CSE-2016 → Paper II

6) (c) Let $f(x) = e^{2x}\cos 3x$, for $x \in [0,1]$. Estimate the value of $f(0.5)$ using Lagrange interpolating polynomial of degree 3 over the nodes $x=0$, $x=0.3$, $x=0.6$ and $x=1$. Also, compute the error bound over the interval $[0,1]$ and the actual error $E(0.5)$.

⇒ ∴ here we form the table as,

x	0	0.3	0.6	1
$f(x)$	1	1.1326	−0.7543	−7.3151

we know Lagrange's Interpolation Formula is,

$$L(x) = \omega(x)\sum_{r=0}^{n} \frac{f(x_r)}{(x-x_r)\,\omega'(x_r)} = \omega(x)\sum_{r=0}^{n}\frac{y_r}{D_r}$$

where, $\omega(x) = (x-x_0)(x-x_1)(x-x_2) - - - (x-x_n)$

and $D_r = (x-x_r)(x_r-x_0) - - (x_r-x_{r-1})(x_r-x_{r+1}) - - - (x_r-x_n)$

				D_r	y_r	y_r/D_r
x	−0.3	−0.6	−1	$-0.24x$	1	$-1/0.18x$
0.3	$x-0.3$	−0.3	−0.7	$0.063(x-0.3)$	1.1326	$1.1326/0.063(x-0.3)$
0.6	0.3	$x-0.6$	−0.4	$-0.072(x-0.6)$	−0.7543	$0.7543/0.072(x-0.6)$
1	0.7	0.4	$x-1$	$0.28(x-1)$	−7.3151	$-7.3151/0.28(x-1)$

here $\omega(x) = x(x-0.3)(x-0.6)(x-1)$

$$\therefore f(x) = x(x-0.3)(x-0.6)(x-1)\left[-\frac{1}{0.18x} + \frac{1.1326}{0.063(x-0.3)} + \frac{0.7543}{0.072(x-0.6)} - \frac{7.3151}{0.28(x-1)}\right]$$

$$\therefore f(0.5) = 0.005 \times [-11.1111 + 89.8889 - 104.7639 + 52.2507]$$

$$= 0.131323$$

actual value of $f(x)$ at $x=0.5$ is $= 0.19228$

$\therefore$ Actual error, $E(0.5) = |0.19228 - 0.133131|$

$= 0.060957$

To find error bound over $[0,1]$

$$E(x,f) = \frac{f^{IV}(\xi)}{4!}(x)(x-0.3)(x-0.6)(x-1)$$

we know, $f(x)$ is a decreasing function from 0 to 1.

Now, $f(x) = e^{2x}\cos 3x$

$$f'(x) = -3e^{2x}\sin 3x + 2e^{2x}\cos 3x$$

$$f''(x) = -6e^{2x}\sin 3x - 9e^{2x}\cos 3x + 4e^{2x}\cos 3x - 6e^{2x}\sin 3x$$

$$= -12e^{2x}\sin 3x - 5e^{2x}\cos 3x$$

$$f'''(x) = -24e^{2x}\sin 3x - 36e^{2x}\cos 3x - 10e^{2x}\cos 3x + 15e^{2x}\sin 3x$$

$$= -9e^{2x}\sin 3x - 46e^{2x}\cos 3x$$

$$f^{IV}(x) = -18e^{2x}\sin 3x - 27e^{2x}\cos 3x - 92e^{2x}\cos 3x + 138e^{2x}\sin 3x$$

$$= 120e^{2x}\sin 3x - 119e^{2x}\cos 3x$$

Now $f^{IV}(x)$ is increasing function from 0 to 1.

$$\therefore \max_{x\in[0,1]} |f^{IV}(x)| \le 995.6273$$

$$\therefore \text{Error bound} = \left| \frac{f^{IV}(\xi)}{4!} x(x-0.3)(x-0.6)(x-1) \right|$$

$$= \left| \frac{995.6273}{24} \times 0.0051 \right|$$

$$= 0.2074223542$$

7) (c) For an integral $\int_{-1}^{1} f(x)dx$, show that the two-point Gauss quadrature rule is given by,

$$\int_{-1}^{1} f(x)dx = f\left(\frac{1}{\sqrt{3}}\right) + f\left(-\frac{1}{\sqrt{3}}\right)$$. Using the rule estimate $\int_{2}^{4} 2xe^{x}dx$.

⇒ Gaussian formula imposes a restriction on the limits of integration to be form -1 to 1. In general limit of integral $\int_a^b f(x)dx$ are changed to -1 to 1 by means of the transformation, $x = \frac{1}{2}(b-a)u + \frac{1}{2}(b+a)$

Let us consider the Gauss formula,

$$\int_{-1}^{1} f(x)dx = \omega_1 f(x_1) + \dots + \omega_n f(x_n) = \sum_{i=1}^{n} \omega_i f(x_i) \quad \text{--- (1)}$$

In equation (1), there are $2n$ arbitrary constants (i.e, weight & abssica 'n' weights & 'n' abscissa) and therefore can be determine such that the formula is exact when f(x) is a polynomial of degree not exceeding $(2n-1)$.

Hence, we consider

$$f(x) = C_0 + C_1 x + \dots + C_{2n-1}x^{2n-1} \quad \text{--- (2)}$$

∴ from (1),

$$\int_{-1}^{1} f(x)dx = \int_{-1}^{1}\left[C_0 + C_1 x + \dots + C_{2n-1}x^{2n-1}\right]dx$$

$$= 2C_0 + \frac{2}{3}C_2 + \frac{2}{5}C_4 + \dots \quad \text{--- (3)}$$

Now, we put $x = x_i$ in (2), we obtain

$$\int_{-1}^{1} f(x)dx = \sum_{i=1}^{n} \omega_i f(x_i) = \omega_1 f(x_1) + \omega_2 f(x_2) + \dots + \omega_n f(x_n)$$

$$= \omega_1\left[C_0 + C_1 x_1 + C_2 x_2 + \dots + C_{2n-1}x_1^{2n-1}\right] + \dots$$

$$\dots + \omega_n\left[C_0 + C_1 x_n + C_2 x_n + \dots + C_{2n-1}x_n^{2n-1}\right] \quad \text{--- (4)}$$

But the equations (3) & (4) are identical for all values of C_i, hence comparing co-efficient of C_i, we obtain $2n$ ~~equation~~ unknown ω_i and x_i $(i = 1, 2, \dots n)$

$$\therefore \left.\begin{aligned} &\omega_1 + \omega_2 + \cdots + \omega_n = 2 \\ &\omega_1 x_1 + \omega_2 x_2 + \cdots + \omega_n x_n = 0 \\ &\omega_1 x_1^2 + \omega_2 x_2^2 + \cdots + \omega_n x_n^2 = 2/3 \\ &\quad \vdots \\ &\omega_1 x_1^{2n-1} + \omega_2 x_2^{2n-1} + \cdots + \omega_n x_n^{2n-1} = 0 \end{aligned}\right\} \quad \text{---} (5)$$

<u>Now</u>, <u>one point formula</u> $\Rightarrow (n=1)$

$$\int_{-1}^{1} f(x)\,dx = \omega_1 f(x_1) \quad \text{---} (6)$$

there are two unknows ω_1 & x_1

$\therefore$ from (5) $\Rightarrow \omega_1 = 2$ & $\omega_1 x_1 = 0 \Rightarrow x_1 = 0$

Now from (6) $\Rightarrow \int_{-1}^{1} f(x) = 2f(0)$ --- (7)

<u>Now</u>, <u>two point formula</u> $\Rightarrow (n=2)$

$$\int_{-1}^{1} f(x)\,dx = \omega_1 f(x_1) + \omega_2 f(x_2) \quad \text{---} (8)$$

Here, four unknows are ω_1, ω_2, & x_1, x_2

$\therefore$ from (5),

$$\left.\begin{aligned} &\omega_1 + \omega_2 = 2 \\ &\omega_1 x_1 + \omega_2 x_2 = 0 \\ &\omega_1 x_1^2 + \omega_2 x_2^2 = 2/3 \\ &\omega_1 x_1^3 + \omega_2 x_2^3 = 0 \end{aligned}\right\} \quad \text{---} (9)$$

solving these 4 we get, $\omega_1 = \omega_2 = 1$, $x_1 = -\frac{1}{\sqrt{3}}$, $x_2 = \frac{1}{\sqrt{3}}$

$\therefore$ from (8) we get,

$$\int_{-1}^{1} f(x)\,dx = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) \quad \boxed{\text{proved}}$$

<u>2nd part</u>,

let $f(x) = 2xe^x$

we have $x = \frac{1}{2}(b-a)u + (b+a)$

$= u + 3$

$x = 2 \Rightarrow u = -1$ & $x = 4 \Rightarrow u = 1$

& $f(u) = 2(u+3)e^{u+3}$ & $dx = du$

$\therefore \int_2^4 2x e^x dx = \int_{-1}^{1} 2(u+3) e^{u+3} du$

$= 2\left(-\frac{1}{\sqrt{3}}+3\right)e^{-\frac{1}{\sqrt{3}}+3} + 2\left(\frac{1}{\sqrt{3}}+3\right)e^{\frac{1}{\sqrt{3}}+3}$

$\left[\because f\left(-\frac{1}{\sqrt{3}}\right) = 2\left(-\frac{1}{\sqrt{3}}+3\right)e^{-\frac{1}{\sqrt{3}}+3} \right.$

$\left. f\left(\frac{1}{\sqrt{3}}\right) = 2\left(\frac{1}{\sqrt{3}}+3\right)e^{\frac{1}{\sqrt{3}}+3} \right]$