Previous Years' Papers (Solved)

IFS MATHEMATICS EXAM., 2012

PAPER-I

Instructions: *Candidates should attempt Question Nos. 1 and 5 which are compulsory and any THREE of the remaining questions, selecting at least ONE question from each Section. All questions carry equal marks. Marks allotted to parts of a question are indicated against each. Answers must be written in ENGLISH only. Assume suitable data, if considered necessary, and indicate the same clearly. Unless indicated otherwise, symbols and notations carry their usual meaning.*

Section-A

1. (a) Let $V = \mathbb{R}^3$ and $\alpha_1 = (1, 1, 2)$, $\alpha_2 = (0, 1, 3)$, $\alpha_3 = (2, 4, 5)$ and $\alpha_4 = (-1, 0, -1)$ be the elements of V. Find a basis for the intersection of the subspace spanned by $\{\alpha_1, \alpha_2\}$ and $\{\alpha_3, \alpha_4\}$. (8)

(b) Show that the set of all functions which satisfy the differential equation

$$\frac{d^2 f}{dx^2} + 3\frac{df}{dx} = 0$$ is a vector space. (8)

(c) If the three thermodynamic variables P, V, T are connected by a relation, $f(P, V, T) = 0$

show that, $$\left(\frac{\partial P}{\partial T}\right)_V \left(\frac{\partial T}{\partial V}\right)_P \left(\frac{\partial V}{\partial P}\right)_T = -1$$ (8)

(d) If $u = Ae^{-gx} \sin(nt - gx)$, where A, g, n are positive constants, satisfies the heat conduction equation, $\frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial x^2}$ then show that $g = \sqrt{\left(\frac{n}{2\mu}\right)}$. (8)

(e) Find the equations to the lines in which the plane $2x + y - z = 0$ cuts the cone $4x^2 - y^2 + 3z^2 = 0$. (8)

2. (a) Let $f : \mathbb{R} \to \mathbb{R}^3$ be a linear transformation defined by $f(a, b, c) = (a, a + b, 0)$. Find the matrices A and B respectively of the linear transformation f with respect to the standard basis (e_1, e_2, e_3) and the basis (e'_1, e'_2, e'_3) where $e'_1 = (1, 1, 0)$, $e'_2 = (0, 1, 1)$, $e'_3 = (1, 1, 1)$.
Also, show that there exists an invertible matrix P such that

$$B = P^{-1}AP$$ (10)

(b) Verify Cayley–Hamilton theorem for the matrix $A = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ and find its inverse. Also express $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ as a linear polynomial in A. (10)

(c) Find the equations of the tangent plane to the ellipsoid $2x^2 + 6y^2 + 3z^2 = 27$ which passes through the line $x - y - z = 0 = x - y + 2z - 9$. (10)

(d) Show that there are three real values of λ for which the equations: $(a - \lambda) x + by + cz = 0$, $bx + (c - \lambda) y + az = 0$, $cx + ay + (b - \lambda) z = 0$ are simultaneously true and that the product of these values of λ is $D = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$. (10)

3. (a) Find the matrix representation of linear transformation T on V_3 ($\mathbb{R}$) defined as $T(a, b, c) = (2b + c, a - 4b, 3a)$ corresponding to the basis $B = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$. (10)

(b) Find the dimensions of the rectangular box, open at the top, of maximum capacity whose surface is 432 sq. cm. (10)

(c) If $2c$ is the shortest distance between the lines $\frac{x}{l} - \frac{z}{n} = 1$, $y = 0$.

and $\frac{y}{m}+\frac{z}{n}=1,\ x=0$

then show that

$\frac{1}{l^2}+\frac{1}{m^2}+\frac{1}{n^2}=\frac{1}{c^2}$. (10)

(d) Show that the function defined as

$$f(x)=\begin{cases}\frac{\sin 2x}{x} & \text{when } x\neq 0\\ 1 & \text{when } x=0\end{cases}$$

has removable discontinuity at the origin. (10)

4. (a) Find by triple integration the volume cut off from the cylinder $x^2+y^2=ax$ by the planes $z=mx$ and $z=nx$. (10)

(b) Show that all the spheres, that can be drawn through the origin and each set of points where planes parallel to the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$ cut the coordinate axes, form a system of spheres which are cut orthogonally by the sphere $x^2+y^2+2fx+2gy+2hz=0$ if $af+bg+ch=0$. (10)

(c) A plane makes equal intercepts on the positive parts of the axes and touches the ellipsoid $x^2+4y^2+9z^2=36$. Find its equation. (10)

(d) Evaluate the following in terms of Gamma function:

$$\int_0^a \sqrt{\left(\frac{x^3}{a^3-x^3}\right)}\,dx.$$ (10)

Section-B

5. (a) Solve $\frac{dy}{dx}-\frac{\tan y}{1+x}=(1+x)e^x\sec y$. (8)

(b) Solve and find the singular solution of $x^3p^2+x^2py+a^3=0$. (8)

(c) A particle is projected vertically upwards from the earth's surface with a velocity just sufficient to carry it to infinity. Prove that the time it takes to reach a height h is

$$\frac{1}{3}\sqrt{\left(\frac{2a}{g}\right)}\left[\left(1+\frac{h}{a}\right)^{3/2}-1\right].$$ (8)

(d) A triangle ABC is immersed in a liquid with the vertex C in the surface and the sides AC, BC equally inclined to the surface. Show that the vertical C divides the triangle into two others, the fluid pressures on which are as $b^3+3ab^2 : a^3+3a^2b$ where a and b are the sides BC and AC respectively. (8)

(e) If $u=x+y+z$, $v=x^2+y^2+z^2$, $w=yz+zx+xy$, prove that grad u, grad v and grad w are coplanar. (8)

6. (a) Solve:

$$x^2y\frac{d^2y}{dx^2}+\left(x\frac{dy}{dx}-y\right)^2=0.$$ (10)

(b) Find the value of $\iint_s(\overline{\nabla}\times\overline{F})\cdot\overline{ds}$ taken over the upper portion of the surface $x^2+y^2+z^2-2ax+az=0$ and the bounding curve lies in the plane $z=0$, when

$$\overline{F}=\left(y^2+z^2-x^2\right)\bar{i}+\left(z^2+x^2-y^2\right)\bar{j}+\left(x^2+y^2-z^2\right)\bar{k}.$$ (10)

(c) A particle is projected with a velocity u and strikes at right angle on a plane through the plane of projection inclined at an angle β to the horizon. Show that the time of flight is $\frac{2u}{g\sqrt{(1+3\sin^2\beta)}}$, range on the plane is $\frac{2u^2}{g}\cdot\frac{\sin\beta}{1+3\sin^2\beta}$ and the vertical height of the point struck is $\frac{2u^2\sin^2\beta}{g(1+3\sin^2\beta)}$ above the point of projection. (10)

(d) Solve $\frac{d^4y}{dx^4}+2\frac{d^2y}{dx^2}+y=x^2\cos x$. (10)

7. (a) A particle is moving with central acceleration $\mu\left[r^5-c^4r\right]$ being projected from an apse at a distance c with velocity $c^3\sqrt{\left(\frac{2\mu}{3}\right)}$, show that its path is a curve, $x^4+y^4=c^4$. (13)

(b) A thin equilateral rectangular plate of uniform thickness and density rests with one end of its base on a rough horizontal plane and the other against a small vertical wall. Show that the least angle, its base can make with the horizontal plane is given by

$$\cot\theta = 2\mu + \frac{1}{\sqrt{3}}$$

μ, being the coefficient of friction. (14)

(c) A semicircular area of radius a is immersed vertically with its diameter horizontal at a depth b. If the circumference be below the centre, prove that the depth of centre of pressure is

$$\frac{1}{4}\,\frac{3\pi\left(a^2+4b^2\right)+32ab}{4a+3\pi b}. \qquad (13)$$

8. (a) Solve $x = y\dfrac{dy}{dx} - \left(\dfrac{dy}{dx}\right)^2$. (10)

(b) Find the value of the line integral over a circular path given by $x^2 + y^2 = a^2$, $z = 0$, where the vector field,

$\vec{F} = (\sin y)\,\vec{i} + x(1+\cos y)\,\vec{j}$. (10)

(c) A heavy elastic string, whose natural length is $2\pi a$, is placed round a smooth cone whose axis is vertical and whose semi vertical angle is α. If W be the weight and λ the modulus of elasticity of the string, prove that it will be in equilibrium when in the form of a circle whose radius is

$$a\left(1+\frac{W}{2\pi\lambda}\cot\alpha\right). \qquad (10)$$

(d) Solve $x^2\dfrac{d^2y}{dx^2} + 3x\dfrac{dy}{dx} + y = (1-x)^{-2}$. (10)

PAPER-II

Instructions: *Candidates should attempt Question Nos. 1 and 5 which are compulsory, and any THREE of the remaining questions, selecting at least ONE question from each Section. All questions carry equal marks. The number of marks carried by each part of a question is indicated against each. Answers must be written in ENGLISH only. Assume suitable data, if considered necessary, and indicate the same clearly. Symbols and notations have their usual meanings, unless indicated otherwise.*

Section-A

1. Answer the following:

(a) Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1, & x \text{ is irrational} \\ -1, & x \text{ is rational} \end{cases}$$

is discontinuous at every point in $\mathbb{R}$. (10)

(b) Show that every field is without zero divisor. (10)

(c) Evaluate the integral

$$\int_{2-i}^{4+i} (x + y^2 - ixy)\,dz$$

along the line segment AB joining the points A(2, –1) and B(4, 1). (10)

(d) Show that the functions:

$u = x^2 + y^2 + z^2$

$v = x + y + z$

$w = yz + zx + xy$

are not independent of one another. (10)

2. (a) Show that in a symmetric group S_3, there are four elements σ satisfying σ^2 = Identity and three elements satisfying σ^3 = Identity. (13)

(b) If

$$u = x^2\tan^{-1}\left(\frac{y}{x}\right) - y^2\tan^{-1}\left(\frac{x}{y}\right),$$

show that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 2u. \qquad (13)$$

(c) Solve the following problem by Simplex Method. How does the optimal table indicate that the optimal solution obtained is not unique?

Maximize $z = 8x_1 + 7x_2 - 2x_3$

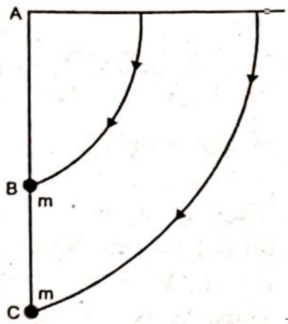
subject to the constraints

$x_1 + 2x_2 + 2x_3 \le 12$

(c) Draw a flow chart for interpolation using Newton's forward difference formula. (14)

8. (a) Solve
$(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$
using Lagrange's Method. (13)

(b) A weightless rod ABC of length $2a$ is movable about the end A which is fixed and carries two particles of mass m each one attached to the mid-point B of the rod and the other attached to the end C of the rod. If the rod is held in the horizontal position and released from rest and allowed to move, show that the angular velocity of the rod when it is vertical is $\sqrt{\frac{6g}{5a}}$. (13)



(c) Using Euler's Modified Method, obtain the solution of

$$\frac{dy}{dx} = x + \left|\sqrt{y}\right|,\ y(0) = 1$$

for the range $0 \le x \le 0.6$ and step size 0.2. (14)

ANSWERS

PAPER-I

Section-A

1. (a) $\alpha_1 = (1, 1, 2), \alpha_2 = (0, 1, 3), \alpha_3 = (2, 4, 5), \alpha_4 = (-1, 0, -1)$ are the elements of V.
Let a, b be two scalars such that the subspace generated is
$= a\,\alpha_1 + b\,\alpha_2 = a(1, 1, 2) + b(0, 1, 3)$
$= (a, a + b, 2a + 3b)$
Now similarly, subspace spanned by (α_3, α_4) is
$= c\,\alpha_3 + d\,\alpha_4 = c(2, 4, 5) + d(-1, 0, -1)$
$= (2c - d, 4c, 5c - d)$
According to the question,
intersection of the subspace spanned by $\{\alpha_1, \alpha_2\}$ and $\{\alpha_3, \alpha_4\}$ is given by
$(a, a + b, 2a + 3b) = (2c - d, 4c, 5c - d)$
$\Rightarrow (a = 2c - d), (a + b = 4c),$
$(2a + 3b = 5c - d)$
$d = 2c - a,\ b = 4c - a,\ 3b = 5c - d - 2a$

1. (b) Let W be the set of all functions which satisfy the differential equation,

$$\frac{d^2 f}{dx^2} + 3\frac{df}{dx} = 0$$

$$\therefore\ W = \left\{ f : \frac{d^2 f}{dx^2} + 3\frac{df}{dx} = 0 \right\}$$

Let $y = f(x)$
Obviously $f(x) = 0$ or $y = 0$ satisfy the given differential equation and as such it belongs to W and thus $W \ne \phi$
Now let $y_1, y_2 \in W$, then

$$\frac{d^2 y_1}{dx^2} + 3\frac{dy_1}{dx} = 0 \qquad ...(1)$$

and $$\frac{d^2 y_2}{dx^2} + 3\frac{dy_2}{dx} = 0 \qquad ...(2)$$

Let $a, b \in R$. If W is to be a subspace then we should show that $ay_1 + by_2$ also belongs to W *i.e.,* it is a solution of the given differential equation.
We have

$$\frac{d^2}{dx^2}(ay_1 + by_2) + 3\frac{d}{dx}(ay_1 + by_2)$$

$$= a\frac{d^2 y_1}{dx^2} + b\frac{d^2 y_2}{dx^2} + 3a\frac{dy_1}{dx} + 3b\frac{dy_2}{dx}$$

$$= a\left(\frac{d^2y_1}{dx^2}+3\frac{dy_1}{dx}\right)+b\left(\frac{d^2y_2}{dx^2}+3\frac{dy_2}{dx}\right)$$

$= a(0) + b(0)$ [using (1) and (2)]

$= 0$

Thus $ay_1 + by_2$ is a solution of the given differential equation and so it belongs to W.

Hence, W is the subspace.

Thus, W is a vector space.

1. (c) It is given that three thermodynamic variables P, V, T are connected by a relation $f(P, V, T) = 0$

It means we can take the equation of an ideal gas *i.e.,*

$$PV = nRT \quad ...(1)$$

$$\Rightarrow PV - nRT = 0$$

$$\Rightarrow f(P, V, T) = 0$$

Now from (1)

$$\left(\frac{\partial P}{\partial T}\right)_V = \frac{nR}{V} \quad ...(2)$$

Rearranging (1) to find $\left(\frac{\partial T}{\partial V}\right)_P$ as follows

$$T = \frac{PV}{nR}$$

$$\left(\frac{\partial T}{\partial V}\right)_P = \frac{P}{nR} \quad ...(3)$$

Again, Rearranging (1) to find $\left(\frac{\partial V}{\partial P}\right)_T$ as follows

$$V = \frac{nRT}{P}$$

$$\left(\frac{\partial V}{\partial P}\right)_T = \frac{-nRT}{P^2} \quad ...(4)$$

Multiplying (2), (3), (4) we get

$$\left(\frac{\partial P}{\partial T}\right)_V\left(\frac{\partial T}{\partial V}\right)_P\left(\frac{\partial V}{\partial P}\right)_T$$

$$= \frac{nR}{V}\times\frac{P}{nR}\times\left(\frac{-nRT}{P^2}\right) = \frac{-nRT}{PV} = -1$$

[using (1)]

1. (d) $u = Ae^{-gx}\sin(nt - gx)$, where A, g, n are positive constants. This expression u satisfies the heat conduction equation *i.e.,*

$$\frac{\partial u}{\partial t} = \mu\frac{\partial^2 u}{\partial x^2} \quad ...(1)$$

First finding $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$ from given expression of u, we get

$$\frac{\partial u}{\partial t} = nAe^{-gx}\cos(nt-gx) \quad ...(2)$$

and

$$\frac{\partial u}{\partial x} = A\left(-ge^{-gx}\cos(nt-gx) - ge^{-gx}\sin(nt-gx)\right)$$

$$= -Age^{-gx}\left[\cos(nt-gx)+\sin(nt-gx)\right]$$

$$\therefore \frac{\partial^2 u}{\partial x^2} = -Ag\begin{pmatrix} e^{-gx}[(g\sin(nt-gx) \\ -g\cos(nt-gx)] \\ -ge^{-gx}[\cos(nt-gx) \\ +\sin(nt-gx)] \end{pmatrix}$$

$$\frac{\partial^2 u}{\partial x^2} = -Ag^2e^{-gx}\left[\sin(nt-gx) - \cos(nt-gx)-\sin(nt-gx)-\cos(nt-gx)\right]$$

$$\frac{\partial^2 u}{\partial x^2} = 2Ag^2e^{-gx}\cos(nt-gx) \quad ...(3)$$

Substituting values of $\frac{\partial u}{\partial t}$ and $\frac{\partial^2 u}{\partial x^2}$ from (2) and (3) in (1) we get

$nAe^{-gx}\cos(nt - gx)$

$= 2Ag^2e^{-gx}\mu[\cos(nt - gx)$

$n = 2\mu g^2$

$$\therefore g = \sqrt{\left(\frac{n}{2\mu}\right)}.$$

1. (e) Let one of the lines of intersection of the plane

$$2x + y - z = 0 \quad ...(1)$$

and the cone

$$4x^2 - y^2 + 3z^2 = 0 \quad ...(2)$$

be $$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \quad ...(3)$$

The line (3) lies in the plane (1) and on the cone (2)

$\therefore \quad 2l + m - n = 0 \quad ...(4)$

and $4l^2 - m^2 + 3n^2 = 0 \quad ...(5)$

Eliminating n between (4) and (5) we get

$4l^2 - m^2 + 3(2l + m)^2 = 0$

$\Rightarrow 16l^2 + 12lm + 2m^2 = 0$

$\Rightarrow \quad 8l^2 + 6lm + m^2 = 0$

$\Rightarrow \quad (4l + m)(2l + m) = 0$

$4l + m = 0, \ 2l + m = 0$

$m = -4l, \ m = -2l$

when $m = -4l$, then from (4), $n = -2l$

and when $m = -2l$, then from (4), $n = 0$

Hence, In first case we rearrange as

$$\frac{l}{1} = \frac{m}{-4} = \frac{n}{-2}$$

and in second case, we rearrange as

$$\frac{l}{1} = \frac{m}{-2} = \frac{n}{0}$$

Thus, the equation of the lines in which the given plane cuts the given cone are:

$$\frac{x}{1} = \frac{y}{-4} = \frac{z}{-2} \text{ and } \frac{x}{1} = \frac{y}{-2} = \frac{z}{0}$$

2. (a) We are given the linear transformation $f(a, b, c) = (a, a + b, 0)$

The standard basis is $\{e_1, e_2, e_3\}$

where, $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$

Now $f(e_1) = f(1, 0, 0) = (1, 1, 0)$

$f(e_2) = f(0, 1, 0) = (0, 1, 0)$

$f(e_3) = f(0, 0, 1) = (0, 0, 0)$

Now expressing $f(e_1), f(e_2)$ and $f(e_3)$ as a linear combination of the vectors in the basis $\{e_1, e_2, e_3\}$

i.e., $(1, 1, 0) = p(1, 0, 0) + q(0, 1, 0) + r(0, 0, 1)$

$\therefore p = 1, q = 1, r = 0$

i.e., in other words,

$f(e_1) = 1(e_1) + 1(e_2) + 0(e_3)$

similarly, $f(e_2) = 0(e_1) + 1(e_2) + 0(e_3)$

and $f(e_3) = 0(e_1) + 0(e_2) + 0(e_3)$

$\therefore$ Matrix A of the linear transformation with respect to standard basis is

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad ...(I)$$

Now the other basis is $\{e'_1, e'_2, e'_3\}$

where $e'_1 = (1, 1, 0)$, $e'_2 = (0, 1, 1)$, $e'_3 = (1, 1, 1)$

By the definition of 'f' we have

$f(e'_1) = f(1, 1, 0) = (1, 2, 0)$

$f(e'_2) = f(0, 1, 1,) = (0, 1, 0)$

$f(e'_3) = f(1, 1, 1) = (1, 2, 0)$

Now, expressing $f(e'_1), f(e'_2)$ and $f(e'_3)$ as a linear combination of the vectors in the basis $\{e'_1, e'_2, e'_3\}$.

Therefore,

$f(e'_1) = (1, 2, 0) = p(1, 1, 0) + q(0, 1, 1) + r(1, 1, 1)$

$\Rightarrow 1 = p + r, 2 = p + q + r, 0 = q + r$

$\therefore$ we get

$p = 2, q = 1, r = -1$

$\therefore f(e'_1) = 2e'_1 + e'_2 - e'_3 \quad ...(1)$

Now, $f(e'_2) = (0, 1, 0) = p(1, 1, 0) + q(0, 1, 1) + r(1, 1, 1)$

$\Rightarrow 0 = p + r, 1 = p + q + r, 0 = q + r$

$p = 1, q = 1, r = -1.$

$\therefore$ we get $p(e'_2) = e'_1 + e'_2 - e'_3 \quad ...(2)$

Now $f(e'_3) = (1, 2, 0) = P(1, 1, 0) + q(0, 1, 1) + r(1, 1, 1)$

$p + r = 1, p + q + r = 2, q + r = 0$

$\therefore p = 2, q = 1, r = -1$

$\therefore f(e'_3) = 2e'_1 + e'_2 - e'_3 \quad ...(3)$

Hence, matrix B of the transformation with respect to basis $\{e'_1, e'_2, e'_3\}$ is

$$B = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -1 \\ 2 & 1 & -1 \end{bmatrix} \quad ...(II)$$

Now, to prove that there exists an invertible matrix P such that $B = P^{-1}AP$

we must prove that the matrices A and B are similar matrices.

In other words, the characteristic equation or the characteristic roots of A and B are same.

Now, from I we get the eigen values of A are 1 and 1

from II, we see that the first and the third rows are identical.

$$\therefore \text{ Reducing B to } = \begin{bmatrix} 2 & 1 & -1 \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

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Now operating $R_1 \to R_1 - R_2$ we get

$$B \sim \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

(operate $C_1 \to C_1 - C_2$)

$$B \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence, now we get the eigen values of B are 1 and 1.
Thus, A and B have same eigen values.
Therefore, A and B are similar matrices.
Hence, we can say that there exists an invertible matrix P such that $B = P^{-1}AP$.

2. (b) Cayley–Hamilton theorem states that every square matrix satisfies its characteristic equation.

Now, for matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

The characteristic equation is

$$|A - \lambda I| = 0$$

$$\Rightarrow \quad \begin{vmatrix} 1-\lambda & 4 \\ 2 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(3-\lambda) - 8 = 0$$

$$\Rightarrow \quad \lambda^2 - 4\lambda - 5 = 0 \qquad \text{...(1)}$$

By Cayley-Hamilton theorem the matrix A must satisfy (1)

$\therefore$ We have to verify that

$$A^2 - 4A - 5I = 0 \qquad \text{...(2)}$$

Now $\quad A^2 = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix}$$

Now $A^2 - 4A - 5I$

$$= \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - 4\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - 5\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 & 16 \\ 8 & 17 \end{bmatrix} - \begin{bmatrix} 4 & 16 \\ 8 & 12 \end{bmatrix} - \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Hence, $A^2 - 4A - 5I = 0$
Thus, Cayley–Hamilton theorem is verified.
Now we have to compute A^{-1}.
Multiply (2) by A^{-1} we get

$$A - 4I - 5A^{-1} = 0$$

$$\Rightarrow \quad A^{-1} = \frac{1}{5}(A - 4I)$$

$$= \frac{1}{5}\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} - \frac{4}{5}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{5} & \frac{4}{5} \\ \frac{2}{5} & \frac{3}{5} \end{bmatrix} - \begin{bmatrix} \frac{4}{5} & 0 \\ 0 & \frac{4}{5} \end{bmatrix}$$

$$\therefore \quad A^{-1} = \begin{bmatrix} \frac{-3}{5} & \frac{4}{5} \\ \frac{2}{5} & \frac{-1}{5} \end{bmatrix}$$

Now from (2), we get

$$A^2 = 4A + 5I \qquad \text{...(3)}$$

Multiplying both sides of (3) by A, we get

$$A^3 = 4A^2 + 5A \qquad \text{...(4)}$$

$$\therefore \quad A^4 = 4A^3 + 5A^2 \qquad \text{...(5)}$$

$$\text{and} \quad A^5 = 4A^4 + 5A^3 \qquad \text{...(6)}$$

Now, $A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$

Substituting for A^5 from (6)

$= (4A^4 + 5A^3) - 4A^4 - 7A^3 + 11A^2 - A - 10I$

$= -2A^3 + 11A^2 - A - 10I$

$= -2(4A^2 + 5A) + 11A^2 - A - 10I$ [using (4)]

$= 3A^2 - 11A - 10I$

$= 3(4A + 5I) - 11A - 10I$ [using (3)]

$= A + 5I$, which is a linear polynomial in A.

2. (c) The equation of the plane passing through the line $x - y - z = 0 = x - y + 2z - 9$ is

$$x - y - z + k(x - y + 2z - 9) = 0$$

$$\Rightarrow (1 + k)x - (1 + k)y + (2k - 1)z - 9k = 0 \qquad \text{...(1)}$$

Compare it with the general equation of the plane $lx + my + nz = p$

we get $l = 1 + k$, $m = -(1 + k)$

$$n = 2k - 1, \; p = 9k$$

Now, using the condition of tangency to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ by the plane $lx + my + nz = p$, is

$a^2l^2 + b^2m^2 + c^2n^2 = p^2$ ...(2)

Here, we are given the equation of the ellipsoid as $2x^2 + 6y^2 + 3z^2 = 27$

$$\Rightarrow \frac{x^2}{\left(\frac{27}{2}\right)} + \frac{y^2}{\left(\frac{27}{6}\right)} + \frac{z^2}{\left(\frac{27}{3}\right)} = 1$$

$$\therefore a^2 = \frac{27}{2},\ b^2 = \frac{27}{6},\ c^2 = \frac{27}{3}$$

On substituting the values in (2), we get

$$\frac{27}{2}(1+k)^2 + \frac{27}{6}[-(1+k)]^2 + \frac{27}{3}(2k-1)^2 = (9k)^2$$

$\Rightarrow 18(1 + k)^2 + 9(2k - 1)^2 = 81k^2$

$\Rightarrow 2(1 + k)^2 + (2k - 1)^2 = 9k^2$

$\Rightarrow 2 + 2k^2 + 4k + 4k^2 + 1 - 4k = 9k^2$

$\Rightarrow 3k^2 = 3 \ \Rightarrow k = \pm 1$

Putting the values of k in (1), we get two equations of the tangent planes to the given ellipsoid as

when $k = 1$

$\Rightarrow 2x - 2y + z - 9 = 0$

when $k = -1$

$\Rightarrow -3z + 9 = 0$

$\Rightarrow z = 3$

2. (d) The given equations are:

$(a - \lambda)x + by + cz = 0$

$bx + (c - \lambda)y + az = 0$

$cx + ay + (b - \lambda)z = 0$

The above system of equations are simultaneously true when the determinant of the coefficient matrix is zero *i.e.*,

$$\begin{vmatrix} a-\lambda & b & c \\ b & c-\lambda & a \\ c & a & b-\lambda \end{vmatrix} = 0$$

$R_1 \rightarrow R_1 + R_2 + R_3$, we get

$$\begin{vmatrix} a+b+c-\lambda & a+b+c-\lambda & a+b+c-\lambda \\ b & c-\lambda & a \\ c & a & b-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (a+b+c-\lambda)\begin{vmatrix} 1 & 1 & 1 \\ b & c-\lambda & a \\ c & a & b-\lambda \end{vmatrix} = 0$$

$C_1 \rightarrow C_1 - C_2$

$C_2 \rightarrow C_2 - C_3$, we get

$$\Rightarrow (a+b+c-\lambda)\begin{vmatrix} 0 & 0 & 1 \\ \lambda+b-c & c-a-\lambda & a \\ c-a & \lambda+a-b & b-\lambda \end{vmatrix} = 0$$

$\Rightarrow (a + b + c - \lambda)\ [(\lambda + b - c)\ (\lambda + a - b) - (c - a)\ (c - a - \lambda)] = 0$

i.e., $a + b + c - \lambda = 0$

$\cup\ (\lambda + b - c)\ (\lambda + a - b) - (c - a)\ (c - a - \lambda) = 0$

$\cup\ \lambda^2 + (b - c + a - b)\lambda + (a - b)(b - c) - (c - a)^2 + \lambda(c - a) = 0$

$\cup\ \lambda^2 + (a + b - c + c - b - a)\lambda + ab - b^2 - ac + bc - c^2 - a^2 + 2ac = 0$

$\cup\ \lambda^2 - (a^2 + b^2 + c^2 - ab - bc - ca) = 0$

Hence, combining both the above factors, we get

$$[\lambda - (a+b+c)][\lambda^2 - (a^2+b^2+c^2 - ab - bc - ca)] = 0$$

$\Rightarrow \lambda^3 - (a + b + c)\ \lambda^2 - (a^2 + b^2 + c^2 - ab - bc - ca)\lambda + (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca) = 0$

This is a cubic equation in λ

Hence, product of its roots $= \lambda_1\ \lambda_2\ \lambda_3$

$$= \frac{(-1)^3\ (\text{Constant term})}{(\text{Coefficient of } \lambda^3)}$$

$$= \frac{-(a+b+c)(a^2+b^2+c^2-ab-bc-ca)}{(1)}$$

(using the fact that in $Ax^3 + Bx^2 + Cx + D = 0$

product of roots $= (-1)^3\frac{D}{A}$)

$\therefore\ \lambda_1\ \lambda_2\ \lambda_3 = -(a^3 + b^3 + c^3 - 3abc)$.
$= 3abc - a^3 - b^3 - c^3$

$$\therefore D = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}.$$

3. (a) Given basis $B = \{(1,1,1), (1,1,0), (1,0,0)\}$

and $T(a, b, c) = (2b + c, a - 4b, 3a)$

Let $\alpha_1 = (1, 1, 1), \alpha_2 = (1, 1, 0), \alpha_3 = (1, 0, 0)$

By definition of T we have

$T(\alpha_1) = T(1, 1, 1) = (2(1) + 1, 1 - 4, 3)$

$\Rightarrow T(\alpha_1) = (3, -3, 3)$

Similarly, $T(\alpha_2) = T(1, 1, 0) = (2, -3, 3)$

and $T(\alpha_3) = T(1, 0, 0) = (0, 1, 3)$

Now our aim is to express $T(\alpha_1)$, $T(\alpha_2)$ and $T(\alpha_3)$ as linear combination of the vectors in the basis $B[\alpha_1, \alpha_2, \alpha_3]$

Let $(x, y, z) = p\alpha_1 + q\alpha_2 + r\alpha_3$

$(x, y, z) = p(1, 1, 1) + q(1, 1, 0) + r(1, 0, 0)$

$(x, y, z) = (p + q + r, p + q, p)$

$\therefore$ we get $\quad x = p + q + r$

$y = p + q$

$z = p$

Solving these equations, we get

$p = z, q = y - z, r = x - y$

Putting $x = 3, y = -3, z = 3$, we get

$p = 3, q = -6, r = 6$

$\therefore T(\alpha_1) = 3\alpha_1 - 6\alpha_2 + 6\alpha_3$ ...(1)

Similarly, on putting $x = 2, y = -3, z = 3$, we get

$p = 3, q = -6, r = 5$

$\therefore T(\alpha_2) = 3\alpha_1 - 6\alpha_2 + 5\alpha_3$ ...(2)

Similarly, on putting

$x = 0, y = 1, z = 3$, we get

$p = 3, q = -2, r = -1$

$\therefore T(\alpha_3) = 3\alpha_1 - 2\alpha_2 - \alpha_3$ ...(3)

From (1), (2) and (3), we see that the matrix of T relative to the basis

$$\{\alpha_1, \alpha_2, \alpha_3\} = \begin{bmatrix} 3 & -6 & 6 \\ 3 & -6 & 5 \\ 3 & -2 & -1 \end{bmatrix}$$

3. (b) Let the dimensions of the rectangular box be x, y and z where these represent length, breadth and height respectively

Then volume, $V = xyz$

and the surface area of the rectangular box (open at the top)

$= xy + 2z(x + y) = 432$ (given)

Define a Lagrangian function

$F = xyz + \lambda(xy + 2z(x + y) - 432)$

Then for extremum value $dF = 0$

$$\Rightarrow dF = [yz + \lambda(y + 2z)]dx + [xz + \lambda(x + 2z)]dy + [xy + \lambda(2(x + y))]dz = 0$$

Now equating the coefficients, we get

$yz + \lambda(y + 2z) = 0$ ...(1)

$xz + \lambda(x + 2z) = 0$ ...(2)

$xy + 2\lambda(x + y) = 0$ ...(3)

Subtracting (2) from (1) we get

$\Rightarrow (y - x)z + \lambda(y - x) = 0$

$\Rightarrow (y - x)(z + \lambda) = 0$

$\Rightarrow y - x = 0$, other factors cannot be zero.

$\therefore \quad y = x$

Now multiplying equation (2) by 2 and then subtracting the resulting equation from equation (3) we get

$x(y - 2z) + 2\lambda(x + y - x - 2z) = 0$

$\Rightarrow (x + 2\lambda)(y - 2z) = 0$

$\Rightarrow y = 2z$

$\therefore$ The dimensions of the box are of the form

$$x = y = 2z$$

Also, $xy + 2z(x + y) = 432$

$\Rightarrow \quad 12z^2 = 432$

$\Rightarrow \quad z^2 = 36$

$z = 6$

Hence, the dimensions of the box are (12, 12, 6) cm respectively.

3. (c) The equations of the given lines are:

$$\frac{x}{l} - \frac{z}{n} = 1, \; y = 0 \quad ...(1)$$

and $$\frac{y}{m} + \frac{z}{n} = 1, \; x = 0 \quad ...(2)$$

The equation of the line (1) being put in symmetrical form as

$$\frac{x - l}{l} = \frac{y}{0} = \frac{z}{n} \quad ...(I)$$

The equation of any plane through the line (2) is

$$\left(\frac{y}{m} + \frac{z}{n} - 1\right) + \lambda x = 0$$

$$\Rightarrow \lambda x + \left(\frac{1}{m}\right)y + \left(\frac{1}{n}\right)z - 1 = 0 \quad ...(3)$$

If the plane (3) is parallel to the line (I), then the normal to the plane (3) whose d.c.'s are $\lambda, \frac{1}{m}, \frac{1}{n}$ will be perpendicular to the line (I), and so we have

$$l\lambda + 0\left(\frac{1}{m}\right) + n\left(\frac{1}{n}\right) = 0$$

$$\lambda = \frac{-1}{l}$$

Putting this value of λ in (3), the equation of the plane containing the line (2) and parallel to the line (I) is

$$-\frac{x}{l} + \frac{y}{m} + \frac{z}{n} - 1 = 0$$

or $$\frac{x}{l} - \frac{y}{m} - \frac{z}{n} + 1 = 0 \qquad ...(4)$$

Clearly, $(l, 0, 0)$ is a point on the line (I) [*i.e.*, (1)]

Hence, the length $2c$ or shortest distance = perpendicular distance of $(l, 0, 0)$ from the plane (4).

$$\therefore \quad 2c = \frac{\left| l\left(\frac{1}{l}\right) - 0 - 0 + 1 \right|}{\sqrt{\left(\frac{1}{l}\right)^2 + \left(\frac{1}{m}\right)^2 + \left(\frac{1}{n}\right)^2}}$$

$$= \frac{2}{\sqrt{\frac{1}{l^2} + \frac{1}{m^2} + \frac{1}{n^2}}}$$

$$\Rightarrow \quad \sqrt{\frac{1}{l^2} + \frac{1}{m^2} + \frac{1}{n^2}} = \frac{1}{c}$$

Hence, $\frac{1}{l^2} + \frac{1}{m^2} + \frac{1}{n^2} = \frac{1}{c^2}$.

3. (d) $$f(x) = \begin{cases} \frac{\sin 2x}{x}, & \text{when } x \neq 0 \\ 1, & \text{when } x = 0 \end{cases}$$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{\sin 2x}{x}$$

$$= \lim_{x \to 0} \frac{\sin 2x}{2x} \cdot 2 = 2$$

so that $\lim_{x \to 0} f(x) \neq f(0)$

Hence, the limit exists but is not equal to the value of the function at the origin. Thus, the function has a removable discontinuity at the origin.

4. (a) The required volume cut off from the cylinder $x^2 + y^2 = ax$ by the planes $z = mx$ and $z = nx$ is:

$$= 4\int_0^{x_1} \int_0^{\sqrt{ax - x^2}} \int_{mx}^{nx} dz\, dy\, dx$$

$$= 4\int_0^{x_1} \int_0^{\sqrt{ax - x^2}} (n - m)x\, dy\, dx$$

$$= 4(n - m)\int_0^{x_1} \int_0^{\sqrt{ax - x^2}} x\, dy\, dx$$

$$= 4(n - m)\int_0^{x_1} x\sqrt{ax - x^2}\, dx$$

$$x^2 + y^2 = ax$$

$$\Rightarrow \quad x^2 - ax + y^2 = 0$$

$$\Rightarrow \quad \left(x - \frac{a}{2}\right)^2 + y^2 = \frac{a^2}{4}$$

$\therefore$ Polar equation can be formed as

$$x - \frac{a}{2} = \frac{a}{2}\cos\theta$$

$$y = \frac{a}{2}\sin\theta$$

Changing into polar coordinates we get the result as

$$= 4(n - m)\pi \int_0^a r^2 dr = 4(n - m)\pi \left[\frac{r^3}{3}\right]_0^a$$

$$= \frac{4\pi}{3}(n - m)a^3$$

4. (b) The equation of spheres passing through the origin is

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz = 0 \qquad ...(1)$$

Now, the planes parallel to the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 0$ is given as

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = k \quad \text{(where } k \text{ is any constant)}$$

The x-intercept of the above plane is given as

$$\frac{x_{\text{intercept}}}{a}+0+0 = k$$

$$x_{\text{intercept}} = ak$$

$\therefore$ Coordinates of the point is $(ak, 0, 0)$. Similarly, y intercept is bk and z intercept is ck.

Thus, the four points through which the set of spheres passes are $(0, 0, 0)$, $(ak, 0, 0)$, $(0, bk, 0)$ $(0, 0, ck)$.

Putting these values one by one in equation (1) we get

$$u = \frac{-ak}{2}, \ v = \frac{-bk}{2}, \ w = \frac{-ck}{2}$$

Hence, the equation of a system–spheres passing through the origin and each set of points where planes parallel to the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 0$ cut the coordinate axes is

$$x^2 + y^2 + z^2 - k(ax + by + cz) = 0$$

The equation of other sphere cut orthogonally by the above system of spheres is given as

$$x^2 + y^2 + 2fx + 2gy + 2hz = 0$$

Thus, by the condition of orthogonally, *i.e.*,

$$2u_1u_2 + 2v_1v_2 + 2w_1w_2 = d_1 + d_2$$

Putting the values, we get

$$2\left(\frac{-ak}{2}\right)(f)+2\left(\frac{-bk}{2}\right)(g)+2\left(\frac{-ck}{2}\right)(h) = 0+0$$

$\Rightarrow -afk - bgk - chk = 0$

$\Rightarrow k(af + bg + ch) = 0$

either $k = 0$ or $af + bg + ch = 0$

But $k \neq 0$. (as it will represent the given plane itself, not the plane parallel to the given plane.)

Hence, $af + bg + ch = 0$.

4. (c) Let the equation of the plane, making equal intercepts on the positive parts of the axes, be

$$x + y + z = k \quad \text{...(1)}$$

(where $k > 0$ and indicate the value of the intercept)

Now, it is given that the above plane touches the ellipsoid

$$x^2 + 4y^2 + 9z^2 = 36$$

Therefore, by using the condition of tangency,

(*i.e.*, when the plane $lx + my + nz = p$ touches the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$) is given by

$$a^2l^2 + b^2m^2 + c^2n^2 = p^2 \quad \text{...(2)}$$

we have [from (1)]

Here, $l = m = n = 1$ and $p = k$.

Also, rearranging the given equation of ellipsoid as

$$\frac{x^2}{36}+\frac{4y^2}{36}+\frac{9z^2}{36} = 1$$

$$\frac{x^2}{(6)^2}+\frac{y^2}{(3)^2}+\frac{z^2}{(2)^2} = 1$$

$\therefore$ We get the values as $a = 6$, $b = 3$, $c = 2$

Now, putting values in equation (2) we get

$36(1) + 9(1) + 4(1) = k^2$

$\Rightarrow \quad k^2 = 49$

$\Rightarrow \quad k = \pm 7$

But $\quad k \neq -7 \quad$ (as $k > 0$)

$\therefore \quad k = 7$

Hence, the equation of the required plane is

$$x + y + z = 7$$

4. (d) Let $\mathrm{I} = \int_0^a \sqrt{\frac{x^3}{a^3 - x^3}}\, dx$

Let $x^3 = a^3 \sin^2\theta \quad$ when $x \to 0, \theta \to 0$

$\Rightarrow \quad x = a \sin^{2/3}\theta \quad$ when $x \to a, \theta \to \frac{\pi}{2}$

$$\therefore \quad dx = \frac{2}{3} a \sin^{-1/3}\theta \cos\theta \, d\theta$$

$$\therefore \quad \mathrm{I} = \int_0^{\pi/2} \sqrt{\frac{a^3 \sin^2\theta}{a^3 - a^3 \sin^2\theta}}$$

$$\frac{2a}{3}\sin^{-1/3}\theta\cos\theta d\theta$$

$$= \int_0^{\pi/2} \frac{\sin\theta}{\cos\theta}\frac{2}{3}a\sin^{-1/3}\theta\cos\theta\, d\theta$$

$$= \frac{2}{3}a\int_0^{\pi/2} \sin^{2/3}\theta\, d\theta$$

Now, using formula

$$\int_0^{\pi/2} \sin^p\theta\cos^q\theta\, d\theta = \frac{\sqrt{\left(\frac{p+1}{2}\right)}\sqrt{\left(\frac{q+1}{2}\right)}}{2\sqrt{\left(\frac{p+q+2}{2}\right)}}$$

$$\therefore \mathrm{I} = \frac{2}{3}a\frac{\sqrt{\left(\frac{\frac{2}{3}+1}{2}\right)}\sqrt{\left(\frac{0+1}{2}\right)}}{2\sqrt{\left(\frac{\frac{2}{3}+0+2}{2}\right)}}$$

[*i.e.*, putting $p = \frac{2}{3}$ and $q = 0$]

$$\mathrm{I} = \frac{2}{3}a\frac{\sqrt{\frac{5}{6}}\sqrt{\frac{1}{2}}}{2\sqrt{\frac{4}{3}}}$$

$$= \frac{\frac{\sqrt{\pi}a}{3}\sqrt{\frac{5}{6}}}{\sqrt{\left(\frac{1}{3}+1\right)}} \quad \left(\text{using } \sqrt{\frac{1}{2}} = \sqrt{\pi}\right)$$

$$= \frac{a\sqrt{\pi}}{3}\frac{\sqrt{\frac{5}{6}}}{\frac{1}{3}\sqrt{\frac{1}{3}}} \quad \left(\text{using } \sqrt{n+1} = n\sqrt{n}\right)$$

$$\therefore \mathrm{I} = a\sqrt{\pi}\,\frac{\sqrt{\frac{5}{6}}}{\sqrt{\frac{1}{3}}}.$$

Section-B

5. (a) $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y$

This is the general form of first degree linear differential equation.

It can be rearranged in the form of

$\frac{dy}{dx} + \mathrm{P}y = \mathrm{Q}$ where P and Q are function of x or constants.

Dividing by (sec y) to both sides, we get

$$\frac{1}{\sec y}\frac{dy}{dx} - \frac{\tan y}{\sec y}\left(\frac{1}{1+x}\right) = (1+x)e^x$$

$$\Rightarrow \cos y\frac{dy}{dx} - \sin y\left(\frac{1}{1+x}\right) = e^x(1+x) \quad ...(1)$$

Let $\sin y = t$

On differentiation, we get $\cos y\frac{dy}{dx} = \frac{dt}{dx}$

Putting in equation (1) we get

$$\frac{dt}{dx} - \frac{1t}{(1+x)} = e^x(1+x) \quad ...(2)$$

which is the general form of first order and first degree linear differential equation.

Now, solving this linear differential equation

Integrating factor (I.F.)

$$= e^{\int \frac{-1}{(1+x)}dx} = e^{-\ln|1+x|}$$

$$\text{I.F.} = \frac{1}{(1+x)}$$

$\therefore$ Solution of the differential equation (2) is given as

$$t(\text{I.F.}) = \int \mathrm{Q}(\text{I.F.})\,dx + \mathrm{C}$$

where C is a constant of integration and Q is the right side of equation (2).

Putting values of Q and I.F. we get

$$\frac{t}{1+x} = \int e^x(1+x)\cdot\frac{1}{(1+x)}dx + \mathrm{C}$$

$$= \int e^x\,dx + \mathrm{C} = e^x + \mathrm{C}$$

Since, the original differential equation is a function of x and y.

$\therefore$ Replace t by a function of y (which we let)

Hence, $\dfrac{\sin y}{1+x} = e^x + C$

Thus, the required solution is

$$\frac{\sin y}{1+x} - e^x = C$$

5. (b) The given equation is

$$x^3p^2 + x^2py + a^3 = 0 \qquad ...(1)$$

solving for y, $y = -xp - \dfrac{a^3}{x^2p}$ ...(2)

Differentiating (2) with respect to (x) writing p for $\dfrac{dy}{dx}$, we have

$$p = -p - x\frac{dp}{dx} - a^3\left(\frac{-2}{x^3p} - \frac{1}{x^2p^2}\frac{dp}{dx}\right)$$

$$\Rightarrow 2p + x\frac{dp}{dx} - \frac{2a^3}{x^3p} - \frac{a^3}{x^2p^2}\frac{dp}{dx} = 0$$

$$\Rightarrow 2p\left(1 - \frac{a^3}{x^3p^2}\right) + x\frac{dp}{dx}\left(1 - \frac{a^3}{x^3p^2}\right) = 0$$

$$\text{or} \quad \left(1 - \frac{a^3}{x^3p^2}\right)\left(2p + x\frac{dp}{dx}\right) = 0$$

Omitting the first factor since it does not involve $\dfrac{dp}{dx}$, we get

$$2p + x\frac{dp}{dx} = 0$$

$$\Rightarrow \frac{1}{p}dp + \frac{2}{x}dx = 0$$

Integrating, we get $\log p + 2\log x = \log C$ (where $\log C$ is an integration constant)

$$\Rightarrow \log(px^2) = \log C$$

$$\Rightarrow px^2 = C$$

$$\text{or} \quad p = \frac{C}{x^2} \qquad ...(3)$$

Eliminating p between (1) and (3), the required general solution is

$$x^3\frac{C^2}{x^4} + x^2y\left(\frac{C}{x^2}\right) + a^3 = 0$$

$$\Rightarrow \frac{C^2}{x} + Cy + a^3 = 0$$

$$\Rightarrow C^2 + xyC + a^3x = 0 \qquad ...(4)$$

By (4), C-discriminant relation is

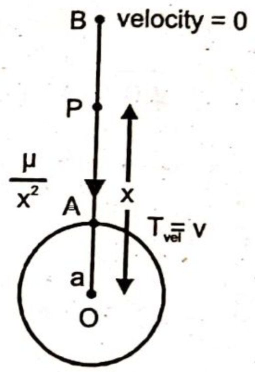
$$(xy)^2 - 4(1)(a^3x) = 0$$

$$\Rightarrow x(xy^2 - 4a^3) = 0$$

Now, $x = 0$ and $xy^2 - 4a^3 = 0$ both satisfy equation (1) and hence required singular solutions are

$x = 0$ and $xy^2 - 4a^3 = 0$.

5. (c)



Let O be the centre of the earth and A be the point of projection on the earth's surface.

If P be the position of the particle at any time t, such that OP = x, then the acceleration at P = $\dfrac{\mu}{x^2}$ directed towards O.

$\therefore$ The equation of motion of the particle at P is

$$\frac{d^2x}{dt^2} = \frac{-\mu}{x^2}$$

(Negative sign indicates that acceleration acts in the direction of x decreasing.)

But at the point A, on the surface of the earth,

$$x = a \text{ and } \frac{d^2x}{dt^2} = -g$$

$$\therefore \quad -g = \frac{-\mu}{a^2} \quad \text{or } \mu = a^2g$$

$$\therefore \quad \frac{d^2x}{dt^2} = \frac{-a^2g}{x^2}$$

Multiplying by $2\left(\frac{dx}{dt}\right)$ and integrating with respect to (t) we get

$$\left(\frac{dx}{dt}\right)^2 = \frac{2a^2g}{x} + C$$

where C is a constant

But when $x \to \infty$, $\frac{dx}{dt}$ (velocity) $\to 0$

$\therefore \quad C = 0$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2 = \frac{2a^2g}{x}$$

(Here +ve sign is taken because the particle is moving in the direction of x increasing)

$$\Rightarrow \quad \frac{dx}{dt} = a\sqrt{\frac{2g}{x}}$$

Separating the variables, we have

$$dt = \frac{1}{a\sqrt{2g}}\sqrt{x}\,dx$$

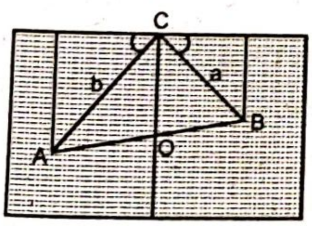
Integrating between the limits $x = a$ to $x = a + h$, the required time t to reach height h is given by

$$t = \frac{1}{a\sqrt{2g}}\int_a^{a+h}\sqrt{x}\,dx = \frac{1}{a\sqrt{2g}}\left[\frac{2}{3}x^{3/2}\right]_a^{a+h}$$

$$= \frac{1}{3a}\sqrt{\frac{2}{g}}\left[(a+h)^{3/2} - a^{3/2}\right]$$

$$= \frac{1}{3}\sqrt{\frac{2a}{g}}\left[\left(1+\frac{h}{a}\right)^{3/2} - 1\right].$$

5. (d)



Let the vertical through C meets AB at O.

then $\angle ACO = \angle BCO = \frac{1}{2}\angle C$

Area of $\Delta AOC = \frac{1}{2}AC \,.\, OC \sin \angle ACO$

Area of $\Delta BOC = \frac{1}{2}BC \,.\, OC \sin \angle BCO$

The depth of the centre of gravity (C.G.) of ΔAOC below the surface of the liquid

$$= \frac{1}{3}(AC \cos \angle ACO + OC)$$

and the depth of the C.G of ΔBOC below the surface of the liquid

$$= \frac{1}{3}(BC \cos \angle BCO + OC)$$

$$\therefore \quad \frac{\text{Pressure on } \Delta AOC}{\text{Pressure on } \Delta BOC}$$

$$= \frac{\frac{1}{2}AC\cdot OC \sin \angle ACO\cdot\frac{1}{3}(AC\cos\angle ACO + OC)\cdot w}{\frac{1}{2}BC\cdot OC \sin \angle BCO\cdot\frac{1}{3}(BC\cos\angle BCO + OC)\cdot w}$$

$$= \frac{\left(\frac{1}{2}b\,OC\sin\frac{C}{2}\right)\left(\frac{1}{3}\left(b\cos\frac{C}{2}+OC\right)\right)}{\left(\frac{1}{2}a\,OC\sin\frac{C}{2}\right)\left(\frac{1}{3}\left(a\cos\frac{C}{2}+OC\right)\right)}$$

$$= \frac{b\left(b\cos\frac{C}{2}+OC\right)}{a\left(a\cos\frac{C}{2}+OC\right)}$$

From Δ's BCO and ACO, we have

$$\frac{CO}{\sin B} = \frac{OB}{\sin\frac{C}{2}} \text{ and } \frac{CO}{\sin A} = \frac{AO}{\sin\frac{C}{2}} \quad ...(1)$$

$$\text{Also } \frac{AO}{b} = \frac{OB}{a} = \frac{AO+OB}{b+a} = \frac{c}{b+a} \quad ...(2)$$

$\therefore$ The required ratio

$$= \frac{b\left(b\cos\frac{C}{2} + \frac{OB\sin B}{\sin\frac{C}{2}}\right)}{a\left(a\cos\frac{C}{2} + \frac{AO\sin A}{\sin\frac{C}{2}}\right)} \quad \text{[using (1)]}$$

$$= \frac{b(b\sin C+2OB\sin B)}{a(a\sin C+2OA\sin A)}$$

$$\left(\because \frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}\right)$$

$$= \frac{b\left(b\sin C+2OB\frac{b\sin C}{c}\right)}{a\left(a\sin C+2OA\,a\frac{\sin C}{c}\right)}$$

$$= \frac{b^2}{a^2}\cdot\left(\frac{c+2OB}{c+2OA}\right)=\frac{b^2}{a^2}\cdot\frac{\left(c+\frac{2ac}{b+a}\right)}{\left(c+\frac{2bc}{b+a}\right)}$$

[using (2)]

$$= \frac{b^2}{a^2}\cdot\left[\frac{c(a+b)+2ac}{c(a+b)+2bc}\right]$$

$$= \frac{b^2(3a+b)}{a^2(a+3b)} = \frac{b^3+3ab^2}{a^3+3a^2b}.$$

5. (e) Given $u = x + y + z$, $v = x^2 + y^2 + z^2$, $w = yz + zx + xy$

$$\text{grad } u=\nabla u=\left(\hat{i}\frac{\partial}{\partial x}+\hat{j}\frac{\partial}{\partial y}+\hat{k}\frac{\partial}{\partial z}\right)(x+y+z)$$

$$= \hat{i}\frac{\partial}{\partial x}(x+y+z)+\hat{j}\frac{\partial}{\partial y}(x+y+z)+\hat{k}\frac{\partial}{\partial z}(x+y+z)$$

$$\nabla u=\hat{i}+\hat{j}+\hat{k}$$

Now,

$$\text{grad } v = \hat{i}\frac{\partial}{\partial x}(x^2+y^2+z^2)+\hat{j}\frac{\partial}{\partial y}(x^2+y^2+z^2)+\hat{k}\frac{\partial}{\partial z}(x^2+y^2+z^2)$$

$$\nabla v=2x\hat{i}+2y\hat{j}+2z\hat{k}$$

Now,

$$\text{grad } w = \hat{i}\frac{\partial}{\partial x}(yz+zx+xy)+\hat{j}\frac{\partial}{\partial y}(yz+zx+xy)+\hat{k}\frac{\partial}{\partial z}(yz+zx+xy)$$

$$\nabla w=(y+z)\hat{i}+(z+x)\hat{j}+(x+y)\hat{k}$$

To prove that ∇u, ∇v and ∇w coplanar, we must have the following condition to be true.

$$i.e., \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & z+x & x+y \end{vmatrix}=0$$

On carrying out operations on LHS, we get

$C_1 \to C_1 - C_2$

$C_2 \to C_2 - C_3$, we get

$$= \begin{vmatrix} 0 & 0 & 1 \\ 2(x-y) & 2(y-z) & 2z \\ y-x & z-y & x+y \end{vmatrix}$$

Solving the determinant we get

$$= 1[2(x-y)(z-y)-2(y-z)(y-x)]$$

$$= 2[(x-y)(z-y)-(x-y)(z-y)]$$

$= 0 =$ RHS

Hence, we can say that grad u, grad v and grad w are coplanar.

6. (a) $x^2y\dfrac{d^2y}{dx^2}+\left(x\dfrac{dy}{dx}-y\right)^2=0.$

The given equation can be rewritten as

$$x^2\left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right]-\left[2xy\frac{dy}{dx}-y^2\right]=0$$

$$\Rightarrow \left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right]-\frac{\left[2xy\left(\frac{dy}{dx}\right)-y^2\right]}{x^2}=0$$

$$\text{or } \frac{d}{dx}\left(y\frac{dy}{dx}\right)-\frac{d}{dx}\left(\frac{y^2}{x}\right)=0$$

Integrating, we get

$$y\frac{dy}{dx}-\frac{y^2}{x}=C_1 \quad ...(1)$$

This is Bernoulli form

$\therefore$ Putting $y^2 = v$, so that $2y\dfrac{dy}{dx}=\dfrac{dv}{dx}$

$\therefore$ (1) becomes

$$\frac{1}{2}\frac{dv}{dx}-\frac{v}{x}=C_1 \Rightarrow \frac{dv}{dx}-\frac{2v}{x}=2C_1$$

This is the first order linear differential equation.

Its I.F. $= e^{-\int \frac{2}{x}dx} = e^{-2\ln(x)} = \frac{1}{x^2}$

Hence, solution is

$$v\left(\frac{1}{x^2}\right) = 2C_1\int \frac{1}{x^2}dx + C_2$$

$$\frac{y^2}{x^2} = \frac{-2C_1}{x} + C_2$$

$$\Rightarrow \quad y^2 = x(C_2x - 2C_1).$$

6. (b) The surface $x^2 + y^2 + z^2 - 2ax + az = 0$ meets the plane $z = 0$ in the circle C given by $x^2 + y^2 - 2ax = 0$, $z = 0$.

The polar equation of the circle C lying in the xy-plane is $r = 2a\cos\theta$, $0 \le \theta < \pi$ or the equation $x^2 + y^2 - 2ax = 0$ can be written as $(x-a)^2 + y^2 = a^2$. Therefore, the parametric equations of the circle C can be taken as

$$x = a + a\cos t,\ y = a\sin t,\ z = 0 \qquad 0 \le t < 2\pi$$

Method 1: By application of double integral:

Let s denote the portion of the surface $x^2 + y^2 + z^2 - 2ax + az = 0$ lying above the plane $z = 0$

and s_1 denote the plane region bounded by the circle C. By an application of divergence theorem, we have

$$\iint_s \text{curl}\,\hat{F}\cdot\hat{n}\,ds = \iint_{s_1} \text{curl}\,\vec{F}\cdot\hat{k}\,ds$$

Now, curl $\hat{F}\cdot\hat{k} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}\cdot\hat{k}$

where $\vec{F} = F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$

Given

$$\vec{F} = (y^2+z^2-x^2)\hat{i} + (z^2+x^2-y^2)\hat{j} + (x^2+y^2-z^2)\hat{k}$$

$$= F_1\hat{i} + F_2\hat{j} + F_3\hat{k}$$

Now, solving curl $\vec{F}\cdot\hat{k} = (\nabla\times\vec{F})\cdot\hat{k}$ as follows:

$(\nabla\times\vec{F})\cdot\hat{k}$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2+z^2-x^2 & z^2+x^2-y^2 & x^2+y^2-z^2 \end{vmatrix}\cdot\hat{k}$$

$$= \left(\frac{\partial}{\partial x}(z^2+x^2-y^2) - \frac{\partial}{\partial y}(y^2+z^2-x^2)\right)(\hat{k}\cdot\hat{k})$$

$$= 2(x-y) \qquad (\because \hat{i}\cdot\hat{k} = \hat{j}\cdot\hat{k} = 0)$$

$$\therefore \iint_s \text{curl}\,\vec{F}\cdot n\,ds = \iint_{s_1} \text{curl}\,\vec{F}\cdot k\,ds$$

$$= \iint_{s_1} 2(x-y)\,ds$$

$$= 2\int_{\theta=0}^{\pi}\int_{r=0}^{2a\cos\theta} (r\cos\theta - r\sin\theta)r\,d\theta\,dr$$

[Changing to polars, we get $x = r\cos\theta$, $y = r\sin\theta$ and $ds = r\,d\theta\,dr$]

$$= 2\int_{\theta=0}^{\pi}\left[(\cos\theta-\sin\theta)\int_{r=0}^{2a\cos\theta} r^2dr\right]d\theta$$

$$= 2\int_{\theta=0}^{\pi}(\cos\theta-\sin\theta)\left(\frac{r^3}{3}\right)_0^{2a\cos\theta}d\theta$$

$$= \frac{2\times 8a^3}{3}\int_0^{\pi}\left(\cos^4\theta - \cos^3\theta\sin\theta\right)d\theta$$

$$= \frac{16a^3}{3}\int_0^{\pi}\cos^4\theta\,d\theta \quad \left[\because \int_0^{\pi}\cos^3\theta\sin\theta\,d\theta = 0\right]$$

$$= 2\times\frac{16a^3}{3}\int_0^{\pi/2}\cos^4\theta\,d\theta$$

$$= 2\times\frac{16a^3}{3}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2} = 2\pi a^3$$

Method 2: By the application of line integral

$$\int_C F\cdot ds = \int_C\left[(y^2+z^2-x^2)dx + (z^2+x^2-y^2)dy + (x^2+y^2-z^2)dz\right]$$

$$= \int_C (y^2 - x^2)dx + (x^2 - y^2)dy$$

($\because$ as on C, $z = 0$ and $dz = 0$)

$$= \int_0^{2\pi} (x^2 - y^2)\left(\frac{dy}{dt} - \frac{dx}{dt}\right)dt$$

$$= \int_0^{2\pi} \left[(a + a\cos t)^2 - a^2\sin^2 t\right] (a\cos t + a\sin t)dt$$

$$= a^3 \int_0^{2\pi} (1 + \cos^2 t + 2\cos t - \sin^2 t) (\cos t + \sin t)dt$$

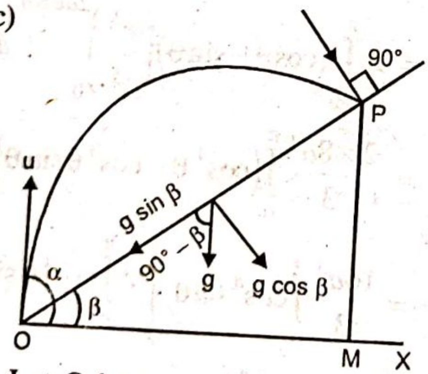
$$= a^3 \int_0^{2\pi} 2\cos^2 t\, dt, \text{ the other integral, vanish}$$

$$= 2a^3 \times 4 \int_0^{\pi/2} \cos^2 t\, dt = 8a^3 \times \frac{1}{2} \times \frac{\pi}{2} = 2\pi a$$

Note: We have seen that method 1 and method 2 gives the same result.
In other words, stokes's theorem verified

or $\iint_s \text{curl } \vec{F} \cdot \hat{n}\, ds = \int_C \vec{F} \cdot ds$

6. (c)



Let O be the point of projection, u be the velocity of projection, α be the angle of projection and P be the point where the particle strikes the plane at right angles. Let T be the time of flight from O to P. Then by the formula for the time of flight on an inclined plane, we have

$$T = \frac{2u\sin(\alpha - \beta)}{g\cos\beta} \quad ...(1)$$

Since the particle strikes the inclined plane at right angles at P, therefore the velocity of the particle at P along the inclined plane is zero.

Also the resolved part of the velocity of the particle at O along the inclined plane is $u\cos(\alpha - \beta)$ upwards and the resolved part of the acceleration g along the inclined plane is $g\sin\beta$ downwards.

So, considering the motion of the particle from O to P along the inclined plane and using the formula

$v = u + at$, we have

$0 = u\cos(\alpha - \beta) - g\sin\beta T$

$$\text{or, } T = \frac{u\cos(\alpha - \beta)}{g\sin\beta} \quad ...(2)$$

Equating the values of T from (1) and (2), we have

$$\frac{2u\sin(\alpha - \beta)}{g\cos\beta} = \frac{u\cos(\alpha - \beta)}{g\sin\beta}$$

$$\text{or, } \tan(\alpha - \beta) = \frac{1}{2}\cot\beta \quad ...(3)$$

The condition for striking the plane at right angles.

(*i*) To prove $T = \dfrac{2u}{g\sqrt{1 + 3\sin^2\beta}}$

Proof: From (2) we have

$$T = \frac{u}{g\sin\beta}\cos(\alpha - \beta) = \frac{u}{g\sin\beta\sec(\alpha - \beta)}$$

$$= \frac{u}{g\sin\beta\sqrt{1 + \tan^2(\alpha - \beta)}}$$

$$= \frac{u}{g\sin\beta\sqrt{1 + \frac{1}{4}\cot^2\beta}}$$

[substituting value from (3)]

$$= \frac{2u\sin\beta}{g\sin\beta\sqrt{4\sin^2\beta + \cos^2\beta}}$$

$$= \frac{2u}{g\sqrt{\sin^2\beta + \cos^2\beta + 3\sin^2\beta}}$$

$$\therefore T = \frac{2u}{g\sqrt{1+3\sin^2\beta}}.$$

(*ii*) Range, R on the plane $= \dfrac{2u^2}{g}\dfrac{\sin\beta}{1+3\sin^2\beta}$

Proof: Let R be the range on the inclined plane then R = OP considering the motion from O to P along the inclined plane and using the formula $v^2 = u^2 + 2as$, we have

$0 = u^2\cos^2(\alpha - \beta) - 2g\sin\beta\, R$

$$\text{or, } R = \frac{u^2\cos^2(\alpha-\beta)}{2g\sin\beta} = \frac{u^2}{2g\sin\beta\sec^2(\alpha-\beta)}$$

$$= \frac{u^2}{2g\sin\beta\left[1+\tan^2(\alpha-\beta)\right]}$$

$$= \frac{u^2}{2g\sin\beta\left[1+\frac{1}{4}\cot^2\beta\right]} \qquad \text{[From (3)]}$$

$$= \frac{4u^2\sin^2\beta}{2g\sin\beta(4\sin^2\beta+\cos^2\beta)}$$

Hence, Range, $R = \dfrac{2u^2\sin\beta}{g(1+3\sin^2\beta)}$.

(*iii*) The vertical height of the point struck is

$$\frac{2u^2\sin^2\beta}{g(1+3\sin^2\beta)}$$

Proof: The vertical height of P above O = PM

$$= OP\sin\beta = R\sin\beta = \frac{2u^2\sin^2\beta}{g(1+3\sin^2\beta)}.$$

6. (d) Solve $\dfrac{d^4y}{dx^4}+2\dfrac{d^2y}{dx^2}+y = x^2\cos x$

Let $D \equiv \dfrac{d}{dx}$, then the given differential equation becomes

$(D^4 + 2D^2 + 1)y = x^2\cos x$

This equation is the differential equation of first order with constant coefficients. It is solved by the following method.

The auxiliary equation is

$$m^4 + 2m^2 + 1 = 0$$

$$\Rightarrow \quad (m^2+1)^2 = 0$$

$$\Rightarrow \quad m = \pm i$$

Thus the complementary function is given by

$y = (C_1 + C_2x)\cos x + (C_3 + C_4x)\sin x$

where C_1, C_2, C_3 and C_4 are arbitrary constants.

Now, the particular integral is given by

$$y = \frac{1}{(1+2D^2+D^4)}x^2\cos x$$

$$= \frac{1}{(D^2+1)^2}x^2\cos x$$

$$y = \text{Real part of}\left(\frac{1}{(D^2+1)^2}x^2e^{ix}\right)$$

$$\left[\because e^{ix} = \cos x + i\sin x\right]$$

Now, solving $\dfrac{1}{(D^2+1)^2}x^2e^{ix}$ as

$$= e^{ix}\frac{1}{\left[(D+i)^2+1\right]^2}x^2$$

$$\left(\text{Using formula } \frac{1}{f(D)}e^{ax}V = e^{ax}\cdot\frac{1}{f(D+a)}V\right)$$

where, V is any function of x

Here $V = x^2$

$f(D) = (D^2+1)^2$

$a = i$

$$= e^{ix}\frac{1}{\left[D^2+i^2+2iD+1\right]^2}x^2$$

$$= e^{ix}\frac{1}{(D^2+2iD)^2}x^2 \qquad (\because i^2 = -1)$$

$$= e^{ix}\frac{1}{(2iD)^2\left[1+\frac{D^2}{2iD}\right]^2}x^2$$

$$= e^{ix}\frac{1}{-4D^2}\left[1+\frac{D}{2i}\right]^{-2}x^2$$

$$= \frac{-1}{4}e^{ix}\frac{1}{D^2}\left(1+(-2)\left(\frac{D}{2i}\right) + \frac{(-2)(-2-1)}{2!}\left(\frac{D}{2i}\right)^2 + \dots\right)x^2$$

[using expansion of $(1+x)^n = 1+nx+\frac{n(n-1)x^2}{2!}+\dots$]

$$= \frac{-1}{4}e^{ix}\frac{1}{D^2}\left(1-\frac{D}{i}-\frac{3}{4}D^2+\dots\right)x^2$$

$$= \frac{-e^{ix}}{4}\frac{1}{D^2}\left[x^2-\frac{1}{i}(2x)-\frac{3}{4}(2)+0+0+\dots\right]$$

$$= \frac{-e^{ix}}{4}\frac{1}{D^2}\left[\left(x^2-\frac{3}{2}\right)+i(2x)\right]$$

$$\left[\because \frac{1}{i}=\frac{i}{i^2}=-i\right]$$

$$= \frac{-e^{ix}}{4}\left[\frac{1}{D}\int\left(x^2-\frac{3}{2}\right)dx+2i\frac{1}{D}\int x\,dx\right]$$

$$\left[\because \frac{1}{D}=\int dx\right]$$

$$= \frac{-e^{ix}}{4}\left[\int\left(\frac{x^3}{3}-\frac{3x}{2}\right)dx+2i\int\frac{x^2}{2}dx\right]$$

$$= \frac{-e^{ix}}{4}\left[\frac{x^4}{12}-\frac{3x^2}{4}+2i\left(\frac{x^3}{6}\right)\right]$$

$$= \frac{-e^{ix}}{4}\left[\frac{x^4}{12}-\frac{3x^2}{4}+\frac{ix^3}{3}\right] \quad \text{...(1)}$$

Note: While we want the real part of (1), we must open e^{ix} as $(\cos x + i \sin x)$

$\therefore$ (1) equation can be arranged as

$$= \frac{-1}{4}(\cos x+i\sin x)\left[\frac{x^4-9x^2}{12}+\frac{i}{3}x^3\right]$$

$$= \left(\frac{9x^2-x^4}{48}-\frac{i}{12}x^3\right)(\cos x+i\sin x)$$

$$= \left[\left(\frac{9x^2-x^4}{48}\right)\cos x+\frac{1}{12}x^3\sin x\right]+ i\left[\frac{-1}{12}x^3\cos x+\frac{\sin x}{48}(9x^2-x^4)\right]$$

The real part of this is the particular integral.

$\therefore$ Particular Integral,

$$y = \frac{x^2}{48}\cos x(9-x^2)+\frac{1}{12}x^3\sin x$$

Thus, the general solution is given by

$y = \text{C.F} + \text{P.I}$

$\therefore y = (C_1 + C_2x)\cos x + (C_3 + C_4x)\sin x + \frac{x^2}{48}(9 - x^2)\cos x + \frac{x^3}{12}\sin x$ is the required solution.

7. (a) Here, the central acceleration,

$$p = \mu[r^5 - c^4r] = \mu\left[\frac{1}{u^5}-\frac{c^4}{u}\right] \quad \left(\because r=\frac{1}{u}\right)$$

$\therefore$ The differential equation of the path is

$$h^2\left[u+\frac{d^2u}{d\theta^2}\right] = \frac{p}{u^2} = \frac{\mu}{u^2}\left[\frac{1}{u^5}-\frac{c^4}{u}\right]$$

$$\Rightarrow u^2 = h^2\left[u+\frac{d^2u}{d\theta^2}\right] = \frac{p}{u^2} = \mu\left[\frac{1}{u^7}-\frac{c^4}{u^3}\right]$$

Multiplying both sides by $2\left(\frac{du}{d\theta}\right)$, we get

$$h^2\left[2\left(\frac{du}{d\theta}\right)u+2\left(\frac{du}{d\theta}\right)\frac{d^2u}{d\theta^2}\right] = \frac{2p}{u^2}\left(\frac{du}{d\theta}\right)$$

$$\frac{h^2d}{d\theta}\left[u^2+\left(\frac{du}{d\theta}\right)^2\right] = \frac{2p}{u^2}\left(\frac{du}{d\theta}\right)$$

Now, integrating above equation with respect to 'θ' we have

$$h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right] = 2\int\frac{p}{u^2}du+A$$

where A is a constant

$$\text{or, } v^2 = h\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=2\mu\int\left(\frac{1}{u^7}-\frac{c^4}{u^3}\right)+A$$

or, $v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right]$

$= \mu\left(\frac{-1}{3u^6} + \frac{c^4}{u^2}\right) + A$ ...(1)

But initially when $r = c$ *i.e.*,

$u = \frac{1}{c},\ \frac{du}{d\theta} = 0$ (at apse)

and $v = c^3\sqrt{\frac{2\mu}{3}}$.

$\therefore$ From (1) we have

$$\frac{2\mu c^6}{3} = h^2 \cdot \frac{1}{c^2} = \mu\left[\frac{-c^6}{3} + c^6\right] + A$$

$\therefore h^2 = \frac{2}{3}\mu c^8,\ A = 0$

Substituting the values of h^2 and A, in (1) we have

$$\frac{2}{3}\mu c^8\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\left[\frac{-1}{3u^6} + \frac{c^4}{u^2}\right]$$

$$\text{or, } c^8\left(\frac{du}{d\theta}\right)^2 = \frac{-1}{2u^6} + \frac{3c^4}{2u^2} - c^8u^2$$

$$= \frac{1}{u^6}\left[\frac{-1}{2} + \frac{3}{2}c^4u^4 - c^8u^8\right]$$

$$\Rightarrow c^8\left(\frac{du}{d\theta}\right)^2 = \frac{1}{u^6}\left[\frac{-1}{2} - \left(c^8u^8 - \frac{3}{2}c^4u^4\right)\right]$$

$$= \frac{1}{u^6}\left[\frac{-1}{2} - \left(c^4u^4 - \frac{3}{4}\right)^2 + \frac{9}{16}\right]$$

$$c^8\left(\frac{du}{d\theta}\right)^2 = \frac{1}{u^6}\left[\left(\frac{1}{4}\right)^2 - \left(c^4u^4 - \frac{3}{4}\right)^2\right]$$

$$\therefore c^4u^3\frac{du}{d\theta} = \sqrt{\left(\frac{1}{4}\right)^2 - \left(c^4u^4 - \frac{3}{4}\right)^2}$$

$$\text{or, } d\theta = \frac{c^4u^3du}{\sqrt{\left(\frac{1}{4}\right)^2 - \left(c^4u^4 - \frac{3}{4}\right)^2}}$$

Putting $c^4u^4 - \frac{3}{4} = z$, so that $4c^4u^3du = dz$, we have

$$4d\theta = \frac{dz}{\sqrt{\left(\frac{1}{4}\right)^2 - z^2}}$$

Integrating, $4\theta + B = \sin^{-1}\left(\frac{z}{1/4}\right)$

$\Rightarrow\ 4\theta + B = \sin^{-1}(4z)$

where B is a constant

$\Rightarrow\ 4\theta + B = \sin^{-1}(4c^4u^4 - 3)$

But initially when $u = \frac{1}{c},\ \theta = 0$

$\therefore\quad B = \sin^{-1}(1)$

$\Rightarrow\quad B = \frac{\pi}{2}$

$\therefore\quad 4\theta + \frac{\pi}{2} = \sin^{-1}(4c^4u^4 - 3)$

$\Rightarrow \sin\left(\frac{\pi}{2} + 4\theta\right) = 4c^4u^4 - 3$

$\Rightarrow \cos 4\theta = 4c^4u^4 - 3$

$\Rightarrow\ 4c^4u^4 = 3 + \cos 4\theta$

$\Rightarrow\ \frac{4c^4}{r^4} = 3 + \cos 4\theta$

$\Rightarrow\ 4c^4 = r^4\,[3 + 2\cos^2 2\theta - 1]$

$= 2r^4\,[1 + \cos^2 2\theta]$

$= 2r^4\left[(\cos^2\theta + \sin^2\theta)^2 + (\cos^2\theta - \sin^2\theta)^2\right]$

$= 4r^4\,(\cos^4\theta + \sin^4\theta)$

$\therefore\quad c^4 = r^4\,(\cos^4\theta + \sin^4\theta)$

$\Rightarrow\quad c^4 = (r\cos\theta)^4 + (r\sin\theta)^4$

$\Rightarrow\quad c^4 = x^4 + y^4$

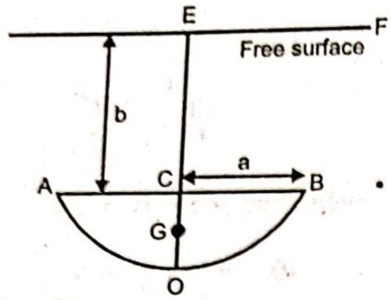
($\because x = r\cos\theta$ and $y = r\sin\theta$)

Hence, $x^4 + y^4 = c^4$ is the equation of path.

7. (c) Depth of the centre of pressure of the semi-circular area $= \frac{k^2}{h}$

where k is the radius of gyration about the line EF on the free surface and

h = depth of the CG. of the lemina below
EF = EG
k^2 = "k^2" about paralleloxi through G + $(EG)^2$



Now, $CG = \frac{4a}{3\pi}$ and hence $EG = b+\frac{4a}{3\pi}$

$$\Rightarrow EG = h = \frac{4a+3b\pi}{3\pi} \quad ...(1)$$

$\therefore k^2$ = "k^2" about AB – $(CG)^2$ + $(EG)^2$

$$= \frac{a^2}{4}-\left(\frac{4a}{3\pi}\right)^2+\left(\frac{4a+3b\pi}{3\pi}\right)^2$$

$$= \frac{9\pi^2a^2+36b^2\pi^2+96ab\pi}{36\pi^2}$$

$$\therefore k^2 = \frac{3\pi(a^2+4b^2)+32ab}{12\pi} \quad ...(2)$$

From (1) and (2) we get
Depth of the centre of pressure

$$= \frac{k^2}{h} = \left(\frac{3\pi(a^2+4b^2)+32ab}{12\pi}\right)\bigg/\left(\frac{4a+3b\pi}{3\pi}\right)$$

$$= \frac{1}{4}\left(\frac{3\pi(a^2+4b^2)+32ab}{4a+3\pi b}\right)$$

8. (a) $x = y\frac{dy}{dx}-\left(\frac{dy}{dx}\right)^2$

Let $\frac{dy}{dx}$ = P, we get

$$x = Py - P^2 \quad ...(1)$$

Differentiating (1) with respect to 'y' we get

$$\frac{1}{P} = P+y\left(\frac{dP}{dy}\right)-2P\left(\frac{dP}{dy}\right)$$

$$\Rightarrow \frac{1-P^2}{P} = (y-2P)\frac{dP}{dy}$$

$$\Rightarrow \frac{dy}{dP} = \frac{P(y-2P)}{(1-P^2)}$$

$$\Rightarrow \frac{dy}{dP}-\frac{Py}{(1-P^2)} = \frac{-2P^2}{(1-P^2)} \quad ...(2)$$

which is linear with I.F. = $e^{\int P dP}$

where $P = \frac{-P}{(1-P^2)}$

Now, $\int P dP = -\int \frac{P}{1-P^2}dP$

$$= \frac{1}{2}\int\frac{(-2P)}{1-P^2}dP = \frac{1}{2}\log(1-P^2)$$

$$= \log\left(1-P^2\right)^{1/2}$$

$$\therefore \text{I.F. of (2)} = e^{\int \log(1-P^2)^{1/2}} = \left(1-P^2\right)^{1/2}$$

Therefore, solution of (2) is

$$y\left(1-P^2\right)^{1/2} = -\int\frac{2P^2}{(1-P^2)}\left(1-P^2\right)^{1/2}dP+C$$

$$= -2\int\frac{P^2}{\left(1-P^2\right)^{1/2}}dP+C$$

$$= 2\left[\int\frac{\left(1-P^2\right)dP}{\left(1-P^2\right)^{1/2}}-\int\frac{1\,dP}{\left(1-P^2\right)^{1/2}}\right]+C$$

$$= 2\int\left(1-P^2\right)^{1/2}dP-2\int\frac{dP}{\left(1-P^2\right)^{1/2}}+C$$

$$= 2\left[\frac{P}{2}\sqrt{1-P^2}+\frac{1}{2}\sin^{-1}P\right]-2\sin^{-1}P+C$$

$$= P\left(1-P^2\right)^{1/2}+\sin^{-1}P-2\sin^{-1}P+C$$

$$\Rightarrow y\left(1-P^2\right)^{1/2} = P\left(1-P^2\right)^{1/2}-\sin^{-1}P+C$$

$$\Rightarrow y = \left(C-\sin^{-1}P\right)\left(1-P^2\right)^{-1/2}+P \quad ...(3)$$

Substituting the above value of y, in (1) we have

$$x = P\left[\left(C - \sin^{-1} P\right)\left(1 - P^2\right)^{-1/2} + P\right] - P^2$$

$$\therefore\ x = P\left(C - \sin^{-1} P\right)\left(1 - P^2\right)^{-1/2} \quad ...(4)$$

Hence (3) and (4) together give the solution in parametric form, P being the parameter.

8. (b) The line integral over a circular path given by C over vector field $\vec{F} = \int_C \vec{F} \cdot dr$

Here, C is given as $x^2 + y^2 = a^2$, $z = 0$

and $\vec{F} = (\sin y)\hat{i} + x(1 + \cos y)\hat{j}$

As we know that $\vec{r}$ is a position vector and is given as

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\therefore\quad d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

Thus, the required integral value

$$= \oint_C \left[\sin y\hat{i} + x(1 + \cos y)\hat{j}\right] \cdot (dx\hat{i} + dy\hat{j} + dz\hat{k})$$

$$= \oint_C \sin y\,dx + x(1 + \cos y)\,dy$$

$$= \oint_C M\,dx + N\,dy$$

Now by Green's theorem in plane we have

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \oint_C M\,dx + N\,dy$$

Here $M = \sin y$, $N = x(1 + \cos y)$

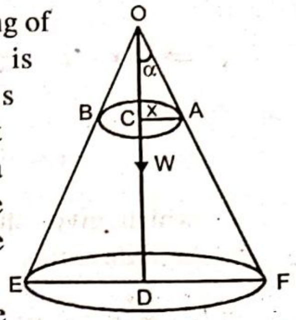
$$\therefore\quad \frac{\partial M}{\partial y} = \cos y, \quad \frac{\partial N}{\partial x} = 1 + \cos y$$

Hence, the given line integral is equal to

$$= \iint_R (1 + \cos y - \cos y)\,dx\,dy = \iint_R dx\,dy$$

= Area of the circle C = πa^2

8. (c) OEF is a smooth fixed cone of semi-vertical angle α, the axis OD of the cone being vertical.



A heavy elastic string of natural length $2\pi a$ is placed round this cone and suppose it rests in the form of a circle whose centre is C and whose radius CA is x.

The weight W of the string acts at its centre of gravity C.

Let T be the tension in this string.

Give the string a small displacement in which x changes to $x + \delta x$. The point O remains fixed, the point C is slightly displaced. $\angle\alpha$ is fixed and the length of the string slight changed.

We have the length of the string AB in the form of a circle of radius x is $2\pi x$ and so the work done by the tension T of this string is $-T\delta(2\pi x)$.

Also, the depth of the point of application C of the weight W below the fixed point O $= OC = AC \cot\alpha = x\cot\alpha$

and so the work done by the weight W during this small displacement $= W\delta\,(x\cot\alpha)$

Since the reactions at the various points of contact do work, we have by the **Principle of virtual work,**

$$-T\delta(2\pi x) + W\delta(x\cot\alpha) = 0$$

$$\Rightarrow -2\pi\, T\delta x + W\cot\alpha\,\delta x = 0$$

$$\text{or}\quad (-2\pi T + W\cot\alpha)\delta x = 0$$

$$\Rightarrow\quad -2\pi T + W\cot\alpha = 0\ (\because \delta x \neq 0)$$

$$\text{or}\quad T = \frac{W\cot\alpha}{2\pi}$$

Now, by Hooke's law the tension T in the elastic string AB is given by

$$T = \lambda\frac{(2\pi x - 2\pi a)}{2\pi a}$$

$$T = \lambda\frac{x - a}{a}$$

Equating the two values of T, we get

$$\frac{W\cot\alpha}{2\pi} = \lambda\frac{(x - a)}{a}$$

$$\Rightarrow \quad x - a = \frac{a}{2\pi\lambda} W \cot \alpha$$

$$\Rightarrow \quad x = a\left(1 + \frac{W}{2\pi\lambda}\cot\alpha\right)$$

which gives the radius of the string in equilibrium.

8. (d) $x^2 \frac{d^2y}{dx^2} + 3x\frac{dy}{dx} + y = (1-x)^{-2}$

Let $\frac{d}{dx} \equiv D$, we get

$(x^2D^2 + 3xD + 1)y = (1-x)^{-2}$ ...(1)

The above equation is also known as linear homogeneous form or Euler-Cauchy form of differential equation.

It is solved by putting $x = e^z$, then reducing the above equation in the form of y and z.

Since $x = e^z \Rightarrow z = \log x$

then $\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}$

or $\frac{xdy}{dx} = \frac{dy}{dz}$

or $xD = D_1 \quad \left(\text{where } D_1 = \frac{d}{dz}\right)$

Again $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dz}\left(\frac{dy}{dx}\right)\frac{dz}{dx}$

$$= \frac{d}{dz}\left(\frac{1}{x}\frac{dy}{dz}\right)\frac{1}{x} = \frac{1}{x}\left[\frac{1}{x}\frac{d^2y}{dz^2} - \frac{1}{x}\frac{dy}{dz}\right]$$

or $\frac{x^2d^2y}{dx^2} = \frac{d^2y}{dz^2} - \frac{dy}{dz}$

or $x^2D^2 = D_1^2 - D_1$

$\Rightarrow x^2D^2 = D_1(D_1 - 1)$

Putting these values in equation (1), we get

$$(D_1(D_1 - 1) + 3D_1 + 1)y = \frac{1}{(1-x)^2}$$

$$(D_1^2 + 2D_1 + 1)y = \frac{1}{(1-x)^2}$$

The auxiliary equation is given by

$m^2 + 2m + 1 = 0$

$\Rightarrow (m + 1)^2 = 0$

$\Rightarrow m = -1, -1$

Hence, the complementary function is given by

$$y = (C_1 + C_2z)e^{-z} = (C_1 + C_2 \log x)\cdot\frac{1}{x}$$

where C_1, C_2 are arbitrary constants

Now, Particular Integral is given by

$$y = \frac{1}{(D_1^2 + 2D_1 + 1)}(1-x)^{-2}$$

$$= \frac{1}{(D_1+1)^2}(1-x)^{-2}$$

$$= \frac{1}{(D_1+1)}\cdot\frac{1}{(D_1+1)}(1-x)^{-2}$$

$$= \frac{1}{D_1+1}x^{-1}\int x^{1-1}(1-x)^{-2}dx$$

Using Formula $\frac{1}{D_1+a}X = x^{-a}\int x^{a-1}X dx$

(**Note:** Here RHS cannot be solved in an usual way, hence, the above formula helps in this special case)

$$= \frac{1}{D_1+1}x^{-1}\int(1-x)^{-2}dx$$

$$= \frac{1}{(D_1+1)}x^{-1}(1-x)^{-1}$$

$$= x^{-1}\int x^{1-1}x^{-1}(1-x)^{-1}dx = x^{-1}\int\frac{dx}{x(1-x)}$$

$$= x^{-1}\int\left(\frac{1}{x} + \frac{1}{1-x}\right)dx$$

$$= x^{-1}[\log x - \log(1-x)] = x^{-1}\log\left[\frac{x}{(1-x)}\right]$$

$\therefore$ The solution is y = C.F. + P.I.

$$= (C_1 + C_2\log x)\frac{1}{x} + \frac{1}{x}\log\left[\frac{x}{(1-x)}\right]$$