LINEAR ALGEBRA

: CSE-2010 :

1(a) If $\lambda_1, \lambda_2, \lambda_3$ are eigen values of the matrix $A = \begin{bmatrix} 26 & -2 & 2 \\ 2 & 21 & 4 \\ 4 & 2 & 28 \end{bmatrix}$ then show that $\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2} \leq \sqrt{1949} \rightarrow \sqrt{1925}$ [correction]

$\rightarrow$ Char. eq^n of of $A = |A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 26-\lambda & -2 & 2 \\ 2 & 21-\lambda & 4 \\ 4 & 2 & 28-\lambda \end{vmatrix} = 0$

$(26-\lambda)\left[(21-\lambda)(28-\lambda) - 8\right] + 2\left[2(28-\lambda) - 16\right] + 2\left[4 - 4(21-\lambda)\right] = 0$

$\Rightarrow (26-\lambda)\left[580 - 49\lambda + \lambda^2\right] + 2\left[40 - 2\lambda\right] + 2\left[-80 + 4\lambda\right] = 0$

$\Rightarrow 15080 - 580\lambda - 1274\lambda + 49\lambda^2 + 26\lambda^2 - \lambda^3 + 80 - 4\lambda - 160 + 8\lambda = 0$

$\Rightarrow \lambda^3 - 75\lambda^2 + 1850\lambda - 15000 = 0$

$\Rightarrow \lambda_1 = 20, \ \lambda_2 = 30, \ \lambda_3 = 25.$

Then $\sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2} = \sqrt{400+900+625}$

$= \sqrt{1925}. \leq \cancel{\sqrt{1949}} \ \sqrt{1925}.$

$\therefore \sqrt{\lambda_1^2+\lambda_2^2+\lambda_3^2} \leq \sqrt{1925}$

1 (b) What is the nullspace of the differentiation transformation $\frac{d}{dx} : P_n \to P_n$ where P_n is a space of all polynomials of degree $\leq n$ over the real numbers? What is the null space of the second derivative as a transformation of P_n? What is the nullspace of k^{th} derivative?

→ (i) $\frac{d}{dx} : P_n \to P_n$: $N_A\left(\frac{d}{dx}\right) = \{ p(x) \in P_n \mid \frac{d}{dx} p(x) = 0 \}$

Let $p(x) = a_0$ where $a_0 \in \mathbb{R}$. Then,

$\frac{d}{dx} p(x) = \frac{d}{dx} a_0 = 0$.

Hence, all constant polynomials lie in the null space of $\frac{d}{dx}$.

Let $p(x) = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n$ where $a_i \in \mathbb{R}$, $i \in [0, n]$.

Then, $\frac{d}{dx} p(x) = a_1 + 2a_2 x + \dots + n a_n x^{n-1}$.

Hence, no polynomial of degree > 0 lies in the nullspace of $\frac{d}{dx}$.

$\therefore N_A\left(\frac{d}{dx}\right) = \{ a \mid a \in \mathbb{R} \}$

(ii) $\frac{d^2}{dx^2} : P_n \to P_n$: $N_A\left(\frac{d^2}{dx^2}\right) = \{ p(x) \in P_n \mid \frac{d^2}{dx^2} p(x) = 0 \}$.

WKT $0 = 0 + 0x + 0x^2 + \dots 0x^n \in P_n$. Then

$\frac{d^2}{dx^2} 0 = 0$. $\Rightarrow 0 \in N_A\left(\frac{d^2}{dx^2}\right) \Rightarrow N_A\left(\frac{d^2}{dx^2}\right) \neq \phi$.

Now: Let $p(x) = a_0$ where $a_0 \in \mathbb{R}$, then.

$\frac{d^2}{dx^2} a_0 = \frac{d}{dx}\left(\frac{d}{dx} a_0\right) = \frac{d}{dx} 0 = 0$.

$\therefore$ All constant polynomials lie in the null space of $\frac{d^2}{dx^2}$

Also, let $p(x) = a_0 + a_1 x$; $a_0, a_1 \in \mathbb{R}$, then

$$\frac{d^2}{dx^2} p(x) = \frac{d}{dx}\left(\frac{d}{dx}(a_0 + a_1 x)\right) = \frac{d}{dx}(a_1) = 0$$

$\therefore$ All polynomials of degree 1 also lie in the null space of $\frac{d^2}{dx^2}$.

Now, taking any other higher degree polynomial,
$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$, $a_i \in \mathbb{R}$ $\forall i \in [0,n]$, then

$$\frac{d^2}{dx^2}(p(x)) = \frac{d}{dx}\left(\frac{d}{dx} p(x)\right) = \frac{d}{dx}\left(a_1 + 2a_2 x + \cdots + n a^n x^{n-1}\right)$$

$$= 2a_2 + \cdots + n(n-1) a^n x^{n-2} \neq 0.$$

$\therefore$ No other polynomial belong to its nullspace.

$\therefore N_A\left(\frac{d^2}{dx^2}\right) = \{a_0 + a_1 x \mid a_0, a_1 \in \mathbb{R}\}$

Generalising, for $n = k$, the null space of $\frac{d^k}{dx^k}$ is given by

$$N_A\left(\frac{d^k}{dx^k}\right) = \{a_0 + a_1 x + \cdots + a_{k-1} x^{k-1} \mid a_i \in \mathbb{R};\ i \in [0, k-1]\}$$

2(a) Let $M = \begin{pmatrix} 4 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$. Find the unique linear Transformation $T: \mathbb{R}^3 \to \mathbb{R}^2$ so that M is the matrix of T wrt the basis $\beta = \{v_1 = (1,0,0), v_2 = (1,1,0), v_3 = (1,1,1)\}$ of $\mathbb{R}^3$ and $\beta' = \{w_1 = (1,0), w_2 = (1,1)\}$ of $\mathbb{R}^2$. Also find $T(x,y,z)$.

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$\Rightarrow T(x,y,z) = (4x+2y+z, y+3z)$.

$\rightarrow$ By the matrix, It is clear that

$T(v_1) = 4w_1 + 0w_2$, $T(v_2) = 2w_1 + w_2$, $T(v_3) = w_1 + 3w_2$.

$T(1,0,0) = 4(1,0) + 0(1,1) = (4,0)$

$T(1,1,0) = 2(1,0) + 1(1,1) = (3,1)$

$T(1,1,1) = 1(1,0) + 3(1,1) = (4,3)$

Now $(x,y,z) = a(1,0,0) + b(1,1,0) + c(1,1,1)$
$= (a+b+c, b+c, c)$

$\Rightarrow$ Comparing both sides: $c = z$, $b+c = y$
$b = y - z$

$a+b+c = x$
$a + y - z + z = x$
$a = x - y$

$\therefore (x,y,z) = (x-y)(1,0,0) + (y-z)(1,1,0) + z(1,1,1)$

Now $T(x,y,z) = T((x-y)(1,0,0) + (y-z)(1,1,0) + z(1,1,1))$
$= (x-y)T(1,0,0) + (y-z)T(1,1,0) + zT(1,1,1)$
$= (x-y)(4,0) + (y-z)(3,1) + z(4,3)$ [T is a L.T.]
$= (4(x-y) + 3(y-z) + 4z, y - z + 3z)$
$= (4x - y + z, y + 2z)$

3(a) Let A & B be $n \times n$ ~~eigen values~~ matrices over $\mathbb{R}$. Show that I-BA is invertible if I-AB is invertible. Deduce that AB and BA have the same eigen values.

$\rightarrow$ Let $(I-AB)^{-1} = X$ [Since $I-AB$ is invertible].

Then, expanding left side, we have

$\Rightarrow I + AB + AB \cdot AB + AB \cdot AB \cdot AB + \cdots = X$

$\Rightarrow X = I + AB + AB^2 + \cdots$

Premultiplying with B & postmultiplying with A on both sides, we get

$BXA = BAB + BA \cdot BA + B(AB)(AB)A + \cdots$

$BXA = BA + (BA)(BA) + (BA)(BA)(BA) + \cdots$ [By Asso.]

$BXA = BA + BA^2 + BA^3 + \cdots$

$I + BXA = I + BA + BA^2 + \cdots = (I-BA)^{-1}$

$\Rightarrow (I-BA)^{-1} = I + BXA$ where $X = (I-AB)^{-1}$

$\therefore$ $I-BA$ is invertible if $I-AB$ is invertible.

If X is the eigen vector of AB wrt eigen value λ,

then $(AB)X = \lambda X$

$\Rightarrow B(ABX) = \lambda BX$

$\Rightarrow (BA)(BX) = \lambda(BX)$,

i.e. BX is the eigen vector of BA wrt the same eigen value λ. Hence, AB & BA have the same eigen values

4(a)(i) In the n-space $\mathbb{R}^n$, determine whether or not; the set $\{e_1-e_2, e_2-e_3, \ldots, e_{n-1}-e_n, e_n-e_1\}$ is linearly independent.

Let $\{e_1, e_2, \ldots e_n\}$ be the standard basis of $\mathbb{R}^n$.

$\longrightarrow$ Let $a_1, a_2 \ldots a_n \in \mathbb{R}$ such that

$$a_1(e_1-e_2) + a_2(e_2-e_3) + \cdots + a_n(e_n-e_1) = 0$$

$$e_1(a_1-a_n) + e_2(a_2-a_1) + e_3(a_3-a_2) + \cdots + e_n(a_n-a_{n-1}) = 0 \quad \text{—①}$$

Then, ~~if $a_1=a_2=\cdots=a_n$, then the expression~~

Since $\{e_1, e_2, \ldots e_n\}$ is the standard basis of $\mathbb{R}^n$, then they are linearly independent.

Then $a_1-a_n=0,\ a_2-a_1=0, \ \ldots\ a_n-a_{n-1}=0$

$\Rightarrow a_1=a_2=\cdots=a_n=k$ where $k \in \mathbb{R}$

$\therefore$ There exist infinite number of values such that

$$a_1(e_1-e_2) + a_2(e_2-e_3) + \cdots + a_n(e_n-e_1) = 0$$

$\therefore$ The set is linearly dependent

4(a)(ii) Let T be a linear transformation from a vectorspace V over reals into V such that $T-T^2=I$. Show that T is invertible.

→ We have:

Nullspace of $T \equiv \ker(T) = \{\alpha \in V \mid T(\alpha) = 0\}$.

$0 \in \ker(T)$ since $T(0) = \hat{0}$.

$\therefore \ker(T) \neq \phi$.

Given that $T - T^2 = I$. Let $\alpha \in \ker(T)$. $\Rightarrow T(\alpha) = 0$ — ①

$\Rightarrow (T - T^2)(\alpha) = I(\alpha) = \alpha$

$\Rightarrow T(\alpha) - T^2(\alpha) = \alpha$

$\Rightarrow 0 - T^2(\alpha) = \alpha$ [$T(\alpha) = 0$ from ①].

$\Rightarrow 0 - T(T(\alpha)) = \alpha$

$\Rightarrow 0 - T(0) = \alpha$

$\Rightarrow 0 - 0 = \alpha$

$\Rightarrow \alpha = 0$.

$\therefore \ker(T) = \{0\}$.

$\therefore T$ is non-singular $\Rightarrow T$ is invertible.