Mains Test Series - 2020

Test - 8, Paper - II (full Syllabus)

Answer key (Batch-I)

1(a) Let $GL(2,\mathbb{R})$ be the group of all non-singular 2×2 matrices over $\mathbb{R}$. Show that is a subgroup of $GL(2,\mathbb{R})$.

$$H = \left\{\begin{bmatrix} a & b \\ -b & a\end{bmatrix} \in GL(2,\mathbb{R}) \mid \text{either } a \text{ or } b \neq 0\right\}.$$

Sol^n: $\begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix} \in H$, Hence $H \neq \phi$

Let $A = \begin{bmatrix} a & b \\ -b & a\end{bmatrix}$ and $B = \begin{bmatrix} c & d \\ -d & c\end{bmatrix}$ be two elements of H.

Since a and b cannot be both zero.
we find that $a^2+b^2 \neq 0$. Similarly $c^2+d^2 \neq 0$.

Now
$$B^{-1} = \begin{bmatrix} \frac{c}{c^2+d^2} & \frac{-d}{c^2+d^2} \\ \frac{d}{c^2+d^2} & \frac{c}{c^2+d^2}\end{bmatrix}$$

Hence
$$AB^{-1} = \begin{bmatrix} a & b \\ -b & a\end{bmatrix}\begin{bmatrix} \frac{c}{c^2+d^2} & \frac{-d}{c^2+d^2} \\ \frac{d}{c^2+d^2} & \frac{c}{c^2+d^2}\end{bmatrix}$$

$$= \begin{bmatrix} \frac{ac+bd}{c^2+d^2} & \frac{-ad+bc}{c^2+d^2} \\ \frac{-bc+ad}{c^2+d^2} & \frac{bd+ac}{c^2+d^2}\end{bmatrix} = \begin{bmatrix} x & y \\ -y & x\end{bmatrix}$$

where $x = \dfrac{ac+bd}{c^2+d^2}$ and $y = \dfrac{-ad+bc}{c^2+d^2}$

Now $x^2+y^2 = \dfrac{(ac+bd)^2+(-ad+bc)^2}{(c^2+d^2)^2}$

$$= \frac{a^2c^2+b^2d^2+a^2d^2+b^2c^2}{(c^2+d^2)^2}$$

$$= \frac{(a^2+b^2)(c^2+d^2)}{(c^2+d^2)^2}$$

$$= \frac{(a^2+b^2)}{(c^2+d^2)} \neq 0$$

Hence either $x \neq 0$ (or) $y \neq 0$. So, we find that $AB^{-1} \in H$.

Hence H is a subgroup of $GL(2, \mathbb{R})$.

1(b) If R is a division ring, show that $\{0\}$ and R are only ideals of R. Is the converse true? Justify your answer.

Sol^n: let $(R, +, \times)$ be a division

let S be an ideal of R.

Case(i) when $S = \{0\}$, there is nothing to prove

Case(ii) when $S \neq \{0\}$, S contains non-zero elements.

let $\underset{\neq 0}{a} \in S (\subseteq R)$ then a^{-1} exists in R

$\therefore aa^{-1} \in S (\subseteq R)$

$\Rightarrow 1 \in S$ (by inverse prop. in R)

let $x \in R$ then $x = x \cdot \underset{\in S}{1}$

$\therefore R \subseteq S$

$\therefore S = R$

$\therefore$ R has only two ideals say $\{0\}$ & R.

The converse of the above need not be true.

i.e If the ring $(R, +, \times)$ has $\{0\}$ and R only ideals of R the R need not be a division ring.

Try Yourself.

1(c) → Show that the function $f(x) = \frac{1}{x^2}$ is uniformly continuous on $[a, \infty[$, where $a > 0$, but not uniformly continuous on $]0, \infty[$.

Sol'n: To show that $f(x) = \frac{1}{x^2}$ is uniformly continuous on $[a, \infty)$ where $a > 0$.

For $x, y \geq a > 0$, we obtain

$$|f(x) - f(y)| = \left|\frac{1}{x^2} - \frac{1}{y^2}\right|$$

$$= \left|\frac{1}{x} - \frac{1}{y}\right| \left|\frac{1}{x} + \frac{1}{y}\right|$$

$$\Rightarrow |f(x) - f(y)| = \left(\frac{y-x}{xy}\right)\left(\frac{1}{x} + \frac{1}{y}\right)$$

$$\leq \frac{2|x-y|}{a|xy|} \qquad \left(\because x, y \geq a > 0 \Rightarrow \frac{1}{x} < \frac{1}{a} \;\&\; \frac{1}{y} < \frac{1}{a} \Rightarrow \frac{1}{x} + \frac{1}{y} < \frac{2}{a}\right)$$

$$\leq \frac{2}{a^3}|x-y| \qquad \left(\because x \geq a, y \geq a \Rightarrow xy \geq a^2 \Rightarrow \frac{1}{xy} \leq \frac{1}{a^2}\right)$$

Let $\epsilon > 0$ be given

Let $\delta = \frac{\epsilon a^3}{2}$.

Then $|f(x) - f(y)| < \epsilon$, when $|x - y| < \delta \;\forall\, x, y \geq a$

Hence, f is uniformly conituous on $[a, \infty)$

Now we show that f is not uniformly continuous in $(0, \infty)$.

Let $\epsilon = \frac{1}{2}$ and δ be any positive number. We can always choose a positive integer n such that $n > \frac{1}{2\delta}$ (or) $\frac{1}{2n} < \delta$ — ①

Let $x_1 = \frac{1}{\sqrt{n}}$ and $x_2 = \frac{1}{\sqrt{n+1}} \in (0, \infty)$

Then $|f(x_1) - f(x_2)| = \left|\frac{1}{x_1^2} - \frac{1}{x_2^2}\right| = |n - (n+1)| = 1 > \epsilon$

and $|x_1 - x_2| = \left|\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}\right| = \frac{|\sqrt{n+1} - \sqrt{n}|}{\sqrt{n}\sqrt{(n+1)}} = \frac{1}{\sqrt{n}\sqrt{n+1}[\sqrt{n+1} + \sqrt{n}]}$

$$< \frac{1}{\sqrt{n}\cdot 2\sqrt{n}} \qquad (\because \sqrt{n}\sqrt{n+1} > \sqrt{n}\sqrt{n} \;\&\; \sqrt{n+1} + \sqrt{n} > 2\sqrt{n})$$

$$= \frac{1}{2n} < \delta \text{ by ①}$$

Thus $|f(x_1) - f(x_2)| > \epsilon$, when $|x_1 - x_2| < \delta$.

Hence, f is not uniformly continuous on $(0, \infty)$

1(d) Show that the function defined by

$$f(z)=\begin{cases}\dfrac{x^3y^5(x+iy)}{x^6+y^{10}}, & z\neq 0\\ 0, & z=0\end{cases}$$

is not analytic at the origin though it satisfies Cauchy's Riemann equations at the origin.

Sol'n: Here $u+iv=\dfrac{x^3y^5(x+iy)}{x^6+y^{10}}$

$$\therefore u=\frac{x^4y^5}{x^6+y^{10}},\quad v=\frac{x^3y^5}{x^6+y^{10}}$$

At the origin,

$$u_x=\lim_{x\to 0}\frac{u(x,0)-u(0,0)}{x}=\lim_{x\to 0}\frac{0-0}{x}=0.$$

$$u_y=\lim_{y\to 0}\frac{u(0,y)-u(0,0)}{y}=\lim_{y\to 0}\frac{0-0}{y}=0$$

Similarly $v_x=0,\ v_y=0$

Hence Cauchy-Riemann equations are satisfied at the origin.

But
$$\lim_{z\to 0}\frac{f(z)-f(0)}{z}=\lim_{\substack{x\to 0\\ y\to 0}}\left[\frac{x^3y^5(x+iy)}{x^6+y^{10}}-0\right]\frac{1}{x+iy}$$

$$=\lim_{\substack{x\to 0\\ y\to 0}}\frac{x^3y^5}{x^6+y^{10}}$$

$$=\lim_{x\to 0}\frac{x^3m^5x^5}{x^6+m^{10}x^{10}}$$ if $z\to 0$ along the radius vector $y=mx$

$$=\lim_{x\to 0}\frac{m^5x^2}{1-m^{10}x^4}=0$$

and $$=\lim_{x\to 0}\frac{x^3x^3}{x^6+x^6}$$ if $z\to 0$ along the curve $y^5=x^3$

$=\frac{1}{2}$

showing that $f'(0)$ does not exist.

Hence $f(z)$ is not analytic at origin although Cauchy-Riemann equations are satisfied there.

2(a) Show that $Z[\sqrt{2}] = \{m+n\sqrt{2} : m,n \in Z\}$ is a Euclidean domain.

Sol^n: we know that $Z[\sqrt{2}]$ is an integral domain with unity $1 = 1+\sqrt{2}.0$

Let us define a mapping
$$d : Z[\sqrt{2}]-\{0\} \to Z \text{ by}$$
$$d(m+n\sqrt{2}) = |m^2-2n^2| \quad \forall \quad m+n\sqrt{2} \in Z[\sqrt{2}]-\{0\}$$
we have $m\neq 0$ (or) $n \neq 0$.

$\therefore d(m+n\sqrt{2})$ is a +ve integer for each $m+n\sqrt{3} \in Z[\sqrt{2}]-\{0\}$
$$\therefore d(m+n\sqrt{2}) \geq 0.$$
Now let $a = m+n\sqrt{2} \neq 0$, $b = m_1+n_1\sqrt{2} \neq 0$ in $Z[\sqrt{2}]$.
$m\neq 0$ or $n\neq 0$; $m_1 \neq 0$ or $n_1 \neq 0$.

then we have
$$ab = (mm_1+2nn_1) + (mn_1+m_1n)\sqrt{2}$$
and $d(ab) = |(mm_1+2nn_1)^2 - 2(mn_1+m_1n)^2|$ (by defn)
$$= |m^2m_1^2 + 4n^2n_1^2 - 2(m^2n_1^2+m_1^2n^2)|$$
$$= |(m^2-2n^2)(m_1^2-2n_1^2)|$$
$$= |m^2-2n^2|\,|m_1^2-2n_1^2|$$
$$\geq |m^2-2n^2| \quad (\because |m_1^2-2n_1^2| \geq 1)$$
$$= d(a)$$
$$\therefore d(a) \leq d(ab).$$
Now we have
$$\frac{a}{b} = \frac{m+n\sqrt{2}}{m_1+n_1\sqrt{2}} = \frac{(m+n\sqrt{2})(m_1-n_1\sqrt{2})}{(m_1+n_1\sqrt{2})(m_1-n_1\sqrt{2})}$$

$$= \left(\frac{mm_1 - 2nn_1}{m_1^2 - 2n_1^2}\right) + \left(\frac{m_1 n - mn_1}{m_1^2 - 2n_1^2}\right)\sqrt{2}$$

$$= p + q\sqrt{2}$$

where $p = \dfrac{mm_1 - 2nn_1}{m_1^2 - 2n_1^2}$ & $q = \dfrac{m_1 n - mn_1}{m_1^2 - 2n_1^2}$ are rational numbers.

Corresponding to the rational numbers p and q, we can find two integers p' and q' such that $|p' - p| \le \frac{1}{2}$ and $|q' - q| \le \frac{1}{2}$.

Let $t = p' + q'\sqrt{2}$

Then $t \in Z[\sqrt{2}]$

we have $\frac{a}{b} = \lambda$, where $\lambda = p + q\sqrt{2}$

$$\Rightarrow a = \lambda b = (\lambda - t)b + tb$$
$$= tb + r, \text{ where } r = (\lambda - t)b$$

Now $a, b, t \in Z[\sqrt{2}]$

$$\Rightarrow a - tb \in Z[\sqrt{2}]$$
$$\Rightarrow r \in Z[\sqrt{2}]$$

$\therefore \exists\ t, r \in Z[\sqrt{2}]$ such that $a = tb + r$; where $r = 0$ (or)

$$d(r) = d\{(\lambda - t)b\}$$
$$= d\{(p + q\sqrt{2}) - (p' + q'\sqrt{2})\} d(b)$$
$$= d\{(p - p') + (q - q')\sqrt{2}\} d(b)$$
$$\le |(p - p')^2 + 2(q - q')^2| \, d(b)$$
$$\le \left(\frac{1}{4} + \frac{2}{4}\right) d(b)$$
$$= \frac{3}{4} d(b)$$
$$< d(b)$$

$\therefore Z[\sqrt{2}]$ is a Euclidean domain.

2(b) (i) The union of an infinite number of closed sets in $\mathbb{R}$ is not necessarily a closed set.

Sol'n: Let us consider the sets F_i :

where $F_1 = \{x \in \mathbb{R} \,/\, -1 \le x \le 1\}$

$F_2 = \{x \in \mathbb{R} \,/\, -\frac{1}{2} \le x \le \frac{1}{2}\}$

$\vdots$

$F_n = \{x \in \mathbb{R} \,/\, -\frac{1}{n} \le x \le \frac{1}{n}\}$

$\vdots$

For each F_i is a closed set.

$\bigcup_{i=1}^{\infty} F_i = F_1$

$\therefore$ clearly F_1 is a closed set.

Let us consider the sets F_i, where

$F_1 = \{x \in \mathbb{R} \,/\, 1 \le x \le 3-1\}$

$F_2 = \{x \in \mathbb{R} \,/\, \frac{1}{2} \le x \le 3-\frac{1}{2}\}$

$\vdots$

$F_n = \{x \in \mathbb{R} \,/\, \frac{1}{n} \le x \le 3-\frac{1}{n}\}$

Here each F_i is closed set.

$\bigcup_{i=1}^{\infty} F_i = \{x \in \mathbb{R} \,/\, 0 < x < 3\}$

Clearly it is not a closet.

$\therefore$ These two examples establish that the union of an infinite number of closed sets in $\mathbb{R}$ is not a closet.

$\therefore$ These two examples establish that the union of an infinite number of closed sets in $\mathbb{R}$ is not necessarily a closed set.

2(b)(ii) Prove that the function f defined on $[0,1]$ as

$f(x) = 2n$, if $x = 1/n$ where $n = 1, 2, ----$.

$= 0$, otherwise

is not Riemann-integrable on $[0,1]$.

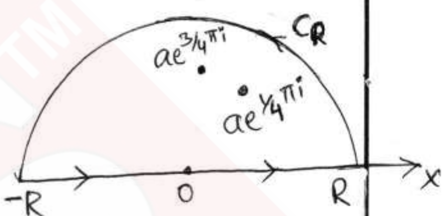
Sol'n: $\lim_{x \to 0} f(x) = \infty$, 'f' is not bounded above and hence not Riemann integrable on $[0,1]$.

2(c)(ii) Use the method of contour integration evaluate $\int_0^\infty \frac{dx}{x^4+a^4}$ $(a>0)$.

sol^n: Consider the integral

$\int_C f(z)\,dz$, where $f(z)=\frac{1}{z^4+a^4}$,

taken round a closed contour C consisting of the upper half of a large circle $|z|=R$ and the real axis from $-R$ to R.



The poles of $f(z)$ are given by

$$z^4+a^4=0$$

i.e. $z^4=-a^4=a^4e^{\pi i}=a^4e^{2n\pi i+\pi i}$

$\Rightarrow z=ae^{\frac{1}{4}(2n+1)\pi i}$

$\therefore z=ae^{\frac{1}{4}\pi i}$ and $z=ae^{\frac{3}{4}\pi i}$ (for $n=0$, and 1).

are the only two poles which lie within the contour.

Let α denote any one of these poles than $-a^4=\alpha^4$.

Residue of $f(z)$ at $ae^{\frac{1}{4}\pi i}$ is $\left[\frac{1}{\frac{d}{dz}(z^4+a^4)}\right]_{z=ae^{\frac{1}{4}\pi i}}$

$$=\frac{-1}{4a^3}e^{\frac{1}{4}\pi i}$$

Similarly residue at $ae^{3i\pi/4}=\frac{e^{-i\pi/4}}{4a^3}$.

$\therefore$ Sum of residues $=-\frac{1}{2a^3}\,\frac{e^{i\pi/4}-e^{-i\pi/4}}{2}$

$$=\frac{-1}{2a^3}\cdot i\sin\frac{\pi}{4}=-\frac{i}{2\sqrt{2}a^3}$$

Hence by Cauchy's residue theorem, we have

$\int_C f(z)\,dz=2\pi i\times$ sum of residues within C.

i.e. $\int_{-R}^{R} f(x)\,dx+\int_{C_R} f(z)\,dz=2\pi i\left(\frac{-1}{2\sqrt{2}a^3}\right)$

$$\Rightarrow \int_{-R}^{R} \frac{dx}{x^4+a^4} + \int_{C_R} \frac{dz}{z^4+a^4} = \frac{\pi}{2\sqrt{2}a^3} \quad \text{--- (1)}.$$

Now, $$\left| \int_{C_R} \frac{1}{z^4+a^4} dz \right|$$

$$\leq \int_{C_R} \frac{|dz|}{|z^4+a^4|}$$

$$\leq \int_{C_R} \frac{|dz|}{|z^4|-|a^4|}$$

$$= \int_0^{\pi} \frac{R\,d\theta}{R^4-a^4}$$

$$= \frac{\pi R}{R^4-a^4} \quad \text{which} \to 0 \text{ as } R\to\infty$$

Hence by making $R\to\infty$, relation (1) becomes

$$\int_{-\infty}^{\infty} \frac{dx}{x^4+a^4} = \frac{\pi}{2\sqrt{2}a^3}$$

$$\int_0^{\infty} \frac{dx}{R^4+a^4} = \frac{\pi}{2\sqrt{2}a^3} = \frac{\pi\sqrt{2}}{4a^3}$$

Particularly $$\int_0^{\infty} \frac{dx}{x^4+1} = \frac{\pi\sqrt{2}}{4}.$$

3(a) (i) Let $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix}$ and $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 3 & 5 & 1 \end{pmatrix}$ in S_5.

Find a permutation γ in S_5 such that $\alpha\gamma = \beta$.

Sol^n: Here $\alpha, \beta, \gamma \in S_5$

Given $\alpha\gamma = \beta$

$$\alpha^{-1}\alpha\gamma = \alpha^{-1}\beta$$

$$\Rightarrow \gamma = \alpha^{-1}\beta$$

$$\Rightarrow \gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix}^{-1} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 3 & 5 & 1 \end{pmatrix}$$

$$\Rightarrow \gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 1 & 5 & 4 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 3 & 5 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 1 & 4 & 3 \end{pmatrix}$$

$$\therefore \gamma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 5 & 1 & 4 & 3 \end{pmatrix}$$

3(a)(ii) Let F be the field of integers modulo 5. Show that the polynomial x^2+2x+3 is irreducible over F. Use this to construct a field containing 25 elements.

Sol^n: We have $F = Z_5 = \{0,1,2,3,4\}$

Let $x^2+2x+3 = (x+a)(x+b)$; $a, b \in F$

Comparing the coefficients of x and constants on both the sides, we get
$$2 = a+b \quad \text{——} (1)$$
$$3 = ab \quad \text{——} (2)$$

(1) is satisfied for $(a,b) = (0,2), (1,1), (3,4), (2,0), (4,3)$.
For these values of a and b, $ab = 0, 1, 2, 0, 2$ i.e. (2) is never satisfied.

Consequently, x^2+2x+3 is irreducible over F.

Hence, $\dfrac{F[x]}{\langle x^2+2x+3\rangle}$ is a field.

Any element of this field is $f(x)+A$, where $f(x) \in F[x]$, $A = \langle x^2+2x+3\rangle$.

By division algorithm in $F[x]$, for $f(x) \in F[x]$, $x^2+2x+3 \in F[x]$ there exist $t(x), r(x) \in F[x]$ such that
$$f(x) = (x^2+2x+3)\,t(x) + r(x), \quad \text{——} (1)$$
where $r(x) = 0$ (or) $\deg r(x) < \deg(x^2+2x+3) = 2$

we may take $r(x) = \alpha x + \beta$ where $\alpha, \beta \in F$
$$\therefore f(x)+A = r(x) + (x^2+2x+3)\,t(x) + A, \text{ by } (1)$$
$$\Rightarrow f(x)+A = r(x)+A = \alpha x + \beta + A \quad \text{——} (2)$$
Since $(x^2+2x+3)\,t(x) \in A = \langle x^2+2x+3\rangle$.

In (2), we see that $\alpha, \beta \in F = Z_5$ and $o(Z_5) = 5$.
Consequently, each of α and β can be selected in (5) ways.
Hence, by (2), the number of elements of the field $\dfrac{F[x]}{\langle x^2+2x+3\rangle}$ is $5^2 = 25$.

3(b) (i) Show that the sequence of functions $\langle f_n \rangle$, where $f_n(x) = \frac{n^2 x}{1+n^2x^2}$ is non-uniformly convergent on $[0,1]$.

(ii) Evaluate the integral $\int_0^1 \frac{x^\alpha - 1}{\log x}\,dx$ $(\alpha > -1)$ by applying differentiating under the integral sign.

Sol^n: (i) when $x=0$, $f_n(x)=0 \ \forall n$

when $0<x\le 1$, $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{n^2 x}{1+n^2x^2}$

$$= \lim_{n\to\infty} \frac{x}{\frac{1}{n^2}+x^2} = \frac{1}{x}$$

$$\therefore f(x) = \begin{cases} \frac{1}{x}, & \text{if } 0<x\le 1 \\ 0, & \text{if } x=0 \end{cases}$$

when $0<x\le 1$,

$$|f_n(x)-f(x)| = \left|\frac{n^2x}{1+n^2x^2} - \frac{1}{x}\right| = \frac{1}{x(1+n^2x^2)}$$

Let $y = \frac{1}{x(1+n^2x^2)}$, then y is maximum when $x = 1/n$ and maximum value of y is $n/2$.

Also $x = \frac{1}{n} \to 0$ as $n\to\infty$.

$$M_n = \max_{x\in[0,1]} |f_n(x)-f(x)| = \max_{x\in[0,1]} \left[\frac{1}{x(1+n^2x^2)}\right] = \frac{n}{2}$$

which does not tend to zero as $n\to\infty$.

Hence the sequence $\langle f_n \rangle$ is not uniformly convergent on $[0,1]$.

3(b)(ii) Sol'n: Let $F(\alpha) = \int_0^1 \frac{x^\alpha - 1}{\log x} dx$ —— (1)

Differentiating bothsides w.r.t α, we get

$$F'(\alpha) = \int_0^1 \frac{\partial}{\partial \alpha}\left[\frac{x^\alpha - 1}{\log x}\right] dx$$

$$= \int_0^1 \frac{1}{\log x} x^\alpha \log x \, dx$$

$$= \int_0^1 x^\alpha dx$$

$$= \left[\frac{x^{\alpha+1}}{\alpha+1}\right]_0^1 \text{ for } \alpha > -1$$

$$= \frac{1}{1+\alpha}$$

Integrating bothsides w.r.t α

$$F(\alpha) = \log(1+\alpha) + C \quad \text{—— (2)}$$

From (1), when $\alpha = 0$

$$F(0) = \int_0^1 \frac{1-1}{\log x} dx = 0$$

$\therefore$ from (2), $0 = \log 1 + C = 0 + C \Rightarrow C = 0$

$\therefore F(\alpha) = \log(1+\alpha)$ where $\alpha > -1$.

Hence $\int_0^1 \frac{x^\alpha - 1}{\log x} dx = \log(1+\alpha)$.

3(c) → Solve the following LPP by using simplex method.

Max $z = 5x_1 - 2x_2 + 3x_3$

subject to

$2x_1 + 2x_2 - x_3 \geq 2$

$3x_1 - 4x_2 \leq 3$

$x_1 + 3x_3 \leq 5$

and $x_1, x_2, x_3 \geq 0$.

Sol: Introducing the surplus variable $s_1 \geq 0$, slack variables $s_2 \geq 0$, $s_3 \geq 0$ and an artificial variable $a_1 \geq 0$, the constraints of the problem become:

$2x_1 + 2x_2 - x_3 - s_1 + a_1 = 2$

$3x_1 - 4x_2 + s_2 = 3$

$x_2 + 3x_3 + s_3 = 5$

and using big M technique, objective function becomes

$\text{Max} Z = 5x_1 - 2x_2 + 3x_3 + 0s_1 + 0s_2 + 0s_3 - Ma_1$.

Now the IBFS is

$x_1 = x_2 = x_3 = s_1 = 0$ (non-basic)

$a_1 = 2,\ s_2 = 3,\ s_3 = 5$. (basic)

for which $Z = -2M$.

Now, we put the above information in the simplex tableau

	c_j	5	−2	3	0	0	0	−M		
C_B	Basis	x_1	x_2	x_3	s_1	s_2	s_3	a_1	b	θ
−M	a_1	$\boxed{2}$	2	−1	−1	0	0	1	2	$\frac{2}{2}=1$ →
0	s_2	3	−4	0	0	1	0	0	3	$\frac{3}{3}=1$
0	s_3	0	1	3	0	0	1	0	5	—

$z_j = \sum C_B a_{ij}$	$-2M$	$-2M$	M	M	0	0	$-M$	$-2M$
$C_j = c_j - z_j$	$5+2M$	$-2+2M$	$3-M$	$-M$	0	0	0	

From the table, x_1 is the entering variable; further, since the minimum ratio in the last column of above table is 1 for both the 1st and 2nd rows, therefore either a_1 or s_2 tends to leave the basis. This is an indication of the existence of degeneracy. But, a_1 being an artificial variable will be preferred to leave the basis. and (2) is the key element and all other elements in its column equal to zero.

Continuing the simplex routine, the computations are presented in the following tabular form.

C_B	Basis	x_1 ($c_j=5$)	x_2 (-2)	x_3 (3)	s_1 (0)	s_2 (0)	s_3 (0)	b	θ
5	x_1	1	1	$-1/2$	$-1/2$	0	0	1	—
0	s_2	0	-7	$\boxed{3/2}$	$3/2$	1	0	0	0 →
0	s_3	0	1	3	0	0	1	5	$5/3$
$z_j = \sum C_B a_{ij}$		5	5	$-\frac{5}{2}$	$-\frac{5}{2}$	0	0	5	
$C_j = c_j - z_j$		0	-7	$11/2$ ↑	$5/2$	0	0		
5	x_1	1	$-4/3$	0	0	$1/3$	0	1	—
3	x_3	0	$-14/3$	1	1	$2/3$	0	0	—
0	s_3	0	$\boxed{15}$	0	-3	-2	1	5	$\frac{1}{3}$ →
$z_j = \sum C_B a_{ij}$		5	$-\frac{62}{3}$	3	3	$\frac{11}{3}$	0	5	
$C_j = c_j - z_j$		0	$56/3$ ↑	0	-3	$-\frac{11}{3}$	0		
5	x_1	1	0	0	$-4/15$	$7/45$	$4/45$	$13/9$	—
3	x_3	0	0	1	$\boxed{1/15}$	$2/45$	$14/45$	$14/9$	$\frac{70}{3}$ →
-2	x_2	0	1	0	$-1/5$	$-2/15$	$1/15$	$1/3$	—

$z_j = \sum c_B a_{ij}$	5	-2	3	$-11/15$	$53/45$	$56/45$	101/9
$c_j = c_j - z_j$	0	0	0	$\frac{11}{15}\uparrow$	$-53/45$	$-\frac{56}{45}$	

	c_j	5	-2	3	0	0	0		
C_B	Basis	x_1	x_2	x_3	s_1	s_2	s_3	b	θ
5	x_1	1	0	4	0	$\frac{1}{3}$	$\frac{4}{3}$	$23/3$	
0	s_1	0	0	15	1	$\frac{2}{3}$	$\frac{14}{3}$	$70/3$	
-2	x_2	0	1	3	0	0	1	5	
$z_j = \sum a_{ij} c_B$		5	-2	14	0	$\frac{5}{3}$	$\frac{14}{3}$	$85/3$	
$c_j = c_j - z_j$		0	0	-11	0	$-\frac{5}{3}$	$-\frac{14}{3}$		

Since all c_j's are ≤ 0.

$\therefore$ The optimum solution is

$x_1 = 23/3$, $x_2 = 5$, $x_3 = 0$.

$\text{Max } Z = \frac{85}{3}$.

4(a) let H be a subgroup of a group G such that $(G:H)=2$ then prove that H is a normal subgroup of G.
Is converse true? Justify your answer.

Sol let $(G,\times)$ be a group and $H \leq G$ such that $(G:H)=2$

To p.T $H \trianglelefteq G$.

$\because (G:H)=2$

There are two distinct left & right cosets of H in G.

let H & Ha ; H & aH be two distinct right cosets & left cosets of H in G; $a \in G$

then $G = H \cup Ha = H \cup aH$ ——(1)

$\because a \in G \Rightarrow a \in H$ or $a \notin H$

Case(i) when $a \in H$

$\therefore Ha = H = aH$

$\Rightarrow Ha = aH$.

case(ii) when $a \notin H$

$\therefore Ha \neq H$ & $aH \neq H$

$\because G = H \cup Ha$ & $G = H \cup aH$

$\therefore H \cup Ha = H \cup aH$

$\Rightarrow Ha = aH$

$\therefore Ha = aH \ \forall a \in G$.

$\therefore H \trianglelefteq G$.

The converse of the above need not be true.

ie If $H \trianglelefteq G$ then $(G:H) \neq 2$

for example:

let $G = \{1, -1, i, -i, j, -j, k, -k\}$ be a group

let $H = \{1, -1\}$

clearly $H \trianglelefteq G$

but $(G:H) = 4$

$\neq 2$.

4(b) Examine the convergence of the integrals.

(i) $\int_0^1 \frac{x^n \log x}{(1+x)^2}$ (ii) $\int_0^\infty \frac{\cos\alpha x - \cos\beta x}{x} dx,\ a>0.$

Sol^n: (i) $\underset{x\to 0+}{\mathrm{Lt}} \frac{x^n \log x}{(1+x)^2} = 0$ if $n>0$

$\therefore \int_0^1 \frac{x^n \log x}{(1+x)^2} dx$ is proper and, hence, convergent so long as $n>0$.

If $n=0$, let $f(x) = -\frac{\log x}{(1+x)^2}$

0 is the only point of infinite discontinuity.

Take $g(x) = \frac{1}{x^p}$

$$\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} -\frac{x^p \log x}{(1+x)^2} = 0 \text{ if } p>0.$$

Taking p between 0 and 1, $\int_0^1 g(x)dx$ is convergent.

$\Rightarrow \int_0^1 f(x)\,dx$ is convergent.

$\Rightarrow \int_0^1 \frac{x^n \log x}{(1+x)^2} dx$ is Convergent.

If $n<0$, let $n=-m$ where $m>0$

Let $f(x) = -\frac{x^n \log x}{(1+x)^2} = -\frac{\log x}{x^m (1+x)^2}$

Take $g(x) = \frac{1}{x^q}$

$$\lim_{x\to 0+} \frac{f(x)}{g(x)} = \lim_{x\to 0+} -\frac{x^{q-m}\log x}{(1+x)^2} = 0 \text{ if } q-m>0.$$

Taking $0<q<1$ and also $q-m>0$ i.e. $q>m$
$\Rightarrow 0<m<q<1$
$\Rightarrow m<1 \Rightarrow n>-1$

$\int_0^1 g(x)dx$ is convergent and hence $\int_0^1 f(x)dx$ is convergent.

$\therefore \int_0^1 \frac{x^n \log x}{(1+x)^2} dx$ is convergent for all $n>-1$.

4(b)(ii) Sol^n: $\int_a^\infty \frac{\cos\alpha x - \cos\beta x}{x}\,dx = \int_a^\infty \frac{\cos\alpha x}{x}\,dx - \int_a^\infty \frac{\cos\beta x}{x}\,dx$ —— (1)

$= I_1 - I_2$ (say)

Consider $I_1 = \int_a^\infty \frac{\cos\alpha x}{x}\,dx$

Let $f(x) = \cos\alpha x$ and $g(x) = \frac{1}{x}$

Since $\left|\int_a^t f(x)\,dx\right| = \left|\int_a^t \cos\alpha x\,dx\right| = \left|\frac{\sin\alpha t - \sin\alpha a}{\alpha}\right|$

$\le \frac{1}{|\alpha|}\left(|\sin\alpha t| + |\sin\alpha a|\right)$

$\le \frac{1}{|\alpha|}(1+1) = \frac{2}{|\alpha|}$

$\therefore \int_a^t f(x)\,dx$ is bounded for all $t \ge a$.

Also $g(x)$ is a bounded and monotonically decreasing function tending to 0 as $x \to \infty$.

$\therefore$ By Dirichlet's test $\int_a^\infty f(x)\,g(x)\,dx = I_1$ is convergent.

Similarly I_2 is convergent. Hence, from (1) the given integral is convergent.

4(c) If the function $f(z)$ is analytic and one valued in $|z-a|<R$, Prove that for $0<r<R$

$$f'(a) = \frac{1}{\pi r}\int_0^{2\pi} P(\theta)e^{-i\theta}d\theta, \text{ where } P(\theta) \text{ real part of } (a+re^{i\theta})$$

__Sol'n__: Since $f(z)$ is regular in $|z-a|<R$, therefore it must be regular in $|z-a|=r$, $(r<R)$

Hence $f(z)$ can be expanded in a Taylor's Series within the Circle $|z-a|=r$

i.e $\quad f(z) = \sum_0^{\infty} a_m (z-a)^m$

$$= \sum_0^{\infty} a_m r^m (e^{i\theta})^m \qquad \text{Since } z-a = re^{i\theta}$$

So that $\overline{f(z)} = \sum_0^{\infty} \bar{a}_m r^m (e^{i\theta})^m$

Now, consider the integral $\int_C \overline{f(z)} \cdot \frac{dz}{(z-a)^{n+1}}$

$$\int_C \overline{f(z)} \cdot \frac{dz}{(z-a)^{n+1}} = \int_0^{2\pi} \sum_0^{\infty} \bar{a}_m r^m e^{-im\theta} \frac{re^{i\theta} \cdot i d\theta}{r^{n+1} e^{i(n+1)\theta}}$$

$$= \sum_0^{\infty} \bar{a}_m \cdot r^{m-n} \cdot i \int_0^{2\pi} e^{-i(m+n)\theta} = 0 \quad \text{for all values of } n$$

Particularly $\int_C \overline{f(z)} \frac{dz}{(z-a)^2} = 0$ —— (1)

we know that $f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^2}$

$$= \frac{1}{2\pi i}\int_C \frac{f(z)+\overline{f(z)}}{(z-a)^2} dz \quad \text{from (1)}$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} \frac{f(a+re^{i\theta})+\overline{f(a+re^{i\theta})}}{r^2 e^{2i\theta}} re^{i\theta} \cdot i d\theta$$

since $z-a = re^{i\theta}$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{2\operatorname{Re} f(a+re^{i\theta})}{r^2 e^{2i\theta}} re^{i\theta} \cdot i d\theta$$

since $z+\bar{z} = \operatorname{Re}(z)$

$$= \frac{1}{\pi r}\int_0^{2\pi} P(\theta)e^{-i\theta}d\theta. \quad [\text{Since } P(\theta) = \operatorname{Re} f(a+re^{i\theta})]$$

This Proves the result.

Q. 5.a) → Find the equation of the integral surface of the differential equation $(x^2-yz)p+(y^2-zx)q=z^2-xy$ which passes through the line $x=1,\ y=0$.

Sol: Given $(x^2-yz)p+(y^2-zx)q=z^2-xy$ ——— (1)

Here the Lagrange's auxiliary equations are

$$\frac{dx}{x^2-yz}=\frac{dy}{y^2-zx}=\frac{dz}{z^2-xy} \quad \text{——— (A)}$$

choosing $1,-1,0$ and $0,1,-1$ as multipliers in turn, each fraction of (A)

$$\frac{dx-dy}{x^2-y^2+z(x-y)}=\frac{dy-dz}{(y-z)(y+z+x)}$$

so that $$\frac{dx-dy}{(x-y)(x+y+z)}=\frac{dy-dz}{(y-z)(y+z+x)}$$

or $$\frac{d(x-y)}{x-y}-\frac{d(y-z)}{y-z}=0$$

Integrating,

$$\log(x-y)-\log(y-z)=\log c_2$$

or $(x-y)/(y-z)=c_1$ ——— (2)

choosing x,y,z as multipliers, each fraction of (A)

$$=\frac{x\,dx+y\,dy+z\,dz}{x^3+y^3+z^3-3xyz}$$

$$=\frac{x\,dx+y\,dy+z\,dz}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)} \quad \text{——— (3)}$$

Again, choosing $1,1,1$ as multipliers, each fraction of (A)

$$= \frac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx} \quad \text{——— (4)}$$

From (3) and (4),

$$\frac{x\,dx+y\,dy+z\,dz}{x+y+z} = dx+dy+dz$$

or $\quad 2(x+y+z)\,d(x+y+z) - (2x\,dx+2y\,dy+2z\,dz)=0$

Integrating,

$$(x+y+z)^2-(x^2+y^2+z^2) = 2c_2$$

or $\quad (x^2+y^2+z^2+2xy+2yz+2zx)-(x^2+y^2+z^2) = 2c_2$

or $\quad xy+yz+zx = c_2 \quad$ ——— (5)

The given curve is represented by $x=1, y=0$ ——— (6)

using (6) in (2) and (5), we obtain

$$-1/z = c_1 \quad \text{and} \quad z = c_2$$

So that $(-1/z)z = c_1c_2$

or $\quad c_1c_2+1=0 \quad$ ——— (7)

putting the value of c_1 and c_2 from (2) and (5) in (7), the required integral surface is

$$[(x-y)/(y-z)](xy+yz+zx)+1=0$$

or $\quad (x-y)(xy+yz+zx)+y-z=0$

Q.5.b) → Find a complete, singular and general integrals of $(p^2+q^2)y = qz$.

Sol: Here the given equation is

$$f(x,y,z,p,q) = (p^2+q^2)y - qz = 0 \quad \text{——} (1)$$

∴ charpits auxiliary equations are

$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

or $$\frac{dp}{-pq} = \frac{dq}{p^2} = \frac{dz}{-2p^2y+qz-2q^2y} = \frac{dx}{-2py} = \frac{dy}{-2qy+z} \quad \text{——} (2)$$

Taking the first two fractions, we get

$$p\,dp + q\,dq = 0$$

Integrating, $p^2+q^2 = a^2$, (say) ——— (3)

using (3), (1) gives

$$a^2y = qz \quad \text{or} \quad q = a^2y/z$$

Putting this value of q in (3), we get

$$p = \sqrt{(a^2-q^2)} = \sqrt{\left(a^2-\frac{a^4y^2}{z^2}\right)} = \frac{a}{z}\sqrt{(z^2-a^2y^2)}$$

Now putting these values of p and q in

$$dz = p\,dx + q\,dy,$$

$$dz = \frac{a}{z}\sqrt{(z^2-a^2y^2)}\,dx + \frac{a^2y}{z}dy$$

or $$\frac{z\,dz - a^2y\,dy}{\sqrt{(z^2-a^2y^2)}} = a\,dx.$$

Integrating, $(z^2-a^2y^2)^{1/2} = ax+b$

or $z^2-a^2y^2=(ax+b)^2$ ——— (4)

which is a required complete integral.

Singular Integral: Differentiating (4) partially w.r.t. a and b, we have

$$0 = 2ay^2 + 2(ax+b)x \quad \text{——— (5)}$$

and $0 = 2(ax+b)$ ——— (6)

Eliminating 'a' and 'b' between (4), (5) and (6), we get $z=0$ which clearly satisfies (1) and hence it is the singular integral.

General Integral: Replacing b by $\phi(a)$ in (4), we get

$$z^2-a^2y^2=[ax+\phi(a)]^2 \quad \text{——— (7)}$$

Differentiating (7) partially w.r.t. a, we get

$$-2ay^2 = 2[ax+\phi(a)]\cdot[x+\phi'(a)]. \quad \text{——— (8)}$$

G.I. is obtained by eliminating 'a' from (7) and (8)

5(c) Find the positive root of $\log_e x = \cos x$ nearest to five places of decimal by Newton-Raphson method.

Ans: 1·3030.
(Try yourself)

5(d) Find the equivalent of numbers given in a specified number system to the system mentioned against them

(i) $(41.6875)_{10}$ to binary numbers

(ii) $(10111011001.101110)_2$ to octal.

(iii) $(100011110000.00101100)_2$ to hexadecimal system.

(iv) $(CAF2)_{16}$ to decimal system.

Sol^n: Taking the integer part first:

(i)

$$\begin{array}{r|l} 2 & 41 \\ \hline 2 & 20 - 1 \\ \hline 2 & 10 - 0 \\ \hline 2 & 5 - 0 \\ \hline 2 & 2 - 1 \\ \hline 2 & 1 - 0 \\ \hline & 0 - 1 \end{array} \uparrow$$

$$(41)_{10} = (101001)_2$$

Taking the fractional part:

fraction	fraction × 2	Remainder new fraction	Integer
0.6875	1.375	0.375	1
0.375	0.75	0.75	0
0.75	1.5	0.5	1
0.5	1	0	1

$$(0.6875)_{10} = (1011)_2.$$

$$\therefore (41.6875)_{10} = (101001.1011)_2.$$

(ii) Binary number can be converted into equivalent octal number by making groups of 3 bits starting from LSB and moving towards MSB for integer part of the number and then replacing each group of three bits by its octal representation for fractional part, the grouping of 3 bits are made from the binary point.

Given $(10111011001.101110)_2 = (010\,111\,011\,001.101\,110)_2$

$$= \underbrace{010}_{2}\ \underbrace{111}_{7}\ \underbrace{011}_{3}\ \underbrace{001}_{1}.\underbrace{101}_{5}\ \underbrace{110}_{6}$$

$$= (2731.56)_8$$

(iii) Similarly, binary number can be converted into equivalent hexadecimal number by making groups of 4 bits.

$(1000111110000.00101100)_2$

$$= \underset{1}{0001}\ \underset{1}{0001}\ \underset{F}{1111}\ \underset{0}{0000}.\underset{2}{0010}\ \underset{C}{1100}$$

$$= (11F0.2C)_{16}.$$

(iv) $(CAF2)_{16} = C\times16^3 + 4\times16^2 + F\times16^1 + 2\times16^0$

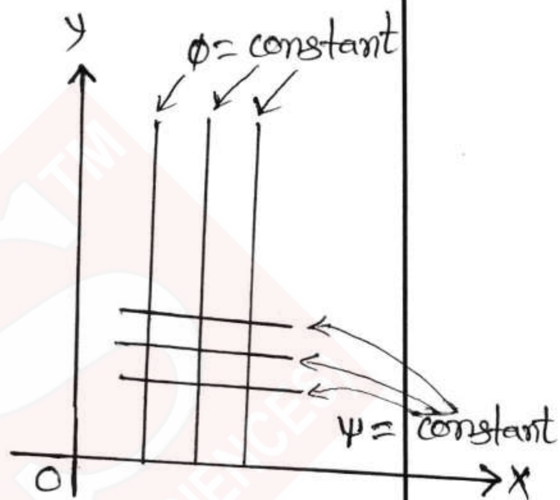
$$= 12\times16^3 + 4\times16^2 + 15\times16 + 2\times1$$

$$= 49152 + 1024 + 240 + 2$$

$$= (50,418)_{10}$$

8.5.e) Find the stream function ψ for a given velocity potential $\phi = cx$, where c is a constant. Also, draw a set of streamlines and equipotential lines.

Sol: The velocity component u and v in x and y directions are given by



$$u = -\frac{\partial \phi}{\partial x} = -c \text{ and } v = \frac{-\partial \phi}{\partial y} = 0 \quad \text{---} (1)$$

$$\therefore u = \frac{-\partial \psi}{\partial y} \text{ and } v = \frac{\partial \psi}{\partial x}.$$

$$\Rightarrow \frac{\partial \psi}{\partial y} = c \text{ and } \frac{\partial \psi}{\partial x} = 0 \quad \text{---} (2)$$

Then, $\partial \psi = \left(\frac{\partial \psi}{\partial x}\right) dx + \left(\frac{\partial \psi}{\partial y}\right) dy = c\, dy$

Integrating, $\psi = cy + d$ ——— (3)

where d is constant of integration.

Now,

ϕ = constant $\Rightarrow cx$ = constant $\Rightarrow x$ = constant,

showing that the lines of equipotential are parallel to y-axis.

Next,

ψ = constant $\Rightarrow cy + d$ = constant $\Rightarrow y$ = constant.

showing that the streamlines are parallel to x-axis as shown in the above figure.

Q. 6a(i) → Form a partial differential equation by eliminating the arbitrary functions f and g from $z = y f(x) + x g(y)$

Sol: Given $z = y f(x) + x g(y)$ ——— (1)

Differentiating (1) partially w.r.t. 'x' and 'y', we get

$$\frac{\partial z}{\partial x} = y f'(x) + g(y) \quad \text{——— (2)}$$

and $$\frac{\partial z}{\partial y} = f(x) + x g'(y) \quad \text{——— (3)}$$

Differentiating (3) with respect to x, we have

$$\frac{\partial^2 z}{\partial x \partial y} = f'(x) + g'(y) \quad \text{——— (4)}$$

From (2) and (3),

$$f'(x) = \frac{1}{y}\left[\frac{\partial z}{\partial x} - g(y)\right], \quad g'(y) = \frac{1}{x}\left[\frac{\partial z}{\partial y} - f(x)\right]$$

Substituting these values in (4), we have

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{1}{y}\left[\frac{\partial z}{\partial x} - g(y)\right] + \frac{1}{x}\left[\frac{\partial z}{\partial y} - f(x)\right]$$

or $$xy \frac{\partial^2 z}{\partial x \partial y} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - \{x g(y) + y f(x)\}$$

or $$xy \frac{\partial^2 z}{\partial x \partial y} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - z. \quad (\text{by } (2))$$

Q. 6a.(ii) → Solve $(3D^2 - 2D'^2 + D - 1)z = 4e^{x+y}\cos(x+y)$.

Sol: since $(3D^2 - 2D'^2 + D - 1)$ cannot be resolved into linear factors in D and D', hence as usual

$$C.F. = \sum Ae^{hx+ky}, \text{ where } 3h^2 - 2k^2 + h - 1 = 0$$

Now,
$$P.I. = \frac{1}{3D^2 - 2D'^2 + D - 1} 4e^{x+y}\cos(x+y)$$

$$= 4e^{x+y}\frac{1}{3(D+1)^2 - 2(D'+1)^2 + (D+1) - 1}\cos(x+y)$$

$$= 4e^{x+y}\frac{1}{3D^2 + 7D - 2D'^2 - 4D' + 1}\cos(x+y)$$

$$= 4e^{x+y}\frac{1}{3(-1^2) + 7D - 2(-1^2) - 4D' + 1}\cos(x+y)$$

$$= 4e^{x+y}\frac{1}{7D - 4D'}\cos(x+y)$$

$$= 4e^{x+y}\frac{7D + 4D'}{49D^2 - 16D'^2}\cos(x+y)$$

$$= 4e^{x+y}\frac{7D + 4D'}{49(-1^2) - 16(-1^2)}\cos(x+y)$$

$$= -\frac{4}{33}e^{x+y}(7D + 4D')\cos(x+y)$$

$$= -\frac{4}{33}e^{x+y}\left[7D\cos(x+y) + 4D'\cos(x+y)\right]$$

$$= -\frac{4}{33}e^{x+y}\left[-7\sin(x+y) - 4\sin(x+y)\right]$$

$$= (4/3)e^{x+y}\sin(x+y)$$

Hence solution is $z = \sum A e^{hx+ky} + (4/3)e^{x+y}\sin(x+y)$, where A and h are arbitrary constants related by $3h^2 - 2k^2 + h - 1 = 0$.

6(b) Reduce $x^2\left(\frac{\partial^2 z}{\partial x^2}\right) - y^2\left(\frac{\partial^2 z}{\partial y^2}\right) = 0$ to canonical form and hence solve it.

Sol^n (a) Re-writing the given equation $x^2 r - y^2 t = 0$. ↳ (1)

comparing (1) with $Rr + Ss + Tt + f(x,y,z,p,q) = 0$

here $R = x^2$, $S = 0$ and $T = -y^2$ so that

$S^2 - 4RT = 4x^2y^2 > 0$ for $x \neq 0$ $y \neq 0$ and

hence (1) is hyperbolic. The λ-Quadrate equation

$R\lambda^2 + S\lambda + T = 0$ reduces to $\lambda^2 x^2 - y^2 = 0$ So that

$\lambda = \frac{y}{x}, -y/x$ and hence the corresponding characteristic

equations become $\left(\frac{dy}{dx}\right) + (y/x) = 0$ and

$\left(\frac{dy}{dx}\right) - (y/x) = 0$.

Integrating these $xy = c_1$ and $\frac{x}{y} = c_2$

In order to reduce (1) to its canonical form, we choose

$$u = xy \text{ and } v = \frac{x}{y} \longrightarrow (2)$$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\cdot\frac{\partial v}{\partial x} = y\frac{\partial z}{\partial u} + \frac{1}{y}\frac{\partial z}{\partial v} \rightarrow (3)$$

$$Q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = x\frac{\partial z}{\partial u} - \frac{x}{y^2}\frac{\partial z}{\partial v} \rightarrow (4)$$

$$r = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(y\frac{\partial z}{\partial u} + \frac{1}{y}\frac{\partial z}{\partial v}\right)$$

$$= y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \frac{1}{y}\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right] +$$

$$\frac{1}{y}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]$$

$$= y\left(\frac{\partial^2 z}{\partial u^2}\, y+\frac{\partial^2 z}{\partial v \partial u}\cdot\frac{1}{y}\right)+\frac{1}{y}\left(\frac{\partial^2 z}{\partial v\partial u}\cdot y + \frac{\partial^2 z}{\partial v^2}\cdot\frac{1}{y}\right)$$

using (1)

$$r = y^2\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{1}{y^2}\frac{\partial^2 z}{\partial v^2} \to (5)$$

$$t = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(x\frac{\partial z}{\partial u}-\frac{x}{y^2}\frac{\partial z}{\partial v}\right)$$

$$= x\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)-\left[-\frac{2x}{y^3}\frac{\partial z}{\partial v}+\frac{x}{y^2}\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\right]$$

$$= x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}\right]+\frac{2x}{y^3}\frac{\partial z}{\partial v}-\frac{x}{y^2}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$= x\left[\frac{\partial^2 z}{\partial u^2}\cdot x+\frac{\partial^2 z}{\partial v\partial u}\cdot\left(\frac{-x}{y^2}\right)\right]+\frac{2x}{y^3}\frac{\partial z}{\partial v}-\frac{x}{y^2}\left[\frac{\partial^2 z}{\partial u\partial v}\cdot x+\frac{\partial^2 z}{\partial v^2}\left(\frac{-x}{y}\right)\right]$$

$$t = x^2\frac{\partial^2 z}{\partial u^2}-\frac{2x^2}{y^2}\frac{\partial^2 z}{\partial u\partial v}+\frac{2x}{y^3}\frac{\partial z}{\partial v}+\frac{x^2}{y^4}\frac{\partial^2 z}{\partial v^2} \to (6)$$

putting these values of r and t in (1) we get

$$x^2\left(y^2\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{1}{y^2}\frac{\partial^2 z}{\partial v^2}\right)-y^2\times\left(x^2\frac{\partial^2 z}{\partial u^2}-\frac{2x^2}{y^2}\frac{\partial^2 z}{\partial u\partial v}+\frac{2x}{y^3}\frac{\partial z}{\partial v}+\frac{x^2}{y^4}\frac{\partial^2 z}{\partial v^2}\right)=0$$

(or) $4x^2 \dfrac{\partial^2 z}{\partial u \partial v} - \dfrac{2x}{y}\dfrac{\partial z}{\partial v} = 0$ (or)

$$2xy \frac{\partial^2 z}{\partial u \partial v} - \frac{\partial z}{\partial v} = 0$$

(or) $2u\left(\dfrac{\partial^2 z}{\partial u \partial v}\right) - \left(\dfrac{\partial z}{\partial v}\right) = 0$ using ② $\longrightarrow$ (7)

This is the required canonical form of (1)

we now proceed to find solution of (1).

Multiplying both sides of (7) by v, we get

$$2uv \frac{\partial^2 z}{\partial u \partial v} - v\frac{\partial z}{\partial v} = 0 \quad \text{(or)}$$

$$(2uv DD' - vD')z = 0 \longrightarrow (8)$$

where $D \equiv \dfrac{\partial}{\partial u}$ and $D' \equiv \dfrac{\partial}{\partial v}$. we now reduce (8) to a linear equation with constant co-efficients by using the usual method.

Let $u = e^x$ and $v = e^y$ so that $x = \log u$ and $y = \log v$. $\longrightarrow$ (9)

Let $D_1 \equiv \dfrac{\partial}{\partial x}$ and $D_1' \equiv \partial/\partial y$ then (8) reduces to

$(2D_1 D_1' - D_1')z = 0$ (or) $D_1'(2D_1 - 1)z = 0$

its general sol^n is given by

$$z = e^{x/2}\phi_1(y) + \phi_2(x) = u^{1/2}\phi_1(\log v) + \phi_2(\log u)$$

$$= u^{1/2}\psi_1(v) + \psi_2(u) = (xy)^{1/2}\psi_1(x/y) + \psi_2(xy)$$

$$= x\left(\frac{y}{x}\right)^{1/2}\psi_1(x/y) + \psi_2(xy)$$

$$= x f(x/y) + \psi_2(xy)$$

where f and ψ_2 are arbitrary functions

———— Ans.

6(c) → Obtain temperature distribution $y(x,t)$ in a uniform bar of unit length whose one end is kept at $10°C$ and the other end is insulated. further it is given that.

$$y(x,0) = 1-x \quad , \quad 0<x<1.$$

Sol^n Suppose the bar be placed along the x-axis with its one end (which is at $10°c$) at origin and the other end at $x=1$ (which is insulated so that flux $-K(\partial y/\partial x)$ is zero there, K being the thermal conductivity).

Then we are to solve

$$\frac{\partial y}{\partial t} = k\left(\frac{\partial^2 y}{\partial x^2}\right) \quad \text{——} (1)$$

with B.C. $y_x(1,t)=0$, $y(0,t)=10$ ——(2)

and I.C. $y(x,0) = 1-x$, $0<x<1$ ——(3)

Let $\quad y(x,t) = u(x,t)+10$ ——(4)

ie. $\quad u(x,t) = y(x,t)-10$ ——(5)

Using (4) or (5), (1), (2) and (3) reduce to

$$\frac{\partial u}{\partial t} = k\left(\frac{\partial^2 u}{\partial x^2}\right) \quad \text{——} (6)$$

$$u_x(1,t)=0 \quad , \quad u(0,t)=0 \quad \text{——}(7)$$

$$u(x,0) = y(x,0)-10 = -(x+9) \quad \text{——}(8)$$

Suppose that (6) has solutions of the form

$$u(x,t) = X(x)T(t) \quad \text{——} (9)$$

Substituting this value of u in (6), we get.

$$XT' = kX''T \quad \text{or} \quad X''/X = T'/kT \quad \text{——}(10)$$

Since x and t are independent variables, (5) can only be true if each side is equal to the same constant, say μ.

$$\therefore \quad X'' - \mu X = 0 \quad \text{——— (11)}$$

$$T'' = \mu k T \quad \text{——— (12)}$$

using (7) and (9) —

$$X'(1)T(t) = 0 \text{ and } X(0)T(t) = 0 \quad \text{——— (13)}$$

Since $T(t) = 0$, leads to $u \equiv 0$, so we suppose that $T(t) \neq 0$

$$\therefore \text{ from (13)} \Rightarrow X'(1) = 0 \text{ and } X(0) = 0 \quad \text{——— (14)}$$

We now solve (11) under B.C. (14). Three cases arise -

<u>Case - I:</u> Let $\mu = 0$, then solution of (11) is

$$X(x) = Ax + B \quad \text{——— (15)}$$

from (15) $\quad X'(x) = A \quad$ ——— (15′)

Using B.C. (14), (15) and (15′) give $0 = A$ and $0 = B$

So from (15), $X(x) \equiv 0$, which leads to $u \equiv 0$.

So reject $\mu = 0$.

<u>Case. II:</u> Let $\mu = \lambda^2$, $\lambda \neq 0$. Then solution of (11) is -

$$X(x) = Ae^{\lambda x} + Be^{-\lambda x} \quad \text{——— (16)}$$

so that $\quad X'(x) = A\lambda e^{\lambda x} - B\lambda e^{-\lambda x} \quad$ ——— (16′)

Using B.C. (14), (16) and (16′) give

$$0 = A\lambda e^{\lambda} - B\lambda e^{-\lambda} \text{ and } 0 = A + B$$

These give $A = B = 0$ So that $X(x) \equiv 0$ and Hence $u(x) \equiv 0$ and hence $u(x) \equiv 0$.

So we reject $\mu = \lambda^2$.

<u>Case - III:</u> - Let $\mu = -\lambda^2$, $\lambda \neq 0$. The solution of (11)

$$X(x) = A\cos\lambda x + B\sin\lambda x \quad \text{——— (17)}$$

so that $\quad X'(x) = -A\lambda \sin\lambda x + B\lambda \cos\lambda x \quad$ ——— (17′)

Using B.C. (14), (17) and (17') give.

$$0 = -A\lambda \sin\lambda + B\lambda \cos\lambda \text{ and } 0 = A$$

These give $A = 0$ and $\cos\lambda = 0$; ——— (18)

where we have taken $B \neq 0$, since otherwise $X(x) \equiv 0$ and hence $u \equiv 0$

Now $\cos\lambda = 0 \Rightarrow \lambda = \frac{1}{2}(2n-1)\pi$, $n = 1, 2, 3, \ldots$

$$\therefore \quad \mu = -\lambda^2 = -\frac{1}{4}(2n-1)^2\pi^2 \quad \text{——— (19)}$$

Hence non-zero solutions $X_n(x)$ of (17) are given by,

$$X_n(x) = B_n \sin\left\{\frac{1}{2}(2n-1)\pi x\right\}$$

Again using (19), (12) reduces to

$$\frac{dT}{dt} = \frac{-(2n-1)^2\pi^2 k}{4} T \quad \text{or} \quad \frac{dT}{T} = -C_n^2\, dt \quad \text{——— (20)}$$

where $C_n^2 = \frac{1}{4}(2n-1)^2\pi^2 k$ ——— (21)

Solving (20), $T_n(t) = D_n e^{-C_n^2 t}$ ——— (22)

So, $u_n(x,t) = X_n T_n = E_n \sin\frac{(2n-1)\pi x}{2} e^{-C_n^2 t}$

are solutions of (6), satisfying (7). Here.

$E_n (= B_n D_n)$ in another arbitrary constant. In order to obtain a solution also satisfying (8), we consider more general solution.

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} E_n \sin\frac{(2n-1)\pi x}{2} e^{-C_n^2 t} \quad \text{——— (23)}$$

putting $t = 0$ in (23) and using (8), we have

$$-(x+9) = \sum_{n=1}^{\infty} E_n \sin\frac{(2n-1)\pi x}{2} \quad \text{——— (24)}$$

Multiplying both sides of (24) by $\sin\left\{\frac{1}{2}(2m-1)\pi x\right\}$ and then integrating with respect to x from 0 to 1, we get,

$$-\int_0^1 (x+9)\sin\left\{\frac{1}{2}(2m-1)\pi x\right\}dx$$

$$= \sum_{n=1}^{\infty} E_n \int_0^1 \sin\frac{(2n-1)\pi x}{2}\sin\frac{(2m-1)\pi x}{2}\,dx \quad \text{---} (25)$$

But $\int_0^1 \sin\frac{(2n-1)\pi x}{2}\sin\frac{(2m-1)\pi x}{2}\,dx.$

$$= \begin{cases} 0, & \text{if } m \neq n \\ 1, & \text{if } m = n \end{cases} \quad \text{---} (26)$$

Using (26), (25) gives $-\int_0^1 (x+9)\sin\frac{(2m-1)\pi x}{2}\,dx = E_m$

$$\therefore E_n = -\int_0^1 (x+9)\sin\frac{(2n-1)\pi x}{2}.$$

$$= -2\left[(x+9)\left\{\frac{-\cos\frac{(2n-1)\pi x}{2}}{\frac{(2n-1)\pi}{2}}\right\} - (1)\left\{\frac{-\sin\frac{(2n-1)\pi x}{2}}{\frac{(2n-1)^2\pi^2}{4}}\right\}\right]_0^1$$

(on using chain rule of integration by parts]

$$= \frac{8(-1)^n}{(2n-1)^2\pi^2} - \frac{36}{(2n-1)\pi} \quad \left\{\begin{array}{l} \because \cos\frac{(2n-1)\pi}{2} = 0 \\ \text{and } \sin\frac{(2n-1)\pi}{2} = (-1)^{n-1} \end{array}\right\} \text{-}(27)$$

Using (23) and (4) the required solution is given by

$$y(x,t) = 10 + \sum_{n=1}^{\infty} E_n \sin\frac{(2n-1)\pi x}{2}\, e^{-C_n^2 t}.$$

where C_n and E_n are given by (21) and (27) respectively.

7(b) The velocity v of a particle at distance s from a point on its path is given by the table:

s ft :	0	10	20	30	40	50	60
v ft/sec :	47	58	64	65	61	52	38

Estimate the time taken to travel 160ft by using Simpson's $\frac{1}{3}$ rule. Compare the result with Simpson's $3/8$ rule.

Sol'n: We know,

$$\text{velocity } v = \frac{ds}{dt}$$

$$\therefore dt = \frac{ds}{v}$$

$$\therefore t = \int dt = \int_{s_0}^{s} \frac{ds}{v} = \int_{s_0}^{s} y\, ds \quad \text{---} (1)$$

where $y = \frac{1}{v}$

s	0	10	20	30	40	50	60
v	47	58	64	65	61	52	38
$y=\frac{1}{v}$	0.0213	0.0172	0.0156	0.0154	0.0164	0.0192	0.0263
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

Using Simpson's $\frac{1}{3}$rd rule.

$$I = \frac{h}{3}\left[y_0 + 4(y_1+y_3+y_5)+2(y_2+y_4+y_6)\right]$$

$$= 1.06 \text{ sec}$$

Using Simpson's $3/8$th rule.

$$I = \frac{3h}{8}\left[y_0 + 3y_1 + 3y_2 + 2y_3 + 3y_4 + 3y_5 + y_6\right]$$

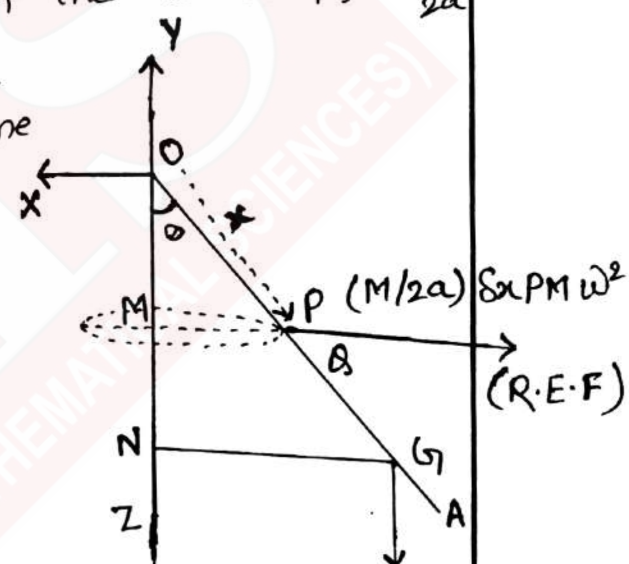
$$= 1.064 \text{ sec}.$$

& Difference between $\frac{1}{3}$rd rule & $3/8$ rule $= 0.00087$ sec

8(a). A uniform rod OA, of length 2a, free to turn about its end O, revolves with uniform angular velocity ω about the vertical OZ through O, and is inclined at a constant angle α to OZ, find the value of α.

Sol'n: Let the rod OA of length 2a and mass M revolve with uniform angular velocity ω about the vertical OZ through O, making a constant angle α to OZ. Let $PQ = \delta x$ be an element of the rod at a distance x from O. The mass of the element PQ is $\frac{M}{2a}\delta x$

This element PQ will make a circle in the horizontal plane with radius PM ($= x\sin\alpha$) and centre at M. Since the rod revolve with uniform angular velocity, the only effective force on this element is $\frac{M}{2a}\delta x \cdot PM \cdot \omega^2$ along PM.



Thus the reversed effective force on the element PQ is $\frac{M}{2a}\delta x \cdot x\sin\alpha \cdot \omega^2$ along MP.

Now by D'Alembert's principle all the reversed effective forces acting at different points of the rod, and the external forces, weight mg and reaction at O are in equilibrium. To avoid reaction at O, taking moment about O, we get

$$\Sigma\left(\frac{M}{2a}\delta x \cdot \omega^2 \cdot \sin\alpha\right) \cdot OM - Mg \cdot NG = 0$$

(or) $$\int_0^{2a} \frac{M}{2a}\omega^2 x^2 \sin\alpha \cos\alpha \, dx - Mg \cdot a\sin\alpha = 0, \quad (\because OM = x\cos\alpha)$$

(or) $\frac{M}{2a}\omega^2 \cdot \left\{\frac{1}{3}(2a)^3\right\} \cdot \sin\alpha\cos\alpha - Mg\, a\sin\alpha = 0$

(or) $Mg\, a\sin\alpha\left(\frac{4a}{3g}\omega^2\cos\alpha - 1\right) = 0$

$\therefore$ either $\sin\alpha = 0$ i.e. $\alpha = 0$

or $\frac{4a}{3g}\omega^2\cos\alpha - 1 = 0$

i.e. $\cos\alpha = \frac{3g}{4a\omega^2}$

Hence, the rod is inclined at an angle zero or $\cos^{-1}\left(\frac{3g}{4a\omega^2}\right)$.

Q. 8 b.) → Use Hamilton's equations to find the equations of the motion of a particle in a plane reffered to moving axes.

Sol: Let OX and OY be the fixed axes the in the x-y plane which is the plane of motion. Let OA and OB be the moving axes in x-y plane turning with angular velocity ω.

If (x, y) are the coordinates of a particle m reffered to OX and OY then the components of the velocity of the particle reffered to the moving axes OA and OB are

$$\dot{x} - \omega y \text{ and } \dot{y} + \omega x$$

$$\therefore \text{K.E.}, T = \frac{1}{2}m\left[(\dot{x} - \omega y)^2 + (\dot{y} + \omega x)^2\right]$$

$$\therefore L = T - V = \frac{1}{2}m\left[(\dot{x} - \omega y)^2 + (\dot{y} + \omega x)^2\right] - V$$

where V is the potential and is a function of x, y and t only.

Here x and y are the generalised coordinates.

$$\therefore p_x = \frac{\partial L}{\partial \dot{x}} = m(\dot{x} - \omega y), \quad p_y = \frac{\partial L}{\partial \dot{y}} = m(\dot{y} + \omega x) \quad \text{——— (1)}$$

Since L does not contain t explicitely,

$$\therefore H = -L + (p_x\dot{x} + p_y\dot{y})$$

$$= -\frac{1}{2}m\left[(\dot{x} - \omega y)^2 + (\dot{y} + \omega x)^2\right] + p_x\dot{x} + p_y\dot{y} + V$$

Eliminating $\dot{x}$ and $\dot{y}$, using relations (1), we have

$$H = -\frac{1}{2m}(p_x^2 + p_y^2) + p_x\left(\frac{1}{m}p_x + \omega y\right) + p_y\left(\frac{1}{m}p_y - \omega x\right) + V$$

or

$$H = \frac{1}{2m}(p_x^2 + p_y^2) + p_x\omega y - p_y\omega x + V$$

Hence the four Hamilton's equations are

$$\dot{p}_x = -\frac{\partial H}{\partial x} = \omega p_y - \frac{\partial V}{\partial x} \quad \text{———} \quad (H_1)$$

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{1}{m} p_x + \omega y \quad \text{———} \quad (H_2)$$

$$\dot{p}_y = -\frac{\partial H}{\partial y} = -\omega p_x - \frac{\partial V}{\partial y} \quad \text{———} \quad (H_3)$$

$$\dot{y} = \frac{\partial H}{\partial p_y} = \frac{1}{m} p_y - \omega x \quad \text{———} \quad (H_4)$$

Differentiating (H_2) and using (H_1), we have

$$m\ddot{x} = \dot{p}_x + m\omega\dot{y} = \left(\omega p_y - \frac{\partial V}{\partial x}\right) + m\omega\dot{y}$$

$$= \omega \cdot m(\dot{y} + \omega x) + m\omega\dot{y} - \frac{\partial V}{\partial x}, \text{ from}(1)$$

or $m(\ddot{x} - 2\omega\dot{y} - \omega^2 x) = -\frac{\partial V}{\partial x}$

Similarly, $m(\ddot{y} + 2\omega\dot{x} - \omega^2 y) = -\frac{\partial V}{\partial y}$ ——— (3)

Equations (2) and (3) are the required equations of motion.

Q. 8.C.) → Test whether the motion specified by $q = \frac{K^2(xj - yi)}{x^2+y^2}$ (K = const), is a possible motion for an incompressible fluid. If so, determine the equation of the streamlines. Also test whether the motion is of the potential kind and if so determine the velocity potential.

Sol: Let $q = ui + vj + wK$ then here

$$u = -\frac{K^2 y}{x^2+y^2}, \quad v = \frac{K^2 x}{x^2+y^2}, \quad w = 0 \quad \text{——— (1)}$$

The equation of continuity for an incompressible fluid is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \text{——— (2)}$$

From (1), $\frac{\partial u}{\partial x} = \frac{2K^2xy}{(x^2+y^2)^2}$, $\frac{\partial v}{\partial y} = -\frac{2K^2xy}{(x^2+y^2)^2}$, $\frac{\partial w}{\partial z} = 0$

Hence (2) is satisfied and so the motion specified by given q is possible.

The equation of the streamlines are:

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

i.e. $$\frac{dx}{-K^2y/(x^2+y^2)} = \frac{dy}{K^2x/(x^2+y^2)} = \frac{dz}{0} \quad \text{——— (3)}$$

Taking the last fraction,

$$dz = 0 \quad \text{so that} \quad z = c_1 \quad \text{——— (4)}$$

Taking the first two fractions in (3) and simplifying, we get

$$dx/(-y) = dy/x \quad \text{or} \quad 2x\,dx + 2y\,dy = 0$$

Integrating,

$$x^2+y^2 = c_2, \quad \text{——— (5)}$$

c_2 being an arbitrary constant.

(4) and (5) together give the streamlines. clearly, the streamlines are circles whose centres are on the z-axis, their planes being perpendicular to this axis.

Again

$$\text{curl}\, q = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -\frac{k^2 y}{x^2+y^2} & \frac{k^2 x}{x^2+y^2} & 0 \end{vmatrix}$$

$$= k^2 \left\{ \frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2} \right\} k = 0$$

Hence the flow is of the potential kind and we can find velocity potential $\phi(x,y,z)$ such that $q = -\nabla\phi$

Thus, we have

$$\frac{\partial\phi}{\partial x} = -u = \frac{k^2 y}{x^2+y^2} \quad \text{——— (6)}$$

$$\frac{\partial\phi}{\partial y} = -v = -\frac{k^2 x}{x^2+y^2} \quad \text{——— (7)}$$

$$\frac{\partial\phi}{\partial z} = -w = 0 \quad \text{——— (8)}$$

Equation (8) shows that the velocity potential ϕ is function of x and y only so that $\phi = \phi(x,y)$

Integrating (6),

$$\phi(x,y) = k^2 \tan^{-1}(x/y) + f(y), \quad \text{——— (9)}$$

where $f(y)$ is an arbitrary function.

From (9),
$$\frac{\partial \phi}{\partial y} = f'(y) - \frac{K^2 x}{x^2+y^2} \quad \text{———} \quad (10)$$
comparing (7) and (10), we have

$$f'(y) = 0 \quad \text{so that} \quad f(y) = \text{constant}.$$

Since the constant can be omitted while writting velocity potential, the required velocity potential can be taken as [refer eq.(9)]

$$\phi(x,y) = K^2 \tan^{-1}(x/y) \quad \text{———} \quad (11)$$

The equipotentials are given by

$$K^2 \tan^{-1}(x/y) = \text{constant} = K^2 \tan^{-1} c$$

or

$$x = cy, \ c \text{ being a constant.}$$

which are planes through the z-axis. They are intersected by the streamlines as shown in the figure. Dotted lines represent equipotentials and ordinary lines represent streamlines.