ANALYTIC GEOMETRY

:IFoS - 2012:

① (e) Find the equations to the lines in which the plane $2x+y-z=0$ cuts the cone $4x^2-y^2+3z^2=0$

→ The plane & the cone pass through the origin. Hence, their lines of intersection pass through the origin.

Let the equation of the line in which the given plane cuts the given cone be $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ —— ①

Then, this line is a generator to the cone & also, it lies on the given plane.

$\therefore 2l+m-n=0$ and $4l^2-m^2+3n^2=0$

$n=2l+m$. $\qquad 4l^2-m^2+3(2l+m)^2=0$

$4l^2-m^2+12l^2+3m^2+12lm=0$

$8\left(\frac{l}{m}\right)^2+6\frac{l}{m}+1=0$

$8\left(\frac{l}{m}\right)^2+4\frac{l}{m}+2\frac{l}{m}+1=0$

$\Rightarrow \frac{4l}{m}+1=0$ & $\frac{2l}{m}+1=0$

$\Rightarrow \frac{l}{-1}=\frac{m}{4}$ and $\frac{l}{-1}=\frac{m}{2}$

$n=2l+m$: (i) $m=-4l$:

$n=2l-4l$

$n=-2l$

$\frac{l}{-1}=\frac{n}{2}$

(ii) $m=-2l$:

$n=2l-2l$

$n=0$

$\therefore \frac{l}{-1}=\frac{m}{4}=\frac{n}{2}$ and $\frac{l}{-1}=\frac{m}{2}=\frac{n}{0}$.

$\therefore$ Reqd lines are: $\frac{x}{-1}=\frac{y}{4}=\frac{z}{2}$ and $\frac{x}{-1}=\frac{y}{2}=\frac{z}{0}$

② (c) Find the equations of the tangent plane to the ellipsoid $2x^2+6y^2+3z^2=27$ which pass through the line $x-y-z=0=x-y+2z-9$

→ The given ellipsoid can be written as $\frac{x^2}{27/2}+\frac{y^2}{27/6}+\frac{z^2}{27/3}=1$ —— ①

Any plane passing through the given line is:

$x-y-z+\lambda(x-y+2z-9)=0$

$\Rightarrow x(1+\lambda)+y(-1-\lambda)+z(-1+2\lambda)-9\lambda=0$ —— ②

Comparing ① with $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \Rightarrow a^2=\frac{27}{2},\ b^2=\frac{27}{6}$ and $c^2=\frac{27}{3}$

Comparing ② with $lx+my+nz=p \Rightarrow l=1+\lambda,\ m=-(1+\lambda),\ n=2\lambda-1,\ p=9\lambda$

Condn for tangency in case of ellipsoid is $p^2=a^2l^2+b^2m^2+c^2n^2$

$\Rightarrow 81\lambda^2 = 27\left[\frac{1}{2}(1+\lambda)^2+\frac{1}{6}(1+\lambda)^2+\frac{1}{3}(2\lambda-1)^2\right]$

$\Rightarrow 3\lambda^2 = \frac{2}{3}(1+\lambda^2+2\lambda)+\frac{1}{3}(4\lambda^2+1-4\lambda)$

$\Rightarrow 9\lambda^2 = 2+2\lambda^2+\cancel{4\lambda}+4\lambda^2+1-\cancel{4\lambda}$

$\Rightarrow 3\lambda^2 = 3 \Rightarrow \lambda=\pm 1$

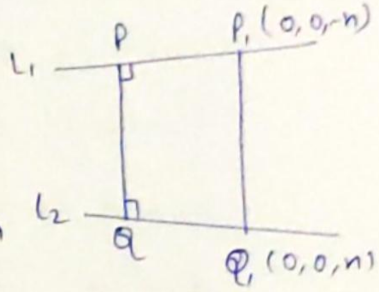
∴ Reqd tangent planes are:

$2x-2y+z-9=0$ and $z=3$

③(c) If 2C is the shortest distance between the lines $\frac{x}{l}-\frac{z}{n}=1, y=0$ and $\frac{y}{m}+\frac{z}{n}=1, x=0$, then show that $\frac{1}{l^2}+\frac{1}{m^2}+\frac{1}{n^2}=\frac{1}{c^2}$

→ $L_1 \equiv \frac{x}{l}=\frac{y}{0}=\frac{z+n}{n}$ $L_2:\ \frac{x}{0}=\frac{y}{-m}=\frac{z-n}{n}$

Let the SD line be PQ. Let PQ have dcs l_1,m_1,n_1, $PQ \perp L_1$ & $PQ \perp L_2$.



∴ $l l_1 + 0 m_1 + n n_1 = 0$

$0 l_1 - m m_1 + n n_1 = 0 \Rightarrow \frac{l_1}{mn}=\frac{m_1}{-ln}=\frac{n_1}{lm}$

∴ $\frac{l_1}{mn}=\frac{m_1}{-nl}=\frac{n_1}{lm}=\frac{1}{\sqrt{m^2n^2+n^2l^2+l^2m^2}}$

∴ $SD = PQ = 2C = \frac{1}{\sqrt{m^2n^2+n^2l^2+l^2m^2}}\left[mn(0-0)-nl(0-0)+lm(n+n)\right]$

$\Rightarrow 2C = \frac{2lmn}{\sqrt{m^2n^2+n^2l^2+l^2m^2}}$

$\Rightarrow \frac{1}{c^2} = \frac{m^2n^2+n^2l^2+l^2m^2}{l^2m^2n^2} = \frac{1}{l^2}+\frac{1}{m^2}+\frac{1}{n^2}$

$\Rightarrow \boxed{\frac{1}{l^2}+\frac{1}{m^2}+\frac{1}{n^2}=c^2}$

(1)(b) Show that all the spheres that can be drawn through the origin and each set of points where planes parallel to the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$ cut the coordinates axes form a system of spheres which are cut orthogonally by the sphere $x^2+y^2+2fx+2gy+2hz=0$ if $af+bg+ch=0$

→ Any sphere through the origin is

$x^2+y^2+z^2+2ux+2vy+2wz=0$ — ①

Plane parallel to given plane is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=k$

It cuts axes at $(ak,0,0)$, $(0,bk,0)$, $(0,0,ck)$.

The sphere ① passes through these points.

$\therefore u=-\frac{ak}{2}$, $v=-\frac{bk}{2}$, $\omega=-\frac{ck}{2}$

$\therefore$ ①: $x^2+y^2+z^2-k[ax+by+cz]=0$ — ②

The system of spheres given is $x^2+y^2+2fx+2gy+2hz=0$ — ③

Centre of sphere ② is $C_1\left(\frac{ak}{2},\frac{bk}{2},\frac{ck}{2}\right)$.

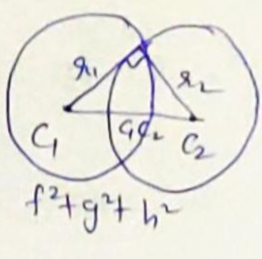
Radius of sphere ② is $r_1=\sqrt{\frac{k^2}{4}[a^2+b^2+c^2]}=\frac{k}{2}\sqrt{a^2+b^2+c^2}$

Centre of sphere ③ is $(-f,-g,-h)$

Radius of sphere ③ is $r_2=\sqrt{f^2+g^2+h^2}$

If the two systems of spheres cut orthogonally, then,

$C_1C_2^2=r_1^2+r_2^2$



$\Rightarrow \left(\frac{ak}{2}+f\right)^2+\left(\frac{bk}{2}+g\right)^2+\left(\frac{ck}{2}+h\right)^2$

$=\frac{k^2}{4}a^2+\frac{k^2}{4}b^2+\frac{k^2}{4}c^2+f^2+g^2+h^2$

$\Rightarrow akf+bkg+ckh=0$

$\Rightarrow \boxed{af+bg+ch=0}$ which is the required condition.

④(c) A plane makes equal intercepts on the positive parts of the axes and touches the ellipsoid $x^2+4y^2+9z^2=36$. Find its eq^n.

$\frac{x^2}{36}+\frac{4y^2}{36}+\frac{9z^2}{36}=1$

→ Given ellipsoid: $\frac{x^2}{(6)^2}+\frac{y^2}{(3)^2}+\frac{z^2}{2^2}=1$ — ①

comparing with $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, we have $a^2=36$, $b^2=9$, $c^2=4$

Any plane that makes equal intercepts on the +ve side of axes is $\frac{x}{a_1}+\frac{y}{a_1}+\frac{z}{a_1}=1 \Rightarrow x+y+z=a_1$.

Comparing with $lx+my+nz=p$, $l=1, m=1, n=1, p=a_1$ ②

The plane ② is tangent plane to the ellipsoid ① the condition for which is

$p^2=a^2l^2+b^2m^2+c^2n^2 \Rightarrow a_1^2=36[1]+9[1]+4[1]$

$\Rightarrow a_1^2=49 \Rightarrow a_1=\pm 7$

since a_1 is +ve as the plane cuts axes on the +ve portion

$a_1=7$.

∴ Reqd plane is: $x+y+z=7$