Date : 8/9/18

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A CONSOLIDATED QUESTION PAPER-CUM-ANSWER BOOKLET

IMS™
(INSTITUTE OF MATHEMATICAL SCIENCES)

MAINS TEST SERIES-2019

(JUNE-2019 *to* SEPT.-2019)

Under the guidance *of* K. Venkanna

MATHEMATICS

212/250

PAPER - I : FULL SYLLABUS

TEST CODE: TEST-17: IAS(M)/08-SEPT.-2019

Time : 3 Hours **Maximum Marks : 250**

INSTRUCTIONS

1. This question paper-cum-answer booklet has 48 pages and has 35 PART/SUBPART questions. Please ensure that the copy of the question paper-cum-answer booklet you have received contains all the questions.
2. Write your Name, Roll Number, Name of the Test Centre and Medium in the appropriate space provided on the right side.
3. A consolidated Question Paper-cum-Answer Booklet, having space below each part/sub part of a question shall be provided to them for writing the answers. Candidates shall be required to attempt answer to the part/sub-part of a question strictly within the pre-defined space. Any attempt outside the pre-defined space shall not be evaluated. "
4. Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
5. Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any **THREE** of the remaining questions selecting at least **ONE** question from each Section.
6. The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
7. Symbols/notations carry their usual meanings, unless otherwise indicated.
8. All questions carry equal marks.
9. All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
10. All rough work should be done in the space provided and scored out finally.
11. The candidate should respect the instructions given by the invigilator.
12. The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name: SHIKHAR PRADHAN

Roll No.

Test Centre: ORN

Medium: English

Do not write your Roll Number or Name anywhere else in this Question Paper-cum-Answer Booklet.

I have read all the instructions and shall abide by them

Shikhar Pradhan

Signature of the Candidate

I have verified the information filled by the candidate above

Signature of the invigilator

IMPORTANT NOTE:
Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed. Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

P.T.O.

DO NOT WRITE ON THIS SPACE

INDEX TABLE

QUESTION	No.	PAGE NO.	MAX. MARKS	MARKS OBTAINED	
1	(a)			08	38
	(b)			08	
	(c)			08	
	(d)			06	
	(e)			08	
2	(a)				
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	(c)				
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3	(a)				
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4	(a)			14	41
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5	(a)			09	43
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			Total Marks		

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SECTION – A

1. (a) Find the characteristic polynomial of the matrix $A=\begin{pmatrix}1 & 1\\ -1 & 3\end{pmatrix}$ Hence find A^{-1} and A^6.

[10]

Characteristic equation: $|A-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix}1-\lambda & 1\\ -1 & 3-\lambda\end{vmatrix} = 0 \Rightarrow (1-\lambda)(3-\lambda)+1=0$$

$$\Rightarrow \lambda^2-4\lambda+4=0 \rightarrow (\lambda-2)^2=0 \Rightarrow \lambda=2,2$$

$\therefore$ Characteristic equation: $\cancel{A}$ $\lambda^2-4\lambda+4=0$

Characteristic polynomial: $\boxed{A^2-4A+4I=0}$

$$A^{-1}(A^2-4A+4I) = A^{-1}0 = 0$$

$$A-4I+4A^{-1}=0 \Rightarrow A^{-1} = \frac{4I-A}{4}$$

$$\therefore A^{-1} = \frac{4\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}-\begin{bmatrix}1 & 1\\ -1 & 3\end{bmatrix}}{4} = \frac{1}{4}\begin{bmatrix}3 & -1\\ 1 & 1\end{bmatrix}$$

08

$$\therefore \boxed{A^{-1} = \begin{bmatrix}3/4 & -1/4\\ 1/4 & 1/4\end{bmatrix}}$$

$$A^2 = 4A-4I = 4\left\{\begin{bmatrix}1 & 1\\ -1 & 3\end{bmatrix}-\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right\} = \begin{bmatrix}0 & 4\\ -4 & 8\end{bmatrix}$$

$$A^4 = \begin{bmatrix}0 & 4\\ -4 & 8\end{bmatrix}\begin{bmatrix}0 & 4\\ -4 & 8\end{bmatrix} = \begin{bmatrix}-16 & 32\\ -32 & 48\end{bmatrix}$$

$$A^6 = A^2 \cdot A^4 = \begin{bmatrix}0 & 4\\ -4 & 8\end{bmatrix}\begin{bmatrix}-16 & 32\\ -32 & 48\end{bmatrix} = \begin{bmatrix}-128 & 192\\ -192 & 256\end{bmatrix}$$

$$\therefore \boxed{A^6 = \begin{bmatrix}-128 & 192\\ -192 & 256\end{bmatrix}}$$

1. (b) Let T be the linear transformation from IR^3 to IR^4 defined by
$(2x_1 + x_2 + x_3, x_1 + x_2, x_1 + x_3, 3x_1 + x_2 + 2x_3)$
for each $(x_1, x_2, x_3) \in IR^3$

Determine a basis for the Null space of T. What is the dimension of the Range space of T. **[10]**

Let $\alpha \in N(T)$ s.t. $\alpha = (x_1, x_2, x_3)$

$\Rightarrow 2x_1 + x_2 + x_3 = 0;\ x_1 + x_2 = 0;\ x_1 + x_3 = 0;\ 3x_1 + x_2 + 2x_3 = 0.$

$\therefore x_2 = -x_1;\ x_3 = -x_1$

$\Rightarrow \alpha = (x_1, x_2, x_3) = (x_1, -x_1, -x_1) = x_1(1, -1, -1)$

$\therefore$ N(T) is generated by $(1, -1, -1)$

$\therefore N(T) = L(S);\ \boxed{S = \{(1, -1, -1)\}}$ where S is LI

$\Rightarrow$ S is a basis for N(T)

can directly use Rank Nullity Thm!

Let $(\alpha, \beta, \gamma, \delta) \in R(T)$;

$\alpha = 2x_1 + x_2 + x_3;\ \beta = x_1 + x_2;\ \gamma = x_1 + x_3;\ \delta = 3x_1 + x_2 + 2x_3$

for some $(x_1, x_2, x_3) \in IR^3$

Clearly, $\alpha = \beta + \gamma;\ \delta = \beta + 2\gamma$

$\therefore (\alpha, \beta, \gamma, \delta) = (\beta + \gamma, \beta, \gamma, \beta + 2\gamma) = \beta(1, 1, 0, 1) + \gamma(1, 0, 1, 2)$

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$\therefore R(T) = L(S_1);\ S_1 = \{(1, 1, 0, 1), (1, 0, 1, 2)\}$

Clearly, S_1 is linearly independant (since neither vector is a scaler multiple of the other)

$\therefore S_1$ is a basis for $R(T)$ & $\boxed{\dim(R(T)) = 2}$

1. (c) Let $f(x, y) = xy / \sqrt{x^2+y^2}, (x,y) \neq (0,0)$
$f(0, 0) = 0$.
Show that f is countinuous at the origin (0, 0), possesses partial derivatives thereat but is not differentiable at origin. **[10]**

$$f(x,y) = \begin{cases} xy/\sqrt{x^2+y^2} & ; (x,y) \neq (0,0) \\ 0 & ; (x,y) = (0,0) \end{cases}$$

Now, $\underset{(x,y)\to(0,0)}{Lt} f(x,y) = \underset{(x,y)\to(0,0)}{Lt} \frac{xy}{\sqrt{x^2+y^2}}$.

$$|f(x,y) - 0| = \left|\frac{xy}{\sqrt{x^2+y^2}}\right| \leq \left|\frac{\sqrt{x^2+y^2}\,\sqrt{x^2+y^2}}{\sqrt{x^2+y^2}}\right| \quad \left[\because x \leq \sqrt{x^2+y^2},\ y \leq \sqrt{x^2+y^2}\right.$$

$$\leq \left|\sqrt{x^2+y^2}\right| < \epsilon$$

whenever $\sqrt{x^2+y^2} < \delta = \frac{\epsilon}{1}$

$\therefore f(x,y)$ is continuous at $(0,0)$

$$f_x(0,0) = \underset{h\to0}{Lt} \frac{f(h,0)-f(0,0)}{h} = \underset{h\to0}{Lt} \frac{0-0}{h} = 0$$

$$f_y(0,0) = \underset{k\to0}{Lt} \frac{f(0,k)-f(0,0)}{k} = \underset{k\to0}{Lt} \frac{0-0}{k} = 0$$

$\therefore f(x,y)$ possesses partial derivatives at $(0,0)$

Let $f(h,k) - f(0,0) = h f_x(0,0) + k f_y(0,0) + \sqrt{h^2+k^2}\,\phi(h,k)$

$$\frac{hk}{\sqrt{h^2+k^2}} = \sqrt{h^2+k^2}\,\phi(h,k) \Rightarrow \phi(h,k) = \frac{hk}{h^2+k^2}.$$

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Now, $\underset{(h,k)\to(0,0)}{Lt} \phi(h,k) = \underset{(h,k)\to(0,0)}{Lt} \frac{hk}{h^2+k^2} = \underset{h\to0}{Lt} \frac{h(mh)}{h^2+m^2h^2}$ (taking path $k=mh$)

$$= \underset{h\to0}{Lt} \frac{m h^2}{h^2(1+m^2)} = \frac{m}{1+m^2} \rightarrow \text{Limit doesn't tend to } 0$$

$\therefore f(x,y)$ is not differentiable at $(0,0)$

1. (d) Evaluate $\int_0^1 (x \ln x)^3 \, dx$. **[10]**

Let $\ln x = z$; $x = e^z$; $\frac{1}{x} dx = dz \Rightarrow dx = e^z dz$

$$\therefore I = \int_0^1 (x \ln x)^3 dx = \int_{-\infty}^{0} (z e^z)^3 e^z dz$$

$$= \int_{-\infty}^{0} z^3 e^{4z} dz \quad ; \quad \text{Let } z = -t$$

$$= \int_{\infty}^{0} -\cancel{t^3 e^{-4t}} dt = \int_0^{\infty} e^{-4t} t^3 dt$$

Let $v = +4t$; $dv = 4dt$

$$\therefore I = \int_0^{\infty} e^{-v} \left(\frac{v}{4}\right)^3 \frac{dv}{4} = \frac{1}{256} \int_0^{\infty} e^{-v} v^3 dv$$

06

$$= \frac{1}{256} \Gamma(4) = \frac{\lfloor 3}{256} = \boxed{\frac{-3}{128}}$$

1. (e) Find the equation of the plane which passes through the points (0,1,1) and(2,0,–1) and is parallel to the line joining the points (–1,1,–2), (3,–2,4). Find also the distance between the line and the plane. **[10]**

Any plane through $(0,1,1)$: $\pi : A(x-0)+B(y-1)+C(z-1)=0$

$\rightarrow \pi : Ax + B(y-1)+C(z-1) = 0.$

Since π passes through $(2,0,-1)$:

$$\boxed{2A - B - 2C = 0} \quad - \text{①}$$

Since π is parallel to the given line, the normal to π is perpendicular to the line.

Direction ratios of line: $(4, -3, 6)$

$$\therefore \boxed{4A - 3B + 6C = 0} \quad - \text{②}$$

from ① & ② : $\dfrac{A}{-12} = \dfrac{B}{-20} = \dfrac{C}{-2}$

$$\Rightarrow \cancel{\boxed{\frac{A}{6} = \frac{B}{+10} = \frac{C}{-5}}} \Rightarrow \boxed{\frac{A}{6} = \frac{B}{10} = \frac{C}{1}}$$

08

$\therefore \pi : 6x + 10(y-1) + (z-1) = 0$

$$\boxed{6x + 10y + z = 11}$$ – which is the required plane

distance between π and line = distance between π and any point on line

$$\therefore d = \frac{|6(-1) + 10(1) + 1(-2) - 11|}{\sqrt{6^2+10^2+1^2}} = \boxed{\frac{9}{\sqrt{137}}}$$

2. (a) Find the condition on a, b, and c so that the following system in unknowns x, y and z has a solution.

$$x + 2y - 3z = a$$
$$2x + 6y - 11z = b$$
$$x - 2y + 7z = c$$

[10]

4. (a) Examine whether the matrix $A=\begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{bmatrix}$ is diagonalizable. Find all eigen values. Then obtain a matrix P such that $P^{-1}AP$ is a diagonal matrix. **[15]**

Characteristic eqn of A: $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix}-2-\lambda & 2 & -3\\ 2 & 1-\lambda & -6\\ -1 & -2 & -\lambda\end{vmatrix}=0 \Rightarrow (-2-\lambda)[-\lambda(1-\lambda)-12]$$
$$+2[6+2\lambda]-3[-4+1-\lambda]=0$$

$$\Rightarrow (\lambda+2)[\lambda(1-\lambda)+12]+(12+4\lambda)+(9+3\lambda)=0$$

$$-\lambda^3-\lambda^2+14\lambda+24+12+4\lambda+9+3\lambda=0.$$

$$\Rightarrow \cancel{\lambda^3+\lambda^2-21\lambda+45=0} \quad \boxed{\lambda^3+\lambda^2-21\lambda-45=0}$$

$$\therefore \boxed{\lambda=5,-3,-3} \rightarrow \text{Eigenvalues}$$

~~Since the eigenvalues are distinct, A is diagonalizable.~~

Eigenvectors :

Corresponding to $\lambda = 5$; $(A-5I)X = 0 \Rightarrow \begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} X = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\sim \begin{bmatrix} -1 & -2 & -5 \\ 2 & -4 & -6 \\ -7 & 2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \sim \begin{bmatrix} -1 & -2 & -5 \\ 0 & -8 & -16 \\ 0 & 16 & 32 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$\sim \begin{bmatrix} -1 & -2 & -5 \\ 0 & -8 & -16 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow x_2 + 2x_3 = 0 \Rightarrow x_2 = -2x_3$

$-x_1 - 2x_2 - 5x_3 = 0 \Rightarrow -x_1 + 4x_3 - 5x_3 = 0$

$\Rightarrow x_1 = -x_3$

$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix} \Rightarrow X_1 = \begin{bmatrix} -1 \\ -2 \\ 1 \end{bmatrix}$

Corresponding to $\lambda = -3$:

$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$\therefore x_1 + 2x_2 - 3x_3 = 0 \Rightarrow x_1 = -2x_2 + 3x_3$

$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -2x_2 + 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = x_2 X_2 + x_3 X_3$

$\therefore X_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}; \; X_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$.

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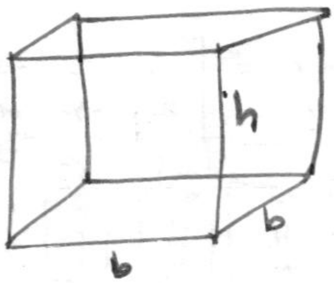
$\therefore$ geometric multiplicity of each eigenvalue = arithmetic multiplicity

$\therefore$ A is diagonalizable

$$\boxed{P = \begin{bmatrix} -1 & -2 & 3 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}$$ and $D = P^{-1}AP = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$.

4. (b) (i) An open tank is to be constructed with a square base and vertical sides to hold a given quantity of water. Find the ratio of its depth to the width so that the cost of lining the tank with lead is least.

(ii) Examine if the improper integral $\int_0^3 \frac{2x\,dx}{(1-x^2)^{2/3}}$ exists. **[20]**



let the ~~height~~ depth of the tank be h & base be a square of side b

Volume $= b^2 h$ (constant) $= V$ — ①

Surface Area $(S) = b^2 + 4bh$ (to be minimised) — ②

from ①, $h = \frac{V}{b^2} \Rightarrow S = b^2 + 4b\frac{V}{b^2} = b^2 + \frac{4V}{b}$

$\therefore \frac{dS}{db} = 2b - \frac{4V}{b^2}$

At stationary point, $\frac{dS}{db} = 0 \Rightarrow 2b - \frac{4V}{b^2} = 0$

$\Rightarrow 2b^3 = 4V \Rightarrow b^3 = 2V$.

$\frac{d^2S}{db^2} = 2 + \frac{8V}{b^3} > 0 \Rightarrow$ minima at stationary point

$\therefore$ for minimum surface area, $V = \frac{b^3}{2} = b^2 h$

$\Rightarrow \boxed{\frac{h}{b} = \frac{1}{2}}$ which is the required ratio of depth to width ii). $I = \int_0^3 \frac{2x\,dx}{(1-x^2)^{2/3}}$

Let $x^2 = t$; $2x\,dx = dt$; $I = \int_0^{\sqrt{3}} \frac{dt}{(1-t)^{2/3}}$

point of discontinuity : $t = 1$

$I = \underbrace{\int_0^1 \frac{dt}{(1-t)^{2/3}}}_{I_1} + \underbrace{\int_1^{\sqrt{3}} \frac{dt}{(1-t)^{2/3}}}_{I_2}$

both I_1 & I_2 are convergent

16 since $n = \frac{2}{3} < 1$

$\therefore$ the given improper integral exists

4. (c) A variable plane is parallel to the plane

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$$

and meets the axes in A, B, C respectively. Prove that the circle ABC lies on the cone

$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0.$$

[15]

Any plane parallel to given plane : $\pi : \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=k$

$\therefore$ $A=(ka,0,0)$; $B=(0,kb,0)$; $C=(0,0,kc)$

$\therefore$ circle ABC = Sphere OABC + plane ABC

sphere OABC : $x^2+y^2+z^2-(ka)x-(kb)y-(kc)z=0$

$\Rightarrow (x^2+y^2+z^2)=k(ax+by+cz)$

Cone with given circle as base curve:

$(x^2+y^2+z^2)=\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)(ax+by+cz)$

$(x^2+y^2+z^2)=(x^2+y^2+z^2)+\left(\frac{a}{b}xy+\frac{a}{c}zx+\frac{b}{a}xy\right.$
$\left.+\frac{b}{c}yz+\frac{c}{a}zx+\frac{c}{b}yz\right)$

$\Rightarrow$ Cone : $\boxed{yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0}$

Since the give cone is the cone with the circle as the base curve, the circle lies on the cone.

SECTION – B

5. (a) Solve the ordinary differential equation

$$\cos 3x \frac{dy}{dx} - 3y\sin 3x = \frac{1}{2}\sin 6x + \sin^2 3x,\ 0 < x < \frac{\pi}{2}.$$ **[10]**

$$\frac{dy}{dx} - (3\tan 3x)\, y = \frac{\cancel{2}\sin 3x \cancel{\cos 3x}}{\cancel{2}\cancel{\cos 3x}} + \frac{\sin^2 3x}{\cos 3x}$$

$$\rightarrow \frac{dy}{dx} - (3\tan 3x)\, y = \sin 3x + \sin 3x \tan 3x$$

Integrating factor : $IF = e^{\int -3\frac{\sin 3x}{\cos 3x}dx}$

$\left(\cos 3x = t;\ -3\sin 3x\, dx = dt;\ \int \frac{dt}{t} = \ln t\right)$

$\therefore IF = e^{\log \cos 3x} = \cos 3x$

$\therefore$ general solution : g S : $y\cos 3x = \int (\sin 3x \cos 3x + \sin^2 3x)\, dx$

$$y\cos 3x = \int \frac{1}{2}\sin 6x + \frac{1-\cos 6x}{2}\,dx$$

$$= -\frac{\cos 6x}{12} + \frac{x}{2} - \frac{\sin 6x}{12} + C$$

$$\therefore \boxed{y = -\frac{\cos 6x}{12\cos 3x} + \frac{x}{2\cos 3x} - \frac{\sin 6x}{12\cos 3x} + C\sec 3x}$$

-09

5. (b) Obtain the equation of the orthogonal trajectory of the family of curves represented by $r^n = a \sin n\theta$, (r, θ) being the plane polar coordinates. **[10]**

$$r^n = a\sin n\theta$$

$$\cancel{n} r^{n-1}\frac{dr}{d\theta} = -a\cancel{n}\cos n\theta$$

$$a = \frac{r^n}{\sin n\theta}$$

$$\therefore \boxed{r^{n-1}\frac{dr}{d\theta} = -\frac{r^n}{\sin n\theta}\cos n\theta} \rightarrow \text{differential equation of given family of curves}$$

$$\boxed{(r^{n-1})\left(-r^2\frac{d\theta}{dr}\right) = \frac{-r^n}{\sin n\theta}\cos n\theta} \rightarrow \text{differential equation of orthogonal trajectories}$$

$$-r^n \, r\frac{d\theta}{dr} = -r^n \frac{\cos n\theta}{\sin n\theta}$$

$$\Rightarrow r\frac{d\theta}{dr} = \frac{\cos n\theta}{\sin n\theta} \Rightarrow \frac{\sin n\theta}{\cos n\theta}\, d\theta = \frac{dr}{r}$$

Let $\cos n\theta = t$; $-n\sin n\theta \, d\theta = dt$

$$\Rightarrow -\frac{dt}{t} = \frac{dr}{r} \Rightarrow -\log t = \log r - \log c$$

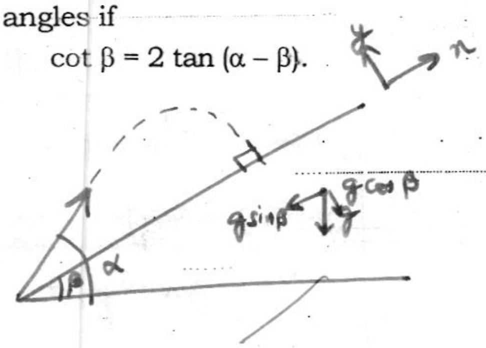
$$\Rightarrow tr = c \Rightarrow \boxed{r\cos n\theta = c}$$

—which is the equation of the required orthogonal trajectories

—09

5. (c) A particle is projected at an angle α with the horizontal from the foot of the plane, whose inclination to the horizontal is β. Show that it will strike the plane at right angles if

$\cot\beta = 2\tan(\alpha - \beta)$. **[10]**



$u_x = u\cos(\alpha-\beta)$

$u_y = u\sin(\alpha-\beta)$

$a_x = -g\sin\beta$

$a_y = -g\cos\beta$

$v_x = u_x + a_x T$ (T : time of flight)

for time of flight: $y = u_y t + \frac{1}{2} a_y t^2$

$$\therefore u_y T + \frac{1}{2} a_y T^2 = 0 \Rightarrow u\sin(\alpha-\beta)T - \frac{1}{2} g\cos\beta T^2 = 0$$

$$\Rightarrow \boxed{T = \frac{2u\sin(\alpha-\beta)}{g\cos\beta}}$$

Now, $v_x = u_x + a_x T$

for particle to strike perpendicular to plane, $\boxed{v_x = 0}$

$\therefore u_x + a_x T = 0 \Rightarrow u\cos(\alpha-\beta) - g\sin\beta\left(\dfrac{2u\sin(\alpha-\beta)}{g\cos\beta}\right) = 0$

$\Rightarrow \cancel{u}\cos(\alpha-\beta) = \dfrac{\cancel{g}\tan\beta}{\cancel{g}}\left(2\cancel{u}\sin(\alpha-\beta)\right)$.

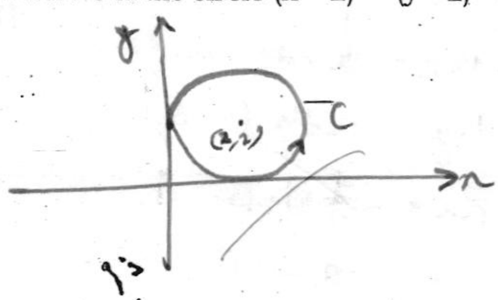
$\Rightarrow \boxed{2\tan(\alpha-\beta) = \cot\beta}$ which is the required result

08

5. (d) Apply Greens theorem to evaluate the line integral .

$\oint\limits_C (4x-2y)\,dx + (2x-4y)\,dy$

Where C is the circle $(x-2)^2 + (y-2)^2 = 4$ **[10]**



Green's theorem:

$\oint\limits_C P\,dx + Q\,dy = \iint (Q_x - P_y)\,dx\,dy$

$\therefore \oint\limits_C (4x-2y)\,dx + (2x-4y)\,dy$

$= \iint 2-(-2)\;dx\,dy = \iint 4\,dx\,dy$

$= 4\iint dx\,dy = 4 \times$ Area of circle enclosed by C

$= 4 \times (\pi \times 4) = \boxed{16\pi}$

-08

5. (e) Find f(r) such that $\nabla f = \frac{\vec{r}}{r^5}$ and $f(1) = 0$. **[10]**

$\nabla f = f'(r) \nabla r = f'(r) \frac{\bar{r}}{r}$

Given that $\nabla f = \frac{\bar{r}}{r^5} = \frac{f'(r)}{r} \bar{r}$

$\Rightarrow \frac{f'(r)}{r} = \frac{1}{r^5} \Rightarrow f'(r) = \frac{1}{r^4}$.

$\Rightarrow f(r) = \int \frac{1}{r^4} dr = \frac{-1}{3r^3} + c$.

Given that $f(1) = 0 \Rightarrow -\frac{1}{3} + c = 0 \Rightarrow \boxed{c = \frac{1}{3}}$

-09 $\therefore f(r) = \frac{1}{3} - \frac{1}{3r^3} = \boxed{\frac{1}{3}\left(1 - \frac{1}{r^3}\right)}$

which is the required function

6. (a) Find the Wronskian of the set of functions $\{3x^3, |3x^3|\}$ on the interval $[-1, 1]$ and determine whether the set is linearly dependent on $[-1, 1]$. **[10]**

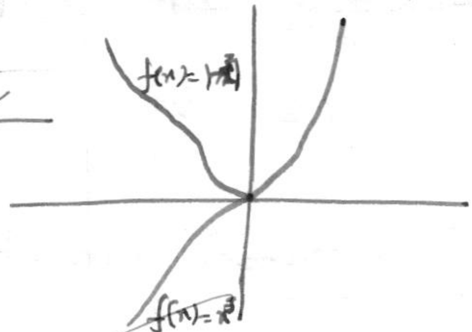
$$W(3x^3, |3x^3|) = \begin{cases} W(3x^3, 3x^3) ; & x \in [0,1] \\ W(3x^3, -3x^3) ; & x \in [-1,0) \end{cases}$$

$$= \begin{cases} \begin{vmatrix} 3x^3 & 3x^3 \\ 9x^2 & 9x^2 \end{vmatrix} ; & x \in [0,1] \\ \begin{vmatrix} 3x^3 & -3x^3 \\ 9x^2 & -9x^2 \end{vmatrix} ; & x \in [-1,0) \end{cases}$$

$$= \begin{cases} 0 ; & x \in [0,1] \\ 0 ; & x \in [-1,0) \end{cases} = 0 \quad \forall x \in [-1,1].$$

$\therefore$ The Wronskian of the given functions is 0 on the interval $[-1,1]$

Graphically,



Hence, graphically, the given functions are linearly independant on $[-1,1]$ despite their Wronskian being 0 on the given interval

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6. (b) Solve the following differential equation :

$$x^3\frac{d^3y}{dx^3}+3x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+8y=65\cos(\log_e x)$$ **[12]**

(Given equation is Cauchy Euler eq^n)

Let $x = e^z$

$dx = e^z dz$

The equation reduces to:

i.e. $\left(D_1(D_1-1)(D_1-2)+3D_1(D_1-1)+D_1+8\right)z = 65\cos z$

$\left(D_1^3-\cancel{3D_1^2}+\cancel{2D_1}+\cancel{3D_1^2}-\cancel{2D_1}+8\right)y = 65\cos z$

$(D_1^3+8)y = 65\cos z$

Auxilliary equation: $m^3+8=0 \Rightarrow (m+2)(m^2+4-2m)=0.$

$m = \frac{2\pm\sqrt{-12}}{2}, -2 = \boxed{1\pm i\sqrt{3}, -2}$

$\therefore y_c(z) = e^z(c_1\cos\sqrt{3}z + c_2\sin\sqrt{3}z) + c_3e^{-2z}$

$$y_p(z) = \frac{1}{D_1^3+8} 65\cos z = 65 \frac{1}{-D_1+8}\cos z$$

$$= 65 \frac{8+D_1}{64-D_1^2}\cos z = \frac{65}{65}(8+D_1)\cos z$$

$$= 8\cos z - \sin z$$

$$\therefore y(z) = e^z\left(c_1\cos\sqrt{3}\,z + c_2\sin\sqrt{3}\,z\right) + c_3 e^{-2z} + 8\cos z - \sin z$$

$$\therefore \boxed{y(x) = x\left(c_1\cos\sqrt{3}\log x + c_2\sin\sqrt{3}\log x\right) + \frac{c_3}{x^2} + 8\cos\log x - \sin\log x}$$

which is the required solution

6. (c) Solve the differential equation

$$x^2y'' - 4xy' + 6y = x^4 \sec^2 x$$

by variation of parameters. [13]

Standard form: $y'' - \frac{4}{x}y' + \frac{6y}{x^2} = x^2\sec^2 x$

$\therefore R = x^2\sec^2 x$

Homogenous part: $x^2y'' - 4xy' + 6y = 0$

Let $x = e^z$; $(D_1(D_1-1) - 4D_1 + 6)y = 0$.

$(D_1^2 - 5D_1 + 6)y = 0 \rightarrow (D_1-2)(D_1-3)y = 0$.

$m = 2, 3 \Rightarrow y_c(z) = c_1e^{2z} + c_2e^{3z}$

$\therefore \boxed{y_c(x) = c_1x^2 + c_2x^3}$

Let $u = x^2$; $v = x^3$; $W(u,v) = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = x^4 \neq 0$.

Let $y_p(x) = Au(x) + Bv(x)$

Applying method of variation of parameters,

$A = -\int \frac{Rv}{W}\,dx = -\int \frac{(x^2\sec^2 x)\,x^3}{x^4}\,dx$

$= -\int x\sec^2 x\,dx = -\left[x\tan x - \int (1)\cdot\tan x\,dx\right]$

$= -x\tan x + \log\cos x$

$B = \int \frac{Ru}{W}\,dx = \int \frac{(x^2\sec^2 x)\,x^2}{x^4}\,dx = \int \sec^2 x\,dx = \tan x$

$\therefore y_p(x) = -x^3 \tan x + x^2 \log \cos x + x^3 \tan x$

$= x^2 \log \cos x$

$\therefore \quad y(x) = y_c(x) + y_p(x) = \boxed{c_1 x^2 + c_2 x^3 + x^2 \log(\cos x)}$

— which is the required solution

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6. (d) Solve the initial value problem

$$\frac{d^2y}{dt^2} + y = 8e^{-2t}\sin t,\ y(0)=0, y'(0)=0$$

by using Laplace-transform **[15]**

Applying laplace transform on either side :

$p^2 L\{y\} - p\,y(0) - y'(0) + L\{y\} = L\{8e^{-2t}\sin t\}$

$\rightarrow (p^2+1)\,L\{y\} = 8\,\dfrac{1}{(p+2)^2+1}$

$\therefore L\{y\} = \dfrac{8}{((p+2)^2+1)(p^2+1)} = \dfrac{8}{(p^2+4p+5)(p^2+1)}$

Using partial fractions : $\dfrac{8}{(p^2+4p+5)(p^2+1)} = \dfrac{Ap+B}{p^2+4p+5} + \dfrac{Cp+D}{p^2+1}$.

$= \dfrac{(Ap^3 + Bp^2 + Ap + B) + (Cp^3 + (D+4C)p^2 + (5C+4D)p + 5D)}{(p^2+4p+5)(p^2+1)}$

$\Rightarrow A+C=0,\; B+D+4C=0,\; A+5C+4D=0,\; B+5D=8$

$C=-A\;;\; -4A+4D=0 \Rightarrow D=A\;;\; B-3A=0 \Rightarrow B=3A$

$\therefore\; 8A=8 \Rightarrow \boxed{A=1}\;;\; \boxed{B=3}\;;\; \boxed{C=-1}\;;\; \boxed{D=1}$

$$\therefore\; \frac{8}{((p+2)^2+1)(p^2+1)} = \frac{p+3}{(p+2)^2+1} + \frac{-p+1}{p^2+1}$$

$$= \frac{p+2}{(p+2)^2+1} + \frac{1}{(p+2)^2+1} + \frac{-p}{p^2+1} + \frac{1}{p^2+1}$$

$$\therefore\; y(t) = e^{-2t}\cos t + e^{-2t}\sin t - \cos t + \sin t$$

$$\therefore\; \boxed{y(t) = \cos t\,(e^{-2t}-1) + \sin t\,(e^{-2t}+1)}$$

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7. (a) Two equal uniform rods are firmly jointed at one end so that the angle between them is α, and they rest in a vertical plane on a smooth sphere of radius r. Show that they are in a stable or unstable equilibrium according as the length of rod is > or < 4r cosec α. **[17]**

7. (c) A particle is projected with velocity V from the cusp of a smooth inverted cycloid down the arc. Show that the time of reaching the vertex is

$$2\sqrt{\frac{a}{g}}\cot^{-1}\left(\frac{V}{2\sqrt{ag}}\right)$$

where a is the radius of the generating circle. **[17]**

8. (a) (i) Show that

$$curl\left(\hat{k}\times grad\frac{1}{r}\right)+grad\left(\hat{k}\cdot grad\frac{1}{r}\right)=0$$

where r is the distance from the origin and $\hat{k}$ is the unit vector in the direction OZ.

(ii) Find the values of constants a, b, and c such that the maximum value of directional derivative of $f = axy^2 + byz + cx^2z^2$ at $(1,-1,1)$ is in the direction parallel to y axis and has magnitude 6. **[12]**

i). $\nabla\left(\frac{1}{r}\right) = \frac{-1}{r^2}\nabla r = \frac{-1}{r^2}\frac{\vec{r}}{r} = \frac{-\vec{r}}{r^3} = -\frac{(x\hat{i}+y\hat{j}+z\hat{k})}{r^3}$

$\therefore \hat{k}\times\nabla\left(\frac{1}{r}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 0 & 1 \\ \frac{-x}{r^3} & \frac{-y}{r^3} & \frac{-z}{r^3} \end{vmatrix} = \frac{y}{r^3}\hat{i} - \frac{x}{r^3}\hat{j}$

$= \frac{y\hat{i} - x\hat{j}}{r^3}$

$\therefore \nabla \times \left(\hat{k} \times \nabla\left(\frac{1}{r}\right)\right) = \nabla \times \left[\frac{y\hat{i} - x\hat{j}}{r^3}\right] = \nabla\left(\frac{1}{r^3}\right) \times (y\hat{i} - x\hat{j})$

$+ \frac{1}{r^3} \nabla \times (y\hat{i} - x\hat{j})$

$= \frac{-3\bar{r}}{r^4 r} \times (y\hat{i} - x\hat{j}) + \frac{1}{r^3} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & -x & 0 \end{vmatrix}$

$= \frac{-3}{r^5} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ y & -x & 0 \end{vmatrix} + \frac{1}{r^3}(-2\hat{k}) = \frac{-3}{r^5}\left[xz\hat{i} + yz\hat{j} - (x^2+y^2)\hat{k}\right]$

$- \frac{2\hat{k}}{r^3}$

$\hat{k} \cdot \nabla \frac{1}{r} = \frac{-z}{r^3}$; $\nabla\left(\frac{-z}{r^3}\right) = -\left[\frac{r^3 \nabla z - z \nabla r^3}{r^6}\right] = -\left[\frac{r^3\hat{k} - 3r^2 z \frac{\bar{r}}{r}}{r^6}\right]$

$= \frac{-\hat{k}}{r^3} + \frac{3\bar{r}z}{r^5}$

$\therefore \nabla \times \left(\hat{k} \times \nabla\left(\frac{1}{r}\right)\right) + \nabla\left(\hat{k} \cdot \nabla \frac{1}{r}\right) = \frac{-3z}{r^5}(x\hat{i} + y\hat{j} + z\hat{k}) + \frac{3}{r^5}(r^2)\hat{k}$

$- \frac{2\hat{k}}{r^3} - \frac{\hat{k}}{r^3} + \frac{3z\bar{r}}{r^5} = \boxed{0}$

Hence the result

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ii) $\nabla f = (ay^2 + 2cxz^2)\hat{i} + (2axy + bz)\hat{j} + (by + 2cx^2z)\hat{k}$

at $(1,-1,1)$: $\nabla f = (a+2c)\hat{i} + (-2a+b)\hat{j} + (-b+2c)\hat{k}$

Given that $\nabla f_{max}(1,-1,1)$ ~~is~~ along $\hat{j}$ $\Rightarrow a + 2c = 0$; $-b + 2c = 0$

$\Rightarrow a = -2c; b = 2c$

$\therefore \nabla f(1,-1,1) = (4c+2c)\hat{j} = 6c\hat{j}$

Given that $\nabla f_{max}(1,-1,1) = 6\hat{j} \Rightarrow \boxed{c = 1; \ a = -2; \ b = 2}$

which are the required values of the constants

8. (b) (i) Prove that

$$\nabla^2 r^n = n(n+1)r^{n-2}$$

and that $r^n \vec{r}$ is irrotational, where $r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$.

(ii) Find the value of a if $A = a\hat{i} + \hat{j} + \sqrt{5}\hat{k}$ subtends and angle of 60° with $4\hat{i} - 5\hat{j} + \sqrt{5}\hat{k}$

[14]

i) $\nabla^2 r^n = \nabla \cdot (\nabla r^n) = \nabla \cdot [n r^{n-1} \nabla r] = \nabla \cdot [n r^{n-1} \frac{\bar{r}}{r}]$

$= \nabla \cdot [n r^{n-2} \bar{r}] = n(\nabla r^{n-2}) \cdot \bar{r} + n r^{n-2} \nabla \cdot \bar{r}$

$= n(n-2) r^{n-3} \frac{\bar{r}}{r} \cdot \bar{r} + n r^{n-2} (3)$

$= n(n-2) r^{n-2} + 3n r^{n-2} = (n^2 - 2n + 3n) r^{n-2}$

$= (n^2+n) r^{n-2} = \boxed{n(n+1) r^{n-2}}$

which is the required result

$\nabla \times (r^n \bar{r}) = r^n (\nabla \times \bar{r}) + (\nabla (r^n)) \times \bar{r}$

$= r^n (\bar{0}) + n r^{n-1} \frac{\bar{r}}{r} \times \bar{r}$ $(\because \nabla \times \bar{r} = \bar{0})$

$(\because \bar{r} \times \bar{r} = \bar{0})$

$= \bar{0} + n r^{n-2} (\bar{0}) = \boxed{\bar{0}}$

$\therefore r^n \bar{r}$ is irrotational

ii) $\cos\theta = \dfrac{4a - 5 + 5}{\sqrt{a^2+1+5}\sqrt{16+25+5}} = \cos 60 = \dfrac{1}{2}$.

$\Rightarrow \dfrac{4a}{\sqrt{a^2+6}\sqrt{46}} = \dfrac{1}{2}$. $\Rightarrow 8a = \sqrt{a^2+6}\sqrt{46}$

$\Rightarrow 64a^2 = 46a^2 + 276$

$\Rightarrow 18a^2 = 276 \Rightarrow a^2 = \frac{46}{3} \Rightarrow \boxed{a = \pm\sqrt{\frac{46}{3}}}$

$\therefore$ $\boxed{a = \sqrt{\frac{46}{3}}}$ (-ve value is discarded since the angle is acute)

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8. (c) For two vectors $\bar{a}$ and $\bar{b}$ given respectively by

$\vec{a} = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$ *and* $\vec{b} = \sin t\hat{i} - \cos t\hat{j}$

Determine: $(i)\frac{d}{dt}(\vec{a}\cdot\vec{b})$ and $(ii)\frac{d}{dt}(\vec{a}\times\vec{b})$ **[08]**

$\bar{a}\cdot\bar{b} = 5t^2 \sin t - t\cos t$

$\therefore \frac{d}{dt}(\bar{a}\cdot\bar{b}) = 10t\sin t + 5t^2\cos t + t\sin t - \cos t$

$= \boxed{11t\sin t + (5t^2 - 1)\cos t}$

$\bar{a}\times\bar{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5t^2 & t & -t^3 \\ \sin t & -\cos t & 0 \end{vmatrix} = -t^3\cos t\,\hat{i} - t^3\sin t\,\hat{j} - (5t^2\cos t + t\sin t)\hat{k}.$

$\therefore \frac{d}{dt}(\bar{a}\times\bar{b}) = (t^3\sin t - 3t^2\cos t)\hat{i} + (-t^3\cos t - 3t^2\sin t)\hat{j}$
$-(-5t^2\sin t + 10t\cos t + t\cos t + \sin t)\hat{k}$

$$= \boxed{\begin{aligned}&(t^3 \sin t - 3t^2 \cos t)\, \hat{i} - (t^3 \cos t + 3t^2 \sin t)\, \hat{j} \\ &+ [(5t^2-1)\sin t - 11 t \cos t]\, \hat{k}\end{aligned}}$$

which is the required result

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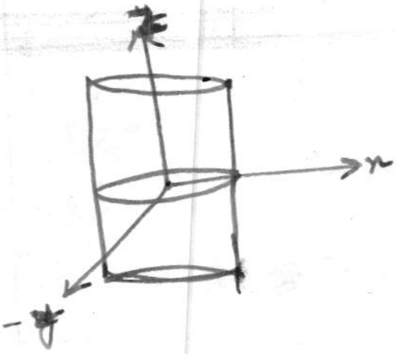
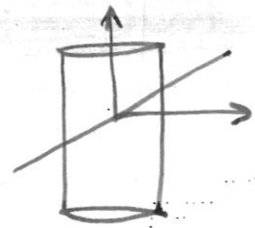
8. (d) State Stokes' theorem. Verify the Stokes' theorem for the function $\bar{f} = x\hat{i} + z\hat{j} + 2y\hat{k}$, where c is the curve obtained by the intersection of the plane z = x and the cylinder $x^2 + y^2 = 1$ and S is the surface inside the intersected one. **[16]**

Stokes' theorem: $\boxed{\oint_C \bar{F} \cdot d\bar{r} = \iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, ds}$

Stokes' theorem states that the line integral of a function around a curve C is equal in value to the surface integral of the curl of the function over any surface which is bounded by C.





Surface integral: $\hat{n} = \frac{-\hat{i}+\hat{k}}{\sqrt{2}}$.

$$\nabla \times \bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x & z & 2y \end{vmatrix} = \boxed{\hat{i}}$$

$$\therefore \iint_S (\nabla \times \bar{F}) \cdot \hat{n}\, dS = \iint -\frac{1}{\sqrt{2}}\, dS = -\frac{1}{\sqrt{2}} \iint dS$$

$$= -\frac{1}{\sqrt{2}} \iint \frac{dx\, dy}{|n \cdot \hat{k}|} = -\frac{1}{\sqrt{2}} \iint \frac{dx\, dy}{\frac{1}{\sqrt{2}}}$$

$$= -\iint dx\, dy = \boxed{-\pi}$$

Line integral: $z = t$; $x = t$; $y = \sqrt{1-t^2}$; $dx = dt$, ~~$dy = -2t\,dt$~~ $dz = dt$

$dy = \frac{1}{2}\frac{-2t}{\sqrt{1-t^2}}\, dt = \frac{-t}{\sqrt{1-t^2}}\, dt$

$$\cancel{\therefore \int \bar{F} \cdot d\bar{r} = \int_{t=-1}^{1} (t\, dt) - 2t^2\, dt + 2(1-t^2)\, dt}$$

$$\cancel{= \int_{-1}^{1} 2 + t - 4t^2\, dt =}$$

$$\therefore \int \bar{F} \cdot d\bar{r} = \int_{-1}^{1} t\, dt - \frac{t^2}{\sqrt{1-t^2}}\, dt + 2\sqrt{1-t^2}\, dt$$

$$= \int_{-1}^{1} t\, dt + \frac{2-3t^2}{\sqrt{1-t^2}}\, dt = \int_{-1}^{1} t\, dt + \frac{2}{\sqrt{1-t^2}}\, dt + \frac{3(1-t^2)}{\sqrt{1-t^2}}\, dt - \frac{3}{\sqrt{1-t^2}}\, dt$$

$$= \frac{t^2}{2} + \frac{3t}{2}\sqrt{1-t^2} + \frac{3}{2}\sin^{-1} t - \sin^{-1} t \Big|_{-1}^{1} = \frac{1}{2}(\sin^{-1} 1 - \sin^{-1}(-1)) = \pi$$

ROUGH SPACE

$\int \frac{\sin x}{\cos x} dx$; $\cos x = t$; $-\sin x\, dx = dt$

$\int -\frac{dt}{t} = -\log t = -\log \cos x = \log \sec x$

$\cos\theta = \frac{\sqrt{a^2-x^2}}{a}$

$\int \sqrt{a^2-x^2}\, dx$; $x = a\sin\theta$ $\quad \sin\theta = \frac{x}{a}$

$\int (a\cos\theta)\, a\cos\theta\, d\theta = \int a^2\cos^2\theta\, d\theta = a^2 \int \frac{1+\cos 2\theta}{2} d\theta$

$= a^2 \frac{\theta}{2} + \frac{a^2}{2}\frac{\sin 2\theta}{2}$

$= \frac{a^2}{2}\sin^{-1}\frac{x}{a} + \frac{a^2}{4} 2\left(\frac{x}{a}\frac{\sqrt{a^2-x^2}}{a}\right)$

$= \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}.$

$\lambda^3+3\lambda^2-2\lambda^2-6\lambda-15\lambda-45=0.$

$\lambda^2(\lambda+3)-2\lambda(\lambda+3)-15(\lambda+3)=0.$

$(\lambda^2-2\lambda-15)(\lambda+3)=0$

$(\lambda+3)(\lambda-5)(\lambda+3)=0.$

$\frac{3t}{2}\sqrt{1-t^2} + \frac{3}{2}\sin^{-1}t$