Previous Years' Papers (Solved)

IFS MATHEMATICS MAIN EXAM., 2014

PAPER-I

Instructions: *Candidates should attempt Question Nos. 1 and 5 which are compulsory and any THREE of the remaining questions, selecting at least ONE question from each Section. All questions carry equal marks. Marks allotted to parts of a question are indicated against each. Answers must be written in ENGLISH only. Assume suitable data, if considered necessary, and indicate the same clearly. Unless indicated otherwise, symbols and notations carry their usual meaning.*

Section-A

1. (*a*) Show that $u_1 = (1, -1, 0)$, $u_2 = (1, 1, 0)$ and $u_3 = (0, 1, 1)$ form a basis for R^3. Express $(5, 3, 4)$ in terms of u_1, u_2 and u_3. 8

(*b*) For the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Prove that

$A^n = A^{n-2} + A^2 - 1,\ n \geq 3.$ 8

(*c*) Show that the function given by

$$f(x) = \begin{cases} \dfrac{x\left(e^{1/x} - 1\right)}{\left(e^{1/x} + 1\right)}, & x \neq 0 \\ 0 & , x = 0 \end{cases}$$

is continuous but not differentiable at $x = 0$. 8

(*d*) Evaluate $\iint\limits_R y \dfrac{\sin x}{x} \, dx\, dy$ over R, where $R = \{(x, y) : y \leq x \leq \pi/2,\ 0 \leq y \leq \pi/2\}$. 8

(*e*) Prove that the locus of a variable line which intersects the three lines:
$y = mx, z = c; y = -mx, z = -c; y = z, mx = -c$
is the surface $y^2 - m^2x^2 = z^2 - c^2$. 8

2. (*a*) Let $B = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$. Find all eigen values and corresponding eigen vectors of B viewed as a matrix over:

(*i*) the real field $\mathbb{R}$

(*ii*) the complex field C. 10

(*b*) If $xyz = a^3$, then show that the minimum value of $x^2 + y^2 + z^2$ is $3a^2$. 10

(*c*) Prove that every sphere passing through the circle $x^2 + y^2 - 2ax + r^2 = 0, z = 0$ cut orthogonally every sphere through the circle $x^2 + z^2 = r^2, y = 0$. 10

(*d*) Show that the mapping $T : V_2(\bar{R}) \to V_3(\bar{R})$ defined as $T(a, b) = (a + b, a - b, b)$ is a linear transformation. Find the range, rank and nullity of T. 10

3. (*a*) Examine whether the matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ is diagonalizable. Find all eigen values. Then obtain a matrix P such that $P^{-1}AP$ is a diagonal matrix. 10

(*b*) A moving plane passes through a fixed point (2, 2, 2) and meets the coordinate axes at the points A, B, C, all away from the origin O. Find the locus of the centre of the sphere passing through the points O, A, B, C. 10

(*c*) Evaluate the integral

$$I = \int_0^\infty 2^{-ax^2} dx$$

using Gamma function. 10

(*d*) Prove that the equation:
$4x^2 - y^2 + z^2 - 3yz + 2xy + 12x - 11y + 6z + 4 = 0$ represents a cone with vertex at $(-1, -2, -3)$. 10

4. (*a*) Let f be a real valued function defined on $[0, 1]$ as follows:

$$f(x) = \begin{cases} \frac{1}{a^{r-1}}, & \frac{1}{a^r} < x \le \frac{1}{a^{r-1}}, r = 1, 2, 3, \ldots \\ 0 & x = 0 \end{cases}$$

where a is an integer greater than 2. Show that $\int_0^1 f(x)\,dx$ exists and is equal to $\frac{a}{a+1}$. 10

(*b*) Prove that the plane $ax + by + cz = 0$ cuts the cone $yz + zx + xy = 0$ in perpendicular lines if $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$. 10

(*c*) Evaluate the integral $\iint_R \frac{y}{\sqrt{x^2 + y^2 + 1}}\,dx\,dy$ over the region R bounded between $0 \le x \le \frac{y^2}{2}$ and $0 \le y \le 2$. 10

(*d*) Consider the linear mapping $F : R^2 \to R^2$ given as $F(x, y) = (3x + 4y, 2x - 5y)$ with usual basis.
Find the matrix associated with the linear transformation relative to the basis $S = \{u_1, u_2\}$ where $u_1 = (1, 2)$, $u_2 = (2, 3)$. 10

Section-B

5. (*a*) Solve the differential equation:

$$y = 2px + p^2 y, \; p = \frac{dy}{dx}$$

and obtain the non-singular solution. 8

(*b*) Solve:

$$\frac{d^4 y}{dx^4} - 16y = x^4 + \sin x.$$ 8

(*c*) A particle whose mass is m, is acted upon by a force $m\mu\left(x + \frac{a^4}{x^3}\right)$ towards the origin. If it starts from rest at a distance 'a' from the origin, prove that it will arrive at the origin in time $\frac{\pi}{4\sqrt{\mu}}$. 8

(*d*) A hollow weightless hemisphere filled with liquid is suspended from a point on the rim of its base. Show that the ratio of the thrust on the plane base to the weight of the contained liquid is $12 : \sqrt{73}$. 8

(*e*) For three vectors show that:

$$\bar{a} \times (\bar{b} \times \bar{c}) + \bar{b} \times (\bar{c} \times \bar{a}) + \bar{c} \times (\bar{a} \times \bar{b}) = 0.$$ 8

6. (*a*) Solve the following differential equation:

$$\frac{dy}{dx} = \frac{2y}{x} + \frac{x^3}{y} + x \tan\frac{y}{x^2}.$$ 10

(*b*) An engine, working at a constant rate H, draws a load M against a resistance R. Show that the maximum speed is H/R and the time taken to attain half of this speed is $\frac{MH}{R^2}\left(\log 2 - \frac{1}{2}\right)$. 10

(*c*) Solve by the method of variation of parameters:
$y'' + 3y' + 2y = x + \cos x$. 10

(*d*) For the vector $\bar{A} = \frac{x\hat{i} + y\hat{j} + z\hat{k}}{x^2 + y^2 + z^2}$ examine if $\bar{A}$ is an irrotational vector. Then determine ϕ such that $\bar{A} = \nabla\phi$. 10

7. (*a*) A solid consisting of a cone and a hemisphere on the same base rests on a rough horizontal table with the hemisphere in contact with the table. Show that the largest height of the cone so that the equilibrium is stable is $\sqrt{3} \times$ radius of hemisphere. 15

(*b*) Evaluate $\iint_S \nabla \times \bar{A} \cdot \bar{n}\, dS$ for

$\bar{A} = (x^2 + y - 4)\hat{i} + 3xy\,\hat{j} + (2xz + z^2)\hat{k}$ and S is the surface of hemisphere $x^2 + y^2 + z^2 = 16$ above xy plane. **15**

(*c*) Solve the D.E. :

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2y = e^x + \cos x. \quad \textbf{10}$$

8. (*a*) A semi circular disc rests in a vertical plane with its curved edge on a rough horizontal and equally rough vertical plane. If the coeff. of friction is μ, prove that the greatest angle that the bounding diameter can make with the horizontal plane is:

$$\sin^{-1}\left(\frac{3\pi}{4}\,\frac{\mu + \mu^2}{1 + \mu^2}\right). \quad \textbf{15}$$

(*b*) A body floating in water has volumes V_1, V_2 and V_3 above the surface when the densities of the surrounding air are ρ_1, ρ_2, ρ_3 respectively. Prove that:

$$\frac{\rho_2 - \rho_3}{V_1} + \frac{\rho_3 - \rho_1}{V_2} + \frac{\rho_1 - \rho_2}{V_3} = 0. \quad \textbf{10}$$

(*c*) Verify the divergence theorem for $\bar{A} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ over the region $x^2 + y^2 = 4$, $z = 0$, $z = 3$. **15**

PAPER II

Now two lines will be at right angle

if $\quad l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

i.e.
$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$$

5. (*a*) We have

$$y = 2xp + yp^2 \qquad ...(i)$$

$$\Rightarrow \quad 2xp = y - yp^2$$

$$\Rightarrow \quad x = \frac{y}{2p} - \frac{py}{2} \qquad ...(ii)$$

Differentiating (*ii*) w.r.t. *y*, we get

$$\frac{dx}{dy} = \frac{1}{2}\left(\frac{1}{p}\cdot 1 + y\cdot -\frac{1}{p^2}\frac{dp}{dy}\right)$$

$$-\frac{1}{2}\left(p\cdot 1 + y\frac{dp}{dy}\right)$$

$$\Rightarrow \quad \frac{1}{p} = \frac{1}{2p} - \frac{y}{2p^2}\frac{dp}{dy} - \frac{p}{2} - \frac{y}{2}\frac{dp}{dy}$$

$$\Rightarrow \quad \frac{1}{2p} - \frac{p}{2} = \left(\frac{y}{2} - \frac{y}{2p^2}\right)\frac{dp}{dy}$$

$$\Rightarrow \quad -\left(\frac{p}{2} - \frac{1}{2p}\right) = \frac{y}{p}\left(\frac{p}{2} - \frac{1}{2p}\right)\frac{dp}{dy}$$

$$\Rightarrow \quad -1 = \frac{y}{p}\frac{dp}{dy}$$

$$\Rightarrow \quad \frac{dp}{p} + \frac{dy}{y} = 0$$

Integrating,

$$\log p + \log y = \log c$$

$$\Rightarrow \quad py = c$$

$$\Rightarrow \quad p = c/y$$

Putting the value of *p* in (*i*), we get

$$y = 2x\left(\frac{c}{y}\right) - y\left(\frac{c}{y}\right)^2$$

$$\Rightarrow \quad y^2 = 2cx - c^2$$

$$\Rightarrow y^2 - 2cx + c^2 = 0.$$

5. (*c*) Given $\dfrac{d^2x}{dt^2} = -\mu\left[x + \dfrac{a^4}{x^3}\right]$, $\qquad ...(i)$

the –ve sign being taken because the force is attractive.

Integrating it after multiplying throughout by 2 (*dx*/*dt*), we get

$$\left(\frac{dx}{dt}\right)^2 = \mu\left[-x^2 + \frac{a^4}{x^2}\right] + C.$$

When $x = a$, $dx/dt = 0$, so that $C = 0$.

$$\therefore \quad \left(\frac{dx}{dt}\right)^2 = \mu\left[\frac{a^4 - x^4}{x^2}\right]$$

or
$$\frac{dx}{dt} = -\frac{\sqrt{\mu}\sqrt{(a^4 - x^4)}}{x}, \qquad ...(ii)$$

the –ve sign is taken because the particle is moving in the direction of *x* decreasing. If t_1 be the time taken to reach the origin, then integrating (*ii*), we get

$$t_1 = -\frac{1}{\sqrt{\mu}}\int_a^0 \frac{x}{\sqrt{(a^4 - x^4)}}dx = \frac{1}{\sqrt{\mu}}\int_0^a \frac{x\,dx}{\sqrt{(a^4 - x^4)}}.$$

Put $x^2 = a^2 \sin\theta$ so that $2x\,dx = a^2\cos\theta\,d\theta$.

When $x = 0$, $\theta = 0$ and when $x = a$, $\theta = \dfrac{\pi}{2}$.

$$\therefore \quad t_1 = \frac{1}{\sqrt{\mu}}\int_0^{\pi/2} \frac{\frac{1}{2}a^2\cos\theta\,d\theta}{a^2\cos\theta}$$

$$= \frac{1}{2\sqrt{\mu}}\int_0^{\pi/2} d\theta = \frac{1}{2\sqrt{\mu}}[\theta]_0^{\pi/2}$$

$$= \frac{1}{2\sqrt{\mu}}\cdot\frac{\pi}{2}$$

$$= \frac{\pi}{4\sqrt{\mu}}.$$

5. (*e*) $\bar{a}\times(\bar{b}\times\bar{c})+\bar{b}\times(\bar{c}\times\bar{a})+\bar{c}\times(\bar{a}\times\bar{b}) = 0$

$$(\bar{a}\cdot\bar{c})\cdot\bar{b}-(\bar{a}\cdot\bar{b})\cdot\bar{c}+(\bar{b}\cdot\bar{a})\cdot\bar{c}-(\bar{b}\cdot\bar{c})\cdot\bar{a}$$

$$+(\bar{c}\cdot\bar{b})\cdot\bar{a}-(\bar{c}\cdot\bar{a})\cdot\bar{b} = 0$$

7. (*b*) The boundary C of the surface S is the circle $x^2 + y^2 = a^2$, $z = 0$. Suppose $x = a\cos t$, $y = a\sin t$, $z = 0$, $0 \le t \le 2\pi$ are the parametric equations of C. By Stoke's theorem, we have

$$\iint_S (\nabla\times \bar{A})\cdot\bar{n}\,ds = \int_C \bar{A}\cdot\bar{dr}$$

$$= \int_C (x^2+y-4)\hat{i}+3xy\hat{j}+(2xz+z^2)\hat{k}\cdot(\hat{i}dx+\hat{j}dy+\hat{k}dz)$$

$$= \int_C (x^2+y-4)dx+3xy\cdot dy+(2xz+z^2)\cdot dz$$

$$= \int_C (x^2+y-4)dx+3xy\cdot dy \quad [\because z=0 \therefore dz=0]$$

$$= \int_0^{2\pi}(a^2\cos^2 t+a\sin t-4)(-a\sin t)\cdot dt$$

$$+3a\cos t\cdot a\sin t(a\cos t).dt$$

$$= \int_0^{2\pi}\left[2a^3\cdot\cos^2 t\cdot\sin t-\frac{a^2}{2}(1-\cos 2t)+4a\sin t\right].dt$$

$$= \frac{-2a^3}{3}(\cos^3 t)_0^{2\pi}-\frac{a^2}{2}[t]_0^{2\pi}+\frac{a^2}{4}[\sin 2t]_0^{2\pi}-4a[\cos t]_0^{2\pi}$$

$$= 0 - a^2.\pi + 0 - 0 = -16\pi$$

$$(\because a = 4 \Rightarrow a^2 = 16)$$

8. (*c*) The divergence theorem is

$$\iiint_V \nabla\bullet A dv = \iint_S A\bullet\hat{n}ds.$$

Now volume integral

$$= \iiint_V\left[\left\{\frac{\partial}{\partial x}\hat{i}+\frac{\partial}{\partial x}\hat{j}+\frac{\partial}{\partial x}\hat{k}\right\}\bullet\{4x\hat{i}-2y^2\hat{j}+z^2\hat{k}\}dx\,dy\,dz\right]$$

$$= \iiint_V\left(\frac{\partial}{\partial x}(4x)+\frac{\partial}{\partial y}(-2y^2)+\frac{\partial}{\partial z}(z^2)\right)dx\,dy\,dz$$

$$= \int_{x=-2}^{2}\int_{-y_1}^{y_1}\int_{z=0}^{3}(4-4y+2z)dx\,dy\,dz$$

where $y_1 = \sqrt{(4-x^2)}$

$$= \int_{x=-2}^{2}\int_{z=0}^{3}\int_{y=-\sqrt{(4-x^2)}}^{\sqrt{(4-x^2)}}(4-4y+2z)dx\,dz\,dy$$

$$= \int_{x=-2}^{2}\int_{z=0}^{3}\left[4y-2y^2+2zy\right]_{-\sqrt{(4-x^2)}}^{\sqrt{(4-x^2)}}dx\,dy$$

$$= 2\int_{-2}^{2}\int_0^3[(4+2z)y]_0^{\sqrt{(4-x^2)}}dx\,dz$$

$$= 2\int_{-2}^{2}\int_0^3\left\{(4+2z)\sqrt{(4-x^2)}\right\}dx\,dz$$

$$= 4\int_0^2\int_0^3(4+2z)\sqrt{(4-x^2)}\,dx\,dz$$

$$= 4\left[\left\{4z+z^2\right\}_0^3\left\{\frac{x}{2}\sqrt{(4-x^2)}+2\sin^{-1}\left(\frac{x}{2}\right)\right\}_0^2\right]$$

$= 4[(12 + 9)(\pi)] = 84\pi$.

Now we proceed to find the surface integral.

The surface S of the cylinder consists of a base $S_1(z = 0)$, the top $S_2(z = 3)$ and the convex portion $S_3(x^2 + y^2 = 4)$.

The given surface integral can be written as

$$\iint_S A\bullet\hat{n}ds$$

$(\because A_x = 4xi - 2y^2j$ and $n_1 = k)$

$$\iint A_y\bullet n_2 ds_2 = \iint(4xi-2y^2j+3^2k)\bullet k ds_2$$

$(\because A_x = 4xi - 2y^2j + 3^2k$ and $n_2 = k)$

$$= 9\iint_{S_2}ds_2 = 9S_2 = 9.4\pi = 36\pi$$

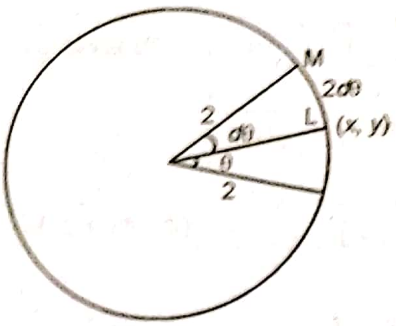
$(\because S_2 =$ circumference of the circle of radius $2 = 4\pi)$

$x = 2\cos\theta$

$y = 2\sin\theta$ $h = (x, y)$

Hence, surface area of element at L.

(width 2 $d\theta$ and height dz) is



$2d\theta\, dz = ds_3$ and $\iint A_z \cdot n_3\, ds_3$

$$= \iint_{S_3} \left[\{4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}\}\cdot\left\{\frac{2x\hat{i} + 2y\hat{j}}{\sqrt{16}}\right\}\right] ds_3$$

$$= \iint_{S_3}\left\{\frac{8x^2 - 4y^2}{4}\right\} ds_3 = \iint_{S_3}(2x^2 - y^3)(2d\theta\, dz)$$

$$= \int_{\theta=0}^{2\pi}\int_{z=0}^{2}\left\{2(2\cos\theta)^2 - 2^3\sin^2\theta\right\}(2d\theta\, dz)$$

$$= 48\int_{\theta=0}^{2\pi}(\cos^2\theta - \sin^3\theta)\,d\theta = 48\int_{\theta=0}^{2\pi}\cos^2\theta\, d\theta$$

$$= 48.4\int_{\theta=0}^{\frac{\pi}{2}}\cos^2\theta\, d\theta = 48.4\cdot\frac{\pi}{4} = 48\pi. \quad ...(iii)$$

Adding (*i*), (*ii*), (*iii*), we get

$$\iint A\cdot\hat{n}\, ds = 36\pi + 48\pi = 84\pi.$$

Thus the divergence theorem is verified.