1(a) Consider the set V of all $n \times n$ real magic squares. Show that V is a vector space over R. Give examples of two distinct 2×2 magic squares. (10)

A squared matrix is called a magic square if the sum of the elements alongs any row, column and both the diagonals is a constant.
(Note that entries may repeat).

$V =$ set of all $n \times n$ real magic squares.

Hence zero matrix of order $n \times n$ is a magic square.

Let $M(s)$ be a $n \times n$ magic square from V, with line sum 's'.

First we show: V is an abelian group

Let $M(s), M(t) \in V$

$$\therefore M(s) + M(t) = M(s+t) \in V$$

Matrix Addition is associative.

$$M(0) \in V$$

and $M(s) + M(-s) = M(0)$

$$\therefore -M(s) = M(-s).$$

$$M(s) + M(t) = M(t) + M(s) \quad \text{, abelian}$$

Now, let $r \in R$, field

then $r\,M(s) = M(rs) \in V$

$$r(M(s) + M(t)) = M(rs) + M(rt)$$
$$= M(rs + rt) \in V$$

for $r_1, r_2 \in R$.

$$(r_1 + r_2) M(s) = M[(r_1+r_2)s]$$
$$= M(r_1 s + r_2 s)$$
$$= M(r_1 s) + M(r_2 s)$$
$$= r_1 M(s) + r_2 M(s)$$

$$(r_1 r_2) M(s) = M(r_1 r_2 s) = r_1 M(r_2 s)$$
$$= r_1 (r_2 M(s))$$

$$1 \cdot M(s) = M(s)$$

Hence V forms a vector space over R.

Examples of 2×2 Magic squares — (according to our definition)

$$M(2) = \begin{array}{|c|c|} \hline 1 & 1 \\ \hline 1 & 1 \\ \hline \end{array} \in V.$$

$$M(4) = \begin{array}{|c|c|} \hline 2 & 2 \\ \hline 2 & 2 \\ \hline \end{array} \in V$$

DATE

1(b) Let $M_2(R)$ be the vector space of all 2×2 real matrices.
Let $B = \begin{bmatrix} 1 & -1 \\ -4 & 4 \end{bmatrix}$.
Suppose $T: M_2(R) \to M_2(R)$ is a L.T. defined by $T(A) = BA$.

Find the rank and nullity of T.
Find a matrix A which maps to the null ~~space~~ matrix.
(10)

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(R)$

$$T(A) = \begin{bmatrix} 1 & -1 \\ -4 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a-c & b-d \\ -4a+4c & -4b+4d \end{bmatrix} \quad — (*)$$

$$= (a-c)\begin{bmatrix} 1 & 0 \\ -4 & 0 \end{bmatrix} + (b-d)\begin{bmatrix} 0 & 1 \\ 0 & -4 \end{bmatrix}$$

$$= k_1\begin{bmatrix} 1 & 0 \\ -4 & 0 \end{bmatrix} + k_2\begin{bmatrix} 0 & 1 \\ 0 & -4 \end{bmatrix}$$

$\begin{bmatrix} 1 & 0 \\ -4 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & -4 \end{bmatrix}$ are linearly independent as they are not multiple of each other.

$\therefore$ Rank(T) = 2

Let ~~A ∈~~ $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in$ Null Space

$\therefore T(A) = 0 \Rightarrow a = c, \; b = d$
[using (*)]

Hence Null Space

$$N = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} : T(A) = 0 \right\}$$

$$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} ; a = c, b = d \right\}$$

$$= \left\{ \begin{bmatrix} a & b \\ a & b \end{bmatrix} : \right\}$$

$$= \left\{ a \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$$

$$\text{Nullity}(T) = 2$$

Take $a = 1$, $b = 1$

$$\therefore A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$T(A) = BA = \begin{bmatrix} 1 & -1 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$\therefore$ A is mapped to null matrix under T.

DATE

1(c) Evaluate $\lim_{x \to \frac{\pi}{4}} (\tan x)^{\tan 2x}$ (10)

Let $l = \lim_{x \to \frac{\pi}{4}} (\tan x)^{\tan 2x}$

$$\log l = \lim_{x \to \pi/4} \tan 2x \cdot \log(\tan x)$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{\log(\tan x)}{\cos 2x} \cdot \sin 2x \quad \left(\frac{0}{0} \text{ form}\right)$$

$$= \lim_{x \to \frac{\pi}{4}} \frac{\frac{1}{\tan x} \cdot \sec^2 x}{-2\sin 2x} \cdot \lim_{x \to \frac{\pi}{4}} \sin 2x$$

(L-Hospital)

$$= \lim_{x \to \frac{\pi}{4}} \frac{-1}{(\sin 2x)^2} \cdot 1$$

$$= -1$$

$$\therefore \boxed{l = \lim_{x \to \frac{\pi}{4}} (\tan x)^{\tan 2x} = e^{-1}}$$

DATE

1(d) Find all the asymptotes of the curve $(2x+3)y = (x-1)^2$. (10)

$$(2x+3)y = (x-1)^2$$

$$x^2 - 2xy - 2x - 3y + 1 = 0 \quad \text{—①}$$

Asymptotes parallel to the coordinates axes:

Coeff of highest power of x is constant hence there is no asymptotes parallel to x-axis.

Coeff of highest power of y is $(-2x)-3$

$\therefore$ $2x+3=0$ is asymptote parallel to y-axis.

Oblique Asymptote:

Putting $x=1$, $y=m$ in second and first degree terms in ①

$$\phi_2(m) = 1-2m$$

$$\phi_1(m) = -2-3m = -2-3m$$

Slopes of the asymptotes are real roots of eqn $\phi_2(m)=0$

i.e. $1-2m=0 \Rightarrow m = \frac{1}{2}$

$$c = -\frac{\phi_1(m)}{\phi_2'(m)} = \frac{-(-2-3m)}{-2} = \frac{-(3m+2)}{2}$$

For $m = \frac{1}{2}$, $c = -\frac{1}{2}\left(3\cdot\frac{1}{2}+2\right) = -\frac{7}{4}$

$\therefore$ Asymptote: $y = mx+c \Rightarrow \boxed{y = \frac{1}{2}x - \frac{7}{4}}$

DATE

1(e) Find the equations of the tangent plane to the ellipsoid $2x^2+6y^2+3z^2=27$

which passes through the line

$$x-y-z=0=x-y+2z-9$$

(10)

Plane passing through this line is

$$(x-y+2z-9)+\lambda(x-y-z)=0$$

$$(1+\lambda)x-(1+\lambda)y+(2-\lambda)z=9 \quad \text{—(1)}$$

Tangent plane to ellipsoid at point (a,b,c)

$$2x^2+6y^2+3z^2=27$$

is given by : $2ax+6by+3cz=27$ —(2)

If (1) and (2) are same, then

$$\frac{2a}{1+\lambda}=\frac{6b}{-(1+\lambda)}=\frac{3c}{-(\lambda-2)}=\frac{\cancel{27}^{\,3}}{\cancel{9}}$$

$$a=\frac{3}{2}(1+\lambda),\quad b=-\frac{1}{2}(1+\lambda),\quad c=-(\lambda-2)$$

(a,b,c) lies on ellipsoid

$$\therefore\ \cancel{2}\cdot\frac{9}{\cancel{4}_{\,2}}(1+\lambda)^2+\cancel{6}^{\,3}\cdot\frac{1}{\cancel{4}_{\,2}}(1+\lambda)^2+3(\lambda-2)^2=27$$

$$9(1+\lambda)^2+3(1+\lambda)^2+6(\lambda-2)^2=27$$

$$\Rightarrow 3\lambda^2=3 \quad \Rightarrow \quad \boxed{\lambda=1,-1}$$

Putting in (1)

$$2x-2y+z=9,$$

$$z=3$$

are required equations of tangent planes.