7

Definite Integrals

§ 7·1. Definition.

Sometimes in geometrical and other applications of integral calculus it becomes necessary to find the difference in the values of an integral of a function $f(x)$ for two given values of the variable x, say, a and b. This difference is called the *definite integral* of $f(x)$ from a to b or between the *limits* a and b.

This definite integral is denoted by

$$\int_a^b f(x)\,dx$$

and is read as *"the integral of $f(x)$ with respect to x between the limits a and b"*.

It is often written thus :

$$\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b = F(b) - F(a),$$

where $F(x)$ is an integral of $f(x)$, $F(b)$ is the value of $F(x)$ at $x = b$, and $F(a)$ is the value of $F(x)$ at $x = a$.

The number a is called the *lower limit* and the number b, the *upper limit* of integration. The interval (a, b) is called the *range of integration*.

**§ 7·2. Fundamental properties of Definite integrals.

Property 1. We have $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$ *i.e., the value of a definite integral does not change with the change of variable of integration (also called* **'argument'**) *provided the limits of integration remain the same.*

Proof. Let $\int f(x)\,dx = F(x)$; then $\int f(t)\,dt = F(t)$.

Now $\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b = F(b) - F(a)$, ...(1)

and $\int_a^b f(t)\,dt = \Big[F(t)\Big]_a^b = F(b) - F(a)$. ...(2)

From (1) and (2), we see that $\int_a^b f(x)\,dx = \int_a^b f(t)\,dt$.

Property 2. *We have* $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$ *i.e., interchanging the limits of a definite integral does not change the absolute value but changes only the sign of the integral.*

Proof. Let $\int f(x)\,dx = F(x)$. Then

$$\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b = F(b) - F(a) \quad ...(1)$$

Also $-\int_b^a f(x)\,dx = -\Big[F(x)\Big]_b^a = -[F(a) - F(b)] = F(b) - F(a).$

...(2)

From (1) and (2), we see that $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$.

Property 3. *We have* $\int_a^b \mathbf{f(x)\,dx} = \int_a^c \mathbf{f(x)\,dx} + \int_c^b \mathbf{f(x)\,dx}$.

Proof. Let $\int f(x)\,dx = F(x)$. Then the R.H.S.

$= \Big[F(x)\Big]_a^c + \Big[F(x)\Big]_c^b = \{F(c) - F(a)\} + \{F(b) - F(c)\}$

$= F(b) - F(a) = \int_a^b f(x)\,dx =$ L.H.S.

Note 1. This property also holds true even if the point c is exterior to the interval (a, b).

Note 2. In place of one additional point c, we can take several points. Thus

$$\int_a^b f(x)\,dx = \int_a^{c_1} f(x)\,dx + \int_{c_1}^{c_2} f(x)\,dx + \int_{c_2}^{c_3} f(x)\,dx + \dots + \int_{c_{r-1}}^{c_r} f(x)\,dx + \dots + \int_{c_n}^{b} f(x)\,dx.$$

****Property 4.** *We have* $\int_0^a \mathbf{f(x)\,dx} = \int_0^a \mathbf{f(a-x)\,dx}$.

(Lucknow 1980; Meerut 74, 84 S; Alld. 73; Bihar 74; Ranchi 74; Vikram 76; Kanpur 77; Magadh 77; Kashmir 71)

Proof. Let $I = \int_0^a f(x)\,dx$.

Put $x = a - t$, so that $dx = -dt$.

When $x = 0$, $t = a$ and when $x = a$, $t = 0$.

$\therefore \quad I = \int_a^0 f(a-t)(-dt) = \int_0^a f(a-t)\,dt,$ [by property 2]

$= \int_0^a f(a-x)\,dx.$ [by property 1]

Property 5. $\int_{-a}^{a} \mathbf{f(x)\,dx = 0}$ **or** $\mathbf{2}\int_0^a \mathbf{f(x)\,dx}$,

according as $f(x)$ *is an odd or an even function of* x. **(Kanpur 1980)**

Proof. Odd and even functions. A function $f(x)$ is said to be

(i) *an odd function* of x if $f(-x) = -f(x)$,

(ii) *an even function* of x if $f(-x) = f(x)$.

Now $\int_{-a}^{a} f(x)\,dx = \int_{-a}^{0} f(x)\,dx + \int_0^a f(x)\,dx$, **by property 3.** ...(1)

Let $u = \int_{-a}^{0} f(x)\,dx$. In the integral u, put $x = -t$ so that $dx = -dt$.

Also $t = a$, when $x = -a$ and $t = 0$ when $x = 0$.

$\therefore \quad u = \int_{a}^{0} f(-t)(-dt) = \int_{0}^{a} f(-t)\,dt,$ [by property 2]

$= \int_{0}^{a} f(-x)\,dx,$ [by property 1]

$= -\int_{0}^{a} f(x)\,dx$, if $f(x)$ is an odd function of x,

or $= \int_{0}^{a} f(x)\,dx$, if $f(x)$ is an even function of x.

$\therefore$ from (1), we get

$$\int_{-a}^{a} f(x)\,dx = -\int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 0,$$

if $f(x)$ is an odd function of x

and $$\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx,$$

if $f(x)$ is an even function of x.

****Property 6.** $\int_{0}^{2a} \mathbf{f(x)\,dx} = 2\int_{0}^{a} \mathbf{f(x)\,dx}$, **if** $\mathbf{f(2a - x) = f(x)}$

and $\int_{0}^{2a} \mathbf{f(x)\,dx} = 0$, **if** $\mathbf{f(2a - x) = -f(x)}$.

(Meerut 1981; Lucknow 77; Kanpur 78, 75, 74; Vikram 75; Bihar 74; Jiwaji 71)

Proof. We have $\int_{0}^{2a} f(x)\,dx = \int_{0}^{a} f(x)\,dx + \int_{a}^{2a} f(x)\,dx$

$= \int_{0}^{a} f(x)\,dx - \int_{a}^{0} f(2a - y)\,dy,$ [putting $x = 2a - y$ in the second integral and changing the limits]

$= \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(2a - y)\,dy,$

interchanging the limits in the second integral

$= \int_{0}^{a} f(x)\,dx + \int_{0}^{a} f(2a - x)\,dx,$

changing the argument from y to x in the second integral

$= 2\int_{0}^{a} f(x)\,dx$, if $f(2a - x) = f(x)$

or $= 0$, if $f(2a - x) = -f(x)$.

Cor. $\int_{0}^{2a} \mathbf{f(x)\,dx} = \int_{0}^{a} \mathbf{f(x)\,dx} + \int_{0}^{a} \mathbf{f(2a - x)\,dx}$.

(Kashmir 1974, 72)

Remember :

(i) $\int_{-\pi/2}^{\pi/2} f(\sin x)\,dx = 2\int_{0}^{\pi/2} f(\sin x)\,dx$ or $= 0$

as if, $f(\sin x)$ is an *even* or an *odd* function respectively.

(ii) $\int_0^{\pi} f(\sin x)\,dx = 2\int_0^{\pi/2} f(\sin x)\,dx$,

[by property 6, because $\sin(\pi - x) = \sin x$]

(iii) $\int_{-\pi/2}^{\pi/2} f(\cos x)\,dx = 2\int_0^{\pi/2} f(\cos x)\,dx$, [by property 5]

(iv) $\int_0^{\pi} f(\cos x)\,dx = 2\int_0^{\pi/2} f(\cos x)\,dx$ or $= 0$,

as if, $f(\cos x)$ is an *even* or an *odd* function respectively.

(v) $\int_0^{\pi/2} f(\sin x)\,dx = \int_0^{\pi/2} f\{\sin(\frac{1}{2}\pi - x)\}\,dx$, [by property 4]

$$= \int_0^{\pi/2} f(\cos x)\,dx.$$

(vi) $\int_0^{\pi} \sin^m x \cos^n x\,dx = 2\int_0^{\pi/2} \sin^m x \cos^n x\,dx$ or $= 0$,

according as n is an even or an odd integer, (by property 6).

Solved Examples

Ex. 1. *If* $f(x) = f(a + x)$, *prove that*

$$\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx.$$

(K. U. 1977)

Sol. We have $\int_0^{na} f(x)\,dx = \int_0^a f(x)\,dx +$

$$\int_a^{2a} f(x)\,dx + \int_{2a}^{3a} f(x)\,dx + \ldots + \int_{(n-1)a}^{na} f(x)\,dx. \quad ...(1)$$

[by prop. 3]

Now $\int_a^{2a} f(x)\,dx = \int_0^a f(a+t)\,dt$,

putting $x = a + t$ and changing the limits

$= \int_0^a f(a+x)\,dx$, [by prop. 1]

$= \int_0^a f(x)\,dx$, since $f(a+x) = f(x)$, as given ...(2)

Also $\int_{2a}^{3a} f(x)\,dx = \int_a^{2a} f(a+t)\,dt$,

putting $x = a + t$ and changing the limits

$= \int_a^{2a} f(a+x)\,dx = \int_a^{2a} f(x)\,dx$

$= \int_0^a f(x)\,dx$, from (2).

Thus we can show that each integral of the R.H.S. of (1) is equal to $\int_0^a f(x)\,dx$ and these being n in number, therefore

we have $\int_0^{na} f(x)\,dx = n\int_0^a f(x)\,dx$.

Note. Similarly we can prove that

$$\int_a^{na} f(x)\,dx = (n-1)\int_0^a f(x)\,dx.$$

Ex. 2 (a). *Evaluate* $\int_0^\pi \cos^{2n} x\,dx$.

(Kanpur 1974; Jiwaji 71)

Sol. We have $\int_0^\pi \cos^{2n} x\,dx = 2\int_0^{\pi/2} \cos^{2n} x\,dx$,

$\left[\because \int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx \text{ if } f(2a-x) = f(x).\right.$

Here taking $f(x) = \cos^{2n} x$, we see that

$\left. f(\pi - x) = \cos^{2n}(\pi - x) = (-\cos x)^{2n} = \cos^{2n} x = f(x)\right]$

$= 2\cdot\dfrac{(2n-1)(2n-3)\ldots\ldots 3.1}{2n(2n-2)(2n-4)\ldots\ldots 4.2}\cdot\dfrac{\pi}{2}$, by Walli's formula

$= \dfrac{(2n-1)(2n-3)\ldots 3.1}{2^n.n!}\cdot\pi.$

Ex. 2 (b). *Evaluate* $\int_0^\pi \cos^6 x\,dx$.

Sol. Let $f(x) = \cos^6 x$. Then

$f(\pi - x) = \cos^6(\pi - x) = (-\cos x)^6 = \cos^6 x = f(x)$.

Now $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$, if $f(2a-x) = f(x)$.

Refer property 6

$\therefore \int_0^\pi \cos^6 x\,dx = 2\int_0^{\pi/2} \cos^6 x\,dx$

$= 2\dfrac{5.3.1}{6.4.2}\cdot\dfrac{\pi}{2}$, by Walli's formula

$= 5\pi/16.$

Ex. 2 (c). *Evaluate* $\int_0^\pi \sin^3 x\,dx$.

Sol. Let $f(x) = \sin^3 x$.

Then $f(\pi - x) = \sin^3(\pi - x) = \sin^3 x = f(x)$.

Now $\int_0^{2a} f(x)\,dx = 2\int_0^a f(x)\,dx$, if $f(2a-x) = f(x)$.

$\therefore \int_0^\pi \sin^3 x\,dx = 2\int_0^{\pi/2} \sin^3 x\,dx$

$= 2\cdot\dfrac{2}{3.1}\cdot 1 = \dfrac{4}{3}$, by Walli's formula.

Ex. 2 (d). *Evaluate* $\int_0^\pi \theta\sin^3\theta\,d\theta$.

Sol. Let $I = \int_0^\pi \theta.\sin^3\theta\,d\theta$. ...(1)

Then $I = \int_0^\pi (\pi - \theta)\cdot\sin^3(\pi - \theta)\,d\theta$,

$$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx, \text{ refer prop. 4}\right]$$

$$= \int_0^\pi (\pi - \theta)\sin^3\theta\,d\theta. \quad ...(2)$$

Adding (1) and (2), we get

$$2I = \int_0^\pi [\theta\sin^3\theta + (\pi-\theta)\sin^3\theta]\,d\theta$$

$$= \int_0^\pi (\theta + \pi - \theta)\sin^3\theta\,d\theta$$

$$= \int_0^\pi \pi\sin^3\theta\,d\theta = \pi\int_0^\pi \sin^3\theta\,d\theta$$

$$= 2\pi\int_0^{\pi/2} \sin^3\theta\,d\theta,$$

by a property of definite integrals; refer prop. 6

$$= 2\pi\cdot\frac{2}{3.1}\cdot 1, \quad \text{by Walli's formula}$$

$$= 4\pi/3.$$

$\therefore \quad I = \frac{2}{3}\pi.$

Ex. 3 (a). *Evaluate the following integrals :*

(*i*) $\displaystyle\int_{-1}^{1} \frac{x^2 \sin^{-1} x}{\sqrt{(1-x^2)}}\,dx,$ (*ii*) $\displaystyle\int_{-a}^{a} x\sqrt{(a^2-x^2)}\,dx,$

(*iii*) $\displaystyle\int_{-1}^{1} \frac{x \sin^{-1} x}{\sqrt{(1-x^2)}}\,dx.$

Sol. (i) Let $I = \displaystyle\int_{-1}^{1} \frac{x^2 \sin^{-1} x}{\sqrt{(1-x^2)}}\,dx.$

Put $\sin^{-1} x = t$, so that $\{1/\sqrt{(1-x^2)}\}\,dx = dt$ and $x = \sin t$. When $x = -1$, $t = \sin^{-1}(-1) = -\pi/2$ and when $x = 1$, $t = \sin^{-1} 1 = \pi/2$.

$\therefore \quad I = \displaystyle\int_{-\pi/2}^{\pi/2} t\sin^2 t\,dt.$

Now let $f(t) = t\sin^2 t$. Then $f(-t) = (-t)\sin^2(-t)$ $= -t(-\sin t)^2 = -t\sin^2 t = -f(t)$.

Therefore $f(t)$ is an odd function of t.

$\therefore \quad I = \displaystyle\int_{-\pi/2}^{\pi/2} t\sin^2 t\,dt = 0.$ Refer property 5

(ii) Let $\quad I = \displaystyle\int_{-a}^{a} x\sqrt{(a^2-x^2)}\,dx.$

Let $\quad f(x) = x\sqrt{(a^2-x^2)}$. Then $f(-x) = -x\sqrt{\{a^2-(-x)^2\}}$ $= -x\sqrt{(a^2-x^2)} = -f(x)$. Therefore $f(x)$ is an odd function of x.

$\therefore \quad I = \displaystyle\int_{-a}^{a} x\sqrt{(a^2-x^2)}\,dx = 0.$ Refer property 5.

(iii) Let $I = \int_{-1}^{1} \frac{x \sin^{-1} x}{\sqrt{(1-x^2)}} dx$.

Proceeding as in part (i) of this question, we have

$I = \int_{-\pi/2}^{\pi/2} t \sin t \, dt$.

Let $f(t) = t \sin t$. Then $f(-t) = (-t) \sin(-t)$
$= t \sin t$, so that $f(t)$ is an even function of t.

$\therefore \quad I = 2 \int_0^{\pi/2} t \sin t \, dt$, Refer property 5

$= 2 \left[t(-\cos t) \right]_0^{\pi/2} - 2 \int_0^{\pi/2} 1 . (-\cos t) \, dt$

$= 2 \times 0 + 2 \int_0^{\pi/2} \cos t \, dt = 2 \left[\sin t \right]_0^{\pi/2} = 2[1-0] = 2.$

Ex. 3 (b). *Prove without performing integration that*

$$\int_{-a}^{2a} \frac{x \, dx}{x^2 + p^2} = \int_a^{2a} \frac{x \, dx}{x^2 + p^2}.$$

Sol. We have

$$\int_{-a}^{2a} \frac{x \, dx}{x^2 + p^2} = \int_{-a}^{a} \frac{x \, dx}{x^2 + p^2} + \int_a^{2a} \frac{x \, dx}{x^2 + p^2}. \quad ...(1)$$

But if $f(x) = \frac{x}{x^2 + p^2}$, then $f(-x) = \frac{-x}{x^2 + p^2} = -f(x)$.

Therefore $f(x)$ is an odd function of x.

$\therefore \quad \int_{-a}^{a} \frac{x \, dx}{x^2 + p^2} = 0$. So from (1), we get

$$\int_{-a}^{2a} \frac{x \, dx}{x^2 + p^2} = \int_a^{2a} \frac{x \, dx}{x^2 + p^2}.$$

Ex. 4. *Evaluate the following integrals :*

(*i*) $\int_0^{\pi} \frac{dx}{a + b \cos x}$ (*ii*) $\int_0^{2\pi} \frac{dx}{a + b \cos x + c \sin x}$.

Sol. (i) Let $I = \int_0^{\pi} \frac{dx}{a + b \cos x}$. ...(1)

Then $I = \int_0^{\pi} \frac{dx}{a + b \cos(\pi - x)}$ (prop. 4) $= \int_0^{\pi} \frac{dx}{a - b \cos x}$. ...(2)

Adding (1) and (2), we get

$2I = \int_0^{\pi} \frac{2a}{a^2 - b^2 \cos^2 x} dx$

$$= 2a \cdot 2\int_0^{\pi/2} \frac{dx}{a^2 - b^2 \cos^2 x}.$$ Refer property 6

$$\therefore \quad I = 2a \int_0^{\pi/2} \frac{dx}{a^2 - b^2 \cos^2 x} = 2a \int_0^{\pi/2} \frac{\sec^2 x \, dx}{a^2 \sec^2 x - b^2}$$

$$= 2a \int_0^{\pi/2} \frac{\sec^2 x \, dx}{a^2 (1 + \tan^2 x) - b^2} = 2a \int_0^{\pi/2} \frac{\sec^2 x \, dx}{a^2 \tan^2 x + a^2 - b^2}.$$

Now let $a > b > 0$.

Put $a \tan x = t$, so that $a \sec^2 x \, dx = dt$. The new limits for t are 0 to ∞.

$$\therefore \quad I = 2\int_0^{\infty} \frac{dt}{t^2 + \{\sqrt{(a^2 - b^2)}\}^2}$$

$$= 2 \frac{1}{\sqrt{(a^2 - b^2)}} \left[\tan^{-1} \frac{t}{\sqrt{(a^2 - b^2)}}\right]_0^{\infty}$$

$$= \frac{2}{\sqrt{(a^2 - b^2)}} [\tan^{-1} \infty - \tan^{-1} 0]$$

$$= \frac{2}{\sqrt{(a^2 - b^2)}} \cdot \left(\frac{\pi}{2} - 0\right) = \frac{\pi}{\sqrt{(a^2 - b^2)}}.$$

(ii) Let $\quad I = \displaystyle\int_0^{2\pi} \frac{dx}{a + b \cos x + c \sin x}.$

Let $b = r \cos \alpha$ and $c = r \sin \alpha$, so that $r = \sqrt{(b^2 + c^2)}$ and $\tan \alpha = c/b$.

Then $\quad b \cos x + c \sin x = r (\cos \alpha \cos x + \sin \alpha \sin x)$

$$= r \cos (x - \alpha).$$

$$\therefore \quad I = \int_0^{2\pi} \frac{dx}{a + r \cos (x - \alpha)}.$$

Put $x - \alpha = t$, so that $dx = dt$. When $x = 0$, $t = -\alpha$ and when $x = 2\pi$, $t = 2\pi - \alpha$.

$$\therefore \quad I = \int_{-\alpha}^{2\pi - \alpha} \frac{dt}{a + r \cos t}$$

$$= \int_{-\alpha}^{0} \frac{dt}{a + r \cos t} + \int_0^{2\pi - \alpha} \frac{dt}{a + r \cos t}. \quad ...(1)$$

[Refer property 3]

Now put $t = z - 2\pi$ in the first integral on the R.H.S. of (1). Then $dt = dz$ and the limits for z are $2\pi - \alpha$ to 2π.

$$\therefore \quad I = \int_{2\pi - \alpha}^{2\pi} \frac{dz}{a + r \cos (z - 2\pi)} + \int_0^{2\pi - \alpha} \frac{dt}{a + r \cos t}$$

$$= \int_0^{2\pi - \alpha} \frac{dt}{a + r\cos t} + \int_{2\pi - \alpha}^{2\pi} \frac{dz}{a + r\cos z}$$

$$= \int_0^{2\pi - \alpha} \frac{dt}{a + r\cos t} + \int_{2\pi - \alpha}^{2\pi} \frac{dt}{a + r\cos t},$$ because a definite integral does not change by changing the variable

$$= \int_0^{2\pi} \frac{dt}{a + r\cos t},$$ Refer property 3

$$= 2\int_0^{\pi} \frac{dt}{a + r\cos t},$$ Refer property 6.

Now proceeding as in part (i) of this question, we get

$$\int_0^{\pi} \frac{dt}{a + r\cos t} = \frac{\pi}{\sqrt{(a^2 - r^2)}},$$ provided $a > r > 0$.

$$\therefore \quad I = \frac{2\pi}{\sqrt{[a^2 - (b^2 + c^2)]}},$$ provided $a > \sqrt{(b^2 + c^2)} > 0$

$$= \frac{2\pi}{\sqrt{(a^2 - b^2 - c^2)}}.$$

***Ex. 5 (a).** *Evaluate* $\displaystyle\int_0^{\pi} \frac{x\,dx}{a^2 \cos^2 x + b^2 \sin^2 x}$.

(Delhi 1983; Meerut 81, 84 P)

Sol. Let $$I = \int_0^{\pi} \frac{x\,dx}{a^2 \cos^2 x + b^2 \sin^2 x}. \quad \text{...(1)}$$

Then $$I = \int_0^{\pi} \frac{(\pi - x)\,dx}{a^2 \cos^2 (\pi - x) + b^2 \sin^2 (\pi - x)},$$

$$\left[\because \int_0^a f(x)\,dx = \int_0^a f(a - x)\,dx\right]$$

$$= \int_0^{\pi} \frac{(\pi - x)\,dx}{a^2 \cos^2 x + b^2 \sin^2 x}. \quad \text{...(2)}$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi} \frac{x + (\pi - x)}{a^2 \cos^2 x + b^2 \sin^2 x}\,dx$$

$$= \pi \int_0^{\pi} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x}$$

$$= 2\pi \int_0^{\pi/2} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x},$$ by a property of definite integrals. Refer prop. 6.

$$\therefore \quad I = \pi \int_0^{\pi/2} \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x}$$

$$= \pi \int_0^{\pi/2} \frac{\sec^2 x \, dx}{a^2 + b^2 \tan^2 x},$$ dividing the numerator and the denominator by $\cos^2 x$.

Now put $b \tan x = t$. Then $b \sec^2 x \, dx = dt$.
Also when $x = 0, t = 0$ and when $x \to \pi/2, t \to \infty$.

$$\therefore \quad I = \frac{\pi}{b} \int_0^{\infty} \frac{dt}{a^2 + t^2} = \frac{\pi}{b} \cdot \frac{1}{a} \left[\tan^{-1} \frac{t}{a}\right]_0^{\infty}$$

$$= \frac{\pi}{ab} [\tan^{-1} \infty - \tan^{-1} 0]$$

$$= \frac{\pi}{ab} \left[\frac{\pi}{2} - 0\right] = \frac{\pi^2}{2ab}.$$

Ex. 5 (b). *Show that*

$$\int_0^{\pi} \frac{x \, dx}{(a^2 \cos^2 x + b^2 \sin^2 x)^2} = \frac{\pi^2 (a^2 + b^2)}{4a^3 b^3}.$$

(Meerut 1983 S, 81)

Sol. Let $$I = \int_0^{\pi} \frac{x \, dx}{[a^2 \cos^2 x + b^2 \sin^2 x]^2}$$

$$= \int_0^{\pi} \frac{(\pi - x) \, dx}{[a^2 \cos^2 (\pi - x) + b^2 \sin^2 (\pi - x)]^2},$$ (Refer prop. 4)

$$= \int_0^{\pi} \frac{(\pi - x) \, dx}{[a^2 \cos^2 x + b^2 \sin^2 x]^2} = \int_0^{\pi} \frac{\pi \, dx}{[a^2 \cos^2 x + b^2 \sin^2 x]^2} - I.$$ (Note)

$$\therefore \quad 2I = \int_0^{\pi} \frac{\pi \, dx}{[a^2 \cos^2 x + b^2 \sin^2 x]^2}$$

$$= 2 \int_0^{\pi/2} \frac{\pi \, dx}{[a^2 \cos^2 x + b^2 \sin^2 x]^2},$$ Refer prop. 6

or $$I = \pi \int_0^{\pi/2} \frac{\sec^4 x \, dx}{(a^2 + b^2 \tan^2 x)^2},$$ dividing the Nr. and the Dr. by $\cos^4 x$

$$= \pi \int_0^{\pi/2} \frac{(1 + \tan^2 x) \sec^2 x}{(a^2 + b^2 \tan^2 x)^2} dx.$$

Now put $b \tan x = a \tan \theta$, so that $b \sec^2 x \, dx = a \sec^2 \theta \, d\theta$. Also when $x = 0, \theta = 0$ and when $x = \frac{1}{2}\pi, \theta = \frac{1}{2}\pi$.

$$\therefore \quad I = \pi \int_0^{\pi/2} \frac{\{1 + (a^2/b^2) \tan^2 \theta\} . (a/b) \sec^2 \theta \, d\theta}{a^4 \sec^4 \theta}$$

$$= \frac{\pi}{a^3 b^3} \int_0^{\pi/2} (b^2 \cos^2 \theta + a^2 \sin^2 \theta) \, d\theta.$$

$$=\frac{\pi}{a^3b^3}\left[b^2\cdot\frac{1}{2}\cdot\frac{\pi}{2}+a^2\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right],$$ by Walli's formula

$$=\frac{\pi^2}{4a^3b^3}(a^2+b^2).$$

Ex. 6. *Show that* $\int_0^{\pi}\frac{x\,dx}{a^2-\cos^2 x}=\frac{\pi^2}{2a\sqrt{(a^2-1)}}$, $(a>1)$.

(Meerut 1981 S, 84, 90 S; Luck. 76; Kanpur 72)

Sol. Let $I=\int_0^{\pi}\frac{x\,dx}{a^2-\cos^2 x}=\int_0^{\pi}\frac{(\pi-x)\,dx}{a^2-\cos^2(\pi-x)}$,

(Refer prop. 4)

$$=\pi\int_0^{\pi}\frac{dx}{a^2-\cos^2 x}-I.$$

$$\therefore\quad 2I=\int_0^{\pi}\frac{\pi\,dx}{a^2-\cos^2 x}=2\int_0^{\pi/2}\frac{\pi\,dx}{a^2-\cos^2 x},\ \text{(Refer prop. 6)}$$

or $I=\pi\int_0^{\pi/2}\frac{\sec^2 x\,dx}{a^2\sec^2 x-1}$, dividing the Nr. and the Dr. by $\cos^2 x$

$$=\pi\int_0^{\pi/2}\frac{\sec^2 x\,dx}{a^2(1+\tan^2 x)-1}=\pi\int_0^{\pi/2}\frac{\sec^2 x\,dx}{(a^2-1)+a^2\tan^2 x}.$$

Now put $a\tan x=t$, so that $a\sec^2 x\,dx=dt$. Also $t=0$ when $x=0$ and $t\to\infty$ when $x\to\frac{1}{2}\pi$.

$$\therefore\quad I=\frac{\pi}{a}\int_0^{\infty}\frac{dt}{(a^2-1)+t^2}=\frac{\pi}{a}\cdot\frac{1}{\sqrt{(a^2-1)}}\left[\tan^{-1}\left\{\frac{t}{\sqrt{(a^2-1)}}\right\}\right]_0^{\infty}$$

$$=\frac{\pi}{a\sqrt{(a^2-1)}}[\tan^{-1}\infty-\tan^{-1}0]$$

$$=\frac{\pi}{a\sqrt{(a^2-1)}}\left[\frac{\pi}{2}-0\right]=\frac{\pi^2}{2a\sqrt{(a^2-1)}}.$$

Ex. 7. *Evaluate* $\int_0^{\pi}\frac{x\,dx}{1+\cos^2 x}$. **(Ranchi 1972)**

Sol. Let $I=\int_0^{\pi}\frac{x\,dx}{1+\cos^2 x}=\int_0^{\pi}\frac{(\pi-x)\,dx}{1+\cos^2(\pi-x)}$,

[Refer prop. 4]

$$=\int_0^{\pi}\frac{(\pi-x)\,dx}{1+\cos^2 x}=\int_0^{\pi}\frac{\pi\,dx}{1+\cos^2 x}-\int_0^{\pi}\frac{x\,dx}{1+\cos^2 x}$$

$$=\pi\int_0^{\pi}\frac{dx}{1+\cos^2 x}-I.$$

$$\therefore \quad 2I = \pi \int_0^{\pi} \frac{dx}{1+\cos^2 x} = 2\pi \int_0^{\pi/2} \frac{dx}{1+\cos^2 x},$$ [Refer prop. 6]

or $I = \pi \int_0^{\pi/2} \frac{\sec^2 x \, dx}{\sec^2 x + 1}$, dividing the Nr. and the Dr. by $\cos^2 x$

$$= \pi \int_0^{\pi/2} \frac{\sec^2 x \, dx}{2 + \tan^2 x}.$$

Now put $\tan x = t$, so that $\sec^2 x \, dx = dt$. Also $t = 0$ when $x = 0$ and $t \to \infty$ when $x \to \frac{1}{2}\pi$.

$$\therefore \quad I = \pi \int_0^{\infty} \frac{dt}{2+t^2} = \pi \cdot \frac{1}{\sqrt{2}} \left[\tan^{-1} \frac{t}{\sqrt{2}}\right]_0^{\infty}$$

$$= \frac{\pi}{\sqrt{2}} [\tan^{-1} \infty - \tan^{-1} 0]$$

$$= \frac{\pi}{\sqrt{2}} \left[\frac{\pi}{2} - 0\right] = \frac{\pi^2}{2\sqrt{2}} = \frac{\pi^2 \sqrt{2}}{4}.$$

Ex. 8 (a). *Evaluate* $\int_0^{\pi/2} \frac{\cos x - \sin x}{1 + \sin x \cos x} dx.$

Sol. Let $I = \int_0^{\pi/2} \frac{\cos x - \sin x}{1 + \sin x \cos x} dx.$

Then $I = \int_0^{\pi/2} \frac{\cos(\frac{1}{2}\pi - x) - \sin(\frac{1}{2}\pi - x)}{1 + \sin(\frac{1}{2}\pi - x)\cos(\frac{1}{2}\pi - x)} dx$, [Refer prop. 4]

$$= \int_0^{\pi/2} \frac{\sin x - \cos x}{1 + \cos x \sin x} dx = -\int_0^{\pi/2} \frac{\cos x - \sin x}{1 + \sin x \cos x} dx = -I.$$

$\therefore \quad 2I = 0 \quad$ or $\quad I = 0.$

Ex. 8 (b). *Evaluate* $\int_0^{\pi/2} (\sin x - \cos x) \log(\sin x + \cos x) \, dx.$

Sol. Let $I = \int_0^{\pi/2} (\sin x - \cos x) \log(\sin x + \cos x) \, dx.$

Then $I = \int_0^{\pi/2} (\cos x - \sin x) \log(\sin x + \cos x) \, dx$

$\left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx.\right.$

Note that $\sin(\frac{1}{2}\pi - x) = \cos x$

and $\left.\cos(\frac{1}{2}\pi - x) = \sin x\right]$

$$= -\int_0^{\pi/2} (\sin x - \cos x) \log(\sin x + \cos x) \, dx = -I.$$

$\therefore \quad 2I = 0 \quad$ or $\quad I = 0.$

Ex. 8 (c). *Evaluate* $\int_0^{\pi/2} \sin 2x \log \tan x \, dx$.

Sol. Let $I = \int_0^{\pi/2} \sin 2x \log \tan x \, dx$. ...(1)

Then $I = \int_0^{\pi/2} \sin 2\left(\frac{1}{2}\pi - x\right) \log \tan\left(\frac{1}{2}\pi - x\right)$, [Refer prop. 4]

$= \int_0^{\pi/2} \sin(\pi - 2x) \log \cot x \, dx$

$= \int_0^{\pi/2} \sin 2x \log \cot x \, dx$. ...(2)

Adding (1) and (2), we get

$2I = \int_0^{\pi/2} \sin 2x \, (\log \tan x + \log \cot x) \, dx$

$= \int_0^{\pi/2} \sin 2x \log(\tan x \cot x) \, dx$

$= \int_0^{\pi/2} (\sin 2x) \, . \log 1 \, dx$

$= \int_0^{\pi/2} 0 \, . \sin 2x \, dx$

$= 0 \times \int_0^{\pi/2} \sin 2x \, dx = 0$.

$\therefore \quad I = 0$.

Ex. 8 (d). *Show that* $\displaystyle\int_0^{\pi/2} \frac{\sin x - \cos x}{\sin x + \cos x} dx = 0$. (Meerut 1989 S)

Sol. Proceed as in Ex. 8 (a).

Ex. 9. *Evaluate* $\displaystyle\int_0^{\pi} \frac{x \, dx}{1 + \sin x}$. (Kanpur 1979)

Sol. Let $I = \displaystyle\int_0^{\pi} \frac{x \, dx}{1 + \sin x} = \int_0^{\pi} \frac{(\pi - x) \, dx}{1 + \sin(\pi - x)}$,

[Refer prop. 4]

$= \displaystyle\int_0^{\pi} \frac{(\pi - x)}{1 + \sin x} dx = \int_0^{\pi} \frac{\pi}{1 + \sin x} dx - \int_0^{\pi} \frac{x}{1 + \sin x} dx$

$= \pi \displaystyle\int_0^{\pi} \frac{1}{1 + \sin x} dx - I$.

$\therefore \quad 2I = \pi \displaystyle\int_0^{\pi} \frac{dx}{1 + \sin x} = 2\pi \int_0^{\pi/2} \frac{dx}{1 + \sin x}$, [Refer prop. 6]

or $I = \pi \displaystyle\int_0^{\pi/2} \frac{dx}{1 + \sin x} = \pi \int_0^{\pi/2} \frac{dx}{1 + \sin\left(\frac{1}{2}\pi - x\right)}$,

[Refer prop. 4]

$$= \pi \int_0^{\pi/2} \frac{dx}{1+\cos x} = \pi \int_0^{\pi/2} \frac{dx}{2\cos^2 \frac{1}{2}x} = \pi \int_0^{\pi/2} \tfrac{1}{2}\sec^2 \tfrac{1}{2}x\, dx$$

$$= \pi \left[\tan \tfrac{1}{2}x\right]_0^{\pi/2} = \pi\, [\tan \tfrac{1}{4}\pi - \tan 0] = \pi\,(1-0) = \pi.$$

Ex. 10. *Prove that* $\displaystyle\int_0^{\pi} \frac{x \sin x}{1+\sin x}\, dx = \pi\left(\frac{\pi}{2} - 1\right).$

Sol. Let $\displaystyle I = \int_0^{\pi} \frac{x \sin x}{1+\sin x}\, dx = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\sin(\pi - x)}\, dx,$

[Refer prop. 4]

$$= \int_0^{\pi} \frac{(\pi - x)\sin x}{1+\sin x}\, dx = \pi \int_0^{\pi} \frac{\sin x}{1+\sin x}\, dx - \int_0^{\pi} \frac{x\sin x}{1+\sin x}\, dx$$

$$= \pi \int_0^{\pi} \frac{\sin x}{1+\sin x}\, dx - I.$$

$$\therefore \quad 2I = \pi \int_0^{\pi} \frac{\sin x}{1+\sin x}\, dx = 2\pi \int_0^{\pi/2} \frac{\sin x}{1+\sin x}\, dx,$$

[Refer prop. 6]

or $\displaystyle I = \pi \int_0^{\pi/2} \frac{(1+\sin x) - 1}{1+\sin x}\, dx = \pi \int_0^{\pi/2} \left[1 - \frac{1}{1+\sin x}\right] dx$

$$= \pi \int_0^{\pi/2} dx - \pi \int_0^{\pi/2} \frac{dx}{1+\sin x}$$

$$= \pi \left[x\right]_0^{\pi/2} - \pi \int_0^{\pi/2} \frac{dx}{1+\sin(\frac{1}{2}\pi - x)} \qquad \text{[Refer prop. 4]}$$

$$= \pi\left(\frac{\pi}{2} - 0\right) - \pi \int_0^{\pi/2} \frac{dx}{1+\cos x} = \frac{\pi^2}{2} - \pi \int_0^{\pi/2} \frac{dx}{2\cos^2 \frac{1}{2}x}$$

$$= \frac{\pi^2}{2} - \pi \int_0^{\pi/2} \tfrac{1}{2}\sec^2 \tfrac{1}{2}x\, dx = \frac{\pi^2}{2} - \pi \left[\tan \tfrac{1}{2}x\right]_0^{\pi/2}$$

$$= \frac{\pi^2}{2} - \pi\, [\tan \tfrac{1}{4}\pi - \tan 0] = \frac{\pi^2}{2} - \pi = \pi\left(\frac{\pi}{2} - 1\right).$$

****Ex. 11.** *Evaluate* $\displaystyle\int_0^{\pi} \frac{x \sin x}{(1+\cos^2 x)}\, dx.$

(Meerut 1983, 90, 91S; Kanpur 71; Gorakhpur 71; Agra 80; Rohilkhand 79)

Sol. Let $\displaystyle I = \int_0^{\pi} \frac{x \sin x}{1+\cos^2 x}\, dx,$ ...(1)

Then $\displaystyle I = \int_0^{\pi} \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)}\, dx,$

$$= \int_0^{\pi} \frac{(\pi - x)\sin x}{1 + \cos^2 x}\, dx. \qquad \left[\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right] \quad ...(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi} \frac{\pi \sin x}{1 + \cos^2 x}\, dx = \pi \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x}\, dx$$

$$= 2\pi \int_0^{\pi/2} \frac{\sin x}{1 + \cos^2 x}\, dx, \qquad \text{[Refer prop. 6]}$$

or $$I = \pi \int_0^{\pi/2} \frac{\sin x}{1 + \cos^2 x}\, dx.$$

Now put $\cos x = t$, so that $-\sin x\, dx = dt$. Also $t = 1$ when $x = 0$ and $t = 0$ when $x = \frac{1}{2}\pi$.

$$\therefore \quad I = \pi \int_1^0 \frac{-dt}{1 + t^2} = \pi \int_0^1 \frac{dt}{1 + t^2} = \pi \left[\tan^{-1} t\right]_0^1$$

$$= \pi (\tan^{-1} 1 - \tan^{-1} 0) = \pi (\tfrac{1}{4}\pi - 0) = \tfrac{1}{4}\pi^2.$$

***Ex. 12.** *Show that* $\displaystyle\int_0^{\pi} \frac{x \tan x}{\sec x + \cos x}\, dx = \tfrac{1}{4}\pi^2.$

Sol. The given integral $\displaystyle I = \int_0^{\pi} \frac{x.(\sin x/\cos x)}{(1/\cos x) + \cos x}\, dx$

$$= \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x}\, dx = \frac{\pi^2}{4}. \qquad \text{[Proceed as in Ex. 11]}$$

****Ex. 13.** *Show that* $\displaystyle\int_0^{\pi} \frac{x \tan x\, dx}{\sec x + \tan x} = \pi (\tfrac{1}{2}\pi - 1).$

(Meerut 1984, 85 S, 87, 88 S; Gorakhpur 82; Allahabad 80; Delhi 82; Luck. 79; 77, 74; K.U. 77; Kashmir 75)

Sol. Let

$$I = \int_0^{\pi} \frac{x \tan x}{\sec x + \tan x}\, dx = \int_0^{\pi} \frac{x.(\sin x/\cos x)}{(1/\cos x) + (\sin x/\cos x)}\, dx$$

$$= \int_0^{\pi} \frac{x \sin x}{1 + \sin x}\, dx.$$

Now proceed as in Ex. 10.

***Ex. 14.** *Evaluate* $\int_0^{\pi} x \sin^6 x \cos^4 x\, dx.$ **(Meerut 1983, 74)**

Sol. Let $I = \int_0^{\pi} x \sin^6 x \cos^4 x\, dx$

$$= \int_0^{\pi} (\pi - x) \sin^6 (\pi - x) \cos^4 (\pi - x)\, dx, \qquad \text{(Refer prop. 4)}$$

$$= \int_0^{\pi} (\pi - x) \sin^6 x \cos^4 x \, dx$$
$$= \int_0^{\pi} \pi \sin^6 x \cos^4 x \, dx - \int_0^{\pi} x \sin^6 x \cos^4 x \, dx$$
$$= \pi \int_0^{\pi} \sin^6 x \cos^4 x \, dx - I.$$
$$\therefore \quad 2I = \pi \int_0^{\pi} \sin^6 x \cos^4 x \, dx = 2\pi \int_0^{\pi/2} \sin^6 x \cos^4 x \, dx,$$
[Refer prop. 6]

or $I = \pi \int_0^{\pi/2} \sin^6 x \cos^4 x \, dx$

$$= \pi \frac{5.3.1.3.1}{10.8.6.4.2} \cdot \frac{\pi}{2} = \frac{3\pi^2}{512},$$
by Walli's formula.

Ex. 15. *Evaluate the following integrals :*

(*i*) $\int_0^{\pi} \sin^3 \theta \, (1 + 2 \cos \theta) \, (1 + \cos \theta)^2 \, d\theta.$ (Agra 1982)

(*ii*) $\int_0^{\pi} \sin^5 x \, (1 - \cos x)^3 \, dx.$

Sol. **(i)** Let $I = \int_0^{\pi} \sin^3 \theta \, (1 + 2 \cos \theta) \, (1 + \cos \theta)^2 \, d\theta$

$$= \int_0^{\pi} \sin^3 \theta \, (1 + 2 \cos \theta) \, (1 + 2 \cos \theta + \cos^2 \theta) \, d\theta$$
$$= \int_0^{\pi} \sin^3 \theta \, (1 + 4 \cos \theta + 5 \cos^2 \theta + 2 \cos^3 \theta) \, d\theta$$
$$= \int_0^{\pi} (\sin^3 \theta + 4 \sin^3 \theta \cos \theta + 5 \sin^3 \theta \cos^2 \theta + 2 \sin^3 \theta \cos^3 \theta) \, d\theta.$$

Now $\int_0^{\pi} \sin^m \theta \cos^n \theta \, d\theta = 2 \int_0^{\pi/2} \sin^m \theta \cos^n \theta \, d\theta$ or $= 0$,

according as n is an even or an odd integer. [Refer prop. 6]

$$\therefore \quad I = 2 \int_0^{\pi/2} \sin^3 \theta \, d\theta + 5 \times 2 \int_0^{\pi/2} \sin^3 \theta \cos^2 \theta \, d\theta,$$

because the integrals containing odd powers of $\cos \theta$ vanish

$$= 2 \cdot \frac{2}{3.1} + 10 \cdot \frac{2.1}{5.3.1} = \frac{4}{3} + \frac{4}{3} = \frac{8}{3}.$$

(ii) Let $I = \int_0^{\pi} \sin^5 x \, (1 - \cos x)^3 \, dx$

$$= \int_0^{\pi} \left(2 \sin \frac{x}{2} \cos \frac{x}{2}\right)^5 \left(2 \sin^2 \frac{x}{2}\right)^3 dx$$
$$= 2^8 \int_0^{\pi} \sin^{11} \frac{x}{2} \cos^5 \frac{x}{2} \, dx.$$

Now put $x/2 = t$, so that $dx = 2 \, dt$.
When $x = 0, t = 0$ and when $x = \pi, t = \frac{1}{2}\pi$.

$$\therefore \quad I = 2 \times 2^8 \int_0^{\pi/2} \sin^{11} t \cos^5 t \, dt$$

$$= 2 \times 2^8 \cdot \frac{10.8.6.4.2.4.2}{16.14.12.10.8.6.4.2} = \frac{32}{21}.$$

***Ex. 16.** *Show that* $\int_0^{\pi/2} \log(\tan x)\, dx = 0.$

(Meerut 1986 S)

Sol. Let $I = \int_0^{\pi/2} \log(\tan x)\, dx = \int_0^{\pi/2} \log\tan(\frac{1}{2}\pi - x)\, dx,$

[Refer prop. 4]

$= \int_0^{\pi/2} \log\cot x\, dx = \int_0^{\pi/2} \log(\tan x)^{-1}\, dx$

$= -\int_0^{\pi/2} \log\tan x\, dx = -I.$

$\therefore \quad 2I = 0$ *i.e.*, $I = 0.$

****Ex. 17.** *Show that* $\int_0^{\pi/2} \log\sin x\, dx = -\frac{1}{2}\pi\log 2$ *or* $\frac{1}{2}\pi\log\frac{1}{2}$.

(Meerut 1982, 84 P, 84 S, 87 P, 88, 89 S, 91; Luck. 77, 75; Kanpur 78; Gorakh. 78; Delhi 81, 77; Rohilkhand 79; Allahabad 79; Agra 78; K.U. 77)

Sol. Let $I = \int_0^{\pi/2} \log\sin x\, dx.$ ...(1)

Then $I = \int_0^{\pi/2} \log\sin(\frac{1}{2}\pi - x)\, dx,$

$$\left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$= \int_0^{\pi/2} \log\cos x\, dx.$ ...(2)

Adding (1) and (2), we get

$2I = \int_0^{\pi/2} \log\sin x\, dx + \int_0^{\pi/2} \log\cos x\, dx$

$= \int_0^{\pi/2} \log(\sin x\cos x)\, dx$ (Note)

$= \int_0^{\pi/2} \log\left\{\frac{\sin 2x}{2}\right\} dx = \int_0^{\pi/2} (\log\sin 2x - \log 2)\, dx$

$= \int_0^{\pi/2} \log\sin 2x\, dx - \int_0^{\pi/2} \log 2\, dx$

$= \int_0^{\pi/2} \log\sin 2x\, dx - (\log 2)\left[x\right]_0^{\pi/2}$

$= \int_0^{\pi/2} \log\sin 2x\, dx - \frac{\pi}{2}\log 2.$

Now put $2x = t$, so that $2\, dx = dt$. Also $t = 0$ when $x = 0$ and $t = \pi$ when $x = \frac{1}{2}\pi$.

$\therefore \quad 2I = \frac{1}{2}\int_0^{\pi} \log\sin t\, dt - \frac{\pi}{2}\log 2$

[Refer prop. 1]

$= \frac{1}{2}\int_0^{\pi} \log\sin x\, dx - \frac{\pi}{2}\log 2,$

$= \frac{1}{2} . 2 \int_0^{\pi/2} \log \sin x \, dx - \frac{\pi}{2} \log 2,$ [Refer prop. 6]

$= I - \frac{1}{2}\pi \log 2.$

Therefore $2I - I = -\frac{1}{2}\pi \log 2$ or $I = -\frac{1}{2}\pi \log 2 = \frac{1}{2}\pi \log (2)^{-1}$

$= \frac{1}{2}\pi \log \frac{1}{2}$.

Ex. 18 (a). *Prove that* $\int_0^1 \log \sin \left(\frac{\pi}{2} y\right) dy = \log \frac{1}{2}$.

Sol. Let $u = \int_0^1 \log \sin \left(\frac{\pi}{2} y\right) dy$.

Put $\frac{1}{2}\pi y = x$, so that $\frac{1}{2}\pi \, dy = dx$.

When $y = 0, x = 0$ and when $y = 1, x = \frac{1}{2}\pi$.

$\therefore \quad u = \int_0^{\pi/2} (\log \sin x) \cdot \frac{2}{\pi} dx = \frac{2}{\pi} \int_0^{\pi/2} \log \sin x \, dx.$

Now let $I = \int_0^{\pi/2} \log \sin x \, dx$.

Then proceeding as in Ex. 17, we get

$$I = \frac{1}{2}\pi \log \frac{1}{2}.$$

$\therefore \quad u = \frac{2}{\pi} I = \frac{2}{\pi} \cdot \frac{1}{2}\pi \log \frac{1}{2} = \log \frac{1}{2}$.

Ex. 18 (b). *Evaluate* $\int_0^{\pi} x \log \sin x \, dx$.

(Meerut 1988, 89 P, 91 S; Gorakh. 76)

Sol. Let $I = \int_0^{\pi} x \log \sin x \, dx$.

Then $I = \int_0^{\pi} (\pi - x) \log \sin (\pi - x) \, dx,$ [Refer prop. 4]

$= \int_0^{\pi} (\pi - x) \log \sin x \, dx$

$= \int_0^{\pi} \pi \log \sin x \, dx - \int_0^{\pi} x \log \sin x \, dx$

$= \pi \int_0^{\pi} \log \sin x \, dx - I.$

$\therefore \quad 2I = \pi \int_0^{\pi} \log \sin x \, dx = 2\pi \int_0^{\pi/2} \log \sin x \, dx,$

[Refer prop. 6]

or $I = \pi \int_0^{\pi/2} \log \sin x \, dx.$

Now let $u = \int_0^{\pi/2} \log \sin x \, dx$. Then proceeding as in Ex. 17, we have $u = \frac{1}{2}\pi \log \frac{1}{2}$.

$\therefore \quad I = \pi u = \pi . \frac{1}{2}\pi \log \frac{1}{2} = \frac{1}{2}\pi^2 \log \frac{1}{2}$.

Ex. 19. *Evaluate* $\int_0^{\pi/2} \log \cos x \, dx$.

(Meerut 1985 P; Gorakh. 75; Vikram 75)

Sol. Let $I = \int_0^{\pi/2} \log \cos x \, dx$. ...(1)

Then $I = \int_0^{\pi/2} \log \cos (\frac{1}{2}\pi - x) \, dx$ [Refer prop. 4]

$= \int_0^{\pi/2} \log \sin x \, dx$. ...(2)

Adding (1) and (2), we get

$$2I = \int_0^{\pi/2} (\log \cos x + \log \sin x) \, dx = \int_0^{\pi/2} \log (\sin x \cos x) \, dx.$$

Now proceed as in Ex. 17 and get $I = \frac{1}{2}\pi \log \frac{1}{2}$.

Ex. 20. *Evaluate* $\int_0^{\pi/2} \log \sin 2x \, dx$.

(Agra 1982)

Sol. Let $I = \int_0^{\pi/2} \log \sin 2x \, dx$.

Put $2x = t$, so that $2 \, dx = dt$. Also $t = 0$ when $x = 0$ and $t = \pi$ when $x = \frac{1}{2}\pi$.

$$\therefore \quad I = \frac{1}{2}\int_0^{\pi} \log \sin t \, dt = \frac{1}{2} . 2 \int_0^{\pi/2} \log \sin t \, dt, \quad \text{[Refer prop. 6]}$$

$$= \int_0^{\pi/2} \log \sin t \, dt.$$

Now proceeding as in Ex. 17, we get $I = \frac{1}{2}\pi \log \frac{1}{2}$.

Ex. 21 (a). *Show that* $\int_0^{\pi/2} x \cot x \, dx = \frac{1}{2}\pi \log 2$.

Sol. Let $I = \int_0^{\pi/2} x \cot x \, dx$. Integrating by parts taking $\cot x$ as the second function, we get

$$I = \left[x \log \sin x\right]_0^{\pi/2} - \int_0^{\pi/2} 1 . \log \sin x \, dx$$

$$= \left[\frac{\pi}{2} \log 1 - \lim_{x \to 0} x \log \sin x\right] - \int_0^{\pi/2} \log \sin x \, dx$$

$$= 0 - \lim_{x \to 0} x \log \sin x - \int_0^{\pi/2} \log \sin x \, dx.$$

Now $\lim_{x \to 0} x \log \sin x = \lim_{x \to 0} \dfrac{\log \sin x}{1/x}$, $\quad \left[\text{form } \frac{\infty}{\infty}\right]$

$$= \lim_{x \to 0} \frac{(1/\sin x) \cos x}{-1/x^2} = \lim_{x \to 0} \frac{-x^2 \cos x}{\sin x}, \quad \left[\text{form } \frac{\infty}{\infty}\right]$$

$$= \lim_{x \to 0} \frac{-2x \cos x + x^2 \sin x}{\cos x} = \frac{0}{1} = 0.$$

$$\therefore \quad I = 0 - \int_0^{\pi/2} \log \sin x \, dx = -\int_0^{\pi/2} \log \sin x \, dx.$$

Now let $u = \int_0^{\pi/2} \log \sin x \, dx$.

Then proceeding as in Ex. 17, we have $u = -\frac{1}{2}\pi \log 2$.

$\therefore \quad I = -u = \frac{1}{2}\pi \log 2$.

Ex. 21 (b). *Evaluate* $\displaystyle\int_0^{\infty} \frac{\tan^{-1} x \, dx}{x(1+x^2)}$. **(Agra 1983)**

Sol. Let $\quad I = \displaystyle\int_0^{\infty} \frac{\tan^{-1} x}{x(1+x^2)} dx$.

Put $\tan^{-1} x = t$, so that $\{1/(1+x^2)\} \, dx = dt$ and $x = \tan t$. Also when $x = 0$, $t = 0$ and when $x = \infty$, $t = \pi/2$.

$$\therefore \quad I = \int_0^{\pi/2} \frac{t \, dt}{\tan t} = \int_0^{\pi/2} t \cot t \, dt.$$

Now proceeding as in Ex. 21 (a), we get $I = \frac{1}{2}\pi \log 2$.

Ex. 22 (a). *Show that* $\displaystyle\int_0^{\pi/2} \left(\frac{\theta}{\sin \theta}\right)^2 d\theta = \pi \log 2$. **(Gorakh. 1972)**

Sol. Let $\quad I = \displaystyle\int_0^{\pi/2} \left(\frac{\theta}{\sin \theta}\right)^2 d\theta = \int_0^{\pi/2} \theta^2 \operatorname{cosec}^2 \theta \, d\theta$.

Integrating by parts taking $\operatorname{cosec}^2 \theta$ as the second function, we get

$$I = \left[\theta^2(-\cot\theta)\right]_0^{\pi/2} - \int_0^{\pi/2} 2\theta.(-\cot\theta) \, d\theta$$

$$= -\left(\frac{\pi}{2}\right)^2 \cot\frac{\pi}{2} + \lim_{\theta \to 0} \theta^2 \cot\theta + 2\int_0^{\pi/2} \theta \cot\theta \, d\theta$$

$$= 0 + \lim_{\theta \to 0} \theta^2 \cot\theta + 2\int_0^{\pi/2} \theta \cot\theta \, d\theta.$$

Now $\quad \displaystyle\lim_{\theta \to 0} \theta^2 \cot\theta = \lim_{\theta \to 0} \frac{\theta^2}{\tan\theta}$, $\quad \left[\text{form } \frac{0}{0}\right]$

$$= \lim_{\theta \to 0} \frac{2\theta}{\sec^2\theta} = \frac{0}{1} = 0.$$

$\therefore I = 2\displaystyle\int_0^{\pi/2} \theta \cot\theta \, d\theta$. Now proceed as in Ex. 21 (a).

Ex. 22 (b). *Evaluate* $\displaystyle\int_0^{\infty} (\cot^{-1} x)^2 \, dx$.

Sol. Let $\quad I = \displaystyle\int_0^{\infty} (\cot^{-1} x)^2 \, dx$.

Put $\cot^{-1} x = \theta$ *i.e.*, $x = \cot\theta$, so that $dx = -\operatorname{cosec}^2\theta \, d\theta$. The new limits for θ are $\frac{1}{2}\pi$ to 0.

$$\therefore \quad I = \int_{\pi/2}^{0} \theta^2 . (-\operatorname{cosec}^2\theta) \, d\theta = \int_0^{\pi/2} \theta^2 \operatorname{cosec}^2\theta \, d\theta.$$

Now proceed as in Ex. 22 (a).

Ex. 23. *Show that* $\int_0^1 \frac{\sin^{-1} x}{x} dx = \frac{1}{2}\pi \log 2.$

(Agra 1977; Rohilkhand 77)

Sol. Let $I = \int_0^1 \frac{\sin^{-1} x}{x} dx.$

Put $x = \sin t$, so that $dx = \cos t\, dt$. Also $t = 0$ when $x = 0$ and $t = \frac{1}{2}\pi$ when $x = 1$.

$\therefore \quad I = \int_0^{\pi/2} \frac{t}{\sin t} \cos t\, dt = \int_0^{\pi/2} t \cot t\, dt.$

Now proceed as in Ex. 21 (a).

Ex. 24. *Show that* $\int_0^{\pi} \log(1 + \cos x)\, dx = \pi \log \frac{1}{2}.$

(Agra 1974; Gorakhpur 72; Ranchi 75)

Sol. Let $I = \int_0^{\pi} \log(1 + \cos x)\, dx.$ ...(1)

Then $I = \int_0^{\pi} \log\{1 + \cos(\pi - x)\}\, dx,$ [Refer prop. 4]

$= \int_0^{\pi} \log(1 - \cos x)\, dx.$ ...(2)

Adding (1) and (2), we get

$2I = \int_0^{\pi} [\log(1 + \cos x) + \log(1 - \cos x)]\, dx$

$= \int_0^{\pi} \log\{(1 + \cos x)(1 - \cos x)\}\, dx$

$= \int_0^{\pi} \log(1 - \cos^2 x)\, dx = \int_0^{\pi} \log \sin^2 x\, dx = 2\int_0^{\pi} \log \sin x\, dx.$

$\therefore \quad I = \int_0^{\pi} \log \sin x\, dx = 2\int_0^{\pi/2} \log \sin x\, dx.$ [Refer prop. 6]

Now let $u = \int_0^{\pi/2} \log \sin x\, dx$. Then proceeding as in Ex. 17 on page 253, we have $u = \frac{1}{2}\pi \log \frac{1}{2}$.

$\therefore \quad I = 2u = 2 \cdot \frac{1}{2}\pi \log \frac{1}{2} = \pi \log \frac{1}{2}.$

****Ex. 25.** *Show that* $\int_0^{\infty} \log\left(x + \frac{1}{x}\right) \frac{dx}{1 + x^2} = \pi \log 2.$

(Lucknow 1989; Meerut 90 P; Agra 81; Allahabad 80; Rajasthan 77; Gorakhpur 71)

Sol. Put $x = \tan\theta$, so that $dx = \sec^2\theta\, d\theta$.

Also when $x = 0$, $\theta = 0$ and when $x \to \infty$, $\theta \to \pi/2$.

$\therefore \quad I = \int_0^{\infty} \log\left(x + \frac{1}{x}\right) \frac{dx}{1 + x^2}$

$$= \int_0^{\pi/2} \log \left(\tan\theta + \frac{1}{\tan\theta}\right) \frac{\sec^2\theta}{\sec^2\theta}\, d\theta$$

$$= \int_0^{\pi/2} \log \left(\frac{1+\tan^2\theta}{\tan\theta}\right) d\theta = \int_0^{\pi/2} \log \frac{\sec^2\theta}{\tan\theta}\, d\theta$$

$$= \int_0^{\pi/2} \log \left(\frac{1}{\sin\theta\cos\theta}\right) d\theta = \int_0^{\pi/2} \log (\sin\theta\cos\theta)^{-1}\, d\theta$$

$$= -\int_0^{\pi/2} \log\sin\theta\, d\theta - \int_0^{\pi/2} \log\cos\theta\, d\theta.$$

Now let $u = \int_0^{\pi/2} \log\sin\theta\, d\theta$. Then proceeding as in Ex. 17, we have $u = \int_0^{\pi/2} \log\cos\theta\, d\theta = -\frac{\pi}{2}\log 2$.

$\therefore \quad I = -u - u = -2u = -2\left(-\frac{1}{2}\pi\log 2\right) = \pi\log 2.$

Ex. 26. *Show that* $\int_0^{\infty} \frac{\log(1+x^2)\,dx}{(1+x^2)} = \pi \log 2.$

(Meerut 1982 S; Luck. 76)

Sol. Put $x = \tan\theta$; so that $dx = \sec^2\theta\, d\theta$. The new limits for θ are 0 to $\frac{1}{2}\pi$.

$$\therefore \quad I = \int_0^{\infty} \frac{\log(1+x^2)}{(1+x^2)}\, dx = \int_0^{\pi/2} \frac{\log(1+\tan^2\theta)\sec^2\theta\, d\theta}{\sec^2\theta}$$

$$= \int_0^{\pi/2} \log\sec^2\theta\, d\theta, \qquad (\because 1+\tan^2\theta = \sec^2\theta)$$

$$= 2\int_0^{\pi/2} \log\sec\theta\, d\theta = 2\int_0^{\pi/2} \log(\cos\theta)^{-1}\, d\theta$$

$$= -2\int_0^{\pi/2} \log\cos\theta\, d\theta.$$

Now let $u = \int_0^{\pi/2} \log\cos\theta\, d\theta$. Then proceeding as in Ex. 19, we have $u = -\frac{1}{2}\pi\log 2$.

$\therefore \quad I = -2u = -2.\left(-\frac{1}{2}\pi\log 2\right) = \pi\log 2.$

****Ex. 27.** *Show that* $\int_0^{\pi/4} \log(1+\tan\theta)\, d\theta = \frac{\pi}{8}\log 2.$

(Allahabad 1981; Lucknow 81; Delhi 76; Meerut 80, 88P, 89, 90; Bihar 73; Agra 71; Gorakhpur 71; Kanpur 77)

Sol. Let $I = \int_0^{\pi/4} \log(1+\tan\theta)\, d\theta$.

Then $\quad I = \int_0^{\pi/4} \log\{1+\tan\left(\frac{1}{4}\pi - \theta\right)\}\, d\theta,$

$$\left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx\right]$$

$$= \int_0^{\pi/4} \log\left[1 + \frac{(1 - \tan\theta)}{(1 + \tan\theta)}\right] d\theta = \int_0^{\pi/4} \log\left\{\frac{2}{1 + \tan\theta}\right\} d\theta$$

$$= \int_0^{\pi/4} \log 2 \,.\, d\theta - \int_0^{\pi/4} \log(1 + \tan\theta)\, d\theta$$

$$= \log 2 \,.\, \Big[\theta\Big]_0^{\pi/4} - I.$$

$\therefore \quad 2I = \frac{1}{4}\pi \log 2 \quad$ or $\quad I = \frac{1}{8}\pi \log 2.$

****Ex. 28.** *Show that* $\displaystyle\int_0^1 \frac{\log(1+x)}{1+x^2}\, dx = \frac{1}{8}\pi \log 2.$

(Meerut 1983 S, 82 P, 89 P, 91 P; Allahabad 82; Lucknow 83, 80, 79; Kanpur 78; Agra 76, 73; Gorakh. 72; Jiwaji 72)

Sol. Put $x = \tan\theta$, so that $dx = \sec^2\theta\, d\theta$.

And the new limits are $\theta = 0$ to $\theta = \pi/4$.

$$\therefore \quad I = \int_0^1 \frac{\log(1+x)}{1+x^2}\, dx = \int_0^{\pi/4} \frac{\log(1+\tan\theta)}{\sec^2\theta} \sec^2\theta\, d\theta$$

$$= \int_0^{\pi/4} \log(1 + \tan\theta)\, d\theta = \frac{1}{8}\pi \log 2. \quad \text{[Proceeding as in Ex. 27]}$$

***Ex. 29 (a).** *Show that* $\displaystyle\int_0^{\pi/2} \frac{\sin x\, dx}{\sin x + \cos x} = \frac{\pi}{4}.$

(Meerut 1984; Rohilkhand 76; Magadh 71)

Sol. Let $\displaystyle I = \int_0^{\pi/2} \frac{\sin x\, dx}{\sin x + \cos x}.$ ...(1)

Then $\displaystyle I = \int_0^{\pi/2} \frac{\sin(\frac{1}{2}\pi - x)}{\sin(\frac{1}{2}\pi - x) + \cos(\frac{1}{2}\pi - x)}$ [Refer prop. 4]

$$= \int_0^{\pi/2} \frac{\cos x\, dx}{\cos x + \sin x}. \qquad \text{...(2)}$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi/2} \frac{\sin x\, dx}{\sin x + \cos x} + \int_0^{\pi/2} \frac{\cos x\, dx}{\sin x + \cos x}$$

$$= \int_0^{\pi/2} \left[\frac{\sin x}{\sin x + \cos x} + \frac{\cos x}{\sin x + \cos x}\right] dx$$

$$= \int_0^{\pi/2} 1 \,.\, dx = \Big[x\Big]_0^{\pi/2} = \frac{\pi}{2}.$$

$\therefore \quad I = \frac{1}{4}\pi.$

Similarly we can prove that

$$\int_0^{\pi/2} \frac{\cos x\, dx}{\sin x + \cos x} = \frac{\pi}{4}.$$

***Ex. 29 (b).** *Evaluate* $\int_0^{\pi/2} \frac{dx}{1 + \tan x}$. **(Meerut 1986, 90 S)**

Sol. We have $I = \int_0^{\pi/2} \frac{dx}{1 + \tan x} = \int_0^{\pi/2} \frac{dx}{1 + (\sin x/\cos x)}$

$= \int_0^{\pi/2} \frac{\cos x\, dx}{\cos x + \sin x}$ [Proceeding an in Ex. 29 (a)].

$= \frac{\pi}{4}$.

Ex. 29 (c). *Evaluate* $\int_0^{\pi/2} \frac{dx}{1 + \cot x}$. **(Kashmir 1971)**

Sol. We have $\int_0^{\pi/2} \frac{dx}{1 + \cot x}$

$= \int_0^{\pi/2} \frac{dx}{1 + (\cos x/\sin x)} = \int_0^{\pi/2} \frac{\sin x\, dx}{\sin x + \cos x}$

$= \frac{\pi}{4}$. [Proceed as in Ex. 29 (a)]

Ex. 29 (d). *Show that* $\int_0^{\infty} \frac{x\, dx}{(1 + x)(1 + x^2)} = \frac{\pi}{4}$. **(Meerut 1985, 91 P)**

Sol. Put $x = \tan \theta$, so that $dx = \sec^2 \theta\, d\theta$.
Also when $x = 0, \theta = 0$ and when $x \to \infty, \theta \to \pi/2$.

$\therefore$ the given integral $I = \int_0^{\pi/2} \frac{\tan \theta \sec^2 \theta\, d\theta}{(1 + \tan \theta)(1 + \tan^2 \theta)}$

$= \int_0^{\pi/2} \frac{\tan \theta\, d\theta}{1 + \tan \theta} = \int_0^{\pi/2} \frac{\sin \theta/\cos \theta}{1 + (\sin \theta/\cos \theta)} d\theta$

$= \int_0^{\pi/2} \frac{\sin \theta\, d\theta}{\cos \theta + \sin \theta} = \frac{\pi}{4}$. [Proceed as in Ex. 29 (a)]

Ex. 29 (e). *Show that* $\int_0^{a} \frac{dx}{x + \sqrt{(a^2 - x^2)}} = \frac{\pi}{4}$. **(Meerut 1983)**

Sol. Put $x = a \sin \theta$, so that $dx = a \cos \theta\, d\theta$.
Also the new limits are $\theta = 0$ to $\theta = \pi/2$.

$\therefore$ the given integral $I = \int_0^{\pi/2} \frac{a \cos \theta\, d\theta}{a \sin \theta + a \cos \theta}$

$= \int_0^{\pi/2} \frac{\cos \theta\, d\theta}{\sin \theta + \cos \theta} = \frac{\pi}{4}$. [See Ex. 29 (a)]

****Ex. 30 (a).** *Show that* $\int_0^{\pi/2} \frac{\sqrt{(\sin x)}}{\sqrt{(\sin x)} + \sqrt{(\cos x)}} dx = \frac{\pi}{4}$.

(Lucknow 1982; Meerut 89 P, 90 P, 91; Vikram 78; Ranchi 74; Alld. 73; Gorakh. 73; Bihar 74)

Sol. Let $I=\int_0^{\pi/2}\frac{\sqrt{(\sin x)}}{\sqrt{(\sin x)}+\sqrt{(\cos x)}}dx.$ ...(1)

Then $I=\int_0^{\pi/2}\frac{\sqrt{[\sin(\frac{1}{2}\pi-x)]}\,dx}{\sqrt{[\sin(\frac{1}{2}\pi-x)]}+\sqrt{[\cos(\frac{1}{2}\pi-x)]}},$

$$=\int_0^{\pi/2}\frac{\sqrt{(\cos x)}\,dx}{\sqrt{(\cos x)}+\sqrt{(\sin x)}}.$$ (Refer prop. 4) ...(2)

Adding (1) and (2), we get

$$2I=\int_0^{\pi/2}\left[\frac{\sqrt{(\sin x)}}{\sqrt{(\sin x)}+\sqrt{(\cos x)}}+\frac{\sqrt{(\cos x)}}{\sqrt{(\sin x)}+\sqrt{(\cos x)}}\right]dx$$

$$=\int_0^{\pi/2}\frac{\sqrt{(\sin x)}+\sqrt{(\cos x)}}{\sqrt{(\sin x)}+\sqrt{(\cos x)}}dx=\int_0^{\pi/2}1\,.\,dx=\frac{\pi}{2}.$$

$\therefore\quad I=\frac{1}{4}\pi.$

Ex. 30. (b). *Show that* $\int_0^{\pi/2}\frac{dx}{1+\sqrt{(\tan x)}}=\frac{\pi}{4}.$ (Agra 1970)

Sol. We have

$$I=\int_0^{\pi/2}\frac{dx}{1+\sqrt{(\tan x)}}=\int_0^{\pi/2}\frac{dx}{1+\sqrt{(\sin x/\cos x)}}$$

$$=\int_0^{\pi/2}\frac{\sqrt{(\cos x)}\,dx}{\sqrt{(\cos x)}+\sqrt{(\sin x)}}.$$

Now proceeding exactly as in Ex. 30 (a) we get the result.

Ex. 30 (c). *Show that* $\int_0^{\pi/2}\frac{\sqrt{(\tan x)}\,dx}{1+\sqrt{(\tan x)}}=\frac{\pi}{4}.$ (Kanpur 1972)

Sol. We have

$$I=\int_0^{\pi/2}\frac{\sqrt{(\tan x)}\,dx}{1+\sqrt{(\tan x)}}=\int_0^{\pi/2}\frac{\sqrt{(\sin x)}\,dx}{\sqrt{(\cos x)}+\sqrt{(\sin x)}}.$$

($\because \tan x=\sin x/\cos x$)

$=\frac{\pi}{4}.$ [Proceed as in Ex. 30 (a)]

Ex. 30 (d). *Prove that* $\int_0^{\pi/2}\frac{\sqrt{(\tan x)}}{\sqrt{(\tan x)}+\sqrt{(\cot x)}}dx=\frac{\pi}{4}.$ (Meerut 1984 S)

Sol. Here $I=\int_0^{\pi/2}\frac{\sin x}{\sin x+\cos x}dx$, changing to $\sin x$ and $\cos x$. Now procced as in Ex. 29 (a).

Ex. 31. (a) *Show that* $\int_0^{\pi/2} \frac{\sin^2 x\, dx}{(\sin x + \cos x)} = \frac{1}{\sqrt{2}} \log(\sqrt{2}+1)$.

(Vikram 1978, 75; Meerut 72, 84 S, 87 S, 88 S)

Sol. Let $I = \int_0^{\pi/2} \frac{\sin^2 x\, dx}{(\sin x + \cos x)}$. ...(1)

Then $I = \int_0^{\pi/2} \frac{[\sin(\frac{1}{2}\pi - x)]^2\, dx}{\sin(\frac{1}{2}\pi - x) + \cos(\frac{1}{2}\pi - x)}$, [Refer prop. 4]

or $I = \int_0^{\pi/2} \frac{\cos^2 x\, dx}{\cos x + \sin x}$. ...(2)

Adding (1) and (2), we get

$$2I = \int_0^{\pi/2} \frac{\sin^2 x\, dx}{\sin x + \cos x} + \int_0^{\pi/2} \frac{\cos^2 x\, dx}{\cos x + \sin x}$$

$$= \int_0^{\pi/2} \frac{(\sin^2 x + \cos^2 x)}{(\sin x + \cos x)}\, dx = \int_0^{\pi/2} \frac{dx}{(\sin x + \cos x)}$$

$$= \int_0^{\pi/2} \frac{(1/\sqrt{2})\, dx}{(1/\sqrt{2}) \sin x + (1/\sqrt{2}) \cos x} \quad \text{(Note)}$$

$$= \frac{1}{\sqrt{2}} \int_0^{\pi/2} \frac{dx}{\cos(x - \frac{1}{4}\pi)}, \quad \left[\because \cos\frac{\pi}{4} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}\right]$$

$$= \frac{1}{\sqrt{2}} \int_0^{\pi/2} \sec(x - \tfrac{1}{4}\pi)\, dx$$

$$= \frac{1}{\sqrt{2}} \log\left[\sec\left(x - \frac{\pi}{4}\right) + \tan\left(x - \frac{\pi}{4}\right)\right]_0^{\pi/2}$$

$$= \frac{1}{\sqrt{2}} [\log(\sec\tfrac{1}{4}\pi + \tan\tfrac{1}{4}\pi) - \log\{\sec(-\tfrac{1}{4}\pi) + \tan(-\tfrac{1}{4}\pi)\}]$$

$$= \frac{1}{\sqrt{2}} \log\left[\frac{\sec\frac{1}{4}\pi + \tan\frac{1}{4}\pi}{\sec(-\frac{1}{4}\pi) + \tan(-\frac{1}{4}\pi)}\right] = \frac{1}{\sqrt{2}} \log\left[\frac{\sqrt{2}+1}{\sqrt{2}-1}\right]$$

$$= \frac{1}{\sqrt{2}} \log\left[\frac{(\sqrt{2}+1)(\sqrt{2}+1)}{(\sqrt{2}-1)(\sqrt{2}+1)}\right] = \frac{1}{\sqrt{2}} \log(\sqrt{2}+1)^2$$

$$= (1/\sqrt{2}) \cdot 2 \log(\sqrt{2}+1).$$

$$\therefore \quad I = \frac{1}{\sqrt{2}} \log(\sqrt{2}+1).$$

Ex. 31. (b) *Evaluate* $\int_0^{\pi/2} \frac{\cos^2 x}{(\sin x + \cos x)}\, dx$. **(Kanpur 1972)**

Sol. Proceed exactly as in Ex. 31 (a). The answer is the same as in Ex. 31 (a).

Ex. 31. (c) *Evaluate* $\int_0^a \frac{a\,dx}{\{x+\sqrt{(a^2-x^2)}\}^2}$.

Sol. Put $x = a \sin\theta$, so that $dx = a\cos\theta\,d\theta$.
When $x = 0$, $\theta = 0$ and when $x = a$, $\theta = \pi/2$.

$\therefore$ the given integral $I = \int_0^{\pi/2} \frac{a\,.\,a\cos\theta\,d\theta}{a^2(\sin\theta+\cos\theta)^2}$

$$= \int_0^{\pi/2} \frac{\cos\theta\,d\theta}{(\sin\theta+\cos\theta)^2}. \quad ...(1)$$

Also $I = \int_0^{\pi/2} \frac{\cos(\frac{1}{2}\pi-\theta)\,d\theta}{[\sin(\frac{1}{2}\pi-\theta)+\cos(\frac{1}{2}\pi-\theta)]^2}$, [Refer prop. 4]

$$= \int_0^{\pi/2} \frac{\sin\theta\,d\theta}{(\cos\theta+\sin\theta)^2}. \quad ...(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi/2} \frac{\cos\theta+\sin\theta}{(\sin\theta+\cos\theta)^2}\,d\theta = \int_0^{\pi/2} \frac{d\theta}{\sin\theta+\cos\theta}$$

$= (1/\sqrt{2})\,.2\log(\sqrt{2}+1)$, proceeding as in Ex. 31 (a).

$\therefore$ $I = (1/\sqrt{2})\log(\sqrt{2}+1)$.

***Ex. 31. (d)** *Evaluate* $\int_0^{\pi/2} \frac{x\,dx}{\sin x+\cos x}$.

Sol. Let $I = \int_0^{\pi/2} \frac{x\,dx}{\sin x+\cos x}$. ...(1)

Then $I = \int_0^{\pi/2} \frac{(\frac{1}{2}\pi-x)\,dx}{\sin(\frac{1}{2}\pi-x)+\cos(\frac{1}{2}\pi-x)}$, [Refer prop. 4]

$$= \int_0^{\pi/2} \frac{(\frac{1}{2}\pi-x)\,dx}{\sin x+\cos x}. \quad ...(2)$$

Adding (1) and (2), we get

$$2I = \int_0^{\pi/2} \frac{\frac{1}{2}\pi\,dx}{\sin x+\cos x} = \frac{\pi}{2}\int_0^{\pi/2} \frac{dx}{\sin x+\cos x}.$$

$\therefore$ $I = \frac{\pi}{4}\int_0^{\pi/2} \frac{dx}{\sin x+\cos x}$.

Now proceeding as in Ex. 31 (a), we have

$$\int_0^{\pi/2} \frac{dx}{\sin x+\cos x} = \frac{1}{\sqrt{2}}\cdot 2\log(\sqrt{2}+1).$$

$\therefore$ $I = \frac{\pi}{4}\cdot\frac{1}{\sqrt{2}}\cdot 2\log(\sqrt{2}+1) = \frac{\pi}{2\sqrt{2}}\log(\sqrt{2}+1)$.

Ex. 32. *Show that*

$$\int_0^{\pi/2} \phi(\sin 2x) \sin x\,dx = \int_0^{\pi/2} \phi(\sin 2x) \cos x\,dx$$
$$= \sqrt{2}\int_0^{\pi/4} \phi(\cos 2x) \cos x\,dx.$$

Sol. We have $\int_0^{\pi/2} \phi(\sin 2x) . \sin x\,dx$

$= \int_0^{\pi/2} \phi\left[\sin 2\left(\frac{\pi}{2} - x\right)\right] \cdot \sin\left(\frac{\pi}{2} - x\right) dx,$ (Refer prop. 4)

$= \int_0^{\pi/2} \phi[\sin(\pi - 2x)] \cos x\,dx$

$= \int_0^{\pi/2} \phi(\sin 2x) \cos x\,dx$. **The first part proved.**

Now to prove the second part, let

$$I = \int_0^{\pi/2} \phi(\sin 2x) \sin x\,dx.$$

Put $x = \frac{1}{4}\pi + t$, so that $dx = dt$.

When $x = 0$, $t = -\pi/4$ and when $x = \pi/2$, $t = \pi/4$.

$\therefore \quad I = \int_{-\pi/4}^{\pi/4} \phi[\sin 2(\frac{1}{4}\pi + t)] \sin(\frac{1}{4}\pi + t)\,dt$

$= \int_{-\pi/4}^{\pi/4} \phi[\sin(\frac{1}{2}\pi + 2t)] \sin(\frac{1}{4}\pi + t)\,dt$

$= \int_{-\pi/4}^{\pi/4} \phi(\cos 2t)(\sin\frac{1}{4}\pi \cos t + \cos\frac{1}{4}\pi \sin t)\,dt$

$= \frac{1}{\sqrt{2}}\int_{-\pi/4}^{\pi/4} \phi(\cos 2t) \cos t\,dt + \frac{1}{\sqrt{2}}\int_{-\pi/4}^{\pi/4} \phi(\cos 2t) \sin t\,dt.$

Now $\int_{-\pi/4}^{\pi/4} \phi(\cos 2t) \sin t\,dt = 0$, because $\phi(\cos 2t) \sin t$ is an odd function of t

and $\int_{-\pi/4}^{\pi/4} \phi(\cos 2t) \cos t\,dt = 2\int_0^{\pi/4} \phi(\cos 2t) \cos t\,dt$

because $\phi(\cos 2t) \cos t$ is an even function of t.

$\therefore \quad I = \frac{2}{\sqrt{2}}\int_0^{\pi/4} \phi(\cos 2t) \cos t\,dt$

$= \sqrt{2}\int_0^{\pi/4} \phi(\cos 2x) \cos x\,dx,$

because a definite integral does not change by changing the variable.

Ex. 33. *Show that* $\int_0^{\pi} \sin^m x \cos^{2m+1} x\,dx = 0.$

Sol. Let $\quad I = \int_0^{\pi} \sin^m x \cos^{2m+1} x\,dx.$

Then $\quad I = \int_0^{\pi} \sin^m(\pi - x) . \cos^{2m+1}(\pi - x)\,dx$, [Refer prop. 4]

$= \int_0^{\pi} \sin^m x(-\cos x)^{2m+1}\,dx$

$$= -\int_0^{\pi} \sin^m x \cos^{2m+1} x \, dx = -I.$$

$\therefore \quad 2I = 0 \quad$ or $\quad I = 0.$

Ex. 34. *Show that* $\displaystyle\int_0^{\pi} \frac{x^2 \sin 2x \sin (\frac{1}{2}\pi \cos x)}{2x - \pi} dx = \frac{8}{\pi}.$

(Lucknow 1977; Gorakh. 74; Raj. 72)

Sol. Let $\displaystyle I = \int_0^{\pi} \frac{x^2 \sin 2x . \sin (\frac{1}{2}\pi \cos x)}{2x - \pi} dx.$

Put $x = \frac{1}{2}\pi - t$, so that $dx = -dt$.

Also $t = \frac{1}{2}\pi$ when $x = 0$ and $t = -\frac{1}{2}\pi$ when $x = \pi$.

$$\therefore \quad I = \int_{\pi/2}^{-\pi/2} \frac{(\frac{1}{2}\pi - t)^2 \sin 2 (\frac{1}{2}\pi - t) . \sin \{\frac{1}{2}\pi \cos (\frac{1}{2}\pi - t)\}}{2 (\frac{1}{2}\pi - t) - \pi} (-dt)$$

$$= \int_{-\pi/2}^{\pi/2} \frac{(\frac{1}{2}\pi - t)^2 \sin 2t . \sin (\frac{1}{2}\pi \sin t)}{-2t} dt$$

$$= -\frac{1}{2} \int_{-\pi/2}^{\pi/2} \frac{(\frac{1}{4}\pi^2 - \pi t + t^2) \sin 2t . \sin (\frac{1}{2}\pi \sin t)}{t} dt.$$

Now $\dfrac{\frac{1}{4}\pi^2 \sin 2t . \sin (\frac{1}{2}\pi \sin t)}{t}$ and $t \sin 2t . \sin (\frac{1}{2}\pi \sin t)$ are both odd functions of t while $\sin 2t . \sin (\frac{1}{2}\pi \sin t)$ is an even function of t.

$$\therefore \quad I = -\frac{1}{2} . 2 \int_0^{\pi/2} (-\pi) \sin 2t . \sin (\tfrac{1}{2}\pi \sin t) \, dt, \text{ [Refer prop. 5]}$$

$$= \pi \int_0^{\pi/2} 2 \sin t \cos t . \sin (\tfrac{1}{2}\pi \sin t) \, dt.$$

Now put $\frac{1}{2}\pi \sin t = z$, so that $\frac{1}{2}\pi \cos t \, dt = dz$.

Also $z = 0$ when $t = 0$ and $z = \frac{1}{2}\pi$ when $t = \frac{1}{2}\pi$.

$$\therefore \quad I = \pi \int_0^{\pi/2} \frac{2 . 2z}{\pi} \cdot \sin z \cdot \frac{2}{\pi} dz = \frac{8}{\pi} \int_0^{\pi/2} z \sin z \, dz$$

$$= \frac{8}{\pi} \left[\left\{ z (-\cos z) \right\}_0^{\pi/2} - \int_0^{\pi/2} 1 . (-\cos z) \, dz \right]$$

$$= \frac{8}{\pi} \left[0 + \int_0^{\pi/2} \cos z \, dz \right] = \frac{8}{\pi} \left[\sin z \right]_0^{\pi/2} = \frac{8}{\pi} (1 - 0) = \frac{8}{\pi}.$$

**§ 7·3. The definite integral as the limit of a sum.

So far integration has been defined as the inverse process of differentiation. But it is also possible to regard a definite integral as the limit of the sum of certain number of terms, when the number of terms tends to infinity and each term tends to zero.

Definition. *Let $f(x)$ be a single valued continuous function defined in the interval (a, b) where $b > a$ and let the interval (a, b) be divided into n equal parts each of length h, so that $nh = b - a$; then we define*

$$\int_a^b \mathbf{f(x)\,dx = \lim h\,[f(a) + f(a+h) + f(a+2h) + \ldots + f\{a + (n-1)\,h\}]},$$

when $\mathbf{n \to \infty, h \to 0}$ and $\mathbf{nh \to b - a}$.

Thus $\int_a^b f(x)\,dx = \lim_{h \to 0} h \sum_{r=0}^{n-1} f(a + rh)$, where $n \to \infty$ as $h \to 0$ and nh remains equal to $b - a$. We call $\int_a^b f(x)\,dx$ as the definite integral of $f(x)$ w.r.t. x between the limits a and b.

****Ex. 35.** *Evaluate $\int_a^b x^2\,dx$ directly from the definition of the integral as the limit of a sum.*

(Meerut 1981, 82 P, 85 P; Kanpur 79; Kashmir 75; Rohiikhand 77; K.U. 77)

Sol. From the definition of a definite integral as the limit of a sum, we know that

$$\int_a^b f(x)\,dx = \lim_{n \to \infty} h\,[f(a) + f(a+2h) + \ldots + f\{a + (n-1)\,h\}],$$

where $h \to 0$ as $n \to \infty$ and $nh \to b - a$.

Here $f(x) = x^2$; $\therefore$ $f(a), f(a+h), f(a+2h)$, etc. will be a^2, $(a+h)^2$, $(a+2h)^2$, ..., respectively.

$$\therefore \quad \int_a^b x^2\,dx = \lim_{n \to \infty} h\,[a^2 + (a+h)^2 + (a+2h)^2 + \ldots + \{a + (n-1)\,h\}^2],$$

where $h \to 0$ as $n \to \infty$ and $nh \to b - a$

$$= \lim_{n \to \infty} h\,[na^2 + 2ah\,\{1 + 2 + 3 + \ldots + (n-1)\} + h^2\,\{1^2 + 2^2 + 3^2 + \ldots + (n-1)^2\}].$$

But we know that

$$\Sigma n = \frac{n\,(n+1)}{2} \quad \text{and} \quad \Sigma n^2 = \frac{n\,(n+1)\,(2n+1)}{6}.$$

Taking $n = (n-1)$ in the above results, we get

$$\int_a^b x^2\,dx = \lim_{n \to \infty} h\left[na^2 + 2ah \cdot \frac{(n-1)\,n}{2} + \frac{h^2}{6}(n-1)\,n\,(2n-1)\right]$$

$$= \lim_{n \to \infty} [(nh)\,a^2 + a\,(nh)\,(n-1)\,h + \tfrac{1}{6}(nh)\,(n-1)\,h\,(2n-1)\,h]$$

$$= \lim_{n \to \infty} \left[(nh)\,a^2 + a\,(nh)^2\left(1 - \frac{1}{n}\right)\right.$$

$$+\frac{1}{6}\cdot 2\,(nh)^3\left(1-\frac{1}{n}\right)\left(1-\frac{1}{2n}\right)\Big].$$

Now as $n \to \infty$, $h \to 0$ and $nh \to b - a$.

$\therefore \quad \int_a^b x^2\,dx = (b-a)\,a^2 + a\,(b-a)^2 + \frac{1}{3}(b-a)^3$

$= \frac{1}{3}(b-a)\,\{3a^2 + 3\,(b-a)\,a + b^2 - 2ab + a^2\}$

$= \frac{1}{3}(b-a)\,(a^2 + ab + b^2) = \frac{1}{3}(b^3 - a^3)$.

Ex. 36 (a). *Find by summation the value of* $\int_a^b x\,dx$.

Sol. Here $f(x) = x$; $\therefore$ $f(a) = a, f(a+h) = a+h$, $f(a+2h) = a+2h$, etc.

Now

$$\int_a^b f(x)\,dx = \lim_{h\to 0} h\,[f(a) + f(a+h) + \dots + f\{a+(n-1)\,h\}],$$

where $n \to \infty$ and $nh \to b - a$ as $h \to 0$.

$$\therefore \quad \int_a^b x\,dx = \lim_{h\to 0} h\,[a + (a+h) + (a+2h) + \dots + \{a+(n-1)\,h\}]$$

$= \lim_{h\to 0} h\left[\frac{n}{2}\{2a + (n-1)\,h\}\right]$, summing the A.P.

$= \lim_{h\to 0} \frac{nh}{2}[2a + nh - h]$

$= \lim_{h\to 0} \frac{b-a}{2}[2a + (b-a) - h]$, $\quad [\because nh = b - a]$

$= \frac{1}{2}(b-a)\,[2a + (b-a)] = \frac{1}{2}(b-a)\,(b+a) = \frac{1}{2}(b^2 - a^2)$.

Ex. 36 (b). *Evaluate by summation* $\int_1^2 x\,dx$.

Sol. Proceed as in Ex. 36 (a). Here $b = 2$, $a = 1$.

Thus proceeding as above, we get

$$\int_1^2 x\,dx = \tfrac{1}{2}(4-1) = \tfrac{3}{2}.$$

Ex. 36 (c). *Evaluate by summation* $\int_0^2 x^3\,dx$.

Sol. Here $f(x) = x^3$ and $a = 0$, $b = 2$; $\therefore nh = 2 - 0 = 2$.

$$\therefore \quad \int_0^2 x^3\,dx = \lim_{h\to 0} h\,[0^3 + h^3 + 2^3h^3 + 3^3h^3 + \dots + (n-1)^3h^3]$$

$= \lim_{h\to 0} h^4\,[1^3 + 2^3 + 3^3 + \dots + (n-1)^3]$, where $nh = 2$

$= \lim_{h\to 0} h^4\left[\frac{(n-1)^2\,\{(n-1)+1\}^2}{4}\right]$, summing up the series using the formula $\Sigma n^3 = \left[\frac{n(n+1)}{2}\right]^2$

$$= \lim_{h \to 0} \tfrac{1}{4} h^4 (n-1)^2 n^2, \text{ where } nh = 2$$

$$= \lim_{h \to 0} \tfrac{1}{4} (nh - h)^2 (nh)^2 = \tfrac{1}{4} (2-0)^2 . 2^2 = 4.$$

*Ex. 37. *Show that* $\int_a^b x^m \, dx = \dfrac{b^{m+1} - a^{m+1}}{(m+1)}$.

Sol. Here $f(x) = x^m$; $\therefore f(a) = a^m, f(a+h) = (a+h)^m$, etc.

$$\therefore \int_a^b x^m \, dx = \lim_{h \to 0} h \left[a^m + (a+h)^m + \ldots + \{a + (n-1)h\}^m\right],$$

where $b - a = nh$.

Now $\displaystyle\lim_{h \to 0} \frac{(t+h)^{m+1} - t^{m+1}}{h} = \frac{d}{dt} t^{m+1} = (m+1) t^m.$

$$\therefore \lim_{h \to 0} \frac{(t+h)^{m+1} - t^{m+1}}{h \cdot t^m} = (m+1), \text{ i.e., a constant.} \qquad \text{...(1)}$$

Putting $t = a, (a+h), (a+2h)$, etc., in (1), we get

$$\lim_{h \to 0} \frac{(a+h)^{m+1} - a^{m+1}}{h . a^m} = \lim_{h \to 0} \frac{(a+2h)^{m+1} - (a+h)^{m+1}}{h (a+h)^m} = \ldots$$

$$\ldots = \lim_{h \to 0} \frac{(a+nh)^{m+1} - \{a + (n-1)h\}^{m+1}}{h \{a + (n-1)h\}^m} = (m+1)$$

i.e., a constant. ...(2)

Also we know that if $\dfrac{a}{b} = \dfrac{c}{d} = \dfrac{e}{f} = \ldots\ldots$, then each of these ratios

is equal to $\dfrac{a + c + e + \ldots.}{b + d + f + \ldots}$ ...(3)

Now we apply the property (3) to various limits given in (2). Thus forming a new numerator and denominator by adding the numerators and denominators of the various ratios in (2), we get

$$\lim_{h \to 0} \frac{(a+nh)^{m+1} - a^{m+1}}{h \left[a^m + (a+h)^m + \ldots + \{a + (n-1)h\}^m\right]} = (m+1)$$

$$\text{or} \quad \lim_{h \to 0} \frac{[a + (b-a)]^{m+1} - a^{m+1}}{h \left[a^m + (a+h)^m + \ldots + \{a + (n-1)h\}^m\right]} = (m+1).$$

$[\because nh = b - a]$

$$\text{or} \quad \lim_{h \to 0} h \left[a^m + (a+h)^m + \ldots + \{a + (n-1)h\}^m\right]$$

$$= \frac{b^{m+1} - a^{m+1}}{m+1}. \qquad \therefore \int_a^b x^m \, dx = \frac{b^{m+1} - a^{m+1}}{(m+1)}.$$

Ex. 38. *From the definition of a definite integral as the limit of a sum, evaluate $\int_a^b e^x dx$.* **(Kashmir 1983; Meerut 83; K.U. 77; Ranchi 76; Magadh 76; Luck. 81, 79, 74)

Sol. Here $f(x) = e^x$; $\therefore$ $f(a) = e^a$, $f(a+h) = e^{a+h}$, etc.

$$\therefore \int_a^b e^x\,dx = \lim_{h \to 0} h\,\{e^a + e^{a+h} + e^{a+2h} + \ldots + e^{a+(n-1)h}\},$$

where $nh = b - a$ and $n \to \infty$ as $h \to 0$

$$= \lim_{h \to 0} he^a\,\{1 + e^h + e^{2h} + \ldots + e^{(n-1)h}\}$$

$$= \lim_{h \to 0} he^a \left\{\frac{(e^h)^n - 1}{e^h - 1}\right\}, \text{ summing the G.P.}$$

$$= \lim_{h \to 0} he^a \left[\frac{e^{nh} - 1}{e^h - 1}\right] = \lim_{h \to 0} he^a \left[\frac{e^{b-a} - 1}{e^h - 1}\right], [\because nh = (b - a)]$$

$$= e^a\,(e^{b-a} - 1), \qquad \left\{\because \lim_{h \to 0} \frac{h}{e^h - 1} = \lim_{h \to 0} \frac{1}{e^h} = 1\right\}$$

$$= e^b - e^a.$$

Ex. 39. (a). *Evaluate by summation $\int_a^b \sin x\,dx$.*

Sol. Here $f(x) = \sin x$; $\therefore$ $f(a) = \sin a$, $f(a+h) = \sin(a+h)$, etc.

$$\therefore \int_a^b \sin x\,dx = \lim_{h \to 0} h\,[\sin a + \sin(a+h) + \ldots + \sin\{a + (n-1)h\}],$$

where $nh = b - a$ and $n \to \infty$ as $h \to 0$

$$= \lim_{h \to 0} h \left[\frac{\sin(\frac{1}{2}nh)}{\sin\frac{1}{2}h} . \sin\{a + \tfrac{1}{2}(n-1)h\}\right],$$

from Trigonometry

$$= \lim_{h \to 0} 2 . \frac{\frac{1}{2}h}{\sin\frac{1}{2}h} . \sin\left(\frac{b-a}{2}\right) . \sin\left(a + \frac{b-a-h}{2}\right),$$

$[\because nh = b - a]$

$$= 2 . 1 . \sin\left(\frac{b-a}{2}\right) \sin\left(a + \frac{b-a}{2}\right), \qquad \left\{\because \lim_{\theta \to 0} \frac{\theta}{\sin\theta} = 1\right\}$$

$$= 2\sin\frac{b-a}{2}\sin\frac{a+b}{2} = \cos a - \cos b.$$

Ex. 39 (b). *Using the definition of integral as the limit of a sum, show that $\int_a^b \cos x\,dx = \sin b - \sin a$.* **(Meerut 83 S)**

Sol. Procced exactly as in Ex. 39 (a).

Ex. 39 (c). *Evaluate by summation* $\int_0^{\pi/2} \sin x \, dx$.

(Bihar 1976, 71; Ranchi 75; Magadh 71)

Sol. Here $f(x) = \sin x$; $a = 0$ and $b = \pi/2$,

$nh = b - a = \frac{1}{2}\pi - 0 = \frac{1}{2}\pi$.

Proceeding exactly as in Ex. 39 (a), we get

$$\int_0^{\pi/2} \sin x \, dx = \cos 0 - \cos \tfrac{1}{2}\pi = 1 - 0 = 1.$$

Ex. 39 (d). *Evaluate by summation* $\int_0^{\pi/2} \cos x \, dx$.

Sol. Here $f(x) = \cos x$; $a = 0$ and $b = \pi/2$,

$nh = b - a = \frac{1}{2}\pi - 0 = \frac{1}{2}\pi$.

Proceeding exactly as in Ex. 39 (a), we get

$$\int_0^{\pi/2} \cos x \, dx = \sin \tfrac{1}{2}\pi - \sin 0 = 1 - 0 = 1.$$

Ex. 40. *Evaluate by summation* $\displaystyle\int_a^b \frac{1}{x^2} dx$.

(Lucknow 1977)

Sol. First do Ex. 37 and then put $m = -2$.

§ 7·4. Summation of series with the help of definite integrals.

The definition of a definite integral as the limit of a sum (§ 7·3) helps us to evaluate the limit of the sums of some special types of series. We know that

$$\int_a^b f(x) \, dx = \lim_{n \to \infty} h \left[f(a) + f(a+h) + \ldots + f\{a + (n-1)h\} \right]$$

$$= \lim_{n \to \infty} h \sum_{r=0}^{n-1} f(a + rh), \text{ where } nh = b - a.$$

Putting $a = 0$ and $b = 1$, so that $h = (1/n)$, we get

$$\int_0^1 \mathbf{f(x) \, dx} = \lim_{\mathbf{n \to \infty}} \frac{\mathbf{1}}{\mathbf{n}} \sum_{\mathbf{r=0}}^{\mathbf{n-1}} \mathbf{f}\left(\frac{\mathbf{r}}{\mathbf{n}}\right).$$

Thus the limit of the sum of a series can be expressed in the form of a definite integral provided the series has the following properties:

(a) Each term of the series should have $(1/n)$ as a common factor which tends to zero as $n \to \infty$.

(b) The general term of the series should be the product of $1/n$ and a function $f(r/n)$ of r/n, so that the various terms of the series can be obtained from it by giving different values to r, say $r = 0, 1, 2 \ldots, n-1$.

(c) There should be n terms in the series, but if however the number of terms differs by a finite number from n, then the required limit does not change because each term tends to zero. Thus

$$\lim_{n \to \infty} \frac{1}{n} \sum_{r=p}^{n+q} f\left(\frac{r}{n}\right) = \int_0^1 f(x)\, dx,$$

if p and q are independent of n.

Working Rule :

(i) Write down the general term [say rth term or $(r-1)$th term etc., as convenient] of the series. Take out $(1/n)$ as a factor from the general term and thus write the series in the form $\frac{1}{n} \sum_{r=0}^{n-1} f\left(\frac{r}{n}\right)$. We may have some other limits of r in the summation; for example, r may vary from 1 to n or from 0 to $2n$, etc..

(ii) Now to evaluate $\lim_{n \to \infty} \frac{1}{n} \sum_{r=0}^{n-1} f\left(\frac{r}{n}\right)$, replace r/n by x, $1/n$ by dx and $\lim_{n \to \infty} \Sigma$ by the sign of integration *i.e.*, by $\int$.

(iii) To find the limits of integration of x first note carefully the limits of r in the summation $\Sigma f(r/n)$. Divide these limits by n to get the values of r/n. Take limits of these values of r/n as $n \to \infty$ and get the limits of integration of x.

***Ex. 41.** *Show that the limit of the sum*

$$\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{3n},$$

when n is indefinitely increased is $\log 3$.

(Lucknow 1980; Delhi 77; Agra 80; Meerut 89)

Sol. Here the general term of the series is $\frac{1}{n+r}$ and r varies from 0 to $2n$.

Now we have to find $\lim_{n \to \infty} \sum_{r=0}^{2n} \frac{1}{n+r}$.

We have $\lim_{n \to \infty} \sum_{r=0}^{2n} \frac{1}{n+r} = \lim_{n \to \infty} \sum_{r=0}^{2n} \frac{1}{n\{1+(r/n)\}}$, expressing the general term in the form $(1/n) f(r/n)$

$= \lim_{n \to \infty} \frac{1}{n} \sum_{r=0}^{2n} \frac{1}{1+(r/n)}$, taking $\frac{1}{n}$ outside the sign of summation.

Now $\lim_{n \to \infty} \frac{1}{n} \sum_{r=0}^{2n} \frac{1}{1+(r/n)}$ is of the form $\lim_{n \to \infty} \frac{1}{n} \Sigma f\left(\frac{r}{n}\right)$,

where $f\left(\frac{r}{n}\right) = \frac{1}{1+(r/n)}$. The limits of r in this summation are 0 to $2n$. When $r = 0$, $\frac{r}{n} = \frac{0}{n} = 0$ and when $r = 2n$, $\frac{r}{n} = \frac{2n}{n} = 2$. As

$n \to \infty$, these values of $\frac{r}{n}$ tend to 0 and 2 respectively, giving us the limits of integration.

Now replacing r/n by x, $1/n$ by dx, $\lim_{n \to \infty} \Sigma$ by the sign of integration $\int$, taking the limits of integration of x from 0 to 2, we get

$$\lim_{n \to \infty} \frac{1}{n} \sum_{r=0}^{2n} \frac{1}{1+(r/n)} = \int_0^2 \frac{1}{1+x}\,dx$$

$$= \Big[\log(1+x)\Big]_0^2 = \log 3 - \log 1 = \log 3 - 0 = \log 3.$$

****Ex. 42.** *Evaluate the following :*

$$\lim_{n \to \infty} \left[\frac{1}{n+1} + \frac{1}{n+2} + \dots + \frac{1}{2n}\right].$$

(Agra 1975; Ranchi 74; Bhagalpur 73; Kanpur 70)

Sol. Here the general term (*i.e.*, the rth term) $= \dfrac{1}{n+r}$ and r varies from 1 to n. Thus we have to find $\lim_{n \to \infty} \sum_{r=1}^{n} \dfrac{1}{n+r}$.

We have $\lim_{n \to \infty} \sum_{r=1}^{n} \dfrac{1}{n+r} = \lim_{n \to \infty} \sum_{r=1}^{n} \dfrac{1}{n\{1+(r/n)\}}$

$= \lim_{n \to \infty} \dfrac{1}{n} \sum_{r=1}^{n} \dfrac{1}{1+(r/n)}$.

The limits of r in this summation are 1 to n. Therefore the lower limit of integration $= \lim_{n \to \infty} \dfrac{1}{n} = 0$,

and the upper limit of integration $= \lim_{n \to \infty} \dfrac{n}{n} = \lim_{n \to \infty} 1 = 1$.

Hence the required limit

$$= \int_0^1 \frac{1}{1+x}\,dx = \Big[\log(1+x)\Big]_0^1 = \log 2.$$

Ex. 43. *Evaluate the following :*

$$\lim_{n \to \infty} \left[\frac{1}{n+m} + \frac{1}{n+2m} + \dots + \frac{1}{n+nm}\right].$$

Sol. Here the rth term $= \dfrac{1}{n+rm} = \dfrac{1}{n}\left\{\dfrac{1}{1+(r/n)\,m}\right\}$

and r varies from 1 to n.

$\therefore$ the given limit $= \lim_{n \to \infty} \sum_{r=1}^{n} \dfrac{1}{n} \cdot \left\{\dfrac{1}{1+(r/n)\,m}\right\}$

$$= \lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{n} \frac{1}{1 + (r/n)\, m}.$$

Also the lower limit of integration

$$= \lim_{n \to \infty} (1/n) = 0, \qquad [\because r = 1 \text{ for the first term }]$$

and the upper limit of integration $= \lim_{n \to \infty} (n/n) = 1$,

$$[\because r = n \text{ for the last term}].$$

$\therefore$ the required limit $= \int_0^1 \frac{1}{1 + mx}\, dx$

$$= \left[\frac{1}{m} \log (1 + mx)\right]_0^1 = (1/m) \log (1 + m).$$

Ex. 44. *Find the limit, when* $n \to \infty$, *of the series*

$$\frac{n}{(n+1)^2} + \frac{n}{(n+2)^2} + \dots + \frac{n}{(n+n)^2}.$$

(Meerut 1976, 91P)

Sol. Here the rth term $= \frac{n}{(n+r)^2} = \frac{n}{n^2 \{1 + (r/n)\}^2}$

$= \frac{1}{n} \cdot \frac{1}{\{1 + (r/n)\}^2}$, and r varies from 1 to n.

$\therefore$ we have to find

$$\lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{n} \cdot \frac{1}{\{1 + (r/n)\}^2}.$$

The lower limit of integration

$$= \lim_{n \to \infty} \left(\frac{1}{n}\right) = 0, \qquad [\because r = 1 \text{ for the 1st term}]$$

and the upper limit

$$= \lim_{n \to \infty} \left(\frac{n}{n}\right) = 1, \qquad [\because r = n \text{ for the last term}].$$

$\therefore$ the required limit $= \int_0^1 \frac{1}{(1+x)^2}\, dx = \left[-\frac{1}{(1+x)}\right]_0^1$

$$= -\tfrac{1}{2} - (-1) = -\tfrac{1}{2} + 1 = \tfrac{1}{2}.$$

Ex. 45. *Evaluate the following limits :*

(i) $\lim_{n \to \infty} [\{\sqrt{(n+1)} + \sqrt{(n+2)} + \dots + \sqrt{(2n)}\}/n\sqrt{n}]$.

(ii) $\lim_{n \to \infty} n \left[\frac{1}{(n+1)(n+2)} + \frac{1}{(n+2)(n+4)} + \frac{1}{(n+3)(n+6)} + \dots + \frac{1}{6n^2}\right]$.

(Agra 1983)

Sol. (i) The given limit

$$= \lim_{n \to \infty} \sum_{r=1}^{n} \frac{\sqrt{(n+r)}}{n\sqrt{n}} = \lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{n} \sqrt{\left(1+\frac{r}{n}\right)}$$

$$= \int_0^1 \sqrt{(1+x)}\, dx = \left[\frac{(1+x)^{3/2}}{3/2}\right]_0^1$$

$$= \frac{2}{3}\left[(1+x)^{3/2}\right]_0^1 = \frac{2}{3}\left[2^{3/2}-1\right] = \frac{2}{3}\left[2\sqrt{2}-1\right].$$

(ii) The given limit

$$= \lim_{n \to \infty} n \sum_{r=1}^{n} \frac{1}{(n+r)(n+2r)}$$

$$= \lim_{n \to \infty} \frac{n}{n^2} \sum_{r=1}^{n} \frac{1}{(1+r/n)(1+2r/n)}$$

$$= \lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{n} \frac{1}{(1+r/n)(1+2r/n)} = \int_0^1 \frac{1}{(1+x)(1+2x)}\, dx.$$

Let $\dfrac{1}{(1+x)(1+2x)} \equiv \dfrac{A}{1+x} + \dfrac{B}{1+2x}$.

Then $A = -1$, $B = 2$.

$\therefore$ the required limit

$$= \int_0^1 \left[\frac{-1}{1+x} + \frac{2}{1+2x}\right] dx$$

$$= \left[-\log(1+x) + \log(1+2x)\right]_0^1$$

$$= \left[\log\left(\frac{1+2x}{1+x}\right)\right]_0^1 = \log\frac{3}{2} - \log 1 = \log\frac{3}{2}.$$

Ex. 46. (a). *Evaluate*

$$\lim_{n \to \infty} \left[\frac{n}{n^2} + \frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \ldots + \frac{n}{n^2+(n+1)^2}\right].$$

(Meerut 1981; Lucknow 81; Vikram 73; Gorakhpur 70)

Sol. Here, the rth term $= \dfrac{n}{n^2+(r-1)^2}$. As the rth term contains $(r-1)$, we consider the $(r+1)$th term.

The $(r+1)$th term $= \dfrac{n}{n^2+r^2} = \dfrac{n}{n^2\{1+(r/n)^2\}}$

$$= \frac{1}{n} \cdot \left\{\frac{1}{1+(r/n)^2}\right\},$$ and r varies from 0 to $n+1$.

$\therefore$ the given limit $= \displaystyle\lim_{n \to \infty} \sum_{r=0}^{n+1} \frac{1}{n}\left[\frac{1}{1+(r/n)^2}\right].$

Also the lower limit of integration

$$= \lim_{n \to \infty} \left(\frac{0}{n}\right) = \lim_{n \to \infty} 0 = 0, \qquad [\because r = 0 \text{ for the 1st term}]$$

and the upper limit

$$= \lim_{n \to \infty} \left(\frac{n+1}{n}\right) = \lim_{n \to \infty} \left(1 + \frac{1}{n}\right) = 1,$$

$\because r = (n+1)$ for the last term.

$\therefore$ the required limit $= \displaystyle\int_0^1 \frac{1}{1+x^2}\,dx$

$$= \left[\tan^{-1} x\right]_0^1 = \tan^{-1} 1 = \frac{\pi}{4}.$$

Ex. 46 (b). *Evaluate*

$$\lim_{n \to \infty} \left[\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \dots + \frac{1}{2n}\right].$$

(Meerut 1990 P)

Sol. Here the rth term $= \dfrac{n}{n^2+r^2}$, and r varies from 0 to n.

Proceeding as in Ex. 46 (a), we get the required limit $= \pi/4$.

Ex. 47. *Evaluate*

$$\lim_{n \to \infty} \left[\frac{n+1}{n^2+1^2} + \frac{n+2}{n^2+2^2} + \dots + \frac{1}{n}\right].$$

(Delhi 1978; Meerut 90)

Sol. Here the rth term

$$= \frac{n+r}{n^2+r^2} = \frac{1+(r/n)}{n\,\{1+(r/n)^2\}} = \frac{1}{n} \cdot \left\{\frac{1+(r/n)}{1+(r/n)^2}\right\},$$

and r varies from 1 to n.

$\therefore$ the given limit $= \displaystyle\lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{n} \cdot \left\{\frac{1+(r/n)}{1+(r/n)^2}\right\}$

$$= \int_0^1 \frac{x+1}{x^2+1}\,dx = \int_0^1 \left[\frac{x}{x^2+1} + \frac{1}{x^2+1}\right] dx$$

$$= \left[\tfrac{1}{2}\log(x^2+1) + \tan^{-1} x\right]_0^1 = \frac{1}{2}\log 2 + \frac{\pi}{4}.$$

Ex. 48. *Evaluate*

$$\lim_{n \to \infty} \left[\frac{1}{n^3}(1+4+9+16+\dots+n^2)\right].$$

(Meerut 1991S)

Sol. Here the rth term $= \dfrac{1}{n^3}(r^2) = \dfrac{1}{n} \cdot \left(\dfrac{r}{n}\right)^2,$

and r varies from 1 to n.

$\therefore$ the given limit $= \displaystyle\lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{n} \cdot \left(\frac{r}{n}\right)^2$

$$= \int_0^1 x^2\,dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}.$$

****Ex. 49.** *Prove that*

$$\lim_{n \to \infty} \left[\frac{1^2}{1^3+n^3} + \frac{2^2}{2^3+n^3} + \dots + \frac{n^2}{n^3+n^3}\right] = \tfrac{1}{2} \log 2.$$

(Lucknow 1983; Kanpur 79; Agra 73; Jiwaji 72; Meerut 84, 84S, 85S, 91)

Sol. Here the rth term

$$= \frac{r^2}{r^3+n^3} = \frac{1}{n^3}\left\{\frac{r^2}{(r/n)^3+1}\right\} = \frac{1}{n} \cdot \left\{\frac{(r/n)^2}{(r/n)^3+1}\right\},$$

and r varies from 1 to n.

$\therefore$ the given limit $= \lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{n} \cdot \left\{\frac{(r/n)^2}{(r/n)^3+1}\right\}$

$$= \int_0^1 \frac{x^2\,dx}{x^3+1} = \left[\tfrac{1}{3} \log (x^3+1)\right]_0^1$$

$$= \tfrac{1}{3} \log 2 - \tfrac{1}{3} \log 1 = \tfrac{1}{3} \log 2.$$

****Ex. 50.** *Evaluate*

$$\lim_{n \to \infty} \left[\frac{1}{n} + \frac{n^2}{(n+1)^3} + \frac{n^2}{(n+2)^3} + \dots + \frac{1}{8n}\right].$$

(Meerut 1990; Gorakh. 75, 73; Vikram 77; Kashmir 71)

Sol. Here the general term

$$= \frac{n^2}{(n+r)^3} = \frac{n^2}{n^3\{1+(r/n)\}^3} = \frac{1}{n} \cdot \frac{1}{\{1+(r/n)\}^3},$$

and r varies from 0 to n.

$\therefore$ the given limit $= \lim_{n \to \infty} \sum_{r=0}^{n} \frac{1}{n} \cdot \frac{1}{\{1+(r/n)\}^3}$

$$= \int_0^1 \frac{1}{(1+x)^3}\,dx = \left[-\frac{1}{2(1+x)^2}\right]_0^1 = -\tfrac{1}{8} + \tfrac{1}{2} = 3/8.$$

Ex. 51. *Evaluate*

$$\lim_{n \to \infty} \left[\frac{n^{1/2}}{n^{3/2}} + \frac{n^{1/2}}{(n+3)^{3/2}} + \frac{n^{1/2}}{(n+6)^{3/2}} + \dots + \frac{n^{1/2}}{\{n+3(n-1)\}^{3/2}}\right]$$

(Meerut 1980, 84; Gorakhpur 71)

Sol. Here the general term $= \dfrac{n^{1/2}}{(n+3r)^{3/2}}$

$$= \frac{1}{n\{1+(3r/n)\}^{3/2}},$$

and r varies from 0 to $n-1$.

$\therefore$ the given limit $= \lim_{n \to \infty} \sum_{r=0}^{n-1} \frac{1}{n} \cdot \frac{1}{\{1+(3r/n)\}^{3/2}}$

$$= \int_0^1 \frac{dx}{(1+3x)^{3/2}} = -\frac{2}{3} \cdot \left[\frac{1}{(1+3x)^{1/2}}\right]_0^1$$

$= -\frac{2}{3}[\frac{1}{2} - 1] = \frac{1}{3}$.

Ex. 52. *Evaluate*

$$\lim_{n \to \infty} \left[\frac{1}{n} + \frac{1}{\sqrt{(n^2 - 1^2)}} + \frac{1}{\sqrt{(n^2 - 2^2)}} + \dots + \frac{1}{\sqrt{\{n^2 - (n-1)^2\}}}\right].$$

(Meerut 1983 S, 89; Lucknow 80; Rohilkhand 79)

Sol. Here the general term

$$= \frac{1}{\sqrt{(n^2 - r^2)}} = \frac{1}{n} \cdot \frac{1}{\sqrt{\{1 - (r/n)^2\}}},$$ and r varies from 0 to $(n-1)$.

$\therefore$ the given limit $= \lim_{n \to \infty} \sum_{r=0}^{n-1} \frac{1}{n} \cdot \frac{1}{\sqrt{\{1 - (r/n)^2\}}}$

$$= \int_0^1 \frac{1}{\sqrt{(1 - x^2)}} dx = \left[\sin^{-1} x\right]_0^1$$

$$= \sin^{-1} 1 - \sin^{-1} 0 = \tfrac{1}{2}\pi.$$

Ex. 52 (a). *Evaluate*

$$\lim_{n \to \infty} \left[\frac{1}{\sqrt{(n^2 - 1^2)}} + \frac{1}{\sqrt{(n^2 - 2^2)}} + \dots + \frac{1}{\sqrt{\{n^2 - (n-1)^2\}}}\right].$$

(Meerut 1983 P, 83 S)

Sol. Proceed exactly as in Ex. 52.

Ex. 53. *Evaluate*

$$\lim_{n \to \infty} \left[\frac{1}{n^2} \sec^2 \frac{1}{n^2} + \frac{2}{n^2} \sec^2 \frac{4}{n^2} + \frac{3}{n^2} \sec^2 \frac{9}{n^2} + \dots + \frac{1}{n} \sec^2 1\right].$$

(Garhwal 1983; Meerut 77, 84 P, 87 P, 88; Agra 78, 72)

Sol. Here the rth term

$$= \frac{r}{n^2} \sec^2 \frac{r^2}{n^2} = \frac{1}{n} \cdot \left\{\frac{r}{n} \sec^2 \frac{r^2}{n^2}\right\},$$ and r varies from 1 to n.

$\therefore$ the given limit $= \lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{n} \cdot \left\{\frac{r}{n} \sec^2 \frac{r^2}{n^2}\right\}$

$= \int_0^1 x \sec^2 x^2 \, dx = \frac{1}{2} \int_0^1 \sec^2 t \, dt$, putting $x^2 = t$ so that

$2x \, dx = dt$ and the limits for t are 0 to 1

$$= \tfrac{1}{2}\left[\tan t\right]_0^1 = \tfrac{1}{2}(\tan 1 - \tan 0) = \tfrac{1}{2}\tan 1.$$

Ex. 54. *Evaluate*

$$\lim_{n \to \infty} \frac{1}{n}\left[\sin^{2k} \frac{\pi}{2n} + \sin^{2k} \frac{2\pi}{2n} + \sin^{2k} \frac{3\pi}{2n} + \dots + \sin^{2k} \frac{\pi}{2}\right].$$

(Lucknow 1982, 77)

Sol. Here the rth term $= \frac{1}{n} \cdot \sin^{2k} \frac{r\pi}{2n}$, and r varies from 1 to n.

$\therefore$ the given limit $= \lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{n} \cdot \sin^{2k} \frac{r\pi}{2n}$

$= \int_0^1 \sin^{2k}\left(\frac{\pi}{2} \cdot x\right) dx = \frac{2}{\pi} \int_0^{\pi/2} \sin^{2k} t \, dt$, putting $\frac{\pi x}{2} = t$

so that $\frac{1}{2}\pi \, dx = dt$ and the limits for t are 0 to $\pi/2$

$= \frac{2}{\pi} \cdot \frac{(2k-1)}{2k} \cdot \frac{(2k-3)}{(2k-2)} \dots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$, by Walli's formula

$= \frac{(2k-1)(2k-3)\dots 3.1}{2k.(2k-2)\dots 4.2}$.

***Ex. 55.** *Evaluate* $\lim_{n \to \infty} \sum_{r=1}^{n} \frac{r^3}{r^4 + n^4}$. **(Meerut 1983, 90 P)**

Sol. Here $\lim_{n \to \infty} \sum_{r=1}^{n} \frac{r^3}{r^4 + n^4} = \lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{n^4} \left\{\frac{r^3}{(r/n)^4 + 1}\right\}$

$= \lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{n} \cdot \left\{\frac{(r/n)^3}{(r/n)^4 + 1}\right\}$ (Note)

$= \int_0^1 \frac{x^3}{x^4 + 1} dx = \frac{1}{4}\left[\log(1 + x^4)\right]_0^1 = \frac{1}{4}\log 2.$

Ex. 56. *Prove that* $\lim_{n \to \infty} \sum_{r=0}^{n-1} \frac{n}{r^2 + n^2} = \frac{\pi}{4}$.

Sol. See Ex. 46.

***Ex. 57.** *Evaluate* $\lim_{n \to \infty} \sum_{r=0}^{n-1} \frac{1}{n} \cdot \sqrt{\left(\frac{n+r}{n-r}\right)}$.

(Meerut 1981 S, 88, 89 P, 90; Ranchi 73)

Sol. Here $\lim_{n \to \infty} \sum_{r=1}^{n-1} \frac{1}{n} \sqrt{\left(\frac{n+r}{n-r}\right)}$

$= \lim_{n \to \infty} \sum_{r=1}^{n-1} \frac{1}{n} \cdot \sqrt{\left\{\frac{1 + (r/n)}{1 - (r/n)}\right\}}$

$= \int_0^1 \sqrt{\left(\frac{1+x}{1-x}\right)} dx = \int_0^1 \frac{(1+x)}{\sqrt{(1-x^2)}} dx.$ (Note)

Now put $x = \sin\theta$ so that $dx = \cos\theta \, d\theta$.
Also $\theta = 0$ when $x = 0$ and $\theta = \pi/2$ when $x = 1$.

$\therefore$ the required limit $= \int_0^{\pi/2} \frac{1 + \sin\theta}{\cos\theta} \cos\theta \, d\theta$

$= \int_0^{\pi/2} (1 + \sin\theta) \, d\theta = \left[\theta - \cos\theta\right]_0^{\pi/2}$

$= \left(\frac{\pi}{2} - 0\right) - (0 - 1) = \frac{\pi}{2} + 1.$

Ex. 58. *Evaluate* $\lim_{n \to \infty} \sum_{r=1}^{n} \frac{\sqrt{n}}{\sqrt{r}\,.(3\sqrt{r} + 4\sqrt{n})^2}$.

(Meerut 1983, 82 P)

Sol. The given limit

$$= \lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{n} \cdot \frac{1}{\sqrt{(r/n)}\,.\{3\sqrt{(r/n)} + 4\}^2}$$

$$= \int_0^1 \frac{dx}{\sqrt{x}\,.(3\sqrt{x} + 4)^2} \cdot \text{ Now put } 3\sqrt{x} + 4 = t,$$

so that $3\,.(1/2\sqrt{x})\,dx = dt$.

The limits for t are 4 to 7.

$\therefore$ the required limit

$$= \frac{2}{3}\int_4^7 \frac{dt}{t^2} = -\frac{2}{3}\left[\frac{1}{t}\right]_4^7 = -\frac{2}{3}\left[\frac{1}{7} - \frac{1}{4}\right] = \frac{1}{14}.$$

Ex. 59. *Evaluate* $\lim_{n \to \infty} \sum_{r=1}^{n} \frac{n^2}{(n^2 + r^2)^{3/2}}$.

(Kashmir 1974)

Sol. The given limit

$$= \lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{n} \cdot \frac{1}{[1 + (r/n)^2]^{3/2}}$$

$$= \int_0^1 \frac{dx}{(1 + x^2)^{3/2}} = \int_0^{\pi/4} \frac{\sec^2\theta\, d\theta}{\sec^3\theta}, \text{ putting } x = \tan\theta$$

so that $dx = \sec^2\theta\, d\theta$ and the limits for θ are 0 to $\pi/4$

$$= \int_0^{\pi/4} \cos\theta\, d\theta = \Big[\sin\theta\Big]_0^{\pi/4} = \sin\frac{\pi}{4} - \sin 0 = \frac{1}{\sqrt{2}}.$$

Ex. 60. *Prove that*

$$\lim_{n \to \infty} \left[\frac{1}{\sqrt{(2n - 1^2)}} + \frac{1}{\sqrt{(4n - 2^2)}} + \dots + \frac{1}{n}\right] = \frac{\pi}{2}.$$

Sol. Here the rth term

$$= \frac{1}{\sqrt{(2nr - r^2)}} = \frac{1}{n} \cdot \frac{1}{\sqrt{\{2(r/n) - (r/n)^2\}}},$$

and r varies from 1 to n.

$\therefore$ the given limit

$$= \lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{n} \cdot \frac{1}{\sqrt{\{2(r/n) - (r/n)^2\}}}$$

$$= \int_0^1 \frac{dx}{\sqrt{(2x - x^2)}} = \int_0^1 \frac{dx}{\sqrt{\{1 - (x - 1)^2\}}} = \Big[\sin^{-1}(x - 1)\Big]_0^1$$

$$= \sin^{-1} 0 - \sin^{-1}(-1) = 0 + \sin^{-1} 1 = \pi/2.$$

***Ex. 61.** *Find the limit as* $n \to \infty$ *of the series*

$$\frac{n}{(n+1)\sqrt{(2n+1)}} + \frac{n}{(n+2)\sqrt{\{2(2n+2)\}}} + \dots + \frac{n}{2n\sqrt{(n.3n)}}.$$

Sol. Here the rth term $= \dfrac{n}{(n+r)\sqrt{\{r(2n+r)\}}}$

$$= \frac{n}{(n+r)\,n\sqrt{[(r/n)\{2+(r/n)\}]}}$$

$$= \frac{1}{n}\cdot\frac{1}{\{1+(r/n)\}\sqrt{[(r/n)\{2+(r/n)\}]}}.$$

Also r varies from 1 to n.

$\therefore$ the given limit

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\cdot\frac{1}{\{1+(r/n)\}\sqrt{[(r/n)\{2+(r/n)\}]}}$$

$$= \int_0^1 \frac{dx}{(1+x)\sqrt{\{x(2+x)\}}} = \int_0^1 \frac{dx}{(1+x)\sqrt{\{(1+x)^2-1\}}}$$

$$= \left[\sec^{-1}(1+x)\right]_0^1 = \sec^{-1}2 - \sec^{-1}1 = \frac{\pi}{3} - 0 = \frac{\pi}{3}.$$

Ex. 62. *Evaluate the limit of the following sum as $n\to\infty$:*

$$\frac{(n-m)^{1/3}}{n} + \frac{(2^2 n - m)^{1/3}}{2n} + \frac{(3^2 n - m)^{1/3}}{3n} + \dots + \frac{(n^3-m)^{1/3}}{n^2}.$$

Sol. Here the rth term $= \dfrac{(r^2 n - m)^{1/3}}{rn}$

$$= \frac{n\left[\frac{r^2}{n^2} - \frac{m}{n^3}\right]^{1/3}}{r\,.\,n} = \frac{1}{n}\cdot\frac{1}{(r/n)}\cdot\left[\frac{r^2}{n^2}-\frac{m}{n^3}\right]^{1/3}.$$

$\therefore$ the given limit $= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\cdot\frac{1}{(r/n)}\cdot\left[\frac{r^2}{n^2}-\frac{m}{n^3}\right]^{1/3}$

$$= \int_0^1 \frac{(x^2-0)^{1/3}}{x}\,dx = \int_0^1 x^{-1/3}\,dx = \left[\frac{3}{2}x^{2/3}\right]_0^1 = \frac{3}{2}.$$

Ex. 63. *Evaluate*

$$\lim_{n\to\infty}\left[\frac{1}{na} + \frac{1}{na+1} + \frac{1}{na+2} + \dots + \frac{1}{nb}\right].$$

Sol. Here the last term $= \dfrac{1}{nb} = \dfrac{1}{na + n(b-a)}$. (Note)

Now the rth term $= \dfrac{1}{na+r}$,

and r varies from 0 to $n(b-a)$.

$\therefore$ the given limit $= \lim_{n\to\infty}\sum_{r=0}^{n(b-a)}\frac{1}{na+r}$

$$= \lim_{n \to \infty} \sum_{r=0}^{n(b-a)} \frac{1}{n} \cdot \left\{ \frac{1}{a + (r/n)} \right\}.$$

Also the lower limit of integration

$$= \lim_{n \to \infty} \left(\frac{r}{n} \right), \text{ for the 1st term}$$

$$= 0, \qquad (\because r = 0 \text{ for the 1st term})$$

and upper limit $= \lim_{n \to \infty} \left(\frac{r}{n} \right)$, for the last term

$$= \lim_{n \to \infty} \frac{n(b-a)}{n}, \qquad [\because r = n(b-a) \text{ for the last term}]$$

$$= (b-a).$$

$\therefore$ the required limit $= \int_0^{(b-a)} \frac{1}{a+x}\, dx$

$$= \left[\log(a+x) \right]_0^{(b-a)} = \log b - \log a = \log(b/a).$$

Ex. 64 (a). *Evaluate*

$$\lim_{n \to \infty} \frac{1 + 2^{10} + 3^{10} + \dots + n^{10}}{n^{11}}.$$

Sol. The given limit may be written as

$$\lim_{n \to \infty} \left[\frac{1}{n} \cdot \left\{ \left(\frac{1}{n} \right)^{10} + \left(\frac{2}{n} \right)^{10} + \left(\frac{3}{n} \right)^{10} + \dots + \left(\frac{n}{n} \right)^{10} \right\} \right]. \qquad \text{(Note)}$$

Now the rth term $= \frac{1}{n} \cdot \left(\frac{r}{n} \right)^{10}$, and r varies from 1 to n.

$\therefore$ the given limit

$$= \lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{n} \cdot \left(\frac{r}{n} \right)^{10} = \int_0^1 x^{10}\, dx = \left[\frac{x^{11}}{11} \right]_0^1 = \frac{1}{11}.$$

Ex. 64 (b). *Prove that*

$$\lim_{n \to \infty} \frac{1^m + 2^m + 3^m + \dots + n^m}{n^{m+1}} = \frac{1}{m+1}, \ (m > 1).$$

Sol. The given limit

$$= \lim_{n \to \infty} \sum_{r=1}^{n} \frac{r^m}{n^{m+1}} = \lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{n} \left(\frac{r}{n} \right)^m$$

$$= \int_0^1 x^m\, dx = \left[\frac{x^{m+1}}{m+1} \right]_0^1 = \frac{1}{m+1}.$$

Limit of a product by integration. Let the limit of a product of n factors as $n \to \infty$ be P. Then the limit can easily be evaluated by taking logarithms of both sides as is clear from the following examples :

Ex. 65. *Find the limit, as $n \to \infty$, of the product*

$$\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)^{1/2}\left(1+\frac{3}{n}\right)^{1/3}\ldots\left(1+\frac{n}{n}\right)^{1/n}.$$

(Agra 1979; Lucknow 77; Rohilkhand 76; Kanpur 78; Jiwaji 70)

Sol. Let $P=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)^{1/2}\ldots\left(1+\frac{n}{n}\right)^{1/n}.$

Then $\log P=\lim_{n\to\infty}\left[\log\left(1+\frac{1}{n}\right)+\frac{1}{2}\log\left(1+\frac{2}{n}\right)+\frac{1}{3}\log\left(1+\frac{3}{n}\right)+\ldots+\frac{1}{n}\log\left(1+\frac{n}{n}\right)\right]$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{r}\log\left(1+\frac{r}{n}\right) \qquad \text{(Note)}$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\cdot\frac{1}{(r/n)}\log\left(1+\frac{r}{n}\right)$$

$$=\int_0^1\frac{1}{x}\log(1+x)\,dx=\int_0^1\frac{1}{x}\left[x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\ldots\right]dx$$

$$=\int_0^1\left(1-\frac{x}{2}+\frac{x^2}{3}-\frac{x^3}{4}+\ldots\right)dx=\left[x-\frac{x^2}{4}+\frac{x^3}{9}-\frac{x^4}{16}+\ldots\right]_0^1$$

or $\log P=1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\ldots\infty=\frac{\pi^2}{12}$, from trigonometry.

$\therefore\quad P=e^{\pi^2/12}.$

Ex. 66. (a). *Evaluate*

$$\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)\ldots\left(1+\frac{n}{n}\right)\right]^{1/n}.$$

(Agra 1979; Delhi 75; Allahabad 73)

Sol. Let

$$P=\lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)\ldots\left(1+\frac{n}{n}\right)\right]^{1/n}.$$

$$\log P=\lim_{n\to\infty}\frac{1}{n}\left[\log\left(1+\frac{1}{n}\right)+\log\left(1+\frac{2}{n}\right)+\ldots+\log\left(1+\frac{n}{n}\right)\right]$$

$$=\lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\log\left(1+\frac{r}{n}\right)$$

$$=\int_0^1\log(1+x)\,dx=\int_0^1\{\log(1+x)\}.\,1\,dx$$

$$=\left[\{\log(1+x)\}.x\right]_0^1-\int_0^1\frac{x}{1+x}\,dx,$$

integrating by parts taking 1 as the 2nd function

$$= \log 2 - \int_0^1 \frac{(1+x)-1}{1+x}\,dx = \log 2 - \int_0^1 dx + \int_0^1 \frac{1}{1+x}\,dx$$

$$= \log 2 - \left[x\right]_0^1 + \left[\log(1+x)\right]_0^1 = \log 2 - 1 + \log 2$$

$$= 2\log 2 - 1 = \log 2^2 - \log e = \log(4/e).$$

$\therefore \quad P = 4/e.$

***Ex. 66. (b).** *Evaluate*

$$\lim_{n \to \infty} \left[\frac{(n+1)(n+2)(n+3)\ldots(n+n)}{n^n}\right]^{1/n}.$$

(Meerut 1988 P)

Sol. The given limit

$$= \lim_{n \to \infty} \left[\left(\frac{n+1}{n}\right)\left(\frac{n+2}{2}\right)\cdots\left(\frac{n+n}{n}\right)\right]^{1/n}$$

$$= \lim_{n \to \infty} \left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{n}{n}\right)\right]^{1/n}.$$

Now proceed as in Ex. 66 (a).

****Ex. 67.** *Evaluate*

$$\lim_{n \to \infty} \left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\left(1+\frac{3^2}{n^2}\right)\cdots\left(1+\frac{n^2}{n^2}\right)\right]^{1/n}.$$

(Lucknow 1982; Kanpur 80, Agra 81, Meerut 88 S, 89 S, 91 P)

Sol. Let

$$P = \lim_{n \to \infty} \left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\left(1+\frac{3^2}{n^2}\right)\cdots\left(1+\frac{n^2}{n^2}\right)\right]^{1/n}.$$

$$\therefore \quad \log P = \lim_{n \to \infty} \frac{1}{n}\left[\log\left(1+\frac{1}{n^2}\right) + \log\left(1+\frac{2^2}{n^2}\right) + \log\left(1+\frac{3^2}{n^2}\right) + \ldots + \log\left(1+\frac{n^2}{n^2}\right)\right]$$

$$= \lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{n} \log\left(1+\frac{r^2}{n^2}\right)$$

$$= \int_0^1 \log(1+x^2)\,dx = \int_0^1 \log(1+x^2)\,.1\,dx$$

$$= \left[x\log(1+x^2)\right]_0^1 - \int_0^1 \frac{2x\,.x\,dx}{1+x^2},$$

integrating by parts taking 1 as the 2nd function

$$= \log 2 - 2\int_0^1 \frac{(1+x^2)-1}{1+x^2}\,dx$$

$$= \log 2 - 2\int_0^1 \left[1 - \frac{1}{1+x^2}\right] dx$$

$= \log 2 - 2\left[x - \tan^{-1} x\right]_0^1$

$= \log 2 - 2\left[1 - \frac{1}{4}\pi\right].$

Thus $\log P = \log 2 + \frac{1}{2}(\pi - 4)$, or $\log (P/2) = \frac{1}{2}(\pi - 4)$

or $\quad P = 2e^{(\pi - 4)/2}.$

Ex. 68. *Evaluate*

$$\lim_{n \to \infty} \left[\left(1 + \frac{1}{n^4}\right)\left(1 + \frac{2^4}{n^4}\right)^{1/2}\left(1 + \frac{3^4}{n^4}\right)^{1/3} \cdots \left(1 + \frac{n^4}{n^4}\right)^{1/n}\right].$$

Sol. Let

$$P = \lim_{n \to \infty} \left[\left(1 + \frac{1}{n^4}\right)\left(1 + \frac{2^4}{n^4}\right)^{1/2}\left(1 + \frac{3^4}{n^4}\right)^{1/3} \cdots \left(1 + \frac{n^4}{n^4}\right)^{1/n}\right].$$

$$\therefore \quad \log P = \lim_{n \to \infty} \left[\log\left(1 + \frac{1}{n^4}\right) + \frac{1}{2}\log\left(1 + \frac{2^4}{n^4}\right)\right.$$

$$\left. + \frac{1}{3}\log\left(1 + \frac{3^4}{n^4}\right) + \ldots + \frac{1}{n}\log\left(1 + \frac{n^4}{n^4}\right)\right]$$

$$= \lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{r}\log\left(1 + \frac{r^4}{n^4}\right) = \lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{n} \cdot \frac{n}{r}\left(1 + \frac{r^4}{n^4}\right)$$

$$= \int_0^1 \frac{1}{x}\log(1 + x^4)\,dx = \int_0^1 \frac{1}{x}\left[x^4 - \frac{x^8}{2} + \frac{x^{12}}{3} - \ldots \infty\right]dx$$

$$= \int_0^1 \left[x^3 - \frac{x^7}{2} + \frac{x^{11}}{3} - \ldots \infty\right]dx$$

$$= \left[\frac{x^4}{4} - \frac{x^8}{16} + \frac{x^{12}}{36} - \ldots \infty\right]_0^1 = \frac{1}{4} - \frac{1}{16} + \frac{1}{36} - \ldots \infty$$

$$= \frac{1}{4}\left[1 - \frac{1}{4} + \frac{1}{9} - \ldots \infty\right]$$

$$= \frac{1}{4}\left[1 - \frac{1}{2^2} + \frac{1}{3^2} - \ldots \infty\right]$$

$= \dfrac{1}{4} \cdot \dfrac{\pi^2}{12}$, from trigonometry.

Thus $\quad \log P = \dfrac{\pi^2}{48}; \quad \therefore \quad P = e^{\pi^2/48}.$

****Ex. 69.** *Find the limit of* $\left\{\dfrac{n!}{n^n}\right\}^{1/n}$ *when n tends to infinity.*

(Meerut 1984 S, 85, 87, 90 S; Lucknow 79; Agra 77; K.U. 77; Magadh 72; Kanpur 80; Delhi 76)

Sol. Let $P = \lim_{n \to \infty} \left\{\frac{n!}{n^n}\right\}^{1/n} = \lim_{n \to \infty} \left\{\frac{1.2.3.4 \ldots n}{n.n.n.n \ldots n}\right\}^{1/n}$

$= \lim_{n \to \infty} \left\{\frac{1}{n} \cdot \frac{2}{n} \cdot \frac{3}{n} \cdots \frac{n}{n}\right\}^{1/n}.$

$\therefore \quad \log P = \lim_{n \to \infty} \frac{1}{n}\left[\log\left(\frac{1}{n}\right) + \log\left(\frac{2}{n}\right) + \log\left(\frac{3}{n}\right) + \ldots + \log\left(\frac{n}{n}\right)\right]$

$= \lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{n} \log\left(\frac{r}{n}\right) = \int_0^1 \log x \, dx = \int_0^1 (\log x) .1 \, dx$

$= \left[(\log x) . x\right]_0^1 - \int_0^1 \frac{1}{x} \cdot x \, dx$, integrating by parts

$= 0 - \left[x\right]_0^1 = -1.$

$\therefore \quad P = e^{-1} = 1/e.$

Ex. 70. *Evaluate*

$$\lim_{n \to \infty} \left[\sin\frac{\pi}{2n} \sin\frac{2\pi}{2n} \sin\frac{3\pi}{2n} \ldots \sin\frac{n\pi}{2n}\right]^{1/n}.$$

(Meerut 1991S; Agra 76)

Sol. Let $P = \lim_{n \to \infty} \left[\sin\frac{\pi}{2n} \sin\frac{2\pi}{2n} \ldots \sin\frac{n\pi}{2n}\right]^{1/n}.$

$\therefore \quad \log P = \lim_{n \to \infty} \frac{1}{n}\left[\log\left(\sin\frac{\pi}{2n}\right) + \ldots + \log\left(\frac{\sin n\pi}{2n}\right)\right]$

$= \lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{n} \log\left(\sin\frac{r\pi}{2n}\right)$

$= \int_0^1 \log\left(\sin\frac{\pi}{2}x\right) dx$

$= \frac{2}{\pi}\int_0^{\pi/2} \log(\sin\theta)\, d\theta,$

putting $\frac{\pi}{2}x = \theta$ and changing the limits

$= \frac{2}{\pi} \cdot \left(-\frac{\pi}{2}\log 2\right), \quad \because \int_0^{\pi/2} \log\sin\theta \, d\theta = -\frac{\pi}{2}\log 2.$

(See Ex. 17 page 253)

$= -\log 2 = \log(2^{-1}) = \log(1/2).$

Thus $\log P = \log\left(\frac{1}{2}\right)$, or $P = \frac{1}{2}.$

Ex. 71. *Evaluate*

$$\lim_{n \to \infty} \left[\tan \frac{\pi}{2n} \tan \frac{2\pi}{2n} \tan \frac{3\pi}{2n} \dots \tan \frac{n\pi}{2n} \right]^{1/n}.$$

(Kanpur 1979; Meerut 91)

Sol. Let $P = \lim_{n \to \infty} \left[\tan \frac{\pi}{2n} \tan \frac{2\pi}{2n} \dots \tan \frac{n\pi}{2n} \right]^{1/n}.$

Then $\log P = \lim_{n \to \infty} \frac{1}{n} \left[\log \left(\tan \frac{\pi}{2n} \right) + \log \left(\tan \frac{2\pi}{2n} \right) + \dots + \log \left(\tan \frac{n\pi}{2n} \right) \right]$

$$= \lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{n} \log \left(\tan \frac{r\pi}{2n} \right)$$

$$= \int_0^1 \log \left\{ \tan \frac{\pi}{2} x \right\} dx = \frac{2}{\pi} \int_0^{\pi/2} \log (\tan \theta)\, d\theta,$$

putting $(\pi x/2) = \theta$ and changing the limits accordingly

$$= \frac{2}{\pi} \int_0^{\pi/2} \log \left(\frac{\sin \theta}{\cos \theta} \right) d\theta$$

$$= \frac{2}{\pi} \int_0^{\pi/2} \log \sin \theta \, d\theta - \frac{2}{\pi} \int_0^{\pi/2} \log \cos \theta \, d\theta$$

$$= \frac{2}{\pi} \int_0^{\pi/2} \log \sin \left(\frac{\pi}{2} - \theta \right) d\theta - \frac{2}{\pi} \int_0^{\pi/2} \log \cos \theta \, d\theta,$$

$$\left[\because \int_0^a f(x)\, dx = \int_0^a f(a-x)\, dx \right]$$

$$= \frac{2}{\pi} \int_0^{\pi/2} \log \cos \theta \, d\theta - \frac{2}{\pi} \int_0^{\pi/2} \log \cos \theta \, d\theta = 0.$$

$\therefore \quad P = e^0 = 1.$

Ex. 72. *Evaluate the limit*

$$\lim_{n \to \infty} \left(1 + \frac{1}{n^2}\right)^{2/n^2} \left(1 + \frac{2^2}{n^2}\right)^{4/n^2} \left(1 + \frac{3^2}{n^2}\right)^{6/n^2} \dots \left(1 + \frac{n^2}{n^2}\right)^{2n/n^2}.$$

Sol. Let the required limit be P. Then

$$\log P = \lim_{n \to \infty} \left[\frac{2}{n^2} \log \left(1 + \frac{1}{n^2}\right) + \frac{4}{n^2} \log \left(1 + \frac{2^2}{n^2}\right) + \dots + \frac{2n}{n^2} \log \left(1 + \frac{n^2}{n^2}\right) \right]$$

$$= \lim_{n \to \infty} \sum_{r=1}^{n} \frac{2r}{n^2} \log \left(1 + \frac{r^2}{n^2}\right)$$

$$= \lim_{n \to \infty} \frac{1}{n} \sum_{r=1}^{n} 2\left(\frac{r}{n}\right) \log\left\{1 + \left(\frac{r}{n}\right)^2\right\} = \int_0^1 2x \log(1 + x^2)\, dx.$$

Now put $1 + x^2 = t$, so that $2x\, dx = dt$.
When $x = 0$, $t = 1$ and when $x = 1$, $t = 2$.

$\therefore \quad \log P = \int_1^2 \log t\, dt$

$$= \Big[t \log t\Big]_1^2 - \int_1^2 t \cdot \frac{1}{t}\, dt,$$

integrating by parts taking 1 as the second function

$$= \left(2 \log 2 - \log 1\right) - \int_1^2 dt = 2 \log 2 - \Big[t\Big]_1^2$$

$= \log 2^2 - (2 - 1) = \log 4 - 1 = \log 4 - \log e = \log(4/e).$

$\therefore \quad P = 4/e.$

$\square$