LINEAR ALGEBRA

: IFoS 2019 :

1) Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be a Linear operator on $\mathbb{R}^3$ defined by ~~T(x~~
$T(x,y,z) = (2y+z, x-4y, 3x)$. Find the matrix of T in the basis $\{(1,1,1), (1,1,0), (1,0,0)\}$.

$\longrightarrow$ $(x,y,z) = a(1,1,1) + b(1,1,0) + c(1,0,0)$ where $a,b,c \in \mathbb{R}$

$(x,y,z) = (a+b+c, a+b, a)$

On comparing both sides:- $a = z$, $a+b = y \Rightarrow b = y-z$

and $a+b+c = x \Rightarrow c = x-a-b$

$\Rightarrow c = x-z-(y-z)$

$\Rightarrow c = x-y$

$\therefore (x,y,z) = z(1,1,1) + (y-z)(1,1,0) + (x-y)(1,0,0)$

Now $T(1,1,1) = (3,-3,3) = 3(1,1,1) + (-6)(1,1,0) + 6(1,0,0)$

$T(1,1,0) = (2,-3,3) = 3(1,1,1) + (-6)(1,1,0) + (5)(1,0,0)$

$T(1,0,0) = (0,1,3) = 3(1,1,1) + (-2)(1,1,0) + (-1)(1,0,0)$

$\therefore$ Matrix of T wrt given basis is $A = \begin{bmatrix} 3 & 3 & 3 \\ -6 & -6 & -2 \\ 6 & 5 & -1 \end{bmatrix}$

2) The eigen values of a real symmetric matrix A are $-1, 1$ and -2. The corresponding eigen vectors are $\frac{1}{\sqrt{2}}(-1,1,0)^T$, $(0,0,1)^T$ and $\frac{1}{\sqrt{2}}(-1,-1,0)^T$ respectively. Find the matrix A^4.

$\longrightarrow$ Since A is a real symmetric matrix and has distinct eigen values, $\Rightarrow$ A is diagonalizable.

Let P be the transformation matrix [where P is non-singular]

and D be the diagonal matrix such that [Since A is a real symmetric matrix]

$D = P^{-1}AP \Rightarrow A = PDP^{-1}$ where $P^{-1} = P^T$.

$P = [X_1 \; X_2 \; X_3]$ where X_1, X_2, X_3 are eigen vectors corresponding to the eigen values $-1, 1, -2$. respectively.

$\therefore P = \begin{bmatrix} -1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow P = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \end{bmatrix}$

Hence $\quad A = PDP^{-1}$

$$\Rightarrow A \cdot A \cdot A \cdot A = (PDP^{-1})(PDP^{-1})(PDP^{-1})(PDP^{-1})$$
$$= PD(P^{-1}P)D(P^{-1}P)D(P^{-1}P)DP^{-1}$$
$$= PD \cdot D \cdot D \cdot DP^{-1} = PD^4P^{-1}$$

$$\Rightarrow A^4 = PD^4P^{-1} \quad \text{where } D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \text{ \& } P^{-1} = P^T$$

$$= \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} \begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \\ -1 & 1 & 0 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 8.5 & 7.5 & 0 \\ 7.5 & 8.5 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Q3 Consider the singular matrix $A = \begin{bmatrix} -1 & 3 & -1 & 1 \\ -3 & 5 & 1 & -1 \\ 10 & -10 & -10 & 14 \\ 4 & -4 & -4 & 8 \end{bmatrix}$.

Given that one eigen value of A is 4 and one eigen vector that does not correspond to eigen value 4 is $(1\,1\,0\,0)^T$. Find all the eigen values of A other than 4 and hence, also find the real numbers p, q, r that satisfy the matrix equation $A^4 + pA^3 + qA^2 + rA = 0$

→ Given that $X_1 = (1\,1\,0\,0)^T$ is an eigen vector of A corr. to an unknown eigenvalue λ_1. Then,

$$AX_1 = \lambda_1 X_1 \Rightarrow \begin{bmatrix} -1 & 3 & -1 & 1 \\ -3 & 5 & 1 & -1 \\ 10 & -10 & -10 & 14 \\ 4 & -4 & -4 & 8 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \lambda_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \lambda_1 = 2.$$

$\therefore X_1 = (1\,1\,0\,0)^T$ corresponds to eigen value 2.

We have: Determinant (A) = Product of eigen values.

Let $\lambda_1 = 2, \lambda_2 = 4, \lambda_3, \lambda_4$ be the eigen values. Then,

Since A is a singular matrix, $\det A = 0$.

$$\Rightarrow \lambda_1 \lambda_2 \lambda_3 \lambda_4 = 0 \Rightarrow 2 \cdot 4 \cdot \lambda_3 \lambda_4 = 0$$

$$\Rightarrow \lambda_3 \cdot \lambda_4 = 0 \quad \text{——— (1)}$$

Trace (A) = Sum of eigen values.

$\Rightarrow (-1)+5+(-10)+8 = \lambda_1+\lambda_2+\lambda_3+\lambda_4 = 2+4+\lambda_3+\lambda_4$

$\rightarrow \lambda_3+\lambda_4 = -4 \Rightarrow \lambda_3 = -(4+\lambda_4).$

Putting in (1): $\lambda_3\lambda_4 = 0 \Rightarrow -(4+\lambda_4)\lambda_4 = 0$

$\Rightarrow \lambda_4 = 0$ or $\lambda_4 = -4.$

Then $\lambda_3 = -4$ or $\lambda_3 = 0.$

$\therefore$ ~~There can be two~~ sets of four eigen values. $\{2, 4, -4, 0\}$ ~~and~~

Now: The characteristic polynomial is given as

$(x-2)(x-4)(x+4)(x-0) = 0$

$\rightarrow (x^2-6x+8)(x^2+4x) = 0$

$\rightarrow x^4-6x^3+8x^2+4x^3-24x^2+32x = 0$

$\rightarrow x^4-2x^3-16x^2+32x = 0.$

By Cayley-Hamilton's Theorem, A satisfies this eqn.

$\therefore A^4-2A^3-16A^2+32A = 0$

Comparing with $A^4+pA^3+qA^2+rA = 0$, we have

$p=-2,\ q=-16,\ r=32$

Q4 Consider the vectors $x_1=(1,2,1,-1)$, $x_2=(2,4,1,1)$, $x_3=(-1,-2,0,-2)$ and $x_4=(3,6,2,0)$ in $\mathbb{R}^4$. Justify that the linear span of the set $S=\{x_1,x_2,x_3,x_4\}$ is a subspace of $\mathbb{R}^4$ defined as $\{(\xi_1,\xi_2,\xi_3,\xi_4)\in\mathbb{R}^4 / 2\xi_1-\xi_2=0, 2\xi_1-3\xi_3-\xi_4=0\}$

Can this subspace be written as ~~$\{(\alpha, 2\alpha$~~ $\{(\alpha, 2\alpha, \beta, 2\alpha-3\beta) / \alpha, \beta\in\mathbb{R}\}$

What is the dimension of this subspace

→ Let $A = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 2 & 4 & 1 & 1 \\ -1 & -2 & 0 & -2 \\ 3 & 6 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & -1 & 3 \end{bmatrix}$ $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 + R_1$, $R_4 \to R_4 - 3R_1$

$\sim \begin{bmatrix} 1 & 2 & 1 & -1 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $R_3 \to R_3 + R_2$, $R_4 \to R_4 - R_2$. $\sim \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $R_1 \to R_1 + R_2$

Then the vectors $(1,2,0,2)$ and $(0,0,-1,3)$ form the basis of the span of $\{x_1, x_2, x_3, x_4\}$ since these two are L.I.

<u>Now</u>: Span of $(1,2,0,2)$ & $(0,0,-1,3)$ is

$\{a(1,2,0,2) + b(0,0,-1,3) \mid a, b \in \mathbb{R}\}$

$\{(a, 2a, -b, 2a+3b) \mid a, b \in \mathbb{R}\}$. Let $a = \xi_1$, $2a = \xi_2$, $-b = \xi_3$ & $2a+3b = \xi_4$

$\Rightarrow \{(a, 2a, -b, 2a+3b) \mid a, b \in \mathbb{R}\}$

$\Rightarrow \{(\xi_1, \xi_2, \xi_3, \xi_4) \mid \xi_2 = 2\xi_1,\ 2\xi_1 - 3\xi_3 - \xi_4 = 0\}$

$\left[\text{since } 2a - 3(-b) - (2a+3b) = 0\right]$.

<u>Now</u>: Let us put $\alpha = a$ & $\beta = -b$, then the span is

$\{(\alpha, 2\alpha, \beta, 2\alpha - 3\beta) \mid \alpha, \beta \in \mathbb{R}\}$

Since there are only two vectors in the basis, its dimension is 2

<u>Q5</u> Using elementary row operations, reduce the matrix $A = \begin{bmatrix} 2 & 1 & 3 & 0 \\ 3 & 0 & 2 & 5 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 3 \end{bmatrix}$ to echelon form and find the inverse of A and hence solve the system of linear equations $AX = b$ where $X = (x, y, z, u)^T$ and $b = (2, 1, 0, 4)^T$.

→ $[A \mid I] = \left[\begin{array}{cccc|cccc} 2 & 1 & 3 & 0 & 1 & 0 & 0 & 0 \\ 3 & 0 & 2 & 5 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 2 & 1 & 1 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 3 & 0 & 2 & 5 & 0 & 1 & 0 & 0 \\ 2 & 1 & 3 & 0 & 1 & 0 & 0 & 0 \\ 2 & 1 & 1 & 3 & 0 & 0 & 0 & 1 \end{array}\right]$ $R_1 \leftrightarrow R_3$

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 2R_1$, $R_4 \to R_4 - 2R_1$ $\qquad$ $R_2 \leftrightarrow R_3$

$\sim \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -3 & -1 & 2 & 0 & 1 & -3 & 0 \\ 0 & -1 & 1 & -2 & 1 & 0 & -2 & 0 \\ 0 & -1 & -1 & 1 & 0 & 0 & -2 & 1 \end{array}\right] \sim \left[\begin{array}{cccc|cccc} 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & -1 & 1 & -2 & 1 & 0 & -2 & 0 \\ 0 & -3 & -1 & 2 & 0 & 1 & -3 & 0 \\ 0 & -1 & -1 & 1 & 0 & 0 & -2 & 1 \end{array}\right]$

$R_1 \to R_1 + R_2,\ R_3 \to R_3 - 3R_2,\ R_4 \to R_4 - R_2$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 2 & -1 & 1 & 0 & -1 & 0 \\ 0 & -1 & 1 & -2 & 1 & 0 & -2 & 0 \\ 0 & 0 & -4 & 8 & -3 & 1 & 3 & 0 \\ 0 & 0 & -2 & 3 & -1 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 \div -1,\ R_3 \to R_3 \div -4$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 2 & -1 & 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 2 & -1 & 0 & 2 & 0 \\ 0 & 0 & 1 & -2 & 3/4 & -1/4 & -3/4 & 0 \\ 0 & 0 & -2 & 3 & -1 & 0 & 0 & 1 \end{array}\right]$$

$R_1 \to R_1 - 2R_3,\ R_2 \to R_2 + R_3,\ R_4 \to R_4 + 2R_3$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 3 & -1/2 & 1/2 & 1/2 & 0 \\ 0 & 1 & 0 & 0 & -1/4 & -1/4 & 5/4 & 0 \\ 0 & 0 & 1 & -2 & 3/4 & -1/4 & -3/4 & 0 \\ 0 & 0 & 0 & -1 & 1/2 & -1/2 & -3/2 & 1 \end{array}\right]$$

$R_1 \to R_1 + 3R_4,\ R_3 \to R_3 - 2R_4$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & -4 & 3 \\ 0 & 1 & 0 & 0 & -1/4 & -1/4 & 5/4 & 0 \\ 0 & 0 & 1 & 0 & -1/4 & 3/4 & 9/4 & -2 \\ 0 & 0 & 0 & -1 & 1/2 & -1/2 & -3/2 & 1 \end{array}\right]$$

$R_4 \to R_4 \div -1$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & -1 & -4 & 3 \\ 0 & 1 & 0 & 0 & -1/4 & -1/4 & 5/4 & 0 \\ 0 & 0 & 1 & 0 & -1/4 & 3/4 & 9/4 & -2 \\ 0 & 0 & 0 & 1 & -1/2 & 1/2 & 3/2 & -1 \end{array}\right] \Rightarrow A^{-1} = \begin{bmatrix} 1 & -1 & -4 & 3 \\ -1/4 & -1/4 & 5/4 & 0 \\ -1/4 & 3/4 & 9/4 & -2 \\ -1/2 & 1/2 & 3/2 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{4}\begin{bmatrix} 4 & -4 & -16 & 12 \\ -1 & -1 & 5 & 0 \\ -1 & 3 & 9 & -8 \\ -2 & 2 & 6 & -4 \end{bmatrix}$$

Now! The solution to the system of given linear equations is

$$X = A^{-1}B \Rightarrow \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 4 & -4 & -16 & 12 \\ -1 & -1 & 5 & 0 \\ -1 & 3 & 9 & -8 \\ -2 & 2 & 6 & -4 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 0 \\ 4 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix} = \begin{bmatrix} 13 \\ -0.75 \\ -7.75 \\ -4.5 \end{bmatrix}$$

$\therefore x = 13,\ y = -0.75,\ z = -7.75,\ u = -4.5$