1

Areas of Curves
(Quadrature)

§ 1. Definition.

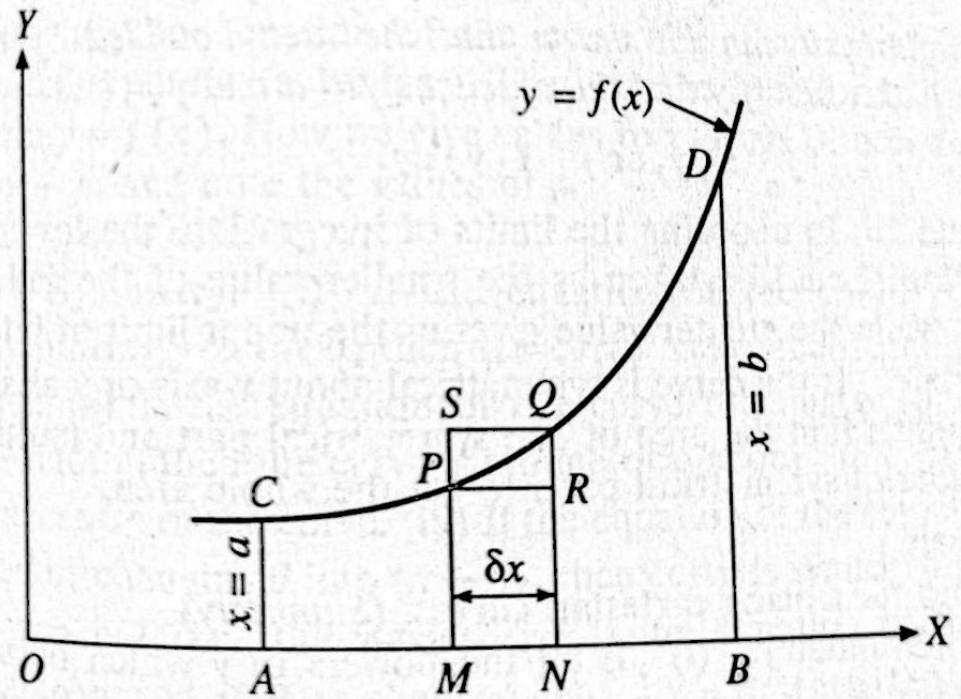
The process of finding the area of any bounded portion of a curve is called **quadrature.**

§ 2. Areas of curves given by cartesian equations.

If $f(x)$ is a continuous and single valued function of x, then the area bounded by the curve $y = f(x)$, the axis of x and the ordinates $x = a$ and $x = b$ is

$$\int_a^b y\,dx, \text{ or } \int_a^b f(x)\,dx.$$

Proof. Let CD be the arc of the curve $y = f(x)$ and AC and BD be the two ordinates $x = a$ and $x = b$.



Consider $P(x, y)$ and $Q(x + \delta x, y + \delta y)$, the two neighbouring points on the curve. Draw PM and QN perpendiculars to the axis of x, then

$$PM = y,\ QN = y + \delta y \text{ and } MN = \delta x.$$

Draw PR and QS perpendiculars to NQ and MP produced respectively. The area $AMPC$ depends upon the position of P on the curve. Let A denote the area $AMPC$ and $A + \delta A$ be the area $ANQC$. Then the area $MNQP$ = area $ANQC$ − area $AMPC$

$$= A + \delta A - A = \delta A.$$

But clearly this area δA (*i.e.*, the area $MNQP$) lies in magnitude between the areas of the rectangles $MNRP$ and $MNQS$.

Thus, we have

Area of the rectangle $MNQS > \delta A >$ area of the rectangle $MNRP$

i.e., $$(y + \delta y)\,\delta x > \delta A > y\,\delta x$$

or $$y + \delta y > \frac{\delta A}{\delta x} > y.$$

Now as $Q \to P$, $\delta x \to 0$ and $\delta y \to 0$. Therefore we have

$$\frac{dA}{dx} = y = f(x), \quad \text{or} \quad dA = y\,dx.$$

Integrating both sides between the limits $x = a$ and $x = b$, we have

$$\int_{x=a}^{x=b} dA = \int_a^b y\,dx$$

or $$\Big[A\Big]_{x=a}^{x=b} = \int_a^b y\,dx$$

or (Area A when $x = b$) $-$ (Area A when $x = a$) $= \int_a^b y\,dx$

or Area $ABDC - 0 = \int_a^b y\,dx$

or **Area ABDC** $= \int_a^b \mathbf{y\,dx} = \int_a^b \mathbf{f(x)\,dx}$.

Similarly, it can be shown that the area bounded by the curve $x = f(y)$, *the axis of* y *and the abscissae* $y = a$ *and* $y = b$ *is*

$$\int_a^b \mathbf{x\,dy}, \text{ or } \int_a^b \mathbf{f(y)\,dy}.$$

Note 1. In choosing the limits of integration, the lower limit of integration should be taken as the smaller value of the independent variable while the greater value gives us the upper limit of integration.

Note 2. If the curve is symmetrical about x-axis or y-axis or both, then we shall find the area of one symmetrical part and multiply it by the number of symmetrical parts to get the whole area.

Remember.

1. Rules for tracing cartesian curves. (*Summary*)

(a) Symmetry. (i) If all the powers of y which occur in the equation of the curve are even, the curve is symmetrical about the axis of x. **(ii)** If all the powers of x are even the curve is symmertical about the axis of y. **(iii)** If the powers of x and y are all even, the curve is symmetrical about both the axes. **(iv)** If on interchanging x and y the equation of the curve remains unchanged, then the curve is symmetrical about the straight line $y = x$. **(v)** If on putting $-x$ for x and $-y$ for y the equation of the curve is unaltered, then there is symmetry in opposite quadrants.

(b) **Location at the orign.** Observe whether the curve passes through the orign or not. If the point $(0, 0)$ satisfies the equation of the curve, it passes through the origin. If the curve passes through the origin, then the tangents at the origin are obtained by equating to zero the lowest degree terms occurring in the equation of the curve.

(c) **Some points on the curve.** Find the points where the curve cuts the co-ordinate axes. In the equation of the curve, put $y = 0$ and solve for x. Again put $x = 0$ and solve for y. Thus we get the points where the curve cuts the coordinate axes. Also find out the tangents at these points. Giving other suitable values to x find out some other points on the curve.

(d) **Asymptotes parallel to the axes.** The asymptotes parallel to the x-axis can be obtained by equating to zero the coefficient of the highest power of x in the equation of the curve. Similarly, the asymptotes parallel to the y-axis can be obtained by equating to zero the coefficient of the highest power of y in the equation of the curve.

(e) **Points of inflexion.** While tracing the curve, if it appears that the curve possesses some points of inflexion, then they can be clearly located by putting (d^2y/dx^2) or (d^2x/dy^2) equal to zero and solving the resulting equation.

(f) Solve the equation of the curve for y or x whichever is convenient. Suppose we solve for y. Then put the equation of the curve in the form $y = f(x)$. Now we give values to x from 0 to ∞ and then from 0 to $-\infty$ and note the values of y.

II. Tracing of a polar curve.

(a) **Symmetry.** (i) If the equation of the curve does not change by putting $-\theta$ for θ, then the curve is symmetrical about the initial line. (ii) If the equation of the curve remains unchanged by putting $-r$ for r, then the curve is symmetrical about the pole and the pole is the centre of the curve. (iii) If the equation of the curve remains unchanged by changing θ into $(\pi - \theta)$, then curve is symmetrical about the line $\theta = \pi/2$ (*i.e.*, the y-axis). (iv) If the equation of the curve remains unchanged if θ is changed into $\frac{1}{2}\pi - \theta$, then the curve is symmetrical about the line $\theta = \pi/4$.

(b) The curve will pass through the pole if for some value of θ the value of r comes out to be zero. Also if $r = 0$ when $\theta = \alpha$, then usually the line $\theta = \alpha$ will be a tangent to the curve at the pole.

We should find the values of θ for which $r = 0$, or r is maximum, or r is minimum, or $r \to \infty$.

If r vanishes for more than one values of θ, the curve is said to have a loop. The limits for a loop are obtained by finding two consecutive values of θ for which r is zero.

(c) Plot certain points on the curve. For this, give certain values to θ and find the corresponding values of r.

For this we must remember the following table :

$\theta =$	0°	30°	60°	90°	120°	150°	180°	210°	240°	270°
$\sin\theta =$	0	$\frac{1}{2}$	$\sqrt{3}/2$	1	$\sqrt{3}/2$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\sqrt{3}/2$	-1
$\cos\theta =$	1	$\sqrt{3}/2$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$-\sqrt{3}/2$	-1	$-\sqrt{3}/2$	$-\frac{1}{2}$	0

(d) If the curve possesses an infinite branch we find its asymptotes.

(e) Find $\tan\phi = r\,(d\theta/dr)$. This gives the direction of the tangent at any point on the curve.

Solved Examples

Ex. 1. *Find the area bounded by the axis of x, and the following curves and the given ordinates :*

(i) $y = c\cosh(x/c)$; $x = 0, x = a$.

(ii) $y = \sin^2 x$; $x = 0, x = \frac{1}{2}\pi$.

(iii) $y = \log x$; $x = a, x = b\ (b > a > 1)$.

(iv) $xy = c^2$; $x = a, x = b, (a > b > 0)$.

Sol. (i) The required area $= \int_0^a y\,dx$,

($\because$ limits are from $x = 0$ to $x = a$)

$$= \int_0^a c\cosh\left(\frac{x}{c}\right)dx = c^2\left[\sinh\frac{x}{c}\right]_0^a = c^2\sinh\frac{a}{c}.$$

(ii) The required area $= \int_0^{\pi/2} y\,dx = \int_0^{\pi/2} \sin^2 x\,dx$,

($\because y = \sin^2 x$)

$$= \tfrac{1}{2}\int_0^{\pi/2}(1-\cos 2x)\,dx = \tfrac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^{\pi/2}$$

$$= \tfrac{1}{2}\left[\frac{\pi}{2} - \tfrac{1}{2}\times 0\right] = \frac{\pi}{4}.$$

(iii) The required area $= \int_a^b y\,dx = \int_a^b \log x\,dx$

$$= \left[(\log x).x\right]_a^b - \int_a^b \frac{1}{x}.x\,dx, \text{ integrating by parts}$$

$= [b \log b - a \log a] - [b - a] = (b \log b - b) - (a \log a - a)$

$= b \log (b/e) - a \log (a/e)$, $[\because \log e = 1]$.

(iv) The required area $= \int_b^a y\,dx = \int_b^a (c^2/x)\,dx$, $[\because y = c^2/x]$

$= c^2 . \left[\log x\right]_b^a = c^2 (\log a - \log b) = c^2 \log (a/b)$.

Ex. 2. *Find the area bounded by the curve $y = x^3$, the y-axis and the lines $y = 1$ and $y = 8$.*

Sol. Here the curve is bounded between the axis of y and the lines parallel to x-axis. Hence the required area

$$= \int_{y=1}^{8} x\,dy \qquad \text{(Note)}$$

$$= \int_1^8 y^{1/3}\,dy, \qquad [\because x^3 = y \text{ or } x = y^{1/3}]$$

$$= \left[\frac{3}{4} y^{4/3}\right]_1^8 = \frac{3}{4}\left[8^{4/3} - 1^{4/3}\right] = \frac{3}{4}\left[2^4 - 1\right] = \frac{45}{4}.$$

Ex. 3 (a). *Find the area bounded by the ellipse*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

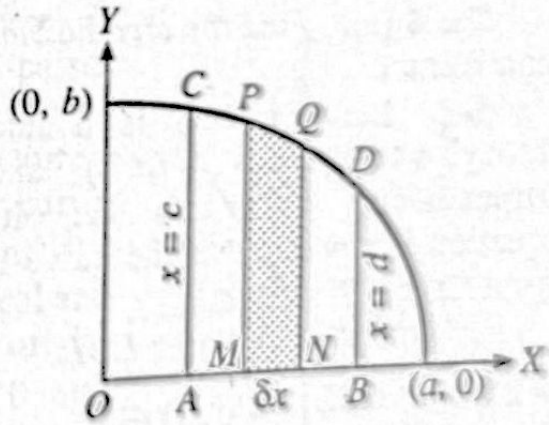
the ordinates $x = c$, $x = d$ and the x-axis.

Sol. Equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

or $$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

giving $$y = \frac{b}{a}\sqrt{(a^2 - x^2)}. \quad ...(1)$$



$\therefore$ the required area = the area $ABDC = \int_c^d y\,dx$

$$= \int_c^d \frac{b}{a}\sqrt{(a^2 - x^2)}\,dx,$$

from (1)

$$= \frac{b}{a}\left[\tfrac{1}{2} x \sqrt{(a^2 - x^2)} + \tfrac{1}{2} a^2 \sin^{-1}\left(\frac{x}{a}\right)\right]_c^d$$

$$= \frac{b}{2a}\left[d\sqrt{(a^2 - d^2)} - c\sqrt{(a^2 - c^2)} + a^2\left(\sin^{-1}\frac{d}{a} - \sin^{-1}\frac{c}{a}\right)\right].$$

Ex. 3 (b). *Find the area of the quadrant of an ellipse.* (Agra 1983; Rohilkhand 76)

Sol. See Fig. of Ex. 3 (a). Here the required area (*i.e.*, the area of a quadrant) lies between the limits $x = 0$ and $x = a$.

$\therefore$ area of the quadrant $= \int_0^a y\,dx$

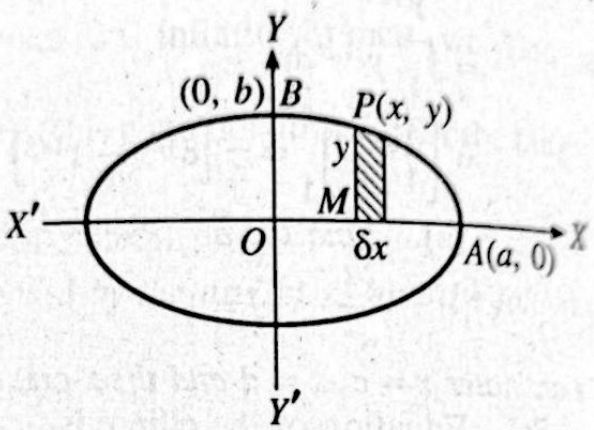
$= \frac{b}{a}\int_0^a \sqrt{(a^2-x^2)}\,dx,$ from (1) of Ex. 3 (a)

$= \frac{b}{a}\left[\frac{1}{2}x\sqrt{(a^2-x^2)} + \frac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right)\right]_0^a$

$= \frac{b}{a}\left[\frac{1}{2}(0-0) + \frac{1}{2}a^2\left(\frac{1}{2}\pi - 0\right)\right] = \frac{\pi ab}{4}.$

Ex. 3 (c). *Find the whole area of the ellipse* $(x^2/a^2) + (y^2/b^2) = 1$. **(Meerut 1982, 96; Bhopal 73; Kashmir 72)**

Sol. Clearly the area of an ellipse is 4 times the area of a quadrant.



$\therefore$ The required area of the ellipse

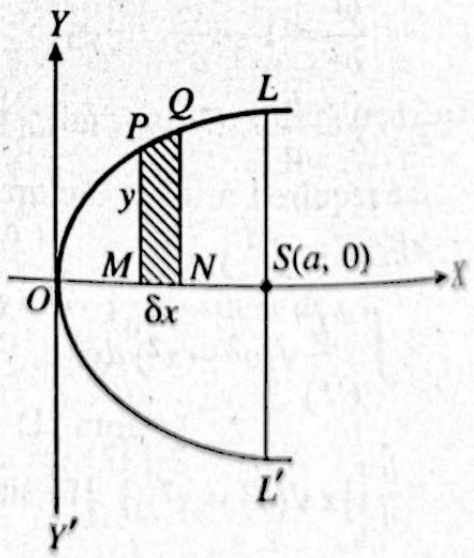
$= 4\int_0^a y\,dx$

$= 4.\frac{\pi ab}{4},$ from Ex. 3 (b)

$= \pi ab.$

Ex. 4 (a). *Find the area bounded by the parabola* $y^2 = 4ax$ *and its latus rectum.*

Sol. Latus rectum is a line through the focus $S(a, 0)$ and perpendicular to x-axis *i.e.*, its equation is $x = a$. Also the curve is symmetrical about x-axis.



$\therefore$ the required area LOL'

$= 2 \times$ area $OSL = 2.\int_0^a y\,dx$

$= 2\int_0^a \sqrt{(4ax)}\,dx,$

$[\because y^2 = 4ax, i.e., y = \sqrt{(4ax)}]$

$= 2\sqrt{(4a)}\left[\frac{2}{3}x^{3/2}\right]_0^a$

$= \frac{8}{3}\sqrt{(a)}.a^{3/2} = \frac{8}{3}a^2.$

Ex. 4 (b). *Show that the area cut off a parabola by any double ordinate is two third of the corresponding rectangle contained by that double ordinate and its distance from the vertex.*

Sol. Let the parabola be $y^2 = 4ax$. Also let $x = b$ be any double ordinate. Since the curve is symmetrical about x-axis, therefore the area cut off the parabola $y^2 = 4ax$ by the double ordinate $x = b$ is $2 \times$ (area included between the x-axis, $x = b$ and curve in the +ive quadrant).

$\therefore$ the required area $= 2\int_{x=0}^{b} y\,dx$

$$= 2\int_0^b \sqrt{(4ax)}\,dx, \qquad [\because y^2 = 4ax]$$

$$= 4\sqrt{(a)}\left[\frac{2}{3}x^{3/2}\right]_0^b = \frac{8}{3}\sqrt{(a)}.\,b^{3/2}. \qquad ...(1)$$

Again, at $x = b$, from $y^2 = 4ax$, we have

$$y^2 = 4ab \quad \text{or} \quad y = 2\sqrt{(ab)}.$$

$\therefore$ length of the double ordinate $= 2y = 2.2\sqrt{(ab)} = 4\sqrt{(ab)}$.

Now area of the rectangle contained by the double ordinate and its distance from the vertex $= 2y \times x = 2y.b$

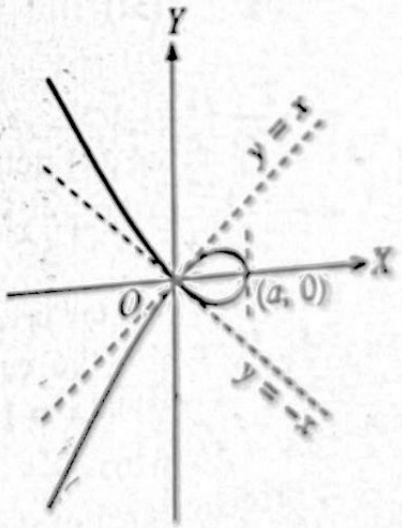
$$= 4\sqrt{(ab)}.\,b = 4a^{1/2}b^{3/2}.$$

Its two third $= \frac{2}{3}.4a^{1/2}b^{3/2} = \frac{8}{3}a^{1/2}b^{3/2}$

= area cut off a parabola by the double ordinate, from (1).

Ex. 5. *Trace the curve $ay^2 = x^2(a - x)$ and show that the area of its loop is $8a^2/15$.* (Meerut 1972; G.N.U. 75)

Sol. Tracing. (i) The curve is symmetrical about x-axis and passes through the origin.



(ii) Equating to zero the lowest degree terms in the equation of the curve, we get $y^2 - x^2 = 0$ *i.e.*, $y = \pm x$ as the tangents at origin and these being real and distinct the node is expected at the origin.

(iii) At $y = 0$, we get $x = 0$ and $x = a$ *i.e.*, the curve crosses the x-axis at $(0, 0)$ and $(a, 0)$. Also when $x > a$, y^2 is negative, *i.e.*, y is imaginary. Hence the curve does not exist for values of $x > a$. Also as x decreases from 0 to $-\infty$, y increases from 0 to ∞.

(iv) No asymptotes. Thus the shape of the curve is as shown in the figure. Clearly the loop is formed between $x = 0$ and $x = a$.

$\therefore$ required area of the loop $= 2\int_0^a y\,dx$,

[$\because$ curve is symmetrical about x-axis]

$$= 2\int_0^a \frac{x\sqrt{(a-x)}}{\sqrt{a}}\,dx, \text{ putting for } y \text{ from the equation of the curve}$$

$$= 2\int_0^{\pi/2} \frac{a\sin^2\theta\sqrt{(a)}.\cos\theta}{\sqrt{a}}\,2a\sin\theta\cos\theta\,d\theta,$$

[putting $x = a\sin^2\theta$, $dx = 2a\sin\theta\cos\theta\,d\theta$]

$$= 4a^2\int_0^{\pi/2}\sin^3\theta\cos^2\theta\,d\theta$$

$$= 4a^2\frac{\Gamma 2.\Gamma\frac{3}{2}}{2\Gamma\frac{7}{2}} = 4a^2.\frac{\frac{1}{2}\sqrt{\pi}}{2.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\sqrt{\pi}} = \frac{8}{15}a^2.$$

Ex. 6 (a). *Find the area of the loop of the curve*

$$3ay^2 = x(x-a)^2.$$

Sol. The curve is symmetrical about x-axis. Putting $y = 0$, we get $x = 0$ and $x = a$, *i.e.*, the loop is formed between $x = 0$ and $x = a$.

$\therefore$ required area of the loop

$= 2\int_0^a y\,dx,$ [$\because$ curve is symmetrical about x-axis].

Now for the portion of the loop lying in the first quadrant, y is +ive and x lies between 0 and a. Therefore for this portion of the loop, we have $y = \{1/\sqrt{(3a)}\}.\sqrt{(x)}.(a-x)$.

$\therefore$ required area of the loop

$$= 2\int_0^a \frac{(a-x).\sqrt{x}}{\sqrt{(3a)}}, \text{ putting for } y \text{ from the given equation of the curve}$$

$$= \frac{2}{\sqrt{(3a)}}\int_0^a (ax^{1/2} - x^{3/2})\,dx$$

$$= \frac{2}{\sqrt{(3a)}}\left[\frac{2}{3}ax^{3/2} - \frac{2}{5}x^{5/2}\right]_0^a = 8a^2/(15/\sqrt{3}).$$

Ex. 6 (b). *Find the area of the loop of the curve* $y^2 = x(x-1)^2$.

Sol. Here also the curve is symmetrical about x-axis. Putting $y = 0$, we get $x = 0$ and $x = 1$ *i.e.*, the loop is formed between $x = 0$ and $x = 1$.

$\therefore$ required area of the loop $= 2\int_0^1 y\,dx$

$= 2\int_0^1 (1-x).\sqrt{x}\,dx$, putting for y from the equation of the curve.

[Note that we have taken, $y = \sqrt{(x)}.(1-x)$]

$$= 2\int_0^1 (x^{1/2} - x^{3/2})\,dx = 2\left[\frac{2}{3}x^{3/2} - \frac{2}{5}x^{5/2}\right]_0^1$$

$$= 2\left[\frac{2}{3} - \frac{2}{5}\right] = \frac{8}{15}.$$

Ex. 7. *Find the area of a loop of the curve*

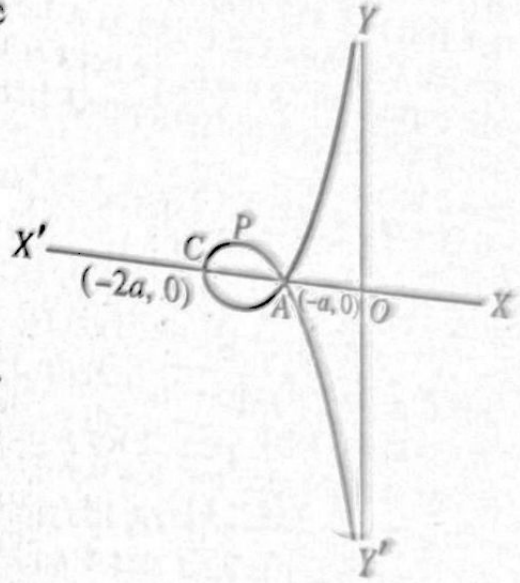
$$xy^2 + (x+a)^2(x+2a) = 0.$$

Sol. The curve is symmetrical about x-axis.

Putting $y = 0$, we get $x = -a$ and $x = -2a$.

The loop is formed between $x = -2a$ and $x = -a$.

To find the area of the loop, we first shift the origin to the point $(-a, 0)$. The equation of the curve then becomes



$$(x-a)y^2 + \{(x-a)+a\}^2 (x-a+2a) = 0$$

or $\quad y^2(x-a) + x^2(x+a) = 0$

or $\quad y^2 = \dfrac{x^2(a+x)}{a-x} \qquad \ldots(1)$

Note that the shifting of the origin only changes the equation of the curve and has no effect on its shape. Now the origin being at the point A, the new limits for the loop are $x = -a$ to $x = 0$.

$\therefore$ required area of the loop $= 2 \times$ area CPA

$= 2\int_{-a}^{0} y\,dx$, [the value of y to be put from (1)]

$$= 2\int_{-a}^{0} \left\{ -x\sqrt{\left(\frac{a+x}{a-x}\right)} \right\} dx,$$

[Note that in the equation (1), for the portion CPA, $y = -x\sqrt{\{(a+x)/(a-x)\}}$]

$= 2\displaystyle\int_{-a}^{0} \frac{-x.(a+x)}{\sqrt{(a^2-x^2)}}\,dx$, multiplying the numerator and the denominator by $\sqrt{(a+x)}$

$= 2\displaystyle\int_{\pi/2}^{0} \frac{-(-a\sin\theta)(a-a\sin\theta)}{a\cos\theta} \cdot (-a\cos\theta)\,d\theta,$

putting $x = -a\sin\theta$ and $dx = -a\cos\theta\,d\theta$

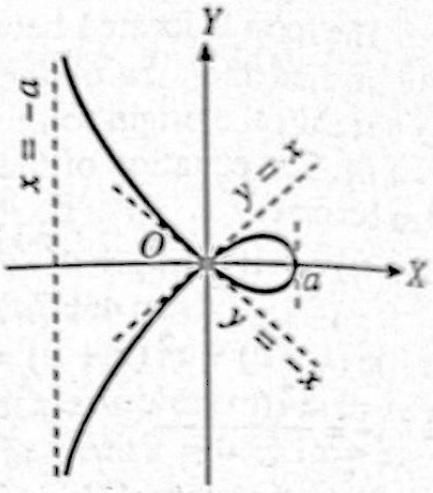
$= -2a^2\displaystyle\int_{\pi/2}^{0} (\sin\theta - \sin^2\theta)\,d\theta = 2a^2\int_0^{\pi/2} (\sin\theta - \sin^2\theta)\,d\theta$

$= 2a^2[1 - \frac{1}{2}\cdot\frac{1}{2}\pi]$, by Walli's formula

$= 2a^2(1 - \frac{1}{4}\pi)$.

Ex. 8. *Find the area (i) of the loop of the curve* $x(x^2+y^2) = a(x^2-y^2)$ or $y^2(a+x) = x^2(a-x)$.

(ii) *of the portion bounded by the curve and its asymptotes.* (Meerut 1983 S, 86 S, 93)



Sol. The curve is symmetrical about x-axis. The tangents at origin are $a(y^2 - x^2) = 0$ *i.e.*, $y = \pm x$. Since there are two real and distinct tangents at the origin, therefore the origin is a node on the curve. Putting $y = 0$, we get $x = 0$ and $x = a$ *i.e.*, the loop is formed between $x = 0$ and $x = a$.

$\therefore$ required area of the loop

$= 2\int_0^a y\,dx$, by symmetry

$$= 2\int_0^a x\sqrt{\left(\frac{a-x}{a+x}\right)}\,dx,$$

putting for y from the given equation of the curve

$$= 2\int_0^a \frac{x(a-x)}{\sqrt{(a^2-x^2)}}\,dx,$$ multiplying the numerator and the denominator by $\sqrt{(a-x)}$

$$= 2\int_0^{\pi/2} \frac{a\sin\theta\,(a - a\sin\theta)}{\sqrt{(a^2 - a^2\sin^2\theta)}}\,.\,a\cos\theta\,d\theta,$$ putting $x = a\sin\theta$ so that $dx = a\cos\theta\,d\theta$

$$= 2a^2\int_0^{\pi/2} \sin\theta\,(1 - \sin\theta)\,d\theta$$

$$= 2a^2\left[\int_0^{\pi/2} \sin\theta\,d\theta - \int_0^{\pi/2} \sin^2\theta\,d\theta\right]$$

$= 2a^2\,[1 - \frac{1}{2}.\frac{1}{2}\pi]$, by Walli's formula

$= 2a^2\,(1 - \frac{1}{4}\pi) = \frac{1}{2}a^2\,(4 - \pi)$.

(ii) The line $x = -a$ is the asymptote of the curve.

Now the area lying between the curve and its asymptote

$= 2\int_{-a}^0 y\,dx$, the value of y to be put from the equation of the curve

$$= 2\int_{-a}^0 -x\sqrt{\left(\frac{a-x}{a+x}\right)}\,dx,$$

[Note that for the arc of the curve lying in the second quadrant x is $-$ve and y is $+$ive so that $y = -x\sqrt{\{(a-x)/(a+x)\}}$ for this arc].

$$= 2\int_{-a}^0 \frac{-x\,(a-x)}{\sqrt{(a^2-x^2)}}\,dx$$

$$= 2\int_{\pi/2}^{0} \frac{-(-a\sin\theta)(a + a\sin\theta)}{\sqrt{(a^2 - a^2\sin^2\theta)}} \cdot (-a\cos\theta)\, d\theta,$$

putting $x = -a\sin\theta$ so that $dx = -a\cos\theta\, d\theta$

$$= -2a^2 \int_{\pi/2}^{0} \sin\theta\,(1 + \sin\theta)\, d\theta = 2a^2 \int_0^{\pi/2} (\sin\theta + \sin^2\theta)\, d\theta$$

$= 2a^2\,[1 + \frac{1}{2}\cdot\frac{1}{2}\pi]$, by Walli's formula.

$= 2a^2\,(1 + \frac{1}{4}\pi) = \frac{1}{2}a^2\,(4 + \pi)$.

Ex. 9. *Trace the curve $y^2(2a - x) = x^3$ and find the entire area between the curve and its asymptotes.*

(Meerut 1975, 87 S; G.N.U. 73; Gorakhpur 72)

Sol. Tracing of the curve $y^2(2a - x) = x^3$.

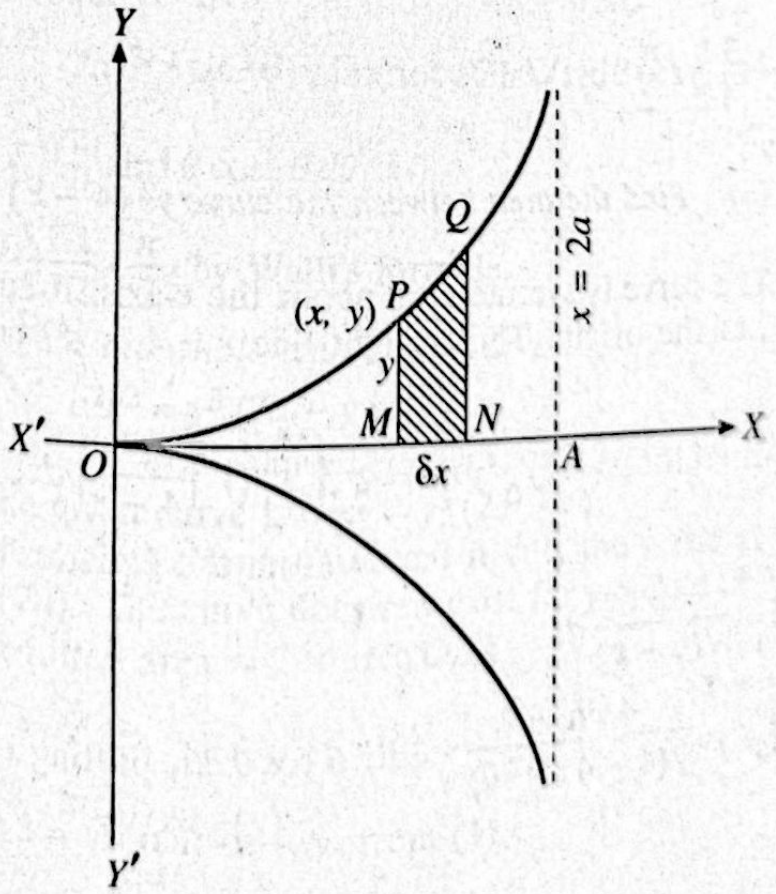
(i) Since in the equation of the curve the powers of y that occur are all even, therefore the curve is symmetrical about the axis of x.

(ii) The curve passes through the origin. Equating to zero the lowest degree terms in the equation of the curve, we get the tangents at the origin as $2ay^2 = 0$ *i.e.*, $y = 0$, $y = 0$ are two coincident tangents at the origin. Therefore the origin may be a cusp.

(iii) The curve cuts the coordinate axes only at the origin.

(iv) Solving the equation of the curve for y, we get

$$y^2 = \frac{x^3}{2a - x}.$$



When $x = 0, y^2 = 0$.

When $x \to 2a$, $y^2 \to \infty$. Therefore $x = 2a$ is an asymptote of the curve.

When $0 < x < 2a$, y^2 is +ive *i.e.*, y is real. Therefore the curve exists in this region.

When $x > 2a$, y^2 is –ive *i.e.*, y is imaginary. Therefore the curve does not exist in the region $x > 2a$.

When $x < 0$, y^2 is –ive. Therefore the curve does not exist in the region $x < 0$.

Combining all these facts, we see that the shape of the curve is as shown in the figure.

Now the required area $= 2 \times$ area in the first quadrant

$$= 2\int_0^{2a} y\,dx$$

$$= 2\int_0^{2a} \frac{x^{3/2}}{\sqrt{(2a-x)}}\,dx, \qquad \left[\because y^2 = \frac{x^3}{2a-x}\right]$$

Now put $x = 2a\sin^2\theta$, so that $dx = 4a\sin\theta\cos\theta\,d\theta$.

$\therefore$ the required area

$$= 2\int_0^{\pi/2} \frac{(2a\sin^2\theta)^{3/2}}{\sqrt{(2a - 2a\sin^2\theta)}} \cdot 4a\sin\theta\cos\theta\,d\theta$$

$$= 16a^2\int_0^{\pi/2} \frac{\sin^3\theta}{\cos\theta}\sin\theta\cos\theta\,d\theta = 16a^2\int_0^{\pi/2} \sin^4\theta\,d\theta$$

$$= 16a^2 \cdot \frac{3.1}{4.2} \cdot \frac{\pi}{2}, \text{ by Walli's formula}$$

$$= 3\pi a^2.$$

Ex. 9 (a). *Find the area between the curve $y^2(4-x) = x^2$ and its asymptote.* (Ranchi 1976)

Sol. The curve is symmetrical about the x-axis. It cuts the x-axis at $x = 0$ *i.e.*, at the origin. The straight line $x = 4$ is the asymptote of the curve.

$\therefore$ required area $= 2\int_0^4 y\,dx = 2\int_0^4 \sqrt{\left(\frac{x^2}{4-x}\right)}\,dx$, putting for y from the equation of the curve

$$= 2\int_0^4 \frac{x}{\sqrt{(4-x)}}\,dx$$

$$= 2\int_0^{\pi/2} \frac{4\sin^2\theta}{\sqrt{(4-4\sin^2\theta)}} \cdot 8\sin\theta\cos\theta\,d\theta, \text{ putting } x = 4\sin^2\theta$$

so that $dx = 8\sin\theta\cos\theta\,d\theta$

$$= 32 \int_0^{\pi/2} \sin^3\theta \, d\theta = 32 \cdot \frac{2}{3.1}, \text{ by Walli's formula}$$

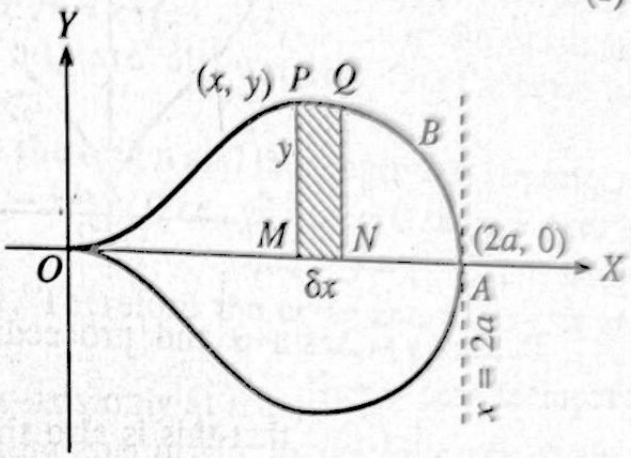
$$= \frac{64}{3} \text{ units of area.}$$

Ex. 10. *Find the whole area of the curve* $a^2y^2 = x^3(2a - x)$.

(Ranchi 1974; Meerut 69, 93P, 98)

Sol. The given curve is $a^2y^2 = x^3(2a - x)$. ...(1)

It is symmetrical about x-axis and it cuts the x-axis at the points $(0, 0)$ and $(2a, 0)$. The curve does not exist for $x > 2a$ and $x < 0$. Thus the curve consists of a loop lying between $x = 0$ and $x = 2a$.



$\therefore$ the required area

$= 2 \times$ area OBA

$$= 2\int_0^{2a} y\,dx$$

$$= 2\int_0^{2a} \frac{x^{3/2}\sqrt{(2a - x)}}{a}\,dx, \text{ from (1).}$$

Now put $x = 2a\sin^2\theta$ so that $dx = 4a\sin\theta\cos\theta\,d\theta$.

When $x = 0$, $\theta = 0$ and when $x = 2a$, $\theta = \frac{1}{2}\pi$.

$\therefore$ the required area

$$= \frac{2}{a}\int_0^{\pi/2} (2a)^{3/2}\sin^3\theta \,.\sqrt{(2a)}\,.\cos\theta\,.\,4a\sin\theta\cos\theta\,d\theta$$

$$= 32a^2\int_0^{\pi/2} \sin^4\theta\cos^2\theta\,d\theta$$

$$= 32a^2 . \frac{3.1.1}{6.4.2} \cdot \frac{\pi}{2}, \text{ by Walli's formula.}$$

Ex.11. *Find the whole area of the curve*

$$a^2x^2 = y^3(2a - y).$$

(Gorakhpur 1975, 70; Vikram 73; Indore 73; Ranchi 70)

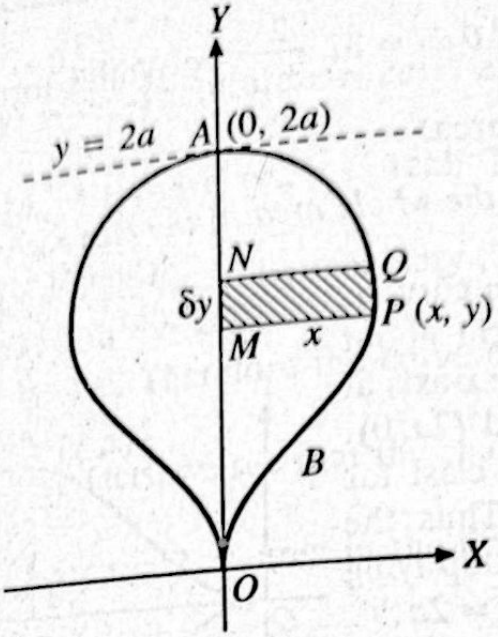
Sol. The given curve is $a^2x^2 = y^3(2a - y)$. ...(1)

It is symmetrical about y-axis and it cuts the y-axis at the points $(0,0)$ and $(0, 2a)$. The curve does not exist for $y > 2a$ and $y < 0$.

$\therefore$ the required area $= 2 \times$ area OBA

$$= 2\int_0^{2a} x\,dy$$

$$= 2\int_0^{2a} \frac{y^{3/2}\sqrt{(2a - y)}}{a}\,dy, \text{ from (1).}$$

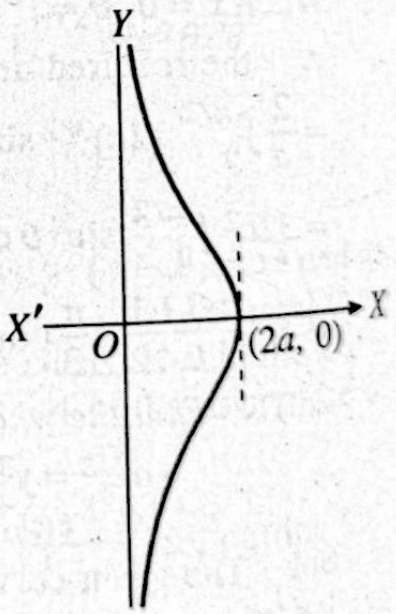


Putting $y = 2a \sin^2\theta$ and proceeding as in Ex. 10, we get the required area $= \pi a^2$.

(Note that this is also the area of a circle of radius a).

Ex. 12. *Find the area bounded by the curve $xy^2 = 4a^2(2a - x)$ and its asymptote.* **(Kanpur 1977; Gorakhpur 77, 73; Jiwaji 72)**

Sol. The given curve is symmetrical about the x-axis and cuts the x-axis at the point $(2a, 0)$.



Equating to zero the coefficient of highest power of y in the equation of the curve we get $x = 0$ *i.e.*, the y-axis as the asymptote of the curve parallel to y-axis.

Hence the required area $= 2\int_0^{2a} y\,dx$

$$= 2\int_0^{2a} \frac{2a\sqrt{(2a-x)}}{\sqrt{x}}\,dx,$$

[$\because$ from the given equation of the curve, $y^2 = 4a^2(2a - x)/x$].

Putting $x = 2a\sin^2\theta$ so that $dx = 4a\sin\theta\cos\theta\,d\theta$, we get the required area

$$= 4a\int_0^{\pi/2} \frac{\sqrt{(2a)}\cos\theta\,.4a\sin\theta\cos\theta\,d\theta}{\sqrt{(2a)}\,.\sin\theta}$$

$$= 16a^2\int_0^{\pi/2} \cos^2\theta\,d\theta$$

$$= 16a^2\,.\tfrac{1}{2}\,.\tfrac{1}{2}\pi, \text{ by Walli's formula}$$

$$= 4\pi a^2.$$

Ex. 12 (a). *Find the area enclosed by the curve $xy^2 = a^2(a-x)$ and y-axis.* **(Meerut 1973; Gorakhpur 71)**

Sol. Proceed exactly as in Ex. 12. Here also y-axis is the asymptote and in place of $2a$ we have a *i.e.*, replace $a/2$ in place of a.

The required area $= 4\pi.(a/2)^2 = \pi a^2$.

Ex. 13. *Trace the curve $a^2y^2 = a^2x^2 - x^4$ and find the whole area within it.* **(Magadh 1974; Lucknow 70; Meerut 81)**

Sol. The given curve is $a^2y^2 = x^2(a^2 - x^2)$. Since in the equation of the curve, the powers of x and y are all even, therefore the curve is symmetrical about both the axes.

The curve passes through the origin and the tangents at the origin are $a^2y^2 - a^2x^2 = 0$ *i.e.*, $a^2(y^2 - x^2) = 0$ *i.e.*, $y^2 - x^2 = 0$ *i.e.*, $y = \pm x$.

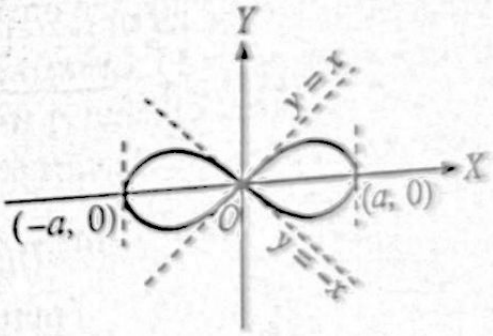
The curve cuts the axis of x where $y = 0$ *i.e.*, where $x^2(a^2 - x^2) = 0$ or $x = 0, \pm a$. Therefore the curve cuts the x-axis at $(0,0)$, $(a, 0)$ and $(-a, 0)$.

The curve intersects the y-axis only at the origin.

Tangent at $(a, 0)$. Shifting the origin to the point $(a, 0)$ the equation of the curve becomes

$$a^2y^2 = (x+a)^2\{a^2 - (x+a)^2\} = (x+a)^2(-x^2 - 2ax).$$

Equating to zero the lowest degree terms, we get $x = 0$ as the tangent at the new origin. Thus new y-axis is tangent at the new origin. Solving the equation of the curve for y, we get



$$y^2 = \frac{x^2(a^2 - x^2)}{a^2}.$$

When $x = 0$, $y^2 = 0$.

When $x = a$, $y^2 = 0$.

When $0 < x < a$, y^2 is +ive.

Therefore the curve exists in the region $0 < x < a$.

When $x > a$, y^2 is − ive. Therefore the curve does not exist in the region $x > a$.

Hence the curve is as shown in the figure and it consists of two equal loops.

By symmetry, the whole area within the curve

$= 4 \times$ area of half a loop

$$= 4\int_0^a y\,dx = 4\int_0^a \frac{x\sqrt{(a^2-x^2)}}{a}\,dx,$$

putting for y from the given equation of the curve

$$= 4\int_0^{\pi/2} \sin\theta\,.a\cos\theta\,.a\cos\theta\,d\theta,$$

putting $x = a\sin\theta$ so that $dx = a\cos\theta\,d\theta$

$$= 4a^2\int_0^{\pi/2} \cos^2\theta\sin\theta\,d\theta$$

$$= 4a^2\,.\frac{1}{3.1}.1, \text{ by Walli's formula } = \frac{4}{3}a^2.$$

Ex. 13 (a). *Find the area of the curve* $y^2 = x^2(4-x^2)$.

Sol. Proceed as in Ex. 13. The required area

$$= 4\int_0^2 x\sqrt{(4-x^2)}\,dx = \frac{32}{3}.$$

Ex. 13 (b). *Prove that the area of a loop of he curve* $a^4y^2 = x^4(a^2-x^2)$ *is* $\pi a^2/8$.

(Calicut 1975; Kashmir 74; Andhra 72)

Sol. The curve is symmetrical about both the axes. Putting $y = 0$ in the given equation of the curve, we get $x^4(a^2-x^2) = 0$ *i.e.*, $x = 0, x = \pm a$. Thus the above curve will have a loop between $x = 0$ and $x = a$. By symmetry, the area of a loop

$$= 2\int_0^a y\,dx = 2\int_0^a \frac{x^2\sqrt{(a^2-x^2)}}{a^2}\,dx,$$

putting for y from the given equation of the curve

$$= \frac{2}{a^2}\int_0^{\pi/2} a^2\sin^2\theta\sqrt{(a^2-a^2\sin^2\theta)}\,.\,a\cos\theta\,d\theta$$

putting $x = a\sin\theta$ so that $dx = a\cos\theta\,d\theta$

$$= 2a^2\int_0^{\pi/2} \sin^2\theta\cos^2\theta\,d\theta$$

$$= 2a^2\cdot\frac{1.1}{4.2}\cdot\tfrac{1}{2}\pi, \text{ by Walli's formula}$$

$$= \pi a^2/8.$$

Ex. 14. *Show that the whole area of the curve* $a^4y^2 = x^5(2a-x)$ *is to that of the circle whose radius is a, as 5 to 4.*

Sol. The given curve is symmetrical about x-axis. It passes through the origin and the tangents at the origin are $a^4y^2 = 0$ *i.e.*, $y^2 = 0$ *i.e.*, $y = 0, y = 0$.

The curve cuts the x-axis at the points $(0, 0)$ and $(2a, 0)$. It intersects the y-axis only at the origin. When $0 < x < 2a$, y^2 is +ive so that the curve exists in this region. When $x > 2a$, y^2 is –ive so that the

curve does not exist in this region. When $x < 0$, y^2 is –ive so that the curve does not exist in this region.

Thus the given curve consists of a loop lying between $x = 0$ and $x = 2a$. Hence the whole area of this curve

$$= 2\int_0^{2a} y\,dx = 2\int_0^{2a} \frac{x^{5/2}(2a-x)^{1/2}}{a^2}\,dx,$$

putting for y from the given equation of the curve

$$= 2\int_0^{\pi/2} \frac{(2a)^{5/2}\sin^5\theta\,.(2a)^{1/2}\cos\theta\,.4a\sin\theta\cos\theta\,d\theta}{a^2},$$

putting $x = 2a\sin^2\theta$ so that $dx = 2a\,.2\sin\theta\cos\theta\,d\theta$

$$= 64a^2\int_0^{\pi/2} \sin^6\theta\cos^2\theta\,d\theta$$

$$= 64a^2\frac{\Gamma(\frac{7}{2})\,\Gamma(\frac{3}{2})}{2\Gamma(5)} = 64a^2\frac{\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\sqrt{\pi}\,.\frac{1}{2}\sqrt{\pi}}{2.4.3.2.1} = \frac{5\pi a^2}{4}.$$

Also the area of the circle of ratius a is πa^2.

$$\therefore \quad \frac{\text{Area of the curve}}{\text{Area of the circle}} = \frac{5\pi a^2/4}{\pi a^2} = \frac{5}{4}.$$

Ex. 15. *Find the area between the curve $y^2(a-x) = x^3$ (cissoid) and its asymptotes. Also find the ratio in which the ordinate $x = a/2$ divides the area.*

Sol. The figure of the curve is similar to Ex. 9, page 11. Equating to zero the coefficient of the highest power of y, we get $a - x = 0$ *i.e.*, $x = a$ as an asymptote of the curve parallel to the y-axis.

Let A be the whole area between the curve and its asymptote. Then

$A = 2 \times$ (area in the first quadrant)

$$= 2\int_0^a y\,dx = 2\int_0^a \frac{x^{3/2}}{\sqrt{(a-x)}}\,dx,$$

putting for y from the given equation of the curve

$$= 2\int_0^{\pi/2} \frac{a^{3/2}\sin^3\theta\,.2a\sin\theta\cos\theta}{\sqrt{(a)}\,.\cos\theta}\,d\theta,$$

putting $x = a\sin^2\theta$ so that $dx = a\,.2\sin\theta\cos\theta\,d\theta$

$$= 4a^2\int_0^{\pi/2} \sin^4\theta\,d\theta = 4a^2.\frac{3.1}{4.2}\cdot\frac{\pi}{2} = \frac{3\pi a^2}{4}. \qquad \text{...(1)}$$

Now let A_1 be the area of the portion between $x = 0$ and $x = \frac{1}{2}a$. Then

$$A_1 = 2\int_0^{a/2} y\,dx = 2\int_0^{a/2} \frac{x^{3/2}}{\sqrt{(a-x)}}\,dx,$$

putting for y from the given equation of the curve

$$= 4a^2 \int_0^{\pi/4} \sin^4\theta \, d\theta, \text{ putting } x = a\sin^2\theta, \text{ etc.}$$

$$= a^2 \int_0^{\pi/4} (2\sin^2\theta)^2 \, d\theta = a^2 \int_0^{\pi/4} (1 - \cos 2\theta)^2 \, d\theta \qquad \text{(Note)}$$

$$= \tfrac{1}{2} a^2 \int_0^{\pi/2} (1 - \cos\phi)^2 \, d\phi,$$

[putting $2\theta = \phi$ so that $2\,d\theta = d\phi$ and adjusting the limits]

$$= \tfrac{1}{2} a^2 \int_0^{\pi/2} (1 - 2\cos\phi + \cos^2\phi) \, d\phi$$

$$= \tfrac{1}{2} a^2 \left[\int_0^{\pi/2} d\phi - 2\int_0^{\pi/2} \cos\phi \, d\phi + \int_0^{\pi/2} \cos^2\phi \, d\phi \right]$$

$$= \tfrac{1}{2} a^2 \left[\left\{ \phi \right\}_0^{\pi/2} - 2\left\{ \sin\phi \right\}_0^{\pi/2} + \frac{1}{2} \cdot \frac{\pi}{2} \right] = \tfrac{1}{2} a^2 \left[\tfrac{1}{2}\pi - 2.1 + \tfrac{1}{4}\pi \right]$$

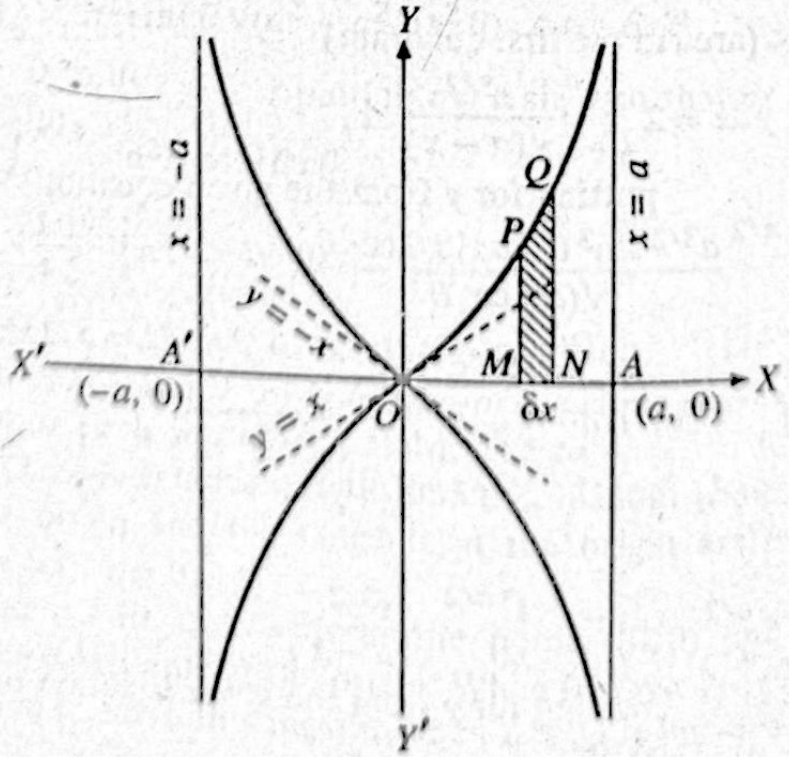
$$= \tfrac{1}{2} a^2 \left[\frac{3\pi}{4} - 2 \right] = \frac{1}{8} a^2 (3\pi - 8).$$

Now let A_2 be the area of the portion of the curve lying between $x = \frac{1}{2}a$ and $x = a$. Then $A_2 = A - A_1 = \frac{3}{4}\pi a^2 - \frac{1}{8}a^2(3\pi - 8)$ $= \frac{1}{8}a^2(3\pi + 8)$.

$$\therefore \quad \text{required ratio} = \frac{A_1}{A_2} = \frac{(a^2/8)(3\pi - 8)}{(a^2/8)(3\pi + 8)} = \frac{3\pi - 8}{3\pi + 8}.$$

**Ex. 16. *Find the whole area between the curve $x^2y^2 = a^2(y^2 - x^2)$ and its asymptotes.* (Meerut 1970 S)

Sol. The given curve is symmetrical about both the axes and passes through the origin. The tangents at $(0, 0)$ are given by $y^2 - x^2 = 0$ *i.e.*, $y = \pm x$ are the tangents at the origin.



Equating to zero the coefficient of the highest power of y (*i.e.*, of y^2) the asymptotes parallel to y-axis are given by $x^2 - a^2 = 0$ *i.e.*, $x = \pm a$.

The asymptotes parallel to x-axis are given by $y^2 + a^2 = 0$ which gives two imaginary asymptotes.

$\therefore$ the required area $= 4 \times$ area lying in the first quadrant

$$= 4\int_0^a y\,dx = 4\int_0^a \sqrt{\left(\frac{a^2x^2}{a^2 - x^2}\right)}\,dx,$$

[$\because$ from the equation of the given curve, $y^2 = a^2x^2/(a^2 - x^2)$]

$$= 4\int_0^a \frac{ax\,dx}{\sqrt{(a^2 - x^2)}} = -2a\int_0^a \frac{-2x\,dx}{\sqrt{(a^2 - x^2)}} = -2a\left[\frac{(a^2 - x^2)^{1/2}}{1/2}\right]_0^a$$

$$= -4a\,[0 - a] = 4a^2.$$

Ex.17. *Find the area of the loop of the curve*

$$y^2(a - x) = x^2(a + x).$$

Sol. The given curve is symmetrical about the x-axis and cuts the x-axis at the points $(0, 0)$ and $(-a, 0)$.

The tangents at $(0, 0)$ are $y^2 = x^2$ *i.e.*, $y = \pm x$.

Clearly there is a loop which lies between $x = -a$ and $x = 0$.

$\therefore$ the required area of the loop $= 2 \times$ area of the upper half of the loop

$$= 2\int_{-a}^0 y\,dx = 2\int_{-a}^0 -x \cdot \frac{\sqrt{(a + x)}}{\sqrt{(a - x)}}\,dx,$$

putting for y for the upper half of the loop from the given equation of the curve

$$= 2\int_{-a}^0 -x\frac{(a + x)}{\sqrt{(a^2 - x^2)}}\,dx,$$

multiplying the numerator and the denominator by $\sqrt{(a + x)}$.

Now put $x = -a\sin\theta$ so that $dx = -a\cos\theta\,d\theta$.

When $x = -a$, $\theta = \pi/2$ and when $x = 0$, $\theta = 0$.

$\therefore$ the required area

$$= 2\int_{\pi/2}^0 -(-a\sin\theta)\frac{(a - a\sin\theta)}{\sqrt{(a^2 - a^2\sin^2\theta)}} \cdot (-a\cos\theta)\,d\theta$$

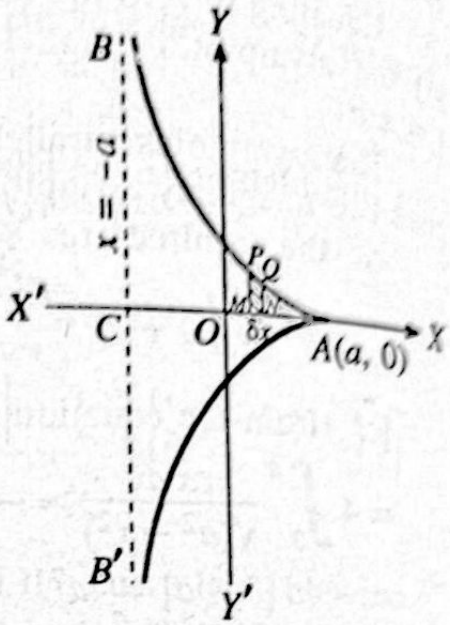
$$= -2a^2\int_{\pi/2}^0 \sin\theta\,(1 - \sin\theta)\,d\theta$$

$$= 2a^2\int_0^{\pi/2} (\sin\theta - \sin^2\theta)\,d\theta$$

$$= 2a^2\left[1 - \tfrac{1}{2}\cdot\tfrac{1}{2}\pi\right], \text{ by Walli's formula}$$

$$= 2a^2\left(1 - \tfrac{1}{4}\pi\right) = \tfrac{1}{2}a^2(4 - \pi).$$

Ex. 18. *Trace the curve* $4y^2(a+x) = (a-x)^3$ *and find the area between the curve and its asymptotes.*



Sol. The curve is symmetrical about x-axis. It does not pass through the origin. Putting $x = 0$ in the equation of the curve, we get $y = \pm a/2$ and putting $y = 0$ in it we get $x = a$. Thus the curve cuts the y-axis at the points $(0, \pm \frac{1}{2}a)$ and it cuts the x-axis at the point $(a, 0)$.

Equating to zero the coefficient of the highest power of y the asymptote parallel to y-axis is $a + x = 0$ *i.e.*, $x = -a$.

The equation of the curve can be written as $y^2 = (a-x)^3/\{4(a+x)\}$ which shows that for $x > a$, y is imaginary *i.e.*, the curve does not exist for $x > a$. Thus the shape of the curve is as shown in the figure. Now the required area $= 2 \times$ area lying above the x-axis

$$= 2\int_{-a}^{a} y\,dx = 2\int_{-a}^{a} \frac{(a-x)^{3/2}}{\sqrt{(a+x)}}\,dx,$$

putting for y from the equation of the curve

$$= \int_{-a}^{a} \frac{(a-x)^2}{\sqrt{(a^2-x^2)}}\,dx,$$

multiplying the Nr. and the Dr. by $\sqrt{(a-x)}$

$$= \int_{-\pi/2}^{\pi/2} \frac{(a - a\sin\theta)^2}{\sqrt{(a^2 - a^2\sin^2\theta)}} \cdot a\cos\theta\,d\theta,$$

putting $x = a\sin\theta$ so that $dx = a\cos\theta\,d\theta$

$$= a^2\int_{-\pi/2}^{\pi/2} (1-\sin\theta)^2\,d\theta$$

$$= a^2\int_{-\pi/2}^{\pi/2} (1 - 2\sin\theta + \sin^2\theta)\,d\theta$$

$$= a^2\int_{-\pi/2}^{\pi/2} \{1 - 2\sin\theta + \tfrac{1}{2}(1-\cos 2\theta)\}\,d\theta$$

$$= a^2\left[\frac{3}{2}\theta + 2\cos\theta - \tfrac{1}{4}\sin 2\theta\right]_{-\pi/2}^{\pi/2}$$

$$= a^2\left(\frac{3\pi}{2}\right) = \frac{3}{2}\pi a^2.$$

Ex. 18 (a). *Trace the curve* $y^2(a+x) = (a-x)^3$. *Find the area between the curve and its asymptote.*

Sol. Proceed as above. The required area $= 3\pi a^2$.

Area bounded by two curves. (Cartesian equations)

Ex. 19 (a). *Find the area included between the curves $y^2 = 4ax$ and $x^2 = 4by$* (Magadh 1973)

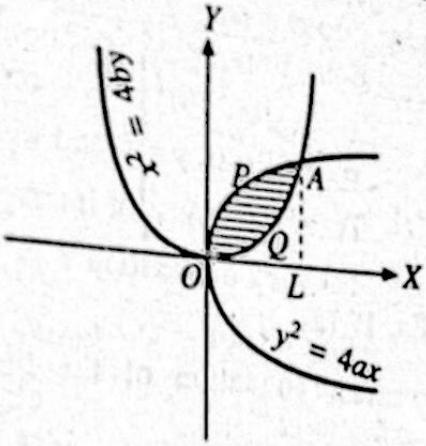
Sol. Solving the equations of the two given curves, we have

$$y^4 = 16a^2(4by) = 64a^2 by.$$

$$\therefore \quad y(y^3 - 64a^2b) = 0,$$

giving $y = 0, 4a^{2/3}b^{1/3}$.

When $y = 0$, $x = 0$ and when $y = 4a^{2/3}b^{1/3}$, $x = 4a^{1/3}b^{2/3}$.



Hence, the points of intersection of the given curves are $O(0, 0)$ and $A(4a^{1/3}b^{2/3}, 4b^{1/3}a^{2/3})$.

$\therefore$ the required area (*i.e.*, the shaded area)

$$= \text{area } OPAL - \text{area } OQAL$$

$$= \int_0^{4a^{1/3}b^{2/3}} y\,dx, \quad \text{from the curve } y^2 = 4ax$$

$$- \int_0^{4a^{1/3}b^{2/3}} y\,dx, \quad \text{from the curve } x^2 = 4by$$

(Note that for the required area x varies from 0 to $4a^{1/3}b^{2/3}$)

$$= \int_0^{4a^{1/3}b^{2/3}} \sqrt{(4ax)}\,dx - \int_0^{4a^{1/3}b^{2/3}} \left(\frac{x^2}{4b}\right) dx$$

$$= 2\sqrt{a}\left[\frac{2x^{3/2}}{3}\right]_0^{4a^{1/3}b^{2/3}} - \frac{1}{4b}\left[\frac{x^3}{3}\right]_0^{4a^{1/3}b^{2/3}}$$

$$= \frac{4\sqrt{a}}{3}[8\sqrt{(a)}\,.\,b] - \frac{1}{12b}(64\,ab^2)$$

$$= \frac{32}{3}ab - \frac{16}{3}ab = \frac{16}{3}ab.$$

**Ex. 19 (b). *Find the common area between the curves $y^2 = 4ax$ and $x^2 = 4ay$.* (Meerut 1984, 92, 94P, 95BP, 96P, 97; Bhopal 80; Agra 79, 73)

Sol. Proceed exactly as in the above Ex. Here $b = a$. Solving the given equations, we get the points of intersection as $(0, 0)$ and $(4a, 4a)$ and hence the required area

$$= \frac{16}{3}a^2.$$

By double integration. The required area

$$= \int_{x=0}^{4a} \int_{y=(x^2/4a)}^{2\sqrt{(ax)}} dx\,dy = \int_0^{4a} \left[y\right]_{(x^2/4a)}^{2\sqrt{(ax)}} dx$$

$$= \int_0^{4a} \left[2\sqrt{(ax)} - \frac{x^2}{4a}\right] dx = 2\sqrt{(a)} \left[\frac{2}{3}x^{3/2}\right]_0^{4a} - \frac{1}{12a}\left[x^3\right]_0^{4a}$$

$$= \frac{4}{3}\sqrt{(a)}.(4a)^{3/2} - \left[\frac{(4a)^3}{12a}\right] = \frac{32}{3}a^2 - \frac{16}{3}a^2 = \frac{16}{3}a^2.$$

Ex. 20. *The curves $y = 4x^2$ and $y^2 = 2x$, meet at the origin O and the point A, forming a loop. Show that the straight line OA divides the loop into two parts of equal area.*

Sol. Solving the equations of the two given curves, we have

$$16x^4 = 2x \quad \text{or} \quad 16x^4 - 2x = 0 \text{ i.e., } x = 0 \text{ and } x = \tfrac{1}{2}.$$

When $x = 0, y = 0$ and when $x = \frac{1}{2}, y = 1$.

Thus, the points of intersection are $(0, 0)$ and $(\frac{1}{2}, 1)$.

The area of the loop is just like the shaded portion in the Fig. of Ex. 19 (a).

The equation of the line OA is $y - 0 = \dfrac{1-0}{\frac{1}{2}-0}(x - 0)$ i.e., $y = 2x$.

Now the area between the parabola $y^2 = 2x$ and the line $y = 2x$

$= \int_0^{1/2} y\,dx$, from the parabola $y^2 = 2x$

$- \int_0^{1/2} y\,dx$, from the line $y = 2x$

$$= \int_0^{1/2} \sqrt{(2x)}\,dx - \int_0^{1/2} 2x\,dx = \left[\frac{2\sqrt{2}}{3}x^{3/2} - x^2\right]_0^{1/2}$$

$$= \frac{2\sqrt{2}}{3}\cdot\frac{1}{2\sqrt{2}} - \frac{1}{4} = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}. \qquad \text{...(1)}$$

Again the area between the parabola $y = 4x^2$ and the line $y = 2x$

$= \int_0^{1/2} y\,dx$, from the line $y = 2x$

$- \int_0^{1/2} y\,dx$, from the parabola $y = 4x^2$

$$= \int_0^{1/2} 2x\,dx - \int_0^{1/2} 4x^2\,dx = \left[x^2\right]_0^{1/2} - \frac{4}{3}\left[x^3\right]_0^{1/2}$$

$$= \frac{1}{4} - \frac{1}{6} = \frac{1}{12}, \qquad \text{...(2)}$$

From (1) and (2) we observe that the straight line OA divides the loop into two parts of equal area.

Ex. 21 (a). *Find the area included between $y^2 = 4ax$ and $y = mx$.*

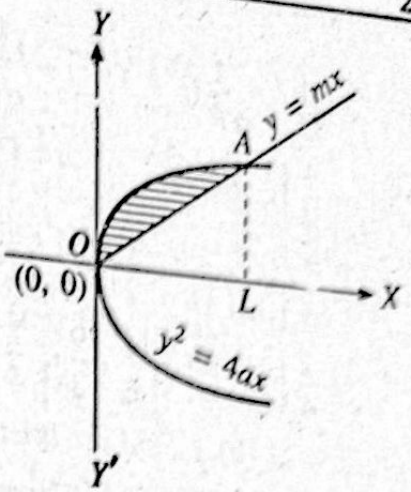
Sol. Solving the equation of the parabola $y^2 = 4ax$ and the equation of the line $y = mx$ for x, we get

$$m^2x^2 = 4ax$$

or $x(m^2x - 4a) = 0.$

This gives $x = 0$ or $x = 4a/m^2$.

Thus the two curves cut at the points where $x = 0$ and $x = 4a/m^2$.



$\therefore$ the required area $= \int_0^{4a/m^2} y\,dx$, from the curve $y^2 = 4ax$

$- \int_0^{4a/m^2} y\,dx$, from the st. line $y = mx$

$$= \int_0^{4a/m^2} \sqrt{(4ax)}\,dx - \int_0^{4a/m^2} mx\,dx$$

$$= 2\sqrt{(a)}\left[\frac{2}{3}x^{3/2}\right]_0^{4a/m^2} - m\left[\frac{1}{2}x^2\right]_0^{4a/m^2}$$

$$= \frac{4\sqrt{a}}{3}\left(\frac{4a}{m^2}\right)^{3/2} - \frac{1}{2}m\left(\frac{4a}{m^2}\right)^2 = \frac{32a^2}{3m^3} - \frac{8a^2}{m^3} = \frac{8a^2}{3m^3}.$$

Ex. 21 (b). *Find the area included between* $y^2 = 9x$ and $y = x$.

Sol. Proceed exactly as in Ex. 21 (a). Here $m = 1$ and $a = 9/4$.

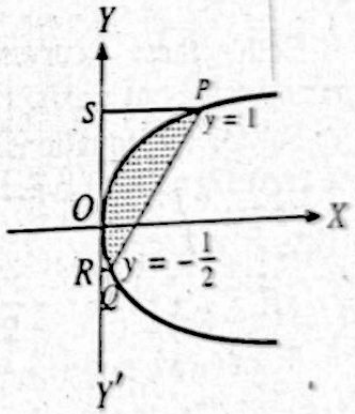
Ex. 22 (a). *Find the area of the segment cut off from the parabola* $y^2 = 2x$ *by the straight line* $y = 4x - 1$.

Sol. The given curves are

$$y^2 = 2x, \qquad ...(1)$$

and

$$y = 4x - 1. \qquad ...(2)$$



The two curves have been shown in the figure. Solving (1) and (2) for y we have

$$y^2 = 2 \cdot \tfrac{1}{4}(y + 1)$$

or

$$2y^2 - y - 1 = 0$$

or

$$(y - 1)(2y + 1) = 0.$$

$\therefore$

$$y = -\tfrac{1}{2}, 1.$$

Thus the curves (1) and (2) intersect at the points where $y = -\frac{1}{2}$ and $y = 1$.

Now the required area of the segment POQ (*i.e.*, the dotted area) = the area bounded by the st. line $y = 4x - 1$ and the y-axis from $y = -\frac{1}{2}$ to $y = 1$

− the area bounded by the parabola $y^2 = 2x$ and the y-axis from $y = -\frac{1}{2}$ to $y = 1$

$= \int_{-1/2}^{1} x\,dy$, from (2) $- \int_{-1/2}^{1} x\,dy$, from (1)

$$= \int_{-1/2}^{1} \frac{1}{4}(y+1)\,dy - \int_{-1/2}^{1} \frac{1}{2}y^2\,dy$$

$$= \frac{1}{4}\left[\frac{1}{2}y^2 + y\right]_{-1/2}^{1} - \frac{1}{6}\left[y^3\right]_{-1/2}^{1}$$

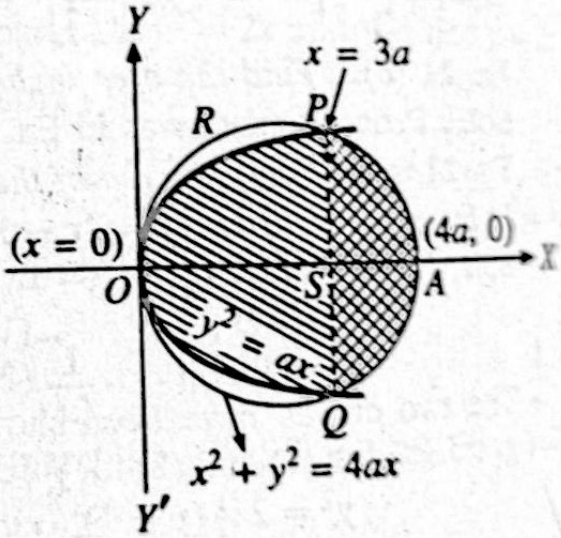
$$= \frac{1}{4}\left[\frac{3}{2} - \left(\frac{1}{8} - \frac{1}{2}\right)\right] - \frac{1}{6}\left(1 + \frac{1}{8}\right)$$

$$= \frac{1}{4}\left(\frac{3}{2} + \frac{3}{8}\right) - \frac{1}{6}\cdot\frac{9}{8} = \frac{1}{4}\cdot\frac{15}{8} - \frac{1}{6}\cdot\frac{9}{8} = \frac{15}{32} - \frac{3}{16} = \frac{9}{32}.$$

Ex. 22 (b). *Find the area of the segment cut off from the parabola $y^2 = 4x$ by the line $y = 8x - 1$.*

Sol. Proceed exactly as in Ex. 22 (a). Here the points of intersection are $(\frac{1}{16}, -\frac{1}{2})$ and $(\frac{1}{4}, 1)$ and hence the required area $= \frac{9}{64}$.

Ex. 23 (a). *Find the area common to the two curves $y^2 = ax$, $x^2 + y^2 = 4ax$.* **(Meerut 1982, 87 P; Agra 83; Bhopal 81)**



Sol. $y^2 = ax$ is a parabola with vertex at the origin and axis along x-axis and latus rectum a, and $x^2 + y^2 = 4ax$ is a circle with centre $(2a, 0)$ and radius $2a$.

Both these curves are symmetrical about x-axis.

Solving the equations of the two curves for x we have

$$x^2 + ax = 4ax$$

or $\quad x^2 - 3ax = 0$

or $\quad x(x - 3a) = 0.$

Therefore $x = 0, 3a$.

Thus the two curves intersect at the points where $x = 0$ and $x = 3a$.

Also A is the point $(4a, 0)$.

The area common to the parabola and the circle (*i.e.*, the shaded area)

$= 2\,[\text{Area } OPS + \text{Area } PSA]$, (by symmetry)

$$= 2\left[\int_0^{3a} y\,dx, \text{ from the parabola } y^2 = ax + \int_{3a}^{4a} y\,dx, \text{ from the circle } x^2 + y^2 = 4ax\right]$$

$$= 2\left[\int_0^{3a} \sqrt{(ax)}\,dx + \int_{3a}^{4a} \sqrt{(4ax - x^2)}\,dx\right]$$

$$= 2\sqrt{a}\int_0^{3a} x^{1/2}\,dx + 2\int_{3a}^{4a} \sqrt{[4a^2-(x-2a)^2]}\,dx$$

$$= 2\sqrt{a}\left[\frac{2}{3}x^{3/2}\right]_0^{3a}$$

$$+ 2\left[\tfrac{1}{2}(x-2a)\sqrt{\{4a^2-(x-2a)^2\}} + \frac{4a^2}{2}\sin^{-1}\frac{x-2a}{2a}\right]_{3a}^{4a}$$

$$= 4a^2\sqrt{3} + 2\left[\{0 - \tfrac{1}{2}a\sqrt{3}\,.a\} + 2a^2\{(\pi/2)-(\pi/6)\}\right]$$

$$= 4\sqrt{3}\,a^2 - a^2\sqrt{3} + \tfrac{4}{3}\pi a^2 = 3\sqrt{3}a^2 + \tfrac{4}{3}\pi a^2$$

$$= a^2(3\sqrt{3} + \tfrac{4}{3}\pi).$$

Ex. 23 (b). *Find the area lying above x-axis and included between the circle $x^2 + y^2 = 2ax$ and the parabola $y^2 = ax$.*

Sol. The figure is similar to that of Ex. 23 (a). Here we are required to find the unshaded area *ORP* in the figure of Ex. 23 (a). Solving the given equations $y^2 = ax$ and $x^2 + y^2 = 2ax$ for x, we get $x = 0$ and $x = a$.

$\therefore$ the required area $= \int_0^a y\,dx$, from the curve $x^2 + y^2 = 2ax$

$- \int_0^a y\,dx$, from the curve $y^2 = ax$

$$= \int_0^a \sqrt{(2ax - x^2)}\,dx - \int_0^a \sqrt{(ax)}\,dx$$

$$= \int_0^a \{a^2 - (x-a)^2\}\,dx - \sqrt{a}\,.\left[\frac{2}{3}x^{3/2}\right]_0^a$$

$$= \left[\frac{1}{2}(x-a)\sqrt{(2ax-x^2)} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right)\right]_0^a - \frac{2}{3}a^2$$

$$= \left[0 - \frac{a^2}{2}\sin^{-1}(-1)\right] - \frac{2}{3}a^2 = \frac{a^2}{2}\cdot\frac{\pi}{2} - \frac{2}{3}a^2 = a^2\left[\frac{\pi}{4} - \frac{2}{3}\right].$$

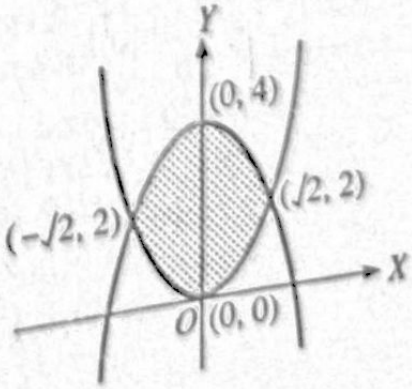
Ex. 23 (c). *Prove that the area of the region bounded by the parabolas $y = x^2$ and $y = 4 - x^2$ is $\frac{16}{3}\sqrt{2}$.* (Meerut 1993 P)

Sol. The given parabolas are

$$y = x^2 \qquad ...(1)$$

and

$$y = 4 - x^2 \qquad ...(2)$$



Both the curves (1) and (2) are symmetrical about y-axis.

The equation of the parabola (2) can be written as $x^2 = -(y-4)$ which shows that its vertex is the point $(0, 4)$.

Solving (1) and (2) for x, we get

$x^2 = 4 - x^2$ or $2x^2 = 4$ or $x^2 = 2$ or $x = \pm\sqrt{2}$.

Putting $x = \sqrt{2}$ in (1), we get $y = 2$.

$\therefore$ the curves (1) and (2) meet at the points $(-\sqrt{2}, 2)$ and $(\sqrt{2}, 2)$.

Required area bounded by the two parabolas

$$= 2\left[\int_0^{\sqrt{2}} \underset{\text{From (2)}}{y\,dx} - \int_0^{\sqrt{2}} \underset{\text{From (1)}}{y\,dx}\right]$$

$$= 2\left[\int_0^{\sqrt{2}} (4 - x^2)\,dx - \int_0^{\sqrt{2}} x^2\,dx\right]$$

$$= 2\int_0^{\sqrt{2}} (4 - 2x^2)\,dx$$

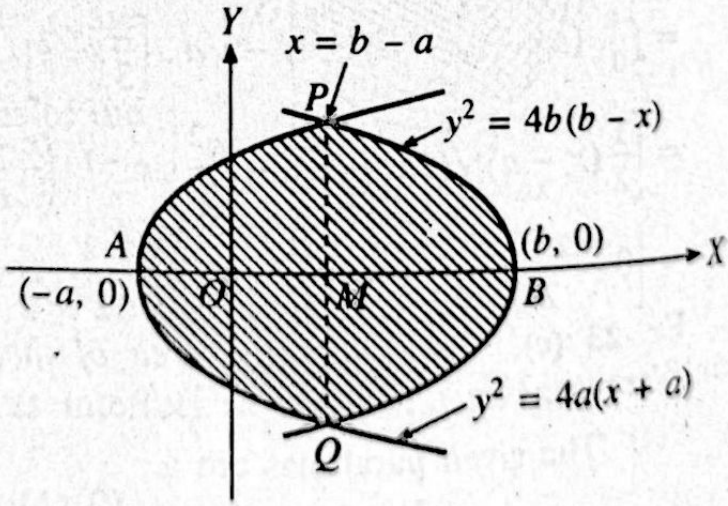
$$= 2\left[4x - \tfrac{2}{3}x^3\right]_0^{\sqrt{2}}$$

$$= 2\left[4\sqrt{2} - \tfrac{2}{3}\cdot 2\sqrt{2}\right]$$

$$= 2.4\sqrt{2}\left[1 - \tfrac{1}{3}\right] = 2.4\sqrt{2}\cdot\tfrac{2}{3} = \tfrac{16}{3}\sqrt{2}.$$

Ex. 24 (a). *Show that the area included between the parabolas* $y^2 = 4a(x + a)$, $y^2 = 4b(b - x)$ *is* $\frac{8}{3}(a + b)\sqrt{(ab)}$.

Sol. $y^2 = 4a(x + a)$ represents a parabola whose vertex is $(-a, 0)$ and latus rectum is $4a$. Also $y^2 = 4b(b - x)$ represents a parabola whose vertex is $(b, 0)$ and latus rectum $4b$. Both the curves have been shown in the figure.



Equating the values of y^2 from the two given equations of parabolas, we get $4a(x + a) = 4b(b - x)$ or $x = b - a$ *i.e.*, the abscissa of the point of intersection P is $b - a$.

Now both the curves are symmetrical about x-axis.

$\therefore$ the required area $= 2$ [Area APM + Area PMB], by symmetry

$$= 2\left[\int_{-a}^{b-a} y\,dx, \text{ for the parabola } y^2 = 4a(x + a)\right.$$

$$\left. + \int_{b-a}^{b} y\,dx, \text{ for the parabola } y^2 = 4b(b - x)\right]$$

$$= 2\left[\int_{-a}^{b-a} \sqrt{\{4a(x+a)\}}\,dx + \int_{b-a}^{b} \sqrt{\{4b(b-x)\}}\,dx\right]$$

$$= 4\sqrt{a}\int_{-a}^{b-a} (x+a)^{1/2}\,dx + 4\sqrt{b}\int_{b-a}^{b} \sqrt{(b-x)}\,dx$$

$$= 4\sqrt{a}\left[\frac{2}{3}(x+a)^{3/2}\right]_{-a}^{b-a} - 4\sqrt{b}\left[\frac{2}{3}(b-x)^{3/2}\right]_{b-a}^{b}$$

$$= \tfrac{1}{3}[8\sqrt{a}\,.b^{3/2}] + \tfrac{1}{3}[8\sqrt{b}\,.a^{3/2}]$$

$$= \tfrac{8}{3}\sqrt{(ab)}\,.(b+a) = \tfrac{8}{3}(a+b)\sqrt{(ab)}.$$

Ex. 24 (b). *Show that the area common to the ellipses*

$$a^2x^2 + b^2y^2 = 1,\ b^2x^2 + a^2y^2 = 1, \text{ where } 0 < a < b,$$

is $4(ab)^{-1}\tan^{-1}(a/b)$.

(Agra 1972; G.N.U. 75)

Sol. The given equations of the two ellipses are

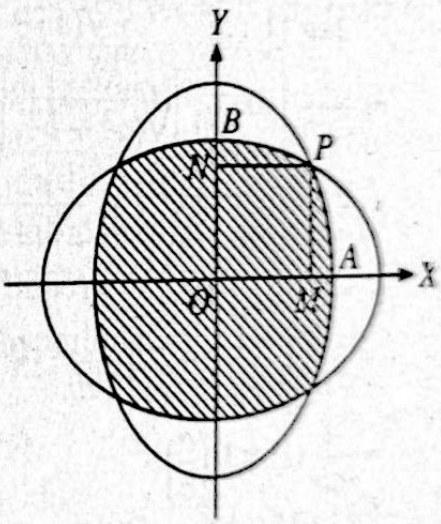
$$a^2x^2 + b^2y^2 = 1 \qquad ...(1)$$

and

$$b^2x^2 + a^2y^2 = 1. \qquad ...(2)$$

Since $0 < a < b$, therefore $(1/a) > (1/b)$.

The ellipse (1) cuts the x-axis on the positive side at the point $(1/a, 0)$ and it cuts the y-axis on the positive side at the point $B\,(0, 1/b)$. The ellipse (2) cuts the x-axis on the positive side at the point $A\ (1/b, 0)$. Both the ellipses are symmetric about both the axes.



Solving (1) and (2) we have the coordinates of the point of intersection P in the first quadrant as $(1/\sqrt{\{a^2+b^2\}}, 1/\sqrt{\{a^2+b^2\}})$.

Draw PM and PN perpendiculars to the axis of x and the axis of y respectively.

Now the area common to the two ellipses (*i.e.*, the shaded area)

$= 4 \times$ (common area in the Ist quadrant) $= 4$ area $OAPB$

$= 4$ [area $OMPB$ + area APM]

$= 4$ [{area of the square $OMPN$ + area BPN} + area APM] ...(3)

$= 4$ [area of the square $OMPN$ + 2 area APM]

[$\because$ area BPN = area APM on account of the symmetrical situation of the area $OAPB$ about OP]

Now the area of the square $OMPN = OM\,.ON$

$$= \frac{1}{\sqrt{(a^2+b^2)}} \cdot \frac{1}{\sqrt{(a^2+b^2)}} = \frac{1}{(a^2+b^2)}.$$

Also the area APM

$$= \int_{1/\sqrt{(a^2+b^2)}}^{1/b} y\,dx, \text{ from the ellipse } b^2x^2 + a^2y^2 = 1$$

$$= \int_{1/\sqrt{(a^2+b^2)}}^{1/b} \frac{(1-b^2x^2)^{1/2}}{a}\,dx = \frac{b}{a}\int_{1/\sqrt{(a^2+b^2)}}^{1/b} \sqrt{\left(\frac{1}{b^2}-x^2\right)}\,dx$$

$$= \frac{b}{a}\left[\frac{x}{2}\sqrt{\left(\frac{1}{b^2}-x^2\right)} + \frac{1}{2}\cdot\frac{1}{b^2}\sin^{-1}\left(\frac{x}{1/b}\right)\right]_{1/\sqrt{(a^2+b^2)}}^{1/b}$$

$$= \frac{b}{2a}\left[0 + \frac{1}{b^2}\sin^{-1}1 - \frac{1}{\sqrt{(a^2+b^2)}}\cdot\frac{a}{b\sqrt{(a^2+b^2)}} - \frac{1}{b^2}\sin^{-1}\frac{b}{\sqrt{(a^2+b^2)}}\right]$$

$$= \frac{1}{2ab}\left[\frac{\pi}{2} - \frac{ab}{a^2+b^2} - \sin^{-1}\frac{b}{\sqrt{(a^2+b^2)}}\right]$$

$$= \frac{1}{2ab}\left[\left\{\frac{\pi}{2} - \sin^{-1}\frac{b}{\sqrt{(a^2+b^2)}}\right\} - \frac{ab}{a^2+b^2}\right]$$

$$= \frac{1}{2ab}\left[\cos^{-1}\left\{\frac{b}{\sqrt{(a^2+b^2)}}\right\} - \frac{ab}{a^2+b^2}\right]$$

$$= \frac{1}{2ab}\left[\tan^{-1}\left(\frac{a}{b}\right) - \frac{ab}{a^2+b^2}\right].$$

Hence from (3), the required area

$$= 4\left[\frac{1}{a^2+b^2} + 2\left\{\frac{1}{2ab}\tan^{-1}\frac{a}{b} - \frac{1}{2(a^2+b^2)}\right\}\right]$$

$$= \frac{4}{ab}\tan^{-1}\left(\frac{a}{b}\right).$$

Ex. 25 (a). *If $P(x,y)$ be any point on the ellipse $x^2/a^2 + y^2/b^2 = 1$ and S be the sectorial area bounded by the curve, the x-axis and the line joining the origin to P, show that $x = a\cos(2S/ab)$, $y = b\sin(2S/ab)$.*

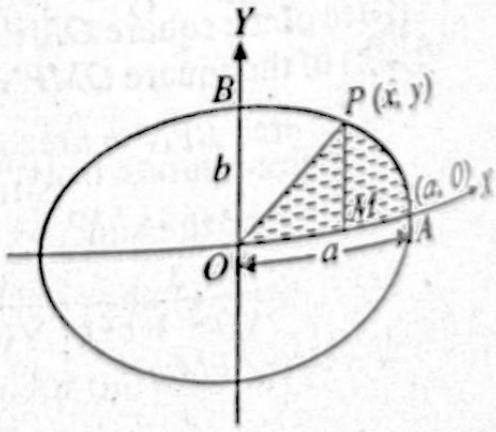
Sol. The given ellipse is shown in the figure. We have

$S =$ the sectorial area OAP (*i.e.*, the dotted area)

$=$ the area of the $\Delta\, OMP$ + the area PMA

$= \frac{1}{2}OM\,.\,MP + \int_x^a y\,dx$, for the ellipse

$= \frac{1}{2}xy + \int_x^a \frac{b}{a}\sqrt{(a^2-x^2)}\,dx,$



[$\because$ from the equation of the ellipse, $y = (b/a)\sqrt{(a^2 - x^2)}$]

$$= \tfrac{1}{2}x \cdot \frac{b}{a}\sqrt{(a^2 - x^2)} + \frac{b}{a}\left[\frac{x}{2}\sqrt{(a^2 - x^2)} + \tfrac{1}{2}a^2 \sin^{-1}\frac{x}{a}\right]_x^a$$

$$= \frac{bx}{2a}\sqrt{(a^2 - x^2)} + \frac{b}{a}\left[0 + \tfrac{1}{2}a^2 \cdot \frac{\pi}{2} - \frac{x}{2}\sqrt{(a^2 - x^2)} - \tfrac{1}{2}a^2 \sin^{-1}\frac{x}{a}\right]$$

$$= \frac{bx}{2a}\sqrt{(a^2 - x^2)} + \frac{b}{a} \cdot \tfrac{1}{2}a^2\left(\frac{\pi}{2} - \sin^{-1}\frac{x}{a}\right) - \frac{bx}{2a}\sqrt{(a^2 - x^2)}$$

$$= \frac{ab}{2}\left(\frac{\pi}{2} - \sin^{-1}\frac{x}{a}\right) = \frac{ab}{2}\cos^{-1}\frac{x}{a}.$$

Thus $S = \dfrac{ab}{2}\cos^{-1}\dfrac{x}{a}$;

$$\therefore \quad \cos^{-1}\frac{x}{a} = \frac{2S}{ab} \quad \text{or} \quad \frac{x}{a} = \cos\frac{2S}{ab} \quad \text{or} \quad x = a\cos\frac{2S}{ab}.$$

Also $y = \dfrac{b}{a}\sqrt{(a^2 - x^2)} = \dfrac{b}{a}\sqrt{\{a^2 - a^2\cos^2(2S/ab)\}} = b\sin\dfrac{2S}{ab}$.

Ex. 25 (b). *If A is the vertex, O the centre and P any point (x, y) on the hyperbola $x^2/a^2 - y^2/b^2 = 1$, show that*

$$x = a\cosh(2S/ab),\ y = b\sinh(2S/ab),$$

where S is the sectorial area OPA.

Sol. The given hyperbola is shown is the figure. We have

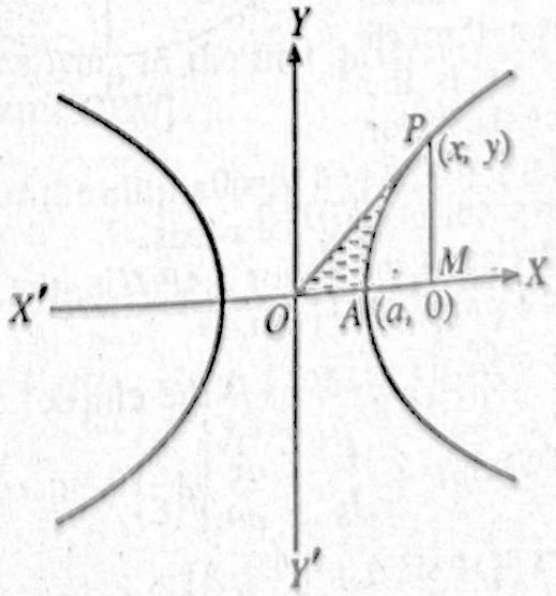
S = the sectorial area OAP (*i.e.*, the dotted area)

= the area of the ΔOMP − the area PAM

$= \tfrac{1}{2}OM \cdot MP - \int_a^x y\,dx$, for the hyperbola

$$= \tfrac{1}{2}xy - \int_a^x \frac{b}{a}\sqrt{(x^2 - a^2)}\,dx,$$

[$\because$ from the equation of the hyperbola $y = (b/a)\sqrt{(x^2 - a^2)}$]



$$= \tfrac{1}{2}x \cdot \frac{b}{a}\sqrt{(x^2-a^2)} - \frac{b}{a}\left[\frac{x}{2}\sqrt{(x^2-a^2)} - \tfrac{1}{2}a^2\cosh^{-1}\frac{x}{a}\right]_a^x$$

$$= \frac{bx}{2a}\sqrt{(x^2-a^2)} - \frac{b}{a}\left[\frac{x}{2}\sqrt{(x^2-a^2)} - \tfrac{1}{2}a^2\cosh^{-1}\frac{x}{a} - (0-0)\right]$$

$$= \frac{bx}{2a}\sqrt{(x^2-a^2)} - \frac{bx}{2a}\sqrt{(x^2-a^2)} + \frac{ab}{2}\cosh^{-1}\frac{x}{a} = \frac{ab}{2}\cosh^{-1}\frac{x}{a}.$$

Thus $S = \dfrac{ab}{2}\cosh^{-1}\dfrac{x}{a}$;

$$\therefore \quad \cosh^{-1}\frac{x}{a} = \frac{2S}{ab} \quad \text{or} \quad x = a\cosh\frac{2S}{ab}.$$

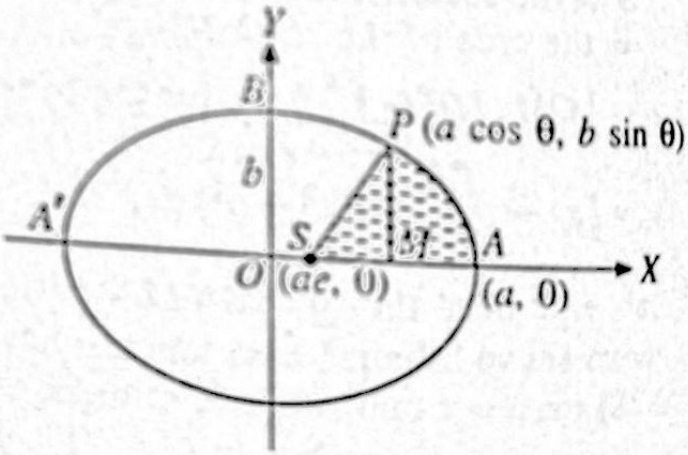
$$\text{Also } y = \frac{b}{a}\sqrt{(x^2-a^2)} = \frac{b}{a}\sqrt{\left\{a^2\cosh^2\frac{2S}{ab} - a^2\right\}}$$

$$= \frac{b}{a}.a\sqrt{\left\{\cosh^2\frac{2S}{ab} - 1\right\}} = b\sinh\frac{2S}{ab}.$$

Ex. 25 (c). *Prove that the area of a sector of the ellipse of semi-axes a and b between the major axis and a radius vector from the focus is $\frac{1}{2}ab(\theta - e\sin\theta)$, where θ is the eccentric angle of the point to which the radius vector is drawn.*

Sol. Let the equation of the ellipse be $x^2/a^2 + y^2/b^2 = 1$. Let O be its centre and S be its focus $(ae, 0)$. Let θ be the eccentric angle of any point $P(x, y)$ on the ellipse. Then

$x = a\cos\theta,\ y = b\sin\theta.$



Now SP is the radius vector of P drawn through the focus S and SA is the radius vector along the major axis. At the point A, $x = a$ and $\theta = 0$.

Draw PM perpendicular to the x-axis.

The required area of the sector SAP (*i.e.*, the dotted area)

$= \text{area of the } \Delta\, SMP + \text{area } PMA$

$= \tfrac{1}{2}SM \cdot MP + \int_{a\cos\theta}^{a} y\,dx$, for the ellipse

$= \tfrac{1}{2}(OM - OS) \cdot MP + \displaystyle\int_{\theta}^{0} y\frac{dx}{d\theta}\,d\theta$

$= \tfrac{1}{2}(a\cos\theta - ae)\,b\sin\theta + \int_{\theta}^{0} b\sin\theta.(-a\sin\theta)\,d\theta,$

$[\because x = a\cos\theta \text{ and } y = b\sin\theta]$

$$= \tfrac{1}{2}ab\,(\cos\theta - e)\sin\theta + \int_0^{\theta} ab\,.\tfrac{1}{2}(1-\cos 2\theta)\,d\theta$$

$$= \tfrac{1}{2}ab\,(\cos\theta - e)\sin\theta + \tfrac{1}{2}ab\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\theta}$$

$$= \tfrac{1}{2}ab\,(\cos\theta - e)\sin\theta + \tfrac{1}{2}ab\,(\theta - \tfrac{1}{2}.2\sin\theta\cos\theta)$$

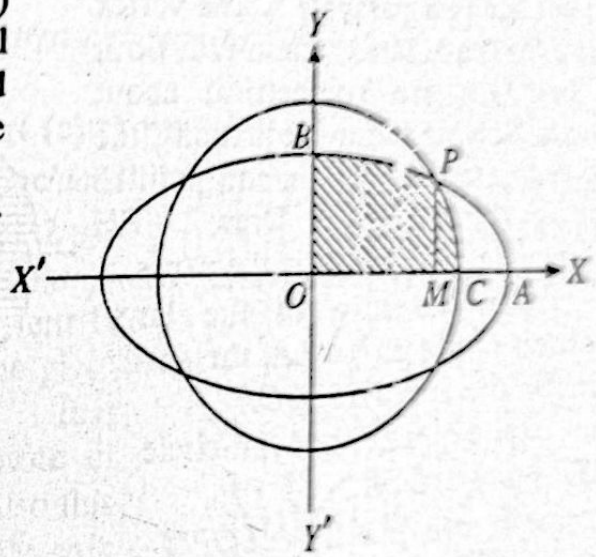
$$= \tfrac{1}{2}ab\,[\cos\theta\sin\theta - e\sin\theta + \theta - \sin\theta\cos\theta] = \tfrac{1}{2}ab\,(\theta - e\sin\theta).$$

Ex. 26. *Find the area common to the circle $x^2 + y^2 = 4$ and the ellipse $x^2 + 4y^2 = 9$.*

Sol. The equation of the circle is $x^2 + y^2 = 4$, ...(1)

and the equation of the ellipse is $x^2 + 4y^2 = 9$. ...(2)

Both the curves (1) and (2) are symmetrical about both the axes and have been shown in the figure.



Solving (1) and (2) for x, we have

$\therefore\ x^2 + 4(4 - x^2) = 9$

or $3x^2 = 7$ or $x^2 = 7/3$.

$\therefore$ the x-coordinate of the point of intersection P lying in the first quadrant is $\sqrt{(7/3)}$.

Also putting $y = 0$ in $x^2 + y^2 = 4$, we get $x = 2$ at C.

Now the required area is symmetrical about both the axes.

$\therefore$ the required area (*i.e.*, the area common to the circle and the ellipse)

$= 4 \times$ (common area lying in the first quadrant) $= 4 \times$ area $OCPB$

$= 4$ [area $OBPM$ + area CPM]

$$= 4\left[\int_0^{\sqrt{(7/3)}} y\,dx, \text{ for the ellipse} + \int_{\sqrt{(7/3)}}^{2} y\,dx, \text{ for the circle}\right]$$

$$= 4\left[\int_0^{\sqrt{(7/3)}} \tfrac{1}{2}\sqrt{(9-x^2)}\,dx + \int_{\sqrt{(7/3)}}^{2} \sqrt{(4-x^2)}\,dx\right],$$

[$\because$ for the ellipse, $y = \tfrac{1}{2}\sqrt{(9-x^2)}$ and for the circle, $y = \sqrt{(4-x^2)}$]

$$= 2\left[\frac{x\sqrt{(9-x^2)}}{2} + \frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right]_0^{\sqrt{(7/3)}}$$

$$+ 4\left[\frac{x\sqrt{(4-x^2)}}{2} + 2\sin^{-1}\left(\frac{x}{2}\right)\right]_{\sqrt{(7/3)}}^{2}$$

$$= 2\left[\frac{1}{2}\cdot\sqrt{\left(\frac{7}{3}\right)}\sqrt{\left(\frac{20}{3}\right)} + \frac{9}{2}\sin^{-1}\left\{\frac{1}{3}\sqrt{\left(\frac{7}{3}\right)}\right\}\right]$$

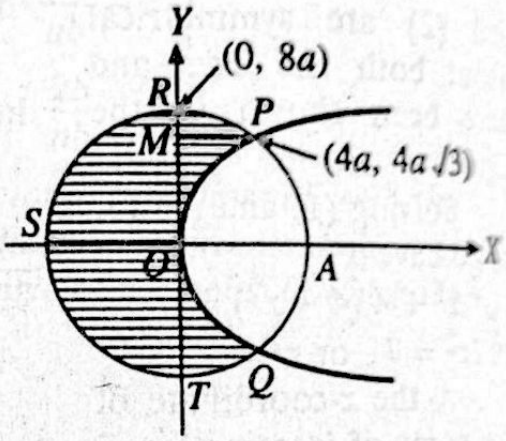
$$+ 4\left[2\sin^{-1}(1) - \tfrac{1}{2}\sqrt{(\tfrac{7}{3})}\sqrt{(\tfrac{5}{3})} - 2\sin^{-1}\{\tfrac{1}{2}\sqrt{(\tfrac{7}{3})}\}\right]$$

$$= \frac{2\sqrt{(35)}}{3} + 9\sin^{-1}\frac{\sqrt{7}}{3\sqrt{3}} + 4\pi - \frac{2}{3}\sqrt{(35)} - 8\sin^{-1}\frac{\sqrt{7}}{2\sqrt{3}}$$

$$= 4\pi + 9\sin^{-1}\{\tfrac{1}{3}.\sqrt{(7/3)}\} - 8\sin^{-1}\{\tfrac{1}{2}.\sqrt{(7/3)}\}.$$

Ex. 27 (a). *Show that the larger of the two areas into which the circle $x^2 + y^2 = 64a^2$ is divided by the parabola $y^2 = 12ax$ is $\frac{16}{3}a^2[8\pi - \sqrt{3}]$.* **(Magadh 1977; Meerut 85)**

Sol. $x^2 + y^2 = 64a^2$ is a circle with centre (0, 0) and radius $8a$ and $y^2 = 12ax$ is a parabola whose vertex is at (0, 0) and latus rectum $12a$. Both the curves are symmetrical about x-axis. Solving the two equations, the co-ordinates of the common point P are $(4a, 4a\sqrt{3})$. Draw PM perpendicular from P to the y-axis.



Now the area of the larger portion of the circle (*i.e.*, the shaded area) = the area $PRSTQOP$

= the area of the semi-circle RST + 2 area OPR

$$= \tfrac{1}{2}.\pi(8a)^2 + 2\,[\text{area } OPM + \text{area } MPR]$$

$$= \tfrac{1}{2}\pi(8a)^2 + 2\int_0^{4a\sqrt{3}} x\,dy, \quad \text{for } y^2 = 12ax$$

$$+ 2\int_{4a\sqrt{3}}^{8a} x\,dy, \quad \text{for } x^2 + y^2 = 64a^2$$

$$= 32\pi a^2 + 2\int_0^{4a\sqrt{3}} \frac{y^2}{12a}\,dy + 2\int_{4a\sqrt{3}}^{8a} \sqrt{(64a^2 - y^2)}\,dy$$

$$= 32\pi a^2 + \frac{1}{6a}\left[\frac{y^3}{3}\right]_0^{4a\sqrt{3}}$$

$$+ 2\left[\tfrac{1}{2}y\sqrt{(64a^2 - y^2)} + \frac{64a^2}{2}\sin^{-1}\frac{y}{8a}\right]_{4a\sqrt{3}}^{8a}$$

$$= 32\pi a^2 + \frac{1}{6a}\left[\frac{64\times 3\sqrt{3}a^3}{3}\right]$$

$$+ 2\left[\{0 - 8a^2\sqrt{3}\} + 32a^2\{\sin^{-1}1 - \sin^{-1}(\sqrt{3}/2)\}\right]$$

$$= 32\pi a^2 + \frac{32\sqrt{3}a^2}{3} - 16a^2\sqrt{3} + \frac{32}{3}a^2\pi$$

$$= \frac{128}{3} a^2\pi - \frac{16}{3} a^2 \sqrt{3} = \frac{16}{3} a^2 (8\pi - \sqrt{3}).$$

Note 1. The required area can also be found as follows :

the required area $= 2 \left[\{ \int_{-8a}^{4a} y\,dx, \text{ for the circle} \right.$

$$\left. - \int_0^{4a} y\,dx, \text{ for the parabola} \right].$$

Note 2. The required area of the larger portion can also be evaluated by finding out the area of the circle and then subtracting from it the area $APOQA$ which is equal to

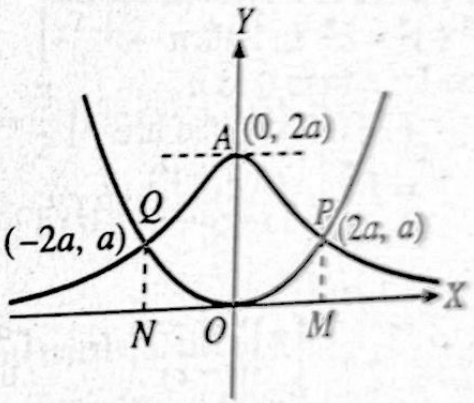
$$2 \left[\int_0^{4a} y\,dx, \text{ for the parabola} + \int_{4a}^{8a} y\,dx, \text{ for the circle} \right].$$

Ex. 27 (b). *Find the area common to the circle $x^2 + y^2 = 9$ and the parabola $x^2 = 8y$.* (Ranchi 1975)

Sol. Proceed as in part (a) above.

Ex. 28. *Find the area included between the parabola $x^2 = 4ay$ and the curve $y = 8a^3/(x^2 + 4a^2)$.* (Meerut 1983, 96 BP)

Sol. The curve $y(x^2 + 4a^2) = 8a^3$ is symmetrical about y-axis. Equating to zero the coefficient of the highest power of x in the given equation, we find that $y = 0$ *i.e.*, x-axis is an asymptote of the curve parallet to x-axis. Also this curve cuts the y-axis at $(0, 2a)$.



Solving the two given equations $x^2 = 4ay$ and $y = 8a^3/(x^2 + 4a^2)$ we get their points of intersection as $(\pm 2a, a)$.

Also both the curves are symmetrical about y-axis.

Now the required area $OPAQO = 2 \times$ area OPA (by symmetry)

$= 2 \times$ [area $OAPM$ − area OPM]

$$= 2 \left[\int_0^{2a} y\,dx, \text{ for } y = 8a^3/(x^2 + 4a^2) - \int_0^{2a} y\,dx, \text{ for } x^2 = 4ay \right]$$

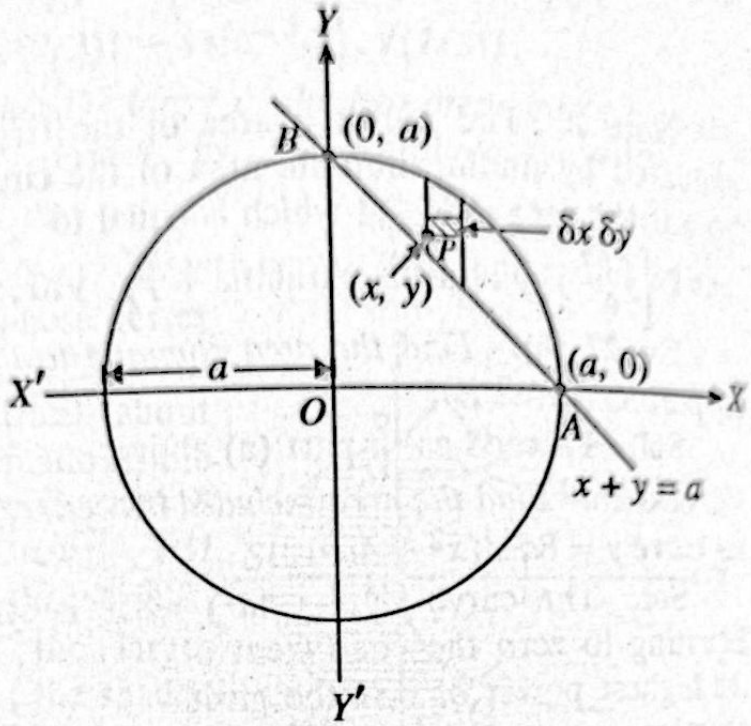
$$= 2 \int_0^{2a} \frac{8a^3}{x^2 + 4a^2} dx - 2 \int_0^{2a} \frac{x^2}{4a} dx$$

$$= 16a^3 . \frac{1}{2a} \left[\tan^{-1} \frac{x}{2a} \right]_0^{2a} - \frac{1}{2a} \left[\frac{x^3}{3} \right]_0^{2a} = 2\pi a^2 - \frac{4a^2}{3}$$

$$= [2\pi - \tfrac{4}{3}]\, a^2.$$

Ex. 29. *Find by double integration the area of the region enclosed by the curves $x^2 + y^2 = a^2$, $x + y = a$ (in the first quadrant).* (Garhwal 1983; Meerut 81S, 78)

Sol. The given equations of the circle $x^2+y^2=a^2$ [centre $(0,0)$ and radius a] and of the straight line $x+y=a$ (with equal intercepts a on both the axes) can be easily traced as shown in the figure.



The required area is the area bounded by the arc AB and the line AB. To find it with the help of double integration take any point $P(x,y)$ in this portion and consider an elementary area $\delta x\,\delta y$ at P. The required area can now be covered by first moving y from the straight line $x+y=a$ to the arc of the circle $x^2+y^2=a^2$ and then moving x from 0 to a.

$\therefore$ the required area

$$=\int_{x=0}^{a}\int_{y=(a-x)}^{\sqrt{(a^2-x^2)}} dx\,dy,$$ the first integration to be performed w.r.t. y whose limits are variable

$$=\int_0^a \Big[y\Big]_{(a-x)}^{\sqrt{(a^2-x^2)}} dx=\int_0^a [\sqrt{(a^2-x^2)}-(a-x)]\,dx$$

$$=\left[\left\{\tfrac{1}{2}x\sqrt{(a^2-x^2)}+\frac{1}{2}a^2\sin^{-1}(x/a)\right\}-ax+\frac{1}{2}x^2\right]_0^a$$

$$=\tfrac{1}{2}a^2.(\tfrac{1}{2}\pi)-a^2+\tfrac{1}{2}a^2=\tfrac{1}{2}a^2(\tfrac{1}{2}\pi-1)=\tfrac{1}{4}a^2(\pi-2).$$

Note. The required area can also be covered by first moving x from the st. line $x+y=a$ to the arc of the circle $x^2+y^2=a^2$ and then moving y from 0 to a.

Ex. 30. *Find by double integration the area bounded by the curves $y(x^2+2)=3x$ and $4y=x^2$.* (Meerut 1976, 95)

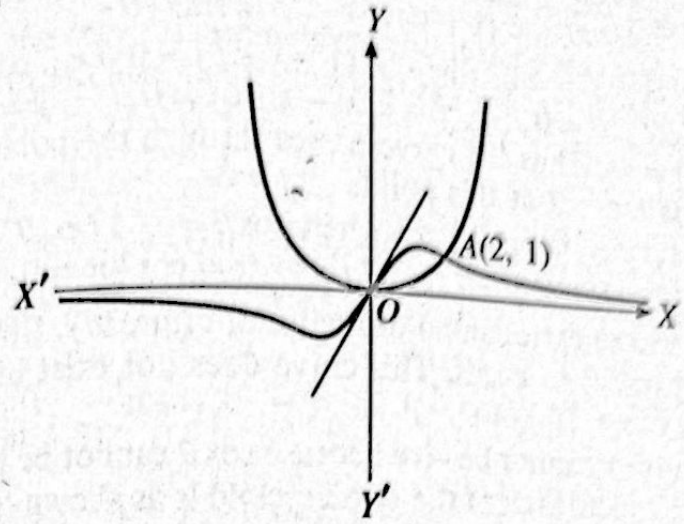
Sol. Eliminating y from the given equations, we get

$$3x/(x^2+2)=x^2/4 \quad \text{or} \quad 12x=x^2(x^2+2)$$

or $\quad x(12-x^3-2x)=0$, giving $x=0$ and $x=2$.

Thus the two curves intersect at the points where $x=0$ and $x=2$.

Both the curves have been shown in the figure.



The required area

$$= \int_{x=0}^{2} \int_{y=x^2/4}^{3x/(x^2+2)} dx\, dy,$$

the first integration to be performed w.r.t. y

$$= \int_0^2 \left[y\right]_{x^2/4}^{3x/(x^2+2)} dx = \int_0^2 \left[\frac{3x}{x^2+2} - \frac{x^2}{4}\right] dx$$

$$= \frac{3}{2}\int_0^2 \frac{2 \,.\, x\, dx}{x^2+2} - \frac{1}{4}\int_0^2 x^2\, dx = \frac{3}{2}\left[\log(x^2+2)\right]_0^2 - \frac{1}{4}\left[\frac{x^3}{3}\right]_0^2$$

$$= \tfrac{3}{2}[\log 6 - \log 2] - \tfrac{1}{12}.[8] = \tfrac{3}{2}\log\left(\tfrac{6}{2}\right) - \tfrac{2}{3}$$

$$= \tfrac{3}{2}\log 3 - \tfrac{2}{3}.$$

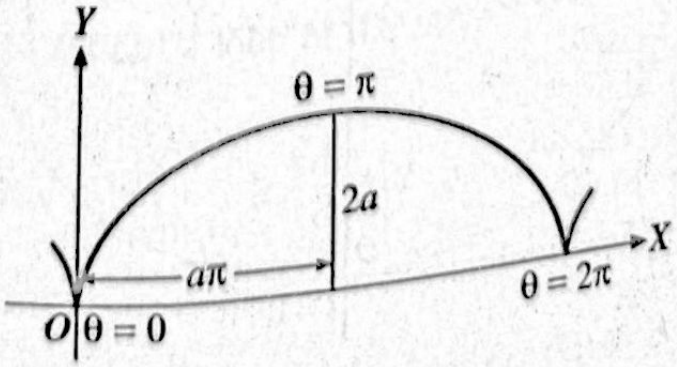
Areas of curves given by parametric equations.

Ex. 31 (a). *Find the area included between the cycloid* $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$ *and its base.*

(Kumayun 1983; Meerut 82 S, 87, 87S, 98; Vikram 75)

Sol. The parametric equations of the given cycloid are $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$.

We have $dx/d\theta = a(1 - \cos\theta)$, $dy/d\theta = a\sin\theta$.



$$\therefore \quad \frac{dy}{dx} = \frac{dy\,d\theta}{dx/d\theta} = \frac{a\sin\theta}{a\,(1-\cos\theta)} = \frac{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\sin^2\frac{1}{2}\theta} = \cot\tfrac{1}{2}\theta.$$

In this curve $y = 0$ when $a\,(1-\cos\theta) = 0$ *i.e.*, $\cos\theta = 1$ *i.e.*, $\theta = 0$. When $\theta = 0$, $x = a\,(0 - \sin 0) = 0$, $y = 0$ and $dy/dx = \cot 0 = \infty$. Thus the curve passes through the point $(0, 0)$ and the axis of y is tangent at this point.

In this curve y is maximum when $\cos\theta = -1$ *i.e.*, $\theta = \pi$. When $\theta = \pi, x = a\,(\pi - \sin\pi) = a\pi, y = 2a$, $dy/dx = \cot\frac{1}{2}\pi = 0$. Thus at the point $\theta = \pi$, whose cartesian co-ordinates are $(a\pi, 2a)$, the tangent to the curve is parallel to x-axis. This curve does not exist in the region $y > 2a$.

In this curve y cannot be –ive because $\cos\theta$ cannot be greater than 1. Thus one ~~complete~~ arch of the given cycloid is as shown in the figure.

Now this cycloid is symmetrical with respect to the line $x = a\pi$ (axis of the cycloid) and its base is the x-axis. Therefore the required area

$$= 2\int_{x=0}^{a\pi} y\,dx = 2\int_{\theta=0}^{\pi} y\frac{dx}{d\theta}\cdot d\theta$$

$$= 2\int_0^{\pi} a\,(1-\cos\theta)\,.\,a\,(1-\cos\theta)\,d\theta = 2a^2\int_0^{\pi}(1-\cos\theta)^2\,d\theta$$

$$= 2a^2\int_0^{\pi}(2\sin^2\tfrac{1}{2}\theta)^2\,d\theta = 8a^2\int_0^{\pi}\sin^4\tfrac{1}{2}\theta\,d\theta$$

$$= 8a^2\int_0^{\pi/2}\sin^4\phi\,2d\phi, \text{ putting } \tfrac{1}{2}\theta = \phi \text{ so that } \tfrac{1}{2}d\theta = d\phi$$

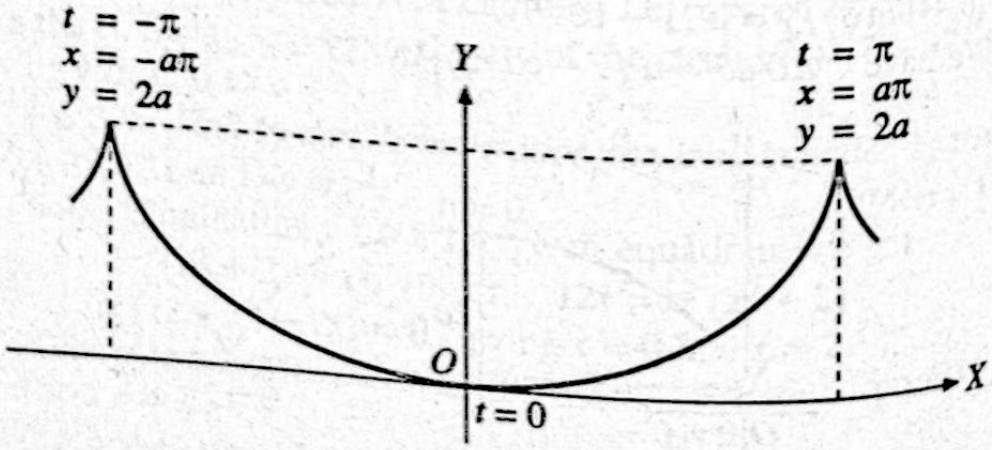
$$= 16a^2\int_0^{\pi/2}\sin^4\phi\,d\phi = 16a^2\cdot\frac{3.1}{4.2}\cdot\frac{\pi}{2}, \text{ by Walli's formula}$$

$$= 3\pi a^2.$$

***Ex. 31 (b).** *Find the area included between the curve $x = a\,(t + \sin t)$, $y = a\,(1 - \cos t)$ and its base.*

(Rohilkhand 1983; Meerut 82 S, 84 R; Kanpur 76)

Sol. The given cycloid has been shown in the figure. [To trace one complete arch of this cycloid proceed as in Ex. 31 (a)].



Since the curve is symmetrical about the y-axis and its base is the line $y = 2a$, therefore the required area

$$= 2\int_{y=0}^{2a} x\,dy = 2\int_{t=0}^{\pi} x \cdot \frac{dy}{dt} \cdot dt$$

$$= 2\int_0^{\pi} a\,(t + \sin t)\,a \sin t\,dt = 2a^2 \int_0^{\pi} (t \sin t + \sin^2 t)\,dt$$

$$= 2a^2 \int_0^{\pi} t \sin t\,dt + 2a^2 \int_0^{\pi} \sin^2 t\,dt$$

$$= 2a^2 \left[\left\{t \cdot (-\cos t)\right\}_0^{\pi} - \int_0^{\pi} 1 \cdot (-\cos t)\,dt\right] + 4a^2 \int_0^{\pi/2} \sin^2 t\,dt$$

$$= 2a^2 \left[(\pi - 0) + \int_0^{\pi} \cos t\,dt\right] + 4a^2 \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

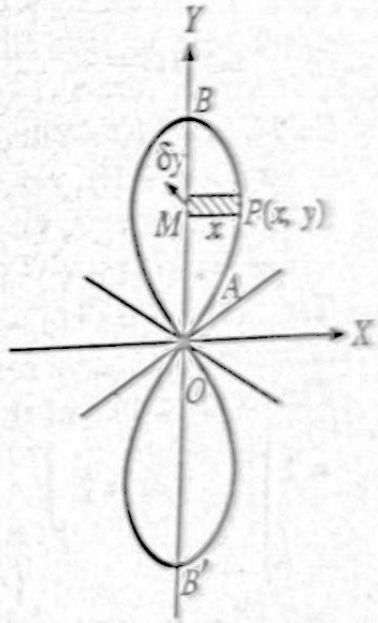
$$= 2a^2 (\pi + 0) + \pi a^2, \quad \left[\because \int_0^{\pi} \cos t\,dt = 0\right]$$

$$= 3\pi a^2.$$

Ex. 32. *Find the area of a loop of the curve*

$$x = a \sin 2t,\ y = a \sin t \text{ or } a^2x^2 = 4y^2(a^2 - y^2).$$

Sol. To trace the given curve, we first find its cartesian equation by eliminating t. We have



$$x = a \sin 2t = 2a \sin t \cos t.$$

$$\therefore \quad x^2 = 4a^2 \sin^2 t \cos^2 t$$

$$= 4a^2 \sin^2 t\,(1 - \sin^2 t)$$

$$= 4a^2 (y^2/a^2)\,\{1 - (y^2/a^2)\},$$

$$[\because y = a \sin t]$$

or $a^2x^2 = 4y^2(a^2 - y^2)$ is the cartesian equation of the given curve.

Now we trace the curve from its cartesian equation. The curve is as shown in the figure.

At O, $x = 0$, $y = 0$ and so $t = 0$. Again at B, $x = 0$, $y = a$ and so $t = \frac{1}{2}\pi$.

The required area of a loop of the curve

$= 2 \times$ area OAB [By symmetry]

$$= 2\int_{y=0}^{a} x\,dy = 2\int_{t=0}^{\pi/2} x \cdot \frac{dy}{dt} \cdot dt$$

$$= 2\int_0^{\pi/2} a \sin 2t \,.\, a \cos t\,dt$$

$$= 2a^2 \int_0^{\pi/2} \sin 2t \cos t\,dt = 4a^2 \int_0^{\pi/2} \sin t \cos^2 t\,dt$$

$= 4a^2 \frac{1}{3.1} \cdot 1$, by Walli's formula

$= 4a^2/3$.

Ex. 33. *Show that the area bounded by the cissoid*

$$x = a \sin^2 t, y = (a \sin^3 t)/\cos t$$

and its asymptote is $3\pi a^2/4$.

(Agra 1982; Meerut 85 S, 86, 88; Kanpur 73)

Sol. Eliminating t from the given parametric equations, we get

$$y^2 = a^2 \frac{\sin^6 t}{\cos^2 t} = \frac{a^2 (x^3/a^3)}{1 - \sin^2 t} = \frac{x^3/a}{1 - (x/a)} = \frac{x^3}{(a - x)}.$$

Therefore the **Cartesian equation of the given curve** is $\mathbf{y^2 (a - x) = x^3}$. To trace this curve see Ex. 9 page 11.

The curve is symmetrical about x-axis, passes through the origin, the tangent there being $y = 0$. Also there are no real points of the curve if $x < 0$ or if $x > a$. The line $x = a$ is an asymptote of the curve. When $x = 0, t = 0$ and when $x = a, t = \pi/2$.

$\therefore$ the required area

$$= 2 \int_{x=0}^{a} y\, dx = 2 \int_{t=0}^{\pi/2} y \cdot \frac{dx}{dt} \cdot dt = 2 \int_0^{\pi/2} \frac{a \sin^3 t}{\cos t} \cdot 2a \sin t \cos t\, dt$$

$$= 4a^2 \int_0^{\pi/2} \sin^4 t\, dt = 4a^2 \frac{\Gamma \frac{5}{2} \Gamma \frac{1}{2}}{2 \Gamma 3} = 4a^2 \frac{\frac{3}{2} \cdot \frac{1}{2} \sqrt{\pi} \cdot \sqrt{\pi}}{2.2.1} = \frac{3\pi a^2}{4}.$$

Ex. 34. *Find the area of the loop of the curve*

$$x = a(1 - t^2), y = at(1 - t^2), \text{ where } -1 \le t \le 1.$$

Sol. Eliminating t, we have

$$y^2 = a^2 t^2 (1 - t^2)^2 = x^2 t^2 = x^2 \{1 - (x/a)\} = x^2 (a - x)/a.$$

Therefore $ay^2 = x^2 (a - x)$ is the cartesian equation of the given curve. To trace this curve see Ex. 5, page 7.

The required area of the loop

$$= 2 \int_{x=0}^{a} y\, dx = 2 \int_{t=1}^{0} y \cdot \frac{dx}{dt} \cdot dt,$$

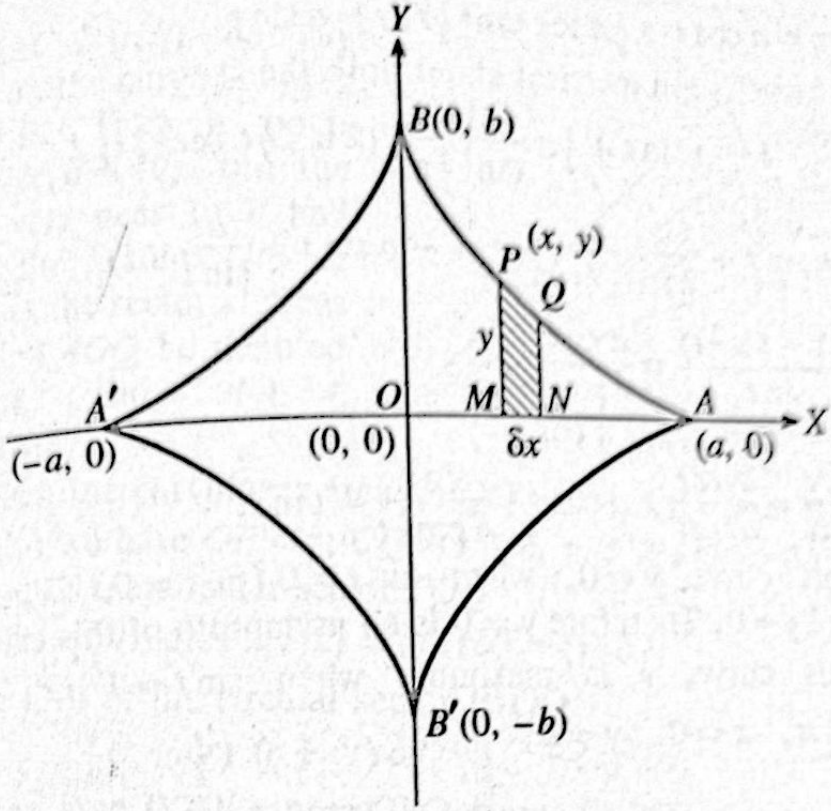
[$\because$ when $x = 0, t = 1$ and when $x = a, t = 0$]

$$= 2 \int_1^0 at(1 - t^2) \cdot (-2at)\, dt = 4a^2 \int_0^1 (t^2 - t^4)\, dt$$

$$= 4a^2 \left[\frac{1}{3} t^3 - \frac{1}{5} t^5\right]_0^1 = 4a^2 \left[\frac{1}{3} - \frac{1}{5}\right] = \frac{8}{15} a^2.$$

Ex. 35 (a). *Find the whole area of the curve (hypocycloid) given by the equations* $x = a \cos^3 t, y = b \sin^3 t$. **(Jodhpur 1976; Meerut 85)**

Sol. Eliminating t from the given equations the cartesian equation of the curve is obtained as $(x/a)^{2/3} + (y/b)^{2/3} = 1$ *i.e.*, $\{(x/a)^2\}^{1/3} + \{(y/b)^2\}^{1/3} = 1$.



Since the powers of x and y are all even, the curve is symmetrical about both the axes. It does not pass through the origin. It cuts the axis of x at the points $(\pm a, 0)$ and the axis of y at the points $(0, \pm b)$. The tangent at the point $(a, 0)$ is x-axis. At the point B, $x = 0$ and $t = \frac{1}{2}\pi$. At the point A, $x = a$ and $t = 0$.

$\therefore$ the required area $= 4 \times$ area OAB

$$= 4\int_{x=0}^{a} y\,dx = 4\int_{t=\pi/2}^{0} y \cdot \frac{dx}{dt} \cdot dt$$

$$= 4\int_{\pi/2}^{0} b \sin^3 t\,.(-3a\cos^2 t \sin t)\,dt, \quad \text{(putting for } y \text{ and } dx/dt\text{)}$$

(Note)

$$= 12ab\int_{0}^{\pi/2} \sin^4 t \cos^2 t\,dt$$

$$= 12ab \cdot \frac{3.1.1}{6.4.2} \cdot \frac{\pi}{2} = \frac{3}{8}\pi ab.$$

Ex. 35 (b). *Find the area of the astroid*

$$x^{2/3} + y^{2/3} = a^{2/3} \quad \text{or} \quad x = a\cos^3 t,\ y = a\sin^3 t.$$

(Meerut 1992)

Sol. Proceed exactly as in Ex. 35 (a). Put $b = a$ in the above example. The required area $= (3/8)\pi a^2$.

Ex. 36. *Prove that the whole area between the four infinite branches of the tractrix* $x = a\cos t + \frac{1}{2}a\log\tan^2\frac{1}{2}t,\ y = a\sin t$ *is equal to the area of a circle of radius a.*

(Meerut 1988 P)

Sol. The parametric equations of the given curve are

$$x = a\cos t + \tfrac{1}{2}a\log\tan^2\tfrac{1}{2}t,\ y = a\sin t.$$

This curve is symmetrical about both the axes. (Note)

We have $\dfrac{dx}{dt} = -a\sin t + \tfrac{1}{2}a\dfrac{1}{\tan^2\frac{1}{2}t}\cdot(2\tan\tfrac{1}{2}t.\sec^2\tfrac{1}{2}t)\cdot\tfrac{1}{2}$

$$= -a\sin t + \frac{a}{2\sin\frac{1}{2}t\cos\frac{1}{2}t} = -a\sin t + \frac{a}{\sin t}$$

$$= \frac{a(1-\sin^2 t)}{\sin t} = \frac{a\cos^2 t}{\sin t}.$$

Also $\quad dy/dt = a\cos t.$

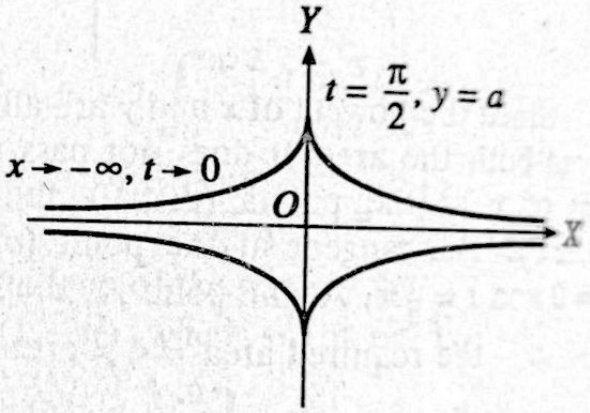
$$\therefore\quad \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = a\cos t\cdot\frac{\sin t}{a\cos^2 t} = \tan t.$$

In this curve $y = 0$, when $\sin t = 0$ *i.e.*, $t = 0$. When $t\to 0$ $x\to -\infty$ and $y\to 0$. Therefore $y = 0$ is an asymptote of this curve.

In this curve, y is maximum when $\sin t = 1$ *i.e.*, $t = \tfrac{1}{2}\pi$. When $t = \tfrac{1}{2}\pi$, $x = 0$, $y = a$, $dy/dx = \infty$.

Therefore at the point $(0, a)$ the tangent to the curve is perpendicular to the x-axis.

The shape of the curve is as shown in the figure.



The required area $= 4\times$ area lying in the second quadrant

$$= 4\int_{-\infty}^{0} y\,dx = 4\int_0^{\pi/2} y\cdot\frac{dx}{dt}\,dt = 4\int_0^{\pi/2} a\sin t\cdot\frac{a\cos^2 t}{\sin t}\,dt$$

$$= 4a^2\int_0^{\pi/2}\cos^2 t\,dt = 4a^2\cdot\frac{1}{2}\cdot\frac{\pi}{2},\ \text{by Walli's formula}$$

$= \pi a^2 =$ the area of a circle of radius a.

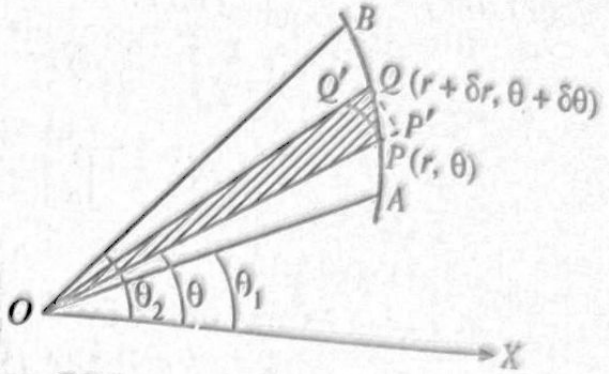
§ 3. Areas of Curves given by Polar equations.

If $r = f(\theta)$ be the equation of a curve in polar coordinates where $f(\theta)$ is a single valued continuous function of θ, then the area of the sector enclosed by the curve and the two radii vectors $\theta = \theta_1$ and $\theta = \theta_2$ $(\theta_1 < \theta_2)$, is equal to

$$\frac{1}{2}\int_{\theta=\theta_1}^{\theta_2} r^2\,d\theta.$$

Proof. Let OAB be the area of the curve $r = f(\theta)$ between the radii vectors $\theta = \theta_1$ and $\theta = \theta_2$.

Let $P(r, \theta)$ be any point on the curve between A and B. Take a point $Q(r + \delta r, \theta + \delta\theta)$ on the curve very near to P and draw the radius vector OQ. Let the sectorial areas AOP and AOQ be denoted by A and $A + \delta A$ respectively.



Then the curvilinear area $OPQO = A + \delta A - A = \delta A$.

Also we have $OP = r$; $OQ = r + \delta r$ and $\angle POQ = \delta\theta$.

The area of the circular sector POQ'

$$= \tfrac{1}{2}\,(\text{radius} \times \text{arc}) = \tfrac{1}{2} r \,.\, r\delta\theta = \tfrac{1}{2} r^2 \delta\theta,$$

and the area of the circular sector $P'OQ$

$$= \tfrac{1}{2}(r + \delta r)\,.(r + \delta r)\,\delta\theta = \tfrac{1}{2}(r + \delta r)^2 \delta\theta.$$

Now, area $POQ' <$ area $OPQ <$ area $P'OQ$,

i.e., $\quad \tfrac{1}{2} r^2 \delta\theta < \delta A < \tfrac{1}{2}(r + \delta r)^2 \delta\theta,$

i.e., $\quad \tfrac{1}{2} r^2 < \delta A / \delta\theta < \tfrac{1}{2}(r + \delta r)^2.$

Proceeding to limits as $\delta\theta \to 0$, we get

$$\frac{dA}{d\theta} = \tfrac{1}{2} r^2 \quad \text{or} \quad dA = \tfrac{1}{2} r^2 \, d\theta.$$

$$\therefore \qquad \Big[A\Big]_{\theta_1}^{\theta_2} = \int_{\theta_1}^{\theta_2} \tfrac{1}{2} r^2 \, d\theta.$$

Now the L.H.S. = the value of A for θ equal to θ_2 − the value of A for θ equal to θ_1 = (the area AOB) − 0 = area AOB.

Hence the required area AOB

$$= \tfrac{1}{2} \int_{\theta_1}^{\theta_2} r^2 \, d\theta.$$

Note. In some cases it is more convenient to find the required area by using double integration. In that case the area is given by

$$\int_{\theta = \theta_1}^{\theta_2} \int_{r=0}^{f(\theta)} r \, d\theta \, dr, \ (\theta_1 < \theta_2).$$

Remember. The number of loops in $r = a \cos n\theta$ or $r = a \sin n\theta$ is n or $2n$ according as n is odd or even.

Ex. 37. *Find the area between the following curves and the given radii vectors :* (Meerut 1971)

(i) *The spiral* $r\theta^{1/2} = a$; $\theta = \alpha, \theta = \beta$.

(ii) *The equiangular spiral* $r = ae^{m\theta}$; $\theta = \alpha, \theta = \beta$.

(iii) *The parabola* $l/r = 1 + \cos\theta$; $\theta = 0, \theta = \alpha$.

Sol. (i) The required area

$$= \tfrac{1}{2}\int_{\theta=\alpha}^{\beta} r^2\,d\theta = \tfrac{1}{2}\int_{\alpha}^{\beta} \frac{a^2}{\theta}\,d\theta, \qquad (\because r\theta^{1/2} = a)$$

$$= \frac{a^2}{2}\int_{\alpha}^{\beta} \frac{1}{\theta}\,d\theta = \frac{a^2}{2}\Big[\log\theta\Big]_{\alpha}^{\beta} = \frac{a^2}{2}\Big[\log\beta - \log\alpha\Big]$$

$$= \tfrac{1}{2}a^2\log(\beta/\alpha).$$

(ii) The required area of the given curve $r = ae^{m\theta}$

$$= \tfrac{1}{2}\int_{\theta=\alpha}^{\beta} r^2\,d\theta = \tfrac{1}{2}a^2\int_{\alpha}^{\beta} e^{2m\theta}\,d\theta = \tfrac{1}{2}a^2\left[\frac{e^{2m\theta}}{2m}\right]_{\alpha}^{\beta}$$

$$= (a^2/4m)\,(e^{2m\beta} - e^{2m\alpha}).$$

(iii) The required area

$$= \tfrac{1}{2}\int_0^{\alpha} r^2\,d\theta = \tfrac{1}{2}\int_0^{\alpha} \frac{l^2}{(1+\cos\theta)^2}\,d\theta,$$

putting for r from the given equation of the curve

$$= \tfrac{1}{2}l^2\int_0^{\alpha} \frac{d\theta}{(2\cos^2\frac{1}{2}\theta)^2} = \tfrac{1}{2}l^2\int_0^{\alpha} \frac{d\theta}{4\cos^4\frac{1}{2}\theta}$$

$$= \frac{1}{8}l^2\int_0^{\alpha} \sec^4\frac{\theta}{2}\,d\theta.$$

Now put $\theta/2 = t$ so that $\frac{1}{2}d\theta = dt$. Also when $\theta = 0$, $t = 0$ and when $\theta = \alpha$, $t = \frac{1}{2}\alpha$.

$\therefore$ required area $= \frac{1}{4}l^2\int_0^{\alpha/2} \sec^4 t\,dt$

$$= \tfrac{1}{4}l^2\int_0^{\alpha/2} \sec^2 t.\sec^2 t\,dt = \tfrac{1}{4}l^2\int_0^{\alpha/2} (1+\tan^2 t)\sec^2 t\,dt$$

$$= \tfrac{1}{4}l^2\int_0^{\alpha/2} \{\sec^2 t + (\tan^2 t)\sec^2 t\}\,dt$$

$$= \tfrac{1}{4}l^2\left[\tan t + \tfrac{1}{3}\tan^3 t\right]_0^{\alpha/2}, \qquad \left[\because \int (\tan t)^2\sec^2 t\,dt = \tfrac{1}{3}(\tan t)^3\right]$$

$$= \tfrac{1}{4}l^2\left[\tan\tfrac{1}{2}\alpha + \tfrac{1}{3}\tan^3\tfrac{1}{2}\alpha\right].$$

Ex. 38. *Find the area of the loop of the curve* $r = a\,\theta\cos\theta$ *between* $\theta = 0$ *and* $\theta = \pi/2$. (Meerut 1986)

Sol. The required area

$$= \tfrac{1}{2}\int_0^{\pi/2} r^2\,d\theta = \tfrac{1}{2}\int_0^{\pi/2} a^2\theta^2\cos^2\theta\,d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/2} \theta^2\cos^2\theta\,d\theta = \frac{a^2}{4}\int_0^{\pi/2} \theta^2(1+\cos 2\theta)\,d\theta$$

$$= \frac{a^2}{4}\int_0^{\pi/2} \theta^2\, d\theta + \frac{a^2}{4}\int_0^{\pi/2} \theta^2 \cos 2\theta\, d\theta$$

$$= \frac{a^2}{4}\left[\frac{\theta^3}{3}\right]_0^{\pi/2} + \frac{a^2}{4}\left[\theta^2 \frac{\sin 2\theta}{2}\right]_0^{\pi/2} - \frac{a^2}{4}\int_0^{\pi/2} 2\theta \cdot \frac{\sin 2\theta}{2}\, d\theta$$

$$= \frac{a^2}{12}\cdot\frac{\pi^3}{8} + \frac{a^2}{8}\cdot 0 - \frac{a^2}{4}\left[\frac{\theta(-\cos 2\theta)}{2}\right]_0^{\pi/2}$$

$$+ \frac{a^2}{4}\int_0^{\pi/2} 1 \cdot \left[\frac{-\cos 2\theta}{2}\right] d\theta$$

$$= \frac{\pi^3 a^2}{96} + \frac{a^2}{8}\left[\frac{\pi}{2}(-1)\right] - \frac{a^2}{8}\left[\frac{\sin 2\theta}{2}\right]_0^{\pi/2}$$

$$= \frac{\pi^3 a^2}{96} - \frac{\pi a^2}{16} = \frac{\pi a^2}{96}(\pi^2 - 6).$$

Ex. 39. *Find the area of one loop of $r = a \cos 4\theta$.* (Lucknow 1970)

Sol. The given curve is $r = a \cos 4\theta$.

It is symmetrical about the initial line.

One loop is obtained by two consecutive values of θ for which r is zero. We have $r = 0$ when $\cos 4\theta = 0$ *i.e.*, $4\theta = -\frac{1}{2}\pi, \frac{1}{2}\pi$ *i.e.*, $\theta = -\pi/8, \pi/8$. Thus two consecutive values of θ for which r is zero are $-\pi/8$ and $\pi/8$. Therefore one loop of the curve lies between $\theta = -\pi/8$ and $\pi/8$ and this loop is symmetrical about the initial line $\theta = 0$.

Hence the area of a loop $= 2\int_0^{\pi/8} \frac{1}{2} r^2\, d\theta$

$$= \int_0^{\pi/8} a^2 \cos^2 4\theta\, d\theta, \qquad [\because r = a\cos 4\theta]$$

$$= \tfrac{1}{4} a^2 \int_0^{\pi/2} \cos^2 t\, dt, \text{ putting } 4\theta = t \text{ so that } 4\, d\theta = dt$$

$$= \tfrac{1}{4} a^2 \cdot \tfrac{1}{2} \cdot \tfrac{1}{2}\pi, \text{ by Walli's formula}$$

$$= \tfrac{1}{16}\pi a^2.$$

Ex. 40. *Find the area of a loop of the curve $r = a \sin 3\theta$.*

(Agra 1975, 74; Gorakhpur 72; Indore 72)

Sol. The given curve is not symmetrical about the initial line. We have $r = 0$ when $\sin 3\theta = 0$ *i.e.*, $3\theta = 0, \pi$ *i.e.*, $\theta = 0, \frac{1}{3}\pi$. Thus two consecutive values of θ for which r is zero are 0 and $\frac{1}{3}\pi$. Therefore one loop of the curve lies between $\theta = 0$ and $\frac{1}{3}\pi$. In all there are three loops as shown in the figure. For the first loop θ varies from $\theta = 0$ to $\theta = \frac{1}{3}\pi$.

Hence the area of a loop

$$= \tfrac{1}{2}\int_0^{\pi/3} r^2\, d\theta$$

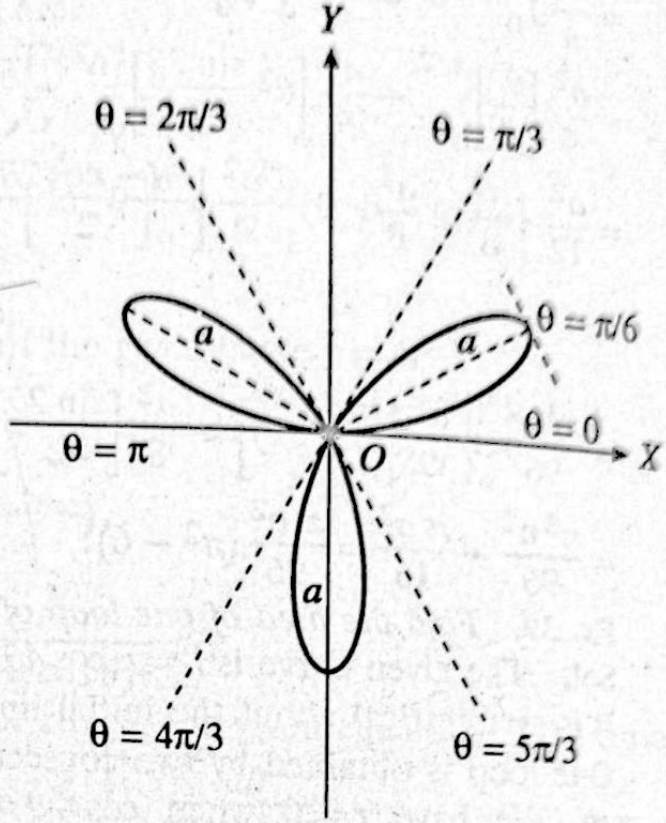
$$= \tfrac{1}{2}\int_0^{\pi/3} a^2 \sin^2 3\theta\, d\theta,$$

$$[\because r = a \sin 3\theta]$$

$$= \frac{a^2}{4}\int_0^{\pi/3} (1 - \cos 6\theta)\, d\theta$$

$$= \tfrac{1}{4}a^2 \left[\theta - \frac{1}{6}\sin 6\theta\right]_0^{\pi/3}$$

$$= \frac{a^2}{4}\left[\frac{\pi}{3}\right] = \frac{\pi a^2}{12}.$$



Note : Also whole area of the curve $r = a \sin 3\theta$

$= 3 \times$ area of one loop

$= 3 \times (1/12)\, \pi a^2 = \tfrac{1}{4}\pi a^2.$

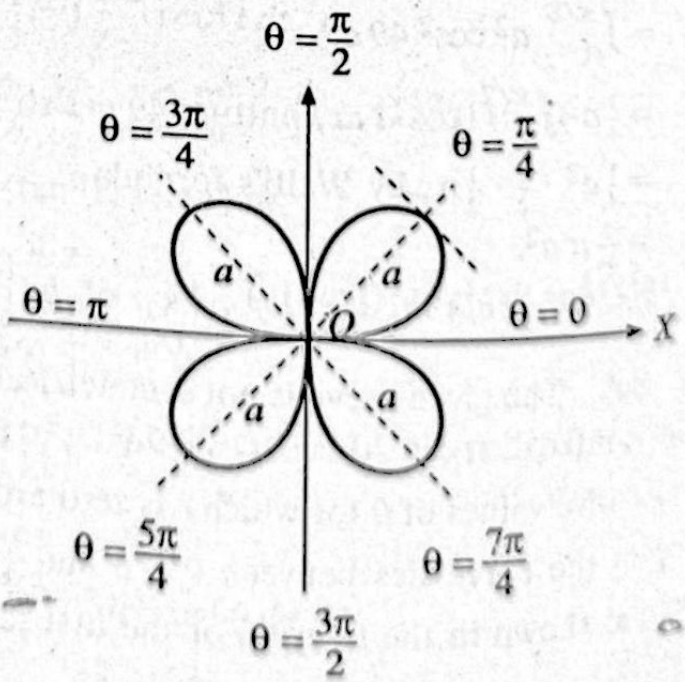
Important : The above curve is a particular case of the curves of the type $r = a \sin n\theta$ which have n loops when n is odd and $2n$ loops when n is even.

Ex. 41. *Find the whole area of the curve $r = a \sin 2\theta$.*

(Kashmir 1973)

Sol. Here the given curve is $r = a \sin 2\theta$.

Comparing with $r = a \sin n\theta$ we observe that $n = 2$ (*i.e.*, even), therefore the curve has four loops. This curve is not symmetrical about the initial line. Putting $r = 0$, we get $\sin 2\theta = 0$ *i.e.*, $2\theta = 0, \pi$ *i.e.*, $\theta = 0, \tfrac{1}{2}\pi$. Thus two consecutive values of θ for which r is zero are 0 and $\tfrac{1}{2}\pi$. Therefore for one loop of the curve θ varies from 0 to $\tfrac{1}{2}\pi$.



Now the whole area of the curve $= 4 \times$ area of one loop

$$= 4\int_0^{\pi/2} \tfrac{1}{2} r^2\, d\theta = 2\int_0^{\pi/2} a^2 \sin^2 2\theta\, d\theta$$

$$= a^2 \int_0^{\pi/2} 2\sin^2 2\theta\, d\theta = a^2 \int_0^{\pi/2} (1 - \cos 4\theta)\, d\theta$$

$$= a^2 \left[\theta - \frac{\sin 4\theta}{4}\right]_0^{\pi/2} = \tfrac{1}{2}\pi a^2.$$

Ex. 42. *Find the whole area of the curve $r = a \cos 2\theta$.*

Sol. Comparing the given equation with $r = a \cos n\theta$ we observe than $n = 2$ (*i.e.*, even), therefore the number of loops

$$= 2n = 2 \times 2 = 4.$$

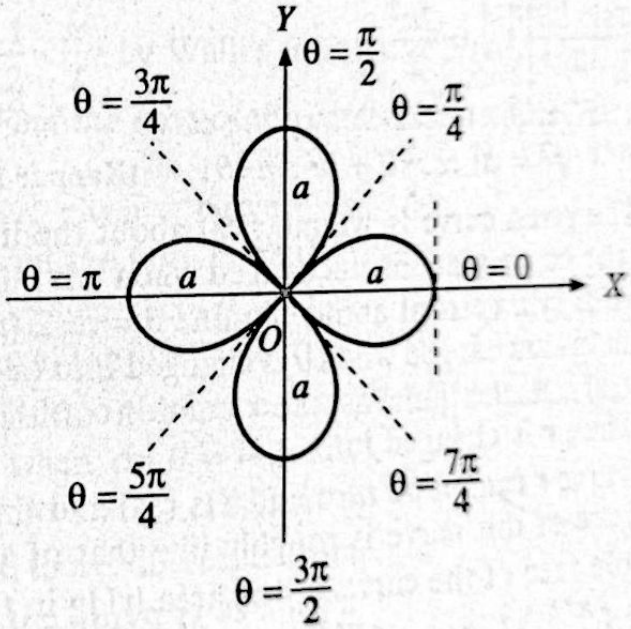
This curve is symmetrical about the inital line.

Putting $r = 0$, we get $\cos 2\theta = 0$

or $2\theta = \pm \frac{1}{2}\pi$ or $\theta = \pm \frac{1}{4}\pi$ *i.e.*, for the first loop θ varies from $-\pi/4$ to $\pi/4$ and this loop is symmetrical about the initial line $\theta = 0$.

$\therefore$ whole area of the curve $= 4 \times$ area of one loop

$$= 4 \times 2\int_0^{\pi/2} \tfrac{1}{2} r^2\, d\theta = 4\int_0^{\pi/4} a^2 \cos^2 2\theta\, d\theta.$$



Now put $2\theta = t$ so that $2\, d\theta = dt$. When $\theta = 0$, $t = 0$ and when $\theta = \frac{1}{4}\pi$, $t = \frac{1}{2}\pi$.

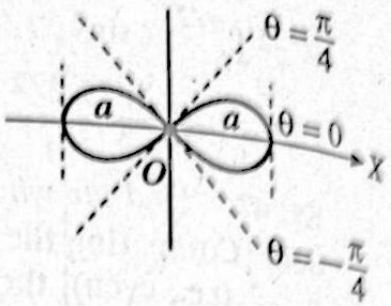
$\therefore$ the required area $= 4a^2 \int_0^{\pi/2} \cos^2 t \cdot \frac{1}{2} dt = 2a^2 \int_0^{\pi/2} \cos^2 t\, dt$

$$= 2a^2 \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{1}{2}\pi a^2.$$

Ex. 43. *Find the area of the curve $r^2 = a^2 \cos 2\theta$.* (Meerut 1984 S)

Sol. The given curve is symmetrical about the initial line $\theta = 0$ and about the pole. Putting $r = 0$ in the given equation of the curve,

we get $\cos 2\theta = 0$ or $2\theta = \pm \frac{1}{2}\pi$ or $\theta = \pm \frac{1}{4}\pi$. Thus two consecutive values of θ for which r is zero are $-\frac{1}{4}\pi$ and $\frac{1}{4}\pi$. Therefore for one loop of the curve θ varies from $-\pi/4$ to $\pi/4$.



When $\frac{1}{2}\pi < 2\theta < \frac{3}{2}\pi$ *i.e.*, $\frac{1}{4}\pi < \theta < \frac{3}{4}\pi$, r^2 is negative *i.e.*, r is imaginary. Therefore this curve does not exist in the region $\frac{1}{4}\pi < \theta < \frac{3}{4}\pi$.

Hence this curve has only two loops as shown in the figure.

$\therefore$ whole area of the curve $= 2 \times$ area of one loop

$$= 2\int_{-\pi/4}^{\pi/4} \tfrac{1}{2} r^2\, d\theta = \int_{-\pi/4}^{\pi/4} a^2 \cos 2\theta\, d\theta, \qquad [\because r^2 = a^2 \cos 2\theta]$$

$$= 2a^2 \int_0^{\pi/4} \cos 2\theta\, d\theta, \qquad \text{[by a property of definite integrals]}$$

$$= 2a^2 \left[\frac{\sin 2\theta}{2}\right]_0^{\pi/4} = \frac{2a^2}{2} = a^2.$$

Ex. 44. *Find the whole area of the curve*

$$r^2 = a^2 \cos^2\theta + b^2 \sin^2\theta.$$ **(Kanpur 1977; Lucknow 79)**

Sol. The given curve is symmetrical about the initial line ($\because$ the equation of the curve remains unchanged when θ is changed into $-\theta$), and the curve is symmetrical about the line $\theta = \pi/2$ (*i.e.*, y-axis) as the equation remains unchanged when θ is changed into $(\pi - \theta)$. Also there is symmetry about the pole as the equation of the curve remains unchanged when r is changed into $-r$.

In this curve r cannot be zero and r is real and finite for all values of θ. The figure of this curve is roughly like that of an ellipse.

$\therefore$ whole area of the curve $= 4 \times$ area lying in the first quadrant

$$= 4 \times \tfrac{1}{2}\int_{\theta=0}^{\pi/2} r^2\, d\theta = 2\int_0^{\pi/2} (a^2 \cos^2\theta + b^2 \sin^2\theta)\, d\theta$$

$$= 2a^2 \int_0^{\pi/2} \cos^2\theta\, d\theta + 2b^2 \int_0^{\pi/2} \sin^2\theta\, d\theta$$

$$= 2a^2 \cdot \tfrac{1}{2} \cdot \tfrac{1}{2}\pi + 2b^2 \cdot \tfrac{1}{2} \cdot \tfrac{1}{2}\pi = \tfrac{1}{2}\pi\,(a^2 + b^2).$$

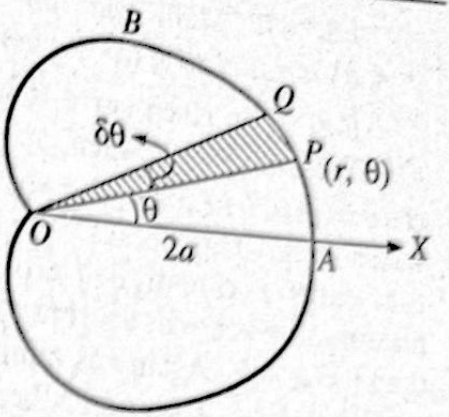
Ex. 45. *Find the area of the cardioid* $r = a\,(1 + \cos\theta)$.

(Bhopal 1983; Rohilkhand 82; Meerut 85 P, 96 P)

Sol. The given curve is symmetrical about the initial line since its equation remains unaltered when θ is changed into $-\theta$.

We have $r = 0$, when $\cos\theta = -1$ *i.e.*, $\theta = \pi$. Therefore the line $\theta = \pi$ is tangent at the pole to the curve. Also r is maximum when $\cos\theta = 1$ *i.e.*, $\theta = 0$ and then $r = 2a$.

When θ increases from 0 to π, r decreases from $2a$ to 0. Thus the curve is as shown in the figure.



Now the required area $= 2 \times$ area of the upper half of the curve

$$= 2\int_0^{\pi} \tfrac{1}{2} r^2\, d\theta$$

$$= 2\int_0^{\pi} \tfrac{1}{2} a^2 (1+\cos\theta)^2\, d\theta,$$

$$[\because r = a(1+\cos\theta)]$$

$$= a^2 \int_0^{\pi} (2\cos^2 \tfrac{1}{2}\theta)^2\, d\theta = 4a^2 \int_0^{\pi} \cos^4 \tfrac{1}{2}\theta\, d\theta.$$

Now put $\frac{1}{2}\theta = \phi$ so that $\frac{1}{2} d\theta = d\phi$.

Also when $\theta = 0$, $\phi = 0$ and when $\theta = \pi$, $\phi = \pi/2$.

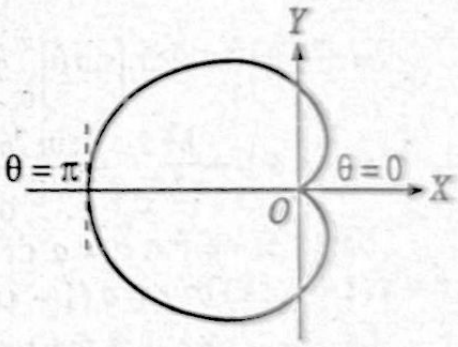
$\therefore$ the required area $= 8a^2 \int_0^{\pi/2} \cos^4 \phi\, d\phi$

$= 8a^2 \cdot \dfrac{3.1}{4.2} \cdot \dfrac{\pi}{2}$, by Walli's formula $= 3\pi a^2/2$.

Ex. 46. *Find the area of the cardioid $r = a(1 - \cos\theta)$.*

(Meerut 1974, 98)

Sol. The given curve is symmetrical about the initial line. We have $r = 0$, when $\cos\theta = 1$ *i.e.*, $\theta = 0$. Therefore the line $\theta = 0$ is tangent at the pole to the curve. Also r is maximum when $\cos\theta = -1$ *i.e.*, $\theta = \pi$ and then $r = 2a$. When θ increases from 0 to π, r increases from 0 to $2a$. Thus the curve is as shown in the figure.



The required area

$$= 2\int_0^{\pi} \tfrac{1}{2} r^2\, d\theta = \int_0^{\pi} a^2 (1-\cos\theta)^2\, d\theta$$

$$= a^2 \int_0^{\pi} (2\sin^2 \tfrac{1}{2}\theta)^2\, d\theta = 4a^2 \int_0^{\pi} \sin^4 \tfrac{1}{2}\theta\, d\theta$$

$$= 8a^2 \int_0^{\pi/2} \sin^4 \phi\, d\phi,$$

[putting $\frac{1}{2}\theta = \phi$ so that $\frac{1}{2} d\theta = d\phi$ and adjusting the limits]

$$= 8a^2 \cdot \frac{3.1}{4.2} \cdot \frac{\pi}{2} = \frac{3\pi a^2}{2}.$$

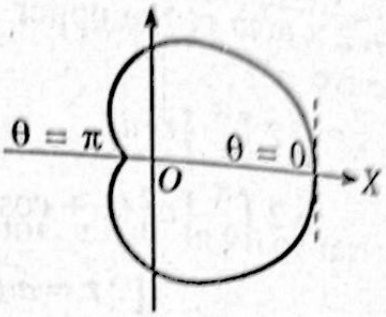
Ex. 47. *Show that the area of the limacon* $r = a + b\cos\theta$, $(b < a)$ *is equal to* $\pi\,(a^2 + \frac{1}{2}b^2)$.

(Meerut 1998)

Sol. The given curve is symmetrical about the initial line.

We have $r = 0$, when

$\cos\theta = -a/b$ *i.e.*, $\theta = \cos^{-1}(-a/b)$ which is not real because $a > b$. Thus in this curve r cannot be zero. Also r is maximum when $\cos\theta = 1$ *i.e.*, $\theta = 0$ and then $r = a + b$. Again r is minimum when $\cos\theta = -1$ *i.e.*, $\theta = \pi$ and then $r = a - b$ which is positive. Some values of r and θ are as follows :



θ	0	$\frac{1}{3}\pi$	$\frac{1}{2}\pi$	$\frac{2}{3}\pi$	π
r	$a + b$	$a + \frac{1}{2}b$	a	$a - \frac{1}{2}b$	$a - b$

Hence the curve is as shown in the figure.

$\therefore$ the required area $= 2 \times$ area above the initial line

$$= 2\int_0^{\pi} \tfrac{1}{2}r^2\,d\theta = \int_0^{\pi} (a + b\cos\theta)^2\,d\theta,$$

putting for r from the given equation of the curve

$$= \int_0^{\pi} (a^2 + 2ab\cos\theta + b^2\cos^2\theta)\,d\theta$$

$$= a^2\left[\theta\right]_0^{\pi} + 2ab\left[\sin\theta\right]_0^{\pi} + \frac{b^2}{2}\int_0^{\pi} (1 + \cos 2\theta)\,d\theta$$

$$= a^2\pi + 0 + \frac{b^2}{2}\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi} = a^2\pi + \frac{b^2}{2}\pi = \pi\,(a^2 + \tfrac{1}{2}b^2).$$

Note. If we put $b = a$ or $-a$, we get the area of the cardioid $r = a\,(1 + \cos\theta)$ or $r = a\,(1 - \cos\theta)$.

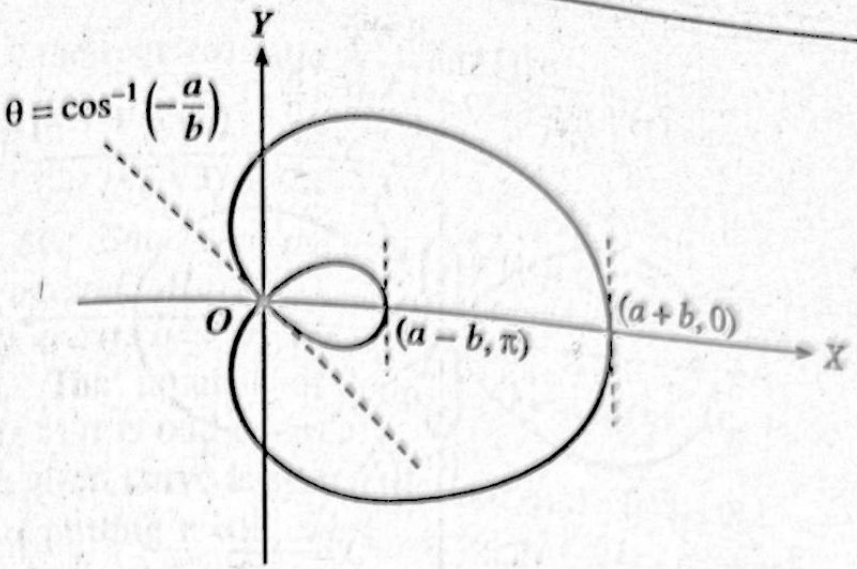
Ex. 48. *Prove that the sum of the areas of the two loops of the limacon* $r = a + b\cos\theta$, $(b > a)$ *is equal to* $\pi\,(2a^2 + b^2)/2$.

(Garhwal 1983; Meerut 86S)

Sol. The given curve is symmetrical about the initial line. We have $r = 0$, when $\cos\theta = -a/b$ *i.e.*, $\theta = \cos^{-1}(-a/b) = \alpha$, (say). Since $\cos\alpha$ is –ive, therefore $\frac{1}{2}\pi < \alpha < \pi$.

Now r is maximum when $\cos\theta = 1$ *i.e.*, $\theta = 0$ and then $r = a + b$. Also r is minimum when $\cos\theta = -1$ *i.e.*, $\theta = \pi$ and then $r = a - b$ which is negative because $b < a$. Some values of r and θ are as follows :

θ	0	$\frac{1}{3}\pi$	$\frac{1}{2}\pi$	α	$\alpha < \theta < \pi$	π
r	$a + b$	$a + \frac{1}{2}b$	a	0	–ive	$a - b$



Thus the curve is as shown in the figure. For the upper half of the larger loop θ varies from 0 to α *i.e.*, $\cos^{-1}(-a/b)$ and for the lower half of the smaller loop θ varies from α to π.

$\therefore$ required sum of the areas of the two loops

$$= 2\left[\int_0^{\alpha} \tfrac{1}{2}r^2\,d\theta + \int_{\alpha}^{\pi} \tfrac{1}{2}r^2\,d\theta\right]$$

$$= 2\int_0^{\pi} \tfrac{1}{2}r^2\,d\theta, \text{ by a property of definite integrals}$$

$$= \int_0^{\pi} r^2\,d\theta = \int_0^{\pi} (a + b\cos\theta)^2\,d\theta$$

$$= \int_0^{\pi} (a^2 + 2ab\cos\theta + b^2\cos^2\theta)\,d\theta$$

$$= \int_0^{\pi} a^2\,d\theta + 2ab\int_0^{\pi} \cos\theta\,d\theta + b^2\int_0^{\pi} \cos^2\theta\,d\theta$$

$$= a^2\left[\theta\right]_0^{\pi} + 0 + 2b^2\int_0^{\pi/2} \cos^2\theta\,d\theta$$

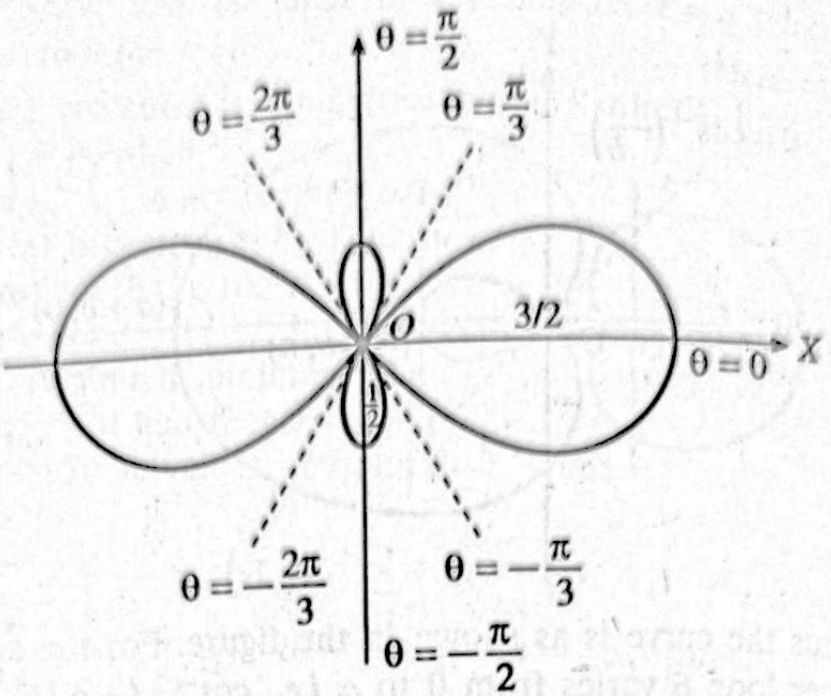
$$= a^2\pi + 2b^2\cdot\tfrac{1}{2}\cdot\tfrac{1}{2}\pi = a^2\pi + \tfrac{1}{2}b^2\pi$$

$$= \tfrac{1}{2}\pi(2a^2 + b^2).$$

Ex. 49. *Calculate the ratio of the area of the larger to the area of the smaller loop of the curve* $r = \frac{1}{2} + \cos 2\theta$.

Sol. The given curve is symmetrical about the initial line. In the given equation of the curve $r = \frac{1}{2} + \cos 2\theta$ putting $r = 0$, we get $\cos 2\theta = -\frac{1}{2}$ *i.e.*, $2\theta = \pm 2\pi/3$ or $\pm 4\pi/3$ *i.e.*, $\theta = \pm\pi/3$ or $\pm 2\pi/3$.

The greatest radius vector of the loop lying between $\theta = -\frac{1}{3}\pi$ and $\theta = \frac{1}{3}\pi$ is given by $\theta = 0$ and it is equal to $\frac{3}{2}$. The greatest radius vector of the loop lying between $\theta = \frac{1}{3}\pi$ and $\theta = \frac{2}{3}\pi$ is given by $\theta = \frac{1}{2}\pi$ and its numerical value is $\frac{1}{2}$.



Thus we ovserve that the larger loop lies between $\theta = -\pi/3$ and $\theta = \pi/3$ and it is symmetrical about the initial line $\theta = 0$.

Also the smaller loop lies between $\theta = \pi/3$ and $\theta = 2\pi/3$.

Hence area of the larger loop

$$= 2\int_0^{\pi/3} \tfrac{1}{2}r^2\,d\theta = \int_0^{\pi/3} (\tfrac{1}{2} + \cos 2\theta)^2\,d\theta,$$

putting for r from the given equation of the curve

$$= \int_0^{\pi/3} (\tfrac{1}{4} + \cos 2\theta + \cos^2 2\theta)\,d\theta$$

$$= \int_0^{\pi/3} [\tfrac{1}{4} + \cos 2\theta + \tfrac{1}{2}(1 + \cos 4\theta)]\,d\theta$$

$$= \left[\frac{3}{4}\theta + \frac{1}{2}\sin 2\theta + \frac{1}{2}\frac{\sin 4\theta}{4}\right]_0^{\pi/3}$$

$$= \left[\frac{\pi}{4} + \frac{\sqrt{3}}{4} - \tfrac{1}{2}\frac{\sqrt{3}}{8}\right] = \tfrac{1}{4}\left[\pi + \frac{3\sqrt{3}}{4}\right] = \frac{1}{16}(4\pi + 3\sqrt{3}).$$

Area of the smaller loop (lying between $\theta = \pi/3$ and $\theta = 2\pi/3$)

$$= \int_{\pi/3}^{2\pi/3} \tfrac{1}{2}r^2\,d\theta = \tfrac{1}{2}\int_{\pi/3}^{2\pi/3} (\tfrac{1}{2} + \cos 2\theta)^2\,d\theta$$

$$= \frac{1}{2}\left[\frac{3}{4}\theta + \frac{1}{2}\sin 2\theta + \frac{1}{2}\cdot\frac{\sin 4\theta}{4}\right]_{\pi/3}^{2\pi/3}, \text{ as in the 1st case}$$

$$= \tfrac{1}{2}\left[\frac{\pi}{4} + \frac{1}{2}\left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right) + \frac{1}{8}\left\{\frac{\sqrt{3}}{2} - \left(-\frac{\sqrt{3}}{2}\right)\right\}\right]$$

$$= \tfrac{1}{2}\left[\frac{\pi}{4} - \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{8}\right] = \tfrac{1}{2}\left[\frac{\pi}{4} - \frac{3\sqrt{3}}{8}\right] = \frac{(2\pi - 3\sqrt{3})}{16}.$$

$\therefore$ the required ratio $= \dfrac{\text{Area of the larger loop}}{\text{Area of the smaller loop}}$

$$= \frac{\frac{1}{16}(4\pi + 3\sqrt{3})}{\frac{1}{16}(2\pi - 3\sqrt{3})} = \frac{4\pi + 3\sqrt{3}}{2\pi - 3\sqrt{3}}.$$

Ex. 50. *Show that the area of a loop of $r = a\cos n\theta$ is $\pi a^2/4n$, n being integral. Also prove that the whole area is $\pi a^2/4$ or $\pi a^2/2$ according as n is odd or even.*

Sol. The number of loops in $r = a\cos n\theta$ will be n or $2n$ according as n is odd or even.

The given curve is symmetrical about the initial line.

Also putting $r = 0$, we have $\cos n\theta = 0$ *i.e.*, $n\theta = -\frac{1}{2}\pi, \frac{1}{2}\pi$ *i.e.*, $\theta = -\pi/2n, \pi/2n$. Thus two consecutive values of θ for which r is zero are $-\pi/2n$ and $\pi/2n$.

$\therefore$ one loop lies between $\theta = -\pi/2n$ and $\theta = \pi/2n$ and it is symmetrical about the initial line $\theta = 0$.

$\therefore$ area of one loop $= 2.\int_0^{\pi/2n} \frac{1}{2}r^2\,d\theta$, (By symmetry)

$$= \int_0^{\pi/2n} r^2\,d\theta = \int_0^{\pi/2n} a^2\cos^2 n\theta\,d\theta.$$

Now put $n\theta = t$ so that $n\,d\theta = dt$. Also when $\theta = 0$, $t = 0$ and when $\theta = \pi/2n$, $t = \pi/2$.

$\therefore$ the area of one loop

$$= \frac{a^2}{n}\int_0^{\pi/2} \cos^2 t\,dt = \frac{a^2}{n}\cdot\frac{1}{2}\cdot\frac{\pi}{2} = \frac{\pi a^2}{4n}.$$

Now if n is odd, the total number of loops will be n and the whole area $= n \times$ area of one loop $= n(\pi a^2/4n) = \frac{1}{4}\pi a^2$.

If n is even, the total number of loops will be $2n$ and then the whole area $= 2n \times$ area of one loop $= 2n \times (\pi a^2/4n) = \frac{1}{2}\pi a^2$.

Ex. 51. *Find the area of a loop of the curve*

$$r = a\cos 3\theta + b\sin 3\theta.$$ (Meerut 1986 P)

Sol. In the given equation of the curve

put $a = k\cos\alpha,\ b = k\sin\alpha$

so that $k = \sqrt{(a^2 + b^2)}$ and $\alpha = \tan^{-1}(b/a)$.

Thus the given equation reduces to

$$r = k\cos 3\theta\cos\alpha + k\sin 3\theta\sin\alpha$$

or

$$r = k\cos(3\theta - \alpha) = k\cos 3(\theta - \tfrac{1}{3}\alpha).$$ (Note)

Now rotating the initial line through an angle $\alpha/3$, the given equation of the curve becomes (Note)

$$r = k\cos 3(\theta + \tfrac{1}{3}\alpha - \tfrac{1}{3}\alpha) = k\cos 3\theta.$$

It should be noted that the rotation of the initial line changes only the equation of the curve and has no effect on its shape. Therefore the area of a loop of the given curve is the same as the area of a loop of the curve $r = k \cos 3\theta$.

The curve $r = k \cos 3\theta$ is symmetrical about the initial line.

Putting $r = 0$ in it we have,

$$\cos 3\theta = 0 \text{ i.e., } 3\theta = \pm \pi/2 \text{ i.e., } \theta = \pm \pi/6.$$

$\therefore$ one loop of this curve lies between $\theta = -\pi/6$ and $\theta = +\pi/6$ and it is symmetrical about the initial line.

$$\therefore \quad \text{the required area} = 2 \cdot \int_0^{\pi/6} \tfrac{1}{2} r^2 \, d\theta, \qquad \text{(By symmetry)}$$

$$= \int_0^{\pi/6} k^2 \cos^2 3\theta \, d\theta.$$

Now put $3\theta = t$, so that $3\,d\theta = dt$. Also when $\theta = 0$, $t = 0$ and when $\theta = \pi/6$, $t = \pi/2$.

$$\therefore \quad \text{the required area} = \frac{k^2}{3} \int_0^{\pi/2} \cos^2 t \, dt = \frac{k^2}{3} \cdot \tfrac{1}{2} \cdot \tfrac{1}{2}\pi = \frac{k^2}{12}\pi$$

$$= (a^2 + b^2)\,\pi/12, \qquad (\because k^2 = a^2 + b^2).$$

Ex. 52. *Trace the curve $r = \sqrt{3} \cos 3\theta + \sin 3\theta$, and find the area of a loop.*

Sol. The given equation of the curve is

$$r = \sqrt{3} \cos 3\theta + \sin 3\theta$$

$$= 2\,\{(\sqrt{3}/2) \cos 3\theta + \tfrac{1}{2} \sin 3\theta\}, \qquad \text{(Note)}$$

$$= 2\,\{\sin \pi/3 \cos 3\theta + \cos \pi/3 \sin 3\theta\},$$

$$[\because \sin \pi/3 = \sqrt{3}/2 \text{ and } \cos \pi/3 = \tfrac{1}{2}]$$

$$= 2 \sin (3\theta + \pi/3) = 2 \sin 3\,(\theta + \pi/9).$$

Now turning the initial line through an angle $-\pi/9$ the given equation of the curve becomes

$$r = 2 \sin 3\,(\theta - \pi/9 + \pi/9) = 2 \sin 3\theta.$$

Now we shall trace the curve $r = 2 \sin 3\theta$.

This curve $r = 2 \sin 3\theta$ will have 3 loops and is not symmetrical about the initial line. Also putting $r = 0$, we get

$$\sin 3\theta = 0 \text{ i.e., } 3\theta = 0, \pi \text{ i.e., } \theta = 0, \pi/3.$$

Therefore the lines $\theta = 0$ and $\theta = \pi/3$ are tangents to the curve at the pole and one loop of this curve lies between these two lines. For this loop r is greatest when $\sin 3\theta = 1$ i.e., $3\theta = \pi/2$ i.e., $\theta = \pi/6$. Thus this loop bends at $\theta = \pi/6$ and there r is equal to 2.

Here one loop of the curve lies in the region $0 < \theta < \pi/3$, one loop lies in the region $\pi/3 < \theta < 2\pi/3$ and one loop lies in the region $2\pi/3 < \theta < \pi$. If θ increases beyond π to 2π the same branches of the curve are repeated and we do not get any new branch.

The shape of the curve is similar to that shown in Ex. 40 (page 44). Here $a = 2$.

Now to get the location of the given curve turn the initial line back to its original position *i.e.*, in the figure of Ex. 40, turn the initial line through an angle $\pi/9$.

The required area of a loop

$$= 2 \times \int_0^{\pi/6} \tfrac{1}{2} r^2 \, d\theta = \int_0^{\pi/6} 4 \sin^2 3\theta \, d\theta$$

$$= 2 \int_0^{\pi/6} (1 - \cos 6\theta) \, d\theta = 2 \left[\theta - \frac{\sin 6\theta}{6} \right]_0^{\pi/6} = \frac{\pi}{3}.$$

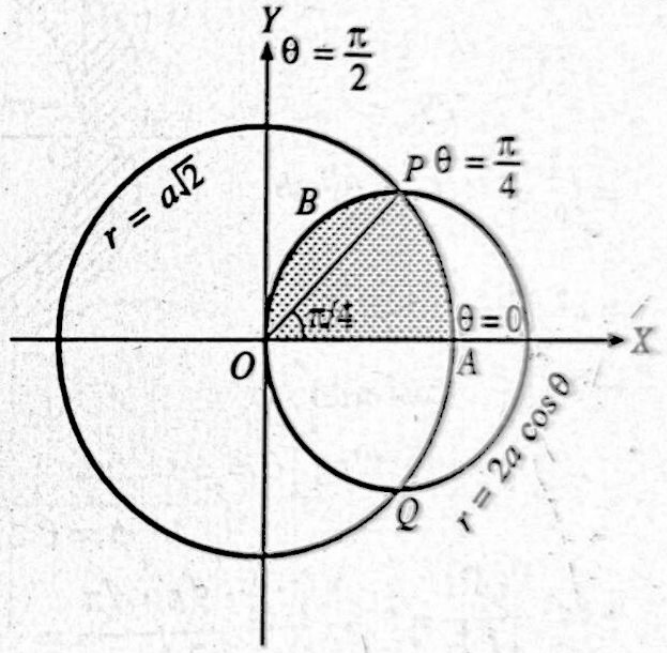
Area bounded by two curves (Polar equations).

Ex. 53. *Find the area common to the circles*

$$r = a\sqrt{2} \quad \textit{and} \quad r = 2a \cos\theta.$$

(Meerut 1993; Agra 76; Gorakhpur 75; Kanpur 74; Vikram 74; Indore 70)

Sol. The given equations of circles are $r = a\sqrt{2}$ and $r = 2a\cos\theta$. The first equation represents a circle with centre at pole and radius $a\sqrt{2}$. The second equation represents a circle passing through the pole and the diameter through the pole as the initial line. Both these circles are symmetrical about the initial line. Eliminating r between the two equations, we have at the points of intersection



$$a\sqrt{2} = 2a\cos\theta, \textit{ i.e.}, \cos\theta = 1/\sqrt{2}, \textit{i.e.}, \theta = \pm\pi/4.$$

Thus at P, $\theta = \pi/4$. For the circle $r = 2a\cos\theta$, at O, $r = 0$ and so $\cos\theta = 0$ *i.e.*, $\theta = \frac{1}{2}\pi$.

Now the required area = Area $OQAPBO$

$= 2$ (area $OAPBO$), (by symmetry)

$= 2$ [Area OAP + Area $OPBO$]

$$= 2 \left[\tfrac{1}{2} \int_0^{\pi/4} r^2 \, d\theta, \text{ for the circle } r = a\sqrt{2} + \tfrac{1}{2} \int_{\pi/4}^{\pi/2} r^2 \, d\theta, \text{ for the circle } r = 2a\cos\theta \right]$$

$$= \int_0^{\pi/4} (a\sqrt{2})^2 \, d\theta + \int_{\pi/4}^{\pi/2} (2a\cos\theta)^2 \, d\theta$$

$$= 2a^2 \left[\theta\right]_0^{\pi/4} + 2a^2 \int_{\pi/4}^{\pi/2} (1 + \cos 2\theta)\, d\theta$$

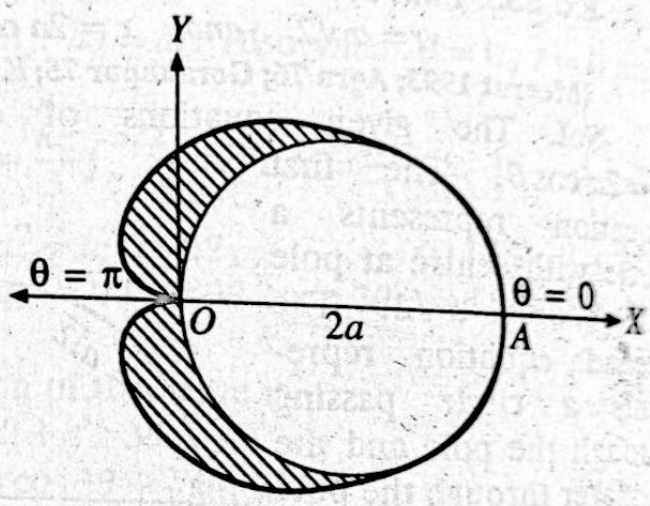
$$= 2a^2 \left(\frac{\pi}{4}\right) + 2a^2 \left[\theta + \frac{\sin 2\theta}{2}\right]_{\pi/4}^{\pi/2} = \frac{\pi a^2}{2} + 2a^2 \left[\frac{\pi}{2} - \frac{\pi}{4} - \frac{1}{2}\right]$$

$$= \tfrac{1}{2}\pi a^2 + \tfrac{1}{2}\pi a^2 - a^2 = \pi a^2 - a^2 = a^2(\pi - 1).$$

Ex. 54. *Find the area outside the circle $r = 2a \cos\theta$ and inside the cardioid $r = a(1 + \cos\theta)$.*

Sol. The two curves intersect where $2a\cos\theta = a(1 + \cos\theta)$ *i.e.*, $\cos\theta = 1$ *i.e.*, $\theta = 0$. Besides this the two curves also intersect at the pole $r = 0$. Since for all values of θ,
$2a\cos\theta$ *i.e.*, $a\cos\theta + a\cos\theta \le a(1 + \cos\theta)$, therefore the circle lies entirely within the cardioid.

Hence the required area
= Area of the cardioid
− Area of the circle. ...(1)



Now area of the cardioid

$$= 2\int_0^{\pi} \tfrac{1}{2} r^2\, d\theta$$

$$= \int_0^{\pi} a^2 (1 + \cos\theta)^2\, d\theta$$

$$= a^2 \int_0^{\pi} 4\cos^4 \tfrac{1}{2}\theta\, d\theta$$

$$= 8a^2 \int_0^{\pi/2} \cos^4\phi\, d\phi,$$

(putting $\frac{1}{2}\theta = \phi$ so that $\frac{1}{2}d\theta = d\phi$, also when $\theta = 0$, $\phi = 0$ and when $\theta = \pi$, $\phi = \frac{1}{2}\pi$)

$$= 8a^2 \frac{\Gamma\frac{5}{2}\Gamma\frac{1}{2}}{2\Gamma 3} = 8a^2 \cdot \frac{\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}\cdot\sqrt{\pi}}{2.2.1} = \frac{3\pi a^2}{2}.$$

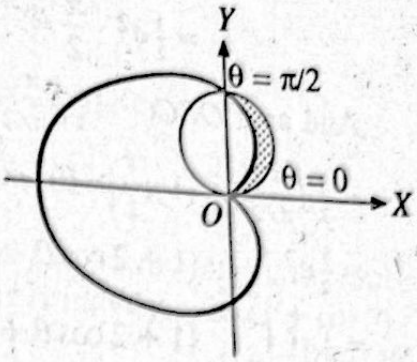
And area of the circle $= \pi a^2$, because the radius of the circle is a.

$\therefore$ From (1), the required area $= \frac{3}{2}\pi a^2 - \pi a^2 = \frac{1}{2}\pi a^2$.

Ex. 55. *Find the total area inside $r = \sin\theta$ and outside $r = 1 - \cos\theta$.*

Sol. Eliminating r between the given equations, we have
$\sin\theta = 1 - \cos\theta$, or $\sin^2\theta = (1 - \cos\theta)^2 = 1 + \cos^2\theta - 2\cos\theta$,
or $1 - \cos^2\theta = 1 + \cos^2\theta - 2\cos\theta$, or $2\cos\theta(\cos\theta - 1) = 0$.
$\therefore \cos\theta = 0$ or $\cos\theta = 1$ *i.e.*, $\theta = 0$ or $\pi/2$. Thus the two curves intersect at the points where $\theta = 0$ and $\theta = \pi/2$.

Draw the two curves in the same figure. The first curve is a circle passing through the pole and the diameter through the pole along the line $\theta = \pi/2$. The second curve is a cardioid.



$\therefore$ required area $= \frac{1}{2}\int_0^{\pi/2} r^2\, d\theta$, for the curve $r = \sin\theta$

$$- \frac{1}{2}\int_0^{\pi/2} r^2\, d\theta, \text{ for the curve } r = 1 - \cos\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} \sin^2\theta\, d\theta - \frac{1}{2}\int_0^{\pi/2} (1-\cos\theta)^2\, d\theta$$

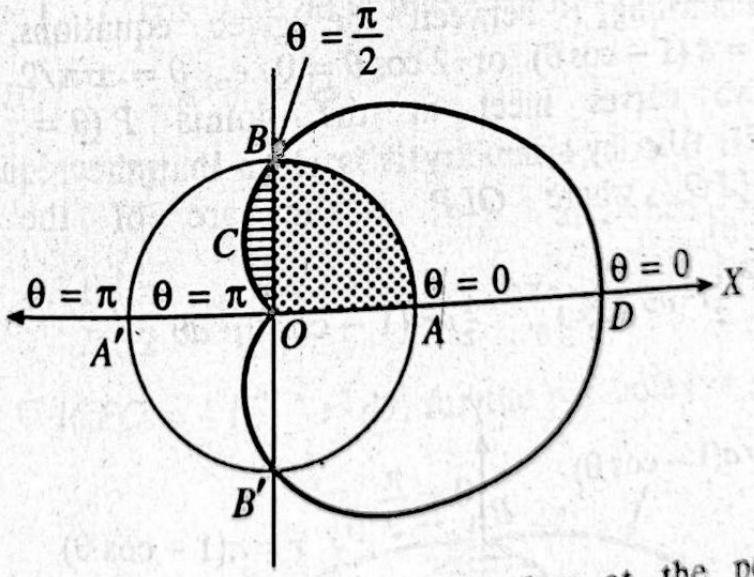
$$= \frac{1}{2}\int_0^{\pi/2} \left\{\sin^2\theta - (1-\cos\theta)^2\right\} d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} 2(\cos\theta - \cos^2\theta)\, d\theta = \left[\sin\theta\right]_0^{\pi/2} - \frac{1}{2}\cdot\frac{\pi}{2} = \left(1 - \frac{\pi}{4}\right).$$

Ex. 56. *Find the area common to the circle $r = a$ and the cardioid $r = a(1 + \cos\theta)$.* (Meerut 1996, 97)

Sol. Eliminating r between the given equations, we get

$$a(1 + \cos\theta) = a \quad \text{or} \quad \cos\theta = 0 \quad \text{or} \quad \theta = \pm\pi/2.$$



Thus the two curves cut each other at the point where $\theta = \pm\pi/2$.

Both the curves are symmetrical about the initial line and have been shown in the same figure.

Hence the required area $= 2 \times$ area $ABCOA$, (By symmetry) ...(1)

$= 2 \times$ (Area $OABO$ + Area $OBCO$)

Now area $OABO = \frac{1}{2}\int_{\theta=0}^{\pi/2} r^2\, d\theta$, for $r = a$

$$= \frac{1}{2}a^2\int_0^{\pi/2} d\theta = \frac{1}{2}a^2\left[\theta\right]_0^{\pi/2}$$

$$= \tfrac{1}{2} a^2 \cdot \frac{\pi}{2} = \frac{\pi a^2}{4}.$$

And area $OBCO = \frac{1}{2}\int_{\pi/2}^{\pi} r^2\, d\theta$, for $r = a\,(1 + \cos\theta)$

$$= \tfrac{1}{2}\int_{\pi/2}^{\pi} a^2 (1 + \cos\theta)^2\, d\theta$$

$$= \tfrac{1}{2} a^2 \int_{\pi/2}^{\pi} (1 + 2\cos\theta + \cos^2\theta)\, d\theta$$

$$= \tfrac{1}{2} a^2 \int_{\pi/2}^{\pi} \{1 + 2\cos\theta + \tfrac{1}{2}(1 + \cos 2\theta)\}\, d\theta$$

$$= \tfrac{1}{2} a^2 \int_{\pi/2}^{\pi} (\tfrac{3}{2} + 2\cos\theta + \tfrac{1}{2}\cos 2\theta)\, d\theta$$

$$= \frac{a^2}{2}\left[\frac{3\theta}{2} + 2\sin\theta + \frac{\sin 2\theta}{4}\right]_{\pi/2}^{\pi} = \frac{a^2}{2}\left(\frac{3\pi}{2} - \frac{3\pi}{4} - 2\right)$$

$$= \frac{a^2}{8}(3\pi - 8).$$

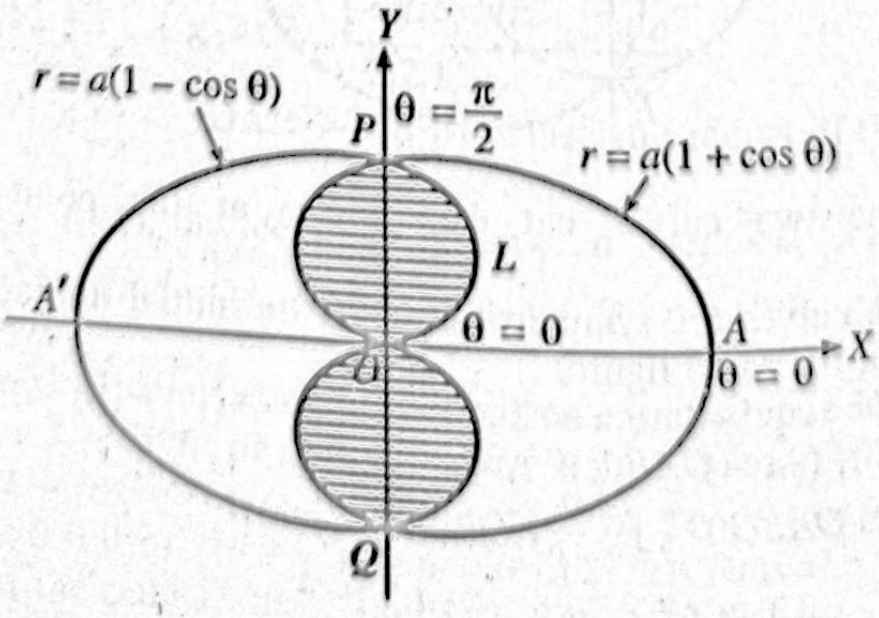
Hence from (1), the required area

$$= 2\left[\frac{\pi a^2}{4} + \frac{a^2}{8}(3\pi - 8)\right] = a^2\left(\frac{5\pi}{4} - 2\right).$$

Ex. 57. *Find the area of the portion included between the cardioids $r = a\,(1 + \cos\theta)$ and $r = a\,(1 - \cos\theta)$.* (Allahabad 1974)

Sol. Eliminating r between the given equations, we get $a\,(1 + \cos\theta) = a\,(1 - \cos\theta)$ or $2\cos\theta = 0$ *i.e.*, $\theta = \pm\pi/2$, showing that the two curves meet at the points $P\,(\theta = \pi/2)$ and $Q\,(\theta = -\pi/2)$. Also by symmetry it is clear that the required area $= 4 \times$ area $OLPO$, where OLP is an arc of the cardioid $r = a\,(1 - \cos\theta)$

$$= 4\int_0^{\pi/2} \tfrac{1}{2} r^2\, d\theta = 4\int_0^{\pi/2} \tfrac{1}{2} a^2 (1 - \cos\theta)^2\, d\theta$$



$$= 2a^2 \int_0^{\pi/2} (1 - 2\cos\theta + \cos^2\theta)\, d\theta$$

$$= 2a^2 \int_0^{\pi/2} [1 - 2\cos\theta + \tfrac{1}{2}(1 + \cos 2\theta)]\, d\theta$$

$$= 2a^2 \left[\frac{3}{2}\theta - 2\sin\theta + \tfrac{1}{4}\sin 2\theta\right]_0^{\pi/2} = 2a^2\left(\frac{3}{4}\pi - 2\right).$$

Ex. 58. *Find the ratio of the two parts into which the parabola $2a = r(1 + \cos\theta)$ divides the area of the cardioid $r = 2a(1 + \cos\theta)$.*

(Jiwaji 1971)

Sol. Eliminating r between the given equations of the curves, we get

$$2a(1 + \cos\theta) = 2a/(1 + \cos\theta) \text{ or } (1 + \cos\theta)^2 = 1$$

or $\cos\theta(\cos\theta + 2) = 0$ or $\cos\theta = 0$, $[\because \cos\theta \neq -2]$

or $\theta = \pm\pi/2$. Thus at the point of intersection P of the two curves, $\theta = \pi/2$.

Now area of the whole cardioid

$$= 2 \times \tfrac{1}{2}\int_0^{\pi} r^2\, d\theta, \text{ (by symmetry)}$$

$$= \int_0^{\pi} 4a^2(1 + \cos\theta)^2\, d\theta$$

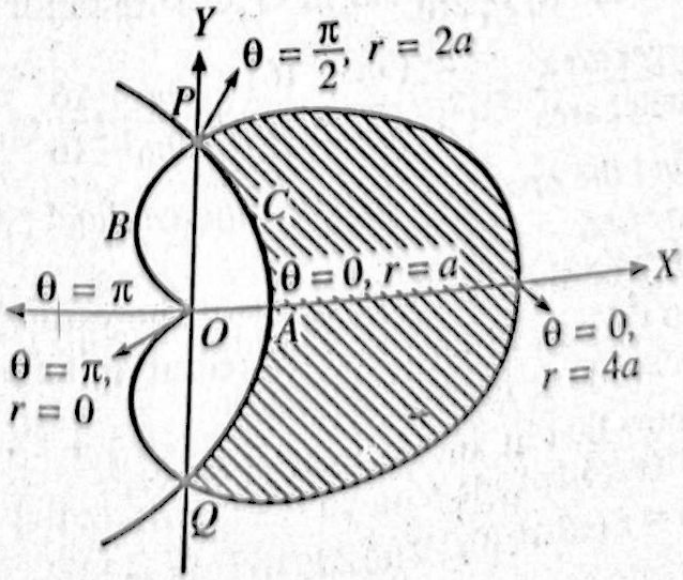
$$= 4a^2 \int_0^{\pi} (2\cos^2\tfrac{1}{2}\theta)^2\, d\theta = 16a^2 \int_0^{\pi} \cos^4\tfrac{1}{2}\theta\, d\theta$$

$$= 16a^2 \int_0^{\pi/2} \cos^4\phi\, .2\, d\phi,$$

(putting $\tfrac{1}{2}\theta = \phi$ so that $\tfrac{1}{2}d\theta = d\phi$; also when $\theta = 0, \phi = 0$ and when $\theta = \pi, \phi = \tfrac{1}{2}\pi$)

$$= 32a^2 \frac{3.1}{4.2} \cdot \frac{\pi}{2} = 6\pi a^2. \quad \text{...(1)}$$

Area $OACPO = \tfrac{1}{2}\int_0^{\pi/2} r^2\, d\theta$, for the parabola $r = \dfrac{2a}{1 + \cos\theta}$



$$= \tfrac{1}{2} \cdot 4a^2 \int_0^{\pi/2} \frac{d\theta}{(1+\cos\theta)^2} = \frac{a^2}{2} \int_0^{\pi/2} \sec^4 \tfrac{1}{2}\theta \, d\theta$$

$$= \tfrac{1}{2} a^2 \int_0^{\pi/2} (1 + \tan^2 \tfrac{1}{2}\theta) \sec^2 \tfrac{1}{2}\theta \, d\theta$$

$$= a^2 \left[\tan \tfrac{1}{2}\theta + \tfrac{1}{3} \tan^3 \tfrac{1}{2}\theta\right]_0^{\pi/2} = a^2 (1 + \tfrac{1}{3}) = \frac{4a^2}{3}. \qquad ...(2)$$

Also area $OPBO = \frac{1}{2} \int_{\pi/2}^{\pi} r^2 d\theta$, for the cardioid

$$r = 2a(1 + \cos\theta)$$

$$= \tfrac{1}{2} \int_{\pi/2}^{\pi} 4a^2 (1 + \cos\theta)^2 d\theta$$

$$= 2a^2 \int_{\pi/2}^{\pi} [1 + 2\cos\theta + \cos^2\theta] \, d\theta$$

$$= 2a^2 \int_{\pi/2}^{\pi} (1 + 2\cos\theta + \tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta) \, d\theta$$

$$= 2a^2 \left[\tfrac{3}{2}\theta + 2\sin\theta + \tfrac{1}{4}\sin 2\theta\right]_{\pi/2}^{\pi}$$

$$= \frac{3}{2}\pi a^2 - 4a^2. \qquad ...(3)$$

Adding (2) and (3) and multiplying by 2, we get the whole area included between the two curves *i.e.*, the area of the smaller portion of the cardioid

$$= 2 \times \left[\frac{4}{3}a^2 + \left(\frac{3}{2}\pi a^2 - 4a^2\right)\right] = a^2 \left[3\pi - \frac{16}{3}\right]$$

$$= \tfrac{1}{3} a^2 [9\pi - 16]. \qquad ...(4)$$

Also the shaded area (*i.e.*, the area of the larger portion of the cardioid)

= (Area of the whole cardioid) − (unshaded area)

i.e., = (1) − (4)

$$= 6\pi a^2 - \tfrac{1}{3} a^2 (9\pi - 16) = \tfrac{1}{3} a^2 (9\pi + 16). \qquad ...(5)$$

∴ Ratio of the two parts

$$= \frac{\text{Larger area}}{\text{Smaller area}} = \frac{\frac{1}{3} a^2 (9\pi + 16)}{\frac{1}{3} a^2 (9\pi - 16)} = \frac{9\pi + 16}{9\pi - 16}.$$

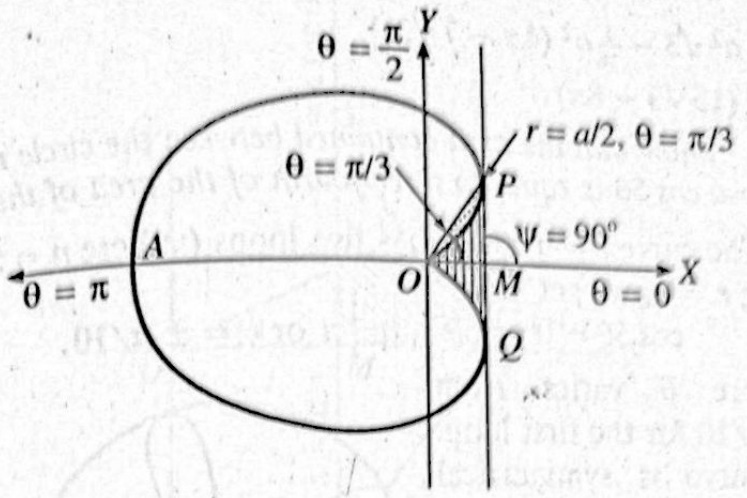
Ex. 59. *Find the area lying between the cardioid $r = a(1 - \cos\theta)$ and its double tangent.* **(Kanpur 1976)**

Sol. Let PQ be the double tangent of the cardioid. Clearly it is perpendicualr to OX *i.e.*, it must be inclined at an angle of 90° to the initial line *i.e.*, $\psi = 90°$ at P.

Also we know that at any point of a curve,

$$\psi = \theta + \phi. \qquad ...(1)$$

Now $\tan\phi = r(d\theta/dr) = r/(dr/d\theta)$



$= a(1-\cos\theta)/(a\sin\theta)$, $\quad [\because r = a(1-\cos\theta)]$

$$= \frac{2\sin^2\frac{1}{2}\theta}{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta} = \tan\tfrac{1}{2}\theta.$$

$\therefore \quad \phi = \frac{1}{2}\theta.$

Putting the value of ϕ in (1), we get $\psi = \theta + \frac{1}{2}\theta = \frac{3}{2}\theta$. Since at $P, \psi = \frac{1}{2}\pi$, therefore at P, $\frac{1}{2}\pi = \frac{3}{2}\theta$ or $\theta = \pi/3$.

$\therefore$ the vectorial angle of the point of contact P of the double tangent is $\pi/3$ *i.e.*, 60°. Substituting this value of θ in the equation of the curve, we get the radius vector $OP = a(1-\cos 60^\circ) = a/2$.

Thus in the triangle OPM,

$OP = \frac{1}{2}a$, $\angle POM = 60^\circ$, $\angle PMO = 90^\circ$.

$\therefore \quad OM = \frac{1}{2}a\cos 60^\circ = \frac{1}{2}a \cdot \frac{1}{2}$ and $PM = \frac{1}{2}a\sin 60^\circ = \frac{1}{2}a(\sqrt{3}/2)$.

$\therefore$ area of the traingle $OPM = \frac{1}{2}OM \,.\, PM = \frac{1}{2}(\frac{1}{4}a)(\sqrt{3}a/4)$

$= (1/32)\,a^2\sqrt{3}.$

Also the sectorial area OPO of the cardioid $r = a(1-\cos\theta)$ *i.e.*, the dotted area

$$= \tfrac{1}{2}\int_0^{\pi/3} r^2\,d\theta = \tfrac{1}{2}\int_0^{\pi/3} a^2(1-\cos\theta)^2\,d\theta$$

$$= \tfrac{1}{2}a^2\int_0^{\pi/3}(1 - 2\cos\theta + \tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta)\,d\theta$$

$$= \tfrac{1}{2}a^2\left[\frac{3}{2}\theta - 2\sin\theta + \frac{1}{4}\sin 2\theta\right]_0^{\pi/3}$$

$$= \tfrac{1}{2}a^2(\tfrac{1}{2}\pi - \sqrt{3} + \tfrac{1}{8}\sqrt{3}) = \tfrac{1}{16}a^2(4\pi - 7\sqrt{3}).$$

Hence the required area (*i.e.*, the area shaded by vertical lines)

$= 2$ [area of $\Delta\, OPM$ − area of sector OPO]

$= 2\left[\frac{1}{32}a^2\sqrt{3} - \frac{1}{16}a^2(4\pi - 7\sqrt{3})\right]$

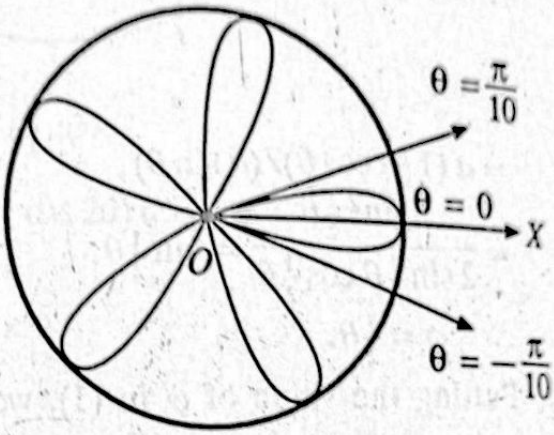
$= \frac{1}{16}a^2(15\sqrt{3} - 8\pi)$.

Ex. 60. *Show that the area contained between the circle $r = a$ and the curve $r = a\cos 5\theta$ is equal to three-fourth of the area of the circle.*

Sol. The curve $r = a\cos 5\theta$ has five loops ($\because$ here $n = 5$ is odd). Also putting $r = 0$, we get

$$\cos 5\theta = 0 \text{ or } 5\theta = \pm\tfrac{1}{2}\pi \text{ or } \theta = \pm\pi/10.$$

Therefore θ varies from $-\pi/10$ to $\pi/10$ for the first loop. Also the curve is symmetrical about the initial line. Further giving values to θ from 0 to 2π in $r = a\cos 5\theta$, we observe that the maximum value of r is a and hence the curve $r = a\cos 5\theta$ lies completely inside the circle $r = a$ as shown in the figure.



Now area of the five loops of the curve $r = a\cos 5\theta$

$= 5$ times area of one loop

$$= 5\int_{-\pi/10}^{\pi/10} \tfrac{1}{2}r^2\,d\theta = \frac{5}{2}\int_{-\pi/10}^{\pi/10} a^2\cos^2 5\theta\,d\theta$$

$$= \frac{5}{2}\cdot 2\int_0^{\pi/10} a^2\cos^2 5\theta\,d\theta = a^2\int_0^{\pi/2}\cos^2\phi\,d\phi,$$

[putting $5\theta = \phi$ so that $5\,d\theta = d\phi$; also when $\theta = 0$, $\phi = 0$ and when $\theta = \pi/10$, $\phi = \pi/2$]

$= a^2\cdot\frac{1}{2}\cdot\frac{1}{2}\pi = \frac{1}{4}\pi a^2$.

Also area of the circle $= \pi a^2$.

$\therefore$ area contained between the two curves

= area of the circle − area of the curve $r = a\cos 5\theta$

$= \pi a^2 - \frac{1}{4}\pi a^2$

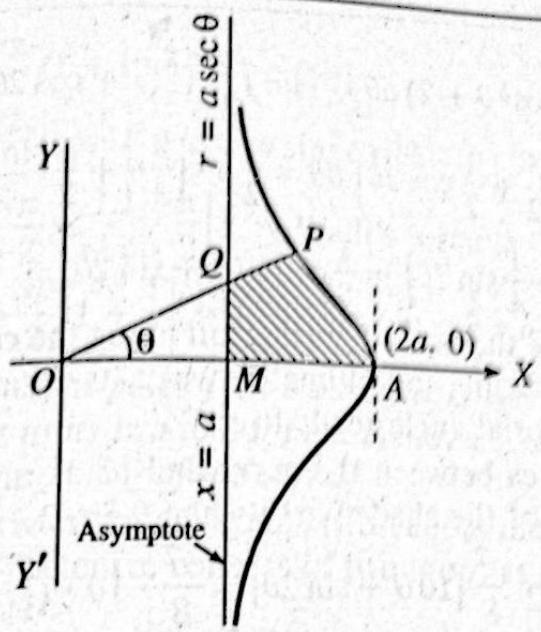
$= (3/4)\pi a^2 = (3/4)$ of the area of the circle.

Ex. 61. *Find the area between the curve $r = a(\sec\theta + \cos\theta)$ and its asymptote.* **(U.P. P.C.S. 1997)**

Sol. The given curve is symmetrical about the initial line. When $\theta = 0$, $r = 2a$.

The given equation of the curve can be written as

$$r = a\{(1/\cos\theta) + \cos\theta\} = a(1 + \cos^2\theta)/\cos\theta$$



or $$\frac{1}{r} = \frac{\cos\theta}{a(1+\cos^2\theta)} = f(\theta), \text{ say.}$$

Now $f(\theta) = 0$ gives $\cos\theta = 0$ *i.e.*, $\theta = \pi/2$.

Also $$f'(\theta) = \frac{1}{a}\,\frac{-\sin\theta(1+\cos^2\theta) - \cos\theta(-2\sin\theta\cos\theta)}{(1+\cos^2\theta)^2}. \quad ...(1)$$

Putting $\theta = \pi/2$ in (1), we get $f'(\pi/2) = -1/a$.

$\therefore$ asymptote of the curve is $r\sin(\theta - \frac{1}{2}\pi) = 1/(-1/a)$

or $r\cos\theta = a$ *i.e.*, $r = a\sec\theta$.

The cartesian equation of the asymptote is $x = a$.

To find the area between the curve and its asymptote.

Let OQP be a radius vector cutting the curve at P and the asymptote $r = a\sec\theta$ at Q. Let $\angle XOP = \theta$.

Then the shaded area $MAPQM$ = area $OAPO$ − area $OMQO$.

Now area $OAPO = \int_0^\theta \frac{1}{2}r^2\,d\theta$, for the given curve

$$= \frac{1}{2}\int_0^\theta a^2(\sec\theta + \cos\theta)^2\,d\theta$$

$$= \frac{1}{2}\int_0^\theta a^2(\sec^2\theta + \cos^2\theta + 2)\,d\theta.$$

Also area $OMQO = \int_0^\theta \frac{1}{2}r^2\,d\theta$, for the st. line $r = a\sec\theta$

$$= \frac{1}{2}\int_0^\theta a^2\sec^2\theta\,d\theta.$$

$\therefore$ the shaded area $MAPQM$

$$= \frac{1}{2}\int_0^\theta a^2(\sec^2\theta + \cos^2\theta + 2)\,d\theta - \frac{1}{2}\int_0^\theta a^2\sec^2\theta\,d\theta$$

$$= \tfrac{1}{2}\int_0^\theta a^2(\cos^2\theta + 2)\,d\theta = \tfrac{1}{2}a^2\int_0^\theta \{\tfrac{1}{2}(1+\cos 2\theta) + 2\}\,d\theta$$

$$= \tfrac{1}{2}a^2\int_0^\theta \left(\frac{5}{2} + \frac{1}{2}\cos 2\theta\right)d\theta = \frac{1}{2}a^2\left[\frac{5}{2}\theta + \frac{1}{2}\frac{\sin 2\theta}{2}\right]_0^\theta$$

$$= \tfrac{1}{2}a^2\left(\frac{5}{2}\theta + \frac{1}{4}\sin 2\theta\right) = \frac{1}{8}a^2(10\theta + \sin 2\theta).$$

Now we move the point P further off along the curve. Its vectorial angle goes on increasing and ultimately when its distance from O tends to infinity its vectorial angle tends to $\pi/2$.

Hence the area between the curve and its asymptote lying above the x-axis = limit of the shaded area when $\theta \to \pi/2$

$$= \lim_{\theta\to\pi/2} \frac{a^2}{8}[10\theta + \sin 2\theta] = \frac{a^2}{8}\cdot 10\cdot(\tfrac{1}{2}\pi) = \frac{5}{8}\pi a^2.$$

$\therefore$ by symmetry, the total area between the curve and the asymptote $= 2.(5\pi a^2/8) = 5\pi a^2/4$.

Aliter. The above area can also be easily found by changing the equation of the curve to cartesian form.

The equation of the curve can be written as

$$r\cos\theta = a(1+\cos^2\theta) \quad \text{or} \quad r\cos\theta - a = a\cos^2\theta$$

or $\quad r^2(r\cos\theta - a) = a r^2\cos^2\theta$, multiplying both sides by r^2.

Now putting $r\cos\theta = x$ and $r^2 = x^2 + y^2$, we get

$$(x^2+y^2)(x-a) = ax^2 \text{ or } y^2(x-a) = ax^2 - x^2(x-a)$$

or $y^2(x-a) = x^2(2a-x)$, which is the cartesian equation of the given curve.

Now trace the curve with the help of this cartesian equation. The curve is symmetrical about x-axis. It meets the x-axis at the point $(2a, 0)$ and the line $x = a$ is an asymptote of the curve. Here origin is a conjugate point because we get imaginary tangents at the origin. The curve does not exist in the regions $x > 2a$ and $x < a$. It exists only in the region $a < x \le 2a$.

Hence the required area

$$= 2\int_a^{2a} y\,dx = 2\int_a^{2a} x\sqrt{\left(\frac{2a-x}{x-a}\right)}\,dx,$$

(putting for y from the given equation of the curve).

Now put $x = a + a\sin^2\theta$ so that $dx = 2a\sin\theta\cos\theta\,d\theta$.

Also when $x = a$, $\sin\theta = 0$ or $\theta = 0$ and when $x = 2a$, $\sin\theta = 1$ or $\theta = \pi/2$.

$\therefore$ the required area

$$= 2\int_0^{\pi/2} (a + a\sin^2\theta)\frac{\cos\theta}{\sin\theta}\cdot 2a\sin\theta\cos\theta\,d\theta$$

$$= 4a^2 \int_0^{\pi/2} (1 + \sin^2\theta) \cos^2\theta \, d\theta$$

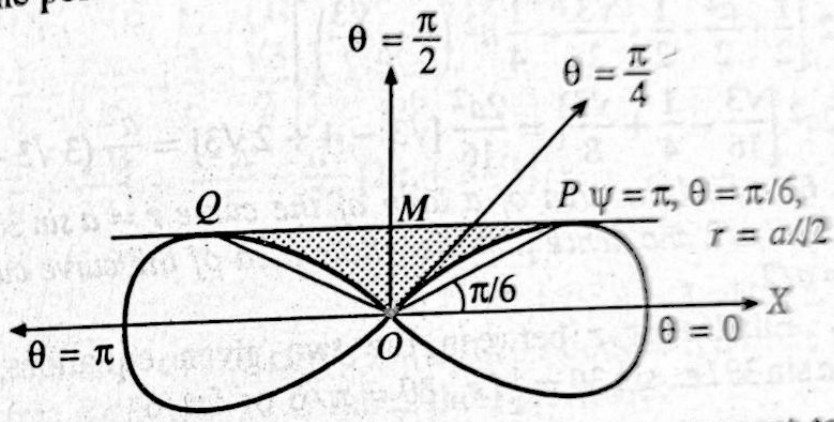
$$= 4a^2 \int_0^{\pi/2} (\cos^2\theta + \cos^2\theta \sin^2\theta) \, d\theta$$

$$= 4a^2 \left[\frac{1}{2} \cdot \frac{\pi}{2} + \frac{1.1}{4.2} \cdot \frac{\pi}{2}\right], \text{ by Walli's formula}$$

$$= 4a^2 \cdot \frac{1}{2} \cdot \frac{\pi}{2} \left[1 + \frac{1}{4}\right] = 4a^2 \cdot \frac{1}{4}\pi \cdot \frac{5}{4} = \frac{5}{4}\pi a^2.$$

Ex. 62. *O is the pole of the lemniscate $r^2 = a^2 \cos 2\theta$ and PQ is a common tangent to its two loops. Find the area bounded by the line PQ and the arcs OP and OQ of the curve.*

Sol. The given curve is symmetrical about the initial line and also about the pole. The curve consists of two loops as shown in the figure.



From the figure it is clear that the common tangent to the two loops of the curve *i.e.*, the line PQ is parallel to the axis of x. Thus the tangent PQ at the point P makes an angle π with the x-axis *i.e.*, $\psi = \pi$ at P.

Now differentiating the given equation $r^2 = a^2 \cos 2\theta$, we get

$$2r\frac{dr}{d\theta} = a^2(-2\sin 2\theta), \qquad \text{or} \quad \frac{dr}{d\theta} = -\frac{a^2 \sin 2\theta}{r}.$$

$$\therefore \quad \tan\phi = r\frac{d\theta}{dr} = -\frac{r \times r}{a^2 \sin 2\theta} = -\frac{a^2 \cos 2\theta}{a^2 \sin 2\theta}$$

$$= -\cot 2\theta = \tan\left(\tfrac{1}{2}\pi + 2\theta\right).$$

$$\therefore \quad \phi = \tfrac{1}{2}\pi + 2\theta.$$

But for any point of the curve,

$$\psi = \theta + \phi = \theta + \left(\tfrac{1}{2}\pi + 2\theta\right) = \tfrac{1}{2}\pi + 3\theta.$$

Since at P, $\psi = \pi$, therefore at P, $\pi = \frac{1}{2}\pi + 3\theta$ or $\theta = \pi/6$.

Thus the vectorial angle θ of the point P is $\pi/6$ and the radius vector OP is given by $OP^2 = a^2 \cos\{2.(\pi/6)\} = a^2/2$.

$$\therefore \quad OP = a/\sqrt{2}.$$

Also putting $r = 0$ in the equation $r^2 = a^2 \cos 2\theta$, we get

$$\cos 2\theta = 0 \quad \text{or} \quad 2\theta = \pm \pi/2 \quad \text{or} \quad \theta = \pm \pi/4.$$

Thus $\theta = \pm \pi/4$ are the tangents at the pole to the curve.

Now the required area (*i.e.*, the dotted area)

$= 2$ [area of the $\Delta\, OPM$ − area of the segment $OPSO$ of the curve $r^2 = a^2 \cos 2\theta$]

$$= 2\left[\tfrac{1}{2} OM.OP - \int_{\pi/6}^{\pi/4} \tfrac{1}{2} r^2\, d\theta, \text{ for the curve } r^2 = a^2 \cos 2\theta\right]$$

$$= 2\left[\tfrac{1}{2}(OP \sin \tfrac{1}{6}\pi).(OP \cos \tfrac{1}{6}\pi) - \int_{\pi/6}^{\pi/4} \tfrac{1}{2} a^2 \cos 2\theta\, d\theta\right]$$

$$= 2\left[\tfrac{1}{2} OP^2 \sin \tfrac{1}{6}\pi \cos \tfrac{1}{6}\pi - \tfrac{1}{2} a^2 \left(\frac{\sin 2\theta}{2}\right)_{\pi/6}^{\pi/4}\right]$$

$$= 2\left[\frac{1}{2} \cdot \frac{a^2}{2} \cdot \frac{1}{2} \cdot \frac{\sqrt{3}}{2} - \frac{1}{4} a^2 \left(1 - \frac{\sqrt{3}}{2}\right)\right]$$

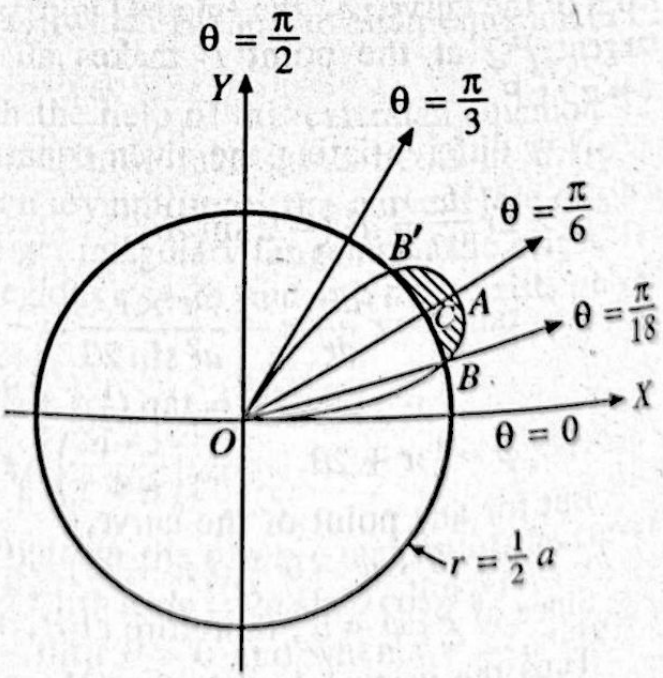
$$= 2a^2\left[\frac{\sqrt{3}}{16} - \frac{1}{4} + \frac{\sqrt{3}}{8}\right] = \frac{2a^2}{16}[\sqrt{3} - 4 + 2\sqrt{3}] = \frac{a^2}{8}(3\sqrt{3} - 4).$$

Ex. 63. *Find the area of a loop of the curve $r = a \sin 3\theta$ outside the circle $r = a/2$ and hence find the whole area of the curve outside the circle $r = a/2$.*

Sol. Eliminating r between the two given equations, we get $(a/2) = a \sin 3\theta$ *i.e.*, $\sin 3\theta = \frac{1}{2}$ *i.e.*, $3\theta = \pi/6$ or $5\pi/6$

i.e., $\theta = \pi/18$ or $5\pi/18$ *i.e.*, $\theta = 10^\circ$ or 50°.

Thus the loop of the curve $r = a \sin 3\theta$ lying between $\theta = 0$ and $\theta = \pi/3$ intersects the circle $r = a/2$ at the points B and B' where $\theta = 10^\circ$ at B and $\theta = 50^\circ$ at B'. This loop is symmetrical about OA and $\theta = \pi/6$ at A.



Now the required area of a loop of the curve $r = a \sin 3\theta$ lying outside the circle $r = a/2$

$=$ the area $BAB'CB$

(*i.e.*, the shaded area)

$= 2 \times$ area $BACB$, (by symmetry)

$= 2 \times$ [(area of the curve $r = a \sin 3\theta$ between the radii vectors OB and OA *i.e.*, $\theta = \pi/18$ and $\theta = \pi/6$) − (area of the circle $r = a/2$ between the radii vectors OB and OC *i.e.*, $\theta = \pi/18$ and $\theta = \pi/6$)]

$$= 2\left[\tfrac{1}{2}\int_{\pi/18}^{\pi/6} r^2\,d\theta, \quad \text{for the curve } r = a\sin 3\theta\right.$$

$$\left. - \tfrac{1}{2}\int_{\pi/18}^{\pi/6} r^2\,d\theta, \quad \text{for the circle } r = \frac{a}{2}\right]$$

$$= \int_{\pi/18}^{\pi/6} a^2\sin^2 3\theta\,d\theta - \int_{\pi/18}^{\pi/6} \frac{a^2}{4}\,d\theta$$

$$= \frac{a^2}{2}\int_{\pi/18}^{\pi/6} (1-\cos 6\theta)\,d\theta - \frac{a^2}{4}\left[\theta\right]_{\pi/18}^{\pi/6}$$

$$= \frac{a^2}{2}\left[\theta - \frac{\sin 6\theta}{6}\right]_{\pi/18}^{\pi/6} - \frac{a^2}{4}\left[\frac{\pi}{6} - \frac{\pi}{18}\right]$$

$$= \frac{a^2}{2}\left[\left\{\frac{\pi}{6} - \frac{\sin\pi}{6}\right\} - \left\{\frac{\pi}{18} - \frac{1}{6}\sin\frac{\pi}{3}\right\}\right] - \frac{a^2}{4}\cdot\frac{\pi}{9}$$

$$= \frac{a^2}{2}\left[\frac{\pi}{6} - \left\{\frac{\pi}{18} - \frac{1}{6}\cdot\frac{\sqrt{3}}{2}\right\}\right] - \frac{a^2\pi}{36}$$

$$= \frac{a^2}{2}\left[\frac{\pi}{9} + \frac{\sqrt{3}}{12}\right] - \frac{a^2\pi}{36} = \frac{a^2}{72}[2\pi + 3\sqrt{3}].$$

Again the curve $r = a\sin 3\theta$ has 3 equal loops

($\because n = 3$ which is odd).

$\therefore$ whole area of the curve $r = a\sin 3\theta$ outside the circle $r = a/2$

$= 3 \times$ area $BAB'CB$ *i.e.*, 3 times the shaded area

$$= 3 \times \frac{1}{72}a^2[2\pi + 3\sqrt{3}] = \frac{1}{24}a^2[2\pi + 3\sqrt{3}].$$

Cartesian equations changed to polar form.

Sometimes it is convenient to find the required area if the given cartesian equation of the curve is changed to polar form by putting $x = r\cos\theta$ and $y = r\sin\theta$.

**Ex. 64. *Find the area of a loop of the folium*

$$x^3 + y^3 = 3axy.$$

(Garhwal 1983; Bhopal 82; Rohilkhand 82; Gorakhpur 76, 73; Meerut 71, 85, 88S, 97; Bihar 72)

Also prove that the area included between the folium $x^3 + y^3 = 3axy$ and its asymptote is equal to the area of its loop.

(Allahabad 73)

Sol. Changing the equation of the curve $x^3 + y^3 = 3axy$ into polar from by putting $x = r\cos\theta$ and $y = r\sin\theta$, we have

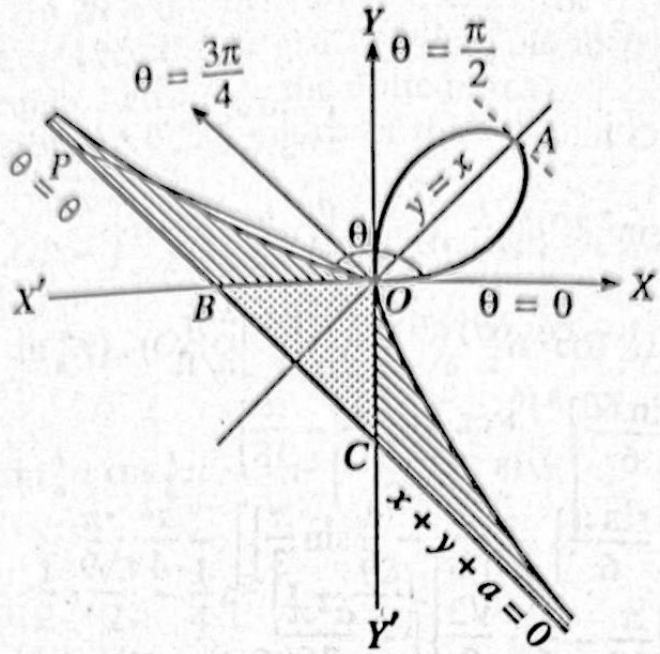
$$(r\cos\theta)^3 + (r\sin\theta)^3 = 3a(r\cos\theta).(r\sin\theta) \quad \text{...(1)}$$

or $\quad r = 3a\cos\theta\sin\theta/(\cos^3\theta + \sin^3\theta)$.

From (1), $r = 0$ when $\theta = 0$ and when $\theta = \pi/2$,

$\therefore$ the loop lies between $\theta = 0$ and $\theta = \pi/2$.

Hence the required area of the loop



$$= \tfrac{1}{2}\int_0^{\pi/2} r^2\,d\theta = \tfrac{1}{2}\int_0^{\pi/2}\left(\frac{3a\cos\theta\sin\theta}{\cos^3\theta+\sin^3\theta}\right)^2 d\theta,$$

putting for r from (1)

$$= \frac{9a^2}{2}\int_0^{\pi/2}\frac{\cos^2\theta\sin^2\theta}{(\cos^3\theta+\sin^3\theta)^2}\,d\theta$$

$$= \frac{9a^2}{2}\int_0^{\pi/2}\frac{\tan^2\theta\sec^2\theta}{(1+\tan^3\theta)^2}\,d\theta,$$

dividing the numerator and the denominator by $\cos^6\theta$.

Now put $1+\tan^3\theta = t$ so that $3\tan^2\theta\sec^2\theta\,d\theta = dt$.

Also when $\theta = 0$, $t = 1$ and when $\theta \to \pi/2$, $t \to \infty$.

$$\therefore \quad \text{area of the loop} = \frac{9a^2}{2}\int_1^\infty \frac{1}{t^2}\cdot\frac{dt}{3} = \frac{3a^2}{2}\left[-\frac{1}{t}\right]_1^\infty = \frac{3a^2}{2}.$$

Ist result.

Now to find asymptotes of the curve $x^3+y^3 = 3axy$, putting $y = m$ and $x = 1$ in the highest degree terms, we get $\phi_3(m) = m^3+1$. The only real root of the equation $\phi_3(m) = 0$ *i.e.*, $m^3+1 = 0$ is, $m = -1$. Also $\phi_2(m) = -3am$. For $m = -1$, c is given by $c(3m^2) - 3am = 0$. Therefore $c = -a$ when $m = -1$.

Hence $y = -x-a$ is the only real asymptote of the curve. Changing to polars, the equation of the asymptote $x+y+a = 0$ becomes $r = -a/(\cos\theta+\sin\theta)$. Considering its cartesian equation we find that the asymptote makes an angle $3\pi/4$ with the axis of x.

We shall now find the area between the curve and its asymptote. We shall first find the area between the curve and its asymptote lying in the second quadrant.

Let P be any point on the asymptote for which the vectorial angle θ lies between $3\pi/4$ and π. Join OP. As the point P tends to infinity along the asymptote, $\theta \to 3\pi/4$. Now the area included between the folium (1) and its asymptote lying in the second quadrant

$= \lim_{\theta \to 3\pi/4}$ [{sectorial area between the asymptote and the radii vectors $\theta = \theta$ and $\theta = \pi$ } − {sectorial area between the curve and the radii vectors $\theta = \theta$ and $\theta = \pi$}]

$$= \lim_{\theta \to 3\pi/4} \left[\left\{\tfrac{1}{2}\int_\theta^\pi r^2\,d\theta, \text{ for the asymptote}\right\} - \left\{\tfrac{1}{2}\int_\theta^\pi r^2\,d\theta, \text{ for the curve}\right\}\right]$$

$$= \lim_{\theta \to 3\pi/4} \left[\frac{a^2}{2}\int_\theta^\pi \frac{d\theta}{(\cos\theta + \sin\theta)^2} - \frac{9a^2}{2}\int_\theta^\pi \frac{\sin^2\theta\cos^2\theta\,d\theta}{(\cos^3\theta + \sin^3\theta)^2}\right]$$

$$= \lim_{\theta \to 3\pi/4} \left[\frac{a^2}{2}\int_\theta^\pi \frac{\sec^2\theta\,d\theta}{(1+\tan\theta)^2} \text{ (dividing Nr. \& Dr. by } \cos^2\theta)\right.$$

$$\left. - \frac{3a^2}{2}\int_\theta^\pi \frac{3\tan^2\theta\sec^2\theta\,d\theta}{(1+\tan^2\theta)^2} \text{ (dividing Nr. \& Dr. by } \cos^6\theta)\right]$$

$$= \lim_{\theta \to 3\pi/4} \left[\frac{a^2}{2}\left(-\frac{1}{1+\tan\theta}\right)_\theta^\pi - \frac{3a^2}{2}\left(-\frac{1}{1+\tan^3\theta}\right)_\theta^\pi\right] \text{ (Note)}$$

$$= \lim_{\theta \to 3\pi/4} \left[\frac{a^2}{2}\left(-1 + \frac{1}{1+\tan\theta}\right) - \frac{3a^2}{2}\left(\frac{1}{1+\tan^3\theta} - 1\right)\right]$$

$$= \lim_{\theta \to 3\pi/4} \left[\frac{a^2}{2}\left(\frac{-\tan\theta}{1+\tan\theta}\right) - \frac{3a^2}{2}\left(-\frac{\tan^3\theta}{1+\tan^3\theta}\right)\right]$$

$$= \lim_{\theta \to 3\pi/4} \left[\frac{a^2}{2}\cdot\frac{-\tan\theta\,(1-\tan\theta+\tan^2\theta) + 3\tan^3\theta}{(1+\tan^3\theta)}\right]$$

$$= \lim_{\theta \to 3\pi/4} \left[\frac{a^2}{2}\cdot\frac{2\tan^3\theta + \tan^2\theta - \tan\theta}{1+\tan^3\theta}\right]$$

$$= \lim_{\theta \to 3\pi/4} \left[\frac{a^2}{2}\cdot\frac{(1+\tan\theta)\,(2\tan^2\theta - \tan\theta)}{(1+\tan\theta)\,(1-\tan\theta+\tan^2\theta)}\right]$$

$$= \lim_{\theta \to 3\pi/4} \left[\frac{a^2}{2}\cdot\frac{\tan\theta\,(2\tan\theta - 1)}{1-\tan\theta+\tan^2\theta}\right] = \frac{a^2}{2}\cdot\left[\frac{3}{3}\right] = \frac{a^2}{2}.$$

Since the given curve is symmetrical about the line $y = x$, therefore the area between the curve and the asymptote lying in the fourth quadrant is also equal to $a^2/2$. Also the area lying in the third quadrant, being that of a triangle, is $\frac{1}{2} OB.OC$ *i.e.*, $\frac{1}{2}a^2$.

Hence the required area between the curve and its asymptote $= \frac{1}{2}a^2 + \frac{1}{2}a^2 + \frac{1}{2}a^2 = (3/2)\,a^2 =$ the area of the loop of the curve.

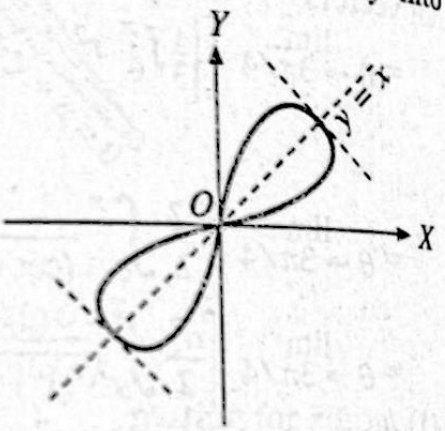
Ex. 65. *Find the area of a loop of the curve* $x^4 + y^4 = 4a^2xy$.

(Rohilkhand 1977; Ranchi 75; Gorakhpur 74)

Sol. Changing the equation of the curve $x^4 + y^4 = 4a^2xy$ into polar from by putting $x = r\cos\theta$, $y = r\sin\theta$, we have

$r^4(\cos^4\theta + \sin^4\theta) = 4a^2r^2\cos\theta\sin\theta$

or $r^2 = 4a^2\sin\theta\cos\theta/(\cos^4\theta + \sin^4\theta)$. ...(1)



From (1), $r = 0$, when $\sin\theta = 0$ or $\cos\theta = 0$ *i.e.*, $\theta = 0$ or $\pi/2$.

$\therefore$ a loop lies between $\theta = 0$ and $\theta = \pi/2$.

Hence the required area of a loop

$$= \frac{1}{2}\int_0^{\pi/2} r^2\,d\theta = \frac{1}{2}\int_0^{\pi/2} \frac{4a^2\sin\theta\cos\theta}{\cos^4\theta + \sin^4\theta}\,d\theta, \text{ from (1)}$$

$$= 2a^2\int_0^{\pi/2} \frac{\tan\theta\sec^2\theta}{1 + \tan^4\theta}\,d\theta, \text{ dividing the Nr. and Dr. by } \cos^4\theta.$$

Now put $\tan^2\theta = t$ so that $2\tan\theta\sec^2\theta\,d\theta = dt$. Also when $\theta = 0$, $t = 0$ and when $\theta \to \pi/2$, $t \to \infty$.

$$\therefore \quad \text{the required area} = a^2\int_0^{\infty} \frac{dt}{1 + t^2} = a^2\left[\tan^{-1}t\right]_0^{\infty}$$

$$= a^2[\tan^{-1}\infty - \tan^{-1}0] = a^2(\pi/2).$$

Ex. 66 (a). *Find the area of a loop of the curve*

$$(x^2 + y^2)^2 = 4axy^2.$$

Sol. Changing to polars by putting $x = r\cos\theta$, $y = r\sin\theta$, the equation of the curve becomes

$$(r^2)^2 = 4a\,(r\cos\theta)\,(r\sin\theta)^2 \text{ or } r = 4a\cos\theta\sin^2\theta. \quad ...(1)$$

From (1), $r = 0$ when $\sin^2\theta = 0$ or $\cos\theta = 0$ *i.e.*, $\theta = 0$ or $\pi/2$.

$\therefore$ a loop lies between $\theta = 0$ and $\theta = \pi/2$.

Hence the required area of a loop

$$= \frac{1}{2}\int_0^{\pi/2} r^2\,d\theta = \frac{1}{2} \times 16a^2\int_0^{\pi/2} \cos^2\theta\sin^4\theta\,d\theta, \text{ from (1)}$$

$$= 8a^2 \cdot \frac{1.3.1}{6.4.2} \cdot \frac{\pi}{2} = \frac{\pi a^2}{4}.$$

Ex. 66 (b). *Find the area of a loop of the curve*

$$(x^2 + y^2)^2 = 4xy^2.$$

Sol. Proceed exactly as in Ex. 66 (a) above. Here $a = 1$.

Ex. 67. *Prove that the area of a loop of the curve*

$$x^6 + y^6 = a^2x^2y^2 \text{ is } \pi a^2/12.$$ (Meerut 1996 BP)

Sol. Put $x = r\cos\theta$ and $y = r\sin\theta$ to change the equation of the curve into polar form. Then the curve becomes

$$r^6(\cos^6\theta + \sin^6\theta) = a^2r^4 . \cos^2\theta \sin^2\theta$$

or $$r^2 = a^2 \sin^2\theta \cos^2\theta/(\cos^6\theta + \sin^6\theta). \quad ...(1)$$

From (1), $r = 0$ when $\sin^2\theta = 0$ or $\cos^2\theta = 0$ *i.e.*, $\theta = 0$ or $\pi/2$.

$\therefore$ a loop lies between $\theta = 0$ and $\theta = \pi/2$.

Hence the required area of the loop

$$= \tfrac{1}{2}\int_0^{\pi/2} r^2\,d\theta = \tfrac{1}{2}\int_0^{\pi/2} \frac{a^2\sin^2\theta\cos^2\theta\,d\theta}{\cos^6\theta + \sin^6\theta} \text{ from (1)}$$

$$= \frac{a^2}{2}\int_0^{\pi/2} \frac{\tan^2\theta\sec^2\theta}{1 + \tan^6\theta}\,d\theta, \text{ dividing the Nr. and Dr. by } \cos^6\theta.$$

Now put $\tan^3\theta = t$ so that $3\tan^2\theta\sec^2\theta\,d\theta = dt$. Also when $\theta = 0, t = 0$ and when $\theta \to \pi/2, t \to \infty$.

$$\therefore \quad \text{the required area} = \frac{a^2}{6}\int_0^{\infty} \frac{dt}{1+t^2} = \frac{a^2}{6}\left[\tan^{-1}t\right]_0^{\infty}$$

$$= \frac{1}{6}a^2\left[\tan^{-1}\infty - \tan^{-1}0\right] = \frac{1}{6}a^2 \cdot (\pi/2) = \frac{1}{12}\pi a^2.$$

Ex. 68. *Find the area of a loop of the curve*

$$x^4 + 3x^2y^2 + 2y^4 = a^2xy.$$ (Guru Nanak 1973)

Sol. Put $x = r\cos\theta$ and $y = r\sin\theta$ to change the equation of the curve into polar form. Then the curve becomes

$$r^4(\cos^4\theta + 3\cos^2\theta\sin^2\theta + 2\sin^4\theta) = a^2r^2\cos\theta\sin\theta$$

or $$r^2 = \frac{a^2\sin\theta\cos\theta}{\cos^4\theta + 3\cos^2\theta\sin^2\theta + 2\sin^4\theta}. \quad ...(1)$$

From (1), $r = 0$ when $\sin\theta = 0$ or $\cos\theta = 0$ *i.e*, $\theta = 0$ or $\pi/2$.

$\therefore$ a loop lies between $\theta = 0$ and $\theta = \pi/2$.

Hence the required area of the loop

$$= \tfrac{1}{2}\int_0^{\pi/2} r^2\,d\theta = \tfrac{1}{2}\int_0^{\pi/2} \frac{a^2\sin\theta\cos\theta\,d\theta}{\cos^4\theta + 3\cos^2\theta\sin^2\theta + 2\sin^4\theta}, \text{ from (1)}$$

$$= \frac{a^2}{2}\int_0^{\pi/2} \frac{\tan\theta\sec^2\theta\,d\theta}{1 + 3\tan^2\theta + 2\tan^4\theta},$$

dividing the Nr. and Dr. by $\cos^4\theta$.

Now put $\tan^2\theta = t$ so that $2\tan\theta\sec^2\theta\, d\theta = dt$ and the new limits are $t = 0$ to $t = \infty$.

$$\therefore \quad \text{the required area} = \frac{a^2}{4}\int_0^\infty \frac{dt}{1+3t+2t^2}$$

$$= \frac{a^2}{4}\int_0^\infty \frac{dt}{(1+t)(1+2t)}$$

$$= \frac{a^2}{4}\int_0^\infty \left(\frac{2}{1+2t} - \frac{1}{1+t}\right) dt, \text{ by partial fractions}$$

$$= \frac{a^2}{4}\Big[\log(1+2t) - \log(1+t)\Big]_0^\infty = \frac{a^2}{4}\left[\log\left(\frac{1+2t}{1+t}\right)\right]_0^\infty$$

$$= \frac{a^2}{4}\lim_{t\to\infty}\log\left\{\frac{(1/t)+2}{(1/t)+1}\right\} - \frac{a^2}{4}\left[\log\left(\frac{1+2t}{1+t}\right)\right]_{t=0} \quad \text{(Note)}$$

$$= \tfrac{1}{4}a^2[\log(2/1)] - \tfrac{1}{4}a^2[\log(1/1)] = \tfrac{1}{4}a^2\log 2.$$

Ex. 69. *Prove that the area of a loop of the curve $x^5 + y^5 = 5ax^2y^2$ is five times the area of one loop of the curve $r^2 = a^2\cos 2\theta$.*

Sol. Changing the equation of the curve $x^5 + y^5 = 5ax^2y^2$ into polar form by putting $x = r\cos\theta$ and $y = r\sin\theta$, we get

$$r^5(\cos^5\theta + \sin^5\theta) = 5ar^4\cos^2\theta\sin^2\theta$$

or

$$r = a\sin^2\theta\cos^2\theta/(\cos^5\theta + \sin^5\theta). \quad \text{...(1)}$$

From (1), $r = 0$ when $\sin^2\theta = 0$ or $\cos^2\theta = 0$ *i.e.*, when $\theta = 0$ or $\theta = \pi/2$.

$\therefore$ the loop of the curve (1) lies between $\theta = 0$ and $\theta = \pi/2$.

Hence the required area of the loop of $x^5 + y^5 = 5ax^2y^2$

$$= \frac{1}{2}\int_0^{\pi/2} r^2\, d\theta = \tfrac{1}{2}\int_0^{\pi/2}\frac{25a^2\sin^4\theta\cos^4\theta}{(\sin^5\theta + \cos^5\theta)^2}\, d\theta, \text{ from (1)}$$

$$= \frac{25a^2}{2}\int_0^{\pi/2}\frac{\tan^4\theta\sec^2\theta\, d\theta}{(\tan^5\theta + 1)^2}, \text{ dividing the Nr. and Dr. by } \cos^{10}\theta.$$

Now put $\tan^5\theta + 1 = t$ so that $5\tan^4\theta\sec^2\theta\, d\theta = dt$. Also when $\theta = 0$, $t = 1 + \tan 0 = 1$ and when $\theta \to \pi/2$, $t \to \infty$.

$\therefore$ the area of the loop

$$= \frac{25a^2}{2}\int_1^\infty \frac{1}{t^2}\frac{dt}{5} = \frac{5a^2}{2}\int_1^\infty \frac{1}{t^2}\, dt = \frac{5a^2}{2}\left[\frac{-1}{t}\right]_1^\infty = \frac{5a^2}{2}.$$

Also from Ex. 43, page 46, the area of one loop of the curve $r^2 = a^2\cos 2\theta = \frac{1}{2}a^2$. (Deduce it here).

Thus the area of a loop of $x^5 + y^5 = 5ax^2y^2$

$$= \tfrac{5}{2}a^2 = 5.(\tfrac{1}{2}a^2) = 5.\,[\text{area of a loop of } r^2 = a^2\cos 2\theta].$$

Ex. 70. *Prove that the area of the loop of the curve $x^3 + y^3 = 3axy$ is three times the area of loop of the curve $r^2 = a^2 \cos 2\theta$ (lemniscate).* (Agra 1977)

Sol. The area of a loop of $x^3 + y^3 = 3axy$ as obtained in Ex. 64, page 66, is $\frac{3}{2}a^2$. (Find it here)

And area of a loop of the lemniscate, $r^2 = a^2 \cos 2\theta$, as obtained in Ex. 43, page 46, is $\frac{1}{2}a^2$. (Find it here)

Clearly area of the loop of $x^3 + y^3 = 3axy$

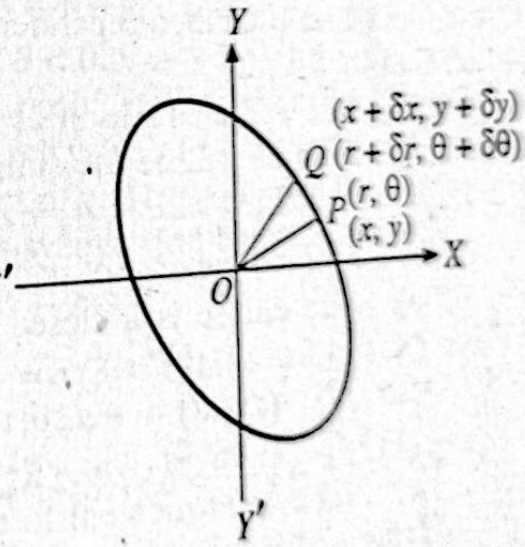
$$= \tfrac{3}{2}a^2 = 3 \cdot (\tfrac{1}{2}a^2) = 3 \text{ (area of a loop of } r^2 = a^2 \cos 2\theta).$$

Area of the closed curve.

If a closed curve be given by the equations $x = f(t), y = \phi(t)$, t being the parameter, then its area is given by

$$\frac{1}{2}\int_{t_1}^{t_2}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt,$$

where t_1 and t_2 are the values of t such that the point travels completely once round the curve in anti-clockwise direction.



Suppose that the curve does not cut itself. Let P and Q be two neighbouring points on the curve. Let the cartesian co-ordinates of P and Q be (x, y) and $(x + \delta x, y + \delta y)$ respectively and their polar co-ordinates be (r, θ) and $(r + \delta r, \theta + \delta\theta)$ respectively. Suppose that a point has to move anticlokwise in travelling from P to Q along the curve *i.e.*, the area falls on the left of an observer travelling along the curve from P to Q.

Now area of $\Delta OPQ = \frac{1}{2} OP \cdot OQ \cdot \sin \angle POQ$

$$= \tfrac{1}{2} r \cdot (r + \delta r) \sin \delta\theta$$

$$= \tfrac{1}{2} r^2 \delta\theta, \text{ neglecting higher powers of } \delta r \text{ and } \delta\theta.$$

Again the area of triangle $OPQ = \frac{1}{2}\begin{vmatrix} 1 & 0 & 0 \\ 1 & x & y \\ 1 & x + \delta x & y + \delta y \end{vmatrix}$

$$= \tfrac{1}{2}[x(y + \delta y) - y(x + \delta x)] = \tfrac{1}{2}[x\,\delta y - y\,\delta x].$$

Thus the area of $\Delta OPQ = \frac{1}{2} r^2 \delta\theta = \frac{1}{2}(x\,\delta y - y\,\delta x)$.

Hence the required area $= \frac{1}{2}\int r^2\,d\theta = \frac{1}{2}\int (x\,dy - y\,dx)$

$$= \frac{1}{2}\int \left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) dt,$$

the limits of integration are those values of t so that the point $P(x, y)$ returns to its initial position.

Note 1. The area is positive or negative according as the direction of rotation is anticlockwise or clockwise when point P moves from P to Q *i.e.*, the area is +ive or –ive according as the area lies to the left or right of the point moving on the curve in the direction in which t increases.

Note 2. **Suppose the curve crosses itself** so as to form a figure of eight as shown in the figure (i). Here two loops are described; one loop lies on the left while the second lies on the right of the point moving along the curve in the direction in which t increases. Thus the above formula will give us the difference of the areas of the two loops. Hence to find the whole area in this case, we should find each area separately and then find their sum.

Fig. (i)

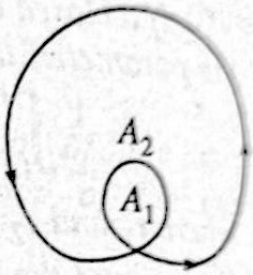


Fig. (ii)

But if the curve cuts itself as in [fig. (ii)] then the area given by the above formula will be the sum of the areas of the two loops *i.e.*, the area so obtained will be $A_1 + A_2$.

Ex. 71. *Find the area of the ellipse $x = a\cos t$, $y = b\sin t$.*

Sol. The ellipse is a closed curve and is completely described when t varies from 0 to 2π.

Also $\quad (dx/dt) = -a\sin t$ and $(dy/dt) = b\cos t$.

$$\therefore \quad x\frac{dy}{dt} - y\frac{dx}{dt} = ab(\cos^2 t + \sin^2 t) = ab.$$

Hence the required area

$$= \frac{1}{2}\int_0^{2\pi} \left(x\frac{dy}{dt} - y\frac{dx}{dt}\right) dt$$

$$= \frac{1}{2}\int_0^{2\pi} ab\,dt = \frac{1}{2}ab\left[t\right]_0^{2\pi} = \frac{1}{2}ab\,.2\pi = \pi\,ab.$$

Ex. 72. *Show that the area of the loop of the curve*

$x = a(1 - t^2)$, $y = at(1 - t^2)$, $-1 \le t \le 1$, *is* $8a^2/15$.

Sol. Here x and y both vanish when $t = -1$ and then again when $t = 1$. When $-1 < t < 1$, x is always positive, but y is positive when t is +ive, and negative when t is –ive. When $t = 0$, $x = a$ and $y = 0$. Also

by giving values to t equal in magnitude but opposite in sign we observe that for each value of x between 0 and a there are two equal and opposite values of y.

Thus a loop (symmetrical about x-axis) is obtained as t varies from -1 to 1.

$\therefore$ the required area of the loop

$$= \tfrac{1}{2}\int_{-1}^{1}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt$$

$$= \tfrac{1}{2}\int_{-1}^{1}\left[a(1-t^2)(a-3at^2) - at(1-t^2)(-2at)\right]dt$$

$$= \frac{a^2}{2}\int_{-1}^{1}\left[1-3t^2-t^2+3t^4+2t^2-2t^4\right]dt$$

$$= \frac{a^2}{2}\int_{-1}^{1}(1-2t^2+t^4)\,dt = \frac{a^2}{2}\left(t-\frac{2t^3}{2}+\frac{t^5}{5}\right)_{-1}^{1}$$

$$= \frac{a^2}{2}\left(\frac{16}{15}\right) = \frac{8a^2}{15}.$$

Ex. 73. *Find the area of the loop of the curve*

$$x = \frac{a\sin 3\theta}{\sin\theta},\ y = \frac{a\sin 3\theta}{\cos\theta}.$$

Sol. Here x and y both vanish when $\theta = -\pi/3$ and then again when $\theta = \pi/3$. When $-\pi/3 < \theta < 0$, x is +ive and y is –ive, when $0 < \theta < \pi/3$, x and y are both positive; and when $\theta \to 0$, $x \to 3a$ and $y \to 0$. Also we observe that for each value of x between 0 and $3a$, there are two equal and opposite values of y. Thus a loop is formed between $\theta = -\pi/3$ and $\theta = \pi/3$ and it is symmetrical about x-axis.

[**Note.** Obviously $\tan\theta = y/x$. Also

$$x = a\frac{3\sin\theta - 4\sin^3\theta}{\sin\theta} = a(3-4\sin^2\theta) = a\left(3-4\frac{y^2}{x^2+y^2}\right)$$

or $\quad x(x^2+y^2) = a(3x^2-y^2) \quad$ or $\quad y^2(a+x) = x^2(3a-x)$.

Thus the cartesian equation of the above curve is $y^2(a+x) = x^2(3a-x)$ and a loop is formed between $x = a$ and $x = 3a$, and $x = -a$ is an asymptote]

$\therefore$ the required area of the loop

$$= \tfrac{1}{2}\int_{-\pi/3}^{\pi/3}\left(x\frac{dy}{d\theta} - y\frac{dx}{d\theta}\right)d\theta$$

$$= \tfrac{1}{2}\int_{-\pi/3}^{\pi/3}\left[\frac{a\sin 3\theta}{\sin\theta}\times a\left\{\frac{\cos\theta.(3\cos 3\theta)+\sin 3\theta\sin\theta}{\cos^2\theta}\right\} - \frac{a\sin 3\theta}{\cos\theta}\times a\left\{\frac{\sin\theta(3\cos 3\theta)-\sin 3\theta.\cos\theta}{\sin^2\theta}\right\}\right]d\theta$$

$$= \frac{a^2}{4}\int_{-\pi/3}^{\pi/3} \frac{\sin^2 3\theta}{(\cos^2\theta \sin^2\theta)}\, d\theta = \frac{a^2}{4}\int_{-\pi/3}^{\pi/3} \frac{(3\sin\theta - 4\sin^3\theta)^2}{\sin^2\theta\cos^2\theta}\, d\theta$$

$$= \frac{a^2}{4}\int_{-\pi/3}^{\pi/3} \frac{(3-4\sin^2\theta)^2}{\cos^2\theta}\, d\theta = \frac{a^2}{4}\int_{-\pi/3}^{\pi/3} \frac{\{3-4(1-\cos^2\theta)\}^2}{\cos^2\theta}\, d\theta$$

$$= \frac{a^2}{4}\int_{-\pi/3}^{\pi/3} \frac{(4\cos^2\theta - 1)^2}{\cos^2\theta}\, d\theta = \frac{a^2}{4}\int_{-\pi/3}^{\pi/3} (16\cos^2\theta - 8 + \sec^2\theta)\, d\theta$$

$$= \frac{a^2}{4}\cdot 2\int_0^{\pi/3} [8(1+\cos 2\theta) - 8 + \sec^2\theta]\, d\theta$$

$$= \frac{a^2}{2}\int_0^{\pi/3} [8\cos 2\theta + \sec^2\theta]\, d\theta$$

$$= \frac{a^2}{2}\Big[4\sin 2\theta + \tan\theta\Big]_0^{\pi/3}$$

$$= \frac{a^2}{2}\left[4\cdot\frac{\sqrt{3}}{2} + \sqrt{3}\right] = \frac{3\sqrt{3}a^2}{2}.$$

Ex. 74. *Prove that area of the curve*

$x = a\cos\theta + b\sin\theta + c,\ y = a'\cos\theta + b'\sin\theta + c'$ *is equal to* $\pi(ab' - a'b)$.

Sol. To find the shape of the curve shift the origin to the point (c, c'). Then the equations of the curve become

$$x = a\cos\theta + b\sin\theta,\ y = a'\cos\theta + b'\sin\theta.$$

These equations are linear in $\cos\theta$ and $\sin\theta$, and if solved, the values of $\cos\theta, \sin\theta$ will involve x and y in the first degree and will be homogenous in x and y. Hence on squaring and adding, the equation to the curve will be of the form $Ax^2 + 2Hxy + By^2 = 1$ which is a conic with its centre at the new origin.

From the given equations of the curve we observe that x and y are both finite for all values of θ. Thus the given conic is a closed curve and it is completely traced from $\theta = 0$ to $\theta = 2\pi$.

$\therefore$ the required area $= \frac{1}{2}\int_0^{2\pi}\left(x\frac{dy}{d\theta} - y\frac{dx}{d\theta}\right) d\theta$

$$= \tfrac{1}{2}\int_0^{2\pi} [(a\cos\theta + b\sin\theta + c)(-a'\sin\theta + b'\cos\theta)$$
$$- (a'\cos\theta + b'\sin\theta + c')(-a\sin\theta + b\cos\theta)]\, d\theta$$

$$= \tfrac{1}{2}\int_0^{2\pi} [ab'(\cos^2\theta + \sin^2\theta) - a'b(\sin^2\theta + \cos^2\theta)$$
$$+ c(b'\cos\theta - a'\sin\theta) + c'(a\sin\theta - b\cos\theta)]\, d\theta$$

$$= \tfrac{1}{2}\int_0^{2\pi} (ab' - a'b)\, d\theta + \tfrac{1}{2}(cb' - c'b)\int_0^{2\pi} \cos\theta\, d\theta$$
$$+ \tfrac{1}{2}(c'a - ca')\int_0^{2\pi} \sin\theta\, d\theta$$

$$= \tfrac{1}{2}(ab' - a'b)\left[\theta\right]_0^{2\pi} + \tfrac{1}{2}(cb' - c'b)\left[\sin\theta\right]_0^{2\pi}$$

$$+ \tfrac{1}{2}(c'a - ca')\left[-\cos\theta\right]_0^{2\pi}$$

$$= \pi\,(ab' - a'b).$$

Ex. 75 (a). *If $P(x, y)$ be any point on the ellipse $x^2/a^2 + y^2/b^2 = 1$ and S be the sectorial area bounded by the curve, the x-axis and the line joining the origin to P, show that*

$$x = a\cos(2S/ab),\ y = b\sin(2S/ab).$$

Sol. For figure see Ex. 25 (a), page 28.

The parametric equations of the ellipse are

$$x = a\cos t,\ y = b\sin t. \qquad \text{...(1)}$$

At the vertex A, $t = 0$ and at P, $t = t$.

$$S = \text{sectorial area } AOP = \tfrac{1}{2}\int_{t=0}^{t}\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt$$

$$= \tfrac{1}{2}\int_0^t [a\cos t\,.\,b\cos t - b\sin t\,.(-a\sin t)]\,dt$$

$$= \tfrac{1}{2}ab\int_0^t (\cos^2 t + \sin^2 t)\,dt = \tfrac{1}{2}ab\int_0^t dt = \tfrac{1}{2}ab\left[t\right]_0^t = \tfrac{1}{2}abt.$$

$\therefore\quad t = 2S/ab$. Putting this value of t in (1), we get

$$x = a\cos(2S/ab) \text{ and } y = b\sin(2S/ab).$$

Ex. 75 (b). *If A is the vertex, O is the centre and P any point on the hyperbola $x^2/a^2 - y^2/b^2 = 1$, show that $x = a\cosh(2S/ab)$ and $y = b\sinh(2S/ab)$, where S is the sectorial area OPA.*

Sol. For figure see Ex. 25 (b), page 29.

The parametric equations of the hyperbola are

$$x = a\cosh t,\ y = b\sinh t. \qquad \text{...(1)}$$

At the vertex A, $t = 0$ and at P, $t = t$.

$\therefore\quad S =$ the sectorial area AOP

$$= \tfrac{1}{2}\int_0^t\left(x\frac{dy}{dt} - y\frac{dx}{dt}\right)dt$$

$$= \tfrac{1}{2}\int_0^t (a\cosh t\,.\,b\cosh t - b\sinh t\,.\,a\sinh t)\,dt$$

$$= \tfrac{1}{2}ab\int_0^t (\cosh^2 t - \sinh^2 t)\,dt = \tfrac{1}{2}ab\int_0^t dt = \tfrac{1}{2}ab\,t.$$

$\therefore\quad t = (2S/ab)$. Putting this value of t in (1), we get

$$x = a\cosh(2S/ab),\ y = b\sinh(2S/ab).$$

Ex. 76. *If the pedal equation of a curve be $p = f(r)$, prove that the area bounded by the curve and two radii vectors is $\frac{1}{2}\int \frac{pr\,dr}{\sqrt{(r^2 - p^2)}}$, taken between suitable limits.*

Sol. The required area $= \frac{1}{2}\int r^2\,d\theta$, between the suitable limits.

Now $\tan\phi = \dfrac{r\,d\theta}{dr}$. Also $\sin\phi = \dfrac{p}{r} \Rightarrow \tan\phi = \dfrac{p}{\sqrt{(r^2 - p^2)}}$.

$$\therefore \quad \frac{r\,d\theta}{dr} = \frac{p}{\sqrt{(r^2 - p^2)}} \text{ i.e., } r^2\,d\theta = \frac{pr\,dr}{\sqrt{(r^2 - p^2)}}.$$

$$\therefore \quad \text{required area} = \frac{1}{2}\int r^2\,d\theta = \frac{1}{2}\int \frac{pr\,dr}{\sqrt{(r^2 - p^2)}},$$

between the suitable limits.

□