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QUESTION BANK SERIES

PAPER 1 : 01 LINEAR ALGEBRA & MATRIX

Content:

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01 PROBLEMS ON MATRIX

(1) Every invertible matrix possesses a unique inverse.
(or) The inverse of a matrix if it exists is unique.

(2) If A, B are invertible matrices of the same order, then
(i) $(AB)^{-1} = B^{-1}A^{-1}$ (ii) $(A')^{-1} = (A^{-1})'$

(3) Every square matrix can be expressed as the sum of a symmetric and skew – symmetric matrices in one and only way (uniquely).

(4) Prove that inverse of a non – singular symmetric matrix A is symmetric.

(5) If A is a symmetric matrix, then prove that adj A is also symmetric.

(6) If A, B are orthogonal matrices, each of order n then AB and BA are orthogonal matrices.

(7) Prove that the inverse of an orthogonal matrix is orthogonal and its transpose is also orthogonal.

(8) Evaluate $A^2 - 3A + 9I$ where $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$ and I is a unit matrix.

(9) Express the matrix A as a sum of symmetric and skew – symmetric matrix where $A = \begin{bmatrix} 3 & -2 & 6 \\ 2 & 7 & -1 \\ 5 & 4 & 0 \end{bmatrix}$

(10) Find the adjoint and inverse of $A = \begin{bmatrix} 2 & 3 & 4 \\ 4 & 3 & 1 \\ 1 & 2 & 4 \end{bmatrix}$

(11) Compute the adjoint and inverse of the matrix A = $\begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & 0 \\ 0 & 1 & 2 \end{bmatrix}$.

(12) Find the inverse of the matrix Diag [a, b, c], a$\neq 0, b \neq 0, c \neq 0$.

(13) Find the values of 'x' such that the matrix 'A' is singular where

A = $\begin{bmatrix} 3-x & 2 & 2 \\ 2 & 4-x & 1 \\ -2 & -4 & -(1+x) \end{bmatrix}$

(14) Prove that $\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & \frac{-2}{3} \\ \frac{2}{3} & \frac{-2}{3} & \frac{1}{3} \end{bmatrix}$ is orthogonal.

(15) Determine the values of a, b , c when $\begin{bmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{bmatrix}$ is orthogonal.

(16) Prove that the following matrix is orthogonal $\begin{bmatrix} -2/3 & 1/3 & 2/3 \\ 2/3 & 2/3 & 1/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix}$.

(17) Show that A is $\frac{1}{2}\begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix}$ is orthogonal.

(18) Solve the equations 3x + 4y + 5z =18, 2x –y + 8z =13 and 5x -2y +7x = 20 by matrix inversion method.

(19) Solve the system of equations by matrix method

$x_1 + x_2 + x_3 = 2$; $4x_1 - x_2 + 2x_3 = -6$; $3x_1 + x_2 + x_3 = -18$.

(20) Solve the equations 2x + y - z =1, x – y + z =2, 5x + 5y - 4z =3 by cramer's rule.

(21) If A = $\begin{bmatrix} 3 & 7-4i & -2+5i \\ 7+4i & -2 & 3+i \\ -2-5i & 3-i & 4 \end{bmatrix}$ then show that A is Hermitian and iA is Skew – Hermitian.

(22) If A and B are Hermitian matrices, prove that AB –BA is a Skew – Hermitian.

(23) Prove that ½ $\begin{bmatrix} 1+i & -1+i \\ 1+i & 1-i \end{bmatrix}$ is a unitary matrix.

(24) Show that A = $\begin{pmatrix} a+ic & -b+id \\ b+id & a-ic \end{pmatrix}$ is a unitary if $a^2+b^2+c^2+d^2 = 1$.

(25) If A is a Hermitian matrix, prove that iA is a Skew – Hermitian matrix.

(26) If A is a Skew – Hermitian matrix, prove that iA is a Hermitian matrix.

(27) Show that every square matrix is uniquely expressible as the sum of a Hermitian matrix and a Skew – Hermitian matrix.

(28) Prove that every Hermitian matrix can be written as A +iB, where A is real and symmetric, and B is real and skew – symmetric.

(29) Express the matrix $\begin{bmatrix} 1+i & 2 & 5-5i \\ 2i & 2+i & 4+2i \\ -1+i & -4 & 7 \end{bmatrix}$ as the sum of Hermitian matrix and Skew – Hermitian matrix.

(30) Given that A = $\begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$, show that

$(I-A)(I+A)^{-1}$ is a unitary matrix.

(31) Show that the inverse of a unitary matrix is unitary.

(32) Prove that the product of two unitary matrices is unitary.

(33) Prove that the transpose of a unitary matrix is unitary.

(34) Find the Hermitian form H for A = $\begin{bmatrix} 0 & i & 0 \\ -i & 1 & -2i \\ 0 & 2i & 2 \end{bmatrix}$ with X = $\begin{bmatrix} i \\ 1 \\ -i \end{bmatrix}$

(35) Find the Hermitian form of A = $\begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$ with X = $\begin{bmatrix} 1 \\ i \end{bmatrix}$.

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02 RANK NORMAL FORM

(1)Reduce the matrix A = $\begin{pmatrix} 1 & 2 & 3 & 0 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 5 \end{pmatrix}$ into echelon form and find rank.

(2)Reduce the matrix $\begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$ into Echelon form and find its rank.

(3)Reduce to Echelon form and find its rank $\begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$

(4)If A = $\begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$, B = $\begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$ find the ranks of A, B, A+B, AB and BA.

(5)For what value of K the matrix $\begin{bmatrix} 4 & 4 & -3 & 1 \\ 1 & 1 & -1 & 0 \\ K & 2 & 2 & 2 \\ 9 & 9 & K & 3 \end{bmatrix}$ has rank 3.

(6)Find the value of k such that the rank of $\begin{bmatrix} 1 & 2 & 3 \\ 2 & k & 7 \\ 3 & 6 & 10 \end{bmatrix}$ is 2.

(7)Find the value of **k** , so that Rank A is 2, A = $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & k & 0 \end{bmatrix}$

(8)Find the value of **k** so that the rank of A = $\begin{bmatrix} 1 & 1 & -1 & 1 \\ 1 & -1 & k & -1 \\ 3 & 1 & 0 & 1 \end{bmatrix}$ is 2.

(9)Find the rank of $\begin{pmatrix} 1 & 4 & 3 & -2 & 1 \\ -2 & -3 & -1 & 4 & 3 \\ -1 & 6 & 7 & 2 & 9 \\ -3 & 3 & 6 & 6 & 12 \end{pmatrix}$

(10)Find the rank of $\begin{pmatrix} 1 & 4 & 3 & -2 & 1 \\ -2 & -3 & -1 & 4 & 3 \\ -1 & 6 & 7 & 2 & 9 \\ -3 & 3 & 6 & 6 & 12 \end{pmatrix}$

(11)Find the rank of A = $\begin{bmatrix} 2 & -2 & 0 & 6 \\ 4 & 2 & 0 & 2 \\ 1 & -1 & 0 & 3 \\ 1 & -2 & 1 & 2 \end{bmatrix}$ by reducing it to canonical.

(12)Find the rank of A = $\begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -3 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ by reduce it to normal form.

(13)Find the rank of $\begin{bmatrix} 1 & 4 & 3 & -2 & 1 \\ -2 & -3 & -1 & 4 & 3 \\ -1 & 6 & 7 & 2 & 9 \\ -3 & 3 & 6 & 6 & 12 \end{bmatrix}$

(14)Obtain non – singular matrices P and Q such that PAQ is of the form $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ where A = $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 0 & -1 & -1 \end{bmatrix}$ and hence obtain its rank.

(15)If A = $\begin{bmatrix} 3 & 2 & -1 & 5 \\ 5 & 1 & 4 & -2 \\ 1 & -4 & 11 & -19 \end{bmatrix}$, obtain non – singular matrices P and Q such that PAQ = $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$ by suitable elementary row and column operations.

(16)Find non – singular matrices P and Q so that PAQ is of the normal form, where A = $\begin{bmatrix} 1 & -2 & 3 & 4 \\ -2 & 4 & -1 & -3 \\ -1 & 2 & 7 & 6 \end{bmatrix}$

(17)Find the non – singular matrices P and Q such that the normal form of A is PAQ where A = $\begin{bmatrix} 1 & 3 & 6 & -1 \\ 1 & 4 & 5 & 1 \\ 1 & 5 & 4 & 3 \end{bmatrix}$. Hence find its rank.

(18)Find P and Q such that the normal form of

A = $\begin{bmatrix} 1 & -1 & -1 \\ 1 & 1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ is PAQ. Hence find the rank of A.

(19)If A = $\begin{bmatrix} 2 & 1 & -3 & -6 \\ 2 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$ find non – singular matrices such that PAQ is in normal form.

(20)Find the non – singular matrices P and Q such that PAQ is in the normal form of the matrix and find the rank of matrix A = $\begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}$.

(21)Reduce the matrix $\begin{bmatrix} 0 & 1 & 2 & -2 \\ 4 & 0 & 2 & 6 \\ 2 & 1 & 3 & 1 \end{bmatrix}$ to normal form and hence find the rank.

03 PROBLEMS ON MATRIX INVERSE

(1) Find the inverse of A = $\begin{bmatrix} -2 & 1 & 3 \\ 0 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}$ using elementary row operations (Gauss – Jordan method).

(2) Find the inverse of the matrix A using elementary operations (i.e., using Gauss –Jordan method).A = $\begin{bmatrix} -1 & -3 & 3 & -1 \\ 1 & 1 & -1 & 0 \\ 2 & -5 & 2 & -3 \\ -1 & 1 & 0 & 1 \end{bmatrix}$

(3) Find the inverse by elementary row operations $\begin{bmatrix} 1 & 1 & 3 \\ 1 & 3 & -3 \\ -2 & -4 & -4 \end{bmatrix}$

(4) Given A = $\begin{bmatrix} 1 & -2 & -3 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$, find its inverse.

(5) If A = $\begin{pmatrix} 4 & -1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 3 \end{pmatrix}$, find A^{-1}.

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04 LINEAR EQUATIONS

(1) Write the following equations in matrix form AX =B and solve for X by finding A^{-1}: $x + y - 2z = 3$, $2x - y + z = 0$, $3x + y - z=8$.

(2) Show that the equations $x + y + z = 4$, $2x + 5y - 2z =3$, $x + 7y - 7z = 5$ are not consistent.

(3) Solve the equations $x + y + z =9$; $2x + 5y + 7z = 52$ and $2x + y - z =0$.

(4) Solve the system of linear equations by matrix method.

$x + y + z = 6$, $2x + 3y - 2z = 2$, $5x + y + 2z = 13$.

(5) Discuss for what value of λ, μ the simultaeneous

Equations $x + y + z =6$, $x + 2y + 3z = 10$, $x + 2y + \lambda z=\mu$ have (i) no solution (ii) a unique solution (iii) an infinite number of solutions.

(6) Find for what values of λ the equations $x + y + z =1$, $x + 2y + 4z = \lambda$, $x + 4y + 10z =\lambda^2$ have a solution and solve them in each case.

(7) Discuss for all values of λ, the system of equations $x + y + 4z = 6$; $x + 2y - 2z = 6$; $\lambda x + y + z =6$ with regard to consistency.

(8) If $a + b + c \neq 0$, show that the system of equations $-2x + y + z = a$, $X - 2y + z = b$, $x + y - 2z = c$ has no solution. If $a + b + c =0$, show that it has infinitely many solutions.

(9) Find the values of 'a' and 'b' for which the equations $x + y + z = 3$, $x + 2y + 2z = 6$, $x + ay + 3z = b$ have (i) no solution (ii) a unique solution (iii) infinite number of solutions.

(10) Show that the equations $x + y + z = 6$, $x + 2y + 3z =14$, $x + 4y + 7z = 30$ are consistent and solve them.

(11) Find whether the following system of equations are consistent. If so solve them. $x+2y+2z=2$, $3x-2y-z=5$; $2x-5y+3z=-4$, $x+4y+6z=0$.

(12) Find the value of λ for which the system of equations $3x-y+4z=3$, $x+2y-3z=-2$, $6x+5y+\lambda z=-3$ will have infinite number of solutions and solve them with that λ value.

(13) Find whether the following set of equations are consistent if so, solve them

$x_1+x_2+x_3+x_4=0$

$x_1+x_2+x_3-x_4=4$

$x_1+x_2-x_3+x_4=-4$

$x_1-x_2+x_3+x_4=2$

(14) Prove that the following set of equations are consistent and solve them.

$3x+3y+2z=1$

$x+2y=4$

$10y+3z=-2$

$2x-3y-z=5$

(15) Show that the equations $x-4y+7z=14$, $3x+8y-2z=13$, $7x-8y+26z=5$ are not consistent.

(16) Test for the consistency of $x+y+z=1$, $x-y+2z=1$, $x-y+2z=5$, $2x-2y+3z=1$, $3x+y+z=2$.

(17) Find the values of p and q so that the equations $2x+3y+5z=9$, $7x+3y+2z=8$, $2x+3y+pz=q$ have

(i) No solution (ii) unique solution

(ii) An infinite number of solutions.

(18) Find the values of 'a' and 'b' for which the equations, $x+y+z=3$, $x+2y+2z=6$, $x+9y+az=b$ have

(i) No solution

(ii) A unique solution

(iii) Infinite number of solutions

(19) If consistent, solve the system of equations, $x+y+z+t=4$, $x-z+2t=2$, $y+z-3t=-1$, $x+2y-z+t=3$.

(20) Test for consistency and if consistent solve the system, $5x+3y+7t=4$, $3x+26y+2t=9$, $3x+26y+2t=9$.

(21) Solve the equations $2x+3y+5z=9$, $7x+3y-2z=8$, $2x+3y+\lambda z=u$.

(22) Find the values of λ and μ so that the system of equations $2x+3y+5z=9$, $7x+3y-2z=8$, $2x+3y+\lambda z=\mu$ has (i) unique solution (ii) No solution (iii) infinite no. of solutions.

(23) Solve completely the system of equations $x+y-2z+3w=0$, $x-2y+z-w=0$, $4x+y-5z+8w=0$, $5x-7y+2z-w=0$.

(24) Determine the values of λ, for which the following set of equations may possess non – trivial solution:

$3x_1+x_2-\lambda x_3=0$, $4x_1-2x_2-3x_3=0$, $2\lambda x_1+4x_2+\lambda x_3=0$.

For each permissible value of λ, determine the general solution.

(25) Solve the system of equations $x+y-3z+2w=0$, $2x-y+2z-3w=0$, $3x-2y+z-4w=0$, $-4x+y-3z+w=0$.

(26) Solve $x_1+2x_3-2x_4=0$, $2x_1-x_2-x_4=0$, $x_1+2x_3-x_4=0$, $4x_1-x_2+3x_3-x_4=0$.

(27) Solve the system of equations $x+2y+(2+k)z=0$, $2x+(2+k)y+4z=0$, $7x+13y+(18+k)z=0$ for all values of k.

(28) Solve the equations $x+y-z+t=0$, $x-y+2z-t=0$, $3x+y+t=0$.

(29) Solve the system $\lambda x+y+z=0$, $x+\lambda y+z=0$, $x+y+\lambda z=0$ if the system has non –zero solution only.

(30) Show that the only real number λ for which the system $x+2y+3z=\lambda x$, $3x+y+2z=\lambda y$, $2x+3y+z=\lambda z$ has non –zero solution is 6 and solve them, when $\lambda=6$.

(31) Solve: $2x+3ky+(3k+4)z=0$, $x+(k+4)y+(4k+2)z=0$, $x+2(k+l)y+(3k+4)z=0$.

(32) Find the values of λ, for which the equations

$$(\lambda-1)x+(3\lambda+1)y+2\lambda z=0$$

$$(\lambda-1)x+(4\lambda-2)y+(\lambda+3)z=0$$

$$2x+(3\lambda+1)y+3(\lambda-1)z=0$$

Are consistent and find the ratio of x:y:z when λ has the smallest of these values. What happens when λ has the greater of these values?

(33) Determine whether the following equations will have a non –trivial solution if so solve them, $4x+2y+z+3w=0$, $6x+3y+4z+7w=0$, $2x+y+w=0$.

(34) Solve the system of equations $x+y+w=0$, $y+z=0$, $x+y+z+w=0$, $x+y+2z=0$.

(35) Solve the system $2x-y+3z=0$, $3x+2y+z=0$ and $x-4y+5z=0$

(36) Find all the solutions of the system of equations: $x+2y-z=0$, $2x+y+z=0$, $x-4y+5z=0$.

(37) Solve completely the system of equations: $x+3y-2z=0$, $2x-y+4z=0$, $x-11y+14z=0$.

(38) Solve completely the equations $3x+4y-z-6w=0$, $2x+3y+2z-3w=0$, $2x+y-14z-9w=0$, $x+3y+13z+3w=0$.

(39) Solve the equations $2x_1+x_2+x_3=10$, $3x_1+2x_2+3x_3=18$, $x_1+4x_2+9x_3=16$ using Gauss- Elimination method.

(40) Solve the equations $x+y+z=6$, $3x+3y+4z=20$, $2x+y+3z=13$ using partial pivoting Gaussian elimination method.

(41) Solve the equations $3x+y+2z=3$, $2x-3y-z=-3$, $x+2y+z=4$ using Gauss elimination method.

(42) Express the following system in matrix form and solve by Gauss Elimination method.

$2x_1+x_2+2x_3+x_4=6$, $6x_1-6x_2+6x_3+12x_4=36$,

$4x_1+3x_2+3x_3-3x_4=-1$, $2x_1+2x_2-x_3+x_4=10$

(43) Using Gauss- Jordan method, solve the system $2x+y+z=10$, $3x+2y+3z=18$, $x+4y+9z=16$.

(44) Solve the equations $10x+y+z=12$, $2x+10y+z=13$ and $x+y+5z=7$ by Gauss – Jordan method.

(45) Solve the equations $10x_1+x_2+x_3=12$, $x_1+10x_2-x_3=10$ and $x_1-2x_2+10x_3=9$ by Gauss – Jordan method.

(46) Find all the solutions of the following system of equations: $3x+4y-z-6w=0$,

$$2x+3y+2z-3w=0,$$
$$2x+y-14z-9w=0,$$
$$x+3y+13z+3w=0.$$

(47) Solve the equations

$$\lambda x+2y-2z-1=0,$$
$$4x+2\lambda y-z-2=0,$$
$$6x+6y+\lambda z-3=0.$$

Considering specially the case when $\lambda=2$

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05 PROBLEMS ON DIAGONALIZATION

(1) Determine the modal matrix P for A = $\begin{pmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{pmatrix}$ and hence diagonalize A.

(2) Determine the modal matrix P of A = $\begin{pmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{pmatrix}$. Verify that $P^{-1}AP$ is a diagonal matrix.

(3) If A = $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ -4 & 4 & 3 \end{bmatrix}$ find (a) A^8 (b) A^4

(4) (a) Find a matrix P which transform the matrix

A = $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ to diagonal form. Hence calculate A^4.

Find the eigen values and eigen vectors of A.

(b) Determine the eigen values of A^{-1}.

(c) Diagonalize A = $\begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ and hence find A^8.

(5) Diagonalize the matrix A = $\begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$

(6) Diagonalize the matrix $\begin{bmatrix} -1 & 2 & -2 \\ 1 & 2 & 1 \\ -1 & -1 & 0 \end{bmatrix}$

(7) Show that the matrix A = $\begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ is diagonalizable. Also find the diagonal form and a diagonalize matrix P.

(8) Show that the matrix A = $\begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ cannot be diagonalized.

(9) Find an orthogonal matrix that will diagonalize the real symmetric matrix A = $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$. Also find the resulting diagonal matrix.

(10) Diagonalize the matrix where A = $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & -1 \\ 0 & -1 & 3 \end{bmatrix}$, by orthogonal reduction.

(11) Find the diagonal matrix orthogonally similar to the following real symmetric matrix. Also obtain the transforming matrix.
A = $\begin{bmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ -4 & -1 & -8 \end{bmatrix}$

(12) Determine the diagonal matrix orthogonally similar to the following symmetric matrix A = $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

(13) Diagonalize the matrix by an orthogonal transformation $\begin{bmatrix} 2 & 0 & 1 \\ 0 & 6 & 0 \\ 4 & 0 & 2 \end{bmatrix}$.

Also find the matrix of the transformation.

(14) An n × n matrix is diagonalizable if and only if it possesses n linearly independent eigenvectors.

(15) Similar matrices have the same determinant.

(16) Similar matrices have the same characteristic polynomial and hence the same eigenvalues. If X is an eigenvector of A corresponding to the eigenvalue λ, then $P^{-1}X$ is an eigenvector of B corresponding to the eigenvalue λ where B = $P^{-1}AP$.

(17) If A is similar to a diagonal matrix D, the diagonal elements of D are the eigenvalues of A.

(18) If the eigenvalues of an $n \times n$ matrix are all distinct then it is always similar to a diagonal matrix.

(19) The necessary and sufficient condition for a square matrix to be similar to a diagonal matrix is that the geometric multiplicity of each of its eigenvalues coincides with the algebraic multiplicity.

(20) Show that the rank of every matrix similar to A is same as that of A.

(21) Let A and B be n – rowed square matrices and let A be non – singular. Show that the matrices $A^{-1}B$ and BA^{-1} have the same eigenvalues.

(22) If A and B are non – singular matrices of order n, show that the matrices AB and BA are similar.

(23) A and B are two $n \times n$ matrices with the same set of n distinct eigenvalues. Show that there exist two matrices P and Q (one of them non – singular) such tha A = PQ, B = QP.

(24) Prove that if A is similar to a diagonal matrix, then A^r is similar to A.

(25) Show that the matrix A = $\begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ is diagonalizable. Also find the diagonal form and a diagonalizing matrix P.

(26) Show that the matrix A = $\begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ is diagonalizable. Also find the transforming matrix and diagonal matrix.

(27) Show that the following matrices are not similar to diagonal matrices:

(i) $\begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ (ii) $\begin{bmatrix} 2 & -1 & 1 \\ 2 & 2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$

(28) Any two eigenvectors corresponding to two distinct eigenvalues of a real symmetric matrix are orthogonal.

(29) Find an orthogonal matrix that will diagonalize the real symmetric matrix A = $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$. Also write the resulting diagonal matrix.

(30) Determine diagonal matrices orthogonally similar to the following real symmetric matrices, obtaining also the transforming matrices:

(i) A = $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ (ii) A = $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

(iii)A = $\begin{bmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ -4 & -1 & -8 \end{bmatrix}$ (iv) A = $\begin{bmatrix} 7 & 0 & -2 \\ 0 & 5 & -2 \\ -2 & -2 & 6 \end{bmatrix}$

(31) If P is a real orthogonal matrix and D a real diagonal matrix such that $P^{-1}AP=D$, show that A is a real symmetric matrix.

(32) Any two eigenvectors corresponding to two distinct eigenvalues of a Hermitian matrix are orthogonal.

(33) Determine the diagonal matrix unitarily similar to the Hermitian matrix A = $\begin{bmatrix} 2 & 1-2i \\ 1+2i & -2 \end{bmatrix}$, obtaining also the transformation matrix.

(34) Show that if P is unitary and $P^{-1}AP$ is a real diagonal matrix, then A is Hermitian.

(35) Prove that Hermitian, real symmetric, unitary, real orthogonal, skew – Hermitian, and real skew – symmetric matrices are normal.

(36) Prove that any diagonal matrix over the complex field is normal.

(37) A triangular matrix is normal if and only if it is diagonal.

(38) Prove that a square matrix A is normal if and only if it can be expressed as B +iC, where B and C are commutative Hermitian matrices.

(39) If A is normal and non – singular, prove that so also is A^{-1}.

(40) If A is normal, then show that A is similar to A^T.

(41) Let A be a normal matrix. Show that

(i) if all the characteristic roots of A are real, then A is Hermitian.

(ii) if all the characteristic roots of A are of modulus 1, then A is unitary.

(42) If A, B are square matrices each of order n and I is the corresponding unit matrix, show that the equation AB –BA =I can never hold.

(43) Prove that the trace of a matrix is equal to the sum of the characteristic roots.

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06 CAYLEY HAMILTON PROBLEMS

(1) **THE CAYLEY-HAMILTON THEOREM**

Theorem: Every square matrix satisfies its own characteristic equation.

(2) If A = $\begin{bmatrix} 2 & 1 & 2 \\ 5 & 3 & 3 \\ -1 & 0 & -2 \end{bmatrix}$ verify Cayley – Hamilton theorem and find A^{-1}.

(3) Find the inverse of $\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 1 \\ 2 & 1 & 2 \end{bmatrix}$ by using Cayley – Hamilton theorem.

(4) Find the characteristic polynomial of the matrix A = $\begin{bmatrix} 3 & 1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 5 \end{bmatrix}$. Verify Cayley – Hamilton theorem and hence find A^{-1}.

(5) If A = $\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$, express $A^6-4A^5+8A^4-12A^3+14A^2$ as a polynomial in A.

(6) Show that the matrix A = $\begin{bmatrix} 0 & c & -b \\ -c & 0 & a \\ b & -a & 0 \end{bmatrix}$ satisfies Cayley – Hamilton theorem.

(7) If A = $\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$, find the value of the matrix

$A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I$.

(8) Find the eigen values of A and hence find A^n(n is a +ve integer) if A = $\begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$.

(9) Verify Cayley – Hamilton theorem for the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}.$$ Hence find A^{-1}.

(10) If $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$ verify Cayley – Hamilton theorem.
Find A^4 and A^{-1} using Cayley – Hamilton.

(11) Verify Cayley – Hamilton theorem and hence find A^{-1} for the matrix $A = \frac{1}{4}\begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$.

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07 PROBLEMS ON QUADRATICS

(1) Find the symmetric matrix corresponding to the quadratic form $x^2+2y^2+3z^2+4xy+5yz+6zx$.

(2) Find the quadratic form corresponding to the matrix A = $\begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix}$.

(3) Write down the quadratic form corresponding to the matrix $\begin{bmatrix} 0 & 5 & -1 \\ 5 & 1 & 6 \\ -1 & 6 & 2 \end{bmatrix}$.

(4) Find the quadratic form relating to the matrix diag $[\lambda_1\ \lambda_2\ \lambda_n]$.

(5) Find the Quadratic form corresponding to the matrix $\begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{bmatrix}$

(6) Find the inverse transformation $y_1=2x_1+x_2+x_3$, $y_2=x_1+x_2+2x_3$, $y_3= x_1-2x_3$.

(7) Show that the transformation $y_1 = x_1 \cos\theta + x_2\sin\theta$, $y_2= x_1 \sin\theta + x_2 \cos\theta$ *is orthogonal*.

(8) Identify the nature of the quadratic form $x_1^2+4x_2^2+x_3^2-4x_1x_2+2x_1x_3-4x_2x_3$.

(9) Discuss the nature of the quadratic form $x^2+4xy+6xz-y^2+2yz+4z^2$.

(10) Reduce the matrix A = $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ to diagonal form and interpret the result in terms of quadratic form.

(11) Find the nature of the quadratic form $2x^2+2y^2+2z^2+2yz$.

(12) Find the rank, signature and index of the quadratic form $2x_1^2+x_2^2-3x_3^2+12x_1x_2-4x_1x_3-8x_2x_3$ by reducing it to canonical form or normal form. Also write the linear transformation which brings about the normal reduction.

(13) Reduce the following quadratic form to canonical form and find its rank and signature. $x^2+4y^2+9z^2+t^2-12yz+6zx-4xy-2xt-6zt$.

(14) Reduce the quadratic form $7x^2+6y^2+5z^2-4xy-4yz$ to the canonical form.

(15) Find nature of the quadratic form, index and signature of $10x^2+2y^2+5z^2-4xy-10xz+6yz$.

(16) Find the transformation which will transform $4x^2+3y^2+z^2-8xy-6yz+4zx$ into a sum of squares and find the reduced form.

(17) Reduce the quadratic form to the canonical form $x^2+y^2+2z^2-2xy+4zx+4yz$.

(18) Reduce the quadratic form to the canonical form $3x^2-3y^2-5z^2-2xy-6yz-6xz$.

(19) Find the rank and signature of the quadratic form $x_1x_2-4x_1x_4-2x_2x_3+12x_3x_4$.

(20) Reduce the quadratic form $3x^2+2y^2+3z^2-2xy-2yz$ to the normal form by orthogonal transformation.

(21) Find the eigen vectors of the matrix $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ and hence reduce $6x^2+3y^2+3z^2-2yz+4zx-4xy$ to a sum of squares.

(22) Reduce the quadratic form $3x^2+5y^2+3z^2-2yz+2zx-2xy$ to the canonical form by orthogonal reduction.

(23) Reduce the quadratic form $3x_1^2 + 3x_2^2 + 3x_2^3 + 2x_1x_2 + 2x_1x_3 - 2x_2x_3$ into sum of squares form by an orthogonal transformation and give the matrix of transformation.

(24) Reduce the quadratic form to canonical form by an orthogonal reduction and state the nature of the quadratic form $2x^2+2y^2+2z^2-2xy-2yz-2zx$.

(25) Reduce the quadratic form to the canonical form $2x^2+2y^2+2z^2-2xy+2zx-2yz$.

(26) Reduce the following quadratic form by orthogonal reduction and obtain the corresponding transformation. Find the index, signature and nature of the quadratic form $q = 2xy+2yz+2zx$.

(27) Reduce the quadratic form, $q = 3x^2-2y^2-z^2-4xy+12yz+8xz$ to the canonical form by orthogonal reduction. Find its rank, index and signature. Find also the corresponding transformation.

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08 EXTRA PROBLEMS ON MATICES

(1) Let A and B be two square matrices of order n. If $\rho(A) = \rho(B) = n$, then prove that $\rho(AB) = n$ and conversely.

(2) Prove that every skew – symmetric matrix of odd order has rank less than its order.

(3) If A be a non – zero column and B is a non – zero row matrix, then show that $\rho(AB) = 1$.

(4) If A is n – rowed square matrix of rank n-1, then show that adj A is non – zero matrix. Since the rank of A is n-1, i.e., $\rho(A) = n - 1$, then there exists a non – zero (n-1) minor of A, therefore there exists at least one element of adj A which is non – zero, hence adj A is non – zero matrix.

(5) Let A be a square matrix of order n. Show that $\rho(\text{adj.} A)$ is n or 0 in accordance with $\rho(A)$is n or less than n-1.

(6) Show that the following matrices are not similar to diagonal matrices:

(i) $\begin{bmatrix} \mathbf{2} & \mathbf{3} & \mathbf{4} \\ \mathbf{0} & \mathbf{2} & \mathbf{-1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$ **(ii)** $\begin{bmatrix} \mathbf{2} & \mathbf{-1} & \mathbf{1} \\ \mathbf{2} & \mathbf{2} & \mathbf{-1} \\ \mathbf{1} & \mathbf{2} & \mathbf{-1} \end{bmatrix}$

(7) If A and B are two matrices of the same type, then $\rho(A + B) \leq \rho(A) + \rho(B)$

(8) If A and B are two n – rowed square matrices, then $\rho(AB) \geq \rho(A) + \rho(B) - n$.

(9) If A be any non – singular matrix and B a matrix such that AB exists, then show that $\rho(AB) = \rho(B)$

(10) If A is a square matrix of order $n \times n$ such that $A^2 = A$, then show that $\rho(A) + \rho(I_n - A) = n$.

(11) If A is a square matrix of order $n \times n$ and $\rho(A)$ =n-1, show that $\rho(\text{adj. A}) = 1$.

(12) Find the rank of the matrix A = $\begin{bmatrix} 1 & a & b & 0 \\ 0 & c & d & 1 \\ 1 & a & b & 0 \\ 0 & c & d & 1 \end{bmatrix}$

(13) Find two non – singular matrices P and Q such that PAQ is in the normal form where A = $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 3 & 1 & 1 \end{bmatrix}$. Also find the rank of the matrix A.

(14) Determine non – singular matrices P and Q such PAQ is in the normal form $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$, where A = $\begin{bmatrix} 3 & 2 & -1 & 5 \\ 5 & 1 & 4 & -2 \\ 1 & -4 & 11 & -19 \end{bmatrix}$

(15) The necessary and sufficient conditions for a square matrix to be similar to a diagonal matrix is that the geometric multiplicity of each of its eigenvalues coincides with the algebraic multiplicity.

(16) A square matrix is invertible if and only if it is non – singular.

(17) If A is an invertible matrix, then (adj.A)' = adj(A').

(18) The adjoint of a symmetric matrix is also a symmetric matrix.

(19) If A and B are non – singular matrices of the same order, then adj.(AB) = (adj.B) (adj.A)

(20) If the product of two non null square matrices is a null matrix, then both of them must be singular.

(21) By using elementary row- transformations find the inverse of the following matrices:

(i) $\begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix}$ (ii) $\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

(22) Find the inverse of the matrix A = $\begin{bmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 1 & 1 & 2 \end{bmatrix}$ by using elementary row – transformation.

(23) Using elementary transformation, find the inverse of the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 7 \\ -2 & -4 & -5 \end{bmatrix}$$

(24) Find the inverse of the matrix A = $\begin{bmatrix} 0 & 1 & 2 & 2 \\ 1 & 1 & 2 & 3 \\ 2 & 2 & 2 & 3 \\ 2 & 3 & 3 & 3 \end{bmatrix}$ by using elementary transformations.

(25) For what value of η the equations $x + y + z = 1$, $x+2y+4z = \eta$, $x+4y+10z = \eta^2$ have a solution? Solve them completely in each case.

(26) Show that similar matrices have the same eigenvalues.

(27) Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 2 & 0 & 1 & -3 \\ 0 & 2 & 10 & 4 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

(28) If A be a non- singular matrix of order $n \times n$ and its characteristic polynomial is $|A-\lambda I|=(-1)^n[\lambda^n+a_1\lambda^{n-1}+\ldots+a_{n-1}\lambda+a_n]$ then det(A) = $(-1)^n a_n$.

(29) If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of a square matrix of order $n \times n$, then $\det(A) = \lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n$.

(30) Let A be an $n \times n$ matrix with characteristic polynomial

$f(t) = (-1)^n[t^n+a_1t^{n-1}+\ldots+a_{n-1}t+ a_n]$
Then A is invertible if $a_n \neq 0$ and its inverse is
$A^{-1}= \left(\frac{-1}{a_n}\right)[A^{n-1}+a_1A^{n-2}+\ldots+a_{n-1}t +a_nI]$.

(31) If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalue of a matrix A of order $n \times n$, then $\mathrm{Tr}(A) = \text{Trace of } A = \sum_{i=1}^{n} \lambda_i$

(32) If the characteristic equation of a matrix A of order $n \times n$ is $|A-\lambda I| = (-1)^n[\lambda^n + a_1\lambda^{n-1} + a_2\lambda^{n-2} + \ldots + a_{n-1}\lambda + a_n] = 0$ then

$\mathrm{Tr}(A) = -a_1$

(33) **(Rank-Multiplicity Theorem)**

The geometric multiplicity of an eigenvalue cannot exceed its algebraic multiplicity.

09 VECTOR SPACES

(1) Show that the set of all polynomials over a field F is a vector space.

(2) Show that the set of all convergent sequences is a vector space over the field of real numbers.

(3) Let V be the set of all pairs (x,y) of real numbers, and let F be the field of real numbers. Define $(x,y)+(x_1,y_1)=(x+x_1,0)$ and $c(x,y) = (cx,0)$. Is V, with these operations, a vector space over the field of real number?

(4) Let V be the set of all pairs (x,y) of real numbers and let F be the field of real numbers. Define $(x,y)+(x_1,y_1)=(3y+3y_1,-x-x_1)$
$c(x,y) = (3cy, -cx)$
Verify that V, with these operations, is not a vector space over the field of real numbers.

(5) Show that the set of all real valued continuous functions defined on [0,1] is a vector space over field of reals.

(6) Let V be the set of all pairs (x,y) of real numbers, and let F be the field of real numbers. Examine in each of the following cases whethe V is a vector space over the field of real numbers or not?
(i) $(x,y) +(x_1,y_1)= (x+x_1,y+y_1)$; $c(x,y)=(|c|x,|c|y)$
(ii) $(x,y) +(x_1,y_1) = (x+x_1, y+y_1)$: $c(x,y) = (0,cy)$
(iii) $(x,y) +(x_1,y_1) = (x+x_1, y+y_1)$: $c(x,y) = (c^2x,c^2y)$

(7) How many elements are there in the vector space of polynomials of degree at most n in which the coefficients are the elements of the field Z(p), the integer modulo p over the field Z(p), p being a prime number?

(8) Let K = Z_3, the integers modulo 3. How many elements are in the vector space V = K^4?

(9) Is Z_7 a vector space over Z_5 ?

(10) The necessary and sufficient conditions for a non – empty subset W of V(F) to be a subspace are that:

(i) a $\in W, \beta \in W \Rightarrow \alpha - \beta \in W$

(ii) a $\in F, \alpha \in W \Rightarrow a\alpha \in W$

(11) The necessary and sufficient condition for a non – empty subset of W of a vector space V(F) to be a subspace of V is a,b $\in F, \alpha, \beta \in W \Rightarrow a\alpha + b\beta \in W$

(12) Let V be the vector space of all 2×2 matrices over the real field R. Show that W is not a subspace of V, where

(i) W consists of all matrices with zero determinant,

(ii) W consists of all matrices A from which A^2=A.

(13) The intersection of any two subspaces of a vector space is a subspace.

(14) The union of two subspaces of a vector space is a subspace iff one is contained in the other.

(15) Show that the set W ={(a,b,c): a-3b+4c=0} is a subspace of the 3 – tuple space R^3(R).

(16) Show that the set W = {(a_1,a_2,0)}: a_1,$a_2 \in F$} is a subspace of V_3(F).

(17) Let W be the collection of all elements from the space M_2(F) of the form $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$. Show that W is a subspace of M_2(F).

(18) Which of the following sets of vectors α = (a_1,a_2,…a_n) $\in R^n$ are subsapces of R^n(n≥ 3)?

(i) All α such that $a_1 \leq 0$.

(ii) All α such that a_3 is an integer.

(iii) All α such that $a_2+4a_3=0$.

(iv) All α such that $a_1+a_2+\ldots a_n$=k (a given constant) .

(19) If a_1,a_2,a_3 are fixed elements of a field F, then the set W of all ordered triads (x_1,x_2,x_3) of elements of field F, such that $a_1x_1+a_2x_2+a_3x_3$ =0 is a subspace of V_3(F).

(20) Let R be the field of real numbers. Which of the following are subspaces of $V_3(R)$?

(i) $W_1 = \{(x,x,y): x \in R\}$

(ii) $W_2 = \{(x,y,z): x,y,z$ are rational numbers$\}$

(21) Let V be the vector space of all 2×2 matrices over the field R. Show that W is not a subspace of V, where W contains all 2×2 matrices with zero determinant.

(22) Let $M_n(F)$ be the vector space of all $n \times n$ matrices over the field F. Let W be the subset of $M_n(F)$ consisting of all symmetric matrices. Show that W is a subspace of $M_n(F)$.

(23) Let V(F) be the vector space of all $n \times 1$ *matrices over* the field F. Let A be an m $\times n$ *matrix over* F. *Then the* set W of all n $\times$ 1 *matrices* X *over* F *such that* $AX = O$ *is a subspace of* V, *here* O *is a null matrix of the type* $m \times 1$.

(24) Let V be the vector space of all polynomials in an indeterminate x over a field F, i.e., V = F(x). Let W be a subset of V consisting of all polynomials of degree $\leq n$. Then W is a subspace of V.

(25) Prove that the set of all solutions (a,b,c) of the equation a+b+2c =0 is a subspace of vector space $V_3(R)$.

(26) If $V = R^3(R)$ be the real vector space and let $W_1 = \{(0,y,z): y,z \in R\}$, $W_2 = \{(x,y,0): x,y \in R\}$. *What is* $W_1 \cap W_2$? Is it subspace of V? Is $W_1 \cup W_2$ subspace of V?

(27) Let V be the (real) vector space of all functions f from R into R. Which of the following sets of functions are subspaces of V?

(i) All f such that $f(x^2) = [f(x)]^2$

(ii) All f which are continuous.

(28) The linear sum of two subspaces of a vector space is also a subspace.

(29) The necessary and sufficient condition for a vector space V to be the direct sum of two of its subspaces W_1 and W_2 are:

(i) $V = W_1+W_2$
(ii) $W_1 \cap W_2 = \{0\}$.

(30) In $V = R^3$. Let W_1 be the xy – plane and let W_2 be the z – plane given by $W_1=\{(x,y,z):x,y\in R\}$ *and* $W_2 =\{(0,y,z):y,z\in R\}$. Show that $V = \{W_1 \Phi W_2\}$.

(31) In $V = R^3$ and W_1 be the xy – plane and let W_2 be the yz – plane: $W_1 = \{(x,y,0):x,y \in R\}$ *and* $W_2 =\{(0,y,z):y,z\in R\}$ then show that V is not the direct sum of W_1 and W_2.

(32) If $V_3(R)$ is a vector space and $W_1 =\{(a,0,c):a,c \in R\}$ and $W_2 =\{(0,b,c):b,c \in R\}$ *are two subspaces of* $V_3(R)$, then show that $V = W_1+W_2$ and $V \neq W_1 \Phi W_2$.

(33) Let V be the vector space of all functions from R into R; let V_e be the subset of even functions,such that f(-x) = f(x); Let V_0 be the subset of odd functions f(-x) = -f(x). Prove that

(a) V_e and V_0 are subspaces of V.
(b) $V_e+V_0 = V$
(c) $V_e \cap V_0 = \{0\}$

(34) Let R be the field of real numbers, show that the set W = {(x,2y,3z):x,y,z∈ R is a subspace of $V_3(R)$.

(35) Show that the set of all real valued continuous functions defined on [0,1] is a vector space over field of reals.

(36) Show that $R^2(R)$ is not a vector space when addition and scalar multiplication composition are defined by

$(a_1,a_2) +(b_1,b_2) = (a_1+b_1,a_2+b_2)$ and $a(a_1,a_2) = (aa_1,a_2)$,
$\forall\ a, a_1, a_2, b_1, b_2 \in R$.

(37) Prove the solution set W of the differential equation $2\frac{d^2y}{dx^2} - 9\frac{dy}{dx} + 2y = 0$ is a subspace of vector space of all real valued functions of R.

(38) Let V be the vector space of all the functions from the real field R into R. Show that the set W ={f:f(7) =2+f(1)} is not a subspace of V.

(39) Let V be the vector space of all square $n \times n$ matrices over a field of reals R. Show that W is a subspace of V, where

(i) W consists of the symmetric matrices.

(ii) W consists of all matrices which commute with a given matrix M, i.e., W = $\{A \in V : AM = MA\}$.

(40) Let V = R^3. Show that set W = $\{(a,b,c): a^2+b^2+c^2 \leq 1\}$ is not a subspace of V.

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10 LINEAR DEPENDENCE

(1) Is the vector (2,-5,3) in the subspace of R^3 spanned by the vectors (1,-3,2), (2,-4,-1),(1,-5,7)?

(2) In the vector space R^3 express the vector (1,-2,5) as a linear combination of the vectors (1,1,1), (1,2,3) and (2,-1,1)

(3) Write the polynomial $f(x) = x^2+4x-3$ over R as a linear combination of the polynomials

$f_1(x) = x^2-2x+5$, $f_2(x) = 2x^2-3x$ and $f_3(x) = x+3$

(4) Find a condition on a,b,c such that $\alpha = (a,b,c)$ is a linear combination of vectors (1,-3,2) and (2,-1,1).

(5) Show that the system of three vectors (1,3,2), (1,-7,-8),(2,1,-1) of $V_3(R)$ is linearly dependent.

(6) If α, β, γ are linearly independent vectors of a vector space V (F) where F is any field of complex numbers, then so also are $\alpha+\beta, \beta+\gamma, \gamma+\alpha$.

(7) In $V_3(R)$, where R is the field of real numbers, examine each of the following sets of vectors for linear dependence:

(i) {(1,3,2),(1,-7,-8),(2,1,-1)}
(ii) {(0,2,-4),(1,-2,-1),(1,-4,3)}
(iii) {(1,2,0),(0,3,1),(-1,0,1)}
(iv) {(-1,2,1),(3,0,-1)(-5,4,3)}
(v) {(2,3,5),(4,9,25)}
(vi) {(2,1,2),(8,4,8)}

(8) Prove that in R [x_2], the vector space of all polynomials in x over R, the system of $p(x) = 1+x+2x^2$, $q(x) = 2-x+x^2$, $r(x) = -4+5x+x^2$ is linearly dependent.

(9) Show that the set $\{1, x, 1+x+x^2\}$ is a linearly independent set of vectors in the vector space of all polynomials over the field of real numbers.

(10) Show that (1,1,1),(0,1,1) and (0,1,-1) generate R^3. In order to show that (1,1,1),(0,1,1) and (0,1,-1) generate R^3, we have to show that any vector of R^3 is a linear combination of (1,1,1),(0,1,1) and (0,1,-1).

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11 PROBLEMS ON BASIS

(1) If W_1 and W_2 are two finite dimensional subspaces of a vector space V(F), then W_1+W_2 is finite dimensional and dim.W_1+ dim.W_2 = dim.($W_1\cap W_2$)+ dim.(W_1+W_2).

(2) If a finite dimensional vector space V(F) be the direct sum of its two subspaces W_1 and W_2, then dim.V = dim.W_1+dim.W_2.

(3) Let V be the vector space of ordered pairs of complex numbers over the real field R, i.e., let V be the vector space $C^2(R)$. Show that the set S = {(1,0),(i,0),(0,1),(0,i) is a basis for V.

(4) Let α = (1,2,1), β = (2,9,0) and γ = (3,3,4). Show that the set S =$\{\alpha,\beta,\gamma\}$ is a basis of R^3.

(5) Show that the vectors α_1= (1,0,-1),α_2 = (1,2,1), α_3= (0,-3,2) form a basis of R^3. Express each of the standard basis vectors as a linear combination of $\alpha_1,\alpha_2,\alpha_3$.

(6) Given that each set S below spans R^3, find a basis of R^3 which is contained in S:

(i) {(1,0,2),(0,1,1),(2,1,5),(1,1,3),(1,2,1)}

(ii) {(2,6,-3),(5,15,-8)(3,9,-5),(1,3,-2),(5,3,-2)}

Let S = {(1,0,2),(0,1,1),(2,1,5),(1,1,3),(1,2,1)

(7) Given that the set S is a basis of R^4 and that T is linearly independent. Extent T to a basis of R^4, where

S = {(1,0,0,0),(0,0,1,0),(5,1,11,0),(-4,0,6,1)} and

T = {(1,0,1,0),(0,2,0,3)}.

(8) Extend the linearly independent subset {(1,0,1),(0,-1,1)} of $V_3(R)$ to form a basis of $V_3(R)$.

(9) Extend the linearly dependent subset {(1,-1,0,0),(1,1,1,0)} of $V_4(R)$ to form a basis of $V_4(R)$.

(10) Let W be the subspace of $V_4(R)$ generated by the vectors (1,-2,5,-3),(2,3,1,-4),(3,8,-3,-5), then
(i) Find a basis and dimension of W.
(ii) Extend the basis of W to a basis of $V_4(R)$.

(11) Let W be the subspace of $V_4(R)$ generated by the set of vectors S = {(1,1,0,-1),(1,2,3,0),(2,3,3,-1)} and W_2 the subspace of $V_4(R)$ generated by the set of vectors
T = {(1,2,2,-2),(2,3,2,-3),(1,3,4,-3)}
Find: (i) dim.(W_1+W_2) (ii) dim.$(W_1 \cap W_2)$

(12) Let W be the subspace of R^3 defined by W = {(a,b,c);a+b+c =0}. Find a basis and dimension of W.

(13) Let W be the subspace of R^3 defined by W = {(a,b,c): a=b=c}. Find a basis and dimension of W.

(14) Let W_1 and W_2 be distinct subspaces of V and dim. W_1 =4, dim. W_2=4 and dim.V =6. Find the possible dimensions of $W_1 \cap W_2$.

(15) Let W_1 and W_2 be subspaces of R^3 for which dim. W_1 =1, dim.W_2 = 2 and $W_1 \nsubseteq W_2$. Show that $R^3 = W_1 \oplus W_2$.

(16) Show that if S = $\{\alpha, \beta, \gamma\}$ is a basis of $C^3(C)$, then the set S' = $\{\alpha + \beta, \beta + \gamma, \gamma + \alpha\}$ is also a basis of $C^3(C)$.

(17) Give a basis for each of the following vector space over the indicated fields: (i)$R(\sqrt{2})$ $over\ R$ (ii) Q $(2^{1/4})$ over Q
Where Q,R are field of rational and real numbers.

(18) Determine dim V/W where V = C(R) and W = R (R).

(19) Let V = $C^2(R)$ and W = $R^2(R)$, find dim (V/W).

(20) Under what conditions on the scalar 'a', do the vectors (1,1,1) and $(1,a,a^2)$ form a basis of $C^3(C)$.

SuccessClap : Question Bank for Practice

12 EIGEN VALUES

(1) Find the eigen values and the corresponding eigen vectors of $\begin{pmatrix} 5 & 4 \\ 1 & 2 \end{pmatrix}$.

(2) Find the eigen values and eigen vectors of A = $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & -2 \end{bmatrix}$.

(3) Find the eigen values and Eigen vectors of A= $\begin{bmatrix} 5 & -2 & 0 \\ -2 & 6 & 2 \\ 0 & 2 & 7 \end{bmatrix}$.

(4) Find the characteristic roots of the matrix $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$ and the corresponding eigen vectors.

(5) Find the eigen values and eigen vectors of $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

(6) Verify that the sum of eigen values is equal to the trace of 'A' for the matrix A = $\begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$ and find the corresponding eigen vectors.

(7) Determine the characteristic roots and the corresponding characteristic vectors of the matrix A = $\begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$.

(8) Find the eigen values and the corresponding eigen vectors of $\begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$.

(9) Find the eigen values and eigen vectors of $\begin{bmatrix} 3 & 2 & 2 \\ 1 & 2 & 2 \\ -1 & -1 & 0 \end{bmatrix}$.

(10) The sum of the eigen values of a square matrix is equal to its trace and product of the eigen values is equal to its determinant.

i. e., if A is an $n \times n$ matrix and $\lambda_1, \lambda_2, \ldots \lambda_n$ are its n eigen values, then $\lambda_1 + \lambda_2 + \ldots \lambda_n = Tr(A)$ and $\lambda_1.\lambda_2.\lambda_3.\ldots\ldots.\lambda_n = \det(A)$

(11) A square matrix A and its transpose A^T have the same eigen values.

(12) If A and B are n rowed square matrices and if A is invertible show that $A^{-1}B$ and BA^{-1} have same eigen values.

(13) If λ is an eigen value of the matrix A then λ+k is an eigen value of the matrix A +kI.

(14) If $\lambda_1, \lambda_2, \ldots.\lambda_n$ are the eigen values of A, find the eigen values of the matrix $(A-\lambda I)^2$.

(15) If λ is an eigen value of a non – singular matrix A corresponding to the eigen vector X, then λ^{-1} is an eigen value of A^{-1} and corresponding eigen vector X itself.

(OR)

Prove that the eigen values of A^{-1} are the reciprocals of the eigen values of A.

(16) If λ is an eigen value of a non – singular matrix A, then $\frac{|A|}{\lambda}$ is an eigen value of the matrix adj A.

(17) If λ is an eigen value of an orthogonal matrix then $\frac{1}{\lambda}$ is also an eigen value.

(18) Suppose that A and P be square matrices of order n such that P is non – singular. Then A and P^{-1} AP have the same eigen values.

(19) The eigen values of a real symmetric matrix are always real (or real numbers).

(20) For a real symmetric matrix, the eigen vectors corresponding to two distinct eigen values are orthogonal.

(21) Prove that the two eigen vectors corresponding to the two different eigen values are linearly independent.

(22) Determine the eigen values and eigen vectors of

B = $2A^2 - \frac{1}{2} A + 3I$ $where\ A = \begin{pmatrix} 8 & -4 \\ 2 & 2 \end{pmatrix}$.

(22) For the matrix A = $\begin{bmatrix} 1 & 2 & -3 \\ 0 & 3 & 2 \\ 0 & 0 & -2 \end{bmatrix}$ find the eigen values of $3A^3+5A^2$-6A+2I.

(24) The Eigen values of a Hermitian matrix are all real.

(25) The eigen values of a real symmetric matrix are all real.

(26) The eigen values of a Skew – Hermitian matrix are either purely imaginary.

(27) The eigen values of a Skew – Hermitian matrix are purely imaginary or zero.

(28) The Eigen values of an unitary matrix have absolute value 1.

(29) Prove that transpose of a unitary matrix is unitary.

(30) Find the eigen values of the following matrices:

(i) A = $\begin{bmatrix} 4 & 1-3i \\ 1+3i & 7 \end{bmatrix}$ (ii) B = $\begin{bmatrix} 3i & 2+i \\ -2+i & -i \end{bmatrix}$ (iii) C = $\begin{bmatrix} \frac{1}{2}i & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2}i \end{bmatrix}$

(31) Find the eigen values and eigen vectors of the Hermitian matrix $\begin{bmatrix} 2 & 3+4i \\ 3-4i & 2 \end{bmatrix}$.

(32) Prove that $\frac{1}{2}\begin{bmatrix} i & \sqrt{3} \\ \sqrt{3} & i \end{bmatrix}$ is a unitary matrix. Find its eigen values.

(33) Prove that the matrix $\frac{1}{\sqrt{3}}\begin{bmatrix} 1 & 1+i \\ 1-i & -1 \end{bmatrix}$ is unitary and determine the eigen values and eigen vectors.

(34) Prove that the determinant of a unitary matrix is of uint modulus.

(35) Show that the matrix $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$ is Skew – Hermitian and hence find eigen values and eigen vectors.

(36) Find the Eigen vectors of the Hermitian matrix A = $\begin{bmatrix} a & b+ic \\ b-ic & k \end{bmatrix}$.

(37) Show that A = $\begin{bmatrix} i & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{bmatrix}$ is a Skew – Hermitian and also Unitary. Find eigen values and the corresponding eigen vectors of A.

(38) Prove that the modulus of each latent root (eigen value) of a Unitary matrix is unity.

(39) Verify the matrix A = $\frac{1}{2}\begin{bmatrix} 1+i & 1-i \\ 1-i & 1+i \end{bmatrix}$ has eigen values with unit modulus.

(40) Find the eigen vectors of the Skew – Hermitian matrix A = $\begin{bmatrix} 2i & 3i \\ 3i & 0 \end{bmatrix}$

(41) If X is a characteristic vector of a matrix A, then X cannot correspond to more than one characteristic values of A.

(42) Determine the characteristic roots and the corresponding characteristic vectors of the matrix A = $\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$.

(43) Determine the eigenvectors of the matrix A = $\begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$.

(44) Show that 0 is a characteristic root of a matrix if and only if the matrix is singular.

(45) If α is a characteristic root of a non – singular matrix A, then prove that $\frac{|A|}{\alpha}$ is a characteristic root of Adj A.

(46) Find the characteristic roots of the 2 – rowed orthogonal matrix $\begin{bmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{bmatrix}$ and verify that they are of unit modulus.

(47) Show that the roots of the equation $\begin{bmatrix} a+x & h & g \\ h & b+x & f \\ g & f & c+x \end{bmatrix}$ =0 are real; a,b,c,f,g,h being real numbers.

(48) Find the characteristic roots and characteristics vectors of the matrix A = $\begin{bmatrix} 3 & 10 & 5 \\ -2 & -3 & -4 \\ 3 & 5 & 7 \end{bmatrix}$. Also verify the fact that the geometric multiplicity of a characteristic root cannot exceed its algebraic multiplicity.

(49) Suppose S is an n – rowed real skew – symmetric matrix and I is the unit marix of order n. Then show that
(i) I – S is non – singular
(ii) A = (I+S) $(I-S)^{-1}$ is orthogonal,
(iii) A = $(I-S)^{-1}$ (I+S),
(iv) If X is a characteristic vector of S corresponding to the characteristic root λ, then X is also a characteristic vector of A and (1+λ) (1-λ) is the corresponding characteristic root.

(50)If A be an orthogonal matrix with the property that -1 is not a characteristic root, tehm A is expressible as (I+S)$(I-S)^{-1}$ for some suitable real skew – symmetric matrix S.

(51) If S is a skew – Hermitian matrix, show that the matries I-S and I+S are both non – singular. Also show that A = (I+S) $(I-S)^{-1}$ is a unitary matrix.

(52) Prove that ± 1 can be the only real characteristic roots of an orthogonal matrix.

(53) If A is both real symmetric and orthogonal, prove that all its eigenvalues are +1 or -1.

(54) If A and B are two square matrices of the same order, then AB and BA have the same characteristic roots.

(55) A real matrix is unitary if and only if it is orthogonal.

(56) (a) If P is unitary so are P^r, P^-, P^θ and P^{-1}.
(b) If P and Q are unitary so is PQ.
(c) If P is unitary, then |P| is of unit modulus.
(d) Any two eigenvectors corresponding to the distinct eigenvalues of a unitary matrix are orthogonal.

(57) Prove that the eigenvalues of A^θ are the conjugates of the eigenvalues of A. If k_1 and k_2 are distinct eigenvalues of A, prove that any eigenvector of A corresponding to k_1 is orthogonal to any eigenvector of A^θ corresponding to k_2.

(58) Show that the matrix A = $\begin{bmatrix} a+ic & -b+id \\ b+id & a-ic \end{bmatrix}$ is unitary if and only if $a^2+b^2+c^2+d^2=1$.

(59) Show that the matrix A = $\begin{bmatrix} 0 & 2m & n \\ l & m & -n \\ l & -m & n \end{bmatrix}$, l = $\frac{1}{\sqrt{2}}$, m = $\frac{1}{\sqrt{6}}$, n = $\frac{1}{\sqrt{3}}$ is orthogonal.

(60) Show that if A is Hermitian and P orthogonal, then $P^{-1}AP$ is symmetric.

(61) Show that if A is Hermitian and P unitary, then $P^{-1}AP$ is Hermitian.

SuccessClap : Question Bank for Practice

13 LINEAR TRANSFORMATION

(1). Let T_1 and T_2 be linear operators of R^2 defined as follows:
$T_1(x_{1,}x_2) = (x_2, x_1)$ and $T_2(x_{1,}x_2) = (x_2, 0)$
Show that $T_2T_1 \neq T_2T_1$.

(2). Let V(R) be the vector space of all polynomial function in x with coefficients in the field R of real numbers. Let D and T be two linear operators on V defined by $D[f(x)] = \frac{d}{dx}f(x)$

$$Tf(x) = \int_0^x f(x)dx$$

For every $f(x) \in V$. Then show that $DT = I$ (Identity operator) and $TD \neq I$.

(3). Let V(R) be the vector space of all polynomials in x with coefficients in the field R. Let D and let T be two linear transformations on V defined by

$$D[f(x)] = \frac{d}{dx}f(x) \ \forall \ f(x) \in V$$

And $T[f(x)] = xf(x) \ \forall \ f(x) \in$
Then show that $DT \neq TD$. Also, show that $DT - TD = I$.

(4). Describe explicitly a linear transformation from $V_3(R)$ into $V_3(R)$ which has its range the subspace spanned by $(1,0,-1)$ and $(1,2,2)$.

(5). Let T be linear operator on a vector space V(F). if $T^2 = 0$, what can you say about the relation of the range of T to the null of T? Give an example of a linear operator on $V_2(R)$ such that $T^2 = 0$ but $T \neq 0$.

(6). If T: $R^3 \to R^3$ is a linear operator defined by $T(x, y, z) = (x + z, x - z, y)$. Show that T is invertible and find $T^{-1}(2,4,6)$.

(7). A linear transformation T is defined on $V_2(C)$ by $T(a, b) = (\alpha a + \beta b + \gamma a + \delta b)$,where $\alpha, \beta, \gamma, \delta$ are fixed elements of C. prove that T is invertible if and only if $\alpha\delta - \beta\gamma \neq 0$.

(8). Let V be finite dimensional vector space and T be a linear operator on V. Suppose that rank $(T^2) = \text{rank}(T)$. prove that the range and null space of T are disjoint, i.e. have only the zero vector in common.

(9). Let $T: R^2(R) \to R^2(R)$, where for any $(x, y) \in R^2$, $T(x, y) = \left(2x, \frac{1}{2}y\right)$. Find the matrix associated with T with respect to the ordered basis $\{(1,0), (0,1)\}$.

(10). Find the matrix of the linear Transformation T on $V_3(R)$ defined as $T(a, b, c) = (2b + c, a - 4b, 3a)$ with respect to the ordered basis B and also with respect to the ordered basis B' where
a) $B = (1,0,0), (0,1,0), (0,0,1)$
b) $B' = (1,1,1), (1.1.0). (1,0,0)$.

(11). Let T be a linear operator on R^2 defined by $T(x, y) = (2y, 3x - y)$.find the matrix representation of T relative to the basis $(1,3), (2,5)$.

(12). Let T be a linear operator on R^3 defined by

$$T(x, y, z) = (3z + z, -2x + y, -x + 2y + 4z)$$

Prove that T is invertible and find a formula for T^{-1}.

(13). Consider the vector space $V(R)$ of all 2×2 matrices over the field R of real numbers. Let T be the linear transformation on V sending each matrix X onto AX, where A=$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Find the matrix of T with respect to the ordered basis $B = \{E_1, E_2, E_3, E_4\}$, for V where

$$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, E_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

(14). Let a linear map $T: P_3 \to P_2$ be defined by

$$T(a_0 + a_1x + a_2x^2 + a_3x^3) = a_3 + (a_2 + a_3)x + (a_0 + a_1)x^2$$

Where $P_n[x] =$ set of all polynomials of degree $\leq n$. Find the matrix of T with respect to the ordered bases $B = \{1, (x-1), (x-1)^2, (x-1)^3\}$ and $B' = \{1, x, x^2\}$.

(15). How that the mapping T defined by $T(a, b) = (\alpha+\beta, \alpha-\beta, \beta), \forall(\alpha, \beta) \in V_2(R)$ is a linear transformation.

(16). Which of the following T from R^2 into R^2 are linear transformation?

a) $T(x_{1,}x_2) = ({x_1}^2, x_2)$

b) $T(x_{1,}x_2) = (\sin x_1, x_2)$

c) $T(x_{1,}x_2) = ({x_1}^2 - x_2, 0)$.

(17). Show that the mapping $T: R^3 \to R^2$, defined by

$$T(\alpha, \beta, \gamma) = (\alpha, \beta), \forall(\alpha, \beta, \gamma) \in R^3$$

Is a homomorphism (linear transformation) of the vector space $R^3(R)$ onto $R^2(R)$.

(18). Let F be the field of complex numbers and let T be the function from R^3onto R^3 defined by $T(a_1, a_2, a_3) = (a_1 - a_2 + 2a_3, 2a_1 + a_1 - a_3, -a_3 - a_1, -2a_2)$. Verify that T is a linear transformation. Describe the null space of T.

(19). Describe explicitly the linear transformation $T: R^2 \to R^3$such that $T(2,3) = (4,5)$ and $T(1,0) = (0,0)$.

(20). If a map $T: V_2(R) \to V_3(R)$ defined by $T(a, b) = (a+b), (a-b), b)$ is a linear transformation. Find the range, rank, null-space and nullify of T. Determine of range of T, i. e. R_T and rank:

(21). Find a linear transformation $T: R^2 \to R^2$ such that $T(1,0) = (1,1)$ and $T(0,1) = (-1,2)$. Prove that T maps the square with vertices $(0,0), (1,0), (1,1)$ and $(0,1)$ into a parallelogram.

(22). Let $T: V_3(R) \to V_3(R)$ be a linear transformation defined by

$$T(a, b, c) = (3a, a - b, 2a + b + c), \forall a, b, c \in R.$$

Prove that T is invertible and find T^{-1}, also prove that $(T^2 - 1)(T - 3I) = 0$.

(23). If T is a linear transformation on a vector space V such that $T^2 - T + I = 0$, then show that T is invertible.

(24). Let $T: R^2(R) \to R^2(R)$, where for any $(x, y) \in R^2, T(x, y) = \left(2x, \frac{1}{2}y\right)$. Find the matrix associated with T w.r.t. the ordered basis {(1,0), (0,1)}.

(25). Find the matrix representation of a linear map $T: R^3 \to R^3$ defined $T(x, y, z) = (z, y + z, x + y + z)$ relative to the basis {(1,0,1), (-1,2,1), (2,1,1)}

(26). Let T be linear operator in R^3 defined by

$$T(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_2 + 2x_2 + 4x_3)$$

Find the matrix of T in the ordered basis $(\alpha_1, \alpha_2, \alpha_3)$, where

$$\alpha_1 = (1,0,1), \alpha_2 = (-1,2,1), \alpha_3 = (2,1,1)$$

(27). If the matrix of a linear transformation T on a vector space $V_2(C)$ w.r.t the ordered basis $B = \{(1,0, (0, -1)\}$ is

$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ what is the matrix w.r.t the ordered basis $B' = (1,1), (1, -1)$.

(28). Let $B = \{(1,0, (0,1)\}$ $B' = (1,2), (2,3)$ be any two bases of R^2

a) Find the transition matrices P from B to B'

b) Verify that $[\alpha]_B = P^{-1}[\alpha]_B, \forall \alpha \in R^2$

c) Verify that $P^{-1}[T]_B P = [T]_B$, where $T(x, y) = (2x - 3y, x + y)$.

(29). Let V be the vector spaces of all $n \times n$ matrices over the field F, and let B be a fixed $n \times n$ matrix, if $T(A) = AB - BA \forall A \in V$

Verify that T is a linear transformation from V into V.

(30). Let T: $R^3 \rightarrow R^2$ be defined as $T(\alpha_1) = (1,0), T(\alpha_2) = (2,-1), \quad T(\alpha_3) = (4,3)$ Find $T(2,-3,5)$, where $\alpha_1 = (1,1,1), \quad \alpha_2 = (1,1,0), \alpha_3(1,0,0)$ and $(\alpha_1, \alpha_2, \alpha_3)$ forms a basis of R^3

14 PROBLEM SET 1

(1). Let W_1 and W_2 be two subspaces of R^4 given by
$W_1 = \{(a,b,c,d): b - 2c + d = 0\}$, $W_2 = \{(a,b,c,d): a = d, b = 2c\}$,
Find the basis and dimension of a) W_1 , (b) W_2 (c) $W_1 \cap W_2$
And hence find $\dim(W_1 + W_2)$.

(2). If W is the subspace of $V_4(R)$ generated by the vectors $(1,-2,5,-3), (2,3,1,-4)$ and $(3,8,-3,-5)$. Find a basis of W and its dimension.

(3). V is the vector space of polynomials over R. W_1 and W_2 are the subspaces generated by
$\{x^3 + x^2 - 1, x^3 + 2x^2 - 3x, 2x^3 + 3x^2 + 3x - 1\}$ and

$\{x^3 + 2x^2 + 2x - 2, 2x^3 + 3x^2 + 2x - 3, x^3 + 3x^2 + 4x - 3\}$
respectively find (a) $\dim(W_1 + W_2)$ (b) $\dim(W_1 \cap W_2)$

(4). If $U = (1,2,1), (0,1,2)$, $W = (1,0,0), (0,1,0)$ determine the dimension of $U + W$.

(5). Let V be the vector space of polynomials in the variable in the x over R. Let $f(x) \in V(R)$; show that
(a) $D; V \to V$defined by $Df(x) = \frac{df(x)}{dx}$.
(b) $I: V \to D$ defined by $I\, f(x) = \int_0^x f(x)\, dx$ are linear transformations

(6). Let $P_n(R)$ be the vector space of all polynomials of degree n over a field R. If a linear operator T on $P_n(R)$ is such that $Tf(x) = f(x+1), f(x) \in P_n(R)$.
Show that $T = 1 + \frac{D}{1!} + \frac{D^2}{2!} + \frac{D^3}{3!} + \cdots \ldots + \frac{D^n}{n!}$

(7). Is the mapping $T: R^3 \to R^2$ defined by $T(x, y, z) = (|x|, 0)$ a linear transformation?

(8). Let T be a linear transformation on a vector space U into V. Prove that the vectors $x_1, x_2, \ldots\ldots x_n \in U$ are linearly independent in $T(x_1), T(x_2), \ldots\ldots T(x_n)$, are $L.I.$

(9). Let V be a vector space of $n \times n$ matrices over the field F. M is a fixed matrix in V. The mapping $T: V \to V$ is defined by $T(A) = AM + MA$ where $A \in V$. show that T is linear.

(10). Describe explicitly the linear transformation $T: R^2 \to R^2$ such that $T(2,3) = (4,5)$ and $T(1,0) = (0,0)$.

(11). Find $T(x, y, z)$ where $T: R^2 \to R$ is defined by $T(1,1,1) = 3, \quad T(0,1,-2) = 1, T(0,0,1) = -2.$

(12). Let $T: U \to V$ be a linear transformation $\{(1,2,1), (2,1,0), (1,-1,-2)\}$ and $\{(1,0,0), (0,1,0), (1,1,1)\}$ are the basis of U and V. Find T for the transformation of the basis of U to the basis of V.

(13). Let $T: V_3(R) \to V_2(R)$ and $H: V_3(R) \to V_2(R)$ be the two lines transformation defined by $T(x, y, z) = (x - y, y + z)$ and $H(x, y, z) = (2x, y - z)$ find. (a) $H + T$ and (b) aH

(14). Let $G: V_3 \to V_3$ and $H: V_3 \to V_3$ be two linear operators defined by $G(e_1) = e_1 + e_2, G(e_2) = e_{3,} G(e_3) = e_2 - e_3$ and $H(e_1) = e_3, H(e_2) = 2e_2 - e_3, H(e_1) = 0$ where $\{e_1, e_2, e_3\}$ is the standard basis of $V_3 R$.

a. $G + H.$

b. $2G.$

(15). Let $T: R^3 \to R^2$ and $H: R^3 \to R^2$ be defined by $T(x, y, z) = (3x, y + z)$ and $H(x, y, z) = (2x - x, y)$ compare

a. $T + H$

b. $4T - 5H$

c. TH
d. HT

(16). Let $T_1: R^3 \rightarrow R^2$ and $T_2: R^3 \rightarrow R^2$ are two linear transformation defined by $T_1(x, y, z) = (3x, 4y - z), T_2(x, y) = (-x, y)$. Compare T_1T_2 and T_2T_1.

(17). Let $P(R)$ be the vector space of all polynomials in x and D, T be two linear operators on P defined by $D[f(x) = \frac{df}{dx}$ and $T[f(x)] = xf(x) \forall f(x) \in V$. Show that

a. $TD \neq DT$
b. $(TD)^2 = T^2D^2 + TD$

(18). If $T: V_4(R) \rightarrow V_3(R)$ is a linear transformation defined by $T(a, b, c, d) = (a - b + c + d, a + 2c - d, a + b + 3c - 3d)$ for $a, b, c, d \in R$ then verify $p(T) + v(T) = \dim V_4(R)$.

(19). Find a linear transformation $T: R^3 \rightarrow R^3$ whose range is spanned by $(1,2,0,-4), (2,0,-1,-3)$.

(20). Find $F: R^3 \rightarrow R^4$ be linear transformation whose range is spanned by $(1,-1,2,3), (2,3,-1,0)$.

(21). Let V be a vector space of 2×2 matrices over reals. Let P be a fixed matrix of $V: P = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$ and $T: V \rightarrow V$ be linear operators defined by $T(A) = PA, A \in V$. Find the nullity T.

(22). Find the null space, range, rank and nullity of transformation $T: R^2 \rightarrow R^3$ defined by $T(x, y) = (x + y, x - y, y)$.

(23). Verify the Rank-nullity theorem for the linear map $T: V_4 \rightarrow V_3$ defined by $T(e_1) = f_1 + f_2 + f_3, T(e_2) = f_1 - f_2 + f_3, T(e_3) = f_1, T(e_4) = f_1 + f_3$ When $\{e_1, e_2, e_3\}$ and $\{f_1, f_2 f_3\}$ are standard basis V_4 and V_3 respectively.

(24). Verify Rank-nullity theorem for the linear transformation $T: R^3 \to R^3$ defined by $T(x, y, z) = (x - y, 2y + z, x + y + z)$.

(25). Find the kernel of the linear transformation $T: R^2 \to R^2$ defined as $T(1,0) = (1,1)$ are $T(0,1) = (-1,2)$.

(26). If $T: R^3 \to R^3$ be a linear transformation defined by $T(x, y, z) = (x - y, y - z, z - x)$ then show that T is a linear transformation and find its rank.

(27). If $T: R^3 \to R^3$ is invertible operator defined by $T(x, y, z) = (2x, 4x - y, 2x + 3y - z)$. Find T^{-1}.

(28). The set $\{e_1, e_2, e_3\}$ is standard basis of $V_3(R)$ $T: V_3(R) \to V_3(R)$ is a linear operator by $T(e_1) = e_1 + e_2, T(e_2) = e_2 + e_3, T(e_3) = e_1 + e_2 + e_3$. Show that T is non-singular and find its inverse.

(29). A linear transformation T is defined on $V_3(C)$ by $T(a, b) = (a\alpha + b\beta + b\delta)$ where $\alpha, \beta, \gamma, \delta$ are fixed elements of C. prove that T is invertible if and only if $\alpha\beta - \beta\gamma \neq 0$.

(30). Let $T: V_2 \to V_3$ be defined by $T(x, y) = (x + y, 2x - y, 7y)$ find $[T: B_1, B_2]$ where $B_1$and B_2 are the standard bases of V_2 and V_3

(31). Let $T: R^3 \to R^2$ be the linear transformation defined by $T(x, y, z) = (3x + 2y - 4x, x - 5y + 3x)$. Find the matrix of T relative to the bases $B_1 = \{(1,1,1,), (1,10), (1,0,0)\}$,
$B_2 = \{(1,3), (2,5)\}$.

(32). If $C(R)$ is a vector space having the bases $B_1 = 1, i$ and $B_2 = 1 + i, 1 + 2i$,find the transition matrix of T from B_1 to B_2.

(33). If the matrix of a linear transformation T on $V_3(R)$ w.r.t to the standard basis is $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$ what is the matrix of T w.r.t to the basis $(0,1,-1),(1,-1,1),(-1,1,0)$.

(34). Let $D: P_3 \rightarrow P_2$ be the polynomial different transformation $D(p) = \frac{dp}{dx}$, find the matrix of D relative to the standard bases $B_1 = \{1, x, x^2, x^3\}$ and $B_2 = \{1, x, x^2\}$.

(35). $T: R^2 \rightarrow R^2$ such that $T(1,1) = (2,-3), T(1,-1) = (4,7)$. Find the matrix of T relative to the basis $S = \{(1,0),(0,1)\}$.

(36). Let T be the linear operator on R^2 defined by
$T(x,y) = (4x - 2y, 2x + y)$. Find the matrix of T w.r.t to the basis
$T(1,1), (-1,0)$
Also verify $[T]_B[\alpha]_B = [T][\alpha]_B \;\; \forall \alpha \in R^2$.

(37). Let T be a linear operator on $V_3(R)$ defined by
$T(x,y,z) = (3x + z, -2x + y, -x + 2y + z)$ prove that T is invertible and find T^{-1}.

(38). Let $B = \{(1,0),(0,1)$ and $B' = \{(1,3),(2,5)$ be the bases of R^2 find the transition matrices from B to B' and B' to B.

(39). Let T be the linear operator on $V_3(R)$ defined by
$T(x,y,z) = (2y + z, x - 4y, 3x)$.

a) Find the matrix of T relative to the basis $B = (1,1,1),(1,1,0),(1,0,0)$.
b) Verify $[T(\alpha)]_B = [T]_B[\alpha]_B$.

(40). Let $T: R^3 \rightarrow R^2$ be a linear transformation defined by
$T(x,y,z) = (2x + y - z, 3x - y + 4z)$.

a) Obtain the matrix of T to the bases
b) $B_1 = (1,1,1),(1,1,0),(1,0,0); B_2 = (1,3),(1,4)$
c) Verify for any vector $\alpha \in R^3$.

(41). Let V(F) be a vector space of polynomials in x of degree at most 3 and D be the different operator on V.