IFoS(M) MATHS PAPER-I (2017)

## Section-A

Q1. (a) Let $A$ be a square matrix of order 3 such that each of its diagonal elements is 'a' and each of its off-diagonal elements is 1.
If $B = bA$ is orthogonal, determine values of 'a' and 'b'. (8)

Sol: Given,

$$A = \begin{bmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{bmatrix} \quad \therefore B = bA = \begin{bmatrix} ba & b & b \\ b & ba & b \\ b & b & ba \end{bmatrix}$$

$B$ is orthogonal ie. $BB^T = I$

$$\therefore \begin{bmatrix} ba & b & b \\ b & ba & b \\ b & b & ba \end{bmatrix} \begin{bmatrix} ba & b & b \\ b & ba & b \\ b & b & ba \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore \; b^2a^2 + 2b^2 = 1$ and $2b^2a + b^2 = 0$

$b^2(a^2+2) = 1$ and $b^2(2a+1) = 0$

$\therefore \; b^2 = 0$ or $(2a+1) = 0$, ~~not possible~~

But $b = 0$ is not possible, as first equation will not be satisfied

$\therefore \; a = -\frac{1}{2} \Rightarrow b^2\left(\frac{1}{4}+2\right) = 1 \Rightarrow b = \pm \frac{2}{3}$.

1(b) Let V be a vector space of all $2\times 2$ matrices over the field R. Show that W is not a subspace of V, where

i) W contains all $2\times 2$ matrices with zero determinant. (8)

Sol:

$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{R} \right\} = M_2(\mathbb{R})$$

It is a vector space over field $\mathbb{R}$.

W = Set of all $2\times 2$ matrices with determinant zero.

Let $w_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $w_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

$\det(w_1) = 0 = \det(w_2) \quad \therefore w_1, w_2 \in W$

But $w_1 + w_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\det(w_1 + w_2) = 1 \neq 0$

$\therefore w_1 + w_2 \notin W$

Hence, W is not a subspace of V.

ii) W Consists of all $2\times 2$ matrices A such that $A^2 = A$.

Let, $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $B^2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$\therefore$ $A, B \in W$

But $A+B = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$

$$(A+B)^2 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix} \neq (A+B)$$

$\therefore$ $(A+B) \notin W$

Hence, W is not a subspace of V.

1.(c) Using the Mean Value Theorem, show that

i) $f(x)$ is constant in $[a,b]$, if $f'(x) = 0$ in $[a,b]$.

Sol: Let $x_1$ and $x_2$ be two distinct points in interval $[a,b]$, and let $x_1 < x_2$.

$\therefore [x_1, x_2] \subseteq [a,b]$

Then $f$ is continuous on $[x_1, x_2]$ and $f$ is differential on $(x_1, x_2)$

Using LMVT, there exist some, $c \in (x_1, x_2)$ such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$$

ie. $$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0 \qquad \left[\because f'(x) = 0 \ \forall x \in [a,b]\right)$$

$$\Rightarrow \quad f(x_2) - f(x_1) = 0$$

ie $$f(x_1) = f(x_2)$$

Hence, $f(x)$ is constant function.

as $x_1$ and $x_2$ were arbitrary in $[a,b]$.

ii) $f(x)$ is decreasing function in $(a,b)$ if $f'(x)$ exists and is $<0$ everywhere in $(a,b)$.

Sol: Let $x_1$ and $x_2$ be any two distinct points in $[a,b]$ and $x_1 < x_2$, ~~∴ f~~

$$\therefore [x_1, x_2] \subseteq [a,b]$$

$f$ is differentiable on $(a,b)$, hence it is differentiable on $(x_1, x_2) \subseteq (a,b)$ and continuous also.

Using LMVT, there exist some $c \in (x_1, x_2)$ such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$$

$$f(x_2) - f(x_1) = f'(c) \cdot (x_2 - x_1)$$

Now, since $x_2 > x_1$ $\therefore (x_2 - x_1) > 0$

and $f'(x) < 0$ on $(a,b)$ $\therefore f'(c) < 0$ as $c \in (x_1, x_2) \subseteq (a,b)$

$$\therefore f(x_2) - f(x_1) < 0$$

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$$

$\therefore f(x)$ is decreasing function on $(a,b)$.

1(d) Let $u(x,y) = ax^2 + 2hxy + by^2$ and
$v(x,y) = Ax^2 + 2Hxy + By^2$.

Find the Jacobian $J = \dfrac{\partial(u,v)}{\partial(x,y)}$, and hence, show that $u$ and $v$ are independent unless, $\dfrac{a}{A} = \dfrac{b}{B} = \dfrac{h}{H}$ (8)

Sol:

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 2ax+2hy & 2hx+2by \\ 2Ax+2Hy & 2Hx+2By \end{vmatrix}$$

$$= 4(ax+2hy)(Hx+By) - 4(Ax+Hy)(hx+by)$$

$$= 4\left[(aHx^2 + xy(hH+aB) + hBy^2) - (Ahx^2 + xy(Hh+Ab) + bHy^2)\right]$$

$$= 4\left[(aH-Ah)x^2 + (aB-Ab)xy + (Bh-bH)y^2\right]$$

$u$, and $v$ are independent, if $J=0$

ie. $aH - Ah = 0$ ; $aB - Ab = 0$, $Bh - bH = 0$

$\therefore \dfrac{a}{A} = \dfrac{h}{H}$ ; $\dfrac{a}{A} = \dfrac{b}{B}$, $\dfrac{h}{H} = \dfrac{b}{B}$

ie. $\dfrac{a}{A} = \dfrac{b}{B} = \dfrac{h}{H}$

1(e) Find the equation of the planes parallel to the plane $3x-2y+6z+8=0$ and at a distance 2 from it. (8)

Sol: Equation of any plane parallel to given plane is

$$3x-2y+6z+k=0$$

Distance between two planes.

$$\frac{|k-8|}{\sqrt{9+4+36}} = 2$$

ie $|k-8| = 14$

$\therefore$ $k-8=14$ or $k-8=-14$

$k=22$ or $k=-6$

Hence, required equations of planes are

$3x-2y+6z+22=0$ or

$3x-2y+6z-6=0$.

2.(a) State Cayley – Hamilton Theorem. Verify the Theorem for matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$ Hence find $A^{-1}$. (10)

<u>Sol:</u> Every square matrix satisfies its characteristic equation, given by, $|A - \lambda I| = 0$

for the given matrix

$$\begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & -1-\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

$$\therefore \quad (1-\lambda)(\lambda^2 + \lambda - 1) = 0$$

$$\Rightarrow \quad \lambda^3 - 2\lambda + 1 = 0 \quad \text{——— (1)}$$

Now

$$A^2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & -1 \\ 0 & -1 & +1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 4 \\ 0 & -3 & 2 \\ 0 & 2 & -1 \end{bmatrix}$$

$$A^3 - 2A + I = \begin{bmatrix} 1 & 0 & 4 \\ 0 & -3 & 2 \\ 0 & 2 & -1 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= O$$

Hence, A satisfies its characteristic equation, given by (1).

2.(b) Show that

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta = \frac{1}{2} \frac{\Gamma\left(\frac{p+1}{2}\right)\cdot\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}, \quad p,q>-1.$$

Hence evaluate, (10)

i) $\int_0^{\pi/2} \sin^4 x \cdot \cos^5 x \, dx$

ii) $\int_0^{1} x^3 (1-x^2)^{5/2} \, dx$

iii) $\int_0^{1} x^4 (1-x)^3 \, dx$

Sol: We define, $\Gamma(m) = \int_0^{\infty} x^{m-1} e^{-x} \, dx, \quad m>0$

$$\beta(m,n) = \int_0^1 x^{m-1} (1-x)^{n-1} \, dx, \quad m,n>0.$$

We put, $\sin^2\theta = x \quad \therefore 2\sin\theta\cos\theta \, d\theta = dx$

$$\int_0^{\pi/2} \sin^p\theta \cos^q\theta \, d\theta$$

$$= \int_0^{\pi/2} (\sin^2\theta)^{\frac{p-1}{2}} \cdot (\cos^2\theta)^{\frac{q-1}{2}} \cdot \sin\theta\cos\theta \, d\theta$$

$$= \int_0^{1} x^{\frac{p-1}{2}} \cdot (1-x)^{\frac{q-1}{2}} \cdot \frac{dx}{2}$$

$$= \frac{1}{2} \beta\left(\frac{p+1}{2}, \frac{q+1}{2}\right)$$

$$= \frac{1}{2} \cdot \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)} \quad \left[\because \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\right]$$

i) $$\int_0^{\pi/2} \sin^4 x \cos^5 x \, dx = \frac{1}{2} \cdot \frac{\Gamma\left(\frac{4+1}{2}\right)\Gamma\left(\frac{5+1}{2}\right)}{\Gamma\left(\frac{4+5+2}{2}\right)}$$

$$= \frac{1}{2} \cdot \frac{\Gamma\left(\frac{5}{2}\right) \cdot \Gamma(3)}{\Gamma\left(\frac{11}{2}\right)}$$

$$\boxed{\begin{array}{c} \Gamma(m) = (m-1)! \\ \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi} \\ \Gamma(m) = (m-1)\Gamma(m-1) \end{array}}$$

$$= \frac{\cancel{\frac{1}{2}} \times \cancel{\frac{3}{2}} \times \cancel{\frac{1}{2}} \times \cancel{\Gamma\left(\frac{1}{2}\right)} \cdot 2!}{\frac{9}{2} \times \frac{7}{2} \times \frac{5}{2} \times \cancel{\frac{3}{2}} \times \cancel{\frac{1}{2}} \times \cancel{\Gamma\left(\frac{1}{2}\right)}}$$

$$= \frac{8}{315}$$

ii) Put, $x^2 = y \Rightarrow 2x\,dx = dy$ or $dx = \frac{1}{2x} dy$

$$\int_0^1 x^3 (1-x^2)^{5/2} dx = \int_0^1 y (1-y)^{5/2} \frac{dy}{2}$$

$$= \frac{1}{2} \beta\left(1+1, \frac{5}{2}+1\right) = \frac{1}{2} \beta\left(2, \frac{7}{2}\right)$$

$$= \frac{1}{2} \frac{\Gamma(2) \cdot \Gamma\left(\frac{7}{2}\right)}{\Gamma\left(2+\frac{7}{2}\right)} = \frac{1}{2} \cdot \frac{\Gamma(2) \cdot \Gamma\left(\frac{7}{2}\right)}{\Gamma\left(\frac{11}{2}\right)}$$

$$= \frac{1}{2} \cdot \frac{(1)! \; \cancel{\Gamma(7/2)}}{\frac{9}{2} \cdot \frac{7}{2} \cdot \cancel{\Gamma(7/2)}} = \frac{2}{63}$$

iii) $$\int_0^1 x^4 (1-x)^3 dx = \beta(4+1, 3+1) = \beta(5, 4)$$

$$= \frac{\Gamma(5)\,\Gamma(4)}{\Gamma(5+4)} = \frac{\cancel{\Gamma(5)} \cdot \cancel{3!}}{8 \times 7 \times \cancel{6} \times 5 \cdot \cancel{\Gamma(5)}}$$

$$= \frac{1}{280}$$

2(c) Find the maxima and minima for the function
$f(x,y) = x^3 + y^3 - 3x - 12y + 20.$
Also find the saddle point (if any) for the fn.
(10)

Sol: $f_x = 3x^2 - 3$ , $f_y = 3y^2 - 12$

for critical points, $f_x = 0$ and $f_y = 0$

$\therefore x = \pm 1$ , $y = \pm 2$

The function has four stationary points
$(1,2)$ , $(-1,2)$ , $(1,-2)$ , $(-1,-2)$

Now, $f_{xx} = 6x$ , $f_{yy} = 6y$ , $f_{xy} = 0$

✓ At $(1,2)$, $f_{xx} = +6 > 0$, $f_{yy} = 12 > 0$, $f_{xy} = 0$

$\therefore f_{xx} \cdot f_{yy} - f_{xy}^2 = 6 \cdot 12 - 0 = 72 > 0$

$\therefore (1,2)$ is point of minima.

✓ At $(-1,2)$,

$f_{xx} = -6 < 0$, $f_{yy} = 12 > 0$ , $f_{xy} = 0$

$f_{xx} \cdot f_{yy} - f_{xy}^2 = -6 \times 12 = -72 < 0$

function is neither maximum, nor minimum at $(-1,2)$

✓ At $(1,-2)$

$f_{xx} = 6 > 0$, $f_{yy} = -12 < 0$, $f_{xy} = 0$

$f_{xx} \cdot f_{yy} - f_{xy}^2 = -72 < 0$

function is neither maximum, nor min at $(1,-2)$

At $(-1,-2)$

$f_{xx} = -6 < 0, \quad f_{yy} = -12 < 0, \quad f_{xy} = 0$

$f_{xx} \cdot f_{yy} - f_{xy}^2 = 72 > 0$

$(-1,-2)$ is point of maxima.

Point of Maxima is $(-1,-2)$
Point of Minima is $(1,2)$
Stationary points like $(-1,2)$ and $(1,-2)$ which are not extreme points are saddle points.

[Result Used:

$f(a,b)$ is an extreme value of $f(x,y)$,

if $f_x(a,b) = 0 = f_y(a,b)$, and

$f_{xx} \cdot f_{yy} - f_{xy}^2 > 0$ at $(a,b)$.

and this extreme value is

maximum if $f_{xx}(a,b) < 0$ or

minimum if $f_{xx}(a,b) > 0$. ]

2.(d) Show that the angle between the planes given by the equation

$$2x^2 - y^2 + 3z^2 - xy + 7xz + 2yz = 0 \text{ is } \tan^{-1}\frac{\sqrt{50}}{4}.$$

(10)

Sol: Let the equations of two planes be

$a_1x + b_1y + z = 0$ and $a_2x + b_2y + 3z = 0$

($\because$ planes passes through origin)

Now the combined equation is

$$(a_1x + b_1y + z)(a_2x + b_2y + 3z) = 0$$

$$a_1a_2x^2 + b_1b_2y^2 + 3z^2 + xy(a_1b_2 + a_2b_1)$$
$$+ xz(3a_1 + a_2) + yz(b_2 + 3b_1) = 0$$

Comparing the coefficients in the given eqn

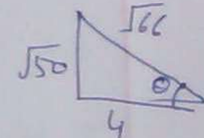
$a_1a_2 = 2$, $b_1b_2 = -1$, $a_1b_2 + a_2b_1 = -1$

$3a_1 + a_2 = 7$, $3b_1 + b_2 = 2$

$\Rightarrow a_1 = 2$, $a_2 = 1$ ; $b_1 = 1$, $b_2 = -1$

Equations of planes are

$2x + y + z = 0$ ; $x - y + 3z = 0$




Angle between the planes

$$\cos\theta = \frac{2.1 + 1(-1) + 1(3)}{\sqrt{4+1+1}\,\sqrt{1+1+9}} = \frac{4}{\sqrt{6}\cdot\sqrt{11}} = \frac{4}{\sqrt{66}}$$

$\therefore \tan\theta = \frac{\sqrt{50}}{4}$ $\therefore \theta = \tan^{-1}\frac{\sqrt{50}}{4}$.

3(a) Reduce the following matrix to row echelon form and hence find its rank.

$$A = \begin{bmatrix} -1 & 2 & -1 & 0 \\ 2 & 4 & 4 & 2 \\ 0 & 0 & 1 & 5 \\ 1 & 6 & 3 & 2 \end{bmatrix}$$

(10)

Sol: $R_2 \to R_2 + 2R_1$ ; $R_4 \to R_4 + R_1$

$$A \sim \begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & 8 & 2 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 8 & 2 & 2 \end{bmatrix}$$

$R_4 \to R_4 - R_2$

$$\sim \begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & 8 & 2 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Above form is row echelon form of A.

We have 3 non-zero rows

$\therefore$ Rank(A) = 3.

3(b). Given the set $\{u, v, w\}$ is linearly Independent. Examine the sets.

i) $\{u+v,\ v+w,\ w+u\}$

ii) $\{u+v,\ u-v,\ u-2v+2w\}$

for linear independence. (10)

Sol: Set of vectors $\{v_1, v_2, \dots v_n\}$ is L.I. if

$c_1 v_1 + c_2 v_2 + \dots + c_n v_n = 0$ implies that

$c_1 = c_2 = \dots = c_n = 0$.

i) Let $a, b, c \in \mathbb{R}$

Consider, $a(u+v) + b(v+w) + c(w+u) = 0$

i.e. $(a+c)u + (a+b)v + (b+c)w = 0$

Since, $\{u, v, w\}$ are L.I.

$\therefore a+c=0,\ a+b=0,\ b+c=0$

Adding these three, $2(a+b+c)=0$

Hence, $a=0,\ b=0,\ c=0$

$\therefore$ Given set is L.I.

ii) Again let $a, b, c \in \mathbb{R}$ and consider

$a(u+v) + b(u-v) + c(u-2v+2w) = 0$

$(a+b+c)u + (a-b-2c)v + 2cw = 0$

$\Rightarrow a+b+c=0$ $[\because \{u, v, w\}$ are L.I.$]$

$a-b-2c=0$

$2c=0$ $\therefore c=0 \Rightarrow a=0;\ b=0$

Hence, given set is L.I.

3.(c) Evaluate the integral $\int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$ by changing to polar co-ordinates. Hence, show that, $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$. (10)

Sol: The region of integration is first quadrant of xy-plane. Hence 'r' varies from 0 to $\infty$ and $\theta$ varies from 0 to $\frac{\pi}{2}$.

$$I = \int_0^\infty\int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$$

$$= \int_0^{\pi/2}\int_0^\infty e^{-r^2}\cdot r\,dr\,d\theta = -\frac{1}{2}\int_0^{\pi/2}\left[e^{-r^2}\right]_0^\infty d\theta$$

$$= \frac{-1}{2}\int_0^{\pi/2}(0-1)\,d\theta = \frac{1}{2}\times\frac{\pi}{2} = \frac{\pi}{4}$$

Also, $$I = \int_0^\infty e^{-x^2}\,dx \times \int_0^\infty e^{-y^2}\,dy$$

$$= \left(\int_0^\infty e^{-x^2}\,dx\right)^2 = \frac{\pi}{4}$$

$$\therefore \int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$

3(d) Find the angle between the lines whose direction cosines are given by the relations $l+m+n=0$ and
$2lm+2ln-mn=0$. (10)

Sol: Eliminating $n$ between the given relations

$$n=-(l+m)$$

$$2lm-2l(l+m)+m(l+m)=0$$

$$2l\cancel{m}-2l^2-2l\cancel{m}+ml+m^2=0$$

or $$2l^2-lm-m^2=0$$

$$2l^2-2lm+lm-m^2=0$$

$$2l(l-m)+m(l-m)=0$$

$$(l-m)(2l+m)=0$$

If $2l+m=0$, then from $l+m+n=0$, $n=l$

$$\therefore \quad \frac{l}{1}=\frac{m}{-2}=\frac{n}{1} \qquad -(1)$$

If $l-m=0$, then from $l+m+n=0$,
$n=-2m$

$$\therefore \quad \frac{l}{1}=\frac{m}{1}=\frac{n}{-2} \qquad -(2)$$

Hence, angle between the lines with direction ratios given by (1) and (2) is

$$\cos\theta=\frac{1\cdot1+(-2)\cdot1+1\cdot(-2)}{\sqrt{1+4+1}\cdot\sqrt{1+1+4}}=\frac{-3}{6}=\frac{-1}{2}$$

$$\cos\theta=\frac{-1}{2} \qquad \therefore \quad \underline{\theta=120^\circ}$$

4(a) Find eigenvalues and eigenvectors for matrix, $A = \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix}$. Examine whether matrix A is diagonizable. Obtain matrix D such that $D = P^{-1}AP$. (10)

Sol: Characteristic eqn, $|A - \lambda I| = 0$

$$\begin{vmatrix} -\lambda & -2 \\ 1 & 3-\lambda \end{vmatrix} = 0 \Rightarrow \begin{array}{l} \lambda^2 - 3\lambda + 2 = 0 \\ (\lambda-1)(\lambda-2) = 0 \end{array}$$

Eigen values, $\lambda = 1$, $\lambda = 2$

For $\lambda = 1$, Let eigenvector be, $v = \begin{bmatrix} x \\ y \end{bmatrix}$ $\therefore Av = \lambda v$

ie $(A - \lambda I)v = 0$

$$\therefore \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

ie. $x + 2y = 0$ $\therefore$ Eigen-vector is $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$

For $\lambda = 2$, $\begin{bmatrix} -2 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

ie $x + y = 0$ $\therefore$ Eigen vector is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Here, Algebraic multiplicity of each eigen value is equal to geometric multiplicity.

$\therefore$ A is diagonizable.

$P = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}$, $P^{-1} = \begin{bmatrix} -1 & -1 \\ -1 & -2 \end{bmatrix}$

$$\therefore D = P^{-1}AP = \begin{bmatrix} -1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

4.(b) A function $f(x,y)$ is defined as

$$f(x,y) = \begin{cases} \dfrac{x^2 y^2}{x^2+y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

Show that $f_{xy}(0,0) = f_{yx}(0,0)$ (10)

Sol:

$$f_{xy}(0,0) = \lim_{h\to 0} \frac{f_y(0+h,0) - f_y(0,0)}{h} \quad \text{— (1)}$$

$$\cancel{= \lim_{h\to 0} \frac{0-0}{k} = 0}$$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k\to 0} \frac{0}{k} = 0$$

$$f_y(h,0) = \lim_{k\to 0} \frac{f(h,k) - f(h,0)}{k} = \lim_{k\to 0} \frac{1}{k}\cdot\frac{h^2k^2}{(h^2+k^2)}$$

$$= 0$$

Hence, from ①, $f_{xy}(0,0) = \lim_{h\to 0} \frac{0-0}{h} = 0.$

Now, $$f_{yx}(0,0) = \lim_{k\to 0} \frac{f_x(0,k) - f_x(0,0)}{k} \quad \text{— (2)}$$

$$f_x(0,k) = \lim_{h\to 0} \frac{f(h,k) - f(0,k)}{h} = \lim_{h\to 0} \frac{1}{h}\cdot\frac{h^2k^2}{h^2+k^2}$$

$$= 0$$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

Hence, from (2), $f_{yx} = \lim_{k\to 0} \frac{0-0}{k} = 0$

$\therefore f_{xy}(0,0) = f_{yx}(0,0).$

4.(c) Find the equation of the right circular cone with vertex at the origin and whose axis makes equal angles with the coordinate axes and the generator is the line passing through the origin with direction ratios (1, −2, 2). (10)

Sol: The vertex of the cone is O (0,0,0) and since its axis makes equal angles with the coordinate axes, so the equations of its axis can be taken as

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{1} \quad (\because l = m = n)$$

Also, d.r.'s of its generator are (1, −2, 2)

If θ is the semi-vertical angle of the cone, then

$$\cos\theta = \frac{1\cdot 1 + 1\cdot(-2) + 1\cdot 2}{\sqrt{1+1+1}\cdot\sqrt{1+4+4}} = \frac{1}{3\sqrt{3}} \quad -(1)$$

If P(x, y, z) is any general point on the cone, then OP is a generator and d.r.'s of OP are (x−0, y−0, z−0) ie (x, y, z)

Also OP makes an angle θ with the axis.

$$\therefore \cos\theta = \frac{x\cdot 1 + y\cdot 1 + z\cdot 1}{\sqrt{1+1+1}\sqrt{x^2+y^2+z^2}} \quad -(2)$$

From (1) and (2),

$$\frac{x+y+z}{\cancel{\sqrt{3}}\cdot\sqrt{x^2+y^2+z^2}} = \frac{1}{3\sqrt{3}} \quad \text{or} \quad 9(x+y+z)^2 = x^2+y^2+z^2$$

or $\quad 4(x^2+y^2+z^2) + 9(xy+yz+zx) = 0.$

4.(d) Find the shortest distance and the equation of the line of the shortest distance between the lines $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$ and $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$. (10)

Sol: Any point on first line $P(3r+3, -r+8, r+3)$

" " Second line $Q(-3t-3, 2t-7, 4t+6)$

D.r.'s of PQ are

$(3r+3t+6, -r-2t+15, r-4t-3)$ —(i)

If PQ is the shortest distance (SD) between the given lines, then PQ is $\perp$ to both lines.

$\therefore 3(3r+3t+6) - 1(-r-2t+15) + 1(r-4t-3) = 0$

and $-3(3r+3t+6) + 2(-r-2t+5) + 4(r-4t-3) = 0$

or $11r + 7t = 0$ and $7r + 29t = 0$

Solving, we get $r = 0$, $t = 0$

$\therefore$ Points, $P(3, +8, 3)$ and $Q(-3, -7, 6)$

D.r's of PQ are $(6, 15, -3)$ or $(2, 5, -1)$

$$SD = PQ = \sqrt{(3+3)^2 + (8+7)^2 + (3-6)^2}$$
$$= \sqrt{36 + 225 + 9} = 3\sqrt{30}$$

Also, PQ is line through $P(3,8,3)$ and with d.r's $(2, 5, -1)$, so its equation is

$$\frac{x-3}{2} = \frac{y-8}{5} = \frac{z-3}{-1}$$

SEC-B

5.(a) Solve, $(2D^3-7D^2+7D-2)y = e^{-8x}$, where $D=\frac{d}{dx}$ (8)

Sol: Let $2D^3-7D^2+7D-2 = f(D)$ — (1)

$\therefore \; f(D)y = e^{-8x}$ — (2)

Auxialiary equation of (2) is, $f(D)=0$

or $2D^3-7D^2+7D-2=0$

roots are, $D = 1, 2, \frac{1}{2}$.

Complementary function (CF) of (2) is

$$y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{x/2} \qquad \text{— (3)}$$

For finding Particular Integral (PI) —

$f(D)y = \phi(x)$, where $\phi(x) = e^{-8x}$

$$PI = \frac{1}{f(D)}\phi(x) = \frac{1}{(D-1)(D-2)(D-\frac{1}{2})2}\left(e^{-8x}\right)$$

$$= \frac{1}{(-8-1)(-8-2)(-16-1)} \cdot e^{-8x}$$

$$= \frac{1}{-1530} e^{-8x} \qquad \text{— (4)}$$

General Solution, $y = y_c + y_p$

$$y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{x/2} - \frac{e^{-8x}}{1530}.$$

5(b) Solve the DE

$$x^2 \frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^4 \quad -(1) \qquad (8)$$

Sol: Using the substitution, $x = e^z$ ie $z = \log x$

The equation becomes

$$(D_1(D_1-1) - 2D_1 - 4)\, y = e^{4z}, \quad D_1 = \frac{d}{dz}$$

$$\therefore (D_1^2 - 3D_1 - 4)\, y = e^{4z} \quad -(2)$$

Auxialiary equation of (2) is

$$D_1^2 - 3D_1 - 4 = 0 \quad \therefore D_1 = 4, -1$$

$$\therefore C.F. = y_c = c_1 e^{-z} + c_2 e^{4z} \quad -(3)$$

$$P.I. = \frac{1}{D_1^2 - 3D_1 - 4}\left(e^{4z}\right)$$

$$= \frac{1}{(D_1-4)(D_1+1)}\left(e^{4z}\right) = \frac{1}{(D_1-4)} \cdot \frac{e^{4z}}{(4+1)}$$

$$= \frac{z}{5} \cdot e^{4z} \quad -(4)$$

$$\therefore y = y_c + y_p$$

$$= c_1 e^{-z} + c_2 e^{4z} + \frac{z e^{4z}}{5}$$

$$\cancel{= c_1 x^{-1} + c_2 e}$$

$$= c_1 e^{-\log x} + c_2 e^{4\log x} + \frac{\log x \cdot e^{4\log x}}{5}$$

$$= \frac{c_1}{x} + c_2 x^4 + \frac{1}{5} x^4 \cdot \log x \,.$$

5.(c) A particle is undergoing SHM of period $T$ about a centre O and it passes through the position P ($OP=b$) with velocity $v$ in the direction OP. Prove that the time that elapses before it returns to P is $\frac{T}{\pi}\tan^{-1}\left(\frac{vT}{2\pi b}\right)$. (8)

Sol: $x = a\sin\omega t$, $v = a\omega\cos\omega t$, $\omega = \frac{2\pi}{T}$

At $x = b$,

$b = a\sin\omega t_b$, $v = a\omega\cos\omega t_b$

$\Rightarrow t_b = \frac{1}{\omega}\sin^{-1}\frac{b}{a}$ $\therefore v = a\omega(1-\sin^2\omega t_b)^{1/2}$

$= \frac{T}{2\pi}\sin^{-1}\frac{b}{a}$ $= a\omega\left(1-\frac{b^2}{a^2}\right)^{1/2}$

$= \omega\sqrt{a^2-b^2}$

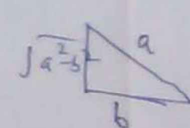
$\Rightarrow a^2 = \frac{v^2}{\omega^2}+b^2$

$t_a = \frac{T}{4} = \frac{2\pi}{\omega}\cdot\frac{1}{4}$ $= \left(\frac{vT}{2\pi}\right)^2+b^2$

Time required,

$$t = 2(t_a-t_b) = 2\left(\frac{T}{4}-\frac{T}{2\pi}\sin^{-1}\frac{b}{a}\right)$$

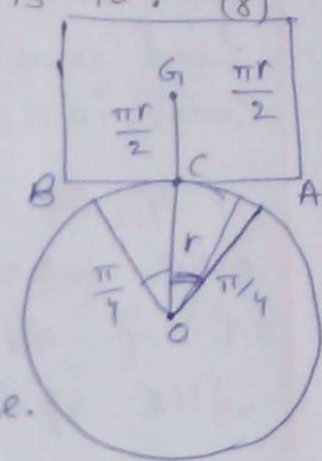
$$= \frac{T}{\pi}\left(\frac{\pi}{2}-\sin^{-1}\frac{b}{a}\right) = \frac{T}{\pi}\cos^{-1}\frac{b}{a}$$




$$= \frac{T}{\pi}\tan^{-1}\frac{\sqrt{a^2-b^2}}{b}$$

$$= \frac{T}{\pi}\tan^{-1}\left(\frac{Tv}{2\pi b}\right) \quad \left[\because a^2-b^2 = \left(\frac{vT}{2\pi}\right)^2\right]$$

5.(b) A heavy uniform cube balances on the highest point of a sphere whose radius is '$r$'. If the sphere is rough enough to prevent sliding and if the side of the cube be $\frac{\pi r}{2}$, the prove that the total angle through which the cube can swing without falling is $90^\circ$. (8)




Sol: If G is the centre of gravity of the cube, then for equilibrium the line OCG must be vertical.

First we show that the equilibrium of the cube is stable.

Here, $\rho_1$ = the radius of curvature of the upper body at the point of contact $= \infty$

and, $\rho_2$ = the radius of curvature of the lower body at the point of contact $= r$

$h$ = height of the centre of gravity, G of the upper body above the point C

= half of the edge of cube $= \frac{\pi r}{4}$

The equilibrium will be stable if

$$\frac{1}{h} > \frac{1}{\rho_1} + \frac{1}{\rho_2} \quad \text{ie} \quad \frac{1}{\pi r/4} > \frac{1}{\infty} + \frac{1}{r}$$

$$\text{ie.} \quad \frac{4}{\pi r} > \frac{1}{r} \quad \text{ie} \quad 4 > \pi$$

which is true.

Hence the equilibrium is stable. So, if the cube is slightly displaced, it will tend to come back to its original position of equilibrium.

During a swing to the right, the cube will not fall down till the right hand corner A of the lowest edge comes in contact with the sphere.

If $\theta$ is the angle through which the cube turns when the right hand corner A of the lowest edge comes in contact with the sphere,

$$\therefore r\theta = \text{half the edge of the cube} = \frac{\pi r}{4}$$

$$\therefore \theta = \pi/4$$

Similarly the cube can turn through an angle of $\pi/4$ to the left side on the sphere. Hence the total angle through which the cube can swing (or rock) without falling is $2 \cdot \frac{\pi}{4}$ ie $\pi/2$.

5.(e) Prove that

$$\nabla^2 r^n = n(n+1)\, r^{n-2}$$

and that $r^n \vec{r}$ is irrotational, where

$$r = |\vec{r}| = \sqrt{x^2+y^2+z^2}. \qquad (8)$$

Sol:

$$\nabla^2 r^n = \nabla\cdot(\nabla r^n) = \operatorname{div}(\operatorname{grad} r^n)$$

$$= \operatorname{div}(n r^{n-1} \operatorname{grad} r)$$

$$= \operatorname{div}\left(n r^{n-1} \frac{\vec{r}}{r}\right) = \operatorname{div}(n r^{n-2} \vec{r})$$

$$= (n r^{n-2}) \operatorname{div}\vec{r} + \vec{r}\cdot(\operatorname{grad} n r^{n-2})$$

$$= 3n r^{n-2} + \vec{r}\cdot[n(n-2) r^{n-3} \operatorname{grad} r]$$

$$= 3n r^{n-2} + \vec{r}\cdot\left[n(n-2) r^{n-3}\cdot \frac{\vec{r}}{r}\right]$$

$$= 3n r^{n-2} + \vec{r}\cdot[n(n-2) r^{n-4} \vec{r}]$$

$$= 3n r^{n-2} + n(n-2) r^{n-2} \qquad (\because \vec{r}\cdot\vec{r} = r^2)$$

$$= n r^{n-2}(3+n-2)$$

$$= n(n+1) r^{n-2}$$

Now, $|\vec{r}| = \sqrt{x^2+y^2+z^2}$

$$r^n \vec{r} = r^n \{xi + yj + zk\}$$

$$\operatorname{curl}(\vec{r}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= i(0) + j(0) + k(0) = 0$$

$[\because \operatorname{curl}(\phi A) = (\operatorname{grad}\phi)\times A + \phi \operatorname{curl} A)]$

Hence, $r^n \vec{r}$ is irrotational.

6(a) Solve the DE,

$$\left(\frac{dy}{dx}\right)^2 + 2\cdot\frac{dy}{dx}\cdot y\cot x = y^2 \quad \text{—(1)} \qquad (15)$$

Sol: Put $\frac{dy}{dx} = p$,

$$p^2 + 2py\cot x = y^2$$

$$\Rightarrow p^2 + 2py\cot x + y^2\cot^2 x = y^2 + y^2\cot^2 x$$

$$\Rightarrow (p + y\cot x)^2 = y^2\operatorname{cosec}^2 x$$

$$\therefore p + y\cot x = \pm y\operatorname{cosec} x$$

ie.

$$\left.\begin{aligned} &\frac{dy}{dx} + y(\cot x - \operatorname{cosec} x) = 0 \\ &\frac{dy}{dx} + y(\cot x + \operatorname{cosec} x) = 0 \end{aligned}\right\} \text{ components of Eqn.}$$

$$\frac{dy}{y} + (\cot x - \operatorname{cosec} x)\,dx = 0$$

Integrating,

$$\log y + \log\sin x - \log\left(\tan\frac{x}{2}\right) = \log c$$

$$\therefore \log y = \log c + \log\tan\frac{x}{2} - \log\sin x$$

$$y = \frac{c \times \tan x/2}{2\sin x} = \frac{c\cdot\sin x/2}{\cos\frac{x}{2}\cdot 2\sin\frac{x}{2}\cos\frac{x}{2}}$$

$$y = \frac{c}{2\cos^2\frac{x}{2}} = \frac{c}{1+\cos x}$$

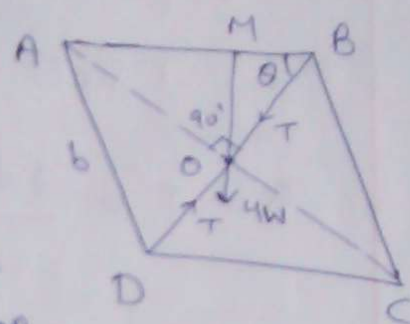
ie $y(1+\cos x) = c$ is one solution.

Similarly, solving the second equation, we get, $y(1-\cos x) = c$

$\therefore$ Gen Solution of (1) is, $[y(1+\cos x) - c][y(1-\cos x) - c] = 0.$

6(a) A string of length a, forms the shorter diagonal of a rhombus formed of four uniform rods, each of length 'b' and weight W, which are hinged together. If one of the rods is supported in a horizontal position, then prove that the tension of the string is

$$\frac{2W(2b^2-a^2)}{b\sqrt{4b^2-a^2}}. \qquad (10).$$




Sol: Let T be the tension in the string BD.
The total weight 4W of the rods AB, BC, CD and DA can be taken as acting at the point of intersection O of the diagonals AC and BD.
We have, $\angle AOB = 90^\circ$

Let $\angle ABO = \theta$, Draw OM ⊥ to AB.

Give the system a small symmetrical displacement in which θ changes ~~from~~ to $\theta + \delta\theta$. The line AB remains fixed. The points O, C and D change. ~~The~~ The angle AOB will remain 90°.

$$BD = 2BO = 2AB\cos\theta = 2b\cos\theta$$

($\because$ length BD = a at equilibrium. It changes during displacement, and depends on angle θ )

The depth of O below the fixed line AB is $= MO = (BO)\sin\theta = (AB\cos\theta)\sin\theta$

ie $MO = b\sin\theta\cos\theta$

By the principle of virtual work,

$$-T\delta(2b\cos\theta) + 4W\delta(b\sin\theta\cos\theta) = 0$$

or $2bT\sin\theta\,\delta\theta + 4bW(\cos^2\theta - \sin^2\theta)\delta\theta = 0$

or $2b[T\sin\theta - 2W(\sin^2\theta - \cos^2\theta)]\delta\theta = 0$

or $T\sin\theta - 2W(\sin^2\theta - \cos^2\theta) = 0 \quad (\because \delta\theta \neq 0)$

or $$T = \frac{2W(\sin^2\theta - \cos^2\theta)}{\sin\theta} = \frac{2W(1-2\cos^2\theta)}{\sqrt{1-\cos^2\theta}}$$

In the position of equilibrium,

$BD = a$ or $BO = \frac{a}{2}$.

$$\therefore \cos\theta = \frac{BO}{AB} = \frac{a}{2b}$$

$$\therefore T = \frac{2W\left(1 - 2\cdot\frac{a^2}{4b^2}\right)}{\sqrt{1 - a^2/4b^2}}$$

$$\left[\; T = \frac{2W(2b^2 - a^2)}{b\sqrt{4b^2 - a^2}} \;\right]$$

6(c). Using Stockes' theorem, evaluate

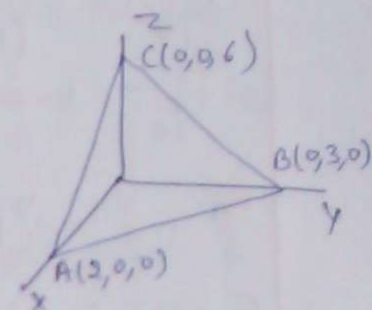
$$\oint_C [(x+y)\,dx + (2x-z)\,dy + (y+z)\,dz],$$

where C is the boundary of the triangle with vertices at (2,0,0), (0,3,0) and (0,0,6).

(15)

Sol: The given integral is of the form

$\oint_C F \cdot dr$, where




$$F = (x+y)i + (2x-z)j + (y+z)k$$

$$dr = i\,dx + j\,dy + k\,dz$$

C: Boundary of Triangle ABCA

S: Area of Triangle ABC

$$\text{curl}(F) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix} = 2i + k \quad \text{---(1)}$$

Using Stoke's theorem,

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\text{curl}\,\vec{F} \cdot \hat{n})\,dS$$

Here $\hat{n}$ is unit normal vector to $\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1$

$$\hat{n} = \frac{6}{\sqrt{14}}\left(\frac{i}{2} + \frac{j}{3} + \frac{k}{6}\right) = \frac{1}{\sqrt{14}}(3i + 2j + k) \quad \text{---(2)}$$

$$\text{curl}(\vec{F}) \cdot \hat{n} = \frac{1}{\sqrt{14}}(6+1) = \frac{7}{\sqrt{14}} \quad [\text{from (1) \& (2)}]$$

$$\therefore \oint_C \vec{F} \cdot d\vec{r} = \frac{7}{\sqrt{14}} \iint_S dS = \frac{7}{\sqrt{14}}(\text{Area of } \Delta ABC)$$

$$= \frac{7}{\sqrt{14}} \times 3\sqrt{14} = 21.$$

[Area ($\Delta ABC$) $\ni$ $\Delta^2 = \Delta_x^2 + \Delta_y^2 + \Delta_z^2$

$\Delta^2 = \left(\frac{1}{2} \times 3 \cdot 6\right)^2 + \left(\frac{1}{2} \cdot 2 \cdot 6\right)^2 + \left(\frac{1}{2} \cdot 2 \cdot 3\right)^2 = 126$ ]

7.(a) Solve the DE

$$e^{3x}\left(\frac{dy}{dx}-1\right)+\left(\frac{dy}{dx}\right)^3 e^{2y}=0. \qquad (10)$$

Sol: Take, $\frac{dy}{dx}=p$, The DE equation becomes

$$e^{3x}(p-1)+p^3\cdot e^{2y}=0 \qquad —(1)$$

ie $\quad (1-p)\,e^{3x}=p^3\cdot e^{2y}$

ie $\quad 1-p=p^3\cdot e^{2y-3x}$

$$e^y(1-p)=\left(p\cdot e^{y-x}\right)^3 \qquad —(2)$$

Eqn (2) is in the form of

$$e^{by}(a-bp)=f\left(pbe^{by-ab}\right),$$

Here, $a=1$, $b=1$

Let $e^x=X$, $\quad e^y=Y \quad \therefore e^x dx=dX$

$e^y dy=dY$

ie $\frac{dY}{dX}=\frac{e^y}{e^x}\cdot\frac{dy}{dx}=\frac{e^y}{e^x}\cdot p$

$$\therefore \quad p=\frac{e^x}{e^y}\cdot\frac{dY}{dX}=\frac{X}{Y}\cdot\frac{dY}{dX}=\frac{X}{Y}P$$

Hence, Eqn (1) becomes,

$$X^3\left(\frac{X}{Y}P-1\right)+\left(\frac{X}{Y}P\right)^3\cdot Y^2=0$$

ie $\quad XP-Y+P^3=0$

$\therefore \quad Y=XP+P^3$, which is clairat's form.

$y=xp+f(p)$.

General solution is,

$Y=XC+C^3$ ie $\boxed{e^y=e^x c+c^3}$

7.(b) A planet is describing an ellipse about the Sun as a focus. Show that its velocity away from the Sun is the greatest when the radius vector to the planet is at a right angle to the major axis of path and that the velocity then is $\dfrac{2\pi ae}{T\sqrt{1-e^2}}$, where $2a$ is the major axis, $e$ is the eccentricity and $T$ is the periodic time.

Sol: The polar equation of the elliptic orbit is

$$\frac{l}{r} = 1 + e\cos\theta \quad \text{or} \quad lu = 1 + e\cos\theta \quad -(1)$$

We know,

$$h^2 = r^2\dot{\theta} \quad \text{or} \quad \dot{\theta} = hu^2. \qquad \left(u = \frac{1}{r}\right) \quad -(2)$$

Also, $$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}(hu^2) = -h\frac{du}{d\theta}$$

for maximum value of $\frac{dr}{dt}$, we have

$$\frac{d}{d\theta}\left(\frac{dr}{dt}\right) = 0 \quad \text{or} \quad \frac{d}{d\theta}\left(-h\frac{du}{d\theta}\right) = 0 \quad \text{or} \quad \frac{d^2u}{d\theta^2} = 0$$

($\because$ $h$ is constant)

From (1),

$$\frac{du}{d\theta} = -\frac{e}{l}\sin\theta \quad \& \quad \frac{d^2u}{d\theta^2} = -\frac{e}{l}\cos\theta$$

$$\therefore \frac{d^2u}{d\theta^2} = 0 \Rightarrow -\frac{e}{l}\cos\theta = 0 \Rightarrow \cos\theta = 0 \text{ ie } \theta = \frac{\pi}{2}.$$

This proves the first part.

For maximum value of $\frac{dr}{dt}$,

$$\frac{du}{d\theta} = -\frac{e}{l}\sin\frac{\pi}{2} = -\frac{e}{l} \quad \text{--- (3)}$$

From (2) & (3), $\left(\frac{dr}{dt}\right)_{max} = \frac{he}{l} = \sqrt{\mu l}\cdot\frac{e}{l} = \sqrt{\frac{\mu}{l}}\,e$ --- (4)

As, $l = \frac{b^2}{a} = a(1-e^2)$ and $T = \frac{2\pi a^{3/2}}{\sqrt{\mu}}$

$\therefore \sqrt{l} = \sqrt{a(1-e^2)}$ and $\sqrt{\mu} = \frac{2\pi a^{3/2}}{T}$

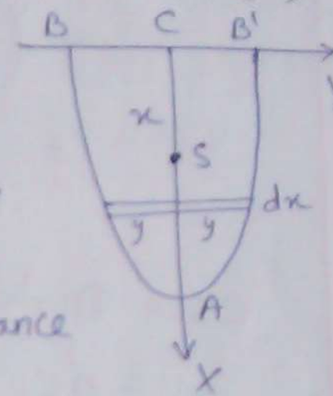
Substituting in (4)

$$\left(\frac{dr}{dt}\right)_{max} = \frac{2\pi a^{3/2}\cdot e}{T\sqrt{a(1-e^2)}} = \frac{2\pi a e}{T\sqrt{1-e^2}}.$$

[ $l$ = Semi-latus rectum

7(c) A semi-ellipse bounded by its minor-axis is just immersed in a liquid, the density of which varies as the depth. If the minor axis lies on the surface, then find the eccentricity in order that the focus may be the centre of pressure. (10)

Sol: BAB′ is the semi-ellipse immersed in a liquid with minor axis BB′ in the surface.




Consider the elementary strip of width $dx$ at a distance $x$ from C.

$$\therefore \quad p = \rho g x = kx \cdot gx = kgx^2$$

$$ds = 2y\,dx$$

But, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ ie. $y = \dfrac{b}{a}\sqrt{a^2 - x^2}$

$$\therefore \quad ds = \frac{2b}{a}\sqrt{a^2 - x^2}\,dx$$

$$\therefore \quad \bar{x} = \frac{\int_0^a x p\,ds}{\int_0^a p\,ds} = \frac{\int_0^a x \cdot kgx^2 \cdot \frac{2b}{a}\sqrt{a^2-x^2}\,dx}{\int_0^a kgx^2 \cdot \frac{2b}{a}\sqrt{a^2-x^2}\,dx}$$

~~Put~~ $$= \frac{\int_0^a x^3\sqrt{a^2-x^2}\,dx}{\int_0^a x^2\sqrt{a^2-x^2}\,dx}$$

Put, $x = a\sin\theta$, $dx = a\cos\theta\, d\theta$

$$\therefore \bar{x} = \frac{\int_0^{\pi/2} a^3 \sin^3\theta \cdot a^2\cos^2\theta\, d\theta}{\int_0^{\pi/2} a^2\sin^2\theta \cdot a^2\cos^2\theta\, d\theta}$$

$$= \left(\frac{2\cdot 1}{5\cdot 3}a\right) \Big/ \left(\frac{1\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2}\right)$$

$$= \frac{32}{15\pi}a$$

CS = Distance of focus from C

$$= ae = \frac{32a}{15\pi}$$

$$\therefore \boxed{e = \frac{32}{15\pi}}$$

7.(d) Evaluate, $\iint_S (\nabla \times \vec{f}) \cdot \hat{n}\, dS$,

where S is the surface of the cone,

$z = 2 - \sqrt{x^2+y^2}$ above xy- plane and

$\vec{f} = (x-z)\hat{i} + (x^3+yz)\hat{j} - 3xy^2 \hat{k}$. (10)

Sol: The xy-plane cuts the surface S of cone in the circle, C whose eqn is $x^2+y^2 = 4;\ z=0$.

parametric eqn : $x = 2\cos t,\quad y = 2\sin t$

By stoke's theorem,

$$\iint_S (\nabla \times \vec{f}) \cdot \hat{n}\, dS = \oint_C \vec{F} \cdot d\vec{r}$$

$$= \int_C (x-z)\,dx + (x^3+yz)\,dy + (-3xy^2)\,dz$$

$$= \int_C x\,dx + x^3\,dy \qquad (\because z = dz = 0)$$

$$= \int_{t=0}^{2\pi} \left[ x\frac{dx}{dt} + x^3 \frac{dy}{dt} \right] dt$$

$$= \int_{t=0}^{2\pi} \left[ 2\cos t(-2\sin t) + 8\cos^3 t \cdot 2\cos t \right] dt$$

$$= \int_{t=0}^{2\pi} \left( -2(\sin 2t) + 16\cos^4 t \right) dt$$

$$= -2\int_0^{2\pi} \sin 2t\, dt + 16 \int_0^{2\pi} \left( \frac{1+\cos 2t}{2} \right)^2 dt$$

$$= \left[ \cos 2t \right]_0^{2\pi} + 16 \left\{ \frac{1}{32}\left[\sin 4x\right]_0^{2\pi} - \frac{1}{8}\left[x\right]_0^{2\pi} + \frac{1}{2}\left[x\right]_0^{2\pi} + \frac{1}{4}\left[\sin 2x\right]_0^{2\pi} \right\}$$

$$= 0 + 16\left\{ 0 - \frac{2\pi}{8} + \frac{2\pi}{2} + 0 \right\}$$

$$= 12\pi$$

8.(a) Solve, $\frac{d^2y}{dx^2} + 4y = \tan 2x$ by using the method of variation of parameter. (10)

Sol Comparing with, $y_2 + Py_1 + Qy = R$

$P = 0, \quad Q = 4, \quad R = \tan 2x$

Auxiliary eqn : $(D^2+4) = 0 \quad \therefore D = \pm 2i$

$\therefore y_c = c_1 \cos 2x + c_2 \sin 2x \quad$ —— (1)

Let $y_p = A\cos 2x + B \sin 2x$ be PI of DE

where A and B are functions of $x$.

Then, $u(x) = \cos 2x$, $v(x) = \sin 2x$, $R(x) = \tan 2x$

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

$$A = \int \frac{-vR}{W} dx = \int \frac{-\sin 2x \cdot \tan 2x}{2} dx$$

$$= \int \frac{-\sin^2 2x}{2\cos 2x} dx = \frac{-1}{2} \int \frac{1-\cos^2 2x}{\cos 2x} dx$$

$$= \frac{-1}{2} \int (\sec 2x - \cos 2x) dx$$

$$= \frac{-1}{4} \left( \log|\sec 2x + \tan 2x| - \sin 2x \right)$$

and, $$B = \int \frac{uR}{W} dx = \int \frac{\cos 2x \cdot \tan 2x}{2} dx$$

$$= \frac{1}{2} \int \sin 2x \, dx = -\frac{1}{4} \cos 2x$$

$$\therefore y_p = \frac{-1}{4} (\log|\sec 2x + \tan 2x| - \sin 2x) \cos 2x - \frac{1}{4} \cos 2x \cdot \sin 2x \quad \text{—— (2)}$$

from (1) & (2)

$y = y_c + y_p$ gives the general solution of DE.

8.(b) A particle moves in a straight line, its acceleration directed towards a fixed point O in the line and is alway equal to $\mu\left(\frac{a^5}{x^2}\right)^{1/3}$ when it is at a distance $x$ from O. If it starts from rest at a distance 'a' from O, then prove that it will arrive at O with a velocity $a\sqrt{6\mu}$ after time $\frac{8}{15}\sqrt{\frac{6}{\mu}}$. (10)

Sol: Acceleration, $\frac{d^2x}{dt^2} = -\mu \cdot \frac{a^{5/3}}{x^{2/3}}$

$$\int_0^{v_0} 2\left(\frac{d^2x}{dt^2}\right)\cdot\frac{dx}{dt} = -2\mu\,\frac{a^{5/3}}{x^{2/3}}\cdot\frac{dx}{dt}$$

Integrating both sides w.r.t. $\left(\frac{dx}{dt}\right)$ from rest to final point (0).

$$\left(\frac{dx}{dt}\right)^2\Bigg|_0^{v_0} = -2\mu a^{5/3}\,\frac{x^{1/3}}{1/3}\Bigg|_a^0$$

$$v_0^2 = 6\mu a^{5/3}\cdot a^{1/3} = 6\mu a^2$$

$$v_0 = a\sqrt{6\mu}$$

Also, $$\left(\frac{dx}{dt}\right)^2\Bigg|_0^{dx/dt} = -6\mu a^{5/3}\cdot x^{1/3}\Bigg|_a^x$$

$$\left(\frac{dx}{dt}\right)^2 = 6\mu a^{5/3}\left(a^{1/3} - x^{1/3}\right)$$

$$\frac{dx}{\sqrt{a^{1/3} - x^{1/3}}} = -\sqrt{6\mu a^{5/3}}\, dt$$

$$\int_a^0 \frac{dx}{(a^{1/3} - x^{1/3})^{1/2}} = -\int_0^{t_0} \sqrt{6\mu a^{5/3}}\, dt$$

Put, $x^{1/3} = a^{1/3}\sin^2\theta \Rightarrow x = a\sin^6\theta$

$dx = 6a\sin^5\theta\cos\theta\, d\theta$

$x = a \to \theta = \pi/2$

$x = 0 \to \theta = 0$

$$\int_{\pi/2}^{0} \frac{6a\sin^5\theta\cos\theta\, d\theta}{a^{1/6}(1-\sin^2\theta)^{1/2}}\, d\theta = -\int_0^{t_0} \sqrt{6\mu a^{5/3}}\, dt$$

$$\sqrt{6\mu a^{5/3}}\; t_0 = \int_0^{\pi/2} 6a^{5/6} \cdot \sin^5\theta\, d\theta$$

$$\sqrt{6\mu}\; a^{5/6} t_0 = 6a^{5/6} \cdot \frac{4 \cdot 2}{1 \cdot 3 \cdot 5} = \frac{16}{5} a^{5/6}$$

$$\therefore \quad t_0 = \frac{16}{5} \cdot \frac{1}{\sqrt{6\mu}}$$

$$\boxed{t_0 = \frac{8}{15} \cdot \sqrt{\frac{6}{\mu}}}$$

8.(c) Find the curvature and torsion of the circular helix, $\vec{r} = a(\cos\theta, \sin\theta, \theta\cot\beta)$, $\beta$ is the constant angle at which it cuts its generators. (10)

Sol: Curvature,
$$\kappa = \frac{\left|\frac{d\vec{r}}{d\theta} \times \frac{d^2\vec{r}}{d\theta^2}\right|}{\left|\frac{d\vec{r}}{d\theta}\right|^3} \quad -(1)$$

Torsion,
$$\tau = \frac{\left[\frac{d\vec{r}}{d\theta} \quad \frac{d^2\vec{r}}{d\theta^2} \quad \frac{d^3\vec{r}}{d\theta^3}\right]}{\left|\frac{d\vec{r}}{d\theta} \times \frac{d^2\vec{r}}{d\theta^2}\right|^2} \quad -(2)$$

$$\vec{r} = a(\cos\theta\, i + \sin\theta\, j + \theta\cot\beta\, k)$$

$$\frac{d\vec{r}}{d\theta} = a(-\sin\theta\, i + \cos\theta\, j + \cot\beta\, k)$$

$$\frac{d^2\vec{r}}{d\theta^2} = a(-\cos\theta\, i - \sin\theta\, j)$$

$$\frac{d^3\vec{r}}{d\theta^3} = a(\sin\theta\, i - \cos\theta\, j)$$

$$\frac{d\vec{r}}{d\theta} \times \frac{d^2\vec{r}}{d\theta^2} = \begin{vmatrix} i & j & k \\ -a\sin\theta & a\cos\theta & a\cot\beta \\ -a\cos\theta & -a\sin\theta & 0 \end{vmatrix}$$

$$= i(a^2\sin\theta\cot\beta) - j(a^2\cos\theta\cot\beta) + k(a^2\sin^2\theta + a^2\cos^2\theta)$$

$$= a^2\left[(\sin\theta\cot\beta)i - (\cos\theta\cot\beta)j + k\right]$$

$$\left|\frac{d\vec{r}}{d\theta}\times\frac{d^2\vec{r}}{d\theta^2}\right| = a^2\sqrt{(\sin\theta\cot\beta)^2+(\cos\theta\cot\beta)^2+1^2}$$

$$= a^2\sqrt{1+\cot^2\beta} = a^2\operatorname{cosec}\beta$$

$$\left|\frac{d\vec{r}}{d\theta}\right| = a\sqrt{\sin^2\theta+\cos^2\theta+\cot^2\beta} = a\operatorname{cosec}\beta$$

$$\therefore\ K = \frac{a^2.\operatorname{cosec}\beta}{(a\operatorname{cosec}\beta)^3} = \frac{1}{a}\sin^2\beta$$

For torsion, scalar triple product is

$$\left[\frac{d\vec{r}}{d\theta}\ \frac{d^2\vec{r}}{d\theta^2}\ \frac{d^3\vec{r}}{d\theta^3}\right] = \begin{vmatrix} -a\sin\theta & a\cos\theta & a\cot\beta \\ -a\cos\theta & -a\sin\theta & 0 \\ a\sin\theta & -a\cos\theta & 0 \end{vmatrix}$$

$$= a\cot\beta(a^2\cos^2\theta+a^2\sin^2\theta)$$

$$= a^3\cot\beta$$

$$\tau = \frac{a^3\cot\beta}{(a^2\operatorname{cosec}\beta)^2} = \frac{1}{a}\cdot\frac{\cos\beta}{\sin\beta}\times\sin^2\beta$$

$$= \frac{1}{a}\sin\beta\cos\beta$$

8.(d) If the tangent to a curve makes a constant angle $\alpha$, with a fixed line, then prove that $\kappa\cos\alpha \pm \tau\sin\alpha = 0$.

Conversely, if $\frac{\kappa}{\tau}$ is constant, then show that the tangent makes a constant angle with a fixed direction.

Sol: Let e, be the unit vector parallel to the given fixed line so that, as given

$$t.e = \cos\alpha \qquad \text{— (1)}$$

Differentiating, we get

$$\frac{dt}{ds}.e = 0 \quad \text{or} \quad \kappa\, n.e = 0 \qquad (\text{Frenet's First})$$

$$\therefore \quad n.e = 0 \qquad \text{— (2)}$$

Hence, n is $\perp$ to e. Thus the vectors b, t, e are coplanar.

$$\therefore \quad b.e = \pm\sin\alpha \qquad \text{— (3)}$$

Differentiating (2) and applying Frenet's second

$$\frac{dn}{ds}.e = 0 \quad \text{ie} \quad -(\kappa t + \tau b).e = 0$$

$$\therefore \quad \kappa\cos\alpha \pm \tau\sin\alpha = 0 \qquad \text{From (1) \& (3)}$$

<u>Conversely:</u>

Let $\frac{\kappa}{\tau} = \frac{1}{a}$, a is some scalar constant

or $\frac{1}{\kappa} = \frac{a}{\tau}$ ie $\sigma = a\rho$

As,

$$\frac{dt}{ds} = \frac{1}{\rho} n \quad \text{and} \quad \frac{db}{ds} = \frac{1}{\sigma} n$$

$$\therefore \quad \rho \frac{dt}{ds} = n = \sigma \frac{db}{ds}$$

or

$$\frac{dt}{ds} = \frac{\sigma}{\rho} \cdot \frac{db}{ds} = a \frac{db}{ds}$$

Integrating, we get

$$t = a\mathbf{b} + c,$$

where c is a constant vector

multiplying scalarly with t, we get

$$t \cdot t = ab \cdot t + c \cdot t$$

$$1 = 0 + ct \quad \text{ie} \quad t \cdot c = 1$$

Hence, the tangent makes a constant angle with the direction of the fixed vector c.