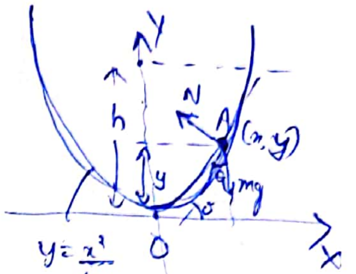
~~2007~~

[5(c)] Q → A smooth parabolic tube is placed with vertex downwards
2008 in a vertical plane. A particle slides down the tube from rest under the influence of gravity. Prove that in any position, the reaction to tube is equal to $2w(h+a)/\rho$, where ω is the weight of the particle, ρ is radius of curvature of the tube, $4a$ the latus rectum and h is the initial vertical height of the particle above the vertex of the tube.



$y = \frac{x^2}{4a}$

$\Rightarrow y' = \frac{x}{2a}, \quad y'' = \frac{1}{2a}$

Soln – Now, Radius of Curvature $\rho = \left|\frac{(1+y'^2)^{3/2}}{y''}\right|$

$$= \frac{(4a^2+x^2)^{3/2}}{4a^2} = \frac{2}{\sqrt{a}}(y+a)^{3/2}$$

Now at any instant of time, say at point A,

$$v = \sqrt{2g(h-y)}$$

Now at point A,

$$N - mg\cos\theta = \frac{mv^2}{\rho}$$

where $\cos\theta = \sqrt{\frac{1}{1+\tan^2\theta}} = \sqrt{\frac{1}{1+y'^2}} = \sqrt{\frac{a}{a+y}}$

i.e. $\cos\theta = \frac{2(a+y)}{\rho}$

$$\Rightarrow N = \frac{2mg(a+y)}{\rho} + \frac{m}{\rho}\left[2g(h-y)\right]$$

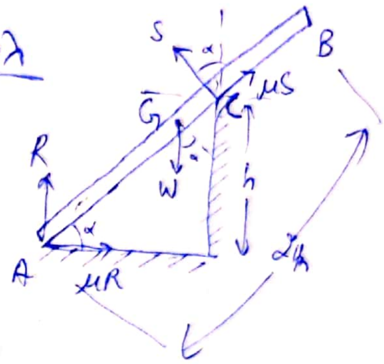
$$\boxed{\text{Reaction} \Rightarrow N = \frac{2mg}{\rho}(h+a) = \frac{\omega(h+a)}{\rho}}$$

2008

[5(d)]

Q- A straight uniform beam of length '2h' rests in limiting equilibrium, in contact with a rough vertical wall of height 'h', with one end on a rough horizontal ~~wall~~ plane and the other end projecting beyond the wall. If both the wall and the plane are equally rough, prove that 'λ' the angle of friction is given by $\sin 2\lambda = \sin\alpha \sin 2\alpha$, α being inclination of beam to the horizon.

Soln- Let μ be coefficient of friction, then $\mu = \tan\lambda$



Taking moments about A

$$\Rightarrow S\cdot(AC) = W\cos\alpha\cdot(AG)$$

$$\Rightarrow S\cdot(h\,\mathrm{cosec}\,\alpha) = W\cos\alpha\cdot(h)$$

$$\Rightarrow S = W\sin\alpha\cos\alpha = \frac{W}{2}\sin 2\alpha \quad -①$$

In horizontal direction,

$$\mu R + \mu S\cos\alpha = S(\sin\alpha)$$

$$\Rightarrow R = S\left(\frac{\sin\alpha}{\mu} - \cos\alpha\right) \quad -②$$

In vertical direction,

$$R + S\cos\alpha + \mu S(\sin\alpha) = W$$

$$\Rightarrow R + S(\cos\alpha + \mu\sin\alpha) = W$$

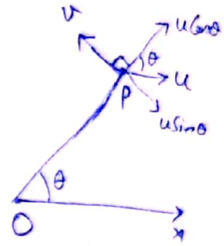
Using ① & ② $\Rightarrow \frac{W}{2}\sin\alpha\sin 2\alpha\left(\mu + \frac{1}{\mu}\right) = W.$

$$\Rightarrow \frac{\sin\alpha \sin 2\alpha}{2}\left(\frac{\sin\lambda}{\cos\lambda} + \frac{\cos\lambda}{\sin\lambda}\right) = 1$$

$$\Rightarrow \boxed{\sin\alpha \sin 2\alpha = \sin 2\lambda}$$

6(a)

Q - A particle P moves in a plane in such a way that it is acted upon by two constant velocities u and v along direction Ox and along the direction perpendicular to OP. Show that the path traversed by P is a conic section with focus at O and eccentricity u/v.



Soln - Let $P(r, \theta)$ be position of particle at time t.

Then radial velocity $= \frac{dr}{dt} = u\cos\theta$ — (1)

Transverse velocity $= \frac{r\,d\theta}{dt} = v - u\sin\theta$ — (2)

Dividing (1) by (2)

$$\Rightarrow \frac{dr}{r\,d\theta} = \frac{u\cos\theta}{v - u\sin\theta}$$

$$\Rightarrow \frac{dr}{r} = \frac{u\cos\theta}{v - u\sin\theta}\,d\theta$$

Integrating $\Rightarrow \log r = -\log(v - u\sin\theta) + \log c$

$$\Rightarrow \log\left(\frac{c}{r}\right) = \log(v - u\sin\theta)$$

$$\Rightarrow \frac{c}{r} = v + u\cos\left(\frac{\pi}{2} + \theta\right)$$

$$\Rightarrow \frac{c/v}{r} = 1 + \frac{u}{v}\cos\left(\frac{\pi}{2} + \theta\right)$$

which is of the for $\frac{l}{r} = 1 + e\cos\theta$ whose focus is pole O

and eccentricity $= e = \frac{u}{v}$.

[7(a)]

Q— A particle moves under a force $m\mu\{3au^4 - 2(a^2-b^2)u^5\}$ and is projected from an apse at a distance $(a+b)$ with velocity $\sqrt{\frac{\mu}{a+b}}$. Show that eqn of the path is $r = a + b\cos\theta$

Soln — Central acceleration $P = \mu\{3au^4 - 2(a^2-b^2)u^5\}$

Eqn of path is — $\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2} = \frac{\mu}{u^2h^2}\{3au^4 - 2(a^2-b^2)u^5\}$

$$\Rightarrow \frac{d^2u}{d\theta^2} + u = \frac{\mu}{h^2}\{3au^2 - 2(a^2-b^2)u^3\}$$

Multiply both sides by $2\left(\frac{du}{d\theta}\right)$ and integrating

$$\Rightarrow \underbrace{h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right]}_{v^2} = \mu\{2au^3 - (a^2-b^2)u^4\} + A \quad -(1)$$

At apse, $r = a+b$, $u = \frac{1}{a+b}$, $\frac{du}{d\theta} = 0$ and $v = \sqrt{\frac{\mu}{a+b}}$

$$\Rightarrow \frac{\mu}{(a+b)^2} = h^2\left(\frac{1}{(a+b)^2}\right) = \mu\left[\frac{2a}{(a+b)^3} - \frac{(a^2-b^2)}{(a+b)^4}\right] + A$$

$$\Rightarrow h^2 = \mu \text{ and } A = 0$$

Putting these values in (1)

$$\Rightarrow \left(\frac{-1}{r^2}\frac{dr}{d\theta}\right)^2 = \frac{-1}{r^2} + \frac{2a}{r^3} - \frac{(a^2-b^2)}{r^4}$$

$$\Rightarrow \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{1}{r^4}\left[-r^2 + 2ar - (a^2-b^2)\right]$$

$$\Rightarrow \left(\frac{dr}{d\theta}\right)^2 = b^2 - (r-a)^2$$

$$\Rightarrow \frac{dr}{d\theta} = \sqrt{b^2-(r-a)^2} \Rightarrow d\theta = \frac{dr}{\sqrt{b^2-(r-a)^2}}$$

Integrating,

$$\theta + B = \sin^{-1}\left(\frac{r-a}{b}\right) \quad -(2)$$

Initially, $r = a+b$ and let $\theta = 0°$.

$$\Rightarrow \quad B = \sin^{-1}(1) = \pi/2$$

Putting it in (2)

$$\Rightarrow \theta + \frac{\pi}{2} = \sin^{-1}\left(\frac{r-a}{b}\right) \Rightarrow r - a = b\sin\left(\frac{\pi}{2}+\theta\right)$$

$\Rightarrow \boxed{r = a + b\cos\theta}$ is eqn of the path.

[7(b)]

Q- A shell lying in a straight smooth horizontal tube suddenly breaks into two portions of masses m_1 and m_2. If S is the distance apart, in the tube of the masses after a time t, Show that the work done by the explosion is $\frac{1}{2}\cdot\frac{m_1 m_2}{m_1+m_2}\cdot\frac{S^2}{t^2}$

Soln- Let u_1 and u_2 be velocities of masses m_1 and m_2 resp. Then relative velocity of the masses after explosion is u_1+u_2. u_1+u_2 will remain constant

$$\Rightarrow \quad (u_1+u_2)t = S \quad -(1)$$

By principle of conservation of linear momentum

$$m_1u_1 - m_2u_2 = 0$$

$$\Rightarrow m_1u_1 = m_2u_2 \quad \Rightarrow u_2 = \frac{m_1u_1}{m_2}$$

Putting u_2 in (1)

$$\Rightarrow u_1\left(1+\frac{m_1}{m_2}\right)t = S$$

$$\Rightarrow u_1 = \frac{m_2 S}{(m_1+m_2)t}$$

$$\therefore u_2 = \frac{m_1 S}{(m_1+m_2)t}$$

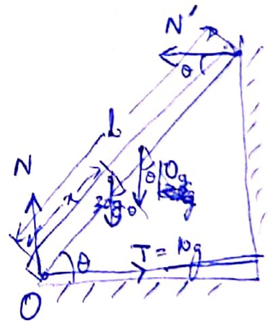
Workdone by explosion = Kinetic energy released

$$= \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2$$

$$= \frac{1}{2} \frac{s^2}{t^2} \frac{1}{(m_1+m_2)^2} (m_1 m_2^2 + m_1^2 m_2)$$

$$= \frac{1}{2} \frac{s^2}{t^2} \frac{m_1 m_2}{m_1+m_2}.$$

Q- A ladder of weight 10kg rests on a smooth horizontal ground leaning against a smooth vertical wall at an inclination $\tan^{-1}(2)$ with the horizontal and is prevented from slipping by a string attached to it at the lower end and to the junction of floor and the wall. A body of weight 30kg begins to ascend the ladder. If the string can bear a tension of 10kg wt, how far along the ladder can the boy rise with safety?



Soln- Along the horizontal,

$$T = N' = 10g.$$

Along the vertical,

$$30g + 10g = N$$

$$\Rightarrow N = 40g$$

Taking moments with reference to point O,

$$\Rightarrow N \sin\theta \cdot l = 30g \cos\theta (x) + 10g \cos\theta \cdot \left(\frac{l}{2}\right)$$

$$\Rightarrow 40g \sin\theta \cdot l = 30g \cos\theta (x) + 10g \cos\theta \left(\frac{l}{2}\right)$$

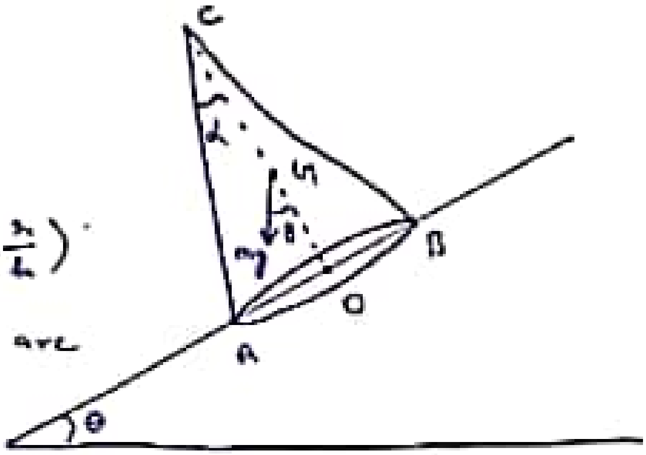
$\sin\theta = \frac{2}{\sqrt{5}}$, $\cos\theta = \frac{1}{\sqrt{5}}$

$\Rightarrow \boxed{x = \frac{1}{2}}$ $\therefore$ Body can travel halfway up the ladder.

IAS 2008 → A solid Right circular cone whose height is h & radius of whose base is r, is placed on an inclined plane. It is prevented from sliding. If the inclination of the plane θ (to the horizontal) be gradually decreased, find when cone will topple over. For a cone whose semi-vertical angle is 30°, determine the critical value of θ which when exceeded, the cone will topple over.

Solⁿ] The above question is partially wrong [most possibly typing error]
The correct question should say "gradually increased"
So we should solve the question by assuming this -

Let θ be the inclination, after which if we increase the angle, the cone will topple over.



Radius of Cone = r] given
height of Cone = h]
Semi-Vertical angle = $\alpha = \tan^{-1}\left(\frac{r}{h}\right)$

The forces acting on the cone are gravitational force & the Normal Reaction by the inclined plane.

At the pt of critical stage (θ after which the cone will topple) the normal reaction will pass through A & the components of mg will keep the system in equilibrium by balancing the torque so we write

$$mg\cos\theta \times OA = mg\sin\theta \times OG$$

Putting $OA = r$ & $OG = \frac{h}{4}$ where G = centre of gravity of cone

$\Rightarrow \boxed{\theta = \tan^{-1}\left(\frac{4r}{h}\right)}$.

After this cone will topple

If $\alpha = 30° \Rightarrow \tan 30° = \frac{1}{\sqrt{3}} = \frac{r}{h}$

Putting this in result we get $\boxed{\theta = \tan^{-1}\left(\frac{4}{\sqrt{3}}\right)}$