ANALYTIC GEOMETRY

CSE - 2011

(1)(e) Find the equations of the straight line through the point (3,1,2) to intersect the straight line $x+4 = y+1 = 2(z-2)$ and parallel to the plane $4x+y+5z=0$.

→ Any line through (3,1,2) is $\frac{x-3}{l} = \frac{y-1}{m} = \frac{z-2}{n}$ — (1)

It intersects the line $\frac{x+4}{2} = \frac{y+1}{2} = \frac{z-2}{1}$ — (2)

The cond^n for intersection is $\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$

$$\begin{vmatrix} -4-3 & -1-1 & 2-2 \\ l & m & n \\ 2 & 2 & 1 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} -7 & -2 & 0 \\ l & m & n \\ 2 & 2 & 1 \end{vmatrix} = 0 \Rightarrow -7[m-2n] - 2[2n-l] = 0$$

$$\Rightarrow 7m - 14n + 4n - 2l = 0$$

$$\Rightarrow 2l - 7m + 10n = 0 \quad \text{— (3)}$$

The line (1) is parallel to the plane $4x+y+5z=0$ — (4)

⇒ The line (1) is ⊥ar to normal to the plane (4) whose drs are 4, 1, 5.

$\therefore 4l + m + 5n = 0$ — (5) from (3) & (5): $\frac{l}{-3} = \frac{m}{2} = \frac{n}{2}$.

∴ Reqd eq^n of line is (1): $\frac{x-3}{-3} = \frac{y-1}{2} = \frac{z-2}{2}$.

(1)(f) Show that the equation of sphere which touches the sphere $4(x^2+y^2+z^2) + 10x - 25y - 2z = 0$ at the point (1,2,-2) and passes through the point (-1,0,0) is $x^2+y^2+z^2 + 2x - 6y + 1 = 0$

→ The given equation of sphere is $4(x^2+y^2+z^2) + 10x - 25y - 2z = 0$ — (1)

Tangent plane to sphere (1) at (1,2,-2) is

$$4(1\cdot x + 2\cdot y + (-2)\cdot z) + 5(x+1) - \frac{25}{2}(y+2) - (z-2) = 0$$

$$\Rightarrow 9x - \frac{9}{2}y - 9z - 18 = 0$$

$$\Rightarrow 2x - y - 2z - 4 = 0 \quad \text{— (2)}$$

Any sphere through the circle of intersection of sphere ① & plane ② is given by

$$4(x^2+y^2+z^2)+10x-25y-2z+\lambda(2x-y-2z-4)=0$$

$$\Rightarrow 4(x^2+y^2+z^2)+(10+2\lambda)x+(-25-\lambda)y+(-2-2\lambda)z-4\lambda=0 \quad \text{— ③}$$

It passes through $(-1,0,0)$.

$$\therefore 4[(-1)^2+0^2+0^2]-(10+2\lambda)+0+0-4\lambda=0$$

$$\Rightarrow 4-10=6\lambda \Rightarrow \lambda=-1$$

$$\therefore ③ \equiv 4(x^2+y^2+z^2)+(10-2)x+(-25+1)y+(-2+2)z+4=0$$

$$\Rightarrow \boxed{x^2+y^2+z^2+2x-6y+1=0}$$

④(a) Three points P, Q, R are taken on the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ so that the lines joining P, Q, R to the origin are mutually perpendicular. Prove that the plane PQR touches a fixed sphere.

→ Let the equation of plane PQR be $lx+my+nz=1$ — ①

The equation of a cone with vertex at origin and ~~passing~~ the ellipsoid as guiding curve along with intersection of plane ① is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=(lx+my+nz)^2.$$

It has 3 mutually ⊥ar generators OP, OQ & OR.

$\therefore$ coeff of x^2 + coeff of y^2 + coeff of z^2 = 0

$$\therefore l^2+m^2+n^2-\frac{1}{a^2}-\frac{1}{b^2}-\frac{1}{c^2}=0$$

$$\rightarrow l^2+m^2+n^2=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{\lambda^2}\ (\text{say}) \quad \text{— ②}$$

The distance of plane ① from origin is $\frac{1}{\sqrt{l^2+m^2+n^2}}$

If the plane ① touches the sphere $x^2+y^2+z^2=\lambda^2$, then, the length of perpendicular from the centre $(0,0,0)$ to the plane ① must be equal to the radius λ of the sphere.

$$\therefore \lambda=\frac{1}{\sqrt{l^2+m^2+n^2}} \Rightarrow l^2+m^2+n^2=\frac{1}{\lambda^2}$$ which is same as condn ②

$\therefore$ The plane ① touches the sphere $x^2+y^2+z^2=\lambda^2$.

④ (b) Show that the cone $yz+zx+xy=0$ cuts the sphere $x^2+y^2+z^2=a^2$ in two equal circles & find their area.

$\rightarrow$ $x^2+y^2+z^2=a^2$ & $xy+yz+zx=0$

$\therefore x^2+y^2+z^2+2(xy+yz+zx)=a^2$

$\Rightarrow (x+y+z)^2 = a^2 \Rightarrow x+y+z=\pm a$.

$\therefore$ The cone $xy+yz+zx=0$ cuts the sphere $x^2+y^2+z^2=a^2$ into two circles $x^2+y^2+z^2=a^2$, $x+y+z=a$ and $x^2+y^2+z^2=a^2$, $x+y+z=-a$ which are evidently equal

Distance of the plane $x+y+z=\pm a$ from the origin is $d=\dfrac{|\pm a|}{\sqrt{1^2+1^2+1^2}} = \dfrac{a}{\sqrt{3}}$

radius of given sphere is $r=a$.

Then radius of circles ~~are~~ is $R^2=r^2-d^2=a^2-\dfrac{a^2}{3}$

$\Rightarrow R=\sqrt{\dfrac{2}{3}}\,a$

$\therefore$ Area of these circles is $= \pi R^2 = \dfrac{2\pi a^2}{3}$

④ (c) Show that the generators through any of the ends of an equiconjugate diameter of the principal elliptic section of the hyperboloid $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}-\dfrac{z^2}{c^2}=1$ are inclined to each other at an angle $60°$ if $a^2+b^2=6c^2$. Find also the condition for the generators to be $\perp$ar to each other.

$\rightarrow$ ~~The end points of equiconjugate diameters can be taken as $(a\cos\theta, b\sin\theta, 0)$, $(-a\sin\theta, b\cos\theta, 0)$.~~

The ~~end~~ points of diameter can be taken as $(a\cos\theta, b\sin\theta, 0)$

Then eqn of generators through the diameter is

$$\frac{x-a\cos\theta}{a\sin\theta} = \frac{y-b\sin\theta}{-b\cos\theta} = \frac{z}{\pm c}$$

Angle between these generators be α. Then,

$$\cos\alpha = \frac{a\sin\theta \cdot a\sin\theta + (-b\cos\theta)(+b\cos\theta) + c\cdot(-c)}{\sqrt{a^2\sin^2\theta + b^2\cos^2\theta + c^2}\sqrt{a^2\sin^2\theta + b^2\cos^2\theta + c^2}}$$

$$\Rightarrow \cos\alpha = \frac{a^2\sin^2\theta + b^2\cos^2\theta - c^2}{a^2\sin\theta + b^2\cos^2\theta + c^2} \quad \text{— (1)}$$

Here, $\theta = 45°$ since equiconjugate diameters means equal length of conjugate diameters i.e. $\sqrt{a^2\cos^2\theta + b^2\sin^2\theta} = \sqrt{a^2\sin^2\theta + b^2\cos^2\theta}$.

$$\therefore \cos\alpha = \frac{a^2\sin^2 45° + b^2\cos^2 45° - c^2}{a^2\sin^2 45° + b^2\cos^2 45° + c^2}$$

$$\Rightarrow \cos\alpha = \frac{a^2+b^2-2c^2}{a^2+b^2+2c^2}$$

<u>If $\alpha = 60°$</u>. Then, $\frac{1}{2} = \frac{a^2+b^2-2c^2}{a^2+b^2+2c^2}$

$$\Rightarrow \boxed{a^2+b^2 = 6c^2}$$

For the generators to be $\perp$ar, $\alpha = 90°$.

$$\therefore \cos 90 = \frac{a^2+b^2-2c^2}{a^2+b^2+2c^2} \Rightarrow \frac{a^2+b^2-2c^2}{a^2+b^2+2c^2} = 0$$

$$\Rightarrow \boxed{a^2+b^2 = 2c^2}$$