भारतीय वन सेवा परीक्षा....2011....

Indian Forest Services Examination........

7311

D-VSF-L-ZNA

MATHEMATICS

Paper I

Time Allowed : Three Hours

Maximum Marks : 200

INSTRUCTIONS

Candidates should attempt Questions No. 1 and 5 which are compulsory, and any THREE of the remaining questions, selecting at least ONE question from each Section.

All questions carry equal marks.

Marks allotted to parts of a question are indicated against each.

Answers must be written in ENGLISH only.

Assume suitable data, if considered necessary, and indicate the same clearly.

Unless indicated otherwise, symbols & notations carry their usual meaning.

SECTION A

1. Answer any ***four*** of the following :

   (a) Let V be the vector space of 2×2 matrices over the field of real numbers $\mathbf{R}$. Let $W = \{A \in V \mid \text{Trace } A = 0\}$. Show that W is a subspace of V. Find a basis of W and dimension of W. *10*

(b) Find the linear transformation from $\mathbf{R}^3$ into $\mathbf{R}^3$ which has its range the subspace spanned by $(1, 0, -1)$, $(1, 2, 2)$. *10*

(c) Show that the function defined by

$$f(x, y) = \begin{cases} \dfrac{x^3 + y^3}{x - y}, & x \neq y \\ 0, & x = y \end{cases}$$

is discontinuous at the origin but possesses partial derivatives f_x and f_y thereat. *10*

(d) Let the function f be defined by

$$f(t) = \begin{cases} 0, & \text{for } t < 0 \\ t, & \text{for } 0 \leq t \leq 1 \\ 4, & \text{for } t > 1. \end{cases}$$

(i) Determine the function $F(x) = \int_0^x f(t)\, dt$.

(ii) Where is F non-differentiable ? Justify your answer. *10*

(e) A variable plane is at a constant distance p from the origin and meets the axes at A, B, C. Prove that the locus of the centroid of the tetrahedron OABC is $\dfrac{1}{x^2} + \dfrac{1}{y^2} + \dfrac{1}{z^2} = \dfrac{16}{p^2}$. *10*

2. (a) Let $V = \{(x, y, z, u) \in \mathbf{R}^4 : y + z + u = 0\}$,

$W = \{(x, y, z, u) \in \mathbf{R}^4 : x + y = 0,\ z = 2u\}$

be two subspaces of $\mathbf{R}^4$. Find bases for V, W, V + W and $V \cap W$. *10*

(b) Find the characteristic polynomial of the matrix

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$$

and hence compute A^{10}. *10*

(c) Let $A = \begin{pmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{pmatrix}$.

Find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix. *10*

(d) Find an orthogonal transformation to reduce the quadratic form $5x^2 + 2y^2 + 4xy$ to a canonical form. *10*

3. (a) Show that the equation $3^x + 4^x = 5^x$ has exactly one root. *8*

(b) Test for convergence the integral $\int_0^\infty \sqrt{x\, e^{-x}}\ dx$. *8*

(c) Show that the area of the surface of the sphere $x^2 + y^2 + z^2 = a^2$ cut off by $x^2 + y^2 = ax$ is $2(\pi - 2)a^2$. *12*

(d) Show that the function defined by

$$f(x, y, z) = 3 \log (x^2 + y^2 + z^2) - 2x^2 - 2y^3 - 2z^3, \quad (x, y, z) \neq (0, 0, 0)$$

has only one extreme value, $\log \left(\frac{3}{e^2} \right)$. *12*

4. (a) Find the equation of the right circular cylinder of radius 2 whose axis is the line $\frac{x-1}{2} = \frac{y-2}{1} = \frac{z-3}{2}$. *10*

(b) Find the tangent planes to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ which are parallel to the plane $lx + my + nz = 0$. *10*

(c) Prove that the semi-latus rectum of any conic is a harmonic mean between the segments of any focal chord. *8*

(d) Tangent planes at two points P and Q of a paraboloid meet in the line RS. Show that the plane through RS and middle point of PQ is parallel to the axis of the paraboloid. *12*

SECTION B

5. Answer any ***four*** of the following :

(a) Find the family of curves whose tangents form an angle $\pi/4$ with hyperbolas $xy = c$. *10*

(b) Solve :

$$\frac{d^2y}{dx^2} - 2\tan x \frac{dy}{dx} + 5y = \sec x \cdot e^x.$$ *10*

(c) The apses of a satellite of the Earth are at distances r_1 and r_2 from the centre of the Earth. Find the velocities at the apses in terms of r_1 and r_2. *10*

(d) A cable of length 160 meters and weighing 2 kg per meter is suspended from two points in the same horizontal plane. The tension at the points of support is 200 kg. Show that the span of the cable is $120 \cosh^{-1}\left(\frac{5}{3}\right)$ and also find the sag. *10*

(e) Evaluate the line integral

$\oint_C (\sin x \, dx + y^2 dy - dz)$, where C is the circle

$x^2 + y^2 = 16$, $z = 3$, by using Stokes' theorem. *10*

6. (a) Solve :

$$p^2 + 2\, py \cot x = y^2,$$

where $p = \frac{dy}{dx}$. *10*

(b) Solve :

$$\{x^4D^4 + 6x^3D^3 + 9x^2D^2 + 3xD + 1\}y = (1 + \log x)^2,$$

where $D \equiv \frac{d}{dx}$. *15*

(c) Solve :

$(D^4 + D^2 + 1)y = ax^2 + be^{-x} \sin 2x$,

where $D \equiv \frac{d}{dx}$. 15

7. (a) One end of a uniform rod AB, of length 2a and weight W, is attached by a frictionless joint to a smooth wall and the other end B is smoothly hinged to an equal rod BC. The middle points of the rods are connected by an elastic cord of natural length a and modulus of elasticity 4W. Prove that the system can rest in equilibrium in a vertical plane with C in contact with the wall below A, and the angle between the rod is $2 \sin^{-1} \left(\frac{3}{4}\right)$. 13

(b) AB is a uniform rod, of length 8a, which can turn freely about the end A, which is fixed. C is a smooth ring, whose weight is twice that of the rod, which can slide on the rod, and is attached by a string CD to a point D in the same horizontal plane as the point A. If AD and CD are each of length a, fix the position of the ring and the tension of the string when the system is in equilibrium.

Show also that the action on the rod at the fixed end A is a horizontal force equal to $\sqrt{3}$ W, where W is the weight of the rod. 14

(c) A stream is rushing from a boiler through a conical pipe, the diameter of the ends of which are D and d; if V and v be the corresponding velocities of the stream and if the motion is supposed to be that of the divergence from the vertex of cone, prove that

$$\frac{v}{V} = \frac{D^2}{d^2} e^{\left(v^2 - V^2\right)/2K}$$

where K is the pressure divided by the density and supposed constant. *13*

8. (a) Find the curvature, torsion and the relation between the arc length S and parameter u for the curve :

$$\vec{r} = \vec{r}(u) = 2 \log_e u \, \hat{i} + 4u \, \hat{j} + (2u^2 + 1) \hat{k}$$ *10*

(b) Prove the vector identity :

$$\text{curl } (\vec{f} \times \vec{g}) = \vec{f} \text{ div } \vec{g} - \vec{g} \text{ div } \vec{f} + (\vec{g} \cdot \nabla) \vec{f} - (\vec{f} \cdot \nabla) \vec{g}$$

and verify it for the vectors $\vec{f} = x \hat{i} + z \hat{j} + y \hat{k}$ and $\vec{g} = y \hat{i} + z \hat{k}$. *10*

(c) Verify Green's theorem in the plane for

$$\oint_C [(3x^2 - 8y^2)dx + (4y - 6xy)dy],$$

where C is the boundary of the region enclosed by the curves $y = \sqrt{x}$ and $y = x^2$. *10*

(d) The position vector $\vec{r}$ of a particle of mass 2 units at any time t, referred to fixed origin and axes, is

$$\vec{r} = (t^2 - 2t)\hat{i} + \left(\frac{1}{2}t^2 + 1\right)\hat{j} + \frac{1}{2}t^2\hat{k}.$$

At time t = 1, find its kinetic energy, angular momentum, time rate of change of angular momentum and the moment of the resultant force, acting at the particle, about the origin. *10*