2018 IFoS (PDE)

Find the pde of all ~~sp~~ planes which are at a constant distance a from the origin.

Let $lx + my + nz = a$ —— (1)

be the equation of the given plane where l, m, n are direction cosines of the normal to the plane so that

$l^2 + m^2 + n^2 = 1$; l, m, n being parameters. —— (2)

Differentiating (1) partially w.r.t. x and y, we have

$$l + np = 0 \text{ and } m + nq = 0; \text{ where}$$

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}$$

$$\Rightarrow l = -np \text{ and } m = -nq$$

$\Rightarrow$ (2) becomes

$$n^2(p^2+q^2+1) = 0 \Rightarrow n = (p^2+q^2+1)^{-\frac{1}{2}}$$

$$\therefore l = np = -p(p^2+q^2+1)^{-1/2} \text{ and}$$

$$m = -nq = -q(p^2+q^2+1)^{-1/2}$$

Using these in (1), we get

$$-px(p^2+q^2+1)^{-1/2} - qy(p^2+q^2+1)^{-1/2} + z(p^2+q^2+1)^{-1/2} = a$$

$\Rightarrow z = px + qy + a(p^2+q^2+1)^{1/2}$, which is the required pde.

Air, obeying Boyle's law, is in motion in a uniform tube of small section. Prove that if ρ be the density and v be the velocity at a distance x from a fixed point at time t, then

$$\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2}{\partial x^2}\{\rho(v^2+k)\}$$

Air obeying Boyle's law,

$P = k\rho$, where k being positive constant
P is the pressure

From ideal gas equation,

$PV = \text{constant}$

Pressure is inversely proportional to volume

$\therefore P \propto \frac{1}{V}$

Air is in motion of uniform tube of small cross section.

$\therefore F = \frac{mv^2}{3L}$ [3 degrees of freedom]

Here L = length travelled by air molecules in uniform tube

Pressure, $P = \frac{Nmv^2}{3V}$

$\rho = \frac{P}{k}$, k being constant

$k\rho = P = \frac{Nmv^2}{3V}$

But density, $\rho = \frac{\text{mass}}{\text{volume}}$

$\therefore k\rho = \frac{N\rho v^2}{3}$ When we differentiate

$k\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial^2}{\partial x^2}\{\rho(v^2+k)\}$

or $\frac{\partial^2 \rho}{\partial t^2} = \frac{\partial}{\partial x^2}\{\rho(v^2+k)\}$

Find the complete integral of the pde $(p^2+q^2)x = zp$ and deduce the solution which passes through the curve $x=0$, $z^2=4y$. Here, $p=\frac{\partial z}{\partial x}$, $q=\frac{\partial z}{\partial y}$.

We have

$$(p^2+q^2)x = zp \quad \text{—— (1)}$$

Let $F \equiv (p^2+q^2)x - zp = 0$

By Charpit's subsidiary equations are

$$\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}}$$

$$= \frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}}$$

$$\Rightarrow \frac{dx}{-(2px-z)} = \frac{dy}{-2qx} = \frac{dp}{q^2} = \frac{dq}{-pq}$$

From the last two ratios,

$$\frac{dp}{q} = \frac{dq}{-p}$$

$$\Rightarrow p\,dp + q\,dq = 0$$

Integrating we get

$$p^2+q^2 = a^2, \quad a = \text{arbitrary constant}$$

Using $p^2+q^2=a^2$ in the given equation, we get $a^2x = zp$

$$\Rightarrow p = \frac{a^2x}{z}$$

and $q = \sqrt{a^2-p^2}$

$$= \sqrt{a^2 - \left(\frac{a^2x}{z}\right)^2} = \frac{a}{z}\sqrt{z^2-a^2x^2}$$

Now, $dz = p\,dx + q\,dy$

$$= \frac{a^2 x}{z} dx + \frac{a}{z}\sqrt{z^2 - a^2x^2}\, dy$$

$$\Rightarrow z\,dz - a^2 x\,dx = a\sqrt{z^2 - a^2x^2}\, dy$$

$$\Rightarrow \frac{z\,dz - a^2x\,dx}{\sqrt{z^2 - a^2x^2}} = a\,dy$$

$$\Rightarrow \frac{d(z^2 - a^2x^2)}{2\sqrt{z^2 - a^2x^2}} = a\,dy$$

Integrating, we get

$$\sqrt{z^2 - a^2x^2} = ay + b$$

$\Rightarrow z^2 = a^2x^2 + (ay+b)^2$, which is the required complete integral. — (2)

Here, given curve is $x = 0,\ z^2 = 4y$

The parametric equations are given by

$$x = 0,\ y = s^2,\ z = 2s$$

Now eqn (2) in this case becomes

$$4s^2 = (as^2 + b)^2 \quad — (3)$$

Differentiating (3) with respect to s, we get

$$2 = a(as^2 + b) \quad — (4)$$

On eliminating s between (3) and (4), we get

$$ab = 1 \Rightarrow b = \frac{1}{a}$$

Using this in (2), the one-parametric sub-family of the complete integral is

$$z^2 = a^2x^2 + \left(ay + \frac{1}{a}\right)^2$$

$$\Rightarrow a^4(x^2+y^2) + a^2(2y - z^2) + 1 = 0 \quad — (5)$$

And its envelope is obtained by eliminating 'a' between (5) and

$$2a^2(x^2+y^2)+(2y-z^2)=0$$

The envelope is

$$(2y-z^2)^2=4(x^2+y^2)$$

$$\Rightarrow z^2 = 2(y \pm \sqrt{x^2+y^2})$$

Since $\sqrt{x^2+y^2} \geq y$, the minus sign is to be discarded and then

$$z^2 = 2(y+\sqrt{x^2+y^2}),$$

which is the required integral surface.

Solve $(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx$, where $p = \frac{\partial z}{\partial x}$, $q = \frac{\partial z}{\partial y}$

If the solution of the above represents a sphere, what will be the coordinates of its centre.

We have,

$$(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx$$

Lagrange's subsidiary equations are

$$\frac{dx}{z^2 - 2yz - y^2} = \frac{dy}{xy + zx} = \frac{dz}{xy - zx}$$

Using multipliers x, y, z; we get

$$x\,dx + y\,dy + z\,dz = 0$$

Integrating, we get

$$x^2 + y^2 + z^2 = c_1$$

Again from the last two ratios, we get

$$\frac{dy}{y+z} = \frac{dz}{y-z}$$

$$\Rightarrow (y-z)dy - (y+z)dz = 0$$

$$\Rightarrow y\,dy - (z\,dy + y\,dz) - z\,dz = 0$$

Integrating, we get

$$\frac{y^2}{2} - yz - \frac{z^2}{2} = c \Rightarrow y^2 - 2yz - z^2 = c_2$$

$\therefore \phi(x^2 + y^2 + z^2, y^2 - z^2 - 2yz) = 0$, ϕ being an arbitrary function.

Thus, it represents a sphere with centre at origin.

Find a real function V of x and y, satisfying $\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = -4\pi(x^2+y^2)$ and reducing to zero, when $y=0$.

The given equation can be written as

$$(D^2+D'^2)V = -4\pi(x^2+y^2) \quad -(1)$$

The auxiliary equation is given by

$$m^2+1=0 \Rightarrow m = -i, i$$

$\therefore$ C.F. $= \phi_1(y-ix) + \phi_2(y+ix)$, where ϕ_1 and ϕ_2 are arbitrary functions

Now, P.I. $= \frac{1}{D^2+D'^2}\left(-4\pi(x^2+y^2)\right)$

$$= -4\pi \frac{1}{D^2+D'^2}(x^2+y^2)$$

$$= -4\pi \frac{1}{D^2}\left(1+\frac{D'^2}{D^2}\right)^{-1}(x^2+y^2)$$

$$= -4\pi \cdot \frac{1}{D^2}\left[1-\frac{D'^2}{D^2}+\cdots\right](x^2+y^2)$$

$$= -4\pi \frac{1}{D^2}\left\{(x^2+y^2) - \frac{1}{D^2}\cdot 2\right\}$$

$$= -4\pi \cdot \frac{1}{D^2}\left[(x^2+y^2) - 2\cdot\frac{x^2}{2}\right]$$

$$= -4\pi \cdot \frac{1}{D^2}\cdot y^2$$

$$= -4\pi y^2 \cdot \frac{x^2}{2}$$

$$= -2\pi x^2 y^2$$

$\therefore$ The required solution is

$$V = \phi_1(y-ix) + \phi_2(y+ix) - 2\pi x^2 y^2$$