8. *Show that the function* $e^x (\cos y + i \sin y)$ *is holomorphic and find its derivative.*

Let
$$f(z) = u + iv$$
$$= e^x \cos y + i . e^x \sin y,$$

We have
$$\frac{\partial u}{\partial x} = e^x \cos y$$
$$\frac{\partial u}{\partial y} = -e^x \sin y$$
$$\frac{\partial v}{\partial x} = e^x \sin y, \ \frac{\partial v}{\partial y} = e^x \cos y.$$

These relations show that
$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

so that u, v satisfy Cauchy-Riemann equations.

Also u and v are clearly continuous functions of x, y, for all finite values of x, y. Hence $f(z)$ is regular.

Now
$$f'(z) = \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$
$$= e^x \cos y + i e^x \sin y$$
$$= e^x (\cos y + i \sin y) = e^{x+iy}$$
$$= e^z.$$

Here the derivative is identical with the given function. This result is similar to that of the real function e^x.

10. *Show that the function*

$$f(z)=u+v,$$

where
$$f(z)=\frac{x^3(1+i)-y^3(1-i)}{x^2+y^2}\ (z\neq 0),$$
$$f(0)=0.$$

is continuous and that the Cauchy-Riemann equations are satisfied at the origin, yet $f'(0)$ does not exist. **(Agra 1956, '59, '62)**

Here $u=\dfrac{x^3-y^3}{x^2+y^2}$ and $v=\dfrac{x^3+y^3}{x^2+y^2}$.

When $z\neq 0$, u and v are rational functions of x and y with non-zero denominators. It follows that they are continuous when $z\neq 0$. To test them for continuity at $z=0$, we get on changing, to polars,

$$u=r(\cos^3\theta-\sin^3\theta) \text{ and } v=r(\cos^3\theta+\sin^3\theta).$$

each of which tends to zero as $r\to 0$ whatever value θ may have.

Now the actual values of u and v at the origin are zero since $f(0)=0$.

Since the actual and limiting values of u and v are equal at the origin they are continuous there. Hence $f(z)$ is a continuous function for all values of z.

Now at the origin,

$$\frac{\partial u}{\partial x}=\lim_{x\to 0}\frac{u(x,0)-u(0,0)}{x}$$

$$=\lim_{x\to 0}\frac{x-0}{x}=1.$$

Similarly $\dfrac{\partial u}{\partial y}=\lim\limits_{y\to 0}\dfrac{-y-0}{y}=-1,$

$$\frac{\partial v}{\partial x}=\lim_{x\to 0}\frac{x-0}{x}=1$$

$$\frac{\partial v}{\partial y}=\lim_{y\to 0}\frac{y-0}{y}=1.$$

Hence
$$\frac{\partial u}{\partial x}=\frac{\partial v}{\partial}$$

and
$$\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}.$$

The Cauchy-Riemann equotions are therefore satisfied.

Again $f'(0)=\lim\limits_{z\to 0}\dfrac{f(z)-f(0)}{z}$

$$=\lim_{z\to 0}\frac{(x^3-y^3)+i\,(x^3+y^3)}{x^2+y^2}\cdot\frac{1}{x+iy}.$$

Now let $z\to 0$ along $y=x$; then

$$f'(0)=\lim_{x\to 0}\frac{2i}{2\,(1+i)}=\tfrac{1}{2}\,(1+i).$$

Again let $z\to 0$ along x-axis, then

$$f'(0)=\lim_{x\to 0}\frac{x^3+ix^3}{x^3}\quad [\because\ \ y=0]$$

$$=(1+i).$$

Since the two limits obtained above are different, the function $f(z)$ is not differentiable at $z=0$.

4. *Prove that*

$$|z_1+z_2|^2+|z_1-z_2|^2=2|z_1|^2+2|z_2|^2.$$

Interpret the result geometrically and deduce that

$$|\alpha+\sqrt{(\alpha^2-\beta^2)}|+|\alpha-\sqrt{(\alpha^2-\beta^2)}|$$
$$=|\alpha+\beta|+|\alpha-\beta|,$$

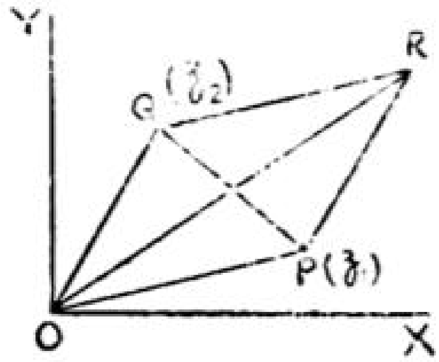
all the numbers involved being complex. **(Agra 1963)**

We have

$$|z_1+z_2|^2+|z_1-z_2|^2$$
$$=(\bar{z}_1+\bar{z}_2)(z_1+z_2)+(z_1-z_2)(\bar{z}_1-\bar{z}_2)$$
$$=2z_1\bar{z}_1+2z_2\bar{z}_2=2|z_1|^2+2|z_2|^2. \quad \ldots(1)$$

Geometrical Interpretation. Let P and Q be the points of affix z_1 and z_2 respectively.

Complete the parallelogram $OPRQ$. Then we have



$$|z_1|=OP,\ |z_2|=OQ,$$
$$|z_1+z_2|=OR,\ |z_1-z_2|=QP.$$

Now from a property of a parallelogram, we have

$$OR^2+QP^2=2OP^2+2OQ^2$$

or $$|z_1+z_2|^2+|z_1-z_2|^2=2|z_1|^2+2|z_2|^2.$$

Deduction.

Now let $z_1=\alpha+\sqrt{(\alpha^2-\beta^2)}$ and $z_2=\alpha-\sqrt{(\alpha^2-\beta^2)}$.

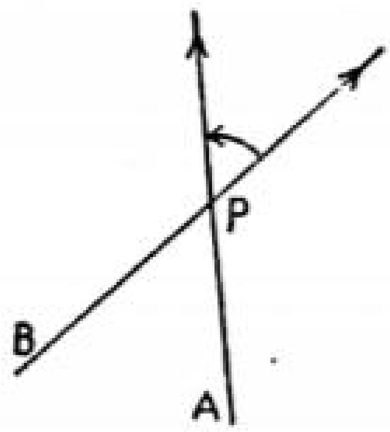
We then have

$$|z_1|^2+|z_2|^2=\tfrac{1}{2}|z_1+z_2|^2+\tfrac{1}{2}|z_1-z_2|^2 \text{ from (1)}$$
$$=\tfrac{1}{2}|2\alpha|^2+\tfrac{1}{2}|2\sqrt{(\alpha^2-\beta^2)}|^2$$
$$=2|\alpha|^2+2|\alpha^2-\beta^2|$$

and so $$[|z_1|+|z_2|]^2=|z_1|^2+|z_2|^2+2|z_1z_2|$$
$$=2|\alpha|^2+2|\alpha^2-\beta^2|+2|\beta|^2,$$
$$=|\alpha+\beta|^2+|\alpha-\beta|^2+2|\alpha^2-\beta^2| \quad \text{using (1)]}$$
$$=[|\alpha+\beta|+|\alpha-\beta|]^2.$$

Hence $|\alpha+\sqrt{(\alpha^2-\beta^2)}|+|\alpha-\sqrt{(a^2-\beta^2)}|=|\alpha+\beta|+|\alpha-\beta|.$

5. *Show that arg* $\dfrac{z-a}{z-b}$ *is the angle between the lines joining the points a to z and b to z on the Argand plane.*



Let A, B, P be the points on the Argand diagram representing the complex numbers a, b and z respectively.

Then the complex numbers $z-a$ and $z-b$ are represented by the vectors $\overrightarrow{AP}$ and $\overrightarrow{BP}$ respectively.

[See equation (2) of § 1·8]

Hence the principal value of arg $\dfrac{z-a}{z-b}$ is the angle θ, where $-\pi \leqslant \theta \leqslant \pi$ through which the vector $\overrightarrow{BP}$ has to rotate to coincide with the direction of the vector $\overrightarrow{AP}$. For the adjoining figure it is clear that this argument is positive.

Thus arg $\dfrac{z-a}{z-b}$ is the angle between the lines joining a to z and b to z taken in the proper sense.

Note. If AP is perpendicular to BP, then

$$\arg \frac{z-a}{z-b} = \pm \frac{\pi}{2}$$

so that $\dfrac{z-a}{z-b}$ is purely imaginary.

If AP coincides with BP, then

$$\arg \frac{z-a}{z-b} = 0 \text{ or } \pi$$

so that $\dfrac{z-a}{z-b}$ is purely real.

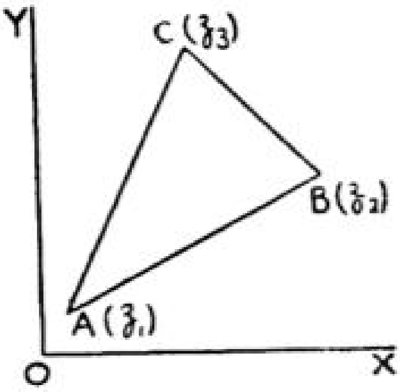
It follows that if $\dfrac{z-a}{z-b}$ is purely real, the points A, B, P are collinear.

6. *Prove that the area of the triangle whose vertices are the points z_1, z_2, z_3 on the Argand diagram is*

$$\Sigma \{(z_2-z_3) \mid z_1 \mid^2/4iz_1\}.$$

Show also that the triangle is equilateral if

$$z_1^2+z_2^2+z_3^2=z_1z_2+z_2z_3+z_3z_1. \qquad \textbf{(Agra 60)}$$



Let z_1, z_2, z_3 represent the points A, B, C on the Argand diagram.

Also let $z_1=x_1+iy_1$,
$z_2=x_2+iy_2$,
$z_3=x_3+iy_3$.

Then the required area

$$=\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$=\frac{1}{2i}\begin{vmatrix} x_1 & x_1+iy_1 & 1 \\ x_2 & x_2+iy_2 & 1 \\ x_3 & x_3+iy_3 & 1 \end{vmatrix}=\frac{1}{2i}\begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix}$$

$$=\frac{1}{2i}\ \Sigma\, x_1\,(z_2-z_3)=\frac{1}{2i}\ \Sigma\ \tfrac{1}{2}\,(z_1+\bar{z}_1)\,(z_2-z_3)$$

$$=\frac{1}{4i}\ \Sigma\, z_1\,(z_2-z_3)+\frac{1}{4i}\ \Sigma\, \bar{z}_1\,(z_2-z_3)$$

$$=0+\frac{1}{4i}\ \Sigma\ \frac{z_1\bar{z}_1}{z_1}\ \ (z_2-z_3)=\Sigma\ \frac{\mid z_1 \mid^2 (z_2-z_3)}{4iz_1}.$$

Now the triangle ABC will be equilateral if

$$AB=BC=CA$$

i.e. if $\qquad \mid z_1-z_2 \mid=\mid z_2-z_3 \mid=\mid z_3-z_1 \mid$

i.e. if $\qquad \mid z_1-z_2 \mid^2=\mid z_2-z_3 \mid^2=\mid z_3-z_1 \mid^2$

i.e. if $\quad (z_1-z_2)\,(\bar{z}_1-\bar{z}_2)=(z_2-z_3)\,(\bar{z}_2-\bar{z}_3)$

$$=(z_3-z_1)\,(\bar{z}_3-\bar{z}_1). \qquad \ldots(1)$$

From first two of (1), we get

$$\frac{z_1-z_2}{\bar{z}_2-\bar{z}_3}=\frac{z_2-z_3}{\bar{z}_1-\bar{z}_2}=\frac{(z_1-z_2)+(z_2-z_3)}{(\bar{z}_2-\bar{z}_3)+(\bar{z}_1-\bar{z}_2)}$$

or
$$\frac{z_1-z_2}{\bar{z}_2-\bar{z}_3}=\frac{z_1-z_3}{\bar{z}_1-\bar{z}_3}. \qquad \ldots(2)$$

Again from last two of (1), we get

$$(z_2-z_3)\,(\bar{z}_2-\bar{z}_3)=(z_3-z_1)\,(\bar{z}_3-\bar{z}_1).$$

Multiplying (2) and (3), we get

$$(z_1-z_2)\,(z_2-z_3)=(z_1-z_3)^2$$

or
$$z_1z_2-z_1z_3-z_2^2+z_2z_3=z_1^2+z_3^2-2z_1z_3$$

or
$$z_1^2+z_2^2+z_3^2=z_1z_2+z_2z_3+z_3z_1,$$

which is the required condition.

Theorem II. *The continuous one valued function $f(z)$ is regular in a domain D if the four partial derivatives u_x, v_x, u_y, v_y exist, are continuous and satisfy the Cauchy-Riemann equations at each point of D.*

We have

$$u = u(x, y) \text{ and } u + \Delta u = u(x + \Delta x, y + \Delta y),$$

so that

$$\begin{aligned} \Delta u &= u(x + \Delta x, y + \Delta y) - u(x, y) \\ &= u(x + \Delta x, y + \Delta y) - u(x + \Delta x, y) \\ &\qquad + u(x + \Delta x, y) - u(x, y) \\ &= \Delta y \, u_y (x + \Delta x, y + \theta \Delta y) \\ &\qquad + \Delta x \, u_x (x + \theta' \Delta x, y), \qquad \ldots(1) \end{aligned}$$

where $\qquad 0 < \theta < 1 ; 0 < \theta' < 1,$

by the mean value theorem*.

---

*Mean value theorem states that if (i) $f(x)$ is continuous in $a \leqslant x \leqslant b$, (ii) differentiable in $a < x < b$, then $f(a+h) - f(a) = hf'(a + \theta h)$, where $0 < \theta < 1$.

Now $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ are continuous in the given region, and therefore from the property of uniform continuty, we have

$$|u_y(x+\Delta x, y+\theta\Delta y)-u_y(x, y)|<\epsilon$$

and $$|u_x(x+\theta'\Delta x, y)-u_x(x, y)|<\epsilon$$

provided that $$|\Delta x|<\delta, |\Delta y|<\delta,$$

i. e. $$u_y(x+\Delta x, y+\theta\Delta y)-u_y(x, y)=\alpha$$

and $$u_x(x+\theta'\Delta x, y)-u_x(x, y)=\beta$$

where $$|\alpha|<\epsilon \text{ and } |\beta|<\epsilon.$$

Then, we have from (1),

$$\Delta u=[u_y(x, y)+\alpha]\Delta y+[u_x(x, y)+\beta]\Delta x.$$

Similarly, we shall get

$$\Delta v=[v_y(x, y)+\alpha']\Delta y+[v_x(x, y)+\beta']\Delta x,$$
$$|\alpha'|<\epsilon' \text{ and } |\beta'|<\epsilon'.$$

Hence, we get

$$\frac{\Delta w}{\Delta z}=\frac{\Delta u+i\Delta v}{\Delta x+i\Delta y}$$
$$=\frac{(u_y\Delta y+u_x\Delta x+\alpha\Delta y+\beta\Delta x)+i(v_y\Delta y+v_x\Delta x+\alpha'\Delta y+\beta'\Delta x)}{\Delta x+i\Delta y}$$
$$=\frac{-v_x\Delta y+u_x\Delta x+iu_x\Delta y+iv_x\Delta x+\alpha\Delta y+\beta\Delta x+i\alpha'\Delta y+i\beta'\Delta x}{\Delta x+i\Delta y}$$

$$[\because \quad u_x=v_y \text{ and } v_x=-u_y]$$

or $$\frac{\Delta w}{\Delta z}=\frac{(u_x+iv_x)(\Delta x+i\Delta y)+\alpha\Delta y+\beta\Delta x+i\alpha'\Delta y+i\beta'\Delta x}{\Delta x+i\Delta y}$$
$$=(u_x+iv_x)+\frac{(\alpha+i\alpha')\Delta y}{\Delta x+i\Delta y}+\frac{(\beta+i\beta')\Delta x}{\Delta x+i\Delta y}$$

or $$\left|\frac{\Delta w}{\Delta z}-(u_x+iv_x)\right|$$
$$\leqslant\frac{|\alpha+i\alpha'||\Delta y|}{|\Delta x+i\Delta y|}+\frac{|\beta+i\beta'||\Delta x|}{|\Delta x+i\Delta y|}$$
$$\leqslant|\alpha|+|\alpha'|+|\beta|+|\beta'|,$$

since $$|\Delta x|\leqslant|\Delta x+i\Delta y|$$

and $$|\Delta y|\leqslant|\Delta x+i\Delta y|$$

$$\therefore \quad \left|\frac{\Delta w}{\Delta z}-(u_x+iv_x)\right|<2\epsilon+2\epsilon'.$$

Hence $$\lim_{\Delta z\to 0}\frac{\Delta w}{\Delta z}=u_x+iv_x. \qquad \ldots(2)$$

Alternative. Let $w=f(z)=u(x, y)+iv(x, y)$.

Since $u(x, y)$ is continuous and differentiable in domain D, we have by the mean value theorem* for functions of two variables,

$$\Delta u = u(x+\Delta x, y+\Delta y) - u(x, y)$$
$$= [u_x(x, y)+\epsilon]\,\Delta x + [u_y(x, y)+\eta]\,\Delta y, \qquad \ldots(1)$$

where ϵ and η tend to zero as Δx and Δy tend to zero.

Similarly applying this theorem for $v(x, y)$, we get

$$\Delta v = v(x+\Delta x, y+\Delta y) - v(x, y)$$
$$= [v_x(x, y)+\epsilon']\,\Delta x + [v_y(x, y)+\eta']\,\Delta y,$$

where ϵ' and η' also tend to zero.

$$\therefore \quad \Delta w = \Delta u + i\Delta v$$
$$= [u_x(x, y) + iv_x(x, y)]\,\Delta x + [u_y(x, y) + iv_y(x, y)]\,\Delta y$$
$$+ (\epsilon + i\epsilon')\,\Delta x + (\eta + i\eta')\,\Delta y$$
$$= [u_x(x, y) + iv_x(x, y)]\,\Delta x + [-v_x(x, y) + iu_x(x, y)]\,\Delta y$$
$$+ (\epsilon + i\epsilon')\,\Delta x + (\eta + i\eta')\,\Delta y \quad \text{[using Cauchy Riemann equations } u_x = v_y,\ u_y = -v_x]$$
$$= [u_x(x, y) + iv_x(x, y)]\,(\Delta x + i\Delta y)$$
$$+ (\epsilon + i\epsilon')\,\Delta x + (\eta + i\eta')\,\Delta y.$$

We then have

$$\frac{\Delta w}{\Delta z} = \frac{\Delta u + i\Delta v}{\Delta x + i\Delta y}$$
$$= u_x(x, y) + iv_x(x, y) + \frac{(\epsilon + i\epsilon')\,\Delta x}{\Delta x + i\Delta y} + \frac{(\eta + i\eta')\,\Delta y}{\Delta x + i\Delta y},$$
$$\left|\frac{\Delta w}{\Delta z} - \{u_x(x, y) + iv_x(x, y)\}\right|$$
$$\leqslant \frac{|\epsilon + i\epsilon'|\,|\Delta x|}{|\Delta x + i\Delta y|} + \frac{|\eta + i\eta'|\,|\Delta y|}{|\Delta x + i\Delta y|}$$
$$< |\epsilon| + |\epsilon'| + |\eta| + |\eta'|,$$

since $\quad |\Delta x| \leqslant |\Delta x + i\Delta y|$

and $\quad |\Delta y| \leqslant |\Delta x + i\Delta y|$.

Hence $\quad \displaystyle\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = u_x(x, y) + iv_x(x, y).$

Thus $f(z)$ is differentiable at each point of D.

Note. We have

$$\frac{dw}{dz} = u_x + iv_x = \frac{\partial w}{\partial x}.$$

* For proof see § 154 page 278 of Hardy's Pure Mathematics'.

Also $$\frac{dw}{dz}=v_y-iu_y$$

$$=\frac{1}{i}(u_y+iv_y)$$

$$=\frac{1}{i}\frac{\partial w}{\partial y},$$

so that $f'(z)$ is either equal to

$$\frac{\partial w}{\partial x} \quad \text{or} \quad \frac{1}{i}\frac{\partial w}{\partial y}.$$

9. *Shew that the function*

$$f(z)=\sqrt{|xy|}$$

is not regular at the origin, although the Cauchy-Riemann equations are satisfied at that point. **(Delhi 1959)**

Let $f(z)=u(x, y)+iv(x, y)$ so that $u(x, y)=\sqrt{|xy|}$ and $v(x, y)=0$.

We tnen have at the origin,

$$\frac{\partial u}{\partial x}=\lim_{x\to 0}\frac{u(x, 0)-u(0, 0)}{x},$$

$$=\lim_{x\to 0}\frac{0-0}{x}=0.$$

Similarly $\frac{\partial u}{\partial y}=\lim_{y\to 0}\frac{u(0, y)-u(0, 0)}{y}$

$$\lim_{y\to 0}=\frac{0-0}{y}=0$$

and
$$\frac{\partial v}{\partial x}=\lim_{x\to 0}\frac{0-0}{x}=0,$$

$$\frac{\partial v}{\partial y}=\lim_{y\to 0}\frac{0-0}{y}=0.$$

Hence Cauchy-Riemann equations are satisfied at the origin

Now $f'(0)=\lim_{z\to 0}\frac{f(z)-f(0)}{z}$

$$=\lim_{z\to 0}\frac{\sqrt{|xy|}-0}{(x+iy)}.$$

Now if $z\to 0$ along $y=mx$, we get

$$f'(0)=\lim_{x\to 0}\frac{\sqrt{|mx^2|}}{x(1+im)}$$

$$=\frac{\sqrt{|m|}}{(1+im)}.$$

Now this limit is not unique since it depends on m. Hence $f'(0)$ does not exist.

17. *If $f(z)$ is a regular function of z, prove that*

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2=4\,|f'(z)|^2.$$

Let
$$\phi=|f(z)|^2=u^2+v^2,$$
then
$$\frac{\partial \phi}{\partial x}=2u\,\frac{\partial u}{\partial x}+2v\,\frac{\partial v}{\partial x},$$
$$\frac{\partial^2 \phi}{\partial x^2}=2\left[u\,\frac{\partial^2 u}{\partial x^2}+\left(\frac{\partial u}{\partial x}\right)^2\right]+2\left[v\,\frac{\partial^2 v}{\partial x^2}+\left(\frac{\partial v}{\partial x}\right)^2\right].$$

Similarly, we have
$$\frac{\partial^2 \phi}{\partial y^2}=2\left[u\,\frac{\partial^2 u}{\partial y^2}+\left(\frac{\partial u}{\partial y}\right)^2\right]+2\left[v\,\frac{\partial^2 v}{\partial y^2}+\left(\frac{\partial v}{\partial y}\right)^2\right].$$

Hence
$$\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}=2u\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+2v\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$
$$+2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]$$
$$=4\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right]=4\,|u_x+iv_x|^2.$$

using Cauchy-Riemann equations and the condition that u, v satisfy Laplace's equation.

Hence
$$\frac{\partial^2 \phi}{\partial x^2}+\frac{\partial^2 \phi}{\partial y^2}=4\,|f'(z)|^2.$$

Alternative. Since $x+iy=z$ and $x-iy=\bar{z}$, we have
$$x=\frac{1}{2}(z+\bar{z}) \text{ and } y=-\frac{i}{2}(z-\bar{z})$$
so that
$$\frac{\partial}{\partial z}=\frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial z}+\frac{\partial}{\partial y}\cdot\frac{\partial y}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\,\frac{\partial}{\partial y}\right)$$

17. *If $f(z)$ is a regular function of z, prove that*

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2=4\,|f'(z)|^2.$$

Let $$\phi=|f(z)|^2=u^2+v^2,$$

then $$\frac{\partial\phi}{\partial x}=2u\,\frac{\partial u}{\partial x}+2v\,\frac{\partial v}{\partial x},$$

$$\frac{\partial^2\phi}{\partial x^2}=2\left[u\,\frac{\partial^2 u}{\partial x^2}+\left(\frac{\partial u}{\partial x}\right)^2\right]+2\left[v\,\frac{\partial^2 v}{\partial x^2}+\left(\frac{\partial v}{\partial x}\right)^2\right].$$

Similarly, we have

$$\frac{\partial^2\phi}{\partial y^2}=2\left[u\,\frac{\partial^2 u}{\partial y^2}+\left(\frac{\partial u}{\partial y}\right)^2\right]+2\left[v\,\frac{\partial^2 v}{\partial y^2}+\left(\frac{\partial v}{\partial y}\right)^2\right].$$

Hence $$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=2u\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)+2v\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

$$+2\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2+\left(\frac{\partial v}{\partial y}\right)^2\right]$$

$$=4\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2\right]=4\,|u_x+iv_x|^2.$$

using Cauchy-Riemann equations and the condition that u, v satisfy Laplace's equation.

Hence $$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=4\,|f'(z)|^2.$$

Alternative. Since $x+iy=z$ and $x-iy=\bar{z}$, we have

$$x=\frac{1}{2}(z+\bar{z})\ \text{ and }\ y=-\frac{i}{2}(z-\bar{z})$$

so that $$\frac{\partial}{\partial z}=\frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial z}+\frac{\partial}{\partial y}\cdot\frac{\partial y}{\partial z}=\frac{1}{2}\left(\frac{\partial}{\partial x}-i\,\frac{\partial}{\partial y}\right)$$

and $$\frac{\partial}{\partial \bar{z}}=\frac{\partial}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}}+\frac{\partial}{\partial y}\cdot\frac{\partial y}{\partial \bar{z}}=\frac{1}{2}\left(\frac{\partial}{\partial x}+i\,\frac{\partial}{\partial y}\right),$$

x and y being treated as functions of two independent variables z and $\bar{z}$.

$$\therefore\quad \frac{\partial}{\partial z}\cdot\frac{\partial}{\partial \bar{z}}=\frac{1}{4}\left(\frac{\partial}{\partial x}-i\,\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial x}+i\,\frac{\partial}{\partial y}\right)$$

or $$4\,\frac{\partial^2}{\partial z\,\partial \bar{z}}=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}.$$

Hence $$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2=4\,\frac{\partial^2}{\partial z\,\partial \bar{z}}\,[f(z)\,f(\bar{z})]$$

$$=4\,\frac{\partial}{\partial z}\,[f(z)\,f'(\bar{z})]$$

$$=4f'(z)\,f'(\bar{z})=4\,|f'(z)|^2.$$

Example 3: Plot the complex number $z=-\sqrt{3}+i$ in the complex plane and then write it in its polar form.

Solution:

Find r

$$r=\sqrt{a^2+b^2}$$
$$r=\sqrt{\left(-\sqrt{3}\right)^2+(1)^2}$$
$$r=\sqrt{3+1}$$
$$r=\sqrt{4}$$
$$r=2$$

Find θ

$$\tan\theta = \frac{b}{a}$$
$$\tan\theta = \frac{1}{-\sqrt{3}}$$
$$\tan\theta = -\frac{\sqrt{3}}{3}$$

$\tan\frac{\pi}{6} = \frac{\sqrt{3}}{3}$ so the reference angle of $\frac{\pi}{6}$ would be subtracted from π to get the value of θ.

$$\theta = \pi - \frac{\pi}{6}$$
$$\theta = \frac{5\pi}{6}$$

Write the complex number in its polar form

$$z = r\,(\cos\theta + i\sin\theta)$$
$$z = 2\,(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6})$$

Example 4: Write the complex number $z = 5(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$ in its rectangular form and then plot it in the complex plane.

Solution:

Evaluate cos and sin at the value of theta

$$\cos\frac{\pi}{3} = \frac{1}{2}$$

$$\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$$

Substitute in the exact values of cos and sin to find the rectangular form

$$z = 5(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3})$$
$$z = 5(\frac{1}{2} + i\frac{\sqrt{3}}{2})$$
$$z = \frac{5}{2} + \frac{5\sqrt{3}}{2}i$$

Plot the complex number

Find the polar form of $-4 + 4i$.

Solution

First, find the value of r.

$$
\begin{aligned}
r &= \sqrt{x^2 + y^2} \\
r &= \sqrt{(-4)^2 + (4^2)} \\
r &= \sqrt{32} \\
r &= 4\sqrt{2}
\end{aligned}
$$

Find the angle θ using the formula:

$$
\begin{aligned}
\cos\theta &= \frac{x}{r} \\
\cos\theta &= \frac{-4}{4\sqrt{2}} \\
\cos\theta &= -\frac{1}{\sqrt{2}} \\
\theta &= \cos^{-1}\left(-\frac{1}{\sqrt{2}}\right) \\
&= \frac{3\pi}{4}
\end{aligned}
$$

Thus, the solution is $4\sqrt{2}\, cis \left(\frac{3\pi}{4}\right)$.

Convert the polar form of the given complex number to rectangular form:

$$z = 12\left(\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right)$$

Solution

We begin by evaluating the trigonometric expressions.

$$\cos\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} \text{ and } \sin(\frac{\pi}{6}) = \frac{1}{2}$$

After substitution, the complex number is

$$z = 12\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$

We apply the distributive property:

$$\begin{aligned} z &= 12\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right) \\ &= (12)\frac{\sqrt{3}}{2} + (12)\frac{1}{2}i \\ &= 6\sqrt{3} + 6i \end{aligned}$$

The rectangular form of the given point in complex form is $6\sqrt{3} + 6i$.

Find the rectangular form of the complex number given $r = 13$ and $\tan\theta = \frac{5}{12}$.

Solution

If $\tan\theta = \frac{5}{12}$, and $\tan\theta = \frac{y}{x}$, we first determine $r = \sqrt{x^2 + y^2} = \sqrt{122 + 52} = 13$. We then find $\cos\theta = \frac{x}{r}$ and $\sin\theta = \frac{y}{r}$.

$$\begin{aligned} z &= 13\left(\cos\theta + i\sin\theta\right) \\ &= 13\left(\frac{12}{13} + \frac{5}{13}i\right) \\ &= 12 + 5i \end{aligned}$$

The rectangular form of the given number in complex form is $12 + 5i$.

3.37. Prove that in polar form the Cauchy–Riemann equations can be written

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Solution

We have $x = r\cos\theta$, $y = r\sin\theta$ or $r = \sqrt{x^2+y^2}$, $\theta = \tan^{-1}(y/x)$. Then

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial u}{\partial r}\left(\frac{x}{\sqrt{x^2+y^2}}\right) + \frac{\partial u}{\partial \theta}\left(\frac{-y}{x^2+y^2}\right) = \frac{\partial u}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial u}{\partial \theta}\sin\theta \tag{1}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y} = \frac{\partial u}{\partial r}\left(\frac{y}{\sqrt{x^2+y^2}}\right) + \frac{\partial u}{\partial \theta}\left(\frac{x}{x^2+y^2}\right) = \frac{\partial u}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial u}{\partial \theta}\cos\theta \tag{2}$$

Similarly,

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial x} = \frac{\partial v}{\partial r}\cos\theta - \frac{1}{r}\frac{\partial v}{\partial \theta}\sin\theta \tag{3}$$

$$\frac{\partial v}{\partial y} = \frac{\partial v}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial v}{\partial \theta}\frac{\partial \theta}{\partial y} = \frac{\partial v}{\partial r}\sin\theta + \frac{1}{r}\frac{\partial v}{\partial \theta}\cos\theta \tag{4}$$

From the Cauchy–Riemann equation $\partial u/\partial x = \partial v/\partial y$ we have, using (1) and (4),

$$\left(\frac{\partial u}{\partial r} - \frac{1}{r}\frac{\partial v}{\partial \theta}\right)\cos\theta - \left(\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right)\sin\theta = 0 \tag{5}$$

From the Cauchy–Riemann equation $\partial u/\partial y = -(\partial v/\partial x)$ we have, using (2) and (3),

$$\left(\frac{\partial u}{\partial r} - \frac{1}{r}\frac{\partial v}{\partial \theta}\right)\sin\theta + \left(\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta}\right)\cos\theta = 0 \tag{6}$$

Multiplying (5) by $\cos\theta$, (6) by $\sin\theta$ and adding yields

$$\frac{\partial u}{\partial r} - \frac{1}{r}\frac{\partial v}{\partial \theta} = 0 \quad \text{or} \quad \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$$

.

Multiplying (5) by $-\sin\theta$, (6) by $\cos\theta$ and adding yields

$$\frac{\partial v}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \theta} = 0 \quad \text{or} \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

.