2013

IFoS
S&D·

37 (5b)

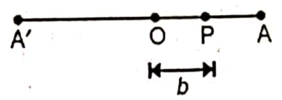
5(b) A particle is performing a simple harmonic motion of period T about centre O and it passes through a point P, where OP = b with velocity v in the direction of OP. Find the time which elapses before it returns to P. 8

on of the given equations. In other
a, b, c should satisfy the relation

$\frac{b}{}=\frac{c}{4}$ for (a, b, c) to be in Ker T.

-2, 4, 3) is one member of Ker T and

5. (*b*) We have to find time taken from P to A and then A to P



$t = 2$ (time from A to P)

$$= 2\int_0^t dt = 2\int_a^p \frac{dx}{\sqrt{u}\sqrt{a^2-x^2}}$$ 5

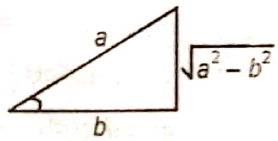
(Ignoring –ve sign) $\left(\frac{dx}{dt} = \sqrt{u}\sqrt{a^2-x^2}\right)$

$$= \frac{2}{\sqrt{\mu}}\left[\cos^{-1}\frac{x}{a}\right]_a^b$$

$$= \frac{2}{\sqrt{\mu}}\left[\cos^{-1}\frac{b}{a} - \cos^{-1}\frac{a}{b}\right]$$

$$= \frac{2}{\sqrt{\mu}}\cos^{-1}\frac{b}{a}$$

$$\Rightarrow \quad t = \frac{2}{\sqrt{\mu}}\tan^{-1}\left(\frac{\sqrt{a^2-b^2}}{b}\right)$$



$$= \frac{2}{\sqrt{\mu}}\tan^{-1}\left(\frac{v}{b\sqrt{\mu}}\right)$$

$$= \frac{2}{\frac{2\pi}{T}}\tan^{-1}\left[\frac{v}{b\left(\frac{2\pi}{T}\right)}\right]$$

$$= \frac{T}{\pi}\tan^{-1}\left[\frac{vT}{2\pi b}\right].$$

$$\left[\begin{aligned} \because \quad v^2 &= \mu(a^2-b^2) \\ \Rightarrow \quad v &= \sqrt{a}\sqrt{(a^2-b^2)} \\ \Rightarrow \quad \frac{v}{\sqrt{\mu}} &= \sqrt{a^2-b^2} \end{aligned}\right]$$

$$\left[T = \frac{2\pi}{\sqrt{\mu}} \Rightarrow \sqrt{\mu} = \frac{2\pi}{T}\right]$$

Proved.

$\frac{}{192} \cdot \frac{}{7} = \frac{}{56}$.

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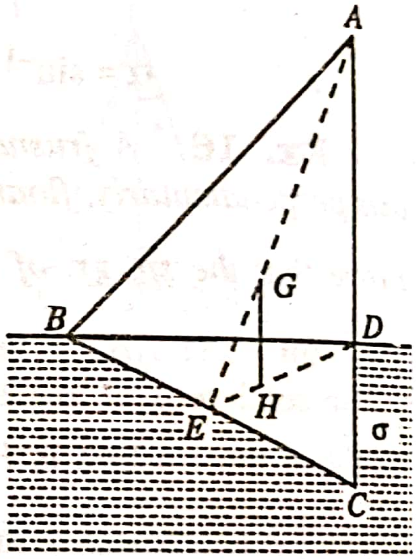
Ex. 17. *A triangular lamina ABC of density ρ floats in a liquid of density σ with its plane vertical, the angle B being in the surface of the liquid and the angle A not immersed. Show that*

$$\frac{\rho}{\sigma} = \frac{\sin A \cos C}{\sin B} = \frac{a^2 + b^2 - c^2}{2b^2},$$

a, b, c being the lengths of the sides of the triangle. **(Rohilkhand 1998, 99, 2000, 2004; Garhwal 2004)**

Sol. The portion BCD of the ΔABC is immersed in the liquid with BD in contact with the surface. Let G and H be the centres of gravity and buoyancy respectively.



E is the mid-point of BC.

The conditions of equilibrium are :

(i) The line GH must be vertical.

(ii) The weight of the lamina must be equal to the weight of the liquid displaced.

Since $EG = \frac{1}{3} EA$, $EH = \frac{1}{3} ED$, GH is parallel to AD.

But GH is vertical from the first condition so AC must be vertical.

From the second condition of equilibrium, we have

$$\Delta ABC\, \rho g = \Delta BDC\, \sigma g.$$

$$\therefore \quad \frac{\rho}{\sigma} = \frac{\Delta BDC}{\Delta ABC} = \frac{\frac{1}{2} BD \cdot DC}{\frac{1}{2} BD \cdot AC} = \frac{DC}{AC} = \frac{BC \cos C}{AC}.$$

But $\quad \frac{AC}{\sin B} = \frac{BC}{\sin A} \quad$ or $\quad BC = \frac{AC \sin A}{\sin B}$.

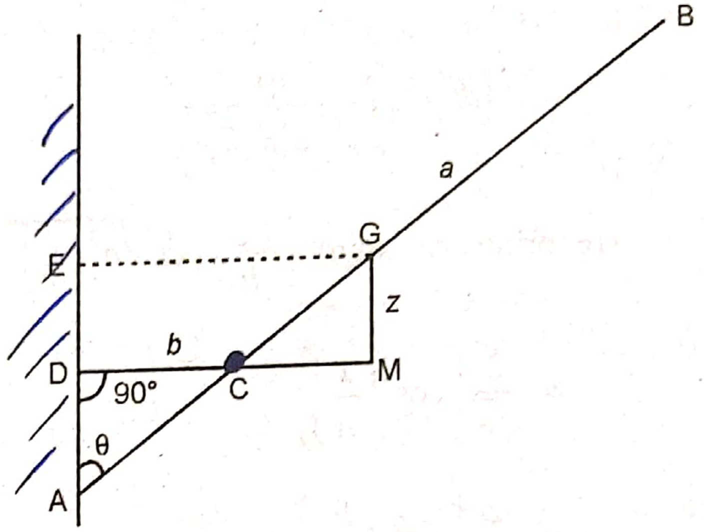
Hence $\quad \frac{\rho}{\sigma} = \frac{AC \sin A \cos C}{AC \sin B} = \frac{\sin A \cos C}{\sin B} = \frac{a}{b} \cdot \frac{a^2 + b^2 - c^2}{2ab} = \frac{a^2 + b^2 - c^2}{2b^2}.$

Ex. 18. A uniform prism whose cross-section ... vertical

39 (5e)

5(e) A heavy uniform rod rests with one end against a smooth vertical wall and with a point in its length resting on a smooth peg. Find the position of equilibrium and discuss the nature of equilibrium. 8

5(e) The end A rests against smooth wall and rod rests on a smooth peg C, and CD = b.



Suppose the rod makes an angle θ with the wall. The centre of gravity of the rod is at its middle point G. Let z be the height of G above the fixed peg C, *i.e.*, GM = z. We shall express z in terms of θ. We have,

$$z = GM = ED = AE - AD$$
$$= AG\cos\theta - CD\cot\theta$$
$$= a\cos\theta - b\cot\theta.$$

$\therefore \quad dz/d\theta = -a\sin\theta + b\,\text{cosec}^2\,\theta$

and $d^2z/d\theta^2 = -a\cos\theta - 2b\,\text{cosec}^2\theta\cot\theta$

For equilibrium of the rod, we have $dz/d\theta = 0$

i.e., $-a\sin\theta + b\,\text{cosec}^2\theta = 0$

or $\quad a\sin\theta = b\,\text{cosec}^2\theta,$

or $\quad \sin^3\theta = b/a$

or $\quad \sin\theta = (b/a)^{1/3},$

or $\quad \theta = \sin^{-1}(b/a)^{1/3}.$

This gives the position of equilibrium of the rod.

Again $d^2z/d\theta^2 = -(a\cos\theta + 2b\,\text{cosec}^2\theta\cot\theta)$

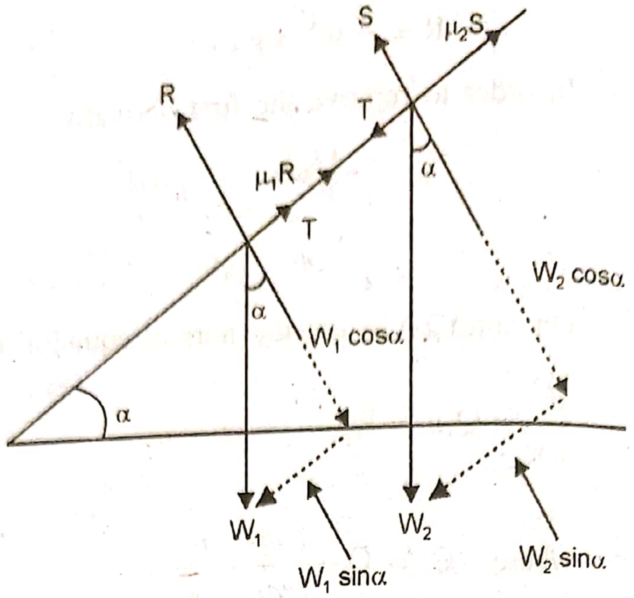
= negative for all actute values of θ.

Thus $d^2z/d\theta^2$ is negative in the position of equilibrium and so z is maximum. Hence the equilibrium is unstable.

40 (6c)

6(c) Two bodies of weights w_1 and w_2 are placed on an inclined plane and are connected by a light string which coincides with a line of greatest slope of the plane; if the coefficient of friction between the bodies and the plane are respectively μ_1 and μ_2, find the inclination of the plane to the horizontal when both bodies are on the point of motion, it being assumed that smoother body is below the other. 14

6. (*c*)



R and S are normal reactions and $\mu_1 R$ and $\mu_2 S$ are forces of friction. Let T be the tension in the string. Let α be the inclination of plane to the horizontal.

For W_1 : For limiting equilibrium,

<u>Horizontally</u>

$$\mu_1 R + T = W_1 \sin\alpha$$

$$\Rightarrow \quad T = W_1 \sin\alpha - \mu_1 R \quad \text{....}(i)$$

<u>Vertically</u>

$$R = W_1 \cos\alpha \quad \text{....}(ii)$$

From (*i*) and (*ii*), we get

$$T = W_1 \sin\alpha - \mu_1 W_1 \cos\alpha \quad \text{...}(iii)$$

For W_2 : For limiting equilibrium,

<u>Horizontally</u>

$$T + W_2 \sin\alpha = \mu_2 S$$

$$\Rightarrow \quad T = \mu_2 S - W_2 \sin\alpha \quad \text{...}(iv)$$

<u>Vertically,</u>

$$S = W_2 \cos\alpha \quad \text{...}(v)$$

From (*iv*) and (*v*), we get

$$T = \mu_2 W_2 \cos\alpha - W_2 \sin\alpha \quad \text{...}(vi)$$

From (*iii*) and (*vi*), we get,

$$W_1 \sin\alpha - \mu_1 W_1 \cos\alpha = \mu_2 W_2 \cos\alpha - W_2 \sin\alpha$$

$$\Rightarrow W_1 \sin\alpha + W_2 \sin\alpha = \mu_1 W_1 \cos\alpha + \mu_2 W_2 \cos\alpha$$

$$\Rightarrow \tan\alpha = \frac{\mu_1 W_1 + \mu_2 W_2}{W_1 + W_2}$$

Proceed as in Ex. 11.

Ex. 13. *A sphere of density σ floats just immersed in three liquids. The densities of the liquids in ascending order are $\rho, 4\rho, 9\rho$ and the thickness of two upper liquids are each one-third of the sphere. Prove that $27\sigma = 122\rho$.*

Sol. Proceed as in Ex. 11.

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Ex. 14. *A body floating in water has volumes V_1, V_2, V_3 above the surface, when the densities of the surrounding air are respectively ρ_1, ρ_2, ρ_3. Prove that*

$$\frac{\rho_2 - \rho_3}{V_1} + \frac{\rho_3 - \rho_1}{V_2} + \frac{\rho_1 - \rho_2}{V_3} = 0. \qquad \textbf{(Rohilkhand 1991, 93)}$$

Sol. Let V be the volume and W the weight of the body. Then the volumes immersed in water in the three cases are

$$(V - V_1), (V - V_2) \text{ and } (V - V_3).$$

Let ρ be the density of water.

For equilibrium, wt. of the body = wt. of water displaced + wt. of air displaced

$$\therefore \qquad W = (V - V_1)\rho g + V_1 \rho_1 g \quad \text{or} \quad W - V\rho g = V_1 g(\rho_1 - \rho)$$

or
$$\frac{W - V\rho g}{V_1} = g(\rho_1 - \rho) \qquad \ldots(1)$$

Similarly
$$\frac{W - V\rho g}{V_2} = g(\rho_2 - \rho) \qquad \ldots(2)$$

and
$$\frac{W - V\rho g}{V_3} = g(\rho_3 - \rho) \qquad \ldots(3)$$

Multiplying (1) by $(\rho_2 - \rho_3)$, (2) by $(\rho_3 - \rho_1)$ and (3) by $(\rho_1 - \rho_2)$ and adding, we get

$$(W - V\rho g)\left[\frac{\rho_2 - \rho_3}{V_1} + \frac{\rho_3 - \rho_1}{V_2} + \frac{\rho_1 - \rho_2}{V_3}\right] = 0$$

or $$\frac{\rho_2 - \rho_3}{V_1} + \frac{\rho_3 - \rho_1}{V_2} + \frac{\rho_1 - \rho_2}{V_3} = 0.$$

Note. The above result can be put in the form

$$V_2 V_3 (\rho_2 - \rho_3) + V_3 V_1 (\rho_3 - \rho_1) + V_1 V_2 (\rho_1 - \rho_2) = 0$$

or $$\rho_1 V_1 (V_2 - V_3) + \rho_2 V_2 (V_3 - V_1) + \rho_3 V_3 (V_1 - V_2) = 0.$$

42 (7c)

7(c) A particle is projected vertically upwards with a velocity u, in a resisting medium which produces a retardation kv^2 when the velocity is v. Find the height when the particle comes to rest above the point of projection. 14

7. (*c*) Equation of Motion is

$$\ddot{x} = -g - kv^2$$

For maximum/minimum velocity

$$\ddot{x} = 0 \Rightarrow v = \sqrt{g/k}$$

$$\therefore \qquad \ddot{x} = -g\left(1+\frac{v^2}{V^2}\right)$$

$$\text{or,} \qquad v.\frac{dv}{dx} = -g\left(1+\frac{v^2}{V^2}\right)$$

$$\frac{2g}{V^2}.x = \int\frac{2v.dv}{v^2+V^2}+c$$

$$= -\text{Log}\,(v^2 + V^2) + c$$

For, $x = 0$, $v = u$

$$\therefore \qquad c = \text{Log}\,(u^2 + V^2)$$

$$i.e., \quad \frac{2g}{V^2}.x = \text{Log}\,(u^2 + V^2) - \text{Log}\,(v^2 + V^2)$$

$$= \text{Log}\left(\frac{u^2+V^2}{v^2+V^2}\right)$$

At highest point, $v = 0$, therefore the greatest height

$$h = \frac{V^2}{2g}.\text{Log}\left(\frac{u^2+V^2}{V^2}\right)$$

8.a=?⇒ $$= \frac{V^2}{2g}.\text{Log}\left(1+\frac{u^2}{V^2}\right).$$

1. (b) *A particle is projected with velocity V along a smooth horizontal plane in a medium whose resistance per unit mass is μ times the cube of the velocity. Show that the distance it has described in time t is*

$$\frac{1}{\mu V}\left[\sqrt{(1+2\mu t V^2)}-1\right]$$

and that its velocity then is $V/\sqrt{(1-2\mu t V^2)}$. [Meerut 1976]

Sol. Here since particle is moving in a horizontal plane, the weight mg of the particle will not act. Hence the only force acting on the particle is that due to resistance and is equal to $-m\mu v^3$.

The equation of motion of the particle is

$$m\,(dv/dt)=-m\mu v^3 \quad \text{or} \quad -(dv/v^3)=\mu\, dt.$$

Integrating ; $\dfrac{1}{2v^2}=\mu t+C$, where C is a constant of integration.

Initially when $t=0$, $v=V$: $\therefore \quad C=1/2V^2$.

$$\therefore \quad \frac{1}{2v^2}=\mu t+\frac{1}{2V^2} \quad \text{or} \quad \frac{1}{v^2}=\frac{2\mu t V^2+1}{V^2}$$

or
$$v=V/\sqrt{(1+2\mu t V^2)}. \qquad \ldots(1)$$

If x be the distance described by the particle in time t, then equation (1) may be written as

$$\frac{dx}{dt}=\frac{V}{\sqrt{(1+2\mu t V^4)}} \quad \text{or} \quad dx=\frac{V}{\sqrt{(1+2\mu t V^2)}}\,dt.$$

Integrating, $x=\dfrac{V\,(1+2\mu t V^2)^{1/2}}{2\mu V^2 \times \frac{1}{2}}+C'$

or
$$x=\frac{1}{\mu V}\sqrt{(1+2\mu t V^2)}+C''. \qquad \ldots(2)$$

Initially when $t=0$, $x=0$; $\therefore \quad C'=-1/\mu V$.

Hence equation (2) becomes

$$x=\frac{1}{\mu V}\sqrt{(1+2\mu t V^2)}-\frac{1}{\mu V}$$

or
$$x=\frac{1}{\mu V}[\sqrt{(1+2\mu t V^2)}-1]. \qquad \ldots(3)$$

Equations (1) and (3) give required results.