MAINS TEST SERIES-2021

TEST-13 (BATCH-I)

FULL SYLLABUS (PAPER-I)

Answer Key

1.(a) → A single linear equation can be treated as a very simple system of equations. Describe all solutions of the homogeneous "system" $10x_1 - 3x_2 - 2x_3 = 0$.

Solⁿ: Given $10x_1 - 3x_2 - 2x_3 = 0$ ——— ①

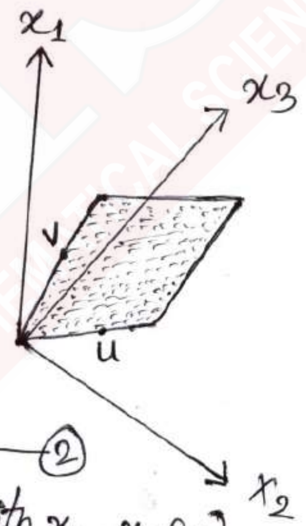
There is no need for matrix notation. Solve for the basic variable x_1 in terms of the free variables. The general solution is $x_1 = .3x_2 + .2x_3$, with x_2 and x_3 free. As a vector, the general solution is

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} .3x_2 + .2x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= \begin{bmatrix} .3x_2 \\ x_2 \\ 0 \end{bmatrix} + \begin{bmatrix} .2x_3 \\ 0 \\ x_3 \end{bmatrix}$$

$$= x_2 \underbrace{\begin{bmatrix} .3 \\ 1 \\ 0 \end{bmatrix}}_{u} + x_3 \underbrace{\begin{bmatrix} .2 \\ 0 \\ 1 \end{bmatrix}}_{v} \text{ ——— ②}$$

(with x_2, x_3 free)



This calculation shows that every solution of ① is a linear combination of the vectors u and v, shown in ②. That is, the solution set is span{u, v}. Since neither u nor v is a scalar multiple of the other, the solution set is a plane through the origin.

1.(b) → Find a basis for the null space of the matrix

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

Solⁿ: first, write the solution of $AX=0$ in parametric vector form:

$$[A \;\; 0] \sim \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad \begin{aligned} x_1 - 2x_2 - x_4 + 3x_5 &= 0 \\ -x_3 + 2x_4 - 2x_5 &= 0 \\ 0 &= 0 \end{aligned}$$

The general solution is $x_1 = 2x_2 + x_4 - 3x_5$, $x_3 = -2x_4 + 2x_5$, with x_2, x_4 and x_5 free.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 + x_4 - 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}}_{u} + x_4 \underbrace{\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}}_{v} + x_5 \underbrace{\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}}_{w}$$

$$= x_2 u + x_4 v + x_5 w \quad \text{——— (1)}$$

equation (1) shows that Nul A coincides with the set of all linear combinations of u, v and w. That is $\{u, v, w\}$ generates Nul A. In fact, this construction of u, v, w automatically makes them linearly Independent, because equation (1) shows that $0 = x_2 u + x_4 v + x_5 w$ only if the weights x_2, x_4 and x_5 are all zero.

So $\{u, v, w\}$ is a basis for Nul A.

Q)1(d) find the area of the portion of the sphere $x^2+y^2+z^2=9$ lying inside the cylinder $x^2+y^2=3y$.

Sol^n

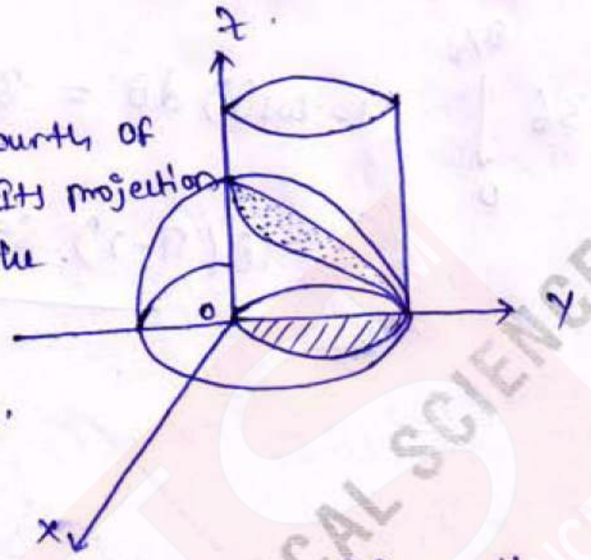


figure shows one-fourth of the required area. Its projection on the xy-plane is the semicircle $x^2+y^2=3y$ bounded by the y-axis.

for the sphere

$x^2+y^2+z^2=9$, $\frac{\partial z}{\partial x} = \frac{-x}{z}$ and $\frac{\partial z}{\partial y} = \frac{-y}{z}$.

$$\therefore \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1 = (x^2+y^2+z^2)/z^2$$

$$= \frac{9}{9-x^2-y^2} = \frac{9}{9-r^2}$$ when $x = r\cos\theta$, $y = r\sin\theta$.

using polar co-ordinates, the required area is found by integrating $\frac{3}{\sqrt{9-r^2}}$ over the semi-circle $r=3\sin\theta$, for which r varies from 0 to $3\sin\theta$ and θ varies from 0 to $\pi/2$.

Hence the required surface area

$$= 4\int_0^{\pi/2}\int_0^{3\sin\theta} \frac{3}{\sqrt{9-r^2}}\, r.d\theta\, dr$$

$$= -6 \int_0^{\pi/2} \left| \frac{\sqrt{9-r^2}}{1/2} \right|_0^{3\sin\theta} d\theta.$$

$$= 36 \int_0^{\pi/2} (1-\cos\theta)\, d\theta = 36 \left| \theta - \sin\theta \right|_0^{\pi/2}.$$

$$= 18(\pi - 2) \text{ sq. units.}$$

1(e). Show that the straight line whose direction cosines are given by the equations: $ul+vm+wn=0$, $al^2+bm^2+cn^2=0$ are (i) perpendicular if $u^2(b+c)+v^2(c+a)+w^2(a+b)=0$ and (ii) parallel if $(u^2/a)+(v^2/b)+(w^2/c)=0$.

Sol'n: The d.c's of the lines are given by

$ul+vm+wn=0$ and $al^2+bm^2+cn^2=0$

Eliminating n between these, we get

$$al^2+bm^2+c[-(ul+vm)/w]^2=0$$

$$\Rightarrow (aw^2+cu^2)l^2+(bw^2+cv^2)m^2+2cuvlm=0$$

$$\Rightarrow (aw^2+cu^2)(l/m)^2+2cuv(l/m)+(bw^2+cv^2)=0 \quad \text{---(1)}$$

dividing each term by m^2

(i) Its two roots are l_1/m_1 and l_2/m_2, if the d.c's of the two lines be taken as (l_1,m_1,n_1) and (l_2,m_2,n_2)

$\therefore$ from (1) we have

$$\frac{l_1}{m_1}\cdot\frac{l_2}{m_2} = \text{product of the roots} = \frac{bw^2+cv^2}{cu^2+aw^2}$$

$$\Rightarrow \frac{l_1l_2}{bw^2+cv^2} = \frac{m_1m_2}{cu^2+aw^2} = \frac{n_1n_2}{av^2+bu^2}; \text{ by symmetry}$$

$\therefore$ If the two lines are $\perp$ar, then we have

$l_1l_2+m_1m_2+n_1n_2=0$

i.e. $(bw^2+cv^2)+(cu^2+aw^2)+(av^2+bu^2)=0$

$\Rightarrow u^2(b+c)+v^2(c+a)+w^2(a+b)=0$. Hence proved

(ii) If the two lines are parallel, then their d.c's are equal are consequently the roots of (1) are equal, the condition for the same being

$b^2=4ac$

i.e. $(2cuv)^2=4(aw^2+cu^2)(bw^2+cv^2)$

$$c^2u^2v^2 = abw^4+acw^2v^2+bcu^2w^2+c^2u^2v^2$$

$$\Rightarrow abw^4+acw^2v^2+bcu^2w^2=0$$

$$\Rightarrow abw^2+acv^2+bcu^2=0$$

$$\Rightarrow \frac{w^2}{c}+\frac{v^2}{b}+\frac{u^2}{a}=0. \quad \text{dividing each term by } abc$$

2.(a) → Let us pose the following problem. Let W be the subspace of R^4 spanned by the vectors

$$\alpha_1 = (1,2,2,1)$$
$$\alpha_2 = (0,2,0,1)$$
$$\alpha_3 = (-2,0,-4,3)$$

(i) Prove that $\alpha_1, \alpha_2, \alpha_3$ form a basis for W, i.e., that these vectors are linearly independent.

(ii) Let $\beta = (b_1, b_2, b_3, b_4)$ be a vector in W. What are the coordinates of β relative to the ordered basis $\{\alpha_1, \alpha_2, \alpha_3\}$?

(iii) Let
$$\alpha'_1 = (1,0,2,0)$$
$$\alpha'_2 = (0,2,0,1)$$
$$\alpha'_3 = (0,0,0,3)$$
Show that $\alpha'_1, \alpha'_2, \alpha'_3$, form a basis for W.

(iv) If β is in W, let X denote the coordinates matrix of β relative to the α-basis and X' the coordinate matrix of β relative to the α'-basis. Find the 3×3 matrix P such that $X = PX'$ for every such β.

Solⁿ: To answer these questions by the first method we form the matrix A with row vectors $\alpha_1, \alpha_2, \alpha_3$, find the row-reduced echelon matrix R which is row-equivalent to A and simultaneously perform the same operations on the identity to obtain the invertible matrix Q such that $R = QA$:

$$\begin{bmatrix} 1 & 2 & 2 & 1 \\ 0 & 2 & 0 & 1 \\ -2 & 0 & -4 & 3 \end{bmatrix} \rightarrow R = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow Q = \frac{1}{6}\begin{bmatrix} 6 & -6 & 0 \\ -2 & 5 & -1 \\ 4 & -4 & 2 \end{bmatrix}$$

(i) Clearly R has rank 3, so α_1, α_2 and α_3 are independent.

(ii) Which vectors $\beta = (b_1, b_2, b_3, b_4)$ are in W?
We have the basis for W given by P_1, P_2, P_3 the row vectors of R. One can see at a glance that the span of P_1, P_2, P_3 consists of the vectors β for which $b_3 = 2b_1$. For such a β we have

$$\beta = b_1P_1 + b_2P_2 + b_4P_3$$
$$= [b_1, b_2, b_4]R$$
$$= [b_1\ b_2\ b_4]QA$$
$$= x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3$$

where $x_i = [b_1\ b_2\ b_4]Q_i$:

$$\left.\begin{aligned} x_1 &= b_1 - \tfrac{1}{3}b_2 + \tfrac{2}{3}b_4 \\ x_2 &= -b_1 + \tfrac{5}{6}b_2 - \tfrac{2}{3}b_4 \\ x_3 &= \quad -\tfrac{1}{6}b_2 + \tfrac{1}{3}b_4. \end{aligned}\right\} \text{——} (1)$$

(iii) The vectors $\alpha_1', \alpha_2', \alpha_3'$ are all of the form (y_1, y_2, y_3, y_4) with $y_3 = 2y_1$ and thus they are in W. and they are independent.

(iv) The matrix P has for its columns

$$P_j = [\alpha'_j]_{\mathcal{B}}$$

where $\mathcal{B} = \{\alpha_1, \alpha_2, \alpha_3\}$. The equations ① tell us how to find the coordinate matrices for $\alpha'_1, \alpha'_2, \alpha'_3$. for example with $\beta = \alpha'_1$ we have $b_1 = 1, b_2 = 0,$ $b_3 = 2, b_4 = 0$ and

$$x_1 = 1 - \frac{1}{3}(0) + \frac{2}{3}(0) = 1$$

$$x_2 = -1 + \frac{5}{6}(0) - \frac{2}{3}(0) = -1$$

$$x_3 = \quad -\frac{1}{6}(0) + \frac{1}{3}(0) = 0.$$

Thus $\alpha'_1 = \alpha_1 - \alpha_2$. Similarly we obtain $\alpha'_2 = \alpha_2$ and $\alpha'_3 = 2\alpha_1 - 2\alpha_2 + \alpha_3$.

Hence
$$P = \begin{bmatrix} 1 & 0 & 2 \\ -1 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

Now let us see how we would answer the questions by the second method which we described. We form the 4×3 matrix B with column vectors $\alpha_1, \alpha_2, \alpha_3$:

$$B = \begin{bmatrix} 1 & 0 & -2 \\ 2 & 2 & 0 \\ 2 & 0 & -4 \\ 1 & 1 & 3 \end{bmatrix}$$

We inquire for which y_1, y_2, y_3, y_4 the system $BX = Y$ has a solution.

$$\begin{bmatrix} 1 & 0 & -2 & y_1 \\ 2 & 2 & 0 & y_2 \\ 2 & 0 & -4 & y_3 \\ 1 & 1 & 3 & y_4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -2 & y_1 \\ 0 & 2 & 4 & y_2 - 2y_1 \\ 0 & 0 & 0 & y_3 - 2y_1 \\ 0 & 1 & 5 & y_4 - y_1 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & -2 & y_1 \\ 0 & 0 & -6 & y_2 - 2y_4 \\ 0 & 1 & 5 & y_4 - y_1 \\ 0 & 0 & 0 & y_3 - 2y_1 \end{bmatrix}$$

$$\longrightarrow \begin{bmatrix} 1 & 0 & 0 & y_1 - \frac{1}{3}y_2 + \frac{2}{3}y_4 \\ 0 & 0 & 1 & \frac{1}{6}(2y_4 - y_2) \\ 0 & 1 & 0 & -y_1 + \frac{5}{6}y_2 - \frac{2}{3}y_4 \\ 0 & 0 & 0 & y_3 - 2y_1 \end{bmatrix}$$

Thus the condition that the system $BX = Y$ have a solution is $y_3 = 2y_1$. So $\beta = (b_1, b_2, b_3, b_4)$ is in W if and only if $b_3 = 2b_1$. If β is in W, then the coordinates (x_1, x_2, x_3) in the orderded basis $\{\alpha_1, \alpha_2, \alpha_3\}$ can be read off from the last matrix above. We obtain once again the formulas (1) for these coordinates.

The questions (iii) and (iv) are now answered as before.

2.(b)(i) Find $\lim_{x\to-\infty}(x^2 \operatorname{sgn}(\cos x))$.

Sol^n: Let $x = -2n\pi$, so when $x \to -\infty$, $n \to \infty$

Now

$$x^2 \operatorname{sgn}(\cos x) = (-2n\pi)^2 \operatorname{sgn}\cos(-2n\pi) = 4n^2\pi^2$$

$$\therefore \lim_{x\to-\infty} x^2 \operatorname{sgn}(\cos x) = \infty$$

Again, let $x = -(2n+1)\pi$

$$\therefore x^2 \operatorname{sgn}(\cos x) = (-(2n+1)\pi)^2 \operatorname{sgn}\cos(-(2n+1)\pi) = -(2n+1)^2\pi^2$$

and so

$$\lim_{x\to-\infty} x^2 \operatorname{sgn}(\cos x) = -\infty$$

Hence $\lim_{x\to-\infty} x^2 \operatorname{sgn}(\cos x)$ does not exist

2.c(i) → Through a point $P(\alpha, \beta, \gamma)$ a plane is drawn at right angles to OP to meet the axes in A, B, C. Prove that the area of the triangle ABC is $p^5/(2\alpha\beta\gamma)$, where $OP = p$.

Sol^n: The equation of the plane through $P(\alpha, \beta, \gamma)$ at right angles to the line OP is

$$\alpha(x-\alpha)+\beta(y-\beta)+\gamma(z-\gamma)=0$$

or $x\alpha + y\beta + z\gamma = \alpha^2+\beta^2+\gamma^2 = p^2$,

$$\because OP^2 = p^2 = \alpha^2+\beta^2+\gamma^2.$$

or $$\frac{x}{(p^2/\alpha)} + \frac{y}{(p^2/\beta)} + \frac{z}{(p^2/\gamma)} = 1.$$

If this plane meets the coordinates axes in A, B, C, then the co-ordinates of A, B and C are

$$\left(\frac{p^2}{\alpha}, 0, 0\right), \left(0, \frac{p^2}{\beta}, 0\right) \text{ and } \left(0, 0, \frac{p^2}{\gamma}\right)$$

Let $\Delta_x, \Delta_y, \Delta_z$ be the projections of ΔABC on the yz, zx and xy-planes, we have

$$\Delta_x = \text{area of } \Delta BCO = \frac{1}{2}\cdot OB\cdot OC = \frac{1}{2}\cdot\frac{p^2}{\beta}\cdot\frac{p^2}{\gamma};$$

$$\Delta_y = \text{area of } \Delta COA = \frac{1}{2}\cdot OC\cdot OA = \frac{1}{2}\cdot\frac{p^2}{\gamma}\cdot\frac{p^2}{\alpha};$$

and $$\Delta_z = \text{area of } \Delta AOB = \frac{1}{2}\cdot OA\cdot OB = \frac{1}{2}\cdot\frac{p^2}{\alpha}\cdot\frac{p^2}{\beta}.$$

$\therefore$ The required area of ΔAOB

$$= \sqrt{[\Delta_x^2+\Delta_y^2+\Delta_z^2]}$$

$$= \sqrt{\left[\left(\frac{p^4}{2\beta\gamma}\right)^2+\left(\frac{p^4}{2\gamma\alpha}\right)^2+\left(\frac{p^4}{2\alpha\beta}\right)^2\right]}$$

$$= \frac{p^4}{2}\sqrt{\left[\frac{1}{(\beta\gamma)^2}+\frac{1}{(\gamma\alpha)^2}+\frac{1}{(\alpha\beta)^2}\right]}$$

$$= \frac{p^4}{2\alpha\beta\gamma}\sqrt{[\alpha^2+\beta^2+\gamma]}$$

$$= \frac{p^4}{2\alpha\beta\gamma}\sqrt{(p^2)}, \quad \because p^2=\alpha^2+\beta^2+\gamma^2$$

$$= p^5/(2\alpha\beta\gamma).$$

4(c) … tegration.

2.(c)(ii) Reduce the equation

$$9x^2 - 7y^2 + 2z^2 - 10yz - 8zx - 10xy + 6x + 12y - 6z + 5 = 0$$

to the standard form.

What does it represent?

Sol^n: Comparing the given equation $F(x,y,z)=0$ with the equation

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$$

we have $a=2,\ b=-7,\ c=2,\ f=-5,\ g=-4,\ h=-5,\ u=3,\ v=6,\ w=-3,\ d=5$

Now coordinates of the centre (x_1,y_1,z_1) of the given surface are given by

$$\frac{\partial f}{\partial x}=0,\quad \frac{\partial f}{\partial y}=0 \text{ and } \frac{\partial f}{\partial z}=0$$

$$4x_1-8z_1-10y_1+6=0 \Rightarrow 2x_1-5y_1-4z_1+3=0 \quad \text{---(1)}$$

$$-14y_1-10z_1-10x_1+12=0 \Rightarrow 5x_1+7y_1+5z_1-6=0 \quad \text{---(2)}$$

$$4z_1-10y_1-8x_1-6=0 \Rightarrow 4x_1+5y_1-2z_1+3=0 \quad \text{---(3)}$$

solving (1), (2) and (3) we get

$$x_1=1/3,\ y_1=-1/3,\ z_1=4/3$$

$\therefore$ Centre of the given surface is $(1/3, -1/3, 4/3)$

Also $d'=ux_1+vy_1+wz_1+d$

$$=3(1/3)+6(-1/3)+(-3)(4/3)+5$$

$$=1-2-4+5=0 \quad \text{---(4)}$$

Now the discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix}=0 \Rightarrow \begin{vmatrix} 2-\lambda & -5 & -4 \\ -5 & -7-\lambda & -5 \\ -4 & -5 & 2-\lambda \end{vmatrix}=0$$

$\Rightarrow (2-\lambda)[-(7+\lambda)(2-\lambda)-25]+5[-5(2-\lambda)-20]$
$-4[25-4(7+\lambda)]=0$

$\Rightarrow \lambda^3+3\lambda^2-90\lambda+216=0$

$\Rightarrow (\lambda-3)(\lambda^2+6\lambda-72)=0$

$\Rightarrow (\lambda-3)(\lambda-6)(\lambda+12)=0$

$\Rightarrow \lambda=3,6,-12.$

$\therefore$ Let $\lambda_1=3,\ \lambda_2=6,\ \lambda_3=-12$

$\therefore$ By rotation of axes the given equation transforms to

$\lambda_1x^2+\lambda_2y^2+\lambda_3z^2+d'=0$

substituting the values of $\lambda_1, \lambda_2, \lambda_3, d'$

$\Rightarrow 3x^2+6y^2-12z^2+0=0$

$\Rightarrow x^2+2y^2-4z^2=0$

which is the required standard form and represents a cone.

Also the vertex of the cone is

$\left(\frac{1}{3}, -\frac{1}{3}, \frac{4}{3}\right).$

3(a) → Let $A = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix}$, $u = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$, $b = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix}$, $c = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix}$, and

define a transformation $T: \mathbb{R}^2 \to \mathbb{R}^3$ by $T(x) = Ax$,

so that

$$T(x) = A(x) = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 3x_2 \\ 3x_1 + 5x_2 \\ -x_1 + 7x_2 \end{bmatrix}$$

(i) Find $T(u)$, the image of u under the transformation T.

(ii) Find an x in $\mathbb{R}^2$ whose image under T is b.

(iii) Is there more than one x whose image under T is b?

(iv) Determine if c is in the range of the transformation T.

Solⁿ: (i) compute

$$T(u) = Au = \begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \\ -9 \end{bmatrix}$$

(ii) solve $T(x) = b$ for x. That is, solve $Ax = b$, or

$$\begin{bmatrix} 1 & -3 \\ 3 & 5 \\ -1 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -5 \end{bmatrix} \quad \text{--- (1)}$$

using the method, row reduce the augmented matrix:

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & -.5 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1.5 \\ 0 & 1 & -.5 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{--- (2)}$$

Hence $x_1 = 1.5$, $x_2 = -.5$, and $\begin{bmatrix} 1.5 \\ -.5 \end{bmatrix}$. The image of this x under T is the given vector b.

(iii) Any x whose image under T is b must satisfy equation (1), from (2), it is clear that equation (1) has a unique solution. So there is exactly one x whose image is b.

(iv) The vector c is in the range of T if c is the image of some x in $\mathbb{R}^2$, that is, if $c = T(x)$ for some x. This is just another way of asking if the system $Ax = c$ is consistent. To find the answer, row reduce the augmented matrix:

$$\begin{bmatrix} 1 & -3 & 3 \\ 3 & 5 & 2 \\ -1 & 7 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 14 & -7 \\ 0 & 4 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 14 & -7 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & -35 \end{bmatrix}$$

The third equation, $0 = -35$, shows that the system is inconsistent. So c is not in the range of T.

3(c) → A cone has as base the circle $x^2+y^2+2ax+2by=0$, $z=0$ and passes through the fixed point $(0,0,c)$. If the section of the cone by zx-plane is a rectangular hyperbola, Prove that the vertex lies on a fixed circle.

Sol^n: Let (α,β,γ) be the vertex of the cone. Any line through (α,β,γ) is $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ —— ①

It meets the plane $z=0$ in $\left(\alpha-\frac{l\gamma}{n}, \beta-\frac{m\gamma}{n}, 0\right)$ and if this point lies on the given conic, we have

$$\left(\alpha-\frac{l\gamma}{n}\right)^2+\left(\beta-\frac{m\gamma}{n}\right)^2+2a\left(\alpha-\frac{l\gamma}{n}\right)+2b\left(\beta-\frac{m\gamma}{n}\right)=0 \quad \text{— ②}$$

Eliminating l,m,n between ① & ②, the equation of the cone is

$$\left[\alpha-\left(\frac{x-\alpha}{z-\gamma}\right)\gamma\right]^2+\left[\beta-\left(\frac{y-\beta}{z-\gamma}\right)\gamma\right]^2+2a\left[\alpha-\left(\frac{x-\alpha}{z-\gamma}\right)\gamma\right]$$
$$+2b\left[\beta-\left(\frac{y-\beta}{z-\gamma}\right)\gamma\right]=0$$

$$(\alpha z - x\gamma)^2 + (\beta z - \gamma y)^2 + 2a(\alpha z - xy)(z-\gamma) + 2b(\beta z - y\gamma)(z-\gamma) = 0$$

If this cone passes through $(0,0,c)$, then

$$(\alpha c)^2 + (\beta c)^2 + 2a(\alpha c)(c-\gamma) + 2b(\beta c)(c-\gamma) = 0 \quad \text{--- (3)}$$

Again the section of the cone by zx-plane

i.e. $y=0$ is

$$(\alpha z - \gamma x)^2 + (\beta z)^2 + 2a(\alpha z - x\gamma)(z-\gamma) + 2b(\beta z)(z-\gamma) = 0$$

and if this section is a rectangular hyperbola in the zx-plane, then the sum of the coefficients of x^2 and z^2 should be zero

i.e. $\gamma^2 + (\alpha^2 + \beta^2 + 2a\alpha + 2b\beta) = 0$ --- (4)

$\therefore$ The locus of (α, β, γ) from (3) and (4) is

$$c(x^2+y^2) + 2ax(c-z) + 2by(c-z) = 0 \quad \text{--- (5)}$$

and $x^2 + y^2 + z^2 + 2ax + 2by = 0$ --- (6)

Multiplying (6) by c and subtracting (5) from the result so obtained, we get

$$cz^2 + 2azx + 2byz = 0 \text{ (or) } 2ax + 2by + cz = 0 \quad \text{--- (7)}$$

which is the equation of a plane.

$\therefore$ the required locus of the vertex is given by (6) and (7) which taken together represent a circle.

4.(a)(i) Let V be a two-dimensional vector space over the field F, and let β be an ordered basis for V. If T is a linear operator on V and $[T]_\beta = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

prove that $T^2 - (a+d)T + (ad-bc)I = 0$

Sol^n Given $[T]_\beta = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ — ①

$$T^2(x) = T(T(x)) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}(x)$$

$$= \begin{bmatrix} a^2+bc & ab+bd \\ ac+dc & bd+d^2 \end{bmatrix}(x) \quad \text{— ②}$$

$T^2 - (a+d)T + (ad-bc)I$

$$= \begin{bmatrix} a^2+bc & ab+bd \\ ac+dc & bd+d^2 \end{bmatrix} - (a+d)\begin{bmatrix} a & b \\ c & d \end{bmatrix} + (ad-bc)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$ [Using equation ① and ②]

$$= \begin{bmatrix} a^2+bc-a^2-ad & ab+bd-(a+d)b \\ ac+dc-(a+d)c & bd+d^2-(a+d)d \end{bmatrix} - \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

Hence proved

4.a(ii) Let $A=\begin{bmatrix} 7 & 2 \\ -4 & 1 \end{bmatrix}$. Find a formula for A^K, given that $A = PDP^{-1}$, where $P=\begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}$ and $D=\begin{bmatrix} 5 & 0 \\ 0 & 3 \end{bmatrix}$.

Sol^n: The standard formula for the inverse of a 2×2 matrix yields

$P^{-1}=\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$ Then, by associativity of matrix multiplication,

$$A^2 = (PDP^{-1})(PDP^{-1}) = PD\underbrace{(P^{-1}P)}_{1}DP^{-1} = PDDP^{-1}$$

$$= PD^2P^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5^2 & 0 \\ 0 & 3^2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

Again,

$$A^3 = (PDP^{-1})A^2 = (PD\underbrace{P^{-1})P}_{1}D^2P^{-1} = PDD^2P^{-1} = PD^3P^{-1}$$

In general, for $K \geq 1$,

$$A^K = PD^KP^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} 5^K & 0 \\ 0 & 3^K \end{bmatrix}\begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2\cdot5^K - 3^K & 5^K - 3^K \\ 2\cdot3^K - 2\cdot5^K & 2\cdot3^K - 5^K \end{bmatrix}$$

A square matrix A is said to be diagonalizable if A is similar to a diagonal matrix, that is, if $A = PDP^{-1}$ for some invertible matrix P and some diagonal matrix D.

4(b)(i) show that if $a>1$, $\int_0^\infty \frac{x^a}{a^x}dx = \frac{\Gamma(a+1)}{(\log a)^{a+1}}$

(ii) If $v = At^{-1/2}e^{-x^2/4a^2t}$, Prove that $\frac{\partial v}{\partial t} = a^2 \frac{\partial^2 v}{\partial x^2}$.

Sol^n: (i) Since $a = e^{\log a}$

$$\therefore a^x = e^{x\log a}$$

$$\therefore \int_0^\infty \frac{x^a}{a^x}dx = \int_0^\infty \frac{x^a}{e^{x\log a}}dx = \int_0^\infty e^{-x\log a}.x^a\,dx \quad \text{—— (1)}$$

put $x\log a = z$

$$\Rightarrow x = \frac{z}{\log a}$$

$$\Rightarrow dx = \frac{dz}{\log a}$$

Limits when $x=0$, $z=0$;
when $x\to\infty$, $z\to\infty$

$$\therefore (1) \equiv \int_0^\infty \frac{x^a}{a^x}dx = \int_0^\infty e^{-z}\frac{z^a}{(\log a)^a}\cdot\frac{dz}{\log a}$$

$$= \frac{1}{(\log a)^{a+1}}\int_0^\infty e^{-z}.z^a\,dz$$

$$= \frac{1}{(\log a)^{a+1}}\int_0^\infty e^{-z}z^{(a+1)-1}dz$$

$$= \frac{1}{(\log a)^{a+1}}\Gamma(a+1)$$

$\left[\because \text{By defn of Gamma function } \Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx\right]$

$$= \frac{\Gamma(a+1)}{(\log a)^{a+1}}$$

4(b)(ii) $v = At^{-1/2} e^{-x^2/4a^2t}$ — ①

Diff ① partially w.r.t t we get

$$\frac{\partial v}{\partial t} = A\left[\left(-\frac{1}{2}\right)t^{-3/2} e^{-\frac{x^2}{4a^2t}} + t^{-1/2} e^{-x^2/4a^2t} \cdot \frac{x^2}{4a^2t^2}\right]$$

$$= A e^{-x^2/4a^2t}\, t^{-1/2}\left[-\frac{1}{2t} + \frac{x^2}{4a^2t^2}\right] = v\left[\frac{-1}{2t} + \frac{x^2}{4a^2t^2}\right] \text{ — ②}$$

Diff ① partially w.r.t x, we get

$$\frac{\partial v}{\partial x} = At^{-1/2} e^{-x^2/4a^2t}\left(-\frac{2x}{4a^2t}\right)$$

$$= v\left(\frac{-x}{2a^2t}\right) \text{ — ③}$$

Again differentiating ③ w.r.t x, we get

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial x}\left(-\frac{x}{2a^2t}\right) + v\left(-\frac{1}{2a^2t}\right)$$

$$= v\left(\frac{-x}{2a^2t}\right)\left(\frac{-x}{2a^2t}\right) - v\left(\frac{1}{2a^2t}\right) \quad (\text{from } ③)$$

$$= v\left[\frac{x^2}{4a^4t^2} - \frac{1}{2a^2t}\right]$$

$$= \frac{v}{a^2}\left[\frac{x^2}{4a^2t^2} - \frac{1}{2t}\right]$$

$$\Rightarrow a^2\frac{\partial^2 v}{\partial x^2} = v\left[\frac{-1}{2t} + \frac{x^2}{4a^2t^2}\right] \text{ — ④}$$

$\therefore$ from ② and ④ we have

$$\boxed{\frac{\partial v}{\partial t} = a^2\frac{\partial^2 v}{\partial x^2}}$$

4(c) CP, CQ are any two conjugate semi-diameters of the ellipse $x^2/a^2 + y^2/b^2 = 1$, $z = c$, CP', CQ' are the conjugate diameters of the ellipse $x^2/a^2 + y^2/b^2 = 1$, $z = -c$ drawn in the same directions as CP and CQ, prove that the hyperboloid $2x^2/a^2 + 2y^2/b^2 - z^2/c^2 = 1$ is generated by either PQ' or P'Q'.

Sol'n The Coordinates of P, Q, P' and Q' are given by

$P(a\cos\theta, b\sin\theta, c)$, $Q(-a\sin\theta, b\cos\theta, c)$,

$P'(a\cos\theta, b\sin\theta, -c)$ and $Q'(-a\sin\theta, b\cos\theta, -c)$

$\therefore$ Equations to PQ' are

$$\frac{x-a\cos\theta}{-a\sin\theta-a\cos\theta} = \frac{y-b\sin\theta}{b\cos\theta-b\sin\theta} = \frac{z-c}{-c-c} = r$$

$\therefore x-a\cos\theta = r[-a(\sin\theta+\cos\theta)]$,

$y-b\sin\theta = r[b(\cos\theta-\sin\theta)]$ and

$z-c = -2cr$

$\Rightarrow x/a = \cos\theta - r(\sin\theta+\cos\theta)$,

$y/b = \sin\theta + r(\cos\theta-\sin\theta)$ and

$z = c(1-2r)$ ——— (i)

Eliminating r from these we get,

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = \left[\cos\theta - r(\sin\theta-\cos\theta)\right]^2 + \left[\sin\theta + r(\cos\theta-\sin\theta)\right]^2$$

$$= 1+r^2\{(\sin\theta+\cos\theta)^2 + (\cos\theta-\sin\theta)^2\}$$
$$-2r[\cos\theta(\sin\theta+\cos\theta) - \sin\theta(\cos\theta-\sin\theta)\}$$

$$= 1+r^2(2)-2r(1) = 2r^2-2r+1$$

or $2\left(\frac{x^2}{a^2}\right)+2\left(\frac{y^2}{b^2}\right)-\left(\frac{z^2}{c^2}\right) = 1$ Which is a hyperboloid

5.(a) → Solve $x = py + p^2$.

Soln: Given $x = py + p^2$ ——— ①

Differentiating ① w.r.t. 'y',

$$1/p = p + y(dp/dy) + 2p(dp/dy)$$

or $(1-p^2)/p = (y+2p)(dp/dy)$

or $dy/dp = p(y+2p)/(1-p^2)$

or $dy/dp - \{p/(1-p^2)\}y = 2p^2/(1-p^2)$, ——— ②

which is linear with I.F. $e^{\int P dx}$, where $P = -p/(1-p^2)$.

Now, $\int P\,dx = -\int \frac{p}{1-p^2}dp = \frac{1}{2}\int \frac{(-2p)dp}{1-p^2}$

$$= \frac{1}{2}\log(1-p^2) = \log(1-p^2)^{1/2}$$

$\therefore$ I.F. of ② $= e^{\log(1-p^2)^{1/2}} = (1-p^2)^{1/2}$ and so solution of ② is

$$y(1-p^2)^{1/2} = \int \frac{2p^2}{1-p^2}(1-p^2)^{1/2}dp + C$$

$$= C + 2\int \frac{1-(1-p^2)}{(1-p^2)^{1/2}}dp$$

$$= C + 2\int \frac{dp}{(1-p^2)^{1/2}} - 2\int (1-p^2)^{1/2}dp.$$

$$= C + 2\sin^{-1}p - 2\{(p/2)\times(1-p^2)^{1/2} + (1/2)\times\sin^{-1}p\}$$

or $y = C(1-p^2)^{-1/2} + (1-p^2)^{-1/2}\sin^{-1}p - p$.

or $y = (C + \sin^{-1}p)(1-p^2)^{-1/2} - p$ ——— ③

Substituting the value of y in (1), we have

$$x = p\{C(1-p^2)^{-1/2} + (1-p^2)^{-1/2}\sin^{-1}p - p\} + p^2$$

or $\quad x = Cp(1-p^2)^{-1/2} + p(1-p^2)^{-1/2}\sin^{-1}p$

or $\quad x = p(C + \sin^{-1}p)(1-p^2)^{-1/2}$ ———— (4)

(3) and (4) together give the solution is parametric form, p being the parameter.

5. b(i) If $L\left\{2\sqrt{\left(\frac{t}{\pi}\right)}\right\} = \frac{1}{s^{3/2}}$, show $\frac{1}{s^{1/2}} = L\left\{\frac{1}{\sqrt{(\pi t)}}\right\}$.

Sol^n: Let $F(t) = 2\sqrt{(t/\pi)}$. Then $F(0) = 2\sqrt{(0/\pi)} = 0$ ——— (1)

Again $F'(t) = \frac{d}{dt}\left[\frac{2}{\sqrt{\pi}}t^{1/2}\right] = \frac{2}{\sqrt{\pi}}\cdot\frac{1}{2}t^{-1/2} = \frac{1}{\sqrt{(\pi t)}}$ ——— (2)

Now, we know that

$$L\{F'(t)\} = sL\{F(t)\} - F(0) \quad \text{——— (3)}$$

Substituting values of $F'(t)$, $F(t)$ and $F(0)$ in (3), we get

$$L\left\{\frac{1}{\sqrt{\pi t)}}\right\} = sL\left\{2\sqrt{\left(\frac{t}{\pi}\right)}\right\} - 0$$

$$= s\cdot\frac{1}{s^{3/2}} = \frac{1}{s^{1/2}}.$$

$\because$ given that
$L\{2\sqrt{t/\pi}\} = 1/s^{3/2}$.

5.b(ii) → Find a function $F(t)$ for which $F(t) = L^{-1}\left\{\frac{3}{s} - \frac{4e^{-s}}{s^2} + \frac{4e^{-3s}}{s^2}\right\}$

Solⁿ: $F(t) = L^{-1}\left\{\frac{3}{s} - \frac{4e^{-s}}{s^2} + \frac{4e^{-3s}}{s^2}\right\}$

or $F(t) = 3L^{-1}\left\{\frac{1}{s}\right\} - 4L^{-1}\left\{e^{-s}\frac{1}{s^2}\right\} + 4L^{-1}\left\{e^{-3s}\cdot\frac{1}{s^2}\right\}$ —— (1)

But $L^{-1}\left\{\frac{1}{s}\right\} = 1$ and $L^{-1}\left\{\frac{1}{s^2}\right\} = \frac{t}{1!} = t$.

Hence by second shifting theorem, we have

$L^{-1}\left\{e^{-s}\cdot\frac{1}{s^2}\right\} = (t-1)H(t-1)$ and

$L^{-1}\left\{\frac{e^{-3s}}{s^2}\right\} = (t-3)H(t-3)$

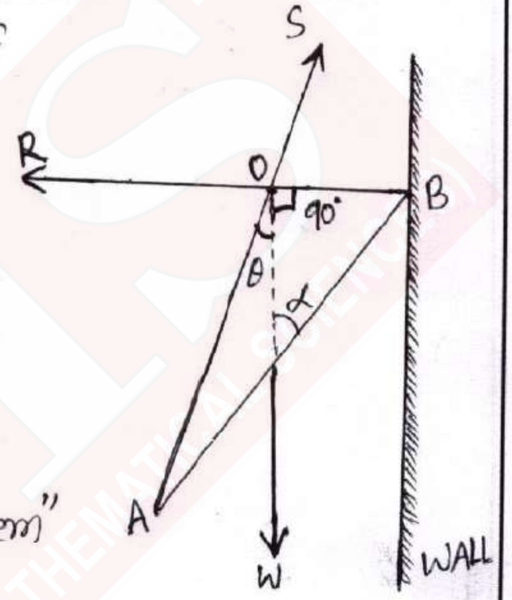
$\therefore$ (1) $\Rightarrow$ $F(t) = 3 - 4(t-1)H(t-1) + 4(t-3)H(t-3)$

5.(c) → A uniform rod AB movable about a hinge at A, rests with the other end against a smooth vertical wall. If α be the inclination of the rod to the vertical, Prove that the magnitude of the reaction of the hinge is $\frac{1}{2}W\sqrt{4+\tan^2\alpha}$.

Solⁿ: Let W be the weight of the rod acting vertically downwards through G, the mid point of AB. Let R be the normal reaction of the wall.

Let the line of action of W and R meet in O.

Then S, the reaction of the hinge at A must also pass through O and act along AO.



Let $\angle AOG = \theta$ and $\angle OGB = \alpha$. By "m-n theorem" in ΔAOB

$$(a+a)\cot\alpha = a\cot\theta - a\cot 90^\circ$$

$$\Rightarrow 2\cot\alpha = \cot\theta$$

$$\therefore \tan\theta = \frac{1}{2}\tan\alpha \quad \text{—— (1)}$$

By Lami's Theorem to forces at O, we have

$$\frac{S}{\sin 90^\circ} = \frac{W}{\sin(90^\circ+\theta)}$$

$$\therefore S = \frac{W}{\cos\theta} = W\sec\theta = W\sqrt{1+\tan^2\theta} = W\sqrt{1+\frac{1}{4}\tan^2\alpha} \quad (\because \text{of } (1))$$

$$= \frac{W}{2}\sqrt{4+\tan^2\alpha} \qquad \left[\text{or } \frac{W}{2}\sqrt{3+\sec^2\alpha}\right]$$

5.(d) → Transform the function $f = \rho e_\rho + \rho e_\phi$ from cylindrical to cartesian system.

Solⁿ: Given $f = \rho e_\rho + \rho e_\phi$

cylindrical and cartesian co-ordinates are related by

$$x = \rho\cos\phi,\ y = \rho\sin\phi,\ z = z$$

and $e_\rho = \cos\phi\, i + \sin\phi\, j$,

$$e_\phi = -\sin\phi\, i + \cos\phi\, j$$

$$\therefore\ f = \rho(\cos\phi\, i + \sin\phi\, j) + \rho(-\sin\phi\, i + \cos\phi\, j)$$

$$= xi + yj - yi + xj$$

$$= (x-y)i + (x+y)j$$

5.(e) Show that $F = (\sin y + z)i + (x\cos y - z)j + (x-y)K$ is irrotational and find a function ϕ such that $F = \nabla\phi$.

Sol^n: $\text{curl } F = \begin{vmatrix} i & j & K \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ \sin y + z & x\cos y - z & x - y \end{vmatrix}$

$$= i(-1+1) - j(1-1) + K(\cos y - \cos y) = 0.$$

$\therefore$ The given vector is irrotational and so

$$F = \nabla\phi.$$

Hence, $(\sin y + z)i + (x\cos y - z)j + (x-y)K$

$$= \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}K$$

$\therefore \frac{\partial\phi}{\partial x} = \sin y + z$, hence, $\phi = x\sin y + xz + f_1(y,z)$ —— (1)

$\frac{\partial\phi}{\partial y} = x\cos y - z$, hence, $\phi = x\sin y - yz + f_2(x,z)$ —— (2)

$\frac{\partial\phi}{\partial z} = x - y$, hence, $\phi = xz - yz + f_3(x,y)$ —— (3)

(1), (2) and (3) each represents ϕ. These agree if we choose $f_1(y,z) = -yz$, $f_2(x,z) = xz$ and $f_3(x,y) = x\sin y$. Hence, the required ϕ is given by

$$\phi = x\sin y + xz - yz + C,$$

where C being a constant.

6.a(i) → Determine the constant A such that the equation $\left(\frac{1}{x^2}+\frac{1}{y^2}\right)dx+\left(\frac{Ax+1}{y^3}\right)dy=0$ is exact and Solve the resulting exact equation.

Solⁿ: Suppose $M(x,y)=\frac{1}{x^2}+\frac{1}{y^2}$ and

$$N(x,y)=\frac{Ax+1}{y^3}. \text{ Then,}$$

We obtain $\frac{\partial M(x,y)}{\partial y}=-\frac{2}{y^3}$ and $\frac{\partial N(x,y)}{\partial x}=\frac{A}{y^3}$.

In order to make the differential equation become exact differential equation, it must be

$$\frac{\partial M(x,y)}{\partial y}=\frac{\partial N(x,y)}{\partial x}=-\frac{2}{y^3}.$$

Thus, $\frac{A}{y^3}=-\frac{2}{y^3} \Leftrightarrow A=-2.$

Therefore, we obtain that the differential equation $\left(\frac{1}{x^2}+\frac{1}{y^2}\right)dx+\left(\frac{-2x+1}{y^3}\right)dy=0$ is exact differential equation. furthermore, we must find F such that

$$\frac{\partial F(x,y)}{\partial x}=M(x,y)=\frac{1}{x^2}+\frac{1}{y^2} \quad \text{and}$$

$$\frac{\partial F(x,y)}{\partial y}=N(x,y)=\frac{-2x+1}{y^3}.$$

from the first of these,

$$F(x,y)=\int M(x,y)\partial x+h(y)$$

$$= \int \left(\frac{1}{x^2} + \frac{1}{y^2} \right) \partial x + h(y) = -\frac{1}{x} + \frac{x}{y^2} + h(y).$$

Then $$\frac{\partial F(x,y)}{\partial y} = -\frac{2x}{y^3} + \frac{dh(y)}{dy}.$$

But we must have

$$\frac{\partial F(x,y)}{\partial y} = N(x,y) = \frac{-2x+1}{y^3}.$$

Thus $$\frac{-2x+1}{y^3} = -\frac{2x}{y^3} + \frac{dh(y)}{dy} \Leftrightarrow \frac{1}{y^3} = \frac{dh(y)}{dy}.$$

Thus $h(y) = -\frac{1}{2y^2} + C_0$, where C_0 is an arbitrary constant, and so

$$F(x,y) = -\frac{1}{x} + \frac{x}{y^2} - \frac{1}{2y^2} + C_0.$$

Hence a one-parameter family of solution is

$$F(x,y) = C_1,$$

or $$-\frac{1}{x} + \frac{x}{y^2} - \frac{1}{2y^2} + C_0 = C_1$$

combining the constants C_0 and C_1 we may write this solution as

$$-\frac{1}{x} + \frac{x}{y^2} - \frac{1}{2y^2} = C$$

where $C = C_1 - C_0$ is an arbitrary constant.

So, we conclude that the general solution of the exact differential equation

$$\left(\frac{1}{x^2} + \frac{1}{y^2}\right)dx + \left(\frac{-2x+1}{y^3}\right)dy = 0 \text{ is } -\frac{1}{x} + \frac{x}{y^2} - \frac{1}{2y^2} = C.$$

6.a(ii) Find the value of n such that the curves $x^n + y^n = c_1$ are orthogonal trajectories of the family $y = \frac{x}{1-c_2 x}$.

Solⁿ: Step 1. we first find the differential equation of the given family.

$$y = \frac{x}{1-c_2 x} \quad \text{--- (1)}$$

Differentiating, we obtain

$$\frac{dy}{dx} = \frac{1}{(1-c_2 x)^2} \quad \text{--- (2)}$$

Eliminating the parameter c_2 between equation (1) and (2), we obtain the differential equation of the family (1) in the form

$$\frac{dy}{dx} = \frac{1}{\left[1-\left(1-\frac{x}{y}\right)\right]^2} \text{ or } \frac{dy}{dx} = \frac{y^2}{x^2} \quad \text{--- (3)}$$

Step 2. we now find the differential equation of the orthogonal trajectory by replacing y^2/x^2 in (3) by its negative reciprocal, obtaining

$$\frac{dy}{dx} = -\frac{x^2}{y^2} \quad \text{--- (4)}$$

Step 3. we now solve the differential equation (4).

Seperating variables, we have

$$y^2 dy = -x^2 dx.$$

Integrating, we obtain the one-parameter family of solutions of (1) in the form

$$\frac{1}{3}y^3 + \frac{1}{3}x^3 = c_2$$

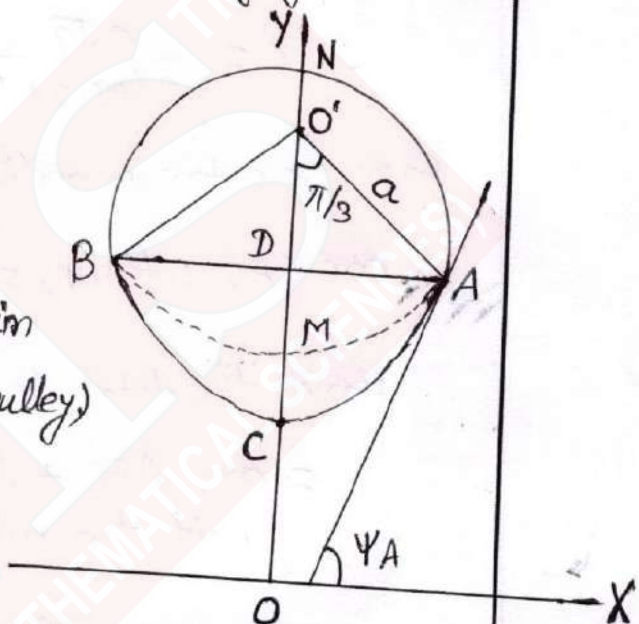
or $x^3 + y^3 = c_1$, where c_2 and c_1 are arbitray constant.

Therefore, the value of n that we want is 3,

$$n = 3.$$

6.(b) → Find the length of an endless chain which will hang over a circular pulley of radius a. so as to be in contact with the two thirds of the circumference of the pulley.

Soln: Let ANBMA be the circular pulley of radius a and ANBCA the endless chain hanging over it.



Since the chain is in contact with the two-thirds of the circumference of the pulley, hence the length of this portion ANB of the chain

$= \frac{2}{3}$(circumference of the pulley)

$= \frac{2}{3}(2\pi a) = \frac{4}{3}\pi a.$

Let the remaining portion of the chain hang in the form of the catenary ACB, with AB horizontal. C is the lowest point i.e., the vertex, CO'N the axis and OX the directrix of this catenary.

Let $OC = c =$ the parameter of the catenary. The tangent at A will be perpendicular to the radius O'A.

∴ If the tangent at A is inclined at an angle ψ_A to the horizontal, then

$$\psi_A = \angle AO'D = \frac{1}{2}(\angle AO'B) = \frac{1}{2}\left(\frac{1}{3}\cdot 2\pi\right) = \frac{1}{3}\pi.$$

From the triangle $AO'D$, we have

$$DA = O'A \sin\tfrac{1}{3}\pi = a\sqrt{3}/2.$$

$\therefore$ from $x = c\log(\tan\psi + \sec\psi)$, for the point A, we have

$$x = DA = c\log(\tan\psi_A + \sec\psi_A)$$

or $$\frac{a\sqrt{3}}{2} = c\log\left(\tan\frac{\pi}{3} + \sec\frac{\pi}{3}\right) = c\log(\sqrt{3}+2).$$

$$\therefore\ c = \frac{a\sqrt{3}}{2\log(2+\sqrt{3})}$$

From $s = c\tan\psi$ applied for the point A, we have

$$\text{arc } CA = c\tan\psi_A = c\tan\tfrac{1}{3}\pi = c\sqrt{3} = \frac{3a}{2\log(2+\sqrt{3})}$$

Hence the total length of the chain

$= \text{arc } ABC +$ length of the chain in contact with the pulley.

$$= 2\cdot(\text{arc } CA) + \frac{4}{3}\pi a$$

$$= 2\cdot\frac{3a}{2\log(2+\sqrt{3})} + \frac{4}{3}\pi a$$

$$= a\left\{\frac{3}{\log(2+\sqrt{3})} + \frac{4\pi}{3}\right\}.$$

6.(c) Find the directional derivative of $\phi = x^2yz + 4xz^2$ at $(1,-2,-1)$ in the direction $2i-j-2K$. in which direction the directional derivative will be maximum and what is its magnitude. Also find a unit normal to the surface $x^2yz + 4xz^2 = 6$ at the point $(1,-2,-1)$. Find the equation of tangent plane and normal at the point $(1,-2,-1)$.

Solⁿ: We have

$$\frac{\partial\phi}{\partial x} = 2xyz + 4z^2, \quad \frac{\partial\phi}{\partial y} = x^2 z, \quad \frac{\partial\phi}{\partial z} = x^2y + 8xz.$$

$$\text{grad}\,\phi = \Sigma i\,\frac{\partial\phi}{\partial x} = (2xyz + 4z^2)i + x^2zj + (x^2y + 8xz)K.$$

or, grad $\phi = 8i - j - 10K$ at the point $(1,-2,-1)$.

Now, if $\hat{a}$ be a unit vector then directional derivative of ϕ along the direction of $\hat{a}$ is $\hat{a}\cdot\text{grad}\,\phi$.

A unit vector in the direction $(2i-j-2K)$ is

$$\frac{2i-j-2K}{\sqrt{(4+1+4)}} = \frac{1}{3}(2i-j-2K)$$

$\therefore$ Directional derivative

$$= \frac{1}{3}(2i-j-2K)\cdot(8i-j-10K)$$

$$= \frac{1}{3}(16+1+20) = \frac{37}{3}.$$

We know that the directional derivative is maximum in the direction of normal which is the direction of grad ϕ.

Hence, directional derivative is maximum along grad $\phi = 8i - j - 10k$. Maximum value of this directional derivative is $|\text{grad}\,\phi|$

$$= \sqrt{(64+1+100)} = \sqrt{(165)}.$$

A unit normal to the surface $= \dfrac{\text{grad}\,\phi}{|\text{grad}\,\phi|}$

$$= \frac{8i - j - 10k}{\sqrt{(165)}}.$$

<u>Tangent plane to the surface at $(1,-2,-1)$.</u>

Let P be any point on the tangent plane at A $(1,-2,-1)$ and the position vectors of P and A be r and r_0.

$$\therefore r = xi + yj + zk \quad \text{and} \quad r_0 = i - 2j - k$$

$$\overrightarrow{AP} = r - r_0 = (x-1)i + (y+2)j + (z+1)k.$$

grad ϕ is normal to the surface and as such it is perpendicular to AP.

$$\therefore \text{grad}\,\phi \cdot \overrightarrow{AP} = 0.$$

$$\Rightarrow (8\hat{i} - j - 10k) \cdot [(x-1)i + (y+2)j + (z+1)k] = 0.$$

$$\Rightarrow 8(x-1) - (y+2) - 10(z+1) = 0$$

$$\Rightarrow 8x - y - 10z = 20.$$

<u>Normal to the surface.</u>

If P be any point on the normal line at A, then $(r - r_0)$ lies along the normal, i.e. along grad ϕ, hence

$$(r - r_0) \times \text{grad}\,\phi = 0$$

$$\Rightarrow \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x-1 & y+2 & z+1 \\ 8 & -1 & -10 \end{vmatrix} = 0$$

$$\Rightarrow \hat{i}\{(z+1)-10(y+2)\}+\hat{j}\{10(x-1)+8(z+1)\}$$
$$-\hat{k}\{(x-1)+8(y+2)\}=0$$

Equating to zero the coefficients of $\hat{i}, \hat{j}$ and $\hat{k}$, we get

$$\frac{z+1}{10}=\frac{y+2}{1};\ \frac{z+1}{10}=\frac{x-1}{-8};\ \frac{x-1}{-8}=\frac{y+2}{1}$$

$$\therefore \quad \frac{x-1}{-8}=\frac{y+2}{1}=\frac{z+1}{10}.$$

<u>Note:</u> It could be written directly as the equation of the line through $(1,-2,-1)$ and with direction ratios as $(8,-1,-10)$.

7.a(i) Solve $(d^4y/dx^4) + 6(d^3y/dx^3) + 11(d^2y/dx^2) + 6(dy/dx) = 20e^{-2x}\sin x.$

Solⁿ: Re-writing the given equation,

$$(D^4+6D^3+11D^2+6D)y = 20e^{-2x}\sin x.$$

Its auxiliary equation is

$$D^4+6D^3+11D^2+6D=0$$

or $D(D+1)(D+2)(D+3)=0$

Solving it we get $D=0,-1,-2,-3$.

$$\therefore C.F. = c_1e^{0\cdot x}+c_2e^{-x}+c_3e^{-2x}+c_4e^{-3x},$$

c_1, c_2, c_3 and c_4 being arbitrary constants.

$$P.I. = \frac{1}{D^4+6D^3+11D^2+6D}20e^{-2x}\sin x$$

$$= 20e^{-2x}\frac{1}{(D-2)^4+6(D-2)^3+11(D-2)^2+6(D-2)}\sin x$$

$$= 20e^{-2x}\frac{1}{D^4+8D^3+24D^2+32D+16+6(D^3-6D^2+12D-8)+11(D^2-4D+4)+6(D-2)}\sin x$$

$$= 20e^{-2x}\frac{1}{D^4-2D^3-D^2+2D}\sin x$$

$$= 20e^{-2x}\frac{1}{(D^2)^2-2D(D^2)-D^2+2D}\sin x$$

$$= 20e^{-2x}\frac{1}{(-1^2)^2-2(-1^2)D-(-1^2)+2D}\sin x$$

$$= 20e^{-2x}\frac{1}{2(1+2D)}\sin x$$

$$= 10e^{-2x}(1-2D)\frac{1}{1-4D^2}\sin x$$

$$= 10e^{-2x}(1-2D)\frac{1}{1-4(-1^2)}\sin x$$

$$= 2e^{-2x}(1-2D)\sin x$$

$$= 2e^{-2x}(\sin x - 2\cos x)$$

$\therefore$ The required Solution is

$$y = c_1 + c_2e^{-x} + c_3e^{-2x} + c_4e^{-3x} - 2e^{-2x}(\sin x - 2\cos x)$$

7.a(ii) Solve by the method of variation of parameters
$x(dy/dx)-y=(x-1)(d^2y/dx^2-x+1)$

Sol^n: Re-writing the given equation, we have
$xy_1-y=(x-1)y_2-(x-1)^2$

or $y_2-\{x/(x-1)\}y_1+\{1/(x-1)\}y=x-1$ ——— (1)

consider $y_2-\{x/(x-1)\}y_1+\{1/(x-1)\}y=0$ ——— (2)

Comparing (2) with $y_2+Py_1+Qy=0$, here

$P=(-x)/(x-1)$ and $Q=1/(x-1)$. Then,

$P+Qx=(-x)/(x-1)+x/(x-1)=0$,

$1+P+Q=1+(-x)/(1-x)+1/(x-1)=0$.

Hence by working rule, we see that x and e^x are integrals of C.F. of (1) or solutions of (2).
Again the wronikian W of x and e^x is given by

$$W=\begin{vmatrix} x & e^x \\ dx/dx & d(e^x)/dx \end{vmatrix}=\begin{vmatrix} x & e^x \\ 1 & e^x \end{vmatrix}$$

$$=e^x(x-1)\neq 0, \quad \text{——— (3)}$$

Showing that x and e^x are linearly indepent solutions of (2).

Hence, the general solution of (2) is $y=ax+be^x$ and therefore C.F. of (1) is $ax+be^x$, a and b being arbitrary constants. Now comparing (1) with $y_2+Py_1+Qy=R$. here $R=x-1$. ——— (4)

Let $u = x$ and $v = e^{x}$ ——— (5)

Then, P.I. of (1) $= uf(x) + vg(x)$. ——— (6)

Where $f(x) = -\int \frac{vR}{W}dx = -\int \frac{e^{x}(x-1)}{e^{x}(x-1)}dx$

$= -\int dx = -x$, using (2), (4) and (5).

and $g(x) = \int \frac{uR}{W}dx = \int \frac{x(x-1)}{e^{x}(x-1)}dx$

$= \int xe^{-x}dx$, by (2), (4) and (5)

$= x(-e^{-x}) - \int 1\cdot(-e^{-x})dx$

$= -xe^{-x} - e^{-x} = -e^{-x}(x+1)$

Substituting the above values of $u, v, f(x)$ and $g(x)$ in (6), we have

P.I. of (1) $= x(-x) + e^{x}\{-e^{-x}(x+1)\}$

$= -(x^2 + x + 1)$

Hence the general solution of (1) is

$y = C.F. + P.I$

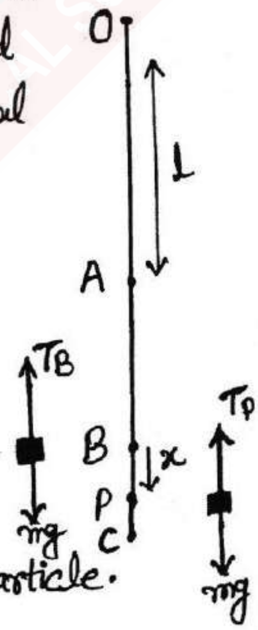
$y = ax + be^{x} - (x^2 + x + 1)$.

7.(b) A light elastic string of natural length l is hung by one end and to the other end are tied successively particles of masses m_1 and m_2. If t_1 and t_2 be the periods and c_1, c_2 the statical extensions corresponding to these two weights, prove that

$$g(t_1^2 - t_2^2) = 4\pi^2(c_1 - c_2)$$

Sol: One end of a string OA of natural length l is attached to a fixed point O. Let B be the position of equilibrium of a particle of mass m attached to the other end of the string. Then AB is the statical extension in the string corresponding to this particle of mass m.

Let $AB = d$.



In the equilibrium position of the particle of mass m at B, the tension $T_B = \lambda(d/l)$ in the string OB balances the weight mg of the particle.

$$\therefore \frac{\lambda d}{l} = mg \quad \text{or} \quad \frac{\lambda}{lm} = \frac{g}{d} \quad \text{——— (1)}$$

Now suppose the particle at B is slightly pulled down upto C and then let go. Let P be the position of the particle at any time t where $BP = x$. When the particle is at P, the tension T_P in the string P is $\lambda \frac{d+x}{l}$, acting vertically upwards.

By Newton's second law of motion, the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2} = -\frac{\lambda(d+x)}{l} + mg,$$

[NOTE that the weight mg of the particle has been taken with the +ive sign because it is acting vertically downwards i.e., in the direction of x-increasing.]

or $$m\frac{d^2x}{dt^2} = -\frac{\lambda d}{l} - \frac{\lambda x}{l} + mg$$

$$= -\frac{\lambda x}{l}, \quad \left[\because \frac{\lambda d}{l} = mg\right]$$

$$\therefore \frac{d^2x}{dt^2} = -\frac{\lambda}{lm}\cdot x = -\frac{g}{d}x, \quad [\text{from } ①]$$

Hence the motion of the particle is simple harmonic about the centre B and its period is $\frac{2\pi}{\sqrt{(g/d)}}$ i.e., $2\pi\sqrt{\left(\frac{d}{g}\right)}$

But according to the question, the period is t_1 when $d = c_1$ and the period is t_2 when $d = c_2$

$$\therefore t_1 = 2\pi\sqrt{(c_1/g)} \text{ and } t_2 = 2\pi\sqrt{(c_2/g)},$$

so that $t_1^2 - t_2^2 = (4\pi^2/g)(c_1 - c_2)$

or $g(t_1^2 - t_2^2) = 4\pi^2(c_1 - c_2)$.

7.(c) Let $F = 2xz\,i - x\,j + y^2 K$. Evaluate $\iiint F\,dV$ where V is the region bounded by the surfaces $x=0$, $y=0$, $y=6$, $z=x^2$, $z=4$.

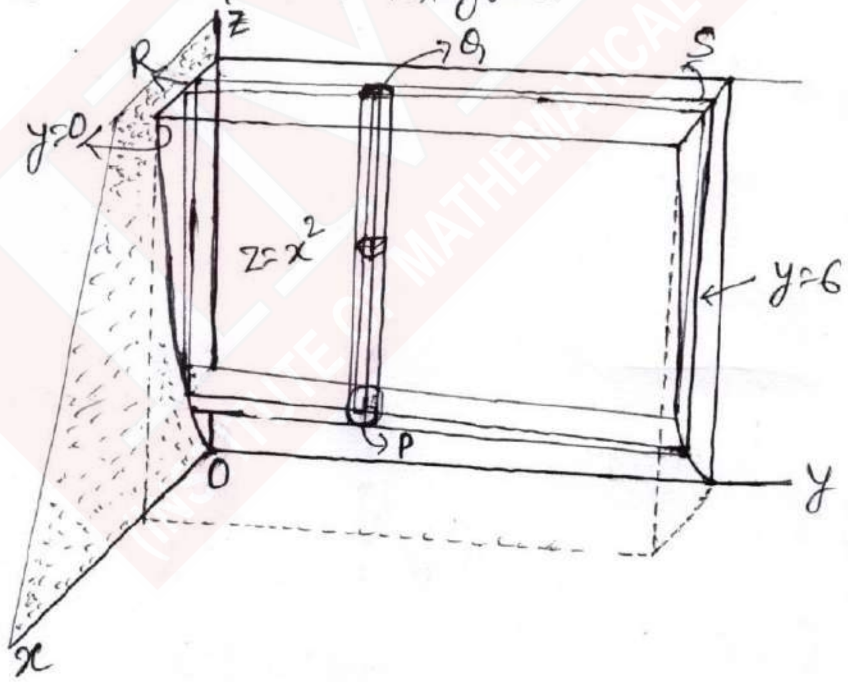
Sol^n: The region V is covered

(a) by keeping x and y fixed and integrating from $z=x^2$ to $z=4$ (base to top of column PQ).

(b) then by keeping x fixed and integrating from $y=0$ to $y=6$ (R to S in the slab).

(c) finally integrating from $x=0$ to $x=2$ (where $z=x^2$ meets $z=4$).

Then the required integral is



$$\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4}(2xz\,i - x\,j + y^2 K)\,dz\,dy\,dx$$

$$= i\int_0^2\int_0^6\int_{x^2}^4 2xz\,dz\,dy\,dx -$$

$$j\int_0^2\int_0^6\int_{x^2}^4 x\,dz\,dy\,dx + K\int_0^2\int_0^6\int_{x^2}^4 y^2\,dz\,dy\,dx.$$

$$= 128i - 24j + 384K.$$

8.(a) → Solve $(D^2+n^2)y = a\sin(nt+\alpha)$, if $y = Dy = 0$ when $t=0$.

Soln: Given that $y''+n^2y = a\sin(nt+\alpha)$

i.e., $y''+n^2y = a(\sin nt\cos\alpha + \cos nt\sin\alpha)$ —— (1)

with initial conditions:

$y(0)=0$ and $y'(0)=0$ —— (2)

Taking Laplace transform of both sides of (1), we have

$L\{y''\}+n^2L\{y\} = a\cos\alpha\, L\{\sin nt\} + a\sin\alpha\, L\{\cos nt\}$

or $s^2L\{y\}-sy(0)-y'(0)+n^2L\{y\} = (an\cos\alpha)/(s^2+n^2) + (as\sin\alpha)/(s^2+n^2)$

or $(s^2+n^2)L\{y\} = (an\cos\alpha + as\sin\alpha)/(s^2+n^2)$, by (2)

or $L\{y\} = (an\cos\alpha)/(s^2+n^2)^2 + (as\sin\alpha)/(s^2+n^2)^2$ —— (3)

Taking inverse Laplace transform of both side of (3), we get

$$y = an\cos\alpha\, L^{-1}\left\{\frac{1}{(s^2+n^2)^2}\right\} + a\sin\alpha\, L^{-1}\left\{\frac{s}{(s^2+n^2)^2}\right\} \quad \text{—— (4)}$$

Now, $L^{-1}\left\{\frac{s}{(s^2+n^2)^2}\right\} = -\frac{1}{2}L^{-1}\left\{\frac{d}{ds}\left(\frac{1}{s^2+n^2}\right)\right\}$

$= -\frac{1}{2}(-1)^1 t\, L^{-1}\left\{\frac{1}{s^2+n^2}\right\}$ by Inverse Laplace transform of derivatives.

Thus, $L^{-1}\{s/(s^2+n^2)^2\} = (t/2n)\sin nt$ —— (5)

Let $f(s) = \frac{1}{(s^2+n^2)}$ and $g(s) = \frac{1}{(s^2+n^2)}$ —— (6)

Then, $F(t) = L^{-1}\{f(s)\} = L^{-1}\{1/(s^2+n^2)\} = (1/n)\sin nt$
and $G(t) = L^{-1}\{g(s)\} = L^{-1}\{1/(s^2+n^2)\} = (1/n)\sin nt$ —— (7)

Now, by the convolution theorem, we have

$$L^{-1}\{f(s)\, g(s)\} = \int_0^t F(u)\, G(t-u)\, du$$

or $$L^{-1}\{1/(s^2+n^2)^2\} = \int_0^t \frac{\sin nu}{n} \cdot \frac{\sin n(t-u)}{n}\, du, \text{ by (6) \& (7)}$$

$$= \frac{1}{2n^2}\int_0^t \left[\cos n(t-2u) - \cos nt\right] du$$

$$= \frac{1}{2n^2}\left[\frac{\sin n(t-2u)}{-2n} - u\cos nt\right]_0^t$$

$$= \frac{1}{2n^2}\left[\frac{\sin nt}{2n} - t\cos nt + \frac{\sin nt}{2n}\right]$$

$$= \frac{1}{2n^2}\left[\frac{\sin nt}{n} - t\cos nt\right] \quad \text{—— (8)}$$

using (5) and (8), (4) reduces to

$$y = an\cos\alpha \cdot \frac{1}{2n^2}\left(\frac{\sin nt}{n} - t\cos nt\right) + a\sin\alpha \cdot \frac{1}{2n} t\sin nt.$$

or $y = (a/2n^2)\cos\alpha \sin nt - (at/2n)(\cos\alpha\cos nt - \sin\alpha \sin nt)$

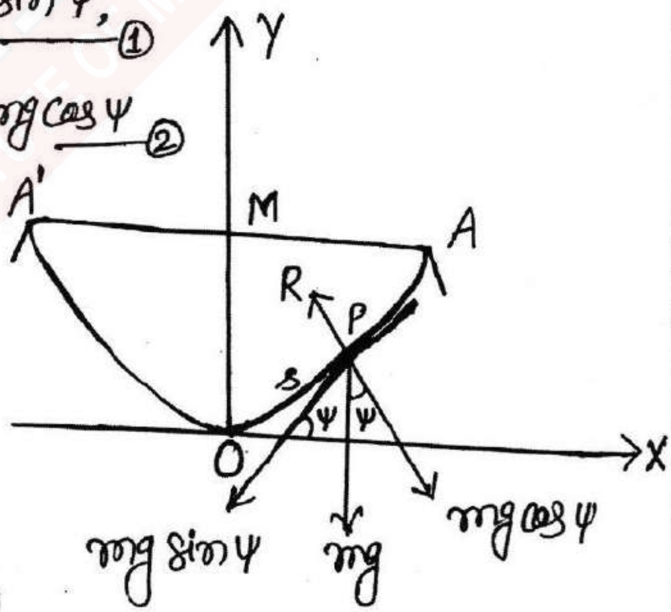
or $y = (a/2n^2)[\cos\alpha \sin nt - nt\cos(\alpha + nt)]$.

8.(b) A particle slides down the arc of a smooth cycloid whose axis is vertical and vertex lowest, starting at rest from the cusp. Prove that the time occupied in falling down the first half of the vertical height is equal to the time of falling down the second half.

Sol: Let a particle start from rest from the cusp A of the cycloid. The velocity v of the particle at any point P, at time t, the equations of motion along the tangent and normal are

$$m\frac{d^2s}{dt^2} = -mg\sin\psi, \quad \text{--- (1)}$$

and $m\frac{v^2}{\rho} = R - mg\cos\psi$ --- (2)



for the cycloid, $s = 4a\sin\psi$ --- (3)

From ① and ③, we have

$$\frac{d^2s}{dt^2} = -\frac{g}{4a}s.$$

Multiplying both sides by $2(ds/dt)$ and then Integrating, we have

$$v^2 = \left(\frac{ds}{dt}\right)^2 = -\frac{g}{4a}s^2 + A.$$

Initially at the cusp A, $s = 4a$ and $ds/dt = 0$.

$$\therefore A = \frac{g}{4a}\cdot(4a)^2 = 4ag.$$

$$\therefore v^2 = \left(\frac{ds}{dt}\right)^2 = -\frac{g}{4a}s^2 + 4ag$$

$$= \frac{g}{4a}(16a^2 - s^2),$$

or $\frac{ds}{dt} = -\frac{1}{2}\sqrt{(g/a)}\cdot\sqrt{(16a^2 - s^2)}$,

the negative sign is taken because the particle is moving in the direction of s decreasing.

Seperating the variables, we have

$$\therefore dt = -2\sqrt{(a/g)}\cdot\frac{ds}{\sqrt{(16a^2 - s^2)}} \quad \text{——— ④}$$

The vertical height of the cycloid is $2a$. At the point where the particle has fallen down the first half of the vertical height of the cycloid, we have $y=a$. putting $y=a$ in the equation $s^2=8ay$, we get $s^2=8a^2$ or $s=2\sqrt{2}a$.

$\therefore$ Integrating (4) from $s=4a$ to $s=2\sqrt{2}a$, the time t_1 taken in falling down the first half of the vertical height of the cycloid is given by

$$t_1=-2\sqrt{(a/g)}\int_{s=4a}^{2\sqrt{2}a}\frac{ds}{\sqrt{(16a^2-s^2)}}$$

$$=2\sqrt{(a/g)}\left[\cos^{-1}(s/4a)\right]_{4a}^{2\sqrt{2}a}$$

$$=2\sqrt{(a/g)}\left[\cos^{-1}\frac{2\sqrt{2}a}{4a}-\cos^{-1}1\right]$$

$$=2\sqrt{(a/g)}\left[\cos^{-1}\frac{1}{\sqrt{2}}-\cos^{-1}1\right]$$

$$=2\sqrt{(a/g)}\left[\frac{1}{4}\pi-0\right]$$

$$=\frac{1}{2}\pi\sqrt{(a/g)}.$$

Again Integrating ④ from $s = 2\sqrt{2}a$ to $s = 0$, the time t_2 taken in falling down the second half of the vertical height of the cycloid is given by

$$t_2 = -2\sqrt{(a/g)} \int_{s=2\sqrt{2}a}^{0} \frac{ds}{\sqrt{(16a^2 - s^2)}}$$

$$= 2\sqrt{(a/g)} \cdot \left[\cos^{-1}\left(\frac{s}{4a}\right)\right]_{2\sqrt{2}a}^{0}$$

$$= 2\sqrt{(a/g)} \cdot \left[\cos^{-1} 0 - \cos^{-1}\frac{1}{\sqrt{2}}\right]$$

$$= 2\sqrt{(a/g)}\left[\frac{1}{2}\pi - \frac{1}{4}\pi\right]$$

$$= \frac{1}{2}\pi\sqrt{(a/g)}.$$

Hence $t_1 = t_2$ i.e., the time occupied in falling down the first half of the vertical height is equal to the time of falling down the second half.

8.(c) → Applying stoke's theorem to prove that

$$\int_C (y\,dx + z\,dy + x\,dz) = -2\sqrt{2}\pi a^2,$$

where 'C' is the curve given by

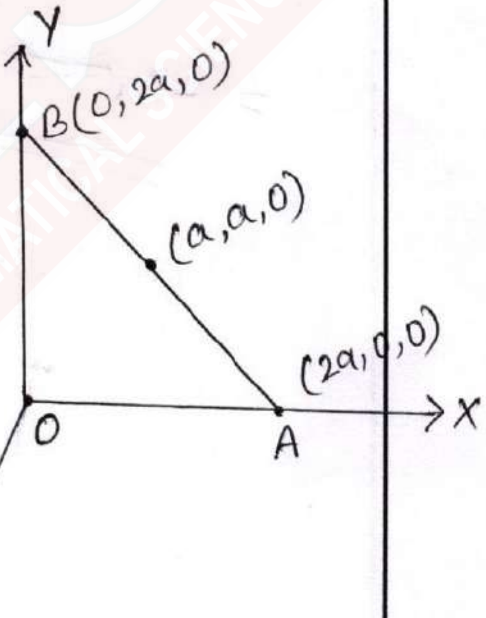
$$x^2+y^2+z^2-2ax+2ay=0,\quad x+y=2a.$$

and begins at the point $(2a,0,0)$ and goes at first below the z-plane.

Sol^n: The centre of the sphere $x^2+y^2+z^2-2ax-2ay=0$ is the point $(a,a,0)$, therefore the circle C is great circle of this sphere.

∴ Radius of the circle C

= radius of the sphere = $\sqrt{(a^2+a^2)} = a\sqrt{2}$



Now $\int_C (y\,dx + z\,dy + x\,dz)$

$$= \int_C (y\mathbf{i} + z\mathbf{j} + x\mathbf{k})\cdot d\mathbf{r}$$

$$= \iint_S [\text{curl}(y\mathbf{i}+z\mathbf{j}+x\mathbf{k})]\cdot \mathbf{n}\,dS,$$

where S is any surface of which circle C is boundary.

Now $\text{curl}(y\mathbf{i}+z\mathbf{j}+x\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & z & x \end{vmatrix}$

$$= -\mathbf{i}-\mathbf{j}-\mathbf{k} = -(\mathbf{i}+\mathbf{j}+\mathbf{k}).$$

Let us take S as the surface of the plane $x+y=2a$ bounded by the circle C. Then a vector normal to S is $\text{grad}(x+y) = i+j$.

$\therefore$ n = unit normal to $S = \frac{1}{\sqrt{2}}(i+j)$

$\therefore \int_C (y\,dx + z\,dy + x\,dz)$

$= \iint_S -(i+j+k)\cdot\left(\frac{1}{\sqrt{2}}i + \frac{1}{\sqrt{2}}j\right)ds$

$= -\frac{2}{\sqrt{2}}\iint_S ds = -\frac{2}{\sqrt{2}}$ (area of the circle of radius $a\sqrt{2}$)

$= -2\sqrt{2}\pi a^2$