2014

1

(b) Let f be defined on $[0, 1]$ as

$$f(x) = \begin{cases} \sqrt{1-x^2}, & \text{if } x \text{ is rational} \\ 1-x \ , & \text{if } x \text{ is irrational} \end{cases}$$

Find the upper and lower Riemann integrals of f over $[0, 1]$.

2

(b) Show that the function $f(x) = \sin\frac{1}{x}$ is continuous but not uniformly continuous on $(0, \pi)$.

3

(b) Change the order of integration and evaluate $\int_{-2}^{1}\int_{y^2}^{2-y} dx\, dy$.

4

(a) Show that the function $f(x) = \sin x$ is Riemann integrable in any interval $[0, t]$ by taking the partition $P = \left\{0, \frac{t}{n}, \frac{2t}{n}, \frac{3t}{n}, \ldots, \frac{nt}{n}\right\}$ and $\int_0^t \sin x\, dx = 1 - \cos t$.

IFOS-2014

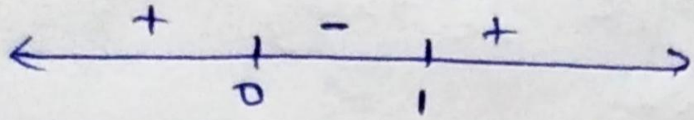
1 Given $f(x) = \begin{cases} \sqrt{1-x^2} & , \text{ if } x \text{ is rational} \\ 1-x & , \text{ if } x \text{ is rational} \end{cases}$

$\forall\, x \in (0,1)$

Now. $(1-x)^2 - \left(\sqrt{1-x^2}\right)^2$

$= x^2 - 2x + 1 - 1 + x^2$

$= 2x^2 - 2x \quad = 2x(x-1)$



$\therefore$ In $(0,1)$ $\quad (1-x^2)^2 - \left(\sqrt{1-x^2}\right)^2 \not< 0$

$\Rightarrow \quad \sqrt{1-x^2} > (1-x) \qquad \forall\, x \in (0,1)$

$\therefore$ Sup $f(x) = \sqrt{1-x^2}$

Inf $f(x) = 1-x$

Upper Reimann integral $= \int_0^1 \sup f(x)\, dx$

$= \int_0^1 \sqrt{1-x^2}\, dx$

$= \left| \frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x \right|_0^1$

$= \left(0 + \frac{1}{2}\sin^{-1} 1\right) - \left(0 + \frac{1}{2}\sin^{-1} 0\right)$

$= \pi/4$

Lower Reimann integral $= \int_0^1 \text{Inf} f(x)\, dx$

$$= \int_0^1 (1-x)\, dx = \left| x - \frac{x^2}{2} \right|_0^1$$

$$= \left(1 - \frac{1}{2}\right) - (0 - 0) = \frac{1}{2}$$

$\therefore$ Upper Reimann integral $= \pi/24$

Lower " " $= 1/2$

Q2 sol. Given $f(x) = \sin\frac{1}{x}$ in interval $(0, \pi)$

Let $g(x) = \sin x$, $g(x)$ is trigonometric function and continuous in the interval $(0, \pi)$

$h(x) = \frac{1}{x}$, is a rational function defined in $(0, \pi)$

$\therefore$ continuous in interval $(0, \pi)$

$f(x) = g \circ h(x) = \sin\left(\frac{1}{x}\right)$ is composition of two continuous functions $\therefore$ $f(x)$ is continuous in $(0, \pi)$

If $f(x) = \sin\frac{1}{x}$ is uniform continuous then ~~then~~ for every $\varepsilon > 0$, there exist δ such that

$$|f(x) - f(y)| < \varepsilon \quad \forall \quad |x-y| < \delta$$

Let $x, y \in (0, \pi)$ such that $x = \frac{2}{(4n+1)\pi}$ and

$y = \frac{1}{2n\pi}$ then

$$|x-y| = \left|\frac{2}{(4n+1)\pi} - \frac{1}{2n\pi}\right| = \left|\frac{1}{2n(4n+1)\pi}\right|$$

Let δ be any +ve number such that

$$|x-y| = \left|\frac{1}{2n(4n+1)\pi}\right| < \delta$$

Taking $\varepsilon = \frac{1}{4}$

$$|f(x) - f(y)| = \left|\sin (4n+1)\frac{\pi}{2} - \sin 2n\pi\right|$$

$$= |1-0| = 1 > \varepsilon \left(= \tfrac{1}{4}\right)$$

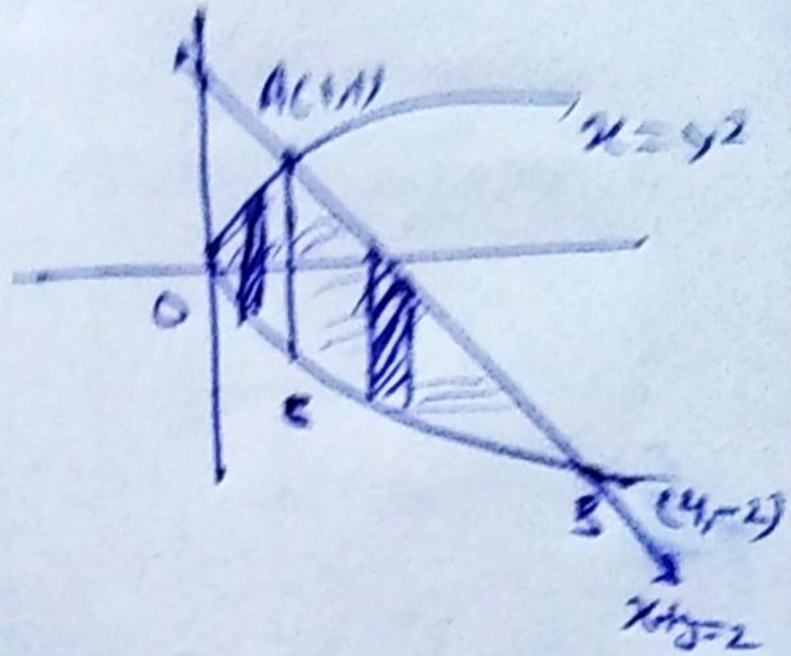
Thus we have $\delta > 0$ and given $\varepsilon = \frac{1}{4}$

such that

$$|f(x) - f(y)| > \varepsilon \text{ for } |x-y| < \delta$$

$\Rightarrow f(x) = \sin \frac{1}{x}$ is not uniformly continuous in the interval $(0, \pi)$

Q3 Let $I = \int_{-2}^{1}\int_{y^2}^{2-y} dx\,dy$



Changing order of integration

$$I = \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} dx\,dy + \int_1^4 \int_{-\sqrt{x}}^{2-x} dx\,dy$$

(OAC) (ABC)

$$= \int_0^1 |y|_{-\sqrt{x}}^{\sqrt{x}}\, dx + \int_1^4 |y|_{-\sqrt{x}}^{2-x}\, dx$$

$$= \int_0^1 2\sqrt{x}\, dx + \int_1^4 (2-x+\sqrt{x})\, dx$$

$$= \left|\frac{4}{3}x^{3/2}\right|_0^1 + \left|2x - \frac{x^2}{2} + \frac{2}{3}x^{3/2}\right|_1^4$$

$$= \left(\frac{4}{3} - 0\right) + \left(2(4-1) - \left(-\frac{1}{2} + 8\right) + \frac{2}{3}(8-1)\right)$$

$$= \frac{4}{3} + \left(6 - 8 + \frac{1}{2} + \frac{16}{3} - \frac{2}{3}\right) = \boxed{\frac{9}{2} \text{ units}}$$

Q4 Sol Given $f(x) = \sin x$.

$f(x)$ is continues and bounded ($\because |\sin x| \le 1$) in the interval $[0, t]$

$\therefore$ $\sin x$ is Reimann integrable in $[0,t]$

Taking Partition $P = \left\{0, \frac{t}{n}, \frac{2t}{n}, \ldots, \frac{nt}{n}\right\}$

Length of each subinterval $= \frac{t}{n}$ (δx)

$\|P\| \to 0$ as $n \to \infty$ and $x_r = 0 + \frac{rt}{n} = \frac{rt}{n}$

Using integral as limit of sum

$$\int_0^t \sin x \, dx = \lim_{\|P\| \to 0} \sum_{r=1}^{n} \sin(x_r)\,\delta x$$

$$= \lim_{n \to \infty} \sum_{r=1}^{n} \sin\left(\frac{rt}{n}\right) \frac{t}{n}$$

$$= \lim_{n \to \infty} \frac{t}{n}\left[\sin\frac{t}{n} + \sin\frac{2t}{n} + \ldots + \sin\frac{nt}{n}\right]$$

$$= \lim_{n \to \infty} \frac{t}{n} \frac{\sin\left(\frac{t}{n} + \left(\frac{n-1}{2}\right)\frac{t}{n}\right)\sin\left(\frac{n}{2}\frac{t}{n}\right)}{\sin(t/2n)}$$

$$= \lim_{n \to \infty} \frac{t}{n} \frac{\sin\left(\frac{n+1}{2} \cdot \frac{t}{n}\right)\sin\frac{t}{2}}{\sin(t/2n)}$$

$$= \lim_{n \to \infty} 2 \cdot \frac{t/2n}{\sin(t/2n)} \cdot \sin\left(t\left(\frac{1}{2} + \frac{1}{2n}\right)\right) \cdot \sin t/2$$

$$\Rightarrow 2 \cdot (1) \cdot \sin\frac{t}{2} \sin t/2$$

$$= 2\sin^2 t/2 = 1 - \cos t$$

$$\therefore \Rightarrow \boxed{\int_0^t \sin x \, dx = 1 - \cos t}$$

Hence Proved.