ANALYTIC GEOMETRY

: CSE - 2017 :

①(d) Find the equation of the tangent plane at point (1,1,1) to the conicoid $3x^2 - y^2 = 2z$

→ Any line through (1,1,1) is $\frac{x-1}{l} = \frac{y-1}{m} = \frac{z-1}{n} = r$ (say) — ①

Any point on this line is $(1+lr, 1+mr, 1+nr)$.

This point lies on the given conicoid. Then,

$3(1+lr)^2 - (1+mr)^2 = 2(1+nr)$

$\Rightarrow (3l^2 - m^2)r^2 + 2r(3l - m - n) = 0$.

$\Rightarrow r[(3l^2 - m^2)r + 2(3l - m - n)] = 0$

For this line ① to be tangent to the conicoid, both 'r' values are same. Since one value of r is zero, the other is also zero.

$\therefore 3l - m - n = 0$ — ②

Reqd. tangent plane is obtained by eliminating l, m, n among ① & ②

$3(x-1) - (y-1) - (z-1) = 0$

$3x - 3 - y + 1 - z + 1 = 0 \Rightarrow 3x - y - z = 1$.

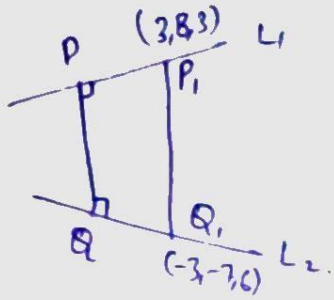
①(e) Find the shortest distance between the skew lines

$\frac{x-3}{3} = \frac{8-y}{1} = \frac{z-3}{1}$ and $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$.

→ Given lines are: $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$ — ① and $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$ — ②

Let the S.D. line be PQ. Then,

$PQ \perp L_1$ and $PQ \perp L_2$. Let l, m, n be drs of PQ. Then, it is ⊥ar to L_1 & L_2 whose drs are 3, -1, 1 and -3, 2, 4.



$\therefore 3l - m + n = 0$ $[l_1 l_2 + m_1 m_2 + n_1 n_2 = 0]$

$-3l + 2m + 4n = 0$

By cross-multiplication: $\frac{l}{-6} = \frac{m}{-15} = \frac{n}{3} \Rightarrow \frac{l}{2} = \frac{m}{5} = \frac{n}{-1} = \frac{1}{\sqrt{30}}$

$\Rightarrow l = \frac{2}{\sqrt{30}}, \quad m = \frac{5}{\sqrt{30}}, \quad n = \frac{-1}{\sqrt{30}}$

Required shortest distance is the projection of line joining P_1 and Q_1 on the line whose dcs are $l = \frac{2}{\sqrt{30}}$, $m = \frac{5}{\sqrt{30}}$, $n = \frac{-1}{\sqrt{30}}$.

$\therefore$ S.D $= l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$

$= \frac{2}{\sqrt{30}}(3+3) + \frac{5}{\sqrt{30}}(8+7) - \frac{1}{\sqrt{30}}(3-6) = 3\sqrt{30}$ units

2(c) A plane passes through a fixed point (a,b,c) and cuts the coordinates axes at A, B and C respectively. Find the locus of centres of spheres which passes through the origin and A, B, C

$\rightarrow$ Let $A(p,0,0)$, $B(0,q,0)$ and $R(0,0,r)$. Then, let the equation of sphere passing through the origin $O(0,0,0)$, and A, B, C be

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$ —— ①

It passes through the origin: $d=0$.

Now: It passes through $A(p,0,0)$: $p^2+2up=0 \Rightarrow u = -\frac{p}{2}$

Sly: $v = -\frac{q}{2}$, $w = -\frac{r}{2}$. Therefore, the sphere through O, A, B and C is $x^2+y^2+z^2-px-qy-rz=0$ —— ②

The centre of this sphere is $\left(\frac{p}{2}, \frac{q}{2}, \frac{r}{2}\right)$ —— ③

An equation of plane through A, B, C is $\frac{x}{p}+\frac{y}{q}+\frac{z}{r}=1$ —— ④

It passes through $(a,b,c) \Rightarrow \frac{a}{p}+\frac{b}{q}+\frac{c}{r}=1$

Multiplying both sides by 2 $\Rightarrow \frac{2a}{p}+\frac{2b}{q}+\frac{2c}{r}=2$

$\Rightarrow \frac{a}{(p/2)} + \frac{b}{(q/2)} + \frac{c}{(r/2)} = 2.$

$\therefore$ Reqd locus of centre $(p/2, q/2, r/2)$ is

$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2$

2(c) Show that the plane $2x-2y+z+12=0$ touches the sphere $x^2+y^2+z^2-2x-4y+2z-3=0$. Find the point of contact

$\rightarrow$ Tangent plane to the given sphere at any point (α,β,γ) is

$$\alpha x+\beta y+\gamma z-(x+\alpha)-2(y+\beta)+(z+\gamma)-3=0$$

$$\Rightarrow (\alpha-1)x+(\beta-2)y+(\gamma+1)z-(\alpha+2\beta-\gamma+3)=0 \quad \text{— ①}$$

If the given plane $\underbrace{2x-2y+z+12=0}_{②}$ is tangent plane at (α,β,γ) to the sphere, then, ① & ② are the same. Therefore,

$$\frac{\alpha-1}{2}=\frac{\beta-2}{-2}=\frac{\gamma+1}{1}=\frac{-(\alpha+2\beta-\gamma+3)}{12}$$

$\Rightarrow$ $12\alpha-12=-2\alpha-4\beta+2\gamma-6$, $12\beta-24=2\alpha+4\beta-2\gamma+6$

and $12\gamma+12=-\alpha-2\beta+\gamma-3$

$$\Rightarrow \left.\begin{array}{l} 14\alpha+4\beta-2\gamma-6=0 \\ 2\alpha-8\beta-2\gamma+30=0 \\ \alpha+2\beta+11\gamma+15=0 \end{array}\right\} \Rightarrow \alpha=-1,\ \beta=4,\ \gamma=-2$$

Putting $(\alpha,\beta,\gamma)\equiv(-1,4,-2)$ in the given sphere,

$$(-1)^2+4^2+(-2)^2-2(-1)-4(4)+2(-2)-3$$

$$\Rightarrow 1+16+4+2-16-4-3=0.$$

$\therefore$ The point $(-1,4,-2)$ satisfies the sphere.

$\therefore$ The given plane is tangent plane to the sphere and touches the sphere at the point $\underline{\underline{(-1,4,-2)}}$.

3(d) Find the locus of the point of intersection of three mutually perpendicular tangent planes to the conicoid $ax^2+by^2+cz^2=1$.

Let the tangent plane to the given conicoid be $lx+my+nz=p$ — ①

The condition for tangency is $p^2 = \frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c} \Rightarrow p = \pm\sqrt{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}}$

Then, let the three ⊥ar tangent planes be given by

$l_1x+m_1y+n_1z = \sqrt{\frac{l_1^2}{a}+\frac{m_1^2}{b}+\frac{n_1^2}{c}}$ — ②, $l_2x+m_2y+n_2z = \sqrt{\frac{l_2^2}{a}+\frac{m_2^2}{b}+\frac{n_2^2}{c}}$ — ③

and $l_3x+m_3y+n_3z = \sqrt{\frac{l_3^2}{a}+\frac{m_3^2}{b}+\frac{n_3^2}{c}}$ — ④

Squaring and adding ② ③ & ④

$$(l_1x+m_1y+n_1z)^2 + (l_2x+m_2y+n_2z)^2 + (l_3x+m_3y+n_3z)^2$$

$$= \frac{l_1^2}{a}+\frac{m_1^2}{b}+\frac{n_1^2}{c}+\frac{l_2^2}{a}+\frac{m_2^2}{b}+\frac{n_2^2}{c}+\frac{l_3^2}{a}+\frac{m_3^2}{b}+\frac{n_3^2}{c}$$

$$\Rightarrow x^2\Sigma l_1^2 + y^2\Sigma m_1^2 + z^2\Sigma n_1^2 + 2xy\,\Sigma l_1m_1 + 2yz\,\Sigma m_1n_1 + 2xz\,\Sigma n_1l_1$$

$$= \frac{\Sigma l_1^2}{a} + \frac{\Sigma m_1^2}{b} + \frac{\Sigma n_1^2}{b} \quad \text{—— ⑤}$$

where $\Sigma l_1^2 = l_1^2+l_2^2+l_3^2$, $\Sigma m_1^2 = m_1^2+m_2^2+m_3^2$, $\Sigma n_1^2 = n_1^2+n_2^2+n_3^2$

$\Sigma l_1m_1 = l_1m_1+l_2m_2+l_3m_3$, $\Sigma m_1n_1 = m_1n_1+m_2n_2+m_3n_3$

$\Sigma n_1l_1 = n_1l_1+n_2l_2+n_3l_3$.

Since the three tangent planes are ⊥ar, then

$$\Sigma l_1^2 = \Sigma m_1^2 = \Sigma n_1^2 = 1 \ \& \ \Sigma l_1m_1 = \Sigma m_1n_1 = \Sigma n_1l_1 = 0$$

∴ from ⑤ :

$$x^2+y^2+z^2 = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$ which is the required locus