

* Linear programming *

Introduction:

The linear programming originated during world war II (1939–1945), when the British and American Military management called upon a group of scientists to study and plan the war activities, so that maximum damages could be inflicted on the enemy camps at minimum cost and loss. Because of the success in military operations, it quickly spread in all phases of industry and government organisations.

It was first coined in 1940 by Mc Closky and Treften (by using the term operations Research) in a small town, Bowdsey, of the United Kingdom.

In India, it came into existence in 1949, with opening of an operations research unit at the regional research laboratory at Hyderabad.

Linear programming problems:

In the competitive world of business and industry, the decision maker wants to utilize his limited resources in a best possible manner. The limited resources may include material, money, time, man power, machine capacity etc. Linear programming can be viewed as a scientific approach that has evolved as an aid to a decision maker in business, industrial, agri-cultural, hospital, government and military organisations.

→ Now, suppose a vendor has a sum of Rs. 350 with which he wishes to purchase two types of tape, say, red and blue. Red tape costs Rs. 2 per metre and blue tape costs Rs. 3 per metre. He doesnot want to buy more than 40 metres of red tape. The question arises, " How many metre of red and blue tapes can he buy ?" Assume that he buys x metres of red tape and y metres of blue tape. The above problem can be stated mathematically as follows:

find x and y such that

$$2x + 3y \le 350 \quad \text{—(i)}$$
$$x \le 40 \quad \text{—(ii)}$$
$$x \ge 0,\ y \ge 0 \quad \text{—(iii)}$$

There can be a number of solution pairs (x, y). Now, further suppose that the vendor sells red tape at a profit of Rs. 0.75 per metre while blue tape at a profit of Rs. 1 per metre. Obviously, vendor likes to pick up a pair (x, y) which gives him the maximum profit. Now, the problem arises to find out the pair (x, y) which give maximum profit to the vendor, i.e, which will maximize $0.75x + 1\cdot y$.

The above kind of problem is called a linear programming problem.

→ In a linear programming problem, we have constraints expressed in the form of linear inequalities. Therefore, to study linear programming, we must know the system of linear inequalities particularly their graphical solutions. Now we shall confine our discussion to the graphical solutions of inequalities.

— Closely linked with the system of linear inequalities is the theory of convex sets. This theory has very important applications not only in linear programming but also in Economics, Game theory etc. Due to these applications, a great deal of work has been done to develope the theory of convex sets.

Thus, now we discuss the inequalities and convex sets. In addition, we need the notion of extreme points, Hyper-plane and Half spaces. These notions will be defined and explained with the help of some simple examples.

Inequalities and their graphs:

We know that a general equation of a line is

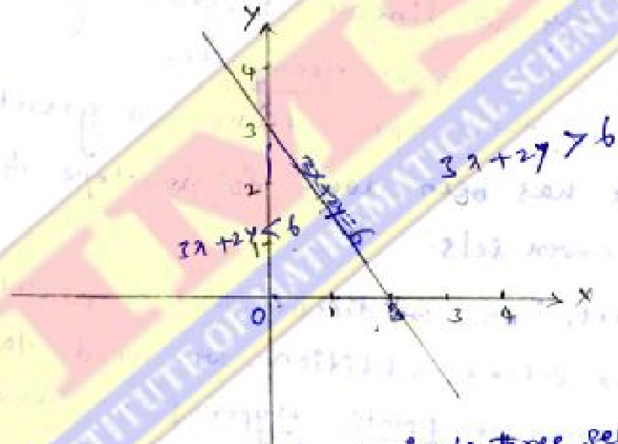
$ax + by = c$,

where a, b, c are real constants.

It is also called a linear eqn in two variables x and y.

If we put $y=0$, we get $x = c/a$, provided $a \neq 0$.
$x = c/a$ is the intercept of the line on x-axis.
Similarly on taking $x=0$, we get
$y = c/b$, $b \neq 0$.
as the intercept on y-axis.
By joining the points $\left(\frac{c}{a}, 0\right)$ & $\left(0, \frac{c}{b}\right)$, $a \neq 0, b \neq 0$
we can trace the line.

For example,
Consider the line $3x+2y=6$.
Draw this line as shown in the figure



This line divides the plane into three sets or regions as shown in the figure.
These Regions may be described as follows:
(i) The set of points (x, y) such that
$3x+2y=6$
i.e, those points which lie on the line.
(ii) The set of points (x, y) such that
$3x+2y<6$.
– The set of points (x,y) for which $3x+2y<6$ is called the half plane bounded by the line $3x+2y=6$

(iii) The set of points (x, y) such that $3x+2y>6$. the other half plane bounded by the line $3x+2y=6$

- The inequality $3x+2y \le 6$ represents the set of points (x,y) which either lie on the line $3x+2y=6$ or belong to the half-plane

$$3x+2y<6$$

Similarly, the inequality $3x+2y \geq 6$ represents the set of points (x,y) which either lie on the line $3x+2y=6$ or belong to the half-plane $3x+2y>6$.

- Most of the inequalities that we study here will be of the form

$$ax+by \le c \quad \text{or} \quad ax+by \geq c.$$

- In general we can say that a line $ax+by=c$ divides the xy-plane into three regions viz.

(i) the set of points (x, y) such that $ax+by=c$, that is the line itself.

(ii) the set of points (x, y) such that $ax+by<c$ i.e., one of the half-planes bounded by the line.

(iii) the set of points (x, y) such that $ax+by>c$ the other half plane bounded by the line.

→ Draw the graph of the inequality $15x + 8y \geq 60$

First consider the line.

$$15x + 8y = 60$$

If we take $y = 0$, then $x = 4$.

If $x = 0$, then $y = 15/2$.

$\therefore$ we can trace the line by joining the points $(4,0)$ and $(0, 15/2)$.

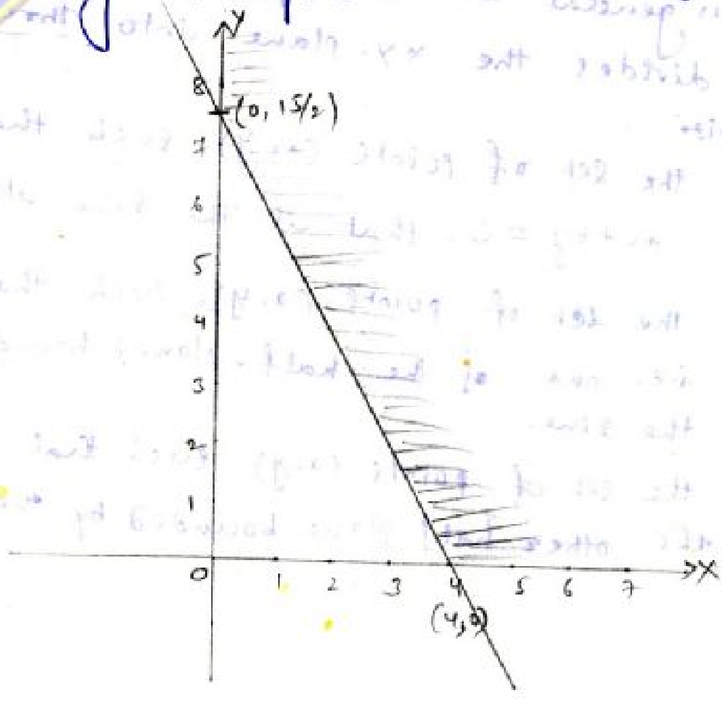
Let us now determine the location of the half plane.

For this, we put

$x = 0$ and $y = 0$

$15(0) + 9(0) = 0 \leq 60$

This shows that $15x + 8y \geq 60$ is that half plane in which origin doesnot lie. Hence the shaded region as shown in the figure, represents the $15x + 8y \geq 60$

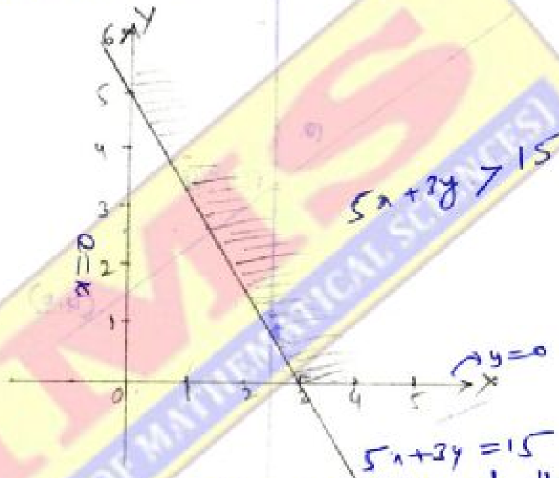


→ Graph the set of points (x, y) satisfying the following three inequalities.

$A = \{(x, y) / 5x + 3y \geq 15\}$

$B = \{(x, y) / x \geq 0\}$ and $C = \{(x, y) / y \geq 0\}$

Sol^n: The desired set is the intersection of the three sets. The set $A \cap B \cap C$ is the area which is shaded and bounded by the line $5x + 3y = 15$



Note: The shaded region consists of the points lying only in the first quadrant due to the restrictions $x \geq 0$ and $y \geq 0$.

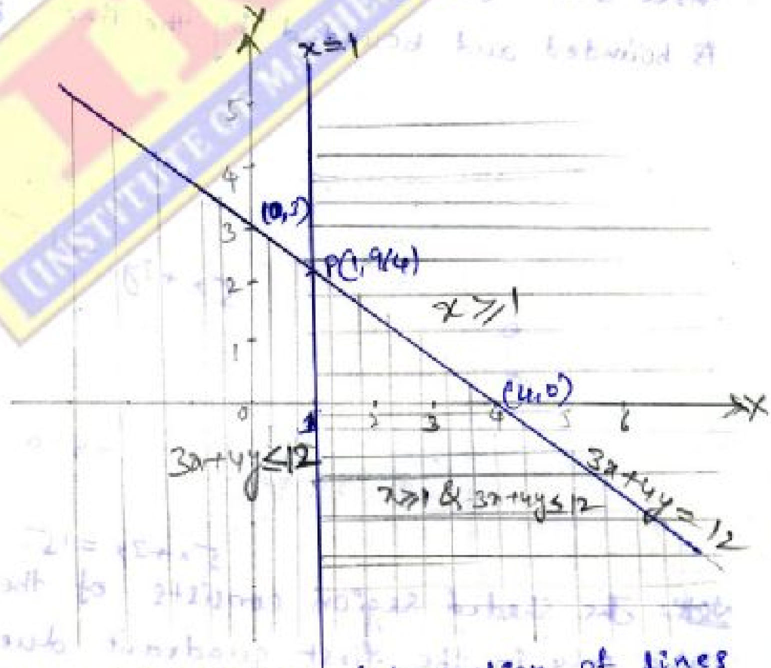
→ Graph the set of points which satisfy the inequalities $x \geq 1$ and $3x + 4y \leq 12$

Sol^n: Draw the line $x = 1$, which is a vertical line through the point $(1, 0)$.

The line $3x + 4y = 12$ is the line joining the points $(4, 0)$ and $(0, 3)$.

The set $A = \{(x,y) \mid x \geq 1\}$ is shaded with horizontal lines and the set $B = \{(x,y) / 3x+4y \leq 12\}$ is the set with vertical shading.

Now the set of points which satisfy both the inequalities i.e., the set $A \cap B$ of points is the cross-hatched region show in the fig.



The corner point $P(1, \frac{9}{4})$ is the intersection of lines $x=1$ and $3x+4y=12$.

Convex sets and their Geometry

Notion of convex sets:

Let x_1 and x_2 be any two points in the Euclidean space E^n. Consider a line passing through the points x_1, x_2 $(x_1 \neq x_2)$ in E^n defined as the set

$$S = \{ x / x = \lambda x_2 + (1-\lambda) x_1, \text{ all real } \lambda \}.$$

By giving different values to λ we will get the corresponding different points on the line. Suppose λ is chosen such that $0 \leq \lambda \leq 1$. Then for $\lambda = 0$, we get $x = x_1$ and for $\lambda = 1$, we get $x = x_2$.

Thus we get various points between x_1 and x_2. The line joining these points corresponding to the values of λ between 0 and 1, is often called a line segment.

In other words, the line segment joining the points x_1, x_2 in E^n corresponding to the values of λ between 0 and 1. is a set of points denoted by S,

where $S = \{ x / x = \lambda x_2 + (1-\lambda) x_1, \ 0 \leq \lambda \leq 1 \}$

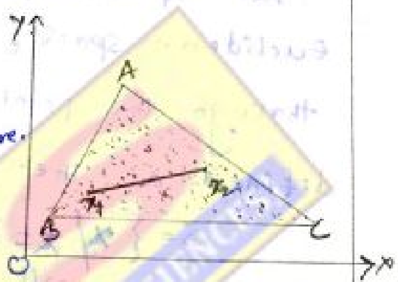
Convex set

Defn: A set 'S' is said to be Convex if for any two points x_1, x_2 in the set, the line segment joining these points is also in the set.

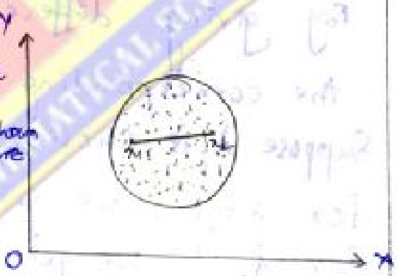
In other words, a set S is said to be convex if for any elements

$x_1, x_2 \in S$, $\lambda x_2 + (1-\lambda)x_1 \in S$ for $0 \le \lambda \le 1$.

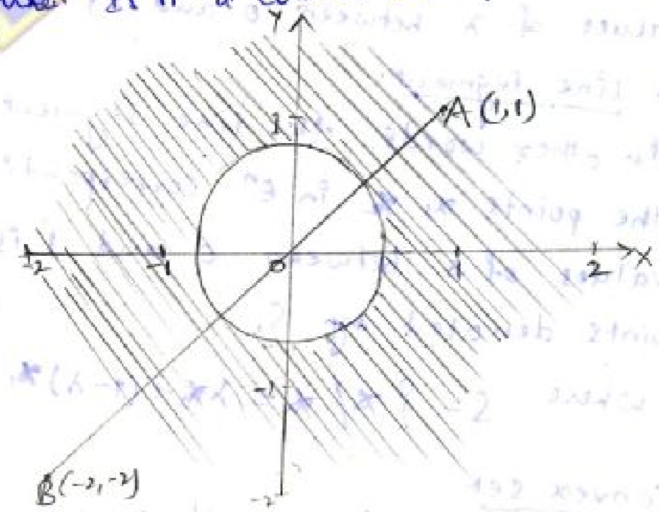
→ Consider a triangle ABC with its interior is a convex set as shaded in figure.



→ consider a set $S = \{(x,y) / x^2+y^2 \le 1\}$ which is the circle with its interior as shown in figure.
Clearly it is a convex set.



→ However, consider the set $S = \{(x,y) / x^2+y^2 \ge 1\}$ which is the circle with its exterior as shown in figure. Is it a convex set?



Sol^n: Let the points $A(1,1)$ and $B(-2,-2)$.

Since $1^2+1^2 > 1$ and $(-2)^2+(-2)^2 = 8 > 1$.

$\therefore$ Both the points A and B belong to S.

Take $\lambda = \frac{2}{3}$. Then $\lambda A + (1-\lambda)B$

$$= \frac{2}{3}(1,1) + \left(\frac{1}{3}\right)(-2,-2)$$

gives the point $(0,0)$.

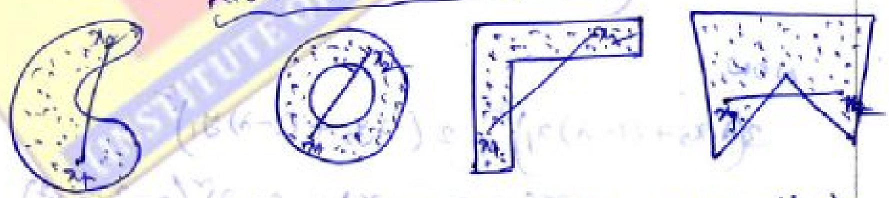
but $(0,0)$ doesnot satisfy $x^2+y^2 \geq 1$.

This shows that S is not a convex set.

Few more graphs of the sets which are convex and non-convex are as given below.

convex sets

Non-convex sets



→ If S_1 and S_2 are convex sets, then their intersection is also a convex set.

Soln: Suppose $S_3 = S_1 \cap S_2$.

Let x_1, x_2 be any two points in S_3.

Then $x_1, x_2 \in S_1$ and $x_1, x_2 \in S_2$.

Since S_1 and S_2 are convex,

therefore $\lambda x_2 + (1-\lambda)x_1 \in S_1$ for $0 \leq \lambda \leq 1$

and $\lambda x_2 + (1-\lambda)x_1 \in S_2$ for $0 \leq \lambda \leq 1$

Hence $\lambda x_2+(1-\lambda)x_1 \in S_1\cap S_2$ for $0\le\lambda\le1$.

$\Rightarrow \lambda x_2+(1-\lambda)x_1 \in S_3$, $0\le\lambda\le1$.

$\therefore S_3$ is convexset.

→ Show that the set $S=\{(x,y)/\ 3x^2+2y^2\le 6\}$ is convex.

Sol^n: Suppose $p=(x_1,y_1)\in S$, $q=(x_2,y_2)\in S$ are any two points. Then

$$\left.\begin{array}{l}3x_1^2+2y_1^2\le 6\\ 3x_2^2+2y_2^2\le 6\end{array}\right\}\text{—①}$$

S will be convex if

$\lambda(x_2,y_2)+(1-\lambda)(x_1,y_1)\in S$; $0\le\lambda\le1$.

i.e. $\big(\lambda x_2+(1-\lambda)x_1,\ \lambda y_2+(1-\lambda)y_1\big)\in S$; $0\le\lambda\le1$.

Now,

$$3(\lambda x_2+(1-\lambda)x_1)^2+2(\lambda y_2+(1-\lambda)y_1)^2$$

$$=\lambda^2(3x_2^2+2y_2^2)+(1-\lambda)^2(3x_1^2+2y_1^2)+2\lambda(1-\lambda)[3x_2x_1+2y_2y_1]$$

$$\le 6\lambda^2+6(1-\lambda)^2+2\lambda(1-\lambda)[3x_1x_2+2y_1y_2] \text{ —②}$$

(from ①)

Now consider

$$3(x_1-x_2)^2+2(y_2-y_1)^2\ge 0$$

$\Rightarrow 3[x_1^2+x_2^2-2x_1x_2]+2[y_2^2+y_1^2-2y_1y_2]\ge0$

$\Rightarrow (3x_1^2+2y_1^2)+(3x_2^2+2y_2^2)-2(3x_1x_2+2y_1y_2)\ge0$

$\Rightarrow 2(3x_1x_2+2y_1y_2)\le(3x_1^2+2y_1^2)+(3x_2^2+2y_2^2)$

$\le 6+6$ ($\because$ from ①)

$\Rightarrow 3x_1x_2 + 2y_1y_2 \le 6.$ —— (3)

from (2) and (3), we get

$$3\left[\lambda x_2 + (1-\lambda)x_1\right]^2 + 2\left(\lambda y_2 + (1-\lambda)y_1\right)^2 \le 6\lambda^2 + 6(1-\lambda)^2 + 12\lambda(1-\lambda)$$

$$= 6\left[\lambda + (1-\lambda)\right]^2$$

$$= 6.$$

Hence $3\left(\lambda x_2 + (1-\lambda)x_1\right)^2 + 2\left(\lambda y_2 + (1-\lambda)y_1\right)^2 \le 6$

$\therefore \left(\lambda x_2 + (1-\lambda)x_1, \lambda y_2 + (1-\lambda)y_1\right) \in S.$

$\therefore$ S is a convex set.

→ Show that the set $S = \{(x,y) / xy \le 1, x \ge 0, y \ge 0\}$ is not convex.

Sol: In order to show that S is not convex we will take two points in S and show that their convex combination doesnot belong to S.

Clearly $(3, \frac{1}{3})$ & $(\frac{1}{2}, 2)$ belong to S.

Consider the combination of these points.

i.e $\lambda(3, \frac{1}{3}) + (1-\lambda)(\frac{1}{2}, 2)$, $0 \le \lambda \le 1$.

$\Rightarrow \left(3\lambda + \frac{1}{2}(1-\lambda), \frac{\lambda}{3} + 2(1-\lambda)\right)$, $0 \le \lambda \le 1$

$\Rightarrow \left(\frac{1}{2} + \frac{5}{2}\lambda, 2 - \frac{5}{3}\lambda\right)$, $0 \le \lambda \le 1$.

S will be convex if

$\left(\frac{1}{2} + \frac{5}{2}\lambda, 2 - \frac{5}{3}\lambda\right) \in S$, $\forall \lambda \in [0,1]$

i.e, $\left(\frac{1}{2}+\frac{5}{2}\lambda\right)\left(2-\frac{5}{3}\lambda\right) \le 1$ for $\lambda \in [0,1]$

$\Rightarrow 1+5\lambda-\frac{5}{6}\lambda-\frac{25}{6}\lambda^2 \le 1$ for $\lambda \in [0,1]$

$\Rightarrow \frac{25}{6}\lambda-\frac{25}{6}\lambda^2 \le 0$ for $\lambda \in [0,1]$.

This inequality should hold for all values of λ such that $0 \le \lambda \le 1$.

But, if we take $\lambda = \frac{1}{2}$, then we get

$$\frac{25}{6}\lambda - \frac{25}{6}\lambda^2 = \frac{25}{12} - \frac{25}{24} = \frac{25}{24} > 0.$$

Thus the inequality is not satisfied for $\lambda = \frac{1}{2}$.

This contradiction shows that S is not convex.

Defn:

Extreme point or vertex of a convex set:

An extreme point (or vertex) of a convex set is a point of the set which doesnot lie on any segment joining two other points of the set.

(or) Let S be a convex set. A point $x \in S$ is an extreme point of the convex set S if and only if there donot exist points x_1, x_2 $(x_1 \neq x_2)$ in the set S such that

$$x = \lambda x_1 + (1-\lambda)x_2, \; 0<\lambda<1.$$

Note that the point $x = \lambda x_1 + (1-\lambda)x_2$, $0<\lambda<1$ is a point in between x_1 and x_2 $(x_1 \neq x_2)$.

According to the above definition an extreme point fails to satisfy this property.

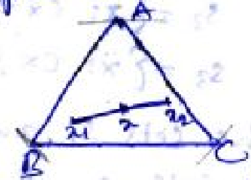
Consider a convex set 'S' formed by a triangle ABC and its interior. If x is a point inside the Δ^{le} ABC, then it is possible to find points x_1 and x_2 in S such that

$$x = \lambda x_1 + (1-\lambda)x_2, \quad 0<\lambda<1,$$

— If x is a boundary point in S different from points A, B and C, even then we can find points x_1 and x_2 in S such that

$$x = \lambda x_1 + (1-\lambda)x_2, \quad 0<\lambda<1.$$

But for the points A, B and C this is not possible. That is why the points A, B and C are called the extreme points.



Note:- we should note the strict inequality imposed on λ.

— An extreme point is a boundary point of the set; however, not all the boundary points of a convex set are necessarily extreme points.

— If we consider circle, then every point on the circumference of the circle is an extreme point.

Hyper plane and Half Spaces:

A hyper plane in E^n is defined to be a set of points

$$S = \{ x \in E^n / c_1 x_1 + c_2 x_2 + \cdots\cdots + c_n x_n = d \}$$

i.e., $S = \{ x \in E^n / cx = d \}$.

where $c = [c_1 \; c_2 \; \cdots \; c_n]$ $\quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

W.K.T $c_1 x_1 + c_2 x_2 = d$ $(c_1, c_2, d$ constants) is the eqn of the straight line in E^2 and also this line divides the plane into three parts.

Similarly, hyper plane $cx = d$ in E^n divides E^n into three mutually exclusive and exhaustive regions. These are denoted by the sets

$S_1 = \{ x : cx < d \}$.

$S_2 = \{ x : cx = d \}$

$S_3 = \{ x : cx > d \}$

The sets S_1 and S_3 are called open half spaces.

The sets $S_4 = \{ x / cx \leq d \}$ and $S_5 = \{ x / cx \geq d \}$ are called closed half spaces.

Note: $S_4 \cap S_5 = S_2$ which is the hyper-plane $cx = d$.

→ A hyper plane is a convex set

(or) prove that the collection $S = \{ x \in E^n / cx = d \}$ is a convex set.

Solⁿ: If x_1, x_2 are any two points on the hyper-plane $cx = d$, then $cx_1 = d$ and $cx_2 = d$ (9)

The hyper plane will be convex if the point

$x = \lambda x_2 + (1-\lambda)x_1$ for $0 \le \lambda \le 1$.

lies on the hyperplane.

we have

$$cx = c[\lambda x_2 + (1-\lambda)x_1]$$
$$= c\lambda x_2 + c(1-\lambda)x_1$$
$$= \lambda cx_2 + (1-\lambda)cx_1$$
$$= \lambda d + (1-\lambda)d = d$$

$\therefore cx = d$.

Hence the hyperplane is a convex set.

→ A closed half space is a convex set.

Solⁿ: Consider the closed half space.

$$S_4 = \{x : cx \le d\}$$

Suppose $x_1, x_2 \in S_4$ then $cx_1 \le d$, $cx_2 \le d$.

Consider $x = \lambda x_2 + (1-\lambda)x_1$, $0 \le \lambda \le 1$.

Now $cx = \lambda cx_2 + (1-\lambda)cx_1$, $0 \le \lambda \le 1$.

$\le \lambda d + (1-\lambda)d = d$.

i.e, $cx \le d$.

Hence S_4 is a convex set.

→ Similarly, we can show that S_1, S_3 & S_5 are convex sets.

Convex combination:

Let $x_1, x_2, \dots x_m$ be a finite number of points in a Euclidean space E^n. A convex combination of points $x_1, x_2, \dots x_m$ is defined as a point

$$x = \sum_{i=1}^{m} \mu_i x_i \ , \ \mu_i \geq 0 \ , \ i = 1, 2, 3, \ldots m$$

where $\sum_{i=1}^{m} \mu_i = 1$

→ The set of all convex combinations of a finite number of points $x_1, x_2, \ldots x_m$ in E^n is a convex set that is, the set

$$S = \{ x \mid x = \sum_{i=1}^{m} \mu_i x_i \ , \ \mu_i \geq 0 \ , \ \sum_{i=1}^{m} \mu_i = 1 \}$$ is convex.

→ **Convex Hull:**

Suppose A is a set which is not convex. Then the smallest convex set which contains A is called the Convex Hull of A. i.e, the convex Hull of a set A is the intersection of all Convex sets which contain A.

For example: The convex Hull of the set

$$A = \{ (x,y) / x^2 + y^2 = 1 \}$$ is the set

$$S = \{ (x,y) / x^2 + y^2 \leq 1 \}.$$

Here S is convex and it contains A. Observe that S is the smallest convex set containing A. In this way we can say that the convex Hull of the point on the circumference of a circle is the circumference plus the interior of the circle. This is the smallest convex set containing the circumference.

→ The convex Hull of a finite number of points $x_1, x_2, \ldots x_m$ in E^n is the set of all convex combination of $x_1, x_2, \ldots x_m$.

i.e, the convex hull of $x_1, x_2, \ldots x_m$ is the set

$$S = \{ x \in E^n / x = \sum_{i=1}^{m} \mu_i x_i \ , \ \mu_i \geq 0 \ , \ \sum_{i=1}^{m} \mu_i = 1 \}.$$

→ The convex Hull of the finite number of points is called Convex polyhedron spanned by these points.

Optimization in two variables

We now initiate the study of Linear programming by taking a few examples involving only two variables and giving its mathematical formulation. Thereafter, we discuss its solution by Graphical Method and simultaneously give the intutive idea of its feasible and optimal solutions. Finally we describe bounded and unbounded sets through the graphical method for solving a linear programming problem in two variables.

* Different Areas of Applications of LPP.

We shall now discuss some important areas of applications of LPP.

(i) Manufacturing problems:

In these problems, we determine the number of units of different products which should be produced and sold by a firm when each product requires a fixed manpower, machine hours, labour hours per unit of the product, warehouse, space per unit of the output etc. in order to make maximum profit

(2). Diet problems:

In these problems, we determine the amount of different kinds of constituents/nutrients which should be included in a diet so as to minimize the cost of desired diet such that it contains a certain minimum amount of each constituent/nutrient.

(3) Investment problems:

In these problems, we determine the amount which should be invested in a number of fixed income securities to maximize the return on investment.

(4) Transportation problems:

In these problems, we determine a transportation schedule in order to find the cheapest way of transporting a product from plants or factories situated at different locations to different markets.

(5) Blending problems:

In these problems, we have to determine optimum amount of several constituents to be used in producing a set of products while determining the optimum quantity of each product to be produced.

(6) Advertising media selection problems:

In these problems, we find the optimum allocation of advertisements in different media in order to maximize the total effective audience / customers.

Basic concepts of LPP:

The term programming means planning and refers to a process of determining a particular plan of action from amongst several alternatives. The term 'linear' stands for indicating that all relationships involved in a particular problem are linear.

The general linear programming problem (LPP) calls for optimizing (maximizing or minimizing) a linear function for variables called the 'objective function' subject to a set of linear equations and/or inequalities called the constraints or restrictions.

General formulation of LPP:

Maximize or minimize

$$Z = c_1 x_1 + c_2 x_2 + c_3 x_3 + \cdots\cdots + c_n x_n$$

subject to

$$\left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots\cdots + a_{1n}x_n \ (\leq, =, \geq)\ b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots\cdots + a_{2n}x_n \ (\leq, =, \geq)\ b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots\cdots + a_{mn}x_n \ (\leq, =, \geq)\ b_m \end{array}\right\} \text{—①}$$

$$x_1, x_2, x_3, \cdots\cdots x_n \geq 0 \quad \text{—②}$$

where

(i) the linear function Z which is to be maximized or minimized is the objective function of the LPP.

(ii) $x_1, x_2, \cdots\cdots x_n$ are the decision variables.

(iii) the equations/inequations ① are the constraints of LPP.

(iv) In the set of constraints ① the expression $(\leq, =, \geq)$ means that each constraint may take any one of the three signs.

(v) c_j $(j = 1, 2, \cdots\cdots n)$ represents per unit profit (or) cost to the j^{th} variable.

(vi) the set of inequations ② is the set of non negative restrictions of the general linear programming problem.

(vii) b_i $(i=1,2,\ldots m)$ is the requirement (or) availability of the i^{th} constraint.

(viii) a_{ij} $(i=1,2\ldots m,\ j=1,2\ldots n)$ is Referred to as the technological coefficient.

Mathematical formulation of a linear programming problem:

The procedure for mathematical formulation of LPP consists of the following major steps:

Step 1: write down the decision variables and assign symbols $x_1, x_2, \ldots x_n$ to them. These decision variables are those quantities whose values we wish to determine.

Step 2: Formulate the objective function to be optimized (maximizing profit or minimizing cost) as a linear function of the decision variables.

Step 3: Formulate the other conditions of the form such as resource limitations, market, constraints, inter-relation between variables etc. as linear equations or inequations in terms of the decision variables.

Step 4: add the non-negativity constraint on decision variables, as in the physical problems, negative values of decision variables have no valid Interpretation.

The objective function, the set of constraints and the non-negative constraints together form a Linear programming problem (LPP).

Example

A furniture dealer deals in only two items, viz, tables and chairs. He has Rs 10,000 to invest and a space to store at most 60 pieces. A table costs him Rs 500 and a chair Rs 200. He can sell a table at a profit of Rs 50 and a chair at a profit of Rs 15. Assume that he can sell all the items that he buys. Formulate this problem as an LPP, so that he can maximize the profit.

Soln Let x and y denote the number of tables and chairs, respectively, (x and y are decision variables).

The cost of x tables = Rs $500x$.
The cost of y tables = Rs $200y$.

∴ The total cost of x tables and y chairs = Rs $500x + 200y$ which cannot be more than 10,000.

∴ Thus $500x + 200y \leq 10000$ (constraint)

Also $x + y \leq 60$ (Constraint)

as the dealer has the space to store at the most 60 items.

It is obvious that $x \geq 0, y \geq 0$ (non-negative restriction) as the number of chairs & tables cannot be negative.

profit on x tables $= 50x$

profit on y chairs $= 15y$

Hence, the profit on x tables and y chairs $=$ Rs. $50x + 15y$. (objective function)

obviously, dealer wishes to maximize the profit $Z = 50x + 15y$.

∴ The mathematical formulation of the LPP is

Maximize $Z = 50x + 15y$.

Subject to the constraints

$$5x + 2y \leq 100$$

$$x + y \leq 60$$

$$x \geq 0, y \geq 0.$$

→ A firm manufacturers two products A and B. One unit of product A needs 2 hours on machine I and 3 hours on machine II. One unit of the product B needs 3 hours on machine I and 1 hour on machine II. Daily capacity of machines I and II are 12 hours and 8 hours perday respectively. Profits obtained on selling one unit of A and one unit of B are Rs. 4/- and Rs 3/- respectively. The problem is to determine the daily level of products A and B so as to maximize the profit.

→ A manufacturer has 3 machines I, II and III installed in his factory. Machines I and II are capable of being operated for at the most 12 hours, whereas machine III must be operated atleast for 5 hours a day. He produces only two items, each requiring the use of the 3 machines.

The number of hours required for producing 1 unit of each of the items A and B on the 3 machines are given in the following table:

Items	Number of hours required on the machines		
	I	II	III
A	1	2	1
B	2	1	5/4

He makes a profit of Rs. 60 on item A and Rs. 40 on item B. Assuming that he can sell all that he produces, how many of each item should he produce so as to maximize his profit? Formulate the above problem as a linear programming problem.

Sol: Let x be the number of items A and y be the number of items B produced.

The total profit on the production is Rs. $60x+40y$

The objective of manufacturer is to maximize the profit.

∴ The formulation of the problem is

Maximize $Z = 60x + 40y$.

Subject to the constraints

$x+2y \le 12$ (constraint on machine I)
$2x+y \le 12$ (constraint on machine II)
$x+\frac{5}{4}y \ge 5$ (constraints on machine III)
$x \ge 0, y \ge 0.$ (non-negativity restriction)

→ A retired person wants to invest an amount of upto Rs 20,000. His broker recommends investing in two types of bonds A and B, bond A yielding 10% return on the amount invested and bond B yielding 15% return on the amount invested. After some consideration, he decides to invest atleast Rs 5000 in bond A and no more than Rs 8000 in bond B. He also wants to invest atleast as much in bond A as in bond B. What should his broker suggest if he wants to maximize his return on investments. Formulate LPP.

Sol^n: Let x be the amount (in Rs) invested in bond A and y be the amount (in Rs) invested in bond B.

$100 \to 10$
$x \to ?$
$\frac{x(10)}{100} = 0.10x$

His objective is to maximize his return on investment. i.e,

Maximize $Z = 0.10x + 0.15y$

subject to the constraints

$x+y \le 20000$ (sum of the investments)
$x \ge 5000$ (constraint on investment in bond A)
$y \le 8000$ (constraint on investment in bond B)
$x \ge y$ (Relation between investments)
$x \ge 0, y \ge 0$ (investment cannot be negative).

H.W. → A dealer wants to purchase a number of fans and sewing machines. He has only Rs. 5760 to invest and has space for atmost 20 items. A fan costs him Rs. 360 and a sewing machine Rs. 240. His expectation is that he can sell a fan at a profit of Rs. 22 and a sewing machine at a profit of Rs. 18. Assuming that he can sell all the items that he can buy, how should he invest his money in order to maximize his profit?

→ Vitamins A and B are found in two different foods F_1 and F_2. One unit of food F_1 contains 2 units of vitamin A and 3 units of vitamin B. One unit of food F_2 contains 4 units of vitamin A and 2 units of vitamin B. One unit of food F_1 cost Rs. 3 and one unit of food F_2 costs Rs. 2.50. The minimum daily requirement for a person of vitamin A and B is 40 and 50 respectively. Assuming that anything in excess of daily minimum requirement of vitamin A and B is not harmful, find out the mixture of food F_1 and F_2 at the minimum cost which meets the daily minimum requirement of vitamins A and B.

Sol: Let x = number of units of food F_1
y = number of units of food F_2

Number of units of vitamin A in x units of food F_1 and y units of food F_2 is $2x+4y$.

The minimum daily requirement of vitamin A is 40 units.

$\therefore \quad 2x+4y \geq 40.$

Similarly, the number of units of vitamin B in F_1 and F_2 is $3x+2y$.

$\therefore$ The daily minimum requirement of vitamin B is 50 units.

$\therefore \quad 3x+2y \geq 50.$

As the costs of one unit of F_1 and F_2 are Rs. 3 and Rs. 2.50 respectively.

$\therefore$ The total cost of purchasing x unit of food F_1 and y units of food F_2 (in Rs) is

$$Z = 3x+2.5y$$

which is the objective function.

$\therefore$ The mathematical formulation of the problem is

Minimize $Z = 3x+2.5y$

Subject to the constraints

$$2x+4y \geq 40$$
$$3x+2y \geq 50$$
$$x, y \geq 0.$$

→ A house wife wishes to mix two types of food F_1 and F_2 in such a way that the vitamin contents of the mixture contain atleast 8 units of vitamin A and 11 units of vitamin B, food F_1 costs Rs 60/kg and food F_2 costs Rs 80/kg. Food F_1 contains 3 units/kg of vitamin A and 5 units/kg of vitamin B while food F_2 contains 4 units/kg of vitamin A and 2 units/kg of vitamin B. formulate the above problem as a linear programming problem to minimize the cost of mixture.

→ Some Important Definitions Related with the general LPP:-

Solution: A set of values of decision variables $x_1, x_2, \ldots, x_n$ which satisfy all the constraints of a General LPP is called a solution to General LPP.

Feasible solution: Any solution to a General LPP which also satisfies the non-negative restrictions of the problem, is called a feasible solution to the General LPP.

Optimal feasible solution:

Any feasible solution which optimizes (minimizes or maximizes) the objective function of General LPP is called an optimal feasible solution to the General LPP. or simply optimal solution to the General LPP.

Note: The term optimum solution is also used for optimal solution.

Feasible Region: The common region determined by all the constraints and non-negativity restriction of an LPP is called a feasible region.

Convex region: A region is said to be convex, if the line segment joining any two arbitrary points of the region lies entirely with in the region.

– feasible region of an LPP always a convex-region.

→ The Graphical method of solving an LPP:

The graphical method of solving an LPP is used when there are two variables.
If the problem has three or more variables, the graphical method is not suitable.
In that case, a very powerful method called simplex method is used.

There are two techniques of solving an LPP by graphical method.
These are:
(1) Corner point method
(2) Iso profit method. (or) Iso-cost method.

(1) Corner Point Method:

The optimal solution to a LPP, if it exists, occurs at an extreme point (corner) of the feasible region.

The method comprises of the following steps:

(i) Find the feasible region of the LPP.
(ii) Find the co-ordinates of each vertex of the feasible region

The co-ordinates of the vertex can be obtained either by inspection or by solving the two equations of the lines intersecting at that point.

(iii) Evaluate the value of the objective function Z at each corner point obtained in step (ii).

If the problem is of maximization (minimization), the solution corresponding to largest (smallest) value of z is the optimal solution of the LPP and the value of z is the optimum value.

Note: finding all the corners is a long and tedious process for the problem having more constraints.

(2) Iso-profit or Iso-cost method

– This is an alternative and more general method of finding the optimal solution of an LPP.

In this method, we first give any suitable constant value, say z_1, to the objective function and draw the corresponding line of the objective function.

This line is called (equal) Iso-profit (or) Iso-cost line. Since every point on this line will give the same profit or cost z_1.

Various steps of the method are as follows:

(i) Find the feasible region of LPP.

(ii) Assign a constant value z_1 to z and draw the corresponding line of the objective function.

(iii) Assign another value z_2 to z and draw the corresponding line of the objective function.

(iv) If $z_1 < z_2$, $(z_1 > z_2)$, then in case of maximization (minimization) move the line P_1Q_1 corresponding to z_1 to the line P_2Q_2 corresponding to z_2 parallel to itself as far as possible, until the farthest point with in the feasible region is touched by this line. The co-ordinates of the point give maximum (minimum) value of the objective function. The extremum may be either a point or a straight line. If it is a corner point, this yields the optimal solution. If it is a straight line, we have infinite number of optimal solutions.

Note: In case of unbounded region, it either finds an optimal solution or declare an unbounded solution.

Unbounded solutions are not considered as optimal solutions. In real world problems unlimited profit or loss is not possible.

Problems

(1) Solve the following LPP graphically.

Maximize $z = 60x + 15y$ ——— ①

Subject to the constraints

$x + y \leq 50$ ——— ②

$3x + y \leq 90$ ——— ③

$x, y \geq 0$ ——— ④

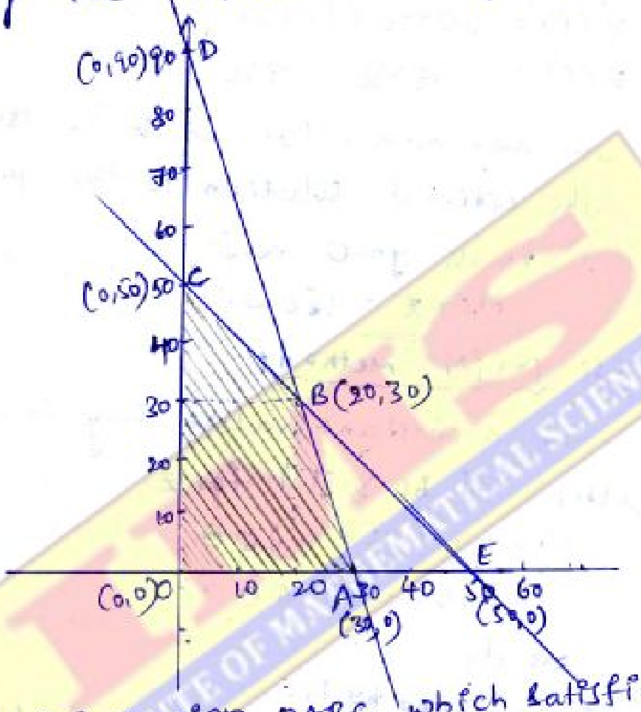
Solⁿ: Let us consider the constraint $x + y \leq 50$ as equality $x + y = 50$, represents a line which passes through $(0, 50)$ and $(50, 0)$ in xy-plane.

The origin $(0, 0)$ gives $0 + 0 = 0 < 50$.

$\therefore$ All the points below and on this line satisfy the inequation $x + y \leq 50$.

Let us consider the inequality $3x + y \leq 90$ as equality $3x + y = 90$, represents a line which passes through (0, 90), (30, 0) in xy-plane.

∴ All the points below and on this line satisfy the inequation $3x+y \le 90$.



The shaded region OABC which satisfies the equations (2), (3) and (4).

∴ The shaded region OABC is called as solution space or feasible region for the given problem.

solving $x+y=50$ and
$3x+y=90$.
we get $x=20$, $y=30$.
∴ The required point $B=(20,30)$.

(i) Corner-point Method:

The vertices of the feasible region are $O(0,0)$, $A=(30,0)$, $B=(20,30)$ and $C=(0,50)$.

(The vertices O, A, B, C are also known as corners or extreme points.)

The values of the objective function

$z = 60x + 15y$ at these vertices are

$z(O) = 0$

$z(A) = 60 \times 30 + 0 = 1800$

$z(B) = 60 \times 20 + 15 \times 30 = 1650$

$z(C) = 15 \times 50 = 750.$

The maximum value of z is 1800 at $A = (30, 0)$.

$\therefore$ The optimal solution to the problem is

$x = 30,\ y = 0$. and

$\text{Max } z = 1800.$

(ii) Iso-profit method:

Take a constant value say 600 (common multiple of 60 & 15) for z.

$\therefore 60x + 15y = 600$

$\Rightarrow 4x + y = 40$

we draw this line which is represented by P_1Q_1.

we take another constant value 1200 for z.

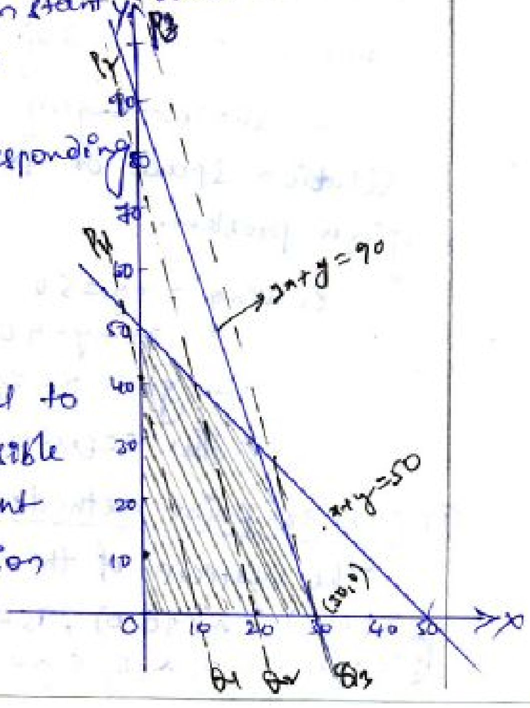
$\therefore 60x + 15y = 1200$

$\Rightarrow 4x + y = 80.$

Draw the corresponding line P_2Q_2.

As $600 < 1200$,

we move P_1Q_1 from P_1Q_1 to P_2Q_2 parallel to itself as far as possible until the farthest point within the feasible region is touched by the line



(15)

$\therefore$ The farthest point $A = (30, 0)$ with in the feasible region is touched by the line.

$\therefore$ At the point $A = (30, 0)$ we obtain the maximum value of z.

$\therefore$ optimal solution is $x = 30$, $y = 0$ and the optimal value of $z = 60 \times 30 + 0$

$= 1800$

(2) $\rightarrow$ Solve the following LPP Graphically.

Maximize $z = 4x_1 + 5x_2$

Subject to the constraints:

$2x_1 + 3x_2 \le 12$

$3x_1 + x_2 \le 8$

$x_1, x_2 \ge 0$.

Sol^n: Since every point which satisfies the condition $x_1 \ge 0$ & $x_2 \ge 0$ lies in the first quadrant only.

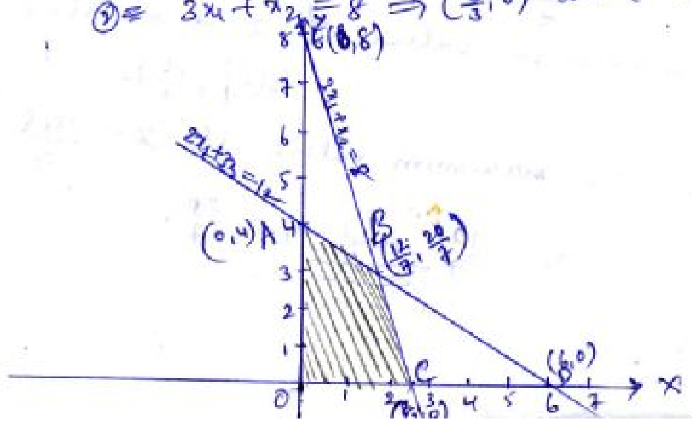
$\therefore$ The desired pair (x_1, x_2) is restricted to the points of the first quadrant only.

Let $2x_1 + 3x_2 = 12$ —— (1)

$3x_1 + x_2 = 8$ —— (2)

(1) $\equiv$ $2x_1 + 3x_2 = 12 \Rightarrow (6, 0)$ and $(0, 4)$

(2) $\equiv$ $3x_1 + x_2 = 8 \Rightarrow (\frac{8}{3}, 0)$ and $(0, 8)$



The set of points satisfying

$x_1 \geq 0$, $x_2 \geq 0$ and $2x_1 + 3x_2 \leq 12$

is represented by the shaded area given by the Δ^{le} AOD.

Similarly, the set of points satisfying

$x_1 \geq 0$, $x_2 \geq 0$ and $3x_1 + x_2 \leq 8$ is represented by the shaded area given by the Δle COE.

The feasible region or the solution space is the area of the graph which contains all pair of values that satisfy all the constraints.

The feasible region is bounded by the two axes and two lines $2x_1 + 3x_2 = 12$ and $3x_1 + x_2 = 8$ and it is the common shaded portion OABC.

$\therefore$ The four corners or extreme points of the polygon are

$O(0,0)$, $A(0,4)$, $B\left(\frac{12}{7}, \frac{20}{7}\right)$, $C\left(\frac{8}{3}, 0\right)$

The values of the objective function $z = 4x_1 + 5x_2$ at these extreme points are

$z(O) = 0$

$z(A) = 20$

$z(B) = \frac{148}{7}$

$z(C) = \frac{32}{3}$

The maximum value of z is at the point $B\left(\frac{12}{7}, \frac{20}{7}\right)$.

and the maximum value is $z = \frac{148}{7}$.

$\therefore x_1 = \frac{12}{7}$ and $x_2 = \frac{20}{7}$.

→ Solve the following LPP Graphically

→ Max $Z = 3x_1 + 4x_2$

Subject to

$4x_1 + 2x_2 \le 80$

$2x_1 + 5x_2 \le 180$

$x_1, x_2 \geq 0$.

→ (SFS 2007) Max $Z = 5x_1 + 7x_2$

Subject to

$x_1 + x_2 \le 4$

$3x_1 + 8x_2 \le 24$

$10x_1 + 7x_2 \le 35$.

$x_1, x_2 \geq 0$.

→ Max $Z = 2x + y$

Subject to

$5x + 10y \le 50$

$x + y \geq 1$

$y \le 4$

$x - y \le 0$

$x, y \geq 0$.

→ Max $Z = 22x_1 + 18x_2$

Subject to

$360x_1 + 240x_2 \le 5760$

$x_1 + x_2 \le 20$

$x_1, x_2 \geq 0$

(3) → A house wife wishes to mix together two kinds of food, I & II, in such a way that the mixture contains atleast 10 units of vitamin A, 12 units of vitamin B and 8 units of vitamin C. The vitamin contents of one kg of food is given below.

	vitamin A	vitamin B	vitamin C
Food I	1	2	3
Food II	2	2	1

One kg of food I costs Rs 6 and 1kg of food II costs Rs 10. formulate the above problem as linear programming problem to find the least cost of the mixture which will produce the diet.

Soln: Let the mixture contain x kg of food I and y kg of food II. The formulation of the above problem is as follows.

Minimize $Z = 6x + 10y$.

Subject to the constraints

$x + 2y \geq 10$

$2x + 2y \geq 12$

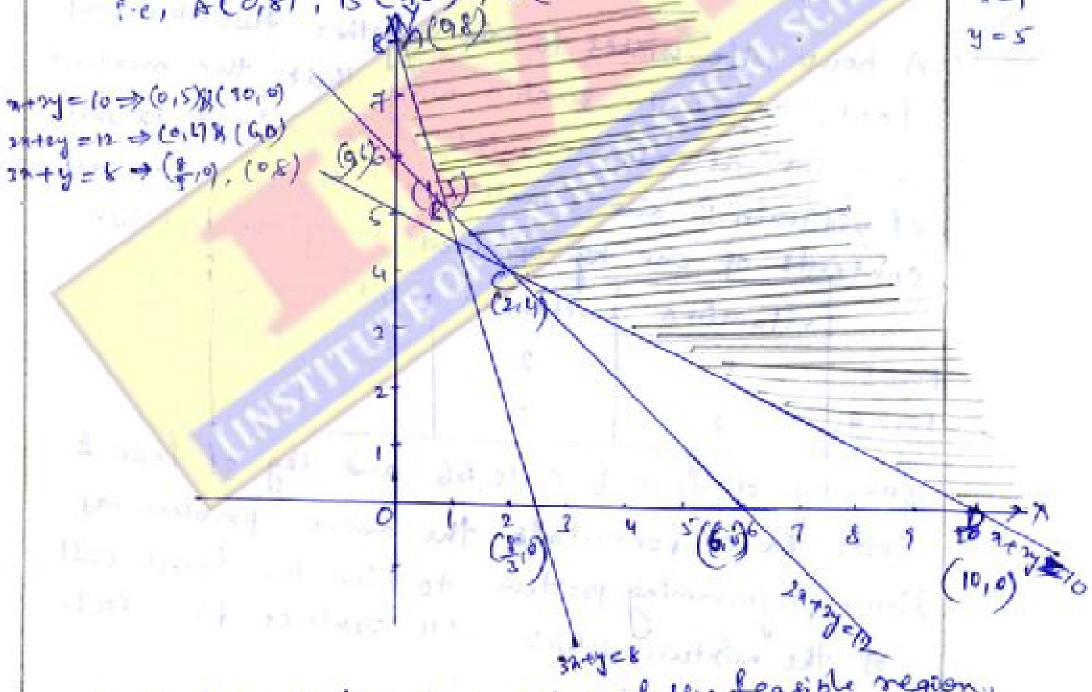
$3x+y \geq 8$

$x, y \geq 0$

we draw the lines $x+2y=10$, $2x+2y=12$ & $3x+y=8$.
Since each of the constraints is greater than or equal to type, the points (x,y) satisfying all of them will form the region that falls towards the right of each of these straight lines.
Here the feasible region is open with vertices

$3x+2y=12$
$x+y=6$
$3x+y=8$
$-2x=-2$
$x=1$
$y=5$

A, B, C and D.
i.e, A(0,8), B(1,5), C(2,4) and D(10,0).



Note that (0,0) is not a point of the feasible region.
The values of the objective function $Z = 6x+10y$ at the extreme points are

$Z(0,8) = 80$
$Z(1,5) = 56$
$Z(2,4) = 52$
$Z(10,0) = 100$
The minimum value of Z is at the

point C (2,4). and the minimum value is

$Z = 52$.

$\therefore x = 2$; $y = 4$

Iso-cost Method:

we give a constant value, say 30, to Z and draw the line P_1Q_1 corresponding to $6x+10y=30$. Now, we give another value 60 to Z and draw the line P_2Q_2 corresponding to $6x+10y=60$. As $30<60$, we move from P_2Q_2 to P_1Q_1 parallel to itself until the farthest point C of the feasible region is touched by this line. The point C so obtained gives the optimal value of Z. The point C is the intersection of the equations

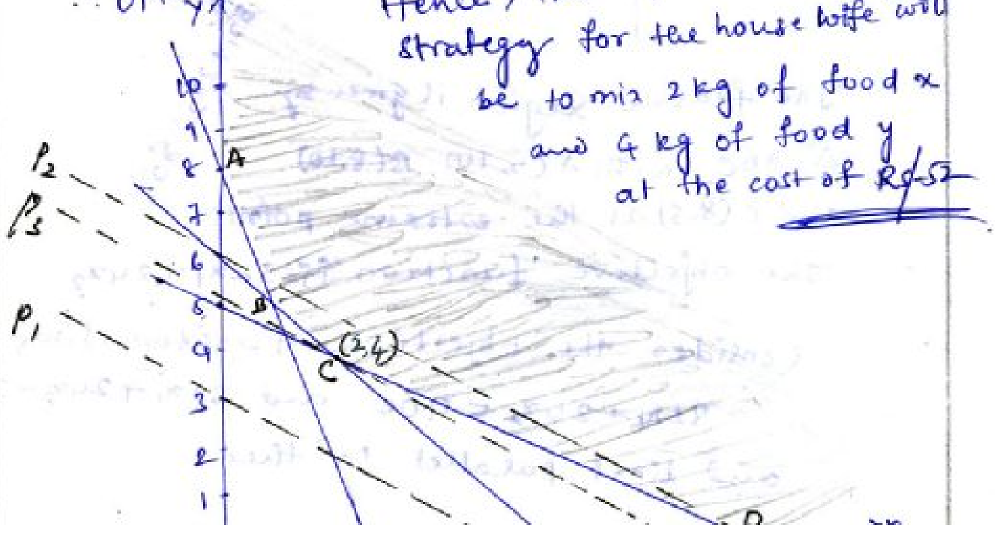
$x+2y=10$ and $2x+2y=12$.

solving these equations, we find co-ordinates of C as (2,4).

$\therefore$ The optimal value of $Z = 6\times2+10\times4 = 52$.

$\therefore$ optimal solution is $x=2$ $y=4$

Hence, the optimal mixing strategy for the house wife will be to mix 2 kg of food x and 4 kg of food y at the cost of Rs 52



④ → Minimize $z = 32x_1 + 24x_2$

Subject to the constraints

$x_1 \leq 8$

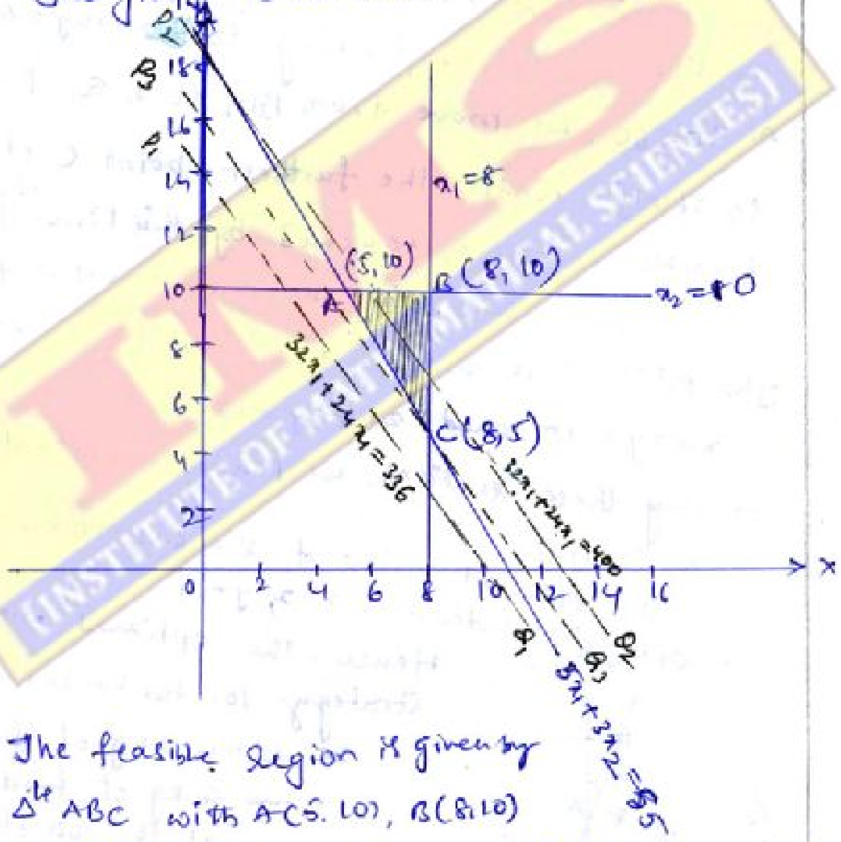
$x_2 \leq 10$

$5x_1 + 3x_2 \geq 55$

$x_1, x_2 \geq 0.$

Sol^n: Let $x_1 = 8$, $x_2 = 10$ and $5x_1 + 3x_2 = 55 \Rightarrow (0, \frac{55}{3})$ and $(11, 0)$

The graph is as shown in the figure.



The feasible region is given by Δ^{le} ABC with A(5, 10), B(8, 10) and C(8, 5) as the extreme points.

The objective function is $Z = 32x_1 + 24x_2$

Consider the objective function lines $32x_1 + 24x_2 = 400$ and $32x_1 + 24x_2 = 336$ and lines parallel to them.

we find that the line corresponding to the minimum value of z passes through the point C.

Therefore, minimum z is obtained at the point $C(8,5)$. and minimum value of $z = 376$.

We can verify minimum value of $z = 376$ by calculating the value of z at the extreme points.

At $A(5,10)$ $z = 400$

At $B(8,10)$ $z = 496$

At $C(8,5)$ $z = \underline{376}$.

$\therefore x = 8$ and $y = 5$

and $\min z = 376$

→ Solve the following LPP Graphically:

→ Minimize $z = 3x_1 + 2.5x_2$

subject to the constraints

$2x_1 + 4x_2 \geq 40$

$3x_1 + 2x_2 \geq 50$

$x_1, x_2 \geq 0$.

→ Minimize $z = 4x + 2x_2$

Subject to the constraints:

$x_1 + 2x_2 \geq 2$

$3x_1 + x_2 \geq 3$

$4x_1 + 3x_2 \geq 6$.

$x_1, x_2 \geq 0$.

→ Minimize $z = 20x_1 + 10x_2$

Subject to the constraints

$x_1 + 2x_2 \leq 40$

$3x_1 + x_2 \geq 30$

$4x_1 + 3x_2 \geq 60$

$x_1, x_2 \geq 0$.

→ Minimize $z = -x_1 + 2x_2$

Subject to the constraints

$-x_1 + 3x_2 \leq 10$

$x_1 + x_2 \leq 6$

$x_1 - x_2 \leq 2$

$x_1, x_2 \geq 0$

— In the previous problems, we have obtained a feasible region in each case. This feasible region in each case represents a convex set. Convex sets may be bounded or unbounded.

— Refer to the graph of the problems discussed in (1) and (2). The convex polyhedron OABC bounded.

In problem (3), the feasible region is not bounded. i.e, it is unbounded, because x_1 and x_2 both can go upto infinity. Since the problem is of minimization, the minimum value of Z exists, but Max $Z \to \infty$.

In problem (4), the feasible region is the Δ^{le} ABC and it is bounded.

Now consider an example where the feasible region is unbounded and the maximum Z is infinite.

→ Max $Z = 2x_1 + 2x_2$

Subject to the constraints

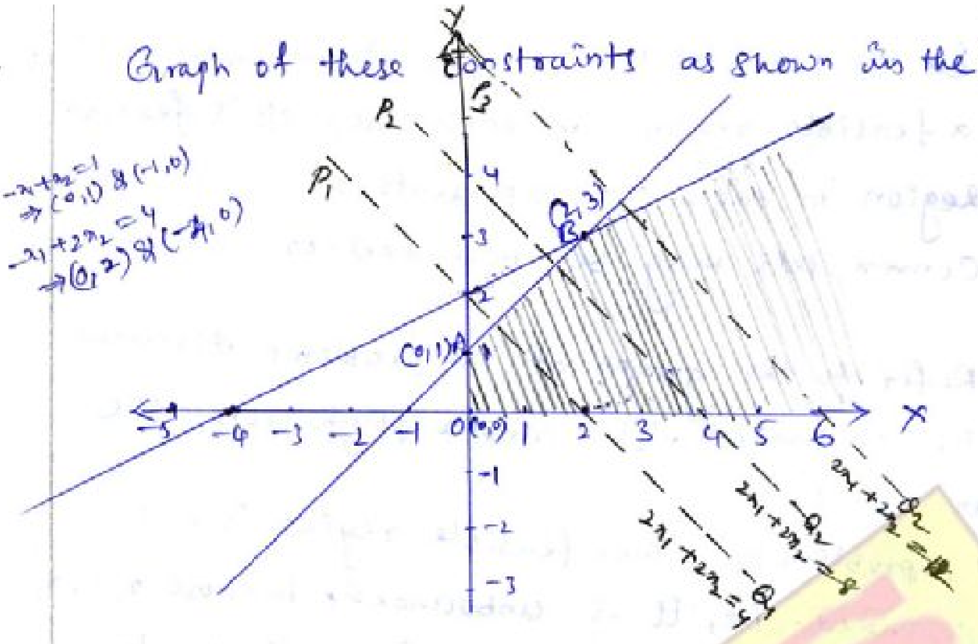
$$x_1 - x_2 \geq -1$$
$$-x_1 + 2x_2 \leq 4$$
$$x_1, x_2 \geq 0.$$

Solⁿ: First constraint can be written as
$$-x_1 + x_2 \leq 1.$$

∴ we have Max $Z = 2x_1 + 2x_2$

Subject to $-x_1 + x_2 \leq 1$
$-x_1 + 2x_2 \leq 4$
$x_1, x_2 \geq 0.$

Graph of these constraints as shown in the figure:



The shaded portion is the feasible region and is unbounded. The extreme points are $O(0,0)$, $A(0,1)$ and $B(2,3)$.

At these points, the value of Z is finite. As x_1 and x_2 both can approach to infinity, therefore the value of Z approaches to ∞.

$\therefore$ we say that the given problem is unbounded.

The objective function lines indicate that $Z \to \infty$.

H.W. $\rightarrow$ Maximize $Z = 3x_1 + 2x_2$

Subject to the constraints

$x_1 - x_2 \leq 1$

$x_1 + x_2 \geq 3$

$x_1 \geq 0$ and $x_2 \geq 0$

H.W. $\rightarrow$ Max $Z = x_1 + 0.75x_2$

Subject to the constraints:

$x_1 - x_2 \geq 0$

$-x_1 + 2x_2 \leq 2$

$x_1, x_2 \geq 0$.

→ Now consider Linear programming problem which has no feasible region. In such a situation we say that the given problem is infeasible.

No feasible solution:

when there is no feasible region formed by the constraints in conjunction with non-negativity conditions, then no solution to the linear programming problem exists.

→ Maximize $z = 3x_1 - 2x_2$

Subject to the constraints

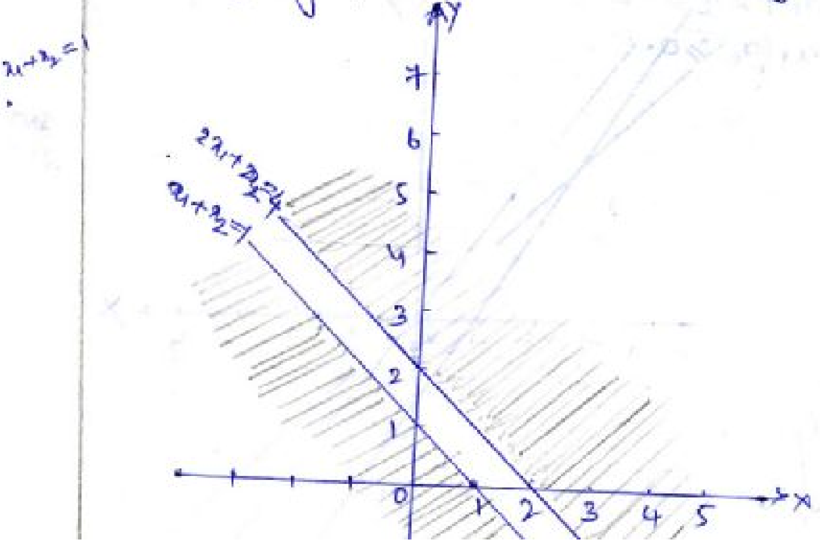
$x_1 + x_2 \leq 1$

$2x_1 + 2x_2 \geq 4$

$x_1, x_2 \geq 0.$

Sol^n: Clearly, the desired number pairs (x_1, x_2) lies in the first quadrant only.

its graph is given in the figure.



In the graph, there is no point (x_1, x_2) which satisfy both the constraints. In this case the two constraints are inconsistent. There is no feasible region and hence the given problem is infeasible.

→ Solve the following LPP:

Maximize $z = x_1 + x_2$

Subject to the constraints

$x_1 + x_2 \le 1$

$-3x_1 + x_2 \geq 3$

$x_1 \geq 0,\ x_2 \geq 0.$

H.W. → Maximize $z = 5x_1 + 2x_2$

subject to the constraints

$x_1 + x_2 \le 2$

$3x_1 + 3x_2 \geq 12$

$x_1, x_2 \geq 0.$

→ Consider another example where the constraints are consistent but still the problem is infeasible.

→ Maximize $z = 3x_1 - 2x_2$

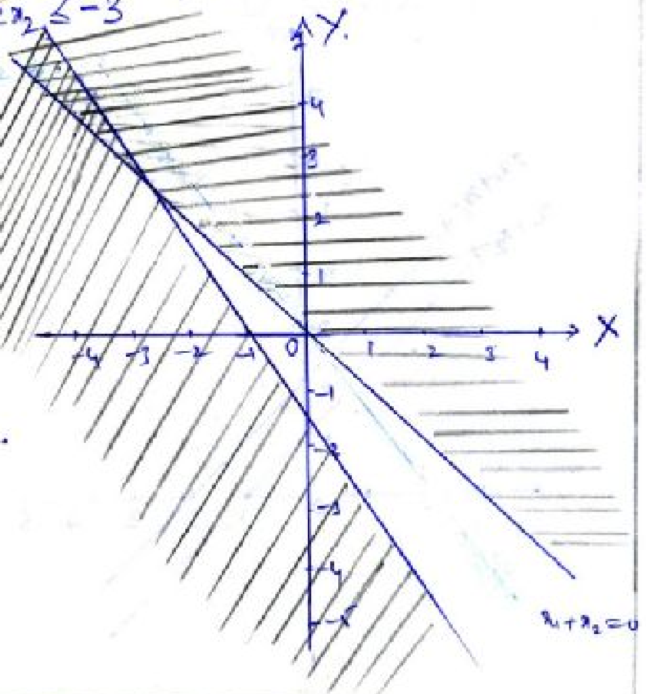
Subject to the constraints

$x_1 + x_2 \geq 0$

$3x_1 + 2x_2 \le -3$; $x_1, x_2 \geq 0$

Sol:

Graph of the given constraints as shown in the figure.



$x_1+x_2=0$ (0,0)

$3x_1+2x_2=-3$

$\Rightarrow (0,-\frac{3}{2})$ & $(-1,0)$

In this case there is a common region which satisf the constraints but doesnot satisfy the non-negativity restriction $x_1 \geq 0$, $x_2 \geq 0$.

Hence there is no feasible region and the given problem is infeasible.

H.W → Max $z = 10x_1 + 15x_2$

Subject to the constraints

$x_1 \geq 5$

$x_2 \leq 10$

$x_1 + 2x_2 \leq 10$

$x_1, x_2 \geq 0$

→ **An Aternative optimum Solution.**

A linear programming problem may have more than one ptimal solution. This happens when the objective function line is parallel to a binding constraint (i.e, a constraint that is satisfied in the equality sense by the optimal Solut

for example:

Maximize $z = 15x_1 + 30x_2$

subject to the constraints

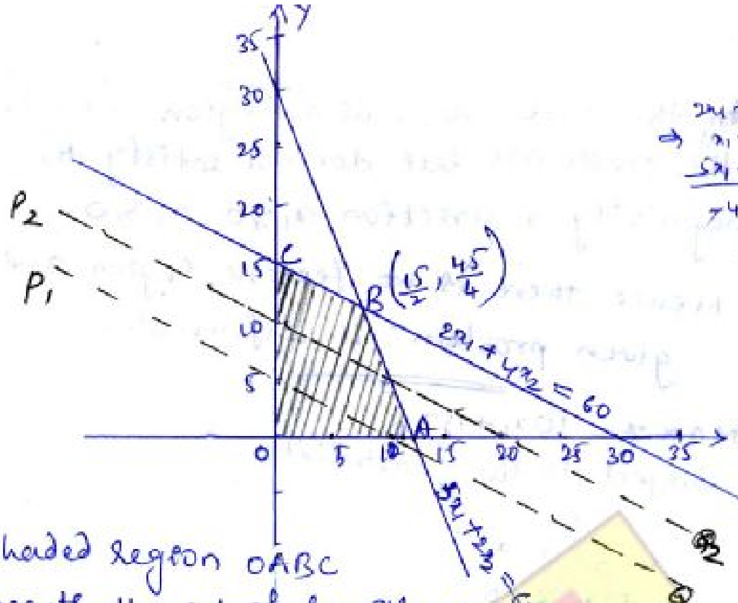
$2x_1 + 4x_2 \leq 60$

$5x_1 + 2x_2 \leq 60$.

$x_1, x_2 \geq 0$.

Solⁿ: Graph of the given constraints as shown in the figure.

$2x_1 + 4x_2 = 60$
$\Rightarrow (0,15) \& (30,0)$
$5x_1 + 2x_2 = 60$
$\Rightarrow (0,30) \& (12,0)$



$2x_1 + 4x_2 = 60$
$\Rightarrow x_1 + 2x_2 = 30$
$5x_1 + 2x_2 = 60$
$-4x_1 = -30$
$\boxed{x_1 = \frac{15}{2}}$
$4x_2 = 45$
$\boxed{x_2 = \frac{45}{4}}$

The shaded region OABC represents the set of feasible solutions.

To locate a point on the feasible region which maximizes the objective function.

If we give a constant value, say 150 to z, then the objective function may be written as

$15x_1 + 30x_2 = 150$.

The line is given by P_1Q_1.

Now, we give another value 300 to z,

i.e, $15x_1 + 30x_2 = 300$.

The line is P_2Q_2.

The lines P_1Q_1 and P_2Q_2 both lie within the feasible region.

Increasing the value of z and drawing different lines, we find that the objective function line coincides with the line CB.

Hence every point on the line segment CB of the feasible region provides the optimal value of z.

The extreme points of CB are

$C(0,15)$ and $B\left(\frac{15}{2}, \frac{45}{4}\right)$.

and the value of Z at $(0,15)$ is

$$15(0) + 30(15) = 450$$

value of Z at $\left(\frac{15}{2}, \frac{45}{4}\right)$ is

$$15\left(\frac{15}{2}\right) + 30\left(\frac{45}{4}\right) = 450.$$

i.e, maximum value of Z at both the extreme point is same and it is 450. If we take any point on the line segment joining C and B, then the value of Z at that point will also be the same.

with the help of the extreme points. also we can see that the optimal value of Z is 450.

At $O(0,0)$, $Z = 0$

At $A(12,0)$, $Z = 180$

At $B\left(\frac{15}{2}, \frac{45}{4}\right)$, $Z = 450$

At $C(0,15)$, $Z = 450$.

$\therefore$ Maximum value of $Z = 150$ at both the points $(0,15)$ and $\left(\frac{15}{2}, \frac{45}{4}\right)$

HW $\rightarrow$ Max $Z = 2x_1 + 2x_2$

subject to the constraints

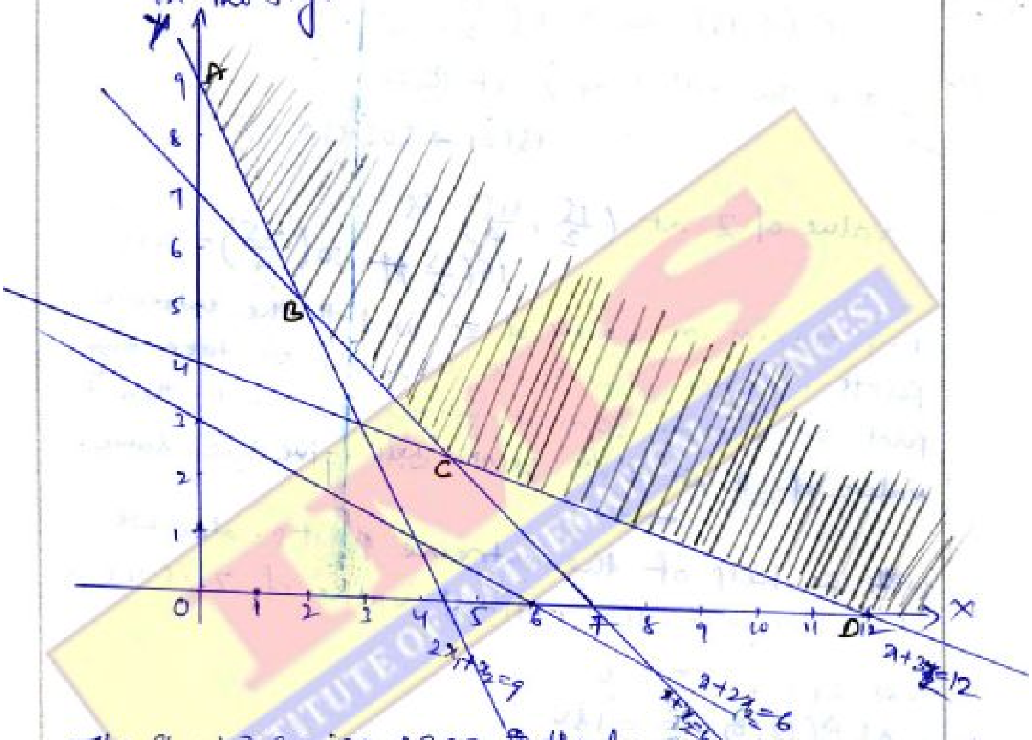
$$x_1 + x_2 \le 4$$
$$x_1 + 2x_2 \le 6$$
$$x_1 \le 3.$$
$$x_1, x_2 \ge 0.$$

$\rightarrow$ Minimize $Z = 5x_1 + 8x_2$

subject to the constraints

$$x_1 + 2x_2 \ge 6$$
$$x_1 + x_2 \ge 7$$
$$x_1 + 3x_2 \ge 12$$
$$2x_1 + x_2 \ge 9 \; ; \; x_1, x_2 \ge 0.$$

Solⁿ The graph of the given constraints as shown in the fig.



The shaded region ABCD is the feasible region corresponding to the above constraints. The constraint $x_1+2x_2 \geq 6$ doesnot intersect the feasible region.

If we remove the constraint $x+2x_2 \geq 6$, the feasible region remains unchanged. Such a constraint is called Redundant Constraint. Hence, the constraint $x_1+2x_2 \geq 6$ is redundant.

The objective function is $Z = 5x_1+8x_2$

consider the objective function lines

$5x_1+8x_2 = 60$, it is given by P_1Q_1

and $5x_1+8x_2 = 40$, it is given by P_2Q_2.

As $40 < 60$, we move from P_1Q_1 to P_2Q_2 parallel to itself until the point C of the feasible region is touched by this line.

The point so obtained gives the optimal value of Z.

$\therefore$ Minimum Z is obtained at the point C(4.5, 2.5).

and the minimum value of $Z = 42.50$

Optimization in more than two variables

We now extend the method of a linear program problem in more than two variables and further try the same methods for a L.P.P in more than three variables. This will lead us to discover the limitations of the graphical method in solving the LPP in more than three variables and hence the need for a more scientific algebraic method which is known as Simplex method to be discussed later.

Mathematical formulation in three variables:

→ A small scale industrialist produces three types of machine components M_1, M_2 and M_3 made of steel and brass. The amounts of steel, brass required and the number of man-weeks for each component and to manufacture and assemble of labour required to manufacture and assemble one unit of each component are as follows.

	M_1	M_2	M_3	Availability.
steel	6	5	3	100 kg
Brass	3	4	9	75 kg
Man-weeks	1	2	1	20 weeks.

This labour is restricted to 20 man-weeks, steel is restricted to 100 kg per week and the brass to 75 kg per week. The industrialist's profit on each unit of M_1, M_2 and M_3 is Rs/- 6, Rs/- 4 and Rs/- 7 respectively. Give its mathematical formulation as a LPP such that the total profit is maximum.

Sol^n: Suppose x_j be the number of units of component M_j produces per week, $j = 1, 2, 3$.

we want to find the values of x_1, x_2, x_3 which maximize the total profit.

Since the amount of steel, amount of brass and the labour are limited, we cannot arbitrarily increase the out put of any component.

Consider first the restriction imposed by the availability of steel.

Amount of steel used is

$6x_1 + 5x_2 + 3x_3$ per week.

because, 6 kg. are required for each unit of component M_1, 5 kg are required for each unit of component M_2 and 3 kg. for each unit of component M_3.

Since the total amount of steel available is restricted to 100 kg.

$\therefore 6x_1 + 5x_2 + 3x_3 \le 100$.

Similarly for brass,

$3x_1 + 4x_2 + 9x_3 \le 75$.

As the labour is restricted to 20 man-weeks.

$\therefore \quad x_1 + 2x_2 + x_3 \le 20$

Since we cannot produce negative quantities.

$\therefore x_1 \geq 0, \; x_2 \geq 0$ and $x_3 \geq 0$

Ex :—

slack variables ($\le$) $3x_1 + 8x_2 \le 10$

x_3 slack variables converted into

$3x_1 + 8x_2 + x_3 = 10$

surplus variables ($\geq$)

$2x_1 + 3x_2 \geq 11$

x_3 surplus variable converted into

$2x_1 + 3x_2 - x_3 = 11.$

Note :- (Unrestricted variables)

~~A~~ variable which is unrestricted in sign (i.e positive, negative or zero) is equivalent to the difference between two non-negative variables.

Thus if x_j is unrestricted in sign, it can be replaced by $(x_j' - x_j'')$, where x_j' and x_j'' are both non-negative

i.e $x_j = x_j' - x_j''$, where $x_j' \geq 0$ and $x_j'' \geq 0$

* Generating an initial feasible solution :-

After all linear constraints (with non-negative right-hand sides) have been transformed into equalities by introducing slack and surplus variables where necessary, add a new variable, called an artificial variable, to the left-hand side of each constraint equation that does not contain a slack variable. Each constraint equation will then contain either one slack variable or one artificial variable. A non-negative initial solution to this new set of constraints is obtained by setting each slack variable and each artificial variable equal to the right-hand side of equation in which it appears and setting all other variables, including the surplus variables, equal to zero.

Ex: The constraints

$x_1 + 2x_2 \leq 3$

$4x_1 + 5x_2 \geq 6$

$7x_1 + 8x_2 = 15$, is transformed into a system of equations by adding a slack variable x_3, to the left-hand side of the first constraint and subtracting a surplus variable x_4, from the LHS of the second constraints.

The new system is

$$\left.\begin{aligned} x_1+2x_2+x_3 &= 3 \\ 4x_1+5x_2-x_4 &= 6 \\ 7x_1+8x_2 &= 15 \end{aligned}\right\} \text{———} (1)$$

If now artificial variables x_5 and x_6 are respectively added to the L.H.S of the last two constraints in system (1), the constraints without a slack variable, the result is

$$\left.\begin{aligned} x_1+2x_2+x_3 &= 3 \\ 4x_1+5x_2-x_4+x_5 &= 6 \\ 7x_1+8x_2+x_6 &= 15 \end{aligned}\right\} \text{———} (2)$$

A non-negative initial solution to the system (2) is $x_3 = 3$, $x_5 = 6$, $x_6 = 15$ and $x_1 = x_2 = x_4 = 0$.

* **Penalty costs:**

The introduction of slack and surplus variables alters neither the nature of the constraints nor the objective.

Accordingly, such variables are incorpo-rated into the objective function with zero coefficients.
Artificial variables, however, do change the nature of the constraints. Since they added to only one side of an inequality, the new system is equivalent to the old system of constraints if and only if the artificial variables are zero. To guarantee such assignments in the optimal solution (in contrast to the initial solution), artificial variables are incorporated into the objective function with very large positive coefficients in a minimization program (or) very large negative coefficients in a maximization program. These coefficients, denoted by either M or −M, where M is under-stood to be a large positive number, represent the (severe) penalty incurred in making a unit assignment to the artificial variables.

* Matrix form of L.P.P. in the standard form:

~~A~~ linear programming is in standard form if the constraints are all modeled as equalities and if one feasible solution is known.

In matrix notation, standard form is

$$\text{optimize (max or min) } Z = C^T X$$
$$\text{subject to: } AX = B$$
$$\text{and } X \geq 0$$

where X is the column vector of unknowns, including all slack, surplus, and artificial variables; C^T is ~~the row~~ vector of the corresponding costs. A is the coefficient matrix of the constraints equations; and B is the column vector of the right-hand sides of the constraints equations.

If X_0 denotes the vector of slack and artificial variables only, then the initial feasible solution is given by $X_0 = B$, where it is understood that all variables in X not included in X_0 are assigned zero values.

Problems :

① → put the following program in standard form Maximize $Z = 80x_1 + 60x_2$

subject to

$$0.20x_1 + 0.32x_2 \leq 0.25$$

$$x_1 + x_2 = 1$$

and $x_1 \geq 0, x_2 \geq 0$.

and hence obtain the initial feasible solution.

Sol: To convert the first constraint into an equality, add a slack variable x_3 to the LHS.

Since the second constraint equation, does not contain a slack variable, add an artificial variable x_4 to its LHS. Both new variables are included in the objective function, the slack variable with a zero cost coefficient and the artificial variable with a very large negative cost coefficient,

yielding the program

Maximize $Z = 80x_1 + 60x_2 + 0x_3 - Mx_4$

Subject to

$$0.20x_1 + 0.32x_2 + x_3 = 0.25$$

$$x_1 + x_2 + x_4 = 1$$

and $x_i \geq 0$ $i = 1, 2, 3, 4$.

This program is in standard form, with an initial feasible solution

$x_3 = 0.25$, $x_4 = 1$ and

$x_1 = x_2 = 0$.

② → put the following program in standard form

HW

$$\text{Max } z = 5x_1 + 2x_2$$

subject to $6x_1 + x_2 \geq 6$

$4x_1 + 3x_2 \geq 12$

$x_1 + 2x_2 \geq 4$

and $x_1 \geq 0$, $x_2 \geq 0$.

and hence obtain an initial feasible solution.

③ → Put the following program in standard matrix form.

$$\text{Max } z = x_1 + x_2$$

Subject to $x_1 + 5x_2 \leq 5$

$2x_1 + x_2 \leq 4$.

$x_1 \geq 0$, $x_2 \geq 0$.

Solⁿ: Adding slack variables x_3 and x_4 respectively to the LHS of the constraints, and including these new variables with zero cost coefficients in the objective function.

we have

$$\max z = x_1 + x_2 + 0x_3 + 0x_4$$

$$\text{subject to} \quad \left.\begin{array}{l} x_1 + 5x_2 + x_3 = 5 \\ 2x_1 + x_2 + x_4 = 4 \end{array}\right\} \text{---} (1)$$

$$x_i \geq 0, \; i = 1, 2, 3, 4.$$

Since each constraint equation has a slack variable, no artificial variables are required.

An initial feasible solution is

$x_3 = 5$, $x_4 = 4$; $x_1 = x_2 = 0$

System (1) is in the standard matrix form if we define

$$X = [x_1, x_2, x_3, x_4]^T ; \; C = [1, 1, 0, 0]^T \Rightarrow C^T = [1\,1\,0\,0]$$

$$A = \begin{bmatrix} 1 & 5 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix}, \; B = \begin{bmatrix} 5 \\ 4 \end{bmatrix} \quad X_0 = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}$$

→ put the following program in standard matrix form

$$\min Z = x_1 + 2x_2 + 3x_3$$

$$\text{Subject to} \quad 3x_1 + 4x_3 \leq 5$$

$$5x_1 + x_2 + 6x_3 = 7$$

$$8x_1 + 9x_3 \geq 2$$

$$x_1 \geq 0, \; x_2 \geq 0, \; x_3 \geq 0$$

Sol: Adding a slack variable x_4 to the LHS of the first constraint, subtracting a surplus variable x_5 from the LHS of the third constraint, and then adding an artificial variable x_6 only to the LHS of the third constraint.

We have

$$\min z = x_1 + 2x_2 + 3x_3 + 0x_4 + 0x_5 + Mx_6$$

Subject to
$$\left.\begin{aligned} 3x_1 + 4x_3 + x_4 &= 5 \\ 5x_1 + x_2 + 6x_3 &= 7 \\ 8x_1 + 9x_3 - x_5 + x_6 &= 2 \end{aligned}\right\} \text{—(1)}$$

$x_i \geq 0$, $i = 1, 2, 3, 4, 5, 6$.

This program is in standard form, with an initial feasible solution $x_4 = 5$, $x_2 = 7$, $x_6 = 2$, $x_1 = x_3 = x_5 = 0$.

System (1) has standard matrix form, if we define

$$X = [x_1, x_2, x_3, x_4, x_5, x_6]^T \quad C = [1, 2, 3, 0, 0, M]^T$$

$$A = \begin{bmatrix} 3 & 0 & 4 & 1 & 0 & 0 \\ 5 & 1 & 6 & 0 & 0 & 0 \\ 8 & 0 & 9 & 0 & -1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 5 \\ 7 \\ 2 \end{bmatrix} \quad X_0 = \begin{bmatrix} x_4 \\ x_2 \\ x_6 \end{bmatrix}$$

Note:- In this case, x_2 can be used to generate the initial solution rather than adding an artificial variable to the second constraint to achieve the same result.

In general, whenever a variable appears in one and only one constraint equation, and there with a +ve coefficient, that variable can be used to generate part of the initial solution by first dividing the constraint equation by the +ve coefficient and then setting the variable equal to the RHS of the equation; an artificial variable need not be added to the equation.

2005 Put the following program in standard form.

minimize: $z = 25x_1 + 30x_2$

Subject to: $4x_1 + 7x_2 \geq 1$

$8x_1 + 5x_2 \geq 3$

$6x_1 + 9x_2 \geq -2$

and hence obtain an initial feasible solution.

Sol:

Since both x_1 and x_2 are unrestricted,

we write $x_1 = x_1' - x_1''$

$x_2 = x_2' - x_2''$

where all four new variables are required to be non negative.

Substituting these quantities in to the given program and then multiply the last constraint by -1 to force a non-ve RHS,

we obtain the equivalent program:

minimize: $z = 25x_1' - 25x_1'' + 30x_2' - 30x_2''$.

Subject to: $4x_1' - 4x_1'' + 7x_2' - 7x_2'' \geq 1$

$8x_1' - 8x_1'' + 5x_2' - 5x_2'' \geq 3$

$-6x_1' + 6x_1'' - 9x_2' - 9x_2'' \leq 2$

with : all variables non -ve.

This program is converted into standard form by subtracting surplus variables x_3 and x_4 respectively, from the left-hand sides of the first two constraints; adding a slack variable x_5 to the LHS of the third constraint; and then adding artificial variables x_6 and x_7 respectively, to the LHS of the first two constraints.

we have

minimize: $Z = 25x_1' - 25x_1'' + 30x_2' - 30x_2'' + 0x_3 + 0x_4 + 0x_5 + Mx_6 + Mx_7$.

Subject to: $4x_1' - 4x_1'' + 7x_2' - 7x_2'' - x_3 \qquad + x_6 = 1$

$8x_1' - 8x_1'' + 5x_2' - 5x_2'' \qquad - x_4 \qquad + x_7 = 3$

$-6x_1' + 6x_1'' + 9x_2' + 9x_2'' \qquad + x_5 \qquad = 2$

with : all variables non -ve.

An initial feasible solution to the program in standard form is

$x_6 = 1,\ x_7 = 3,\ x_5 = 2,\ x_1' = x_1'' = x_2' = x_2'' = x_3 = x_4 = 0$

H.W → Reduce the following problem to the standard form:

maximize $Z = 3x_1 + 5x_2 + 8x_3$

Subject to $2x_1 - 5x_2 \le 6$

$3x_1 + 2x_2 + x_3 \ge 5$

$3x_1 + 4x_3 \le 3$

with $x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0$.

H.W → Express the following L.P.P in the standard form

Minimize $Z = 3x_1 + 2x_2 + 5x_3$

Subject to: $-5x_1 + 2x_2 \le 5$

$2x_1 + 3x_2 + 4x_3 \ge 7$

$2x_1 + 5x_3 \le 3$

with $x_1, x_2, x_3 \ge 0$

H.W → Convert the following L.P.P. to standard form:

Maximize $Z = 3x_1 - 2x_2 + 4x_3$

Subject to: $x_1 + 2x_2 + x_3 \le 8$

$2x_1 - x_2 + x_3 \ge 2$

$4x_1 - 2x_2 - 3x_3 = -6$

with $x_1, x_2 \ge 0$.

---

if not, the procedure of jumping from one extreme point to another is repeated.

- Since the number of vertices is finite, simplex method leads to an optimal vertex in a finite number of steps.

If at any stage, the procedure leads us to a vertex which has an edge leading to infinity and if the objective function value can be further improved by moving along that edge, the simplex method tells us that there is an unbounded solution.

However, to discuss Simplex Method, we need to know some basic concepts and results.

Now we shall introduce two important forms of a general linear programming problem, namely the standard form and canonical form. Also, we shall introduce some special types of variables namely the slack and surplus variables. and identify the nature of solutions such as feasible solutions, Basic solutions, Basic feasible solutions, optimal solutions etc.

Slack and surplus variables

- Let the constraints of General LPP be $\sum_{j=1}^{n} a_{ij} x_j \le b_i$; $i = 1, 2, \ldots k$

Then, the non-negative variables x_{n+i} which satisfy $\sum_{j=1}^{n} a_{ij} x_j + x_{n+i} = b_i$, $i = 1, 2, \ldots k$, are called slack variables

— Let the constraints of a General L.P.P be $\sum_{j=1}^{n} a_{ij} x_j \geq b_i$; $i = k+1, k+2, \ldots\ldots l$.

Then, the non-negative variables x_{n+i} which satisfy $\sum_{j=1}^{n} a_{ij} x_j - x_{n+i} = b_i$; $i = k+1, k+2, \ldots\ldots l$

are called surplus variables.

* Two form of L.P.P.

— Canonical form: The general L.P.P. in the form

Maximize $Z = c_1 x_1 + c_2 x_2 + \ldots\ldots + c_n x_n$

subject to the constraints:

$a_{i1} x_1 + a_{i2} x_2 + \ldots\ldots + a_{in} x_n \leq b_i$; $i = 1, 2, \ldots\ldots m$

$x_1, x_2, \ldots\ldots x_n \geq 0$ is known as in canonical form.

The characteristics of this form are:

(i) The objective function is of the maximization type.

The minimization of a function $f(x)$, is equivalent to the maximization of the negative expression of this function $-f(x)$.

i.e minimize $f(x) = +$ Maximize $\{-f(x)\}$.

Ex:- Minimize $Z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$.

$\Rightarrow$ Maximize Z' $(=-Z)$

$= -c_1x_1 - c_2x_2 - \cdots - c_nx_n$

(ii) All the constraints are of the "$\leq$" type except for the non-negative restrictions.

— An inequality of "$\geq$" type can be changed to an inequality of the "$\leq$" type by multiplying both sides of the inequality by -1.

Ex:- $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \geq b_i$

$\Rightarrow -a_{i1}x_1 - a_{i2}x_2 - \cdots - a_{in}x_n \leq -b_i$

— An equation may be replaced by two weak inequalities in opposite directions.

Ex:- $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n = b_i$

$\Rightarrow a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \leq b_i$

and $a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n \geq b_i$

(iii) All decision variables are non-negative.

Note:- The canonical form is a format for a L.P.P. which finds its use in the Duality theory.

* Standard form :

The general L.P.P in the form

Maximize or minimize $Z = c_1x_1 + c_2x_2 + \dots\dots + c_nx_n$.

subject to the constraints :

$$a_{i1}x_1 + a_{i2}x_2 + \dots\dots + a_{in}x_n = b_i$$

$$i = 1, 2, \dots\dots m$$

$$x_1, x_2, x_3, \dots\dots x_n \geq 0$$

is known as in standard form.

The characteristics of this form are:

(i) All the constraints are expressed in the form of equations,

(ii) The right hand side of each constraint equation is non-negative.

(iii) All variables are non-negative type.

(iv) The objective function is of the maximization or minimization.

The inequality constraints can be changed into equation by introducing a non-negative variable on the left hand side of such constraint. It is to be added (slack variable) if the constraint is of "$\leq$" type and subtracted (surplus variable) if the constraint is of "$\geq$" type.

→ Convert the following LPP to the standard

Min $z = 2x_1 + x_2 + 4x_3$

Subject to $-2x_1 + 4x_2 \leq 4$

$x_1 + 2x_2 + x_3 \geq 5$

$2x_1 + 3x_3 \leq 2$

$x_1, x_2 \geq 0$ and x_3 unrestricted.

→ Reduce the following LPP to its stand. form

Max $z = x_1 - 3x_2$

Subject to $-x_1 + 2x_2 \leq 15$

$x_1 + 3x_2 = 10$

x_1 and x_2 being unrestricted.

→ Reduce the following LPP to the standard form:

Determine $x_1 \geq 0, x_2 \geq 0, x_3 \geq 0$ so as to

max $z = 2x_1 + x_2 + 4x_3$

Subject to $-2x_1 + 4x_2 \leq 4$

$x_1 + 2x_2 + x_3 \geq 5$

$2x_1 + 3x_3 \leq 2$

→ Reduce the following LPP in the stand form

Min $z = -3x_1 + x_2 + x_3$

Subject to $x_1 - 2x_2 + x_3 \leq 11$

$-4x_1 + x_2 + 2x_3 \geq 3$

$2x_1 - x_3 = -1$

$x_1, x_2 \geq 0$, $x_3 \geq$ or < 0

(i.e. x_3 is unrestricted i

⇒ Put each of the following programs in ma standard form.

→ Min $z = 2x_1 - x_2 + 4x_3$

Subject to $5x_1 + 2x_2 - 3x_3 \geq -7$

$2x_1 - 2x_2 + x_3 \leq 8$

$x_i \geq 0.$

→ $\max z = 10x_1 + 11x_2$

Subject to $x_1 + 2x_2 \leq 150$

$3x_1 + 4x_2 \leq 200$

$6x_1 + x_2 \leq 175$

$x_1, x_2 \geq 0.$

→ $\min z = 3x_1 + 2x_2 + 6x_3 + 6x_4$

Subject to $x_1 + 2x_2 + x_3 + x_4 \geq 1000$

$2x_1 + x_2 + 3x_3 + 7x_4 \geq 1500$

$x_i \geq 0 \quad i = 1, 2, 3, 4.$

→ $\min z = 6x_1 + 3x_2 + 4x_3$

Subject to $x_1 + 6x_2 + x_3 = 10$

$2x_1 + 3x_2 + x_3 = 15$

$x_i \geq 0 \quad i = 1, 2, 3.$

→ $\max z = 7x_1 + 2x_2 + 3x_3 + x_4$

Subject to $2x_1 + 7x_2 = 7$

$5x_1 + 8x_2 + 2x_4 = 10$

$x_1 + x_3 = 11$

$x_1 \geq 0, \; x_2 \geq 0, \; x_3 \geq 0$

→ $\min z = 10x_1 + 2x_2 - x_3$

Subject to $x_1 + x_2 \leq 50$

$x_1 + x_2 \geq 10$

$x_2 + x_3 \leq 30$

$x_2 + x_3 \geq 7$

$x_1 + x_2 + x_3 = 60$

$x_1, x_2, x_3 \geq 0$

— If one or more of the basic variables equal to zero, the basic feasible solution is degenerate.

— If all the basic variables are positive, the basic feasible solution is non-degenerate.

— optimal basic feasible solution is that basic feasible solution which maximizes (or minimizes) the objective function of the L.P.P.

Note: The total number of basic solutions $= {}^{n}C_{m}$

where 'n' is the number of unknown variables and 'm' is the number of constraints.

→ A linear program is in standard form

optimize $Z = c^T x$

Subject to $Ax = B$ —①

with $x \geq 0$ —②

Denote the columns of the $m \times n$ coefficient matrix A in System ① by $A_1, A_2, \ldots \ldots A_n$ respectively. Then the matrix constraint eqn $Ax = B$ can be rewritten in the vector form

$$x_1 A_1 + x_2 A_2 + \cdots \cdots + x_n A_n = B$$

the A-vectors and B are known m-dimensional vectors.

→ The set of all feasible solutions to a linear programming problem is a convex set.

Sol^n: W.K.T the constraints of a L.P.P can be converted into equations by means of introduction of slack or surplus variables.

∴ Let us consider the constraint system of any LPP of the form

$$Ax = B, \quad x \geq 0$$

where A is an $m \times n$ matrix, x is $n \times 1$ matrix and B is an $m \times 1$ matrix.

Let the set K be the set of all feasible solutions of the LPP $Ax = B$.

$$\therefore K = \{x \mid Ax = B, \ x \geq 0\}$$

Now to prove that K is a convex set.

Let $x_1, x_2 \in K$.

Then we have

$$Ax_1 = B, \quad x_1 \geq 0$$

$$Ax_2 = B, \quad x_2 \geq 0.$$

Now consider $\lambda x_1 + (1-\lambda) x_2$ for $0 \leq \lambda \leq 1$.

and $A[\lambda x_1 + (1-\lambda) x_2] = \lambda A x_1 + (1-\lambda) A x_2$

$$= \lambda B + (1-\lambda) B$$

$$= B.$$

Since x_1, x_2, λ and $1-\lambda$ are all ≥ 0.

$$\therefore \lambda x_1 + (1-\lambda) x_2 \geq 0$$

Thus $\lambda x_1 + (1-\lambda) x_2 \in K$ for $0 \leq \lambda \leq 1$.

which implies that the set K is a convex set.

→ A basic feasible solution to a LPP corresponds to an extreme point of the convex set K of feasible solutions and conversely, every extreme point of K corresponds to a basic feasible solution to a LPP.

For example:

Maximize $Z = 4x_1 + 5x_2$

subject to

$$\left.\begin{aligned} 2x_1 + 3x_2 &\le 12 \\ 3x_1 + x_2 &\le 8 \end{aligned}\right\} \longrightarrow (1)$$

$$x_1, x_2 \geq 0.$$

Adding the slack variables, the constraints become

$$\left.\begin{aligned} 2x_1 + 3x_2 + x_3 &= 12 \\ 3x_1 + x_2 + x_4 &= 8 \end{aligned}\right\} \longrightarrow (2)$$

from (2): There are 4 variables and two constraint equations, a basic solution can be obtained by setting any two $(4-2=2)$ variables equal to zero and then solving the resulting equations.

Also the total number of basic solutions $= {}^4C_2 = 6$.

All basic feasible solutions are given by

$x^1 = (0, 0, 12, 8)$, $x^2 = \left(\frac{8}{3}, 0, \frac{20}{3}, 0\right)$

$x^3 = (0, 4, 0, 4)$ and $x^4 = \left(\frac{12}{7}, \frac{20}{7}, 0, 0\right)$

and other two basic solutions $x^5 = (0, 8, -12, 0)$ and $x^6 = (6, 0, 0, -10)$ are not feasible.

From the graphical solution of the eqn (1) the extreme points of the convex set K of feasible solutions are

$(0,0)$, $(\frac{8}{3},0)$, $(0,4)$, $(\frac{12}{7},\frac{20}{7})$.

Thus, Clearly we can see the correspondence between basic feasible solutions and extreme points of the Convex set of feasible solution

x^1 corresponds to $(0,0)$

x^2 corresponds to $(\frac{8}{3},0)$

x^3 corresponds to $(0,4)$

x^4 corresponds to $(\frac{12}{7},\frac{20}{7})$

Also, Conversely, every extreme point of K corresponds to some basic feasible solutions.

2007 12M. → Put the following in slack form and describe which of the variables are 0 at each of the vertices of the constraint set and hence determine the vertices algebraically:

Maximize $u = 4x+3y$

subject to

$$\left.\begin{array}{l} x+y\le 4 \\ -x+y\le 2 \end{array}\right\} \text{——} (1)$$

$x, y \ge 0.$

Sol: Adding the slack variables to the constraints of the given LPP.

then ...

① ≡ $$\left.\begin{array}{l} x+y+x_1 = 4 \\ -x+y+x_2 = 2 \end{array}\right\} \text{—— ②}$$

② ≡ There are 4 variables and two constraint equations, a basic solution can be obtained by setting any two variables equal to zero and then solving the resulting equations.

Also the total number of basic solutions $= {}^4C_2 = 6$

Let $x=0,\ y=0$ in ②
we get $x_1=4$
$x_2=2$.

$A=(0,0,4,2)$

Let $x=0,\ x_1=0$
we get $y=4$ and $x_2=-2$

$\boxed{y=4}$
$y+x_2=+2$
$\Rightarrow x_2=2-4$
$\boxed{x_2=-2}$

$B=(0,4,0,-2)$

Let $y=0,\ x_1=0$
we get $x=4$ and $x_2=6$

$\boxed{x=4}$
$-x+x_2=2$
$x_2=2+4$
$\boxed{x_2=6}$

$C=(4,0,0,6)$

Let $x=0,\ x_2=0$
we get $x_1=2,\ y=2$

$\boxed{y=2}$
$y+x_1=4$
$x_1=4-2=2$
$\boxed{x_1=2}$

$D=(0,2,2,0)$

Let $y=0,\ x_2=0$
we get $x_1=6$
$x=-2$

$-x=2 \Rightarrow \boxed{x=-2}$
$x+x_1=4$
$\Rightarrow x_1=4-x$
$=\cancel{4}4+2=6$
$\boxed{x_1=6}$

$E=(-2,0,6,0)$

Let $x_1=0,\ x_2=0$
we get $y=3$ & $x=1$

$\cancel{x}+y=4$
$-\cancel{x}+y=2$
$2y=6 \Rightarrow y=3$
$x=1$

$F=(1,3,0,0)$.

(Continue in this way)

problems.

→ write the constraint equations of the linear program in the vector form

Minimize $Z = 2x_1 + 3x_2 + x_3 + 0x_4 + Mx_5 + 0x_6$

subject to $x_1 + 2x_2 + 2x_3 - x_4 + x_5 = 3$
$2x_1 + 3x_2 + 4x_3 + x_6 = 6$

with all variables non-negative.

sol. In matrix notation,

standard form is

Minimize $Z = C^T X$

subject to $AX = B$ ——— (1)

with $X \geq 0$ ——— (2)

where X is column vector of unknowns, including all slack, surplus and artificial variables.

i.e $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$,

C^T is the row vector of the corresponding costs i.e $C = [2, 3, 1, 0, M, 0]^T$,

A is coefficient matrix of the constraint equations

i.e $A = \begin{bmatrix} 1 & 2 & 2 & -1 & 1 & 0 \\ 2 & 3 & 4 & 0 & 0 & 1 \end{bmatrix}$

and B is the column vector of the right-hand sides of the constraint equations.

i.e $B = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$

Denote the columns of the 2×6 coefficient matrix A in system (1) by $A_1, A_2, A_3, A_3, A_4, A_5$ & A_6 respectively.

Then the constraint equation $AX=B$ can be rewritten in the vector form

$$x_1\underset{A_1}{\begin{bmatrix}1\\2\end{bmatrix}} + x_2\underset{A_2}{\begin{bmatrix}2\\3\end{bmatrix}} + x_3\underset{A_3}{\begin{bmatrix}2\\4\end{bmatrix}} + x_4\underset{A_4}{\begin{bmatrix}-1\\0\end{bmatrix}} + x_5\underset{A_5}{\begin{bmatrix}1\\0\end{bmatrix}} + x_6\underset{A_6}{\begin{bmatrix}0\\1\end{bmatrix}} = \underset{B}{\begin{bmatrix}3\\6\end{bmatrix}}$$

HW → Write the constraint equations for the following linear program in vector form

Maximize $z = x_1 + 2x_2 + 3x_3 + 4x_4 + 0x_5 + 0x_6 + 0x_7$

subject to $x_1 + 2x_2 + x_3 + 3x_4 + x_5 = 9$

$2x_1 + x_2 + 3x_4 + x_6 = 9$

$-x_1 + x_2 + x_3 + x_7 = 0$

with all variables non-negative.

→ Obtain all the basic feasible solns to the following system of linear equation:

$$x_1 + 2x_2 + x_3 = 4$$
$$2x_1 + x_2 + 5x_3 = 5$$

Sol The given system of equations can be written in the matrix form as

$$Ax = b$$

where $A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 5 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and

$b = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$.

Since the rank of A is 2 (i.e $m = 2$), the maximum number of linearly independent columns of A is 2

Thus we consider any of the 2x2 sub-matrices as basis matrix B:

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 2 & 5 \end{bmatrix} \text{ and } \begin{bmatrix} 2 & 1 \\ 1 & 5 \end{bmatrix}.$$

The variables not associated with the columns of B are x_3, x_2 and x_1 respectively in three different cases.

Let $B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

A basic solution to the system is obtained by taking $x_3 = 0$ and solving the system $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

$$\Rightarrow \left.\begin{matrix} x_1 + 2x_2 = 4 \\ 2x_1 + x_2 = 5 \end{matrix}\right\} \Rightarrow x_1 = 2;\ x_1 = 1.$$

∴ The basic (non-basic) solution to the given system is:

Basic: $x_1 = 2, x_2 = 1$; Non-basic $x_3 = 0$

and all $x_j \geq 0$ $(j = 1, 2, ---)$

∴ The solution is feasible.

∴ it is a basic feasible solution.

Similarly. the other two basic and non-basic solutions are:

Basic $x_1 = 5, x_3 = -1$; Non-basic $x_2 = 0$

clearly which is not a feasible solution. $(\because x_3 \ngtr 0)$

and Basic $x_2 = 5/3, x_3 = 2/3$; non-basic $x_1 = 0$

clearly which is a basic feasible solution.

Note:- The above first and third basic solutions are non-degenerate solutions.

→ Find all the basic solutions of the following system of equations identifying in each case of the basic and non-basic variables:

$$2x_1 + x_2 + 4x_3 = 1$$
$$3x_1 + x_2 + 5x_3 = 14.$$

Investigate whether the basic solutions are degenerate basic solutions or not. Hence find the basic feasible solution of the system.

Sol: Since there are 3 variables and two constraints, a basic solution can be obtained by setting any one variable equal to zero. and then solving resulting equation.

Also the total number of basic solutions $= {}^3C_2 = 3$

The characteristics of the various basic solutions are as given below.

No. of Basic solutions	Basic variables	Non basic variables	values of basic variables	Is the solution feasible? (Are all $x_j \geq 0$?)	Is the solution degenerate?
1	x_1, x_2	$x_3 = 0$	$2x_1 + x_2 = 11$ $3x_1 + x_2 = 14$ $\therefore x_1 = 3,\ x_2 = 5$	Yes	No.
2	x_2, x_3	$x_1 = 0$	$x_2 + 4x_3 = 11$ $x_2 + 5x_3 = 14$ $\therefore x_2 = -1, x_3 = 3$	No.	No
3	x_1, x_3	$x_2 = 0$	$2x_1 + 4x_3 = 11$ $3x_1 + 5x_3 = 14$ $\therefore x_1 = 1/2, x_3 = 5/2$	Yes	No.

The basic feasible solutions are:

(i) $x_1 = 3,\ x_2 = 5,\ x_3 = 0$

(ii) $x_1 = \frac{1}{2},\ x_2 = 0,\ x_3 = 5/2.$

The second solution is non-degenerate which is not feasible solution.

→ Find an optimal solution to the following LPP by computing all basic solutions and then finding one that maximizes the objective function.

$$\text{Max } Z = 2x_1 + 3x_2 + 4x_3 + 7x_4$$

$$\text{subject to } 2x_1 + 3x_2 - x_3 + 4x_4 = 8$$
$$x_1 - 2x_2 + 6x_3 - 7x_4 = -3$$
$$x_1, x_2, x_3, x_4 \geq 0.$$

Sol^n: Since there are four variables and two constraints, a basic solution can be obtained by setting any two variables equal to zero and then solving the resulting equations. Also the total number of basic solutions $= {}^4C_2 = 6$

The characteristics of the various basic solutions are given below.

No. of basic solutions	Basic variables	Non-basic variables	values of basic variables	Is the solution feasible? (are all $x_j \geq 0$?)	value of Z	Is the solution optimal?
1.	x_1, x_2	$x_3 = 0$, $x_4 = 0$	$2x_1 + 3x_2 = 8$, $x_1 - 2x_2 = -3$, $\therefore x_1 = 1,\ x_2 = 2$	Yes	8	NO
2.	x_1, x_3	$x_2 = 0$, $x_4 = 0$	$2x_1 - x_3 = 8$, $x_1 + 6x_3 = -3$, $\therefore x_1 = -\frac{14}{13},\ x_3 = -\frac{27}{13}$ $\frac{45}{13}$	NO	—	—
3.	x_1, x_4	$x_2 = 0$, $x_3 = 0$	$2x_1 + 8x_4 = 8$, $x_1 - 7x_4 = -3$, $\therefore x_1 = 22/9,\ x_4 = 7/9$	Yes	10.3	NO
4.	x_2, x_3	$x_1 = 0$, $x_4 = 0$	$3x_2 - x_3 = 8$, $-2x_2 + 6x_3 = -3$, $\therefore x_2 = \frac{45}{16},\ x_3 = \frac{7}{16}$	Yes	10.2	NO
5.	x_2, x_4	$x_1 = 0$, $x_3 = 0$	$3x_2 + 4x_4 = 8$, $-2x_2 - 7x_4 = -3$, $\therefore x_2 = \frac{132}{39},\ x_4 = \frac{-7}{13}$	NO	—	—
6.	x_3, x_4	$x_1 = 0$, $x_2 = 0$	$-x_3 + 4x_4 = 8$, $6x_3 - 7x_4 = -3$, $\therefore x_3 = \frac{44}{17},\ x_4 = \frac{45}{17}$	Yes	28.9	Yes.

Hence the optimal basic feasible solution is

$x_1 = 0,\ x_2 = 0,\ x_3 = \frac{44}{17},\ x_4 = \frac{45}{17}.$

and the maximum value of $z = 28.9$

2001 → Compute all basic feasible solutions of the linear programming problem.

$$\text{Max } z = 2x_1 + 3x_2 - x_3 = 8$$
$$x_1 - 2x_2 + 6x_3 = -3$$
$$x_1, x_2, x_3 \geq 0.$$

and hence indicate the optimal solution.

2002 → For the following system of equations

(1)

$$x_1 + x_2 + x_3 = 3$$
$$2x_1 - x_2 + 3x_3 = 4$$

Determine i) all basic solutions

(ii) all basic feasible solutions

(iii) a ~~feasible~~ basic solution which is not a basic feasible solution.

H.W. → Find all the basic solutions to the following problem:

(2)

Maximize $z = x_1 + 3x_2 + 3x_3$

Subject to: $x_1 + 2x_2 + 3x_3 = 4$

$$2x_1 + 3x_2 + 5x_3 = 7$$

with $x_1, x_2, x_3 \geq 0$

which of the basic solutions are

a) non-degenerate basic feasible   b) optimal basic feasible ?

→ For the following system of equations

$$x_1 + 2x_2 + x_3 = 4$$
$$2x_1 + x_2 + 5x_3 = 5.$$

Determine (i) all basic solutions.

(ii) all basic feasible solutions.

→ Find all the basic solutions of the following system of equations identifying in each case the basic and non-basic variables $2x_1+x_2-x_3=2$

$3x_1+2x_2+x_3=3$.

Investigate whether the basic solutions are degenerate basic solutions (or) not. Hence find the basic feasible solution of the system.

(or)

Show that the following system of linear equations has two degenerate basic feasible solutions and the non-degenerate basic solution is not feasible.

$2x_1+x_2-x_3=2,\quad 3x_1+2x_2+x_3=3.$

→ Find all the basic feasible solutions of the equations
$2x_1+6x_2+2x_3+x_4=3,$
$6x_1+4x_2+4x_3+6x_4=2$

Note:- If there is a feasible solution to the system of constraints
$Ax=B$
$x\geq 0$, then there also exists a basic feasible solution to the system.

For example

Consider the system of equations

$2x_1+3x_2-x_3=4$
$-5x_1+6x_2+x_3=2$

A feasible solution is $x_1=1, x_2=1, x_3=1$.
Reduce this feasible solution to a basic feasible solution.

46(i)

sol

The above given system of eqns may be put in matrix notations as

$$\begin{bmatrix} 2 & 3 & -1 \\ -5 & 6 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

$\Rightarrow AX = B$

where $A = \begin{bmatrix} 2 & 3 & -1 \\ -5 & 6 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$ and $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

Let the columns of A be denoted by

$A_1 = \begin{bmatrix} 2 \\ -5 \end{bmatrix}$, $A_2 = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$ & $A_3 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Here $\rho(A) = 2$

$\therefore$ a basic solution to the given system of eqns exist with not more than two variables different from zero.

Also, the column vectors A_1, A_2, A_3 are linear dependent (LD) (we can easily verify)

$\therefore \exists$ scalars $\lambda_1, \lambda_2, \lambda_3$ not all zero s.t

$A_1\lambda_1 + A_2\lambda_2 + A_3\lambda_3 = 0.$

$\Rightarrow \begin{bmatrix} 2 \\ -5 \end{bmatrix}\lambda_1 + \begin{bmatrix} 3 \\ 6 \end{bmatrix}\lambda_2 + \begin{bmatrix} -1 \\ 1 \end{bmatrix}\lambda_3 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow \left.\begin{array}{l} 2\lambda_1 + 3\lambda_2 - \lambda_3 = 0 \\ -5\lambda_1 + 6\lambda_2 + \lambda_3 = 0. \end{array}\right]$

$\therefore$ This is a system of two eqns in three unknowns $\lambda_1, \lambda_2, \lambda_3$.

Let us choose one of the λ's arbitrarily say $\lambda_1 = 2$.

H.W. Let $x_1 = 2, x_2 = 4$ & $x_3 = 1$ be a feasible soln to the system of eqns $2x_1 - x_2 + 2x_3 = 2$, $x_1 + 4x_2 = 18$. Reduce the given feasible soln.

$$\therefore \left.\begin{aligned} 3\lambda_2 - \lambda_3 &= -4 \\ 6\lambda_2 + \lambda_3 &= 10 \end{aligned}\right\}$$

solving, we get $\boxed{\lambda_2 = 2/3}$, $\boxed{\lambda_3 = 6.}$

To reduce the number of +ve variables, the variable to be driven to zero is found by choosing r for which

$$\frac{x_r}{\lambda_r} = \min_i \left\{ \frac{x_i}{\lambda_i} \Big/ \lambda_i > 0 \right\}.$$

$$= \min_i \left\{ \frac{x_1}{\lambda_1}, \frac{x_2}{\lambda_2}, \frac{x_3}{\lambda_3} \right\}.$$

$$= \min \left\{ 1/2, \frac{1}{2/3}, 1/6 \right\} = \min\left\{\frac{1}{2}, 3/2, 1/6\right\}.$$

$$= 1/6.$$

Thus, we can remove vector A_3 for which $\frac{x_3}{\lambda_3} = 1/3$ and obtain new solution with not more than two non-negative (non-zero) variables.

The values of new variables are given by

$$\hat{x}_1 = x_1 - \frac{x_3}{\lambda_3} \lambda_1 = 1 - \frac{1}{6}\cdot(2) = \frac{2}{3}.$$

$$\hat{x}_2 = x_2 - \frac{x_3}{\lambda_3} \lambda_2 = 1 - \frac{1}{6} \cdot 2/3 = 8/9.$$

$$\hat{x}_3 = x_3 - \frac{x_3}{\lambda_3} \lambda_3 = 0.$$

Obviously, columns A_1, A_2 of A corresponding to these non-zero variables are LI.

Hence the basic feasible solution to given system of eqns is given by

$x_1 = 2/3$, $x_2 = 8/9$, $x_3 = 0$.

Note

If A_1 or A_2 are eliminated instead of A_3, then x_1 or x_2 are driven to zero. In these cases we will find that new solutions are not feasible.

HW 5 → Consider the system of equations

$$2x_1 + x_2 + 4x_3 = 11$$
$$3x_1 + x_2 + 5x_3 = 14.$$

A feasible solution is $x_1 = 2,\ x_2 = 3,\ x_3 = 1$.
Reduce this feasible solution to a basic feasible solution.

→ For the system of equations

$$x_1 + 2x_2 + 4x_3 + x_4 = 7$$
$$2x_1 - x_2 + 2x_3 - 2x_4 = 3.$$

Here $(1,1,1,0)$ is a feasible solution.
Find a basic feasible solution.

→ Write the constraint equations of the following linear program in the vector form

minimise $z = 2x_1 + 3x_2 + x_3 + 0x_4 + Mx_5 + 0x_6$.

subject to $x_1 + 2x_2 + 2x_3 - x_4 + x_5 = 3$
$2x_1 + 3x_2 + 4x_3 + x_6 = 6$

with: all variables non-negative.

(1) Determine whether $[1,0,1,0,0,0]^T$ is a basic feasible solution to the linear program.

(2) Determine whether $[1,0,0,0,2,4]^T$ is a basic feasible solution to the linear program.

Solⁿ: The vector form of the given linear program becomes

$$x_1\underbrace{\begin{bmatrix}1\\2\end{bmatrix}}_{A_1}+x_2\underbrace{\begin{bmatrix}2\\3\end{bmatrix}}_{A_2}+x_3\underbrace{\begin{bmatrix}2\\4\end{bmatrix}}_{A_3}+x_4\underbrace{\begin{bmatrix}-1\\0\end{bmatrix}}_{A_4}+x_5\underbrace{\begin{bmatrix}1\\0\end{bmatrix}}_{A_5}+x_6\underbrace{\begin{bmatrix}0\\1\end{bmatrix}}_{A_6}=\underbrace{\begin{bmatrix}3\\6\end{bmatrix}}_{B}$$

$Ax=B$ form

(1) $[1,0,1,0,0,0]^T$ is not basic feasible solution, although all its components are non-negative.

because: the vectors A_1 and A_3 associated with the x-variables not set equal to zero are not linearly independent.

(2). The coefficient matrix A, comprising the column vectors A_1 through A_6, has order 2×6.

Therefore a basic feasible solution must have atleast $6-2=4$ zero components (variables), which is not the case here.