Numerical solution of ordinary Differential Equations

Set-VI

In the field of science and technology, a number of problems can be formulated into differential equations. Such problems in this field are reduced to the problem of solving differential equations satisfying certain given conditions. The analytical methods of solving differential equations are applicable only to a limited class of equations. Quite often differential equations appearing in physical problems do not belong to any of these familiar types and one is obliged to resort to numerical methods.

These methods are of even greater importance when we realise that computing machines are now readily available which reduce numerical work considerably.

The solution of an ordinary differential equation means finding an explicit expression for y in terms of a finite number of elementary functions of x. Such a solution of a differential equation is known as the closed or finite form of solution. In the absence of such a solution, we have recourse to numerical methods of solution.

the solution of ordinary differential equations, let us consider the first order differential equation

$$\frac{dy}{dx} = f(x, y), \quad \text{——} (1)$$

with the initial condition

$$y(x_0) = y_0.$$

The methods so developed can, in general, be applied to the solution of systems of first order equations, and will yield the solution of one of the two forms.

(i) A series for y in terms of powers of x, from which the value of y can be obtained by direct substitution.

(ii) A set of tabulated values of x and y.

The methods of Taylor and Picard belong to class (i). In these methods, y in (1) approximated by a truncated series, each term of which is a function of x. The information about the curve at one point is utilized and the solution is not iterated. As such, these are referred to as single step methods.

The methods of Euler, Runge-Kutta, Milne, Adams-Bashforth etc. belong to the latter class solutions (i.e, class (ii)). In these methods, the next point on the curve is evaluated in short steps ahead, by performing iterations till sufficient accuracy is achieved. As such, these methods are called step by step methods. (or) multistep methods.

Euler and Runge-Kutta methods are used for computing y over a limit range of x values whereas Milne and Adams methods may be applied for finding y over a wider range of x-values.

The methods Milne and Adams require starting values which are found by Picard's Taylor's series or Runge-Kutta methods.

Initial and boundary conditions:

An ordinary differential equation of the nth order is of the form

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots \frac{d^ny}{dx^n}\right) = 0. \quad \text{—— (2)}$$

Its general solution contains 'n' arbitrary constants and is of the form

$$\phi(x, y, c_1, c_2, \dots c_n) = 0 \quad \text{—— (3)}.$$

To obtain its particular solution, n conditions must be given so that the constants $c_1, c_2, \dots c_n$ can be determined.

If these conditions are prescribed at one point only (say: x_0), then the differential equation together with the conditions constitute an initial value problem of the n^{th} order.

If the conditions are prescribed at two or more points, then the problem is termed as boundary value problem.

we shall first describe methods for solving initial value problems of the type

$$\frac{dy}{dx} = f(x, y) \ ; \quad y(x_0) = y_0.$$

and at the end of the chapter we will outline methods for solving boundary value problems for second order differential equations.

Solution by Taylor's series method.

The Taylor series method provides a solution of the equation.

$$\frac{dy}{dx} = f(x, y) \; ; \; y(x_0) = y_0 \quad \text{——— (1)}$$

we assume that $f(x, y)$ is sufficiently differentiable with respect to x and y.

If $y(x)$ is the exact solution of (1), then the Taylor's series for $y(x)$ around $x = x_0$ in power of $(x - x_0)$ is given by

$$y(x) = y_0 + (x - x_0) y_0' + \frac{(x - x_0)^2}{2!} y_0'' + \cdots$$

(or) $y(x_0 + h) = y_0 + h y_0' + \frac{h^2}{2!} y_0'' + \cdots \cdots$ where $h = x - x_0 \Rightarrow x = x_0 + h$

Since, the solution is not known, the derivatives in the above expansion are not known explicitly. However, f is assumed to be sufficiently differentiable and therefore, the derivatives can be obtained directly from the given differential equation itself. Noting that f is an implicit function of y, we have.

$$y' = \frac{dy}{dx} = f(x, y) = f$$

$$y'' = \frac{d}{dx}(y') = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

$$= f_x + f_y \cdot f.$$

$$y''' = \frac{d}{dx}(y'')$$

$$= \frac{d}{dx}[f_x + f_y f]$$

$$= \frac{\partial}{\partial x} f_x \frac{dx}{dx} + \frac{\partial f_x}{\partial y} \frac{dy}{dx} + f\left[\frac{\partial f_y}{\partial x}\frac{dx}{dx} + \frac{\partial f_y}{\partial y}\frac{dy}{dx}\right] + f_y\left[f_x + f_y f\right]$$

$$= f_{xx} + \underline{f_{yx}} f + f[\underline{f_{xy}} + f_{yy} f] + f_y [f_x + f_y f]$$

$$= f_{xx} + 2ff_{xy} + f^2 f_{yy} + f_y (f_x + f_y f)$$

$$y^{iv} = f_{xxx} + 3ff_{xxy} + 3f^2 f_{xyy} + f_y(f_{xx} + 2ff_{xy} + f^2 f_{yy}) + 3(f_x + ff_y)(f_{xy} + ff_{yy}) + f_y^2(f_x + ff_y)$$

etc.

By continuing in this manner, we can express the derivative of y in terms of $f(x,y)$ and its partial derivatives.

The method is best understood by the following example.

→ From the Taylor series for $y(x)$, find $y(0.1)$ correct to four decimal places if $y(x)$ satisfies $y' = x - y^2$ and $y(0) = 1$.

Sol^n: The Taylor series for $y(x)$ is given by

$$y(x) = 1 + xy_0' + \frac{x^2}{2} y_0'' + \frac{x^3}{6} y_0''' + \frac{x^4}{24} y_0^{iv} + \frac{x^5}{120} y_0^{v} + \cdots$$

The derivatives $y_0', y_0'', \ldots$ etc are

$y'(x) = x - y^2 \Rightarrow y_0' = -1$

$y''(x) = 1 - 2yy' \Rightarrow y_0'' = 3$

$y'''(x) = -2yy'' - 2y'^2 \Rightarrow y_0''' = -8$

$y^{iv}(x) = -2yy''' - 6y'y'' \Rightarrow y_0^{iv} = 34$

$y^{v}(x) = -2yy^{iv} - 8y'y''' - 6y''^2 \Rightarrow y_0^{(v)} = -186.$

Using these values the Taylor series becomes

$$y(x) = 1 - x + \frac{3}{2}x^2 - \frac{4}{3}x^3 - \frac{17}{12}x^4 - \frac{31}{20}x^5 + \cdots$$

To obtain the value of $y(0.1)$ correct to four decimal places, it is found that the terms upto x^4 should be considered, and we have $\boxed{y(0.1) = 0.9138}$

Suppose that we wish to find the range of values of 'x' for which the above series, truncated after the term containing x^4, can be used to compute the values of y correct to four decimal places.

we need only to write

$$\frac{31}{20}x^5 \leq 0.00005.$$

so that $x \leq 0.126$

→ find $y(1.1)$, given $y' = 2x - y$ and $y(1) = 3$

Sol^n: Given $y' = 2x - y$, $x_0 = 1$, $y_0 = 3$, $x_1 = 1.1$,

$h = 0.1$

Taylor series is given by

$$y_1 = y_0 + h y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots\cdots \quad \text{---} (1)$$

$$y' = 2x - y \Rightarrow y_0' = 2x_0 - y_0$$
$$= 2(1) - 3$$
$$= -1$$

$$y'' = 2 - y' \Rightarrow y_0'' = 2 - y_0'$$
$$= 2 - (-1)$$
$$= 3$$

$$y''' = -y'' \Rightarrow y_0''' = -y_0''$$
$$= -3$$

∴ Equation (1) becomes

$$y_1 = 3 + (0.1)(-1) + \frac{(0.1)^2}{2}(3) + \frac{(0.1)^3}{6}(-3) + \cdots\cdots$$

$$= 3 - 0.1 + 0.015 - 0.0005 + \cdots$$

$$= 2.9145$$

Example 8.1 Using Taylor's series method, find the solution of the initial value problem

$$\frac{dy}{dt} = t + y, \qquad y(1) = 0$$

at $t = 1.2$, with $h = 0.1$ and compare the result with the closed form solution.

Solution Let us compute the first few derivatives from the given differential equation as follows:

$$y' = t + y, \qquad y'' = 1 + y', \qquad y''' = y'', \qquad y^{\text{IV}} = y''', \qquad y^{\text{V}} = y^{\text{IV}} \tag{1}$$

Prescribing the initial condition, that is, at $t_0 = 1$, $y_0 = y(t_0) = 0$, we have

$$y_0' = 1, \qquad y_0'' = 2, \qquad y_0''' = y_0^{\text{IV}} = y_0^{\text{V}} = 2$$

Now, using Taylor's series method, we have

$$y(t) = y_0 + (t - t_0)y_0' + \frac{(t-t_0)^2}{2}y_0'' + \frac{(t-t_0)^3}{6}y_0'''$$

$$+ \frac{(t-t_0)^4}{24}y_0^{\text{IV}} + \frac{(t-t_0)^5}{120}y_0^{\text{V}} + \cdots \tag{2}$$

Substituting the above values of the derivatives, and the initial condition, into (2), we obtain

$$y(1.1) = 0 + (0.1)(1) + \frac{0.01}{2}(2) + \frac{0.001}{6}(2) + \frac{0.0001}{24}(2) + \frac{0.00001}{120}(2) + \cdots$$

$$= 0.1 + 0.01 + \frac{0.001}{3} + \frac{0.0001}{12} + \frac{0.00001}{60} + \cdots$$

$$= 0.1 + 0.01 + 0.000333 + 0.0000083 + 0.0000001 + \cdots$$

$$\cong 0.1103414.$$

Therefore,

$$y(1.1) = y_1 = 0.1103414 \cong 0.1103$$

Taking $y_1 = 0.1103$ at $t = 1.1$, the values of the derivatives as computed from Eq. (1) are

$$y_1' = 1.1 + 0.1103 = 1.2103$$

$$y_1'' = 1 + 1.2103 = 2.2103$$

$$y_1''' = y_1^{\text{IV}} = y_1^{\text{V}} = 2.2103$$

Substituting the value of y_1 and its derivatives into Taylor's series expansion (2) we get, after retaining terms up to fifth derivative only

$$y(1.2) = y_1 + (t - t_1)y_1 + \frac{(t-t_1)^2}{2}y_1'' + \frac{(t-t_1)^3}{6}y_1''' + \frac{(t-t_1)^4}{24}y_1^{\text{IV}} + \frac{(t-t_1)^5}{120}y_1^{\text{V}}$$

$$= 0.1103 + 0.12103 + 0.0110515 + 0.0003683 + 0.000184 + 0.0000003$$

Hence,

$$y(1.2) = 0.2429341 \cong 0.2429 \tag{3}$$

To obtain the closed form solution, we rewrite the given IVP as

$$\frac{dy}{dt} - y = x \quad \text{or} \quad d(ye^{-t}) = te^{-t}$$

On integration, we get

$$y = -e^t\,(te^{-t} + e^{-t}) + ce^t = \underline{ce^t - t - 1}$$

Using the initial condition, we get

$$0 = ce - 2 \quad \text{or} \quad c = \frac{2}{e}$$

Therefore, the closed form solution is

$$y = -t - 1 + 2e^{t-1}$$

when $t = 1.2$, the closed form solution becomes

$$y(1.2) = -1.2 - 1 + 2(1.2214028) = -2.2 + 2.4428056 = 0.2428 \tag{4}$$

Comparing the results (3) and (4), obtained numerically and in closed form, we observe that they agree up to three decimals.

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EXAMPLE 7.2 Given $\frac{dy}{dx} = 3x + \frac{y}{2}$ and $y(0) = 1$. Find the values of $y(0.1)$ and $y(0.2)$, using the Taylor series method.

Solution

Given

$$y' = 3x + y/2 \text{ and } x_0 = 0,\ y_0 = 1$$

Taylor series is given by

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{iv} + \ldots \tag{i}$$

To find y(0.1):

$$y' = 3x + \frac{y}{2} \quad \Rightarrow \quad y_0' = 3x_0 + \frac{y_0}{2}$$

$$= 3(0) + \frac{1}{2}$$

$$= 0.5$$

$$y'' = 3 + \frac{y'}{2} \quad \Rightarrow \quad y_0'' = 3 + \frac{y_0'}{2}$$

$$= 3 + \frac{0.5}{2}$$

$$= 3.25$$

$$y''' = \frac{y''}{2} \quad \Rightarrow \quad y_0''' = \frac{y_0''}{2}$$

$$= \frac{3.25}{2}$$

$$= 1.625$$

$$y^{iv} = \frac{y'''}{2} \quad \Rightarrow \quad y_0^{iv} = \frac{y_0'''}{2}$$

$$= \frac{1.625}{2}$$

$$= 0.8125$$

$\therefore$ Equation (i) $\Rightarrow$

$$y_1 = 1 + (0.1)(0.5) + \frac{(0.1)^2}{2}(3.25) + \frac{(0.1)^3}{6}(1.625) + \frac{(0.1)^4}{24}(0.8125) + \ldots$$

$$= 1.0665$$

$$y_1 = y(0.1)$$

$$= 1.0665$$

To find y(0.2):

Taylors series formula for $y(0.2)$ is

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \frac{h^4}{4!}y_1^{iv} + \ldots \qquad \text{(ii)}$$

$$y' = 3x + \frac{y}{2} \quad \Rightarrow \quad y_1' = 3x_1 + \frac{y_1}{2}$$

$$= 3(0.1) + \frac{1.0665}{2}$$

$$= 0.83325$$

$$y'' = 3 + \frac{y'}{2} \quad \Rightarrow \quad y_1'' = 3 + \frac{y_1'}{2}$$

$$= 3 + \frac{0.83325}{2}$$

$$= 3.416625$$

$$y''' = \frac{y''}{2} \quad \Rightarrow \quad y_1''' = \frac{y_1''}{2}$$

$$= \frac{3.416625}{2}$$

$$= 1.7083125$$

$$y^{iv} = \frac{y'''}{2} \quad \Rightarrow \quad y_1^{iv} = \frac{y_1'''}{2}$$

$$= \frac{1.7083125}{2}$$

$$= 0.85415625$$

Equation (ii) $\Rightarrow$

$$y_2 = 1.0665 + (0.1)(0.83325) + \frac{(0.1)^2}{2}(3.416625) + \frac{(0.1)^3}{6}(1.7083125) + \frac{(0.1)^4}{24}(0.85415625) + \ldots$$

$$= 1.167196.$$

$$y_2 = y(0.2) = 1.167196.$$

EXAMPLE 7.3 Obtain $y(4.2)$ and $y(4.4)$, given

$$\frac{dy}{dx} = \frac{1}{x^2 + y}, \; y(4) = 4, \text{ taking } h = 0.2$$

Solution

Given $y' = \dfrac{1}{x^2 + y}$, $x_0 = 4$, $y_0 = 4$, $x_1 = 4.2$, $x_2 = 4.4$, $h = 0.2$.

Taylor series is given by

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \ldots \tag{i}$$

To find y(4.2):

$$y' = \frac{1}{x^2 + y} \quad \Rightarrow \quad y_0' = \frac{1}{x^2 + y_0}$$

$$= \frac{1}{4^2 + 4}$$

$$= 0.05$$

$$y'' = -1\,(x^2 + y)^{-2}\, y' \quad \Rightarrow \quad y_0'' = \frac{-y_0'}{(x_0^2 + y_0)^2}$$

$$= \frac{-0.05}{(20)^2}$$

$$= -0.000125$$

$$y''' = 2(x^2 + y)^{-3}(y')^2 + (-1)(x^2 + y)^{-2} y''$$

$$y_0''' = \frac{2(y_0')^2}{(x_0^2 + y_0)^3} - \frac{y_0''}{(x_0^2 + y_0)^2}$$

$$= \frac{2(0.05)^2}{(20)^3} + \frac{0.000125}{(20)^2}$$

$$= 0.0000009375.$$

$\therefore$ Equation (i) $\Rightarrow$

$$y_1 = 4 + (0.2)(0.05) + \frac{(0.2)^2}{2}(-0.000125) + \frac{(0.2)^3}{6}(0.0000009375) + \ldots$$

$$= 4 + 0.01 - 0.0000025 + 0.00000000125$$

$$= 4.0099975$$

$$y_1 = y(4.2)$$

$$= 4.009998$$

To find y(4.4):

Taylor series is given by

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \ldots \tag{ii}$$

$$y' = \frac{1}{x^2 + y} \quad \Rightarrow \quad y_1' = \frac{1}{x_1^2 + y_1}$$

$$= \frac{1}{(4.2)^2 + 4.009998}$$

$$= 0.046189$$

$$y_1'' = \frac{-y_1'}{(x_1^2 + y_1)^2} = \frac{-0.046189}{[(4.2)^2 + 4.009998]^2}$$

$$= -0.000098542$$

$$y_1''' = \frac{2(y_1')^2}{(x_1^2 + y_1)^3} - \frac{y_1''}{(x_1^2 + y_1)^2}$$

$$= \frac{2(0.046189)^2}{[(4.2)^2 + 4.009998]^3} + \frac{0.000098542}{[(4.2)^2 + 4.009998]^2}$$

$$= 0.000000420469 + 0.00000021024$$

$$= 0.000000630704$$

$\therefore$ Equation (ii) $\Rightarrow$

$$y_2 = 4.009998 + (0.2)\,(0.046189) + \frac{(0.2)^2}{2}\,(-0.000098542)$$

$$+ \frac{(0.2)^3}{6}\,(0.000000630704)$$

$$= 4.009998 + 0.0092378 - 0.00000197084 + 0.00000000094094$$

$$= 4.019234$$

EXAMPLE 7.4 Find $y(0.1)$, given $y' = x^2y - 1$, $y(0) = 1$.

Solution

Given $y' = x^2y - 1$, $x_0 = 0$, $y_0 = 1$, $x_1 = 0.1$, $h = 0.1$

Taylor series is given by

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}\,y_0'' + \frac{h^3}{3!}\,y_0''' + \ldots \qquad \text{(i)}$$

$$y' = x^2y - 1 \quad \Rightarrow \quad y_0' = x_0^{\,2}\,y_0 - 1$$

$$= (0)\,(1) - 1$$

$$= -1$$

$$y'' = x^2y' + 2xy \quad \Rightarrow \quad y_0'' = x_0^{\,2}\,y_0' + 2x_0y_0$$

$$= 0$$

$$y_0''' = x_0^2(y_0')^2 + 2x_0\, y_0' + 2x_0 y_0' + 2y_0$$
$$= 2$$

$\therefore$ Equation (i) $\Rightarrow$

$$y_1 = 1 + (0.1)(-1) + \frac{(0.1)^2}{2!}(0) + \frac{(0.1)^3}{6}(2) + \ldots$$
$$= 1 - 0.1 + 0.0003333 + \ldots$$
$$= 0.900333$$
$$y_1 = y(0.1)$$
$$= 0.900333$$

7.1.1 Taylor Series Method for Simultaneous First-order Differential Equations

The simultaneous first-order differential equations of the form

$$\frac{dy}{dx} = f_1(x, y, z)$$

and
$$\frac{dz}{dx} = f_2(x, y, z)$$

with initial values $y(x_0) = y_0$ and $z(x_0) = z_0$
To solve this system of equations at an interval h, the increments in y and z are obtained by using the formulae

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \ldots$$

and
$$z_1 = z_0 + hz_0' + \frac{h^2}{2!}z_0'' + \frac{h^3}{3!}z_0''' + \ldots$$

EXAMPLE 7.5 Solve the differential equations using the Taylor series

$$\frac{dy}{dx} = 1 + xz, \quad \frac{dz}{dx} = -xy, \quad \text{for } \overset{h}{x} = 0.3$$

given that $x = 0$, $y = 0$, $z = 1$.

Solution

$$\frac{dy}{dx} = 1 + xz$$

$$\frac{dz}{dx} = -xy$$

$$x = 0,\ y = 0,\ z = 1,\ h = 0.3$$

Taylors series for y' is given by

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \dots \quad \text{(i)}$$

$$y' = 1 + xz \Rightarrow y_0' = 1 + x_0 z_0$$
$$= 1 + (0)(1)$$
$$= 1$$

$$y'' = xz' + z \Rightarrow y_0'' = x_0 z_0' + z_0$$
$$= x_0(-x_0 y_0) + z_0$$
$$= 0 + 1$$
$$= 1$$

$$y''' = x(z')^2 + z' + z' \Rightarrow y_0''' = x_0 (z_0')^2 + 2z_0'$$
$$= x_0(-x_0 y_0)^2 + 2(x_0 y_0)$$
$$= 0$$

$\therefore$ Equation (i) $\Rightarrow$

$$y_1 = 0 + (0.3)(1) + \frac{(0.3)^2}{2}(1) + \frac{(0.3)^3}{6}(0) + \dots$$
$$= 0.3 + 0.045$$
$$= 0.345$$
$$y_1 = y(0.3)$$
$$= 0.345$$

Taylor series for z' is given by

$$z_1 = z_0 + hz_0' + \frac{h^2}{2!}z_0'' + \frac{h^3}{3!}z_0''' + \dots \quad \text{(ii)}$$

$$z' = -xy \Rightarrow z_0' = -x_0 y_0 = 0$$

$$z'' = -(xy' + y) \Rightarrow z_0'' = -x_0 y_0' + y_0$$
$$= -(0)(1) + 0$$
$$= 0$$

$$z''' = -[x y'' + y' + y'] \Rightarrow z_0''' = -x_0 y_0'' + 2y_0'$$
$$= -(0)(1) + 2(1)$$
$$= 2.$$

$\therefore$ Equation (ii) $\Rightarrow$

$$z_1 = 1 + (0.3)(0) + \frac{(0.3)^2}{2}(0) + \frac{(0.3)^3}{6}(2) + \dots$$

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$$= 1 + 0.009$$
$$= 1.009$$
$$z_1 = z(0.3)$$
$$= 1.009$$

EXAMPLE 7.6 Find $y(0.1)$, $y(0.2)$, $z(0.1)$, $z(0.2)$, given

$$\frac{dy}{dx} = x + z, \frac{dz}{dx} = x - y^2 \text{ and } y(0) = 2, z(0) = 1$$

Solution

Given

$$y' = x + z, z' = x - y^2, x_0 = 0, y_0 = 2, z_0 = 1, h = 0.1$$

To find y(0.1):

Taylor series is given by

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \dots \quad \text{(i)}$$

$$y' = x + z, \qquad z' = x - y^2$$
$$y'' = 1 + z', \qquad z'' = 1 - 2yy'$$
$$y''' = z''$$
$$z''' = -[2yy'' + 2(y')^2]$$

$$\therefore \quad y' = x + z \quad \Rightarrow \quad y_0' = x_0 + z_0$$
$$= 0 + 1$$
$$= 1$$
$$y'' = 1 + z' = 1 + x - y^2 \quad \Rightarrow \quad y_0'' = 1 + x_0 - y_0^2$$
$$= 1 - 2^2$$
$$= -3$$

$\therefore$ Equation (i) $\Rightarrow$

$$y_1 = 2 + (0.1)(1) + \frac{(0.1)^2}{2}(-3) + \frac{(0.1)^3}{6}(-3) + \dots$$
$$= 2 + 0.1 - 0.015 - 0.0005 \dots$$
$$= 2.0845$$

To find z(0.1):

Taylor series is given by

$$z_1 = z_0 + hz_0' + \frac{h^2}{2!} z_0'' + \frac{h^3}{3!} z_0''' + \dots \quad \text{(ii)}$$

$$z_0' = x_0 - y_0^2$$
$$= 0 - 2^2$$

$$z_0'' = 1 - 2y_0\, y_0'$$
$$= 1 - 2(2)\,(1)$$
$$= -3$$

$$z_0''' = -[2y_0 y_0'' + 2(y_0')^2]$$
$$= -[2(2)\,(-3) + 2(1)^2]$$
$$= -[-12 + 2]$$
$$= 10$$

$\therefore$ Equation (ii) $\Rightarrow$

$$z_1 = 1 + (0.1)\,(-4) + \frac{(0.1)^2}{2}(-3) + \frac{(0.1)^3}{6}(10) + \ldots$$
$$= 1 - 0.4 + (-0.015) + 0.0016667$$
$$= 1 - 0.4 - 0.015 + 0.001667$$
$$= 0.586667$$

To find y(0.2):
Taylor series is given by

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \ldots \qquad \text{(iii)}$$

$$y_1' = x_1 + z_1$$
$$= 0.1 + 0.586667$$
$$= 0.686667$$

$$y_1'' = 1 + z_1'$$
$$= 1 + x_1 - y_1^2$$
$$= 1 + 0.1 - 2.0845^2$$
$$= -3.24514$$

$$y_1''' = z_1''$$
$$= 1 - 2y_1 y_1'$$
$$= 1 - 2(2.0845)\,(0.686667)$$
$$= -1.862415$$

$\therefore$ Equation (iii) $\Rightarrow$

$$y_2 = 2.0845 + (0.1)\,(0.686667) + \frac{(0.1)^2}{2}(-3.24514)$$
$$+ \frac{(0.1)^3}{6}(-1.862715) + \ldots$$

$$= 2.0845 + 0.0686667 - 0.01623 - 0.00031045$$
$$= 2.13663$$

To find z(0.2):

Taylor series is given by

$$z_2 = z_1 + hz_1' + \frac{h^2}{2!} z_1'' + \frac{h^3}{3!} z_1''' + \ldots$$

$$z_1' = x_1 - y_1^2$$
$$= 0.1 - 2.0845^2$$
$$= -4.24514$$

$$z_1'' = 1 - 2y_1 y_1'$$
$$= 1 - 2(2.0845)(0.686667)$$
$$= -1.862715$$

$$z_1''' = -[2y_1 y_1'' + 2(y_1')^2]$$
$$= -[2(2.0845)(-3.24514) + 2(0.686664)^2]$$
$$= -[-13.52899 + 0.943023]$$
$$= 12.585967$$

$\therefore$ Equation $\Rightarrow$

$$z_2 = (0.1) + (0.1)(-4.24514) + \frac{(0.1)^2}{2}(-1.862715)$$
$$+ \frac{(0.1)^3}{6}(12.585967) + \ldots$$
$$= 0.1 - 0.424514 - 0.009313575 + 0.00209766$$
$$= -0.33173$$

7.1.2 Taylor Series Method for Second-order Differential Equations

The differential equation of the second-order can be solved by reducing it to a lower-order differential equation. A second-order differential equation can be reduced to a first-order differential equation by transformation $y' = z$.

Suppose $$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right)$$

i.e., $$y'' = f(x, y, y') \tag{7.2}$$

is the differential equation together with initial conditions.

$$y(x_0) = y_0 \quad \text{and} \quad y'(x_0) = y_0' \tag{7.3}$$

where y_0, y_0' are known values.

Setting $p = y'$, we get $y'' = p'$

The equation (7.2) becomes

$$p' = f(x, y, p) \text{ with initial conditions}$$

$$y(x_0) = y_0 \tag{7.4}$$

$$y'(x_0) = y_0' \tag{7.5}$$

where y_0, y_0' are known values.

By putting $y' = p$, $y'' = p'$, Eq. (7.2) becomes

$$p' = f(x, y, p)$$

with initial conditions

$$y(x_0) = y_0$$

and $\quad y'(x_0) = y_0' \quad \Rightarrow \quad p(x_0) = p_0$

Solving p' by using Eqs. (7.4) and (7.5), we get

$$p_1 = p_0 + hp_0' + \frac{h^2}{2!} p_0'' + \frac{h^3}{3!} p_0''' + \ldots \tag{7.6}$$

where $h = x_1 - x_0$

Since $p = y'$, we get Eq. (7.6) as

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \ldots$$

Similarly, proceeding in similar manner, we get

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \ldots$$

$\therefore$ We calculate $y_1, y_2 \ldots$

EXAMPLE 7.7 Solve $y'' = y + xy'$, given $y(0) = 1$, $y'(0) = 0$ and calculate $y(0.1)$.

Solution

$$x_0 = 0$$

$$y_0 = 1$$

$$y_0' = 0$$

$$y'' = y + xy'$$

$$\Rightarrow \quad y_0'' = y_0 + x_0 y_0' = 1 + (0)(0) = 1$$

Differentiating with respect to x

$$y''' = y' + y' + xy'' = 2y' + xy''$$

$$\Rightarrow \quad y_0''' = 2y_0' + x_0 y_0''$$
$$= 2(0) + 0(1) = Q$$
$$y^{iv} = 2y'' + y'' + xy'''$$
$$= 3y'' + xy'''$$
$$\Rightarrow \quad y^{iv} = 3y_0'' + x_0 y_0'''$$
$$= 3(1) + (0)(0)$$
$$= 3$$
$$y^{v} = 4y''' + xy^{iv}$$
$$y^{v} = 4y_0''' + x_0\, y_0^{iv}$$
$$= 4(0) + 0(3)$$
$$= 0$$

We know that Taylor series is given by

$$y(x) = y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \frac{h^4}{4!}y_0^{iv} + \ldots$$
$$= 1 + (0.1)(0) + \frac{(0.1)^2}{2}(1) + \frac{(0.1)^3}{6}(0) + \frac{(0.1)^4}{24}(3) + \ldots$$
$$= 1 + \frac{(0.1)^2}{2} + \frac{(0.1)^4}{24}(3) + \ldots$$
$$= 1.0050125$$

EXAMPLE 7.8 Find $y(0.2)$, given $y'' + y = 0$, $y(0) = 1$, $y'(0) = 0$.

Solution

$$y'' = -y,\ x_0 = 0,\ y_0 = 1,\ y_0' = 0,\ h = 0.2.$$

To find y(0.2)

We know that Taylor series is given by

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \frac{h^3}{3!}y_0''' + \ldots \tag{i}$$

$$y'' = -y \quad \Rightarrow \quad y_0'' = -y_0 = -1$$
$$y''' = -y' \quad \Rightarrow \quad y_0''' = -y_0' = 0$$
$$y^{iv} = -y'' \quad \Rightarrow \quad y_0^{iv} = -y_0'' = -(-1) = 1$$
$$y^{v} = -y''' \quad \Rightarrow \quad y_0^{v} = -y_0''' = 0$$

$\therefore$ Equation (i) $\Rightarrow$

$$y_1 = 1 + (0.2)(0) + \frac{(0.2)^2}{2}(-1) + \frac{(0.2)^3}{6}(6) + \frac{(0.2)^4}{24}(1) + \ldots$$
$$= 1 + \frac{(0.2)^2}{2}(-1) + \frac{(0.2)^4}{24}(1) + \ldots$$

$$= 1 - 0.02 + 0.00006667$$
$$= 0.9800667$$

EXERCISES

7.1 Solve $\dfrac{dy}{dx} = x + y$, given $y(1) = 0$ and get $y(1.1)$, $y(1.2)$ by Taylor series method.

[*Ans.* $y(1.1) = 0.110342$, $y(1.2) = 0.24281$]

7.2 Using the Taylor series method, find correct to four decimal places, the value of $y(0.1)$, given

$$\frac{dy}{dx} = x^2 + y^2 \text{ and } y(0) = 1.$$

[*Ans.* $y(0.1) = 1.11145$]

7.3 Using the Taylor method, compute $y(0.2)$ and $y(0.4)$ correct to four decimal places, given

$$\frac{dy}{dx} = 1 - 2xy \text{ and } y(0) = 0.$$

[*Ans.* $y(0.2) = 0.19475$, $y(0.4) = 0.359884$]

7.4 Given $\dfrac{dy}{dx} = 3x + \dfrac{y}{2}$ and $y(0) = 1$. Find the values of $y(0.1)$ and $y(0.2)$, using the Taylor series method.

[*Ans.* $y(0.1) = 1.0665$, $y(0.2) = 1.167196$]

7.5 Solve by the Taylor series method of third-order the problem

$$\frac{dy}{dx} = (x^3 + xy^2)e^{-x},\ y(0) = 1$$

to find y, for $x = 0.1, 0.2, 0.3$.

[*Ans.* $y(0.1) = 1.0047$, $y(0.2) = 1.01812$, $y(0.3) = 1.03995$]

7.6 Solve by the Taylor series method (of fourth-order)

$$\frac{dy}{dx} = xy^2 + 1, \quad y(0) = 1 \text{ at } x = 0.2, 0.4.$$

[*Ans.* $y(0.2) = 1.226$, $y(0.4) = 1.54205$]

7.7 Using the Taylor series method, solve

$$\frac{dy}{dx} = x^2 - y,\ y(0) = 1 \text{ at } x = 0.1, 0.2, 0.3 \text{ and } 0.4.$$

[*Ans.* $y(0.1) = 0.9052$, $y(0.2) = 0.8213$,
$y(0.3) = 0.7492$, $y(0.4) = 0.6897$]

7.8 Find $y(0.1)$, given $\dfrac{dy}{dx} = x + y$, $y(0) = 1$.

[*Ans.* $y(0.1) = 1.1103$]

7.9 Find $y(0.1)$, $y(0.2)$, $y(0.3)$, given

$$y' = \frac{x^3 + xy^2}{e^x}, \; y(0) = 1.$$

[*Ans.* $y(0.1) = 1.0047$, $y(0.2) = 1.01812$, $y(0.3) = 1.03995$]

7.10 Solve $\dfrac{dy}{dx} = y + x^3$, for $x = 1.1, 1.2, 1.3$, given $y(1) = 1$.

[*Ans.* $y(1.1) = 1.225$, $y(1.2) = 1.512$, $y(1.3) = 1.874$]

7.11 Find $y(0.1)$, $y(0.2)$, $z(0.1)$, $z(0.2)$, given

$$\frac{dy}{dx} = x + z, \frac{dz}{dx} = x - y^2 \text{ and } y(0) = 2, z(0) = 1.$$

[*Ans.* 2.0845, 2.1367, 0.5867, 0.1550]

7.12 Evaluate $x(0.1)$, $y(0.1)$, $x(0.2)$, $y(0.2)$, given

$$\frac{dx}{dt} = t_y + 1, \frac{dy}{dt} = -t_x$$

given $x = 0$, $y = 1$ at $t = 0$.

[*Ans.* $x(0.1) = 0.105$, $y(0.1) = 0.9987$, $x(0.2) = 0.21998$, $y(0.2) = 0.9972$]

7.13 Find $y(0.3)$, $z(0.3)$, given

$$\frac{dz}{dx} = -xy, \frac{dy}{dx} = 1 + xz$$

where $y(0) = 0$, $z(0) = 1$.

[*Ans.* $y(0.3) = 0.3448$, $z(0.3) = 0.991$]

7.14 Solve for x and y

$$\frac{dx}{dt} = x + y + t, \quad \frac{dy}{dt} = 2x - t$$

given $x = 0$, $y = 1$ at $t = 1$.

$$\left[\textit{Ans. } x = 2t + t^2 + \frac{5}{6}t^3 + \dots \right.$$
$$\left. -y = 1 - t + \frac{3}{2}t^2 + \frac{2}{3}t^3 + \dots \right]$$

7.15 Solve numerically, using the Taylor series method find approximate values of y and z corresponding to $x = 0.1, 0.2$ given that

$$y(0) = 2,\ z(0) = 1 \text{ and } \frac{dy}{dx} = x + z, \frac{dz}{dx} = x - y^2$$

[*Ans.* $y(0.1) = 2.0845$, $z(0.1) = 0.5867$
$y(0.2) = 2.1367$, $z(0.2) = 0.15497$]

7.16 Find the value of $y(1.1)$ and $y(1.2)$ from $\dfrac{d^2y}{dx^2} + y^2 \dfrac{dy}{dx} = x^3$, $y(1) = 1$, $y'(1) = 1$ by using the Taylor series method.

[*Ans.* $y(1.1) = 1.1002$, $y(1.2) = 1.2015$]

7.17 Given $\dfrac{d^2y}{dx^2} - x\left(\dfrac{dy}{dx}\right)^2 + y^2 = 0$ with $y(0) = 1$, $y'(0) = 0$, obtain the values of $y(0.1)$ and $y(0.2)$, correct to 3 decimal places, using the Taylor series method.

[*Ans.* $y(0.1) = 0.995$, $y(0.2) = 0.981$]

7.18 Using the Taylor series method, find $y(0.1)$, $y(0.2)$, given $y'' + xy = 0$ and $y(0) = 1$, $y'(0) = 0.5$.

[*Ans.* $y(0.1) = 1.0498$, $y(0.2) = 1.0986$]

Euler's Method:

we have so far discussed the methods which yield the solution of a differential equation in the form of a powerseries. we will now describe the methods which give the solution in the form of a set of tabulated values.

consider the differential equation of first order

$$\frac{dy}{dx} = f(x, y) \quad \text{——— (1)}$$

with the initial condition $y(x_0) = y_0$.

suppose we want to find the approximate value of y say y_n when $x = x_n$.

we divide the interval $[x_0, x_n]$ into n-subintervals of equal length say h, with the division points $x_0, x_1, \ldots\ldots x_n$, where $x_r = x_0 + rh$ $(r = 1, 2, \ldots n)$

Let us assume that

$$f(x, y) \approx f(x_{r-1}, y_{r-1}) \quad \text{——— (2)}$$

in $[x_{r-1}, x_r]$.

Integrating equation (1) in $[x_{r-1}, x_r]$, we get

$$\int_{x_{r-1}}^{x_r} dy = \int_{x_{r-1}}^{x_r} f(x, y)\, dx$$

$$\Rightarrow [y_r - y_{r-1}] = \int^{x_r} f(x, y)\, dx$$

$$\Rightarrow y_r \simeq y_{r-1} + f(x_{r-1}, y_{r-1}) \int_{x_{r-1}}^{x_r} dx$$

$\Rightarrow \cancel{y_r \approx}$

$\left[\because f(x,y) \approx f(x_{r-1}, y_{r-1}) \text{ in } [x_{r-1}, x_r]\right]$

$$\Rightarrow y_r \simeq y_{r-1} + f(x_{r-1}, y_{r-1})(x_r - x_{r-1})$$

$$\therefore y_r \simeq y_{r-1} + h f(x_{r-1}, y_{r-1}) \longrightarrow (2)(i)$$

which is known as Euler's iteration formula.

Taking $r = 0, 1, 2 \ldots n$ in (2)(i), we get the successive approximate values of y as follows

$$y_1 = y(x_1) = y_0 + h f(x_0, y_0)$$
$$y_2 = y(x_2) = y_1 + h f(x_1, y_1)$$
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$
$$y_n = y(x_n) = y_{n-1} + h f(x_{n-1}, y_{n-1}).$$

<u>Note:</u> Euler's Method has limited usage because of the large error that is accumulated as the process proceeds. The process is very slow and to obtain reasonable accuracy with Euler's method we have to take a smaller value of h.

Further, the method should not be used for a larger range of 'x' as the values found by this method go on becoming farther and farther away from the true values. So the method is unsuitable for practical use and a modification of it, known as the modified Euler's method which gives more accurate

Modified Euler's Method: (14)

From Euler's formula we know that

$$y_r \approx y_{r-1} + h f(x_{r-1}, y_{r-1}) \quad \text{——— (1)}$$

Let $y(x_r) = y_r$ denote the initial value using (1) an approximate value of $y_r^{(0)}$ can be calculated as

$$y_r^{(0)} = y_{r-1} + \int_{x_{r-1}}^{x_r} f(x, y)\, dx.$$

$$\Rightarrow y_r^{(0)} \approx y_{r-1} + h f(x_{r-1}, y_{r-1})$$

Replacing $f(x,y)$ by $f(x_{r-1}, y_{r-1})$ in $x_{r-1} \le x < x_r$.

Using Trapezoidal rule in $[x_{r-1}, x_r]$, we can write

$$y_r^{(0)} = y_{r-1} + \frac{h}{2}\left[f(x_{r-1}, y_{r-1}) + f(x_r, y_r)\right]$$

Replacing $f(x_r, y_r)$ by its approximate value $f(x_r, y_r^{(0)})$ at the end point of the interval $[x_{r-1}, x_r]$, we get

$$y_r^{(1)} = y_{r-1} + \frac{h}{2}\left[f(x_{r-1}, y_{r-1}) + f(x_r, y_r^{(0)})\right]$$

where $y_r^{(1)}$ is the first approximation to $y_r = y(x_r)$ proceeding as above we get the iteration formula

$$y_r^{(n)} = y_{r-1} + \frac{h}{2}\left[f(x_{r-1}, y_{r-1}) + f(x_r, y_r^{(n-1)})\right]$$

where $y_r^{(n)}$ denote the n^{th} approximation to y_r.

$\therefore$ we have

$$y_r \approx y_r^{(n)} = y_{r-1} + \frac{h}{2}\left[f(x_{r-1}, y_{r-1}) + f(x_r, y_r^{(n-1)})\right]$$

Example : Solve the equation $\frac{dy}{dx} = 1 - y$, with the initial condition $x = 0$, $y = 0$, using Euler's algorithm and tabulate the solutions at $x = 0.1, 0.2, 0.3$.

Solution : Given $\frac{dy}{dx} = 1 - y$, with the initial condition $x = 0$, $y = 0$

$$\therefore f(x, y) = 1 - y$$

we have

$$h = 0.1$$
$$\therefore x_0 = 0, y_0 = 0$$
$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$
$$x_2 = 0.2, x_3 = 0.3.$$

Taking $n = 0$ in

$$y_{n+1} = y_n + hf(x_n, y_n)$$

we get

$$y_1 = y_0 + hf(x_0, y_0)$$
$$= 0 + (0.1)(1 - 0) = 0.1$$
$$\therefore y_1 = 0.1 \text{ i.e. } y(0.1) = 0.1,$$
$$y_2 = y_1 + hf(x_1, y_1)$$
$$y_2 = 0.1 + (0.1)(1 - y1)$$
$$= 0.1 + (0.1)(1 - 0.1)$$
$$= 0.19$$
$$\therefore y_2 = y(0.2) = 0.19,$$
$$y_3 = y_2 + hf(x_2, y_2)$$
$$\therefore y_3 = 0.19 + (0.1)(1 - y_2)$$
$$= 0.19 + (0.1)(1 - 0.19)$$
$$= 0.19 + (0.1)(0.81)$$
$$= 0.271$$
$$\therefore y_3 = y(0.3) = 0.271.$$

x	Solution by Euler's method
0	0
0.1	0.1
0.2	0.19
0.3	0.271

y(0.02) (16

Example : Given $\frac{dy}{dx} = x^3 + y$, $y(0) = 1$, compute $y(0.2)$ by Euler's method taking $h = 0.01$.

Solution : Given

$$\frac{dy}{dx} = x^3 + y,$$

with the initial condition $y(0) = 1$.

$\therefore$ We have

$$f(x, y) = x^3 + y$$

$$x_0 = 0, y_0 = 1, h = 0.01$$

$$x_1 = x_0 + h = 0 + 0.01 = 0.01,$$

$$x_2 = x_0 + 2h = 0 + 2(0.01) = 0.02.$$

Applying Euler's formula we get

$$y_1 = y_0 + hf(x_0, y_0)$$

$$\therefore\ y_1 = 1 + (0.01)(x_0^3 + y_0)$$

$$= 1 + (0.01)(0^3 + 1)$$

$$= 1.01$$

$$\therefore\ y_1 = y(0.01) = 1.01,$$

$$y_2 = y_1 + hf(x_1, y_1)$$

$$= 1.01 + (0.01)[x_1^3 + y_1]$$

$$= 1.01 + (0.01)[(0.01)^3 + 1.01] = 1.0201$$

$$\therefore\ y_2 = y(0.02) = 1.0201.$$

Example : Solve by Euler's method the following differential equation $x = 0.1$ correct to four decimal places $\frac{dy}{dx} = \frac{y - x}{y + x}$ with the initial condition $y(0) = 1$.

Solution : Here

$$\frac{dy}{dx} = \frac{y - x}{y + x}$$

$$\Rightarrow f(x, y) = \frac{y - x}{y + x},$$

the initial condition is $y(0) = 1$

Taking $h = 0.02$, we get

$$x_1 = 0.02,$$
$$x_2 = 0.04,$$
$$x_3 = 0.06,$$
$$x_4 = 0.08,$$
$$x_5 = 0.1.$$

Using Euler's formula we get

$$y_1 = y(0.02) = y_0 + hf(x_0, y_0)$$

$$= y_0 + h\left(\frac{y_0 - x_0}{y_0 + x_0}\right)$$

$$= 1 + (0.02)\left(\frac{1-0}{1+0}\right)$$

$$= 1.0200$$

$$\therefore y(0.02) = 1.0200,$$

$$y_2 = y(0.04) = y_1 + hf(x_1, y_1)$$

$$= y_1 + h\left(\frac{y_1 - x_1}{y_1 + x_1}\right)$$

$$= 1.0200 + (0.02)\left(\frac{1.0200 - 0.02}{1.0200 + 0.02}\right)$$

$$= 1.0392$$

$$y_2 = y(0.04) = 1.0392,$$

$$y_3 = y(0.06) = y_2 + h\left(\frac{y_2 - x_2}{y_2 + x_2}\right)$$

$$= 1.0392 + (0.02)\left[\frac{1.0392 - 0.04}{1.0392 + 0.04}\right]$$

$$\therefore y_3 = y(0.06) = 1.0577,$$

$$y_4 = y(0.08) = y_3 + hf(x_3, y_3)$$

$$= y_3 + h\left(\frac{y_3 - x_3}{y_3 + x_3}\right)$$

$$= 1.0577 + (0.02)\left[\frac{1.0577 - 0.06}{1.0577 + 0.06}\right]$$

$$= 1.0756$$

$$\therefore y_4 = y(0.08) = 1.0756$$

$$y_5 = y(0.1) = y_4 + hf(x_4, y_4)$$

$$= y_4 + h\left(\frac{y_4 - x_4}{y_4 + x_4}\right)$$

$$= 1.0756 + (0.02)\left[\frac{1.0756 - 0.08}{1.0756 + 0.08}\right]$$

$$= 1.0928$$

$\therefore\ y(0.1) = 1.0928.$

T.I.'13

Example : Solve by the Euler's modified method the following differential equation for $x = 0.02$ by taking $h = 0.01$ $\dfrac{dy}{dx} = x^2 + y$, $y = 1$, when $x = 0$.

Solution : Here we have

$$f(x, y) = x^2 + y$$

$$h = 0.01,\ x_0 = 0,\ y_0 = y(0) = 1$$

$$x_1 = 0.01,\ x_2 = 0.02$$

$x_r \to x_0 + rh$

$x_1 = 0 + 1\,(0.01)$

$x_2 =$

we get

$$\therefore\ y_1^{(0)} = y_0 + hf(x_0, y_0)$$

$$= 1 + (0.01)(x_0^2 + y_0)$$

$$= 1 + (0.01)(0^2 + 1) = 1.01$$

$$\therefore\ y_1^{(0)} = 1.01.$$

Applying Euler's modified formula we get

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f\left(x_0, y_0\right) + f\left(x_1, y_1^{(0)}\right)\right]$$

$$= 1 + \frac{0.01}{2}\left[0^2 + 1 + (0.01)^2 + 1.01\right]$$

$$= 1.01005$$

$$\therefore\ y_1^{(1)} = 1.01005,$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f\left(x_0, y_0\right) + f\left(x_1, y_1^{(1)}\right)\right]$$

$$= 1 + \frac{0.01}{2}\left[0^2 + 1 + (0.01)^2 + 1.1005\right]$$

$$= 1.01005$$

$\therefore\ y_1^{(1)} = y_1^{(2)} = 1.01005,$

$\therefore\ y_2^{(0)} = y_1 + hf(x_1,\ y_1)$

$= 1.01005 + (0.01)(x_1^2 + y_1)$

$= 1.01005 + (0.01)((0.01)^2 + 1.01005)$

$= 1.02015,$

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f\left(x_1, y_1\right) + f\left(x_2, y_2^{(0)}\right)\right]$$

$$= 1.01005 + \frac{0.01}{2}\left[(0.01)^2 + (1.01005) + (0.02)^2 + (1.02015)\right]$$

$= 1.020204,$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f\left(x_1, y_1\right) + f\left(x_2, y_2^{(1)}\right)\right]$$

$$= 1.01 + \frac{0.01}{2}\left[(0.01)^2 + (1.01005) + (0.02)^2 + (1.020204)\right]$$

$\therefore\ y_2 = 1.020204$

$\therefore\ y_2 = y(0.02) = 1.020204.$

Exercise

1. Given $\dfrac{dy}{dx} = \dfrac{1}{x^2 + y}$, $y(4) = 4$ find $y(4.2)$ by Taylory's series method, taking $h = 0.1$.

2. Given that $\dfrac{dy}{dx} = x + y^2$, $y(0) = 1$, find $y(0.2)$.

3. Solve $\dfrac{dy}{dx} = 3x + y^2$, $y = 1$, when $x = 0$, numerically upto $x = 0.1$ by Taylor's series method.

4. Apply Taylor's algorithm to $y^1 = x^2 + y^2$, $y(0) = 1$. Take $h = 0.5$ and determine approximations to $y(0.5)$. Carry the calculations upto 3 decimals.

5. Find $y(1)$ by Euler's method from the differential equation $\dfrac{dy}{dx} = -\dfrac{-y}{1+x}$, when $y(0.3) = 2$. Convert upto four decimal places taking step length

6. Find $y(4.4)$, by Euler's modified method taking $h = 0.2$ from the differential equation $\frac{dy}{dx} = \frac{2-y^2}{5x}$, $y = 1$, when $x = 4$.

7. Given $\frac{dy}{dx} = x^2 + y$, with $y(0) = 1$, evaluate $y(0.02)$, $y(0.04)$ by Euler's method.

8. Given $\frac{dy}{dx} = y - x$, where $y(0) = 2$, find $y(0.1)$ and $y(0.2)$ by Euler's method upto two decimal places.

9. Given $\frac{dy}{dx} = -\frac{y-x}{1+x}$, with boundary condition $y(0) = 1$, find approximately y for $x = 0.1$, by Euler's method (five steps).

10. Use modified Euler's method with one step to find the value of y at $x = 0.1$ to five significant figures, where $\frac{dy}{dx} = x^2 + y$, $y = 0.94$, when $x = 0$.

11. Solve $y^1 = x - y^2$, by Euler's method for $x = 0.2$ to 0.6 with $h = 0.2$ initially $x = 0$, $y = 1$.

12. Determine $y(0.02)$, $y(0.04)$ and $y(0.06)$ using Euler's modified method.

Answers

1. 4.0098 **2.** 1.2375 **3.** 1.12725 **4.** 1.052 **5.** 1.2632 **6.** 1.01871 **7.** 1.0202, 1.0408, 1.0619 **8.** 2.42, 2.89 **9.** 1.0928 **10.** 1.039474 **11.** $y(0.2) = 0.8512$, $y(0.4) = 0.7798$, $y(0.6) = 0.7260$ **12.** $y(0.02) = 1.0202$, $y(0.04) = 1.0408$, $y(0.06) = 1.0619$

Runge-Kutta method:

As already, we considered the initial value problem $y' = f(x,y)$; $y(x_0) = y_0$ — ①

and developed Taylor series method and Euler's method for its solution.

As mentioned earlier, Euler's method being a first order method, requires a very small step size for reasonable accuracy and therefore may require lot of computations.

Higher order Taylor series methods require evaluation of higher order derivatives either manually or computationally for complicated functions, finding second, third and higher order total derivatives is tedious. Hence Taylor series methods of higher order are not of much practical use in finding the solution of initial value problems of the form given by equation ①.

However there is a class of methods known as Runge-kutta methods which do not require the calculations of higher order derivatives and give greater accuracy. The Runge-kutta formulae possess the advantage of requiring only the function values at some selected points. These methods agree with taylor's series solution upto method and is called the order of that method.

(i) First Order R-K Method:

we have seen that Euler's method gives

$$y_1 = y_0 + h f(x_0, y_0).$$

$$= y_0 + h y_0' \qquad (\because \; y' = f(x, y))$$

Expanding LHS by Taylor's series

$$y_1 = y(x_0+h) = y_0 + h y_0' + \frac{h^2}{2!} y_0'' + \cdots$$

It follows that the Euler's method agrees with the taylor's series solution upto the term in h.

Hence, Euler's method is the Runge-Kutta method of the first order.

(ii) Second order R-K method:

The modified Euler's method gives

$$y_1 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_0+h, y_1^{(0)})\right] \quad \text{--- (1)}$$

where $y_1^{(0)} = y_0 + h f(x_0, y_0)$.

Substituting $y_1^{(0)} = y_0 + h f(x_0, y_0)$ on the R.H.S of (1), we obtain

$$y_1 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_0+h, y_0 + h f(x_0, y_0))\right]$$

$$= y_0 + \frac{h}{2}\left[f_0 + f(x_0+h, y_0 + h f_0)\right] \quad \text{--- (2)}$$

where $f_0 = f(x_0, y_0)$.

(20)

Expanding LHS by taylor's series, we get

$$y_1 = y(x_0+h) = y_0 + h y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots \quad \text{--- (3)}$$

Expanding $f(x_0+h,\ y_0+hf_0)$ by Taylor's series for a function of two variables, eqn (2) gives

$$y_1 = y_0 + \frac{h}{2}\left[f_0 + \left\{ f(x_0, y_0) + h\left(\frac{\partial f}{\partial x}\right)_0 + hf_0 \left(\frac{\partial f}{\partial y}\right)_0 + O(h^2) \right] \right.$$

where $O(h^2)$ – terms containing second and higher powers of h and is read as order of h^2.

$$\left[\because f(x,y) = f(x_0, y_0) + \left(\frac{\partial f}{\partial x}\right)_0 (x-x_0) + \left(\frac{\partial f}{\partial y}\right)_0 (y-y_0) + \cdots \right.$$

$$= f(x_0, y_0) + \left(\frac{\partial f}{\partial x}\right)_0 h + \left(\frac{\partial f}{\partial y}\right)_0 h + \cdots$$

putting $x = x_0+h,\ y = y_0+h$]

$$= y_0 + \frac{1}{2}\left[hf_0 + hf_0 + h^2\left\{ \left(\frac{\partial f}{\partial x}\right)_0 + f_0\left(\frac{\partial f}{\partial y}\right)_0 \right\} + O(h^3) \right]$$

$$= y_0 + \frac{1}{2}\left[2hf_0 + h^2 f_0' + O(h^3) \right]$$

$$\left[\because \frac{d}{dx} f(x,y) = \frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y} \right.$$

$$\left[\frac{d}{dx} f(x,y)\right]_{(x_0,y_0)} = \left(\frac{\partial f}{\partial x}\right)_0 + f_0\left(\frac{\partial f}{\partial y}\right)_0$$

$$= y_0 + hf_0 + \frac{h^2}{2} f_0' + O(h^3).$$

$$= y_0 + h y_0' + \frac{h^2}{2} y_0'' + O(h^3) \quad \text{--- (4)}$$

$[\because y' = f(x,y)]$

Comparing (3) & (4) it follows that the modified Euler's method agrees with the Taylor's series solution up to the term of h^2

Hence the modified Euler's method is the Runge-kutta method of the second order.

∴ The second order Runge-kutta formula is

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

where $k_1 = hf(x_0, y_0)$

and $k_2 = hf(x_0 + h, y_0 + k_1)$.

The third order Runge-kutta formula is

$$y_1 = y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3)$$

where $k_1 = hf(x_0, y_0)$,

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$

$k_3 = hf(x_0 + h, y_0 + k')$

where $k' = hf(x_0 + h, y_0 + k_1)$

(iv) Fourth order R-K Method:

This method is most commonly used and is often referred to as Runge-kutta method only.

working rule: for finding the increment k of y corresponding to an increment h of x by Runge-kutta method

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

is as follows.

calculate successively

$k_1 = hf(x_0, y_0)$

$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$

$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$

and $k_4 = h f(x_0 + h, y_0 + k_3)$

finally compute $k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

which gives the required approximate value as $\underline{y_1 = y_0 + k}$.

Note: One of the advantages of these methods is that the operation is identical whether the differential equation is linear or non-linear

→ Apply Runge-Kutta fourth order method to find an approximate value of y when $x = 0.2$ given that $\frac{dy}{dx} = x + y$ and $y = 1$ when $x = 0$.

Soln:

we have

$x_0 = 0, \; y_0 = 1$

$f(x, y) = x + y$ and $h = 0.2$

$f(x_0, y_0) = x_0 + y_0$
$= 0 + 1 = 1$

$k_1 = h f(x_0, y_0)$
$= 0.2 \times 1 = 0.2$

$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$
$= (0.2) f(0.1, 1 + 0.1)$
$= (0.2) f(0.1, 1.1)$
$= (0.2)(1.2)$
$= 0.24$

$$k_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= (0.2) f(0.1, 1 + 0.12)$$

$$= (0.2) f(0.1, 1.12)$$

$$= (0.2)(1.22) = 0.2440.$$

$$k_4 = hf(x_0 + h, y_0 + k_3)$$

$$= (0.2) f(0.2, 1 + 0.244)$$

$$= (0.2) f(0.2, 1.244)$$

$$= (0.2)(1.444) = 0.2888$$

$$\therefore k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}(0.2 + 2(0.24) + 2(0.244) + 0.2888)$$

$$= \frac{1}{6}(1.4568)$$

$$= 0.2428.$$

Hence the Required approximate value of y is $= y_0 + k$

$$= 1 + 0.2428$$

$$= \underline{\underline{1.2428}}$$

Example : Use Runge – Kutta method to approximate y when $x = 0.1$, given that $y = 1$, when

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$$x = 0 \text{ and } \frac{dy}{dx} = x + y.$$

Solution : We have

$$x = 0, y_0 = 1$$

$$f(x, y) = x + y, \text{ and } h = 0.1.$$

$$\therefore f(x_0, y_0) = x_0 + y_0 = 0 + 1 = 1,$$

we get

$$k_1 = h f(x_0, y_0)$$
$$= 0.1 \times 1$$
$$= 0.1,$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$
$$= (0.1)(f(0 + 0.05), 1 + 0.05)$$
$$= (0.1)f(0.05, 1.05)$$
$$= (0.1)(0.05 + 1.05)$$
$$= 0.11,$$

$$k_3 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$= (0.1)(f(0 + 0.05), 1 + 0.055)$

$= (0.1)(0.05 + 1.055)$

$= (0.1)(1.105)$

$= 0.1105,$

$k_4 = f(x_0 + h, y_0 + k_3)$

$= (0.1)f(0 + 0.01, 1 + 0.1105)$

$= (0.1)f(0.1, 1.1105)$

$= (0.1)(1.2105)$

$= 0.12105,$

$$\therefore\ \Delta y = \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$= \frac{1}{6}\left(0.1 + 0.22 + 0.2210 + 0.12105\right)$$

$$= 0.11034.$$

We get

$x_1 = x_0 + h = 0 + 0.1 = 0.1$

$y_1 = y_0 + \Delta y = 1 + 0.11034$

$= 1.11034.$

Example 1 : Using Runge–Kutta method, find an approximate value of y for $x = 0.2$, if $\frac{dy}{dx} = x + y^2$, given that $y = 1$ when $x = 0$.

Solution : Taking step–length $h = 0.1$, we have

$$x_0 = 0\ ,\ y_0 = 1\ ,\ \frac{dy}{dx} = f\left(x, y\right) = x + y^2\ .$$

Now

$k_1 = h\,f(x_0, y_0) = (0.1)(0 + 1) - 0.1,$

$$k_2 = h\,f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$= (0.1)(0.05 + 1.1025)$

$= (0.1)(1.1525)$

$= 0.11525$

$$k_3 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= (0.1)(0.05 + 1.1185)$$

$$= (0.1)(1.1685)$$

$$= 0.11685$$

$$k_4 = h\, f(x_0 + h, y_0 + k_3)$$

$$= (0.1)(0.01 + 1.2474)$$

$$= (0.1)(1.3474)$$

$$= 0.13474,$$

$$\therefore \Delta y = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$= \frac{1}{6}\left(0.1 + 2(0.11525) + 2(0.11685) + 0.13474\right)$$

$$= \frac{1}{6}(0.6991) = 0.1165\,.$$

We get

$$y_1 = y_0 + \Delta y = 1 + 0.1165$$

$$\therefore y(0.1) = 1.1165.$$

For the second step, we have

$$x_0 = 0.1,\ y_0 = 1.1165,$$

$$k_1 = (0.1)(0.1 + 1.2466) = 0.1347,$$

$$k_2 = (0.1)(0.15 + 1.4014) = (0.1)(1.55S14)$$

$$= 0.1551,$$

$$k_3 = (0.1)(0.15 + 1.4259) = (0.1)(1.5759)$$

$$= 0.1576,$$

$$k_4 = (0.1)(0.2 + 1.6233) = (0.1)(1.8233)$$

$$= 0.1823,$$

$$\Delta y = \frac{1}{6}(0.9424) = 0.1571\,,$$

$$\therefore y(0.2) = 1.1165 + 0.1571 = 1.2736$$

$$\therefore y(0.1) = 1.1165 \text{ and } y(0.2) = 1.2736$$

Exercise

1. Solve the equation $\frac{dy}{dx} = x - y^2$, $y(0) = 1$ for $x = 0.2$ and $x = 0.4$ to 3 decimal places by Runge – Kutta fourth order method.

2. Use the Runge–Kutta method to approximate y at $x = 0.1$ and $x = 0.2$ for the equation $\frac{dy}{dx} = x + y$, $y(0) = 1$.

3. For the equation $\frac{dy}{dx} = 3x + \frac{y}{2}$, $y(0) = 1$. Find y at the following points with the given step – lengths.

4. Use Runge – Kutta method to solve $y^{I} = xy$ for $x = 1.4$, initially $x = 1, y = 2$ (by taking step – length $h = 0.2$).

5. $\frac{dy}{dx} = \frac{y^2 - 2x}{y^2 + x}$, use Runge – Kutta method to find y at $x = 0.1, 0.2, 0.3$ and 0.4, given that $y = 1$ when $x = 0$.

6. Use Runge – Kutta method to obtain y when $x = 1.1$ given that $y = 1.2$ when $x = 1$ and y satisfies the equation $\frac{dy}{dx} = 3x + y^2$.

Answers

1. 0.851, 0.780 **2.** 1.1103, 1.2428

3.

x	h	y
0.1	0.1	1.0665242
0.2	0.2	1.1672208
0.4	0.4	1.4782

4. 2.99485866 **5.** $y(0.1) = 1.0874$, $y(0.2) = 1.1557$, $y(0.3) = 1.2104$, $y(0.4) = 1.2544$

6. $y(1.1) = 1.7271$

Runge-Kutta Method for a System of Equations:

The fourth-order Runge-kutta method can be extended to numerically solve the higher order ordinary differential equations - linear or non-linear.

Let us consider a second order ordinary differential equation of the form

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right) \quad \text{with the initial condition } y(x_0)=y_0 \;\&\; y'(x_0)=y_0'. \quad \text{—— (1)}$$

By writing $\frac{dy}{dx} = p$, it can be reduced to two first order differential equations as given below.

$$\frac{dy}{dx} = p = f_1(x, y, p), \quad \frac{dp}{dx} = f_2(x, y, p)$$

with the initial conditions

$$y(x_0) = y_0$$

$$\text{and } y'(x_0) = y_0' \Rightarrow p(x_0) = p_0 .$$

Now, the Runge-Kutta method is applied as follows: Starting at (x_0, y_0, p_0) and taking the step-sizes for x, y, z to be h, k, l respectively,

then we define,

$$k_1 = hf_1(x_0, y_0, p_0) \quad ; \quad l_1 = hf_2(x_0, y_0, p_0)$$

$$k_2 = hf_1\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}, p_0+\frac{l_1}{2}\right) ; \quad l_2 = hf_2\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}, p_0+\frac{l_1}{2}\right)$$

$$k_3 = hf_1\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}, p_0+\frac{l_2}{2}\right) ; \quad l_3 = hf_2\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}, p_0+\frac{l_2}{2}\right)$$

$$k_4 = hf_1(x_0+h, y_0+k_3, p_0+l_3) ; \quad l_4 = hf_2(x_0+h, y_0+k_3, p_0+l_3).$$

Hence $y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$P_1 = P_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

To compute y_2 and P_2, we simply replace x_0, y_0, P_0 by x_1, y_1, P_1 in the above formulae.

→ Using Runge-kutta method, solve $y'' = xy'^2 - y^2$ for $x = 0.2$ correct to 4 decimal places. Initial conditions are $x = 0$, $y = 1$, $y' = 0$.

Sol: Let $\frac{dy}{dx} = p = f_1(x, y, p)$

Then $\frac{dp}{dx} = xp^2 - y^2 = f_2(x, y, p)$

we have $x_0 = 0$, $y_0 = 1$, $P_0 = 0$, $h = 0.2$

∴ Runge-kutta formulae become.

$k_1 = h f_1(x_0, y_0, P_0)$
$= (0.2)(0) = 0$

$l_1 = h f_2(x_0, y_0, P_0)$
$= (0.2)(x_0 P_0^2 - y_0^2)$
$= (0.2)[(0)(0) - 1^2]$
$= (0.2)(-1) = -0.2$

$k_2 = h f_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, P_0 + \frac{l_1}{2}\right)$
$= (0.2) f_1(0.1, 1, -0.1)$
$= (0.2)(-0.1) = -0.02$

$l_2 = h f_2 \left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, p_0 + \frac{l_1}{2}\right)$

$= (0.2)(0.1, 1, -0.1)$

$= (0.2)\left[(0.1)(-0.1)^2 - 1^2\right]$

$= (0.2)\left[0.001 - 1\right]$

$= (0.2)(-0.999) = -0.1998$

$k_3 = h f_1 \left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, p_0 + \frac{l_2}{2}\right)$

$= (0.2) f_1 (0.1, 0.99, -0.0999)$

$= (0.2)(-0.0999)$

$= -0.01998$

$l_3 = h f_2 \left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, p_0 + \frac{l_2}{2}\right)$

$= (0.2) f_2 (0.1, 0.99, -0.0999)$

$= (0.2)\left[(0.1)(-0.0999)^2 - (0.99)^2\right)$

$= (0.2)(-0.9791)$

$= -0.1958$

$k_4 = h f_1 (x_0 + h, y_0 + k_3, p_0 + l_3)$

$= (0.2) f_1 (0.2, 0.98002, -0.1958)$

$= (0.2)(-0.1958)$

$= -0.03916 = -0.0392$

$l_4 = h f_2 (x_0 + h, y_0 + k_3, p_0 + l_3)$

$= (0.2) f_2 (0.2, 0.98002, -0.1958)$

$= (0.2)\left[(0.2)(-0.1958)^2 - (0.98002)^2\right]$

$= (0.2)(-0.9527) = -0.1905$

$$\therefore K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$$
$$= \frac{1}{6}\left[0.00 + 2(-0.02) + 2(-0.01998) - 0.0392\right]$$
$$= \frac{1}{6}(-0.11916)$$
$$= -0.01986 = -0.0199$$

$$l = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$$
$$= \frac{1}{6}\left[-0.2 + 2(-0.1998) + 2(-0.1958) - 0.1905\right]$$
$$= \frac{1}{6}(-1.1817) = -0.19695$$
$$= -0.1970$$

Hence at $x = 0.2$

$$y = y_0 + K = 1 - 0.0199$$
$$= 0.9801$$

and $y' = p = p_0 + l = 0 - 0.1970$
$$= -0.1970.$$

→ Using Runge-Kutta method of order 4, Solve $y'' = y + xy'$, $y(0) = 1$, $y'(0) = 0$ to find $y(0.2)$ and $y'(0.2)$.

→ Solve the differential equations

$\frac{dy}{dx} = 1 + xp$, $\frac{dp}{dx} = -xy$ for $x = 0.3$. using fourth order Runge-Kutta method. Initial values are $x = 0$, $y = 0$, $p = 1$.