DATE

6(a) Solve the following DE.

$$\frac{dy}{dx} = \frac{2y}{x} + \frac{x^3}{y} + x\tan\frac{y}{x^2}.$$

(10)

Put

$$\frac{y}{x^2} = t \quad \text{i.e.} \quad y = tx^2$$

$$\frac{dy}{dx} = 2tx + x^2\frac{dt}{dx}$$

Now DE becomes

$$\cancel{2tx} + x^2\frac{dt}{dx} = \frac{2tx^{\cancel{2}}}{\cancel{x}} + \frac{x^{\cancel{3}}}{t\cancel{x^2}} + x\tan t$$

$$x\frac{dt}{dx} = \frac{1}{t} + \quad = \frac{\cos t + t\sin t}{t\cos t}$$

$$\int \frac{t\cos t\,dt}{\cos t + t\sin t} = \int \frac{dx}{x}$$

$$\log(\cos t + t\sin t) = \log x + \log c$$

$$\Rightarrow \quad \cos t + t\sin t = cx$$

i.e. $\boxed{\cos\left(\frac{y}{x^2}\right) + \frac{y}{x^2}\sin\left(\frac{y}{x^2}\right) = cx}$

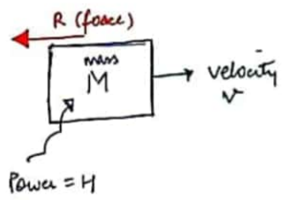
which is the required solution.

classmate

Date

Page

6b) An engine, working at a constant rate H, draws a load M against a resistance R. Show that the maximum speed is H/R and the time taken to attain half of this speed is $\frac{MH}{R^2}\left(\log 2 - \frac{1}{2}\right)$. (10m)



Energy equation for time dt

Energy supplied = $H\,dt$

Energy lost due to ~~due~~ resistance

$= $ Force × distance

$= R v\,dt$

Assuming no change of PE, $\Delta PE = 0$

$\sum$ energy supplied − $\sum$ energy lost = $\Delta(K.E.)$.

$$H\,dt - Rv\,dt = d\cdot\left(\tfrac{1}{2}mv^2\right)$$

$$H\,dt - Rv\,dt = mv\,dv$$

$$H - Rv = mv\frac{dv}{dt}$$

for max. velocity; $\frac{dv}{dt} = 0$; ⇒ acceleration $= 0$

$$H - Rv = 0$$

$$\boxed{V_{max} = H/R}$$

now integrating $\quad H - Rv = mv\frac{dv}{dt}$

$$dt = m\left(\frac{v\,dv}{\cancel{H}-Rv}\right) = \frac{m}{R}\left(\frac{Rv}{H-Rv}\right)dv$$

$$dt = \frac{m}{R}\left(\frac{Rv-H+H}{H-Rv}\right)dv = \frac{m}{R}\left(\frac{H}{H-Rv} - 1\right)dv$$

$$\int_0^t dt = \int_0^{V_{max}/2} \frac{m}{R}\cdot\left(\left(\frac{H}{H-Rv}\right)dv - dv\right)$$

$$t = \int_0^{\frac{H}{2R}} \frac{m}{R}\left(\frac{\ln(H-Rv)H}{-R} - v\right) = \boxed{\frac{MH}{R^2}\left(-\ln\left(\tfrac{1}{2}\right) - \tfrac{1}{2}\right)} \#$$

$\frac{V_{max}}{2} = \frac{H}{2R}$

DATE

(c) Solve by the method of variation of parameters

$$y'' + 3y' + 2y = x + \cos x \qquad (10)$$

D.E. $\Rightarrow$ $(D^2 + 3D + 2)y = x + \cos x$

Auxiliary Eqn: $D^2 + 3D + 2 = 0$

$$(D+1)(D+2) = 0$$

$$D = -1, -2$$

$$C.F. = c_1' e^{-x} + c_2' e^{-2x}$$

$$y_1 = e^{-x}, \quad y_2 = e^{-2x}$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix}$$

$$= -2e^{-3x} + e^{-3x} = -e^{-3x} \neq 0$$

$\therefore$ y_1 & y_2 are linearly independent.

To get complete solution by variation of parameters, we replace c_1' and c_2' in C.F. by function A and B.

$$y = Ae^{-x} + Be^{-2x}$$

$$= Ay_1 + By_2$$

$$A = -\int \frac{Ry_2}{W} dx = -\int \frac{(x + \cos x)e^{-2x}}{-e^{-3x}} dx$$

$$= \int e^x(x + \cos x)\, dx = \int x \cdot e^x dx + \int e^x \cos x\, dx$$

$$= xe^x - \int 1 \cdot e^x dx + \frac{1}{2} e^x(\cos x + \sin x) + c_1$$

$$= xe^x - e^x + \frac{e^x}{2}(\cos x + \sin x) + c_1$$

classmate PAGE

DATE

$$\left[\int e^{ax}\sin bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx)\right.$$

$$\left.\int e^{ax}\cos bx\,dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)\right]$$

$$B = \int \frac{y_1 R}{W}\,dx$$

$$= \int \frac{e^{-x}(x+\cos x)}{-e^{-3x}}\,dx$$

$$= -\left[\int e^{-2x} x\,dx + \int e^{-2x}\cos x\,dx\right]$$

$$= -\left[x.\frac{e^{-2x}}{-2} - \int 1.\frac{e^{-2x}}{-2}dx + \frac{e^{-2x}}{4+1}(-2\cos x+\sin x)\right]$$

$$= \frac{x}{2}e^{-2x} + \frac{1}{4}e^{-2x} + \frac{e^{-2x}}{5}(-2\cos x+\sin x)+C_2$$

Hence, Complete solution is given by

$$y = Ay_1 + By_2$$

$$y = \left[xe^{x} - e^{x} + \frac{e^{x}}{2}(\cos x+\sin x)+C_1\right]e^{-x}$$

$$+\left[\frac{x}{2}e^{-2x} + \frac{e^{-2x}}{4} + \frac{e^{-2x}}{5}(-2\cos x+\sin x)+C_2\right]e^{2x}$$

DATE

6(a) For the vector $\vec{A} = \frac{xi + yj + zk}{(x^2+y^2+z^2)}$

examine if $\vec{A}$ is an irrotational vector.

Then determine ϕ such that $\vec{A} = \nabla\phi$. (10)

$\vec{A}$ is irrotational if $\nabla \times \vec{A} = 0$

$$\nabla \times \vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{r^2} & \frac{y}{r^2} & \frac{z}{r^2} \end{vmatrix} \quad ; \; r^2 = x^2+y^2+z^2$$

$$= i\left(\frac{-2z}{r^3}\cdot 2y + \frac{2y}{r^3}\cdot 2z\right) + j\left(\frac{-2x}{r^3}\cdot 2z + \frac{2z}{r^3}\cdot 2x\right)$$

$$+ k\left(\frac{-2y}{r^3}\cdot 2x + \frac{2x}{r^3}\cdot 2y\right) = 0$$

$\therefore \vec{A}$ is irrotational.

Let $\vec{A} = \nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$

i.e. $$\left.\begin{aligned} \frac{\partial\phi}{\partial x} &= \frac{x}{x^2+y^2+z^2} \\ \frac{\partial\phi}{\partial y} &= \frac{y}{x^2+y^2+z^2} \\ \frac{\partial\phi}{\partial z} &= \frac{z}{x^2+y^2+z^2} \end{aligned}\right\} \Rightarrow \phi = \frac{1}{2}\log(x^2+y^2+z^2) + C$$

$\therefore$ scalar potential ϕ is such that

$$\vec{A} = \nabla\phi.$$