2019 IFoS

(1d) Using Cauchy's integral formula, evaluate

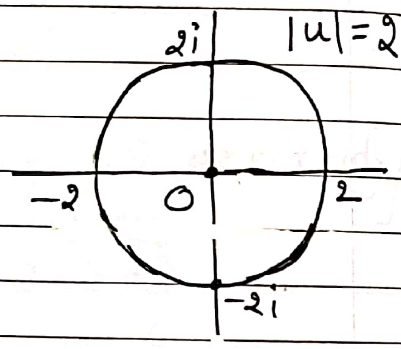
$$\oint_C \frac{dz}{(z^2+4)^2} \quad \text{where } C: |z-i| = 2.$$

Cauchy integral formula:
If $f(z)$ is analytic on a region bounded by closed curve C and z_0 is any point lying in this region, then

$$\int_C \frac{f(z)}{(z-z_0)^{n+1}}\,dz = \frac{2\pi i}{n!} f^n(z_0)$$

Let $z-i=u$, $(z^2+4)^2 = (z+2i)^2(z-2i)^2 = (u+3i)^2(u-i)^2$

$$\boxed{\cancel{\frac{1}{(z^2+4)^2} = \frac{1}{[(z+2i)(z-2i)]^2}}}$$



$$\frac{dz}{(z^2+4)^2} = \frac{du}{(u+3i)^2(u-i)^2}$$

Set, $f(u) = \dfrac{1}{(u+3i)^2}$

Note that $f(u)$ is analytic on the region bounded by closed curve, $|u| = 2$

$u = i$ is only singular point lying inside curve, $|u| = 2$.

$$\therefore \int_C \frac{dz}{(z^2+4)^2} = \int_C \frac{1}{(u+3i)^2} \cdot \frac{du}{(u-i)^2}$$

$$= \frac{2\pi i}{1!} u'(i) = 2\pi i \times \left.\frac{-2}{(u+3i)^3}\right|_{u=i}$$

$$= \frac{-4\pi i}{(4i)^3} = \frac{\pi}{16}$$

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If $f(z)$ is analytic in a domain D and $|f(z)|$ is a non-zero constant in D, then show that $f(z)$ is constant in D.

Let $f(z) = u + iv$

given, $|f(z)| = \sqrt{u^2+v^2}$ is constant

i.e. $u^2 + v^2 = K$

$\Rightarrow 2u u_x + 2v v_x = 0 \Rightarrow u \cdot u_x + v \cdot v_x = 0$

& $2u u_y + 2v v_y = 0 \qquad u \cdot u_y + v \cdot v_y = 0$ — (1)

Also,

As $f(z)$ is analytic, it satisfies C-R equations

i.e. $u_x = v_y$,

$u_y = -v_x$ — (2)

using (2) in (1)

$$u \cdot u_x + v \cdot v_x = 0. \quad \text{— (3)}$$

$$u(-v_x) + v(u_x) = 0$$

$$\text{i.e. } v \cdot u_x - u \cdot v_x = 0 \quad \text{— (4)}$$

Solving (3) & (4)

$$u \cdot u_x + v \cdot \frac{v \cdot u_x}{u} = 0.$$

$$u_x \left(\frac{u^2 + v^2}{u} \right) = 0$$

$$\Rightarrow u_x = 0 \quad \therefore \quad v_x = 0$$

$$\frac{df}{dz} = u_x + i v_x = 0$$

Integrating, we get $f(z) = C$ (constant).

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4(b) Classify the singular point $z=0$ of the function $f(z) = \frac{e^z}{z+\sin z}$ and obtain the principal part of the Laurent series expansion of $f(z)$.

$$\lim_{z\to 0} (z-0)f(z) = \lim_{z\to 0} \frac{z \cdot e^z}{z+\sin z}$$

$$= \lim_{z\to 0} \frac{e^z}{1+\frac{\sin z}{z}} = \frac{1}{1+1} = \frac{1}{2}$$

Hence, $z=0$ is a pole of order 1.

Now let us find Laurent series expansion of $f(z) = \frac{e^z}{z+\sin z}$

We know, $e^z = 1+z+\frac{z^2}{2!}+\dots$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots$$

$$\therefore f(z) = \frac{1+z+\frac{z^2}{2!}+\dots}{z+\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}+\dots\right)}$$

$$= \frac{1+z+\frac{z^2}{2!}+\dots}{2z\left(1-\frac{z^2}{12}+\frac{z^5}{120}+\dots\right)}$$

$$= \frac{1}{2z}\left[1+z+\frac{z^2}{2!}+\dots\right]\left[1+\left(\frac{-z^2}{12}+\frac{z^5}{240}+\dots\right)\right]^{-1}$$

$$= \frac{1}{2z}\left[1+z+\frac{z^2}{2!}+\dots\right]\left[1+\frac{z^2}{12}+O(3)\right]$$

$$= \frac{1}{2z}\left[1+z+\frac{7z^2}{12}+O(z^3)\right]$$

$$= \boxed{\frac{1}{2z}} + \frac{1}{2} + \frac{7z}{24} + O(z^2)$$

Principle part →