Finals 2014

① Solve the initial value problem.

$$\frac{d^2y}{dt^2} + y = 8e^{-2t}\sin t \quad y(0) = 0,\ y'(0) = 0.$$

taking laplase on both sides we get,

$$p^2 L(y(t)) - p\,y(0) - y'(0) + L(y(t)) = 8\cancel{\frac{1}{p^2+1}}$$

$$= 8\frac{1}{1+(p+2)^2}.$$

$$L(y(t))[p^2+1] = \frac{8}{1+(p+2)^2}.$$

$$L(y(t)) = \frac{8}{(1+p^2)(1+(p+2)^2)}$$

$$\frac{Ap+B}{1+p^2} + \frac{Cp+D}{(1+(p+2)^2)}$$

$$(Ap+B)(1+(p+2)^2) + ((Cp+D)(1+p^2)$$

$\underline{p=0}.$ $B(1+4) + D = 8$ $\qquad D = 8-5B$

$5B + D = 8$ — ①.

$\underline{p=-2}$ $-2A+B+(D-2C)(5) = 8.$

$-2A+B-10C+5D = 8$ — ②

$\underline{p=1}$ $(A+B)(1+9) + (C+D)(2) = 8.$

$\cancel{10A}$ $5A+5B+C+D = 4$ — ③

$(-A+B)(2) + (-C+D)(2) = 8.$

$\underline{p=-1}.$ $-A+B-C+D = 4$ — ④

$$-2A+B-10C+5(8-5B)=8.$$

$$-2A-24B-10C=8(-4)$$

$$A+12B+5C=16 \quad -(1)$$

$$5A+\cancel{5B}+C+8-\cancel{5B}=4.$$

$$5A+C=-4 \quad -(2)$$

$$-A+B-C+8-5B=4.$$

$$-A-4B-C=-4.$$

$$A+4B+C=4 \quad -(3)$$

$$A \Rightarrow -1 \quad B=1 \quad C=1 \qquad D=8-5(1)$$

$$D=3.$$

$$\frac{-p+1}{1+p^2} + \frac{p+3}{1+(p+2)^2}.$$

$$\frac{1}{1+p^2} \quad \frac{-p}{1+p^2} + \frac{p+2}{1+(p+2)^2} + \frac{1}{1+(p+2)^2}.$$

$$\boxed{\cancel{\sin x}\ \sin x - \cos x + e^{-2x}\cos x + e^{-2x}\sin x}.$$

(3) $x\frac{d^2y}{dx^2} - 2(x+1)\frac{dy}{dx} + (x+2)y = (x-2)e^{2x}$.

$y = uv$

C.F $y = e^x$ is one solution.

Changing independent variable we have.

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)dv = \left(\frac{x-2}{x}\right)\frac{e^{2x}}{e^x}.$$

$$\frac{d^2v}{dx^2} + (e^x + 2)\frac{dv}{dx} = \left(\frac{x-2}{x}\right)e^x.$$

$$\frac{dv}{dx} = t \qquad \frac{d^2v}{dx^2} = \frac{dt}{dx}.$$

$$\frac{dt}{dx} + (e^x + 2)t = \left(\frac{x-2}{x}\right)e^x.$$

$$e^{\int (e^x+2)\,dx}.$$

$$e^{e^x + 2x}.$$

$$\frac{d^2v}{dx^2} + \left(-2 - \frac{2}{x} + 2\right)\frac{dv}{dx} = \left(1 - \frac{2}{x}\right)e^x.$$

$$\frac{d^2v}{dx^2} + \left(-\frac{2}{x}\right)\frac{dv}{dx} = \left(1 - \frac{2}{x}\right)e^x.$$

$$e^{\int -\frac{2}{x}dx} = e^{-2\int\frac{dx}{x}} \Rightarrow e^{-2\ln x} = \frac{1}{x^2}.$$

$$\frac{dv}{dx} = q.$$

$$\frac{dq}{dx} + \left(-\frac{2}{x}\right)q = \left(1 - \frac{2}{x}\right)e^x.$$

$$\frac{(q)}{x^2} = \int\left(\frac{x-2}{x^3}\right)e^x + C$$

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$$\frac{q}{x^2} = \int \left(\frac{x-2}{x^3}\right) e^x + C.$$

$$= \int x^{-2} e^x - 2\int \underset{\text{II}}{x^{-3}}\, \underset{\text{I}}{e^x} + C.$$

$$= \int x^{-2} e^x - 2\left[e^x \frac{x^{-3+1}}{-3+1} - \int e^x \frac{x^{-3+1}}{-3+1}\right] + C -$$

$$= \int \cancel{x^{-2} e^x} + e^x x^{-2} - \cancel{\int x^{-2} e^x} + C$$

$$\frac{q}{x^2} = e^x x^{-2} + C$$

$$q = e^x + Cx^2.$$

$$\frac{du}{dx} = e^x + cx^2,$$

$$\boxed{u = e^x + \frac{c_1 x^3}{2} + c_2}$$

$$\boxed{y = e^x\left(e^x + \frac{c_1 x^3}{2} + c_2\right)}$$

(4) $x^3\frac{d^3y}{dx^3} + 3x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + 8y = 65\cos(\log_e x)$

$x = e^z$.

$((D)(D-1)(D-2) + 3(D)(D-1) + D + 8)y = 0$.

$((D^2-D)(D-2) + 3D^2 - 3D + D + 8)y = 0$

$(D^3 - 2\cancel{D^2} - \cancel{D^2} + \cancel{2D} + \cancel{3D^2} - \cancel{2D} + 8)y = 0$.

$(D^3+8)y = 0$

$(D+2)(D^2+4-2D)y = 0$.

$\frac{2\pm\sqrt{4-4\cdot4}}{2}$.

$\frac{2\pm\sqrt{4(-3)}}{2}$

C.F. $-2,\ 1\pm\sqrt{3}i$

$$\boxed{y = Ae^{-2z} + Be^{z}\cos(\sqrt{3}z + C)}$$

P.I. $(D^3+8)y = 65\,\cancel{\log}\,z$.

$y = \frac{65z}{D^3+8}$.

$y = \frac{65}{8}\left(1+\left(\frac{D}{2}\right)^3\right)^{-1} z$.

$y = \frac{65}{8}z$.

$$\boxed{y = \cancel{Ae}\ \frac{A}{x^2} + Bx\cos(\sqrt{3}\log x + C) + \frac{65}{8}\log x}$$

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⑤. $\frac{dy}{dx} - 5y = \sin x$ by variation of parameter.

$(D-5)y = 0$

$\cancel{y = Ae^{5x} + B\cdot 0}$

$\cancel{u = e^{5x}}$ $\quad \cancel{v = 0}$

$\begin{vmatrix} u & v \\ u' & v' \end{vmatrix}$

$$\frac{d^2y}{dx^2} - 5\frac{dy}{dx} = \cos x$$

$(D^2 - 5D) = 0$ $\quad \boxed{D = 0,\ D = 5}$

$D(D-5) = 0$

$y = A + Be^{5x}$ $\quad u(x) = 1 \quad v(x) = e^{5x}$

$\cancel{y = \frac{\cos x}{D^2 - 5D}}$ $\quad \begin{vmatrix} 1 & e^{5x} \\ 0 & 5e^{5x} \end{vmatrix} = 5e^{5x} \neq 0$

Variation of parameter

$y_p = a\,u(x) + b\,v(x)$

$$a = -\int \frac{v(x)R(x)}{uv' - vu'} = -\int \frac{R(x)\,v(x)}{uv' - vu'}$$

$$a = -\int \frac{(\cos x)\,\cancel{e^{5x}}}{5\cancel{e^{5x}}} \Rightarrow -\frac{\sin x}{5}.$$

$$b = \int \frac{R(x)\,u(x)}{uv' - vu'} = \int \frac{(\cos x)\cdot 1}{e^{5x}} \Rightarrow \int \underset{\mathrm{I}}{e^{-5x}}\ \underset{\mathrm{II}}{\cos x}\,dx.$$

$$e^{-5x}\sin x - \int -5e^{-5x}\sin x\,dx$$

$$= e^{-5x}\sin x + 5\int \underset{\mathrm{I}}{e^{-5x}}\ \underset{\mathrm{II}}{\sin x}\,dx$$

$$= e^{-5x}\sin x + 5\left[-e^{-5x}\cos x - \int +5e^{-5x}\cos x\,dx\right]$$

$$\int e^{-5x}\cos x = e^{-5x}\sin x - 5e^{-5x}\cos x - 25\int e^{-5x}\cos x\,dx$$

$$26\int e^{-5x}\cos x\,dx = e^{-5x}\left[\sin x - 5\cos x\right]$$

$$\boxed{\int e^{-5x}\cos x\,dx = \frac{e^{-5x}}{26}\left[\sin x - 5\cos x\right]}$$

$$y_p = u(x)a + v(x)b$$

$$y_p = \frac{-\sin x}{5} + \frac{1}{26}(\sin x - 5\cos x)$$

$$\boxed{y = A + Be^{5x} - \frac{\sin x}{5} + \frac{1}{26}(\sin x - 5\cos x)}$$

$\partial y \quad \partial x$

(6)

let $f(x+y)$ be the IF for

$M(x,y)\,dx + N(x,y)\,dy = 0$.

then $f(x+y)\,M(x,y)\,dx + f(x+y)\,N(x,y)\,dy = 0$

$M'(x,y)\,dx + N'(x,y)\,dy = 0$

$$\frac{\partial M'}{\partial y} = f'(x+y)\,M(x,y) + f(x+y)\frac{\partial M}{\partial y}(x,y).$$

$$\frac{\partial N'}{\partial x} = f'(x+y)\,N(x,y) + f(x+y)\,\frac{\partial N}{\partial x}(x,y)$$

for exact $\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}$

$$f'(x+y)M(x,y) + f(x+y)\frac{\partial M}{\partial y}(x,y) = f'(x+y)\,N(x,y) + f(x+y)\frac{\partial N}{\partial x}(x,y)$$

$$f(x+y)\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = f'(x+y)\left[N(x,y) - M(x,y)\right]$$

$$\frac{f'(x+y)}{f(x+y)} = \frac{1}{(N-M)}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right]$$

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$$\ln f(x+y) = \int \frac{1}{N-m}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] dx\, dy$$

$$f(x+y) = e^{\int \frac{1}{N-m}\left[\frac{\partial m}{\partial y} - \frac{\partial N}{\partial x}\right]}.$$

Since $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. and $f(x+y)$ has to be a function of $(x+y)$.

Now $M = x^2 + xy$ and $N = y^2 + xy$.

$$\frac{\partial M}{\partial y} = x \qquad \frac{\partial N}{\partial x} = y.$$

$$\frac{1}{N-m}\left(\frac{\partial m}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{-1}{x+y}.$$

$$F(x+y) = e^{\int \frac{-1}{(x+y)} d(x+y)}$$

$$\boxed{f(x+y) \Rightarrow \frac{1}{x+y}}.$$

$$\frac{(x^2+xy)dx}{x+y} + \frac{(y^2+xy)dy}{x+y} = 0$$

$$x\,dx + y\,dy = 0.$$

$$\boxed{x^2 + y^2 = c}$$

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⑦ $[y + x f(x^2+y^2)]dx + [y f(x^2+y^2) - x]dy = 0$.

$$\frac{\partial M}{\partial y} = 1 + 2xy f'(x^2+y^2)$$

$$\frac{\partial N}{\partial x} = 2xy f'(x^2+y^2) - 1$$

$$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x} \quad \text{not exact.}$$

$$M' = \frac{y + x f(x^2+y^2)}{x^2+y^2}.$$

$$\frac{\partial M'}{\partial y} = \frac{(x^2+y^2)(1 + 2xy f'(x^2+y^2)) - (2y)(y + x f(x^2+y^2))}{(x^2+y^2)^2}.$$

$$\frac{\partial M'}{\partial y} = \frac{x^2 - y^2 + 2xy[-f(x^2+y^2) + (x^2+y^2) f'(x^2+y^2)]}{(x^2+y^2)^2}.$$

$$\frac{\partial N'}{\partial x} = \frac{(x^2+y^2)(2xy f'(x^2+y^2) - 1) - (2x)(y f(x^2+y^2) - x)}{(x^2+y^2)^2}.$$

$$\frac{\partial N'}{\partial x} = \frac{x^2 - y^2 + 2xy[(x^2+y^2) f'(x^2+y^2) - f(x^2+y^2)]}{(x^2+y^2)^2},$$

$$\boxed{\frac{\partial M'}{\partial y} = \frac{\partial N'}{\partial x}}$$

hence $\frac{1}{x^2+y^2}$ is the I.F.

When $f(x^2+y^2) = (x^2+y^2)^2$.

$$\left(y + x(x^2+y^2)^2\right)dx + \left(y(x^2+y^2)^2 - x\right)dy = 0$$

I.F is $\frac{1}{x^2+y^2}$.

$$\left[\frac{y}{x^2+y^2} + x(x^2+y^2)\right]dx + \left[y(x^2+y^2) - \frac{x}{x^2+y^2}\right]dy = 0$$

$$\frac{y\,dx - x\,dy}{x^2+y^2} + (x^2+y^2)(x\,dx + y\,dy) = 0$$

$$\tan^{-1}\left(\frac{x}{y}\right) + \frac{1}{2}\frac{(x^2+y^2)^2}{2} = c.$$

$$\boxed{\tan^{-1}\left(\frac{x}{y}\right) + \frac{1}{4}(x^2+y^2)^2 = c}$$

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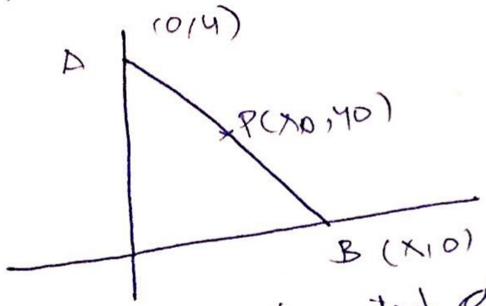
(7.) Find the curve for which the part of the tangent cut off by the axis is bisected at the point of tangency.



Let the tangent by bisected at (x_0, y_0).

$B \Rightarrow (x, 0)$

then $A \Rightarrow (0, y)$

$\frac{x}{2} = x_0$ $\quad\quad$ $\frac{y}{2} = y_0$

$y = 2y_0$.

$x = 2x_0$ $\quad\quad$ $B \Rightarrow (2x_0, 0)$.

$A \Rightarrow (0, 2y_0)$

Slope of $AB \Rightarrow \frac{2y_0 - 0}{0 - 2x_0} = -\frac{y_0}{x_0}$.

$\frac{dy}{dx} = \frac{-y}{x}$ if P is a general point.

$\ln y = -\ln x + c$.

$xy = c$. is the curve.