# LINEAR ALGEBRA

## : IFoS 2016 :

① Let $T: \mathbb{R}^3 \to \mathbb{R}^4$ be given by $T(x,y,z) = (2x-y, 2x+z, x+2z, x+y+z)$. Find the matrix of $T$ w.r.t the standard basis of $\mathbb{R}^3$ and $\mathbb{R}^4$. Examine if $T$ is a linear map.

$\longrightarrow$ Let $S_1 = \{e_1, e_2, e_3\}$ where $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ & $e_3 = (0,0,1)$

$S_2 = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ where $\alpha_1 = (1,0,0,0)$, $\alpha_2 = (0,1,0,0)$, $\alpha_3 = (0,0,1,0)$, $\alpha_4 = (0,0,0,1)$.

be the standard basis of $\mathbb{R}^3$ and $\mathbb{R}^4$.

$$T(e_1) = T(1,0,0) = (2,2,1,1) = 2(1,0,0,0) + 2(0,1,0,0) + 1(0,0,1,0) + 1(0,0,0,1)$$
$$= 2\alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4.$$

$$T(e_2) = T(0,1,0) = (-1,0,0,1) = (-1)(1,0,0,0) + 0(0,1,0,0) + 0(0,0,1,0) + 1(0,0,0,1)$$
$$= (-1)\alpha_1 + 0\alpha_2 + 0\alpha_3 + \alpha_4.$$

$$T(e_3) = T(0,0,1) = (0,1,2,1) = 0(1,0,0,0) + 1(0,1,0,0) + 2(0,0,1,0) + 1(0,0,0,1)$$
$$= 0\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4.$$

Hence, the matrix of $T$ is given by: $A = \begin{bmatrix} 2 & -1 & 0 \\ 2 & 0 & 1 \\ 1 & 0 & 2 \\ 1 & 1 & 1 \end{bmatrix}$

<u>Now</u>: let $\alpha = (x_1, y_1, z_1)$ and $\beta = (x_2, y_2, z_2)$ where $\alpha, \beta \in \mathbb{R}^3$.

then $x_1, x_2, y_1, y_2, z_1, z_2 \in \mathbb{R}$. Let $a, b \in \mathbb{R}$. Then

$$T(a\alpha + b\beta) = T(a(x_1,y_1,z_1) + b(x_2,y_2,z_2))$$
$$= T(ax_1 + bx_2, ay_1 + by_2, az_1 + bz_2)$$
$$= (2(ax_1 + bx_2) - (ay_1 + by_2), 2(ax_1 + bx_2) + az_1 + bz_2, (ax_1 + bx_2) + 2(az_1 + bz_2), (ax_1 + bx_2) + (ay_1 + by_2) + (az_1 + bz_2))$$
$$= ((2ax_1 - ay_1) + (2bx_2 - by_2), (2ax_1 + az_1) + (2bx_2 + bz_2), (ax_1 + 2az_1) + (bx_2 + 2bz_2), (ax_1 + ay_1 + az_1) + b(x_2 + y_2 + z_2))$$
$$= a(2x_1 - y_1, 2x_1 + z_1, x_1 + 2z_1, x_1 + y_1 + z_1) + b(2x_2 - y_2, 2x_2 + z_2, x_2 + 2z_2, x_2 + y_2 + z_2)$$
$$= aT(\alpha) + bT(\beta).$$

$\therefore$ $T$ is a linear map

② For the matrix $A = \begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$, obtain the eigen value and get the value of $A^4+3A^3-9A^2$.

→ Char. eqn of A is given by $|A-\lambda I| = 0$

$$\begin{vmatrix} -1-\lambda & 2 & 2 \\ 2 & -1-\lambda & 2 \\ 2 & 2 & -1-\lambda \end{vmatrix} = 0 \Rightarrow (-1-\lambda)\left[(1+\lambda)^2-4\right]+2\left[4+2(1+\lambda)\right]+2\left[4+2(1+\lambda)\right]=0$$

$$\Rightarrow (-1-\lambda)\left[\lambda^2+2\lambda-3\right]+4\left[6+2\lambda\right]=0$$

$$\Rightarrow -\lambda^2-\lambda^3-2\lambda-2\lambda^2+3+3\lambda+24+8\lambda=0$$

$$\Rightarrow \lambda^3+3\lambda^2-9\lambda-27=0 \quad \text{——①}$$

$$\Rightarrow \lambda = 3, -3, -3.$$

Hence, the eigen values of A are $3, -3, -3$.

By Cayley-Hamilton's Theorem, A satisfies its char. eqn①.

Therefore: $A^3+3A^2-9A-27I=0$

Premultiplying both sides with A, we get

$$A\cdot A^3+3A\cdot A^2-9A\cdot A-27A\cdot I = A\cdot 0$$

$$\Rightarrow A^4+3A^3-9A^2 = 27A = 27\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} = \begin{bmatrix} -27 & 54 & 54 \\ 54 & -27 & 54 \\ 54 & 54 & -27 \end{bmatrix}$$

③ Let T be a linear map such that $T: V_3 \to V_2$ defined by $T(e_1) = 2f_1 - f_2$, $T(e_2) = f_1 + 2f_2$, $T(e_3) = 0f_1 + 0f_2$, where $e_1, e_2, e_3$ and $f_1, f_2$ are the standard basis of $V_3$ & $V_2$. Find the matrix of T relative to these bases. Further take two other bases ~~$B_1 = \{(1,0,0)$~~, $B_1 = \{(1,1,0),(1,0,1),(0,1,1)\}$ and $B_2 = \{(1,1),(1,-1)\}$. Obtain the matrix $T_1$ relative to $B_1$ & $B_2$.

→ $T(e_1) = 2f_1 - f_2$, $T(e_2) = f_1 + 2f_2$, $T(e_3) = 0f_1 + 0f_2$.

Therefore, matrix of T w.r.t the standard bases of $V_2$ and $V_3$ is given by $A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 2 & 0 \end{bmatrix}$.

The other bases are given as.

$B_1 = \{(1,1,0),(1,0,1),(0,1,1)\}$ and $B_2 = \{(1,1),(1,-1)\}$

Let $(x,y) \in V_2$ and $a,b \in \mathbb{R}$. Then,

$$(x,y) = a(1,1) + b(1,-1)$$

$$(x,y) = (a+b, a-b) \Rightarrow a+b = x, \quad a-b = y$$

$$\Rightarrow 2a = x+y \qquad b = -y+a$$

$$a = \frac{1}{2}(x+y) \qquad b = \cancel{a} - y + \frac{1}{2}x + \frac{1}{2}y$$

$$b = \frac{1}{2}(x-y)$$

$$\therefore (x,y) = \frac{1}{2}(x+y)(1,1) + \frac{1}{2}(x-y)(1,-1)$$

Now: $T(1,1,0) = T(e_1 + e_2 + 0e_3) = T(e_1) + T(e_2) + T(0)$ [T is a L.T.]

$$= 2f_1 - f_2 + f_1 + 2f_2 + 0$$

$$= 3f_1 + f_2 = (3,1) = 2(1,1) + 1(1,-1)$$

$$T(1,0,1) = T(e_1 + 0e_2 + e_3) = T(e_1) + T(0) + T(e_3)$$

$$= 2f_1 - f_2 + 0f_1 + 0f_2 = 2f_1 - f_2 = (2,-1)$$

$$= \frac{1}{2}(1,1) + \frac{3}{2}(1,-1)$$

$$T(0,1,1) = T(0.e_1 + e_2 + e_3) = T(0) + T(e_2) + T(e_3)$$

$$= 0 + f_1 + 2f_2 + 0f_1 + 0f_2 = f_1 + 2f_2 = (1,2)$$

$$= \frac{3}{2}(1,1) - \frac{1}{2}(1,-1)$$

$$\therefore \text{Matrix } T_1 = \begin{bmatrix} 2 & 1/2 & 3/2 \\ 1 & 3/2 & -1/2 \end{bmatrix}$$

(4) For the matrix $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, find two non singular matrices $P$ & $Q$ such that $PAQ = I$. Hence find $A^{-1}$!

→ <u>Theorem</u>: Any elementary row (or column) transformation on a matrix can be effected by pre-multiplication (or post multiplication) with the corresponding elementary matrix.

We can write $A$ as $A = IAI$.

$$\Rightarrow \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_1 \to R_1 - R_2$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\cancel{R_3 \to 3R_3 - R_2}$ $C_3 \to C_3 + C_2$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

$C_2 \to C_2 + 3C_3$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 3 & 1 \end{bmatrix}$$

$R_2 \leftrightarrow R_3$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ -2 & 3 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 3 & 1 \end{bmatrix}$$

$\cdot R_2 \to R_2 \div -1$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & -1 \\ -2 & 3 & 0 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 3 & 1 \end{bmatrix}$$

Let $P = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & -1 \\ -2 & 3 & 0 \end{bmatrix}$ and $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 3 & 1 \end{bmatrix}$.

Then $PAQ = I$. Now for $A^{-1}$: we have $PAQ = I$

$$\Rightarrow A = P^{-1}Q^{-1}$$
$$\Rightarrow A^{-1} = (P^{-1}Q^{-1})^{-1} = (Q^{-1})^{-1}(P^{-1})^{-1}$$
$$\Rightarrow A^{-1} = QP.$$

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 1 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & -1 \\ -2 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$$