

IAS MATHEMATICS (OPT.)-2017

PAPER - II : SOLUTIONS

1(a) \rightarrow Let $x_1 = 2$ and $x_{n+1} = \sqrt{x_n + 20}$,
 201.7 $n = 1, 2, 3, \dots$. Show that the sequence
 x_1, x_2, x_3, \dots is convergent.

Given that $x_1 = 2$, $x_{n+1} = \sqrt{x_n + 20}$
 $n = 1, 2, 3, \dots$

$$\text{if } n=1: x_2 = \sqrt{x_1 + 20} = \sqrt{2+20} \quad (1)$$

$$= \sqrt{22} > 2 = x_1$$

$$\therefore x_2 > x_1$$

Let us assume that $x_{k+1} > x_k$
 we have

$$x_{k+1} + 20 > x_k + 20$$

$$\Rightarrow \sqrt{x_{k+1} + 20} > \sqrt{x_k + 20}$$

$$\Rightarrow x_{k+2} > x_{k+1}$$

∴ By mathematical induction,
 $x_{n+1} > x_n \forall n \in \mathbb{N}$

i.e. $x_n < x_{n+1} \forall n \in \mathbb{N}$.

∴ (x_n) is an increasing sequence.

We have $x_1 = 2 < 5$

$$x_2 = \sqrt{x_1 + 20} = \sqrt{2+20} = \sqrt{22} < 5 \quad (2)$$

Let us assume that

$$x_k < 5 \quad (3)$$

We have

$$x_k + 20 < 20 + 5$$

$$\Rightarrow \sqrt{x_k + 20} < \sqrt{25} = 5$$

$$\Rightarrow x_{k+1} < 5$$

∴ By mathematical induction,
 $x_n < 5 \forall n \in \mathbb{N}$.

- ∴ The given sequence
 (x_n) is monotonically increasing
and bounded above
- ∴ (x_n) is convergent
and converges to its
least upper bound.

(Q) Let G be a group of order n :
 show that G is isomorphic to a
 subgroup of the permutation group S_n .

Sol let G be a group of order n

let $G_1 = A(G)$ be the group of all
 permutations of the set G and is
 of order $n!$ and degree n .

For any $a \in G$, define a map

$$f_a: G \rightarrow G \text{ s.t. } f_a(x) = ax$$

$$\forall a, y \in G \text{ s.t. } a=y \Rightarrow ax=ay \Rightarrow f_a(x) = f_a(y)$$

$\therefore f_a: G \rightarrow G$ is well-defined.

Again $f_a(x) = f_a(y) \Rightarrow a \cdot x = a \cdot y$

$$\Rightarrow a \cdot x = a \cdot y \text{ (by using cancellation laws in } G)$$

$\therefore f_a: G \rightarrow G$ is 1-1.

Also, for any $y \in G$, $\exists a^{-1}y \in G$

$$\text{s.t. } f_a(a^{-1}y) = a(a^{-1}y) = f_a^{-1}(y)$$

$\therefore f_a: G \rightarrow G$ is an onto.

$\therefore f_a \in A(G)$.

Let $K = \{f_a / f_a: G \rightarrow G : f_a(x) = ax\} \subseteq A(G)$.

To P.T. $K \leq A(G)$

clearly $K \neq \emptyset$ as $f_e \in K$.

Let $f_a, f_b \in K$

$$\text{we have } (f_b f_{b^{-1}})(a) = f_b(f_b^{-1}a)$$

$$= f_b(b^{-1}a) = b(a) = a$$

$$= (b^{-1})a = a = a \quad \checkmark \text{ in } G.$$

$$= f_a(a)$$

$\therefore f_b f_{b^{-1}} = f_a$. (identity element in G)

$$\Rightarrow f_{b^{-1}} = (f_b)^{-1}$$

$$\text{Also } (f_a f_b)(a) = f_a(f_b(a)) = f_a(ba) \quad (\text{by } ①)$$

$$= a(ba)$$

$$= (ab)a$$

$$= f_{ab}(a)$$

$$\therefore f_a f_b = f_{ab}$$

$$\therefore f_a(f_b)^{-1} = f_a f_{b^{-1}} = f_{ab^{-1}} \in K.$$

$$\therefore \underline{\underline{K \leq A(G)}}$$

let us define a mapping $\psi: G \rightarrow K$
such that $\psi(a) = f_a \quad \forall a \in G$ ②

To s.t. ψ is well-defined, i.e.

$$\forall a, b \in G, \quad a = b$$

$$\Leftrightarrow a_n = b_n$$

$$\Leftrightarrow f_a(a) = f_b(b) \quad \checkmark \text{ in } G$$

$$\Leftrightarrow f_a = f_b$$

$$\Leftrightarrow \psi(a) = \psi(b)$$

ψ is obviously onto. we have

$$\text{To s.t. } \psi \text{ is homomorphism: we have } \psi(ab) = f_{ab} = f_a f_b = \psi(a) \psi(b).$$

$\therefore \psi$ is 1-1 onto homomorphism $\therefore G$ isomorphic to K .

1C find the supremum and infimum
2AS of $\frac{x}{\sin x}$ on the interval $[0, \frac{\pi}{2}]$.
2017

SOL let $f(x) = \frac{x}{\sin x} \quad x \in (0, \frac{\pi}{2})$.

then $f'(x) = \frac{\sin x - x \cos x}{\sin^2 x} \quad \text{--- (1)}$

let $g(x) = \sin x - x \cos x$

then $g'(x) = \cos x - (\cos x - x \sin x)$
 $= x \sin x > 0 \quad x \in (0, \frac{\pi}{2})$

$\therefore g(x)$ is an increasing function
 on $(0, \frac{\pi}{2})$.

$\therefore x \in (0, \frac{\pi}{2}) \Rightarrow 0 < x \leq \frac{\pi}{2}$

$\Rightarrow g(0) < g(x) \leq g(\frac{\pi}{2})$

$\Rightarrow 0 < g(x) \leq 1$

$\therefore f'(x) > 0 \quad x \in (0, \frac{\pi}{2})$.

f is an increasing on $(0, \frac{\pi}{2})$.

$\therefore \text{infimum } f = \lim_{x \rightarrow 0} \frac{x}{\sin x} = 1$

$\text{supremum } f = f(\frac{\pi}{2}) = \frac{\frac{\pi}{2}}{\sin \frac{\pi}{2}} = \frac{\pi}{2}$

Q. 1(d) Determine all entire functions $f(z)$ such that 0 is a removable singularity of $f(1/z)$?

Sol:- As the function $f(z)$ has no singularity in the finite part of the plane; it can be expanded in Taylor's series in any circle $|z|=k$; where k is arbitrarily large.

$$\therefore f(z) = \sum_{r=0}^{\infty} A_r z^r$$

Also, if $f(z)$ has no singularity at $z=\infty$; $f(1/z)$ has none at $z=0$. Moreover, since $f(z)$ has no singularity in finite part of plane; we have

$$f(z) = \sum_{n=0}^{\infty} A_n z^n \quad \therefore f(1/z) = \sum_{n=0}^{\infty} A_n z^{-n}$$

Since; $f(z)$ has a pole of order 'n' at infinity, $f(1/z)$ has a pole of order 'n' at zero (0).

$$\therefore f\left(\frac{1}{z}\right) = \sum_{s=1}^{\infty} \frac{B_s}{z^s} + \phi(z).$$

where, $\phi(z)$ is a regular function of z .

Hence; $\phi(z) + \sum_{s=1}^{\infty} \frac{B_s}{z^s} = \sum_{r=0}^{\infty} A_r z^r$

$$\therefore \phi(z) = A_0 = \text{a constant}$$

and $A_{n+1} = A_{n+2} = \dots = 0$

$$f\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} A_n z^{-n} = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \dots$$

$f\left(\frac{1}{z}\right)$ is a removable singularity of degree n .

1(e)
2017Using Graphical method, find the maximum value of $2x+3y$ Subject to $4x+3y \leq 12$

$$4x+y \leq 8$$

$$4x-y \leq 8$$

$$x, y \geq 0$$

Soln:

1(e) Use graphical max = $2x+3y$

Subject to

$$4x+3y \leq 12$$

$$4x+y \leq 8$$

$$4x-y \leq 8$$

$$x, y \geq 0$$

By converting all constraints into eqn.

$$4x+3y = 12 \quad \text{(i)}$$

$$4x+y = 8 \quad \text{(ii)}$$

$$4x-y = 8 \quad \text{(iii)}$$

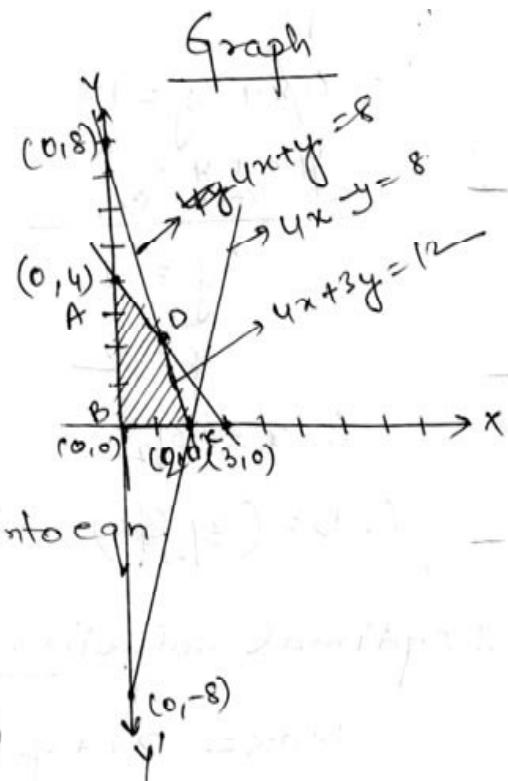
From eqn - (i)

$$4x+3y = 12$$

Putting $x=0 ; y=4$ Putting $y=0 ; x=3$

From eqn - (ii)

$$4x+y = 8$$

Putting $x=0 ; y=8$ Putting $y=0 ; x=2$ 

x	0	3
y	4	0

x	0	2
y	8	0

From eqn (ii)

$$4x + y = 8$$

Putting $x=0; y=8$ Putting $y=0; x=2$

x	0	2
y	8	0

From eqn (iii)

$$4x - y = 8$$

Putting $x=0; y=-8$ Putting $y=0; x=2$

x	0	2
y	-8	0

Now our shaded region ABCD which satisfies the given condition.

From eqn (i) and (ii) we find D.

$$\begin{aligned} \therefore 4x + 3y &= 12 \\ 4x + y &= 8 \\ \hline 2y &= 4 \\ y &= 2 \end{aligned}$$

$$\therefore x = 3/2$$

$$\therefore D = (3/2, 2)$$

Optimal Solution

$$\text{Max} = 2x + y.$$

$$\text{at point } a(0, 4) = 2(0) + 4 = 4$$

$$b(0, 0) = 2(0) + 0 = 0$$

$$c(2, 0) = 2(2) + 0 = 4$$

$$d(3/2, 2) = 2\left(\frac{3}{2}\right) + 2 = 5.$$

\therefore Max. value of z is at point $D(3/2, 2)$ and its value is 5.

Q. 2 (a) Let $f(t) = \int_0^t [x] dx$

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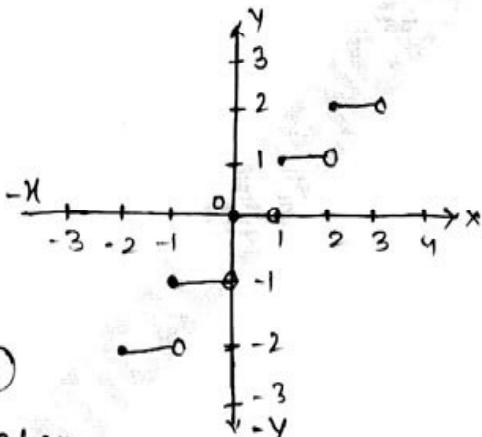
where $[x]$ denotes the largest integer less than or equal to x .

(i) Determine all the real numbers t at which f' is differentiable.

(ii) Determine all the real numbers t at which f' is continuous but not differentiable.

Sol:-

$$[x] = \begin{cases} 1 & ; 1 \leq x < 2 \\ 0 & ; 0 \leq x < 1 \\ -1 & ; -1 \leq x < 0 \\ -2 & ; -2 \leq x < -1 \\ \vdots & \end{cases}$$



①

Note that, we have an open circle on the right of each step, where the function "jumps up" to the next step. Thus, the function $[x]$ is discontinuous at every integer point, and continuous at non-integer points.

Now, $f(t) = \int_0^t [x] dx$ is defined by integrating the greatest integer function $[x]$ from 0 to t .

for example:

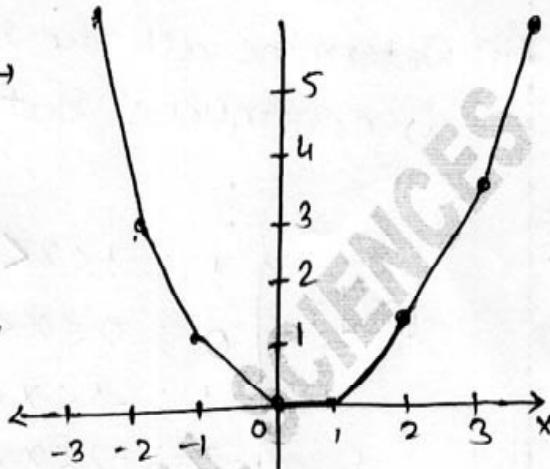
$$\begin{aligned} f(3.5) &= \int_0^1 [x] dx + \int_1^2 [x] dx + \int_2^3 [x] dx + \int_3^{3.5} [x] dx \\ &= \int_0^0 dx + \int_1^1 dx + \int_2^2 dx + \int_3^{3.5} 3 dx \\ &= 0 + 1 + 2 + 3(0.5) = 4.5. \end{aligned}$$

$$f(x) = \begin{cases} \vdots & \\ x-1 & ; \text{ if } 1 \leq x < 2 \\ 0 & ; \text{ if } 0 \leq x < 1 \\ -x & ; \text{ if } -1 \leq x < 0 \\ -2x-1 & ; \text{ if } -2 \leq x < -1 \\ \vdots & \end{cases}$$

(2)

graphing this function, →

we know that,

there are corners in
this graph at integer
values of x ;

Hence, the given function,

$$f(t) = \int_0^t [x] dx$$

is continuous $\forall x \in \mathbb{R}$, but is not differentiable at the points where the integrand is discontinuous;
ie at all integer points.

Q(5) IAS-2017

~~Q(6)~~

Prove that $\int_0^\infty \frac{x \sin mx}{x^2 + a^2} dx = \frac{\pi}{2} e^{-ma}$ by contour integration.

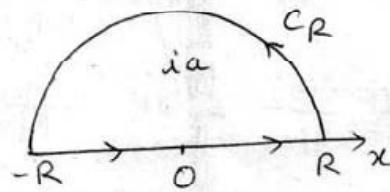
Solⁿ→

Consider the integral $\int_C f(z) dz$,

$$\text{where } f(z) dz = \frac{z e^{imz}}{z^2 + a^2}$$

taken around the closed the contour C consisting of real axis from $-R$ to R , and the upper half of the large circle $|z|=R$.

Poles of $f(z)$ are given by $z^2 + a^2 = 0 \Rightarrow z = \pm ia$



Hence by making $R \rightarrow \infty$, relation ① reduces to .

$$\int_{-\infty}^{\infty} \frac{x e^{imx} dx}{x^2 + a^2} = 2\pi i \times \frac{e^{-am}}{2}$$

Equating imaginary parts we get -

$$\int_{-\infty}^{\infty} \frac{x \sin mx}{x^2 + a^2} dx = \frac{\pi}{2} e^{-am}$$

Q(1)

2017

Let F be a field and $F[x]$ denote the ring of polynomials over F in a single variable x . For $f(x), g(x) \in F[x]$ with $g(x) \neq 0$, show that there exist $q(x), r(x) \in F[x]$ such that $\deg r(x) < \deg g(x)$ and $f(x) = q(x) \cdot g(x) + r(x)$.

Sol'n: Let us consider the set

$$S = \{ f(x) - h(x)g(x) / h(x) \in F[x] \}$$

For $0(x) \in F[x]$,

$$f(x) = f(x) - 0(x)g(x) \in S$$

$$\therefore S \neq \emptyset$$

Let $0 \in S$. Then by definition of S ,

$$\exists q(x) \in F[x] \text{ so that } 0 = f(x) - q(x)g(x)$$

$$\text{i.e. } f(x) = q(x)g(x) + 0(x)$$

$$\text{i.e. } f(x) = q(x)g(x) + r(x) \text{ where } r(x) = 0$$

\therefore the theorem is proved.

Let $0(x) \notin S$. Then every polynomial in S is a non-zero polynomial and hence non-negative degree.

Let $r(x) \in S$ be a polynomial of least degree.

By definition of S , there exist $q(x) \in F[x]$ so that

$$r(x) = f(x) - q(x)g(x)$$

$$\text{i.e. } f(x) = g(x)q(x) + r(x) \quad \text{--- (1)}$$

$$\text{Let } g(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n, a_n \neq 0$$

$$\text{so that } \deg g(x) = n$$

Now we have to prove that $\deg r(x) < n$,
i.e. $\deg r(x) < \deg g(x)$.

if possible, suppose that $m = \deg r(x) \geq n$.

Let $r(x) = c_0 + c_1x + c_2x^2 + \dots + c_m x^m$, $c_m \neq 0$.

Now we have

$$c_m a_n^{-1} x^{m-n} g(x) = c_m a_n^{-1} x^{m-n} [a_0 + a_1 x + a_2 x^2 + \dots + a_{n-1} x^{n-1} + a_n x^n]$$

$$= c_m a_n^{-1} a_0 x^{m-n} + c_m a_n^{-1} a_1 x^{m-n+1} + \dots + c_m a_n^{-1} a_{n-1} x^{m-1} + c_m a_n^{-1} a_n x^m.$$

$$\therefore r(x) = c_m a_n^{-1} x^{m-n} g(x)$$

$$= (c_{m-1} x^{m-1} + \dots + c_0) -$$

$$(c_m a_n^{-1} a_{n-1} x^{m-1} + \dots + c_m a_n^{-1} a_0 x^{m-n})$$

$$\therefore r(x) = c_m a_n^{-1} x^{m-n} g(x) + \alpha(x) \quad \text{--- } ②$$

$$\text{where } \alpha(x) = (c_{m-1} - c_m a_n^{-1} a_{n-1}) x^{m-1} + \dots + c_0$$

$$\Rightarrow \deg \alpha(x) \leq m-1$$

$$\text{i.e. } \deg \alpha(x) \leq \deg r(x) - 1$$

$$\text{i.e. } \deg \alpha(x) < \deg r(x)$$

\therefore from ① & ②

$$\alpha(x) = f(x) - g(x) \{ q(x) + c_m a_n^{-1} x^{m-n} \}$$

$$= f(x) - g(x) \beta(x)$$

where $\beta(x) = q(x) + c_m a_n^{-1} x^{m-n} \in F[x]$

$\therefore \alpha(x) \in S$

Now we have $\alpha(x), r(x) \in S$ and $\deg \alpha(x) < \deg r(x)$

This is a contradiction. Since $r(x)$ is the polynomial of least degree in S .

\therefore our supposition is wrong.

Hence $\deg r(x) < n$

i.e. $\deg r(x) < \deg g(x)$

Uniqueness of $q(x)$ and $r(x)$:

If possible, suppose that

$$f(x) = q'(x)g(x) + r'(x)$$

where $r'(x) = 0$ (or) $\deg r'(x) < \deg g(x)$.

$$\text{then } q(x)g(x) + r(x) = q'(x)g(x) + r'(x)$$

$$\text{i.e. } (q(x) - q'(x))g(x) = r'(x) - r(x)$$

If $g(x) - g'(x) \neq 0$ then

$$\deg(q(x) - q'(x))g(x) = \deg(q(x) - q'(x)) + \deg g(x)$$

$$\text{i.e. } \deg (\gamma'(x) - \delta(x)) \geq \deg g(x)$$

This is a contradiction because

$\deg r(x) < \deg g(x)$ and

$$\deg g'(x) < \deg g(x)$$

$$\therefore q(x) - q'(x) = 0 \text{ and } r'(x) - r(x) = 0$$

$$\Rightarrow q'(x) = q(x) \quad \& \quad \gamma'(x) = \gamma(x)$$

Hence $g_1(z), g_2(z) \in F[x]$ are unique.

Note: (1) The polynomials $q(x)$ and $r(x)$ of the above theorem are called the quotient and the remainder.

(2) In the above theorem if $\sigma(x) = 0$ then we say that $q(x)$ divides $f(x)$ (or) $g(x)$ is a factor of

$$f(x)$$

3(G)
DAT
-2017

Show that the groups $\mathbb{Z}_5 \times \mathbb{Z}_7$
and \mathbb{Z}_{35} are isomorphic.

Sol let $(\mathbb{Z}_{35} = \{0, 1, 2, 3, \dots, 34\}, +)$
be a cyclic group of order 35.

let $\mathbb{Z}_5 \times \mathbb{Z}_7 = \{(a, b) / a \in \mathbb{Z}_5, b \in \mathbb{Z}_7\}$
be a cyclic ($\because \gcd(5, 7) = 1$)

where $(\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}, +)$

and $(\mathbb{Z}_7 = \{0, 1, 2, 3, 4, 5, 6\}, +)$

are two cyclic groups such that

$$\text{o}(\mathbb{Z}_5) = 5 \text{ and } \text{o}(\mathbb{Z}_7) = 7$$

Define a mapping $f: \mathbb{Z}_{35} \rightarrow \mathbb{Z}_5 \times \mathbb{Z}_7$

such that $f(a) = (a \oplus_5 0, a \oplus_7 0)$

Now we shall show that
 f is well-defined!

let $a, b \in \mathbb{Z}_5$ s.t. $a = b$

$$\Rightarrow a \oplus_5 0 = b \oplus_5 0$$

$\therefore f$ is well-defined.

To S.T 1-1: let $(a \oplus_5 0, a \oplus_7 0) = (b \oplus_5 0, b \oplus_7 0)$

$$\Rightarrow a = b$$

$\therefore f$ is 1-1

To show onto:

for every $(a \oplus_5 0, a \oplus_7 0) \in \mathbb{Z}_5 \times \mathbb{Z}_7$

$\exists a \in \mathbb{Z}_{35}$ such that

$$f(a) = (a \oplus_5 0, a \oplus_7 0) \text{ by defn'ly.}$$

$\therefore f$ is onto.

To show homomorphism:

Let $a, b \in \mathbb{Z}_{35}$, then

$$f(a \oplus_{35} b) = ((a \oplus_{35} b) \oplus_5 0, (a \oplus_{35} b) \oplus_7 0)$$

$$= (a \oplus_{35} b, a \oplus_{35} b)$$

$$= (a \oplus_5 0, a \oplus_7 0) \oplus_{35} (b \oplus_5 0, b \oplus_7 0)$$

$$= f(a) \oplus_{35} f(b)$$

$\therefore f$ is a homomorphism

$$\therefore \mathbb{Z}_5 \times \mathbb{Z}_7 \cong \mathbb{Z}_{35}$$

3(b)
2017

Let $f = u + iv$ be an analytic function on the unit disc.

$$D = \{z \in \mathbb{C} / |z| < 1\} \text{ Show that}$$

$$\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = 0 = \frac{\partial v}{\partial x} + i \frac{\partial u}{\partial y} \text{ at all}$$

points of D.

Sol'n: Given that the function $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain.

$$\therefore f'(z) = \frac{\partial f}{\partial z} \quad \text{--- (1)} \quad \text{or} \quad f'(z) = -i \frac{\partial f}{\partial y} \quad \text{--- (2)}$$

Since the analytic function has derivatives of all orders.

$$\text{--- (1)} \equiv f''(z) = \frac{\partial}{\partial x} (f'(z))$$

$$= \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial z} \right)$$

$$= \frac{\partial^2 f}{\partial z^2} \quad \text{--- (3)}$$

$$\text{--- (2)} \equiv f''(z) = -i \frac{\partial}{\partial y} (f'(z))$$

$$= -i \frac{\partial}{\partial y} \left(-i \frac{\partial f}{\partial y} \right) = (-1) \frac{\partial^2 f}{\partial y^2} \quad \text{--- (4)}$$

from (3) & (4) we have

$$\frac{\partial^2 f}{\partial z^2} = -\frac{\partial^2 f}{\partial y^2}$$

$$\Rightarrow \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0$$

$$\Rightarrow \nabla^2 f = 0 \quad \text{--- (5)}$$

which is valid for any analytic function $f(z)$
i.e. If $f(z) = u(x,y) + iv(x,y)$ is an analytic
in a domain D then from ⑤

$$\nabla^2 f = 0$$

$$\Rightarrow \nabla^2(u+iv) = 0$$

$$\Rightarrow \nabla^2 u + i \nabla^2 v = 0$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0; \quad \underline{\underline{\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0}}.$$

3(C)

2017

Solve the following linear programming problem by simplex method:

maximize

$$Z = 3x_1 + 5x_2 + 4x_3$$

Subject to

$$2x_1 + 3x_2 \leq 8$$

$$2x_2 + 5x_3 \leq 10$$

$$3x_1 + 2x_2 + 4x_3 \leq 15$$

$$x_1, x_2, x_3 \geq 0$$

Soln: Simplex method:

$$\text{Max } Z = 3x_1 + 5x_2 + 4x_3$$

Subject to:

$$2x_1 + 3x_2 \leq 8$$

$$2x_2 + 5x_3 \leq 10$$

$$3x_1 + 2x_2 + 4x_3 \leq 15$$

$$x_1, x_2, x_3 \geq 0$$

Soln 1 - Standard form of given problem.

$$Z = 3x_1 + 5x_2 + 4x_3 + 0s_1 + 0s_2 + 0s_3$$

S.C.

$$2x_1 + 3x_2 + 0x_3 + s_1 + 0s_2 + 0s_3 = 8 \quad (i)$$

$$0x_1 + 2x_2 + 5x_3 + 0s_1 + s_2 + 0s_3 = 10 \quad (ii)$$

$$3x_1 + 2x_2 + 4x_3 + 0s_1 + 0s_2 + s_3 = 15 \quad (iii)$$

where $x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$ or, $s_1, s_2, s_3 \rightarrow$ slack variables.

For initial basic feasible soln

free variable = 6 - 3 = 3

From (i) we get, put $x_1, x_2, x_3 = 0$

$$S_1 = 8, S_2 = 10, S_3 = 15$$

$$\text{IBFS } (x_1, x_2, x_3, S_1, S_2, S_3) = (0, 0, 0, 8, 10, 15)$$

Now,

	C_j	3	5	4	0	0	0		
CB	Basic.	x_1	x_2	x_3	S_1	S_2	S_3	b	Q
0	S_1	2	(3)	0	1	0	0	8	$\frac{8}{3} \leftarrow$
0	S_2	0	2	5	0	1	0	10	5
0	S_3	3	2	4	0	0	1	15	$\frac{15}{2}$
$Z_j = \sum_{CB} a_{ij}$		0	0	0	0	0	0		
$C_j = C_j - Z_j$		3	5	4	0	0	0		

$\therefore 3$ is key element

\therefore incoming variable = x_2 and outgoing variable = x_1

	C_j	3	5	4	0	0	0		
CB	Basic	x_1	x_2	x_3	S_1	S_2	S_3	B	Q
S	x_2	$\frac{2}{3}$	1	0	$\frac{1}{3}$	0	0	$\frac{8}{3}$	$\frac{14}{15} \leftarrow$
0	S_2	$-\frac{2}{3}$	0	($\frac{5}{2}$)	$-\frac{1}{3}$	$\frac{1}{2}$	0	$\frac{7}{3}$	$\frac{29}{12}$
0	S_3	$\frac{5}{6}$	0	2	$-\frac{1}{3}$	0	$\frac{1}{2}$	$\frac{29}{6}$	
$Z_j = \sum_{CB} a_{ij}$		10/3	5	0	5/3	0	0		
$C_j = C_j - Z_j$		-1/3	0	4	-5/3	0	0		

$\therefore \frac{5}{2}$ = key element

\therefore incoming variable = x_3 \therefore outgoing variable = S_2

	C_j	3	5	4	0	0	0		
CB	Basic	x_1	x_2	x_3	S_1	S_2	S_3		
S	x_2	$\frac{2}{3}$	1	0	$\frac{1}{3}$	0	0	$\frac{8}{3}$	4
4	x_3	$-\frac{4}{15}$	0	1	$-\frac{2}{15}$	$\frac{1}{45}$	0	$\frac{14}{15}$	-
0	S_3	($\frac{41}{60}$)	0	0	$-\frac{1}{5}$	$-\frac{1}{5}$	$\frac{1}{2}$	$\frac{89}{60}$	$\frac{89}{41} \leftarrow$

$$\begin{array}{l}
 z_j = \sum c_{0j} a_{ij} \\
 C_j = f_j - z_j
 \end{array}
 \left| \begin{array}{cccccc}
 34/15 & 5 & 4 & 37/15 & 4/5 & 0 \\
 11/15 & 0 & 0 & -37/15 & -4/5 & 0 \\
 \downarrow & & & & &
 \end{array} \right|$$

$\therefore 4/5 = \text{key element}$ $\therefore B.V. = x_1$ and $O.V. = s_3$

CB	Basic	x_1	x_2	x_3	s_1	s_2	s_3	B	O
5	x_2	0	$3/2$	0	$94/41$	$12/41$	$-39/41$	$75/41$.
4	x_3	0	0	$15/4$	$-65/82$	$75/164$	$39/41$	$465/82$.
3	x_1	1	0	0	$-12/11$	$-12/41$	$36/41$	$89/41$.
	$Z_j = \sum c_{0j} a_{ij}$	3	$15/2$	15	$304/41$	$99/41$	$60/41$	$1572/41 \Rightarrow Z_{\max}$.
	$C_j = f_j - z_j$	0	$-5/2$	-11	$-304/41$	$-99/41$	$-60/41$	$x_1 = 89/41$	$. \quad . \quad . \quad . \quad . \quad .$
								$x_2 = 75/41$	$. \quad . \quad . \quad . \quad . \quad .$
								$x_3 = 465/82$	$. \quad . \quad . \quad . \quad . \quad .$

From above table all c_j 's ≤ 0 \therefore optimal BFS = $x_1 = 89/41$
 $x_2 = 75/41$ $x_3 = 465/82$ $\therefore \text{Max } Z = \frac{1572}{41}$

4.(a) For a function $f : \mathbb{C} \rightarrow \mathbb{C}$ and $n \geq 1$, let $f^{(n)}$ denote the n^{th} derivative of f and $f^{(0)} = f$. Let f be an entire

function such that for some $n \geq 1$, $f^{(n)}\left(\frac{1}{k}\right) = 0$ for all $k = 1, 2, 3, \dots$, show that f is a polynomial.

Sol ":

YOUR SELF

~~4(b)~~ → find the initial basic feasible solution of the following transportation problem using vogel's approximation method and find the cost.

Destinations:

	D ₁	D ₂	D ₃	D ₄	D ₅	
O ₁	4	7	0	3	6	14
Origins O ₂	1	2	-3	3	8	9
O ₃	3	1	4	0	5	17
	8	3	8	13	8	40

sol Demand
clearly the TPP is a balanced problem and is of 3×5 order matrix.
 \therefore There are $m+n-1 = 3+5-1 = 7$ basic cells
 \therefore feasible solutions.

let us find initial basic feasible solution by using vogel's approximation method;

4	7	0	3	6	7(3)(1)(1)(2)(2)(6)
7	0	0	0	7	7(1)
1	2	-3	3	8	9(4)(1)(2)(7)
1	0	0	0	0	7(1)(1)(3)(1)(6)(5)
3	1	4	0	5	7(1)
0	3	0	13	1	
8	3	8	13	8	

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(2)✓ (3)✓ (3)✓ (3)✓ (1)✓
 (2)✓ (3)✓ (3)✓ (1)✓
 (2)✓ (3)✓ (3)✓ (1)✓ (0)

$\therefore \text{DBPS is } (7, 0, 0, 0, 7; 1, 0, 8, 0, 0, 0, 3, 0, 13, 1)$

$\therefore \text{Minimum cost}$

$$\begin{aligned}
 &= 7(4) + 1(0) + 3(-1) + 8(-3) \\
 &+ 13(0) + 7(6) + 1(5) \\
 &= 49,
 \end{aligned}$$

Let us find the cost by using modified method:-

$$\Delta_{ij} = u_i + v_j - c_{ij} \quad (1)$$

u_i 's & v_j 's are helping variables corr. to rows and columns respectively, and c_{ij} 's costs.

$\Delta_{ij} = 0$ corr. to basic cells.

$$\begin{array}{l}
 \Delta_{11} = 0 \Rightarrow u_1 + v_1 = 4 \vee \left. \begin{array}{l} \text{let } u_2 = 0 \\ \text{then } v_4 = 0 \end{array} \right\} \\
 \Delta_{15} = 0 \Rightarrow u_1 + v_5 = 6 \vee \left. \begin{array}{l} v_2 = -1 \\ v_4 = -1 \end{array} \right\} \\
 \Delta_{21} = 0 \Rightarrow u_2 + v_1 = 1 \vee \left. \begin{array}{l} v_3 = 5 \\ v_5 = 5 \end{array} \right\} \\
 \Delta_{23} = 0 \Rightarrow u_2 + v_3 = -3 \vee \left. \begin{array}{l} u_1 = 1 \\ u_3 = 1 \end{array} \right\} \\
 \Delta_{32} = 0 \Rightarrow u_3 + v_2 = -1 \vee \left. \begin{array}{l} u_1 = 3 \\ u_2 = -2 \end{array} \right\} \\
 \Delta_{34} = 0 \Rightarrow u_3 + v_4 = 0 \vee \left. \begin{array}{l} v_3 = -1 \\ v_5 = -1 \end{array} \right\} \\
 \Delta_{35} = 0 \Rightarrow u_3 + v_5 = 5 \vee
 \end{array}$$

Let us find Δ_{ij} corr. to non-basic cells:

$$\begin{array}{ll}
 \Delta_{12} = -7 & \Delta_{24} = -5 \\
 \Delta_{13} = 0 & \Delta_{25} = -5 \\
 \Delta_{14} = -2 & \Delta_{31} = 0 \quad \therefore \text{optimality} \\
 \Delta_{22} = -5 & \Delta_{33} = -5 \quad \text{has been obtained}
 \end{array}$$

$\therefore \Delta_{ij} \leq 0$
 $\therefore \text{OFS is same as DBPS.}$

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$\therefore \text{Min. cost } 49.$

Q1C) Let $\sum_{n=1}^{\infty} x_n$ be a conditionally convergent series of real numbers. Show that there is a rearrangement $\sum_{n=1}^{\infty} x_{\pi(n)}$ of the series $\sum_{n=1}^{\infty} x_n$ that converges to 100.

Sol: we wish to show that, we can re-arrange the terms of $\sum x_n$ to form a series whose sum converges to 100.

Add together, in order, just enough of the first positive terms of $\sum x_n$ so that their sum exceeds 100.

Say we need, n_1 terms to do so, then the partial sum $S_{n_1} > 100$.

We can always do so no matter how large the sum (100) is, since the series of positive terms diverges to infinity (∞).

To this sum S_{n_1} , in order, just enough of the first negative terms of $\sum x_n$ to make the resulting sum less than 100. Say we need n_2 negative terms, then the partial sum

$$S_{n_1+n_2} < 100$$

We can always do so no matter how far the +ve terms took us to right of 100, since the series of negative terms diverges to $-\infty$ (infinity).

Now, Add just enough of the next positive terms, say n_3 +ve terms to get the sum to exceed

100 again. we know have the partial sum

$$S_{n_1+n_2+n_3} > 100$$

Now, again add just enough of the next negative terms say n_4 negative terms, so that the sum is less than 100 again, we now have the partial sum -

$$S_{n_1+n_2+n_3+n_4} < 100$$

we continue to repeat this process, adding each time just enough new positive terms to make the sum exceed 100 and then just enough new negative terms to make the sum less than 100.

Now, we notice, that all these partial sums differ from 100 by, atmost, 1+re or 1-re sum. These partial sums must be closing in on 100 - since the original series $\sum x_n$ converges, its terms x_n go to zero (0) as $n \rightarrow \infty$, [i.e $x_n \rightarrow 0 \text{ as } n \rightarrow \infty$]

we can get the partial sums as close to 100 as we wish (within any ϵ), if we go out far enough in the series (to where all the remaining terms have their absolute values less than ϵ). So, the partial sums converges to 100, which proves that the series of re-arranged terms converges to 100.

Note: The given question is a restatement of Riemann's Re-arrangement Theorem .

Q5.(a) Solve $(D^2 - 2DD' + D'^2)z = e^{x+2y} + x^3 + \sin 2x$,
 where $D = \frac{\partial}{\partial x}$, $D' = \frac{\partial}{\partial y}$, $D^2 = \frac{\partial^2}{\partial x^2}$, $D'^2 = \frac{\partial^2}{\partial y^2}$. (10)

Sol: Given Eqn can be written as

$$(D - D')^2 z = e^{x+2y} + x^3 + \sin 2x \quad (1)$$

$$\text{Its auxiliary Eqn } \Rightarrow (m-1)^2 = 0 \Rightarrow m=1, 1$$

$$\therefore C.F. = \phi_1(y+x) + x\phi_2(y+x),$$

where ϕ_1 and ϕ_2 are arbitrary functions.

→ PI corresponding to e^{x+2y}

$$= \frac{1}{(D - D')^2} e^{x+2y} = \frac{1}{(1-2)^2} e^{x+2y} = e^{x+2y}$$

→ PI corresponding to x^3 ,

$$= \frac{1}{(D - D')^2} x^3 = \frac{1}{D^2 (1 - \frac{D'}{D})^2} x^3 = \frac{1}{D^2} \left(1 - \frac{D'}{D}\right)^{-2} x^3$$

$$= \frac{1}{D^2} \left(1 + \frac{D'}{D} + \dots\right) x^3 = \frac{1}{D^2} x^3 = \frac{1}{D} \frac{x^4}{4} = \frac{x^5}{20}$$

→ PI corresponding to $\sin 2x$

$$= \frac{1}{(D - D')^2} \sin 2x = \frac{1}{(D - D')^2} \sin(2x + 0y)$$

$$= \frac{1}{(2-0)^2} \iint \sin v dv dv, \text{ where } v = 2x + 0y$$

$$= -\frac{1}{4} \int \cos v dv = -\frac{1}{4} \sin v = -\frac{1}{4} \sin 2x$$

∴ Gen Sol

$$Z = \phi_1(y+x) + x\phi_2(y+x) + e^{x+2y} + \frac{x^5}{20} - \frac{1}{4} \sin 2x.$$

Q1) Explain the main steps of the Gauss-Jordan method and apply this method to find the inverse of the matrix?

$$\begin{bmatrix} 2 & 6 & 6 \\ 2 & 8 & 6 \\ 2 & 6 & 8 \end{bmatrix}$$

Q2) Let us explain the main steps of the Gauss-Jordan's method.

- This is done by augmenting the matrix A by the identity matrix I of the order same as that of A.
- Using elementary row operations on the augmented matrix $[A|I]$
- We reduce the matrix A to the form I and in the process the matrix I is transformed to A^{-1} .

i.e. $[A|I] \xrightarrow[\text{Gauss-Jordan}]{} [I|A^{-1}]$

Let us find the inverse of

$$A = \begin{bmatrix} 2 & 6 & 6 \\ 2 & 8 & 6 \\ 2 & 6 & 8 \end{bmatrix}$$

we have

$$[A|I] = \left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 2 & 8 & 6 & 0 & 1 & 0 \\ 2 & 6 & 8 & 0 & 0 & 1 \end{array} \right]$$

$$= \left[\begin{array}{ccc|ccc} 2 & 6 & 6 & 1 & 0 & 0 \\ 0 & 2 & 0 & -1 & 1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array} \right] \begin{matrix} R_2 \rightarrow R_2 - R_1 \\ R_3 \rightarrow R_3 - R_1 \end{matrix}$$

$$= \left[\begin{array}{ccc|ccc} 2 & 0 & 6 & 4 & -3 & 0 \\ 0 & 2 & 0 & -1 & -1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array} \right] \begin{matrix} R_1 \rightarrow R_1 - 3R_2 \end{matrix}$$

$$= \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 7 & -3 & -3 \\ 0 & 2 & 0 & -1 & -1 & 0 \\ 0 & 0 & 2 & -1 & 0 & 1 \end{array} \right] \begin{matrix} R_1 \rightarrow R_1 - 3R_3 \end{matrix}$$

$$= \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 7/2 & -3/2 & -3/2 \\ 0 & 1 & 0 & -1/2 & 1/2 & 0 \\ 0 & 0 & 1 & -1/2 & 0 & 1/2 \end{array} \right] \begin{matrix} R_1 \rightarrow \frac{1}{2}R_1 \\ R_2 \rightarrow \frac{1}{2}R_2 \\ R_3 \rightarrow \frac{1}{2}R_3 \end{matrix}$$

$$= [I | \bar{A}^{-1}]$$

$$\therefore \bar{A}^{-1} = \frac{1}{2} \left[\begin{array}{ccc} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{array} \right]$$

5(c). Write the Boolean expression.

$$z(y+z)(x+y+z)$$

in its simplest form using Boolean postulate rules. Mention the rules used during simplification Verify your result by constructing the truth table for the given expression and for its simplest form?

Ques. Let $A = z(y+z)(x+y+z)$

$$\begin{aligned} &= (zy+z^2)(x+y+z) \\ &= zyx + zy^2 + z^2y + z^2x + z^2y + z^3 \\ &= zyx + zy + zy + zx + zy + z^3 \\ &\quad [\because A^2 = A ; A^3 = A] \\ &= zy(x+1) + zy + zx + zy + z \\ &= zy + zy + zx + z(y+1) \\ &= zy + zx + z \quad [\because 1+A = 1] \\ &= zy + z(x+1) \\ &= zy + z \\ &= z(y+1) \quad [\because 1+y = 1] \\ A &= z. \end{aligned}$$

x	y	z	$z(y+z)$	$(x+y+z)$	$z(y+z)(x+y+z)$	z
0	0	0	0	0	0	0
0	0	1	1	1	1	1
0	1	0	0	1	0	0
0	1	1	1	1	1	1
1	0	0	0	1	0	0
1	0	1	1	1	1	1
1	1	0	0	1	0	0
1	1	1	1	1	1	1

$\therefore z(y+z)(x+y+z) = z$ Hence proved.

5(a)

Let Γ be a closed curve in XY plane and let S denote the region bounded by the curve Γ . Let

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = f(x,y) \quad \forall (x,y) \in S.$$

If f be prescribed at each point (x,y) of S and w is prescribed on the boundary Γ of S , then prove that any solution $w = w(x,y)$, satisfying these conditions, is unique?

Sol:

Let $w = w_1(x,y)$; $w = w_2(x,y)$ be two solutions satisfying

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = f(x,y) \quad \forall (x,y) \in S \quad \text{--- (1)}$$

in S together with the prescribed boundary conditions on Γ .

$$\text{let } w(x,y) = w_1(x,y) - w_2(x,y) \quad \text{--- (2)}$$

$$\text{Then } \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = f - f = 0 \quad \text{--- (3)}$$

Also; on Γ ; $w=0$; since; $w_1=w_2$ on Γ

Now;

$$\mathbb{I} = \iint_S \left\{ \left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \right\} dx dy$$

$$\mathbb{I} = \iint_S \left\{ \left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 + w \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) \right\} dx dy$$

Using (3)

$$\mathbb{I} = \iint_S \left\{ \frac{\partial}{\partial x} \left(w \frac{\partial w}{\partial x} \right) + \frac{\partial}{\partial y} \left(w \frac{\partial w}{\partial y} \right) \right\} dx dy$$

$$= \phi_{\Gamma} \left(w \frac{\partial w}{\partial x} dy - w \frac{\partial w}{\partial y} dx \right) \\ = 0 \quad (\text{since; } w=0 \text{ on } \Gamma)$$

Now; $\left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 \geq 0$

But $I=0$; $\left(\frac{\partial w}{\partial x} \right)^2 + \left(\frac{\partial w}{\partial y} \right)^2 = 0$ in S

which will be true only if

$$\frac{\partial w}{\partial x} = 0 = \frac{\partial w}{\partial y} \text{ at each point of } S$$

Now; shows that $w=\text{constant}$ in S .

Since, $w=0$ on Γ , we infer from continuity of w ; that $w=0$ throughout S .

This gives $w_1=w_2$; which establishes the result.

5(e) Show that the moment of inertia of an elliptic area of mass M and semi-axis a and b about a semi-diameter of length γ is $\frac{1}{4}M \frac{a^2 b^2}{\gamma^2}$. Further prove that the moment of inertia about a tangent is $\frac{5M}{4} p^2$, where p is the star distance from the centre of the ellipse to the tangent.

Sol'n: (i) Let PP' be semi-diameter of length γ of an elliptic area of mass M and semi-axes a and b.

Equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \quad \text{--- (1)}$$

If PP' make an angle θ with ox then coordinates of P are $(\gamma \cos \theta, \gamma \sin \theta)$.

Since P lies on equation (1)

$$\therefore (\frac{\gamma^2}{a^2}) \cos^2 \theta + (\frac{\gamma^2}{b^2}) \sin^2 \theta = 1$$

$$\Rightarrow b^2 \cos^2 \theta + a^2 \sin^2 \theta = \frac{a^2 b^2}{\gamma^2} \quad \text{--- (2)}$$

Now M.I of the ellipse about $OX = A = \frac{1}{4} M b^2$

and M.I of the ellipse about $OY = B = \frac{1}{4} M a^2$

Also P.I of the ellipse about OX and $OY = F = 0$ (By symmetry)

\therefore M.I of the ellipse about the diameter PP'

$$= A \cos^2 \theta + B \sin^2 \theta - F \sin 2\theta$$

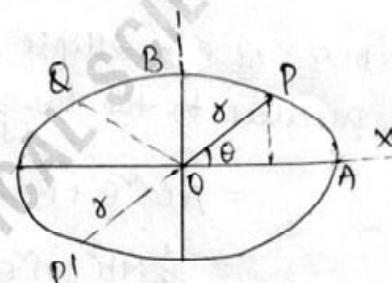
$$= \frac{1}{4} M b^2 \cos^2 \theta + \frac{1}{4} M a^2 \sin^2 \theta - 0$$

$$= \frac{1}{4} M (b^2 \cos^2 \theta + a^2 \sin^2 \theta)$$

$$= \frac{1}{4} M \cdot \frac{a^2 b^2}{\gamma^2}$$

(2) Equation of the tangent to the ellipse is

$$y = mx + \sqrt{a^2 m^2 + b^2}$$

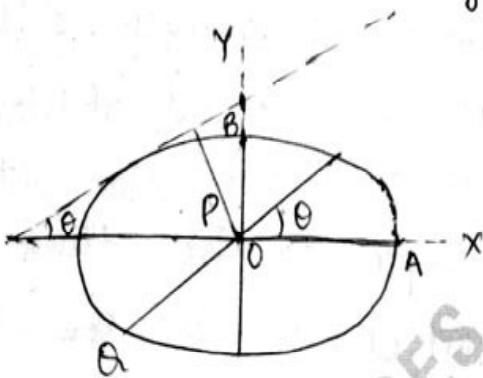


where $m = \tan\theta$, if tangent is inclined at an angle θ to the axis of x .

If P is the \perp ar from the centre $(0,0)$ on the tangent

① then

$$\begin{aligned} P &= \frac{\sqrt{(a^2m^2+b^2)}}{\sqrt{(1+m^2)}} \\ &= \frac{\sqrt{(a^2\tan^2\theta+b^2)}}{\sqrt{(1+\tan^2\theta)}} \\ &= \sqrt{(a^2\sin^2\theta+b^2\cos^2\theta)} \end{aligned}$$



M.I of the ellipse about the diameter PQ which is parallel to the tangent :

$$\begin{aligned} &= A\cos^2\theta + B\sin^2\theta - F\sin 2\theta \\ &= \frac{1}{4}Mb^2\cos^2\theta + \frac{1}{4}Ma^2\sin^2\theta - 0 \\ &= \frac{1}{4}M(b^2\cos^2\theta + a^2\sin^2\theta) \\ &= \frac{1}{4}MP^2, \text{ from ②} \end{aligned}$$

∴ M.I of the ellipse about the tangent

⇒ M.I about the parallel line through C.G 'O'

+ M.I of mass M at θ about the tangent –

$$= \frac{1}{4}MP^2 + MP^2 = \underline{\underline{\frac{5}{4}MP^2}}$$

6.(a)

Find a complete integral of the PDE

$$2(pq + y\dot{p} + q\dot{x}) + x^2 + y^2 = 0 \quad (15)$$

Sol.

charpit's auxiliary eqns are

$$\frac{dp}{\frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z}} = \frac{dz}{-p \frac{\partial f}{\partial p} - q \frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

Here, $f(x, y, z, p, q) = 2(pq + y\dot{p} + q\dot{x}) + x^2 + y^2 = 0 \quad (1)$

$$\begin{aligned} \therefore \frac{dp}{2q + 2x} &= \frac{dq}{2p + 2y} = \frac{dz}{-p(2q + 2y) - q(2p + 2x)} = \frac{dx}{-(2q + 2y)} = \frac{dy}{-(2p + 2x)} \\ &= \frac{dp + dq + dx + dy}{(2q + 2x) + (2p + 2y) - (2q + 2y) - (2p + 2x)} \\ &= \frac{d(p + q + x + y)}{0} \end{aligned}$$

So that $(p+x) + (q+y) = a \cdot \quad (2)$

Re-writing (1),

$$2(p+x)(q+y) + (x-y)^2 = 0$$

$$\text{or } (p+x)(q+y) = -\frac{(x-y)^2}{2} \quad (3)$$

$$\begin{aligned} \text{Now, } (p+x) - (q+y) &= \sqrt{(p+x) + (q+y)]^2 - 4(p+x)(q+y)} \\ &= \sqrt{a^2 + 2(x-y)^2} \quad \text{by (2) & (3)} \end{aligned}$$

$$(2) + (4) \Rightarrow 2(p+x) = a + \sqrt{a^2 + 2(x-y)^2} \quad (4)$$

$$(2) - (4) \Rightarrow 2(q+y) = a - \sqrt{a^2 + 2(x-y)^2}$$

From these we get p & q as

$$p = -x + \frac{a}{2} + \frac{1}{2} \sqrt{a^2 + 2(x+y)^2}, \quad q = -y + \frac{a}{2} - \frac{1}{2} \sqrt{a^2 + 2(x-y)^2}$$

$\therefore dz = p dx + q dy$ becomes

$$= -(x dx + y dy) + \frac{a}{2} (dx + dy) + \frac{1}{2} \sqrt{a^2 + 2(x+y)^2} \\ \times (dx - dy)$$

$$\therefore dz = -\frac{1}{2} d(x^2 + y^2) + \frac{a}{2} d(x+y) + \sqrt{2} \times \frac{1}{2} \sqrt{\frac{a^2}{2} + (x-y)^2} d(x-y)$$

Put $x-y=t$ s.t. $d(x-y)=dt$ (5)

$$\therefore dz = -\frac{1}{2} d(x^2 + y^2) + \frac{a}{2} d(x+y) + \frac{1}{\sqrt{2}} \sqrt{\left(\frac{a}{\sqrt{2}}\right)^2 + t^2} dt$$

$$\therefore z = -\frac{x^2 + y^2}{2} + a \cdot \frac{x+y}{2} + \frac{1}{\sqrt{2}} \left[\pm \sqrt{\left(\frac{a}{\sqrt{2}}\right)^2 + t^2} \right. \\ \left. + \frac{(ay/\sqrt{2})^2}{2} \log \left[t + \sqrt{\left(\frac{a}{\sqrt{2}}\right)^2 + t^2} \right] \right] + b$$

Putting back the value of t , the required complete integral is

$$z = -\frac{(x^2 + y^2)}{2} + \frac{a(x+y)}{2} + \frac{1}{2\sqrt{2}} \left[(x-y) \sqrt{\frac{a^2}{2} + (x-y)^2} \right. \\ \left. + \frac{a^2}{2} \log \left[x-y + \sqrt{\frac{a^2}{2} + (x-y)^2} \right] \right] + b.$$

Q6
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for given equidistant
 u_{-1}, u_0, u_1 and $u_{2,1} \sim$ value
 is interpolated by Lagrange's
 formula. Show that it
 can be written in the form

$$u_x = y u_0 + x u_1 + y \frac{(y-1)}{3!} \Delta^{u_1}$$

$$+ \frac{x(x-1)}{3!} \Delta^x \quad \text{where } x+y=1.$$

R.H.S.

$$y u_0 + x u_1 + y \frac{(y-1)}{3!} \Delta^{u_1} \\ + \frac{x(x-1)}{3!} \Delta^x u_0$$

$$= u_0 - x u_0 + x u_2 + (1-x) \left[(1-x)^{-1} \right] \Delta^{u_1} \\ + \frac{x(x-1)}{3!} \Delta^x u_0$$

$$= u_0 - x u_0 + x u_2 + (1-x) \left[x^{-2} \right] \Delta (\Delta u_1) \\ + \frac{x(x-1)}{3!} \Delta (\Delta u_0)$$

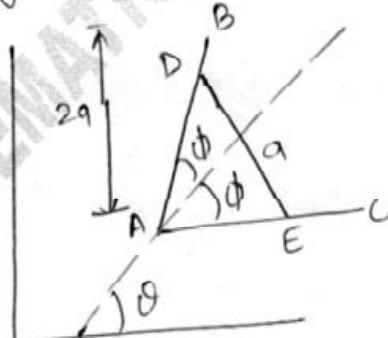
$$\begin{aligned}
 &= u_0 - \lambda u_0 + \lambda u_2 + \frac{\lambda(1-\lambda)(\lambda-2)}{3!} (u_0 - u_1) \\
 &\quad + \frac{\lambda(\lambda^2-1)}{3!} \Delta(u_1 - u_0) \\
 &= u_0 - \lambda u_0 + \lambda u_2 + \frac{\lambda(1-\lambda)(\lambda-2)}{3!} [(u_1 - u_0) - (u_2 - u_1)] \\
 &\quad + \frac{\lambda(\lambda^2-1)}{3!} [(u_2 - u_1) - (u_0 - u_1)] \\
 &= u_0 - \frac{\lambda u_0}{1} + \lambda u_2 + \frac{\lambda(1-\lambda)(\lambda-2)}{3!} (u_1 - 2u_0 + u_1) \\
 &\quad + \frac{\lambda(\lambda^2-1)}{3!} [u_2 - 2u_1 + u_0] \\
 &= u_0 \left[(1-\lambda) - \frac{1}{3!} \lambda(1-\lambda)(\lambda-2) + \frac{\lambda(\lambda^2-1)}{3!} \right] \\
 &\quad + u_2 \left[\lambda + \frac{\lambda(\lambda^2-1)}{3!} \right] + u_1 \left[\frac{\lambda(1-\lambda)(\lambda-2)}{3!} - \frac{1}{3!} \lambda(\lambda^2-1) \right] \\
 &\quad + \frac{\lambda(1-\lambda)(\lambda-2)}{3!} u_1. \\
 &= \frac{\lambda(1-\lambda)(\lambda-2)}{3!} u_1 + \underbrace{\left[\frac{\lambda^3 - 2\lambda^2 + \lambda + 2}{2} \right]}_{2} u_0 \\
 &\quad + \frac{\lambda^2(1-\lambda)}{2} u_1 + \frac{\lambda(\lambda^2+5)}{6} u_2 \\
 &= u_n
 \end{aligned}$$

Q.6(c) Two uniform rods AB, AC, each of mass m and length ' $2a$ ', are smoothly hinged together at 'A' and move on a horizontal plane. At time 't', the mass centre of rods is at the points (ξ, η) referred to fixed perpendicular axes Ox, Oy in the plane, and the rods make angles $\theta \pm \phi$ with Ox . Prove that the kinetic energy of the system is

$$m \left[\dot{\xi}^2 + \dot{\eta}^2 + \left(\frac{1}{3} + \sin^2 \phi \right) a^2 \dot{\theta}^2 + \left(\frac{1}{3} + \cos^2 \phi \right) a^2 \dot{\phi}^2 \right].$$

Also derive Lagrange's equations of motion for the system if an external force with components $[X, Y]$ along the axes acts at 'A'.

Sol: Let, two uniform rods AB, AC each of mass m and length = $2a$, smoothly hinged together at 'A'.



For the two rods:

$$T = \frac{1}{2} m \left[\dot{x}_B^2 + \dot{y}_B^2 \right] + \frac{1}{2} \left[\frac{m 4a^2}{12} \right] (\dot{\theta} + \dot{\phi})^2 + \frac{1}{2} m \left[\dot{x}_E^2 + \dot{y}_E^2 \right] + \frac{1}{2} \left[\frac{m 4a^2}{12} \right] (\dot{\theta} - \dot{\phi})^2 \quad (1)$$

$$G = \{ \xi, \eta \} = \text{centre of mass}$$

From the diagram

$$\begin{aligned} x_D &= \xi - a \sin \phi \sin \theta & y_D &= \eta + a \sin \phi \cos \theta \\ x_E &= \xi + a \sin \phi \sin \theta & y_E &= \eta - a \sin \phi \cos \theta \end{aligned}$$

$$\begin{aligned}\therefore \dot{x}_D &= \dot{\xi} - a \cos \phi \sin \theta \cdot \dot{\phi} - a \sin \phi \cos \theta \cdot \dot{\theta} \\ \dot{x}_E &= \dot{\xi} + a \cos \phi \sin \theta \cdot \dot{\phi} + a \sin \phi \cos \theta \cdot \dot{\theta} \\ \dot{y}_D &= \dot{\eta} + a \cos \phi \cos \theta \cdot \dot{\phi} - a \sin \phi \sin \theta \cdot \dot{\theta} \\ \dot{y}_E &= \dot{\eta} - a \cos \phi \cos \theta \cdot \dot{\phi} + a \sin \phi \sin \theta \cdot \dot{\theta}\end{aligned}$$

So;

$$\begin{aligned}\dot{x}_D^2 + \dot{x}_E^2 &= 2 \dot{\xi}^2 + 2(a \cos \phi \sin \theta \cdot \dot{\phi} + a \sin \phi \cos \theta \cdot \dot{\theta})^2 \\ \dot{x}_D^2 + \dot{x}_E^2 &= 2 \dot{\xi}^2 + 2a^2 \cos^2 \phi \sin^2 \theta \dot{\phi}^2 + 2a^2 \sin^2 \phi \cos^2 \theta \dot{\theta}^2 \\ &\quad + 4a^2 \sin \phi \cos \phi \sin \theta \cdot \cos \theta \cdot \dot{\phi} \dot{\theta}. \quad \text{--- (1)}\end{aligned}$$

Similarly;

$$\begin{aligned}\dot{y}_D^2 + \dot{y}_E^2 &= 2 \dot{\eta}^2 + 2a^2 \cos^2 \phi \cos^2 \theta \dot{\phi}^2 + 2a^2 \sin^2 \phi \sin^2 \theta \cdot \dot{\theta}^2 \\ &\quad - 4a^2 \sin \phi \cos \phi \sin \theta \cdot \cos \theta \cdot \dot{\phi} \dot{\theta}. \quad \text{--- (2)}\end{aligned}$$

 \therefore from (1) + (2)

$$\dot{x}_D^2 + \dot{x}_E^2 + \dot{y}_D^2 + \dot{y}_E^2 = 2 [\dot{\xi}^2 + \dot{\eta}^2 + a^2 \cos^2 \phi \dot{\phi}^2 + a^2 \sin^2 \phi \dot{\theta}^2] \quad \text{L (3)}$$

Rearranging (1)

$$T = \frac{1}{2} m [\dot{x}_D^2 + \dot{y}_D^2 + \dot{x}_E^2 + \dot{y}_E^2] + \frac{1}{3} \frac{m a^2}{2} [2 \dot{\phi}^2 + 2 \dot{\theta}^2]$$

using (2) in above equation

$$T = m [\dot{\xi}^2 + \dot{\eta}^2 + \left(\frac{1}{3} + \sin^2 \phi\right) a^2 \dot{\phi}^2 + \left(\frac{1}{3} + \cos^2 \phi\right) a^2 \dot{\theta}^2] \quad \text{proved.}$$

Now, constant force $[x, y]$ acts at A

$$\therefore V = \int_A F \cdot d\mathbf{r}.$$

$$\text{co-ordinates of } A = [\xi - a \cos \phi \cos \theta, \eta - a \cos \phi \sin \theta]$$

$$\therefore V = X(a \cos \phi \cos \theta - \xi) + Y(a \cos \phi \sin \theta - \eta) + \text{const}$$

$$L = T - V$$

$$L = m \left[\dot{\xi}^2 + \dot{\eta}^2 + \left(\frac{1}{3} + \sin^2 \phi \right) a^2 \dot{\phi}^2 + \left(\frac{1}{3} + \cos^2 \phi \right) a^2 \dot{\theta}^2 \right] \\ + X [\dot{\xi} - a \cos \phi \cos \theta] + Y [\dot{\eta} - a \cos \phi \sin \theta] + \text{const.}$$

Equation of motion.

$$\frac{d}{dt} \left(\frac{\partial L}{\partial q_a} \right) - \frac{\partial L}{\partial q_a} = 0$$

For ξ $\frac{d}{dt} [2m \ddot{\xi}] - X = 0$

$$\Rightarrow \ddot{\xi} - \frac{X}{2m} = 0 \quad \text{--- (I)}$$

For η \rightarrow $\ddot{\eta} - \frac{Y}{2m} = 0 \quad \text{--- (II)}$

For ϕ $\rightarrow \frac{d}{dt} \left[\left(\frac{1}{3} + \cos^2 \phi \right) 2a^2 \dot{\phi} \right] - \left[2 \sin \phi \cos \phi a^2 \dot{\theta}^2 - 2 \sin \phi \cos \phi a^2 \dot{\phi}^2 + X a \sin \phi \cos \theta + Y a \sin \phi \sin \theta \right] = 0$

$$\Rightarrow 2a^2 \left[\frac{1}{3} + \cos^2 \phi \right] \ddot{\phi} - a^2 (\sin 2\phi \dot{\theta}^2 - \sin 2\phi \dot{\phi}^2) - a \sin \phi (X \cos \theta + Y \sin \theta) = 0$$

$$\Rightarrow 2a^2 \left(\frac{1}{3} + \cos^2 \phi \right) \ddot{\phi} - a^2 \sin 2\phi (\dot{\theta}^2 - \dot{\phi}^2) - a \sin \phi (X \cos \theta + Y \sin \theta) = 0 \quad \text{--- (III)}$$

For θ :

$$\frac{d}{dt} \left[\left(\frac{1}{3} + \sin^2 \phi \right) 2a^2 \dot{\theta} \right] - (X a \cos \phi \sin \theta - Y a \cos \phi \cos \theta) = 0$$

$$= 2a^2 \left(\frac{1}{3} + \sin^2 \phi \right) \ddot{\theta} - a \cos \theta (X \sin \theta - Y \cos \theta) = 0 \quad \text{--- (IV)}$$

7.(a) Reduce the Eqn

$$y^2 - \frac{\partial^2 z}{\partial x^2} - 2xy \frac{\partial^2 z}{\partial x \partial y} + x^2 \frac{\partial^2 z}{\partial y^2} = \frac{y^2}{x} \frac{\partial z}{\partial x} + \frac{x^2}{y} \frac{\partial z}{\partial y}$$

to canonical form and hence solve it. (15) —①

Sol: Given Eqn can be re-written as-

$$y^2 R - 2xyz + x^2 t - \frac{y^2}{x} p - \frac{x^2}{y} q = 0 \quad (2)$$

Comparing (2) with

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0$$

Here, $R = y^2$, $S = -2xyz$, $T = x^2$

$S^2 - 4RT = 0 \Rightarrow$ PDE ① is parabolic.

The λ -quadratic equation, $R\lambda^2 + S\lambda + T = 0$

reduces to, $y^2\lambda^2 - 2xy\lambda + x^2 = 0$

$$\text{or } (y\lambda - x)^2 = 0 \Rightarrow \lambda = \frac{x}{y}, \frac{x}{y}$$

The corresponding characteristic eqn is -

$$\frac{dy}{dx} + \frac{x}{y} = 0 \quad \text{or} \quad xdx + ydy = 0$$

So that, $\frac{x^2}{2} + \frac{y^2}{2} = c_1$, c_1 being arbitrary constant.

Let us choose,

$$u = \frac{x^2}{2} + \frac{y^2}{2} \quad \text{and} \quad v = \frac{x^2}{2} - \frac{y^2}{2} \quad —③$$

in such a way that u and v are independent functions of x, y as verified by Jacobian —

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} x & y \\ x & -y \end{vmatrix} = -xy \neq 0.$$

Now, $p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$

$$= x \left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right) \quad \text{using (2)} \quad \text{--- (4)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y} = y \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} \right) \quad \text{--- (5)}$$

$$\begin{aligned} r &= \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial u} \left\{ x \left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right) \right\} \\ &= 1 \left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v} \right) + x \cdot \frac{\partial}{\partial u} \left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right) \\ &= \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v \partial u} + x \left[\frac{\partial}{\partial u} \left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right) \frac{\partial u}{\partial x} + \frac{\partial}{\partial v} \left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \right) \frac{\partial v}{\partial x} \right] \\ &= \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} + x^2 \left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \right) \quad \text{by (3)} \quad \text{--- (6)} \end{aligned}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} + y^2 \left(\frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \right) \quad \text{--- (7)}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = xy \left(\frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial v^2} \right) \quad \text{--- (8)}$$

using above values in (2) and simplifying

$$4x^2y^2 \left(\frac{\partial^2 z}{\partial v^2} \right) = 0 \Rightarrow \frac{\partial^2 z}{\partial v^2} = 0. \quad \text{--- (9)}$$

which is required canonical form.

Integrating (9) partially w.r.t v , $\frac{\partial z}{\partial v} = \phi(u)$

Again " " " , $z = v \cdot \phi(u) + \psi(u)$

$$\therefore z = \frac{(x^2 - y^2)}{2} \phi\left(\frac{x^2 + y^2}{2}\right) + \psi\left(\frac{x^2 + y^2}{2}\right).$$

7(b) → Derive the formula

$$\int_a^b y \, dx = \frac{3h}{8} \left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \dots + y_{n-1}) + 2(y_3 + y_6 + \dots + y_{n-3}) \right]$$

Is there any restriction on n ? state that condition.
what is the error bound in the case of Simpson's $\frac{3}{8}$ rule?

Sol'n: we know that the

Newton's-Cotes formula (A general quadrature for equidistant ordinates)

$$I = \int_a^b f(x) \, dx = h \left[ny_0 + \frac{n^2}{2} \Delta y_0 + \frac{1}{2} \left(\frac{n^3}{2} - \frac{n^2}{2} \right) \Delta^2 y_0 + \frac{1}{6} \left(\frac{n^4}{4} - n^3 + n^2 \right) \Delta^3 y_0 + \dots \right]$$

putting $n=3$ in the quadrature formula and taking the curve through (x_i, y_i) ; $i=0, 1, 2, 3$ as a polynomial of third order so that differences above the third order vanish.

we get

$$\int_{x_0}^{x_3} f(x) \, dx = \int_{x_0}^{x_0+3h} f(x) \, dx$$

$$= 3h \left[y_0 + \frac{3}{2} \Delta y_0 + \frac{3}{4} \Delta^2 y_0 + \frac{1}{8} \Delta^3 y_0 \right]$$

$$= \frac{3h}{8} (y_0 + 3y_1 + 3y_2 + 3y_3).$$

Similarly,

$$\int_{x_0+3h}^{x_0+6h} f(x) \, dx = \frac{3h}{8} (y_3 + 3y_4 + 3y_5 + y_6)$$

and so on.

Adding all these integrals from x_0 to x_0+nh .
where n is a multiple of 3.

we obtain

$$\int_{x_0}^{x_0+nh} f(x) dx = \int_{x_0}^{x_0+3h} f(x) dx + \int_{x_0+3h}^{x_0+6h} f(x) dx + \dots + \int_{x_0+(n-3)h}^{x_0+nh} f(x) dx$$

$$= \frac{3h}{8} \left[(y_0 + 3y_1 + 3y_2 + y_3) + (y_3 + 3y_4 + 3y_5 + y_6) \right. \\ \left. + \dots + (y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n) \right]$$

$$= \frac{3h}{8} \left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + \dots + y_{n-2} + y_{n-1}) \right. \\ \left. + 2(y_3 + y_6 + \dots + y_{n-3}) \right]$$

which is known as Simpson's $\frac{3}{8}$ rule.

Note: While applying Simpson's $\frac{3}{8}$ rule, the number of sub-intervals should be taken as multiple of 3.

→ The error in Simpson's $\frac{3}{8}$ rule is given by

$$E = -\frac{3h^5}{80} y^{IV}(\bar{x}), \text{ where } y^{IV}(\bar{x}) \text{ is}$$

the largest value of the fourth order derivatives.

Note! It may be noted that the errors in Simpson's $\frac{1}{3}$ and $\frac{3}{8}$ rules are of the same order. However, if we consider the magnitudes of the error terms,

Simpson's $\frac{3}{8}$ rule is superior to Simpson's $\frac{1}{3}$ rule.

Hence Simpson's $\frac{3}{8}$ rule, is not so accurate as

Simpson's rule, the dominant term in the error of this

formula being $-\frac{3}{80} h^5 y^{IV}(\bar{x})$

7(C)

A stream is rushing from a boiler through a conical pipe, the diameters of the ends of which are D and d . If v and V be the corresponding velocities of the stream and if the motion is assumed to be steady and diverging from the vortex of the cone, then Prove that

$$\frac{v}{V} = \frac{D^2}{d^2} e^{(v^2 - V^2)/2k}$$

where k is the pressure divided by the density and is constant.

Sol'n: Let u be the velocity at a distance x from the end A, the equation of motion is

$$u \frac{du}{dx} = 0 - \frac{1}{\rho} \frac{\partial p}{\partial x}$$



(Since the motion is steady)

$$\Rightarrow \frac{\partial}{\partial x} \left(\frac{1}{2} u^2 \right) = - \frac{k}{\rho} \frac{\partial p}{\partial x} \quad \text{as } p = kp$$

$$\text{Integrating } \frac{1}{2} u^2 = -k \log p + C$$

$$\Rightarrow \log p - \log A = - \frac{u^2}{2k}$$

$$\Rightarrow p = A_1 e^{-u^2/2k} \quad \text{--- (1)}$$

Boundary conditions are

$$(i) \quad p = p_1 \text{ when } u = V, \quad (ii) \quad p = p_2 \text{ when } u = v$$

Substituting (i) & (ii) we obtain $p_1 = A_1 e^{-V^2/2k}$ and $p_2 = A_1 e^{-v^2/2k}$

$$\text{This } \Rightarrow \frac{p_1}{p_2} = e^{(V^2 - v^2)/2k} \quad \text{--- (2)}$$

By the equation of continuity

flux at A = flux at B

$$\pi \left(\frac{d}{2}\right)^2 v \cdot l_1 = \pi \left(\frac{D}{2}\right)^2 \cdot V \cdot l_2 \Rightarrow \frac{l_1}{l_2} = \frac{V}{v} \cdot \frac{D^2}{d^2}$$

Now (2) becomes $\frac{V}{v} \cdot \frac{D^2}{d^2} = e^{(V^2 - v^2)/2k}$

$$\Rightarrow \frac{v}{V} = \frac{D^2}{d^2} e^{(v^2 - V^2)/2k}$$

8.(a)

Given the one-dimensional wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}; t > 0 \text{ where } c^2 = \frac{T}{m}, \quad \textcircled{1}$$

T is the constant tension in the string and m is the mass per unit length of the string.

- Find the appropriate solution of the above wave equation.
- Find also the solution under the conditions

$$y(0, t) = 0, y(l, t) = 0 \text{ for all } t. \text{ and}$$

$$\left. \frac{dy}{dt} \right|_{t=0} = 0, y(x, 0) = a \sin \frac{\pi x}{l}, 0 < x < l, a > 0. \quad \textcircled{2}$$

Sol:

- Assume that the solution of $\textcircled{1}$ is of the form

$$y(x, t) = X(t) T(t).$$

$$\text{Then, } \frac{\partial^2 y}{\partial t^2} = X \cdot T'' \text{ and } \frac{\partial^2 y}{\partial x^2} = X'' T$$

$$\text{Substituting in } \textcircled{1}, X T'' = c^2 X'' T$$

$$\text{i.e. } \frac{X''}{X} = \frac{1}{c^2} \cdot \frac{T''}{T} \quad \textcircled{2}$$

Clearly the LHS of $\textcircled{2}$ is a function of x only and the ~~right~~ RHS is a function of t only. Since x and t are independent variables, $\textcircled{2}$ can hold good if each side is equal to a constant k (say).

Then $\textcircled{2}$ leads to the ODEs —

$$\frac{d^2 X}{dx^2} - kX = 0 \quad \textcircled{3} \qquad \frac{d^2 T}{dt^2} - k c^2 T = 0 \quad \textcircled{4}$$

Solving (3) and (4), we get

i) When $k > 0$ and equal to p^2 , then

$$x = c_1 e^{px} + c_2 e^{-px} ; T = c_3 e^{cpt} + c_4 e^{-cpt}$$

ii) When $k < 0$ and equal to $-p^2$, then

$$x = c_5 \cos px + c_6 \sin px$$

$$T = c_7 \csc cpt + c_8 \sin cpt$$

iii) When $k = 0$, $x = c_9 x + c_{10}$; $T = c_{11} t + c_{12}$.

Thus various possible solutions of wave-eqn ① are

$$y = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{cpt} + c_4 e^{-cpt}) \quad \text{--- } ⑤$$

$$y = (c_5 \cos px + c_6 \sin px)(c_7 \csc cpt + c_8 \sin cpt) \quad \text{--- } ⑥$$

$$y = (c_9 x + c_{10})(c_{11} t + c_{12}) \quad \text{--- } ⑦$$

Of these three solutions, we choose that solution which is consistent with the physical nature of the problem. As the problem deals with vibrations, y must be a periodic function of x and t .

Hence, required solution is —

$$y = (c_5 \cos px + c_6 \sin px)(c_7 \csc cpt + c_8 \sin cpt). \quad \text{--- } ⑧$$

ii) Given boundary Conditions

$$y(0, t) = 0 \quad \text{--- (ii)} \quad y(l, t) = 0 \quad \text{--- (iii)}$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \quad \text{--- (iv)} ; \quad y(x, 0) = a \sin \frac{\pi x}{l} \quad \text{--- (v)}$$

$$y(x,t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos pt + c_4 \sin pt)$$

(ii) gives, $y(0,t) = c_1(c_3 \cos pt + c_4 \sin pt) = 0$

for this to be true for all time, $c_1 = 0$

$$\therefore y(x,t) = c_2 \sin px \cdot c_3 \sin pt + c_2 \sin px(c_3 \cos pt + c_4 \sin pt) \quad (\text{iii})$$

$$\frac{\partial y}{\partial t} = c_2 \sin px [c_3(-\cos pt) + c_4(\sin pt)]$$

$$\text{By (iv)} \Rightarrow \left. \frac{\partial y}{\partial t} \right|_{t=0} = c_2 \sin px \cdot (c_4 \cos 0) = 0$$

$$\Rightarrow c_2 c_4 \cos 0 = 0$$

If $c_2 = 0$, (iii) will give trivial sol, $y(x,t) = 0$.

\therefore the only possibility is $c_4 = 0$

$$(v) \text{ becomes, } y(x,t) = c_2 c_3 \sin px \cos pt \quad (\text{vii})$$

$$\therefore \text{By (iii), } y(l,t) = c_2 c_3 \sin pl \cos pt = 0 \quad \forall t$$

Since c_2 and $c_3 \neq 0$, $\therefore \sin pl = 0$

$$\Rightarrow pl = n\pi \quad \therefore p = \frac{n\pi}{l}, n \in \mathbb{Z}.$$

Hence, (vii) reduces to,

$$y(x,t) = c_2 c_3 \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}$$

$$\text{By condition (v), } y(x,0) = c_2 c_3 \sin \frac{n\pi x}{l} = a \sin \frac{n\pi x}{l}$$

which will get satisfied if we take

$$c_2 c_3 = a \text{ and } n = 1.$$

Hence the required solution is —

$$y(x,t) = a \sin \frac{\pi x}{l} \cdot \cos \frac{\pi ct}{l}.$$

Q8(b)

Write an Algorithm in the form of a flow chart for Newton-Raphson Method. Describe the Cases of failure of this Method.

Solution

To find a Root of Eqⁿ $f(x) = 0$

The Iterative Scheme for Newton-Raphson Method is given by.

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

This Method can be fail in following Cases :-

(i) If $f'(x) = 0$

(ii) If $f(x)$ is not Continuously differentiable.

(iii) If starting point x_0 is outside the Range of Guaranteed Convergence.

$f(x) = 0$ is the Equation whose Root is to be found

$f'(x)$ is derivative of $f(x)$ w.r.t. x

x_0 is the value of n^{th} Iteration

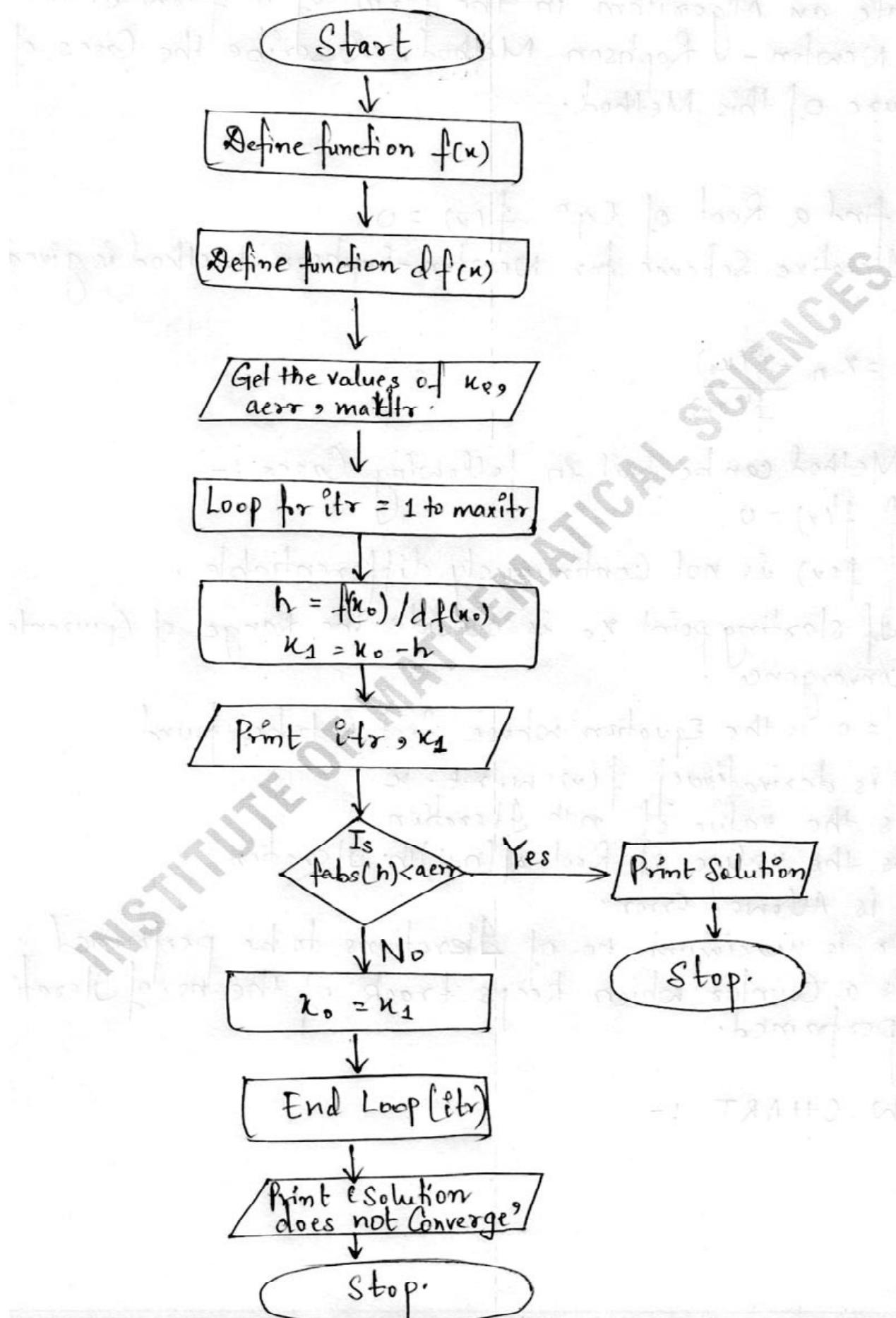
x_1 is the value of Root of $(n+1)^{th}$ Iteration

err is Allowed Error

maxitr is maximum no. of Iterations to be performed

itr is a Counter which keeps track of the no. of Iterations performed.

FLOW CHART :-



8(C)

2017

If the velocity of an incompressible fluid at the point (x, y, z) is given by

$$\left(\frac{3xz}{\gamma^5}, \frac{3yz}{\gamma^5}, \frac{3z^2 - \gamma^2}{\gamma^5} \right), \gamma^2 = x^2 + y^2 + z^2$$

then prove that the liquid motion is possible and that the velocity potential is $\frac{\gamma^2}{\gamma^3}$. Further, determine the streamlines.

$$\text{Soln: Here } u = \frac{3xz}{\gamma^5}, v = \frac{3yz}{\gamma^5}, w = \frac{3z^2 - \gamma^2}{\gamma^5} = \frac{3z^2}{\gamma^5} - \frac{1}{\gamma^3} \quad \text{--- (1)}$$

$$\text{where } \gamma = x^2 + y^2 + z^2 \quad \text{--- (2)}$$

$$\text{from (2), } \frac{\partial \gamma}{\partial x} = \frac{x}{\gamma}, \frac{\partial \gamma}{\partial y} = \frac{y}{\gamma}, \frac{\partial \gamma}{\partial z} = \frac{z}{\gamma} \quad \text{--- (3)}$$

from (1), (2) and (3), we have

$$\frac{\partial u}{\partial x} = 3z \left[\frac{1}{\gamma^5} + (-5x)\gamma^{-6} \frac{\partial \gamma}{\partial x} \right] = \frac{3z}{\gamma^5} - \frac{15x^2 z}{\gamma^7}$$

$$\frac{\partial v}{\partial y} = 3z \left[\frac{1}{\gamma^5} + (-5y)\gamma^{-6} \frac{\partial \gamma}{\partial y} \right] = \frac{3z}{\gamma^5} - \frac{15y^2 z}{\gamma^7}$$

$$\begin{aligned} \frac{\partial w}{\partial z} &= \frac{6z}{\gamma^5} - 15z^2 \gamma^{-6} \frac{\partial \gamma}{\partial z} + 3\gamma^{-4} \frac{\partial \gamma}{\partial z} = \frac{6z}{\gamma^5} - \frac{15z^2}{\gamma^6} \cdot \frac{z}{\gamma} + \frac{3}{\gamma^4} \cdot \frac{z}{\gamma} \\ &= \frac{9z}{\gamma^5} - \frac{15z^3}{\gamma^7} \end{aligned}$$

$$\begin{aligned} \therefore \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} &= \frac{15z}{\gamma^5} - \frac{15z}{\gamma^7} (x^2 + y^2 + z^2) \\ &= \frac{15z}{\gamma^5} - \frac{15z}{\gamma^7} \gamma^2 = 0 \end{aligned}$$

Since the equation of continuity is satisfied by the given values of u, v and w , the motion is possible. Let ϕ be the required velocity potential. Then

$$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz = -(udx + vdy + wdz), \text{ by definition of } \phi.$$

$$= \left[\frac{3xz}{\gamma^5} dx + \frac{3yz}{\gamma^5} dy + \frac{3z^2 - \gamma^2}{\gamma^5} dz \right]$$

$$= \frac{\delta^2 dz + 3z(xdx + ydy + zdz)}{z^5} = \frac{z^3 dz - 3z^2 z dz}{(z^3)^2} = d\left(\frac{z}{z^3}\right)$$

using ②

Integrating, $\boxed{\phi = \frac{z}{z^3}}$

[omitting constant of integration, for it has no significance in ϕ]
 In spherical polar coordinates (r, θ, ϕ) , we know that
 $z = r \cos \theta$. Hence required potential is given by

$$\phi = \frac{(r \cos \theta)}{r^3} = \frac{\cos \theta}{r^2}$$

We now obtain the streamlines. The equations of streamlines
 are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \text{ ie, } \frac{dx}{3z^2/z^5} = \frac{dy}{3y^2/z^5} = \frac{dz}{(3z^2-z^2)/z^5}$$

$$\Rightarrow \frac{dx}{3z^2} = \frac{dy}{3y^2} = \frac{dz}{3z^2-z^2} \quad \text{--- ③}$$

Taking the first two members of ③ and simplifying
 we get

$$dx/x = dy/y \Rightarrow \frac{dx}{x} - \frac{dy}{y} = 0$$

Integrating $\log x - \log y = \log C_1$, i.e. $x/y = C_1$, C_1 being a constant.

$$\text{Now, each member in ③} = \frac{x dx + y dy + z dz}{3x^2 z + 3y^2 z + 3z^3 - z^2 z}$$

$$= \frac{x dx + y dy + z dz}{3z(x^2 + y^2 + z^2) - z^2 z}$$

$$= \frac{x dx + y dy + z dz}{3z(x^2 + y^2 + z^2) - 2(x^2 + y^2 + z^2)} = \frac{x dx + y dy + z dz}{2z(x^2 + y^2 + z^2)}, \text{ by ②} \quad \text{--- ⑤}$$

Taking the first member in ③ and ⑤, we get.

$$\frac{2}{3} \frac{dx}{x} = \frac{2}{3} \frac{x dx + y dy + z dz}{x^2 + y^2 + z^2}$$

Integrating $\frac{2}{3} \log x = \frac{1}{3} \log(x^2 + y^2 + z^2) + \log C_2$
 $x^{2/3} = C_2(x^2 + y^2 + z^2)^{1/2}$, C_2 being an arbitrary constant.

The required streamlines are the curves of intersection of ④ and ⑥

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