Chapter 5

2016

5.1 Section-A

Question-1(a) **Let** $\mathrm{T} : \mathrm{B}^3 \rightarrow \mathrm{B}^4$ **be given by** $T(x, y, z) = (2x - y, 2x + z, (z) + 2z, x + y + z)$. **Find the matrix of** T **with respect to standard basis of** EB^3 **and** R^4 **(i.e.,** $(1, 0, 0)$**,** $(0, 1, 0)$**, etc. Examine if** T **is a linear map.**

[8 Marks]

Solution: Given $T : R^3 \rightarrow R^4$,

$$T(x, y, z) = (2x - y, 2x + z, x + 2z, x + y + z)$$

$$\begin{aligned} T(1,0,0) &= (2,2,1,1) \\ &= 2(1,0,0,0) + 2(0,1,0,0) + 1(0,0,1,0) + 1(0,0,0,1) \\ T(0,1,0) &= (-1,0,0,1) \\ &= -1(1,0,0,0) + 0(0,1,0,0) + 0(0,0,1,0) + 1(0,0,0,1) \end{aligned}$$

$$\begin{aligned} T(0,0,1) &= (0,1,2,1) \\ &= 0(1,0,0,0) + 1(0,1,0,0) + 2(0,0,1,0) + 1(0,0,0,1) \end{aligned}$$

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$$\therefore [T]_\alpha^\beta = \begin{bmatrix} 2 & 2 & 1 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 2 & 1 \end{bmatrix}^{\top} = \begin{bmatrix} 2 & -1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

Let $a = (x_1, y_1, z_1)$, $\quad b = (x_2, y_2, z_2) \quad \& \quad k$ is constant.

$$\begin{aligned} T(a+b) =& T(x_1 + x_2, y_1 + y_2, z_1 + z_2) \\ =& \left[\begin{array}{l} 2(x_1 + x_2) - (y_1 + y_2), 2(x_1 + x_2) + (z_1 + z_2), \\ (x_1 + x_2) + 2(z_1 + z_2), (x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2) \end{array} \right] \end{aligned}$$

$$= \left[\begin{array}{l} (2x_1 - y_1) + (2x_2 - y_2), (2x_1 + z_1) + (2x_2 + z_2) \\ (x_1 + 2z_1) + (x_2 + 2z_2) + (x_1 + y_1 + z_1) + (x_2 + y_2 + z_2) \end{array} \right]$$

$$= T(x_1, y_1, z_1) + T(x_2, y_2, z_2) = T(a) + T(b)$$

Similary,

$$T(kx_1) = k \cdot T(x_1).$$

Hence T is linear.

Question-1(b) Show that $\dfrac{x}{(1+x)} < \log(1+x) < x$ **for** $x > 0$.

[8 Marks]

Solution: Consider the function,

$$f(x) = \log(1+x) - \frac{x}{1+x}$$

$$f'(x) = \frac{1}{1+x} - \frac{(1+x)-x}{(1+x)^2} = \frac{1}{1+x^2} > 0$$

$\therefore f(x)$ is increasing function,

$$\therefore \quad If \quad x > 0 \Rightarrow f(x) > f(0)$$

ie

$$\log(1+x) - \frac{x}{1+x} > \log(1+0) - \frac{0}{1+0}$$

ie

$$\log(1+x) > \frac{x}{1+x} - (1)$$

Again, let

$$g(x) = x - \log(1+x)$$

$$g'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x} > 0 \quad \forall x > 0$$

$\therefore g(x)$ is increasing function $\therefore$

$$for, x > 0 \quad \Rightarrow \quad f(x) > f(0)$$

ie

$$x - \log(1+x) > 0 - \log(1+0)$$
$$x > \log(1+x) - (2)$$

Combining (1) and (2),

$$\frac{x}{1+x} < \log(1+x) < x$$

Question-1(c) Examine if the function $f(x,y) = \dfrac{xy}{x^2+y^2}, (x,y) \neq (0,0)$ **and** $f(0,0) = 0$ **is continuous at** $(0,0)$**. Find** $\dfrac{\partial f}{\partial x}$ **and** $\dfrac{\partial f}{\partial y}$ **at points other than origin.**

[8 Marks]

Solution:

$$f(x)=\begin{cases}\dfrac{xy}{x^2+y^2}, & (x,y)\neq(0,0)\\ 0 & ,(x,y)=(0,0)\end{cases}$$

We show that limit does not exist at $(0,0)$.
Along the curve $y=mx$,

$$\lim_{(x,y)\to(0,0)}\frac{x(mx)}{x^2+(mx)^2}=\frac{m}{1+m^2}$$

Which is different for different values of x. Hence, limit does not exist and to f(x) is not continuous at (0,0).
For the points, other than origin

$$\frac{\partial F}{\partial x}=\frac{\partial}{\partial x}\left(\frac{xy}{x^2+y^2}\right)=\frac{y\left(x^2+y^2\right)-2x(xy)}{\left(x^2+y^2\right)^2}$$

$$=\frac{y^3-x^2y}{\left(x^2+y^2\right)^2}=\frac{y\left(y^2-x^2\right)}{\left(x^2+y^2\right)^2}$$

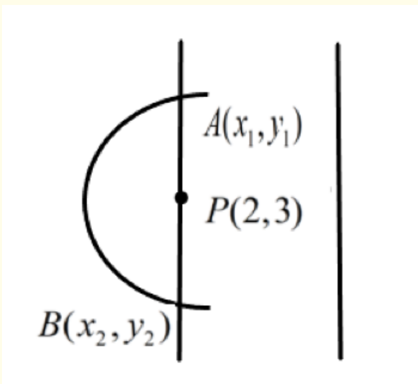
Similarly,

$$\frac{\partial F}{\partial y}=\frac{x\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2}$$

Question-1(d) If the point (2,3) is the mid-point of a chord of the parabola $y^2=4x$, then obtain the equation of the chord.

[8 Marks]

Solution: Let two points on the parabola be $A\left(x_1,y_1\right)$ & $B\left(x_2,y_2\right)$ where chord cut the parabola and $P(2,3)$ be the mid-point.



$$\therefore\quad y_1^2=4x_1-(1)\quad\&\quad y_2^2=4x_2-(2)$$

$$\frac{x_1+x_2}{2}=2\quad,\quad\frac{y_1+y_2}{2}=3$$

As

$$y_2^2-y_1^2=4x_2-4x_1$$

$$\left(y_1+y_2\right)\left(y_2-y_1\right)=4\left(x_2-x_1\right)$$

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{4}{y_1 + y_2}$$
$$= \frac{4}{6} = \frac{2}{3}$$

Slope of

$$AB = \frac{y_2 - y_1}{x_2 - x_1}$$
$$= \frac{2}{3}$$

$\therefore$ Eqn of Chord:

$$y - 3 = 2/3(x - 2)$$
$$3y - 9 = 2x - 4$$
$$2x - 3y + 5 = 0$$

Question-1(e) For the matrix $A = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$, obtain the eigenvalue and get the value of $A^4 + 3A^3 - 9A^2$.

[8 Marks]

Solution: Here, $|A - \lambda I| = 0$ gives

$$\begin{vmatrix} -1-\lambda & 2 & 2 \\ 2 & -1-\lambda & 2 \\ 2 & 2 & -1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \quad \lambda^3 + 3\lambda^2 - 9\lambda - 27 = 0$$
$$\Rightarrow \quad (\lambda + 3)\left(\lambda^2 - 9\right) = 0$$

$\therefore \lambda = -3, 3, 3$ are the eigenvalues. By Cayley-Hamilton Theorem.

$$A^3 + 3A^2 - 9A - 27I = 0$$
$$\Rightarrow \quad A^4 + 3A^3 - 9A^2 - 27A = 0$$

$$\therefore \quad A^4 + 3A^3 - 9A^2 = 27A = \begin{bmatrix} -27 & 54 & 54 \\ 54 & -27 & 54 \\ 54 & 54 & -27 \end{bmatrix}$$

Question-2(a) After changing the order of integration of $\int_0^\infty \int_0^\infty e^{-xy} \sin nx dx dy$ show that $\int_0^\infty \frac{\sin nx}{x} dx = \frac{\pi}{2}$.

[10 Marks]

Solution:

$$I = \int_0^\infty \int_0^\infty \sin nx \cdot e^{-xy} \cdot dydx$$
$$= \int_0^\infty \sin nx \cdot \left(\frac{e^{-xy}}{-x}\right)_{y=0}^{\infty} dx$$
$$= \int_0^\infty \sin nx \left(0 + \frac{1}{x}\right) dx = \int_0^\infty \frac{\sin nx}{x} dx - (1)$$

Now, first integrating w.r.t x,

$$I = \int_0^\infty \left[-\frac{1}{y} e^{-xy} \cdot \sin nx \Big|_{x=0}^{\infty} + \int_0^\infty \frac{1}{y} e^{-xy} \cdot ncosnxdx \right] dy$$
$$= \int_0^\infty \left[\frac{n}{y} \left(-\frac{1}{y} e^{-xy} \cos nx \Big|_{x=0}^{\infty} - \int_0^\infty \frac{e^{-xy}}{y} n \sin nx \right) \right] dy$$
$$= \int_0^\infty \frac{n}{y} \left(0 + \frac{1}{y} - \frac{n}{y} I' \right) dy$$
$$= \int_0^\infty \left(\frac{n}{y^2} - \frac{n^2}{y^2} I \right) dy$$
$$\therefore \qquad \frac{n}{y^2} - \frac{n^2}{y^2} I = I \Rightarrow I \left(1 + \frac{n^2}{y^2}\right) = \frac{n}{y^2}$$
$$I = \frac{n}{n^2 + y^2}$$
$$\therefore \int_0^\infty \frac{n}{n^2 + y^2} dy = \frac{1}{n} \cdot n \, tan^{-1} \frac{y}{n} \Big|_0^\infty = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$
$$\therefore I = \int_0^\infty \frac{\sin nx}{x} = \pi/2$$

Question-2(b) A perpendicular is drawn from the centre of ellipse Q $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to any tangent. Prove that the locus of the foot of the perpendicular is given-by $(x^2 + y^2)^2 = a^2x^2 + b^2y^2$.

[10 Marks]

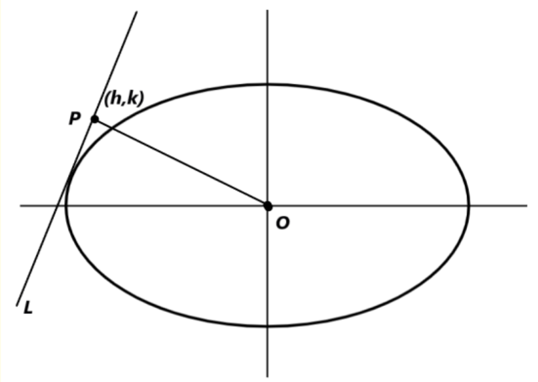
Solution: The tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

is

$$y = mx \pm \sqrt{a^2m^2 + b^2} - (1)$$

for any value of m.



Slope of line $OP = \dfrac{k-0}{h-0} = \dfrac{k}{h}$, Slope of tangent line $= -\dfrac{h}{k} \quad (OP \perp L)$

$\therefore$ Eqn of tangent line

$$\begin{aligned} y - k &= -\frac{h}{k}(x-h) \\ y &= -\frac{h}{k}x + \frac{h^2}{k} + k \\ y &= -\frac{h}{k}x + \left(\frac{h^2+k^2}{k}\right) - (2) \end{aligned}$$

Comparing Eqn (1) with (2)

$$\pm\sqrt{a^2m^2+b^2} = \frac{h^2+k^2}{k}$$

$$\left(a^2\left(\frac{-h}{k}\right)^2 + b^2\right) = \left(\frac{h^2+k^2}{k}\right)^2 \quad \left(\because m = \frac{-h}{k}\right)$$

$$\therefore a^2h^2 + b^2k^2 = \left(h^2+k^2\right)^2$$

Hence required locus:

$$\left(x^2+y^2\right)^2 = a^2x^2 + b^2y^2$$

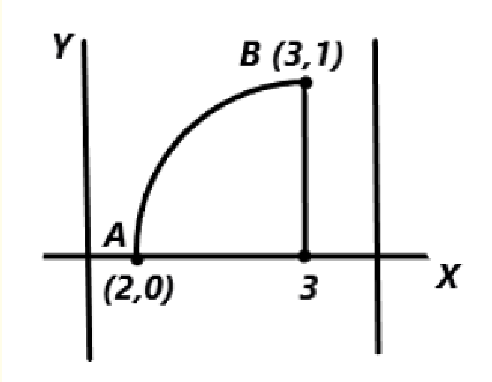
Question-2(c) Using mean value theorem, find a point on the curve $y = \sqrt{x-2}$, defined on $[2,3]$, where the tangent (is, parallel to the chord joining the end points of the curve.

[10 Marks]

Solution:

$$y = \sqrt{x-2}, \quad x \in [2,3]$$

$$y^2 = x - 2$$



End points are $A(2,0)$ and $B(3,1)$

$$y = \sqrt{x-2}, \text{is continuous on}[2,3]$$

$$y = \sqrt{x-2}, \text{is differentiable on}(2,3)$$

Hence, by Lagrange's mean value theorem (LMVT), there exists some $c \in (2,3)$ s.t.

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\frac{1}{2\sqrt{c-1}} = \frac{f(3)-f(2)}{3-2} = \frac{1-0}{1}$$

$$\Rightarrow 2\sqrt{c-1} = 1$$

$$\text{ie. } c-2 = \frac{1}{4} \Rightarrow c = \frac{9}{4}$$

Hence, at

$$x = 9/4, y = \sqrt{\frac{9}{4}-2} = \frac{1}{2},$$

tangent to the curve is parallel to the chord joining the end points as slopes are equal there.

Question-2(d) Let T be a linear map such that $\mathrm{T} : \mathrm{v}_3 \to \mathrm{v}_2$ defined by

$$\mathrm{T}(\mathrm{e}_1) = 2\mathrm{f}_1 - \mathrm{f}_2,$$

$$\mathrm{T}(\mathrm{e}_2) = \mathrm{f}_1 + 2\mathrm{f}_2,$$

$$\mathrm{T}(\mathrm{e}_3) = 0\mathrm{f}_1 + 0\mathrm{f}_2,$$

where $\mathrm{e}_1, \mathrm{e}_2, \mathrm{e}_3$ and $\mathrm{f}_1, \mathrm{f}_2$ are standard basis in V_3 and V_2.
Find the matrix of T relative to these basis.
Further take two other basis $\mathrm{B}_1[(1,1,0),(1,0,1),(0,1,1)]$ and $\mathrm{B}_2[(1,1),(1,-1)]$. Obtain the matrix T_1 relative to B_1 and B_2.

[10 Marks]

Solution:

$$T(e_1) = 2f_1 - f_2$$

$$T\left(e_2\right)=f_1+2f_2$$
$$T\left(e_3\right)=0f_1+0f_2$$
$$T=\begin{bmatrix}2&-1\\1&2\\0&0\end{bmatrix}^{\top}=\begin{bmatrix}2&1&0\\-1&2&0\end{bmatrix}$$
$$T(a,b,c)=\begin{bmatrix}2&1&0\\-1&2&0\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}2a+b\\-a+2b\end{bmatrix}$$
$$T(1,1,0)=(3,1)=x_1(1,1)+y_1(1,-1)$$
$$T(1,0,1)=(2,-1)=x_2(1,1)+y_2(1,-1)$$
$$T(0,1,1)=(1,2)=x_3(1,1)+y_3(1,-1)$$
$$\therefore\quad x_1=2,y_1=1,\quad x_2=\frac{1}{2},y_2=\frac{3}{2},\quad x_3=\frac{3}{2},y_3=\frac{-1}{2}$$
$$\therefore\quad [T]_{B_1}^{B_2}=\begin{bmatrix}2&1\\1/2&3/2\\3/2&-1/2\end{bmatrix}^{\top}=\begin{bmatrix}2&1/2&3/2\\1&3/2&-1/2\end{bmatrix}$$

Question-3(a) For the matrix $A=\begin{bmatrix}3&-3&4\\2&-3&4\\0&-1&1\end{bmatrix}$**, find two non-singular matrices** P **and** Q **such that** $\mathrm{PAQ}=\mathrm{I}$. **Hence find** A^{-1}.

[10 Marks]

Solution:

$$IAI=A$$
$$\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}A\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}3&-3&4\\2&-3&4\\0&-1&1\end{bmatrix}$$
$$R_2\rightarrow R_2-\frac{2}{3}R_1$$
$$\begin{bmatrix}1&0&0\\-2/3&1&0\\0&0&1\end{bmatrix}A\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}3&-3&4\\0&-1&4/3\\0&-1&1\end{bmatrix}$$
$$R_3\rightarrow R_3-R_2$$
$$\begin{bmatrix}1&0&0\\-2/3&1&0\\2/3&-1&1\end{bmatrix}A\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}3&-3&4\\0&-1&4/3\\0&0&-1/3\end{bmatrix}$$
$$C_2\rightarrow C_2+C_1,C_3\rightarrow C_3-4/3C_1$$
$$\begin{bmatrix}1&0&0\\-2/3&1&0\\2/3&-1&1\end{bmatrix}A\begin{bmatrix}1&1&-4/3\\0&1&0\\0&0&1\end{bmatrix}=\begin{bmatrix}3&0&0\\0&-1&4/3\\0&0&-1/3\end{bmatrix}$$

$$C_3 \to C_3 + \frac{4}{3}C_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2/3 & 1 & 0 \\ 2/3 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4/3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1/3 \end{bmatrix}$$

$$R_1 \to R_1/3, \quad R_2 \to -R_2, R_3 \to -3R_3$$

$$\begin{bmatrix} 1/3 & 0 & 0 \\ 2/3 & -1 & 0 \\ -2 & 3 & -3 \end{bmatrix} A \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4/3 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & +1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$PAQ = I$$

$$A = P^{-1}Q^{-1}$$

$$A^{-1} = QP$$

$$\Rightarrow A^{-1} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 4/3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/3 & 0 & 0 \\ 2/3 & -1 & 0 \\ -2 & 3 & -3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$

Question-3(b) Using Lagrange's method of multipliers, find the point on the plane $2x + 3y + 4z = 5$ which is closest to the point $(1, 0, 0)$.

[10 Marks]

Solution: Let the required point be (x, y, z). Now we have to maximize

$$f(x, y, z) = (x-1)^2 + y^2 + z^2 - (1)$$

subject to

$$2x + 3y + 4z = 5 \quad -(2)$$

Let

$$g(x, y, z) = 2x + 3y + 4z - 5$$

Let λ be the Lagrange's multiplier,

$$f + \lambda g = F(x, y, z)$$

For critical points, $\partial F = 0$

$$dx = 2(x-1) + 2\lambda = 0 \quad \Rightarrow \quad x = -\lambda + 1$$

$$dy = 2y + 3\lambda = 0 \quad \Rightarrow \quad y = -\frac{3\lambda}{2}$$

$$dz = 2z + 4\lambda = 0 \quad \Rightarrow \quad z = -2\lambda$$

Using Eqn (2)

$$2(-\lambda + 1) + 3\left(-\frac{3\lambda}{2}\right) + 4(-2\lambda) = 5$$

$$\frac{-29}{2}\lambda = 3 \Rightarrow \lambda = -\frac{6}{29}$$

$$\therefore x = \frac{6}{29} + 1 = \frac{35}{29}, \quad y = \frac{9}{29}, z = \frac{12}{29}$$

Hence, the required point is

$$\left(\frac{35}{29}, \frac{9}{29}, \frac{12}{29}\right)$$

(which is the foot of the $\perp$ also).

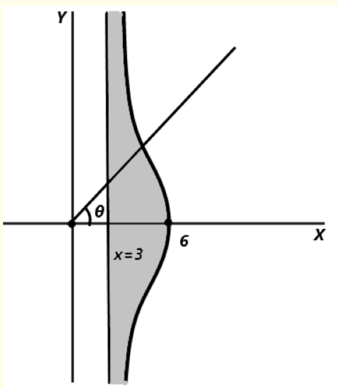
Question-3(c) Obtain the area between the curve $x = 3(\sec\theta + \cos\theta)$ and its asymptote $x = 3$.

[10 Marks]

Solution: The curve is symmetrical about the initial line and has an asymptote

$$r = 3\sec\theta$$

In the upper half of the curve θ varies from 0 to $\pi/2$.



$\therefore$The required area

$$= 2\int_0^{\pi/2}\int_{3sec\theta}^{3(sec\theta+cos\theta)} r dr d\theta$$

$$= 2\int_0^{\pi/2} \left.\frac{r^2}{2}\right|_{3\sec\theta}^{3(\sec\theta+cos\theta)} d\theta$$

$$= 2\cdot\frac{9}{2}\int_0^{\pi/2} (\sec\theta + cos\theta)^2 - \sec^2\theta d\theta$$

$$= 9\int_0^{\pi/2} \left(2 + \cos^2\theta\right) d\theta$$

$$= 9\left[(2\theta)_0^{\pi/2} + \frac{1}{2}\cdot\frac{\pi}{2}\right] = 9\cdot\frac{\pi}{2}\left(2 + \frac{1}{2}\right)$$

$$= \frac{45}{4}\pi \text{ sq. unit.}$$

Question-3(d) Obtain the equation of the sphere on which the intersection of the plane $5x - 2y + 4z + 7 = 0$ with the sphere which has $(0, 1, 0)$ and $(3, -5, 2)$ as the end points of its diameter is a great circle.

[10 Marks]

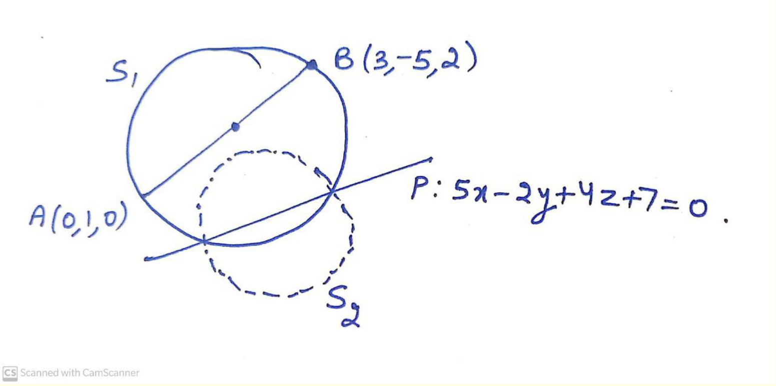
Solution:

$$r = \sqrt{\frac{9}{4} + 9 + 1} = \frac{7}{2}$$

Equation of S_1

$$\left(x - \frac{3}{2}\right)^2 + (y + 2)^2 + (z - 1)^2 = \frac{49}{4}$$

$$x^2 + y^2 + z^2 - 3x + 4y - 2z - 5 = 0$$



Equation of S_2 is :$S_1 + \lambda P = 0$

$$\left(x^2 + y^2 + z^2 - 3x + 4y - 2z - 5\right) + \lambda(5x - 2y + 4z + 7) = 0$$

$$x^2 + y^2 + z^2 + (-3 + 5\lambda)x + (4 - 2\lambda)y$$
$$+(-2 + 4\lambda)z - 5 + 7\lambda = 0$$

Centre

$$\left(\frac{3 - 5\lambda}{2}, -2 + \lambda, 1 - 2\lambda\right)$$

lies on P

$$5\left(\frac{3 - 5\lambda}{2}\right) - 2(-2 + \lambda) + 4(1 - 2\lambda) + 7 = 0$$

$$\lambda = 1$$

$\therefore$ Eqn of S_2

$$x^2 + y^2 + z^2 + 2x + 2y + 2z + 2 = 0$$

with centre (-1,-1,-1) and radius 1 .

Question-4(a) Examine whether the real quadratic form $4x^2 - y^2 + 2z^2 + 2xy - 2yz - 4xz$ is a positive definite or not. Reduce it to its diagonal form and determine its signature.

[10 Marks]

Solution: The given quadratic form can be written is:

$$\left(4x^2 + xy - 2xz\right) + \left(yx - y^2 - yz\right) + \left(-2zx - zy + 2z^2\right)$$

The matrix of this quadratic form is:

$$A = \begin{bmatrix} 4 & 1 & -2 \\ 1 & -1 & -1 \\ -2 & -1 & 2 \end{bmatrix}$$ Which is a symmetric square matrix of order 3×3

First we reduce it to its diagonal (canonical) form by writing $A = IAI$

$$\begin{bmatrix} 4 & 1 & -2 \\ 1 & -1 & -1 \\ -2 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

To avoid fraction,

$$R_2 \to 4R_2, R_3 \to 2R_3$$

$$\begin{bmatrix} 4 & 1 & -2 \\ 4 & -4 & -4 \\ -4 & -2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Perform corresponding column operations,

$$C_2 \to 4C_2, C_3 \to 2C_3$$

$$\begin{bmatrix} 4 & 4 & -4 \\ 4 & -16 & -8 \\ -4 & -8 & 8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Apply,

$$R_2 \to R_2 - R_1, R_3 \to R_3 + R_1 \quad 4 \quad C_2 \to C_2 - C_1, C_3 \to C_3 + C_1$$

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & -20 & -4 \\ 0 & -4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 4 & 0 \\ -1 & 0 & 2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 1 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$R_3 \to R_3 - \frac{1}{5}R_2, \quad C_3 \to C_3 - C_2/5$$

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & -20 & 0 \\ 0 & 0 & 24/5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 4 & 0 \\ 6/5 & -4/5 & 2 \end{bmatrix} A \begin{bmatrix} 1 & -1 & 6/5 \\ 0 & 4 & -4/5 \\ 0 & 0 & 2 \end{bmatrix}$$

Diagonal form,

$$4x^2 - 20y^2 + \frac{24}{5}z^2$$

Rank (r) of given quadratic form= No. of non zero terms in diagonal form(canonical/normal form)=3.
Signature (S) of given quadratic form = No. of positive terms − No. of negative terms=2-1=1
The index of the given quadratic form= No. of positive terms in normal form=2
Since, $r = S$ here, the given quadratic form is not positive definite.

Question-4(b) Show that the integral $\int_0^\infty \mathrm{e}^{-x}\mathrm{x}^{\alpha-1}\mathrm{dx}, \alpha > 0$ exists, by separately taking the cases for $\alpha \geq 1$ and $0 < \alpha < 1$.

[10 Marks]

Solution:

$$I = \int_0^\infty e^{-x} \cdot x^{\alpha-1} dx = \int_0^1 e^{-x} x^{\alpha-1} dx (\text{Let } I_1) + \int_1^\infty e^{-x} \cdot x^{\alpha-1} dx (\text{Let } I_2)$$

For $\alpha \geq 1, I_1$ is a proper integral
while I_2 is improper

$$I_2 = \int_1^\infty e^{-x} \cdot \alpha^{\alpha-1} dx,$$

let

$$f(x) = x^{\alpha-1} \cdot e^{-x}$$

and take

$$g(x) = \frac{1}{x^2}$$

$$\therefore \lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^{\alpha-1} \cdot e^{-x}}{1/x^2} = \lim_{x\to\infty} x^{\alpha+1} \cdot e^{-x}$$

$$= \lim_{x\to\infty} \frac{x^{\alpha+1}}{e^x} \left(\frac{\infty}{\infty} form\right)$$

$$= \frac{(\alpha+1)!}{e^x} = 0, \quad \Rightarrow convergent$$

pence I exists for $\alpha \geq 1$. For $0 < \alpha < 1$ I_1 is an improper integral & I_2 is an improper integral & point of non-convergence, $x = 0$

$$I_1 = \int_0^1 e^{-x} \cdot x^{\alpha-1} dx,$$

let

$$f(x) = \frac{e^{-x}}{x^{1-\alpha}}$$

& $\quad g(x) = \dfrac{1}{x^{1/2}}$ where $\int_0^1 \dfrac{1}{x^u} du$ is congt for $0 < u < 1$

$$\lim_{x\to\alpha} \frac{f(x)}{g(x)} = \lim_{n\to\infty} \frac{e^{-x}}{x^{1-\alpha}} x^u = \lim_{x\to\infty} \frac{e^{-x}}{x^{1-\alpha-4}}$$

$$= 0$$

$\therefore$ The integral is convergent

$$I_2 = \int_0^\infty e^{-x} \cdot x^{\alpha-1} dx, 0 < \alpha < 1$$

take

$$g(x) = \frac{1}{x^2}$$

$$\lim_{x\to\infty}\frac{e^{-x}}{x^{1-\alpha}}x^2=\frac{e^{-x}}{x^{1-\alpha-2}}=x^{1+\alpha}\cdot e^{-x}$$
$$=\frac{x^{1+\alpha}}{e^x}=\frac{(1+\alpha)x^{\alpha}}{e^x}=0\quad\left(\frac{0}{0}form\right)$$

Hence we get it convergent by Comparision Test hence integral exist for $0<\alpha<1$

Question-4(c) **Prove that** $\Gamma(2z)=\dfrac{2^{2z-1}}{\sqrt{\pi}}\Gamma(z)\Gamma\left(z+\dfrac{1}{2}\right)$

[10 Marks]

Solution: We know that

$$\beta(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)},m>0,n>0$$

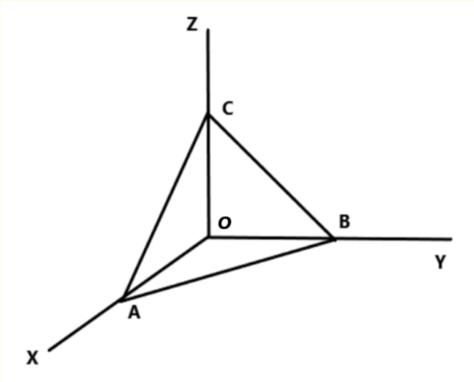
Take $m=n$

$$\beta(n,n)=\frac{(\Gamma(n))^2}{\Gamma(2n)}=2\int_0^{\pi/2}\sin^{2n-1}\theta\cos^{2n-1}\theta d\theta$$
$$=\int_0^1 x^{n-1}(1-x)^{n-1}dx\quad\left[\begin{array}{c}x=\sin^2\theta\\ dx=\sin 2\theta d\theta\end{array}\right]$$
$$B(n,n)=2\int_0^{\pi/2}(\sin\theta\cdot\cos\theta)^{2n-1}d\theta=\frac{2}{2^{2n-1}}\int_0^{\pi/2}(\sin 2\theta)^{2n-1}d\theta$$
$$=\frac{1}{2^{2n-1}}\int_0^{\pi}(\sin\alpha)^{2n-1}d\alpha\quad\left[\begin{array}{c}\text{let } 2\theta=\alpha\\ 2d\theta=d\alpha\end{array}\right]$$
$$=\frac{2}{2^{2n-1}}\cdot\int_0^{\pi/2}\sin^{2n-1}\alpha\cdot d\alpha\left[\begin{array}{c}\int_0^{2a}f(x)dx=2\int_0^a f(x)dx\\ \text{if } f(2a-x)=f(x)\end{array}\right]$$
$$=\frac{1}{2^{2n-2}}\int_0^{\pi/2}\sin^{2n-1}\alpha\cdot\cos^0\alpha d\alpha$$
$$=\frac{1}{2^{2n-2}}\cdot\frac{\Gamma(n)\cdot\Gamma(1/2)}{2\Gamma(n+1/2)}\quad\left[\ 2n-1=0\Rightarrow n=1/2\ \right]$$
$$\therefore\frac{\Gamma(n)\cdot\Gamma(n)}{\Gamma(2n)}=\frac{1}{2^{2n-2}\cdot 2}\cdot\frac{\sqrt{\pi}\cdot\Gamma(n)}{\Gamma(n+1/2)}$$
$$\therefore\quad\Gamma(2n)=\Gamma(n)\cdot\Gamma\left(n+\frac{1}{2}\right)\frac{2^{2n-1}}{\sqrt{\pi}}$$

Question-4(d) **A plane** $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=a_2$ **cuts the coordinate plane at** A, B, C. **Find the equation of the cone with vertex at origin and guiding curve as the circle passing through** A, B, C.

[10 Marks]

Solution: Let $A(a,0,0)$ $B(0,b,0)$, $C(0,0,c)$ Let Eqn of sphere passing through O,A,B,C be



$$x^2+y^2+z^2+2ux+2vy+2wz+d=0$$

$$\therefore d=0; \quad u=-\frac{a}{2}, \quad v=-\frac{b}{2}, \quad w=-\frac{c}{2}$$

$$\therefore \quad x^2+y^2+2^2-ax-by-cz=0 \qquad \ldots(1)$$

plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}$ $\qquad \ldots(2)$.

The equation of the required cone is obtained by making eqn (1) homogeneous with the help of eqn (2).

$$x^2+y^2+z^2-(ax+by+cz)\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)=0$$

$$x^2+y^2+z^2-\left(x^2+\frac{a}{b}xy+\frac{a}{c}zx+\frac{b}{a}xy+y^2+\frac{b}{c}yz\right.$$

$$\left.+\frac{c}{a}zx+\frac{c}{b}zy+z^2\right)=0$$

$$\Rightarrow \quad xy\left(\frac{a}{b}+\frac{b}{a}\right)+yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{a}{c}+\frac{c}{a}\right)=0$$

which is the required eqn of cone.

5.2 Section-B

Question-5(a) **Obtain the curve which passes through (1,2) and has a slope** $=\frac{-2xy}{x^2+1}$. **Obtain one asymptote to the curve.**

[8 Marks]

Solution: Given, $\dfrac{dy}{dx} = -\dfrac{2xy}{x^2+1}$ and curve passes through (1,2) separate variables

$$\frac{dy}{y} = -\frac{2x}{x^2+1}dx$$

Integrate on both sides

$$\int \frac{dy}{y} = -\int \frac{2x}{x^2+1}dx + c$$

$$\text{Put,} \quad x^2+1 = t$$
$$2xdx = dt$$
$$\therefore \quad \log y = -\int \frac{dt}{t} + c$$
$$\log y = -\log t + c$$
$$\log y = -\log\left(x^2+1\right) + c$$

Put,

$$x = 1, y = 2$$
$$\log 2 = -\log(2) + c$$
$$\Rightarrow \quad c = 2\log 2 = \log 4$$
$$\therefore \quad \log y = -\log\left(x^2+1\right) + \log 4$$
$$\Rightarrow \quad y = \frac{4}{x^2+1}$$

Question-5(b) Solve the ode to get the particular integral of

$$\frac{\mathrm{d}^4\mathrm{y}}{\mathrm{dx}^4} + 2\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{dx}^2} + \mathrm{y} = \mathrm{x}^2\cos\mathrm{x}$$

[8 Marks]

Solution: Sol. The auxiliary equation is $m^4 + 2m^2 + 1 = 0$, or

$$\left(m^2+1\right)^2 = 0$$

giving

$$m = \pm i, \pm i$$

$$\therefore \quad \text{C.F.} = (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x, \because e^{0x} = 1$$

And

$$P.I. = \frac{1}{D^4+2D^2+1}x^2\cos x$$

$$= \text{ Real part of } \frac{1}{\left(D^2+1\right)^2}x^2 e^{ix}, \quad \left[\because e^{ix} = \cos x + i\sin x\right]$$

$$= \text{ R.P. of } e^{ix}\frac{1}{\left\{(D+i)^2+1\right\}^2}x^2$$

$$= \text{ R.P. of } e^{ix}\frac{1}{\left(D^2+2iD\right)^2}x^2 \quad \left[\because i^2 = -1\right]$$

$$= \text{ R.P. of } e^{ix}\frac{1}{4i^2D^2[1+(D/2i)]^2}x^2$$

$$= \text{ R.P. of } -\frac{1}{4}e^{ix}\frac{1}{D^2}\left[1+\frac{D}{2i}\right]^{-2}x^2 \quad [\because i^2=-1]$$

$$= \text{ R.P. of } -\frac{1}{4}e^{ix}\frac{1}{D^2}\left[1-\frac{1}{2}iD\right]^{-2}x^2, \quad \left[\because \frac{1}{i}=-i\right]$$

$$= \text{ R.P. of } -\frac{1}{4}e^{ix}\frac{1}{D^2}\left[1+2\cdot\frac{1}{2}iD+3\cdot\frac{1}{4}i^2D^2+\ldots\right]x^2$$

(Expanding by binomial theorem)

$$= \text{R.P. of } -\frac{1}{4}e^{ix}\frac{1}{D^2}\left[1+iD-\frac{3}{4}D^2+\ldots\right]x^2$$

$$= \text{ R.P. of } -\frac{1}{4}e^{ix}\left[\frac{1}{D^2}+\frac{i}{D}-\frac{3}{4}+\text{ terms in } D, D^2 \text{ and so on}\right]x^2$$

$$= \text{ RP of } -\frac{1}{4}e^{ix}\left[\frac{1}{3}\frac{x^4}{4}+i\frac{1}{3}x^3-\frac{3}{4}x^2+\text{terms in } x^1, x^0\right]$$

($\because 1/D$ stands for integration w.r.t)x

$$= \text{ R.P. of } -\frac{1}{4}(\cos x+i\sin x)\left\{(1/12)x^4+\frac{1}{3}ix^3-\frac{3}{4}x^2+\text{ terms in } x^1, x^0\right\}$$

$$= -\frac{1}{4}\left\{(1/12)x^4-(3/4)x^2\right\}\cos x+\frac{1}{4}\left(\frac{1}{3}x^3\right)\sin x+\text{ terms already included in the C. F.}$$

$$= (-1/48)\left(x^4-9x^2\right)\cos x+(1/12)x^3\sin x$$

(neglecting the terms already included in the C.F.)
Hence the complete solution is

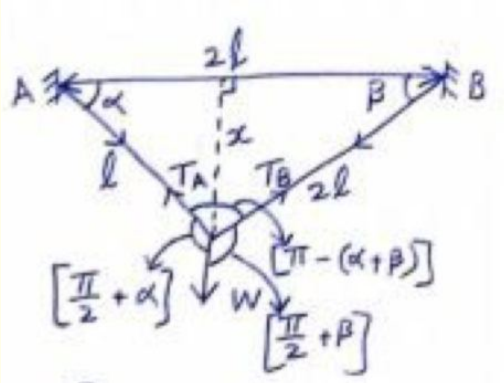
$$y = (\text{C.F.}) + (\text{P.I.})$$

$$y = (c_1+c_2x)\cos x+(c_3+c_4x)\sin x$$
$$-(1/48)\left(x^4-9x^2\right)\cos x+(1/12)x^3\sin x$$

Question-5(c) A weight W is hanging with the help of two strings of length l and $2l$ in such a way that the other ends A and B of those strings lie on a horizontal line at a distance $2l$. Obtain the tension in the two strings.

[8 Marks]

Solution:



Lami's theorem,

$$\frac{w}{\sin(\pi-(\alpha+\beta))} = \frac{T_A}{\sin\left(\frac{\pi}{2}+\beta\right)} \cdot \frac{T_B}{\sin\left(\frac{\pi}{2}+\alpha\right)}$$

$$\Rightarrow \frac{W}{\sin(\alpha+\beta)} = \frac{T_A}{\cos\beta} = \frac{T_B}{\cos\alpha} \qquad \ldots(1)$$

Using the Sine rule,

$$\frac{\sin\alpha}{2l} = \frac{\sin\beta}{\ell} = \frac{\sin(\alpha+\beta)}{2l} \qquad \ldots(2)$$

Also,

$$\cos\alpha = \frac{(2l)^2 + l^2 - (2l)^2}{2(2l)(l)} = \frac{1}{4} \Rightarrow \sin\alpha = \frac{\sqrt{15}}{4}$$

$$\cos\beta = \frac{(2l)^2 + (2l)^2 - l^2}{2(2l)(2l)} = \frac{3}{8} \Rightarrow \sin\beta = \frac{\sqrt{55}}{8}$$

$\therefore$ From (2),

$$\sin(\alpha+\beta) = \sin\alpha = \frac{\sqrt{15}}{4}$$

Putting above values in (1), we get

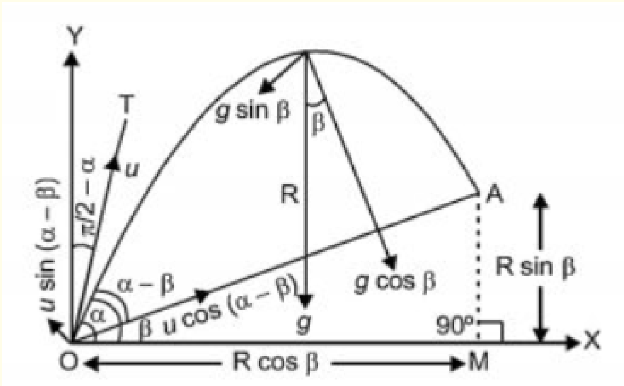
$$\Rightarrow T_A = \frac{\frac{3}{8}W}{\sqrt{15/4}} = \frac{1}{2}\sqrt{\frac{3}{5}}W$$

$$T_B = \frac{\frac{1}{4}W}{\sqrt{15/4}} = \frac{W}{\sqrt{15}}$$

Question-5(d) From a point in a smooth horizontal plane, a particle is projected with velocity u at angle α to the horizontal from the foot of a plane, inclined at an angle β with respect to the horizon. Show that it will strike the plane at right angles, if $\cot\beta = 2\tan(\alpha-\beta)$.

[8 Marks]

Solution: Suppose the particle strike the inclined plane at A. Let $OA = R$. Let T be the time of flight from O to A. As shown in the figure, the components of initial velocity of the particle along and perpendicular to the inclined plane are $u\cos(\alpha-\beta)$ and $u\sin(\alpha-\beta)$ respectively. Again, the component of g along the inclined is $g\sin\beta$ (down the plane)



and the component of g perpendicular to the inclined plane is $g\sin\beta$ (along the downward normal to the plane OA). Let time taken from O to A be T. While moving from O to A, the displacement of the particle perpendicular to OA is zero. So, considering motion of the particle from O to A perpendicular to OA and using the formula

$$s = ut + (1/2)ft^2$$

We have

$$s = u.t + \frac{1}{2}a.t^2$$

$$0 = u\sin(\alpha-\beta)\cdot T - (1/2)g\cos\beta\cdot T^2 \text{ or } T\{g$$
$$\cos\beta\cdot T - 2u\sin(\alpha-\beta)\} = 0$$

Since $T = 0$ gives time from O to O, hence time from O to A is given by $\therefore T =$ time of flight up the inclined plane

$$= \frac{2u\sin(\alpha-\beta)}{g\cos\theta} - (1)$$

Since the particle strikes the plane OA at right angles at A, hence the direction of velocity of the particle at A is perpendicular to OA and so the component of velocity of the particle at A along OA is zero. So, considering the motion of the particle from O to A along OA and using the formula.

$$V = u + a.t$$
$$O = u\cos(\alpha-\beta) - g\sin\beta.T$$
$$T = \frac{u}{g}\cdot\frac{\cos(\alpha-\beta)}{\sin\beta} \quad -(ii)$$

From (i) and (ii), we have

$$\frac{2u}{g}\cdot\frac{\sin(\alpha-\beta)}{\cos\beta} = \frac{u}{g}\cdot\frac{\cos(\alpha-\beta)}{\sin\beta}$$
$$2\tan(\alpha-\beta) = \cot\beta$$

Question-5(e) If E be the solid bounded by the xy plane and the paraboloid $z = 4 - x^2 - y^2$, then evaluate $\iint_S \overline{\mathrm{F}} \cdot d\,\mathrm{S}$, where S is the surface bounding the volume E and $\bar{F}$=$(zx\sin yz + x^3)\,\hat{i} + \cos yz\hat{j} + \left(3zy^2 - e^{\lambda^2+y^2}\right)\hat{k}$.

[8 Marks]

Solution: Given that

$$\overrightarrow{F} = \left(zx\sin yz + x^3\right)\hat{i}$$
$$+\cos yz\hat{j} + \left(3zy^2 - e^{x^2+y^2}\right)\hat{k}$$

$$divF = \frac{\partial}{\partial x}\left(xz\sin(yz) + x^3\right) + \frac{\partial}{\partial y}(\cos(yz))$$
$$+\frac{\partial}{\partial z}\left(3zy^2 - e^{x^2+y^2}\right)$$
$$= \left(z\sin(yz) + 3x^2\right) + (-z\sin(yz))$$
$$+\left(3y^2\right) = 3x^2 + 3y^2$$

Thus, we have from the divergence theorem

$$\iint_S F.dS = \iiint_E \operatorname{div} F dV$$
$$= \iint_D \int_0^{4-x^2-y^2} \left(3x^2 + 3y^2\right) dzdA$$

where D is the disk $x^2 + y^2 \leq 4$ in the xy -plane. Thus, we'll use polar coordinates for this double integral, or cylindrical coordinates for the triple integral:

$$\iint_S F.dS = \int_0^{2\pi}\int_0^2\int_0^{4-r^2}\left(3r^2\right) rdzdrd\theta$$
$$= \int_0^{2\pi}\int_0^2\left(12r^3 - 3r^5\right) drd\theta$$
$$= \int_0^{2\pi}\left[3r^4 - \frac{1}{2}r^6\right]_0^2 d\theta$$
$$= \int_0^{2\pi}(48 - 32)d\theta = 32\pi$$

Question-6(a) A stone is thrown vertically with the velocity which would just carry it to a height of 40 m. Two seconds later another stone is projected vertically from the same place with the same velocity. When and where will they meet?

[10 Marks]

Solution: Let u be the initial velocity of projection. since the greatest height is $40m$, we have

$$0 = u^2 - 2g \cdot 40$$
$$\therefore u = \sqrt{2g \times 40} = 28m$$

Let T be the time after the first stone starts before the two stones meet. Then, the distance traversed by the first stone in time T = distance traversed by the second stone in time $(T-2)$

$$\begin{aligned}\therefore 28T - \frac{1}{2}gT^2 &= 28(T-2) - \frac{1}{2}g(T-2)^2 \\ &= 28T - 56 - \frac{1}{2}g\left(T^2 - 4T + 4\right) \\ \therefore \quad 56 &= \frac{1}{2}g(4T-4) = 4.9(4T-4) \\ \therefore \quad T &= 3\frac{6}{7} \text{ seconds.}\end{aligned}$$

Also, the height at which they meet

$$\begin{aligned}&= 28 \times \frac{27}{7} - \frac{1}{2} \times 9.8 \times \left(\frac{27}{7}\right)^2 \\ &= 108 - 72.9 = 35.1m\end{aligned}$$

The first stone will be coming down and the second stone going upwards.

Question-6(b) Using the method of variation of parameters, solve

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = x^2e^x$$

[10 Marks]

Solution: Let, $y = x^m$

$$\frac{dy}{dx} = mx^{m-1}$$
$$\text{and } \frac{d^2y}{dx^2} = m(m-1)x^{m-2}$$

Now,

$$x^2 \cdot \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = 0$$
$$x^2 \cdot m(m-1) \cdot x^{m-2} + x \cdot mx^{m-1} - x^m = 0$$
$$x^m\{m(m-1) + m - 1\} = 0$$
$$x^m\left\{m^2 - 1\right\} = 0$$
$$m^2 - 1 = 0 \Rightarrow m = \pm 1$$

The general solution is then

$$y = c_1e^{-x} + c_2 \cdot e^x$$

Question-6(c) Water is flowing through a pipe of 80 mm diameter under a gauge pressure of 60kPa, with a mean velocity of 2 m/s. Find the total head, if the pipe is 7 m above the datum line.

[10 Marks]

Solution: Given Data: Diameter of pipe:

$$d = 80mm = 0.08m$$

Gauge pressure of water:

$$p = 60kPa = 60 \times 10^3 pa \text{ or } N/m^2$$

Mean velocity of water:

$$V = 2m/s$$

Datum head:

$$z = 7m$$

According to Bernoulli's equation: Total head of water:

$$\begin{aligned} H &= \frac{p}{\rho g} + \frac{V^2}{2g} + z \\ &= \frac{60 \times 10^3}{1000 \times 9.81} + \frac{(2)^2}{2 \times 9.81} + 7 \\ &= 6.11 + 0.20 + 7 \\ &= 13.31m \text{ of water} \end{aligned}$$

Question-6(d) Evaluate $\iint_S (\nabla \times \bar{f}), \hat{n} dS$ for $\bar{f} = (2x - y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$ where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ bounded by its projection on the xy plane.

[10 Marks]

Solution:

$$\begin{aligned} \int_C F \cdot dr &= \oint_C (F_x dx + F_y dy + E_z dz) \\ &= \oint_C \{(2x - y)dx - yz^2 dy - y^2 z dz\} \end{aligned}$$

But the boundary C of S is a circle in the xy -plane of radius unity and centre at (0,0,0) Hence the parametric equations of C are $x = \cos\theta, y = \sin\theta, z = 0$ where θ varies from 0 to 2π. Thus,

$$\begin{aligned}\int_C F\cdot dr &= \int_{\theta=0}^{2\pi}\{(2\cos\theta-\sin\theta)(-\sin\theta d\theta)-0-0\}\\ &= \int_0^{2\pi}(2\cos\theta-\sin\theta)\sin\theta d\theta\\ &= \int_0^{2\pi}\left(\sin 2\theta-\sin^2\theta\right)d\theta\\ &= \int_0^{2\pi}\left\{\sin 2\theta-\frac{1-\cos 2\theta}{2}\right\}d\theta\\ &= -\left[\frac{\cos 2\theta}{2}-\frac{\theta}{2}+\frac{\sin 2\theta}{2}\right]_0^{2\pi}=\pi\end{aligned}$$

Further

$$\nabla\times A=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ (2x-y) & -yz^2 & -y^2z\end{vmatrix}=k$$

Hence,

$$\iint_S(\nabla\times A)\cdot ds=\iint_S k.ds=\iint_R dxdy$$

where R is the projection of S on $xy-plane$ and $k\cdot ds=dxdy=$ projection of ds on xy -plane. Thus, R is $x^2+y^2=1$

$$\begin{aligned}\therefore\iint_R dxdy &= 4\int_0^1\int_0^1\sqrt{(1-x^2)\,dxdy}\\ &= 4\int_0^1\sqrt{(1-x^2)}dx\\ &= 4\left[\frac{x}{2}\sqrt{(1-x^2)}+\frac{1}{2}\sin^{-1}x\right]_0^1\\ &= 4\left[\frac{\pi}{4}\right]=\pi\end{aligned}$$

Thus, from above, we have $\int_C A.dr=\iint_S(\nabla\times A)\cdot ds$ and hence Stokes' Theorem is verified.

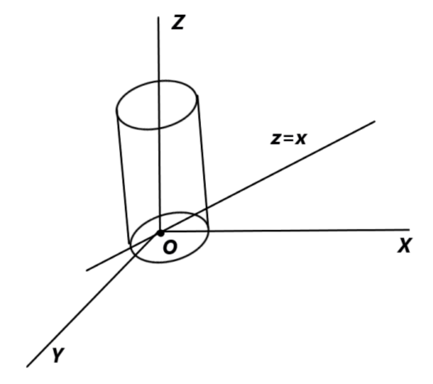
Question-7(a) State Stokes' theorem. Verify the Stokes' theorem for the function $\bar{\text{f}}=x\hat{i}+z\hat{j}+2y\hat{k}$, where c is the curve obtained by the intersection of the plane $z=x$ and the cylinder $x^2+y^2=1$ and S is the surface inside the intersected cone.

[15 Marks]

Solution: Stokes' Theorem: Let S be a closed surface, bounded by curve C, then

$$\oint_c\vec{F}\cdot dr=\iint_S(\nabla\times\vec{F})\cdot\hat{n}ds$$

$\hat{n}$ is outward unit normal the surface.



Here,

$$\vec{F} = xi + zj + 2yk$$
$$\vec{r} = xi + yj + zk$$
$$d\vec{r} = dxi + dyj + dzk$$
$$\vec{F} \cdot d\vec{r} = xdx + zdy + 2ydz$$

Surface S is intersection of cylinder $x^2 + y^2 = 1$ and plane $x = 2$ (passing through y -axis)
Boundary curve

$$C : x^2 + y^2 = 1 \quad \& z = x$$

parameterizing

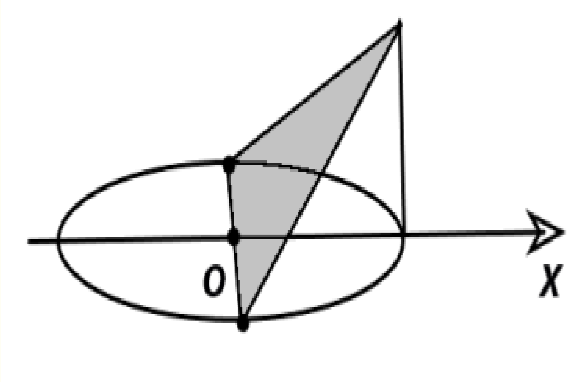
$$C : x = \cos\theta, y = \sin\theta$$
$$0 \leq \theta < 2\pi$$

$$\begin{aligned}
\oint_C F \cdot dx &= \oint xdx + zdy + 2ydz \\
&= \int_1^{2\pi} (\cos\theta)(-\sin\theta)d\theta + \cos\theta \cdot \cos\theta d\theta + 2\sin\theta(-\sin\theta)d\theta \\
\int \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} \left[\frac{-1}{2}\sin 2\theta + \left(\frac{1+\cos 2\theta}{2}\right) - 2\left(\frac{1-\cos 2\theta}{2}\right)\right] d\theta \\
&= \int_0^{2\pi} \left(\frac{-1}{2}\sin 2\theta + \frac{3}{2}\cos 2\theta - \frac{1}{2}\right) d\theta \\
&= \left[\frac{1}{4}\cos 2\theta + \frac{3}{4}\sin 2\theta - \frac{\theta}{2}\right]_0^{2\pi} \\
&= -\pi
\end{aligned}$$

Now,

$$\nabla \times \vec{F} = \begin{vmatrix} 1 & j & k \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & z & 2y \end{vmatrix}$$

$$= i(2-1) + j(0-0) + k(0-0)$$
$$= i$$



S: $x - z = 0$

$$\hat{n} = \frac{\frac{\nabla S}{|\nabla S|}}{= \frac{1}{\sqrt{2}}(i-k)}$$

$$\iint_S (\nabla \times F) \cdot \hat{n} dS$$

$$= \iint_D i \cdot \left(\frac{i-k}{\sqrt{2}}\right) \frac{dxdy}{(\hat{n} \cdot k)}$$

(Taking Projection on xy -plane)

$$D : x^2 + y^2 \leq 1$$

$$= \int\int_D \frac{1}{\sqrt{2}} \cdot \frac{dxdy}{-1/\sqrt{2}} = -\iint_D dxdy$$

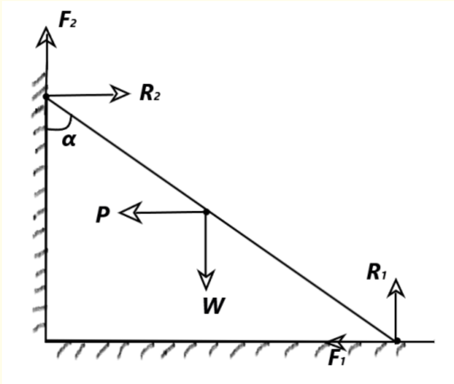
=-Area of unit circle D

$$= -\pi(1)^2 = -\pi$$

Question-7(b) A uniform rod of weight W is resting against an equally rough horizon and a wall, at an angle α with the wall. At this condition, a horizontal force P is stopping them from sliding, implemented at the mid-point of the rod. Prove that $P = W\tan(\alpha - 2\lambda)$, where λ is the angle of friction. Is there any condition on λ and α?

[15 Marks]

Solution: $\mu = \tan\lambda$ Let length (say) $F_1 = \mu R_1 - (1)$ $F_2 = \mu R_2 - (2)$



Force:

$$R_1 + F_2 = W - (3)$$
$$F_1 + P = R_2 - (4)$$

Moments about O:

$$\rightarrow R_1(a\sin\alpha) = R_2(a\cos\alpha) + F_1(a\cos\alpha) + F_2(\sin\alpha)$$

$$\Rightarrow R_1(\sin\alpha - \mu\cos\alpha) = R_2(\cos\alpha + \mu\sin\alpha)$$

From Eqn (3) and (4)

$$\Rightarrow R_2 = R_1 \times \frac{(\tan\alpha - \mu)}{(1 + \mu\tan\alpha)}$$
$$\Rightarrow R_2 = R_1\tan(\alpha - \lambda) - (5)(\because \mu = \tan\lambda)$$
$$(3) \equiv R_1 + \mu R_2 = W$$

and

$$(4) \equiv \mu R_1 + P = R_2$$

Using (5),

$$\Rightarrow R_1 + \mu\tan(\alpha - \lambda)R_1 = W - (6)$$
$$\& \quad \mu R_1 + P = R_1\tan(\alpha - \lambda)$$
$$\Rightarrow P = R_1(\tan(\alpha - \lambda) - \mu) - (7)$$
$$\frac{(7)}{(6)} \Rightarrow \frac{P}{W} = \frac{(\tan(\alpha - \lambda) - \mu)}{1 + \mu\tan(\alpha - \lambda)}$$
$$\Rightarrow \quad P = W\tan(\alpha - 2\lambda) \quad (\because \mu = \tan\lambda)$$

condition is that P should be the +ve

$$\Rightarrow \alpha > 2\lambda$$

Question-7(c) Obtain the singular solution of the differential equation

$$y^2 - 2pxy + p^2\left(x^2 - 1\right) = m^2, p = \frac{dy}{dx}$$

[10 Marks]

Solution:

$$y^2 - 2pxy + p^2x^2 = m^2 + p^2$$
$$(y - px)^2 = p^2 + m^2$$
$$y = px \pm \sqrt{p^2 + m^2}$$

It is in Clairaut's form: $y = px + f(p)$ To get the solution, we replace p by arbitrary constant c.

$$y = cx \pm \sqrt{c^2 + m^2}$$

or

$$y^2 - 2cxy + c^2x^2 = c^2 + m^2$$
$$c^2\left(x^2 - 1\right) - 2cxy + y^2 - m^2 = 0$$

C-Discriminant:

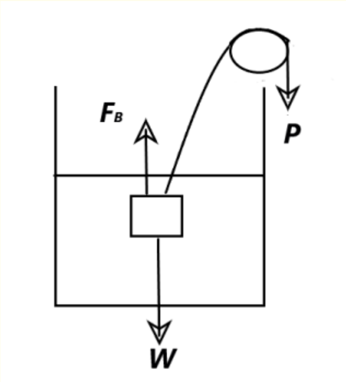
$$B^2 - 4AC$$
$$A = x^2 - 1, \quad B = -2xy, C = y^2 - m^2$$
$$\begin{aligned} \therefore B^2 - 4AC &= (-2xy)^2 - 4\left(x^2 - 1\right)\left(y^2 - m^2\right) \\ &= 4x^2y^2 - 4x^2y^2 + 4y^2 + 4x^2m^2 - 4m^2 \\ &= 4\left(y^2 + m^2\left(x^2 - 1\right)\right) \end{aligned}$$

$B^2 - 4AC = 0$ i.e. $y^2 + m^2\left(x^2 - 1\right)$ is the required singular solution of the given $D \cdot E$.

Question-8(a) A body immersed in a liquid is balanced by a weight P to which it is attached by a thread passing over a fixed pulley and when half immersed, is balanced in the same manner by weight 2P. Prove that the density of the body and the liquid are in the ratio 3 : 2?

[10 Marks]

Solution: Let ρ_s = density of body, V = volume of body. ρ_l = density of liquid. W = Weigh of body $= \rho_s V g$ & F_B = Buyount force Body immersed in liquid

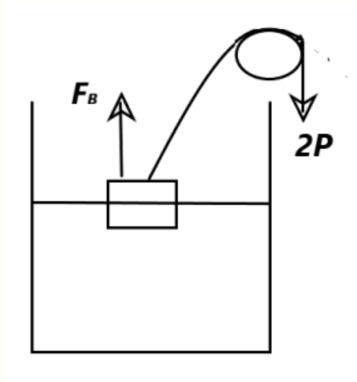


$$W = \rho_s V g$$
$$F_B = \rho_l V g$$

Balancing Forces

$$\rho_l V g + P = \rho_s V g - (1)$$

Body half immersed in liquid



$$W = \rho_s V g$$
$$F_B = \rho_l \frac{V}{2} g$$

Balancing Forces

$$\rho_l \frac{V}{2} g + 2P = \rho_s V g - (2)$$

Subtract(1) by (2)

$$P = \rho_l \frac{V}{2} g - (3)$$

Putting (3)in (1)

$$3\rho_l \frac{V}{2} g = \rho_s V g - (2)$$
$$\therefore \frac{\rho_s}{\rho_l} = \frac{3}{2}$$

Hence, Proved.

Question-8(b) Solve the differential equation

$$\frac{dy}{dx} - y = y^2(\sin x + \cos x)$$

[10 Marks]

Solution:

$$\Rightarrow \quad \frac{-1}{y^2}\frac{dy}{dx} + \frac{1}{y} = \sin x + \cos x$$

It is Bernoulli's equation. Let

$$\frac{1}{y} = z \quad , \quad \frac{-1}{y^2} \cdot \frac{dy}{dx} = \frac{dz}{dx}$$
$$\therefore \quad \frac{dz}{dx} + z = \sin x + \cos x$$

I.F.$= e^{\int 1dx} = e^x$ solution:

$$z \cdot e^x = \int e^x(\sin x + \cos x)dx$$

$$ze^x = \int e^x \sin x dx + \int e^x \cos x dx$$
$$= (\sin x)e^x - \int (\cos x)e^x dx + \int e^x \cos x dx$$

(integrating by parts)

$$= e^x \sin x + c$$
$$z = \sin x + ce^{-x}$$

i.e.

$$y\left(\sin x + ce^{-x}\right) - 1 = 0$$

is the required general solution of ODE.

Question-8(c) Prove that $\bar{a} \times (\bar{b} \times \bar{c}) = (\bar{a} \times \vec{b}) \times \vec{c}$, if and only if either $\bar{b} = \bar{0}$ or $\bar{c}$ is collinear with $\bar{a}$ or $\bar{b}$ is perpendicular to both $\bar{a}$ and $\bar{c}$.

[10 Marks]

Solution:

$$(A \times B) \times C = (A \cdot C)B - (B \cdot C)A$$
$$A \times (B \times C) = (A \cdot C)B - (A \cdot B)C$$

First, If $b = 0$, then $a \times (b \times c) = 0$ and $(a \times b) \times c = 0$, hence true. If c is collinear with a i.e. $c = \lambda a$

$$a \times (b \times c) = a \times [b \times (\lambda a)]$$
$$= [a \cdot (\lambda a)]b - [a \cdot b](\lambda a)$$
$$= \lambda\left[|a|^2 b - (a \cdot b)a\right]$$
$$(a \times b) \times c = (a \times b) \times (\lambda a)$$
$$= (a \cdot (\lambda a))b - (b \cdot (\lambda a))a$$
$$= \lambda\left(|a|^2 b - (a \cdot b)a\right]$$

$\therefore \quad a \times (b \times c) = (a \times b) \times c$. If b is $\perp$ to a and c both

$$b \cdot a = 0, \quad b \cdot c = 0$$

$$a \times (b \times c) = (a \cdot c)b - (a \cdot b)c$$
$$= (a \cdot c)b$$
$$(a \times b) \times c = (a \cdot c)b - (b \cdot c)a$$
$$= (a \cdot c)b$$
$$\therefore (a \times b) \times c = a \times (b \times c)$$

Conversely, Let

$$(a \times b) \times c = a \times (b \times c)$$

ie.

$$(a \cdot c)b - (a \cdot b)c = (a \cdot c)b - (b \cdot c)a$$
$$(b.c)a - (a \cdot b)c = 0$$

$$b \times (a \times c) = 0$$

This is possible, when either of the condition is met.
i) $b = 0$
ii) c is collinear with a, then $a \times c = 0$
iii) $b \cdot a = 0$ & $b \cdot c = 0$ i.e. b is perpendicular to both a and c.

Question-8(d) A particle is acted on a force parallel to the axis of y whose acceleration is λy, initially projected with a velocity $a\sqrt{\lambda}$ parallel to x-axis at the point where $y = a$. Prove that it will describe a catenary.

[10 Marks]

Solution: Given,

$$\frac{d^2y}{dt^2} = \lambda y$$

$$\Rightarrow \quad 2\frac{dy}{dt} \cdot \frac{d^2y}{dt^2} = 2\lambda \cdot y\frac{dy}{dt}$$

[multiplying by $2\frac{dy}{dt}$ and integrating]

$$\left(\frac{dy}{dt}\right)^2 = \lambda y^2 + C_1$$

When $t = 0$, $\frac{dy}{dt} = 0$ and $y = a$ (initial velocity is 0 in y-direction)

$$\therefore \quad C_1 = -\lambda a^2$$

$$\left(\frac{dy}{dt}\right)^2 = \lambda\left(y^2 - a^2\right)$$

$$\frac{dy}{dt} = \sqrt{\lambda}\sqrt{y^2 - a^2} - (1)$$

Also, In x -direction, $\frac{d^2x}{dt^2} = 0$ [No acceleration in x -direction]

$$\frac{dx}{dt} = C_2; t = 0, \frac{dx}{dt} = a\sqrt{\lambda} \Rightarrow C_2 = a\sqrt{\lambda}$$

$$\therefore \frac{dx}{dt} = a\sqrt{\lambda} - (2)$$

Dinding (1) by (2),

$$\frac{dy}{dx} = \frac{\sqrt{y^2 - a^2}}{a}$$

ie

$$\frac{dy}{\sqrt{y^2 - a^2}} = \frac{dx}{a} \Rightarrow \cosh^{-1}\frac{y}{a} = \frac{x}{a} + C_3$$

Initially, $x = 0$ and

$$y = a \Rightarrow c_3 = \cosh^{-1}(1) = 0$$

$$\therefore \quad y = a\cosh(x/a)$$

Eqn of catenary.