PDE-II

Non-linear Eqns:-

The integrals or solutions of the non-linear partial differentiable eqns of the first order:

w.k.T the relation of the type $f(x,y,z,a,b)=0$ — ①
gives rise to a PDE of the first order of the form
$F(x,y,z,p,q)=0$ — ②
on the elimination of arbitrary constants a & b. Here x, y are independent variables and z is dependent variable.

→ If ① has been derived from ② then ① is a solution of ②.

Any such relation ① which contains as many arbitrary constants as there are independent variables, is called the complete integral or complete solution of ②.

→ Any particular integral of ② is obtained by giving particular values to a & b in ①.

Singular Integral (S.I):

The singular integral is obtained by eliminating a & b b/w the three eqns $f(x,y,z,a,b)=0$, $\frac{\partial f}{\partial a}=0$ and $\frac{\partial f}{\partial b}=0$.

General Integral (G.I):

If in the eqn ①, one of the constants is a function of the other say $b=\phi(a)$ then ① becomes
$f(x,y,z,a,\phi(a))=0$ — ③.
It is a one-parameter subfamily of the family ①.
The eqn of the envelope of the family of surfaces represented by ③ is also a solution of the eqn ②.
It is called the general integral of ② corresponding to the complete integral ①.

The eqn of the envelope of the surfaces represented by ③ is obtained by eliminating 'a' between the eqns
$f(x,y,z,a,\phi(a))=0$ and $\frac{\partial f}{\partial a}=0$.

Charpit's Method:

We now give a general method due to Charpit for finding the complete integral of a non-linear differential eqn of the first order.

Let the given eqn be $f(x, y, z, p, q) = 0$ —— ①

Since z depends on x & y, we have

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$\Rightarrow dz = p\,dx + q\,dy \quad \text{—— ②}$$

The fundamental idea in Charpit's method is the introduction of another PDE of the first order

$$g(x, y, z, p, q, a) = 0 \quad \text{—— ③}$$

which contains arbitrary constant 'a' and

(i) we can solve the eqns ① & ③ for

$$p = p(x, y, z, a) \ \& \ q = q(x, y, z, a)$$

(ii) Substituting these values of p & q in ②, the eqn ② becomes

$$dz = p(x, y, z, a)\,dx + q(x, y, z, a)\,dy \quad \text{—— ④}$$

This gives the solution, provided ④ is integral.

If such a relation ③ has been found, the solution of the eqn ④

$$\phi(x, y, z, a, b) = 0 \quad \text{—— ⑤}$$

containing two arbitrary constants a & b will be a solution of eqn ①.

Also it is a complete integral of the eqn ①.

How to determine g

Differentiating ① & ③ w.r.t x, we get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial f}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}\cdot\frac{\partial q}{\partial x} = 0$$

and $$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial x} + \frac{\partial g}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial g}{\partial q}\frac{\partial q}{\partial x} = 0$$

$$\left.\begin{aligned}\Rightarrow\ & \frac{\partial f}{\partial x}+\frac{\partial f}{\partial z}p+\frac{\partial f}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial f}{\partial q}\cdot\frac{\partial q}{\partial x}=0\\ \text{and }\ & \frac{\partial g}{\partial x}+\frac{\partial g}{\partial z}p+\frac{\partial g}{\partial p}\frac{\partial p}{\partial x}+\frac{\partial g}{\partial q}\cdot\frac{\partial q}{\partial x}=0\end{aligned}\right\}\text{———}(6)$$

Again diff (1) & (3) w.r.t y, we get

$$\left.\begin{aligned}& \frac{\partial f}{\partial y}+\frac{\partial f}{\partial z}q+\frac{\partial f}{\partial p}\cdot\frac{\partial p}{\partial y}+\frac{\partial f}{\partial q}\cdot\frac{\partial q}{\partial y}=0\\ \text{and }\ & \frac{\partial g}{\partial y}+\frac{\partial g}{\partial z}q+\frac{\partial g}{\partial p}\frac{\partial p}{\partial y}+\frac{\partial f}{\partial q}\frac{\partial q}{\partial y}=0\end{aligned}\right\}\text{———}(7)$$

Now eliminating $\frac{\partial p}{\partial x}$ from the eqns in (6) & $\frac{\partial q}{\partial y}$ from the eqns in (7), we get

$$\left(\frac{\partial f}{\partial x}\cdot\frac{\partial g}{\partial p}-\frac{\partial f}{\partial p}\frac{\partial g}{\partial x}\right)+p\left(\frac{\partial f}{\partial z}\frac{\partial g}{\partial p}-\frac{\partial f}{\partial p}\frac{\partial g}{\partial z}\right)+\frac{\partial q}{\partial x}\left(\frac{\partial f}{\partial q}\frac{\partial g}{\partial p}-\frac{\partial f}{\partial p}\frac{\partial g}{\partial q}\right)=0\text{———}(8)$$

and
$$\left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial q}-\frac{\partial f}{\partial q}\frac{\partial g}{\partial y}\right)+q\left(\frac{\partial f}{\partial z}\frac{\partial g}{\partial q}-\frac{\partial f}{\partial q}\frac{\partial g}{\partial z}\right)+\frac{\partial p}{\partial y}\left(\frac{\partial f}{\partial p}\frac{\partial g}{\partial q}-\frac{\partial f}{\partial q}\frac{\partial g}{\partial p}\right)=0\text{———}(9)$$

Since
$$\begin{aligned}\frac{\partial q}{\partial x}&=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\\ &=\frac{\partial^2 z}{\partial x\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\\ &=\frac{\partial p}{\partial y}\end{aligned}$$

$$\therefore\ \frac{\partial q}{\partial x}=\frac{\partial p}{\partial y}$$

(8) + (9) =

$$\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial p}-\frac{\partial f}{\partial p}\frac{\partial g}{\partial x}\right)+p\left(\frac{\partial f}{\partial z}\frac{\partial g}{\partial p}-\frac{\partial f}{\partial p}\frac{\partial g}{\partial z}\right)+\left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial q}-\frac{\partial f}{\partial q}\frac{\partial g}{\partial y}\right)+q\left(\frac{\partial f}{\partial z}\frac{\partial g}{\partial q}-\frac{\partial f}{\partial q}\frac{\partial g}{\partial z}\right)=0$$
$$\left(\because\ \frac{\partial p}{\partial y}=\frac{\partial q}{\partial x}\right)$$

$$\Rightarrow\left(\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}\right)\frac{\partial g}{\partial p}+\left(\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}\right)\frac{\partial g}{\partial q}+\left(-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}\right)\frac{\partial g}{\partial z}+\left(-\frac{\partial f}{\partial p}\right)\frac{\partial g}{\partial x}+\left(-\frac{\partial f}{\partial q}\right)\frac{\partial g}{\partial y}=0\text{———}(10)$$

Clearly it is a linear PDE of the first order with x, y, z, p, q as independent variables and g as a dependent variable.

The Lagrange's auxiliary eqns are

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\partial f/\partial q} = \frac{dg}{0} \quad \text{(11)}$$

$P\frac{\partial z}{\partial x}+Q\frac{\partial z}{\partial y}=R.$ $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$

These eqns are known as charpit's auxiliary eqns. Any of integrals of (11) satisfies (10). If such an integral contains p or q (or both). It can be taken the required second PDE (3).

Note: It should be noted that not all of charpit's eqns (11) need be used, but that p or q must occur in the solution obtained.

Working Rule of Charpit's Method:

Step 1: Transfer terms of the given eqn to LHS and denote the entire expression by f.

Step 2: write down the charpit's auxiliary eqns (11).

Step 3: using the value of f in step 1, write down the values of $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ etc. occuring in step (2) and put these values in Charpit's auxiliary eqns (11).

Step 4: After simplifying step (3) select two proper fractions so that the resulting integral may come out to be the simplest relation involving atleast one of p and q.

Step 5: The simplest relation of step (4) is solved along with the given eqn to determine p & q.

Step 6: put these values of p & q in

$$dz = p\,dx + q\,dy$$

which on integration gives the complete integral of the given eqn.

Problems:

→ Find the complete integral of $px + qy = pq$.

Soln: Given that $px + qy = pq$

$$\Rightarrow px + qy - pq = 0$$

Let $f(x, y, z, p, q) = px + qy - pq = 0$ —— ①

Charpit's auxiliary eqns are

$$\frac{dx}{-\partial f/\partial p} = \frac{dy}{-\partial f/\partial q} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}}$$

$$\Rightarrow \frac{dx}{-x+q} = \frac{dy}{-y+p} = \frac{dz}{-p(x-q) - q(y-p)} = \frac{dp}{p + p(0)} = \frac{dq}{q} \quad \text{—— ②}$$

Taking last two fractions of ②, we get

$$\frac{dp}{p} = \frac{dq}{q}.$$

$$\log p = \log q + \log a$$

$$\Rightarrow \boxed{p/q = a} \Rightarrow \boxed{p = qa} \quad \text{—— ③}$$

$$① \equiv \quad qax + qy - (qa) = 0$$

$$\Rightarrow q[ax + y - qa] = 0$$

$$\Rightarrow ax + y - qa = 0 \quad (\because q \neq 0)$$

$$\Rightarrow ax + y = qa$$

$$\Rightarrow \boxed{q = \frac{ax+y}{a}} \quad \text{—— ④}$$

$$③ \equiv \quad p = \left(\frac{ax+y}{a}\right)a$$

$$\Rightarrow \boxed{p = ax + y} \quad \text{—— ⑤}$$

putting these values p and q in $dz = p\,dx + q\,dy$,

we get $dz = (ax+y)dx + \left(\frac{ax+y}{a}\right)dy$

$$\Rightarrow a\,dz = (ax+y)(a\,dx + dy)$$

$$\Rightarrow a\,dz = (ax+y)\,d(ax+y)$$

$$\Rightarrow az = \frac{(ax+y)^2}{2} + b$$

which is the complete integral of ①.

2000 → Solve by Charpit's method eqn

$$p^2x(x-1)+2pqxy+q^2y(y-1)-2pxz-2qyz+z^2=0$$

Sol^n Let $f(x,y,z,p,q)=$

$$p^2x(x-1)+2pqxy+q^2y(y-1)-2pxz-2qyz+z^2=0 \quad \text{--- (1)}$$

The Charpit's auxiliary eqns are

$$\frac{dx}{-f_p}=\frac{dy}{-f_q}=\frac{dz}{-pf_p-qf_q}=\frac{dp}{f_x+pf_z}=\frac{dq}{f_y+qf_z} \quad \text{--- (2)}$$

From (1), $f_x=p^2(2x-1)+2pqy-2pz$

$f_y=2pqx+q^2(2y-1)-2qz$

$f_z=-2px-2qy+2z$

$f_p=2px(x-1)+2qxy-2xz$

$f_q=2pxy+2qy(y-1)-2yz$

and $f_x+pf_z=-p^2$; $f_y+qf_z=-q^2$

$$\therefore (2) \equiv \frac{dx}{-(2px^2-2px+2qxy-2xz)}=\frac{dy}{-(2pxy+2qy^2-2qy-2yz)}$$

$$=\frac{dz}{-p[2px(x-1)+2qxy-2xz]-q[2pxy+2qy(y-1))-2yz]}$$

$$=\frac{dp}{-p^2}=\frac{dq}{-q^2} \quad \text{--- (3)}$$

each fraction of (3) $=\dfrac{\frac{1}{p}dp}{-p}=\dfrac{\frac{1}{q}dq}{-q}=\dfrac{\frac{1}{p}dp-\frac{1}{q}dq}{-p+q}$ --- (4)

Also each fraction of (3) $=\dfrac{\frac{1}{x}dx-\frac{1}{y}dy}{-\cancel{2px}+2p-\cancel{2qy}+\cancel{2z}+\cancel{2px}+\cancel{2qy}-2q-\cancel{2z}}$ --- (5)

$\therefore$ (4) & (5)

$$\Rightarrow \frac{\frac{1}{p}dp-\frac{1}{q}dq}{-(p-q)}=\frac{\frac{1}{x}dx-\frac{1}{y}dy}{2(p-q)}$$

$$\Rightarrow \frac{1}{2}\left(\frac{1}{x}dx-\frac{1}{y}dy\right)=\frac{1}{q}dq-\frac{1}{p}dp$$

Integrating, we get

$$\frac{1}{2}(\log x-\log y)=\log q-\log p+\log a.$$

$\Rightarrow \left(\frac{x}{y}\right)^{1/2} = \frac{q}{p}\cdot a$

$\therefore \boxed{p = \frac{a y^{1/2} q}{x^{1/2}}}$; a is arbitrary constant. —— (6)

$\therefore (1) \equiv (px+qy-z)^2 = p^2x + q^2y$

$\Rightarrow px+qy-z = \pm\sqrt{p^2x+q^2y}$ —— (7)

Taking +ve sign in (7).

$px+qy-z = \sqrt{p^2x+q^2y}$ —— (8)

$\Rightarrow \left(\frac{a y^{1/2} q}{x^{1/2}}\right)x + qy - z = \sqrt{\left(\frac{a^2 y q^2}{x}\right)(x) + q^2y}$ (by (6))

$\Rightarrow aq(xy)^{1/2} + qy - z = \sqrt{ya^2q^2 + q^2y} = qy^{1/2}(1+a^2)^{1/2}$

$\Rightarrow q\left[y + a(xy)^{1/2} - (1+a^2)^{1/2}y^{1/2}\right] = z$

$\Rightarrow \boxed{q = \frac{z}{y + a(xy)^{1/2} - (1+a^2)^{1/2}y^{1/2}}}$ —— (9)

putting these values in $dz = pdx + qdy$

$$dz = \frac{az\,dx}{x^{1/2}\left[y^{1/2}+ax^{1/2}-(1+a^2)^{1/2}\right]} + \frac{z\,dy}{y^{1/2}\left[y^{1/2}+ax^{1/2}-(1+a^2)^{1/2}\right]}$$

$$\Rightarrow \frac{dz}{z} = \frac{ay^{1/2}dx + x^{1/2}dy}{(xy)^{1/2}\left[y^{1/2}+ax^{1/2}-(1+a^2)^{1/2}\right]}$$

Integrating

$\Rightarrow \log z = 2\log\left[y^{1/2}+ax^{1/2}-(1+a^2)^{1/2}\right] + \log b$

$\Rightarrow z = b\left[y^{1/2}+ax^{1/2}-(1+a^2)^{1/2}\right]^2$; b is an arbitrary constant.

2002 → Solve $z = \frac{1}{2}(p^2+q^2) + (p-x)(q-y)$

2002 → find two complete integrals of the PDE $x^2p^2+y^2q^2-4=0$

Sol: Given that $x^2p^2+q^2y^2-4=0$

Let $f(x,y,z,p,q) = x^2p^2 + y^2q^2 - 4 = 0$ —— ①

∴ Charpits A·Es are

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z}$$

$$\Rightarrow \frac{dx}{-2x^2p} = \frac{dy}{-2y^2q} = \frac{dz}{-p(2x^2p) - q(2y^2q)} = \frac{dp}{2xp^2} = \frac{dq}{2yq^2} \quad \text{—— ②}$$

To find first Complete integral:

Taking the first & fourth fraction of ②,

we get $\frac{dx}{-2x^2p} = \frac{dp}{2xp^2} \Rightarrow \frac{dx}{-x} = \frac{dp}{p}$

$\Rightarrow \log x = -\log p + \log c$

$\Rightarrow \log(px) = \log c$

$\Rightarrow px = c \Rightarrow \boxed{p = \frac{c}{x}}$

① ≡ $x^2\frac{c^2}{x^2} + y^2q^2 = 4$

$\Rightarrow c^2 + y^2q^2 = 4 \Rightarrow q^2 = \frac{4-c^2}{y^2}$

$\Rightarrow \boxed{q = \frac{\sqrt{4-c^2}}{y}}$

Substituting the values of p & q in

$dz = pdx + qdy$, we get

$\Rightarrow dz = \frac{c}{x}dx + \frac{\sqrt{4-c^2}}{y}dy$

Integrating

$$\boxed{z = c\log x + \sqrt{4-c^2}\log y + \log b}$$

Taking 2nd & last fractions of ②, we get

$\frac{dy}{-2y^2q} = \frac{dq}{2yq^2} \Rightarrow \frac{dy}{-y} = \frac{dq}{q}$

$\Rightarrow \log y = -\log q + \log d$

$\Rightarrow yq = d$

$\Rightarrow \boxed{q = \frac{d}{y}}$

∴ ① ≡ $x^2p^2 + d^2 = 4$.

$\Rightarrow p^2 = \frac{4-d^2}{x^2} \Rightarrow \boxed{p = \frac{\sqrt{4-d^2}}{x}}$

substituting the values of p & q in
$$dz = p\,dx + q\,dy.$$
$$\Rightarrow dz = \frac{\sqrt{4-d^2}}{x}dx + \frac{d}{y}dy$$
$$\Rightarrow \boxed{z = \sqrt{4-d^2}\log x + d\log y}$$

→ Find three complete integrals of $pq = px+qy$.

2004 → Find a complete, singular, and general integrals of $(p^2+q^2)y = qz$

Sol^n: Given that $(p^2+q^2)y - qz = 0$

Let $f(x,y,z,p,q) = (p^2+q^2)y - qz = 0$ ——— ①

A. Eqns are
$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{f_x+pf_z} = \frac{dq}{f_y+qf_z}$$
$$\Rightarrow \frac{dx}{-2py} = \frac{dy}{-2qy+z} = \frac{dz}{-2p^2y+qz-2q^2y} = \frac{dp}{-pq} = \frac{dq}{p^2} \text{ ——— ②}$$

Taking last two fractions of ②
$$\frac{dp}{-pq} = \frac{dq}{p^2} \Rightarrow \frac{dp}{-q} = \frac{dq}{p}$$
$$\Rightarrow p\,dp + q\,dq = 0$$
$$\Rightarrow \boxed{p^2+q^2 = a^2} \text{ ——— ③}$$

① ≡ $a^2y = qz$
$$\Rightarrow \boxed{q = \frac{a^2y}{z}}$$

③ ≡ $p^2 + \frac{a^4y^2}{z^2} = a^2$
$$\Rightarrow p^2 = a^2 - \left(\frac{a^2y}{z}\right)^2$$
$$\Rightarrow p = \sqrt{a^2-\left(\frac{a^2y}{z}\right)^2}$$
$$\Rightarrow \boxed{p = \frac{a}{z}\sqrt{z^2-a^2y^2}}$$

putting these p and q in $dz = p\,dx + q\,dy$

$$\Rightarrow dz = \frac{a}{z}\sqrt{z^2-a^2y^2}\,dx + \frac{a^2y}{z}\,dy$$

$$\Rightarrow \frac{z\,dz - a^2y\,dy}{\sqrt{z^2-a^2y^2}} = a\,dx$$

$$\Rightarrow (z^2-a^2y^2)^{1/2} = ax+b$$

$$\Rightarrow z^2-a^2y^2 = (ax+b)^2 \quad \text{--- (4)}$$

which is the required complete integral.

Singular integral:

Diff (4) w.r.t a & b, we get

$$-2ay^2 = 2(ax+b)x$$

$$\Rightarrow \boxed{ax^2+bx+ay^2=0} \quad \text{--- (5)}$$

and $2(ax+b)=0$

$$\Rightarrow \boxed{ax+b=0} \quad \text{--- (6)}$$

Now eliminating a & b from (4), (5) & (6), we get

(5) $\equiv x(0)+ay^2=0$ (by (6))

$$\Rightarrow \boxed{a=0} \quad (\because y\neq 0)$$

(6) $\equiv \boxed{b=0}$

$\therefore$ (4) $\equiv z=0$ which clearly satisfies (1)

$\therefore$ It is the required singular solution of (1).

General Integral:

Let $b=\phi(a)$ in (5), then $z^2-a^2y^2=[ax+\phi(a)]^2$ --- (7)

Diff (7) partially w.r.t a, we get

$$-2ay^2 = 2(ax+\phi(a))(x+\phi'(a)) \quad \text{--- (8)}$$

$\therefore$ G.I is obtained by eliminating 'a' from (7) & (8).

1997 → Find a complete integral of $z^2(p^2z^2+q^2)=1$

1996 → Find a complete integral of $z=px+qy+p^2+q^2$

1994 → Find a complete integral of $16p^2z^2+9q^2z^2+4z^2-4=0$

1995 → " " " " $2x(z^2q^2+1)=pz$

1999 → " " " " $p^2+q^2-2px-2qy+1=0$

Special Types of equations:

we shall consider some special types of first-order partial differential eqns whose solutions may be obtained easily by Charpit's method.

Type 1: Equations involving only p & q :-

for eqns of the type $f(p,q)=0$ —— ①.

Charpit's auxiliary eqns are

$$\frac{dx}{-\partial f/\partial p} = \frac{dy}{-\partial f/\partial q} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}}$$

$$\Rightarrow \frac{dx}{-\partial f/\partial p} = \frac{dy}{-\partial f/\partial q} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dp}{0} = \frac{dq}{0}$$

Taking third & fourth fractions, we get

$dp = 0 \Rightarrow \boxed{p=a}$ (constant) —— ②

① $\equiv f(a,q)=0$

$\Rightarrow q = \text{constant}$ —— ③

$= \phi(a)$ (say)

putting these values in $dz = p\,dx + q\,dy$

$\Rightarrow dz = a\,dx + \phi(a)\,dy$

integrating

$z = ax + \phi(a)y + b$ —— ④

where b is constant

which is a complete integral of ①

it contains two arbitrary constants a & b

General Integral:

putting $b = \psi(a)$ in ④,

where ψ is arbitrary function

we get,

$z = ax + \phi(a)y + \psi(a)$ —— ⑤

Diff ⑤ partially w.r.t 'a', we get

$0 = x + \phi'(a)y + \psi'(a)$ —— (6)

eliminating a b/w (5) & (6).

Singular Integral: The singular integral, if ~~its~~ it exists, is obtained by eliminating a & b between the complete integral (4) and the eqns formed by differentiating (4) partially w.r.t a & b.

i.e, b/w the eqns

$z = ax + \phi(a)y + b$,

$0 = x + \phi'(a)y$ and $0 = 1$

Since $1 = 0$ is inconsistent (meaningless)

$\therefore$ In this case there is no singular solution.

→ Solve $pq = k$, where k is constant.

Soln: The given eqn is $pq = k$ —— (1)

where k is constant.

Clearly the eqn (1) is of the form $f(p, q) = 0$

$\therefore$ Its complete integral is

$z = ax + \phi(a)y + b$ —— (2)

Taking $a = p$; $\phi(a) = q$.

$\therefore$ (1) $\equiv$ $a\phi(a) = k$

$\Rightarrow \phi(a) = \frac{k}{a}$

$\therefore$ (2) $\equiv$ $z = ax + \frac{k}{a}y + b$ —— (3)

where a & b are arbitrary constants.

$\therefore$ It is the required complete integral.

To find singular integral:

Diff (3) partially w.r.t a & b, we get

$0 = x - \frac{k}{a^2}y$

$0 = 1$ which is meaningless

$\therefore$ The given eqn ① has no singular integral.

General integral:

putting $b = \phi(a)$ in ③, we get

$$z = ax + \frac{k}{a}y + \phi(a) \quad \text{—④}$$

Diff ④ partially w.r.t a, we get

$$0 = x - \frac{ky}{a^2} + \phi'(a) \quad \text{—⑤}$$

$\therefore$ The required G.I is obtained by eliminating a between ④ & ⑤.

→ Solve $q = 3p^2$

→ Solve $q = e^{-p/\alpha}$ where α is a constant.

→ find the complete integral of $\sqrt{p} + \sqrt{q} = 1$

Equations reducible to type 1:

→ Find the complete integral of

$$z^2p^2y + 6zpxy + 2zqx^2 + 4x^2y = 0 \quad \text{—①}$$

Soln ① $\equiv$ $z^2y\left(\frac{\partial z}{\partial x}\right)^2 + 6zxy\left(\frac{\partial z}{\partial x}\right) + 2zx^2\left(\frac{\partial z}{\partial y}\right) + 4x^2y = 0$

Dividing by x^2y, we get

$$\Rightarrow \frac{z^2}{x^2}\left(\frac{\partial z}{\partial x}\right)^2 + 6\frac{z}{x}\left(\frac{\partial z}{\partial x}\right) + \frac{2z}{y}\left(\frac{\partial z}{\partial y}\right) + 4 = 0$$

$$\Rightarrow \left(\frac{z}{x}\frac{\partial z}{\partial x}\right)^2 + 6\left(\frac{z}{x}\frac{\partial z}{\partial x}\right) + 2\left(\frac{z}{y}\frac{\partial z}{\partial y}\right) + 4 = 0 \quad \text{—②}$$

putting $x\,dx = dX$; $y\,dy = dY$; $z\,dz = dZ$

$$\Rightarrow \frac{x^2}{2} = X, \quad \frac{y^2}{2} = Y; \quad \frac{z^2}{2} = Z$$

② $\equiv$ $\left(\frac{\partial Z}{\partial X}\right)^2 + 6\left(\frac{\partial Z}{\partial X}\right) + 2\left(\frac{\partial Z}{\partial Y}\right) + 4 = 0$

$$\Rightarrow P^2 + 6P + 2Q + 4 = 0$$

where $P = \frac{\partial Z}{\partial X}$, $Q = \frac{\partial Z}{\partial Y}$.

clearly ③ is in the form of $f(P,Q)=0$

$\therefore$ Its complete integral is of the form

$$Z = ax + \phi(a)Y + b \quad \text{—— (4)}$$

where $a=P$ & $\phi(a)=Q$.

$$③ \equiv a^2+6a+2\phi(a)+4=0$$

$$\Rightarrow \phi(a) = -\frac{(a^2+6a+4)}{2}$$

$$④ \equiv Z = ax - \left(\frac{a^2+6a+4}{2}\right)Y + b$$

where a & b are arbitrary constants.

$$\therefore \frac{z^2}{2} = a\left(\frac{x^2}{2}\right) - \left(\frac{a^2+6a+4}{2}\right)\left(\frac{y^2}{2}\right) + b$$

which is the required complete integral of ①

Solve $x^2p^2+y^2q^2=z^2$ —— ①

$$\Rightarrow \frac{x^2}{z^2}\left(\frac{\partial z}{\partial x}\right)^2 + \frac{y^2}{z^2}\left(\frac{\partial z}{\partial y}\right)^2 = 1$$

$$\Rightarrow \left(\frac{x}{z}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{z}\frac{\partial z}{\partial y}\right)^2 = 1$$

$$\Rightarrow \left(\frac{\frac{1}{z}\partial z}{\frac{1}{x}\partial x}\right)^2 + \left(\frac{\frac{1}{z}\partial z}{\frac{1}{y}\partial y}\right)^2 = 1 \quad \text{—— (2)}$$

putting $\frac{1}{x}dx = dX$; $\frac{1}{y}dy = dY$; $\frac{1}{z}dz = dZ$

$$\Rightarrow \boxed{\log x = X};\ \boxed{\log y = Y};\ \boxed{\log z = Z}$$

$$\therefore ② \equiv \left(\frac{\partial Z}{\partial X}\right)^2 + \left(\frac{\partial Z}{\partial Y}\right)^2 = 1$$

$$\Rightarrow P^2+Q^2=1 \quad \text{—— (3)}$$

$\therefore$ It is of the form $f(P,Q)=0$

Its complete integral is of the form

$$Z = aX + \phi(a)Y + b \quad \text{—— (4)}$$

Taking $a = P$ & $\phi(a) = Q$

$\therefore$ (3) $\equiv$ $a^2 + [\phi(a)]^2 = 1$

$\Rightarrow [\phi(a)]^2 = 1 - a^2$

$\Rightarrow \phi(a) = \sqrt{1-a^2}$

$\therefore$ (4) $\equiv$ $Z = ax + (\sqrt{1-a^2})y + b$

where a & b are arbitrary constants.

$\Rightarrow \log z = a \log x + (\sqrt{1-a^2}) \log y + b$

which is the required complete integral.

If we take

$a = \cos\alpha, \quad b = \log c$

then complete integral is

$\log z = \cos\alpha \log x + \sin\alpha \log y + \log c.$

$\Rightarrow \boxed{z = x^{\cos\alpha} y^{\sin\alpha} c.}$ —— (5)

where α & c are arbitrary constants.

Singular integral:

Diff (5) partially w.r.t α & c, we get.

$0 = c\left[x^{\cos\alpha}\left(y^{\sin\alpha} \log y\right)\cos\alpha + y^{\sin\alpha} x^{\cos\alpha} \log x \cdot (-\sin\alpha)\right]$

$\Rightarrow x^{\cos\alpha} y^{\sin\alpha}\left(\cos\alpha \log y - \sin\alpha \log x\right) = 0$ —— (6)

and $0 = x^{\cos\alpha} y^{\sin\alpha}$. —— (7).

Eliminating α, c from (1), (5), (6), (7), we get

$z = 0$

which is the required singular solution.

G.I: putting $c = \phi(\alpha)$

(5) $\equiv$ $z = x^{\cos\alpha} y^{\sin\alpha} \phi(\alpha)$ —— (8)

Diff (8) partially w.r.t α.

$$0 = x^{\cos\alpha}\phi(\alpha)\left[y^{\sin\alpha}\log y\cdot\cos\alpha\right] + x^{\cos\alpha}y^{\sin\alpha}\phi'(\alpha) \quad \text{--- (9)}$$

Eliminating α from (8) & (9), we get the required G.S of (1).

→ Find a complete integral of

(i) $pq = x^m y^n z^{2l}$ (ii) $pq = x^m y^n z^l$

Sol^n (i) Given that

$$pq = x^m y^n z^{2l} \quad \text{--- (1)}$$

$$\Rightarrow \frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y} = x^m y^n z^{2l}$$

$$\Rightarrow \left(\frac{z^{-2l}}{x^m y^n}\right)\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right) = 1$$

$$\Rightarrow z^{-l}dz = dZ \; ; \; x^m dx = dX \; ; \; y^n dy = dY$$

$$\Rightarrow \boxed{Z = \frac{z^{-l+1}}{-l+1}} \; ; \; \boxed{X = \frac{x^{m+1}}{m+1}} \; ; \; \boxed{Y = \frac{y^{n+1}}{n+1}}$$

Type 2:

Eqns not involving the independent variables:

If the partial diff. eqn is of the type $f(z, p, q) = 0$ --- (1)

Charpit's auxiliary eqns reduce to

$$\frac{dx}{\partial f/\partial p} = \frac{dy}{\partial f/\partial q} = \frac{dz}{p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q}} = \frac{dp}{-p\frac{\partial f}{\partial z}} = \frac{dq}{-q\frac{\partial f}{\partial z}} \quad \text{--- (2)}$$

Taking last two fractions, we get

$$\frac{dq}{q} = \frac{dp}{p} \Rightarrow \frac{q}{p} = a$$

$$\Rightarrow \boxed{q = pa} \quad \text{--- (3)}$$

where a is arbitrary constant

$\therefore$ from $dz = p\,dx + q\,dy$, we get

$$dz = p(dx + a\,dy)$$

$\Rightarrow dz = p\,d(x+ay)$

$\Rightarrow dz = p\,dX$ where $X = x+ay$

$\Rightarrow \boxed{\frac{dz}{dX} = p}$

$\therefore$ ③ $\equiv \boxed{q = a\frac{dz}{dX}}$

$\therefore$ ① $\equiv f\left(z, \frac{dz}{dX}, a\frac{dz}{dX}\right) = 0$

which is an ordinary diff. eqn of the first order and solving it, a complete integral can be obtained.

Working rule:-

Step(1): write down the given eqn $f(p,q,z)=0$

Step(2): put $p = \frac{dz}{dX}$ & $q = a\frac{dz}{dX}$ where $X = x+ay$

Step(3): solving the resulting ODE in the variables z & X then substitute $X = x+ay$. This gives the complete integral.

Note: Some times using transformations eqns to the form of type (II).

→ Find a complete integral of $9(p^2z+q^2)=4$

Sol: Given that $9(p^2z+q^2)=4$ —— ①

Clearly it is of the form $f(p,q,z)=0$

where $p = \frac{dz}{dX}$, $q = a\frac{dz}{dX}$

① $\equiv 9\left[\left(\frac{dz}{dX}\right)^2 z + a^2\left(\frac{dz}{dX}\right)^2\right] = 4$

where $X = x+ay$ —— ②

$\Rightarrow 9(z+a^2)\left(\frac{dz}{dX}\right)^2 = 4$

$$\Rightarrow \frac{dz}{dx} = \frac{2}{3\sqrt{z+a^2}}$$

$$\Rightarrow \frac{3}{2}\sqrt{z+a^2}\,dz = dx$$

$$\Rightarrow \frac{3}{2}\frac{(z+a^2)^{3/2}}{3/2} = x+b$$ where b is arbitrary constant

$$\Rightarrow (z+a^2)^{3/2} = (x+ay)+b \quad (\text{by } ②)$$

$$\Rightarrow (z+a^2)^3 = (x+ay+b)^2,$$ where a & b are arbitrary constants.

which is the required complete integral.

→ Find the complete integral of the eqn $p^2z^2+q^2=1$

1991 → Find a complete integral of $p^3+q^3-3pqz=0$

1993 → Find a complete integral of $z^2(p^2+q^2+1)=k^2$ where k is constant.

→ Find a complete integral of $p^2y^2 = z(z-px)$. ①

Hint: $\left(y\frac{\partial z}{\partial y}\right)^2 = z\left(z - x\frac{\partial z}{\partial x}\right)$

$$\Rightarrow \left(\frac{\partial z}{\frac{1}{y}\partial y}\right)^2 = z\left(z - \frac{\partial z}{\frac{1}{x}\partial x}\right) \quad ②$$

Taking $\frac{1}{x}dx = dX$ || $\frac{1}{y}dy = dY$

$\log x = X$ || $\log y = Y$.

$$\therefore ② \equiv \left(\frac{\partial z}{\partial Y}\right)^2 = z\left(z - \frac{\partial z}{\partial X}\right)$$

$$\Rightarrow z(z-P) = Q^2$$ where $P = \frac{\partial z}{\partial X}$; $Q = \frac{\partial z}{\partial Y}$

and proceed.

Type(III)

Seperable equations

Eqns not involving z and it happens the terms containing p & x can be seperated from those containing q & y.

i.e, they have the form

$$f_1(x, p) = f_2(y, q) \quad \text{——— (1)}$$

corresponding Charpit's auxiliary eqns are

$$\frac{dx}{\frac{\partial f_1}{\partial p}} = \frac{dy}{-\frac{\partial f_2}{\partial q}} = \frac{dp}{-\frac{\partial f_1}{\partial x}} = \frac{dq}{-\frac{\partial f_2}{\partial y}} \quad \text{——— (2)}$$

$$\frac{\partial f_1}{\partial p}dp + \frac{\partial f_1}{\partial x}dx = 0$$

$$\Rightarrow df_1 = 0 \qquad \left(\because df_1 = \frac{\partial f_1}{\partial x}dx + \frac{\partial f_1}{\partial p}dp\right)$$

$$\Rightarrow f_1 = \text{constant}$$

$$\Rightarrow f_1(x, p) = a \text{ (say)} \quad \text{——— (3)}$$

$$\therefore (1) \equiv f_2(y, q) = f_1(x, p)$$

$$\Rightarrow f_2(y, q) = a \quad \text{——— (4)}$$

solving (3) & (4) for p & q, we get

$$p = F_1(x, a), \quad q = F_2(y, a).$$

putting these values of p & q in $dz = p\,dx + q\,dy$,

we get $dz = F_1(x, a)\,dx + F_2(y, a)\,dy$.

Integrating we get

$$z = \int F_1(x, a)\,dx + \int F_2(y, a)\,dy + b$$

where b is an arbitrary constant.

$\therefore$ which is the required complete integral.

working rule:

Step 1: write the given eqn in the form $f_1(x, p) = f_2(y, q)$.

Step 2: putting both sides of the above eqn equal to an arbitrary constant. we get the two eqns.

Step 3: Solving them for p & q.

Substitute the values of p & q in

$$dz = p\,dx + q\,dy.$$

Integrate, we get the complete integral of ①

Note: Some times using transformations eqns reduce to the form of type III

→ Find a complete integral of $p^2+q^2=x+y$

Soln: Given that $p^2+q^2 = x+y$ — ①

$$\Rightarrow p^2 - x = y - q^2 \quad \text{— ②}$$

$$\Rightarrow p^2 - x = y - q^2 = a \ (\text{say}).$$

$$\Rightarrow p^2 - x = a \quad \text{— ③}$$

$$\& \ y - q^2 = a \quad \text{— ④}$$

③ ≡ $p = \sqrt{x+a}$ & ④ ≡ $q = \sqrt{y-a}$

∴ putting these values of p & q in $dz = p\,dx + q\,dy$

$$\Rightarrow dz = \sqrt{x+a}\,dx + \sqrt{y-a}\,dy$$

Integrating

$$z = \frac{2}{3}(x+a)^{3/2} + \frac{2}{3}(y-a)^{3/2} + b.$$

which is the required complete integral.

1989 → Find a complete integral of $z^2(p^2+q^2) = x^2+y^2$.

Soln: i.e $z^2\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right] = x^2+y^2$ — ①

$$\Rightarrow \left(z\frac{\partial z}{\partial x}\right)^2 + \left(z\frac{\partial z}{\partial y}\right)^2 = x^2+y^2 \quad \text{— ②}$$

Taking $z\,dz = dZ$

② ≡ $\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2 = x^2+y^2$

$\Rightarrow P^2+Q^2 = x^2+y^2$ where $P=\frac{\partial z}{\partial x}$ $Q=\frac{\partial z}{\partial y}$

and proceed.

89 → find a complete integral of $z(p^2-q^2)=x-y$

2001 → find the complete integral of the PDE

$$2p^2q^2+3x^2y^2 = 8x^2q^2(x^2+y^2).$$

Sol^n: Given that

$$2p^2q^2+3x^2y^2 = 8x^2q^2(x^2+y^2) \quad \text{—①}$$

$$\Rightarrow 2q^2(p^2-4x^4) = x^2y^2(8q^2-3)$$

$$\Rightarrow \frac{p^2-4x^4}{x^2} = \frac{y^2(8q^2-3)}{2q^2} = 4a^2 \text{ (say) (constant)}$$

$$\Rightarrow \frac{p^2-4x^4}{x^2}=4a^2 \;;\quad \frac{y^2(8q^2-3)}{2q^2}=4a^2$$

$$\Rightarrow p^2=4x^2(a^2+x^2) \;;\quad 8q^2(y^2-a^2)=3y^2$$

$$\Rightarrow p=2x(a^2+x^2)^{1/2} \;;\quad q^2=\frac{3y^2}{8(y^2-a^2)}=\left(\frac{1}{4}\right)\left(\frac{3}{2}\right)\frac{y^2}{y^2-a^2}$$

$$q=\left(\frac{3}{2}\right)^{1/2}\left(\frac{y}{2}\right)\frac{1}{(y^2-a^2)^{1/2}}$$

Substituting the values of p & q in

$$dz = p\,dx+q\,dy$$

$$\Rightarrow dz = 2x(a^2+x^2)^{1/2}dx+\left(\frac{3}{2}\right)^{1/2}\left(\frac{y}{2}\right)(y^2-a^2)^{-1/2}$$

Integrating

$$\Rightarrow z = \frac{4}{3}(a^2+x^2)^{3/2}+\left(\frac{3}{2}\right)^{1/2}(y^2-a^2)^{1/2}+b.$$

Type(IV) Clairaut eqn:-

A first order PDE is said to be Clairaut form if it is in the form $z = px + qy + f(p,q)$ —— (1)

The corresponding Charpit's auxiliary eqns are

$$\frac{dx}{x+f_p} = \frac{dy}{y+f_q} = \frac{dz}{px+qy+pf_p+qf_q} = \frac{dp}{0} = \frac{dq}{0}$$

$\Rightarrow p = a$ & $q = b$

$\therefore$ (1) $\equiv$ $z = ax + by + f(a,b)$ —— (2)

which is the required complete integral.

→ To find the general integral

put $b = \phi(a)$ in (2), where ϕ is an arbitrary function.

then $z = ax + y\phi(a) + f\{a, \phi(a)\}$ —— (3)

Diff (3) partially w.r.t a, we get

$0 = x + y\phi'(a) + f'(a)$ —— (4)

Eliminating a b/w (3) & (4), we get G.I of (1).

→ To find the S.I, eliminate a & b b/w the three eqns $z = ax + by + f(a,b)$

$x + \frac{\partial f}{\partial a} = 0$ and $y + \frac{\partial f}{\partial b} = 0$

Note: Some times, using the transformations eqns reduce to the form of type IV.

Problems: Find the singular integral of $z = px + qy + c\sqrt{1+p^2+q^2}$

Soln: Given $z = px + qy + c\sqrt{1+p^2+q^2}$

It is of the Clairaut's form.

$\therefore$ Its complete integral is $\boxed{z = ax + by + c\sqrt{1+a^2+b^2}}$ —— (1)

Singular integral:

Diff ① partially w.r.t a and b,

we get $0 = x + \frac{ac}{\sqrt{1+a^2+b^2}}$ —②, $0 = y + \frac{bc}{\sqrt{1+a^2+b^2}}$ —③

from ② & ③

$$x^2+y^2 = \frac{a^2c^2+b^2c^2}{1+a^2+b^2}$$

$$\Rightarrow c^2-x^2-y^2 = \frac{c^2}{1+a^2+b^2}$$

$$\Rightarrow 1+a^2+b^2 = \frac{c^2}{c^2-x^2-y^2} \quad \text{—④}$$

from ② $a = -\frac{x\sqrt{1+a^2+b^2}}{c}$

$$= \frac{-x}{\sqrt{c^2-x^2-y^2}} \quad (\text{by ④}) \quad \text{—⑤}$$

and from ③ $b = -\frac{y\sqrt{1+a^2+b^2}}{c}$

$$= \frac{-y}{\sqrt{c^2-x^2-y^2}} \quad \text{—⑥}$$

putting the values from ④, ⑤ & ⑥ in ①,

the singular solution is

$$z = -\frac{x^2}{\sqrt{c^2-x^2-y^2}} - \frac{y^2}{\sqrt{c^2-x^2-y^2}} + \frac{c^2}{\sqrt{c^2-x^2-y^2}}$$

$$= \frac{c^2-x^2-y^2}{\sqrt{c^2-x^2-y^2}} = \sqrt{c^2-x^2-y^2}$$

$$\Rightarrow z^2 = c^2-x^2-y^2$$

$$\Rightarrow \boxed{x^2+y^2+z^2=c^2}$$

→ Find a complete and singular integral of
$4xyz = pq + 2px^2y + 2qxy^2$.

Sol^n: Given that $4xyz = pq + 2p x^2 y + 2q xy^2$ —— ①

$$\Rightarrow z = \frac{1}{4xy}\left(\frac{\partial z}{\partial x}\right)\left(\frac{\partial z}{\partial y}\right) + \frac{1}{2xy}\left(\frac{\partial z}{\partial x}\right)x^2y + \frac{1}{2xy}\left(\frac{\partial z}{\partial y}\right)xy^2$$

$$\Rightarrow z = \left(\frac{1}{2x}\cdot\frac{\partial z}{\partial x}\right)\left(\frac{1}{2y}\frac{\partial z}{\partial y}\right) + \left(\frac{1}{2x}\frac{\partial z}{\partial x}\right)x^2 + \left(\frac{1}{2y}\frac{\partial z}{\partial y}\right)y^2 \quad \text{—— ②}$$

Taking $2x\,dx = dX$; $2y\,dy = dY$

$\Rightarrow \boxed{x^2 = X}$; $\Rightarrow \boxed{y^2 = Y}$

② ≡ $$z = \left(\frac{\partial z}{\partial X}\right)\left(\frac{\partial z}{\partial Y}\right) + \left(\frac{\partial z}{\partial X}\right)X + \left(\frac{\partial z}{\partial Y}\right)Y$$

$$\Rightarrow z = PX + QY + PQ \quad \text{—— ③}$$

where $P = \frac{\partial z}{\partial X}$; $Q = \frac{\partial z}{\partial Y}$.

Clearly which is in Clairaut's form.

∴ The complete integral of ③ is

$z = aX + bY + ab$. (by putting $P = a$ & $Q = b$)

$$\Rightarrow \boxed{z = ax^2 + by^2 + ab} \quad \text{—— ④}$$

which is the required complete integral of ①.

Singular integral:

Differentiating ④ w.r.t a & b, we get

$0 = x^2 + b \Rightarrow \boxed{b = -x^2}$ —— ⑤

and $0 = y^2 + a \Rightarrow \boxed{a = -y^2}$ —— ⑥

∴ ④ ≡ $z = -y^2x^2 - x^2y^2 + x^2y^2$

$$\Rightarrow \boxed{z = -x^2y^2}$$

which is the required singular integral of ①

2008 / 15M → Find complete and singular integrals of
$2xz - px^2 - 2qxy + pq = 0$
using Charpit's method.

Solutions satisfying given conditions :-

We shall consider the determination of surfaces which satisfy the PDE $f(x,y,z,p,q)=0$ —— ①
and which satisfy some other condition as passing through given curve (or) circumscribing a given surface. we shall also consider how to derive one complete integral from another.

I → first of all, we shall discuss how to Determine the solution of ① which passes through a given curve 'c' which has parametric eqns

$x=x(t), y=y(t), z=z(t)$ —— ②

where t is parameter.

If there is an integral surface of the eqn ① through the curve 'c', then it is ;

(a) A particular case of the complete integral
$f(x,y,z,a,b)=0$ —— ③
obtained by giving particular values to a or b.

(or)

(b) A particular case of the general integral corresponding to ③
i.e, the envelope of a one-parameter subfamily of (3).

(or)

(c) The envelope of the two-parameter system ③.

→ Now the points of intersection of the surface ③ and the curve 'c' are determined interms of the parameter 't' by the eqn $f(x(t), y(t), z(t), a, b)=0$ —— ④
and the condition that the curve 'c' should touch the

surface ③ is that the eqn ④ must have two equal roots (i.e, $b^2-4ac=0$)

(or) the eqn ④ and

the eqn $\frac{\partial}{\partial t} f(x(t), y(t), z(t), a, b) = 0$ —— ⑤

should have a common root.

Now eliminating 't' from ④ & ⑤, we get the relation b/w a & b of the type $\psi(a,b)=0$ —— ⑥

The eqn ⑥ may be factorised into a set of alternative eqns.

$b=\phi_1(a),\ b=\phi_2(a), \ldots$ —— ⑦

∴ Each of which defines a subsystem of one-parameter.

The envelope of each of these one-parameter subsystems is a solution of the problem.

problems

2004 → Find a complete integral of the PDE $(p^2+q^2)x = pz$ and deduce the solution which passes through the curve $x=0,\ z^2=4y$.

Sol: Given that $(p^2+q^2)x = pz$.

Let $f(x,y,z,p,q) = (p^2+q^2)x - pz = 0$ —— ①

By Charpit's method its complete integral

$$z^2 = a^2x^2 + (ay+b)^2 \quad \text{——} ②$$

and the given curve is $x=0,\ z^2=4y$ —— ③

Now taking 't' as parameter in ③.

we get $x=0,\ y=t^2,\ z=2t$ —— ④

The intersection of ② & ④ is

$$(2t)^2 = a^2(0) + (at^2+b)^2$$

$$\Rightarrow 4t^2 = a^2t^4 + b^2 + 2abt^2$$

$$\Rightarrow a^2(t^2)^2 + (2ab-4)t^2 + b^2 = 0 \quad \text{——} ⑤$$

This has equal roots

if $(2ab-4)^2-4a^2b^2=0$

$\Rightarrow \quad 4-4ab=0$

$\Rightarrow \quad \boxed{ab=1}$ —— (6)

$\therefore \boxed{b=\frac{1}{a}}$

$\therefore (2) \equiv \quad z^2 = a^2x^2+\left(ay+\frac{1}{a}\right)^2$ —— (7)

$\therefore$ which is the one parameter subsystem of (2).

(7) $\equiv x^2a^2+y^2a^2+\frac{1}{a^2}+2y-z^2=0$

$\Rightarrow (x^2+y^2)a^4+(2y-z^2)a^2+1=0$ —— (8)

This has equal roots

if $(2y-z^2)^2-4(x^2+y^2)=0$ —— (9)

which required envelope of (8)

(9) $\equiv \quad 2y-z^2 = 2\sqrt{x^2+y^2}$

$\Rightarrow z^2 = 2y-2\sqrt{x^2+y^2}$

which is the required solution of the eqn (1).

→ Find a complete integral of the eqn $p^2x+q^2y=z$ and hence derive the eqn of an integral surface of which the line $y=1$, $x+z=0$ is a generator

→ Show that integral surface of the eqn $z(1-q^2)=2(px+qy)$ which pass through the line $x=1$, $y=hz+k$ has the eqn $(y-kx)^2=z^2\{(1+h^2)x-1\}$.

II The problem of deriving one complete integral from another may be treated in a very similar way.

Suppose we know that $f(x,y,z,a,b)=0$ —— (1)

is complete integral of $F(x,y,z,p,q)=0$ —— (2)

and we want to show that another relation

$g(x,y,z,h,k)=0$ —— (3)

where h & k are arbitrary constants

is also complete integral of ②.

we choose on the surface ③ a curve c in whose eqns the constants h,k appear as independent parameters and then find the envelope of the one-parameter subfamily of ① touching the curve C.

Since this solution contains two arbitrary constants, it is a complete integral.

→ show that the equation $xpq + yq^2 = 1$ has complete integrals (a) $(z+b)^2 = 4(ax+y)$

(b) $kx(z+h) = k^2y + x^2$

and deduce (b) from (a)

Soln: Given that $xpq + yq^2 = 1$ ——①

Charpit's auxiliary eqns are

$$\frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dz}{-pf_p - qf_q} = \frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z}$$

$$\Rightarrow \frac{dx}{-xq} = \frac{dy}{-[xp+2yq]} = \frac{dz}{p(xq) + q(xp+2yq)} = \frac{dp}{pq} = \frac{dq}{q^2} \quad \text{——②}$$

Now taking last two fractions from ②, we have

$$\boxed{p = qa}$$

①≡ $xq^2a + yq^2 = 1$

$\Rightarrow q^2(xa+y) = 1$

$$\Rightarrow \boxed{q = \frac{1}{\sqrt{xa+y}}}$$

$$\boxed{\therefore p = \frac{a}{\sqrt{xa+y}}}$$

substituting the values of p and q in $dz = pdx + qdy$

$$\Rightarrow dz = \frac{1}{\sqrt{xa+y}}[adx + dy]$$

$$\Rightarrow dz = \frac{d(ax+y)}{\sqrt{xa+y}}$$

$$\Rightarrow z = 2(xa+y)^{1/2} + b$$

$$\Rightarrow \boxed{(z+b)^2 = 4(ax+y)} \text{ ——— (a)}$$

Taking first & last fractions of (2), we get

$$\frac{dx}{-x} = \frac{dq}{q}$$

$$\Rightarrow xq = k \Rightarrow \boxed{q = \frac{k}{x}}$$

$$\therefore (1) \equiv \quad pk + y\frac{k^2}{x^2} = 1$$

$$\Rightarrow \boxed{p = \frac{1}{k}\left(\frac{x^2 - yk^2}{x^2}\right)}$$

$$\therefore dz = \left(\frac{x^2 - yk^2}{x^2 k}\right) dx + \frac{k}{x} dy$$

$$dz = \frac{1}{k} dx - \frac{yk}{x^2} dx + \frac{k}{x} dy$$

$$\Rightarrow dz = \frac{1}{k} dx + d\left(y \cdot \frac{k}{x}\right)$$

$$\Rightarrow z = \frac{x}{k} + \frac{yk}{x} + h$$

$$\Rightarrow (z + h) = \frac{x^2 + yk^2}{kx}$$

$$\Rightarrow kx(z+h) = x^2 + yk^2$$

$$\Rightarrow \boxed{kx(z+h) = k^2 y + x^2} \text{ ——— (b)}$$

Now consider the curve from (b)

$$y = 0 \; ; \; x = k(z+h) \text{ ——— (3)}$$

where h, k are independent parameters.

Now taking 't' as parameter in (3)

we get $z = t \; ; \; x = k(t+h) \; ; \; y = 0$ ——— (4)

The intersection of (a) & (4) is

$$(t+b)^2 = 4[ak(t+h)]$$

$$\Rightarrow t^2 + (2b - 4ak)t + b^2 - 4akh = 0$$

This has equal roots if $(2b-4ak)^2-4(1)(b^2-4akh)=0$

$\Rightarrow b^2+4a^2k^2-4abk-b^2+4akh=0$

$\Rightarrow a^2k^2-abk+akh=0$

$\Rightarrow ak[ak-b+h]=0$

$\Rightarrow \boxed{b=h+ak}$ $(\because ak\neq 0)$

④ $\equiv [z+(h+ak)]^2=4(ax+y)$ —— ⑤

which is the one-parameter subsystem of ①

⑤ $\equiv z^2+(h+ak)^2+2z(h+ak)=4ax+4y$

$\Rightarrow z^2+h^2+a^2k^2+2hak+2hz+2kza-4ax-4y=0$

$\Rightarrow k^2a^2+(2hk+2zk-4x)a+\underbrace{z^2+2hz+h^2}-4y=0$

This has equal roots

if $(2hk+2zk-4x)^2-4k^2((z+h)^2-4y)=0$

$\Rightarrow (hk+zk-2x)^2=k^2[(z+h)^2-4y)]$

$\Rightarrow \cancel{h^2k^2}+\cancel{z^2k^2}+4x^2+\cancel{2hzk^2}-4xzk-4xhk$

$=\cancel{2hzk^2}+\cancel{k^2z^2}+\cancel{k^2h^2}-4yk^2$

$\Rightarrow 4x^2-4xzk-4xhk=-4yk^2$

$\Rightarrow x^2+yk^2=xzk+xhk$

$\Rightarrow kx(z+h)=k^2y+x^2$

→ Show that the diff. eqn $2xz+q^2=x(xp+yq)$ has a complete integral $z+a^2x=axy+bx^2$ and deduce that $x(y+hx)^2=4(z-kx^2)$ is also a complete integral

→ Find the complete integral of diff. eqn $xp(1+q)=(y+z)q$ corresponding to that integral of Charpit's eqns which involving only q & x, and deduce that $(z+hx+k)^2=4hx(k-y)$ is also a complete integral.

$\boxed{\text{III}}$ The determination of surfaces which satisfy the PDE $F(x, y, z, p, q) = 0$ —— ①

and which satisfy some other condition such as circumscribing a given surface.

→ Two surfaces are said to be circumscribe each other if they touch along curve.

→ Now we shall suppose that $f(x, y, z, a, b) = 0$ —— ②

is a complete integral of ①.

Now we wish to find, by using ②, an integral surface of ①, which circumscribes the surface 'Σ' whose eqn is $\psi(x, y, z) = 0$ —— ③

If we have a surface E; $u(x, y, z) = 0$ —— ④

of the required kind then it will be one of the three kinds:

(a) A particular case of the complete integral $f(x, y, z; a, b) = 0$

obtained by giving particular values to a or b.

(b) A particular case of the general integral corresponding to ②

i.e, the envelope of a one-parameter subfamily of ②.

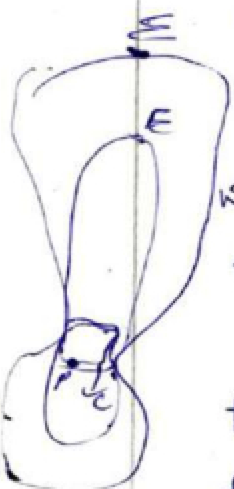
(c) The envelope of two-parameter system ②.

→ We now to find the surface ② which touch Σ and see if they provide a solution of the problem.

→ The surface ② touches the surface ④ iff the eqns ②, ③ and $\frac{f_x}{\psi_x} = \frac{f_y}{\psi_y} = \frac{f_z}{\psi_z}$ —— ⑤ are consistent

Now eliminating x, y and z from these eqns, we get the relation b/w a & b of the form $\xi(a,b)=0$ —— ⑥

This relation may be factorized into set of alternative eqns $b=\phi_1(a)$, $b=\phi_2(a), \ldots\ldots$ —— ⑦ each of which defines a subsystem of ② whose members touch ③.



The points of contact lie on the surface whose eqn is obtained by eliminating a & b from the eqns ⑥ & ⑦.

The curve C is the intersection of this surface with Σ. Each of the relations ⑦ defines a subsystem whose envelope E touches Σ along C.

→ Show that the only integral surface of the eqn $2q(z-px-qy)=1+q^2$ which is circumscribed about the paraboloid $2x=y^2+z^2$ is the enveloping cylinder which touches it along its section by the plane $y+1=0$.

Sol^n: Given that $2q(z-px-qy)=1+q^2$ —— ①

$$\Rightarrow z-px-qy = \frac{1+q^2}{2q}$$

$$\Rightarrow z = px+qy+\frac{1+q^2}{2q} \quad \text{—— ②}$$

Clearly which is in clairaut's form

$$z = px+qy+f(p,q)$$

The required complete integral of ① is

$$z = ax+by+\frac{1+b^2}{2b} \quad (\text{by putting } p=a,\ q=b)$$

Let $f(x,y,z,a,b) = z - ax - by - \frac{1+b^2}{2b} = 0$ —— (3)

and given that integral surface of (1) which circumscribes about the paraboloid $2x = y^2 + z^2$

Let $\psi(x,y,z) = 2x - y^2 - z^2$ —— (4)

Now $\frac{f_x}{\psi_x} = \frac{f_y}{\psi_y} = \frac{f_z}{\psi_z}$

$\Rightarrow \frac{a}{2} = \frac{b}{-2y} = \frac{-1}{-2z}$ —— (5)

$\Rightarrow \frac{a}{2} = \frac{b}{-2y}$ & $\frac{a}{2} = \frac{1}{2z}$

$\Rightarrow \boxed{y = -\frac{b}{a}}$ & $\boxed{z = \frac{1}{a}}$ —— (6)

Now eliminating x b/w (3) & (4), we get

$2z = a(y^2+z^2) + 2by + 2\left(\frac{b^2+1}{2b}\right)$

$\Rightarrow 2bz = aby^2 + abz^2 + 2b^2y + b^2+1$ —— (7)

and eliminating y & z from (7) by using (6), we get

$(b-a)(b^2+1) = 0$

$\Rightarrow \boxed{b = a}$ $(\because b^2+1 \neq 0)$

which defines a subsystem of (3) whose envelope is a surface of the required kind.

$\therefore$ The envelope of the subsystem

$[2(x+y)+1]a^2 - 2az + 1 = 0$ is

$4z^2 - 4[2(x+y)+1]1 = 0$

$\Rightarrow z^2 = 2(x+y)+1$ —— (8)

Since the surface (4) touches the surface (8)

$2x - y^2 = 2(x+y)+1$

$\Rightarrow -y^2 = 2y+1$

$\Rightarrow y^2 + 2y + 1 = 0$

$$\Rightarrow (y+1)^2 = 0$$

$$\Rightarrow y+1 = 0$$

→ find the integral surface of the PDE $(y+zq)^2 = z(1+p^2+q^2)$ circumscribed about the surface $x^2 - z^2 = 2y$.

→ Show that the integral surface of the eqn $2y(1+p^2) = pq$ which is circumscribed about the cone $x^2+z^2 = y^2$ has eqn

$$z^2 = y^2(4y^2+4x+1).$$

Jacobi's Method

Working rule for solving PDE's with three (or) more than three independent variables:

Step 1: Suppose the given eqn with three independent variables is $f(x_1, x_2, x_3, p_1, p_2, p_3) = 0$ —— ① in which the dependent variable z does not appear;

x_1, x_2, x_3 are independent variables and $p_i = \frac{\partial z}{\partial x_i}$; $i = 1, 2, 3$.

Step 2: write down the Jacobi's auxiliary eqns

$$\frac{dx_1}{-\partial f/\partial p_1} = \frac{dx_2}{-\partial f/\partial p_2} = \frac{dx_3}{-\partial f/\partial p_3} = \frac{dp_1}{\partial f/\partial x_1} = \frac{dp_2}{\partial f/\partial x_2} = \frac{dp_3}{\partial f/\partial x_3}$$

solving these eqns, we obtain two additional eqns

$$F_1(x_1, x_2, x_3, p_1, p_2, p_3) = a_1 \quad \text{—— ②}$$

$$F_2(x_1, x_2, x_3, p_1, p_2, p_3) = a_2 \quad \text{—— ③}$$

where a_1 & a_2 arbitrary constants.

Step 3: verify that relations ② & ③ satisfy the condition

$$(F_1, F_2) = \sum_{r=1}^{3} \left(\frac{\partial F_1}{\partial x_r} \frac{\partial F_2}{\partial p_r} - \frac{\partial F_1}{\partial p_r} \frac{\partial F_2}{\partial x_r} \right) = 0$$

$$\Rightarrow (F_1, F_2) = \sum_{r=1}^{3} \frac{\partial(F_1, F_2)}{\partial(x_r, p_r)} = 0 \quad \text{—— ④}$$

If ④ is satisfied then solve (1), (2) & (3) for p_1, p_2, p_3 in terms of x_1, x_2, x_3.

∴ substitute these values in

$$dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3$$

It gives the complete integral of the given eqn and containing three arbitrary constants.

Note: while solving a PDE with four independent variables, Step 1: The given eqn is of the form

$$f(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = 0 \quad \text{--- (1)}$$

Step 2: write down the Jacobi's auxiliary eqns

$$\frac{dx_1}{-\partial f/\partial p_1} = \frac{dx_2}{-\partial f/\partial p_2} = \frac{dx_3}{-\partial f/\partial p_3} = \frac{dx_4}{-\partial f/\partial p_4} = \frac{dp_1}{\partial f/\partial x_1} = \frac{dp_2}{\partial f/\partial x_2} = \frac{dp_3}{\partial f/\partial x_3} = \frac{dp_4}{\partial f/\partial x_4}$$

Solving these eqns, we obtain three additional

eqns $F_1(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = a_1$ --- (2)

$F_2(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = a_2$ --- (3)

$F_3(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = a_3$ --- (4)

where a_1, a_2, a_3 are arbitrary constants.

Step 3: verify the relations (2) (3), & (4) satisfy the following three conditions.

$$(F_1, F_2) = \sum_{r=1}^{4} \frac{\partial(F_1, F_2)}{\partial(x_r, p_r)} = 0 \quad \text{--- (5)}$$

$$(F_2, F_3) = \sum_{r=1}^{4} \frac{\partial(F_2, F_3)}{\partial(x_r, p_r)} = 0 \quad \text{--- (6)}$$

and $$(F_3, F_1) = \sum_{r=1}^{4} \frac{\partial(F_3, F_1)}{\partial(x_r, p_r)} = 0 \quad \text{--- (7)}$$

If (5), (6) & (7) are satisfied then solve (1), (2), (3) & (4) for p_1, p_2, p_3 & p_4 interms of x_1, x_2, x_3 & x_4 and substitute these values in $dz = p_1dx_1 + p_2dx_2 + p_3dx_3 + p_4dx_4$

which gives the complete integral of (1) and containing four arbitrary constants.

97 → Find a complete integral of $p_1^3 + p_2^2 + p_3 = 1$

Sol^n: Let the given eqn be

$$f(x_1, x_2, x_3, p_1, p_2, p_3) = p_1^3 + p_2^2 + p_3 - 1 = 0 \quad \text{--- (1)}$$

Now Jacobi's AE's are

$$\frac{dx_1}{-\partial f/\partial p_1} = \frac{dx_2}{-\partial f/\partial p_2} = \frac{dx_3}{-\partial f/\partial p_3} = \frac{dp_1}{\partial f/\partial x_1} = \frac{dp_2}{\partial f/\partial x_2} = \frac{dp_3}{\partial f/\partial x_3}$$

$$\Rightarrow \frac{dx_1}{-3p_1^2} = \frac{dx_2}{-2p_2} = \frac{dx_3}{-1} = \frac{dp_1}{0} = \frac{dp_2}{0} = \frac{dp_3}{0}.$$

from the ~~fourth~~ 4^{th} & 5^{th} fractions, we get

$$dp_1 = 0 \text{ and } dp_2 = 0$$

$$\Rightarrow p_1 = a_1 \text{ and } p_2 = a_2$$

Here $F_1(x_1, x_2, x_3, p_1, p_2, p_3) = p_1 - a_1 = 0$ —— ②

$F_2(x_1, x_2, x_3, p_1, p_2, p_3) = p_2 - a_2 = 0$ —— ③

Now $(F_1, F_2) = \sum_{r=1}^{3} \frac{\partial(F_1, F_2)}{\partial(x_r, p_r)}$

$$= \frac{\partial(F_1,F_2)}{\partial(x_1,p_1)} + \frac{\partial(F_1,F_2)}{\partial(x_2,p_2)} + \frac{\partial(F_1,F_2)}{\partial(x_3,p_3)}$$

$$= \frac{\partial F_1}{\partial x_1}\frac{\partial F_2}{\partial p_1} - \frac{\partial F_1}{\partial p_1}\frac{\partial F_2}{\partial x_1} + \frac{\partial F_1}{\partial x_2}\frac{\partial F_2}{\partial p_2} - \frac{\partial F_1}{\partial p_2}\frac{\partial F_2}{\partial x_2} + \frac{\partial F_1}{\partial x_3}\frac{\partial F_2}{\partial p_3} - \frac{\partial F_1}{\partial p_3}\frac{\partial F_2}{\partial x_3}$$

$$= 0$$

$\therefore (F_1, F_2) = 0$

$\therefore$ The eqns (2) & (3) taken as additional eqns.

$\therefore$ solving ①, ② & ③ for p_1, p_2 & p_3.

we have $p_1 = a_1$, $p_2 = a_2$ & $p_3 = 1 - a_1^3 - a_2^2$.

$\therefore$ putting these values in $dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3$

we have

$$dz = a_1 dx_1 + a_2 dx_2 + (1 - a_1^3 - a_2^2) dx_3$$

Integrating, we get

$$z = a_1 x_1 + a_2 x_2 + (1 - a_1^3 - a_2^2) x_3 + a_3$$

where a_1, a_2, a_3 are arbitrary constants.

which is the required complete integral.

1998. → Find a complete integral of $2p_1 x_1 x_3 + 3p_2 x_3^2 + p_2^2 p_3 = 0$

→ find a complete integral of $p_1 p_2 p_3 = z^3 x_1 x_2 x_3$

i.e, $\frac{\partial z}{\partial x_1}\frac{\partial z}{\partial x_2}\frac{\partial z}{\partial x_3} = z^3 x_1 x_2 x_3$

Soln. $\left(\frac{1}{z}\frac{dz}{\partial x_1}\right)\left(\frac{1}{z}\frac{\partial z}{\partial x_2}\right)\left(\frac{1}{z}\frac{\partial z}{\partial x_3}\right) = x_1x_2x_3$ ——— (1)

Taking $\frac{1}{z}dz = dZ$

$\Rightarrow \boxed{\log z = Z}$

(1) ≡ $\left(\frac{\partial Z}{\partial x_1}\right)\left(\frac{\partial Z}{\partial x_2}\right)\left(\frac{\partial Z}{\partial x_3}\right) = x_1x_2x_3$

Let $f(x_1,x_2,x_3,P_1,P_2,P_3) = P_1P_2P_3 - x_1x_2x_3 = 0$ ——— (2)

where $P_1 = \frac{\partial Z}{\partial x_1}$ $P_2 = \frac{\partial Z}{\partial x_2}$ $P_3 = \frac{\partial Z}{\partial x_3}$

Now the Jacobi's auxiliary eqns are

$$\frac{dx_1}{-P_2P_3} = \frac{dx_2}{-P_1P_3} = \frac{dx_3}{-P_1P_2} = \frac{dP_1}{-x_2x_3} = \frac{dP_2}{-x_1x_3} = \frac{dP_3}{-x_1x_2} \text{ ——— (3)}$$

(2) ≡ $P_2P_3 = \frac{x_1x_2x_3}{P_1}$

∴ first and fourth fractions of (3) give

$$\frac{dx_1}{-\frac{x_1x_2x_3}{P_1}} = \frac{dP_1}{-x_2x_3} \Rightarrow \frac{dP_1}{P_1} = \frac{dx_1}{x_1}$$

Integrating, we get

$\boxed{P_1 = a_1x_1}$

Let $F_1(x_1,x_2,x_3,P_1,P_2,P_3) = P_1 - a_1x_1$ ——— (4)

Similarly we have $F_2(x_1,x_2,x_3,P_1,P_2,P_3) = P_2 - a_2x_2$ ——— (5)

$$(F_1,F_2) = \sum_{r=1}^{3}\frac{\partial(F_1,F_2)}{\partial(x_r,P_r)} = \frac{\partial(F_1,F_2)}{\partial(x_1,P_1)} + \frac{\partial(F_1,F_2)}{\partial(x_2,P_2)} + \frac{\partial(F_1,F_2)}{\partial(x_3,P_3)}$$

$$= \frac{\partial F_1}{\partial x_1}\frac{\partial F_2}{\partial P_1} - \frac{\partial F_1}{\partial P_1}\frac{\partial F_2}{\partial x_1} + \frac{\partial F_1}{\partial x_2}\frac{\partial F_2}{\partial P_2} - \frac{\partial F_1}{\partial P_2}\frac{\partial F_2}{\partial x_2} + \frac{\partial F_1}{\partial x_3}\frac{\partial F_2}{\partial P_3} - \frac{\partial F_1}{\partial P_3}\frac{\partial F_2}{\partial x_3}$$

$= 0$

∴ The eqns (4) & (5) taken as additional eqns.

Solving (2), (4) & (5), we get $P_1 = a_1x_1$, $P_2 = a_2x_2$,

$P_3 = \frac{x_3}{a_1a_2}$

Putting these values in $dZ = P_1dx_1 + P_2dx_2 + P_3dx_3$

$dZ = a_1x_1dx_1 + a_2x_2dx_2 + \frac{x_3}{a_1a_2}dx_3$

Integrating, we get

$Z = \frac{1}{2}a_1x_1^2 + \frac{1}{2}a_2x_2^2 + \frac{1}{2a_1a_2}x_3^2 + a_3$

Taking $Z = \log z$

∴ $2\log z = a_1x_1^2 + a_2x_2^2 + \frac{x_3^2}{a_1a_2} + a_3$, which is the required complete integral.

Cauchy's Method of Characteristics:

for solving non-linear differential eqns:

working rule:

Let us consider the non-linear PDE

$f(x, y, z, p, q) = 0$ —— ①

Suppose we wish to find the solution of ① which passes through a given curve whose parametric eqns are

$x = f_1(\lambda),\ y = f_2(\lambda),\ z = f_3(\lambda)$ —— ②

where λ is a parameter.

then in the solution

$$\left.\begin{aligned} x &= x(p_0, q_0, x_0, y_0, z_0, t_0, t) \\ y &= y(p_0, q_0, x_0, y_0, z_0, t_0, t) \text{ and} \\ z &= z(p_0, q_0, x_0, y_0, z_0, t_0, t) \end{aligned}\right\} \text{—— ③}$$

of the characteristics eqns of ① are

$x'(t) = f_p,\ y'(t) = f_q,\ z'(t) = pf_p + qf_q$

$p'(t) = -f_x - pf_z$ and $q'(t) = -f_y - qf_z$ —— ④

where $x'(t) = \frac{dx}{dt}$ etc.

and $f_p = \frac{\partial f}{\partial p}$ etc.

we shall assume that

$x_0 = f_1(\lambda),\ y_0 = f_2(\lambda),\ z_0 = f_3(\lambda)$ as the initial values of x, y, z respectively. then the corresponding initial values of p_0, q_0 are determined by the following relations

$f_3'(\lambda) = p_0 f_1'(\lambda) + q_0 f_2'(\lambda)$ &

$f(f_1(\lambda), f_2(\lambda), f_3(\lambda), p_0, q_0) = 0$

if these values of x_0, y_0, z_0, p_0, q_0 and the appropriate value of 'to' substitute in the eqn ③.

we find that x, y, z can be expressed interms of the two parameters t & λ of the form

$x = \phi_1(t,\lambda)$, $y = \phi_2(t,\lambda)$ & $z = \phi_3(t,\lambda)$ —— ④

which are known as characteristic strips of ①

Finally by eliminating λ & t from ④,

we get the relation of the form $\psi(x,y,z)=0$ which is the required integral surface of ① passing through the given curve ②.

2002 → Find the solution of the eqn

$z = \frac{1}{2}(p^2+q^2) + (p-x)(q-y)$ which passes through the x-axis.

Soln: Given that $z = \frac{1}{2}(p^2+q^2) + (p-x)(q-y)$

Let $f(x,y,z,p,q) = \frac{1}{2}(p^2+q^2) + (p-x)(q-y) - z$ —— ①

we are to find the integral surface of ① which passes through x-axis whose parametric eqns are $x=\lambda$, $y=0$, $z=0$

where λ is the parameter.

i.e, $x = f_1(\lambda) = \lambda$, $y = f_2(\lambda) = 0$, $z = f_3(\lambda) = 0$ —— ②

Let the initial values x_0, y_0, z_0, p_0, q_0 of x, y, z, p, q be taken as $x_0 = f_1(\lambda) = \lambda$; $y_0 = f_2(\lambda) = 0$, $z_0 = f_3(\lambda) = 0$.

Now we find the initial values p_0 & q_0 by the following relations

$f_3'(\lambda) = p_0 f_1'(\lambda) + q_0 f_2'(\lambda)$ &

$f(f_1(\lambda), f_2(\lambda), f_3(\lambda), p_0, q_0) = 0$

i.e., $f(x_0, y_0, z_0, p_0, q_0) = 0$

$\Rightarrow 0 = p_0(1) + q_0(0)$ &

$\frac{1}{2}(p_0^2 + q_0^2) + (p_0 - x_0)(q_0 - y_0) - z_0 = 0$

$\Rightarrow \boxed{p_0 = 0}$ & $\frac{1}{2}q_0^2 - x_0(q_0 - y_0) - z_0 = 0$ $(\because p_0 = 0)$

$\Rightarrow \frac{1}{2}q_0^2 - \lambda(q_0 - 0) - 0 = 0$

$\Rightarrow q_0(\frac{1}{2}q_0 - \lambda) = 0$

$\Rightarrow \frac{1}{2}q_0 = \lambda \quad (\because q_0 \neq 0)$

$\Rightarrow \boxed{q_0 = 2\lambda}$

$\therefore x_0 = \lambda, y_0 = 0, z_0 = 0, p_0 = 0,$ & $q_0 = 2\lambda.$ & $t = t_0$ —— (3)

Now the characteristic eqns of (1) are

$x'(t) = \frac{\partial f}{\partial p} = p + (q - y)$ —— (4)

$y'(t) = \frac{\partial f}{\partial q} = q + (p - x)$ —— (5)

$z'(t) = p[p + q - y] + q[q + p - x]$ —— (6)

$p'(t) = -\frac{\partial f}{\partial x} - p\frac{\partial f}{\partial z}$

$= (q - y) - p(-1)$

$= q - y + p$ —— (7)

$q'(t) = -\frac{\partial f}{\partial y} - q\frac{\partial f}{\partial z}$

$= (p - x) - q(-1) = p - x + q$ —— (8)

from (4) & (7), we have $x'(t) = p'(t)$

$\Rightarrow \frac{dx}{dt} = \frac{dp}{dt}$

$\Rightarrow dx = dp$

$\Rightarrow \boxed{x = p + c_1}$ —— (9)

from (5) & (8), we get

$$y'(t) = q'(t)$$

$$\Rightarrow dy = dq$$

$$\Rightarrow \boxed{y = q + c_2} \longrightarrow (10)$$

using the initial values in (9) & (10)

(9) ≡ $\lambda = 0 + c_1$; (10) ≡ $0 = 2\lambda + c_2$

$\Rightarrow \boxed{c_1 = \lambda}$ $\qquad \Rightarrow \boxed{c_2 = -2\lambda}$

∴ from (9) & (10), we have

$$\boxed{x = p + \lambda} \quad \& \quad \boxed{y = q - 2\lambda} \longrightarrow (11)$$

from (4), (7) & (8), we get

$$\frac{dp}{dt} + \frac{dq}{dt} - \frac{dx}{dt} = p + q - x$$

$$\Rightarrow \frac{d}{dt}(p+q-x) = p+q-x$$

$$\Rightarrow \frac{d(p+q-x)}{p+q-x} = dt$$

$$\Rightarrow \log(p+q-x) = t + \log c_3$$

$$\Rightarrow \boxed{p+q-x = c_3 e^t.} \longrightarrow (12)$$

from (5), (7) & (8), we get

$$\frac{dp}{dt} + \frac{dq}{dt} - \frac{dy}{dt} = p+q-y$$

$$\Rightarrow \frac{d(p+q-y)}{p+q-y} = dt$$

$$\Rightarrow \boxed{p+q-y = c_4 e^t} \longrightarrow (13)$$

using the initial values in (12) & (13), we get

$$p_0 + q_0 - x_0 = c_3 e^{t_0} \quad \& \quad p_0 + q_0 - y_0 = c_4 e^{t_0}$$

Taking $t = t_0 = 0$

$$\Rightarrow p_0 + q_0 - x_0 = c_3 \quad \& \quad 0 + 2\lambda - 0 = c_4$$

$\Rightarrow \boxed{\lambda = c_3}$ & $\boxed{c_4 = 2\lambda}$

from (12) & (13), we get

$p+q-x = \lambda e^t$ —— (14) & $p+q-y = 2\lambda e^t$ —— (15)

Now from (11), (14) & (15), we get

$p+q-(p+\lambda) = \lambda e^t$; $p+q-(q-2\lambda) = 2\lambda e^t$

$\Rightarrow q-\lambda = \lambda e^t$ $\qquad \boxed{p = 2\lambda(e^t-1)}$

$\Rightarrow \boxed{q = \lambda(1+e^t)}$

$\therefore$ from (11), we have

$x = \lambda(2e^t-1)$ —— (16) & $y = \lambda(e^t-1)$ —— (17)

(6) $\equiv \dfrac{dz}{dt} = p[p+q-y] + q[p+q-x]$

$= 2\lambda(e^t-1)(2\lambda e^t) + \lambda(1+e^t)(\lambda e^t)$ (from (14) & (15))

$= 5\lambda^2 e^{2t} - 3\lambda^2 e^t$

$dz = \lambda^2(5e^{2t}-3e^t)\,dt$

$z = \lambda^2\left(\frac{5}{2}e^{2t} - 3e^t\right) + c_5$

By using the initial values.

$z_0 = \lambda^2\left(\frac{5}{2}e^{2t_0} - 3e^{t_0}\right) + c_5$

$\Rightarrow 0 = \lambda^2\left(\frac{5}{2} - 3\right) + c_5$

$\Rightarrow \boxed{c_5 = \frac{\lambda^2}{2}}$

$\therefore \boxed{z = \lambda^2\left(\frac{5}{2}e^{2t} - 3e^t\right) + \frac{\lambda^2}{2}}$ —— (18)

$\therefore$ The required characteristic strips of (1) are given by

$x = \lambda(2e^t-1)$ —— (i), $y = \lambda(e^t-1)$ —— (ii) & $z = \lambda^2\left(\frac{5}{2}e^{2t} - 3e^t\right) + \frac{\lambda^2}{2}$ —— (iii)

—— (19)

Now eliminating t & λ from (19)

Now solving (i) & (ii) from (19) for λ & e^t, we get

$$x = 2\lambda\left(\frac{y+\lambda}{\lambda}\right) - \lambda \qquad \left(\because \text{from (ii)} \; e^t = \frac{y+\lambda}{\lambda}\right)$$

$$\Rightarrow x = 2y + 2\lambda - \lambda$$

$$\Rightarrow x = 2y + \lambda$$

$$\Rightarrow \boxed{\lambda = x - 2y}$$

(ii) $\equiv$ $y = (x-2y)(e^t - 1)$

$$\Rightarrow e^t - 1 = \frac{y}{x-2y}$$

$$\Rightarrow \boxed{e^t = \frac{y}{x-2y} + 1} \Rightarrow \boxed{e^t = \frac{x-y}{x-2y}}$$

(iii) $\equiv$ $z = (x-2y)^2\left\{\left[\frac{5}{2}\left(\frac{x-y}{x-2y}\right)^2 - 3\left(\frac{x-y}{x-2y}\right)\right] + \frac{1}{2}\right\}$

which is the required solution of (1) passing through the given curve.

1999 → Find characteristics of the eqn $pq = z$ and the integral surface which passes through the parabola $x = 0, y^2 = z$.

Sol^n: Given that $pq = z$

$$\Rightarrow f(x, y, z, p, q) = pq - z = 0 \quad \text{—— (1)}$$

Now we are to find the integral surface of (1) which is passing through the parabola.

$x = 0, y^2 = z$

whose parametric eqns are

$x = 0$, $y = \lambda$ & $z = \lambda^2$.

i.e, $x = f_1(\lambda)$, $y = f_2(\lambda)$. & $z = f_3(\lambda)$ —— (2)

Let the initial values x_0, y_0, z_0, p_0, q_0 of x, y, z, p, q be taken as

$$x_0 = f_1(\lambda) = 0, \quad y_0 = f_2(\lambda) = \lambda, \quad z_0 = f_3(\lambda) = \lambda^2$$

Now we find the initial values p_0 & q_0 by the following relations

$$f_3'(\lambda) = p_0 f_1'(\lambda) + q_0 f_2'(\lambda) \quad \&$$

$$f(f_1(\lambda), f_2(\lambda), f_3(\lambda), p_0, q_0) = 0$$

i.e, $f(0, \lambda, \lambda^2, p_0, q_0) = 0$

$\Rightarrow 2\lambda = p_0(0) + q_0(1)$ & $p_0 q_0 - \lambda^2 = 0$

$\Rightarrow 2\lambda = q_0$ $\quad\quad \Rightarrow p_0(2\lambda) - \lambda^2 = 0$

$\Rightarrow \boxed{q_0 = 2\lambda}$ $\quad\quad \Rightarrow p_0 = \frac{\lambda}{2}$

$$\therefore x_0 = 0, \; y_0 = \lambda, \; z_0 = \lambda^2, \; q_0 = 2\lambda, \; p_0 = \lambda/2 \quad \text{--- (3)}$$

Now the characteristic eqns of (1) are

$$x'(t) = \frac{\partial f}{\partial p} = q \quad \text{--- (4)}$$

$$y'(t) = \frac{\partial f}{\partial q} = p \quad \text{--- (5)}$$

$$z'(t) = pq + pq = 2pq \quad \text{--- (6)}$$

$$p'(t) = -\frac{\partial f}{\partial x} - p\frac{\partial f}{\partial z} = -p(-1) = p \quad \text{--- (7)}$$

$$q'(t) = -\frac{\partial f}{\partial y} - q\frac{\partial f}{\partial z} = -q(-1) = q \quad \text{--- (8)}$$

Now from (4) & (8), we have

$$x'(t) = q'(t)$$

$$\Rightarrow dx = dq$$

$$\Rightarrow \boxed{x = q + c_1} \quad \text{--- (9)}$$

from (5) & (7), we have

$y'(t) = p'(t) \Rightarrow dy = dp$

$\Rightarrow \boxed{y = p + c_2}$ —— (10)

Now using the initial values in (9) & (10),

we get $x_0 = q_0 + c_1$ & $y_0 = p_0 + c_2$

$\Rightarrow 0 = 2\lambda + c_1$ & $\lambda = \frac{\lambda}{2} + c_2$

$\Rightarrow \boxed{c_1 = -2\lambda}$ $\Rightarrow \boxed{c_2 = \frac{\lambda}{2}}$

$\therefore$ from (9) & (10), we have

$\boxed{x = q - 2\lambda}$ —— (11) $\boxed{y = p + \frac{\lambda}{2}}$ —— (12)

(7) $\equiv$ $\frac{dp}{dt} = p$

$\Rightarrow \frac{dp}{p} = dt \Rightarrow \log p = t + \log c_3$

$\Rightarrow \boxed{p = c_3 e^t}$ —— (13)

(8) $\equiv$ $q'(t) = q \Rightarrow \frac{dq}{q} = dt$

$\Rightarrow \log q = t + \log c_4$

$\Rightarrow \boxed{q = c_4 e^t}$ —— (14)

Using the initial values in (13) & (14), we get

$p_0 = c_3 e^{t_0}$ & $q_0 = c_4 e^{t_0}$

$\Rightarrow \frac{\lambda}{2} = c_3$ & $2\lambda = c_4$ $(\because t_0 = 0)$

from (13) & (14), we have

$p = \frac{\lambda}{2} e^t$ & $q = 2\lambda e^t$ —— (15)

(11) $\equiv$ $x = 2\lambda e^t - 2\lambda \Rightarrow x = 2\lambda(1 - e^t)$ —— (16)

(12) $\equiv$ $y = \frac{\lambda}{2} e^t + \frac{\lambda}{2} \Rightarrow y = \frac{\lambda}{2}(e^t + 1)$ —— (17)

(6) $\equiv$ $z'(t) = 2pq$

$\Rightarrow \frac{dz}{dt} = 2\left(e^t \frac{\lambda}{2}\right)(2\lambda e^t)$

$\Rightarrow dz = 2\lambda^2 e^{2t} dt$

$\Rightarrow z = \lambda^2 e^{2t} + c_5$ ——— (18)

using the initial values in (18), we get

$z_0 = \lambda^2 e^{2t_0} + c_5$

$\Rightarrow \lambda^2 = \lambda^2(1) + c_5$

$\Rightarrow \boxed{c_5 = 0}$

$\therefore (18) \equiv \boxed{z = \lambda^2 e^{2t}}$

$\therefore$ The required characteristics of (1) are given by

$x = 2\lambda(e^t - 1)$ ——(19), $y = \frac{\lambda}{2}(e^t + 1)$ ——(20), $z = \lambda^2 e^t$ ——(21)

Now eliminating e^t & λ from (19), (20) & (21)

$(19) \equiv x = \frac{4y}{e^t+1}(e^t - 1)$

$\Rightarrow xe^t + x = 4ye^t - 4y$

$\Rightarrow e^t(x - 4y) = -(x + 4y)$

$\Rightarrow \boxed{e^t = -\frac{(x+4y)}{x-4y}}$

$(20) \equiv \lambda = \frac{2y}{e^t+1} = \frac{2y}{-\frac{(x+4y)}{x-4y}+1} = \frac{2y(x-4y)}{-8y}$

$\boxed{\lambda = \frac{x-4y}{-4}}$

$\therefore z = \left(\frac{x-4y}{-4}\right)^2 \frac{(x+4y)^2}{(x-4y)^2}$

$\boxed{z = \frac{(x+4y)^2}{16}}$

which is the required integral surface of (1).

2006 P.T for the eqn $z+px+qy-1-pqx^2y^2=0$ the characteristic strips are given by

$$x=\frac{1}{B+Ce^{-t}},\quad y=\frac{1}{A+De^{-t}},\quad z=E-(AC+BD)e^{-t}$$

$$p=A(B+Ce^{-t})^2,\quad q=B(A+Be^{-t})^2$$

where A, B, C, D and E are arbitrary constants. Hence find the integral surface which passes through the line $z=0$, $x=y$.

Soln: Given that $z+px+qy-1-pqx^2y^2=0$

Let $f(x,y,z,p,q)=z+px+qy-1-pqx^2y^2=0$ —— ①

we are to find integral surface of ① which passes through the line $x=y$, $z=0$

whose parametric eqns are

$x=\lambda$, $y=\lambda$ & $z=0$

i.e, $x=f_1(\lambda)=\lambda$; $y=f_2(\lambda)=\lambda$ & $z=f_3(\lambda)=0$ —— ②

Let the initial values x_0, y_0, z_0, p_0, q_0 of x, y, z, p, q be taken as

$x_0=f_1(\lambda)=\lambda$, $y_0=f_2(\lambda)=\lambda$, $z_0=f_3(\lambda)=0$

Now we find the initial values p_0 & q_0 by the relations

$f_3'(\lambda)=p_0f_1'(\lambda)+q_0f_2'(\lambda)$ &

$f(f_1(\lambda), f_2(\lambda), f_3(\lambda), p_0, q_0)=0$

i.e, $f(\lambda,\lambda,0,p_0,q_0)=0$

$\Rightarrow 0=p_0(1)+q_0(1)$ & $0+p_0(\lambda)+q_0(\lambda)-1-p_0q_0\lambda^4=0$

$\Rightarrow \boxed{p_0+q_0=0}$ —— (i) $\Rightarrow \lambda(p_0+q_0)-1-p_0q_0\lambda^4=0$

$\Rightarrow \lambda(0)-p_0q_0\lambda^4=0$

$\Rightarrow \boxed{p_0q_0=-\frac{1}{\lambda^4}}$

Now $(p_0-q_0)^2 = (p_0+q_0)^2 - 4p_0q_0$

$$= 0 + \frac{4}{\lambda^4}$$

$$\therefore \boxed{p_0 - q_0 = \frac{2}{\lambda^2}} \quad \text{--- (ii)}$$

from (i) & (ii) $\quad 2p_0 = \frac{2}{\lambda^2}$

$$\Rightarrow \boxed{p_0 = \frac{1}{\lambda^2}} \quad \& \quad \boxed{q_0 = -\frac{1}{\lambda^2}}$$

$\therefore x_0 = \lambda,\ y_0 = \lambda,\ z_0 = 0,\ p_0 = \frac{1}{\lambda^2}$ & $q_0 = -\frac{1}{\lambda^2}$ --- (3)

Now the characteristic eqns of (1) are

$x'(t) = f_p = x - qx^2y^2$ --- (4)

$y'(t) = f_q = y - px^2y^2$ --- (5)

$z'(t) = p[x - qx^2y^2] + q[y - px^2y^2]$

$= px + qy - 2pqx^2y^2$ --- (6)

$p'(t) = -[p - 2pqxy^2] - p(1)$

$= -2p[1 - qxy^2]$ --- (7)

$q'(t) = -[q - 2pqx^2y] - q(1)$

$= -2q[1 - px^2y]$ --- (8)

from (4) & (7), we have

$$x'(t) = x\left(\frac{p'(t)}{-2p}\right)$$

$$\Rightarrow -\frac{2}{x}dx = \frac{1}{p}dp$$

$$\Rightarrow -2\log x = \log p + \log c_1$$

$$\Rightarrow \boxed{x^{-2} = pc_1} \quad \text{--- (9)}$$

from (5) & (8), we have

$$y'(t) = y\left(\frac{-q'(t)}{2q}\right)$$

$$\Rightarrow -\frac{2}{y}dy = \frac{1}{q}dq.$$

$\Rightarrow -2\log y = \log q + \log C_2$

$\Rightarrow \boxed{y^{-2} = qC_2}$ —— (10)

Using the initial values in (9) & (10), we get

$x_0^{-2} = p_0 C_1$ & $y_0^{-2} = q_0 C_2$

$\Rightarrow \lambda^{-2} = \frac{1}{\lambda^2} C_1$ & $\lambda^{-2} = -\frac{1}{\lambda^2} C_2$

$\Rightarrow \boxed{C_1 = 1}$ & $\boxed{C_2 = -1}$

$\therefore$ from (9) & (10), we have

$\boxed{\frac{1}{x^2} = p}$ & $\boxed{-\frac{1}{y^2} = q}$ —— (11)

Continuing in this way.

(1) 2000 Find the characteristic strips of the eqn $xp + yq - pq = 0$ and then find the eqn of integral surface through curve $z = \frac{x}{2}, y = 0$.

(2) Write down and integrate completely, the equations for the characteristics of $(1+q^2)z = px$. Expressing x, y, z and p interms of ϕ, where $q = \tan\phi$ and determine the integral surface which passes through parabola $x^2 = 2z, y = 0$.

(3) Determine the characteristics of the equation $z = p^2 - q^2$ and find the integral surface which passes through the parabola $4z + x^2 = 0, y = 0$.

(4) Integrate the eqns of the characteristics of the eqn $p^2 + q^2 = 4z$.
Expressing x, y, z and p interms of q and then find the solutions of this equation which reduce to $z = x^2 + 1, y = 0$