LINEAR ALGEBRA

: CSE-2013 :

1(a) : Find the ~~sa~~ inverse of the matrix $A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & -1 & 7 \\ 3 & 2 & -1 \end{bmatrix}$ by using elementary row operations. Hence solve the system of linear equations

$$\begin{aligned} x + 3y + z &= 10 \\ 2x - y + 7z &= 21 \\ 3x + 2y - z &= 4 \end{aligned}$$

$\rightarrow$ Let $[A|I] = \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 2 & -1 & 7 & 0 & 1 & 0 \\ 3 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$

$R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 3R_1$

$\sim \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -7 & 5 & -2 & 1 & 0 \\ 0 & -7 & -4 & -3 & 0 & 1 \end{array}\right]$

$R_1 \to 7R_1 + 3R_2,\ R_3 \to R_3 - R_2$

$\sim \left[\begin{array}{ccc|ccc} 7 & 0 & 22 & 1 & 3 & 0 \\ 0 & -7 & 5 & -2 & 1 & 0 \\ 0 & 0 & -9 & -1 & -1 & 1 \end{array}\right]$

$R_1 \to 9R_1 + 22R_3,\ R_2 \to 9R_2 + 5R_3$

$\sim \left[\begin{array}{ccc|ccc} 63 & 0 & 0 & -13 & 5 & 22 \\ 0 & -63 & 0 & -23 & 4 & 5 \\ 0 & 0 & -9 & -1 & -1 & 1 \end{array}\right]$

$R_1 \to R_1 \div 63,\ R_2 \to R_2 \div (-63),\ R_3 \to R_3 \div (-9)$

$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -13/63 & 5/63 & 22/63 \\ 0 & 1 & 0 & 23/63 & -4/63 & -5/63 \\ 0 & 0 & 1 & 1/9 & 1/9 & -1/9 \end{array}\right]$

$\therefore A^{-1} = \frac{1}{63}\begin{bmatrix} -13 & 5 & 22 \\ 23 & -4 & -5 \\ 7 & 7 & -7 \end{bmatrix}$

The given system of equations can be rewritten as $AX = B$ where $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 10 \\ 21 \\ 4 \end{bmatrix}$.

Then, the unique solution to the equations can be given by $X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{1}{63}\begin{bmatrix} -13 & 5 & 22 \\ 23 & -4 & -5 \\ 7 & 7 & -7 \end{bmatrix}\begin{bmatrix} 10 \\ 21 \\ 4 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$\therefore x = 1,\ y = 2,\ z = 3.$

1(b) Let A be a square matrix and A^* be its adjoint. Show that the eigen values of matrices AA^* and A^*A are real. Further, show that the trace $(AA^*) =$ trace (A^*A).

→ We have, $A \cdot A^* = |A|I$ ~~$= \lambda I$. where $\lambda = |A|$~~

Let X be the eigen vector of AA^* and λ be the corr. eigen value. Then, $X \neq 0$ and

$$AA^* X = \lambda X \Rightarrow X^* AA^* X = \lambda X^* X \quad \text{---} (1)$$

Taking adjoint ~~Tranjugate~~ both sides, $(X^* AA^* X)^* = (\lambda X^* X)^*$

$$\Rightarrow X^* AA^* X = \bar{\lambda} X^* X. \quad (2)$$

$$(2) = (1) \Rightarrow \lambda X^* X = \bar{\lambda} X^* X.$$

$$\Rightarrow (\lambda - \bar{\lambda}) X^* X = 0 \Rightarrow \lambda - \bar{\lambda} = 0 \qquad [\text{Since } X \neq 0 \Rightarrow X^* X \neq 0]$$

$$\Rightarrow \lambda = \bar{\lambda}$$

Since $\lambda = \bar{\lambda}$, the eigen values λ is real.

Similarly, the eigen values of A^*A are real

Also: Since $AA^* = |A|I \Rightarrow AA^* =$
or $A^*A = |A|I \Rightarrow A^*A =$

$$\begin{bmatrix} |A| & 0 & 0 & \cdots & 0 \\ 0 & |A| & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & |A| \end{bmatrix}$$

$\therefore$ Trace of $AA^* = |A| + |A| + \cdots + |A| =$ Trace of A^*A^*

$\therefore Tr(A^*A) = Tr(AA^*)$

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2(a) (i) Let P_n denote the vector space of all real polynomials of degree atmost n and $T: P_2 \to P_3$ be a linear transformation given by $T(P(x)) = \int_0^x p(t)\,dt$; $p(x) \in P(2)$.
find the matrix of T wrt the bases $\{1, x, x^2\}$ and $\{1, x, 1+x^2, 1+x^3\}$ of P_2 and P_3 respectively. Also, find the nullspace of T.

$$T(1) = \int_0^x 1\,dt = [t]_0^x = x - 0 = x.$$
$$= 0.1 + 1.x + 0.(1+x^2) + 0.(1+x^3)$$

$$T(x) = \int_0^x t\,dt = \left[\frac{t^2}{2}\right]_0^x = \frac{x^2}{2} - \frac{0}{2} = \frac{x^2}{2}$$
$$= \frac{-1}{2}.1 + 0.x + \frac{1}{2}(1+x^2) + 0.(1+x^3)$$

$$T(x^2) = \int_0^x t^2\,dt = \left[\frac{t^3}{3}\right]_0^x = \frac{x^3}{3} - \frac{0}{3} = \frac{x^3}{3}$$
$$= \frac{-1}{3}.1 + 0.x + 0.(1+x^2) + \frac{1}{3}(1+x^3)$$

$\therefore$ Matrix of T wrt given bases is $A = \begin{bmatrix} 0 & -1/2 & -1/3 \\ 1 & 0 & 0 \\ 0 & 1/2 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$

Null space of $T = N_A(T) = \{ p(x) \in P_2 \mid T(p(x)) = 0 \}$.

Let $p_1(x) \in N_A(T) \Rightarrow T(p_1(x)) = 0$ where $p_1(x) = a_0 + a_1 x + a_2 x^2$.

$$\Rightarrow \int_0^x p_1(t)\,dt = 0 \Rightarrow \int_0^x (a_0 + a_1 t + a_2 t^2)\,dt$$
$$\Rightarrow \quad = \left(a_0 t + a_1\frac{t^2}{2} + a_2\frac{t^3}{3}\right)_0^x$$
$$= a_0 x + a_1\frac{x^2}{2} + a_2\frac{x^3}{3} - 0$$
$$= a_0 x + a_1\frac{x^2}{2} + a_2\frac{x^3}{3} = 0 = 0 + 0x + 0x^2 + 0x^3$$

Comparing the coeff on both sides.

$a_0 = 0, \ a_1 = 0, \ a_2 = 0$

$\therefore p(x) = 0$

$\therefore$ Null space of T has only zero polynomial

2(a) (ii) · ·

Let V be a n-dimensional vector space and $T: V \to V$ be an investible linear operator. If $\beta = \{X_1, X_2, \dots X_n\}$ is a basis of V, show that $\beta' = \{TX_1, TX_2, TX_3, \dots TX_n\}$ is also a basis of V.

→ T is invertible $\Rightarrow$ T is one-one and onto. Then,

$T(\alpha) = 0 \Rightarrow$ The value of α is zero i.e. $\alpha = 0$

Let $\beta' = \{TX_1, TX_2, \dots, TX_n\}$.

Let $a_1, a_2, \dots a_n$ be n scalars such that

$a_1 TX_1 + a_2 TX_2 + \dots + a_n TX_n = 0$ where $0 \in V$

$\Rightarrow T(a_1X_1) + T(a_2X_2) + \dots + T(a_nX_n) = 0$ [since T is a L.T.]

$\Rightarrow T[a_1X_1 + a_2X_2 + \dots + a_nX_n] = 0$

$\Rightarrow a_1X_1 + a_2X_2 + \dots + a_nX_n = T^{-1}(0) = 0$

$\therefore a_1X_1 + a_2X_2 + \dots + a_nX_n = 0$

Then, since $X_1, X_2, \dots X_n$ are L.I. $\Rightarrow a_1 = a_2 = \dots = a_n = 0$

$\therefore$ ~~F~~ $a_1TX_1 + a_2TX_2 + \dots + a_nTX_n = 0 \Rightarrow a_1 = a_2 = \dots = a_n = 0$.

$\therefore \{TX_1, TX_2, \dots TX_n\}$ is a linearly independent set.

Since dimension of V is n, then, any subset of V containing 'n' ~~ind~~ linearly independent vectors is a basis of V. Therefore, the set $\beta' = \{T.X_1, TX_2, \dots TX_n\}$ is a basis of V if $\{X_1, X_2, \dots X_n\}$ is a basis of V.

2(b)(i) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix}$ where $\omega (\neq 1)$ is a cube root of the unity. If $\lambda_1, \lambda_2, \lambda_3$. denote the eigen values of A^2, then show that $|\lambda_1| + |\lambda_2| + |\lambda_3| \leq 9$.

→ Since $1, \omega, \omega^2$ are the cube root of unity, then

$1 + \omega + \omega^2 = 0$

The characteristic equation of A^2 is $|A^2 - \lambda I| = 0$

$$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix} = \begin{bmatrix} 1+1+1 & 1+\omega+\omega^2 & 1+\omega+\omega^2 \\ 1+\omega+\omega^2 & 1+\omega+\omega^2 & 1+1+1 \\ 1+\omega+\omega^2 & 1+1+1 & 1+\omega+\omega^2 \end{bmatrix} \quad [\because \omega^3 = 1]$$

$$= \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix}$$

Then, $|A^2 - \lambda I| = 0 \Rightarrow \begin{vmatrix} 3-\lambda & 0 & 0 \\ 0 & -\lambda & 3 \\ 0 & 3 & -\lambda \end{vmatrix} = (3-\lambda)(\lambda^2 - 9) = 0$

$\Rightarrow \lambda = 3, 3, -3.$

$\therefore$ Let $\lambda_1 = 3$, $\lambda_2 = 3$, $\lambda_3 = -3$. Then, $|\lambda_1| + |\lambda_2| + |\lambda_3| = 3+3+3 = 9$

$\therefore |\lambda_1| + |\lambda_2| + |\lambda_3| \le 9.$

2(b) (ii): Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 8 & 12 \\ 3 & 5 & 8 & 12 & 17 \\ 5 & 8 & 12 & 17 & 23 \\ 8 & 12 & 17 & 23 & 30 \end{bmatrix}$

$\longrightarrow$ Let us convert A into echelon form

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 8 & 12 \\ 3 & 5 & 8 & 12 & 17 \\ 5 & 8 & 12 & 17 & 23 \\ 8 & 12 & 17 & 23 & 30 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -1 & -1 & 0 & 2 \\ 0 & -1 & -1 & 0 & 2 \\ 0 & -2 & -3 & -3 & -2 \\ 0 & -4 & -7 & -9 & -10 \end{bmatrix} \quad \begin{matrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \\ R_4 \to R_4 - 5R_1 \\ R_4 \to R_4 - 8R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -3 & -6 \\ 0 & 0 & -3 & -9 & -18 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 - R_2 \\ R_4 \to R_4 - 2R_2 \\ R_5 \to R_4 - 4R_2 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -3 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R_5 \to R_5 - 3R_4$$

$$\sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -1 & -1 & 0 & 2 \\ 0 & 0 & -1 & -3 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \leftrightarrow R_4 \quad \text{—— (1)}$$

The matrix (1) is in echelon form. It has three non-zero rows. Therefore $\boxed{\text{Rank of } A = 3}$

2(c) (i) Let A be a Hermitian matrix having all distinct eigen values $\lambda_1, \lambda_2, \ldots \lambda_n$. If $X_1, X_2, \ldots X_n$ are the corresponding eigen vectors, then show that the $n \times n$ matrix C whose kth column consists of the vector X_k is non-singular.

$\longrightarrow$ $C = [X_1 \; X_2, \ldots \; X_n]$. The matrix is non-singular iff $X_1, X_2, \ldots X_n$ are linearly independent.

Let the eigen vectors $X_1, X_2, \ldots X_n$ corresponding to $\lambda_1, \lambda_2 \ldots \lambda_n$. be assumed to be Linearly Dependent. Then, we can find 'r' such that $0<r<n$ and $r \in \mathbb{Z}^+$ such that $X_1, X_2, \ldots X_r$ vectors are Linearly independent and $X_1, X_2, \ldots X_r, X_{r+1}$ are Linearly Dependent.

Since $X_1, X_2, \ldots X_r, X_{r+1}$ are Linearly Dependent, $\exists$ scalars $a_1, a_2, \ldots a_r, a_{r+1} \in F$ not all zeroes such that

$$a_1 X_1 + a_2 X_2 + \ldots + a_r X_r + a_{r+1} X_{r+1} = 0 \quad \text{—— (1)}$$

Premultiplying both sides with A,

$$a_1 AX_1 + a_2 AX_2 + \ldots + a_r AX_r + a_{r+1} AX_{r+1} = 0 \quad \left[\because AX_i = \lambda_i X_i \;\; \forall\, i \in [1,n]\right]$$

$$\Rightarrow a_1 \lambda_1 X_1 + a_2 \lambda_2 X_2 + \ldots + a_r \lambda_r X_r + a_{r+1} \lambda_{r+1} X_{r+1} = 0 \quad \text{(2)}$$

(2) $- \lambda_{r+1} \cdot$ (1) :

$$a_1(\lambda_1 - \lambda_{r+1}) X_1 + a_2(\lambda_2 - \lambda_{r+1}) X_2 + \ldots + a_r(\lambda_r - \lambda_{r+1}) X_r + a_{r+1}(\lambda_{r+1} - \lambda_{r+1}) X_{r+1} = 0$$

$$\Rightarrow a_1(\lambda_1 - \lambda_{r+1}) X_1 + a_2(\lambda_2 - \lambda_{r+1}) X_2 + \ldots + a_n(\lambda_r - \lambda_{r+1}) X_r = 0$$

Since λ_i are distinct and $X_1, X_2, \ldots X_r$ are Linearly Independent

$\therefore a_1 = a_2 = \ldots = a_r = 0$.

Putting in (1) we have $a_{r+1} X_{r+1} = 0$

$\Rightarrow a_{r+1} = 0$ since $X_{r+1} \neq 0$.

which is a contradiction to our assumption that the scalars are not all zeroes. Our assumption that the vectors $X_1, X_2, \ldots X_n$ are L.D. is wrong.

Hence, $X_1, X_2, \ldots X_n$ are L.I.

Then, $C = [X_1 \; X_2, \ldots X_n]$ is a non-singular matrix

2 (c) (ii) Show that the vectors $X_1 = (1, 1+i, i)$, $X_2 = (i, -i, 1-i)$ and $X_3 = (0, 1-2i, 2-i)$ in $\mathbb{C}^3$ are L.I. over the field of real numbers but are linearly dependent over the field of complex numbers.

→ Let $a, b, c \in \mathbb{R}$. such that

$aX_1 + bX_2 + cX_3 = 0$

$\Rightarrow a(1, 1+i, i) + b(i, -i, 1-i) + c(0, 1-2i, 2-i) = (0,0,0)$

$\Rightarrow (a+ib, (a+c)+i(a-b-2c), (b+2c)+i(a-b-c)) = (0,0,0)$ — ①

Comparing both sides:

$a+ib = 0 \Rightarrow a+ib = 0 + i \cdot 0$ — ②

Comparing both sides of ②: we get $a=0, b=0$

$a+c+i(a-b-2c) = 0$

$0+c+i(0-0-2c) = 0$

$\Rightarrow c - 2ic = 0 \Rightarrow c(1-2i) = 0$.

Since $1-2i \neq 0$, $c = 0$.

$\therefore a = b = c = 0$. Hence, X_1, X_2, X_3 is L.I. over $\mathbb{R}$

~~Let $p, q, r \in \mathbb{R}$. such that~~

If $a, b, c \in \mathbb{C}$, then, from ①, we have.

$a+ib = 0 \Rightarrow a = -ib$.

$a+c+i(a-b-2c) = 0$

$-ib+c+i(-ib-b-2c) = 0$

$-ib+c+b-ib-i2c = 0$

$\Rightarrow c(1-2i) + b(1-2i) = 0$

$\Rightarrow c+b = 0 \Rightarrow b = -c$.

Hence a and c depend on b. Therefore, the vectors X_1, X_2, X_3 are L.D over $\mathbb{C}$.