Mains Test Series - 2019

Test - 07 (Paper - I Full Syllabus)

Answer Key

SECTION - A

1.(a) (i) → The rank of a product of two matrices cannot exceed the rank of either matrix.

Solution:

Let A and B be two matrices of orders $m \times n$ and $n \times p$ respectively.

Let $\rho(A) = r_1$, $\rho(B) = r_2$ and $\rho(AB) = r$.

We know that $\exists$ a non-singular matrix P such that

$$PA = \begin{bmatrix} G \\ O \end{bmatrix}, \text{ where } G \text{ is of order } r_1 \times n$$

and O is a zero matrix of order $(m - r_1) \times n$.

Now by post-multiplying both sides by B, we have $PAB = \begin{bmatrix} G \\ O \end{bmatrix} B$

$\therefore \rho(PAB) = \rho(AB) = r$

$\therefore$ rank of the matrix $\begin{bmatrix} G \\ 0 \end{bmatrix} B$ is r.

Since the matrix G has only r_1 non-zero rows.

$\therefore \begin{bmatrix} G \\ 0 \end{bmatrix} B$ cannot have more than r_1 non-zero rows.

$\therefore$ Rank of the matrix $\begin{bmatrix} G \\ 0 \end{bmatrix} B \leq r_1$.

$\Longrightarrow r \leq r_1$.

i.e. $\rho(AB) \leq \rho(A)$ (i.e. A is the pre-factor) —— (i)

Again, $\rho(AB) = [\rho(AB)']$

$= \rho[B'A']$

$\leq \rho(B')$ [by using (i) i.e. $\rho(AB) \leq \rho(A)$]

$= \rho(B)$ [$\because \rho(B') = \rho(B)$]

$= r_2$

$\therefore r \leq r_2$

i.e. $\rho(AB) \leq \rho(B)$ —— (ii)

$\therefore$ from (i) & (ii), we have

$\rho(AB) \leq \rho(A)$ and $\rho(AB) \leq \rho(B)$

Hence Proved.

1.(a) (ii) → Prove : The zero vector $\mathbf{0} = (0, 0, \cdots, 0)$ is a solution (the zero solution) of any homogeneous system $AX = 0$.

<u>Solution</u>:

Consider any homogeneous system $AX = 0$ where

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}; \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}; \quad O = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{m\times 1}$$

where the matrix A is called the co-efficient matrix, X is called the variable matrix and O is called the constant matrix.

Consider,

AX

$$= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1}$$

Put $X = \mathbf{0}$

i.e. x_i's $= 0 \quad \forall \; i = 1$ to n.

$$\Rightarrow \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{22} \\ \vdots & & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n\times 1}$$

$$\Rightarrow \begin{bmatrix} a_{11}(0) + a_{12}(0) + \cdots + a_{1n}(0) \\ a_{21}(0) + a_{22}(0) + \cdots + a_{22}(0) \\ \vdots \\ a_{m1}(0) + a_{m2}(0) + \cdots + a_{mn}(0) \end{bmatrix}_{m\times 1}$$

$$\Rightarrow \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{m\times 1}$$

$$= \mathbf{0}$$

i.e. $AX = \mathbf{0}$ when $X = \mathbf{0}$

i.e. The zero vector $\mathbf{0} = (0, 0, \cdots, 0)$ is a solution (the zero solution) of any homogeneous system $AX = \mathbf{0}$.

Hence, proved.

1.(b) → Define a basis of a vector space over a field F and the dimension of vector space. What is the dimension of complex vector space over the field of complex numbers? Give an example of a vector space the dimension of which is not finite.

Solution:

(i) Definition of basis of a vector space:

Let $V(F)$ be a vector space and let S be a subset of $V(F)$.

If (i) S is L.I. subset of $V(F)$

(ii) $L(S) = V$, then 'S' is called a basis of $V(F)$.

(ii) Definition of dimension of a vector space:

Let $V(F)$ be a vector space. If there exists a finite subset 'S' of V such that $L(S) = V$, then V is called a finite-dimensional vector space over the fields.

The number of elements in any basis of a finite dimensional vector space $V(F)$ is called the dimension of the vector space.

(iii) Dimension of $\mathbb{C}(\mathbb{C})$:

Let $\mathbb{C} = \{a+ib \,/\, a, b \in \mathbb{R},\ i = \sqrt{-1}\}$ be the complex vector space over the field '$\mathbb{C}$'.

Let $\alpha \in \mathbb{C}$ such that

$$\alpha = a+ib \quad : a, b \in \mathbb{R}$$
$$= a(1) + b(i)$$
$$\in L(S) \quad : \text{where } S = \{1, i\} \subseteq \mathbb{C}$$

$\therefore\ \alpha \in \mathbb{C} \Rightarrow \alpha \in L(S)$

$\therefore\ \mathbb{C} \subseteq L(S)$

Clearly, $L(S) \subseteq \mathbb{C}$

$\therefore\ \underline{L(S) = \mathbb{C}}$

Since $i = i(1)$: $i \in \mathbb{C}$ (field)

$\therefore$ S is Linearly Dependent set of $\mathbb{C}(\mathbb{C})$.

$\therefore$ Let $S' = \{1\}$

$\therefore\ S' \subseteq S \subseteq \mathbb{C}(\mathbb{C})$

Clearly S' is Linearly Independent and S' spans $\mathbb{C}(\mathbb{C})$

i.e. $L(S') = \mathbb{C}(\mathbb{C})$

$\therefore$ S' is a basis of $\mathbb{C}(\mathbb{C})$.

$\therefore\ \boxed{\dim(\mathbb{C}(\mathbb{C})) = 1}$

(iv) Example of Vector Space of infinite dimension:

Let $F[x]$ = Vector space of all polynomials over field F.

Clearly $F[x]$ is an infinite dimensional vector space because there exists no finite subset 'S' of $F[x]$ which spans $F[x]$.

Alternatively.
The real field $\mathbb{R}$ is an infinite dimensional vector space over the rational field $\mathbb{Q}$.

Hence, the result.

1.(c) → If $z = (x+y) + (x+y)\,\phi(y/x)$, prove that

$$x\left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y \partial x}\right) = y\left(\frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x \partial y}\right)$$

Solution:

As z is a homogeneous function of degree 1, then

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z \quad \text{——(1)}$$

$$x^2\frac{\partial^2 z}{\partial x^2} + y^2\frac{\partial^2 z}{\partial y^2} - 2xy\frac{\partial^2 z}{\partial x \partial y} = 0 \quad \text{——(2)}$$

$$\frac{\partial z}{\partial x} = 1 + \phi(y/x) + (x+y)\,\phi'(y/x)(-y/x^2)$$

$$= 1 + \phi(y/x) - \frac{y(x+y)}{x^2}\,\phi'(y/x)$$

$$\frac{\partial z}{\partial y} = 1 + \phi(y/x) + \left(\frac{x+y}{x}\right)\phi'(y/x)$$

Substituting in (1) and (2), we have

$$x + y + \phi(y/x)(x+y) = z$$

$$\frac{\partial^2 z}{\partial x^2} = \phi'(y/x)(-y/x^2) + \frac{y(x+y)}{x^2}(y/x^2)\,\phi''(y/x)$$
$$+ \phi'(y/x)\left(y/x^2 - \frac{2y^2}{x^3}\right)$$

$$\frac{\partial^2 z}{\partial y^2} = \phi'(y/x)(1/x) + \frac{1}{x}\phi'(y/x) + \frac{(x+y)}{x^2}\,\phi'(y/x)$$

$$\frac{\partial^2 z}{\partial x \partial y} = -y/x^2\,\phi'(y/x) - y/x^2\,\phi'(y/x) - \frac{(x+y)}{x}\frac{y}{x^2}\cdot\phi''(y/x)$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{1}{x}\phi'(y/x) - \frac{(x+2y)}{x^2}\phi'(y/x) - y\frac{(x+y)}{x^3}\phi''(x)$$

$$x\left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y \partial x}\right) = \phi'(y/x)\left(\frac{2y^2}{x^2} + \frac{2y}{x}\right) + \phi''(y/x)\left(\frac{y^3}{x^3} + \frac{2y^2}{x^2} + y/x\right) \quad \text{--- (3)}$$

$$y\left(\frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x \partial y}\right) = \phi'\left(\frac{y}{x}\right)\left(\frac{2y^2}{x^2} + \frac{2y}{x}\right) + \phi''(y/x)\left(\frac{y^3}{x^3} + \frac{2y^2}{x^2} + \frac{y}{x}\right) \quad \text{--- (4).}$$

$\therefore$ from (3) & (4), we conclude that,

$$\boxed{x\left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y \partial x}\right) = y\left(\frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x \partial y}\right)}$$

Hence Proved.

1.(d) → Show that the right circular cylinder of the given surface and maximum volume is such that its height is equal to the diameter of its base.

Solution :

Let the right circular cylinder of the given surface and maximum volume have the following

(i) Surface area $S = 2\pi r^2 + 2\pi r h$

(ii) Volume $V = \pi r^2 h$

Let $\phi = \pi r^2 h + \lambda(2\pi r^2 + 2\pi r h)$

$\therefore \frac{\partial \phi}{\partial r} = 0$

$\Rightarrow 2\pi r h + \lambda(4\pi r + 2\pi h) = 0$

$\Rightarrow \boxed{rh + \lambda(2r + h) = 0}$ —— (1)

Also, $\frac{\partial \phi}{\partial h} = 0$

$\Rightarrow \pi r^2 + \lambda(2\pi r) = 0$

$\Rightarrow r^2 + 2\lambda r = 0$

$\Rightarrow r + 2\lambda = 0$ $[\because r \neq 0]$

$\Rightarrow \boxed{\lambda = -\frac{r}{2}}$ —— (2)

Substituting the value of λ obtained in (2) in (1), we have,

$$rh + \left(-\frac{r}{2}\right)(2r+h) = 0$$

$$\Rightarrow r\left[h - r - \frac{h}{2}\right] = 0$$

$$\Rightarrow 2h - 2r - h = 0 \qquad [\because r \neq 0]$$

$$\Rightarrow h - 2r = 0$$

$$\Rightarrow \boxed{h = 2r}$$

i.e. The height of the right circular cylinder is equal to the diameter of its base.

Hence, proved.

1.(e) → Prove that the plane $x+2y-z=4$ cuts the sphere $x^2+y^2+z^2-x+z+2=0$ in a circle of radius unity and find the equations of the sphere which has this circle for one of its great circles.

Solution :

The centre of the given sphere is $\left(\frac{1}{2}, 0, -\frac{1}{2}\right)$

and its radius $= \sqrt{\left[\left(\frac{1}{2}\right)^2+(0)^2+\left(-\frac{1}{2}\right)^2-(-2)\right]}$

$$= \sqrt{\left(\frac{5}{2}\right)} = R \text{ (say)}$$

Also length of perpendicular from $\left(+\frac{1}{2}, 0, -\frac{1}{2}\right)$ to $x+2y-z=4$ is

$$\frac{\frac{1}{2}+2(0)-\left(-\frac{1}{2}\right)-4}{\sqrt{\left[2^2+2^2+(-1)^2\right]}} = \frac{1}{2}\sqrt{6} = p \text{ (say)}$$

Then radius of the circle $= \sqrt{(R^2-p^2)}$

$$= \sqrt{\frac{5}{2}-\frac{6}{4}}$$

$$= 1 \qquad \text{—— (i)}$$

Hence, proved.

The equations of the circle are $x^2+y^2+z^2-x+z-2=0$, $x+2y-z-4=0$.

$\therefore$ The equation of a sphere through this circle is

$$(x^2+y^2+z^2-x+z-2)+\lambda(x+2y-z-4)=0$$

i.e. $x^2+y^2+z^2+(\lambda-1)x+2\lambda y+(1-\lambda)z-(2+4\lambda)=0$ —— (1)

Its centre is $\left[-\frac{1}{2}(\lambda-1), -\lambda, -\frac{1}{2}(1-\lambda)\right]$.

If this circle is a great circle of the sphere (1), then the centre of (1) should lie on the plane of the circle i.e. the plane $x+2y-z-4=0$.

$\therefore$ $-\frac{1}{2}(\lambda-1)+2(-\lambda)+\frac{1}{2}(1-\lambda)-4=0$

$\Rightarrow -3\lambda-3=0$

$\Rightarrow \lambda=-1$

$\therefore$ from (1), the equation of the required sphere is

$$\boxed{x^2+y^2+z^2-2x-2y+2=0.}$$

2.(a) → Let W be the solution space of the homogeneous system

$$x + 2y - 3z + 2s - 4t = 0$$
$$2x + 4y - 5z + s - 6t = 0$$
$$5x + 10y - 13z + 4s - 16t = 0$$

Find the dimension and a basis for W.

Solution:

The given system can be written as

$$AX = \begin{bmatrix} 1 & 2 & -3 & 2 & -4 \\ 2 & 4 & -5 & 1 & -6 \\ 5 & 10 & -13 & 4 & -16 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ s \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = B$$

Consider the matrix A and reduce it into echelon form.

$$\begin{bmatrix} 1 & 2 & -3 & 2 & -4 \\ 2 & 4 & -5 & 1 & -6 \\ 5 & 10 & -13 & 4 & -16 \end{bmatrix} \underset{R_3 \to R_3 - 5R_1}{\overset{R_2 \to R_2 - 2R_1}{\sim}} \begin{bmatrix} 1 & 2 & -3 & 2 & -4 \\ 0 & 0 & 1 & -3 & 2 \\ 0 & 0 & 2 & -6 & 4 \end{bmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\sim \begin{bmatrix} 1 & 2 & -3 & 2 & -4 \\ 0 & 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Clearly which is in echelon form having 2 non-zero rows.

Now, writing the echelon form of matrix in the form of the given system, we have,

$$\begin{bmatrix} 1 & 2 & -3 & 2 & -4 \\ 0 & 0 & 1 & -3 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ s \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \; z - 3s + 2t = 0 \; ; \; x + 2y - 3z + 2s - 4t = 0$

$\Rightarrow \; z = 3s - 2t \; ; \; x + 2y - 3(3s - 2t) + (2s - 4t) = 0$

$\Rightarrow \; x + 2y - 7s + 2t = 0$

$\Rightarrow \; x = -2y + 7s - 2t$

$\therefore (x, y, z, s, t) = (-2y + 7s - 2t, \; y, \; 3s - 2t, \; s, \; t)$

$= y(-2, 1, 0, 0, 0) + s(7, 0, 3, 1, 0) + t(-2, 0, -2, 0, 1)$

$\in L(S)$

where

$$\boxed{S = \{(-2, 1, 0, 0, 0), \; (7, 0, 3, 1, 0), \; (-2, 0, -2, 0, 1)\}}$$

Let $M = \begin{bmatrix} -2 & 1 & 0 & 0 & 0 \\ 7 & 0 & 3 & 1 & 0 \\ -2 & 0 & -2 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} -2 & 1 & 0 & 0 & 0 \\ 0 & 7/2 & 3 & 1 & 0 \\ 0 & -1 & -2 & 0 & 1 \end{bmatrix}$

$\sim \begin{bmatrix} -2 & 1 & 0 & 0 & 0 \\ 0 & 7/2 & 3 & 1 & 0 \\ 0 & 0 & -8/9 & 2/9 & 1 \end{bmatrix} \Rightarrow S$ is Linearly Independent.

$\Rightarrow \boxed{S \text{ is a basis of } W \text{ and } \dim W = 3.}$

Hence, the result.

2.(b) → Show that the function

$$f(x,y) = \begin{cases} x^2y/(x^2+y^2), & \text{when } x^2+y^2 \neq 0 \\ 0, & \text{when } x^2+y^2 = 0 \end{cases}$$

is continuous but not differentiable at (0,0).

Solution:

Putting $x = r\cos\theta$, $y = r\sin\theta$; we get

$$|f(x,y) - f(0,0)| = \left| \frac{r^2\cos^2\theta \cdot r\sin\theta}{r^2} - 0 \right|$$

$$= r|\cos\theta||\cos\theta||\sin\theta|$$

$$\leq r = \sqrt{x^2+y^2}$$

Let $\varepsilon > 0$ be given.

Choose $\delta = \varepsilon$.

Then $|f(x,y) - f(0,0)| < \varepsilon$ if $\sqrt{x^2+y^2} < \delta$.

Hence f is continuous at the origin. —(i)

Now,

$$f_x(0,0) = \lim_{h\to 0} \left[\frac{f(h,0) - f(0,0)}{h}\right]$$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0$$

Similarly, $f_y(0,0) = 0$.

Let, if possible, f be differentiable at $(0,0)$.

Then $f(h,k) - f(0,0) = Ah + Bk + \sqrt{h^2+k^2}\, g(h,k)$

where $A = f_x(0,0)$,
$B = f_y(0,0)$ and
$g(h,k) \to 0$ as $(h,k) \to (0,0)$ ——— (1)

$$\therefore \frac{h^2k}{h^2+k^2} = \sqrt{h^2+k^2}\, g(h,k)$$

$$\Rightarrow g(h,k) = \frac{h^2k}{(h^2+k^2)^{3/2}}$$

Now, $\lim\limits_{h\to 0} g(h,mh) = \frac{m}{(1+m^2)^{3/2}}$ $[k = mh]$

$\therefore \lim\limits_{(h,k)\to(0,0)} g(h,k) = \frac{m}{(1+m^2)^{3/2}}$, which depends on m and so the limit does not exist.

This contradicts (1).

Hence, f is not differentiable at $(0,0)$. ——— (ii)

Hence, the result.

2.(c) → Evaluate $\int_0^{\pi/2} \frac{\sin^2 x}{1+\sin x \cos x}\,dx$

Solution:

$$\text{Let } I = \int_0^{\pi/2} \frac{\sin^2 x}{1+\sin x\cdot\cos x}\,dx \quad \text{—(i)}$$

$$\therefore I = \int_0^{\pi/2} \frac{\sin^2(\pi/2 - x)}{1+\sin(\pi/2-x)\cos(\pi/2-x)}\,dx$$

$$= \int_0^{\pi/2} \frac{\cos^2 x}{1+\cos x\cdot\sin x}\,dx \quad \text{—(ii)}$$

Adding (i) and (ii), we obtain

$$2I = \int_0^{\pi/2} \frac{\sin^2 x+\cos^2 x}{1+\sin x\cdot\cos x}\,dx = \int_0^{\pi/2} \frac{dx}{1+\sin x\cdot\cos x}$$

$$= 2\int_0^{\pi/2} \frac{dx}{2+2\sin x\cdot\cos x} = 2\int_0^{\pi/2} \frac{dx}{1+\sin 2x}$$

Put $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$$\Rightarrow dx = \frac{dt}{(1+t^2)}$$

$$\therefore 2I = 2\int_{t=0}^{\infty} \frac{dt}{(1+t^2)\left[2+\frac{2t}{1+t^2}\right]}$$

$$\left(\because \ \sin 2x = \frac{2\tan x}{1+\tan^2 x} = \frac{2t}{1+t^2} \right)$$

$$= \int_0^\infty \frac{dt}{t^2+t+1} = \int_0^\infty \frac{dt}{\left(t+\frac{1}{2}\right)^2 + \left(1-\frac{1}{4}\right)}$$

$$= \int_0^\infty \frac{dt}{\left(t+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = \frac{2}{\sqrt{3}} \left| \tan^{-1} \frac{t+1/2}{\sqrt{3}/2} \right|_0^\infty$$

$$= \frac{2}{\sqrt{3}} \left| \tan^{-1}\left(\frac{2t+1}{\sqrt{3}}\right) \right|_0^\infty$$

$$= \frac{2}{\sqrt{3}} \left(\frac{\pi}{2} - \frac{\pi}{6} \right)$$

$$= \frac{2\pi}{3\sqrt{3}}$$

i.e. $2I = \dfrac{2\pi}{3\sqrt{3}}$

$$\Rightarrow I = \frac{\pi}{3\sqrt{3}}$$

i.e. $\boxed{\displaystyle\int_0^{\pi/2} \frac{\sin^2 x}{1+\sin x \cos x} dx = \frac{\pi}{3\sqrt{3}}}$

Hence, the result.

2.(d) (i) → The plane $lx + my = 0$ is rotated through an angle α about its line of intersection with the plane $z = 0$. Prove that equation to the plane in its new position is $lx + my \pm z\sqrt{(l^2+m^2)}\tan\alpha = 0$.

Solution:

The equation of any plane through the line of intersection of the plane $lx + my = 0$ and $z = 0$ is $lx + my + \lambda z = 0$ ——— (1)

It is given that the angle between the plane $lx + my = 0$ and (1) is α, so the angle between their normal is $(\pi - \alpha)$.

Also, the d.c.'s of their normals are $l, m, 0$ and l, m, λ respectively.

$$\therefore \tan(\pi - \alpha) = \frac{\pm\sqrt{[\Sigma(m_1n_2 - m_2n_1)]}}{\Sigma l_1 l_2}$$

$$= \frac{\pm\sqrt{[(m\lambda - 0)^2 + (0 - l\lambda)^2 + (lm - ml)^2]}}{l^2 + m^2 + 0}$$

$$= \frac{\pm\sqrt{[\lambda^2(l^2+m^2)]}}{(l^2+m^2)} = \pm\frac{\lambda}{\sqrt{(l^2+m^2)}}$$

$$\Rightarrow \quad -\tan\alpha = \pm\frac{\lambda}{\sqrt{(l^2+m^2)}}$$

or $\lambda = \pm\sqrt{(l^2+m^2)}\tan\alpha$.

$\therefore$ from (1), the required equation is

$$\boxed{lx + my \pm z\sqrt{(l^2+m^2)}\tan\alpha = 0}$$

Hence proved.

2.(d)(ii) → Find the distance of the point $(3, 8, 2)$ from the line $\frac{1}{2}(x-1) = \frac{1}{4}(y-3) = \frac{1}{3}(z-2)$ measured parallel to the plane $3x+2y-2z+15=0$.

Solution:

Let the given point $(3, 8, 2)$ be P.
Any point N on the given line is

$$(1+2r,\ 3+4r,\ 2+3r) \quad \text{---(1)}$$

$\therefore$ The direction ratios of the line PN are

$(1+2r)-3,\ (3+4r)-8,\ (2+3r)-2$

i.e., $2r-2,\ 4r-5,\ 3r.$ ---(2)

If PQ is parallel to the plane $3x+2y-2z+15=0$, then PQ is perpendicular to the normal to this plane, consequently

$$3(2r-2)+2(4r-5)-2(3r)=0$$

$\Rightarrow$ $\underline{\underline{r=2}}$.

$\therefore$ from (1), the coordinates of N are

$$[1+2(2),\ 3+4(2),\ 2+3(2)]$$

i.e. $(5, 11, 8)$

$\therefore$ Required distance = distance between $P(3,8,2)$ and $N(5,11,8)$

$$=\sqrt{[(5-3)^2+(11-8)^2+(8-2)^2}$$

$$=7.$$

$\therefore$ $\boxed{\text{The required distance is 7 units.}}$

Hence, the result.

3.(a)(i) Consider the bases $B = \{(1,2), (3,-1)\}$ and $B' = \{(1,0), (0,1)\}$ of $\mathbb{R}^2$. If u is a vector such that $u_B = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, find $u_{B'}$.

Solution:

We express the vectors of B in terms of the vectors of B' to get the transition matrix.

$$(1,2) = 1(1,0) + 2(0,1)$$
$$(3,-1) = 3(1,0) - 1(0,1)$$

The co-ordinate vectors of (1,2) and (3,-1) are $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ -1 \end{bmatrix}$. The transition matrix P is thus

$$P = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}$$

We can also observe that columns of P are the vectors of the basis B.

Now, we get

$$u_{B'} = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 15 \\ 2 \end{bmatrix}$$

$\therefore$ $\boxed{u_{B'} = \begin{bmatrix} 15 \\ 2 \end{bmatrix}}$ is the required vector.

3.(a)(ii) → Consider the bases $B = \{(1,2), (3,-1)\}$ and $B' = \{(3,1), (5,2)\}$ of $\mathbb{R}^2$. Find the transition matrix from B to B'. If $\mathbf{u}$ is a vector such that $\mathbf{u}_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, find $\mathbf{u}_{B'}$.

Solution :

We use the standard basis $S = \{(1,0), (0,1)\}$ as an intermediate basis. The transition matrix P from B to S and the transition matrix P' from B' to S are

$$P = \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix}, \quad P' = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$$

The transition matrix from S to B' will be $(P')^{-1}$. The transition matrix from B to B' (by way of S) is $\underline{(P')^{-1}P}$. Thus, the transition matrix from B to B' is

$$\begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -5 \\ -1 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} -8 & 11 \\ 5 & -6 \end{bmatrix}$$

This gives $\boxed{\mathbf{u}_{B'} = \begin{bmatrix} -8 & 11 \\ 5 & -6 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 4 \end{bmatrix}}$

which is the required vector.

3.(a) (iii) → Prove that similar matrices have the same eigenvalues but not a converse.

Solution :

Let A and B be similar matrices. Hence, there exists a matrix C such that $B = C^{-1}AC$.
The characteristic polynomial of B is $|B - \lambda I|$.
Substituting for B and using the multiplicative properties of determinants, we get

$$
\begin{aligned}
|B - \lambda I| &= |C^{-1}AC - \lambda I| \\
&= |C^{-1}(A - \lambda I)C| \\
&= |C^{-1}||A - \lambda I||C| \\
&= |A - \lambda I||C^{-1}||C| \\
&= |A - \lambda I||C^{-1}C| \\
&= |A - \lambda I||I| \\
&= |A - \lambda I|
\end{aligned}
$$

The characteristic polynomials of A and B are identical ⇒ A and B have same eigenvalues.

Hence proved.

Converse :

Consider the matrices $A \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and

$B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

The eigenvalues of A and B are same, i.e. 1, 1.

But they are not similar matrices identity matrix can be similar to no other matrix as $CIC^{-1} = I$.

Also, we know that two matrices with same eigenvalues are similar if they are non-singular.

Hence the result.

3.(b). → A space probe in the shape of the ellipsoid $4x^2+y^2+4z^2=16$ enters the earth's atmosphere and its surface begins to heat. After one hour, the temperature at the point (x,y,z) on the probe's surface is $T(x,y,z) = 8x^2+4yz-16z+600$. Find the hottest point on the probe's surface.

Solution:

$$T(x,y,z) = 8x^2+4yz-16z+600$$

$$g(x,y,z) = 4x^2+y^2+4z^2-16=0$$

$$\nabla T = 16x\hat{i}+4z\hat{j}+(4y-16)\hat{k} \text{ and}$$

$$\nabla g = 8x\hat{i}+2y\hat{j}+8z\hat{k}$$

so that $\nabla T = \lambda \nabla g$

$$\Rightarrow 16x\hat{i}+4z\hat{j}+(4y-16)\hat{k} = \lambda(8x\hat{i}+2y\hat{j}+8z\hat{k})$$

$$\Rightarrow 16x = 8x\lambda, \quad 4y-16 = 8z\lambda, \quad 4z=2y$$

$$\Rightarrow \boxed{\lambda=2} \text{ or } \boxed{x=0}$$

Case (1):

$$\lambda=2 \Rightarrow 4z=2y(2) \Rightarrow z=y$$

Then $4z-16=16z \Rightarrow z=-4/3$

$$\Rightarrow y=-4/3$$

$\therefore\ 4x^2 + y^2 + 4z^2 = 4x^2 + (-4/3)^2 + 4(-4/3)^2 = 16$

$\Rightarrow\ x = \pm 4/3$

Case (2):

$x = 0 \Rightarrow \lambda = \frac{2z}{y} \Rightarrow 4y - 16 = 8z\left(\frac{2z}{y}\right)$

$\Rightarrow y^2 - 4y = 4z^2$

$\Rightarrow 4(0)^2 + y^2 + (y^2 - 4y) - 16 = 0$

$\Rightarrow y^2 - 2y - 8 = 0$

$\Rightarrow (y-4)(y+2) = 0$

$\Rightarrow y = 4$ (or) $y = -2$.

Now $y = 4 \Rightarrow 4z^2 = 4^2 - 4(4) \Rightarrow z = 0$

and $y = -2 \Rightarrow 4z^2 = (-2)^2 - 4(-2) = 4 + 8 = 12$

$\therefore\ z = \pm\sqrt{3}$

The temperatures are $T = (\pm 4/3, -4/3, -4/3)$
$= 642\ 2/3$

$T(0,4,0) = 600^\circ$, $T(0,-2,\sqrt{3}) = (600 - 2\sqrt{3})^\circ$

and $T(0,-2,-\sqrt{3}) = (600 + 24\sqrt{3})^\circ \approx 641.6^\circ$

$\therefore$ $T(\pm 4/3, -4/3, -4/3)$ are the hottest points on the space probe.

Hence, the result.

3.(c) → Show that the locus of the points from which three mutually perpendicular tangents can be drawn to the paraboloid $ax^2+by^2=2z$ is given by $ab(x^2+y^2)-2(a+b)z-1=0$.

Solution:

The equations of a line through (α,β,γ) are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \quad \text{——— (1)}$$

Any point on this line is $(\alpha+lr, \beta+mr, \gamma+nr)$ ——— (2)

If the line (1) meets the given paraboloid at a distance r from the point (α,β,γ), then the point given by (2) must lie on the given paraboloid and so we have

$$a(\alpha+lr)^2+b(\beta+mr)^2=2(\gamma+nr)$$

$$\Rightarrow r^2(al^2+bm^2)+2r(al\alpha+bm\beta-n)+(a\alpha^2+b\beta^2-2\gamma)=0 \quad \text{——— (3)}$$

If the line (1) is a tangent of the given paraboloid, then the line (1) should meet the paraboloid in two coincident points, the condition for the same is that the roots of (3) are equal.

i.e., $B^2=4AC$

$$\Rightarrow 4(al\alpha+bm\beta-n)^2=4(al^2+bm^2)(a\alpha^2+b\beta^2-2\gamma) \quad \text{——— (4)}$$

The locus of the line (1) which is tangent to the given paraboloid is obtained by eliminating l, m, n between (1) and (4), and is

$$[a\alpha(x-\alpha)+b\beta(y-\beta)-(z-\gamma)]^2 = [a(x-\alpha)^2+b(y-\beta)^2](a\alpha^2+b\beta^2-2\gamma) \quad \text{—(5)}$$

If $S \equiv ax^2+by^2-2z$, $S_1 = a\alpha^2+b\beta^2-2\gamma$ and $T \equiv a\alpha x+b\beta y-(z+\gamma)$, then (5) can be written as

$$(T-S_1)^2 = (S+S_1-2T)S_1$$

$$\Rightarrow T^2+\cancel{S_1^2}-2TS_1 = SS_1+\cancel{S_1^2}-\cancel{2TS_1}$$

$$\Rightarrow SS_1 = T^2$$

$$\Rightarrow (ax^2+by^2-2z)(a\alpha^2+b\beta^2-2\gamma) = [a\alpha x+b\beta y-(z+\gamma)]^2 \quad \text{—(6)}$$

which is the equation of the enveloping cone of the given paraboloid.

Now, we apply the condition that the enveloping cone of the given paraboloid, with vertex at (α, β, γ) may have three mutually perpendicular generators and the condition for the same is that the sum of the co-efficients of x^2, y^2 and z^2 in the equation of the cone is zero.

$\therefore$ from (6), we get,

$$\left[a(a\alpha^2+b\beta^2-2\gamma)-a^2\alpha^2\right]+\left[b(a\alpha^2+b\beta^2-2\gamma)-b^2\beta^2\right]-1^2=0$$

$$\Rightarrow \quad ab\beta^2-2a\gamma+ba\alpha^2-2b\gamma-1^2=0$$

$$\Rightarrow \quad ab(\alpha^2+\beta^2)-2(a+b)\gamma-1=0$$

Hence, the required locus of the point (α,β,γ) is

$$\boxed{ab(x^2+y^2)-2(a+b)z-1=0}$$

Thus, proved.

4.(a) (i) Let A be an $n \times n$ matrix.
If A has n linearly independent eigenvectors it is diagonalizable. The matrix C whose columns consist of n linearly independent eigenvectors can be used in a similarity transformation $C^{-1}AC$ to give a diagonal matrix D. The diagonal elements of D will be the eigenvalues of A.

Solution:

Let A have eigenvalues $\lambda_1, \dots, \lambda_n$, (which need not be distinct), with corresponding linearly independent eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$.
Let C be the matrix having $\mathbf{v}_1, \dots, \mathbf{v}_n$ as column vectors.

$$C = [\mathbf{v}_1 \dots \mathbf{v}_n].$$

Since $A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1, \dots, A\mathbf{v}_n = \lambda_n \mathbf{v}_n$, matrix multiplication in terms of columns gives

$$\begin{aligned} AC &= A[\mathbf{v}_1 \dots \mathbf{v}_n] \\ &= [A\mathbf{v}_1 \dots A\mathbf{v}_n] \\ &= [\lambda_1 \mathbf{v}_1 \dots \lambda_n \mathbf{v}_n] \\ &= [\mathbf{v}_1 \dots \mathbf{v}_n] \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} \end{aligned}$$

$$= C\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

Since the columns of C are linearly independent, C is non-singular.

Thus,

$$C^{-1}AC = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$$

$\therefore$ If an $n\times n$ matrix A has n linearly independent eigenvectors then these eigenvectors can be used as the columns of a matrix C that diagonalizes A. The diagonal matrix has the eigenvalues of A as diagonal elements.

Hence proved.

4.(a). (ii) → Let A be an $n\times n$ matrix.
If A is diagonalizable then it has n linearly independent eigenvectors.

Solution:

Let us assume that C is a matrix $[v_1 \dots v_n]$ that diagonalizes A.

Thus, there exist scalars $\gamma_1, \gamma_2, \ldots, \gamma_n$, such that

$$C^{-1}AC = \begin{bmatrix} \gamma_1 & & 0 \\ & \ddots & \\ 0 & & \gamma_n \end{bmatrix}$$

$$\Rightarrow \quad AC = C\begin{bmatrix} \gamma_1 & & 0 \\ & \ddots & \\ 0 & & \gamma_n \end{bmatrix}$$ — [Pre-multiplying both sides by C]

$$\Rightarrow \quad AC = \begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix}\begin{bmatrix} \gamma_1 & & 0 \\ & \ddots & \\ 0 & & \gamma_n \end{bmatrix}$$

$$\Rightarrow \quad AC = \begin{bmatrix} \gamma_1 \mathbf{v}_1 & \cdots & \gamma_n \mathbf{v}_n \end{bmatrix}$$

i.e.,

$$A\begin{bmatrix} \mathbf{v}_1 & \cdots & \mathbf{v}_n \end{bmatrix} = \begin{bmatrix} \gamma_1 \mathbf{v}_1 & \cdots & \gamma_n \mathbf{v}_n \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} A\mathbf{v}_1 & \cdots & A\mathbf{v}_n \end{bmatrix} = \begin{bmatrix} \gamma_1 \mathbf{v}_1 & \cdots & \gamma_n \mathbf{v}_n \end{bmatrix}$$

$$\Rightarrow A\mathbf{v}_1 = \gamma_1 \mathbf{v}_1, \ \ldots, \ A\mathbf{v}_n = \gamma_n \mathbf{v}_n$$

$\Rightarrow \mathbf{v}_1, \ldots, \mathbf{v}_n$ are eigenvectors of A.

$\because$ C is non-singular, these vectors (i.e. column vectors of C) are linearly independent.

$\therefore$ If an $n \times n$ matrix A is diagonalizable, it has n linearly independent eigenvectors.

Hence, Proved.

4.(b) find the volume bounded by the elliptic paraboloids $z = x^2 + 9y^2$ and $z = 18 - x^2 - 9y^2$.

Solution:

Equating the two expressions for z gives

$$x^2 + 9y^2 = 18 - (x^2 + 9y^2)$$

$$\Rightarrow 2(x^2 + 9y^2) = 18$$

$$\Rightarrow x^2 + 9y^2 = 9$$

and thus, here $z = 9$.

The volume of intersection can be divided into two parts:

$0 \le z \le 9$ where the restrictions on x and y are $x^2 + 9y^2 \le z$, and

$9 \le z \le 18$ where the restrictions on x and y are $18 - (x^2 + 9y^2) \geq z$.

Now,

For lower region, $x^2 + 9y^2 \le z$ for $0 \le z \le 9$, we see that for a given z that is an ellipse with major axis $a = \sqrt{z}$ over the x-coordinate and minor axis $b = \frac{\sqrt{z}}{3}$ over the y-coordinate.

Area of an ellipse $A = \pi ab$

$$= \pi(\sqrt{z})\left(\frac{\sqrt{z}}{3}\right)$$

$$= \frac{\pi z}{3}.$$

Now, volume of that region is given as,

$$V_1 = \int_0^9 \frac{\pi z}{3}\, dz = \frac{81\pi}{6} \quad \text{(1)}$$

Further,
for the upper region, $x^2+9y^2 \leq 18-z$ for $9 \leq z \leq 18$, we see that for a given z that is an ellipse with major axis $a = \sqrt{18-z}$ over the x-coordinate and minor axis $b = \frac{\sqrt{18-z}}{3}$ over the y-coordinate.

Area of an ellipse $A = \pi ab$

$$= \pi\left(\sqrt{18-z}\right)\left(\frac{\sqrt{18-z}}{3}\right)$$

$$= \frac{\pi(18-z)}{3}$$

Now, Volume of this region is given as,

$$V_2 = \int_9^{18} \frac{\pi(18-z)}{3}\, dz = \frac{\pi}{3}\left[18z - \frac{z^2}{2}\right]_9^{18}$$

$$= \frac{\pi}{3}\left[\cancel{324} - \cancel{162} - \cancel{162} + \frac{81}{2}\right]$$

$$= \frac{\pi}{3}\left[\frac{81}{2}\right]$$

$$\therefore V_2 = \frac{81\pi}{6} \quad \text{——— (2)}$$

Now, total Volume $V = V_1 + V_2$

$$= \frac{81\pi}{6} + \frac{81\pi}{6}$$

$$= \frac{81\pi}{3}$$

$$\therefore \boxed{V = 27\pi \text{ cubic units.}}$$

Hence, the result.

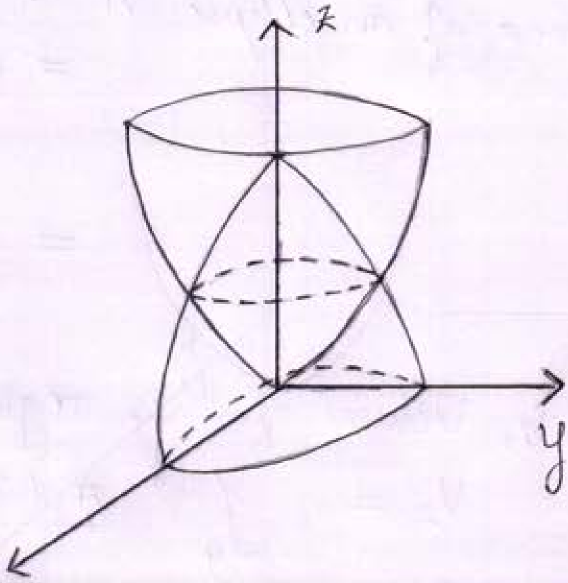
Alternatively,

Intersection of two paraboloid is

$x^2 + 9y^2 = 18 - x^2 - 9y^2$

$\Rightarrow x^2 + 9y^2 = 9$

which is an ellipse with centre $(0,0)$.



$$V = \int_{-1}^{1}\int_{-\sqrt{9-9y^2}}^{\sqrt{9-9y^2}}\int_{x^2+9y^2}^{18-x^2-9y^2} dz\, dx\, dy$$

$$V = \int_{-1}^{1}\int_{-\sqrt{9-9y^2}}^{\sqrt{9-9y^2}} (18-2x^2-18y^2)\, dx\, dy$$

$$= 2\int_{-1}^{1}\int_{0}^{\sqrt{9-9y^2}} (18-2x^2-18y^2)\, dx\, dy$$

[since integrand is an even function of x]

$$V = 2\int_{-1}^{1}\left\{18(1-y^2)\times 3\sqrt{1-y^2} - \frac{2}{3}(9-9y^2)\sqrt{9-9y^2}\right\} dy$$

$$= 2\times 54\times \frac{2}{3}\int_{-1}^{1}(1-y^2)^{3/2}\, dy$$

$$= \frac{8\times 54}{3}\int_{0}^{\pi/2}(1-y^2)^{3/2}\, dy$$

$$= \frac{8\times 54}{3}\int_{0}^{\pi/2}\cos^4\theta\, d\theta \quad [\text{Taking } y=\sin\theta]$$

$$\therefore V = \frac{4\times 54}{3}\times 2\int_{0}^{\pi/2}\cos^4\theta\, d\theta$$

Using Gamma function the integral

$$2\int_0^{\pi/2} \cos^4\theta \, d\theta = \frac{\Gamma\frac{1}{2}\,\Gamma\frac{5}{2}}{\Gamma 3} = \frac{3}{8}\pi$$

$$\therefore V = \frac{\cancel{4} \times \overset{27}{\cancel{54}}}{\cancel{3}} \times \frac{\cancel{3}}{\underset{\cancel{2}}{\cancel{8}}}\pi$$

$$\therefore V = 27\pi \text{ cubic units.}$$

Hence, the result.

4.(c) → Prove that in general two generators of the hyperboloid $(x^2/a^2)+(y^2/b^2)-(z^2/c^2)=1$ can be drawn to cut a given generator at right angles. Also show that if they meet the plane $z=0$ in P and Q, PQ touches the ellipse $(x^2/a^6)+(y^2/b^6)=c^4/(a^4b^4)$.

Solution:

We know that for the given hyperboloid, the generator belonging to λ-system is given by

$$\frac{x}{a}-\frac{z}{c}=\lambda\left(1-\frac{y}{b}\right) \text{ and } \frac{x}{a}+\frac{z}{c}=\frac{1}{\lambda}\left(1+\frac{y}{b}\right) \quad \text{—— (1)}$$

$$\Rightarrow \frac{x}{a}+\frac{\lambda y}{b}-\frac{z}{c}=\lambda \text{ and } \lambda\frac{x}{a}-\frac{y}{b}+\lambda\frac{z}{c}=1$$

$\therefore$ If l_1, m_1, n_1 be the d.r.'s of the generator (1), then

$$\frac{l_1}{a}+\frac{\lambda m_1}{b}-\frac{n_1}{c}=0 \text{ and } \frac{\lambda l_1}{a}-\frac{m_1}{b}+\frac{\lambda n_1}{c}=0$$

Solving these simultaneously, we get

$$\frac{l_1/a}{\lambda^2-1}=\frac{m_1/b}{-\lambda-\lambda}=\frac{n_1/c}{-1-\lambda^2}$$

$$\Rightarrow \frac{l_1}{-a(\lambda^2-1)}=\frac{m_1}{2\lambda b}=\frac{n_1}{c(1+\lambda^2)} \quad \text{—— (2)}$$

Similarly the direction ratios l_2, m_2, n_2 of the generator belonging to μ-system.

i.e. $\frac{x}{a} - \frac{z}{c} = \mu\left(1+\frac{y}{b}\right)$ and $\frac{x}{a}+\frac{z}{c} = \frac{1}{\mu}\left(1-\frac{y}{b}\right)$ —— (3)

are given by

$$\frac{l_2}{a(\mu^2-1)} = \frac{m_2}{2b\mu} = \frac{n_2}{-c(\mu^2+1)} \quad \text{—— (4)}$$

If these two generators given by (1) and (3) are perpendicular then

$$-a^2(\lambda^2-1)(\mu^2-1) - 4b^2\mu\lambda - c^2(1+\lambda^2)(\mu^2+1) = 0 \quad \text{—— (5)}$$

Now if λ-generator is given, ~~by~~ then λ is constant and (5) will be a quadratic equation in μ which gives two values of μ and this shows that there will be two generators of μ-system which will be perpendicular to a generator of λ-system.

Now let the generators of μ-system meet the plane $z=0$ in the points $P(a\cos\alpha, b\sin\alpha, 0)$ and $Q(a\cos\beta, b\sin\beta, 0)$

$\therefore$ The generator of the μ-system through these points are given by

$$\frac{x - a\cos\alpha}{a\sin\alpha} = \frac{y - b\sin\alpha}{-b\cos\alpha} = \frac{z}{c} \quad \text{—(6)}$$

and $$\frac{x - a\cos\beta}{a\sin\beta} = \frac{y - b\sin\beta}{-b\cos\beta} = \frac{z}{c} \quad \text{—(7)}$$

These two generators intersect at right angles a generator of λ-system through any point $(a\cos\theta, b\sin\theta, 0)$ say whose equations are

$$\frac{x - a\cos\theta}{a\sin\theta} = \frac{y - b\sin\theta}{-b\cos\theta} = \frac{z}{-c} \quad \text{— (8)}$$

As (6) and (7) are both perpendicular to (8),

so $a^2\sin\alpha\sin\theta + b^2\cos\alpha\cos\theta - c^2 = 0$

and $a^2\sin\beta\sin\theta + b^2\cos\beta\cos\theta - c^2 = 0$.

Solving these simultaneously for $a^2\sin\theta$, $b^2\cos\theta$ and $-c^2$, we get

$$\frac{a^2\sin\theta}{\cos\alpha - \cos\beta} = \frac{b^2\cos\theta}{\sin\beta - \sin\alpha} = \frac{-c^2}{\sin\alpha\cos\beta - \cos\alpha\sin\beta}$$

$$\Rightarrow \frac{a^2 \sin\theta}{2\sin\frac{\alpha+\beta}{2}\sin\frac{\beta-\alpha}{2}} = \frac{b^2\cos\theta}{2\cos\frac{\alpha+\beta}{2}\cdot\sin\frac{\beta-\alpha}{2}}$$

$$= \frac{-c^2}{\sin(\alpha-\beta)} = \frac{-c^2}{2\sin\frac{\alpha-\beta}{2}\cos\frac{\alpha-\beta}{2}}$$

$$\Rightarrow \frac{a^2\sin\theta}{c^2} = \frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}},$$

$$\frac{b^2\cos\theta}{c^2} = \frac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}} \quad \text{---(9)}$$

Also equation of the line joining P and Q is $\frac{x}{a}\cos\frac{\alpha+\beta}{2} + \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha-\beta}{2}$,

$z = 0$

$$\Rightarrow \frac{x}{a}\left(\frac{b^2\cos\theta}{c^2}\right) + \frac{y}{b}\left(\frac{a^2\sin\theta}{c^2}\right) = 1, \quad z=0 \quad \text{---(10)}$$

using the results of (9).

Now in order to find its envelope, we differentiate (10) with respect to θ and then eliminate θ.

Differentiating (10) w.r.t. θ, we get,

$$-\frac{xb^2}{ac^2}\sin\theta + \frac{ya^2}{bc^2}\cos\theta = 0, \quad z=0 \qquad (11)$$

Squaring and adding (10) and (11), θ is eliminated and we get the required envelope of PQ as $\frac{x^2b^4}{a^2c^4} + \frac{y^2a^4}{b^2c^4} = 1$, $z=0$

$$\Rightarrow \quad \frac{x^2}{a^6} + \frac{y^2}{b^6} = \frac{c^4}{a^4b^4}, \quad z=0$$

which represents an ellipse on the plane $z=0$.

Hence, proved.

SECTION - B

5.(a) Solve $(D^4 - 4D^2 - 5)y = e^x(x + \cos x)$.

Solution:

The auxiliary equation is

$$D^4 - 4D^2 - 5 = 0$$

$$\Rightarrow (D^2 - 5)(D^2 + 1) = 0$$

$$\Rightarrow D^2 = 5 \text{ or } -1$$

$$\Rightarrow D = \pm\sqrt{5}, \pm i$$

$$\therefore \text{C.F.} = c_1 \cosh x\sqrt{5} + c_2 \sinh x\sqrt{5} + c_3 \cos x + c_4 \sin x$$

P.I. corresponding to xe^x

$$= \frac{1}{D^4 - 4D^2 - 5} xe^x = e^x \frac{1}{(D+1)^4 - 4(D+1)^2 - 5} x$$

$$= e^x \frac{1}{D^4 + 4D^3 + 2D^2 - 4D - 8} x$$

$$= -\frac{e^x}{8} \frac{1}{1 + D/2 - D^2/4 - D^3/2 - D^4/8} x$$

$$= -\frac{e^x}{8}\left[1 + \frac{D}{2} - \frac{D^2}{4} - \frac{D^3}{2} - \frac{D^4}{8}\right]^{-1} x$$

$$= -\left(e^x/8\right) \times \left\{1 - \left(D/2 - D^2/4 - D^3/2 - D^4/8\right) + \dots\right\} x$$

$$= -\left(e^x/8\right) \times \left(x - 1/2\right)$$

$$= -\frac{e^x}{16} \times (2x-1)$$

P.I. corresponding to $e^x \cos x = \frac{1}{D^4 - 4D^2 - 5} e^x \cos x$

$$= e^x \frac{1}{(D+1)^4 - 4(D+1)^2 - 5} \cos x$$

$$= e^x \frac{1}{D^4 + 4D^3 + 2D^2 - 4D - 8} \cos x$$

$$= e^x \frac{1}{(D^2)^2 + 4D^2 \cdot D + 2D^2 - 4D - 8} \cos x$$

$$= e^x \frac{1}{(-1)^2 + 4(-1^2) D + 2(-1^2) - 4D - 8} \cos x$$

$$= e^x \frac{1}{9 + 8D} \cos x = -e^x \frac{9 - 8D}{(9+8D)(9-8D)} \cos x$$

$$= -e^x \frac{9 - 8D}{81 - 64D^2} \cos x = -e^x \frac{9-8D}{81 - 64(-1^2)} \cos x$$

$$= -\frac{1}{145} e^x (9\cos x + 8\sin x)$$

$\therefore$ Required solution is $y = c_1 \cosh x\sqrt{5} + c_2 \sinh x\sqrt{5} + c_3 \cos x + c_4 \sin x - (e^x/16) \times (2x-1) - (1/145)\, e^x (9\cos x + 8\sin x)$; c_1, c_2, c_3, c_4 being arbitrary constants.

Hence, the result.

5.(b) Solve the differential equation $(D^2-2D+2)y = e^x \tan x$, $D \equiv d/dx$ by method of variation of parameters.

Solution:

Given $(D^2-2D+2)\,y = e^x \tan x$

or $y_2 - 2y_1 + 2y = e^x \tan x$ —(1)

Comparing (1) with $y_2 + Py_1 + Qy = R$, we have $R = e^x \tan x$.

Consider $y_2 - 2y_1 + 2y = 0$ or $(D^2-2D+2)y = 0$ —(2)

Auxiliary equation for (2) is $D^2 - 2D + 2 = 0$

$\Rightarrow D = 1 \pm i$

$\therefore$ C.F. of (1) $= e^x(c_1 \cos x + c_2 \sin x)$, c_1 and c_2 being arbitrary constants —(3)

Let $u = e^x \cos x$ and $v = e^x \sin x$. Also, $R = e^x \tan x$ —(4)

Here, $W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x(\cos x - \sin x) & e^x(\cos x + \sin x) \end{vmatrix}$

$\Rightarrow W = e^{2x}\{\cos x(\cos x + \sin x) - \sin x(\cos x - \sin x)\}$ —(5)

$= e^{2x} \neq 0$

$\therefore$ P.I. of (1) $= u\,f(x) + v\,g(x)$, —(6)

where,

$$f(x) = -\int \frac{vR}{W}dx = -\int \frac{(e^x \sin x) \times (e^x \tan x)}{e^{2x}} dx$$

(using (4) and (5))

$$= -\int \frac{1-\cos^2 x}{\cos x} dx = \int (\cos x - \sec x) dx$$

$$= \sin x - \log(\sec x + \tan x) \quad \text{---(7)}$$

and
$$g(x) = \int \frac{uR}{W} dx = \int \frac{(e^x \cos x) \times (e^x \tan x)}{e^{2x}} dx,$$

(using (4) and (5))

$$= \int \sin x \, dx = -\cos x \quad \text{---(8)}$$

Using (6), (7) and (8), we have

$$\text{P.I. of (1)} = e^x \cos x \{\sin x - \log(\sec x + \tan x)\} + (e^x \sin x) \times (-\cos x)$$

$$= -e^x \cos x \log(\sec x + \tan x)$$

$\therefore$ The required solution of (1) is

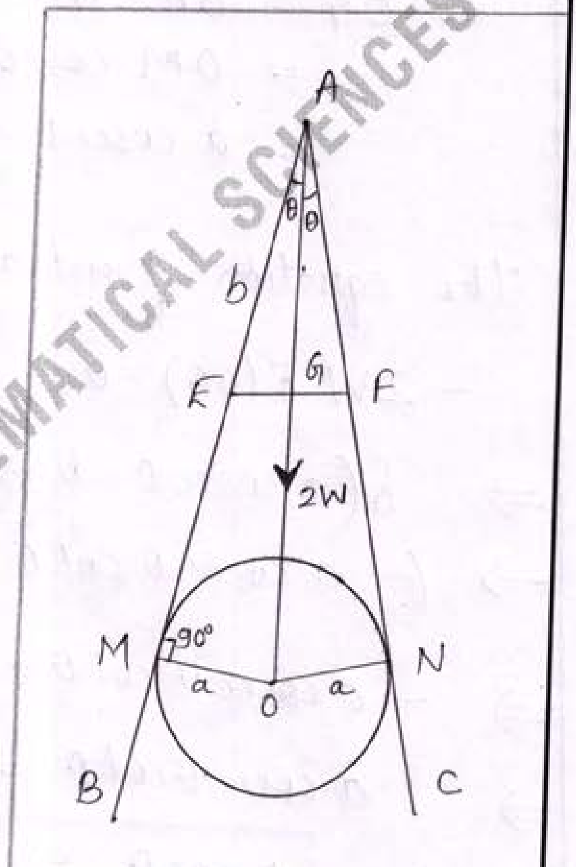
$$\boxed{y = e^x(c_1 \cos x + c_2 \sin x) - e^x \cos x \log(\sec x + \tan x)}$$

Hence, the result.

5.(c) → Two equal rods, AB and AC, each of length $2b$, are freely jointed at A and rest on a smooth vertical circle of radius a. Show that if 2θ be the angle between them, then $b\sin^3\theta = a\cos\theta$.

Solution:

Let O be the centre of the given fixed circle and W be the weight of each of the rods AB and AC. If E and F are the middle points of AB and AC, then the total weight 2W of the two rods can be taken acting at G, the middle point of EF. The line AO is vertical. We have



$\angle BAO = \angle CAO = \theta$

Also $AB = 2b$, $AE = b$. If the rod AB touches the circle at M, then $\angle OMA = 90°$ and OM = the radius of the circle $= a$.

We give the rods, a small symmetrical displacement in which θ changes to $\theta + \delta\theta$.

The point O remains fixed and the point G is slightly displaced.

The $\angle AMO$ remains 90°. We have,

the height of G above the fixed point O

$= OG = OA - GA$

$= OM \operatorname{cosec}\theta - AE\cos\theta$

$= a\operatorname{cosec}\theta - b\cos\theta.$

The equation of virtual work is

$-2W\,\delta(OG) = 0$ or $\delta(OG) = 0$

$\Rightarrow \delta(a\operatorname{cosec}\theta - b\cos\theta) = 0$

$\Rightarrow (-a\operatorname{cosec}\theta\cot\theta + b\sin\theta)\,\delta\theta = 0$

$\Rightarrow -a\operatorname{cosec}\theta\cot\theta + b\sin\theta = 0 \quad (\because \delta\theta \neq 0)$

$\Rightarrow a\operatorname{cosec}\theta\cot\theta = b\sin\theta$

$\Rightarrow \boxed{a\cos\theta = b\sin^3\theta}$

Hence, proved.

5.(d) → Show that the frenet - Serret formulae can be written in the form $\frac{d\mathbf{T}}{ds} = \boldsymbol{\omega} \times \mathbf{T}$, $\frac{d\mathbf{N}}{ds} = \boldsymbol{\omega} \times \mathbf{N}$, $\frac{d\mathbf{B}}{ds} = \boldsymbol{\omega} \times \mathbf{B}$ and determine $\boldsymbol{\omega}$.

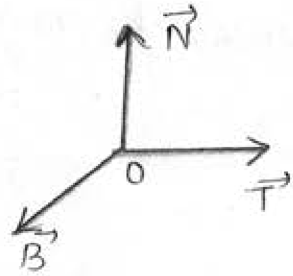
Solution:

From frenet - Serret formulae, we have

$$\boxed{\frac{d\vec{T}}{ds} = k\vec{N}} \quad —(1) \quad ; \quad \boxed{\frac{d\vec{B}}{ds} = -\tau\vec{N}} \quad —(2) \quad ; \quad \boxed{\frac{d\vec{N}}{ds} = \tau\vec{B} - k\vec{T}} \quad —(3)$$

Consider the vector,

$\vec{\omega} = \tau\vec{T} + k\vec{B}$



Now,

(i) $\vec{\omega} \times \vec{T} = (\tau\vec{T} + k\vec{B}) \times \vec{T}$

$= \tau\cancel{(\vec{T} \times \vec{T})}^{0} + k(\vec{B} \times \vec{T})$

$= k\vec{N} = \frac{d\vec{T}}{ds}$ [from (1)]

(ii) $\vec{\omega} \times \vec{N} = (\tau\vec{T} + k\vec{B}) \times \vec{N}$

$= \tau(\vec{T} \times \vec{N}) + k(\vec{B} \times \vec{N})$

$$= \tau\vec{B} - k\vec{T} = \frac{d\vec{N}}{ds} \quad [\text{from (3)}]$$

(iii) $\vec{\omega} \times \vec{B} = (\tau\vec{T} + k\vec{B}) \times \vec{B}$

$$= \tau(\vec{T} \times \vec{B}) + k\cancel{(\vec{B} \times \vec{B})}^{\,\vec{0}}$$

$$= -\tau\vec{N} = \frac{d\vec{B}}{ds} \quad [\text{from (2)}]$$

Determination of $\vec{\omega}$:

$\vec{\omega} = \tau\vec{T} + k\vec{B}$ (can be deduced)

Hence, the result.

5.(e) → Show that

$(y^2z^3\cos x - 4x^3z)dx + 2z^3y\sin x\, dy + (3y^2z^2\sin x - x^4)\,dz$ is an exact differential of some function ϕ and find this function.

Solution:

Let $\vec{F} = (y^2z^3\cos x - 4x^3z)\hat{i} + 2z^3y\sin x\,\hat{j} + (3y^2z^2\sin x - x^4)\hat{k}$.

We have curl $\vec{F}$

$$= \begin{vmatrix} \hat{i} & \hat{i} & \hat{j} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2z^3\cos x - 4x^3z & 2z^3y\sin x & 3y^2z^2\sin x - x^4 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(3y^2z^2\sin x - x^4) - \frac{\partial}{\partial z}(2z^3y\sin x)\right]$$

$$-\hat{j}\left[\frac{\partial}{\partial x}(3y^2z^2\sin x - x^4) - \frac{\partial}{\partial z}(y^2z^3\cos x - 4x^3z)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}(2z^3y\sin x) - \frac{\partial}{\partial y}(y^2z^3\cos x - 4x^3z)\right]$$

$$= (6yz^2\sin x - 6z^2y\sin x)\hat{i} - \left[(3y^2z^2\cos x - 4x^3) - (3z^2y^2\cos x - 4x^3)\right]\hat{j} + (2z^3y\cos x -$$

$2yz^3\cos x)\ \hat{k}$

$= 0\hat{i} - 0\hat{j} + 0\hat{k}$

$= 0$

$\therefore$ there exists a scalar function $\phi(x, y, z)$ such that $F = \nabla\phi$

$\therefore\ \vec{F} \cdot d\vec{r} = \nabla\phi \cdot d\vec{r}$

$\Rightarrow (y^2z^3\cos x - 4x^3z)\,dx + 2z^3y\sin x\,dy +$

$(3y^2z^2\sin x - x^4)\,dz = d\phi.$

Hence $(y^2z^3\cos x - 4x^3z)\,dx + 2z^3y\sin x\,dy$
$+ (3y^2z^2\sin x - x^4)\,dz$ is an exact differential of some function ϕ. —— (i)

Now, $\vec{F} = \nabla\phi$

$\Rightarrow \vec{F} = \dfrac{\partial\phi}{\partial x}\hat{i} + \dfrac{\partial\phi}{\partial y}\hat{j} + \dfrac{\partial\phi}{\partial z}\hat{k}$

Equating the co-efficients of $\hat{i}, \hat{j}, \hat{k}$ on both sides, we get,

$\frac{\partial \phi}{\partial x} = y^2 z^3 \cos x - 4x^3 z$ whence

$$\phi = y^2 z^3 \sin x - x^4 z + f_1(y,z) \quad \text{——(1)}$$

$\frac{\partial \phi}{\partial y} = 2z^3 y \sin x$ whence $\phi = z^3 y^2 \sin x + f_2(z,x)$ ——(2)

$\frac{\partial \phi}{\partial z} = 3y^2 z^2 \sin x - x^4$ whence $\phi = y^2 z^3 \sin x - x^4 z + f_3(x,y)$ ——(3)

(1), (2), (3) each represents ϕ.

These agree if we choose

$f_1(y,z) = 0$, $f_2(z,x) = -x^4 z$, $f_3(x,y) = 0$.

$\therefore \phi = y^2 z^3 \sin x - x^4 z$ to which may be added any constant.

Hence, $\boxed{\phi = y^2 z^3 \sin x - x^4 z + C}$ ——(ii)

Thus, the result.

6.(a) → Solve $(3y^2-7x^2+7)\,dx + (7y^2-3x^2+3)\,dy = 0$. —— (1)

$x\,dx$ $\quad$ $y\,dy$

Solution:

Let $X = x^2$, $Y = y^2$ so that
$dX = 2x\,dx$, $dY = 2y\,dy$

∴ from (1),

$$\frac{dY}{dX} = \frac{(7X-3Y-7)}{(7Y-3X+3)}$$

Now, putting $X = u+h$, $Y = v+K$
$dX = du$, $dY = dv$ so that

$$\frac{dv}{du} = \frac{7u-3v+(7h-3k-7)}{7v-3u+(7k-3h+3)} \quad \text{—— (2)}$$

choose h, k so that $(7h-3k-7)=0$ and
$(7k-3h+3)=0$ —— (3)

Solving (3), we get $h=1$, $k=0$.

∴ $X = u+1$, $Y = v+0$
$\Rightarrow u = X-1$, $v = Y$

Eq^n (2) becomes

$$\frac{dv}{du} = \frac{7u-3v}{7v-3u} = \frac{7-3(v/u)}{7(v/u-3)} \quad \text{—— (4)}$$

Taking $\frac{v}{u} = w \Rightarrow v = uw$

$$\frac{dv}{du} = w + u\frac{dw}{du} \quad \text{—— (5)}$$

From (4) and (5)

$$w + u\frac{dw}{du} = \frac{7-3w}{7w-3}$$

$$\Rightarrow u\frac{dw}{du} = \frac{7-3w}{7w-3} - w$$

$$\Rightarrow u\frac{dw}{du} = \frac{7-7w^2}{7w-3}$$

$$\Rightarrow \frac{7w-3}{7(1-w^2)}\, dw = \frac{du}{u}$$

$$\Rightarrow \frac{du}{u} = \frac{1}{7}\left[-5.\frac{dw}{w+1} - 2\frac{dw}{w-1}\right]$$

$$\Rightarrow \log u + \log c' = \frac{1}{7}\left[-5\log(w+1) - 2\log(w-1)\right]$$

$$\Rightarrow 7\log uc' = -\left[\log (w+1)^5.(w-1)^2\right]$$

$$\Rightarrow \log (uc')^7 = -\left[\log(w+1)^5.(w-1)^2\right]$$

$$\Rightarrow \log u^7.c + \log\left[(w+1)^5.(w-1)^2\right] = 0$$

$$\Rightarrow cu^7.(w+1)^5.(w-1)^2 = 0$$

$$\Rightarrow c = \frac{1}{(w+1)^5.(w-1)^2.u^7}$$

$$\Rightarrow \quad c = \frac{1}{(v/u+1)^5 (v/u-1)^2 u^7}$$

$$\Rightarrow \quad c = \frac{1}{\left(\frac{v+u}{u}\right)^5 \left(\frac{v-u}{u}\right)^2 \cdot u^7}$$

$$\Rightarrow \quad c = \frac{1}{\frac{(v+u)^5}{\cancel{u^5}} \cdot \frac{(v-u)^2}{\cancel{u^2}} \cdot \cancel{u^7}}$$

$$\Rightarrow \quad c = \frac{1}{(v+u)^5 (v-u)^2}$$

$$\Rightarrow \quad c = \frac{1}{(Y-X+1)^2 (Y+X-1)^5}$$

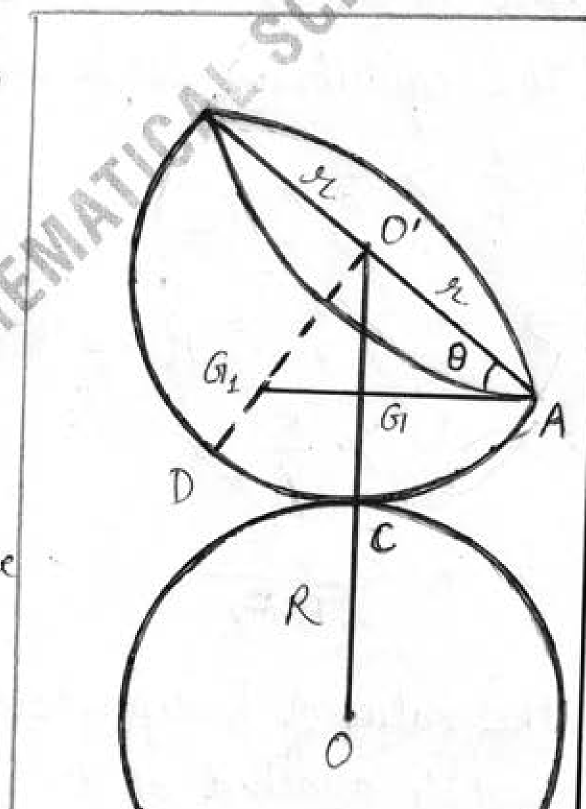
$$\Rightarrow \quad \boxed{c = \frac{1}{(y^2-x^2+1)^2 (y^2+x^2-1)^5}}$$ which

is the required solution.

Hence, the result.

6.(b) → A heavy hemispherical shell of radius r has a particle attached to a point on the rim, and rests with the curved surface in contact with a rough sphere of radius R at the highest point. Prove that if $R/r > \sqrt{5}-1$, the equilibrium is stable, whatever be the weight of the particle.

Solution :



Let O' be the centre of the base of the hemispherical shell of radius r.

Let a weight be attached to the rim of the hemi-spherical shell at A. The centre of gravity G_1 of the hemispherical shell is on its symmetrical radius $O'D$ and $O'G_1 = \frac{1}{2}O'D = \frac{1}{2}r$.

Let G be the centre of gravity of the combined body consisting of the hemispherical shell and the weight at A. Then G lies on the line AG_1.

The hemispherical shell rests with its curved surface in contact with a rough sphere of radius R and centre O at the highest point C.

For equilibrium the line $OCGO'$ must be vertical but AG_1 need not be horizontal.

Let $CG = h$

Also here, $\rho_1 = r$ and $\rho_2 = R$.

The equilibrium will be stable if

$$\frac{1}{h} > \frac{1}{\rho_1} + \frac{1}{\rho_2}$$

$$\Rightarrow \frac{1}{h} > \frac{1}{r} + \frac{1}{R}$$

$$\Rightarrow \frac{1}{h} > \frac{R+r}{rR}$$

$$\Rightarrow h < \frac{rR}{R+r} \qquad \text{—— (1)}$$

The value of h depends on the weight of the particle attached at A. So the equilibrium will be stable, whatever be the weight of the particle attached at A, if the relation (1) holds even for the maximum value of h.

Now h will be maximum if $O'G$ is minimum i.e., if $O'G$ is perpendicular to AG_1 or if $\Delta AO'G$ is right angled.

Let $\angle O'AG = \theta$.

Then from right angled $\Delta AO'G_1$,

$$\tan\theta = \frac{O'G_1}{O'A} = \frac{\frac{1}{2}r}{r} = \frac{1}{2}.$$

$$\therefore \quad \sin\theta = \frac{1}{\sqrt{5}}$$

$\therefore$ the minimum value of $O'G = O'A\sin\theta$

$$= r\left(1/\sqrt{5}\right)$$

$$= r/\sqrt{5}$$

$\therefore$ the maximum value of $h = r -$ the minimum value of $O'G$.

$$= r - \frac{r}{\sqrt{5}}$$

$$= \frac{r(\sqrt{5}-1)}{\sqrt{5}}$$

Hence, the equilibrium will be stable, whatever be the weight of the particle at A, if

$$\frac{r(\sqrt{5}-1)}{\sqrt{5}} < \frac{rR}{R+r}$$

$\Rightarrow$ if $\dfrac{\sqrt{5}-1}{\sqrt{5}} < \dfrac{R}{R+r}$

$\Rightarrow$ if $(\sqrt{5}-1)R + (\sqrt{5}-1)r < R\sqrt{5}$

$\Rightarrow$ if $(\sqrt{5}-1)r < R$

i.e. if $\dfrac{R}{r} > \sqrt{5}-1$.

Hence, Proved.

6.(c) (i) Find the most general differentiable function $f(r)$ so that $f(r)\,\mathbf{r}$ is solenoidal.

Solution:

Given that $f(r)\vec{r}$ is solenoidal

i.e. $\text{div}\,(f(r)\cdot\vec{r}) = 0$

$\Rightarrow f(r)\,\text{div}\,\vec{r} + \vec{r}\cdot \text{grad}\, f(r) = 0$

$\Rightarrow f(r)(3) + \vec{r}\cdot \text{grad}\, f(r) = 0$ — (1)

Now, $\text{grad}\, f(r) = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) f(r)$

$$= \hat{i} f'(r)\frac{\partial r}{\partial x} + \hat{j} f'(r)\frac{\partial r}{\partial y} + \hat{k} f'(r)\frac{\partial r}{\partial z}$$

$$= f'(r)\left[\hat{i}\cdot\frac{x}{r} + \hat{j}\frac{y}{r} + \hat{k}\frac{z}{r}\right]$$

$$= \frac{f'(r)}{r}\left[x\hat{i} + y\hat{j} + z\hat{k}\right]$$

$$= \frac{f'(r)}{r}\cdot\vec{r}$$

$\therefore$ from (1), we have,

$$3f(r) + \vec{r}\cdot\left(\frac{f'(r)}{r}\cdot\vec{r}\right) = 0$$

$$\Rightarrow 3f(r) + \frac{f'(r)}{r}(\vec{r}\cdot\vec{r}) = 0$$

$$\Rightarrow 3f(r) + f'(r)\cdot\frac{r^2}{r} = 0$$

$$\Rightarrow 3f(r) + rf'(r) = 0$$

$$\Rightarrow \frac{f'(r)}{f(r)} = \frac{-3}{r}$$

$$\Rightarrow \log f(r) = -3\log r + \log c$$

$$\Rightarrow \log f(r) + \log r^3 = \log c$$

$$\Rightarrow f(r)\cdot r^3 = c$$

$\Rightarrow \boxed{f(r) = c/r^3}$, where c is an arbitrary constant.

Hence, the result.

6.(c) (ii) → Show that $E = \mathbf{r}/r^2$ is irrotational. Find ϕ such that $E = -\nabla\phi$ and such that $\phi(a) = 0$ where $a>0$.

Solution:

(a) $\vec{E}$ is irrotational if $\nabla \times \vec{E} = 0$

Now, $\nabla \times \vec{E} = \nabla \times \left(\frac{\vec{r}}{r^2}\right)$

$$= \nabla\left(\frac{1}{r^2}\right) \times \vec{r} + \frac{1}{r^2}(\nabla \times \vec{r})$$

$[\because \nabla \times (\phi A) = (\nabla\phi) \times A + \phi(\nabla \times A)]$

$$= -\frac{2}{r^4}(\vec{r} \times \vec{r}) + 0$$

$[\because \nabla \times \vec{r} = 0]$

$$= -\frac{2}{r^4}(0) \qquad [\because \vec{r} \times \vec{r} = 0]$$

$$= 0$$

i.e. $\nabla \times \vec{E} = 0$

$\Rightarrow$ E is irrotational. —— (1)

(b) Given $\vec{E} = -\nabla\phi = \frac{\vec{r}}{r^2}$

$$\Rightarrow \nabla\phi = -\vec{r}/r^2$$

$$\Rightarrow \hat{i}\frac{\partial \phi}{\partial x} + \hat{j}\frac{\partial \phi}{\partial y} + \hat{k}\frac{\partial \phi}{\partial z} = -\frac{\vec{r}}{r^2} \quad \text{--- (1)}$$

$$\frac{\partial \phi(r)}{\partial x} = \phi'(r)\frac{\partial r}{\partial x}$$

$$\Rightarrow \frac{\partial \phi(r)}{\partial x} = \phi'(r)\cdot\left(\frac{x}{r}\right) \qquad \left[\begin{array}{l} \because r^2 = x^2+y^2+z^2 \\ 2r\frac{\partial r}{\partial x} = 2x \\ \frac{\partial r}{\partial x} = \frac{x}{r} \end{array}\right]$$

Similarly, $\frac{\partial \phi}{\partial y} = \phi'(r)\cdot\left(\frac{y}{r}\right)$ & $\frac{\partial \phi}{\partial z} = \phi'(r)\cdot\left(\frac{z}{r}\right)$

$\therefore$ from (1),
we have,

$$\phi'(r)\left(\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right) = -\frac{\vec{r}}{r^2}$$

$$\Rightarrow \phi'(r)\cdot\frac{\vec{r}}{r} = -\frac{\vec{r}}{r^2}$$

$$\Rightarrow \phi'(r) = -\frac{1}{r}$$

Integrating, we get

$$\phi(r) = -\log r + \log c$$

$$\Rightarrow \phi(r) = \log\left(\frac{c}{r}\right) \quad \text{--- (2)}$$

given $\phi(a) = 0$

$\therefore$ from (2), $\phi(a) = -\log a + \log c$

$\Rightarrow \quad 0 = -\log a + \log c$

$\Rightarrow \quad \log c = \log a$

$\Rightarrow \quad \boxed{c = a}$

$\therefore$ from (2),

$$\boxed{\phi(r) = \log \frac{a}{r}}$$

Hence, the result.

6.(d) → Prove that

$$\text{grad}(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} + (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{B}\times\text{curl}\,\mathbf{A} + \mathbf{A}\times\text{curl}\,\mathbf{B}.$$

Solution:

We have,

$$\text{grad}(\mathbf{A}\cdot\mathbf{B}) = \nabla(\mathbf{A}\cdot\mathbf{B}) = \sum \mathbf{i}\frac{\partial}{\partial x}(\mathbf{A}\cdot\mathbf{B})$$

$$= \sum \mathbf{i}\left(\mathbf{A}\cdot\frac{\partial \mathbf{B}}{\partial x} + \frac{\partial \mathbf{A}}{\partial x}\cdot\mathbf{B}\right)$$

$$= \sum\left\{\left(\mathbf{A}\cdot\frac{\partial \mathbf{B}}{\partial x}\right)\mathbf{i}\right\} + \sum\left\{\left(\mathbf{B}\cdot\frac{\partial \mathbf{A}}{\partial x}\right)\mathbf{i}\right\} \quad \text{——— (1)}$$

Now, we know that

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

$$\therefore (\mathbf{a}\cdot\mathbf{b})\mathbf{c} = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - \mathbf{a}\times(\mathbf{b}\times\mathbf{c})$$

$$\therefore \left(\mathbf{A}\cdot\frac{\partial \mathbf{B}}{\partial x}\right)\mathbf{i} = (\mathbf{A}\cdot\mathbf{i})\frac{\partial \mathbf{B}}{\partial x} - \mathbf{A}\times\left(\frac{\partial \mathbf{B}}{\partial x}\times\mathbf{i}\right)$$

$$= (\mathbf{A}\cdot\mathbf{i})\frac{\partial \mathbf{B}}{\partial x} + \mathbf{A}\times\left(\mathbf{i}\times\frac{\partial \mathbf{B}}{\partial x}\right).$$

Thus, $\sum\left\{\left(A\cdot\frac{\partial B}{\partial x}\right)i\right\} = \sum\left\{(A\cdot i)\frac{\partial B}{\partial x}\right\} + \sum\left\{A\times\left(i\times\frac{\partial B}{\partial x}\right)\right\}$

$$= \left\{A\cdot\sum i\frac{\partial}{\partial x}\right\}B + A\times\sum\left(i\times\frac{\partial B}{\partial x}\right)$$

$$= (A\cdot\nabla)B + A\times(\nabla\times B) \quad \text{---(2)}$$

Similarly,

$$\sum\left\{\left(B\cdot\frac{\partial A}{\partial x}\right)i\right\} = (B\cdot\nabla)A + B\times(\nabla\times A) \quad \text{---(3)}$$

Putting the values from (2) and (3), we get

$$\boxed{\text{grad}(A\cdot B) = (A\cdot\nabla)B + A\times(\nabla\times B) + (B\cdot\nabla)A + B\times(\nabla\times A)}$$

Hence, Proved.

7. (a) (i) → Solve $6\cos^2 x\,(dy/dx) - y\sin x + 2y^4\sin^3 x = 0$.

Solution:

Re-writing the given equation,

$$-y^{-4}(dy/dx) + (\sin x/6\cos^2 x)\,y^{-3} = (1/3)\sin x \times \tan^2 x$$

Putting $y^{-3} = v$ and $-3y^{-4}(dy/dx) = dv/dx$, the above equation yields

$$\frac{1}{3}\frac{dv}{dx} + \frac{\sin x}{6\cos^2 x}v = \frac{1}{3}\sin x\tan^2 x$$

i.e. $$\frac{dv}{dx} + \frac{1}{2}(\tan x\sec x)v = \sin x\tan^2 x$$

which is linear differential equation whose integrating factor is

$$e^{\int (1/2)\times \sec x\tan x\,dx}$$

i.e. $$e^{(1/2)\times\sec x}$$

and hence its solution is given by

$$v.\,e^{(1/2)\times\sec x} = \int e^{(1/2)\times\sec x}\sin x.\tan^2 x\,dx + c,$$

c being an arbitrary constant.

i.e. $$\boxed{y^{-3}\times e^{(1/2)\sec x} = \int e^{(1/2)\times\sec x}.\sin x.\tan^2 x\,dx + C}$$

which is the required result.

7.(a)(ii) → Reduce the equation $x^2p^2 + py(2x+y) + y^2 = 0$ where $p = dy/dx$ to Clairaut's form and find its complete primitive and its singular solution.

Solution:

Given equation is $x^2p^2 + py(2x+y) + y^2 = 0$ —— (1)

Let $y = u$ and $xy = v$ —— (2)

Differentiating (2), $dy = du$ and $x\,dy + y\,dx = dv$

$$\therefore \frac{x\,dy + y\,dx}{dy} = \frac{dv}{du}$$

$$\Rightarrow x + y\frac{dx}{dy} = \frac{dv}{du}$$

$$\Rightarrow x + \frac{y}{p} = P$$

$$\Rightarrow y/p = P - x \quad \Rightarrow p = y/(P-x),$$

where $p = dy/dx$, $P = dv/du$.

Putting $p = y/(P-x)$ in (1), we have

$$\frac{x^2y^2}{(P-x)^2} + \frac{y^2}{(P-x)}(2x+y) + y^2 = 0$$

$$\Rightarrow x^2 + (P-x)(2x+y) + (P-x)^2 = 0$$

$$\Rightarrow Py - xy + P^2 = 0$$

$$\Rightarrow v = uP + P^2, \quad \text{using (2)} \quad \text{—— (3)}$$

(3) is in Clairaut's form.
So replacing P by c, its general solution is
$\boxed{v = uc + c^2 \quad \text{or} \quad xy = yc + c^2}$, c being an arbitrary constant. —— (4)

Thus,
(4) represents the complete primitive of (1) which is a quadratic equation in c and hence c-discriminant relation is

$y^2 - 4.1.(-xy) = 0$

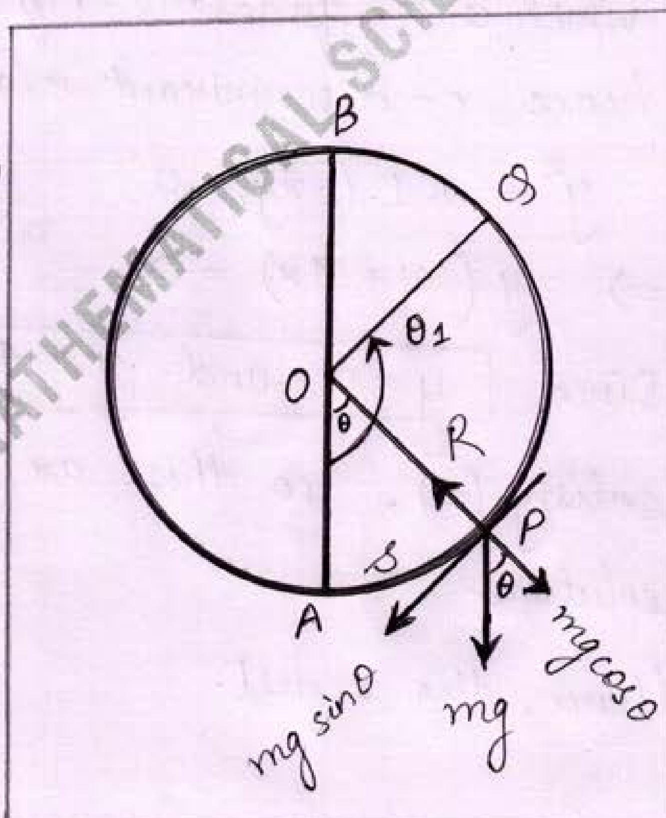
$\Rightarrow y(y + 4x) = 0$

Since $\boxed{y = 0 \text{ and } y + 4x = 0}$ both satisfy (1), ~~so~~ these are both singular solutions.

Hence, the result.

7.(b) → A particle is free to move on a smooth vertical circular wire of radius a. It is projected from the lowest point with velocity just sufficient to carry it to the highest point. Show that the reaction between the particle and the wire is zero after a time $\sqrt{(a/g)} \cdot \log(\sqrt{5}+\sqrt{6})$.

Solution:



Let a particle of mass m be projected from the lowest point A of a vertical circle of radius a with velocity u which is just sufficient to carry it to the highest point B.

If P is the position of the particle at any time t such that $\angle AOP = \theta$ and arc $AP = s$, then the equations of motion of the particle along the tangent and normal are

$$m\frac{d^2s}{dt^2} = -mg\sin\theta \quad \text{—— (1)}$$

and $$m\frac{v^2}{a} = R - mg\cos\theta \quad \text{—— (2)}$$

Also $$s = a\theta \quad \text{—— (3)}$$

from (1) and (3), we have $a\dfrac{d^2\theta}{dt^2} = -g\sin\theta.$

Multiplying both sides by $2a\,(d\theta/dt)$ and integrating, we have

$$v^2 = \left(a\frac{d\theta}{dt}\right)^2 = 2ag\cos\theta + A$$

But according to the demand of the question, $v = 0$ at the highest point B, where $\theta = \pi$.

$\therefore\ 0 = 2ag\cos\pi + A$

$\Rightarrow A = 2ag$

$$\therefore\ v^2 = \left(a\frac{d\theta}{dt}\right)^2 = 2ag\cos\theta + 2ag. \quad \text{—— (4)}$$

from (2) and (4), we have

$$R = \frac{m}{a}\left(v^2 + ag\cos\theta\right) = \frac{m}{a}\left(2ag + 3ag\cos\theta\right) \quad \text{—— (5)}$$

If the reaction $R=0$ at the point C where $\theta=\theta_1$, then from (5), we have

$$0 = \frac{m}{a}\left(2ag + 3ag\cos\theta_1\right)$$

$$\Rightarrow \cos\theta_1 = -\frac{2}{3} \qquad \text{—— (6)}$$

From (4), we have

$$\left(a\frac{d\theta}{dt}\right)^2 = 2ag\left(\cos\theta + 1\right)$$

$$= 2ag \cdot 2\cos^2\theta/2$$

$$= 4ag\cos^2\theta/2$$

$\therefore \dfrac{d\theta}{dt} = 2\sqrt{(g/a)}\cos\theta/2$, the positive sign being taken before the radical sign because θ increases as t increases.

$$\Rightarrow dt = \frac{1}{2}\sqrt{(a/g)}\sec\theta/2\, d\theta$$

Integrating from $\theta=0$ to $\theta=\theta_1$, the required time t is given by

$$t = \frac{1}{2}\sqrt{(a/g)}\int_{\theta=0}^{\theta=\theta_1}\sec\theta/2\, d\theta$$

$$\Rightarrow \quad t = \sqrt{(a/g)}\left[\log\left(\sec \theta/2 + \tan \theta/2\right)\right]_0^{\theta_1}$$

$$\Rightarrow \quad t = \sqrt{(a/g)}\, \log\left(\sec \frac{\theta_1}{2} + \tan \frac{\theta_1}{2}\right) \quad \text{——— (7)}$$

from (6), we have

$$2\cos^2 \frac{\theta_1}{2} - 1 = -\frac{2}{3}$$

$$\Rightarrow \quad 2\cos^2 \frac{\theta_1}{2} = 1 - \frac{2}{3} = \frac{1}{3}$$

$$\Rightarrow \quad \cos^2 \frac{\theta_1}{2} = \frac{1}{6}$$

$$\Rightarrow \quad \sec^2 \frac{\theta_1}{2} = 6.$$

$$\therefore \quad \sec \frac{\theta_1}{2} = \sqrt{6}$$

and $\tan \frac{\theta_1}{2} = \sqrt{\left(\sec^2 \frac{\theta_1}{2} - 1\right)} = \sqrt{6-1} = \sqrt{5}.$

Substituting in (7), the required time is given by

$$\boxed{t = \sqrt{(a/g)}\, \log\left(\sqrt{6} + \sqrt{5}\right).}$$

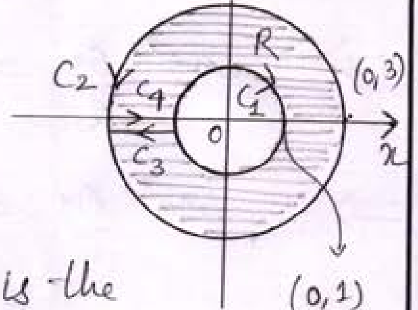
Hence, proved.

7.(c) → Verify Green's theorem in the plane for $\oint_C (2x-y^3)\,dx - xy\,dy$, where C is the boundary of the region enclosed by the circles $x^2+y^2=1$ and $x^2+y^2=9$.

<u>Solution:</u>

By Green's theorem, we have,

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$



The boundary of the curve C is given by

$C: C_1 + C_2 + C_3 + C_4$ and R is the region bounded by the circles $x^2+y^2=1$ and $x^2+y^2=3$.

$$\therefore \int_C M\,dx + N\,dy = \int_{C_1+C_2+C_3+C_4} M\,dx + N\,dy$$

We also observe that in this question, along C_3 and C_4 i.e. $\int_{C_3+C_4} M\,dx + N\,dy = 0$.

$\because$ C_3 & C_4 are in opposite direction.

Along C_3 & C_4 : $y = 0 \Rightarrow dy = 0$

$$\therefore \int_{C_3} 2x\,dx + \int_{C_4} 2x\,dx = \int_1^3 2x\,dx + \int_3^1 2x\,dx$$

$$= \int_1^3 2x\,dx - \int_1^3 2x\,dx = 0$$

Now,

$$L.H.S = \int_C M\,dx + N\,dy = \int_{C_1+C_2} M\,dx + N\,dy$$

$$= \int_{C_1} M\,dx + N\,dy + \int_{C_2} M\,dx + N\,dy \quad \text{——(1)}$$

Let

$$\int_{C_2} M\,dx + N\,dy = \int_{C_2} (2x - y^3)\,dx - xy\,dy$$

Putting $x = 3\cos\theta$, $y = 3\sin\theta$
$dx = -3\sin\theta$, $dy = 3\cos\theta$.

$$\int_{C_2} M\,dx + N\,dy = \int_0^{2\pi} (6\cos\theta - 27\sin^3\theta)(-3\sin\theta)\,d\theta$$

$$- 27\cos^2\theta \sin\theta\,d\theta$$

$$= \int_0^{2\pi} \left(-18\cos\theta\sin\theta + 81\sin^4\theta - \right.$$

$27 \sin\theta \cdot \cos^2\theta)\, d\theta$

$$= 0 + 81\cdot(4)\cdot \int_0^{\pi/2} \sin^4\theta\, d\theta + 27 \left.\frac{\cos^3\theta}{3}\right|_0^{2\pi}$$

$$= 81\cdot \cancel{4}\cdot \frac{3}{\cancel{4}}\cdot\frac{1}{2}\cdot\frac{\pi}{2} + 0$$

$$= \frac{243\pi}{4}$$

$$\int_{C_1} M\,dx + N\,dy = \int_{2\pi}^{0} \left(2\cos\theta - \sin^3\theta\right)(-\sin\theta\, d\theta) - \cos^2\theta \sin\theta\, d\theta$$ by putting $x = \cos\theta$, $y = \sin\theta$.

$\theta : 2\pi \to 0.$

$$= 0 + \int_{2\pi}^{0} \sin^4\theta\, d\theta - 0$$

$$= -4\int_0^{\pi/2} \sin^4\theta\, d\theta = -\cancel{4}\cdot\frac{3}{\cancel{4}}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

$$= -\frac{3\pi}{4}$$

$\therefore$ from (1),

$$\int_C M\,dx + N\,dy = -\frac{3\pi}{4} + \frac{243\pi}{4} = \frac{240\pi}{4}$$

$$\therefore \boxed{\int_C M\,dx + N\,dy = 60\pi}$$ (A)

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (3y^2 - y)\,dx\,dy$$

Using the polar co-ordinates,
$x = r\cos\theta$, $y = r\sin\theta$.
$dx\,dy = r\,dr\,d\theta$.

$$= \int_{\theta=0}^{2\pi} \int_{r=1}^{3} (3r^2\sin^2\theta - r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi} \left[\frac{3r^4}{4}\sin^2\theta - \frac{r^3}{3}\sin\theta\right]_1^3 d\theta$$

$$= \int_0^{2\pi} \left(60\sin^2\theta - \frac{26}{3}\sin\theta\right) d\theta$$

$$= 60\left[\frac{\theta - \frac{\sin 2\theta}{2}}{2}\right]_0^{2\pi}$$

$$= 60 \cdot \frac{2\pi}{2} = 60\pi$$

$$\therefore \boxed{\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = 6\pi} \quad \text{—— (B)}$$

$\therefore$ from (A) & (B), Green's theorem is verified.

Hence, the result.

8.(a) → By using Laplace transform method, solve $(D^2+n^2)y = a\sin(nt+\alpha)$, $y = Dy = 0$ when $t=0$.

Solution:

The given equation can be written as

$$(D^2+n^2)y = a(\sin nt\cos\alpha + \cos nt\sin\alpha).$$

$$\therefore L\{y''\} + n^2 L\{y\} = a\cos\alpha \, L\{\sin nt\} + a\sin\alpha \, L\{\cos nt\}$$

$$\Rightarrow p^2 L\{y\} - p\,y(0) - y'(0) + n^2 L\{y\} = a\cos\alpha \cdot \frac{n}{p^2+n^2} + a\sin\alpha \cdot \frac{p}{p^2+n^2}$$

$$\Rightarrow (p^2+n^2) L\{y\} = \frac{n\,a\cos\alpha}{(p^2+n^2)} + \frac{ap\sin\alpha}{(p^2+n^2)}$$

$$\Rightarrow L\{y\} = \frac{na\cos\alpha}{(p^2+n^2)^2} + \frac{ap\sin\alpha}{(p^2+n^2)^2}$$

$$\therefore y = an\cos\alpha \, L^{-1}\left\{\frac{1}{(p^2+n^2)^2}\right\} + a\sin\alpha \cdot L^{-1}\left\{\frac{p}{(p^2+n^2)^2}\right\}$$

$$= na\cos\alpha \cdot \int_0^t \left(\frac{1}{n}\sin nx\right)\frac{1}{n}\sin n(t-x)\,dx$$

$$- \frac{a\sin\alpha}{2} \cdot L^{-1}\left\{\frac{d}{dp}\frac{1}{(p^2+n^2)}\right\}$$

by the convolution theorem, since $L^{-1}\left\{\frac{1}{p^2+n^2}\right\} = \frac{1}{n}\cdot\sin nt$

$$= \frac{a\cos\alpha}{2n}\int_0^t \{\cos n(t-2x) - \cos nt\}\,dx +$$

$$\frac{a\sin\alpha}{2}\, t\, L^{-1}\left\{\frac{1}{p^2+n^2}\right\}$$

$$= \frac{a\cos\alpha}{2n}\left[-\frac{1}{2n}\sin n(t-2x) - x\cos nt\right]_0^t$$

$$+ \frac{at\sin\alpha}{2n}\sin nt$$

$$= \frac{a\cos\alpha}{2n}\left[\frac{\sin nt}{2n} - t\cos nt + \frac{\sin nt}{2n}\right] +$$

$$\frac{at\sin\alpha\sin nt}{2n}$$

$$= \frac{a\cos\alpha\sin nt}{2n^2} - \frac{at}{2n}(\cos\alpha\cos nt - \sin\alpha\sin nt)$$

$$= \frac{a\cos\alpha\sin nt}{2n^2} - \frac{at}{2n}\cos(\alpha+nt)$$

$$= \boxed{\frac{a}{2n^2}\left[\cos\alpha\sin nt - nt\cos(\alpha+nt)\right]}$$

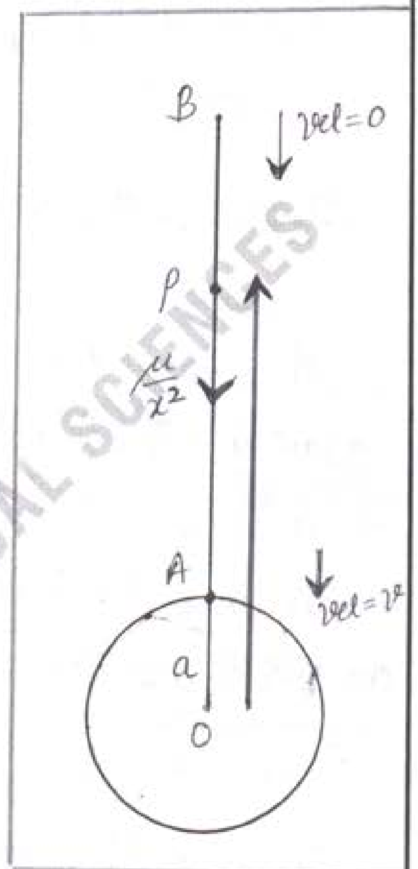
which is the required solution.

8.(b) Assuming that a particle falling freely under gravity can penetrate the earth without meeting any resistance, show that a particle falling from rest at a distance b $(b>a)$ from the centre of the earth would on reaching the centre acquire a velocity $\sqrt{[ga(3b-2a)/b]}$ and the time to travel from the surface to the centre of the earth is $\sqrt{\left(\frac{a}{g}\right)}\sin^{-1}\sqrt{\left[\frac{b}{(3b-2a)}\right]}$, where a is the radius of the earth and g is the acceleration due to gravity on the earth's surface.

Solution :

Let the particle fall from rest from the point B such that $OB = b$, where O is the centre of the earth. Let P be the position of the particle at any time t measured from the instant it starts falling from B and let $OP = x$.

Acceleration at $P = \mu/x^2$ towards O. The equation of motion of P is



$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^2},$$

which holds good for the motion from B to A i.e., outside the surface of the earth.

But at the point A (on one earth's surface) $x = a$ and $d^2x/dt^2 = -g$.

$$\therefore -g = -\mu/a^2$$

$$\Rightarrow \mu = a^2 g.$$

$$\therefore \frac{d^2x}{dt^2} = -\frac{a^2 g}{x^2} \quad \text{---} (1)$$

Multiplying both sides of (1) by $2(dx/dt)$ and then integrating w.r.t. 't', we have

$$\left(\frac{dx}{dt}\right)^2 = \frac{2a^2 g}{x} + A, \text{ where } A \text{ is a constant.}$$

But at B, $x = OB = b$ and $dx/dt = 0$.

$$\therefore 0 = \frac{2a^2 g}{b} + A \Rightarrow A = -\frac{2a^2 g}{b}$$

$$\therefore \left(\frac{dx}{dt}\right)^2 = 2a^2 g\left(\frac{1}{x} - \frac{1}{b}\right). \quad \text{---} (2)$$

If V is the velocity of the particle at the point A, then at A, $x = OA = a$ and $(dx/dt)^2 = V^2$

$$\therefore V^2 = 2a^2 g\left(\frac{1}{a} - \frac{1}{b}\right) \quad \text{——— (3)}$$

Now the particle starts moving through a hole from A to O with velocity V at A.

Let x, $(x<a)$, be the distance of the particle from the centre of the earth at any time t measured from the instant the particle starts penetrating the earth at A. The acceleration at this point will be λx towards O, where λ is a constant.

The equation of motion (inside the earth) is

$\frac{d^2x}{dt^2} = -\lambda x$, which holds good for the motion from A to O.

At A, $x = a$ and $d^2x/dt^2 = -g$.

$$\therefore \lambda = g/a.$$

$$\therefore \frac{d^2x}{dt^2} = -\frac{g}{a}x.$$

Multiplying both sides by $2(dx/dt)$ and then integrating w.r.t. 't', we have,

$$\left(\frac{dx}{dt}\right)^2 = -\frac{g}{a}x^2 + B \quad \text{where } B \text{ is a constant.}$$

$$\text{——— (4)}$$

But at A, $x = OA = a$ and $\left(\frac{dx}{dt}\right)^2 = V^2 = 2a^2 g\left(\frac{1}{a} - \frac{1}{b}\right)$

(from (3))

$$\therefore \; 2a^2 g\left(\frac{1}{a} - \frac{1}{b}\right) = -\frac{g}{a} a^2 + B$$

$$\Rightarrow \quad B = ag\left(\frac{3b-2a}{b}\right).$$

Substituting the value of B in (4), we have

$$\left(\frac{dx}{dt}\right)^2 = ag\left(\frac{3b-2a}{b}\right) - \frac{g}{a}x^2 \quad \text{——} \quad (5)$$

Putting $x = 0$ in (5), we get the velocity on reaching the centre of the earth as $\sqrt{\left[\frac{ga(3b-2a)}{b}\right]}$

Again from (5), we have

$$\left(\frac{dx}{dt}\right)^2 = \frac{g}{a}\left[a^2 \frac{(3b-2a)}{b} - x^2\right]$$

$$= \frac{g}{a}(c^2 - x^2), \text{ where } c^2 = \frac{a^2}{b}(3b-2a).$$

$$\therefore \; \frac{dx}{dt} = -\sqrt{\left(\frac{g}{a}\right)} \cdot \sqrt{(c^2 - x^2)}$$, the $-$ve sign being taken because the particle is moving in the direction of x decreasing.

$\Rightarrow \quad dt = -\sqrt{\left(\frac{a}{g}\right)} \cdot \frac{dx}{\sqrt{(c^2-x^2)}}$, separating the variables.

Integrating from A to O, the required time t is given by

$$t = -\sqrt{\left(\frac{a}{g}\right)} \int_{x=a}^{0} \frac{dx}{\sqrt{(c^2-x^2)}}$$

$$= \sqrt{\left(\frac{a}{g}\right)} \int_{0}^{a} \frac{dx}{\sqrt{(c^2-x^2)}}$$

$$= \sqrt{\left(\frac{a}{g}\right)} \left[\sin^{-1}\frac{x}{c}\right]_0^a$$

$$= \sqrt{\left(\frac{a}{g}\right)} \cdot \sin^{-1}\left(\frac{a}{c}\right) = \sqrt{\left(\frac{a}{g}\right)} \cdot \sin^{-1}\left[\frac{a}{a\sqrt{\left\{\frac{(3b-2a)}{b}\right\}}}\right]$$

substituting for c.

$$\therefore t = \sqrt{\left(\frac{a}{g}\right)} \cdot \sin^{-1}\sqrt{\frac{b}{(3b-2a)}}$$

$$\therefore \text{Time } \boxed{t = \sqrt{\left(\frac{a}{g}\right)} \cdot \sin^{-1}\sqrt{\left[\frac{b}{(3b-2a)}\right]}}$$

Hence, proved.

8.(c) Evaluate $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$, where

$\mathbf{F} = (x^2 + y - 4)\,\mathbf{i} + 3xy\,\mathbf{j} + (2xy + z^2)\,\mathbf{k}$ and S is the surface of the paraboloid $z = 4 - (x^2 + y^2)$ above the xy-plane.

Solution:

The surface $z = 4 - (x^2 + y^2)$ meets the plane $z = 0$ in a circle C given by

$x^2 + y^2 = 4,\ z = 0.$

Let S_1 be the plane region bounded by the circle C. If S' is the surface consisting of the surfaces S and S_1, then S' is a closed surface.

Let V be the volume bounded by S'.

If $\mathbf{n}$ denotes the outward drawn (drawn outside the region V) unit normal vector to S', then on the plane surface S_1, we have $\mathbf{n} = -\mathbf{k}$.

Also, $\mathbf{k}$ is a unit vector normal to S_1 drawn into the region V.

By Gauss-divergence theorem, we have

$$\iint_{S'} (\text{curl}\, \vec{F}) \cdot \hat{n}\, dS = \iiint_V [\text{div curl}\, \vec{F}]\, dV = 0$$

since $\operatorname{div}\operatorname{curl}\vec{F} = 0$.

$$\therefore \iint_S (\operatorname{curl}\vec{F})\cdot\hat{n}\,dS + \iint_{S_1}(\operatorname{curl}\vec{F})\cdot\hat{n}\,dS = 0$$

[$\because$ S' consists of S and S_1]

$$\Rightarrow \iint_S (\operatorname{curl}\vec{F})\cdot\hat{n}\,dS - \iint_{S_1}(\operatorname{curl}\vec{F})\cdot\hat{k}\,dS = 0$$

[$\because$ on S_1, $\hat{n} = -\hat{k}$]

$$\Rightarrow \iint_S (\operatorname{curl}\vec{F})\cdot\hat{n}\,dS = \iint_{S_1}(\operatorname{curl}\vec{f})\cdot\hat{k}\,dS.$$

Now, $\operatorname{curl}\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2+y-4 & 3xy & 2xz+z^2 \end{vmatrix}$

$$= \hat{i}\left[\frac{\partial}{\partial y}(2xz+z^2) - \frac{\partial}{\partial z}(3xy)\right] -$$

$$\hat{j}\left[\frac{\partial}{\partial x}(2xz+z^2) - \frac{\partial}{\partial z}(x^2+y-4)\right]$$

$$+\hat{k}\left[\frac{\partial}{\partial x}(3xy) - \frac{\partial}{\partial y}(x^2+y-4)\right]$$

$$= 0\hat{i} - 2z\hat{j} + (3y-1)\hat{k}$$

$\therefore (\text{curl } \vec{F}) \cdot \hat{k} = [-2z\hat{j} + (3y-1)\hat{k}] \cdot \hat{k}$

$= 3y - 1$ over the surface S_1 bounded by the circle $x^2+y^2=4, z=0$.

Hence, $\iint_S (\text{curl } \vec{F}) \cdot \hat{n}\, dS = \iint_{S_1} (3y-1)\, dS$

$$= \int_{x=-2}^{2} \int_{y=-\sqrt{(4-x^2)}}^{\sqrt{(4-x^2)}} (3y-1)\, dx\, dy$$

$$= 2\int_{x=-2}^{2} \int_{y=0}^{\sqrt{(4-x^2)}} (-1)\, dx\, dy \quad [\because 3y \text{ is an odd function of } y]$$

$$= -2\int_{x=-2}^{2} [y]_{y=0}^{\sqrt{(4-x^2)}} dx$$

$$= -2\int_{-2}^{2} \sqrt{(4-x^2)}\, dx = -4\int_{0}^{2} \sqrt{(4-x^2)}\, dx$$

$$= -4\left[\frac{x}{2}\sqrt{(4-x^2)} + 2\sin^{-1}\frac{x}{2}\right]_0^2 = -4\left[2 \cdot \frac{\pi}{2}\right] = -4\pi.$$

$$\therefore \boxed{\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, dS = -4\pi}$$ Hence, the result.