Set - I

* REAL NUMBER SYSTEM *

→ The set of natural numbers
$N = \{1, 2, 3, \ldots\}$

→ The set of whole numbers
$\omega = \{0, 1, 2, 3, \ldots\}$

→ The set of integers
$I = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

→ The set of +ve integers
$I^+ = \{1, 2, 3, \ldots\}$

→ The set of −ve integers
$I^- = \{\ldots, -3, -2, -1\}$

→ The set of rational numbers
$Q = \left\{ \frac{p}{q} \,/\, p, q \in I,\ q \neq 0 \right\}$

→ The set of irrational numbers
Q' = The numbers which cannot be expressed in the form of p/q $(q \neq 0)$ are known as irrational numbers.

Ex: $\sqrt{2}, \sqrt{5}, e, \pi$ etc.

Note:

→ (1) The rational number can be expressed either as a terminating decimal (or) non-terminating recurring decimal.

→ (2) An irrational number can be expressed as non-terminating non-recurring decimal.

The set of real numbers
$\mathbb{R} = Q \cup Q'$. i.e the set of real numbers $\mathbb{R}$ which contains the set of rational and irrational numbers.

Note:

(1) $N \subset W \subset I \subset Q \subset \mathbb{R}$ and $Q' \subset \mathbb{R}$.

(2) Between any two distinct consequtive integers, there exists no integer.

(3) Between any two distinct rational numbers, there lie infinitely many rational numbers.

(4) Between any two rational numbers there lie infinitely many irrational numbers.

(5) Between any two irrational numbers there lie infinitely many irrational numbers as well as infinitely many rational numbers.

(6) Between any two real numbers there lie infinitely many real numbers.

Note: The symbols $\exists$ and $\forall$ are known as Quantifies and the symbols $\Rightarrow, \Leftrightarrow$ as connectives.

→ Some important properties of real numbers in the form of axioms.

These axioms can be divided into three types.

1. Field axioms 2. Order axioms
3. completeness axiom.

→ **Field axioms:**

Let $\mathbb{R}$ be the set of real numbers then the algebraic structure $(\mathbb{R}, +, \cdot)$ satisfies the following axioms.

(I) $(\mathbb{R}, +)$ is an abelian group.

i.e. (i) Closure property:

$$\forall\, a, b \in \mathbb{R} \Rightarrow a+b \in \mathbb{R}.$$

(ii) Associative Property:

$$\forall\, a, b \in \mathbb{R} \Rightarrow a+(b+c) = (a+b)+c$$

(iii) Existence of identity:

$\forall\, a \in \mathbb{R}$, $\exists\, 0 \in \mathbb{R}$ such that

$$a+0 = 0+a = a$$

The real number '0' is called the additive identity of $\mathbb{R}$.

(iv) Existence of Inverse:

$\forall\, a \in \mathbb{R}$, $\exists\, b \in \mathbb{R}$ such that

$$a+b = 0 = b+a$$

The real number 'b' is called the additive inverse of 'a'.

(V) Commutative Property:

$$\forall\, a, b \in \mathbb{R} \Rightarrow a+b = b+a$$

(II) $(\mathbb{R}, \cdot)$ is an abelian group.

i.e. (i) Closure Property:

$$\forall\, a, b \in \mathbb{R} \Rightarrow a \cdot b \in \mathbb{R}$$

(ii) Associative Property:

$$\forall\, a, b, c \in \mathbb{R} \Rightarrow a \cdot (b \cdot c) = (a \cdot b) \cdot c$$

(iii) Existence of Identity

$\forall\, a \in \mathbb{R}$, $\exists\, 1 \in \mathbb{R}$ such that

$$a \cdot 1 = 1 \cdot a = a$$

The real number '1' is called the multiplicative identity of $\mathbb{R}$.

(iv) Existence of Inverse:

$\forall\, a \in \mathbb{R}$, $a \neq 0$, $\exists\, b \in \mathbb{R}$ such that $a \cdot b = b \cdot a = 1$.

The real number 'b' is called the multiplicative inverse of 'a' and is denoted by a^{-1}.

(V) Commutative Property:

$$\forall\, a, b \in \mathbb{R} \Rightarrow a \cdot b = b \cdot a.$$

III Distributivity :-

multiplication is distributive with respect to addition in $\mathbb{R}$.

i.e. $\forall\ a,b,c \in \mathbb{R}$

$\Rightarrow a\cdot(b+c) = a\cdot b + a\cdot c$ (L.D.L)

and $(b+c)\cdot a = b\cdot a + c\cdot a$ (R.D.L)

→ A non-empty set 'S' is said to be field if it possesses the two compositions $+^n$ & $\times^n$ and satisfied all the above axioms.

Ex:- $(\mathbb{Q}, +, \cdot)$ is a field but $(\mathbb{Z}, +, \cdot)$ & $(\mathbb{N}, +, \cdot)$ are not fields.

2. Order axioms :-

The order relation '>' between pairs of real numbers $\mathbb{R}$ satisfies the following axioms:

Let $a,b,c \in \mathbb{R}$ then

O_1 : for $a,b \in \mathbb{R}$, exactly one of the following holds

(i) $a>b$ (ii) $a=b$ and (iii) $b>a$

which is known as the law of trichotomy.

O_2 : For $a,b,c \in \mathbb{R}$;

$a>b,\ b>c \Rightarrow a>c$

which is known as the law of transitivity.

O_3 :- $\forall\ a,b,c \in \mathbb{R}$;

$a>b \Rightarrow a+c > b+c$

which is known as the monotone property for $+^n$.

O_4 : $\forall\ a,b,c \in \mathbb{R}$;

$a>b$ and $c>0 \Rightarrow ac > bc$

which is known as the monotone property for $\times^n$.

→ A field satisfying the above properties, is called an ordered field.

Hence $(\mathbb{R}, +, \cdot)$ is an ordered field.

Note: $(\mathbb{Q}, +, \cdot)$ is an ordered field

* Some more definitions :-

→ Less than relation : For $a,b \in \mathbb{R}$;

$a<b \Leftrightarrow b>a$.

→ +ve real numbers :

$a \in \mathbb{R}$ is said to be +ve if $a>0$ and is denoted by $\mathbb{R}^+$.

→ **-ve real numbers:**

$a \in \mathbb{R}$ is said to be -ve if $a<0$ and is denoted by $\mathbb{R}^-$

$\therefore \mathbb{R} = \mathbb{R}^- \cup \{0\} \cup \mathbb{R}^+$.

→ If $a \in \mathbb{R}^+$ and $b \in \mathbb{R}^-$ then $a>b$.

→ A real number 'a' is said to be greater than (or) equal to 'b' (i.e. $a \geq b$) if either $a>b$ (or) $a=b$

→ A real number 'a' is said to be less than (or) equal to 'b' (i.e. $a \leq b$) if either $a<b$ (or) $a=b$.

* → **Some Properties of order relation:**

→ $a \in \mathbb{R}^+ \Leftrightarrow a>0$ and $a \in \mathbb{R}^- \Leftrightarrow a<0$

→ $\forall a,b \in \mathbb{R}^+ \Rightarrow a+b \in \mathbb{R}^+$ and $ab \in \mathbb{R}^+$ i.e. $a>0, b>0 \Rightarrow a+b>0 \;\&\; ab>0$

→ $\forall a,b \in \mathbb{R}^- \Rightarrow a+b \in \mathbb{R}^-$ and $ab \in \mathbb{R}^+$.

i.e. $a<0, b<0 \Rightarrow a+b<0 \;\&\; ab>0$.

→ $a<b$ and $b<c \Rightarrow a<c$ (law of transitivity)

→ $a<b \Leftrightarrow a+c < b+c$ & $a<b$ and $c<0 \Rightarrow ac>bc$.

→ $a<0 \Leftrightarrow -a>0$ &

$a>0 \Leftrightarrow -a<0$.

→ $a>b \Leftrightarrow (a-b)>0$ &

$a<b \Leftrightarrow (a-b)<0$.

→ $a>b \Leftrightarrow -a<-b$

→ $a>0 \Leftrightarrow 1/a>0$.

→ $a>b>0 \Rightarrow \frac{1}{b}>\frac{1}{a}>0$.

→ $a \neq 0 \Rightarrow a^2>0$.

→ $a>b>0 \Rightarrow a^2>b^2$ and $a<b<0 \Rightarrow a^2>b^2$.

→ The relations $\geq$ and $\leq$ are known as the weak inequalities and the relations $>$ and $<$ are known as the strict inequalities.

* Intervals :—

Intervals are two types:
① finite intervals
② Infinite intervals.

1. Finite Intervals :

Let $a, b \in \mathbb{R}$ with $a<b$ then

(i) the set $\{x / x \in \mathbb{R}, a \leq x \leq b\}$ is called a closed interval and is denoted by $[a,b]$.

a and b are called the end points of the interval.

a is called the left end point while b is called the right end point.

Here both the end points a&b belong to the interval.

(ii) the set $\{x | x \in \mathbb{R}; a < x < b\}$ is called an open interval and is denoted by (a,b) or $]a,b[$

Here both the end points donot belong to the interval.

(iii) the set $\{x | x \in \mathbb{R}, a \leq x < b\}$ is called left-half closed interval (or right-half open interval). and is denoted by $[a,b)$ or $[a,b[$.

Here the left end point 'a' belongs to the interval and right end point 'b' doesnot belong to the interval.

(iv) the set $\{x | x \in \mathbb{R}, a < x \leq b\}$ is called right-half closed interval (or left-half open interval) and is denoted by $(a,b]$ or $]a,b]$

Note:— If $a=b$

$(a,a) = \phi$ and $[a,a] = \{a\}$.

2. Infinite intervals:

Let $a \in \mathbb{R}$ then

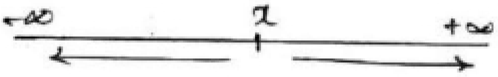
(i) the set $\{x | x \in \mathbb{R}, \underset{\text{i.e. } a \leq x}{x \geq a}\}$ is called closed right ray and is denoted by $[a, \infty)$.

(ii) The set $\{x | x \in \mathbb{R}, a < x\}$ is called open right ray and is denoted by (a, ∞).

(iii) the set $\{x | x \in \mathbb{R}, x \leq a\}$ is called closed left ray and is denoted by $(-\infty, a]$

(iv) the set $\{x | x \in \mathbb{R}, x < a\}$ is called open left ray and is denoted by $(-\infty, a)$.

(v) The set $\{x \mid x \in \mathbb{R}\}$ is also called an interval and has no end points. It is denoted by $(-\infty, \infty)$.



→ **Length of an interval:-**

For each interval whose end points are any real numbers a & b such that $a<b$, the length of the interval is $b-a$.

Obviously the length of each of the intervals $[a,b]$, $]a,b[$, $]a,b]$ and $[a,b[$ is $b-a$. These intervals are called finite intervals because the length of each of them is finite.

The intervals $[a,\infty[$, $]a,\infty[$, $]-\infty,a]$, $]-\infty,a[$ and $]-\infty,\infty[$ are called infinite intervals because the length of each of them is infinite.

Note:- (i) Every interval is an infinite set but every infinite set need not be an interval.

Ex:- ① $\mathbb{N}$ is not an interval
② $\mathbb{Z}$ is not an interval
③ $\mathbb{Q}$ is not an interval
④ $\mathbb{R}-\mathbb{Q}$ is not an interval.
⑤ ϕ, $\mathbb{R}$ sets are intervals.

2. A finite interval is also an infinite set because the word finite only signifies that the length of the interval is finite.

3. A ray is an infinite interval.

The extended real number system ($\mathbb{R}^*$):-

To extend the real number system by adjoining two "ideal points" denoted by $+\infty$ and $-\infty$. The enlarged set is called the set of extended real numbers.

Note: $\mathbb{R}$ is denoted by $(-\infty, \infty)$ and $\mathbb{R}^*$ by $[-\infty, \infty]$

If $x \in \mathbb{R}$ then $-\infty < x < \infty$,

$$x+\infty = \infty + x = -x+\infty = \infty - x = \infty;$$

$$x-\infty = -\infty + x = -\infty - x = -x-\infty = -\infty;$$

$$\frac{x}{\infty} = 0$$

$$\frac{\infty}{x} = \infty \times x = x \times \infty = \begin{cases} \infty & \text{if } x>0 \\ -\infty & \text{if } x<0 \end{cases}$$

Further $\infty \times \infty = (-\infty)\times(-\infty) = \infty + \infty = \infty$

$\infty \times (-\infty) = (-\infty) \times \infty = -\infty - \infty = -\infty$

The following combinations are meaningless.

$\infty - \infty$, $-\infty + \infty$, $0 \times \infty$, $\infty \times 0$, $\frac{\infty}{\infty}$

Bounds of Set:-

Lower bound of a subset of $\mathbb{R}$:

Let S be a non-empty subset of $\mathbb{R}$. If there exists a number $u \in \mathbb{R}$ such that $u \leq x \ \forall x \in S$ then u is called a lower bound of S.

Ex: (1) $N = \{\underbrace{1, 2, 3, \ldots\ldots}_{x}\} \subseteq \mathbb{R}$.

$1 \leq x \ \forall x \in N$.

$\therefore$ 1 is called the lower bound of N.

(2) The set $S = \{0, 1, 2, 3, \ldots\} \subseteq \mathbb{R}$

$0 \leq x \ \forall x \in S$

$\therefore$ 0 is the lower bound.

Bounded below set:

A non-empty subset S of $\mathbb{R}$ (i.e, $S \subseteq \mathbb{R}$) is said to be bounded below if it has lower bound.

Ex: (1) $S = \{1^2, 2^2, 3^2, 4^2, \ldots\} \subseteq \mathbb{R}$ is bounded below.

Since 1 is lower bound.

(2) $\mathbb{R}^+ = \{x / x > 0\} = (0, \infty)$ is bounded below.

Since 0 is lower bound & $0 \notin \mathbb{R}^+$.

(3) $S = \{x / x \geq 0\} = [0, \infty)$ is bounded below.

Since 0 is lower bound & $0 \in S$.

Note: If 'u' is lower bound of S then every real number smaller than 'u' is also a lower bound of S.

i.e, if a set S is bounded below then the set of all such real numbers that are lower bounds of S is infinite.

Greatest lower bound (glb) or infimum:

Let S be a non-empty subset of $\mathbb{R}$. If a set 'S' is bounded below and if the set of all lower bounds of 'S' has a greatest member, say 't' then 't' is called greatest lower bound or infimum of 'S'

(Or)

If 't' is a lower bound of S and any real number greater than 't' is not lower bound of S then t is called the greatest lower bound or infimum of S.

(or)

If S is bounded below, then a number t is said to be greatest lower bound or infimum of S if it satisfies the conditions

1. t is lower bound of S and
2. if w is any lower bound of S then $w \le t$.

Ex:- $S = \{\underbrace{1, 2, 3 \cdots}_{x}\} \subseteq \mathbb{R}$.

Since $-1 < x \ \forall x \in S$

$\therefore$ -1 is lower bound of S but -1 is not greatest lower bound of S.

Since $0 < x \ \forall x \in S$

$\therefore$ 0 is a lower bound of S but 0 is not greatest lower bound of S.

Since $0.9 < x \ \forall x \in S$

$\therefore$ 0.9 is a lower bound of S. but $.9$ is not greatest lower bound of S.

$1 \le x \ \forall x \in S$

$\therefore$ 1 is a lower bound of S. and is greatest lower bound of S.

because, the greatest of all lower bounds of S is 1.

Note :- If t is infimum of S then for each $\epsilon > 0$ (however small), the number $t+\epsilon$ is not a lower bound of S, there exists atleast one member $x \in S$ such that $t \le x < t+\epsilon$.

Upper bound :- Let S be a non-empty subset of $\mathbb{R}$. If there exists a number $v \in \mathbb{R}$ such that

$x \leq v \ \forall x \in S$ then v is called an upper bound of S.

Ex:- $S = \{\ldots -3, -2, -1\} \subseteq \mathbb{R}$

$x \leq -1 \ \forall x \in S$

$\therefore -1$ is called the upper bound of S.

Bounded Above set

A non-empty subset S of $\mathbb{R}$ (i.e. $S \subseteq \mathbb{R}$) is said to be bounded above if it has an upper bound.

Ex:- (1) $\mathbb{R}^- = \{x \in \mathbb{R} : x < 0\} = (-\infty, 0)$ is bounded above

Since 0 is an upper bound and $0 \notin \mathbb{R}^-$

(2). $S = \{x \in \mathbb{R} : x \leq 0\} = (-\infty, 0]$ is bounded above.

Since 0 is an upper bound and $0 \in S$.

Note:- If v is an upper bound of a set 'S' then every real number greater than v is also an upper bound of S. i.e. If a set 'S' is bounded above then set of all such numbers that are upper bounds of S is infinite.

*** Least upper bound (lub) (or) Supremum :—**

Let S be a non-empty subset of $\mathbb{R}$. If a set 'S' is bounded above and if the set of all upperbounds of S has a least member say 't_1' then 't_1' is called least upper bound (or) supremum of S.

(or)

If t_1 is an upper bound of S and any real number less than t_1 is not an upper bound of S then t_1 is called least upper bound (or) Supremum of S.

(or)

If S is bounded above then a number t_1 is said to be least upper bound (or) supremum of S if it satisfies the following conditions.

(1) t_1 is an upper bound of S and

(2) If w_1 is any upper bound of S then $t_1 \leq w_1$.

Ex:- $S = \{\underbrace{\ldots 48, 49, 50}_{x}\} \subseteq \mathbb{R}$

Since $x < 51 \ \forall x \in S$

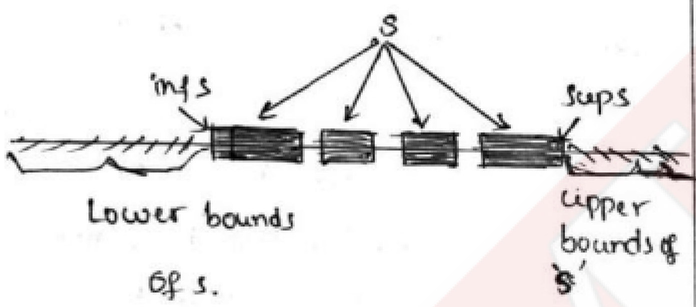
$\therefore 51$ is an upper bound of S but is not supremum of S.

Since $x < 50.5 \ \forall x \in S$

$\therefore 50.5$ is an upper bound of S. but is not supremum of S.

$x \leq 50 \ \forall x \in S$

Note:— If t_1 is supremum of S then for each $\epsilon>0$ (however small), the number $t_1-\epsilon$ is not an upper bound of S, there exists at least one member $x\in S$ such that $t_1-\epsilon<x\leq t_1$



→ Find the infimum & supremum of the following sets and also find whether they are belong to set or not.

(1). $S=\{3,4,7\}\subseteq\mathbb{R}$

$\inf S=3\in S$ $\quad\sup S=7\in S$

(2). $S=\{\frac{1}{n}/n\in N\}=\{\frac{1}{1},\frac{1}{2},\frac{1}{3},\cdots\}\subseteq\mathbb{R}$

since $n\in N$, $n>0$

$\Rightarrow 0<\frac{1}{n}\leq 1\ \forall n\in N.$

$\Rightarrow 0<x\leq 1\ \forall x\in S$

$\therefore \inf S=0\notin S$ &

$\sup S=1\in S.$

(3). $S=\{-1/n\ /n\in N\}=\{-1,-\frac{1}{2},-\frac{1}{3},-\frac{1}{4},\cdots\}\subseteq\mathbb{R}$

since $n\in N$; $n>0$.

$\Rightarrow 0<\frac{1}{n}\leq 1$

$\Rightarrow -1\leq -1/n<0.$

$\therefore \inf=-1\in S$ & $\sup=0\notin S.$

(4) $S=\{\frac{1}{3n}/n\in N\}=\{\frac{1}{3},\frac{1}{3^2},\frac{1}{3^3},\cdots\}$

$\sup=\frac{1}{3}\in S$, $\inf=0\notin S.$

(5) $S=\{\frac{(-1)^n}{n}/n\in N\}$

$=\{-1,\frac{1}{2},-\frac{1}{3},\frac{1}{4},-\frac{1}{5},\cdots\}$

$=\{-1,-\frac{1}{3},-\frac{1}{5},\cdots,\frac{1}{2},\frac{1}{4},\cdots\}$

$\inf=-1\in S$; $\sup=\frac{1}{2}\in S.$

(6) $S=\{a+\frac{1}{n}/n\in N\}=\{a+1,a+\frac{1}{2},\cdots\}$

since $n\in N$, $n>0$

$\Rightarrow 0<\frac{1}{n}\leq 1$

$\Rightarrow a<a+1/n\leq a+1$

$\therefore \inf=a\notin S$; $\sup=a+1\in S$

(7) $S=\{1\}$

$\inf=\sup=1\in S$

(8) $S=\{\frac{3n+2}{2n+1}/n\in N\}\subseteq\mathbb{R}$

$=\{5/3,8/5,\cdots\}$

$\sup=5/3\in S$; $\inf=\text{Lt } S$

$=3/2\notin S.$

(9). $S=\{1/5n\ /n\in Z;\ n\neq 0\}$

$=\{\pm 1/5,\pm 1/10,\pm 1/15,\cdots\}$

$\inf=-1/5\in S$; $\sup=1/5\in S.$

⑩. $S = \{2^n / n \in N\} = \{2^1, 2^2, 2^3, \ldots\}$

$\inf = 2 \in S$; $\sup = \lim_{n\to\infty} 2^n = \infty$.

$\therefore$ supremum doesnot exist.

⑪ $S = \{1 - 1/n \,/\, n \in N\}$

⑫ $S = \{x \,/\, -5 < x < 3\}$

⑬ $S = \{x / x = (-1)^n ; n \in N\}$

⑭ $S = \{x / x = (-1)^n . n ; \forall n \in N\}$

$= \{-1, 2, -3, 4, -5, 6, -7, \ldots\}$

$= \{-1, -3, -5, \ldots ; 2, 4, 6, \ldots\}$

$\therefore$ inf S = does not exist, &

sup S = doesnot exist.

* Bounded Subset 's' of real Numbers :—

A subset 's' of $\mathbb{R}$ is said to be bdd if it is bdd below as well as bdd above.

i.e. A set 's' is bdd iff there exist two real numbers u, v such that $u \le x \le v \;\; \forall x \in S$.

i.e. $x \in [u, v]$; $x \in S$

i.e. $S \subset [u, v]$

i.e. S is a subset of $[u, v]$.

Ex:- (1). Every finite set is bdd and it has inf & sup.

(2). The null set ϕ is bdd but inf & sup of ϕ does not exist.

Because :

The null set ϕ is bdd above if 'u' is any real numbes then 'u' is an upper bound for ϕ obviously the condition $x \le u$ for all $x \in \phi$ is vaccuously satisfied because ϕ has no elements.

Thus every real number is an upper bound for ϕ. Since the set of all real numbers has no smallest member.

$\therefore$ Sup ϕ does not exist

Similarly inf ϕ doesnot exist.

(3). $N = \{1, 2, 3, \ldots\}$ is bdd below but not bdd above.

(4). $\mathbb{R}^+ = \{x \in \mathbb{R} \,|\, x > 0\}$ is bdd below but not bdd above.

(5). $\mathbb{R}^- = \{x \in \mathbb{R} \,|\, x < 0\}$ is bdd above but not bdd below.

* Greatest & Least members of a Subset of $\mathbb{R}$:—

→ If the supremum of a subset 's' of $\mathbb{R}$ is a member of S (i.e 's' attains its supremum) then this supremum is called greatest member of S.

→ If the inf of a subset 's' of $\mathbb{R}$ is a member of 's' (i.e. 's' attains its infimum) then this infimum is least member of 's'.

→ Ex:- (1) $S =]2,3[$

i.e. $S = \{x / 2 < x < 3\}$

Sup $S = 3 \notin S$ & inf $S = 2 \notin S$.

∴ 3 is not greatest member & 2 is not least member.

(2) $S = [2,3]$

i.e. $S = \{x \mid 2 \leq x \leq 3\}$

(3). $S = [1,2)$ i.e. $S = \{x / 1 \leq x < 2\}$

(4). $S = (1,2]$ i.e. $S = \{x / 1 < x \leq 2\}$

(5). The unbounded intervals are

$[a, \infty)$, $]a, \infty[$

, $]-\infty, a]$, $]-\infty, a[$, $]-\infty, \infty[$

Now suppose

$S = [a, \infty) = \{x / a \leq x\}$

inf $= a \in S$ & Sup = does not exist.

∴ least member of $S = a$.

* <u>Note</u> :-(1). Every finite set has two bounds

(2) Every infinite set may or may not have bounds.

(3). The bounds of a set may or may not belong to the set.

(4). Supremum & infimum of a bounded set need not belong to the set.

(5). Every greatest member of a set 's' is the supremum of 's' but every sup. of S need not be the greatest member of S.

(6). Every least member of a set 's' is the infimum of 's' but every inf of 's' need not be the least member. S.

* <u>Completeness property of $\mathbb{R}$</u> <u>(or Completeness axiom)</u>:—

Every non-empty subset of real numbers which is bounded above has the supremum (or l.u.b) in $\mathbb{R}$.

i.e. If 'S' is any non-empty subset of $\mathbb{R}$ which is bounded above, then the set of all upper bounds of S must have smallest member i.e. 'S' must possess the least upper bound which is a member of $\mathbb{R}$.

this property of real numbers is known as completeness. (This property is also called the supremum property of $\mathbb{R}$).

(or)

→ Every non-empty subset of real numbers which is bounded below has the infimum (or glb) in $\mathbb{R}$. This property of real numbers is known as completeness. This property is also called infimum property of $\mathbb{R}$.

* Complete Ordered Field:-

An ordered field F is said to be a complete ordered field if every non-empty subset S of F (i.e. $S \subseteq F$) which is bounded above has the supremum (or least upper bound) in F.

Ex:- 1. The set $\mathbb{R}$ of real numbers is complete ordered field.

Because $\mathbb{R}$ satisfies

① field axioms

② order axioms and

③ completeness axiom.

Ex:-(2). The set Q of rational numbers is an ordered field but not completeness.

→ Now we shall show that the ordered field of rational numbers is not a complete ordered field.

For this we are enough to show that there exists a non-empty subset of Q which is bounded above but which does not have a supremum in Q.

i.e. no rational numbers exists which can be the supremum.

Let us consider the set S of all those +ve rational numbers whose squares are less than 2

i.e. let $S = \{x : x \in Q^+ \text{ and } 0 < x^2 < 2\}$

Since $1 \in S$

$\therefore S \neq \phi$

i.e. S is non-empty.

Clearly 2 is an upper bound of S.

$\therefore$ S is bounded above.

$\therefore$ S is a non-empty subset of Q and is bounded above.

If possible suppose that the rational number k be its least upper bound.

Clearly k is +ve.

By law of trichotomy, which holds good in Q one and only one of (i) $k^2<2$ (ii) $k^2=2$ (iii) $k^2>2$ holds.

(i) $k^2<2$

Let us consider the +ve rational number $y = \frac{4+3k}{3+2k}$

then $k-y = k-\left(\frac{4+3k}{3+2k}\right)$

$= \frac{2(k^2-2)}{3+2k}$

<0 ($\because k^2<2$ i.e. $k^2-2<0$).

[$S=\{1/n \mid n\in N\}$ $=\{x \mid x\in Q^+, 0<x\leq 1\}\subseteq Q$ $k<y$]

$\boxed{\therefore k<y}$

Also $2-y^2 = 2-\left(\frac{4+3k}{3+2k}\right)^2$

$= \frac{2-k^2}{(3+2k)^2}$

>0 ($\because k^2<2$ i.e. $2-k^2>0$)

$\therefore 2-y^2>0$

$\Rightarrow y^2<2$

$\Rightarrow \boxed{y\in S}$

$\therefore$ The member y of S is greater than k so that k cannot be an upper bound of S.

$\therefore$ which is contradiction.

(ii) $k^2=2$, we know that there exists no rational number whose square is equal to 2.

$\therefore$ This case is not possible.

(iii) $k^2>2$

Let us consider the +ve rational number.

$$y = \frac{4+3k}{3+2k} \ (>0)$$

then $k-y = k-\left(\frac{4+3k}{3+2k}\right)$

$= \frac{2(k^2-2)}{3+2k}$

>0 ($\because k^2>2 \Rightarrow k^2-2>0$)

$\therefore k-y>0$

$\Rightarrow \boxed{k>y}$

Also $2-y^2 = 2-\left(\frac{4+3k}{3+2k}\right)^2$

$= \frac{2-k^2}{(3+2k)^2}<0$ ($\because 2-k^2<0$)

$\therefore 2-y^2<0$

$\Rightarrow 2<y^2$

$\Rightarrow \boxed{y^2>2}$

$\therefore y<k$ & $y^2>2 \Rightarrow y^2<k^2$ & $y^2>2$

$\Rightarrow 2<y^2<k^2$

If x is any member of 'S' then

$0<x^2<2<y^2<k^2$

$\Rightarrow 0<x<y<k$.

which shows K & y are both upper bounds of S.

But $y < K$

$\therefore$ K cannot be the supremum.

$\therefore$ Since K is any rational number, we conclude that no rational number can be the supremum of S.

* The Archimedian Property:-

If a & b be any two real numbers and if $a > 0$ then there exists a +ve integer n such that $na > b$.

Proof:- Let a & b be any two real numbers and $a > 0$.

Now if possible suppose that, for all +ve integers n

i.e. $n \in I^+$, $na \le b$

Let $S = \{na \mid n \in I^+\}$ then S is bounded above by 'b' (i.e. b is an upper bound of S).

$\therefore$ By completeness property of the ordered field of real numbers, the set S must have a supremum M (say)

$\therefore na \le M \quad \forall\, n \in I^+$

$\Rightarrow (n+1)a \le M \quad \forall\, n \in I^+$

$\Rightarrow na \le M - a \quad \forall\, n \in I^+$

$\therefore$ $M-a$ is an upper bound of S.

The number $M-a$ is less than the supremum M (least upper bound) is an upper bound of S.

$\therefore$ which is a contradiction.

$\therefore$ Our supposition is wrong.

Hence theorem.

* Absolute Value (modulus of a real number):—

If $x \in \mathbb{R}$ then the modulus (or absolute value or numerical value) of x is denoted by $|x|$ and defined as $|x| = \begin{cases} x & \text{if } x \ge 0 \\ -x & \text{if } x < 0 \end{cases}$

Properties:

Prove that (i) $|x| = \max\{x, -x\}$ (ii) $|x|^2 = x^2$ (iii) $x \le |x|$ and $-x \le |x|$ (iv) $|x| = |-x|$.

Proof:- (i) Since $x \in \mathbb{R}$, either $x \ge 0$ or $x < 0$.

If $x \ge 0$ then $|x| = x$ and $x \ge -x$.

and if $x < 0$ then $|x| = -x$ and $-x > x$

$\therefore |x|$ is greater of two numbers x & $-x$.

$\therefore |x| = \max\{x, -x\}$.

(ii) Since $|x| = x$ if $x \geq 0$
$= -x$ if $x < 0$

$\therefore |x|^2 = x^2$ (or) $(-x)^2$
$= x^2$

$\therefore \boxed{|x|^2 = x^2}$

(iii) Since $|x| = \max\{x, -x\}$

$\therefore |x| \geq x$ or $-x$

$\therefore x \leq |x|$ and $-x \leq |x|$.

(iv) Since $|x| = \max\{x, -x\}$,

and $|-x| = \max\{-x, x\} = \max\{x, -x\}$

$\therefore \boxed{|x| = |-x|}$.

Note :- $|x|^2 = x^2$

$\Rightarrow |x| = \pm\sqrt{x^2}$

Since $|x| \geq 0$.

$\therefore$ rejecting the $-$ve sign,

we have $\boxed{|x| = \sqrt{x^2}}$

(iv) $|x| = \sqrt{x^2}$

$\therefore |(-x)| = \sqrt{(-x)^2}$

$= \sqrt{x^2}$

$= |x|$

$\therefore \boxed{|x| = |-x|}$

$\rightarrow$ If x & y are any two real numbers.

then (a) $|x+y| \leq |x| + |y|$

(b) $|x-y| \geq ||x| - |y||$

(c) $|xy| = |x|\,|y|$

(d) $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$, if $y \neq 0$.

Proof:

(a) $|x+y| = \sqrt{(x+y)^2}$

$= \sqrt{x^2+y^2+2xy}$

$\leq \sqrt{x^2+y^2+2|x||y|}$ ($\because x < |x|$ & $y < |y|$)

$= \sqrt{|x|^2+|y|^2+2|x||y|}$

$= \sqrt{[|x|+|y|]^2}$

$= \big||x|+|y|\big|$ ($\because |x| = \sqrt{x^2}$)

$= |x|+|y|$ ($\because |x|+|y| \geq 0$ i.e., $|a| = a$ if $a > 0$.)

$\therefore \boxed{|x+y| \leq |x|+|y|}$

(b) $|x-y| = \sqrt{(x-y)^2}$

$= \sqrt{x^2+y^2-2xy}$

$\geq \sqrt{x^2+y^2-2|x||y|}$ ($\because x \leq |x|$ & $y \leq |y|$ $\Rightarrow xy \leq |x||y|$ $\Rightarrow -xy \geq -|x||y|$.)

$= \sqrt{|x|^2+|y|^2-2|x||y|}$

$= \sqrt{(|x|-|y|)^2}$

$= \big||x|-|y|\big|$ ($\because |x| = \sqrt{x^2}$)

$\therefore \boxed{|x-y| \geq \big||x|-|y|\big|}$

(c) $|xy| = \sqrt{(xy)^2}$

$= \sqrt{x^2y^2}$

$= \sqrt{x^2}\cdot\sqrt{y^2}$

$= |x|\,|y|$.

(d) $\left|\frac{x}{y}\right| = \sqrt{\left(\frac{x}{y}\right)^2}$

$= \frac{\sqrt{x^2}}{\sqrt{y^2}} = \frac{|x|}{|y|}$ provided $y \neq 0$.

→ If $r>0$ then Prove that

(a) $|x|<r \Leftrightarrow -r<x<r$

(b) $|x-a|<r \Leftrightarrow a-r<x<a+r$.

Sol^n:- (a) $|x|<r \Leftrightarrow \text{Max}\{x,-x\}<r$

$\Leftrightarrow x<r$ and $-x<r$

$\Leftrightarrow +x<+r$ and $x>-r$

$\Leftrightarrow -r<x$ and $x<r$

$\Leftrightarrow -r<x<r$

(b) $|x-a|<r \Leftrightarrow \text{Max}\{x-a,-(x-a)\}<r$

$\Leftrightarrow x-a<r$ and $-(x-a)<r$

$\Leftrightarrow x<a+r$ and $-x<r-a$

$\Leftrightarrow x<a+r$ and $x>a-r$

$\Leftrightarrow a-r<x$ and $x<a+r$

$\Leftrightarrow a-r<x<a+r$

→ Prove that (i) $|x-y| \le |x|+|y|$

(ii) $a<x<b \Leftrightarrow \left|x-\left(\frac{a+b}{2}\right)\right|<\frac{b-a}{2}$

Proof:- (i) $|x-y| = |x+(-y)|$

$\le |x|+|-y|$

$= |x|+|y|$

$(\because |-y|=|y|)$

$\therefore |x-y| \le |x|+|y|$.

(ii) $a<x<b$

Adding throughout $-\left(\frac{a+b}{2}\right)$, we get

$\Leftrightarrow a-\left(\frac{a+b}{2}\right)<x-\left(\frac{a+b}{2}\right)<b-\left(\frac{a+b}{2}\right)$

$\Leftrightarrow \frac{a-b}{2}<x-\left(\frac{a+b}{2}\right)<\frac{b-a}{2}$

$\Leftrightarrow -\left(\frac{b-a}{2}\right)<x-\left(\frac{a+b}{2}\right)<\left(\frac{b-a}{2}\right)$

$\Leftrightarrow \left|x-\left(\frac{a+b}{2}\right)\right|<\frac{b-a}{2}$

$(\because |x|<r \Leftrightarrow -r<x<r)$

* Neighbourhood of a point:-

If a is any real number and $\delta>0$ (however small), then the open interval $(a-\delta, a+\delta)$ is called a δ- neighbourhood of 'a' and is denoted by $N_\delta(a)$ or $N(\delta,a)$. i.e. $N_\delta(a) = (a-\delta, a+\delta)$.

— shortly written as ~~neighbourhood~~ nbd.

$x \in (a-\delta, a+\delta)$

→ If from the neighbourhood of a point, the point itself is excluded, we get the deleted neighbourhood of that point.

i.e. $N_\delta(a)-\{a\}$ is a deleated neighbourhood of a point 'a'. and is denoted by $N_\delta d(a)$

i.e. $N_\delta d(a) = N_\delta(a)-\{a\}$.

Ex:- If $a=5$, $\delta=0.2>0$ then $(4.8, 5.2)$ is a neighbourhood of 5

Now $x \in (4.8, 5.2)-\{5\} \Rightarrow x \in (4.8, 5.2)$; $x \ne 5$ is a deleted neighbourhood of 5.

Note:- $x \in N_\delta(a)$

$\Leftrightarrow x \in (a-\delta, a+\delta)$

$\Leftrightarrow a-\delta < x < a+\delta$

$\Leftrightarrow -\delta < x-a < \delta$

$\Leftrightarrow |x-a| < \delta$

$(\because |x| < r \Leftrightarrow -r < x < r)$

and $x \in N_{\delta}d(a)$

$\Leftrightarrow x \in (a-\delta, a+\delta) - \{a\}$

$\Leftrightarrow x \in (a-\delta, a+\delta); x \neq a$

$\Leftrightarrow a-\delta < x < a+\delta \ ; \ x \neq a$

$\Leftrightarrow |x-a| < \delta \ ; \ x \neq a$

$\Leftrightarrow 0 < |x-a| < \delta$

* Neighbourhood of a set :-

→ A subset S of IR (i.e. $S \subseteq \mathbb{R}$) is said be neighbourhood of a point $a \in \mathbb{R}$ if there exists a $\delta > 0$ (however small) such that

$(a-\delta, a+\delta) \subset S$

Note:- If S is a neighbourhood of a point 'a' then $S - \{a\}$ is called deleted neighbourhood of 'a'.

Ex:- (1) If $a \in \mathbb{R} \subseteq \mathbb{R}$ then IR is a neighbourhood of 'a' because $a \in (a-\delta, a+\delta) \subset \mathbb{R}$.

(2) If $a \in \mathbb{Q} \subseteq \mathbb{R}$ then Q is not neighbour hood of a. because $a \in (a-\delta, a+\delta) \not\subset \mathbb{Q}$.

(3). If $a \in \mathbb{Z} \subset \mathbb{R}$ then Z is not neighbourhood of 'a' because $a \in (a-\delta, a+\delta) \not\subset \mathbb{Z}$

(4). If $a \in \mathbb{N} \subset \mathbb{R}$ then N is not neighbourhood of a because $a \in (a-\delta, a+\delta) \not\subset \mathbb{N}$.

Problems

→ Any open interval is a neighbourhood of each of its points.

Sol'n :- Let $S = (a,b)$

Let P be any point of (a,b)

i.e. $P \in (a,b)$

$\Rightarrow a < P < b$

Let $\epsilon = \text{Min}\{P-a, b-P\} > 0$

$\Rightarrow \epsilon \leq P-a \ ; \ \epsilon \leq b-P$

$\Rightarrow a \leq P-\epsilon \ ; \ b \geq P+\epsilon$

$\Rightarrow a \leq \underbrace{P-\epsilon < P < P+\epsilon}\leq b$

$\Rightarrow P \in (P-\epsilon, P+\epsilon) \subset (a,b)$

$\therefore (a,b)$ is a neighbourhood of P.

→ A closed interval $[a,b]$ is a neighbourhood of each of its points except the two end points a & b.

Sol'n :- Let $S = [a,b]$

Let $P \in [a,b]$.

$\Rightarrow a \leq P \leq b$

$\Rightarrow$ (i) $a < P < b$

(ii) $P = a$ &

(iii) $P = b$

Let $\epsilon = \text{Min}\{P-a, b-P\} > 0$

(i) $P \in (P-\epsilon, P+\epsilon) \subset (a,b) \subset [a,b]$

$\therefore P \in (P-\epsilon, P+\epsilon) \subset [a,b]$

$\therefore [a,b]$ is a neighbourhood of P.

i.e. $[a,b]$ is a neighbourhood of each $P \in (a,b)$.

(ii) $P = a$

$\Rightarrow (P-\epsilon, P+\epsilon) = (a-\epsilon, a+\epsilon) \not\subset [a,b]$

$\therefore [a,b]$ is not neighbourhood of a.

(iii) $P = b$

$\Rightarrow (P-\epsilon, P+\epsilon) = (b-\epsilon, b+\epsilon) \not\subset [a,b]$

$\therefore [a,b]$ is not a neighbourhood of b.

$\rightarrow$ $[a,b)$ is a neighbourhood of each of its points except 'a'

$\rightarrow$ $(a,b]$ is a neighbourhood of each of its points except b.

$\rightarrow$ A non-empty finite set cannot be a neighbourhood any of its points.

Sol'n :- Let S be any non-empty finite set.

Let P be any point of S

Let $\epsilon > 0$ (however small)

then $(P-\epsilon, P+\epsilon)$ is an infinite set.

$\therefore (P-\epsilon, P+\epsilon) \not\subset S$

$\therefore S$ is not a neighbourhood of P.

$\rightarrow$ Empty set ϕ is a neighbourhood of each of its points.

Sol'n :- The empty set ϕ is a neighbourhood of each of its points.

because there is no point at all in ϕ

and so there is no point in ϕ of which it is not a neighbourhood.

$\rightarrow$ Show that the set 'N' of all natural is not a neighbourhood of any of its points.

Sol'n :- Let $P \in N$

and let $\epsilon > 0$.

then $(p-\epsilon, p+\epsilon)$ contains infinitely many rational and irrational numbers.

$\therefore (p-\epsilon, p+\epsilon) \not\subset N$

$\therefore N$ is not a neighbourhood of any point $p \in N$.

Similarly, the set 'W' of all whole numbers is not a neighbourhood of any of its points.

and the set 'I' of integers is not a neighbourhood of any of its points.

→ Show that the set Q of all rational numbers is not a neighbourhood of any of its points.

Sol^n:- Let $p \in Q$ and let $\epsilon > 0$. (however small)

then $(p-\epsilon, p+\epsilon)$ contains infinitely many irrational numbers which are not members of Q.

$\therefore (p-\epsilon, p+\epsilon) \not\subset Q$.

$\therefore Q$ is not a neighbourhood of any point $p \in Q$.

Similarly, the set Q' of all irrational numbers is not nbd of any of its points.

→ S.T the set of real numbers is a nbd of each of its points.

Sol^n: Let $p \in \mathbb{R}$ and let $\epsilon > 0$.

then $(p-\epsilon, p+\epsilon)$ contains infinitely many real numbers.

$p \in (p-\epsilon, p+\epsilon) \subset \mathbb{R}$

$\therefore$ The set $\mathbb{R}$ of real numbers is a nbd of its points.

→ Any set 'S' cannot be a nbd of any point of the set $\mathbb{R}-S$.

Sol^n: Let $p \in \mathbb{R}-S$

then $p \notin S$.

let $\epsilon > 0$.

$\Rightarrow (p-\epsilon, p+\epsilon) \not\subset S$.

$\therefore S$ is not a nbd of any point $p \in \mathbb{R}-S$.

→ Every superset of nbd of a point 'p' is also nbd of p.

Sol^n: Let S be a nbd of p.

$\Rightarrow (p-\epsilon, p+\epsilon) \subset S$.

If T is any superset of S, then $S \subset T$.

$\therefore (p-\epsilon, p+\epsilon) \subset S \subset T$

$\Rightarrow (p-\epsilon, p+\epsilon) \subset T$.

$\therefore T$ is a nbd of p.

→ The intersection of two nbds of a point is also a nbd of that point.

Sol^n: Let M_1 and M_2 be two nbds of P.

$\therefore \exists\ \epsilon_1$ and $\epsilon_2 > 0$ (however small) such that

$P \in (P-\epsilon_1, P+\epsilon_1) \subset M_1$ and $P \in (P-\epsilon_2, P+\epsilon_2) \subset M_2$.

Let $\epsilon = \min\{\epsilon_1, \epsilon_2\}$

$\therefore (P-\epsilon, P+\epsilon) \subset (P-\epsilon_1, P+\epsilon_1) \subset M_1$

and $(P-\epsilon, P+\epsilon) \subset (P-\epsilon_2, P+\epsilon_2) \subset M_2$

$\therefore P \in (P-\epsilon, P+\epsilon) \subset M_1 \cap M_2$.

$\therefore M_1 \cap M_2$ is also a nbd of P.

→ If M_1 is a nbd of P (or) M_2 is a nbd of P then $M_1 \cup M_2$ is also a nbd of P.

Interior point of a set:-

Let $S \subset \mathbb{R}$, $P \in S$ is called an interior point of a set S if S is a nbd of P

i.e., if $\exists\ \epsilon > 0$ (however small) such that $(P-\epsilon, P+\epsilon) \subset S$.

Ex: (1) Every point of an open interval (a,b) is an interior point of the interval.

(2) Every point of a closed interval $[a,b]$ is an interior point of the interval except the end points a and b.

(3) Every point of a semi closed interval $[a,b)$ is an interior point of the interval

except the left end point 'a'.

→ Every point of a semi open interval $(a, b]$ is an interior point of the interval except the right end point 'b'.

→ Every point of the empty set is an interior point.

→ Every non-empty finite set has no interior point.

→ $\mathbb{N}$ has no interior point.

Similarly $W, \mathbb{Z}, \mathbb{Q}, \mathbb{R}-\mathbb{Q}$.

→ Every point of a real number set is an interior point of $\mathbb{R}$.

Interior of a set:

The set of all interior points of a set 'S' is called interior of a set 'S' and is denoted by S^o (or) $\text{int}(S)$.

Ex: (1) If $S = (a, b)$ then $S^o = S$ because every point of S is an interior point of S

(2) If $S = [a, b]$ then $S^o = (a, b)$ because every point of S is an interior point of S except the end points a & b.

(3) If $S = [a, b)$ then $S^o = (a, b)$

(4) If $S = (a, b]$ then $S^o = (a, b)$.

(5) $\mathbb{R}^o = \mathbb{R}$ because every point of $\mathbb{R}$ is an interior point.

(6) If S is a non-empty finite set then $S^o = \phi$.

(7) $\mathbb{N}^o = \phi$, $\mathbb{Z}^o = \phi$, $\mathbb{Q}^o = \phi$, $(\mathbb{R}-\mathbb{Q})^o = \phi$ & $W^o = \phi$. because $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}-\mathbb{Q}, W$ are not nbd of any points and therefore no point is an interior point of $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}-\mathbb{Q}$ or W.

(8) If $S = \phi$ then $S^o = \phi$.

→ Find the interior of the following sets.

(i) $\{1,2,3,4,5\}$ (ii) $[0,1]$ (iii) $[0,1]\cup[3,5]$ (iv) $\{\frac{1}{n} / n\in N\}$

Sol: (i) Let $A=\{1,2,3,4,5\}$.

then A is a non-empty finite set.

$\Rightarrow$ A is not nbd of any point.

$\therefore$ no point is an interior point of A.

$\Rightarrow A^0=\phi$.

(iii) Let $A=[0,1]\cup[3,5]$.

then A is nbd of each point of $(0,1)\cup(3,5)$

$\therefore A^0=(0,1)\cup(3,5)$.

(iv) Let $A=\{\frac{1}{n} / n\in N\}$

$=\{1,\frac{1}{2},\frac{1}{3},\ldots\}$

Let p be any point of A

i.e, $p\in A$.

then $\exists$ no $\epsilon>0$ such that

$(p-\epsilon, p+\epsilon)\subset A$

$\Rightarrow$ p is not an interior point of A.

i.e, A has no interior point

$\therefore A^0=\phi$.

Open set: A subset S of $\mathbb{R}$ is said to be an openset if S is a nbd of each of its points.

i.e, if for each $p \in S$ $\exists$ an $\epsilon > 0$ such that

$(p-\epsilon, p+\epsilon) \subset S$.

(or)

If 'S' is a subset of $\mathbb{R}$ is said to be open if every point of S is an interior point of S.

i.e, S is open $\Leftrightarrow$ $S^0 = S$.

Ex:- (1) Every open interval is an openset.

Sol^n: Let $S = (a, b)$

then $S^0 = (a, b)$

$\therefore S^0 = S$

$\Rightarrow S$ is openset.

(2) $S = [a, b]$ then $S^0 = (a, b)$

$\therefore S \neq S^0$

$\Rightarrow S$ is not an open set.

Similarly $[a, b)$, $(a, b]$ are not opensets.

(3) $S = \mathbb{N}$ then $S^0 = \phi$.

$\therefore S \neq S^0$

$\Rightarrow S$ is not open set.

Similarly $\mathbb{W}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}-\mathbb{Q}$ are not opensets.

(4) $S = \mathbb{R}$. then $S^0 = \mathbb{R}$.

$\therefore S = S^0$

$\Rightarrow S$ is open.

(5) $S = \mathbb{R}^+ = (0, \infty)$ is an open set.

because let $x \in S$

$\exists$ an $\epsilon > 0$ (however small) such that $x \in (x-\epsilon, x+\epsilon) \subset S$

$\therefore S = S^{\circ}.$

(6) $S = \mathbb{R}^{-} = (-\infty, 0)$ is an open set.

(7) Every non-empty finite set is not an open set.

because for every nbd of a point contains infinitely many points.

(8) $S = \{\ \}$ is an open set.

because $S^{\circ} = \phi$
$\Rightarrow S^{\circ} = S.$

(9) $S = \{\frac{1}{n} \mid n \in N\}$ is not an open set.

since $S \neq S^{\circ}.$

→ Union of two open sets is an open set.

Sol^n^: Let S_1 and S_2 be two open sets.

Let $S = S_1 \cup S_2$

Let $x \in S \Rightarrow x \in S_1 \cup S_2$
$\Rightarrow x \in S_1$ or $x \in S_2$

If $x \in S_1$ then $\exists\ \epsilon > 0$ such that
$x \in (x-\epsilon, x+\epsilon) \subset S_1 \subset S_1 \cup S_2$ ($\because S_1$ is open)

If $x \in S_2$ then $\exists\ \epsilon > 0$ such that
$x \in (x-\epsilon, x+\epsilon) \subset S_2 \subset S_1 \cup S_2$ ($\because S_2$ is open)

$\therefore x \in (x-\epsilon, x+\epsilon) \subset S_1 \cup S_2 = S$

$\therefore x$ is an interior point of $S_1 \cup S_2 = S$

$\therefore S = S_1 \cup S_2$ is an open set.

→ The union of an arbitrary family of open sets is an open set.

→ The intersection of two open sets is an open set.

Sol^n: Let S_1 & S_2 are two open sets.

To S.T $S_1 \cap S_2$ is also an openset.

Let $S = S_1 \cap S_2$.

Let $x \in S \Rightarrow x \in S_1 \cap S_2$

$\Rightarrow x \in S_1$ and $x \in S_2$

$\Rightarrow x \in (x-\epsilon_1, x+\epsilon_1) \subset S_1$ and

$x \in (x-\epsilon_2, x+\epsilon_2) \subset S_2$

($\because S_1$ & S_2 are two open sets)

Choosing $\epsilon = \min\{\epsilon_1, \epsilon_2\} > 0$

$x \in (x-\epsilon, x+\epsilon) \subset (x-\epsilon_1, x+\epsilon_1) \subset S_1$ and

$x \in (x-\epsilon, x+\epsilon) \subset (x-\epsilon_2, x+\epsilon_2) \subset S_2$

$\Rightarrow x \in (x-\epsilon, x+\epsilon) \subset S_1 \cap S_2 = S$

$\therefore S = S_1 \cap S_2$ is an open set.

→ The intersection of a finite collection of open sets is an openset.

→ The intersection of an infinite collection of opensets need not be an open set.

Sol^n: (i) Let $S_n = \left(-\frac{1}{n}, \frac{1}{n}\right)$ $\forall n \in N$.

$\Rightarrow S_1 \cap S_2 \cap S_3 \cap \dots\dots = (-1, 1) \cap \left(-\frac{1}{2}, \frac{1}{2}\right) \cap \dots\dots$

$= \{0\}$

which is not an openset.

because $(0-\epsilon, 0+\epsilon) \not\subset \{0\}$.

$\therefore$ The intersection of an infinite collection of opensets is not an open set.

(ii) Let $S_n = (0, n) \subseteq \mathbb{R} \ \forall n \in \mathbb{N}$.

Then $S_1 \cap S_2 \cap \ldots\ldots = (0,1) \cap (0,2) \cap \ldots\ldots$
$= (0,1)$
which is an open set.

$\therefore$ The intersection of an infinite collection of open sets need not be an open set.

Note: Every open interval is an open set. but every open set need not be an open interval.

for example:
Let $S_1 = (1,2)$; $S_2 = (3,4)$ are two open sets.

$S_1 \cup S_2 = (1,2) \cup (3,4)$ is an open set.

but $(1,2) \cup (3,4)$ is not an open interval.

* Limit point of a subset S of $\mathbb{R}$:—

A point $p \in \mathbb{R}$ is said to be a limit point of a subset S of $\mathbb{R}$ if every nbd of 'p' has a point of S other than p itself.

(or)

A point $p \in \mathbb{R}$ is said to be a limit point of a subset S of $\mathbb{R}$ if every nbd of p has an infinite number of points of 'S'

(or)

A point $p \in \mathbb{R}$ is said to be a limit point of subset 'S' of $\mathbb{R}$ if every nbd of p contains atleast one point of S other than p.

i.e, p is a limit point of S $\Leftrightarrow$

$$(p-\epsilon, p+\epsilon) \cap S - \{p\} \neq \phi.$$

Note: (1) Limit point is also called cluster point (or) condensation point (or) accumulation point.

(2) A limit point of 'S' may or may not belong to the set 'S'.

(3) A set may have no limit point, a unique limit point or a finite or infinite number of limit points.

(4) $p \in \mathbb{R}$ is not a limit point of a subset 'S' of $\mathbb{R}$ if there exists a nbd of 'p' which doesnot contain any point of 'S'.

(5) p is not a limit point of 'S' if for some $\epsilon > 0$, $(p-\epsilon, p+\epsilon) \cap S = \phi$ (or)

$$(p-\epsilon, p+\epsilon) \cap S = \{p\}.$$

→ A finite set has no limit point.

Sol: Let 'A' be a finite set.

If possible suppose that p is a limit point of 'A' and let $\epsilon > 0$.

Then $(p-\epsilon, p+\epsilon)$ contains infinite number of points of 'A'.

$\therefore$ A is infinite.

It is a contradiction.

$\therefore$ 'A' has no limit points

$\therefore$ A finite set 'A' has no limit points

→ $N = \{1, 2, 3, \ldots\} \subseteq \mathbb{R}$ has no limit points.

→ $A = \{\ldots -3, -2, -1\}$ has no limit points.

→ Every point of the set $\mathbb{R}$ of all real numbers is a limit point of $\mathbb{R}$.

→ Let $S = \mathbb{R}$

$p \in \mathbb{R}$; $\epsilon > 0$.

$(p-\epsilon, p+\epsilon) \cap \mathbb{R}$ = infinite number of real numbers.

→ Every real number is a limit point of the $\mathbb{Q}$ of all rational numbers.

Let $S = \mathbb{Q}$

let $p \in \mathbb{R}$: $\epsilon > 0$.

$(p-\epsilon, p+\epsilon) \cap S$ = infinite number of rational numbers.

Similarly, $S = \mathbb{Q}'$ or $\mathbb{R} - \mathbb{Q}$.

→ The empty set ϕ has no limit points.

i.e, let $S = \phi$.

$p \in \mathbb{R}$: $\epsilon > 0$.

$(p-\epsilon, p+\epsilon) \cap S = \phi$
not an infinite set.

→ $S = (a, b)$.

every point of S is a limit point of S.

→ $S = [a, b]$, $(a, b]$, $[a, b)$.

Every point of S is a limit point of S.

→ $S = \{\frac{1}{n} \mid n \in N\} = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\} \subseteq \mathbb{R}$

Let $0 \in \mathbb{R}$; $\epsilon > 0$

$(0-\epsilon, 0+\epsilon) \cap S$ = infinite set.

$\therefore$ 0 is a limit point of S.

(or)

$\lim_{n \to \infty} \frac{1}{n} = \frac{1}{\infty} = 0 \notin S$. (i.e, 0 is not a member of S)

→ $S = \left\{ \frac{n}{n+1} \middle/ n \in N \right\} = \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots \right\}$

(or)

$= \left\{ 1 - \frac{1}{n+1} \middle/ n \in N \right\}$.

Since $1 \in \mathbb{R}$, $\epsilon > 0$ such that $(1-\epsilon, 1+\epsilon)$ contains infinitely many points of S.

$\therefore$ 1 is a limit point of S and $1 \notin S$.

(or)

$\underset{n\to\infty}{Lt} S = 1 \notin S$. is a limit point.

→ $S = \left\{ 1 - \frac{1}{n} \middle/ n \in N \right\}$

$\underset{n\to\infty}{Lt} S = 1 \notin S$ is a limit point of S.

→ $S = \left\{ (-1)^n / n \in N \right\} = \{-1, +1, -1, +1, \ldots\}$

Since $\underset{n\to\infty}{Lt} S = \underset{n\to\infty}{Lt} (-1)^n = \begin{cases} -1 & \text{if } n \text{ is odd} \\ +1 & \text{if } n \text{ is even} \end{cases}$

$\therefore$ S has two limit points -1 and $+1$ which are members of S.

→ $S = \left\{ (-1)^n n / n \in N \right\} = \{-1, +2, -3, +4, -5, \ldots\}$

has no limit points.

since $\underset{n\to\infty}{Lt} S = \underset{n\to\infty}{Lt} (-1)^n n = \begin{cases} -\infty & \text{if } n \text{ is odd} \\ +\infty & \text{if } n \text{ is even} \end{cases}$

<u>Note</u>: (1) Every finite set has no limit points

(2) Every infinite set may or may not have limit points.

(3) Every interior point is a limit point, but every limit point need not be an interior point.

<u>ex</u>: Let $S = (a, b)$, then $D(S) = [a, b]$ & $S^0 = (a, b)$.

$\therefore$ a, b are limit points but not interior points

→ If the supremum of a set does not belong to the set, then it is a limiting point of the set. 16

Sol Let S be the non-empty subset of real number set R.
and has supremum but not belong to the set S.

Let it be 'u'
i.e u = supremum of S but $u \notin S$.

Now we have to prove that 'u' is a limiting point of a set 'S'.

for this we have to prove that every nbd of the point u contains a point of S other than u.

Let $(u-\epsilon, u+\epsilon)$ be any nbd of the point 'u'. where $\epsilon > 0$.

Since u = l.u.b (supremum) of S.
$\therefore u-\epsilon$ is not an upperbound of S.

$\therefore \exists$ some $x \in S$ s.t $x > u-\epsilon$ —— (1)

Also $x < u < u+\epsilon$ —— (2). ($\because x \in S$ and $u \notin S$ $\therefore x \neq u$)

from (1) and (2), we have
$u-\epsilon < x < u+\epsilon$ where $x \neq u$.

$\Rightarrow (u-\epsilon, u+\epsilon)$ contains a point $x \in S$

$\Rightarrow$ 'u' is a limiting point of the set S.

→ If the supremum of a set does not belong to the set, then it is a limit point of a set.

For example:

① $S = (-\infty, a) \subseteq \mathbb{R}$

$\therefore$ S is bdd above

and $\sup S = a \notin S$

$\therefore$ 'a' is a limiting point of S.

② $S = (a, \infty)$

$\therefore$ S is bdd below by 'a'

and $\inf S = a \notin S$

$\therefore$ 'a' is a limiting point of S.

* **Isolated point:**~

A point $p \in S$ is called an isolated point of S if p is not a limit point of S.

i.e if $\exists$ a nbd of 'p' which contains no points of S other than 'p' itself.

— A set 'S' is called a discrete set if all its points are isolated points.

For example: Let $S = \{1, \frac{1}{2}, \frac{1}{3}, \ldots\}$

Since all the points of the set 'S' are its isolated points and so it is a discrete set.

Derived Set:

The set of all limit points of a subset 'S' of $\mathbb{R}$ is called the derived set of S and is denoted by S' or $D(S)$.

i.e, $D(S)$ or $S' = \{x \in \mathbb{R} \,/\, x \text{ is a limit point of } S\}$

— Again the derived set of $D(S)$ is called the second derived set of S and is denoted by $D^2(S)$ or S''.

In general, the n^{th} derived set of S denoted by $D^{(n)}(S)$ or $S^{(n)}$

→ A set is said to be of first species if it has only a finite number of derived sets.

It is said to be of second species if the number of derived sets is infinite.

Note: ① If the set S is finite, then 'S' has no limit point and consequently, $D(S) = \phi$.

② If a set 'S' is of first species, then its last derived set must be empty.

③ A set whose n^{th} derived set is a finite set, so that its $(n+1)^{th}$ derived set is empty is called a set of n^{th} order.

* **Adherent point :—**

A real number 'p' is called an adherent point of a set $S \subseteq \mathbb{R}$ if every nbd of p contains a point of S.

i.e point $p \in \mathbb{R}$ is an adherent point of $S \subset \mathbb{R}$
$\Leftrightarrow$ for each nbd N of p, $N \cap S \neq \phi$.

Note:- Due to a close resemblance between the definitions of an adherent point of a set and a limit point of a set, the distinction between the two should be carefully noted.

For a point 'p' to be a limit of a set S, every nbd N of 'p' must contain a point of S other than p.

i.e $N \cap S - \{p\} \neq \phi$.

For a point 'p' to be an adherent point of a set S, every nbd of p must contain a point of S which can be 'p' itself.

i.e $N \cap S \neq \phi$.

— If $p \in S$, then 'p' is an adherent point of S, since every nbd of p contains p which belongs to S.

— If $p \in D(S)$ then p is a limit point of S and, therefore, every nbd of 'p' contains a point of S other than p.

Thus p is also an adherent point of S.

Clearly, a real number p is an adherent point $\Leftrightarrow$ either $p \in S$ or $p \in D(S)$.

— Every point of S is an adherent of S.

— Every limit point of S is an adherent point.

But an adherent point of S need not be a limit point of S.

Closure of a set :-

The set of all adherent points of a set 'S' is called the closure of S and is denoted by Cl S or $\bar{S}$.

Thus $\bar{S} = S \cup D(S)$.

Dense set:

A subset 'S' of $\mathbb{R}$ is said to be dense (or dense in $\mathbb{R}$ or everywhere dense) if every point of $\mathbb{R}$ is a point of S or a limit point of S or both.

(or)

Let $S \subseteq \mathbb{R}$ then 'S' is said to be dense if $\bar{S} = \mathbb{R}$.

Dense in itself:

A set 'S' is said to be dense-in-itself if every point of S is a limit point of 'S'.

(or)

A subset 'S' of $\mathbb{R}$ is said to be dense-in-itself if $S \subseteq D(S)$

(or)

A subset 'S' of $\mathbb{R}$ is said to be dense-in-itself if it possesses no isolated points.

perfect set:

A set 'S' is said to be perfect set if $S = D(S)$.

(or)

A set 'S' is said to be perfect set if it is dense-in-itself and if it contains all its limit points.

→ The set Q of rational numbers.

Let $S = Q \subseteq \mathbb{R}$

Let x be any real number. Then for each $\epsilon > 0$ (however small),

$(x-\epsilon, x+\epsilon)$ is a nbd of x

and it contains infinitely many rational number other than 'x'.

i.e $(x-\epsilon, x+\epsilon) \cap Q - \{x\} \neq \phi$.

$\Rightarrow$ x is a limit point of $S = Q$

$\Rightarrow$ every real number is a limit point of Q.

Hence the set of the limit points of Q is the set of all real numbers $\mathbb{R}$

$\therefore \boxed{D(Q) = \mathbb{R}}$.

Also $\bar{S} = S \cup D(S)$

$= Q \cup \mathbb{R}$

$\boxed{\bar{S} = \mathbb{R}}$

clearly 'S' is dense in $\mathbb{R}$ and $S \subseteq D(S)$

$\therefore S = Q$ is dense-in-itself

Since $S \neq D(S)$

i.e $Q \neq D(Q)$

$\therefore S = Q$ is not a perfect set.

→ The set $\mathbb{R} - Q$ of irrational numbers.

Let $S = \mathbb{R} - Q \subseteq \mathbb{R}$ then

$D(S) = \mathbb{R}$.

→ The set N of natural numbers.

Let $S = N \subseteq \mathbb{R}$

Let $x \in \mathbb{R}$

If $x \in N$, the nbd of x (i.e $(x-\epsilon, x+\epsilon)$) contains no point of N other than x

$\therefore$ x is not limit point of N of natural numbers.

If $x \notin N$, then the nbd of 'x' does not contain any point of 'S'

$\therefore$ x is not limit point of the set N of natural numbers.

$\therefore$ N has no limit points.

$\therefore D(N) = \phi$.

Since no point of N is a limit point of N.

$\therefore$ All the points of N are isolated points.

Hence $\boxed{N \text{ is descrete set}}$.

Also $\boxed{N \text{ is of first species}}$. $(\because D(N) = \phi)$.

H.W → The set W of all whole numbers.

H.W → The set I of all integers.

→ Let $S = \phi \subseteq \mathbb{R}$.

Let $x \in \mathbb{R}$ then for each $\epsilon > 0$, (however small)

$(x-\epsilon, x+\epsilon) \cap \phi = \phi$

$\Rightarrow$ x is not a limit point of $S = \phi$.

$\Rightarrow$ No real number is a limit of ϕ.

$\therefore \boxed{D(\phi) = \phi}$.

$\bar{S} = S \cup D(S)$

$= \phi \cup D(\phi)$

$\boxed{\bar{S} = \phi}$

Since $\phi \subseteq D(\phi)$

$\therefore$ S is dense-in-itself.

Also $S = D(S)$ i.e $\phi = D(S)$

$\therefore$ ϕ is perfect set.

→ The set $\mathbb{R}$ of all real numbers.

Let $x \in \mathbb{R} \subseteq \mathbb{R}$

Then for each $\epsilon > 0$, (however small)

the nbd of x (i.e $(x-\epsilon, x+\epsilon)$)

contains infinitely many real numbers other than 'x'.

$\therefore$ x is a limit point of $\mathbb{R}$

$\Rightarrow$ Every real number is a limit point of $\mathbb{R}$.

$\therefore$ $\boxed{D(\mathbb{R}) = \mathbb{R}}$.

$\bar{\mathbb{R}} = \mathbb{R} \cup D(\mathbb{R})$

$\boxed{\bar{\mathbb{R}} = \mathbb{R}}$.

$\therefore$ $\mathbb{R}$ is dense set

and it is dense-in-itself

Also it is perfect set $(\because \mathbb{R} = D(\mathbb{R}))$.

Note:- we have $D(\mathbb{R}) = \mathbb{R}$, $D^2(\mathbb{R}) = D(\mathbb{R}) = \mathbb{R}$

$D^3(\mathbb{R}) = \mathbb{R}$ and so on.

$\therefore$ for every +ve integer 'n', $D^{(n)}(\mathbb{R}) = \mathbb{R}$

$\therefore$ The number of derived sets of $\mathbb{R}$ is infinite

$\therefore$ $\mathbb{R}$ is of the second species.

→ $S = (a, b)$.

Sol: If $x \in [a, b]$.

then $x = a$ or $x = b$ or $x \in (a, b)$

If $x = a$, then for every $\epsilon > 0$,

$(x-\epsilon, x+\epsilon) = (a-\epsilon, a+\epsilon)$ contains

infinitely many points of (a, b) to the right of 'a'.

If $x=b$, then for every $\epsilon>0$,

$(x-\epsilon, x+\epsilon) = (b-\epsilon, b+\epsilon)$ contains infinitely many points of (a,b) to the left of 'b'.

If $x \in (a,b)$, then for every $\epsilon>0$,

$(x-\epsilon, x+\epsilon)$ contains infinitely many points of (a,b).

Thus, if $x \in [a,b]$, then for every $\epsilon>0$, $(x-\epsilon, x+\epsilon)$ is a nbd of 'x' containing infinitely many points of (a,b).

$\Rightarrow$ every point of $[a,b]$ is a limit point of (a,b).

$\therefore \boxed{D((a,b)) = [a,b]}$.

Since $\bar{S} = S \cup D(S)$

$= (a,b) \cup [a,b]$

$\boxed{\bar{S} = [a,b] \subseteq \mathbb{R}}$

Since $S \subseteq D(S)$ i.e. $(a,b) \subseteq [a,b]$.

$\therefore$ S is dense-in-itself.

Since $S \neq D(S)$

$\therefore$ S is not a perfect set.

$\rightarrow$ $S = [a,b)$

$\rightarrow$ $S = [a,b]$

$\rightarrow$ $S = [a,b]$.

$\rightarrow$ $S = \{\frac{1}{n} \mid n \in \mathbb{N}\} \subseteq \mathbb{R}$

Here $S = \{\frac{1}{n} \mid n \in \mathbb{N}\} \subset (0,1] \subseteq \mathbb{R}$

Let $p \in \mathbb{R}$, then for each $\epsilon>0$ (however small),

If $p=0$, then $(0-\epsilon, 0+\epsilon)$ is a nbd of '0'.

and it contains infinitely many points of S other than '0'

$\therefore$ 0 is the limit point of S

Now we shall show that no other real number p other than '0' can be a limit point of S.

The following cases arise:

case (i) If $p<0$ then $(-\infty, 0)$ is a nbd of p which contains no point of S. i.e., $(-\infty, 0)\cap S=\phi$

$\therefore$ p is not a limit point of S.

Case(ii): If $p>1$ then $(1,\infty)$ is a nbd of p which doesnot contain any point of S. i.e, $(1,\infty)\cap S=\phi$

$\therefore$ p is not a limit point of S

Case(iii): If $p=1$, then $(\frac{1}{2},\infty)$ is a nbd of p which contains no point of 'S' otherthan p. i.e, $(\frac{1}{2},\infty)\cap S-\{1\}=\phi$

$\therefore$ p is not limit point of 'S'

Case(iv):

If $0<p<1$, then $\frac{1}{p}>0$.

$\therefore$ $\exists$ a unique natural number 'n' such that $n\leq\frac{1}{p}<n+1$.

$$\Rightarrow \frac{1}{n}\geq p>\frac{1}{n+1}$$

$$\Rightarrow \frac{1}{n+1}<p\leq\frac{1}{n}<\frac{1}{n-1}$$

$\Rightarrow$ The nbd $\left(\frac{1}{n+1},\frac{1}{n-1}\right)$ of p contains only one point $\frac{1}{n}$ of S.

$\therefore$ p is not limit point of 'S'.

Hence 0 is the only limit point of 'S'

$$\therefore \boxed{D(S)=\{0\}}$$

Also $D^2(S) = D(0) = \phi$.

$\therefore$ The set S is of the first species and of first order.

and $\bar{S} = S \cup D(S)$
$= \{\frac{1}{n} / n \in N\} \cup \{0\}$

$\rightarrow$ Find S' i.e $D(S)$
where $S = \{\frac{1}{n} \mid n \in Z, n \neq 0\}$.

Let $S = \{\frac{1}{n} \mid n \in Z, z \neq 0\} \subseteq [-1, 1] \subseteq \mathbb{R}$

Let $p = 0 \in \mathbb{R}$ then the nbd of '0' contains infinitely many numbers.

$\therefore$ 0 is a limit of S.

Now we shall show that no real number p other than 0 can be a limit point of S.

The following cases will arise:

Case (i) If $p < -1$ then $(-\infty, -1)$ is a nbd of p which contains no point of S
i.e $(-\infty, -1) \cap S = \phi$.
$\therefore$ p is not a limit point of S.

Case (ii) If $p > 1$ then $(1, \infty)$ is a nbd of p which contains no point of S
i.e $(1, \infty) \cap S = \phi$

Case (iii): If $p = 1$ then $(\frac{1}{2}, \infty)$ is a nbd of p which does not contain any point of S other than p
i.e $(\frac{1}{2}, \infty) \cap S - \{1\} = \phi$.
$\therefore$ p is not a limit point of S.

Case (iv) If $p = -1$ then $(-\infty, -\frac{1}{2})$ is a nbd of p which does not contain any point of S other than p
i.e $(-\infty, -\frac{1}{2}) \cap S - \{-1\} = \phi$

$(0, 1)$
$(-1, 3)$
$(\frac{1}{2}, \infty)$

$\therefore -1$ is not a limit point of S.

Case (v) If $0 < p < 1$, then $\left(\frac{1}{n+1}, \frac{1}{n-1}\right)$ is a nbd of p which contains only one point $\frac{1}{n}$ of S

i.e. a finite number of points of S

$\therefore p$ is not a limit point of S

Case (vi) If $-1 < p < 0$ so that $0 < -p < 1$ and $-\frac{1}{p} > 0$, $\exists$ a unique $n \in N$ s.t.

$n \le -\frac{1}{p} < n+1$

$\Rightarrow -\frac{1}{n} \le p < -\frac{1}{n+1}$

$\Rightarrow -\frac{1}{n-1} < -\frac{1}{n} \le p < \frac{-1}{n+1}$

$\Rightarrow$ The nbd $\left(\frac{-1}{n-1}, \frac{-1}{n+1}\right)$ of p contains only one point $-\frac{1}{n}$ of S.

$\therefore p$ is not a limit point of S.

Hence '0' is the only limit point of S'.

$\therefore D(S) = \{0\}$.

and $\bar{S} = S \cup \{0\}$.

→ Find the derived set of each of the following:

(i) $(1, \infty)$ (ii) $(-\infty, -1)$ (iii) $\left\{\frac{n}{n+1} / n \in N\right\}$.

(iv) $\left\{a + \frac{1}{n} / a \in \mathbb{R}, n \in N\right\}$

(v) $\left\{\frac{1+(-1)^n}{n} / n \in N\right\}$, (vi) $\left\{\frac{1}{2^n} / n \in N\right\}$ (vii) $\left\{\frac{1}{3^n} / n \in N\right\}$.

Sol

Now we shall show that no other real number p other than '0' can be a limit point of S:

The following cases arise:

Case(i) If $p<0$ then $(-\infty,0)$ is a nbd of 'p' which contains no point of S

i.e $(-\infty,0)\cap S=\phi$.

$\therefore$ p is not a limit point of S.

Case(ii) If $p>1$ then $(1,\infty)$ is a nbd of p which does not contain any point of S

i.e $(1,\infty)\cap S=\phi$

$\therefore$ p is not a limit point of S.

Case(iii): If $p=1$, then $(\frac{1}{2},\infty)$ is a nbd of p which contains no point of S other than p

i.e $(\frac{1}{2},\infty)\cap S-\{1\}=\phi$

$\therefore$ 'p' is not a limit

Case(iv) If $0<p<1$, then $\frac{1}{p}>0$.

$\therefore$ $\exists$ a unique natural number 'n' such that $n\le\frac{1}{p}<n+1$

$\Rightarrow \frac{1}{n}\ge p>\frac{1}{n+1}$

$\Rightarrow \frac{1}{n+1}<p\le\frac{1}{n}<\frac{1}{n-1}$

$\Rightarrow$ The nbd $\left(\frac{1}{n+1},\frac{1}{n-1}\right)$ of p contains only one point $\frac{1}{n}$ of S.

$\therefore$ p is not limit point of S

Hence '0' is the only limit point of S

$\therefore D(S)=\{0\}$.

(i) Let $S = (1,\infty) \subseteq \mathbb{R}$

Let x be any real number.

If $x<1$, then for $0<\epsilon<1-x$,

$$(x-\epsilon, x+\epsilon) \cap (1,\infty) = \phi.$$

$\Rightarrow$ Any real number <1 is not a limit point of $(1,\infty)$.

If $x \in [1,\infty)$, then for every $\epsilon>0$, $(x-\epsilon, x+\epsilon)$ contains infinitely many points of $(1,\infty)$ to the right of 1.

$\Rightarrow$ Every elt of $[1,\infty)$ is a limit point of $(1,\infty)$

$$\therefore (1,\infty)' = [1,\infty).$$

(ii) Ans: $(-\infty,-1)' = (-\infty,-1]$.

(v) Let $S = \left\{ \frac{1+(-1)^n}{n} \,/\, n \in \mathbb{N} \right\} \subseteq \mathbb{R}$.

when 'n' is odd,

$$\frac{1+(-1)^n}{n} = \frac{1-1}{n} = 0.$$

when 'n' is even,

$$\frac{1+(-1)^n}{n} = \frac{1+1}{n} = \frac{2}{n}.$$

$$\therefore S = \{0\} \cup \left\{ \frac{2}{n} \,/\, n \in \mathbb{N} \text{ and } n \text{ is even} \right\}$$

$$= \{0\} \cup \left\{ \frac{2}{2}, \frac{2}{4}, \frac{2}{6}, \ldots \right\}$$

$$= \{0\} \cup \left\{ 1, \tfrac{1}{2}, \tfrac{1}{3}, \ldots \right\} \subset [0,1] \subseteq \mathbb{R}.$$

Let $p \in \mathbb{R}$

If $p = 0$ then for each $\epsilon>0$ (however small), $(0-\epsilon, 0+\epsilon)$ is a nbd of '0' and it contains infinitely many points of S other than '0'

$\therefore$ 0 is the limit point of S.

→ $S = \{\cos(\frac{n\pi}{2}) / n \in I\} \subseteq \mathbb{R}$

$= \{\ldots -1, 0, -1, 0, 1, 0, -1, 0, 1, \ldots\}$

clearly $-1, 0, 1$ are limit points of S

$\therefore D(S) = \{-1, 0, 1\}$.

→ Let $S = \{\sin(\frac{n\pi}{2}) / n \in N\} \subseteq \mathbb{R}$

then $D(S) = \{-1, 0, 1\}$.

→ Let $S = \{\cos(\frac{n\pi}{3}) / n \in I\} \subseteq \mathbb{R}$.

then $D(S) = \{-\frac{1}{2}, -1, \frac{1}{2}, 1\}$.

→ Let $S = \{\sin\frac{n\pi}{3} / n \in N\} \subseteq \mathbb{R}$ then

$D(S) = \{-\frac{\sqrt{3}}{2}, \frac{\sqrt{3}}{2}\}$.

* Existence of limit points of a set:
Bolzano-Weierstrass theorem:

We have seen that a finite set has no limit points. Also we have observed that an infinite set may or may not have a limit point.

for example:

The infinite set N of natural numbers has no limit point whereas the infinite set $S = \{\frac{1}{n} / n \in N\}$ has '0' as its limit point.

We observe that the set S is bdd.

Now we shall give a theorem which gives us a set of sufficient conditions for a set to have a limit point. This theorem is known as Bolzano-Weierstrass theorem.

Statement:

Every infinite bounded set of real numbers has a limit point.

Note: The converse of the above need not be true. i.e, An infinite set has a limit point, then the set is not bounded.

for example:

Ex-1 $[0, \infty)$ is an infinite set and has limit points but it is not bounded.

Ex:2 $S = \mathbb{Q}, \mathbb{R} - \mathbb{Q}, \mathbb{R}$

Some results on Derived sets:

→ If A and B be any two subsets of $\mathbb{R}$, then

(1) $A \subset B \Rightarrow D(A) \subseteq D(B)$

(2) $D(A \cup B) = D(A) \cup D(B)$

(3) $D(A \cap B) \subseteq D(A) \cap D(B)$

(4) $D(D(A)) \subseteq D(A)$

→ $D\left(\bigcup_{i \in N} A_i\right) = D(A_1) \cup D(A_2) \cup \cdots\cdots$

→ $D\left(\bigcap_{i \in N} A_i\right) \subseteq D(A_1) \cap D(A_2) \cap \cdots\cdots$

Note:

(1) The derived set of any bounded set is bounded.

(2) Every infinite bounded set has the greatest and the smallest limit points.

i.e, the derived set of any infinite bounded set attains its bounds.

(3). The smallest and the greatest members l and u of the derived set $D(S)$ of an infinite and bounded set S always exist.

They are usually denoted by $\underline{\lim}\, S$ and $\overline{\lim}\, S$ respectively and are called the inferior (or lower) limit of S and the superior (or upper) limit of S.

Also $\underline{\lim}\, S \leq \overline{\lim}\, S$

(4) The supremum (or infimum) of a bounded set S is always members of $\bar{S}$

(5) If S is bounded then $\bar{S}$ is also bounded.

Closed Set:

A subset S of $\mathbb{R}$ is said to be closed if its complement (i.e., $S^c = \mathbb{R} - S$) is an open set.

(or)

A set $S \subset \mathbb{R}$ is said to be closed if every limit point of the set S is a member of the set S.

(or)

A subset S of $\mathbb{R}$ is said to be closed if $D(S) \subseteq S$

(i) S is closed set $\Leftrightarrow D(S) \subseteq S$

(ii) S is closed set $\Leftrightarrow S^c$ is open.

(iii) S is open set $\Leftrightarrow S^c$ is closed.

(iv) S is closed set $\Leftrightarrow \bar{S} = S$

Note: If S is a closed set then every limit point of S is a member of S., but every point of S is not limit point.

Ex: (1) If $S=\{a\}$ then $S^c = \mathbb{R}-S$

$$= \mathbb{R}-\{a\}$$
$$= (-\infty, a) \cup (a, \infty)$$

Since union of two open sets is again open

$\therefore$ S^c is open.

$\Rightarrow$ S is closed.

(or)

$S=\{a\}$ then $D(S)=\phi \subseteq S$

i.e, $D(S) \subseteq S$

$\therefore$ S is closed.

(or)

$S=\{a\}$ then $D(S)=\phi$.

$\therefore \bar{S} = S \cup D(S)$

$$= \{a\} \cup \phi = \{a\}$$
$$= S$$

$\therefore \bar{S} = S$

$\therefore$ S is closed.

(2) If S be a non-empty finite set.

then $D(S)=\phi \subseteq S$

i.e, $D(S) \subseteq S$

$\therefore$ S is closed

(3) If $S=N$ then $D(S)=\phi \subseteq S$

$\Rightarrow D(S) \subseteq S$

$\therefore$ S is closed

Similarly, $S = W, I$.

(4) $S = \mathbb{Q}$

then $D(S) = \mathbb{R} \not\subseteq S$.

$\Rightarrow S$ is not closed

(5) $S = \mathbb{R} - \mathbb{Q}$

then $D(S) = \mathbb{R} \not\subseteq S$

$\Rightarrow S$ is not closed.

(6) $S = (a, b)$ then $D(S) = [a, b] \not\subseteq S$

$\therefore S$ is not closed.

(7) $S = [a, b)$, $(a, b]$

then $D(S) = [a, b] \not\subseteq S$

$\therefore S$ is not closed

(8) $S = [a, b]$

then $D(S) = [a, b] \subseteq S$

$\therefore S$ is closed.

(9) $S = \mathbb{R}$ then $D(S) = \mathbb{R}$.

$\therefore D(S) \subseteq \mathbb{R}$

$\therefore S$ is closed.

(10) $S = \mathbb{R}^+$

i.e., $S = \{x / x > 0\}$

$= (0, \infty)$

$\Rightarrow D(S) = [0, \infty)$

$\therefore D(S) \not\subseteq S$

$\therefore S$ is not closed

(11) $S = \mathbb{R}^-$

$= (-\infty, 0) \Rightarrow D(S) = (-\infty, 0] \not\subseteq S$

$\therefore S$ is not closed.

(12) $S = \{ \frac{1}{n} / n \in N \}$

then $D(S) = \{0\} \not\subseteq S$ $\quad (\because 0 \notin S)$

$\therefore S$ is not closed.

(13) $S = \{ \frac{1}{n} / n \in \mathbb{Z} \}$ then $D(S) = \{0\} \not\subseteq S$

$\therefore S$ is not closed.

<u>Note</u>: If a set has no limit point then $\underline{\bar{S} = S}$.

→ The intersection of an arbitrary family of closed sets is a closed set.

Sol^n: Let $S_1, S_2, S_3, \ldots$ be closed sets.

then $S_1^c, S_2^c, S_3^c \ldots$ be the open sets.

Let $S = S_1 \cap S_2 \cap S_3 \cap \ldots$

$\Rightarrow S^c = (S_1 \cap S_2 \cap S_3 \cap \ldots)^c$

$= S_1^c \cup S_2^c \cup S_3^c \cup \ldots$

Since the union of arbitrary family of open sets is open.

$\Rightarrow S^c$ is open

$\Rightarrow S$ is closed. set

→ The union of a finite collection of closed sets is a closed set.

Sol^n: Let $S_1, S_2, \ldots S_n$ be closed sets.

then $S_1^c, S_2^c, S_3^c, \ldots S_n^c$ be the opensets.

Let $S = S_1 \cup S_2 \cup \ldots \cup S_n$

$\Rightarrow S^c = (S_1 \cup S_2 \cup \ldots \cup S_n)^c$

$\Rightarrow = S_1^c \cap S_2^c \cap \ldots \cap S_n^c$

Since intersection of finite collection of open sets is open.

$\therefore S^c$ is an open set

$\Rightarrow S$ is closed. set

→ The union of an infinite collection of closed sets need not be a closed set.

Solⁿ: Let $S_n = [\frac{1}{n}, 1]$ $\forall n \in \mathbb{N}$

Then each S_n is a closed set.

Now $\bigcup_{n=1}^{\infty} S_n = S_1 \cup S_2 \cup S_3 \cup \ldots\ldots$

$= \{1\} \cup [\frac{1}{2}, 1] \cup [\frac{1}{3}, 1] \cup \ldots\ldots$

$= (0, 1] = S$ (say)

which is not a closed set ($\because D(S) \not\subseteq S$)

$\therefore$ The union of an infinite collection of closed sets need not be a closed set.

→ Let 'A' be a closed set and B be an open set.

then (i) A−B is closed (ii) B−A is open.

Solⁿ:

Since A is closed $\Rightarrow A^c$ is open

B is open $\Rightarrow B^c$ is closed

(i) $B - A = B \cap A^c$.

Since B and A^c are open.

$\Rightarrow B \cap A^c$ open.

$\therefore B - A$ is open

(ii) $A - B = A \cap B^c$.

Since A and B^c are closed.

$\Rightarrow A \cap B^c$ is closed

$\therefore A - B$ is closed.

Compact set:

A non-empty subset of $\mathbb{R}$ is said to be compact if it is closed and bounded.

Ex: (1) $S = \phi$.

$D(S) = \phi \subseteq S$

$\Rightarrow S$ is closed and bounded

$\Rightarrow \therefore S$ is compact. not

(2) $S = [a, b]$.

$D(S) = [a, b] \subseteq S$

$\therefore$ S is closed and bounded.

$\therefore$ S is compact.

(3) $S = [-1, 1] \cup [2, 3]$,

Since the union of two closed sets is closed and bounded.

$\therefore$ S is compact.

(4) $S = N$

$D(S) = \phi \subseteq N$

$\therefore$ S is closed but not bounded.

$\therefore$ S is not compact.

Similarly, $S = W, Z$.

(5) $S = Q \Rightarrow D(S) = \mathbb{R} \not\subseteq Q$.

i.e, $D(S) \not\subseteq S$.

$\therefore$ S is not closed and not bound.

$\therefore$ S is not compact.

(6) $S = \mathbb{R} - Q$.

$\Rightarrow D(S) = \mathbb{R} \not\subseteq \mathbb{R} - Q$,

i.e, $D(S) \not\subseteq S$.

$\therefore$ S is not closed and bounded

$\therefore$ S is not compact.

(7) $S = \mathbb{R}$, $D(S) = \mathbb{R}$.

$\therefore D(S) \subseteq \mathbb{R}$.

$\therefore$ S is closed but not bounded.

$\therefore$ S is not ~~closed~~ compact.

(8) $S = (a, b) \Rightarrow D(S) = [a, b] \not\subseteq S$

i.e, $D(S) \not\subseteq S$

$\therefore$ S is not closed but S is bounded

$\therefore$ S is not compact.

Similarly $S = [a, b)$, $(a, b]$.

(9) $S = \{x : a \le x\}$.
$= [a, \infty)$.
$\Rightarrow D(S) = [a, \infty) \subseteq S$.
$\therefore D(S) \subseteq S$.
$\therefore S$ is closed but is not bounded.
$\therefore S$ is not compact.

(10) $S = \{1^2, 2^2, 3^2, \ldots\ldots (123)^2\}$.
Since S is finite.
$\therefore D(S) = \phi \subseteq S$
$\Rightarrow D(S) \subseteq S$.
$\therefore S$ is closed and bounded
$\therefore S$ is compact.

→ The union of finite family of compact sets is compact.

Soln: Let $S_1, S_2, \ldots S_n$ be compact sets.
Then $S_1, S_2, S_3, \ldots S_n$ be closed and bounded

Let $S = \bigcup_{i=1}^{n} S_i$
Since the union of finite collection of closed sets is a closed.
S is closed.

Now we have to show that S is bounded.
Also $S_i \subset [a_i, b_i]$, $1 \le i \le n$.

If $a = \min\{a_1, a_2, \ldots a_n\}$
and $b = \max\{b_1, b_2, \ldots b_n\}$
then $S = \bigcup_{i=1}^{n} S_i \subset [a, b]$
$\therefore S$ is bounded.

S is closed and bounded.
$\therefore S$ is compact.

→ The intersection of an arbitrary family of Compact sets, Containing atleast one point in common, is compact.

Sol: Let $S_1, S_2, \ldots S_n, \ldots$ be arbitrary family of compact sets.

Then $S_1, S_2, \ldots S_n, \ldots$ be closed and bounded.

Let $S = \bigcap_{i=1}^{\infty} S_i$

Since the intersection of arbitrary family of closed sets is closed.

$\therefore$ S is closed.

Also $S \subset S_i$ for each i.

and each S_i bounded.

$\therefore$ S is bounded.

$\therefore$ S is closed and bounded.

$\therefore$ S is compact.

Cover of a set:

→ Let 'S' be a set and $\{G_\alpha\}$ be a family of sets.

We say that $\{G_\alpha\}$ is a cover of S, if the union of members of $\{G_\alpha\}$ contains S, as a subset.

i.e., if every point of S belongs to some member of the family $\{G_\alpha\}$.

→ We say that $\{G_\alpha\}$ is an open cover if every member of $\{G_\alpha\}$ is an open set.

(or)

Let 'S' be a set and $\{G_\alpha\}$ be a collection of some open subsets of $\mathbb{R}$ such that $S \subseteq \cup G_\alpha$.

Then $\{G_\alpha\}$ is called an open cover of S.

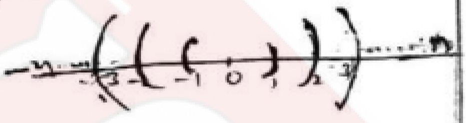
→ S.T $G = \{(-n, n) / n \in \mathbb{N}\}$ is an open cover of the set $\mathbb{R}$.

Sol^n: Given $G = \{(-n, n) / n \in \mathbb{N}\}$

$= \{\underset{G_1}{(-1, 1)}, \underset{G_2}{(-2, 2)}, \underset{G_3}{(-3, 3)}, \ldots\ldots\}$.

Since every $x \in \mathbb{R}$ belongs to atleast one of the open interval in G.

$\therefore G$ is an open cover of $\mathbb{R}$.



Also $\mathbb{R} = \bigcup_{n=1}^{\infty} G_n$, where $G_n = (-n, n)$

Similarly,

→ $G_1 = \{(-2n, 2n) / n \in \mathbb{N}\}$

$G_2 = \{(n, n+2) / n \in \mathbb{Z}\}$

$G_3 = \{(n, n+1) / n \in \mathbb{Z}\}$ are open covers of $\mathbb{R}$.

→ Show that $G_1 = \left\{\left(\frac{1}{4}, \frac{5}{4}\right), \left(\frac{3}{4}, \frac{7}{4}\right), \left(\frac{5}{3}, \frac{9}{4}\right)\right\}$ is open cover of the interval $[1,2]$ whereas $G_2 = \left\{\left(\frac{1}{2}, \frac{5}{4}\right), \left(\frac{3}{2}, \frac{9}{4}\right)\right\}$ is not an open cover of the interval $[1,2]$.

Sol^n: Given $G_1 = \left\{\left(\frac{1}{4}, \frac{5}{4}\right), \left(\frac{3}{4}, \frac{7}{4}\right), \left(\frac{5}{3}, \frac{9}{4}\right)\right\}$

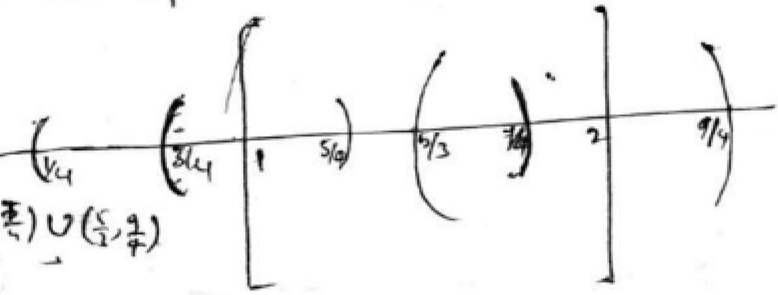
is an open cover of the interval $[1,2]$

Since every element of the set $S = [1,2] = \{x / 1 \le x \le 2\}$

belongs to atleast one of the subsets of G_1

and each of the subsets of G_1 is an open set.

$\therefore G_1$ is an open cover of $S = [1,2]$



i.e., $[1,2] \subset \left(\frac{1}{4}, \frac{5}{4}\right) \cup \left(\frac{3}{4}, \frac{7}{4}\right) \cup \left(\frac{5}{3}, \frac{9}{4}\right)$

$\longrightarrow$ $G_2 = \left\{\left(\frac{1}{2}, \frac{5}{4}\right), \left(\frac{3}{2}, \frac{9}{4}\right)\right\}$ is not an open cover of the interval $S = [1,2] = \{x / 1 \leq x \leq 2\}$.

because none of points in the interval $\left[\frac{5}{4}, \frac{3}{2}\right]$ belongs to any of the subsets of G_2.

i.e., $S = [1,2]$ is not covered by union of opensets $\left(\frac{1}{2}, \frac{5}{4}\right)$ & $\left(\frac{3}{2}, \frac{9}{4}\right)$

$\therefore$ G_2 is not an open cover of S.

Subcover and finite subcover of a set:

Let G be an open cover of a set S. A subcollection E of G is called a subcover of S if E too is a cover of S.

Further, if there are only a finite number of ~~(sets in E is a finite number of)~~ sets in E, then we say that E is a finite subcover of the open cover G of S.

Thus if G is an open cover of a set S, then a collection E is a finite subcover of the open cover G of S provided the following three conditions hold.

(i) E is contained in G

(ii) E is a finite collection

(iii) E is itself a cover of S.

Heine-Borel property:

A subset S of $\mathbb{R}$ is said to have the Heine-Borel property if every open cover of S has a finite sub-cover.