Date : ....

A CONSOLIDATED QUESTION PAPER-CUM-ANSWER BOOKLET

IMS™
(INSTITUTE OF MATHEMATICAL SCIENCES)

MAINS TEST SERIES-2020

(OCT. TO JAN.-2020-21)

IAS/IFoS

MATHEMATICS

Under the guidance *of* K. Venkanna

Common Test
Test-13 for Batch-I
&
Test-5 for Batch-II

FULL SYLLABUS (PAPER-I)

DATE : 22-NOV.-2020

Time: 3 Hours **Maximum Marks: 250**

INSTRUCTIONS

1. This question paper-cum-answer booklet has 50 pages and has 36 PART/SUBPART questions. Please ensure that the copy of the question paper-cum-answer booklet you have received contains all the questions.
2. Write your Name, Roll Number, Name of the Test Centre and Medium in the appropriate space provided on the right side.
3. A consolidated Question Paper-cum-Answer Booklet, having space below each part/sub part of a question shall be provided to them for writing the answers. Candidates shall be required to attempt answer to the part/sub-part of a question strictly within the pre-defined space. Any attempt outside the pre-defined space shall not be evaluated. "
4. Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
5. Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any **THREE** of the remaining questions selecting at least **ONE** question from each Section.
6. The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
7. Symbols/notations carry their usual meanings, unless otherwise indicated.
8. All questions carry equal marks.
9. All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
10. All rough work should be done in the space provided and scored out finally.
11. The candidate should respect the instructions given by the invigilator.
12. The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any page from this booklet.

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name

Roll No.

Test Centre

Medium

Do not write your Roll Number or Name anywhere else in this Question Paper-cum-Answer Booklet.

I have read all the instructions and shall abide by them

Signature of the Candidate

I have verified the information filled by the candidate above

Signature of the invigilator

IMPORTANT NOTE:
Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed. Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

P.T.O.

DO NOT WRITE ON THIS SPACE

INDEX TABLE

QUESTION	No.	PAGE NO.	MAX. MARKS	MARKS OBTAINED
1	(a)			
	(b)			
	(c)			
	(d)			
	(e)			
2	(a)			
	(b)			
	(c)			
	(d)			
3	(a)			
	(b)			
	(c)			
	(d)			
4	(a)			
	(b)			
	(c)			
	(d)			
5	(a)			
	(b)			
	(c)			
	(d)			
	(e)			
6	(a)			
	(b)			
	(c)			
	(d)			
7	(a)			
	(b)			
	(c)			
	(d)			
8	(a)			
	(b)			
	(c)			
	(d)			
			Total Marks	

DO NOT WRITE ON THIS SPACE

SECTION – A

1. (a) Let W_1 and W_2 be subspaces of a vector space V such that the set-theoretic union of W_1 and W_2 is also a subspace. Prove that one of the spaces W_i is contained in the other. **[10]**

1. (b) Let $T : M_{22} \to M_{22}$ be defined by $T(A) - A^T$. Give M_{22} the standard basis

$$S=\left\{\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix},\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix},\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right\}=\{e_1, e_2, e_3, e_4\}$$

and find the matrix for T with respect to S. **[10]**

1. (c) If $u = At^{-1/2}e^{-x^2/4a^2t}$, Prove that $\dfrac{\partial u}{\partial t} = a^2 \dfrac{\partial^2 u}{\partial x^2}$. **[10]**

1. (d) Find the altitude and the semi-vertical angle of a cone of least volume which can be circumscribed to a sphere of radius a. **[10]**

1. (e) Prove that the circles $x^2 + y^2 + z^2 - 2x + 3y + 4z - 5 = 0$, $5y + 6z + 1 = 0$ and $x^2 + y^2 + z^2 - 3x - 4y + 5z - 6 = 0$, $x + 2y - 7z = 0$ lies on the same sphere and find its equation. Also find the value of a for which $x + y + z = a\sqrt{3}$ touches the sphere.

[10]

2. (a) (i) Let V be the space of 2×2 matrices over F. Find a basis $\{A_1, A_2, A_3, A_4\}$ for V such that $A^2_i = A_i$ for each i.

(ii) Let V be a vector space over a subfield F of the complex numbers. Suppose α, β, and γ are linearly independent vectors in V. Prove that $(\alpha + \beta)$, $(\beta, + \gamma)$, and $(\gamma, + \alpha)$ are linearly independent. **[10+10=20]**

2. (b) Show that the function $f : R^2 \to R$ defined by setting

$$f(x,y)=\begin{cases} x\sin\left(\frac{1}{x}\right)+y\sin\left(\frac{1}{y}\right), & \text{when } xy\neq 0 \\ x\sin\frac{1}{x}, & \text{when } x\neq 0,\ y=0 \\ y\sin\frac{1}{y}, & \text{when } x=0,\ y\neq 0 \\ 0, & \text{when } x=y=0 \end{cases}$$

is continuous but not differentiable at (0, 0). **[14]**

2. (c) (i) The plane $x - 2y + 3z = 0$ is rotated through a right angle about its line of intersection with the plane $2x + 3y - 4z - 5 = 0$, find the equation of the plane in its new position.

(ii) Find the S. D. between lines

$$\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1} \text{ and } \frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}$$

Find also its equations and the points in which it meets the given lines.

[6+10=16]

3. (a) We consider the 5 × 5 matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 2 & 4 & 1 & 10 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and the following problems concerning A

(a) Find an invertible matrix P such that PA is a row-reduced echelon matrix R.

(b) Find a basis for the row space W of A.

(c) Say which vectors $(b_1, b_2, b_3, b_4, b_5)$ are in W.

(d) Find the coordinate matrix of each vector $(b_1, b_2, b_3, b_4, b_5)$ in W in the ordered basis chosen in (b).

(e) Write each vector $(b_1, b_2, b_3, b_4, b_5)$ in W as a linear combination of the rows of A.

(f) Give an explicit description of the vector space V of all 5 × 1 column matrices X such that AX = 0. **[16]**

3. (b) (i) Test for convergence the integrals

$$\int_0^\infty \frac{x\tan^{-1}x}{\left(1+x^4\right)^{1/3}}dx$$

(ii) Let E = { (x, y) ∈$\mathbf{R}^2$/ 0 < x < y } . Then evaluate

$$\iint_E ye^{-(x+y)}dxdy$$

[18]

3. (c) If the feet of the three normals from P to the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ lie on the plane $x/a + y/b + z/c = 1$ prove that the feet of the other three lie on the plane $x/a + y/b + z/c + 1 = 0$ and P lies on the line $a(b^2 - c^2)x = b(c^2 - a^2)y = c(a^2 - b^2)z$. **[16]**

4. (a) (i) Let V be the (real) vector space of all polynomial functions from **R** into **R** of degree 2 or less, i.e., the space of all functions f of the form $f(x) = c_0 + c_1x + c_2x^2$.

Let t be a fixed real number and define

$g_1(x) = 1, \; g_2(x) = x + t, \; g_3(x) = (x + t)^2$.

Prove that $B = \{g_1, g_2, g_3\}$ is a basis for V. If

$f(x) = c_0 + c_1x + c_2x^2$

what are the coordinates of f in this ordered basis B ?

(ii) Let $T : p_1 \to p_2$ be defined by $T(a + bx) = ax + (b/2)x^2$. Give p_1 and p_2 the standard bases $S = \{1, x\}$ and $\tau = \{1, x, x^2\}$, respectively. Find the matrix of T with respect to these bases. Do the same for $L : p_2 \to p_1$ defined by $L(a + bx + cx^2) = b + 2cx$.

[10+10=20]

4. (b) The ellipsoid with equation $x^2 + 2y^2 + z^2 = 4$ is heated so that its temperature at (x, y, z) is given by $T(x, y, z) = 70 + 10(x+ z)$. find the hottest and coldest points on the ellipsoid. **[15]**

4. (c) Find the locus of the points from which three mutually perpendicular tangents can be drawn to the parabooid

$(x^2/a^2) - (y^2/b^2) = 2z$ **[15]**

SECTION – B

5. (a) (i) Solve $dy/dx = (x + y - 2)/(y - x - 4)$

(ii) Solve $(2xy^4 e^y + 2xy^3 + y)\,dx + (x^2y^4 e^y - x^2y^2 - 3x)\,dy = 0.$ **[10]**

5. (b) (i) Find L {F (t)}, where

$$F(t)=\begin{bmatrix} \cos\left(t-\frac{2}{3}\pi\right), & t>\frac{2\pi}{3} \\ 0 \quad , & t<\frac{2\pi}{3} \end{bmatrix}$$

(ii) Find $L^{-1}\left\{\frac{(p+1)e^{-\pi p}}{p^2+p+1}\right\}$ **[10]**

5. (c) A heavy uniform cube balances on the highest point of a sphere whose radius is r. If the sphere is rough enough to prevent sliding and if the side of the cube be $\pi r/2$, show that the cube can rock through a right angle without falling. **[10]**

5. (d) A point moves in a straight line so that its distance s from a fixed point at any time t is proportional to t^n. If v be the velocity and f the acceleration at any time t, show that $v^2 = nfs/(n - 1)$. **[10]**

5. (e) (i) Prove that $F = (y^2 \cos x + z^3)\,\mathbf{i} + (2y \sin x - 4)\mathbf{j} + (3xz^2 + 2)\mathbf{k}$ is a conservative force field.

(ii) Find the scalar potential for F.

(iii) Find the work done in moving an object in this field from $(0, 1, -1)$ to $(\pi/2, -1, 2)$.

[10]

6. (a) (i) Find the orthogonal trajectories of the family of curves $r = a\,(1 + \cos\theta)$, where a is the parameter.

(ii) Solve : $p^3 - 4xyp + 8y^2 = 0$

(iii) Find the values of λ for which all solutions of $x^2\,(d^2y/dx^2) - 3x\,(dy/dx) - \lambda y = 0$ tend to zero $x \to \infty$. **[20]**

6. (b) A uniform chain of length l hangs between two points A and B which are at a horizontal distance a from one another, with B at a vertical distance b above A. Prove that the parameter of the catenary is given by

$$2c\ \sinh(a/2c) = \sqrt{(l^2 - b^2)}\ .$$

Prove also that, if the tensions at A and B are T_1 and T_2 respectively,

$$T_1 + T_2 = W\sqrt{\left(1 + \frac{4c^2}{l^2 - b^2}\right)} \text{ and } T_2 - T_1 = Wb/l,$$

where W is the weight of the chain. **[15]**

6. (c) (i) Given the space curve x = t, y = t^2, $z=\frac{2}{3}t^3$, find (i) the curvature κ, (ii) the torsion τ.

(ii) Evaluate by Green's theorem in plane $\int_C \left(e^{-x}\sin y\,dx + e^{-x}\cos y\,dy\right)$, where C is the rectangle with vertices (0, 0) (π, 0), $\left(\pi,\frac{1}{2}\pi\right),\left(0,\frac{1}{2}\pi\right)$. **[15]**

7. (a) Apply the method of variation of parameters to solve the equation
$(x + 2)y_2 - (2x + 5)y_1 + 2y = (x + 1)\, e^x$. **[16]**

7. (b) A particle is projected with velocity V from the cusp of a smooth inverted cycloid down the arc, show that the time of reaching the vertex is
$2\sqrt{(a/g)}\tan^{-1}\left[\left(\sqrt{4ag}\right)/V\right]$. **[16]**

7. (c) (I) Find the angle of intersection at $(4, -3, 2)$ of spheres $x^2 + y^2 + z^2 = 29$ and $x^2 + y^2 + z^2 + 4x - 6y - 8z - 47 = 0$.

(II) (i) Prove that $r^n \mathbf{r}$ is an irrotational vector for any value of n but is solenoidal only if $n + 3 = 0$.

(ii) If $u = (1/r)\,\mathbf{r}$, show that $\nabla \times u = 0$.

(iii) if $u = (1/r)\,\mathbf{r}$ find grad (div $\mathbf{u}$). **[18]**

8. (a) By using Laplace transform method solve the initial value problem $(D^3 - 2D^2 + 5D)\, y = 0$ if $y(0) = 0$, $y'(0) = 1$, $y(\pi/8) = 1$. **[15]**

8. (b) A particle describes the curve $r^n = a^n \cos n\theta$ under a force to the pole. Find the law of force. **[17]**

8. (c) If $F = (y^2 + z^2 - x^2)\,\mathbf{i} + (z^2 + x^2 - y^2)\,\mathbf{j} + (x^2 + y^2 - z^2)\mathbf{k}$, evaluate $\iint \mathrm{curl}\,F \bullet n\,dS$ taken over the portion of the surface $x^2 + y^2 + z^2 - 2ax + az = 0$ above the plane $z = 0$, and verify Stroke's theorem. **[18]**

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