ABCDEF is a regular hexagon. Let $\vec{AB} = \mathbf{a}$ *and* $\vec{BC} = \mathbf{b}$. *Find the vectors determined by the other four sides taken in order. Also express the vectors* $\vec{AC}$, $\vec{AD}$, $\vec{AF}$, $\vec{AE}$, $\vec{CE}$ *in terms of* **a** *and* **b**.

Solution. $\vec{AC} = \vec{AB} + \vec{BC} = \mathbf{a} + \mathbf{b}$

$\because$ AD is parallel and double of BC,

$\therefore$ $\vec{AD} = 2\mathbf{b}$.

In $\Delta\, ACD$,

$$\vec{AC} + \vec{CD} = \vec{AD}$$

$$\Rightarrow \quad \vec{CD} = \vec{AD} - \vec{AC}$$

$$= 2\mathbf{b} - (\mathbf{a} + \mathbf{b})$$

$$= \mathbf{b} - \mathbf{a}.$$

$$\vec{AF} = \vec{CD} = \mathbf{b} - \mathbf{a}.$$

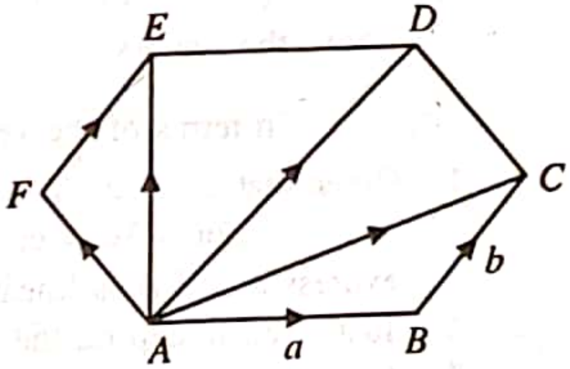

Fig. 1.20

Now, $\vec{DE} = \vec{BA} = -\mathbf{a}$

$$\vec{EF} = \vec{CB} = -\mathbf{b}$$

$$\vec{FA} = \vec{DC} = -(\mathbf{b} - \mathbf{a}) = \mathbf{a} - \mathbf{b}$$

Again, $\vec{AE} = \vec{AD} + \vec{DE} = 2\mathbf{b} + (-\mathbf{a}) = 2\mathbf{b} - \mathbf{a}$

and $\vec{CE} = \vec{CD} + \vec{DE} = \mathbf{b} - \mathbf{a} + (-\mathbf{a})$

$$= \mathbf{b} - 2\mathbf{a}$$

Example 2. *Examine whether the vectors* $5\mathbf{a} + 6\mathbf{b} + 7\mathbf{c}$, $7\mathbf{a} - 8\mathbf{b} + 9\mathbf{c}$ *and* $3\mathbf{a} + 20\mathbf{b} + 5\mathbf{c}$, ($\mathbf{a}, \mathbf{b}, \mathbf{c}$ *being non-coplanar vectors*) *are linearly independent or dependent.*

Solution. If possible, let txe linearly dependent. Then there exist scalars x_1, x_2, x_3, not all zero, such that

$$x_1 (5\mathbf{a} + 6\mathbf{b} + 7\mathbf{c}) + x_2 (7\mathbf{a} - 8\mathbf{b} + 9\mathbf{c}) + x_3 (3\mathbf{a} + 20\mathbf{b} + 5\mathbf{c}) = \mathbf{0} \quad ...(i)$$

$$\Rightarrow \quad (5x_1 + 7x_2 + 3x_3)\, \mathbf{a} + (6x_1 - 8x_2 + 20x_3)\, \mathbf{b} + (7x_1 + 9x_2 + 5x_3)\, \mathbf{c} = 0$$

As $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are non-coplanar vectors,

$$\Rightarrow \quad \begin{aligned} 5x_1 + 7x_2 + 3x_3 &= 0 \\ 6x_1 - 8x_2 + 20x_3 &= 0 \\ 7x_1 + 9x_2 + 5x_3 &= 0 \end{aligned}$$

From first two equations, we get

$$\frac{x_1}{2} = \frac{x_2}{-1} = \frac{x_3}{-1} = k \quad \text{(say)}$$

$$\therefore \quad x_1 = 2k, \; x_2 = -k, \; x_3 = -k.$$

These values also satisfy the third equation.

Example 4. *If* $\hat{\mathbf{a}}$ and $\hat{\mathbf{b}}$ *are unit vectors and* θ *is the angle between them, show that* $\sin(\theta/2) = \frac{1}{2}|\hat{\mathbf{a}} - \hat{\mathbf{b}}|$.

Solution. $|\hat{\mathbf{a}} - \hat{\mathbf{b}}|^2 = (\hat{\mathbf{a}} - \hat{\mathbf{b}}).(\hat{\mathbf{a}} - \hat{\mathbf{b}})$

$$= \hat{\mathbf{a}}.\hat{\mathbf{a}} - \hat{\mathbf{a}}.\hat{\mathbf{b}} - \hat{\mathbf{b}}.\hat{\mathbf{a}} + \hat{\mathbf{b}}.\hat{\mathbf{b}}$$

$$= 1 - \cos\theta - \cos\theta + 1$$

$$= 2(1 - \cos\theta) = 4\sin^2\frac{\theta}{2}$$

$\therefore$ $$\sin\frac{\theta}{2} = \frac{1}{2}|\hat{\mathbf{a}} - \hat{\mathbf{b}}|.$$

Example 2. *A line makes angles* α, β, γ, δ *with the diagonals of a cube; show that*

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}.$$

Solution.

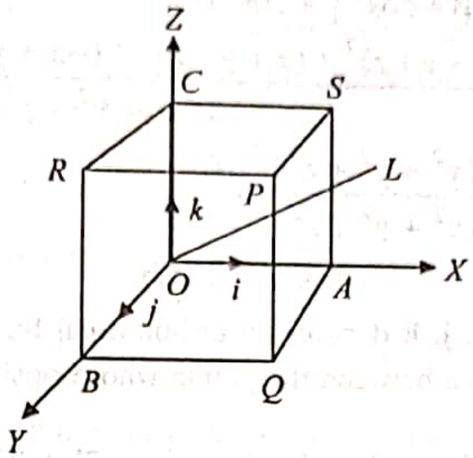

Fig. 3.19

Since the angle will remain unchanged for any size of cube, consider a unit cube. Represent the coterminous edges $\vec{OA}$, $\vec{OB}$, $\vec{OC}$ by unit vectors **i, j, k.**

Then $\vec{OA} = \mathbf{i}$, $\vec{OB} = \mathbf{j}$, $\vec{OC} = \mathbf{k}$,

$$\vec{OP} = \vec{OA} + \vec{AQ} + \vec{QP} = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

$$\vec{CQ} = -\mathbf{k} + \mathbf{i} + \mathbf{j},\ \vec{AR} = -\mathbf{i} + \mathbf{k} + \mathbf{j}$$

and $\vec{BS} = \mathbf{i} - \mathbf{j} + \mathbf{k}$.

$\therefore$ $OP = CQ = AR = BS = \sqrt{3}.$

Let any line OL be given by

$$\vec{OL} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

If α be the angle between OL and OP, then

$$\vec{OL}\,.\,\vec{OP} = OL \times OP \times \cos\alpha$$

$\Rightarrow$ $(x\mathbf{i} + y\mathbf{j} + z\mathbf{k})\,.\,(\mathbf{i} + \mathbf{j} + \mathbf{k}) = \sqrt{(x^2 + y^2 + z^2)}.\sqrt{3}\cos\alpha$

$\Rightarrow$ $\cos\alpha = \dfrac{(x + y + z)}{\sqrt{3}\sqrt{(x^2 + y^2 + z^2)}}$

Similarly,

$$\cos\beta = \frac{(x + y - z)}{\sqrt{3}\sqrt{(x^2 + y^2 + z^2)}}$$

$$\cos \gamma = \frac{(-x+y+z)}{\sqrt{3}\sqrt{(x^2+y^2+z^2)}}$$

and
$$\cos \delta = \frac{(x-y+z)}{\sqrt{3}\sqrt{(x^2+y^2+z^2)}}$$

$$\therefore \quad \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma + \cos^2 \delta$$

$$= \frac{(x+y+z)^2 + (x+y-z)^2 + (-x+y+z)^2 + (x-y+z)^2}{3(x^2+y^2+z^2)}$$

$$= \frac{4(x^2+y^2+z^2)}{3(x^2+y^2+z^2)} = \frac{4}{3}.$$

Example 4. *Find the value of p so that the vectors* $2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ *and* $3\mathbf{i} + p\mathbf{j} + 5\mathbf{k}$ *are coplanar.* *(Avadh 2000)*

Solution. Given vectors will be coplanar if

$$\begin{vmatrix} 2 & -1 & 1 \\ 1 & 2 & -3 \\ 3 & p & 5 \end{vmatrix} = 0$$

$$\Rightarrow \qquad 2\,(10 + 3p) + 1\,(5 + 9) + 1\,(p - 6) = 0$$

$$\Rightarrow \qquad 7p + 28 = 0 \quad \Rightarrow \quad p = -4.$$

Example 5. *Prove that*

$$\mathbf{a} \times \mathbf{b} = [(\mathbf{i} \times \mathbf{a}) \,.\, \mathbf{b}]\,\mathbf{i} + [(\mathbf{j} \times \mathbf{a}) \,.\, \mathbf{b}]\,\mathbf{j} + [(\mathbf{k} \times \mathbf{a}) \,.\, \mathbf{k}]\,\mathbf{k}.$$

Solution. For any vector $\mathbf{r}$, we have

$$\mathbf{r} = (\mathbf{i} \,.\, \mathbf{r})\,\mathbf{i} + (\mathbf{j} \,.\, \mathbf{r})\,\mathbf{j} + (\mathbf{k} \,.\, \mathbf{r})\,\mathbf{k}$$

Replacing $\mathbf{r}$ by $(\mathbf{a} \times \mathbf{b})$, we get

$$(\mathbf{a} \times \mathbf{b}) = [\mathbf{i} \,.\, (\mathbf{a} \times \mathbf{b})]\,\mathbf{i} + [\mathbf{j} \,.\, (\mathbf{a} \times \mathbf{b})]\,\mathbf{j} + [\mathbf{k} \,.\, (\mathbf{a} \times \mathbf{b})]\,\mathbf{k}$$

$$= [(\mathbf{i} \times \mathbf{a}) \,.\, \mathbf{b}]\,\mathbf{i} + [(\mathbf{j} \times \mathbf{a}) \,.\, \mathbf{b}]\,\mathbf{j} + [(\mathbf{k} \times \mathbf{a}) \,.\, \mathbf{b}]\,\mathbf{k}.$$

since the position of the dot and cross can be interchanged in a scalar triple product.

Example 5. *Find the volume of the tetrahedron the rectangular cartesian co-ordinates of whose vertices are*

$$(0,\ 1,\ 2),\quad (3,\ 0,\ 1),\quad (4,\ 3,\ 6),\quad (2,\ 3,\ 2).$$

Solution. Let, as usual, **i, j, k** denote unit vectors along the three rectangular axes. If A, B, C, D denote the given vertices, we have

$$\vec{OA} = \mathbf{j} + 2\mathbf{k}, \qquad \vec{OB} = 3\mathbf{i} + \mathbf{k},$$

$$\vec{OC} = 4\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}, \qquad \vec{OD} = 2\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}.$$

The required volume is

$$\frac{1}{6} \mid \vec{AB} \times \vec{AC} \cdot \vec{AD} \mid.$$

We have

$$\vec{AB} = \vec{OB} - \vec{OA} = 3\mathbf{i} - \mathbf{j} - \mathbf{k}.$$

$$\vec{AC} = \vec{OC} - \vec{OA} = 4\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}.$$

$$\vec{AD} = \vec{OD} - \vec{OA} = 2\mathbf{i} + 2\mathbf{j}.$$

$$\therefore \quad \vec{AB} \times \vec{AC} = (3\mathbf{i} - \mathbf{j} - \mathbf{k}) \times (4\mathbf{i} + 2\mathbf{j} + 4\mathbf{k})$$

$$= -\ 2\mathbf{i} - 16\mathbf{j} + 10\mathbf{k}.$$

$$\therefore \quad \vec{AB} \times \vec{AC} \cdot \vec{AD} = (-2\mathbf{i} - 16\mathbf{j} + 10\mathbf{k}) \cdot (2\mathbf{i} + 2\mathbf{j}) = -36.$$

Thus, the required volume $= 36$.

Example 9. *Evaluate* $\int \mathbf{a} \cdot \left(\mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2}\right) dt$

Solution. We have $\frac{d}{dt}\left(\mathbf{r} \times \frac{d\mathbf{r}}{dt}\right) = \frac{d\mathbf{r}}{dt} \times \frac{d\mathbf{r}}{dt} + \mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} = \mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2}$

$$\therefore \quad \int \left(\mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2}\right) dt = \mathbf{r} \times \frac{d\mathbf{r}}{dt} + \mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} + \mathbf{c}.$$

where $\mathbf{c}$ is an arbitrary constant vector.

Now,
$$\int \mathbf{a} \cdot \left(\mathbf{r} \cdot \frac{d^2\mathbf{r}}{dt^2}\right) dt = \mathbf{a} \cdot \int \mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2}\, dt$$

$$= \mathbf{a} \cdot \left[\mathbf{r} \times \frac{d\mathbf{r}}{dt} + \mathbf{c}\right] = \mathbf{a} \cdot \mathbf{r} \times \frac{d\mathbf{r}}{dt} + d$$

where $d = \mathbf{a} \cdot \mathbf{c}$ is an arbitrary constant scalar.

Example 2. *If* $\phi(x, y, z) = xy^2z$ *and* $\mathbf{A} = xz\,\mathbf{i} - xy\,\mathbf{j} + yz^2\,\mathbf{k}$ *find* $\dfrac{\partial^3(\phi\mathbf{A})}{\partial x^2 \partial z}$ *at* $(2, -1, 1)$.

Solution. $$\phi\mathbf{A} = xy^2z\,(xz\mathbf{i} - xy\mathbf{j} + yz^2\mathbf{k})$$

$$= x^2y^2z^2\,\mathbf{i} - x^2y^2z\,\mathbf{j} + xy^3z^3\,\mathbf{k}$$

Now

$$\frac{\partial}{\partial z}(\phi\mathbf{A}) = 2x^2y^2z\,\mathbf{i} - x^2y^3\mathbf{j} + 3xy^3z^2\mathbf{k}$$

$$\frac{\partial^2}{\partial x \partial z}(\phi\mathbf{A}) = 4xy^2z\,\mathbf{i} - 2xy^3\mathbf{j} + 3y^3z^2\mathbf{k}$$

and $$\frac{\partial^3}{\partial x^2 \partial z}(\phi\mathbf{A}) = 4y^2z\,\mathbf{i} - 2y^3\mathbf{j}$$

$$= 4(-1)^2(1)\,\mathbf{i} - 2(-1)^3\,\mathbf{j}$$

$$= 4\mathbf{i} + 2\mathbf{j} \text{ at the point } (2, -1, 1).$$

Example 1. *Find grad log* $|\mathbf{r}|$. *(Rohilkhand, 2000, Garhwal 2000)*

Solution. We have $r = \sqrt{(x^2 + y^2 + z^2)}$

$$\log |\mathbf{r}| = \frac{1}{2}\log(x^2 + y^2 + z^2)$$

Now, $\dfrac{\partial}{\partial x}\log|\mathbf{r}| = \dfrac{1}{2}\cdot\dfrac{2x}{(x^2 + y^2 + z^2)} = \dfrac{x}{r^2}$

Similarly, $\dfrac{\partial}{\partial y}\log|\mathbf{r}| = \dfrac{y}{r^2}$, $\dfrac{\partial}{\partial z}\log|\mathbf{r}| = \dfrac{z}{r^2}$

$\therefore$ $\quad$ $\text{grad} \log |\mathbf{r}| = \dfrac{1}{r^2}(\mathbf{i}x + \mathbf{j}y + \mathbf{k}z) = \dfrac{\mathbf{r}}{r^2}$

Example 2. *Evaluate*

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

where $\mathbf{F} = (x^2 + y^2)\,\mathbf{i} - 2xy\mathbf{j}$ *and the curve C is the rectangle in the xy-plane bounded by* $y = 0,\ x = a,\ y = b,\ x = 0$.

Solution. In the xy-plane $z = 0$.

$\therefore \quad \mathbf{r} = x\mathbf{i} + y\mathbf{j}$

$d\mathbf{r} = dx\mathbf{i} + dy\mathbf{j}$

$\therefore \quad \displaystyle\int_C \mathbf{F} \cdot d\mathbf{r}$

$$= \int_C \left\{ \left(x^2 + y^2\right) dx - 2xy\, dy \right\} \qquad \text{...(1)}$$

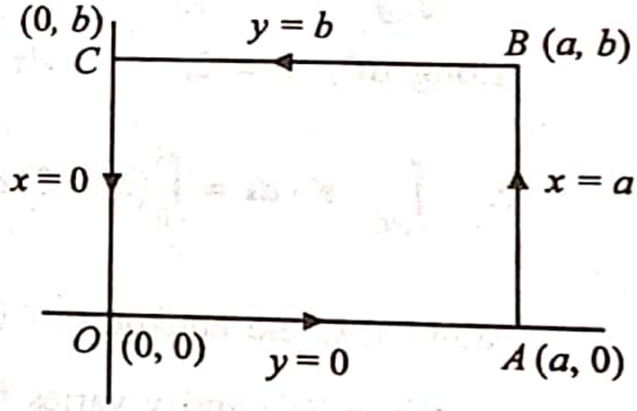

Fig. 11.5.

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{OA} \mathbf{F}\cdot d\mathbf{r} + \int_{AB} \mathbf{F}\cdot d\mathbf{r} + \int_{BC} \mathbf{F}\cdot d\mathbf{r} + \int_{CO} \mathbf{F}\cdot d\mathbf{r} \quad \text{...(2)}$$

Along OA, $y = 0$ $\therefore$ $dy = 0$ and x varies from 0 to a.

Along AB, $x = a$ $\therefore$ $dx = 0$ and y varies from 0 to b.

Along BC, $y = b$ $\therefore$ $dy = 0$ and x varies from a to 0.

Along CO, $x = 0$ $\therefore$ $dx = 0$ and y varies from b to 0.

Hence, from (1) and (2),

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^a x^2 dx - \int_0^b 2ay\, dy + \int_a^0 \left(x^2 + b^2\right) dx + \int_b^0 0.dy$$

$$= \frac{a^3}{3} - 2a\frac{b^2}{2} + \left[\frac{x^3}{3} + b^2 x\right]_a^0$$

$$= \frac{a^3}{3} - ab^2 - \frac{a^3}{3} - b^2 a = -2ab^2.$$

Example 4. *Find the circulation of* $\mathbf{F}$ *round the curve C, where* $\mathbf{F} = (2x + y^2)\,\mathbf{i} + (3y - 4x)\,\mathbf{j}$ *and C is the curve* $y = x^2$ *from (0, 0) to (1, 1) and the curve* $y^2 = x$ *from (1, 1) to (0, 0).*

Solution.

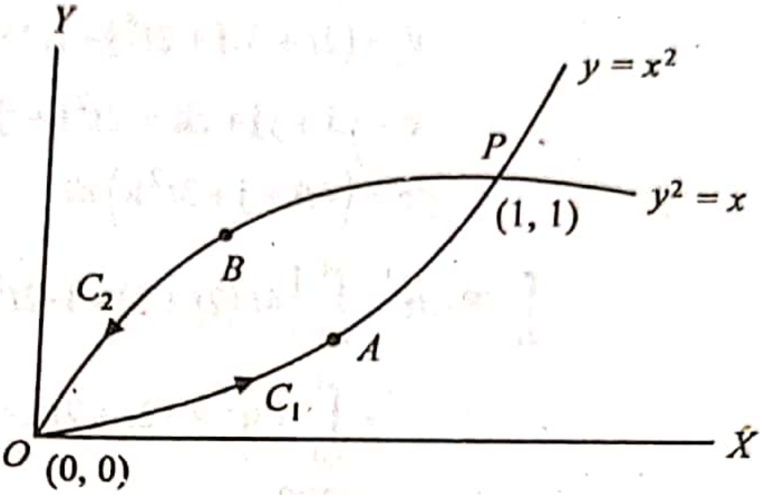

Fig. 11.7.

$$\int_C \mathbf{F}\cdot d\mathbf{r}$$

$$= \int_C \left[(2x + y^2)\,dx + (3y - 4x)\,dy\right]$$

We have to evaluate this integral along two curves C_1 and C_2.

For C_1, $y = x^2$ $\therefore$ $dy = 2xdx$ and x varies from 0 to 1.

For C_2, $x = y^2$ $\therefore$ $dx = 2ydy$ and y varies from 1 to 0.

$$\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_0^1 (2x + x^4)\,dx + (3x^2 - 4x)\,2xdx$$

$$= \int_0^1 (2x - 8x^2 + 6x^3 + x^4)\,dx$$

$$= 2\cdot\frac{1}{2} - 8\cdot\frac{1}{3} + 6\cdot\frac{1}{4} + \frac{1}{5} = \frac{1}{30}$$

$$\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_1^0 (2y^2 + y^2)\,2y\,dy + (3y - 4y^2)\,dy$$

$$= \int_1^0 (3y - 4y^2 + 6y^3)\,dy$$

$$= -\left[\frac{3}{2} - \frac{4}{3} + \frac{6}{4}\right] = \frac{5}{3}$$

$$\therefore \quad \int_C \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} + \int_{C_2} \mathbf{F}\cdot d\mathbf{r}$$

$$= \frac{1}{30} - \frac{5}{3} = -\frac{49}{30}.$$

Example 6. *Evaluate*

$$\int_S \frac{\mathbf{r}}{r^3} \cdot d\mathbf{a},$$

where S denotes the sphere of radius a with centre at the origin.

Solution. At any point of S,

$$\mathbf{n} = \frac{\mathbf{r}}{r},$$

for the normal lies along the line joining the centre to the point.

$$\therefore \quad \int_S \frac{\mathbf{r}}{r^3} \cdot d\mathbf{a} = \int_S \frac{\mathbf{r}}{r^3} \cdot \frac{\mathbf{r}}{r}\, dS$$

$$= \int_S \frac{\mathbf{r}^2}{r^4}\, dS = \frac{1}{a^2} \int_S dS = \frac{1}{a^2} \times 4\pi a^2 = 4\pi.$$

Example 6. *Prove that*

$$\int_V (\mathbf{g} \cdot curl\ curl\ \mathbf{f} - \mathbf{f} \cdot curl\ curl\ \mathbf{g})\, dV$$

$$= \int_S \{(\mathbf{f} \times \text{curl } \mathbf{g}) - (\mathbf{g} \times \text{curl } \mathbf{f})\} \cdot d\mathbf{a}$$

Solution. $\int_S (\mathbf{f} \times \text{curl } \mathbf{g}) \cdot d\mathbf{a} = \int_S (\mathbf{f} \times \text{curl } \mathbf{g}) \cdot \mathbf{n}\, dS$

$$= \int_V \text{div}\,[\mathbf{f} \times \text{curl } \mathbf{g}] \cdot dV$$

$$= \int_V [\text{curl } \mathbf{g} \cdot \text{curl } \mathbf{f} - \mathbf{f} \cdot \text{curl curl } \mathbf{g}]\, dV \quad ...(1)$$

$$\int_S (\mathbf{g} \times \text{curl } \mathbf{f}) \cdot d\mathbf{a} = \int_S (\mathbf{g} \times \text{curl } \mathbf{f}) \cdot \mathbf{n} \cdot dS$$

$$= \int_V \text{div}\,(\mathbf{g} \times \text{curl } \mathbf{f})\, dV$$

$$= \int [\text{curl } \mathbf{f} \cdot \text{curl } \mathbf{g} - \mathbf{g} \cdot \text{curl curl } \mathbf{f}]\, dV \quad ...(2)$$

Subtracting (1) from (2), we get the required result.

Example 1. *Verify Stoke's theorem for the function*

$$\mathbf{F} = x\,(\mathbf{i}x + \mathbf{j}y),$$

integrated round the square in the plane $z = 0$ whose sides are along the lines

$$x = 0, \quad y = 0, \quad x = a, \quad y = a.$$

Solution. We have

curl $x\,(\mathbf{i}x + \mathbf{j}y) = \mathbf{k}y$.

$$\therefore \qquad \int_S \text{curl}\, x\,(\mathbf{i}x + \mathbf{j}y) \cdot d\mathbf{a}$$

$$= \int_0^a \int_0^a \mathbf{k} y . \mathbf{k} \, dx \, dy.$$

$$= \int_0^a \int_0^a y \, dx \, dy = \frac{a^3}{2}$$

Again

$$+ \int_{BC} \mathbf{F} \cdot d\mathbf{r} + \int_{CO} \mathbf{F} \cdot d\mathbf{r}.$$

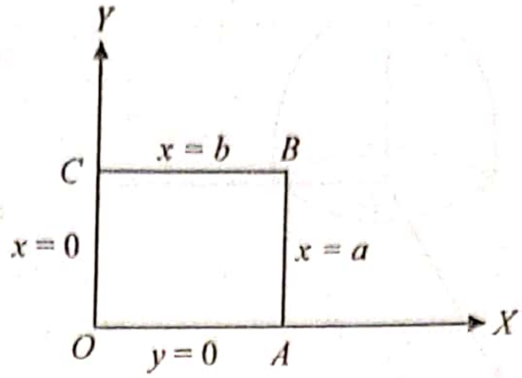

Fig. 11.14.

$$+ \int_{BC} \mathbf{F} \cdot d\mathbf{r} + \int_{CO} \mathbf{F} \cdot d\mathbf{r}.$$

Now

$$\int_{OA} \mathbf{F} \cdot d\mathbf{r} = \int_0^a x\left(\mathbf{i}x + \mathbf{j}y\right) \cdot \mathbf{i} dx = \int_0^a x^2 dx = \frac{1}{3}a^3,$$

$$\int_{AB} \mathbf{F} \cdot d\mathbf{r} = \int_0^a x\left(\mathbf{i}x + \mathbf{j}y\right) \cdot \mathbf{j} dy = \int_0^a ay \, dy = \frac{1}{2}a^3,$$

$$\int_{BC} \mathbf{F} \cdot d\mathbf{r} = \int_a^0 x\left(\mathbf{i}x + \mathbf{j}y\right) \cdot \mathbf{i} dx = -\int_0^a x^2 dx = -\frac{1}{3}a^3,$$

$$\int_{CO} \mathbf{F} \cdot d\mathbf{r} = \int_a^0 x\left(\mathbf{i}x + \mathbf{j}y\right) \cdot \mathbf{j} dy = 0.$$

$$\therefore \quad \int_C \mathbf{F} \cdot d\mathbf{r} = \frac{1}{3}a^3 + \frac{1}{2}a^3 - \frac{1}{3}a^3 + 0 = \frac{1}{2}a^3.$$

Hence the verification.

Example 2. *Find the value of*

$$\int \text{curl } \mathbf{F} \cdot d\mathbf{a},$$

taken over the portion of the surface

$$x^2 + y^2 - 2ax + az = 0,$$

for which $z \geq 0$, when

$$\mathbf{F} = \left(y^2 + z^2 - x^2\right)\mathbf{i} + \left(z^2 + x^2 - y^2\right)\mathbf{j} + \left(x^2 + y^2 - z^2\right)\mathbf{k}.$$

Solution. Rewriting the equation

$$x^2 + y^2 - 2ax + az = 0$$

as

$$(x - a)^2 + y^2 = -a(z - a),$$

we see that the surface is a paraboloid with its vertex at (a, o, a) and axis parallel to z-axis and turned towards the negative direction of the same. It meets the plane $z = 0$ in the circle C, given by

$$x^2 + y^2 - 2ax = 0, \quad z = 0.$$

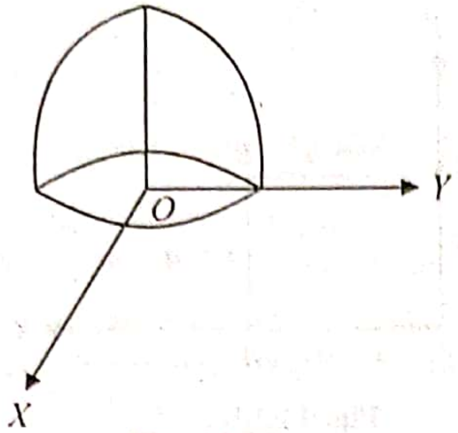

Fig. 11.15.

(In the Fig., O is the point (a, o, o), and ox, oy, oz, are the lines parallel to the co-ordinate axes).

By Stoke's theorem, the given surface integral is equal to the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

The circle C is given by

$$x = a\ (1 + \cos\theta),\quad y = a \sin\theta,\quad z = 0.$$

Along C,

$$\mathbf{F} = \left[a^2 \sin^2\theta - a^2(1+\cos\theta)^2\right]\mathbf{i} + \left[a^2(1+\cos\theta)^2 - a^2\sin^2\theta\right]\mathbf{j} + \left[a^2(1+\cos\theta)^2 + a^2\sin^2\theta\right]\mathbf{k}$$

$$\therefore \quad \mathbf{F}\cdot\left(\frac{dx}{d\theta}\mathbf{i} + \frac{dy}{d\theta}\mathbf{j} + \frac{dz}{d\theta}\mathbf{k}\right)$$

$$\therefore \quad = [a^2 \sin^2\theta - a^2\ (1 + \cos\theta)^2]\ [- a \sin\theta - a \cos\theta].$$

$$\therefore \quad \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}\cdot\frac{d\mathbf{r}}{d\theta}\,d\theta = 2a^3\pi.$$

Another method. By a further application of Stoke's theorem, we see that the given integral

$$= \int_{S_1} \text{curl } \mathbf{F}\cdot\mathbf{n}\, dS,$$

where S_1 is the plane region bounded by the circle C.

Here

$$\mathbf{n} = \mathbf{k}$$

Thus, $\qquad \text{curl } \mathbf{F} \,.\, \mathbf{n} = 2(x - y).$

$\therefore$ The integral $= 2\iint (x - y)\, dx\, dy,$

taken over S_1. Changing to polar co-ordinates, so that

$$x = a + r \cos\theta,\quad y = r \sin\theta,$$

we see that the integral

$$= 2\int_0^a \int_0^{2\pi} (a + r\cos\theta - r\sin\theta)\, r d\theta\, dr$$

$$= 2a^3\pi.$$

Example 5. *Show that*

$$\mathbf{F} = (\sin y + z)\,\mathbf{i} + (x \cos y - z)\,\mathbf{j} + (x - y)\,\mathbf{k}$$

is irrotational and find a function ϕ *such that* $\mathbf{F} = \nabla\phi$. *(Rohilkhand 2005)*

Solution. $$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \sin y + z & x \cos y - z & x - y \end{vmatrix}$$

$$= \mathbf{i}(-1+1) - \mathbf{j}(1-1) + \mathbf{k}(\cos y - \cos y) = 0.$$

$\therefore$ The given vector is irrotational and so $\mathbf{F} = \nabla\phi$. [§ 11.9]

Hence, $$(\sin y + z)\,\mathbf{i} + (x \cos y - z)\,\mathbf{j} + (x - y)\,\mathbf{k} = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

$$\therefore \quad \frac{\partial\phi}{\partial x} = \sin y + z, \text{ hence, } \phi = x \sin y + xz + f_1(y, z) \qquad ...(1)$$

$$\frac{\partial\phi}{\partial y} = x \cos y - z, \text{ hence, } \phi = x \sin y - yz + f_2(x, z) \qquad ...(2)$$

$$\frac{\partial\phi}{\partial z} = x - y, \text{ hence, } \phi = xz - yz + f_3(x, y) \qquad ...(3)$$

(1), (2) and (3) each represents ϕ. These agree if we choose $f_1(y, z) = -yz$, $f_2(x, z) = xz$ and $f_3(x, y) = x \sin y$. Hence, the required ϕ is given by

$$\phi = x \sin y + xz - yz + c,$$

c being a constant.