$$\text{Sol'n: } \lim_{x \rightarrow 0^+} \left( \frac{1}{x} - \frac{1}{\sin x} \right) \quad | \begin{array}{l} \infty - \infty \\ \text{form} \end{array}$$

$$= \lim_{x \rightarrow 0^+} \frac{\sin x - x}{x \sin x} \quad | \begin{array}{l} \text{form } \frac{0}{0} \\ 0 \end{array}$$

$$= \lim_{x \rightarrow 0^+} \frac{\cos x - 1}{x \cos x + \sin x} \quad | \begin{array}{l} \text{form } \frac{0}{0} \\ 0 \end{array}$$

$$= \lim_{x \rightarrow 1^+} \frac{\sin x}{-x \sin x + 2 \cos x}$$

$$= \frac{0}{2} = 0$$

$$\text{H.W. } \lim_{x \rightarrow 0} \left( \frac{1}{x^2} - \frac{1}{x \tan x} \right)$$

$$\text{H.W. } \lim_{x \rightarrow 0} \left( \frac{1}{x} - \cot x \right)$$

$$\text{H.W. } \lim_{x \rightarrow \pi/2} \left( \sec x - \tan x \right)$$

$$\text{H.W. } \lim_{x \rightarrow 0} \left[ \frac{1}{x} - \frac{1}{x^2} \log(1+x) \right]$$

$$\rightarrow \lim_{x \rightarrow 0} \left[ \frac{\pi}{4x} - \frac{\pi}{2x(e^{\pi x} + 1)} \right]$$

$$\rightarrow \lim_{x \rightarrow 1} \left[ \frac{2}{x^2 - 1} - \frac{1}{x-1} \right]$$

$$\rightarrow \lim_{x \rightarrow 4} \left[ \frac{1}{\log(x-3)} - \frac{1}{x-4} \right]$$

$$\rightarrow \lim_{x \rightarrow 0} \left[ \frac{1}{e^x - 1} - \frac{1}{x} \right]$$

$$\rightarrow \lim_{x \rightarrow 0} \left[ \cot^2 x - \frac{1}{x^2} \right]$$

$$\text{Sol'n: } \lim_{x \rightarrow 0} \left( \cot^2 x - \frac{1}{x^2} \right) \quad | \begin{array}{l} \infty - \infty \\ \text{form} \end{array}$$

$$= \lim_{x \rightarrow 0} \left( \frac{\cot^2 x}{\sin^2 x} - \frac{1}{x^2} \right)$$

$$= \lim_{x \rightarrow 0} \frac{x^2 \cos^2 x - \sin^2 x}{x^2 \sin^2 x}$$

$$= \lim_{x \rightarrow 0} \frac{x^2 \cos^2 x - \sin^2 x}{x^4} \cdot \frac{x^2}{\sin^2 x}$$

$$= \lim_{x \rightarrow 0} \frac{x^2 \cos^2 x - \sin^2 x}{x^4} \left( \lim_{x \rightarrow 0} \frac{x}{\sin x} \right)^2$$

$$= \lim_{x \rightarrow 0} \frac{x^2 \cos^2 x - \sin^2 x}{x^4} \cdot (1)^2$$

$$= \lim_{x \rightarrow 0} \frac{x^2 \left( \frac{1 + \cos 2x}{2} \right) - \left( \frac{1 - \cos 2x}{2} \right)}{x^4}$$

$$= \lim_{x \rightarrow 0} \frac{x^2 (1 + \cos 2x) - (1 - \cos 2x)}{2x^4}$$

$$= \lim_{x \rightarrow 0} \frac{(x^2 - 1) + (x^2 + 1)(\cos 2x)}{2x^4} \quad | \begin{array}{l} \frac{0}{0} \\ \text{form} \end{array}$$

$$\lim_{x \rightarrow 0} \frac{2x - (x^2 + 1) \left( \frac{3 \ln 2x}{2} \right) + 2x \cos 2x}{8x^3}$$

$$\lim_{x \rightarrow 0} \frac{2 + 2 \cos 2x - 4x \sin 2x - 4x \ln 2x - 4(x^2 + 1) \cos 2x}{8x^3}$$

$\frac{0}{0}$  form

$$\lim_{x \rightarrow 0} \frac{-8 \sin 2x - 16x \cos 2x - 8x \cos 2x + 2(4x^2 + 2) \sin 2x}{48x}$$

$$\lim_{x \rightarrow 0} \frac{-24x \cos 2x + (8x^2 - 4) \sin 2x}{48x} \quad \text{from } \frac{0}{0}$$

$$\lim_{x \rightarrow 0} \frac{-24 \cos 2x + 48x \sin 2x + 16x \sin 2x + 2(8x^2 - 4) \cos 2x}{48}$$

$$= \frac{-24 + 0 + 0 - 8}{48} = \frac{-32}{48} = -\frac{2}{3}$$

Form  $0 \times \infty$ :

Evaluate the following limits:

$$\lim_{x \rightarrow 0+} x \ln x$$

$$\text{Soln: } \lim_{x \rightarrow 0+} x \ln x \quad \text{from } \frac{0}{\infty} \text{ form}$$

$$= \lim_{x \rightarrow 0+} \frac{\log x}{\frac{1}{x}}$$

$$= \lim_{x \rightarrow 0+} \frac{\frac{1}{x}}{-\frac{1}{x^2}}$$

$$= \lim_{x \rightarrow 0+} \frac{-x^2}{x}$$

$$= \lim_{x \rightarrow 0+} (-x)$$

$$= 0$$

$$\lim_{x \rightarrow 0} x^3 \ln x$$

$$\lim_{x \rightarrow 0} \frac{1}{x(\ln x)^2}$$

$$\text{Soln: } \lim_{x \rightarrow 0} \frac{1}{x(\ln x)^2} \quad \text{from } \frac{0}{0}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{1}{x}}{(\ln x)^2} \quad \text{from } \frac{\infty}{\infty}$$

$$= \lim_{x \rightarrow 0} \frac{-\frac{1}{x^2}}{2(\ln x)^{\frac{1}{x}}}$$

$$= \lim_{x \rightarrow 0} \frac{-\frac{1}{x}}{2 \ln x} \quad \text{from } \frac{\infty}{\infty} \text{ form}$$

$$= \lim_{x \rightarrow 0} \frac{\frac{1}{x^2}}{2(\ln x)}$$

$$= \lim_{x \rightarrow 0} \frac{1}{2x}$$

$$= \infty$$

$$\lim_{x \rightarrow \infty} \frac{x^3}{e^x} = l.$$

Forms:  $0^0, 1^\infty, \infty^\infty$

$$\lim_{x \rightarrow 0+} x^x = ?$$

$$\text{Soln: } \lim_{x \rightarrow 0+} x^x = e \quad \text{from } 0^0 \text{ form}$$

$$\Rightarrow \log \left[ \lim_{x \rightarrow 0+} x^x \right] = \log e$$

$$\Rightarrow \lim_{x \rightarrow 0+} [\log(x^x)] = \log e$$

$$\Rightarrow \log e = \lim_{x \rightarrow 0+} [x \ln x] \quad \text{from } 0 \times \infty \text{ form}$$

$$= \lim_{x \rightarrow 0+} \frac{\log x}{\frac{1}{x}} \quad \text{from } \frac{\infty}{\infty}$$

$$= \lim_{x \rightarrow 0^+} \frac{1/x}{-1/x^2}$$

$$= \lim_{x \rightarrow 0^+} \left( \frac{-x^2}{x} \right)$$

$$= \lim_{x \rightarrow 0^+} (-x) = 0$$

$$\therefore \log l = 0$$

$$\Rightarrow l = e^0$$

$$\Rightarrow \boxed{l=1}$$

$$\rightarrow \lim_{x \rightarrow 0^+} (1 + 1/x)^x = ?$$

$$\underline{\text{Sol'n:}} \quad \lim_{x \rightarrow 0^+} (1 + 1/x)^x \quad | \infty^0 \text{ form}$$

$$\text{Let } l = \lim_{x \rightarrow 0^+} (1 + 1/x)^x$$

$$\Rightarrow \log l = \lim_{x \rightarrow 0^+} [\log(1 + 1/x)^x]$$

$$= \lim_{x \rightarrow 0^+} [x \log(1 + 1/x)]$$

|  $\infty \times \infty$   
- form

$$= \lim_{x \rightarrow 0^+} \left[ \frac{\log(1 + 1/x)}{1/x} \right]$$

$$= \lim_{x \rightarrow 0^+} \left[ \frac{\frac{1}{1+1/x} \cdot -\frac{1}{x^2}}{-1/x^2} \right]$$

$$= \left( \frac{1}{1 + \frac{1}{0}} \right)$$

$$= 0$$

$$\rightarrow (a) \lim_{x \rightarrow \infty} \frac{\ln x}{x^2} \quad (0, \infty)$$

$$(b) \lim_{x \rightarrow \infty} \frac{\ln x}{\sqrt{x}} \quad (0, \infty)$$

$$(c) \lim_{x \rightarrow 0} x \ln \sin x \quad (0, \pi)$$

$$(d) \lim_{x \rightarrow \infty} \frac{x + \ln x}{x \ln x} \quad (0, \infty)$$

$$\rightarrow (e) \lim_{x \rightarrow 0^+} x^{2/x} \quad (0, \infty)$$

$$(f) \lim_{x \rightarrow 0} (1 + 3/x)^x; \quad (0, \infty)$$

$$(g) \lim_{x \rightarrow \infty} (1 + 3/x)^x; \quad (0, \infty)$$

$$= \lim_{x \rightarrow \infty} \frac{x}{1+x} \quad | \infty / \infty \text{ form}$$

$$= \lim_{x \rightarrow \infty} \frac{1}{1/x + 1} = 1$$

$$\therefore \log l = 1$$

$$\Rightarrow \boxed{1 \cdot e^1}$$

$\rightarrow$  (a)  $\lim_{x \rightarrow \infty} x^{\sqrt{x}}$ ;  $(0, \infty)$

(b)  $\lim_{x \rightarrow 0+} (\sin x)^x$ ;  $(0, \pi)$

(c)  $\lim_{x \rightarrow 0+} x^{\sin x}$ ;  $(0, \infty)$

(d)  $\lim_{x \rightarrow \frac{\pi}{2}-} (\sec x - \tan x)$ ;  $(0, \pi/2)$

