Mains Test Series - 2018

Test - 11 (Paper - I)

Answer Key

1(a). Let W be the subspace of R^4 generated by vectors $(1,-2,5,-3)$, $(2,3,1,-4)$ and $(3,8,-3,-5)$. Find a basis and dimension of W. Extend this basis of W to a basis of R^4.

Sol^n: Form the matrix A whose rows are the given vectors and row reduce A to an echelon form.

$$A = \begin{bmatrix} 1 & -2 & 5 & -3 \\ 2 & 3 & 1 & -4 \\ 3 & 8 & -3 & -5 \end{bmatrix}.$$

$$\sim \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 14 & -18 & 4 \end{bmatrix} \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & -2 & 5 & -3 \\ 0 & 7 & -9 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 - 2R_1$$

The non-zero rows $(1,-2,5,-3)$, $(0,7,-9,2)$ of the echelon matrix form a basis of the row space of A which is W.

$\therefore \dim W = 2$

We seek four independent vectors which include the above two vectors. The vectors $(1,-2,5,-3)$, $(0,7,-9,2)$, $(0,0,1,0)$, $(0,0,0,1)$ are independent. [Since they form an echelon matrix), and so they form a basis of $\mathbb{R}^4$ which is an extension of the basis of W.

1(b) Find the characteristic equation of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and hence, find the matrix represented by $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$.

Sol^n: The characteristic equation of the matrix A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)\left[(1-\lambda)(2-\lambda)\right] - 1(0-0) + 1\left[0-(1-\lambda)\right] = 0$$

$$\Rightarrow (1-\lambda)\left[(2-\lambda)^2 - 1\right] = 0$$

$$\Rightarrow (1-\lambda)\left[\lambda^2 - 4\lambda + 3\right] = 0$$

$$\Rightarrow \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$$

By Cayley-Hamilton theorem, the matrix A must satisfy its characteristic equation. so we must have

$$A^3 - 5A^2 + 7A - 3I = 0 \quad \text{——} \quad (1)$$

To evaluate

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$$

$$= A^5(A^3 - 5A^2 + 7A - 3I) + A(A^3 - 5A^2 + 7A - 3I) + A^2 + A + I$$

$$= A^5(0) + A(0) + A^2 + A + I \quad (\because \text{from } (1))$$

$$= A^2 + A + I$$

$$\therefore A^2 + A + I = \begin{bmatrix} 5 & 4 & 4 \\ 0 & 1 & 0 \\ 4 & 4 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

which is the required matrix.

1(c) Examine the convergence of the integrals

(i) $\int_1^2 \frac{dx}{(1+x)\sqrt{2-x}}$ (ii) $\int_0^\infty \frac{x^2}{\sqrt{x^5+1}}\,dx$

Sol^n: (i) Here $f(x) = \frac{1}{(1+x)\sqrt{2-x}}$

2 is the only point of infinite discontinuity of f on $[1,2]$

Take $g(x) = \frac{1}{\sqrt{2-x}}$, then

$$\lim_{x\to 2-} \frac{f(x)}{g(x)} = \lim_{x\to 2-} \frac{1}{1+x} = \frac{1}{3}$$

which is non zero and finite.

$\therefore$ By Comparison test, $\int_1^2 f(x)\,dx$ and $\int_1^2 g(x)\,dx$ converge or diverge together.

But $\int_1^2 g(x)\,dx = \int_1^2 \frac{dx}{\sqrt{2-x}}$ {form $\int_a^b \frac{dx}{(b-x)^n}$ with $b=2$ converges. ($\because n=\frac{1}{2}<1$)

$\therefore \int_1^2 f(x)\,dx = \int_1^2 \frac{dx}{(1+x)\sqrt{2-x}}$ is convergent.

(ii) $$\int_0^\infty \frac{x^2}{\sqrt{x^5+1}}\,dx = \int_0^1 \frac{x^2}{\sqrt{x^5+1}}\,dx + \int_1^\infty \frac{x^2}{\sqrt{x^5+1}}\,dx \quad \text{——— (1)}$$

The first integral on the right is a proper integral and therefore convergent.

Let $f(x) = \frac{x^2}{\sqrt{x^5+1}} = \frac{x^2}{x^{5/2}\sqrt{1+\frac{1}{x^5}}} = \frac{1}{\sqrt{1+\frac{1}{x^5}}}$

Take $g(x) = \frac{1}{x^{1/2}}$ etc.

The second integral on the right is divergent.

$\therefore$ from (1), $\int_0^\infty \frac{x^2}{\sqrt{x^5+1}}\,dx$ is divergent.

1(d) Show that the function $f(x,y) = \begin{cases} x^2y/(x^2+y^2), & \text{when } x^2+y^2 \neq 0 \\ 0, & \text{when } x^2+y^2 = 0 \end{cases}$

is continuous but not differentiable at (0,0)

Sol'n – Putting $x = r\cos\theta$, $y = r\sin\theta$; we get

$$|f(x,y) - f(0,0)| = \left|\frac{r^2\cos^2\theta \, r\sin\theta}{r^2} - 0\right|$$

$$= r|\cos\theta|\,|\cos\theta|\,|\sin\theta|$$

$$\leq r = \sqrt{x^2+y^2}$$

Let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon$. Then

$|f(x,y) - f(0,0)| < \varepsilon$ if $\sqrt{x^2+y^2} < \delta$

Hence f is continuous at the origin.

Now $f_x(0,0) = \lim_{h\to 0} [f(h,0) - f(0,0)]/h$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0.$$

Similarly $f_y(0,0) = 0$.

Let, if possible, f be differentiable at (0,0) then

$f(h,k) - f(0,0) = Ah + Bk + \sqrt{h^2+k^2}\, g(h,k)$

where $A = f_x(0,0)$, $B = f_y(0,0)$ and $g(h,k) \to 0$ as $(h,k) \to (0,0)$ —— ①

$$\therefore \frac{h^2k}{h^2+k^2} = \sqrt{h^2+k^2}\, g(h,k) \Rightarrow g(h,k) = \frac{h^2k}{(h^2+k^2)^{3/2}}$$

Now $\lim_{h\to 0} g(h, mh) = \frac{m}{(1+m^2)^{3/2}}$ $(k = mh)$

$\therefore \lim_{(h,k)\to(0,0)} g(h,k) = \frac{m}{(1+m^2)^{3/2}}$, which depends on m and

so the limit does not exist. This contradicts ①. Hence f is not differentiable at (0,0).

1(e) A sphere S has points $(0,1,0)$, $(3,-5,2)$ at opposite ends of diameter. Find the eqn of the sphere having the intersection of the sphere S with the plane $5x-2y+4z+7=0$ as a great circle.

Sol^n:- Equation of the sphere S on joining the two given points $(0,1,0)$, $(3,-5,2)$ is given by

$$S=(x-0)(x-3)+(y-1)(y+5)+(z-0)(z-2)=0$$

$$\Rightarrow x^2+y^2+z^2-3x+4y-2z-5=0$$

Equation of any sphere having the intersection of the sphere S with the plane $P \equiv 5x-2y+4z+7=0$ is $S+\lambda P=0$

i.e, $(x^2+y^2+z^2-3x+4y-2z-5)+\lambda(5x-2y+4z+7)=0$ —— (1)

$$\Rightarrow x^2+y^2+z^2+(-3+5\lambda)x+(4-2\lambda)y+(-2+4\lambda)z$$
$$+(-5+7\lambda)=0$$

Its centre is $\left(\frac{3-5\lambda}{2}, \frac{2\lambda-4}{2}, \frac{2-4\lambda}{2}\right)$ —— (2)

Now if the given circle (which is the section of the sphere $x^2+y^2+z^2-3x+4y-2z-5$ by the plane $5x-2y+4z+7=0$) is a great circle of the sphere (1), the centre of the sphere (1) must lie on the plane

$5x-2y+4z+7=0.$

$\Rightarrow 5\left(\frac{3-5\lambda}{2}\right)-2\left(\frac{2\lambda-4}{2}\right)+4\left(\frac{2-4\lambda}{2}\right)+7=0$

$\Rightarrow 15-25\lambda-4\lambda+8+8-16\lambda+14=0.$

$\Rightarrow 45-45\lambda=0$

$\Rightarrow \boxed{\lambda=1}.$

$\therefore$ From (1), we get

$x^2+y^2+z^2-3x+4y-2z-5+(5x-2y+4z+7)=0$

$\Rightarrow x^2+y^2+z^2+2x+2y+2z+2=0$

which is the required equation of the sphere

2(a) Show that the vectors $x_1 = (1, 1+i, i)$, $x_2 = (i, -i, 1-i)$ and $x_3 = (0, 1-2i, 2-i)$ in $\mathbb{C}^3$ are linearly independent over the field of real numbers but are linearly dependent over the field of complex numbers.

Sol^n: Let $S = \{(1, 1+i, i), (i, -i, 1-i), (0, 1-2i, 2-i)\}$

The set S is linearly independent over the field of real numbers, since for any $\alpha, \beta, \gamma \in \mathbb{R}$.

$\alpha(1, 1+i, i) + \beta(i, -i, 1-i) + \gamma(0, 1-2i, 2-i) = 0 + i0$

$\Rightarrow \alpha = 0, \beta = 0.$

However, the set S is linearly dependent over the field of complex numbers, since

$\alpha(1, 1+i, i) + \beta(i, -i, 1-i) + \gamma(0, 1-2i, 2-i) = 0 + i0$

$\Rightarrow \alpha = -i, \beta = 1, \gamma = -1$

①

2(b) Show that the transformation $T(ax^2+bx+c) = 2ax+b$ of $P_2 \to P_1$ is linear. Find the image of $3x^2-2x+1$. Determine another element of P_2 that has the same image.

Soln: Let $T : P_2 \to P_1$ defined by

$$T(ax^2+bx+c) = 2ax+b \quad \text{—(1)}$$

To show that T is L.T:

Let $\alpha, \beta \in P_2$ such that

$$\alpha = a_1x^2+b_1x+c_1 \; ; \; \beta = a_2x^2+b_2x+c_2.$$

Let $a, b \in F$

then we have

$$a\alpha+b\beta = (aa_1+ba_2)x^2+(ab_1+bb_2)x + ac_1+bc_2 \in P_2$$

($\because P_2$ is a vector space over F)

$$\therefore T(a\alpha+b\beta) = 2[aa_1+ba_2]x+(ab_1+bb_2)$$

$$= a[2a_1x+b_1]+b[2a_2x+b_2]$$

$$= aT(\alpha)+bT(\beta)$$

$\therefore T : P_2 \to P_1$ is a L.T

To find the image of $3x^2-x+2$:

$$T(3x^2-2x+1) = 2(3)x+(-2) = 6x-2 \quad \text{(by (1))}$$

Let ax^2+bx+c be any other element in P_2 having the image $6x-2$.

then $T(ax^2+bx+c) = 6x-2$

$\Rightarrow 2ax+b = 6x-2$.

$\Rightarrow a=3, \; b=-2$.

$\therefore$ other element is of the type $3x^2-2x+c$ having the same image; where $c \in \mathbb{R}$.

2(c) Let P_n denote the vector space of all real polynomials of degree atmost n and $T: P_2 \to P_3$ be a linear transformation given by $T(p(x)) = \int_0^x p(t)\,dt$, $p(x) \in P_2$. Find the matrix of T with respect to the bases $\{1, x, x^2\}$ and $\{1, x, 1+x^2, 1+x^3\}$ of P_2 and P_3 respectively. Also find the null space of T.

Sol'n: P_n: vector space of all polynomials of degree $\le n$.

Let $T: P_2 \to P_3$ be linear transformation given by $T(p(x)) = \int_0^x p(t)\,dt$, $p(x) \in P_2$.

$T(1) = \int_0^x 1\,dt = x = 0\cdot 1 + 1\cdot x + 0\cdot(1+x^2) + 0\cdot(1+x^3)$

$T(x) = \int_0^x t\,dt = \frac{x^2}{2} = -\frac{1}{2}\cdot 1 + 0\cdot x + \frac{1}{2}(1+x^2) + 0(1+x^3)$

$T(x^2) = \int_0^x t^2 dt = \frac{x^3}{3} = -\frac{1}{3}\cdot 1 + 0\cdot x + 0\cdot(1+x^2) + \frac{1}{3}(1+x^3)$

$\therefore$ Matrix of T w.r.t bases B_1 and B_2 is

$$[T : B_1, B_2] = \begin{bmatrix} 0 & -\frac{1}{2} & -\frac{1}{3} \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

Null space of T will be given by

$\int_0^x p(t)\,dt = 0$ i.e., if $p(x) = a_0 + a_1 x + a_2 x^2$.

$$\begin{bmatrix} 0 & -\frac{1}{2} & -\frac{1}{3} \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow a_0 = 0$; $a_1 = 0$; $a_2 = 0$

$a_0 = a_1 = a_2 = 0$

$\therefore p(x) = 0$

Null space of T contains only a single element $\{0\}$.

3(a)(i). If $\phi(x) = f(x) + f(1-x)$ and $f''(x) \le 0 \ \forall\, x \in [0,1]$ show that ϕ increases in $[0,\frac{1}{2}]$ and decreases in $[\frac{1}{2},1]$, Hence or otherwise Prove that $\pi < \frac{\sin \pi x}{x(1-x)} \le 4$ when $0 < x < 1$.

Sol^n. we have $\phi'(x) = f'(x) + f'(1-x)(-1)$

and $\phi''(x) = f''(x) + f''(1-x)$

Now as x varies from 0 to 1, $1-x$ varies from 1 to 0 so that $f''(1-x)$ is also negative $\forall x \in [0,1]$.

Hence $\phi''(x)$ is negative in $[0,1]$ and

Consequently $\phi'(x)$ is monotonic decreasing in $[0,1]$.

Again $\phi'(0) = f'(0) - f'(1)$, $\phi'(\frac{1}{2}) = 0$

and $\phi'(1) = f'(1) - f'(0)$

Since $f''(x) < 0 \ \forall x \in [0,1]$,

therefore $f'(x)$ is monotonic decreasing in $[0,1]$ and so $f'(0) > f'(1)$

$\therefore \phi'(0) > 0, \ \phi'(\frac{1}{2}) = 0, \ \phi'(1) < 0$.

But $\phi'(x)$ is monotonic decreasing in $[0,1]$

$\therefore \phi'(x)$ is positive in $[0, \frac{1}{2})$ and negative in $(\frac{1}{2}, 1]$.

$\therefore \phi(x)$ is monotonic increasing in $[0, \frac{1}{2}]$ and decreasing in $[\frac{1}{2}, 1]$ there being a maximum at $x = \frac{1}{2}$.

Now let $\phi(x) = \dfrac{\sin \pi x}{x(1-x)}$

$= \dfrac{2 \sin \pi x/_2 \ \cos \pi x/_2}{x(1-x)}$

$$= 2\,\frac{\sin\frac{\pi x}{2}}{x}\;\frac{\sin\frac{1}{2}\pi(1-x)}{1-x}$$

Then $\log\left[\frac{1}{2}\phi(x)\right] = f(x) + f(1-x)$

where $f(x) = \log\left(\sin\frac{\pi x}{2}\right) - \log x$.

Since $f''(x) = -\frac{1}{4}\pi^2 \operatorname{cosec}^2 \frac{\pi}{2}x + x^{-2}$

$$= \frac{-\frac{1}{4}\pi^2 x^2 + \sin^2\frac{\pi}{2}x}{x^2 \sin^2\frac{\pi x}{2}} < 0$$

when $0<x<1$

we deduce that $\log\left[\frac{1}{2}\phi(x)\right]$ or $\phi(x)$ increases in $\left(0,\frac{1}{2}\right]$ and decreases in $\left[\frac{1}{2},1\right)$.

But $\phi(x) \to \pi$ as $x \to 0+$ or $x \to 1-$

and $\phi\left(\frac{1}{2}\right) = 4$

$\therefore \pi < \phi(x) \le 4$ in $\left(0,\frac{1}{2}\right]$ and

$4 \geq \phi(x) > \pi$ in $\left[\frac{1}{2},1\right)$

Hence $\pi < \frac{\sin \pi x}{x(1-x)} \le 4$ when $0<x<1$

3(ii) Show that $\int_0^\infty \log\left(x+\frac{1}{x}\right)\frac{dx}{1+x^2} = \pi\log 2$

Sol^n: Let $x=\tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

$$\therefore I = \int_0^\infty \log\left(x+\frac{1}{x}\right)\frac{dx}{1+x^2} = \int_0^{\pi/2}\log\left(\tan\theta+\frac{1}{\tan\theta}\right)\frac{\sec^2\theta\, d\theta}{1+\tan^2\theta}$$

$$= \int_0^{\pi/2}\log\left(\frac{\tan^2\theta+1}{\tan\theta}\right)d\theta$$

$$= \int_0^{\pi/2}\log\left(\frac{\sec^2\theta}{\tan\theta}\right)d\theta$$

$$= \int_0^{\pi/2}\log\left(\frac{1}{\sin\theta\cos\theta}\right)d\theta$$

$$= -\int_0^{\pi/2}\log\sin\theta\, d\theta - \int_0^{\pi/2}\log\cos\theta\, d\theta$$

$$= -\left(-\frac{\pi}{2}\log 2\right) - \left[-\frac{\pi}{2}\log 2\right]$$

$$= \frac{\pi}{2}\log 2 + \frac{\pi}{2}\log 2 \qquad \left(\because \int_0^{\pi/2}\log\sin x = \int_0^{\pi/2}\log\cos x = -\frac{\pi}{2}\log 2\right)$$

$$= \pi\log 2$$

3(b). A rectangular box, open at the top, is to have a volume of 32 cubic feet, what ~~is the~~ must be the dimensions so that the total surface is a minimum?

Sol. Let x, y and z ft. be the edges of the box and S be its surface.

Then $S = xy + 2yz + 2zx$ —— (1)

and $xyz = 32$ —— (2)

Let us consider a function F of independent variables x, y, z

where $F = (xy + 2yz + 2zx) + \lambda(xyz - 32)$

$\therefore dF = (y + 2z + \lambda yz)dx + (x + 2z + \lambda xz)dy + (2y + 2x + \lambda xy)$ $(\because dF = f_x dx + f_y dy + f_z dz)$

At stationary points, $dF = 0$

$f_x = 0 \Rightarrow y + 2z + \lambda yz = 0$ —— (3)

$f_y = 0 \Rightarrow x + 2z + \lambda xz = 0$ —— (4)

$f_z = 0 \Rightarrow 2y + 2x + \lambda xy = 0$ —— (5)

multiplying (3) by x and (4) by y and subtracting, we get

$2zx - 2zy = 0$

$\Rightarrow 2z(x - y) = 0$

$\Rightarrow z = 0$ (or) $x = y$

The value $z = 0$ is neglected, as it will not satisfy equation (2).

$\therefore \boxed{x = y}$ —— (6)

Again multiplying (4) by y and (5) by z and subtracting, we get $\boxed{y = 2z}$ — (7)

From (6) & (7),

$$x = y = 2z$$

from (2), $xyz = 32$

$\Rightarrow x(x)\left(\frac{x}{2}\right) = 32$ $\quad (\because x = y \ \& \ x = 2z)$

$\Rightarrow x^3 = 64$

$\Rightarrow x = 4.$

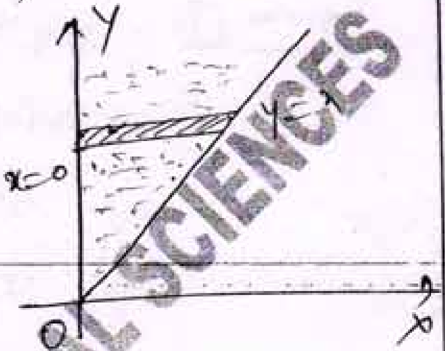
$\therefore x = y = 4$ and $z = 2.$

$\therefore$ The dimensions of the box are

$x = y = 4$ ft and $z = 2$ ft.

3(c) Let $E=\{(x,y)\in\mathbb{R}^2 / 0<x<y\}$. Then evaluate $\iint_E y e^{-(x+y)}\,dx\,dy$.

Sol: Let $I=\iint_E y e^{-(x+y)}\,dx\,dy$ where $E=\{(x,y)\in\mathbb{R}^2/0<x<y\}$



$$=\int_{y=0}^{\infty}\int_{x=0}^{y} y e^{-(x+y)}\,dx\,dy$$

$$=\int_{y=0}^{\infty} y\left(\frac{e^{-x}}{-1}\right)_0^y e^{-y}\,dy$$

$$=\int_{y=0}^{\infty} y e^{-y}\left(e^{-x}\right)_y^0\,dy$$

$$=\int_0^{\infty} y e^{-y}\left(1-e^{-y}\right)dy$$

$$=\int_0^{\infty}\left(ye^{-y}-ye^{-2y}\right)dy$$

$$=\int_0^{\infty} e^{-y}y^{2-1}\,dy-\int_0^{\infty}e^{-2y}y^{2-1}\,dy \qquad \left(\because \int_0^{\infty} e^{-ax}x^{n-1}\,dx=\frac{\Gamma n}{a^n}\right)$$

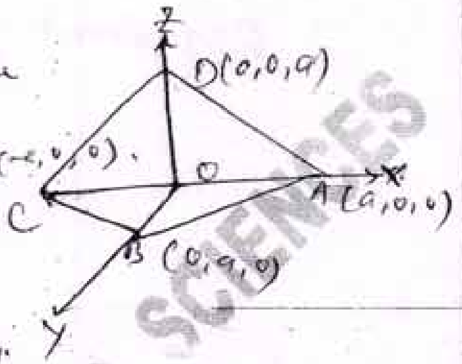
$$=\frac{\Gamma 2}{1^2}-\frac{\Gamma 2}{2^2}$$

$$=1-\frac{1}{4} \qquad \left(\because \Gamma(n+1)=n!\right)$$

$$=\frac{3}{4}$$

4(a) A square ABCD of diagonal 2a is folded along the diagonal AC so that the planes DAC, BAC are at right angles. find the S.D. between DC and AB.

Sol^n: Let O, the centre of the square, be taken as the origin and OA, OB and OD be taken as x, y and z-axes respectively.



Then the co-ordinates of A, B, C and D are (0,0,0), (0,a,0), (a,0,0) and (0,0,a) respectively.

Therefore equations of AB are

$$\frac{x-a}{a} = \frac{y-0}{-a} = \frac{z-0}{0} \quad \text{——— (1)}$$

Equations of DC are $\frac{x-0}{a} = \frac{y-0}{0} = \frac{z-a}{a}$ — (2)

Now any point on the line DC is (0,0,a)

And the equation of the plane through the line AB and parallel to DC

i.e., through (1) and parallel to (2) is

$$\begin{vmatrix} x-a & y-0 & z-0 \\ a & -a & 0 \\ a & 0 & a \end{vmatrix} = 0$$

$\Rightarrow (x-a)(-a)(a) - y(a\cdot a) + z(a\cdot a) = 0$

$\Rightarrow -(x-a) - y + z = 0$

$\Rightarrow \quad x+y-z-a=0$ —— (3)

$\therefore$ Required S.D = length of perpendicular from $(0,0,a)$ to the plane (3)

$$= \left|\frac{0+0-a-a}{\sqrt{1^2+1^2+(-1)^2}}\right|$$

$$= \frac{2a}{\sqrt{3}}$$

4(b). Find the equation of the right circular cylinder which passes through the circle $x^2+y^2+z^2=9$, $x-y+z=3$

Sol^n: The direction ratios of the axis of the cylinder, which is perpendicular to the plane of the circle given by $x-y+z=3$ are $1,-1,1$.

So let one of the generators of the cylinder passing through any point (α,β,γ) on the cylinder be

$$\frac{x-\alpha}{1} = \frac{y-\beta}{-1} = \frac{z-\gamma}{1}.$$

Any point on this generator at a distance r from (α, β, γ) is $(\alpha+r, \beta+r, \gamma+r)$.

If this point on the given circle, then we have

$$(\alpha+r)^2+(\beta-r)^2+(\gamma+r)^2=9,$$
$$(\alpha+r)-(\beta-r)+(\gamma+r)=3$$

$$\Rightarrow \alpha^2+\beta^2+\gamma^2+2r(\alpha-\beta+\gamma)+3r^2=9,$$
$$\alpha-\beta+\gamma+3r=3$$

Eliminating r we get

$$\alpha^2+\beta^2+\gamma^2+2(\alpha-\beta+\gamma)\left(\frac{3-\alpha+\beta-\gamma}{3}\right)$$
$$+3\left[\frac{1}{3}(3-\alpha+\beta-\gamma)\right]^2=9$$

$$\Rightarrow 3(\alpha^2+\beta^2+\gamma^2)+2(\alpha-\beta+\gamma)(3-\alpha+\beta-\gamma)$$
$$+(3-\alpha+\beta-\gamma)^2=27$$

$$\Rightarrow 3(\alpha^2+\beta^2+\gamma^2)+(3-\alpha+\beta-\gamma)(3+\alpha-\beta+\gamma)=27$$

$$\Rightarrow 3(\alpha^2+\beta^2+\gamma^2)+9-(\alpha-\beta+\gamma)^2=27$$

$$\Rightarrow \alpha^2+\beta^2+\gamma^2+\alpha\beta-\alpha\gamma+\beta\gamma-9=0.$$

$\therefore$ The equation of the cylinder i.e., the locus of $P(x_1, y_1, z_1)$ is

$$x^2+y^2+z^2+xy-xz+yz-9=0$$

4(c) Show that the plane $2x-4y-z+3=0$ touches the paraboloid $x^2-2y^2=3z$ and find the point of contact.

Sol: Let the plane $2x-4y-z=-3$ — ①

touch the paraboloid $x^2-2y^2=3z$ — ②

at the point (α,β,γ).

The equation of the tangent plane to ② at (α,β,γ) is

$$\alpha x - 2\beta y = \frac{3}{2}(z+\gamma)$$

$$\Rightarrow 2\alpha x - 4\beta y - 3z = 3\gamma \quad \text{—} ③$$

If the plane ① touches ② at (α,β,γ), then ① and ③ represent the same plane, and so comparing ① and ③, we get $\frac{2\alpha}{2} = \frac{-4\beta}{-4} = \frac{-3}{-1} = \frac{3\gamma}{-3}$.

which gives $\alpha=3,\ \beta=3,\ \gamma=-3$ — ④

Also (α,β,γ) lies on ②, $\alpha^2-2\beta^2=3\gamma$ — ⑤

Since values of α,β,γ given by ④ satisfy ⑤, so the plane ① touches the paraboloid ② at (α,β,γ).

Also from ④, the co-ordinates of the point of contact are $(3,3,-3)$.

4(d) Find the equations to the generating lines of the hyperboloid $\frac{x^2}{4}+\frac{y^2}{9}-\frac{z^2}{16}=1$ which pass through the points $(2,3,-4)$ and $(2,-1,\frac{4}{3})$

Sol^n: Any line through $(2,3,-4)$ and $(2,-1,4/3)$

is $\frac{x-2}{l} = \frac{y-3}{m} = \frac{z+4}{n} = r$ (say) —— ①

Any point on this line is

$(lr+2, mr+3, nr-4)$ and it lies

on the given hyperboloid if

$$\frac{(lr+2)^2}{4} + \frac{(mr+3)^2}{9} - \frac{(nr-4)^2}{16} = 1$$

$$r^2\left(\frac{l^2}{4} + \frac{m^2}{9} - \frac{n^2}{16}\right) + 2r\left(\frac{2l}{4} + \frac{3m}{9} + \frac{4n}{16}\right) = 0 \quad \text{—— ②}$$

If the line ① is a generator of the given hyperboloid, then ① lies wholly on the hyperboloid and the conditions for which from ② are

$$\frac{l^2}{4} + \frac{m^2}{9} - \frac{n^2}{16} = 0 \quad \text{and} \quad \frac{2l}{4} + \frac{3m}{9} + \frac{4n}{16} = 0$$

$$\Rightarrow \frac{l^2}{4} + \frac{m^2}{9} - \frac{n^2}{16} = 0 \quad \text{and} \quad \frac{l}{2} + \frac{m}{3} = -\frac{n}{4} \quad \text{—— ③}$$

Eliminating n, we get

$$\frac{l^2}{4} + \frac{m^2}{9} - \left(\frac{l}{2} + \frac{m}{3}\right)^2 = 0$$

$$\Rightarrow -\frac{1}{3} lm = 0$$

$\Rightarrow$ either $l = 0$ or $m = 0$

when $l = 0$, from ③, we get $\frac{m}{3} = -\frac{n}{4}$

when $m = 0$, from ③, we get $\frac{l}{2} = -n/4$

$\Rightarrow \frac{l}{1} = -\frac{n}{2}$

Hence from (1), equations of the required generator through $(2,3,-4)$

are $\frac{x-2}{0} = \frac{y-3}{3} = \frac{z+4}{-4}$

and $\frac{x-2}{1} = \frac{y-3}{0} = \frac{z+4}{-2}$

Similarly, we can find the generators through the point $(2,-1,4/3)$ are

$\frac{x-2}{0} = \frac{y+1}{3} = \frac{z-(4/3)}{-4}$

and $\frac{x-2}{3} = \frac{y+1}{6} = \frac{z-(4/3)}{10}$

5(a) Solve $\frac{dy}{dx}+\frac{y}{(1-x^2)^{3/2}}=\frac{x+\sqrt{(1-x^2)}}{(1-x^2)^2}$

Sol: Comparing the given equation with $\frac{dy}{dx}+Py=Q$, here $P=\frac{1}{(1-x^2)^{3/2}}$ & $Q=\frac{x+\sqrt{1-x^2}}{(1-x^2)^2}$

Hence $\int P\,dx=\int\frac{1}{(1-x^2)^{3/2}}dx$

$$=\int\frac{\cos\theta\,d\theta}{\cos^3\theta\,d\theta}, \text{ putting } x=\sin\theta$$

$$=\int\sec^2\theta\,d\theta=\tan\theta$$

$$=\frac{\sin\theta}{\cos\theta}=\frac{x}{\sqrt{1-x^2}}$$

$\therefore$ I.F $=e^{\int P\,dx}=e^{\frac{x}{\sqrt{1-x^2}}}$

Solution of the given differential equation is

$$y(\text{I.F})=\int Q(\text{I.F})+c$$

Now, $\int Q(\text{I.F})=\int\frac{x+\sqrt{1-x^2}}{(1-x^2)^2}e^{\frac{x}{(1-x^2)^{1/2}}}dx$ — (1)

put $\frac{x}{\sqrt{1-x^2}}=t$

$$\Rightarrow \frac{\sqrt{1-x^2}\cdot 1-x\cdot\frac{1}{2}(1-x^2)^{-\frac{1}{2}}(-2x)}{(1-x^2)}dx=dt$$

$$\Rightarrow \frac{\sqrt{(1-x^2)}+[x^2/\sqrt{1-x^2}]}{1-x^2}dx=dt$$

$$\Rightarrow \frac{1}{(1-x^2)^{3/2}}dx=dt$$

from ①,

$$\int Q(I.F)\,dx = \int \frac{\left[x/\sqrt{1-x^2}+1\right]}{(1-x^2)^{3/2}} e^{\frac{x}{(1-x^2)^{1/2}}}\,dx$$

$$= \int (t+1)e^t\,dt$$

$$= (t+1)e^t - \int e^t\,dt$$

$$= te^t.$$

$$\therefore \int Q\cdot = \frac{x}{\sqrt{1-x^2}} e^{\frac{x}{\sqrt{1-x^2}}}$$

$$\therefore y e^{x/(1-x^2)^{1/2}} = \frac{x}{\sqrt{1-x^2}} e^{\frac{x}{\sqrt{1-x^2}}} + c$$

$$\Rightarrow \boxed{y = \frac{x}{\sqrt{1-x^2}} + c e^{-\frac{x}{\sqrt{1-x^2}}}}$$

which is the required solution

5(b) Solve $\left[(3x+2)^2D^2+3(3x+2)D-36\right]y = 3x^2+4x+1$

Sol: Let $3x+2 = v$

$$\Rightarrow \frac{dy}{dx} = 3\frac{dy}{dv} \text{ and } \frac{d^2y}{dx^2} = \frac{3d^2y}{dv^2}$$

Hence

$$v^2 3^2 \frac{d^2y}{dv^2} + 3v.3.\frac{dy}{dv} - 36y = 3\left(\frac{v-2}{3}\right)^2 + 4\left(\frac{v-2}{3}\right)+1$$

$$\Rightarrow v^2\frac{d^2y}{dv^2} + v\frac{dy}{dv} - 4y = \frac{1}{27}(v^2-1) \quad \text{—— ①}$$

which is homogeneous in y and v

So let $v = e^z \Rightarrow z = \log v$ and $D_1 = \frac{d}{dz}$

Then ① gives

$$\left[D_1(D_1-1)+D_1-4\right]y = \frac{e^{2z}-1}{27}$$

$\Rightarrow (D_1^2-4)y = \dfrac{e^{2z}-1}{27}$

Here A.E is $D_1^2-4=0$

$\Rightarrow D_1 = \pm 2$

and hence C.F. $= c_1 e^{2z} + c_2 e^{-2z}$

$= c_1 v^2 + c_2 v^{-2}$ $\quad (\because v = e^z)$

C.F $= c_1 (3x+2)^2 + c_2 (3x+2)^{-2}$ $\quad (\because v = 3x+2)$

$$P.I = \frac{1}{27} \frac{1}{D_1^2-4}\left(e^{2z}-1\right)$$

$$= \frac{1}{27}\left[\frac{1}{(D_1-2)(D_1+2)} e^{2z} - \frac{1}{D_1^2-4} e^{0z}\right]$$

$$= \frac{1}{27}\left[\frac{1}{(D_1-2)}\frac{1}{4} e^{2z} - \frac{1}{0-4} e^{0z}\right]$$

$$= \frac{1}{27\times 4}\left[\frac{1}{D_1-2} e^{2z} + 1\right]$$

$$= \frac{1}{108}\left[\frac{z e^{2z}}{1!} + 1\right]$$

$$= \frac{1}{108}\left(z e^{2z}+1\right)$$

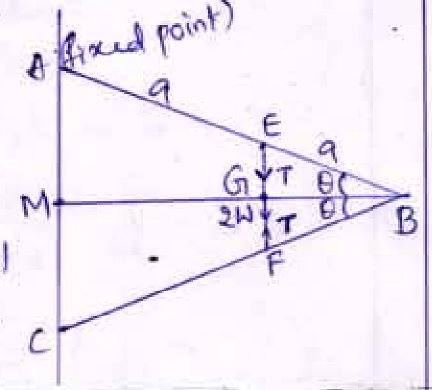
$$= \frac{1}{108}\left[v^2 \log v + 1\right] = \frac{1}{108}\left[(3x+2)^2 \log(3x+2)+1\right]$$

$\therefore$ Required general solution is

$$y = c_1(3x+2)^2 + c_2(3x+2)^{-2} + \frac{1}{108}\left[(3x+2)^2\log(3x+2)+1\right]$$

5(C) One end of a uniform rod AB, of length $2a$ and weight W, is attached by a frictionless joint to a smooth vertical wall, and the other end B is smoothly jointed to an equal rod BC. The middle points of the rods are joined by an elastic string, of natural length a and modulus of elasticity $4W$. Prove that the system can rest in equilibrium in a vertical plane with C in contact with the wall below A, and the angle b/w the rods is $2\sin^{-1}(3/4)$.

Sol'n: AB and BC are two rods each of length 2a and weight w smoothly joined together at B. The end A of the rod AB is attached to a smooth vertical wall and the end C of the rod BC is in contact with the wall. The middle points E and F of the rods AB and BC are connected by an elastic string of natural length a.



Let T be the tension in the string EF.
The total weight 2W of the two rods can be taken acting at the middle point of EF. The line BG is horizontal and meets AC at its middle point M.

Let $\angle ABM = \theta = \angle CBM$.

Give the system a small symmetrical displacement about BM in which θ changes $\theta + \delta\theta$. The point A remains fixed. The point G is slightly displaced, the length EF changes, the lengths of rods AB and BC do not change.

We have $EF = 2EG = 2EB\sin\theta = 2a\sin\theta$

Also the depth of G below the fixed point A

$$= AM = AB\sin\theta = 2a\sin\theta$$

The equation of virtual work is

$$-T\delta(2a\sin\theta) + 2W\delta(2a\sin\theta) = 0$$

$$\Rightarrow (-2aT\cos\theta + 4aW\cos\theta)\delta\theta = 0$$

$$\Rightarrow 2a\cos\theta(-T+2W)\delta\theta = 0$$

$$\Rightarrow -T + 2W = 0 \qquad [\because \delta\theta \neq 0 \text{ and } \cos\theta \neq 0]$$

$$\Rightarrow T = 2W$$

Also by Hooke's law the tension T in the elastic string EF is given by $T = \lambda \dfrac{2a\sin\theta - a}{a}$

where λ is the modulus of elasticity of the string

$$= 4W(2\sin\theta - 1) \qquad [\because \lambda = 4W]$$

Equating the two values of T, we have

$$2W = 4W(2\sin\theta - 1)$$

$$\Rightarrow 1 = 2(2\sin\theta - 1), \text{ (or) } 1 = 4\sin\theta - 2$$

$$\Rightarrow 4\sin\theta = 3, \text{ (or) } \sin\theta = 3/4$$

$$\Rightarrow \theta = \sin^{-1}(3/4)$$

$\therefore$ In equilibrium the whole angle b/w AB and BC

$$= 2\theta = 2\sin^{-1}(3/4)$$

5(d) A Body moving in a straight line OAB with S.H.M has zero velocity they at the points A and B whose distances from O are a and b respectively, and has velocity v when half way between them. Show that the complete period is $\pi(b-a)/v$.

Sol^n: from the figure; A and B are the positions of instaneous rest in a S.H.M. Let C be the middle point of AB. Then C is the centre of the motion. Also it is given that $OA = a$, $OB = b$.

The amplitude of the motion $= \frac{1}{2}AB = \frac{1}{2}(OB - OA) = \frac{1}{2}(b-a)$

Now in a S.H.M the velocity at the centre $= \sqrt{\mu} \times$ amplitude

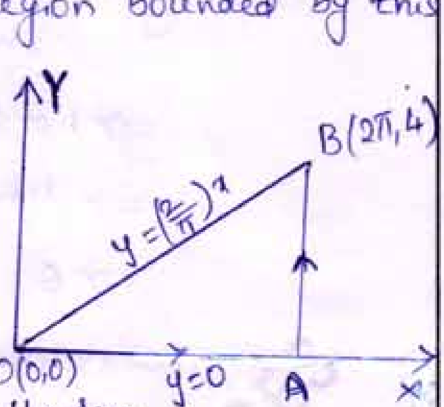
Since in this case the velocity at the centre is given to be v.

$$\therefore v = \frac{1}{2}(b-a)\cdot\sqrt{\mu} \Rightarrow \sqrt{\mu} = \frac{2v}{(b-a)}$$

Hence the time period $T = \frac{2\pi}{\sqrt{\mu}} = \frac{2\pi(b-a)}{2v} = \frac{\pi(b-a)}{v}$.

5(e) Apply Green's theorem in the plane to evaluate $\int_C \{(y - \sin x)dx + \cos x\, dy\}$, where C is the triangle enclosed by the lines $y = 0$, $x = 2\pi$, $\pi y = 2x$.

Sol^n: Here C is the closed curve traversed in positive direction by ΔOAB and R is the region bounded by this Curve C.



we have $\int_C \{(y-\sin x)dx + \cos x\, dy\}$

$= \int_C M dx + N dy$, where $M = y - \sin x$, $N = \cos x$

$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx dy$ by Green's theorem.

$= \iint_R \left[\frac{\partial}{\partial x}(\cos x) - \frac{\partial}{\partial y}(y - \sin x)\right] dx dy$

$= \int_0^{2\pi}\int_{y=0}^{(2/\pi)x} (-\sin x - 1)\, dx dy$ ($\because$ for the region R, y varies from 0 to $\frac{2}{\pi}x$ and x varies from 0 to $\pi/2$)

$= \int_0^{2\pi} \Big[-y\sin x - y\Big]_{y=0}^{y=\left(\frac{2}{\pi}\right)x}$

$= \int_0^{2\pi} \left(-\frac{2}{\pi}x\sin x - \frac{2}{\pi}x\right) dx$

$= -\frac{2}{\pi}\int_0^{2\pi} (x + \sin x)\, dx$

$= -\frac{2}{\pi}\left(\frac{x^2}{2}\right)_0^{2\pi} - \frac{2}{\pi}\Big[x(1-\cos x)\Big]_0^{2\pi} - \frac{2}{\pi}\int_0^{2\pi}\cos x\, dx$

$= \frac{-2}{\pi}\left(\frac{4\pi^2}{2}\right) - \frac{2}{\pi}\left[2\pi(-\cos 2\pi) - 0\right] - 0$

$= 4 - 4\pi$

6(a) Solve $(xp-y)^2 = (x^2-y^2)\sin^{-1}(y/x)$

Sol'n: Given equation $(xp-y)^2 = (x^2-y^2)\sin^{-1}(y/x)$ —— (1)

putting $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$\Rightarrow p = v + xP \text{ where } P = \frac{dv}{dx}$$

$\therefore$ (1) $\equiv$

$$\left[x(v+xP) - y\right]^2 = (x^2 - x^2v^2)\sin^{-1}v$$

$$\Rightarrow (y + x^2P - y)^2 = x^2(1-v^2)\sin^{-1}v$$

$$\Rightarrow (x^2P)^2 = x^2(1-v^2)\sin^{-1}v$$

$$\Rightarrow x^2P^2 = (1-v^2)\sin^{-1}v$$

$$\Rightarrow P^2 = \frac{1-v^2}{x^2}\sin^{-1}v$$

$$\Rightarrow P = \pm\frac{\sqrt{1-v^2}}{x}\sqrt{\sin^{-1}v}$$

$$\Rightarrow \frac{dv}{dx} = \pm\frac{\sqrt{1-v^2}}{x}\sqrt{\sin^{-1}v}$$

$$\Rightarrow \int\frac{1}{\sqrt{1-v^2}\sqrt{\sin^{-1}v}}dv = \pm\int\frac{dx}{x} + c$$

$$\sin^{-1}v = t$$

$$\Rightarrow \frac{1}{\sqrt{1-v^2}}dv = dt$$

$$\int\frac{dt}{\sqrt{t}} = \pm\log x + c$$

$$\Rightarrow 2t^{1/2} = \pm\log x + c$$

$$\Rightarrow 4t = (\pm\log x + c)^2$$

$$\Rightarrow 4\sin^{-1}v = (\pm\log x + c)^2$$

$$\Rightarrow 4\sin^{-1}(y/x) = (\pm\log x + c)^2$$

which is the required Solution.

6(b) Use Wronskian to show that the functions x, x^2, x^3 are independent. Determine the differential equation with these as independent solutions.

Sol^n: Let $y_1(x) = x,\ y_2(x) = x^2,\ y_3(x) = x^3$ —— (1)

the wronskian $W(x)$ of y_1, y_2 and y_3 is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 3x^2 \\ 0 & 2 & 6x \end{vmatrix}, \text{ using (1)}$$

$\Rightarrow W(x) = x(12x^2 - 6x^2) - 1(6x^3 - 2x^3) = 2x^3$

which is not identically equal to zero. Hence the functions y_1, y_2 and y_3 are linearly independent.

To form the differential equation. The general solution of the required differential equation may be written as

$$y = Ay_1 + By_2 + Cy_3 = Ax + Bx^2 + Cx^3 \quad \text{—— (2)}$$

Differentiating (2), $y' = A + 2Bx + 3Cx^2$ —— (3)

Diff. (3), $y'' = 2B + 6Cx$ —— (4)

Diff. (4) $y''' = 6C$ —— (5)

To find the required equation, we now eliminate A, B, C from (2), (3), (4) and (5)

from (5), $C = (1/6)y'''$. Then from (4) $B = \frac{1}{2}(y'' - xy''')$ —— (6)

Multiplying bothsides of (3) by x, $xy' = Ax + 2Bx^2 + 3Cx^3$ —— (7)

Substracting (7) from (2), $y - xy' = -Bx^2 - 2Cx^3$

$\Rightarrow y - xy' = (-1/2)x^2(y'' - xy''') - (2x^3)\frac{1}{6}y'''$, using (6)

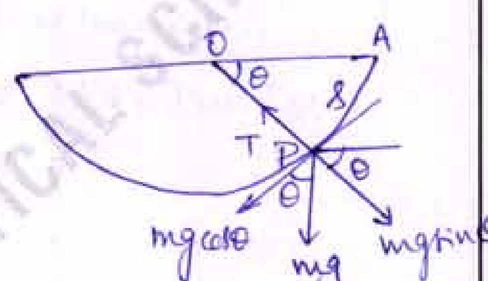
$\Rightarrow 6y - 6xy' = -3x^2y'' + 3x^3y''' - 2x^3y'''$

$\Rightarrow x^3y''' - 3x^2y'' + 6xy' - 6y = 0.$

which is the required differential equation.

6(C). A particle attached to a fixed peg O by a string of length l, is lifted up with the string horizontal and then let go. Prove that when the string makes an angle θ with the horizontal, the resultant acceleration is $g\sqrt{1+3\sin^2\theta}$.

Sol^n: Let a particle of mass m be attached to a string of length l whose other end is attached to a fixed peg O. Initially let the string be horizontal in the position $OA=l$. The particle starts from A and moves in a circle whose centre is O and radius is l. Let P be the position of the particle at any time t such that $\angle AOP=\theta$ and arc $AP=s$. The forces acting on the particle at P are:



(i) its weight mg acting vertically downwards and

(ii) the tension T in the string along PO.

$\therefore$ the equations of motion of the particle along the tangent and normal at P are

$$m\frac{d^2s}{dt^2}=mg\cos\theta \quad \text{——— (1)}$$

and $$m\frac{v^2}{l}=T-mg\sin\theta \quad \text{——— (2)}$$

Also $s=l\theta$

from (1) and (3), we have $l\frac{d^2\theta}{dt^2}=g\cos\theta$.

Multiplying bothsides by $2l(d\theta/dt)$ and integrating, we have

$$v^2=\left(l\frac{d\theta}{dt}\right)^2=2lg\sin\theta+A$$

But initially at the point A, $\theta=0$, $v=0$, $\therefore A=0$.

$\therefore v^2=2lg\sin\theta$

The resultant acceleration of the particle at P

$$=\sqrt{(\text{Tangential accel.})^2+(\text{Normal accel})^2}$$

$$=\sqrt{\left[\left(\frac{d^2s}{dt^2}\right)^2+\left(\frac{v^2}{l}\right)^2\right]} \qquad \left[\because \text{normal acc}=\frac{v^2}{\rho}=\frac{v^2}{l}\right]$$

$$= \sqrt{\left[(g\cos\theta)^2 + \left(\frac{2lg\sin\theta}{l}\right)^2\right]}$$

$$= g\sqrt{[1-\sin^2\theta + 4\sin^2\theta]}$$

$$= g\sqrt{(1+3\sin^2\theta)}$$

5(d) Show that $\text{div. curl curl}(\vec{a}\phi) + \nabla^2 \text{div}(\vec{a}\phi) = \vec{a}\cdot\text{grad}\,\nabla^2\phi$.
where ϕ is a scalar point function.

Sol^n: we know that $\text{div}(\text{curl}\,\vec{F}) = 0$

$\therefore \text{div curl curl}(\vec{a}\phi) = 0$ —— ①

$$\nabla^2 \text{div}(\vec{a}\phi) = \nabla^2(\nabla\cdot\vec{a}\phi)$$
$$= \nabla^2(\nabla\phi\cdot\vec{a} + \phi(\nabla\cdot\vec{a}))$$
$$= \nabla^2(\nabla\phi\cdot\vec{a}) + 0 \quad (\because \nabla\cdot\vec{a} = 0)$$
$$= \nabla^2(\nabla\phi\cdot\vec{a})$$
$$= \vec{a}\cdot[\nabla^2(\nabla\phi)]$$
$$= \vec{a}\cdot[\nabla(\nabla^2\phi)]$$
$$= \vec{a}\cdot\text{grad}(\nabla^2\phi) \text{ —— ②}$$

$\therefore$ from ① and ②

$\text{div. curl curl}(\vec{a}\phi) + \nabla^2 \text{div}(\vec{a}\phi) = \vec{a}$

7(a) Solve by the method of variation of parameters

$$x\frac{dy}{dx} - y = (x-1)\left(\frac{d^2y}{dx^2} - x + 1\right)$$

Sol^n: Re-writing the given equation, we have

$$xy_1 - y = (x-1)y_2 - (x-1)^2.$$

$$\Rightarrow y_2 - \{x/(x-1)\}y = x-1 \quad \text{--- (1)}$$

Consider $y_2 - \left\{\frac{x}{x-1}\right\}y_1 + \left\{\frac{1}{(x-1)}\right\}y = 0$ --- (2)

Comparing (2) with $y_2 + Py_1 + Qy = 0$ here $P = \frac{(-x)}{(x-1)}$

and $Q = \frac{1}{(x-1)}$

Then $P + Qx = (-x)/(x-1) + \frac{x}{(x-1)} = 0$

$$1 + P + Q = 1 + \frac{(-x)}{(1-x)} + \frac{1}{(x-1)} = 0$$

therefore, we see that x and e^x are integrals of C.F of (1) (or) solutions of (2). Again the Wronskian W of x and e^x is given by

$$W = \begin{vmatrix} x & e^x \\ dx/dx & d(e^x)/dx \end{vmatrix} = \begin{vmatrix} x & e^x \\ 1 & e^x \end{vmatrix} = e^x(x-1) \neq 0 \quad \text{--- (3)}$$

showing that x and e^x are linearly independent solutions of (2).

Hence, the general solution of (2) is $y = ax + be^x$ and therefore C.F of (1) is $ax + be^x$, a & b being arbitrary constants.

Comparing (1) with $y_2 + Py_1 + Qy = R$, here $R = x-1$ --- (4)

Let $u = x$ and $v = e^x$ --- (5)

Then, P.I of ① $= u\,f(x) + v\,g(x)$. —— ⑥

where $f(x) = -\int \frac{vR}{W}\,dx = -\int \frac{(x-1)}{e^x(x-1)}\,dx$

$= -\int dx = -x$, using ②, ④ and ⑤

and $g(x) = \int \frac{uR}{W}\,dx = \int \frac{x(x-1)}{e^x(x-1)}\,dx = \int x e^{-x}\,dx$ by ②, ④ and ⑤

$= x(-e^{-x}) - \int 1.\,(-e^{-x})\,dx$

$= -xe^{-x} - e^{-x} = -e^{-x}(x+1)$

Substituting the above value of $u, v, f(x)$ & $g(x)$ in ⑥, we have

P.I of ① $= x(-x) + e^x\{-e^{-x}(x+1)\} = -(x^2+x+1)$

Hence the general solution of ① is

$y = C.F + P.I$

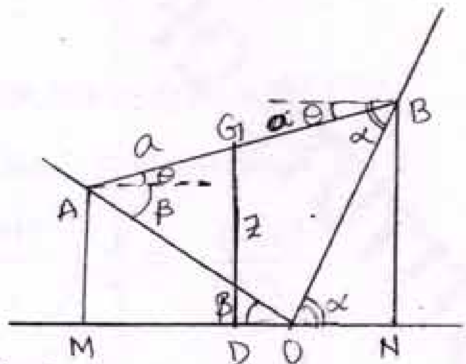
$y = ax + be^x - (x^2+x+1)$

7(b) A uniform beam of length $2a$ rests with its ends on two smooth planes which intersect in horizontal line. If the inclinations of the planes to the horizontal are α and β $(\alpha > \beta)$, show that the inclination of the beam to the horizontal in one of the equilibrium positions given by

$$\tan\theta = \frac{1}{2}\left(\cot\beta - \cot\alpha\right).$$

Sol^n: Let AB be a uniform beam of length $2a$ resting with its ends A and B on two smooth inclined planes OA and OB. Suppose the beam makes an angle θ with the horizontal. we have

$\angle AOM = \beta$ and $\angle BON = \alpha$



The centre of gravity of the beam AB is its middle point G

Let z be the height of G above the fixed horizontal line MN.

We shall express z as a function of θ.

we have, $z = GD = \frac{1}{2}(AM + BN)$

$$= \frac{1}{2}\left(OA\sin\beta + OB\sin\alpha\right)$$

Now in the triangle OAB, $\angle OAB = \beta + \theta$, $\angle OBA = \alpha - \theta$ and $\angle AOB = \pi - (\alpha+\beta)$. Applying the sine theorem for the ΔOAB, we have

$$\frac{OA}{\sin(\alpha-\theta)} = \frac{OB}{\sin(\beta+\theta)} = \frac{AB}{\sin[\pi-(\alpha+\beta)]} = \frac{2a}{\sin(\alpha+\beta)}$$

$$\therefore OA = \frac{2a\sin(\alpha-\theta)}{\sin(\alpha+\beta)}, \quad OB = \frac{2a\sin(\beta+\theta)}{\sin(\alpha+\beta)}$$

Substituting for OA and OB in (1), we have

$$= \frac{1}{2}\left[\frac{2a\sin(\alpha-\theta)}{\sin(\alpha+\beta)}\sin\beta + \frac{2a\sin(\beta+\theta)}{\sin(\alpha+\beta)}\sin\alpha\right]$$

$$= \frac{a}{\sin(\alpha+\beta)}\left[\sin(\alpha-\theta)\sin\beta + \sin(\beta+\theta)\sin\alpha\right]$$

$$= \frac{a}{\sin(\alpha+\beta)}\left[(\sin\alpha\cos\theta - \cos\alpha\sin\theta)\sin\beta + (\sin\beta\cos\theta + \cos\beta\sin\theta)\sin\alpha\right]$$

$$= \frac{a}{\sin(\alpha+\beta)}\left[\sin\theta(\sin\alpha\cos\beta - \cos\alpha\sin\beta) + 2\cos\theta\sin\alpha\sin\beta\right]$$

$\therefore$ For equilibrium of the beam, we have $\frac{dz}{d\theta} = 0$

i.e, $\cos\theta(\sin\alpha\cos\beta - \cos\alpha\sin\beta) - 2\sin\theta\sin\alpha\sin\beta = 0.$

i.e, $2\sin\theta\sin\alpha\sin\beta = \cos\theta(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$

(or) $$\frac{\sin\theta}{\cos\theta} = \frac{1}{2}\left(\frac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\sin\alpha\sin\beta}\right)$$

$$\tan\theta = \frac{1}{2}(\cot\beta - \cot\alpha).$$

This gives the required Position of equilibrium of the beam.

7(C). By using Divergence theorem of Gauss, evaluate the surface integral.

$\iint (a^2x^2+b^2y^2+c^2z^2)^{-1/2} ds$, where S is the surface of the ellipsoid $ax^2+by^2+cz^2=1$, a, b & c being all +ve constants.

Sol'n: Let us first put the integral

$$\iint_S (a^2x^2+b^2y^2+c^2z^2)^{-1/2} ds \text{ in the form}$$

$$\iint_S F.n\, ds,$$

where n is a unit normal vector to the closed surface S whose equation is $ax^2+by^2+cz^2=1$.

The normal vector to $\phi(x,y,z) \equiv ax^2+by^2+cz^2-1=0$ is

$$= \nabla\phi = 2ax\hat{i}+2by\hat{j}+2cz\hat{k}$$

$$\therefore n = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2ax\hat{i}+2by\hat{j}+2cz\hat{k}}{\sqrt{(4a^2x^2+4b^2y^2+4c^2z^2)}} = \frac{ax\hat{i}+by\hat{j}+cz\hat{k}}{\sqrt{(a^2x^2+b^2y^2+c^2z^2)}}$$

Here we are to choose F such that

$$F.n = \frac{1}{\sqrt{a^2x^2+b^2y^2+c^2z^2}} \text{ on } S.$$

Obviously $F = x\hat{i}+y\hat{j}+z\hat{k}$, because then

$$F.n = \frac{ax^2+by^2+cz^2}{\sqrt{a^2x^2+b^2y^2+c^2z^2}} = \frac{1}{\sqrt{(a^2x^2+b^2y^2+c^2z^2)}} \text{ on } S.$$

Note that on S, $ax^2+by^2+cz^2=1$

Now $\iint_S \frac{1}{\sqrt{(a^2x^2+b^2y^2+c^2z^2)}} ds$

$= \iint_S F.n\, ds$, where $F = x\hat{i} + y\hat{j} + z\hat{k}$

$= \iiint_V (\nabla . F)\, dv$ by divergence theorem; V is the volume enclosed by surface S.

$= \iiint_V \left[\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)\right] dv$

$= \iiint_V 3\, dv$

$= 3V = 3 \cdot \frac{4}{3}\pi \cdot \frac{1}{\sqrt{a}} \cdot \frac{1}{\sqrt{b}} \cdot \frac{1}{\sqrt{c}}$

$= \frac{4\pi}{\sqrt{abc}}$

8(a) By using Laplace transform method solve $(D^3+1)y=1$, $t>0$
$y=Dy=D^2y=0$ when $t=0$.

Sol:- Given $(D^3+1)y=1$

Taking Laplace transform of the given equation, we have

$L(y''')+L\{y\}=L\{1\}$

$p^3L\{y\}-p^2y(0)-p\,y'(0)-y''(0)+L\{y\}=\frac{1}{p}$

$\Rightarrow (p^3+1)L\{y\}=\frac{1}{p}$

$\Rightarrow L\{y\}=\frac{1}{p(p^3+1)}=\frac{1}{p(p+1)(p^2-p+1)}$

$=\frac{1}{p}-\frac{1}{3(p+1)}-\frac{2p-1}{3(p^2-p+1)}$

$L\{y\}=\frac{1}{p}-\frac{1}{3(p+1)}-\frac{2(p-\frac{1}{2})}{3\{(p-\frac{1}{2})^2+\frac{3}{4}\}}$

$\therefore y=L^{-1}\{\frac{1}{p}\}-\frac{1}{3}L^{-1}\{\frac{1}{p+1}\}$
$-\frac{2}{3}L^{-1}\left\{\frac{p-\frac{1}{2}}{(p-\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}\right\}$

$=1-\frac{1}{3}e^{-t}-\frac{2}{3}e^{t/2}L^{-1}\left\{\frac{p}{p^2+(\frac{\sqrt{3}}{2})^2}\right\}$

$y=1-\frac{1}{3}e^{-t}-\frac{2}{3}e^{t/2}\cos\left(\frac{\sqrt{3}}{2}t\right)$

which is the required solution.

8(b) $\to$ A particle moves with a central acceleration which varies inversely as the cube of the distance. If it be projected from an apse at a distance a from the origin with a velocity which is $\sqrt{2}$ times the velocity for a circle of radius a, show that the equation to its path is $r\cos(\theta/\sqrt{2}) = a$.

Sol'n: Here the central acceleration varies inversely as the cube of the distance i.e, $P = \mu/r^3 = \mu u^3$, where μ is a constant.

If V is the velocity for a circle of radius a, then

$$\frac{V^2}{a} = [P]_{r=a} = \frac{\mu}{a^3}$$

$$\Rightarrow V = \sqrt{(\mu/a^2)}$$

$\therefore$ the velocity of projection $v_1 = \sqrt{2}V = \sqrt{2\mu/a^2}$.

The differential equation of the path is

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{\mu u^3}{u^2} = \mu u$$

Multiplying both sides by $2(du/d\theta)$ and integrating, we have

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu u^2 + A \quad \text{— (1)}$$

where A is a constant.

But initially when $r = a$ i.e. $u = 1/a$, $du/d\theta = 0$ (at an apse)

and $v = v_1 = \sqrt{2\mu/a^2}$.

$\therefore$ from (1), we have,

$$2\mu\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu u^2 + \frac{\mu}{a^2}$$

$\therefore\ h^2 = 2\mu$ and $A = \mu/a^2$

Substituting the values of h^2 and A in (1), we have

$$2\mu\left[u^2+\left(\frac{du}{d\theta}\right)^2\right] = \mu u^2 + \frac{\mu}{a^2}$$

$$\Rightarrow 2\left(\frac{du}{d\theta}\right)^2 = \frac{1}{a^2} + u^2 - 2u^2 = \frac{1-a^2u^2}{a^2}$$

$$\Rightarrow \sqrt{2}\,a\,\frac{du}{d\theta} = \sqrt{(1-a^2u^2)}$$

$$\Rightarrow \frac{d\theta}{\sqrt{2}} = \frac{a\,du}{\sqrt{(1-a^2u^2)}}$$

Integrating, $\theta/\sqrt{2} + B = \sin^{-1}(au)$, where B is constant

But initially; when $u = 1/a$, $\theta = 0$.

$$\therefore\ B = \sin^{-1} 1 = \tfrac{1}{2}\pi$$

$$\therefore\ \theta/\sqrt{2} + \tfrac{1}{2}\pi = \tfrac{1}{2}\sin^{-1}(au)$$

$$\Rightarrow au = a/r = \sin\left\{\tfrac{1}{2}\pi + (\theta/\sqrt{2})\right\}$$

$\Rightarrow a = r\cos(\theta/\sqrt{2})$, which is the required equation of the path.

8(c) verify stokes theorem for $\vec{F} = -y^3\hat{i} + x^3\hat{j}$ where S is the circular disc $x^2+y^2 \le 1$, $z=0$.

Sol'n: The boundary C of S is a circle in xy-plane of radius one and centre at origin.

Suppose $x = \cos t$, $y = \sin t$, $z = 0$, $0 \le t < 2\pi$ are parametric equations of C.

Then $\oint_C F.dr = \oint_C (-y^3\hat{i} + x^3\hat{j}).(dx\hat{i} + dy\hat{j} + dz\hat{k})$

$$= \oint_C \left(-y^3 dx + x^3 dy\right)$$

$$= \int_{t=0}^{2\pi} \left(-y^3 \frac{dx}{dt} + x^3 \frac{dy}{dt}\right) dt$$

$$= \int_0^{2\pi} \left[-\sin^3 t(-\sin t) + \cos^3 t(\cos t)\right] dt$$

$$= \int_0^{2\pi} (\cos^4 t + \sin^4 t)\, dt$$

$$= 4\int_0^{\pi/2} (\cos^4 t + \sin^4 t)\, dt$$

$$= 4\left\{\frac{3}{2}\cdot\frac{1}{2}\cdot\frac{\pi}{2} + \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right\} = \frac{3\pi}{2}.$$

Also $\nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -y^3 & x^3 & 0 \end{vmatrix} = (3x^2 + 3y^2)\hat{k}$.

Here $n = \hat{k}$ because the surface S is the xy-plane.

$$(\nabla \times F)\cdot\hat{n} = (3x^2 + 3y^2)\hat{k}\cdot\hat{k} = 3(x^2+y^2)$$

$$\therefore \iint_S (\nabla \times F)\cdot\hat{n}\, dS = 3\iint_S (x^2+y^2)\, dS$$

$$= 3\int_{\theta=0}^{2\pi}\int_{r=0}^{1} r^2 r\, d\theta\, dr$$

by changing to polars

$$= \frac{3}{4}\int_0^{2\pi} d\theta$$

$$= \frac{3}{4}(2\pi) = \frac{3\pi}{2}.$$

Hence the theorem is verified.