2013 - ODE - IFoS

Q 1 ⇒

Solve $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$

Sol^n

Given Eq^n $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$

$$\Rightarrow \sec^2 y \frac{dy}{dx} + 2x \tan y = x^3$$

let $\tan y = t$

replace t-

$\sec^2 y \frac{dy}{dx} = \frac{dt}{dx}$

$$\frac{dt}{dx} + 2xt = x^2$$

Compare with $\frac{dt}{dx} + P(x)t = Q(x)$

$P(x) = 2x$; $Q(x) = x^2$

$$I.F. = e^{\int P dx} = e^{x^2}$$

$$t.\, e^{x^2} = \int x^3 e^{x^2} dx + c$$

$x^2 = p$

$2x \frac{dx}{dp} = dp$

$x dx = \frac{dp}{2}$

$$t.\, e^{x^2} = \frac{1}{2} \int p e^p dp + c$$

$$t.\, e^{x^2} = \frac{1}{2} [p e^p - e^p] + c$$

$$t.\, e^{x^2} = \frac{1}{2} [x^2 - 1] e^{x^2} + c$$

$$\boxed{\tan y \, e^{x^2} = \frac{e^{x^2}}{2}(x^2 - 1) + c}$$

Q 2 ⇒

Solve the differential equation $\frac{d^2y}{dx^2} - 4x \frac{dy}{dx} + (4x^2 - 1)y = -3e^{x^2} \sin 2x$ by changing the dependent variable.

Sol^n :-

Given Eq^n $\frac{d^2y}{dx^2} - 4x \frac{dy}{dx} + (4x^2 - 1)y = -3e^{x^2} \sin 2x$

Comparing Given Eq^n with

$$\frac{d^2y}{dx^2} + P(x) \frac{dy}{dx} + Q(x) y = R(x)$$

$R(x) = -3e^{x^2} \sin 2x$

$P(x) = -4x$; $Q(x) = (4x^2 - 1)$

$$u = e^{-1/2 \int P dx} = e^{x^2}$$

$$I = Q - \frac{P^2}{4} - \frac{1}{2}\frac{dP}{dx} = 1.$$

$$S = \frac{R}{u} = \frac{-3e^{x^2}\sin 2x}{e^{x^2}} = -3\sin 2x$$

$$\frac{d^2v}{dx^2} + Iv = S$$

$$\frac{d^2v}{dx^2} + v = -3\sin 2x$$

$$(D^2+1)v = -3\sin 2x$$

$$m = \pm i$$

$$V_c = [c_1\cos x + c_2 \sin x]$$

$$V_p = -3\frac{\sin 2x}{D^2+1} = \frac{-3}{(-4+1)}\sin 2x = \sin 2x.$$

$$V = V_c + V_p = (c_1\cos x + c_2\sin x + \sin 2x)$$

$$y = uv \Rightarrow \boxed{y = e^{x^2}(c_1\cos x + c_2\sin x + \sin 2x)}$$

Q3⇒ Solve $(D^3+1)y = e^{x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)$; where $D = \frac{d}{dx}$

Sol^n Given Eq^n $(D^3+1)y = e^{x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)$

C.F. Auxillary Eq^n $m^3+1=0$

$$(m+1)(m^2-m+1)=0$$

$$m_1 = -1,\ m_2, m_3 = \frac{1\pm\sqrt{3}}{2}$$

$$y_c = c_1e^{-x} + e^{x/2}\left\{c_2\cos\frac{\sqrt{3}}{2}x + c_3\sin\frac{\sqrt{3}}{2}x\right\}$$

P.I.

$$y_p = \frac{e^{x/2}\sin\left(\frac{\sqrt{3}}{2}x\right)}{D^3+1} = e^{x/2}\frac{1}{(D+\frac{1}{2})^3+\frac{1}{2}}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{x/2}\frac{1}{(D^3+\frac{1}{8}+\frac{3}{2}D^2+\frac{3}{4}D+\frac{1}{2})}\sin\left(\frac{\sqrt{3}}{2}x\right)$$

$$= e^{x/2}\frac{\sin\left(\frac{\sqrt{3}}{2}x\right)}{-\frac{3}{4}\cancel{D}+\frac{1}{8}-\frac{9}{8}+\frac{1}{2}+\frac{3}{4}\cancel{D}}$$

$$y_p = -2\left(e^{x/2}\sin\frac{\sqrt{3}}{2}x\right)$$

$$y = y_c + y_p$$

$$\boxed{y = c_1 e^{-x} + e^{x/2}\left\{c_2 \cos\frac{\sqrt{3}}{2}x + c_3 \sin\frac{\sqrt{3}}{2}x - 2\sin\frac{\sqrt{3}}{2}x\right\}}$$

Q4→ Apply the method of variation of parameters to Solve $\frac{d^2y}{dx^2} - y = 2(1+e^x)^{-1}$.

Sol^n Given Eq^n $\frac{d^2y}{dx^2} - y = 2(1+e^x)^{-1}$

Its homogeneous Eq^n.

$$(D^2-1)y = 0 \qquad D = \pm 1$$

$$y_c = c_1 e^x + c_2 e^{-x}$$

Comparing with

$$y_c = c_1 u(x) + c_2 v(x)$$

$$u(x) = e^x, \quad v(x) = e^{-x}$$

$$w = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2 = uv' - u'v$$

$$A = \int \frac{-vR}{uv'-vu'}dx = \int \frac{-e^{-x} \times 2(1+e^x)^{-1}}{-2}dx = \int \frac{e^{-x}}{(1+e^x)^{-1}}$$

$$A = \int \frac{e^{-x}}{e^{x}(e^{-x}+1)^{-1}}dx \qquad 1+e^{-x} = t \qquad -e^{-x}dx = dt$$
$$e^{-x}dx = -dt$$

$$A = -\int \frac{t}{(t-1)}dt = - \int \frac{(t-1)+1}{t-1}dt$$

$$A = -\int \left(1 + \frac{1}{t-1}\right)dt = -\left[t + \log(t-1)\right]$$

$$A = -\left[1+e^{-x} + \log e^{-x}\right] = -\left[1+e^{-x} - x\right]$$

$$A = x - 1 - e^{-x}$$

$$B = \int \frac{uR}{uv'-vu'} = -\int \frac{e^x}{1+e^x} = -\log(1+e^x).$$

$$\boxed{y = \left(c_1 e^x + c_2 e^{-x}\right) + (x-1-e^{-x})e^x - \log(1+e^x)\,e^{-x}}$$