2011 IFoS

1. Family of curves whose tangents form an angle $\frac{\pi}{4}$ with hyperbolas $xy = c$

slope of hyperbolas $xy = c$ is

$$x\frac{dy}{dx} + y = 0 \Rightarrow \boxed{\frac{dy}{dx} = -\frac{y}{x} = -\frac{c}{x^2}}$$

let slope of desired family of curves be m_1 then angle b/w them is given by -

$$\tan\theta = \frac{m_1 - m_2}{1 + m_1 m_2}$$

$\theta = \frac{\pi}{4}$, $m_2 = -\frac{c}{x^2}$ we get,

$$\tan\frac{\pi}{4} = \frac{m_1 + \frac{c}{x^2}}{1 - \frac{m_1 c}{x^2}}$$

$$\Rightarrow m_1 + \frac{c}{x^2} = \frac{x^2 - m_1 c}{x^2}$$

$$\Rightarrow m_1 x^2 + c = x^2 - m_1 c$$

$$\Rightarrow m_1(x^2 + c) = x^2 - c$$

$$m_1 = \frac{x^2 - c}{x^2 + c}$$

i.e. $\frac{dy}{dx} = \frac{x^2 - c}{x^2 + c}$

$$\Rightarrow dy = \left(\frac{x^2 + c}{x^2 + c} - \frac{2c}{x^2 + c}\right)dx$$

$$\Rightarrow y = x - 2c \times \frac{1}{\sqrt{c}} \tan^{-1}\frac{x}{\sqrt{c}} + c_2$$

Again putting $xy = c$ we get

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$$\boxed{y = x - 2\sqrt{xy}\tan^{-1}\sqrt{\frac{x}{y}} + c'}$$

2 $\frac{d^2y}{dx^2} - 2\tan x\frac{dy}{dx} + 5y = \sec x \cdot e^x$ → (1)

comparing with standard 2^{nd} order linear differential equation →

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = \sec x \cdot e^x$$

$P(x) = -2\tan x$, $Q(x) = 5$, $R(x) = \sec x e^x$

let $u = e^{-\frac{1}{2}\int P dx} = e^{\frac{-1}{2}\int -2\tan x dx}$

$= e^{\log \sec x} = \sec x$. let $y = uv$ be so to (1)

then (1) transforms to

$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$$

$$\frac{d^2v}{dx^2} + \left(-2\tan x + 2\cos x \times \sec x \tan x\right)\frac{dv}{dx} = e^x$$

ie $\frac{d^2v}{dx^2} = e^x \Rightarrow \frac{dv}{dx} = e^x + C_1$

$$\Rightarrow \boxed{v = e^x + C_1 x + C_2}$$

so gs of (1) is $y = uv$

$= \sec x \times (e^x + C_1 x + C_2)$

$$\boxed{y = e^x \sec x + C_1 x \sec x + C_2 \sec x}$$

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3. $p^2 + 2py\cot x = y^2$ $\quad p = \frac{dy}{dx}$

This equation is soluable for p

so $p = \frac{-2y\cot x \pm \sqrt{4y^2\cot^2 x + 4y^2}}{2}$

$p = -y\cot x \pm y\,\mathrm{cosec}\,x$

i.e. $p = -y\cot x + y\,\mathrm{cosec}\,x$, $-y\cot x - y\,\mathrm{cosec}\,x$

(i) $p = \frac{dy}{dx} = -y\cot x + y\,\mathrm{cosec}\,x$

$\frac{dy}{dx} = -y\frac{\cos x}{\sin x} + \frac{y}{\sin x} = \frac{y(1-\cos x)}{\sin x}$

$\frac{dy}{y\,dx} = \frac{2\sin^2\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}} = \tan\frac{x}{2}$

$\Rightarrow$ ~~Integrating~~ $\boxed{\log y = 2\log\sec\frac{x}{2} + \log c}$ — (1)

$\Rightarrow \boxed{y = c\sec^2\frac{x}{2}}$

(ii) using $p = \frac{dy}{dx} = -y\cot x - y\,\mathrm{cosec}\,x$

$\frac{dy}{dx} = \frac{-y\cos x - y}{\sin x} = \frac{-y(1+\cos x)}{\sin x}$

$= -y \times \frac{\cancel{2}\cos^{\cancel{2}}\frac{x}{2}}{\cancel{2}\sin\frac{x}{2}\cancel{\cos\frac{x}{2}}} = -y\cot\frac{x}{2}$

Integrating,

$\log y = \cancel{2}\log\mathrm{cosec}\frac{x}{2} + \log c'$

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$\Rightarrow \boxed{y = c_1\,\mathrm{cosec}^2\frac{x}{2}}$ — (2)

using (1) & (2) generalised sol is

ROUGH

$$\left(y\cos^2\frac{x}{2}-c\right)\left(y\sin^2\frac{x}{2}-c\right)=0$$

Q-4 $(x^4D^4+6x^3D^3+9x^2D^2+3xD+1)y=(1+\log x)^2$

put $x=e^z$ then by Euler-cauchy equation -

$D_1=\frac{\partial}{\partial z}$ Then $x^4D^4=D_1(D_1-1)(D_1-2)(D_1-3)$

$x^3D^3=D_1(D_1-1)(D_1-2)$

$xD=D_1$

$$[D_1(D_1-1)(D_1-2)(D_1-3)+6D_1(D_1-1)(D_1-2)+9D_1(D_1-1)+3D_1+1]y=(1+z)^2$$

$$[D_1^4-3D_1^3+2D_1^2-3D_1^3+9D_1^2-6D_1+6[D_1^3-D_1^2-2D_1^2+2D_1]+9(D_1^2-D_1)+3D_1+1]y=(1+z)^2$$

$$[D_1^4+2D_1^2+1]y=(1+z)^2$$

AE is $m^4+2m^2+1=0$

$(m^2+1)^2=0 \Rightarrow m=i,i,-i,-i$

CE is $y_c=(C_1+C_2z)\cos z+(C_3+C_4z)\sin z$

PI

$$y_p=\frac{1}{(D_1^4+2D_1^2+1)}(1+z)^2$$

$$=(1+(2D_1^2+D_1^4))^{-1}(z^2+2z+1)$$

$$=(1-2D_1^2+\cdots)(z^2+2z+1)$$

$$=z^2+2z+1-4=z^2+2z-3$$

so GS is $y=y_c+y_p$

$$y=(C_1+C_2\log x)(\cos\log x)+(C_3+C_4\log x)\sin(\log x)+(\log x)^2+2\log x-3$$

ROUGH

Q-5 solve $(D^4+D^2+1)y = ax^2 + be^{-x}\sin 2x$

AE is $m^4+m^2+1=0$ ie

$$\left(m^2+\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = 0$$

$$\left(m^2+\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\left(m^2+\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=0$$

$$m^2+\frac{1}{2}-\frac{\sqrt{3}}{2}i=0, \quad m^2+\frac{1}{2}+\frac{\sqrt{3}}{2}i=0$$

$$m^2+e^{i\left(2\pi-\frac{\pi}{3}\right)}=0; \quad m^2+e^{i\frac{\pi}{3}}=0$$

$$m=\pm\, i e^{i\left(\pi-\frac{\pi}{6}\right)}; \quad m=\pm\, i e^{i\frac{\pi}{6}}$$

$$=\pm i\left(-\frac{\sqrt{3}}{2}+\frac{i}{2}\right); \quad m=\pm i\left[\frac{\sqrt{3}}{2}+\frac{i}{2}\right]$$

$$=\pm\left(-\frac{\sqrt{3}}{2}i-\frac{1}{2}\right); \quad m=\pm\left[\frac{\sqrt{3}}{2}i-\frac{1}{2}\right]$$

$\Rightarrow$ CF $y_c = e^{-\frac{1}{2}x}\left[c_1\cos\frac{\sqrt{3}}{2}x + c_2\sin\frac{\sqrt{3}}{2}x\right]$

$$+ e^{\frac{1}{2}x}\left[c_3\cos\frac{\sqrt{3}}{2}x + c_4\sin\frac{\sqrt{3}}{2}x\right]$$

PI, $y_p = \dfrac{1}{D^4+D^2+1}\left(ax^2+be^{-x}\sin 2x\right)$

$$= \left(1+(D^2+D^4)\right)^{-1}(ax^2)$$

$$= \quad + \; b\,\frac{e^{-x}}{(D-1)^4+(D-1)^2+1}\sin 2x$$

$$= (1-D^2)(ax^2) + be^{-x}\,\frac{1}{\begin{array}{c}D^4-4D^3+6D^2-4D+1\\ +\;D^2+1-2D+1\end{array}}\sin 2x$$

$$= ax^2 - 2a + be^{-x}\,\frac{1}{D^4-4D^3+7D^2-6D+2}\sin 2x$$

$$= \;'' \;+ be^{-x}\,\frac{1}{(-4)^2-4D\times-4+7\times-4-6D+2}\sin 2x$$

$$\left(\text{using } \frac{1}{f(D^2)}\sin ax = \frac{1}{f(D^2=-a^2)}\sin ax\right)$$

$$= \;'' \;+ be^{-x}\,\frac{1}{16+16D-28-6D+2}\sin 2x$$

$$\times\,\frac{1}{10D-10}\sin 2x$$

$$\times\,\frac{be^{-x}}{10}\left(\frac{1}{D-1}\times\frac{D+1}{D-1}\sin 2x\right)$$

$$\times\,\frac{be^{-x}}{10}\left(\frac{D+1}{D^2-1}\sin 2x\right)$$

$$\times\,\frac{be^{-x}}{10}(D+1)\left(\frac{\sin 2x}{-5}\right)$$

$$y_p = \frac{be^{-x}}{-50}(2\cos 2x + \sin 2x)$$

So GS is $y = y_c + y_p$