IFOS - 2017

Q1 Solve $(2D^3-7D^2+7D-2)y = e^{-8x}$ where $D = \frac{d}{dx}$

sol. $(2D^3-7D^2+7D-2)y = e^{-8x}$

Auxiliary eqn is : $2m^3-7m^2+7m-2 = 0$

$m=2$ is root of the equation (By inspection)

$\Rightarrow \quad 2m^3-4m^2-3m^2+6m+m-2 = 0$

$\Rightarrow \quad 2m^2(m-2)-3m(m-2)+1(m-2) = 0$

$\Rightarrow \quad (m-2)(2m^2-3m+1) = 0$

$\Rightarrow \quad (m-2)(2m^2-2m-m+1) = 0$

$(m-2)(2m-1)(m-1) = 0$

$\therefore \quad m = \frac{1}{2}, 1, 2$

$C.F. = c_1 e^{x} + c_2 e^{x/2} + c_3 e^{2x}$

$P.I. = \frac{1}{2D^3-7D^2+7D-2} e^{-8x} = e^{-8x} \frac{1}{2(-8)^3-7(8)^2+7(-8)-2}$

$= \frac{e^{-8x}}{-1530}$

$\therefore$ General solution is $C.F + P.I = \boxed{c_1 e^{x}+c_2 e^{x/2}+c_3 e^{2x} - \frac{e^{-8x}}{1530}}$

Q2 Solve $x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^4$

sol. Let $x = e^z$ then $x^2\frac{d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$

$x\frac{dy}{dx} = \frac{dy}{dz}$

$\therefore$ On substitution we get $\left(\frac{d^2y}{dz^2}-\frac{dy}{dx}\right)-2\left(\frac{dy}{dz}\right)-4y = (e^z)^4$

$\Rightarrow \quad \frac{d^2y}{dz^2} - 3\frac{dy}{dz} - 4y = e^{4z}$

Auxiliary eqn is $m^2-3m-4 = 0$

$\Rightarrow \quad m^2-4m+m-4 = 0$

$\Rightarrow \quad (m-4)(m+1) = 0$

$\therefore \quad m = 4, -1$

Hence, $C.F. = c_1 e^{4z} + c_2 e^{-z}$

$C.F. = c_1 x^4 + \frac{c_2}{x}$

$$P.I. = \frac{1}{D^2-3D-4} e^{4z} = \frac{1}{(D-4)(D+1)} e^{4z}$$

$$= \frac{1}{(D-4)(4+1)} e^{4z} = \frac{1}{5} \cdot \frac{1}{(D-4)} e^{4z}$$

$$= \frac{z e^{4z}}{5} = \frac{(\ln x) x^4}{5}$$

$$\therefore \quad y = C.F + P.I$$

$$\boxed{y = C_1 x^4 + \frac{C_2}{x} + \frac{x^4 (\ln x)}{5}}$$

Q 3 Solve the differential equation $\left(\frac{dy}{dx}\right)^2 + 2\frac{dy}{dx} \cdot y \cot x = y^2$.

sol: $\cancel{\left(\frac{dy}{dx}\right)\left[\frac{dy}{dx} + 2y\cot x\right] = y^2}$

Let $\frac{dy}{dx} = p$, then eqn becomes $p^2 + 2py\cot x - y^2 = 0$

$$\therefore \quad p = \frac{-2y\cot x \pm \sqrt{4y^2\cot^2 x - 4(-y^2)}}{2}$$

$$p = \frac{-2y\cot x \pm \sqrt{4y^2(1+\cot^2 x)}}{2}$$

$$p = \frac{-2y\cot x \pm \sqrt{4y^2\operatorname{cosec}^2 x}}{2}$$

$$p = \frac{-2y\cot x \pm 2y\operatorname{cosec} x}{2}$$

$$p = -y\cot x \pm y\operatorname{cosec} x$$

$$\frac{dy}{dx} = -y(\cot x \pm \operatorname{cosec} x)$$

$$\int \frac{dy}{y} = -\int (\cot x \pm \operatorname{cosec} x)\, dx$$

$$\ln y = -[\ln(\sin x) \pm (\ln|\operatorname{cosec} x + \cot x|)] + \log k$$

$$\ln y = -[\ln(\sin x) - \ln|\operatorname{cosec} x + \cot x|]$$

$$\ln y = -\ln\left|\frac{\sin x}{\operatorname{cosec} x + \cot x}\right|$$

$$y = e^{-\ln\left|\frac{\sin x}{\operatorname{cosec} x + \cot x}\right| + \log k}$$

$$y = k\frac{\operatorname{cosec} x + \cot x}{\sin x}$$

$$y = k(\operatorname{cosec}^2 x + \cot x \operatorname{cosec} x)$$

$$\ln y = -[\ln(\sin x) + \ln|\operatorname{cosec} x + \cot x|] + \log k$$

$$\ln y = -[\ln(1+\cos x)] + \log k$$

$$y = e^{-\ln(1+\cos x) + \log k}$$

$$y = \frac{k}{1+\cos x}$$

$\therefore$ $y = K(\operatorname{Cosec}^2 x + \operatorname{Cosec} x \operatorname{Cot} x)$

and $y = \frac{K}{1+\operatorname{Cos} x}$ are required solutions.

Q4. Solve the differential equation $e^{3x}\left(\frac{dy}{dx}-1\right) + \left(\frac{dy}{dx}\right)^3 e^{2y} = 0$

sol. Let $\frac{dy}{dx} = p$. then

$e^{3x}(p-1) + p^3 e^{2y} = 0$

Multiply the equation by e^y

$e^{3x}(pe^y - e^y) + p^3 e^{3y} = 0$ — (1)

Let $e^x = u \Rightarrow e^x dx = du$

$e^y = v \Rightarrow e^y dy = dv$

$\therefore \frac{dv}{dx} = \frac{e^y dy}{e^x dx} = e^{y-x}\left(\frac{dy}{dx}\right) = pe^{y-x} = p'$

$\therefore$ $\boxed{p = p' e^{x-y}}$ substituting this in (1)

$e^{3x}(p'e^{x-y} \cdot e^y - e^y) + p'^3 e^{3x-3y} \cdot e^{3y} = 0$

$\Rightarrow e^{3x}(p'e^x - e^y) + p'^3 e^{3x} = 0$

$\Rightarrow p'e^x - e^y + p'^3 = 0$

$\Rightarrow p'u - v + p'^3 = 0$

$\Rightarrow$ $\boxed{v = p'u + p'^3}$ Clairaut's form

$\therefore$ solution is

$v = cu + c^3$

$\Rightarrow$ $\boxed{e^y = ce^x + c^3}$ is the solution.

Q Solve $\frac{d^2y}{dx^2} + 4y = \tan 2x$ using method of variation of parameters.

sol. The homogeneous part of eqn is $\frac{d^2y}{dx^2} + 4y = 0$

$\therefore$ Auxiliary eqn is $m^2 + 4 = 0$

$\Rightarrow m = \pm 2i$

Hence, C.F. $= C_1 \operatorname{Cos} 2x + C_2 \operatorname{Sin} 2x$

$\therefore$ $u = \cos 2x$ & $v = \sin 2x$ are solutions of homogeneous equation.

$$W = \begin{vmatrix} u & u' \\ v & v' \end{vmatrix} = \begin{vmatrix} \cos 2x & -2\sin 2x \\ \sin 2x & 2\cos 2x \end{vmatrix} = 2\cos^2 2x + 2\sin^2 2x$$
$$= 2(\cos^2 2x + \sin^2 2x)$$
$$\therefore \quad W = 2 \neq 0$$

$\therefore$ u & v are independent.

Variation of Parameters

Let $y_p = Au + Bv$ where A & B are parameters of x. & $R = \tan 2x$.

then,
$$A = -\int \frac{vR}{W}\,dx = -\int \frac{(\sin 2x)(\tan 2x)}{2}\,dx$$
$$= -\int \frac{\sin^2 2x}{2\cos 2x}\,dx = -\frac{1}{2}\int \frac{1-\cos^2 2x}{\cos 2x}\,dx$$
$$= \frac{1}{2}\int (\cos 2x - \sec 2x)\,dx$$
$$= \frac{1}{2}\left[\frac{\sin 2x}{2} - \frac{1}{2}\log|\sec 2x + \tan 2x|\right]$$
$$= \frac{1}{4}\left[\sin 2x - \log|\sec 2x + \tan 2x|\right]$$

And
$$B = \int \frac{uR}{W} = \int \frac{(\cos 2x)(\tan 2x)}{2}\,dx = \int \frac{\sin 2x}{2}\,dx$$
$$= -\frac{\cos 2x}{4}$$

$$\therefore y_p = Au + Bv = (\cos 2x)\left[\frac{1}{4}\sin 2x - \frac{1}{4}\log|\sec 2x + \tan 2x|\right]$$
$$+ (\sin 2x)\left(-\frac{\cos 2x}{4}\right)$$
$$= \frac{1}{4}\sin 2x\cos 2x - \frac{\cos 2x}{4}\log|\sec 2x + \tan 2x| - \frac{1}{4}\sin 2x\cos 2x$$
$$= -\frac{\cos 2x}{4}\log|\sec 2x + \tan 2x|$$

$\therefore$ Solution of $y = C.F. + y_p = C_1\cos 2x + C_2\sin 2x - \frac{\cos 2x}{4}\log|\sec 2x + \tan 2x|$

$$\boxed{y = C_1\cos 2x + C_2\sin 2x - \frac{\cos 2x}{4}\log|\sec 2x + \tan 2x|}$$