7(a) Find the general solution of the DE

$$\ddot{x} + 4x = \sin^2 2t$$

Hence find the particular solution satisfying the conditions

$$x(\pi/8) = 0 \quad \text{and} \quad \dot{x}(\pi/8) = 0$$ (15)

Let $D = \frac{d}{dt}$, $D^2 = \frac{d^2}{dt^2}$

$$(D^2+4)x = \sin^2 2t$$

Auxiliary Eqn : $D^2+4 = 0$

$$D = \pm 2i$$

$$C.F. = c_1 \cos 2t + c_2 \sin 2t$$

$$P.I. = \frac{1}{D^2+4} \sin^2 2t$$

$$= \frac{1}{D^2+4}\left(\frac{1-\cos 4t}{2}\right)$$

$$= \frac{1}{2}\frac{1}{D^2+4}\cdot 1 - \frac{1}{2}\frac{1}{D^2+4}\cos 4t$$

$$= \frac{1}{2}\frac{1}{D^2+4}e^{0t} - \frac{1}{2}\cdot\frac{\cos 4t}{(-16)+4}$$

$$= \frac{1}{8} + \frac{1}{24}\cos 4t$$

General solution : $x = C.F. + P.I.$

$$x = c_1 \cos 2t + c_2 \sin 2t + \frac{1}{24}(3 + \cos 4t)$$

$$x\left(\frac{\pi}{8}\right) = 0 \Rightarrow \quad c_1 + c_2 = \frac{-\sqrt{2}}{8}$$

$$\dot{x}\left(\frac{\pi}{8}\right) = 0 \Rightarrow \quad c_2 - c_1 = \frac{\sqrt{2}}{12}$$

classmate $\therefore\ c_1 = \frac{-5\sqrt{2}}{48}, \quad c_2 = \frac{-\sqrt{2}}{48}$

PAGE

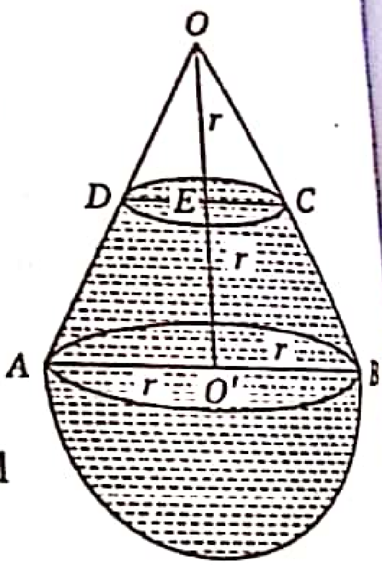
From (1) and (2), we get the required result.

IFoS 2019

Ex. 11. *A vessel in the shape of a hollow hemisphere surmounted by a cone is held with the axis vertical and vertex uppermost. If it be filled with a liquid so as to submerge half the axis of the cone in the liquid, and height of the cone be double the radius of its base, show that the resultant downward thrust of the liquid on the vessel is $\frac{15}{8}$ times the weight of the liquid that the hemisphere can hold.*

Sol. Let r be the radius of the base of the hemisphere or cone so that the height of the surmounting cone is $2r$.

The vessel is filled upto CD so as to submerge half the axis of the cone in the liquid.



From similar triangles OEC and $OO'B$, we have

$$\frac{EC}{O'B} = \frac{OE}{OO'} = \frac{r}{2r} = \frac{1}{2}.$$

$$\therefore \quad EC = \frac{1}{2} OB' = \frac{1}{2} r.$$

The resultant downward thrust of the liquid on the vessel

= weight of the liquid contained in the vessel

= wt. of the liquid in the hemisphere

+ wt. of the liquid in the frustum

$$= \frac{2}{3}\pi r^3 w + \left[\frac{1}{3}\pi r^2 . 2r - \frac{1}{3}\pi\left(\frac{r}{2}\right)^2 . r\right] w$$

$$= \frac{2}{3}\pi r^3 w + \frac{1}{3}\pi r^3 w\left(2 - \frac{1}{4}\right) = \frac{1}{3}\pi r^3 w\left(2 + \frac{7}{4}\right) = \frac{1}{3}\pi r^3 w \cdot \frac{15}{4}$$

$$= \frac{15}{8}\left(\frac{2}{3}\pi r^3 w\right)$$

DATE

7(c) Derive the Frenet-Serret formulae. Verify the same for the space curve $x = 3\cos t$, $y = 3\sin t$, $z = 4t$ (10)

i) $\frac{dT}{ds} = kN$ ii) $\frac{dB}{ds} = -\tau N$ iii) $\frac{dN}{ds} = \tau B - kT$

where

T, N, B are unit vectors along tangent, principal normal and binormal directions.

$|T| = 1 \Rightarrow T \cdot T = 1$

$\Rightarrow 2T \cdot \frac{dT}{ds} = 0 \Rightarrow \frac{dT}{ds}$ is $\perp$ to T.

Also, $\frac{dT}{ds}$ lies in osculating plane.

$\therefore dT/ds$ is parallel to N

$\therefore \boxed{\frac{dT}{ds} = kN}$

ii) since, $|B| = 1$, unit vector

$\therefore B \cdot B = 1 \Rightarrow 2B \cdot \frac{dB}{ds} = 0$

$\Rightarrow dB/ds$ is $\perp$ to B —①

W.k.T $\frac{dB}{ds}$ lies in osculating plane.

also, since B and T are $\perp$

$\therefore B \cdot T = 0$

$\Rightarrow B \cdot \frac{dT}{ds} + T \cdot \frac{dB}{ds} = 0$

$B \cdot (kN) + T \cdot \frac{dB}{ds} = 0$

$(B \cdot N)k + \frac{dB}{ds} \cdot T = 0 \Rightarrow \frac{dB}{ds} \cdot T = 0$

($\because$ B is $\perp$ N)

i.e. $\frac{dB}{ds}$ is $\perp$ to T. —②

classmate

PAGE

from ① and ②,

$\frac{dB}{ds}$ is parallel to N

$$\Rightarrow \boxed{\frac{dB}{ds} = -\tau N} \qquad \tau = \text{Torsion}$$

iii) $B \times T = N$

$$B \times \frac{dT}{ds} + \frac{dB}{ds} \times T = \frac{dN}{ds}$$

$$B \times (\kappa N) + (-\tau N) \times T = \frac{dN}{ds}$$

$$\kappa(-T) - \tau(-B) = dN/ds$$

$$\therefore \boxed{\frac{dN}{ds} = \tau B - \kappa T}$$

Here, $x = 3\cos t$, $y = 3\sin t$, $z = 4t$

$$\vec{r} = (3\cos t)i + (3\sin t)j + (4t)k$$

$$\frac{d\vec{r}}{dt} = (-3\sin t)i + (3\cos t)j + 4k$$

$$\left|\frac{dr}{dt}\right| = \sqrt{9\sin^2 t + 9\cos^2 t + 16} = 5$$

Let s be length of arc from $t=0$ to any point t on the curve, then

$$s = \int_0^t \left|\frac{d\vec{r}}{dt}\right| dt = \int_0^t 5\,dt = 5t$$

$$\therefore \vec{r} = \left(3\cos\frac{s}{5}\right)i + \left(3\sin\frac{s}{5}\right)j + \left(\frac{4}{5}s\right)k$$

classmate PAGE

Scanned with CamScanner

DATE

$$T = \frac{d\vec{r}}{ds} = \left(\frac{-3}{5}\sin\frac{s}{5}\right)i + \left(\frac{3}{5}\cos\frac{s}{5}\right)j + \frac{4}{5}k$$

$$\frac{dT}{ds} = \left(\frac{-3}{25}\cos\frac{s}{5}\right)i - \frac{3}{25}\sin\frac{s}{5}\,j + 0$$

Principal Normal, N is parall to $\dot{\vec{r}} \times (\dot{\vec{r}} \times \ddot{\vec{r}})$

$$\dot{\vec{r}} \times \ddot{\vec{r}} = \begin{vmatrix} i & j & k \\ -3\sin t & 3\cos t & 4 \\ -3\cos t & -3\sin t & 0 \end{vmatrix}$$

$$= i(0 + 12\sin t) + j(-12\cos t + 0) + k(+9\sin^2 t + 9\cos^2 t)$$

$$= (12\sin t)i - (12\cos t)j + 9k$$

$$\dot{\vec{r}} \times (\dot{\vec{r}} \times \ddot{\vec{r}}) = \begin{vmatrix} i & j & k \\ -3\sin t & 3\cos t & 4 \\ 12\sin t & -12\cos t & 9 \end{vmatrix}$$

$$= i(27\cos t + 48\cos t) + j(48\sin t + 27\sin t) + k(36\sin t\cos t - 12\sin t\cos t)$$

$$= (75\cos t)\,i + (75\sin t)\,j$$

$$\therefore\ N = \pm\frac{1}{75}(75\cos t\, i + 75\sin t\, j) = -(\cos t)i + -(\sin t)j$$

$$N = -\left(\cos\frac{s}{5}\right)i + \left(-\sin\frac{s}{5}\right)j$$ (Taking −ve sign)

$$\frac{dN}{ds} = -\frac{1}{5}\sin\frac{s}{5}\,i + \frac{1}{5}\cos\frac{s}{5}\,j$$

Binormal vector B, is parallel to $\dot{\vec{r}} \times \ddot{\vec{r}}$

$$B = \frac{1}{\sqrt{144+81}}\left[12\sin t\, i - 12\cos t\, j + 9k\right]$$

DATE

$$B = \frac{12}{15}\sin t\, i - \frac{12}{15}\cos t\, j + \frac{9}{15}k$$

$$B = \frac{4}{5}\sin\frac{s}{5} i - \frac{4}{5}\cos\frac{s}{5} j + \frac{3}{5}k$$

$$\frac{dB}{ds} = \frac{4}{25}\cos\frac{s}{5} i + \frac{4}{25}\sin\frac{s}{5} j$$

$$\kappa = \left|\frac{dT}{ds}\right| = \frac{3}{25}, \quad \tau = \left|\frac{dB}{ds}\right| = \frac{4}{25}$$

Taking, $N = -\cos\frac{s}{5} i - \sin\frac{s}{5} j$

i) $\kappa N = \frac{3}{25}\left(-\cos\frac{s}{5} i - \sin\frac{s}{5} j\right) = \frac{dT}{ds}$

ii) $-\tau N = \frac{-4}{25}\left(-\cos\frac{s}{5} i - \sin\frac{s}{5} j\right) = \frac{dB}{ds}$

iii) $\tau B - \kappa T = \frac{4}{25}\left(\frac{4}{5}\sin\frac{s}{5} i - \frac{4}{5}\cos\frac{s}{5} j + \frac{3}{5}k\right)$

$- \frac{3}{25}\left(\frac{-3}{5}\sin\frac{s}{5} i + \frac{3}{5}\cos\frac{s}{5} j + \frac{4}{5}k\right)$

$= \frac{1}{5}\sin\frac{s}{5} i - \frac{1}{5}\cos\frac{s}{5} j$

$= \frac{dN}{ds}$

Hence, we see that frenet-serret formulae are satisfied by the given curve in space.