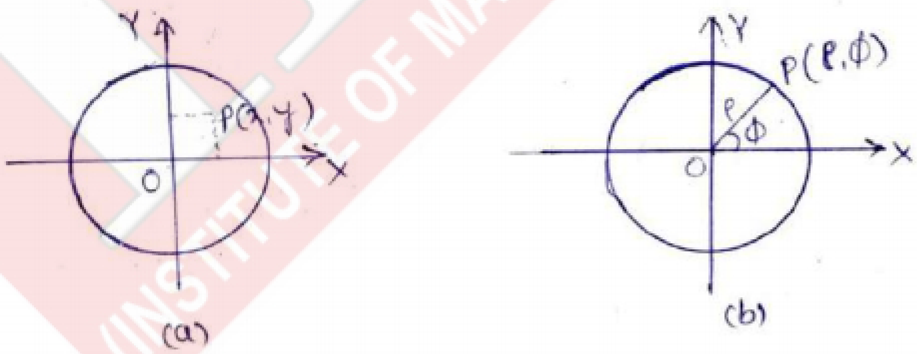
Curvilinear Co-ordinate Systems

(I)

- Here we are studying Non-Cartesian co-ordinate systems.
- The non-Cartesian co-ordinate systems are special cases of the general orthogonal curvilinear co-ordinates.
- We have introduced curvilinear co-ordinates by linking them with Cartesian co-ordinates.
- The expressions for gradient, divergence, curl and Laplacian operator are derived first in curvilinear and then in polar co-ordinates.
- These expressions will be useful in next section.

(II)

* Plane Polar Co-ordinate System:



(a) (b)

From the fig (a). It models a flat circular dinner plate. To specify any point P on its surface. we have to draw two fixed coordinate axes at right angles to each other passing through O, the centre of the plate. This is the familiar Cartesian co-ordinate system. we can locate the point of interest by giving its distance from the two axes.

the position of point P can also be uniquely defined by measuring its radial distance from the origin and the angle ϕ. between the x-axis and the line joining the point to the origin as shown fig (b).

point O is called the pole.

The distance ρ is called the radius vector of the point P and ϕ is its polar angle. These two coordinates taken together are called plane coordinates of P.

Now we can specify the position of the point P as $P(\rho, \phi)$. we would now like to know how x and y are called related to ρ and ϕ.

To relate the x & y coordinates of P to its ρ & ϕ coordinates, we can easily write referring to fig (b).

$$x = \rho\cos\phi \qquad (-\infty < x < \infty)$$

$$\text{and} \quad y = \rho\sin\phi \qquad (-\infty < y < \infty) \qquad \text{—— (1)}$$

It is quite easy to change from Cartesian coordinates to plane polar coordinates. By squaring these relations and adding, we will get

$$\rho = \sqrt{x^2+y^2} \quad \text{——} \quad (2)$$

and on dividing one by the other, we can write

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) \quad \text{——} \quad (3)$$

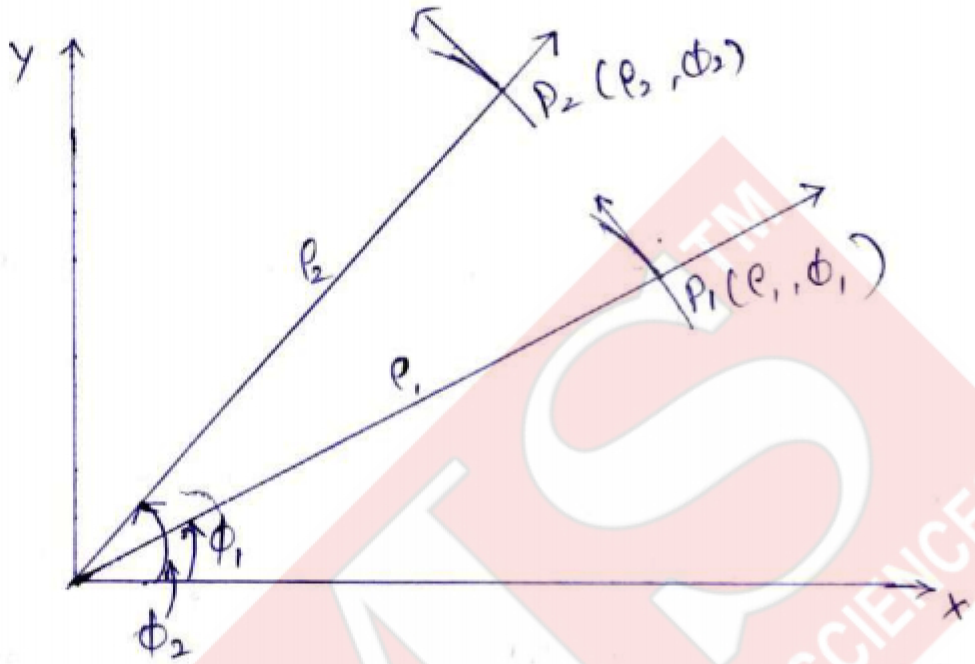
we take ρ to be +ve when it is measured from the origin along the line OP. Similarly, ϕ is taken to be +ve in the anticlockwise direction from x-axis. The range of variation of these coordinates is given by

$$0 < \rho < \infty, \quad 0 \leq \phi \leq 2\pi$$

we know that Cartesian coordinate system is orthogonal, i.e. the x and y-axes meet at right angles at any point in the plane and their directions are fixed.

we will note that each concurrent line cuts the concentric circles at one point only. when we draw a tangent at that point, we will note that the coordinate axes defining the directions of increasing ρ and increasing ϕ are at right angles. That is, the polar coordinates form an orthogonal coordinate system in the plane. But for two different points

$P_1(\rho_1, \phi_1)$ and $P_2(\rho_2, \phi_2)$, the directions of ρ and ϕ coordinate axes are not the same.

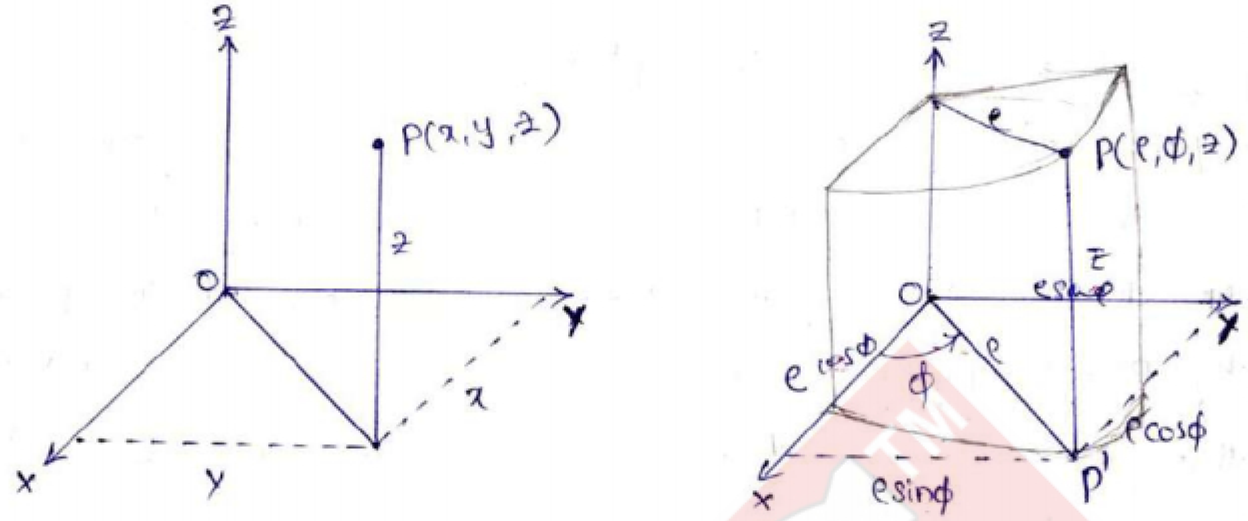


From this we may conclude that unlike in the Cartesian system, the directions of plane polar coordinate axes vary from point to point.

Cylindrical Co-ordinate System:

(M1) we know that in Cartesian co-ordinate system, the position of a point P in space is denoted by $P(x, y, z)$. we denote the cylindrical polar co-ordinates of this point by $P(\rho, \phi, z)$. By referring to below figure, we will note that ρ and z respectively denote the distance of 'P' from the z-axis, and the xy-plane while ϕ is the angle which the line joining the origin with the projection of P on xy-plane makes with the x-axis. It is called the azimuthal angle.

If we recall the equation $x^2+y^2=\rho^2$ with ρ = constant defines a circle in a plane. This suggests that if ~~you~~ we add a Cartesian z-axis to the plane polar co-ordinate system, so that z denotes the perpendicular distance from the xy-plane, we obtain a cylinder. That is, cylindrical polar co-ordinate system extends the plane polar co-ordinate system to three dimensions. So if a point has Cartesian co-ordinates (x, y, z) and cylindrical polar co-ordinates (ρ, ϕ, z).



<u>Representation of a point in Cartesian and Cylindrical co-ordinates.</u>

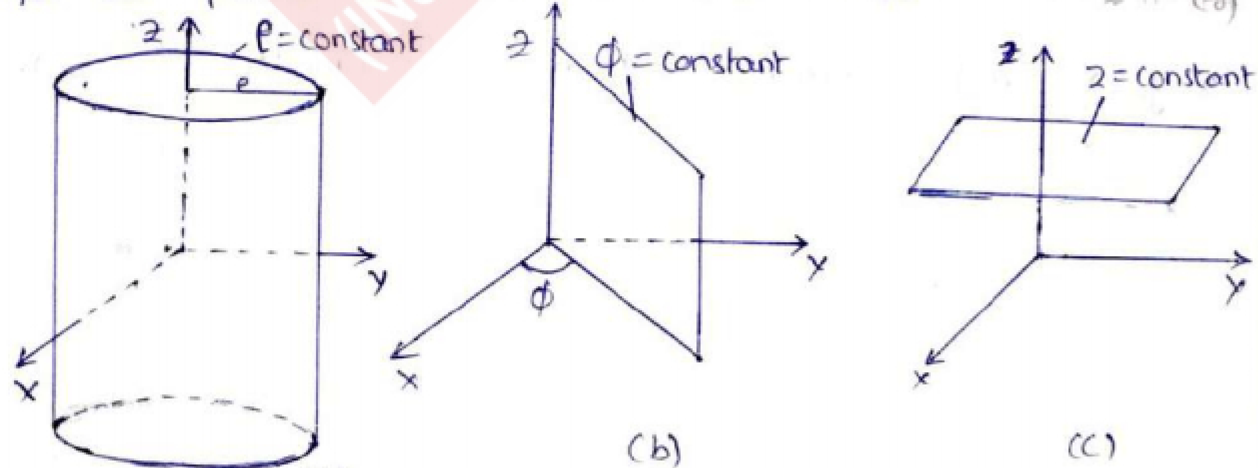
$$x = \rho\cos\phi \qquad (-\infty < x < \infty)$$
$$y = \rho\sin\phi \qquad (-\infty < y < \infty) \quad \text{———} \quad (1)$$
$$z = z \qquad (-\infty < z < \infty)$$

$$\rho = \sqrt{x^2+y^2} \qquad (0 < \rho < \infty)$$
$$\phi = \tan^{-1}(y/x) \qquad (0 \le \phi \le 2\pi)$$
$$z = z \qquad (-\infty < z < \infty) \quad \text{———} \quad (2)$$

$\theta = \tan^{-1}(y/x) = \tan^{-1}(0/0)$

In case of plane polar coordinates, ϕ is undefined at the origin. But in cylindrical co-ordinates ϕ is undefined for all points on the z-axis ($x = 0 = y$).

$\phi = \tan^{-1}(y/x) = \tan^{-1}(0/0)$



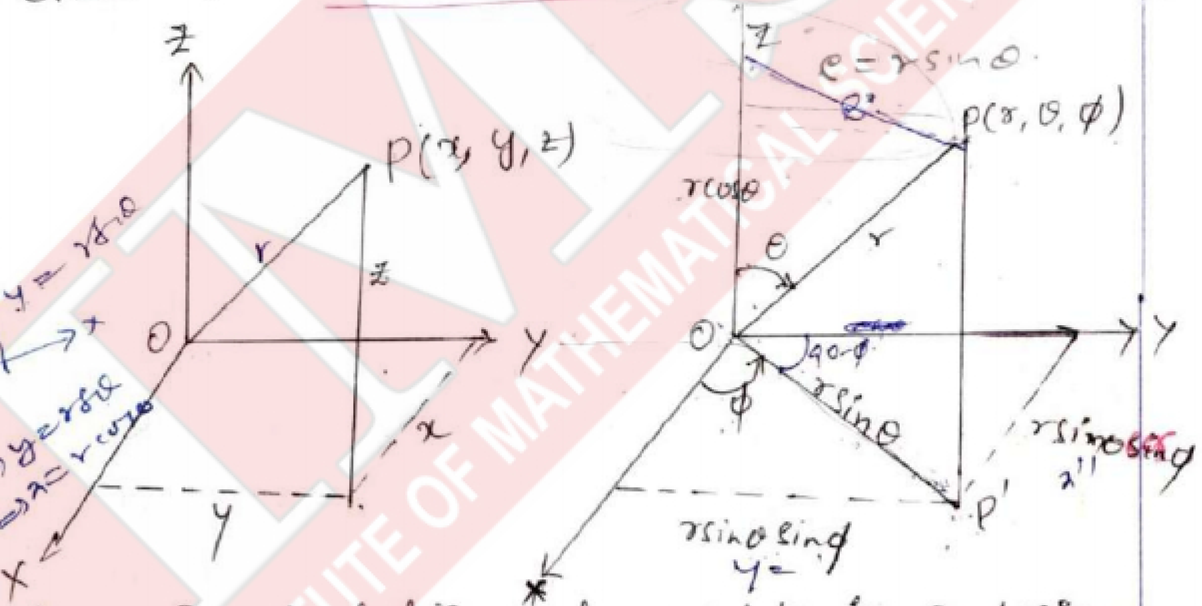
(a) (b) (c)

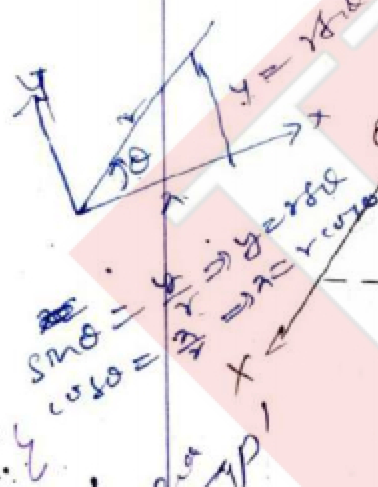
Now refer to fig (a), which shows a cylinder of radius ρ and whose axis of symmetry is along the z-axis. we will note that for any point on the surface of cylinder, ρ is constant. That is, ρ = constant defines a circular cylindrical surface. This is also called the ϕ-z surface. For different values of ρ, we will obtain co-axial right circular cylinders. Their common axis of symmetry is z-axis.

- The ρz-surface defined by ϕ = constant is half-plane bounded on one edge by the z-axis (fig b)
- But the $\rho\phi$-surface given by z = constant is a plane parallel to the xy plane (fig c) just as in the Cartesian co-ordinate system.

(IV) Spherical polar co-ordinate System:

In spherical polar co-ordinate system, the position of a point is specified by the radial distance r, the polar angle θ and the azimuthal angle ϕ as shown in the figure. We can see that while θ is measured in the clockwise direction from the z-axis, ϕ is measured in the anticlockwise direction from the x-axis.





Representation of a point in Cartesian and Spherical polar co-ordinate system.

Referring to this figure we can see that the projection OP onto the xy-plane, $OP' = r\sin\theta$ while its projection on z-axis is $r\cos\theta$.

The components of OP' along x and y-axes are $OP'\cos\phi$ and $OP'\sin\phi$. (i.e, $r\sin\theta\cos\phi$ and $r\sin\theta\sin\phi$)

$OM = OP'\cos\phi$
$= r\sin\theta\cos\phi$

Hence, for a point p having cartesian co-ordinates (x, y, z) and spherical polar co-ordinate (r, θ, ϕ),

we can write

$$\left.\begin{aligned} x &= op'\cos\phi = r\sin\theta\cos\phi \quad (-\infty < x < \infty) \\ y &= op'\sin\phi = r\sin\theta\sin\phi \quad (-\infty < y < \infty) \\ z &= op\cos\theta = r\cos\theta \quad (-\infty < z < \infty) \end{aligned}\right\}$$

$$\left.\begin{aligned} r &= \sqrt{x^2+y^2+z^2} \quad (0 \le r < \infty) \\ \theta &= \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) \quad (0 \le \theta \le \pi) \\ \text{and } \phi &= \tan^{-1}\left(\frac{y}{x}\right). \quad (0 \le \phi \le 2\pi) \end{aligned}\right\}$$

$\left[\because x^2+y^2 = r^2\sin^2\theta(\sin^2\phi + \cos^2\phi) = r^2\sin^2\theta\right.$

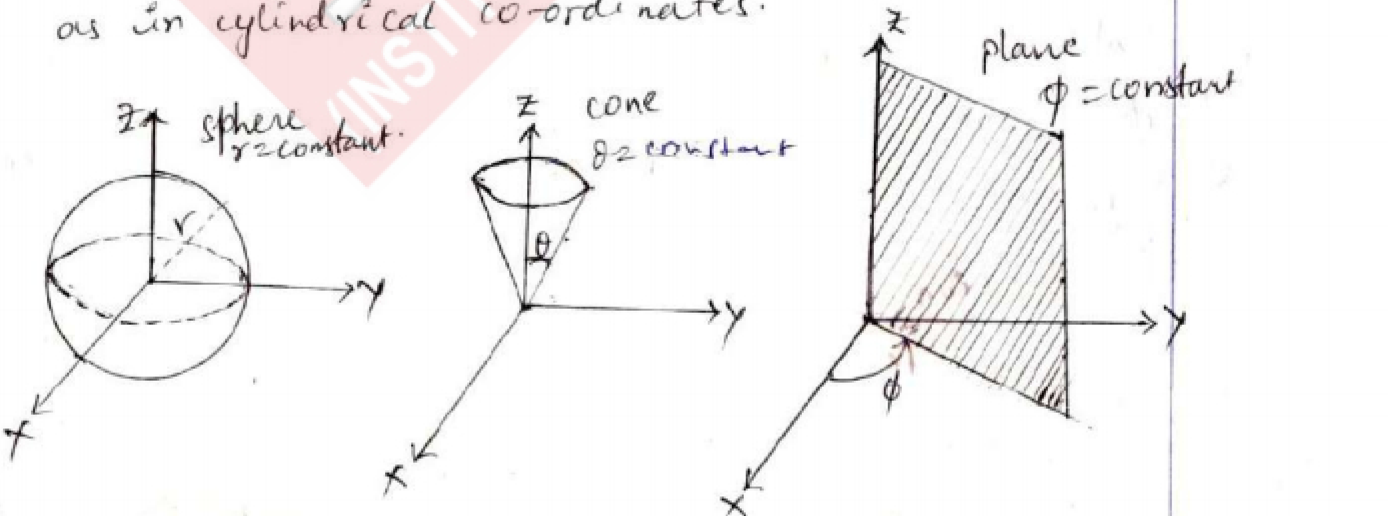
$\Rightarrow r\sin\theta = \sqrt{x^2+y^2}$

$\therefore \frac{r\sin\theta}{r\cos\theta} = \frac{\sqrt{x^2+y^2}}{z}$

$\Rightarrow \tan\theta = \frac{\sqrt{x^2+y^2}}{z}$

$\left.\Rightarrow \theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)\right]$

The surfaces defined by r = constant, θ = constant and ϕ = constant are as shown below. It is instructive to note that in spherical co-ordinates, ϕ = constant is the half plane as in cylindrical co-ordinates.



The co-ordinate surfaces are

$r =$ constant, say, c_1 ; spheres having centre at the origin (or origin if constant $= 0$ i.e., $c_1 = 0$)

$\theta =$ constant, say, c_2 ; cones having vertex at the origin (lines if $c_2 = 0$ or π, and the xy-plane if $c_2 = \pi/2$)

$\phi =$ constant, say, c_3 ; planes through the z-axis.

Expressing a vector in polar coordinates:

A vector A in cartesian co-ordinates can be written as

$$A = A_x\hat{i} + A_y\hat{j} + A_z\hat{k} \quad \text{——} (1)$$

where $\hat{i}$, $\hat{j}$, and $\hat{k}$ are unit vectors along the x, y and z axes respectively. This means that to express A in polar co-ordinates, we must relate $\hat{i}, \hat{j}, \hat{k}$ with the unit vectors associated with the system of interest.

Cylindrical co-ordinate system:

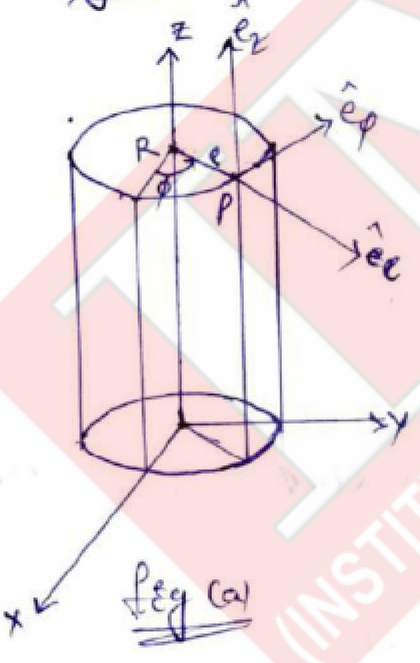


fig (a)

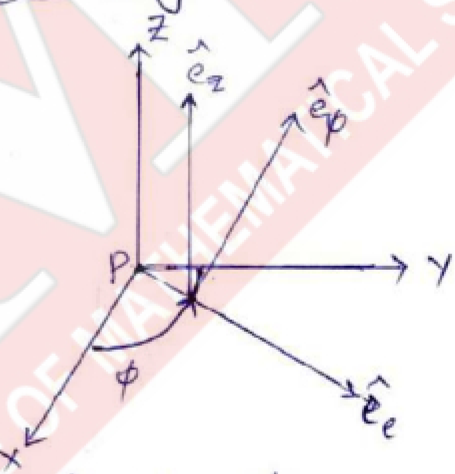


fig (b)

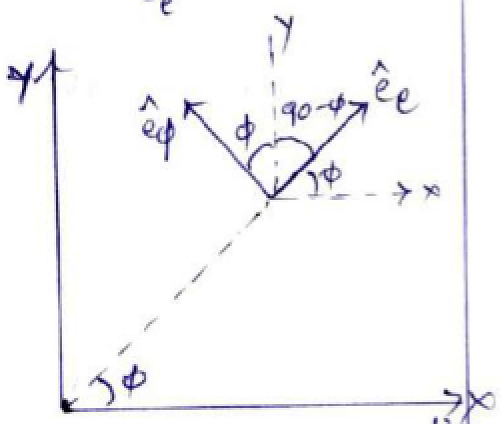


fig (c)

for cylindrical co-ordinate system, we define the unit vectors $\hat{e}_\rho$, $\hat{e}_\phi$ and $\hat{e}_z$ at a given point P as follows: Let R be the point on the z-axis with the same z-co-ordinate as P. Then we define $\hat{e}_\rho$ to be the unit vector P which is normal to the cylindrical surface $\rho =$ constant through P. So $\hat{e}_\rho$ will be in the direction of RP, i.e., along the direction of increasing ρ, as shown in fig (a).

Similarly, we define $\hat{e}_\phi$ to be the unit vector normal to the half plane $\phi =$ constant through P in the direction of increasing ϕ.

The unit vector $\hat{e}_z$ is defined normal to the plane $z =$ constant through P in the direction of increasing z.

we have $\hat{e}_\rho = \frac{RP}{|RP|}$

$= \frac{x\hat{i}+y\hat{j}}{\rho}$

on substituting for $x = \rho\cos\phi$, $y = \rho\sin\phi$

we get

$$\hat{e}_\rho = \frac{\rho\cos\phi\,\hat{i} + \rho\sin\phi\,\hat{j}}{\rho}$$

$$\boxed{\hat{e}_\rho = \cos\phi\,\hat{i} + \sin\phi\,\hat{j}} \quad \text{——} (2)$$

Since $\hat{e}_\phi$ makes an angle $\left(\frac{\pi}{2}+\phi\right)$ with the x-axis and an angle ϕ with the y-axis, we can write

(8)

$$\hat{e}_\phi = \cos\left(\frac{\pi}{2}+\phi\right)\hat{i} + \sin\left(\frac{\pi}{2}+\phi\right)\hat{j}$$

$$\boxed{\hat{e}_\phi = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}} \quad \text{--- (3)}$$

Since $\hat{e}_z$ is parallel to z-axis, its projection along the x and y-axes will be zero.

Hence $\boxed{\hat{e}_z = \hat{k}.}$ --- (4)

These relations can readily be inverted. To this end, we have to multiply (2) by $\cos\phi$ and (3) by $-\sin\phi$.

Add the resulting expressions and use the identity $\cos^2\phi + \sin^2\phi = 1$.

we get

$$\boxed{\hat{i} = \cos\phi\,\hat{e}_\rho - \sin\phi\,\hat{e}_\phi}$$

Similarly we write

$$\boxed{\hat{j} = \sin\phi\,\hat{e}_\rho + \cos\phi\,\hat{e}_\phi}$$

and $\boxed{\hat{k} = \hat{e}_z.}$

We know that directions of the Cartesian unit vectors $(\hat{i}, \hat{j}, \hat{k})$ are uniquely fixed.

But for the unit vectors $\hat{e}_\rho$, $\hat{e}_\phi$ and $\hat{e}_z$ the directions vary from point to point.

Computing their dot and cross products, we can easily check that these vectors are normal to each other and form a right handed system.

→ In cartesian co-ordinates, the position vector is given by $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Express it in terms of cylindrical co-ordinates (ρ, ϕ, z) and the associated unit vectors $\hat{e}_\rho, e_\phi, \hat{e}_z$.

Sol^n: The position vector is given by

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

substituting $x = \rho\cos\phi$, $y = \rho\sin\phi$, $z = z$

and
$$\hat{i} = \cos\phi\,\hat{e}_\rho - \sin\phi\,\hat{e}_\phi$$
$$\hat{j} = \sin\phi\,\hat{e}_\rho + \cos\phi\,\hat{e}_\phi$$
$$\hat{k} = \hat{e}_z$$

we get

$$\vec{r} = \rho\cos\phi(\cos\phi\,\hat{e}_\rho - \sin\phi\,\hat{e}_\phi) + \rho\sin\phi(\sin\phi\,\hat{e}_\rho + \cos\phi\,\hat{e}_\phi) + z\hat{e}_z$$

$$= \rho(\cos^2\phi + \sin^2\phi)\hat{e}_\rho + z\hat{e}_z$$

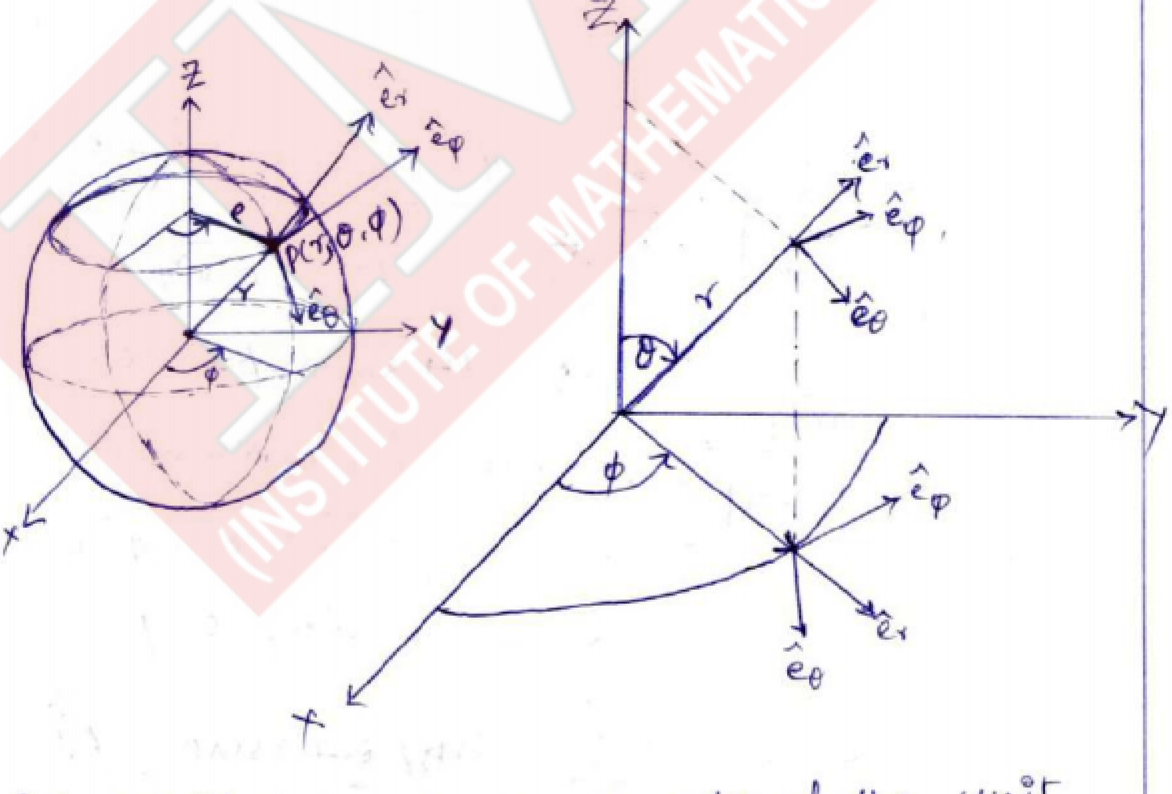
$$= \rho\hat{e}_\rho + z\hat{e}_z.$$

Spherical co-ordinate system

For the spherical polar co-ordinates, we define the unit vectors as follows:

At the point $P(r, \theta, \phi)$, the unit vector $\hat{e}_r$ is normal to the surface $r =$ constant, $\hat{e}_\theta$ is normal to the surface $\theta =$ constant and $\hat{e}_\phi$ is normal to the surface defined by $\phi =$ constant through the given point.

Their directions are along increasing r, θ and ϕ respectively as shown in the figure.



By considering the components of the unit vectors $\hat{e}_r$, $\hat{e}_\theta$, $\hat{e}_\phi$ in the directions of unit vectors $\hat{i}$, $\hat{j}$, $\hat{k}$.

we have

$$\hat{e}_r = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}$$

$$\hat{e}_\theta = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}$$

$$\hat{e}_\phi = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}.$$

Conversely

$$\hat{i} = \sin\theta\cos\phi\,\hat{e}_r + \cos\theta\cos\phi\,\hat{e}_\theta - \sin\phi\,\hat{e}_\phi.$$

$$\hat{j} = \sin\theta\sin\phi\,\hat{e}_r + \cos\theta\sin\phi\,\hat{e}_\theta + \cos\phi\,\hat{e}_\phi$$

$$\hat{k} = \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta.$$

(OR)

At any point P, we have $x = r\sin\theta\cos\phi$,
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$.

so that $\vec{r} = r\sin\theta\cos\phi\,\hat{i} + r\sin\theta\sin\phi\,\hat{j} + r\cos\theta\,\hat{k}$.

If $\hat{e}_r, \hat{e}_\theta, \hat{e}_\phi$ be the unit vectors at P

then

$$\hat{e}_r = \frac{\frac{\partial\vec{r}}{\partial r}}{\left|\frac{\partial\vec{r}}{\partial r}\right|} = \frac{\sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k}}{\sqrt{\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi + \cos^2\theta}}$$

$$= \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi + \cos\theta\,\hat{k}$$

$$\hat{e}_\theta = \frac{\partial\vec{r}/\partial\theta}{|\partial\vec{r}/\partial\theta|} = \frac{r\cos\theta\cos\phi\,\hat{i} + r\cos\theta\sin\phi\,\hat{j} - r\sin\theta\,\hat{k}}{r\sqrt{\cos^2\theta\cos^2\phi + \cos^2\theta\sin^2\phi + \sin^2\theta}}$$

$$= \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}.$$

$$\hat{e}_\phi = \frac{\partial\vec{r}/\partial\phi}{|\partial\vec{r}/\partial\phi|} = \frac{-r\sin\theta\sin\phi\,\hat{i} + r\sin\theta\cos\phi\,\hat{j}}{r\sin\theta}$$

$$= -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}.$$

Differential Element of vector Length:

(10)

In spherical co-ordinates, the position vector of a point moving through space is given by

$$\vec{r} = r\hat{e}_r \quad \text{—— (1)}$$

$[\because \vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

substituting $x = r\sin\theta\cos\phi$
$y = r\sin\theta\sin\phi$
$z = r\cos\theta$

and $\hat{i} = \sin\theta\cos\phi\,\hat{e}_r + \cos\theta\cos\phi\,\hat{e}_\theta - \sin\phi\,\hat{e}_\phi$

$\hat{j} = \sin\theta\sin\phi\,\hat{e}_r + \cos\theta\sin\phi\,\hat{e}_\theta + \cos\phi\,\hat{e}_\phi$

$\hat{k} = \cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta$.

then

$\vec{r} = r\sin\theta\cos\phi(\sin\theta\cos\phi\,\hat{e}_r + \cos\theta\cos\phi\,\hat{e}_\theta - \sin\phi\,\hat{e}_\phi)$
$+ r\sin\theta\sin\phi(\sin\theta\sin\phi\,\hat{e}_r + \cos\theta\sin\phi\,\hat{e}_\theta + \cos\phi\,\hat{e}_\phi)$
$+ r\cos\theta(\cos\theta\,\hat{e}_r - \sin\theta\,\hat{e}_\theta)$

$= \hat{e}_r(r\sin^2\theta\cos^2\phi + r\sin^2\theta\sin^2\phi + r\cos^2\theta)$
$+ \hat{e}_\theta(r\sin\theta\cos^2\phi\cos\theta + r\sin\theta\cos\theta\sin^2\phi$
$- r\cos\theta\sin\theta) +$
$\hat{e}_\phi(-r\sin\theta\cos\phi\sin\phi + r\sin\theta\sin\phi\cos\phi)$

$= \hat{e}_r[r\sin^2\theta(\cos^2\phi + \sin^2\phi) + r\cos^2\theta]$

$= r\hat{e}_r.]$

substituting the value of $\hat{e}_r$ in (1)

we get $\vec{r} = r(\hat{i}\sin\theta\cos\phi + \sin\theta\sin\phi\,\hat{j} + \cos\theta\,\hat{k})$

To compute $d\vec{r}$, we would recall that r, θ, and ϕ will change as $\vec{r}$ changes.

Hence we can write

$$d\vec{r} = dr(\hat{i}\sin\theta\cos\phi + \hat{j}\sin\theta\sin\phi + \hat{k}\cos\theta)$$
$$+ r(\hat{i}\cos\theta\cos\phi\, d\theta - \hat{i}\sin\theta\sin\phi\, d\phi +$$
$$\hat{j}\sin\theta\cos\phi\, d\phi + \hat{j}\cos\theta\sin\phi\, d\theta - \hat{k}\sin\theta\, d\theta)$$

In spherical co-ordinates, small line elements along $\hat{e}_r, \hat{e}_\theta$ and $\hat{e}_\phi$ are given by dr, $r\,d\theta$ and $r\sin\theta\, d\phi$ respectively.

On collecting coefficients of dr, $r\,d\theta$ and $r\sin\theta\, d\phi$, this expression becomes

$$d\vec{r} = (\hat{i}\sin\theta\cos\phi + \hat{j}\sin\theta\sin\phi + \hat{k}\cos\theta)\,dr$$
$$+ (\hat{i}\cos\theta\cos\phi + \hat{j}\cos\theta\sin\phi - \hat{k}\sin\theta)\,r\,d\theta$$
$$+ (-\hat{i}\sin\phi + \hat{j}\cos\phi)\,r\sin\theta\, d\phi \quad \text{——} \ (2)$$

Since $\hat{e}_r = \sin\theta\cos\phi\,\hat{i} + \sin\theta\sin\phi + \cos\theta\,\hat{k}$

$\hat{e}_\theta = \cos\theta\cos\phi\,\hat{i} + \cos\theta\sin\phi\,\hat{j} - \sin\theta\,\hat{k}$

$\hat{e}_\phi = -\sin\phi\,\hat{i} + \cos\phi\,\hat{j}$

$\therefore$ From (1) we find that differential element of vector length in spherical polar co-ordinates can be written as

$$d\vec{r} = \hat{e}_r\,dr + \hat{e}_\theta\, r\,d\theta + \hat{e}_\phi\, r\sin\theta\, d\theta. \quad \text{——} \ (3)$$

→ In cylindrical polar co-ordinates, the position vector of a point moving through space is given by $\vec{r} = \rho\hat{e}_\rho + z\hat{e}_z$

Substituting the value of $\hat{e}_\rho$ in $\vec{r}$, we get

$$\vec{r} = \rho(\hat{i}\cos\phi + \hat{j}\sin\phi) + \hat{k}z$$

A small change in $\vec{r}$ gives rise to changes in ρ, ϕ and z.

So we can write

$$d\vec{r} = d\rho(\hat{i}\cos\phi + \hat{j}\sin\phi) + \rho(-\hat{i}\sin\phi + \hat{j}\cos\phi)d\phi + \hat{k}dz$$

$$= \hat{e}_\rho d\rho + \rho d\phi\,\hat{e}_\phi + \hat{e}_z dz. \quad \text{—— (4)}$$

$$(\because \hat{i}\cos\phi + \hat{j}\sin\phi = \hat{e}_\rho\ ;\ -\hat{i}\sin\phi + \hat{j}\cos\phi = \hat{e}_\phi \text{ and } \hat{k} = \hat{e}_z)$$

where $d\rho$, $\rho d\phi$ and dz denote small line element in the increasing directions of ρ, ϕ and z respectively.

→ Differential Element of vector Area

From (2), we can identify that in spherical co-ordinates dr, $rd\theta$ and $r\sin\theta d\phi$ are respectively, analogous to dx, dy and dz in Cartesian co-ordinate system.

Similarly from eqn (4), we can identify ~~that~~ in cylindrical co-ordinates $d\rho$, $\rho d\phi$ and dz analogous to dx, dy and dz, respectively.

We can use this analogy to obtain expressions for components of differential element of vector area.

In Cartesian co-ordinate system, the area of the face normal to x-axis is given by

$$dA_x = dy\, dz$$

We can use the same definition for non-Cartesian co-ordinates as well.

In terms of cylindrical co-ordinates, we can write

$$dA_\rho = (dr)_\phi \,(dr)_z$$
$$= \rho\, d\phi\, dz \quad \text{—— (i)}$$

Similarly

$$dA_\phi = (dr)_\rho\, (dr)_z$$
$$= d\rho\, dz \quad \text{—— (ii)}$$

and $dA_z = (dr)_\rho\, (dr)_\phi$
$$= \rho\, d\rho\, d\phi \quad \text{—— (iii)}$$

On combining eqns (i), (ii) and (iii), we can express the differential element of vector area in cylindrical co-ordinates as

$$dA = dA_\rho\, \hat{e}_\rho + dA_\phi\, \hat{e}_\phi + dA_z\, \hat{e}_z.$$

(or) $dA = \rho\, d\phi\, dz\, \hat{e}_\rho + d\rho\, dz\, \hat{e}_\phi + \rho\, d\rho\, d\phi\, \hat{e}_z$ —— (5)

we can follow the same procedure and show that differential element of vector area in spherical co-ordinates can be expressed as

$$dA = r^2 \sin\theta\, d\theta\, d\phi\, \hat{e}_r + r\sin\theta\, dr\, d\phi\, \hat{e}_\theta + r\, d\theta\, dr\, \hat{e}_\phi.$$

→ In Cartesian co-ordinates, an element of volume is defined as

$$dv = dx\, dy\, dz.$$

we use the same definition in non-Cartesian co-ordinates.

In cylindrical co-ordinates

$x = r\cos\theta$
$y = r\sin\theta$
~~$dx = r\, d\theta\, d\theta$~~
$dx\, dy$ -/.

$$dv = (dr)_\rho (dr)_\phi (dr)_z$$
$$= (d\rho)(\rho\, d\phi)\, dz$$
$$= \rho\, d\rho\, d\phi\, dz \quad \text{—— (iv)}$$

In spherical co-ordinates

$$dv = (dr)_r (dr)_\theta (dr)_\phi$$
$$= (dr)(r\, d\theta)(r\sin\theta\, d\phi)$$
$$= r^2 dr \sin\theta\, d\theta\, d\phi. \quad \text{—— (v)}$$