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Q) $f : \mathbb{R} \longrightarrow \mathbb{R}$

$f(x+y) = f(x) \times f(y)$; $x, y \in \mathbb{R}$

and $f(x) \neq 0$ for $x \in \mathbb{R}$

show that $f'(x) = f(x)$ for all $x \in \mathbb{R}$

given that $f'(0) = f(0)$ and function

is differentiable for all $x \in \mathbb{R}$

Soln: $f(x+y) = f(x)f(y)$

$\Rightarrow f(0+0) = f(0) \cdot f(0)$

$\Rightarrow f(0)\,[f(0)-1] = 0$

$\Rightarrow f(0) = 0$ or $f(0) = 1$

If $f(0) = 0$

$\Rightarrow f(x+0) = f(x)f(0)$

$\Rightarrow f(x) = 0$

So, $f'(x) = 0 = f(x)$ for all $x \in \mathbb{R}$

If $f(0) = 1$:

Then $f(a-a) = f(a)f(-a)$

$\Rightarrow f(-a) = [f(a)]^{-1}$

and $f(m) = f(\underbrace{1 + \dots + 1}_{\leftarrow m \text{ times} \rightarrow})$

$= [f(1)]^m$

$= a^m$ where $a = f(1)$

Then $f(-m) = a^{-m}$

So, $f(n) = a^n$ for $n \in \mathbb{N}$

Now, $f(\frac{1}{q} + \dots q \text{ times}) = [f(\frac{1}{q})]^q$

$\Rightarrow f\left(\frac{1}{q}\right) = f(1)^{1/q}$

So, $f\left(\frac{p}{q}\right) = f(1)^{p/q}$

$\Rightarrow f(x) = a^x$ for $x \in \mathbb{Q}$

Since $\mathbb{Q}$ [illegible] the set of $\mathbb{R}$ is dense

$\Rightarrow f(x) = a^x$ for $x \in \mathbb{R}$

So, we get $f(x) = a^x$

$\Rightarrow f'(x) = a^x \log a$

$f'(0) = a^0 \log a = \log a$

$f(0) = a^0 = 1$

$\because f'(0) = f(0)$

$\Rightarrow \log a = 1$

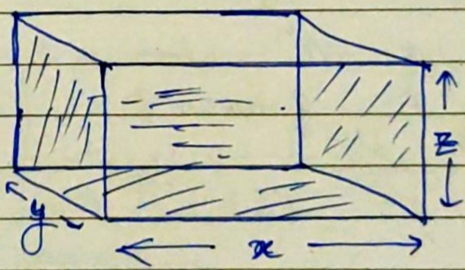
$\Rightarrow a = e$

So, $f(x) = e^x$

$\Rightarrow f'(x) = e^x$

Q) Rectangular box open at the top is to have a surface area of 12 sq units. Find the dimensions of the box so that volume is maximum.

Soln :



Surface Area $= xy + 2xz + 2yz = 12$ — (A)

Let $V = xyz$

Consider

$F = xyz + \lambda(xy + 2xz + 2yz - 12)$

Applying Lagrange's Method of Multipliers:

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$$

$\Rightarrow \quad yz + \lambda(y + 2z) = 0 \quad$ — (I)

$xz + \lambda(x + 2z) = 0 \quad$ — (II)

$xy + \lambda(2x + 2y) = 0 \quad$ — (III)

(I) $\times x$ + (II) $\times y$ + (III) $\times z$

$\Rightarrow 3V + \lambda(2xy + 4xz + 4yz) = 0$

$\Rightarrow 3V + \lambda(24) = 0$

$\Rightarrow V + 8\lambda = 0$

So, $\lambda = -\frac{V}{8} \quad$ — (IV)

From (I), (II), (III), (IV)

$$\frac{y + 2z}{yz} = \frac{x + 2z}{xz} = \frac{2x + 2y}{xy} = \frac{V}{8}$$

$$\Rightarrow \frac{1}{z} + \frac{2}{y} = \frac{1}{z} + \frac{2}{x} = \frac{2}{y} + \frac{2}{x} = \frac{V}{8}$$

The first two equations give us

$$\frac{2}{x} = \frac{2}{y} \Rightarrow x = y$$

Using first and third terms:

$$\frac{4}{x} = \frac{1}{z} + \frac{2}{x} \Rightarrow \frac{1}{z} = \frac{2}{x} \Rightarrow x = 2z$$

So, $x = y = 2z$

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Putting $x = y = 2z$ in (A)

$(2z)^2 + 2(2z)z + 2(2z)z = 12$

$\Rightarrow 4z^2 + 4z^2 + 4z^2 = 12$

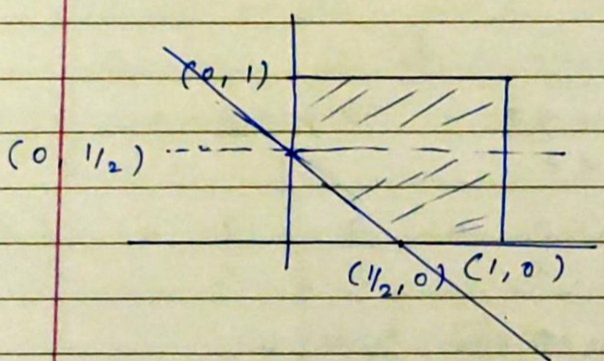
$\Rightarrow z = 1$

So, $x = y = 2$

And Volume $= 2 \times 2 \times 1 = 4$ cube units

Q) $\iint_R (x - y + 1)\, dx\, dy$

R is the region inside unit square given by $x + y \geq 1/2$



So,

$$I = \int_0^{1/2} \int_{\frac{1}{2}-y}^{1} (x - y + 1)\, dx\, dy$$

$$+ \int_{1/2}^{1} \int_0^{1} (x - y + 1)\, dx\, dy$$

$$\Rightarrow I = \int_0^{1/2} \left[\frac{x^2}{2} - x(y-1)\right]\Bigg|_{\frac{1}{2}-y}^{1} dy$$

$$+ \int_{1/2}^{1} \left[\frac{x^2}{2} - x(y-1)\right]_0^1 dy$$

$$\Rightarrow I = \int_0^{1/2} \left[\left(\frac{1}{2} - y + 1\right) - \frac{1}{2}\left(\frac{1}{2} - y\right)^2 + \left(\frac{1}{2} - y\right)(y-1)\right] dy$$

$$+ \int_{1/2}^{1} \left(\frac{1}{2} - y + 1\right) dy$$

$$\Rightarrow I = \int_0^1 \left(\frac{3}{2} - y\right) dy - \frac{1}{2}\int_0^{1/2} \left(y - \frac{1}{2}\right)^2 dy$$

$$- \frac{1}{2}\int_0^{1/2} (y-1)(2y-1)\, dy$$

$$\Rightarrow I = \left[\frac{3}{2}y - \frac{y^2}{2}\right]_0^1 - \frac{1}{2}\, \frac{(y - 1/2)^3}{3}\Bigg|_0^{1/2}$$

$$- \frac{1}{2}\int_0^{1/2} (2y^2 - 3y + 1)\, dy$$

$$\Rightarrow I = \cancel{R}\ 1 + \frac{1}{6} \times \frac{1}{8} - \frac{5}{48} = \frac{11}{12}$$

IFoS - Real - 2011

Q) Determine whether

$f(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$

is Riemann Integrable on $[0, 1]$

Soln: $\sin \frac{1}{x}$ is continuous on $(0, 1]$

$\Rightarrow x \sin \frac{1}{x}$ is continuous on $(0, 1]$

Also $\lim_{x \to 0} x \sin \frac{1}{x} = 0$

And $\lim_{x \to 0} \cos \frac{1}{x}$ doesn't exist

But $\cos \frac{1}{x}$ is continuous on $(0, 1]$

So, $f(x)$ is continuous on $[0, 1]$ except at $x = 0$, so

$f(x)$ is Riemann integrable on $[0, 1]$

$$\int_0^1 f(x)\, dx = \left[x^2 \sin \frac{1}{x} \right]_0^1$$

$$= \sin 1$$

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Q)
$$f(x) = \begin{cases} \dfrac{1}{2^t} ; & \dfrac{1}{2^{t+1}} < x \leq \dfrac{1}{2^t} \quad t = 0, 1, 2, \ldots \\ 0 ; & x = 0 \end{cases}$$

Is f Riemann Integrable on $[0,1]$?
If yes, evaluate $\int_0^1 f \, dx$

Soln:
$$f(x) = \begin{cases} 1 ; & \frac{1}{2} < x \leq 1 \\ \frac{1}{2} ; & \frac{1}{2^2} < x \leq \frac{1}{2} \\ \vdots & \\ 0 & \end{cases}$$

So, f is monotonically decreasing function on $[0, 1]$ which is discontinuous at $\left\{ \frac{1}{2}, \frac{1}{2^2}, \ldots \frac{1}{2^t} \right\}$ and this set has a finite limit point $\{0\}$
$\therefore$ $f(x)$ is Riemann Integrable.

Now,
$$\int f \, dx = \int_{1/2}^{1} (1) \, dx + \int_{1/2^2}^{1/2} \left(\frac{1}{2}\right) dx + \int_{1/2^3}^{1/2^2} \left(\frac{1}{2^2}\right) dx + \ldots \infty \text{ terms}$$

$$= 1 - \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2^2}\right) + \frac{1}{2^2}\left(\frac{1}{2^2} - \frac{1}{2^3}\right) + \ldots \infty \text{ terms}$$

$$= \left(1 + \frac{1}{2^2} + \frac{1}{2^4} + \ldots\right) - \frac{1}{2}\left(1 + \frac{1}{2^2} + \frac{1}{2^4} + \ldots\right)$$

$$= \frac{1}{2}\left(1 + \frac{1}{2^2} + \frac{1}{2^4} + \ldots\right)$$

$$= \frac{1}{2} \times \frac{1}{1 - 1/4} = \frac{1}{\cancel{2}} \times \frac{\cancel{4}\,2}{3} = \frac{2}{3}$$

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Q) Examine the convergence of

$\int_0^\infty \frac{dx}{(1+x)\sqrt{x}}$ and evaluate, if possible.

Soln: For $f(x) = \frac{1}{(1+x)\sqrt{x}}$

$$\int_0^\infty f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^\infty f(x)\,dx$$

Consider $\int_0^1 \frac{1}{(1+x)\sqrt{x}}\,dx$

Taking $g(x) = \frac{1}{x^{1/2}}$

Then $\lim_{x\to 0} \frac{f(x)}{g(x)} = \lim_{x\to 0} \frac{1}{1+x} = 1$

and $\int_0^1 g(x)\,dx$ is convergent

$\Rightarrow \int_0^1 f(x)\,dx$ is convergent

Consider $\int_1^\infty \frac{1}{(1+x)\sqrt{x}}\,dx$

Take $g(x) = \frac{1}{x^{3/2}}$

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^{3/2}}{x^{1/2}+x^{3/2}}$$

$$= \lim_{x\to\infty} \frac{1}{(1+1/x)} = 1$$

and $\int_1^\infty g(x)\,dx$ is convergent

$\Rightarrow \int_1^\infty f(x)\,dx$ is convergent

So, $\int_0^\infty \frac{dx}{(1+x)\sqrt{x}}$ is convergent.

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Now, $I = \int_0^\infty \frac{dx}{(1+x)\sqrt{x}}$

Put $x = \tan^2\theta$

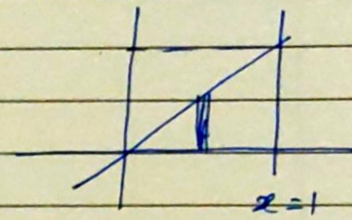
$dx = 2\tan\theta\sec^2\theta\, d\theta$

$\Rightarrow I = \int_0^{\pi/2} \frac{2\tan\theta\sec^2\theta\, d\theta}{\sec^2\theta \cdot \tan\theta}$

$\Rightarrow I = 2\int_0^{\pi/2} d\theta = \pi$

Q) Evaluate $\iint \sqrt{4x^2 - y^2}\, dx\, dy$ over the triangle $y = 0,\ x = 1,\ y = x$

Soln:
$$\int_0^1 \int_0^x \sqrt{4x^2 - y^2}\, dy\, dx$$



Let $I_1 = \int_0^x \sqrt{4x^2 - y^2}\, dy$

Using $\int \sqrt{a^2 - y^2}\, dy = \frac{y}{2}\sqrt{a^2 - y^2} + \frac{a^2}{2}\sin^{-1}\frac{y}{a}$

$$\Rightarrow \int_0^x \sqrt{4x^2 - y^2}\, dy = \left[\frac{2x}{2}\sqrt{4x^2 - y^2} + 2x^2 \sin^{-1}\frac{y}{2x}\right]_0^x$$

$$= \frac{x^2}{2}\sqrt{3} + 2x^2\frac{\pi}{6}$$

So,
$$\int_0^1 \int_0^x \sqrt{4x^2 - y^2}\, dy\, dx$$

$$= \int_0^1 \left(\frac{\sqrt{3}}{2}x^2 + \frac{\pi}{3}x^2\right) dx$$

$$= \left(\frac{\pi}{3} + \frac{\sqrt{3}}{2}\right)\frac{1}{3}$$