Q133. Show that the function

$$f(x,y) = \begin{cases} \dfrac{x^2 - y^2}{x - y}, & (x,y) \neq (1,-1),(1,1) \\ 0, & (x,y) \neq (1,-1),(1,1) \end{cases}$$

Is continous and differntiable at (-1, 1).

(Year 2019)

(10 Marks)

Q134. Evaluate

$$\int_0^\infty \frac{\tan^{-1}(ax)}{x(1+x^2)} dx, a > 0, a \neq 1$$

(Year 2019)

(10 Marks)

Q135. Discuss the uniform convergence of

$$f_n(x) = \frac{nx}{1+n^2x^2}, \forall\, x \in \mathbb{R}\,(-\infty,\infty), n = 1,2,3,...$$

(Year 2019)

(15 Marks)

Q136. Find the maximum value of $f(x,y,z) = x^2y^2z^2$ subject to the subsidiary condition

$x^2 + y^2 + z^2 = c^2, (x,y,z > 0)$.

(Year 2019)

(15 Marks)

Q137. Discuss the convergence of $\int_1^2 \frac{\sqrt{x}}{\ln x} dx$

(Year 2019)

(15 Marks)

→ Show that the function

$$f(x,y) = \begin{cases} \dfrac{x^2-y^2}{x-y} & , (x,y) \neq (1,-1), (1,1) \\ 0 & , (x,y) = (1,1), (1,-1) \end{cases}$$

is continuous and differentiable at $(1,-1)$.

Solution:

Given expression can be written as

$$f(x,y) = \begin{cases} x+y & , (x,y) \neq (1,-1), (1,1) \\ 0 & , (x,y) = (1,1), (1,-1) \end{cases}$$

$\because \lim\limits_{(x,y)\to(1,-1)} f(x,y) = \lim\limits_{(x,y)\to(1,-1)} (x+y) = 1+(-1) = 0 = f(1,-1)$

$\Rightarrow$ $f(x,y)$ is continuous at $(1,-1)$

Since; $f_x(x,y) = 1$ and $f_y(x,y) = 1$ which are continuous everywhere including $(1,-1)$

Therefore, f is differentiable everywhere including $(1,-1)$.

$\Rightarrow$ $f(x,y)$ is differentiable at $(1,-1)$.

QED.

→ Evaluate

$$\int_0^{\infty} \frac{\tan^{-1}(ax)}{x(1+x^2)}\,dx \,, \quad a>0 \,, \quad a \neq 1.$$

Solution:

Let $$f(a) = \int_0^{\infty} \frac{\tan^{-1}(ax)}{x(1+x^2)}\,dx \quad \text{——— (1)}$$

Differentiating both sides w.r.t. 'a', we get

$$F'(a) = \int_0^{\infty} \frac{\partial}{\partial a}\left[\frac{\tan^{-1}(ax)}{x(1+x^2)}\right] dx$$

$$= \int_0^{\infty} \frac{1}{x(1+x^2)} \cdot \frac{1}{1+a^2x^2} \cdot x \, dx$$

$$= \int_0^{\infty} \frac{dx}{(1+x^2)(1+a^2x^2)}$$

$$= \frac{1}{1-a^2} \int_0^{\infty} \left[\frac{1}{1+x^2} - \frac{a^2}{1+a^2x^2}\right] dx$$

$$= \frac{1}{1-a^2}\left[\tan^{-1}x\right]_0^{\infty} - \frac{a^2}{1-a^2}\int_0^{\infty} \frac{dx}{1+a^2x^2}$$

$$= \frac{1}{1-a^2}\left[\tan^{-1}\infty - \tan^{-1}0\right] - \frac{a^2}{1-a^2}\cdot\frac{1}{a^2}\int_0^\infty \frac{dx}{x^2+\frac{1}{a^2}}$$

$$= \frac{1}{1-a^2}\cdot\frac{\pi}{2} - \frac{1}{1-a^2}\cdot\frac{1}{\frac{1}{a}}\left[\tan^{-1}\frac{x}{\frac{1}{a}}\right]_0^\infty$$

$$= \frac{1}{1-a^2}\cdot\frac{\pi}{2} - \frac{a}{1-a^2}\left[\tan^{-1}\infty - \tan^{-1}0\right]$$

$$= \frac{1}{1-a^2}\left[\frac{\pi}{2} - a\cdot\frac{\pi}{2}\right]$$

$$= \frac{1}{1-a^2}\cdot\frac{\pi}{2}\left[1-a\right]$$

$$= \frac{\pi}{2(1+a)}$$

Integrating both sides w.r.t. 'a':

$$F(a) = \frac{\pi}{2}\log(1+a) + c \quad \text{——(2)}$$

from (1),

when $a=0$, $F(0)=0$

$\therefore$ from (2),

$$0 = \frac{\pi}{2}\log 1 + c$$

$$\Rightarrow \quad 0 = 0 + c$$

$$\Rightarrow \quad c = 0$$

$$\therefore \quad F(a) = \frac{\pi}{2}\log(1+a).$$

Thus, $\boxed{\int_0^\infty \frac{\tan^{-1}(ax)}{x(1+x^2)}\,dx = \frac{\pi}{2}\log(1+a)}$

where $a>0$, $a\neq 1$

is the required result.

QED.

3.(a) → Discuss the uniform convergence of

$$f_n(x) = \frac{nx}{1+n^2x^2}, \quad \forall x \in \mathbb{R}\ (-\infty, \infty)$$

$$n = 1, 2, 3, \ldots$$

Solution:

Let $f_n(x) = \dfrac{nx}{1+n^2x^2}$, $x \in \mathbb{R}$

Suppose that $\{f_n\}$ is uniformly convergent in $(-\infty, \infty)$.

Also, the point-wise limit f is given as

$$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{nx}{1+n^2x^2} = \lim_{n\to\infty} \frac{x}{nx^2 + 1/n}$$

$$= 0, \quad \forall x \in \mathbb{R}.$$

$$\Rightarrow f(x) = 0 \quad \forall x \in \mathbb{R}.$$

Now, from our assumption, $\{f_n\}$ is uniformly convergent in $(-\infty, \infty)$ so that we have the point-wise limit f is also the uniform limit.

Let $\varepsilon > 0$ be given. Then there exists m such that $\forall x \in (-\infty, \infty)$ and $\forall n \geq m$.

$$\left| \frac{nx}{1+n^2x^2} - 0 \right| < \varepsilon.$$

We take $\varepsilon = 1/4$.

Now there exists an integer k such that

$$k \geqslant m \quad \text{and} \quad 1/k \in (-\infty, \infty).$$

Taking $n = k$ and $x = 1/k$, we have

$$\frac{nx}{1+n^2x^2} = \frac{1}{2}$$ which is not less than $1/4$.

Thus, we arrive at a contradiction and so.

the sequence $f_n(x) = \dfrac{nx}{1+n^2x^2}$, $\forall x \in$ 19 / 65

$n = 1, 2, 3, \dots$

is not uniformly convergent with 0, in the interval $(-\infty, \infty)$ even though it is point-wise convergent.

Hence, the result.

Find the maximum value of $f(x,y,z) = x^2y^2z^2$ subject to the subsidiary condition $x^2+y^2+z^2 = c^2$, $(x, y, z > 0)$.

Solution:

On the spherical surface $x^2+y^2+z^2 = c^2$, the function must assume the greatest value, since the surface is a bounded and closed set.

According to the Method of Undetermined Multipliers, we form the expression

$$F = x^2y^2z^2 + \lambda(x^2+y^2+z^2-c^2)$$

and by differentiation, we obtain

$$2xy^2z^2 + 2\lambda x = 0,$$

$$2x^2yz^2 + 2\lambda y = 0,$$

$$2x^2y^2z + 2\lambda z = 0.$$

The solutions with $x=0$, $y=0$, or $z=0$ can be excluded, for at these points the function takes on its least value, zero.

The other solutions of the equation are $x^2 = y^2 = z^2$, $\lambda = -x^4$. Using the subsidiary condition, we obtain the values

$$x = \pm \frac{c}{\sqrt{3}}, \quad y = \pm \frac{c}{\sqrt{3}}, \quad z = \pm \frac{c}{\sqrt{3}}$$

for the required coordinates.

At all these points, the function assumes the same value $\underline{c^6/27}$, which accordingly is the maximum.

Hence, any triad of numbers satisfies the relation

$$\sqrt[3]{x^2 y^2 z^2} \leq \frac{c^2}{3} = \frac{x^2 + y^2 + z^2}{3},$$

which states that the geometric mean of three nonnegative numbers x^2, y^2, z^2 is never greater than their arithmetic mean.

4.(c) → Discuss the convergence of $\int_1^2 \frac{\sqrt{x}}{\ln x}\,dx$.

Solution:

Let $f(x) = \frac{\sqrt{x}}{\ln x}$

1 is the only point of infinite discontinuity of 'f' on $[1,2]$.

Take $g(x) = \frac{1}{(x-1)^n}$

$$\therefore \lim_{x\to 1^+} \frac{f(x)}{g(x)} = \lim_{x\to 1^+} \frac{(x-1)^n\sqrt{x}}{\ln x} \quad \left(\frac{0}{0} \text{ form}\right)$$

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$$= \lim_{x\to 1^+} \frac{n(x-1)^{n-1}\sqrt{x} + (x-1)^n \frac{1}{2\sqrt{x}}}{1/x}$$

$$= \lim_{x\to 1^+} (x-1)^{n-1}\left[nx^{3/2} + \left(\frac{x-1}{2}\right)\sqrt{x}\right]$$

$$= 1 \quad \text{if} \quad n = 1.$$

($\therefore$ a non-zero finite number)

$\therefore$ By Comparison test,

$\int_1^2 f(x)\,dx$ & $\int_1^2 g(x)\,dx$ are convergent (or) divergent together. But $\int_1^2 g(x)\,dx$ diverges ($\because n = 1$).

$\therefore \int_1^2 f(x)\,dx$ diverges

i.e. $\int_1^2 \frac{\sqrt{x}}{\ln x}\,dx$ diverges.