1

System of Co-ordinates

INTRODUCTION

In a plane the position of a point is determined by an ordered pair (x, y) of real numbers, obtained with reference to two straight lines in the plane generally at right angles. The position of a point in *space* is, however, determined by an ordered triad (x, y, z) of real numbers. We now proceed to explain as to how this is done.

1.1 CO-ORDINATES OF A POINT IN SPACE

Let $X'OX, Z'OZ$ be two perpendicular straight lines determining the *XOZ*-plane. Through *O*, their point of intersection, called the *origin*, draw the line $Y'OY$ perpendicular to the *XOZ*-plane so that we have three mutually perpendicular straight lines

$$X'OX, Y'OY, Z'OZ$$

known as *Rectangular Co-ordinate Axes**. The positive directions of the axes are indicated by arrow heads. these three axes, taken in pairs, determine the three planes,

$$XOY, YOZ \text{ and } ZOX$$

or briefly the *XY, YZ, ZX* planes mutually at right angles, known as *Rectangular Co-ordinate Planes.*

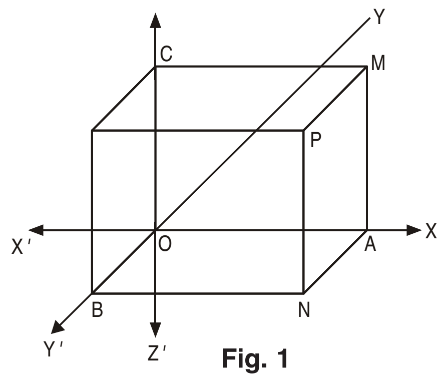

Fig. 1

Through *any* point, *P*, in space, draw three planes parallel to the three co-ordinate planes (being also perpendicular to the corresponding axes) to meet the axes in *A, B, C*.

Let $OA = x,\ OB = y$ and $OC = z.$

These three numbers *x, y, z* taken in this order determined by the point *P*, are called the co-ordinates of the point *P*.

We refer to the ordered triad (x, y, z) formed of the co-ordinates of the point *P* as the point *P* itself.

Any one of these *x, y, z* will be positive or negative according as it is measured from *O*, along the corresponding axis, in the positive or the negative direction.

Conversely, given an ordered triad (x, y, z) of numbers, we can find the point whose co-ordinates are *x, y, z*. To do this, we proceed as follows :

(i) Measure OA, OB, OC along OX, OY, OZ equal to *x, y, z* respectively.

(ii) Through the points *A, B, C* draw planes paralell to the co-ordinate planes *YZ, ZX, XY* respectively.

The point of intersection of these three planes is the required point *P*.

* The plane *XOZ* containing the lines $X'OX$ and $Z'OZ$ may be imagined as the plane of the paper; the line *OY* as pointing towards the reader and OY' behind the paper.

Note. The three co-ordinate planes divide the whole space in eight compartments which are known as *eight octants* and since each of the co-ordinates of a point may be positive or negative, there are $2^3\ (=8)$ points whose co-ordinates have the same numerical values and which lie in the eight octants, one in each.

1.1.1 Further Explanation about Co-ordinates

In § 1.1 above, we have learnt that in order to obtain the co-ordinates of a point *P*, we have to draw three planes through *P* respectively parallel to the three co-ordinate planes. The three planes through *P* and the three co-ordinate planes determined a parallelopiped whose consideration leads to three other useful constructions for determining the co-ordinates of *P*.

The parallelopiped, in questions, has six rectangular faces consisting of three pairs of parallel planes, *viz.,*

$$PMAN,\ LCOB;\ PNBL,\ MAOC,\ PLCM,\ NBOA \qquad \text{(See fig. 1)}$$

(i) We have

$x = OA = CM = LP =$ perpendicular from *P* on the *YZ*-plane;

$y = OB = AN = MP =$ perpendicular from *P* on the *ZX*-plane;

$z = OC = AM = NP =$ perpendicular from *P* on the *XY*-plane.

Thus, the co-ordinates x, y, z of a point P, are the perpendicular distances of P from the three rectangular co-ordinate planes YZ, ZX and XY respectively.

(ii) As the line *PA* lies in the plane *PMAN* which is perpendicular to the line *OA**, we have

$$PA \perp OA.$$

Similarly $\qquad PB \perp OB$ and $PC \perp OC$

Thus, the co-ordinates x, y, z of a point P are also the distances from the origin O of the feet A, B, C of the perpendiculars from the point P to the co-ordinate axes $X'X, Y'Y$ *and* $Z'Z$ *respectively.*

Ex. What are the perpendicular distances of a point (x, y, z) from the co-ordinate axes ?

[**Ans.** $\sqrt{y^2+z^2}, \sqrt{z^2+x^2}, \sqrt{x^2+y^2}$]

(iii) We have (Fig. 1)

$NP = AM = OC = z;$

$AN = OB = y;$

$OA = x.$

Thus, (Fig. 2) if we draw the line PN perpendicular to the XY-plane meeting it at N and the line NA parallel to the line, OY meeting OX at A, we have

$OA = x,\ AN = y,\ NP = z.$

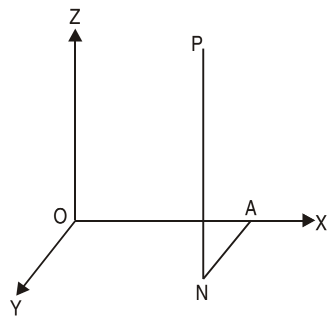

Fig. 2

EXAMPLE

In which octant the following points lie

(i) $(-1, -2, -3)$, (ii) $(a, b, -c)$, (iii) (a, b, c), (iv) $(-a, -b, c)$?

Sol. (i) Since all the three co-ordinates are negative hence, $(-1, -2, -3)$ lies in octant $OX'\ Y'Z'$ (Fig. 1).

(ii) Similarly $(a, b, -c)$ is a point in the octant $OXYZ'$.

(iii) It is a point in the octant *OXYZ.*

(iv) It is a point in the octant $OX'Y'Z$.

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* A line perpendicular to a plane is perpendicular to every line is the plane.

EXERCISES

1. In Fig. 1 write down the co-ordinates of the point *A, B,C; L, M, N* when the co-ordinates of *P* are (x, y, z).

2. Show that for every point (x, y, z) on the *ZX*-plane, $y = 0$.

3. Show that for every point (x, y, z) on the *Y*-axis, $x = 0, z = 0$.

4. What is the locus of a point (x, y, z) for which

(i) $x = 0$, (ii) $y = 0$, (iii) $z = 0$,

(iv) $x = a$, (v) $y = b$, (vi) $z = c$.

5. What is the locus of a point (x, y, z) for which

(i) $y = 0, z = 0$, (ii) $z = 0, x = 0$, (iii) $x = 0, y = 0$

(iv) $y = b, z = c$, (v) $z = c, x = a$, (vi) $x = a, y = b$.

6. *P* is any point (x, y, z), and α, β, γ are the angles which *OP* makes with *X*-axis, *Y*-axis and *Z*-axis respectively, show that

$$\cos\alpha = x/r, \cos\beta = y/r, \cos\gamma = z/r,$$

where $r = OP$.

7. Find the length of the edges of the rectangular parallopiped formed by planes drawn through the points (1, 2, 3) and (4, 7, 6) parallel to the co-ordinate planes. [**Ans.** 3, 5, 3]

1.2 DISTANCE BETWEEN TWO POINTS

To find the distance between two given points $P(x_1, y_1, z_1)$ *and* $Q(x_2, y_2, z_2)$.

Through the points *P, Q* draw planes parallel to the co-ordinate planes to form a rectangular parallelopiped whose one diagonal is *PQ*.

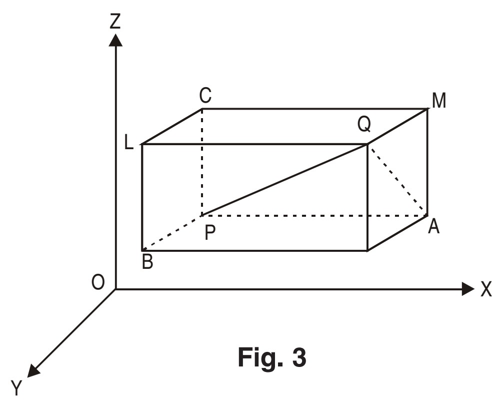


Fig. 3

Then

$$APCM, NBLQ; LCPB, QMAN; BPAN, LCMQ$$

are the three pairs of parallel faces of this parallelopiped.

Now, *PA* is the distance between the planes drawn through the points *P* and *Q* parallel to the *YZ*-plane and is, therefore, equal to the difference between their *x*-co-ordinates.

$$\left.\begin{aligned} \therefore \quad & PA = x_2 - x_1, \\ \text{Similarly} \quad & AN = y_2 - y_1, \\ \text{and} \quad & NQ = z_2 - z_1. \end{aligned}\right\} \quad ...(i)$$

The line *AQ* lies in the plane $QMAN \perp PA$

$$\Rightarrow \qquad AQ \perp PA$$

$$\Rightarrow \qquad PQ^2 = PA^2 + AQ^2 \qquad \text{...(ii)}$$

$\angle ANQ$ is a rt. angle

$$\Rightarrow \qquad AQ^2 = AN^2 + NQ^2 \qquad \text{...(iii)}$$

From (i), (ii) and (iii), we obtain

$$PQ^2 = PA^2 + AQ^2 = PA^2 + AN^2 + NQ^2$$

$$= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2$$

Thus, the distance between the points (x_1, y_1, z_1) *and* (x_2, y_2, z_2) *is*

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Cor. Distance from the origin. When P coincides with the origin O, we have $x_1 = y_1 = z_1 = 0$ so that we obtain,

$$OQ^2 = x_2^2 + y_2^2 + z_2^2$$

Note. The reader should notice the similarity of the formula obtained above for the distance between two points with the corresponding formula in plane co-ordinate geometry. Also refer 1.3.

EXERCISES

1. Find the distance between the points $(4, 3, -6)$ and $(-2, 1, -3)$. [**Ans.** 7]
2. Show that the point $(0, 7, 10)$, $(-1, 6, 6)$, $(-4, 9, 6)$ form an isosceles right-angled triangle.
3. Show that the three points $(-2, 3, 5)$, $(1, 2, 3)$, $(7, 0, -1)$ are collinear.
4. Show that the pints $(3, 2, 2)$, $(-1, 1, 3)$, $(0, 5, 6)$, $(2, 1, 6)$ lie on a sphere whose centre is $(1, 3, 4)$. Find also the radius of the sphere. [**Ans.** 3]
5. Find the co-ordinates of the point equidistant from the four points $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$ and $(0, 0, 0)$. $\left[\textbf{Ans. } \frac{1}{2}a, \frac{1}{2}b, \frac{1}{2}c\right]$

1.3 DIVISION OF THE JOIN OF TWO POINTS

To find the co-ordinates of the point dividing the segment joining the points

$$P(x_1, y_1, z_1) \text{ and } Q(x_2, y_2, z_2)$$

in the ratio m : n.

Let $R(x, y, z)$ be the piont dividing the segment PQ in the ratio $m : n$.

Draw PL, QM, RN perpendicular to the XY-plane.

The line PL, QM, RN clearly lie in one plane so that the points L, M, N lie in the straight line which is the intersection of this plane with the XY-plane.

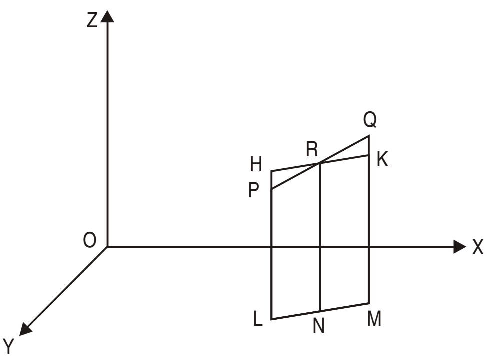


Fig. 4

The line throguh R parallel to the line LM shall lie in the same plane. Let it intersect PL and QM at H and K respectively.

The triangles HPR and QRK are similar.

$$\Rightarrow \qquad \frac{m}{n} = \frac{PR}{RQ} = \frac{PH}{KQ} = \frac{NR - LP}{MQ - NR} = \frac{z_2 - z_1}{z_2 - z}$$

$$\Rightarrow \qquad z = \frac{mz_2 + nz_1}{m + n}.$$

Similarly, by drawing perpendiculars to the *XY* and *YZ*-planes, we obtain

$$y = \frac{my_2 + ny_1}{m + n} \quad \text{and} \quad x = \frac{mx_2 + nx_1}{m + n}$$

The point *R* divides *PQ* internally or externally according as the ratio $m:n$ is positive or negative.

Thus, the co-ordinates of the point which divides the join of the points (x_1, y_1, z_1) *and* (x_2, y_2, z_2) *in the ratio m : n are*

$$\left(\frac{mx_2 + nx_1}{m + n}, \frac{my_2 + ny_1}{m + n}, \frac{mz_2 + nz_1}{m + n}\right)$$

Cor. 1. Co-ordinates of the middle point. In case *R* is the middle point of *PQ*, we have

$$m:n::1:1$$

$$\Rightarrow \qquad x = \frac{1}{2}(x_1 + x_2),\ y = \frac{1}{2}(y_1 + y_2),\ z = \frac{1}{2}(z_1 + z_2).$$

Cor. 2. Co-ordinates of a point on the join of two points : Putting *k* for $m/n,$ we see that the co-ordinates of the point *R* which divides *PQ* in the ratio $k:1$ are

$$\left(\frac{kx_2 + x_1}{1 + k}, \frac{ky_2 + y_1}{1 + k}, \frac{kz_2 + z_1}{1 + k}\right)$$

To every value of $k \neq -1,$ ther corresponds a point *R* on the line *PQ* and to every point *R* on the line *PQ* corresponds some value of *k, viz.,* PR/RQ.

Thus, we see that the point

$$\left(\frac{kx_2 + x_1}{1 + k}, \frac{ky_2 + y_1}{1 + k}, \frac{kz_2 + z_1}{1 + k}\right) \qquad \text{...(i)}$$

lies on the line *PQ* whatever value $k \neq -1$ may have and *conversely any* given point on the line *PQ* is obtained by giving *some* suitable value to *k* other than – 1. This idea is sometimes expressed by saying that (i) is the *general co-ordiantes* of a point on the line joining $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$.

The set of points of the line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\left[\left(\frac{kx_2 + x_1}{1 + k}, \frac{ky_2 + y_1}{1 + k}, \frac{kz_2 + z_1}{1 + k}\right); k \neq -1\right]$$

In other words, the line joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is the set

$$\left[\left(\frac{kx_2 + x_1}{1 + k}, \frac{ky_2 + y_1}{1 + k}, \frac{kz_2 + z_1}{1 + k}\right); k \neq -1\right]$$

EXAMPLES

1. *Find the ratio in which the lne joining the points* $(2, 4, 5), (3, 5, -4)$ *is divided by the xy-plane.*

Sol. Co-ordinates of a point that divides the line joining the given point in the ratio $k:1$ is

$$\left(\frac{3k + 2}{k + 1}, \frac{5k + 4}{k + 1}, \frac{-4k + 5}{k + 1}\right)$$

For a point on *xy*-plane, $z = 0$, *i.e.*,

$$\frac{-4k+5}{k+1} = 0, \text{ or } k = \frac{5}{4}.$$

Hence, the *xy*-plane divides the line in the ratio 5 : 4.

Putting $k = \frac{5}{4}$, the co-ordinates of the point are $\left(\frac{23}{9}, \frac{41}{9}, 0\right)$.

2. *Given that* $P(3, 2, -4), Q(5, 4, -6), R(9, 8, -10)$ *are collinear, find the ratio in which Q divides PR.*

Sol. Let *Q divides PR* in the ratio $k:1$. Now *x* co-ordinate of *Q* should be

$$\frac{9k+3}{k+1} = 5, \text{ or } k = \frac{1}{2}.$$

Hence, the requied ratio is 1 : 2.

3. *From any point* $(1, -2, 3)$, *lines are drawn to meet the sphere* $x^2 + y^2 + z^2 = 4$ *and they are divided in the ratio* $2:3$. *Prove that the points of section lie on a sphere.*

Sol. Let the line through $(1, -2, 3)$ meets the sphere in point (x_1, y_1, z_1). Hence,

$$x_1^2 + y_1^2 + z_1^2 = 4 \qquad ...(1)$$

Let the point (α, β, γ) divide the line joining the points $(1, -2, 3)$ and (x_1, y_1, z_1) in the ratio $2:3$.

Then
$$\alpha = \frac{2 . x_1 + 3 . 1}{2+3} \Rightarrow x_1 = \frac{5\alpha - 3}{2}$$

$$\beta = \frac{2 \cdot y_1 + 3(-2)}{2+3} \Rightarrow y_1 = \frac{5\beta + 6}{2}$$

$$\gamma = \frac{2 \cdot z_1 + 3 \cdot 3}{2+3} \Rightarrow z_1 = \frac{5\gamma - 9}{2}$$

From (1), we have

$$\frac{(5\alpha - 3)^2}{4} + \frac{(5\beta + 6)^2}{4} + \frac{(5\gamma - 9)^2}{4} = 4$$

$$\Rightarrow \quad 25\alpha^2 + 25\beta^2 + 25\gamma^2 - 30\alpha + 60\beta - 90\gamma + 210 = 0$$

$$\Rightarrow \quad 5\alpha^2 + 5\beta^2 + 5\gamma^2 - 6\alpha + 12\beta - 18\gamma + 22 = 0$$

Hence, locus of (α, β, γ) will be

$$x^2 + y^2 + z^2 - \frac{6}{5}x + \frac{12}{5}y - \frac{18}{5}z + \frac{22}{5} = 0,$$

which is a sphere.

EXERCISES

1. Find the co-ordinates of the points which divides the line joining the points $(2, -4, 3)$, $(-4, 5, -6)$ in the ratios.

   (i) $(1:-4)$ and (ii) (2 : 1). [**Ans.** (i) $(4, -7, 6)$; (ii) $(-2, 2, -3)$]

2. $A(3, 2, 0), B(5, 3, 2), C(-9, 6, -3)$ are three points forming a traingle, *AD*, the bisector of the angle *BAC,* meets *BC* at *D*. Find the co-ordinates of the point *D*.

   $\left[\textbf{Ans. } \frac{38}{16}, \frac{57}{16}, \frac{97}{16}\right]$

3. Find the ratio in which the line joining the point

   $(2, 4, 5), (3, 5, -4)$

   is divided by the *YZ*-plane. [**Ans.** – 2 : 3]

4. Find the ratio in which the XY-plane divides the join of

$$(-3, 4, -8) \text{ and } (5, -6, 4).$$

Also obtain the point of intersection of the line with the plane. [**Ans.** 2; $(7/3, -8/3, 0)$]

5. The three points $A(0, 0, 0)$, $B(2, -3, 3)$, $C(-2, 3, -3)$ are collinear. Find the ratio in which each point divides the segment joining the other two.

[**Ans.** $AB/BC = -1/2$, $BC/CA = -2$, $CA/AB = 1$]

6. Show that the following triads of points are collinear :

(i) $\{(2, 5, -4), (1, 4, -3), (4, 7, -6)\}$ (ii) $\{(5, 4, 2), (6, 2, -1), (8, -2, -7)\}$

7. Find the ratios in which the join of the points (3, 2, 1), (1, 3, 2) is divided by the locus of the equation $3x^2 - 72y^2 + 128z^2 = 3.$ [**Ans.** $-2:1; 1:-2$]

8. $A(4, 8, 12)$, $B(2, 4, 6)$, $C(3, 5, 4)$, and $D(5, 8, 5)$ are four points; show that the lines AB and CD intersect.

9. Show that the point $(1, -1, 2)$, is common to the lines which join $(6, -7, 0)$ to $(16, -19, -4)$ and $(0, 3, -6)$ to $(2, -5, 10)$.

10. Show that the set of points on the plane determined by the three points $(x_1, y_1, z_1), (x_2, y_2, z_2)$ and (x_3, y_3, z_3) is

$$\left[\left(\frac{lx_1 + mx_2 + nx_3}{l+m+n}, \frac{ly_1 + my_2 + ny_3}{l+m+n}, \frac{lz_1 + mz_2 + nz_3}{l+m+n}\right); l + m + n \neq 0\right].$$

11. Show that the centroid of the triangle with vertices (x_r, y_r, z_r), $r = 1, 2, 3$ is

$$\left(\frac{x_1 + x_2 + x_3}{3}, \frac{y_1 + y_2 + y_3}{3}, \frac{z_1 + z_2 + z_3}{3}\right).$$

1.4 ANGLE BETWEEN TWO LINES

The meaning of the angle between two intersecting, *i.e.,* coplanar lines, is already known to the student. We now give the definition of the angle between two non-coplanar lines, also sometimes called *skew* lines.

Def. *The angle between two* **non-coplanar,** *i.e., non-intersecting lines is the angle between two intersecting lines drawn from any point parallel to each of the given lines.*

Note 1. To justify the definition of angle between two non-coplanar lines, as given above, it is necessary to show that this angle is independent of the position of the point through which the parallel lines are drawn, but here we simply assume this result.

Note 2. The angle between a given line and the co-ordinate axes are the angles which the line drawn through the origin parallel to the given lines makes with the axes.

1.5 DIRECTION COSINES OF A LINE

Let α, β, γ be the angles which any line makes with the positive directions of the co-ordinates axes. Then $\cos\alpha, \cos\beta, \cos\gamma$ are called the *direction cosines* of the given line and are generally denoted by *l, m, n* respectively.

Ex. What are the direction cosines of the axes of co-ordinates ?

[**Ans.** 1, 0, 0; 0, 1, 0; 0, 0, 1]

1.5.1 A Useful Relation

If O the origin of co-ordinates and (x, y, z) the co-ordinates of a point P, then

$$x = lr,\ y = mr,\ z = nr,$$

l, m, n being the direction cosines of the line OP and r, the length of the segment OP.

Through the point P draw the line PL perpendicular to the X-axis so that $OL = x$.

From the right-angled triangle OPL, we have

$$\frac{OL}{OP} = \cos \angle LOP \Rightarrow \frac{x}{r} = l \Rightarrow x = lr.$$

Similarly, we have

$$y = mr,\ z = nr.$$

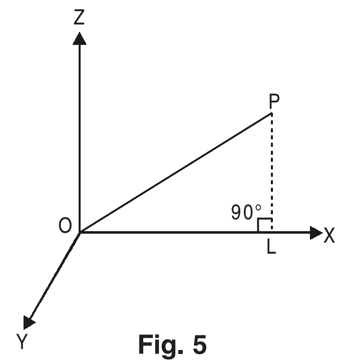


Fig. 5

1.6 RELATION BETWEEN DIRECTION COSINES

If l, m, n are the direction cosines of a line, then

$$\boldsymbol{l^2 + m^2 + n^2 = 1,}$$

i.e., the sum of the squres of the direction cosines of every line is one.

Let OP be drawn through the origin parallel to the given lines so that l, m, n are the cosines of the angles which the line OP makes with the co-ordinate axes OX, OY, OZ respectively (Refer fig. 5).

Let (x, y, z) be the co-ordinates of any point P on this line.

Let $OP = r$.

We have $x = lr,\ y = mr,\ z = nr.$

Squaring and adding, we obtain

$$x^2 + y^2 + z^2 = (l^2 + m^2 + n^2)\, r^2$$

$$\Rightarrow \quad r^2 = OP^2 = x^2 + y^2 + z^2 = (l^2 + m^2 + n^2)\, r^2$$

$$\Rightarrow \quad l^2 + m^2 + n^2 = 1.$$

Cor. If a, b, c be three numbers *proportional* to the direction cosines l, m, n of a line, we have

$$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \pm \frac{\sqrt{l^2 + m^2 + n^2}}{\sqrt{a^2 + b^2 + c^2}} = \pm \frac{1}{\sqrt{a^2 + b^2 + c^2}}$$

$$\Rightarrow \quad l = \pm \frac{a}{\sqrt{a^2 + b^2 + c^2}},\ m = \pm \frac{b}{\sqrt{a^2 + b^2 + c^2}},\ n = \pm \frac{c}{\sqrt{a^2 + b^2 + c^2}}$$

where the same sign, positive or negative, is to be chosen throughout.

Direction Ratios : From above, we see that a set of three numbers which are proportional to the direction cosines of a line are sufficient to specify the direction of a line. Such numbers are called the **direction ratios** or **direction numbers** of the line. Thus, if a, b, c be the direction ratios of a line, its direction cosines are

$$\pm \frac{a}{\sqrt{\Sigma a^2}},\ \pm \frac{b}{\sqrt{\Sigma a^2}},\ \pm \frac{c}{\sqrt{\Sigma a^2}}.$$

Note. It is easy to see that if a line OP through the origin O makes angles α, β, γ with OX, OY, OZ then the line OP' obtained by producing OP backwards through O will make angles $\pi - \alpha,\ \pi - \beta,\ \pi - \gamma$ with the axes $OX, OY, OZ.$ Thus, if

$$\cos \alpha = l, \cos \beta = m, \cos \gamma = n$$

are the direction cosines of OP, then

$$\cos (\pi - \alpha) = -l, \cos (\pi - \beta) = -m, \cos (\pi - \gamma) = -n$$

are the direction cosines of OP' *i.e.*, of the line OP produced backwards.

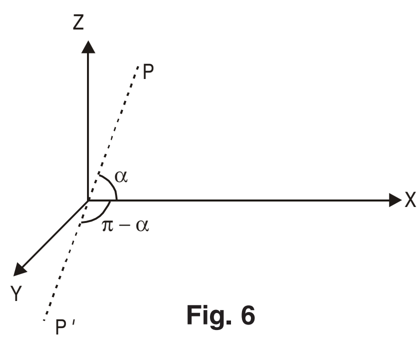


Fig. 6

Thus, if we ignore the two senses of a line, we can think of the direction cosines l, m, n or $-l, -m, -n,$ determining the direction of one and the same line. This explains the ambiguity in the sign obtained above.

Note. The student should always make a distinction between direction cosines and direction ratios. It is only when l, m, n are direction cosines, that we have the relation

$$l^2 + m^2 + n^2 = 1.$$

EXAMPLES

1. *If* α, β, γ *be the angles which a line makes with the positive direction of the axes, prove that*

$$\sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = 2.$$

Sol. We have

$$l = \cos \alpha, m = \cos \beta, n = \cos \gamma$$

$$\therefore \quad \cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$$

$$\Rightarrow \quad 1 - \sin^2 \alpha + 1 - \sin^2 \beta + 1 - \sin^2 \gamma = 1$$

$$\Rightarrow \quad \sin^2 \alpha + \sin^2 \beta + \sin^2 \gamma = 2.$$

2. *The direction cosines l, m, n, of two lines are connected by the relations*

$$l + m + n = 0 \quad \text{...(1)}$$

$$2lm + 2\ln - mn = 0 \quad \text{...(ii)}$$

Find them.

Sol. We shall solve the two given equations one of which is of the first degree and the other of second degree in *l, m, n*.

Eliminating *n* between (i) and (ii), we get

$$2l^2 - lm - m^2 = 0$$

$$\Rightarrow \quad 2\left(\frac{l}{m}\right)^2 - \frac{l}{m} - 1 = 0 \quad \text{...(iii)}$$

This equation gives two values of l/m implying that there are two lines. Let l_1, m_1, n_1; l_2, m_2, n_2 be the direction cosines of these lines.

The two roots of the quadratic equation (iii) in l/m are 1 and $-1/2$.

Also $\quad l_1 + m_1 + n_1 = 0 \Rightarrow \frac{l_1}{m_1} + 1 + \frac{n_1}{m_1} = 0 \Rightarrow \frac{n_1}{m_1} = -2$

$$l_2 + m_2 + n_2 = 0 \Rightarrow \frac{l_2}{m_2} + 1 + \frac{n_2}{m_2} = 0 \Rightarrow \frac{n_2}{m_2} = -\frac{1}{2}$$

Thus, we have

$$\frac{l_1}{1} = \frac{m_1}{1} = \frac{n_1}{-2} = \frac{1}{\sqrt{6}} \Rightarrow l_1 = \frac{1}{\sqrt{6}}, m_1 = \frac{1}{\sqrt{6}}, n_1 = -\frac{2}{\sqrt{6}}$$

$$\frac{l_2}{1} = \frac{m_2}{-2} = \frac{n_2}{1} = \frac{1}{\sqrt{6}} \Rightarrow l_2 = \frac{1}{\sqrt{6}}, m_2 = -\frac{2}{\sqrt{6}}, n_2 = -\frac{1}{\sqrt{6}}.$$

EXERCISES

1. 6, 2, 3 are direction ratios of a line. What are the direction cosines ? [**Ans.** 6/7, 2/7, 3/7]

2. What are the direction cosines of lines equally inclined to the axes ? How many such lines are there ? [**Ans.** $(1/\sqrt{3}, 1/\sqrt{3}, 1/\sqrt{3})$; 4]

3. The co-ordinates of a point *P* are (3, 12, 4). Find the direction cosines of the line *OP*. [**Ans.** 3/13, 12/13, 4/13]

4. The direction cosines of two lines are determined by the relations

(i) $l - 5m + 3n = 0, 7l^2 + 5m^2 - 3n^2 = 0;$

(ii) $l + m - n = 0, mn + 6ln - 12lm = 0.$

$$\left[\begin{array}{l}\textbf{Ans. } (i) \quad \dfrac{1}{\sqrt{14}}, \dfrac{2}{\sqrt{14}}, \dfrac{3}{\sqrt{14}}; -\dfrac{1}{\sqrt{6}}, \dfrac{1}{\sqrt{6}}, \dfrac{2}{\sqrt{6}}; \\ \qquad (ii) \quad \dfrac{1}{\sqrt{26}}, \dfrac{3}{\sqrt{26}}, \dfrac{4}{\sqrt{26}}; \dfrac{1}{\sqrt{14}}, \dfrac{2}{\sqrt{14}}, \dfrac{3}{\sqrt{14}}\end{array}\right]$$

1.7 PROJECTION ON A STRAIGHT LINE

1.7.1 Projection of a Point on a Line

The foot of the perpendicular from a given point on a given straight line is called the orthogonal projection (or simply projection) of the point on the line. This projection is the same point where the plane through the given point and perpendicular to the given line meets the line.

Thus, in Fig. 1, page 1, the point A is the projection of the point P on X-axis; also the points B and C are the projections of the point P on Y-axis and Z-axis respectively.

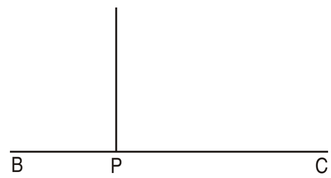


Fig. 7

1.7.2 Projection of a Segment on Another Line

The projection of a segment AB on a line CD is the segment $A'B'$ where A', B' are the projections of points A, B respectively on the line CD.

Clearly the projection $A'B'$ of the segment AB is the intercept made on CD by planes perpendicular to the line CD through the points A and B.

Ex. The co-ordinates of a point P are (x, y, z). What are the projections of the segment OP on the co-ordinate axes ?

Theorem. *The projection of a segment AB on a line CD is* $AB \cos \theta$, *where* θ *is the angle between the lines AB and CD.*

Let the planes through the points A and B perpendicular to the line CD meet the line CD in A', B' respectively so that $A'B'$ is the projection of AB. Through the point A draw a line $AP \parallel CD$ to meet the plane through the point B at P.

Now $\qquad AP \parallel CD$

$\Rightarrow \qquad \angle PAB = \theta.$

Also BP lies in the plane which is perpendicular to AP

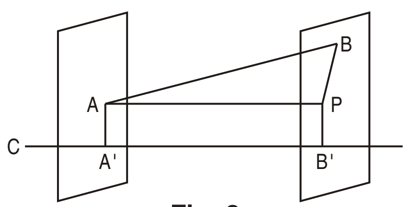


Fig. 8

$\Rightarrow \qquad \angle APB = 90°.$

Hence, $\qquad AP = AB \cos \theta$

Clearly $A'B'PA$ is a rectangle implying that

$$AP = A'B'$$

Hence, $$A'B' = AB \cos \theta.$$

Cor. Direction cosines of the join of two points.

To find the direction cosines of the line joining the two points

$$P(x_1, y_1, z_1) \text{ and } Q(x_2, y_2, z_2).$$

Let the points *L, M* be the feet of the perpendicular drawn from the points P, Q to the *X*-axis respectively so that we have

$$OL = x_1, OM = x_2$$

Projection of the segment *PQ* on *X*-axis $= LM$

$$= OM - OL = x_2 - x_1.$$

Also if l, m, n be the direction cosines of the line *PQ*, the projection of *PQ* on *X*-axis $= l \cdot PQ$.

$$l \cdot PQ = x_2 - x_1.$$

Similarly projecting *PQ* on *Y*-axis and *Z*-axis, we get

$$m \cdot PQ = y_2 - y_1,$$

$$n \cdot PQ = z_2 - z_1.$$

From these we obtain the relations*,

$$\frac{x_2 - x_1}{l} = \frac{y_2 - y_1}{m} = \frac{z_2 - z_1}{n} = PQ.$$

Thus, the direction cosines of the line joining the two points

$$(x_1, y_1, z_1) \text{ and } (x_2, y_2, z_2)$$

are proportional to

$$x_2 - x_1, y_2 - y_1, z_2 - z_1.$$

EXAMPLE

The projection of a line on the line on the axes are 2, 3, 6. *What is the length of the line ?*

Sol. Let PQ be the length of the line and $[l, m, n]$ be the direction cosines. Then,

projection on *x*-axes, $PQ \cdot l = 2,$

proejction on *y*-axes, $PQ \cdot m = 3,$

projection on *z*-axes, $PQ \cdot n = 6.$

Squaring and adding, we get

$$PQ^2(l^2 + m^2 + n^2) = 2^2 + 3^2 + 6^2$$

$$\Rightarrow \quad PQ^2 = 49 \Rightarrow PQ = 7.$$

EXERCISES

1. Find the direction cosines of the lines joining the points

(i) $(4, 3, -5)$ and $(-2, 1, -8)$. [**Ans.** 6/7, 2/7, 3/7]

(ii) $(7, -5, 9)$ and $(5, -3, 8)$. [**Ans.** 2/3, – 2/3, 1/3]

2. Show that the points $(1, -2, 3), (2, 3, -4), (0, -7, 10)$ are collinear.

3. The projections of a line on the co-ordinate axes are 12, 4, 3. Find the length and the direction cosines of the line. [**Ans.** 13; (12/13, 4/13, 3/13)]

1.7.3 Projection of a Broken Line

If $P_1, P_2, P_3, ..., P_n$ *be a number of points in space, then the sum of the projections of the segments*

$$P_1P_2, P_2P_3, ..., P_{n-1}P_n$$

on any line is equal to the projection of the segment P_1P_n *on the same line.*

Let $$Q_1, Q_2, Q_3, ..., Q_n$$

---

* The relations can be given in this form only if none of *l, m, n* is zero.

be the projections of the points

$$P_1, P_2, P_3, ..., P_n$$

on the given line. Then,

$$Q_1Q_2 = \text{projection of } P_1P_2$$

$$Q_2Q_3 = \text{projection of } P_2P_3$$

and so on.

Also $$Q_1Q_n = \text{projection of } P_1P_n.$$

As $Q_1, Q_2, Q_3, ..., Q_n$ lie on the same line, we have for all relative positions of these points on the line, the relation

$$Q_1Q_2 + Q_2Q_3 + ... + Q_{n-1}Q_n = Q_1Q_n.$$

Hence, the result.

1.7.4 Projection of the Join of Two Points on a Line

To show that the projection of the segment joining the points

$$P\,(x_1, y_1, z_1) \ \textit{and} \ Q(x_2, y_2, z_2)$$

on a line with direction cosines, l, m, n is

$$(x_2 - x_1)\,l + (y_2 - y_1)\,m + (z_2 - z_1)\,n.$$

Through P, Q draw planes parallel to the co-ordinate planes to form a rectangular parallelopiped whose one diagonal is PQ (See Fig. 3, page 3).

Now $$PA = x_2 - x_1,\ AN = y_2 - y_1,\ NQ = z_2 - z_1.$$

The lines PA, AN, NQ are respectively parallel to X-axis, Y-axis, Z-axis. Therefore, their respective projections on the line with direction cosines l, m, n are

$$(x_2 - x_1)\,l,\ (y_2 - y_1)\,m,\ (z_2 - z_1)\,n.$$

As the projection of the segment PQ on any line is equal to the sum of the projections of the segments PQ, AN, NQ on that line, the required projection is

$$\boldsymbol{(x_2 - x_1)\,l + (y_2 - y_1)\,m + (z_2 - z_1)\,n.}$$

EXERCISES

1. $A\,(6, 3, 2), B\,(5, 1, 4), C\,(3, -4, 7), D\,(0, 2, 5)$ are four points. Find the projections of the segment AB on the line CD and the segment CD on the line AB. [**Ans.** $-13/7; -13/3$]
2. Show by projection that if P, Q, R, S are the points $(6, -6, 0), (-1, -7, 6), (3, -4, 4), (2, -9, 2)$ respectively, then $PQ \perp RS$.

1.8 ANGLE BETWEEN TWO LINES

To find the angle between lines whose direction cosines are (l_1, m_1, n_1) *and* (l_2, m_2, n_2).

Let OP_1, OP_2 be lines through the origin parallel to the given line so that the cosines of the angles which OP_1 and OP_2 make with the axes are l_1, m_1, n_1 and l_2, m_2, n_2 respectively and the angle between the given lines is the angle between OP_1 and OP_2. Let this angle be θ.

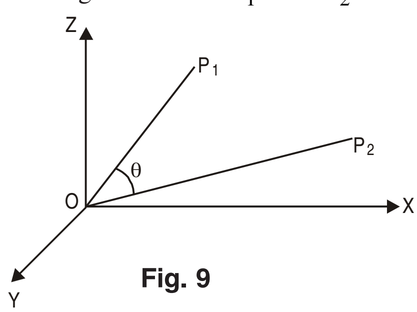


Fig. 9

Let the co-ordinates of P_2 be (x_2, y_2, z_2).

The proejction of the segment OP_2 joining

$$O\,(0, 0, 0) \text{ and } P_2\,(x_2, y_2, z_2)$$

on the line OP_1 with direction cosines

$$l_1, m_1, n_1$$

is $$(x_2 - 0)\,l_1 + (y_2 - 0)\,m_1 + (z_2 - 0)\,n_1 = l_1 x_2 + m_1 y_2 + n_1 z_2.$$

Also this projection is $OP_2 \cos\theta$.

$\therefore$ $$OP_2 \cos\theta = l_1 x_2 + m_1 y_2 + n_1 z_2.$$

But $$x_2 = l_2 \cdot OP_2,\ y_2 = m_2 \cdot OP_2,\ z_2 = n_2 \cdot OP_2 \qquad \text{...(1.5.1)}$$

$\therefore$ $$OP_2 \cos\theta = (l_1 l_2 + m_1 m_2 + n_1 n_2)\, OP_2$$

$\Rightarrow$ $$\boldsymbol{\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2.}$$

Second Method. Suppose $OP_1 = r_1, OP_2 = r_2$.

Let the co-ordinates of the points P_1, P_2 be (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively, so that

$$x_1 = r_1 l_1,\ \ y_1 = r_1 m_1,\ \ z_1 = r_1 n_1 \qquad \text{...(1.5.2)}$$
$$x_2 = r_2 l_2,\ \ y_2 = r_2 m_2,\ \ z_2 = r_2 n_2$$

We have

$$\begin{aligned} P_1P_2^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 \\ &= (x_2^2 + y_2^2 + z_2^2) + (x_1^2 + y_1^2 + z_1^2) - 2\,(x_1 x_2 + y_1 y_2 + z_1 z_2) \\ &= r_2^2 + r_1^2 - 2 r_2 r_1\,(l_1 l_2 + m_1 m_2 + n_1 n_2) \qquad \text{...(i)} \end{aligned}$$

Also from the cosine rule in Trigonometry, we have

$$P_1P_2^2 = r_1^2 + r_2^2 - 2 r_1 r_2 \cos\theta \qquad \text{...(ii)}$$

Therefore, from (i) and (ii), we obtain

$$r_1^2 + r_2^2 - 2 r_1 r_2 \cos\theta = P_1P_2^2 = r_1^2 + r_2^2 - 2 r_1 r_2\,(l_1 l_2 + m_1 m_2 + n_1 n_2)$$

$\Rightarrow$ $$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2.$$

Cor. 1. sin θ and tan θ The expressions for $\sin\theta$ and $\tan\theta$ in a convenient form are obtained as follows :

$$\begin{aligned} \sin^2\theta &= 1 - \cos^2\theta = 1 - (l_1 l_2 + m_1 m_2 + n_1 n_2)^2 \\ &= (l_1^2 + m_1^2 + n_1^2)(l_2^2 + m_2^2 + n_2^2) - (l_1 l_2 + m_1 m_2 + n_1 n_2)^2 \\ &= (l_1 m_2 - l_2 m_1)^2 + (m_1 n_2 - m_2 n_1)^2 + (n_1 l_2 - n_2 l_1)^2 \end{aligned}$$

$\Rightarrow$ $$\sin\theta = \pm\sqrt{\Sigma\,(l_1 m_2 - l_2 m_1)^2}$$

Also $$\tan\theta = \frac{\sin\theta}{\cos\theta} = \pm\frac{\sqrt{\Sigma\,(l_1 m_2 - l_2 m_1)^2}}{\Sigma\, l_1 l_2}$$

Cor. 2. If the direction cosines of two lines be proportional to a_1, b_1, c_1 and a_2, b_2, c_2, so that their actual values are

$$\pm\frac{a_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}, \pm\frac{b_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}}, \pm\frac{c_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}};$$
$$\pm\frac{a_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}, \pm\frac{b_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}, \pm\frac{c_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}},$$

and if θ be the angle between the given lines, we have

$$\cos\theta = \pm \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\,\sqrt{a_2^2 + b_2^2 + c_2^2}},$$

$$\sin\theta = \pm \frac{\sqrt{(a_1b_2 - a_2b_1)^2 + (b_1c_2 - b_2c_1)^2 + (c_1a_2 - c_2a_1)^2}}{\sqrt{a_1^2 + b_1^2 + c_1^2}\,\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

$$\tan\theta = \pm \frac{\sqrt{\Sigma\,(a_1b_2 - a_2b_1)^2}}{\Sigma\, a_1a_2}.$$

The expression for $\tan\theta$ is the same whether we use direction cosines or direction ratios.

Cor. 3. Conditions for perpendicular and parallelism.

(i) The given lines are perpendicular

$\Rightarrow$ $\theta = 90°$

$\Rightarrow$ $\cos\theta = 0$

$\Rightarrow$ $\boldsymbol{a_1a_2 + b_1b_2 + c_1c_2 = 0.}$

(ii) The given lines are parallel

$\Rightarrow$ the lines through the origin drawn parallel to the lines coincide.

$\Rightarrow$ the direction cosines of the lines are the same.

$\Rightarrow$ the direction ratios of the lines are proportional.

EXAMPLES

1. *$l_1, m_1, n_1; l_2, m_2, n_2$ are the direction cosines of two mutually perpendicular lines. Show that the direction cosines of the line perpendicular to them both are*

$$m_1n_2 - m_2n_1,\ n_1l_2 - n_2l_1,\ l_1m_2 - l_2m_1$$

Sol. If *l, m,n* be the direction cosines of the required line, we have

$$ll_1 + mm_1 + nn_1 = 0$$

$$ll_2 + mm_2 + nn_2 = 0$$

$$\Rightarrow \quad \frac{l}{m_1n_2 - m_2n_1} = \frac{m}{n_1l_2 - n_2l_1} = \frac{n}{l_1m_2 - l_2m_1} = \frac{\sqrt{\Sigma l^2}}{\sqrt{\Sigma\,(m_1n_2 - m_2n_1)^2}} = \frac{1}{\sin\theta},$$

where θ is the angle between the given lines. As $\theta = 90°$, we have $\sin\theta = 1$. Hence, the result.

2. *A line makes angles $\alpha, \beta, \gamma, \delta$ with the four diagonals of a cube, prove that*

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}.$$

Sol. Let a be the length of each side of the cube. Taking three coterminous edges *OA, OB, OC* as axes, the co-ordinates of various vertices will be $A\,(a, 0, 0)$, $L\,(a, a, 0)$, $B\,(0, a, 0)$, $N\,(0, a, a)$, $C\,(0, 0, a)$, $M\,(a, 0, a)$, $P\,(a, a, a)$ and $O\,(0, 0, 0)$.

d.c.s of diagonal *OP* are

$$\left[\frac{a}{\sqrt{a^2 + a^2 + a^2}}, \frac{a}{\sqrt{a^2 + a^2 + a^2}}, \frac{a}{\sqrt{a^2 + a^2 + a^2}}\right]$$

$$\Rightarrow \quad \left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]$$

Similarly, d.c.'s of *AN* are $\left[-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right]$

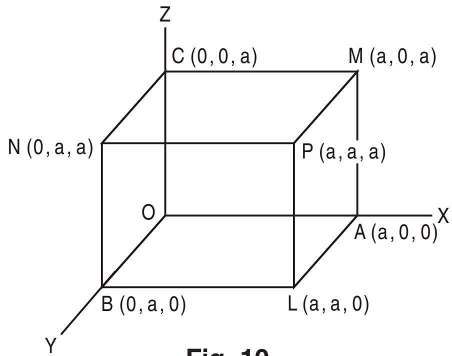


Fig. 10

of BM are $\left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right]$

and of CL are $\left[\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right]$.

If $[l, m, n]$ be the d.c.'s of a line which makes angles $\alpha, \beta, \gamma, \delta$ with these four diagonals of the cube, then

$$\cos\alpha = \frac{l+m+n}{\sqrt{3}}, \cos\beta = \frac{-l+m+n}{\sqrt{3}}, \cos\gamma = \frac{l-m+n}{\sqrt{3}} \text{ and } \cos\delta = \frac{l+m-n}{\sqrt{3}}$$

Hence,

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$$
$$= \frac{1}{3}[(l+m+n)^2 + (-l+m+n)^2 + (l-m+n)^2 + (l+m-n)^2]$$
$$= \frac{4}{3}.$$

3. *Show that the straight lines whose direction cosines are given by the equations*

$$al + bm + cn = 0, ul^2 + vm^2 + wn^2 = 0$$

are perpendicular or parallel according as

$$a^2(v+w) + b^2(w+u) + c^2(u+v) = 0 \text{ or } a^2/u + b^2/v + c^2/w = 0.$$

Sol. Eliminating l, between the given relations, we have

$$\frac{u\,(bm+cn)^2}{a^2} + vm^2 + wn^2 = 0$$

$$\Rightarrow \quad (b^2u + a^2v)\,m^2 + 2ubcmn + (c^2u + a^2w)\,n^2 = 0. \quad \text{...(i)}$$

If the lines be parallel, their direction cosines are equal so that the two values of m/n must be equal. The condition for this is

$$u^2b^2c^2 = (b^2u + a^2v)(c^2u + a^2w)$$

$$\Rightarrow \quad \frac{a^2}{u} + \frac{b^2}{v} + \frac{c^2}{w} = 0.$$

Again, if l_1, m_1, n_1 and l_2, m_2, n_2 be the direction cosines of the two lines then equation (i) gives

$$\frac{m_1}{n_1} \cdot \frac{m_2}{n_2} = \frac{m_1m_2}{n_1n_2} = \frac{c^2u + a^2w}{b^2u + a^2u}$$

$$\Rightarrow \quad \frac{m_1m_2}{c^2u + a^2w} = \frac{n_1n_2}{b^2u + a^2v}$$

Similarly the elimination of n, gives, (or by symmetry)

$$\frac{l_1 l_2}{b^2 w + c^2 v} = \frac{m_1 m_2}{a^2 w + c^2 u}$$

Thus, we have

$$\frac{l_1 l_2}{b^2 w + c^2 w} = \frac{m_1 m_2}{a^2 w + c^2 u} = \frac{n_1 n_2}{b^2 u + a^2 v} = k, \text{ say}.$$

$\Rightarrow$ $$l_1 l_2 + m_1 n_2 + n_1 n_2 = k\,(b^2 w + c^2 v + a^2 w + c^2 u + b^2 u + a^2 v)$$

For perpendicular lines

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0.$$

Thus, the condition for perpendicularity is

$$a^2 (v + w) + b^2\ (w + u) + c^2 (u + v) = 0.$$

5. *If* $l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3$ *are the direction cosines of three mutually perpendicular lines, show that the line whose direction cosines are proportional to* $l_1 + l_2 + l_3, m_1 + m_2 + m_3,$ $n_1 + n_2 + n_3$ *makes equal angles with them.*

Sol. Since the given lines are mutaully perpendicular, hence

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0 \qquad ...(1)$$

$$l_2 l_3 + m_2 m_3 + n_2 n_3 = 0 \qquad ...(2)$$

$$l_1 l_3 + m_1 m_3 + n_1 n_3 = 0 \qquad ..(3)$$

Let θ be the angle between the line whsoe d.c.s are $[l_1, m_1, n_1]$ and the line whose d.c.'s are proportional to $l_1 + l_2 + l_3, m_1 + m_2 + m_3,\ \ n_1 + n_2 + n_3;$ then

$$\cos\theta = \frac{l_1\,(l_1 + l_2 + l_3) + m_1\,(m_1 + m_2 + m_3) + n_1\,(n_1 + n_2 + n_3)}{\sqrt{(l_1 + l_2 + l_3)^2 + (m_1 + m_2 + m_3)^2 + (n_1 + n_2 + n_3)^2}}$$

$$= \frac{1}{\sqrt{3}}, \text{ from (1), (2) and (3)}$$

which is independent of $l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3$.

Hence, the given line makes equal angles with the three lines.

6. *If a variable line in two adjacent positions had direction cosines l, m, n;* $l + \delta l, m + \delta m, n + \partial n,$ *show that the small angle* $\delta\theta$ *between two positions is given by*

$$\delta\theta^2 = \delta l^2 + \delta m^2 + \delta n^2.$$

Sol. Since $[l, m, n]$ and $[l + \delta l, m + \delta m, n + \delta n]$ are d.c.'s, hence

$$l^2 + m^2 + n^2 = 1 \qquad ...(1)$$

and $$(l + \delta l)^2 + (m + \delta m)^2 + (n + \delta n)^2 = 1$$

$\Rightarrow$ $$\delta l^2 + \delta m^2 + \delta n^2 = -2\,(l\,\delta l + m\,\delta m + n\,\delta n) \qquad ...(2)$$

Now, $$\cos\delta\theta = l\,(l + \delta l) + m\,(m + \delta m) + n\,(n + \delta n)$$

$$= l^2 + m^2 + n^2 + l\,\delta l + m\,\delta m + n\,\delta n$$

$$= 1 - \frac{1}{2}\{\delta l^2 + \delta m^2 + \delta n^2\}, \text{ from (1) and (2).}$$

$\Rightarrow$ $$\delta l^2 + \delta m^2 + \delta n^2 = 2\,(1 - \cos\delta\theta) = 2 \cdot 2\sin^2\frac{\delta\theta}{2}$$

$$= 4\left(\frac{1}{2}\,\delta\theta\right)^2 = \delta\theta^2$$

EXERCISES

1. Find the angles between the lines whose direction ratios are

(i) $5, -12, 13; -3, 4, 5.$ [**Ans.** $\cos^{-1}(1/65)$]

(ii) $1, 1, 12; \sqrt{3}-1, -\sqrt{3}-1, 4.$ [**Ans.** $\pi/3$]

2. If A, B, C, D are the points $(3, 4, 5), (4, 6, 3), (-1, 2, 4)$ and $(1, 0, 5)$, find the angle between CD and AB. [**Ans.** $\cos^{-1}(4/9)$]

3. Find the angle between any two diagonals of a cube. [**Ans.** $\cos^{-1}(1/3)$]

4. Show that a line can be found perpendicular to the three lines with direction cosines proportional to $(2, 1, 5), (4, -2, 2), (-6, 4, -1)$. Hence, show that if these three lines be concurrent, they are also coplanar.

5. Find the direction cosines of a line which is perpendicular to the lines whose direction ratios are $1, 2, 3; -1, 3, 5.$ [**Ans.** $1/\sqrt{90}, -8/\sqrt{90}, 5/\sqrt{90}$]

6. l_1, m_1, n_1 and l_2, m_2, n_2 are the direction ratios of two intersecting lines. Show that lines through the intersection of these two with direction ratio

$$l_1 + kl_2, m_1 + km_2, n_1 + kn_2$$

are coplanar with them; k being a number whatsoever.

(Show that they all have a common perpendicular direction)

7. Show that three concurrent lines with direction cosines

$$(l_1, m_1, n_1), (l_2, m_2, n_2), (l_3, m_3, n_3)$$

are coplanar if and only if

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0$$

8. Show that the join of points (1, 2, 3), (4, 5, 7) is parallel to the join of the points $(-4, 3, -6), (2, 9, 2)$.

9. Show that the points $(4, 7, 8), (2, 3, 4), (-1, -2, 1), (1, 2, 5)$ are the vertices of a parallelogram.

10. Show that the points $(5, -1, 1), (7, -4, 7), (1, -6, 10), (-1, -3, 4)$ are the vertices of a rhombus.

11. Show that the points $(0, 4, 1), (2, 3, -1), (4, 5, 0), (2, 6, 2)$ are the vertices of a square.

12. $A(1, 8, 4), B(0, -11, 4), C(2, -3, 1)$ are three points and D is the foot of the perpendicular from A on BC. Find the co-ordinates of D. [**Ans.** $4, 5, -2$]

13. Find the point in which the join of $(-9, 4, 5)$ and $(11, 0, -1)$ is met by the perpendicular from the origin. [**Ans.** 1, 2, 2]

14. $A(-1, 2, -3), B(5, 0, -6), C(0, 4, -1)$ are three points. Show that the direction cosines of the bisectors of the angle BAC are proportional to (25, 8, 5) and $(-11, 20, 23)$.

[**Hint.** Find the co-ordinates of the points which divide BC in the ratio $AB : AC$]

15. Find the angle between the lines whose direction cosines are given by the equations $3l + m + 5n = 0$ and $6mn - 2nl + 5lm = 0$. [**Ans.** $\cos^{-1} 1/6$]

16. Show that the pair of lines whose direction cosines are given by $3lm - 4ln + mn = 0$, $l + 2m + 3n = 0$ are perpendicular.

17. Find the angle between the lines whose direction cosines satisfy the equations $l + m + n = 0$ and $2nl + 2lm - mn = 0$.

18. Show that the straight lines whose d.c.'s are given by $l+m+n=0$, $2mn+3nl-5lm=0$ are perpendicular to each other.

19. Find the angle between the lines $l+m+n=0$, $\dfrac{mn}{q-r}+\dfrac{nl}{r-p}+\dfrac{lm}{p-q}=0$. [**Ans.** $\pi/3$]

20. Show that the straight lines whose d.c.'s are given by $a^2l+b^2m+c^2n=0$, $mn+nl+lm=0$ will be parallel if $a+b+c=0$.

21. Show that the straight lines whose direction cosines are given by

$$al+bm+cn=0,\quad fmn+gnl+hlm=0,$$

are perpendicular if

$$f/a+g/b+h/c=0,$$

are parallel if

$$\sqrt{af}\pm\sqrt{bg}\pm\sqrt{ch}=0.$$

22. If, in a tetrahedron *OABC,*

$$OA^2+BC^2=OB^2+CA^2=OC^2+AB^2$$

then its pairs of opposite edges are at right angles.

23. l_1, m_1, n_1 and l_2, m_2, n_2 are two directions inclined at an angle φ, to each other. Show that

$$\frac{l_1+l_2}{2\cos\frac{1}{2}\varphi},\ \frac{m_1+m_2}{2\cos\frac{1}{2}\varphi},\ \frac{n_1+n_2}{2\cos\frac{1}{2}\varphi}$$

are the direction cosines of the line which bisects the angle between these two directions.

24. Show that the direction equally inclined to the three mutually perpendicular directions

$$l_1, m_1, n_1;\ l_2, m_2, n_2;\ l_3, m_3, n_3$$

is given by the direction cosines

$$\frac{l_1+l_2+l_3}{\sqrt{3}},\ \frac{m_1+m_2+m_3}{\sqrt{3}},\ \frac{n_1+n_2+n_3}{\sqrt{3}}.$$

25. Show that the area of the triangle whose vertices are the origin and the points (x_1, y_1, z_1) and (x_2, y_2, z_2) is

$$\frac{1}{2}\sqrt{(y_1z_2-y_2z_1)^2+(z_1x_2-z_2x_1)^2+(x_1y_2-x_2y_1)^2}.$$

❑❑❑

2

The Plane

GENERAL EQUATION OF FIRST DEGREE

An equation of the first degree in *x, y, z* is of the form

$$ax + by + cz + d = 0$$

where *a, b, c* are given real numbers and *a, b, c* are not all zero. The condition that *a, b, c* are not all zero is equivalent to the single condition $a^2 + b^2 + c^2 \neq 0$.

We are now interested in the locus of the points whose co-ordinates satsify an equation of first degree *viz.,*

$$ax + by + cz + d = 0, a^2 + b^2 + c^2 \neq 0.$$

It will be shown that this locus is a plane. To show this, we make use of the characteristic property of a plane which we given below :

A geometrical locus is a plane if it is such that if P and Q are any two points on the locus then every point of the line PQ is also a point on the locus.

2.1 THEOREM

Every equation of the first degree in x, y, z represents a plane.

Consider the equation

$$ax + by + cz + d = 0, a^2 + b^2 + c^2 \neq 0.$$

The locus of this equation will be a plane if *every* point of the line joining *any* two points on the locus also lies on the locus.

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$

be two points on the locus, so that we have

$$ax_1 + by_1 + cz_1 + d = 0 \quad \text{...(i)}$$

$$ax_2 + by_2 + cz_2 + d = 0 \quad \text{...(ii)}$$

Multiplying (ii) by k and adding to (i), we get

$$a(x_1 + kx_2) + b(y_1 + ky_2) + c(z_1 + kz_2) + d(1 + k) = 0$$

$$\Rightarrow \quad a\frac{x_1 + kx_2}{1 + k} + b\frac{y_1 + ky_2}{1 + k} + c\frac{z_1 + kz_2}{1 + k} + d = 0$$

Assuming that $k \neq -1$, the relation (ii) shows that the point

$$\left(\frac{x_1 + kx_2}{1 + k}, \frac{y_1 + ky_2}{1 + k}, \frac{z_1 + kz_2}{1 + k}\right)$$

is also a point on the locus for every value of $k \neq -1$.

Thus, every point on the straight line joining any two *arbitrary points* on the locus also lies on the locus. The given equation, therefore, represents a plane. Hence, every equation of the first degree in *x, y, z* represents a kplane.

Ex. Find the co-ordiantes of the point where the plane

$$ax + by + cz + d = 0, a^2 + b^2 + c^2 \neq 0$$

meets the three co-ordinate axes.

2.2 CONVERSE OF THE PRECEDING THEOREM

We shall now show that *the equation of every plane is of the first degree i.e.,* is of the form

$$ax + by + cz + d = 0,$$

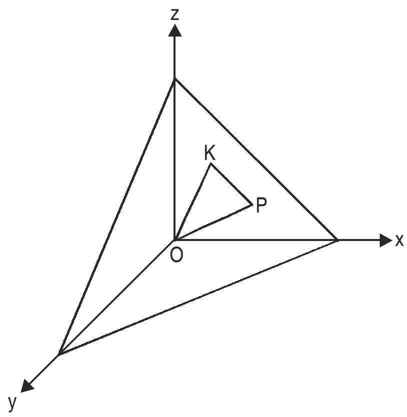
where $$a^2 + b^2 + c^2 \neq 0.$$

Consider any plane. Let p be the length of the perpendicular from the origin to the plane and let l, m, n be the directrion cosines of this perpendicular.

We shall show that for any point (x, y, z) on the plane, we have the relation

$$lx + my + nz = p$$

implying that the equation of the plane is of the first degree.



Let K be the foot of the perpendicular from the origin O to the plane. Let $OK = p$ and let l, m, n be its direction cosines. Take any point $P(x, y, z)$ on the plane.

Now PK lies in the plane

$\Rightarrow$ $$PK \perp OK$$

$\Rightarrow$ the projection of OP on $OK = OK = p.$

Also the projection of the segment OP joining the points

$$O(0, 0, 0) \text{ and } P(x, y, z)$$

on the line OK with direction cosines

$$l, m, n$$

is $$l(x - 0) + m(y - 0) + n(z - 0) = lx + my + nz.$$

It follows that

$$\boldsymbol{lx + my + nz = p}.$$

This equation, being satisfied by the co-ordiantes of any point $P(x, y, z)$ on the given plane, is the equation of the plane.

Note 1. The equation

$$lx + my + nz = p,$$

is called the **normal** form of the equation of a plane.

Note 2. The plane whose equation is

$$ax + by + cz + d = 0$$

is referred to as

$$ax + by + cz + d = 0$$

itself *i.e.,* we often refer to an equation of the plane itself as the plane.

Ex. Find the equation of the plane containing the lines through the origin with direction cosiens proportional to $(1, -2, 2)$ and $(2, 3, -1)$. [**Ans.** $4x - 5y - 7z = 0$]

2.3 TRANSFORMATION TO THE NORMAL FORM

To transform the equation

$$ax + by + cz + d = 0, a^2 + b^2 + c^2 \neq 0$$

to the normal form

$$lx + my + nz = p.$$

As these two equations represent the same plane, we have

$$-\frac{d}{p} = \frac{a}{l} = \frac{b}{m} = \frac{c}{n} = \pm\frac{\sqrt{a^2 + b^2 + c^2}}{\sqrt{l^2 + m^2 + n^2}} = \pm\sqrt{a^2 + b^2 + c^2}$$

Thus, $d/p = \pm\sqrt{a^2 + b^2 + c^2}$. As p, according to our convention, is always positive, we shall take positive or negative sign with the radical according as, d, is negative or positive.

Thus, if d be positive, we have

$$l = -\frac{a}{\sqrt{\Sigma a^2}}; m = -\frac{b}{\sqrt{\Sigma a^2}}; n = -\frac{c}{\sqrt{\Sigma a^2}}; p = +\frac{d}{\sqrt{\Sigma a^2}};$$

and if d be negative, we have

$$l = \frac{a}{\sqrt{\Sigma a^2}}; m = \frac{b}{\sqrt{\Sigma a^2}}; n = \frac{c}{\sqrt{\Sigma a^2}}; p = -\frac{d}{\sqrt{\Sigma a^2}}$$

Thus, the normal form of the equation $ax + by + cz + d = 0$ is

$$-\frac{a}{\sqrt{\Sigma a^2}} x - \frac{b}{\sqrt{\Sigma a^2}} y - \frac{c}{\sqrt{\Sigma a^2}} z = \frac{d}{\sqrt{\Sigma a^2}},$$

if d be positive, and

$$+\frac{a}{\sqrt{\Sigma a^2}} x + \frac{b}{\sqrt{\Sigma a^2}} y + \frac{c}{\sqrt{\Sigma a^2}} z = -\frac{d}{\sqrt{\Sigma a^2}}$$

if d be negative.

2.3.1 Direction Cosines of the Normal to a Plane

From above we deduce that *the direction cosines of the normal to a plane are proportioanl to the coefficients of x, y, z in its equation or* that the coefficients of *x, y, z* are direction ratios of the normal *to the plane.*

Thus,

$$a, b, c$$

are direction ratios of the normal to the plane

$$ax + by + cz + d = 0.$$

Ex. 1. Find the direction cosines of the normals to the planes

(i) $2x - 3y + 6z = 7$, (ii) $x + 2y + 2z - 1 = 0$.

[**Ans.** (i) $2/7, -3/7, 6/7$; (ii) 1/3, 2/3, 2/3]

Ex. 2. Show that the normals to the planes

$$x - y + z = 1, 3x + 2y - z + 2 = 0$$

are perpendicular to each other.

2.3.2. Angle Between Two Planes

Angle between two planes is equal to the angle between the normals to them from any point.

It follows that the angle between the two planes

$$ax + by + cz + d = 0, a_1x + b_1y + c_1z + d_1 = 0$$

being equal to the angle between the lines with direction ratios

$$a, b, c; a_1, b_1, c_1$$

is
$$\cos^{-1}\left\{\frac{aa_1 + bb_1 + cc_1}{\sqrt{\Sigma a^2}\sqrt{\Sigma a_1^2}}\right\}.$$

Cor. Parallelism and perpendicularity of two planes. Two planes are parallel or perpendicular according as the normals to them are parallel or perpendicular.

Thus, the two planes

$$ax + by + cz + d = 0, \ a_1x + b_1y + c_1z + d_1 = 0$$

will be parallel, if

$$a, b, c \quad \textit{and} \quad a_1, b_1, c_1$$

are direction ratios of the same line and will be perpendicular, if

$$\boldsymbol{aa_1 + bb_1 + cc_1 = 0.}$$

EXERCISES

1. Find the angles between the following pairs of planes

(i) $2x - y + 2z = 3$; $3x + 6y + 2z = 4$ [**Ans.** $\cos^{-1}(4/21)$]

(ii) $2x - y + z = 6$; $x + y + 2z = 7$ [**Ans.** $\pi/3$]

(iii) $3x - 4y + 5y = 0$; $2x - y - 2z = 5$ [**Ans.** $\pi/2$]

2. Show that the equations

$$ax + by + r = 0, by + cz + p = 0, cz + ax + q = 0$$

represents planes respectively perpendicular to the *XY, YZ, ZX* planes.

3. Show that $ax + by + cz + d = 0$ represents planes, perpendicular respectively to *YZ, ZX, XY* planes, if *a, b, c* separetely vanish (Similar to Ex. 2).

4. Show that the plane

$$x + 2y - 3z + 4 = 0$$

is perpendicular to each of the planes

$$2x + 5y + 4z + 1 = 0, 4x + 7y + 6z + 2 = 0.$$

2.4 DETERMINATION OF A PLANE UNDER GIVEN CONDITIONS

The *general* equation $ax + by + cz + d = 0$ of a plane contains *three* arbitrary constrants (ratios of the coefficients *a, b, c, d*) and, therefore, a plane can be found to satisfy three conditions each giving rise to only one relation between the constants. The three constants can then be determined from the three resulting relations.

We give below a few sets of conditions which determine a plane :

(i) passing through *three* non-collinear points;

(ii) passing through *two* given points and perpendicular to a given plane;

(iii) passing through a given point and perpendicular to *two* given planes.

2.4.1. Intercept Form of the Equation of a Plane

To find the equation of a plane in terms of the intercepts a, b, c which it makes on the axes.

The intercept of a plane on any co-ordinate axis is the distance of the point where the plane meets the axis from the origin taken with the appropriate sign.

We of course, suppose here that the plane does not pass through the origin so that none of *a, b, c* is zero.

Let the equation of the plane be

$$Ax + By + Cz + D = 0. \quad \text{...(i)}$$

The plane not passing through the origin, we have

$$D \neq 0.$$

The points $(a, 0, 0), (0, b, 0), (0, 0, c)$ lying on the plane (i), we have

$$aA + D = 0 \Rightarrow -\frac{A}{D} = \frac{1}{a}$$

$$bB + D = 0 \Rightarrow -\frac{B}{D} = \frac{1}{b}$$

$$cC + D = 0 \Rightarrow -\frac{C}{D} = \frac{1}{c}.$$

The equation (1) can be rewritten as

$$-\frac{A}{D}x - \frac{B}{D}y - \frac{C}{D}z = 1$$

so that after substitution, we obtain

$$\boldsymbol{\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1,}$$

as the required equation of the plane.

2.4.2. Plane Through Three Points

To find the equation of the plane through the three non-collinear points

$$(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$$

Let the required equation of the palne be

$$ax + by + cz + d = 0. \quad \text{...(i)}$$

As the given points lie on the plane (i), we have

$$ax_1 + by_1 + cz_1 + d = 0, \quad ...(ii)$$
$$ax_2 + by_2 + cz_2 + d = 0, \quad ...(iii)$$
$$ax_3 + by_3 + cz_3 + d = 0. \quad ...(iv)$$

Eliminating *a, b, c, d* from (i) – (iv), we have

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

which is the required equation of the plane.

Cor. The equation of the plane which makes intercepts *a, b, c* respectively on the three co-ordiante axes is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

in that this is the plane through the 3 points $(a, 0, 0), (0, b, 0), (0, 0, c)$.

Note. In actual numerical exercises, the student would find it more convenient to follow the method of the first example below.

EXAMPLES

1. *Find the equation of the plane through the points*

$$P\,(2, 2, -1), Q\,(3, 4, 2), R\,(7, 0, 6)$$

Sol. The general equation of a plane through $P\,(2, 2, -1)$ is

$$a\,(x - 2) + b\,(y - 2) + c\,(z + 1) = 0 \quad ...(i)$$

It will pass through *Q* and *R*, if

$$a + 2b + 3x = 0$$
$$5a - 2b + 7c = 0.$$

These give

$$\frac{a}{20} = \frac{b}{8} = \frac{c}{-12} \quad \text{or} \quad \frac{a}{5} = \frac{b}{2} = \frac{c}{-3}.$$

Substituting these values in (i), we have

$$5\,(x - 2) + 2\,(y - 2) - 3\,(z + 1) = 0$$
$$\Rightarrow \quad 5x + 2y - 3z - 17 = 0$$

as the required equation.

2. *Find the equation of the plane through the points*

$$(2, 2, 1) \ \textit{and} \ (9, 3, 6),$$

and perpendicular to the plane

$$2x + 6y + 6z = 9.$$

Sol. Any plane through (2, 2, 1) is

$$a\,(x - 2) + b\,(y - 2) + c\,(z - 1) = 0. \quad ...(i)$$

It passes through (9, 3, 6)

$$\Rightarrow \quad a\,(9 - 2) + b\,(3 - 2) + c\,(6 - 1) = 0$$
$$\Rightarrow \quad 7a + b + 5c = 0 \quad ...(ii)$$

The plane (i) is perpendicular to the given plane

$$\Rightarrow \quad 2a + 6b + 6c = 0 \quad ...(iii)$$

From (ii) and (iii), we have

$$\frac{a}{-24} = \frac{b}{-32} = \frac{c}{40} \Rightarrow \frac{a}{3} = \frac{b}{4} = \frac{c}{-5}.$$

Substituting in (i), we see that the equation of the required plane is

$$3\,(x - 2) + 4\,(y - 2) - 5\,(z - 1) = 0 \Leftrightarrow 3x + 4y - 5z = 9.$$

EXERCISES

1. Find the equation of the plane through the three points $(1, 1, 1), (1, -1, 1), (-7, -3, -5)$ and show that it is perpendicular to the *XZ* plane. [**Ans.** $3x - 4z + 1 = 0$]
2. Obtain the equation of the plane passing through the point $(-2, -2, 2)$ and containing.
3. If, from the point $P(a, b, c)$, perpendiculars *PL, PM* be drawn to *YZ* and *ZX* planes, find the equation of the plane *OLM*. [**Ans.** $bcx + cay - abz = 0$]
4. Show that the four points $(-6, 3, 2), (3, -2, 4), (5, 7, 3)$ and $(-13, 17, -1)$ are coplanar.
5. Show that the points $(6, -4, 4), (0, 0, -4)$ intersects the join of $(-1, -2, -3), (1, 2, -5)$.
6. Show that $(-1, 4, -3)$ is the circumcentre of the triangle formed by the points $(3, 2, -5), (-3, 8, -5), (-3, 2, 1)$.
7. Show that the equations of the three planes passing through the points, $(1, -2, 4)$, $(3, -4, 5)$ and perpendicular to the *XY, YZ, ZX* planes are $x + y + 1 = 0; x - 2z + 7 = 0;$ $y + 2z = 6$ respectively.
8. Obtain the equation of the plane which passes through the point $(-1, 3, 2)$ and is perpendicular to each of the two planes $x + 2y + 2z = 5; 3x + 2y + 2z = 8$.
[**Ans.** $2x - 4y + 3z + 8 = 0$]
9. Find the equation of the plane which passes through $A(-1, 1, 1)$ and $B(1, -1, 1)$ and is perpendicular to the plane $x + 2y + 2z = 5$. [**Ans.** $2x + 2y - 3z + 3 = 0$]
10. Find the intercepts of the plane $2x - 3y + z = 12$ on the co-ordinate axes. [**Ans.** $6, -4, 12$]
11. A plane meets the co-ordiante axes *A, B, C* such that the centroid of the triangle *ABC* is the point (a, b, c), show that the equation of the plane is $x/a + y/b + z/c = 3$.
12. Find the equations of the two planes which pass through the points $(0, 4, -3), (6, -4, 3)$ other than the plane through the origin, which cut off from the axes intercepts wehose sum is zero. [**Ans.** $2x - 3y - 6z = 6; 6x + 3y - 2z = 18$]
13. A variable plane is at a constant distance *p* from the origin and meets the co-ordinate axes in *A, B, C*. Show that the locus of the centroid of the tetrahedron *OABC* is $x^{-2} + y^{-2} + z^{-2} = 16p^{-2}$.

2.5 SYSTEMS OF PLANES

The equation of a plane satisfying two conditions will involve one arbitrary constant which can be chosen in an infinite number of ways, thus giving rise to an infinite number of planes, called a *System of planes.*

The arbitrary constant which is different for different members of the system is called a *Parameter.*

Similarly the equation of a plane satisfying one condition will involve two parameters.

The following are the equations of a few systems of planes involving one or two parameters :

1. The equation

$$ax + by + cz + k = 0$$

represents the system of planes parallel to a given plane

$$ax + by + cz + d = 0,$$

k being the parameter.

Thus, the set of planes parallel to a given plane

$$ax + by + cz + d = 0$$

is $\{ax + by + cz + k = 0; k \text{ is any number}\}$.

2. The equation

$$ax + by + cz + d = 0$$

represents the system of planes perpendicular to given line with direction ratios *a, b, c, d* being the parameter.

3. The equation

$$(ax + by + cz + d) + k\,(a_1x + b_1y + c_1z + d_1) = 0 \qquad ...(1)$$

represents the system of planes through the line of intersection of the plane

$$ax + by + cz + d = 0, \qquad ...(2)$$

$$a_1x + b_1y + c_1z + d_1 = 0; \qquad ...(3)$$

k being the parameter, for the equation (1), being of the first degree in x, y, z represents a plane; and it is evidently satisfied by the co-ordinates of the points which satisfy (2) and (3), whatever value k may have.

4. The equation

$$A\,(x - x_1) + B\,(y - y_1) + C\,(z - z_1) = 0,$$

represents the system of planes passing through the point (x_1, y_1, z_1) where the required *two* parameters are the two ratios of the coefficients A, B, C; for the equation is of the first degree and is clearly satisfied by the point (x_1, y_1, z_1) whatever be the ratios of the coefficients.

EXAMPLES

1. *Find the equation of the plane passing through the lines of intersection of the planes*

$$2x - y = 0 \;\; \textit{and} \;\; 3z - y = 0$$

and perpendicular to the plane

$$4x + 5y - 3z = 8.$$

Sol. The plane

$$2x - y + k\,(3z - y) = 0 \iff 2x - (1 + k)\,y + 3kz = 0$$

passes through the line of intersection of the given planes whatever value k may have. This plane is perpendicular to

$$4x + 5y - 3z = 8$$

$$\Rightarrow \qquad 2\cdot 4 - (1 + k)\cdot 5 + 3k\,(-3) = 0 \Rightarrow 14k = 3 \Rightarrow k = 3/14.$$

Thus, the required equation is

$$2x - y + \left(\frac{3}{14}\right)(3z - y) = 0 \iff 28x - 17y + 9z = 0.$$

2. *A point P moves on a fixed plane* $x/a + y/b + z/c = 1$. *The plane through P perpendicular to OP meets the axes in A, B, C. The planes through A, B, C parallel to co-ordinate planes intersect in Q. Show that the locus of Q is*

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{ax} + \frac{1}{by} + \frac{1}{cz}.$$

Sol. Let the point be $P = (\alpha, \beta, \gamma)$. Hence

$$\frac{\alpha}{a} + \frac{\beta}{b} + \frac{\gamma}{c} = 1 \qquad ...(1)$$

Equation of the plane perpendicular to OP is

$$\alpha x + \beta y + \gamma z = d.$$

But it passes through $P\,(\alpha, \beta, \gamma)$, we have

$$d = \alpha^2 + \beta^2 + \gamma^2$$

Hence, equation of plane through P and perpendicular to OP is

$$\alpha x + \beta y + \gamma z = \alpha^2 + \beta^2 + \gamma^2 \qquad ...(2)$$

$$\Rightarrow \qquad OA = \frac{\alpha^2 + \beta^2 + \gamma^2}{\alpha}, OB = \frac{\alpha^2 + \beta^2 + \gamma^2}{\beta}, OC = \frac{\alpha^2 + \beta^2 + \gamma^2}{\gamma}.$$

So the planes through A, B, C parallel to the planes YOZ, ZOX, XOY intersect in the point Q whose co-ordinates are

$$x = \frac{\alpha^2 + \beta^2 + \gamma^2}{\alpha}, \; y = \frac{\alpha^2 + \beta^2 + \gamma^2}{\beta}, \; z = \frac{\alpha^2 + \beta^2 + \gamma^2}{\gamma}.$$

With the help of (1),

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{\alpha^2 + \beta^2 + \gamma^2}$$

and $$\frac{1}{ax} + \frac{1}{by} + \frac{1}{cz} = \frac{\frac{\alpha}{a} + \frac{\beta}{b} + \frac{\gamma}{c}}{\alpha^2 + \beta^2 + \gamma^2} = \frac{1}{\alpha^2 + \beta^2 + \gamma^2}, \text{ from (1).}$$

Hence, required locus is

$$\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2} = \frac{1}{ax} + \frac{1}{by} + \frac{1}{cz}.$$

3. *The plane* $lx + my = 0$ *is rotated about its line of intersection with the plane* $z = 0$ *through an angle* α. *Prove that the equation of the plane in its new position is*

$$lx + my \pm z\sqrt{l^2 + m^2}\,\tan\alpha = 0.$$

Sol. The equation of a plane through the line of intersection of the palnes $lx + my = 0$ and $z = 0$, is

$$lx + my + \lambda z = 0.$$

This plane makes an angle α with the plane $lx + my = 0$.

$$\therefore \qquad \cos\alpha = \frac{l^2 + m^2}{\sqrt{(l^2 + m^2)(l^2 + m^2 + \lambda^2)}}$$

$$\therefore \qquad \cos^2\alpha = \frac{l^2 + m^2}{(l^2 + m^2 + \lambda^2)}$$

$$\Rightarrow \qquad \lambda = \pm\sqrt{(l^2 + m^2)}\,\tan\alpha$$

Hence, required plane is

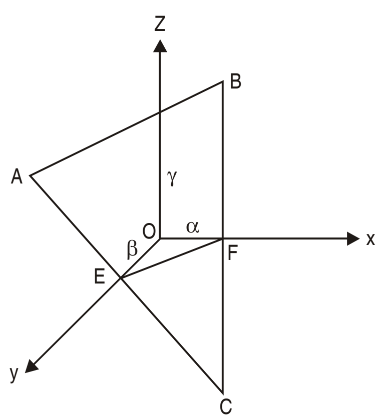
$$lx + my \pm z\sqrt{l^2 + m^2}\,\tan\alpha = 0.$$

4. *A triangle, the lengths of whose sides are a, b, and c is placed so that the middle points of the sides are on the axes. Show that the equation to the plane is*

$$x/\alpha + y/\beta + z/\gamma = 1.$$

where $\alpha^2 = \dfrac{(b^2 + c^2 - a^2)}{8}, \beta^2 = \dfrac{(c^2 - a^2 - b^2)}{8}, \gamma^2 = \dfrac{(a^2 + b^2 - c^2)}{8}.$

Sol. Let α, β, γ be the intercepts that the plane makes with the axes. E and F are the mid-points of AC and BC. Therefore, EF is parallel to and half of AB.



$$\therefore \qquad EF^2 = OE^2 + OF^2 = \alpha^2 + \beta^2.$$

But $$EF = \frac{AB}{2} = \frac{c}{2} \Rightarrow \alpha^2 + \beta^2 = \frac{c^2}{4}$$

Similarly, $\beta^2 + \gamma^2 = a^2/4$ and $\gamma^2 + \alpha^2 = b^2/4$.

Adding, $$\alpha^2 + \beta^2 + \gamma^2 = \frac{a^2 + b^2 + c^2}{8}$$

$$\Rightarrow \quad \gamma^2 = \frac{a^2 + b^2 + c^2}{8} - \frac{c^2}{4} = \frac{a^2 + b^2 - c^2}{8}$$

Similarly,

$$\alpha^2 = \frac{b^2 + c^2 - a^2}{8}, \beta^2 = \frac{c^2 + a^2 - b^2}{8}.$$

Hence, the equation of plane is

$$x/\alpha + y/\beta + z/\gamma = 1, \text{ where } \alpha^2, \beta^2, \gamma^2 \text{ as given below.}$$

EXERCISES

1. Obtain the equation of the plane through the intersection of the planes

$$x + 2y + 3z + 4 = 0 \quad \text{and} \quad 4x + 3y + 2z + 1 = 0$$

and the origin. [**Ans.** $3x + 2y + z = 0$]

2. Find the equation of the plane which is perpendicular to the plane

$$5x + 3y + 6z + 8 = 0$$

and which contains the line of intersection of the planes

$$x + 2y + 3z - 4 = 0, 2x + y - z + 5 = 0.$$

[**Ans.** $51x + 15y - 50z + 173 = 0$]

3. The plane $x - 2y + 3z = 0$ is rotated through a right angle about the line of intersection wit the plane $2x + 3y - 4z - 5 = 0,$ find the equation of the plane in its new position.

[**Ans.** $22x + 5y - 4z - 35 = 0$]

4. Find the equation of the plane through the line of intersection of the planes

$$ax + by + cz + d = 0, \; a_1x + b_1y + c_1z = d_1 = 0$$

and perpendicular to the *XY* plane.

[**Ans.** $x(ac_1 - a_1c) + y(bc_1 - b_1c) + z(dc_1 - d_1c) = 0$]

5. Find the equation of the plane through the point (2, 3, 4) and parallel to the plane $5x - 6y + 7z = 3.$ [**Ans.** $5x - 6y + 7z = 20$]

6. Find the equation of the plane through $(2, 3, -4)$ and $(1, -1, 3)$ parallel to the *x*-axis.

[**Ans.** $7y + 4z - 5 = 0$]

7. A variable plane is at a constant distance $3p$ from the origin and meets the axes in *A, B* and *C*. Show that the locus of the centroid of the triangle *ABC* is $x^{-2} + y^{-2} + z^{-2} = p^{-2}.$

8. Find the equation of the plane that passes through the point $(3, -3, 1)$ and is normal to the line joining the points $(3, 4, -1)$ and $(2, -1, 5).$ [**Ans.** $x + 5y - 6z + 18 = 0$]

9. Obtain the equation of the plane that bisects the segment joining the points (1, 2, 3), (3, 4, 5), at right angles. [**Ans.** $x + y + z = 9$]

10. A variable plane passes through a fixed point (a, b, c) and meets the co-ordinate axes in *A, B, C*. Show that the locus of the point common to the planes through *A, B, C* parallel to the co-ordinate plane is

$$a/x + b/y + c/z = 1.$$

2.6 TWO SIDES OF A PLANE

Consider any plane. Two points *P* and *Q* which do not lie on the plane lie on the different or the same side of plane according as the segment *PQ* has or does not have a point in common with the plane.

We proceed to determine a criterion for two given points to lie on the same or different sides of a given plane and show that :

Two points $A(x_1, y_1, z_1), B(x_2, y_2, z_2)$ *lie on the same or different sides of the plane*

$$ax + by + cz + d = 0,$$

according as the expressions

$$ax_1 + by_1 + cz_1 + d, ax_2 + by_2 + cz_2 + d$$

are of the same or different signs.

Let the line AB meets the given plane in the point P and let P divides AB in the ratio $r:1$ so that r is positive or negative according as P divides AB internally or externally, *i.e.,* according as A and B lie on the opposite or the same side of the plane.

Since the point P whose co-ordinates are

$$\left(\frac{rx_2 + x_1}{r+1}, \frac{ry_2 + y_1}{r+1}, \frac{rz_2 + z_1}{r+1}\right)$$

lies on the same plane, we have

$$a\,\frac{rx_2 + x_1}{r+1} + b\,\frac{ry_2 + y_1}{r+1} + c\,\frac{rz_2 + z_1}{r+1}\,d = 0$$

$$\Rightarrow \qquad r\,(ax_2 + by_2 + cz_2 + d) + (ax_1 + by_1 + cz_1 + d) = 0,$$

$$\Rightarrow \qquad r = -\frac{ax_1 + by_1 + cz_1 + d}{ax_2 + by_2 + cz_2 + d}$$

This shows that r is negative or positive according as

$$ax_1 + by_1 + cz_1 + d,\ ax_2 + by_2 + cz_2 + d$$

are of the same or different signs.

Thus, the theorem is proved.

Ex. Show that the origin and the point $(2, -4, 3)$ lie on different sides of the plane $x + 3y - 5z + 7 = 0.$

2.7 LENGTH OF THE PERPENDICULAR FROM A POINT TO A PLANE

To find the perpendicular distance of the point

$$P\,(x_1, y_1, z_1)$$

from the plane

$$lx + my + nz = p.$$

The equation of the plane which passes through the point

$$P\,(x_1, y_1, z_1)$$

and is parallel to the given plane is

$$lx + my + nz = p_1$$

where $\qquad lx_1 + my_1 + nz_1 = p_1.$

Let OKK' be the perpendicular from the origin O to the two parallel planes meeting them in K and K' so that

$$OK = p \quad \text{and} \quad OK' = p_1$$

Draw the line PL perpendicular to the given plane. We have

$$LP = OK' - OK$$

$$= p_1 - p = lx_1 + my_1 + nz_1 - p.$$

Cor. *To find the length of the perpendicular from the point*

$$(x_1, y_1, z_1)$$

to the plane as $ax + by + cz + d = 0.$

The normal form of the given equation being

$$\pm\frac{a}{\sqrt{\Sigma a^2}}\,x \pm \frac{b}{\sqrt{\Sigma a^2}}\,y \pm \frac{c}{\sqrt{\Sigma a^2}}\,z \pm \frac{d}{\sqrt{\Sigma a^2}} = 0.$$

the required length of the perpendicular is

$$\pm\frac{ax_1 + by_1 + cz_1 + d}{\sqrt{(a^2 + b^2 + c^2)}}.$$

EXAMPLES

1. *Find the locus of a point, the sum of the squares of whose distances from the planes* $x+y+z=0,\ x-z=0,\ x-2y+z=0$ *is 9.*

Sol. Let the co-ordinates of the point be (α, β, γ). Its distances from the given planes are

$$\frac{\alpha+\beta+\gamma}{\sqrt{3}}, \frac{\alpha-\gamma}{\sqrt{2}}, \frac{\alpha-2\beta+\gamma}{\sqrt{6}}$$

We are given that

$$\left(\frac{\alpha+\beta+\gamma}{\sqrt{3}}\right)^2+\left(\frac{\alpha-\gamma}{\sqrt{2}}\right)^2+\left(\frac{\alpha-2\beta+\gamma}{\sqrt{6}}\right)^2=9$$

$$\Rightarrow \qquad 6\alpha^2+6\beta^2+6\gamma^2=54 \Rightarrow \alpha^2+\beta^2+\gamma^2=9$$

Hence, the locus of (α, β, γ) is

$$x^2+y^2+z^2=9.$$

2. *Two systems of rectangular axes have the same origin. If a plane cuts them at distances a, b,c and* a', b', c' *respectively from the origin, prove that*

$$1/a^2+1/b^2+1/c^2=1/a'^2+1/b'^2+1/c'^2.$$

Sol. Equation s of the plane w.r.t. two systems are

$$x/a+y/b+z/c=1$$

and
$$x/a'+y/b'+z/c'=1.$$

Since origin is common to both, hence the perpendicular distances of these planes from the origin must be equal. Hence,

$$\frac{1}{\sqrt{1/a^2+1/b^2+1/c^2}}=\frac{1}{\sqrt{1/a'^2+1/b'^2+1/c'^2}}$$

or
$$1/a^2+1/b^2+1/c^2=1/a'^2+1/b'^2+1/c'^2.$$

EXERCISES

1. Find the distances of the points (2, 3, 4) and (1, 1, 4) from the plane

$$3x-6y+2z+11=0$$ [**Ans.** 1; 16/7]

2. Show that distances between the parallel planes

$$2x-2y+z+3=0 \text{ and } 4x-4y+2z+5=0$$

is 1/6.

(The distance between two parallel planes is the distance of any point on one from the other).

3. Find the locus of the point whose distance from the origin is three times its distance from the plane $2x+2z=3$. [**Ans.** $3x^2+3z^2-4xy+8xz-4yz-12x+6y-12z+9=0$]

4. Show that (1/8, 1/8, 1/8) is the incentre of the tetrahedron formed by the four planes $x=0, y=0, z=0, x+2y+2z=1$.

5. Sum of the distances of any number of fixed points from a variable plane is zero; show that the plane passes through a fixed point.

2.7.1 Bisectors of Angles Between Two Planes

Just as we have two bisectors between two given lines, we also have two bisectors between two given planes. Of course, the bisectors are now planes. We have proceed *to find the equations of the bisectors of the angle between the planes*

$$ax+by+cz+d=0$$
$$a_1x+b_1y+c_1z+d_1=0.$$

If (x, y, z) be a point on any one of the planes bisecting the angles between the planes, then the perpendiculars from this point to the two planes must be equal (in magnitude) so that

$$\frac{ax+by+cz+d}{\sqrt{(a^2+b^2+c^2)}} = \pm \frac{a_1x+b_1y+c_1z+d_1}{\sqrt{(a_1^2+b_1^2+c_1^2)}}$$

are the equations of the two bisecting planes.

Of these two bisecting planes, one bisects the acute and the other the obtuse angle between the given planes.

The bisector of the acute angle makes with either of the planes an angle which is less than $45°$ and the bisector of the obtuse angle makes with either of them an angle which is greater than $45°$. this gives a test for determining which angle, acute or obtuse, each bisecting plane bisects.

EXAMPLE

Find the equations of the planes bisecting the angles between the planes

$$x+2y+2z-3=0 \quad \text{...(i)}$$

$$3x+4y+12z+1=0 \quad \text{...(ii)}$$

and specify the one which bisects the acute angle.

Sol. The equations of the two bisecting planes are

$$\frac{x+2y+2z-3}{3} = \pm \frac{3x+4y+12z+1}{13}$$

$$\begin{cases} 2x+7y-5z-21=0, & \text{...(iii)} \\ 11x+19y+31z-18=0 & \text{...(iv)} \end{cases}$$

If θ be the angle between the planes (i) and (iii), we have

$$\cos\theta = \frac{2}{\sqrt{78}}$$

$$\Rightarrow \qquad \tan\theta = \frac{\sqrt{74}}{2} > 1$$

$\Rightarrow$ θ is greater than $45°$.

$\Rightarrow$ the plane (iii) bisects the obtuse angle.

$\Rightarrow$ (iv) bisects the acture angle.

EXERCISES

1. Find the bisector of the acute angle between the planes

$$2x-y-2z+3=0, 3x-2y+6z+8=0.$$

[**Ans.** $23x-13y+32z+45=0$]

2. Show that the plane

$$14x-8y+13=0$$

bisects the obtuse angle between the planes

$$3x+4y-5z+1=0, 5x+12y-13z=0.$$

3. Find the bisector of that angle between the planes

$$3x-6y+2z+5=0, 4x-12y+3z-3=0$$

which contains the origin. [**Ans.** $67x-162y+47z+44=0$]

2.8 JOINT EQUATION OF TWO PLANES

Consider any two planes. We are interested in finding an equation which represents the two planes simultaneously. Thus, we propose to find an equation which will be satisfied if and only if a point lies on *either* of the two planes *i.e.,* either on one plane or the other or both.

Let

$$ax+by+cz+d=0 \quad \text{...(i)}$$

and $$a_1x+b_1y+c_1z+d_1=0 \quad \text{...(ii)}$$

be the equations of two planes.

Consider the equation

$$(ax + by + cz + d)\,(a_1x + b_1y + c_1z + d_1) = 0. \qquad \text{...(iii)}$$

A point (x_1, y_1, z_1) lies on (i)

$\Rightarrow$ $$ax_1 + by_1 + cz_1 + d = 0$$

$\Rightarrow$ $$(ax_1 + by_1 + cz_1 + d)\,(a_1x_1 + b_1y_1 + c_1z_1 + d_1) = 0.$$

A point (x_1, y_1, z_1) lies on (ii)

$\Rightarrow$ $$(ax_1 + by_1 + cz_1 + d)\,(a_1x_1 + b_1y_1 + c_1z_1 + d_1) = 0.$$

A point (x_1, y_1, z_1) lies on (iii)

$\Rightarrow$ $$ax_1 + by_1 + cz_1 + d = 0 \quad \text{or} \quad a_1x_1 + b_1y_1 + c_1z_1 + d_1 = 0.$$

$\Rightarrow$ (x_1, y_1, z_1) lies on the plane (i) or on the plane (ii).

Thus, we have the following :

A point (x, y, z) either lies on the plane (i) or on the plane (ii)

$\Leftrightarrow$ $$(ax + by + cz + d)\,(a_1x + b_1y + c_1z + d_1) = 0.$$

We thus say that

$$(ax + by + cz + d)\,(a_1x + b_1y + c_1z + d_1) = 0$$

is the equation of the two planes.

2.8.1

Condition for the homogeneous second degree equation

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \qquad \text{...(i)}$$

to represent two planes.

We suppose that the given equation represents two planes.

Let the equation of the two planes separately be

$$lx + my + nz = 0,$$

and $$l'x + m'y + n'z = 0.$$

There cannot appear constant terms in the separate equations of the planes, for, otherwise, their joint equation will not be homogeneous.

We have

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = (lx + my + nz)\,(l'x + m'y + n'z)$$

$$\Rightarrow \quad \begin{cases} a = ll', b = mm', c = nn' \\ 2f = m'n + mn', 2g = ln' + l'n, 2h = lm' + l'm. \end{cases}$$

The required condition which is essentially the condition for the consistency of these equations is obtained on eliminating $l, m, n;\ l', m', n'$ from the above six relations and this can be easily effected as follows. We have

$$0 = \begin{vmatrix} l & l' & 0 \\ m & m' & 0 \\ n & n' & 0 \end{vmatrix} \times \begin{vmatrix} l' & l & 0 \\ m' & m & 0 \\ n' & n & 0 \end{vmatrix} = \begin{vmatrix} ll' + l'l & l'm + lm' & l'n + ln' \\ lm' + l'm & mm' + m'm & m'n + mn' \\ n'l + nl' & n'm + nm' & n'n + nn' \end{vmatrix}$$

$$= 8\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 8\,(abc + 2fgh - af^2 - bg^2 - ch^2).$$

Thus, if the equation

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0,$$

represents two planes, we have the condition,

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0.$$

Cor. angle between planes. If θ be the angle between the planes represented by the equation (i), we have if $ll' + mm' + nn' \neq 0$

$$\tan\theta = \frac{\sqrt{(mn' - m'n)^2 + (nl' - n'l)^2 + (lm' - l'm)^2}}{ll' + mm' + nn'}$$

$$= \frac{2\sqrt{f^2 + g^2 + h^2 - ab - bc - ca}}{a + b + c}.$$

The planes will be at right angles if

$$ll' + mm' + nn' = 0 \iff a + b + c = 0.$$

Ex. *Prove that the equation* $2x^2 - 6y^2 - 12z^2 + 18yz + 2zx + xy = 0$ *represents a pair of planes. Also find the angle between them.*

Sol. $a = 2, b = -6, c = -12, f = 9, g = 1, h = 1/2.$ These values satisfy the condition

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0.$$

$$\tan\theta = \frac{2\sqrt{f^2 + g^2 + h^2 - ab - bc - ca}}{a + b + c} = \frac{2\sqrt{185}}{2(-16)} = -\frac{\sqrt{185}}{16}$$

EXERCISES

Show that the following equations represent pairs of planes and also find the angles between each pair.

(i) $12x^2 - 2y^2 - 6z^2 - 2xy + 7yz + 6zx = 0.$ [**Ans.** $\cos^{-1}(4/21)$]

(ii) $2x^2 - 2y^2 + 4z^2 + 6xz + 2yz + 3xy = 0.$ [**Ans.** $\cos^{-1}(4/9)$]

(iii) Show that the equation

$$\frac{a}{y - z} + \frac{b}{z - x} + \frac{c}{x - y} = 0$$

represents a pair of planes.

2.9 ORTHOGONAL PROJECTION ON A PLANE

Corresponding to the notion of projection on a line, we also have that of projection on a plane whcih we now proceed to consider.

Def. Orthogonal projection on a plane. *The foot of the perpendicular from a point to a given plane is called the orthogonal projection of the point on the plane.*

This plane on which we project is called the plane of the projection.

Thus, (Fig. 1, page 1) *L, M, N* are respectively the orthogonal projections of the point *P* on the *YZ, ZX* and *XY* planes.

The proejction of a curve on a plane is the locus of the projections on the plane of any point on the curve.

The projection on a given plane of the area enclosed by a plane curve is the area enclosed by the projection of the curve on the plane.

In particular, the projection of a straight line on a given plane is the locus of the feet of the perpendiculars drawn from points on the line on the plane.

2.9.1

The following simple results in *Pure Solid Geometry* are assumed without proof :

(1) The projection of a straight line is a straight line.

(2) If a line *AB* in a plane be perpendicular to the line of intersection of this plane with the plane of projection, then the length of its projection is $AB\cos\theta; \theta$ being the angle between the two planes.

In case a segment *AB* is parallel to the plane of projection, then the length of the projection is the same as that of *AB*.

(3) The projection of the area, *A*, enclosed by a curve in a plane is $A\cos\theta; \theta$ being the angle between the plane of the curve containing the given area and the plane of projection.

Theorem. *If* A_x, A_y, A_z *be the areas of the projections of an area, A, on the three co-ordinate planes, then*

$$A^2 = A_x^2 + A_y^2 + A_z^2$$

Let l, m, n be the direction cosines of the normal to the plane of the area A.

Since l is the cosine of the angle between the YZ plane and the plane of the area A, therefore,

$$A_x = lA.$$

Similarly, $$A_y = mA,$$

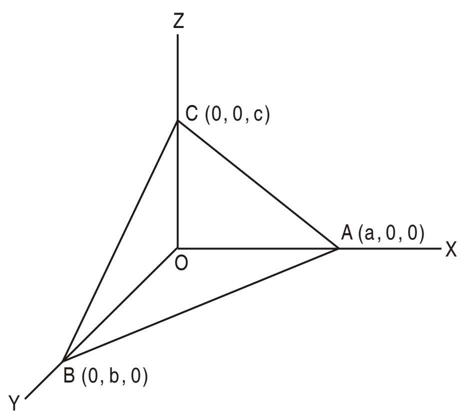
and $$A_z = nA.$$

Hence $$A_x^2 + A_y^2 + A_2^2 = A^2\,(l^2 + m^2 + n^2) = A^2.$$

EXAMPLE

A plane makes intercepts $OA = a, OB = b$ *and* $OC = c$ *respectively on the co-ordinate axes. Find the area of* $\Delta\, ABC$.

Sol. Co-ordinates of the points A, B, C are $(a, 0, 0), (0, b, 0)$ and $(0, 0, c)$. Now, if A_x, A_y, A_z be the projections of the area of $\Delta\, ABC$ on the planes $x = 0, y = 0, z = 0$ respectively, then



$$A_x = \text{area of } \Delta\, OBC$$

$$= \frac{1}{2}\, OB \cdot OC = \frac{1}{2}\, bc$$

Similarly,

$$A_y = \frac{1}{2}\, ac \text{ and } A_z = \frac{1}{2}\, ab.$$

$\therefore$ Area of

$$\Delta\, ABC = \sqrt{A_x^2 + A_y^2 + A_z^2}$$

$$= \frac{1}{2}\sqrt{a^2b^2 + b^2c^2 + c^2a^2}.$$

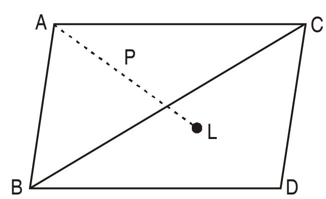
EXERCISES

1. Find the areas of the triangles whose vertices are the points :
   (i) $(a, 0, 0), (0, b, 0), (0, 0, c)$. (ii) $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3)$.
2. From a point $P(x', y', z')$ a plane is drawn at right angles to OP to meet the co-ordinate axes at A, B, C; prove that the area of the triangle ABC is $r^5 / 2x'y'z'$, where r is the measure of OP.

2.10 VOLUME OF A TETRAHEDRON

To find the volume of a tetrahedron in terms of the co-ordinates

$(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4)$ *of its vertices A, B, C, D.*



Let V be the volume of the tetrahedron $ABCD$

Then

$$V = \frac{1}{3}\,\Delta\, p, \qquad \text{...(i)}$$

where p is the length of the perpendicular AL from a vertex A to the opposite face BCD; and Δ is the area of the triangle BCD.

The equation of the plane BCD is

$$\begin{vmatrix} x & y & z & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

$$\Leftrightarrow \quad x\begin{vmatrix} y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \\ y_4 & z_4 & 1 \end{vmatrix} - y\begin{vmatrix} x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \\ x_4 & z_4 & 1 \end{vmatrix} + z\begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix} - \begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix} = 0 \qquad \text{...(i)}$$

$$\therefore \quad \frac{x_1\begin{vmatrix} y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \\ y_4 & z_4 & 1 \end{vmatrix} - y_1\begin{vmatrix} x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \\ x_4 & z_4 & 1 \end{vmatrix} + z_1\begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix} - \begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix}}{\left\{\begin{vmatrix} y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \\ y_4 & z_4 & 1 \end{vmatrix}^2 + \begin{vmatrix} x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \\ x_4 & z_4 & 1 \end{vmatrix}^2 + \begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}^2\right\}^{1/2}} \qquad \text{...(iii)}$$

The numerator of $p = \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$.

If $\Delta_x, \Delta_y, \Delta_z$ be the areas of the projections of the triangle on the YZ, ZX, XY planes respectively, we obtain

$$2\Delta_x = \begin{vmatrix} y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \\ y_4 & z_4 & 1 \end{vmatrix}, 2\Delta_y = \begin{vmatrix} x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \\ x_4 & z_4 & 1 \end{vmatrix}, 2\Delta_3 = \begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x_4 & y_4 & 1 \end{vmatrix}$$

Therefore the denominator of $p = [4(\Delta_x^2 + \Delta_y^2 + \Delta_z^2)]^{1/2} = 2\Delta$.

From (i) and (ii), we deduce that the required volume is

$$\frac{1}{3}\Delta p = \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}.$$

EXAMPLES

1. *Prove that the four planes* $my + nz = 0, nz + lx = 0, lx + my = 0, lx + my + nz = p$ *form a tetrahedron whose volume is* $\dfrac{2p^3}{3lmn}$.

Sol. Solving the given equations taking three planes at a time, we get the vertices of the tetrahedron as

$$(0, 0, 0), \left(\frac{-p}{l}, \frac{p}{m}, \frac{p}{n}\right), \left(\frac{p}{l}, \frac{-p}{m}, \frac{p}{n}\right) \text{ and } \left(\frac{p}{l}, \frac{p}{m}, \frac{-p}{n}\right).$$

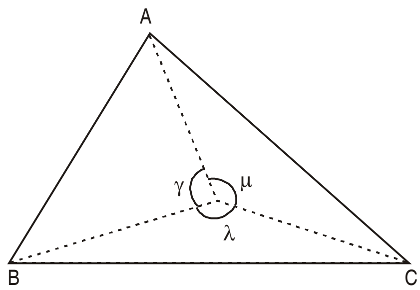
With these points as vertices, the volume V of the terahedron is given by

$$V = \frac{1}{6}\begin{vmatrix} 0 & 0 & 0 & 1 \\ -p/l & p/m & p/n & 1 \\ p/l & -p/m & p/n & 1 \\ p/l & p/m & -p/n & 1 \end{vmatrix} = \frac{-p^3}{6lmn}\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= \frac{p^3}{6lmn}(4) = \frac{2}{3}\frac{p^3}{lmn}.$$

2. *Find the volume of a tetrahedron in terms of the lengths of the three edges which meet in a point and of the angles which these edges make with each other in pairs.*

Sol. Let *OABC* be a tetrahedron.



Let $OA = a, OB = b, OC = c.$

Let $\angle BOC = \lambda, \angle COA = \mu, \angle AOB = \nu.$

We take *O* as origin and any system of three mutually perpendicular lines through *O* as co-ordinate axes.

Let the direction cosines of the lines *OA, OB, OC* be

$$l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3.$$

Thus, the co-ordinates of *A, B, C* are

$$(l_1a, m_1a, n_1a); (l_2b, m_2b, n_2b); (l_3c, m_3c, n_3c)$$

Therefore, the volume of the tetrahedron *OABC*

$$= \frac{1}{6}\begin{vmatrix} 0 & 0 & 0 & 1 \\ l_1a & m_1a & n_1a & 1 \\ l_2b & m_2b & n_2b & 1 \\ l_3c & m_3c & n_3c & 1 \end{vmatrix} = \frac{1}{6}\begin{vmatrix} l_1a & m_1a & n_1a \\ l_2b & m_2b & n_2b \\ l_3c & m_3c & n_3c \end{vmatrix} = \frac{abc}{6}\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$$

Now
$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}^2 = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \times \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix}$$

$$= \begin{vmatrix} \Sigma l_1^2 & \Sigma l_1 l_2 & \Sigma l_1 l_2 \\ \Sigma l_1 l_2 & \Sigma l_2^2 & \Sigma l_2 l_3 \\ \Sigma l_3 l_1 & \Sigma l_3 l_2 & \Sigma l_3^2 \end{vmatrix} = \begin{vmatrix} 1 & \cos\nu & \cos\mu \\ \cos\nu & 1 & \cos\lambda \\ \cos\mu & \cos\lambda & 1 \end{vmatrix}$$

Thus, the volume of the tetrahedron *OABC*

$$= \frac{abc}{6}\begin{vmatrix} 1 & \cos\nu & \cos\mu \\ \cos\nu & 1 & \cos\lambda \\ \cos\mu & \cos\lambda & 1 \end{vmatrix}^{1/2}$$

EXERCISES

1. The vertices of a tetrahedron are (0, 1, 2), (3, 0, 1), (4, 3, 6), (2, 3, 2); show that its volume is 6.

2. *A, B, C* are three fixed points and a variable point *P* moves so that the volume of the tetrahedron *PABC* is constant; show that the locus of the point *P* is a plane parallel to the plane *ABC*.

3. A variable plane makes with the co-ordinate planes a tetrahedron of constant volume $64k^3$. Find

(i) the locus of the centroid of the tetrahedron. [**Ans.** $xyz = 6k^3$]

(ii) the locus of the foot of the perpendicular from the origin to the plane.

[**Ans.** $(x^2 + y^2 + z^2)^3 = 384k^3xyz$]

4. If the volume of the tetrahedron whose vertices are $(a, 1, 2), (3, 0, 1), (4, 3, 6),$ (2, 3, 2) is 6. Find the value of *a*. [**Ans.** 0]

5. Find the volume of the tetrahedron formed by planes whose equations are

$$y + z = 0, z + x = 0, x + y = 0 \text{ and } x + y + z = 1.$$ [**Ans.** 2/3]

❑❑❑

3

The Straight Line

3.1 REPRESENTATION OF LINE

In this chapter, it is proposed to discuss the manner in which a straight line can be represented. We introduce the method analytically as follows :

Consider any two of the co-ordinate planes say *YOZ* and *ZOX,* whose equations are $x = 0$ and $y = 0$ respectively. These two planes intersect in *Z*-axis.

A point (x, y, z) lies on the *Z*-plane

$\Leftrightarrow$ {the point (x, y, z) lies on the *YOZ* plane **and** the point (x, y, z) lies on the *ZOX* plane

$\Leftrightarrow$ $x = 0$ **and** $y = 0.$

Thus, we see that a point (x, y, z) lies on the *Z*-axis if and only if, we simultaneously have $x = 0,\ y = 0.$ We are thus, led to say that $x = 0,\ y = 0$ are the two equations of *Z*-axis.

Consider now any line whatsoever and any two planes through the line. Let

$$ax + by + cz + d = 0 \text{ and } a_1 x + b_1 y + c_1 z + d_1 = 0$$

be the equations of these two planes. Clearly, we have the following statement :

A point (x, y, z) lies on the given line if and only if we simultaneously have

$$ax + by + cz + d = 0 \textbf{ and } a_1 x + b_1 y + c_1 z + d_1 = 0$$

Thus, we say that

$$ax + by + cz + d = 0 \textbf{ and } a_1 x + b_1 y + c_1 z + d_1 = 0$$

are the two equations of the line.

It follows that a straight line is represented by **two** *equatons of the first degree in x, y, z.*

Of course any given line can be represented by *different pairs* of first degree equations, for we may take *any* pair of planes through the line and the equations of the same will constitute the equations of the line.

In particular, as the *X*-axis is the intersection of the *XZ* and *XY* planes, $y = 0, z = 0$ taken together are its equations. Similarly $x = 0, z = 0$ are the equations of the *Y*-axis and $x = 0, y = 0$ are the equations of the *Z*-axis.

EXERCISES

1. What is the locus of the point (x, y, z) which satisfies the following conditions :

(i) $2x + 3y - 4z + 1 = 0$ and $3x - y + z + 2 = 0$

(ii) $2x + 3y - 4z + 1 = 0$ or $3x - y + z + 2 = 0$

(iii) $2x - 3y + 5z + 4 = 0$ and $2x + y + z - 8 = 0$

(iv) $2x - 3y + 5z + 4 = 0$ or $2x + y + z - 8 = 0.$

2. Find the intersection of the line

$$x - 2y + 4z + 4 = 0,\ x + y + z - 8 = 0.$$

with the plane

$$x - y + 2z + 1 = 0.$$

[**Ans.** 2, 5, 1]

3.1.1. Equaton of the Line Through a Given Point Drawn in a Given Direction

To find the equations of the line passing through a given point $A(x_1, y_1, z_1)$ *and having direction cosiens* $l, m, n.$

Let $P(x, y, z)$ be a point on the given line and let $AP = r.$

Projecting the segment AP on the co-ordinate axes, we obtain

$$x - x_1 = lr,\ y - y_1 = mr,\ z - z_1 = nr \qquad \text{...(i)}$$

so that for all points (x, y, z) on the given line, we have

$$x = x_1 + lr,\ y = y_1 + mr,\ z = z_1 + nr.$$

Thus, the set of points on the given line is

$$\{(x_1 + lr,\ y_1 + mr,\ z_1 + nr)\};$$

r being any number.

In case none of l, m, n is zero, we have

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r.$$

Thus, if $l \neq 0, m \neq 0, n \neq 0,$ or equivalently $lmn \neq 0,$

$$\boldsymbol{\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}} \qquad \text{...(ii)}$$

are the *two* required equations of the line.

Clearly the equations (ii) of the line are not altered if we replace the direction cosines l, m, n by the three numbers proportional to them, so that it suffices to use direction ratios in place of direction cosines while writing down the equation of a line.

Cor. *From the relation (i),* we have

$$x = x_1 + lr,\ y = y_1 + mr,\ z = z_1 + nr$$

so that the set of points on the line through the point (x_1, y_1, z_1) and having direction ratios l, m, n is

$$\{(x_1 + lr,\ y_1 + mr,\ z_1 + nr);\ r \text{ being any number}\}.$$

This statement does not depend upon the vanishing or otherwise of any of l, m, n

We may remark that r is what is known as the parameter here.

Note. The equation

$$\frac{x - x_1}{l} = \frac{y - y_1}{m}$$

of first degree, being free of z, represents the plane through the line drawn perpendicular to the XOY plane. Similar statements may be made about the equations

$$\frac{y - y_1}{m} = \frac{z - z_1}{n},\ \frac{z - z_1}{n} = \frac{x - x_1}{l}.$$

The two equations

$$(x - x_1)/l = (y - y_1)/m,\ (y - y_1)/m = (z - z_1)/n$$

represents a pair of planes through the given line.

3.1.2. Equation of a Line Through Two Points

To find the equations of the line through two points

$$(x_1, y_1, z_1) \text{ and } (x_2, y_2, z_2)$$

Since $x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1$

are proportioanl to the direction cosines of the line, the required equations are

$$\boldsymbol{\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}}$$

Here we have assumed that none of

$$x_2 - x_1,\ y_2 - y_1,\ z_2 - z_1$$

is zero.

EXAMPLES

1. *If the axes are rectangular and if $l_1, m_1, n_1; l_2, m_2, n_2$ are direction cosines, show that the equations to the planes through the lines which bisect the angle between*

$$x/l_1 = y/m_1 = z/n_1;\ x/l_2 = y/m_2 = z/n_2$$

and at right angles to the plane containing them are

$$(l_1 \pm l_2)\, x + (m_1 \pm m_2)\, y + (n_1 \pm n_2)\, z = 0.$$

Sol. The given lines pass through the origin. Co-ordinates of any two points, each of them at a ditance r from the origin are (rl_1, rm_1, rn_1) and (rl_2, rm_2, rn_2). The co-ordinates of the middle point P of the line joining these two points are $\frac{1}{2} r\,(l_1 + l_2), \frac{1}{2} r\,(m_1 + m_2), \frac{1}{2} r\,(n_1 + n_2)$.

The point P clearly lies on one of the bisectors and since the two bisectors are at right angles to each other, hence, OP is normal to the plane passing throguh the other bisectors. The d.c.s of OP are proportional to

$$\frac{1}{2}(l_1 + l_2), \frac{1}{2}(m_1 + m_2), \frac{1}{2}(n_1 + n_2).$$

Hence, one of the required planes is

$$\frac{1}{2}(l_1 + l_2)\, x + \frac{1}{2}(m_1 + m_2)\, y + \frac{1}{2}(n_1 + n_2)\, z = 0.$$

i.e., $$(l_1 + l_2)\, x + (m_1 + m_2)\, y + (n_1 + n_2)\, z = 0.$$

Similarly, if P lies on the other bisector, its co-ordiantes will then be

$$\frac{1}{2}(l_1 - l_2)\, r, \frac{1}{2}(m_1 - m_2)\, r, \frac{1}{2}(n_1 - n_2)\, r.$$

The corresponding plane, therefore, will be

$$(l_1 - l_2)\, x + (m_1 - m_2)\, y + (n_1 - n_2)\, z = 0.$$

2. *Find the image of the point* $P\,(1, 3, 4)$ *in the plane*

$$2x - y + z + 3 = 0.$$

Sol. If two points P, Q be such that the line is bisected perpendicularly by a plane, then either of the points is the image of the other in the plane.

the line throguh P perpendicular to the given plane is

$$\frac{x-1}{2} = \frac{y-3}{-1} = \frac{z-4}{1},$$

so that the co-ordinates of Q are of the from

$$(2r + 1, -r + 3, r + 4).$$

Making use of the fact that the mid-point

$$\left(r + 1, -\frac{1}{2} r + 3, \frac{1}{2} r + 4\right)$$

of PQ lies on the given plane, we see that

$$r = -2$$

so that the image of P is $(-3, 5, 2)$.

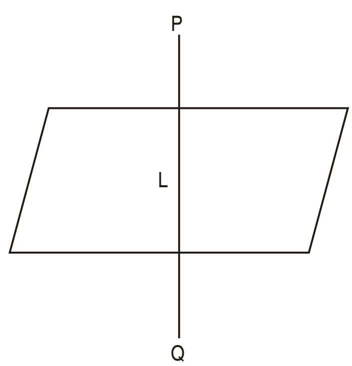


Fig. 1

EXERCISES

1. Find k so that the lines

$$\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$$

$$\frac{x-1}{3k} = \frac{y-5}{1} = \frac{z-6}{-5}$$

may be perpendicular to each other. [**Ans.** – 10/7]

2. Find two points on the line

$$\frac{x-2}{1} = \frac{y+3}{-2} = \frac{z-5}{2}$$

on either side of $(2, -3, -5)$ and at a distance 3 from it. [**Ans.** $(3, -5, -3); (1, -1, -7)$]

3. Find the point where the line joining $(2, -3, 1), (3, -4, -5)$ cuts the plane

$$2x + y + z = 7.$$ [**Ans.** 1, –2, 7]

4. Find the distance of the point $(-1, -5, -10)$ from the point of intersection of the line $\frac{1}{2}(x-2) = \frac{1}{4}(y+1) = \frac{1}{12}(z-2)$ and the plane $x - y + z = 5$.

5. Find the distance of the point $(3, -4, 5)$ from the plane

$$2x + 5y - 6z = 16$$

measured along a line with direction cosines proportional to $(2, 1, -2)$. [**Ans.** 60/7]

6. Find the equations to the line through $(-1, 3, 2)$ and perpendicular to the plane $x + 2y + 2z = 3$, the length of the perpendicular and the co-ordinates of its foot.

[**Ans.** 2; $(-5/3, 5/3, 2/3)$]

7. Find the co-ordinates of the foot of the perpendicular drawn from the origin to the plane $2x + 3y - 4z + 1 = 0$; also find the co-ordinates of the point which is the image of the origin in the plane. [**Ans.** $(-2/29, -3/29, 4/29)$; $(-4/29, -6/29, 8/29)$]

8. Find the equations to the line thrrough (x_1, y_1, z_1) perpendicular to the plane $ax + by + cz + d = 0$ and the co-ordinates of its foot. Deduce the expression for the perpendicular distance of the given point from the given plane.

[**Ans.** $(ar + x_1, br + y_1, cr + z_1)$, where $r = -(ax_1 + by_1 + cz_1 + d)/(a^2 + b^2 + c^2)$]

9. Show that the line

$$\frac{1}{2}(x-7) = -(y+3) = (z-4)$$

intersects the planes

$$6x + 4y - 5z = 4 \quad \text{and} \quad x - 5y + 2z = 12$$

in the same point and deduce that the line is coplanar with the line of intersection of the plane.

10. P is a point on the plane $lx + my + nz = p$ and a point Q is taken on the line OP such that $OP.\,OQ = p^2$; show that the locus of the point Q is $p\,(lx + my + nz) = x^2 + y^2 + z^2$.

11. A variable plane makes intercepts on the co-ordinate axes the sum of whose squares is constant and equal to k^2. Find the locus of the foot of the perpendicular from the origin to the plane. [**Ans.** $(x^{-2} + y^{-2} + z^{-2})(x^2 + y^2 + z^2)^2 = k^2$]

12. Show that the equations of the lines bisecting the angles between the lines

$$\frac{x-3}{2} = \frac{y+4}{-1} = \frac{z-5}{-2}, \frac{x-3}{4} = \frac{y+4}{-12} = \frac{z-5}{3}$$

are

$$\frac{x-3}{38} = \frac{y+4}{-49} = \frac{z-5}{-17}, \frac{x-3}{14} = \frac{y+14}{23} = \frac{z-5}{-35}.$$

3.1.3. Two Forms of the Equation of a Line

It has been seen in 3.1.1, 3.1.2, that the equations of a straight line which we generally employ are of two forms.

One is the form deduced from the consideration that a straight line is completely determined when we know its direction ratios and the co-ordinates of any one point on it, or when any two points on the line are given. This is sometimes referred to as the **Symmetrical form** of the equations of a line.

The second form a is deduced from the consideration that a straight line is the locus of points common to any two planes through it. This is sometimes referred to as the **Unsymmetrical form** of the equations of a line.

In fact the symmetrical form takes note only of a special pair of planes through this line, *viz.*, the pair of planes through the line perpendicular to two of the co-ordinate planes.

In the next section, it will be seen how one form of equations can be transferred into the other.

3.1.4. Transformation from the Unsymmetrical to the Symmetrical Form

To transform the equations

$$ax + by + cz + d = 0,\ a_1x + b_1y + c_1z + d_1 = 0,$$

of a line to the symmetrical form.

To transform these equations to the symmetrical form, we require :

(i) the direction ratios of the line, and

(ii) the co-ordinates of any one point on it.

Let *l, m, n* be the direction ratios of the line. Since the line lies in both the planes

$$ax + by + cz + d = 0 \text{ and } a_1x + b_1y + c_1z + d_1 = 0,$$

it is perpendicular to the normals to both of them. The direction ratios of the normals to the planes being

$$a, b, c; a_1, b_1, c_1,$$

we have

$$\begin{cases} al + bm + cn = 0, \\ a_1l + b_1m + c_1n = 0, \end{cases}$$

$$\Rightarrow \quad \frac{l}{bc_1 - b_1c} = \frac{m}{ca_1 - c_1a} = \frac{n}{ab_1 - a_1b}.$$

Now, we require the co-ordinates of *any one* point on the line and there is an infinite number of points from which to choose. We, for the sake of convenience, find the point of intersection of the line with the plane $z = 0$. This point which is given by the equations

$$ax + by + d = 0 \text{ and } a_1x + b_1y + d_1 = 0,$$

is

$$\left(\frac{bd_1 - b_1d}{ab_1 - a_1b}, \frac{a_1d - ad_1}{ab_1 - a_1b}, 0\right).$$

Thus, in the symmetrical form, the equations of the given line are

$$\frac{x - (bd_1 - b_1d)/(ab_1 - a_1b)}{bc_1 - b_1c} = \frac{y - (a_1d - ad_1)/(ab_1 - a_1b)}{ca_1 - c_1a} = \frac{z - 0}{ab_1 - a_1b}.$$

EXAMPLES

1. *Find the equation of the line through the point* $(1, 2, 3)$ *parallel to the line*

$$x - y + 2z = 5, 3x + y + z = 6.$$

Sol. Let *l, m, n* be the direction ratios of the required line. Since it is parallel to the given line, the direction ratios of the given line are also *l, m, n*. But the given line is the intersection of the two planes $x - y + 2z = 5$ and $3x + y + z = 6,$ and hence, lies in both the planes and is perpendicular to the normals of these planes.

$$l \cdot 1 - m \cdot 1 + n \cdot 2 = 0$$

and $$l \cdot 3 + m \cdot 1 + n \cdot 1 = 0$$

$$\Rightarrow \quad \frac{l}{-3} = \frac{m}{5} = \frac{n}{4}$$

Thus, the equations of the line in symmetrical form are

$$\frac{x - 1}{-3} = \frac{y - 2}{5} = \frac{z - 3}{4}.$$

2. *Prove that the equations to the line through* (α, β, γ) *at right angles to the lines*

$$\frac{x}{l_1} = \frac{y}{m_1} = \frac{z}{n_1}; \frac{x}{l_2} = \frac{y}{m_2} = \frac{z}{n_2}$$

are $$\frac{x - \alpha}{m_1n_2 - m_2n_1} = \frac{y - \beta}{n_1l_2 - n_2l_1} = \frac{z - \gamma}{l_1m_2 - l_2m_1}.$$

Sol. Let the dc's of the required line be l, m, n. Since it is perpendicular to the given lines, hence

$$ll_1 + mm_1 + nn_1 = 0 \text{ and } ll_2 + mm_2 + nn_2 = 0.$$

Solving, we get

$$l/(m_1n_2 - m_2n_1) = m/(n_1l_2 - n_2l_1) = n/(l_1m_2 - l_2m_1).$$

Hence, the eqations of the required line are

$$\frac{x-\alpha}{(m_1n_2 - m_2n_1)} = \frac{y-\beta}{(n_1l_2 - n_2l_1)} = \frac{z-\gamma}{(l_1m_2 - l_2m_1)}.$$

EXERCISES

1. Find, in a symmetrical form, the equations of the line

$$x + y + z = 1 = 0, 4x + y - 2z + 2 = 0$$

and find its direction cosines. $\left[\textbf{Ans. } \frac{x+1/3}{1} = \frac{y+2/3}{-2} = \frac{z}{1}; \frac{1}{\sqrt{6}} - \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right]$

2. Obtain the symmetrical form of the equations of the line

$$x - 2y + 3z = 4, 2x - 3y + 4z = 5.$$

$\left[\textbf{Ans. } (x+2) = \frac{1}{2}(y+3) = z\right]$

3. Find the points of intersection of the line

$$x + y - z + 1 = 0 = 14x + 9y - 7z - 1$$

with the XY and YZ planes, and hence put down the symmetrical form of its equations.

[**Ans.** $-(x)/2 = (y-4)/7 = (z-5)/5$]

4. Find the equation of the plane through the point (1, 1 1) and perpendicular to the line

$$x - 2y + z = 2, 4x + 3y - z + 1 = 0.$$

[**Ans.** $x - 5y - 11z + 15 = 0$]

5. Find the equation of the line through the point (1, 2, 4) parallel to the line

$$3x + 2y - z = 4, x - 2y - 2z = 5.$$

[**Ans.** $(x-1)/6 = (2-y)/5 = (z-4)/8$]

6. Find the angle between the lines in which the planes

$$3x - 7y - 5z = 1, 5x - 13y + 3z + 2 = 0$$

cut the plane $8x - 11y + z = 0.$ [**Ans.** 90°]

7. Find the angle between the lines

$$3x + 2y + z - 5 = 0 = x + y - 2z - 3,$$
$$2x - y - z = 0 = 7x + 10y - 8z.$$

[**Ans.** 90°]

8. Show that the condition for the lines

$$x = az + b,\ y = cz + d;\ x = a_1z + b_1,\ y = c_1z + d_1,$$

to be perpendicular is

$$aa_1 + cc_1 + 1 = 0.$$

3.2 ANGLE BETWEEN A LINE AND A PLANE

To find the angle between the line

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$$

and the plane

$$ax + by + cz + d = 0.$$

The angle between a line and a plane is the complement of the angle between the line and the normal to the plane.

Since the direction cosines of the normal to the given plane and of the given line are proportional to a, b, c and l, m, n respectively, we have

$$\sin\theta = \frac{al + bm + cn}{\sqrt{(a^2 + b^2 + c^2)}\sqrt{(l^2 + m^2 + n^2)}},$$

where θ is the required angle.

The straight line is *parallel to the plane*

$$\Rightarrow \qquad \theta = 0$$

$$\Rightarrow \qquad \boldsymbol{al + bm + cn = 0.}$$

This codition is also evident from the fact that a *line will be parallel to a plane if and only if it is perpendicular to the normal to it.*

EXERCISES

1. Show that the line $\frac{1}{3}(x-2) = \frac{1}{4}(y-3) = \frac{1}{5}(z-4)$ is parallel to the plane

$$2x + y - 2z = 3.$$

2. Find the equations of the line through the point $(-2, 3, 4)$ and parallel to the planes $2x + 3y + 4z = 5$ and $3x + 4y + 5z = 6.$

$\left[\textbf{Ans. } (x+2) = -\frac{1}{2}(y-3) = (z-4)\right]$

[**Hint.** The direction ratios, *l, m, n* of the line are given by the relations $2l + 3m + 4n = 0 = 3l + 4m + 5n$.]

3. Find the equation of the plane through the points

$$(1, 0, -1), (3, 2, 2)$$

and parallel to the line.

$$(x-1) = (1-y)/2 = (z-2)/3.$$

[**Ans.** $4x - y - 2z = 6$]

4. Show that the equations of the plane parallel to the join of

$$(3, 2, -5) \text{ and } (0, -4, -11)$$

and passing throguh the points

$$(-2, 1, -3) \text{ and } (4, 3, 3)$$

is

$$4x + 3y - 5z = 10.$$

5. Find the equation of the plane containing the line

$$2x - 5y + 2z = 6, 2x + 3y - z = 5$$

and parallel to the line $x = -y/6 = z/7.$ [**Ans.** $6x + y - 16 = 0$]

6. Show that the equation of the plane through the line

$$u_1 \equiv a_1x + b_1y + c_1z + d_1 = 0, u_2 \equiv a_2x + b_2y + c_2z + d_2 = 0$$

and parallel to the line

$$x/l = y/m = z/n$$

is

$$u_1(a_2l + b_2m + c_2n) = u_2(a_1l + b_1m + c_1n).$$

7. Find the equation of the plane through the point (f, g, h) and parallel to the lines $x/l_r = y/m_r = z/n_r; r = 1, 2.$ [**Ans.** $\Sigma(x-f)(m_1n_2 - m_2n_1) = 0$]

8. Find the equations of the two planes through the origin which are parallel to the line

$$(x-1)/2 = -(y+3) = -(z+1)/2$$

and distant 5/3 from it; show that the two planes are perpendicular.

[**Ans.** $2x + 2y + z = 0, x - 2y + 2z = 0$]

3.3 CONDITIONS FOR A LINE TO LIE IN A PLANE

To find the conditions for the line

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

to lie in the plane

$$ax + by + cz + d = 0.$$

The line would lie in the given plane if and only if every point of the line is a point of the plane, *i.e.,* the point

$$(lr + x_1, mr + y_1, nr + z_1)$$

lies on the plane for all values of r implying that the equation

$$r(al + bm + cn) + (ax_1 + by_1 + cz_1 + d) = 0$$

is true for every value of r.

This implies that

$$\begin{cases} \mathbf{al + bm + cn = 0} \\ \mathbf{ax_1 + by_1 + cz_1 + d = 0} \end{cases}$$

which are the required *two* conditions.

These coditions, when geometrically interpreted, state that a line lies in a given plane, if

(*i*) *the normal to the plane is perpendicular to the line, and*

(*ii*) *any one point on the line lies in the plane.*

Cor. *The general equation of a plane containing the line*

$$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n} \qquad ...(i)$$

is $$A(x - x_1) + B(y - y_1) + C(z - z_1) = 0$$

where $$Al + Bm + Cn = 0. \qquad ...(ii)$$

In other words, the set of planes containing the line (i) is

$$\{A(x - x_1) + B(y - y_1) + C(z - z_1) = 0,\ Al + Bm + Cn = 0\}$$

EXAMPLES

1. *Prove that the plane through* (α, β, γ) *and the line* $x = py + q = rz + s$ *is given by*

$$\begin{vmatrix} x & py + q & rz + s \\ \alpha & p\beta + q & r\gamma + s \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

Sol. The given line can be written as

$$x/1 = y + q/p/1/p = z + s/r/1/r \qquad ...(1)$$

Let equation of any plane be

$$Ax + By + Cz + D = 0 \qquad ...(2)$$

It will pass through line (1), if

$$A.0 + B(-q/p) + C(-s/r) + D = 0 \qquad ...(3)$$

and $$A \cdot 1 + B \cdot 1/p + C \cdot 1/r = 0 \qquad ...(4)$$

The plane will pass through (α, β, γ) if

$$A \cdot \alpha + B \cdot \beta + C \cdot \gamma + D = 0 \qquad ...(5)$$

Subtracting (3) from (2) and (5), we get

$$Ax + B(y + q/p) + C(z + s/r) = 0 \qquad ...(6)$$

$$A \cdot \alpha + B(\beta + q/p) + C(\gamma + s/r) = 0 \qquad ...(7)$$

Eliminating *A, B, C* from (6), (7) and (4)

$$\begin{vmatrix} x & y + q/p & z + s/r \\ \alpha & \beta + q/p & \gamma + s/r \\ 1 & 1/p & 1/r \end{vmatrix} = 0$$

$$\Rightarrow \quad \begin{vmatrix} x & py + q & rz + s \\ \alpha & p\beta + q & r\gamma + s \\ 1 & 1 & 1 \end{vmatrix} = 0.$$

2. *The axes are rectangular and the plane* $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ *meets them in A, B, C. Prove that the equations to BC are* $x/0=y/0=z-c/-c$; *that the equation to the plane through OX at right angles to BC is* $by=cz$; *that the three planes throguh OX, OY, OZ at right angles to BC, CA, AB respectively pass through the line* $ax=by=cz$; *and that the co-ordinates of the orthocentre of the triangle ABC are*

$$\left[\frac{a^{-1}}{a^{-2}+b^{-2}+c^{-2}},\frac{b^{-1}}{a^{-2}+b^{-2}+c^{-2}},\frac{c^{-1}}{a^{-2}+b^{-2}+c^{-2}}\right].$$

Sol. The given plane meets the axes in points $A\,(a, 0, 0)$; $B\,(0, b, 0)$ and $C\,(0, 0, c)$. Equations of the line through B and C are

$$\frac{x}{0}=\frac{y}{b}=\frac{z-c}{-c} \qquad \text{...(1)}$$

Equation of any plane through OX is

$$y+\lambda z=0.$$

If BC is perpendicular to above plane,

then, $\qquad b/1=-c/\lambda \Rightarrow \lambda=-c/b$

Hence, the plane is $\qquad by=cz.$

Similarly the planes through OY and OZ and at right angles to CA and AB respectively are

$$cz=ax,\ ax=by.$$

Hence, the three planes pass through the line

$$ax=by=cz. \qquad \text{...(2)}$$

The orthocentre of the triangle ABC lies where the line (2) meets the given plane.

Any point on (2) is $(r/a, r/b, r/c)$.

If it lies on the given plane, then

$$r=\frac{1}{a^{-2}+b^{-2}+c^{-2}}.$$

Hence, the co-ordinates of the orthocentre are

$$\left[\frac{a^{-1}}{a^{-2}+b^{-2}+c^{-2}},\frac{b^{-1}}{a^{-2}+b^{-2}+c^{-2}},\frac{c^{-1}}{a^{-2}+b^{-2}+c^{-2}}\right].$$

EXERCISES

1. Show that the line $x+10=(8-y)/2=z$ lies in the plane

$$x+2y+3z=6$$

and the line

$$\frac{1}{3}(x-2)=-(y+2)=\frac{1}{4}(z-3) \text{ in the plane}$$

$$2x+2y-z+3=0.$$

2. Find the equation of the plane containing the line

$$\frac{1}{2}(x+2)+\frac{1}{3}(y+3)=-\frac{1}{2}(z-4)$$

and the point (0, 6, 0). **[Ans.** $3x+2y+6z-12=0$]

3. $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ and $\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$ and two straight lines. Find the equation of the plane containing the first line and parallel to the second.

[Ans. $\Sigma(x-x_1)(m_1n_2-m_2n_1)=0$]

4. Find the equation to the plane containing the line $y/b + z/c = 1, x = 0$ and parallel to the line $x/a + z/c = 1, y = 0$. [**Ans.** $x/a - y/b - z/c + 1 = 0$]

5. Find the equation to the plane which passes through the z-axis and is perpendicular to the line

$$\frac{x-1}{\cos\theta} = \frac{y+2}{\sin\theta} = \frac{z-3}{0}.$$ [**Ans.** $x\cos\theta + y\sin\theta = 0$]

6. Show that the equation of the plane which passes through the line

$$\frac{x-1}{3} = \frac{y+6}{4} = \frac{z+1}{2}$$

and is parallel to the line

$$\frac{x-2}{2} = \frac{y-1}{-3} = \frac{z+4}{5},$$

is $26x - 11y - 17z - 109 = 0$ and show that the point $(2, 1, -4)$ lies on it. What is the geometrical relation between the two lines and the plane ?

7. Find the equation of the plane containing the line

$$-\frac{1}{3}(x+1) = \frac{1}{2}(y-3) = (z+2)$$

and the point $(0, 7, -7)$ and show that the line

$$x = \frac{1}{3}(7-y) = \frac{1}{2}(z+7)$$

lies in the same plane. [**Ans.** $x + y + z = 0$]

3.4 COPLANAR LINES, CONDITION FOR THE COPLANARITY OF LINES

To find the condition that two given straight lines

$$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1} \quad ...(1)$$

$$\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2} \quad ...(2)$$

are coplanar.

Sol. First Method : Equation of *any* plane containing the line (1) is

$$A(x-x_1) + B(y-y_1) + C(z-z_1) = 0; \quad ...(i)$$

A, B, C being numbers not all zero satisfying the condition

$$Al_1 + Bm_1 + Cn_1 = 0. \quad ...(ii)$$

The plane (i) will contain the line (2) if

(a) the point (x_2, y_2, z_2) lies on it

$$\Rightarrow \quad A(x_2-x_1) + B(y_2+y_1) + C(z_2-z_1) = 0 \quad ...(iii)$$

(b) the lines is perpendicular to the normal to the plane

$$\Rightarrow \quad Al_2 + Bm_2 + Cn_2 = 0. \quad ...(iv)$$

The two lines will be coplanar if the three linear homogeneous equations (ii), (iii), (iv) in *A, B, C* are consistent so that

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0 \quad ...(A)$$

which is thus, the required condition for the lines to intersect. Assuming this condition is satisfied, we see that the required equation of the plane is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0.$$

This is the equation of the plane contianing the two lines.

Second Method. Two lines are coplanar if and only if they intersect or are parallel. We first consider the case of intersection. The condition for intersection may also be obtained as follows :

$$(l_1 r_1 + x_1, m_1 r_1 + y_1, n_1 r_1 + z_1) \quad \text{and} \quad (l_2 r_2 + x_2, m_2 r_2 + y_2, n_2 r_2 + z_2)$$

are the general co-ordinates of the points on the lines (1) and (2) respectively for all values of r_1 and r_2.

In case the lines intersect, these points should coincide for some values of r_1 and r_2. This requires that the following three equations

$$(x_1 - x_2) + l_1 r_1 - l_2 r_2 = 0,$$
$$(y_1 - y_2) + m_1 r_1 - m_2 r_2 = 0,$$
$$(z_1 - z_2) + n_1 r_1 - n_2 r_2 = 0.$$

in r_1, r_2 are consistent, so that we have the condition

$$\begin{vmatrix} x_1-x_2 & l_1 & l_2 \\ y_1-y_2 & m_1 & m_2 \\ z_1-z_2 & n_1 & n_2 \end{vmatrix} = 0 \Leftrightarrow \begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

which is the same condition as (A).

This condition is clearly satisfied if the lines are parallel.

Note 1. In general, the equation

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

represents the plane which passes through the line (1) and is parallel to the line (2), and the equation

$$\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

represents the plane which passes throgh the line (2) and is parallel to the line (1).

In case the lines are coplanar, the condition (A) shows that the point (x_2, y_2, z_2) lies on the first plane and the point (x_1, y_1, z_1) on the second. These two equations are then identical.

Thus, the plane containing two coplanar *lines* is the one which passes through one line and is parallel to the other *or,* through one line and any point on the other.

Note 2. Two lines will intersect if and only if, there exists a point whose co-ordinates satisfy the *four* equations, two of each line so that for intersection, we require that the four linear equations in three unknowns should be *consistent.*

It is sometimes comparatively more convenient to follow this method to obtain the condition of intersection or to prove the fact of intersection of two lines.

Note 3. The condition for the lines whose equations, given in the unsymmetrical form, are

$$a_1 x + b_1 y + c_1 z + d_1 = 0, \; a_2 x + b_2 y + c_2 z + d_2 = 0;$$
$$a_3 x + b_3 y + c_3 z + d_3 = 0, \; a_4 x + b_4 y + c_4 z + d_4 = 0;$$

to intersect, is the condition for the consistency of these four equations, *i.e.,*

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = 0.$$

In case, this condition is satisfied, the co-ordinates of the point of intersection are obtained by solving any three of the four equations simultaneously.

EXAMPLES

1. *Show that the lines*

$$\frac{x+3}{2} = \frac{y+5}{3} = \frac{z-7}{-3}, \frac{x+1}{4} = \frac{y+1}{5} = \frac{z+1}{-1}$$

are coplanar and find the equation of the plane containing them.

Sol. The equation of the plane which contains the first line and is parallel to the second is

$$\begin{vmatrix} x+3 & y+5 & z-7 \\ 2 & 3 & -3 \\ 4 & 5 & -1 \end{vmatrix} = 0 \Leftrightarrow 6x - 5y - z = 0.$$

This plane, as may be easily seen, passes through the point $(-1, -1, -1)$ on the second line so that it also contains the second line.

Thus, the two lines are coplanar and the equation of the plane containing them is

$$6x - 5y - z = 0.$$

2. *If OA, OB, OC have direction ratios $l_r, m_r, n_r, r = 1, 2, 3$ and OA', OB', OC' bisect the angles BOC, COA, AOB, the planes AOA', BOB', COC' pass through the line*

$$\frac{x}{l_1 + l_2 + l_3} = \frac{y}{m_1 + m_2 + m_3} = \frac{z}{n_1 + n_2 + n_3}.$$

Sol. Let O be the origin, equations of OB and OC are

$$y / l_2 = y / m_2 = z / n_2$$

and $$x / l_3 = y / m_3 = z / n_3$$

Points on these lines at unit distance are (l_2, m_2, n_2) and (l_3, m_3, n_3).

Corresponding point on bisector OA' is

$$\left[\frac{1}{2}(l_2 + l_3), \frac{1}{2}(m_2 + m_3), \frac{1}{2}(n_2 + n_3)\right]$$

$\therefore$ Equations of OA' are

$$\frac{x}{l_2 + l_3} = \frac{y}{m_2 + m_3} = \frac{z}{n_2 + n_3}$$

Now, equation of the plane containing OA and OA', *i.e.*, AOA' is

$$\begin{vmatrix} x & y & z \\ l_1 & m_1 & n_1 \\ l_2 + l_3 & m_2 + m_3 & n_2 + n_3 \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x & y & z \\ l_1 & m_1 & n_1 \\ l_1 + l_2 + l_3 & m_1 + m_2 + m_3 & n_1 + n_2 + n_3 \end{vmatrix} \quad \text{(Operating } R_3 + R_2\text{)}$$

This plane clearly passes through the line

$$\frac{x}{l_1 + l_2 + l_3} = \frac{y}{m_1 + m_2 + m_3} = \frac{z}{n_1 + n_2 + n_3}.$$

Similarly plane BOB' and COC' pass through the same line.

3. *A, A', B, B', C, C' are points on the axes. Show that the lines of intersection of the planes $A'BC$, $AB'C$, $B'CA$, $BC'A'$, $C'AB$, $CA'B'$ are coplanar.*

Sol. Let the points A, A', B, B', C, C' be $(a, 0, 0), (a', 0, 0), (0, b, 0), (0, b', 0)$, $(0, 0, c)$, $(0, 0, c')$ respectively.

Equations of the planes $A'BC$ and $AB'C'$ are

$$y/a' + y/b + z/c = 1 \text{ and } x/a + y/b' + z/c' = 1.$$

These equations taken together represent the line of intersection of the planes $A'BC$ and $AB'C'$.

Any plane through this line is

$$(x/a' + y/b + z/c - 1) + \lambda_1 (x/a + y/b' + z/c' - 1) = 0 \quad ...(1)$$

Similarly, planes through the lines of intersection of $B'CA$, $BC'A'$; $C'AB$, $CA'B'$ are respectively

$$(x/a + y/b' + z/c - 1) + \lambda_2 (x/a' + y/b + z/c' - 1) = 0$$

and

$$(x/a + y/b + z/c' - 1) + \lambda_3 (x/a' + y/b' + z/c - 1) = 0$$

In case the three lines are coplanar then for some value of $\lambda_1, \lambda_2, \lambda_3$, the above equations must represent the same plane. This is obviously so, when $\lambda_1, \lambda_2, \lambda_3$, and then the plane becomes

$$x(1/a + 1/a') + y(1/b + 1/b') + z(1/c + 1/c') = 2.$$

EXERCISES

1. Show that the lines

$$\frac{1}{3}(x+4) = \frac{1}{5}(y+6) = -\frac{1}{2}(z-1)$$
$$3x - 2y + z + 5 = 0 = 2x + 3y + 4z - 4$$

are coplanar. Find also the co-ordinates of their point of intersection and the equation of the plane in which they lie. [**Ans.** $(2, 4, -3)$; $45x - 17y + 25z + 53 = 0$]

2. Prove that the lines

$$\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8}; \frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}$$

intersect. Find also their point of intersection and the plane through them.

[**Ans.** $(5, -7, 6)$; $11x = 6y + 5z + 67$]

3. Prove that the lines

$$\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7}; \frac{x-2}{1} = \frac{y-4}{3} = \frac{z-6}{5}$$

intersect. Find their point of intersection and the plane in which they lie.

[**Ans.** $(1/2, -1/2, -3/2)$; $x - 2y + z = 0$]

4. Show that the lines

$$x + 2y - 5z + 9 = 0 = 3x - y + 2z - 5;$$
$$2x + 3y - z - 3 = 0 = 4x - 5y + z + 3$$

are coplanar.

5. Prove that the lines

$$x - 3y + 2z + 4 = 0 = 2x + y + 4z + 1;$$
$$3x + 2y + 5z - 1 = 0 = 2y + z$$

intersection and find the co-ordinates of their point of intersection. [**Ans.** $(3, 1, -2)$]

6.
$$x - 2y - z - 3 = 0, 3x - y + 2z - 1 = 0,$$
$$2x - 2y + 3z - 2 = 0, x - y + z + 1 = 0$$

are two given pairs of planes. Show that the line of intersection of the first pair is coplanr with the line of intersection of the latter.

7. Show that the line of intersection of the planes

$$7x - 4y + 7z + 16 = 0, 4x + 3y - 2z + 3 = 0$$

is coplanar with the line of intersection of planes

$$x - 3y + 4z + 6 = 0, x - y + z + 1 = 0.$$

Obtain the equation of the plane through the two lines. [**Ans.** $3x - 7y + 9z + 13 = 0$]

8. Prove that the lines

$$\frac{x-a}{a'} = \frac{y-b}{b'} = \frac{z-c}{c'} \text{ and } \frac{x-a'}{a} = \frac{y-b'}{b} = \frac{z-c'}{c}$$

intersect and find the co-ordinates of the point of intersection and the equation of the plane in which by they lie. [**Ans.** $(a + a', b + b', c + c')$; $\Sigma x\,(bc' - b'c) = 0$]

3.5 NUMBER OF ARBITRARY CONSTANTS IN THE EQUATIONS OF A STRAIGHT LINE

We have already seen that the general equation of a plane contains **three** arbitrary constants and it will now be shown *that there are four arbitrary constants in the equations of a straight line*.

A given line PQ can be regarded as the intersection of *any* two planes through it. In particular, we may take the two planes perpendicular to two of the co-ordinate planes, say, YZ and ZX planes.

The equations of the planes through a line PQ perpendicular to the YZ and ZX planes are respectively of the forms

$$z = cy + d \text{ and } z = ax + b$$

which are, therefore, the equations of the line PQ and contain four arbitrary constants a, b, c, d.

Hence, the *equations of a straight line involve four arbitrary contants* as it is always possible to express them in the above form.

The fact that the general equations of a straight line contain four arbitrary constants may also be seen as follows :

We see that the equations

$$\frac{x - x_1}{l} = \frac{y - y_1}{m}; \frac{y - y_1}{m} = \frac{z - z_1}{n}$$

are equivalent to

$$x = \frac{l}{m} y + \frac{(mx_1 - ly_1)}{m}, y = \frac{m}{n} z + \frac{(ny_1 - mz_1)}{n}$$

respectively, so that

$$\frac{l}{m}, \frac{m}{n}, \frac{mx_1 - ly_1}{m}, \frac{ny_1 - mz_1}{n}$$

are the *four* arbitrary constants or parameters.

3.5.1 Determination of Lines Satisfying Given Conditions

We now consider the various *sets of conditions* which determine a line.

We know that the equations of a straight line involve four arbitrary constants and as such any four geometrical conditions, each of which gives rise to one relation between the constants, fix a straight line.

It may be noted that the conditions for a line to intersect a given line or be perpendicular to it separately involve one relation between the constants and hence, three more relations are required to fix the line.

A given conditions may sometimes give rise to two relations between the constants as, for instance, the conditions that the required line

(i) passes through a given point; (ii) has a given direction.

In such cases only two more relations will be required to fix the straight line.

We have already considered equations of a line which

(i) pass through a given point and have a given direction;

(ii) pass through two given points;

(iii) pass through a point and are parallel to two given planes;

(iv) pass through a point and pependicular to two given lines.

Some further sets of conditions which determine a line are given below :

(v) passing throguh a given point and intersecting two given lines;

(vi) intersecting two given lines and having a given direction;

(vii) intersecting a given line at right angles and passing through a given point;

(viii) intersecting two given lines at right angles;

(ix) intersecting a given line parallel to a given line and passing through a given point;

(x) passing through a given point and perpendicular to two given lines; and so on.

An Important Note : *If*

$$u_1 = 0 = v_1 \quad \textit{and} \quad u_2 = 0 = v_2$$

be two straight lines, then the general equations of a straight line intersecting them both are

$$u_1 + \lambda_1 v_1 = 0 = u_2 + \lambda_2 v_2,$$

where λ_1, λ_2 *are any two numbers.*

The line $u_1 + \lambda_1 v_1 = 0 = u_2 + \lambda_2 v_2$ lies in the plane $u_1 + \lambda_1 v_1 = 0$ which again contains the line $u_1 = 0 = v_1$.

The two lines

$$u_1 + \lambda_1 v_1 = 0 = u_2 + \lambda_2 v_2 ; u_1 = 0 = v_1$$

are, therefore, coplanar and hence they intersect.

Similarly, the same line intersects the line $u_2 = 0 = v_2$.

This conclusion will be found very helpful in what follows.

For the sake of illustration, we give below a few examples.

EXAMPLES

1. *Find the equations of the line which passes through the point* $(2, -1, 1)$ *and intersects the lines*

$$2x + y - 4 = 0 = y + 2z; x + 3z = 4, 2x + 5z = 8.$$

Sol. The line

$$2x + y - 4 + \lambda_1 (y + 2z) = 0, x + 3z - 4 + \lambda_2 (2x + 5z - 8) = 0$$

intersects the two given lines for all values of λ_1, λ_2.

The line will pass through the point $(2, -1, 1)$, if

$$-1 + \lambda_1 = 0 \text{ and } 1 + \lambda_2 = 0,$$

$$\Rightarrow \qquad \lambda_1 = 1, \lambda_2 = -1.$$

The required equations, therefore, are

$$x + y + z = 2 \text{ and } x + 2z = 4.$$

2. *Find the equations of the line which passes through the point* $(3, -1, 11)$ *and is perpendicular to the line*

$$\frac{1}{2} x = \frac{1}{3} (y - 2) = \frac{1}{4} (z - 3).$$

Obtain also the foot of the perpendicular.

Sol. The co-ordinates of any point on the given line are

$$2r, 3r + 2, 4r + 3.$$

This will be the required foot of the perpendicular if the line joining it to the point $(3, -1, 11)$ be perpendicular to the given line. This requires

$$2(2r - 3) + 3(3r + 2 + 1) + 4(4r + 3 - 11) = 0 \Rightarrow r = 1.$$

Therefore, the required foot is (2, 5, 7) and the required equations of the perpendiculars are

$$\frac{x - 3}{1} = \frac{y + 1}{-6} = \frac{z - 11}{4}.$$

EXERCISES

1. Find the equations of the perpendicular from

(i) $(2, 4, -1)$ to $(x+5) = \frac{1}{4}(y+3) = \frac{1}{9}(z-6)$,

(ii) $(-2, 2, -3)$ to $(x-3) = \frac{1}{2}(y+1) = -\frac{1}{4}(z-2)$,

(iii) $(0, 0, 0)$ to $2x + y + z - 7 = 0 = 4x + z - 14$.

Obtain also the feet of the perpendiculars.

[**Ans.** (i) $\frac{1}{6}(x-2) = \frac{1}{3}(y-4) = \frac{1}{2}(z+1), (-4, 1, -3)$

(ii) $\frac{1}{6}(x+2) = -(y-2) = (z+3), (4, 1, -2)$

(iii) $-x/2 = y = z/4, (2/3, -1/3, -4/3)$

(iv) $\frac{1}{6}(x+2) = -(y-2) = (z+3), (4, 1, -2)$]

2. A line with direction cosines proportional to $(7, 4, -1)$ is drawn to intersect the lines

$$\frac{x-1}{3} = \frac{y-3}{-1} = \frac{z+2}{1}, \frac{x+3}{-3} = \frac{y-3}{2} = \frac{z-5}{4}.$$

Find the points of intersection and the length intercepted on it.

[**Ans.** $(7, 5, 0), (0, 1, 1), \sqrt{66}$]

3. Find the line which intersects the lines

$$x + y + z = 1, 2x - y - z = 2; x - y - z = 3, 2x + 4y - z = 4$$

and passes through the point (1, 1, 1). Find also the points of intersection.

$\left[\textbf{Ans. } x = 1, (y-1)/1 = (z-1)/3; \left(1, \frac{1}{2}, -\frac{1}{2}\right); (1, 0, -2)\right]$

4. Find the equations of the line which passes through the point $(-4, 3, 1)$, is parallel to the plane $x + 2y - z = 5$ and intersects the line

$$-(x+1)/3 = (y-3)/2 = -(x-2)$$

Find also the point of intersection. [**Ans.** $(x+4)/3 = -(y-3) = (z-1); (2, 1, 3)$]

5. Find the distance of the point $(-2, 3, -4)$ from the line

$$(x+2)/3 = (2y+3)/4 = (3z+4)/5$$

measured parallel to the plane

$$4x + 12y - 3z + 1 = 0.$$ [**Ans.** 17/2]

6. Find the equations of the straight line through the point (2, 3, 4) perpendicular to the *X*-axis and intersecting the line $x = y = z$. [**Ans.** $x = 2, 2y - z = 2$]

3.6 THE SHORTEST DISTANCE BETWEEN TWO LINES

To show that the shortest distance between two lines lies along the line meeting them both at right angles.

Let *AB, CD* be two given lines.

A line is completely determined if it intersects two lines at right angles [See 3.5.1, Case (viii)].

Thus, there is one and only one line which intersects the two given lines at right angles, say, at *G* and *H*.

GH is, then, the shortest distance between the two lines for, if *A, C* be *any* two points, one on each of the two given lines,

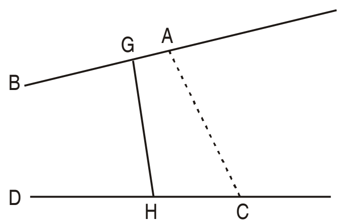


Fig. 2

GH is the projection of *AC* on itself

$\Rightarrow \quad GH = AC\cos\theta$

$\Rightarrow \quad GH < AC$

θ, being the angle between GH and AC. Thus, GH is the shortest distance (S.D.) between the two lines AB and CD.

3.6.1

To find the magnitude and the equations of the line of shortest distance between two straight lines.

Let AB, CD be the two given lines, and GH, the line which meets them both at right angles at G and H. Then GH is the line of shortest distance between the given lines; the length of GH being the magnitude.

Let the equations of the given lines be

$$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}, \qquad \text{...(i)}$$

$$\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2} \qquad \text{...(ii)}$$

and let the shortest distance lie along the line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}, \qquad \text{...(iii)}$$

Line (iii) is perpendicular to both the lines (i) and (ii)

$$\Rightarrow \quad \begin{cases} ll_1+mm_1+nn_1=0, \\ ll_2+mm_2+nn_2=0, \end{cases}$$

$$\Rightarrow \quad \frac{l}{m_1n_2-m_2n_1}=\frac{m}{n_1l_2-n_2l_1}=\frac{n}{l_1m_2-l_2m_1};$$

$$=\frac{1}{\sqrt{\Sigma\,(m_1n_2-m_2n_1)^2}} \qquad \text{...(iv)}$$

The line of shortest distance is perpendicular to both the lines. Therefore, the magnitude of the shortest distance is the projection on the line of shortest distance of the line joining *any* two points, one on each of the given lines (i) and (ii).

Taking the projection of the join of $(x_1, y_1, z_1), (x_2, y_2, z_2)$ on the line with direction cosines $l, m, n;$ we see that the shortest distance

$$=(x_2-x_1)\,l+(y_2-y_1)\,m+(z_2-z_1)\,n,$$

where l, m, n have the values as given in (iv).

To find the equations of the line of shortest distance, we observe that it is coplanar with both the given lines.

The equations of the plane containing the coplanar lines (i) and (iii) is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l & m & n \end{vmatrix}=0 \qquad \text{...(v)}$$

and that of the plane containing the coplanar lines (ii) and (iii) is

$$\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_2 & m_2 & n_2 \\ l & m & n \end{vmatrix}=0. \qquad \text{...(vi)}$$

Thus, (v) and (vi) are the two equations of the line of shortest distance, where l, m, n are given in (iv).

Note. Other methods of determining the shortest distance are given below where an example has been solved by three different methods.

EXAMPLES

1. *Find the magnitude and the equations of the line of shortest distance between the lines :*

$$\frac{x-8}{3}=\frac{y+9}{-16}=\frac{z-10}{7}, \qquad \text{...(i)}$$

$$\frac{x-15}{3}=\frac{y-29}{8}=\frac{z-5}{-5}. \qquad \text{...(ii)}$$

Sol. First Method. Let *l, m, n* be the direction cosines of the line of shortest distance.

As it is perpendicular to the two lines, we have

$$\begin{cases} 3l-16m+7n=0, \\ 3l+8m-5n=0. \end{cases}$$

$$\Rightarrow \qquad \frac{l}{24}=\frac{m}{36}=\frac{n}{72},$$

$$\Rightarrow \qquad \frac{l}{2}=\frac{m}{3}=\frac{n}{6},$$

$$\Rightarrow \qquad l=\frac{2}{7}, m=\frac{3}{7}, n=\frac{6}{7}.$$

The magnitude of the shortest distance is the projection of the join of the points $(8,-9,10)$, $(15, 29, 5)$, on the line of the shortest distance and is, therefore,

$$=7\cdot\frac{2}{7}+38\cdot\frac{3}{7}-5\cdot\frac{6}{7}=14$$

Again, the equation of the plane containing the first of the two given lines and the line of shortest distance is

$$\begin{vmatrix} x-8 & y+9 & z-10 \\ 3 & -16 & 7 \\ 2 & 3 & 6 \end{vmatrix}=0 \Leftrightarrow 117x+4y-41z-490=0.$$

Also the equation of the plane containing the second line and the shortest distance line is

$$\begin{vmatrix} x-15 & y-29 & z-5 \\ 3 & 8 & -5 \\ 2 & 3 & 6 \end{vmatrix}=0 \Leftrightarrow 9x-4y-z=14.$$

Hence, the equations of the shortest distance line are

$$117x+4y-41z-490=0=9x-4y-z=14.$$

Second Method

$$P\,(3r+8,-16r-9,7r+10),\ P'\,(3r'+15,8r'+29,5r'+5)$$

are the general co-ordinates of the points on the two lines respectively. The direction cosines of PP' are proportional to

$$3r-3r'-7,-16r-8r'-38,7r+5r'+5.$$

Now PP' will be required line of shortest distance, if it is perpendicular to both are given lines, which requires

$$\begin{cases} 3\,(3r-3r'-7)-16\,(-16r-8r'-38)+7\,(7r+5r'+5)=0, \\ 3\,(3r-3r'-7)+8\,(-16r-8r'-38)-5\,(7r+5r'+5)=0. \end{cases}$$

$$\Rightarrow \qquad 157r+77r'+311=0 \text{ and } 11r+7r'+25=0,$$

$$\Rightarrow \qquad r=-1, r'=-2.$$

Therefore, the co-ordinates of the point P and P' are

$$(5,7,3) \text{ and } (9,13,15).$$

Hence, the shortest distance $PP'=14$ and its equations are

$$\frac{x-5}{2}=\frac{y-7}{3}=\frac{z-3}{6}$$

This method is sometimes very convenient and is specially useful when we require also the points where the line of shortest distance meets the two lines.

2. *Find the shortest distance between the axis of z and the line*

$$ax + by + cz + d = 0, a'x + b'y + c'z + d' = 0$$

Sol. Now the general equation of the plane through the second given line is

$$ax + by + cz + d + k\ (a'x + b'y + c'z + d') = 0$$

$$\Leftrightarrow \qquad (a + ka')\ x + (b + kb')\ y + (c + kc')\ z + (d + kd') = 0 \qquad \text{...(i)}$$

k being the parameter.

It will be paralell to z-axis whose direction cosines are 0, 0, 1 if the normal to the plane is perpendicular to the z-axis, *i.e.,* if

$$0.(a + ka') + 0.(b + kb') + 1.(c + kc') = 0$$

$$\Rightarrow \qquad k = - c / c'.$$

Substituting this value of k in (i), we see that the equation of the plane through the second line parallel to the first is

$$(ac' - a'c)\ x + (bc' - b'c)\ y + (dc' - d'c) = 0 \qquad \text{...(ii)}$$

The required S.D. is the distance of *any* point on the z-axis from the plane (ii) so that

S.D. = perpendicular from (0, 0, 0), (a point on z-axis)

$$= \pm \frac{dc' - d'c}{\sqrt{(ac' - a'c)^2 + (bc' - b'c)^2}}.$$

3. *Prove that the S.D. between the diagonals of rectangular parallelopiped and the edges not meeting it are*

$$\frac{bc}{\sqrt{(b^2 + c^2)}}, \frac{ca}{\sqrt{(c^2 + a^2)}}, \frac{ab}{\sqrt{(a^2 + b^2)}}$$

where a, b, c are the lengths of the edges.

Sol. Let coterminous edges OA, OB, OC be taken as the axes of reference. We will find S.D. between the diagonal OP and edge BL (which does not meet OP). Equations of OP and BL are

$$\frac{x}{a} = \frac{y}{b} = \frac{z}{c} \text{ and } \frac{x}{a} = \frac{y - b}{0} = \frac{z}{0}.$$

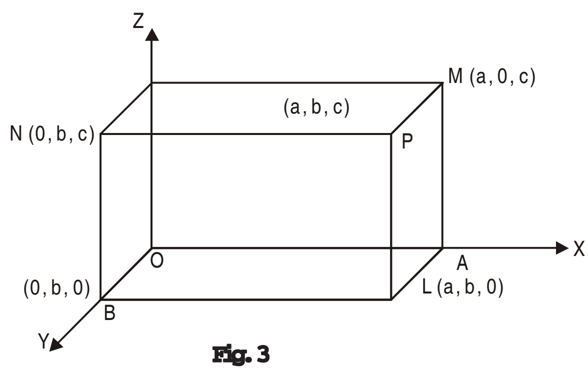


Fig. 3

Let l, m, n be the dc's of S.D., then

$$al + bm + cn = 0$$

and
$$al + 0 \cdot m + 0 \cdot n = 0$$

$$\Rightarrow \qquad l = 0, m = \frac{c}{\sqrt{b^2 + c^2}}, n = \frac{-b}{\sqrt{b^2 + c^2}}$$

$\therefore$ S.D. = Projection of OB on the line of S.D.

$$= (0-0)\, l + (b-0)\, m + (0-0)\, n = bm = \frac{bc}{\sqrt{b^2+c^2}}.$$

Similarly, S.D.'s between OP and AL and OP and MC are $ca/\sqrt{c^2+a^2}$ and $ab/\sqrt{a^2+b^2}$.

4. *Show that the equation of the plane containing the line*

$$\frac{y}{b}+\frac{z}{c}=1,\ x=0$$

and parallel to the line $$\frac{x}{a}-\frac{z}{c}=1,\ y=0$$

is $$\frac{x}{a}-\frac{y}{b}-\frac{z}{c}+1=0$$

and if 2d is the S.D. show that $d^{-2}=a^{-2}+b^{-2}+c^{-2}$.

Sol. The equation of the plane containing the line

$$\frac{y}{b}+\frac{z}{c}-1+\lambda x=0$$

is $$\left(\frac{y}{b}+\frac{z}{c}-1\right)+\lambda x=0$$

$$\Rightarrow \quad \lambda x+(1/b)y+(1/c)z-1=0 \qquad \text{...(i)}$$

If it is parallel to the line

$$\frac{x}{a}-\frac{z}{c}=1,\ y=0,\ i.e.,\ \frac{x-a}{a}=\frac{y}{0}=\frac{z}{c},$$

then the normal to the plane (i) must be perpendicular to the line and so we have

$$\lambda\cdot(a)+(1/b)\cdot 0+(1/c)\cdot c=0 \Rightarrow \lambda=-1/a.$$

$\therefore$ From (1) the equation of the required palne is

$$\left(\frac{y}{b}+\frac{z}{c}-1\right)-\frac{1}{a}x=0 \Rightarrow \frac{x}{a}-\frac{y}{b}-\frac{z}{c}+1=0 \qquad \text{...(ii)}$$

Now, any point on the line $\dfrac{x-a}{a}=\dfrac{y}{0}=\dfrac{z}{c}$ is $(a, 0, 0)$.

Therefore,

$2d =$ S.D. beween the given lines

= perpendicular distance of the point $(a, 0, 0)$ from the plane (i)

$$= \frac{a\cdot(1/a)-0\cdot(1/b)-0\cdot(1/c)+1}{\sqrt{(1/a)^2+(-1/b)^2+(-1/c)^2}} = \frac{2}{\sqrt{(a^{-2}+b^{-2}+c^{-2})}}$$

$$\Rightarrow \quad d^{-2}=a^{-2}+b^{-2}+c^{-2}.$$

5. *A square ABCD of diagonal 2a is folded along the diagonal AC, so that planes DAC, BAC are at right angles. Show that the shortest distance betweden DC and AB is then* $2a/\sqrt{3}$.

Sol. Let O be the centre of square and OA axis of x. Planes DAC and BAC are mutually at right angles. Take OB and OD as axes of y and z.

Then co-ordinates of A, B, C, D are $(a, 0, 0)$, $(0, a, 0)$, $(-a, 0, 0)$ and $(0, 0, a)$.

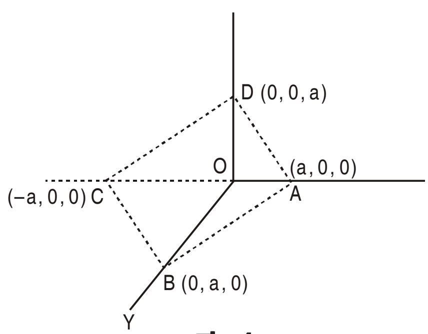


Fig. 4

Equations of AB are $(x-a)/a = y/-a = z/0$ and of DC $x/a = y/0 = (z-a)/a$. Thus, a plane containing DC and parallel to AB is

$$\begin{vmatrix} x & y & z-a \\ a & 0 & a \\ a & -a & 0 \end{vmatrix} = 0$$

$$\Rightarrow \qquad x+y+z+a=0$$

S.D. = Perpendicular distance of this plane from a point $(a, 0, 0)$ on AB

$$= \frac{a+a}{\sqrt{(1+1+1)}} = \frac{2a}{\sqrt{3}}.$$

EXERCISES

1. Find the magnitude and the equations of the line of shortest distance between the two lines :

(i) $\dfrac{x-3}{2} = \dfrac{y+15}{-7} = \dfrac{z-9}{5}; \dfrac{x+1}{2} = \dfrac{y-1}{1} = \dfrac{z-9}{-3}.$

(ii) $\dfrac{x-3}{-1} = \dfrac{y-4}{2} = \dfrac{z+2}{1}; \dfrac{x-1}{1} = \dfrac{y+7}{3} = \dfrac{z+2}{2}.$

[**Ans.** (i) $x = y = z; 4\sqrt{3}$ (ii) $(x-4) = (y-2)/3 = -(z+3)/5; \sqrt{35}$]

2. Find the length and the equations of the shortest distance line between

$$5x - y - z = 0 \qquad x - 2y + z + 3 = 0;$$
$$7x - 4y - 2z = 0, \qquad x - y + z - 3 = 0.$$

[**Hint.** Transform the equations to the symmetrical form]

[**Ans.** $17x + 20y - 19z - 39 = 0 = 8x + 5y - 31z + 67; 13/\sqrt{75}$]

3. Find the magnitude and the position of the line of shortest distance between the lines

(i) $2x + y - z = 0, x - y + 2z = 0; x + 2y - 3z = 4, 2x - 3y + 4z = 5$

(ii) $\dfrac{x}{4} = \dfrac{y+1}{3} = \dfrac{z-2}{2}; 5x - 2y - 3z + 6 = 0; x - 3y + 2z - 3 = 0.$

[**Ans.** (i) $3x + z = 0 = 22x - 5y + 4z - 67, 2\sqrt{14}/7.$

(ii) $7x - 2y - 11z + 20 = 0 = 13x - 13z + 24; 17\sqrt{6/39}$]

4. Obtain the co-ordinates of the points where the shortest distance line between the lines

$$\frac{x-23}{-6} = \frac{y-19}{-4} = \frac{z-25}{3}, \frac{x-12}{-9} = \frac{y-1}{4} = \frac{z-5}{2}$$

meets them. [**Ans.** (11, 11, 31) and (3, 5, 7)]

5. Find the co-ordinates of the points on the join of $(-3, 7, -13)$ and $(-6, 1, -10)$ which is nearest to the intersection of the planes

$3x - y - 3z + 32 = 0$ and $3x + 2y - 15z - 8 = 0.$ [**Ans.** $(-7, -1, -9)$]

6. Show that the shortest distance between the lines

$$x + a = 2y = -12z \text{ and } x = y + 2a = 6z - 6a$$

is $2a$.

3.7 LENGTH OF THE PERPENDICULAR FROM A POINT TO A LINE

To find the length of the perpendicular from a given point $P(x_1, y_1, z_1)$ to a given line

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}.$$

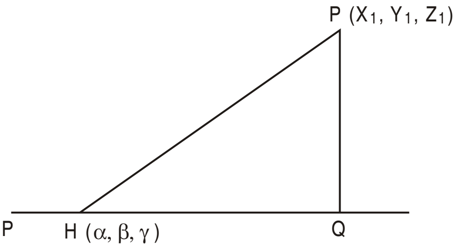


Fig. 5

Let H be the point (α, β, γ) on the given line and Q the foot of the perpendicular from the point P on it.

We have $$PQ^2 = HP^2 - HQ^2.$$

Also $$HP^2 = (x_1 - \alpha)^2 + (y_1 - \beta)^2 + (z_1 - \gamma)^2$$

and $$HQ = \text{projection of } HP \text{ on the given line}$$
$$= l\,(x_1 - \alpha) + m(y_1 - \beta) + n\,(z_1 - \gamma)$$

provided l, m, n are the actual direction cosines.

It follows that

$$PQ^2 = (x_1 - \alpha)^2 + (y_1 - \beta)^2 + (z_1 - \gamma)^2 - [l\,(x_1 - \alpha) + m\,(y_1 - \beta) + n\,(z_1 - \gamma)]^2.$$

EXAMPLE

Find the perpendicular distance of $P\,(1, 2, 3)$ *from the line* $\dfrac{x-6}{3} = \dfrac{y-7}{2} = \dfrac{z-7}{-2}$.

Sol. First Method. $A\,(6, 7, 7)$ will be a point on the line. Let the perpendicular from P meets the line in N.

Then,

$$AP^2 = (6-1)^2 + (7-2)^2 + (7-3)^2 = 66$$

AN = Projection of AP on the given line

$$= (6-1)\frac{3}{\sqrt{17}} + (7-2)\,.\,\frac{2}{\sqrt{17}} + (7-3)\left(-\frac{2}{\sqrt{17}}\right)$$

$$= \sqrt{17}$$

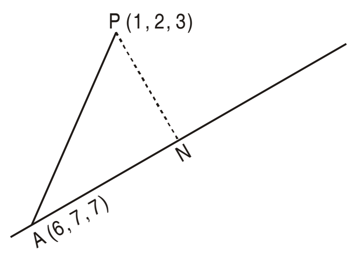


Fig. 6

$\therefore$ $$PN^2 = AP^2 - AN^2 = 66 - 17 = 49$$

$\Rightarrow$ $$PN = 7.$$

Second Method. The given line can be written as

$$\frac{x-6}{3/\sqrt{17}} = \frac{y-7}{2/\sqrt{17}} = \frac{z-7}{-2/\sqrt{17}}$$

$\therefore$ $$PN^2 = \left\{\frac{2}{\sqrt{17}}(1-6) - \frac{3}{\sqrt{17}}(2-7)\right\}^2 + \left\{\frac{-2}{\sqrt{17}}(2-7) - \frac{2}{\sqrt{17}}(3-7)\right\}^2 + \left\{\frac{3}{\sqrt{17}}(3-7) + \frac{2}{\sqrt{17}}(1-6)\right\}^2 = 49$$

$\Rightarrow$ $$PN = 7.$$

Third Method. Any point N on the line is $(3r + 6, 2r + 7, -2r + 7)$. Let this be the foot of the perpendicular.

Then PN whose dr's are $3r + 5, 2r + 5, -2r + 4,$ will be perpendicular to the given line.

$\therefore$ $$3\,(3r + 5) + 2\,(2r + 5) - 2\,(-2r + 4) = 0$$

$\Rightarrow$ $$r = -1.$$

Thus, the foot of perpendicular N is (3, 5, 9) and hence $PN = 7$.

EXERCISES

1. Find the length of the perpendicular from the point $(4, -5, 3)$ to the line

$$\frac{x-5}{3} = \frac{y+2}{-4} = \frac{z-6}{5}.$$ $\left[\textbf{Ans. } \dfrac{\sqrt{457}}{5}\right]$

2. Find the locus of the point whcih moves so that its distance from the line $x = y = z$ is twice its distance from the plane $x + y + z = 1$.

[**Ans.** $x^2 + y^2 + z^2 + 5xy + 5yz + 5zx - 4x - 4y - 4z + 2 = 0$]

3. Find the length of the perpendicular from the point $P\,(5, 4, -1)$ upon the line

$$\frac{1}{2}(x-1) = \frac{1}{9}\,y = \frac{1}{5}\,z.$$ [**Ans.** $\sqrt{2109/110}$]

3.8 INTERSECTION OF THREE PLANES

Given three distinct planes such that no two of them are parallel.

We have the following three possibilities in respect of their intersection.

The three planes may :

(i) have only one point in common (Fig. 7a);

(ii) have a line in common so that the three planes are coaxial (Fig. 7b);

(iii) form a triangular prism (Fig. 7c).

We shall in the following find conditions for each of these three possibilities.

Three planes are said to form a triangular prism if the three lines of intersection of the three planes, taken in pairs, are distinct and parallel.

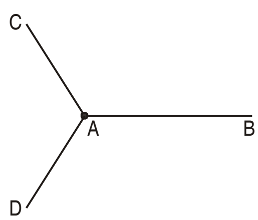
Clearly, the three planes will form a triangular prism if the line of intersection of two of them be parallel to the third.

3.8.1

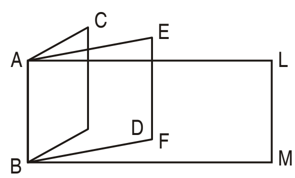
To find the condition that the three planes

$$a_r x + b_r y + c_r z + d_r = 0;\ (r = 1, 2, 3)$$

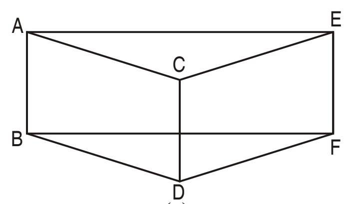
should form a prism or intersect in a line.



(a)



(b)



(c)

Fig. 7

We assume that the first two planes are *not* parallel.

The line of intersection of the first two planes is

$$\frac{x - (b_1 d_2 - b_2 d_1)/(a_1 b_2 - a_2 b_1)}{b_1 c_2 - b_2 c_1} = \frac{y - (a_2 d_1 - a_1 d_2)/(a_1 b_2 - a_2 b_1)}{a_2 c_1 - a_1 c_2} = \frac{z}{a_1 b_2 - a_2 b_1} \quad \text{...(i)}$$

The three planes will form a triangular prism if this line is parallel to the third plane without lying in the same.

The line (i) will be parallel to the third plane, if

$$a_3(b_1 c_2 - b_2 c_1) + b_3(c_1 a_2 - c_2 a_1) + c_3(a_1 b_2 - a_2 b_1) = 0$$

$$\Rightarrow \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0 \quad \text{...(ii)}$$

Again, the planes will intersect in a line if and only if the line (i) lies in the plane

$$a_3 x + b_3 y + c_3 z + d_3 = 0.$$

This rquires :

(1) This line is parallel to the third plane *i.e.,* (ii) is satisfied, and

(2) the point $\left(\dfrac{b_1d_2 - b_2d_1}{a_1b_2 - a_2b_1}, \dfrac{a_2d_1 - a_1d_2}{a_1b_2 - a_2b_1}, 0\right)$ lies on it implying that

$$a_3(b_1d_2 - b_2d_1) + b_3(a_2d_1 - a_1d_2) + d_3(a_1b_2 - a_2b_1) = 0$$

$$\Rightarrow \qquad \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} = 0 \qquad \text{...(iii)}$$

Thus, the three planes will intersect in a line, if the condition

(ii) and (iii) hold

and will form a triangular prism, if

(ii) holds but (iii) does not hold.

The three planes will intersect in a unique finite point if the conditon (ii) does not hold.

EXAMPLES

1. *Prove that the planes* $x = cy + bz,\ y = az + cx,\ z = bx + ay$ *pass through one line if* $a^2 + b^2 + c^2 + 2abc = 1.$ *Show that the equations of this line are*

$$\frac{x}{\sqrt{1-a^2}} = \frac{y}{\sqrt{1-b^2}} = \frac{z}{\sqrt{1-c^2}}.$$

Sol. The three planes can be written as

$$x - cy - bz = 0 \qquad \text{...(1)}$$

$$cx - y + az = 0 \qquad \text{...(2)}$$

$$bx + ay - z = 0 \qquad \text{...(3)}$$

Let (l, m, n) be the dc's of the line of intersection of (1) and (2); then

$$l - cm - bn = 0$$

$$cl - m + an = 0$$

$$\Rightarrow \qquad \frac{l}{ac + b} = \frac{m}{bc + a} = \frac{n}{1 - c^2}$$

Planes (1) and (2) both passes throguh origin, hence, their line of intersection will also pass throguh (0, 0, 0). Thus, equation of line of intersection of (1) and (2) is

$$\frac{x}{ac + b} = \frac{y}{bc + a} = \frac{z}{1 - c^2} \qquad \text{...(4)}$$

Now the three planes will intersect in a line if (4) lies in (3). The point (0, 0, 0) of (4) already satisfies (3). Hence, the required condition is

$$b\,(ac + b) + a\,(bc + a) - (1 - c^2) = 0$$

$$\Rightarrow \qquad a^2 + b^2 + c^2 + 2abc = 1 \qquad \text{...(5)}$$

We have

$$ac + b = \sqrt{(ac + b)^2} = \sqrt{a^2c^2 + b^2 + 2abc} = \sqrt{a^2c^2 + (1 - a^2 - c^2)} \qquad \text{[From (5)]}$$

$$= \sqrt{(1 - a^2)\,(1 - c^2)}.$$

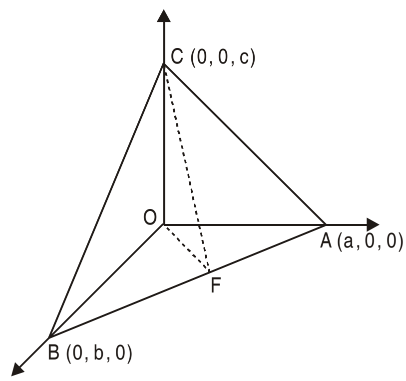
Similarly, $bc + a = \sqrt{(1 - b^2)\,(1 - c^2)}.$

Putting these in (4), we get

$$\frac{x}{\sqrt{1-a^2}} = \frac{y}{\sqrt{1-b^2}} = \frac{z}{\sqrt{1-c^2}}.$$

2. *The plane* $x/a + y/b + z/c = 1$ *meets the axes OX, OY, OZ which are rectangular, in A, B, C. Prove that the plane through the axes and the internal bisector of the angles of the triangle ABC pass through the line*

$$\frac{x}{a\sqrt{b^2+c^2}} = \frac{y}{b\sqrt{c^2+a^2}} = \frac{z}{c\sqrt{a^2+b^2}}.$$



Sol. We know that bisector of any angle of a triangle meets the opposite side at a point which divides it in the ratio of other two sides. Hence, if CF is the internal bisector of angle ABC, then

$$\frac{AF}{BC} = \frac{AC}{BC} = \frac{\sqrt{(a^2+c^2)}}{\sqrt{(b^2+c^2)}}$$

Thus, co-ordinates of F will become

$$\left(\frac{a\sqrt{(b^2+c^2)}}{\sqrt{(a^2+c^2)}+\sqrt{(b^2+c^2)}}, \frac{b\sqrt{a^2+c^2}}{\sqrt{(a^2+c^2)}+\sqrt{(b^2+c^2)}}, 0\right)$$

Plane through z-axis is

$$x + \lambda y = 0$$

If it passes through F, then

$$\lambda = -\frac{a\sqrt{(b^2+c^2)}}{b\sqrt{(a^2+c^2)}}$$

Hence, equation of plane OCF is

$$x - \frac{a\sqrt{(b^2+c^2)}}{b\sqrt{(a^2+c^2)}}\, y = 0$$

i.e.,
$$\frac{x}{a\sqrt{(b^2+c^2)}} = \frac{y}{b\sqrt{(a^2+c^2)}}.$$

Similarly, other planes through axes and lines bisecting the angles A and B are

$$\frac{y}{b\sqrt{a^2+c^2}} = \frac{z}{c\sqrt{(a^2+b^2)}} \quad \text{and} \quad \frac{x}{a\sqrt{(b^2+c^2)}} = \frac{z}{c\sqrt{(a^2+b^2)}}$$

The planes clearly pass through the line

$$\frac{x}{a\sqrt{(b^2+c^2)}} = \frac{y}{b\sqrt{(a^2+c^2)}} = \frac{z}{c\sqrt{(a^2+b^2)}}.$$

EXERCISES

1. Show that the following sets of planes intersect in lines :

(i) $4x + 3y + 2z + 7 = 0, 2x + y - 4z + 1 = 0, x - 7z - 7 = 0$

(ii) $2x + y + z + 4 = 0, y - z + 4 = 0, 3x + 2y + z + 8 = 0.$

2. Show that the following sets of planes form triangular prisms :

(i) $x + y + z + 3 = 0, 3x + y - 2z + 2 = 0, 2x + 4y + 7z - 7 = 0.$

(ii) $x - z - 1 = 0, x + y - 2z - 3 = 0, x - 2y + z - 3 = 0.$

3. Examine the nature of intersection of the following sets of planes :

(i) $4x - 5y - 2z - 2 = 0, 5x - 4y + 2z + 2 = 0, 2x + 2y + 8z - 1 = 0.$

(ii) $2x + 3y - z - 2 = 0, 3x + 3y + z - 4 = 0, x - y + 2z - 5 = 0.$

(iii) $5x + 3y + 7z - 4 = 0, 3x + 26y + 2z - 9 = 0, 7x + 2y + 10z - 5 = 0.$

(iv) $2x + 6y + 11 = 0, 6x + 20y - 6z + 3 = 0, 6y - 18z + 1 = 0.$

[**Ans.** (i) prism, (ii) point, (iii) line, (iv) prism]

4. Show that the planes

$$bx - ay = n,\ cy - bz = l,\ az - cx = m,$$

will intersect in a line if

$$al + bm + cn = 0,$$

and the direction ratios of the line, then, are a, b, c.

5. Prove that the three planes

$$bz - cy = b - c,\ cx - az = c - a,\ ay - bx = a - b,$$

pass through one line (say l), and the three planes

$$(c - a)\ z - (a - b)\ y = b + c,$$
$$(a - b)\ x - (b - c)\ z = c + a,$$
$$(b - c)\ y - (c - a)\ z = a + b,$$

pass through another line (say l'). Show that the lines l and l' are at right angles to each other.

6. If the planes $x = y + z,\ y = az + x,\ z = x + ay$ pass through one line, find the value of a.

[**Ans.** $a = -1$]

7. Prove that the planes $ny - mz = \lambda, lz - nx = \mu$ and $mx - ly = \nu$ have a common line if $l\lambda + m\mu + n\nu = 0.$ Show also that the distance of the line from the origin is

$$\left(\frac{\lambda^2 + \mu^2 + \nu^2}{l^2 + m^2 + n^2}\right)^{1/2}$$

❑❑❑

4

The Sphere

4.1 DEFINITION

A **sphere** *is the locus of a point which remains at a constant distance from a fixed point.* The constant distance is called the *Radius* and the fixed point the *Centre* of the sphere.

4.1.1 Equation of a Sphere

Let (a, b, c) be the centre and r the radius of a given sphere.

Equating the radius r to the distance of any point (x, y, z) on the sphere from its centre (a, b, c) we have

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = r^2$$

$$\Leftrightarrow \quad x^2 + y^2 + z^2 - 2ax - 2by - 2cz + (a^2 + b^2 + c^2 - r^2) = 0 \qquad \text{...(A)}$$

which is the required equation of the given sphere.

Thus, the sphere whose centre is the point (a, b, c) and whose radius is r is the set

$$\{(x, y, z) = x^2 + y^2 + z^2 - 2ax - 2by - 2cz + (a^2 + b^2 + c^2 - r^2) = 0\}$$

We note the following *characteristics* of the equation (A) of the sphere :

1. It is of the second degree in $x, y, z;$
2. The coefficient of x^2, y^2, z^2 are all equal;
3. The product terms xy, yz, zx are absent.

Conversely, we consider the equation.

$$ax^2 + ay^2 + az^2 + 2ux + 2vy + 2wz + d = 0, a \neq 0 \qquad \text{...(B)}$$

having the above three characteristics; a, u, v, w, d being given constants and $a \neq 0$.

The equation (B) can be written as

$$\left(x + \frac{u}{a}\right)^2 + \left(y + \frac{v}{a}\right)^2 + \left(z + \frac{w}{a}\right)^2 = \frac{u^2 + v^2 + w^2 - ad}{a^2}$$

This manner of rewriting shows that the distance between the variable point (x, y, z) and the fixed point

$$\left(-\frac{u}{a}, -\frac{v}{a}, -\frac{w}{a}\right)$$

is

$$\frac{\sqrt{u^2 + v^2 + w^2 - ad}}{|a|}, u^2 + v^2 + w^2 - ad \geq 0$$

and is, therefore, constant.

The locus of the equation (B) is thus a sphere, if

$$u^2 + v^2 + w^2 - ad \geq 0$$

4.1.2 General Equation of a Sphere

We write the equation (B) in the form

$$x^2 + y^2 + z^2 + \frac{2u}{a}x + \frac{2v}{a}y + \frac{2w}{a}z + \frac{d}{a} = 0, a \neq 0$$

$$\Leftrightarrow \quad x^2 + y^2 + z^2 + 2u'x + 2v'y + 2w'z + d' = 0$$

which is taken as the *general equation of a sphere.*

The family of spheres is thus given by the equation

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

where *u, v, w, d* are parameters such that $u^2 + v^2 + w^2 - d \geq 0$.

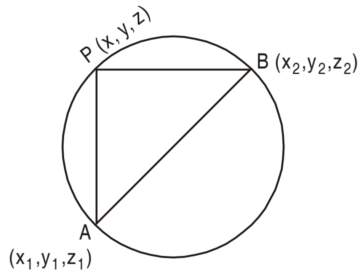
The radius of the sphere is '0' if

$$u^2 + v^2 + w^2 - d = 0$$

In this case, the sphere is what we may call a *Point sphere.*

4.1.3 Equation to a sphere on line joining $(x_1, y_1, z_1), (x_2, y_2, z_2)$ as diameter.

Let $P(x, y, z)$ be a point on the sphere. Then APB is a right-angled triangle right-angled at P.



Now direction cosines of *AP* are proportional to $x - x_1, y - y_1, z - z_1$ and direction cosines of *BP* are proportional to $x - x_2, y - y_2, z - z_2$.

But *AP* and *BP* are at right angles to each other

$$\therefore \quad (x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$$

is the required equation of the sphere.

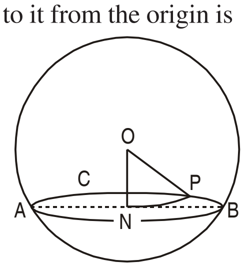
EXAMPLE

1. *A plane passes through a fixed point (a, b, c); show that the locus of the foot of the perpendicular to it from the origin is the sphere* $x^2 + y^2 + z^2 - ax - by - cz = 0$.

Sol. Any plane through *(a, b, c)* is

$$l(x - a) + m(y - b) + n(z - c) = 0 \qquad \text{....(1)}$$

and the line perpendicular to it from the origin is



$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \qquad \text{....(2)}$$

The foot of the perpendicular is the point of intersection of (1) and (2). Thus, to find the locus of the foot of perpendicular, one should eliminate *l, m, n* between (1) and (2), *i.e.,*

$$x(x - a) + y(y - b) + z(z - c) = 0.$$

$$\Rightarrow \quad x^2 + y^2 + z^2 - ax - by - cz = 0$$

EXERCISES

1. Find the centres and the radii of the following spheres :

(i) $x^2 + y^2 + z^2 - 6x + 8y - 10z + 1 = 0$,

(ii) $x^2 + y^2 + z^2 + 2x - 4y - 6z + 5 = 0$,

(iii) $2x^2 + 2y^2 + 2z^2 - 2x + 4y + 2z + 3 = 0.$

$$\left[\textbf{Ans}. (i)\ (3, -4, 5);\ 7\ (ii)\ (-1, 2, 3);\ 3\ (iii) \left(\frac{1}{2}, -1, -\frac{1}{2}\right);\ 0\right]$$

2. Obtain the equation of the sphere described on the join of the points

$$A\,(2, -3, 4)\ B\,(-5, 6, -7)$$

as diameter. [**Ans.** $x^2 + y^2 + z^2 + 3(x - y + z) - 56 = 0$]

3. Prove that the equation $ax^2 + ay^2 + az^2 + 2ux + 2vy + 2wz + d = 0$ represents a sphere. Find its radius and centre.

$$\left[\textbf{Ans}. \frac{\sqrt{\Sigma u^2 - ad}}{a};\ \left(-\frac{u}{a}, -\frac{v}{a}, -\frac{w}{a}\right)\right]$$

4. Through a point P three mutually perpendicular straight lines are drawn; one passes through a fixed point C on the z-axis, while the others intersect the x-axis and y-axis, respectively; show that the locus of P is a sphere of which C is the centre.

4.2 THE SPHERE THROUGH FOUR GIVEN POINTS

The general equation of a sphere contains *four* parameters and, as such a sphere can be uniquely determined so as to satisfy four conditions, each of which is such that it gives rise to one linear relation between the constants.

In particular, we can find a sphere through four non-coplanar points

$$(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4)$$

Let $$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \qquad \text{...(i)}$$

be the equation of the sphere through the four given points.

We have then the linear equation

$$x_1^2 + y_1^2 + z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d = 0 \qquad \text{...(ii)}$$

and three more similar equations corresponding to the remaining three points so that we obtain a system of four linear equations in four unknowns *u, v, w, d.* We solve these equations and substituting the values thus obtained for *u, v, w, d* in (i); we get the required equation.

EXAMPLES

1. *Find the equation to the sphere through the points (0, 0, 0), (0, 1, –1), (–1, 2, 0), (1, 2, 3).*

Sol. Let the equation of the sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \qquad \text{...(i)}$$

As it passes through given points, we have

$$d = 0;$$
$$2 + 2v - 2w + d = 0;$$
$$2 + 2v - 2w + d = 0;$$
$$4 + 2u + 4v + 6w + d = 0;$$

yielding $u = -\frac{15}{14}, v = -\frac{25}{14}, w = -\frac{11}{14}$ and $d = 0.$

Hence, the equation of the sphere becomes

$$x^2 + y^2 + z^2 - \frac{15}{7}x - \frac{25}{7}y - \frac{11}{7}z = 0$$

or $$7(x^2 + y^2 + z^2) - 15x - 25y - 11z = 0$$

2. *Prove that the centres of the spheres which touch the lines* $y = mx, z = c;\ y = -mx, z = -c;$ *lie upon the conicoid* $mxy + cz(1 + m^2) = 0.$

Sol. Let the equation of the sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

Line $y = mx, z = c$ meets it where

$$x^2 + m^2x^2 + c^2 + 2ux + 2vmx + 2wc + d = 0$$

or $$(1 + m^2)\, x^2 + 2\,(u + vm)\, x + (c^2 + 2wc + d) = 0$$

The line is given to be tangent to the sphere; hence the two values of x given by above equation must be coincident, which gives

$$(u + vm)^2 = (1 + m^2)\,(c^2 + 2wc + d) \qquad \text{...(1)}$$

In the same way, line $y = - mx, z = - c$ will touch the sphere, if

$$(u - vm)^2 = (1 + m^2)\,(c^2 - 2wc + d) \qquad \text{...(2)}$$

Subtracting (2) from (1), we have

$$4uvm = 4wc\,(1 + m^2)$$

or $$uvm - wc\,(1 + m^2) = 0$$

Hence, the locus of centre $(-u, -v, -w)$ will be

$$xym + zc\,(1 + m^2) = 0$$

3. *A variable plane through a fixed point (a, b, c) cuts the co-ordinate axes in the point A, B, C. Show that the locus of the centres of the sphere OABC is*

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2$$

Sol. Let the sphere *OABC* be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz = 0$$

so that *u, v, w* are different for different spheres. The points *A, B, C* where it cuts the three axes are $(-2u, 0, 0), (0, -2v, 0), (0, 0, -2w)$. The equation of the plane *ABC* is

$$\frac{x}{-2u} + \frac{y}{-2v} + \frac{z}{-2w} = 1$$

Since this plane passes through *(a, b, c)* we have

$$\frac{a}{-2u} + \frac{b}{-2u} + \frac{c}{-2w} = 1 \qquad \text{...(2)}$$

If *x, y, z* be the centre of the sphere (1), then

$$x = -u, y = -v, z = -w \qquad \text{...(3)}$$

From (2) and (3), we obtain

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 2$$

as the required locus.

4. *A sphere of constant radius k passes through the origin and cuts the axes in A, B and C. Find the locus of the centroid of the triangle ABC.*

Sol. Let the co-ordinates of *A, B, C* be $(a, 0, 0), (0, b, 0)$ and $(0, 0, c)$ respectively. The sphere also passes through the origin (0, 0, 0).

Let the equation of the sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

As it passes through $(0, 0, 0), (a, 0, 0), (0, b, 0)$ and $(0, 0, c)$, we have $d = 0$,

$$a^2 + 2ua + d = 0 \;\Rightarrow\; u = -\frac{1}{2}a,\; v = -\frac{1}{2}b,\; w = -\frac{1}{2}c$$

$\therefore$ Required equation of sphere is

$$x^2 + y^2 + z^2 - ax - by - cz = 0$$

Its radius $= \sqrt{\left(\frac{1}{2}a\right)^2 + \left(\frac{1}{2}b\right)^2 + \left(\frac{1}{2}c\right)^2} = k$

$$\Rightarrow \quad a^2 + b^2 + c^2 = 4k^2$$

If (x_1, y_1, z_1) be the co-ordinates of the centroid of ΔABC, then

$$x_1 = \frac{1}{3}a, \ y_1 = \frac{1}{3}b, \ z_1 = \frac{1}{3}c \ \Rightarrow \ a = 3x_1, b = 3y_1, c = 3z_1$$

Substituting these values in (ii), and generalizing, the required locus is

$$9(x^2 + y^2 + z^2) = 4k^2$$

EXERCISES

1. Find the equation of the sphere through the four points

$$(4, -1, 2), (0, -2, 3), (1, -5, -1), (2, 0, 1)$$

[**Ans.** $x^2 + y^2 + z^2 - 4x + 6y - 2z + 5 = 0$]

2. Find the equation of the sphere through the four points.

$$(0, 0, 0)\ (-a, b, c)\ (a, -b, c)\ (a, b, -c)$$

and determine its radius.

$$\left[\textbf{Ans. } \frac{x^2 + y^2 + z^2}{a^2 + b^2 + c^2} - \frac{x}{a} - \frac{y}{b} - \frac{z}{c} = -0\,;\ \frac{1}{2}(a^2 + b^2 + c^2)\sqrt{a^{-2} + b^{-2} + c^{-2}}\right]$$

3. Find the equation of the sphere passing through the origin and the points (1, 0, 0), (0, 2, 0) and (0, 0, 3). [**Ans.** $x^2 + y^2 + z^2 - x - 2y - 3z = 0$]

4. Obtain the equation of the sphere circumscribing the tetrahedron whose faces are

$$x = 0, \ y = 0, \ z = 0, \ \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

5. Obtain the equation of the sphere which passes through the three points

$$(1, 0, 0), (0, 1, 0), (0, 0, 1)$$

and has its radius as small as possible. [**Ans.** $3(x^2 + y^2 + z^2) - 2(x + y + z) - 1 = 0$]

6. Show that the equation of the sphere passing through the three points (3, 0, 2), (–1, 1, 0), (2,–5, 4) and having its centre on the plane $2x + 3y + 4z = 6$ is :

$$x^2 + y^2 + z^2 + 4y - 6z = 1$$

7. Obtain the sphere having its centre on the line $5y + 2z = 0 = 2x - 3y$ and passing through the two points $(0, -2, -4), (2, -1, -1)$. [**Ans.** $x^2 + y^2 + z^2 - 6x - 4y + 10z + 12 = 0$]

8. Find the equation to a sphere passing through the points $(1, -3, 4), (1, -5, 2), (1, -3, 0)$ and having centre on the plane $x + y + z = 0$. [**Ans.** $x^2 + y^2 + z^2 - 2x + 6y - 4z + 10 = 0$]

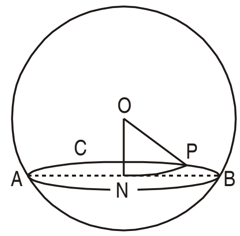
9. A sphere of constant radius r passes through the origin O and cuts the axes in A, B, C. Find the locus of the foot of the perpendicular from O to the plane ABC.

[**Ans.** $(x^2 + y^2 + z^2)^2 (x^{-2} + y^{-2} + z^{-2}) = 4r^2$]

10. If O be the centre of a sphere of radius unity and A, B be two points in a line with O such that $OA \cdot OB = 1$, and if P be a variable point no the sphere, show that

$$PA : PB = \text{constant}$$

4.3.1 Plane Section of a Sphere



Consider a sphere and a plane. We suppose that the sphere and the plane have points on common, *i.e.,* intersect. The set of points common to a sphere and a plane, assuming that the sphere and the plane intersect, is called a *Plane Section* of a sphere. We show that *the locus of points common to a sphere and a plane is a circle, i.e., a plane section of a sphere is a circle.*

Let O be the centre of the sphere and P, a point on the plane section. Let ON be the perpendicular to the given plane; N being the foot of the perpendicular.

As ON is perpendicular to the plane which contains the line NP, we have

$$ON \perp NP \;\Rightarrow\; NP^2 = OP^2 - ON^2$$

Now, O and N being fixed points, this relation shows that NP is constant for all positions of P on the section.

Hence, the locus of P is a circle whose centre is the point N, *viz.,* the locus of the perendicular from the centre of the sphere to the plane.

The section of a sphere by a plane through its centre is known as a *great-circle.*

The centre and radius of a great circle are the same as those of the sphere.

There exist points of intersection of a sphere and a plane if and only if the distance of the centre of the sphere from the plane is less than or equal to the radius of the sphere.

Thus, the sphere

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

will intersect the plane.

$$lx + my + nz = p$$

if and only if

$$(ul + vm + wn + p)^2 \le (l^2 + m^2 + n^2)(u^2 + v^2 + w^2 - d)$$

Cor. *The circle through three given points lies entirely on any sphere through the same three points.*

Thus, the condition of a sphere containing a given circle is equivalent to that of its passing through any three of its points.

4.3.2 Intersection of Two Spheres

We now consider two spheres and assume that the given spheres have points in common, *i.e.,* intersect. Assuming that two given spheres intersect, we show that *the locus of the points of intersection of two spheres is a circle.*

The co-ordinates of points, if any, common to the two spheres

$$S_1 \equiv x^2 + y^2 + z^2 + 2u_1 x + 2w_1 z + d_1 = 0$$

$$S_2 \equiv x^2 + y^2 + z^2 + 2u_2 x + 2v_2 y + 2w_2 z + d_2 = 0$$

satisfy both these equations and therefore, they also satisfy the equation

$$S_1 - S_2 \equiv 2x(u_1 - u_2) + 2y(v_1 - v_2) + 2z(w_1 - w_2) + (d_1 - d_2) = 0$$

which, being of the first degree, represents a plane.

Thus, the points of intersection of the two spheres are the same as those of any one of them and this plane and, therefore, they lie on a circle.

4.3.3 Sphere With a Given Diameter

To find the equation of the sphere described on the segment joining the points

$$A(x_1, y_1, z_1),\; B(x_2, y_2, z_2)$$

as a diameter

Let $P(x, y, z)$ be a point on the sphere described on the segment AB as diameter.

Since the section of the required sphere by the plane through the three points P, A, B is a great circle having AB as diameter, the point P lies on a semi-circle and, therefore,

$$PA \perp PB$$

The direction cosines of PA, PB being proportional to

$$x - x_1, y - y_1, z - z_1 \text{ and } x - x_2, y - y_2, z - z_2$$

they will be perpendicular, if

$$(x - x_1)(x - x_2) + (y - y_1) + (z - z_1)(z - z_2) = 0$$

which is the required equation of the sphere.

4.4 EQUATIONS OF A CIRCLE

A circle is the intersection of its plane with some sphere through it. As such, a circle can be represented by two equation, representing a sphere and the other a plane.

Thus, the two equations

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0, \quad lx + my + nz = p$$

taken together represent a circle.

A circle can also be represenetd by the equations of any two spheres through it.

Note : The reader may note that the equations

$$x^2 + y^2 + z^2 + 2fy + c = 0, \quad z = 0$$

also represented a circle which is the intersection of the cylinder.

$$x^2 + y^2 + z^2 + 2fy + c = 0$$

with the plane.

EXAMPLES

1. *A variable plane is parallel to the given plane* $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 0$ *and meets the axes in A, B, C. Prove that the circle ABC lies on the cone.*

$$yz\left(\frac{b}{c} + \frac{c}{b}\right) + zx\left(\frac{c}{a} + \frac{a}{c}\right) + xy\left(\frac{a}{b} + \frac{b}{a}\right) = 0$$

Sol. Let the variable plane, which is parallel to the given plane, be

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = k$$

This meets the axes in $A(ak, 0, 0), B(0, bk, 0)$ and $C(0, 0, ck)$. Hence, equation of the sphere $OABC$ is

$$x^2 + y^2 + z^2 - akx - bky - ckz = 0$$

or $$x^2 + y^2 + z^2 - k(ax + by + cz) = 0 \qquad ...(2)$$

The circle ABC lies on both, the plane (1) and the sphere (2). Hence, (1) and (2) together represented the circle ABC and the locus of the circle ABC will be obtained by eliminating k from (1) and (2). Thus, the locus of circle ABC is

$$(x^2 + y^2 + z^2) - \left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right)(ax + by + cz) = 0$$

or $$yz\left(\frac{b}{c} + \frac{c}{b}\right) + zx\left(\frac{c}{a} + \frac{a}{c}\right) + xy\left(\frac{a}{b} + \frac{b}{a}\right) = 0$$

2. *If r be the radius of the circle*

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0, \; lx + my + nz = 0$$

prove that

$$(r^2 + d)(l^2 + m^2 + d^2) = (mw - nv)^2 + (nu - hv)^2 + (lv - mu)^2$$

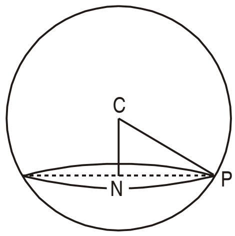
Sol. The equation of the given sphere is

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \qquad ...(1)$$

having centre at $(-u, -v, -w)$ and radius $CP = \sqrt{u^2 + v^2 + w^2 - d}$.

Now distance CN of centre of sphere from the plane is length of perpendicular from centre of sphere on the plane

$$lx + my + nz = 0$$



Hence,
$$CN = \frac{lu + mv + nu}{\sqrt{l^2 + m^2 + n^2}}$$

$$CP = \sqrt{u^2 + v^2 + w^2 - d}\ , NP = r$$

$$r^2 = CP^2 - CN^2 = u^2 + v^2 + w^2 - d - \frac{(lu + mv + nw)^2}{l^2 + m^2 + n^2}$$

$$\Rightarrow \quad (r^2 + d)(l^2 + m^2 + n^2) = (u^2 + v^2 + w^2)(l^2 + m^2 + n^2) - (lu + mv + nw)^2$$

$$\Rightarrow \quad (r^2 + d)(l^2 + m^2 + n^2) = (mw - nv)^2 + (nu - lw)^2 + (lv - mu)^2$$

by Lagrange's identity.

3. *A is a point on OX and B on OY so that the angle OAB is constant (= α). On AB as diameter a circle is described whose plane is parallel to OZ. Prove that as AB varies, the circle generates the cone* $2xy - z^2 \sin 2\alpha = 0$.

Sol. Let A be the point $(a, 0, 0)$ and $B(0, b, 0)$.

Then since $\angle OAB = \alpha$, we have

$$\tan\alpha = \frac{b}{a}. \qquad ...(1)$$

Now a sphere on AB as diameter is

$$(x - a)x + (y - b)y + z^2 = 0$$

$$\Rightarrow \quad x^2 + y^2 + z^2 = ax + by \qquad ...(2)$$

A plane through AB parallel to OZ is

$$\frac{x}{a} + \frac{y}{b} = 1 \qquad ...(3)$$

The required circle is given by intersection of (2) and (3). Now to determine the locus of the circle, we will eliminate a and b from (2), with the help of (1) and (3). So, we have

$$x^2 + y^2 + z^2 = (ax + by)\left(\frac{x}{a} + \frac{y}{b}\right)$$

$$\Rightarrow \quad x^2 + y^2 + z^2 = x^2 + y^2 + xy\left(\frac{a}{b} + \frac{b}{a}\right)$$

$$\Rightarrow \quad z^2 = \frac{2xy}{\sin 2\alpha}$$

$$\Rightarrow \quad 2xy - z^2 \sin 2\alpha = 0$$

EXERCISES

1. Find the centre and the radius of the circle

$$x + 2y + 2z = 15, \quad x^2 + y^2 + z^2 - 2y - 4z = 11$$ [**Ans.** $(1, 3, 4), \sqrt{7}$]

2. Find the equations of that section of the sphere

$$x^2 + y^2 + z^2 = a^2$$

of which a given internal point (x_1, y_1, z_1) is the centre.

[**Hint :** The plane through (x_1, y_1, z_1) drawn perpendicular to the line joining this point to the centre (0, 0, 0) of the sphere determines the required section.]

[**Ans.** $x^2 + y^2 + z^2 = a^2$, $xx_1 + yy_1 + zz_1 = x_1^2 + y_1^2 + z_1^2$]

3. Obtain the equations of the circle lying on the sphere

$$x^2 + y^2 + z^2 - 2x + 4y - 6z + 3 = 0$$

and having its centre at (2, 3, –4).

[**Ans.** $x^2 + y^2 + z^2 - 2x + 4y - 6z + 3 = 0 - x + 5y - 7z - 45$]

4. O is the origin and A, B, C are the points.

$$(4a, 4b, 4c), (4b, 4c, 4a), (4c, 4a, 4b)$$

Show that the sphere

$$x^2 + y^2 + z^2 - 2(x + y + z)(a + b + c) + 8(bc + ca + ab) = 0$$

passes through the nine point circles of the faces of the tetrahedron $OABC$.

5. Find the equation of the diameter of the sphere $x^2 + y^2 + z^2 = 29$ such that a rotation about it will transfer the point (4, –3, 2) to the point (5, 0, –2) along a great circle of the sphere. Find also the angle through which the sphere must be so rotated.

$$\left[\textbf{\textit{Ans.}}\ \frac{x}{2} = \frac{y}{6} = \frac{z}{6}, \cos^{-1}\left(\frac{16}{29}\right)\right]$$

6. Show that the following sets of points are concyclic :

(i) (5, 0, 2), (2, –6, 0), (7, –3, 8), (4, –9, 6)

(ii) (–8, 5, 2), (–5, 2, 2), (–7, 6, 6), (–4, 3, 6).

4.4.1 Sphere Through a Given Circle

The equation

$$S + kU = 0$$

represents a sphere through the circle with equations

$$S = 0, \ U = 0$$

where $S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d$

$U \equiv lx + my + nz - p$

Thus, the set of spheres through the circle

$$S = 0, U = 0,$$

is $\{S + kU = 0;\ k \text{ is the parameter}\}$

Also the equation

$$S + kS' = 0$$

represents a sphere through the circle with equations

$$S = 0, S' = 0,$$

where

$$S' \equiv x^2 + y^2 + z^2 + 2u'x + 2v'y + 2w'z + d'$$

for all values of k,

The set of spheres through the circle

$$S = 0, \quad S' = 0$$

is thus $\{S + kS' = 0;\ k \text{ is the parameter}\}$.

Here k is the a parameter which may be so chosen that these equations fulfil one more conditions.

Note 1. We notice that the equation of the plane of the circle through the two given spheres

$$S = 0, S' = 0$$

is $S - S' = 2(u - u')x + 2(v - v')y + 2(w - w')z + d - d' = 0$

From this we see that the equation of any sphere through the circle

$$S = 0, S' = 0$$

is also of the form $\quad S + k(S - S') = 0$

k, being the parameter.

This from sometimes proves comparatively more convenient.

Note 2. It is important to remember that the general equation of a sphere through the circle

$$x^2 + y^2 + 2gx + 2fy + c = 0, z = 0$$

is $\quad x^2 + y^2 + z^2 + 2gx + 2fy - 2kz + c = 0,$

where k is the parameter.

EXAMPLES

1. *Find the equation of the sphere through the circle*

$$x^2 + y^2 + z^2 = 9, \ 2x + 3y + 4z = 5$$

and the point (1, 2, 3).

Sol. The sphere

$$x^2 + y^2 + z^2 - 9 + k(2x + 3y + 4z - 5) = 0$$

passes through the given circle for all values of k.

It will pass through (1, 2, 3) if

$$5 + 15k = 0 \quad \Rightarrow \quad k = -\frac{1}{3}$$

The required equation of the sphere, therefore, is

$$3(x^2 + y^2 + z^2) - 2x - 3y - 4z - 22 = 0$$

2. *Show that the two circles*

$$x^2 + y^2 + z^2 - y + 2z = 0, x - y + z - 2 = 0;$$

$$x^2 + y^2 + z^2 + x - 3y + z - 5 = 0, 2x - y + 4z - 1 = 0;$$

lie on the same sphere and find its equation.

Sol. The equation of any sphere through the first circle is

$$x^2 + y^2 + z^2 - y + 2z + k(x - y \ \ z - 2) = 0,$$

and that of *any* sphere through the second circle is

$$x^2 + y^2 + z^2 + x - 3y + z - 5 + k'(2x - y + 4z - 1) = 0$$

The equations (i) and (ii) will represent the same sphere, if k, k' can be chosen so as to satisfy the four lines equations.

$$k = 2k' + 1, -1 - k = -k' - 3$$

$$2 + k = 4k' + 1, \ -2k = -k' - 5$$

The first two of these equations give $k = 3, k' = 1$, and these values clearly satisfy the remaining two equations also. These four equations in k, k' being consistent, the two circles lie on the same sphere, viz.,

$$x^2 + y^2 + z^2 - y + 2z + 3(x - y + z - 2) = 0$$

$$\Rightarrow \quad x^2 + y^2 + z^2 + 3x - 4y + 5z - 6 = 0$$

3. *Prove that the plane* $x + 2y - z = 4$ *cuts the sphere* $x^2 + y^2 + z^2 - x + z - 2 = 0$ *in a circle of radius unity and find the equation of sphere which has this circle for one of its great circle.*

Sol. The centre of the given sphere is (1/2, 0, –1/2) and its radius

$$= \sqrt{\left(\frac{1}{2}\right)^2 + 0^2 + \left(-\frac{1}{2}\right)^2 - (-1)} = \sqrt{\frac{5}{2}} = r.$$

Length of perpendicular from (1/2, 0, –1/2) to the plane is

$$\frac{1}{2}\sqrt{6} = p \quad \text{(say)}$$

$\therefore$ Radius of circle $= \sqrt{r^2 - p^2} = \sqrt{\frac{5}{2} - \frac{6}{4}} = 1.$

Now, equation of a sphere through given circle is

$$x^2 + y^2 + z^2 - x + z - 2 + \lambda\,(x + 2y - z - 4) = 0$$

or
$$x^2 + y^2 + z^2 + (\lambda - 1)\,x + 2\lambda y + (1 - \lambda)\,z - (2 + 4\lambda) = 0 \qquad ...(1)$$

Its centre is $[-(\lambda - 1)/2, -\lambda, -(1 - \lambda)/2]$

If the circle is a great circle of the sphere (1), then its centre should lie on the plane

$$x + 2y - z - 4 = 0$$

of the circle.

$$-\frac{1}{2}(\lambda - 1) + 2\,(-\lambda) + \frac{1}{2}(1 - \lambda) - 4 = 0$$

or
$$-3\lambda - 3 = 0 \quad \text{or} \quad \lambda = -1.$$

From (1), the equation of required sphere is

$$x^2 + y^2 + z^2 - 2x - 2y + 2z + 2 = 0$$

EXERCISES

1. Find the equation of the sphere through the circle

$$x^2 + y^2 + z^2 + 2x + 3y + 6 = 0,\ \ x - 2y + 4z - 9 = 0,$$

and the centre of the sphere

$$x^2 + y^2 + z^2 - 2x + 4y - 6z + 5 = 0$$

[**Ans.** $x^2 + y^2 + z^2 + 7y - 8z + 24 = 0$]

2. Find the equation to the sphere which passes through point (α, β, γ) and the circle $x^2 + y^2 = a^2, z = 0.$ [**Ans.** $(x^2 + y^2 + z^2 - a^2)\,\gamma + (a^2 - \alpha^2 - \beta^2 - \gamma^2)\,z = 0$]

3. Show that the equation of the sphere having its centre on the plane

$$4x - 5y - z = 3$$

and passing through the circle with equations

$$x^2 + y^2 + z^2 - 2x - 3y + 4z + 8 = 0,\ x^2 + y^2 + z^2 + 4x + 5y - 6z + 2 = 0$$

is
$$x^2 + y^2 + z^2 + 7x + 9y - 11z - 1 = 0.$$

4. Obtain the equation of the sphere having the circle

$$x^2 + y^2 + z^2 + 10y - 4z - 8 = 0,\ \ x + y + z = 3$$

as the great circle.

[**Hint :** The centre of the required sphere lies on the plane $x + y + z = 3$]

[**Ans.** $x^2 + y^2 + z^2 - 4x + 6y - 8z + 4 = 0$]

5. A sphere S has point (0, 1, 0), (3, –5, 2) at opposite ends of a diameter. Find the equation of the sphere having the intersection of the sphere S with the plane

$$5x - 2y + 4z + 7 = 0$$

as a great circle. [**Ans.** $x^2 + y^2 + z^2 + 2x + 2y + 2z + 2 = 0$]

6. Obtain the equation of the sphere which passes through the circle $x^2 + y^2 = 4, z = 0$ and is cut by the plane $x + 2y + 2z = 0$ in a circle of radius 3.

[**Ans.** $x^2 + y^2 + z^2 = 6z - 4 = 0$]

7. Show that the two circles

$$2(x^2+y^2+z^2)+8x-13y+17z-17=0,\ 2x+y-3z+1=0;$$
$$x^2+y^2+z^2+3x-4y+3z=0,\ x-y+2z-4=0;$$

lie on the same sphere and find its equation.

[**Ans**. $x^2+y^2+z^2+5x-6y+7z-8=0$]

8. Prove that the circles

$$x^2+y^2+z^2-2x+3y+4z-5=0;\ 5y+6z+1=0;$$
$$x^2+y^2+z^2-3x-4y+5z-6=0;\quad x+2y-7z=0$$

lie on the same sphere and find its equation.

[**Ans**. $x^2+y^2+z^2-2x-2y-2z-6=0$]

4.5 INTERSECTION OF A SPHERE AND A LINE

Let $$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad ...(1)$$

and $$\frac{x-\alpha}{1}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}, \quad ...(2)$$

be the equations of a sphere and a line respectively.

The point $(lr+\alpha, mr+\beta, nr+\gamma)$ which lies on the given line (2) for all values of r, will also lie on the given sphere (1), for those of the values of r which satisfy the equation

$$r^2(l^2+m^2+n^2)+2r[l(\alpha+u)+m(\beta+v)+n(\lambda+w)]$$
$$+(\alpha^2+\beta^2+\gamma^2+2u\alpha+2v\beta+2w\gamma+d)=0 \quad ...(A)$$

and this latter being a quadratic equation in r, gives two values say r_1, r_2 of r. We suppose that the equation has real roots so that r_1, r_2 are real. Then

$$(lr_1+\alpha, mr_1+\beta, nr_1+\gamma), (lr_2+\alpha, mr_2+\beta, nr_2+\gamma)$$

are the two points of intersection.

Example : Find the co-ordinates of the points where the line

$$\frac{1}{4}(x+3)=\frac{1}{3}(y+4)=-\frac{1}{5}(z-8)$$

intersects the sphere

$$x^2+y^2+z^2+2x-10y=23$$ [**Ans**. (1, – 1, 3); (5, 2, – 2)]

4.5.1 Power of a Point

Let l, m, n be the direction cosines of the given line (2) in 6.5, so that $l^2+m^2+n^2=1$. Then, r_1, r_2 are the distances of the points $A(\alpha,\beta,\gamma)$ from the points of intersection P and Q and we have

$$AP.AQ=r_1r_2=\alpha^2+\beta^2+\gamma^2+2u\alpha+2v\beta+2w\gamma+d$$

which is independent of the direction cosines l, m, n.

Thus, *if from a fixed point A, chords be drawn in any direction to intersect a given sphere in P and Q, then AP.AQ is constant.* This constant is called the *Power* of the point A with respect to the sphere.

EXAMPLE

1. *Show that the sum of the squares of the intercepts made by a given sphere on any three mutually perpendicular straight lines through a fixed point is constant.*

Sol. Take the fixed point O as the origin and *any* three mutually perpendicular lines through it as the co-ordinate axes. With this choice of axes, let the equation of the given sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

The X-axis, $(y = 0 = z)$ meets the sphere in points given by

$$x^2 + 2ux + d = 0$$

so that if x_1, x_2 be its roots, the two points of intersection are $(x_1, 0, 0), (x_2, 0, 0)$.

Also we have

$$x_1 - x_2 = -2u, \quad x_1x_2 = d$$

$$(\text{intercept on } X\text{-axis})^2 = (x_1 - x_2)^2 = (x_1 + x_2)^2 - 4x_1x_2 = 4(u^2 - d)$$

Similarly,

$$(\text{intercept on } Y\text{-axis})^2 = 4(v^2 - d)$$

$$(\text{intercept on } Z\text{-axis})^2 = 4(w^2 - d)$$

The sum of the squares of the intecepts

$$= 4(u^2 + v^2 + w^2 - 3d)$$

$$= 4(u^2 + v^2 + w^2 - d) - 8d = 4r^2 - 8p,$$

where r is the radius of the given sphere and p is the power of the given point with respect to the sphere.

Since the sphere and the point are both given, r and p are both constants.

Hence, the result.

Note : The coefficeints u, v, w and d in the equation of the sphere will be different for different sets of mutually perpendicular lines through O as axes.

Since the sphere is fixed and the point O is also fixed the expression

$$r^2 = u^2 + v^2 + w^2 - d$$

for the square of the radius and

$$p = d,$$

for the power of the point, with respect to the sphere will be invariant.

EXERCISES

1. Find the locus of a point whose powers with respect, to two given spheres are in a constant ratio.
2. Show that the locus of the mid-points of a system of parallel chords of a sphere is a plane through its centre perpendicular to the given chords.

4.6 EQUATION OF A TANGENT PLANE

To find the equation of the tangent plane at any point (α, β, γ) of the sphere

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

The point (α, β, γ) lies on the sphere.

$$\Rightarrow \quad \alpha^2 + \beta^2 + \gamma^2 + 2u\alpha + 2v\beta + 2w\gamma + d = 0 \quad \text{...(i)}$$

The points of intersection of any line

$$\frac{x - \alpha}{1} = \frac{y - \beta}{m} = \frac{z - \gamma}{n} = r \quad \text{...(ii)}$$

through (α, β, γ) with the given sphere are

$$(lr + \alpha, mr + \beta, nr + \gamma)$$

where the values of r are the roots of the quadratic equation

$$r^2(l^2 + m^2 + n^2) + 2r[l(\alpha + u) + m(\beta + v) + n(\gamma + w)]$$
$$+ (\alpha^2 + \beta^2 + \gamma^2 + 2u\alpha + 2v\beta + 2w\gamma + d) = 0.$$

By virtue of the condition (i), one root of this quadratic equation is zero so that one of the points of intersection coincides with (α, β, γ).

In order that the second point of intersection may also coincide with (α, β, γ) the second value of r must also vanish and this requires,

$$l(\alpha + y) + m(\beta + v) + n(\gamma + w) = 0 \qquad \text{...(iii)}$$

Thus, the line

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$$

meets the sphere in two coincident points at (α, β, γ) and so is a *tangent line* to it thereat for any set of values *l, m, n* which satisfy the condition (iii).

The locus of the tangent lines at (α, β, γ) obtained by eliminating *l, m, n* between the condition (iii) and the equations (ii) of the line is

$$(x-\alpha)(\alpha+u) + (y-\beta)(\beta+v) + (z-\gamma)(\gamma+w) = 0$$

$$\Leftrightarrow \quad \alpha x + \beta y + \gamma z + u(x+\alpha) + v(y+\beta) + w(z+\gamma) + d$$

$$= \alpha^2 + \beta^2 + \gamma^2 + 2u\alpha + 2v\beta + 2w\gamma + d = 0 \qquad \text{[From (i)]}$$

which is a plane known as the *tangent plane* at (α, β, γ).

It follows that

$$(\alpha+u)x + (\beta+v)y + (\gamma+w)z + (u\alpha + v\beta + w\gamma + d) = 0$$

is the equation of the tangent plane to the given sphere at the given point (α, β, γ).

Cor. 1. *The line joining the centre of a sphere to any point on it is perpendicular to the tangent plane threat,* for the direction cosines of the line joining the centre $(-u, -v, -w)$ and the point (α, β, γ) on the sphere are proportional to

$$(\alpha + u, \beta + v, \gamma + w)$$

which are also the coefficients of *x, y, z* in the equation of the tangent plane at (α, β, γ). Hence, the result.

Cor. 2. If a plane or a line touches a sphere, then the length of the perpendicular from its centre to the plane or the line is equal to its radius.

Note : Any line in the tangent plane through its plane of contact touches the section of the sphere by any plane through the line.

EXAMPLES

1. *Show that the plane* $lx + my + nz = p$ *will touch the sphere*

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

$$(ul + vm + wn + p)^2 = (l^2 + m^2 + n^2)(u^2 + v^2 + w^2 - d)$$

Sol. Equating the radius $\sqrt{u^2 + v^2 + w^2 - d}$ of the sphere to the length of the perpendicular from the centre $(-u, -v, -w)$ to the plane

$$lx + my + nz = p,$$

2. *Find the equation of the sphere which touches the sphere*

$$x^2 + y^2 + z^2 - x + 3y + 2z - 3 = 0$$

at the point (1, 1, –1) and passes through the origin.

Sol. The tangent plane to the given sphere at (1, 1, –1) is

$$x + 5y - 6 = 0$$

The equation of the required sphere is, therefore,

$$x^2 + y^2 + z^2 - x + 3y + 2z - 3 + k(x + 5y - 6) = 0$$

where k is a suitably chosen number.

This will pass through the origin if $k = -1/2$.

Thus, the required equation is

$$2(x^2+y^2+z^2)-3x+y+4z=0$$

3. *Find the equation of the sphere through the circle,*

$$x^2+y^2+z^2=1,\ 2x=4y+5z=6$$

and touching the plane $z=0$.

Sol. The sphere

$$x^2+y^2+z^2-1+\lambda(2x+4y-5z-6)=0$$

passes through the given circle for all values of λ.

Its centre is $\left(-\lambda,-2\lambda,-\frac{5}{2}\lambda\right)$, and radius is $\left(\lambda^2+4\lambda^2+\frac{25}{4}\lambda^2+1+6\lambda\right)^2$

Since it touches $z=0$, we have by Cor. 2,

$$-\frac{5}{2}\lambda=\pm\left(5\lambda^2-\frac{25}{4}\lambda^2+1-6\lambda\right)$$

$$\Rightarrow \qquad 5\lambda^2+6\lambda+1=0,$$

This gives $\qquad \lambda=-1$ or $-\frac{1}{5}$

The two required spheres, therefore, are

$$x^2+y^2+z^2-2x-4y-5z+5=0$$

$$5(x^2+y^2+z^2)-2x-4y-5z+1=0$$

4. *If the tangent plane to the sphere* $x^2+y^2+z^2=r^2$ *makes intercepts a, b, c on the co-ordinate axes, show that*

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{r^2}.$$

Sol. The equation to the tangent plane at (α,β,γ) to the given sphere is

$$x\alpha+y\beta+z\gamma=r^2 \qquad \text{...(i)}$$

Given that a is the intercept made by the palne (i) on x-axis. So,

$$a\alpha=r^2 \quad\Rightarrow\quad \alpha=\frac{r^2}{a}$$

Similarly, $\qquad \beta=\frac{r^2}{b}$ and $\gamma=\frac{r^2}{c}$.

Also, as (α,β,γ) is a point on the sphere, so we have

$$\alpha^2+\beta^2+\gamma^2=r^2$$

$$\Rightarrow \qquad \left(\frac{r^2}{a}\right)^2+\left(\frac{r^2}{b}\right)^2+\left(\frac{r^2}{c^2}\right)^2=r^2$$

$$\Rightarrow \qquad \frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{r^2}.$$

EXERCISES

1. Find the equation of the tangent plane to the sphere

$$3(x^2+y^2+z^2)-2x-3y-4z-22=0$$

and the point (1, 2, 3). [**Ans.** $4x+9y+14z-64=0$]

2. Find the equation of the tangent line to the circle

$$x^2+y^2+z^2+5x-7y+2z-8=0,\quad 3x-2y+4z+3=0$$

and the point (–3, 5, 4). [**Ans.** $(x+3)/32=(y-5)/34=-(z-4)/7$]

3. Find the values of a for which the plane

$$x + y + z = a\sqrt{3}$$

touches the sphere $\quad x^2 + y^2 + z^2 - 2x - 2y - 2z - 6 = 0$ [**Ans.** $\pm\sqrt{3}$]

4. Find the equation of the tangent planes to the sphere

$$x^2 + y^2 + z^2 + 2x - 4y + 6z - 7 = 0$$

which intersects the line

$$6x - 3y - 2z = 0 = 3z + 2$$ [**Ans.** $2x - y + 4z = 5, 4x - 2y - z = 16$]

5. Show that the plane $2x - 2y + z + 12 = 0$ touches the sphere

$$x^2 + y^2 + z^2 - 2x - 4y + 2z = 3$$

and find the point of contact. [**Ans.** (–1, 4, –2)]

[**Hint :** The point of contact of a tangent plane is the point where the line through the centre perpendicular to the plane meets the sphere.]

6. Find the co-ordinates of the points on the sphere

$$x^2 + y^2 + z^2 - 4x + 2y = 4$$

the tangent planes at which are parallel to the plane

$$2x - y + 2z = 1$$ [**Ans.** (4, –2, 2), (0, 0, –2)]

7. Show that the equation of the sphere which touches the sphere

$$4(x^2 + y^2 + z^2) + 10x - 25y - 2z = 0$$

at the point (1, 2, –2) and passes through the point (–1, 0, 0) is

$$x^2 + y^2 + z^2 + 2x - 6y + 1 = 0$$

8. Obtain the equations of the tangent planes to the sphere

$$x^2 + y^2 + z^2 + 6x - 2z + 1 = 0,$$

which pass through the line

$$3(16 - x) = 3z = 2y + 30$$

[**Ans.** $2x + 2y - z - 2 = 0, x + 2y - 2z + 14 = 0$]

9. Obtain the equations of the sphere which pass through the circle

$$x^2 + y^2 + z^2 - 2x + 2y + 4z - 3 = 0, \quad 2x + y + z = 4$$

and touches the plane $3x + 4y = 14$.

[**Ans.** $x^2 + y^2 + z^2 + 2x + 4y + 6z - 11 = 0, x^2 + y^2 + z^2 - 2x + 2y + 4z - 3 = 0$]

10. Find the equation of the sphere which has its centre at the origin and which touches the line $2(x + 1) = 2 - y = z + 3$. [**Ans.** $9(x^2 + y^2 + z^2) = 5$]

11. Find the equation of the spheres of radius r which touch the three co-ordinates axes. How many such spheres are there ?

[**Ans.** $2(x^2 + y^2 + z^2) + 2\sqrt{2(\pm x \pm y \pm z)}\, r + r^2 = 0$; eight]

12. Prove that the equation of the sphere which lies in the octant $OXYZ$ and touches the co-ordinate planes is of the form

$$x^2 + y^2 + z^2 - 2\lambda(x + y + z) + 2\lambda^2 = 0.$$

Show that, in general, two spheres can be drawn throgh a given point to touch the co-ordinate planes and find for what positions of the point the spheres are (a) real, (b) coincident.

The distances of the centre from the co-ordinate planes are all equal to the radius so that we may suppose that λ is the radius and $(\lambda, \lambda, \lambda)$ is the centre, λ being the parameter.

4.6.1 Plane of Contact

To find the locus of the points of contact of the tangent planes which pass through a given point (α, β, γ) *and touch the sphere.*

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

The tangent plane

$$x(x' + u) + y(y' + v) + z(z' + w) + (ux' + vy' + wz' + d) = 0$$

at (x', y', z') will pass through the point (α, β, γ), if

$$\alpha(x' + u) + \beta(y' + v) + \gamma(z' + w) + (ux' + vv' + wz' + d) = 0$$

$$\Leftrightarrow \qquad x'(\alpha + u) + y'(\beta + v) + z'(\gamma + w) + (u\alpha + v\beta + w\gamma + d) = 0$$

which is the condition that the point (x', y', z') should lie on the plane

$$x(\alpha + u) + y(\beta + v) + z(\gamma + w) + (u\alpha + v\beta + w\gamma + d) = 0$$

It is called the *plane of contact* for the point (α, β, γ).

Thus, the locus of points of contact is the circle in which the plane cuts the sphere.

Ex. 1. Show that the line joining any point P to the centre of a given sphere is perpendicular to the plane of contact of P and if OP meets it in Q, then

$$OP.OQ = (\text{radius})^2$$

Ex. 2. Show that the planes of contact of all points on the line

$$\frac{x}{2} = \frac{(y-a)}{3} = \frac{(z+3a)}{4}$$

with respect to the sphere $x^2 + y^2 + z^2 = a^2$ pass through the line

$$-(2x + 3a)/13 = (y - a)/3 = z/1.$$

4.6.2 The Polar Plane

If a line drawn through a fixed point A meets a given sphere in points P, Q and a point R is taken on this line such that the segment AR is divided internally and externally by the points P, Q in the same ratio, then **the locus of R** *is a plane called the* **Polar Plane** *of A w.r.t. the sphere,*

Consider the sphere $\qquad x^2 + y^2 + z^2 = a^2$

and let A be the point (α, β, γ).

Let $R(x, y, z)$ be the co-ordinates of the point R on any line through A. The co-ordinates of the point dividing AR in the ratio $\lambda : 1$ are

$$\left[\left(\frac{\lambda x + \alpha}{\lambda + 1}\right), \left(\frac{\lambda y + \beta}{\lambda + 1}\right), \left(\frac{\lambda z + \gamma}{\lambda + 1}\right)\right]$$

This point will be on the sphere (1) for values of λ which are roots of the quadratic equation

$$\left(\frac{\lambda x + \alpha}{\lambda + 1}\right)^2 + \left(\frac{\lambda y + \beta}{\lambda + 1}\right)^2 + \left(\frac{\lambda z + \gamma}{\lambda + 1}\right)^2 = a^2,$$

$$\Leftrightarrow \qquad \lambda^2 (x^2 + y^2 + z^2 - a^2) + 2\lambda(\alpha x + \beta y + \gamma z - a^2) + (\alpha^2 + \beta^2 + \gamma^2 - a^2) = 0 \qquad \text{...(2)}$$

Its roots λ_1 and λ_2 are the ratios in which the points P, Q divide the segment AR.

Since P, Q divide th segment AR internally and externally in the same ratio, we have

$$\lambda_1 + \lambda_2 = 0$$

Thus, from (2), we have $\qquad \alpha x + \beta y + \gamma z - a^2 = 0 \qquad$ ...(3)

which is the relation satisfied by the co-ordinates (x, y, z) of R.

Hence, (3) is the locus of R. Clearly, it is a plane.

Thus, we have seen here that the equation of the polar plane of the point (α, β, γ) with respect to the sphere

$$x^2 + y^2 + z^2 = a^2$$

is $$\alpha x + \beta y + \gamma z = a^2$$

It may similarly be shown that the polar plane of (α, β, γ) with respect to the sphere

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

is the plane $$(x + u)\, x + (\beta + v)\, y + (\gamma + w)\, z + (u\alpha + v\beta + w\gamma + d) = 0.$$

On comparing the equation of the polar plane with that of the tangent plane (4.6) and the plane of contact (4.6.1), we see that the polar plane of a point lying on the sphere is the tangent plane at the point and that of a point, lying outside it, is its plane of contact.

Pole of a Plane : Def. *If π be the polar plane of a point P, then P is called the pole of the plane π.*

4.6.3 Pole of a Plane

To find the pole of the plane $$lx + my + nz = p \qquad \text{...(i)}$$

with respect to the sphere $$x^2 + y^2 + z^2 = a^2$$

If (α, β, γ) be the tangent pole, then we see that the equation (i) is identical with

$$\alpha x + \beta y + \gamma z = a^2 \qquad \text{...(ii)}$$

so that, on comparing (i) and (ii), we obtain

$$\frac{\alpha}{I} = \frac{\beta}{m} = \frac{\gamma}{m} = \frac{a^2}{p},$$

$$\Rightarrow \qquad \alpha = \frac{a^2 I}{p}, \beta = \frac{a^2 m}{p}, \gamma = \frac{a^2 n}{p}.$$

Thus, the point

$$\left(\frac{a^2 l}{p}, \frac{a^2 m}{p}, \frac{a^2 n}{p}\right)$$

is the pole of the plane $lx + my + nz = p$, w.r.t. the sphere $x^2 + y^2 + z^2 = a^2$.

4.6.4 Some Results Concerning Poles and Polars

In the following discussion, we shall always take the equation of a sphere in the form

$$x^2 + y^2 + z^2 = a^2$$

1. *The line joining the centre O of a sphere to any point P is perpendicular to the polar plane of P.*

The direction ratios of the line joining the centre O (0, 0, 0) to the point $P(\alpha, \beta, \gamma)$ are α, β, γ and these are also the direction ratios of the normal to the polar plane $\alpha x + \beta y + \gamma z = a^2$ of $P(\alpha, \beta, \gamma)$.

2. *If the line joining the centre O of a sphere to a point P meets the polar plane of P in Q, then*

$$OP.OQ = a^2,$$

where a is the radius of the sphere.

We have $$OP = \sqrt{\alpha^2 + \beta^2 + \gamma^2}$$

Also, OQ, which is the length of perpendicular from the centre $O(0, 0, 0)$ to the polar plane $\alpha x + \beta y + \gamma z = a^2$ of P is given by

$$OQ = \frac{a^2}{\sqrt{\alpha^2 + \beta^2 + \gamma^2}}$$

Hence, the result.

3. *If the polar plane of a point P passes through a point Q, then the polar plane Q passes through P.*

The condition that the polar plane

$$\alpha_1 x + \beta_1 y + \gamma_1 z = a^2$$

of $P(\alpha_1, \beta_1, \gamma_1)$ passes through $Q(\alpha_2, \beta_2, \gamma_2)$ is

$$\alpha_1\alpha_2 + \beta_1\beta_2 + \gamma_1\gamma_2 = a^2$$

which is also, by symmetry or directly the condition that the polar plane of Q passes through P.

Conjugate Points : *Two points such that the polar plane of either passes through the other are called conjugate points.*

4. *If the pole of a plane π_1 lies on another plane π_2, then the pole of π_2 also lies on π_1.*

The condition that the pole

$$\left(\frac{a^2 l_1}{p_1}, \frac{a^2 m_1}{p_1}, \frac{a^2 n_1}{p_1}\right)$$

of the plane π_1 $\quad l_1 x + m_1 y + n_1 z = p_1$

lies on the plane π_2 $\quad l_2 x + m_2 y + n_2 z = p_2$

is $\quad a^2 (l_1 l_2 + m_1 m_2 + n_1 n_2) = p_1 p_2$

which is also, clearly, the condition that the pole

$$\left(\frac{a^2 l_2}{p_2}, \frac{a^2 m_2}{p_2}, \frac{a^2 n_2}{p_2}\right)$$

of π_2 lies on π_1.

Conjugate planes : *Two planes such that the pole of either lies on the other are called conjugate planes.*

5. *The polar planes of all the points on a line 1 pass through another line l'.*

The polar plane of any point

$$(lr + \alpha, mr + \beta, nr + \gamma)$$

on the line *l*.

$$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n}$$

is $\quad (lr + \alpha)\, x + (mr + \beta)\, y + (nr + \gamma)\, z = a^2$

$\Leftrightarrow \quad (\alpha x + \beta y + \gamma z - a^2) + r\,(lx + my + nz) = 0$

which clearly passes through the line

$$\alpha x + \beta y + \gamma z - a^2 = 0,\ \ lx + my + nz = 0,$$

whatever value, *r,* may have. Hence, the result.

Let this line be l'. We shall now prove that the polar plane of every point on l' also passes through the line *l*.

Now, as the polar plane of any arbitrary point *P* on *l* passes through every point of l', therefore, the polar plane of every point of l', passes through the point *P* on *l* and as, *P* is arbitrary, it passes through every point of *l, i.e.,* it passes through *l*.

Polar Lines : *Two lines such that the polar plane of every point on either passes through the other are called* **Polar Lines.**

EXAMPLE

1. *Find the polar line of* $(x-1)/2=(y-2)/3=(z-3)/4$ *w.r.t. the sphere* $x^2+y^2+z^2=16.$

Sol. Any point on given line is $(2r+1, 3r+2, 4r+3)$. Polar plane of this point with respect to sphere is

$$x(2r+1)+y(3r+2)+z(4r+3)=16$$

i.e.,
$$(x+2y+3z-16)+r(2x+3y+4z)=0$$

This clearly, passes through the line

$$x+2y+3z-16=0=2x+3y+4z$$

which is the required polar line.

EXERCISES

1. Show that the polar line of

$$\frac{(x+1)}{2}=\frac{(y-2)}{3}=(z+3)$$

with respect to the sphere

$$x^2+y^2+z^2=1;$$

is the line

$$\frac{7x+3}{11}=\frac{2-7y}{5}=\frac{z}{-1}.$$

2. Show that if a line l is coplanar with the polar line of a line l' then l' is coplanar with the polar line of l.

3. If PA, QB be drawn perpendicular to the planes of Q and P respectively, with respect to a sphere, with centre O, then

$$\frac{PA}{QB}=\frac{OP}{OQ}.$$

4. Show that, for a given sphere, there exist an unlimited number of tetraheda such that each vertex is the pole of the opposite face with respect to the sphere.
(Such a tetrahedron is known as a *self-conjugate* or *self-polar* tetrahedron).

4.7 ANGLE OF INTERSECTION OF TWO SPHERES

Def. *The angle of intersection of two spheres at a common point is the angle between the tangent planes to them at that point* and is, therefore, also equal to the angle between the radii of the spheres to the common point, the radii being perpendicular to the respective tangent planes at the point.

The angle of intersection at every common point of the spheres is the same, for if P, P' be any two common points of C, C' the centres of the spheres, the triangles $CC'P$ and $CC'P'$ are congruent and accordingly

$$\angle CPC' = \angle CP'C'$$

The spheres are said to be **orthogonal** if the angle of intersection of two spheres is a right angle. In this case

$$CC'^2 = CP^2 + C'P^2$$

4.7.1 Condition for the Orthogonality of Two Spheres

To find the condition for the two spheres

$$x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0$$
$$x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0$$

to be orthogonal.

The spheres will be orthogonal if the *square of the distance between their centres is equal to the sum of the squares of their radii* and this requires

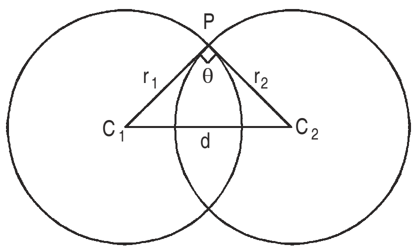
$$(u_1 - u_2)^2 + (v_1 - v_2)^2 + (w_1 - w_2)^2 = (u_1^2 + v_1^2 + w_1^2 - d_1) + (u_2^2 + v_2^2 + w_2^2 - d_2)$$

$$\Leftrightarrow \quad 2u_1u_2 + 2v_1v_2 + 2w_1w_2 = d_1 + d_2$$

EXAMPLES

1. *If d is the distance between the centres of two spheres of radii* r_1 *and* r_2, *prove that the angle between them is* $\cos^{-1}\{(r_1^2 + r_2^2 - d^2)/2r_1r_2\}$.

Sol. The angle of intersection, *i.e.,* the angle between the tangents at *P* is the angle between the radii of the two spheres joining *P*. Thus, $\angle C_1PC_2 = \theta$.



Applying cosine formulae to ΔC_1PC_2, we have

$$d^2 = r_1^2 + r_2^2 - 2r_1r_2 \cos\theta$$

$$\therefore \quad \cos\theta = (r_1^2 + r_2^2 - d^2)/2r_1r_2.$$

2. *Two spheres of radii* r_1 *and* r_2 *cut orthogonally. Prove that the radius of the common circle is* $r_1r_2/\sqrt{r_1^2 + r_2^2}$.

Sol. Let the common circle be

$$x^2 + y^2 = a^2, \; z = 0$$

The sphere $$x^2 + y^2 + z^2 + 2kz - a^2 = 0$$

passes through this circle for all values of *k*. Let the two given spheres through the circle be

$$x^2 + y^2 + z^2 + 2k_1z - a^2 = 0, \; x^2 + y^2 + z^2 + 2k_2z - a^2 = 0$$

We have

$$r_1^2 = k_1^2 + a^2, \; r_2^2 = k_2^2 + a^2 \quad \text{...(i)}$$

Since the spherses cut orthogonally, we have

$$2k_1k_2 = a^2 + a^2 = 2a^2 \quad \text{...(ii)}$$

From (i) and (ii), eliminating k_1, k_2 we have

$$(r_1^2 - a^2)(r_2^2 - a^2) = a^4 \Leftrightarrow a^2 = r_1^2r_2^2/(r_1^2 + r_2^2).$$

Hence, the result.

EXERCISES

1. Find the equation of the sphere that passes through the circle

$$x^2 + y^2 + z^2 - 2x + 3y - 4z + 6 = 0, \; 3x - 4y + 5z - 15 = 0$$

and cuts the sphere

$$x^2 + y^2 + z^2 + 2x + 4y - 6z + 11 = 0$$

orthogonally. [**Ans.** $5(x^2 + y^2 + z^2) - 13x + 19y - 25z + 45 = 0$]

2. Find the equation of the sphere that passes through the two points
$$(0, 3, 0), (-2, -1, 4)$$
and cuts orthogonally the two spheres
$$x^2 + y^2 + z^2 + x - 3z - 2 = 0, \ 2(x^2 + y^2 + z^2) + x + 3y + 4 = 0$$
[**Ans**. $x^2 + y^2 + z^2 + 2x - 2y + 4z - 3 = 0$]

3. Find the equation of the sphere which touches the plane
$$3x + 2y - z + 2 = 0$$
at the point (1, –2, 1) and cuts orthogonally the sphere
$$x^2 + y^2 + z^2 - 4x + 6y + 4 = 0$$
[**Ans**. $x^2 + y^2 + z^2 + 7x + 10y - 5z + 12 = 0$]

4. Show that every sphere through the circle
$$x^2 + y^2 - 2ax + r^2 = 0, \quad z = 0$$
cuts orthogonally every sphere through the circle
$$x^2 + z^2 = r^2, \ y = 0$$

5. Two points P, Q are conjugate with respect to a sphere S; show that the sphere on PQ as diameter cuts S orthogonally.

6. If two spheres S_1 and S_2 are orthogonal, the polar plane of any point on S_1 with respect to S_2 passes through the other end of the diameter of S_1 through P.

4.8 RADICAL PLANE

The locus of a point whose powers with respect to two spheres are equal is a plane perpendicular to the line joining their centres :

The powers of the point $P(x, y, z)$ with respect to the spheres
$$S_1 \equiv x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1 = 0;$$
$$S_2 \equiv x^2 + y^2 + z^2 + 2u_2x + 2v_2y + 2w_2z + d_2 = 0,$$
are $$x^2 + y^2 + z^2 + 2u_1x + 2v_1y + 2w_1z + d_1,$$
and $$x^2 + y^2 + z^2 + 2u_2x + 2v_2x + 2w_2z + d_2,$$
respectively.

Equating these, we obtain
$$\mathbf{2x\,(u_1 - u_2) + 2y\,(v_1 - v_2) + 2z\,(w_1 - w_2) + (d_1 - d_2) = 0,}$$
as the required locus. This locus being of the first degree in (x, y, z), represents a plane which is obviously perpendicular to the line joining the centres of the two spheres. This plane is called the *Radical plane* of the two spheres.

Thus, *the radical plane of the spheres*
$$S_1 = 0, \ S_2 = 0,$$
in both of which the coefficients of the second degree terms are equal to unity, is
$$S_1 - S_2 = 0$$
In case the two spheres intersect, the plane of their common circle is their radical plane (§4.3.2).

4.8.1 Radical Line

The three radical planes of three spheres intersect in a line,

If $$S_1 = 0, \ S_2 = 0, \ S_3 = 0$$
be the three spheres, their radical planes
$$S_1 - S_2 = 0, \quad S_2 - S_3 = 0, \ S_3 - S_1 = 0$$

clearly meet in the line

$$S_1 = S_2 = S_3$$

$$\Leftrightarrow \quad S_1 - S_2 = 0, \ S_2 - S_3 = 0$$

This line is called the *Radical line* of the three spheres.

4.8.2 Radical Centre

The four radical lines of four spheres taken three by three intersect at a point.

The point common to the three planes

$$S_1 = S_2 = S_3 = S_4$$

is clearly common to the radical lines, taken three by three, of the four spheres

$$S_1 = 0, S_2 = 0, S_3 = 0, S_4 = 0$$

This point is the intersection of the two lines

$$S_1 - S_2 = 0, S_2 - S_3 = 0; \ S_1 - S_3 = 0, S_2 - S_4 = 0.$$

This point is called the *Radical centre* of the four spheres.

4.8.3 Theorem

If $S_1 = 0, S_2 = 0$, *be two spheres, then the equation*

$$S_1 + \lambda S_2 = 0$$

λ *being the parameter, represents a system of spheres such that any two members , of the system have the same radical plane.*

Let $$S_1 + \lambda_1 S_2 = 0 \text{ and } S_1 + \lambda_2 S_2 = 0$$

be any two members of the system.

Making the coefficients of second degree terms unity, we write these equations in the form

$$\frac{S_1 + \lambda_1 S_2}{1 + \lambda_1} = 0, \ \frac{S_1 + \lambda_2 S_2}{1 + \lambda_2} = 0.$$

The radical plane of these two spheres is

$$\frac{S_1 + \lambda_1 S_2}{1 + \lambda_1} - \frac{S_1 + \lambda_2 S_2}{1 + \lambda_2} = 0,$$

$$\Leftrightarrow \quad S_1 - S_2 = 0.$$

Since this equation is indepenent of λ_1 and λ_2, we see that every two members of the system have the same radical plane.

Co-axel Systems : Def. *A system of spheres any two members of which have the same radical plane is called a co-axal system of spheres.*

Thus, the system of spheres

$$S_1 + \lambda S_2 = 0$$

is co-axal and we say that it is determined, by the two spheres

$$S_1 = 0, S_2 = 0$$

The common radical plane is

$$S_1 - S_2 = 0$$

This co-axal system is also given by the equation

$$S_1 + k_2 (S_1 - S_2) = 0$$

Refer Note 1, § 4.4.1.

Note : It can similarly be proved that the system of spheres.

$$S + \lambda U = 0$$

is co-axal; $S = 0$ being a sphere and $U = 0$ a plane; the common radical plane is $U = 0$.

Cor. *The locus of the centres of spheres of a co-axal system is a line.*

For, if *(x, y, z)* be the centre of the sphere

$$S_1 + \lambda S_2 = 0,$$

we have

$$x = -\frac{u_1 + \lambda u_2}{1+\lambda}, \quad y = -\frac{v_1 + \lambda v_2}{1+\lambda}, \quad z = -\frac{w_1 + \lambda w_2}{1+\lambda}$$

On eliminating λ, we find that it lies on the line

$$\frac{x + u_1}{u_1 - u_2} = \frac{y + v_1}{v_1 - v_2} = \frac{z + w_1}{w_1 - w_2}.$$

This result is also otherwise clear as the line joining the centres of any two spheres is perpendicular to their common radical plane.

4.9 A SIMPLIFIED FORM OF THE EQUATION OF TWO GIVEN SPHERES

By taking the line joining the centres of two given spheres as *X*-axis, their equations take the form

$$x^2 + y^2 + z^2 + 2u_1 x + d_1 = 0, \; x^2 + y^2 + z^2 + 2u_2 x + d_2 = 0 .$$

The radical plane of these spheres is

$$2x\,(u_1 - u_2) + (d_1 - d_2) = 0$$

Further, if we take this radical plane as the *YZ* plane, *i.e.,* $x = 0$, we have $d_1 = d_2 = d$ (say).

Thus, by taking the line joining the centres of two given spheres as *X*-axis and their radical plane as the *YZ* plane, their equations take the form

$$x^2 + y^2 + z^2 + 2u_1 x + d = 0, \; x^2 + y^2 + z^2 + 2u_2 x + d = 0,$$

where u_1, u_2 are different.

Cor. 1. The equation

$$x^2 + y^2 + z^2 + 2kx + d = 0; \; k, \text{ is a parameter.}$$

represents a co-axel system of spheres for different values of *k, d* being constant. The *YZ* plane is the common radical plane and *X*-axis the line of centres.

Cor. 2. Limiting Points : The equation

$$x^2 + y^2 + z^2 + 2kx + d = 0$$

can be written as

$$(x + k)^2 + y^2 + z^2 = k^2 - d$$

For $k = \pm\sqrt{d}$, we get spheres of the system with radius zero and thus the *system includes the two points spheres*

$$(-\sqrt{d}, 0, 0), (\sqrt{d}, 0, 0)$$

These two points, called the *limiting points,* exist only when *d* is positive, *i.e.,* when the spheres do not meet the radical plane is a real circle.

Def. Limiting points *of a co-axel system of spheres are the point spheres of the system.*

EXAMPLES

1. *Find the limiting points of the co-axal system defined by the sphere*

$$x^2 + y^2 + z^2 + 3x - 3y + 6 = 0, \qquad x^2 + y^2 + z^2 - 6y - 6z + 6 = 0$$

Sol. The equation of the plane of the circle through the two given spheres is

$$3x + 3y + 6z = 0 \;\Rightarrow\; x + y + 2z = 0;$$

Then the equation of the co-axal system determined by the given shpere is

$$x^2 + y^2 + z^2 + 3x - 3y + 6 + \lambda\,(x + y + 2z) = 0,$$

$$\Leftrightarrow \qquad x^2 + y^2 + z^2 + (3 + \lambda)\,x + (\lambda - 3)\,y + 2\lambda + 6 = 0. \qquad ...(1)$$

λ being a parameter.

The centre of (1) is $\left(-\frac{3+\lambda}{2}, -\frac{\lambda-3}{2}, -\lambda\right)$

and its radius is $\left[\left(\frac{3+\lambda}{2}\right)^2 + \left(\frac{\lambda-3}{2}\right)^2 + \lambda^2 - 6\right]^{1/2}$

Equating this radius to zero, we obtain

$$6\lambda^2 - 6 = 0 \Leftrightarrow \lambda = \pm 1$$

The spheres corresponding to these values of λ become point spheres coinciding with their centres and are the limiting points of the given system of spheres.

The limiting points, therefore, are

$$(-1, 2, 1) \text{ and } (-2, 1, -1)$$

2. *Show that spheres which cut two given spheres along great circles all pass through two fixed points.*

Sol. With proper choice of axes, the equations of the given spheres take the form

$$x^2 + y^2 + z^2 + 2u_1x + d = 0 \qquad \text{...(i)}$$

$$x^2 + y^2 + z^2 + 2u_2x + d = 0 \qquad \text{...(ii)}$$

The equation of any other sphere is

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + c = 0 \qquad \text{...(iii)}$$

where u, v, w are different for different spheres.

The plane

$$2x(u - u_1) + 2vy + 2wz + (c - d) = 0$$

If the circle common to the spheres (i) and (iii) will pass through the centre

$$(-u_1, 0, 0)$$

of (i) if, $$-2u_1(u - u_1) + (c - d) = 0 \qquad \text{...(iv)}$$

which is thus the condition for the sphere (iii) to cut the sphere (i) along a great circle.

Similarly, $$-2u_1(u - u_2) + (c - d) = 0 \qquad \text{...(v)}$$

is the condition for the sphere (iii) to cut the sphere (ii) along the great circle.

Solving the linear equations (iv) and (v) for u and $c,$ we get

$$u = u_1 + u_2\ ; \quad c_1 = 2u_1u_2 + d,$$

so that u, c are constants, being dependent on u_1, u_2 d only.

The spheres (iii) cuts X-axis at points whose x-co-ordinates are the roots of the equation

$$x^2 + 2ux + c = 0$$

The roots of this equation are constant, depending as they do upon the constants u and c only.

Thus, every sphere (iii) meets the X-axis at the same two points and hence, the result.

EXERCISES

1. Show that the sphere

$$x^2 + y^2 + z^2 + 2vy + 2wz - d = 0$$

passes through the limiting points of the co-axal system

$$x^2 + y^2 + z^2 + 2kx + d = 0$$

and cuts every member of the system orthogonally, whatever be the values of $v, w.$ Hence, deduce the every sphere that passes through the limiting points of a co-axel system cuts every sphere of the system orthogonally.

2. Show that the locus of the point spheres of the system

$$2x^2 + y^2 + z^2 + 2vy + 2wz - d = 0$$

is the common circle of the system

$$x^2 + y^2 + z^2 + 2ux + d = 0$$

u, v, w being the parameters and d a constant.

3. Show that the sphere which cuts two spheres orthogonally will cut every member of the co-axal system determined by them orthogonally.

4. Find the limiting points of the co-axal system of spheres

$$x^2 + y^2 + z^2 - 20x + 30y - 40z + 29 + \lambda(2x - 3y - 4z) = 0$$

[**Ans.** $(2, -3, -4), (-2, 3, -4)$]

5. Three spheres of radii r_1, r_2, r_3 have their centres A, B, C at the points $(a, 0, 0)$, $(0, b, 0)$, $(0,0, c)$ and $r_1^2 + r_2^2 + r_3^2 = a^2 + b^2 + c^2$. A fourth sphere passes through the origin and the points A, B, C. Show that the radical centre of the four spheres lies on the plane $ax + by + cz = 0$.

6. Show that the locus of a point from which equal tangents may be drawn to the three spheres

$$x^2 + y^2 + z^2 + 2x + 2y + 2z + 2 = 0, \quad x^2 + y^2 + z^2 + 4x + 4z = 0,$$

$$x^2 + y^2 + z^2 + x + 6y - 4z - 2 = 0$$

is the straight line

$$x/2 = (y-1)/5 = z/3$$

7. Show that there are, in general, two spheres of a co-axal system which touch a given plane. Find the equation to the two spheres of the co-axal system

$$x^2 + y^2 + z^2 - 5 + \lambda(2x + y + 3z - 3) = 0$$

which touch the plane $3x + 4y = 15$

[**Ans.** $x^2 + y^2 + z^2 + 4x + 2y - 6z - 11 = 0, 5(x^2 + y^2 + z^2) - 8x - 4y - 12z - 13 = 0$]

8. P is a variable point on a given line and A, B, C are its projections on the axes. Show that the $OABC$ passes through a fixed circle.

9. Show that the radical planes of the spheres of a co-axal system and of a given sphere pass through a line.

❑❑❑

5

Cones, Cylinders

5.1 DEFINITIONS

A **cone** *is a surface generated by a straight line which passes through a fixed point and satisfies one more condition; for instance,* it may intersect a given curve or touch a given surface.

The fixed point is called the *Vertex* and the given curve the *Guiding curve* of the cone.

An individual straight line on the surface of a cone is called its *Generator.*

Thus, a cone is essentially a set of lines called *Generators* through a given point. Also, we may say that a cone is a set of points on its generators.

Whereas we can have cones with equations of any degree whatsoever depending upon the condition to be satisfied by its generators, we shall in this book be concerned only with *Quadratic cones, i.e.,* cones with second degree equations.

It will be seen that the degree of the equation of a cone whose generators intersect a given conic or touch a given sphere is of the second degree.

5.1.1 Equation of a Cone with a Conic as Guiding Curve

To find the equation of the cone whose vertex is the point

$$(\alpha, \beta, \gamma)$$

are whose generators intersect the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0, \ z = 0 \qquad \text{...(i)}$$

We have to find the locus of points on lines which pass through the given point (α, β, γ) and intersect the given curve.

The equations to *any* line through (α, β, γ) are

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \qquad \text{...(ii)}$$

This line will be a generator of the cone if and only if it intersects the given curve.

This line meets the plane $z = 0$ in the point

$$\left(\alpha - \frac{l\gamma}{n}, \beta - \frac{m\gamma}{n}, 0\right)$$

which will lie on the given conic, if

$$a\left(\alpha - \frac{l\gamma}{n}\right)^2 + 2h\left(\alpha - \frac{l\gamma}{n}\right)\left(\beta - \frac{m\gamma}{n}\right) + b\left(\beta - \frac{m\gamma}{n}\right)^2$$
$$+ 2g\left(\alpha - \frac{l\gamma}{n}\right) + 2f\left(\beta - \frac{m\gamma}{n}\right) + c = 0 \qquad \text{...(iii)}$$

This is the condition for the line (ii) to intersect the conic (i). Eliminating *l, m, n* between (ii) and (iii), we get

$$a\left(\alpha - \frac{x-\alpha}{z-\gamma}\right)^2 + 2h\left(\alpha - \frac{x-\alpha}{z-\gamma}\gamma\right)\left(\beta - \frac{y-\beta}{z-\gamma}\gamma\right) + b\left(\beta - \frac{y-\beta}{z-\gamma}\gamma\right)^2$$
$$+ 2g\left(\alpha - \frac{x-\alpha}{z-\gamma}\gamma\right) + 2f\left(\beta - \frac{y-\beta}{z-\gamma}\gamma\right) + c = 0$$

$$\Rightarrow \qquad a(\alpha z - x\gamma)^2 + 2h(\alpha z - x\gamma)(\beta z - y\gamma) + (\beta z - y\gamma)^2$$
$$+ 2g(\alpha z - x\gamma)(z-\gamma) + 2f(\beta z - z\gamma)(z-\gamma) + c(z-\gamma)^2 = 0$$

which is the required equation of the cone.

EXAMPLES

1. *Find the equation of the cone whose vertex is* (α, β, γ) *and base* $ax^2 + by^2 = l, z = 0.$

Solution. Any line through (α, β, γ) is

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \qquad \text{...(i)}$$

This cuts $z = 0,$ where

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{-\gamma}{n}$$

$$\Rightarrow \qquad \left(\alpha - \frac{l\gamma}{n}, \beta - \frac{m\gamma}{n}, 0\right)$$

which will lie on the given conic, if

$$a\left(\alpha - \frac{l\gamma}{n}\right)^2 + b\left(\beta - \frac{m\gamma}{n}\right)^2 = 1$$

Eliminating *l, m, n* with the help of (i), we get

$$a\left(\alpha - \frac{x-\alpha}{z-\gamma}\gamma\right)^2 + b\left(\beta - \frac{y-\beta}{z-\gamma}\right)^2 = 1$$

or
$$a(\alpha z - \gamma x)^2 + b(\beta z - z\gamma)^2 = (z-\gamma)^2$$

This is the required cone.

2. *The vertex of cone is (a, b, c) and the yz-plane cuts it in the curve F (y, z) = 0, x = 0, show that xz-plane cuts it in the curve,*

$$y = 0, \; F\left[\frac{bx}{x-a}, \frac{cx-az}{x-a}\right] = 0,$$

Solution. Any line through *(a, b, c)* is

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} \qquad \text{...(i)}$$

It meets $x = 0$ in the point $\left[0, b - \frac{am}{l}, c - \frac{an}{l}\right]$ and if it lies on the given curve $F(y, z) = 0,$ then

$$F\left[b - \frac{am}{l}, c - \frac{an}{l}\right] = 0$$

Eliminating *l, m, n* between (i) and (ii), we get

$$F\left[b - a\left(\frac{y-b}{x-a}\right), c - a\left(\frac{z-c}{x-a}\right)\right] = 0$$

$$\Rightarrow \qquad F\left[\frac{bx-ay}{x-a}, \frac{cx-az}{x-a}\right] = 0$$

It meets *zx*-plane, *i.e., y = 0* in the curve.

$$F\left[\frac{bx}{x-a}, \frac{cx-az}{x-a}\right] = 0, \; y = 0.$$

3. *Find the equation of the cone with vertex at (2a, b, c) and passing through the curve* $x^2 + y^2 = 4a^2$ *and z = 0. Find b and c if the cone also passes through the curve* $y^2 = 4a(z+a), x = 0.$ *Also, show that the cone is cut by the plane y = 0 in two straight lines and the angle* θ *between them is given by* $\tan\theta = 2.$

Solution. Any line through $(2a, b, c)$ is

$$\frac{x-2a}{l} = \frac{y-b}{m} = \frac{z-c}{n} \quad \text{...(i)}$$

It meets $z=0$ in the point $\left[2a - \frac{lc}{n}, b - \frac{mc}{n}, 0\right]$ and if it lies on the curve $x^2 + y^2 = 4a^2, z = 0$ then we get

$$\left(2a - \frac{lc}{n}\right)^2 + \left(b - \frac{mc}{n}\right)^2 = 4a^2 \quad \text{...(ii)}$$

Eliminating *l, m, n* between (i) and (ii) we get the required cone as

$$\left[2a - \left(\frac{x-2a}{z-c}\right)c\right]^2 + \left[b - \left(\frac{y-b}{z-c}\right)c\right]^2 = 4a^2$$

$$\Rightarrow \quad (2az - cx)^2 + (bz - yc)^2 = 4a^2(z-c)^2 \quad \text{...(iii)}$$

If this cone passes through $y^2 = 4a(z+a), x = 0,$ then putting $x = 0$ in (iii), we get

$$(2az)^2 + (bz - yc)^2 = 4a^2(z-c)^2$$

$$\Rightarrow \quad b^2z^2 + c^2y^2 - 2bcyz - 4a^2c^2 + 8a^2cz = 0 \quad \text{...(iv)}$$

If it is same as $y^2 = 4a(z+a),$ then comparing this with (iv), we have

$$b^2 = 0 \Rightarrow b = 0 \text{ which reduces (iv) to}$$

$$e^2y^2 = 4a^2c^2 - 8a^2zc$$

$$\Rightarrow \quad y^2 = -\left(\frac{8a^2}{c}\right)\left(z - \frac{1}{2}c\right)$$

This gives $\quad -\frac{8a^2}{c} = 4a$ and $-\frac{1}{2}c = a$

$$\Rightarrow \quad c = -2a.$$

Hence, $b = 0, c = -2a.$

Substituting these values of b and c in (iii), the equation of cone intersecting the given cone reduces to

$$(2ax + 2az)^2 + 4a^2y^2 = 4a^2(z+2a)^2$$

$$\Rightarrow \quad x^2 + y^2 + 2zx - 4az - 4a^2 = 0$$

The plane $y = 0$ cuts cone (v) in

$$x^2 + 2zx - 4az - 4a^2 = 0, \ y = 0$$

$$\Rightarrow \quad (x^2 - 4a^2) + 2z(x - 2a) = 0, \ y = 0$$

$$\Rightarrow \quad (x-2a)(x + 2a + 2z) = 0, \ y = 0$$

$$\Rightarrow \quad x - 2a = 0, \ y = 0$$

and $\quad x + 2a + 2z = 0, \ y = 0$

This gives the required lines. These lines lie in the plane $y=0$ and their combined equation is

$$y=0,\ x^2+2zx-4az-4a^2=0$$

If θ be the angle between these lines, then

$$\tan\theta=\frac{2\sqrt{1^2-0}}{1+0}=2.$$

EXERCISES

1. Find the equation of the cone whose generators pass through the point (α,β,γ) and have their direction cosines satisfying the relation $al^2+bm^2+cn^2=0.$

[**Ans.** $a(x-\alpha)^2+b(y-\beta)^2+c(z-\gamma)^2=0$]

2. Find the equation of the cone whose vertex is the point (1, 1, 0) and whose guiding curve is $y=0, x^2+z^2=4.$ [**Ans.** $x^2-3y^2+z^2-2xy+8y-4=0$]

3. Obtain the locus of the lines which pass through a point (α,β,γ) and through points of the conic.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z=0$$

$$\left[\textbf{Ans.}\left(\frac{\alpha z-x\gamma}{a}\right)^2+\left(\frac{\beta z-y\gamma}{b}\right)^2=(z-\gamma)^2\right]$$

4. Show that he equation of the cone whose vertex is the origin and whose base is the circle through the three points $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$ is $\Sigma a(b^2+c^2)\,yz=0.$

5. Find the equation of the cone which vertex at (1, 2, 3) and guiding curve

$$x^2+y^2+z^2=4,\ x+y+z=1$$

[**Ans.** $5x^2+3y^2+z^2-2xy-6yz-4zx+6x+8y+10z-26=0$]

6. Find the equation of the cone whose vertex is at the point (–1, 1, 2) and whose guiding curve is $3x^2-y^2=1, z=0.$

7. Find the equation of cone whose vertex is (1, 2, 3) and base is $y^2=4ax, z=0.$

5.1.2 Enveloping Cone of a Sphere

To find the equation of the cone whose vertex is at the point (α,β,γ) *and whose generators touch the sphere*

$$x^2+y^2+z^2=a^2 \qquad \text{...(i)}$$

The equation to *any* line through (α,β,γ) are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad \text{...(ii)}$$

This line will be a generator of the given curve if and only if it touches the given sphere.

The points of intersection of the line (ii) with the sphere (i) are given by

$$r^2(l^2+m^2+n^2)+2r(l\alpha+m\beta+n\gamma)+(\alpha^2+\beta^2+\gamma^2-a^2)$$

so that the line will touch the sphere, if the two roots of the quadratic equation in r are equal and this requires

$$(l\alpha+m\beta+n\gamma)^2=(l^2+m^2+n^2)(\alpha^2+\beta^2+\gamma^2) \qquad \text{...(iii)}$$

This is the condition for the line (ii) to touch the sphere (i).

Eliminating *l, m, n* between (ii) and (iii), we get

$$[\alpha (x-\alpha) + \beta (y-\beta) + \gamma (z-\gamma)]^2$$
$$= [(x-\alpha) + (y-\beta) + (z-\gamma)]^2 \, (\alpha^2 + \beta^2 + \gamma^2 - a^2) \quad \text{...(iv)}$$

which is the required equation of the cone.

If we write

$$S \equiv x^2 + y^2 + z^2 - a^2, S_1 \equiv \alpha^2 + \beta^2 + \gamma^2 - a^2, T \equiv \alpha x + \beta y + \gamma z - a^2$$

the equation (iv) can be rewritten as

$$(T - S_1)^2 = (S - 2T + S_1)\, S$$

$$\Leftrightarrow \qquad SS_1 = T^2$$

$$\Leftrightarrow \qquad (x^2 + y^2 + z^2 - a^2)(\alpha^2 + \beta^2 + \gamma^2 - a^2) = (\alpha x + \beta y + \gamma z - a^2)^2$$

Def. Enveloping Cone : *The cone formed by the tangent lines to a surface, drawn from a given point is called the Enveloping Cone of the surface with given point as its vertex.*

EXERCISES

1. Find the enveloping cone of the sphere

$$x^2 + y^2 + z^2 - 2x + 4z = 1$$

with its vertex at (1,1, 1). [**Ans.** $4x^2 + 3y^2 - 5z^2 - 6yz - 8x + 16z - 4 = 0$]

2. Show that the plane $z = 0$ cuts the enveloping cone of the sphere $x^2 + y^2 + z^2 = 11$ which has its vertex (2, 4, 1) in a rectangular hyperbola.

5.1.3 Quadratic Cones with Vertex at Origin

The equation of a cone whose vertex is the origin is homogeneous and conversely.

We take up the general equation

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$$

of the second degree and show that it represents a cone with its vertex at the origin, if and only if

$$u = v = w = d = 0$$

Let the equation represents a cone with its vertex at the origin.

Let $P(x', y', z')$ be a point on the cone represented by equation (1). Then

$$(rx', ry', rz')$$

are the general co-ordinates of a point on the line *OP* joining the point *P* on the origin *O*.
Since the line *OP*is a generator of the cone (i), the point

$$(rx', ry', rz')$$

lies on it for every value of *r* implying that the equation

$$r^2 (ax'^2 + by'^2 + cz'^2 + 2fy'z' + 2gz'x' + 2hx'y') + 2r(ux' + vy' + wz') + d = 0$$

is true for every value of *r*.
This implies that we have

$$ax'^2 + by'^2 + cz'^2 + 2fy'z' + 2gz'x' + 2hx'y' = 0 \quad \text{...(i)}$$
$$ux' + vy' + wz' = 0 \quad \text{...(ii)}$$
$$d = 0. \quad \text{...(iii)}$$

From (ii), we see that if *u, v, w* be not all zero, then the co-ordinates x', y', z' of any point on the cone satisfy an equation of the first degree *viz.,*

$$ux + vy + wz = 0$$

so that the surface is a plane and we have a contradiction. Thus,

$$u = v = w = 0; \ d = 0$$

so that the equation of a cone with its veretx at the origin is necessarily homogeneous.

Conversely : *We show that every homogenous equation of the second degree represents cone with its vertex at the origin.*

It is clear from the nature of the equation that if the co-ordinates x', y', z' satisfy it, then so do also rx', ry', rz' for all values of *r*.

Hence, if any point *P* lies on the surface then every point on the line *OP* lies on it.

Thus, the surface is generated by lines through the origin *O* and hence, by definition is a cone with its vertex at *O*.

Note : A homogeneous equation of the second degree will represent a pair of planes, if the homogeneous expression can be factorized into linear factors. The condition for this has already been obtained in Chapter 2. A pair of intersecting planes can thus be thought of as a cone with any point on the line of intersection as a vertex thereof.

Cor. 1. If *l, m, n* be the direction ratios of any generator of the cone

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \qquad ...(1)$$

so that the point *(lr, mr, nr)* lies on it for every value of *r,* we have

$$al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm = 0 \qquad ...(2)$$

Conversely, it is obvious that if the result (2) be true then the line with direction ratios *l, m, n* is a generator of the cone whose equation is (1). The proof of this statement is straightforward.

Cor. 2. The general equation of a cone with its vertex at the point (α, β, γ) is

$$a(x-\alpha)^2 + b(y-\beta)^2 + c(z-\gamma)^2$$
$$+ 2f(z-\gamma)(y-\beta) + 2g(x-\alpha)(z-\gamma) + 2h(x-\alpha)(y-\beta) = 0,$$

an can easily be verified by transfering the origin to the point (α, β, γ).

EXAMPLES

1. *Find the equation of the cone whose vertex is at the origin and which passes through the curve given by the equations*

$$ax^2 + by^2 + cz^2 = l, \ lx + my + nz = p$$

Solution. The required equation is the hoemogenous equation of the second degree satisfied by points satisfying the two given equations.

We have $$lx + my + nz = p \ \Rightarrow \ \frac{lx + my + nz}{p} = 1$$

Thus, the required equation is

$$ax^2 + by^2 + cz^2 = \left(\frac{lx + my + nz}{p}\right)^2$$

$$\Leftrightarrow \qquad \Sigma(ap^2 - l^2)x^2 - 2\Sigma lmxy = 0$$

2. *Show that the equation of the cone whose vertex is the origin and base curve* $z = k$, $f(x, y) = 0$.

$$f\left(\frac{xk}{z}, \frac{yk}{z}\right) = 0$$

Solution. Let $$f(x, y) = ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0 \qquad ...(i)$$

By making (i) homogeneous with the help of $z = k$, we get the equation of required cone as

$$ax^2 + by^2 + 2hxy + 2gx\left(\frac{z}{k}\right) + 2fy\left(\frac{z}{k}\right) + c\left(\frac{z}{k}\right)^2 = 0$$

Multiplying by $\frac{k^2}{z^2}$, we get

$$a\left(\frac{xk}{z}\right)^2 + b\left(\frac{yk}{z}\right)^2 + 2h\left(\frac{xk}{z}\right)\left(\frac{yk}{z}\right) + 2g\left(\frac{xk}{z}\right) + 2f\left(\frac{yk}{z}\right) + c = 0$$

$\Rightarrow$ $$f\left(\frac{xk}{z}, \frac{yk}{z}\right) = 0$$

EXERCISES

1. Find the equation of the cone whose vertex is at the origin and the direction cosines of whose generators satisfy the relation

$$3l^2 - 4m^2 + 5n^2 = 0$$ [**Ans.** $3x^2 - 4y^2 + 5z^2 = 0$]

2. Find the equation to the cones with vertex at the origin and which pass through the curves given by the equations

(i) $z = 2, x^2 + y^2 = 4$

(ii) $ax^2 + by^2 = 2z, lx + my + nz = p$

(iii) $ax^2 + by^2 + cz^2 = 1, \alpha x^2 + \beta y^2 = 2z$

(iv) $x^2 + y^2 + z^2 + x - 2y + 3z = 4; x^2 + y^2 + z^2 + 2x - 3y + 4z = 5$

[**Ans.** (*i*) $x^2 + y^2 + z^2 = 0$, (*ii*) $p(ax^2 + by^2) = 2z(lx + my + nz)$,

(*iii*) $(ax^2 + by^2 + cz^2)\, 4z^2 = (\alpha x^2 + \beta y^2)^2$, (*iv*) $2x^2 + y^2 - 5xy - 3yz + 4zx = 0$]

3. A sphere S and a plane α have, respectively, the equations

$$\varphi + u + c = 0; \quad v = 1$$

where $\varphi = x^2 + y^2 + z^2$, u and v are homogeneous linear functions of *x, y, z* and c is a constant. Find the equation of the cone whose generators join the origin O to the points of intersection of S and α.

Show that this cone meets S again in points lying on a plane β and find the equation of β in terms of *u, v* and *c.*

If the radius of S varies, while its centre, the plane α, at the point O remains fixed, prove that β passes through a fixed line.

[The required cone, C, is given by

$$C = \varphi + uv + cv^2.$$

Now $$C - S \equiv (\varphi + uv + cv^2) - (\varphi + u + c) = (v - 1)(u + cv + c)$$

so that we see that the cone C meets the sphere S again in points lying on the palne $\beta \equiv u + cv + c = 0.$

Since the radius of S varies and its centre remains fixed, we see that u is constant while c varies. Also v is constant. This shows that the plane $\beta \equiv u + c(v + 1)$ passes through the line of intersection of the fixed planes $u = 0, v + 1 = 0.$]

5.1.4 Determination of Quadratic Cones Under Given Conditions

As a general equation of quadratic cone with a given vertex contains *five arbitrary constants,* it follows that five conditions determine such a cone provided each condition gives rise to a single linear relation between the constants. For instance, *a cone can be determined so as to have any given five concurrent lines as generators,* no three of them being coplanar.

EXAMPLES

1. *Show that the general equation to a cone which passes through the three axes is*

$$fyz + gzx + hxy = 0$$

f, g, h being parameters.

Solution. The general equation of a cone with its vertex at the origin is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

Now X-axis is a generator.

$\Rightarrow$ its direction cosines (1, 0, 0) satisfy (i) $\Rightarrow$ $a = 0$.

Similarly, $b = c = 0$.

2. *The plane* $\frac{x}{a} + \frac{y}{b} + \frac{c}{z} = l$ *meets the co-ordinate axes in A, B, C. Prove that the equation to the cone generated by lines drawn from O to meet the circle ABC is*

$$yz\left(\frac{b}{c} + \frac{c}{b}\right) + zx\left(\frac{c}{a} + \frac{a}{c}\right) + xy\left(\frac{a}{b} + \frac{b}{a}\right) = 0$$

Solution. Points A, B, C, are $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$ respectively. Equation of the sphere $OABC$ is

$$x^2 + y^2 + z^2 - ax - by - cz = 0 \qquad \text{...(i)}$$

The circle ABC is obtained by intersection of given plane with (i).

Making (i) homogeneous with the help of given plane, the required cone is

$$(x^2 + y^2 + z^2) - (ax + by + cz)\left(\frac{x}{a} + \frac{y}{b} + \frac{z}{c}\right) = 0$$

3. *Planes through OX and OY include an angle* α. *Show that their line of intersection lies on the cone* $z^2(x^2 + y^2 + z^2) = x^2y^2 \tan^2 \alpha$.

Solution. Any plane through $OX (y = 0, z = 0)$ is

$$y + \lambda z = 0$$

Also a plane through OY is

$$x + \mu z = 0$$

The angle between two planes is α, *i.e.*,

$$\cos \alpha = \frac{0.1 + 1.0 + \lambda\mu}{\sqrt{1 + \lambda^2}\sqrt{1 + \mu^2}} = \frac{\mu\lambda}{\sqrt{1 + \lambda^2 + \mu^2 + \lambda^2\mu^2}}$$

so that
$$\tan^2 \alpha = \frac{1 + \lambda^2 + \mu^2 + \lambda^2\mu^2}{\lambda^2\mu^2} - 1 = \frac{1 + \lambda^2 + \mu^2}{\lambda^2\mu^2}$$

Eliminating λ and μ from (i), (ii) and (iii), the required cone is

$$\tan^2 \alpha = \frac{1 + \frac{y^2}{z^2} + \frac{x^2}{z^2}}{\left(\frac{y^2}{z^2}\right)\left(\frac{x^2}{z^2}\right)} = \frac{z^2(x^2 + y^2 + z^2)}{x^2y^2}$$

$$\Rightarrow \qquad z^2(x^2 + y^2 + z^2) = x^2y^2 \tan^2 \alpha.$$

EXERCISES

1. Find the equation to the cone which passes through the three co-ordinate axes as well as the two lines

$$\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}, \quad \frac{x}{3} = \frac{y}{-1} = \frac{z}{1}$$

[**Ans.** $3yz + 16zx + 15xy = 0$]

2. Find the equation of the cone which contains the three co-ordinate axes and the two lines through the origin with direction coisines (l_1, m_1, n_1) and (l_1, m_1, n_1).

[**Ans.** $\Sigma l_1 l_2\,(m_1 n_2 - m_2 n_1)\,yz = 0$]

3. Find the equation of the quadratic cone which passes through the three co-ordinate axes and the three mutually perpendicular lines

$$\frac{1}{2}x = y - z,\;\; x = \frac{1}{3}y = \frac{1}{5}z,\;\; \frac{1}{8}z = -\frac{1}{11}y = \frac{1}{5}z$$

[**Ans.** $16yz - 33zx - 25xy = 0$]

4. Show that the lines drawn through the point (α, β, γ) whose direction cosines satisfy $al^2 + bm^2 + cn^2 = 0$ generate the cone $a(x-\alpha)^2 + b(y-\beta)^2 + c(z-\gamma)^2 = 0.$

5.2 CONDITION THAT THE GENERAL EQUATION OF THE SECOND DEGREE SHOULD REPRESENT A CONE, CO-ORDINATES OF THE VERTEX

We have seen that the equation of a cone with its vertex at the origin is necessarily homogeneous and conversely. Thus, any given equation of the second degree will represent a cone if, and only if there exists a point such on transfering the origin to the same the equation becomes homogeneous.

Let $f(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2hxy + 2ux + 2wz + d = 0$ ...(1)

represent a cone having its vertex at (x', y', z').

Shift the origin to the vertex (x', y', z') so that we change

$$x \text{ to } x + x',\; y \text{ to } y + y' \text{ and } z \text{ to } z + z'$$

The transformed equation is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2\,[x(ax' + by' + gz' + u)$$
$$+ y(hx' + by' + fz' + v) + z(gx' + fy' + cz' + w)] + f(x', y', z') = 0 \quad ...(2)$$

The equation (2) represents a cone with its vertex at the origin and must, therefore, be homogenoeus. This gives

$$ax' + by' + gz' + u = 0 \quad ...(i)$$
$$hx' + by' + fz' + v = 0 \quad ...(ii)$$
$$gx' + fy' + cz' + w = 0 \quad ...(iii)$$
$$f(x', y', z') = 0 \quad ....(iv)$$

Also, $f(x', y', z') \equiv x'(ax' + by' + gz' + u) + y'(hx' + by' + fz' + v)$
$+ z'(gx' + fy' + cz' + w) + (ux' + vy' + wz' + d)$

Thus, with the help of (i), (ii) and (iii), we see that (iv) is equivalent to

$$ux' + vy' + wz' + d = 0 \quad ...(v)$$

The system of equations (i), (ii), (iii), (iv) is equivalent to the system (i), (ii), (iii), (iv).

Thus, if the given equation represent a cone, there exist (x', y', z') satisfying the equations (i), (ii), (iii), (v) implying that these four equations are consistent. The condition of consistency of the system (i), (ii), (iii) and (v) of four linear equation is

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} = 0$$

This is the condition for the equation (1) of the second degree to represent a cone.

If the condition is satisfied, the co-ordinates (x', y', z') *of the vertex are obtained by solving simultaneously the three linear equations (i), (ii) and (iii).*

The point (x', y', z') is such that if we shift the origin to this point, the new equation will be homogeneous and as such will represent a cone.

Cor. *If* $F(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$ *represents a cone, the co-ordinates of its vertex satisfy the equations*

$$F_x = 0,\ F_y = 0,\ F_z = 0,\ F_t = 0$$

where 't' is used to make F (x, y, z) homogeneous and is put equal to unity after differentiations.

Making $F(x, y, z)$ homogeneous, we write

$$F(x, y, z, t) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2uxt + 2vyt + 2wzt + dt^2$$

We have

$$F_x = 2(ax + hy + gz + ut),\ F_y = 2(hx + by + fz + vt)$$

$$F_z = 2(gx + fy + cz + wt),\ F_t = 2(ux + vy + wz + dt)$$

Putting $t = 1$, we see from (i), (ii), (iii) and (iv) that the vertex (x_1, y_1, z_1) satisfies the four linear equations.

$$F_x = 0,\quad F_y = 0,\quad F_z = 0,\quad F_t = 0$$

Note : The student should note that the coefficients of second degree term in the transformed equation (2) are the same as those in the original equation (1).

Note : The equation $F(x, y, z) = 0$ represents a cone if, and only, if the four linear equations $F_x = 0, F_y = 0, F_z = 0, F_t = 0$ are consistent. In the case of consistency the vertex is given any three of these.

In case we have

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} = 0 \text{ as well as } \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

the equation will represent a pair of planes.

EXAMPLE

1. *Prove that the equation*

$$ax^2 + by^2 + cz^2 + 2ux + 2vy + 2wz + d = 0$$

represents a cone if

$$\frac{u^2}{a} + \frac{v^2}{b} + \frac{w^2}{c} = d$$

Solution. Let

$$f(x, y, z, t) \equiv ax^2 + by^2 + cz^2 + 2uxt + 2vyt + 2vyt + 2wzt + dt^2 = 0$$

$\therefore \quad \dfrac{\partial F}{\partial x} = 0$ for $t = 1$ gives

$$2ax + 2u = 0 \quad \text{or} \quad x = -\frac{u}{a} \qquad ...(1)$$

Similarly, $\dfrac{\partial F}{\partial y} = 0$ for $t = 1$ gives $y = -\dfrac{y}{b}$ ...(2)

$\dfrac{\partial F}{\partial y} = 0$ for $t = 1$ gives $z = -\dfrac{w}{c}$ ...(3)

and $$\frac{\partial F}{\partial y} = 0 \quad \text{for} \quad t = 1 \quad \text{gives} \quad ux + vy + wz + d = 0 \qquad \text{...(4)}$$

Substituting the values of x, y, z from (1), (2), (3) in (4), we get the required condition as

$$u\left(-\frac{u}{a}\right) + v\left(-\frac{v}{b}\right) + w\left(-\frac{w}{c}\right) + d = 0$$

$$\Rightarrow \qquad \frac{u^2}{a} + \frac{v^2}{b} + \frac{w^2}{c} = d.$$

EXERCISES

1. Prove that

$$2x^2 + 2y^2 + 7x^2 - 10yz - 10zx + 2x + 2y + 26z - 17 = 0$$

represents a cone with vertex at (2, 2, 1).

2. Show that the equation

$$x^2 - 2y^2 + 3z^2 - 4xy + 5yz - 6zx + 8x - 19y - 2z - 20 = 0$$

represents a cone with vertex (1, –2, 3).

3. Show that the equation

$$2y^2 - 8yz - 4zx - 8xy + 6x - 4y - 2z + 5 = 0$$

represents a cone whose vertex is $\left(-\frac{7}{6}, \frac{1}{3}, \frac{5}{6}\right)$.

EXAMPLE

1. *Find the equations to the lines in which the plane*

$$2x + y - z = 0,$$

cuts the cone $$4x^2 - y^2 + 3z^2 = 0$$

Solution. Let $$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$$

be the equations of any one of the two lines in which the given plane meets the given cone so that we have

$$2l + m - n = 0, \quad 4l^2 - m^2 + 3n^2 = 0$$

These two equations are now to be solved for l, m, n. Eliminating n, we have

$$4l^2 - m^2 + 3(2l + m)^2 = 0$$

$$8l^2 + 6lm + m^2 = 0$$

$$\Rightarrow \qquad \frac{l}{m} = \frac{-6 \pm \sqrt{36 - 32}}{16} = -\frac{1}{4} \quad \text{or} \quad -\frac{1}{2}$$

We also have

$$2\frac{l}{m} + 1 - \frac{n}{m} = 0$$

$$\frac{l}{m} = -\frac{1}{4} \quad \Rightarrow \quad \frac{n}{m} = \frac{1}{2}$$

and $$\frac{l}{m} = -\frac{1}{2} \quad \Rightarrow \quad \frac{n}{m} = 0$$

Now $$\frac{l}{m} = -\frac{1}{4}, \; \frac{n}{m} = \frac{1}{2} \quad \Rightarrow \quad \frac{l}{-l} = \frac{m}{4} = \frac{0}{2}$$

and $$\frac{l}{m} = -\frac{1}{2}, \; \frac{n}{m} = 0 \quad \Rightarrow \quad \frac{l}{-l} = \frac{m}{2}; n = 0$$

Thus, the two required lines are

$$\frac{x}{-1}=\frac{y}{4}=\frac{z}{2};\ \frac{x}{-1}=\frac{y}{2};\ z=0.$$

EXERCISES

1. Find the equation of the lines of intersection of the following planes and cones :

   (i) $x+3y-2z=0,\quad x^2+9y^2-4z^2=0.$

   (ii) $3x+4y+z=0,\quad 15x^2-32y^2-7z^2=0$

   (iii) $x+7y-5z=0,\quad 3yz+14zx-30xy=0$

   [**Ans.** (i) $x=2z,\ y=0,\ 3y=2z,\ x=0$ (ii) $\frac{x}{-3}=\frac{y}{2}=\frac{z}{1},\ \frac{x}{2}=\frac{y}{-1}=\frac{z}{-2}$, (iii) $\frac{x}{1}=\frac{y}{2}=\frac{z}{3},\ \frac{x}{3}=\frac{y}{1}=\frac{z}{2}$]

2. Show that the equation of the quadric cone which contains the three co-ordinate axes and the lines in which the plane $x-5y-3z=0$ cuts the cone

$$7x^2+5y^2-3z^2=0 \text{ is } yz+10zx-18xy=0$$

3. Find the angles between the lines of intersection of

   (i) $x-3y+z=0$ and $x^2-5y^2+z^2=0$

   (ii) $10x+7y-6z=0$ and $20x^2+7y^2-108z^2=0$

   (iii) $4x-y-5z=0$ and $8yz+3zx-5xy=0$

   (iv) $x+y+z=0$ and $6xy+3yz-2zx=0$

   (v) $x+y+z=0$ and $x^2-yz+xy-3z^2=0$

   [**Ans.** (i) $\cos^{-1}(5/6)$, (ii) $\cos^{-1}(16/21)$, (iii) $\pi/2$, (iv) $\pi/3$, (v) $\pi/6$]

5.3 CONE AND A PLANE THROUGH ITS VERTEX

To find the angle between the lines of intersection of the plane

$$ux+vy+wz=0$$

and
$$f(x,y,z)\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \qquad ...(1)$$

The plane
$$ux+vy+wz=0 \qquad ...(2)$$

will cut the cone (1) in two lines passing through the origin.

Let one of these lines be

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \qquad ...(3)$$

Thus, line (3) in plane (2), therefore,

$$ul+vm+wn=0 \qquad ...(4)$$

Also, line (3) lies on (1) hence it is generator of the cone, *i.e.,* its d.c.'s satisfy the equation of the cone, hence

$$al^2+bm^2+cn^2+2fmn+2gnl+2hlm=0 \qquad ...(5)$$

Putting $n=-\frac{ul+vm}{w}$ from (4) in (5), we have

$$al^2+bm^2+c\left(-\frac{ul+vm}{w}\right)^2+(2fm+2gl)\left(-\frac{ul+vm}{w}\right)+2hlm=0$$

i.e., $\quad l^2(aw^2+cu^2-2gwu)+2lm(cuv-fwu-gvw+hw^2)+m^2(bw^2+cv^2-2fvw)=0$

i.e., $$\frac{l^2}{m^2}(aw^2 + cu^2 - 2gwu) + 2\frac{l}{m}(cuv - fwu - gvw + hw^2)$$
$$+ (bw^2 + cv^2 + 2fvw) = 0 \qquad ...(6)$$

Now (6) is quadratic equation in l/m and shows that plane (2) cuts cone in two lines. If their direction ratios are l_1, m_1, n_1 and l_2, m_2, n_2 then we have

$$\frac{l_1}{m_1} \cdot \frac{l_2}{m_2} = \frac{bw^2 + cv^2 - 2fvw}{aw^2 + cu^2 - 2gwu}$$

i.e., $$\frac{l_1 l_2}{bw^2 + cv^2 - 2fvw} = \frac{m_1 m_2}{aw^2 + cu^2 - 2gwu} = \frac{n_1 n_2}{bu^2 + av^2 - 2huv} \text{ (similarly)}$$

$$= \frac{l_1 l_2 + m_1 m_2 + n_1 n_2}{(b + c)u^2 + (c + a)v^2 + (a + b)w^2 - 2fvw - 2gwu - 2huv}$$

$$= \frac{l_1 l_2 + m_1 m_2 + n_1 n_2}{(a + b + c)(u^2 + v^2 + w^2) - f(u, v, w)}$$

Also, sum of the roots of (6) gives

$$\frac{l_1}{m_1} + \frac{l_2}{m_2} = -\frac{2(cuv - fwu - gvw + hw^2)}{aw^2 + cu^2 - 2gwu}$$

i.e., $$\frac{l_1 m_2 + l_2 m_1}{-2(cuv - fwu - gvw - hw^2)} = \frac{m_1 m_2}{aw^2 + cu^2 - 2gwu}$$

$$= \frac{l_1 l_2}{bw^2 + cv^2 - 2fvw} = \frac{n_1 n_2}{av^2 + bu^2 - 2huv}$$

$$= \frac{[(l_1 m_2 + l_2 m_1)^2 - 4 l_1 l_2 m_1 m_2]^{1/2}}{[4(cuv - fwu - gvw + hw^2)^2 - 4(bw^2 + cv^2 - 2fvw)(aw^2 + cu^2 - 2gwu)]^{1/2}}$$

$$= \frac{l_1 m_2 - l_2 m_2}{\pm 2wP} \text{ where } P^2 = \begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & 0 \end{vmatrix}$$

$$= \frac{m_1 n_2 - m_2 n_1}{\pm 2uP} = \frac{n_1 l_2 - n_2 l_1}{\pm 2vP}$$

$$= \frac{[\Sigma(m_1 n_2 - m_2 n_1)^2]^{1/2}}{\pm 2P(u^2 + v^2 + w^2)^{1/2}}$$

If θ be the angle between the lines, then

$$\tan\theta = \frac{[\Sigma(m_1 n_2 - m_2 n_1)^2]^{1/2}}{l_1 l_2 + m_1 m_2 + n_1 n_2}$$

or $$\boldsymbol{\tan\theta = \frac{2P(u^2 + v^2 + w^2)^{1/2}}{(a + b + c)(u^2 + v^2 + w^2) - f(u, v, w)}}$$

Cor. Condition of Perpendicularity

To find the condition, so that lines in which plane $ux + vy + wz = 0$ *cuts a cone*

$$f(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

may be at right angles.

The angle θ between the two lines is given by

$$\tan \theta = \pm \frac{2P(u^2 + v^2 + w^2)^{1/2}}{(a+b+c)(u^2 + v^2 + w^2) - f(u, v, w)}$$

If $\theta = 90°$, $\tan \theta = \infty$

i.e., $$(a+b+c)(u^2 + v^2 + w^2) - f(u, v, w) = 0$$

This is the required condition.

5.3.1 Mutually Perpendicular Generators of a Cone

To show that the cone

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \qquad \text{...(iii)}$$

Equation of the plane through the origin perpendicular to the line (ii) is

$$\lambda x + \mu y + vz = 0 \qquad \text{(iv)}$$

If *(l, m, n)* be the direction cosines of any one of the two generators in which the plane cuts the given cone, we have

$$al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm = 0 \qquad \text{...(v)}$$

and $$l\lambda + m\mu + nv = 0 \qquad \text{...(vi)}$$

Eliminating n between (v) and (vi), we obtain

$$l^2(av^2 + c\lambda^2 - 2g\lambda v) + 2lm(c\lambda\mu + hv^2 - g\mu v + f\lambda v) + m^2(bv^2 + c\mu^2 - 2f\mu v) = 0$$

which, being a quadratic in $l : m$, we see that the plane (iv) cuts the given cone in two generators.

Hence, if (l_1, m_1, n_1), (l_2, m_2, n_2) be the direction cosines of these two generators, we have

$$\frac{l_1 l_2}{m_1 m_2} = \frac{bv^2 + c\mu^2 - 2f\mu v}{av^2 + c\lambda^2 - 2g\lambda v}$$

$$\Rightarrow \qquad \frac{l_1 l_2}{bv^2 + c\mu^2 - 2f\mu v} = \frac{m_1 m_2}{av^2 + c\lambda^2 - 2g\lambda v}$$

From symmetry, each of these is further

$$= \frac{n_1 n_2}{a\mu^2 + b\lambda^2 - 2h\lambda\mu} = k, \text{ (say)}$$

Thus, we have

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = k\,[a(\mu^2 + \nu^2) + b(\nu^2 + \lambda^2) + c(\lambda^2 + \mu^2) - 2f\mu\nu + 2g\nu\lambda - 2h\lambda\mu]$$
$$= k(a + b + c) + (\lambda^2 + \mu^2 + \nu^2) \qquad \text{...(viii)}$$

with the help of (iii).

The two generators in which the plane (iv) intersects the curve (ii) will be at right angles if and only if

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

i.e., if and only if

$$a + b + c = 0$$

We note that

$$\frac{x}{\lambda} = \frac{y}{\mu} = \frac{z}{\nu}$$

is an arbitrary generator of the cone and the condition that the planes through the vertex and perpendicular to the generators meet the cone in two perpendicular generators is independent of λ, μ, v.

Also we see that the two generators will themselves be perpendicular to the first generator so that the three generators will be perpendicular in pairs.

It follows that the cone (i) *admits of an infinite number of sets of three mutually perpendicular genaerators if and only if*

$$a + b + c = 0$$

In fact if this condition is satisfied, then the plane perpendicular to *any* generator OP of the cone cuts the same in two perpendicular generators OQ, OR so that OP, OQ, OR is a set of three mutually perpendicular generators.

Note : If the general equation

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0$$

represents a cone having sets of three mutually perpendicular generators, then also

$$a + b + c = 0$$

for, on shifting the origin to its vertex, the coefficients of the second degree term remain unaffected.

EXAMPLES

1. *Prove that the plane* $ax + by + cz = 0$ *cuts the cone* $yz + zx + xy = 0$ *in perpendicular to lines if* $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0.$

Solution. Let one of the lines of intersection be

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$$

The line lies on given cone and plane, hence

$$mn + nl + lm = 0 \qquad \text{...(i)}$$

and

$$al + bm + cn = 0 \qquad \text{...(ii)}$$

Putting the values of n from (ii) in (i), we get

$$(m + l)\left(-\frac{al + bm}{c}\right) + lm = 0$$

$$\Rightarrow \qquad al^2 + (a + b - c)\, lm + bm^2 = 0$$

$$\Rightarrow \qquad a\left(\frac{l}{m}\right)^2 + (a + b - c)\frac{l}{m} + b = 0$$

Let $\frac{l_1}{m_1}, \frac{l_2}{m_2}$ be the two roots, then

$$\frac{l_1}{m_1} \cdot \frac{l_2}{m_2} = \frac{b}{a}$$

$$\Rightarrow \qquad \frac{l_1 l_2}{1/a} = \frac{m_1 m_2}{1/b} = \frac{n_1 n_2}{1/c} \quad \text{(by symmetry)}$$

The angle between the lines will be a right angle if

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0 \;\; \Rightarrow \;\; \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$$

2. *Prove that the angle between the lines given by* $x + y + z = 0$, $ayz + bzx + cxy = 0$ *is* $\pi/2$ *if* $a + b + c = 0$ *and* $\pi/3$ *if* $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0.$

Solution. Let the plane $x + y + z = 0$ cuts the cone $ayz + bzx + cxy = 0$ in a line

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$$

Then $\qquad l + m + n = 0$ and $amn + bnl + clm = 0$

Eliminating n between these relations, we get

$$(am + bl)(-l - m) + clm = 0 \Rightarrow bl^2 + (a + b - c) + lm + am^2 = 0$$

$$\Rightarrow \qquad b\left(\frac{l}{m}\right)^2 + (a + b - c)\left(\frac{l}{m}\right) + a = 0. \qquad \text{...(i)}$$

If the roots of this equation are $\frac{l_1}{m_1}$ and $\frac{l_2}{m_2}$, then

$$l_1l_2 + m_1m_2 + n_1n_2 = 0 \quad \Rightarrow \quad a + b + c = 0$$

Again, from (i), we get

$$\frac{l_1}{m_1} + \frac{l_2}{m_2} = \frac{c - b - a}{b}$$

$$\Rightarrow \qquad \frac{l_1m_2 + l_2m_1}{m_1m_2} = \frac{c - b - a}{b} \quad \Rightarrow \quad \frac{l_1m_2 + l_2m_1}{c - b - a} = \frac{m_1m_2}{b} = k.$$

Now, $\quad (l_1m_2 - l_2m_1)^2 = (l_1m_2 + l_2m_1)^2 - 4l_1l_2.m_1m_2$

$$= k^2(c - b - a)^2 - 4ak.bk = k^2[(c - b - a)^2 - 4ab]$$

$$= k^2(a^2 + b^2 + c^2 - 2ab - 2bc - 2ca)$$

Now, $\quad \tan\theta = \dfrac{\sqrt{\Sigma (l_1m_2 - l_2m_1)^2}}{l_1l_2 + m_1m_2 + n_1n_2} = \dfrac{\sqrt{k^2(3(a^2 + b^2 + c^2 - 2bc - 2ca - 2ab)]}}{k(a + b + c)}$

If $\theta = \pi/3$, then

$$\tan^2\frac{\pi}{3} = \frac{3(a^2 + b^2 + c^2 - 2bc - 2ca - 2ab)}{(a + b + c)^2}$$

$$\Rightarrow \qquad 3(a + b + c)^2 = 3(a^2 + b^2 + c^2 - 2bc - 2ca - 2ab) \qquad [\because \tan(\pi/3) = \sqrt{3}]$$

$$\Rightarrow \qquad 4(bc + ca + ab) = 0 \quad \Rightarrow \quad \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0.$$

EXERCISES

1. Prove that the plane $lx + my + nz = 0$ cuts the cone

$$(b - c)x^2 + (c - a)y^2 + (a - b)z^2 + 2fyz + 2gzx + 2hxy = 0$$

in perpendicular lines if

$$(b - c)l^2 + (c - a)m^2 + (a - b)n^2 + 2fmn + 2gnl + 2hlm = 0$$

2. If $\qquad x = \frac{1}{2}y = z$

represents one of a set of three mutually perpendicular generators of the cone

$$11yz + 6zx - 14xy = 0$$

find the equations of the other two. $\qquad \left[\textbf{Ans. } \frac{x}{2} = \frac{y}{-3} = \frac{z}{4}; \frac{x}{-11} = \frac{y}{2} = \frac{z}{7}\right]$

3. Find the angle between the lines given by

$$x + y + z = 0, \quad \frac{yz}{b - c} + \frac{zx}{c - a} + \frac{xy}{a - b} = 0$$

4. If the plane $2x - y + cz = 0$ cuts the cone $yz + zx + xy = 0$ in perpendicular lines, find the value of c. [**Ans.** $c = 2$]

5. Find the equations of the lines in which the plane $2x + y - z = 0$ cuts the cone

$$4x^2 - y^2 + 3z^2 = 0 \qquad \left[\textbf{Ans. } \frac{x}{-1} = \frac{y}{4} = \frac{z}{2}; \frac{x}{-1} = \frac{y}{2} = \frac{z}{0}\right]$$

6. If $\frac{x}{1} = \frac{y}{1} = \frac{z}{2}$ be one of a set of three mutually perpendicular generators of the cone $3yz - 2zx - 2xy = 0$, find the equations of other two generators.

$$\left[\textbf{Ans. } \frac{x}{2} = \frac{y}{(-4)} = \frac{z}{1}; \frac{x}{3} = \frac{y}{1} = \frac{z}{(-2)}\right]$$

7. Show that the cone whose vertex is the origin and which passes through the curve of intersection of the surface $2x^2 - y^2 + 2z^2 = 3d^2$ and any plane at a distance d, from the origin has three mutually perpendicular generators.

8. Find the locus of a point from which three mutually perpendicular lines can be drawn to intersect the central conic

$$ax^2 + by^2 = 1; z = 0 \qquad [\textbf{Ans. } a(x^2 + z^2) + b(y^2 + z^2) = 1]$$

9. Show that the mutually perpendicular tangent lines can be drawn to the sphere

$$x^2 + y^2 + z^2 = r^2$$

from any point on the surface

$$2(x^2 + y^2 + z^2) = 3r^2$$

10. Three points P, Q, R are taken on the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

so that the lines joining P, Q, R to the origin the mutually perpendicular. Prove that the plane PQR touches a fixed sphere.

5.4 INTERSECTION OF A LINE WITH A CONE

To find the points of intersection of the line

$$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n} \qquad \text{...(i)}$$

and the cone.

$$f(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fz + 2gzx + 2hxy = 0 \qquad \text{...(ii)}$$

The point $(lr + \alpha, mr + \beta, nr + \gamma)$ which lies on the line (i) for all values of r will lie on the cone (ii) for values of r given by the equation

$$a(lr + \alpha)^2 + b(mr + \beta)^2 + c(nr + \gamma)^2 + 2f(mr + \beta)(nr + \gamma) + 2g(lr + \alpha)(nr + \gamma) + 2h(lr + \alpha)(mr + \beta) = 0,$$

$$\Leftrightarrow \quad r^2(al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm) + 2r[l(a\alpha + h\beta + g\gamma) + m(h\alpha + b\beta + f\gamma) + n(g\alpha + f\beta + c\gamma)] + f(\alpha, \beta, \gamma) = 0 \qquad \text{...(A)}$$

Let r_1, r_2 be the roots of this quadratic equation in r. The two points of intersection are

$$(lr_1 + \alpha, mr_1 + \beta, nr_1 + \gamma), (lr_2 + \alpha, mr_2 + \beta, nr_2 + \gamma)$$

Cor. A plane section of a quadratic cone is a conic, as every line in the plane meets the curve of intersection in two points.

Note : The equation (A) gives the distances of the points of intersection P and Q from the points (α, β, γ); if (l, m, n) are direction cosines.

EXERCISES

1. Show that the locus of mid-points of chords of the cone

$$ax^2 + by^2 + cz^2 + 2fyz + 2gx + 2hxy = 0$$

drawn parallell to the line

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$$

is the plane

$$x(al + hm + gn) + y(hl + bm + fn) + z(gl + fm + cn) = 0$$

[**Hint :** If (α, β, γ) be the middle pionts of any such chords

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n},$$

the two roots of the equation (A) are equal and opposite and as such their sum is zero.]

2. Find the locus of the chords of a cone which are bisected at a fixed point.

5.4.1 The tangent Lines and Tangent Plane at a Point

Let $$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{m} \qquad \text{...(i)}$$

be a line through a point (α, β, γ) of the cone

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \qquad \text{...(ii)}$$

so that $$a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta = 0$$

Thus, one of the values of r given by the equation (A) of 5.4 is zero and as such one of the two points of intersection coincides with (α, β, γ). The second point of intersection will also coincide with (α, β, γ) if the second root of the same equation is also zero. This requires

$$l(a\alpha + h\beta + g\gamma) + m(h\alpha + b\beta + f\gamma) + n(g\alpha + f\beta + c\gamma) = 0 \qquad \text{...(iii)}$$

The line (i) corresponding to the set of values of l, m, n satisfying the relation (iii) is *an tangent line* at (α, β, γ) to the cone (ii).

Eliminating l, m, n between (i) and (ii), we obtain the locus of all the tangent lines through (α, β, γ), *viz.,*

$$(x-\alpha)(a\alpha + h\beta + g\gamma) + (y-\beta)(h\alpha + b\beta + f\gamma) + (z-\gamma)(g\alpha + f\beta + c\gamma) = 0$$

$$\Leftrightarrow \quad x(a\alpha + h\beta + g\gamma) + y(h\alpha + b\beta + f\gamma) + z(g\alpha + f\beta + c\gamma)$$
$$= a\alpha^2 + f\beta^2 + c\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta = 0$$

which is a plane known as the **tangent plane.**

Clearly, the tangent plane at any point of a cone passes through its vertex.

Cor. The tangent plane at *any* point $(k\alpha, k\beta, k\gamma)$ on the generator through the point (α, β, γ) is the same as the tangent plane at (α, β, γ).

Thus, we see that the *tangent plane at any point on a cone touches the cone at all points of the generator through that point and we say that the plane touches the cone along the generator.*

EXAMPLE

1. *Show that* $$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$$

the line of intersection of the tangent planes to the cone

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

along the lines in which it is cut by the plane

$$x(al + hm + gn) + y(hl + bm + fn) + z(gl + fm + cn) = 0$$

Sol. The tangent plane at any point (α, β, γ) of the given cone is

$$x(a\alpha + h\beta + g\gamma) + y(h\alpha + b\beta + f\gamma) + z(g\alpha + f\beta + c\gamma) = 0$$

It will contain the line

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$$

$$l(a\alpha + h\beta + g\gamma) + m(h\alpha + b\beta + f\gamma) + n(g\alpha + f\beta + c\gamma) = 0$$

$$\Leftrightarrow \quad \alpha(al + hm + gn) + \beta(hl + bm + fn) + \gamma(gl + fm + cn) = 0$$

Thus, the point (α, β, γ) lies on the plane

$$x(al + hm + gn) + y(hl + bm + fn) + z(gl + fm + cn) = 0$$

Hence, the result.

5.4.2 Condition for Tangency

To find the condition that the plane

$$lx + my + nz = 0, \qquad ...(1)$$

should touch the cone

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \qquad ...(2)$$

If (α, β, γ) be the point of contact, the tangent plane

$$x(a\alpha + h\beta + g\gamma) + y(h\alpha + b\beta + f\gamma) + z(g\alpha + f\beta + c\gamma) = 0$$

thereat should be the same as the plane (1).

$$\therefore \quad \frac{a\alpha + h\beta + g\gamma}{l} = \frac{h\alpha + b\beta + f\gamma}{m} = \frac{g\alpha + f\beta + c\gamma}{n} = k, \text{ (say)}$$

Hence,

$$a\alpha + h\beta + g\gamma - lk = 0 \qquad ...(i)$$

$$h\alpha + b\beta + f\gamma - mk = 0 \qquad ...(ii)$$

$$g\alpha + f\beta + c\gamma - nk = 0 \qquad ...(iii)$$

Also, since (α, β, γ) lies on the plane (1), we have

$$l\alpha + m\beta + n\gamma = 0 \qquad ...(iv)$$

Eliminating, α, β, γ, k between (i), (ii), (iii), (iv), we obtain

$$\begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & 0 \end{vmatrix} = 0,$$

as the required condition.

The determinant on expansion, gives

$$Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0$$

where $A, B, C, F, G, H,$ are the co-factors of a, b, c, f, g, h respectively in the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

We may see that

$$A = bc - f^2, \quad B = ca - g^2, \quad C = ab - h^2,$$

$$F = gh - af, \quad G = hf - bg, \quad H = fg - ch$$

5.4.3 Reciprocal Cones

To find the locus of lines through the vertex of the cone

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \qquad ...(1)$$

perpendicular to its tangent planes,

Let $$lx + my + nz = 0 \qquad ...(2)$$

be a tangent plane to the cone (1) so that we have

$$Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0 \qquad ...(3)$$

The line through the vertex perpendicular to the tangent palne (2) is

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \qquad ...(4)$$

Eliminating *l, m, n* between (3) and (4) , we get

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0 \qquad ...(5)$$

as the required locus which is again a quadric cone with its vertex at the origin.

If we now find the locus of lines through the origin perpendicular to the tangent planes to the cone (5), we have to substitute for *A, B, C, F, G, H* in its equation the corresponding co-factors in the determinant

$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$$

Since, we have, by actual manipulation,

$$BC - F^2 = aD, \quad CA - G^2 = bD, \quad AB - H^2 = cD;$$
$$GH - AF = fD, \quad HF - BG = gD, \quad FG - CH = hD;$$

where $$D \equiv abc + 2fgh + af^2 - bg^2 - ch^2$$

It follows that the required locus for the cone (5) is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

which is the same as (1).

The two cones (1) and (5) are, therefore, such that each is the locus of the normals drawn through the origin to the tangent planes to the other and they are, on this account, called *reciprocal cones.*

We have supposed that $D \neq 0$ implying that the equation (i) does not represent a pair of planes (Refer 2.8).

Cor. *The condition for the cone*

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \qquad ...(i)$$

passes through mutually perpendicular tangent planes is

$$A + B + C = 0$$

The cone (i) will clearly passes through three mutually perpendicular tangent planes, if its reciprocal cone

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0$$

has three mutually perpendicular generators and this will be so if

$$A + B + C = 0 \quad \Leftrightarrow \quad bc + ca + ab = f^2 + g^2 + h^2.$$

EXAMPLES

1. *Show that the general equation of a cone which touches the three co-ordinate plane is*

$$\sqrt{fx} \pm \sqrt{gy} \pm \sqrt{hz} = 0,$$

f, g, h being parameters.

Solution. The reciprocal of a cone touching the three co-ordinate planes is a cone with three co-ordinate axes as three of its generators. Now, the general equation of a cone through the three axes is

$$fyz + gzx + hxy = 0$$

Its reciprocal cone is

$$-f^2x^2 - g^2y^2 - h^2z^2 + 2ghyz + 2hfzx + 2fgxy = 0$$

$\Leftrightarrow$ $(fx + gy - hz)^2 = 4fgxy$

$\Leftrightarrow$ $fx + gy - hz = \pm 2\sqrt{fgxy}$

$\Leftrightarrow$ $fx + gy \pm 2\sqrt{fgxy} = hz$

$\Leftrightarrow$ $(\sqrt{fx} \pm \sqrt{gh})^2 = hz,$

$\Leftrightarrow$ $\sqrt{fx} \pm \sqrt{gy} \pm \sqrt{hz} = 0$

2. *Prove that the cones* $ax^2 + by^2 + cz^2 = 0$ *and* $\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0$ *are reciprocal.*

Solution. The reciprocal cone of

$$ax^2 + by^2 + cz^2 = 0$$

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0 \quad ...(1)$$

where $$\Delta = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$$

and $$A = \frac{\partial x}{\partial a} = bc,\ B = \frac{\partial \Delta}{\Delta b} = ac,\ C = \frac{\partial \Delta}{\partial c} = ab,$$

$$F = \frac{1}{2}\frac{\partial \Delta}{\partial f} = 0,\ G = \frac{1}{2}\frac{\partial \Delta}{\partial g} = 0,\ H = \frac{1}{2}\frac{\partial \Delta}{\partial h} = 0.$$

By putting these values, (1) becomes

$$bcx^2 + cay^2 + abz^2 = 0$$

or $$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0 \ \frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0$$

3. *Prove that the equation* $\sqrt{fx} \pm \sqrt{gy} \pm \sqrt{hz} = 0$ *represents a cone that touches the co-ordinate planes; and that the equation to the reciprocal cone is* $fyz + gzx + hxy = 0$.

Solution. The given equation can be written as

$$\sqrt{fx} \pm \sqrt{gy} = \mp \sqrt{hz}$$

$\Rightarrow$ $fx + gy \pm 2\sqrt{fgxy} = hz$

$\Rightarrow$ $(fx + gy - hz)^2 = 2fgxy$

$\Rightarrow$ $f^2x^2 + g^2y^2 + h^2z^2 - 2ghyz - 2hfzx - 2fgxy = 0 \quad ...(i)$

The equation is a homogeneous equation of second degree, hence it represents a quadratic cone.

The co-ordinate plane $x = 0$ meets (i) where

$$g^2y^2 + h^2z^2 - 2ghyz = 0 \ \Rightarrow \ (gy - hz)^2 = 0$$

which being a perfect square it follows that the plane $x = 0$ touches it. Similarly, we can show that $y = 0,\ z = 0$ also touch the cone (i).

Again for the cone (i), we have

$$`a' = f^2, `b' = g^2, `c' = h^2, `f' = -gh, `g' = -hf, `h' = -fg$$

$\therefore$ $$A = bc - f^2 = g^2h^2 - (-gh)^2 = 0$$

Similarly, $B = C = 0,$

$$F = gh - af = (-hf)(-fg) - f^2(-gh) = 2f^2gh$$

Similarly, $G = 2g^2hf,\ H = 2h^2fg$

$\therefore$ The required equation of the cone reciprocal to (i) is

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0$$

$\Rightarrow$ $$2f^2ghyz + 2g^2hfzx + 2h^2fgxy = 0$$

$\Rightarrow$ $$fyz + gzx + hxy = 0$$

EXERCISES

1. Find the plane which touches the cone

$$x^2 + 2y^2 - 3z^2 + 2yz - 5zx + 3xy = 0$$

along the generator whose direction ratios are 1, 1, 1.

2. Prove that the perpendiculars drawn from the origin to be tangent planes to the cone $ax^2 + by^2 + cz^2 = 0$ lie on the cone $\dfrac{x^2}{a} + \dfrac{y^2}{b} + \dfrac{c^2}{z} = 0.$

3. Prove that tangent planes to the cone

$$x^2 - y^2 + 2z^2 - 3yz + 4zx - 5xy = 0$$

are perpendicular to the generators of the cone

$$17x^2 + 8y^2 + 29z^2 + 28yz - 46zx - 16xy = 0$$

4. Prove that the cones

$$ayz + bzx + cxy = 0,\ (ax)^{1/2} + (by)^{1/2} + (cz)^{1/2} = 0$$

are reciprocal.

5. Prove that the cones $fyz + gzx + hxy = 0; \sqrt{fx} + \sqrt{gy} + \sqrt{hz} = 0$ are reciprocal.

6. Find the condition that the plane $ux + vy + wz = 0$ may touch the cone.

$$ax^2 + by^2 + cz^2 = 0$$ $\left[\textbf{Ans. } \dfrac{u^2}{a} + \dfrac{v^2}{b} + \dfrac{w^2}{c} = 0\right]$

7. Show that a quadric cone can be found to touch any five planes which meet at a point provided no three of them intersect in a line.
Find the equation of the cone which touches the three co-ordinate planes and the planes

$$x + 2y + 3z = 0,\ 2x + 3y + 4z = 0$$

[**Ans.** $(x)^{1/2} + (-6y)^{1/2} + (6z)^{1/2} = 0$]

5.5 INTERSECTION OF TWO CONES WITH A COMMON VERTEX

Sections of two cones, having a common vertex, by any plane are two coplanar conics which, general, intersect in four points.

The four lines joining the common vertex to the four points of intersection of these two coplanar conics are the four common generators of the two cones.

Therefore, *two cones with a common vertex have, in general, four generators in common.* In case two cones with the same vertex have five common generators, they coincide.

If $$S = 0, S' = 0$$

be the equations of two cones with origin as the common vertex, then

$$S + kS' = 0$$

Clearly, the genreal equation of a cone whose vertex is at the origin and which passes through the four common generators of the cones

$$S = 0, S' = 0$$

If k be so chosen that $S + kS' = 0$ becomes the product of two linear factors, then the corresponding equations obtained by putting the linear factors equal to zero represent a pair of planes through the common generators.

Such values of k are the roots of the k-cubic equation

$$\begin{vmatrix} a+ka' & h+kh' & g+kg' \\ h+kh' & b+kb' & f+kf' \\ g+kg' & f+kf' & c+kc' \end{vmatrix} = 0$$

The three values of k give the three pairs of planes through the four common generators.

EXERCISES

1. Find the equation of the cone which passes through the common generators of the cones

$$-2x^2 + 4y^2 + z^2 = 0 \text{ and } 10xy - 2yz + 5zx = 0$$

and the line with direction cosines proportional to 1, 2, 3.

[**Ans.** $2x^2 - 4y^2 - z^2 + 10xy - yz + 5zx = 0$]

2. Show that the equation of the cone through the intersection of the cones

$$x^2 - 2y^2 + 3z^2 - 4yz + 5zx - 6xy = 0 \text{ and } 2x^2 - 3y^2 + 4z^2 - 5yz + 6zx - 10xy = 0$$

and the line with direction cosines proportional to 1, 1, 1 is

$$y^2 - 2z^2 + 3yz - 4zx + 2xy = 0$$

3. Show that the plane $3x + 2y - 4z = 0$ passes through a pair of common generators of the cones $27x^2 + 20y^2 - 32z^2 = 0$ and $2yz + zx - 4xy = 0$.

4. Show that the plane $3x - 2y - z = 0$ cuts the cones

$$3yz - 2zx + 2xy = 0 \text{ and } 21x^2 - 4y^2 - 5z^2 = 0$$

in the same pair of perpendicular lines.

Also show that the plane $7x + 2y + 5z = 0$ contains the remaining two common generators.

5.6 THE RIGHT CIRCULAR CONE

5.6.1 Definition

A right circular cone is a surface generated by a line which passes through a fixed point, and makes a constant angle with a fixed line through the fixed point.

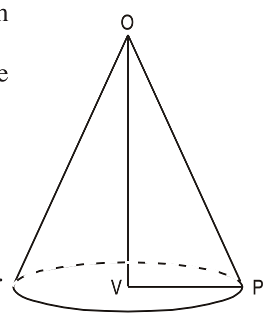
The fixed point is called the *vertex,* the fixed line the *axis* and the fixed angle the *semi-vertical* angle of the cone.

The justification for the name right circular cone is contained in the result obtained below.

Every section of a right circular cone by a plane perpendicular to its axis is a circle.

Let a plane perpendicular to the axis ON of the right circular cone with semi-vertical angle, α, meet it at N.

Let Pbe any point of the section. Since ON is perpendicular to the plane which contains the line NP, we have



$$ON \perp NP$$

$$\Rightarrow \qquad \frac{PN}{ON} = \tan \angle NOP = \tan \alpha$$

$$\Rightarrow \qquad PN = ON \tan \alpha$$

so that NP is constant for every position of the point P of the section.

Hence, the section is a circle with N as its centre.

5.6.2 Equation of a Right Circular Cone

To find the equation of the right circular cone whose vertex is the poitn (α, β, γ) *and whose axis is the line*

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$$

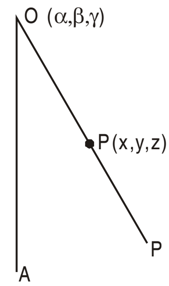
and semi-vertical angle θ.

Let O be the vertex, and, OA the axis of the cone.

The required equation is to be obtained by using the condition that the line joining any point (x, y, z) on the curve to the vertex $O(\alpha, \beta, \gamma)$ makes an angle θ with the axis OA.

Direction cosines of the line OP, being proportional to

$$x-\alpha,\ y-\beta,\ z-\gamma$$



We have

$$\cos\theta = \frac{l(x-\alpha) + m(y-\beta) + n(z-\gamma)}{\sqrt{l^2+m^2+n^2}\ \sqrt{(x-\alpha)^2 + (y-\beta)^2 + (c-\gamma)^2}}$$

The required equation of the cone, therefore, is

$$[l(x-\alpha) + m(y-\beta) + n(z-\gamma)^2] = (l^2+m^2+n^2)$$
$$[(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2]\cos^2\theta$$

Cor. 1. If the vertex be the origin, the equation of the cone becomes

$$(lx+my+nz)^2 = (l^2+m^2+n^2)(x^2+y^2+z^2)\cos^2\theta$$

Cor. 2. If the vertex be the origin and axis of the cone be the Z-axis, then taking

$$z^2 = (x^2+y^2+z^2)\cos^2\theta \iff x^2+y^2 = z^2\tan^2\theta \qquad ...(1)$$

Cor. 3. *The semi-vertical angle of a right circular cone admitting sets of three mutually perpendicular generators is*

$$\tan^{-1}\sqrt{2}$$

for, the sum of the coefficients of x^2, y^2, z^2 in the equation of such a cone must be zero and this means that

$$1+1-\tan^2\theta = 0,\ \textit{i.e.},\ \theta = \tan^{-1}\sqrt{2} \qquad \text{[Refer (1), Cor. 2]}$$

Cor. 4. *The semi-vertical angle of a right circular cone having sets of three mutually perpendicular tangent planes is*

$$\tan^{-1}\sqrt{\frac{1}{2}},$$

for by Cor. to 5.4.3, this will be so if [Refer (1), Cor. 2]

$$1-\tan^2\theta - \tan^2\theta = 0 \ \Rightarrow\ \theta = \tan^{-1}\sqrt{\frac{1}{2}}.$$

EXAMPLES

1. *Find the equation to the right circular cone whose vertex is at origin, the axis along x-axis and semi-vertical angle* α.

Solution. Let $P(x, y, z)$ be any point on the surface of the cone, so that the direction ratios of the line OP are x, y, z; O being the origin. The direction cosines of x-axis are 1, 0, 0.

$$\therefore \qquad \cos\alpha = \frac{x.1 + y.0 + z.0}{\sqrt{x^2+y^2+z^2}}$$

$$\Rightarrow \qquad (x^2+y^2+z^2)\cos^2\alpha = x^2$$

$$\Rightarrow \qquad y^2+z^2 = x^2\tan^2\alpha.$$

2. *Lines are drawn through the origin with direction cosines proportional to (1, 2, 3), (2, 3, 6), (3, 4, 12). Show that the axis of the right circular cone through them has direction cosines*

$$-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$$

and that the semi-vertical angle of the cone is $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

Obtain the equation of the cone also and show that it passes through the co-ordinate axes.

Solution. Let (l, m, n) be the direction cosines of the axes of the right circular cone. Let O be the origin and P, Q, R be the points, so that d.r.'s of OP, OQ, OR are (1, 2, 2), (2, 3, 6), (3, 4, 12) respectively.

Therefore, d.c.'s of $OP$are $\frac{1}{3}, \frac{2}{3}, \frac{2}{3}$, those of OQ are $\frac{2}{7}, \frac{3}{7}, \frac{6}{7}$ and those of OR are $\frac{3}{13}, \frac{4}{13}, \frac{12}{13}$.

Let α be the semi-vertical angle of the cone, then

$$\cos\theta = \frac{1}{3}l + \frac{2}{3}m + \frac{2}{3}n = \frac{2}{7}l + \frac{3}{7}m + \frac{6}{n}n = \frac{3}{13}l + \frac{4}{13}m + \frac{12}{13}n$$

From first two relations

$$l + 5m - 4n = 0 \qquad \text{...(i)}$$

and from first and last, we get

$$2l + 7m - 5n = 0 \qquad \text{...(ii)}$$

Solving, we obtain

$$\frac{l}{-1} = \frac{m}{1} = \frac{n}{1} = \pm\frac{\sqrt{l^2 + m^2 + n^2}}{\sqrt{1+1+1}} = \pm\frac{1}{\sqrt{3}}$$

Therefore, direction cosines of the axis are

$$-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}.$$

Therefore, semi-vertical angle of the cone will be

$$\cos\alpha = \frac{1}{3}\left(-\frac{1}{\sqrt{3}}\right) + \frac{2}{3}.\frac{1}{\sqrt{3}} + \frac{2}{3}.\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

If *(x, y, z)* be any point on the cone, then its equation will be

$$\cos\alpha = \frac{-1.x + 1.y + 1.z}{\sqrt{3}\sqrt{x^2 + y^2 + z^2}} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \qquad yz - zx - xy = 0$$

EXERCISES

1. Find the equation of the right circualr cone with its vertex at the origin, axis along Z-axis and semi-vertical angle α.
2. Show that the equation of the right circular cone with vertex (2, 3, 1), axis parallel to the line $-x = \frac{y}{2} = z$ and one of its generators having direction cosines proportional to (1, –1, 1) is $x^2 + 8y^2 + z^2 + 12yz - 12yz + 6zx + 46x + 36y + 22z - 19 = 0$.
3. Find the equation of the circular cone which passes through the point (1, 1, 2) and has its vertex at the origin and axis the line $\frac{x}{2} = -\frac{y}{4} = \frac{z}{3}$.

   [**Ans.** $4x^2 + 40y^2 + 19z^2 - 48xy - 72yz + 36xz = 0$]

4. Find the equation of the right circular cone whose vertex is origin, axis of the line $x = t,\ y = 2t,\ z = 3t,$ and whose semi-vertical angle is 60°.

[**Ans.** $38x^2 + 26y^2 + 6z^2 - 16xy - 48yz - 24zx = 0$]

5. Find the equation of the right circular cone whose vertex is (1, –2, 1), axis the line

$$\frac{x-1}{3} = \frac{y+2}{4} = \frac{z+1}{5}$$

and semi-vertical angle 60°.

[**Ans.** $7x^2 - 7y^2 - 25z^2 + 48xy + 80yz - 60zx + 22x + 4y + 17x + 78 = 0$]

6. Find the equation of the right circular cone whose vertex is (3, 2, 1) axis the line

$$\frac{x-3}{4} = \frac{y-2}{1} = \frac{z-1}{3}$$

and semi-vertical angle 30°.

7. Find the equation of right circular cone which passes through (1, 1, 1), whose vertex is (1, 1, 1) and axis of cone makes equal angle with co-ordinate axes.

[**Ans.** $xy + yz + zx - x - 2y - z + 1 = 0$]

8. Find the equation of the cone generated by rotating the line

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$$

about the line $\dfrac{x}{a} = \dfrac{y}{b} = \dfrac{z}{c}$ as axis.

[**Ans.** $(al + bm + cn)^2\ (x^2 + y^2 + z^2) = (ax + by + cz)^2\ (l^2 + m^2 + n^2)$]

5.7 THE CYLINDER

Def. *A* **cylinder** *is a surface generated by a straight line which is always parallel to a fixed line and is subject to one more condition; for instance, it may intersect a given curve or touch a given surface.*

The given curve is called the *Guiding curve.*

5.7.1 Equation of a Cylinder

To find the equation of the cylinder whose generators intersect the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0,\ z = 0 \qquad \text{...(i)}$$

and are parallel to the line

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \qquad \text{...(ii)}$$

Let (α, β, γ) be *any* point on the cylinder so that the equations of the generator through the point are

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \qquad \text{...(iii)}$$

As in 5.1.2, the line (iii) will intersect the conic (i), if

$$\left(\alpha - \frac{l\gamma}{n}\right)^2 + 2h\left(\alpha - \frac{l\gamma}{n}\right)\left(\beta - \frac{m\gamma}{n}\right) + b\left(\beta - \frac{m\gamma}{n}\right)^2 + 2g\left(\alpha - \frac{l\gamma}{n}\right) + 2f\left(\beta - \frac{m\gamma}{n}\right) + c = 0$$

But this is the condition that the point (α, β, γ) should lie on the surface

$$a\left(x - \frac{lz}{n}\right)^2 + 2h\left(x - \frac{lz}{n}\right)\left(y - \frac{mz}{n}\right) + b\left(y - \frac{mz}{n}\right)^2 + 2g\left(x - \frac{lz}{n}\right) + 2f\left(y - \frac{mz}{n}\right) + c = 0$$

$$\Rightarrow \quad a\,(nx - lz)^2 + 2h\,(nx - lz)\,(ny - mz) + b\,(ny - nz)^2 + 2gn\,(nx - lz)$$
$$+ 2fn\,(ny - nz) + cn^2 = 0$$

which is, therefore, the required equation of the cylinder.

Cor. If the generators be parallel to Z-axis so that

$$l = 0 = m \text{ and } n = 1$$

the equation of the cylinder becomes

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

as is already known to the reader.

EXAMPLES

1. *Find the equation of the cylinder whose generators are parallel to the line*

$$\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$$

and whose guiding curve is the ellipse $x^2 + 2y^2 = 1, z = 0.$

Solution. Let (α, β, γ) be any point on the cylinder, then equations of a generator through (α, β, γ) are

$$\frac{x-\alpha}{1} = \frac{y-\beta}{-2} = \frac{z-\gamma}{3}$$

This meets the plane $z = 0$ at the point given by

$$\frac{x-\alpha}{1} = \frac{y-\beta}{-2} = -\frac{\gamma}{3}$$

i.e., at $$\left(\alpha - \frac{\gamma}{3}, \beta + \frac{2\gamma}{3}, 0\right)$$

Therefore, the generator intersects the given curve if

$$\left(\alpha - \frac{\gamma}{3}\right)^2 + 2\left(\beta + \frac{2\gamma}{3}\right)^2 = 1$$

Hence, locus of (α, β, γ) is

$$\left(x - \frac{z}{3}\right)^2 + 2\left(y + \frac{2z}{3}\right)^2 = 1$$

$$3x^2 + 6y^2 + 3z^2 - 2zx + 8yz - 3 = 0$$

2. *Find the equation of the cylinder whose generators are parallel to*

$$\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$$

and whose guiding curve is the ellipse $x^2 + 2y^2 = 1, z = 3.$

Solution. Let (α, β, γ) be any point on the surface of the cylinder so that the equations of its generators through this point are

$$\frac{x-\alpha}{1} = \frac{y-\beta}{-2} = \frac{z-\gamma}{3}$$

This line meets the plane $z = 3$ at the point given by

$$\frac{x-\alpha}{1} = \frac{y-\beta}{-2} = \frac{3-\gamma}{3},$$

i.e., $$\left(\alpha + \frac{3-\gamma}{3}, \beta + \frac{2\gamma - 6}{3}, 3\right)$$

This point will lie on the surface

$$x^2 + 2y^2 = 1,$$

if $$\left(\alpha + \frac{3-\gamma}{3}\right)^2 + 2\left(\beta + \frac{2\gamma - 6}{3}\right)^2 = 1$$

or $$(3\alpha - \gamma + 3)^2 + 2\,(3\beta + 2\gamma - 6)^2 = 9$$

Hence, locus of the point (α, β, γ) will be

$$(3x - z + 3)^2 + 2\,(3y + 2z - 6)^2 = 9$$

or $$3x^2 + 6y^2 + 3z^2 + 8yz - 2zx + 6x - 24y - 18z + 24 = 0$$

This is the required equation of the cylinder.

3. *Find the equation of the quadratic cylinder with generators parallel to x-axis and passing through the curve* $ax^2 + by^2 + cz^2 = 1, lx + my + nz = p$.

Solution. The equation of the required cylinder is obtained by eliminating x between the equations

$$ax^2 + by^2 + cz^2 = 1 \text{ and } lx + my + nz = p$$

For this, substituting the value of $x = \dfrac{p - my - nz}{l}$ in the other equation, we get

$$a\left(\frac{p - my - nz}{l}\right)^2 + by^2 + cz^2 = 1$$

or $$a\,(p - my - nz)^2 + bl^2y^2 + cl^2z^2 = l^2$$

or $$(bl^2 + am^2)\,y^2 + (cl^2 - am^2)\,z^2 + 2amyz - 2ampy - 2apnz + ap^2 - l^2 = 0$$

This is the equation of required cylinder.

EXERCISES

1. Find the equation of the cylinder whose generators intersect the curve $ax^2 + by^2 = 2z$, $lx + my + nz = p$ and are parallel to the Z-axis.
   [**Hint :** Eliminate z from the equations.]
2. Find the equation of the cylinder whose generators are parallel to
   $$\frac{x}{1} = \frac{y}{2} = \frac{z}{3}$$
   and guiding curve is $x^2 + y^2 = 16, z = 0$.
3. Find the equation of the cylinder whose generators are parallel to z-axis and guiding curve is given by $ax^2 + by^2 + cz^2 = 1, lx + my + nz = p$.
   [**Ans.** $(an^2 + cl^2)\,x^2 + (bn^2 + cm^2)\,y^2 + 2lcmxy - 2cplx - 2cpmy + (cp^2 - n^2) = 0$]
4. Find the equation of cylinder whose generators is parallel to $y = mx, z = nx$ and which intersect the conic $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1, z = 0$. [**Ans.** $b^2\,(nx - z)^2 + a^2\,(ny - nz)^2 = n^2a^2b^2$]

5.7.2 Enveloping Cylinder

To find the equation of the cylinder whose generators touch the sphere

$$x^2 + y^2 + z^2 = a^2 \qquad \text{...(i)}$$

and are parallel to the line

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \qquad \text{...(ii)}$$

Let (α, β, γ) be any point on the cylinder so that the equations of the generator through it are

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \qquad \text{...(iii)}$$

The line (iii) will touch the sphere (i), if

$$(l\alpha + m\beta + n\gamma)^2 = (l^2 + m^2 + n^2)(\alpha^2 + \beta^2 + \gamma^2 - a^2)$$

But this is the condition that the point (α, β, γ) should lie on the surface

$$(lx + my + nz)^2 = (l^2 + m^2 + n^2)(x^2 + y^2 + z^2 - a^2)$$

which is, therefore, the required equation of the cylinder and is known as *Enveloping cylinder of the sphere (i).*

EXAMPLE

1. *Find the enveloping cylinder of the sphere*

$$x^2 + y^2 + z^2 - 2x + 4y = 1$$

having its generators parallel to $x = y = z$. *Also find its guiding curve.*

Solution. Let (α, β, γ) be any point on the surface of the cylinder so that the equations of its generator through this point are

$$\frac{x-\alpha}{1} = \frac{y-\beta}{1} = \frac{z-\gamma}{1} = r \ \text{(say)} \qquad \text{...(i)}$$

Any point on this line is

$$(\alpha + r, \beta + r, \gamma + r)$$

This point will lie on the sphere

$$x^2 + y^2 + z^2 - 2x + 4y = 1$$

if $\quad (\alpha + r)^2 + (\beta + r)^2 + (\gamma + r)^2 - 2(\alpha + r) + 4(\beta + r) = 1$

or $\quad 3r^2 + 2r(\alpha + \beta + \gamma + 1) + (\alpha^2 + \beta^2 + \gamma^2 - 2\alpha + 4\beta - 1) = 0$

Since the generators (1) touches (2), the roots of this quadratic equation in r must be identical, for which

$$4(\alpha + \beta + \gamma + 1)^2 = 12(\alpha^2 + \beta^2 + \gamma^2 - 2\alpha + 4\beta - 1)$$

or $\quad \alpha^2 + \beta^2 + \gamma^2 - \beta\gamma - \gamma\alpha - \alpha\beta - 4\alpha + 5\beta - \gamma - 2 = 0$

Therefore, the locus of (α, β, γ) is

$$x^2 + y^2 + z^2 - yz - zx - xy - 4x + 5y - z - 2 = 0$$

This is the required equation of the enveloping cylinder.

Now, equation to the plane passing through the centre (1, –2, 0) of the sphere (2) and perpendicular to the generators of the cylinder whose direction cosines are proportional to (1, 1, 1) is

$$1 \cdot (x-1) + 1 \cdot (y+2) + 1 \cdot (z-0) = 0$$

or $\quad x + y + z + 1 = 0 \qquad \text{...(3)}$

Clearly, the guiding curve is the curve of intersection of the sphere (2) and the plane (3), *i.e.,* the equations of the guiding curve are

$$x^2 + y^2 + z^2 - 2x + 4y - 1 = 0,$$
$$x + y + z + 1 = 0.$$

EXERCISES

1. Find the equation of the enveloping cylinder of the conicoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

whose generators are parallel to the line (i) $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$; (*ii*) $x = y = z$.

$$\left[\textbf{Ans.}\ \text{(i)}\left(\frac{lx}{a^2}+\frac{my}{b^2}+\frac{nz}{c^2}\right)^2=\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)\right.$$

$$\left.\text{(ii)}\left(\frac{x}{a^2}+\frac{y}{b^2}+\frac{z}{c^2}\right)^2=\left(\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)\right]$$

2. Obtain the equation of a cylinder whose generators touch the sphere

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

whose generators are parallel to the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$.

$$[\textbf{Ans.}\ \{l\,(x+u)+m\,(y+v)+n\,(z+w)\}^2$$
$$= (l^2+m^2+n^2)\,(x^2+y^2+z^2+2ux+2vy+2wz+d)]$$

3. Find the equation of a right circular cylinder which envelopes a sphere with centre *(a, b, c)* and radius *r* and has the generators parallel to the direction cosines *(l, m, n)*.

$$[\textbf{Ans.}\ \{l\,(a-x)+m(b-y)+n\,(c-z)\}^2$$
$$= (l^2+m^2+n^2)\,\{(a-x)^2+(b-y)^2+(c-z)^2-r^2\}]$$

4. Prove that the enveloping cylinders of ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$ whose generators are parallel to the line $\frac{x}{0}=\frac{y}{\pm\sqrt{a^2-b^2}}=\frac{z}{c}$ meet the plane $z=0$ in circles.

5.8 THE RIGHT CIRCULAR CYLINDER

5.8.1 Definition

A right circular cylinder is a surface generated by a line which intersects a fixed circle, called the guiding circle, and is perpendicular to its plane.

The normal to the plane of the guiding circle through its centre is called the *Axis* of the cylinder.

Section of a right circular cylinder by any plane perpendicular to its axis is called a *Normal Section.*

Clearly all the normal sections of a right circular cylinder are circles having the same radius which is also called the radius of the cylinder.

The length of the perpendicular from any point on a right circular cylinder to its axis is equal to its radius.

5.8.2 Equation of a Right Circular Cylinder

To find the equation of the right circular cylinder whose axis is the line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$$

and whose radius is r.

Let *(x, y, z)* be a point on the cylinder. Equating the perpendicular distance of the point from the axis of the radius *r,* we get

$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2-\frac{[l\,(x-\alpha)+m\,(y-\beta)+n\,(z-\gamma)]^2}{l^2+m^2+n^2}=r^2$$

which is the required equation of the cylinder.

EXAMPLES

1. *Find the equation of a circular cylinder whose guiding curve is* $x^2+y^2+z^2=9,\ x-y+z=3.$

Solution. We know that the radius of a right circular cylinder is equal to the radius of the guiding curve and the axis of the cylinder is a line passing through the centre of the circle and hence of the sphere and perpendicular to the plane of the circle.

Here, radius of the sphere - 3.

Length of the perpendicular from the centre O (0, 0, 0) to the given plane

$$=\frac{-3}{\sqrt{1+1+1}}=-\sqrt{3}$$

$\therefore$ Radius of the circle $=\sqrt{3^2-3}=\sqrt{6}.$

The axis of the cylinder passes through (0, 0, 0) and is perpendicular to the plane $x-y+z=3.$

Hence its equations are

$$\frac{x}{1}=\frac{y}{-1}=\frac{z}{1}.$$

Therefore, the equation of the circular cylinder is

$$\left(\frac{1}{\sqrt{3}}\right)^2\left\{\begin{vmatrix} y & z \\ -1 & 1\end{vmatrix}^2+\begin{vmatrix} z & x \\ 1 & 1\end{vmatrix}^2+\begin{vmatrix} x & y \\ 1 & -1\end{vmatrix}^2\right\}=(\sqrt{6})^2$$

or $$(y+z)^2+(z-x)^2+(-x-y)^2=18$$

or $$2x^2+2y^2+2z^2+2yz-2zx+2xy=18$$

or $$x^2+y^2+z^2+xy+yz-zx-9=0$$

2. *Find the right circular cylinder whose radius is 2 and axis is the line*

$$\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-3}{2}$$

Solution. Let $P(x_1, y_1, z_1)$ be any point on the cylinder. The length of the perpendicular from $P(x_1, y_1, z_1)$ to the given line must be equal to the radius.

$$\therefore\ 2^2\,(2^2+1^2+2^2)=\{2\,(y_1-2)-1\,(z_1-3)\}^2+\{2\,(z_1-3)-2\,(x-1)\}^2+\{1\,(x_1-1)-2\,(y_1-2)\}^2$$

$$\Rightarrow\quad 36=(2y_1-z_1-1)^2+(2z_1-2x_1-4)^2+(x_1-2y_1+3)^2$$

$\therefore$ The required equation of the locus of $P(x_1, y_1, z_1)$ is

$$(2y-z-1)^2+(2z-2x-6)^2+(x-2y+3)^2=36$$

$$\Rightarrow\quad 5x^2+8y^2+5z^2-4xy-4yz-8zx+22x-16y-14z-10=0$$

EXERCISES

1. Find the equation of the right circular cylinder of radius 2 whose axis is the line

$$\frac{x-1}{2}=\frac{y-2}{2}=\frac{z-2}{2}$$

[**Ans.** $5x^2+8y^2+5z^2-4xy-4yz-8zx+22x-16y-14z-10=0$]

2. The axis of a right circular cylinder of radius 2 is

$$\frac{x-1}{2} = \frac{y}{3} = \frac{z-3}{1}$$

show that its equation is $10x^2 + 5y^2 + 13z^2 - 12xy - 6yz - 4zx - 8x + 30y - 74x + 59 = 0.$

3. Find the equation of the right circular cylinder of radius 3 and having for its axis the line

$$\frac{x-1}{2} = \frac{y-3}{2} = \frac{5-z}{7}$$

[**Ans.** $5x^2 + 5y^2 + 8z^2 - 4yz + 4zx - 6x - 42y - 96z + 225 = 0$]

4. Find the equation of the right circular cylinder whose axis is

$$\frac{x-2}{2} = \frac{y-1}{1} = \frac{z}{3}$$

and pass through (0, 0, 3).

[**Ans.** $10x^2 + 13y^2 + 5z^2 - 4xy - 6yz - 12zx - 36x - 18y + 24z + 18 = 0$]

5. Find the equation of the right circular cylinder of radius 5 and having for its axis the line

$$\frac{1}{2}x = \frac{1}{3}y = \frac{1}{6}z.$$

[**Ans.** $45x^2 + 40y^2 + 13z^2 - 12xy - 36yz - 24zx - 1225 = 0$]

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6

Conicoid

6.1 THE GENERAL EQUATION OF THE SECOND DEGREE

The locus of the general equation

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0,$$

of the second degree in x, y, z is called a **Conicoid** or **Quadric.**

It is easy to show that every straight line meets a surface whose equation is of the second degree in two points and consequently every plane section of such a surface is a conic. This property justifies the name "Conicoid" as applied to such a surface.

The general equation of second degree contains *nine* effective constants and, therefore, a conicoid can be determined to satisfy nine conditions each of which gives rise to one relation between the constant, *e.g.,* a conicoid can be determined so as to pass through *nine* given points no four of which are coplanar.

The general equation of the second degree can, by transformation of co-ordinate axes, be reduced to any one of the following forms; the actual reduction being given in Chapter 9. (The name of the particular surface which is the locus of the equation is written along with it).

1. $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, Ellipsoid.

2. $x^2/a^2 + y^2/b^2 + z^2/c^2 = -1$, Imaginary ellipsoid.

3. $x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$, Hyperboloid of one sheet.

4. $x^2/a^2 + y^2/b^2 - z^2/c^2 = -1$, Hyperboloid of two sheets.

5. $x^2/a^2 + y^2/b^2 + z^2/c^2 = 0$, Imaginary cone.

6. $x^2/a^2 + y^2/b^2 - z^2/c^2 = 0$, Cone.

7. $x^2/a^2 + y^2/b^2 = 2z/c$, Elliptic paraboloid.

8. $x^2/a^2 - y^2/b^2 = 2z/c$, Hyperbolic paraboloid.

9. $x^2/a^2 + y^2/b^2 = 1$, Elliptic cylinder.

10. $x^2/a^2 - y^2/b^2 = 1$, Hyperbolic cylinder.

11. $x^2/a^2 + y^2/b^2 = -1$, Imaginary cylinder.

12. $x^2/a^2 - y^2/b^2 = 0$, Pair of intersecting planes.

13. $x^2/a^2 + y^2/b^2 = 0$, Pair of imaginary planes.

14. $y^2 = 4ax$, Parabolic cylinder.

15. $y^2 = a^2$, Two real parallel plans.

16. $y^2 = -a^2$, Two imaginary planes.

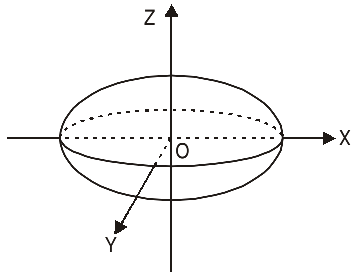
17. $y^2 = 0$, Two coincident planes.

The equations representing cones and cylinders have already been considered and the reader is familiar with the nature of the surface represented by them.

In this chapter we propose to discuss the nature and some of the important geometrical properties of the surfaces represented by the equation, 1, 2, 3, 4, 7, 8.

6.2 SHAPES OF SOME SURFACES

6.2.1 The Ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$



The *following facts* enable us to have an idea of the shape of the surface represented by this equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

(i) We have

(x, y, z) satisfies the equation $\Leftrightarrow$ $(-x, -y, -z)$ satisfies the equation.

The points (x, y, z) $(-x, -y, -z)$ lying on a straight line through the origin and equidistant from the origin, it follows that the origin bisects every chord which passes through it and is, on this account, called the *Centre* of the surface.

(ii) We have

(x, y, z) satisfies the equation $\Leftrightarrow (x, y, -z)$ satisfies the equation.

The line joining the points $(x, y, z), (x, y, -z)$ is bisected at right angle by the *XOY* plane. It follows that the *XOY* plane bisects every chord perpendicular to it and the surface is symmetrical with respect to this plane.

Similarly the surface is symmetrical with respect to the *YOZ* and the *ZOX* planes.

These three planes are called **principal planes** in as much as they bisect all chords perpendicular to them. The three lines of intersection of the three principal planes taken in pairs are called **principal axes.** Co-ordinates axes are the principal axes in the present case.

(iii) x cannot take a value which is numerically greater than a, for otherwise y^2 or z^2 would be negative. Thus, we have $-a \le x \le a$ for every point (x, y, z) on the surface. Similarly, y and z cannot be numerically greater than b and c respectively so that we have for every point (x, y, z) on the surface

$$-a \le x \le a,\ -b \le y \le b, -c \le z \le c$$

Hence, the surface lies between the planes

$$x = a,\ x = -a;\ y = b,\ y = -b;\ z = c,\ z = -c$$

and so is a *closed* surface.

(iv) The *X*-axis meets the surface in the two points $(a, 0, 0)$ and $(-a, 0, 0)$, so that the surface intercepts a length $2a$ on *X*-axis. Similarly, the length intercepted on the *Y*-axis and *Z*-axis are $2b$ and $2c$ respectively.

The lengths $2a, 2b, 2c$ intercepted on the principal axes are called the lengths of the axes of the ellipsoid.

(v) The sections of the surface by the planes $z = k$ which are parallel to the XOY plane are similar ellipses having equations

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1-\frac{k^2}{c^2},\ \ z=k;\ -c\le k\le c. \qquad ...(1)$$

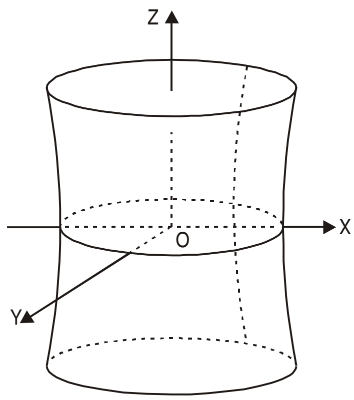
These ellipses have their centres on the Z-axis and diminish in size as k varies from 0 to c. The ellipsoid may, therefore, be generated by the variable ellipse (1) as k varies from $-c$ to c.

It may similarly be shown that the sections by planes parallel to the other co-ordinate planes are also ellipses and the ellipsoid may be supposed to be generated by them.

6.2.2. The Hyperboloid of One Sheet $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1.$

(i) The origin bisects all chords through it and is, therefore, the centre of the surface.

(ii) The co-ordinate planes bisect all chords perpendicular to them and are, therefore, the plane of symmetry or the **principal planes** of the surfaces. The co-ordinate axes are its **principal axes.**



(iii) The X-axis meets the surface in the points $(a, 0, 0)$, $(-a, 0, 0)$ so that the surface intercepts length $2a$ on X-axis. Similarly the length intercepted on Y-axis is $2b$. The Z-axis does not meet the surface.

The sections by the planes $z = k$ which are parallel to the XOY plane are the similar ellipses

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1+\frac{k^2}{c^2},\ z=k \qquad ...(1)$$

whose centres lie on Z-axis and which increase in size as k increases. There is no limit to the increase of k. The surface may, therefore, be generated by the variable ellipse (1) where k varies from $-\infty$ to $+\infty$.

Again, the sections by the planes $x = k$ and $y = k$ are hyperbolas

$$\frac{y^2}{b^2}-\frac{z^2}{c^2}=1-\frac{k^2}{a^2},\quad x=k;\ \frac{x^2}{a^2}-\frac{z^2}{c^2}=1-\frac{k^2}{b^2},\ y=k$$

respectively.

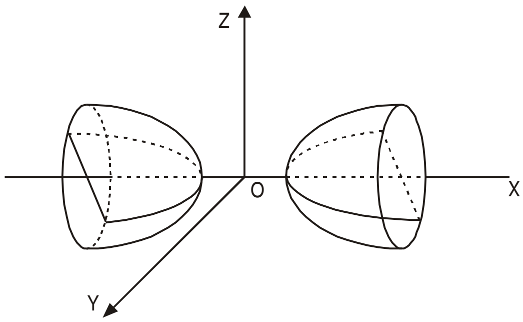
Ex. Trace the surfaces

(i) $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$ (ii) $-\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$

6.2.3. The Hyperboloid of Two Sheets $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$

(i) Origin is the **centre**; co-ordinate planes are the **principal planes**; and co-ordinate axes the **principal axes** of the surface.

(ii) X-axis meets the surface in the points $(a, 0, 0)$ and $(-a, 0, 0)$ whereas the Y and Z-axis do not meet the surface.



(iii) The sections by the planes $z = k$ and $y = k$ are the hyperbolas

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 + \frac{k^2}{c^2}, \quad z = k; \quad \frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 + \frac{k^2}{b^2}; \; y = k$$

respectively.

The plane $x = k$ does not meet the surface if $-a < k < a$ so that there is no portion of the surface between the planes

$$x = -a, \quad x = a$$

when $k^2 > a^2$, *i.e.*, when $k \geq a$ or $k \leq -a$, the plane $x = k$ cuts the surface in the ellipse

$$\frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{k^2}{a^2} - 1, \quad x = k.$$

These ellipses increase in size as k^2 increases.

Ex. Trace the surfaces

(i) $-\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$ (ii) $-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$

6.2.4. Central Conicoids

The equations considered above are of the form

$$ax^2 + by^2 + cz^2 = 1. \qquad ...(1)$$

If a, b, c are all positive, the surface is an ellipsoid; if two are positive and one negative, hyperboloid of one sheet, and finally if two are negative and one positive, hyperboloid of two sheets.

No point (x, y, z) satisfies the equation

$$ax^2 + by^2 + cz^2 = 1$$

if a, b, c are all negative.

All *these surfaces have a centre and three principal planes* and are as such known as *central conicoids.*

On the basis of the preceding discussion, the reader would do well to give precise definitions of

(i) *Centre* (ii) *Principal planes* (iii) *Principal axes* of a central conicoid.

In what follows, we shall consider the equation (1) and the geometrical results deducible from it will, therefore, hold in the case of all central conicoids.

Ex. Show that the surface represented by the equation

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = d$$

is a central conicoid; origin being the centre.

Note. Cone is also a central conicoid, vertex being its centre; this fact is clear from the general equation of a cone with its vertex at the origin.

6.3 INTERSECTION OF A LINE WITH A CONICOID

To find the points of intersection of the line

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \qquad \text{...(i)}$$

with the central conicoid

$$ax^2 + by^2 + cz^2 = l \qquad \text{...(ii)}$$

A point

$$(lr + \alpha, mr + \beta, nr + \gamma)$$

on the line (i) shall also lie on the surface (ii), if and only if,

$$a\,(lz + \alpha)^2 + b\,(mr + \beta)^2 + c\,(nr + \gamma)^2 = 1$$

$$\Leftrightarrow \quad r^2\,(a^2 + bm^2 + cn^2) + 2r\,(al\alpha + bm\beta + cn\gamma) + (a\alpha^2 + b\beta^2 + c\gamma^2 - 1) = 0 \qquad \text{...(A)}$$

Let r_1, r_2 be the two roots of (A), which we suppose to be real. Then

$$(lr_1 + \alpha,\ mr_1 + \beta,\ nr_1 + \gamma),\ (lr_2 + \alpha, mr_2 + \beta, nr_2 + \gamma)$$

are the two points of intersection.

Hence, *every line meets a central conicoid in two points.*

We also see that *a plane section of a central conicoid is a conic for every line in the plane meets the curve of intersection in two points only.*

The two values r_1 and r_2 of r obtained from equation (A) are the measures of the distances of the points of intersection P and Q from the point (α, β, γ) if (l, m, n) are the direction cosines of the line.

Note. The equation (A) of this article will frequently be used in what follows.

Ex. 1. Find the points of intersection of the line

$$-\frac{1}{3}(x+5) = (y-4) = \frac{1}{7}(z-11)$$

with the conicoid

$$12x^2 - 17y^2 + 7z^2 = 7.$$

[**Ans.** (1, 2, – 3), (– 2, 3, 4)]

Ex. 2. Prove that the sum of the squares of the reciprocals of any three mutually perpendicular semi-diameters of a central conicoid is constant.

Ex. 3. Any three mutually orthogonal lines drawn through a fixed point C meets the quadric

$$ax^2 + by^2 + cz^2 = 1$$

in points $P_1, P_2; Q_1, Q_2; R_1, R_2$ respectively; prove that

$$\frac{P_1P_2^2}{CP_1^2 . CP_2^2} + \frac{Q_1Q_2^2}{CQ_1^2 . CQ_2^2} + \frac{R_1R_2^2}{CR_1^2 . CR_2^2} \text{ and } \frac{1}{CP_1 . CP_2} + \frac{1}{CQ_1 . CQ_2} + \frac{1}{CR_1 . CR_2}$$

are constants.

6.3.1. Tangent Lines and Tangent Plane at a Point

Let

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad \text{...(i)}$$

be a line through the point (α, β, γ) of the surface

$$ax^2+by^2+cz^2=1 \qquad \text{...(ii)}$$

Thus, we have

$$a\alpha^2+b\beta^2+c\gamma^2=1 \qquad \text{...(iii)}$$

One root of the equation (A) is § 6.3 is, therefore, zero.

The line (i) will touch the conicoid (ii) at the point (α, β, γ) if both the values of r given by the equation (A) in § 6.3 are zero.

The second value will also be zero, if

$$al\alpha+bm\beta+cn\gamma=0 \qquad \text{...(iv)}$$

which is thus the condition for the line (i) to be a tangent line to the surface (ii) at (α, β, γ).

The locus of the tangent line to the surface, at (α, β, γ) obtained by eliminating *l, m, n* between (i) and (ii), is

$$a\alpha\,(x-\alpha)+b\beta\,(y-\beta)+c\gamma\,(z-\gamma)=0$$

$$\Rightarrow \quad a\alpha x+b\beta y+c\gamma z=a\alpha^2+b\beta^2+c\gamma^2=1$$

which is a plane.

Hence, the tangent lines to the surface (ii) at the point (α, β, γ) lie in the plane

$$a\alpha x+b\beta y+c\gamma z=1$$

which is, therefore, the *tangent plane* at (α, β, γ) to the conicoid

$$ax^2+by^2+cz^2=1$$

Note. A tangent line at any point is a line which meets the surface in two coincident points and the tangent plane at a point is the locus of tangent lines at the point.

6.3.2. Condition of Tangency

To find the condition that the plane

$$lx+my+nz=p, \qquad \text{...}(i)$$

should touch the central conicoid

$$ax^2+by^2+cz^2=1 \qquad \text{...(ii)}$$

If (α, β, γ) be the point of contact, the tangent plane

$$a\alpha x+b\beta y+c\gamma z=1 \qquad \text{...(iii)}$$

there at should be the same as the plane (i).

Comparing the two equations (i) and (iii), we get

$$\alpha=\frac{l}{ap},\ \beta=\frac{m}{bp},\ \gamma=\frac{n}{cp}$$

and since

$$a\alpha^2+b\beta^2+c\gamma^2=1$$

we obtain the required condition

$$\boldsymbol{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p^2.}$$

Also the point of contact then, is,

$$\left(\frac{l}{ap}, \frac{m}{bp}, \frac{n}{cp}\right)$$

Thus, we deduce that the planes

$$\boldsymbol{lx + my + nz = \pm\sqrt{l^2/a + m^2/b + n^2/c}}$$

touch the conicoid (ii) for all values of *l, m, n.*

Cor. There are two tangent planes to a central conicoid parallel to a given plane.

6.3.3. Director Sphere

To show that the locus of the point of intersection of three mutually perpendicular tangent planes to a central conicoid is a sphere concentric with the conicoid.

Let

$$l_1x + m_1y + n_1z = \left(\frac{l_1^2}{a} + \frac{m_1^2}{b} + \frac{n_1^2}{c}\right)^{1/2} \quad \text{...(i)}$$

$$l_2x + m_2y + n_2z = \left(\frac{l_2^2}{a} + \frac{m_2^2}{b} + \frac{n_2^2}{c}\right)^{1/2} \quad \text{...(ii)}$$

$$l_3x + m_3y + n_3z = \left(\frac{l_3^2}{a} + \frac{m_3^2}{b} + \frac{n_3^2}{c}\right)^{1/2} \quad \text{...(iii)}$$

be three mutually perpendicular tangent planes so that

$$\Sigma l_1m_1 = \Sigma m_1n_1 = \Sigma n_1l_1 = 0$$

$$\Sigma l_1^2 = \Sigma m_1^2 = \Sigma n_1^2 = 1 \quad \text{...(iv)}$$

The co-ordinates of the point of intersection satisfy the three equations and its locus is, therefore, obtained by the elimination of $l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3$.

This is easily done by squaring and adding the three equations and using the relations (iv), so that we obtain

$$\boldsymbol{x^2 + y^2 + z^2 = 1/a + 1/b + 1/c}$$

as the required locus which is a concentric sphere called the *Director sphere* of the given quadric. Its centre is the same as that of the central conicoid.

EXAMPLES

1. *Obtain the tangent planes to the ellipsoid*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

which are parallel to the plane

$$lx + my + nz = 0.$$

If 2r is the distance between two parallel tangent planes to the ellipsoid, prove that the line through the origin and perpendicular to the planes lies on the cone

$$x^2\,(a^2 - r^2) + y^2\,(b^2 - r^2) + z^2\,(c^2 - r^2) = 0$$

Sol. The two tangent planes parallel to the plane $\Sigma lx = 0$, are

$$\Sigma lx = \pm\sqrt{\Sigma a^2 l^2} \quad \text{...(1)}$$

The distance between these parallel planes which is twice the distance of either from the origin is

$$2\sqrt{\Sigma a^2 l^2} / \sqrt{\Sigma l^2}$$

Thus, we have

$$\frac{2\sqrt{\Sigma a^2 l^2}}{\sqrt{\Sigma l^2}} = 2r \;\Rightarrow\; \Sigma\,(a^2 - r^2)\,l^2 = 0$$

Hence, the locus of the line

$$x/l = y/m = z/n$$

which is perpendicular to the plane (1), is

$$\Sigma\,(a^2 - r^2)\,x^2 = 0.$$

2. *A tangent plane to the conicoid* $ax^2 + by^2 + cz^2 = 1$, *meets the co-ordinate axes in P, Q and R. Find the locus of the centroid of the triangle PQR.*

Sol. Any tangent plane to the given conicoid is

$$lx + my + nz = \sqrt{\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}} \quad \text{...(i)}$$

Hence,

$$P \equiv \left[\frac{1}{l}\sqrt{\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}},\, 0,\, 0\right]$$

$$Q \equiv \left[0,\, \frac{1}{m}\sqrt{\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}},\, 0\right]$$

$$R \equiv \left[0,\, 0,\, \frac{1}{n}\sqrt{\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}}\right]$$

If (x_1, y_1, z_1) be the centroid of ΔPQR, then

$$x_1 = \frac{1}{3l}\sqrt{\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}},\quad y_1 = \frac{1}{3m}\sqrt{\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}}$$

and $$z_1 = \frac{1}{3n}\sqrt{\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}}$$

$$(3lx_1)^2 = \frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2} = (3my_1)^2 = (3nz_1)^2$$

or $$\frac{9l^2}{a} + \frac{9m^2}{b} + \frac{9n^2}{c} = \left(\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}\right)\left(\frac{1}{ax_1^2} + \frac{1}{by_1^2} + \frac{1}{cz_1^2}\right)$$

$$\Rightarrow \quad \frac{1}{ax_1^2} + \frac{1}{by_1^2} + \frac{1}{cz_1^2} = 9$$

Hence, required locus is $$\frac{1}{ax^2} + \frac{1}{by^2} + \frac{1}{cz^2} = 9.$$

EXERCISES

1. Show that the tangent planes at the extremities of any diameter of a central conicoid are parallel.
2. Show that the plane $3x + 12y - 6z - 17 = 0$ touches the conicoid $3x^2 - 6y^2 + 9z^2 + 17 = 0$, and find the point of contact. [**Ans.** $(-1, 2, 2/3)$]

3. Find the equation of the tangent planes to the curve $x^2 - 2y^2 + 3z^2 = 2$ and parallel to the plane $x - 2y + 3z = 0$. [**Ans.** $x - 2y + 3z = \pm 2$]

4. Find the equations to the tangent planes to the surface $4x^2 - 5y^2 + 7z^2 + 13 = 0$, parallel to the plane $4x + 20y - 21z = 0$. Find their points of contact also.

[**Ans.** $4x + 20y - 21z \neq 13 = 0; (\pm 1, \neq 4, \neq 3)$]

5. Find the equations to the two planes which contain the line given by $7x + 10y - 30 = 0$, and touch the ellipsoid $7x^2 + 5y^2 + 3z^2 = 60$.

[**Ans.** $7x + 5y + 3z - 30 = 0, 14x + 5y + 9z - 60 = 0$]

6. Find the equation to the tangent planes to $2x^2 - 6y^2 + 3z^2 = 5$ which pass through the line $x + 9y - 3z = 0 = 3x - 3y + 6z - 5$. [**Ans.** $2x - 12y + 9z = 5, 4x + 6y + 3z = 5$]

7. *P, Q* are any two points on a central conicoid. Show that the plane through the centre and the line of intersection of the tangent planes at *P, Q* will bisect *PQ*. Also show that if the planes through the centre parallel to the tangent planes at *P, Q* cut the chord in P', Q', then $PP' = QQ'$.

8. Prove that the locus of the foot of the central perpendicular on varying tangent planes of the ellipsoid, $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$, is the surface

$$(x^2 + y^2 + z^2)^2 = a^2x^2 + b^2y^2 + c^2z^2.$$

6.3.4. Normal

Def. *The normal at any point of a quadric is the line through the point perpendicular to the tangent plane there at.*

The equation of the tangent plane at (α, β, γ) to the surface

$$ax^2 + by^2 + cz^2 = 1 \qquad \text{...(i)}$$

is

$$a\alpha x + b\beta y + c\gamma z = 1 \qquad \text{...(ii)}$$

The equations to the normal at (α, β, γ), therefore, are

$$\frac{x - \alpha}{a\alpha} = \frac{y - \beta}{b\beta} = \frac{z - \gamma}{c\gamma} \qquad \text{...(ii)}$$

so that $a\alpha, b\beta, c\gamma$ are the direction ratios of the normal.

If *p,* is the length of the perpendicular from the origin to the tangent plane (ii), we have

$$\frac{1}{a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2} = p^2 \iff (a\alpha p)^2 + (b\beta p)^2 + (c\gamma p)^2 = 1.$$

It follows that $a\alpha p, b\beta p, c\gamma p$ are the actual direction consines of the normal at (α, β, γ).

6.3.5. Number of Normals From a Given Point

We shall now show that *through any given point six normals can be drawn to a central conicoid.*

If the normal (iii) at a point (α, β, γ) passes through a *given* point (f, g, h), we have

$$\frac{f - \alpha}{a\alpha} = \frac{g - \beta}{b\beta} = \frac{h - \gamma}{c\gamma} = r, \text{ (say)}$$

$$\iff \qquad \alpha = \frac{f}{1 + ar}, \beta = \frac{g}{1 + br}, \gamma = \frac{h}{1 + cr} \qquad \text{...(iv)}$$

Since (α, β, γ) lies on the conicoid (i), we have the relation

$$\frac{af^2}{(1+ar)^2} + \frac{bg^2}{(1+br)^2} + \frac{ch^2}{(1+cr)^2} = 1, \qquad \text{...(v)}$$

which, being an equation of the *sixth* degree, gives six values of *r*, to each of which there corresponds a point (α, β, γ), as obtained from (iv).

Therefore, there are *six* points on a central quadric the normal at which pass through a given point, *i.e., through a given point, six normals, in general, can be drawn to a central quadric.*

6.3.6 Cubic Curve Through the Feet of Normals

The feet of the six normals from a given point to a central quadric are the intersections of the quadric with a certain cubic curve.

Consider the curve whose parametric equations are

$$x = \frac{f}{1+ar}, \quad y = \frac{g}{1+br}, \quad z = \frac{h}{1+cr} \qquad \text{...(vi)}$$

r being the *parameter.*

The points (x, y, z) on this curve, arising from those of the values of *r* which are the roots of the equation (v), are the six feet of the normals from the point (f, g, h).

Again, the points of intersection of this curve with any plane

$$Ax + By + Cz + D = 0$$

are given by

$$\frac{Af}{1+ar} + \frac{Bg}{1+br} + \frac{Ch}{1+cr} + D = 0$$

which determines three values of *r*. Hence, the curve (vi) cuts *any* plane in three points and is, as such, a *cubic* curve.

Therefore, the six feet of the normals from (f, g, h) are the intersections of the conicoid and the cubic curve (vi).

6.3.7. Quadric Cone Through Six Concurrent Normals

The six normals drawn from any point to a central quadric are the generators of a quadric cone.

We first prove that the lines drawn from (f, g, h) to intersect the cubic curve (vi) generate a quadric cone.

If *any* line

$$\frac{x-f}{l} = \frac{y-g}{m} = \frac{z-h}{n} \qquad \text{...(vii)}$$

through (f, g, h) intersects the cubic curve, we have

$$\frac{\dfrac{f}{1+ar} - f}{l} = \frac{\dfrac{g}{1+br} - g}{m} = \frac{\dfrac{h}{1+cr} - h}{n}$$

$$\Rightarrow \qquad \frac{af/l}{1+ar} = \frac{bg/m}{1+br} = \frac{ch/m}{1+cr}$$

whence eliminating *r*, we get

$$\frac{af}{l}(b-c) + \frac{bg}{m}(c-a) + \frac{ch}{n}(a-b) = 0$$

which is the condition for the line (vii) to intersect the cubic curve (vi).

Eliminating *l, m, n* between the equations of the line and this condition, we get

$$\frac{af\ (b-c)}{x-f}+\frac{bg\ (c-a)}{y-g}+\frac{ch\ (a-b)}{z-h}=0$$

which represents a cone of the second degree generated by lines drawn from (f, g, h) to intersect the cubic curve.

As the six feet of the normals drawn from a point (f, g, h) to the quadric lie on the cubic curve, the normal are, in particular, the generators of this cone of the second degree.

Note. The importance of this result lies in the fact that while five given concurrent lines determine a unique quadric cone, the six normals through a point lie on a quadric cone, *i.e., the quadric cone through any of the five normals through a point also contains the six normals through the point.*

6.3.8. The General Equation of the Conicoid Through the Six Feet of the Normals

The co-ordinates (α, β, γ) of the foot of any of the six normals from (f, g, h) satisfy the relations

$$\frac{\alpha-f}{a\alpha}=\frac{\beta-g}{b\beta}=\frac{\gamma-h}{c\gamma}$$

so that we see that the feet of the normals lie on three cylinders

$$ax\ (y-g)=by\ (x-f)\ \Leftrightarrow\ (a-b)\ xy-agx+bfy=0$$

$$by\,(z-h)=cz\ (y-g)\ \Leftrightarrow\ (b-c)\ yz-bhy+cgz=0$$

$$cz\ (x-f)=ax\ (z-h)\ \Leftrightarrow\ (c-a)\ zx-cfz+ahx=0$$

The six feet of the normals are the common points of the three cylinders and the conicoid

$$ax^2+by^2+cz^2=1$$

The equation

$$ax^2+by^2+cz^2-1+k_1\,[xy\ (a-b)-agx+bfy]+k_2\ [yz\ (b-c)-bhy+cgz] + k_3\ [zx\ (c-a)-cfz+ahx]=0$$

is satisfied by the *six* feet of the normals and contains *three* arbitrary parameters k_1, k_2, k_3. Therefore, it represents the general equation of the conicoid through them.

EXAMPLES

1. *The normal at any point P of a central conicoid meets the three principal planes at G_1, G_2, G_3 show that PG_1, PG_2, PG_3 are in a constant ratio.*

Sol. The equations of the normal at (α, β, γ) are

$$\frac{x-\alpha}{a\alpha p}=\frac{y-\beta}{b\beta p}=\frac{z-\gamma}{c\gamma p}.$$

Now since $a\alpha p, b\beta p, c\gamma p$ are the direction cosines, each of these fractions represents the distance between the points (α, β, γ) and (x, y, z).

Thus, the distance PG_1, of the point $P\,(\alpha, \beta, \gamma)$ from the point G_1 where the normal meets the co-ordinate plane $x=0$ is $-1/cp$.

Similarly $\qquad PG_2=-1/bp,\ \ PG_3=-1/cp.$

Thus, we have $\qquad PG_1 : PG_2 : PG_3 :: a^{-1} : b^{-1} : c^{-1}.$

2. *Show that the lines drawn from the origin parallel to the normals to the central conicoid*

$$ax^2+by^2+cz^2=1,$$

at its points of intersection with the planes

$$lx+my+nz=p,$$

generate the cone

$$p^2\left(\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}\right)=\left(\frac{lx}{a}+\frac{my}{b}+\frac{nz}{c}\right)^2.$$

Sol. Let (f, g, h) be any point, one the curve of intersection of

$$ax^2+by^2+cz^2=1 \quad \text{and} \quad lx+my+nz=p \qquad ...(1)$$

The normal to the quadric at (f, g, h) is

$$\frac{x-f}{af}=\frac{y-g}{bg}=\frac{z-h}{ch} \qquad ...(2)$$

The line through the origin parallel to this normal is

$$\frac{x}{af}=\frac{y}{bg}=\frac{z}{ch}.$$

Also (f, g, h) satisfies the two equations (1), so that we have

$$af^2+bg^2+ch^2=1,\ lf+mg+nh=p \qquad ...(3)$$

The required locus is obtained by eliminating *f, g, h* between (2) and (3).
The equations (3) give

$$af^2+bg^2+ch^2=\left(\frac{lf+mg+nh}{p}\right)^2 \qquad ...(4)$$

which is a second degree homogeneous expression in *f, g, h*. From (2) and (4), we can easily obtain the required locus.

3. *If P, Q, R and P', Q', R' are the feet of the six normals from a point to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$, and the plane PQR is given by $lx+my+nz=p$, then the plane $P'Q'R'$ is given by*

$$\frac{x}{a^2l}+\frac{y}{b^2m}+\frac{z}{c^2n}+\frac{l}{p}=0.$$

Sol. Let plate $P'Q'R'$ be

$$l'x+m'y+n'z=p'.$$

Then the feet of the six normals from any given point (x', y', z') lie on the locus given by the equation

$$(lx+my+nz-p)\,(l'x+m'y+n'z-p')=0 \qquad ...(i)$$

Let (α, β, γ) be a foot of the normal from (x', y', z').

$$\therefore \quad (l\alpha+m\beta+n\gamma-p)\,(l'\alpha+m'\beta+n'\gamma-p')=0 \qquad ...(ii)$$

Also, the normal at (α, β, γ) passes through (x', y', z').

$$\therefore \quad \frac{x'-\alpha}{\alpha/a^2}=\frac{y'-\beta}{\beta/b^2}=\frac{z'-\gamma}{\gamma/c^2}=\lambda,$$

$$\therefore \quad \alpha=\frac{a^2x'}{a^2+\lambda},\ \beta=\frac{b^2y'}{b^2+\lambda},\ \gamma=\frac{c^2z'}{c^2+\lambda}$$

But (α, β, γ) lies on the given ellipsoid

$$\therefore \quad \frac{a^2x'^2}{(a^2+\lambda)^2}+\frac{b^2y'^2}{(b^2+\lambda)^2}+\frac{c^2z'^2}{(c^2+\lambda)^2}=1 \qquad ...(iii)$$

This equation being a sixth degree equation in λ, gives six values of λ corresponding to the six feet of the normals.

Also putting the values of α, β, γ in (ii), we get

$$\left(\frac{a^2 x'l}{a^2+\lambda}+\frac{b^2 y'm}{b^2+\lambda}+\frac{c^2 z'n}{c^2+\lambda}-p\right)\left(\frac{a^2 x'l'}{a^2+\lambda}+\frac{b^2 y'm'}{b^2+\lambda}+\frac{c^2 z'n'}{c^2+\lambda}-p'\right)=0 \qquad \text{...(iv)}$$

This is also sixth degree equation in λ, gives six values of λ corresponding to the six feet of the normals. Hence, equation (iii) and (iv) are identical.

Comparing coefficients of like terms in (iii) and (iv), we get

$$\frac{a^4 ll'}{a^2}=\frac{b^4 mm'}{b^2}=\frac{c^4 nn'}{c^2}=\frac{-pp'}{1}$$

or $$l'=\frac{-pp'}{a^2 l},\ m'=\frac{-pp'}{b^2 m},\ n'=\frac{-pp'}{c^2 n}$$

Substituting these values, the equation of plane $P'Q'R'$ may be found.

EXERCISES

1. If a point G be taken on the normal at any point P of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

such that $$3PG=PG_1+PG_2+PG_3,$$

show that the locus of G is

$$\frac{a^2x^2}{(2a^2-b^2-c^2)^2}+\frac{b^2y^2}{(2b^2-c^2-a^2)^2}+\frac{c^2z^2}{(2c^2-a^2-b^2)^2}=\frac{1}{9}.$$

2. If a length PQ be taken on the normal at any point P of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

such that $PQ=k^2/p$, where k is a constant and p is the length of the perpendicular from the origin to the tangent planes at p, the locus of Q is

$$\frac{a^2x^2}{(a^2+k^2)^2}+\frac{b^2y^2}{(b^2+k^2)^2}+\frac{c^2z^2}{(c^2+k^2)^2}=1.$$

3. Show that, in general, the normals to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ lie in a given plane. Determine the co-ordinates of the two points on the ellipsoid the normal at which lie in the plane

$$by-cz=\frac{1}{2}(b^2-c^2).$$

$$\left[\textbf{Ans.}\left(\pm\sqrt{\frac{1}{2}}\,a, \frac{1}{2}b, \frac{1}{2}c\right)\right]$$

4. If the feet of the three normals from P to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ lie on the plane

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1,$$

prove that the feet of the other three lie on the plane

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}+1=0$$

and P lies on the line

$$a(b^2-c^2)\,x=b(c^2-a^2)\,y=c(a^2-c^2)\,z.$$

5. Prove that through any point (α, β, γ) six normals can be drawn to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ and that the feet of the normals lie on the curve of intersection of the ellipsoid and the cone

$$\frac{a^2\,(b^2-c^2)\,\alpha}{x}+\frac{b^2\,(c^2-a^2)\,\beta}{y}+\frac{c^2\,(a^2-b^2)\,\gamma}{z}=0.$$

6. Show that the locus of points on a centre quadric, the normals at which intersect a given diameter is the curve of intersection with a cone having the principal axes of the quadric as generators.

6.4 PLANE OF CONTACT

The tangent plane

$$axx'+byy'+czz'=1,$$

at the point (x', y', z') to the quadric $ax^2+by^2+cz^2=1,$ passes through the point (α, β, γ), if

$$a\alpha x'+b\beta y'+c\gamma z'=1.$$

This shows that the points on the quadric the tangent planes at which pass through the point (α, β, γ) lie on the plane

$$a\alpha x+b\beta y+c\gamma z=1$$

which is called the *Plane of contact* for the point (α, β, γ).

6.5. THE POLAR PLANE OF A POINT

If a secant APQ through a given point A meets a conicoid in points P and Q and a point R be taken on this line such that points A and R divide the segment PQ internally and externally in the same ratio, then the locus of the point R is a plane called the polar plane of A.

It may be easily seen that if the points *A* and *R* divide the segment *PQ* internally and externally in the same ratio, then the points *P, Q* divide the segment *AR* also internally and externally in the same ratio.

Let *A,* be a point (α, β, γ) and let (x, y, z) be the co-ordinates of *R.*

The co-ordinates of the point which divides *AR* in the ratio $\lambda : 1$ are

$$\left(\frac{\lambda x+\alpha}{\lambda+1}, \frac{\lambda y+\beta}{\lambda+1}, \frac{\lambda z+\gamma}{\lambda+1}\right)$$

This point will lie on the conicoid

$$ax^2+by^2+cz^2=1$$

for those of the values of λ which are the roots of the equation

$$a\left(\frac{\lambda x+\alpha}{\lambda+1}\right)^2+b\left(\frac{\lambda x+\beta}{\lambda+1}\right)^2+c\left(\frac{\lambda x+\gamma}{\lambda+1}\right)^2=1$$

$$\lambda^2\,(ax^2+by^2+cz^2-1)+2\lambda\,(a\alpha x+b\beta y+c\gamma z-1)+(a\alpha^2+b\beta^2+c\gamma^2-1)=0 \qquad ...(1)$$

The two roots λ_1, λ_2 of this equation are the ratios in which the points *P, Q* divide the segment *AR*. Since *P, Q* divide the segment *AR* internally and externally in the same ratio, we have

$$\lambda_1+\lambda_2=0$$

so that, from (1)

$$a\alpha x+b\beta y+c\gamma z-1=0 \qquad ...(2)$$

Now the equation (2) of the first degree being a relation between the co-ordinates (x, y, z) of the point gives a plane as the locus of the point *R.*

Thus, *the polar plane of the point* (α, β, γ) *with respect to the conicoid*

$$ax^2+by^2+cz^2=1$$

is the plane

$$a\alpha x + b\beta y + c\gamma z = 0.$$

Any point is called the pole of its polar plane.

Note. The reader acquainted with cross-ratios and, in particular, harmonic cross-ratios, would know that the fact that the points *P, Q* divide *AR* internally and externally in the same ratio is also expressed by the statement

$$(AR, PQ) = -1$$

This is further equivalent to the relation

$$\frac{2}{AR} = \frac{1}{AP} + \frac{1}{AQ}.$$

Cor. The polar plane of a point on a conicoid coincides with the tangent plane there at and that of a point outside it coincides with the plane of contact for that point.

Ex. 1. Show that the point of intersection of the tangent planes at three points on a quadric is the pole of the plane formed by their points of contact.

Ex. 2. Find the pole of the plane $lx + my + nz = p$ with respect to the quadric

$$ax^2 + by^2 + cz^2 = 1. \qquad \textbf{[Ans.} \ l/ap, \ m/bp, \ n/cp]$$

6.5.1. Conjugate Points and Conjugate Planes

It is easy to show that if the polar plane of a point *P* passes through point *Q,* then the polar plane of *Q* passes through *P.*

Two such points are called *Conjugate Points.*

Also it can be shown that if the pole of a plane α lies on plane β, then the pole of the plane β lies on the plane α.

Two such planes are called *Conjugate Planes.*

6.5.2. Polar Lines

Consider a line

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}.$$

The polar plane of any point $(lr + \alpha, mr + \beta, nr + \gamma)$ on this line is

$$a\,(lr + \alpha)\,x + b\,(mr + \beta)\,y + c\,(nr - \gamma)\,z = 1$$

$$\Rightarrow \qquad a\alpha x + b\beta y + c\gamma z - 1 + r\,(alx \quad bmy + cnz) = 0$$

which clearly passes through the line of intersection of the planes

$$a\alpha x + b\beta y + c\gamma z - 1 = 0, \ \ alx \quad bmy + cnz = 0$$

for all values of *r.*

Thus, *the polar planes of all the points on a line l pass through another line l'.*

Now, since the polar planes of an arbitrary point *P* on a line *l* pass through every point of *l',* therefore, the polar planes of any point of *l'* will pass through the point *P* on *l* and as *P* is arbitrary, it passes through every point on *l*, *i.e.,* passes through *l.*

It follows that *if the polar plane of any point on a line l passes through the line l', then the polar plane of any point on l' passes through l.*

Two such lines are said to be *Polar Lines* with respect to the conicoid.

To find the polar line of any given line, we have only to find the line of intersection of the polar planes of any two points on it.

6.5.3. Conjugate Lines

Let *l, m* be any two lines and l', m' their polar lines. We suppose that the line *m intersects* the line *l.*

We shall now show that the line l' also intersects the line *m.*

Let P be the point where the lines m' and l intersect.

As P lies on m' and also on l, its polar plane contains the polar lines m and l' of m' and l respectively, *i.e.,* the lines m and l' are coplanar and hence they intersect.

It follows that *if a line l intersects the polar of a line m, then the line m intersects the polar of the line l.*

Two such lines l and m are called *Conjugate Lines.*

EXAMPLE

Find the locus of straight lines drawn through a fixed point (α, β, γ) *at right angles to their polars with respect to the central conicoid* $ax^2 + by^2 + cz^2 = 1$.

Sol. Let
$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \qquad ...(1)$$
be a line perpendicular to its polar line. Now the polar line of (1) is the intersection of the planes
$$a\alpha x + b\beta y + c\gamma z = 1, \quad alx + bmy + cnz = 0$$
If λ, μ, ν be the direction ratios of this line, we have
$$a\alpha\lambda + b\beta\mu + c\gamma\nu = 0, \quad al\lambda + bm\mu + cn\nu = 0$$
$$\Rightarrow \qquad \frac{a\lambda}{n\beta - m\gamma} = \frac{b\mu}{l\lambda - n\alpha} = \frac{c\nu}{m\alpha - \beta l}$$
The perpendicularity of the line (1) to its polar lines implies
$$l\lambda + m\mu + n\nu = 0$$
$$\therefore \qquad \frac{l\,(n\beta - m\gamma)}{a} + \frac{m\,(l\gamma - n\alpha)}{b} + \frac{n\,(m\alpha - l\beta)}{c} = 0$$
$$\Rightarrow \qquad \alpha mn\left(\frac{1}{b} - \frac{1}{c}\right) + \beta nl\left(\frac{1}{c} - \frac{1}{a}\right) + \gamma lm\left(\frac{1}{a} - \frac{1}{b}\right) = 0$$
$$\Rightarrow \qquad \frac{\alpha}{l}\left(\frac{1}{b} - \frac{1}{c}\right) + \frac{\beta}{m}\left(\frac{1}{c} - \frac{1}{a}\right) + \frac{\gamma}{n}\left(\frac{1}{a} - \frac{1}{b}\right) = 0 \qquad ...(2)$$
Eliminating l, m, n between (1) and (2), we see that the required locus is
$$\frac{\alpha}{x-\alpha}\left(\frac{1}{b} - \frac{1}{c}\right) + \frac{\beta}{y-\beta}\left(\frac{1}{c} - \frac{1}{a}\right) + \frac{\gamma}{z-\gamma}\left(\frac{1}{a} - \frac{1}{b}\right) = 0.$$

EXERCISES

1. Prove that the locus of the poles of the tangent planes of the conicoid $ax^2 + by^2 + cz^2 = 1$ with resepct to the conicoid $\alpha x^2 + \beta y^2 + \gamma z^2 = 1$ is the conicoid
$$\frac{\alpha^2 x^2}{a} + \frac{\beta^2 y^2}{b} + \frac{\gamma^2 z^2}{c} = 1.$$
2. Show that the locus of the poles of the plane
$$lx + my + nz = p,$$
with respect to the system of conicoids
$$\frac{x^2}{a^2 + \lambda} + \frac{y^2}{b^2 + \lambda} + \frac{z^2}{c^2 + \lambda} = 1,$$
where λ is the parameter, is a straight line perpendicular to the given plane.
3. Find the locus of straight line drawn through a fixed point (f, g, h) such that its polar lines with respect to the quadrics

$$ax^2 + by^2 + cz^2 = 1 \text{ and } \alpha x^2 + \beta y^2 + \gamma z^2 = 1$$

as coplanar. $\left[\textbf{Ans.} \sum \dfrac{(\alpha - a)(b\gamma - c\beta) f}{x - f} = 0\right]$

4. Find the conditions that the lines

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}, \quad \frac{x-\alpha'}{l'} = \frac{y-\beta'}{m'} = \frac{z-\gamma'}{n'}$$

should be (i) polar, (ii) conjugate with respect to the conicoid

$$ax^2 + by^2 + cz^2 = 1.$$

[**Ans.** (i) $\Sigma a\alpha\alpha' = 1, \Sigma a\alpha' l = 0, \Sigma a\alpha l' = 0, \Sigma all' = 0$

(ii) $(\Sigma a\alpha l')(\Sigma a\alpha' l) = (\Sigma all')(\Sigma a\alpha\alpha' - 1)$]

6.6.1. The Enveloping Cone

Def. *The locus of tangent lines to a quadric through any points is called an enveloping cone.*

To find the enveloping cone of the conicoid

$$ax^2 + by^2 + cz^2 = 1,$$

with its vertex at (α, β, γ).

Any line

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \qquad \text{...(i)}$$

through the point (α, β, γ) will meet the surface in two coincident points if the equation (A) of § 6.3 has equal roots, *i.e.,* if

$$(al\alpha + bm\beta + cn\gamma)^2 = (al^2 + bm^2 + cn^2)(a\alpha^2 + b\beta^2 + c\gamma^2 - 1) \qquad \text{...(ii)}$$

Eliminating *l, m, n* between (i) and (ii), we obtain

$$[a\alpha(x-\alpha) + b\beta(y-\beta) + c\gamma(z-\gamma)]^2$$
$$= [a(x-\alpha)^2 + b(y-\beta)^2 + c(z-\gamma)^2](a\alpha^2 + b\beta^2 + c\gamma^2 - 1)$$

which is the required equation of the **enveloping cone.**

If we write

$$S \equiv ax^2 + by^2 + cz^2 - 1, \; S_1 \equiv a\alpha^2 + b\beta^2 + c\gamma^2 - 1, \; T_1 \equiv a\alpha x + b\beta y + c\gamma z - 1$$

we see that the equation of the enveloping cone can briefly be written as

$$(T_1 - S_1)^2 = (S - 2T_1 + S_1) S_1 \iff SS_1 = T_1^2$$

$$\iff \quad (ax^2 + by^2 + cz^2 - 1)(a\alpha^2 + b\beta^2 + c\gamma^2 - 1) = (a\alpha x + b\beta y + c\gamma z - 1)^2.$$

Note. Obviously the enveloping cone passes through the points common to the conicoid and the polar plane $a\alpha x + b\beta y + c\gamma z = 1$ of the vertex (α, β, γ).

Thus, the enveloping cone may be regarded as a cone whose vertex is the given point and guiding curve the section of the conicoid by its polar plane.

EXERCISES

1. A point P moves so that the section of the enveloping cone of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

with P as vertex by the plane $z = 0$ is a circle, show that P lies on one of the conics

$$\frac{y^2}{b^2 - a^2} + \frac{z^2}{c^2} = 1, \; x = 0; \quad \frac{x^2}{a^2 - b^2} + \frac{z^2}{c^2} = 1, \; y = 0.$$

2. If the section of the enveloping cone of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

whose vertex is P by the plane $z=0$ is a rectangular hyperbola, show that the locus of P is

$$\frac{x^2+y^2}{a^2+b^2}+\frac{z^2}{c^2}=1.$$

3. Find the locus of the points from which three mutually perpendicular tangent lines can be drawn to the conicoid $ax^2+by^2+cz^2=1$.

[**Ans.** $a(b+c)x^2+b(c+a)y^2+c(a+b)z^2=a+b+c$]

4. A pair of perpendicular tangent planes to the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

passes through the fixed point (0, 0, k). Show that their line of intersection lies on the cone

$$x^2(b^2+c^2-k^2)+y^2(c^2+a^2-k^2)+(z-k)^2(a^2+b^2)=0.$$

[**Hint** : The required locus is the locus of the line of intersection of perpendicular tangent planes to the enveloping cone of the given ellipsoid with vertex at (0, 0, k).]

6.6.2. Enveloping Cylinder

Def. *The locus of tangent lines to a quadric parallel to any given line is called an Enveloping cylinder of the quadric.*

To find the enveloping cylinder of the conicoid

$$ax^2+by^2+cz^2=1$$

with its generators parallel to the line

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}.$$

Let (α, β, γ) be a point on the *enveloping* cylinder, so that the equations of the generator through it are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad \text{...(i)}$$

As in § 6.6.1, the line (i) will touch the conicoid, if,

$$(al\alpha+bm\beta+cn\gamma)^2=(al^2+bm^2+cn^2)(a\alpha^2+b\beta^2+c\gamma^2-1)$$

Thus, the locus of (α, β, γ) is the surface

$$(ax^2+by^2+cz^2-1)(al^2+bm^2+cn^2)=(alx+bmy+cnz)^2$$

which is the required equation of the **Enveloping cylinder.**

Note. Equation of enveloping cylinder deduced from that of enveloping cone. Use of elements at infinity. Since each of the lines parallel to the line

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$$

passes through the point $(l, m, n, 0)$ which is, in fact, the point at infinity on each member of this system of parallel lines, we see that the enveloping cylinder is the enveloping cone with vertex $(l, m, n, 0)$.

The homogeneous equation of the surface being

$$ax^2+by^2+cz^2-t^2=0$$

the equation of the enveloping cylinder is

$$(ax^2 + by^2 + cz^2 - t^2)(al^2 + bm^2 + cn^2 - 0) = (alx + bmy + cnz - t.0)^2; \ (SS_1 = T^2)$$

so that in terms of ordinary cartesian co-ordinates, this equation is

$$(ax^2 + by^2 + cz^2 - 1)(al^2 + bm^2 + cn^2) = (alx + bmy + cnz)^2.$$

Note. Clearly the generators of the enveloping cylinder touch the quadric at points where it is met by the plane $alx + bmy + cnz = 0$ which is known as the plane of contact.

EXERCISES

1. Show that the enveloping cylinders of the ellipsoid

$$ax^2 + by^2 + cz^2 = 1,$$

with generators perpendicular to Z-axis meet the plane $z = 0$ in parabolas.

2. Enveloping cylinders of the quadric $ax^2 + by^2 + cz^2 = 1$ meet the plane $z = 0$ in rectangular hyperbola; show that the central perpendiculars to their planes of contact generate the cone $b^2cx^2 + a^2cy^2 + ab(a + b)z^2 = 0.$

3. Prove that the enveloping cylinders of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

whose generators are parallel to the lines

$$x = 0, \ \pm \frac{y}{\sqrt{a^2 - b^2}} = \frac{z}{c}$$

meet the plane $z = 0$ in circles.

6.7.1. Locus of Chords Bisected at a Given Point. Section With a Given Centre

Let the given point be (α, β, γ).

If a chord

$$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n} \qquad ...(1)$$

of the quadric $ax^2 + by^2 + cz^2 = 1$ is bisected at (α, β, γ), the two roots r_1 and r_2 of the equation (A) of § 6.3 are equal and opposite so that $r_1 + r_2 = 0,$ implying

$$al\alpha + bm\beta + cn\gamma = 0 \qquad ...(2)$$

Therefore, the required locus, obtained by eliminating *l, m, n* between (1) and (2), is

$$a\alpha(x - \alpha) + b\beta(y - \beta) + c\gamma(z - \gamma) = 0$$

which is a plane and can briefly be written as

$$T_1 = S_1.$$

The section of the quadric by this plane is a conic whose centre is (α, β, γ); for this point bisects all chords of the conic through it.

Cor. *The plane which cuts the quadric* $ax^2 + by^2 + cz^2 = 1,$ *in a conic whose centre is* (α, β, γ) *is*

$$\Sigma a\alpha x = \Sigma a\alpha^2$$

EXAMPLES

1. *Show that the locus of the centres of sections of a central conicoid which pass through a given line is a conic.*

Sol. Let the central conicoid be

$$ax^2 + by^2 + cz^2 = 1 \quad \text{...(i)}$$

The section of this conicoid whose centre is the point (α, β, γ) is given by

$$a\alpha(x-\alpha) + b\beta(y-\beta) + c\gamma(z-\gamma) = 0$$

This passes through the given line

$$\frac{x-x'}{l} = \frac{y-y'}{m} = \frac{z-z'}{n}$$

if $$a\alpha(x'-\alpha) + b\beta(y'-\beta) + c\gamma(z'-\gamma) = 0$$

and $$a\alpha l + b\beta m + c\gamma n = 0$$

Hence, the locus of centres is given by the equations

$$ax(x'-x) + by(y'-y) + cz(z'-z) = 0$$

and $$alx + bmy + cnz = 0$$

or $$ax^2 + by^2 + cz^2 = axx' + byy' + czz' \quad \text{...(ii)}$$

and $$alx + bmy + cnz = 0 \quad \text{...(iii)}$$

These two equations determine a conic.

2. *Triads of tangent planes at right angles are drawn to the ellipsoid* $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$ *Show that the locus of the centre of section of the surface by the plane through their points of contact is*

$$x^2 + y^2 + z^2 = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)(a^2 + b^2 + c^2).$$

Sol. Suppose that (α, β, γ) is the centre of section of the surface by a plane through the points of contact of a triad of mutually perpendicular tangent planes. The pole of this section must thus be a point of the director sphere

$$x^2 + y^2 + z^2 = a^2 + b^2 + c^2$$

The equation of the section is $T_1 = S_1$, *i.e.*,

$$\frac{\alpha x}{a^2} + \frac{\beta y}{b^2} + \frac{\gamma z}{c^2} = \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} \quad \text{...(i)}$$

If (f, g, h) be its pole, the equation (i) must be the same as

$$\frac{fx}{a^2} + \frac{gy}{b^2} + \frac{hz}{c^2} = 1 \quad \text{...(ii)}$$

Comparing (i) and (ii), we have

$$f = \frac{\alpha}{\Sigma(\alpha^2/a^2)}, \; g = \frac{\beta}{\Sigma(\alpha^2/a^2)}, \; h = \frac{\gamma}{\Sigma(\alpha^2/a^2)}$$

Since $$f^2 + g^2 + h^2 = a^2 + b^2 + c^2$$

we have $$\alpha^2 + \beta^2 + \gamma^2 = [(\Sigma\alpha^2/a^2)]^2 (a^2 + b^2 + c^2).$$

Replacing α, β, γ by *x, y, z* respectively, we have the required result.

EXERCISES

1. Find the equation of the plane which cuts the surface

$$x^2 - 2y^2 + 3z^2 = 4$$

in a conic whose centre is at the point (5, 7, 6). [**Ans.** $5x - 14y + 18z = 35$]

2. Find the centres of the conics

(i) $4x + 9y + 4z = -15, 2x^2 - 3y^2 + 4z^2 = 1;$

(ii) $2x - 2y - 5z + 5 = 0,\ 3x^2 + 2y^2 - 15z^2 = 4.$ [**Ans.** (i) (2, – 3, 1), (ii) (– 2, 3, – 1)]

3. Prove that the plane through the three extremities of the different axes of a central conicoid cuts it in a conic whose centre coincides with the centroid of the triangle formed by those extremities.

4. Show that the centre of the conic

$$lx + my + nz = p,\ ax^2 + by^2 + cz^2 = 1$$

is the point

$$\left(\frac{lp}{ap_0^2}, \frac{mp}{bp_0^2}, \frac{np}{cp_0^2}\right)$$

where $l^2 + m^2 + n^2 = 1$ and $p_0 = \sqrt{\Sigma l^2 / a}$.

5. A variable plane makes intercepts on the axes of a central conicoid whose sum is zero. Show that the locus of the centre of the section determined by it is a cone which has the axes of the conicoid as its generators.

6.7.2. Locus of Mid-Points of a System of Parallel Chords

Let *l, m, n* be proportional to the direction cosines of a given system of parallel chords and let (α, β, γ) be the mid-point of one of them.

As the chord

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$$

of the quadric is bisected at (α, β, γ), we have, as in § 6.7.1.

$$al\alpha + bm\beta + cn\gamma = 0$$

Now, *l, m, n* being fixed, the locus of the mid-points (α, β, γ) of the parallel chords is the plane

$$alx + bmy + cnz = 0,$$

which clearly passes through the centre of the quadric and is known as the *Diametral plane conjugate to the direction l, m, n.*

Conversely, a plane $Ax + By + Cz = 0$ through the centre is the *diametral plane* conjugate to the direction *l, m, n* given by

$$\frac{al}{A} = \frac{bm}{B} = \frac{cn}{C}$$

Thus, *every cetnral plane is a diametral plane conjugate to some direction.*

Note. If *P* be a point on the conicoid, then the plane bisecting chords parallel to the line *OP* is called the *diametral plane of OP*.

Note. Another method. Use of elements of infinity. We know that the mid-point of a line *AB* is the harmonic conjugate of the point at infinity on the line w.r.t. *A* and *B*. Thus, the *locus of the mid-points of a system of parallel chords is the polar plane of the point at infinity common to the chords of the system.*

We know that $(l, m, n, 0)$ is the point at infinity lying on a line with direction ratios, *l, m, n.* Its polar plane w.r.t. the conicoid

$$ax^2 + by^2 + cz^2 - w^2 = 0$$

expressed in cartesian homogeneous co-ordinates, is

$$alx + bmy + cnz + w.0 = 0$$

$$\Leftrightarrow \qquad alx + bmy + cnz = 0.$$

EXERCISES

1. P (1, 3, 2) is a point on the conicoid

$$x^2 - 2y^2 + 3z^2 + 5 = 0.$$

Find the locus of the mid-points of chords drawn parallel to OP. [**Ans.** $x - 6y + 6z = 0$]

2. Find the equation of the chord of the quadric $4x^2 - 5y^2 + 6z^2 = 7$ which passes through the point (2, 3, 4) and is bisected by the plane $2x - 5y + 3z = 0$.

$$\left[\textbf{Ans}.\ (x-2) = \frac{1}{2}(y-3) = (z-4)\right]$$

6.8 CONJUGATE DIAMETERS AND DIAMETRAL PLANES

In what follows, we shall confine our attention to the ellipsoid only.

Let $P(x_1, y_1, z_1)$ be a point on the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

The equation of the diametral plane bisecting chords parallel to the line OP is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} + \frac{zz_1}{c^2} = 1$$

Let $Q(x_2, y_2, z_2)$ be a point on the section of the ellipsoid by this plane so that we have

$$\frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2} + \frac{z_2z_2}{c^2} = 0$$

which is the condition that the diametral plane of OP should pass through Q and, by symmetry, it is also the condition that the diametral plane of OQ should pass through P.

Thus, *if the diametral plane of OP passes through Q, then the diametral plane of OQ also passes through P.*

Let $R(x_3, y_3, z_3)$ be one of the two points where the line of intersection of diametral planes of OP and OQ meets the conicoid.

Since the point R is on the diametral planes of OP and OQ, the diametral plane

$$\frac{xx_3}{a^2} + \frac{yy_3}{b^2} + \frac{zz_3}{c^2} = 0$$

of OR passes through the points P and Q.

Thus, we obtain the following two sets of relations :

$$\left.\begin{aligned} \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} &= 1, \\ \frac{x_2^2}{a^2} + \frac{y_2^2}{b^2} + \frac{z_2^2}{c^2} &= 1, \\ \frac{x_3^2}{a^2} + \frac{y_3^2}{b^2} + \frac{z_3^2}{c^2} &= 1. \end{aligned}\right\} \text{....(A)} \qquad \left.\begin{aligned} \frac{x_2x_3}{a^2} + \frac{y_2y_3}{b^2} + \frac{z_2z_3}{c^2} &= 0, \\ \frac{x_3x_1}{a^2} + \frac{y_3y_1}{b^2} + \frac{z_3z_1}{c^2} &= 0, \\ \frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2} + \frac{z_1z_2}{c^2} &= 0. \end{aligned}\right\} \text{....(B)}$$

The three semi-diameters OP, OQ, OR are called **conjugate semi-diameters** *if the plane containing any two of them is the diametral plane of the third.*

The co-ordinates of the extremities of the conjugate semi-diameters are connected by the relations (A) and (B) above.

Conjugate Planes. *The three diametral planes POQ, QOR, ROP are called conjugate planes if each is the diameteral plane of the line of intersection of the other two.*

We shall now obtain two more sets of relations (C), (D), equivalent to the relations (A), (B).

By virtue of the relations (A), we see that

$$\frac{x_1}{a}, \frac{y_1}{b}, \frac{z_1}{c}; \frac{x_2}{a}, \frac{y_2}{b}, \frac{z_2}{c}; \frac{x_3}{a}, \frac{y_3}{b}, \frac{z_3}{c}$$

can be considered as the direction cosines of some three straight lines and the relations (B) show that these straight lines are also mutually perpendicular.

Hence, we have $$\frac{x_1}{a}, \frac{x_2}{a}, \frac{x_3}{a}; \frac{y_1}{b}, \frac{y_2}{b}, \frac{y_3}{b}; \frac{z_1}{c}, \frac{z_2}{c}, \frac{z_3}{c}$$

are also the direction cosines of three mutually perpendicular straight lines. Therefore, we have

$$\left.\begin{aligned} x_1^2 + x_2^2 + x_3^2 &= a^2, \\ y_1^2 + y_1^2 + y_3^2 &= b^2, \\ z_1^2 + z_2^2 + z_2^3 &= c^2, \end{aligned}\right\} \quad ...(C) \qquad \left.\begin{aligned} y_1 z_1 + y_2 z_2 + y_3 z_3 &= 0, \\ z_1 x_1 + z_2 x_2 + z_3 x_3 &= 0, \\ x_1 y_1 + x_2 y_2 + x_3 y_3 &= 0. \end{aligned}\right\} \quad ...(D)$$

Properties of Conjugate Semi-Diameters

6.8.1. *The sum of the squares of three conjugate semi-diameters is constant.*

Adding the relations (C), we get

$$OP^2 + OQ^2 + R^2 = a^2 + b^2 + c^2$$

which is constant.

6.8.2. *The volume of the parallelopiped formed by three conjugate semi-diameters as coterminous edges is constant.*

The results (B) give

$$\frac{x_1 / a}{\dfrac{y_2 z_3 - y_3 z_2}{bc}} = \frac{y_1 / b}{\dfrac{z_2 x_3 - z_3 x_2}{ca}} = \frac{z_1 / c}{\dfrac{x_2 y_3 - x_3 y_2}{ab}} = \frac{\sqrt{\Sigma x_1^2 / a^2}}{\Sigma\left(\dfrac{y_2 z_3 - y_3 z_2}{bc}\right)} = \pm 1$$

since $\Sigma\left(\dfrac{y_2 z_3 - y_3 z_2}{bc}\right)^2$ is the sine of the angle between two perpendicular lines with direction cosines

$$\frac{x_2}{a}, \frac{y_2}{b}, \frac{z_2}{c} \text{ and } \frac{x_3}{a}, \frac{y_3}{b}, \frac{z_3}{c}$$

We have $$\frac{x_1}{a} = \pm \frac{y_2 z_3 - y_3 z_2}{bc}, \frac{y_1}{b} = \pm \frac{z_2 x_3 - z_3 x_2}{ca}, \frac{z_1}{c} = \pm \frac{x_2 y_3 - x_3 y_2}{ab}$$

Now the volume of the parallelopiped whose coterminous edges are *OP, OQ, OR*

= 6 $\times$ volume of the tetrahedron *OPQR*

$$= \begin{vmatrix} 0 & 0 & 0 & 0 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

$$= x_1 (y_2 z_3 - y_3 z_2) + y_1 (z_2 x_3 - z_3 x_2) + z_1 (x_2 y_3 - x_3 y_2)$$

$$= \pm \frac{bc x_1^2}{a} \pm \frac{ca y_1^2}{b} \pm \frac{ab z_1^2}{c}$$

$$= \pm abc \, \Sigma \frac{x_1^2}{a^2} = \pm abc, \text{ which is a constant.}$$

The same result can also be proved in the following manner :

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} \times \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = \begin{vmatrix} \Sigma\, x_1^2 & \Sigma\, x_1 y_1 & \Sigma\, x_1 z_1 \\ \Sigma\, x_1 y_1 & \Sigma\, y_1^2 & \Sigma\, y_1 z_1 \\ \Sigma\, x_1 z_1 & \Sigma\, y_1 z_1 & \Sigma\, z_1^2 \end{vmatrix}$$

(By the rule of multiplication of determinatants)

6.8.3. *The sum of the squares of the areas of the faces of the parallelopiped formed with any three conjugate semi-diameters as coterminous edges is constant.*

Let A_1, A_2, A_3 be the areas of the triangles *OQR, ORP, OPQ,* and let l_i, m_i, n_i $(i = 1, 2, 3)$ be the direction cosines of the normals to the planes respectively.

Now, the projection of the triangle *OQR* on the *YZ* plane is the triangle with vertices (0, 0, 0), $(0, y_2, z_2), (0, y_3, z_3)$ whose area is $\frac{1}{2}(y_2 z_3 - y_3 z_2)$. Also this is $A_1 l_1$. Thsu, we have

$$A_1 l_1 = \frac{1}{2}(y_2 z_3 - y_3 z_2) = \pm \frac{bcx_1}{2a}$$

Similarly $$A_1 m_1 = \pm \frac{cay_1}{2b},\ A_1 n_1 = \pm \frac{abz_1}{2c}$$

Squaring and adding, we obatain

$$A_1^2 = \frac{b^2 c^2 x_1^2}{4a^2} + \frac{c^2 a^2 y_1^2}{4b^2} + \frac{a^2 b^2 z_1^2}{4c^2}$$

Similarly projecting the areas *ORP* and *OPQ* in the co-ordinate planes, we get

$$A_2^2 = \frac{b^2 c^2 x_2^2}{4a^2} + \frac{c^2 a^2 y_2^2}{4b^2} + \frac{a^2 b^2 z_2^2}{4c^2}$$

$$A_3^2 = \frac{b^2 c^2 x_3^2}{4a^2} + \frac{c^2 a^2 y_3^2}{4b^2} + \frac{a^2 b^2 z_3^2}{4c^2}$$

Adding, we get

$$A_1^2 + A_2^2 + A_3^2 = \frac{1}{4}(b^2 c^2 + c^2 a^2 + a^2 b^2)$$

which is a constant.

6.8.4. *The sum of the projections of three semi-conjugate diameters on any line or plane is constant.*

Let *l, m, n* be the direction cosines of any given line so that the sum of the squares of the projections of three semi-conjugate diameters *OP, OQ, OR* on the line is

$$= (lx_1 + my_1 + nz_1)^2 + (lx_2 + my_2 + nz_2)^2 + (lx_3 + my_3 + nz_3)^2$$

$$= l^2 \Sigma\, x_1^2 + m^2\, \Sigma\, y_1^2 + n^2\, \Sigma\, z_1^2 + 2lm\, \Sigma\, x_1 y_1 + 2mn\, \Sigma\, y_1 z_1 + 2nl\, \Sigma\, z_1 x_1$$

$$= a^2 l^2 + b^2 m^2 + c^2 n^2$$

which is a constant.

Again, let *l, m, n* be the direction cosines of normal of any given plane so that the sum of the squares of the projections of *OP, OQ, OR* on this plane is

$$= OP^2 - (lx_1 + my_1 + nz_1)^2 + OQ^2 - (lx_2 + my_2 + nz_2)^2 + OR^2 - (lx_3 + my_3 + nz_3)^2$$

$$= a^2 + b^2 + c^2 - a^2 l^2 - b^2 m^2 - c^2 n^2$$

$$= a^2 (m^2 + n^2) + b^2 (n^2 + l^2) + c^2 (l^2 + m^2)$$

which is a constant.

EXAMPLES

1. *Show that the equation of the plane through the extremities*

$$(x_k, y_k, z_k), k = 1, 2, 3.$$

of the conjugate semi-diameters of the ellipsoid.

$$x^2 / a^2 + y^2 / b^2 + z^2 / c^2 = l,$$

is $$\frac{x(x_1 + x_2 + x_3)}{a^2} + \frac{y(y_1 + y_2 + y_3)}{b^2} + \frac{z(z_1 + z_2 + z_3)}{c^2} = l$$

Sol. Let $$lx + my + nz = p$$
be the plane through the three extremінties of the given conjugate semi-diameter, so that we have

$$lx_1 + my_1 + nz_1 = p$$
$$lx_2 + my_2 + nz_2 = p$$
$$lx_3 + my_3 + nz_3 = p$$

Multiplying these by x_1, x_2, x_3 respectively and adding we obtain

$$la^2 = p\,\Sigma\, x$$

Similarly $mb^2 = p\,\Sigma\, y_1$ and $nc^2 = p\,\Sigma\, z_1$

Hence, the required equation.

2. *Find the locus of the equal conjugate diameters of the ellipsoid*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Sol. Let *OP, OQ, OR* be three equal conjugate semi-diameters. We have

$$\begin{cases} OP^2 + OQ^2 + OR^2 = a^2 + b^2 + c^2; \\ OP^2 = OQ^2 = OR^2 \end{cases}$$

$$\Rightarrow \qquad OP^2 = \frac{1}{3}(a^2 + b^2 + c^2)$$

Let *P* be the point (x_1, y_1, z_1). We require the locus of the line

$$\frac{x}{x_1} = \frac{y}{y_1} = \frac{z}{z_1} \qquad ...(1)$$

where $$x_1^2 + y_1^2 + z_1^2 = \frac{1}{3}(a^2 + b^2 + c^2) \qquad ...(2)$$

and $$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} = 1 \qquad ...(3)$$

From (2) and (3), we obtain the homogeneous relation

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} + \frac{z_1^2}{c^2} = \frac{3(x_1^2 + y_1^2 + z_1^2)}{(a^2 + b^2 + c^2)} \qquad ...(4)$$

Eliminating x_1, y_1, z_1 from (1) and (4), we obtain the required locus, *viz.,*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = \frac{3(x^2 + y^2 + z^2)}{(a^2 + b^2 + c^2)}.$$

3. *Show that the locus of the foot of the perpendicular from the centre to the plane through the extremities of three conjugate semi-diameters of the ellipsoid*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

is $$a^2x^2 + b^2y^2 + c^2z^2 = 3(x^2 + y^2 + z^2)^2$$

Sol. Let $P(x_1, y_1, z_1), Q(x_2, y_2, z_2), R(x_3, y_3, z_3)$ be the extremities of the three conjugate semi-diameters of the given ellipsoid with centre *O*.

Let (α, β, γ) be the foot of the perpendicular from the centre *O* of the ellipsoid on the plane *PQR*.

Equations of the perpendicular are

$$\frac{x}{\alpha} = \frac{y}{\beta} = \frac{z}{\gamma}.$$

Therefore equation of the plane *PQR* is

$$\alpha(x - \alpha) + \beta(y - \beta) + \gamma(z - \gamma) = 0$$

$$\alpha x + \beta y + \gamma z = \alpha^2 + \beta^2 + \gamma^2$$

Therefore equation of the plane *PQR* is

$$\alpha(x - \alpha) + \beta(y - \beta) + \gamma(z - \gamma) = 0$$

or $$\alpha x + \beta y + \gamma z = \alpha^2 + \beta^2 + \gamma^2$$

P, Q, R lie on this plane. Therefore

$$\alpha x_1 + \beta y_1 + \gamma z_1 = \alpha^2 + \beta^2 + \gamma^2, \quad \text{...(i)}$$

$$\alpha x_2 + \beta y_2 + \gamma z_2 = \alpha^2 + \beta^2 + \gamma^2, \quad \text{...(ii)}$$

and $$\alpha x_3 + \beta y_3 + \gamma z_3 = \alpha^2 + \beta^2 + \gamma^2 \quad \text{...(iii)}$$

Multiplying (*i*) by x_1, (*ii*) by x_2 and (*iii*) by x_3, and adding, we get

$$\alpha(x_1^2 + x_2^2 + x_3^2) + \beta(x_1y_1 + x_2y_2 + x_3y_3) + \gamma(z_1x_1 + z_2x_2 + z_3x_3)$$
$$= (x_1 + x_2 + x_3)(\alpha^2 + \beta^2 + \gamma^2)$$

Using relations (C) and (D) of § 6.8, we have

$$\alpha a^2 = (x_1 + x_2 + x_3)(\alpha^2 + \beta^2 + \gamma^2)$$

or $$\alpha a = \left(\frac{x_1 + x_2 + x_3}{a}\right)(\alpha^2 + \beta^2 + \gamma^2) \quad \text{...(iv)}$$

Similarly $$\beta b = \left(\frac{y_1 + y_2 + y_3}{b}\right)(\alpha^2 + \beta^2 + \gamma^2) \quad \text{...(v)}$$

and $$\gamma c = \left(\frac{z_1 + z_2 + z_3}{c}\right)(\alpha^2 + \beta^2 + \gamma^2) \quad \text{...(vi)}$$

Now squaring and adding these equations, we get

$$\alpha^2a^2 + \beta^2b^2 + \gamma^2c^2 = (\alpha^2 + \beta^2 + \gamma^2)^2\left[\left(\frac{x_1 + x_2 + x_3}{a}\right)^2 + \left(\frac{y_1 + y_2 + y_3}{b}\right)^2 + \left(\frac{z_1 + z_2 + z_3}{c}\right)^2\right]$$

$$= (\alpha^2 + \beta^2 + \gamma^2)^2\left[\frac{x_1^2 + x_2^2 + x_3^2}{a^2} + \frac{y_1^2 + y_2^2 + y_3^2}{b^2} + \frac{z_1^2 + z_2^2 + z_3^2}{c^2} + 2\left(\frac{x_1x_2}{a^2} + \frac{y_1y_2}{b^2} + \frac{z_1z_2}{c^2}\right) + 2\left(\frac{x_3x_1}{a^2} + \frac{y_3y_1}{b^2} + \frac{z_3z_1}{c^2}\right) + 2\left(\frac{x_2x_3}{a^2} + \frac{y_2y_3}{b^2} + \frac{z_2z_3}{c^2}\right)\right]$$

Using relations (B) and (C) of § 6.8, we obtain

$$\alpha^2 a^2 + \beta^2 b^2 + \gamma^2 c^2 = (\alpha^2 + \beta^2 + \gamma^2)^2 \, (1 + 1 + 1)$$

or $$a^2\alpha^2 + b^2\beta^2 + c^2\gamma^2 = 3\,(\alpha^2 + \beta^2 + \gamma^2)^2$$

Hence the locus of the foot of perpendicular is

$$a^2x^2 + b^2y^2 + c^2z^2 = 3\,(x^2 + y^2 + z^2)^2.$$

4. *Prove that the pole of the plane PQR lies on the ellipsoid*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 3,$$

where OP, OQ, OR are the conjugate semi-diameters of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Sol. Equation of the plane PQR through the extremities

$$P\,(x_1, y_1, z_1), Q\,(x_2, y_2, z_2), R\,(x_3, y_3, z_3)$$

is $$\frac{x}{a^2}(x_1 + x_2 + x_3) + \frac{y}{b^2}(y_1 + y_2 + y_3) + \frac{z}{c^2}(z_1 + z_2 + z_3) = 1 \quad \text{...(i)}$$

If (α, β, γ) be the pole of the plane PQR, then its equation is

$$\frac{\alpha x}{x^2} + \frac{\beta y}{b^2} + \frac{\gamma z}{c^2} = 1 \quad \text{...(ii)}$$

(i) and (ii) represent the same plane, therefore,

$$\alpha = x_1 + x_2 + x_3,$$
$$\beta = y_1 + y_2 + y_3,$$
$$\gamma = z_1 + z_2 + z_3.$$

Now multiplying these relations by $1/a, 1/b, 1/c$ respectively; squaring and adding, we get

$$\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} = \left(\frac{x_1 + x_2 + x_3}{a}\right)^2 + \left(\frac{y_1 + y_2 + y_3}{b}\right)^2 + \left(\frac{z_1 + z_2 + z_3}{c}\right)^2$$

On using relations (B) and (C) of § 6.8, this simplifies to

$$\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} = 3$$

Hence the locus of (α, β, γ) is the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 3.$$

EXERCISES

1. Show that the lines

$$\frac{x}{1} = \frac{y}{4} = \frac{z}{3}, \frac{x}{4} = \frac{y}{1} = \frac{z}{-9}, \frac{x}{26} = \frac{y}{-28} = \frac{z}{45},$$

are three mutually conjugate diameters of the ellipsoid

$$\frac{x^2}{2} + \frac{y^2}{4} + \frac{z^2}{9} = 1.$$

2. Find the equations of the diameters in the plane $x + y + z = 0$, conjugate to

$$x = -\frac{1}{2}y = \frac{1}{3}z$$

with respect to the conicoid $3x^2 + y^2 - 2z^2 = 1$. What are the equations of the third conjugate diameter ? $\left[\textbf{Ans. } \frac{x}{4} = \frac{y}{-9} = \frac{z}{5}, \frac{x}{34} = \frac{y}{42} = \frac{z}{3}\right]$

3. Show that for the ellipsoid $x^2 + 4y^2 + 5z^2 = 1$ the two diameters $\frac{1}{3}x = -\frac{1}{2}y = \frac{1}{3}z$ and $x = 0, 2y = 5z$ are conjugate. Obtain the equations of the third conjuate diameter.

[**Ans.** $x/16 = y = -z/2$]

4. If $p_1, p_2, p_3; \pi_1, \pi_2, \pi_3$ be the projections of the three conjugate diameters on any two given lines, then $p_1\pi_1, p_2, \pi_2\ p_3\pi_3$ is constant.

5. If three conjugate diameters vary so that *OP, OQ* lie respectively in the fixed planes

$$\frac{\alpha_1 x}{a^2} + \frac{\beta_1 y}{b^2} + \frac{\gamma_1 z}{c^2} = 0, \frac{\alpha_2 x}{a^2} + \frac{\beta_2 y}{b^2} + \frac{\gamma_2 z}{c^2} = 0$$

show that the locus of *OR* is the cone

$$\Sigma\, \alpha^2 (\beta_1 z - \gamma_1 y)(\beta_2 z - \gamma_2 y) = 0 .$$

[**Hint.** The required locus of *OR* is obtained from the fact that the lines of intersection of the diametral plane of *OR* with the given planes are conjugate lines.]

6. From a fixed point *H* perpendiculars *HA, HB, HC* are drawn to the conjugate diameters *OP, OQ, OR* respectively; show that

$$OP^2.HA^2 + OQ^2.\,HB^2 + OR^2.HC^2$$

is constant.

7. *OP, OQ, OR* are conjugate diameters of an ellipsoid

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = 1 .$$

At *Q* and *R* tangent lines are drawn parallel to *OP* and p_1, p_2 are their distance from *O*. The perpendicular from *O* to the tangent planes at right angles to *OP* is *p*. Prove that

$$p^2 + p_1^2 + p_2^2 = a^2 + b^2 + c^2 .$$

8. Show that the plane $lx + my + nz = p$ will pass through the extremities of conjugate semi-diameters if $a^2l^2 + b^2m^2 + c^2n^2 = 3p^2$.

PARABOLOIDS

6.9. Having discussed the nature and geometrical properties of central conicoids, we now proceed to the consideration of *paraboloids.*

6.9.1. The Elliptic Paraboloids $x^2/a^2 + y^2/b^2 = 2z/c$

We have the following particulars about this surface :

(i) The co-ordinate planes $x = 0$ and $y = 0$ bisect chords perpendicaular to then and are, therefore, its two planes of symmetry or **Principal Planes.**

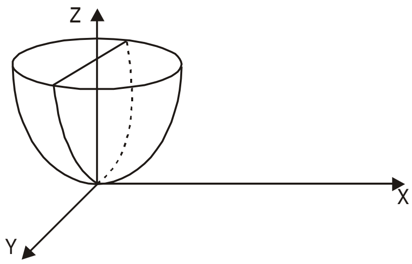
(ii) z cannot be negative, and hence there is no part of the surface on the negative side of the plane $z = 0$. We have taken c positive.

(iii) The sections by the planes $z = k, (k > 0)$, parallel to the *XY* plane, are similar ellipses

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{2k}{c}, z = k \qquad ...(i)$$

whose centres lie on *Z*-axis and which increase in size as k increases; there being no limit to the increase of k. The surface may thus be supposed to be generated by the variable ellipse (*i*).

Hence, the surface is entirely on the positive side of the plane $z = 0$, and extends to infinity.



(*iv*) The section of the surface by planes parallel to the YZ and ZX planes are clearly parabolas. Figure show the nature of the surface.

Ex. Trace the surface $x^2/a^2 + y^2/b^2 = -2z/c$, $(c > 0)$.

6.9.2. The Hyperbolic Paraboloid $x^2/a^2 - y^2/b^2 = 2z/c$

(i) The co-ordinate planes $x = 0$ and $y = 0$ are the two **Principal Planes.**

(ii) The sections by the planes $z = k$ are the similar hyperbolas

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{2k}{c}, \; z = k,$$

with their centres on Z-axis.

If k be positive, the real axis of the hyperbola is parallel to X-axis, and if k be negative, the real axis is parallel to Y-axis.

The section by the plane $z = 0$ is the pair of lines

$$\frac{x}{a} = \frac{y}{b}, \; z = 0 \;\text{ and }\; \frac{x}{a} = -\frac{y}{b}, \; z = 0$$

(iii) The section by the planes parallel to YZ and YX planes are parabolas.

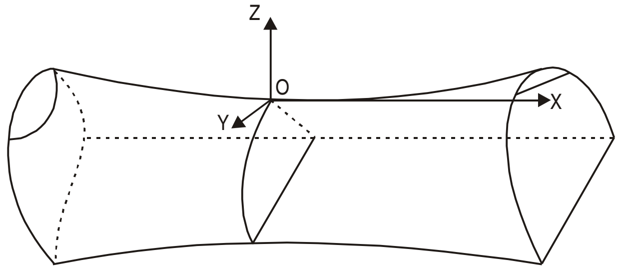


Figure shows the nature of the surface.

Note. The two equations considered in the last two articles are clearly both included in the form

$$ax^2 + by^2 = 2cz$$

This equation represents an elliptic paraboloid if a and b are both positive or both negative, and a hyperbolic paraboloid if one is positive and the other negative.

Hence, for an elliptic paraboloid, ab is positive but, for hyperbolic paraboloid, ab is negative.

The geometrical result deducible from the equation $ax^2 + by^2 = 2cz$ will hold for both the types of paraboloids.

Note. The reader would do well to give precise definitions of (i) vertex, (ii) principal planes, (iii) axis of a paraboloid.

6.9.3 Intersection of a Line With a Paraboloid

The points of intersection of the line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r$$

with the paraboloid $\qquad ax^2+by^2=2cz,$

are $\qquad (lr+\alpha,\ mr+\beta,\ nr+\gamma)$

for the *two* values of r which are the roots of the quadric equation

$$r^2(al^2+bm^2)+2r\,(al\alpha+bm\beta-cn)+(a\alpha^2+b\beta^2-2c\gamma)=0 \qquad \text{...(A)}$$

We thus see that every line meets a paraboloid in two points.

It follows from this that the *plane sections of paraboloids are conics.*

Also, if $l=m=0$, one value of r is infinite and hence any line parallel to Z-axis meets the paraboloid in one point at an infinite distance from (α,β,γ) and so meets it in one finite point only. Such lines are called **Diameters of the paraboloid.**

In particular, Z-axis meets the surface at the origin only.

6.9.4. From the equation (A), § 6.3 above, we deduce certain results similar to those obtained for central conicoids. The proofs of some of them are left as an exercise to the student.

1. The *tangent plane to* $ax^2+by^2=2cz$ at any point (α,β,γ) on the surface is

$$a\alpha x+b\beta y=c\,(z+\gamma).$$

In particular, $z=0$ is the tangent plane at the origin and Z-axis is the normal thereat.

The origin O is called the vertex of the paraboloid and Z-axis, the **axis** of the paraboloid.

2. Condition of Tangency

The plane

$$lx+my+nz=p$$

will touch the paraboloid $\qquad ax^2+by^2=2cz \qquad$ ...(1)

if $\qquad \dfrac{l^2}{a}+\dfrac{m^2}{b}+\dfrac{2np}{c}=0$

and assuming the condition to be satisfied, the point of contact is

$$\left(\frac{-lc}{an},\frac{-mc}{bn},\frac{-p}{n}\right)$$

Thus, *the plane* $\qquad 2n\,(lx+my+nz)+c\,(l^2/a+m^2/b)=0$

touches the surface (1) *for all values of l, m, n.*

3. Locus of the point of intersection of the three mutually perpendicular tangent planes

$$2n_r\ (l_rx+m_ry+n_rz)+c\left(\frac{lr_2^2}{a}+\frac{mr_2^2}{b}\right)=0,\ (r=1,2,3)$$

be three mutually perpendicular tangent planes, the locus of their point of intersection is obtained by eliminating l_r, m_r, n_r, which is done by adding the three equations and is, therefore,

$$2z+c\left(\frac{1}{a}+\frac{1}{b}\right)=0,$$

and is a plane at right angles to the Z-axis; the axis of the paraboloid.

4. Equations of the *normal* at (α,β,γ) are

$$\frac{x-\alpha}{a\alpha}=\frac{y-\beta}{b\beta}=\frac{z-\gamma}{-c}$$

5. The *polar plane* of the point (α, β, γ) is

$$a\alpha x + b\beta y = c\,(\gamma + z).$$

6. The *polar plane* of the *enveloping cone* with (α, β, γ) as its vertex is

$$SS_1 = T_1^2$$

$$\Leftrightarrow \qquad (ax^2 + by^2 - 2cz)\,(a\alpha^2 + b\beta^2 - 2c\gamma) = (a\alpha x + b\beta y - cz - cy)^2$$

Its *plane of contact* with the paraboloid is the polar plane

$$a\alpha x + b\beta y - cz - c\gamma = 0$$

of the vertex (α, β, γ).

7. The equation of the *enveloping cylinder* having its generagtors parallel to the line

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$$

is
$$(ax^2 + by^2 - 2cz)\,(al^2 + bm^2) = (alx + bmy - cn)^2$$

Its *plane of contact* is the plane

$$alx + bmy - cn = 0.$$

8. *The locus of chords bisected at a point* (α, β, γ) *is the plane*

$$T_1 = S_1$$

$$\Leftrightarrow \qquad a\alpha\,(x - \alpha) + b\beta\,(y - \beta) - c\gamma\,(z - \gamma) = 0$$

This plane will meet the paraboloid in a conic whose centre is at (α, β, γ).

9. The *locus of mid-point of a system of parallel chords,* with direction ratios, *l, m, n* is the plane

$$alx + bmy - cn = 0$$

which is parallel to *Z*-axis, the axis of the paraboloid. The plane is called the *Diametral plane* conjugate to the given direction.

A plane $Ax + By + D = 0$ parallel to the axis of the paraboloid is easily seen, by comparison, to be the diametral plane for the system of parallel chords with direction ratios

$$A/a,\ B/b,\ -D/c$$

A plane parallel to the axis of a paraboloid is, thus, a diameteral plane.

EXAMPLES

1. *Two perpendicular tangent planes to the paraboloid* $x^2/a + y^2/b = 2z$ *intersect in a straight line lying in the plane x = 0. Show that the line touches the parabola*

$$x = 0,\ y^2 = (a + b)\,(2z + a).$$

Sol. Equations of any line in the plane $x = 0$ is

$$x = 0,\ my + nz = p \qquad \text{...(i)}$$

Any plane through this line is

$$\lambda x + my + nz - p = 0 \qquad \text{...(ii)}$$

If this is a tangent plane to the paraboloid, then

$$a\lambda^2 + bm^2 + 2np = 0 \qquad \text{...(iii)}$$

This is quadratic in λ and, therefore, gives the values λ_1, λ_2 and thus two tangent planes through the line (*i*) which are

$$\lambda_1 x + my + nz - p = 0$$

and
$$\lambda_2 x + my + nz - p = 0$$

These planes of perpendicular if

$$\lambda_1\lambda_2 + m^2 + n^2 = 0$$

$$\Rightarrow \qquad \frac{bm^2 + 2np}{a} + m^2 + n^2 = 0$$

$$\Rightarrow \qquad (a+b)\, m^2 + 2np + an^2 = 0 \qquad \text{...(iv)}$$

Required parabola is the enveiope of line (i) subject to the condition (iv). Eliminating p between (i) and (iv), we get

$$(a+b)\, m^2 + 2n\,(my + nz) + an^2 = 0,\ x = 0$$

or
$$(a+b)\left(\frac{m}{n}\right)^2 + 2y\left(\frac{m}{n}\right) + (a + 2z) = 0,\ x = 0$$

Therefore, the envelope of the line is given by

$$(2y)^2 - 4\,(a+b)\,(a+2z) = 0,\ x = 0$$

$$\Rightarrow \qquad y^2 = (a+b)\,(a+2x),\ x = 0\cdot$$

2. *Find the locus of the point of intersection of three mutually perpendicular tangent planes to the parboloid* $ax^2 + by^2 = 2cz$.

Sol. The plane $lx + my + nz = p$ touches the paraboloid $ax^2 + by^2 = 2cz$, when

$$\frac{l^2}{a} + \frac{m^2}{b} + \frac{2np}{c} = 0 \quad \text{or} \quad p = -\frac{c}{2n}\left(\frac{l^2}{a} + \frac{m^2}{b}\right)$$

By putting the value of p, we have

$$lx + my + nz + \frac{c}{2n}\,(l^2/a + m^2/b) = 0$$

or
$$2n(lx + my + nz) + c\,(l^2/a + m^2/b) = 0$$

This plane always touches the given paraboloid.

Let three mutually perpendicular tangent planes be

$$2n_r(l_r x + m_r y + n_r z) + c\,(l_r^2/a + m_r^2/b) = 0,\ r = 1, 2, 3 \qquad \text{...(i)}$$

Since the three planes and hence their normals are mutually perpendicular, hence

$$l_1^2 + m_1^2 + n_1^2 = 1,\ l_1^2 + l_2^2 + l_2^3 = 1 \text{ etc.}$$

$$l_1 l_2 + m_1 m_2 + n_1 n_2 = 0,\ l_1 m_1 + l_2 m_2 + l_3 m_3 = 0 \text{ etc.}$$

The locus of the point of intersection is obtained by eliminating l_r, m_r, n_r from the eqns. (i).

For, we add these planes obtained by putting $r = 1, 2, 3$ and use the above relations. Thus, we get

$$2(n_1^2 + n_2^2 + n_3^2)\, z + c\left(\frac{l_1^2 + l_2^2 + l_3^2}{a} + \frac{m_1^2 + m_2^2 + m_3^2}{b}\right) = 0$$

$$\Rightarrow \qquad 2z + c\left(\frac{1}{a} + \frac{1}{b}\right) = 0\,.$$

This is the required plane.

EXERCISES

1. Show that the plane $8x - 6y - z = 5$ touches the paraboloid $x^2/2 - y^2/3 = z$; and find the co-ordinates of the points of contact.
2. Show that the equation to the two tangent planes to the surface

$$ax^2 + by^2 = 2z\,,$$

which passes through the line

$$u \equiv lx + my + nz - p = 0,\ u' \equiv l'x + m'y + n'z - p' = 0$$

is $$u^2\left(\frac{l'^2}{a}+\frac{m'^2}{b}-2n'p'\right)-2uu'\,(ll'+mm'-n'p-n'p)\;+u'^2\left(\frac{l^2}{a}+\frac{m^2}{b}-2nb\right)=0\,.$$

3. Tangent planes at two points P and Q of a paraboloid meet in the line RS; show that the plane through RS and the middle point of PQ is parallel to the axis of the paraboloid.

4. Find the equation of the plane which cuts the paraboloid
$$x^2-\frac{1}{2}y^2=z$$
in a conic with its centre at the point (2, 3, 4). [**Ans.** $4x-3y-z+5=0$]

5. Show that the locus of the centres of a system of parallel plane sections of a paraboloid is a diameter.

6. Show that the centre of the conic
$$ax^2+by^2=2z,\, lx+my+nz=p$$
is the point $$\left(\frac{l}{an},\frac{m}{bn},\frac{k^2}{n^2}\right)$$
where $$k^2=\frac{l^2}{a}+\frac{m^2}{b}+np.$$

6.9.5. Number of Normals From a Given Point

If the normal at (α,β,γ) passes through a given piont (f, g, h), then
$$\frac{f-\alpha}{a\alpha}=\frac{g-\beta}{b\beta}=\frac{h-\gamma}{-c}=r,\ \text{(say)}$$
$$\Leftrightarrow \qquad \alpha=\frac{f}{1+ar},\beta=\frac{g}{1+br},\gamma=h+cr \qquad \text{...(i)}$$

Since (α,β,γ) lies on the paraboloid, we have the relation
$$a\frac{f^2}{(1+ar)^2}+b\frac{g^2}{(1+br)^2}=2c\,(h+cr) \qquad \text{...(ii)}$$
which, being an equation of the fifth degree in r, gives five values of r, to each of which there corresponds a point (α,β,γ), from (i).

Therefore, there are five points, on a paraboloid the normals at which pass through a given point *i.e., through a given point five normals, in general, can be drawn to a paraboloid.*

Cor. I Cubic curve through the feet of the normals

If the normal at (α,β,γ) to the paraboloid
$$ax^2+by^2=2cz$$
passes through a given point (x', y', z'), we have as above
$$\alpha=\frac{x'}{1+a\lambda},\beta=\frac{y'}{1+b\lambda},\gamma=z'+c\lambda\,.$$

Thus, the feet of the normals lie on the curve, defined by the parametric equations, is given by
$$x=\frac{x'}{1+a\lambda},y=\frac{y'}{1+b\lambda},z=z'+c\lambda\,, \qquad \text{...(1)}$$
where λ is the parameter.

The points where this curve meets any given plane, say,
$$ux+vy+wz+d=0 \qquad \text{...(2)}$$

are given by

$$\frac{ux'}{1+a\lambda}+\frac{vy'}{1+b\lambda}+w(z'+c\lambda)+d=0.$$

This is a cubic in λ, giving three values of λ.

Therefore the plane (2) meets the curve (1) in three points, and hence it follows that the curve is a cubic curve.

Cor. II. Cone through the five normals

If the normal at (α, β, γ) to the paraboloid $ax^2+by^2=2cz$ passes through (x', y', z'), then

$$\alpha=\frac{x'}{1+a\lambda}, \beta=\frac{y'}{1+b\lambda}, \gamma=z'+c\lambda$$

Also the direction cosines of the normal at (α, β, γ) are proportional to $a\alpha, b\beta, -c$.

If the line

$$\frac{x-x'}{l}=\frac{y-y'}{m}=\frac{z-z'}{n} \quad \text{...(1)}$$

is a normal at (α, β, γ), then

$$\frac{l}{a\alpha}=\frac{m}{b\beta}=\frac{n}{-c}$$

$$\Rightarrow \quad \frac{l(1+a\lambda)}{ax'}=\frac{m(1+b\lambda)}{by'}=\frac{n}{-c}$$

$$\Rightarrow \quad \frac{l\left(\frac{1}{a}+\lambda\right)}{x'}=\frac{m\left(\frac{1}{b}+\lambda\right)}{y'}=\frac{n}{-c}$$

$$\Rightarrow \quad \frac{\frac{1}{a}+\lambda}{x'/l}=\frac{\frac{1}{b}+\lambda}{y'/m}=\frac{n}{-c}=\frac{\frac{1}{a}-\frac{1}{b}}{\frac{x'}{l}-\frac{y'}{m}}$$

Hence $$n\left(\frac{x'}{l}-\frac{y'}{m}\right)=-c\left(\frac{1}{a}-\frac{1}{b}\right)$$

or $$\frac{x'}{l}-\frac{y'}{m}+\frac{c}{n}\left(\frac{1}{a}-\frac{1}{b}\right)=0$$

Therefore the locus of the normal (1) is

$$\frac{x'}{x-x'}-\frac{y'}{y-y'}+\frac{c}{z-z'}\left(\frac{b-a}{ab}\right)=0$$

which is the equation of a cone.

Hence the five normal from (x', y', z') to the paraboloid are generators of this cone.

6.9.6. Conjugate Diametral Planes

Consider any two *diametral planes*

$$lx+my+p=0 \quad \text{...(i)}$$

and $$l'x+m'y+p'=0 \quad \text{...(ii)}$$

The plane (i) bisects chords parallel to the line

$$\frac{x}{l/a}=\frac{y}{m/b}=\frac{z}{-p/c} \quad \text{...(iii)}$$

which will be parallel to the plane (ii), if

$$\frac{ll'}{a} + \frac{mm'}{b} = 0. \qquad ...(vi)$$

The symmetry of the result shows that the plane (i) is also parallel to the chords bisected by the plane (ii).

Thus, if α *and* β *be two diametral planes,* such that *the plane* α *is parallel to the chords bisected by the plane* β*, then the plane* β *is parallel to the chords bisected by the plane* α.

Two such planes are called *conjugate diametral planes.*

Equation (iv) is the conditon for the diameteral planes (i) (ii) to be conjugate.

EXAMPLES

1. *Show that the planes*

$$x + 3y = 3 \text{ and } 2x - y = 1$$

are conjugate diameter planes of the paraboloid

$$2x^2 + 3y^2 = 4z.$$

Sol. Equation of any diametral plane with respect to the paraboloid $ax^2 + by^2 = 2cz$ is

$$alm + bmy - cn = 0$$

In this case $a = 2, b = 3, c = 2$.

Therefore the equation of the diametral plane is

$$2lx + 3my - 2n = 0 \qquad ...(1)$$

If this plane and $x + 3y = 3$ are the same, then comparing the coefficients, we have

$$\frac{2l}{1} = \frac{3m}{3} - \frac{-2m}{-3}$$

or $$\frac{l}{1/2} = \frac{m}{1} = \frac{n}{3/2}, \text{ i.e., } \frac{l}{1} = \frac{m}{2} = \frac{n}{3},$$

i.e., the direction cosines of the chords which are bisected by the plane $x + 3y = 3$ are proportional to 1, 2, 3.

This shows that these chords are parallel to the plane $2x - y = 1$.

Hence the given planes are conjugate diametral planes.

2. *Prove that any diametral plane of a paraboloid cuts it in a parabola, and that parallel diametral planes cut it in equal parabolas.*

Sol. Let the equation of the parabola be

$$ax^2 + by^2 = 2cz \qquad ...(1)$$

Therefore the diametral plane which bisects chords parallel to the line

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$$

is $$alx + bmy - cm = 0 \qquad ...(2)$$

Now to prove that the section of the paraboloid (1) by the plane (2) is a parabola, it is sufficient to prove that the projection of the section on a co-ordinate plane is a parabola, for the projection of a conic is a conic of the same species. So taking the projection of the section of (1) by the diametral plane (2) on YOZ plane, we have

$$a\left(\frac{cn - bmy}{al}\right)^2 + by^2 = 2cz, x = 0$$

$$\Rightarrow \quad (cn - bmy)^2 + al^2 (by^2 - 2cz) = 0, x = 0$$

$$\Rightarrow \quad b(al^2 + bm^2)y^2 - 2bcmny + n^2 - 2cal^2 z = 0, x = 0$$

$$\Rightarrow \qquad b\,(al^2 + bm^2)\,y^2 - 2bcmny + n^2 = 2cal^2 z,\ x = 0$$

This is obviously a parabola whose latus rectum is

$$\frac{2cal^2}{b\,(al^2 + bm^2)},$$

which is independent of *n*.

Hence it also follows that sections by parallel diametral planes are equal parabolas.

EXERCISES

1. Prove that the diametral planes $2x + 3y = 4$ and $3x - 4y = 7$ are conjugate diametral planes for the paraboloid $x^2 + 2y^2 = 4z$.

2. The plane $3x + 4y = 1$ is a diametral plane of the paraboloid $5x^2 + 6y^2 = 2z$. Find the equation to the chord through (3, 4, 5) which it bisects. $\left[\textbf{Ans. } \frac{x-3}{9} = \frac{y-4}{10} = \frac{z-5}{15}\right]$

3. Show that in general three normals can be drawn from a given point to the paraboloid of revolution $x^2 + y^2 = 2az$ but if the point lies on the surface $27a\,(x^2 + y^2) + 8\,(a - z)^2 = 0$ two of them coincide.

4. Show that the centre of the circle through the feet of the three normals from the point (α, β, γ) to the paraboloid $x^2 + y^2 = 2az$ is

$$\left(\frac{\alpha}{4}, \frac{\beta}{4}, \frac{\gamma + \alpha}{2}\right).$$

❑❑❑

8

Generating Lines of Conicoids

8.1 RULED SURFACES

The surfaces which are generated by a moving straight line are called *ruled surfaces.* For example, cones, cylinders, the hyperboloids of one sheet and hyperbolic paraboloids are ruled surfaces. A ruled surface can also be defined as one through every point of which a straight line can be drawn so as to lie completely on it. The lines which lie on the surfaces are called its *generating lines.*

The ruled surfaces may be divided into two categories : *(i) developable surfaces, (ii) skew surfaces.* A developable surface is one on which the consecutive generators intersect while on a skew surface, the consecutive generating lines do not intersect. The cone is a developable surface as all the generators pass through a common vertex and the cylinder is also a developable surface as consecutive generators touch all along their length. The hyperboloid of one sheet and the hyperbolic paraboloid are skew surfaces.

8.1.1. Condition for a Line to be a Generator of the Conicoid

Let the equation of the line be

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} = r \text{ (say)} \qquad \text{...(i)}$$

and that on the conicoid be

$$ax^2 + by^2 + cz^2 = 1 \qquad \text{...(ii)}$$

Any point on the line (i) is $(lr+\alpha, mr+\beta, nr+\gamma)$.

If it lies on the conicoid (ii), we have

$$r^2 (al^2 + bm^2 + cn^2) + 2r (al\alpha + bm\beta + cn\gamma) + (a\alpha^2 + b\beta^2 + c\gamma^2 - 1) = 0 \qquad \text{...(iii)}$$

If the line (i) is a generator, then it lies wholly on the conicoid, the conditions for which are

$$al^2 + bm^2 + cn^2 = 0 \qquad \text{...(iv)}$$

$$al\alpha + bm\beta + cn\gamma = 0 \qquad \text{...(v)}$$

$$a\alpha^2 + b\beta^2 + c\gamma^2 = 1 \qquad \text{...(vi)}$$

Now, condition (iv) shows that the lines through the centre of the conicoid, *i.e.,* (0, 0, 0) and parallel to the generating lines, *i.e.,* the lines $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ are generators of the cone

$$ax^2 + by^2 + cz^2 = 0,$$

which is called *asymptotic cone.*

The condition (v) shows that the generating lines whose direction cosines are *l, m, n* should lie on the plane,

$$a\alpha x + b\beta y + c\gamma z = 1$$

which is the equation of the tangent plane of the conicoid at the point (α, β, γ).

Equation (iv) and (v) give the *direction ratios of generating lines.*

8.2 GENERATING LINES OF A HYPERBOLOID OF ONE SHEET

It is interesting to see that a hyperboloid of one sheet is a ruled surface in as much it can be thought of as generated by straight lines. The consideration of this aspect of the surface will be taken up in this chapter.

We write the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1, \qquad ...(1)$$

of a hyperboloid of one sheet in the form

$$\frac{x^2}{a^2}-\frac{z^2}{c^2}=1-\frac{y^2}{b^2},$$

$$\Leftrightarrow \qquad \left(\frac{x}{a}-\frac{z}{c}\right)\left(\frac{x}{a}+\frac{z}{c}\right)=\left(1-\frac{y}{b}\right)\left(1+\frac{y}{b}\right)$$

This may again be written in either of the two forms

$$\frac{\frac{x}{a}-\frac{z}{c}}{1-\frac{y}{b}}=\frac{1+\frac{y}{b}}{\frac{x}{a}+\frac{z}{c}} \qquad ...(2)$$

and

$$\frac{\frac{x}{a}-\frac{z}{c}}{1-\frac{y}{b}}=\frac{1-\frac{y}{b}}{\frac{x}{a}+\frac{z}{c}} \qquad ...(3)$$

We consider, now, the two *families* of lines obtained by putting the equal fractions (2) and (3) equal to arbitrary constants λ and μ respectively.

$$\frac{x}{a}-\frac{z}{c}=\lambda\left(1-\frac{y}{b}\right), \quad 1+\frac{y}{b}=\lambda\left(\frac{x}{a}+\frac{z}{c}\right) \qquad ...(A)$$

$$\frac{x}{a}-\frac{z}{c}=\mu\left(1+\frac{y}{b}\right), \quad 1-\frac{y}{b}=\mu\left(\frac{x}{a}+\frac{z}{c}\right) \qquad ...(B)$$

To each value of the constant $\lambda,$ corresponds a member of the family of lines (A) and to each value of the constant $\mu,$ corresponds a member of the family of lines (B).

Now it will be shown that *every point of each of the lines* (A) *and* (B) *lies on the hyperboloid* (1).

If (x_0, y_0, z_0) be a point of a member of the family (A) obtained for some value λ_0 of $\lambda,$ we have

$$\frac{x_0}{a}-\frac{z_0}{c}=\lambda_0\left(1-\frac{y_0}{b}\right), \; 1+\frac{y_0}{b}=\lambda_0\left(\frac{x_0}{a}+\frac{z_0}{c}\right).$$

On eliminating λ_0 from these, we obtain

$$\frac{x_0^2}{a^2}-\frac{z_0^2}{c^2}=1-\frac{y_0^2}{b^2} \quad \Leftrightarrow \quad \frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}-\frac{z_0^2}{c^2}=1$$

which shows that (x_0, y_0, z_0) is a point of the hyperboloid (1).

A similar proof holds for the family of lines (B).

Thus, as λ and μ vary, we get two families of lines (A) and (B) each member of which lies wholly on the hyperboloid. These two families of lines are called *two systems of generating lines (or generators) of the hyperboloid.*

We shall now proceed to discussed some properties of these systems of generating lines.

8.2.1. *Through every point of the hyperboloid there passes one generator of each system.*

Let (x_0, y_0, z_0) be a point of the hyperboloid so that we have

$$\frac{x_0^2}{a^2}+\frac{y_0^2}{b^2}-\frac{z_0^2}{c^2}=1 \qquad ...(4)$$

Now the generator

$$\frac{x}{a} - \frac{z}{c} = \lambda\left(1 - \frac{y}{b}\right),\ 1 + \frac{y}{b} = \lambda\left(\frac{x}{a} + \frac{z}{c}\right)$$

will pass through the point (x_0, y_0, z_0) if and only if λ has a value equal to each of the two fractions

$$\left(\frac{x_0}{a} - \frac{z_0}{c}\right) \div \left(1 - \frac{y_0}{b}\right),\ \left(1 + \frac{y_0}{b}\right) \div \left(\frac{x_0}{a} + \frac{z_0}{c}\right) \qquad ...(5)$$

Also by virtue of the relation (4), these two fractions are equal.

Thus, the member of the system (A) corresponding to either of the equal valuers (5) of λ will pass through the given point (x_0, y_0, z_0). Similarly it can be shown that the member of the system (B) corresponding to either of the equal values

$$\left(\frac{x_0}{a} - \frac{z_0}{c}\right) \div \left(1 + \frac{y_0}{b}\right),\ \left(1 - \frac{y_0}{b}\right) \div \left(\frac{x_0}{a} + \frac{z_0}{c}\right)$$

of μ passes through the given point (x_0, y_0, z_0).

8.2.2. *No two generators of the same system intersect.*

Let

I. (i) $\dfrac{x}{a} - \dfrac{z}{c} = \lambda_1\left(1 - \dfrac{y}{b}\right)$, (ii) $1 + \dfrac{y}{b} = \lambda_1\left(\dfrac{x}{a} + \dfrac{z}{c}\right)$

II. (iii) $\dfrac{x}{a} - \dfrac{z}{c} = \lambda_2\left(1 - \dfrac{y}{b}\right)$, (iv) $1 + \dfrac{y}{b} = \lambda_2\left(\dfrac{x}{a} + \dfrac{z}{c}\right)$

be any two different generators of the λ system.

It will be shown that these four equations in *x, y, z* are not consistent.

Subtracting (iii) from (i), we obtain

$$(\lambda_1 - \lambda_2)\left(1 - \frac{y}{b}\right) = 0 \Rightarrow y = b, \text{ for } \lambda_1 \neq \lambda_2$$

Again, from (ii) and (iv), we obtain

$$\left(\frac{1}{\lambda_1} - \frac{1}{\lambda_2}\right)\left(1 + \frac{y}{b}\right) = 0 \Rightarrow y = -b, \text{ for } \lambda_1 \neq \lambda_2$$

Thus, we see that these four equations are inconsistent and accordingly the two lines do not intersect.

8.2.3. *Any two generators belonging to different systems intersect.*

Let

I. (i) $\dfrac{x}{a} - \dfrac{z}{c} = \lambda\left(1 - \dfrac{y}{b}\right)$, (ii) $1 + \dfrac{y}{b} = \lambda\left(\dfrac{x}{a} + \dfrac{z}{c}\right)$

II. (iii) $\dfrac{x}{a} - \dfrac{z}{c} = \mu\left(1 + \dfrac{y}{b}\right)$, (iv) $1 - \dfrac{y}{b} = \mu\left(\dfrac{x}{a} + \dfrac{z}{c}\right)$

be two generators, one of each system.

It will be shown that these four equations in *x, y, z* are not consistent. Firstly, we solve simultaneously the equations (i), (ii) and (iii). Now, (i) and (iii) give

$$\lambda\left(1 - \frac{y}{b}\right) = \mu\left(1 + \frac{y}{b}\right) \Rightarrow y = b\,\frac{\lambda - \mu}{\lambda + \mu}$$

Substituting this value of *y* in (i) and (ii), we obtain

$$\frac{x}{a} - \frac{z}{c} = \frac{2\lambda\mu}{\lambda + \mu},\quad \frac{x}{a} + \frac{z}{c} = \frac{2}{\lambda + \mu}$$

These given, on adding and subtracting,

$$x = a\,\frac{1+\lambda\mu}{\lambda+\mu},\ \ z = c\,\frac{1-\lambda\mu}{\lambda+\mu}$$

Now, as may easily be seen, these values of x, y, z satisfy (iv) also. Thus, the two lines intersect and the point of intersection is

$$\left(a\,\frac{1+\lambda\mu}{\lambda+\mu},\ b\,\frac{\lambda-\mu}{\lambda+\mu},\ c\,\frac{1-\lambda\mu}{\lambda+\mu}\right) \qquad ...(6)$$

Another method. The planes

$$\frac{x}{a}-\frac{z}{c}-\lambda\left(1-\frac{y}{b}\right)-k\left[1+\frac{y}{b}-\lambda\left(\frac{x}{a}+\frac{z}{c}\right)\right]=0$$

$$\frac{x}{a}-\frac{z}{c}-\mu\left(1+\frac{y}{b}\right)-k'\left[1-\frac{y}{b}-\mu\left(\frac{x}{a}+\frac{z}{c}\right)\right]=0$$

pass through the two lines respectively for all values of k and k'.

Now, obviously these equations becomes identical for $k=\mu$ and $k'=\lambda$.

Thus, the two lines are coplanar and as such they intersect. Also the plane through the two lines, obtained by putting $k=\mu$ or $k'=\lambda$ is

$$\frac{1+\lambda\mu}{\lambda+\mu}\cdot\frac{x}{a}+\frac{\lambda-\mu}{\lambda+\mu}\cdot\frac{y}{b}-\frac{1-\lambda\mu}{\lambda+\mu}\cdot\frac{z}{c}=1 \qquad ...(7)$$

Cor. 1. The plane (7) through two generators of the opposite systems is the tangent plane to the hyperboloid (1) at the point of intersection (6) of the two generators. Since also through every point of the hyperboloid there pass two generators, one of each system, we see that *the tangent plane at a point of hyperboloid meets the hyperboloid in the two generators through the point.*

Cor. 2. *A plane through a generating line is the tangent plane at some point of the generator.* Now like every plane section, the section of the hyperboloid by a plane through a generator is a conic of which the given generator is a part. Thus, the conic is degenerate and the residue must also be a line. At the point of intersection of the lines constituting this degenerate plane section the plane will touch the hyperboloid.

Ex. Prove this result analytical also.

Cor. 3. Parametric equations of the hyperboloid. The co-ordinates (6) show that

$$x = a\,\frac{1+\lambda\mu}{\lambda+\mu},\ \ y = b\,\frac{\lambda-\mu}{\lambda+\mu},\ \ z = c\,\frac{1-\lambda\mu}{\lambda+\mu}$$

are the parametric equations of the hyperboloid; λ, μ being the two parameters. These co-ordinates satisfy the equation of the hyperboloid for all values of the parameters λ and μ.

EXAMPLES

1. *Find the equation to the generating lines of the hyperboloid*

$$\frac{x^2}{4}+\frac{y^2}{9}-\frac{z^2}{16}=1$$

which pass through the point (2, 3, – 4).

Sol. Any line through (2, 3, – 4) is

$$\frac{x-2}{l}=\frac{y-3}{m}=\frac{z+4}{n}=r\ \text{(say)} \qquad ...(i)$$

Any point on this line is $(lr+2, mr+3, nr-4)$ and it lies on the given hyperboloid if

$$\frac{(lr+2)^2}{4}+\frac{(mr+3)^2}{9}-\frac{(nr-4)^2}{16}=1$$

$$\Rightarrow \qquad r^2\left[\frac{l^2}{4}+\frac{m^2}{9}-\frac{n^2}{16}\right]+2r\left[\frac{2l}{4}+\frac{3m}{9}+\frac{4m}{16}\right]=0 \qquad \text{...(ii)}$$

If the line (i) is a generator of the given hyperboloid, then (i) lies wholly on the hyperboloid. The conditions for this are

$$\frac{l^2}{4}+\frac{m^2}{9}-\frac{n^2}{16}=0 \text{ and } \frac{l}{2}+\frac{m}{3}+\frac{n}{4}=0.$$

Eliminating n, we get

$$\frac{l^2}{4}+\frac{m^2}{9}-\left(\frac{l}{2}+\frac{m}{3}\right)^2=0$$

$$\Rightarrow \qquad -\frac{1}{3}lm=0 \quad \Rightarrow \text{ either } l=0 \text{ or } m=0.$$

When $l=0, \frac{m}{3}=-\frac{n}{4}$. When $m=0, \frac{l}{2}=-\frac{n}{4}$.

Hence equations of the required generator are

$$\frac{x-2}{0}=\frac{y-3}{3}=\frac{z+4}{-4} \text{ and } \frac{x-2}{1}=\frac{y-3}{0}=\frac{z+4}{-2}.$$

2. *Find the equations to the generators of the hyperboloid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

which pass through the point $(a\cos\alpha, b\sin\alpha, 0)$.

Sol. Any line through $(a\cos\alpha, b\sin\alpha, 0)$ is given by

$$\frac{x-a\cos\alpha}{l}=\frac{y-b\sin\alpha}{m}=\frac{z}{n}=r \text{ (say)} \qquad \text{...(i)}$$

(i) will meet the hyperboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \qquad \text{...(ii)}$$

if
$$\frac{(a\cos\alpha+lr)^2}{a^2}+\frac{(b\sin\alpha+mr)^2}{b^2}-\frac{(nr)^2}{c^2}=1$$

$$\Rightarrow \qquad r^2\left(\frac{l^2}{a^2}+\frac{m^2}{b^2}-\frac{n^2}{c^2}\right)+2r\left(\frac{l\cos\alpha}{a}+\frac{m\sin\alpha}{b}\right)=0 \qquad \text{...(iii)}$$

(ii) will be a generating line if (iii) is an identity, *i.e.,* if

$$\frac{l^2}{a^2}+\frac{m^2}{b^2}-\frac{n^2}{c^2}=0 \qquad \text{...(iv)}$$

and
$$\frac{l\cos\alpha}{a}+\frac{m\sin\alpha}{b}=0 \qquad \text{...(v)}$$

From (v),
$$\frac{l\cos\alpha}{a}=-\frac{m\sin\alpha}{b}$$

or
$$\frac{l/a}{\sin\alpha}=\frac{m/b}{-\cos\alpha}=\frac{\pm\sqrt{\frac{l^2}{a^2}+\frac{m^2}{b^2}}}{\sqrt{\sin^2\alpha+\cos^2\alpha}}$$

$$\therefore \qquad \frac{l/a}{\sin\alpha} = \frac{m/b}{-\cos\alpha} = \pm\frac{n/c}{1} \text{ [from (iv)]} \qquad ...(vi)$$

$\therefore$ Equations of the required generators are given by

$$\frac{x - a\cos\alpha}{a\sin\alpha} = \frac{y - b\sin\alpha}{-b\sin\alpha} = \frac{z}{\pm c}.$$

3. *CP, CQ are any conjugate diameters of the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, z = c; C'P', C'Q'$ *are the conjugate diameters of the ellipse* $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, z = -c$ *drawn in the same direction as CP and CQ. Prove that hyperboloid* $\frac{2x^2}{a^2} + \frac{2y^2}{b^2} - \frac{z^2}{c^2} = 1$ *is generated by either PQ' or P'Q.*

Sol. Let the co-ordinates of P, Q, P' and Q' are

$P\ (a\cos\theta, b\sin\theta, c)$, $Q\ (-a\sin\theta, b\cos\theta, -c)$,

$P'\ (a\cos\theta, b\sin\theta, -c)$ and $Q'\ (-a\sin\theta, b\cos\theta, -c)$

$\therefore$ Equation to PQ' is

$$\frac{x - a\cos\theta}{-a\sin\theta - a\cos\theta} = \frac{y - b\sin\theta}{b\cos\theta - b\sin\theta} = \frac{z - c}{-c - c} = r \text{ (say)}$$

$$\therefore \qquad \frac{x}{a} = \cos\theta - r(\sin\theta + \cos\theta)$$

$$\frac{y}{b} = \sin\theta + r(\cos\theta - \sin\theta)$$

$$\text{and} \qquad \frac{z}{c} = -2r + 1.$$

Eliminating r, we get

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = r(1+1) + 1 - 2r(\sin\theta\cos\theta + \cos^2\theta - \sin\theta\cos\theta + \sin^2\theta)$$

$$= 2r^2 + 1 - 2r$$

$$\Rightarrow \qquad \frac{2x^2}{a^2} + \frac{2y^2}{b^2} = 4r^2 - 4r + 1 + 1 = (1 - 2r)^2 + 1 = \frac{z^2}{c^2} + 1$$

$$\Rightarrow \qquad \frac{2x^2}{a^2} + \frac{2y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

EXERCISES

1. Write down the equations of the systems of generating lines of the following hyperboloids and determine the pairs of lines of the systems which pass through the given point.

(i) $x^2 + 9y^2 - z^2 = 9$, $(3, 1/3, -1)$ (ii) $x^2/9 - y^2/16 + z^2/4 = 1$, $(-1, 4/3, 2)$

[**Ans.** (i) $x + 3\mu y - z = 3\lambda$, $\lambda x - 3y + \lambda z = 3$; $x + 6y - z = 6$, $2x - 3y + 2z = 3$

$x - 3\mu y - z = 3\lambda$, $\mu x + 3y + \mu z = 3$; $x - 3y - z = 3$, $x + 3y + z = 3$

(ii) $4x - 3y + 6\lambda z = 12\lambda$, $4\lambda x + 3\lambda y - 6z = 12$; $z = 2, 4x + 3y = 0$

$4x - 3y - 6\mu z = 12\mu$, $4\mu x + 3\mu y - 6z = 12$;

$4x - 3y + 2z + 4 = 0$, $4x + 3y - 8z + 36 = 0$]

2. Find the equations to the generating lines of the hyperboloid $yz + 2zx - 3xy + 6 = 0$ which pass through the point (– 1, 0, 3). $\left[\textbf{Ans}. \frac{x+1}{0} = \frac{y-0}{1} = \frac{z-3}{0} \text{ and } \frac{x+1}{1} = \frac{y-0}{-1} = \frac{z-3}{3}\right]$

3. Find equations to the generating lines of hyperboloid $(x + y + z)(2x + y + z) = 6z$, which pass through the point (1, 1, 1). $\left[\textbf{Ans}. \frac{x-1}{4} = \frac{y-1}{-5} = \frac{z-1}{1} \text{ and } \frac{x-1}{1} = \frac{y-1}{-3} = \frac{z-1}{-1}\right]$

4. A point 'm' on the parabola $y = 0, cx^2 = 2a^2 y$ is $(2am, 0, 2cm^2)$ and a point 'n' on the parabola $x = 0, cy^2 = -2b^2 z$ is $(0, 2bn, 2cn^2)$. Find the locus of the line joining the points for which (i) $m = n$, (ii) $m = -n$. $\left[\textbf{Ans}. \frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{2z}{c}\right]$

8.3. *To find the equations of the two generating lines through any point* $(a \cos \theta, b \sin \theta, 0)$, *of the principal elliptic section*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad z = 0,$$

of the hyperboloid by the plane $z = 0$.

Let

$$\frac{x - a\cos\theta}{l} = \frac{y - b\sin\theta}{m} = \frac{z-0}{n}$$

be a generator through the point $(a \cos \theta, b \sin \theta, 0)$.

The point

$$(lr + a\cos\theta,\ mr + b\sin\theta,\ nr)$$

on the generator is a point of the hyperboloid for all values of r so that the equation

$$\frac{(lr + a\cos\theta)^2}{a^2} + \frac{(mr + b\sin\theta)^2}{b^2} - \frac{n^2 r^2}{c^2} = 1$$

$$\Leftrightarrow \qquad \left[\frac{l^2}{a^2} + \frac{m^2}{b^2} - \frac{n^2}{c^2}\right] r^2 + 2r\left[\frac{l\cos\theta}{a} + \frac{m\sin\theta}{b}\right] = 0$$

is true for all values of r. This will be so if

$$\frac{l^2}{a^2} + \frac{m^2}{b^2} - \frac{n^2}{c^2} = 0 \text{ and } \frac{l\cos\theta}{a} + \frac{m\sin\theta}{b} = 0$$

These give

$$\frac{l}{a\sin\theta} = \frac{m}{-b\cos\theta} = \frac{n}{\pm c}$$

Thus, we obtain

$$\frac{x - a\cos\theta}{a\sin\theta} = \frac{y - b\sin\theta}{-b\cos\theta} = \frac{z}{\pm c} \qquad \text{...(C)}$$

as the two required generators.

Note : Since every generator of either system meets the plane $z = 0$ at a point of the principal elliptic section, we see that the two systems of lines obtained from (C) as θ varies from 0 to 2π are the two systems of generators of the hyperboloid. The form (C) of the equations of two systems of generators is often found more useful than the forms (A) and (B) obtained in § 8.2.

Ex. Show that the equations (A) and (B) are equivalent to the equations (C) for

$$\lambda = \tan\left(\frac{1}{4}\pi - \frac{1}{2}\theta\right), \ \mu = \cot\left(\frac{1}{4}\pi - \frac{1}{2}\theta\right).$$

8.4. *To show that the projections of the generators of a hyperboloid on any principal plane are tangents to the section of the hyperboloid by the principal plane.*

Consider a generator

$$\frac{x - a\cos\theta}{a\sin\theta} = \frac{y - b\sin\theta}{-b\sin\theta} = \frac{z}{c}$$

The equation

$$\frac{x - a\cos\theta}{a\sin\theta} = \frac{y - b\sin\theta}{-b\sin\theta}$$

represents the plane through the generator perpendicular to the XOY plane so that the projection of the generator on the XOY plane is

$$\frac{x - a\cos\theta}{a\sin\theta} = \frac{y - b\sin\theta}{-b\sin\theta}, \ z = 0 \quad \Leftrightarrow \quad \frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1, \ z = 0$$

which is clearly the tangent line to the section

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad z = 0$$

of the hyperboloid by the principal plane $z = 0$ at the point $(a\cos\theta, b\sin\theta, 0)$.

Again

$$\frac{x - a\cos\theta}{a\sin\theta} = \frac{z}{c}$$

is the plane through the generator perpendicular to the XOZ plane so that the projection of the generator on the XOZ plane is

$$\frac{x - a\cos\theta}{a\sin\theta} = \frac{z}{c}, \ y = 0 \quad \Leftrightarrow \quad \frac{x\sec\theta}{a} - \frac{z}{c}\tan\theta = 1, \ y = 0$$

which is clearly the tangent to the section

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1, \ y = 0$$

of the hyperboloid by the principal plane $y = 0$ at the point $(a\sec\theta, \ c\tan\theta)$.

Similarly we may show that the projections of the generators on the principal plane $x = 0$ are tangents to the corresponding section.

EXAMPLES

1. *Prove that the equations to the generating lines through the hyperboloid of one sheet are*

$$\frac{x - a\cos\theta\sec\phi}{a\sin(\theta\pm\phi)} = \frac{y - b\sin\theta\sec\phi}{-b\cos(\theta\pm\phi)} = \frac{z - c\tan\theta}{\pm c}.$$

Sol. Let P ("θ, ϕ") be any point on the hyperboloid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1. \quad \text{...(i)}$$

Hence co-ordinates of P would be $(a\cos\theta\sec\phi, b\sin\theta\sec\phi, c\tan\phi)$.

Let equation of tangent plane at this point be

$$\frac{x}{a}\cos\theta\sec\phi + \frac{y}{b}\sin\theta\sec\phi - \frac{z}{c}\tan\phi = 1 \quad \text{...(ii)}$$

This plane meets the plane $z = 0$ in the line given by

$$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = \cos\phi, \ z = 0 \quad \text{...(iii)}$$

If this line meets the section of the surface by $z = 0$ in points A and B whose eccentric angles are α and β respectively, then

$$\frac{a\cos\alpha\cos\theta}{a} + \frac{b\sin\alpha\sin\theta}{b} = \cos\phi$$

or $\quad \cos(\theta - \alpha) = \cos\phi$

and $$\frac{a\cos\beta\cos\theta}{a} + \frac{b\sin\beta\sin\theta}{b} = \cos\phi$$

or $\quad \cos(\theta - \beta) = \cos\phi$

$\therefore \quad \theta - \alpha = -\phi$ and $\theta - \beta = \phi$

so that $\quad \theta = \dfrac{\alpha+\beta}{2}$ and $\phi = \dfrac{\alpha-\beta}{2}$,

i.e., $\quad \alpha = \theta + \phi$ and $\beta = \theta - \phi$ ...(iv)

Since the two generating lines through P are the lines of intersection of the surface and the tangent plane P, AP and BP will be the generators through P such that $\theta + \phi = \alpha$, a constant for all points on the generator AP and $\theta - \phi = \beta$, a constant for all points on the generator BP.

Also, the direction cosines of AP are proportional to

$$a(\cos\alpha - \cos\theta\sec\phi),\ b(\sin\alpha - \sin\theta\sec\phi),\ -c\tan\phi$$

$$\Rightarrow \quad \frac{a(a\cos\alpha\cos\phi - \cos\theta)}{\sin\phi},\ \frac{b(\sin\alpha\cos\phi - \sin\theta)}{\sin\phi},\ -c$$

$$\Rightarrow \quad a\left\{\frac{\cos(\theta+\phi)\cos\phi - \cos\theta}{\sin\phi}\right\},\ b\left\{\frac{\sin(\theta+\phi)\cos\phi - \sin\theta}{\sin\phi}\right\}, -c$$

$$\Rightarrow \quad a\left\{\frac{\cos(\theta+\phi)\cos\phi - \sin(\theta+\phi-\phi)}{\sin\phi}\right\},\ b\left\{\frac{\sin(\theta+\phi)\cos\phi - \sin(\theta+\phi-\phi)}{\sin\phi}\right\}, -c$$

$$\Rightarrow \quad a\sin(\theta+\phi),\ -b\cos(\theta+\phi),\ c$$

$\therefore$ Equation to the generator AP are

$$\frac{x - a\cos\theta\sec\phi}{a\sin(\theta+\phi)} = \frac{y - b\sin\theta\sec\phi}{-b\cos(\theta+\phi)} = \frac{z - c\tan\phi}{c} \quad \text{...(v)}$$

Similarly the generator BP will be

$$\frac{x - a\cos\theta\sec\phi}{a\sin(\theta-\phi)} = \frac{y - b\sin\theta\sec\phi}{-b\cos(\theta+\phi)} = \frac{z - c\tan\phi}{c} \quad \text{...(vi)}$$

2. *The normals to* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} - \dfrac{z^2}{c^2} = 1$ *at points of a generator meet the plane* $z = 0$ *at points lying on a straight line, and for different generators of the same system this line touches a fixed conic.*

Sol. Any generator through

$$(a\cos\theta\sec\phi,\ b\sin\theta\sec\phi,\ c\tan\phi)$$

is $$\frac{x - a\cos\theta\sec\phi}{a\sin(\theta+\phi)} = \frac{y - b\sin\theta\sec\phi}{-b\cos(\theta+\phi)} = \frac{z - c\tan\phi}{c} \quad \text{...(i)}$$

which shows that $\quad (\theta + \phi) = \alpha$ ...(ii)

a constant for a given generator.

Now, the tangent plane at the point "θ, ϕ" is

$$\frac{x}{a}\cos\theta\sec\phi + \frac{y}{b}\sin\theta\sec\phi - \frac{z}{c}\tan\phi = 1$$

$$\Rightarrow \qquad \frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta - \frac{z}{c}\sin\phi = \cos\phi \qquad \text{...(iii)}$$

And the normal at "θ, ϕ" is

$$\frac{x - a\cos\theta\sec\phi}{\dfrac{\cos\theta}{a}} = \frac{y - b\sin\theta\sec\phi}{\dfrac{\sin\theta}{b}} = \frac{z - c\tan\phi}{-\dfrac{\sin\phi}{c}} \qquad \text{...(iv)}$$

which meets the plane $z = 0$ in the point.

$$z = 0,\ x = a\cos\theta\sec\phi + \frac{c^2}{a}\sec\phi\cos\theta = \frac{\cos\theta\sec\phi}{a}(c^2 + a^2)$$

$$y = b\sin\theta\sec\phi + \frac{c^2}{b}\sec\phi\sin\theta = \frac{\sin\theta\sin\phi}{b}(b^2 + c^2)$$

$$\Rightarrow \qquad x = \left(\frac{a^2 + c^2}{a}\right)\frac{\cos(\alpha - \phi)}{\cos\phi} = \left(\frac{a^2 + c^2}{a}\right)(\cos\alpha + \sin\alpha\tan\phi)$$

[since $\theta + \phi = \alpha$, a constant, by (ii)]

$$y = \left(\frac{b^2 + c^2}{b}\right)\frac{\sin(\alpha + \phi)}{\cos\phi} = \left(\frac{b^2 + c^2}{b}\right)(\sin\alpha - \cos\alpha\tan\phi),$$

$$z = 0$$

$$\Rightarrow \qquad \frac{x}{a^2 + c^2} - \cos\alpha = \sin\alpha\tan\phi$$

$$\frac{by}{b^2 + c^2} - \sin\alpha = -\cos\alpha\tan\phi \qquad \text{...(v)}$$

Eliminating ϕ, we get

$$\frac{ax\cos\alpha}{a^2 + c^2} - \cos^2\alpha + \frac{by\sin\alpha}{b^2 + c^2} - \sin^2\alpha = 0,\quad z = 0$$

$$\Rightarrow \qquad \frac{ax\cos\alpha}{a^2 + c^2} + \frac{by\sin\alpha}{b^2 + c^2} = 1,\quad z = 0 \qquad \text{...(vi)}$$

which is a fixed straight line space α is constant.

Again for different generators of the same system, α varies.

$\therefore$ Differentiating (vi) w.r.t. α, we get

$$\frac{-ax\sin\alpha}{a^2 + c^2} + \frac{by\cos\alpha}{b^2 + c^2} = 0,\quad z = 0 \qquad \text{...(vii)}$$

Squaring and adding (vi) and (vii), we get the envelope of (vi) as

$$\frac{a^2x^2}{(a^2 + c^2)^2} + \frac{b^2y^2}{(b^2 + c^2)^2} = 1,\quad z = 0,$$

which is a fixed conic.

3. *Show that the generators through points on the principal elliptic section of*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

such that the eccentric angle of one is double the eccentric angle of the other intersect on the curve given by

$$x = \frac{a(1-3t^2)}{1+t^2}, \quad y = \frac{bt(3-t^2)}{1+t^2}, \quad z = \pm ct.$$

Sol. Let $A(a\cos\theta, b\sin\theta, 0)$ and $B(a\cos\phi, b\sin\phi, 0)$ be the two points on the principal elliptic section by the plane $z = 0$. The points of intersection P and ϕ of the generators of opposite system through them are given by

$$\frac{x}{a} = \frac{\cos\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}, \quad \frac{y}{b} = -\frac{\sin\left(\frac{\theta+\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}, \quad \frac{z}{c} = \pm\frac{\sin\left(\frac{\theta-\phi}{2}\right)}{\cos\left(\frac{\theta-\phi}{2}\right)}$$

Now, we are given that $\phi = 2\theta$.

$$\therefore \qquad \frac{z}{c} = \pm\tan\frac{\theta}{2} \quad \text{or} \quad z = \pm ct.$$

If we take $t = \tan\dfrac{\theta}{2}$.

$$x = a\,.\frac{\cos 3\theta/2}{\cos\theta/2} = a\,.\frac{4\cos^3\theta/2 - 3\cos\theta/2}{\cos\theta/2}$$

$$= a(4\cos^2\theta/2 - 3) = a\,.\frac{4 - 3\sec^2\theta/2}{\sec^2\theta/2}$$

$$= a\,.\frac{4 - 3(1+\tan^2\theta/2)}{1+\tan^2\theta/2} = a\,.\frac{1-3t^2}{1+t^2}$$

$$y = b\,.\frac{\sin 3\theta/2}{\cos\theta/2} = b\,.\frac{3\sin\theta/2 - 4\sin^3\theta/2}{\cos\theta/2}$$

$$= b(3\tan\theta/2 - 4\,.\sin^2\theta/2\tan\theta/2)$$

$$= b\tan\theta/2\left(\frac{3\sec^2\theta/2 - 4\sin^2\theta/2\sec^2\theta/2}{1+\tan^2\theta/2}\right)$$

$$= b\tan\theta/2\left[\frac{3(1+\tan^2\theta/2) - 4\tan^2\theta/2}{1+\tan^2\theta/2}\right] = \frac{bt(1-t^2)}{1+t^2}$$

Hence the generators intersect on the curve

$$x = \frac{a(1-3t^2)}{1+t^2}, \quad y = \frac{bt(3-t^2)}{1+t^2}, \quad z = \pm ct.$$

EXERCISES

1. R, S are the points of intersection of generators of opposite systems drawn at the extremities P, Q of semi-conjugate diameters of the principal elliptic section; show that

(i) the locus of the points R, S are the ellipses $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 2$, $z = \pm c$;

(ii) the perimeter of the skew-quadrilateral $PSQR$ taken in order, is constant and equal to $2(a^2 + b^2 + 2c^2)$;

(iii) $\cot^2\alpha + \cot^2\beta = (a^2 + b^2)/c^2$, where $\angle RPS = 2\alpha$ and $\angle RQS = 2\beta$;

(iv) the volume of the tetrahedro n $PSQR$ is constant and equal to $\dfrac{1}{3}abc$.

2. The generators through a point P on the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ meet the principal elliptic section in points whose eccentric angles differ by a constant 2α; show that the locus of P is the curve of intersection of the hyperboloid with the cone

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{z^2}{c^2}\cos^2\alpha.$$

3. If the generators through a point P on the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ meet the principal elliptic section in two points such that eccentric angle of one is three times that of the other. Prove that P lies on the curve of intersection of the hyperboloid with the cylinder

$$y^2(z^2+c^2)=4b^2z^2.$$

4. Show that the generators through any one of the ends of an equi-conjugate diameter of the principal elliptic section of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ are inclined to each other at an angle 60° if $a^2+b^2=6c^2$. Find also the condition for the generators to be perpendicular to each other. [**Ans.** $a^2+b^2=2c^2$]

5. A variable generator of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ intersects generators of the same system through the extremities of a diameter of the principal elliptic section in points P and P'; show that

$$\frac{x_P x_{P'}}{a^2}=\frac{y_P y_{P'}}{b^2\, z_P z_{P'}}=-c^2.$$

6. Show that the shortest distance between generators of the same system drawn at one end of each of the major and minor axes of the principal elliptic section of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ is $\frac{2abc}{\sqrt{a^2b^2+b^2c^2+c^2a^2}}$.

7. Show that the shortest distance between the generators of the same system drawn at the extremities of the diameters of the principal elliptic section of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$, are parallel to the XOY plane and lie on the surface

$$abz\,(x^2+y^2)=\pm(a^2-b^2)\,cxy.$$

8.5. *To find the locus of the points of intersection of perpendicular generators of the hyperboloid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \qquad ...(1)$$

Let (x_1, y_1, z_1) be a point the generators through which are perpendicular.

The generators are the lines in which the tangent plane

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=1 \qquad ...(2)$$

at the point meets the surface. On making (1) homogeneous with the help of (2), we obtain the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}\right)^2 \qquad ...(3)$$

The curve of intersection of (1) and (2) being a pair of lines, the cone with its vertex at the origin and with the curve of intersection of (1) and (2), as the guiding curve, represented by the equation (3), reduces to a pair of planes.

If *l, m, n* be the direction ratios of either of the two generators, we have, since they lie on the planes (2) and (3),

$$\frac{lx_1}{a^2} + \frac{my_1}{b^2} + \frac{nz_1}{c^2} = 0 \qquad ...(4)$$

and

$$\frac{l^2}{a^2} + \frac{m^2}{b^2} - \frac{n^2}{c^2} = \left(\frac{lx_1}{a^2} + \frac{my_1}{b^2} + \frac{nz_1}{c^2}\right)^2 \qquad ...(5)$$

Now the equation (5) with the help of the equation (4) reduces to

$$\frac{l^2}{a^2} + \frac{m^2}{b^2} - \frac{n^2}{c^2} = 0 \qquad ...(6)$$

Eliminating *n* from (4) and (5), we obtain

$$\frac{l^2}{a^4}(a^2 z_1^2 - c^2 x_1^2) - \frac{2lmc^2 x_1 y_1}{a^2 b^2} + \frac{m^2}{b^4}(b^2 z_1^4 - c^2 y_1^2) = 0$$

If $l_1, m_1, n_1; l_1, m_2, n_2$ be the direction ratios of the two generators, this gives

$$\frac{l_1}{m_1} \cdot \frac{l_2}{m_2} = \frac{b^2 z_1^2 - c^2 y_1^2}{b^4} \cdot \frac{a^4}{a^2 z_1^4 - c^2 x_1^2}$$

$$\Leftrightarrow \qquad \frac{l_1 l_2}{a^4 (b^2 z_1^2 - c^2 y_1^2)} = \frac{m_1 m_2}{b^4 (a^2 z_1^2 - c^2 x_1^2)} = \frac{n_1 n_2}{c^4 (a^2 y_1^2 + c^2 x_1^2)}$$

Since $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0,$ we obtain

$$a^4 (b^2 z_1^2 - c^2 y_1^2) + b^4 (a^2 z_1^2 - c^2 x_1^2) + c^4 (a^2 y_1^2 + b^2 x_1^2) = 0$$

$$\Rightarrow \qquad b^2 c^2 x_1^2 (c^2 - b^2) + a^2 c^2 y_1^2 (c^2 - a^2) + a^2 b^2 z_1^2 (a^2 + b^2) = 0$$

$$\Leftrightarrow \qquad (b^2 - c^2)\frac{x_1^2}{a^2} + (a^2 - c^2)\frac{y_1^2}{b^2} - (a^2 + b^2)\frac{z_1^2}{c^2} = 0$$

We rewrite it as

$$(a^2 + b^2 - c^2)\frac{x_1^2}{a^2} + (a^2 + b^2 - c^2)\frac{y_1^2}{b^2} - (a^2 + b^2 - c^2)\frac{z_1^2}{c^2} = x_1^2 + y_1^2 + z_1^2$$

$$\Leftrightarrow \qquad (a^2 + b^2 - c^2)\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - \frac{z_1^2}{c^2}\right) = x_1^2 + y_1^2 + z_1^2$$

Since now the point (x_1, y_1, z_1) lies on the hyperboloid, this reduces to

$$x_1^2 + y_1^2 + z_1^2 = a^2 + b^2 - c^2$$

Thus, we see that the point of intersection of pairs of perpendicular generators lies on the curve of intersection of the hyperboloid and the director sphere

$$x^2 + y^2 + z^2 = a^2 + b^2 - c^2.$$

Another method : Let *PA, PB* be two perpendicular generators through *P* and *PC* be the normal at *P* so that it is perpendicular to the tangent plane determined by *PA* and *PB*. The lines *PA, PB, PC* are mutually perpendicular and as such the three planes *CPA, APB, BPC* determined by them, taken in pairs, are also mutually perpendicular.

The plane CPA through the generator PA is the tangent plane at some point of PA and the plane CPB through the generator PB is the tangent plane at some point of PB. Also the plane APB is the tangent plane at P.

Thus, the three planes CPA, APB and BPC are mutually perpendicular tangent planes and as such their point of intersection P lies on the director sphere. It follows that the locus of P is the curve of intersection of the hyperboloid with its director sphere.

EXAMPLE

Show that the angle θ *between the generators through any point P of the hyperboloid is given by*

$$\cot\theta = \frac{p\,(r^2 - a^2 - b^2 + c^2)}{2abc}$$

where p is the perpendicular from the centre to the tangent plane at P and r is the distance of P from the centre.

Sol. The tangent plane at $P\,(x_1, y_1, z_1)$ is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} + \frac{zz_1}{c^2} = 1 \qquad \text{...(1)}$$

As in § 8.5 it can be shown that the direction ratios l, m, n of the two generators through this point are given by the equations

$$\frac{lx_1}{a^2} + \frac{my_1}{b^2} + \frac{nz_1}{c^2} = 0, \quad \frac{l^2}{a^2} + \frac{m^2}{b^2} + \frac{n^2}{c^2} = 0.$$

Proceeding as in Example 1, we can show that angle θ between the lines is given by

$$\tan\theta = \frac{\left[-4\left(\frac{x_1^2}{a^4} + \frac{y_1^2}{b^4} + \frac{z_1^2}{c^4}\right)\left(\frac{x_1^2}{a^4b^2c^2} - \frac{y_1^2}{b^4c^2a^2} + \frac{z_1^2}{c^4a^2b^2}\right)\right]^{1/2}}{\frac{1}{a^2}\left(\frac{y_1^2}{b^4} + \frac{z_1^2}{c^4}\right) + \frac{1}{b^2}\left(\frac{z_1^2}{c^4} + \frac{x_1^2}{a^4}\right) - \frac{1}{c^2}\left(\frac{x_1^2}{a^4} + \frac{y_1^2}{b^4}\right)}$$

Now, p, the length of perpendicular from the centre to the tangent plane (1) at (x_1, y_1, z_1), is given by

$$p = \frac{1}{\left[\sum \frac{x_1^2}{a^4}\right]^{1/2}} \quad \Rightarrow \quad \frac{1}{p^2} = \sum \frac{x_1^2}{a^4}$$

Also the denominator of the expression for $\tan\theta$

$$= \frac{1}{a^2b^2c^2}\left[\frac{x_1^2}{a^2}(c^2 - b^2) + \frac{y_1^2}{b^2}(c^2 - a^2) + \frac{z_1^2}{c^2}(a^2 + b^2)\right]$$

$$= \frac{1}{a^2b^2c^2}\left[\frac{x_1^2}{a^2}(c^2 - b^2 - a^2) + \frac{y_1^2}{b^2}(c^2 - a^2 - b^2) + \frac{z_1^2}{c^2}(a^2 + b^2 - c^2)\right.$$

$$\left. + (x_1^2 + y_1^2 + z_1^2)\right]$$

$$= \frac{1}{a^2b^2c^2}\left[r^2 - (a^2 + b^2 - c^2)\left(\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} - \frac{z_1^2}{c^2}\right)\right]$$

$$= \frac{1}{a^2b^2c^2}(r^2 - a^2 - b^2 + c^2)$$

$$\therefore \quad \tan\theta = \frac{\left[-\dfrac{4}{p^2}\left(-\dfrac{1}{a^2b^2c^2}\right)\left(\dfrac{x_1^2}{a^2}+\dfrac{y_1^2}{b^2}-\dfrac{z_1^2}{c^2}\right)\right]^{1/2}}{\dfrac{(r^2-a^2-b^2+c^2)}{a^2b^2c^2}} = \frac{2abc}{p\,(r^2-a^2-b^2+c^2)}.$$

8.6. CENTRAL POINT. LINE OF STRICTION. PARAMETER OF DISTRIBUTION OF A GENERATOR

Def. 1. *The* **central point** *of a given generator, l, is the limiting position of its point of intersection with the line of shortest distance between it and another generator, m, of the same system; the limit being taken when, m, tends to coincide wtih l.*

With some sacrifie of precision, one may say that the central point of a given generator is the point of intersection of the generator and the line of shortest distance between the generator and a consecutive generator of the system.

Def. 2. *The locus of the central points of generators of a hyperboloid is called its line of striction.*

Def. 3. *The* **parameter of distribution** *of a generator, l is*

$$\lim\left(\frac{\Delta s}{\Delta\psi}\right)$$

where, Δs, is the shortest distance and, $\Delta\psi$, the angle between l, and another generator m of the same system, the limit being taken when the generator m tends to coincide with the generator l.

8.6.1. *To determine the central point of a generator.*

We consider generators of the system

$$\frac{x-a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{-b\cos\theta}=\frac{z}{c}$$

Let any generator, *l,* of the system be

$$\frac{x-a\cos\varphi}{a\sin\varphi}=\frac{y-b\sin\varphi}{-b\cos\varphi}=\frac{z}{c} \qquad ...(1)$$

We, now, consider any other generator, *m*

$$\frac{x-a\cos\varphi'}{a\sin\varphi'}=\frac{y-b\sin\varphi'}{-b\cos\varphi'}=\frac{z}{c} \qquad ...(2)$$

of the same system.

Let the shortest distance between these generators meet them in *P* and *Q* respectively so that we have to find the limiting position of the point *P* on the generator *l* when $\varphi'\to\varphi$. Let *C* be the limit of *P*.

Since *PQ* is a chord of the hyperboloid, its limit will be a tangent line *CD* at the point *C*. Let *l, m, n* be the direction ratios of the shortest distance *PQ* and l_0, m_0, n_0 those of its limit. We have

$$\begin{cases} al\sin\varphi - bm\cos\varphi + cn = 0. \\ al\sin\varphi' - bm\cos\varphi' + cn = 0. \end{cases}$$

$$\Rightarrow \qquad \frac{al}{\cos\varphi'-\cos\varphi}=\frac{bm}{\sin\varphi'-\sin\varphi}=\frac{cn}{\sin(\varphi'-\varphi)}$$

$$\Rightarrow \qquad \frac{al}{-\sin\frac{1}{2}(\varphi'+\varphi)}=\frac{bm}{\cos\frac{1}{2}(\varphi'+\varphi)}=\frac{cn}{\cos\frac{1}{2}(\varphi'+\varphi)}$$

Let $\varphi'\to\varphi$.

Thus, we obtain

$$\frac{al_0}{-\sin\varphi} = \frac{bm_0}{\cos\varphi} = \frac{cn_0}{1}$$

Let $$[a\,(r\sin\varphi + \cos\varphi),\ b\,(\sin\varphi - r\cos\varphi),\ cr] \qquad ...(3)$$

be the central point C on the generator (1). The equation of the tangent plance at C is

$$\frac{x\,(r\sin\varphi + \cos\varphi)}{a} + \frac{y\,(\sin\varphi - r\cos\varphi)}{b} - \frac{zr}{c} = 1$$

Since the line CD with direction ratios $l_0, m_0, n_0,$ lies on this tangent plane, we have

$$-\frac{\sin\varphi\,(r\sin\varphi + \cos\varphi)}{a^2} + \frac{\cos\varphi\,(\sin\varphi - r\cos\varphi)}{b^2} - \frac{r}{c^2} = 0$$

$$\Rightarrow \qquad r\left(\frac{\sin^2\varphi}{a^2} + \frac{\cos^2\varphi}{b^2} + \frac{1}{c^2}\right) = \left(\frac{1}{b^2} - \frac{1}{c^2}\right)\sin\varphi\cos\varphi$$

$$\Rightarrow \qquad r = \frac{c^2\,(a^2 - b^2)\sin\varphi\cos\varphi}{(a^2b^2 + a^2c^2\cos^2\varphi + b^2c^2\sin^2\varphi)}$$

so that we have obtained r.

Substituting this value of r in (3), we see that the co-ordinates of the central point $C\,(x, y, z)$ are given by

$$x = \frac{a^3\,(b^2 + c^2)\cos\varphi}{k},\quad y = \frac{b^3\,(c^2 + a^2)\sin\varphi}{k},\quad z = \frac{c^3\,(a^2 - b^2)\sin\varphi\cos\varphi}{k}$$

where $$k = a^2b^2 + a^2c^2\cos^2\varphi + b^2c^2\sin^2\varphi$$

Eliminating $\varphi,$ we see that the *line of striction* is the curve of intersection of the hyperboloid with the cone

$$\frac{a^6\,(b^2 + c^2)^2}{x^2} + \frac{b^6\,(c^2 + a^2)^2}{y^2} - \frac{c^6\,(b^2 - a^2)^2}{z^2} = 0$$

Ex. Find the central point for a generator of the second system and show that the line of striction is the same for either system.

8.6.2. *To determine the parameter of distribution of the generator, l.*

If $\Delta\psi$ be the angle between the generators (1) and (2) of § 8.5.1, we have

$$\tan\Delta\psi = \frac{\sqrt{[b^2c^2\,(\cos\varphi' - \cos\varphi)^2 + c^2a^2\,(\sin\varphi' - \sin\varphi)^2 + a^2b^2\sin^2(\varphi' - \varphi)]}}{a^2\sin\varphi\sin\varphi' + b^2\cos\varphi\cos\varphi' + c^2}$$

$$= 2\sin\frac{1}{2}(\varphi' - \varphi)\,\frac{\sqrt{\left[b^2c^2\sin^2\frac{1}{2}(\varphi' + \varphi) + c^2a^2\cos^2\frac{1}{2}(\varphi' + \varphi) + a^2b^2\cos^2\frac{1}{2}(\varphi' - \varphi)\right]}}{a^2\sin\varphi\sin\varphi' + b^2\cos\varphi\cos\varphi' + c^2}$$

We write $\varphi' = \varphi + \Delta\varphi$ so that $\Delta\varphi \to 0$ as $\varphi' \to \varphi.$ Then, from above, we obtain

$$\frac{d\psi}{d\varphi} = \frac{\sqrt{b^2c^2\sin^2\varphi + a^2c^2\cos^2\varphi + a^2b^2}}{a^2\sin^2\varphi + b^2\cos^2\varphi + c^2}$$

Again we shall now find the S.D., Δs between the two generators. Now the equation of the plane through (1) parallel to (2) is

$$\begin{vmatrix} x - a\cos\varphi & y - b\sin\varphi & z \\ a\sin\varphi & -b\cos\varphi & c \\ a\sin\varphi' & -b\cos\varphi' & c \end{vmatrix} = 0$$

so that cancelling a common factor $\sin \frac{1}{2}(\varphi' - \varphi)$, we obtain

$$-bcx \sin \frac{1}{2}(\varphi' + \varphi) + cay \cos \frac{1}{2}(\varphi' + \varphi) + abz \cos \frac{1}{2}(\varphi' - \varphi) + abc \sin \frac{1}{2}(\varphi' - \varphi) = 0$$

The S.D., Δs, which is the distance of the point $(a \cos \varphi', b \sin \varphi', 0)$ from this plane is given by

$$\Delta s = \frac{2abc \sin \frac{1}{2}(\varphi' - \varphi)}{\sqrt{b^2c^2 \sin^2 \frac{1}{2}(\varphi' - \varphi) + c^2a^2 \cos^2 \frac{1}{2}(\varphi' + \varphi) + a^2b^2 \cos^2 \frac{1}{2}(\varphi' - \varphi)}}$$

Again putting $\varphi' = \varphi + \Delta\varphi$, we obtain

$$\frac{ds}{d\varphi} = \frac{abc}{\sqrt{b^2c^2 \sin^2 \varphi + c^2a^2 \cos^2 \varphi + a^2b^2}}$$

$$\Rightarrow \qquad \frac{ds}{d\psi} = \frac{ds/d\varphi}{d\psi/d\varphi} = \frac{abc\,(a^2 \sin^2 \varphi + b^2 \cos^2 \varphi + c^2)}{b^2c^2 \sin^2 \varphi + c^2a^2 \cos^2 \varphi + a^2b^2}$$

8.7 HYPERBOLIC PARABOLOID

We rewrite the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{2z}{c} \qquad \text{...(1)}$$

of a hyperbolic paraboloid in the form

$$\left[\frac{x}{a} - \frac{y}{b}\right]\left[\frac{x}{a} + \frac{y}{b}\right] = \frac{2z}{c}$$

which may again be rewritten in either of the two forms

$$\frac{\frac{x}{a} - \frac{y}{b}}{\frac{z}{c}} = \frac{2}{\frac{x}{a} + \frac{y}{b}}, \quad \frac{\frac{x}{a} - \frac{y}{b}}{2} = \frac{\frac{z}{c}}{\frac{x}{a} + \frac{y}{b}}$$

Now, as in § 8.2 it can be shown that as λ and μ vary; each member of each of the systems of lines

$$\frac{x}{a} - \frac{y}{b} = \frac{\lambda z}{c}, \quad 2 = \lambda\left[\frac{x}{a} + \frac{y}{b}\right] \qquad \text{...(A)}$$

$$\frac{x}{a} - \frac{y}{b} = 2\mu, \quad \frac{z}{c} = \mu\left[\frac{x}{a} - \frac{y}{b}\right] \qquad \text{...(B)}$$

lies wholly on the hyperbolic paraboloid (1).

Thus, we see that a *hyperbolic paraboloid also admits of two systems of generating lines.*

As in the case of hyperboloid of one sheet, it can be shown that the following results hold good for the two systems of generating lines of a hyperbolic paraboloid also.

1. *Through every point of a hyperbolic paraboloid, there passes a member of each system.*
2. *No two members of the same system intersect.*
3. *Any two generators belonging to the two different systems intersect and the plane through them is the tangent plane at their point of intersection.*
4. *The tangent plane at a point meets the paraboloid in two generators through the point.*
5. *The locus of the point of intersection of perpendicular generator is the curve of intersection of the paraboloid with the plane* $2cz + a^2 - b^2 = 0$.

An Important Note : Since the generator

lies in the plane $$2 = \lambda\left[\frac{x}{a}+\frac{y}{b}\right]$$

which is parallel to the plane $$\frac{x}{a}+\frac{y}{b}=0$$

whatever value λ may have, we deduce that all the generators belonging to one system of the hyperbolic paraboloid

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{2z}{c}$$

are parallel to the plane

$$\frac{x}{a}+\frac{y}{b}=0$$

It may similarly be seen that the generators of the second system are also parallel to a plane, *viz.,*

$$\frac{x}{a}-\frac{y}{b}=0.$$

8.7.1. *Tangent plane at any point meets the paraboloid in two generators through the point.*

The planes passing through the two generators of different systems λ and μ of a hyperbolic paraboloid may be given as

$$\left(\frac{x}{a}-\frac{y}{b}-2\lambda\right)+k\left(\frac{x}{a}+\frac{y}{b}-\frac{z}{c\lambda}\right)=0 \qquad ...(i)$$

and $$\left(\frac{x}{a}+\frac{y}{b}-2\lambda\right)+k'\left(\frac{x}{a}-\frac{y}{b}-\frac{z}{c\mu}\right)=0 \qquad ...(ii)$$

for all values of k and k'.

These planes become identical if $k=\dfrac{1}{k'}=\dfrac{\lambda}{\mu}$.

This shows that the two generators, one of each system, are coplanar and such they intersect. The plane through them being given as [putting $k = 1/k' = \lambda/\mu$ in (i) and (ii)]

$$\mu\left(\frac{x}{a}-\frac{y}{b}-2\lambda\right)+\lambda\left(\frac{x}{a}+\frac{y}{b}-\frac{z}{c\lambda}\right)=0$$

$$\Rightarrow \qquad \frac{x}{a}(\mu+\lambda)-\frac{y}{b}(\mu-\lambda)=\frac{1}{c}(z+2c\lambda\mu) \qquad ...(iii)$$

which is a tangent plane to the hyperbolic paraboloid at the point of intersection of the two generators. Thus, the tangent plane at a point of hyperbolic paraboloid meets it in two generators through the point, the two generators being of different systems λ and μ and the point, their common of intersection.

So we have shown that any plane through a generating line of a hyperbolic paraboloid is a tangent plane at some point of the generator.

8.7.2. *Direction cosines of the generators of the two systems given by*

$$\frac{x}{a}-\frac{y}{b}=2\lambda,\ \frac{x}{a}+\frac{y}{b}=\frac{z}{c\lambda} \qquad ...(i)$$

$$\frac{x}{a}+\frac{y}{b}=2\mu,\ \frac{x}{a}-\frac{y}{b}=\frac{z}{c\mu} \qquad ...(ii)$$

If l_1, m_1, n_1 and l_2, m_2, n_2 are the d.c.'s of (i) and (ii), then we have

$$\frac{l_1}{\frac{1}{bc\lambda}} = \frac{m_1}{\frac{1}{ac\lambda}} = \frac{n_1}{\frac{2}{ab\lambda}},$$

since this generator is the line of intersection of the planes

$$\frac{x}{a} - \frac{y}{b} + 0.z = 2\lambda \quad \text{and} \quad \frac{x}{a} + \frac{y}{b} - \frac{z}{c\lambda} = 0$$

$$\Rightarrow \quad \frac{l_1}{a} = \frac{m_1}{b} = \frac{n_1}{2c\lambda} \qquad \text{...(iii)}$$

Similarly,

$$\frac{-l_2}{\frac{1}{bc\mu}} = \frac{m_2}{\frac{1}{ca\mu}} = \frac{n_2}{\frac{-2}{ab}}$$

$$\Rightarrow \quad \frac{l_2}{a} = \frac{m_2}{-b} = \frac{n_2}{2c\mu} \qquad \text{...(iv)}$$

EXAMPLES

1. *Planes are drawn through the origin O and the generators through any point P of the paraboloid given by* $x^2 - y^2 = az$. *prove that the angle between them is* $\tan^{-1}(2r/a)$, *where r is the length of OP.*

Sol. The two systems of generators of the paraboloid

$$x^2 - y^2 = az \qquad \text{...(i)}$$

are given as

$$x - y = a\lambda, \; x + y = z/\lambda \qquad \text{...(ii)}$$

and

$$x + y = a\mu, \; x - y = z/\mu \qquad \text{...(iii)}$$

Plane through the λ-generator and the origin is

$$x + y = z/\lambda \qquad \text{...(iv)}$$

Plane through the origin and the μ-generator is

$$x - y = z/\mu \qquad \text{...(v)}$$

If θ is the angle between these planes (iv) and (v), then

$$\cos\theta = \frac{1 - 1 + 1/(\lambda\mu)}{\sqrt{1 + 1 + 1/\lambda^2}\,\sqrt{1 + 1 + 1/\mu^2}}$$

$$\Rightarrow \quad \sec^2\theta = (2\lambda^2 + 1)(2\mu^2 + 1)$$

$$\Rightarrow \quad \tan\theta = \sqrt{2\lambda^2 + 2\mu^2 + 4\lambda^2\mu^2} \qquad \text{...(vi)}$$

Also *P*, the point of intersection of the generators (ii) and (iii) is,

$$\left[a\left(\frac{\lambda + \mu}{2}\right), \; a\left(\frac{\mu - \lambda}{2}\right), a\lambda\mu\right]$$

Then

$$OP = r = \frac{1}{2}a\sqrt{(\lambda + \mu)^2 + (\mu - \lambda)^2 + 4\lambda^2\mu^2}$$

$$= \frac{1}{2}a\sqrt{2\lambda^2 + 2\mu^2 + 4\lambda^2\mu^2} = \frac{1}{2}a\tan\theta, \qquad \text{[from (vi)]}$$

$$\therefore \quad \tan\theta = \frac{2r}{a} \quad \Rightarrow \quad \theta = \tan^{-1}(2r/a).$$

2. *Prove that the equation* $2x = ae^{2\phi}, \; y = be^{\phi}\cosh\theta, \; z = ce^{\phi}\sinh\theta$ *determines a hyperbolic paraboloid and that* $(\theta + \phi)$ *is constant for points of a given generator of one system and* $(\theta - \phi)$

is constant for a given generator of the other.

Sol. The parameteric equations of the surface are

$$2x = ae^{2\phi},\ \ y = be^{\phi} \cosh\theta,\ \ z = ce^{\phi}\cosh\theta \qquad \text{...(i)}$$

$$\therefore \qquad \left(\frac{y}{b}\right)^2 - \left(\frac{z}{c}\right)^2 = e^{2\phi} = \frac{2x}{a}$$

$$\Rightarrow \qquad \frac{y^2}{b^2} - \frac{z^2}{c^2} = \frac{2x}{a}, \qquad \text{...(ii)}$$

which is a hyperbolic paraboloid.

The different systems of generators of (ii) are given as

$$\frac{y}{b} + \frac{z}{c} = 2\lambda,\ \ \frac{y}{b} - \frac{z}{c} = \frac{x}{a\lambda} \qquad \text{...(iii)}$$

$$\frac{y}{b} - \frac{z}{c} = 2\mu,\ \ \frac{y}{b} + \frac{z}{c} = \frac{x}{a\mu} \qquad \text{...(iv)}$$

Since both the generators pass through the given point, we have

$$e^{\phi}(\cosh\theta + \sinh\theta) = 2\lambda,\ \ e^{\phi}(\cosh\theta - \sinh\theta) = \frac{e^{2\phi}}{2\lambda} \qquad \text{...(v)}$$

From the second of the relation in (v), we have

$$\cosh\theta - \sinh\theta = \frac{e^{\theta}}{2\lambda} \ \Rightarrow \quad e^{-\theta} = \frac{e^{\phi}}{2\lambda}$$

$$\therefore \qquad \lambda = \frac{e^{(\theta+\phi)}}{2} \qquad \text{...(vi)}$$

Also from (v),

$$e^{\phi}(\cosh\theta + \sinh\theta) = \frac{e^{2\phi}}{2\mu}$$

$$\Rightarrow \qquad \mu = \frac{e^{(\phi-\theta)}}{2} \qquad \text{...(vii)}$$

Relations (vi) and (vii) show that $(\theta + \phi)$ is constant for points of a given generator of the λ-system (since λ was constant for a given generator of the λ-system) and $(\theta - \phi)$ is constant for a given generator of the μ-system.

3. *Show that the angle between the generating lines of* $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2z$ *through* (x, y, z) *is given by*

$$\tan\theta = ab\left(1 + \frac{x^2}{a^4} + \frac{y^2}{b^4}\right)^{1/2}\left(z + \frac{a^2 - b^2}{2}\right)^{-1}$$

Sol. The d.r.'s of different systems of generators are $a, b, 2c\lambda$ and $a, -b, 2c\mu$ respectively. Hence,

$$\tan\theta = \frac{[(2b\mu + 2b\lambda)^2 + (2\lambda a - 2\mu a)^2 + (-2ab)^2]^{1/2}}{a^2 - b^2 + 4\lambda\mu}$$

$$= \frac{[4b^2(\mu+\lambda)^2 + 4a^2(\mu-\lambda)^2 + 4a^2b^2]^{1/2}}{a^2 - b^2 + 4\lambda\mu}$$

$$= \frac{\left[4b^2\left(\frac{x}{a}\right)^2 + 4a^2\left(\frac{y}{b}\right)^2 + 4a^2b^2\right]^{1/2}}{a^2 - b^2 + 2z} = ab\left(1 + \frac{z^2}{a^4} + \frac{y^2}{b^4}\right)^{1/2}\left(z + \frac{a^2 - b^2}{2}\right)^{-1}$$

EXERCISES

1. Obtain equations for the two systems of generating lines on the hyperbolic parabolid $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 4z,$ and hence express the co-ordinates of a point on the surface as functions of two parameters. Find the direction cosines of the generators through $(\alpha, 0, \gamma)$ and show that the cosines of the angle between them is $\frac{(a^2 - b^2 + \gamma)}{(a^2 + b^2 + \gamma)}$.

2. Show that the projections of the generators of a hyperbolic paraboloid on any principal plane are tangents to the s ection by the plane.

3. Find the locus of the perpendiculars from the vertex at the paraboloid

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{2z}{c}$$

to the generators of one system. [**Ans.** $x^2 + y^2 + 2z^2 \neq (a^2 + b^2)xy / ab = 0$]

4. Show that the points of intersection of generators $xy = az$ which are inclined at a constant angle α lie on the curve of intersection of the paraboloid and the hyperboloid

$$x^2 + y^2 - z^2 \tan^2 \alpha + a^2 = 0.$$

5. Through a variable generator $x - y = \lambda,\ x + y = \frac{2z}{\lambda}$ of the paraboloid $x^2 - y^2 = 2z$ a plane is drawn making a constant angle α with the plane $x = y.$ Find the locus of the point at which it touches the paraboloid.

[**Ans.** Curve of intersection of the above surface and paraboloid is $x^2 - y^2 = 2z.$]

8.8 CENTRAL POINT. LINE OF STRICTION. PARAMETER OF DISTRIBUTION

8.8.1. *To determine the central point of any generator of the system of generators.*

$$\frac{x}{a} - \frac{y}{b} = \frac{\lambda z}{c},\ 2 = \lambda\left(\frac{x}{a} - \frac{y}{b}\right).$$

Let a generator, *l,* of this system be

$$\frac{x}{a} - \frac{y}{b} = \frac{pz}{c},\ 2 = p\left(\frac{x}{a} + \frac{y}{b}\right) \quad ...(1)$$

We, now, consider a generator, *m,* of the same system

$$\frac{x}{a} - \frac{y}{b} = \frac{p'z}{c},\ 2 = p'\left(\frac{x}{a} + \frac{y}{b}\right) \quad ...(2)$$

The direction ratios of these generators are $a, -b, 2c/p; a, -b, 2c/p'$.

If *l, m, n* be the direction ratios of the line of S.D., between (1) and (2), we have

$$al - bm + 2cn/p = 0,\quad al - bm + 2cn/p' = 0$$

These give $1/a, 1/b, 0$ as the direction ratios of the line of S.D., being independent of p and p', we see that the line of S.D., is parallel to a fixed line.

Let (x_1, y_1, z_1) be the central point of the generator (1). As in § 8.5.1, the limiting position of the line of S.D., is a line contained in the tangent plane

$$\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{1}{c}(z + z_1)$$

at (x_1, y_1, z_1).

Thus, we have

$$\frac{x_1}{a^3} - \frac{y_1}{b^3} = 0 \qquad ...(3)$$

Also, since (x_1, y_1, z_1) lies on (1), we have

$$\frac{x_1}{a} - \frac{y_1}{b} = \frac{pz_1}{c}, 2 = p\left(\frac{x_1}{a} + \frac{y_1}{b}\right) \qquad ...(4)$$

Solving (3) and (4), we obtain

$$x_1 = \frac{2a^3}{p(a^2 + b^2)}, \quad y_1 = \frac{2b^3}{p(a^2 + b^2)}, \quad z_1 = \frac{2c(a^2 - b^2)}{p(a^2 + b^2)}.$$

Eliminating p, we see that the line of striction is the curve of intersection of the surface with the plane

$$\frac{x}{a^3} + \frac{y}{b^3} = 0.$$

Ex. Find the central point of a generator of the second system and show that the corresponding line of striction is the curve of intersection of the surface with the plane

$$\frac{x}{a^3} + \frac{y}{b^3} = 0.$$

8.8.2. *To determine the parameter of distribution.*

Let $\Delta\psi$ and Δs be the angle of S.D., respectively between the generators (1) and (2).

We have

$$\tan \Delta\psi = \frac{2c\sqrt{(a^2 + b^2)(p' - p)}}{pp'(a^2 + b^2) + 4c^2}$$

Let $p' = p + \Delta p$ so that $\Delta p \to 0$ as $p' \to p$. We have

$$\frac{d\psi}{dp} = \frac{2c\sqrt{a^2 + b^2}}{p^2(a^2 + b^2) + 4c^2} \qquad ...(5)$$

Now the plane through the generator (1) and parallel to the generator (2) is

$$\frac{x}{a} + \frac{y}{b} = \frac{2}{p}.$$

Also taking $z = 0$, we see that $(a/p', b/p', 0)$ is a point on the generator (2).

$$\therefore \qquad \Delta s = \frac{\frac{2}{p} - \frac{2}{p'}}{\left(\frac{1}{a^2} + \frac{1}{b^2}\right)^{1/2}} = \frac{2(p' - p)ab}{pp'\sqrt{a^2 + b^2}} \qquad ...(6)$$

Thus, as before

$$\frac{ds}{dp} = \frac{2ab}{p^2\sqrt{a^2 + b^2}},$$

$$\Rightarrow \qquad \frac{ds}{d\psi} = \frac{ab[p^2(a^2 + b^2) + 4c^2]}{cp^2(a^2 + c^2)}$$

which is the parameter of distribution.

Ex. For the generator of the paraboloid $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2z$ given by

$$\frac{x}{a} - \frac{y}{b} = 2\lambda, \ \frac{x}{a} + \frac{y}{b} = \frac{z}{\lambda},$$

prove that the parameter of distribution is

$$\frac{ab\,(a^2 + b^2 + 4\lambda^2)}{(a^2 + b^2)}$$

and the central point is

$$\left[\frac{2a^3\lambda}{a^2 + b^2}, \frac{-2b^3\lambda}{a^2 + b^2}, \frac{2\,(a^2 - b^2)\,\lambda^2}{a^2 + b^2}\right].$$

Prove that the centtral points of the systems of generators lie on the planes

$$\frac{x}{a^3} \pm \frac{y}{b^3} = 0.$$

8.9. GENERAL CONSIDERATION

We have seen that hyperboloid of one sheet and hyperbolic paraboloid each admit of two systems of generators such that through each point of the surface there passes one member of each system and that two members of opposite systems intersect but no two members of the same system intersect. Also we know that through each point of a cone or a cylinder there passes one generator. Thus, hyperboloids of one sheet, hyperboloid paraboloids, cones and cylinders are *ruled surfaces* in as much as they can be generated by straight lines.

We now proceed to examine the case of the general quadric in relation to the existence of generaotrs.

8.9.1. Condition for a Line to be a Generator

A straight line will be a generator of a quadric if three points of the line lie on the quadric.

Let the quadric be

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2ux + 2vy + 2wz + d = 0 \qquad ...(1)$$

The line $$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n}$$

will be a generator of the quadric, if the point $(lr + \lambda, mr + \beta, nr + \gamma)$ on the line lies on the quadric for all values of r, *i.e.,* the equation obtained on substituted these co-ordinates for x, y, z in (1) is an identity. As this equation is a qudric in r, it will be an identity if it is satisfied for three values of r, *i.e.,* if three points of the line lie on the quadric.

Cor. 1. The quadric equation in r obtained above will be an identity if the coefficients of r^2, r and the constant term are separately zero. This gives

$$al^2 + bm^2 + cn^2 + 2fmn + 2gnl + 2hlm = 0 \qquad ...(2)$$

$$l\,(a\alpha + h\beta + g\gamma) + m\,(h\alpha + b\beta + f\gamma) + n\,(g\alpha + f\beta + c\gamma) = 0 \qquad ...(3)$$

$$a\alpha^2 + b\beta^2 + c\gamma^2 + 2f\beta\gamma + 2g\gamma\alpha + 2h\alpha\beta + 2u\alpha + 3v\beta + 2w\gamma + d = 0 \qquad ...(4)$$

The condition (4) simply means that the point (α, β, γ) lies on the quadric.

Since (2) is a homogeneous quadric equation and (3) is a homogeneous linear equation in l, m, n these two equations will determine *two* sets of values of l, m, n. Thus, we deduce that *through every point on a quadric there pass two lines, real, coincident or imaginary lying wholly on the quadric.*

Cor. 2. *A quadric can be drawn so as to contain* **three** *mutually skew lines as generators,* for the quadric determined by nine points, three on each line, will contain the three lines as generators.

8.10. QUADRICS WITH REAL AND DISTINCT PAIRS OF GENERATING LINES

8.10.1. *Of all real central quadrics, hyperboloid of one sheet only possesses two real and distinct generators through a point.*

Let $$ax^2 + by^2 + cz^2 = 1$$
be any central quadric.

The direction ratios, l, m, n of any generator

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$$

of the quadric through the point (α, β, γ) are given by the equations

$$al^2 + bm^2 + cn^2 = 0, \quad al\alpha + bm\beta + cn\gamma = 0$$

Eliminating n from these, we obtain

$$a\,(a\alpha^2 + c\gamma^2)\,l^2 + 2ab\alpha\beta lm + b\,(b\beta^2 + c\gamma^2)\,m^2 = 0$$

Its roots will be real and distinct if, and only if

$$4a^2b^2\alpha^2\beta^2 - 4ab\,(a\alpha^2 + c\gamma^2)\,(b\beta^2 + c\gamma^2) > 0$$

$$\Leftrightarrow \qquad -4abc\gamma^2\,(a\alpha^2 + b\beta^2 + c\gamma^2) > 0$$

Since $a\alpha^2 + b\beta^2 + c\gamma^2 = 1$, we see that the roots will be real and distinct, if and only, if abc is negative.

Now this will be the case if, a, b, c are all negative or one negative and two positive. In the former case the quadric itself is imaginary and in the latter it is a hyperboloid of one sheet

8.10.2. *Of the two paraboloids, hyperbolic paraboloid only possesses two real and distinct generators through a point.*

In the case of the paraboloid

$$ax^2 + by^2 = 2cz$$

the direction ratios, l, m, n of the generating lines through a point (α, β, γ) of the surface are given by

$$al^2 + bm^2 = 0 \qquad \text{...(1)}$$

$$al\alpha + bm\beta = 0 \qquad \text{...(2)}$$

The equation (1) shows that for real values of l and m, we must have a and b with opposite signs, *i.e.,* the paraboloid must be hyperbolic.

8.11. LINES INTERSECTING THREE LINES

An infinite number of lines can be drawn meeting three given mutually skew lines. For the quadric through the three given mutually skew lines $a, b, c,$ the three lines will be generators of one system and all the other generators of the other system will intersect a, b and c.

In fact the quadric through three given mutually skew lines can be determined as the locus of lines which intersect the three given lines.

Thus, the locus is really the equation of the quadric through the three lines

$$u_r = 0 = v_r; \; r = 1, 2, 3.$$

EXAMPLES

1. *Find the equations of the quadric containing the three lines*

$$y = b, \; z = -c; \; z = c, \; x = -a; \; x = a, \; y = -b.$$

Also obtain the equations of its two systems of generators.

Sol. Any line which intersects the first two lines is given by

$$\left.\begin{array}{l} y - b + \lambda_1 (z + c) = 0, \\ z - c + \lambda_2 (x + a) = 0, \end{array}\right\} \qquad \text{...(i)}$$

for all values of λ_1, λ_2.

This will intersect the third line $x = a,\ y = -b$ if

$$z = c - 2a\lambda_2 = \frac{2b}{\lambda_1} - c$$

$$\Rightarrow \qquad c = \frac{b}{\lambda_1} + a\lambda_2, \qquad \text{...(ii)}$$

which is $f(\lambda_1, \lambda_2) = 0$.

Eliminating λ_1, λ_2 from (i) and (ii), we get

$$c = -\frac{b(z + c)}{y - b} - \frac{a(z - c)}{x + a}.$$

$$\Rightarrow \qquad c(xy - bx + ay - ab) + b(xz + cx + az + ca) + a(yz - cy - bz + bc) = 0$$

$$\Rightarrow \qquad ayz + bzx + cxy + abc = 0 \qquad \text{...(iii)}$$

which is the required quadric containing the three given lines.

To get the two systems of generators of (iii), rewrite (iii) as

$$y(ax + cx) + b(zx + ca) = 0$$

$$\Rightarrow \qquad y(az + cx) + b[(x + a)(z + c) - (az + cx)] = 0$$

$$\Rightarrow \qquad (y - b)(az + cx) + b(x + a)(z + c) = 0 \qquad \text{...(iv)}$$

which may again be written in either of the two forms.

$$\frac{b(x + a)}{(az + cx)} = \frac{-(y - b)}{(z + c)} = \lambda \text{ (say)} \qquad \text{...(v)}$$

and
$$\frac{-(y - b)}{b(x + a)} = \frac{z + c}{(az + cx)} = \mu \text{ (say)} \qquad \text{...(vi)}$$

where λ and μ are arbitrary constants.

Relations (v) and (vi) give system of generators as

$$y - b + \lambda(z + c) = 0,\ b(x + a) - \lambda(az + cx) = 0,$$

$$\mu(az + cx) - (z + c) = 0,\ \mu b(x + a) + (y - b) = 0.$$

2. *Find the locus of the perpendicular from a point on a hyperboloid to the generators of one system.*

Sol. Let the given point O be taken as origin and a generator through O as OX, the x-axis.

Let OZ, the normal at O be taken as the z-axis. Then XOY is the tangent plane at Q, OY being the y-axis.

Then the equation of the hyperboloid of which the x-axis is a generator and the z-axis is the normal, is of the form

$$by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2wz = 0$$

$$\Rightarrow \qquad y(by + 2hx) + z(cz + 2gx + 2fy + 2w) = 0 \qquad \text{...(i)}$$

The system of generators of (i) are given by

$$\lambda y = z,\ (by + 2hx) + \lambda(cz + 2gx + 2fy + 2w) = 0 \qquad \text{...(ii)}$$

$$y = \mu, \quad (cz + 2gx + 2fy + 2w)z + \mu(by + 2hx) = 0 \qquad \text{...(iii)}$$

Again let any line through origin O and perpendicular to the λ-generator be

$$\frac{x}{L} = \frac{y}{M} = \frac{z}{N} = k. \qquad \text{...(iv)}$$

Also the λ-generator is the line of intersection of the plane

$$\lambda y - z = 0$$

and $$2(h + g\lambda)x + (b + 2\lambda f)y + c\lambda z = -2w$$

$\therefore$ The d.c.'s of the λ-generator are proportional to

$$(c\lambda^2 + b + 2\lambda f),\ -2(h + g\lambda),\ -2\lambda(h + g\lambda).$$

Since (iv) is perpendicular to the generator (ii), we have

$$L(c\lambda^2 + b + 2\lambda f) - 2M(h + g\lambda) - 2\lambda N(h + g\lambda) = 0 \qquad \text{...(v)}$$

Eliminating L, M, N and λ between (ii), (iv) and (v), we get

$$\frac{x}{k}\left[c\left(\frac{z}{y}\right)^2 + b - 2\frac{z}{y}f\right] - 2\frac{y}{k}\left(h + g\frac{z}{y}\right) - 2\frac{z}{k}\cdot\frac{z}{y}\left(h + g\cdot\frac{z}{y}\right) = 0$$

$\Rightarrow$ $$x(cz^2 + 2fyz + by^2) - 2y^2(hy + gz) - 2z^2(hy + gz) = 0$$

$\Rightarrow$ $$x(cz^2 + 2fyz + by^2) - 2(y^2 + z^2)(hy + gz) = 0,$$

which is the required locus.

EXERCISES

1. Find the equations of the hyperboloid through the three lines

$$y - z = 1,\ x = 0;\ z - x = 1,\ y = 0;\ x - y = 1,\ z = 0.$$

Also obtain the equations of its two systems of generators.

[**Ans.** $x^2 + y^2 + z^2 - 2xy - 2yz - 2zx = 1;\ x - y - 1 = \lambda z,\ \lambda(x - y + 1) = 2x + 2y - z,$

$x - y - 1 = \lambda(2x + 2y - z),\ \lambda(x - y + 1) = z$]

2. The generators of one system of a hyperbolic paraboloid are parallel to the plane

$$lx + my + nz = 0$$

and the lines $ax + by = 0 = z + c;\ ax - by = 0 = z - c$

are two members of the same system.

Show that the equation of the paraboloid is

$$abc(lx + my + nz) = c(a^2nx + b^2ly + abcn).$$

3. Show that two straight lines can be drawn intersecting four given mutually skew lines.

❑❑❑

9

General Equation of the Second Degree

9.1 REDUATION TO CANONICAL FORMS AND CLASSIFICATION

A quadric has been defined as the locus of a point satisfying an equation of the second degree. Thus, a quadric is the locus of a point satisfying an equation of the type

$$F(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2hxy + 2ux + 2vy + 2wz + d = 0$$

which we may rewrite as

$$\Sigma (ax^2 + 2fyz) + 2\Sigma ux + d = 0 \qquad ...(1)$$

splitting the set of all terms into three homogeneous subsets.

We have considered so far special forms of the equations of the second degree in order to discuss *geometrical* properties of the various types of quadrics. In this chapter we shall see how the general equation of a second degree by means of an appropriate change of co-ordinate system can be reduced to simpler forms and also thus classify the types of quadrics.

Equations of various loci connected with a given quadric : We proceed to determine the equations of various loci associated with a quadric given by a general second degree equation. In this connection, we shall start obtaining a quadric in *r,* which will play a very important role in connection with the determination of the equations of these loci.

Consider a point (α, β, γ) and a line through the same with direction cosines *(l, m, n).* The co-ordinates of the point on this line at a distance r from (α, β, γ) are

$$(lr + \alpha, mr + \beta, nr + \gamma)$$

This point will lie on the quadric

$$F(x, y, z) \equiv \Sigma (ax^2 + 2fyz) + 2\Sigma ux + d = 0$$

for values of r satisfying the equation

$$\Sigma [a (lr + \alpha)^2 + 2f (mr + \beta)(nr + \gamma)] + 2\Sigma u (lr + \alpha) + d = 0$$

$$\Leftrightarrow \quad r^2\Sigma (al^2 + 2fmn) + 2r [l (a\alpha + h\beta + g\gamma + u) + m (h\alpha + b\beta + f\gamma + v)] + n (g\alpha + f\beta + c\gamma + w) + F(\alpha, \beta, \gamma) = 0 \quad ...(2)$$

which is a quadric in r. Thus, if r_1, r_2 be the roots of this quadric, the two points of intersection of the line with the quadric are

$$(lr_1 + \alpha, mr_1 + \beta, nr_1 + \gamma), (lr_2 + \alpha, mr_2 + \beta, nr_2 + \gamma)$$

Note : It may be noted that the equation (2) can be rewritten as

$$r^2\Sigma (al^2 + 2fmn) + r\left(l\frac{\partial F}{\partial \alpha} + \frac{\partial F}{\partial \beta} + \frac{\partial F}{\partial \gamma}\right) + F(\alpha, \beta, \gamma) = 0$$

where $\frac{\partial F}{\partial \alpha} + \frac{\partial F}{\partial \beta} + \frac{\partial F}{\partial \gamma}$ denote the the values of the partial derivatives of $F,$ w.r.t. x, y, z respectively at the point (α, β, γ).

9.1.1 The tangent plane at a point

Suppose that the point (α, β, γ) lies on the quadric so that we have

$$F(\alpha, \beta, \gamma) = 0$$

and one root of the quadric equation (2) is zero. The vanishing of value of r is also a simple consequence of the fact that one of the two points of intersection of the quadric which every line through a point of the quadric coincides with the point in question.

A line through the point (α, β, γ) on the quadric with direction cosines *(l, m, n)* will be a tangent line if the second point of intersection also coincides with (α, β, γ), *i.e,* if the second value of r, as given by (2) is also zero. This will be so if the coefficient of r is also zero, *i.e.,* if

$$l\,(a\alpha + h\beta + g\gamma + u) + m\,(h\alpha + b\beta + f\gamma + v) + n\,(g\alpha + f\beta + c\gamma + w) = 0 \quad ...(3)$$

which is thus the condition for the line

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \quad ...(4)$$

to be a tangent line at the point (α, β, γ). The locus of the tangent lines through (α, β, γ), obtained on eliminating *l, m, n* between (3) and (4) is

$$\Sigma\,(x-a)\,(a\alpha + h\beta + g\gamma + u) = 0$$

$$\Leftrightarrow \qquad \Sigma x\,(a\alpha + h\beta + g\gamma + u) = \Sigma\alpha\,(a\alpha + h\beta + g\gamma + u)$$

Adding $u\alpha + v\beta + w\gamma + d$ to both sides, we get

$$\Sigma x\,(a\alpha + h\beta + g\gamma + u) + (u\alpha + v\beta + g\gamma + d) = F\,(\alpha, \beta, \gamma) = 0$$

Thus, the locus of the tangent line (α, β, γ) is

$$\Sigma x\,(a\alpha + b\beta + g\gamma + u) + (u\alpha + v\beta + w\gamma + d) = 0$$

which is a plane called the *tangent plane* at (α, β, γ).

9.1.2 The normal at a Point

The line through (α, β, γ), perpendicular to the tangent plane thereat, *viz.,*

$$\frac{x-\alpha}{a\alpha + h\beta + g\gamma + u} = \frac{y-\beta}{h\alpha + b\beta + f\gamma + v} = \frac{z-\gamma}{g\alpha + f\beta + c\gamma + w}$$

is the normal at the point (α, β, γ).

9.1.3 Enveloping cone from a Point

Suppose now that (α, β, γ) is a point not necessarily on the quadric. Then any line through (α, β, γ) with direction cosines *(l, m, n)* will touch the quadric, *i.e.,* meet the same in two coincident points, if the two roots of the quadric equation in r, are equal. The condition for this is

$$[\Sigma l\,(a\alpha + h\beta + g\gamma + u)]^2 = \Sigma\,(al^2 + 2fmn)\,F(\alpha, \beta, \gamma) \quad ...(5)$$

The locus of the line

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \quad ...(6)$$

through (α, β, γ) touching the quadric, obtained on eliminating *l, m, n* between (5) and (6), is

$$[\Sigma\,(x-\alpha)\,(a\alpha + h\beta + g\gamma + u)]^2 = [\Sigma a\,(x-\alpha)^2 + 2f\,(y-\beta)\,(z-\gamma)]\,F(\alpha, g, \gamma) \quad ...(7)$$

To put this equation in a convenient form, we write

$$S = F(x, y, z),\ S_1 = F(\alpha, \beta, \gamma),\ T = \Sigma x\,(a\alpha + h\beta + g\gamma + u) + (u\alpha + v\beta + w\gamma + d)$$

Then (7) can be written as

$$(T - S_1)^2 = S_1\,(S + S_1 - 2T)$$

$$\Leftrightarrow \qquad SS_1 = T^2$$

which is the *equation of the Enveloping cone of the quadric S = 0 with the point* (α, β, γ) *as its vertex.*

9.1.4 Enveloping Cylinder

Suppose now that *(l, m, n)* are given and we require the locus of tangent lines with direction cosines *(l, m, n)*. If (α, β, γ) be a point on any such tangent line, we have the condition

$$[\Sigma l\,(a\alpha + h\beta + g\gamma + u)]^2 = [\Sigma\,(al^2 + 2fmn)]\,F(\alpha, \beta, \gamma)$$

as obtained in 9.1.3 above. Thus, the required locus is

$$[\Sigma l\,(ax + hy + gz + u)]^2 = \Sigma(al^2 + 2fmn)\,F(x, y, z)$$

known as Enveloping Cylinder.

This is the equation of the envelopign cylinder of the quadric $F(x, y, z) = 0$ with generators parallel to the line with direction cosines *(l, m, n)*.

9.1.5 Section with a given centre

Suppose now that (α, β, γ) is a given point. Then a chord with direction cosines *(l, m, n)* through the point (α, β, γ) will be bisected thereat if the sum of the two roots of the *r*-quadratic (2) is zero. This will be so, if and only if

$$\Sigma l\,(a\alpha + h\beta + g\gamma + u) = 0 \qquad \text{...(8)}$$

so that the locus of the chord

$$\frac{x - \alpha}{l} = \frac{y - \beta}{m} = \frac{z - \gamma}{n} \qquad \text{...(9)}$$

through the point (α, β, γ) and bisected thereat, obtained on eliminating *l, m, n* from the relations (8) and (9) is

$$\Sigma\,(x - \alpha)\,(a\alpha + h\beta + g\gamma + u) = 0$$

which, we may rewrite as

$$T = S_1.$$

The *plane* $T = S_1$, *meets the quadric in a conic with its centre at* (α, β, γ).

9.2 POLAR PLANE OF A POINT

If a line through a point $A\,(\alpha, \beta, \gamma)$ meets the quadric in points *Q, R* and a point *P* is taken on the line such that the points *A* and *P* divide the segment *QR* internally and externally in the same ratio, then the locus of *P* for different lines through *A* is a plane called the *Polar plane* of the point *A* with respect to the quadric. It is easily seen that if the points *A* and *P* divide the segment *QR* internally and externally in the same ratio, then the points *Q* and *R* also divide the segment *AP* internally and externally in the same ratio.

Consider a line through the point $A\,(\alpha, \beta, \gamma)$ and let *P* be the point *(x, y, z)*. The point dividing the segment *AP* in the ratio $\lambda : 1$ is

$$\left(\frac{\lambda x + \alpha}{\lambda + 1}, \frac{\lambda y + \beta}{\lambda + 1}, \frac{\lambda z + \gamma}{\lambda + 1}\right).$$

This point will lie on the quadric.

$$\Sigma\,(ax^2 + 2fyz) + 2\Sigma ux + d = 0$$

if $$\Sigma\left[a\left(\frac{\lambda x + \alpha}{\lambda + 1}\right)^2 + 2f\left(\frac{\lambda y + \beta}{\lambda + 1}\right)\left(\frac{\lambda z + \gamma}{\lambda + 1}\right)\right] + 2\Sigma u\left(\frac{\lambda x + \alpha}{\lambda + 1}\right) + d = 0$$

$$\Leftrightarrow \quad \lambda^2 F\,(x, y, z) + 2\lambda\,[x\,(a\alpha + h\beta + g\gamma + u) + y\,(h\alpha + b\beta + f\gamma + v)$$
$$+ z\,(g\alpha + f\beta + c\gamma + w) + (u\alpha + v\beta + w\gamma + d) + F(\alpha, \beta, \gamma) = 0$$

The two value of λ give the two ratios in which the points *Q* and *R* divide the segment *AP*. In order that the points *Q* and *R* may divide the segment *AP* internally and externally in the same ratio, the sum of the two values of λ should be zero, *i.e.,*

$$x\,(a\alpha + h\beta + g\gamma + u) + y\,(h\alpha + b\beta + f\gamma + v) + z\,(g\alpha + f\beta + c\gamma + w) + (u\alpha + v\beta + w\gamma + d) = 0 \quad \text{...(10)}$$

which is the required locus of the point $P(x, y, z)$.

Thus, (10) is the required equation of the polar plane.

Note : The notations of *Conjugate points, Conjugate planes, Conjugate lines* and *Polar lines* can be introduced as in the case of particular forms of equations in the preceding chapters.

9.3 DIAMETRAL PLANE CONJUGATE TO A GIVEN DIRECTION

We know that if *(l, m, n)* be the direction cosines of a chord and *(x, y, z)* the mid-points of the same, then we have

$$l\frac{\partial F}{\partial x} + m\frac{\partial F}{\partial y} + n\frac{\partial F}{\partial z} = 0 \quad \text{...(1)}$$

Thus, if *l, m, n* be supposed to be given, then the equation of the locus of the mid-point *(x, y, z)* of parallel chords with direction cosines *(l, m, n)* is given by (1) above. This locus is a plane called the *Diameter plane conjugate to the direction cosines (l, m, n).* We can rewrite *the equation (1) of the diametral plane conjugate to l, m, n as*

$$x\,(al + hm + gn) + y\,(hl + bm + fn) + z\,(gl + fm + cn) + (ul + vm + wn) = 0 \quad \text{...(2)}$$

NOTE : In this connection we should remember that there does not necessarily correspond a diametral plane conjugate to *every* given direction. Thus, we see from above that there is no diametral plane conjugate to the direction cosines *(l, m, n)* if *l, m, n* are such that the coefficients of *x, y, z* in the equation (2) are all zero. Thus there will be no diametral plane corresponding to a direction whose direction cosines *(l, m, n)* satisfy the three relations

$$al + hm + gn = 0$$
$$hl + bm + fn = 0$$
$$gl + fm + cn = 0$$

These three homogeneous linear equations in *l, m, n* will have a non-zero solution, if and only if

$$D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$$

We also denote by *A, B, C* the co-factors of *a, b, c* in this determinant so that we have $A = bc - f^2$, $B = ca - g^2$, $C = ab - h^2$.

9.4 PRINCIPAL DIRECTIONS AND PRINCIPAL PLANES

A direction *l, m, n* is said to be a *Principal direction,* if it is perpendicular to the diametral plane conjugate to the same. Also then the corresponding conjugate diametral plane is called a *Principal plane* in that the chords perpendicular to itself are bisected by it.

Thus, *l, m, n* will be a principal direction if and only if the direction ratios

$$al + hm + gn,\ hl + bm + fn,\ gl + fm + cn$$

of the normal to the corresponding conjugate diametral plane are proportional to *l, m, n i.e.,* if and only if there exists a number λ such that

$$al + hm + gn = l\lambda$$
$$hl + bm + fn = m\lambda$$
$$gl + fm + cn = n\lambda$$

We rewrite these as

$$(a - \lambda)\,l + hm + gn = 0 \quad \text{...(1)}$$
$$hl + (b - \lambda)\,m + fn = 0 \quad \text{...(2)}$$
$$gl + fm + (c - \lambda)\,n = 0 \quad \text{...(3)}$$

These three linear homogeneosu equations in l, m, n will posses a non-zero solution in l, m, n if and only if

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0$$

On expanding this determinant, we see that λ must be a root of the cubic

$$\lambda^3 - \lambda^2 (a+b+c) + \lambda (A+B+C) - D = 0 \qquad ...(4)$$

This cubic is known as the **Discriminating cubic** and each root of the same is called a **Characteristic root.**

The equation (4) has three roots which may not all be real or distinct. Also to each real root of (4) corresponds at least one principal direction l, m, n obtained on solving any two of the equation (1), (2) and (3).

Note : If l, m, n be a principal direction corresponding to a real root λ of the discriminating cubic, then we may easily see that the equation of the corresponding principal plane takes the form

$$\lambda (lx + my + nz) + (ul + vm + wn) = 0$$

This equation shows that we shall have no principal plane corresponding to $\lambda = 0$ if $\lambda = 0$ is a root of the discriminating cubic. In spite of this, however, we shall find it useful to say that l, m, n is a principal direction corresponding to $\lambda = 0$. Thus, every direction l, m, n satsifying the equations (1), (2), (3) corresponding to a root λ of the discriminating cubic (4) will be called a *Principal direction.*

Note 2. In the following, we shall prove some important results concerning the nature of the roots of the discriminating cubic and the **existence** of principal directions and principal planes.

Before taking up this consideration, we give a few preliminary results of algebraic character in the following section.

9.5 SOME PRELIMINARIES TO REDUCTION AND CLASSIFICATION

In this section we shall state some points which will prove useful in relation to the problem of reduction and classification.

In the following discussion, the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

to be denoted by D will play an important part.

We may verify that

$$D = abc + 2fgh - af^2 - bg^2 - ch^2$$

As usual, A, B, C, F, G, H will denote the co-factors of a, b, c, f, g, h respectively in the determinant D, so that we have

$$A = bc - f^2,\ B = ca - g^2,\ C = ab - h^2;$$
$$F = gh - af,\ G = hf - bg,\quad H = fg - ch$$

It can be easily verified that

$$\left.\begin{aligned} &BC - F^2 = aD, CA - G^2 = bD, AB - H^2 = cD; \\ &GH - AF = fD, HF - BG = gD, FD - CH = hD \end{aligned}\right\} \qquad ...(i)$$

Also we have

$$\begin{array}{lll} aA + bH + gG = 0, & hA + bH + fG = 0, & gA + fH + cG = 0; \\ aH + hB + gF = 0, & hH + bB + fF = 0, & gH + fB + cF = 0; \\ aG + hF + gC = 0, & hG + bF + fC = 0, & gG + fF + cC = D. \end{array}$$

9.5.1

If $D = 0$, then from (i), we have

$$BC = F^2, \qquad CA = G^2, \qquad AB = H^2$$
$$GH = AF, \qquad HF = BG, \qquad FG = CH.$$

Ex. Show that

(i) $D = 0$ and $A = 0 \Rightarrow H = 0, G = 0$,

(ii) $D = 0$ and $H = 0 \Rightarrow A = 0, H = 0, C = 0$ or $A = 0, B = 0, F = 0$.

Further prove that if $D = 0, A = 0, B = 0,$ then *F, G, H* must all be zero but *G* may or may not be zero.

9.5.2

If $D = 0$ and $A + B + C = 0$, then

$$A, B, C, F, G, H$$

are all zero.

Now $\qquad D = 0 \Rightarrow BC = F^2, CA = G^2, AB = H^2$

$\Rightarrow \qquad$ *A, B, C* are all of the same sign.

Now *A, B, C* being all of the same sign.

$$A + B + C = 0 \Rightarrow A = 0, B = 0, C = 0$$

Further, *A, B, C* being zero

$$F^2 = BC, B = 0, C = 0$$

$\Rightarrow \qquad F = 0$

Similarly, $\qquad G = 0, H = 0.$

Note : Three homogeneous linear equations

$$a_1x + b_1y + c_1z = 0, \; a_2x + b_2y + c_2z = 0, \; a_3x + b_3y + c_3z = 0$$

will possess a non-zero solution, *i.e.,* a solution wherefore *x, y, z* are not all zero, if and only if

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

9.6 THEOREM (I)

The roots of the discriminating cubic are all real.

We, of course, suppose that the coefficients of the equation $F(x, y, z) = 0$ are all real.

Suppose that λ is a root of the discriminating cubic (4), § 9.4, and *l, m, n* is any non-zero set of values satisfying the corresponding equations (1), (2), (3) § 9.4.

Here it should be remembered that we cannot regard *l, m, n* as real, for λ is not yet proved to be real.

In the following, the complex conjugate of any number will be expressed by putting a bar over the same. Thus, $\bar{l}, \bar{m}, \bar{n}$ will denote the complex conjugate of the numbers *l, m, n* respectively.

Now, we have

$$al + hm + gn = l\lambda, \; hl + bm + fn = n\lambda, \; gl + fm + cn = n\lambda$$

Multiplying these by $\bar{l}, \bar{m}, \bar{n}$ respectively and adding, we obtain

$$\Sigma a l\bar{l} + \Sigma f(\bar{m}n + m\bar{n}) = \lambda \Sigma l\bar{l} \qquad ...(1)$$

Now, *a, b, c, f, g, h* are real. Also,

$$l\bar{l}, m\bar{m}, n\bar{n}$$

being the products of pairs of conjugate complex numbers, are real.

Also we notice that $m\bar{n}$ is the conjugate complex of $\bar{m}n$ so that

$$m\bar{n} + \bar{m}n$$

is real.

Similarly, $$n\bar{l} + \bar{n}l,\ l\bar{m} + \bar{l}n$$

are real.

Finally, $\sum l\bar{l}$ is a non-zero real number.

Thus, λ, being the ratio of two real numbers from (1), is necessarily a real number.

Hence, the ratio of the discriminating cubic are all real. Also, therefore, the numbers *l, m, n* corresponding to each λ, are real.

9.6.1 Theorem (II)

The two principal directions corresponding to any two distinct roots of the discriminating cubic are perpendicular.

Suppose that λ_1, λ_2 are two distinct roots of the discriminating cubic, and

$$l_1, m_1, n_1;\ l_2, m_2, n_2$$

are the two corresponding principal directions.

We then have

(2) $al_1 + hm_1 + gn_1 = \lambda_1 l_1,$ (5) $al_2 + hm_2 + gn_2 = \lambda_2 l_2,$

(3) $hl_1 + bm_1 + fn_1 = \lambda_1 m_1,$ (6) $hl_1 + bm_2 + fn_2 = \lambda_2 m_2,$

(4) $gl_1 + fm_1 + cn_1 = \lambda_1 n_1,$ (7) $gl_2 + fm_2 + cn_2 = \lambda_2 n_2.$

Multiplying (2), (3), (4) by l_2, m_2, n_2 respectively and adding, we obtain

$$\Sigma al_1 l_2 + \Sigma f\ (m_1 n_2 + m_2 n_1) = \lambda_1 \Sigma l_1 l_2 \qquad ...(8)$$

Also multiplying (5), (6), (7) by l_1, m_1, n_1 respectively and adding, we obtain

$$\Sigma al_1 l_2 + \Sigma f\ (m_1 n_2 + m_2 n_1) = \lambda_1 \Sigma l_1 l_2 \qquad ...(9)$$

From (8) and (9), we obtain

$$\lambda_1 \Sigma l_1 l_2 = \lambda_2 \Sigma l_2 l_2$$

$$\Rightarrow \qquad (\lambda_1 - \lambda_2)\ \Sigma l_1 l_2 = 0$$

$$\Rightarrow \qquad \Sigma l_1 l_2 = 0 \text{ for } \lambda_1 - \lambda_2 \neq 0.$$

Thus, the two directions are perpendicular. Hence, the theroem.

9.6.2 Theorem (III)

For every quadric, there exists at least one set of three mutually perpendicular principal directions.

We have to consider the following three cases :

(A) The roots of the discriminating cubic are all distinct.

(B) Two of the roots are equal and third is different from these.

(C) The three roots are all equal.

These three cases will be considered one by one.

(A) Case of three distinct roots : The roots being distinct, there will correspond a principal direction *l, m, n* satisfying the equation (1), (2), (3) on page 233 to each of these.

Also by Theorem II, these three directions will be mutually perpendicular. The three principal directions are unique in this case.

Thus, there exist three principal directions in this case. Moreover, these directions are as well unique in this case.

(B) Case of two equal roots : Let the discriminating cubic have two equal roots and let the third root be different from the same.

Suppose λ is a root of the *D*-cubic repeated twice so that λ satisfies the equation

$$\lambda^3 - \lambda^2\ (a + b + c) + \lambda\ (A + B + C) - D = 0 \qquad ...(10)$$