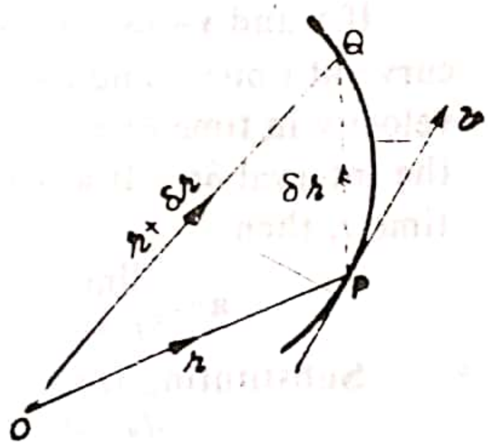
Kinematics in Two Dimensions

§ 1. Kinematics of a particle (Velocity and Acceleration).

Velocity (Definition). *The velocity of a particle moving along a curve is the rate of change of its displacement with respect to time.*

Let P and Q be the positions of a particle moving along a curve at times t and $t+\delta t$ respectively. With respect to O as the origin of vectors, let $\vec{OP}=\mathbf{r}$ and $\vec{OQ}=\mathbf{r}+\delta\mathbf{r}$. Then



$$\vec{PQ}=\vec{OQ}-\vec{OP}=\delta\mathbf{r}$$

represents the displacement of the particle in time δt and $\frac{\delta\mathbf{r}}{\delta t}$ represents the average rate of displacement (or the *average velocity*) during the interval δt. The limiting value of the average velocity $\frac{\delta\mathbf{r}}{\delta t}$ as $\delta t\to 0$ is the velocity— (also known as instantaneous velocity) of the particle at time t. Thus if the vector $\mathbf{r}$ represents the velocity of the particle at time t, then

$$\mathbf{v}=\lim_{\delta t\to 0}\frac{\delta\mathbf{r}}{\delta t}=\frac{d\mathbf{r}}{dt},$$

where $\mathbf{r}$ is the position vector of the particle.

The vector $\frac{d\mathbf{r}}{dt}$ is along the tangent to the path of the particle and consequently the velocity $\mathbf{v}$ is a vector quantity along the tangent at P. The magnitude $v=|\mathbf{v}|$ of the velocity $\mathbf{v}$ is called the *speed* of the particle.

If (x, y, z) are the co-ordinates of the point P and $\mathbf{i}, \mathbf{j}, \mathbf{k}$ the three unit vectors forming the right handed system, the position vector $\mathbf{r}$ of P is given by $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$. ...(2)

$$\therefore\quad \mathbf{v}=\frac{d\mathbf{r}}{dt}=\frac{dx}{dt}\mathbf{i}+\frac{dy}{dt}\mathbf{j}+\frac{dz}{dt}\mathbf{k}. \qquad ...(3)$$

Here $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ are called the components or the *resolved parts* of the velocity v along the axes of x, y and z respectively.

$\therefore$ The speed v of the particle at P is given by

$$v=|\mathbf{v}|=\sqrt{\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right]}. \quad ...(4)$$

Remember : The component dx/dt, of the velocity v along the axis of x, is taken with positive or negative sign according as it is in the direction of x increasing or x decreasing. In the case of dy/dt and dz/dt also the positive and negative signs are taken exactly in the same way.

Acceleration (Definition) : *The* **acceleration vector** *of a particle moving along a curve is defined as the rate of change of its velocity vector.*

If $\mathbf{v}$ and $\mathbf{v}+\delta\mathbf{v}$ are the velocities of a particle moving along a curve at times t and $t+\delta t$ respectively, then $\delta\mathbf{v}$ is the change in velocity in time δt and $\delta\mathbf{v}/\delta t$ is the average acceleration during the interval δt. If $\mathbf{a}$ is the **acceleration vector** of the particle at time t, then

$$\mathbf{a}=\lim_{\delta t\to 0}\frac{\delta\mathbf{v}}{\delta t}=\frac{d\mathbf{v}}{dt}=\frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right)=\frac{d^2\mathbf{r}}{dt^2} \quad ...(5)$$

Substituting for $\mathbf{v}$ from (3), we have

$$\mathbf{a}=\frac{d\mathbf{v}}{dt}=\frac{d^2x}{dt^2}\,\mathbf{i}+\frac{d^2y}{dt^2}\,\mathbf{j}+\frac{d^2z}{dt^2}\,\mathbf{k}. \quad ...(6)$$

Here $\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}$ are called the components of the acceleration $\mathbf{a}$ along the axes of x, y and z respectively.

Also magnitude of acceleration

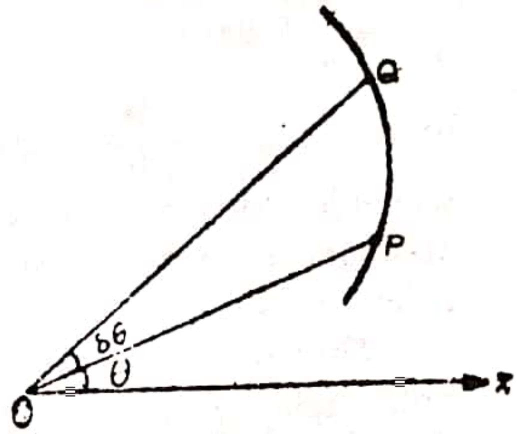
$$=a=|\mathbf{a}|=\sqrt{\left[\left(\frac{d^2x}{dt^2}\right)^2+\left(\frac{d^2y}{dt^2}\right)^2+\left(\frac{d^2z}{dt^2}\right)^2\right]}.$$

Remember : The component $\frac{d^2x}{dt^2}$ of the acceleration $\mathbf{a}$ along the axis of x is taken with positive or negative sign according as it is in the direction of x increasing or x decreasing. The positive and negative signs are taken with $\frac{d^2y}{dt^2}$ and $\frac{d^2z}{dt^2}$ in exactly the same way.

§ 2. Angular Velocity and Acceleration.

Angular Velocity (Definition). *Let P be a point moving in a plane. If O be the* **fixed point** *and OX a* **fixed line** *through O in the plane of motion, then the* **angular velocity** *of the moving point P about O (or of the line OP in the plane XOP) is the rate of change of the angle XOP.*

Let P and Q be the positions of a moving particle at times t and $t+\delta t$ respectively such that $\angle POX=\theta$ and $\angle QOX=\theta+\delta\theta$.



Then the angle turned by the particle in time δt is $\delta\theta$.

$\therefore$ Average rate of change of the angle of P about $O=\dfrac{\delta\theta}{\delta t}$.

$\therefore$ The **angular velocity** of the point P about O

$$=\lim_{\delta t\to 0}\frac{\delta\theta}{\delta t}=\frac{d\theta}{dt}=\dot{\theta},$$

where the dot placed over θ denotes differentiation with respect to the time t.

Since the angular velocity has magnitude as well as direction, it is a vector quantity represented by the vector $\vec{\omega}$. The magnitude of the angular velocity is $\frac{d\theta}{dt}$ $(=\dot{\theta}=\omega)$ and its direction is perpendicular to the plane POQ.

Since the angle θ is measured in radians, the unit of angular velocity is radians/sec.

Angular Acceleration (Definition) : *The rate of change of the angular velocity is called* **angular acceleration.**

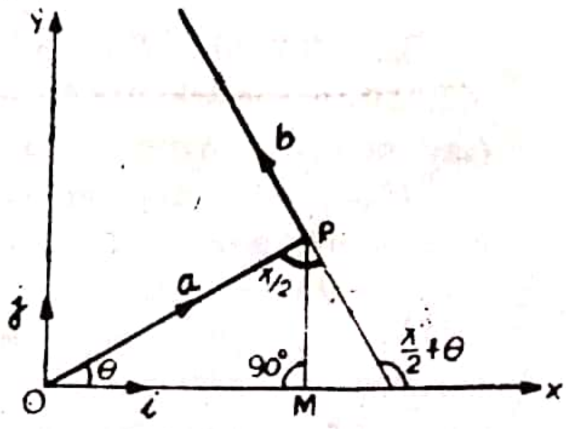
$$\therefore\ \textbf{Angular acceleration}=\frac{d}{dt}\left(\frac{d\theta}{dt}\right)=\frac{d^2\theta}{dt^2}=\ddot{\theta}.$$

The unit of angular acceleration is radians/sec^2.

§ 3. Rate of change of a unit vector in a plane.

Let $\mathbf{i}$, $\mathbf{j}$ be the unit vectors along two mutually perpendicular fixed lines (say the coordinate axes) in the plane.

Let $\mathbf{a}$ denote a **unit vector** $\vec{OP}$ such that $OP=1$ and $\angle POX=\theta$.



Then $\mathbf{a}=\vec{OM}+\vec{MP}$

$=\cos\theta\mathbf{i}+\sin\theta\mathbf{j}$ ...(1)

The vector $\mathbf{a}$ is a function of θ, where θ is a function of the time t. Differentiating (1) w.r.t. t, we have

$$\frac{d\mathbf{a}}{dt}=-\sin\theta\frac{d\theta}{dt}\mathbf{i}+\cos\theta\frac{d\theta}{dt}\mathbf{j}$$ [Note that $\mathbf{i}$ and $\mathbf{j}$ are constant vectors]

$$=\frac{d\theta}{dt}\left[\cos\left(\tfrac{1}{2}\pi+\theta\right)\mathbf{i}+\sin\left(\tfrac{1}{2}\pi+\theta\right)\mathbf{j}\right]$$

or $$\frac{d\mathbf{a}}{dt}=\frac{d\theta}{dt}\mathbf{b}, \qquad \ldots(2)$$

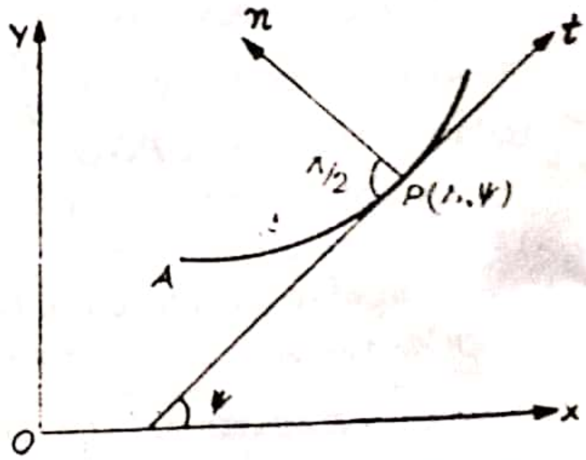
where $\mathbf{b}=\cos\left(\frac{1}{2}\pi+\theta\right)\mathbf{i}+\sin\left(\frac{1}{2}\pi+\theta\right)\mathbf{j}$ is a unit vector inclined at an angle $\frac{1}{2}\pi+\theta$ with OX. Therefore, $\mathbf{b}$ is a unit vector perpendicular to OP in the sense in which θ increases.

Thus remember that if $\mathbf{a}$ is a unit vector which makes a variable angle θ with OX, then

$$\frac{d\mathbf{a}}{dt}=\frac{d\theta}{dt}\mathbf{b},$$

where $\mathbf{b}$ is a unit vector perpendicular to $\mathbf{a}$ in the direction of θ increasing.

Particular case : If $\mathbf{t}$ and $\mathbf{n}$ are the unit vectors along the tangent and normal respectively at any point P of a plane curve (as shown in the figure), then



$$\frac{d\mathbf{t}}{dt}=\frac{d\psi}{dt}\mathbf{n}=\dot{\psi}\mathbf{n},$$

where ψ is the angle which the tangent at the point P makes with OX.

Also $$\frac{d\mathbf{n}}{dt}=-\frac{d\psi}{dt}\mathbf{t}=-\dot{\psi}\mathbf{t}.$$

Here $\mathbf{t}$ is in the direction of s increasing and $\mathbf{n}$ is in the sense in which ψ increases *i.e.*, in the direction of inwards drawn normal.

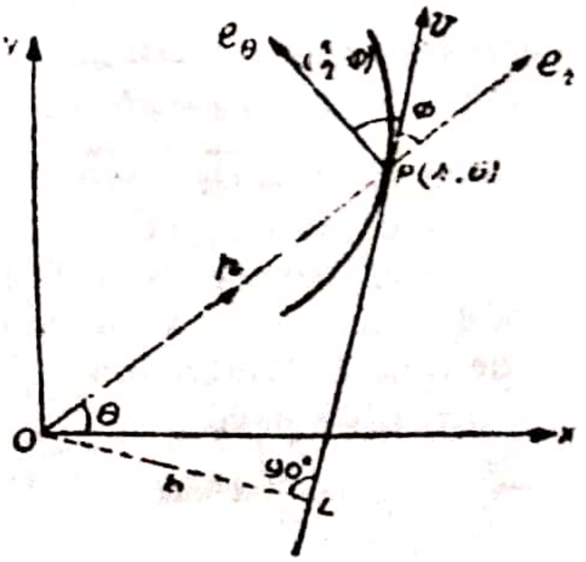
§ 4. Relation between angular and linear velocities.

[Meerut 1978, 79, 83 P; Allahabad 76]

Let OX and OY be two mutually perpendicular fixed lines (say the co-ordinates axes).

If ω is the angular velocity of a moving point P about O, and $\angle POX=\theta$, then

$$\omega=\frac{d\theta}{dt}.$$



Let $\mathbf{r}$ be the position vector of the point P with respect to the origin O and let (r, θ) be the polar co-ordinates of P. If $\mathbf{e}_r$ and $\mathbf{e}_\theta$

are the unit vectors along and perpendicular to OP respectively, then

$$\boxed{\mathbf{r}=|\mathbf{r}|\,\mathbf{e}_r=r\,\mathbf{e}_r} \qquad [\because\ |\mathbf{r}|=|\overrightarrow{OP}|=OP=r]$$

and $$\frac{d\mathbf{e}_r}{dt}=\frac{d\theta}{dt}\,\mathbf{e}_\theta \qquad \text{[Refer § 3.]}$$

Now the linear velocity $\mathbf{v}$ of the point P is along the tangent at P and is given by

$$\mathbf{v}=\frac{d\mathbf{r}}{dt}=\frac{d}{dt}\,(r\mathbf{e}_r)=\frac{dr}{dt}\,\mathbf{e}_r+r\,\frac{d\mathbf{e}_r}{dt}=\frac{dr}{dt}\,\mathbf{e}_r+r\,\frac{d\theta}{dt}\,\mathbf{e}_\theta .$$

Now the component of a vector $\mathbf{a}$ in the direction of a unit vector $\mathbf{b}$ is given by $\mathbf{a}\cdot\mathbf{b}$. If v_θ is the component of the velocity $\mathbf{v}$ in the direction perpendicular to OP, then

$$v_\theta=\mathbf{v}\cdot\mathbf{e}_\theta=\left(\frac{dr}{dt}\,\mathbf{e}_r+r\,\frac{d\theta}{dt}\,\mathbf{e}_\theta\right)\cdot\mathbf{e}_\theta$$

$$=r\,\frac{d\theta}{dt}=r\omega, \qquad [\because\ \mathbf{e}_r\perp\mathbf{e}_\theta \text{ and } |\mathbf{e}_\theta|=1]$$

or $$\omega=\frac{v_\theta}{r}=\frac{\text{Component of the velocity } \mathbf{v} \text{ at } P \text{ perpendicular to } OP}{OP}.$$

(Remember)

Since the angle between $\mathbf{v}$ and $\mathbf{e}_\theta$ is $\frac{1}{2}\pi-\phi$, therefore

$$v_\theta=\mathbf{v}\cdot\mathbf{e}_\theta=v.1.\cos\left(\tfrac{1}{2}\pi-\phi\right)=v\sin\phi .$$

$$\therefore \quad \omega=\frac{v\sin\phi}{r}=\frac{vr\sin\phi}{r^2}$$

$$\omega=\frac{d\theta}{dt}=\frac{vp}{r^2}. \qquad [\because\ p=r\sin\phi]$$

Remark 1. The angular velocity of P about O

$$=\frac{\text{The resolved part of the vel. of } P \perp \text{to } OP}{OP}.$$

Remark 2. If A and B are both in motion, then the angular velocity of B relative to A

$$=\frac{\text{the resolved part of the vel. of } B \text{ relative to } A \perp \text{to } AB}{AB}.$$

§ 5. Components of velocity and acceleration along the co-ordinate axes.

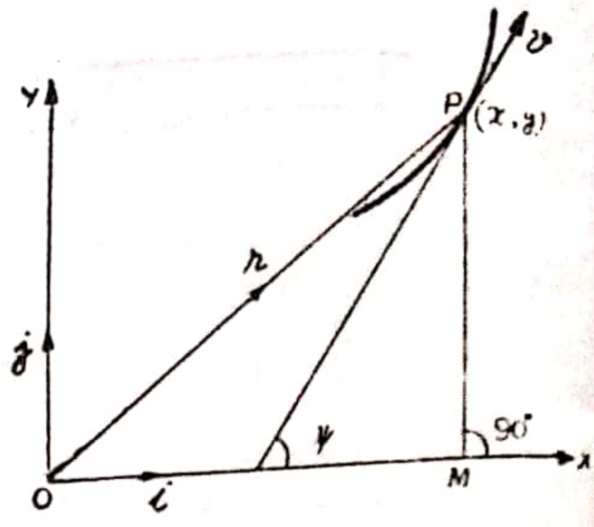
Let $P\,(x, y)$ be the position of a particle moving in a plane curve at any time t. If $\overrightarrow{OP}=\mathbf{r}$, we have

$$\mathbf{r}=\overrightarrow{OP}=\overrightarrow{OM}+\overrightarrow{MP}=x\mathbf{i}+y\mathbf{j}.$$

Let $\mathbf{v}$ be the vector representing the velocity of the particle at P. Then

$$\mathbf{v}=\frac{d\mathbf{r}}{dt}=\frac{d}{dt}(x\mathbf{i}+y\mathbf{j})$$
$$=\frac{dx}{dt}\mathbf{i}+\frac{dy}{dt}\mathbf{j}.$$

Thus the velocity vector $\mathbf{v}$ has been expressed as a linear combination of the vectors $\mathbf{i}$ and $\mathbf{j}$.



$\therefore$ the x-component of the velocity of $P=dx/dt=\dot{x}$, positive in the direction of the vector $\mathbf{i}$ *i.e.*, positive in the direction of x increasing,

and the y-component of the velocity of $P=dy/dt=\dot{y}$, positive in the direction of y increasing.

If v is the resultant velocity of P, we have

$$v=\sqrt{\{(dx/dt)^2+(dy/dt)^2\}}=ds/dt.$$

Also the angle which the direction of v makes with OX

$$=\tan^{-1}\frac{dy/dt}{dx/dt}=\tan^{-1}\frac{dy}{dx}=\tan^{-1}\tan\psi=\psi,$$

showing that the resultant velocity at P is along the tangent at P.

If $\mathbf{a}$ be the acceleration vector of the particle at P, we have

$$\mathbf{a}=\frac{d\mathbf{v}}{dt}=\frac{d}{dt}\left\{\frac{dx}{dt}\mathbf{i}+\frac{dy}{dt}\mathbf{j}\right\}=\frac{d^2x}{dt^2}\mathbf{i}+\frac{d^2y}{dt^2}\mathbf{j}.$$

$\therefore$ the x-component of the acceleration of $P=d^2x/dt^2=\ddot{x}$, positive in the direction of x increasing,

and the y-component of the acceleration of $P=d^2y/dt^2=\ddot{y}$, positive in the direction of y increasing.

The resultant acceleration of $P=\sqrt{\left\{\left(\frac{d^2x}{dt^2}\right)^2+\left(\frac{d^2y}{dt^2}\right)^2\right\}}$.

Illustrative Examples :

Ex. 1. *A particle moves along the curve $x=t^3+1$, $y=t^2$, $z=2t+5$ where t is the time. Find the components of the velocity and acceleration at time $t=1$ in the direction* $\mathbf{i}+\mathbf{j}+3\mathbf{k}$.

Sol. If $\mathbf{r}$ is the position vector of the particle at time t, then

$$\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}=(t^3+1)\,\mathbf{i}+t^2\mathbf{j}+(2t+5)\,\mathbf{k}.$$

$\therefore$ velocity $\mathbf{v}=d\mathbf{r}/dt=3t^2\mathbf{i}+2t\mathbf{j}+2\mathbf{k}$
and acceleration $\mathbf{a}=d\mathbf{v}/dt=6t\mathbf{i}+2\mathbf{j}$.

$\therefore$ at time $t=1$, $\mathbf{v}=3\mathbf{i}+2\mathbf{j}+2\mathbf{k}$ and $\mathbf{a}=6\mathbf{i}+2\mathbf{j}$.

Now the unit vector in the direction of the vector $\mathbf{i}+\mathbf{j}+3\mathbf{k}$ is

$$=\frac{\mathbf{i}+\mathbf{j}+3\mathbf{k}}{\sqrt{(1+1+9)}}=\frac{\mathbf{i}+\mathbf{j}+3\mathbf{k}}{\sqrt{(11)}}.$$

$\therefore$ The components of the velocity and acceleration at time $t=1$ in the direction $\mathbf{i}+\mathbf{j}+3\mathbf{k}$ are

$$(3\mathbf{i}+2\mathbf{j}+2\mathbf{k})\cdot\frac{(\mathbf{i}+\mathbf{j}+3\mathbf{k})}{\sqrt{(11)}}=\frac{3\cdot1+2\cdot1+2\cdot3}{\sqrt{(11)}}=\sqrt{(11)}$$

and $$(6\mathbf{i}+2\mathbf{j})\cdot\frac{(\mathbf{i}+\mathbf{j}+3\mathbf{k})}{\sqrt{(11)}}=\frac{6\cdot1+2\cdot1+0\cdot3}{\sqrt{(11)}}=\frac{8}{\sqrt{(11)}}.$$

Ex. 2. *A particle moves along a curve whose parametric equations are $x=e^{-t}$, $y=a\cos 3t$, $z=b\sin 3t$, where t is the time.—*

(*i*) *Determine its velocity and acceleration at any time t.*

(*ii*) *Find the magnitude of the velocity and acceleration at $t=0$.*

Sol. Here $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}=e^{-t}\,\mathbf{i}+a\cos 3t\mathbf{j}+b\sin 3t\mathbf{k}$.

(i) We have $\mathbf{v}=d\mathbf{r}/dt=-e^{-t}\,\mathbf{i}-3a\sin 3t\mathbf{j}+3b\cos 3t\,\mathbf{k}$,
and acceleration $\mathbf{a}=e^{-t}\,\mathbf{i}-9a\cos 3t\,\mathbf{j}-9b\sin 3t\,\mathbf{k}$.

(ii) At time $t=0$,

$$\mathbf{v}=-\mathbf{i}+3b\,\mathbf{k} \text{ and } \mathbf{a}=\mathbf{i}-9a\,\mathbf{j}.$$

$\therefore$ magnitude of vel. $=\sqrt{(1+9b^2)}$
and magnitude of accel. $=(1+9^2a^2)$.

Ex. 3. *The acceleration of a particle at any time $t\geqslant0$ is given by $\mathbf{a}=12\cos 2t\,\mathbf{i}-8\sin 2t\,\mathbf{j}+16t\,\mathbf{k}$. If the velocity and displacement are zero at $t=0$, find the velocity and displacement at any time.*

Sol. Here, $\mathbf{a}=d\mathbf{v}/dt=12\cos 2t\,\mathbf{i}-8\sin 2t\,\mathbf{j}+16t\mathbf{k}$.

Integrating w.r.t. 't' we have

$$\mathbf{v}=6\sin 2t\,\mathbf{i}+4\cos 2t\,\mathbf{j}+8t^2\,\mathbf{k}+\mathbf{c}_1,$$

where $\mathbf{c}_1$ is a constant vector.

But at $t=0$, $\mathbf{v}=\mathbf{0}$; $\therefore$ $\mathbf{0}=4\mathbf{j}+\mathbf{c}_1$ or $\mathbf{c}_1=-4\mathbf{j}$.

$\therefore$ Velocity $\mathbf{v}=6\sin 2t\,\mathbf{i}+4\cos 2t\,\mathbf{j}+8t^2\,\mathbf{k}-4\mathbf{j}$.

Again $\mathbf{v}=d\mathbf{r}/dt=6\sin 2t\,\mathbf{i}+4\cos 2t\,\mathbf{j}+8t^2\,\mathbf{k}-4\mathbf{j}$.

Integrating, w.r.t. 't', we have

$$\mathbf{r}=-3\cos 2t\,\mathbf{i}+2\sin 2t\,\mathbf{j}+\tfrac{8}{3}t^3\,\mathbf{k}-4t\,\mathbf{j}+\mathbf{c}_2,$$

where $\mathbf{c}_2$ is a constant vector.

But $t=0$, $\mathbf{r}=\mathbf{0}$; $\therefore$ $\mathbf{0}=-3\mathbf{i}+\mathbf{c}_2$ or $\mathbf{c}_2=3\mathbf{i}$.

$\therefore$ Displacement from the origin is given by

$$\mathbf{r}=-3\cos 2t\,\mathbf{i}+2\sin 2t\,\mathbf{j}+\tfrac{8}{3}t^3\,\mathbf{k}-4t\,\mathbf{j}+3\mathbf{i}$$

or $\quad \mathbf{r}=3(1-\cos 2t)\,\mathbf{i}+2(\sin 2t-2t)\,\mathbf{j}+\frac{8}{3}t^3\,\mathbf{k}.$

Ex. 4. *A point moves in a plane, its velocities parallel to the axes of x and y being $u+ey$ and $v+ex$ respectively, show that it moves in a conic section.*

Sol. Here, velocity parallel to x-axis$=\dfrac{dx}{dt}=u+ey$ ...(1)

and velocity parallel to y-axis$=\dfrac{dy}{dt}=v+ex.$ ...(2)

Dividing (1) by (2), we have $\dfrac{dx}{dy}=\dfrac{u+ey}{v+ex}$

or $(v+ex)\,dx=(u+ey)\,dy$, separating the variables.

Integrating, $vx+\frac{1}{2}ex^2=uy+\frac{1}{2}ey^2+C$

or $$e(x^2-y^2)+2vx-2uy-2C=0,$$

which is the equation of a conic section. Hence the particle moves in a conic section.

Note. Remember that a general equation of second degree in x and y represents a conic section.

Ex. 5. *A particle is moving with a constant velocity parallel to the axis of y and a velocity proportional to y parallel to the x-axis prove that it will describe a parabola.*

Sol. Here, velocity parallel to x-axis$=\dfrac{dx}{dt}\propto y$

or $$\frac{dx}{dt}=\lambda y, \qquad ..(1)$$

and velocity parallel to y-axis$=\dfrac{dy}{dt}=\mu,$ ...(2)

where λ and μ are constants.

Dividing (1) by (2), we have $\dfrac{dx}{dy}=\dfrac{\lambda}{\mu}y$

or $$2y\,dy=\frac{2\mu}{\lambda}\,dx=k\,dx,\ \text{ where } \frac{2\mu}{\lambda}=k.$$

Integrating, $y^2=kx+C$, which is the equation of a parabola. Hence, the particle describes a parabola.

Ex. 6. *Prove that the angular acceleration of the direction of motion of a point moving in a plane is*

$$\frac{v}{\rho}\frac{dv}{ds}-\frac{v^2}{\rho^2}\frac{d\rho}{ds}.$$

[Meerut 1988S; Kaupur 79]

Sol. Let P be the position of a moving point at time t. The direction of velocity at P is along the tangent to the path at P. If the tangent at P makes an angle ψ with the axis of x, then

$$\frac{d\psi}{dt}=\frac{d\psi}{ds}\cdot\frac{ds}{dt}=\frac{1}{\rho}v \qquad \left\{\because \frac{ds}{dt}=v \text{ and } \frac{ds}{d\psi}=\rho\right\}$$

Differentiating both sides w.r.t. 't', we have the angular acceleration of the direction of motion

$$=\frac{d^2\psi}{dt^2}=\frac{d}{dt}\left(\frac{1}{\rho}.v\right)=\frac{1}{\rho}\frac{dv}{dt}-\frac{v}{\rho^2}.\frac{d\rho}{dt}$$

$$=\frac{1}{\rho}\frac{dv}{ds}.\frac{ds}{dt}-\frac{v}{\rho^2}\frac{d\rho}{ds}.\frac{ds}{dt}=\frac{1}{\rho}\frac{dv}{ds}.v-\frac{v}{\rho^2}\frac{d\rho}{ds}v$$

$$=\frac{v}{\rho}\frac{dv}{ds}-\frac{v^2}{\rho^2}\frac{d\rho}{ds}.$$

Ex. 7. *A particle describes a parabola with uniform speed; show that its angular velocity about the focus S at any point P, varies inversely as $(SP)^{3/2}$.* **[Kanpur 81; Meerut 79]**

Sol. We know the pedal equation of a parabola referred to the focus S as the pole is given by $p^2=ar$. ...(1)

If v is the velocity (given to be constant) of the particle at any point P, then the angular velocity ω of the particle about S (*i.e.*, about pole) is given by

$$\omega=\frac{vp}{r^2}=\frac{v\sqrt{(ar)}}{r^2}=\frac{v\sqrt{a}}{r^{3/2}}=\frac{v\sqrt{a}}{(SP)^{3/2}}.$$

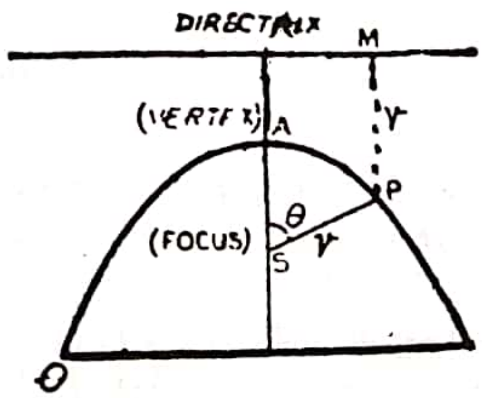
$$\therefore \quad \omega\propto\frac{1}{(SP)^{3/2}}.$$

Ex. 8. *Prove that the angular velocity of a projectile about the focus of its path varies inversely as its distance from the focus.* **[Meerut 75, 80, 83; Alld. 78]**

Sol. We know that the path of a projectile is a parabola whose pedal equation referred to its focus S as the pole is given by $p^2=ar$. ...(1)

Let v be the velocity of the projectile at the point P (r, θ).

Then $MP=PS=r$.



Since the velocity of the projectile at any point of its path is equal to the velocity acquired in falling freely from the directrix to that point, therefore,

$$v=\sqrt{(2g.MP)}=\sqrt{(2gr)}. \quad ...(2)$$

$\therefore$ The angular velocity ω of P about the focus S (*i.e.*, about pole) is given by

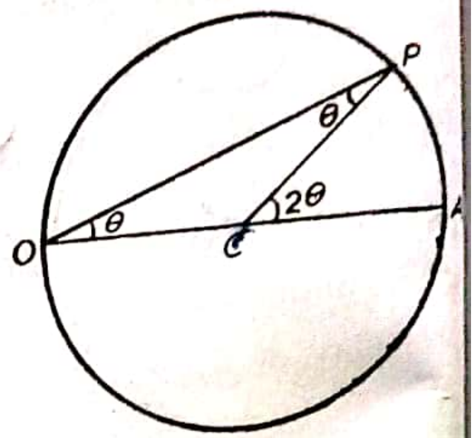
$$\omega=\frac{vp}{r^2}=\frac{\sqrt{(2gr)}.\sqrt{(ar)}}{r^2}=\frac{\sqrt{(2ag)}}{r}=\frac{\sqrt{(2ag)}}{SP}.$$

$$\therefore \quad \omega \propto \frac{1}{SP}.$$

Hence, the angular velocity of P about the focus S varies inversely as its distance from the focus.

Ex. 9. *If a point moves along a circle with constant speed, prove that its angular velocity about any point on the circle is half of that about the centre.* **[Meerut 75]**

Sol. Let O be a point on a circle. Take O as the pole and the diameter through O as the initial line.



Let P be the position of the particle at any time t, such that $\angle POA=\theta$. Then, $\angle PCA=2\theta$, where C is the centre of the circle.

Angular velocity of P about O $=d\theta/dt$ and angular velocity of P about the centre C

$$=\frac{d}{dt}(2\theta)=2\frac{d\theta}{dt}$$

$=2$ (angular velocity of P about O).

Ex. 10 *A body rotates with uniform angular acceleration α. If ω is the angular velocity when the body has turned through an angle θ from rest, show that $\omega^2=2\alpha\theta$.*

Sol. Given that, angular acceleration $=\dfrac{d^2\theta}{dt^2}=\alpha$ ...(1)

and angular velocity $=\dfrac{d\theta}{dt}=\omega$ ...(2)

Multiplying both sides of (1) by $2\dfrac{d\theta}{dt}$ and then integrating,

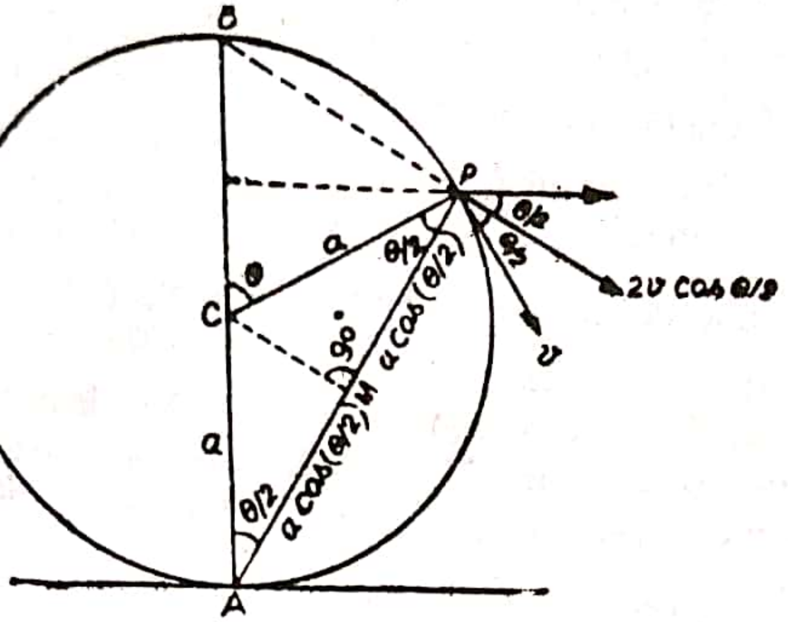
we have $\left(\dfrac{d\theta}{dt}\right)^2=2\alpha\theta+C$, where C is constant of integration.

But initially when $\theta=0, \dfrac{d\theta}{dt}=0$. $\therefore$ $0=0+C$ or $C=0$.

$$\therefore \quad \left(\frac{d\theta}{dt}\right)^2=2\alpha\theta. \qquad ...(3)$$

From (2) and (3), we have $\omega^2=2\alpha\theta$.

Ex. 11. *A wheel rolls along a straight road with constant speed v. Show that the actual velocity of P is $v.(AP/CP)$, where A is the point of contact of the wheel with the road and C is the centre of the wheel. Also find its direction. Find also the angular velocity of P relative to A.* **[Meerut 79]**



Sol. Let P be a point on the wheel. The point P possesses two velocities (i) v parallel to the road, and (ii) v along the tangent at P to the wheel as shown in the figure.

Let $\angle BCP=\theta$. Then the angle between the horizontal line through P and the tangent to the wheel at P is also θ.

The actual velocity of P=the resultant of the two equal velocities v and v at $P=\sqrt{(v^2+v^2+2v^2 \cos \theta)}=2v \cos \frac{1}{2}\theta$

$$=v.\frac{2a \cos \frac{1}{2}\theta}{a}, \text{ where } a \text{ is the radius of the wheel}$$

$$=v.\frac{AP}{CP}.$$

The direction of the actual velocity of P bisects the angle between the two velocities at P and so it makes an angle $\theta/2$ with the horizontal. Now the straight line BP makes an angle $\frac{1}{2}\pi-\frac{1}{2}\theta$ with the vertical BA and so it makes an angle $\frac{1}{2}\theta$ with the horizontal.

Hence the direction of the actual velocity of P is along BP, where B is the highest point of the wheel. The line BP is also perpendicular to AP.

Now the actual velocity of $A=2v \cos \frac{1}{2}\pi=0$. [$\because$ for A, $\theta=\pi$]

The angular velocity of P relative to A

$$=\frac{\text{velocity of } P \text{ relative to } A \text{ in a direction perpendicular to } AP}{AP}.$$

[Refer § 4]

Since the actual velocity of A is zero, therefore the velocity of P relative to A is the actual velocity of P. But as just shown the actual velocity of P is $v.(AP/CP)$ and its direction is perpendicular to AP.

$\therefore$ the angular velocity of P relative to A

$$=\frac{v.(AP/CP)}{AP}=\frac{v}{CP}=\frac{v}{a},$$

where a is the radius of the wheel.

Remark. Velocity of P relative to C is v and is along the tangent to the circle at P *i.e.*, in a direction perpendicular to CP.

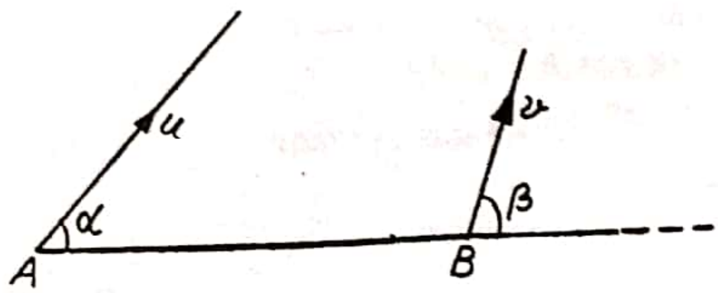
$\therefore$ angular velocity of P relative to $C=\frac{v}{CP}=\frac{v}{a}$.

Ex. 12 (a). *The line joining two points A, B is of constant length a and the velocities of A, B are in directions which make angles α and β respectively with AB. Prove that the angular velocity of AB is $\frac{u \sin(\alpha-\beta)}{a \cos\beta}$, where u is the velocity of A.*

[Meerut 1973, 82 P, 86 S]

Sol. Let v be the velocity of B.

Since the distance AB always remains a, therefore the velocity of A relative to B in the direction AB is zero.



$$\therefore \quad u \cos\alpha - v\cos\beta = 0$$

or
$$u\cos\alpha = v\cos\beta$$

or
$$v=\frac{u\cos\alpha}{\cos\beta}. \qquad \ldots(1)$$

Now the angular velocity of AB

= the relative angular velocity of A with respect to B

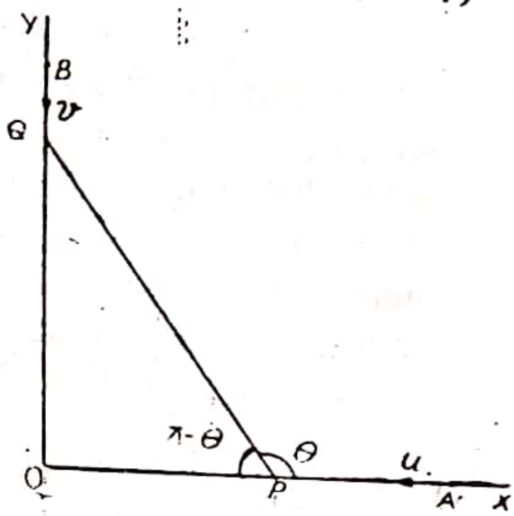
$$=\frac{\text{Velocity of } A \text{ relative to } B \text{ in a direction perpendicuiar to } AB}{AB}$$

$$=\frac{u\sin\alpha-v\sin\beta}{a}=\frac{u\sin\alpha-(u\cos\alpha/\cos\beta)\sin\beta}{a}, \text{ [from (1)]}$$

$$=\frac{u(\sin\alpha\cos\beta-\cos\alpha\sin\beta)}{a\cos\beta}=\frac{u\sin(\alpha-\beta)}{a\cos\beta}.$$

Ex. 12. (b) *Two points are moving with uniform velocities u, v in perpendicular lines OX and OY, the motions being towards O. If initially, their distances from the origin are a and b respectively, calculate the angular velocity of the line joining them at the end of t seconds, and show that it is greatest when $t=(au+bv)/(u^2+v^2)$.*

Sol. Let A and B be the initial positions of the two points moving with uniform velocities u and v in the perpendicular lines OX and OY, towards O, such that $OA=a$ and $OB=b$.



Let P and Q be the positions of the particles after time t. Then

$$AP=ut \text{ and } BQ=vt.$$

$\therefore$ $OP=OA-AP=a-ut$

and $OQ=OB-BQ=b-vt.$

If $\angle QPX=\theta$, then

$$\tan OPQ=\tan(\pi-\theta)=\frac{OQ}{OP}=\frac{b-vt}{a-ut}$$

or $$-\tan\theta=\frac{b-vt}{a-ut}, \quad \text{or} \quad \tan\theta=\frac{vt-b}{a-ut}.$$

$$\therefore \quad \theta=\tan^{-1}\left(\frac{vt-b}{a-ut}\right).$$

$\therefore$ Angular velocity of PQ (line joining the two particles at the end of t seconds)

$$=\omega=\frac{d\theta}{dt}=\frac{1}{1+\{(vt-b)/(a-ut)\}^2}\cdot\frac{v(a-ut)-(vt-b)(-u)}{(a-ut)^2}$$

$$=\frac{av-bu}{(a-ut)^2+(vt-b)^2}.$$

Now ω is greatest when the denominator $(a-ut)^2+(vt-b)^2$ is least.

Let $l=(a-ut)^2+(vt-b)^2$.

Then $\dfrac{dl}{dt}=-2u(a-ut)+2v(vt-b)=2(u^2+v^2)t-2(au+bv)$.

But l is maximum or minimum, if

$$\frac{dl}{dt}=2(u^2+v^2)t-2(au+bv)=0,$$

giving $t=(au+bv)/(u^2+v^2)$.

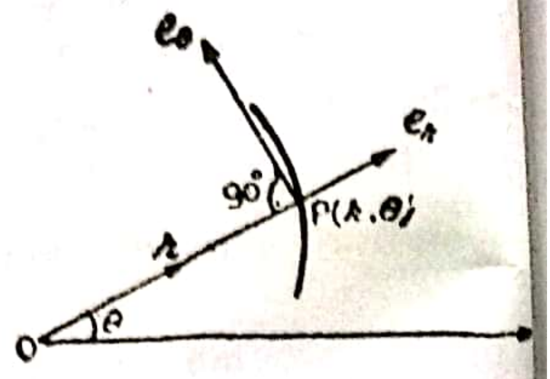
Also $\frac{d^2l}{dt^2}=2(u^2+v^2)$, which is positive.

$\therefore$ l is least or the angular velocity ω is greatest when
$l=(au+bv)/(u^2+v^2)$.

§ 6. **Radial and Transverse Velocities and Accelerations.**
[Meerut 1979, 82P, 82S, 84, 86, 87P, 88S, 89, 90, 90; Agra 80, 87; Gorakh. 75; Ranchi 73; Luck. 77, 8; Kanpur 78, 81; Rohilkhand 87; Allahabad 7

Radial and Transverse Velocities (Definition). Let P be th position of a moving particle at time t and $\mathbf{r}$ its position vect w.r.t. the origin O. *Then the resolved parts of the velocity at along and perpendicular to the radius vector OP are called th* **radial** and **transverse velocities** *of the particle at P. Radial an transverse velocities are taken positive in the directions of r and increasing respectively.*

Let (r, θ) be the polar coordinates of the point P w.r.t. the pole O and the initial line OX. If $\mathbf{e}_r$ and $\mathbf{e}_\theta$ are the unit vectors along and perpendicular to OP, then,



$$\frac{d\mathbf{e}_r}{dt}=\frac{d\theta}{dt}\mathbf{e}_\theta \text{ and } \frac{d\mathbf{e}_\theta}{dt}=-\frac{d\theta}{dt}\mathbf{e}_r \quad ...(1)$$

[Refer § 3]

Now $\overrightarrow{OP}=\mathbf{r}=r\mathbf{e}_r$. $\quad [\because \ |\mathbf{r}|=OP=r$

If $\mathbf{v}$ is the velocity vector of the particle at P, then

$$\mathbf{v}=\frac{d\mathbf{r}}{dt}=\frac{d}{dt}(r\mathbf{e}_r), \quad [\because \ \mathbf{r}=r\mathbf{e}_r]$$

$$=\frac{dr}{dt}\mathbf{e}_r+r\frac{d\mathbf{e}_r}{dt}=\frac{dr}{dt}\mathbf{e}_r+r\frac{d\theta}{dt}\mathbf{e}_\theta. \quad ...(2$$

$$\left[\because \text{ from (1), } \frac{d\mathbf{e}_r}{dt}=\frac{d\theta}{dt}\mathbf{e}_\theta\right]$$

Thus the vector $\mathbf{v}$ has been expressed as a linear combinatio of the vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$.

$\therefore$ **Radial component** of velocity at P = the coeff. of th vector $\mathbf{e}_r$ in (2) $=\frac{dr}{dt}=\dot{r}$, positive in the direction of the vector $\mathbf{e}$ i.e., +ive in the direction of r increasing,

and transverse component of velocity at P

= the coeff. of the vector $\mathbf{e}_\theta$ in (2)

$=r\dfrac{d\theta}{dt}=r\dot{\theta}$, +ive in the direction of θ increasing.

Thus **remember** that at any time t,

the $\boxed{\textbf{radial velocity}=\dfrac{dr}{dt},}$ +ive in the direction of r increasing

and the $\boxed{\textbf{transverse velocity}=r\dfrac{d\theta}{dt},}$ +ive in the direction of θ increasing.

Also the **resultant velocity** $v=\sqrt{\left\{\left(\dfrac{dr}{dt}\right)^2+\left(r\dfrac{d\theta}{dt}\right)^2\right\}}.$

Radial and transverse accelerations (Definition). *The resolved parts of the acceleration at P along and perpendicular to the radius vector OP are called the* **radial** *and* **transverse accelerations** *of the particle at P.*

If $\mathbf{a}$ is the acceleration vector of the particle at P, then

$$\mathbf{a}=\frac{d\mathbf{v}}{dt}=\frac{d}{dt}\left(\frac{dr}{dt}\mathbf{e}_r+r\frac{d\theta}{dt}\mathbf{e}_\theta\right), \quad \left[\text{substituting for } \mathbf{v} \text{ from (2)}\right]$$

$$=\left\{\frac{d}{dt}\left(\frac{dr}{dt}\right)\right\}\mathbf{e}_r+\left(\frac{dr}{dt}\right)\frac{d\mathbf{e}_r}{dt}+\left\{\frac{d}{dt}\left(r\frac{d\theta}{dt}\right)\right\}\mathbf{e}_\theta+\left(r\frac{d\theta}{dt}\right)\frac{d\mathbf{e}_\theta}{dt}$$

$$=\frac{d^2r}{dt^2}\mathbf{e}_r+\frac{dr}{dt}\cdot\frac{d\theta}{dt}\mathbf{e}_\theta+\left(\frac{dr}{dt}\frac{d\theta}{dt}+r\frac{d^2\theta}{dt^2}\right)\mathbf{e}_\theta-r\frac{d\theta}{dt}\cdot\frac{d\theta}{dt}\mathbf{e}_r \text{ [From (1)]}$$

$$=\left\{\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2\right\}\mathbf{e}_r+\left\{2\frac{dr}{dt}\frac{d\theta}{dt}+r\frac{d^2\theta}{dt^2}\right\}\mathbf{e}_\theta.$$

Thus the vector $\mathbf{a}$ has been expressed as a linear combination of the vectors $\mathbf{e}_r$ and $\mathbf{e}_\theta$. The coefficients of $\mathbf{e}_r$ and $\mathbf{e}_\theta$ in this linear combination will give us respectively the radial and transverse accelerations of the particle at P.

$\therefore$ The radial component of acceleration at $P=\dfrac{d^2r}{dt^2}-r\left(\dfrac{d\theta}{dt}\right)^2$

$=\ddot{r}-r\dot{\theta}^2$, +ive in the direction of r increasing,

and the transverse component of acceleration at P

$=2\dfrac{dr}{dt}\dfrac{d\theta}{dt}+r\dfrac{d^2\theta}{dt^2}=\dfrac{1}{r}\dfrac{d}{dt}\left(r^2\dfrac{d\theta}{dt}\right)$, +ive in the direction of θ increasing.

Thus **remember** that at any time t,

the $\boxed{\textbf{radial acceleration}=\dfrac{d^2r}{dt^2}-r\left(\dfrac{d\theta}{dt}\right)^2,}$ +ive in the direction of r increasing

and the $\boxed{\textbf{transverse acceleration}=\dfrac{1}{r}\dfrac{d}{dt}\left(r^2\dfrac{d\theta}{dt}\right),}$ +ive in the direction of θ increasing.

Also the **resultant acceleration**

$$\sqrt{\left[\left\{\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2\right\}^2+\left\{\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)\right\}^2\right]}$$

Illustrative Examples

Ex. 13. *If the angular velocity of a point moving in a plane curve be constant about a fixed origin, show that its transverse acceleration varies as its radial velocity.* **[Lucknow 1976]**

Sol. Here, the angular velocity $=\frac{d\theta}{dt}=\omega$ (const.) ..(1)

$\therefore$ Transverse aeceleration of the point

$$=\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$$

$$=\frac{1}{r}\frac{d}{dt}(r^2\omega)=\frac{2\omega r}{r}\frac{dr}{dt}=2\omega.\text{ (radial velocity)}$$

or transverse acceleration $\propto$ radial velocity.

Ex. 14. *A point P describes, with a constant angular velocity about O, the equiangular spiral $r=ae^\theta$, O being the pole of the spiral. Obtain the radial and transverse accelerations of P.* **[Kanpur 1979]**

Sol. Here given that the path of the particle is $r=ae^\theta$ ...(1)

and the angular velocity $=\frac{d\theta}{dt}=\omega$ (const.) ...(2)

From (1), we have $\frac{dr}{dt}=ae^\theta\cdot\frac{d\theta}{dt}=r\omega$?

and $\frac{d^2r}{dt^2}=\omega\frac{dr}{dt}=\omega.r\omega=r\omega^2$.

$\therefore$ Radial acceleration of $P=\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2$

$$=r\omega^2-r\omega^2=0,$$

and transverse acceleration of $P=\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)=\frac{1}{r}\frac{d}{dt}(r^2\omega)$

$$=\frac{2\omega r}{r}\cdot\frac{dr}{dt}=2\omega.r\omega=2\omega^2r.$$

Ex. 15. *A particle P describes a curve with constant velocity and its angular velocity about a given fixed point O varies inversely as its distance from O; show that the curve is an equiangular spiral.* **[Lucknow 1978]**

Sol. Let the velocity of the particle be equal to v (const.). Given that the angular velocity $d\theta/dt$ of the particle about the fixed point O varies inversely as its distance r from O, we have

$$\frac{d\theta}{dt} \propto \frac{1}{r} \qquad \text{or } \frac{d\theta}{dt}=\frac{\lambda}{r}, \qquad ...(1)$$

where λ is a constant.

Now the velocity v of the particle is the resultant of its radial and transverse velocities.

$\therefore \quad v=\sqrt{[(\text{radial vel.})^2+(\text{trans. vel.})^2]}$

$$=\sqrt{\left[\left(\frac{dr}{dt}\right)^2+\left(r\frac{d\theta}{dt}\right)^2\right]}$$

or $\quad v^2=\left(\frac{dr}{dt}\right)^2+\lambda^2,$ [from (1)]

or $\quad \frac{dr}{dt}=\sqrt{(v^2-\lambda^2)}=\mu,$ where $\sqrt{(v^2-\lambda^2)}=\mu$, a constant

or $$\frac{dr}{d\theta}\cdot\frac{d\theta}{dt}=\mu$$

or $$\frac{dr}{d\theta}\frac{\lambda}{r}=\mu, \qquad \text{[from (1)]}$$

or $$\frac{dr}{r}=\frac{\mu}{\lambda}d\theta.$$

Integrating, $\log r=\frac{\mu}{\lambda}\theta+\log c$, where c is a constant

or $$\log\left(\frac{r}{c}\right)=\frac{\mu}{\lambda}\theta.$$

$\therefore$ $$r=ce^{(\mu/\lambda)\theta}$$

or $\quad r=ce^{k\theta},$ where $\mu/\lambda=k$, a constant.

This is the equation of an equiangular spiral.

Hence the curve is an equiangular spiral.

Ex. 16. *The velocities of a particle along and perpendicular to a radius vector from a fixed origin are λr^2 and $\mu\theta^2$, where λ and μ are constants; find the polar equation of the path of the particle and also its radial and transverse accelerations in terms of r and θ only.*

[Meerut 1974, 77, 84 S; Agra 77; Rohilkhand 88]

Sol. Here, it is given that

radial velocity$=\frac{dr}{dt}=\lambda r^2$ ...(1)

and transverse vel.$=r\frac{d\theta}{dt}=\mu\theta^2.$ ...(2)

To find the equation of the path, we have to eliminate t between (1) and (2).

Dividing (1) by (2), we have

$$\frac{1}{r}\frac{dr}{d\theta}=\frac{\lambda r^2}{\mu\theta^2}$$

or $$\mu\frac{dr}{r^3}=\lambda\frac{d\theta}{\theta^2}.$$

Integrating, we have,

$$-\frac{\mu}{2r^2}=-\frac{\lambda}{\theta}+c,$$ where c is a constant

or $$\frac{\mu}{2r^2}+c=\frac{\lambda}{\theta},$$

which is the equation of the path of the particle.

Now, radial acceleration $=\dfrac{d^2r}{dt^2}-r\left(\dfrac{d\theta}{dt}\right)^2$

$$=\frac{d}{dt}\left(\frac{dr}{dt}\right)-\frac{1}{r}\left(r\frac{d\theta}{dt}\right)^2$$

$$=\frac{d}{dt}\left(\lambda r^2\right)-\frac{1}{r}\left(\mu\theta^2\right)^2$$

[substituting from (1) and (2)]

$$=2\lambda r\frac{dr}{dt}-\frac{\mu^2\theta^4}{r}$$

$$=2\lambda r.\lambda r^2-\frac{\mu^2\theta^4}{r}$$ [from (1)]

$$=2\lambda^2r^3-(1/r)\,\mu^2\theta^4.$$

Again transverse acceleration $=\dfrac{1}{r}\dfrac{d}{dt}\left(r^2\dfrac{d\theta}{dt}\right)$

$$=\frac{1}{r}\frac{d}{dt}\left(r.\mu\theta^2\right)$$

[substituting from (2)]

$$=\frac{\mu}{r}\left(\frac{dr}{dt}\theta^2+2r\theta\frac{d\theta}{dt}\right)$$

$$=\frac{\mu}{r}\left(\lambda r^2\theta^2+2\theta.\mu\theta^2\right)$$

[from (1) and (2)]

$$=\lambda\mu r\theta^2+(1/r).2\mu^2\theta^3.$$

Ex. 17. *The velocities of a particle along and perpendicular to the radius vector are λr and $\mu\theta$; find the path and show that the accelerations along and perpendicula to the radius vector are $\lambda^2r-\mu^2\theta^2/r$ and $\mu\theta\,(\lambda+\mu/r)$.*

[Meerut 1975, 76, 78, 82 S, 84; Kanpur 79; Luck. 81; Rohilkhand 86; Agra 78; Allahabad 78]

Sol. Here, it is given that

$$\text{radial velocity} = \frac{dr}{dt} = \lambda r \quad \ldots(1)$$

and $$\text{transverse velocity} = r\frac{d\theta}{dt} = \mu\theta. \quad \ldots(2)$$

Dividing (1) by (2), we have

$$\frac{dr}{r\,d\theta} = \frac{\lambda r}{\mu\theta}$$

or $$\frac{\mu}{\lambda}\frac{dr}{r^2} = \frac{d\theta}{\theta}.$$

Integrating, $$-\frac{\mu}{\lambda r} = \log\theta + \log c = \log(c\theta).$$

$\therefore$ $$c\theta = e^{-\mu/\lambda r}$$

or $$\theta = ae^{b/r},$$

which is the equation of the path, where a and b are constants.

Now radial acceleration $$= \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2$$

$$= \frac{d}{dt}\left(\frac{dr}{dt}\right) - \frac{1}{r}\left(r\frac{d\theta}{dt}\right)^2$$

$$= \frac{d}{dt}(\lambda r) - \frac{1}{r}(\mu\theta)^2 \quad \text{[from (1) \& (2)]}$$

$$= \lambda\frac{dr}{dt} - \frac{\mu^2\theta^2}{r} = \lambda^2 r - \mu^2\theta^2/r, \text{ [from (1)]}.$$

Again transverse acceleration $$= \frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$$

$$= \frac{1}{r}\frac{d}{dt}(r \cdot \mu\theta) \quad \text{[from (2)]}$$

$$= \frac{\mu}{r}\left(\frac{dr}{dt}\theta + r\frac{d\theta}{dt}\right)$$

$$= \frac{\mu}{r}(\lambda r\theta + \mu\theta)$$

[from (1) and (2)]

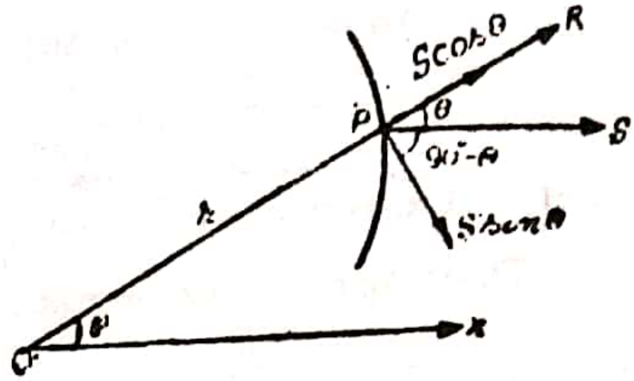
$$= \mu\theta(\lambda + \mu/r).$$

Ex. 18. *The acceleration of a point moving in a plane curve is resolved into two components, one parallel to the initial line and the other along the radius vector; prove that these components are*

$$-\frac{1}{r\sin\theta}\cdot\frac{d}{dt}(r^2\dot\theta) \text{ and } \frac{\cot\theta}{r}\frac{d}{dt}(r^2\dot\theta) + \ddot r - r\dot\theta^2.$$

[Gorakhpur 1975]

Sol. Let $P(r, \theta)$ be the position of a point moving in a plane, at any time t. Let S and R be the components of the acceleration



of the point P parallel to OX and along the radius vector OP respectively.

Then resolving the accelerations of P in the radial and transverse directions, we have radial acceleration, $\ddot{r}-r\dot{\theta}^2$

$$=R+S\cos\theta, \qquad \ldots(1)$$

and transverse acceleration, $\dfrac{1}{r}\dfrac{d}{dt}\left(r^2\dot{\theta}\right)$

$$=-S\sin\theta. \qquad \ldots(2)$$

From (2), we have

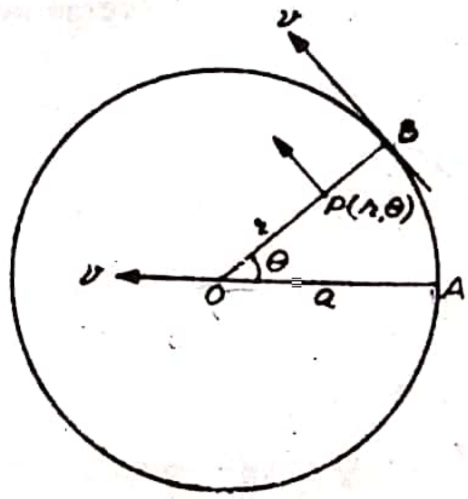
$$S=-\frac{1}{r\sin\theta}\frac{d}{dt}\left(r^2\dot{\theta}\right)$$

Substituting the value of S in (1), we have

$$R=-\cos\theta\left\{-\frac{1}{r\sin\theta}\frac{d}{dt}\,(r^2\dot{\theta})\right\}+\ddot{r}-r\dot{\theta}^2$$

$$=\frac{\cot\theta}{r}\frac{d}{dt}\,(r^2\dot{\theta})+\ddot{r}-r\dot{\theta}^2.$$

Ex. 19. *An insest crawls at a constant rate u along the spoke of a cart wheel of radius a, the cart is moving with velocity v. Find the acceleration along and perpendicular to the spoke.* **[Kanpur 1975]**



Sol. Initially let OA be the position of the spoke of a wheel of radius a. The insect crawls at a constant rate u along the spoke OA starting from O. If OB is the position of the spoke after time t and P the position of the insect at that instant, then $OP=ut$.

Referred to O as pole and OA as initial line, let (r, θ) be the polar coordinates of P. Then $\angle POA=\theta$,

and $$r=OP=ut. \qquad \ldots(1)$$

Since the cart moves with velocity v, therefore, the velocity of any point of the wheel relative to the centre O is also v and is along the tangent to the wheel at that point.

Now the angular velocity of B about $O=\dfrac{d\theta}{dt}$.

But the angular velocity of B relative to O

$$=\frac{\text{the velocity of } B \text{ relative to } O \text{ in a direction perpendicular to } OB}{OB}$$

$$=\frac{v}{OB}=\frac{v}{a}.$$

$$\therefore \quad \frac{d\theta}{dt}=\frac{v}{a}. \qquad \ldots(2)$$

Now the acceleration of ~~the insect along~~ the spoke

$=$the radial acceleration of P

$$=\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2$$

$$=\frac{d^2}{dt^2}(ut)-ut.\left(\frac{v}{a}\right)^2 \qquad \text{[substituting from (1) and (2)]}$$

$$=0-ut\frac{v^2}{a^2}=-ut\frac{v^2}{a^2}.$$

Again the acceleration of the insect perpendicular to the spoke

$=$the transverse acceleration of P

$$=\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right)$$

$$=\frac{1}{ut}\frac{d}{dt}\left(u^2t^2.\frac{v}{a}\right), \qquad \text{[from (1) and (2)]}$$

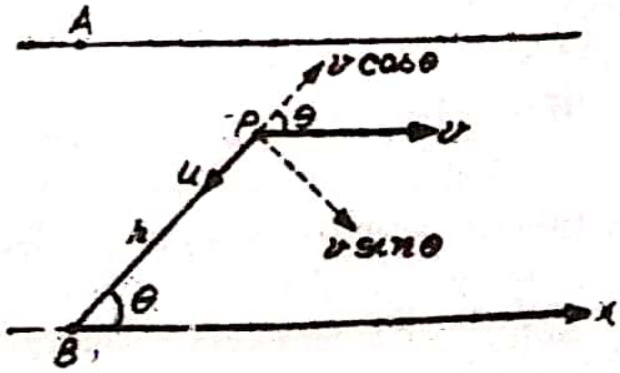
$$=\frac{1}{ut}.\frac{2u^2tv}{a}=\frac{2uv}{a}.$$

Ex. 20. *A boat which is rowed with constant velocity u starts from a point A on the bank of a river which flows with a constant velocity v, and it points always towards a point B on the other bank exactly opposite to A; find the equation of the path of the boat.*

If $v=u$, show that the path is a parabola whose focus is B.

[Meerut 1978]

Sol. Let the boat start with velocity u from the point A of the bank of the river flowing with velocity v. Let B be a point on the other bank of the river exactly opposite to A.



Take the point B as pole and the bank BX as the initial line.

Let $P(r, \theta)$ be the position of the boat after time t. Then the boat at P will have two velocities (i) u, along PB and (ii) v, parallel to BX. Resolving the velocities of P along and perpendicular to the radius vector BP, we have

$$\text{the radial velocity}=\frac{dr}{dt}=v\cos\theta-u \quad ...(1)$$

$$\text{and the transverse velocity}=r\frac{d\theta}{dt}=-v\sin\theta. \quad ...(2)$$

Dividing (1) by (2), we have

$$\frac{dr}{rd\theta}=\frac{v\cos\theta-u}{-v\sin\theta}$$

or $$\frac{dr}{r}=\left(-\cot\theta+\frac{u}{v}\operatorname{cosec}\theta\right)d\theta.$$

Integrating, we have

$$\log r=-\log\sin\theta+\frac{u}{v}\log\tan\frac{\theta}{2}+\log C,$$

where C is a constant

or $\quad \log C-\log r=\log\sin\theta-(u/v)\log\tan\frac{1}{2}\theta$

or $\quad \log(C/r)=\log\{\sin\theta/(\tan\frac{1}{2}\theta)^{u/v}\}$

or $\quad (C/r)=\sin\theta/(\tan\frac{1}{2}\theta)^{u/v}, \quad ...(3)$

which is the equation of the path of the boat.

If $v=u$, then from (3), the equation of the path of the boat is given by

$$\frac{C}{r}=\frac{\sin\theta}{\tan\frac{1}{2}\theta}=\frac{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta.\cos\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}=2\cos^2\frac{1}{2}\theta,$$

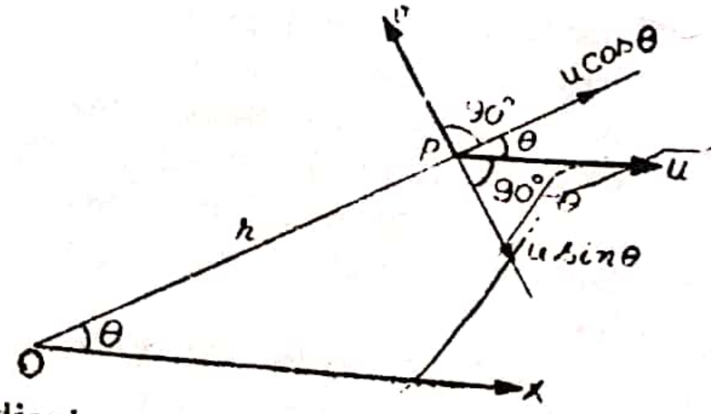
or $$\frac{C}{r}=1+\cos\theta,$$

which is the equation of a parabola with the focus at the pole B.

Ex. 21. *Show that the path of a point P which possesses two constant velocities u and v, the first of which is in a fixed direction and the other is perpendicular to the radius OP drawn from a fixed point O, is a conic whose focus is O and eccentricity is u/v.*

[Meerut 1976, 80, 81, 82, 86, 90; Kanpur 73; Lucknow 75; Gorakhpur 73; Allahabad 77]

Sol. Take the fixed point O as pole and the fixed direction as the initial line OX.



Let $P(r, \theta)$ be the position of the particle at any time t. Then according to the question P possesses two constant velocities : (i) u, in the fixed direction OX and (ii) v, perpendicular to OP as shown in the fig.

Resolving the velocities of P along and perpendicular to the radius vector OP, we have

$$\text{the radial velocity} = \frac{dr}{dt} = u \cos \theta, \qquad \text{...(1)}$$

$$\text{and the transverse velocity} = r\frac{d\theta}{dt} = v - u \sin \theta. \qquad \text{...(2)}$$

Dividing (1) by (2), we have

$$\frac{dr}{r\,d\theta} = \frac{u \cos \theta}{v - u \sin \theta}$$

or
$$\frac{dr}{r} = \frac{u \cos \theta}{v - u \sin \theta}\, d\theta.$$

Integrating, $\log r = -\log (v - u \sin \theta) + \log C$

or
$$\log \left(\frac{C}{r}\right) = \log (v - u \sin \theta)$$

or
$$\frac{C}{r} = v - u \sin \theta$$

or
$$\frac{C}{r} = v + u \cos (\tfrac{1}{2}\pi + \theta)$$

or
$$\frac{C/v}{r} = 1 + \frac{u}{v} \cos (\tfrac{1}{2}\pi + \theta), \qquad \text{...(3)}$$

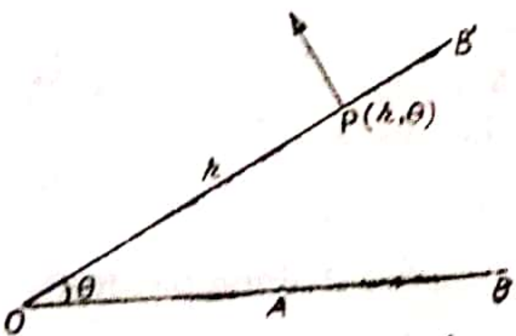
which is the path of the particle.

The equation (3) is of the form $l/r = 1 + e \cos \theta$, which is a conic whose focus is the pole O and eccentricity e is u/v.

Hence, the path of P is a conic whose focus is O and eccentricity is u/v.

Ex. 22. *A straight smooth tube revolves with angular velocity ω in a horizontal plane about one extremity which is fixed. If at zero time a particle inside it be at a distance a from a fixed end and moving with velocity V along the tube, show that its distance at time t is $a \cosh \omega t + (V/\omega) \sinh \omega t$.* **[Lucknow 1977; Meerut 90P]**

Sol. Let the straight smooth tube OB rotate about the fixed end O. Let OB, the initial position of the tube, be taken as the initial line and let A be the initial position of the particle. After time t let OB' be the position of the tube and P that of the particle such that $\angle B'OB = \theta$ and $OP = r$.



The tube being smooth, the only force acting on the particle at P in the plane of motion is the resolved part of the reaction of the tube and is perpendicular to OP. Thus there is no force on the particle at P along the radius vector OP. Therefore the radial acceleration of P is zero

i.e.,
$$\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2=0$$

or
$$\frac{d^2r}{dt^2}-\omega^2r=0, \qquad \ldots(1)$$

because $\frac{d\theta}{dt}=\omega=$constant, as given in the question.

The differential equation (1) may be written as

$$(D^2-\omega^2)\,r=0, \text{ where } D\equiv\frac{d}{dt}.$$

Its general solution is
$$r=A\cosh\omega t+B\sinh\omega t. \qquad \ldots(2)$$

$$\therefore \quad \frac{dr}{dt}=A\omega\sinh\omega t+B\omega\cosh\omega t. \qquad \ldots(3)$$

Initially when $t=0$, $r=a$ and $\frac{dr}{dt}=V$.

$\therefore$ From (2) and (3), we have $A=a$ and $B=V/\omega$.

Putting in (2), the distance r of the particle at time t is given by
$$r=a\cosh\omega t+(V/\omega)\sinh\omega t.$$

Ex. 23. *A ring which can slide on a thin long smooth rod rests at a distance d from one end O. The rod is then set revolving uniformly about O in a horizontal plane; show that in space the ring describes the curve $r=d\cosh\theta$.* **[Allahabad 1978]**

Sol. Let OB be the initial position of the rod and A the initial position of the ring which is at rest such that $OA=d$.

(See figure of Ex. 22)

The rod is set revolving uniformly about O in a horizontal plane. Let OB' be the position of the rod after time t and P the position of the ring at this time such that $OP=r$ and $\angle B'OB=\theta$.

The rod being smooth, there is no force acting on the ring along the radius vector OP. Therefore the radial acceleration of the ring is zero

i.e.,
$$\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2=0. \qquad \ldots(1)$$

Since the rod revolves uniformly,

$$\therefore \quad \frac{d\theta}{dt}=\omega \text{ (constant)}. \qquad \ldots(2)$$

From (1) and (2), we have

$$\frac{d^2r}{dt^2}=r\omega^2.$$

Multiplying both sides by $2\frac{dr}{dt}$ and integrating, we have

$$\left(\frac{dr}{dt}\right)^2=r^2\omega^2+A, \text{ where } A \text{ is a constant.}$$

But initially when $r=d$, $\frac{dr}{dt}=0$.

$$\therefore\quad 0=d^2\omega^2+A \quad\text{or}\quad A=-d^2\omega^2.$$

$$\therefore\quad \left(\frac{dr}{dt}\right)^2=r^2\omega^2-d^2\omega^2=\omega^2\,(r^2-d^2) \qquad \ldots(3)$$

or $\quad \frac{dr}{dt}=\omega\sqrt{(r^2-d^2)}$

or $\quad \frac{dr}{d\theta}\cdot\frac{d\theta}{dt}=\omega\sqrt{(r^2-d^2)}$

or $\quad \frac{dr}{d\theta}\cdot\omega=\omega\sqrt{(r^2-d^2)}$, $\left[\text{substituting } \frac{d\theta}{dt}=\omega \text{ from (2)}\right]$

or $\quad d\theta=\frac{dr}{\sqrt{(r^2-d^2)}}.$

Integrating, we have

$$\theta+B=\cosh^{-1}(r/d), \text{ where } B \text{ is a constant.}$$

But initially $r=d$, $\theta=0$; $\quad\therefore\quad B=\cosh^{-1}1=0.$

$$\therefore\quad \theta=\cosh^{-1}(r/d)$$

or $\quad r=d\cosh\theta,$

which is the equation of the curve described by the ring in space.

Ex. 24. *A small ring is at rest on a smooth straight horizontal rod of length a at distance b from one end of the rod. The rod is then suddenly set rotating in a horizontal plane about the end O with constant angular velocity ω. Prove that the ring will leave the rod with velocity* $\omega\sqrt{(2a^2-b^2)}$ *after a time* $(1/\omega)\cosh^{-1}(a/b)$. **[Meerut 1970]**

Sol. Proceed as in Ex. 23, to get

$$\frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2=0 \quad\text{and}\quad \frac{d\theta}{dt}=\omega \text{ (constant).}$$

$$\therefore\quad \frac{d^2r}{dt^2}=r\omega^2.$$

Multiplying both sides by $2\frac{dr}{dt}$ and integrating, we have

$$\left(\frac{dr}{dt}\right)^2=r^2\omega^2+A, \text{ where } A \text{ is a constant.}$$

But initially $r=b$ and $\frac{dr}{dt}=0$.

$\therefore \quad 0=\omega^2b^2+A \quad$ or $\quad A=-\omega^2b^2$.

$$\therefore \quad \left(\frac{dr}{dt}\right)^2=r^2\omega^2-\omega^2b^2=\omega^2(r^2-b^2). \qquad \ldots(1)$$

The particle will leave the rod when it reaches the other end B of the rod. At B, $r=a$ and so from (1), the radial velocity dr/dt of the particle at B is given by

$$\left(\frac{dr}{dt}\right)^2=\omega^2(a^2-b^2).$$

Also the transverse velocity of the particle at $B=r\frac{d\theta}{dt}=a\omega$.

Therefore, if the particle leaves the rod at B with velocity V, then

$$V=\sqrt{\left[\left(\frac{dr}{dt}\right)^2+\left(r\frac{d\theta}{dt}\right)^2\right]}=\sqrt{\left[\omega^2(a^2-b^2)+a^2\omega^2\right]}$$
$$=\omega\sqrt{(2a^2-b^2)}.$$

Again from (1), we have

$$\frac{dr}{dt}=\omega\sqrt{(r^2-b^2)}$$

or $$dt=\frac{1}{\omega}\frac{dr}{\sqrt{(r^2-b^2)}}.$$

Suppose the particle takes the time t_1 to reach the other end B of the rod. Then

$$\int_0^{t_1} dt=\frac{1}{\omega}\int_b^a \frac{dr}{\sqrt{(r^2-b^2)}}$$

or $$\left[t\right]_0^{t_1}=\frac{1}{\omega}\left[\cosh^{-1}\frac{r}{b}\right]_b^a$$

or $$t_1=\frac{1}{\omega}\left[\cosh^{-1}\frac{a}{b}-\cosh^{-1}1\right]=\frac{1}{\omega}\left[\cosh^{-1}\frac{a}{b}-0\right]$$
$$=(1/\omega)\cosh^{-1}(a/b).$$

Ex. 25 (a). *A small bead sides with constant speed v on a smooth wire in the shape of the cardioid $r=a(1+\cos\theta)$. Show that the angular velocity is $(v/2a)\sec\frac{1}{2}\theta$ and that the radial component of the acceleration is constant.*

[**Kanpur 1974; Luck 80; Meerut 88P**]

Sol. Here, the path of the bead is the curve

$$r=a(1+\cos\theta)=2a\cos^2\tfrac{1}{2}\theta.$$

We have $$\frac{dr}{d\theta}=-4a(\cos\tfrac{1}{2}\theta\sin\tfrac{1}{2}\theta)\cdot\tfrac{1}{2}=-2a\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta. \qquad \ldots(1)$$

$$\therefore \quad \tan\phi=r\frac{d\theta}{dr}=\frac{2a\cos^2\frac{1}{2}\theta}{-2a\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}=-\cot\tfrac{1}{2}\theta=\tan(\tfrac{1}{2}\pi+\tfrac{1}{2}\theta).$$

so that $\phi=\frac{1}{2}\pi+\frac{1}{2}\theta$.

Now $p=r\sin\phi=r\sin(\frac{1}{2}\pi+\frac{1}{2}\theta)=r\cos\frac{1}{2}\theta$. ...(2)

The angular velocity of the bead $=\dfrac{d\theta}{dt}=\dfrac{vp}{r^2}$

$$=\frac{v.r\cos\frac{1}{2}\theta}{r^2} \qquad [\because\ p=r\cos\tfrac{1}{2}\theta]$$

$$=\frac{v\cos\frac{1}{2}\theta}{r}=\frac{v\cos\frac{1}{2}\theta}{2a\cos^2\frac{1}{2}\theta} \qquad [\because\ r=2a\cos^2\tfrac{1}{2}\theta]$$

$$=\frac{v}{2a}\sec\tfrac{1}{2}\theta. \qquad ...(3)$$

Now from (1), $\dfrac{dr}{dt}=-4a\cos\frac{1}{2}\theta.\sin\frac{1}{2}\theta.\frac{1}{2}.\dfrac{d\theta}{dt}$

$$=-2a\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta.\frac{v}{2a}\sec\tfrac{1}{2}\theta$$

[substituting for $d\theta/dt$ from (3)]

$$=-v\sin\tfrac{1}{2}\theta.$$

$$\therefore\quad \frac{d^2r}{dt^2}=\frac{d}{dt}\left(\frac{dr}{dt}\right)=\frac{d}{dt}(-v\sin\tfrac{1}{2}\theta)=-v\cos\tfrac{1}{2}\theta.\tfrac{1}{2}.\frac{d\theta}{dt}$$

[Note that v is constant]

$$=-\tfrac{1}{2}v\cos\tfrac{1}{2}\theta.\frac{v}{2a}\sec\tfrac{1}{2}\theta=-\frac{v^2}{4a}.$$

Now the radial acceleration $=\dfrac{d^2r}{dt^2}-r\left(\dfrac{d\theta}{dt}\right)^2$

$$=-\frac{v^2}{4a}-2a\cos^2\tfrac{1}{2}\theta.\left(\frac{v}{2a}\sec\tfrac{1}{2}\theta\right)^2=-\frac{v^2}{4a}-\frac{v^2}{2a}=-\frac{3v^2}{4a},$$

which is constant.

Ex. 25 (b). *A particle moves along a circle $r=2a\cos\theta$ in such a way that its acceleration towards the origin is always zero. Show that the transverse acceleration varies as the fifth power of cosec θ.* [Meerut 1972, 77]

Sol. The equation of the path is $r=2a\cos\theta$. ..(1)

Now according to the question the acceleration of the particle towards the origin is always zero *i.e.*, the radial acceleration is always zero.

$$\therefore\quad \frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2=0. \qquad ...(2)$$

From (1), $\dfrac{dr}{dt}=-2a\sin\theta\dfrac{d\theta}{dt}$.

and $\dfrac{d^2r}{dt^2}=-2a\sin\theta\dfrac{d^2\theta}{dt^2}-2a\cos\theta\dfrac{d\theta}{dt}\dfrac{d\theta}{dt}$

$$=-2a \sin\theta \frac{d^2\theta}{dt^2}-2a\cos\theta\left(\frac{d\theta}{dt}\right)^2.$$

Substituting the values of $\dfrac{d^2r}{dt^2}$ and r in (2), we have

$$-2a\sin\theta\frac{d^2\theta}{dt^2}-2a\cos\theta\left(\frac{d\theta}{dt}\right)^2-2a\cos\theta\left(\frac{d\theta}{dt}\right)^2=0$$

or $\quad \dfrac{d^2\theta}{dt^2}=-2\dfrac{\cos\theta}{\sin\theta}\left(\dfrac{d\theta}{dt}\right)^2$ [Allāhabad 198

or $\quad \dfrac{d^2\theta/dt^2}{d\theta/dt}=-2\dfrac{\cos\theta}{\sin\theta}\dfrac{d\theta}{dt}$

Integrating with respect to t, we get

$\log(d\theta/dt)=-2\log\sin\theta+\log C$, where C is a constant

$\qquad =\log(C/\sin^2\theta)=\log(C\operatorname{cosec}^2\theta)$.

$\therefore\quad d\theta/dt=C\operatorname{cosec}^2\theta$. ...(3

Now the transverse acceleration $=\dfrac{1}{r}\dfrac{d}{dt}\left(r^2\dfrac{d\theta}{dt}\right)$

$$=\frac{1}{2a\cos\theta}\frac{d}{dt}\left\{4a^2\cos^2\theta . C\operatorname{cosec}^2\theta\right\}$$

[substituting for r and $d\theta/dt$ from (1) and (3)]

$$=\frac{4Ca^2}{2a\cos\theta}\frac{d}{dt}(\cot^2\theta)=\frac{2Ca}{\cos\theta}\,2\cot\theta.(-\operatorname{cosec}^2\theta).\frac{d\theta}{dt}$$

$$=-\frac{4Ca}{\cos\theta}\cot\theta.\operatorname{cosec}^2\theta.C\operatorname{cosec}^2\theta,\quad\left[\because\ \frac{d\theta}{dt}=C\operatorname{cosec}^2\theta\right]$$

$$=-4C^2a\operatorname{cosec}^5\theta.$$

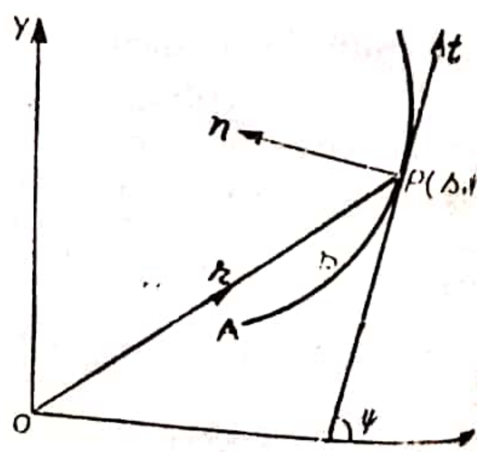
Hence the transverse acceleration $\propto \operatorname{cosec}^5\theta$.

§ 7. Tangential and Normal Velocities and Accelerations.

[Meerut 78, 79, 82, 83 P, 87; Kanpur 75, 76; Rohilkhand 86, 88; Luck. 77, 78; Agra 76; Allahabad 80]

Tangential and Normal Velocities.

Let P be the position of a moving particle at time t and **r** its position vector with respect to the origin O. Let arc $AP=s$ and let ψ be the angle which the tangent at P to the path of the particle makes with OX. Then (s, ψ) are the *intrinsic coordinates* of P.



Let **t** denote the unit vector along the tangent at P in the direction of s increasing and **n** the unit vector along the normal at P in the direction of ψ increasing *i.e.*, in the direction of **inwards drawn normal.**

From vector calculus, we have

$\dfrac{d\mathbf{r}}{ds}=\mathbf{t}$. [Remember that for a curve, $d\mathbf{r}/ds$ denotes the unit tangent vector in the direction of s increasing].

Also $$\frac{d\mathbf{t}}{dt}=\frac{d\psi}{dt}\mathbf{n} \qquad \text{[Refer § 3].} \qquad \ldots(1)$$

If $\mathbf{v}$ is the velocity vector of the particle at P, then

$$\mathbf{v}=\frac{d\mathbf{r}}{dt}=\frac{d\mathbf{r}}{ds}\frac{ds}{dt}=\frac{ds}{dt}\mathbf{t} \qquad \left[\because \frac{d\mathbf{r}}{ds}=\mathbf{t}\right]$$

$$=\frac{ds}{dt}\mathbf{t}+0\mathbf{n}. \qquad \ldots(2)$$

Thus the vector $\mathbf{v}$ has been expressed as a linear combination of the vectors $\mathbf{t}$ and $\mathbf{n}$.

∴ **the tangential velocity** at P=the coefficient of $\mathbf{t}$ in (2)

$=\dfrac{ds}{dt}$, +ive in the direction of s increasing,

and the **normal velocity** at P=the coefficient of $\mathbf{n}$ in (2)

$=0$.

If v is the resultant velocity of the particle at P, then $v=ds/dt$ and is along the tangent at P. Thus remember that *the resultant velocity of a particle is always along the tangent to its path.*

Tangential and normal accelerations. If $\mathbf{a}$ is the acceleration vector of the particle at P, then

$$\mathbf{a}=\frac{d\mathbf{v}}{dt}=\frac{d}{dt}(v\mathbf{t}), \qquad \left[\because \mathbf{v}=\frac{ds}{dt}\mathbf{t}=v\mathbf{t}, \text{ where } v=\frac{ds}{dt}\right]$$

$$=\frac{dv}{dt}\mathbf{t}+v\frac{d\mathbf{t}}{dt}$$

$$=\frac{dv}{dt}\mathbf{t}+v\frac{d\psi}{dt}\mathbf{n}, \qquad \text{[from (1)]}$$

$$=\frac{dv}{dt}\mathbf{t}+v\frac{d\psi}{ds}\frac{ds}{dt}\mathbf{n}$$

$$=\frac{dv}{dt}\mathbf{t}+\left(v\frac{1}{\rho}.v\right)\mathbf{n}, \quad [\because \rho=\text{radius of curvature at } P=ds/d\psi]$$

$\rho=\dfrac{ds}{d\psi}$

$$=\frac{dv}{dt}\mathbf{t}+\frac{v^2}{\rho}\mathbf{n}.$$

$\dfrac{v^2}{\rho}$.

Thus the acceleration vector $\mathbf{a}$ has been expressed as a linear combination of the vectors $\mathbf{t}$ and $\mathbf{n}$. The coefficients of $\mathbf{t}$ and $\mathbf{n}$

in this linear combination give us respectively the tangential and normal accelerations of the particle at P.

$\therefore$ the **tangential acceleration** of $P=\frac{dv}{dt}$, +ive in the direction of s increasing

and the **normal acceleration** of $P=\frac{v^2}{\rho}$, +ive in the direction of inwards drawn normal.

Other expressions for the tangential acceleration :

(i) Tangential acceleration

$$=\frac{dv}{dt}=\frac{d}{dt}\left(\frac{ds}{dt}\right)=\frac{d^2s}{dt^2}.$$

(ii) Tangential acceleration

$$=\frac{dv}{dt}=\frac{dv}{ds}\frac{ds}{dt}=v\frac{dv}{ds}.$$

The resultant acceleration of the particle at time t

$$=\sqrt{\left[\left(\frac{d^2s}{dt^2}\right)^2+\left(\frac{v^2}{\rho}\right)^2\right]}.$$

Illustrative Examples

Ex. 26. *If the velocity of a point moving in a plane curve varies as the radius of curvature, show that the direction of motion revolves with constant angular velocity.*

Sol. It is given that the velocity at any point is proportional to the radius of curvature.

$\therefore$ $v=k\rho$, where k is the constant of proportionality

or $\frac{ds}{dt}=k\frac{ds}{d\psi}$ $\left[\because \rho=\frac{ds}{d\psi}\right]$

or $\frac{ds/dt}{ds/d\psi}=k$ or $\frac{d\psi}{dt}=k.$

Hence the direction of motion *i.e.*, the tangent revolves with constant angular velocity.

Ex. 27. *A point describes the cycloid $s=4a \sin \psi$ with uniform speed v. Find its acceleration at any point.*

[Lucknow 1976]

Sol. The path of the particle is

$$s=4a \sin \psi. \quad ...(1)$$

Since the point describes the cycloid with uniform speed v,

$\therefore$ $\frac{ds}{dt}=v=$ constant.

$$\therefore \quad \frac{d^2s}{dt^2}=0.$$

Differentiating (1), $\rho=\dfrac{ds}{d\psi}=4a\cos\psi$.

Now the resultant acceleration of the particle

$$=\sqrt{[(\text{Tangential accel.})^2+(\text{normal accel.})^2]}$$

$$=\sqrt{\left[\left(\frac{d^2s}{dt^2}\right)^2+\left(\frac{v^2}{\rho}\right)^2\right]}$$

$$=\frac{v^2}{\rho}, \qquad \left[\because \quad \frac{d^2s}{dt^2}=0\right]$$

$$=\frac{v^2}{4a\cos\psi}=\frac{v^2}{4a\sqrt{(1-\sin^2\psi)}}$$

$$=\frac{v^2}{4a\cos\psi}=\frac{v^2}{4a\sqrt{(1-\sin^2\psi)}}$$

$$=\frac{v^2}{4a\sqrt{\left[1-\left(\frac{s}{4a}\right)^2\right]}}, \text{ [substituting for } \sin\psi, \text{ from (1)]}$$

$$=\frac{v^2}{\sqrt{(16a^2-s^2)}}.$$

Ex. 28. *Prove that the acceleration of a point moving in a curve with uniform speed is $(d\psi/dt)^2$.* **[Meerut 1980, 81, 86]**

Sol. Here $v=\dfrac{ds}{dt}=$ constant ; $\therefore \quad \dfrac{d^2s}{dt^2}=0$.

$$\text{Acceleration}=\sqrt{\left[\left(\frac{d^2s}{dt^2}\right)^2+\left(\frac{v^2}{\rho}\right)^2\right]}=\frac{v^2}{\rho}=\rho\left(\frac{v}{\rho}\right)^2$$

$$=\rho\left[\frac{ds/dt}{ds/d\psi}\right]^2 \qquad \left[\because \quad \rho=\frac{ds}{d\psi}\right]$$

$$=\rho\left(\frac{d\psi}{dt}\right)^2=\rho\dot{\psi}^2.$$

Ex. 29. *If the tangential and normal accelerations of a particle describing a plane curve be constant throughout, prove that the radius of curvature at any point t is given by $\rho=(at+b)^2$.* **[Kanpur 1979]**

Sol. Here, given that the tangential and normal accelerations of the particle are constant.

$\therefore$ Let, tangential acceleration $=\dfrac{dv}{dt}=\lambda$ (const.). ...(1)

and normal acceleration $=\dfrac{v^2}{\rho}=\mu$ (const.) ...(2)

Integrating (1) w.r.t. t, we have

$v=\lambda t+c$, where c is a constant.

Substituting the value of v in (2), we have

$$\rho=\frac{v^2}{\mu}=\frac{1}{\mu}(\lambda t+c)^2=\left(\frac{\lambda}{\sqrt{\mu}}t+\frac{c}{\sqrt{\mu}}\right)^2=(at+b)^2,$$

where $$\frac{\lambda}{\sqrt{\mu}}=a \text{ and } \frac{c}{\sqrt{\mu}}=b.$$

Ex. 30. *A particle describes a curve (for which s and ψ vanish simultaneously) with uniform speed v. If the acceleration at any point s be $v^2c/(s^2+c^2)$, find the intrinsic equation of the curve.*

[Agra 1980; Meerut 83, 85P]

Sol. Here, $v=\frac{ds}{dt}=$const. ; $\therefore \frac{d^2s}{dt^2}=0$.

$\therefore$ Acceleration at any point 's'

$$=\sqrt{\left[\left(\frac{d^2s}{dt^2}\right)^2+\left(\frac{v^2}{\rho}\right)^2\right]}=\frac{v^2}{\rho}.$$

But it is given that the acceleration at any point 's'$=\frac{v^2c}{s^2+c^2}$.

$$\therefore \frac{v^2}{\rho}=\frac{v^2c}{s^2+c^2}$$

or $$\frac{1}{\rho}=\frac{d\psi}{ds}=\frac{c}{s^2+c^2} \qquad [\because v^2\neq 0]$$

or $$d\psi=\frac{c}{s^2+c^2}\,ds, \text{ separating the variables.}$$

Integrating, $\psi=\tan^{-1}(s/c)+A$, where A is a constant.

Given that $\psi=0$, when $s=0$.

$\therefore \quad 0=0+A$ or $A=0$.

$\therefore \quad \psi=\tan^{-1}(s/c)$ or $s=c\tan\psi$,

which is the intrinsic equation of the curve and is a catenary.

Ex. 31. (a) *A point moves along the arc of a cycloid in such a manner that the tangent at it rotates with a constant angular velocity. Show that the acceleration of the moving point is constant in magnitude.* **[Allahabad 1978 ; Meerut 84]**

Sol. The intrinsic equation of a cycloid is

$$s=4a\sin\psi. \qquad \ldots(1)$$

It is given that the angular velocity of the tangent is constant

i.e., $$\frac{d\psi}{dt}=c \text{ (constant)}. \qquad \ldots(2)$$

Acceleration of the moving point

$$=\sqrt{\left[\left(\frac{d^2s}{dt^2}\right)^2+\left(\frac{v^2}{\rho}\right)^2\right]}. \qquad \ldots(3)$$

Now, from (1),

$$v=\frac{ds}{dt}=4a\cos\psi.\frac{d\psi}{dt}=4ac\cos\psi, \left[\because \frac{d\psi}{dt}=c\right]$$

$$\frac{d^2s}{dt^2}=-4ac \sin \psi . \frac{d\psi}{dt}=-4ac^2 \sin \psi ,$$

and $\rho=\dfrac{ds}{d\psi}=4ac \cos \psi$.

Substituting in (3), we have the acceleration

$$=\sqrt{\left[\left(-4ac^2 \sin \psi\right)^2+\left\{\frac{(4ac \cos \psi)^2}{4a \cos \psi}\right\}^2\right]}$$

$$=\sqrt{[16a^2c^4 (\sin^2\psi+\cos^2\psi)]}=\sqrt{(16a^2c^4)}=4ac^2=\text{constant}$$

$\frac{d\psi}{dt}=\omega$

Ex. 31 (b). *The rate of change of direction of velocity of a particle moving in a cycloid is constant. Prove that acceleration must be constant in magnitude.*

Sol. We know that the direction of velocity is always along the tangent. Therefore according to the question the angular velocity of the tangent is constant.

Now proceed as in Ex. 31 (a).

Ex. 32. *A point moves in a plane curve so that its tangential acceleration is constant, and the magnitudes of the tangential velocity and normal acceleration are in a constant ratio; find the intrinsic equation of the curve.* **[Meerut 1985, 87]**

Sol. Here, it is given that

$$\text{tangential acceleration}=\frac{dv}{dt}=\lambda \text{ (a constant)}, \quad \ldots(1)$$

and $$\frac{\text{tangential velocity}}{\text{normal accel.}}=\frac{v}{v^2/\rho}=\frac{\rho}{v}=\mu \text{ (a const.)} \quad \ldots(2)$$

From (2), we have $\dfrac{ds/d\psi}{ds/dt}=\mu$ $\quad [\because \rho=ds/d\psi,\ v=ds/dt]$

or $dt/d\psi=\mu$ or $d\psi/dt=1/\mu$. $\quad \ldots(3)$

Now from (1), we have

$$\frac{dv}{d\psi}\cdot\frac{d\psi}{dt}=\lambda \quad \text{or} \quad \frac{dv}{d\psi}\cdot\frac{1}{\mu}=\lambda \quad \left[\because \frac{d\psi}{dt}=\frac{1}{\mu}\right]$$

or $$dv=\mu\lambda d\psi.$$

Integrating, we have $v=\mu\lambda\psi+k$, $\quad \ldots(4)$

where k is constant.

Now from (2), we have

$\rho=\mu v=\mu\ (\mu\lambda\psi+k)$, substituting for v from (4)

or $$\frac{ds}{d\psi}=\mu^2\lambda\ \psi+\mu k \quad \left[\because \rho=\frac{ds}{d\psi}\right]$$

or $$ds=(\mu^2\lambda\psi+k\mu)\ d\psi.$$

Integrating, $s=\frac{1}{2}\mu^2\lambda\psi^2+k\mu\psi+C$

or $s=A\psi^2+B\psi+C$, where $A=\frac{1}{2}\mu^2\lambda$, $B=k\mu$.

Hence the intrinsic equation of the path is

$s=A\psi^2+B\psi+C,$ where A, B, C are constants.

Ex. 33. *A particle moves in a plane in such a manner that its tangential and normal accelerations are always equal and its velocity varies as* $\exp\{\tan^{-1}(s/c)\}$, *s being the length of the arc of the curve measured from a fixed point on the curve. Find the path.*

[Meerut 75; Allahabad 76]

Sol. Given that

the tangential acceleration = normal acceleration

i.e., $$v\frac{dv}{ds}=\frac{v^2}{\rho}. \qquad ...(1)$$

Also velocity $=v=ke^{\tan^{-1}(s/c)}$, $\qquad ...(2)$

where k is a constant.

From (1), we have

$$\frac{dv}{ds}=\frac{v}{\rho} \qquad \left[\because\ v\neq 0\right]$$

or $$\frac{dv}{ds}=v\frac{d\psi}{ds} \qquad \left[\because\ \frac{1}{\rho}=\frac{d\psi}{ds}\right]$$

or $$\frac{dv}{v}=d\psi.$$

Integrating, $\log v=\psi+\log A$, where A is a const.

or $\log(v/A)=\psi$, or $v/A=e^{\psi}$.

or $$v=Ae^{\psi}. \qquad ...(3)$$

Equating the values of v from (2) and (3), we have

$$ke^{\tan^{-1}(s/c)}=Ae^{\psi}.$$

Let $\psi=0$, when $s=0$. Tnen $ke^0=Ae^0$ or $A=k$.

$\therefore$ $ke^{\tan^{-1}(s/c)}=ke^{\psi}$, or $e^{\tan^{-1}(s/c)}=e^{\psi}$

or $\tan^{-1}(s/c)=\psi$, or $s/c=\tan\psi$,

or $s=c\tan\psi$,

which is the required intrinsic equation of the path and is a catenary.

Important remark : Remember that whenever tangential and normal accelerations are given to be equal, we must have $v=Ae^{\psi}$, where A is some constant.

Ex. 34. *A point moves in a plane curve, so that its tangential and normal accelerations are equal and the angular velocity of the tangent is constant. Find the curve.*

[Meerut 84, 86, 87S, 89; Gorakhpur 74; Jiwaji 72; Lucknow 77, 81; Agra 76, 79; Allahabad 75; Rohilkhand 87]

Sol. Here, given that

Tang. Accel. = normal accel.

i.e.,
$$v\frac{dv}{ds}=\frac{v^2}{\rho}, \qquad ...(1)$$

and
$$\frac{d\psi}{dt}=C \text{ (constant)} \qquad ...(2)$$

From (1), we have $\dfrac{dv}{ds}=\dfrac{v}{\rho}=v\dfrac{d\psi}{ds}$

or
$$\frac{dv}{v}=d\psi.$$

Integrating, we have $\log v=\psi+\log A$

or
$$v=\frac{ds}{dt}=Ae^{\psi}$$

or
$$\frac{ds}{d\psi}\cdot\frac{d\psi}{dt}=Ae^{\psi}$$

or
$$\frac{ds}{d\psi}.C=Ae^{\psi}, \qquad \text{[from (2)]}$$

or
$$ds=\frac{A}{C}e^{\psi}.$$

Integrating,
$$s=\frac{A}{C}e^{\psi}+B,$$

which is the equation of the curve.

Ex. 35. *A particle is moving in a parabola with uniform angular velocity about the focus; prove that its normal acceleration at any point is proportional to the radius of curvature of its path at that point.* **[Meerut 76, 78, 88P, 90S; Indore 73; Allahabad 80; Vikram 73; Luck. 73]**

Sol. The pedal equation of a parabola referred to the focus as pole is

$$p^2=ar. \qquad ...(1)$$

Since the particle moves with uniform angular velocity about the focus (*i.e.*, about the pole),

therefore,
$$\frac{d\theta}{dt}=\frac{vp}{r^2}=c=\text{a constant}$$

or
$$v=r^2c/p. \qquad ...(2)$$

From (1), we have

$$2p\frac{dp}{dr}=a, \quad \text{or} \quad \frac{dr}{dp}=\frac{2p}{a}.$$

$\therefore$
$$\rho=r\frac{dr}{dp}=r.\frac{2p}{a}. \qquad ...(3)$$

Now $\qquad \dfrac{\text{Normal acceleration}}{\rho}=\dfrac{v^2/\rho}{\rho}=\dfrac{v^2}{\rho^2}=\dfrac{(r^2c/p)^2}{(2pr/a)^2}$,

substituting for v and ρ from (2) and (3)

$$=\frac{r^2c^2a^2}{4p^4}=\frac{r^2c^2a^2}{4(ar)^2}, \quad \text{substituting for } p^2 \text{ from (1)}$$

$$=\frac{c^2}{4}.$$

Thus normal acceleration$=(c^2/4)\ \rho$.

Hence normal acceleration $\propto \rho$.

Ex. 36. *A particle describes a circle of radius r with a uniform speed v, show that its acceleration at any point of the path is v^2/r and is directed towards the centre of the circle.* **[Madurai 1973]**

Sol. Here $\qquad \dfrac{ds}{dt}=v$ (constant).

$\therefore$ tangential acceleration$=\dfrac{d^2s}{dt^2}=0$.

$\therefore$ the resultant acceleration at any point

$$=\sqrt{\left[\left(\frac{d^2s}{dt^2}\right)^2+\left(\frac{v^2}{\rho}\right)^2\right]}$$

$=v^2/\rho$, along the normal

$=v^2/r$, since for a circle $\rho=$the radius of the circle.

Now in the case of a circle all normals pass through the centre of the circle. Hence the acceleration at any point of the path is v^2/r and is directed towards the centre of the circle.

Ex. 37. *A particle is describing a plane curve. If the tangential and normal accelerations are each constant throughout the motion, prove that the angle ψ, through which the direction of motion turns in time t is given by $\psi=A\log(1+Bt)$.*

[Kanpur 1980; Meerut 83P, 85S, 88]

Sol. Here given that,

the tangential accel.$=\dfrac{dv}{dt}=\lambda$ (const.), $\qquad$ ...(1)

and normal accel.$=\dfrac{v^2}{\rho}=\mu$ (const.) $\qquad$ ...(2)

Integrating (1), we have $\qquad v=\lambda t+c_1$,

where c_1 is a constant.

From (2), we have

$$v.\frac{ds/dt}{ds/d\psi}=\mu \qquad \left[\because\ v=\frac{ds}{dt},\ \rho=\frac{ds}{d\psi}\right]$$

or
$$v\frac{d\psi}{dt}=\mu$$
or
$$d\psi=\frac{\mu}{v}\,dt.$$
Substituting for v from (3), we have
$$d\psi=\frac{\mu}{\lambda t+c_1}\,dt.$$
Integrating, we have $\psi=(\mu/\lambda)\log(\lambda t+c_1)+c_2$, where c_2 is a constant.

Let $\psi=0$, when $t=0$. Then $c_2=-(\mu/\lambda)\log c_1$.

$$\therefore\quad \psi=(\mu/\lambda)\log(\lambda t+c_1)-(\mu/\lambda)\log c_1=(\mu/\lambda)\log\frac{\lambda t+c_1}{c_1}$$
$$=(\mu/\lambda)\log\left(1+\frac{\lambda}{c_1}t\right)$$
$$=A\log(1+Bt),\text{ where } A=\mu/\lambda \text{ and } B=\lambda/c_1.$$

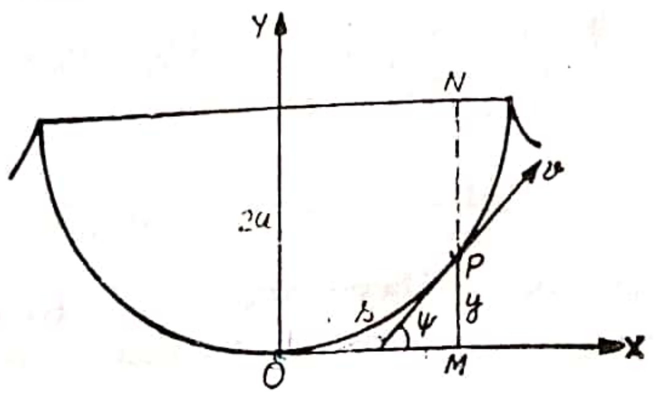
Miscellaneous Examples on the chapter.

Ex. 38. *A particle describes a cycloid with uniform speed. Prove that the normal acceleration at any point varies inversely as the square root of the distance from the base of the cycloid.*

Sol. The intrinsic equation of a cycloid is $s=4a\sin\psi$. ...(1)

Here
$$v=\frac{ds}{dt}=\text{constant}.$$

Normal acceleration
$$=\frac{v^2}{\rho}=\frac{v^2}{ds/d\psi}=\frac{v^2}{4a\cos\psi}=\frac{v^2}{4a\sqrt{(1-\sin^2\psi)}}$$
$$=\frac{v^2}{4a\sqrt{\{1-(s^2/16a^2)\}}},$$
substituting for $\sin\psi$ from (1)
$$=\frac{v^2}{\sqrt{(16a^2-s^2)}}. \qquad ...(2)$$



If y is the distance of any point P of the cycloid from OX (*i.e.*, the tangent at the vertex), then the distance of P from the base of the cycloid $=NP=MN-PM=2a-y$.

If arc $OP=s$, then for the cycloid, we have **[Remember]**

$$s^2=8ay.$$

$\therefore$ from (2), we have

$$\text{normal acceleration}=\frac{v^2}{\sqrt{(16a^2-8ay)}}$$

$$=\frac{v^2}{\sqrt{[8a\,(2a-y)]}}=\frac{v^2}{\sqrt{[8a.NP]}}$$

$$\propto\frac{1}{\sqrt{(NP)}},\quad\text{because }\frac{v^2}{\sqrt{(8a)}}\text{ is constant.}$$

$\therefore$ the normal acceleration varies inversely as the square root of the distance from the base of the cycloid.

Ex. 39. *A particle is acted on by a force parallel to the axis of y whose acceleration is λy and is initially projected with a velocity $a\sqrt{\lambda}$ parallel to the axis of x at a point where $y=a$, prove that it will describe the catenary $y=a\cosh(x/a)$.*

[Lucknow 1981; Allahabad '80]

Sol. Here, we are given that

$$\frac{d^2y}{dt^2}=\lambda y. \quad ...(1)$$

Since there is no force parallel to the axis of x,

$$\therefore\quad \frac{d^2x}{dt^2}=0. \quad ...(2)$$

Multiplying both sides of (1) by $2\dfrac{dy}{dt}$ and then integrating, we have

$$\left(\frac{dy}{dt}\right)^2=\lambda y^2+A, \text{ where } A \text{ is a constant.}$$

Initially, when $y=a$, $\dfrac{dy}{dt}=0$.

[Note that initially there is no velocity parallel to the y-axis].

$$\therefore\quad 0=\lambda a^2+A \quad\text{or}\quad A=-\lambda a^2.$$

$$\therefore\quad \left(\frac{dy}{dt}\right)^2=\lambda\,(y^2-a^2)$$

or
$$\left(\frac{dy}{dt}\right)=\sqrt{\lambda}.\sqrt{(y^2-a^2)}, \quad ...(3)$$

where the square root has been taken with +ive sign because the particle is moving in the direction of y increasing.

Now integrating (2), we have

$$\frac{dx}{dt}=B, \text{ where } B \text{ is a constant.}$$

But initially when $y=a$, $\dfrac{dx}{dt}=a\sqrt{\lambda}$ so that $B=a\sqrt{\lambda}$.

$$\therefore \quad \frac{dx}{dt}=a\sqrt{\lambda}. \qquad \ldots(4)$$

Dividing (3) by (4), we have

$$\frac{dy}{dx}=\frac{\sqrt{(y^2-a^2)}}{a}$$

or
$$\frac{dx}{a}=\frac{dy}{\sqrt{(y^2-a^2)}}.$$

Integrating, we have

$$\frac{x}{a}+C=\cosh^{-1}\frac{y}{a},$$

where C is a constant.

Let us take $x=0$, when $y=a$. Then $C=0$.

$$\therefore \quad x/a=\cosh^{-1}(y/a)$$

or
$$y=a\cosh(x/a), \text{ which is a catenary.}$$

Ex. 40. *A particle is acted on by a force parallel to the axis of y whose acceleration (always towards the axis of x) is μy^{-2} and when $y=a$, it is projected parallel to the axis of x with velocity $\sqrt{(\mu/a)}$. Prove that it will describe a cycloid.* **[Meerut 1981, 83P]**

Sol. Here we are given that

$$\frac{d^2y}{dt^2}=-\mu y^{-2}, \qquad \ldots(1)$$

the negative sign has been taken because the force is in the direction of y decreasing.

Also there is no force parallel to the axis of x. Therefore

$$\frac{d^2x}{dt^2}=0. \qquad \ldots(2)$$

Multiplying both sides of (1) by $2\dfrac{dy}{dt}$ and then integrating w.r.t. t, we have

$$\left(\frac{dy}{dt}\right)^2=\frac{2\mu}{y}+A, \text{ where } A \text{ is a constant.}$$

Initially, when $y=a$, $\dfrac{dy}{dt}=0$. [Note that initially there is no velocity parallel to the y-axis]

$$\therefore \quad A=-\frac{2\mu}{a}.$$

$$\therefore \quad \left(\frac{dy}{dt}\right)^2=\frac{2\mu}{y}-\frac{2\mu}{a}=2\mu\left(\frac{1}{y}-\frac{1}{a}\right)=\frac{2\mu}{a}\left(\frac{a-y}{y}\right)$$

or $$\frac{dy}{dt}=-\sqrt{\left(\frac{2\mu}{a}\right)}\cdot\sqrt{\left(\frac{a-y}{y}\right)} \quad \ldots(3)$$

[$-$ive sign has been taken because the particle is moving in the direction of y decreasing.]

Now integrating (2), we have

$$\frac{dx}{dt}=B, \text{ where } B \text{ is a constant.}$$

Initially, when $y=a$, $\dfrac{dx}{dt}=\sqrt{\left(\dfrac{2\mu}{a}\right)}$,

so that $$B=\sqrt{\left(\frac{2\mu}{a}\right)}.$$

$$\therefore \quad \frac{dx}{dt}=\sqrt{\left(\frac{2\mu}{a}\right)}. \quad \ldots(4)$$

Dividing (3) by (4), we have

$$\frac{dy}{dx}=-\sqrt{\left(\frac{a-y}{y}\right)}$$

or $$dx=-\sqrt{\left(\frac{y}{a-y}\right)}\,dy.$$

Integrating, $$x=-\int\sqrt{\left(\frac{y}{a-y}\right)}\,dy+C$$

$$=2a\int\frac{\cos\theta}{\sin\theta}.\cos\theta\sin\theta\,d\theta+C$$

[putting $y=a\cos^2\theta$, so that $dy=-2a\cos\theta\sin\theta\,d\theta$]

$$=a\int(1+\cos 2\theta)\,d\theta+C=a\,(\theta+\tfrac{1}{2}\sin 2\theta)+C$$

$$=\tfrac{1}{2}a\,(2\theta+\sin 2\theta)+C.$$

Let us take $x=0$, when $y=a$

i.e., when $a\cos^2\theta=a$ *i.e.*, when $\cos\theta=1$ *i.e.*, when $\theta=0$.

Then $0=\frac{1}{2}a\,(0+0)+C$ or $C=0$.

$$\therefore \quad x=\tfrac{1}{2}a\,(2\theta+\sin 2\theta). \quad \ldots(5)$$

Also $$y=a\cos^2\theta=\tfrac{1}{2}a\,(1+\cos 2\theta). \quad \ldots(6)$$

The equations (5) and (6) give us the path of the particle. But these are the parametric equations of a cycloid.

Hence the path is a cycloid.

Ex. 41. *A particle moves in the curve $y=a\log\sec(x/a)$ in such a way that the tangent to the curve rotates uniformly; prove the resultant acceleration of the particle varies as the square of the radius of curvature.*

[Meerut 1975, 80, 81, 87P; Kanpur 74; Luck. 79]

Sol. Since the tangent to the curve rotates uniformly,

$$\therefore \quad d\psi/dt = c \text{ (constant).} \qquad \ldots(1)$$

Equation of the path is

$$y = a \log \sec (x/a).$$

$$\therefore \quad \frac{dy}{dx} = \frac{a}{\sec (x/a)} \cdot \sec \frac{x}{a} \tan \frac{x}{a} \cdot \frac{1}{a}$$

or $$\tan \psi = \frac{dy}{dx} = \tan \frac{x}{a}.$$

$$\therefore \quad \psi = x/a$$

or $$x = a\psi.$$

$$\therefore \quad \frac{dx}{dt} = a \frac{d\psi}{dt} = ac, \text{ and } \frac{d^2x}{dt^2} = 0.$$

$$\text{Now } \frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = \left(\tan \frac{x}{a}\right) . ac.$$

$$\therefore \quad \frac{d^2y}{dt^2} = ac \sec^2 \frac{x}{a} \cdot \frac{1}{a} \cdot \frac{dx}{dt} = ac . \sec^2 \frac{x}{a} \frac{1}{a} . ac = ac^2 \sec^2 \frac{x}{a}.$$

$$\therefore \quad \text{Resultant acceleration} = \sqrt{\left[\left(\frac{d^2x}{dt^2}\right)^2 + \left(\frac{d^2y}{dt^2}\right)^2\right]}$$

$$= \sqrt{\left[(0)^2 + \left(ac^2 \sec^2 \frac{x}{a}\right)^2\right]} = ac^2 \sec^2 \frac{x}{a}. \qquad \ldots(2)$$

Also the radius of curvature $\rho = \dfrac{[1 + (dy/dx)^2]^{3/2}}{d^2y/dx^2}.$

But $\dfrac{dy}{dx} = \tan \dfrac{x}{a}$ implies that $\dfrac{d^2y}{dx^2} = \dfrac{1}{a} \sec^2 \dfrac{x}{a}.$

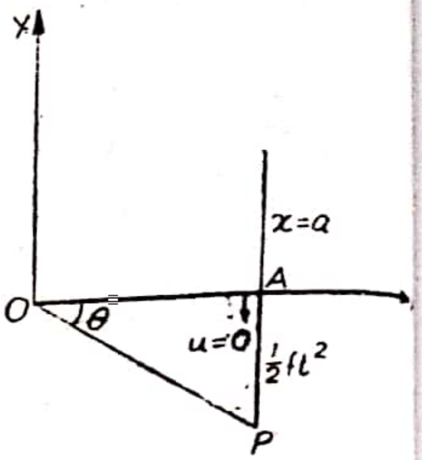
$$\therefore \quad \rho = \frac{[1 + \tan^2 (x/a)]^{3/2}}{(1/a) \sec^2 (x/a)} = \frac{a \sec^3 (x/a)}{\sec^2 (x/a)} = a \sec (x/a)$$

or $$\sec (x/a) = \rho/a.$$

$\therefore$ from (2), the resultant acceleration $= ac^2 (\rho/a)^2 = (c^2/a) \rho^2.$

Hence the resultant acceleration $\propto \rho^2$.

Ex. 42. *A particle falls down a straight line $x = a$, starting from the axis of x. If the distance from the axis of x be $\frac{1}{2} ft^2$ at time t, find the angular velocity and acceleration of the line joining the particle to the origin. How far has the particle dropped when the angular acceleration becomes zero ?*



Sol. Let the particle fall down along the line $x=a$, starting at rest from the point A on the axis of x. If P is the position of the particle at time t, then $AP=\frac{1}{2}ft^2$.

If $\angle AOP=\theta$, then

$$\tan\theta=\frac{AP}{OA}=\frac{1}{2a}ft^2$$

or $$\theta=\tan^{-1}\left(\frac{1}{2a}ft^2\right). \quad \ldots(1)$$

Differentiating (1) w.r.t. 't', the angular velocity of the line OP

$$=\frac{d\theta}{dt}=\frac{1}{1+(ft^2/2a)^2}\cdot\left(\frac{1}{2a}\cdot 2ft\right)=\frac{4\,aft}{4a^2+f^2t^4}. \quad \ldots(2)$$

Differentiating (2) again w.r.t. 't' the angular acceleration of the line OP

$$=\frac{d^2\theta}{dt^2}=\frac{4af.(4a^2+f^2t^4)-4aft.4f^2t^3}{(4a^2+f^2t^4)^2}$$

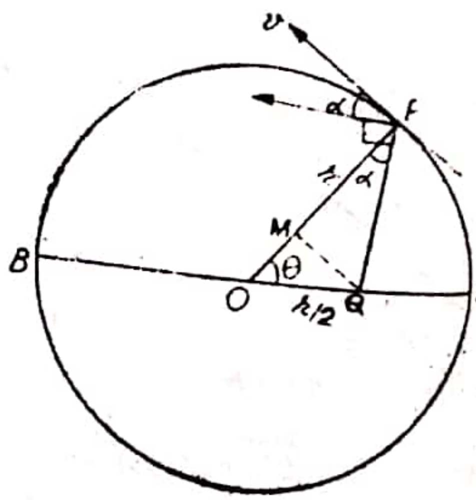
$$=\frac{4af\,(4a^2-3f^2t^4)}{(4a^2+f^2t^4)^2} \quad \ldots(3)$$

Now the angular acceleration becomes zero when

$$\frac{4\,af\,(4a^2-3f^2t^4)}{(4\,a^2+f^2t^4)^2}=0 \text{ or } 4a^2-3f^2t^4=0 \text{ or } t^2=\frac{2a}{f\sqrt{3}}.$$

Also then $AP=\frac{1}{2}ft^2=\frac{1}{2}f.(2a/f\sqrt{3})=a/\sqrt{3}$.

Ex. 43. *A point P describes a circle of radius r with uniform angular velocity ω about the centre. Show that the angular velocity of P about a point Q distance $r/2$ from the centre O fluctuates between 2ω and $2\omega/3$.*



Sol. Let P be the position at time t of a point describing a circle of radius r with uniform angular velocity ω about the centre O. If v is the velocity of the particle at P, then

$$\omega=\frac{vp}{r^2}=\frac{vr}{r^2}=\frac{v}{r}$$

or $$v=\omega r. \quad \ldots(1)$$

Let Q be the point at a distance $r/2$ from the centre O of the circle and let $\angle POQ=\theta$ and $\angle OPQ=\alpha$. Then the angular velocity ω' of P about Q

$$=\frac{\text{velocity of } P \text{ perpendicular to } QP}{PQ}=\frac{v\cos\alpha}{PQ}$$

$$=\frac{\omega r\cos\alpha}{PQ}. \qquad \ldots(2)$$

Now if QM is perpendicular from Q to OP, then

$$OP=r=OM+PM=\tfrac{1}{2}\,r\cos\theta+PQ\cos\alpha.$$

$$\therefore \qquad \cos\alpha=\frac{(2-\cos\theta)\,r}{2\,PQ}.$$

Also from $\triangle OPQ$, we have

$$PQ^2=OP^2+OQ^2-2\,OP.OQ\cos\theta$$

$$=r^2+(\tfrac{1}{2}\,r)^2-2\,r.\tfrac{1}{2}r\cos\theta$$

$$=\frac{5}{4}\,r^2-r^2\cos\theta.$$

$\therefore$ from (2), we have

$$\omega'=\frac{\omega r.(2-\cos\theta)r}{PQ.2PQ}=\frac{\omega r^2\,(2-\cos\theta)}{2.(\frac{5}{4}\,r^2-r^2\cos\theta)}$$

$$=\frac{2\omega\,(2-\cos\theta)}{5-4\cos\theta}=\frac{2\omega}{4}\cdot\frac{(3+5-4\cos\theta)}{5-4\cos\theta}$$

or $$\omega'=\frac{\omega}{2}\left(\frac{3}{5-4\cos\theta}+1\right). \qquad \ldots(3)$$

From (3), it is obvious that ω' is least when $5-4\cos\theta$ is greatest *i.e.*, when $\theta=\pi$.

$\therefore$ the least value of $\omega'=\dfrac{\omega}{2}\left(\dfrac{3}{5-4\cos\pi}+1\right)=2\,\omega/3.$

Also it is obvious that ω' is greatest when $5-4\cos\theta$ is least *i.e.*, when $\theta=0$.

$\therefore$ the greatest value of $\omega'=\dfrac{\omega}{2}\left(\dfrac{3}{5-4\cos 0}+1\right)=2\omega.$

Hence the angular velocity of P about the point Q fluctuates between 2ω and $2\omega/3$.

Ex. 44. *A point describes a circle of radius a with a uniform speed v; show that the radial and transverse accelerations are $-(v^2/a)\cos\theta$ and $-(v^2/a)\sin\theta$, if a diameter is taken as initial line and one end of the diameter as pole.*

[Lucknow 1978; Meerut 83S, 88; Agra 87]

Sol. We know that the equation of a circle passing through the pole and diameter through the pole as initial line is

$$r=2a.\cos\theta, \qquad \ldots(1) \qquad \textbf{[Remember]}$$

where a is the radius of the circle.

Now the particle describes the curve (1) with a constant speed v.

$$\therefore \quad \left(\frac{dr}{dt}\right)^2+\left(r\frac{d\theta}{dt}\right)^2=v^2=\text{constant}. \quad \ldots(2)$$

From (1), $\dfrac{dr}{dt}=-2a\sin\theta\dfrac{d\theta}{dt}$. $\quad \ldots(3)$

Substituting the values of dr/dt and r in (2), we have

$$\left(-2a\sin\theta\frac{d\theta}{dt}\right)^2+\left(2a\cos\theta\frac{d\theta}{dt}\right)^2=v^2$$

or $$4a^2\left(\frac{d\theta}{dt}\right)^2(\sin^2\theta+\cos^2\theta)=v^2$$

or $$\left(\frac{d\theta}{dt}\right)^2=\frac{v^2}{4a^2} \quad \text{or} \quad \frac{d\theta}{dt}=\frac{v}{2a}. \quad \ldots(4)$$

Putting the value of $d\theta/dt$ in (3), we get

$$\frac{dr}{dt}=-2a\sin\theta.\frac{v}{2a}=-v\sin\theta. \quad \ldots(5)$$

$$\frac{d^2r}{dt^2}=-v\cos\theta.\frac{d\theta}{dt}=-v\cos\theta.\frac{v}{2a}, \quad \text{[from (4)]}$$

$$=-\frac{v^2}{2a}\cos\theta.$$

Now the radial acceleration$=\dfrac{d^2r}{dt^2}-r\left(\dfrac{d\theta}{dt}\right)^2$

$$=-\frac{v^2}{2a}\cos\theta-2a\cos\theta\left(\frac{v}{2a}\right)^2=-\frac{v^2}{2a}\cos\theta-\frac{v^2}{2a}\cos\theta=-\frac{v^2}{a}\cos\theta$$

Also the transverse acceleration$=\dfrac{1}{r}\dfrac{d}{dt}\left(r^2\dfrac{d\theta}{dt}\right)$

$$=\frac{1}{r}.\frac{d}{dt}\left(r^2\frac{v}{2a}\right) \qquad \left[\because \frac{d\theta}{dt}=\frac{v}{2a}\right]$$

$$=\frac{v}{2ar}2r\frac{dr}{dt}, \text{ because } v \text{ is constant}$$

$$=\frac{v}{a}(-v\sin\theta) \qquad \left[\because \text{ from (5), } \frac{dr}{dt}=-v\sin\theta\right]$$

$$=-\frac{v^2}{a}\sin\theta.$$

Ex. 45. *A particle moving in a plane, describes the equiangular spiral $r=ae^{\theta\cot\alpha}$. If the radius vector to the particle has a constant angular velocity, show that the resultant acceleration of the particle makes an angle 2α with the radius vector and is of magnitude v^2/r, where v is the speed.*

Sol. The equation of equiangular spiral is given as

$$r=ae^{\theta\cot\alpha}. \quad \ldots(1)$$

Since the angular velocity of the radius vector is given to be constant, therefore

$$d\theta/dt=\omega=\text{constant}. \qquad \ldots(2)$$

Differentiating (1), w.r.t. 't', we have

$$\frac{dr}{dt}=ae^{\theta \cot \alpha}.\cot \alpha.\ \frac{d\theta}{dt}=r\ \omega \cot \alpha, \qquad \ldots(3)$$

[from (1) and (2)]

and so $\dfrac{d^2r}{dt^2}=\omega \cot \alpha\ \dfrac{dr}{dt}=\omega \cot \alpha . r\omega \cot \alpha=r\omega^2 \cot^2 \alpha.$ ...(4)

Now the speed v of the particle is the resultant of its radial and transverse components of velocity.

$\therefore \quad v=\sqrt{(\dot{r})^2+(r\dot{\theta})^2]}=\sqrt{(r^2\omega^2 \cot^2 \alpha+r^2\omega^2)}$

or $\quad v=r\omega \operatorname{cosec} \alpha.$ ...(5)

Now the radial acceleration of the particle$=\ddot{r}-r\dot{\theta}^2$

$=r\omega^2 \cot^2 \alpha-r\omega^2,$ [from (4) and (2)]

$=r\omega^2 (\cot^2 \alpha-1),$

and the transverse acceleration$=\dfrac{1}{r}\dfrac{d}{dt}(r^2\dot{\theta})=\dfrac{1}{r}\dfrac{d}{dt}(r^2\omega)=\dfrac{2r\omega}{r}\dot{r}$

$=2\omega\ r\omega \cot \alpha,$ [from (3)]

$=2\ \omega^2 r \cot \alpha.$

$\therefore$ the resultant acceleration of the particle

$=\sqrt{[(\text{radial acceleration})^2+(\text{transverse acceleration})^2]}$

$=\sqrt{[\{r\omega^2 (\cot^2 \alpha-1)\}^2+(2\ \omega^2 r \cot \alpha)^2]}$

$=r\omega^2 \sqrt{[(\cot^2 \alpha-1)^2+4 \cot^2 \alpha]}$

$=r\omega^2 \sqrt{[(\cot^2 \alpha+1)^2]}\quad =r\omega^2 (\cot^2 \alpha+1)=r\omega^2 \operatorname{cosec}^2 \alpha$

$=\dfrac{(r\omega \operatorname{cosec} \alpha)^2}{r}=\dfrac{v^2}{r}$ [from (5)]

If the resultant acceleration makes an angle β with the radius vector, then

$$\tan \beta=\frac{\text{transverse acceleration}}{\text{radial acceleration}}$$

$$=\frac{2}{r\omega^2}\frac{\omega^2 r \cot \alpha}{(\cot^2 \alpha-1)}=\frac{2 \tan \alpha}{1-\tan^2 \alpha}=\tan 2\alpha.$$

$\therefore \quad \beta=2\alpha$ *i.e.*, the resultant acceleration makes an angle 2α with the radius vector.

Remark. For the curve (1), $\phi=\alpha$ *i.e.*, the tangent at any point P of the curve (1) makes a constant angle α with the radius vector OP. Also the resultant acceleration of P makes an angle 2α with the radius vector OP. Therefore the angle which the resultant acceleration of P makes with the tangent at $P=2\alpha-\alpha=\alpha=$the angle which the radius vector OP makes with the tangent at P.

Ex. 46. *A point starts from the origin in the direction of the initial line with velocity f/ω and moves with constant angular velocity ω about the origin and with constant negative radial acceleration f. Show that the rate of growth of the radial velocity is never positive, but tends to the limit zero, and prove that the equation of the path is $\omega^2 r=f(1-e^{-\theta})$.*

[Agra 1980, 87; Meerut 90S, Rohilkhand 85]

Sol. Here, the angular velocity$=d\theta/dt=\omega=$constant; ...(1)

and radial acceleration$=\dfrac{d^2r}{dt^2}-r\left(\dfrac{d\theta}{dt}\right)^2=-f.$ ...(2)

From (1) and (2), we have

$$\frac{d^2r}{dt^2}=r\omega^2-f. \qquad ...(3)$$

Multiplying both sides of (3) by $2\,(dr/dt)$ and integrating w.r.t. 't', we have

$$(dr/dt)^2=r^2\omega^2-2fr+A, \qquad ...(4)$$

where A is constant.

But the particle starts from the origin in the direction of the initial line with velocity f/ω.

$\therefore$ initially, when $r=0$, $\quad dr/dt=f/\omega$.

$\therefore$ from (4), we have $\quad (f/\omega)^2=A$.

$$\therefore \quad \left(\frac{dr}{dt}\right)^2=r^2\omega^2-2fr+\frac{f^2}{\omega^2}=\left(\frac{f}{\omega}-r\omega\right)^2$$

or $$\frac{dr}{dt}=\frac{f}{\omega}-r\omega$$

[the positive sign is taken because the particle moves in the direction of r increasing]

or $$\frac{dr}{d\theta}\cdot\frac{d\theta}{dt}=\frac{f}{\omega}-r\omega$$

or $$\frac{dr}{d\theta}\cdot\omega=\frac{f}{\omega}-r\omega,$$

or $$\frac{dr}{(f/\omega^2)-r}=d\theta. \qquad \text{[from (1)]}$$

Integrating, we have

$$-\log\left(\frac{f}{\omega^2}-r\right)=\theta+B, \text{ where } B \text{ is a constant.}$$

But initially $\theta=0$ and $r=0$; $\quad \therefore \quad B=-\log(f/\omega^2)$.

$$\therefore \quad -\log(f/\omega^2-r)=\theta-\log(f/\omega^2)$$

or
$$\log\left(\frac{f}{\omega^2}-r\right)-\log\left(\frac{f}{\omega^2}\right)=-\theta$$
or
$$\log\frac{(f/\omega^2-r)}{f/\omega^2}=-\theta$$
or
$$\frac{(f/\omega^2-r)}{f/\omega^2}=e^{-\theta}$$
or
$$f/\omega^2-r=(f/\omega^2)e^{-\theta}$$
or
$$r=(f/\omega^2)(1-e^{-\theta})$$
or
$$\omega^2 r=f(1-e^{-\theta}), \quad \ldots(5)$$
which is the required equation of the path.

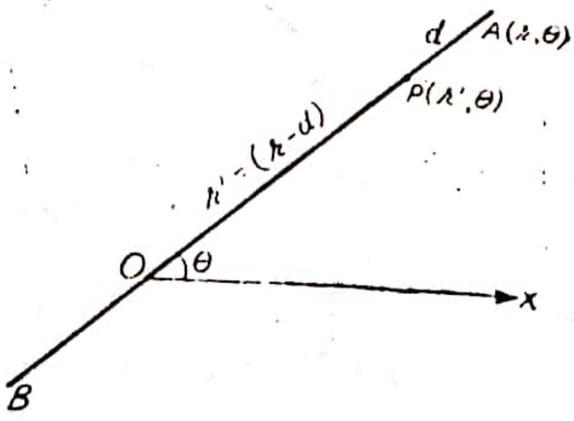
From (3) and (5), we have
$$\frac{d^2r}{dt^2}=f(1-e^{-\theta})-f=-\frac{f}{e^\theta}.$$
Thus the rate of growth of the radial velocity
$$=\frac{d}{dt}\left(\frac{dr}{dt}\right)=\frac{d^2r}{dt^2}=-f/e^\theta,$$
which is never positive because f/e^θ is positive for all values of θ.

Also f/e^θ tends to the limit 0 as $\theta\to\infty$.

Ex. 47. *If a rod which always passes through the origin rotates with uniform angular velocity ω, while one end describes the curve $r=a+be^\theta$, show that radial acceleration of any point of the rod is the same at every instant, and the radial velocity is the same at every point at a given instant.*

Sol. Let AB be the rod which always passes through the origin O and rotates with uniform angular velocity ω.

$\therefore$ $\quad d\theta/dt=\omega=\text{constant}. \quad \ldots(1)$



Let the end A of the rod describe the curve
$$r=a+be^\theta. \quad \ldots(2)$$
$\therefore$
$$\frac{dr}{dt}=be^\theta\frac{d\theta}{dt}=(r-a)\,\omega, \quad \ldots(3)$$
and
$$\frac{d^2r}{dt^2}=\omega\,\frac{dr}{dt}=\omega.(r-a)\,\omega=(r-a)\,\omega^2. \quad \ldots(4)$$
Let P be a point on the rod such that $AP=d$. If (r,θ) are the co-ordinates of A at any time t, then the co-ordinates of P at this instant will be (r',θ), where
$$r'=OP=OA-AP=r-d.$$

Now the radial acceleration of P

$$=\frac{d^2}{dt^2}r'-r'\left(\frac{d\theta}{dt}\right)^2=\frac{d^2}{dt^2}(r-d)-(r-d)\,\omega^2$$

$$=\frac{d^2r}{dt^2}-(r-d)\,\omega^2, \qquad [\because \; d \text{ is a constant}]$$

$$=(r-a)\,\omega^2-(r-d)\,\omega^2, \qquad \text{[substituting from (4)]}$$

$$=(d-a)\,\omega^2, \text{ which is independent of } t.$$

$\therefore$ the radial acceleration of any point of the rod is the same at every instant.

Also the radial velocity of P

$$=\frac{dr'}{dt}=\frac{d}{dt}(r-d)=\frac{dr}{dt} \qquad \text{[from (3)]}$$

$$=(r-a)\,\omega,$$

which is independent of d.

$\therefore$ the radial velocity of every point of the rod is the same at a given instant.

Ex. 48. *If the radial and transverse velocities of a particle are always proportional to each other, (i) show that the path is an equiangular spiral.* **[Kanpur 1978; Meerut 85P]**

(ii) If in addition the radial and transverse accelerations are always proportional to each other, show that the velocity of the particle varies as some power of the radius vector.

[Gorakhpur 1974; Raj. 73]

Sol. Here it is given that the radial velocity $\propto$ the transverse velocity.

$$\therefore \qquad \frac{dr}{dt}=k\left(r\,\frac{d\theta}{dt}\right), \qquad \ldots(1)$$

where k is a constant.

(i) From (1), we have $\dfrac{dr}{r}=k\,d\theta$.

Integrating, $\log r=k\theta+\log A$, where A is a constant.

or $\qquad \log (r/A)=k\theta$.

$\therefore \qquad r=Ae^{k\theta}$.

This is the equation of the path which is an equiangular spiral.

(ii) Here in addition to (1), it is given that the radial acceleration $\propto$ the transverse acceleration.

$$\therefore \quad \frac{d^2r}{dt^2}-r\left(\frac{d\theta}{dt}\right)^2=\lambda.\,\frac{1}{r}\,\frac{d}{dt}\left(r^2\,\frac{d\theta}{dt}\right). \qquad \ldots(2)$$

From (1), $$r\frac{d\theta}{dt}=\frac{1}{k}\frac{dr}{dt}.$$

Putting it in (2), we have

$$\frac{d^2r}{dt^2}-r\left(\frac{1}{kr}\frac{dr}{dt}\right)^2=\frac{\lambda}{r}\frac{d}{dt}\left(r\cdot\frac{1}{k}\frac{dr}{dt}\right)$$

or $$\frac{d^2r}{dt^2}-\frac{1}{k^2r}\left(\frac{dr}{dt}\right)^2=\frac{\lambda}{kr}\left\{r\frac{d^2r}{dt^2}+\left(\frac{dr}{dt}\right)^2\right\}$$

or $$\frac{d^2r}{dt^2}-\frac{1}{k^2r}\left(\frac{dr}{dt}\right)^2=\frac{\lambda}{k}\frac{d^2r}{dt^2}+\frac{\lambda}{kr}\left(\frac{dr}{dt}\right)^2$$

or $$\left(1-\frac{\lambda}{k}\right)\frac{d^2r}{dt^2}=\frac{1}{kr}\left(\frac{1}{k}+\lambda\right)\left(\frac{dr}{dt}\right)^2$$

or $$\frac{d^2r/dt^2}{dr/dt}=\frac{n}{r}\frac{dr}{dt},\quad \text{where}\quad \frac{(1/k+\lambda)}{k\,(1-\lambda/k)}=n \text{ (a const.)}.$$

Integrating with respect to t, we have

$$\log (dr/dt)=n\log r+\log C=\log (Cr^n).$$

$$\therefore\quad dr/dt=Cr^n. \qquad \ldots(3)$$

Now the resultant velocity of the particle

$$=\sqrt{\left\{\left(\frac{dr}{dt}\right)^2+\left(r\frac{d\theta}{dt}\right)^2\right\}}=\sqrt{\left\{\left(\frac{dr}{dt}\right)^2+\left(\frac{1}{k}\frac{dr}{dt}\right)^2\right\}}$$

[from (1)]

$$=\left(1+\frac{1}{k^2}\right)^{1/2}\frac{dr}{dt}=\left(1+\frac{1}{k^2}\right)^{1/2}Cr^n, \qquad \text{[from (3)]}$$

$=Br^n$, where B is some constant.

$\therefore$ the velocity of the particle varies as some power of the radius vector.

Ex. 49. *A curve is described by a particle having a constant acceleration in a direction inclined at a constant angle to the tangent. Show that the curve is an equiangular spiral.*

[Kanpur 1978]

Sol. Let $P\,(s, \psi)$ be the position of the particle at any time t and let f be the constant acceleration of P inclined at a constant angle α to the tangent at P to the path of the particle. Then resolving the acceleration at P along the tangent and normal at P, we have

the tangential acceleration$=v\dfrac{dv}{ds}=f\cos\alpha$, ...(1)

and the normal acceleration$=\dfrac{v^2}{\rho}=f\sin\alpha$. ...(2)

Dividing (1) by (2), we have

$$\frac{1}{v}\frac{dv}{ds}\frac{ds}{d\psi}=\cot\alpha \qquad \left[\because\ \rho=\frac{ds}{d\psi}\right]$$

or $(1/v)\, dv = \cot \alpha \, d\psi$, separating the variables.

Integrating, we have $\log v = \psi \cot \alpha + \log A$,

where A is a constant

or $$\log (v/A) = \psi \cot \alpha$$

or $$v = A e^{\psi \cot \alpha} \qquad \ldots(3)$$

Now from (2), we have

$$\rho = \frac{ds}{d\psi} = \frac{v^2}{f \sin \alpha} = \frac{A^2 e^{2\psi \cot \alpha}}{f \sin \alpha}, \qquad \text{[from (3)]}$$

$$= c e^{2\psi \cot \alpha}, \qquad \text{where } c = A^2/f \sin \alpha = \text{some constant.}$$

$$\therefore \quad ds = c e^{2\psi \cot \alpha}\, d\psi.$$

Integrating, we have

$$s = \frac{c}{2 \cot \alpha} \cdot e^{2\psi \cot \alpha} + B, \text{ where } B \text{ is a constant}$$

or $s = a e^{\lambda \psi} + B$, where a, λ and B are some constants.

This is the intrinsic equation of an equiangular spiral. Hence the curve described by the particle is an equiangular spiral.

Ex. 50. *A particle moves in a catenary $s = c \tan \psi$, the direction of its acceleration at any point makes equal angles with the tangent and normal to the path at the point. If the speed at the vertex (where $\psi = 0$) be u, show that the velocity and acceleration at any other point ψ are ue^{ψ} and $(\sqrt{2}/c)\, u^2 e^{2\psi} \cos^2 \psi$.*

[Meerut 1975, 79; Lucknow 80]

Sol. It is given that the direction of acceleration at any point makes equal angles with the tangent and normal to the path at the point. Therefore the tangential and normal accelerations will be equal at any time t

i.e., $$v \frac{dv}{ds} = \frac{v^2}{\rho} \qquad \ldots(1)$$

or $$\frac{dv}{ds} \cdot \rho = v$$

or $$\frac{dv}{ds} \cdot \frac{ds}{d\psi} = v \qquad \left[\because \rho = \frac{ds}{d\psi}\right]$$

or $$\frac{dv}{v} = d\psi.$$

Integrating, $\log v = \psi + \log A$, where A is a constant.

But at the vertex, where $\psi = 0$, $v = u$.

$$\therefore \quad \log A = \log u \quad \text{or} \quad A = u.$$

$$\therefore \quad \log v = \psi + \log u$$

or $$\log v - \log u = \psi$$

or $\log (v/u)=\psi$

or $v=ue^{\psi}$, ...(2)

which gives the velocity of the particle at any point.

Further it is given that the path of the particle is the catenary

$$s=c \tan \psi.$$

$$\therefore \quad \rho=ds/d\psi=c \sec^2 \psi.$$

$\therefore$ the resultant acceleration of the particle

$$=\sqrt{[(\text{tangential accel.})^2+(\text{normal accel.})^2]}$$

$$=\sqrt{\left[\left(v\frac{dv}{ds}\right)^2+\left(\frac{v^2}{\rho}\right)^2\right]}$$

$$=\sqrt{[(v^2/\rho)^2+(v^2/\rho)^2]}, \qquad \text{[from (1)]}$$

$$=(v^2/\rho).\sqrt{2}$$

$$=\sqrt{2}.\frac{(ue^{\psi})^2}{c \sec^2 \psi}=\frac{\sqrt{2}}{c} u^2 e^{2\psi} \cos^2 \psi.$$

Ex. 51. *A particle, projected with a velocity u, acted on by a force which produces a constant acceleration f in the plane of the motion inclined at a constant angle α with the direction of motion. Obtain the intrinsic equation of the curve described, and show that the particle will be moving in the opposite direction to that of projection at time*

$$\frac{2u}{f \cos \alpha}(e^{\pi \cot \alpha}-1).$$

Sol. Let $P(s, \psi)$ be the position of the particle at any time t. It is given that P possesses a constant acceleration f inclined at a constant angle α to the tangent at P. Therefore resolving the acceleration of P along the tangent and normal at P, we have

the tangential acceleration$=v\dfrac{dv}{ds}=f \cos \alpha$, ...(1)

and the normal acceleration$=\dfrac{v^2}{\rho}=f \sin \alpha$. ...(2)

Dividing (1) by (2), we have

$$\frac{1}{v}\frac{dv}{ds}\frac{ds}{d\psi}=\cot \alpha \qquad \left[\because \quad \rho=\frac{ds}{d\psi}\right]$$

or $(1/v)\, dv=\cot \alpha \, d\psi$. Integrating, we have

$\log v=\psi \cot \alpha+A$, where A is a constant.

Let $\psi=0$ at the point of projection.

Then $\psi=0$, when $v=u$.

$\therefore$ $\log u=A$.

$\therefore$ $\log v=\psi \cot \alpha+\log u$ or $\log (v/u)=\psi \cot \alpha$

or $ds/dt=v=ue^{\psi \cot \alpha}$. ...(3)

Now from (2), we have

$$\frac{ds}{d\psi}=\rho=\frac{v^2}{f\sin\alpha}=\frac{u^2 e^{2\psi\cot\alpha}}{f\sin\alpha}, \qquad \ldots(4)$$

substituting for v from (3).

$$\therefore\ ds=\frac{u}{f\sin\alpha}e^{2\psi\cot\alpha}\,d\psi.$$

Integrating, we have

$$s=\frac{u^2}{(f\sin\alpha)(2\cot\alpha)}e^{2\psi\cot\alpha}+B,$$ where B is a constant

$$=\frac{u^2}{2f\cos\alpha}e^{2\psi\cot\alpha}+B.$$

Let $s=0$, where $\psi=0$.

Then $B=-\dfrac{u^2}{2f\cos\alpha}$.

$$\therefore\ s=\frac{u^2}{2f\cos\alpha}e^{2\psi\cot\alpha}-\frac{u^2}{2f\cos\alpha}=\frac{u^2}{2f\cos\alpha}(e^{2\psi\cot\alpha}-1),$$

which is the required intrinsic equation of the path of the particle.

Now dividing (4) by (3), we have

$$\frac{ds}{d\psi}\cdot\frac{dt}{ds}=\frac{ue^{\psi\cot\alpha}}{f\sin\alpha} \quad\text{or}\quad \frac{dt}{d\psi}=\frac{u}{f\sin\alpha}e^{\psi\cot\alpha}.$$

$$\therefore\ dt=\frac{u}{f\sin\alpha}e^{\psi\cot\alpha}\,d\psi. \qquad \ldots(5)$$

Now the particle will be moving in the direction opposite to the direction of projection (which is $\psi=0$), when $\psi=\pi$. Therefore if the particle takes time t_1 to reach the point where $\psi=\pi$, then integrating (5), we have

$$\int_0^{t_1} dt=\int_0^{\pi}\frac{u}{f\sin\alpha}e^{\psi\cot\alpha}\,d\psi$$

i.e., $$t_1=\frac{u}{(f\sin\alpha)(\cot\alpha)}\Big[e^{\psi\cot\alpha}\Big]_0^{\pi}=\frac{u}{f\cos\alpha}[e^{\pi\cot\alpha}-1],$$

which gives the required time.