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UPSC Mathematics (Optional) STUDY MATERIAL

BOOK- 13

Linear Programming

Total 14 Books

Exam Oriented Study Material

Last Update 25-06-2020 (We Update Regularly)

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1 LINEAR PROGRAMMING — FORMULATION

PRACTICAL STEPS INVOLVED IN THE FORMULATION OF LP PROBLEM

The practical steps involved in the formation of linear programming problem are as follows:

Step 1→ *Identify the **Decision Variables** of interest to the decision maker and express them as* x_1, x_2, x_3 ....

Step 2→ *Ascertain the **Objective** of the decision maker whether he wants to minimize or to maximize.*

Step 3→ *Ascertain the **cost** (in case of minimization problem) or the **profit** (in case of maximization problem) per unit of each of the decision variables.*

Step 4→ *Ascertain the **constraints** representing the maximum availability or minimum commitment or equality and represent them as less than or equal to (≤) type inequality or greater than or equal to (≥) type inequality or 'equal to' (=) type equality respectively.*

Step 5→ *Put **non-negativity restriction** as under:*

$$x_j \geq 0;\ j = 1,\ 2\ ....n \text{ (non-negativity restriction)}$$

Step 6→ *Now **formulate the LP problem as under**:*

Maximize (or Minimize) $Z = c_1x_1 + c_2x_2\ .....c_nx_n$

Subject to constraints:

$a_{11}x_1 + a_{12}x_2\ ,....a_{1n}x_n \leq b_1$ *(Maximum availability)*

$a_{21}x_1 + a_{22}x_2\ ,....a_{2n}x_n \geq b_2$ *(Minimum commitment)*

$a_{31}x_1 + a_{32}x_2\ ,....a_{3n}x_n = b_3$ *(Equality)*

$a_{m1}x_1 + a_{m2}x_2\ ,....a_{mn}x_n \overset{\leq}{\geq} b_m$

$x_1;\ x_2\ ....\ x_n \geq 0$ *(Non-negativity restriction)*

where,

x_j = *Decision Variables i.e. quantity of* j^{th} *variable of interest to the decision maker.*

c_j = *Constant representing per unit contribution (in case of Maximization Problem) or Cost (in case of Minimization Problem) of the* j^{th} *decision variable.*

a_{ij} = *Constant representing exchange coefficients of the* j^{th} *decision variable in the* i^{th} *constant.*

b_i = *Constant representing* i^{th} *constraint requirement or availability.*

PROBLEM 1.1

A firm produces three products A, B and C. It uses two types of raw materials I and II of which 5,000 and 7,500 units respectively are available. The raw material requirements per unit of the products are given below:

Raw Material	*Requirement per unit of Product*		
	A	*B*	*C*
I	3	4	5
II	5	3	5

The labour time for each unit of product A is twice that of Product B and three times that of Product C. The entire labour force of the firm can produce the equivalent of 3,000 units. The minimum demand of the three products is 600, 650 and 500 units respectively. Also the ratios of the number of units produced must be equal to 2 : 3 : 4. Assuming the profits per unit of A, B and C as Rs. 50, 50 and 80 respectively.

Required: Formulate the problem as a linear programming model in order to determine the number of units of each product, which will maximize the profit.

Solution

Let X_1, X_2 and X_3 be the units produced & sold of Product A, B and C respectively.

The labour time for each unit of Product A is twice that of Product B and three times that Product C. Also the entire labour force can produce the equivalent of 3000 units.

$$x_1 + \frac{x_2}{2} + \frac{x_3}{3} \le 3000$$

or $$6x_1 + 3x_2 + 2x_3 \le 18000$$

Since the ratios of the number of units produced must be equal to 2 :3 :4, therefore,

$$\frac{1}{2}x_1 = \frac{1}{3}x_2, \text{ and } \frac{1}{3}x_2 = \frac{1}{4}x_3$$

or $$3x_1 = 2x_2 \text{ and } 4x_2 = 3x_3 \quad \text{...(v)}$$

Since the objective of the firm is to maximize the profit, therefore, the objective function is given by–

Maximise $Z = 50x_1 + 50x_2 + 80x_3$

Subject to the constraints

$3x_1 + 4x_2 + 5x_3 \le 5000$ [Maximum Raw-Material I]

$5x_1 + 3x_2 + 5x_3 \le 7500$ [Maximum Raw-Material II]

$6x_1 + 3x_2 + 2x_3 \le 18000$ [Maximum Labour Time]

$3x_1 = 2x_2$ and $4x_2 = 3x_3$ [Ratio of No. of units Produced]

$x_1 \ge 600$, $x_2 \ge 650$ and $x_3 \ge 500$ [Minimum Demand for Products]

PROBLEM 1.2

In a chemical industry two products A and B are made involving two operations The production of B also results in a by-product C. The product A can be sold at a profit of Rs. 3 per unit and B at a profit of Rs. 8 per unit. The by-product C has a profit of Rs, 2 per unit. Forecasts show that upto 5 units of C can be sold. **The company gets 3 units of C for each unit of B produced.** The manufacturing times are 3 hours per unit for A on each the operation one and two and 4 hours and 5 hours per unit for B on operation one and two respectively. Because the product C results from producing B, no time is used in producing C. The available times are 18 hours and 21 hours of operation one and two respectively. The company desires to know that how much A and B should be produced keeping C in mind to make the highest profit. Formulate LPP model for this problem.

Solution

Let x_1 x_2, x_3 be the number of units produced & sold of products A, B, and C respectively.

In first operation, A takes 3 hours of manufacturer's time and B takes 4 hours of manufacturer's time Therefore, total number of hours required in first operation becomes,

$$3x_1 + 4x_2$$

Further, the company gets three units of by product C for every unit of product B produced, therefore

$$x_3 = 3x_2$$

Now, the allocation problem of the industry can be finally put in the following linear programming problem:

Maximise $Z = 3x_1 + 8x_2 + 2x_3$

Subject to the constraints

$3x_1 + 4x_2 \leq 18$ [Maximum Hours in First Operation]

$3x_1 + 5x_2 \leq 21$ [Maximum Hours in Second Operation]

$x_3 \leq 5,$ [Maximum Sales units of Product C]

$x_3 = 3x_2$ [Ratio of Product C to Product B]

$x_1, x_2, x_3 \leq 0$ [Non-Negativity]

PROBLEM 1.3

A firm buys castings of P and Q type of parts and sells them as finished product after machining, boring and polishing. The purchasing cost for castings are Rs. 3 and Rs. 4 each for parts P and Q and selling prices are Rs. 8 and Rs. 10 respectively. The per hour capacity of machines used for machining, boring and polishing for two products is given below:

Capacity (per hour)	*P*	*Q*
Machining	30	50
Boring	30	45
Polishing	45	30

The running costs for machining, boring and polishing are Rs. 30, Rs, 22.5 and Rs. 22.5 per hour respectively.

Formulate the linear programming problem to find out the product mix to maximize the profit.

Solution

Let the firm buy x units of castings of P and y units of castings of Q which are sold as finished product after machining , boring and polishing. According to the given data, the capacity constraints of machining, boring and polishing machines on per hour basis have been formulated as below:

$\frac{x}{30} + \frac{y}{50} \leq 1$ *or* $50x + 30y \leq 1500$ (*Machining Constraint*) ...(i)

$\frac{x}{30} + \frac{y}{45} \leq 1$ *or* $45x + 30y \leq 1350$ (*Boring Constraint*) ...(ii)

$\frac{x}{45} + \frac{y}{30} \leq 1$ *or* $30x + 45y \leq 1350$ (*Polishing Constraint*) ...(iii)

Calculation of Total Cost and Profit per unit

	Cost/per unit	
	Castings of P type	*Castings of Q type*
Machining	Rs.30/30 = Rs. 1.00	Rs.30/50 = Rs 0.60
Boring	Rs. 22.5/30 = Rs. 0.75	Rs .22.5/45 = Rs. 0.50
Polishing	Rs. 22.5/45 = Rs. 0.50	Rs. 22.5/30 = Rs. 0.75
Purchasing Cost	Rs. 3.00	Rs. 4.00
Total Cost	Rs. 5.25	Rs. 5.85
Sale Price	Rs. 8.00	Rs. 10.00
Profit	Rs. 2.75	Rs. 4.15

The required product mix to maximise the profit of the firm will be given by the following relation:

Maximise $Z = 2.75x + 4.15\,y$

Subject to the Constraints

$$50\,x + 30\,y \leq 1500$$
$$45x + 30y \leq 1350$$
$$30\,x + 45\,y \leq 1350$$

where $x, y, \geq 0$

PROBLEM 1.4

A manufacturer produces three products Y_1, Y_2, Y_3 from three raw materials X_1, X_2, X_3. The cost of raw materials X_1, X_2 and X_3 is Rs. 30, Rs. 50 and Rs. 120 per kg respectively and they are available in a limited quantity viz 20 kg of X_1, 15 kg of X_2 and 10 kg of X_3. The selling price of Y_1, Y_2 and Y_3 is Rs. 90, Rs. 100 and Rs. 120 per kg respectively. In order to produce 1 kg of Y_1, ½ kg of X_1, ¼ kg of X_2 and ¼ kg of X_3 are required. Similarly to produce 1 kg of Y_2, 3/7 kg of X_1, 2/7 kg of X_2 and 2/7 kg of X_3 and to produce 1 kg of Y_3, 2/3 kg of X_2 and 1/3 kg of X_3 will be required.

Formulate the linear programming problem to maximize the profit.

Solution

The information given in the question can be presented of the following tabular form.

	Raw material (in kg) required to produce one kg of product			*Selling*
Products	X_1	X_2	X_3	*Price (per kg)*
Y_1	1/2	1/4	1/4	Rs. 90
Y_2	3/7	2/7	2/7	Rs. 100
Y_3	—	2/3	1/3	Rs. 120
Cost of raw material (per kg)	Rs. 30	Rs. 50	Rs. 120	
Availability of raw material	20 kg	15 kg	10 kg	

From the above table, the cost of producing 1 kg of Y_1, Y_2 and Y_3 can be calculated as given below:

Cost to produce 1 kg of Y_1 = ½ Rs. 30 + ¼ Rs. 50 + ¼ Rs. 120
= Rs. 15 + Rs. 12.50 + Rs. 30 = Rs. 57.50

∴ Profit per kg of Y_1 = Rs. 90 – Rs. 57.50 = Rs. 32.50

Similarly, Cost to produce 1 kg of Y_2 = 3/7 Rs. 30 + 2/7 Rs. 50 + 2/7 Rs. 120
= 1/7(Rs. 90 + Rs. 100 + Rs. 240)
= Rs. 430/7 = Rs. 61.43

Profit per kg of Y_2 = Rs.100 – Rs.61.43 = Rs.38.57

and cost to produce 1 kg of Y_3 = 2/3 Rs.50 + 1/3 Rs.120 = Rs.220/3 = Rs.73.33

Profit per kg of Y_3 = Rs.120 – Rs.73.33 = Rs.46.67

Let the manufacturer produce y_1, y_2 and y_3 units of the products Y_1, Y_2 and Y_3 respectively.

Since the manufacturer wants to maximise the profit, the objective function is given by

Maximise $Z = 32.50\ Y_1 + 38.57\ Y_2 + 46.67\ Y_3$

Subject to the constraints

$$\tfrac{1}{2} Y_1 + 3/7\ Y_2 \le 20 \text{ or } 7\ Y_1 + 6\ Y_2 \le 280$$
$$\tfrac{1}{4} Y_1 + 2/7\ Y_2 + 2/3\ Y_3 \le 15 \text{ or } 21\ Y_1 + 24\ Y_2 + 56\ Y_3 \le 1260$$
$$\tfrac{1}{4} Y_1 + 2/7\ Y_2 + 1/3\ Y_3 \le 10 \text{ or } 21\ Y_1 + 24\ Y_2 + 28\ Y_3 \le 840$$

where Y_1, Y_2 and $Y_3 \ge 0$

PROBLEM 1.5

An agriculturist has a farm with 125 acres. He produces Radish, Muttar and Potato. Whatever he raises is fully sold in the market. He gets Rs. 5 for Radish per kg. Rs.4 for Muttar per kg. and Rs. 5. For Potato per kg. The average yield is 1,500 kg. of Radish per acre, 1,800 kg. of Muttar per acre and 1,200 kg. of Potato per acre. To produce each 100 kg. of Radish and Muttar and to produce each 80 kg. of Potato, a sum of Rs. 12.50 has to be used for manure. Labour required for each acre to raise the crop is 6 man days for Radish and Potato each and 5 man days for Muttar. A total of 500 man days of labour at a rate of Rs. 40 per man day are available.

Formulate this as a Linear Programming model to maximize the Agriculturist's total profit.

Solution

Let x_1, x_2 and x_3 be the number of acres allotted for cultivating radish, mutter and potato respectively.

Sales Revenue/acre (Radish) = Rs. 5 × 1500 = Rs. 7,500
Sales Revenue/acre (Matter) = Rs. 4 × 1800 = Rs. = 7,200
Sales Revenue/acre (Patato) = Rs. 5 × 1200 = Rs. 6,000

Now, the selling price, manure cost, labour cost and profit per acre of land for will be as fallows:

Per acre	Radish	Muttar	Potato
A. Selling Price	Rs. 7,500	Rs. 7,200	Rs. 6,000
B. Manure Cost	$\dfrac{\text{Rs. } 12.50 \times 1500}{100}$ = Rs. 187.50	$\dfrac{\text{Rs. } 12.50 \times 1800}{100}$ = Rs. 225	$\dfrac{\text{Rs. } 12.50 \times 1200}{80}$ = Rs. 187.50
C. Labour Cost	Rs.40 × 6 = Rs.240	Rs.40 × 5 = Rs.200	Rs.40 × 6 = Rs. 240
D. Profit [A – B – C]	= Rs.7,072.50	= Rs.6,775	= Rs. 5,572.50

Since, the agriculturist wants to maximise the total profit, hence the objective function of the problem is given by:

Maximise $Z = 7072.5x_1 + 6775x_2 + 5572.5x_3$

Subject to following constraints:

$x_1 + x_2 + x_3 \le 125$ [Maximum Land Area]

$6x_1 + 5x_2 + 6x_3 \le 500$ [Maximum Man Days]

where x_1, x_2 and $x_3 \ge 0$

PROBLEM 1.6

Three grades of coal A, B and C contains phosphorus and ash as impurities. In a particular industrial process, **fuel up to 100 ton (maximum) is required which could contain ash not more than 3% and phosphorus not more than .03%.** It is desired to maximize the profit while satisfying these conditions. There is an unlimited supply of each grade. The percentage of impurities and the profits of each grade are as follows:

Coal	*Phosphorus (%)*	*Ash (%)*	*Profit in Rs. (per ton)*
A	.02	3.0	12.00
B	.04	2.0	15.00
C	.03	5.0	14.00

You are required to formulate the Linear-programming (LP) model.

Solution

Let X_1, X_2 and X_3 respectively be the amounts in tons of grades A, B, and C used. The constraints are

(i) Phosphorus content must not exceed 0.03%

$$.02\,X_1 + .04\,X_2 + 0.03\,X_3 \le .03\,(X_1 + X_2 + X_3)$$

$$2\,X_1 + 4\,X_2 + 3\,X_3 \le 3\,(X_1 + X_2 + X_3) \text{ or } -X_1 + X_2 \le 0$$

(ii) Ash content must not exceed 3%

$$3\,X_1 + 2\,X_2 + 5\,X_3 \le 3(X_1 + X_2 + X_3) \text{ or } -X_2 + 2\,X_3 \le 0$$

(iii) Total quantity of fuel required is not more than 100 tons. $X_1 + X_2 + X_3 \le 100$

The Mathematical formulation of the problem is

Maximize $Z = 12\,X_1 + 15\,X_2 + 14X_3$

Subject to the constraints:

$$-X_1 + X_2 \le 0$$

$$-X_2 + X_3 \le 0$$

$$X_1 + X_2 + X_3 \le 100$$

$$X_1, X_2, X_3 > 0$$

PROBLEM 1.7

An oil refinery can blend three grades of crude oil to produce quality A and quality B petrol. Two possible blending processes are available. For each production run, the older process uses 5 units of crude Q, 7 units of crude P and 2 units of crude R and produces 9 units of A and 7 units of B. The newer process uses 3 units of crude Q, 9 unit of crude P and 4 units of crude R to produce 5 units of A and 9 units of B.

Because of prior contract commitments, the refinery must produce at least 500 units of A and at least 300 units of B for the next month. It has 1,500 units of crude Q, 1,900 units of crude P and 1,000 of crude R. For each unit of A, refinery receives Rs. 60 while for each unit of B, it receives Rs. 90.

Formulate the problem as linear programming model so as to maximize the revenue.

Solution

Maximize $Z = 60(9x_1 + 5x_2) + 90(7x_1 + 9x_2) = 1170x_1 + 1110x_2$

Subject to $9x_1 + 5x_2 \geq 500$ commitment for A

$7x_1 + 9x_2 \leq 300$ commitment for B

$5x_1 + 3x_2 \leq 1500$ availability of Q

$7x_1 + 9x_2 \leq 1900$ availability of P

$2x_1 + 4x_2 \leq 1000$ availability of R

and $x_1 \geq 0,\ x_2 \geq 0.$

PROBLEM 1.8

A refinery makes 3 grades of petrol (A, B, C) from 3 crude oils (d, e, f,). Crude can be used in any grade but the others must satisfy the following specifications

Grade	*Specification*	*Selling price per* litre
A	Not less than 50% crude d Not more than 25% crude e.	16
B	Not less than 25% crude d Not more than 50% crude e	13
C	No specifications	11

There are capacity limitations on the amounts of the three crude elements that can be used.

Crude	*Capacity*	*Price/litre*
d	2,50,000	19
e	2,50,000	11
f	1,50,000	13

Build a LPP to produce the maximum profit.

Solution

Let there be x_1 litres of d in A
x_2 litres of e in A
x_3 litres of f in A
y_1 litres of d in B
y_2 litres of e in B
y_3 litres of f in B
z_1 litres of d in C
z_2 litres of e in C
z_3 litres of f in C

Calculation of Profit per Litre in each Grade

	x_1	x_2	x_3	y_1	y_2	y_3	z_1	z_2	z_3
Selling Price	16	16	16	13	13	13	11	11	11
Cost	19	11	13	19	11	13	19	11	13
Profit	-3	5	3	-6	2	0	-8	0	-2

Since the objective is to maximize profit, the objective function is given by —

Maximize $Z = -3x_1 + 5x_2 + 3x_3 - 6y_1 + 2y_2 - 8z_1 - 2z_3$

Subject to constraints:

$$\frac{x_1}{x_1 + x_2 + x_3} \geq \frac{1}{2} \qquad \text{i.e. } - x_1 + x_2 + x_3 \leq 0$$

$$\frac{x_1}{x_1 + x_2 + x_3} \geq \frac{1}{4} \qquad \text{i.e. } - x_1 + 3x_2 - x_3 \leq 0$$

$$\frac{y_1}{y_1 + y_2 + y_3} \geq \frac{1}{4} \qquad \text{i.e. } - 3y_1 + y_2 + y_3 \leq 0$$

$$\frac{y_1}{y_1 + y_2 + y_3} \geq \frac{1}{4} \qquad \text{i.e. } - y_1 + y_2 - y_3 \leq 0$$

$$x_1 + y_1 + z_1 \leq 2{,}50{,}000$$

$$x_2 + y_2 + z_2 \leq 2{,}50{,}000$$

$$x_3 + y_3 + z_3 \leq 1{,}50{,}000$$

PROBLEM 1.9

A Mutual Fund Company has Rs. 20 lakhs available for investment in Government Bonds, Blue Chip Stocks, Speculative Stocks and Short-term Bank Deposits. The annual expected return and risk factor are given below:

Type of Investment	*Annual Expected Return %*	*Risk Factor (0 to 100)*
Government Bonds	14	12
Blue Chip Stocks	19	24
Speculative Stocks	23	48
Short-term Deposits	12	6

Mutual fund is required to keep at least 10% of funds in short-term deposits and not to exceed an average risk factor of 42, Speculative stocks must be at most 20 percent of the total amount invested.

Govt. Bonds must be at least 10% of Total Investment. Mutual Fund will not accept on Average Rate of Return below 15%.

How should mutual fund invest the funds so as to maximize its total expected annual return ? Formulate this as a Linear Programming Problem. Do not solve it.

Solution

Let x_1 x_2, x_3 and x_4 denote the amount of funds to be invested in Government Bonds, Blue chip Stocks, Speculative Stocks and Short term Deposits respectively. Let Z denote the Total Expected Return.

The average risk factor is given by—

$$\frac{12x_1 + 24x_2 + 48x_3 + 6x_4}{x_1 + x_2 + x_3 + x_4}$$

$$x_1 \geq 0.10\,(x_1 + x_2 + x_3 + x_4)$$

Since the average risk factor for Mutual Fund should not exceed 42, we get the following . constraint

$$\frac{12x_1 + 24x_2 + 48x_3 + 6x_4}{x_1 + x_2 + x_3 + x_4} \leq 42$$

or $\quad 12\,x_1 + 24\,x_2 + 48\,x_3 + 6\,x_4 \leq 42 \quad (x_1 + x_2 + x_3 + x_4)$

or $\quad 30x_1 - 18x_2 + 6x_3 - 36x_4 \leq 0$

Further, speculative stock must be at most 20 per cent of the total amount invested, hence

$$x_3 \leq 0.20\,(x_1 + x_2 + x_3 + x_4)$$

or $\quad -\,0.2\,x_1 - 0.2\,x_2 + 0.8\,x_3 - 0.2\,x_4 \leq 0$

For Minimum Average Rate of Return

$$0.14x_1 + 0.19x_2 + 0.23x_3 + 0.12x_4 \geq 0.15\,(x_1 + x_2 + x_3 + x_4)$$

$$-\,0.01x_1 + 0.04x_2 + 0.08x_3 + 0.03x_4 \geq 0 \text{ (Minimum Average Rate of Return)}$$

For Minimum Govt. Bonds

$$x_1 \geq 0.10\,(x_1 + x_2 + x_3 + x_4)$$

$$0.9x_1 - 0.1x_2 - 0.1x_3 - 0.1x_4 \geq 0$$

Since, the objective is to maximise the total expected annual return, the objective function for Mutual Fund is given by:

Maximise $\quad Z = 0.14\,x_1 + 0.19\,x_2 + 0.23\,x_3 + 0.12\,x_4$

Subject to the constraints:

$$x_1 + x_2 + x_3 + x_4 \leq 20{,}00{,}000 \text{ [Maximum Total Investment]}$$

$$x_4 \geq 2{,}00{,}000 \text{ [Minimum Short-term Deposit]}$$

$$-\,30x_1 - 18x_2 + 6\,x_3 - 36\,x_4 \leq 0 \text{ [Maximum Average Risk Factor]}$$

$$-\,0.2\,x_1 - 0.2\,x_2 + 0.8\,x_3 - 0.2\,x_4 \leq 0 \text{ [Maximum Speculative Stocks]}$$

$$0.9x_1 - 0.1x_2 - 0.3x_3 - 0.1x_4 \geq 0 \text{ [Maximum Govt. Bonds Investments]}$$

$$-\,0.01x_1 + 0.04x_2 + 0.08x_3 + 0.03x_4 \geq 0 \text{ (Minimum Average Rate of Return)}$$

where $x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0$ *and* $x_4 \geq 0$

PROBLEM 1.10

An investor has money making activities A_1, A_2, A_3 and A_4. He has only Rs. 1 lakh to invest. In order to avoid excessive investment, no more than 50% of the total investment can be placed in Activity A_2 and for Activity A_3. Activity A_1 is very conservative, while Activity A_4 is speculative. To avoid excessive speculation at least Re. 1 must be invested in Activity A_1 for every Rs. 3 invested in Activity A_4. The data on the return on investment are as follows:

Activity	***Anticipated return On investment (%)***
A_1	10
A_2	12
A_3	14
A_4	16

Investor wishes to know how much to invest in each activity to maximize the total return on the investment.

Solution

Let x_1, x_2, x_3 and x_4 represent the percentages of the total fund invested in securities A_1, A_2, A_3 and A_4 respectively. Since the objective of the investor is to maximize total return on the investment,

Let x_1, x_2 and x_3 be the number of acres allotted for cultivating radish, mutter and potato respectively.

Sales Revenue/acre (Radish) = Rs. 5 × 1500 = Rs. 7500

Sales Revenue/acre (Matter) = Rs. 4 × 1800 = Rs. = 7,200

Sales Revenue/acre (Patato) = Rs. 5 × 1200 = Rs. 6,000

Now, the selling price, manure cost, labour cost and profit per acre of land for will be as fallows:

Per acre	*Radish*	*Muttar*	*Potato*
A. Selling Price	Rs. 7,500	Rs. 7,200	Rs. 6,000
B. Manure Cost	$\frac{\text{Rs. } 12.50 \times 1500}{100}$	$\frac{\text{Rs. } 12.50 \times 1800}{100}$	$\frac{\text{Rs. } 12.50 \times 1200}{80}$
	= Rs. 187.50	= Rs. 225	= Rs. 187.50
C. Labour Cost	Rs.40 × 6 = Rs.240	Rs.40 × 5 = Rs.200	Rs.40 × 6 = Rs. 240
D. Profit [A – B – C]	= Rs.7,072.50	= Rs.6,775	= Rs. 5,572.50

Since, the agriculturist wants to maximise the total profit, hence the objective function of the problem is given by:

Maximise $Z = 7072.5x_1 + 6775x_2 + 5572.5x_3$

Subject to following constraints:

$x_1 + x_2 + x_3 \leq 125$ [Maximum Land Area]

$6x_1 + 5x_2 + 6x_3 \leq 500$ [Maximum Man Days]

where x_1, x_2 *and* $x_3 \geq 0$

$$Z = 10\% \text{ (rupees invested in } A_1) + 12\% \text{ (rupees invested in } A_2) + 14\% \text{ (rupees invested in } A_3) + 16\% \text{ (rupees invested in } A_4).$$

$$= 0.10\, x_1 + 0.12\, x_2 + 0.14\, x_3 + 0.16\, x_4$$

Since total money invested in activities is Rs. 1 lakh,

$$x_1 + x_2 + x_3 + x_4 \leq 1{,}00{,}000$$

Also money invested in activity A_2 and activity A_3 is less than or equal to 50% of money invested in four activities, therefore,

$$x_2 + x_3 \leq 0.50\ (x_1 + x_2 + x_3 + x_4)$$

or, $$-0.5x_1 + 0.5x_2 + 0.5x_3 - 0.5x_4 \leq 0$$

or, $$-x_1 + x_2 + x_3 - x_4 \leq 0$$

Further, the money invested in activity A_1 being greater than or equal to one-third of the money invested in activity A_4,

$$x_1 \geq 1/3x_4 \text{ or } \quad 3x_1 - x_4 \geq 0$$

Thus, the appropriate mathematical formulation of the LP problem is:

Maximize $Z = 0.10x_1 + 0.12x_2 + 0.14x_3 + 0.16x_4$

Subject to the constraints:

$x_1 + x_2 + x_3 + x_4 \leq 1{,}00{,}000$ (Funds Constraint)

$-x_1 + x_2 + x_3 - x_4 \leq 0$ (Maximum Constraint)

$3x_1 - x_4 \geq 0$ (Minimum Constraint)

$x_1, x_2, x_3 \geq 0$ (Non-Negativity Constraint)

PROBLEM 1.11

A leading Chartered Accountant is attempting to determine a ‘best’ investment portfolio and is considering six alternative investment proposals. The following table indicates point estimates for the price per share, the annual growth rate in the price per share, the annual dividend per share and a measure of the risk associated with each investment.

PORTFOLIO DATA

	Shares under consideration					
	A	*B*	*C*	*D*	*E*	*F*
Current price per share (Rs.)	80	100	160	120	150	200
Projected annual growth rate	0.08	0.07	0.10	0.12	0.09	0.15
Projected annual dividend per share (Rs.)	4.00	4.50	7.50	5.50	5.75	0.00
Projected risk in return	0.05	0.03	0.10	0.20	0.06	0.08

1. The total amount available for investment is Rs. 25 lakhs and the following conditions are required to be satisfied.
2. The maximum rupee amount to be invested in alternative F is Rs. 2,50,000.
3. No more than Rs. 5,00,000 should be invested in alternatives A and B combined.
4. Total weighted risk should not be greater than 0.10, where

$$\text{Total weighted risk} = \frac{\text{(Amount invested in alternative) (risk of alternative)}}{\text{Total amount invested in all the alternatives}}$$

5. For the sake of diversity, at least 100 shares of each stock should be purchased.
6. At least 10 percent of the total investment should be in alternatives A and B combined.
7. Dividends for the year should be at least Rs. 10,000.

Rupees return per share of stock is defined as price per share one year hence less current price per share PLUS dividend per share. If the objective is to maximise total rupee return, formulate the linear Programming model for determining the optimal number of shares to be purchased in each of the shares under consideration. You may assume that the time horizon for the investment is one year. The formulated LP problem is not required to be solved.

SOLUTION

Let x_1, x_2, x_3, x_4, x_5 and x_6 denote the number of shares to be purchased in each of the six investment proposals A, B, C, D, E and F.

Rupee return per share = Price per share one year hence
Less Current price per share
Plus dividend per share
= Current price per share × Projected annual growth rate
(i.e. Project growth for each share)
Plus dividend per share.

Thus, we get following table of data.

Investment alternatives	*A*	*B*	*C*	*D*	*E*	*F*
No. of shares purchased	x_1	x_2	x_3	x_4	x_5	x_6
Projected growth for each share (Rs.)	6.40	7.00	16.00	14.40	13.50	30.00
Projected annual dividend per share (Rs.)	4.00	4.50	7.50	5.50	5.75	0.00
Rupee return per share	10.40	11.50	23.50	19.90	19.25	30.00

The objective of the Chartered Accountant is to maximise the total rupee return, the objective function of the linear programming problem is given by:

Maximise $Z = 10.40x_1 + 11.50x_2 + 23.50x_3 + 19.90x_4 + 19.25x_5 + 30x_6$

Various constraints are as follows:

Since the total amount available for investment is Rs. 25 lakhs, hence

(I) $80x_1 + 100x_2 + 160x_3 + 120x_4 + 150x_5 + 200x_6 \le 25{,}00{,}000$

(II) $200x_6 \le 2{,}50{,}000$ [from condition (i)]

(III) $80x_1 + 100x_2 \le 5{,}00{,}000$ [from condition (ii)]

(IV) As per condition (iii) of the problem

≤ 0.10

or $4x_1 + 3x_2 + 16x_3 + 24x_4 + 9x_5 + 16x_6 \le 8x_1\ 10x_2 + 16x_3 + 12x_4 + 15x_5 + 20x_6$

or $-4x_1 - 7x_2 + 0x_3 + 12x_4 - 6x_5 - 4x_6 \le 0$

(V) Also $x_i \ge 100$, i = 1, 2, 3, 4, 5, 6 ...[from condition (iv)]

(VI) $80x_1 + 100x_2 \ge 0.10\,(80x_1 + 100x_2 + 160x_3 + 120x_4 + 150x_5 + 200x_6)$

[from condition (v)]

on simplification, we get

$80x_1 + 100x_2 \ge 8x_1 + 10x_2 + 16x_3 + 12x_4 + 15x_5 + 20x_6$

or $72x_1 + 90x_2 - 16x_3 - 12x_4 - 15x_5 - 20x_6 \ge 0$

(VII) $4x_1 + 4.5x_2 + 7.5x_3 + 5.5x_4 + 5.75x_5 \ge 10{,}000$ [from condition (vi)]

Combining all the constraints from I...VII, the linear programming problem becomes

Maximise $Z = 10.40x_1 + 11.50x_2 + 23.50x_3 + 19.90x_4 + 19.25x_5 + 30x_6$

Subject to the constraints:

$80x_1 + 100x_2 + 160x_3 + 120x_4 + 150x_5 + 200x_6 \leq 25{,}00{,}000$ (Maximum Funds)

$200x_6 \leq 2{,}50{,}000$ (Max. Investment in Proposal F)

$80x_1 + 100x_1 \leq 5{,}00{,}000$ (Max. Investment in Proposal A & B)

$-4x_1 - 7x_2 + 0x_3 + 12x_4 - 6x_5 - 4x_6 \leq 0$ (Maximum Risk)

xi, $x_2, x_3, x_4, x_5, x_6, \geq 100$ (Minimum No. of Shares)

$72x_1 + 90x_2 - 16x_3 - 12x_4 - 15x_5 - 20x_6 \geq 0$ (Minimum Investment in Proposal A & B)

$4x_1 + 4.5x_2 + 7.5x_3 + 5.50x_4 + 5.75x_5 + 0x_6 \geq 10{,}000$ (Minimum Dividend)

PROBLEM 1.12

An advertising firm desires to reach two types of audiences - customers with annual income of more than Rs. 40,000 (target audience A) and customers with annual income of less than Rs. 40,000 (target audience B), The total advertising budget is Rs. 2,00,000. One programme of T.V. advertising costs Rs. 50,000 and one programme of Radio advertising costs Rs. 20,000. Contract conditions ordinarily require that there should be at least 3 programmes on T.V. and the number of programmes on Radio must not exceed 5. Survey indicates that a single T.V. programme reaches 7,50,000 customers in target audience A and 1,50,000 in target audience B. One Radio programme reaches 40,000 customers in target audience A and 2,60,000 in target audience B.

Formulate this as a linear programming problem and determine the media mix to maximize the total reach using graphic method.

Solution

Let x_1 be the number of programmes of T.V. advertising and X_2 denote the number of programmes of radio advertising.

Since the advertising firm desires to determine the media mix to maximise the total reach, the objective function is given by

Maximise $Z = (7{,}50{,}000+1{,}50{,}000)\ x_1 + (40{,}000+2{,}60{,}000)\ x_2$

Subject to the constraints

$5\ x_1 + 2\ x_2 < 20$ [Maximum Adv. Budget]

$x_1 \geq 3$, [Minimum T.V. Programmes]

$x_2 \leq 5$ [Maximum Radio Programmes]

where $x_2, x_2 > 0$

Point of intersection for lines $x_1 = 3$ and $5x_1 + 2x_2 = 20$ is P(3, 5/2).

Similarly, lines $x_2 = 5$ and $5x_1 + 2x_2 = 20$ intersect at point (2, 5).

Line $x_1 = 0$ meets $5x_1 + 2x_2 = 20$ at point (0, 10)

Line $x_2 = 0$ meets $5x_1 + 2x_2 = 20$ at point Q(4, 0)

The graphical solution for the problem is given below:

The feasible region is given by the shaded area PQR, and the feasible points are *P*(3,5/2), *Q*(3, 0) and *R*(4, 0).

Value of the objective function Z at P(3, 5/2) is

$$9{,}00{,}000 \times 3 + 3{,}00{,}000 \times 5/2 = 27{,}00{,}000 + 7{,}50{,}000 = 34{,}50{,}000$$

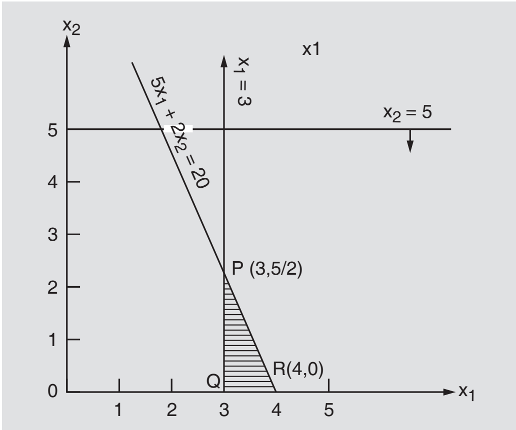


Fig. 1.1

Value of Z at Q(3, 0) is

$$9{,}00{,}000 \times 3 = 27{,}00{,}000$$

and value of Z at R(4, 0) is given by

$$9{,}00{,}000 \times 4 = 36{,}00{,}000$$

It can be seen that the value of Z is maximum at point R (4, 0). Thus, the solution to the given problem is:

$x_1 = 4$, $x_2 = 0$ and Maximum Z = 36,00,000.

In other words, the advertising firm should give 4 programmes on TV and no programme on Radio in order to achieve a maximum reach of 36,00,000 customers.

PROBLEM 1.13

Bharat Advertisers Ltd. is preparing a proposal for an advertising campaign for a client who are manufacturers of law products. An optimal allocation of advertising funds to maximise the total number of exposures has to be made for the client. The characteristics of the three alternative publications are shown in the following table.

	Home Beautiful	***Home and Garden***	***Law Care***
	Rs.	***Rs.***	***Rs.***
Cost Per Advertisement	12,000	16,000	900 0
Maximum no. of advertisements	12	24	12
Minimum no. of advertisements	3	6	2
Characteristics			
Homeowner	80%	70%	20%
Income : Rs. 10,000 or more	70%	80%	60%
Occupation : Gardener	15%	20%	40%
Audience size	1,00,000	50,000	2,00,000

The relative importance of the three characteristics are:

Homeowner, 0.4; income 0.2; gardener, 0.4. The advertising budget is Rs. 4,00,000. Formulate the LPP to find the most effective number of exposures in each magazine.

Solution

Let us first calculate, effective exposure

Media* *A	***Effectiveness Coefficient* *B***	***Audience Size* *C***	***Effective Exposure* *D = B 5 C***
1	0.80 (0.4) + 0.70 (0.2) + 0.15 (0.4) = 0.52	1,00,000	52,000
2	0.70 (0.4) + 0.80 (0.2) + 0.20 (0.4) = 0.52	50,000	26,000
3	0.20 (0.4) + 0.60 (0.2) + 0.40 (0.4) = 0.36	2,00,000	72,000

Let x_1, x_2 and x_3 represent the number of advertisement inserted in the three media 1, 2 and 3 respectively.

Since the objective is to maximize the effective exposure, the objective function is given by:

Maximize $Z = 52{,}000x_1 + 26{,}000x_2 + 72{,}000x_3$

Subject to:

$12{,}000x_1 + 16{,}000x_2 + 9{,}000x_3 \leq 4{,}00{,}000$ (Budget Amount constraint)

$x_1 \geq 3$ (Minimum No. of Advertisement in Home Beautiful constraint)

$x_2 \geq 6$ (Minimum No. of Advertisement in Home and Garden constraint)

$x_3 \geq 2$ (Minimum No. of Advertisement in Law Care constraint)

$x_1 \leq 12$ (Maximum Number of Advertisement in Home Beautiful constraint)

$x_2 \leq 24$ (Maximum Number of Advertisement in Home and Garden constraint)

$x_3 \leq 12$ (Maximum Number of Advertisement in Law Care constraint)

PROBLEM 1.14

The ABC Company, manufacturers of 'ZOLO', a leading brand of hair dye, are planning the media-mix for the next year within their and budget of Rs. 1,00,000. The characteristics of target audience for 'ZOLO' and weightages for each are as follows:

	Characteristics	***Weightage (%)***
Age	Over 35 years	50
Monthly income	Over Rs. 1,200	40
Education	Graduate and above	10

The audience characteristics for the three magazines under consideration are given below:

	Characteristic	***Magazine X(%)***	***Magazine Y(%)***	***Magazine Z(%)***
Age	Over 35 years	50	30	45
Monthly Income	Over Rs. 1,200	70	60	55
Education	Graduate and above	60	50	50

The cost per insertion and the readership for the three magazines are as follows:

Magazine	*Cost per insertion (Rs.)*	*Readership (in 000's)*
X (Monthly)	4,000	170
Y (Fortnightly)	3,500	220
Z (Monthly)	5,000	160

Also at least 2 insertions are necessary in X and Y to create an impact, whilst minimum 3 insertions will be required in case of Z.

Formulate a Linear Programming Model for the given problem to maximize the expected effective exposure.

Solution

Let x_1, x_2 and x_3 represent the number of advertisement inserted in the three magazines X, Y and Z respectively. In order to formulate the objective function, we shall first calculate the effectiveness coefficients as follows:

Media	*Effectiveness Coefficient*
Magazine X	0.50 5 0.50 + 0.70 5 0.40 + 0.60 5 0.10 = 0.59
Magazine Y	0.30 5 0.50 + 0.60 5 0.40 + 0.50 5 0.10 = 0.44
Magazine Z	0.45 5 0.50 + 0.55 5 0.40 + 0.50 5 0.10 = 0.495

The effective exposures for all the three media employed can be computed as follows:

Effective exposure = Effectiveness coefficient $\times$ Audience size, where effectiveness coefficient is a weighted average of audience characteristics. Thus effective exposure for

$$\text{Magazine (X)} = 0.59 \times 170{,}000 = 1{,}00{,}300$$

$$\text{Magazine (Y)} = 0.44 \times 220{,}000 = 96{,}800$$

$$\text{Magazine (Z)} = 0.495 \times 160{,}000 = 79{,}200$$

Now the objective function and constraints of the linear programming model can be written as:

$$\text{Maximize } Z = 1{,}00{,}300x_1 + 96{,}800x_2 + 79{,}200x_3$$

Subject to the constraints:

$4{,}000x_1 + 3{,}500x_2 + 5{,}000x_3 \leq 1{,}00{,}000$ (Budget constraint)

$x_1 \geq 2,\ x_2 \geq 2,\ x_3 \geq 3$ (Minimum number of advertisements allowed constraints)

$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0$ (Non-Negetivity constraints)

PROBLEM 1.15

Computer Company produces three types of models, which are first required to be machined and then assembled. The time (in hours) for these operations for each model is given below:

Model	*Machine Time*	*Assembly Time*
PIII	20	5
P II	15	4
Celeron	12	3

The total available machine time and assembly time are 1,000 hours and 1,500 hours respectively. The selling price and other variable costs for three models are:

	PIII	*PII*	*Celeron*
Selling Price (Rs.)	3,000	5,000	15,000
Labour, Material and other Variable Costs (Rs.)	2,000	4,000	8,000

The company has taken a loan of Rs. 50,000 from a Nationalised Bank, which is required to be repaid on 1.4.2011. In addition, the company has borrowed Rs. 1,00,000 from XYZ Cooperative Bank. However, this bank has given its consent to renew the loan.

The Balance Sheet of the company as on 31.3.2011 is as follows:

Liabilities	*Rs.*	*Assets*	*Rs*
Equity Share Capital	1,00,000	Land	80,000
Capital reserve	20,000	Buildings	50,000
Profit & Loss Account	30,000	Plant & Machinery	1,00,000
Long-term Loan	2,00,000	Furniture etc.	20,000
Loan from XYZ		Vehicles	40,000
Cooperative Bank	1,00,000	Cash	2,10,000
Loan from National Bank	50,000		
	5,00,000		5,00,000

The company is required to pay a sum of Rs. 15,000 towards the salary. Interest on long-term loan is to be paid every month @ 18% per annum. Interest on loan from XYZ Cooperative and Nationalised Banks may be taken as Rs. 1,500 per month. The company has already promised to deliver three P III, Two P II and five Celeron Type of Computer to M/s ABC Ltd. next month. The level of operation in the company is subject to the availability of cash next month.

The Company Manager is willing to know that how many units of each model must be manufactured next month, so as to maximize the profit.

Required: Formulate a linear programming problem for the above.

Solution

Let X_1, X_2 and X_3 denote the number of P Hi, P II and Celeron computers respectively to be manufactured in the company. The following data is given:

	PIII	*PII*	*Celeron*
Selling Price per unit (Rs.)	3,000	5,000	15,000
Labour, Material & Other variable cost per unit (Rs.)	2,000	4,000	8,000
Profit per unit (Rs.)	1,000	1,000	7,000

Calculation of cash availability for the next month from the balance sheet:

Cash balance (Rs. 2,10,000)

Loan to repay to Nationalized bank (Rs. 50,000)

Interest on loan from XYZ cooperative bank and Nationalized bank (Rs. 1500)

Interest on long term loans $\left[\dfrac{0.18 \times 2,00,000}{12}\right] = \text{Rs. } 3,000$

Salary to staff (Rs. 15,000)

Or, Cash availability = Rs. 2,10,000 – (Rs. 50,000 + Rs. 1,500 + Rs. 3,000 + Rs. 15,000)

= Rs. 1,40,500

Since the company wants to maximize the profit, hence the objective function is given by:

Maximize $Z = 1000X_1 + 1000X_2 + 7000X_3 -$ (Rs. 15,000 + Rs. 3,000, + Rs. 1,500) Subject to the constraints:

$20X_1 + 15X_2 + 12X_3 < 1000$ (Machine Time Restriction)

$5X_1 + 4X_2 + 3X_3 < 1500$ (Assembly Time Restriction)

$2000X_1 + 4000X_2 + 8000X_3 <$ Rs. 1,40,500 (Cash Requirements & Availability)

$X_1 \geq 3$, $X_2 > 2$, $X_3 > 5$ (Minimum Deliveries)

X_1, X_2 and $X_3 \geq 0$

PROBLEM 1.16 [Capital Mix Problem]

The most recent audited summarised Balance Sheet of Stop and Shop Financial Services is given below:

Balance Sheet as on March 31, 20X1

Liabilities	*(Rs. in lakhs)*	*Assets*	*(Rs. in lakhs)*
Equity Share Capital	65	**Fixed Assets:**	
Reserves & Surplus	110	— Assets on Leave	
Term Loan from IFCI	80	(original cost: Rs. 550 lakhs)	375
Public Deposits	150	— Other Fixed Assets	50
Bank Borrowings	147	Investments (on wholly owned	
Other Current Liabilities	50	subsidiaries)	20
		Current Assets:	
		— Stock on Hire	80
		— Receivables	30
		— Other Current Assets	35
		Miscellaneous expenditure	
		(not written off)	12
	602		**602**

The company intends to enhance its investment in the lease portfolio by another Rs. 1,000 lakhs. For this purpose it would like to raise a mix of debt and equity in such a way that the overall cost of raising additional funds is minimised. The following constraints apply to the way the funds can be mobilised:

1. Total debt divided by net owned funds, cannot exceed 10.
2. Amount borrowed from financial institutions cannot exceed 25% of the net worth.
3. Maximum amount of bank borrowings cannot exceed three times the net owned funds.
4. The company would like to keep the total public deposit limited to 40% of the total debt.

The post-tax costs of the different sources of finance are as follows :

Equity	:	25%
Term Loans	:	8.5%
Public Deposits	:	7%
Bank Borrowings	:	10%

Formulate the funding problem as a LPP.

Note: (a) Total Debt = Term Loans from Financial Institutions + Public Deposits + Bank Borrowings

(b) Net Worth = Equity Share Capital + Reserve & Surplus

(c) Net Owned Funds= Net Worth – Miscellaneous Expenditures

Solution

Let x_1, x_2, x_3 and x_4 be the quantum of additional funds (all figures here are in lakhs) raised by way of additional equity, additional term loans, additional public deposits and additional bank borrowings respectively. The objective function to minimise the cost of additional funds raised by the company is as follows:

Minimize $Z = 0.25x_1 + 0.085x_2 + 0.07x_3 + 0.1x_4$

subject to the following constraints:

(1) $$\frac{\text{Total debt}}{\text{Net owned funds}} \leq 10$$

or $$\frac{(\text{Existing debt} + \text{Additional debt})}{(\text{Equity share capital} + \text{Reserves \& Surplus})} \leq 10$$

or $$\frac{(80 + 150 + 147 + x_2 + x_3 + x_4)}{(65 + 110 + x_1) - 12} \leq 10$$

or $$\frac{x_2 + x_3 + x_4 + 377}{(x_1 + 163)} \leq 10$$

or $x_2 + x_3 + x_4 + 377 \leq 10x_1 + 1630$

or $-10\,x_1 + x_2 + x_3 + x_4 \leq 1253$

(2) Amount borrowed (financial institutions) ≤ 25% of net worth

or (Existing long term loan from financial institutions + Additional loan) ≤ 25% (Existing Equity Capital + Reserves & Surplus + Addl. Equity Capital)

or $(80 + x_2) \leq 0.25\,(175 + x_1)$

$(80 + x_2) \leq 1/4\,(175 + x_1)$

PROBLEM 1.17

The costs and selling prices per unit of two products manufacturing by a company are as under:

Product	*A (Rs.)*	*B (Rs.)*
Selling Price	500	450
Variable costs:		
Direct Materials @ Rs. 25 per kg.	100	100
Direct Labour @ Rs. 20 per hour	80	40
Painting @ Rs. 30 per hour	30	60
Variable overheads	190	175
Fixed Costs @Rs. 17.50/D.LHr.	70	35
Total Costs	470	410
Profit	30	40

In any month the maximum availability of inputs is limited to the following:

Direct Materials	480 kg.
Direct Labour hours	400 hours
Painting hours	200 hours

Required:

(I) Formulate a linear programme to determine the production plan which maximizes the profits by using graphical approach.

(ii) State the optimal product mix and the monthly profit derived from your solution in (i) above.

If the company can sell the painting time at Rs. 40 per hour as a separate service, show what modification will be required in the formulation of the linear programming problem. You are required to re-formulate the problem but not to solve.

Solution

Contribution Analysis

Products	*A (Rs.)*	*B (Rs.)*
A. Selling Price	500	450
B. Variable Costs:		
Direct Materials	100	100
Direct Labour	80	40
Painting	30	60
Variable Overheads	190	115
Total Variable Costs	400	375
C. Contribution (A – B)	100	175
Direct Material per unit	100/25 = 4 kg.	100/25 = 4 kg.
Direct Labour hour per unit	80/20 = 4 hours	40/20 = 2 hours
Painting hour per unit	30/30 = 1 hour	60/30 = 2 hours

Let A be the units to be produced of Product A and B be the units to be produced of Product B.

LP Problem formulation:

Max Z $= 100A + 75B$ Maximisation of contribution

Subject to:

$4A + 4B \leq 480$ Raw material constraint

$4A + 2B \leq 400$ Direct Labour hour constraint

$A + 2B \leq 200$ Painting hour constraint

$A, B \geq 0$ Non negativity constraint

Raw Material Constraint: Put $B = 0, A = 120$

Put $A = 0, B = 120$

Direct Labour Constraint: Put $B = 0, A = 100$

Put $A = 0, B = 200$

Painting Constraint: Put $B = 0, A = 200$

Put $A = 0, B = 100$

The graphical representation will be as under:

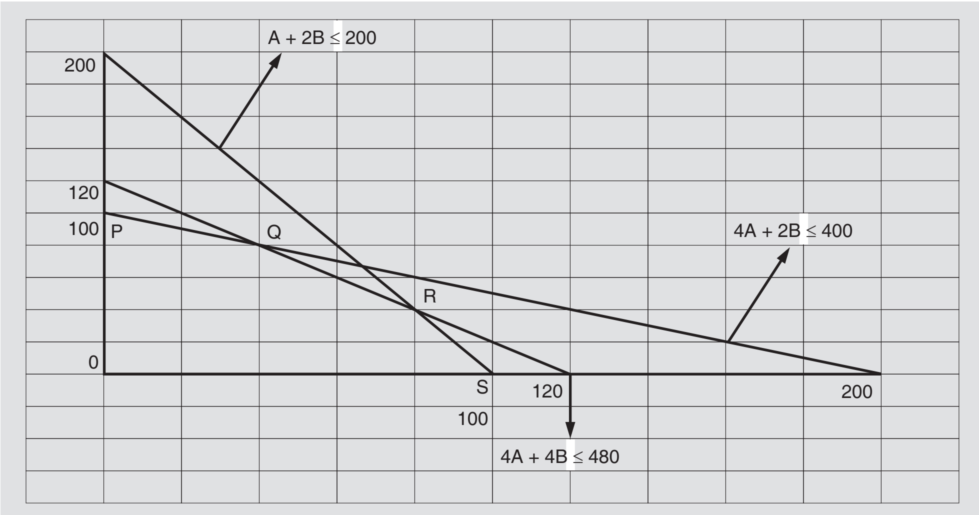


Fig.1.2

Q Intersects	$4A + 2B = 400$	...(1)
and	$4A + 4B = 480$	...(2)

Subtracting (2) from (1), we get $-2B = -80$

$\Rightarrow$ $B = 80/2 = 40$

Putting value of B in (1). we get $4A + 2 \times 40 = 400$

$\Rightarrow$ $A = \dfrac{400 - 80}{4} = 80$

R Intersects	$4A + 4B = 480$	...(3)
and	$A + 26 = 200$	...(4)

Multiplying (4) by (2) and then subtracting from (3), we get

$2A = 80$

$\Rightarrow$ $A = 40$

Putting the value of A in (4), we get $2B = 200 - 40$

$\Rightarrow$ $B = 80.$

Evaluation of Corner Points:

Point	*Products*		*Contribution*		*Total Contribution*
	A	B	A (Rs.) 100 perunit	B (Rs.) 75 per unit	Rs.
P	0	100	0	7,500	7,500
Q	80	40	8,000	3,000	11,000
R	40	80	4,000	6,000	10,000
S	100	0	10,000	0	10,000

Optimal Product Mix is Q

Product	*Units*	*Contribution Rs.*
A	80	8,000
B	40	3,000
Total Contribution		11,000
Less: Fixed costs 400 D.L. Mrs. × Rs. 17.50		7,000
Optimal Profit		4,000

(iii) If the painting time can be sold at Rs. 40 per hour the opportunity cost is calculated as under:

	A (Rs.)	*B (Rs.)*
Income from Sale per hour	40	40
Painting variable Cost per hour	30	30
Opportunity Cost	10	10
Painting hours per unit	1	2
Opportunity Cost	10	20
Revised Contribution	100 – 10 = 90	75 – 20 = 55

Hence, modification is required in the objective function.

Re-formulated problem will be:

Z Max.	$90A + 55B$	Maximisation of contribution Subject to:
	$4A + 4B \leq 480$	Raw Material constraint
	$4A + 2B \leq 400$	Direct Labour hour constraint
	$A + 2B \leq 200$	Painting hour constraint
	$A, B \geq 0$	Non-negativity constraint

PROBLEM 1.18

Transport Ltd. provides tourist vehicles of 3 types - 20-seater vans, 8-seater big cars and 5-seater small cars. These seating capacities are excluding the drivers. The company has 4 vehicles of the 20-seater van type, 10 vehicles of the 8-seatcr big car types and 20 vehicles of the 5-seater small car types. These vehicles have to be used to transport employees of their client company from their residences to their offices and back. All the residences are in the same housing colony. The offices are at two different places, one is the Head Office and the other is the Branch. Each vehicle plies only one round trip per day, if residence to office in the morning and office to residence in the evening. Each day, 180 officials need to be transported in Route I (from residence to Head Office and back) and 40 officials need to be transported in Route II (from Residence to Branch office and back). The cost per round trip for each type of vehicle along each route is given below.

You are required to formulate the information as a linear programming problem, with the objective of minimising the total cost of [hiring vehicles] for the client company, subject to the constraints mentioned above, (only formulation is required. Solution is not needed).

	20-Seater Vans	8-Seater Big Cars	Figs. – Rs. /round trip 5-seater Small Cars
Route I— Residence —Head Office and Back	600	400	300
Route II— Residence—Branch Office and Back	500	300	200

Solution

(a) Type	I 20 – Seater vans	II 8 – Sealer Big cars	III 5 – Seater Small cars	Total no. of Passengers
Route I Residence H.O. Residence	600	400	300	180
Route II Residence Br. Residence	500	300	200	40
No. of Vehicles	4	10	20	
Max. Capacity	80	80	100	220
			260	
No. of Passengers				

Let i be the ith route,
and j be the type of vehicle, so that
S_{11} = no. of vans (vehicles on Route I, Type I)
S_{12} = no. of 8 seater cars on Route I
S_{13} = no. of 5 seater cars on Route I
S_{21} = no. of vans — on Route II
S_{22} = no. of 8 seater cars on Route II
S_{23} = no. of 5 seater cars on Route II

LPP

Minimise Cost $Z = 600\ S_{11} + 400\ S_{12} + 300\ S_{13} + 500\ S_{21} + 300\ S_{22} + 200\ S_{23}$

Subject to

$20\ S_{11} + 8\ S_{12} + 5\ S_{13} = 180$

$20\ S_{21} + 8\ S_{22} + 5\ S_{23} = 40$

$S_{11} + S_{21} \leq 4$

$S_{21} + S_{22} \leq 10$

$S_{31} + S_{32} \leq 20$

All Sij ≥ 0

PROBLEM 1.19

The following matrix gives the unit cost of transporting a product from production plants P_1, P_2 and P_3 to destinations. D_1, D_2 and D_3. Plants P_1, P_2 and P_3 have a maximum production of 65, 24 and 111 units respectively and destinations D_1, D_2 and D_3 must receive at least 60, 65 and 75 units respectively:

To *From*	D_1	D_2	D_3	*Supply*
P_1	400	600	800	65
P_2	1,000	1,200	1,400	24
P_3	500	900	700	111
Demand	60	65	75	200

You are required to formulate the above as a linear programming problem. (Only formulation is needed. Please do not solve).

Solution

(b) Let p_i, d_j, be the variable to denote the number of units of product from the ith plant to this jth destination, so that

$$P_1\,d_1 = \text{transport from plant } P_1 \text{ to } D_1$$

$$P_2\,d_2 = \text{transport from plant } P_2 \text{ to } D_2 \text{ etc.}$$

Objective function

$$\text{Minimize } Z = 400\,p_1d_1 + 600\,p_1d_2 + 800\,p_1d_3 + 1000\,p_2d_1 + 1200\,p_2d_2 + 1400\,p_2d_3 + 500\,p_3d_1 + 900\,p_3d_2 + 700\,p_3d_3.$$

Subject to:

$$\left.\begin{array}{r} p_1d_1 + p_1d_2 + p_1d_3 \le 65 \\ p_2d_1 + p_2d_2 + p_2d_3 \le 24 \\ p_3d_1 + p_3d_2 + p_3d_3 \le 111 \end{array}\right\} \text{(Plant constraints)}$$

and

$$\left.\begin{array}{r} p_1d_1 + p_2d_1 + p_3d_1 \le 60 \\ p_1d_2 + p_2d_2 + p_3d_2 \le 65 \\ p_1d_3 + p_2d_3 + p_3d_3 \le 75 \end{array}\right\} \text{(destination constraints)}$$

all $p_1d_1 \ge 0$

PROBLEM 1.20 [Transportation Problem]

A company is to subcontract work on four assemblies. The five subcontractors have agreed to submit a bid price on each assembly type and a limit on the total number of assemblies (if any combination) for which they are willing to contract. These bids, the contract times, and time requirements for assemblies are given in the following matrix.

		Subcontractor					
		A	***B***	***C***	***D***	***E***	***Assemblies Required***
	1	10	11	12	13	14	500
	2	11	12	11	10	9	300
Assembly	3	12	13	8	9	10	300
	4	13	8	9	10	11	400
Contract Limits		250	280	330	360	380	

Formulate the LP model.

Solution

Let x_{11} be Assembly 1 to Subcontractor A, x_{12} be assembly to Subcontractor B and so on.

		Subcontractor					
		A	***B***	***C***	***D***	***E***	***Assemblies Required***
	1	10 x_{11}	11 x_{12}	12 x_{13}	13 x_{14}	14 x_{15}	500
Assembly	2	11 x_{21}	12 x_{22}	11 x_{23}	10 x_{24}	9 x_{25}	300
	3	12 x_{31}	13 x_{32}	8 x_{33}	9 x_{34}	10 x_{35}	300
	4	13 x_{41}	8 x_{42}	9 x_{43}	10 x_{44}	11 x_{45}	400
Contract Limits		250	280	330	360	380	

It is a minimization problem since the objective is to minimize the cost of sub-contract price of the assembly. Hence, objective function is given by—

$$
\begin{aligned}
\text{Minimum} \quad Z = {} & 10x_{11} + 11x_{12} + 12x_{13} + 13x_{14} + 14x_{15} \\
& + 11x_{21} + 12x_{22} + 11x_{23} + 10x_{24} + 9x_{25} \\
& + 12x_{31} + 13x_{32} + 8x_{33} + 9x_{34} + 10x_{35} \\
& + 13x_{41} + 8x_{42} + 9x_{43} + 10x_{44} + 11x_{45}
\end{aligned}
$$

Subject to the following constraints:

$x_{11} + x_{21} + x_{31} + x_{41} \leq 250$ (No. of Contract limits)

$x_{12} + x_{22} + x_{32} + x_{42} \leq 280$ (No. of Contract limits)

$x_{13} + x_{23} + x_{33} + x_{43} \leq 330$ (No. of Contract limits)

$x_{14} + x_{24} + x_{34} + x_{44} \leq 360$ (No. of Contract limits)

$x_{15} + x_{25} + x_{35} + x_{45} \leq 380$ (No. of Contract limits)

$x_{11} + x_{12} + x_{13} + x_{14} + x_{15} \leq 500$ (No. of Assemblies required)

$x_{21} + x_{22} + x_{23} + x_{24} + x_{25} \leq 300$ (No. of Assemblies required)

$x_{31} + x_{32} + x_{33} + x_{34} + x_{35} \leq 300$ (No. of Assemblies required)

$x_{41} + x_{42} + x_{43} + x_{44} + x_{45} \leq 400$ (No. of Assemblies required)

$x_{ij} \geq 0$ for all i and j (Non-Negativity constraint)

PROBLEM 1.21

North-East Aircraft Company, which operates out of a central terminal, has 8 aircraft of Type I, 15 aircraft of Type II, and 12 aircraft of Type III available for today's flights. The tonnage capacities (in thousands of tons) are 4.5 for Type I, 7 for Type II and 4 for Type III.

The company dispatches its planes to cities A and B. Tonnage requirements (in thousands of tons) are 20 at city A and 30 at city B; excess tonnage capacity supplied to a city has no value. A plane can fly once only during the day.

The cost of sending a plane from the terminal to each city is given by the following table:

	Type I	*Type II*	*Type III*
City A	23	5	1.4
City B	58	10	3.8

Formulate the LP model to minimise the air-transportation cost.

Solution

Let x_{11} be the I planes coming to A, x_{12} the II plane coming to A; and so on.

	Aircraft Type			
	I	*II*	*III*	
	8	15	12	(Nos.)
	4.5	7	4	(Capacity in tons)
Tons A(20)	x_{11}	x_{12}	x_{13}	Cost matrix
	23	5	1.4	
Requirements B(30)	x_{21}	x_{22}	x_{23}	(Cost of sending a plane)
	58	10	3.8	

$$\begin{aligned}
\text{Min. } Z = \; & 23x_{11} + 5x_{12} + 1.4x_{13} + 58x_{21} + 10x_{22} + 3.8x_{23} \\
& 4.5x_{11} + 7x_{12} + 4x_{13} \geq 20 \\
& 4.5x_{21} + 7x_{22} + 4x_{23} \geq 30 \\
& x_{21} + x_{11} \leq 8 \\
& x_{22} + x_{12} \leq 15 \\
& x_{23} + x_{72} \leq 12 \\
& xij \geq 0
\end{aligned}$$

PROBLEM 1.22 [Assignment Problem]

A company has two grades of inspectors, 1 and 2 to undertake quality control inspection. At least 3,500 pieces must be inspected in an 8 hour day. Grade 1 inspector can check 50 pieces in an hour with an accuracy of 95%. Grade 2 inspector checks 25 pieces an hour with an accuracy of 90%.

The daily wages of grade 1 inspectors are Rs. 6 per hour while those of grade 2 inspectors are Rs. 5 per hour. Any error made by an inspector costs Rs. 4 per piece to the company. If there are, in all, 20 grade 1 inspectors and 25 grade 2 inspectors in the company, find the optimal assignment of inspectors that minimizes the daily inspection cost. Formulate the LP problem.

Solution

Let x_1 and x_2 be the number of grade 1 and 2 inspectors

Calculation of Inspection Cost of Inspectors

	Grade 1 (x_1)	Grade 2 (x_2)
A. Wage rate per hour	Rs. 6	Rs. 5
B. Inspection error cost per piece	Rs. 4	Rs. 4
C. Probability of Inspection Error	5%	10%
D. No. of pieces checked per hour	50	25
E. Cost of Inspection Error per hour [B 5 C 5 D]	Rs. 10	Rs. 10
F. Total Inspection cost per hour [A + E]	Rs. 16	Rs. 15
G. Total Inspection cost per day [F 5 8]	Rs. 128	Rs. 120

The above inspection problem can now be formulated in an appropriate mathematical from as follows:

Minimize (Daily inspection cost) $Z = 128x_1 + 120x_2$

Subject to the constraints:

$400x_1 + 200x_2 \geq 3500$ (Inspection pieces constraint)

$x_1 \leq 20$ (Grade 1 inspectors constraint)

$x_2 \leq 25$ (Grade 2 inspectors constraint)

$x_1, \ x_2 \geq 0$ (Non-Negetivity constraint)

PROBLEM 1.23 [Trim Problem]

The Fine Paper Company produces rolls of paper used in cash registers. Each roll of paper is 500 ft. in length and can be produced in widths of 1, 2, 3 and 5 inch. The company's production process results in 500' rolls that are 12 inches in width. Thus, the company must cut its 12 inch roll to the desired widths. It has six basic cutting alternatives as follows:

Cutting Alternative	*No . of Rolls*				*Waste (inches)*
	1"	*2"*	*3"*	*5"*	
1	6	3	0	0	0
2	0	3	2	0	0
3	1	1	1	1	1
4	0	0	2	1	1
5	5	2	1	0	1
6	4	2	1	0	1

The minimum demand requirements for the four rolls are as follows:

Roll Width (inches)	*Demand Requirements (Rolls)*
1	6000
2	4000
3	3000
5	2000

The company wishes to minimise the wastes generated by its production process, while meeting its demand requirements. Formulate the LP model.

Solution

Let x_j be the number of times cutting alternative (j = 1, 2, ... 6) is employed.

Since the objective is to minimize the waste generated, the objective function is given by—

Minimise (waste produced) $Z = 1x_3 + 1x_4 + 1x_5 + 1x_6$

Subject to constraints:

$6x_1 + 1x_3 + 5x_5 + 4x_6 \geq 6000$ (Minimum Demand Requirement of 1” Roll)

$3x_1 + 3x_2 + 1x_3 + 4x_5 + 2x_6 \geq 4000$ (Minimum Demand Requirement of 2” Roll)

$2x_2 + 1x_3 + 2x_4 + 1x_5 + 1x_6 \geq 3000$ (Minimum Demand Requirement of 3” Roll)

$1x_3 + 1x_4 \geq 2000$ (Minimum Demand Requirement of 5” Roll)

$x_j \geq 0$, for all j (Non-Negativity constraint)

PROBLEM 1.24 [Job Schedulings]

A big hospital has the following minimal daily requirements of doctors:

Period	***Clocktime (24 hours.)***	***No. of doctors required***
1	6 a.m. - 10 a.m.	72
2	10 a.m. - 2 p.m.	77
3	2 p.m. - 6 p.m.	85
4	6 p.m. - 10 p.m.	68
5	10 p.m. - 2 a.m.	25
6	2 a.m. - 6 a.m.	23

Doctors report to the hospital at the beginning of each period and work for 8 consecutive hours. Formulate this problem as a linear programming problem to minimise the total number of doctors to meet the needs of the hospital throughout the day.

Solution

Let x_1 represent the number of doctors required at starting period k (k = 1, 2, ..., 6). Then the objective function and constraints can be written as:

Minimise (total number of doctors)

$$Z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$$

Subject to the constraints:

(i) $$\begin{pmatrix}\text{Number of doctors}\\ \text{starting period 1}\end{pmatrix} + \begin{pmatrix}\text{Number of doctors that started}\\ \text{at period 6 and still on duty}\end{pmatrix} \geq 72$$

or, $x_1 + x_6 \geq 72$

(ii) $$\begin{pmatrix}\text{Number of doctors}\\ \text{starting period 2}\end{pmatrix} + \begin{pmatrix}\text{Number of doctors that started}\\ \text{at period 1 and still on duty}\end{pmatrix} \geq 77$$

or, $x_2 + x_1 \geq 77$

(iii) $$\begin{pmatrix}\text{Number of doctors}\\ \text{starting period 3}\end{pmatrix} + \begin{pmatrix}\text{Number of doctors that started}\\ \text{at period 2 and still on duty}\end{pmatrix} \geq 85$$

or, $x_3 + x_2 \geq 85$

(iv) $$\begin{pmatrix}\text{Number of doctors}\\ \text{starting period 4}\end{pmatrix} + \begin{pmatrix}\text{Number of doctors that started}\\ \text{at period 3 and still on duty}\end{pmatrix} \geq 68$$

or, $x_4 + x_3 \geq 68$

(v) $$\begin{pmatrix}\text{Number of doctors}\\ \text{starting period 5}\end{pmatrix} + \begin{pmatrix}\text{Number of doctors that started}\\ \text{at period 4 and still on duty}\end{pmatrix} \geq 25$$

or, $x_5 + x_4 \geq 25$

(vi) $$\begin{pmatrix}\text{Number of doctors}\\ \text{starting period 6}\end{pmatrix} + \begin{pmatrix}\text{Number of doctors that started}\\ \text{at period 5 and still on duty}\end{pmatrix} \geq 23$$

or, $x_6 + x_5 \geq 23$

Alternative Solution:

Minimize $Z = x_1 + x_2 + x_3 + x_4 + x_5 + x_6$

Subject to constraints:

$x_1 + x_2 \geq 72$

$x_2 + x_3 \geq 77$

$x_3 + x_4 \geq 85$

$x_4 + x_5 \geq 68$

$x_5 + x_6 \geq 25$

$x_6 + x_1 \geq 23$

$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0,\ x_4 \geq 0,\ x_5 \geq 0,\ x_6 \geq 0,$

Note: There also exist other alternative solutions.

MISCELLENEOUS SOLVED PROBLEMS

PROBLEM 1.25

Piyush is an aspiring freshman at Delhi University. He realizes that "all work and no play make Piyush a dull boy." As a result, Piyush wants to apportion his available time of about 10 hours a day between work and play. He estimates that play is twice as much fun as work. He also wants to study at least as much as he plays. However, he realizes that if he is going to get all his homework assignments done, he cannot play more than 4 hours a day. How should Piyush allocate his time to miximize his pleasure from both work and play?

Solution

Let x_1 = play hours per day, x_2 = work hours per day

Maximize $Z = 2x_1 + x_2$

Subject to,

$x_1 + x_2 \le 10$ [Maximum Total Time constraint]

$x_1 + x_2 \le 0$ [Minimum Work Hours constraint]

$x_1 \le 4$ [Maximum Play Hours constraint]

$x_1, x_2 \ge 0$ [Non negativity constraint]

PROBLEM 1.26

An assembly line consisting of three consecutive stations produces two radio models: HiFi-1 and HiFi-2. The following table porvides the assembly times for the three workstations.

	Minutes per unit	
Workstation	*HiFi-1*	*HiFi-2*
1	6	4
2	5	5
3	4	6

The daily maintenance for stations 1, 2, and 3 consumes 10%, 14%, and 12%, respectively, of the maximum 480 minutes available for each station each day. Determine the optimal product mix that will minimize the idle (or unused) time in the three workstations.

Solution

Let x_1 = Number of HiFi 1 units, x_2 = Number of HiFi 2 units

Minimize $Z = 1267.2 - (15x_1 + 15x_2)$

As, Total Minutes per unit of HiFi 1 = 6 + 5 + 4 = 15 minutes &

Total Minutes per unit of HiFi 2 = 4 + 5 + 6 = 15 minutes

Total Available Time = 432 + 412.8 + 422.4 = 1267.2 minutes

Subject to,

$6x_1 + 4x_2 \le 432$ Ist constraint [480 – 10% of 480] [Time constraint]

$5x_1 + 5x_2 \le 412.8$ IInd constraint [480 – 14% of 480] [Time constraint]

$4x_1 + 6x_2 \le 422.4$ IIIrd constraint [480 – 12% of 480] [Time constraint]

$x_1, x_2 \ge 0$ [Non negativity constraint]

PROBLEM 1.27

A city will undetake five urban renewal housing projects over the next five years. Each project has a different starting year and a different duration. The following table provides the basic data of the situation:

	Year 1	*Year 2*	*Year 3*	*Year 4*	*Year 5*	*Cost (million $)*	*Annual income (million $)*
Project 1	Start		End			5.0	.05
Project 2		Start			End	8.0	.07
Project 3	Start				End	15.0	.15
Project 4			Start	End		1.2	.02
Budget (million $)	3.0	6.0	7.0	7.0	7.0		

Projects 1 and 4 must be finished completely within their durations. The remaining two projects can be finished partially within budget limitations, if necessary. However, each project must be at least 25% completed within its duration. At the end of each year, the completed section of a project is immediately occupied by tenants and a proportional amount of income is realized.

Solution

Let x_{11} = Portion of project 1 completed in year 1
x_{12} = Portion of project 1 competed in year 2
x_{13} = Portion of project 1 completed in year 3 [Project End]
x_{22} = Portion of project 2 completed in year 2 [As Project 2 starts in year 2]
x_{23} = Portion of project 2 completed in year 3
x_{24} = Portion of project 2 completed in year 4
x_{25} = Portion of project 2 completed in year 5
x_{31} = Portion of project 3 completed in year 1
x_{32} = Portion of project 3 completed in year 2
x_{33} = Portion of project 3 completed in year 3
x_{34} = Portion of project 3 completed in year 4
x_{35} = Portion of project 3 completed in year 5
x_{43} = Portion of project 4 completed in year 3 [As project 4 starts in year 3]
x_{44} = Portion of project 4 completed in year 4 [Project End]

Note: Year 5 will not be considered, for revenue for revenue come in next year and we are considering 5 year horizon. Now optimum function of project 1 is

$$\underbrace{[x_{11} \times 0.05]}_{\text{Year 2}} + \underbrace{[(x_{11} + x_{12})\ 0.05]}_{\text{Year 3}} + \underbrace{[(x_{11} + x_{12} + x_{13})\ 0.05]}_{\text{Year 4}} + \underbrace{[(x_{11} + x_{12} + x_{13})\ 0.05]}_{\text{Year 5}}$$

i.e. $$0.05\,[x_{11} + (x_{11} + x_{12}) + (x_{11} + x_{12} + x_{13}) + (x_{11} + x_{12} + x_{13})]$$

$$= 0.05(4x_{11} + 3x_{12} + 2x_{13})$$

Similarly for project 2, 3 and 4

$$0.07\,(3x_{22} + 2x_{23} + x_{24})$$

$$0.15\,(4x_{31} + 3x_{32} + 2x_{33} + x_{34})$$

$$0.02\,(2x_{43} + x_{44}) \text{ respectively}$$

$$\text{Maximize } Z = 0.05\,(4x_{11} + 3x_{12} + 2x_{13}) + 0.07\,(3x_{22} + 2x_{23} + x_{24}) + 0.15\,(4x_{31} + 3x_{32} + 2x_{33} + x_{34}) + 0.02\,(2x_{43} + x_{44})$$

Subject to

$x_{11} + x_{12} + x_{13} = 1$ I[st] constraint [Project 1 to be completed at given time]

$x_{43} + x_{44} = 1$ II[nd] constraint [Project 4 to be completed at given time]

$x_{22} + x_{23} + x_{24} + x_{25} \geq 0.25$ III[rd] constraint [Minimum completion constraint]

$x_{22} + x_{23} + x_{24} + x_{25} \leq 1$ IV[th] constraint

$x_{31} + x_{32} + x_{33} + x_{34} + x_{35} \geq 0.25$ V[th] constraint [Minimum completion constraint]

$x_{31} + x_{32} + x_{33} + x_{34} + x_{35} \leq 1$ VI[th] constraint

$5x_1 + 15x_{31} \leq 3$ VII[th] constraint [Budget constraint]

$5x_{12} + 8x_{22} + 15x_{32} \leq 6$ VIII[th] constraint [Budget constraint]

$5x_{13} + 8x_{23} + 15x_{33} + 1.2x_{43} \leq 7$ IX[th] constraint [Budget constraint]

$8x_{24} + 15x_{34} + 1.2x_{44} \leq 7$ X[th] constraint [Budget constraint]

$8x_{25} + 15x_{35} \leq 7$ XI[th] constraint [Budget constraints]

$x_{11}, x_{12}, x_{13}; x_{22}, x_{23}, x_{24}; x_{31}, x_{32}, x_{33}, x_{34}; x_{43}, x_{44} \geq 0$ [Non negativity constraint]

PROBLEM 1.28

Prarthna Mittal owns 800 acres of undeveloped land on a scenic lake in the heart of the Ozark Mountains. In the past, little or no regulation was imposed upon new developments around the lake. The lake shores are now dotted with vacation homes, and septic tanks, most of them improperly installed, are in extensive use. Over the years, seepage from the septic tanks led to severe water pollution. To curb further degradation of the lake, county officials have approved stringent ordinances applicable to all future developments: (1) Only single, double, and triple-family homes can be constructed, with single-family homes accounting for at least 50% of the total. (2) To limit the number of septic tanks, minimum lot sizes of 2, 3, and 4 acres are required for single, double, and triple-family homes, respectively. (3) Recreation areas of 1 acre each must be established at the rate of one area per 200 families. (4) To preserve the ecology of the lake, underground water may not be pumped out for house or garden use. The president of Realco is studying the possiblity of developing the 800-acre property. The new development will include single, double, and triple-family homes. It is estimated that 15% of the acreage will be allocated to streets and utility easements. Realco estimates the returns from the different housing units as follows:

Housing unit	*Single*	*Double*	*Triple*
Net return per unit (Rs.)	10,000	12,000	15,000

The cost of connecting water service to the area is proportionate to the number of units constructed. However, the county charges a minimum of Rs. 100,000 for the project. Additionally, the expansion of the water system beyond its present capacity is limited to 200,000 gallons per day during peak periods. The following data summarize the water service connection cost as well as the water consumption, assuming an average size family:

Housing unit	*Single*	*Double*	*Triple*	*Recreation*
Water service connection cost per unit (Rs.)	1000	1200	1400	800
Water consumption per unit (gal/day)	400	600	840	450

Develop an optimal plan for Prarthna Mittal.

Solution

Let x_1 = Single Family Home
x_2 = Double Family Home
x_3 = Triple Family Home
x_4 = Recreation Area

Now,

$(10{,}000 - 1000)x_1 + (12{,}000 - 1200)x_2 + (15000 - 1400)x_3 - 800\,x_4$

i.e. Maximize $Z = 9000x_1 + 10{,}800x_2 + 13{,}600x_3 - 800x_4$

Subject to

$2x_1 + 3x_2 + 4x_3 + x_4 \le 680$ Ist constraint as [800 – 15% of 800] [size constraint]

$x_1 \ge \frac{1}{2}(x_1 + x_2 + x_3)$ ... [at least 50% of total constraint]

$\Rightarrow$ $0.5x_1 - 0.5x_2 - 0.5x_3 \ge 0$ IInd constraint

$x_4 \ge \frac{x_1 + 2x_2 + 3x_3}{200}$ [Recreation area per family constraint]

$\Rightarrow$ $200x_4 - x_1 - 2x_2 - 3x_2 \ge 0$ IIIrd constraint

$400x_1 + 600x_2 + 840x_3 + 450x_4 \le 2{,}00{,}000$ IVth constraint [water capacity constraint]

$x_1, x_2, x_3, x_4 \ge 0$ [Non negativity constraint]

PROBLEM 1.29

Investor Piyush Jindal has Rs. 10,000 to invest in four projects. The following table gives the cash flow for the four investments.

Project	*Cash flow (Rs. 1000) at the start of*				
	Year 1	*Year 2*	*Year 3*	*Year 4*	*Year 5*
1	–1.00	0.50	0.30	1.80	1.20
2	–1.00	0.60	0.20	1.50	1.30
3	0.00	–1.00	0.80	1.90	0.80
4	–1.00	0.40	0.60	1.80	0.95

The information in the table can be interpreted as follows: For project 1, Rs. 1.00 invested at the start of year 1 will yield Rs. 50 at the start of year 2, Rs. .30 at the start of year 3, Rs. 1.80 at the start of year 4, and Rs. 1.20 at the start of year 5. The remaining entries can be interpreted similarly. The entry 0.00 indicates that no transaction is taking place. Doe has the additional option of investing in a bank account that earns 6.5% annually. All funds accumulated at the end of one year can be reinvested in the following year. Formulate the problem as a linear program to determine the optimal allocation of funds to investment opportunities.

Solution

Let x_1 = Amount invested in project 1 in beginning of year 1
x_2 = Amount invested in project 2 in beginning of year 1
x_3 = Amount invested in project 3 in beginning of year 2
x_4 = Amount invested in project 4 in beginning of year 1

y_1 = Amount invested in Bank in beginning of year 1
y_2 = Amount invested in Bank in beginning of year 2
y_3 = Amount invested in Bank in beginning of year 3
y_4 = Amount invested in Bank in beginning of year 4
y_5 = Amount invested in Bank at the end of year 4

Maximize $Z = Y_5$

Subject to

$x_1 + x_2 + x_4 + y_1 \le 10{,}000$ I[st] constraint [Budget constraint]
$0.5x_1 + 0.6x_2 - x_3 + 0.4x_4 + 1.065y_1 - y_2 = 0$ II[nd] constraint [Budget constraint]
$0.3x_1 + 0.2x_2 + 0.8x_3 + 0.6x_4 + 1.065y_2 - y_3 = 0$ III[rd] constraint [Budget constraint]
$1.8x_1 + 1.5x_2 + 1.9x_3 + 1.8x_4 + 1.065y_3 - y_4 = 0$ IV[th] constraint [Budget constraint]
$1.2x_1 + 1.3x_2 + 0.8x_3 + 0.95x_4 + 1.065y_4 - y_5 = 0$ V[th] constraint [Budget constraint]
$x_1, x_2, x_3, x_4; y_1, y_2, y_3, y_4 \ge 0$ [Non negativity constraint]

PROBLEM 1.30

TULSIAN Construction can bid on two 1-year projects. The following table provides the quarterly cash flow (in millions of dollars) for the two projects.

Project	*Cash flow (in millions of Rs.) at*				
	1/1/11	*4/1/11*	*7/1/11*	*10/1/11*	*12/31/011*
I	–1.0	–3.1	–1.5	1.8	5.0
2	–3.0	–2.5	1.5	1.8	2.8

TULSIAN has cash funds of Rs. 1 million at the beginning of each quarter and may borrow at most Rs. 1 million at a 10% nominal annual interest rate. Any borrowed money must be returned at the end of the quarter. Surplus cash can earn quarterly interest at an 8% nominal annual rate. Net accumulation at the end of one quarter is invested in the next quarter.

Assume that TULSIAN is allowed partial or full participation in the two projects. Determine the level of participation that will maximize the net cash accumulated on 31.12.2011.

Solution

Let x_1 = Degree of participation in project 1
x_2 = Degree of participation in project 2
b_1 = Borrowings in beginning of quarter 1
b_2 = Borrowings in beginning of quarter 2
b_3 = Borrowings in beginning of quarter 3
b_4 = Borrowings in beginning of quarter 4
S_1 = Surplus in beginning of quarter 1
S_2 = Surplus in beginning of quarter 2
S_3 = Surplus in beginning of quarter 3
S_4 = Surplus in beginning of quarter 4
S_5 = Surplus in beginning of quarter 4

Maximize S_5

Subject to

$x_1 + 3x_2 + S_1 - b_1 = 1$ I[st] constraint [Budget constraint]
as $[x_1 + 3x_2 + S_1 = 1 - b_1]$
$3.1x_1 + 2.5x_2 + 1.025b_1 - b_2 - 1.02\ S_1 + S_2 = 1$ II[nd] constraint [Budget constraint]
$[\text{as } 3.1x_1 + 2.5x_2 + 1.025b_1 - 1.02\ S_1 + S_2 = 1 + b_1]$

$1.5x_1 - 1.5x_2 + 1.025b_3 - 1.02\ S_2 - b_3 + S_3 = 1$ III^{rd} constraint [Budget constraint]
$-1.8x_1 - 1.8x_2 + 1.025b_3 - 1.02S_3 - b_4 + S_4 = 1$ IV^{th} constrain [Budget constraint]
$5x_1 - 2.8x_2 + 1.025b_4 - 1.02S_4 + S_5 = 0$ V^{th} constraint [Budget constraint]
$b_1 \leq 1,\ b_2 \leq 1,\ b_3 \leq 1,\ b_4 \leq 1$ VI^{th} constraint [As borrowings can not be more than 1]
$x_1, x_2, b_1, b_2, b_3, b_4, S_1, S_2, S_3, S_4, S_5 \geq 0$ [Non negativity constraint]

PROBLEM 1.31

A business executive has the option to invest money in two plans: Plan A guarantees that each dollar invested will earn Rs. .70 a year later, and plan B guarantees that each rupee invested will earn Rs. 2 after 2 years. In plan A, investments can be made annually, and in plan B, investments are allowed for periods that are multiples of two years only. How should the executive invest Rs. 100,000 to maximize the earnings at the end of 3 years?

Solution

Let A_1 = Amount Invested in plan A in beginning of year 1
A_2 = Amount Invested in plan A in beginning of year 2
A_3 = Amount Invested in plan A in beginning of year 3
A_4 = Amount Invested in plan A in beginning of year 3
B_1 = Amount Invested in plan B in beginning of year 1
B_2 = Amount Invested in plan B in beginning of year 2

Maximize A_4
$A_1 + B_1 = 1{,}00{,}000$ I^{st} constraint [Budget constraint]
$-1.70A_1 + A_2 + B_2 = 0$ II^{nd} constraint [Budget constraint]
$-1.70A_2 + A_3 - 3B_1 = 0$ III^{rd} constraint [Budget constraint]
$1.70A_3 - A_4 + 3B_2 = 0$ IV^{th} constraint [Budget constraint]
$A_1, A_2, A_3, A_4, B_1, B_2 \geq 0$ [Non negativity constraint]

PROBLEM 1.32

TUSHAR Manufacturing Company has contracted to deliver home windows over the next 6 months. The demands for each month are 100, 250, 190, 140, 220 and 110 units, respectively. Production cost per window varies from month to month depending on the cost of labor, material, and utilities. Acme estimates the production cost per window over the next 6 months to be \$50, \$45, \$55, \$48, \$52, and \$50, respectively. To take advantage of the fluctuations in manufacturing cost, Acme may elect to produce more than is needed in a given month and hold the excess units for delivery in later months. This, however, will incur storage costs at the rate of \$8 per window per month assessed on end-of-month inventory. Develop a linear program to determine the optimum production schedule.

Solution

Let x_1 = No. of units produced in 1^{st} month
x_2 = No. of units produced in 2^{nd} month
x_3 = No. of units produced in 3^{rd} month
x_4 = No. of units produced in 4^{th} month
x_5 = No. of units produced in 5^{th} month
x_6 = No. of units produced in 6^{th} month
S_1 = Inventory at the end of 1^{st} month
S_2 = Inventory at the end of 2^{nd} month
S_3 = Inventory at the end of 3^{rd} month
S_4 = Inventory at the end of 4^{th} month
S_5 = Inventory at the end of 5^{th} month

Minimize $Z = 50x_1 + 45x_2 + 55x_3 + 48x_4 + 52x_5 + 50x_6 + 8S_1 + 8S_2 + 8S_3 + 8S_4 + 8S_5$

Subject to,

$x_1 - S_1 = 100$ I[st] constraint [Demand constraint]

$x_2 + S_1 - S_2 = 250$ II[nd] constraint [Demand constraint]

$x_3 + S_2 - S_3 = 190$ III[rd] constraint [Demand constraint]

$x_4 + S_3 - S_4 = 140$ IV[th] constraint [Demand constraint]

$x_5 + S_4 - S_5 = 220$ V[th] constraint [Demand constraint]

$x_6 + S_5 = 110$ VI[th] constraint [Demand constraint]

$x_1, x_2, x_3, x_4, x_5, x_6, S_1, S_2, S_3, S_4, S_5 \geq 0$ [Non negativity constraint]

PROBLEM 1.33

A company will manufacture a product for the next four months: March, April, May and June. The demands for each month are 520, 720, 520 and 620 units, respectively. The company has a steady workforce of 10 employees but can meet fluctuating production needs by hiring and firing temporary workers, if necessary. The extra costs of hiring and firing in any month are Rs. 200 and Rs. 400 per worker, respectively. A permanent worker can produce 12 units per month, and a temporary worker, lacking comparable experience, only produce 10 units per month. The company can produce more than needed in any month and carry the surplus over to a succeeding month at a holding cost of Rs. 50 per unit per month. Develop an optimal hiring/firing policy for the company over the four-month planning horizon.

Solution

Let H_1 = Hired labour in month of March
H_2 = Hired labour in month of April
H_3 = Hired labour in month of May
H_4 = Hired labour in month of June
F_2 = Fired labour in month of April
F_3 = Fired labour in month of May
F_4 = Fired labour in month of June
S_1 = Inventory in month of March
S_2 = Inventory in month of April
S_3 = Inventory in month of May

Minimize $Z = 200(H_1 + H_2 + H_3 + H_4) + 400\ (F_2 + F_3 + F_4) + 50(S_1 + S_2 + S_3)$

Subject to

$10H_1 - S_1 = 400$ Ist constraint [Demand for month of March constraint]

as $10H_1 + (12 \times 10) - S_1 = 520$

$\Rightarrow$ $10H_1 - S_1 = 520 - 120 = 400$

Similary

$10H_2 + 10(H_1 - F_2) + S_1 - S_2 = 600$ IInd constraint [Demand for month of April constraint]

$10H_3 + 10(H_2 + H_1 - F_2 - F_3) + S_2 - S_3 = 400$ IIIrd constraint
[Demand for the month of May constraint]

$10H_4 + 10(H_3 + H_2 + H_1 - F_2 - F_3 - F_4) + S_3 = 500$
[Demand for the month of June constraint]

$H_1, H_2, H_3, H_4, F_2, F_3, F_4, S_1, S_2, S_3 \geq 0$ [Non negativity constraint]

PROBLEM 1.34

A firm wants to reach two types of customers: House holds having Rs 5 lakhs or more of annual family income and those having income of less than Rs 5 lakhs. The first category purchases twice as much as the second group. One unit of TV advertisement reaches 2,000 families of first group and 8000 families of second group and costs Rs 20,000 while advertisements in a magazine costs Rs 12,000 and reaches 6,000 and 3,000 families respectively. Not more than 12 advertisements can be given in the magazine and at least 6 TV advertisements have to be given. The advertisement budget is Rs. 1,80,000. Assume that every contracted family purchases the company product. The firm wants to maximize its sales. Formulate the above problem as a linear programming model.

Solution

Determination of Revenue Function

Let,

No. of T.V advertisement to be given x_1

No. of magazine advertisement to be given x_2

As T.V advertisement reaches 2000 families of first group and 8000 families of second group.

Therefore effective reach $[(2000 \times 2) + 8000] = 12000$.

In case of magazine advertisement

$= 6000 \times 2 + 3000 = 15000$.

Therefore Maximise $Z = 12000x_1 + 15000x_2$

Subject to

$x_2 \leq 12$ Ist constraint [Maximum no. of Magazine Advertisement constaint]

$x_1 \geq 6$ IInd constraint [Minimum no. of T.V Advertisement constraint]

$20{,}000\, x_1 + 12{,}000x_2 \leq 1{,}80{,}000$ IIIrd constraint [Budget constraint]

$x_1, x_2 \geq 0$ Non negativity constraint.

PROBLEM 1.35

Industrial design company has been awarded a contract to design a label for a new medicine produced by a pharmaceutical company. The company estimates that 150 hours will be required to complete the project. The firm's three graphic designers available with the company for this project are Leena a senior designer and team leader; Rahul, a senior designer and Sarah, a junior designer. Because Leena has worked on several projects for the company, and that she must be assigned 40% of the total number of hours assigned to the two senior designers. To provide label designing experience for Sarah, she must be assigned at least 15% of the total project time. However, the number of hours assigned to Sarah must not exceed 25% of the total number of hours assigned to the two senior designers. Due to other projects commitments, leena has a maximum of 50 hours available to work on the project.

Hourly wage rate are Rs 1,500 for Leena, Rs 1,250 for Rahul and Rs 900 for Sarah.

Formulate a linear programme problem to determine number of hours each graphic designer should be assigned for the project to minimize the total cost.

Solution

Let x_1 = Work hours of Leena

x_2 = Work hours of Rahul

x_3 = Work hours of Sarah

Minimize $Z = 1500x_1 + 1250x_2 + 900x_3$

Subject to,

$x_1 + x_2 + x_3 = 150$ Ist constraint [Total hours constraint]

also $x_1 \geq \frac{2}{5}[x_1 + x_2]$ [Leena should work 40% i.e. (2/5) of works of senior managers]

i.e. $5x_1 - 2x_1 - 2x_2 \geq 0$

$\Rightarrow$ $3x_1 - 2x_2 \geq 0$ IInd constraint

also, $x_3 \geq \frac{15}{100}(x_1 + x_2 + x_3)$ [Sarah should work at least 15%]

i.e. $100x_3 - 15x_1 - 15x_2 - 15x_3 \geq 0$

$\Rightarrow$ $-15x_1 - 15x_2 + 85x_3 \geq 0$ IIIrd constraint

$x_3 \leq \frac{1}{4}(x_1 + x_2)$ [Sarah can work maximum 25%. i.e. 1/4 of work of senior managers]

i.e. $4x_3 - x_1 - x_2 \leq 0$

$\Rightarrow$ $-x_1 - x_2 + 4x_3 \leq 0$ IVth constraint

$x_1 \leq 50$ Vth constraint [Leena can work maximum 50 hours]

$x_1, x_2, x_3 \geq 0$ [Non negativity constraint]

PROBLEM 1.36

A complete unit of certain product consists of 4 units of component A, 3 units of component B and 2 units of component C. The three components (A, B and C) are manufactured from two different raw materials of which 500 units and 800 units, respectively are available. Four departments are engaged in the production process. The following table give the raw material requirement per production run and the resulting units of each component. The objective is to determine the number of production runs for each department which will maximise the total number of complete units of the final product. Formulate the problem as a linear programming problem:

	Input per run (units)		*Output per run (units)*		
Deptd.	*Raw material 1*	*Raw material 2*	*Component A*	*Component B*	*Component C*
1	6	4	5	3	3
2	3	7	4	7	6
3	1	6	6	2	5
4	2	5	7	3	2

Solution

Let y be the number of find units

x_1 = No. of production run in deptt. 1

x_2 = No. of production run in deptt. 2

x_3 = No. of production run in deptt. 3

x_4 = No. of production run in deptt. 4

Maximise $Z = y$

Subject to constraint

$6x_1 + 3x_2 + x_3 + 2x_4 \leq 500$ [Raw material 1 constraint]

$4x_1 + 7x_2 + 6x_3 + 5x_4 \leq 800$ [Raw material 2 constraint]

$$\frac{5x_1 + 4x_2 + 6x_3 + 7x_4}{4} \geq y$$

$5x_1 + 4x_2 + 6x_3 + 7xy - 4y \geq 0$ (Component units of A constraint)

$$\frac{3x_1 + 7x_2 + 2x_3 + 3x_4}{3} \geq y$$

$3x_1 + 7x_2 + 2x_3 + 3x_4 - 3y \geq 0$ (Component units of B constraint)

$$\frac{3x_1 + 6x_2 + 5x_3 + 2x_4}{2} \geq y$$

$3x_1 + 6x_2 + 5x_3 + 2xy_4 - 2y \geq 0$ (Component units of C constraint)

$x_1, x_2, x_3, x_4, y \geq 0$ (Non negativity constraint).

PROBLEM 1.37

The PQR stone company sells stone secure feam any of three adjacent quaries. The stone sold by the company must conform to the following specifications:

Material X equal to 30%
Material Y equal to less than 40%
Material Z between 30% and 40%

Stone from quarry A costs Rs 100 per tonne and has the following properties:

Material X – 20%

Material Y – 60%

Material Z – 20%

Stone from quarry B costs Rs 120 per tonne and has the following properties:

Material X – 20%
Material Y – 30%
Material Z – 30%

Stone from quarry C costs Rs 150 per tonne and has the following properties:

Material X – 10%
Material Y – 40%
Material Z – 50%

From what quarries should PQR stone company secure rock in order to minimise the cost per tonne of rock.

Solution

Let,

x_1 = Proportion of tonne of rock to be secured from quarry A
x_2 = Proportion of tonne of rock to be secured from quarry B
x_3 = Proportion of tonne of rock to be secured from quarry C

Minimise $Z = 100x_1 + 120x_2 + 150x_3$

Subject to constraints

$0.20x_1 + 0.40x_2 + 0.10x_3 = 0.30$ [Material X constraint]
$0.60x_1 + 0.30x_2 + 0.40x_3 \leq 0.40$ [Material Y constraint]
$0.20x_1 + 0.30x_2 + 0.50x_3 \geq 0.30$ [Minimum material 2 constraint]
$0.20x_1 + 0.30x_2 + 0.50x_3 \leq 0.40$ [Maximum material 2 contraint]
$x_1 + x_2 + x_3 = 1$
$x_1, x_2, x_3 \geq 0$ [Non negativity constraint]

PROBLEM 1.38

A certain forming organisation operates 3 forms of comparable productivity. The output of each farm is limited both by the usable acreage and by the amount of water available for irrigation. The data for upcoming season is as shown below:

Farm	*Usable Acreage*	*Water available (is cubic feet)*
1	400	1,500
2	600	2,000
3	300	900

The organisation is considering planting crops which differ primarily in their expected profit per acre and in their consumption of water. Furthermore, the total acreage that can be devoted to each of te crops is limited by the amount of appropriate harvesting equipment available.

Crop	*Maximum Acreage*	*Water Consumption (in cubic feet)*	*Expected profit per Acre (Rs)*
A	700	5	4,000
B	800	4	3,000
C	300	3	1,000

In order to maintain a uniform work load among the 3 farms, it is the policy of the organisation that the percentage of the usable acreage planted by the same for each form. However any combination of the crops may be grown at any of the farms. The organisation wishes to know how much of each crop should be planted at the respective farms in order to maximise expected profit. Formulate this problem as an LP model in order to maximise the total expected profit.

Solution

Let

x_{11} = Crop A to be planted in farm 1
x_{12} = Crop A to be planted in farm 2
x_{13} = Crop A to be planted in farm 3
x_{21} = Crop B to be planted in farm 1
x_{22} = Crop B to be planted in farm 2
x_{23} = Crop B to be planted in farm 3
x_{31} = Crop C to be planted in farm 1
x_{32} = Crop C to be planted in farm 2
x_{33} = Crop C to be planted in farm 3

Maximise $Z = 4{,}000(x_{11} + x_{12} + x_{13}) + 3{,}000(x_{21} + x_{22} + x_{23}) + 1{,}000(x_{31} + x_{32} + x_{33})$

Subject to constraints

$x_{11} + x_{12} + x_{13} \leq 700$ [Crop A Requirements]
$x_{21} + x_{22} + x_{23} \leq 800$ [Crop B Requirements]
$x_{31} + x_{32} + x_{33} \leq 300$ [Crop C Requirements]
$x_{11} + x_{21} + x_{31} \leq 400$ [Available average for farm 1]
$x_{12} + x_{22} + x_{32} \leq 600$ [Available average for farm 2]
$x_{13} + x_{23} + x_{33} \leq 300$ [Available average for farm 3]
$5x_{11} + 4x_{21} + 3x_{31} \leq 1{,}500$ [Water availability is farm 1]
$5x_{12} + 4x_{22} + 3x_{32} \leq 1{,}500$ [Water availability is farm 2]
$5x_{13} + 4x_{23} + 3x_{33} \leq 900$ [Water availability is farm 3]

$$\frac{x_{11} + x_{21} + x_{31}}{400} = \frac{x_{12} + x_{22} + x_{32}}{600}$$ [Uniform workload constraint]

$$\frac{x_{12} + x_{22} + x_{32}}{600} = \frac{x_{13} + x_{23} + x_{33}}{300}$$ [Uniform workload constraint]

$$\frac{x_{13} + x_{23} + x_{33}}{300} = \frac{x_{11} + x_{21} + x_{31}}{400}$$ [Uniform workload constraint]

$x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23}, x_{31}, x_{32}, x_{33} \geq 0$ [Non negativity constraint.]

2 LINEAR PROGRAMMING — GRAPHICAL METHOD

PRACTICAL STEPS INVOLVED IN SOLVING LPP BY GRAPHICAL METHOD

The practical steps involved in solving linear programming problems by Graphical Method are given below:

Step 1 → *Consider each inequality constraint as equation.*

Step 2 → *Take one variable (say x) in a given equation equal to zero and find the value of other variable (say y) by solving that equation to get one co-ordinate [say (0, y)] for that equation.*

Step 3 → *Take the second variable (say y) as zero in the said equation and find the value of first variable (say x) to get another co-ordinate [say (x, 0)] for that equation.*

Step 4 → *Plot both the co-ordinates so obtained [i.e., (0, y) and (x, 0)] on the graph and join them by a straight line. This straight line shows that any point on that line satisfies the equality and any point below or above that line shows inequality. Shade the feasible region which may be either convex to the origin in case of less than type of inequality (<) or opposite to the origin in case of more than type of inequality (>).*

Step 5 → *Repeat Steps 2 to 4 for other constraints.*

Step 6 → *Find the common shaded feasible region and mark the co-ordinates of its corner points.*

Step 7 → *Put the co-ordinates of each of such vertex in the Objective Function. Choose that vertex which achieves the most optimal solution (i.e., in the case of Maximisation, the vertex that gives the maximum value of 'Z' & in case of Minimisation the vertex that gives the minimum value of 'Z). Solution can be obtained in the following manner:*

Vertex No.	*Co-ordinates of vertices of Common shaded feasible region*	*Value of Z*
1	(x_1, y_1)	Z_1 = ...
2	(x_2, y_2)	Z_2 = ...
3	(x_3, y_3)	Z_3 = ...
4 & so on	(x_4, y_4)	Z_4 = ...

Step 8 → Optimal Solution:

Type of Problem	*Optimal Solution*
(a) In case of Maximisation Problem	The vertex which gives the maximum value of Z is the optimal solution.
(b) In case of Minimisation Problem	The vertex which gives the minimum value of Z is the optimal solution.

PROBLEM 2.1

Sky Ltd. has two products Cloud and Wind. To produce one unit of Cloud, 2 units of material X and 4 units of material Y are required. To produce one unit of Wind, 3 units of material X and 2 units of material Y are required. As the raw material X is in short supply so not more than 16 units of material X can be used. Atleast 16 units of material Y must be used in order to meet the committed sales of Cloud and Wind. Cost per unit of material X and material Y are Rs. 2.50 and Rs. 0.25 respectively. The selling price per unit of cloud and wind are Rs. 12 and Rs. 16 respectively.

You are required:

(i) To formulate mathematical model.
(ii) To solve it for maximum contribution (Graphically).

Solution

Calculation of Contribution per Unit of each Product

	Particulars	*Cloud (Rs.)*		*Wind (Rs.)*	
A	Selling price per unit		12.00		16.00
B	*Less :* cost per unit				
	(i) Material X	(Rs. 2.50 × 2)	5.00	(Rs. 2.50 × 3)	7.50
	(ii) Material Y	(Rs. 0.25 × 4)	1.00	(Rs. 0.25 × 2)	0.50
	Total (i) + (ii)		6.00		8.00
C	Contribution (A – B)		6.00		8.00

Part (i)

Since the objective is to maximise the profit, the objective function is given by:

Maximise $Z = 6x + 8y$

Subject to constraints:

$2x + 3y \leq 16$ (Maximum material X constraint) ...(i)

$4x + 2y \geq 16$ (Minimum material Y constraint) ...(ii)

$x, y \geq 0$ (Non-Negativity constraint)

Part (ii)

Step 1: Finding the vertex for each constraint by treating the constraint of inequality nature as equality.

Constraint (i) in limiting form $2x + 3y = 16$

When $x = 0, y = \frac{16}{3}$

When $y = 0,\ x = \frac{16}{2} = 6$

Thus, the vertices are $(0, \frac{16}{3})$ & (8, 0)

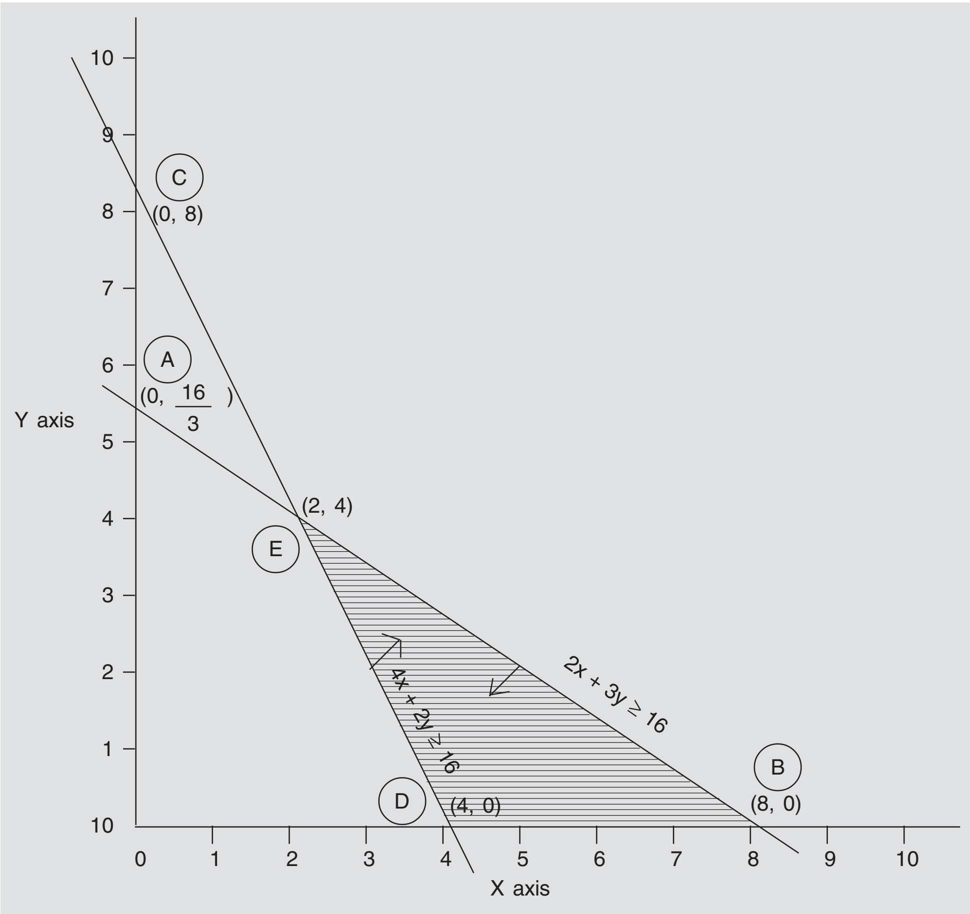
Constraint (ii) in limiting form $4x + 2y = 16$

When $x = 0,\ y = \frac{16}{2}$

When $y = 0,\ x = \frac{16}{4} = 4$

Thus, the vertices are (0, 8) & (4, 0)

Step 2: Plotting both the co-ordinates of the 1st constraint on the graph and joining them by straight line and shading the feasible region which is convex to origin in case of less than type of inequality. Similarly drawing straight line and shading feasible region for other constraint.



Step 3: Reading the co-ordinates of the vertices of common shaded feasible region and putting the co-ordinates of each of such vertex in the objective function. Selecting those vertices which achieve the most optimal solution (i.e. in case of maximisation vertices which give the maximum value of Z).

Set No.	Co-ordinates of vertices of Common shaded feasible region	Value of Z
1	E (2, 4)	$Z_1 = 6 \times 2 + 8 \times 4 = 44$
2	B (8, 0)	$Z_2 = 6 \times 8 + 8 \times 0 = 48$
3	D (4, 0)	$Z_3 = 6 \times 4 + 8 \times 0 = 24$

Optimal Solution: Thus, set no. 2 gives the max. value of 2 with $x = 8$ and $y = 0$.

PROBLEM 2.2

Seed Ltd has two products Rice and Wheat. To produce one unit of Rice 2 units of material X and 4 units of material Y are required. To produce one unit of Wheat 3 units of material X and 2 units of material Y are required. Atleast 16 units of each material must be used in order to meet committed sales of Rice and Wheat. Due to moderate marketing facilities not more than 8 units of product wheat can be sold. Cost per unit of material X and material Y are Rs. 2.50 and Rs. 0.25 respectively. The selling price per unit of Rice and Wheat are Rs. 12 and Rs. 16 respectively.

You are required:

(i) to formulate mathematical model.

(ii) to solve it for maximum contribution by Graphical method.

Solution

Calculation of Contribution per Unit of each Product

Particulars	Rice (Rs.)		Wheat (Rs.)	
A Selling Price		12.00		16.00
B *Less :* Material Cost				
(i) Material X	(Rs. 2.50 × 2)	5.00	(Rs. 2.50 × 3)	7.50
(ii) Material Y	(Rs. 0.25 × 4)	1.00	(Rs. 0.25 × 2)	0.50
Total (i) + (ii)		6.00		8.00
C Contribution (A – B)		6.00		8.00

Part (i)

Since the objective is to maximise the profit, the objective function is given by:

Maximise $Z = 6x + 8y$

Subject to constraints:

$2x + 3y \geq 16$ (Minimum material X constraint) ...(i)

$4x + 2y \geq 16$ (Minimum material Y constraint) ...(ii)

$y \leq 8$ (Maximum sales of Product Wheat) ...(iii)

$x, y \geq 0$ (Non-Negativity constraint)

Part (ii)

Step 1: Finding the vertex for each constraint by treating the constraint of inequality nature as equality.

Constraint (i) in limiting form $2x + 3y = 16$

When $x = 0$, $y = \dfrac{16}{3}$

When $y = 0$, $x = 8$

Thus, the vertices are $(0, \frac{16}{3})$ & $(8, 0)$

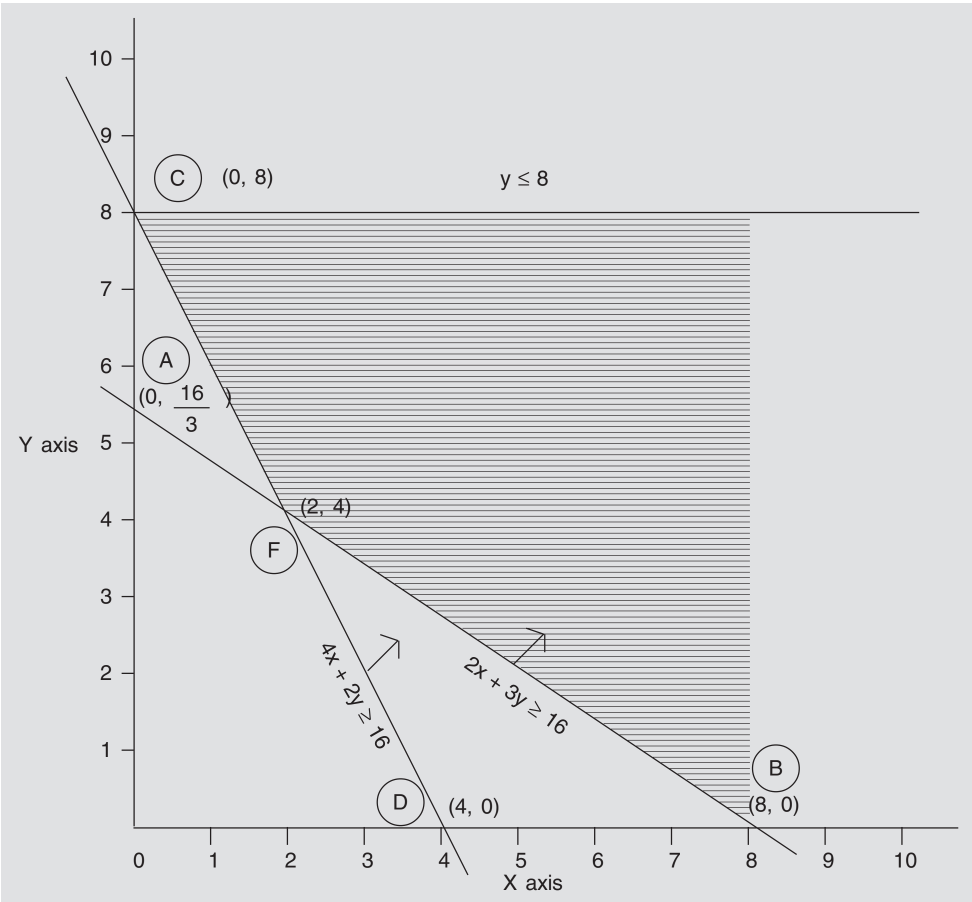
Constraint (ii) in limiting form $4x + 2y = 16$

When $x = 0$, $y = 8$

When $y = 0$, $y = 4$

Thus, the vertices are (0, 8) & (4, 0)

Step 2: Plotting both the co-ordinates of the 1st constraint on the graph and joining them by straight line and shading the feasible region which is opposite to origin in case of more than type of inequality. Similarly drawing straight line and shading feasible region for other constraints.



Step 3: Reading the co-ordinates of the vertices of common shaded feasible region and putting the co-ordinates of each of such vertex in the objective function. Selecting those vertices which achieve the most optimal solution (i.e. in case of maximisation vertices which give the maximum value of Z).

Set No.	*Co-ordinates of vertices of Common shaded feasible region*	*Value of Z*
1	C (0, 8)	$Z_1 = 6 \times 0 + 8 \times 8 = 64$
2	F (2, 4)	$Z_2 = 6 \times 2 + 8 \times 4 = 44$
3	B (8, 0)	$Z_2 = 6 \times 8 + 8 \times 0 = 48$

The solution of this problem is **unbounded**.

PROBLEM 2.3

A company buying scrap metal has two types of scrap available to them. The first type of scrap has 20% of metal A, 10% of impurity and 20% of metal B by weight. The second type of scrap has 30% of metal A, 10% of impurity and 15% of metal B by weight. The company requires at least 120 kg of metal A, at most 40 kg. of impurity and at least 90 kg of metal B. The price for the two scraps are Rs. 200 and Rs. 300 per kg respectively. Determine the optimum quantities of the two scraps to be purchased so that the requirements of the two metals and the restriction on impurity are satisfied at minimum cost.

Solution

Introduce the decision variable x_1 and x_2 indicating the quantity of scrap metal (in kg.) to be purchased. Now, the problem can be formulated as –

Minimise $200x_1 + 300x_2$

subject to:

$$0.2x_1 + 0.3x_2 \geq 120$$
$$0.1x_1 + 0.1x_2 \leq 40$$
$$0.2x_1 + 0.15x_2 \geq 90$$
$$x_1, x_2 \geq 0$$

Multiplying both sides of the inequalities by 10, we obtain

$$2x_1 + 3x_2 \geq 1200$$
$$x_1 + x_2 \leq 400$$
$$2x_1 + 1.5x_2 \geq 900$$

Step 1: Finding the vertex of each constraint by treating the constraint of inequality nature as equality.

Constraint (i) in limiting form $2x_1 + 3x_2 = 1200$

When $x_1 = 0$ $x_2 = 400$

When $x_2 = 0$ $x_1 = 600$

Thus the vertices are (0, 400) & (600, 0).

Constraint (ii) in limiting form $x_1 + x_2 = 400$

When $x_1 = 0$ $x_2 = 400$

When $x_2 = 0$ $x_1 = 400$

Thus the vertices are (0, 400) & (400, 0).

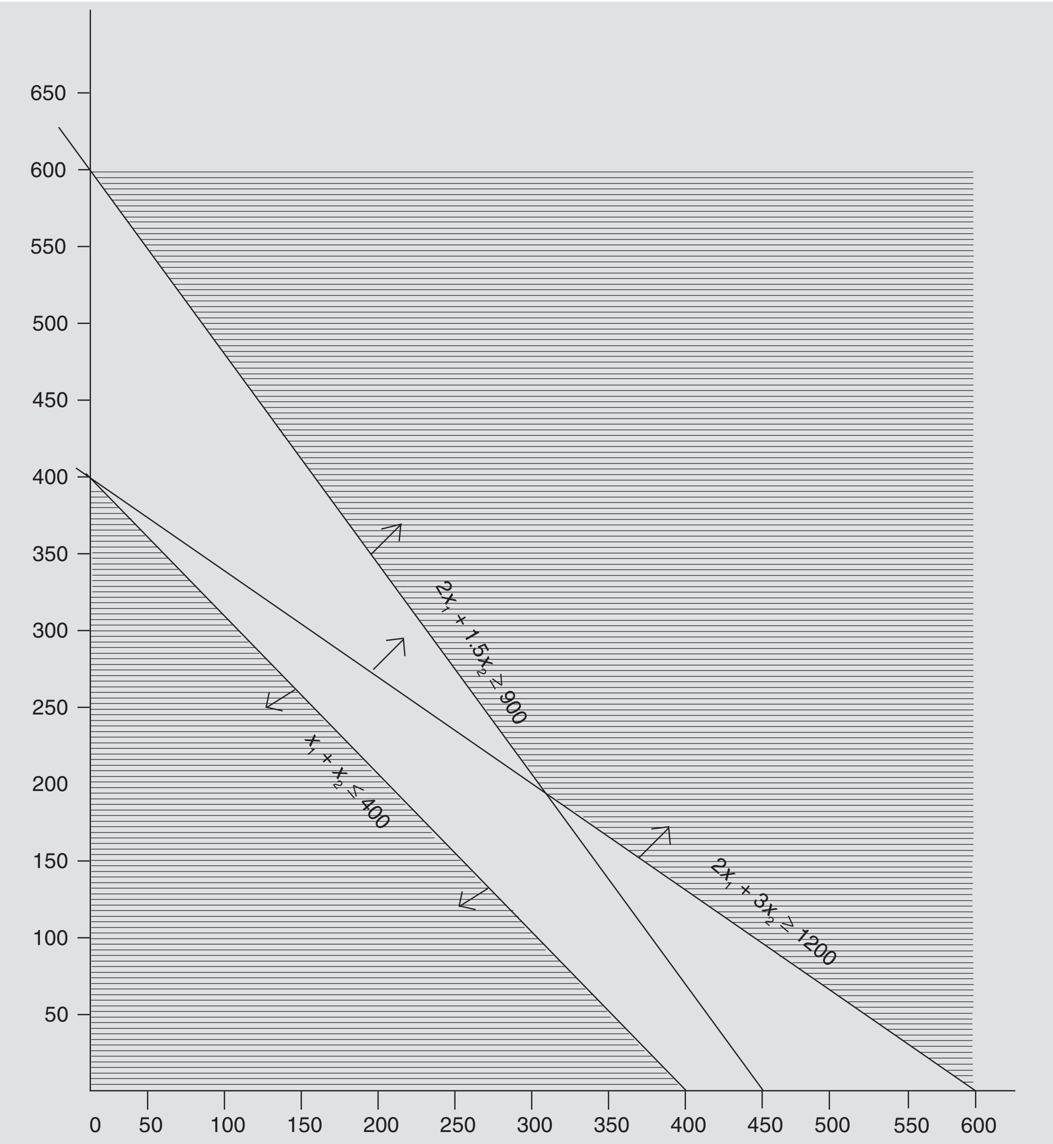
Constraint (iii) in limiting form $2x_1 + 1.5x_2 = 900$

When $x_1 = 0$ $x_2 = 600$

When $x_2 = 0$ $x_1 = 450$

Thus the vertices are (0, 600) & (450, 0)

Step 2: Plotting the co-ordinates of 1st constraint on the graph and joining them by a straight line, and shading the feasible region. Similarly drawing a straight line and shading feasible region for other constraints.



Step 3: There exists **no feasible solution** for this problem since there is no common shaded feasible region.

PROBLEM 2.4

A company buying scrap metal has two types of scrap metals available to him. The first type of scrap metal has 30% of metal A, 20% of metal B and 50% of metal C by weight. The second scrap has 40% of metal A, 10% of metal B and 30% of metal C. The company requires at least 240 kg of metal A, 100 kg of metal B and 290 kg of metal C. The price per kg of the two scraps are Rs. 120 and Rs. 160 respectively. Determine the optimum quantities of the two scraps to be purchased so that the requirements of the three metals are satisfied at a minimum cost.

Solution

We introduce the decision variables x_1 and x_2 indicating the amount of scrap metal to be purchased respectively. Then the problem can be formulated as

Minimise $120x_1 + 160x_2$

subject to:

$$\begin{aligned} 0.3x_1 + 0.4x_2 &\geq 240 \\ 0.2x_1 + 0.1x_2 &\geq 100 \\ 0.2x_1 + 0.3x_2 &\geq 120 \\ 0.5x_1 + 0.3x_2 &\geq 290 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Multiplying both sides of the inequalities by 10, the problem becomes —

Minimise $120x_1 + 160x_2$

subject to:

$$\begin{aligned} 3x_1 + 4x_2 &\geq 2400 \\ 2x_1 + x_2 &\geq 1000 \\ 5x_1 + 3x_2 &\geq 2900 \\ x_1, x_2 &\geq 0 \end{aligned}$$

Step 1: Finding the vertex of each constraint by treating the constraint of inequality in nature as equality.

Constraint (i) in limiting form $3x_1 + 4x_2 = 2400$

When $x_1 = 0$ $x_2 = 600$

When $x_2 = 0$ $x_1 = 800$

Thus the vertices are (0, 600) & (800, 0)

Constraint (ii) in limiting form $2x_1 + x_2 = 1000$

When $x_1 = 0$ $x_2 = 1000$

When $x_2 = 0$ $x_1 = 500$

Thus the vertices are (0, 1000) & (500, 0)

Constraint (iii) in limiting form $5x_1 + 3x_2 = 2900$

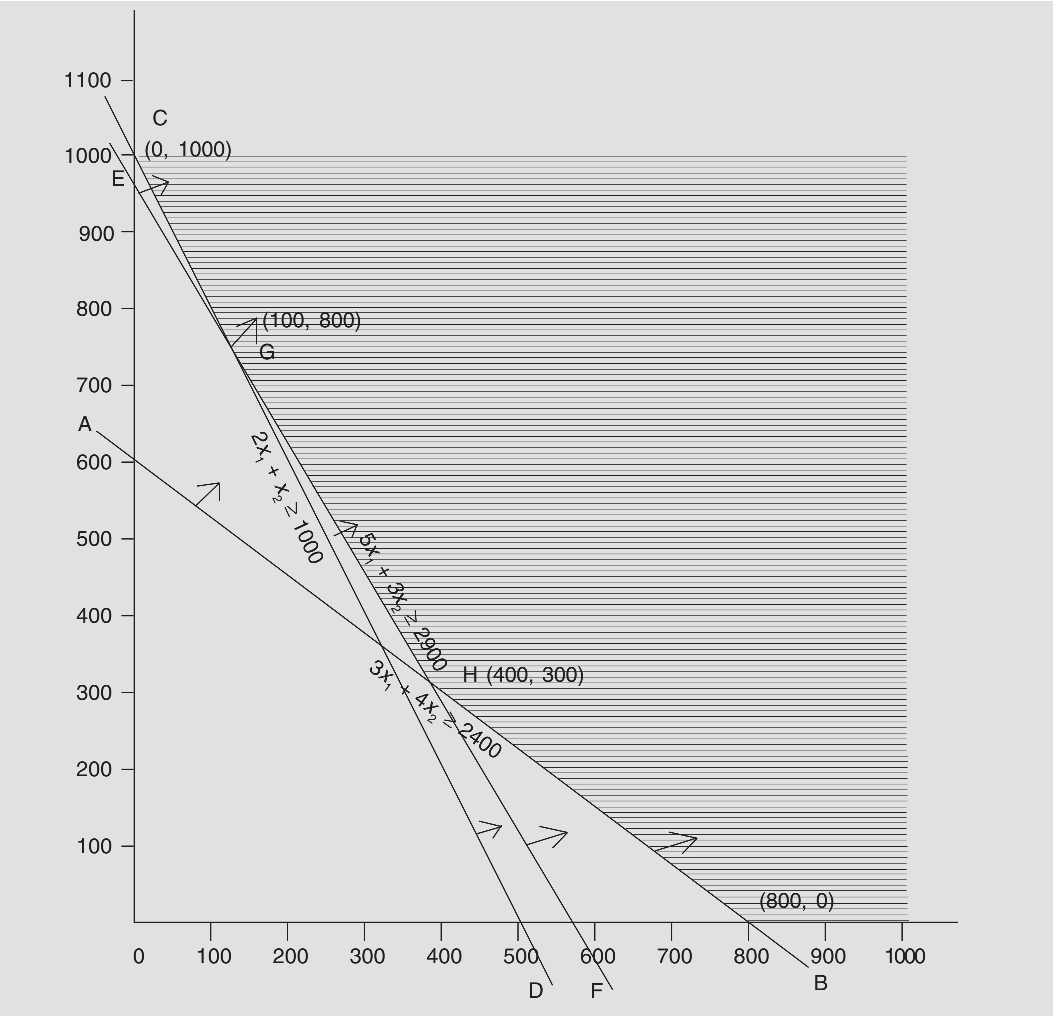
When $x_1 = 0$ $x_2 = 966.67$

When $x_2 = 0$ $x_1 = 580$

Thus the vertices are (0, 966.67) & (580, 0)

Step 2: Plotting the co-ordinates of 1st constraint on the graph and joining them by a straight line, and shading the feasible region. Similarly drawing a straight line and shading feasible region for other constraints.

Step 3: Reading the co-ordinates of the vertices of the common shaded feasible region & putting the co-ordinates of each of such vertex in the objective function. Selecting those vertices which achieve the most optimal solution (i.e. in case of minimisation vertices which give the minimum value of Z).



Set No.	Co-ordinates of vertices of Common shaded feasible region	Value of Z
1	C (0, 1000)	$Z_1 + 120 \times 0 + 160 \times 1000 = 1{,}60{,}000$
2	G (100, 800)	$Z_2 = 120 \times 100 + 160 \times 800 = 1{,}40{,}000$
3	H (400, 300)	$Z_3 = 120 \times 400 + 160 \times 300 = 96{,}000$
4	B (800, 0)	$Z_4 = 120 \times 800 + 160 \times 0 = 96{,}000$

Optimal Solution: Thus at both sets i.e. set No. 3 and 4, value of Z is minimum, at $x_1 = 400$ and $x_2 = 300$ or $x_1 = 800$ and $x_2 = 0$.

PROBLEM 2.5

Solve graphically the following linear programming problems:

Maximise $3x_1 + 2x_2$

subject to:

$$x_1 - x_2 \le 1$$
$$x_1 + x_2 \ge 3$$
$$x_1, x_2 \ge 0$$

Step 1: Finding the vertex for each of the constraint by treating constraint of inequality nature as equality.

Constraint (I) in limiting form $x_1 - x_2 = 1$

When $x_1 = 0 \quad x_2 = -1$

When $x_2 = 0 \quad x_1 = 1$

Thus the vertices are (0, –1) & (1, 0)

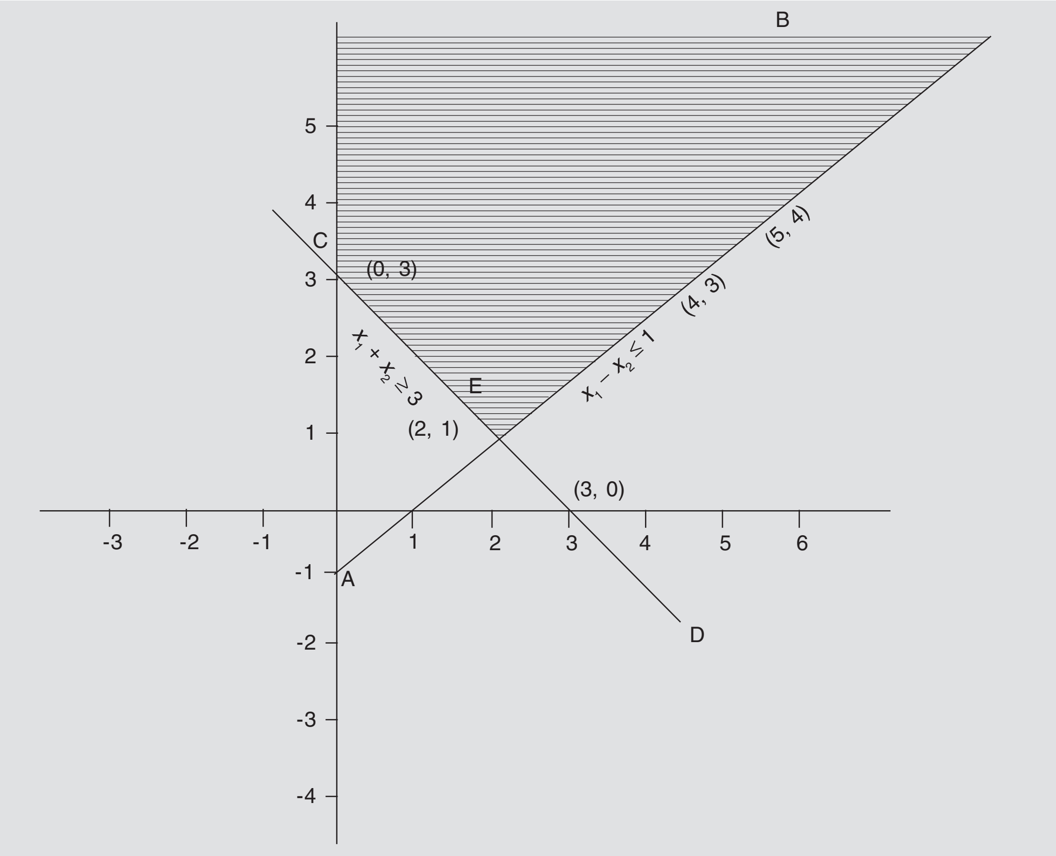
Constraint (ii) in limiting form $x_1 + x_2 = 3$

When $x_1 = 0 \quad x_2 = 3$

When $x_2 = 0 \quad x_1 = 3$

Thus the vertices are (0, 3) & (3, 0).

Step 2: Plotting the co-ordinates of 1st constraint on the graph and joining them by a straight line, and shading the feasible region. Similarly drawing a straight line and shading feasible region for other constraint.



Step 3: Reading the co-ordinates of the vertices of the common shading feasible region & putting the co-ordinates of each of such vertex in the objective function. Selecting those vertices which achieve the most optimal solution (i.e. in case of maxmisation vertices which give the maximum value of Z).

Set No.	*Co-ordinates of vertices of Common shaded feasible region*	*Value of Z*
1	(0, 3)	$Z_2 = 3 \times 0 + 2 \times 3 = 6$
2	(2, 1)	$Z_1 = 3 \times 2 + 2 \times 1 = 8$
3	(4, 3)	$Z_3 = 3 \times 4 + 2 \times 3 = 18$
4	(5, 4)	$Z_4 = 3 \times 5 + 2 \times 4 = 23$

It seems from solution that solution is **unbounded**. As we take, higher values, Z proportionately increases. But this type of problem does not exist in actual practice.

PROBLEM 2.6

A local travel agent is planning a charter trip to a major island resort. The eight day seven-night package includes the fare for round-trip travel, surface transportation, board and lodging and selected tour options. The charter trip is restricted to 200 persons and past experience indicates that there will not be any problem for getting 200 persons. The problem for the travel agent is to determine the number of Deluxe, Standard, and Economy tour packages to offer for this charter. These three plans each differ according to seating and service for the flight, quality of accommodations, meal plans and tour options. The following table summarises the estimated prices for the three packages and the corresponding expenses for the travel agent. The travel agent has hired an aircraft for the flat fee of Rs. 2,00,000 for the entire trip.

Price and Costs for Tour Packages per Person

Tour Plan (Rs.)	*Price (Rs.)*	*Resort Costs (Rs.)*	*Meals & other Expenses*
Deluxe	10,000	3,000	4,750
Standard	7,000	2,200	2,500
Economy	6,500	1,900	2,200

In planning the trip, the following considerations must be taken into account:

(i) At least 10 per cent of the packages must be of the deluxe type.
(ii) At least 35 percent but not more than 70 percent must be of the standard type.
(iii) At least 30 percent must be of the economy type.
(iv) The maximum number of deluxe packages available in any aircraft is restricted to 60.
(v) The resort desires that at least 120 of the tourists should be on the deluxe and standard packages together.

The travel agent wishes to determine the number of packages to offer in each type so as to maximise the total profit.

(a) Formulate the above as a linear programming problem.
(b) Restate the above linear programming problem in terms of two decision variables, taking advantage of the fact that 200 packages will be sold.
(c) Find the optimum solution using graphical methods for the restated linear programming problem and interpret your results.

Solution

Let x_1, x_2, x_3 denote the number of Deluxe, Standard and Economy tour packages to be offered to 200 persons that will maximise the profit. It other words, the total number of tours of three types offered by the concern is restricted to 200 only to maximise its profits.

The contribution (per person) arising out of each type of tour package offered is as follows:

Packages Offered	*Price (Rs.) (1)*	*Resort costs (Rs.) (2)*	*Meals & Other Expenses (Rs.) (3)*	*Net profit (Rs.) (4)=(1) − [(2)+(3)]*
Deluxe	10,000	3,000	4,750	2,250
Standard	7,000	2,200	2,500	2,300
Economy	6,500	1,900	2,200	2,400

The travel agent has to pay the flat fee of Rs. 2,00,000 for the chartered aircraft for the entire trip. Consequently the objective function (profit function) will be:

Maximise Z = Rs. $2{,}250x_1$ + Rs. $2{,}300x_2$ + Rs. $2{,}400x_3$ − Rs. 2,00,000

Part (a)

Condition (1): $x_1 + x_2 + x_3 \le 200$

Condition (2): $x_1 \ge \frac{10}{100}(x_1 + x_2 + x_1)$

or, $x_1 \ge \frac{1}{10}(x_1 + x_1 + x_1)$

or, $9x_1 - x_2 - x_3 \ge 0$

Condition (3): $x_2 \ge \frac{35}{100}(x_1 + x_2 + x_3)$

or, $-35x_1 + 65x_2 - 35x_3 \ge 0$

Condition (4): $x_2 \le \frac{70}{100}(x_1 + x_2 + x_3)$

or, $-70x_1 + 30x_2 - 70x_3 \le 0$

Condition (5): $x_3 \ge \frac{30}{100}(x_1 + x_2 + x_3)$

or, $-30x_1 - 30x_2 + 70x_3 \ge 0$

Condition (6): $x_1 \le 60$

Condition (7): $x_1 + x_2 \ge 120$

Now, the linear problem is:

Maximize Z = $2{,}250x_1 + 2{,}300x_2 + 2{,}400x_3 - 2{,}00{,}000$

Subject to contraints:

$x_1 + x_2 + x_3 \le 200$ (Seat availability constraint)
$9x_1 - x_2 - x_3 \ge 0$ (Minimum number of Delux type)
$-35x_1 + 65x_2 - 35x_3 \ge 0$ (Minimum number of Standard type)
$-70x_1 + 30x_2 - 70x_3 \le 0$ (Maximum number of Standard type)
$-30x_1 - 30x_2 + 70x_3 \ge 0$ (Minimum number of Economy type)
$x_1 \le 60$ (Maximum seat available for Delux type)
$x_1 + x_2 \ge 120$ (Toursit constraint)
$x_1, x_2, x_3 \ge 0$ (Non-Negativity constraint)

Part (b)

Since the value of $x_1 + x_2 + x_3 = 200$, the value of x_3 will be equal to $200 - x_1 - x_2$ and substituting the value of x_3, the LP model will be formulated as under

Maximise $Z = 2{,}250x_1 + 2{,}300x_2 + 2{,}400\ (200 - x_1 - x_2) - 2{,}00{,}000$

or $Z = 2{,}250x_1 + 2{,}300x_2 + 4{,}80{,}000 - 2{,}400x_1 + 2400x_2 - 2{,}00{,}000$

$Z = -150x_1 - 100x_2 + 2{,}80{,}000$

Subject to constraints:

	$x_1 \geq 20$ (i.e. 10% of 200)	(Minimum number of Delux type)
	$x_2 \geq 70$ (i.e. 35% of 200)	(Minimum number of Standard type)
	$x_2 \leq 140$ (i.e. 70% of 200)	(Maximum number of Standard type)
	$x_1 \leq 60$ (i.e. 70% of 200)	(Maximum seat available for Delux type)
	$x_1 + x_2 \geq 120$	(Tourist constraint)
	$x_3 \geq 60$	(Minimum no. of Economy type)
	$-x - x_2 + 200 \geq 60$	
or,	$-x - x_2 \geq -140$	
or,	$x_1 + x_2 \leq 140$	
	$x_1 + x_2 \geq 0$	(Non-Negativity constraint)

Part (c)

Step 1: Finding the vertex of each constraint by treating the constraint of inequality nature as equality.

Constraint (v) in limiting form $x_1 + x_2 = 120$

when $x_1 = 0 \quad x_2 = 120$

when $x_2 = 0 \quad x_1 = 120$

Thus the vertices are (0, 120) & (120, 0)

Constraint (vii) in limiting form $x_1 + x_2 = 140$

when $x_1 = 0 \quad x_2 = 140$

when $x_2 = 0 \quad x_1 = 140$

Thus the vertices are (0, 140) & (140, 0)

Step 2: Plotting the co-ordinates of 1st constraint on the graph and joining them by a straight line, and shading the feasible region. Similarly drawing a straight line and shading feasible region for other constraints. (Graph on next page)

Step 3: Reading the co-ordinates of the vertices of the common shaded feasible region & putting the co-ordinates of each of such vertex in the objective function. Selecting those vertices which achieve the most optimal solution (i.e. in case of maxmisation vertices which give the maximum value of Z).

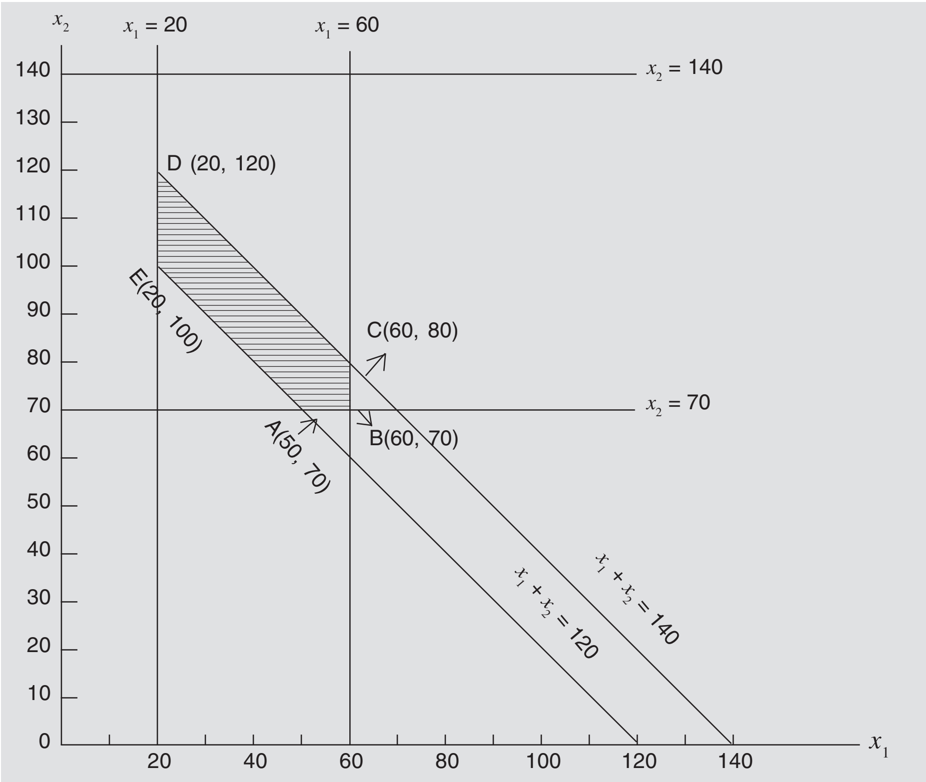
Point	*Co-ordinates of the corner points of the feasible region (values of x_1 and x_2)*	*Values of the objective function $Z = -150x_1 - 100x_2 + 2{,}80{,}000$*
A	(50, 70)	Rs. 2,65,500
B	(60,70)	Rs. 2,64,000
C	(60, 80)	Rs. 2,63,000
D	(20,120)	Rs. 2,65,000
E	(20,100)	Rs. 2,67,000

The maximum profit is attained at the point E, whose coordinates are (20, 100)

Interpretation: The profit of the objective function stated under (a) becomes maximum i.e. Rs. 2,67,000 when

$x_1 = 20$; $x_2 = 100$ and $x_3 = 80$ $\{\therefore\ x_3 = 200 - (x_1 + x_2) = 200 - (20 + 100)\}$

i.e. the travel agent should offer 20 Deluxe, 100 Standard and 80 Economy tour packages so as to get the maximum profit of Rs. 2,67,000.



PROBLEM 2.7

Semicond is an electronics company manufacturing tape recorders and radios. Its per unit labour costs, raw material costs and selling prices are given in Table 1. An extract form from its balance sheet on 31.3.20X1 is shown in Table 2. Its current asset/current liability ratio (called the current ratio) is 2.

Table 1 : Cost Information

Items	*Products*	
	Tape Recorder Rs.	***Radio Rs.***
Selling Price	1,000	900
Labour Cost	500	350
Raw Material Cost	300	400

Table 2 : Extract from Balance Sheet as on 31.3.20X1

	Current Liabilities Rs.	*Current Assets Rs.*
Cash		1,00,000
*Accounts Receivable		30,000
**Inventory		70,000
Short-Term Bank Borrowing	1,00,000	

* Accounts receivable is amount due from customers.

** 100 units of raw material used for tape recorder and 100 units of raw material used for radio.

Semicond must determine how many tape recorders and radios should be produced during April 20X1. Demand is large enough to ensure that all goods produced will be sold. All sales are on credit and payment for goods sold in April 20X1 will not be received until 31.5.20X1. During April 20X1, it will collect Rs. 20,000 in accounts receivable and it must pay off Rs. 10,000. On 30.4.20X1 it will receive a shipment of raw material worth Rs. 20,000, which will be paid for on May 31, 20X1. The management has decided that the cash balance on April 30, 20X1 must be at least Rs. 40,000. Also its banker requires that the ratio of current assets to current liabilities as on April 30, 20X1 be at least 2. In order to maximise the contribution to profit for April 20X1 production it has to find the product mix for April 20X1. Assume that labour costs (wages) monthly rent (Rs. 10,000) are paid in the month in which they are incurred.

Formulate this as a Linear Programming Problem and graphically solve it.

Solution

Let x_1 = Number of tape recorders to be produced

x_2 = Number of tape radios to be produced

Since the objective is to maximise, the objective function is given by —

Maximise Z = $200x_1 + 150x_2$

Condition (1) Regarding funds available for production

$500x_1 + 350x_2 \leq 60,000$

Calculation of Funds Available for Production

A	Balance of cash		1,00,000
B	*Add :* Collection from Debtors		20,000
C	*Less :* Payment of Bank Loan	10,000	
	Payment of monthly rent	10,000	20,000
D	Balance (A + B – C)		1,00,000
E	Minimum Balance required		40,000
F	Funds available (D – E)		60,000

Condition (2) Regarding Maximum Raw Material

$$x_1 \leq 100 \text{ (i.e. } \frac{30,000}{300})$$

$$x_2 \leq 100 \text{ (i.e. } \frac{30,000}{300})$$

Condition (3) Ragarding Ratio of Current Assets to Current Liabilities

Calculation of Current Assets

A	Cash available	100,000
B	*Less :* Utilised for Production	$(800x_1 + 750x_2)$
C	*Add :* (i) Sales Revenue	$1000x_1 + 900x_2$
	(ii) Stock	70,000
	(iii) Debtors	10,000
	(iv) New Stock	20,000
D	Current Assets (A – B + C)	$2,00,000 + 200x_1 + 150x_2$

Calculation of Current Liabilities

A	Bank loan (1,00,000 – 10,000)	90,000
B	Liabilities due to supplies	20,000
C	Current Liability (A + B)	1,10,000

$$\text{Current Ratio} = \frac{\text{Current Assets}}{\text{Current Liabilities}}$$

$$\frac{2,00,000 + 200x_1 + 150x_2}{1,10,000} \geq 2$$

$$2,000,000 + 200x_1 + 150x_2 \geq 2,20,000$$

or $$200x_1 + 150x_2 \geq 20,000$$

Now, LP problem can be shown below:

Maximise $Z = 200x_1 + 150x_2$

Subject to constraints:

$500x_1 + 350x_2 \leq 60,000$ (Funds availability constraints)

$x_1 \leq 100$ (Maximum Raw material available for Tape recorder)

$x_2 \leq 100$ (Maximum Raw material available for Radio)

$200x_1 + 150x_2 \geq 20,000$ (Current Ratio constraint)

$x_1, x_2 \geq 0$ (Non-Negativity constraint)

Step 1: Finding the vertex of each constraint by treating the constraint of inequility value as equality

Constraint (i) in limiting form $500x_1 + 350x_2 = 60,000$

when $x_1 = 0$ $x_2 = 171.43$

when $x_2 = 0$ $x_1 = 120$

Thus the vertices are (0, 171.43) & (120, 0)

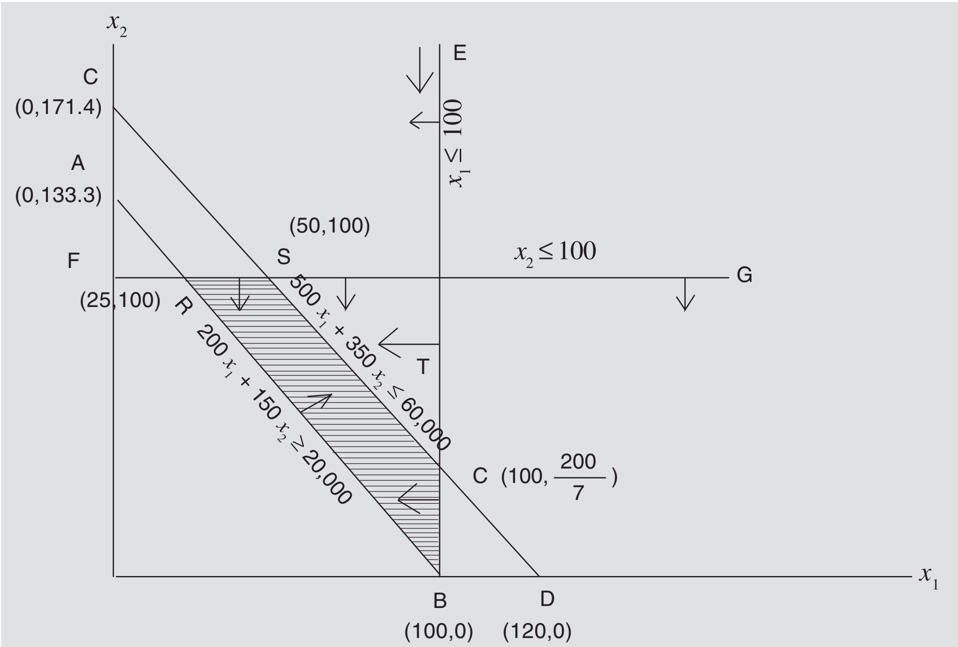
Constraint (iv) in limiting form $200x_1 + 150x_2 = 20,000$

when $x_1 = 0$ $x_2 = 133.33$

when $x_2 = 0$ $x_1 = 100$

Thus the vertices are (0, 133.33) & (100, 0)

Step 2: Plotting the co-ordinates of 1st constraint on the graph and joining them by a straight line, and shading the feasible region. Similarly drawing a straight line and shading feasible region for other constraints.



Step 3: Reading the co-ordinates of the vertices of the common shaded feasible region & putting the co-ordinates of each of such vertex in the objective function. Selecting those vertices which achieve the most optimal solution (i.e. in case of maxmisation vertices which give the maximum value of Z).

Set No.	*Co-ordinates of vertex of Common shaded region*	*Value of Z*
1	R (25, 100)	$Z_1 = 200 \times 25 + 150 \times 100 = 20{,}000$
2	S (50, 100)	$Z_2 = 200 \times 50 + 150 \times 100 = 25{,}000$
3	T (100, 200/7)	$Z_3 = 200 \times 100 + 150 \times 200/7 = 24{,}285.70$
4	D (100, 0)	$Z_4 = 200 \times 100 + 150 \times 0 = 20{,}000$

Optimal Solution: Thus, Set No. 2 gives the maximum value of Z (i.e. Rs. 25,000) at $x_1 = 50$ & $x_2 = 100$.

PROBLEM 2.8

Let us assume that you have inherited Rs. 1,00,000 from your father-in-law that can be invested in a combination of only two stock portfolios, with the maximum investment allowed in either portfolio set at Rs. 75,000. The first portfolio has an average rate of return of 10%, whereas the second has 20%. In terms of risk factors associated with these portfolios, the first has a risk rating of 4 (on a scale from 0 to 10), and the second has 9. Since you wish to maximize your return, you will not accept an average rate of return below 12% or a risk factor above 6. Hence, you then face the important question. How much should you invest in each portfolio?

Formulate this as a Linear Programming Problem and solve it by Graphic Method.

Solution

Let x_1 and x_2 be the amount to be invested in first and second stock portfolio respectively. The average rate of return for first portfolio is 10% and for second portfolio, it ii 20%. Since the company wishes to maximize the return from investment, the objective function is as given below:

Maximise $Z = 0.1x_1 + 0.2x_2$

The maximum amount available for investment is Rs 1,00,000.

Hence, $x_1 + x_2 \leq 1,00,000$... ... (i)

Further, the maximum investment allowed in either portfolio set is Rs. 75,000.

Therefore, $x_1 \leq 75,000$ ...... (ii)

and $x_2 \leq 75,000$ ...... (iii)

The first portfolio has a risk rating of 4 (on a scale from 0 to 10) and the second has 9. The company will not accept a risk factor above 6.

Therefore, $4x_1 + 9x_2 \leq 6\ (x_1 + x_2)$ (iv)

Further, the company will not accept an average rate of return below 12%.

Hence, $0.1x_1 + 0.2\ x_2 \geq 0.12(x_1 + x_2)$ (v)

Also, x_1 and $x_2 \geq 0$. (vi)

The linear programming model for the given problem can now be formulated as follows:

Maximise $Z = 0.1x_1 + 0.2x_2$

Subject to the constraints

$x_1 + x_2 \leq 1,00,000$...... (i)

$x_1 \leq 75,000$...... (ii)

$x_2 \leq 75,000$........ (iii)

$4x_1 + 9x_2 \leq 6(x_1 + x_2)$

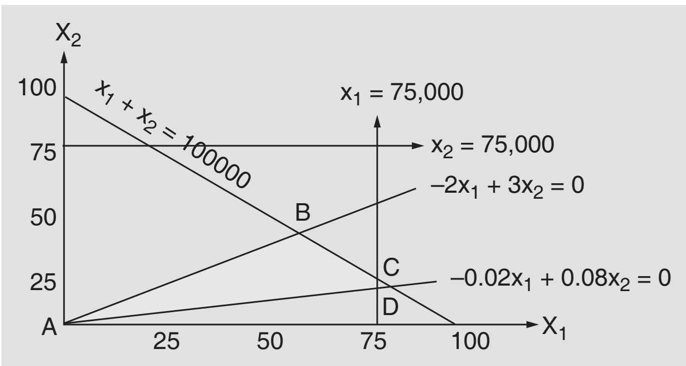
or $-2x_1 + 3x_2 \leq 0$ (iv)

$0.1x_1 + 0.2x_2 \geq 0.12\ (x_1 + x_2)$

or $-0.02x_1 + 0.08x_2 \geq 0$.... (v)

where $x_1, x_2 \geq 0$

The problem is solved graphically below:



The point of intersection for the lines

$$-2x_1 + 3x_2 = 0$$

and $$x_1 + x_2 = 1,00,000$$

is given by B(60,000, 40,000)

The point of intersection for the lines

$$x_1 = 75,000$$

and $$x_1 + x_2 = 1,00,000$$

is given by C(75,000, 25,000)

Similarly, the lines $x_1 = 75,000$

and $$-0.02x_1 + 0.08x_2 = 0$$

intersect at point D(75,000, 18,750)

Thus, the feasible region is bounded by ABCDA and feasible points are A(0, 0); B(60,000, 40,000); C(75,000, 25,000) and D(75,000, 18,750)

Value of the objective function at the above mentioned feasible points is calculated below:

At A, $Z = 0$

At B, $Z = 0.1 \times 60,000 + 0.2 \times 40,000$

$= 6,000 + 8,000 = \text{Rs } 14,000$

At C, $Z = 0.1 \times 75,000 + 0.2 \times 25,000$

$= 7,500 + 5,000 = \text{Rs. } 12,500$

At D, $Z = 0.1 \times 75,000 + 0.2 \times 18,750$

$= 7,500 + 3,750 = \text{Rs. } 11,250$

We find that the value of the objective function is maximum (= Rs. 14,000) at point B(60,000, 40,000).

Hence, the company should invest Rs. 60,000 in first portfolio and Rs. 40,000 in second portfolio to achieve the maximum average rate of return of Rs. 14,000.

PROBLEM 2.9

A Sports Club is engaged in the development of their players by feeding them certain minimum amount of Vitamins (say A, B and C), in addition to their normal diet. In view of this, two types of products X and Y are purchased from the market. The contents of Vitamin constituents per unit, are shown in the following table:

Vitamin Constituents	*Vitamin contents in products*		*Minimum requirement for each player*
	X	*Y*	
A	36	06	108
B	03	12	36
C	20	10	100

The cost of product X is Rs. 20 and that of Y is Rs. 40.

Formulate the linear programming problem for the above and minimize the total cost, and solve problem by using graphic method.

Solution

Let x & y quantity of X & Y product is purchased respectively.

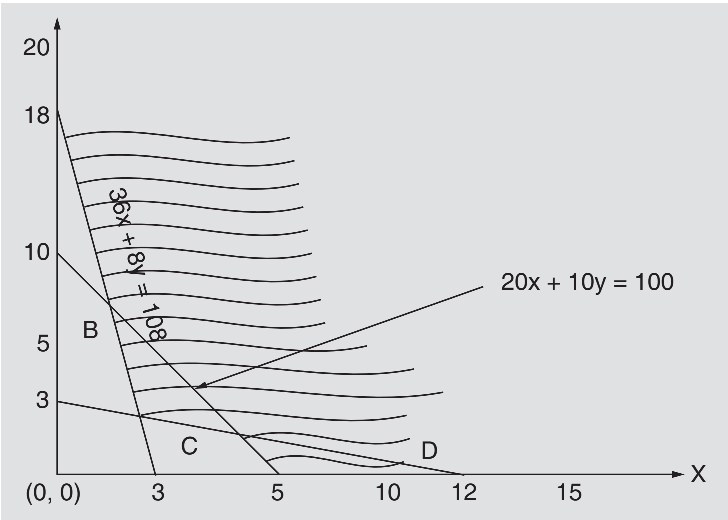
Minimize $Z = 20x + 40y$

Subject to the conditions.

$$36x + 6y \geq 108$$
$$3x + 12y \geq 36$$
$$20x + 10y \geq 100$$

and $x, y \geq 0$

For finding the solution, we plot above equations on x – y plane as shown below.



The feasible region is represented by the shaded area.

The extreme points are A, B, C & D which have the coordinates as

$$A = (0,18),\ B = (2, 6),\ C = (4, 2),\ D = (12, 0)$$

Extreme point	*(x, y)*	*Value of z = 20x + 40y at extreme point*
A	(0, 18)	720
B	(2,6)	280
C	(4, 2)	160 ⇐ Minimum
D	(12,0)	240

Hence, the optimal solution is to purchase 4 units of product X and 2 units of product Y in order to maintain a minimum cost of Rs. 160.

PROBLEM 2.10

A farm is engaged in breeding pigs. The pigs are fed on various products grown in the farm. In view of the need to ensure certain nutrient constituents (call them X, Y and Z), it becomes necessary to buy two additional products say, A and B. One unit of product A contains 36 units ofX, 3 units of Y and 20 units of Z. One unit of product B contains 6 units of X, 12 units of Y and 10 units of Z. The minimum requirement of X, Y and Z is 108 units, 36 units and 100 units respectively. Product A costs Rs. 20 per unit and product B Rs. 40 per unit.

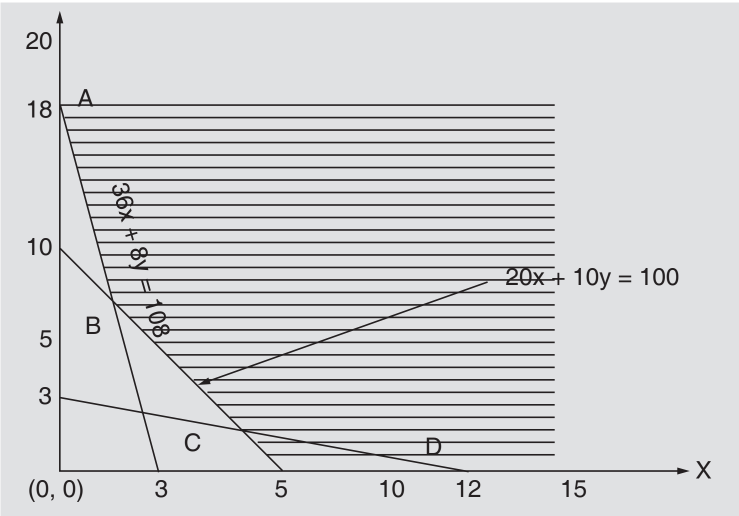
Formulate the above as a linear programming problem to minimize the total cost and solve this problem by using graphic method.

Solution

The data of the given problem can be summarized as under:

Nutrient constituents	*Nutrient contents in products* A	B	*Minimum requirement of nutrient*
X	36	06	108
Y	03	12	36
Z	20	10	100
Cost of product	Rs. 20	Rs. 40	

Let x_1 units of product A and x_2 units of product B are purchased. Making use of the above table, the required mathematical formulation of L.P. problem is as given below:



Minimize $Z = 20x_1 + 40x_2$

Subject to the constraints

$$36x_1 + 6x_2 \geq 108$$

$$3x_1 + 12x_2 \geq 36$$

$$20x_1 + 10x_2 \geq 100$$

and $x_1, x_2 \geq 0$

For solving the above problem graphically, consider a set of rectangular axis in the plane. As each point has the coordinates of type (x_1, x_2), any point satisfying the conditions $x_1 \geq 0$ and $x_2 \geq 0$ lies in the first quadrant only.

The constraints of the given problem as described earlier are plotted by treating them as equations:

$$36x_1 + 6x_2 = 108$$
$$3x_1 + 12x_2 = 36$$
$$20x_1 + 10x_2 = 100$$

or

$$\frac{x_1}{3} + \frac{x_2}{18} = 1$$
$$\frac{x_1}{12} + \frac{x_2}{3} = 1$$
$$\frac{x_1}{5} + \frac{x_2}{10} = 1$$

The area beyond these lines represents the feasible region in respect of these constraints, any point on the straight lines or in the region above these lines would satisfy the constraints. The coordinates of the extreme points of the feasible region are given by—

A = (0, 18), B = (2, 6), C = (4, 2) and D = (12, 0)

The value of the objective function at each of these points can be evaluated as follows:

Extreme point	*(x_1, x_2)*	*$Z = 20x_1 + 40x_2$*
A	(0, 18)	720
B	(2,6)	280
C	(4, 2)	160
D	(12,0)	240

The value of the objective function is minimum at the point C (4, 2).

Hence, the optimum solution is to purchase 4 units of product A and 2 units of product B in order to have minimum cost of Rs. 160.

PROBLEM 2.11

A firm makes two products X and V, and has a total production capacity of 16 tonnes per day. X and Y are requiring the same production capacity. The firm has a permanent contract to supply at least 3 tonnes of X and 6 tonnes of Y per day to another company. Each tonne of X requires 14 machine hours of production time and each tonne of Y requires 20 machine hours of production time, the daily maximum possible number of machine hours is 280. All the firm's output can be sold, and the profit made is Rs. 20 per tonne of X and Rs. 25 per tonne of Y.

Required: Formulate a linear programme to determine the production schedule for maximum profit by using graphical approach and calculate the optimal product mix and profit. ***(Nov., 2010)***

Solution

Maximise Z $20x + 25y$

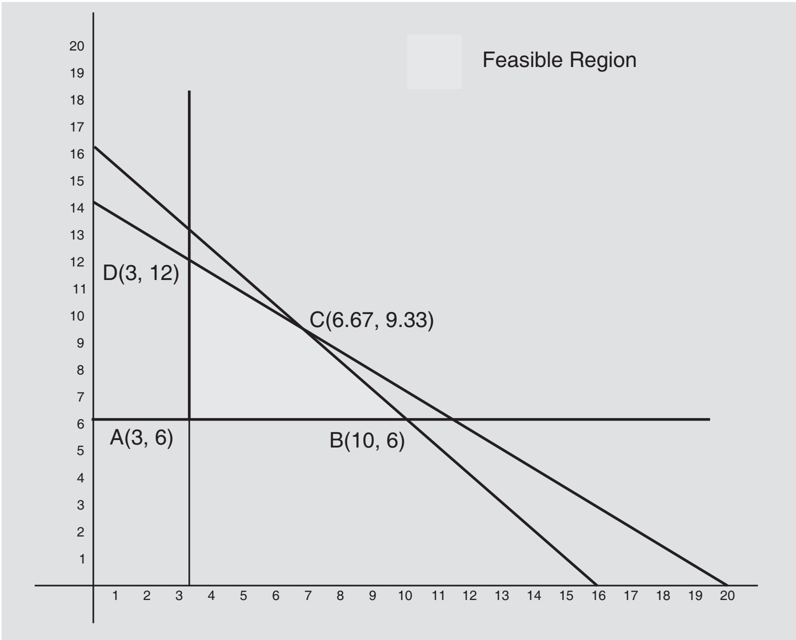
Subject to $x + y \le 16$

$x \ge 3$

$y \ge 6$

$14x + 20y \le 280$

$x, y > 0$



Figure

Z =	20 x +	25y	Total Contribution
Point	X	Y	
A	3	6	210
B	10	6	350
C	6.67	9.33	367 (Optimal)
D	3	12	360

The maximum value of objective function Z= 370 occurs at extreme point C (6.67,9.33).

Hence company should produce x_1 = 6.67 tonnes of product X and x_2 = 9.33 tones of product Y in order to yield a maximum profit of Rs. 367.

3 LINEAR PROGRAMMING — SIMPLEX METHOD

PRACTICAL STEPS INVOLVED IN SOLVING MAXIMISATION PROBLEMS (WHERE ALL CONSTRAINTS HAVE "≤" TYPE INEQUALITIES

Step 1→ Non-Negative Slack Variables—*Add Non-negative Slack Variables (say S_1, S_2 etc.) in each constraint to convert inequalities into equations. A Slack Variable indicates under-utilisation of capacity of the constraint; hence, its contribution to Objective Function is assumed to be zero (or negative, if given). This Slack Variable is also called as Basic Variable. Other variables are called Non-basic Variables.*

Step 2→ Value of Basic Variable—*Assuming the value of each Non-basic Variable equal to 'zero' (i.e. assuming that no performance has taken place), calculate the value of each Basic Variable from the equations.*

Step 3→ Value of Z—*Find the value of 'Z' by putting the value of each basic and non-basic variable in the Objective Function.*

Step 4→ Initial Simplex Table—*Draw the Initial Simplex Table as follows:*

Contribution per unit C_i	Variables	$C_i \rightarrow$ Quantity	C_1 x_1	C_2 x_2	0 S_1	0 S_2	Replacement Ratio Qty/value of key column
C_1	S_1	....	....	....	1	0	
C_2	S_2	....	....	....	0	1	
Total Contribution	Z_j	0	0*	0	....	....	
Opportunity Cost	$(C_i\text{-}Z_j)$		C_1	C_2	0	0	

* **Note:** *The value in the Z_j Row for x_1 variable is computed by the formula = $C_1x_1 + C_2x_1$. The values in the Z_j Row in the column for other variables are computed by the same formula as stated.*

Step 5→ ***Key Column*** *– Mark the Column having **maximum positive value** in $(C_i – Z_j)$ row by ↑ sign representing the Opportunity Cost or Loss of not introducing one unit of the variable of that column. This Column is also known as Key Column. This column indicates the selection of **incoming variable** in the Next Simplex Table.*

3 LINEAR PROGRAMMING — SIMPLEX METHOD

PRACTICAL STEPS INVOLVED IN SOLVING MAXIMISATION PROBLEMS (WHERE ALL CONSTRAINTS HAVE "≤" TYPE INEQUALITIES

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Step 3→ Value of Z—*Find the value of 'Z' by putting the value of each basic and non-basic variable in the Objective Function.*

Step 4→ Initial Simplex Table—*Draw the Initial Simplex Table as follows:*

Contribution per unit		$C_i \rightarrow$	C_1	C_2	0	0	Replacement
C_i	Variables	Quantity	x_1	x_2	S_1	S_2	Ratio Qty/value of key column
C_1 C_2	S_1 S_2	....	....	....	1 0	0 1	
Total Contribution	Z_j	0	0*	0	....	....	
Opportunity Cost	$(C_i\text{-}Z_j)$		C_1	C_2	0	0	

* **Note:** *The value in the Z_j Row for x_1 variable is computed by the formula = $C_1x_1 + C_2x_1$. The values in the Z_j Row in the column for other variables are computed by the same formula as stated.*

Step 5→ ***Key Column*** *– Mark the Column having* ***maximum positive value*** *in $(C_i - Z_j)$ row by ↑ sign representing the Opportunity Cost or Loss of not introducing one unit of the variable of that column. This Column is also known as Key Column. This column indicates the selection of* ***incoming variable*** *in the Next Simplex Table.*

Step 6→ ***Replacement Ratio****–Find out Replacement Ratio (also known as Key Ratio) by dividing value in the Quantity Column of each row by corresponding Key Column value. Replacement Ratio represents how much quantum of variable can be produced based on that row taking the Key Column value.*

Step 7→ ***Key Row****—Mark the Row having* ***minimum non-negative*** *Replacement Ratio by → sign. Minimum Replacement Ratio ensures that no basic variable will ever be negative. (This can be verified by putting higher Replacement Ratio in all constraints). This Row is known as Key Row. This row indicates the selection of* ***outgoing variable*** *from the Current Simplex Table.*

Step 8→ ***Key Element****—Encircle the* ***element at the intersection*** *of Key Row and Key Column. This value is known as Key Element or Pivot Element.*

Step 9→ *Replace the outgoing variable by the incoming variable with its contribution per unit, i.e., C_i in the first Column for Contribution Per Unit.*

Step 10→ *New Values of Key Row—Calculate the New Values of the Key Row as under:*

$$\text{New Values of Key Row} = \frac{\text{Old Values of Key Row}}{\text{Key Element}}$$

Step 11→ **New Values of Non-Key Row—***Calculate the New Values of Other Rows (i.e. Non-Key Rows) as under:*

A	Old Values						
B	New values of Key Row						
C	Key Column Element						
D	Product of B&C						
E	New Values (A–D)						

Step 12→ *Draw another Simplex Table based on new values of each row.*

Step 13→ *Repeat Steps 4 to 12 till all values in $(C_i - Z_j)$ row becomes zero or negative which indicates that any further introduction of additional unit of the variable having negative value in $(C_i - Z_j)$ row will have negative contribution towards Objective Function [no effect in case of zero in $(C_i - Z_j)$ row]. This is also known as condition for optimal solution.*

Step 14→ **Optimal Solution—***Find Optimal Solution from the Final Simplex Table, i.e., the value of Z and quantity of Non-basic Variables.*

$$\text{Value of } Z = C_i\,.(\text{Qty of } x_i) + C_j\,(\text{Qty of } x_j)$$

PROBLEM 3.1

Use simplex method to solve the following L P problem:

Max Z. $= 6x_1 + 8x_2$

Subject to Constraints:

$$2x_1 + 3x_2 \leq 16$$
$$4x_1 + 2x_2 \leq 16$$

Solution

Step 1→ Formulation of LP problem after introducing slack variables:

Max. $Z = 6x_1 + 8x_2 + 0S_1 + 0S_2$

Subject to constraints:

$$2x_1 + 3x_2 + S_1 = 16$$
$$4x_1 + 2x_2 + S_2 = 16$$
$$x_1, x_2, S_1, S_2 \geq 0$$

Step 2→ Preparing Initial Simplex Table:

Simplex Table I

Contribution per unit C_i →			6	8	0	0	Replacement Ratio
↓ C_i	Variable	Qty	x_1	x_2	S_1	S_2	Qty/Value of Key Column
0	S_1	16	2	$\boxed{3}$	1	0	$\frac{16}{3} = 5.33$ ←
0	S_2	16	4	2	0	1	$\frac{16}{2} = 8$
Total Contribution (Z_j)			0	0	0	0	
Opportunity Loss ($C_i - Z_j$)			6	8	0	0	

↑ Key Column Key Element Kew Row ←

Hence, the outgoing variable is S_1 and the incoming variable is x_2

Step 3→ Replacing the outgoing variable (S_1) by incoming variable (x_2) together with its contribution per unit.

Step 4→ Calculating the new values of Key Row as under:

$$\text{New Values of Key Row} = \frac{\text{Old Values of Key Row}}{\text{Key Element}}$$

$$= \frac{16}{3} \quad \frac{2}{3} \quad \frac{3}{3} \quad \frac{1}{3} \quad \frac{0}{3}$$

Step 5→ Calculating the new values of other Rows (i.e. Non-key Row) as under:

A	Old Values	16	4	2	0	1
B	New Values of Key Row	$\frac{16}{3}$	$\frac{2}{3}$	1	$\frac{1}{3}$	0
C	Key Column Element	2	2	2	2	2
D	Product of B & C	$\frac{32}{3}$	$\frac{4}{3}$	2	$\frac{2}{3}$	0
E	New Values (A – D)	$\frac{16}{3}$	$\frac{8}{3}$	0	$-\frac{2}{3}$	1

Step 6→ Preparing Second Simplex Table

Simplex Table II

Contribution per unit $C_i \rightarrow$			6	8	0	0	Replacement Ratio
$C_i \downarrow$	Variable	Qty	x_1	x_2	S_1	S_2	Qty/Value of Key Column
8	x_2	$\frac{16}{3}$	$\frac{2}{3}$	1	$\frac{1}{3}$	0	$\frac{16/3}{2/3} = 8$
0	S_2	$\frac{16}{3}$	$\boxed{\frac{8}{3}}$	0	$-\frac{2}{3}$	1	$\frac{16/3}{8/3} = 2 \leftarrow$
Total Contribution (Z_j)			5.33	8	$\frac{8}{3}$	0	
Opportunity Loss ($C_i - Z_j$)			– 67	0	$-\frac{8}{3}$	0	
			↑ Key Column		Key Element		Key Row

Step 7→ Replacing the outgoing variable (S_2) by incoming variable (x_1) together with its contribution per unit.

Step 8→ Calculating the new values of Key Row as under:

$$\text{New Values of Key Row} = \frac{\text{Old Values of Key Row}}{\text{Key Element}}$$

$$= \frac{16/3}{8/3} \quad \frac{8/3}{8/3} \quad \frac{0}{8/3} \quad \frac{-2/3}{8/3} \quad \frac{1}{8/3}$$

$$= 2 \quad 1 \quad 0 \quad -\frac{1}{4} \quad \frac{2}{3}$$

Step 9→ Calculation of the New Values of Non Key Rows as under:

A	Old Values	$\frac{16}{3}$	$\frac{2}{3}$	1	$\frac{1}{3}$	0
B	New Values of Key Row	2	1	0	$\frac{-1}{4}$	$\frac{3}{8}$
C	Key Column Element	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$	$\frac{2}{3}$
D	Product of B & C	$\frac{4}{3}$	$\frac{2}{3}$	0	$\frac{1}{-6}$	$\frac{1}{4}$
E	New Values (A – D)	4	0	1	$\frac{1}{2}$	$-\frac{1}{4}$

Step 10→ Preparing Simplex Table III

Simplex Table III

Contribution per unit C_i →			6	8	0	0
↓ C_i	Variable	Quantity	x_1	x_2	S_1	S_2
8	x_2	4	0	1	$\frac{1}{2}$	$-\frac{1}{4}$
6	x_1	2	1	0	$-\frac{1}{4}$	$\frac{3}{8}$
Total Contribution (Z_j)		44	6	8	2.5	.25
Opportunity Loss ($C_i - Z_j$)			0	0	–2.5	–.25

Optimal Solution : Since, the values of $C_i - Z_j$ are ≤ 0, the solution is optimal at $x_1 = 2$ and $x_2 = 4$. The optimal value of Z = 2 × 6 + 4 × 8 = Rs. 44.

PRACTICAL STEPS INVOLVED IN SOLVING MINIMISATION PROBLEMS (WHERE ALL CONSTRAINTS HAVE "≥" TYPE INEQUALITIES).

Step 1→ *Deduct Non-negative Surplus Variables (say S_1, S_2, etc.) (also known as Basic Variables) in each constraint to convert inequalities into equations. Suppose the values of Surplus Variables exceed the values of Non-basic Variables; in such case, the total of Left Hand Side of the equation will become negative which is false; hence, we also add Non-negative Artificial Variables (say A_1, A_2, etc.) (also known as Basic Variables) in each constraint to satisfy the equations. We assume that the conribution of each Artificial Variable towards the Objective Function is infinitely large, say 'M'.*

Step 2→ *Assuming the value of each Non-basic Variable and Surplus Variable equal to 'zero', calculate the value of each Artificial Variable from the equations.*

Step 3→ *Find the value of 'Z' by putting the value of each basic and non-basic variable in the Objective Function.*

Step 4→ *Draw the Initial Simplex Table as given below:*

Contribution per unit C_i →			C_1	C_2	0	0	M	M	Replacement
(C_i)	Variables	Quantity	x_1	x_2	S_1	S_2	A_1	A_2	Ratio Qty/Value of column
M	A_1	....	....	....	–1	0	1	0	
M	A_2	....	....	....	0	–1	0	1	
Total Contribution	Z_j	....	....	....	M	M	M	M	
Opportunity Cost ($C_i - Z_j$)			.....	....	–M	–M	0	0	

Note: *The value in the Z_j row for x_1 variable is computed by the formula = $C_1.x_1 + C_2.x_1$. In otherwords, all the values in Z_j Row and ($C_i - Z_j$) Row are computed in the same manner as in the case of Maximisation Problems.*

Step 5→ ***Key Column*** *– Mark the Column having* ***Minimum Value*** *in ($C_i – Z_j$) row by ↑ sign representing the Opportunity Cost or Loss of not introducing one unit of the variable of that column. This Column is also known as Key Column. This column indicates the selection of* ***incoming variable*** *in the Next Simplex Table.*

Step 6→ ***Replacement Ratio*** *– Find out the Replacement Ratio (also known as Key Ratio) by dividing value in the Quantity Column of each row by corresponding Key Column value. Replacement Ratio represents how much quantum of variables can be produced based on that row taking the Key Column value.*

Step 7→ ***Key Row*** *– Mark the Row having* ***Minimum Non-Negative*** *Replacement Ratio by → sign. Minimum Replacement Ratio ensures that no basic variable will ever be negative. (This can be verified by putting higher Replacement Ratio in all constraints). This Row is known as Key Row. This row indicates the selection of* ***outgoing variable*** *from the current simplex table.*

Step 8→ ***Key Element –*** *Encircle the* ***element at the intersection*** *of Key Row and key column. This value is known as Key Element or Pivot Element.*

Step 9→ *Replace the outgoing variable by the incoming variable together with its contribution per unit.*

Step 10→ *Calculate the New Values of the Key Row as under:*

$$\text{New values of key row} = \frac{\text{Old Value of Key Row}}{\text{Key Element}}$$

Step 11→ *Calculate the New Values of other rows (i.e. Non-key rows) as under:*

A Old Values
B New Values of Key Row
C Key Column Element
D Product of B & C
E New Values (A – D)

Step 12→ *Draw another Simplex Table based on new values of each row.*

Step 13→ *Repeat Steps 4 to 12 till all values in ($C_i – Z_j$) row becomes zero or positive which indicates that any further introduction of additional unit of the variable having positive value in ($C_i – Z_j$) row will have negative contribution towards Objective Function [no effect in case of zero in ($C_i – Z_j$) row]. This is also known as condition for optimal solution.*

Step 14→ *Find Optimal Solution from the Final Simplex Table, i.e., the values of Z and quantity of Non-basic Variables.*

PROBLEM 3.2

Use Simplex Method to solve the following LP problem:

Minimize $Z = 3x_1 + 2.5x_2$

Subject to constraints:

$$2x_1 + 4x_2 \geq 40$$
$$-5x_1 - 2x_2 \leq -50$$
$$x_1 ,\ x_2 \geq 0$$

Solution

Step 1→ Formulation of LP problem after introducing Slack and Artificial Variables:

Minimize $3x_1 + 2.5x_2 - 0S_1 - 0S_2 + MA_1 + MA_2$

Subject to constraints:

$$2x_1 + 4x_2 - S_1 + A_1 = 40$$
$$5x_1 + 2x_2 - S_2 + A_2 = 50$$

Step 2→ Preparing Initial Simplex Table I

Simplex Table I

Contribution per unit $C_i \rightarrow$			3	2.5	0	0	M	M	Replacement
↓ C_i	Variable	Quantity	x_1	x_2	S_1	S_2	A_1	A_2	Ratio Qty/Value of Key Column
M	A_1	40	2	4	–1	0	1	0	$\frac{40}{2}=20$
M	A_2	50	$\boxed{5}$ ←	2	0	–1	0	1	$\frac{50}{2}=25$ ←
Total Contribution (Z_j)		90M	7M	6M	–M	–M	M	M	
Opportunity Loss ($C_i - Z_j$)			3–7M ↑	2.5–6M	M	M	0	0	

Key Column Key Element Key Row

Step 3→ Replacing the outgoing variable (A_2) by incoming variable (x_1) together with its contribution per unit.

Step 4→ Calculating the New Values of Key Row as under:

$$\text{New Value of Key Row} = \frac{\text{Old Value of Key Row}}{\text{Key Element}}$$

$$= \frac{50}{5} \quad \frac{5}{5} \quad \frac{2}{5} \quad \frac{0}{5} \quad -\frac{1}{5} \quad \frac{0}{5} \quad \frac{1}{5}$$

$$= 10 \quad 1 \quad \frac{2}{5} \quad 0 \quad -\frac{1}{5} \quad 0 \quad \frac{1}{5}$$

Step 5→ Calculation of the New Values of other Non Key Row:

A	Old Values	40	2	4	–1	0	1	0
B	New Values of Key Row	10	1	$\frac{2}{5}$	0	$-\frac{1}{5}$	0	$\frac{1}{5}$
C	Key Column Element	2	2	2	2	2	2	2
D	Product of B & C	20	2	$\frac{4}{5}$	0	$-\frac{2}{5}$	0	$\frac{2}{5}$
E	New Values (A – D)	20	0	$\frac{16}{5}$	–1	$\frac{2}{5}$	1	$-\frac{2}{5}$

Step 6→ Preparing Second Simplex Table II.

Simplex Table II

Contribution per unit	$C_i \rightarrow$		3	2.5	0	0	M	M	Replacement
↓ C_i	Variable	Quantity	x_1	x_2	S_1	S_2	A_1	A_2	Ratio Qty/Value of Key Column
M	A_1	20	0	$\boxed{\frac{16}{5}}$	–1	$\frac{2}{5}$	1	$-\frac{2}{5}$	$\frac{20}{16/5} = 20$ ←
3	x_1	10	1	$\frac{2}{5}$	0	$-\frac{1}{5}$	0	$\frac{1}{5}$	$\frac{10}{2/5} = 25$
Total Contribution	(Z_j)	20M+30	3	$\frac{16M+6}{5}$	–M	$\frac{2M}{5} - \frac{3}{5}$	M	$-\frac{2}{5}M + \frac{3}{5}$	
Opportunity Loss $(C_i - Z_j)$			0	$25 - \frac{16M+6}{5}$ ↑	M	$\frac{3}{5} - \frac{2M}{5}$	0	$M + \frac{2}{5}M - \frac{3}{5}$	

Key Coloumn Key Element Key Row

Step 7→ Replacing the outgoing variable (A_1) by incoming variable (x_2) together with its contribution per unit.

Step 8→ Calculating the New Values of Key Row as under:

$$\text{New Value of Key Row} = \frac{\text{Old Value of Key Row}}{\text{Key Element}}$$

$$= \frac{20}{16/5} \quad \frac{0}{16/5} \quad \frac{16/5}{16/5} \quad \frac{-1}{16/5} \quad \frac{2/5}{16/5} \quad \frac{1}{16/5} \quad \frac{-2/5}{16/5}$$

$$= 6.5 \quad 0 \quad 1 \quad -\frac{5}{16} \quad \frac{1}{8} \quad \frac{5}{16} \quad -\frac{1}{8}$$

Step 9→ Calculation of the New Values of Non Key Row:

A	Old Values	10	1	$\frac{2}{5}$	0	$-\frac{1}{5}$	0	$\frac{1}{5}$
B	New Values of Key Row	6.25	0	1	$-\frac{5}{16}$	$\frac{1}{8}$	$\frac{5}{16}$	$-\frac{1}{8}$
C	Key Column Element	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{5}$
D	Product of B & C	$\frac{20}{8}$	0	$\frac{2}{5}$	$-\frac{1}{8}$	$\frac{2}{40}$	$\frac{1}{8}$	$-\frac{2}{40}$
E	New Values (A – D)	7.5	1	0	$\frac{1}{8}$	$-\frac{1}{4}$	$-\frac{1}{8}$	$\frac{1}{4}$

Step 10→ Preparing Simplex Table III.

Simplex Table III

Contribution per unit C_i →			3	2.5	0	0	M	M
↓ C_i	Variable	Quantity	x_1	x_2	S_1	S_2	A_1	A_2
2.5	x_2	6.25	0	1	$-\frac{5}{16}$	$\frac{1}{8}$	$\frac{5}{16}$	$-\frac{1}{8}$
3	x_1	7.5	1	0	$\frac{1}{8}$	$-\frac{1}{4}$	$-\frac{1}{8}$	$\frac{1}{4}$
Total Contribution(Z_j)		38.125	3	2-5	-.40625	-.4375	.40625	.4375
Opportunity Loss (C_i–Z_j)			0	0	.40625	.4375	M-.40625	M-.4375

Optimal Solution: Since all $C_i - Z_j$ are ≥ 0, the table provides the optimal solution, i.e. $x_1 = 7.5$, and $x_2 = 6.25$. The optimal value of Z = 7.5 × 3 + 6.25 × 2.5 = 38.125.

PRACTICAL STEPS INVOLVED IN SOLVING MAXIMISATION PROBLEMS (WHERE ANY CONSTRAINT HAS "≥" OR "=" SIGN).

Step 1→

(a) ***Add*** *Non-negative Slack Variables in case of constraints having "≤" sign as usual in case of Maximisation Problems.*

(b) ***Deduct*** *Surplus Variables (say S_1, S_2) and Artificial Variables (say A_1, S_1) in case of constraints having "≥" sign as in case of Minimisation Problems.*

(c) ***Add*** *Artificial Variables (say A_1, S_1) in case of constraints having "=" sign. However, the contribution of Artificial Variables will be deducted in the Objective Function.*

Step 2→ *Follow steps 2 to 14 (given on Page 3.1 & 3.2) as usual in case of Maximisation Problems to obtain the Solution to the given problem.*

PROBLEM 3.3

Use Simplex Method to solve the following LP Problem:

Maximize $Z = 30x_1 + 20x_2$

Subject to constraints:

$$- x_1 - x_2 \geq -8$$
$$-6x_1 - 4x_2 \leq -12$$
$$5x_1 + 8x_2 = 20$$
$$x_1, x_2 \geq 0$$

Solution

Step 1→ Formulation of LP problem after introducing slack and artificial variables:

Max. $Z = 30x_1 + 20x_2 + 0S_1 + 0S_2 - MA_1 - MA_2$

Subject to constraints:

$$x_1 + x_2 + S_1 = 8$$
$$6x_1 + 4x_2 - S_2 + A_1 = 12$$
$$5x_1 + 8x_2 + A_2 = 20$$

Step 2→ Preparing Initial Simplex Table I:

Simplex Table I

Contribution per unit C_i →			30	20	0	0	–M	–M	Replacement
↓ C_i	Variable	Quantity	x_1	x_2	S_1	S_2	A_1	A_2	Ratio Qty/Value of Key Column
0	S_1	8	1	1	1	0	0	0	$\frac{8}{1} = 8$
–M	A_1	12	6	4	0	–1	1	0	$\frac{12}{4} = 3$
–M	A_2	20	5	$\boxed{8}$ ←	0	0	0	1	$\frac{20}{8} = 2.50$ ←
Total Contribution(Z_j)		–32M	–11M	–12M	0	M	–M	–M	
Opportunity Loss (C_i–Z_j)			30+11M	20+12M ↑	0	0	0	0	

Key Coloumn Key Element Key Row

Hence the outgoing variable is A_2 and the incoming variable is x_2.

Step 3→ Replacing the outgoing variable (A_2) by incoming variable (x_2) together with its contribution per unit.

Step 4→ Calculating the New Values of Key Row as under:

$$\text{New Values of Key Row} = \frac{\text{Old Value of Key Row}}{\text{Key Element}}$$

$$= \frac{20}{8} \quad \frac{5}{8} \quad \frac{1}{8} \quad \frac{0}{8} \quad \frac{0}{8} \quad \frac{0}{8} \quad \frac{1}{8}$$

Step 5→ Calculation of the New Values of Other Non Key Rows:

I[st] Non Key Row:

A	Old Values	8	1	1	1	0	0	0
B	New Values of Key Row	2.50	$\frac{5}{8}$	1	0	0	0	$\frac{1}{8}$
C	Key Column Element	1	1	1	1	1	1	1
D	Product of B & C	2.50	$\frac{5}{8}$	1	0	0	0	$\frac{1}{8}$
E	New Values (A – D)	5.50	$\frac{3}{8}$	0	1	0	0	$-\frac{1}{8}$

IInd Non Key Row:

A	Old Values	12	6	4	0	–1	1	0
B	New Values of Key Row	2.50	$\frac{5}{8}$	1	0	0	0	$\frac{1}{8}$
C	Key Column Element	4	4	4	4	4	4	4
D	Product of B & C	10	2.50	4	0	0	0	.50
E	New Values (A – B)	2	3.50	0	0	–1	1	$-\frac{1}{2}$

Step 6→ Preparing Second Simplex Table II.

Simplex Table II

Contribution per unit C_i →			30	20	0	0	–M	–M	Replacement
↓ C_i	Variable	Quantity	x_1	x_2	S_1	S_2	A_1	A_2	Ratio Qty/Value of Key Column
0	S_1	5.50	$\frac{3}{8}$	0	1	0	0	$-\frac{1}{8}$	$\frac{55}{3/8} = 14.67$
–M	A_1	2	$\boxed{3.50}$ ←	0	0	–1	–1	$\frac{1}{-2}$	$\frac{2}{35} = 5.71$ ←
20	x_2	2.50	$\frac{5}{8}$	1	0	0	0	$\frac{1}{8}$	$\frac{2.5}{5/8} = 4$
Total Contribution(Z_j)		50-2M	12.50-3.50M	20	0	M	M	M+2.50	
Opportunity Loss (C_i–Z_j)			17.50+3.50M ↑	0	0	–M	0	–2.50	

Key Coloumn Key Element Key Row

Hence, the outgoing variable is A_1 and the incoming variable is x_1.

Step 7→ Calculating the New Values of Key Row as under:

$$\text{New Values of Key Row} = \frac{\text{Old Values of Key Row}}{\text{Key Element}}$$

$$= \frac{4}{7} \quad 1 \quad 0 \quad 0 \quad -\frac{2}{7} \quad -\frac{2}{7} \quad -\frac{1}{7}$$

Step 8$\rightarrow$ Calculation of the New Values of Non Key Rows:

I[st] Non Key Row:

A	Old Values	5.50	$\frac{3}{8}$	0	1	0	0	$-\frac{1}{8}$
B	New Values of Key Row	$\frac{4}{7}$	1	0	0	$-\frac{2}{7}$	$-\frac{2}{7}$	$-\frac{1}{7}$
C	Key Column Element	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{3}{8}$
D	Product of B & C	$\frac{3}{14}$	$\frac{3}{8}$	0	0	$-\frac{3}{28}$	$-\frac{3}{28}$	$-\frac{3}{56}$
E	New Values (A – D)	$\frac{74}{14}$	0	0	1	$\frac{3}{28}$	$\frac{3}{28}$	$-\frac{1}{14}$

II[nd] Non key Row:

A	Old Values	2.50	$\frac{5}{8}$	1	0	0	0	$\frac{1}{8}$
B	New Values of Key Row	$\frac{4}{7}$	1	0	0	$-\frac{2}{7}$	$-\frac{2}{7}$	$-\frac{1}{7}$
C	Key Column Element	$\frac{5}{8}$	$\frac{5}{8}$	$\frac{5}{8}$	$\frac{5}{8}$	$\frac{5}{8}$	$\frac{5}{8}$	$\frac{5}{8}$
D	Product of B & C	$\frac{5}{14}$	$\frac{5}{8}$	0	0	$-\frac{5}{28}$	$-\frac{5}{28}$	$-\frac{5}{56}$
E	New Values (A – D)	$\frac{30}{14}$	0	1	0	$\frac{5}{28}$	$\frac{5}{28}$	$\frac{3}{14}$

Step 9→ Preparing Simplex Table III.

Simplex Table III

Contribution per unit C_i →			30	20	0	0	–M	–M	Replacement
↓ C_i	Current Variable	Quantity	x_1	x_2	S_1	S_2	A_1	A_2	Ratio Qty/Value of Key Column
0	S_1	$\frac{37}{7}$	0	0	1	$\frac{3}{28}$	$\frac{3}{28}$	$-\frac{1}{14}$	$\frac{37/7}{3/28}=\frac{148}{3}$
30	x_1	$\frac{4}{7}$	1	0	0	$-\frac{2}{7}$	$-\frac{2}{7}$	$-\frac{1}{7}$	$\frac{4/7}{-2/7}=\frac{148}{3}$
20	x_2	$\frac{15}{7}$	0	1	0	$\boxed{\frac{5}{28}}$ ←	$\frac{5}{28}$	$\frac{3}{14}$	$\frac{15/7}{5/28}=12$ ←
Total Contribution(Z_j)		60	30	20	0	$-\frac{20}{28}$	$-\frac{20}{28}$	0	
Opportunity Loss ($C_i - Z_j$)			0	0	0	$\frac{20}{28}$ ↑	$-M+\frac{20}{28}$	–M	

Key Coloumn Key Element Key Row

Hence, the outgoing variable is x_2 and the incoming variable is S_2.

Step 10→ Replacing the outgoing variable (x_2) by incoming variable (S_2) together with its contribution per unit.

Step 11→ Calculating the New Values of Key Row as under:

$$\text{New Values of Key Row} = \frac{\text{Old Values of Key Row}}{\text{Key Element}}$$

$$= \quad 12 \quad 0 \quad \frac{28}{5} \quad 0 \quad 1 \quad 1 \quad \frac{6}{5}$$

Step 12→ Calculation of the New Values of Non Key Rows:

I[st] Non Key Row:

A	Old Values	$\frac{37}{7}$	0	0	1	$\frac{3}{28}$	$\frac{3}{28}$	$-\frac{1}{14}$
B	New Values of Key Row	12	0	$\frac{28}{5}$	0	1	1	$\frac{6}{5}$
C	Key Column Element	$\frac{3}{28}$	$\frac{3}{28}$	$\frac{3}{28}$	$\frac{3}{28}$	$\frac{3}{28}$	$\frac{3}{28}$	$\frac{3}{28}$
D	Product of B & C	$\frac{9}{7}$	0	0	$\frac{3}{5}$	$\frac{3}{28}$	$\frac{3}{28}$	$\frac{3}{70}$
E	New Values (A – D)	4	0	0	$\frac{2}{5}$	0	0	$-\frac{4}{35}$

IInd Non Key Row:

A	Old Values	$\frac{4}{7}$	1	0	0	$-\frac{2}{7}$	$-\frac{2}{7}$	$\frac{1}{7}$
B	New Values of Key Row	12	0	$\frac{28}{5}$	0	1	1	$\frac{6}{5}$
C	Key Column Element	$-\frac{2}{7}$	$-\frac{2}{7}$	$-\frac{2}{7}$	$-\frac{2}{7}$	$-\frac{2}{7}$	$-\frac{2}{7}$	$-\frac{2}{7}$
D	Product of B & C	$-\frac{24}{7}$	0	$-\frac{8}{5}$	0	$-\frac{2}{7}$	$-\frac{2}{7}$	$-\frac{12}{35}$
E	New Values (A – D)	4	1	$\frac{8}{5}$	0	0	0	$\frac{17}{35}$

Step 13→ Preparing Simplex Table IV.

Simplex Table IV

Contribution per unit C_i →			30	20	0	0	–M	–M
↓ C_i	Variable	Quantity	x_1	x_2	S_1	S_2	A_1	A_2
0	S_1	4	0	0	$\frac{2}{5}$	0	0	$-\frac{4}{35}$
30	x_1	4	1	$\frac{8}{5}$	0	0	0	$\frac{17}{35}$
0	S_2	12	0	$\frac{28}{5}$	0	1	1	$\frac{6}{5}$
Total Contribution(Z_j)		120	30	48	0	0	0	$\frac{102}{7}$
Opportunity Loss (C_i– Z_j)			0	–28	0	0	–M	$-M-\frac{102}{7}$

Optimal Solution: Since all $C_i - Z_j$ are ≤ 0 the solution is optimum at $x_1 = 4$ and $x_2 = 0$. Thus, the value of Z = 4 × 30 = 120.

PRACTICAL STEPS INVOLVED IN SOLVING MINIMISATION PROBLEMS (WHERE ANY CONSTRAINT HAS "≤" SIGN OR "=" SIGN)

Step 1→

(a) ***Deduct*** *Non-negative Surplus Variables and* ***Add*** *Artificial Variables in case of constraints having "≥" sign as usual in case of Minimisation Problems.*

(b) ***Add*** *Surplus Variables and Artificial Variables in case of constraints having "≤" sign as in case of Maximisation Problems.*

(c) ***Add*** *Artificial Variables in case of constraints having "=" sign.*

Step 2→ *Follow Steps 2 to 14 (given on Page 3.5 and 3.6) as usual in case of Minimisation Problems to obtain the Solution to the given problem.*

PROBLEM 3.4

Use Simplex Method to solve the following LP problem:

Minimize $Z = 30x_1 + 20x_2$

Subject to constraints:

$$-x_1 - x_2 \geq -8$$
$$-6x_1 - 4x_2 \leq -12$$
$$5x_1 + 8x_2 = 20$$
$$x_1, x_2 \geq 0$$

Solution

Step 1→ Formulation of problem after introducing slack variables:

$$-x_1 - x_2 \geq -8 \text{ or } x_1 + x_2 \leq 8$$
$$-6x_1 - 4x_2 \leq -12 \text{ or } 6x_1 + 4x_2 \geq 12$$

MinimizeZ $= 30x_1 + 20x_2 - 0S_1 - 0S_2 + MA_1 + MA_2$

Subject to constraints:

$$x_1 + x_2 + S_1 = 8$$
$$6x_1 + 4x_2 - S_2 + A_1 = 12$$
$$5x_1 + 8x_2 + A_2 = 20$$

Step 2→ Preparing Initial Simplex Table I:

Simplex Table I

Contribution per unit C_j →			30	20	0	0	–M	–M	Replacement
↓ C_i	Variable	Quantity	x_1	x_2	S_1	S_2	A_1	A_2	Ratio Qty/Value of Key Column
0	S_1	8	1	1	1	0	0	0	$\frac{8}{1} = 8$
M	A_1	12	6	4	0	–1	1	0	$\frac{12}{4} = 3$
M	A_2	20	5	$\boxed{8}$	0	0	0	1	$\frac{28}{8} = 2.5$ ←
Total Contribution (Z_j)			11M	12M	0	–M	M	M	
Opportunity Loss ($C_i - Z_j$)			30–11M	20–12M	0	M	0	0	

Key column   Key Element   Key Row

Step 3→ Replacing the outgoing variable (A_2) by incoming variable (x_2) together with its contribution per unit.

Step 4→ Calculating the New Values of Key Row as under:

$$\text{New Values of Key Row} = \frac{\text{Old Value of Key Row}}{\text{Key Element}}$$

$$= \frac{20}{8} \quad \frac{5}{8} \quad \frac{1}{8} \quad \frac{0}{8} \quad \frac{0}{8} \quad \frac{0}{8} \quad \frac{1}{8}$$

$$= 2.5 \quad .625 \quad 1 \quad 0 \quad 0 \quad 0 \quad .125$$

Step 5→ Calculation of the New Values of Non Key Rows:

I^st Non Key Row:

A	Old Values	8	1	1	1	0	0	0
B	New Values of Key Row	2.5	.625	1	0	0	0	.125
C	Key Column Element	1	1	1	1	1	1	1
D	Product of B & C	2.5	.625	1	0	0	0	.125
E	New Values (A – D)	5.5	.375	0	1	0	0	–.125

II^nd Non Key Row:

A	Old Values	12	6	4	0	–1	1	0
B	New Values of Key Row	2.5	.625	1	0	0	0	.125
C	Key Column Element	4	4	4	4	4	4	4
D	Product of B & C	10	2.5	4	0	0	0	.5
E	New Values (A – D)	2	3.5	0	0	–1	1	–.5

Step 6→ Preparing Second Simplex Table II.

Simplex Table II

Contribution per unit C_i →			30	20	0	0	–M	–M	Replacement
↓ C_i	Variable	Quantity	x_1	x_2	S_1	S_2	A_1	A_2	Ratio Qty/Value of Key Column
20	x_2	2.5	.625	1	0	0	0	.125	$\frac{2.5}{.625} = 4$
0	S_1	5.5	.375	0	1	0	0	–.125	$\frac{5.5}{.375} = 14.67$
M	A_1	2	$\boxed{3.5}$ ←	0	0	–1	1	–.5	$\frac{2}{3.5} = .57$ ←
Total Contribution (Z_j)			12.5+3.5M	20	0	–M	M	.32-.5M	
Opportunity Loss (C_i – Z_j)			17.5-3.5M	0	0	M	0	1.5M-.32	

Key Column Key Element Key Row

Step 7→ Replacing the outgoing variable (A_1) by incoming variable (x_1) together with its contribution per unit.

Step 8→ Calculating the New Values of Key Row as under:

$$\text{New Values of Key Row} = \frac{\text{Old Value of Key Row}}{\text{Key Element}}$$

$$= \frac{2}{3.5} \quad \frac{3.5}{3.5} \quad \frac{0}{3.5} \quad \frac{0}{3.5} \quad \frac{-1}{3.5} \quad \frac{1}{35} \quad \frac{-5}{3.5}$$

$$= .571 \quad 1 \quad 0 \quad 0 \quad .28 \quad -.28 \quad -.4$$

Step 9→ Calculation of the new value of Non Key Rows:

I^st Non Key Row:

A	Old Values	2.5	.625	1	0	0	0	.125
B	New Values of Key Row	.571	1	0	0	-.28	.28	–.14
C	Key Column Element	.625	.625	.625	.625	.625	.625	.625
D	Product of B & C	.356	.625	0	0	–0.175	.175	–.0875
E	New values (A – D)	2.14	0	1	0	.175	–.175	.2125

II^nd Non Key Row:

A	Old Values	5.5	.375	0	1	0	0	–.125
B	New Values of Key Row	.571	1	0	0	–.28	.28	–.14
C	Key Column Element	.375	.375	.375	.375	.375	.375	.375
D	Product of B & C	.21	.375	0	0	–.105	.105	–.0525
E	New values (A – D)	5.285	0	0	1	.105	–.105	–.0725

Step 10→ Preparing Simplex Table III.

Simplex Table III

Contribution per unit C_i →			30	20	0	0	–M	–M
↓ C_i	Variable	Quantity	x_1	x_2	S_1	S_2	A_1	A_2
30	x_1	.571	1	0	0	–.28	.28	–.14
20	x_2	2.14	0	1	0	.175	–.175	.2125
0	S_1	5.285	0	0	1	.105	–.105	–0.0725
Total Contribution (Z_j)			30	20	0	–5	5	0
Opportunity Loss (C_i – Z_j)			0	0	0	5	M–5	M

Optimal Solution: Since all $C_i - Z_j$ are ≥ 0, the table provides the optimal solution, i.e. x_1 = .571 and x_2 = 2.14. The optimal value of Z = (.571 × 30) + (2.14 × 20) = 59.93.

PRACTICAL STEPS INVOLVED IN FORMULATING DUAL PROBLEM FROM THE PRIMAL PROBLEM

The practical steps involved in formulating dual problem from the primal problem are as follows:

Step 1→ *Ensure that in case of a maximization problem all constraints are of '≤' type or in case of a minimization problem, all constraints are of '≥' type.*

If not, adopt the following procedure:

In case of Maximisation Problem

Sign used in Constraints	***Procedure***
1. '≤' type	No change is required.
2. '≥' type	Convert the '≥' type inequality into '≤' type by changing the signs of all values appearing on left hand side and right hand side, (*e.g.* '+' sign into '–' sign, '–' sign into '+' sign).
3. '=' type	(a) Convert equality into two inequalities - one having '≥' sign and the other having '≤' sign. (b) Convert '≥' type inequality into ≤ type inequality by the changing the sign of all values.

In case of Minimization Problem

Sign used in Constraints	***Procedure***
1. '≤' type	Convert '≤' type inequality into '≥' type by changing the signs of all values appearing on left hand side and right hand side. (*e.g.* '+' sign into '–' sign, '–' sign into '+' sign).
2. '≥' type	No change in required.
3. '=' type	(a) Convert equality into two inequalities - one having '≥' sign and the other having '≤' sign. (b) Convert '≤' type inequality into '≥' type by changing the sign of all values.

Step 2→ *Restate the primal problem after taking Step 1.*

Step 3→ ***Change of Objective** – If the primal problem is a maximization problem, the dual problem will be a minimization problem. If the primal problem is a minimization problem, the dual problem will be a maximization problem.*

Step 4→ ***Number of Decision Variables in Dual Problem** – The number of decision variables in the dual problem is equal to the number of constraints in the primal problem.*

Step 5→ ***Coefficient of Decision Variables in Objective Function Dual Problem** – The constraints of the primal problem which appear on the right hand side of the constraints of the primal problem become the coefficients of the decision variables in objective function of the dual problem.*

Step 6→ ***Numbers of Constraints in Dual Problem –*** *The number of constraints in the decision problem is equal to the number of variables in the primal problem.*

Step 7→ *The coefficients of the constraints of the primal problem which appear from left to right are placed from top to bottom in the consraints of the dual problem.*

Step 8→ ***Change of Type of Constraint –*** *If the primal problem has 'less than or equal to' (≤) type constraints, the dual problem will have 'greater than or equal to' (≥) type constraints. If the primal problem has 'greater than or equal to' (≥) type constraints, the dual problem will have 'less than or equal to' (≤) type constraints.*

Step 9→ ***Right hand Side of the Constraints of Dual Problem –*** *The coefficients of objective function of the primal problem appear on the right hand side of the constraints of the dual problem.*

Step 10→ *Non-negativity restriction will also apply to decision variables of dual problem.*

TUTORIAL NOTES:

(i) *The maximum value of the objective function of the primal problem is the minimum value of the objective function of the dual problem.*

(ii) *If the primal problem is in the standard form, the solution of the dual problem can be obtained by multiplying the values of the slack variables in the final Simplex Table by the values ($C_i - Z_j$) appearing on the right hand side of the constraints of the primal problem.*

(iii) *The value of dual variable is referred to as the shadow price or imputed price of a resource. This is the highest price the manufacturer would be willing to pay for the resource. The shadow price of a resource is the unit price that is equal to the increase in profit to be realised by one additional unit of the resource.*

PROBLEM 3.5

Find the dual of the following problem:

Maximize $Z = 30x_1 + 20x_2$

Subject to constraints:

$-x_1 - x_2 \geq -8$ I

$-6x_1 - 4x_2 \leq -12$ II

$5x_1 + 8x_2 = 20$ III

$x_1, x_2 \geq 0$

Solution

Step 1→ *Restating Equality III as two inequalities:*

$5x_1 + 8x_2 \leq 20$ IV

$5x_1 + 8x_2 \geq 20$ V

Step 2→ *Converting '≥' type inequality into '≤' type*

$x_1 + x_2 \leq 8$

$-5x_1 - 8x_2 \leq -20$

Step 3→ *Now, restating the primal as below:*

Maximize $Z = 30x_1 + 20x_2$

Subject to constraints:

$x_1 + x_2 \leq 8$

$$-6x_1 - 4x_2 \le -12$$
$$-5x_1 - 8x_2 \le -20$$
$$5x_1 + 8x_2 \le 20$$
$$x_1,\ x_2 \ge 0$$

Step 4→ *Now formulating the dual as below:*

Minimize $Z = 8y_1 - 12y_2 - 20y_3 + 20y_4$

Subject to constraints:

$$y_1 - 6y_2 - 5y_3 + 5y_4 \ge 30$$
$$y_1 - 4y_2 - 8y_3 + 8y_4 \ge 20$$
$$y_1 \text{ to } y_4 \ge 0 \text{ (non negativity constraint)}$$

PROBLEM 3.6

Find the dual of the following problem:

Minimize $Z = 30x_1 + 20x_2$

Subject to constraints:

$$-x_1 - x_2 \ge -8 \qquad \text{I}$$
$$-6x_1 - 4x_2 \le -12 \qquad \text{II}$$
$$5x_1 + 8x_2 = 20 \qquad \text{III}$$
$$x_1,\ x_2 \ge 0$$

Solution

Step 1→ *Restating equality III as two inequalities:*

$$5x_1 + 8x_2 \le 20 \qquad \text{IV}$$
$$5x_1 + 8x_2 \ge 20 \qquad \text{V}$$

Step 2→ *Converting '≤' type inequality into '≥' type*

$$6x_1 + 4x_2 \ge 12$$
$$-5x_1 - 8x_2 \ge -20$$

Step 3→ *Now, Restating the primal as below:*

Minimize $Z = 30x_1 + 20x_2$

Subject to constraints:

$$-x_1 - x_2 \ge -8$$
$$6x_1 + 4x_2 \ge 12$$
$$-5x_1 - 8x_2 \ge -20$$
$$5x_1 - 8x_2 \ge 20$$
$$x_1,\ x_2 \ge 0$$

Step 4→ *Now the dual will be formulated as below:*

Maximize $Z = -8y_1 + 12y_2 - 20y_3 + 20y_4$

Subject to constraints:

$$-y_1 + 6y_2 - 5y_3 + 5y_4 \le 30$$
$$-y_1 + 4y_2 - 8y_3 + 8y_4 \le 20$$
$$y_1 \text{ to } y_4 \ge 0 \quad \text{(Non-negativity constraint)}$$

PROBLEM 3.7

Formulate the dual for the following linear program:

Maximise $100x_1 + 90x_2 + 40x_3 + 60x_4$

Subject to $6x_1 + 4x_2 + 8x_3 + 4x_4 \leq 140$

$10x_1 + 10x_2 + 2x_3 + 6x_4 \leq 120$

$10x_1 + 12x_2 + 6x_3 + 2x_4 \leq 50$

$x_1, x_2, x_3, x_4, \geq 0$

(Only formulation is required. Please do not solve.) ***(June, 2009)***

Solution

Dual:

Minimise $140y_1 + 120y_2 + 50y_3$

Subject to $6y_1 + 10y_2 + 10y_3 \geq 100$

$4y_1 + 10y_2 + 12y_3 \geq 90$

$8y_1 + 2y_2 + 6y_3 \geq 40$

$4y_1 + 6y_2 + 2y_3 \geq 60$

$y_1, y_2\, y_3\, y_4 \geq 0$

PROBLEM 3.8

The following is a linear programming problem. You are required to set up the initial simplex tableau. (Please do not attempt further iterations or solution):

Maximise $100x_1 = 80x_2$

Subject to $3x_1 + 5x_2 \leq 150$

$x_2 \leq 20$

$8x_1 + 5x_2 \leq 300$

$x_1 + x_2 \geq 25$

$x_1, x_2 \geq 0$ ***(Nov., 2009)***

Solution

Under the usual notations where S_1, S_2, S_3 are Slack Variables, A_4 = Artificial Variable S_4 = Surplus Variable, we have,

Maximise $Z = 100x_1 + 80x_2 + 0S_1 + 0S_2 + 0S_3 + 0S_4 - M\,A_4$.

Subject to $3x_1 + 5x_2 + S_1 = 150$

$x_2 + S_2 = 20$

$8x_1 + 5x_2 + S_3 = 300$

$x_1 + x_2 + - S_4 + A_4 = 25$

		x_1	x_2	S_1	S_2	S_3	S_4	A_4		
Basis	C_1 / C_B	100	80	0	0	0	0	–M		
S_1	0	3	5	1	0	0	0	0	150	√
S_2	0	0	1	0	1	0	0	0	20	√
S_3	0	8	5	0	0	1	0	0	300	√
A_4	–M	1	1	0	0	0	–1	1	25	√
Z_j		–M	–M	0	0	0	M	–M	–25M	√
$C_j - Z_j$		100+M	80+M	0	0	0	–M	0		√

PROBLEM 3.9

Given below is the relevant portion of the first iteration of a linear program under the simplex method, using the usual notations.

			X_1	X_2	S_1	S_2	S_3
Quantity	*Basic Variable*	*Contribution Per unit*	*50*	*40*	*0*	*0*	*0*
150	S_1	0	3	5	1	0	0
20	S_2	0	0	1	0	1	0
296	S_3	0	8	5	0	0	1

(i) Write the initial linear program with the objective function and the in equations.

The following questions are to be answered independent of each other and based on the iteration given above:

(ii) What is the opportunity cost of bringing one unit of x_1 into the solution?

(iii) If we bring 4 units of x_1 into the solution, by how much will the basic variables changes?

(iv) What will be the change in the value of the objective function if 4 units of x_2 are brought into the solution?

(v) What will be the quantity of the incoming variable?

Solution

(i) Maximize $Z = 50x_1 + 40x_2$

Subject to
$$3x_1 + 5x_2 \le 150$$
$$x_2 \le 20$$
$$8x_1 + 5x_2 \le 296$$
$$x_1, x_2 \ge 0$$

(ii) Opportunity Cost of bringing one unit of x_1 into the solution is Rs. 40, (i.e. the contribution lost on not bringing one unit of the next best choice, which is x_2.

(iii)

Change in basic variable per unit of x_1	Change in basic variables for 4 units of x_1	Implication
3	12	S_1 will be reduced by 12
0	0	S_2 will not be impacted
8	32	S_3 will be reduced by 32 units if we bring 4 units of x_1 into the solution

(iv) Objective function value will increase by 4 × 40 = Rs 160 if we bring in 4 units of x_2 into the solution.

(v) x_1 having highest contribution will be the incoming variable. Maximum no of units of x_1 that can come in = Maximum ratio, which is minimum of (150/3, 20/0, 296/8) = Minimum of (150/3, 20/0, 296/8) = Minimum of (50, ∞, 37). Hence quantity of incoming variable x_1 is 37 units.

4 ASSIGNMENT PROBLEMS

PRACTICAL STEPS INVOLVED IN SOLVING MINIMISATION PROBLEMS

Step 1 → ***Dummy Row/Column:*** *See whether Number of Rows are equal to Number of Column. If yes, problem is balanced one; if not, then add a Dummy Row or Column to make the problem a balanced one by alloting zero value or specific value (if any given) to each cell of the Dummy Row or Column, as the case may be.*

Step 2 → ***Row Subtraction:*** *Subtract the minimum element of each row from all elements of that row.*

Note: *If there is zero in each row,there is no need for row substraction.*

Step 3 → ***Column Subtraction:*** *Subtract the minimum element of each column from all elements of that column.*

Note: *If there is zero in each column, there is no need for column substraction.*

Step 4 → ***Minimum No. of Horizontal and/or Vertical Lines:*** *Draw minimum number of Horizontal and/or Vertical Lines to cover all zeros.*

To draw minimum number of lines the following procedure may be followed:

1. *Select a row containing exactly one uncovered zero and draw a vertical line through the column containing this zero and repeat the process till no such row is left.*
2. *Select a column containing exactly one uncovered zero and draw a horizontal line through the row containing the zero and repeat the process till no such column is left.*

Step 5 → ***Subtraction/Addition of Minimum Uncovered Element:*** *If the total lines covering all zeros are equal to the size of the matrix of the Table, we have got the optimal solution; if not,* ***subtract*** *the minimum uncovered element from all uncovered elements and* ***add*** *this element to all elements at the intersection point of the lines covering zeros.*

Step 6 → ***Repeat Steps 4 and 5*** *till minimum number of lines covering all zeros is equal to the size of the matrix of the Table.*

Step 7 → ***Assignment:*** *Select a row containing exactly one unmarked zero and surround it by ' 'and draw a vertical line through the column containing this zero. Repeat this process till no such row is left; then select a column containing exactly one unmarked zero and surround it by ' ' and draw a horizontal line through the row containing this zero and repeat this process till no such column is left.*

Note: *If there are more than one unmarked zero in any row or column, it indicates that an alternative solution exists. In this case, select any one arbitrarily and pass two lines horizontally and vertically.*

Step 8 → *Add up the value attributable to the allocation which shall be the minimum value.*

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Step 6 → ***Repeat Steps 4 and 5*** *till minimum number of lines covering all zeros is equal to the size of the matrix of the Table.*

Step 7 → ***Assignment:*** *Select a row containing exactly one unmarked zero and surround it by ' 'and draw a vertical line through the column containing this zero. Repeat this process till no such row is left; then select a column containing exactly one unmarked zero and surround it by ' ' and draw a horizontal line through the row containing this zero and repeat this process till no such column is left.*

Note: *If there are more than one unmarked zero in any row or column, it indicates that an alternative solution exists. In this case, select any one arbitrarily and pass two lines horizontally and vertically.*

Step 8 → *Add up the value attributable to the allocation which shall be the minimum value.*

Step 9 → ***Alternate Solution:*** *If there are more than one unmarked zero in any row or column, select the other one (i.e., other than the one selected in Step 7) and pass two lines horizontally and vertically. Add up the value attributable to the allocation which shall be the minimum value.*

PROBLEM 4.1

To stimulate interest and provide an atmosphere for intellectual discussion, a finance faculty in a management school decides to hold special seminars on four contemporary topics - leasing, portfolio management, private mutual funds, swaps, and options. Such seminars should be held once per week in the afternoons. However, scheduling these seminars (one for each topic, and not more than one seminar per afternoon) has to be done carefully so that the number of students unable to attend is kept to a minimum. A careful study indicates that the number of students who cannot attend a particular seminar on a specific day is as follows:

	Leasing	*Portfolio Management*	*Private Mutual funds*	*Swaps and Options*
Monday	50	40	60	20
Tuesday	40	30	40	30
Wednesday	60	20	30	20
Thursday	30	30	20	30
Friday	10	20	10	30

Required: Find an optimal schedule of the seminars. Also find out the total number of students who will be missing at least one seminar.

Solution

Step 1 → Introducing a dummy topic to make the problem balanced one by alloting zero students.

	Leasing	*Portfolio Management*	*Private Mutual funds*	*Swaps and Options*	*Dummy*
Monday	50	40	60	20	0
Tuesday	40	30	40	30	0
Wednesday	60	20	30	20	0
Thursday	30	30	20	30	0
Friday	10	20	10	30	0

Step 2 → ***Column Subtraction:*** Subtracting the minimum element of each column from all elements of that column. Since there is zero in each row, there is no need for row subtraction. Drawing minimum number of lines to cover all zeros.

	Leasing	*Portfolio Management*	*Private Mutual funds*	*Swaps and Options*	*Dummy*
Monday	40	20	50	0	0
Tuesday	30	10	30	10	0
Wednesday	50	0	20	0	0
Thursday	20	10	10	10	0
Friday	0	0	0	10	0

Since the number of lines (4) ≠ order of matrix (5), we will have to take step to increase the number of zeros.

Step 3 → Subtracting the minimum uncovered element (10 in this case) from all uncovered elements and adding it to all elements at the intersection point of the above lines and then drawing minimum number of lines to cover all zeros.

	Leasing	*Portfolio Management*	*Private Mutual funds*	*Swaps and Options*	*Dummy*
Monday	30	20	40	0	0
Tuesday	20	10	20	10	0
Wednesday	40	0	10	0	0
Thursday	10	10	0	10	0
Friday	0	10	0	20	10

Since number of lines drawn (5) = order of matrix (5), the above matrix will provide the optimal solution.

Step 4 → ***Assignment:*** Selecting a row containing exactly one unmarked zero and surrounding it by '☐' and draw a vertical line through the column containing this zero. Repeating this process till no such row is left : then selecting a column containing exactly one unmarked zero and surrounding it by '☐' and drawing a horizontal line through the row containing this zero and repeating the process till no such column is left.

	Leasing	*Portfolio Management*	*Private Mutual funds*	*Swaps and Options*	*Dummy*
Monday	30	20	40	$\boxed{0}$	0
Tuesday	20	10	20	10	$\boxed{0}$
Wednesday	40	$\boxed{0}$	10	0	0
Thursday	10	10	$\boxed{0}$	10	0
Friday	$\boxed{0}$	10	0	20	10

Step 5 → Computing minimum number of Students:

Day	***Seminar***	***No. of Students***
Monday :	Swaps and options	20
Tuesday :	No seminar	0
Wednesday :	Portfolio Management	20
Thursday :	Pvt. Mutual funds	20
Friday :	Leasing	10
		70

Thus, the total number of students who will be missing at least one seminar = 70

PROBLEM 4.2

A factory is going to modify of a plant layout to install four new machines M1, M2, M3 and M4. There are 5 vacant place J, K, L, M and available. Because of limite space machine M2 cannot be placed at L and M3 cannot be placed at J. The cost of locating machine to place in Rupees is shown below:

(Rs)

	J	*K*	*L*	*M*	*N*
M1	18	22	30	20	22
M2	24	18	–	20	18
M3	–	22	28	22	14
M4	28	16	24	14	16

Required: Determine the optimal assignment schedule in such a manner that the total cost are kept at a minimum,

Solution

Dummy machine (M5) is inserted to make it a balanced cost matrix and assume its installation cos to zero. Cost of install at cell M3 (J) and M2 (L) is very high marked as é.

	J	*K*	*L*	*M*	*N*
M1	18	22	30	20	22
M2	24	18	é	20	18
M3	é	22	28	22	14
M4	28	16	24	14	16
M5 (Dummy)	0	0	0	0	0

Step 1 → *Subtract the minimum element of each row from each element of that row*

	J	*K*	*L*	*M*	*N*
M1	0	4	12	2	4
M2	6	0	é	2	0
M3	é	8	14	8	0
M4	14	2	10	0	2
M5 (Dummy)	0	0	0	0	0

Step 2 → *Subtract the minimum element of each column from each element of that column*

	J	*K*	*L*	*M*	*N*
M1	0	4	12	2	4
M2	6	0	é	2	0
M3	é	8	14	8	0
M4	14	2	10	0	2
M5 (Dummy)	0	0	0	0	0

Step 3 → *Draw lines to connect the zero as under:*

	J	*K*	*L*	*M*	*N*
M1	0	4	12	2	4
M2	6	0	é	2	0
M3	é	8	14	8	0
M4	14	2	10	0	2
M5 (Dummy)	0	0	0	0	0

There are five lines which are equal to the order of the matrix. Hence the solution is optimal. We may proceed to make the assignment as under:

	J	*K*	*L*	*M*	*N*
M1	$\boxed{0}$	4	12	2	4
M2	6	$\boxed{0}$	é	2	0
M3	é	8	14	8	$\boxed{0}$
M4	14	2	10	$\boxed{0}$	2
M5 (Dummy)	0	0	$\boxed{0}$	0	0

The following is the assignment which keeps the total cost at minimum:

Machines	*Location*	*Cost Rs.*
M1	J	18
M2	K	18
M3	N	14
M4	M	14
M5 (Dummy)	L	0
Total		64

PROBLEM 4.3

Five swimmers are eligible to compete in a relay team which is to consist of four swimmers swimming four different swimming styles; back stroke, breast stroke, free style and butterfly. The time taken for the five swimmers—Anand, Bhaskar, Chandru, Dorai and Easwar—to cover a distance of 100 meters in various swimming styles are given below in minutes: seconds. Anand swims the back stroke in 1 : 09, the breast stroke in 1 : 15, and has never competed in the free style or butterfly. Bhaskar is a free style specialist averaging 1 : 01 for the 100 meters but can also swim the breast stroke in 1 : 16 and butterfly in 1 : 20. Chandru swims all styles - back stroke 1 : 10, butterfly 1 : 12, free style 1 : 05, and breast stroke 1 : 20. Dorai swims only the butterfly 1 : 11 while Easwar swims the back stroke 1 : 20, the breast stroke 1 : 16, the free style 1 : 06 and the butterfly 1 : 10. Which swimmer should be assigned to which swimming style? Who will not be in the relay?

Solution

Step 1 → Let us first create the assignment matrix with time expressed in seconds and Introducing a dummy to make the problem a balanced one by alloting zero.

	Back Stroke	*Breast Stroke*	*Free style*	*Butterfly*	*Dummy*
Anand	69	75	—	—	0
Bhaskar	—	76	61	80	0
Chandru	70	80	65	72	0
Dorai	—	—	—	71	0
Easwar	80	76	66	70	0

Step 2 → ***Column Subtraction:*** Subtracting the minimum element of each column from all elements of that column. Since there is zero in each row, there is no need for row subtraction. Drawing minimum number of Lines to cover all zeros.

	Back Stroke	*Breast Stroke*	*Free style*	*Butterfly*	*Dummy*
Anand	0	0	—	—	0
Bhaskar	—	1	0	10	0
Chandru	1	5	4	2	0
Dorai	—	—	—	1	0
Easwar	11	1	5	0	0

Since the number of lines (4) ≠ order of matrix (5), we will have to take step to increase the number of zeros.

Step 3 → Subtracting the minimum uncovered element (1 in this case) from all uncovered elements and adding it to all elements at the intersection point of the above lines and then drawing minimum numbers of lines to cover all zeros.

	Back Stroke	*Breast Stroke*	*Free style*	*Butterfly*	*Dummy*
Anand	0	0	—	—	1
Bhaskar	—	1	0	10	1
Chandru	0	4	3	1	0
Dorai	—	—	—	0	0
Easwar	11	1	5	0	1

Since the number of lines drawn (5) = the order of the matrix (5), the above matrix will provide the optimal solution.

Step 4 → ***Assignment:*** Selecting a row containing exactly one unmarked zero and surrounding it by '□' and draw a vertical line through the column containing this zero. Repeating this process till no such row is left; then selecting a column containing exactly one unmarked zero and surrounding it by '□' and draw a horizontal line through the row containing this zero and repeating the process till no such column is left.

	Back Stroke	*Breast Stroke*	*Free style*	*Butterfly*	*Dummy*
Anand	0	$\boxed{0}$	—	—	1
Bhaskar	0	1	$\boxed{0}$	10	1
Chandru	$\boxed{0}$	4	3	1	0
Dorai	—	—	—	0	$\boxed{0}$
Easwar	11	1	5	$\boxed{0}$	1

Step 5 → Computing minimum time:

	Time
Anand will be in Breast Stroke	75 seconds
Bhaskar will be in Free stroke	61 seconds
Chandru will be in Back stroke	70 seconds
Dorai will not participate (dummy)	00 seconds
Easwar will be in Butterfly	70 seconds
Total minimum time in the relay	276 (or 4 minutes 36 sec.)

Dorai will be out of the relay.

PROBLEM 4.4

Four Operators O_1, O_2, O_3 and O_4 are available to a manager who has to get four jobs J_1, J_2, J_3 and J_4 done by assigning one job to each operator. Given the time needed by different operators for different jobs in the matrix below:

	J_1	J_2	J_3	J_4
O_1	12	10	10	8
O_2	14	12	15	11
O_3	6	10	16	4
O_4	8	10	9	7

(i) How should manager assign the jobs so that the total time needed for all four jobs is minimum?

(ii) If job J_2 is not to be assigned to operator O_2, what should be the assignment over how much additional total time will be required?

Solution

Step 1 → *Row subtraction:* Subtracting the minimum element of each row from all elements of that row.

Operators	Job			
	J_1	J_2	J_3	J_4
O_1	4	2	2	0
O_2	3	1	4	0
O_3	2	6	12	0
O_4	1	3	2	0

Step 2 → *Column Subtraction:* Subtracting the minimum element of each column from all element of that column and drawing minimum number of lines to cover all zeros.

Operators	Job			
	J_1	J_2	J_3	J_4
O_1	3	1	0	0
O_2	2	0	2	0
O_3	1	5	10	0
O_4	0	2	0	0

Since the number of lines drawn (4) = order of matrix (4), the above matrix will provide the optimal solution.

Step 3 → *Assignment:* Selecting a row containing exactly one unmarked zero and surrounding it by ‘ ’ and draw a vertical line through the column containing this zero. Repeating this process till no such row is left; they selecting a column containing exactly one unmarked zero and surrounding it by ‘ ’ and draw a horizontal line through the row containing this zero and repeating the process till no such column is left.

Operators	Job			
	J_1	J_2	J_3	J_4
O_1	3	1	$\boxed{0}$	0
O_2	2	$\boxed{0}$	2	0
O_3	1	5	10	$\boxed{0}$
O_4	$\boxed{0}$	2	0	0

Step 4 → Computing the Minimum Time:

	Time
Operator O_1 is assigned to Job J_3	10
Operator O_2 is assigned to Job J_2	12
Operator O_3 is assigned to Job J_4	4
Operator O_4 is assigned to Job J_1	8
Total	34

(ii) Step 1 → If Job J_2 is not be assigned to operator O_2 then put a '–' in the cell to obtain the following matrix.

Operators	*Job*			
	J_1	J_2	J_3	J_4
O_1	12	10	10	8
O_2	14	—	15	11
O_3	6	10	16	4
O_4	8	10	9	7

Step 2 → ***Row Subtraction:*** Subtracting the minimum element of each row from all elements of that row.

Operators	*Job*			
	J_1	J_2	J_3	J_4
O_1	4	2	2	0
O_2	3	—	4	0
O_3	2	6	12	0
O_4	1	3	2	0

Step 3 → ***Column Subtraction:*** Subtracting the minimum element of each column from all elements of that column and drawing minimum number of lines to cover all zeros.

Operators	*Job*			
	J_1	J_2	J_3	J_4
O_1	3	0	0	0
O_2	2	—	2	0
O_3	1	4	10	0
O_4	0	1	0	0

Since the number of lines = 3 and order of matrix = 4, we will have to take step to increase the number of zeros.

Step 4 → Subtracting the minimum uncovered element (1 in this case) from all uncovered elements and adding it to all elements at the intersection point of the above lines and then drawing minimum number of lines to cover all zeros.

Operators	*Job*			
	J_1	J_2	J_3	J_4
O_1	3	0	0	1
O_2	1	—	1	0
O_3	0	3	9	0
O_4	0	1	0	1

Since the number of lines drawn (4) = order of matrix (4), the above matrix will provide the optimal solution.

Step 5 → ***Assignment:*** Selecting a row containing exactly one unmarked zero and surrounding it by ' ' and draw a vertical line through the column containing this zero. Repeating this process till no such row is left; then selecting a column containing exactly one unmarked zero and surrounding it by '•' and draw a horizontal line through the row containing this zero and repeating the process till no such column is left.

Operators	Job			
	J_1	J_2	J_3	J_4
O_1	3	$\boxed{0}$	0	1
O_2	1	–	1	$\boxed{0}$
O_3	$\boxed{0}$	3	9	0
O_4	0	1	$\boxed{0}$	1

Step 6 → Computing minimum Time:

	Time
Operator O_1 is assigned Job J_2	10
Operator O_2 is assigned Job J_4	11
Operator O_3 is assigned Job J_1	6
Operator O_4 is assigned Job J_3	9
Total	36

Additional Total Time Required = 36 – 34 = 2 units of time.

PROBLEM 4.5 [Assignments of Job to Typists]

A solicitor firm employs typists on hourly piece basis for their daily work. There are five typists for service and their charges and speeds are different. According to an earlier understanding only one job is given to one typist and the typist is paid for full hours even if he works for a fraction of an hour. Find the least cost allocation for the following data:

Typist	*Rate per hour (Rs.)*	*No. of pages Typed/hour*	*Job*	*No. of Pages*
A	5	12	P	199
B	6	14	Q	175
C	3	8	R	145
D	4	10	S	298
E	4	11	T	178

Solution

Step 1 → The following matrix gives the cost incurred if the typist (i = A, B, C, D, E) executes the job (j = P, Q, R, S, T) which is calculated by following formula:

$$= \frac{\text{Total No. of Pages}}{\text{No. of pages typed/hour}} \text{ (Rounded off to next integer) } \updownarrow \text{ Rate per hour (Rs.)}$$

Typist	*Job*				
	P	*Q*	*R*	*S*	*T*
A	85	75	65	125	75
B	90	78	66	132	78
C	75	66	57	114	69
D	80	72	60	120	72
E	76	64	56	112	68

Step 2 → ***Row Subtraction:*** Subtracting the minimum element of each row from all elements of that row.

Typist	Job				
	P	*Q*	*R*	*S*	*T*
A	20	10	0	60	10
B	24	12	0	66	12
C	18	9	0	57	12
D	20	12	0	60	12
E	20	8	0	56	12

Step 3 → ***Column Subtraction:*** Subtracting the minimum element of each column from all the elements of that column and then drawing the minimum number of lines to cover all zeros.

Typist	Job				
	P	*Q*	*R*	*S*	*T*
A	2	2	0	4	0
B	6	4	0	10	2
C	0	1	0	1	2
D	2	4	0	4	2
E	2	0	0	0	2

Since the number of lines = 4 and order of matrix = 5, we will have to take step to increase the number of zeros.

Step 4 → Subtracting the minimum uncovered element (2 in this case) from all uncovered elements and adding it to all elements at the intersection point of the above lines and then drawing minimum number of lines to cover all zeros.

Typist	Job				
	P	*Q*	*R*	*S*	*T*
A	2	2	2	4	0
B	4	2	0	8	0
C	0	1	2	1	2
D	0	2	0	2	0
E	2	0	2	0	2

Since number of lines = 4 and order of matrix = 5, we will have to take step to increase the number of zeros.

Step 5 → Subtracting the minimum uncovered element (1 in this case) from all uncovered elements and adding it to all elements at the intersection point of the above lines and then drawing the minimum number of lines to cover all zeros.

Typist	Job				
	P	*Q*	*R*	*S*	*T*
A	2	1	2	3	0
B	4	1	0	7	0
C	0	0	2	0	2
D	0	1	0	1	0
E	3	0	3	0	3

Step 6 → *Assignment:* Selecting a row containing exactly one unmarked zero and surrounding it by '☐' and draw a vertical line thorough the column containing this zero. Repeating this process till no such row is left; then selecting a column containing exactly one unmarked zero and surrounding it by '☐' and draw a horizontal line through the row containing this zero and repeating the process till no such column is left.

Typist	*Job*				
	P	*Q*	*R*	*S*	*T*
A	2	1	2	3	$\boxed{0}$
B	4	1	$\boxed{0}$	7	0
C	0	$\boxed{0}$	2	0	2
D	$\boxed{0}$	1	0	1	0
E	3	0	3	$\boxed{0}$	3

Step 7 → Computing minimum cost:

Typist	*Job*		*Cost (Rs.)*
A	T	:	75
B	R	:	66
C	Q	:	66
D	P	:	80
E	S	:	112
	Total	Rs.	399

Alternate Solution:

Typist	*Job*				
	P	*Q*	*R*	*S*	*T*
A	2	1	2	3	$\boxed{0}$
B	4	1	$\boxed{0}$	7	0
C	0	0	2	$\boxed{0}$	2
D	$\boxed{0}$	1	0	1	0
E	3	$\boxed{0}$	3	0	3

Minimum cost:

Typist	*Job*	*Cost (Rs.)*
A	T	75
B	R	66
C	S	114
D	P	80
E	Q	64
	Total	399

PROBLEM 4.6 [Assignment of Rooms]

Welldone Company has taken the third floor of a multistoreyed building for rent with a view to locate one of their zonal offices. There are five main rooms in this floor to be assigned to five managers. Each room has its own advantages and disadvantages. Some have windows, some are closer to the washrooms or to the canteen or secretarial pool. The rooms are of all different sizes and shapes. Each of the five manages were asked to rank their room preferences amongst the rooms 301, 302, 303, 304 and 305. Their preferences were recorded in a table as indicated below.

Manager				
M_1	M_2	M_3	M_4	M_5
302 303 304	302 304 305 301	303 301 304 305 302	302 305 304 303	301 302 304

Most of the managers did not list all the five rooms since they were not satisified with some of these rooms and they have left off these from the list. Assuming that their preferences can be quantified by numbers, find out as to which manager should be assigned to which rooms so that their total preference ranking is a minimum.

Solution

Step 1 → Formulating the preference ranking assignment problem.

	Managers				
Rooms No.	M_1	M_2	M_3	M_4	M_5
301	—	4	2	—	1
302	1	1	5	1	2
303	2	—	1	4	—
304	3	2	3	3	3
305	—	3	4	2	—

Step 2 → ***Row Subtraction:*** Subtracting the minimum element of each row from all elements of that row. Since there is zero in each column, there is no need for column subtraction. Drawing the minimum number of lines to cover all zeros.

	Managers				
Rooms No.	M_1	M_2	M_3	M_4	M_5
301	—	3	1	—	0
302	0	0	4	0	1
303	1	—	0	3	—
304	1	0	1	1	1
305	—	1	2	0	—

Since number of lines = 5 and order of matrix = 5, the above matrix will provide the optimal solution.

Step 3 → ***Assignment:*** Selecting a row containing exactly one unmarked zero and surrounding it by '☐' and draw a vertical line thorough the column containing this zero. Repeating this process till no such row is left; then selecting a column containing exactly one unmarked zero and surrounding it by '☐' and draw a horizontal line through the row containing this zero and repeating the process till no such column is left.

Rooms No.	*Managers*				
	M_1	*M_2*	*M_3*	*M_4*	*M_5*
301	—	3	1	—	$\boxed{0}$
302	$\boxed{0}$	0	4	0	1
303	1	—	$\boxed{0}$	3	—
304	1	$\boxed{0}$	1	1	1
305	—	1	2	$\boxed{0}$	—

Step 4 → Computing minimum ranking

The assignment is given below:

			Ranking
M_1	→	Room No 302	1
M_2	→	Room No 304	2
M_3	→	Room No 303	1
M_4	→	Room No 305	2
M_5	→	Room No 301	1
			7

Thus, the total minimum ranking is 7.

PROBLEM 4.7

A machine operator processes five types of items on his machine each week, and must choose a sequences for them. The set-up cost per change depends on the item presently on the machine and the set-up to be made, according to the following table:

For item	*To item*				
	A	*B*	*C*	*D*	*E*
A	—	4	7	3	4
B	4	—	6	3	4
C	7	6	—	7	5
D	3	3	7	—	7
E	4	4	5	7	—

If he processes each type of item once and only once each week, how should he sequence the items on his machine in order to minimize the total set-up cost?

Solution

Step 1 → *Row Subtraction:* Subtracting the minimum element of each row from all elements of that row.

For item	*To item* A	B	C	D	E
A	—	1	4	0	1
B	1	—	3	0	1
C	2	1	—	2	0
D	0	0	4	—	4
E	0	0	1	3	—

Step 2 → *Column Subtraction:* Subtracting the minimum element of each column from all elements of that column and drawing minimum number of lines to cover all zeros.

For item	*To item* A	B	C	D	E
A	—	1	3	0	1
B	1	—	2	0	1
C	2	1	—	2	0
D	0	0	3	—	4
E	0	0	0	3	—

Since number of lines = 4 and order of matrix = 5, we will have to take step to increase the number of zeros.

Step 3 → Subtracting the minimum uncovered element (1 in this case) from all the uncovered elements and adding it to all the elements at the intersection point of the above lines and drawing minimum number of Lines to cover all zeros.

For item	*To item* A	B	C	D	E
A	—	0	2	0	0
B	0	—	1	0	0
C	2	1	—	3	0
D	0	0	3	—	4
E	0	0	0	4	—

Since the number of lines (5) = order of matrix (5), the above matrix will provide Optimal Solution.

Step 4 → *Assignment:* Selecting a row containing exactly one unmarked zero and surrounding it by '□' and draw a vertical line through the column containing this zero. Repeating this process till no such row is left; then selecting a column containing exactly one unmarked zero and surrounding it by '□' and draw a horizontal line through the row containing this zero and repeating the process till no such column is left.

For item	To item A	B	C	D	E
A	—	0	2	$\boxed{0}$	0
B	$\boxed{0}$	—	1	0	0
C	2	1	—	3	$\boxed{0}$
D	0	$\boxed{0}$	3	—	4
E	0	0	$\boxed{0}$	4	—

Step 5 → Computing Minimum Set-up cost

Item	*Item*	*Cost (Rs.)*
A	D	3
B	A	4
C	E	5
D	B	3
E	C	5
	Total	**20**

Alternative Solution:

For item	To item A	B	C	D	E
A	—	$\boxed{0}$	2	0	0
B	0	—	1	$\boxed{0}$	0
C	2	1	—	3	$\boxed{0}$
D	$\boxed{0}$	0	3	—	4
E	0	0	$\boxed{0}$	4	—

Minimum Set-up cost:

Item	*Item*	*Cost (Rs.)*
A	B	4
B	D	3
C	E	5
D	A	3
E	C	5
	Total	20

PROBLEM 4.8

A large engineering workshop has five shops. Hitherto, they have been fabricating five different types of components, one in each shop. Fabrication of one of these components is to be discontinued. Since the firm will follow the policy one shop-one component, one of the shops will be closed down. Data on the number of units to be manufactured and the unit costs are given below. Recommend an optimal plan as to which component should be produced in which shop and which of the shops be closed down. [Matrix elements are cost of fabrication in Rupees per unit.]

		C_1	C_2	C_3	C_4
S	S_1	6	7	5	8
H	S_2	7	6	5	9
O	S_3	8	7	6	9
P	S_4	8	9	4	8
S	S_5	9	8	6	7
Number of units (000)		8	6	4	5

Solution

This is an Assignment Problem, since each shop will produce only one component.

Because the number of units of the components differ the total cost of fabrication has to be taken into account instead of just the unit cost.

Total cost = Unit cost × No. of units.

Step 1 → Since there are five shops but only four components, a dummy component C_5 [with 0 costs] has to be included to balance the AP.

	C_1	C_2	C_3	C_4	C_5
S_1	48	42	20	40	0
S_2	56	36	20	45	0
S_3	64	42	24	45	0
S_4	64	54	16	40	0
S_5	72	48	24	35	0

Step 2 → ***Column Subtraction:*** Subtracting the minimum element of each column from all elements of that column and drawing the minimum number of lines to cover all zeros.

	C_1	C_2	C_3	C_4	C_5
S_1	0	6	4	5	0
S_2	8	0	4	10	0
S_3	16	6	8	10	0
S_4	16	18	0	5	0
S_5	24	12	8	0	0

Since number of lines drawn (5) = order of matrix = (5), the above matrix will provide optimal solution.

Step 3 → ***Assignment:*** Selecting a row containing exactly one unmarked zero and surrounding it by '☐' and draw a vertical line through the column containing this zero. Repeating this process till no such row is left; then selecting a column containing exactly one unmarked zero and surrounding it by '☐' and draw a horizontal line through the row containing this zero and repeating the process till no such column is left.

	C_1	C_2	C_3	C_4	C_5
S_1	0	6	4	5	0
S_2	8	0	4	10	0
S_3	16	6	8	10	0
S_4	16	18	0	5	0
S_5	24	12	8	0	0

Step 4 → Computing the Minimum:

Shop	*Component*	*Cost (Rs. 000)*
S_1	C_1	48
S_2	C_2	36
S_3	C_5	0 Shop to be closed down
S_4	C_3	16
S_5	C_4	35
	Total cost	135

PROBLEM 4.9 [Flight Scheduling]

Sohil Airlines, operating seven days a week, serves three cities A, B, and C according to the schedule shown in the following table. The layover cost per stop is roughly proportional to the square of the layover time. How should planes be assigned the flights so as to minimize the total layover cost?

Flight No. and Index	*From*	*Departure*	*To*	*Arrival*
A_1 B	A	09 A.M.	B	Noon
A_2 B	A	10 A.M.	B	01 P.M.
A_3 B	A	03 P.M.	B	06 P.M.
A_4 C	A	08 P.M.	C	Midnight
A_5 C	A	10 P.M.	C	02 A.M.
B_1 A	B	04 A.M.	A	07 A.M.
B_2 A	B	11 A.M.	A	02 P.M.
B_3 A	B	03 P.M.	A	06 P.M.
C_1 A	C	07 A.M.	A	11 A.M.
C_2 A	C	03 P.M.	A	07 P.M.

Solution

Assumption:

1. Any plane flying from a station must come back within 24 hours for scheduled trip.
2. Any plane starting from A for B must avail the next opportunity to come back to A.
3. It is not possible for any plane to make more than 2 trips, i.e., going and coming back.
4. Five planes will be operating on the line.

Step 1 → Let us first consider the cost matrix associated with the flights connecting A and C. Any plane through C_1A must return back to C by evening route A_4C or A_5C. At station A the layover for A_4C is 9 hours (11 A.M.–8 P.M.) while at station C, the layover for C_1A is 7 hours (Midnight–7 A.M.). Thus, the layover cost for route C_1A - A_4C is $9^2 + 7^2 = 130$ units. Similarly, the other route costs can be computed and the following cost matrix is obtained:

Table 1

	A_4 C	A_5 C
C_1A	130	146
C_2A	226	178

Step 2 → We now consider the cost associated with the flights connecting A and B. Any plane through the route A_1B cannot return by B_2A, because reaching B at noon, the plane has to return back the next day at 11 A.M. Thus we may consider the cost associated with this flight to be very high say M, to avoid this possibility. The costs for other trips between A and B are easily computed, and the following cost matrix is obtained:

Table 2

	B_1 A	B_2 A	B_3 A
A_1B	260	M	234
A_2B	234	M	260
A_3B	164	290	M

Finding optimal solution for Table 1

Step 3 → ***Row Subtraction:*** Subtracting the minimum element of each row from all elements of that row and drawing a minimum number of lines to cover all zeros.

	A_4 C	A_5 C
C_1A	0	16
C_2A	48	0

Since number of lines (2), = order of matrix = (2), the above matrix will provide optimal solution for table 1.

Step 4 → ***Assignment:*** Selecting a row containing exactly one unmarked zero and surrounding it by '☐' and draw a vertical line thorough the column containing this zero. Repeating this process till no such row is left; then selecting a column containing exactly one unmarked zero and surrounding it by '☐' and draw a horizontal line through the row containing this zero and repeating the process till no such column is left.

	A_4C	A_5C
C_1A	0	16
C_2A	48	0

Finding Solution for Table 2

Step 1 → ***Row Subtraction:*** Subtracting the minimum element of each row from all elements of that row and drawing the minimum number of lines to cover all zeros.

	B_1A	B_2A	B_3A
A_1B	26	M	0
A_2B	0	M	26
A_3B	0	126	M

Since number of lines = 2, and order of matrix = 3, we will have to take step to increase the number of zeros.

Step 2 → Subtracting the minimum uncovered element (26 in this case) from all uncovered elements and adding it to all elements at the intersection point of the above lines and drawing a minimum number of lines to cover all zeros.

	B_1A	B_2A	B_3A
A_1B	52	M	0
A_2B	0	M	0
A_3B	0	100	M

Since number of lines (3), = order of matrix (3), the above matrix will provide optimal solution for Table 2.

Step 3 → ***Assignment:*** Subtracting a row containing exactly one unmarked zero and surrounding it by '•' and draw a vertical line thorough the column containing this zero. Repeating this process till no such row is left; then selecting a column containing exactly one unmarked zero and surrounding it by '•' and draw a horizontal line through the row containing this zero and repeating the process till no such column is left.

	B_1A	B_2A	B_3A
A_1B	52	M	0
A_2B	0	M	0
A_3B	0	0	M

Step 4 → So, Optimal Route Schedule for 5 planes is as follows:

Plane No.	*Departure Route*	*Arrival Route*
1	A_1B	B_3A
2	A_2B	B_1A
3	A_3B	B_2A
4	C_1A	A_4C
5	C_2A	A_5C

PROBLEM 4.10 [Flight Scheduling]

XYZ airline operating 7 days a week has given the following time-table. Crews must have a minimum layover of 5 hours between flights. Obtain the pairing flights that minimizes layover time away from home. For any given pairing the crew will be based at the city that results in the smaller layover.

Flight Number	*Depart.*	*Arrive*	*Flight Number*	*Depart.*	*Arrive*
A_1	6 AM	8 AM	B_1	8 AM	10 AM
A_2	8 AM	10 AM	B_2	9 AM	11 AM
A_3	2 PM	4 PM	B_3	2 PM	4 PM
A_4	8 PM	10 PM	B_4	7 PM	9 PM

Solution

Step 1 → *Formulation of Assignment Problem:*

To begin with, let us first assume that the crew is based at Chennai. The flight A_1, which starts from Chennai at 6 AM, reaches Mumbai at 8 AM. The schedule time for the flight at Mumbai is 8 AM. Since the minimum layover time for crew is 5 hours, this flight can depart only on the next day i.e. the layover time will be 24 hours. Similarly, layover times for other flights are also calculated and given in the following tables.

Table 1: Layover Time in hours at Mumbai

Flight No.	*B_1*	*B_2*	*B_3*	*B_4*
A_1	24	25	6	11
A_2	22	23	28	9
A_3	16	17	22	27
A_4	10	11	16	21

Table 2: Layover Time in hours at Chennai

Flight No.	*B_1*	*B_2*	*B_3*	*B_4*
A_1	20	19	14	9
A_2	22	21	16	11
A_3	28	27	22	17
A_4	10	9	28	23

Now since the crew can be based at either of the places, minimum layover times can be obtained for different flight numbers by selecting the corresponding lower value out of the above two tables. The resulting table is given below:

Table 3: Minimum Layover Time between Flights

Flight No.	*Flight No.*			
	B_1	*B_2*	*B_3*	*B_4*
A_1	20	19	6	9
A_2	22	21	16	9
A_3	16	17	22	17
A_4	10	9	16	21

Step 2 → Subtracting the minimum element of each row from all the elements of that row, we get the following matrix. Since there is a zero in each column, there is no need to perform column reduction and drawing the minimum number of lines to cover all zeros.

Flight No.	*Flight No.*			
	B_1	B_2	B_3	B_4
A_1	14	13	0	3
A_2	13	12	7	0
A_3	0	1	6	1
A_4	1	0	7	12

Since the minimum number of lines to cover all zeros is four which is equal to the order of the matrix, the above table will give the optimal solution.

Step 3 → ***Assignment:*** Subtracting a row containing exactly one unmarked zero and surrounding it by '☐' & draw a vertical line thorough the column containing this zero. Repeating this process till no such row is left; then selecting a column containing exactly one unmarked zero and surrounding it by '☐' and draw a horizontal line through the row containing this zero and repeating the process till no such column is left.

Flight No.	*Flight No.*			
	B_1	B_2	B_3	B_4
A_1	14	13	$\boxed{0}$	3
A_2	13	12	7	$\boxed{0}$
A_3	$\boxed{0}$	1	6	1
A_4	1	$\boxed{0}$	7	12

Step 4 → Computing the Minimum Layover Time :

From Flight No.	*To Flight No.*	*Layover time*
A_1	B_3	6
A_2	B_4	9
A_3	B_1	16
A_4	B_2	9
		40 hours

PROBLEM 4.11 [Flight Scheduling]

A trip from Madras to Bangalore takes six hours by bus. A typical time table of the bus service in both directions is given below:

Departure from Madras	*Route Number*	*Arrival at Bangalore*	*Arrival at Madras*	*Route Number*	*Departure from Bangalore*
06.00	a	12.00	11.30	1	05.30
07.30	b	13.30	15.00	2	09.00
11.30	c	17.30	21.00	3	15.00
19.00	d	1.00	00.30	4	18.30
00.30	e	06.30	06.00	5	00.00

The cost of providing this service by the transport company depends upon the time spent by the bus crew (driver and conductor) away from their places in addition to service times. There are five crews. There is a constraint that every crew should be provided with more than 4 hours of rest before the return trip again and should not wait for more than 24 hours of rest before the return trip again. The company has residential facilities for the crew of Madras as well as at Bangalore. Find which line of service be connected with which other line so as to reduce the waiting time to the minimum.

Solution

Step 1 → *Formulation of Assignment Problem:* As the service time is constant for each line it does not appear directly in the computation. If the entire crew resides at Madras then the waiting times in hours at Bangalore for different route connections are given in the following Table :

Table 1 : Layover Time in Hours at Bangalore

Route	*1*	*2*	*3*	*4*	*5*
a	17.5	21	—	6.5	12
b	16	19.5	—	5	10.5
c	12	15.5	21.5	—	6.5
d	4.5	8	14	17.5	23
e	23	—	8.5	12	17.5

If Route a is combined with Route 1, the crew after arriving at Bangalore at 12 Noon start at 5.30 next morning. Thus the waiting time is 17.5 hours. Some of the assignments are infeasible. Route 3 leaves Bangalore at 15.00 hours. Thus the crew of Route a reaching Bangalore at 12 Noon are unable to take the minimum stipulated rest of four hours if they are asked to leave by Route 3. Hence a_3 is an infeasible assignment. Similarly other infeasible assignments have been markd with '–'.

Similarly, if the crew are assumed to reside at Bangalore then the waiting times of the crew in hours at Madras for different route combinations are given in Table 2.

Table 2 : Layover Time in hours at Madras

Route	*1*	*2*	*3*	*4*	*5*
a	18.5	15	9	5.5	24
b	20	16.5	10.5	7	-
c	24	20.5	14.5	11	5.5
d	7.5	-	22	18.5	13
e	13	9.5	-	24	18.5

As the crew can be asked to reside either at Madras or at Bangalore, minimum waiting time from the above operation can be computed for different route combination by choosing the minimum of the two waiting times. This is presented in Table 5. The asterisk marked waiting times indicates that the crew are based at Madras, otherwise they are based at Bangalore.

Table 3 : Minimum Layover Time in Hours

Route	*1*	*2*	*3*	*4*	*5*
a	17.5	15	9	5.5	12
b	16	16.5	10.5	5	10.5
c	12	15.5	14.5	11	5.5
d	4.5	8	14	17.5	13
e	13	9.5	8.5	12	17.5

Step 2 → ***Row Subtraction:*** Subtracting the minimum element of each row from all the elements of that row.

Route	*1*	*2*	*3*	*4*	*5*
a	12.00	9.5	3.5	0	6.5
b	11.00	11.5	5.5	0	5.5
c	6.5	10.00	9.0	5.5	0
d	0	3.5	9.5	13.0	8.5
e	4.5	1.0	0	3.5	9.0

Step 3 → ***Column Subtraction:*** Subtracting the minimum element of each column of the above matrix from all the elements of that column and then drawing the minimum number of line to cover all zeros.

Route	*1*	*2*	*3*	*4*	*5*
a	12.00	8.5	3.5	0	6.5
b	11.00	10.5	5.5	0	5.5
c	6.5	9.0	9.0	5.5	0
d	0	2.5	9.5	13.0	8.5
e	4.5	0	0	3.5	9.0

Since number of lines drawn (4) and order of matrix (5), we will have to take the step to increase the no. of zeros.

Step 4 → Subtracting the minimum uncovered element (3.5 in this case) from all the uncovered elements and adding to the elements at the intersection points and then drawing the minimum number of lines to cover all zeros.

Route	*1*	*2*	*3*	*4*	*5*
a	8.5	5.0	0	0	3.0
b	7.5	7.0	2.0	0	2
c	6.5	9.0	9.0	9.0	0
d	0	2.5	9.5	16.5	8.5
e	4.5	0	0	7.0	9.0

Since number of lines drawn (5) = order of matrix (5), the above matrix will provide the optimal solution.

Step 5 → ***Assignment:*** Subtracting a row containing exactly one unmarked zero and surrounding it by '☐' and draw a vertical line thorough the column containing this zero. Repeating this process till no such row is left; then selecting a column containing exactly one unmarked zero and surrounding it by '☐' and draw a horizontal line through the row containing this zero and repeating the process till no such column is left.

Route	*1*	*2*	*3*	*4*	*5*
a	8.5	5.0	$\boxed{0}$	$\boxed{0}$	3.0
b	7.5	7.0	2.0	0	2
c	6.5	9.0	9.0	9.0	$\boxed{0}$
d	$\boxed{0}$	2.5	9.5	16.5	8.5
e	4.5	$\boxed{0}$	0	7.0	9.0

Step 6 → Computing the Minimum Layover Time

Route to be Paired	*Residence of the Crew*	*Waiting time*
a – 3	Bangalore	9
b – 4	Madras	5
c – 5	Bangalore	5.5
d – 1	Madras	4.5
e – 2	Bangalore	9.5

The minimum total waiting time is thus 33.5 hours.

PRACTICAL STEPS INVOLVED IN SOLVING MAXIMISATION PROBLEMS

Step 1 → ***Dummy/Row/Column*** *See whether Number of Rows are equal to Number of Columns. If yes, problem is a balanced one; if not, then add a Dummy Row or Column to make the problem a balanced one by alloting zero value or specific value (if any given) to each cell of the Dummy Row or Column, as the case may be.*

Step 2 → *Derive* ***Profit Matrix*** *by deducting Cost from Revenue.*

Step 3 → *Derive* ***Loss Matrix*** *by deducting all elements from the largest element.*

Step 4 → *Follow the same Steps 2 to 9 as involved in solving Minimisation Problems.*

PROBLEM 4.12

Imagine yourself to be the Executive Director of a 5-star Hotel which has four banquet halls that can be used for all functions including weddings. The halls were all about the same size but the facilities in each hall differed. During a heavy marriage season, 4 parties approached you to reserve a hall for the marriage to be celebrated on the same day. These marriage parties were told that the first choice among these 4 halls would cost Rs. 10,000 for the day. They were also required to indicate the second, third and fourth preferences and the price that they would be willing to pay. Marriage party A & D indicated that they won't be interested in Halls 3 & 4. Other particulars are given in the following table:

Marriage Party	*Revenue per Hall*			
	1	*2*	*3*	*4*
A	10,000	9,000	X	X
B	8,000	10,000	8,000	5,000
C	7,000	10,000	6,000	8,000
D	10,000	8,000	X	X

Where *X* indicated that the party does not want that hall.

Required: Decide on an allocation that will maximise the revenue to your hotel.

Solution

Step 1 → **Deriving loss matrix** by deducting all events from the largest element (10,000)

Marriage Party	*Loss Matrix/Hall*			
	1	*2*	*3*	*4*
A	0	1000	X	X
B	2000	0	2000	5000
C	3000	0	4000	2000
D	0	2000	X	X

Step 2 → ***Column Subtraction:*** Subtracting the minimum element of each column from all elements of that column and drawing the minimum number of lines to cover all zeros.

Marriage Party	*Loss Matrix/Hall*			
	1	*2*	*3*	*4*
A	0	1000	X	X
B	2000	0	0	3000
C	3000	0	2000	0
D	0	2000	X	X

Since Number of Lines = 3 and Order of Matrix = 4, the above matrix will provide the optimal solution.

Step 3 → Subtracting the minimum uncovered element (1000 in this case) from all uncovered elements and adding it to all elements at the intersection points of the above lines and drawing the minimum number of lines to cover all zeros.

Marriage Party	*Loss Matrix/Hall*			
	1	*2*	*3*	*4*
A	0	0	X	X
B	3000	0	0	3000
C	4000	0	2000	0
D	0	1000	X	X

Since number of lines (4) = order of matrix (4), the above matrix will provide the optimal solution.

Step 4 → ***Assignment:*** Selecting a row containing exactly one unmarked zero and surrounding it by '□' and draw a vertical line thorough the column containing this zero. Repeating this process till no such row is left; then selecting a column containing exactly one unmarked zero and surrounding it by '□' and draw a horizontal line through the row containing this zero and repeating the process till no such column is left.

Marriage Party	*Loss matrix/Hall*			
	1	*2*	*3*	*4*
A	0	$\boxed{0}$	X	X
B	3000	0	$\boxed{0}$	3000
C	4000	0	2000	$\boxed{0}$
D	$\boxed{0}$	1000	X	X

Step 5 → Computing maximum:

		Revenue
Marriage Party	A – Hall 2	9,000
Marriage Party	B – Hall 3	8,000
Marriage Party	C – Hall 4	8,000
Marriage Party	D – Hall 1	10,000
	Total	35,000

PROBLEM 4.13 [Assignment of Batting Positions]

The captain of a cricket team has to allot five middle batting positions to five batsmen. The average runs scored by each batsman at these positions are as follows:

	Batting positions				
Batsman	*I*	*II*	*III*	*IV*	*V*
P	40	40	35	25	50
Q	42	30	16	25	27
R	50	48	40	60	50
S	20	19	20	18	25
T	58	60	59	55	53

(i) Find the assignment of batsmen to positions, which would give the maximum number of runs.

(ii) If another batsman 'U' with the following average runs in batting positions as given below:

Batting position:	I	II	III	IV	V
Average runs:	45	52	38	50	49

Is added to the team, should he be included to play in the team ? If so, who will be replaced by him?

Solution

Step 1 → Deriving Loss Matrix by deducting all elements from the largest element (60).

	Batting positions				
Batsman	*I*	*II*	*III*	*IV*	*V*
P	20	20	25	35	10
Q	18	30	44	35	33
R	10	12	20	0	10
S	40	41	40	42	35
T	2	0	1	5	7

Step 2 → ***Row Subtraction:*** Subtracting the minimum element of each row from all elements of that row.

	Batting positions				
Batsman	*I*	*II*	*III*	*IV*	*V*
P	10	10	15	25	0
Q	0	12	26	17	15
R	10	12	20	0	10
S	5	6	5	7	0
T	2	0	1	5	7

Step 3 → ***Column Subtraction:*** Subtracting the minimum element of each column from all elements of that column and drawing minimum number of lines to cover all zeros.

Batting positions					
Batsman	*I*	*II*	*III*	*IV*	*V*
P	10	10	14	25	0
Q	0	12	25	17	15
R	10	12	19	0	10
S	5	6	4	7	0
T	2	0	0	5	7

Since number of lines = 4 and order of matrix = 5, we will have to take step to increase the number of zeros.

Step 4 → Subtracting the minimum uncovered element (in this case 4) from all uncovered elements and adding it to all elements at the intersection point of the above lines and drawing minimum number of lines to cover all zeros.

Batting positions					
Batsman	*I*	*II*	*III*	*IV*	*V*
P	6	6	10	25	0
Q	0	12	25	21	19
R	6	8	15	0	10
S	1	2	0	7	0
T	2	0	0	9	11

Since the number of lines drawn (5) = order of matrix (5), the above matrix will provide the optimal solution.

Step 5 → ***Assignment:*** Selecting a row containing exactly one unmarked zero and surrounding it by '☐' and draw a vertical line thorough the column containing this zero. Repeating this process till no such row is left; then selecting a column containing exactly one unmarked zero and surrounding it by '☐' and draw a horizontal line through the row containing this zero and repeating the process till no such column is left.

Batting positions					
Batsman	*I*	*II*	*III*	*IV*	*V*
P	6	6	10	25	$\boxed{0}$
Q	$\boxed{0}$	12	25	21	19
R	6	8	15	$\boxed{0}$	10
S	1	2	$\boxed{0}$	7	0
T	2	$\boxed{0}$	0	9	11

Step 6 → Computing Maximum Runs:

Batsman	*Batting positions*	*Runs*
P	V	50
Q	I	42
R	IV	60
S	III	20
T	II	60
	Total	232

Part (ii)

Step 1 → Including Batsman U in Initial Table. **Introducing a Dummy batting position** to make the problem a balanced one by alloting zero.

	Batting positions					
Batsman	*I*	*II*	*III*	*IV*	*V*	*VI*
P	40	40	35	25	50	0
Q	42	30	16	25	27	0
R	50	48	40	60	50	0
S	20	19	20	18	25	0
T	58	60	59	55	53	0
U	45	52	38	50	49	0

Step 2 → **Deriving Loss Matrix** by deducting all elements from the largest element (60).

	Batting positions					
Batsman	*I*	*II*	*III*	*IV*	*V*	*VI*
P	20	20	25	35	10	60
Q	18	30	44	35	33	60
R	10	12	20	0	10	60
S	40	41	40	42	35	60
T	2	0	1	5	7	60
U	15	8	22	10	11	60

Step 3 → ***Row Subtraction:*** Subtracting the minimum element of each row from all elements of that row.

	Batting positions					
Batsman	*I*	*II*	*III*	*IV*	*V*	*VI*
P	10	10	15	25	0	50
Q	0	12	26	17	15	42
R	10	12	20	0	10	60
S	5	6	5	7	0	25
T	2	0	1	5	7	60
U	7	0	14	2	3	52

Step 4 → ***Column Subtraction:*** Subtracting the minimum element of each column from all element of that column and drawing minimum number of Lines to cover all zeros.

	Batting positions					
Batsman	*I*	*II*	*III*	*IV*	*V*	*VI*
P	10	10	14	25	0	25
Q	0	12	25	17	15	17
R	10	12	19	0	10	35
S	5	6	4	7	0	0
T	2	0	0	5	7	35
U	7	0	13	2	3	27

Since number of lines (6) = order of matrix (6), the above matrix will provide the optimal solution.

Step 5 → ***Assignment:*** Selecting a row containing exactly one unmarked zero and surrounding it by '☐' and draw a vertical line through the column containing this zero. Repeating this process till no such row is left; then selecting a column containing exactly one unmarked zero and surrounding it by '☐' and draw a horizontal line through the row containing this zero and repeating the process till no such column is left.

	Batting positions					
Batsman	*I*	*II*	*III*	*IV*	*V*	*VI*
P	10	10	14	25	$\boxed{0}$	25
Q	$\boxed{0}$	12	25	17	15	17
R	10	12	19	$\boxed{0}$	10	35
S	5	6	4	7	0	$\boxed{0}$
T	2	0	$\boxed{0}$	5	7	35
U	7	$\boxed{0}$	13	2	3	17

Step 6 → Computing Maximum Runs:

Batsman	*Batting Positions*	*Runs*
P	V	50
Q	I	42
R	IV	60
S	VI (dummy)	-
T	III	59
U	II	52
	Total	263

∴ Batsman S will be replaced by Batsman U.

PROBLEM 4.14 [Assignment of New Methods]

A production manager wants to assign one of the five new methods to each of the four operations. The following table summarises the weekly output in units:

	Weekly Output				
Operator	M_1	M_2	M_3	M_4	M_5
A	4	6	11	16	9
B	5	8	16	19	9
C	9	13	21	21	13
D	6	6	9	11	7

Cost per unit in Rs. 10, Selling Price per unit Rs. 35. Find the maximum profit per month.

Solution

Note: Since Profit per unit (Rs 25 i.e., Rs 35 – Rs 10) is same the given problem can be solved using minimization technique.

Step 1 → Introducing a Dummy Operator to make the problem a balanced one by alloting zero output.

	Methods				
Operator	M_1	M_2	M_3	M_4	M_5
A	4	6	11	16	9
B	5	8	16	19	9
C	9	13	21	21	13
D	6	6	9	11	7
Dummy	0	0	0	0	0

Step 2 → Deriving Loss Matrix by subtracting all elements from the largest element (21).

	Methods				
Operator	M_1	M_2	M_3	M_4	M_5
A	17	15	10	5	12
B	16	13	5	2	12
C	12	8	0	0	8
D	15	15	12	10	14
Dummy	21	21	21	21	21

Step 3 → *Row Subtraction:* Subtracting the minimum element of each row from all elements of that row and then drawing the minimum number of lines to cover all zeros.

	Methods				
Operator	M_1	M_2	M_3	M_4	M_5
A	12	10	5	0	7
B	14	11	3	0	10
C	12	8	0	0	8
D	5	5	2	0	4
Dummy	0	0	0	0	0

Since number of lines = 3 and order of matrix = 5, we will have to take step to increase the number of zeros.

Step 4 → Subtracting the minimum uncovered element (4 in this case) from all uncovered elements and adding it to all elements at the intersection point of the above lines and drawing minimum number of lines to cover all zeros.

Operator	Methods				
	M_1	M_2	M_3	M_4	M_5
A	8	6	5	0	3
B	10	7	3	0	6
C	8	4	0	0	4
D	1	1	2	0	0
Dummy	0	0	4	4	0

Since the number of lines = 4 and order of matrix = 5, we will have to take step to increase the number of zeros.

Step 5 → Subtracting the minimum uncovered element (3 in this case) from all uncovered elements and adding it to all elements at the intersection point of the above lines and drawing minimum number of lines to cover all zeros.

Operator	Methods				
	M_1	M_2	M_3	M_4	M_5
A	5	3	2	0	0
B	7	4	0	0	3
C	8	4	0	0	3
D	1	1	2	3	0
Dummy	0	0	4	7	0

Since the number of lines = 4 and order of matrix = 5, we will have to take step to increase the number of zeros.

Step 6 → Subtracting the minimum uncovered elements (1 in this case) from all uncovered elements and adding it to all elements at the intersection point of the above lines and drawing minimum number of lines to cover all zeros.

Operator	Methods				
	M_1	M_2	M_3	M_4	M_5
A	4	2	2	0	0
B	6	3	0	0	3
C	7	3	0	3	4
D	0	0	2	3	0
Dummy	0	0	5	8	1

Since number of lines (5) = order of matrix (5), the above matrix will provide optimal Solution.

Step 7 → ***Assignment:*** Selecting a row containing exactly one unmarked zero and surrounding it by '☐' and draw a vertical line thorough the column containing this zero. Repeating this process till no such row is left; then selecting a column containing exactly one unmarked zero and surrounding it by '☐' and draw a horizontal line through the row containing this zero and repeating the process till no such column is left.

Operator	Methods				
	M_1	M_2	M_3	M_4	M_5
A	4	2	2	0	$\boxed{0}$
B	6	3	0	$\boxed{0}$	3
C	7	3	$\boxed{0}$	3	4
D	$\boxed{0}$	0	2	3	0
Dummy	0	$\boxed{0}$	5	8	1

Step 8 → Computing Optimal Output per month and Maximum Profit:

Operators	*Methods*	*Units Per week*	*Units Per month*
A	M_5	9	36
B	M_4	19	76
C	M_3	21	84
D	M_1	6	24
Dummy	M_2	0	0
	Total Output	55	220

Total sales Revenue for the month @ Rs. 35 per unit	7,700
Less: Total Production Cost for the month @ Rs. 10 per unit	(2,200)
Maximum Profit	5,500

Alternative Solution:

Operator	Methods				
	M_1	M_2	M_3	M_4	M_5
A	4	2	2	0	$\boxed{0}$
B	6	3	0	$\boxed{0}$	3
C	7	3	$\boxed{0}$	3	4
D	0	$\boxed{0}$	2	3	0
Dummy	$\boxed{0}$	0	5	8	1

Operators	*Methods*	*Units Per week*	*Units Per month*
A	M_5	9	36
B	M_4	19	76
C	M_3	21	84
D	M_2	6	24
Dummy	M_1	0	0
	Total Output	55	220

Total sales Revenue for the month @ Rs. 35 per unit	7,700
Less: Total Production Cost for the month @ Rs. 10 per unit	(2,200)
Maximum Profit	5,500

PROBLEM 4.15

The cost matrix giving selling cost per unit of a product by salesman A, B, C and D in regions R_1, R_2, R_3 and R_4 is given below:

	A	*B*	*C*	*D*
R_1	4	12	16	8
R_2	20	28	32	24
R_3	36	44	48	40
R_4	52	60	64	56

(i) Assign one salesman to one region to minimise the selling cost.
(ii) If the selling price of the product is Rs. 2000 per unit and variable cost excluding the selling cost given in the table is Rs. 100 per unit, find the assignment that would maximise the contribution.
(iii) What other conclusion can you make form the above?

Solution

4	12	16	8
20	28	32	24
36	44	48	40
52	60	64	56

Subtracting minimum element – each row.

0	8	12	4
0	8	12	4
0	8	12	4
0	8	12	4

Subtracting minimum element – each column.

0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

Minimum no. of lines to cover all zeros = 4 = order of matrix. Hence optional assignment is possible.

Minimum cost = 4 + 28 + 56 = 136.

$= AR_1 + BR_2 + CR_3 + DR_4$

Since all zeros, the are 24 solution to the assignment problem

Viz.	A	B	C	D	
	R_1	R_2	R_3	R_4	
	R_2	R_3	R_4	R_1	
	R_3	R_4	R_1	R_2	
	R_4	R_1	R_2	R_3	
	R_1	R_3	R_4	R_2	etc.

A can be assigned in 4 ways, B in 3 ways for each of A's 4 ways.

(ii) SP – VC = 100 Rs

	A	*B*	*C*	*D*
R_1	96	88	84	92
R_2	80	72	68	76
R_3	64	56	52	60
R_4	48	40	36	44

Subtracting the highest term

0	8	12	4
16	24	28	20
32	40	44	36
48	56	60	52

Subtracting minimum term of each row

0	8	12	4
0	8	12	4
0	8	12	4
0	8	12	4

Which is the same as the earlier matrix

Maximum contribution = Rs. (96 + 72 + 52 + 44) = Rs. 264.

Alternative Solution:

Maximization of contribution is same as minimizing cost. Hence, same assignments as in (i) will be optional solution

Maximum Contribution Rs. (400 – 136) = Rs 264

(iii) (a) The relative cost of assigning person i to region r does not change by addition or subtraction of a constant from either a row, or column or all elements of the matrix.

(b) Minimizing cost is the same as maximizing contribution. Hence, the assignment solution will be same applying point (i) above.

(c) Many zero's represent many feasible least cost assignment. Here, all zeros mean maximum permutation of a 4 × 4 matrix, viz. 4 × 3 × 2 × 1 = 24 solutions are possible.

PROBLEM 4.16 [Assignment of Machines]

A company is faced with the problem of assigning 4 machines to 6 different jobs (one machine to one job only). The profits are estimated as follows:

	Machine			
Job	*A*	*B*	*C*	*D*
1	3	6	2	6
2	7	1	4	4
3	3	8	5	8
4	6	4	3	7
5	5	2	4	3
6	5	7	6	4

Required: Solve the problem to maximize the total profits.

Solution

Step 1 → Introducing Dummy machines to make the problem a balanced one by alloting zero profit.

	A	*B*	*C*	*D*	*E*	*F*
1	3	6	2	6	0	0
2	7	1	4	4	0	0
3	3	8	5	8	0	0
4	6	4	3	7	0	0
5	5	2	4	3	0	0
6	5	7	6	4	0	0

Step 2 → Derive Loss Matrix by deducting all elements from the largest element (8)

	A	*B*	*C*	*D*	*E*	*F*
1	5	2	6	2	8	8
2	1	7	4	4	8	8
3	5	0	3	0	8	8
4	2	4	5	1	8	8
5	3	6	4	5	8	8
6	3	1	2	4	8	8

Step 3 → ***Row Subtraction:*** Subtracting the minimum element of each row from all elements of that row.

	A	*B*	*C*	*D*	*E*	*F*
1	3	0	4	0	6	6
2	0	6	3	3	7	7
3	5	0	3	0	8	8
4	1	3	4	0	7	7
5	0	3	1	2	5	5
6	2	0	1	3	7	7

Step 4 → *Column Subtraction:* Subtracting the minimum element of each column from all elements of that column and drawing minimum number of lines to cover all zeros.

	A	B	C	D	E	F
1	3	0	3	0	1	1
2	0	6	2	3	2	2
3	5	0	2	0	3	3
4	1	3	3	0	2	2
5	0	3	0	2	0	0
6	2	0	0	3	2	2

Since number of lines = 5 and order of matrix = 6, we will have to take step to increase the number of zeros.

Step 5 → Subtracting the minimum uncovered element (1 in this case) from all uncovered elements and adding it to all elements at the intersection point of the above lines and drawing minimum number of Lines to cover all zeros.

	A	B	C	D	E	F
1	3	0	3	0	0	0
2	0	6	2	3	1	1
3	5	0	2	0	2	2
4	1	3	3	0	1	1
5	1	4	1	3	0	0
6	2	0	0	3	1	1

Since number of lines (6), = order of matrix (6), the above matrix will provide Optimal Solution.

Step 6 → *Assignment:* Selecting a row containing exactly one unmarked zero and surrounding it by '☐' and draw a vertical line thorough the column containing this zero. Repeating this process till no such row is left; then selecting a column containing exactly one unmarked zero and surrounding it by '☐' and draw a horizontal line through the row containing this zero and repeating the process till no such column is left.

	A	B	C	D	E	F
1	3	0	3	0	0	0
2	0	6	2	3	1	1
3	5	0	2	0	2	2
4	1	3	3	0	1	1
5	1	4	1	3	0	0
6	2	0	0	3	1	1

Step 7 → Computing Maximum Profit:

Job	*Machine*	*Profit (Rs.)*
1	E	0
2	A	7
3	B	8
4	D	7
5	F	0
6	C	6
	Total	28

Alternative Solution:

	A	*B*	*C*	*D*	*E*	*F*
1	3	0	3	0	0	$\boxed{0}$
2	$\boxed{0}$	6	2	3	1	1
3	5	$\boxed{0}$	2	0	2	2
4	1	3	3	$\boxed{0}$	1	1
5	1	4	1	3	$\boxed{0}$	0
6	2	0	$\boxed{0}$	3	1	1

Computing Maximum Profit:

Job	*Machine*	*Profit (Rs.)*
1	F	0
2	A	7
3	B	8
4	D	7
5	E	0
6	C	6
	Total	28

PROBLEM 4.17 [Assignment of Production]

A firm is contemplating the introduction of three products 1, 2 and 3, in it's plants A, B and C. Only a single product is decided to be introduced in each of the plants. The unit cost of producing ith product in jth plants, is given in the following matrix:

	Plant		
Product	*A*	*B*	*C*
1	8	12	—
2	10	6	4
3	7	6	6

(i) How should the products be assigned so that the total unit cost is minimised?

(ii) If the quantity of different products is as follows, then what assignment shall minimise the aggregate production cost?

Product	*Quantity (in units)*
1	2,000
2	2,000
3	10,000

(iii) It is expected that the selling prices of the products produced by different plants would be different; as shown in the following table:

	Plant		
Product	*A*	*B*	*C*
1	15	18	—
2	18	16	10
3	12	10	8

Assuming that the quantities mentioned in (ii) above would be produced and sold, how should the products be assigned to the plants to obtain maximum profits ?

Solution

Part (i)

Step 1 → *Row Subtraction:* Subtracting the minimum element of each row from all elements of that row. Since there is zero in each column, there is no need for column subtraction. Drawing minimum number of lines to cover all zeros.

	Plant		
Product	*A*	*B*	*C*
1	0	4	—
2	6	2	0
3	1	0	0

Since number of lines (3) = order of matrix (3), the above matrix will provide the optimal Solution.

Step 2 → *Assignment:* Selecting a row containing exactly one unmarked zero and surrounding it by '☐' and draw a vertical line thorough the column containing this zero. Repeating this process till no such row is left; then selecting a column containing exactly one unmarked zero and surrounding it by '☐' and draw a horizontal line through the row containing this zero and repeating the process till no such column is left.

	Plant		
Product	*A*	*B*	*C*
1	$\boxed{0}$	4	—
2	6	2	$\boxed{0}$
3	1	$\boxed{0}$	0

Step 3 → Computing Minimum Unit Cost:

Products	*Plants*	*Units Cost (Rs.)*
1	A	8
2	C	4
3	B	6
	Total	18

Part (ii)

Step 1 → Deriving Total Production Cost Matrix by multiplying Quantity Matrix by Unit Cost Matrix.

	Plant		
Product	*A*	*B*	*C*
1	16,000	24,000	—
2	20,000	12,000	8,000
3	70,000	60,000	60,000

Step 2 → ***Row Subtraction:*** Subtracting the minimum element of each row from all elements of that row. Since there is zero in column, there is no need for column subtraction. Drawing minimum number of lines to cover all zeros.

	Plant		
Product	*A*	*B*	*C*
1	0	8,000	—
2	12,000	4,000	0
3	10,000	0	0

Since number of lines (3) and order of matrix (3), the above matrix will provide the optimal solution.

Step 3 → ***Assignment:*** Selecting a row containing exactly one unmarked zero and surrounding it by '☐' and draw a vertical line thorough the column containing this zero. Repeating this process till no such row is left; then selecting a column containing exactly one unmarked zero and surrounding it by '☐' and draw a horizontal line through the row containing this zero and repeating the process till no such column is left.

	Plant		
Product	*A*	*B*	*C*
1	$\boxed{0}$	8,000	—
2	12,000	4,000	$\boxed{0}$
3	10,000	$\boxed{0}$	0

Step 4 → Computing Minimum Cost

Products	*Plants*	*Units Cost (Rs.)*
1	A	16,000
2	C	8,000
3	B	60,000
	Total	84,000

Part (iii)

Step 1 → Deriving Sales Matrix by following formula:

Sale = Units × Selling Price Per Unit

	Plant		
Product	*A*	*B*	*C*
1	30,000	36,000	—
2	36,000	32,000	20,000
3	1,20,000	1,00,000	80,000

Step 2 → Deriving Profit Matrix by deducting Cost from revenue by the following formula:

Profit = Total Sales – Total Cost

	Plant		
Product	*A*	*B*	*C*
1	14,000	8,000	—
2	16,000	20,000	12,000
3	50,000	40,000	20,000

Step 3 → Deriving Loss Matrix by deducting all elements from the largest element (50,000).

	Plant		
Product	*A*	*B*	*C*
1	36,000	42,000	—
2	34,000	30,000	38,000
3	0	10,000	30,000

Step 4 → *Row Subtraction:* Subtracting the minimum element of each row from all elements of that row.

	Plant		
Product	*A*	*B*	*C*
1	0	6,000	—
2	4,000	0	8,000
3	0	10,000	30,000

Step 5 → *Column Subtraction:* Subtracting minimum element of each column from all elements of that column and drawing minimum number of Lines to cover all zeros.

Plant			
Product	*A*	*B*	*C*
1	0	6,000	—
2	4,000	0	0
3	0	10,000	22,000

Since number of lines = 2 and order of matrix = 3, we will have to take step to increase the number of zeros.

Step 6 → Subtracting the minimum uncovered element (6,000 in this case) from all uncovered elements and adding it to all elements at the intersection point of the above lines and drawing minimum number of lines to cover all zeros.

Plant			
Product	*A*	*B*	*C*
1	0	0	—
2	10,000	0	0
3	0	4,000	16,000

Since number of lines (3) and order of matrix (3), the above matrix will provide the optimal solution..

Step 7 → *Assignment:* Selecting a row containing exactly one unmarked zero and surrounding it by '☐' and draw a vertical line thorough the column containing this zero. Repeating this process till no such row is left; then selecting a column containing exactly one unmarked zero and surrounding it by '☐' and draw a horizontal line through the row containing this zero and repeating the process till no such column is left.

Plant			
Product	*A*	*B*	*C*
1	0	0	—
2	10,000	0	0
3	0	4,000	16,000

Step 7 → Computing Maximum Profit:

Products	*Plants*	*Profit (Rs.)*
1	B	8,000
2	C	12,000
3	A	50,000
	Total	70,000

PROBLEM 4.18 [Assignment of Sales Territory]

A manufacturing company has four zones A, B, C, D and four sales engineers P, Q, R, S respectively for assignment. Since the zones are not equally rich in sales potential, therefore it is estimated that a particular engineer operating in a particular zone will bring the following sales:

Zone A	:	4,20,000
Zone B	:	3,36,000
Zone C	:	2,94,000
Zone D	:	4,62,000

The engineers are having different sales ability. Working under the same conditions, their yearly sales are proportional to 14, 9, 11 and 8 respectively. The criteria of maximum expected total sales is to be met by assigning the best engineer to the richest zone, the next best to the second richest zone and so on.

Required: Find the optimum assignment and the maximum sales.

Solution

Step 1 → Deriving Revenue Matrix.

Sales Engineer	*Proportion*	*Index taking B as base*
P	14	1
Q	9	$\frac{9}{14}$
R	11	$\frac{11}{14}$
S	8	$\frac{8}{14}$

The problem here is to find the optimum assignment in thefollowing sales table so as to maximise the total sales of the company.

Territory Engineer	*Zones (Sales in thousands of rupees)*			
	I	*II*	*III*	*IV*
P	$420 \times 1 = 420$	$336 \times 1 = 336$	$294 \times 1 = 294$	$462 \times 1 = 462$
Q	$420 \times \frac{9}{14} = 270$	$336 \times \frac{9}{14} = 216$	$294 \times \frac{9}{14} = 189$	$462 \times \frac{9}{14} = 297$
R	$420 \times \frac{11}{14} = 330$	$336 \times \frac{11}{14} = 264$	$294 \times \frac{11}{14} = 231$	$462 \times \frac{11}{14} = 363$
S	$420 \times \frac{8}{14} = 240$	$336 \times \frac{8}{14} = 192$	$294 \times \frac{8}{14} = 168$	$462 \times \frac{8}{14} = 264$

Step 2 → **Deriving Loss Matrix** by deducting all elements from the largest element (462).

Sales Engineer	*Zones (Loss in thousands of rupees)*			
	A	***B***	***C***	***D***
P	42	126	168	0
Q	192	246	273	165
R	132	198	231	99
S	222	270	294	198

Step 3 → ***Row Subtraction:*** Subtracting the minimum element of each row from all elements of that row.

Sales Engineer	*Zones (Loss in thousands of rupees)*			
	A	***B***	***C***	***D***
P	42	126	168	0
Q	27	81	108	0
R	33	99	132	0
S	24	72	96	0

Step 4 → ***Column Subtraction:*** Subtracting the minimum element of each column from all elements of that column and drawing minimum number of lines to cover all zeros.

Sales Engineer	*Zones (Loss in thousands of rupees)*			
	A	*B*	*C*	*D*
P	18	54	72	0
Q	3	9	12	0
R	9	27	36	0
S	0	0	0	0

Since number of lines = 2 and order of matrix = 4, we will have to take step to increase the number of zeros.

Step 5 → Subtracting the minimum uncovered element (in this case 3) from all uncovered elements and adding it to all elements at the intersection point of the above lines and drawing minimum number of lines to cover all zeros.

Sales Engineer	*Zones (Loss in thousands of rupees)*			
	A	*B*	*C*	*D*
P	15	51	69	0
Q	0	6	9	0
R	6	24	33	0
S	0	0	0	3

Since number of lines = 3 and order of matrix = 4, we will have to take step to increase the number of zeros.

Step 6 → Subtracting the minimum uncovered element (in this case 6) from all uncovered elements and adding it to all elements at the intersection point of the above lines and drawing minimum number of lines to cover all zeros.

Sales Engineer	Zones (Loss in thousands of rupees)			
	A	B	C	D
P	15	45	63	0
Q	0	0	3	0
R	6	18	27	0
S	6	0	0	9

Since number of lines = 3 and order of matrix = 4, we will have to take step to increase the number of zeros.

Step 7 → Subtracting the minimum uncovered element (in this case 6) from all uncovered elements and adding it to all elements at the intersection point of the above lines and drawing minimum number of lines to cover all zeros.

Sales Engineer	Zones (Loss in thousands of rupees)			
	A	B	C	D
P	9	39	57	0
Q	0	0	3	6
R	0	12	21	0
S	6	0	0	15

Since the minimum number of lines drawn to cover all the zeros is 4 which is equal to the order of the matrix. Hence, the above table will give the optimum assignment.

Step 8 → ***Assignment:*** Selecting a row containing exactly one unmarked zero and surrounding it by '•' and draw a vertical line thorough the column containing this zero. Repeating this process till no such row is left; then selecting a column containing exactly one unmarked zero and surrounding it by '•' and draw a horizontal line through the row containing this zero and repeating the process till no such column is left.

Sales Engineer	Zones (Loss in thousands of rupees)			
	A	B	C	D
P	9	39	57	$\boxed{0}$
Q	0	$\boxed{0}$	3	6
R	$\boxed{0}$	12	21	0
S	6	0	$\boxed{0}$	15

Step 9 → Computing the Maximum Sales:

Engineers	Zones	Sales (in Rs.)
P	D	4,62,000
Q	B	2,16,000
R	A	3,30,000
S	C	1,68,000
		11,76,000

It can be seen from the above assignments that the best engineer P is assigned to the richest zone D, the next best engineer R is assigned to second richest zone A, the next best engineer Q is assigned to zone B and so on. Hence, the optimum assignment matches the company's criteria of achieving the maximum expected total sales.

PROBLEM 4.19 [Preparation of Production and Profit Matrix]

A firm produces four products. There are four operators who are capable of producing any of these four products. The processing time varies from operator to operator. The firm records 8 hours a day and allows 30 minutes for lunch. The processing time in minutes and the profit for each of the products are given below:

Operators	*Products*			
	A	*B*	*C*	*D*
1	15	9	10	6
2	10	6	9	6
3	25	15	15	9
4	15	9	10	10
Profit (Rs) per unit	8	6	5	4

Find the optimal assignment of products to operators.

Solution

Step 1 → *Deriving Production Matrix:* The firm records 8 hours a day and allows 30 minutes for lunch, hence the net working time available per day is 7 hours and 30 minutes i.e. 450 minutes. The number of units of each product which could be produced in 450 minutes by the four operators is calculated in the table given below by using the formula (i.e. 450 Minutes/ Processing time).

Operators	*Products*			
	A	*B*	*C*	*D*
1	30	50	45	75
2	45	75	50	75
3	18	30	30	50
4	30	50	45	45
Profit per unit	8	6	5	4

Step 2 → Deriving Profit Matrix by multiplying output by profit per unit

Operators	*Profit matrix in Rs. of Products*			
	A	*B*	*C*	*D*
1	240	300	225	300
2	360	450	250	300
3	144	180	150	200
4	240	300	225	180

Step 3 → Deriving Loss Matrix by deducting all the elements from the largest element (450)

Operators	*Loss matrix in Rs. of Products*			
	A	*B*	*C*	*D*
1	210	150	225	150
2	90	0	200	150
3	306	270	300	250
4	210	150	225	270

Step 4 → ***Row Subtraction:*** Subtracting the minimum element of each row from all elements of that row.

Operators	*Loss matrix in Rs. of Products*			
	A	*B*	*C*	*D*
1	60	0	75	0
2	90	0	200	150
3	56	20	50	0
4	60	0	75	120

Step 5 → ***Column Subtraction:*** Subtracting the minimum element of each column from all elements of that column and drawing minimum number of lines to cover all zeros.

Operators	*Loss matrix in Rs. of Products*			
	A	*B*	*C*	*D*
1	4	0	25	0
2	34	0	150	150
3	0	20	0	0
4	4	0	25	120

Since number of lines = 3 and order of matrix = 4, we will have to take step to increase the number of zeros.

Step 6 → Subtracting the minimum uncovered element (in this case 4) from all uncovered elements and adding it to all elements at the intersection point of the above lines and drawing minimum number of lines to cover all zeros.

Operators	*Loss matrix in Rs. of Products*			
	A	*B*	*C*	*D*
1	0	0	21	0
2	30	0	146	150
3	0	24	0	4
4	0	0	21	120

Step 7 → ***Assignment:*** Selecting a row containing exactly one unmarked zero and surrounding it by '☐' and draw a vertical line thorough the column containing this zero. Repeating this process till no such row is left; then selecting a column containing exactly one unmarked zero and surrounding it by '☐' and draw a horizontal line through the row containing this zero and repeating the process till no such column is left.

Operators	Loss matrix in Rs. of Products			
	A	B	C	D
1	0	0	21	$\boxed{0}$
2	30	$\boxed{0}$	146	150
3	0	24	$\boxed{0}$	4
4	$\boxed{0}$	0	21	120

Step 8 → Computing the Maximum Profit:

Operator	Product	Profit (Rs.)
1	D	300
2	B	450
3	C	150
4	A	240
	Total	Rs. 1,140

PROBLEM 4.20

A manager was asked to assign tasks to operators (one task per operator only) so as to minimize the time taken. He was given the matrix showing the hours taken by the operators for the tasks.

First, he preformed the row minimum operation. Secondly, he did the column minimum operation. Then, he realized that there were 4 tasks and 5 operators. At the third step he introduced the dummy row and continued with his fourth step of drawing lines to cover zeros. He drew 2 vertical lines (under operator III and operator IV) and two horizontal lines (aside task T_4 and dummy task T_5). At step 5, he performed the necessary operation with the uncovered element, since the number of lines was less than the order of the matrix. After this, his matrix appeared as follows:

Operators

Tasks	I	II	III	IV	V
T_1	4	2	5	0	0
T_2	6	3	3	0	3
T_3	4	0	0	0	1
T_4	0	0	5	3	0
T_5 (dummy)	0	0	3	3	0

(i) What was the matrix after step II ? Based on such matrix, ascertain (ii) and (iii) given below.

(ii) What was the most difficult task for operators I, II and V?

(iii) Who was the most efficient operators?

(iv) If you are not told anything about the manager's errors, which operator would be denied any task? Why?

(v) Can the manager go ahead with his assignment to correctly arrive at the optional assignment, or should he start afresh after introducing the dummy task at the beginning?

Solution

	01	02	03	04	05
(given)					
T_1	4	2	5	0	0
T_2	6	3	3	0	3
T_3	4	0	0	0	1
T_4	0	0	5	3	0
T_5	0	0	3	3	0
(dummy)					

Junction values at dummy = 3.3 was the minimum uncovered element.

Previous step was

7	5	5	0	3
9	6	3	0	6
7	3	0	0	4
0	0	2	0	0
0	0	0	0	0

(i) At step II the matrix was:

7	5	5	0	3
9	6	3	0	6
7	3	0	0	4
0	0	2	0	0

(ii) For Operator 1, Most difficult task will be indicated by hours = T_2

Operator II T_2

Operator V T_2

(iii) Most efficient operator = Operator 4

(iv) If the Manager's mistake was not known,

4	2	5	0	88
6	3	3	88	3
4	0	88	0	1
0	0	5	3	0
0	0	3	3	0

We continue the assignment; T_1 — 0_5, $T_2 - 0_4$, $T_3 - 0_3$ are fixed.

Between T_4 and T_5, 0_1 or 0_2 Can be allotted.

So, other 0_1 or 0_2 can be denied the job.

(v) Yes, the Manager can go ahead with the optimal assignment.

Row minimum is not affected by when the dummy was introduced.

Column minimum was affected. But in the process, more zeros were generated to provide better solution.

PROBLEM 4.21

A city corporation has decided to carry out road repairs on 4 main roads in the city.

The Government has agreed to make a special grant of Rs 50 lacs towards the cost with the condition that the repairs should be carried out at lowest cost. Five contractors have sent their bids. Only road will be awarded to one contractor. The bids are given below:

	Cost of Repairs (Rs in lacs)				
	Road	R_1	R_2	R_3	R_3
Contractors	C_1	9	14	19	15
	C_2	7	17	20	19
	C_3	9	18	21	18
	C_4	10	12	18	19
	C_5	10	15	21	16

You are informed that C_2 should get R_1 and C_4 should get R_2 to minimize costs,

(i) What is the minimum cost allocation?

(ii) How much is the minimum discount that the eliminated contractor should offer for meriting a contract?

(iii) Independent of (ii) above, if the corporation can negotiate to get a uniform discount rate from each contractor, what is the minimum rate of discount so that the cost is within the grant amount?

Solution

(i) There are 5 rows and 4 columns hence insert a dummy column R_5.

1. C_2 has been allocated to R_1

2. C_4 has been allocated to R_2. Hence the assignment is restricted to

	R_3	R_4	R_5
C_1	19	15	0
C_3	21	18	0
C_5	21	16	0

Column Minimum

	R_3	R_4	R_5
C_1	0	0	0
C_3	2	3	0
C_5	2	1	0

	R_3	R_4	R_5
C_1	0	0	1
C_3	1	2	0
C_5	1	0	0

Hence C_1 has been allotted to R_3, C_3 to R_5 and C_5 to R_4.

Hence the Minimum cost is = 7 + 12 + 19 + 16 + 0 = 54 Lacs

(ii) C_3 should reduce 2 lacs for R_1, 6 lacs for R_2, 2 lacs for R_3 and 2 lacs for R_4

Minimum Discount = 2 Lacs for any of R_1, R_3, R_4

(iii) Minimum rate of Discount (54 – 50) = 4/54 = 7.41%

5 TRANSPORTATION PROBLEMS

PRACTICAL STEPS INVOLVED IN SOLVING TRANSPORTATION PROBLEMS OF MINIMIZATION TYPE

Step 1 → ***Dummy Origin/Destination:*** *See whether Total Requirements are equal to Total Availability; if yes, go to Step 2; if not, introduce a Dummy Origin/Destination, as the case may be, to make the problem a balanced one, taking Transportation Cost per unit as zero for each Cell of Dummy Origin/Destination or as otherwise indicated.*

↓

Step 2 → ***Initial Feasible Solution:*** *Find Initial Feasible Solution by following either the Least Cost Method (or LCM) or North-West Corner Method (or NWCM) or Vogel's Approximation Method (or VAM), (Refer to page 5.2, 5.3 and 5.4).*

Note: *Unless otherwise stated, Use VAM.*

↓

Step 3 → ***Introduction of Infinitely Small Quantity 'e':*** *After obtaining the Initial Feasible Solution Table, see whether Total Number of Allocations are equal to "m + n - 1",; if yes, go to Step 4; if not, introduce an infinitely small quantity 'e' to the Least Cost Independent Cell. (i.e., for which no Loop can be formed).*

Note : *Introduce as many number of 'e' as the total number of allocated cells falls below "m + n -1".*

↓

Step 4 → ***Optimality Test:*** *Carry out the Optimality Test on the Initial Solution Table to find out the optimal solution. (Refer to page 5.27).*

Step 5 → ***Minimum Cost:*** *Calculate the Total Minimum Cost =* $\Sigma (X_{ij} \updownarrow C_{ij})$*,*

where, X = Units Allocated to a Cell;

C = Shipping Cost per Unit of a Cell;

i = Row Number;

j = Column Number

FINDING INITIAL (OR BASIC) FEASIBLE SOLUTION

In general, any basic feasible solution of a transportation problem with **m origins** (such as factories) and **n destinations** (such as warehouses) should have '*m + n - 1*' non zero basic variables.

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where, X = *Units Allocated to a Cell;*

C = *Shipping Cost per Unit of a Cell;*

i = *Row Number;*

j = *Column Number*

FINDING INITIAL (OR BASIC) FEASIBLE SOLUTION

In general, any basic feasible solution of a transportation problem with **m origins** (such as factories) and **n destinations** (such as warehouses) should have '$m + n - 1$' non zero basic variables.

A transport problem is said to be a degenerate transport problem if it has a basic feasible solution with number of non zero basic variables less than $m + n - 1$.

According to Mustafit, "A degenerate basic feasible solution in a transportation problem exists if and only if some partial sum of availabilities (row) is equal to a partial sum of requirements (column)".

Initial feasible solution can be obtained by any of the following three methods:

Method I	:	Least Cost Method (or LCM)
Method II	:	North-West Corner Method (or NWCM)
Method III	:	Vogel's Approximation Method (or VAM)

Let us discuss these methods one by one as under:

METHOD I : Least Cost Method (or LCM)

The practical steps involved in the Least Cost Method are given below:

Step 1 → *Make maximum possible Allocation to the **Least Cost Cell** depending upon the demand/supply for the Column/Row containing that Cell. In case of Tie in the Least Cost Cells, make allocation to the Cell by which maximum demand or capacity is exhausted.*

↓

Step 2 → *Make allocation to the **Second Lowest Cost Cell** depending upon the remaining demand/supply for the Row/Column containing that Cell.*

↓

Step 3 → *Repeat the above Steps till all Rim Requirements are exhausted, i.e., entire demand and supply is exhausted.*

PROBLEM 5.1 [Introduction of Dummy Origin in Case of Contribution Matrix]

Find the basic feasible solution by Least Cost Method.

	W_1	W_2	W_3	W_4	*Supplies*
F_1	48	60	56	58	140
F_2	45	55	53	60	260
F_3	50	65	60	62	360
Demand	200	320	250	210	

Note: Cell entries are the unit contributions.

Solution

Step 1→ Introducing a Dummy factory with zero profit per unit as the total demand is not equal to total supply in order to make the problem balanced one.

	W_1	W_2	W_3	W_4	Supplies
F_1	48	60	56	58	140
F_2	45	55	53	60	260
F_3	50	65	60	62	360
F_4	0	0	0	0	220
Demand	200	320	250	210	

Step 2→ Deriving Loss Matrix by deducting each element from the maximum element (i.e. 65) in order to use minimisation technique and finding initial feasible solution by LCM.

	W_1	W_2	W_3	W_4	Supplies
F_1	17	5	9 $\boxed{140}$	7	140
F_2	20	10	12 $\boxed{90}$	5 $\boxed{170}$	260
F_3	15	0 $\boxed{320}$	5	3 $\boxed{40}$	360
F_4	65 $\boxed{200}$	65	65 $\boxed{20}$	65	220
Demand	200	320	250	210	

METHOD II : North-West Corner Method (or NWCM)

The practical steps involved in the North-West Corner Method are given below:

Step 1→ *Make maximum possible* ***allocation to the Upper-Left Corner Cell*** *(also known as North-West Corner Cell) in the First Row depending upon the availability of supply for that Row and demand requirement for the Column containing that Cell.*
Note: *Unit transportation cost is completely ignored.*

↓

Step 2→ *Move to the* ***Next Cell of the First Row*** *depending upon remaining supply for that Row and the demand requirement for the next Column. Go on till the Row total is exhausted.*

↓

Step 3→ *Move to the the next Row and make allocation to the Cell below the Cell of the preceding Row in which the last allocation was made and follow Steps 1 and 2.*

↓

Step 4→ *Follow Steps 1 to 3 till all Rim requirements are exhausted, i.e., the entire demand and supply is exhausted.*

PROBLEM 5.2 [Introduction of Dummy Origin in Case of Contribution Matrix]

Find initial feasible solution by North West Corner Method in Problem 5.1.

Solution

Initial Feasible Solution by North West Corner Method (NWCM)

	W_1	W_2	W_3	W_4	*Supplies*
F_1	17 [140]	5	9	7	140
F_2	20 [60]	10 [200]	12	5	260
F_3	15	0 [120]	5 [240]	3	360
F_4	65	65	65 [10]	65 [210]	220
Demand	200	320	250	210	

METHOD III : Vogel's Approximation Method (or VAM)

The practical steps involved in Vogel's Approximation Method (*or VAM*) are given below:

Step 1 → ***Row Difference:*** *Find the difference between Smallest and Second Smallest element of each Row, representing the Opportunity Cost of not making the allocation to the Smallest Element Cell, and write the difference on the right-hand side of the concerned Row. In case of tie between two smallest elements, the difference should be taken as zero.*

↓

Step 2 → ***Column Difference:*** *Find the difference between Smallest and Second Smallest element of each column, representing the Opportunity Cost of not making the allocation to the Smallest Element Cell, and write the difference below the concerned Column. In case of tie between two smallest elements, the difference should be taken as zero.*

↓

Step 3 → ***Largest Difference:*** *Mark the Largest Difference amongst all Differences by an arrow indicating the allocation to be made to the row/column having largest difference. Allocate maximum possible quantity to the Least Cost Cell of the Selected row/column depending upon the quantity available. In case of tie between the Differences, select the row or column having least cost cell. However, in case of tie even in case of Least Cost, make allocation to that Cell by which maximum requirements are exhausted.*

↓

Step 4 → *Shade the Row/Column whose availability or requirement is exhausted so that it shall not be considered for any further allocation.*

↓

Step 5 → *Repeat Step 1 and 4 till entire demand and supply is exhausted.*

↓

Step 6 → *Draw the Initial Feasible Solution Table obtained after the above steps.*

PROBLEM 5.3 [Introduction of Dummy Origin in Case of Contribution Matrix]

Find initial feasible solution by VAM in Problem 5.1.

Solution

Initial Feasible Solution by Vogel's Approximation Method (VAM)

	W_1	W_2	W_3	W_4	*Supplies*	D_1	D_2	D_3	D_4	D_5
F_1	17	5	9 (140)	7	140	2	2	8	8	—
F_2	20	10	12 (50)	5 (210)	260	5	7	8	8	8
F_3	15	0 (320)	5 (40)	3	360	3	2	10	—	—
F_4	65 (200)	65	65 (20)	65	220	0	0	0	0	0
Dem.	200	320	250	210						
D_1	2	5	4	2						
D_2	2	—	4	2						
D_3	2	—	4	—						
D_4	3	—	3	—						
D_5	45	—	53	—						

INTRODUCING AN INFINITELY SMALL QUANTITY 'e' IN CASE THE TOTAL NUMBER OF ALLOCATION IS LESS THAN "M + N - 1"

PROBLEM 5.4 [Case when Total Number of Allocations < M + N - 1]

Find the initial basic feasible solution by VAM. Is the number of allocation equal to 'M + N – 1'? If no, how will you deal with this situation?

	W_1	W_2	W_3	W_4	*Supplies*
F_1	1	2	4	4	6
F_2	4	3	2	0	8
F_3	0	2	2	1	10
Demand	4	6	8	6	

Note: Cell entries are the unit transportation costs.

Solution

Initial Feasible Solution by Vogel's Approximation Method

	W_1	W_2	W_3	W_4	*Supplies*	D_1	D_2	D_3
F_1	1	2 (6)	4	4	6	1	1	2
F_2	4	3	2 (2)	0 (6)	8	2	1	1
F_3	0 (4)	2	2 (6)	1	10	1	2	0
Dem.	4	6	8	6				
D_1	1	0	0	1				
D_2	1	0	0	—				
D_3	—	0	0	—				

Since Allocations are 5 which is less than m + n – 1 (i.e. 4 + 3 – 1 = 6), an infinitestimally small allocation e is placed in the least cost and independent cell in such a way that no loop can be formed by the allocated cells (includes the one in which *e* to be allocated)

There are two least cost cells (1, 1) & (3, 4)

Suppose, *e* is allocated in cell (3, 4)

	W_1	W_2	W_3	W_4
F_1	1	2 (6)	4	4
F_2	4	3	2 (2)	0 (6)
F_3	0 (4)	2	2 (6)	1 (e)

Since loop can be formed after allocating '*e*' in the cell (3, 4), '*e*' should not be allocated here.

Let us allocate '*e*' in the cell (1, 1)

	W_1	W_2	W_3	W_4	*Supplies*
F_1	1 (e)	2 (6)	4	4	6
F_2	4	3	2 (2)	0 (6)	8
F_3	0 (4)	2	2 (6)	1	10
	4	6	8	6	

APPLICATION OF OPTIMALITY TEST

The practical steps involved in Optimality Test are given below :

Step 1 →	***U_i & V_j:*** *Compute "U_i" and "V_j" for all Rows and Columns respectively on the basis of Allocated Cells such that $C_{ij} = U_i + V_j$ after taking any U_i or $V_j = 0$,* *where C_{ij} = Shipping Cost per unit of Occupied Cell;* *i = Row Number;* *j = Column Number;* *U_i = Shipping Cost per unit of Supplying Station; and* *V_j = Shipping Cost per unit of Receiving Station* ***Note:*** *While taking any U_i or $V_j = 0$, that row or column which is having maximum allocated cells, should preferably be selected.*
Step 2 →	***Opportunity Cost for Unallocated Cells:*** *Compute Opportunity Cost, (say OC), for Unallocated Cells where OC = $C_{ij} - (U_{ij} + V_{ij})$.*
Step 3 →	***Loop:*** *If 'OC' of each cell is either positive or zero, Initial Feasible Solution is the Optimal Solution. However, if 'OC' for any Cell is negative, Initial Feasible Solution is not optimal. In that case, Find Closed Loop (explained on page 5.9) for the Cell having negative 'OC' and transfer entire quantity from the Allocated Cell having minimum quantity, that is covered by that Loop amongst all Allocated Cells covered by that Loop, to the Unallocated Cell having negative 'OC' (the procedure to transfer the quantity has been explained on page 5.9).* ***Note:*** *The above procedure will be followed even in case 'OC' of any Unallocated Cell is "zero" and 'OC' of other Unallocated Cells is positive to get Alternate Solution.*
Step 4 →	***Infinitely Small Quantity 'e':*** *See whether total number of allocated cells after Step 3 is equal to "m + n –1"; if yes, go to step 5, if not introduce an infinitely small quantity 'e' to the Least Cost Independent Cell, i.e., for which no Loop can be formed.* ***Note:*** *Introduce as many number of 'e' as the total number of Allocated Cells falls below "m + n – 1".*
Step 5 →	*Repeat Steps 1 to 4 till 'OC' of all Unallocated Cells is positive.*

PROBLEM 5.5 [When the Number of Allocation is Less Than 'm + n - 1']

Test the following initial solution for optimality

	W_1	W_2	W_3	W_4	*Supplies*
F_1	1 (e)	2 (6)	4	4	6
F_2	4	3	2 (2)	0 (6)	8
F_3	0 (4)	2	2 (6)	1	10
Demand	4	6	8	6	

Note: Cell entries are transportation costs per unit.

Solution

Step 1 → : *Calculation of U_i and V_j on the basis of costs of allocated cells*

	W_1	W_2	W_3	W_4	U_i
F_1	1	2			$U_0 = 0$
F_2			2	0	$U_1 = -1$
F_3	0		2		$U_2 = -1$
V_j	$V_0 = 1$	$V_1 = 2$	$V_2 = 3$	$V_3 = 1$	

Step 2 → *C_{ij} Matrix of Costs for unallocated cells*

	W_1	W_2	W_3	W_4
F_1			4	4
F_2	4	3		
F_3		2		1

Step 3 → *$U_i + V_j$ Matrix for unallocated Cells*

	W_1	W_2	W_3	W_4
F_1			3	1
F_2	0	1		
F_3		1		0

Step 4 → *Δ_{ij} Matrix where $\Delta_{ij} = C_{ij} - (U_i + V_j)$*

	W_1	W_2	W_3	W_4
F_1			1	3
F_2	4	2		
F_3		1		1

Step 5 → *Since all Δ_{ij} are positive the above solution is optimal. The optimal solution is given below:*

Factory	*Warehouse*	*Qty.*	*Cost per Unit*	*Total cost*
F_1	W_3	6	2	12
F_2	W_3	2	2	4
F_2	W_4	6	0	0
F_3	W_1	4	0	0
F_3	W_3	6	2	12
				28

LOOPING AND REALLOCATION MATRIX

The practical steps involved in the Looping are given below:

Step 1 → ***Most Negative Opportunity Cost Cell:*** *Put a 'Tick' mark in the most negative Opportunity Cost Cell.*
In case of tie, any one may be selected arbitraily, preferably select that one with which corners points are minimum.

↓

Step 2 → ***Draw Loop:*** *Draw at least four lines in the form of a rectangle covering at least four Cells out of which One Cell will be the ticked Cell and the rest will be the Allocated Cells. This is called 'Loop'. In other words all the corners (except the starting corner which lies in most negative unallocated cell) of the Loop will lie in the allocated cells.*

↓

Step 3 → ***Putting '+' & '–' Signs:*** *Put '+' sign on ticked cell and '–' sign on the next allocated corner cell covered by the loop and '+' sign on the next to next allocated corner cell covered by the loop and so on.*
In other words, '+' or '–' sign should be put in each allocated corner cell in alternative order.

↓

Step 4 → ***Transfer of Entire Quantity:*** *Transfer the entire quantity from the Least Allocated Cell (i.e. cell having the least quantity) which has '–', to the Cells containing '+' and deduct the transferred quantity from the other cell containing '–'.*
Note: *A Loop can even contain more than Four Cells; however, Number will always remain 'even'. In that case, one Cell will be the Unallocated Cell having most negative Opportunity Cost and the Others will be the Allocated Cells.*

PROBLEM 5.6 [When there is a Tie between the Two Most Negative Cells]

Prepare a reallocation matrix from the following information:

Initial Allocation Matrix

	1	2	3	4
A	3		6	1
B		4		
C		6		
D				5
E	10	e		

Δ_{ij} (Cell Evaluation) Matrix

	1	2	3	4
A		16		
B	20		35	–8
C	40		20	7
D	13	18	33	
E			10	–8

Solution

Step 1 →Ticking one of the most negative cells (B, 4) and (E, 4) of cell evaluation matrix from where looping is to be started and putting θ quantity therein and drawing lines in such a way that turning point of the line is in the allocated cells. Let us first tick (B, 4) cell.

	1	2	3	4
A	3+		6	1–
B		4–		+
C		6		
D				5
E	10–	e+		

Step 2 → Calculating new quantity of each corner of the loop after subtracting from and adding to 1 quantity (being the least one in negative corner cells) to each of the loop corner quantity.

Corner (B, 4) = 0 + 1 = 0
Corner (A, 4) = 1 – 1 = 0
Corner (A, 1) = 3 + 1 = 4
Corner (E, 1) = 10 – 1= 9
Corner (E, 2) = e + 1 = 1
Corner (B, 2) = 4 – 1 = 3

Step 3 → Now Reallocation Matrix becomes as under:

	1	2	3	4
A	4		6	
B		3		1
C		6		
D				5
E	9	1		

Alternative Solution: Let us tick (E, 4) cell.

Step 1 → Ticking the another most negative cell (E, 4) of cell evaluation matrix from where looping is to be started and putting θ quantity therein and drawing lines in such a way that turning point of the line is in the allocated cells.

	1	*2*	*3*	*4*
A	3+		6	1–
B		4		
C		6		
D				5
E	10–	e		θ +

Step 2 → Calculating new quantity of each corner of the loop after subtracting from and adding to 1 quantity (being the least one in negative corner cells) to each of the loop corner quantity.

Corner (E, 4) = 0 + 1 = 0
Corner (A, 4) = 1 – 1 = 0
Corner (A, 1) = 3 + 1 = 4
Corner (E, 1) = 10 – 1 = 9

Step 3 → Now Reallocation Matrix becomes as under:

	1	*2*	*3*	*4*
A	4		6	
B		4		
C		6		
D				5
E	9	e		1

PROBLEM 5.7 [Use of High Cost Where there is Restriction on Transfer]

Solve the following transportation problem:

GODOWNS

		1	*2*	*3*	*4*	*5*	*6*	*Stock Available*
FACTORY	1	7	5	7	7	5	3	60
	2	9	11	6	11	—	5	20
	3	11	10	6	2	2	8	90
	4	9	10	9	6	9	12	50
Demand		60	20	40	20	40	40	

Note: It is not possible to transport any quantity from factory 2 to godown 5.
Required: State whether the solution derived by you is unique.

Solution

Step 1 → Finding Initial Feasible Solution by following the Vogel's Approximation Method (or VAM).

Factory *Godowns* *Availability* *Diff.*

Factory	1	2	3	4	5	6	Availability	Diff.
1	7	5 (20)	7	7	5	3 (40)	60/40/0	2/4/0
2	9 (10)	11	6 (10)	11	∞	5	20/10/0	1/3
3	11	10	6 (30)	2 (20)	2 (40)	8	90/70/30/0	0/4/2/5
4	9 (50)	10	9	6	9	12	50/0	3/0
Demand	60 50 0	20 0	40 10 0	20 0	40 0	40 0		
Diff.	2	5	0/1	4	3	2		

Step 2 → Since the total number of allocations is less than "M + N − 1" allocations, let us introduce an infinitely small quantity 'e' to the least cost independent cell to make the total number of allocations equal to "M + N − 1" allocations.

Let us calculate "U_i's" and "V_j's" for all rows and columns respectively on the basis of allocated cells such that $C_{ij} = U_i + V_j$ assuming $U_1 = 0$,

Let us calculate Δ_{ij} (Opportunity Cost), for unallocated cells where $\Delta_{ij} = C_{ij} - (U_{ij} + V_{ij})$.

U_i & V_j Matrix

						U_i
	5				3	–2
9		6			5 (e)	0
		6	2	2		0
9						0
V_j 9	7	6	2	2	5	

C_{ij} Matrix for Unallocated Cells

7		7	7	5	
	11		11	∞	
11	10				8
	10	9	6	9	12

$U_i + V_j$ Matrix for Unallocated cells

						U_i
7		4	0	0		-2
	7		2			0
9	7				5	0
	7	6	2	2	5	0
V_j 9	7	6	2	2	5	

$\Delta_{ij} = C_{ij} - (U_i + V_j)$ Matrix

0		3	7	5	
	4		9		
2	3				3
	3	3	4	7	7

Step 3 → Since all Δ_{ij} are positive, the above solution is optimal. The optimal solution is given below:

Factory	*to Godown*	*Unit*	*Cost*	*Value (Rs.)*
1	2	20	5	100
1	6	40	3	120
2	1	10	9	90
2	3	10	6	60
3	3	30	6	180
3	4	20	2	40
3	5	40	2	80
4	1	50	9	450
			Total cost Rs.	1120

Note: Since one of Δ_{ij} is zero, the above solution is not unique. Hence an alternative solution exists. The alternative solution can be found by taking any cell with zero Δ_{ij} as the basic cell.

Students may find that the alternate solution is as given below:

Factory	*to Godown*	*Unit*	*Cost*	*Value (Rs.)*
1	1	10	7	70
1	2	20	5	100
1	6	30	3	90
2	3	10	6	60
2	6	10	5	50
3	3	30	6	180
3	5	40	2	80
3	4	20	2	40
4	1	50	9	450
			Total Cost	Rs. 1120

PROBLEM 5.8

Consider the following data for the transportation problem:

	Destination			
Factory	*(1)*	*(2)*	*(3)*	*Supply to be exhausted*
A	5	1	7	10
B	6	4	6	80
C	3	2	5	15
Demand	75	20	50	

Since there is not enough supply, some of the demands at the three destinations may not be satisfied. For the unsatisfied demands, let the penalty costs be rupees 1, 2 and 3 for destinations (1), (2) and (3) respectively.

Find the optimal allocation that minimizes the transportation and penalty costs

Solution

Step 1 → *Formulating the Transportation Problem.*

Since demand (= 75 + 20 + 50 = 145) is greater than supply (= 10 + 80 + 15 = 105) by 40 units, the given problem is an unbalanced one. We introduce a dummy factory with a supply of 40 units. It is given that for the unsatisfied demands, the penalty cost is rupees 1, 2 and 3 for destinations (1), (2) and (3) respectively. Hence, the transportation problem becomes:

Factory	*Destination* (1)	(2)	(3)	*Supply to be exhausted*
A	5	1	7	10
B	6	4	6	80
C	3	2	5	15
Dummy	1	2	3	40
Demand	75	20	50	

Step 2 → Finding Initial Feasible Solution by following the Vogel's Approximation Method (or VAM).

Factory	*Destination* (1)	(2)	(3)	*Supply*	*Differences*
A	5	1 (10)	7	10/0	4 - -
B	6 (20)	4 (10)	6 (50)	80/70/50/0	2 2 2
C	3 (15)	2	5	15/0	1 1 1
Dummy	1 (40)	2	3	40/0	11 -
Demand	75/35/20/0	20/10/0	50/0		
Difference	2	1	2		
	2	0	2		
	3	2	1		

Step 3 → ***Optimality Test:*** Since the total number of allocations is equal to "M+N–1" allocations, the initial solution is straight away tested for optimality.

Let us calculate "Ui's" and "Vj's" for all rows and columns respectively on the basis of allocated cells such that $C_{ij} = U_i + V_j$ assuming $U_1 = 0$,

Let us calculate Δ_{ij} (Opportunity Cost), for unallocated cells where $\Delta_{ij} = C_{ij} - (U_{ij} + V_{ij})$.

U_i & V_j Matrix

			U_i
	1		–3
6	4	6	0
3			–3
1			–5
V_j	6	4	6

C_{ij} Matrix for Unallocated Cells

5		7
	2	5
	2	3

$U_i + V_j$ Matrix of unallocated cells

			U_i
3		3	−3
			0
	1	3	−3
	−1	1	−5
V_j	6	4	6

$\Delta_{ij} = C_{ij} - (U_i + V_j)$ Matrix

2		4
	1	2
	3	2

Step 4 → Since all Δ_{ij} are positive, the above solution is optimal. The optimal solution is given below:

Factory	*Destination*	*Units*	*Cost*	*Total Cost*	
A	(2)	10	Re 1	Rs. 10	Transportation cost
B	(1)	20	Rs. 6	Rs. 120	
B	(2)	10	Rs. 4	Rs. 40	
B	(3)	50	Rs. 6	Rs. 300	
C	(1)	15	Rs. 3	Rs. 45	
Dummy	(1)	40	Re 1	Rs. 40	Penalty cost
				Rs. 555	

PROBLEM 5.9 [How to deal with cost in fractions and fixed price for destinations]

A company produces a small component for all industrial products and distributes it to five wholesalers at a fixed price of Rs.2.50 per unit. Sales forecasts indicate that monthly deliveries will be 3,000, 3,000, 10,000, 5,000 and 4,000 units to wholesalers 1, 2, 3, 4 and 5 respectively. The monthly production capabilities are 5,000, 10,000, 12,500 at plants 1, 2 and 3 respectively. The direct costs of production of each unit are Rs. 1.00, Re. 0.90 and Re. 0.80 at plants 1, 2 and 3 respectively. The transportation costs of shipping a unit from a plant to a wholesaler are given below:

		Wholesaler				
		1	***2***	***3***	***4***	***5***
	1	0.05	0.07	0.10	0.15	0.15
Plant	2	0.08	0.06	0.09	0.12	0.14
	3	0.10	0.09	0.08	0.10	0.15

Find how many components each plant supplies to each wholesaler in order to maximize profit.

Solution

Since the total capacity of plants is more than the supply to the wholesalers by a quantity 27500 units – 25000 units = 2500 units, so the problem is an unbalanced transportation problem. Introduce a dummy wholesaler supplying 2500 units such that the transportation costs plus the production cost from various plants to this destination are assumed to be zero. Also, the direct costs of production of each unit are given as Re.1, Re.0.90 and Re.0.80 at plants 1, 2 and 3 respectively. The modified balanced transportation problem is now obtained as follows:

		1	*2*	*3*	*4*	*5*	*Dummy*	*Capacity*
	1	1.05	1.07	1.10	1.15	1.15	0	5000
Plant	2	0.98	0.96	0.99	1.02	1.04	0	10000
	3	0.90	0.89	0.88	0.90	0.95	0	12500
		3000	3000	10000	5000	4000	2500	

For simplicity in computations multiply all direct cost *plus* transportation costs in the above table by 100, and consider 100 units = 1 unit of items. Hence, the simplified cost table becomes:

		1	*2*	*3*	*4*	*5*	*Dummy*	*Capacity*
	1	105	107	110	115	115	0	50
Plant	2	98	96	99	102	104	0	100
	3	90	89	88	90	95	0	125
Supply		30	30	100	50	40	25	3

Let us now apply Vogel's Approximation method to find the initial basic feasible solution.

Plant	*Wholesaler*						*Capacity*	*Difference*
	1	*2*	*3*	*4*	*5*	*Dummy*		
1	$\boxed{25}$ 105	107	110	115	115	$\boxed{25}$ 0	50/25/0	105/2/2/2/5/10
2	$\boxed{5}$ 98	$\boxed{30}$ 96	$\boxed{25}$ 99	102	$\boxed{40}$ 104	0	100/70/4/5/5	96/2/2/2/1/6
3	90	89	$\boxed{75}$ 88	$\boxed{50}$ 90	95	0	125/75/0	88/1/–
Supply	30/25/0	30/0	100/25/0	50/0	40/0	25/0		

Diff.						
	8	7	11	12	9	0
	8	7	11	12	9	–
	8	7	11	–	9	–
	7	11	11	–	11	–
	7	–	11		11	

The initial basic feasible solution as obtained by VAM is given below:

Plant	*Wholesaler*					*Capacity*	*Difference*
	1	*2*	*3*	*4*	*5*		
1	$\boxed{25}$ 105	107	110	115	115	25 0	50
2	$\boxed{5}$ 98	$\boxed{30}$ 96	$\boxed{25}$ 99	120	$\boxed{40}$ 104	0	100
3	99	98	$\boxed{75}$ 88	$\boxed{50}$ 90	95	0	120
Supply		30	30	100	50	40	25

We now apply the optimally test to find whether the initial solution found above is optimal or not. The number of allocations is 8 which is equal to the required m + n – 1 (=8) allocations. Also, these allocations are independent. Hence, both the conditions of optimality test are satisfied.

Let us now introduce u_i's, (1 = 1, 2, 3) and v_j's, (j = 1, 2, 3, ... 6) such that $\Delta_{ij} = c_{ij} - (u_i + v_j)$ for allocated cells. We assume that $u_2 = 0$ and remaining u_i's, v_js and Δ_{ij}'s are calculated as below:

Plant	*Wholesaler*					*Dummy*	*u_i's*
	1	*2*	*3*	*4*	*5*		
1	$\boxed{25}$ 105	4 107	4 110	7 115	4 115	$\boxed{25}$ 0	7
2	$\boxed{5}$ 98	$\boxed{30}$ 96	$\boxed{25}$ 99	$\boxed{1}$ 102	$\boxed{40}$ 104	7 0	0
3	3 90	4 89	$\boxed{75}$ 88	$\boxed{50}$ 90	2 95	18 0	–11
v_j	98	96	99	101	104	–7	

Since all Δ_{ij}'s for non-basic cells are positive, therefore, the solution obtained above is an optimal one. The allocation of plant to wholesaler and their (transportation cost + direct production cost) is given below:

Plant		*Warehouse*	*No. of units*	*Cost per unit (Rs.)*	*Total cost Rs.*
1	→	1	2500	1.05	2625
2	→	1	500	0.98	490
2	→	2	3000	0.96	2880
2	→	3	2500	0.99	2475
2	→	5	4000	0.104	4160
3	→	3	7500	0.88	6600
3	→	4	5000	0.90	4500
					23730

Since the company distributes the component at a fixed price of Rs.2.50 per unit to all five wholesalers, the profit is given by

Profit = Rs. 2.50 × 22500 – Rs. 23,730 = Rs. 56,250 – Rs. 23,730 = Rs. 32,520

(**Note:** The problem has been solved using the cost matrix. However, it can be solved using the profit matrix first and then converting it to minimization problem).

PROBLEM 5.10

The following table shows all the necessary information on the available supply to each warehouse, the requirement of each market and the unit transportation cost from each warehouse to each market:

		Market				*Supply*
		I	II	III	IV	
Warehouse	A	5	2	4	3	22
	B	4	8	1	6	15
	C	4	6	7	5	8
Requirement		7	12	17	9	

The shipping clerk has worked out the following schedule from his experience:

12 Units from A to II

1 Unit from A to III

9 Units from A to IV

15 Units from B to III

7 Units from C to I and

1 Unit from C to III.

You are required to answer the following:

(i) Check and see if the clerk has the optimal schedule;

(ii) Find the optimal schedule and minimum total shipping cost; and

(iii) If the clerk is approached by a carrier of route C to II, who offers to reduce his rate in the hope of getting some business, by how much should the rate be reduced before the clerk should consider giving him an order?

Solution

(i) The shipping clerk has worked out the following schedule as initial solution:

		Market				
		I	*II*	*III*	*IV*	*Supply*
	A	5	12 2	1 4	9 3	22
Warehouse	B	4	8	15 1	6	15
	C	7 4	6	1 7	5	8
Requirement		7	12	17	9	

The initial solution is tested for optimality. The total number of independent allocations is 6 which is equal to the desired (m + n – 1) allocations. We introduce u_i's (I = 1, 2, 3) and v_i's (j = 1, 2, 3, 4) such that $\Delta_{ij} = c_{ij} - (u_i + v_j)$. Let us assume $u_i = 0$, remaining u_1's and v_1's are calculated as below:

	I	II	III	IV	u_i
A	4	12	1 $+\theta$	9 $-\theta$	0
	5	2	4	3	
B	6	9	15	6	–3
	4	8	1	6	
C	7	1	1 $-\theta$	–1 θ	3
	4	6	7	5	
v_j	1	2	4	3	

Since one of the Δ_{ij}'s is negative, the schedule worked out by the clerk is not the optimal solution.

(ii) Introduce in the cell with -ve Δ_{ij} [i.e. the cell (C, IV)], an assignment θ. The value of θ and the reallocated solution as obtained from above is given below. The value of u_i's, v_i's are also calculated.

$$\theta = \min(9 - \theta = 0 \ \& \ 1 - \theta = 0) = 1$$

	I	II	III	IV	u_i
A	3 5	12 2	2 4	8 3	0
B	5 4	9 8	15 1	6 6	–3
C	7 4	2 6	1 7	1 5	2
v_j	2	2	4	3	

Since all Δ_{ij}'s for non basic cells are positive, the solution as calculated in the above table is optimal solution. The supply of units from each warehouse to markets, along with the transportation cost is given below:

Warehouse	*Market*	*Units unit(Rs.)*	*Cost per Cost(Rs.)*	*Total*
A	II	12	2	24
A	III	2	4	8
A	IV	8	3	24
B	III	15	1	15
C	I	7	4	28
C	IV	1	5	5
		Minimum Total Shipping Cost		= 104

(iii) If the clerk wants to consider the carrier of route C to II for giving an order, then his transportation cost must be less than the cost of carrier of routes C to I and C to IV i.e. his transportation cost should be at the most Rs. 3 per unit. If the carrier C to II brings down his cost to Rs. 3, he will get an order for 1 unit, and the schedule will be:

Warehouse	*Market*	*Units unit(Rs.)*	*Cost per Cost(Rs.)*	*Total*
A	II	11	2	22
A	III	2	4	8
A	IV	9	3	27
B	III	15	1	15
C	I	7	4	28
C	II	1	3	3
		Minimum Total Shipping Cost		= 103

and the Total Shipping Cost will be Rs. 103

PROBLEM 5.11

A particular product is manufactured in factories A, B, C and D; and is sold at centres 1, 2 and 3. The cost in Rs. of product per unit and capacity in kg per unit time of each plant is given below:

Factory	*Cost (Rs.) per unit*	*Capacity (kg) per unit*
A	12	100
B	15	20
C	11	60
D	13	80

The sale price in Rs. per unit and the demand in kg per unit time are as follows:

Sales centre	*Sale Price (Rs.) per unit*	*Demand (kg) per unit*
1	15	120
2	14	140
3	16	60

Required: Find the optimal sales distribution.

Solution

Step 1 → Deriving profit matrix by equation (i.e. Selling Price – Cost) and Introducing a Dummy factory with zero profit per unit as the total demand is not equal to total supply in order to make the problem balanced one.

Factory	*Sales Centres*			*Capacity (kg) per unit*
	1	*2*	*3*	
A	3	2	4	100
B	0	–1	1	20
C	4	3	5	60
D	2	1	3	80
Dummy	0	0	0	60
Demand (kg)	120	140	60	320

Step 2 → Deriving Loss Matrix by deducting each element from the maximum element (i.e. 5) in order to use minimisation technique and Finding Initial Feasible Solution by VAM.

Factory	*Sales Centre*			*Capacity*	*Difference*
	1	*2*	*3*		
A	2 [100]	3	1	100/0	1 1 -
B	5	6 [20]	4	20/0	1 1 1
C	1	2	0 [60]	60/0	1 - -
D	3 [20]	4 [60]	2	80/60/0	1 1 1
Dummy	5	5 [60]	5	60/0	0 0 0
Demand	120/20/0	140/120/60/0		60/0	
Difference	1	1	1		
	1	1	—		
	2	1	—		

Step 3 → Since the total number of allocations is less than "M + N − 1" allocations, let us introduce an infinitely small quantity 'e' to the least cost independent cell to make the total number of allocations equal to "M + N − 1" allocations.

Let us calculate "U_i's" and "V_j's" for all rows and columns respectively on the basis of allocated cells such that $C_{ij} = U_i + V_j$ assuming $U_1 = 0$,

Let us calculate Δ_{ij} (Opportunity Cost), for unallocated cells where $\Delta_{ij} = C_{ij} - (U_{ij} + V_{ij})$.

U_i & V_j Matrix

				U_i
	2		e 1	*3*
		6		*6*
			0	*2*
	3	4		*4*
		5		*5*
V_j	*−1*	*0*	*−2*	

C_{ij} Matrix for Unallocated cells

	3	
5		4
1	2	
		2
5		5

$U_i + V_j$ Matrix

				U_i
		3		*3*
	5		4	*6*
	1	2		*2*
			2	*4*
	4		3	*5*
V_j	*−1*	*0*	*−2*	

$\Delta_{ij} = C_{ij} - (U_i + V_j)$ Matrix

	0	
0		0
0	0	
		0
1		2

Step 4 → Since all Δ_{ij} are positive, the above solution is optimal. The optimal solution is given below:

From Factory	*To Sales Centre*	*Quantity*	*Profit per unit (Rs.)*	*Total Profit (Rs.)*	
A	1	100	3	300	
B	2	20	−1	−20	
C	3	60	5	300	
D	1	20	2	40	
D	2	60	1	60	
Dummy	2	60	0	0	
			Total Profit =	680	

Note: *Since some of Δ_{ij} are zero, the above solution is not unique. Hence an alternative solution exists. The alternative solution can be found by taking any cell with zero Δ_{ij} as the basic cell.*

PROBLEM 5.12

A company has four factories situated in four different locations in the country and four sales agencies located in four other locations in the country. The cost of production (Rs. per unit), the sale price (Rs. per unit), shipping cost (Rs. per unit) in the cells of matrix, monthly capacities and monthly requirements are given below:

Factory	*Sales Agency*				*Monthly Capacity*	*Cost of*
	1	*2*	*3*	*4*	*(Units)*	*Production*
A	7	5	6	4	10	10
B	3	5	4	2	15	15
C	4	6	4	5	20	16
D	8	7	6	5	15	15
Monthly Requirements (Units)	8	12	18	22		
Sales Price	20	22	25	18		

Find the monthly production and distribution schedule which will maximize profit.

Solution

Step 1 →Deriving Profit Matrix below by equation:

Profit = Sales Price – Cost of Production – Shipping Cost

Factory	*Sales Agency*				*Monthly Capacity (units)*
	1	*2*	*3*	*4*	
A	3	7	9	4	10
B	2	2	6	1	15
C	0	0	5	–3	20
D	–3	0	4	–2	15
Monthly Requirements	8	12	18	22	

Step 2 → Deriving Loss Matrix by deducting each element from the maximum element (i.e. 9) in order to use minimisation technique and Finding Initial Feasible Solution VAM.

Factory	*Sales Agency*				*Monthly Capacity*	*Difference*
	1	2	3	4		
A	6	2 $\boxed{10}$	0	5	10/0	2/3-
B	7	7	3	8 $\boxed{15}$	15/0	4/0/0-
C	9 $\boxed{2}$	9	4 $\boxed{18}$	12	20/2/0	5/0/0/0
D	12 $\boxed{6}$	9 $\boxed{2}$	5	11 $\boxed{7}$	15/13/6/0	4/2/2/2
Monthly Requirement	8/6/0	12/2/0	18/0	22/7		
Difference	1/2/2/3	5/2/2/0	3 - - -	3/4/4/1		

Step 3 → Since the total number of allocations is equal to "M + N − 1" allocations, the initial solution is straight away tested for optimality.

Let us calculate "U_i's" and "V_j's" for all rows and columns respectively on the basis of allocated cells such that $C_{ij} = U_i + V_j$ assuming $U_3 = 0$,

Let us calculate Δ_{ij} (Opportunity Cost), for unallocated cells where $\Delta_{ij} = C_{ij} - (U_{ij} + V_{ij})$.

U_i & V_j Matrix

					U_i
		2			−7
				8	−3
	9		4		−3
	12	9		11	0
V_j	12	9	7	11	

C_{ij} Matrix for Unallocated cells

6		0	5
7	7	3	
	9		12
		5	

$U_i + V_j$ Matrix

					U_i
	5		0	4	−7
	9	6	4		−3
		6		8	−3
			7		0
V_j	12	9	7	11	

$\Delta_{ij} = C_{ij} - (U_i + V_j)$ Matrix

1		0	1
−2	5	−1	
	3		4
		−2	

Step 4 → Since all Δ_{ij} are not positive, the above solution is not optimal. Let us place a small allocation θ in the cell with most negative Δ_{ij} and form a loop including this cell and test the solution for optimality. Next improved solution as obtained from above is shown below which is tested for optimality. The value of "U_i's", "V_j's" and Δ_{ij} are also calculated.

Looping Matrix

	10		
			15
2+		18−	
−6	2	θ +	

Reallocation Matrix

	10		
			15
8		12	
θ	2	6	7

U_i & V_j Matrix

					U_i
		2			−7
				8	−3
	9		4		−1
		9	5	11	0
V_j	10	9	5	11	

C_{ij} Matrix for Unallocated cells

6		0	5
7	7	3	
	9		12
12			

$U_i + V_j$ Matrix

				U_i
3		–2	4	–7
7	6	2		–3
	8		10	–1
10				0
V_j: 10	9	5	11	

$\Delta_{ij} = C_{ij} - (U_i + V_j)$ Matrix

3		2	1
	0	1	1
	1		2
	2		

Step 5 → Since all Δ_{ij} are positive, the above solution is optimal. The optimal solution is given below:

Factory	*Sales Agency*	*Profit*
A	2	10 x Rs 7= Rs. 70
B	4	15 x Rs 1= Rs. 15
C	1	8 x Rs 0= Rs. 0
C	3	12 x Rs 5= Rs. 60
D	2	2 x Rs 0= Rs. 0
D	3	6 x Rs 4= Rs. 24
D	4	7 x Rs –2 = Rs. –14
	Total	= Rs. 155

Note: Since one of Δ_{ij} is zero, the above solution is not unique. Hence an alternative solution exists. The alternative solution can be found by taking cell with zero Δ_{ij} as the basic cell.

PROBLEM 5.13

ABC Enterprises is having three plants manufacturing dry-cells, located at different locations. Production cost differs from plant to plant. There are five sales offices of the company located in different regions of the country. The sales prices can differ from region to region. The shipping cost from each plant to each sales office and other data are given by following table:

Production Data Table

Production cost per unit	*Max. Capacity in No. of units*	*Plant No.*
20	150	1
22	200	2
18	125	3

Shipping Costs Table

	Sales Office 1	*Sales Office 2*	*Sales Office 3*	*Sales Office 4*	*Sales Office 5*
Plant 1	1	1	5	9	4
Plant 2	9	7	8	3	6
Plant 3	4	5	3	2	7

Demand & Sales Prices

	Sales Office 1	Sales Office 2	Sales Office 3	Sales Office 4	Sales Office 5
Demand	80	100	75	45	125
Sales Price	30	32	31	34	29

Find the production and distribution schedule most profitable to the company.

Solution

Step 1 → Deriving the Profit Matrix by equation (i.e. Profit = Selling Price – Cost of Production – Shipping Cost) and Introducing a Dummy Sales Office with zero profit per unit as the total demand is not equal to total supply in order to make the problem balanced one.

9	11	6	5	5	0	150
–1	3	1	9	1	0	200
8	9	10	14	4	0	125
80	100	75	45	125	50	475

Step 2 → Deriving Loss Matrix by deducting each element from the maximum element (i.e. 14) in order to use minimisation technique and finding the initial feasible solution by VAM.

Plant *Sales Centre*

	1	*2*	*3*	*4*	*5*	*6*	*Capacity*	*Difference*
1	5 [50]	3 [100]	8	9	9	14	150	2 2 2 4 - -
2	15	11	13	5 [45]	13 [105]	14 [50]	200	6 2 2 2 1 1
3	6 [30]	5	4 [75]	0	10 [20]	14	125	1 1 1 4 4 4
Demand	80	100	75	45	125	50		
Diff.	1	2	4	5	1	0		
	1	2	4	-	1	0		
	1	2	-	-	1	0		
	1	-	-	-	1	0		
	9	-	-	-	3	0		
	-	-	-	-	3	0		

Step 3 → Since the total number of allocations is equal to "M + N – 1" allocations, the initial solution is straightaway tested for optimality.

Let us calculate "U_i's" and "V_j's" for all rows and columns respectively on the basis of allocated cells such that $C_{ij} = U_i + V_j$ assuming $U_3 = 0$,

Let us calculate Δ_{ij} (Opportunity Cost), for unallocated cells where $\Delta_{ij} = C_{ij} - (U_{ij} + V_{ij})$.

U_i & V_j Matrix | *U_i* | *C_{ij} Matrix for Unallocated cells*

							U_i						
	5	3					*–4*			8	9	9	14
				5	13	14	*0*	15	11	13			
	6		4		10		*–3*		5		0		14
V_j	*9*	*7*	*7*	*5*	*13*	*14*							

$U_i + V_j$ Matrix

						U_i
		3	1	9	10	–4
9	7	7				0
	4		2		11	–3
9	7	7	5	13	14	V_j

$\Delta_{ij} = C_{ij} - (U_i + V_j)$ Matrix

		5	8	0	4
6	4	6			
	1		-2		3

Step 4 → Since all Δ_{ij} are not positive, the above solution is not optimal. Let us place a small allocation θ in the cell with most negative Δ_{ij} and form a loop including this cell and test the solution for optimality. Next improved solution as obtained from above is shown below which is tested for optimality. The value of "U_i's", "V_j's" and Δ_{ij} are also calculated.

Looping Matrix

50	100				
			45–	105+	50
30		75	θ+	20–	

Reallocation Matrix

50	100				
			25	125	50
30		75	20		

U_i & V_j Matrix

						U_i
5	3					–6
			5	13	14	0
6		4	0			–5
11	9	9	5	13	14	V_j

C_{ij} Matrix for Unallocated cells

		8	9	9	14
15	11	13			
	5			10	14

$U_i + V_j$ Matrix

						U_i
		3	–1	7	8	–6
11	9	9				0
	4			8	9	–5
11	9	9	5	13	14	V_j

$\Delta_{ij} = C_{ij} - (U_i + V_j)$ Matrix

		5	10	2	6
4	2	4			
	1			2	5

Step 5 → Since all Δ_{ij} are positive, the above solution is optimal. The optimal solution is given below:

Plant	*Sales Office*	*Units*	*Profit per unit*	*Profit*
1	1	50	9	450
1	2	100	11	1100
2	4	25	9	225
2	5	125	1	125
2	Dummy	50	0	0
3	1	30	8	240
3	3	75	10	750
3	4	20	14	280
			Total	3,170

PROBLEM 5.14 [Maximizing Billing of Assignment]

A leading firm has three auditors. Each auditor can work upto 160 hours during the next month, during which time three projects must be completed. Project 1 will take 130 hours, project 2 will take 140 hours, the project 3 will take 160 hours. The amount per hour that can be billed for assigning each auditor to each project is given below:

Auditor	*Project*		
	1 *Rs.*	*2* *Rs.*	*3* *Rs.*
1	1,200	1,500	1,900
2	1,400	1,300	1,200
3	1,600	1,400	1,500

Required: Formulate this as a transportation problem and find the optimal solution. Also find out the maximum total billings during the next month.

Solution

Step 1 → Introducing a Dummy Project with zero amount per unit as the total hours required is not equal to total hours available in order to make the problem balanced one and divide by Rupees by 100 to ease computations.

Auditor	*Project*			*Dummy*	*Time*
	1 *Rs.*	*2* *Rs.*	*3* *Rs.*	*Rs.*	*available (hours)*
1	12	15	19	0	160
2	14	13	12	0	160
3	16	14	15	0	160
Time required (hrs)	130	140	160	50	480

Step 2 → Deriving Loss Matrix by deducting each element from the maximum element (i.e. 19) in order to use minimisation technique and Finding Initial Feasible Solution VAM.

Auditor	*Project*			*Dummy*	*Time available*	*Difference*
	1	*2*	*3*			
1	7	4	0 $\boxed{160}$	19	160/0	4/-/-/-
2	5	6 $\boxed{110}$	7	19 $\boxed{50}$	160/50/0	1/1/13/13
3	3 $\boxed{130}$	5 $\boxed{30}$	4	19	160/30/0	1/2/14/-
Time Required	130/0	140/110/0	160/0	50/0		
Difference	2/2/-/-	1/1/1/1	4/-/-	0/0/0		

Step 3 → Since the total number of allocations is less than "M + N − 1" allocations, let us introduce an infinitely small quantity 'e' to the least cost independent cell to make the total number of allocations equal to "M + N − 1" allocations.

Let us calculate "U_i's" and "V_j's" for all rows and columns respectively on the basis of allocated cells such that $C_{ij} = U_i + V_j$ assuming $U_1 = 0$, Let us calculate Δ_{ij} (Opportunity Cost), for unallocated cells where $\Delta_{ij} = C_{ij} - (U_{ij} + V_{ij})$.

U_i & V_j Matrix

				U_i
		0	19	–4
	6		19	1
3	5	4 e		0
V_j: 3	5	4	18	

C_{ij} Matrix for Unallocated cells

7	4		19
5		7	
			19

$U_i + V_j$ Matrix

-1	1		14	–4
4		5		1
			18	0
3	5	4	18	

$\Delta_{ij} = C_{ij} - (U_i + V_j)$ Matrix

8	3		5
1		2	
			1

Step 4 → Since all Δ_{ij} are positive, the above solution is optimal. The optimal solution is given below:

Auditor	*Project*	*Billing amount (Rs.)*
1	3	160 × Rs. 1900 = 3,04,000
2	2	110 × Rs. 1300 = 1,43,000
3	1	130 × Rs. 1600 = 2,08,000
3	2	30 × Rs. 1400 = 42,000
	Total billing	= 6,97,000

PROBLEM 5.15

A company has four factories F_1, F_2, F_3 and F_4 manufacturing the same product. Production and raw material costs differ from factory to factory, and are given in the following table in the first two rows. The transportation costs from the factories to sales depots S_1, S_2, S_3 are also given.

The last two columns in the table give the sale price and the total requirement at each depot. The production capacity of each factory is given in the last row.

	F_1	F_2	F_3	F_4	*Sales Price per unit*	*Requirements in units*
Production cost/unit	15	18	14	13		
Raw material cost/unit	10	9	12	9		
S_1	3	9	5	4	34	80
Transportation S_2	1	7	4	5	32	120
Cost per unit : S_3	5	8	3	6	31	150
	10	150	50	100		

Determine the most profitable production and distribution schedule and the corresponding profit. The surplus production should be taken to yield zero profit.

Solution

Step 1 → Deriving the profit matrix below by the following equation:

Profit = Sales Price – Product Cost – Raw Material Cost – Transportation Cost.

	F_1	F_2	F_3	F_4	
S_1	6	–2	3	8	80
S_2	6	–2	2	5	120
S_3	1	–4	2	3	150
	10	150	50	100	

Step 2 → Introducing a Dummy factory with zero profit per unit as the total demand is not equal to total supply in order to make the problem balanced one.

	F_1	F_2	F_3	F_4	*Dummy*	
S_1	6	–2	3	8	0	80
S_2	6	–2	2	5	0	120
S_3	1	-4	2	3	0	150
	10	150	50	100	40	

Step 3 → Deriving Loss Matrix by deducting each element from the maximum element (i.e. 8) in order to use minimisation technique.

	F_1	F_2	F_3	F_4	*Dummy*	
S_1	2	10	5	0	8	80
S_2	2	10	6	3	8	120
S_3	7	12	6	5	8	150
	10	150	50	100	40	

Step 4 → Finding Initial Feasible Solution by following the Vogel's Approximation Method (or VAM).

							Diff
	2	10	5	0 (80)	8	80/0	2
	2 (10)	10 (40)	6 (50)	3 (20)	8	120/110/90/40/0	1/3*/2*/2
	7	12 (110)	6	5	8 (40)	150/110/0	1/1/2/4*
Av.	10/0	150/40/0	50/0	100/20/0	40/0		
	0/5	0/2	1/0	3/2*	0/0		

Step 5 → Since the total number of allocations is equal to "M + N – 1" allocations, the initial solution is straight away tested for optimality.

Let us calculate "U_i's" and "V_j's" for all rows and columns respectively on the basis of allocated cells such that $C_{ij} = U_i + V_j$ assuming $U_1 = 0$,

Let us calculate Δ_{ij} (Opportunity Cost), for unallocated cells where $\Delta_{ij} = C_{ij} - (U_{ij} + V_{ij})$.

U_i & V_j Matrix

						U_i
				0		–3
	2	10	6	3		0
		12			8	2
V_j	2	10	6	3	6	

C_{ij} Matrix for Unallocated cells

2	10	5		8
				8
7		6	5	

$U_i + V_j$ Matrix

						U_i
	–1	7	3		3	–3
					6	0
	4		8	5		2
V_j	2	10	6	3	6	

$\Delta_{ij} = C_{ij} - (U_i + V_j)$ Matrix

3	3	2		5
				2
3		-2	0	

Step 6 → Since all Δ_{ij} are not positive, the above solution is not optimal. Let us place a small allocation θ in the cell with most negative Δ_{ij} and form a loop including this cell and test the solution for optimality. Next improved solution as obtained from above is shown below which is tested for optimality. The value of "U_i's", "V_j's" and Δ_{ij} are also calculated.

Looping Matrix

			80	
10	40+	50–	20	
	110–	θ+		40

Reallocation Matrix

			80	
10	90		20	
	60	50		40

U_i & V_j Matrix

						U_i
				0		–3
	2	10		3		0
		12	6		8	2
V_j	2	10	4	3	6	

C_{ij} Matrix for Unallocated cells

2	10	5		8
		6		8
7			5	

$U_i + V_j$ Matrix

						U_i
	–1	7	1		3	–3
			4		6	0
	4			5		2
V_j	2	10	4	3	6	

$\Delta_{ij} = C_{ij} - (U_i + V_j)$ Matrix

3	3	4		5
		2		2
3			0	

Step 6 → Since all Δ_{ij} are positive, the above solution is optimal. The optimal solution is given below:

Sales Department	*Factory*	*Qty*	*Profit per Unit (Rs.)*	*Total Profit (Rs.)*
S_1	F_4	80	8	640
S_2	F_1	10	6	60
S_2	F_2	90	–2	–180
S_2	F_4	20	5	100
S_3	F_2	60	–4	–240
S_3	F_3	50	2	100
			Total	480

Note: Since one of Δ_{ij} is zero, the above solution is not unique. Hence an alternative solution exists. The alternative solution can be found by taking cell with zero Δ_{ij} as the basic cell.

PROBLEM 5.16 [Portfolio Problem]

XYZ and Co. provided the following data seeking your advice on optimum investment strategy:

Investment Made at the Beginning of year	*Net Return Data (in Paise) of Selected Investments*				*Amount Available*
	P	*Q*	*R*	*S*	*(Lacs)*
1	95	80	70	60	70
2	75	65	60	50	40
3	70	45	50	40	90
4	60	40	40	30	30
Maximum Investment (Lacs)	40	50	60	60	—

The following additional information are also provided :

— P, Q, R and S represent the selected investments.

— The company has decided to have four years investment plan.

— The policy of the company is that amount invested in any year will remain so until the end of the fourth year.

— The values (Paise) in the table represent net return on investment of one Rupee till the end of the planning horizon (for example, a Rupee invested in Investment P at the beginning of year I will grow to Rs. 1.95 by the end of the fourth year, yielding a return of 95 paise).

Using the above, determine the optimum investment strategy.

Solution

Step 1 → Introducing a Dummy Investment with zero return per unit as the total demand is not equal to total supply in order to make the problem balanced one.

Year	*Net Return data (in paise) of investment*					*Amount Available*
	P	*Q*	*R*	*S*	*Dummy*	
1	95	80	70	60	0	70
2	75	65	60	50	0	40
3	70	45	50	40	0	90
4	60	40	40	30	0	30
Maximum Investment	40	50	60	60	20	

Step 2 → Deriving Loss Matrix by deducting each element from the maximum element (i.e. 95) in order to use minimisation technique and then finding Initial Feasible Solution by VAM.

Year	*Loss Matrix-Investment type*					*Amount available*	*Difference*
	P	*Q*	*R*	*S*	*Dummy*		
1	$\boxed{40}$ 0	$\boxed{30}$ 15	25	35	95	70/30/0	15/10 —
2	20	$\boxed{20}$ 30	$\boxed{20}$ 35	45	95	40/20/0	10/5/5/10
3	25	50	$\boxed{40}$ 45	$\boxed{50}$ 55	95	90/50/0	20/5/5/10/40
4	35	55	55	$\boxed{10}$ 65	$\boxed{20}$ 95	30/20/0	20/0/0/10/30
Maximum Investment	40/0	50/20/0	60/40/0	60/10/0	20/0		
Difference	20	15	10	10	0		
	—	15	10	10	0		
	—	20	10	10	0		
	—	—	10	10	0		

Step 3 → Since the total number of allocations is equal to "M + N − 1" allocations, the initial solution is straight away tested for optimality.

Let us calculate "U_i's" and "V_j's" for all rows and columns respectively on the basis of allocated cells such that $C_{ij} = U_i + V_j$ assuming $U_4 = 0$,

Let us calculate Δ_{ij} (Opportunity Cost), for unallocated cells where $\Delta_{ij} = C_{ij} - (U_{ij} + V_{ij})$.

U_i & V_j Matrix

					U_i
0	15				−35
	30	35			−20
		45	55		−10
			65	95	0
V_j 35	50	55	65	95	

C_{ij} Matrix for Unallocated cells

		25	35	95
20			45	95
25	50			95
35	55	55		

$U_i + V_j$ Matrix

					U_i
		20	30	60	−35
15			45	75	−20
25	40			85	−10
35	50	55			0
V_j 35	50	55	65	95	

$\Delta_{ij} = C_{ij} - (U_i + V_j)$ Matrix

		5	5	35
5			0	20
0	10			10
0	5	0		

Step 4 → Since all Δ_{ij} are positive, the above solution is optimal. The optimal solution is given below:

Year A	*Investment* Strategies B	*Investment* C	*Return* D	*Total Return* E = C × D
1	P	40	0.95	38
1	Q	30	0.80	24
2	Q	20	0.65	13
2	R	20	0.60	12
3	R	40	0.50	20
3	S	50	0.40	20
4	S	10	0.30	3
				130

Note: Since some of Δ_{ij} are zero, the above solution is not unique. Hence the alternative solution exist. The alternative solution can be found by taking any cell with zero Δ_{ij} as the basic cell.

PROBLEM 5.17 [Portfolio Problem]

A company wishes to determine an investment strategy for each of the next four years. Five investment types have been selected, investment capital has been allocated for each of the coming four years, and maximum investment levels have been established for each investment type. An assumption is that amounts invested in any year will remain invested until the end of the planning horizon of four years. The following table summarises the data for this problem. The values in the body of the table represent net return on investment of one rupee upto the end of the planning horizon. For example, a rupee invested in investment type B at the beginning of year 1 will grow to Rs. 1.90 by the end of the fourth year, yielding a net return of Re. 0.90.

Investment made At the beginning Of year	*Investment Type* A	B NET	C RETURN	D DATA	E	*Rupees available (in 000's)*
1	0.80	0.90	0.60	0.75	1.00	500
2	0.55	0.65	0.40	0.60	0.50	600
3	0.30	0.25	0.30	0.50	0.20	750
4	0.15	0.12	0.25	0.35	0.10	800
Maximum Rupees Investment (in 000's)	750	600	500	800	1,000	

The objective in this problem is to determine the amount to be invested at the beginning of each year in an investment type so as to maximise the net rupee return for the four-year period.

Solve the above transportation problem and get an optimal solution. Also calculate the net return on investment for the planning horizon of four year period.

Solution

Step 1 → Introducing a Dummy Variable with zero net rupee return as the total amount available is not equal to total investment in order to make the problem balanced one and expressing the Net Return Data in Paise.

Years	*Net Return Data (Re.)*					*Available Rs. (in 000's)*
	A	*B*	*C*	*D*	*E*	
1	80	90		75	100	500
2	55	65	40	60	50	600
3	30	25	30	50	20	750
4	15	12	25	35	10	800
Dummy	0	0	0	0	0	1000
Maximum rupees Investment in ('000)	750	600	500	800	1000	3650

Step 2 → Deriving Loss Matrix by deducting each element from the maximum element (i.e. 1.00) in order to use minimisation technique.

Years	*Net Return Data (Re.)*					*Available Rs. (in 000's)*
	A	*B*	*C*	*D*	*E*	
1	020	010	040	025	0	500
2	045	035	060	040	050	600
3	070	075	070	050	080	750
4	085	088	075	065	090	800
Dummy	100	100	100	100	100	1000
Maximum rupees Investment in ('000)	750	600	500	800	1000	3650

Step 3 → Finding Initial Feasible Solution VAM.

Years	A	B	C	D	E	Rs. available (000's)	Difference				
1	20	10	40 $\boxed{500}$	25	0	500/0	10	-	-	-	-
2	45	35 $\boxed{600}$	60	40	50	600/0	5	5	-	-	-
3	70	75	70	50 $\boxed{750}$	80	750/0	20	20	20	-	-
4	85 $\boxed{250}$	88	75 $\boxed{500}$	65 $\boxed{50}$	90	800/750	10	10	10	10	10
Dummy	100 $\boxed{500}$	100	100	100	100 $\boxed{500}$	1000/500/00		0	0	0	0
Maximum Rupee invest- ment in ('000)	750/ 500/0	600/0	1000/0	800/ 50/0	500/	500/0					

Differences	25	25	20	15	50
	25	40	10	10	30
	15	-	5	15	10
	15	-	25	35	10
	15	-	25	-	10

Step 4 → Since the total number of allocations is less than "M + N − 1" allocations, let us introduce an infinitely small quantity 'e' to the least cost independent cell to make the total number of allocations equal to "M + N − 1" allocations.

Let us calculate "U_i's" and "V_j's" for all rows and columns respectively on the basis of allocated cells such that $C_{ij} = U_i + V_j$ assuming $U_1 = 0$,

Let us calculate Δ_{ij} (Opportunity Cost), for unallocated cells where $\Delta_{ij} = C_{ij} - (U_{ij} + V_{ij})$.

U_i & V_j Matrix

						U_i
		e				
		$\boxed{10}$			0	–85
		35				–60
				50		–15
	85		75	65		0
	100				100	15
V_j	*85*	*95*	*75*	*65*	*85*	

C_{ij} Matrix for Unallocated cells

20		40	25	
45		60	40	50
70	75	70		80
	88			90
	100	100	100	

$U_i + V_j$ Matrix

						U_i
	0		–10	–20		–85
	25		15	5	25	–60
	70	80	60		70	–15
		95			85	0
		110	90	80		15
V_j	*85*	*95*	*75*	*65*	*85*	

$\Delta_{ij} = C_{ij} - (U_i + V_j)$ Matrix

20		50	45	
20		45	35	25
0	–5	10		10
	–7			5
	–10	10	20	

Step 5 → Since all Δ_{ij} are not positive, the above solution is not optimal. Let us place a small allocation θ in the cell with most negative Δ_{ij} and form a loop including this cell and test the solution for optimality. Next improved solution as obtained from above is shown below which is tested for optimality. The value of "U_i's", "V_j's" and Δ_{ij} are also calculated.

Looping Matrix

	e–			+500
	600			
			750	
250		500	50	
500	+θ	500		-500

Reallocation Matrix

				500
	600			
			750	
250		500	50	
500	e			500

U_i & V_j Matrix

						U_i
					0	*–85*
		35				*–50*
				50		*–15*
	85		75	65		*0*
	100	100			100	*15*
V_j	*85*	*85*	*75*	*65*	*85*	

C_{ij} Matrix for Unallocated cells

20	10	40	25	
45		60	40	50
70	75	70		80
	88			90
		100	100	

$U_i + V_j$ Matrix

						U_i
	0	0	–10	20		*–85*
	35		25	15	35	*–50*
	70	70	60		70	*–15*
		85			85	*0*
			90	80		*15*
Vj	*85*	*85*	*75*	*65*	*85*	

$\Delta_{ij} = C_{ij} - (U_i + V_j)$ Matrix

20	10	50	5	
10		35	25	15
0	5	10		10
	3			5
		10	20	

Step 6 → Since all Δ_{ij} are positive, the above solution is optimal. The optimal solution is given below:

In the year	*Investment type*	*Amount (in 000's)*	*Net Return per Re.*	*Total Return*
1	E	500	1.00	500.0
2	B	600	.65	390.0
3	D	750	.50	375.0
4	A	250	.15	37.5
4	C	500	.25	125.0
4	D	50	.35	17.5
			Total	1,445.0

Note: Since one of Δ_{ij} is zero, the above solution is not unique. Hence an alternative solution exists. The alternative solution can be found by taking cell with zero Δ_{ij} as the basic cell.

PROBLEM 5.18

TULSIAN LTD requests you to find optimal solution using the following data:

Factories *Item (per unit)*	F_1	F_2	F_3	F_4	*Sales Price Per unit*	*Requirement*
Conversion Cost	0.15	0.18	0.14	0.13	—	—
Raw Material Cost	0.10	0.09	0.12	0.09	—	—
Transportation Cost S_1	0.03	0.09	0.05	0.04	0.34	80
S_2	0.01	0.07	0.04	0.05	0.32	120
S_3	0.05	0.0	0.03	0.06	0.31	150
Production Capacity	10	150	50	100	—	—

For the unsatisfied demands, let the penally costs be rupee 0.01, 0.02 and 0.03 for Sales Depots S_1, S_2 and S_3 respectively.

Solution

1. To simplity the computations let us multiply Costs and Sales Price in the above table by 100.
2. Profit = Sales Price – Conversion Cost – Raw Material Cost – Transportation Cost.
3. Since the requirement (= 350) is greater than the availability (=310 units), the given problem is an unbalanced one. Let us now convert it into balanced one by introducing a dummy factory with production (=350 – 310 = 40) units and penalty costs.
4. Hence, the balanced transportation profit matrix becomes as follows:

	Factories					
Sales Depots	F_1	F_2	F_3	F_4	*Dummy*	*Requirement*
S_1	6	–2	3	8	–1	80
S_2	6	–2	2	5	–2	120
S_3	1	–4	2	3	–3	150
Availability	10	150	50	100	40	

5. Let us now convert this Profit Matrix into Opportunity Loss Matrix so that standard minimisation transportation technique can be applied to it by subtracting all the elements of the above matrix from the highest element (=8) and apply Vogel's Approximation Method to find the Initial Feasible Solution. The resultant Loss Matrix is given below:

Now the remaining solution is same as given in problem 5.15

PROBLEM 5.19

Goods manufactured at 3 plants, A, B and C are required to be transported to sales outlets X, Y and Z. The unit costs of transporting the goods from the plants to the outlets are given below:

Plants / *Sales outlets*	*A*	*B*	*C*	*Total Demand*
X	3	9	6	20
Y	4	4	6	40
Z	8	3	5	60
Total supply	40	50	30	120

You are required to:

(i) Compute the initial allocation by North-West Corner Rule.
(ii) Compute the initial allocation by Vogel's approximation method and check whether it is optional.
(iii) State your analysis on the optionally of allocation under North-West conrner Rule and Vogel's Approximation method.

Solution

(i) Initial allocation under NW corner rule is as above.

Initial cost:
$20 \times 3 = 60$
$20 \times 4 = 80$
$20 \times 4 = 80$
$30 \times 3 = 90$
$30 \times 5 = 150$
$\overline{\quad 460 \quad}$

(ii) Initial solution by VAM:

$\boxed{20}$ 3	– 9	– 6	20	3		
$\boxed{20}$ 4	– 4	$\boxed{20}$ 6	40	0	0	2
8	$\boxed{50}$ 3	$\boxed{10}$ 5	60	2	2	2
40	50	30				
1	1	1				
4	1	1				
	1	1				

Initial cost:
$20 \times 3 = 60$
$20 \times 4 = 80$
$50 \times 3 = 150$
$20 \times 6 = 120$
$10 \times 5 = 100$
$\overline{\quad 460 \quad}$

Checking for optimality

3			$u_1 = 0$
4		6	$u_2 = 1$
	3	5	$u_3 = 0$
$v_1 = 3$	$v_2 = 3$	$v_3 = 5$	

$u_i + v_i$

	3	5	0
	4		1
3			0
3	3	5	

$\Delta_{ij} = c_{ij} - (u_i + v_i)$

	6	1
	0	
5		

$\Delta_{ij} = c_{ij} - (u_i + v_i)$

Conclusion

The solution under VAM is optimal with a zero in R_2C_2 which means that the cell C_2R_2 which means that the cell C_2R_2 can come into solution, which will be another optimal solution. Under NWC rule the initial allocation had C_2R_2 and the total cost was the same Rs. 460 as the total cost under optimal VAM solution. Thus, in this problem, both methods have yielded the optimal solution under the 1st allocation. If we do an optimally test for the solution, we will get a zero for D_{ij} in C_3R_2 indicating the other optimal solution which was obtained under VAM.

PROBLEM 5.20

The cost per unit of transporting goods from the factories X, Y, Z destination. A, B and C, and the quantities demanded and supplied are tubulated below. As the company is working out the optimum logistics, the Govt.; has announced a fall in oil prices. The revised unit costs are exactly half the costs given in the table. You are required to evaluate the minimum transportation cost.

Destinations / *Factories*	*A*	*B*	*C*	*Supply*
X	15	9	6	10
Y	21	12	6	10
Z	6	18	9	10
Demand	10	10	10	30

Solution

(a) The problem may be treated as an assignment problem. The solution will be the same even if prices are halved. Only at the last stage, calculate the minimum cost and divide it by 2 to account for fall in oil prices.

	A	B	C
X	15	9	6
Y	21	12	6
Z	6	18	9

Subtracting Row minimum, we get

	A	B	C
X	9	3	0
Y	15	6	0
Z	0	12	3

Subtracting Column minimum,

9	$\boxed{6}$	0
15	3	$\boxed{0}$
$\boxed{0}$	8	3

No of lines required to cut Zeros = 3

		Cost/u	*Units*	*Cost*	*Revised Cost being 50%*
Allocation:	X→B	9	10	90	45
	Y→C	6	10	60	30
	Z→A	6	10	60	30
				210	105

Minimum cost = 105 Rs.

PROBLEM 5.21

Following is the profit matrix based on four factories and three sales depots of the company:

		S_1	S_2	S_3	*Availability*
Towns	F_1	6	6	1	10
	F_2	–2	–2	–4	150
	F_3	3	2	2	50
	F_4	8	5	3	100
Requirement		80	120	150	

Determine the most profitable distribution schedule and the corresponding profit, assuming no profit in case of surplus production.

Solution

The given transportation problem is an unbalanced one and it is a maximisation problem. As a first step, we will balance this transportation problem, by adding a dummy factory, assuming no profit in case of surplus production.

Sales Depots

		S_1	S_2	S_3	*Availability*
Factories	F_1	6	6	1	10
	F_2	–2	–2	–4	150
	F_3	3	2	2	50
	F_4	8	5	3	100
	Dummy	0	0	0	40
	Requirement	80	120	150	

We shall now convert the above transportation problem (a profit matrix) into a loss matrix by subtracting all the elements from the highest value in the table i.e. 8. Thereafter, we shall apply the VAM find an initial solution.

Sales depots

		S_1	S_2	S_3	Availability	Diff.
	F_1	2	10 / 2	7	10/0	0.5--
	F_2	10	40 / 10	110 / 12	150/110/0	0222
Factories	F_3	5	50 / 6	6	50/0	1000
	F_4	80 / 0	20 / 3	5	100/20/0	322
	Dummy	8	8	40 / 8	40/0	0000
	Requirement	80/0	120/110/90/40/0	150		
	Diff.	2	1	1		
		–	1	1		
		–	3	1		
		–	2	2		

The initial solution obtained by VAM is given below.

Sales depots

		S_1	S_2	S_3
	F_1	2	10 / 2	7
	F_2	10	40 / 10	110 / 12
Factories	F_3	5	50 / 6	6
	F_4	80 / 0	20 / 3	5
	Dummy	8	8	40 / 8

The initial solution is tested for optimality. The total number of independent allocations is 7 which is equal to the desired (m + n – 1) allocations. We introduce u_i's (i = 1, 2 ... 5) and v_j's (j = 1, 2, 3) such that $\Delta_{ij} = c_{ij} - (u_i + v_j)$. Let us assume $u_2 = 0$, remaining u_i's, v_j's and Δ_{ij}'s are calculated below:

Sales Depots

Factories	S_1	S_2	S_3	u_i
F_1	3 2	$\boxed{10}$ 2	3 7	–8
F_2	3 10	$\boxed{40}$ $+\theta$ 10	$\boxed{110}$ $-\theta$ 12	0
F_3	2 5	$\boxed{50}$ $-\theta$ 6	–2 $+\theta$ 6	–6
F_4	$\boxed{80}$ 0	$\boxed{20}$ 3	0 5	–7
Dummy	5 8	2 8	$\boxed{40}$ 8	–4
v_j	7	10	12	

Since one of the Δ_{ij} is negative, the initial solution is not optimal. Let us introduce in the cell with –ve Δ_{ij} {i.e. the cell (F_3, S_3)}, an assignement θ. The value of θ and the reallocated solution as obtained from about is given below. The values of u_i's, v_j's and Δ_{ij}'s are also calculated.

$$\theta = \min \{(50 - \theta) = (110 - \theta) = 0\} = 50$$

Sales Depots

Factories	S_1	S_2	S_3	u_i
F_1	3 2	$\boxed{10}$ 2	3 7	–8
F_2	3 10	$\boxed{90}$ 10	$\boxed{60}$ 12	0
F_3	4 5	2 6	$\boxed{50}$ 6	–4
F_4	$\boxed{80}$ 0	$\boxed{20}$ 3	0 5	–7
Dummy	11 8	2 8	$\boxed{40}$ 8	–4
v_j	7	10	12	

Since all D_{ij}'s for non basic cells are positive, the solution as calculated in the above table is optimal solution. The distribution schedule from factories to sales depots along with profit is given below:

Factory	*Sales depot*	*No. of units*	*Profit per unit*	*Total profit*
F_1	S_2	10	6	60
F_2	S_2	90	–2	–180
F_2	S_3	60	–4	–240
F_3	S_3	50	2	100
F_4	S_1	80	8	640
F_4	S_2	20	5	100
				480

(**Note:** Since one of the D_{ij}'s = 0, alternate schedule with a profit of Rs. 480 exists.)

PROBLEM 5.22

Three refineries with daily capacities of 6, 5 and 8 million gallons, respectively, supply three distribution areas with daily demands of 4, 8 and 7 million gallons, respectively. Gasoline is transported to the three distribution areas through a network of pipelines. The transportation cost is 10 cents per 1000 gallons per pipeline mile. Table gives the mileage between the refineries and the distribution areas. Refinery 1 is not connected to distribution area 3.

Price Per Million

	City			
	1	2	3	4
Plant	2	600	700	350
	3	320	300	350
	4	500	480	450

Mileage Chart for Problem 5

	Distribution area		
	1	2	3
Refinery 1	120	180	—
Refinery 2	300	100	80
Refinery 3	200	250	120

(a) Construct the associated transportation model.
(b) Determine the optimum shipping schedule in the network.

Solution

Initial solution by applying VAM.

From \ To	1	2	3	Total	D_1	D_2	D_3
1	12 [3]	18 [3]	m	6	6	6	6
2	30	10 [5]	8	5	2		
3	20 [1]	28	12 [7]	8	8	8	8
Total	4	8	7	19			
D_1	8	8	4				
D_2	8	10	m				
D_3	8	10	—				

As number of allocation is equal to 5(i.e. m + n – 1), solution obtained is initial feasible solution.

Checking solution for optimality by applying MODI methods.

From \ To	1	2	3	Total	v_i
1	12 $\boxed{3}$	18 $\boxed{3}$	m m–4	6	12
2	30 26	10 $\boxed{5}$	8 12	5	4
3	20 $\boxed{1}$	28 2	12 $\boxed{7}$	8	20
Total	4	8	7	19	
v_j	0	6	–8		

Since all Δ_{ij} are positive for unallocated cell therefore initial solution obtained is optimal one.

Refinery	*Distribution Area*	*Quantity*	*Total*
1	1	3	36
1	2	3	54
2	2	5	50
3	1	1	20
3	3	7	84
			244

PROBLEM 5.23

In Problem 5.22, suppose that the capacity of refinery 3 is 6 million gallons only and that distribution area 1 must receive all its demand. Additionally, any shortages at areas 2 and 3 will incur a penalty of 5 cents per gallon.

(a) Formulate the problem as a tranportation model.

(b) Determine the optimum shipping schedule.

Solution

Initial solution by applying VAM.

To / From	1	2	3	Total	D_1	D_2	D_3	D_3
1	4 12	2 18	m	6	6	6	6	m–8
2	30	5 10	8	5	2	—	—	—
3	3 20	25	6 12	6	8	8	—	—
Dummy	m	50	1 50	2	0	0	0	0
Total	4	8	7	19				
D_1	8	8	4					
D_2	8	7	38					
D_3	m-12	32	m-50					
D_4	—	32	m-50					

As number of allocated cells is equal to 6 (i.e. m+n–1) hence solution obtained is initial pasible solution.

Now checking solution for aptionality.

To / From	1	2	3	Total	v_i
1	4 12	2 18	m m–18	6	12
2	30 26	5 10 –	+ 8 –2	5	4
3	20 14	25 13	6 12	6	6
Dummy	m m–44	+ 1 50 +	– 1 50 –	2	44
Total	4	8	7	19	
v_i	0	6	6		

Checking the optimality by applying MODI method.

To / From	1	2	3	Total	u_i
1	12 $\boxed{4}$	18 $\boxed{2}$	m m–16	6	12
2	30 26	10 $\boxed{4}$	8 $\boxed{1}$	5	4
3	20 12	25 11	12 $\boxed{6}$	6	6
Dummy	m m–44	50 $\boxed{2}$	50	2	44
Total	4	8	7	19	
v_j	0	6	4		

Since all Δ_{ij} are positive, hence solution obtained is optimal one.
Hence optimal solution is

Refinery	*Distribution Area*	*Quantity*	*Total*
1	1	4	48
1	2	2	36
2	2	4	40
2	3	1	8
3	3	6	72
Dummy	2	2	100
			304

PROBLEM 5.24

Cars are shipped from three distribution centers to five dealers. The shipping cost is based on the mileage between the sources and the destinations, and is independent of whether the truck makes the trip with partial or full loads. Following table summarizes the mileage between the distribution centers and the dealers together with the monthly supply and emand figures given in *number* of cars. A full truckload includes 18 cars. The transportation cost per truck mile is Rs. 25.

(a) Formulate the problem as a tranportation model.

(b) Determine the optimal shipping schedule.

Mileage chart and Supply and Demand

		Dealer 1	2	3	4	5	Supply
	1	100	150	200	140	35	400
Center	2	50	70	60	65	80	200
	3	40	90	100	150	130	150
Demand		100	200	150	160	140	

Solution

To / From	1	2	3	4	5	Total	
1	2,500	3,750	5,000	3,500	875	23	400/18 R. off to next figure
2	1,250	1,750	1,500	1,625	2,000	12	2001/18 R. off to next figure
3	1,000	2,250	2,500	3,750	3,250	9	150/18 R. off to next figure
Total	6	12	9	9	8	44	
	100/18 R. off to next figure	200/18 R. off to next figure	150/18 R. off to next figure	160/18 R. off to next figure	190/18 R. off to next figure		

Initial Solution by Applying VAM

To / From	1	2	3	4	5	Total	D_1	D_2	D_3	D_3
1	2,500	$\boxed{12}$ 3,750	$\boxed{3}$ 5,000	3,500	$\boxed{8}$ 875	23	1625	1625	1250	1250
2	1,250	1,750	$\boxed{3}$ 1,500	$\boxed{9}$ 1,625	2,000	12	250	250	250	250
3	$\boxed{6}$ 1,000 1000	2,250	$\boxed{3}$ 2,500	3,750 1125	3,250 4875	9	1250	1250	1250	250
Total	6	12	9	9	8	44				
D_1	250	500	1000	1875	1125					
D_2	250	500	1000	—	1125					
D_3	250	500	1000	—	—					
D_4	—	500	1000	—	—					

As number of allocations are equal to 7(i.e. m+n–1) so initial solution obtained is initial feasible solution.

Checking the solution for optimality by applying MODI method.

To / From	1	2	3	4	5	Total	D_1	D_2	D_3	D_3
1	2,500	12 3,750	3 5,000	3,500	8 875	23	1625	1625	1250	1250
2	1,250	1,750	3 1,500	9 1,625	2,000	12	250	250	250	250
3	6 1,000 1000	2,250	3 2,500	3,750 1125	3,250 4875	9	1250	1250	1250	250
Total	6	12	9	9	8	44				

D_1	250	500	1000	1875	1125
D_2	250	500	1000	—	1125
D_3	250	500	1000	—	—
D_4	—	500	1000	—	—

Checking the solution for optimality by applying MODI method.

To / From	1	2	3	4	5	Total	u_i
1	2,500 625	12 − 3,750	5,000 1625	3 + 3,500	8 875	23	875
2	1,250 1250	1,750 – 125	6 + 1,500	6 − 1,625	2,000 3000	12	–1000
3	6 1,000	+ 2,250 – 625	3 − 2,500	3,750 1125	3,250 3250	9	0
Total	6	12	9	9	8	44	
v_j	1,000	2,875	2,500	2,625	0		

Checking the solution for optimality by applying MODI method

To / From	1	2	3	4	5	Total	u_i
1	2,500 0	– 9 3,750	– 5,000 1625	+ 6 3,500	8 875	23	875
2	1,250 625	1,750 –125	9 + 1,500	3 – 1,625	2,000 3000	12	–1,000
3	6 1,000	3 2,250	2,500 625	3,750 1750	3,250 3825	9	–625
Total	6	12	9	9	8	44	
v_j	1,625	2,875	2,500	2,625	0		

Checking the solution for optimality by applying MODI method.

To / From	1	2	3	4	5	Total	u_i
1	2,500 0	6 3,750	5,000 1500	9 3,500	8 875	23	875
2	1,250 750	3 1,750	9 1,500	3 1,625 125	2,000 3125	12	–1,125
3	6 1,000	3 2,250	2,500 500	3,750 1750	3,250 3875	9	–625
Total	6	12	9	9	8	44	
v_j	1,625	2,875	2,625	2,625	0		

As all Δ_{ij} are positive and zero, solution obtained is optimal one and also has an alternative solution since $\Delta_{ij} = 0$.

Centre	*Dealer*	*No. of Trucks*	*No. of Cars*	*Cost*
1	2	6	100	22,500
1	4	9	160	31,500
1	5	8	140	7,000
2	2	3	50	5,250
2	3	9	150	13,500
3	1	6	100	6,000
3	2	3	150	6,750
				92,500

PROBLEM 5.25

The following matrix is a minimization problem for transportation cost The unit transportation costs are given at the right hand corners of the cells and the Δij values are encircled.

	D_1	D_2	D_3	Supply
F_1	3	4	4	500
F_2	9 88	6 300	7 88	300
F_3	4 88	6 88	5 200	200
Demand	300	400	300	1000

Find the optimum solution (s) and the minimum cost. ***(May 2011)***

Solution

Δ_{ij} values are given for unallocated cells. Hece, no. of allocated cells = 5, which = 3 + 3 – 1 = no. of columns + no of rows – 1.

Allocating in other than Δ_{ij} cells.

Factory	S_1	D_2	D_3	Supply
	3 300	4 100	4 100	500
	9 88	6 300	7 88	300
	4 88	6 88	5 200	200
	300	400	300	1000

This solution is optional since Δ_{ij} are non-ve. For the other optional solution, which exists since Δ_{ij} = 0 at R_3 C_1, this cell should be brought in with a loop: R_2 C_1 – R_1 C_1 – R_1 C_3 – R_3 C_3.

Working Notes:

Step I : R_1C_1 (Minimum of 300, 500)

Step II: R_2C_2 (Minimum of 300, 400)

Step III: R1C2 balance of C_2 Total: 100, R_1 Total = 100

Step IV: R_1C_3 100 (balance of C_3 Total = 200)

Step V: R_3C_3 200

Solution I

–100				+100
	300	100	100	
		300		
+100	88		200	–100

Solution II

200	100	200
	300	
100		100

	Solution I	*Solution II*
Cost:	3 × 300 = 900	3 × 200 = 600
	4 × 100 = 400	4 × 100 = 400
	4 × 100 = 400	4 × 200 = 800
	6 × 300 = 1,800	6 × 300 = 1,800
	5 × 200 = 1,000	5 × 100 = 500
		4 × 100 = 400
Minimum Cost	4,500	4,500

PROBLEM 5.26

A company has three plants located at A, B and C. The production of these plants is absorbed by four distribution centres located at X, Y, W and Z, the transportation cost per unit has been shown in small cells in the following table:

Factories \ Distribution Centres	X	Y	W	Z	Supply (Units)
A	6	9	13	7	6000
B	6	10	11	5	6000
C	4	7	14	8	6000
Demand (Units)	4000	4000	4500	5000	18000 / 17500

Find the optimum solution of the transportation problem by applying Vogel's Approximation Method. ***(Nov., 2010)***

Solution

Step 1: Initilal Allocatlon based on Least cost cells corresponding to highest differences

	X	*Y*	*W*	*Z*	*Dummy*	*Total*
A		2,000	3,500		500	6,000
B			1,000	5,000		6,000
C	4,000	2,000				6,000
Total	4,000	4,000	4,500	5,000	500	18,000

Step 2: Δ_{ij} Matrix values for Unallocated cells

	X	*Y*	*W*	*Z*	*Total*
A	0			0	
B	2	3			2
C			3	3	2

All Δ_{ij} values > 0. Therefore initial allocation is optimal.

Step 3: Optimal Transportation Cost

	Units	*Costs involved*	*Total*
A to Y	2,000	9	18,000
A to W	3,500	13	45,500
B to W	1,000	11	11,000
B to Z	5,000	5	25,000
C to X	4,000	4	16,000
C to Y	2,000	7	14,000
			Total Minimum Cost = 1,29,500

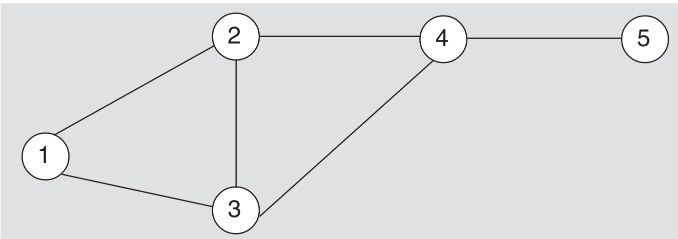
Note: Since there are zeroes in the Δ_{ij} Matrix alternate solutions exist.

6 CRITICAL PATH METHOD — DRAWING NETWORK

NETWORK (OR ARROW DIAGRAM)

A network is a graphical representation of inter-relationship of the various activities of a project. While drawing network, the following rules must be kept in mind.

1. An activity cannot occur untill all activities leading to it are complete.
2. No activity can start untill its tail event is reached.



Activity	*Head Event*	*Tail Event*
1 – 2	2 (Burst Event)	1 (Burst Event)
1 – 3	3 (Merge Event)	1 (Burst Event)
2 – 3	3 (Merge Event)	2 (Burst Event)
2 – 4	4 (Merge Event)	2 (Burst Event)
3 – 4	4 (Merge Event)	3
4 – 5	5	4

The aforesaid graphical representation showing the inter-relationships of the various activities of a project is called '**Network**'.

WORKING METHODOLOGY OF CRITICAL PATH ANALYSIS

The working methodology of Critical Path Analysis (CPA) which includes both CPM and PERT, consists of following five steps:

Step 1: *Analyse and breakdown the project in terms of specific activities and/or events.*
Step 2: *Determine the interdependence and sequence of specific activities and prepare a network.*
Step 3: *Assign estimate of time, cost or both to all the activities of the network.*
Step 4: *Identify the longest or critical path through the network.*
Step 5: *Monitor, evaluate and control the progress of the project by replanning, rescheduling and reassignment of resources.*