2018

1

(b) A function $f : [0, 1] \to [0, 1]$ is continuous on $[0, 1]$. Prove that there exists a point c in $[0, 1]$ such that $f(c) = c$.

Consider the function f defined by

$$f(x, y) = \begin{cases} xy\dfrac{x^2 - y^2}{x^2 + y^2}, & \text{where } x^2 + y^2 \neq 0 \\ 0, & \text{where } x^2 + y^2 = 0 \end{cases}$$

Show that $f_{xy} \neq f_{yx}$ at (0, 0). 10

3

(a) Find the minimum value of $x^2 + y^2 + z^2$ subject to the condition $ax + by + cz = p$. 10

4

(d) Show that the improper integral $\int_0^1 \dfrac{\sin\frac{1}{x}}{\sqrt{x}}\, dx$ is convergent. 10

5,6

(a) Show that

$$\iint_R x^{m-1} y^{n-1} (1 - x - y)^{l-1}\, dx\, dy = \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l + m + n)}; \quad l, m, n > 0$$

taken over R : the triangle bounded by $x = 0$, $y = 0$, $x + y = 1$.

(b) Let $f_n(x) = \dfrac{x}{n + x^2}$, $x \in [0, 1]$. Show that the sequence $\{f_n\}$ is uniformly convergent on [0, 1].

2017

1

1.(b) A function $f : \mathbb{R} \to \mathbb{R}$ is defined as below :

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 1 - x & \text{if } x \text{ is irrational} \end{cases}$$

Prove that f is continuous at $x = \frac{1}{2}$ but discontinuous at all other points in $\mathbb{R}$.

2,3,4,5

$f:[0,1] \to [0,1]$ is continuous.

A cts fun on closed and bounded interval is bounded and it attains its bounds.

Case 1 → If $f(x)$ is constant ie,

$f(x) = c \quad \forall x \in [0,1]$

and since $f(x) \in [0,1]$ so

$c \in [0,1]$

ie. for some $c \in [0,1]$, $f(c) = c$.

Case 2 → if $f(x)$ is not constant,

then consider $g(x) = f(x) - x$

since $f(x), x$ are cts on $[0,1]$

so, $g(x)$ is cts on $[0,1]$.

Now $f(0) \geq 0$, $f(1) \leq 1$

so $g(x) = f(x) - x$

$g(0) \geq 0$

$g(1) = f(1) - 1 \leq 0$.

so, $0 \in [g(1), g(0)]$ and by Inter-mediate value theoream, there exists $c \in [0,1]$ such that

$g(c) = f(c) - c = 0$ so $f(c) = c$

for some $c \in [0,1]$.

Similarly, we can find $f_{xy}(0, 0)$ and $f_{yy}(0, 0)$.

Thus we observe that second order partial derivatives exist at $(0, 0)$ but f is not continuous at $(0, 0)$.

Example 8. ***Give an example of a function $f(x, y)$ for which $f_{xy}(0, 0) \neq f_{yx}(0, 0)$.***

[M.D.U. 2012]

Solution. Consider the function

$$f(x, y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2}, & \text{when } (x, y) \neq (0, 0) \\ 0, & \text{when } (x, y) = (0, 0) \end{cases}$$

When $(x, y) \neq (0, 0)$:

$$f(x, y) = y\left[\frac{x^3 - xy^2}{x^2 + y^2}\right] \qquad ..(1)$$

Differentiating (1) partially w.r.t. x, we have

$$f_x = y\left[\frac{(x^2 + y^2)(3x^2 - y^2) - (x^3 - xy^2)(2x)}{(x^2 + y^2)^2}\right]$$

$$= y\left[\frac{3x^4 + 3x^2y^2 - x^2y^2 - y^4 - 2x^4 + 2x^2y^2}{(x^2 + y^2)^2}\right]$$

$$= \frac{y(x^4 + 4x^2y^2 - y^4)}{(x^2 + y^2)^2}$$

Again, $$f(x, y) = x\left[\frac{x^2y - y^3}{x^2 + y^2}\right] \qquad ...(2)$$

Differentiating (2) partially w.r.t. y, we have

$$f_y = x\left[\frac{(x^2 + y^2)(x^2 - 3y^2) - (x^2y - y^3)(2y)}{(x^2 + y^2)^2}\right]$$

$$= x\left[\frac{x^4 - 3x^2y^2 + x^2y^2 - 3y^4 - 2x^2y^2 + 2y^4}{(x^2 + y^2)^2}\right]$$

$$= \frac{x(x^4 - 4x^2y^2 - y^4)}{(x^2 + y^2)^2}$$

$$f_x(0, 0) = \lim_{h \to 0} \frac{f(h, 0) - f(0, 0)}{h} = \lim_{h \to 0} \frac{0}{h^3} = 0$$

$$f_y(0, 0) = \lim_{k \to 0} \frac{f(0, k) - f(0, 0)}{k} = \lim_{k \to 0} \frac{0}{k^3} = 0$$

As $$f_x(x, y) = \frac{y(x^4 + 4x^2y^2 - y^4)}{(x^2 + y^2)^2}$$

$\therefore$ $$f_x(0, y) = \frac{y(-y^4)}{y^4} = -y$$

As $$f_y(x, y) = \frac{x(x^4 - 4x^2y^2 - y^4)}{(x^2 + y^2)^2}$$

$\therefore$ $$f_y(x, 0) = \frac{x(x^4)}{x^4} = x$$

Now, $$f_{xy}(0, 0) = \lim_{h \to 0} \frac{f_y(h, 0) - f_y(0, 0)}{h}$$

$$= \lim_{h \to 0} \frac{h - 0}{h} = 1 \qquad \text{...(3)}$$

$$f_{yx}(0, 0) = \lim_{k \to 0} \frac{f_x(0, k) - f_x(0, 0)}{k}$$

$$= \lim_{k \to 0} \frac{-k - 0}{k} = -1 \qquad \text{...(4)}$$

$\therefore$ From (3) and (4), $f_{xy}(0, 0) \neq f_{yx}(0, 0)$

Q- $x^2+y^2+z^2$ s.t. $ax+by+cz=p$

Using lagrange multiplier method,

$f(x,y,z) = x^2+y^2+z^2+\lambda(ax+by+cz-p)$

$$\left.\begin{aligned} f_x &= 2x+\lambda a = 0 \\ f_y &= 2y+\lambda b = 0 \\ f_z &= 2z+\lambda c = 0 \end{aligned}\right\} \text{—} (1)$$

$\Rightarrow \quad x = -\dfrac{\lambda a}{2}$

$y = -\dfrac{\lambda b}{2}$

$z = -\dfrac{\lambda c}{2}$

Using $ax+by+cz=p \Rightarrow -\dfrac{\lambda}{2}(a^2+b^2+c^2) = p$

$\Rightarrow \lambda = -\dfrac{2p}{a^2+b^2+c^2}$

Using (1) $\Rightarrow x = \dfrac{ap}{a^2+b^2+c^2}, \; y = \dfrac{bp}{a^2+b^2+c^2}, \; z = \dfrac{cp}{a^2+b^2+c^2}$

Then minimum value of $x^2+y^2+z^2$

$= \dfrac{(a^2+b^2+c^2)p^2}{(a^2+b^2+c^2)^2} = \dfrac{p^2}{a^2+b^2+c^2}$

To prove this is minimum,

$z = \dfrac{1}{c}(p-ax-by)$ Then $x^2+y^2+z^2$

$= x^2+y^2+\dfrac{1}{c^2}(p-ax-by)^2$

$f_x = 2x + \frac{2}{c^2}(b - ax - by)(-a)$

$f_{xx} = 2 - \frac{2a}{c^2}(-a) = 2 + \frac{4a^2}{c^2}$

$f_y = 2y + \frac{2}{c^2}(b - ax - by)(-b)$

$f_{yy} = 2 - \frac{2b}{c^2}(-b)$

$f_{xy} = \frac{2ab}{c^2}$

so, $f_{xx} f_{yy} - f_{xy}^2 = \left(2 + \frac{4a^2}{c^2}\right)\left(2 + \frac{4b^2}{c^2}\right)$

$- \frac{4a^2b^2}{c^2}$

$= 4 + \frac{8b^2}{c^2} + \frac{8a^2}{c^2} + \frac{12a^2b^2}{c^2} > 0$

Also, $f_{xx} = 2 + \frac{4a^2}{c^2} > 0$ so

$f(x, y, z)$ is minimum value.

2009 P-I

Test the convergence of

$$\int_0^1 \frac{\sin 1/x}{\sqrt{x}} dx$$

Sol^n: Let $f(x) = \frac{\sin 1/x}{\sqrt{x}}$

clearly f does not keep the same sign in a bounded of '0'.

Now $|f(x)| = \left| \frac{\sin 1/x}{\sqrt{x}} \right|$

$$= \frac{|\sin 1/x|}{|\sqrt{x}|} \leq \frac{1}{\sqrt{x}} \; \forall x \in (0,1]$$

$[\because |\sin 1/x| \leq 1]$

Since $\int_0^1 \frac{1}{\sqrt{x}} dx$ is convergent at '0'

$(\because n = 1/2 < 1)$

$\therefore$ By Comparison Test

$\int_0^1 |f(x)| dx$ is convergent at '0'.

since absolutely convergence $\Rightarrow$ convergence

$\therefore \int_0^1 f(x) dx$ is convergent.

$$I = \iint_R x^{m-1} y^{n-1} (1-x-y)^{l-1}\, dx\,dy = \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n)}$$

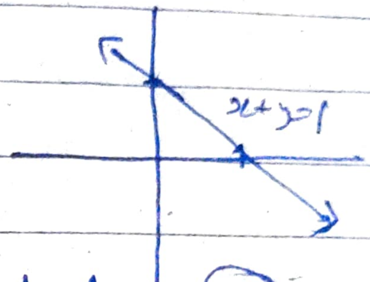
$l, m, n > 0$.

taken over R: the triangle bounded by $x=0,\ y=0,\ x+y=1$.

Take $y = (1-x)t$] ①

$dy = (1-x)\,dt$

i.e. $0 \le t \le 1$.



$$I = \int_{x=0}^{1} \int_{y=0}^{1-x} x^{m-1} y^{n-1} (1-x-y)^{l-1}\, dx\,dy \quad ②$$

Then ① becomes,

$$I = \int_{x=0}^{1} \int_{t=0}^{1} x^{m-1} (1-x)^{n-1} t^{n-1} (1-x)^{l-1} (1-t)^{l-1}\, dx\,dt\, (1-x)$$

$$= \int_0^1 x^{m-1} (1-x)^{n+l-1}\, dx \int_0^1 t^{n-1} (1-t)^{l-1}\, dt$$

$$= B(m, n+l) \times B(n, l)$$

$$= \frac{\Gamma(m)\Gamma(n+l)}{\Gamma(m+n+l)} \times \frac{\Gamma(n)\cdot\Gamma(l)}{\Gamma(n+l)}$$

$$\left[\because B(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \right]$$

$$= \frac{\Gamma(m)\Gamma(n)\Gamma(l)}{\Gamma(m+n+l)}$$

(ii) If n is a +ve integer, $\Gamma(n)=(n-1)!$

. RELATION BETWEEN BETA AND GAMMA FUNCTIONS

To show that

$$B(m, n)=\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)} \quad \text{where } m>0,\ n>0.$$

$= \frac{\Gamma(n-1)!\,\Gamma(n-1)!}{(m+n-1)!}$

(*Agra 1984 ; Meerut 1986, 87, 88 ; Kanpur 1986*)

Proof. We know that for $n>0$,

$$\Gamma(n)=\int_0^{\infty} x^{n-1}\, e^{-x}\, dx$$

Putting $x=az$ so that $dx=a\,dz$, we have

$$\Gamma(n)=\int_0^{\infty} (az)^{n-1}\, e^{-az}\,.\, a\, dz$$

$$=\int_0^{\infty} a^n\, z^{n-1}\, e^{-az}\, dz$$

Replacing z by x,

$$=\int_0^\infty a^n x^{n-1} e^{-ax}\, dx$$

Replacing a by z, we have

$$\Gamma(n)=\int_0^\infty z^n x^{n-1} e^{-zx}\, dx$$

ie. $\Gamma(m)$

Multiplying both sides by $e^{-z} z^{m-1}$, we have

$$\Gamma(n)\,.\, e^{-z} z^{m-1}=\int_0^\infty x^{n-1} z^{m+n-1} e^{-z(1+x)}\, dx$$

Integrating both sides w.r.t. z between the limits 0 to ∞, we have

$$\Gamma(n)\int_0^\infty e^{-z} z^{m-1}\, dx=\int_0^\infty \int_0^\infty x^{n-1} z^{m+n-1} e^{-z(1+x)}\, dx\, dz$$

$$=\int_0^\infty \int_0^\infty x^{n-1} z^{m+n-1} e^{-z(1+x)}\, dz\, dx$$

$$\Rightarrow \quad \Gamma(n)\Gamma(m)=\int_0^\infty x^{n-1}\left[\int_0^\infty z^{m+n-1} e^{-z(1+x)}\, dz\right] dx$$

Putting $\quad z(1+x)=y$

so that $\quad dz=\dfrac{dy}{1+x}$

$$\Gamma(n)\Gamma(m)=\int_0^\infty x^{n-1}\left[\int_0^\infty \left(\frac{y}{1+x}\right)^{m+n-1} e^{-y}\frac{dy}{1+x}\right] dx$$

$$=\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\left[\int_0^\infty y^{m+n-1} e^{-y}\, dy\right] dx$$

$$=\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\,[\Gamma(m+n)]\, dx$$

$$=\Gamma(m+n)\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx$$

$$=\Gamma(m+n)\, B(m, n)$$

$$\left[\because \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\, dx=B(m, n)\right]$$

Hence $B(m, n)=\dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$.

Q. $f_n(x) = \frac{x}{n+x^2}$, $x \in [0,1]$.

$f(x) = \lim_{n\to\infty} f_n(x) = 0. \quad \forall x \in [0,1]$

so, $|f_n(x) - f(x)| = \left|\frac{x}{n+x^2} - 0\right| = \frac{x}{n+x^2}$

$|f_n(x) - f(x)| = \frac{x}{n+x^2} \leq \frac{x}{n} \leq \frac{1}{n}$ on $x \in [0,1]$.

$\left(\because x^2 + n \geq n \Rightarrow \frac{1}{x^2+n} \leq \frac{1}{n}\right)$

$\lim_{n\to\infty} \sup |f_n(x) - f(x)| \leq \lim_{n\to\infty} \frac{1}{n} = 0$

ie. $\sup |f_n(x) - f(x)| \leq 0$

But $|f_n(x) - f(x)| \geq 0$

$\Rightarrow \sup |f_n(x) - f(x)| = 0$

By M_n test $M_n = \sup |f_n(x) - f(x)| = 0$

so $f_n(x) = \frac{x}{n+x^2}$ is uniformly convergent on $[0,1]$.