# Mains Test Series - 2019

## Test number - 12

### Section - A

Ques: 1. a) Find whether the following statements are true or false. Justify your answer.

(i) $Z \times Z$ is a cyclic group.

(ii) $\{a+b\sqrt{2} \in \mathbb{R} \mid a, b$ are rational numbers$\}$ is a cyclic group under usual addition of real numbers.

(iii) $G = \{(1,1), (1,-1), (-1,1), (-1,1)\}$ is a group under the operation $(a,b)(c,d)(ac, bd)$ but not a cyclic group.

(iv) The number of elements of the subgroup $\langle a^{10} \rangle$ in the cyclic group $\langle a \rangle$ of order 30 is 10.

(v) The symmetric group $S_n$ contains a cyclic group of order 'n'.

Solution :-

(i) False :

Suppose, it we cyclic, with generator $(a,b)$ where $a, b \in Z$. Then every element of $Z \times Z$ would be of the form $(na, nb)$ for some $n \in Z$, which is not possible.

ii) False:

Suppose it were cyclic with generator $(m+n\sqrt{2})$ where $m, n \in \mathbb{Q}$, then every element of the set $\{a+b\sqrt{2} \in \mathbb{R} \mid a, b \in \mathbb{Q}\}$ would be of the form $(pm+pn\sqrt{2})$ for some $p \in \mathbb{Q}$ which is not possible.

iii) True:

Consider, composition table for G

| $\cdot$ | (1,1) | (1,-1) | (-1,1) | (-1,-1) |
|---|---|---|---|---|
| (1,1) | (1,1) | (1,-1) | (-1,1) | (-1,-1) |
| (1,-1) | (1,-1) | (1,1) | (-1,-1) | (-1,1) |
| (-1,1) | (-1,1) | (-1,-1) | (1,1) | (1,-1) |
| (-1,-1) | (-1,-1) | (-1,1) | (1,-1) | (1,1) |

clearly, we observe that G is a group. Also, since all elements have order 2 [i.e. no element has order $4 = O(G)$]

Therefore, G is not a cyclic group.

(iv) False:

Since, $O(\langle a^{10} \rangle) = \dfrac{O(a)}{(O(a),10)} = \dfrac{30}{(30,10)} = 3.$

i.e; number of elements of the subgroup $\langle a^{10} \rangle$ in the cyclic group $\langle a \rangle$ of order 30 is 3.

(v) True:

A symmetric group $S_n$ contains a cycle of length 'n'. Also order of a cycle of length $n$ is $n$. Hence, cyclic group of this cycle as a generator has order 'n'.

Ques: 1(b) Let R be commutative ring. If $a, b, \in R$ are nilpotent then show that so are $a \pm b$ and $ar$ for any $r \in \mathbb{R}$. Give an example of a non-commutative ring in which $a, b$ are nilpotent but $a+b$ is not.

Solution:

If a and b are nilpotent in a commutative ring R,

then $a^m = b^n = 0$ for some $n, m \in \mathbb{N}$

(a) Now,

$$(a \pm b)^{m+n} = a^{m+n} \pm {}^{m+n}C_1 a^{m+n-1} b^1 \pm {}^{m+n}C_2 a^{m+n-2} b^2 \pm \ldots \pm {}^{m+n}C_n a^m b^n \pm \ldots \pm b^{m+n}$$

$$= a^m \cdot a^n \pm {}^{m+n}C_1 a^m \cdot a^{n-1} \cdot b^1 \pm {}^{m+n}C_2 a^m b^2 a^{n-2} \pm \ldots \pm {}^{m+n}C_n a^m \cdot b^n + \ldots \pm b^n \cdot b^m$$

[$\because$ R is commutative]

$$= 0. \quad [\because a^m = b^n = 0]$$

Hence, $a \pm b$ are nilpotent.

(b) $(ar)^m = (ar)(ar)(ar)\ldots\ldots(ar)$ (m-times)

$= a^m r^m$ [$\because R$ is commutative]

$= 0$ [$\because a^m = 0$]

Hence, $ar$ is nilpotent, where $a \in R$ is nilpotent and $r \in R$ (arbitrary).

(c) Consider a Ring $R = M_2(\mathbb{C})$

Let $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Clearly, $A^2 = B^2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

$\Rightarrow$ A and B are nilpotent elements of R.

Now; $A + B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

which is not a nilpotent element of R as there does not exist any positive integer 'n' such that $(A+B)^n = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Also, $M_2(\mathbb{C})$ is a non-commutative ring.

Since, $A \cdot B \neq B \cdot A$.

Hence the result

Ques:- 1(c): Test the convergence of the series:-

$$1+\frac{x^2}{2}+\frac{x^4}{4}+\frac{x^6}{6}+\cdots\cdots, \; x>0$$

Solution:-

Given series $-\; 1+\frac{x^2}{2}+\frac{x^4}{4}+\frac{x^6}{6}+\cdots\cdots \;;\; x>0$

Leaving the first term

$$u_n = \frac{x^{2n}}{2n}$$

$$\therefore \quad u_{n+1} = \frac{x^{2n+2}}{2n+2}$$

$$\frac{u_n}{u_{n+1}} = \frac{2n+2}{2n}\cdot\frac{1}{x^2} = \left(1+\frac{1}{n}\right)\cdot\frac{1}{x^2}$$

$$\operatorname*{Lt}_{n\to\infty}\frac{u_n}{u_{n+1}} = \operatorname*{lt}_{n\to\infty}\left(1+\frac{1}{n}\right)\cdot\frac{1}{x^2} = \frac{1}{x^2}$$

$\therefore$ By Ratio test, the series is convergent if

$\frac{1}{x^2}>1$ ; i.e $x^2<1$ i.e if $x<1$

The series is divergent if $\frac{1}{x^2}<1$ i.e $x^2>1$

i.e. $x>1$.

When ; $x^2=1$ i.e $x=1$ ; the Ratio test fails.

However, when $x=1$, the series becomes

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\cdots\cdots$$

Leaving first term

$$\boxed{u_n = \frac{1}{2n}}$$

Taking $v_n = \frac{1}{n}$

$$\Rightarrow \frac{u_n}{v_n} = \frac{1/2n}{1/n} = \frac{1}{2}.$$

$$\Rightarrow \lim_{n\to\infty} \frac{u_n}{v_n} = \frac{1}{2}$$ ; which is finite and non-zero.

$\therefore$ By comparision Test, the series

$\Sigma u_n$ and $\Sigma v_n$ converges or diverges together.

But, the series $\Sigma v_n = \Sigma \frac{1}{n}$ diverges (by p test

as $p = 1$

$\therefore$ $\Sigma u_n$ diverges.

Hence; $\Sigma u_n$ converges when $x < 1$ and diverges when $x \geq 1$.

Ques: 1) d) If $f(z) = u + iv$ is analytic function and $u - v = e^x.(\cos y - \sin y)$, find $f(z)$ in terms of $z$.

Solution:- we have;

$$u - v = e^x.(\cos y - \sin y).$$

and $f(z) = u + iv$

Then; $\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = e^x(\cos y - \sin y)$ —— ①

and $\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = e^x(-\sin y - \cos y)$

$\Rightarrow \frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = -e^x(\sin y + \cos y)$

or $-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} = -e^x(\sin y + \cos y)$

[By Cauchy-Riemann Equations]

or $\frac{\partial v}{\partial x} + \frac{\partial u}{\partial x} = e^x(\sin y + \cos y)$ —— ②

from ① and ②; we get

$\frac{\partial u}{\partial x} = e^x \cos y = \phi_1(x, y).$

$\frac{\partial v}{\partial x} = e^x \sin y = \phi_2(x, y)$

Now; $f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \phi_1(z, 0) + i\phi_2(z, 0)$

$\therefore \quad f(z) = \int [\phi_1(z, 0) + i\phi_2(z, 0)]dz + c$

$f(z) = \int [e^z.\cos 0 + ie^z.\sin 0]dz + c$

$f(z) = \int e^z dz + c \Rightarrow \boxed{f(z) = e^z + c}$

Alternative Solution:-

We have as above

$$\frac{\partial v}{\partial x} = e^x \sin y \quad \text{——— (3)}$$

and $$\frac{\partial u}{\partial x} = e^x \cos y \quad \text{——— (4)}$$

Integrating (3), we get

$$v = e^x \sin y + f(y)$$

Differentiating it w.r.t 'y'

$$\frac{\partial v}{\partial y} = e^x \cos y + f'(y)$$

Also (4) gives. $e^x \cos y = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

Then (5) and (6) give

$$f'(y) = 0 \quad \text{and} \quad f(y) = c$$

$$\therefore \; v = e^x \sin y + c$$

and $u = v + e^x(\cos y - \sin y) = e^x \cos y + c$

$$\therefore \; f(z) = e^x \cos y + c + i(e^x \sin y + c)$$

$$= e^x(\cos y + i \sin y) + c + ic$$

$$= e^x \cdot e^{iy} + \alpha$$

$$f(z) = e^{x+iy} + c_1$$

$\boxed{f(z) = e^z + c_1}$ which is required solution.

Ques: 1.e) A firm has two bottling plants, one located at Coimbatore and, other at Chennai. Each plant produces three drinks, Coca-Cola, Fanta, and Thumps-up named A, B and C respectively. The number of bottles produced per day are, as follows:-

| | Plant at | |
|---|---|---|
| | Coimbatore (E) | Chennai (F) |
| Coca-Cola (A) | 15,000 | 15,000 |
| Fanta (B) | 30,000 | 10,000 |
| Thumps-Up (C) | 20,000 | 50,000 |

A market survey indicates that, during the month of april, there will be a demand of 200,000 bottles of Coca-Cola, 400,000 bottles of Fanta, and 440,000 bottles of Thumps-up. The operating cost per day for plants at coimbatore, and Chennai is 600 and 400 monetary, units respectively. For how many days each plant be run in April so as to, minimize the production cost, while still meeting the market demand?

Solution:-

Let, the plants at Coimbatore and Chennai be run for $x_1$ and $x_2$ days.

The objective function is given by

$$\text{Minimize } Z = 600x_1 + 400x_2.$$

S.C.

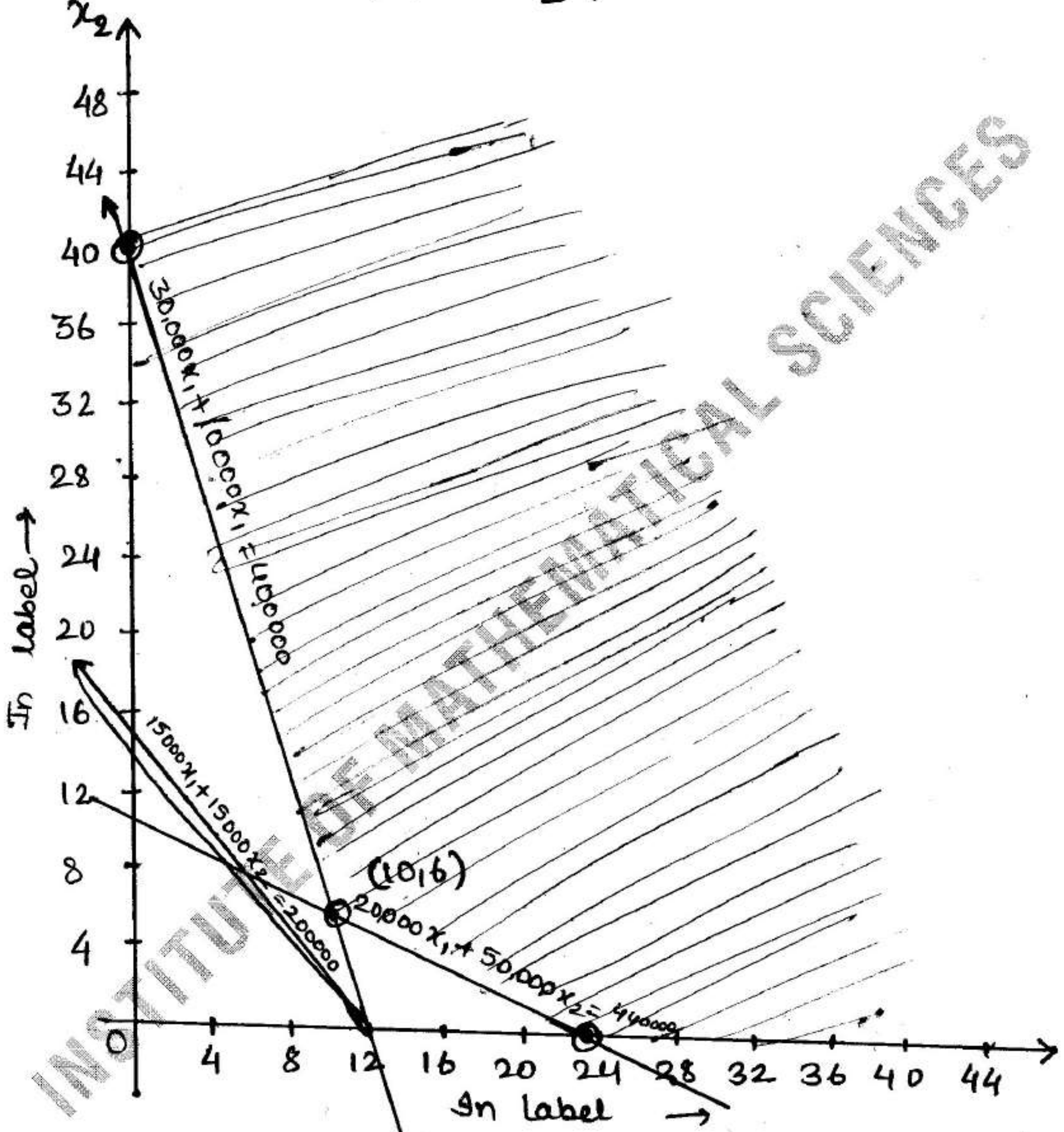
$$15000x_1 + 15000x_2 \geq 200,000$$
$$30000x_1 + 10000x_2 \geq 400,000$$
$$20000x_1 + 50,000x_2 \geq 440,000$$

Subject to constraints are re-written as–

$$15x_1 + 15x_2 \geq 200$$

$$30x_1 + 10x_2 \geq 400$$

$$20x_1 + 50x_2 \geq 440$$




Hence, got three optimal point

$(x_1, x_2) = (0, 40) \Rightarrow Z_{min} = 600 \times 0 + 400 \times 40 = 16000$

$(x_1, x_2) = (10, 6) \Rightarrow Z_{min} = 600 \times 10 + 400 \times 6 = 6000 + 2400$
$= 8400$

$(x_1, x_2) = (24, 0) \Rightarrow Z_{min} = 600 \times 24 + 400 \times 0$
$= 14400$

∴ Optimal feasible Solution.

$$Z_{min}\big|_{at\,(10,6)} = 600 \times 10 + 400 \times 6$$

$$= 6000 + 2400$$

$$\boxed{Z_{min}\big|_{at\,(10,6)} = 8400}$$

Which is required Solution

Ques:- 2(a)(i) Show that any infinite cyclic group is isomorphic to $<Z,+>$ the group of integers.

Solution:- Let, $G = <a>$ be any infinite cyclic group

Define $f: G \longrightarrow Z$, such that

$$f(a^i) = i \qquad i \in Z$$

Since, $G = <a>$ is of infinite order

$$a^i \in G \text{ for all } i \in Z$$

and $a^i = a^j$ for no $i \neq j$.

Thus; $a^i = a^j \Rightarrow i = j$

$\Rightarrow f(a^i) = f(a^j)$ or that $f$ is well defined.

Again; $f(a^i) = f(a^j) \Rightarrow i = j$

$\Rightarrow a^i = a^j \Rightarrow f$ is 1-1

$$f(a^i \cdot a^j) = f(a^{i+j}) = i+j$$

$$\boxed{f(a^i \cdot a^j) = f(a^i) + f(a^j)}$$

Show that '$f$' is a homomorphism.

'$f$' is obviously onto and hence the isomorphism is established.

Cor. : Every subgroup of an infinite cyclic group is an infinite cyclic group which is isomorphism to the group itself.

Ques: 2. a.(ii)) Any finite cyclic group of order 'n' is isomorphic to $Z_n$ the group of integers addition modulo 'n'.

Solution:- Let, $G = \langle a \rangle$ be a cyclic group, such that

$$O(G) = O(a) = n$$

then $G = \{e, a, a^2, \ldots, a^{n-1}\}$

$Z_n = \{0, 1, 2, \ldots, n-1\}$

Define $f : G \to Z_n$ ; such that

f is clearly well defined 1-1 onto mapping:

Again;

$$f(a^i \cdot a^j) = f(a^{i \oplus j}) = i \oplus j$$

$$= f(a^i) \oplus f(a^j)$$

Thus, f is a homomorphism and hence an isomorphism.

Remark:- Any two cyclic groups of same order (finite or infinite) are isomorphism.

Ques: 2(a): iii) Prove that if an ideal 'U' of a ring 'R' contains a unit of R, then U=R.

Solution:-

Consider a Ring R and an ideal U of R which contains a unit of R.

By definition of an ideal,
we know that

$$U \subseteq R \quad \text{——} (1)$$

Further, Let unit of R be 1 (without loss of generality).

we have, $1 \in U$

also if $r \in R$

$1 \cdot r \in U$ [$\because$ U is an ideal of R]

$\Rightarrow r \in U$.

$\therefore R \subseteq U$ —— (2)

from (1) and (2)

$\boxed{U = R}$ Hence proved

Ques: 2 (b) i) Determine the values of a, b, c for which the function:

$$f(x) = \begin{cases} \dfrac{\sin(a+1)x + \sin x}{x} & ; \text{ for } x<0 \\ c & ; \text{ for } x=0 \\ \dfrac{(x+bx^2)^{1/2} - x^{1/2}}{bx^{3/2}} & ; \text{ for } x>0 \end{cases}$$

is continuous at $x=0$

Solution:-

given function; $$f(x) = \begin{cases} \dfrac{\sin(a+1)x + \sin x}{x} & ; \text{ for } x<0 \\ c & ; \text{ for } x=0 \\ \dfrac{(x+bx^2)^{1/2} - x^{1/2}}{bx^{3/2}} & ; \text{ for } x>0 \end{cases}$$

Here; $$f(0+0) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} \frac{(h+bh^2)^{1/2} - h^{1/2}}{bh^{3/2}}$$

$$= \lim_{h\to 0} \frac{\cancel{h^{1/2}}[(1+bh)^{1/2} - 1]}{\cancel{h^{1/2}}\cdot bh}$$

$$= \lim_{h\to 0} \frac{[1+\frac{1}{2}bh+\cdots] - 1}{bh} = \frac{1}{2}.$$

Which is independent of 'b' and so 'b' may have any real value except '0'.

Again; $$f(0-0) = \lim_{h\to 0} f(0-h) = \lim_{h\to 0} \frac{\sin(a+1)(-h) + \sin(-h)}{-h}$$

$$= \lim_{h\to 0} \frac{\sin(a+1)h + \sin h}{h}$$

$$= \lim_{h\to 0} \frac{2\sin(\frac{a}{2}+1)\cdot h\cdot \cos(ah/2)}{h}$$

$$= \lim_{h\to 0} \underbrace{\frac{\sin\{(a+2)/2\}h}{\{(a+2)/2\}h}}_{1} \cdot (a+2) \underbrace{\cos(ah/2)}_{1.}$$

$$= a+2$$

$\therefore$ For continuity at $x=0$,

we have $f(0+0) = f(0-0) = f(0)$

$$\frac{1}{2} = a+2 = c$$

$$\boxed{\therefore c = 1/2 \text{ and } a = -3/2 \text{ and } b \neq 0}$$

Which is required result

Ques:-2(b)-(ii) Show that the function 'f' defined by $f(x) = \frac{1}{x^2}$, $x \neq 0$, is uniformly continuous at $[a,\infty)$ where $a>0$, but not uniformly continuous at $(0,\infty)$?

Solution:- given function: $f(x) = \frac{1}{x^2}$ ; $x \neq 0$

Let, $\varepsilon > 0$ be given.

Let $x_1, x_2$ be any two points of $[a,\infty)$ so that

$$x_1 \geq a \quad \text{and} \quad x_2 \geq a$$

$$|x_1| \geq a \quad \text{and} \quad |x_2| \geq a$$

$$\frac{1}{|x_1|} \leq \frac{1}{a} \quad \text{and} \quad \frac{1}{|x_2|} \leq \frac{1}{a}$$

Now; $|f(x_1) - f(x_2)| = \left|\frac{1}{x_1^2} - \frac{1}{x_2^2}\right|$

$$= \left|\frac{1}{x_1} + \frac{1}{x_2}\right| \left|\frac{1}{x_1} - \frac{1}{x_2}\right|$$

$$\leq \left(\frac{1}{|x_1|} + \frac{1}{|x_2|}\right) \left|\frac{x_2 - x_1}{x_1 x_2}\right|$$

$$\leq \left(\frac{1}{|x_1|} + \frac{1}{|x_2|}\right) \left(\frac{|x_2 - x_1|}{|x_1||x_2|}\right)$$

$$\leq \left(\frac{1}{a} + \frac{1}{a}\right) \frac{|x_1 - x_2|}{a \cdot a}$$

$$= \frac{2}{a^3} |x_1 - x_2| < \varepsilon$$

Whenever ; $|x_1 - x_2| < \frac{a^3}{2} \cdot \varepsilon$

Choose; $\delta = \frac{a^3}{a} \cdot \varepsilon > 0$

then; $|f(x_1) - f(x_2)| < \varepsilon$ ; whenever $|x_1 - x_2| < \delta$

$\Rightarrow$ f is uniformly continuous on $[a, \infty)$.

Now, we shall show that 'f' is not uniformly continuous on $(0, \infty)$.

Let, $\varepsilon > 0$ be given.

we shall show that for each $\delta > 0$ $\exists\, x_1, x_2 \in (0, \infty)$ such that

$$|x_1 - x_2| < \delta \Rightarrow |f(x_1) - f(x_2)| \geq \varepsilon$$

The sequence $\langle a_n \rangle$ defined by

$$a_n = \frac{1}{\sqrt{n+2\varepsilon}} - \frac{1}{\sqrt{n}} \quad ; \text{ converges to } 0$$

Since; $a_n \to 0$ and $n \to \infty$

$\therefore$ Given $\delta > 0$, $\exists$ a positive integer 'm', such that

$$|a_n - 0| < \delta \quad \forall\, n \geq m$$

$$\Rightarrow \left| \frac{1}{\sqrt{n+2\varepsilon}} - \frac{1}{\sqrt{n}} \right| < \delta \quad \forall\, n \geq m$$

Let, us take $x_1 = \frac{1}{\sqrt{m+2\varepsilon}}$ and $x_2 = \frac{1}{\sqrt{m}}$

then $|x_1 - x_2| < \delta$

whereas, $|f(x_1) - f(x_2)| = |(m + 2\varepsilon) - m|$

$$= |\cancel{m} + 2\varepsilon - \cancel{m}| = 2\varepsilon > \varepsilon$$

Hence, f is not uniformly continuous on $(0, \infty)$

Ques:- 2(c)) Evaluate $\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2}$, $a>b>0$

Solution:- Let; $I = \int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2}$ ; $a>b>0$

$$I = \frac{1}{i}\int_c \frac{dz}{z\left[a+\frac{1}{2}b\left(z+\frac{1}{z}\right)\right]^2}$$

$$= \frac{4}{i}\int_c \frac{z\,dz}{(bz^2+2az+b)^2} = \frac{4}{ib^2}\int_c \frac{z\,dz}{(z^2+2a/b\,z+1)^2}$$

$$= \frac{4}{ib^2}\int_c f(z)dz \quad \text{— say.}$$

where, c is the unit circle.

f(z) has poles of order two given by.

$$z^2+2a/b\,z+1=0$$

or. $$z = \frac{1}{2}\left[-\frac{2a}{b} \pm \sqrt{\frac{4a^2}{b^2}-4}\right]$$

$$z = \frac{1}{2}\left[-\frac{2a}{b} \pm \frac{1}{b}\sqrt{4a^2-4b^2}\right]$$

$$z = \frac{-a \pm \sqrt{a^2-b^2}}{b}$$

Taking $\alpha = \frac{-a+\sqrt{a^2-b^2}}{b}$ and $\beta = \frac{-a-\sqrt{a^2-b^2}}{b}$

Thus; $z=\alpha, \beta$ are double poles of $f(z)$.

Since; $a>b>0$ , $|\beta|>1$

Also; we have $|\alpha\beta| = 1$; therefore $|\alpha|<1$.

Thus; $z=\alpha$ is the pole of order two inside c.

We have; $f(z) = \frac{4}{ib^2} \cdot \frac{z}{[z^2+2(a/b)z+1]^2}$

$$= \frac{4z}{ib^2}\left[\frac{1}{(z-\alpha)^2(z-\beta)^2}\right] = \frac{1}{(z-\alpha)^2}\phi(z)$$

Now; $\phi'(z) = [\phi'(z)]_{z=\alpha}$

$$= \frac{4}{ib^2}\left[\frac{(z-\beta)^2 - 2z(z-\beta)}{(z-\beta)^4}\right]_{z=\alpha}$$

$$= \frac{4}{ib^2}\left[\frac{(\alpha-\beta)^2 - 2\alpha(\alpha-\beta)}{(\alpha-\beta)^4}\right]$$

$$= \frac{4}{ib^2}\left[\frac{\alpha^2+\beta^2-\cancel{2\alpha\beta}+2\cancel{\alpha\beta}-2\alpha^2}{(\alpha-\beta)^4}\right]$$

$$= \frac{4}{ib^2}\frac{(\alpha+\beta)}{(\alpha-\beta)^3} = -\frac{ai}{(a^2-b^2)^{3/2}}$$

$\therefore$ Residue at the double pole $z=\alpha$ is

$$\frac{1}{1!}\phi'(\alpha) = -\frac{ai}{(a^2-b^2)^{3/2}}$$

Hence; $\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2} = 2\pi i.$ sum of residues at poles inside c.

$$\boxed{\therefore \int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2} = \frac{2\pi a}{(a^2-b^2)^{3/2}}}$$

Which is required solution.

Q.3(a)(i)> Show that the set of all polynomials with even coefficients is a prime ideal in $Z[x]$?

Solution:-

First, we will show that A is an ideal of $Z[x]$, when A is the set of all polynomials with even integers.

Let; $f(x) = (2kx^2 + 2jx + 2l)$

where $k, j, l \in Z$

Let $g(x) = (ax^2 + bx + c)$ where $a, b, c \in Z$

Since, all the coefficients of $f(x)$ are even, $f(x)$ must be in A. Now.

$$f(x)g(x) = (2kx^2 + 2jx + 2l)(ax^2 + bx + c)$$

$$f(x)g(x) = 2kax^4 + 2kbx^3 + 2kcx^3 + 2jax^3 + 2jbx^2 + 2jxc + 2alx^2 + 2blx + 2cl$$

$$f(x)g(x) = 2(ka)x^4 + 2(kb + ja)x^3 + 2(kc + jb + la)x^2 + 2(jc + bl)x + 2lc.$$

Since, all the coefficient's of the products are even, the result is in A, so A is an Ideal.

Next, we will show that A is a prime ideal.
To do this, we need to show that if two the polynomials are not in 'A', then their product cannot be in A.

Let, there be two polynomials, both with odd coefficients of the form $(2K+1)$.
when finding the product of these two polynomials, one of the first product is

$$((2K+1)x^2) * ((2h+1)x^2)$$

$$= (4Kh + 2K + 2h + 1)x^4$$

$$= (4Kh + 2(K+h) + 1)x^4.$$

Since, the coefficient ends in $+1$, it is an odd number, so the polynomial cannot be in A.

Hence, 'A' is a prime ideal.

Ques: 3(a)(ii)) Let $Z[i]$ denote the set of all complex numbers of the form $a+ib$ where '$a$' and '$b$' are integers. Determine all the prime elements of $Z[i]$.

Solution :-

Given $Z[i] = \{a+bi \,/\, a, b \in Z\}$.

Let us suppose that $a+bi$ is prime.

then $(a+bi) \mid (a+bi)(a-bi) = N(a+bi)$

$$= a^2+b^2$$

Thus, every prime divides a natural number $a^2+b^2$.

[Now, writting this number as a product of primes in $\mathbb{N}$ and keeping in mind that $a+bi$ is a prime in $Z[i]$, we find that $a+bi$ must divide one of the prime factors of $a^2+b^2$.

Therefore, we have the following primes in $Z[i]$:

(i) $1+i$, the prime dividing 2;

(ii) $a+bi$ and $a-bi$, where $p = a^2+b^2 \equiv 1 \bmod 4$;

(iii) rational primes $q \equiv 3 \bmod 4$.

In particular, $Z[i]$ has infinitely many primes.

Hence the result

Ques.3(b)) Show that the differential co-efficient of $\sum_{n=1}^{\infty} \frac{1}{n^3+n^4x^2}$ is $-2x \sum_{n=1}^{\infty} \frac{1}{n^2(1+nx^2)^2}$ for all real $x$.

Solve:- Here; $f_n(x) = \frac{1}{n^3+n^4x^2} = \frac{1}{n^3(1+nx^2)}$

$$\Rightarrow f_n'(x) = \frac{1}{n^3}\left[\frac{-2nx}{(1+nx^2)^2}\right] = \frac{-2x}{n^2(1+nx^2)^2}$$

$$\Rightarrow \sum_{n=1}^{\infty} f_n'(x) = -2x \sum_{n=1}^{\infty} \frac{1}{n^2(1+nx^2)^2}$$

Now, $f_n'(x)$ is maximum when $\frac{d}{dx} f_n'(x) = 0$

i.e. when $-\frac{2}{n^2} \cdot \frac{(1+nx^2)^2 \cdot 1 - x \cdot 2(1+nx^2)\cdot 2nx}{(1+nx^2)^4} = 0$

or when ; $1-3nx^2 = 0$

$$\Rightarrow \boxed{x = \frac{1}{\sqrt{3n}}}$$

$\therefore$ Maximum value of $|f_n'(x)| = \frac{2 \cdot \frac{1}{\sqrt{3n}}}{n^2(1+\frac{1}{3})^2} = \frac{3\sqrt{3}}{8n^{5/2}}$

$$\Rightarrow |f_n'(x)| \leq \frac{3\sqrt{3}}{8n^{5/2}} = \frac{1}{n^{5/2}} \quad \forall x$$

Since; $\sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$ is convergent

Therefore, by Weierstrass's M-Test, the series $\sum_{n=1}^{\infty} f_n'$ is uniformly convergent for all real $x$.

and Hence,

$\sum_{n=1}^{\infty} f_n$ can be differentiated term by term.

$$\therefore \left(\sum_{n=1}^{\infty} f_n\right)' = \sum_{n=1}^{\infty} f_n'$$

$$\Rightarrow \boxed{\left(\sum_{n=1}^{\infty} \frac{1}{n^3+n^4x^2}\right)' = -2x \sum_{n=1}^{\infty} \frac{1}{n^2(1+nx^2)^2}}$$

Which is required result

Ques:- 3(c)) Solve the following LPP by simplex method.

$$\text{Minimize } Z = 4x_1 + x_2$$

Subject to constraints

$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1 + 2x_2 \leq 4$$
$$x_1, x_2 \geq 0$$

Solution:-

Given; Minimize $Z = 4x_1 + x_2$

for simplex, we need to convert Z into Maximize form.

$$Z_{max} = -Z_{min}.$$

$$\therefore Z_{max} = -4x_1 - x_2$$

Now, we write the given LPP in the standard form —

$$\text{Max } Z' = -4x_1 - x_2 + 0S_1 + 0S_2 - MA_1 - MA_2$$

Subject to constraints

$$3x_1 + x_2 + 0S_1 + 0S_2 + A_1 + 0A_2 = 3$$
$$4x_1 + 3x_2 - S_1 + 0S_2 + 0A_1 + A_2 = 6$$
$$x_1 + 2x_2 + 0S_1 + S_2 + 0A_1 + 0A_2 = 4$$
$$x_1, x_2, S_1, S_2, A_1, A_2 \geq 0$$

Where; $S_1$ – surplus variable
$S_2$ – slack variable
$A_1, A_2$ – Artificial variable.

as; $S_1$ is surplus variable, which should not be a basic variable, since its value $S_1 = -6$ when all variables are zero is negative, which is not feasible,

$\therefore S_1$ must be prevented from appearing in initial solution. This is done by taking $S_1=0$

By setting other non-basic variables

$$x_1 = x_2 = 0$$

We obtain the IBFS as

$$x_1 = x_2 = 0,\ S_1 = 0,\ A_1 = 3,\ A_2 = 6 \text{ and } S_2 = 4$$

Thus the initial simplex table is :-

| | $C_j$ | $-4$ | $-1$ | $0$ | $0$ | $-M$ | $-M$ | | |
|---|---|---|---|---|---|---|---|---|---|
| $C_B$ | Basis | $x_1$ | $x_2$ | $S_1$ | $S_2$ | $A_1$ | $A_2$ | $b$ | $\theta$ |
| $-M$ | $A_1$ | 3 | 1 | 0 | 0 | 1 | 0 | 3 | $3/3 = 1 \rightarrow$ |
| $-M$ | $A_2$ | 4 | 3 | $-1$ | 0 | 0 | 1 | 6 | $6/4 = 3/2$ |
| $0$ | $S_2$ | 1 | 2 | 0 | 1 | 0 | 0 | 4 | $4/1 = 4.$ |
| $z_j =$ | $\Sigma C_B a_{ij}$ | $-7M$ | $-4M$ | $M$ | 0 | $-M$ | $-M$ | $-9M$ | |
| $C_j =$ | $c_j - z_j$ | $7M-4$ $\uparrow$ | $4M-1$ | $-M$ | 0 | 0 | 0 | | |

From above table ; $x_1 \rightarrow$ incoming variable
$A_1 \rightarrow$ Outgoing variable.

Here (3) is the key element and convert it into unity and all other elements in its column to zero. Then the new simplex table is :-

| | $C_j$ | $-4$ | $-1$ | $0$ | $0$ | $-M$ | $-M$ | | |
|---|---|---|---|---|---|---|---|---|---|
| $C_B$ | Basis | $x_1$ | $x_2$ | $S_1$ | $S_2$ | $A_1$ | $A_2$ | $b$ | $\theta$ |
| $-4$ | $x_1$ | 1 | $1/3$ | 0 | 0 | $1/3$ | 0 | 1 | $1/_{1/3} = 3$ |
| $-M$ | $A_2$ | 0 | $(5/3)$ | $-1$ | 0 | | 1 | 2 | $2/_{5/3} = \frac{6}{5} \rightarrow$ |
| $0$ | $S_2$ | 0 | $5/3$ | 0 | 1 | | 0 | 3 | $3/_{5/3} = 9/5$ |
| $z_j =$ | $\Sigma C_B a_{ij}$ | $-4$ | $-4/3 - 5/3 M$ | $+M$ | 0 | | $-M$ | $-4-2M$ | |
| $C_j =$ | $c_j - z_j$ | 0 | $(5/3 M - 1/3)$ $\uparrow$ | $-M$ | 0 | | 0 | | |

Outgone

From above table ; $x_2 \to$ Incoming variable
$A_2 \to$ Outgoing variable.

Here $(5/3)$ is the key element and make it unity and all the other elements in its column equal to zero.

Then the revised simplex table is :-

| | $C_j$ | $-4$ | $-1$ | $0$ | $0$ | |
|---|---|---|---|---|---|---|
| $C_B$ | Basis | $x_1$ | $x_2$ | $S_1$ | $S_2$ | $b$ |
| $-4$ | $x_1$ | $1$ | $0$ | $1/5$ | $0$ | $3/5$ |
| $-1$ | $x_2$ | $0$ | $1$ | $-3/5$ | $0$ | $6/5$ |
| $0$ | $S_2$ | $0$ | $0$ | $1$ | $1$ | $1$ |
| $z_j =$ | $\Sigma C_B \cdot a_{ij}$ | $-4$ | $-1$ | $-1/5$ | $0$ | $-18/5$ |
| $C_j =$ | $C_j - z_j$ | $0$ | $0$ | $1/5$ | $0$ | |

From ; the above table all $C_j$'s $\leq 0$ except $S_1$ column, which should not be incoming variable. as $S_1 = 0$.

$\therefore$ The solution is an optimal BFS to the problem and is given by.

$$x_1 = 3/5 \,,\ x_2 = 6/5 \text{ and } S_2 = 1$$

$$\text{Max} \cdot z' = -18/5$$

$$\therefore \text{ Minimize } z = 4x_1 + x_2$$
$$= 4 \times \frac{3}{5} + \frac{6}{5} = \frac{18}{5}.$$

$$\boxed{\therefore \text{ Minimize } z = 18/5}$$

which is required solution.

Ques:- 4(a) Show by an example that in a finite commutative ring, every maximal ideal need not be prime.

Solution:- consider,

the Ring $R = \{0, 2, 4, 6\}$ under addition and multiplication modulo 8.

Let, $M = \{0, 4\}$,

then M is easily seen to be an ideal of R.

Again $2 \otimes 6 = 4 \in M$

but $2, 6 \notin M$

we find M is not a prime ideal.

We show M is maximal.

Let, $M \subseteq N \subseteq R$, where N is an ideal of R

Since $\langle M, + \rangle$ will be a subgroup of $\langle N, + \rangle$

by Lagrange's theorem $O(M) | O(N)$.

Similarly $O(N) | O(R) = 4$.

i.e $2 | O(N)$, $O(N) | 4$

i.e $O(N) = 2$ or $4$

if $O(N) = 2$, then $M = N$ as $M \subseteq N$

if $O(N) = 4$, then $N = R$ as $N \subseteq R$

Hence, M is a maximal ideal of R.

Ques:- 4(b)) Examine the convergence of the integrals:-

(i) $\int_1^{\infty} \frac{\tan^{-1}x}{x^2}dx$ (ii) $\int_0^1 \frac{dx}{(1+x)^2(1-x)^3}$

Solution:-

(i) $$I = \int_1^{\infty} \frac{\tan^{-1}x}{x^2}dx = \lim_{t\to\infty}\int_1^t \frac{\tan^{-1}x}{x^2}dx$$

put $x = \tan\theta$ ; $dx = \sec^2\theta\, d\theta$

$$\Rightarrow \int \frac{\tan^{-1}x}{x^2}dx = \int \frac{\theta}{\tan^2\theta}\cdot \sec^2\theta\, d\theta.$$

$$\Rightarrow \int \theta\cdot\frac{\cancel{\cos^2\theta}}{\sin^2\theta}\cdot\frac{1}{\cancel{\cos^2\theta}}d\theta = \int \theta\, \operatorname{cosec}^2\theta\, d\theta$$

$$\Rightarrow \theta(-\cot\theta) - \int 1(-\cot\theta)d\theta$$

$$\Rightarrow -\theta\cot\theta + \log\sin\theta.$$

$$= -\frac{\tan^{-1}x}{x} + \log\frac{x}{\sqrt{1+x^2}}$$

$$\therefore \int_1^{\infty}\frac{\tan^{-1}x}{x^2}dx = \lim_{t\to\infty}\left[-\frac{\tan^{-1}x}{x} + \log\frac{x}{\sqrt{1+x^2}}\right]_1^t$$

$$= \lim_{t\to\infty}\left[-\frac{\tan^{-1}t}{t} + \log\frac{t}{\sqrt{1+t^2}} + \tan^{-1}1 - \log\frac{1}{\sqrt{2}}\right]$$

$$= 0 + \lim_{t\to\infty}\log\frac{1}{\sqrt{1/t^2+1}} + \frac{\pi}{4} + \frac{1}{2}\log 2$$

$$= \log 1 + \frac{\pi}{4} + \frac{1}{2}\log 2 = \frac{\pi}{4} + \frac{1}{2}\log 2.$$

which is finite.

$\Rightarrow \int_1^\infty \frac{\tan^{-1}x}{x^2}\,dx$ is convergent and its value is

$$\boxed{\int_1^\infty \frac{\tan^{-1}x}{x^2}\,dx = \frac{\pi}{4}+\frac{1}{2}\log 2}$$

(ii) Here; $f(x) = \dfrac{1}{(1+x)^2(1-x)^3}$

1 is the only point of infinite discontinuity of f on $[0,1]$.

Take; $g(x) = \dfrac{1}{(1-x)^3}$

Then; $\lim\limits_{x\to 1^-} \dfrac{f(x)}{g(x)} = \lim\limits_{x\to 1^-} \dfrac{1}{(1+x)^2} = \dfrac{1}{(1+1)^2} = \dfrac{1}{4}$.

Which is non-zero and finite.

$\therefore$ By comparision test,

$\int_0^1 f(x)dx$ and $\int_0^1 g(x)dx$ converges or diverges together.

But $\int_0^1 g(x)\,dx = \int_0^1 \frac{dx}{(1-x)^3}$ [form $\int_a^b \frac{dx}{(b-a)^n}$ with $[\because n=3>1]$ & $b=1$]

$\therefore$ $g(x)$ diverges.

$\therefore \int_0^1 f(x)\,dx = \int_0^1 \frac{dx}{(1+x)^2(1-x)^3}$ is divergent.

Which is required result.

Ques:- 4(c) Expand $f(z) = \dfrac{1}{(z+1)(z+3)}$ in a Laurent's series valid for the regions:

(i) $|z| < 1$ (ii) $1 < |z| < 3$

(iii) $|z| > 3$ (iv) $0 < |z+1| < 2$

Solution:- we have; $f(z) = \dfrac{1}{(z+1)(z+3)}$

Resolving into partial fraction, we get

$$f(z) = \frac{1}{2(z+1)} - \frac{1}{2(z+3)}$$

(i) $|z| < 1$

we have; $f(z) = \frac{1}{2}(z+1)^{-1} - \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$

$$f(z) = \frac{1}{2}\left[1 - z + z^2 - z^3 + \cdots\right] - \frac{1}{6}\left[1 - \frac{z}{3} + \frac{z^2}{9} - \frac{z^3}{27} + \cdots\right]$$

$$f(z) = \left(\frac{1}{2} - \frac{1}{6}\right) - \left(\frac{1}{2} - \frac{1}{18}\right)z + \left(\frac{1}{2} - \frac{1}{54}\right)z^2 + \cdots$$

$$\boxed{f(z) = \frac{1}{3} - \frac{4}{9}z + \frac{13}{27}z^2 - \cdots}$$

(ii) $1 < |z| < 3$

Then, we have; $\frac{1}{|z|} < 1$ and $\frac{|z|}{3} < 1$.

Now, $\dfrac{1}{2(z+1)} = \dfrac{1}{2z(1+1/z)} = \dfrac{1}{2z}\left(1 + 1/z\right)^{-1}$

$$= \frac{1}{2z}\left[1 - \frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^3} + \cdots\right]$$

$$= \frac{1}{2z} - \frac{1}{2z^2} + \frac{1}{2z^3} - \frac{1}{2z^4} + \cdots$$

and $\frac{1}{2(z+3)} = \frac{1}{6(1+z/3)} = \frac{1}{6}\left[1+\frac{z}{3}\right]^{-1}$

$$= \frac{1}{6}\left[1-\frac{z}{3}+\frac{z^2}{9}-\frac{z^3}{27}+\cdots\right)$$

$$= \frac{1}{6}-\frac{1}{18}z+\frac{1}{54}z^2-\frac{1}{162}z^3+\cdots$$

Thus; the Laurent's series valid for the region $1<|z|<3$ is

$$\boxed{\therefore f(z) = \cdots\cdots+\frac{1}{2z^3}-\frac{1}{2z^2}+\frac{1}{2z}+\frac{1}{6}-\frac{1}{18}z+\frac{1}{54}z^2-\cdots}$$

(iii) $|z|>3$

Then; $3/|z|<1$

$$\therefore f(z) = \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1}-\frac{1}{2z}\left(1+\frac{3}{z}\right)^{-1}$$

$$f(z) = \frac{1}{2z}\left[1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\cdots\right]-\frac{1}{2z}\left[1-\frac{3}{z}+\frac{9}{z^2}-\frac{27}{z^3}+\cdots\right]$$

$$f(z) = \left[\frac{1}{2z}-\frac{1}{2z^2}+\frac{1}{2z^3}-\frac{1}{2z^4}+\cdots\right]+\left[\frac{-1}{2z}+\frac{3}{6z^2}-\frac{9}{2z^3}+\frac{27}{2z^4}+\cdots\right]$$

$$\boxed{f(z) = \frac{1}{z^2}-\frac{4}{z^3}+\frac{13}{z^4}-\frac{40}{z^5}+\cdots}$$

(iv) $0<|z+1|<2$

Let $z+1=u$; then $0<|u|<2$

$$\therefore f(z) = \frac{1}{(z+1)(z+3)} = \frac{1}{u(u+2)} = \frac{1}{2u}\left(1+u/2\right)^{-1}$$

$$f(z) = \frac{1}{2u}\left[1 - \frac{u}{2} + \frac{u^2}{4} - \frac{u^3}{8} + \cdots\right]$$

$$f(z) = \frac{1}{2u} - \frac{1}{4} + \frac{u}{8} - \frac{u^2}{16} + \cdots$$

$$\boxed{f(z) = \frac{1}{2(z+1)} - \frac{1}{4} + \frac{z+1}{8} - \frac{(z+1)^2}{16} + \cdots}$$

which is required result

Ques: 4(d)) A department head has four subordinates, and four tasks to be performed. The subordinates differ in efficiency, and the tasks in their intrinsic difficulty. His estimate of the times each man would take to perform each task is given in the effectiveness matrix below. How should the tasks be allocated, one to a man, so as to minimize the total men hours?

Subordinates

$$\text{Task}\quad \begin{array}{c|cccc} & I & II & III & IV \\ \hline A & 8 & 26 & 17 & 11 \\ B & 13 & 28 & 4 & 26 \\ C & 38 & 19 & 18 & 15 \\ D & 19 & 26 & 24 & 10 \end{array}$$

Solution:- Given ; Effective matrix

$$\begin{array}{c|cccc} & I & II & III & IV \\ \hline A & 8 & 26 & 17 & 11 \\ B & 13 & 28 & 4 & 26 \\ C & 38 & 19 & 18 & 15 \\ D & 19 & 26 & 24 & 10 \end{array}$$

Step-I) Subtract the least element of the row ~~and~~ from all the element of that row;

The reduced Table

$$\begin{array}{c|cccc} & I & II & III & IV \\ \hline A & 0 & 18 & 9 & 3 \\ B & 9 & 24 & 0 & 22 \\ C & 23 & 4 & 3 & 0 \\ D & 9 & 16 & 14 & 0 \end{array}$$

Now, check whether all the rows and each column have at least one zero; which is not here; hence, subtract the least element of each column from all other elements of that column.

The reduced table is

| | I | II | III | IV |
|---|---|---|---|---|
| A | 0 | 14 | 9 | 3 |
| B | 9 | 20 | 0 | 22 |
| C | 23 | 0 | 3 | 0 |
| D | 9 | 12 | 14 | 0 |

Now, cover all the zeros using minimum number of Horizontal and verticle lines.
Here; no. of lines = no. of rows = 4.

Hence, optimality acheived.

For optimal solution, the Table to choose only one zero from each row with least no. of zeros at discard others in respective row and column.

| | I | II | III | IV |
|---|---|---|---|---|
| A | 0 | 14 | 9 | 3 |
| B | 9 | 20 | 0 | 22 |
| C | 23 | 0 | 3 | ~~0~~ |
| D | 9 | 12 | 14 | 0 |

$\therefore$ A $\rightarrow$ I
B $\rightarrow$ III
C $\rightarrow$ II
D $\rightarrow$ IV.

$\therefore$ Minimized Men hours $= 8 + 4 + 19 + 10$
$= 31$ hours.

$$\boxed{z_{min} = 31 \text{ hours}}$$

which is required solution.

## Section-B

Ques: 5(a) Find the complete integral of

$$(x+y)(p+q)^2 + (x-y)(p-q)^2 = 1.$$

Solution:-

Let X and Y be two new variables such that

$$X^2 = x+y \quad \text{and} \quad Y^2 = x-y \quad \text{——— (1)}$$

∴ Given equation is $(x+y)(p+q)^2 + (x-y)(p-q)^2 = 1$ — (2)

Now; $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial x} + \frac{\partial z}{\partial Y} \cdot \frac{\partial Y}{\partial x}$

$$p = \frac{1}{2X}\frac{\partial z}{\partial x} + \frac{1}{2Y} \cdot \frac{\partial z}{\partial Y} \quad \text{——— (3)}$$

[∵ from (1), $\frac{\partial X}{\partial x} = \frac{1}{2X}$ and $\frac{\partial Y}{\partial x} = \frac{1}{1/2Y}$

and $q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial X} \cdot \frac{\partial X}{\partial y} + \frac{\partial z}{\partial Y} \cdot \frac{\partial Y}{\partial y}$

$$q = \frac{1}{2X}\frac{\partial z}{\partial X} - \frac{1}{2Y}\frac{\partial z}{\partial Y} \quad \text{——— (4)}$$

[∵ from (1); $\frac{\partial X}{\partial y} = \frac{1}{2X}$ ; $\frac{\partial Y}{\partial y} = \frac{-1}{2Y}$ ]

from (3) & (4) ⇒

$$p+q = \frac{1}{X}\frac{\partial z}{\partial X} \quad \text{and} \quad p-q = \frac{1}{Y}\frac{\partial z}{\partial Y} \quad \text{— (5)}$$

from, using (5) and (1), (2) reduces to

$$\cancel{X^2} \cdot \frac{1}{\cancel{X^2}}\left(\frac{\partial z}{\partial X}\right)^2 + \cancel{Y^2} \cdot \frac{1}{\cancel{Y^2}}\left(\frac{\partial z}{\partial Y}\right)^2 = 1$$

$$\boxed{P^2 + Q^2 = 1} \quad \text{——— (6)}$$

Where; $P = \partial z/\partial X$ and $Q = \partial z/\partial Y$

(4) is of the form $f(P, Q) = 0$

$\therefore$ Solution of (4) is

$$\boxed{z = aX + bY + c} \quad \text{——— (7)}$$

where ; $a^2 + b^2 = 1$

or $b = \sqrt{1-a^2}$

Putting $a$ for $P$ and $b$ for $Q$ in (6)

$\therefore$ from (7), the required complete intergral is

$$z = aX + Y\sqrt{1-a^2} + c$$

$$\boxed{z = a\sqrt{x+y} + \sqrt{x-y}\,\sqrt{1-a^2} + c} \quad \text{——— by (1)}$$

Ques:- 5(b) Solve $(D^2+DD'-6D'^2)z = x^2\sin(x+y)$

Solve:- Given equation

$$(D^2+DD'-6D'^2)z = x^2\sin(x+y)$$

$$\Rightarrow (D+3D')(D-2D')z = x^2\sin(x+y)$$

Its auxillary equation is

$$(m+3)(m-2)=0$$

$$m=-3, 2$$

$$\boxed{\therefore C.F = \phi_1(y-3x)+\phi_2(y+2x)}$$

where, $\phi_1$ and $\phi_2$ being arbitrary function.

$$P.I = \frac{1}{(D+3D')(D-2D')}x^2\sin(x+y)$$

$$= \frac{1}{(D+3D')}\left\{\frac{1}{(D-2D')}x^2\sin(x+y)\right\}$$

$$= \frac{1}{(D+3D')}\int x^2\sin(x+c-2x)\,dx \; ; \quad \text{where } y=c-2x$$

$$= \frac{1}{(D+3D')}\int x^2\sin(c-x)\,dx$$

$$= \frac{1}{(D+3D')}\left[x^2\cos(c-x)-\int 2x\cos(c-x)\,dx\right]$$

$$= \frac{1}{(D+3D')}\left[x^2\cos(c-x)-\left\{-2x\sin(c-x)+\int 2\sin(c-x)\,dx\right\}\right]$$

$$= \frac{1}{(D+3D')}\left[x^2\cos(c-x)+2x\sin(c-x)-2\cos(c-x)\right]$$

$$= \frac{1}{(D+3D')}\left[(x^2-2)(\cos(c-x))+2x\sin(c-x)\right]$$

$$= \frac{1}{(D+3D')}\left[(x^2-2)\cos(x+y)+2x\sin(x+y)\right] \quad \left[\because y=2x-c,\; c=y+2x\right]$$

$$= \int [x^2-2][\cos(x+c'+3x) + 2x\sin(x+c'+3x)]\,dx$$

where; $c' = y-3x$

$$= \int [x^2-2]\cos(4x+c') + 2x\sin(4x+c')\,dx$$

$$= (x^2-2)\frac{\sin(4x+c')}{4} - \int \cancel{2}x\frac{\sin(4x+c')}{\cancel{4}\,2}\,dx + 2\int x\sin(4x+c')\,dx.$$

Integrating by parts 1st integral and keeping the second integral unchanged.

$$= \frac{1}{4}(x^2-2)\sin(4x+c') + \frac{3}{2}\int x\sin(4x+c')\,dx$$

$$= \frac{(x^2-2)}{4}\sin(4x+c') + \frac{3}{2}\left[\frac{-x\cos(4x+c')}{4} + \int\frac{\cos(4x+c')}{4}dx\right]$$

$$= \frac{x^2-2}{4}\sin(4x+c') + \left(\frac{-3}{8}\right)x\cos(4x+c') + \frac{3}{24}\sin(4x+c')$$

$$\Rightarrow \frac{1}{4}(x^2-2)\sin(4x+c') - \frac{3}{8}x\cos(4x+c') + \frac{3}{24}\sin(4x+c')$$

$$\Rightarrow \frac{1}{4}(x^2-2)\sin(x+y) - \frac{3}{8}x\cos(x+y) + \frac{3}{24}\sin(x+y).$$

$$\therefore P.I = \left(\frac{x^2}{4} - \frac{13}{32}\right)\sin(x+y) - \frac{3}{8}x\cos(x+y)$$

$\therefore$ The required solution is $z = C.F + P.I.$

$$\boxed{z = \phi_1(y-3x) + \phi_2(y+2x) + \left(\frac{x^2}{4} - \frac{13}{32}\right)\sin(x+y) - \frac{3}{8}x\cos(x+y)}$$

Required solution.

Ques:-5(c)) The Quadratic Equation $x^4 - 4x^2 + 4 = 0$ has double root at $x = \sqrt{2}$. Starting with $x_0 = 1.5$. compute three successive iterations to the roots by Newton-Raphson method. Does the result converges quadratically or linearly?

Solution:-

The Newton-Raphson method for determining a multiple root of multiplicity 'm' is given by

$$\boxed{x_{k+1} = x_k - \frac{m f_k}{f'_k}}$$

Since, the quadratic equation $x^4 - 4x^2 + 4 = 0$ has double root at $x = \sqrt{2}$ ; i.e $m = 2$, we have

$$x_{n+1} = x_n - \frac{2 f(x_n)}{f'(x_n)} \Rightarrow f(x) = x^4 - 4x^2 + 4$$

$$f'(x) = 4x^3 - 8x$$

and given, $x_0 = 1.5$.

Then, $x_1 = x_0 - \frac{2 f(x_0)}{f'(x_0)} = 1.5 - 2 \frac{f(1.5)}{f'(1.5)}$

$$x_1 = 1.5 - 2 \times \frac{0.0625}{1.5} = 1.5 - 0.08333$$

$$\boxed{x_1 = 1.41667}$$

for 2nd Iteration

$$x_2 = x_1 - \frac{2 f(x_1)}{f'(x_1)} = 1.41667 - 2 \cdot \frac{f(1.41667)}{f'(1.41667)}$$

$$x_2 = 1.41667 - 2\cdot\left[\frac{0.0001}{0.03864}\right]$$

$$x_2 = 1.41667 - 0.005071 = 1.4116$$

for third Iteration

$$x_3 = x_2 - \frac{2f(x_2)}{f'(x_2)} = 1.4116 - \frac{2f(1.4116)}{f'(1.4116)}$$

$$x_3 = 1.4116 - 2\left[\frac{0.00004}{-(0.0417)}\right]$$

$$x_3 = 1.4116 + 0.001918 = 1.41352$$

$$\therefore \boxed{x_1 = 1.41667 \;;\; x_2 = 1.4116 \;;\; x_3 = 1.41352}$$

$$\& \quad \boxed{x_3 = 1.41352 \approx 1.414 = \sqrt{2}}$$

Thus, the method $x_{k+1} = x_k - \frac{m f_k}{f'_k}$ has quadratic rate of convergence.

[$\because$ In the error term

$$\varepsilon_{k+1} = \left(1 - \frac{\alpha}{m}\right)\varepsilon_k + \frac{\alpha \, f^{m+1}(\xi)}{m^2(m+1) f^m(\xi)}\varepsilon_k^2 + O(\varepsilon_k^2)$$

The coefficient of $\varepsilon_k$ vanishes i.e $\alpha = m = 2$ ]

Ques:-5(d) Convert the following:

(i) $(41.6875)_{10}$ to binary number

Solve:-

| | | |
|---|---|---|
| 2 | 41 | |
| 2 | 20 | 1 |
| 2 | 10 | 0 |
| 2 | 5 | 0 |
| 2 | 2 | 1 |
| | 1 | 0 |

$0.6875 \times 2 = \underline{1}.3750 \to 1$
$0.3750 \times 2 = \underline{0}.7500 \to 0$
$0.7500 \times 2 = \underline{1}.5000 \to 1$
$0.5000 \times 2 = \underline{1}.0000 \to 1$
$0.0000$

$$\boxed{\therefore (41.6875)_{10} \longleftrightarrow (101001.1011)_2}$$

(ii) $(101101)_2$ to decimal number.

Solve:- $1\times 2^5 + 0\times 2^4 + 1\times 2^3 + 1\times 2^2 + 0\times 2^1 + 1\times 2^0$

$= 32+0+8+4+0+1$

$= 45.$

$$\boxed{\therefore (101101)_2 \longleftrightarrow (45)_{10}}$$

(iii) $(AF63)_{16}$ to decimal number.

Solve:- $A\times 16^3 + F\times 16^2 + 6\times 16^1 + 3\times 16^0$

$= 10\times 16^3 + 15\times 256 + 6\times 16 + 3$

$= 40960 + 3840 + 96 + 3 = 44899.$

$$\boxed{\therefore (AF63)_{16} \longleftrightarrow (44899)_{10}}$$

(iv) $(101111011111)_2$ to Hexadecimal number

Solve:- $(\underset{B}{1011}\ \underset{D}{1101}\ \underset{F}{1111})_2 \longleftrightarrow (BDF)_{16}$

$$\boxed{\therefore (101111011111)_2 \longleftrightarrow (BDF)_{16}}$$

Ques:- 5 (e)) The x - component of velocity is $u = x^3 + z^4 + 6$ and y component is $v = y^3 + z^4$. find the simplest z component of velocity that satisfy continuity.

Solution:-

given; x - component of velocity is

$$u = x^3 + z^4 + 6$$

y - component of velocity is

$$v = y^3 + z^4.$$

Hence; for simplest z - component of velocity that satisfy equation of continuity.

i.e
$$\boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0} \quad \text{—— (1)}$$

$$\frac{\partial u}{\partial x} = 3x^2 \quad \text{—— (A)}$$

$$\frac{\partial v}{\partial y} = 3y^2 \quad \text{—— (B)}$$

Put (A) & (B) in (1), we get.

$$\frac{\partial w}{\partial z} = -(3x^2 + 3y^2)$$

$$\Rightarrow \quad w = -(3x^2 + 3y^2)z + f(x, y).$$

$\therefore$ Simplest z - component of a velocity

$$\boxed{w = -(3x^2 + 3y^2)z}$$

Ques: 6(a) Find the integral surface of $x^2p+y^2q+z^2=0$, $p=\frac{\partial z}{\partial x}$ and $q=\frac{\partial z}{\partial y}$ which passes through the hyperbola $xy=x+y$, $z=1$.

Solution:-

Given: $x^2p+y^2q+z^2=0$

$$x^2p+y^2q=-z^2 \quad \text{——} (1)$$

Given curve; $xy=x+y$ ; $z=1$ —— (2)

Here Lagrange's auxillary equations are

$$\frac{dx}{x^2}=\frac{dy}{y^2}=\frac{dz}{-z^2} \quad \text{——} (3)$$

from first and third fractions:

$$\frac{dx}{x^2}+\frac{dz}{z^2}=0$$

$$\Rightarrow -\frac{1}{x}-\frac{1}{z}=-c_1 \Rightarrow \frac{1}{x}+\frac{1}{z}=c_1 \quad \text{——} (4)$$

from 2nd and 3rd fractions:-

$$\frac{dy}{y^2}+\frac{dz}{z^2}=0$$

$$\Rightarrow \frac{1}{y}+\frac{1}{z}=c_2 \quad \text{——} (5)$$

Adding (4) and (5), $\frac{1}{x}+\frac{1}{y}+\frac{2}{z}=c_1+c_2$

$$\Rightarrow \frac{x+y}{xy}+\frac{2}{z}=c_1+c_2 \Rightarrow \frac{xy}{xy}+\frac{2}{z}=c_1+c_2$$

(from (2))

$$1+\frac{2}{z}=c_1+c_2 \Rightarrow 1+2=c_1+c_2 \text{ [from (2)]}$$

$$\therefore \frac{1}{x}+\frac{1}{y}+\frac{2}{z}=3 \Rightarrow \boxed{yz+2xy+xz=3xyz}$$ required solution.

Ques: 6(b)) Reduce the equation
$$x^2(y-1)r + (-x)(y^2-1)s + y(y-1)t + xyp - q = 0$$
to canonical form and hence solve it.

Solution:- Given
$$x^2(y-1)r - x(y^2-1)s + y(y-1)t + xyp - q = 0 \quad \text{—(1)}$$
Comparing (1) with
$$Rr + Ss + Tt + f(x,y,z,p,q) = 0, \text{ we get}$$

$$R = x^2(y-1) \quad ; \quad S = -x(y^2-1) \quad ; \quad T = y(y-1)$$
$\therefore$ $\lambda$-quadratic $R\lambda^2 + S\lambda + T = 0$ reduces to
$$x^2(y-1)\lambda^2 - x(y^2-1)\lambda + y(y-1) = 0$$
$$\therefore \lambda_1 = y/x \quad , \quad \lambda_2 = 1/x \text{ (real \& distinct)}$$
so; $(dy/dx) + \lambda_1 = 0$ and $(dy/dx) + \lambda_2 = 0$
$$\Rightarrow \quad \frac{dy}{dx} + \frac{y}{x} = 0$$
$$\frac{dy}{dx} + \frac{1}{x} = 0$$
Integrating these, $xy = c_1$
$$xe^y = c_2$$
So, for canonical form, we take
$$u = xy \quad \text{and} \quad v = xe^y \quad \text{—— (2)}$$
$$\therefore p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$
$$\boxed{p = \frac{\partial z}{\partial x} = y\frac{\partial z}{\partial u} + e^y\cdot\frac{\partial z}{\partial v}} \quad \text{(by 2)} \quad \text{—— (3)}$$
$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x}$$

$$\boxed{q = \frac{\partial z}{\partial y} = x\frac{\partial z}{\partial u} + xe^{y}\cdot\frac{\partial z}{\partial v}} \quad \text{—— (4)}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(y\frac{\partial z}{\partial u} + e^{y}\frac{\partial z}{\partial v}\right) \text{ (by (3)}$$

$$r = y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + e^{y}\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$r = y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\cdot\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right] + e^{y}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\cdot\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\cdot\frac{\partial v}{\partial x}\right]$$

$$\boxed{r = y^2\frac{\partial^2 z}{\partial u^2} + 2ye^{x}\frac{\partial^2 z}{\partial u\partial v} + e^{2y}\frac{\partial^2 z}{\partial y^2}}$$

$$s = \frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(x\cdot\frac{\partial z}{\partial u} + xe^{y}\frac{\partial z}{\partial v}\right)$$

$$s = \frac{\partial z}{\partial u} + x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + e^{y}\frac{\partial z}{\partial v} + xe^{y}\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$s = \frac{\partial z}{\partial u} + e^{y}\frac{\partial z}{\partial v} + x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\cdot\left(\frac{\partial z}{\partial u}\right)\cdot\frac{\partial v}{\partial x}\right] + xe^{y}\left[\frac{\partial}{\partial u}\left[\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]$$

$$\boxed{s = \frac{\partial z}{\partial u} + e^{y}\frac{\partial z}{\partial v} + xy\frac{\partial^2 z}{\partial u^2} + (yxe^{y} + e^{y}x)\frac{\partial^2 z}{\partial u\partial v} + xe^{2y}\frac{\partial^2 z}{\partial v^2}}$$

and $t = \dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{\partial}{\partial y}\left[x\dfrac{\partial z}{\partial u} + xe^{y}\dfrac{\partial z}{\partial v}\right]$

$$t = x\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) + xe^{y}\frac{\partial z}{\partial v} + x\cdot e^{y}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)$$

$$t = x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y}\right] + xe^{y}\frac{\partial z}{\partial v} + xe^{y}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\left(\frac{\partial u}{\partial y}\right) + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$\boxed{t = x^2 \frac{\partial^2 z}{\partial u^2} + 2x^2 e^y \frac{\partial^2 z}{\partial u \partial v} + x^2 e^{2y} \frac{\partial^2 z}{\partial v^2} + x e^y \frac{\partial z}{\partial v}}$$

Substituting the values of $p, q, r, s, t$ in (1) and simplifying, we get

$$\boxed{\frac{\partial^2 z}{\partial u \partial v} = 0}$$ ; which is canonical form of (1)

Integrating it w.r.t 'u' -

$$\boxed{\frac{\partial z}{\partial v} = \phi(v)}$$

Integrating it w.r.t 'v' -

$$z = \phi_1(v) + \phi_2(u)$$

$$\boxed{z = \phi_1(xe^y) + \phi_2(xy)} \quad \text{— by (2)}$$

required solution

Ques: 6(c)) Find the characteristics of the equation $pq = z$ and determine the integral surface which passes through straight line $x=1$, $z=y$.

Solution:-

If the initial data curve is given in parametric form as—

$$x_0(s) = 1 \quad ; \quad y_0(s) = s \quad ; \quad z_0(s) = s$$

then ordinarily the solution is sought in parametric form as—

$$x = x(t,s) \quad ; \quad y = y(t,s) \quad ; \quad z = z(t,s)$$

Thus, using the given data, the differential equation becomes

$$p_0(s)\, q_0(s) - s = 0 = F \qquad \text{—(1)}$$

and the strip condition gives

$$1 = p_0(0) + q_0(1) \quad \text{or} \quad q_0 = 1 \qquad \text{—(2)}$$

Therefore;

$$q_0 = 1 \, , \quad p_0 = s \quad \text{(unique initial strip)} \qquad \text{—(3)}$$

Now, the characteristic equations for the given PDE are,

$$\left.\begin{aligned} \frac{dx}{dt} = q \, , \quad \frac{dy}{dt} = p \, , \quad \frac{dz}{dt} = 2pq \\ \frac{dp}{dt} = p \, , \quad \frac{dq}{dt} = q . \end{aligned}\right\} \text{—(4)}$$

On integration, we get

$$\left.\begin{aligned} p = c_1 \exp(t) \, , \quad q = c_2 \exp(t) \, , \quad x = c_2 \exp(t) + c_3 \\ y = c_1 \exp(t) + c_4 \; ; \quad z = 2c_1 c_2 \exp(2t) + c_5 \end{aligned}\right\} \text{—(5)}$$

Now, taking into account the initial conditions

$$x_0 = 1, \quad y_0 = s, \quad z_0 = s, \quad p_0 = s, \quad q_0 = 1 \quad \text{--- (6)}$$

We can determine the constants of integration and obtain

(Since $c_2 = 1$, $c_3 = 0$)

$$\left.\begin{array}{l} p = s\exp(t), \quad q = \exp(t), \quad x = \exp(t) \\ y = s\exp(t), \quad z = s\exp(2t) \end{array}\right\} \text{--- (7)}$$

Consequently, the required integral surface is obtained from Eq.(7) as

$$\boxed{z = xy}$$

Which is required solution.

Ques: 6(d)) A uniform string of line density '$\rho$' is stretched to tension $\rho c^2$ and executes a small transverse vibration in a plane through the undisturbed line of string. The ends $x=0$, $L$ of the string are fixed. The string is at rest, with the point $x=b$ drawn aside through a small distance '$\varepsilon$' and released at time $t=0$. Find the expression for the displacement $y(x,t)$.

Solution:-

The transverse vibration of the string is described by—

$$\text{P.D.E:} \quad y_{xx} = \frac{1}{c^2} y_{tt} \quad \text{———} \quad (1)$$

The boundary and initial conditions are

$$\text{BCs:} \quad y(0,t) = y(L,t) = 0$$

$$\text{IC:} \quad y_t(x,0) = 0$$

Using the variables separable method, let

$$y(x,t) = X(x)\,T(t)$$

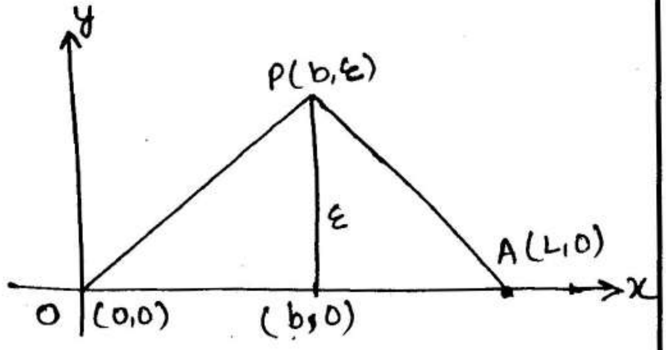
then, we have from eq (1)

$$\frac{X''}{X} = \frac{1}{c^2}\cdot\frac{T''}{T} = \pm\lambda^2$$

The equation of the string is given by

$$y(x,0) = \begin{cases} \dfrac{\varepsilon x}{b}, & 0 \le x \le b \\ \dfrac{\varepsilon(x-L)}{(b-L)}, & b \le x \le L \end{cases}$$

The solution to the given problem, discussed now for various values of $\lambda$.




Case I) Taking the constant $\lambda = 0$, we have

$$X'' = T'' = 0$$

whose general solution is

$$X = Ax + B \quad , \; T = Ct + D$$

Therefore;

$$y(x,t) = (Ax+B)(Ct+D)$$

Using, the BCs at $x = 0, L$,

we observe that $A = B = 0$, implies a trivial solution.

Case II) Taking the constant as $+\lambda^2$, we have

$$X'' - \lambda^2 X = 0 = T'' - c^2\lambda^2 T$$

Thus, the general solution is

$$y(x,t) = (A\cosh\lambda x + B\sinh\lambda x)(C\cosh c\lambda t + D\sinh c\lambda t)$$

Now the BCs:

$$y(0,t) = 0 \quad \text{gives} \quad A = 0$$

and; $\quad y(L,t) = 0$ gives $B\sinh\lambda L = 0$

which is possible only if $B = 0$.

Thus, we are again getting only a trivial solution.

Case (iii) If the constant is $-\lambda^2$, then we have

$$X'' + \lambda^2 X = 0 = T'' + c^2\lambda^2 T = 0$$

In this case, the general solution is

$$y(x,t) = (A\cos\lambda x + B\sin\lambda x)(P\cos c\lambda t + Q\sin c\lambda t)$$

Using the BCs:-

$y(0,t) = 0$ gives $A = 0$

$y(L,t) = 0$ gives $B\sin\lambda L = 0$

For a non-trivial solution, $B \neq 0 \Rightarrow \lambda L = n\pi$.

Therefore, $\lambda = \frac{n\pi}{L}$, $n = 1, 2, 3 \ldots$

Also using the IC: $y_t(x,0) = 0$

we can notice that $Q = 0$.

Hence, the acceptable non-trivial solution is

$$y(x,t) = BP\sin\frac{n\pi x}{L}\cdot\cos\frac{cn\pi t}{L}\ ;\ n = 1,2,3\ldots$$

Using the principle of superposition, we have

$$y(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\cdot\cos\frac{cn\pi t}{L}$$

which gives; $\boxed{y(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}}$

This is half-range sine series, where

$$b_n = \frac{2}{L}\int_0^L y(x,0)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{2}{L}\int_0^b \frac{\varepsilon}{b}x\sin\frac{n\pi x}{L}\,dx + \frac{2}{L}\int_0^L \frac{\varepsilon}{b-L}(x-L)\sin\frac{n\pi x}{L}\,dx$$

$$= \frac{2\varepsilon}{Lb}\left[-\frac{\cos(n\pi x/L)}{n\pi/L}\cdot x\right]_0^b - \frac{2\varepsilon}{Lb}\left[-\frac{\sin(n\pi/L)x}{n^2\pi^2/L^2}\right]_0^L$$

$$+ \frac{2\varepsilon}{L(b-L)}\left[-\frac{\cos(n\pi x/L)}{n\pi/L}(x-L)\right]_0^L - \frac{2\varepsilon}{L(b-L)}\left[-\frac{\sin(n\pi x/L)}{n^2\pi^2/L^2}\right]_0^L$$

$$\boxed{b_n = \frac{2\varepsilon L^2}{n^2\pi^2 b(L-b)}\cdot \sin\frac{n\pi b}{L}}$$

Hence, the subsequent motion of the string is given by

$$\boxed{y(x,t) = \sum_{n=1}^{\infty} \frac{2\varepsilon L^2}{n^2\pi^2 b(L-b)}\cdot \sin\frac{n\pi b}{L}\cdot \sin\frac{n\pi x}{L}\cdot \cos\frac{cn\pi t}{L}}$$

Which is required solution

Ques! 7(a)) Find Lagrange's interpolation polynomial fitting the points $y(1)=-3$, $y(3)=0$, $y(4)=30$, $y(6)=132$. Find $y(5)$.

Solution:-

from given dat.

$y(1)=-3 \qquad y(3)=0$

$y(4)=30 \qquad y(6)=132.$

i.e; $x_0=1$, $x_1=3$, $x_2=4$, $x_3=6$ & $x=5$

$f(x_0)=-3$ ; $f(x_1)=0$ ; $f(x_2)=30$ ; $f(x_3)=132$

By Lagrange's interpolation formula:-

$$f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\cdot f(x_0)+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}f(x_1)+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}f(x_2)+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}\cdot f(x_3)$$

$$f(5)=\frac{(5-3)(5-4)(5-6)}{(1-3)(1-4)(1-6)}\cdot(-3)+\frac{(5-1)(5-4)(5-6)}{(3-1)(3-4)(3-6)}(0)+\frac{(5-1)(5-3)(5-6)}{(4-1)(4-3)(4-6)}\cdot(30)+\frac{(5-1)(5-3)(5-4)}{(6-1)(6-3)(6-4)}\cdot(132)$$

$$f(5)=\frac{\cancel{2}\times1\times-1}{-\cancel{2}\times-\cancel{3}\times-5}\times-\cancel{3}+\frac{\overset{2}{\cancel{4}}\times1\times\cancel{-1}}{\cancel{2}\times\cancel{-1}\times-3}\times0+\frac{4\times\cancel{2}\times-1}{\cancel{3}\times1\times-2}\times\overset{10}{\cancel{30}}+\frac{4\times\cancel{2}\times1}{5\times3\times\cancel{2}}\times132$$

$$f(5) = -\frac{1}{5}\times 1 + \left(-\frac{2}{3}\right)\times 0 + 4\times 10 + \frac{4}{15}\times 132$$

$$f(5) = -\frac{1}{5} + 0 + 40 + \frac{528}{15}$$

$$f(5) = -\frac{3}{15} + \frac{600}{15} + \frac{528}{15}$$

$$f(5) = \frac{600+528-3}{15} = \frac{1125}{15}$$

$$\boxed{y(5) = 75}$$

Which is required result.

Ques: 7(b)) A reservoir discharging water through sluices at a depth 'h' below the water surface has a surface area 'A' for various values of 'h' as given below:

| h(ft.) | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|
| A(sq.ft) | 950 | 1070 | 1200 | 1350 | 1530 |

If it denotes time in minutes, the rate of fall of the surface is given by $\frac{dh}{dt} = -48\sqrt{h}/A$.

Estimate the time taken for the water level to fall from 14 to 10 ft. above the sluices.

Solution:-

Given; The rate of fall of the surface is given by

$$\frac{dh}{dt} = -48\sqrt{h}/A$$

$$\Rightarrow dt = \frac{-1}{48}\frac{A}{\sqrt{h}}dh$$

$$\Rightarrow t = \frac{-1}{48}\int_{14}^{10}\frac{A}{\sqrt{h}}dh.$$

$$t = \frac{1}{48}\int_{10}^{14}\frac{A}{\sqrt{h}}dh.$$

Given table

| height(h)(ft) | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|
| Area (A)- Sq.ft | 950 | 1070 | 1200 | 1350 | 1530 |

Which improves into given table

| h | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|
| $\sqrt{h}$ | 3.1623 | 3.3166 | 3.464 | 3.6056 | 3.7417 |
| A | 950 | 1070 | 1200 | 1350 | 1530 |
| $\frac{A}{48\sqrt{h}}$ | 6.2587 ($y_0$) | 6.7212 ($y_1$) | 7.2169 ($y_2$) | 7.8005 ($y_3$) | 8.519 ($y_4$) |

Using Simpson's $\frac{1}{3}$rd rule

$$t = \frac{h}{3}\left[(y_0+y_4)+4(y_1+y_3)+2y_2\right]$$

$$t = \frac{1}{3}\left[6.2587+8.519)+4(6.7212+7.8005)+2(7.2169)\right]$$

$$t = \frac{1}{3}[87.3123]$$

$$t = 29.1 \approx 29 \text{ minutes.}$$

$\boxed{\therefore \text{ Time required } = 29 \text{ minutes}}$

Ques: 7(c) Use the classical fourth-order Runge-Kutta method with $h=0.2$ to calculate a solution at $x=0.4$ for initial value problem $\frac{du}{dx} = 4-x^2+u$ $u(0)=0$, on the interval $[0, 0.4]$.

Solution:-

given; $u(0)=0$, $h=0.2$

$$\frac{du}{dx} = f(x,u) = 4-x^2+u$$

on interval $[0, 0.4]$.

$u[0.4] = ?$ $x=0$

as $h=0.2$, we need two iterations.

$\therefore$ $u(0.2) = u(0)+K$

$$K = \frac{1}{6}\left[K_1+K_4+2(K_2+K_3)\right]$$

Where $K_1 = hf(x_0,u_0) = 0.2\,f(0,0) = 0.2\times 4$
$= 0.8$

$K_2 = hf(x_0+\frac{1}{2}h, u_0+\frac{1}{2}K_1) = 0.2f(0.1, 0.4)$
$= 0.8782$

$K_3 = hf(x_0+\frac{1}{2}h, u_0+\frac{1}{2}K_2) = 0.2f(0.1, 0.439)$
$= 0.8858$

$K_4 = hf(x_0+h, u_0+K_3) = 0.2\,f(0.2, 0.8858)$
$= 0.96916$

$\therefore K = \frac{1}{6}(0.96916+0.8+2(0.8782+0.8858)]$

$K = 0.8827933 \approx 0.8828$

$u(0.2) = u(0) + K = 0+0.8828 = 0.8828.$

Now, for $y(0.4) = ?$

$y(0.2) = 0.8828$ , $x = 0.2$. , $h = 0.2$

$K_1 = hf(0.2, 0.8828) = 0.2(4 - 0.04 + 0.8828)$
$= 0.96856$

$K_2 = hf(0.3, 1.36708) = 0.2(5.36708 - 0.09)$
$= 1.055416$

$K_3 = hf(0.3, 1.410508) = 0.2(5.410508 - 0.09)$
$= 1.0641016$

$K_4 = hf(0.4, 1.9469) = 0.2(5.9469 - 0.16)$
$= 1.15738$

$K = \frac{1}{6}[K_1 + K_4 + 2(K_2 + K_3)]$

$K = \frac{1}{6}[0.96856 + 1.15738 + 2(1.055416 + 1.0641016)]$

$K = \frac{1}{6}(6.3649352)$

$K = 1.060823 \simeq 1.0608$

$\therefore y(0.4) = y(0.2) + K$

$y(0.4) = 0.8828 + 1.0608$

$\boxed{y(0.4) = 1.9436}$

Which is required solution.

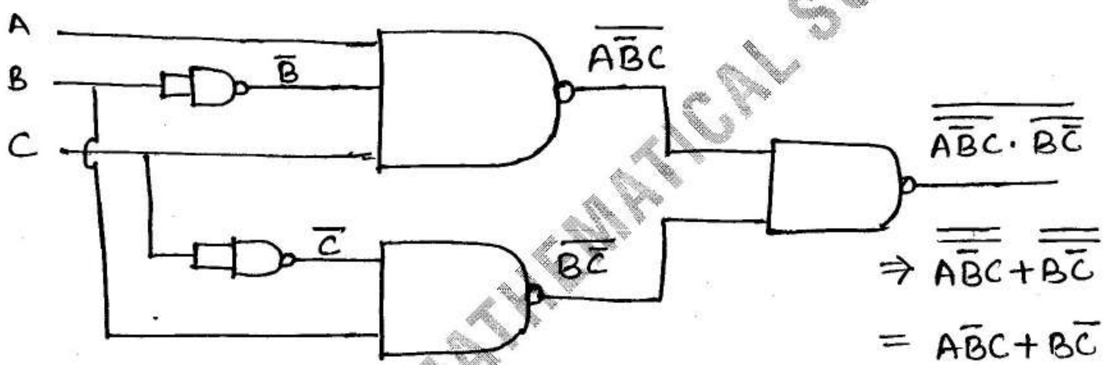
Ques: 7(d)(i) Draw the circuit diagram for $\bar{F} = A\bar{B}C + \bar{C}B$ using NAND to NAND logic long.

Solution:-

$$\bar{F} = A\bar{B}C + \bar{C}B$$

$$\bar{F} = \overline{\overline{A\bar{B}C}} + \overline{\overline{\bar{C}B}}$$

$$\boxed{\bar{F} = \overline{\overline{A\bar{B}C} \cdot \overline{B\bar{C}}}}$$




$$\Rightarrow \overline{\overline{A\bar{B}C}} + \overline{\overline{B\bar{C}}}$$

$$= A\bar{B}C + B\bar{C}$$

$$= \bar{F}$$

(iii) Design a logic circuit having three inputs A, B, C such that output is 1 when A=0 or whenever B=C=1. Also obtain logic circuit using only NAND gates?

Solution

| A | B | C | output = Y | $m_i$ (Output). |
|---|---|---|---|---|
| 0 | 0 | 0 | 1 | $m_0 = \bar{A}\bar{B}\bar{C}$ |
| 0 | 0 | 1 | 1 | $m_1 = \bar{A}\bar{B}C$ |
| 0 | 1 | 0 | 1 | $m_2 = \bar{A}B\bar{C}$ |
| 0 | 1 | 1 | 1 | $m_3 = \bar{A}BC$ |
| 1 | 0 | 0 | 0 | $m_4 = A\bar{B}\bar{C}$ |
| 1 | 0 | 1 | 0 | $m_5 = A\bar{B}C$ |
| 1 | 1 | 0 | 0 | $m_6 = AB\bar{C}$ |
| 1 | 1 | 1 | 1 | $m_7 = ABC$ |

$y = 1$ (output

$y = m_0 + m_1 + m_2 + m_3 + m_7$

$y = \bar{A}\bar{B}\bar{C} + \bar{A}\bar{B}C + \bar{A}B\bar{C} + \bar{A}BC + ABC$

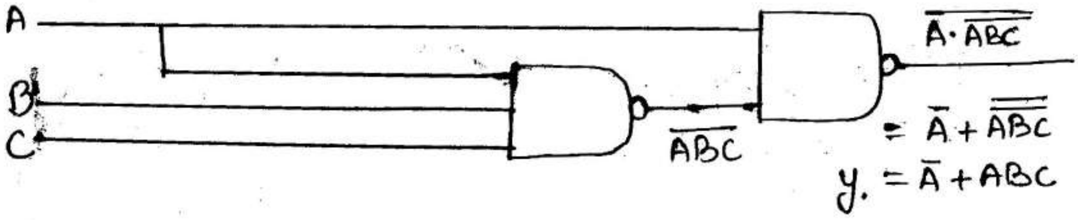
$y = \bar{A}[\bar{B}\bar{C} + \bar{B}C + B\bar{C} + BC] + ABC$

$y = \bar{A}[\bar{B}[\bar{C}+C] + B[\bar{C}+C]] + ABC$

$[\because \bar{C}+C = 1]$

$y = \bar{A}[\bar{B}+B] + ABC \qquad [\because [\bar{B}+B = 1]]$

$$\boxed{y = \bar{A} + ABC}$$




<u>Ques: 7(d) ii)</u> In a Boolean Algebra B, for any a and b prove that $ab' + a'b = 0$ if and only if $a = b$.

<u>Solution:</u> Given $ab' + a'b = 0$

$\Leftrightarrow \quad ab' = (a'b)' \qquad [-(a'b) = (a'b)']$

$ab' = a'' + b'$

$ab' = a + b'$

$ab' - (b') = a$

$b'(a-1) = a$

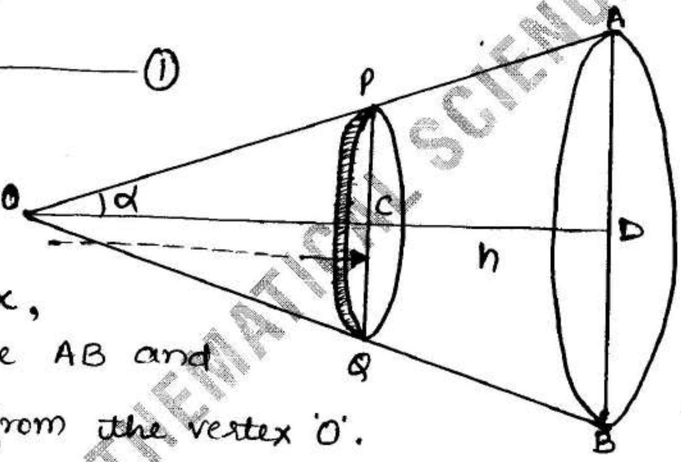
$-(b') = a$

$(b')' = a \Rightarrow \boxed{b = a}$ Hence proved.

Ques: 8(a)) (i) Find the M.I. of a right solid cone of mass M, height 'h' and radius of whose base is 'a', about its axis.

Solution:- Let, 'O' be the vertex of the right solid cone of mass M, height 'h' and radius of whose base is 'a'. If 'α' is the semi-vertical angle and ρ the density of the cone, then

$$\boxed{M = \frac{1}{3}\pi\rho h^3 \tan^2\alpha} \quad \text{——— (1)}$$




Consider an elementary disc PQ of thickness $\delta x$, parallel to the base AB and at a distance '$x$' from the vertex 'O'.

∴ Mass of the disc,

$$\boxed{\delta m = \rho\pi x^2 \tan^2\alpha\,\delta x}$$

M.I. of this elementary disc about axis OD.

$$= \frac{1}{2}\delta m CP^2 = \frac{1}{2}(\rho\pi x^2 \tan^2\alpha\,\delta x)\cdot x^2\tan^2\alpha$$

$$= \frac{1}{2}\rho\pi x^4 \tan^4\alpha\,\delta x$$

∴ M.I. of the cone about the axis OD.

$$= \int_0^h \frac{1}{2}\rho\pi x^4 \tan^4\alpha\,dx = \rho\cdot\frac{\pi}{10}\cdot h^5\tan^4\alpha$$

$$= \frac{3}{10}Mh^2\tan^2\alpha \qquad (\text{from } (1)).$$

∴ $\boxed{\text{M.I of cone about the axis OD} = \frac{3}{10}Ma^2}$ $\left[\because \tan\alpha = \frac{a}{h}\right]$.

Ques:- 8(a) ii) write Hamilton's equations for a particle of mass 'm' moving in a plane under a force which is some function of distance from the origin?

Solution:-

Let, $P(r,\theta)$ be the co-ordinates of a particle of mass 'm' at time 't' referred to the pole at O and OX as initial line. The components of velocity at P along OP and perpendicular to it are $\dot{r}$ and $r\dot{\theta}$ respectively.

$\therefore$ (Vel)$^2$ of mass 'm' at P $\Rightarrow V^2 = \dot{r}^2 + (r\dot{\theta})^2$

$\therefore$ K.E, $T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{r}^2 + (r\dot{\theta})^2)$

Since, the force at 'P' is some function of $\dot{r}$, therefore the potential 'V' is function of r alone and will be independent of $\theta$. i.e $V = V(r)$.

Thus, $L = T - V = \frac{1}{2}m(\dot{r}^2 + (r\dot{\theta})^2) - V(r)$

Here 'r' and '$\theta$' are the generalised co-ordinates,

$$\left.\begin{aligned} \therefore\ p_r &= \frac{\partial L}{\partial \dot{r}} = m\dot{r} \\ p_\theta &= \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta} \end{aligned}\right] \text{———} (1)$$

Since, L does not contain 't' explicitely

$\therefore H = T + V = \frac{1}{2}m(\dot{r}^2 + (r\dot{\theta})^2) + V(r)$

$H = T + V = \frac{1}{2}m\left[\left(p_r/m\right)^2 + r^2\left(p_\theta/mr^2\right)^2\right] + V(r) \rightarrow$ from (1)

$$\boxed{\therefore H = \frac{1}{2m}\left(p_r^2 + p_\theta^2/r^2\right) + V(r)}$$

Hence, four Hamilton's Equations are

$$\dot{p}_r = -\frac{\partial H}{\partial r} = \frac{p_\theta^2}{mr^3} - \frac{dV}{dr} \quad \text{—} \ H_1$$

$$\dot{r} = -\frac{\partial H}{\partial p_r} = -\frac{p_r}{m} \quad \text{—} \ H_2$$

$$\dot{p}_\theta = -\frac{\partial H}{\partial \theta} = 0 \quad \text{—} \ H_3$$

$$\dot{\theta} = \frac{\partial H}{\partial p_\theta} = \frac{p_\theta}{m\dot{r}^2} \quad \text{—} \ H_4.$$

Which is required results.

Ques: 8(b)) A particle of mass 'm' moves in a conservative forces field. Find–

(i) the Lagrangian function and

(ii) the equation of motion in cylindrical co-ordinates $(\rho, \phi, z)$.

Solution:–

Let, P be the position of the particle of mass 'm' whose cylindrical coordinates referred to axes OX, OY, OZ are $(\rho, \phi, z)$.

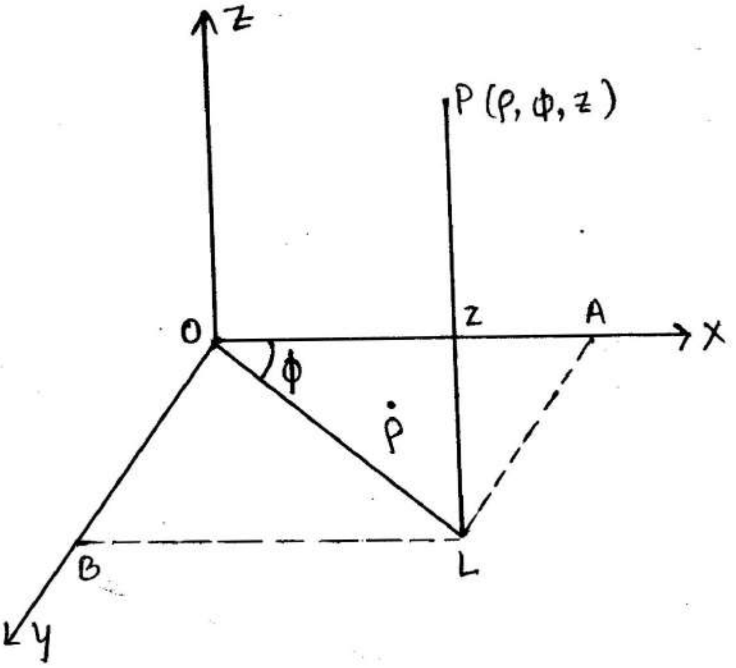
$\therefore$ If $(x, y, z)$ are its cartesian co-ordinates, then

$x = OA = \rho\cos\phi$

$y = OB = \rho\sin\phi$ , $z = z$

If $i, j, k$ are the unit vectors along OX, OY, OZ respectively, then

$$\vec{OP} = r = \rho\cos\phi\,\hat{i} + \rho\sin\phi\,\hat{j} + z\,\hat{k}$$



If $\hat{\rho}_1$ and $\hat{\phi}_1$ are the unit vectors in the directions of $\rho$ and $\phi$ increasing respectively, then

$$\hat{\rho}_1 = \frac{\partial r}{\partial \rho} \Big/ \left|\frac{\partial r}{\partial \rho}\right| = \frac{\cos\phi\,\hat{i} + \sin\phi\,\hat{j}}{\sqrt{\cos^2\phi + \sin^2\phi}}$$

$$\boxed{\hat{\rho}_1 = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}}$$

$$\hat{\phi}_1 = \frac{\partial r}{\partial \phi} \Big/ \left|\frac{\partial r}{\partial \phi}\right| = \frac{-\rho\sin\phi\,\hat{i} + \rho\cos\phi\,\hat{j}}{\sqrt{(\rho^2\sin^2\phi + \rho^2\cos^2\phi)}}$$

$$\hat{\phi}_1 = \frac{-\rho\sin\phi\,\hat{i} + \rho\cos\phi\,\hat{j}}{\rho\sqrt{\sin^2\phi + \cos^2\phi}} = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$$

Now; $v = \dot{r} = (\dot{\rho}\cos\phi - \rho\sin\phi\,\dot{\phi})\hat{i} + (\dot{\rho}\sin\phi + \rho\cos\phi\,\dot{\phi})\hat{j} + \dot{z}\hat{k}$

$$v = \dot{r} = \dot{\rho}(\cos\phi\,\hat{i} + \sin\phi\,\hat{j}) + \rho\dot{\phi}(-\sin\phi\,\hat{i} + \cos\phi\,\hat{j}) + \dot{z}\hat{k}$$

$$v = \dot{r} = (\dot{\rho})\hat{\rho}_1 + \rho\dot{\phi}(\hat{\phi}_1) + \dot{z}K$$

$$\boxed{\therefore\ v^2 = \dot{\rho}^2 + (\rho\dot{\phi})^2 + \dot{z}^2}$$

Total K.E, $\Rightarrow T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2)$

Let; $V = V(\rho, \phi, z)$ be the potential function.

$\therefore$ (i) Lagrangian function :; $L = T - V$

i.e. $\boxed{L = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) - V(\rho, \phi, z)}$

(ii) Lagrange's $\rho$ equation is,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\rho}}\right) - \frac{\partial L}{\partial \rho} = 0$$

or $$\frac{d}{dt}(m\dot{\rho}) - \left(m\rho\dot{\phi}^2 - \frac{\partial V}{\partial \rho}\right) = 0$$

i.e $$m\ddot{\rho} + (-m\rho\dot{\phi}^2) = -\frac{\partial V}{\partial \rho}$$

Lagrange's $\phi$ equation is $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0$

or $$\frac{d}{dt}(m\rho^2\dot{\phi}) - \left(-\frac{\partial V}{\partial \phi}\right) = 0$$

or $$\frac{d}{dt}(m\rho^2\dot{\phi}) = -\frac{\partial V}{\partial \phi}$$

and Lagrange's $z$ equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0$$

or $$\frac{d}{dt}(m\dot{z}) - \left(-\frac{\partial V}{\partial z}\right) = 0$$

or $$\boxed{m\ddot{z} = -\frac{\partial V}{\partial z}}$$ Which is required result.

Ques:- 8(c)) Prove that liquid motion is possible when velocity at $(x, y, z)$ is given by

$$u = \frac{3x^2 - r^2}{r^5}, \quad v = \frac{3xy}{r^5}, \quad w = \frac{3xz}{r^5}$$

where, $r^2 = x^2+y^2+z^2$ are the stream lines are the intersection of the surfaces, $(x^2+y^2+z^2)^3 = c(y^2+z^2)^2$ by the planes passing through $Ox$. Is this motion irrotational.

Solution:-

Step-I) To prove that the liquid motion is possible. for this we have do show that the equation of continuity namely-

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \qquad \text{——} (1)$$

is satisfied.

$$r^2 = x^2+y^2+z^2$$

$$\Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{(6x-2x)r^5 - 5r^3x(3x^2-r^2)}{r^{10}}$$

$$\frac{\partial v}{\partial y} = \frac{3x}{r^{10}}(r^5 - 5r^3y^2)$$

$$\frac{\partial w}{\partial z} = \frac{3x}{r^{10}}(r^5 - 5r^3z^2)$$

This $\Rightarrow \frac{\partial u}{\partial x} = \frac{3x}{r^7}(3r^2 - 5x^2)$

$$\frac{\partial v}{\partial y} = \frac{3x}{r^7}(r^2 - 5y^2)$$

$$\frac{\partial w}{\partial z} = \frac{3x}{r^7}(r^2 - 5z^2)$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0}$$ Hence, the result

Step-II) To determine stream lines

Stream lines are the solutions of $\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$

Putting the values;

$$\frac{dx}{3x^2 - r^2} = \frac{dy}{3xy} = \frac{dz}{3xz}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{x(3x^2 - r^2)} = \frac{y\,dy + z\,dz}{3x(y^2+z^2)}.$$

$$\Rightarrow \frac{dx}{3x^2 - r^2} = \frac{dy}{3xy} = \frac{dz}{3xz} = \frac{x\,dx + y\,dy + z\,dz}{x \cdot 2r^2} = \frac{y\,dy + z\,dz}{3x(y^2+z^2)}$$

$$\Rightarrow \frac{dy}{\cancel{3x}y} = \frac{dz}{\cancel{3x}z}$$

$$\Rightarrow \frac{dy}{y} = \frac{dz}{z} \Rightarrow \log y = \log z + \log c$$

$$\Rightarrow \boxed{y = zc} \quad \text{—— (2)}$$

and $$\frac{x\,dx + y\,dy + z\,dz}{\cancel{x} \cdot 2(x^2+y^2+z^2)} = \frac{y\,dz + z\,dz}{3\cancel{x}(y^2+z^2)} \quad \text{— (3)}$$

Integrating (3), we get.

$\Rightarrow \quad \frac{1}{2}\log(x^2+y^2+z^2) = \frac{1}{3}\log(y^2+z^2)+\frac{1}{6}\log b$

$$\Rightarrow \quad \boxed{(x^2+y^2+z^2)^3 = b(y^2+z^2)^2}$$

which is required result.