G-20 [MATHS]

LINEAR ALGEBRA ERROR FREE IFoS PYQs

All these questions are discussed /solved in Topicwise G-20 Modules

2020

1 (1a)

If A is a skew-symmetric matrix and I + A be a non-singular matrix, then show that $(I - A)(I + A)^{-1}$ is orthogonal. *8*

2 (1b)

By applying elementary row operations on the matrix

$$A = \begin{bmatrix} -1 & 2 & -1 & 0 \\ 2 & 4 & 4 & 2 \\ 0 & 0 & 1 & 5 \\ 1 & 6 & 3 & 2 \end{bmatrix},$$

reduce it to a row-reduced echelon matrix. Hence find the rank of A. *8*

3 (2a)

Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x, y, z) = (2x, -3y, x + y)$, and $B_1 = \{(-1, 2, 0), (0, 1, -1), (3, 1, 2)\}$ be a basis of $\mathbb{R}^3$. Find the matrix representation of T relative to the basis B_1. *10*

4 (3b)

When is a matrix A said to be similar to another matrix B ?

Prove that

(i) if A is similar to B, then B is similar to A.

(ii) two similar matrices have the same eigenvalues.

Further, by choosing appropriately the matrices A and B, show that the converse of (ii) above may not be true. *15*

5 (4b(i))

(i) Verify Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$, hence find its inverse. Also, express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10\,I$ as a linear polynomial in A.

6 (4b(ii))

(ii) Express the vector (1, 2, 5) as a linear combination of the vectors (1, 1, 1), (2, 1, 2) and (3, 2, 3), if possible. Justify your answer. *9+6=15*

2019

7 (1a)

1\. (a) Let $T: R^3 \to R^3$ be a linear operator on R^3 defined by $T(x, y, z) = (2y+z,\ x-4y,\ 3x)$. Find the matrix of T in the basis $\{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$. 8

8 (1b)

(b) The eigenvalues of a real symmetric matrix A are -1, 1 and -2. The corresponding eigenvectors are $\frac{1}{\sqrt{2}}(-1\ 1\ 0)^T$, $(0\ 0\ 1)^T$ and $\frac{1}{\sqrt{2}}(-1\ -1\ 0)^T$ respectively. Find the matrix A^4. 8

9 (2b)

(b) Consider the singular matrix

$$A = \begin{bmatrix} -1 & 3 & -1 & 1 \\ -3 & 5 & 1 & -1 \\ 10 & -10 & -10 & 14 \\ 4 & -4 & -4 & 8 \end{bmatrix}$$

Given that one eigenvalue of A is 4 and one eigenvector that does not correspond to this eigenvalue 4 is $(1 \ 1 \ 0 \ 0)^T$. Find all the eigenvalues of A other than 4 and hence also find the real numbers p, q, r that satisfy the matrix equation $A^4 + pA^3 + qA^2 + rA = 0$. 15

10 (3a)

3. (a) Consider the vectors $x_1 = (1, 2, 1, -1)$, $x_2 = (2, 4, 1, 1)$, $x_3 = (-1, -2, 0, -2)$ and $x_4 = (3, 6, 2, 0)$ in $\mathbb{R}^4$. Justify that the linear span of the set $\{x_1, x_2, x_3, x_4\}$ is a subspace of $\mathbb{R}^4$, defined as

$$\{(\xi_1, \xi_2, \xi_3, \xi_4) \in \mathbb{R}^4 : 2\xi_1 - \xi_2 = 0, \ 2\xi_1 - 3\xi_3 - \xi_4 = 0\}$$

Can this subspace be written as $\{(\alpha, 2\alpha, \beta, 2\alpha - 3\beta) : \alpha, \beta \in \mathbb{R}\}$? What is the dimension of this subspace? 15

11 (4a)

Using elementary row operations, reduce the matrix

$$A = \begin{bmatrix} 2 & 1 & 3 & 0 \\ 3 & 0 & 2 & 5 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 3 \end{bmatrix}$$

to reduced echelon form and find the inverse of A and hence solve the system of linear equations $AX = b$, where $X = (x, y, z, u)^T$ and $b = (2, 1, 0, 4)^T$. 15

2018

12 (1b)

Given that Adj A = $\begin{bmatrix} 2 & 2 & 0 \\ 2 & 5 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and det A = 2. Find the matrix A. *8*

13 (1e)

Prove that the eigenvalues of a Hermitian matrix are all real. *8*

14 (2b)

Show that the matrices

$A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & 1 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 2 & 2 \\ 3 & 2 & 0 \end{bmatrix}$ are congruent. *10*

15 (2d)

Show that the vectors $\alpha_1 = (1, 0, -1)$, $\alpha_2 = (1, 2, 1)$, $\alpha_3 = (0, -3, 2)$ form a basis for R^3. Express each of the standard basis vectors as a linear combination of $\alpha_1, \alpha_2, \alpha_3$. *10*

16 (3c)

Let $T : V_2(R) \to V_2(R)$ be a linear transformation defined by $T(a, b) = (a, a + b)$. Find the matrix of T, taking $\{e_1, e_2\}$ as a basis for the domain and $\{(1, 1), (1, -1)\}$ as a basis for the range. *10*

17 (4d)

If (n + 1) vectors $\alpha_1, \alpha_2, \ldots, \alpha_n, \alpha$ form a linearly dependent set, then show that the vector α is a linear combination of $\alpha_1, \alpha_2, \ldots, \alpha_n$; provided $\alpha_1, \alpha_2, \ldots, \alpha_n$ form a linearly independent set. *10*

2017

18 (1a)

Let A be a square matrix of order 3 such that each of its diagonal elements is 'a' and each of its off-diagonal elements is 1. If B = bA is orthogonal, determine the values of a and b. *8*

19 (1b)

Let V be the vector space of all 2×2 matrices over the field R. Show that W is not a subspace of V, where

(i) W contains all 2×2 matrices with zero determinant.

(ii) W consists of all 2×2 matrices A such that $A^2 = A$. *8*

20 (2a)

State the Cayley-Hamilton theorem. Verify this theorem for the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$ Hence find A^{-1}. *10*

21 (3a)

Reduce the following matrix to a row-reduced echelon form and hence find its rank : *10*

$$A = \begin{bmatrix} -1 & 2 & -1 & 0 \\ 2 & 4 & 4 & 2 \\ 0 & 0 & 1 & 5 \\ 1 & 6 & 3 & 2 \end{bmatrix}$$

22 (3b)

) Given that the set $\{u, v, w\}$ is linearly independent, examine the sets

(i) $\{u + v, v + w, w + u\}$

(ii) $\{u + v, u - v, u - 2v + 2w\}$

for linear independence. *10*

23 (4a)

Find the eigenvalues and the corresponding eigenvectors for the matrix $A = \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix}$. Examine whether the matrix A is diagonalizable. Obtain a matrix D (if it is diagonalizable) such that $D = P^{-1}AP$. *10*

2016

24 (1a)

Let $T : \mathbb{R}^3 \to \mathbb{R}^4$ be given by

$T(x, y, z) = (2x - y, \ 2x + z, \ x + 2z, \ x + y + z)$.

Find the matrix of T with respect to standard basis of $\mathbb{R}^3$ and $\mathbb{R}^4$ (i.e., (1, 0, 0), (0, 1, 0), etc.). Examine if T is a linear map. *8*

25 (1e)

For the matrix $A = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$, obtain the eigen value and get the value of $A^4 + 3A^3 - 9A^2$. *8*

26 (2d)

Let T be a linear map such that $T : V_3 \to V_2$ defined by $T(e_1) = 2f_1 - f_2$, $T(e_2) = f_1 + 2f_2$, $T(e_3) = 0f_1 + 0f_2$, where e_1, e_2, e_3 and f_1, f_2 are standard basis in V_3 and V_2. Find the matrix of T relative to these basis.

Further take two other basis $B_1[(1, 1, 0)\ (1, 0, 1)\ (0, 1, 1)]$ and $B_2[(1, 1)\ (1, -1)]$. Obtain the matrix T_1 relative to B_1 and B_2. *10*

27 (3a)

For the matrix $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, find two non-singular matrices P and Q such that $PAQ = I$. Hence find A^{-1}. *10*

28 (4a)

Q4. (a) Examine whether the real quadratic form $4x^2 - y^2 + 2z^2 + 2xy - 2yz - 4xz$ is a positive definite or not. Reduce it to its diagonal form and determine its signature. *10*

2015

29 (1a)

Find an upper triangular matrix A such that $A^3 = \begin{bmatrix} 8 & -57 \\ 0 & 27 \end{bmatrix}$ 8

30 (1b)

Let G be the linear operator on $\mathbb{R}^3$ defined by

$$G(x, y, z) = (2y + z, x - 4y, 3x)$$

Find the matrix representation of G relative to the basis

$$S = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$$ 8

31 (2a)

) Suppose U and W are distinct four-dimensional subspaces of a vector space V, where $\dim V = 6$. Find the possible dimensions of $U \cap W$. 10

32 (2b)

Find the condition on a, b and c so that the following system in unknowns x, y and z has a solution : 10

$$x + 2y - 3z = a$$
$$2x + 6y - 11z = b$$
$$x - 2y + 7z = c$$

33 (3a)

Find the minimal polynomial of the matrix $A = \begin{pmatrix} 4 & -2 & 2 \\ 6 & -3 & 4 \\ 3 & -2 & 3 \end{pmatrix}$. 10

34 (4a)

Find a 3×3 orthogonal matrix whose first two rows are $\left[\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right]$ and $\left[0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right]$. 10

2014

35 (1a)

Show that $u_1 = (1, -1, 0)$, $u_2 = (1, 1, 0)$ and $u_3 = (0, 1, 1)$ form a basis for $\mathbb{R}^3$. Express $(5, 3, 4)$ in terms of u_1, u_2 and u_3. 8

36 (1b)

For the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Prove that $A^n = A^{n-2} + A^2 - I$, $n \geq 3$. 8

37 (2a)

Let $B=\begin{bmatrix}1 & -1\\ 2 & -1\end{bmatrix}$. Find all eigen values and corresponding eigen vectors of B viewed as a matrix over :

(i) the real field R

(ii) the complex field C. 10

38 (2d)

Show that the mapping $T: V_2(\bar{R}) \to V_3(\bar{R})$ defined as $T(a, b) = (a+b, a-b, b)$ is a linear transformation. Find the range, rank and nullity of T. 10

39 (3a)

Examine whether the matrix $A=\begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{bmatrix}$ is diagonalizable. Find all eigen values.

Then obtain a matrix P such that $P^{-1}AP$ is a diagonal matrix. 10

40 (4d)

Consider the linear mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ given as $F(x, y) = (3x + 4y, 2x - 5y)$ with usual basis.

Find the matrix associated with the linear transformation relative to the basis $S = \{u_1, u_2\}$ where $u_1 = (1, 2)$, $u_2 = (2, 3)$. 10

2013

41 (1a)

Find the dimension and a basis of the solution space W of the system

$x + 2y + 2z - s + 3t = 0,\quad x + 2y + 3z + s + t = 0,\quad 3x + 6y + 8z + s + 5t = 0$ 8

42 (1b)

Find the characteristic equation of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and hence find the matrix represented by $A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$. 8

43 (2a)

) Let V be the vector space of 2×2 matrices over $\mathbb{R}$ and let $M = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$.

Let $F : V \to V$ be the linear map defined by $F(A) = MA$. Find a basis and the dimension of

(i) the kernel of W of F

(ii) the image U of F. 10

44 (2c)

Q. 2(c) Find an orthogonal transformation of co-ordinates which diagonalizes the quadratic form

$q(x, y) = 2x^2 - 4xy + 5y^2$. 10

45 (2d)

Discuss the consistency and the solutions of the equations

$x + ay + az = 1,\ ax + y + 2az = -4,\ ax - ay + 4z = 2$

for different values of a. 10

46 (3c)

Let F be a subfield of complex numbers and T a function from $F^3 \to F^3$ defined by $T(x_1, x_2, x_3) = (x_1 + x_2 + 3x_3, 2x_1 - x_2, -3x_1 + x_2 - x_3)$. What are the conditions on (a, b, c) such that (a, b, c) be in the null space of T ? Find the nullity of T. 10

47 (4b)

Let $H = \begin{bmatrix} 1 & i & 2+i \\ -i & 2 & 1-i \\ 2-i & 1+i & 2 \end{bmatrix}$ be a Hermitian matrix. Find a non-singular matrix P such that $P^t H\overline{P}$ is diagonal and also find its signature. 10

2012

48 (1a)

Let $V = I\!R^3$ and $\alpha_1 = (1, 1, 2)$, $\alpha_2 = (0, 1, 3)$, $\alpha_3 = (2, 4, 5)$ and $\alpha_4 = (-1, 0, -1)$ be the elements of V. Find a basis for the intersection of the subspace spanned by $\{\alpha_1, \alpha_2\}$ and $\{\alpha_3, \alpha_4\}$. 8

49(1b)

(b) Show that the set of all functions which satisfy the differential equation,

$\frac{d^2f}{dx^2} + 3\frac{df}{dx} = 0$ is a vector space. 8

50 (2a)

Let $f: \mathbb{R} \to \mathbb{R}^3$ be a linear transformation defined by $f(a, b, c) = (a, a+b, 0)$.

Find the matrices A and B respectively of the linear transformation f with respect to the standard basis (e_1, e_2, e_3) and the basis (e_1', e_2', e_3') where $e_1' = (1, 1, 0)$, $e_2' = (0, 1, 1)$, $e_3' = (1, 1, 1)$.

Also, show that there exists an invertible matrix P such that

$$B = P^{-1}AP$$ 10

51 (2b)

Verify Cayley–Hamilton theorem for the matrix

$A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ and find its inverse. Also express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ as a linear polynomial in A. 10

52 (2d)

Show that there are three real values of λ for which the equations:

$(a-\lambda)x + by + cz = 0$, $bx + (c-\lambda)y + az = 0$, $cx + ay + (b-\lambda)z = 0$ are simultaneously true and that the product of these values of λ is

$$D = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$ 10

53 (3a)

Find the matrix representation of linear transformation T on $V_3(\mathbb{R})$ defined as $T(a, b, c) = (2b + c, a - 4b, 3a)$ corresponding to the basis

$B = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$. 10

2011

54 (1a)

Let V be the vector space of 2×2 matrices over the field of real numbers $\mathbf{R}$. Let $W = \{A \in V \mid \text{Trace } A = 0\}$. Show that W is a subspace of V. Find a basis of W and dimension of W. *10*

55 (1b)

Find the linear transformation from $\mathbf{R}^3$ into $\mathbf{R}^3$ which has its range the subspace spanned by $(1, 0, -1), (1, 2, 2)$. *10*

56 (2a)

Let $V = \{(x, y, z, u) \in \mathbf{R}^4 : y + z + u = 0\}$,

$W = \{(x, y, z, u) \in \mathbf{R}^4 : x + y = 0,\ z = 2u\}$

be two subspaces of $\mathbf{R}^4$. Find bases for V, W, V + W and $V \cap W$. *10*

57 (2b)

Find the characteristic polynomial of the matrix

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$$

and hence compute A^{10}. *10*

58 (2c)

(c) Let $A = \begin{pmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{pmatrix}$.

Find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix. *10*

59 (2d)

(d) Find an orthogonal transformation to reduce the quadratic form $5x^2 + 2y^2 + 4xy$ to a canonical form. *10*

2010

60 (1a)

Show that the set

$$P[t] = \{at^2 + bt + c \,/\, a, b, c \in \mathbb{R}\}$$

forms a vector space over the field $\mathbb{R}$. Find a basis for this vector space. What is the dimension of this vector space ? *8*

61 (1b)

(b) Determine whether the quadratic form

$$q = x^2 + y^2 + 2xz + 4yz + 3z^2$$

is positive definite. *8*

62 (2a)

Show that the vectors

$\alpha_1 = (1, 0, -1), \quad \alpha_2 = (1, 2, 1), \quad \alpha_3 = (0, -3, 2)$

form a basis for $\mathbb{R}^3$. Find the components of $(1, 0, 0)$ w.r.t. the basis $\{\alpha_1, \alpha_2, \alpha_3\}$. 10

63 (2b)

Find the characteristic polynomial of $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}$. Verify Cayley – Hamilton theorem for this matrix and hence find its inverse. 10

64 (2c)

Let $A = \begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix}$. Find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix. 12

65 (2d)

Find the rank of the matrix

$$\begin{pmatrix} 1 & 2 & 1 & 1 & 2 \\ 2 & 4 & 3 & 4 & 7 \\ -1 & -2 & 2 & 5 & 3 \\ 3 & 6 & 2 & 1 & 3 \\ 4 & 8 & 6 & 8 & 9 \end{pmatrix}$$

8