IMS™

(INSTITUTE OF MATHEMATICAL SCIENCES)

UPSC-CSE 2019

Mains

MATHEMATICS

Optional Paper-I

Solutions

SECTION - A

1.(a) → Let $f: [0, \pi/2] \to \mathbb{R}$ be a continuous function such that

$$f(x) = \frac{\cos^2 x}{4x^2 - \pi^2}, \quad 0 \le x < \pi/2$$

Find the value of $f(\pi/2)$.

Solution:

Given that $f: [0, \pi/2] \to \mathbb{R}$ is a continuous function such that $f(x) = \frac{\cos^2 x}{4x^2 - \pi^2}$,

$0 \le x < \pi/2$.

Now,
since f is continuous on $[0, \pi/2]$
we have $\lim\limits_{x \to \pi/2^-} f(x) = f(\pi/2)$

$$\Rightarrow f(\pi/2) = \lim_{x \to \pi/2^-} \frac{\cos^2 x}{4x^2 - \pi^2} \quad \left(\frac{0}{0}\right)$$

$$= \lim_{x \to \pi/2^-} \frac{2\cos x\,(-\sin x)}{8x}$$

$$= \lim_{x \to \pi/2^-} -\frac{\sin 2x}{8x}$$

$$= -\frac{\sin 2(\pi/2)}{8(\pi/2)}$$

$$= 0$$

$\therefore$ $\boxed{\text{The value of } f(\pi/2) = 0.}$

1.(b) → Let $f: D(\subseteq \mathbb{R}^2) \to \mathbb{R}$ be a function and $(a,b) \in D$. If $f(x,y)$ is continuous at (a,b), then show that the functions $f(x,b)$ and $f(a,y)$ are continuous at $x=a$ and at $y=b$ respectively.

Solution :

Let $f: D(\subseteq \mathbb{R}^2) \to \mathbb{R}$ be a function and $(a,b) \in D$.

If $f(x,y)$ is continuous at (a,b) then by definition,

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

i.e. $f(x,y) \to f(a,b)$ as $(x,y) \to (a,b)$

i.e. for given $\epsilon > 0$ (however small)

$\exists\ \delta > 0\ (\delta(\epsilon))$ such that

$|f(x,y) - f(a,b)| < \epsilon$ whenever $\|(x,y)-(a,b)\| < \delta$.

A point to be particularly noticed is that if a function of more than one variable is continuous at a point, it is continuous at that point when considered as a function of single variable.

To be more specific if a function f of two variables x, y is continuous at (a, b) then $f(x, b)$ is continuous at $x=a$ of x and $f(a, y)$ that of f at $y=b$.

Hence, the result.

Ques: 1(c)) Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map such that $T(2,1) = (5,7)$ and $T(1,2) = (3,3)$. If A is the matrix corresponding of T with respect to the standard bases e_1, e_2, then find Rank (A).

Solution:-

Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear map such that

$$T(2,1) = (5,7) \quad \text{and} \quad T(1,2) = (3,3)$$

Let us find the matrix A corresponding to T with respect to the standard bases e_1, e_2.

where $e_1 = (1,0)$
and $e_2 = (0,1)$.

Then;

$$T(2,1) = (5,7) = 5(1,0) + 7(0,1)$$

$$T(1,2) = (3,3) = 3(1,0) + 3(0,1)$$

$\therefore A = \begin{bmatrix} 5 & 3 \\ 7 & 3 \end{bmatrix}$ is a matrix of T.

Since $|A| = \begin{vmatrix} 5 & 3 \\ 7 & 3 \end{vmatrix} = 15 - 21 = -6 \neq 0$

$\therefore$ Rank of $A = 2$.

Ques: 1) d) If $A=\begin{bmatrix}1 & 2 & 1\\ 1 & -4 & 1\\ 3 & 0 & -3\end{bmatrix}$ and $B=\begin{bmatrix}2 & 1 & 1\\ 1 & -1 & 0\\ 2 & 1 & -1\end{bmatrix}$

then show that $AB = 6I_3$. Use this result to solve the following system of equations:

$$2x+y+z=5$$
$$x-y=0$$
$$2x+y-z=1.$$

Solution:-

Given; $A=\begin{bmatrix}1 & 2 & 1\\ 1 & -4 & 1\\ 3 & 0 & -3\end{bmatrix}$ and $B=\begin{bmatrix}2 & 1 & 1\\ 1 & -1 & 0\\ 2 & 1 & -1\end{bmatrix}$

$$A\cdot B=\begin{bmatrix}1 & 2 & 1\\ 1 & -4 & 1\\ 3 & 0 & -3\end{bmatrix}\begin{bmatrix}2 & 1 & 1\\ 1 & -1 & 0\\ 2 & 1 & -1\end{bmatrix}=\begin{bmatrix}6 & 0 & 0\\ 0 & 6 & 0\\ 0 & 0 & 6\end{bmatrix}$$

$$A\cdot B = \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix} = 6I_3$$

Hence proved.

$$A\cdot B = 6I_3$$
$$A\cdot B\cdot B^{-1} = 6I_3B^{-1}$$
$$A = 6B^{-1}$$
$$B^{-1} = \frac{1}{6}A$$

$$2x+y+z=5$$
$$x-y=0$$
$$2x+y-z=1$$

$$B = \begin{bmatrix} 2 & 1 & 1 \\ 1 & -1 & 0 \\ 2 & 1 & -1 \end{bmatrix}; \quad C = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}; \quad X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

and from the linear equation

$$BX = C \Rightarrow B^{-1}BX = B^{-1}C$$

$$X = B^{-1}C$$

$$X = \frac{1}{6}AC$$

$$X = \frac{1}{6}\begin{bmatrix} 1 & 2 & 1 \\ 1 & -4 & 1 \\ 3 & 0 & -3 \end{bmatrix}\begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 6 \\ 6 \\ 12 \end{bmatrix} = \frac{1}{6} \times 6 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

$\therefore$ $x = 1, y = 1, z = 2.$ is the required result.

1.(e) → Show that the lines

$$\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z+2}{1} \text{ and } \frac{x}{1} = \frac{y-7}{-3} = \frac{z+7}{2}$$

intersect. Find the coordinates of the point of intersection and the equation of the plane containing them.

Solution: Part (a):

Any point on the line

$$\frac{x+1}{-3} = \frac{y-3}{2} = \frac{z+2}{1} \text{ is}$$

$$(-1-3r,\ 3+2r,\ -2+r) \quad \text{—— (1)}$$

Similarly, any point on the line

$$\frac{x}{1} = \frac{y-7}{-3} = \frac{z+7}{2} \text{ is}$$

$$(r',\ 7-3r',\ -7+2r') \quad \text{—— (2)}$$

If the two given lines intersect then for some value of r and r' the two above points (i) and (ii) must coincide

i.e. $-1-3r = r'$;

$3+2r = 7-3r'$;

$-2+r = -7+2r'$

Solving the first two of these equations, we get $r=-1$, $r'=2$.

These values of r and r' satisfy the third equation also, hence the given lines intersect.

Substituting these values r and r' in (1) or (2) we get the required coordinates of the point of intersection as $(2, 1, -3)$. [I]

Part (b):

Also the equation of the plane containing the given lines is

$$\begin{vmatrix} x+1 & y-3 & z+2 \\ -3 & 2 & 1 \\ 1 & -3 & 2 \end{vmatrix} = 0$$

$\Rightarrow (x+1)(4+3)-(y-3)(-6-1)+(z+2)(9-2) = 0$

$\Rightarrow \boxed{x+y+z=0}$

which is the required equation. — [1]

2.(b) → Let A and B be two orthogonal matrices of same order and det A + det B = 0. Show that A + B is a singular matrix.

Solution :

Given that A and B are two orthogonal matrices of same order and $\det A + \det B = 0$.

We know that, all orthogonal matrices are invertible, so we can write

$$A + B = A(I + A^{-1}B) \quad \text{——— (1)}$$

Now, it is enough to show that $(I + A^{-1}B)$ is singular.

[$\because$ the product of matrices is singular as long as atleast one of the matrices is singular.]

Also, $\det(A^{-1} * B) = \det(A^{-1}) * \det B$

$= \det(A^{-1}) * (-\det(A))$

[$\because \det(A) + \det(B) = 0$]

Now,
since the determinant of an orthogonal matrix is either 1 or -1,

$$\det(A)^{-1} * (-\det(A)) = -1.$$

[$\because$ This product is the product of -1 and 1 (in some order)] —— (2)

Using (2), we have

$$\det(I + A^{-1}B) = \det I + \det(A^{-1} * B)$$
$$= \det I + (-1)$$
$$= 1 - 1$$
$$= 0$$

$\Longrightarrow$ $A+B$ is singular matrix

[$\because$ A matrix is singular $\Leftrightarrow$ its determinant is 0].

Hence, proved.

2.(c) →

(i) The plane $x+2y+3z=12$ cuts the axes of co-ordinates in A, B, C. Find the equations of the circle circumscribing the triangle ABC.

Solutions:

The given plane $x+2y+3z=12$ —— (1)

meets the x-axis i.e., $y=0$, $z=0$ in the point A whose coordinates are $(12,0,0)$.

Similarly the co-ordinates of B and C where the given plane meets y-axis are $(0,6,0)$ and z-axis are $(0,0,4)$.

Thus, the point A, B, C are $(12,0,0)$, $(0,6,0)$ and $(0,0,4)$ respectively.

Let the equation of the circle circumscribing the triangle ABC be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad \text{—— (2)}$$

If it passes through A, B, C then we have

$$(12)^2 + 2u(12) + d = 0 \Rightarrow 24u + d + 144 = 0 \quad \text{——(3)}$$

$$(6)^2 + 2v(6) + d = 0 \Rightarrow 12v + d + 36 = 0 \quad \text{——(4)}$$

and $(4)^2 + 2w(4) + d = 0 \Rightarrow 8w + d + 16 = 0 \quad \text{——(5)}$

from (3), (4) and (5) we get

$$2u = -[12 - (d/12)],$$

$$2v = -[6 - (d/6)] \text{ and}$$

$$2w = -[4 - (d/4)].$$

Substituting these values of $2u, 2v, 2w$ in (2), we get the required equation as

$$x^2 + y^2 + z^2 - [12 - (d/12)]x - [6 - (d/6)]y - [4 - (d/4)]z + d = 0$$

where d can take any value.

Hence, the result.

2.(c) (ii) Prove that the plane $z=0$ cuts the enveloping cone of the sphere $x^2+y^2+z^2=11$ which has the vertex at $(2,4,1)$ in a rectangular hyperbola.

Solution:

The equation of the sphere is

$x^2+y^2+z^2-11=0$ and the vertex is $(2, 4, 1)$.

Here,

$$S = x^2+y^2+z^2-11, \quad x_1 = 2, y_1 = 4, z_1 = 1$$

$$S_1 = x_1^2+y_1^2+z_1^2-11 = 4+16+1-11$$

i.e. $S_1 = x_1^2+y_1^2+z_1^2-11 = 10$

and $T = xx_1 + yy_1 + zz_1 - 11$

$$= 2x+4y+z-11.$$

$\therefore$ Equation of the enveloping cone is

$$(x^2+y^2+z^2-11)(10) = (2x+4y+z-11)^2 \quad [\text{Using } SS_1 = T^2]$$

$$\Rightarrow \quad 10(x^2+y^2+z^2-11)-(2x+4y+z-11)^2=0$$

This meets the plane $z=0$ in the curve

$$10(x^2+y^2-11)-(2x+4y-11)^2=0$$

$$\Rightarrow \quad 10(x^2+y^2)-(4x^2+16y^2+\cdots)=0$$

This represents a rectangular hyperbola in the XY-plane if co-eff. of x^2 + co. eff. of $y^2 = 0$

$\Rightarrow$ if $(10-4)+(10-16)=0$

which is true.

Hence, the result.

3.(a) → Find the maximum and the minimum value of the function $f(x) = 2x^3 - 9x^2 + 12x + 6$ on the interval $[2, 3]$.

Solution:

Given $f(x) = 2x^3 - 9x^2 + 12x + 6$

$f'(x) = 6x^2 - 18x + 12$.

Let $f'(x) = 0$

$6x^2 - 18x + 12 = 0$

$\Rightarrow x^2 - 3x + 2 = 0$

$\Rightarrow (x-1)(x-2) = 0$

$\Rightarrow x = 1, 2$.

These are the critical points of the $f(x)$ which $\notin [2,3]$. Thus, $f(x)$ is monotonic on $[2,3]$.

Now, to check whether it is monotonic increasing or decreasing –

$f'(2) = 6(4) - 18(2) + 12 = 24 - 36 + 12 = 0$

$f'(3) = 6(9) - 18(3) + 12 = 54 + 12 - 54 = 12$

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Since, $f'(2)=0$, $f'(3)>0$ and

$f'(3)>f'(2)$.

Hence, $f(x)$ is monotonically increasing on $[2,3]$.

Thus, $$f_{min} = f(x)\Big|_{x=2} = 2\times(2)^3 - 9\times4 + 12\times2 + 6$$
$$= 16-36+24+6$$
$$= 10.$$

$$f_{max} = f(x)\Big|_{x=3} = 2\times(3)^3 - 9\times9 + 12\times3 + 6$$
$$= 54-81+36+6$$
$$= 96-81$$
$$= 15.$$

$\therefore$ $\boxed{f_{min} = 10 \text{ and } f_{max} = 15.}$

Hence, the result.

3.(b) → Prove that, in general, three normals can be drawn from a given point to the paraboloid $x^2+y^2=2az$, but if the point lies on the surface

$$27a(x^2+y^2)+8(a-z)^3=0$$ then

two of the three normals coincide.

Solution:

The equations of the normal at (x_1, y_1, z_1) to the paraboloid $x^2+y^2=2az$ are

$$\frac{x-x_1}{x_1}=\frac{y-y_1}{y_1}=\frac{z-z_1}{-a}$$

This passes through a given point (α, β, γ) if

$$\frac{\alpha-x_1}{x_1}=\frac{\beta-y_1}{y_1}=\frac{\gamma-z_1}{z_1}=\lambda \text{ (say)}$$

These gives $\alpha-x_1=\lambda x_1 \Rightarrow x_1=\alpha/(1+\lambda)$
Similarly, $y_1=\beta/(1+\lambda)$, $z_1=\gamma+a\lambda$ $\quad$ (1)

Also (x_1, y_1, z_1) lies on the given paraboloid, so

$$x_1^2+y_1^2=2az_1 \Rightarrow \left[\frac{\alpha}{1+\lambda}\right]^2+\left[\frac{\beta}{1+\lambda}\right]^2=2a(\gamma+a\lambda),$$

[from (1)]

$$\Rightarrow \alpha^2+\beta^2 = 2a(\gamma+a\lambda)(1+\lambda)^2 \quad \text{——} \quad (2)$$

This being a cubic in λ gives three values of λ and so from (1) there are three points on the paraboloid normals at which pass through (α, β, γ).

The equation (2) can be rewritten as

$$f(\lambda) \equiv 2a(1+\lambda)^2(\gamma+a\lambda) - (\alpha^2+\beta^2) = 0 \quad \text{——} (3)$$

The condition that this equation has two equal roots is obtained by eliminating λ between $f(\lambda)=0$ and $f'(\lambda)=0$.

From (3), $f'(\lambda)=0$ means $2a(1+\lambda)^2(a) +$

$4a(1+\lambda)(\gamma+a\lambda)=0$

$$\Rightarrow a(1+\lambda)+2(\gamma+a\lambda)=0 \quad (\because 1+\lambda \neq 0)$$

$$\Rightarrow (a+2\gamma)+\lambda(3a) = 0$$

$$\Rightarrow \lambda = -(a+2\gamma)/(3a)$$

Substituting this value of λ in (3), we get:

$$2a\left[1-\frac{a+2\gamma}{3a}\right]^2\left[\gamma-\frac{a(a+2\gamma)}{3a}\right] = \alpha^2+\beta^2$$

$$\Rightarrow 2a\left[2(a-\gamma)\right]^2\left[a(\gamma-a)\right] = 27a^3(\alpha^2+\beta^2)$$

$$\Rightarrow 27a(\alpha^2+\beta^2) + 8(a-\gamma)^3 = 0$$

$\therefore$ Locus of the point (α, β, γ) is

$$27a(x^2+y^2) + 8(a-z)^3 = 0.$$

Hence, proved.

3.(c) Let

$$A = \begin{bmatrix} 5 & 7 & 2 & 1 \\ 1 & 1 & -8 & 1 \\ 2 & 3 & 5 & 0 \\ 3 & 4 & -3 & 1 \end{bmatrix}$$

(i) Find the rank of matrix A.

(ii) Find the dimension of the subspace

$$V = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \;\middle/\; A\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = 0 \right\}.$$

Solution:

The given matrix is

$$A = \begin{bmatrix} 5 & 7 & 2 & 1 \\ 1 & 1 & -8 & 1 \\ 2 & 3 & 5 & 0 \\ 3 & 4 & -3 & 1 \end{bmatrix}$$

Now, we use row transformations to obtain an echelon matrix.

$$\sim \begin{bmatrix} 1 & 1 & -8 & 1 \\ 2 & 3 & 5 & 0 \\ 3 & 4 & -3 & 1 \\ 5 & 7 & 2 & 1 \end{bmatrix} \quad \begin{array}{l} R_1 \leftrightarrow R_2; \\ R_2 \leftrightarrow R_3; \\ R_3 \leftrightarrow R_4. \end{array}$$

$$\sim \begin{bmatrix} 1 & 1 & -8 & 1 \\ 0 & 1 & 21 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{array}{l} R_2 \to R_2 - 2R_1. \\ R_3 \to R_3 - (R_1 + R_2) \\ R_4 \to R_4 - (R_2 + R_3) \end{array} \quad \text{--- (I)}$$

which is clearly in the echelon form and the no. of non-zero is 2.

$\therefore \rho(A) = 2.$ — (1)

Let $\mathbb{R}^4 = \{(x_1, x_2, x_3, x_4) / x_1, x_2, x_3, x_4 \in \mathbb{R}\}$ be given vector space.

Let $V = \left\{ (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \middle/ A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = O \right\} \subseteq \mathbb{R}^4$

We have $AX = O$

$$\Rightarrow \begin{bmatrix} 5 & 7 & 2 & 1 \\ 1 & 1 & -8 & 1 \\ 2 & 3 & 5 & 0 \\ 3 & 4 & -3 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & -8 & 1 \\ 0 & 1 & 21 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad [\text{from (I)}]$$

$$\Rightarrow x_1 + x_2 - 8x_3 + x_4 = 0 \quad \text{--- (1)}$$

$$x_2 + 21x_3 - 2x_4 = 0 \quad \text{--- (2)}$$

from (2),

$$x_2 = -21x_3 + 2x_4$$

$\therefore$ from (1)

$$x_1 - 21x_3 + 2x_4 - 8x_3 + x_4 = 0$$

$$\Rightarrow x_1 - 29x_3 + 3x_4 = 0$$

$$\Rightarrow x_1 = 29x_3 - 3x_4$$

$\therefore$ There are two free variables x_3 and x_4 (say).

$\therefore$ dim $V = 2$ and

$$V = \{(29x_3 - 3x_4, -21x_3 + 2x_4, x_3, x_4) \,/\, x_3, x_4 \in \mathbb{R}\} \quad \text{--- (ii)}$$

Hence, the result.

4.(a) → State the Cayley - Hamilton theorem. Use this theorem to find A^{100}, where

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Solution:

Statement:

"Cayley - Hamilton Theorem"

$\boxed{\text{Every square matrix } A \text{ is a zero of its characteristic polynomial.}}$

The given matrix is

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

The characteristic polynomial for the given matrix A $= |A - \lambda I|$

$$= (1-\lambda)(\lambda^2 - 1)$$

$$= -\lambda^3 + \lambda^2 + \lambda - 1$$

By Cayley-Hamilton Theorem, we have

$-A^3 + A^2 + A - I = 0$

$\Rightarrow \quad A^3 = A^2 + A - I \qquad \text{— (1)}$

Now,

$A^4 = A^3 + A^2 - A$

$= (A^2 + A - I) + A^2 - A \qquad \text{— (using (1))}$

$A^4 = 2A^2 - I$

Similarly,

$A^5 = 2A^2 + A - 2I$;

$A^6 = 3A^2 + 3A - 5I \qquad \text{— (2)}$

$\vdots$

$A^8 = 4A^2 + 3A - 6I \qquad \text{— (3)}$

$\vdots$

$A^{10} = 5A^2 + 3A - 7I. \qquad \text{— (4).}$

$\vdots$

By inductive sequence, we have

$A^{2n} = nA^2 + 3A - (n+2)I. \qquad \text{— ( )}$

$\therefore \quad A^{100} = 50A^2 + 3A - 52I.$

$$\therefore \quad A^{100} = 50\begin{bmatrix}1&0&0\\1&1&0\\1&0&1\end{bmatrix} + 3\begin{bmatrix}1&0&0\\1&0&1\\0&1&0\end{bmatrix} - 52\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$= \begin{bmatrix}53&0&0\\53&50&3\\50&3&50\end{bmatrix} - \begin{bmatrix}52&0&0\\0&52&0\\0&0&52\end{bmatrix}$$

$$\therefore \quad \boxed{A^{100} = \begin{bmatrix}1&0&0\\53&-2&3\\50&3&-2\end{bmatrix}}$$

which is the required matrix.

Hence, the result.

4.(b) Find the length of the normal chord through a point P of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$ and prove that

if it is equal to $4PG_3$, where G_3 is the point where the normal chord through P meets the xy-plane, then P lies on the cone

$$\frac{x^2}{a^6}(2c^2-a^2)+\frac{y^2}{b^6}(2c^2-b^2)+\frac{z^2}{c^4}=0.$$

Solution:

Let P be (α,β,γ), then the equations of the normal to the given ellipsoid at $P(\alpha,\beta,\gamma)$ are

$$\frac{x-\alpha}{(p\alpha/a^2)}=\frac{y-\beta}{(p\beta/b^2)}=\frac{z-\gamma}{(p\gamma/c^2)}=r\text{(say)} \quad \text{---} (1)$$

where $$\frac{1}{p^2}=\frac{\alpha^2}{a^4}+\frac{\beta^2}{b^4}+\frac{\gamma^2}{c^4} \quad \text{---} (2)$$

$\therefore$ The co-ordinates of any point Q on the normal (1) are

$$\left(\alpha+\frac{p\alpha}{a^2}r,\ \beta+\frac{p\beta}{b^2}r,\ \gamma+\frac{p\gamma}{c^2}r\right),$$

where r is the distance of Q from P.

If Q lies on the given ellipsoid i.e. PQ is the normal chord, then

$$\frac{1}{a^2}\left(\alpha+\frac{p\alpha}{a^2}r\right)^2+\frac{1}{b^2}\left(\beta+\frac{p\beta}{b^2}r\right)^2+\frac{1}{c^2}\left(\gamma+\frac{p\gamma}{c^2}r\right)^2=1.$$

$$\Rightarrow r^2p^2\left(\frac{\alpha^2}{a^6}+\frac{\beta^2}{b^6}+\frac{\gamma^2}{c^6}\right)+2rp\left(\frac{\alpha^2}{a^4}+\frac{\beta^2}{b^4}+\frac{\gamma^2}{c^4}\right)+\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}\right)=1$$

$$\Rightarrow r^2p^2\left(\frac{\alpha^2}{a^6}+\frac{\beta^2}{b^6}+\frac{\gamma^2}{c^6}\right)+2rp\left(\frac{1}{p^2}\right)=0$$

[from (2) and $\sum\frac{\alpha^2}{a^2}=1$ as $P(\alpha,\beta,\gamma)$ lies on the given conicoid.]

$$\Rightarrow \boxed{r=\frac{-2}{p^3\left(\frac{\alpha^2}{a^6}+\frac{\beta^2}{b^6}+\frac{\gamma^2}{c^6}\right)}=\text{length of normal chord } PQ} \quad \text{——(3)}$$

Also,

Let the normal at $P(\alpha, \beta, \gamma)$ meets the coordinate planes viz. yz, zx and xy planes at G_1, G_2 and G_3, then putting $x=0$, $y=0$ and $z=0$ in succession in the Eq^n (1), we have respectively,

$$PG_1 = -a^2/p, \quad PG_2 = -b^2/p \text{ and } PG_3 = -c^2/p \quad \text{— (4)}$$

Given $PQ = 4PG_3$

$$\Rightarrow \quad PQ = 4(-c^2/p)$$

$$\Rightarrow \frac{-2}{p^3\left(\frac{\alpha^2}{a^6}+\frac{\beta^2}{b^6}+\frac{\gamma^2}{c^6}\right)} = 4\left(-\frac{c^2}{p}\right)$$

$$\Rightarrow 2c^2\left(\frac{\alpha^2}{a^6}+\frac{\beta^2}{b^6}+\frac{\gamma^2}{c^6}\right) = \frac{1}{p^2} = \frac{\alpha^2}{a^4}+\frac{\beta^2}{b^4}+\frac{\gamma^2}{c^4}$$

— from (2)

$$\Rightarrow \frac{\alpha^2}{a^6}(2c^2-a^2)+\frac{\beta^2}{b^6}(2c^2-b^2)+\frac{\gamma^2}{c^6}(2c^2-c^2)=0$$

$\therefore$ The locus of $P(\alpha, \beta, \gamma)$ is

$$\frac{x^2}{a^6}(2c^2-a^2)+\frac{y^2}{b^6}(2c^2-b^2)+\frac{z^2}{c^4}=0$$

Hence, proved.

4.(c)

(i) If $u = \sin^{-1}\sqrt{\dfrac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}}$ then show that $\sin^2 u$ is a homogeneous function of x and y of degree $-1/6$.

Hence show that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \frac{\tan u}{12}\left(\frac{13}{12}+\frac{\tan^2 u}{12}\right)$$

Solution:

Given that $u = \sin^{-1}\sqrt{\dfrac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}}$, we can write

$$\sin u = \left[\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}\right]^{1/2} = \frac{x^{1/6}}{x^{1/4}}\left[\frac{1+(y/x)^{1/3}}{1+(y/x)^{1/2}}\right]$$ 1/2

$$= x^{-1/12} f(y/x)$$

$$\Rightarrow \sin^2 u = \left[\frac{x^{1/3}+y^{1/3}}{x^{1/2}+y^{1/2}}\right] = \frac{x^{1/3}}{x^{1/2}}\left[\frac{1+(y/x)^{1/3}}{1+(y/x)^{1/2}}\right]$$

$$= x^{-1/6} f(y/x)$$

Thus, $z = \sin^2 u$ is a homogeneous function of x and y of degree $-1/6$. —(i)

Now, by Euler's theorem,

$$x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=-\frac{1}{12}z,\quad \text{where } z=\sin u$$

$$\Rightarrow x\cos u\frac{\partial u}{\partial x}+y\cos u\frac{\partial u}{\partial y}=-\frac{1}{12}\sin u$$

$$\Rightarrow x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=-\frac{1}{12}\tan u \quad \text{——(1)}$$

Differentiating (1) partially w.r.t. x and y, respectively

$$x\frac{\partial^2 u}{\partial x^2}+\frac{\partial u}{\partial x}+y\frac{\partial^2 u}{\partial x\partial y}=-\frac{1}{12}\sec^2 u\frac{\partial u}{\partial x} \quad \text{——(2)}$$

$$x\frac{\partial^2 u}{\partial x\partial y}+y\frac{\partial^2 u}{\partial y^2}+\frac{\partial u}{\partial y}=-\frac{1}{12}\sec^2 u\frac{\partial u}{\partial y}. \quad \text{——(3)}$$

Multiply (2) by x, (3) by y and add to get

$$\left(x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}\right)+\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$$

$$=-\frac{1}{12}\sec^2 u\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right) \quad \text{——(4)}$$

From (1) and (4), we get

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=\frac{1}{12}\tan u\left(1+\frac{1}{12}(\sec^2 u)\right)$$

$$=\frac{\tan u}{12}\left(\frac{12+1+\sec^2 u}{12}\right)$$

$$=\frac{\tan u}{12}\left(\frac{13}{12}+\frac{\tan^2 u}{12}\right)$$

Hence, Proved.

4.(c)(ii) Using the Jacobian method, show that if $f'(x)=\frac{1}{1+x^2}$ and $f(0)=0$, then

$$f(x)+f(y)=f\left(\frac{x+y}{1-xy}\right).$$

Solution:

Given that

$$f'(x)=\frac{1}{1+x^2} \text{ and } f(0)=0.$$

Let $u=f(x)+f(y)$ —(1) and $v=\frac{x+y}{1-xy}$ —(2)

We have

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} f'(x) & f'(y) \\ \frac{(1-xy)-(x+y)(-y)}{(1-xy)^2} & \frac{(1-xy)-(x+y)(-x)}{(1-xy)^2} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{1+x^2} & \frac{1}{1+y^2} \\ \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \end{vmatrix}$$

$$= \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2}$$

$$= 0$$

Since, $\frac{\partial(u,v)}{\partial(x,y)} = 0$.

$\therefore$ The given functions are not independent.
i.e. the functions u and v are functionally related.

Let $u = \phi(v)$ then

$$f(x) + f(y) = \phi\left(\frac{x+y}{1-xy}\right)$$

for $y=0$ gives $f(x) = \phi(x)$ $[\because f(0)=0]$

$$\therefore f(x) + f(y) = f\left(\frac{x+y}{1-xy}\right)$$

Hence, the result.

Section - B

Ques: 5(a)) Solve the differential Equation

$$(2y\sin x + 3y^4 \sin x \cos x)dx - (4y^3\cos^2 x + \cos x)dy = 0$$

Solution:-

Given Equations

$$(2y\sin x + 3y^4 \sin x \cos x)dx - (4y^3\cos^2 x + \cos x)dy = 0 \quad \text{---(1)}$$

$$M\,dx + N\,dy = 0$$

$$\frac{\partial M}{\partial y} = 2\sin x + 12y^3 \sin x \cos x.$$

$$\frac{\partial N}{\partial x} = 8y^3 \cos x \sin x + \sin x.$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$ hence it is not exact.

To make it exact, we need to find I.F.

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1(\sin x(1+4y^3\cancel{\cos x}))}{-\cos x(4y^3\cancel{\cos x}+1)}$$

$$= -\tan x$$

$$I.F = e^{\int -\tan x\,dx} = e^{\log \cos x} = \cos x.$$

Then; Multiply I.F to equation (1), we get

$$\Rightarrow (2y\sin x\cos x + 3y^4\sin x\cos^2 x)dx - (4y^3\cos^3 x + \cos^2 x)dy = 0.$$

$$\Rightarrow \left(y\sin 2x + \frac{3}{2}y^4\sin 2x\cos x\right)dx - (4y^3\cos x + 1)\left(\frac{1+\cos 2x}{2}\right)dy = 0$$

$$\Rightarrow \left(y\sin 2x + \frac{3}{2}y^4\sin 2x\cos x\right)dx - \left[2y^3\cos x + 2y^3\cos x\cos 2x + \frac{1}{2} + \frac{\cos 2x}{2}\right]dy = 0$$

$\Rightarrow$ Complete solution $= \int\limits_{y=\text{constant}} M\,dx + \int\limits_{x\text{-terms not included.}} N\,dy$

$$\Rightarrow \int\limits_{y=\text{constant}} \left(y\sin 2x + \frac{3}{2}y^4 \cdot \sin 2x\cos x\right)dx + \int \frac{1}{2}\,dy = c$$

$$\Rightarrow \frac{-y\cos 2x}{2} + \frac{3}{4}y^4 \int (\sin 3x + \sin x)\,dx + \frac{y}{2} = c$$

$$\Rightarrow -\frac{y}{2}\cos 2x + \frac{3}{4}y^4\left(-\frac{\cos 3x}{3} - \cos x\right) + \frac{y}{2} = c.$$

$$\Rightarrow -\left[\frac{y}{2}\cos 2x + \frac{3}{4}y^4\cos x + \frac{1}{4}y^4 \cdot \cos 3x\right] + \frac{y}{2} = c$$

$\Rightarrow$ Multiply whole equation by -4.

$$\Rightarrow 2y\cos 2x + 3y^4\cos x + y^4\cos 3x + (-2y) = c'$$

$$\Rightarrow \boxed{2y\cos 2x + y^4(\cos 3x + 3\cos x) - 2y = c'}$$

required solution.

Ques:5(b)) Determine the complete solution of the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 3x^2e^{2x}\sin 2x.$$

Solution :- Given;

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 3x^2e^{2x}\sin 2x.$$

For the equation, can be written as.

$$y'' - 4y' + 4y = 3x^2e^{2x}\sin 2x. \quad \text{—— (1)}$$

Auxillary equation of $(y''-4y'+4)y = 0$

$$m^2 - 4m + 4 = 0$$

$$(m-2)^2 = 0 \;;\; m = 2, 2$$

$$\therefore\; y_c = (C_1 + xC_2)e^{2x} \quad \text{—— (2)}$$

For, $y_P = \dfrac{1}{D^2-4D+4}(3x^2 \cdot e^{2x}\sin 2x)$

$$= \frac{1}{(D-2)^2}(3x^2e^{2x}\sin 2x)$$

$$= 3e^{2x}\frac{1}{(D+\cancel{2}-\cancel{2})^2}x^2\sin 2x$$

$$= 3e^{2x}\frac{1}{D^2}x^2\sin 2x. \quad \text{—— (2)}$$

Let $I = \dfrac{1}{D^2}x^2\sin 2x = \dfrac{1}{D^2}x^2(\text{I.P. } e^{2xi})$

$$= \text{I.P. of } \frac{1}{D^2}x^2 \cdot e^{2xi}$$

$$= \text{I.P. of } e^{2xi} \frac{1}{(D+2i)^2} x^2$$

$$= \text{I.P. of } e^{2xi} \frac{1}{-4\left(1+\frac{D}{2i}\right)^2} x^2$$

$$= \text{I.P. of } \frac{e^{2xi}}{-4}\left(1+\frac{D}{2i}\right)^{-2} x^2$$

$$= \text{I.P. of } -\frac{1}{4}(e^{2xi})\left(1-\frac{D}{i}+\frac{3D^2}{-4}+\cdots\right)x^2$$

$$= \text{I.P. of } \frac{-1}{4}(\cos 2x + i\sin 2x)\left[x^2 - \frac{2x}{i} - \frac{3}{4}x\,2\right]$$

$$= \text{I.P } \frac{-1}{4}(\cos 2x + i\sin 2x)\left(x^2 + 2xi - \frac{3}{2}\right]$$

$$= \text{I.P } -\frac{x}{2}\cos 2x - \left(x^2 - \frac{3}{2}\right)\frac{1}{4}\cdot\sin 2x.$$

$$\text{I} = -\frac{x}{2}\cos 2x - \frac{x^2}{4}\sin 2x + \frac{3}{8}\sin 2x.$$

$$\therefore y_P = 3e^{2x}\left(-\frac{x}{2}\cos 2x - \frac{x^2}{4}\sin 2x + \frac{3}{8}\sin 2x\right)$$

$$y_P = -3e^{2x}\left[\frac{x}{2}\cos 2x + \frac{x^2}{4}\sin 2x - \frac{3}{8}\sin 2x\right] \quad \text{--- (3)}$$

$$z = y_c + y_P$$

$$z = (c_1 + xc_2)e^{2x} + (-3)e^{2x}\left[\frac{x}{2}\cos 2x + \frac{x^2}{4}\sin 2x - \frac{3}{8}\sin 2x\right]$$

$$\boxed{z = (c_1 + xc_2)e^{2x} - 3e^{2x}\left[\frac{x}{2}\cos 2x + \frac{x^2}{4}\sin 2x - \frac{3}{8}\sin 2x\right]}$$

Required Solution.

Ques: 5(e) Find the directional derivative of the function $xy^2 + yz^2 + zx^2$ along the tangent to the curve $x = t$, $y = t^2$ and $z = t^3$ at the point $(1,1,1)$.

Solution:- Let $\phi(x,y,z) = xy^2 + yz^2 + zx^2$

Then $\text{grad}\,\phi = \left(\frac{\partial \phi}{\partial x}\right)\hat{i} + \left(\frac{\partial \phi}{\partial y}\right)\hat{j} + \left(\frac{\partial \phi}{\partial z}\right)\hat{k}$

$$\text{grad}\,\phi = (y^2 + 2zx)\hat{i} + (z^2 + 2xy)\hat{j} + (x^2 + 2yz)\hat{k}$$

$$\text{grad}\,\phi \Big|_{\text{at }(1,1,1)} = 3\hat{i} + 3\hat{j} + 3\hat{k}$$

Also for the curve $x = t$, $y = t^2$ and $z = t^3$

we have; $\frac{dx}{dt} = 1$, $\frac{dy}{dt} = 2t$, $\frac{dz}{dt} = 3t^2$

At point $(1,1,1)$ on the curve $x = t, y = t^2, z = t^3$ we have $t = 1$.

Now vector along the tangent to the above curve at the point (x, y, z)

$$= \left(\frac{dx}{dt}\right)\hat{i} + \left(\frac{dy}{dt}\right)\hat{j} + \left(\frac{dz}{dt}\right)\hat{k}$$

$$= \hat{i} + 2t\,\hat{j} + 3t^2\,\hat{k}$$

Putting $t = 1$, a vector along the tangent to the curve at the point $(1,1,1)$, we have.

$$= \hat{i} + 2\hat{j} + 3\hat{k}$$

If $\hat{a}$ be the unit vector in the direction of this tangent, then

$$\hat{a} = \frac{\hat{i}+2\hat{j}+3\hat{k}}{|\hat{i}+2\hat{j}+3\hat{k}|} = \frac{\hat{i}+2\hat{j}+3\hat{k}}{\sqrt{1+4+9}}$$

$$\hat{a} = \frac{\hat{i}+2\hat{j}+3\hat{k}}{\sqrt{14}}$$

$\therefore$ The required directional derivative

$$= \hat{a}\cdot \operatorname{grad}\phi \quad \text{at } (1,1,1)$$

$$= \frac{\hat{i}+2\hat{j}+3\hat{k}}{\sqrt{14}}\cdot(3\hat{i}+3\hat{j}+3\hat{k})$$

$$= \frac{3+6+9}{\sqrt{14}} = \frac{18}{\sqrt{14}}.$$

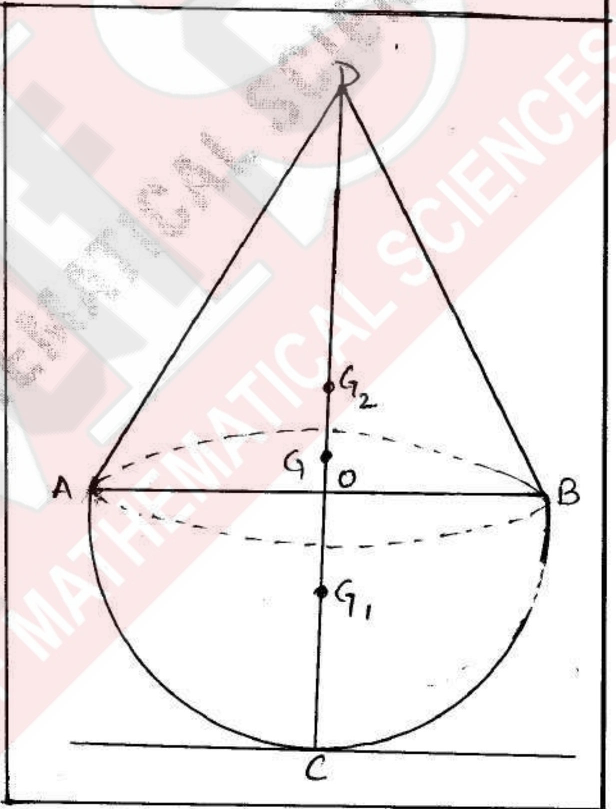
$\therefore$ $\boxed{\hat{a}\cdot \operatorname{grad}\phi = \frac{18}{\sqrt{14}}}$ required Solution

Ques: 6(a) A body consists of a cone and underlying hemisphere. The base of the cone and the top of the hemisphere have same radius 'a'. The whole body rests on a rough horizontal table with hemisphere in contact with the table. Show that the greatest height of the cone, so that the equilibrium may be stable, is $\sqrt{3}a$?

Solution:-

AB is the common base of the hemisphere and the cone and COD is their common axis which must be vertical for equilibrium. The hemisphere touches the table at C.

Let H be the height OD of the cone and 'r' be the radius OA or OC or OB of the hemisphere and the cone respectively.



Let G_1 and G_2 be the centres of gravity of the hemisphere and the cone respectively. Then.

$$OG_1 = 3r/8 \text{ and } OG_2 = H/4$$

If 'h' be the height of the centre of gravity of the combined body composed of the hemisphere and the cone above the point of contact 'C', then using the formula

$x = \dfrac{w_1 x_1 + w_2 x_2}{w_1 + w_2}$, we have

$$h = \frac{\frac{1}{3}\pi r^2 H \cdot CG_2 + \frac{2}{3}\pi r^3 CG_1}{\frac{1}{3}\pi r^2 H + \frac{2}{3}\pi r^3} = \frac{\frac{1}{3}\pi r^2 H\left(r + \frac{1}{4}H\right) + \frac{2}{3}\pi r^3 \cdot \frac{3}{8}r}{\frac{1}{3}\pi r^2 H + \frac{2}{3}\pi r^3}$$

$$\boxed{h = \frac{H\left(r + \frac{1}{4}H\right) + \frac{5}{4}r^2}{H + 2r}}$$

Here; ρ_1 = radius of curvature at the point of contact C of the upper body which is spherical $= r$

ρ_2 = the radius of curvature of the lower body at the point of contact $= \infty$

$\therefore$ the equilibrium will be stable if

$\dfrac{1}{h} > \dfrac{1}{\rho_1} + \dfrac{1}{\rho_2}$ i.e. $\dfrac{1}{h} > \dfrac{1}{r} + \dfrac{1}{\infty}$ i.e $\dfrac{1}{h} > \dfrac{1}{r}$

i.e $h < r$

i.e $\dfrac{H\left(r + \frac{1}{4}H\right) + \frac{5}{4}r^2}{H + 2r} < r$

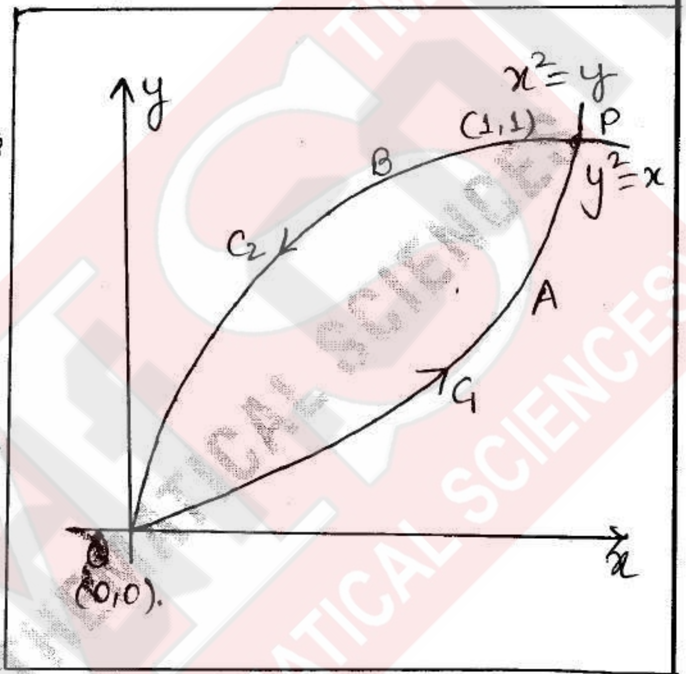
i.e $\cancel{Hr} + \frac{1}{4}H^2 + \frac{5}{4}r^2 < \cancel{Hr} + 2r^2$

i.e $\frac{1}{4}H^2 < \frac{3}{4}r^2$ i.e $H^2 < 3r^2$

i.e $\boxed{H < r\sqrt{3}}$ $r = a$ $\therefore$ $\boxed{H < a\sqrt{3}}$

Hence, the greatest height of the cone consistent with the stable equilibrium of the body is $\sqrt{3}a$ ($\sqrt{3}$ times of the radius) of the hemisphere.

Ques: 6 (b)) Find the circulation of $\vec{F}$ round the curve, where $\vec{F} = (2x+y^2)\hat{i} + (3y-4x)\hat{j}$ and C is the curve $y = x^2$ from (0,0) to (1,1) and the curve $y^2 = x$ from (1,1) to (0,0).



Solution:- Here the closed curve 'C' consists of arcs OAP and PBO

Let C_1 denote the arc OAP and C_2 denote arc PBO

Along C_1, we have $y = x^2$ so that $dy = 2x\,dx$ and x varies from $0 \to 1$

Along C_2, we have $x = y^2$ so that $dx = 2y\,dy$ and y varies from 1 to 0.

Also.

$$F \cdot dr = [(2x+y^2)\hat{i} + (3y-4x)\hat{j}][dx\,\hat{i} + dy\,\hat{j}]$$

$$F \cdot dr = (2x+y^2)dx + (3y-4x)dy.$$

Now circulation of F round C.

$$\therefore \oint_C F \cdot dr = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2} \vec{F} \cdot d\vec{r}$$

$$= \int_{C_1} (2x+y^2)dx + (3y-4x)dy + \int_{C_2} (2x+y^2)dx + (3y-4x)dy$$

$$= \int_{x=0}^{1} (2x+x^4)dx + (3x^2-4x)2x\,dx$$

$$+ \int_{y=1}^{0} (2y^2+y^2)\,2y\,dy + (3y-4y^2)dy$$

$$= \int_0^1 (2x - 8x^2 + 6x^3 + x^4)dx + \int_1^0 (3y - 4y^2 + 6y^3)dy$$

$$= \left[x^2 - \frac{8}{3}x^3 + \frac{3}{2}x^4 + \frac{x^5}{5}\right]_0^1 + \left[\frac{3y^2}{2} - \frac{4}{3}y^3 + \frac{3}{2}y^4\right]_1^0$$

$$= \left[1 - \frac{8}{3} + \frac{3}{2} + \frac{1}{5}\right] - \left[\frac{3}{2} - \frac{4}{3} + \frac{3}{2}\right]$$

$$= \left[\frac{30-80+45+6}{30}\right] - \left[\frac{9-8+9}{6}\right]$$

$$= \frac{1}{30} - \frac{5}{3} = \frac{1-50}{30} = \frac{-49}{30}$$

$$\therefore \boxed{\oint_C \vec{F}\cdot d\vec{r} = -\frac{49}{30}}$$ required solution

$-$ve sign gives the direction.

Ques: 6(c)(i) Solve the differential equation

$$\frac{d^2y}{dx^2}+(3\sin x-\cot x)\frac{dy}{dx}+2y\sin^2 x=e^{-\cos x}\cdot\sin^2 x$$

Solution:- Given that

$$\frac{d^2y}{dx^2}+(3\sin x-\cot x)\frac{dy}{dx}+2y\sin^2 x=e^{-\cos x}\cdot\sin^2 x \quad \text{--- (1)}$$

Clearly it is in the form of

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=R(x),$$

where $P(x)=3\sin x-\cot x$

$Q(x)=2\sin^2 x$

$R(x)=e^{-\cos x}\cdot\sin^2 x.$

let us solve (1) by changing the independent variable 'x' to the new independent variable z, where z is a function of 'x'.

Then the transformed equations is -

$$\frac{d^2y}{dz^2}+\frac{\frac{d^2z}{dx^2}+P\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}\cdot\frac{dy}{dx}+\frac{Q(x)}{\left(\frac{dz}{dx}\right)^2}=\frac{R(x)}{\left(\frac{dz}{dx}\right)^2}.$$

let $\frac{Q(x)}{\left(\frac{dz}{dx}\right)^2}=$ constant

$$\Rightarrow \frac{2\sin^2 x}{\left(\frac{dz}{dx}\right)^2}=2\text{ (say)} \Rightarrow \frac{dz}{dx}=\sin x$$

$$\Rightarrow \boxed{z=\cos x}$$

$\therefore$ from (2)

$$\frac{d^2y}{dz^2}+\frac{-\cos x+(3\sin x-\cot x)(-\sin x)}{\sin^2 x}\left(\frac{dy}{dx}\right)+2y=\frac{e^{-\cos x}\cdot\sin^2 x}{\sin^2 x}$$

$$\Rightarrow \frac{d^2y}{dz^2} + \frac{-\cancel{\cos x} - 3\sin^2 x + \cancel{\cos x}}{\sin^2 x}\left(\frac{dy}{dx}\right) + 2y = e^{-\cos x}$$

$$\Rightarrow \frac{d^2y}{dz^2} - 3\frac{dy}{dx} + 2y = e^{-z}$$

$$(D^2 - 3D + 2)y = e^{-z} \quad ; \quad D = \frac{d}{dz} \quad \text{———} (3)$$

Its A.E. is $m^2 - 3m + 2 = 0$

$$(m-2)(m-1) = 0$$

$$m = 1, 2.$$

$$\therefore \boxed{y_c(z) = ae^{z} + be^{2z}} \quad \text{———} (4)$$

and we have.

$$y_p(z) = \frac{1}{D^2 - 3D + 2} e^{-z} = \frac{1}{1+3+2} e^{-z}$$

$$y_p(z) = \frac{1}{6} e^{-z}.$$

$\therefore$ The general solution of (1) is given by $y(z) = y_c(z) + y_p(z)$

$$\boxed{y(z) = ae^{z} + be^{2z} + \frac{1}{6}e^{-z}}$$

$$\Rightarrow \boxed{y(x) = ae^{\cos x} + be^{2\cos x} + \frac{1}{6}e^{-\cos x}.}$$

which is the required general solution of the given equation.

Ques:- 6(c)(ii) Find the Laplace transforms of $t^{-1/2}$ and $t^{1/2}$. Prove that the laplace transform of $t^{n+1/2}$, where $n \in \mathbb{N}$, is

$$\frac{\Gamma\left(n+1+\frac{1}{2}\right)}{s^{n+1+\frac{1}{2}}}.$$

Solution :-

(a) $f(t) = t^{-1/2}$

$$L\,f(t) = \int_0^\infty e^{-st} \frac{1}{\sqrt{t}}\, dt$$

Now put $st = p \Rightarrow s\,dt = dp$

$$\therefore\ dt = \frac{dp}{s}$$

$$\Rightarrow \int_0^\infty e^{-p} \frac{\sqrt{s}}{\sqrt{p}} \cdot \frac{1}{s}\, dp = \frac{1}{\sqrt{s}} \int_0^\infty e^{-p} \cdot p^{-1/2}\, dp.$$

$$\Rightarrow \frac{1}{\sqrt{s}} \cdot \Gamma\tfrac{1}{2} = \frac{\sqrt{\pi}}{\sqrt{s}} = \sqrt{\frac{\pi}{s}}. \quad \left[\text{By gamma function}\right]$$

(b) $f(t) = t^{1/2}$

$$L\,f(t) = \int_0^\infty e^{-st} \sqrt{t}\, dt$$

put $st = p \Rightarrow s\,dt = dp \Rightarrow dt = \frac{dp}{s}$

$$\Rightarrow \int_0^\infty e^{-p} \frac{\sqrt{p}}{\sqrt{s}} \cdot \frac{dp}{s} = s^{-3/2} \int e^{-p} p^{1/2}\, dp.$$

$$= s^{-3/2} \int e^{-p} \cdot e^{3/2 - 1}\, dp.$$

[By gamma function]

$$\boxed{L(f(t)) = \frac{1}{2s}\sqrt{\frac{\pi}{s}}}$$ Required Solution.

(c) $f(t) = t^n$

$$\therefore f(s) = L(f(t)) = \lim_{A\to\infty} \int_0^A e^{-st} t^n dt$$

$$= \lim_{A\to\infty} \left\{ t^n \cdot \frac{e^{-st}}{-s} \Big|_0^A - \int_0^A \frac{n t^{n-1} \cdot e^{-st}}{-s} dt \right\}$$

$$= 0 + \frac{n}{s} \lim_{A\to\infty} \int_0^A e^{-st} t^{n-1} dt$$

$$L\{t^n\} = \frac{n}{s} L\{t^{n-1}\}$$

So recursive solution

$$L\{t^n\} = \frac{n}{s}\{L(t^{n-1})\}, \forall n$$

So,
$$L\{t^{n-1}\} = \frac{n-1}{s} L\{t^{n-2}\}$$

$$L\{t^{n-2}\} = \frac{n-2}{s} L\{t^{n-3}\}$$

by Mathematical Induction, we got

$$L\{t^n\} = \frac{n}{s} \cdot \frac{(n-1)}{s} \cdot \frac{(n-2)}{s} \cdots \frac{1}{s} L\{1\}$$

$$= \frac{n!}{s^n} \cdot \frac{1}{s} = \frac{n!}{s^{n+1}}$$

$\therefore$ from above formula

$$L\{t^{n+1/2}\} = \frac{\Gamma(n+1+1/2)}{s^{n+1+1/2}}$$

Hence the result.

Ques: 7(a)) Find the linearly independent solutions of the corresponding homogenous differential equation of the equation $x^2y''-2xy'+2y=x^3\sin x$ and then find the general solution of the given equation by the method of variation of parameters.

Solution: Given

$$x^2y''-2xy'+2y=x^3\sin x \quad \text{——} (1)$$

Divide equation (1) by x^2.

$$y''-\frac{2}{x}y'+\frac{2}{x^2}y=x\sin x.$$

$[\because R=x\sin x]$

$$y''-\frac{2}{x}y'+\frac{2}{x^2}y=0 \quad \text{homogeneous equation}$$

$$x^2y''-2xy'+2y=0$$

$$(x^2y''-2xy'+2)y=0$$

$$(x^2D^2-2xD+2)y=0 \quad \left[\because D\equiv\frac{d}{dx}\right]$$

put $x=e^z$ and $D_1\equiv\frac{d}{dz}$

Then, we get

$$(D_1(D_1-1)-2(D_1)+2)y=0$$

$$\Rightarrow (D_1^2-3D_1+2)y=0$$

$$(D_1-2)(D_1-1)=0$$

$$D=2,1.$$

$$y_c=c_1e+c_2e^{2}$$

In terms of x.

$$y_c = c_1 x + c_2 x^2 \quad [\because x = e^z]$$

where $u = x \qquad v = x^2$

$u' = 1 \qquad v' = 2x.$

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2 \neq 0$$

for $P.I = A.u + Bv.$

where $A = -\int \frac{vR}{W} dx$ and $B = \int \frac{uR}{W} dx.$

$$A = -\int \frac{\cancel{x^2}.x\sin x}{\cancel{x^2}} dx$$

$$A = -(-x\cos x + \sin x) = x\cos x - \sin x + c_1$$

$$B = \int \frac{x.x\sin x}{\cancel{x^2}} dx = \int \sin x \, dx$$

$$B = -\cos x + c_2$$

$$P.I = x.[x\cos x - \sin x] + x^2[-\cos x]$$

$$P.I. = x^2 \cancel{\cos x} - x\sin x + (-\cancel{x^2\cos x})$$

$$\boxed{P.I = -x\sin x}$$

$$z = y_c + y_p$$

$$\boxed{z = c_1 x + c_2 x^2 - x\sin x}$$

Required general solution.

Ques: 7(b) Find the radius of curvature and radius of torsion of the helix $x = a\cos u$, $y = a\sin u$, $z = au\tan\alpha$.

Solution: Given; $x = a\cos u$ $\quad y = a\sin u$

$z = au\tan\alpha$

$$r = x\hat{i} + y\hat{j} + z\hat{k}$$

$$r = a\cos u\,\hat{i} + a\sin u\,\hat{j} + au\tan\alpha\,\hat{k} \quad \text{--- (1)}$$

$$\kappa = \frac{\left|\frac{dr}{du} \times \frac{d^2r}{du^2}\right|}{\left|\frac{dr}{du}\right|^3} \quad \text{--- (2)}$$

$$\tau = \frac{\left|\frac{dr}{du} \cdot \frac{d^2r}{du^2} \cdot \frac{d^3r}{du^3}\right|}{\left|\frac{dr}{du} \times \frac{d^2r}{du^2}\right|^2} \quad \text{--- (3)}$$

from (1); differentiate w.r.t 'u' —

$$\frac{dr}{du} = -a\sin u\,\hat{i} + a\cos u\,\hat{j} + a\tan\alpha\,\hat{k}$$

$$\frac{d^2r}{du^2} = -a\cos u\,\hat{i} - a\sin u\,\hat{j} + 0\hat{k}$$

$$\frac{d^3r}{du^3} = +a\sin u\,\hat{i} - a\cos u\,\hat{j}$$

$$\left|\frac{dr}{dt}\right| = \sqrt{(-a\sin u)^2 + (a\cos u)^2 + (a\tan\alpha)^2}$$

$$= \sqrt{a^2(\sin^2 u + \cos^2 u + \tan^2\alpha)}$$

$$= a\sqrt{1 + \tan^2\alpha} = a\sqrt{\sec^2\alpha}$$

$$\boxed{\left|\frac{dr}{dt}\right| = a\sec\alpha}$$

$$\frac{dr}{du} \times \frac{d^2r}{du^2} = \begin{vmatrix} i & j & k \\ -a\sin u & a\cos u & a\tan\alpha \\ -a\cos u & -a\sin u & 0 \end{vmatrix}$$

$$= a^2 \begin{vmatrix} i & j & k \\ -\sin u & \cos u & \tan\alpha \\ -\cos u & -\sin u & 0 \end{vmatrix}$$

$$= a^2 \{ i(-\sin u \cdot \tan\alpha) - \cos u \tan\alpha \hat{j} + \hat{k}]$$

$$\left| \frac{dr}{du} \times \frac{d^2r}{du^2} \right| = \sqrt{a^4 \left(\sin^2 u \tan^2\alpha + \cos^2 u \tan^2\alpha\right) + a^4}$$

$$= a^2 \sqrt{\tan^2\alpha\left(\sin^2 u + \cos^2 u\right) + 1}.$$

$$= a^2 \sqrt{\tan^2\alpha + 1} = a^2 \sec\alpha.$$

$$\text{Curvature} = \kappa = \frac{a^2 \sec\alpha}{(a\sec\alpha)^3} = \frac{a^2 \sec\alpha}{a^3 \cdot \sec^3\alpha}$$

$$\kappa = \frac{1}{a} \cdot \frac{1}{\sec^2\alpha} = \frac{1}{a}\cos^2\alpha.$$

$$\boxed{\text{Radius of curvature} = \frac{1}{\kappa} = a\sec^2\alpha.}$$

$$\left[\frac{dr}{du}\ \frac{d^2r}{du^2}\ \frac{d^3r}{du^3}\right] = \begin{vmatrix} a\sin u & -a\cos u & 0 \\ -a\sin u & a\cos u & a\tan\alpha \\ a\cos u & a\sin u & 0 \end{vmatrix}$$

$$= a^3 \begin{vmatrix} \sin u & -\cos u & 0 \\ -\sin u & \cos u & \tan\alpha \\ \cos u & \sin u & 0 \end{vmatrix}$$

$R_2 \to R_2 + R_1$

$$= a^3 \begin{vmatrix} \sin u & -\cos u & 0 \\ 0 & 0 & \tan\alpha \\ \cos u & \sin u & 0 \end{vmatrix}.$$

$$= a^3 [\sin u [-\sin u \tan\alpha] + \cos u [-\cos u \tan\alpha] + 0]$$

$$= a^3 [-[\sin^2 u \tan\alpha + \cos^2 u \tan\alpha]]$$

$$= -a^3 [\tan\alpha [\sin^2 u + \cos^2 u]].$$

$$= -a^3 \tan\alpha \quad [\because \sin^2 u + \cos^2 u = 1].$$

$$\tau = \frac{|-a^3 \tan\alpha|}{(a^2 . \sec\alpha)^2} = \frac{a^3 \tan\alpha}{a^4 . \sec^2\alpha}.$$

$$\tau = \frac{1}{a} . \frac{1}{2} . \sin 2\alpha . = \frac{1}{2a} . \sin 2\alpha .$$

$\therefore$ radius of Torsion $= \frac{1}{\tau} = \frac{a \sec^2\alpha}{\tan\alpha}.$

$$\boxed{\text{radius of Torsion} = \frac{2a}{\sin 2\alpha}.}$$

required solution.

Ques: 8(a)) Obtain the singular solution of the differential equation

$$\left(\frac{dy}{dx}\right)^2\left(\frac{y}{x}\right)^2\cot^2\alpha - 2\left(\frac{dy}{dx}\right)\left(\frac{y}{x}\right)+\left(\frac{y}{x}\right)^2\operatorname{cosec}^2\alpha = 1$$

Also find the complete primitive of the given differential equation. Give the geometrical interpretations of the complete primitive and singular solution.

Solution :-

Given ; $\left(\frac{dy}{dx}\right)^2\left(\frac{y}{x}\right)^2\cot^2\alpha - 2\left(\frac{dy}{dx}\right)\left(\frac{y}{x}\right)+\left(\frac{y}{x}\right)^2\operatorname{cosec}^2\alpha = 1$

which can be re written as.

$$p^2y^2\cot^2\alpha - 2pxy + y^2\operatorname{cosec}^2\alpha = x^2.$$

$$p^2y^2\frac{\cos^2\alpha}{\sin^2\alpha} - 2pxy + y^2\frac{1}{\sin^2\alpha} = x^2$$

$$p^2y^2\cos^2\alpha - 2pxy\sin^2\alpha + y^2 = x^2\sin^2\alpha.$$

$$\Rightarrow p^2y^2\cos^2\alpha - 2pxy\sin^2\alpha + y^2 - x^2\sin^2\alpha = 0 \quad \text{—(A)}$$

$$\Rightarrow (Py)^2 - (2Py)x\tan^2\alpha + (y^2\sec^2\alpha - x^2\tan^2\alpha) = 0$$

$$\therefore Py = \frac{2x\tan^2\alpha \pm\sqrt{4x^2\tan^4\alpha - 4(y^2\sec^2\alpha - x^2\tan^2\alpha)}}{2.}$$

or $$Py = x\tan^2\alpha \pm\sqrt{x^2\tan^2\alpha(\tan^2\alpha+1) - y^2\sec^2\alpha}$$

$$Py = x\tan^2\alpha \pm \sec\alpha\sqrt{x^2\tan^2\alpha - y^2}$$

$$y\,dy - x\tan^2\alpha\,dx = \pm\sec\alpha\sqrt{(x^2\tan^2\alpha - y^2)}\,dx.$$

$$\Rightarrow \pm\frac{x\tan^2\alpha\,dx - y\,dy}{\sqrt{(x^2\tan^2\alpha - y^2)}} = -\sec\alpha\,dx$$

Integrating, $\pm\sqrt{x^2\tan^2\alpha - y^2} = c - x\sec\alpha$

squaring; $x^2\tan^2\alpha - y^2 = c^2 - 2cx\sec\alpha + x^2\sec^2\alpha$

$$\Rightarrow x^2(\tan^2\alpha - \sec^2\alpha) - y^2 = c^2 - 2cx\sec\alpha$$

$$\Rightarrow -x^2 - y^2 = c^2 - 2cx\sec x \quad [\because -1 = \tan^2 - \sec^2\alpha]$$

$$\Rightarrow x^2 + y^2 - 2cx\sec x + c^2 = 0 \quad \text{—— (B)}$$

From (A)

$$p^2y^2\cos^2\alpha - 2pxy\sin^2\alpha + y^2 - x^2\sin^2\alpha = 0$$

Since the given equation is quadratic in p, the p-disc, relation is –

$$4x^2y^2\sin^4\alpha - 4y^2\cos^2\alpha(y^2 - x^2\sin^2\alpha) = 0$$

$$4y^2[x^2\sin^2\alpha(\sin^2\alpha + \cos^2\alpha) - y^2\cos^2\alpha] = 0$$

$$y^2[x^2\sin^2\alpha - y^2\cos^2\alpha) = 0$$

$$y^2\cos^2\alpha(x^2\tan^2\alpha - y^2) = 0 \quad \text{—— (C)}$$

Now, the general solution of (A) is

$$c^2 - 2cx\sec\alpha + x^2 + y^2 = 0 \quad \text{—— (D)}$$

$\therefore$ The c-disc relation is

$$4x^2\sec^2\alpha - 4(x^2 + y^2) = 0$$

$$x^2(\sec^2\alpha - 1) - y^2 = 0$$

$$x^2 \tan^2\alpha - y^2 = 0$$

$$\therefore (x\tan\alpha - y)(x\tan\alpha + y) = 0 \quad \text{— (D)}$$

So; $x\tan\alpha - y = 0$ $\quad x\tan\alpha + y = 0$

i.e. the lines $y = \pm x\tan\alpha$ are singular solution (envelops) and $y=0$ is a tac-locus.

Now the general solution (B) represents a family of circles all having their centres on x-axis. The circles of the system touch one another on x-axis and $y=0$ i.e. x-axis passes through the points of contact of non-consecutive circles which touch on x-axis. The family of circles is being touched by $y = \pm x\tan\alpha$, which are equally inclined to the line of centres of the circles i.e. x-axis and pass through the origin.

Ques: 8(b)) Prove that the path of a planet, which is moving so that its acceleration is always directed to a fixed point (star) and is equal to $\frac{\mu}{(\text{distance})^2}$, is a conic section. Find the conditions under which the path becomes (i) Ellipse (ii) parabola and (iii) hyperbola.

Solution:-

Here the force is always directed to a fixed point (star), so it is a case of central orbit.

Also given that the central acceleration $P = \frac{\mu}{r^2}$

r = distance

The differential equation of the path (in pedal form) is

$$\frac{h^2}{p^3}\frac{dp}{dr} = P = \frac{\mu}{r^2}$$

Multiplying both sides by -2, we have

$$-\frac{2h^2}{p^3}\,dp = -\frac{2\mu}{r^2}\,dr$$

Integrating, we have

$$v^2 = \frac{h^2}{p^2} = \frac{2\mu}{r} + B \qquad \text{——— (1)}$$

where B is a constant

[Note that in a central orbit, $v = h/p$]

we know that referred to the focus as pole the pedal equations of ellipse, parabola and hyperbola (that branch which is nearer to focus taken as pole) are

$$\frac{b^2}{p^2} = \frac{2a}{r} - 1, \quad p^2 = ar \quad \text{and} \quad \frac{b^2}{p^2} = \frac{2a}{r} + 1$$

respectively, where
in the case of ellipse 2a and 2b are the lengths of major and minor axes,
in the case of parabola 4a is the length of latus rectum, and in the case of hyperbola 2a and 2b are the lengths of transverse and conjugate axes.

Now, since the equation ① can be any of the above three forms, three cases arise here.

Case I: Elliptic Path

Comparing ① with $\frac{b^2}{p^2} = \frac{2a}{r} - 1$, the pedal equation of the ellipse, we have.

$$\frac{h^2}{b^2} = \frac{\mu}{a} = \frac{B}{-1}.$$

$$\therefore \quad h^2 = \frac{\mu b^2}{a} \quad \text{and} \quad B = \frac{-\mu}{a}.$$

Substituting in ①, for elliptical path, we have.

$$v^2 = \frac{2\mu}{r} - \frac{\mu}{a} = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$$

obviously here $\boxed{v^2 < \frac{2\mu}{r}}$

Case II: Parabolic Path:- Comparing ① with $p^2 = ar$, the pedal equation of a parabola, we have

$$\frac{h^2}{1} = \frac{2\mu}{1/a} = \frac{B}{0}$$

$$\therefore \quad h^2 = 2\mu a \quad \text{and} \quad B = 0.$$

Substituting in (1), for parabolic path we have

$$\boxed{v^2 = \frac{2\mu}{r}}.$$

Case III: Hyperbolic path: Comparing (1) with

$\frac{b^2}{p^2} = \frac{2a}{r} + 1$ the pedal equation of a hyperbola, we have

$$\frac{h^2}{b^2} = \frac{\mu}{a} = \frac{B}{1}$$

$$\therefore \quad h^2 = \frac{\mu b^2}{a} \text{ and } B = \mu/a.$$

Substituting in (1), for hyperbolic path, we have

$$v^2 = \mu\left(\frac{2}{r} + \frac{1}{a}\right)$$

obviously here $\boxed{v^2 > \frac{2\mu}{r}}$

Thus from the above three cases, we conclude that the equation (1) always represents a conic section whose focus is at the centre of force. Further the path of the particle is an ellipse, parabola and hyperbola according as B is -ve, zero and positive. The sign of the value of the constant B depends upon the magnitude of the velocity of the particle at any point. we have found that

if (i) $v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)$ or $v^2 < \frac{2\mu}{r}$

then path is elliptic,

if $v^2 = \frac{2\mu}{r}$, then the path is parabolic

if $v^2 > \frac{2\mu}{r}$ or $v^2 = \frac{2\mu}{r} + \frac{\mu}{a}$, then the path is hyperbolic.

It is to be noted that in each of the three cases magnitude of the velocity at any point is independent of the direction of the velocity at that time.

Also, we have found that -

$h^2 = \mu b^2/a = \mu l$ in case of elliptic path

$h^2 = 2\mu a = \mu l$ in case of parabolic path

$h^2 = \mu b^2/a = \mu l$ in case of hyperbolic path.

Thus in all the three cases.

$h = \sqrt{\mu l}$, where 'l' is the length of the semi-latus rectum.

Ques: 8(c)(i) State Gauss divergence theorem. Verify this theorem for $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$, taken over the region bounded by $x^2+y^2=4$, $z=0$ and $z=3$.

Solution:-

Gauss Divergence theorem states that if V is the volume bounded by a closed surface S and 'A' is a vector function of position with continuous derivatives, then

$$\boxed{\iiint_V \nabla\cdot\vec{A}\,dV = \iint_S \vec{A}\cdot\hat{n}\,ds}$$

where $\hat{n}$ is the positive (outward drawn) normal to S.

Now; $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$

taken over the region bounded by $x^2+y^2=4$, $z=0$ and $z=3$.

$\therefore$ By Gauss divergence theorem

$$\iiint_V \nabla\cdot\vec{F}\,dV = \iint_S \vec{F}\cdot\hat{n}\,ds$$

volume integral $= \iiint_V \nabla\cdot\vec{F}\,dV$

$$= \iiint_V \left[\frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2)\right] dV$$

$$= \iiint_V (4-4y+2z)\,dV$$

$$= \int_{x=-2}^{2} \int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{z=0}^{3} (4-4y+2z)\,dz\,dy\,dx$$

$$= \int_{x=-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \left[4z-4yz+z^2\right]_0^3 dy\,dx \quad \left[\begin{array}{l} \because \text{radius of circle} \\ x^2+y^2=4 \quad r=2 \\ y=\pm\sqrt{4-x^2} \end{array}\right]$$

$$= \int_{x=-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} [12-12y+9]\,dy\,dx.$$

using polar coordinates

$x = r\cos\theta$ $\quad$ $y = r\sin\theta$

$dx = r(-\sin\theta)d\theta$ $\quad$ $dy = r\cos\theta\, d\theta$

$\Rightarrow$ $dx\,dy = r\,d\theta\,dr.$

$$\int_{\theta=0}^{2\pi}\int_{0}^{2} (21-12r\sin\theta)\,r\,dr\,d\theta$$

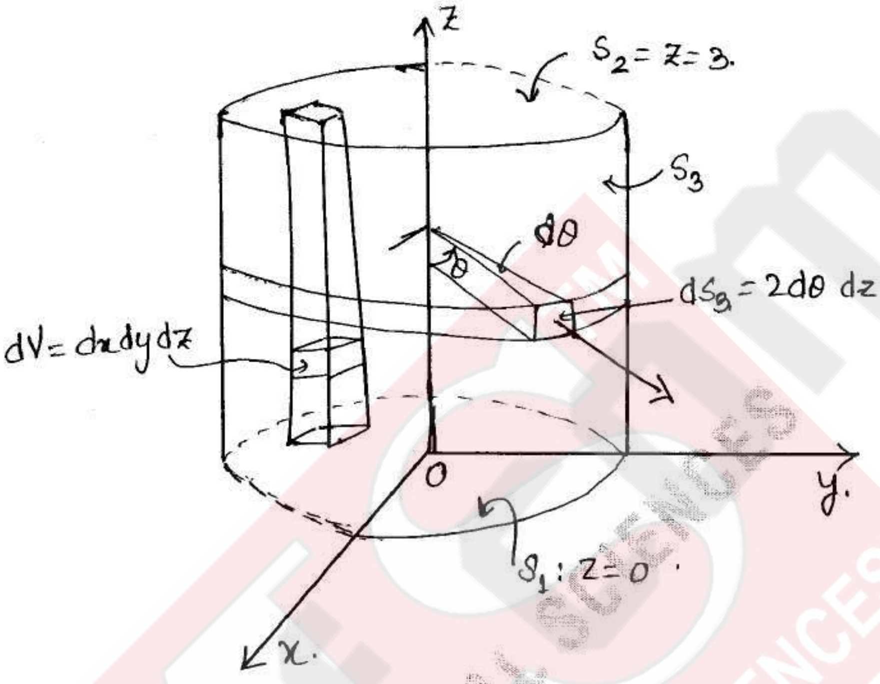
$$= \int_{0}^{2\pi}\int_{0}^{2} (21r - r^2\sin\theta)\,dr\,d\theta$$

$$= \int_{0}^{2\pi} \left[21\frac{r^2}{2} - \frac{r^3}{3}\sin\theta\right]_0^2 d\theta$$

$$= \int_{0}^{2\pi} \left[42 - \frac{8}{3}\sin\theta\right] d\theta = \left[42\theta + \frac{8}{3}\cos\theta\right]_0^{2\pi}$$

$$= \left[84\pi + \left[\frac{8}{\cancel{3}}(1) - \frac{8}{\cancel{3}}(1)\right]\right] = 84\pi.$$

$$\Rightarrow \boxed{\iiint_V \nabla\cdot\vec{F}\,dv = 84\pi}$$



The surface S of the cylinder consists of a base S_1 ($z=0$), the top S_2 ($z=3$) and the convex portion S_3 ($x^2+y^2=4$). Then.

Surface Integral $= \iint_S \vec{F}.\hat{n}\,dS = \iint_{S_1} \vec{F}\cdot\hat{n}\,dS_1 + \iint_{S_2} \vec{F}.\hat{n}\,dS_2 + \iint_{S_3} \vec{F}.\hat{n}\,dS_3$

<u>on $S_1(z=0)$</u> ; $\hat{n} = -k$ $\quad \vec{F} = 4x\hat{i} - 2y^2\hat{j}$

$\vec{F}.\hat{n} = 0$

$\therefore \iint_{S_1} \vec{F}.\hat{n}\,dS_1 = 0.$

<u>On $S_2(z=3)$</u> ; $\hat{n} = k$ $\quad \vec{F} = 4x\hat{i} - 2y^2\hat{j} + 9\hat{k}$

$\vec{F}\cdot\hat{n} = 9.$, so that

$$\iint_{S_2} \vec{F}\cdot\hat{n}\,dS_2 = \iint_{S_2} 9\,dS_2 = 9\cdot\pi r^2$$
$$= 9\pi\cdot(2)^2 \quad [\because r=2]$$

$$\boxed{\iint_{S_2} \vec{F}.\hat{n}\,dS_2 = 36\pi}$$

On $S_3 (x^2+y^2=4)$:

A perpendicular to $x^2+y^2=4$ has the direction

$$\nabla(x^2+y^2) = 2x\hat{i}+2y\hat{j}$$

Then a unit normal is $\hat{n} = \dfrac{2x\hat{i}+2y\hat{j}}{\sqrt{4x^2+4y^2}}$

$$\hat{n} = \frac{\cancel{2}(x\hat{i}+y\hat{j})}{\cancel{2}\sqrt{x^2+y^2}} = \frac{x\hat{i}+y\hat{j}}{2} \quad [\because x^2+y^2=4]$$

$$\vec{F}\cdot\hat{n} = (4x\hat{i}-2y^2\hat{j}+z^2\hat{k})\left(\frac{x\hat{i}+y\hat{j}}{2}\right).$$

$$\boxed{\vec{F}\cdot\hat{n} = 2x^2-y^3.}$$

$$\therefore \iint_{S_3} \vec{F}\cdot\hat{n}\,ds = \int_{\theta=0}^{2\pi}\int_{z=0}^{3}\left[2(2\cos\theta)^2-(2\sin\theta)^3\right]2\,dz\,d\theta$$

$$= \int_{\theta=0}^{2\pi}(48\cos^2\theta-48\sin^3\theta)\,d\theta$$

$$= \int_{\theta=0}^{2\pi} 48\cos^2\theta\,d\theta = 48\pi.$$

$\therefore$ Then the surface integral $= 0+36\pi+48\pi$
$= 84\pi.$

$$\boxed{\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dV = 84\pi}$$

Hence verified.

Ques: 8(c) ii)) Evaluate by stoke's theorem $\oint_C e^x dx + 2y\,dy - dz$, where C is the curve $x^2+y^2=4$, $z=2$.

Solution:-

$\vec{F} = e^x dx + 2y\,dy - dz$ & $\oint_C \vec{F}$

C is the curve $x^2+y^2=4$; $z=2$

Using stokes theorem

$$\oint_C \vec{F}\cdot d\vec{r} = \iint (\nabla\times\vec{F})\cdot\hat{n}\,ds.$$

$$\nabla\times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ e^x & 2y & -1 \end{vmatrix}$$

$$\nabla\times F = \hat{i}(0) - \hat{j}(0) + \hat{k}(0)$$

$$\nabla\times F = 0.$$

$$\therefore \iint (\nabla\times\vec{F})\,\hat{n}\,ds = 0.$$

$$\therefore \boxed{\oint_C e^x dx + 2y\,dy - dz = \iint (\nabla\times\vec{F})\hat{n}\,ds = 0}$$

which is required result.