Set - II

Linear Equations with constant coefficients.

A linear differential eqn of order 'n' of the form

$$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1}\frac{dy}{dx} + a_n y = Q \quad \text{--- (1)}$$

where $a_1, a_2, \ldots a_{n-1}, a_n$ are all constants and Q is any function 'x' is called a linear diff. eqn with constant coefficients.

For our convenience, the operators $\frac{d}{dx}, \frac{d^2}{dx^2}, \ldots \frac{d^{n-1}}{dx^{n-1}}, \frac{d^n}{dx^n}$ are also denoted by $D, D^2, D^3, \ldots D^n$ respectively.

$\therefore$ The equation (1) can be written as

$$D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + \cdots + a_{n-1} Dy + a_n y = Q$$

$$\Rightarrow \left[D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_{n-1} D + a_n\right] y = Q$$

$$\Rightarrow f(D)\, y = Q \quad \text{--- (2)} \quad \text{where } f(D) = D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n.$$

Homogeneous eqn: If $Q=0$ then (2) is called homogeneous eqn with constant coefficients.

i.e, a linear homogeneous eqn of order 'n' is

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_{n-1} D + a_n) y = 0 \quad \text{--- (3)}$$

→ If $y = f(x)$ is the general solution of (3) and $y = p(x)$ is any particular solution of the eqn (2) is not containing any arbitrary constant. Then $y = f(x) + p(x)$ is called the g.s. of (2)

The method of solving a linear eqn is dividing into two parts:

→ first we find the general solution of the eqn (3). It is called the complementary function (C.F).

It must be contain many arbitrary constants as is the order of the given diff-eqn.

→ Next, we find a particular solution of ② which does not contain arbitrary constant. This is called the particular Integral (P.I).

→ If we add (CF) and (P.I) then we get the general solution of (2)

i.e, The general solution of ② is

$$y = C.F + P.I \quad (\text{or})$$
$$y = y_c + y_p.$$

Auxilary Eqn (A.E): Now we consider the diff. eqn $(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_{n-1}D + a_n)y = 0$ — ①

i.e, $f(D)y = 0$

where $f(D) = D^n + a_1 D^{n-1} + \cdots + a_{n-1}D + a_n$.

The eqn $f(m) = 0$ is called the A.E of ① where $m = D$.

∴ A.E of ① is given by

$$m^n + a_1 m^{n-1} + \cdots + a_{n-1}m + a_n = 0$$

Clearly it will have 'n' roots.

These roots may be real (or) complex or surds.

To find the C.F of $f(D)y = 0$:—

Consider the eqn $(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \cdots + a_{n-1}D + a_n)y = 0$

i.e, $f(D)y = 0$

The A.E. of ① is $f(m) = 0$

i.e, $m^n + a_1 m^{n-1} + \cdots + a_{n-1}m + a_n = 0$ — ②

Case(i): When all the roots of ② are real and distinct.

Let $m = m_1, m_2, \ldots, m_n$ be the 'n' real and distinct roots of ②.

Then $y = e^{m_1 x}$, $y = e^{m_2 x}$, ... $y = e^{m_n x}$ are independent solutions of ①.

Hence the g.s of ① is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}.$$

where $c_1, c_2, \dots c_n$ are constants.

Ex: $(D-m_1)y = 0$ — (i)

$\Rightarrow Dy - m_1 y = 0$

$\Rightarrow \frac{dy}{dx} = m_1 y$

$\Rightarrow \frac{dy}{y} = m_1 dx$

$\Rightarrow \log y = m_1 x + \log c$

$\Rightarrow \log(y/c) = m_1 x$

$\Rightarrow y/c = e^{m_1 x}$

$\Rightarrow \boxed{y = ce^{m_1 x}.}$

Case(ii): When two roots of ② are equal and other roots are distinct.

Let $m_1 = m_2$ i.e, $m_1, m_1; m_3, m_4, \dots m_{n-1}, m_n$ be the real and distinct roots of ②.

Then g.s of ① is

$$y = (c_1 + c_2 x)e^{m_1 x} + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}.$$

Ex: $(D-m_1)^2 y = 0$ — ① in which the roots are equal.

$\Rightarrow (D-m_1)(D-m_1)y = 0$

$\Rightarrow (D-m_1)v = 0$ where $v = (D-m_1)y$ — ②

$\Rightarrow v = c_1 e^{m_1 x}$

② $\equiv (D-m_1)y = c_1 e^{m_1 x}$

$\Rightarrow Dy - m_1 y = c_1 e^{m_1 x}$

$\Rightarrow \frac{dy}{dx} - m_1 y = c_1 e^{m_1 x}$ — ③

$\therefore \boxed{I.F = e^{-m_1 x}}$

$\therefore$ G.S is $y e^{-m_1 x} = \int c_1 dx + c_2 = c_1 x + c_2$

$\therefore \boxed{y = (c_1 x + c_2)e^{m_1 x}}$

Case (iii) when three roots are equal.

$\therefore$ G.S of ① is

$$y=(c_1+c_2x+c_3x^2)e^{m_1x}+c_4e^{m_4x}+\cdots\cdots+c_ne^{m_nx}.$$

Case (iv): when all the roots are equal.

$\therefore$ G.S. of ① is

$$y=(c_1+c_2x+c_3x^2+\cdots\cdots+c_{n-1}x^{n-2}+c_nx^{n-1})e^{m_1x}$$

Case (v): when the A.E. of ① has $\alpha\pm i\beta$ as a pair of complex roots.

Let $m_1=\alpha+i\beta$ & $m_2=\alpha-i\beta$

$\therefore$ G.S. of ① is

$$y=c_1e^{(\alpha+i\beta)x}+c_2e^{(\alpha-i\beta)x}$$

$$=c_1e^{\alpha x}e^{i\beta x}+c_2e^{\alpha x}e^{-i\beta x}$$

$$=e^{\alpha x}\left[c_1(\cos\beta x+i\sin\beta x)+c_2(\cos\beta x-i\sin\beta x)\right]$$

$$=e^{\alpha x}\left[(c_1+c_2)\cos\beta x+i(c_1-c_2)\sin\beta x\right]$$

$$=e^{\alpha x}\left[A\cos\beta x+B\sin\beta x\right]$$

where $A=c_1+c_2$; $B=i(c_1-c_2)$;

$i=\sqrt{-1}$

→ If the imaginary roots are repeated, say $\alpha+i\beta$ & $\alpha-i\beta$ occur twice then the solution will be

$$y=e^{\alpha x}\left[(A+Bx)\cos\beta x+(C+Dx)\sin\beta x\right]$$

Note: [1] The expression $e^{\alpha x}(A\cos\beta x+B\sin\beta x)$ can also be written as

$$Ae^{\alpha x}\sin(\beta x+B)\ \text{or}\ Ae^{\alpha x}\cos(\beta x+B).$$

[2] If A.E. of ② has $(\alpha\pm\sqrt{\beta})$ a pair of roots then G.S of ① is

$$y=e^{\alpha x}\left[c_1\cosh\sqrt{\beta}x+c_2\sinh\sqrt{\beta}x\right]$$

some times it may be written as

$$y=c_1e^{\alpha x}\cosh(\sqrt{\beta}x+c_2)$$

→ If the roots $(\alpha \pm \sqrt{\beta})$ is repeated then the G.S is $y = e^{\alpha x}\left[(C_1 + C_2 x)\cosh\sqrt{\beta}x + (C_3 + C_4 x)\sinh\sqrt{\beta}x\right]$

Problems:

→ find C.F. of $(D^2-3D+2)y=0$.

Sol^n: Given that $(D^2-3D+2)y=0$

$\Rightarrow f(D)y = 0$ —— ①

where $f(D) = D^2-3D+2$

A.E. of ① is $f(D)=0$

i.e, $D^2-3D+2=0$

$\Rightarrow (D-1)(D-2)=0$

$\Rightarrow D = 1, 2$

$\therefore$ C.F. of ① is $\boxed{y = C_1e^x + C_2e^{2x}}$

$y=e^x, y=e^{2x}$ are independent sol^n of ①

$\therefore$ CF of ① is $y = C_1e^x + C_2e^{2x}$

→ Solve $\frac{d^3y}{dx^3} - 4\frac{d^2y}{dx^2} + 5\frac{dy}{dx} - 2y = 0$

→ solve $(D^4-81)y=0$

→ solve $(D^4+D^2+1)y=0$

sol^n: A.E. is $f(D)=0$

$\Rightarrow D^4+D^2+1=0$

$\Rightarrow (D^2+1)^2 - D^2 = 0$

$\Rightarrow (D^2+D+1)(D^2+1-D)=0$

$\Rightarrow D^2-D+1=0, \quad D^2+D+1=0$

$D = \frac{1\pm\sqrt{1-4}}{2}; \quad D = \frac{-1\pm\sqrt{1-4}}{2}$

$D = \frac{1\pm\sqrt{3}i}{2}, \quad \frac{-1\pm\sqrt{3}i}{2}$

→ Find C.F. of $(D^2+a^2)y=0$

→ $(D^3+6D^2+12D+8)y=0$

→ $(D^2+D+1)^2y=0$ (or) $(D^4+2D^3+3D^2+2D+1)y=0$

→ $(D^4-7D^3+18D^2-20D+8)y=0$

→ $(D^4-D^3-9D^2-11D-4)y=0$

→ $\left(\frac{d^3y}{dx^3}+y\right)=0$.

→ $(D^2-2D+5)y = 0$ given that (i) $y=0$ when $x=0$
(ii) $\frac{dy}{dx}=4$ when $x=0$.

→ $(D^2+4)y=0$

→ $\frac{d^2y}{dx^2}-3\frac{dy}{dx}+2y=0$ with $y=0$, $x=0$ and $\frac{dy}{dx}=0$.

To find the particular Integral:

Let the given diff eqn be

$(D^n+A_1D^{n-1}+\dots\dots+A_{n-1}D+A_n)y = Q$

where $D=\frac{d}{dx}$.

$\Rightarrow f(D)y=Q$, where $f(D)=D^n+A_1D^{n-1}+\dots+A_{n-1}D+A_n$.

Its g.s. is $\boxed{y = CF+P.I}$

Inverse operator:

→ Since $f(D).\frac{1}{f(D)}Q = Q$.

$\Rightarrow \frac{1}{f(D)}$ is the inverse operator of $f(D)$

→ Since D is the diff operator.

$\frac{1}{D}D=\int D$.

D is the diff operator of D.

$\Rightarrow \frac{1}{D}$ is an integral operator of D.

P.I of $f(D)y=Q$:

Since $y=\frac{1}{f(D)}Q$ satisfies the eqn $f(D)y=Q$.

$\therefore$ P.I of $f(D)y=Q$ is $\frac{1}{f(D)}Q$.

Methods for finding P.I:-

Case(i): To find P.I when $Q=e^{ax}$ when $f(a)\neq0$.

Since $De^{ax}=ae^{ax}$; $D^2e^{ax}=a^2e^{ax}$, ... $D^ne^{ax}=a^ne^{ax}$.

$\Rightarrow f(D)e^{ax}=f(a)e^{ax}$; $f(a)\neq0$.

Now $e^{ax}=\frac{1}{f(D)}f(D)e^{ax} = \frac{1}{f(D)}\{f(a)e^{ax}\}$.

$$\Rightarrow \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax} \quad (\because f(a) \neq 0).$$

Working rule:

If $f(D)y = e^{ax}$; $f(a) \neq 0$

then $P.I = \frac{1}{f(D)} e^{ax}$. where $Q = e^{ax}$

$= \frac{1}{f(a)} e^{ax}$ (putting $D=a$ in $f(D)$).

Problems:

→ Find P.I of $\frac{d^2y}{dx^2} - 7\frac{dy}{dx} + 12y = e^{2x}$

Sol^n: $(D^2 - 7D + 12)y = e^{2x}$

$\Rightarrow f(D)y = Q$ where $f(D) = D^2 - 7D + 12$

and $Q = e^{2x}$.

$\therefore P.I = \frac{1}{f(D)} Q$

$= \frac{1}{D^2 - 7D + 12} e^{2x}$

$= \frac{1}{4 - 14 + 12} e^{2x} = \frac{1}{2} e^{2x}$.

→ Find P.I and solve $(D^2 + D + 1)y = e^{-x}$.

Sol^n Given that $(D^2 + D + 1)y = e^{-x}$.

$\Rightarrow f(D)y = e^{-x}$ —— ①

where $f(D) = D^2 + D + 1$.

A.E. of ① is $f(D) = 0$

$\Rightarrow D^2 + D + 1 = 0$

$\Rightarrow D = \frac{-1 \pm \sqrt{1-4}}{2}$

$= \frac{-1 \pm \sqrt{3}i}{2}$

$= -\frac{1}{2} + \frac{\sqrt{3}}{2} i$

$\therefore y_c = e^{-\frac{1}{2}x}(C_1 \cos \sqrt{3}/2\, x + C_2 \sin \sqrt{3}/2\, x)$

Now $P.I = \frac{1}{f(D)} Q = \frac{1}{D^2 + D + 1}(e^{-x})$

$= \frac{1}{1 - 1 + 1}(e^{-x}) = e^{-x}$.

$\therefore y_p = e^{-x}$.

$\therefore$ G.S. of ① is $y = y_c + y_p$.

$= e^{-\frac{1}{2}x}(C_1 \cos \sqrt{3}/2\, x + C_2 \sin \sqrt{3}/2\, x) + e^{-x}$.

→ solve $(D^2-2D-1)y = \cosh 3x$ $\quad \boxed{\cosh x = \frac{e^{3x}+e^{-3x}}{2}}$

→ solve $(D^2+4D+6)\,y = e^{2x}$

→ Solve $\frac{d^2y}{dx^2} - 13\frac{dy}{dx} + 12y = 2008$

→ solve $(D^3-5D^2+7D-3)\,y = e^{2x}\cosh x.$

$= e^{2x}\,\frac{e^{x}+e^{-x}}{2}$

Case(ii) To find P.I when $Q = \sin(ax)$ or $\cos(ax)$ and $f(-a^2) \neq 0$

Since $D(\sin ax) = a\cos ax$; $D^2(\sin ax) = -a^2\sin ax$

$D^3(\sin ax) = -a^3\cos ax$; $D^4(\sin ax) = (-a^2)^2\sin ax$

$(D^2)^2\sin ax = (-a^2)^2\sin ax.$

- - - - - - - - - - -

$(D^2)^n\sin ax = (-a^2)^n\sin ax.$

$\therefore f(D^2)\sin ax = f(-a^2)\sin ax$
where $f(-a^2) \neq 0.$

NOW $\sin(ax) = \frac{1}{f(D^2)}\,f(D^2)\sin ax$

$= \frac{1}{f(D^2)}\{f(-a^2)\sin ax\}$

$\Rightarrow \boxed{\frac{1}{f(D^2)}\sin ax = \frac{1}{f(-a^2)}\sin ax.}$

Similarly $\boxed{\frac{1}{f(D^2)}\cos ax = \frac{1}{f(-a^2)}\cos ax.}$

working rule:-

To find P.I. $Q = \sin ax$ (or) $\cos ax$ and $f(-a^2) \neq 0.$

$P.I = \frac{1}{f(D^2)}\sin ax.$

$= \frac{1}{f(-a^2)}\sin ax$ (put $D^2 = -a^2$)

write $D^2 = -a^2$, $D^3 = -a^2 D$, $D^4 = (-a^2)^2 = a^4$, - - - -

Note: the linear factor $(D \pm \alpha)$ in the denominator may be removed by multiplying the Nr and Dr with $D \mp \alpha$ and then putting $D^2 = -a^2$

problems:

→ solve $(D^2+4)y = \cos 4x$

→ solve $(D^2-2D+5)y = \sin 3x$

→ solve $\frac{d^3y}{dx^3} + \frac{d^2y}{dx^2} - \frac{dy}{dx} - y = \cos 2x$.

→ solve $(D^2-4D+3)y = \sin 3x \cos 2x$.

$\boxed{2\sin A \sin B = \sin(A+B)+\sin(A-B)}$

→ $(D^2-4)\,y = \sin^2 x$.

→ $(D^2-9)\,y = \cos^2 x$.

$\boxed{\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2}}$

Case(iii): To find P.I. when $Q = x^m$ or polynomial of degree m where m is zero & +ve integer.

$$P.I = \frac{1}{f(D)} x^m \text{ (polynomial)}$$

Takeout common the lowest degree term from $f(D)$. The remaining factor in denominator is of the form $[1+F(D)]$ or $[1-F(D)]$ which is taken in the numerator with negative power.

Now expand $[1 \pm F(D)]^{-1}$ in ascending powers of D by binomial theorem upto D^m and operator upon x^m.

$$\left(\text{i.e, } \frac{1}{f(D)} x^m \Rightarrow \frac{1}{[1 \pm F(D)]} x^m \Rightarrow [1 \pm F(D)]^{-1} x^m.\right)$$

The following expansions by binomial theorem

(1) $(1+x)^{-1} = 1 - x + x^2 - x^3 + x^4 - x^5 + \dots\dots$

(2) $(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots\dots$

(3) $(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots\dots$

4) $(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots\dots$

$$\left[\text{Since } (1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \dots\right]$$

problems

Solve the following diff. eqns.

1 → $\frac{d^2y}{dx^2} - 4y = x^2$

3 · $(D^2+D+1)y = x^3$

2 → $(D^2-3D+2)\,y = 2x^2$

4 · $(D^2-1)y = 2+5x$.

5. $(D^3+8)y = x^4+2x+1$

6. $(D^4-2D^3+D^2)y = x^3$

7. $(D^2+16)y = x^4+e^{3x}+\cos 3x$.

8. $(D^2+D-2)y = x+\sin x$

9. $(D^2-5D+6)y = x+\sin 3x$

10. $(D^3-4D^2+5D)y-2=0$

Case (iv): To find P.I When $Q = e^{ax}.V$

where $V = F(x)$

i.e $V = \sin ax$ (or) $\cos ax$ or x^m.

$$P.I = \frac{1}{f(D)} e^{ax}.V$$

Here take e^{ax} outside after replacing D by $(D+a)$ and operate V by $\frac{1}{f(D+a)}$.

$$\therefore P.I = \frac{1}{f(D)} e^{ax}.V$$

$$= e^{ax} \frac{1}{f(D+a)} V.$$

Problems:

→ Solve the following diff. equations.

1. $(D^2-4)y = x^2 e^{3x}$

2. $(D^3-3D+2)y = x^2 e^x$

3. $(D^2-2D+5)y = e^{2x}\sin x$

4. $(D^4-1)y = e^x \cos x$

5. $(D+1)^3 y = x^2 e^{-x}$

6. $D^2 y = e^x \cos x$

7. $(D^2-4D+3)y = 2xe^{3x}+3e^x.\cos 2x$

8. $(D^3-3D^2+4D-2)y = e^x+\cos x$

9. $(D^3-3D^2+3D-1)y = (x+1)e^x$.

case (v): To find P.I when $Q = e^{ax}$ and $f(a) = 0$

$$P.I = \frac{1}{f(D)} e^{ax}$$

$$= \frac{1}{f(D)} e^{ax} \cdot 1$$

$$= e^{ax} \frac{1}{f(D+a)} \cdot 1.$$

(or)

$$\frac{1}{(D-a)^r} e^{ax} = \frac{x^r}{r!} e^{ax} \text{ where } r = 1, 2, 3, \ldots$$

If $f(a) = 0$. then factorize $f(D)$, first operate on e^{ax} by factor which doesnot vanish by putting 'a' for D and finally the other factor to apply the above formula).

<u>Problems</u>

Find P.I of $[D^3 + D^2 - D - 1]y = e^x$

<u>Sol</u> $P.I = \frac{1}{D^3 + D^2 - D - 1} e^x$

$$= \frac{1}{D^2(D+1) - (D+1)} e^x$$

$$= \frac{1}{(D+1)[D^2 - 1]} e^x$$

$$= \frac{1}{(D+1)^2(D-1)} e^x$$

$$= \frac{1}{(D-1)}\left[\frac{1}{(D+1)^2} e^x\right] = \frac{1}{D-1}\left[\frac{1}{4} e^x\right].$$

$$= \frac{1}{4}\left[\frac{1}{D-1} e^x\right]$$

$$= \frac{1}{4}\left[\frac{x!}{1!} e^x\right]$$

$$= \frac{1}{4} x e^x.$$

→ find P.I of $(D^3+3D^2+3D+1)y = e^{-x}$

Sol P.I $= \frac{1}{D^3+3D^2+3D+1} e^{-x} = \frac{1}{(D+1)^3} e^{-x}$

$$= e^{-x} \frac{1}{(D-1+1)^3}(1)$$

$$= e^{-x} \frac{1}{D^3}(1)$$

$$= e^{-x} \cdot \frac{1}{D^2}[x]$$

$$= e^{-x} \frac{1}{D}\left[\frac{x^2}{2}\right]$$

$$= e^{-x} \frac{x^3}{6}$$

(or)

$$P.I = \frac{1}{[D^3+3D^2+3D+1]} e^{-x} = \frac{x}{3D^2+6D+3}(e^{-x})$$

$$= \frac{x \cdot x}{6D+6}(e^{-x})$$

$$= \frac{x \cdot x \cdot x}{6}(e^{-x})$$

$$= \frac{x^3}{6}(e^{-x}).$$

* solve the following diff. equs:

→ $(D+2)(D-1)^3 y = e^{-x}$

→ $(D^2+D-6)y = e^{2x}$

→ $(D^2-4D+4)y = e^{2x} + \sin 2x$

HW → $(D^2 - 3D + 2)y = \cosh x$

HW → $\frac{d^4 y}{dx^4} - 2\frac{d^3 y}{dx^3} + 5\frac{d^2 y}{dx^2} - 8\frac{dy}{dx} + 4y = e^x$

→ Find P.I. of

Sol $(D^2 + a^2)y = \sin ax$

$$P.I = \frac{1}{D^2+a^2} \sin ax$$

$$= \frac{1}{D^2+a^2} \text{ I.P. of } (\cos ax + i \sin ax)$$

$$= \frac{1}{D^2+a^2} \text{ I.P. of } e^{iax}.$$

$$= \text{I.P. of } \left[\frac{1}{D^2+a^2} e^{iax}\right] \quad \text{—} (1)$$

$$\text{now } \frac{1}{D^2+a^2} e^{(ia)x} = e^{iax} \frac{1}{(D+ia)^2+a^2} (1)$$

$$= e^{iax} \frac{1}{D^2 - a^2 + 2Dia + a^2}$$

$$= e^{iax} \frac{1}{D^2 + 2aiD} (1)$$

$$= e^{iax} \frac{1}{2aiD\left(1 + \frac{D}{2ai}\right)} (1)$$

$$= \frac{e^{iax}}{2aiD} \left(1 + \frac{D}{2ai}\right)^{-1} (1)$$

$$= \frac{e^{iax}}{2aiD} (1) = \frac{e^{iax}}{2ai} (x)$$

$$= -\frac{ix}{2a} (\cos ax + i \sin ax)$$

$$= \frac{x}{2a} (\sin ax - i \cos ax)$$

$$\boxed{\begin{aligned} &\frac{1}{D^2+a^2} e^{iax} \\ &= \frac{x}{2D} e^{iax} \\ &= \frac{x}{2ai} e^{iax}. \end{aligned}}$$

$$= \left[\frac{x \sin ax}{2a} - i\frac{(x \cos ax)}{2a}\right]$$

$\therefore$ I.P of $\left[\frac{1}{D^2+a^2} e^{iax}\right] = \frac{-x \cos ax}{2a}$.

$$\therefore \boxed{\frac{1}{D^2+a^2} \sin ax = \frac{-x \cos ax}{2a}.}$$

or $\frac{1}{D^2+a^2} \sin ax = \frac{-x \cos ax}{2a}$

$$= \frac{x}{2}\int \sin ax\, dx$$

sly $\frac{1}{D^2+a^2} \cos ax = \frac{x}{2}\int \cos ax\, dx$

$$= \frac{x}{2}\left[\frac{\sin ax}{a}\right]$$

$$\therefore \boxed{P.I = \frac{1}{D^2+a^2} \sin ax = \frac{-x \cos ax}{2a}}$$

if $f(-a^2) = 0$

$$\boxed{P.I = \frac{1}{D^2+a^2} \cos ax = \frac{x \sin ax}{2a}}$$

if $f(-a^2) = 0$.

To find P.I when $Q = \sin ax$ (or) $\cos ax$

$\therefore P.I = \frac{1}{D^2+a^2} \sin ax = \frac{-x}{2a} \cos ax$

and $f(-a^2) = 0$

$$= \frac{x}{2}\int \sin ax\, dx.$$

$\therefore P.I = \frac{1}{D^2+a^2} \cos ax = \frac{x}{2a} \sin ax.$

$$= \frac{x}{2}\int \cos ax\, dx.$$

* Solve the following diff. equs:

$\rightarrow (D^2+4)y = \sin^2 x.$ | $\sin^2 x = \frac{1-\cos 2x}{2}$

$\longrightarrow (D^2+1)y = \sin 2x \cos x$ | $\sin 3x + \sin x$.

1991 $\longrightarrow \left(\frac{d^4y}{dx^4} - m^4 y\right) = \sin x$

1995 $\rightarrow$ Find the solution of

$\frac{d^2y}{dx^2} + 4y = 8\cos 2x$. given that

$y = 0$ and $\frac{dy}{dx} = 0$ when $x = 0$.

1992 $\rightarrow$ solve $(D^2+4)y = \sin 2x$, given that when $x=0$, $y=0$ and $\frac{dy}{dx} = 2$.

1993 $\rightarrow$ solve $(D^4+D^2+1)y = e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right)$

Sol Given that

$$(D^4+D^2+1)y = e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right). \quad \text{---(1)}$$

A.E. of (1) is $D^4+D^2+1=0$

$$\Rightarrow (D^2+1)^2 - D^2 = 0$$

$$\Rightarrow (D^2+1-D)(D^2+1+D) = 0$$

$$\Rightarrow D^2-D+1=0 \ \& \ D^2+D+1=0$$

$$\Rightarrow D = \frac{-1\pm\sqrt{3}i}{2}, \frac{1\pm\sqrt{3}i}{2}$$

$$\therefore y_c = e^{-x/2}\left[c_1\cos\left(\tfrac{\sqrt{3}}{2}x\right) + c_2\sin\left(\tfrac{\sqrt{3}}{2}x\right)\right] + e^{x/2}\left[c_3\cos\left(\tfrac{\sqrt{3}}{2}x\right) + c_4\sin\left(\tfrac{\sqrt{3}}{2}x\right)\right]$$

$$P.I = \frac{1}{D^4+D^2+1} e^{-x/2}\cos\left(\tfrac{\sqrt{3}}{2}x\right)$$

$$= e^{-x/2}\frac{1}{(D-1/2)^4 + (D-1/2)^2+1}\cos\left(\tfrac{\sqrt{3}}{2}x\right).$$

$$= e^{-x/2} \frac{1}{D^4 + \frac{1}{16} + D^2 + \frac{D^2}{2} - \frac{D}{2} - 2D^3 + D^2 + \frac{1}{4} - D + 1} \cos(\tfrac{\sqrt{3}}{2}x)$$

$$= e^{-x/2} \frac{1}{D^4 - 2D^3 + \frac{5}{2}D^2 - \frac{3}{2}D + \frac{21}{16}} \cos(\tfrac{\sqrt{3}}{2}x)$$

$$= e^{-x/2} \frac{1}{(D^2 + 3/4)(D^2 - 2D + 7/4)} \cos(\tfrac{\sqrt{3}}{2}x)$$

$$= e^{-x/2} \frac{1}{(D^2 + 3/4)} \left[\frac{1}{-3/4 - 2D + 7/4} \cos(\tfrac{\sqrt{3}}{2}x)\right]$$

$$= e^{-x/2} \frac{1}{D^2 + 3/4} \cdot \frac{1}{(1 - 2D)} \cos(\tfrac{\sqrt{3}}{2}x)$$

$$= e^{-x/2} \frac{1}{D^2 + 3/4} \left[\frac{1 + 2D}{1 - 4D^2} \cos(\tfrac{\sqrt{3}}{2}x)\right]$$

$$= e^{-x/2} \frac{1}{(D^2 + 3/4)} \left[\frac{1 + 2D}{4} \cos(\tfrac{\sqrt{3}}{2}x)\right]$$

$$= \frac{e^{-x/2}}{4} \frac{1}{\left[D^2 + (\frac{\sqrt{3}}{2})^2\right]} \left[\cos(\tfrac{\sqrt{3}}{2}x) - \sqrt{3}\sin(\tfrac{\sqrt{3}}{2}x)\right]$$

$$= \frac{e^{-x/2}}{4} \left[\frac{x}{2(\sqrt{3}/2)} \sin(\tfrac{\sqrt{3}}{2}x) + \frac{\sqrt{3}x}{2(\sqrt{3}/2)} \cos(\tfrac{\sqrt{3}}{2}x)\right]$$

$$= \frac{e^{-x/2}}{4} \left[\frac{x}{\sqrt{3}} \sin\frac{\sqrt{3}}{2}x + x\cos\frac{\sqrt{3}}{2}x\right]$$

$\therefore y = y_c + y_p$ is the g.s of the given diff. eqn.

case (vi) To find P.I

when $Q = xV$ where V is a fn of x

$$P.I = \frac{1}{f(D)}(xV) = x\frac{1}{f(D)}V - \frac{f'(D)}{[f(D)]^2}V$$

Note:- By the repeated use of the above formula $\frac{1}{f(D)}x^m V$ $(m \neq 1)$ can be determined.

but it will more tedious.

first operate on D^2 if possible otherwise operate on Nr then Dr

* Solve the following diff. eqns:

→ $(D^2-1)y = x\sin 3x + \cos x$

→ $(D^2-2D+1)y = x\sin x$

→ $(D^2-2D+1)y = xe^x \sin x$

→ $(D^2+9)y = x\sin x$

→ $(D^2+1)y = e^{-x} + \cos x + x^3 + e^x \sin x$

$\boxed{\text{Note}}$:- If $Q = x^m \sin ax$ (or) $x^m \cos ax$ when the coefficient & of $\sin ax$ (or) $\cos ax$ is x^2 or x^3 or higher power of 'x' then the following method can also be used.

* solve the followeing diff. eqns.

[1] find P.I of $(D^2+1)y = x^2 \sin 2x$

<u>sol</u> $P.I = \frac{1}{D^2+1} x^2 \sin 2x$

$= \frac{1}{D^2+1} x^2 (\text{I.P. of } e^{2ix}$

$= \text{I.P. of } \frac{1}{D^2+1} (e^{2ix} x^2)$ ——— (1)

Now $\frac{1}{D^2+1}(e^{2ix} x^2)$

$= e^{2ix} \frac{1}{(D+2ai)^2+1} x^2$

$= e^{2ix} \left[\frac{1}{D^2+4Di-3} (x^2) \right]$

$= e^{2ix} \left[-\frac{1}{3} \left(1 - \left(\frac{D^2}{3} + \frac{4}{3} Di\right)\right)^{-1} (x^2) \right]$

$= -\frac{1}{3} e^{2ix} \left[\left(1 + \frac{D^2}{3} + \frac{4D}{3} i - \frac{16}{9} D^2\right)(x^2) \right]$

$= -\frac{1}{3} e^{2ix} \left[x^2 + \frac{2}{3} + \frac{4i}{3}(2x) - \frac{16}{9}(2) \right]$

$= -\frac{1}{3} e^{2ix} \left(x^2 + \frac{8}{3} xi - \frac{26}{9} \right).$

$= -\frac{1}{3} (\cos 2x + i \sin 2x) \left(x^2 + \frac{8}{3} xi - \frac{26}{9} \right).$

$\therefore \text{I.P. of } \frac{1}{D^2+1} (e^{2ix} x^2) =$

$-\frac{x^2}{3} (\sin 2x) + \frac{26}{27} \sin 2x$

$- \frac{8x}{9} \cos 2x.$

→ solve $(D^4+2D^2+1)y = x^2 \cos x$

→ Solve $(D^2-4D+4)y = 8x^2 e^{2x} \sin 2x$.

Case(viii).

If Q is a function of x then (i) $\frac{1}{D-\alpha} Q = e^{\alpha x} \int e^{-\alpha x} Q \, dx$.

(ii) $\frac{1}{D+\alpha} Q = e^{-\alpha x} \int e^{+\alpha x} Q \, dx$.

problems.

→ find P.I of $\frac{1}{D^2-4D+3} (x e^{4x})$

Sol Now $P.I = \frac{1}{D^2-4D+3} (x e^{4x})$

$= \frac{1}{(D-1)(D-3)} (x e^{4x})$

$= \frac{1}{(D-1)} \left[\frac{1}{(D-3)} x e^{4x} \right]$

and so on.

→ $\frac{1}{D^2-5D+6} x e^{4x} = ?$

→ $\frac{1}{6D^2-D-2} x e^{-x} = ?$

→ $\frac{1}{D^2-3D+2} \sin(e^{-x}) = ?$

$\rightarrow$ solve $(D^2+a^2)y = \sec ax$

<u>sol</u> $C.F = c_1 \cos ax + c_2 \sin ax.$

$$P.I = \frac{1}{D^2+a^2} \sec ax$$

$$= \frac{1}{2ia}\left[\frac{1}{D-ai} - \frac{1}{D+ai}\right] \sec ax. \quad \text{(1)}$$

Now $\frac{1}{D-ai} \sec ax = e^{iax}\int e^{-iax} \sec ax \, dx$

$$= e^{iax}\left[\int (\cos ax - i \sin ax)\frac{1}{\cos ax} dx\right]$$

$$= e^{iax}\int\left[\cancel{1} - i \tan ax\right] dx$$

$$= e^{iax}\left[x + \frac{i}{a}\log(\cos ax)\right] \quad \text{(2)}$$

Replacing i by $-i$ in (2), we get

$$\frac{1}{D+ia} \sec ax = e^{-aix}\left[x - \frac{i}{a}\log(\cos ax)\right] \quad \text{(3)}.$$

from (1), (2) & (3) we get

$$P.I = \frac{1}{2ia}\left[e^{iax}\left(x + \frac{i}{a}\log(\cos ax)\right) - e^{-iax}\left(x - \frac{i}{a}\log(\cos ax)\right)\right]$$

$$= \frac{x(e^{iax} - e^{-iax})}{2ia} + \frac{1}{a^2}\left(\log(\cos ax)\right).\left(\frac{e^{iax}+e^{-iax}}{2}\right)$$

$= \frac{x}{a} \sin ax + \frac{1}{a^2} \cos ax \log \cos ax.$

$\therefore$ G.S is $y = y_c + y_p.$

→ solve $(D^2 + a^2)y = \tan ax$

Ans: $y = (c_1 \cos ax + c_2 \sin ax)$

$-\left(\frac{1}{a^2}\right) \cos ax \log \tan\left(\frac{1}{4}\pi + \frac{1}{2}ax\right).$

→ solve $(D^2 + a^2)y = \csc ax$

Ans: $P.I = \frac{-x}{a} \cos ax + \frac{1}{a^2} \sin ax \log(\sin ax)$

→ $(D^2 + a^2)y = \cot ax$

Ans: $P.I = \frac{1}{a^2} \sin ax \log\left|\tan\left(\frac{ax}{2}\right)\right|.$

→ find the solution of

$\frac{d^2 i}{dt^2} + \left(\frac{R}{L}\right)\frac{di}{dt} + \left(\frac{i}{LC}\right) = 0$

where $R^2 C = 4L$ and

R, C, L are constant.

Sol Given that $\frac{d^2 i}{dt^2} + \frac{R}{L}\frac{di}{dt} + \frac{i}{LC} = 0$ ——— (1)

$\Rightarrow \left(D^2 + \frac{R}{L} D + \frac{1}{LC}\right) i = 0$ ——— (2)

A.E of (2) is

$D^2 + \frac{R}{L} D + \frac{1}{LC} = 0$

$$\therefore D = \frac{-\frac{R}{L} \pm \sqrt{\frac{R^2}{L^2} - 4(1)\left(\frac{1}{LC}\right)}}{2}$$

$$= \frac{1}{2}\left[\frac{-R}{L} \pm \sqrt{\frac{R^2C - 4L}{L^2C}}\right]$$

$$= \frac{1}{2}\left[-\frac{R}{L} \pm \sqrt{\frac{0}{L^2C}}\right]$$

$(\because R^2C = 4L)$

$$= \frac{-R}{2L}, -\frac{R}{2L}$$

$$\therefore \quad C.F = (C_1 + C_2 t)\, e^{-\frac{R}{2L}t}$$

$\therefore$ G.S of (1) is

$$i = (C_1 + C_2 t)\, e^{-\frac{R}{2L}t}.$$

→ solve the eqn $\frac{d^2y}{dx^2} = a + bx + cx^2$

given that $\frac{dy}{dx} = 0$, when $x = 0$

and $y = d$ when $x = 0$.

<u>Sol</u> Given that $\frac{d^2y}{dx^2} = a + bx + cx^2$

$$\Rightarrow D^2 y = a + bx + cx^2 \quad \text{--- (1)}$$

A.E is $D^2 = 0$

$$\Rightarrow D = 0, 0.$$

$$\therefore \boxed{y_c = c_1 + c_2 x}$$

$$P.I = \frac{1}{D^2}(a + bx + cx^2)$$

$$= \frac{1}{D}\int (a + bx + cx^2)\, dx = \frac{1}{D}\left[ax + \frac{bx^2}{2} + \frac{cx^3}{3}\right]$$

$= \int \left[ax + \frac{bx^2}{2} + \frac{cx^3}{3} \right]$

$= \frac{ax^2}{2} + \frac{bx^3}{6} + \frac{cx^4}{12}.$

$y_p = \frac{ax^2}{2} + \frac{bx^3}{6} + \frac{cx^4}{12}.$

$\therefore$ G.S is $y = y_c + y_p$.

$\Longrightarrow y = (c_1 + c_2 x)e^{0x} + \frac{ax^2}{2} + \frac{bx^3}{6} + \frac{cx^4}{12}$ —— (2)

$\Longrightarrow \frac{dy}{dx} = c_2 + ax + \frac{bx^2}{2} + \frac{cx^3}{3}$ —— (3)

<u>putting</u> $y = d$ and $x = 0$

$d = c_1 + c_2(0) + 0 + 0 + 0$

$\Longrightarrow \boxed{c_1 = d}$

$\therefore$ (2) $\equiv d + \frac{ax^2}{2} + \frac{bx^3}{6} + \frac{cx^4}{12}$

which is reqd soln of (1).

~~1994~~ If $\frac{d^2x}{dt^2} + \frac{g}{b}(x - a) = 0$

(a, b and g being +ve constants)

and $x = a'$ and $\frac{dx}{dx} = 0$ when $t = 0$

then show that $x = a + (a' - a)\cos\left(\sqrt{\frac{g}{b}}\right)t$

<u>Sol</u> Given that $\frac{d^2x}{dt^2} + \frac{g}{b}(x - a) = 0$

$\Rightarrow \frac{d^2x}{dt^2} + \frac{g}{b}x = \frac{ga}{b}$ —— (1)

$\Rightarrow \left(D^2 + \frac{g}{b}\right)(x) = \frac{ga}{b}$ where $D = \frac{d}{dt}$

—— (2)

A.E of (2) is $D^2 + \frac{g}{b} = 0$

$D^2 = -\frac{g}{b} \Rightarrow D = \pm\sqrt{\frac{g}{b}}\,i$

$\therefore \boxed{x_c = c_1 \cos\sqrt{\frac{g}{b}}\,t + c_2 \sin\sqrt{\frac{g}{b}}\,t}$

$P.I = \frac{1}{D^2 + \frac{g}{b}}\left(\frac{ga}{b}\right)$ (continuing next)

→ Find the solution of the eqn $(D^2-1)y = 1$ which vanishes $(y=0)$ when $x=0$ and tends to a finite limit as $x \to -\infty$ and D stands for $\frac{d}{dx}$.

<u>Sol</u> $(D^2-1)y = 1$ —— (1)

A.E is $D^2 - 1 = 0$

$D = \pm 1$

$\therefore \boxed{y_c = c_1 e^{x} + c_2 e^{-x}}$

$P.I = \frac{1}{D^2-1}(1) = \frac{1}{D^2-1}e^{0x}$

$= \frac{1}{-1} = -1$

$\therefore \boxed{y_p = -1}$

G.S is $y = y_c + y_p$

$$\boxed{y = c_1 e^x + c_2 e^{-x} - 1} \longrightarrow ②$$

putting $y=0$ and $x=0$ in ①, we get

$0 = c_1 + c_2 - 1$

$\Rightarrow \boxed{c_1 + c_2 = 1} \longrightarrow ③$

multiplying both sides of ②. by e^x,
we get $y e^x = c_1 (e^x)^2 + c_2 - e^x$

Taking limit on both sides of ②
as $x \longrightarrow -\infty$ we get

$$\lim_{x \to -\infty} y e^x = \lim_{x \to -\infty} c_1 (e^x)^2 + \lim_{x \to \infty} c_2 - \lim_{x \to \infty} e^x$$

$$\therefore y \times 0 = c_1 (0)^2 + c_2 - 0$$
$$(\because \lim_{x \to \infty} e^x = 0)$$

$\Rightarrow \boxed{c_2 = 0}$

$\therefore$ ③ $\equiv \boxed{c_1 = 1}$

② $\equiv \boxed{y = e^x - 1}$

which is the reqd soln.

2001
1997 → solve $(D^2+1)^2 y = 24x \cos x$ given the initial conditions $x=0, y=0$, $D y = 0$, $D^2 y = 0$
$D^3 y = 12$

Ans: $y = 3x^2 \sin x - x^3 \cos x$.

1993, solve $\frac{d^2y}{dx^2} + w_0^2 y = a\cos wt$

and discuss the nature

of solution as $w \to w_0$

1995, Determine all real valued

solutions of the eqn

$y''' - i y'' + y' - i y = 0$;

$y' = \frac{dy}{dx}$.

Sol Given that $y''' - i y'' + y' - i y = 0$

$\Rightarrow \frac{d^3y}{dx^3} - i\frac{d^2y}{dx^2} + \frac{dy}{dx} - iy = 0$.

$\Rightarrow [D^3 - iD^2 + D - i]y = 0$

$D = \frac{d}{dx}$.

——— (1)

A.E is $D^3 - iD^2 + D - i = 0$

$\Rightarrow D^2(D-i) + (D-i) = 0$

$\Rightarrow D = i$; $D = \pm i$

(continuing next)