1.(a) Let H be a subgroup of group G. Then $W = \bigcap_{g \in G} gHg^{-1}$ is a normal subgroup of G.

Sol'n: Let $a = ghg^{-1}$ and $b = gh_1g^{-1}$ be two elements of K. Then $\left[gHg^{-1} = \{ghg^{-1} / h \in H\}\right]$

$$ab^{-1} = ghg^{-1}(gh_1g^{-1})^{-1}$$
$$= ghg^{-1}((g^{-1})^{-1}h_1^{-1}g^{-1})$$
$$= ghg^{-1}gh_1^{-1}g^{-1}$$
$$= ghh_1^{-1}g^{-1}$$

Now $h, h_1 \in H$ and H is a subgroup of G. Hence $hh_1^{-1} \in H$. Then from (*) above

$$ab^{-1} = g(hh_1^{-1})g^{-1} \in gHg^{-1}$$

Hence gHg^{-1} is a subgroup of G for all $g \in G$.

Since the intersection of subgroups is a subgroup, W is a subgroup G. Let $x \in G, w \in W$. Then $w \in gHg^{-1}\ \forall g \in G$. We have to show that $xwx^{-1} \in gHg^{-1}\ \forall g \in G$, which in turn will yield that $xwx^{-1} \in W$.

Let $g \in G$ and let us suppose that $xwx^{-1} \in gHg^{-1}$.

Then $xwx^{-1} = ghg^{-1}$ for some $h \in H$.

thus $g^{-1}xwx^{-1}g = h \in H$.

$$\Rightarrow (g^{-1}x)w(g^{-1}x)^{-1} \in H$$

put $y = x^{-1}g$. Then $g = xy$. Hence in order to show that $xwx^{-1} \in gHg^{-1}$ for a given $g \in G$,

first we need to find $y \in G$ such that $g = xy$.

Since $g = x(x^{-1}g)$,

we can choose $y = x^{-1}g$.

So there exists $y \in G$ such that $g = xy$.

Since $y \in G$, we have $w \in yHy^{-1}$ and $w = yhy^{-1}$ for some $h \in H$.

Hence $xwx^{-1} = x(yhy^{-1})x^{-1}$

$$= xyhy^{-1}x^{-1}$$

$$= (xy)h(xy)^{-1}$$

$$= ghg^{-1} \in gHg^{-1}$$

Since $g \in G$ was arbitrary,

$$xwx^{-1} \in gHg^{-1} \text{ for all } g \in G.$$

Thus, W is a normal subgroup of G.

1.(b) Let $R = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} / a, b \in Z \right\}$ and let ϕ be the mapping that takes $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ to $a-b$.

(i) Show that ϕ is a homomorphism.

(ii) Determine the kernel of ϕ.

(iii) Show that $R/\ker\phi$ is isomorphic to Z.

Soln: Let $\phi : R \to Z$ such that

$$\phi \begin{bmatrix} a & b \\ b & a \end{bmatrix} = a-b.$$

We have

$$\phi\left(\begin{bmatrix} a & b \\ b & a \end{bmatrix} + \begin{bmatrix} a_1 & b_1 \\ b_1 & a_1 \end{bmatrix}\right) = \phi \begin{bmatrix} a+a_1 & b+b_1 \\ b+b_1 & a+a_1 \end{bmatrix}$$

$$= (a+a_1) - (b+b_1)$$

$$= (a-b) + (a_1 - b_1)$$

$$= \phi\begin{pmatrix} a & b \\ b & a \end{pmatrix} + \phi\begin{pmatrix} a_1 & b_1 \\ b_1 & a_1 \end{pmatrix}$$

$\therefore \phi$ is homomorphism.

Now

$$\ker\phi = \left\{ \begin{pmatrix} a & b \\ b & a \end{pmatrix} / \phi\begin{pmatrix} a & b \\ b & a \end{pmatrix} = 0 \right\}$$

since $\phi\begin{pmatrix} a & b \\ b & a \end{pmatrix} = a-b$

$\Rightarrow a-b = 0$

$\Rightarrow \boxed{a=b}$

$$\therefore \ker\phi = \left\{ \begin{pmatrix} a & a \\ a & a \end{pmatrix} / a \in Z \right\}.$$

for each $a \in Z$ $\exists$ $A \in R$ such that

$$\phi(A) = a$$

$\therefore \phi$ is onto.

$\therefore$ By Fundamental theorem of Homomorphism.

$$\frac{R}{\ker\phi} \cong Z.$$

1(c) Let $f(x) = \frac{n}{n+2}$ if $\frac{1}{n+2} \le x \le \frac{1}{n}$, where $n = 1, 2, 3 \ldots$ and $f(0) = 0$. Prove that f is Riemann integrable in $[0,1]$

Sol'n :
$$f(x) = \begin{cases} \frac{n}{n+2} & , \frac{1}{n+2} \le x \le \frac{1}{n}, \ n = 1, 2 \ldots \\ 0 & , x = 0 \end{cases}$$

$$= \tfrac{1}{3} \ ; \ \tfrac{1}{3} \le x \le 1$$

$$= \tfrac{2}{4} \ ; \ \tfrac{1}{4} \le x \le \tfrac{1}{2}$$

$$= 3/5 \ ; \ \tfrac{1}{5} \le x \le \tfrac{1}{3}$$

$$= \tfrac{4}{6} \ ; \ \tfrac{1}{6} \le x \le \tfrac{1}{4}$$

$$\vdots$$

$$= \frac{n-1}{n+1} \ ; \ \frac{1}{n+1} \le x \le \frac{1}{n-1}$$

$$= 0 \ , x = 0.$$

$\Rightarrow f(x)$ is bounded and continuous on $[0,1]$ except at the points $\tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{5}, \tfrac{1}{6} \ldots$

The set of points of discontinuity of f on $[0,1]$ is $\{\tfrac{1}{3}, \tfrac{1}{4}, \tfrac{1}{5}, \tfrac{1}{6} \ldots\}$ which has one limit point '0'.

The set of points of discontinuity of f on $[0,1]$ has a finite no. of limit points.

$\therefore f$ is integrable on $[0,1]$

1.(d) Use Cauchy's theorem and/or Cauchy integral formula to evaluate the following integrals (i) $\int_{|z|=4} \frac{z^4}{(z-1)^3}\,dz$ (ii) $\int_{|z-1-i|=5/4} \frac{z^{1/2}}{z-1}\,dz$.

Sol^n: (i) Given that $\int_{|z|=4} \frac{z^4}{(z-i)^3}\,dz$.

Here $f(z)=z^4$ is analytic in $|z|=4$.

Comparing the given integral with $\int \frac{f(z)}{(z-z_0)^n}\,dz$

since $f(z)=z^4$ is analytic in $|z|=4$.

$\therefore$ we can apply the Cauchy's integral formula

$$\int_{|z|=4} \frac{f(z)}{(z-z_0)^3}\,dz = \frac{2\pi i}{2!} f''(z_0)$$

$$f(z)=z^4 \Rightarrow f'(z)=4z^3$$
$$f''(z)=12z^2$$

$$\int \frac{z^4}{(z-i)^3}\,dz = \frac{2\pi i}{2!} f''(z_0)$$
$$= \frac{2\pi i}{2!}(12)(i)^2 \quad (\because z_0=i)$$
$$= -12\pi i.$$

(ii) Given
$$\int_{|z-1-i|=5/4} \frac{z^{1/2}}{z-1}\,dz.$$

We have $|z-1-i|=\frac{5}{4}$ is a circle with centre at $z=1+i$ and radius $5/4$.

i.e, at the centre $(1,1)$.

Also $z-1=0$
$\Rightarrow z=1$
$\Rightarrow z=(1,0)$ $\quad z=1$ lies inside C.

Let $f(z)=z^{1/2}$ which is clearly analytic at every point within and on C.

$$\therefore \int \frac{f(z)}{(z-1)}\,dz = \int \frac{f(z)}{z-z_0}\,dz, \text{ where } z_0=1.$$
$$= 2\pi i f(z_0) \quad \text{[By Cauchy's integral formula]}$$
$$= 2\pi i (1)^{1/2}$$
$$= 2\pi i.$$

1 (e) Give the dual of the LP problem: $\min z = 2x_1 + 3x_2 + 4x_3$

Subject to the constraints: $2x_1 + 3x_2 + 5x_3 \geq 2$, $3x_1 + x_2 + 7x_3 = 3$, $x_1 + 4x_2 + 6x_3 \leq 5$, $x_1, x_2 \geq 0$ and x_3 is unrestricted.

Soln: Since the variable x_3 is unrestricted in sign, the given LP problem can be transformed into standard primal form by substituting $x_3 = x_3' - x_3''$, where $x_3' \geq 0$, $x_3'' \geq 0$. Therefore standard primal becomes:

$$\max z_x' = -2x_1 - 3x_2 - 4(x_3' - x_3'')$$

Subjected to constraints:

$$-2x_1 - 3x_2 - 5(x_3' - x_3'') \leq -2$$

$$3x_1 + x_2 + 7(x_3' - x_3'') \leq 3$$

$$-3x_1 - x_2 - 7(x_3' - x_3'') \leq -3$$

$$x_1 + 4x_2 + 6(x_3' - x_3'') \leq 5$$

$$x_1, x_2, x_3', x_3'' \geq 0.$$

The dual of the given standard primal is,

$$\min z_w' = -2w_1 + 3(w_2' - w_2'') + 5w_3$$

Subject to the constraints:

$$-2w_1 + 3(w_2' - w_2'') + w_3 \geq -2$$

$$-3w_1 + (w_2' - w_2'') + 4w_3 \geq -3$$

$$-5w_1 + 7(w_2' - w_2'') + 6w_3 \geq -4$$

$$5w_1 - 7(w_2' - w_2'') - 6w_3 \geq 4.$$

$$w_1, w_2', w_2'', w_3 \geq 0$$

Again we may write.

$$\min z_w' = -2w_1 + 3w_2 + 5w_3$$

$$-2w_1 + 3w_2 + w_3 \geq -2$$

$$-3w_1 + w_2 + 4w_3 \geq -3$$

$$5w_1 - 7w_2 - 6w_3 = 4$$

$w_1, w_3 \geq 0$ and w_2 is unrestricted.

2.(a) (i) Let M be the set of all 3X3 matrices of the following form: $\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ b & c & a \end{bmatrix}$

where $a, b, c \in Z_2$. Show that with standard matrix addition and multiplication (over Z_2), M is a commutative ring. Find all the idempotent elements of M.

<u>Solution:-</u> Given Matrix $M = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ b & c & a \end{bmatrix}$; $a, b, c \in Z_2$

Let $A_1 = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_1 & 0 \\ b_1 & c_1 & a_1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} a_2 & 0 & 0 \\ 0 & a_2 & 0 \\ b_2 & c_2 & a_2 \end{bmatrix}$ $a_2, b_2, c_2 \in Z_2$

$a_1, b_1, c_1 \in Z_2$

<u>For Matrix Addition</u>

$$A_1 + A_2 = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_1 & 0 \\ b_1 & c_1 & a_1 \end{bmatrix} + \begin{bmatrix} a_2 & 0 & 0 \\ 0 & a_2 & 0 \\ b_2 & c_2 & a_2 \end{bmatrix} = \begin{bmatrix} a_1+a_2 & 0 & 0 \\ 0 & a_1+a_2 & 0 \\ b_1+b_2 & c_1+c_2 & a_1+a_2 \end{bmatrix}$$

$$A_2 + A_1 = \begin{bmatrix} a_2 & 0 & 0 \\ 0 & a_2 & 0 \\ b_2 & c_2 & a_2 \end{bmatrix} + \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_1 & 0 \\ b_1 & c_1 & a_1 \end{bmatrix} = \begin{bmatrix} a_1+a_2 & 0 & 0 \\ 0 & a_1+a_2 & 0 \\ b_1+b_2 & c_1+c_2 & a_1+a_2 \end{bmatrix}$$

Since $a_1 + a_2 = a_2 + a_1$, $b_1 + b_2 = b_2 + b_1$, $c_1 + c_2 = c_2 + c_1$

as $a_1, a_2, b_1, b_2, c_1, c_2 \in Z_2$

and addition of two is always commutative in Z_2.

Hence, M is commutative in standard matrix addition.

<u>For Matrix Multiplication</u>

$$A_1 \cdot A_2 = \begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_1 & 0 \\ b_1 & c_1 & a_1 \end{bmatrix} \begin{bmatrix} a_2 & 0 & 0 \\ 0 & a_2 & 0 \\ b_2 & c_2 & a_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 & 0 & 0 \\ 0 & a_1a_2 & 0 \\ b_1a_2+a_1b_2 & c_1a_2+a_1c_2 & a_1a_2 \end{bmatrix}$$

M is closed under matrix Multiplication.

$$A_2 \cdot A_1 = \begin{bmatrix} a_2 & 0 & 0 \\ 0 & a_2 & 0 \\ b_2 & c_2 & a_2 \end{bmatrix}\begin{bmatrix} a_1 & 0 & 0 \\ 0 & a_1 & 0 \\ b_1 & c_1 & a_1 \end{bmatrix} = \begin{bmatrix} a_2a_1 & 0 & 0 \\ 0 & a_2a_1 & 0 \\ b_2a_1+a_2b_1 & c_2a_1+a_2c_1 & a_2a_1 \end{bmatrix}$$

$$= \begin{bmatrix} a_1a_2 & 0 & 0 \\ 0 & a_1a_2 & 0 \\ a_1b_2+b_1a_2 & a_1c_2+c_1a_2 & a_1a_2 \end{bmatrix}$$

as multiplication is always commutative in Z_2.

therefore; $\boxed{A_1A_2 = A_2A_1}$

Hence, M is commutative

Idempotent means $\Rightarrow \boxed{M^2 = \quad I}$

$$M^2 = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ b & c & a \end{bmatrix}\begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ b & c & a \end{bmatrix} = \begin{bmatrix} a^2 & 0 & 0 \\ 0 & a^2 & 0 \\ 2ab & 2ac^2 & a^2 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

i.e $a^2 = 1$, $2ab = 0 = 2ac$ i.e $b = c = 0$
$a = \pm 1$.

Hence; Idempotent elements of M are.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \& \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

2.(c)(i). The function f defined by

$$f(z) = u+iv = \begin{cases} \dfrac{Im(z^2)}{\bar{z}} & \text{if } z\neq 0 \\ 0 & \text{if } z=0 \end{cases}$$

Satisfies the C-R equations at the origin, yet it is not differentiable there.

Sol'n: The function f defined by

$$f(z) = u+iv = \begin{cases} \dfrac{Im(z^2)}{\bar{z}} & ;\ z\neq 0 \\ 0 & ;\ z=0 \end{cases}$$

$$= \begin{cases} \dfrac{2xy}{x-iy} & ;\ z\neq 0 \\ 0 & ;\ z=0 \end{cases}$$

$z^2=(x+iy)^2$
$=x^2-y^2+2ixy$
$Im(z^2)=2xy$

$$= \begin{cases} \dfrac{2xy(x-iy)}{x^2+y^2} & ;\ z\neq 0 \\ 0 & ;\ z=0 \end{cases}$$

Here $u=\dfrac{2x^2y}{x^2+y^2}$; $v=\dfrac{2xy^2}{x^2+y^2}$ where $x\neq 0, y\neq 0$.

To show that Cauchy - Riemann equations are satisfied at $z=0$:

Since $f(0)=0$

$\Rightarrow u(0,0)+iv(0,0)=0$

$\Rightarrow u(0,0)=v(0,0)=0$

Now $\left(\dfrac{\partial u}{\partial x}\right)_{(0,0)} = \underset{y\to 0}{Lt}\ \dfrac{u(x,0)-u(0,0)}{x} = \underset{x\to 0}{Lt}\ \dfrac{0-0}{x}=0$

$\left(\dfrac{\partial u}{\partial y}\right)_{(0,0)} = \underset{y\to 0}{Lt}\ \dfrac{u(0,y)-u(0,0)}{y} = \underset{y\to 0}{Lt}\ \dfrac{0-0}{y}=0$

Similarly $\left(\frac{\partial v}{\partial x}\right)_{(0,0)} = 0$ and $\left(\frac{\partial v}{\partial y}\right)_{(0,0)} = 0$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$ at $z=0$

$\therefore$ Cauchy – Riemann equations are satisfied.

To Prove that $f(z)$ docsnot differentiable at $(0,0)$.

$$f'(0) = \underset{z\to 0}{Lt} \frac{f(z)-f(0)}{z}$$

$$= \underset{(x,y)\to 0}{Lt} \frac{2xy\,(x+iy)}{(x^2+y^2)(x+iy)}$$

Let $(x,y) \to (0,0)$ along $y=mx$

$$f'(0) = \underset{x\to 0}{Lt} \frac{2x^2m}{x^2+m^2x^2}$$

$$= \underset{x\to 0}{Lt} \frac{2m}{1+m^2}$$

Clearly which depends on m.

$\therefore f'(0)$ does not exist.

$\therefore f(z)$ does not differentiable at $(0,0)$

$\therefore$ The given statement is true i.e. the given function f satisfies C-R equations. although it is not differentiable at $(0,0)$

2(c)(ii) The integral function $f(z)$ satisfies everywhere the inequality $|f(z)| \leq A|z|^k$ where A and k are positive constants. Prove that $f(z)$ is a polynomial of degree not exceeding k.

Sol'n: Since $f(z)$ is analytic in the finite part of the plane, therefore by Taylor's theorem.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad , \text{ where } |z| < R$$

Now, if $\max |f(z)| = M(r)$ on the circle $|z| = r \, (r < R)$, Then by Cauchy's inequality, we have

$$|a_n| \leq \frac{M(r)}{r^n} \text{ for all } n$$

$$= \frac{A(|z|^k)}{r^n} \quad \text{Since } M(r) = |f(z)| = A|z|^k \text{ when } |z| \to \infty.$$

$$= \frac{Ar^k}{r^n} = Ar^{k-n}$$

Hence as $r \to \infty$, the right handside tends to zero.

Since $n > k$.

i.e. $a_n = 0$ for $n > k$.

i.e. all the coefficients a_n for which $n > k$ becomes zero.

$\therefore f(z) = a_0 + a_1 z + a_2 z^2 + \cdots + a_k z^k$

which is a polynomial of degree k.

3(a) (i) If in a ring R, with unity $(xy)^2 = x^2y^2$ for all $x, y \in R$ then show that R is commutative.

(ii) show that the ring R of real valued continuous functions on $[0,1]$ has zero divisors.

Sol'n: (i) we have $(xy)^2 = x^2y^2 \quad \forall\, x, y \in R$ —— (1)

Replacing y by $y+1 \in R$ in (1), we get

$$[x(y+1)]^2 = x^2(y+1)^2$$

$$\Rightarrow (xy+x)^2 = x^2(y^2+2y+1)$$

$$\Rightarrow (xy+x)(xy+x) = x^2(y^2+2y+1)$$

$$\Rightarrow (xy)^2 + (xy)x + x(xy) + x^2 = x^2y^2 + 2x^2y + x^2 \quad \text{—— (2)}$$

$$\Rightarrow (xy)^2 + (xy)x + x(xy) + x^2 = (xy)^2 + 2x^2y + x^2$$

$$\Rightarrow (xy)x + x(xy) = 2x^2y \quad (\because \text{LCL \& RCL in } (R,+))$$

$$\Rightarrow xyx + x^2y = 2x^2y$$

$$\Rightarrow xyx = x^2y \quad \forall\, x, y \in R \quad (\text{RCL}) \text{ —— (3)}$$

Replacing x by $x+1 \in R$ in (3),

$$(x+1)y(x+1) = (x+1)^2y$$

$$\Rightarrow (x+1)(yx+y) = (x+1)(xy+y)$$

$$\Rightarrow xyx + xy + yx + y = x^2y + xy + xy + y$$

$$\Rightarrow yx = xy \quad \forall\, x, y \in R \quad (\because \text{LCL \& RCL in } (R,+))$$

$\therefore$ R is a commutative ring.

(ii) Consider the functions f and g defined on $[0,1]$ by

$$f(x) = \tfrac{1}{2} - x, \quad 0 \le x \le \tfrac{1}{2}$$

$= 0 \quad , \frac{1}{2} \leq x \leq 1$

and $g(x) = 0 \; , \quad 0 \leq x \leq \frac{1}{2}$

$= x - \frac{1}{2} \; , \; \frac{1}{2} \leq x \leq 1$

then f and g are continuous functions and

$f \neq 0, \; g \neq 0.$

whereas $gf(x) = g(x) f(x) = 0.\left(\frac{1}{2} - x\right)$ if $0 \leq x \leq \frac{1}{2}$

$= \left(x - \frac{1}{2}\right).0 = 0$ if $\frac{1}{2} \leq x \leq 1$

i.e. $gf(x) = 0$ for all x

i.e. $gf = 0$ but $f \neq 0, g \neq 0$.

3(b) Let $f_n(x) = \frac{x}{1+nx^2}$ for all real x. Show that f_n converges uniformly to a function f. What is f? Show that for $x \neq 0$, $f_n'(x) \to f'(x)$ but $f_n'(0)$ does not converge to $f'(0)$. Show that the maximum value $|f_n(x)|$ can take is $\frac{1}{2\sqrt{n}}$.

Sol^n: Here $f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x}{1+nx^2} = 0 \;\forall x \in R$

$$|f_n(x) - f(x)| = \left|\frac{x}{1+nx^2} - 0\right| = \left|\frac{x}{1+nx^2}\right|$$

Let $y = \frac{x}{1+nx^2}$

then $$\frac{dy}{dx} = \frac{(1+nx^2)(1) - x\cdot 2nx}{(1+nx^2)^2} = \frac{1-nx^2}{(1+nx^2)^2}$$

For max. (or) min $\frac{dy}{dx} = 0$

$$\Rightarrow 1-nx^2 = 0 \Rightarrow x = \frac{1}{\sqrt{n}}$$

Also $$\frac{d^2y}{dx^2} = \frac{(1+nx^2)^2(-2nx) - (1-nx^2)\cdot 2(1+nx^2)\cdot 2nx}{(1+nx^2)^4}$$

$$= \frac{-2nx(1+nx^2) - 4nx(1-nx^2)}{(1+nx^2)^3}$$

$$\left.\frac{d^2y}{dx^2}\right]_{x=\frac{1}{\sqrt{n}}} = \frac{-2\sqrt{n}(1+1)}{(1+1)^3} = -\frac{\sqrt{n}}{2} < 0.$$

$\Rightarrow y$ is maximum when $x = \frac{1}{\sqrt{n}}$ and maximum value of y

$$= \frac{1/\sqrt{n}}{1+1} = \frac{1}{2\sqrt{n}}$$

$$\therefore M_n = \max_{x\in[a,b]} |f_n(x)-f(x)| = \max_{x\in[a,b]}\left|\frac{x}{1+nx^2}\right|$$

$$= \frac{1}{2\sqrt{n}} \to 0 \text{ as } n\to\infty$$

Hence $\langle f_n\rangle$ converges uniformly to f on $[a,b]$.

$$\Rightarrow f(x) = 0 \quad \forall\, x\in R$$

$$\Rightarrow f'(x) = 0 \quad \forall\, x\in R$$

when $x\neq 0$,

$$f_n'(x) = \frac{(1+nx^2)\cdot(1-x)\cdot 2nx}{(1+nx^2)^2} = \frac{1-nx^2}{(1+nx^2)^2}$$

$$\lim_{n\to\infty} f_n'(x) = \lim_{n\to\infty}\frac{1-nx^2}{(1+nx^2)^2} \qquad \Big|\text{ form } \frac{\infty}{\infty}$$

$$= \lim_{n\to\infty}\frac{-x^2}{2(1+nx^2)\cdot x^2} = 0 = f'(x)$$

So that if $x\neq 0$, the formula $\lim_{n\to\infty} f_n'(x) = f'(x)$ is true.

At $x=0$,

$$f_n'(0) = \lim_{h\to 0}\frac{f_n(0+h)-f_n(0)}{h}$$

$$= \lim_{h\to 0}\frac{\frac{h}{1+nh^2}-0}{h} = \lim_{h\to 0}\frac{1}{1+nh^2} = 1$$

So that $\lim_{n\to\infty} f_n'(0) = 1 \neq f'(0)$.

Hence at $x=0$, the formula $\lim_{n\to\infty} f_n'(x) = f'(x)$ is false.

3(c) Using the simplex method solve the LPP problem:

Minimize $Z = x_1 + x_2$

subject to

$2x_1 + x_2 \geq 4$

$x_1 + 7x_2 \geq 7$

$x_1, x_2 \geq 0.$

Sol? The objective function of the given LPP is of minimization type.

So, we convert it into maximization type.

$\text{Max}\, Z' = \text{Min}(-Z)$

$= -x_1 - x_2$

Now we write the given LPP in the standard form $\text{Max}\, Z' = -x_1 - x_2 + 0S_1 + 0S_2 - MA_1 - MA_2$

Subject to

$2x_1 + x_2 - S_1 + A_1 = 4$

$x_1 + 7x_2 - S_2 + A_2 = 7$

$A_1, A_2, x_1, x_2, S_1, S_2 \geq 0.$

where S_1, S_2 are the surplus variables

A_1, A_2 are the artificial variables.

Now the IBFS is

$S_1 = S_2 = x_1 = x_2 = 0$ (Non-basic)

$A_1 = 4, A_2 = 7$ (Basic)

Thus the initial simplex table is

C_B	C_j / Basis	-1 x_1	-1 x_2	0 S_1	0 S_2	-M A_1	-M A_2	b	θ
-M	A_1	2	1	-1	0	1	0	4	4
-M	A_2	1	(7)	0	-1	0	1	7	1 →
	$Z_j = \Sigma C_B a_{ij}$	-3M	-8M	M	M	-M	-M	-11M	
	$C_j - Z_j$	-1+3M	-1+8M ↑	-M	-M	0	0		

From the above table,

the variable x_2 is entering variable, A_2 is the outgoing variable and omit column for this variable in the next simplex table. Here (7) is the key element and convert it into unity and all other elements in its column to zero.

Then the new simplex table is:

	C_j	-1	-1	0	0	$-M$		
C_B	Basis	x_1	x_2	s_1	s_2	A_1	b	θ
$-M$	A_1	$(13/7)$	0	-1	$1/7$	1	3	$\frac{21}{13}\longrightarrow$
-1	x_2	$1/7$	1	0	$-1/7$	0	1	7
$z_j=\Sigma a_{ij}c_B$		$-\frac{13M}{7}-\frac{1}{7}$	-1	M	$-\frac{M}{7}+\frac{1}{7}$	$-M$	$-3M-1$	
$C_j=c_j-z_j$		$\frac{13M}{7}-\frac{6}{7}$ ↑	0	$-M$	$\frac{M}{7}-\frac{1}{7}$	0		

From the above table,
x_1 is the entering variable, A_1 is the outgoing variable and omit its column in the next simplex table. Here $(13/7)$ is the key element and make it unity and all other elements in its column equal to zero. Then the revised simplex table is

	c_j	-1	-1	0	0		
C_B	Basis	x_1	x_2	s_1	s_2	b	θ
-1	x_1	1	0	$-7/13$	$\frac{1}{13}$	$21/13$	
-1	x_2	0	1	$1/13$	$-2/13$	$10/13$	
$z_j=\Sigma a_{ij}c_B$		-1	-1	$6/13$	$1/13$	$-31/13$	
$C_j=c_j-z_j$		0	0	$-6/13$	$-1/13$		

From the above table, all C_j's ≤ 0.
There remains no artificial variable in the basis.

$\therefore$ The solution is an optimal BFS to the problem and is given by

$x_1 = 21/13$, $x_2 = 10/13$

$\therefore$ Max $z' = -31/13$

Hence the optimal value of the objective function is Min $Z = -$Max z'
$= 31/13$

4(a) Let H be a subgroup of a group G such that $(G:H)=2$ then prove that H is a normal subgroup of G.
Is converse true? Justify your answer.

Sol Let $(G,\times)$ be a group and $H \leq G$ such that $(G:H)=2$

To P.T $H \trianglelefteq G$.

$\because (G:H)=2$

There are two distinct left & right cosets of H in G.

Let H & Ha; H & aH be two distinct right cosets & left cosets of H in G; $a\in G$

then $G = H\cup Ha = H\cup aH$ —— (1)

$\because a\in G \Rightarrow a\in H$ or $a\notin H$

Case(i) when $a\in H$

$\therefore Ha = H = aH$

$\Rightarrow Ha = aH$.

Case(ii) when $a\notin H$

$\therefore Ha \neq H$ & $aH \neq H$

$\because G = H\cup Ha$ & $G = H\cup aH$

$\therefore H\cup Ha = H\cup aH$

$\Rightarrow Ha = aH$

$\therefore Ha = aH \ \forall\ a\in G$.

$\therefore H \trianglelefteq G$.

The converse of the above need not be true.

ie If $H \trianglelefteq G$ then $(G:H)\neq 2$

For example:

Let $G=\{1,-1,i,-i,j,-j,k,-k\}$ be a group

Let $H=\{1,-1\}$

clearly $H \trianglelefteq G$

but $(G:H)=4$
$\neq 2$.

4(b) → Discuss the convergence of the sequence $\{x_n\}$

where $x_n = \dfrac{\sin\left(\frac{n\pi}{2}\right)}{8}$

Sol'n: let $x_n = \dfrac{\sin\left(\frac{n\pi}{2}\right)}{8} \quad \forall\ n \in N$

Since $\displaystyle \lim_{n\to\infty} x_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1/8 & \text{if } n = 4m+1 \\ -1/8 & \text{if } n = 4m+3 \end{cases}$

does not exist

$\therefore (x_n)$ does not convergent.

4.(C) Use the method of contour integration to Prove that

$$\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta+c\sin\theta)^2} = \frac{2\pi a}{\sqrt[3]{a^2-b^2-c^2}}, \quad a^2>b^2+c^2.$$

Sol'n: $I = \int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta+c\sin\theta)^2}$

$$= \int_0^{2\pi} \frac{d\theta}{\left[a+\frac{1}{2}(e^{i\theta}+e^{-i\theta})+\frac{c}{2i}(e^{i\theta}+e^{-i\theta})\right]^2}$$

write $e^{i\theta}=z \Rightarrow d\theta = \frac{dz}{iz}$

$$I = \int_C \frac{dz/iz}{\left[a+\frac{b}{2}\left(z+\frac{1}{z}\right)+\frac{c}{2i}\left(z-\frac{1}{z}\right)\right]^2}$$

$$= \int_C \frac{4iz\,dz}{\left[2iza+bi(z^2+1)+c(z^2-1)\right]^2}$$

$$= \int_C \frac{4iz\,dz}{\left[z^2(bi+c)+2iza+(bi-c)\right]^2}$$

$$= \frac{4i}{(bi+c)^2}\int_C \frac{z\,dz}{\left[z^2+\frac{2iaz}{bi+c}+\frac{bi-c}{bi+c}\right]^2} = \frac{4i}{(bi+c)^2}\int_C f(z)\,dz \quad \text{—(1)}$$

where C is the unit circle.

f(z) has poles of order 2 given by

$$z^2+\frac{2ia}{bi+c}z+\frac{bi-c}{bi+c}=0.$$

$$\Rightarrow z = \frac{-\frac{2ia}{bi+c} \pm \sqrt{\frac{4a^2}{(bi+c)^2} - \frac{4(bi-c)}{bi+c}}}{2}$$

$$\Rightarrow z = \frac{1}{2}\left[-\frac{2ia}{bi+c} \pm \frac{2i\sqrt{a^2-b^2-c^2}}{bi+c}\right]$$

$$\Rightarrow z = \frac{i}{bi+c}\left[-a \pm \sqrt{a^2-b^2-c^2}\right]$$

$$z = \frac{i}{bi+c}\left(-a+\sqrt{a^2-b^2-c^2}\right) = \alpha \text{ (say)} \quad \& \quad z = \frac{i}{(bi+c)}\left(-a-\sqrt{a^2-b^2-c^2}\right) = \beta \text{ (say)}$$

only $z=\alpha$ lies inside C of order 2.

Residue at $z=\alpha$ is

$$\lim_{z\to\alpha} \frac{d}{dz}\left\{(z-\alpha)^2 \frac{z}{(z-\alpha)^2(z-\beta)^2}\right\}$$

$$= \lim_{z\to\alpha} \frac{d}{dz}\left[\frac{z}{(z-\beta)^2}\right]$$

$$= \lim_{z\to\alpha} \frac{-(z+\beta)}{(z-\beta)^3}$$

$$= -\frac{(\alpha+\beta)}{(\alpha-\beta)^3}$$

$$= \frac{-\frac{i}{(bi+c)}(-2a)}{\left(\frac{2i}{bi+c}\right)^3\left(\sqrt{a^2-b^2-c^2}\right)^3} = \frac{a}{\frac{-4}{(bi+c)^2}\left(\sqrt{a^2-b^2-c^2}\right)^3}$$

Then from (1).

$$I = \frac{4i}{(bi+c)^2}[2\pi i]\left(\text{Residue at } z=\alpha\right)$$

$$= \frac{4i}{(bi+c)^2}(2\pi i)\frac{-a}{\frac{4}{(bi+c)^2}(a^2-b^2-c^2)^{3/2}}$$

$$= \frac{2\pi a}{(a^2-b^2-c^2)^{3/2}}$$

$$\therefore \int_0^{2\pi} \frac{d\theta}{[a+b\cos\theta+c\sin\theta]^2} = \frac{2\pi a}{\sqrt[3]{a^2-b^2-c^2}}$$

4.(d) Make a graphical representation of the set of constraints of the following LPP. Find the extreme points of the feasible region. Finally, solve the problem graphically.

Max. $Z = 2x_1 + x_2$

Subject to $x_1 + x_2 \geq 5$

$2x_1 + 3x_2 \leq 20$

$4x_1 + 3x_2 \leq 25$

$x_1, x_2 \geq 0$.

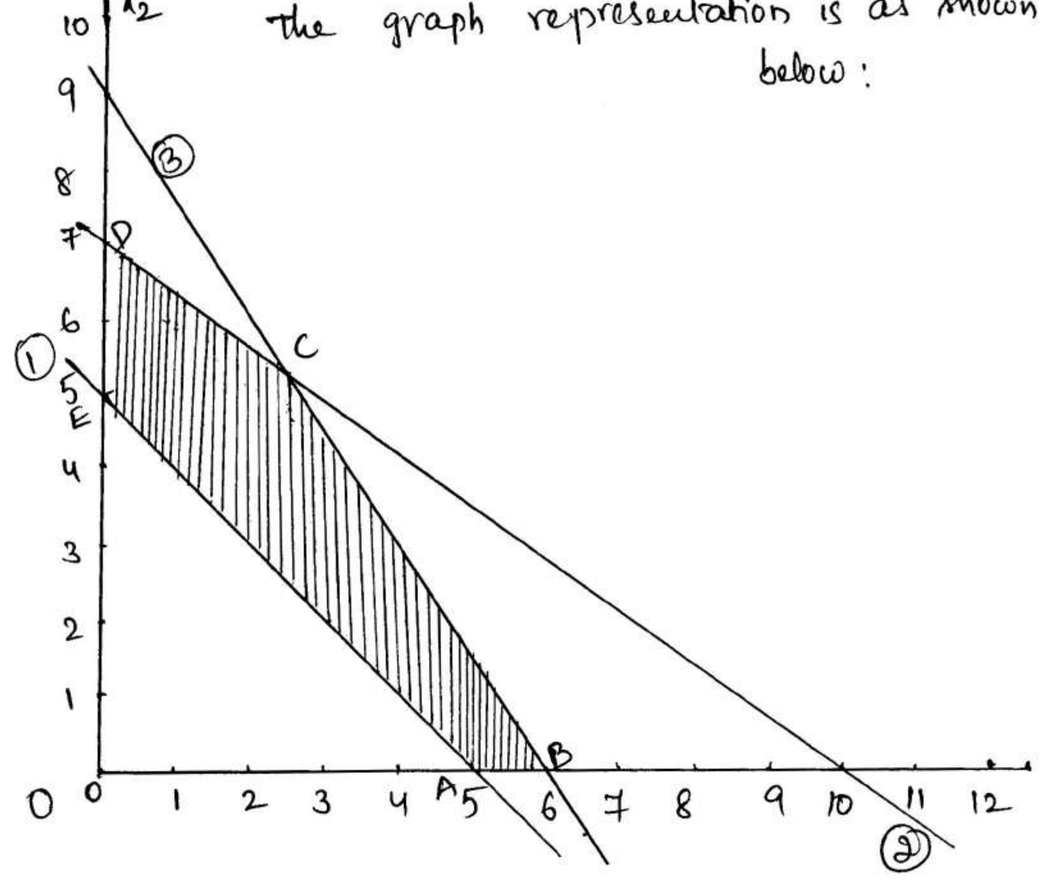
Sol'n: Let us consider the constraint $x_1 + x_2 \geq 5$ as equality $x_1 + x_2 = 5$ represents a line which passes through $(0,5)$ & $(5,0)$ in the XY-plane.

Similarly, we get

$2x_1 + 3x_2 = 20 \Rightarrow (10,0)$ & $\left(0, \frac{20}{3}\right)$

$4x_1 + 3x_2 = 25 \Rightarrow \left(\frac{25}{4}, 0\right)$ & $\left(0, \frac{25}{3}\right)$

The graph representation is as shown below:



The feasible region is given by ABCDE which satisfy all the constraints simultaneously coordinates of extreme points are
(corner points)

$A(5,0)$, $B\left(\frac{25}{4},0\right)$, $C(5/2,5)$, $D\left(0,\frac{20}{3}\right)$, $E(0,5)$

To maximize $z = 2x_1 + x_2$

The values of the objective function $z = 2x_1 + x_2$ at these extreme points are

$z(0,5) = 5$

$z(5,0) = 10$

$z\left(\frac{25}{4},0\right) = 25/2$

$z(5/2,5) = 10$

$z(0,20/3) = 20/3$

$\therefore$ The maximum value of z is at the point

$(25/4, 10)$

and the maximum value is

$z = 25/2$

$\therefore$ The optimal solution is

$x_1 = 25/4$, $x_2 = 0$ and $z = 25/2$.

5.(a) Solve $(D_x^3 - 7D_xD_y^2 - 6D_y^3)z = \sin(x+2y) + e^{3x+y}$.

Sol^n: Here note that D_x and D_y stand for D and D' respectively.

$\therefore$ Auxiliary equation is $m^3 - 7m - 6 = 0$ so that $m = -1, -2, 3$.

$\therefore$ C.F $= \phi_1(y-x) + \phi_2(y-2x) + \phi_3(y+3x)$, ϕ_1, ϕ_2, ϕ_3 being arbitrary functions.

Now P.I corresponding to $\sin(x+2y)$

$$= \frac{1}{D_x^3 - 7D_xD_y^2 - 6D_y^3}\sin(x+2y)$$

$$= \frac{1}{1^3 - 7.1.2^2 - 6.2^3}\iiint \sin v\, dv dv dv$$

$$= \left(-\frac{1}{75}\right)\iint(-\cos v)dvdv = \left(-\frac{1}{75}\right)\int(-\sin v)dv$$

$$= \left(-\frac{1}{75}\right)\cos v = \left(-\frac{1}{75}\right)\cos(x+2y)$$

and P.I corresponding to e^{3x+y}

$$= \frac{1}{D_x^3 - 7D_xD_y^2 - 6D_y^3}e^{3x+y} = \frac{1}{D_x - 3D_y}\left[\frac{1}{(D_x+D_y)(D_x+2D_y)}e^{3x+y}\right]$$

$$= \frac{1}{D_x - 3D_y}\cdot\frac{1}{(3+1)(3+2)}\iint e^v dvdv, \text{ where } v = 3x+y$$

$$= \frac{1}{20}\frac{1}{D_x - 3D_y}\int e^v dv = \frac{1}{20}\frac{1}{D_x - 3D_y}e^v$$

$$= \frac{1}{20}\frac{1}{(D_x - 3D_y)^1}e^{3x+y}$$

$$= \frac{1}{20}\cdot\frac{x}{1^1.1!}e^{3x+y} = \frac{x}{20}e^{3x+y}$$

$\boxed{\text{using formula } \frac{1}{(bD - aD')^n}\phi(ax+by) = \frac{x^n}{b^n.n!}\phi(ax+by)}$

with $a = 3$, $b = 1$, $n = 1$

Hence the required general solution is $z = C.F + P.I$

$$z = \phi_1(y-x) + \phi_2(y-2x) + \phi_3(y+3x) - \frac{1}{75}\cos(x+2y) + \frac{1}{20}xe^{3x+y}$$

5(b) Find a surface satisfying $r+s=0$, i.e, $(D^2+DD')Z=0$ and touching the elliptic paraboloid $z=4x^2+y^2$ along its section by the plane $y=2x+1$.

Sol^n: Given $(D^2+DD')z=0$ —— ①

Here auxiliary equation is $m^2+m=0$ so that $m=0,-1$

$\therefore$ solution of ① is $z=C.F=\phi_1(y)+\phi_2(y-x)$ —— ②

Since ② touches the curve given by

$$z=4x^2+y^2 \quad \text{—— ③}$$

and $y=2x+1$ —— ④.

values of $p\left(=\frac{\partial z}{\partial x}\right)$ and $q=\left(=\frac{\partial z}{\partial y}\right)$ obtained from ② and ③ must be equal for any point on ④

$$-\phi_2'(y-x)=8x \text{ for } y=2x+1$$

$$\Rightarrow \phi_2'(x+1)=-8x \quad \text{—— ⑤}$$

and $\phi_1'(y)+\phi_2'(y-x)=2y$ for $y=2x+1$

$$\Rightarrow \phi_1'(2x+1)+\phi_2'(x+1)=4x+2 \quad \text{—— ⑥}$$

from ⑤ $\phi_2'(x)=8-8x$

Integrating it, $\phi_2(x)=8x-4x^2+c_1$ —— ⑦

Subtracting ⑥ − ⑤ $\Rightarrow \phi_1'(2x+1)=12x+2=6(2x+1)-4$

so that $\phi_1'(x)=6x-4$

Integrating it $\phi_1(x)=3x^2-4x+c_2$ —— ⑧

from ⑧ $\phi_1(y)=3y^2-4y+c_2$

and from ⑦ $\phi_2(y-x)=8(y-x)-4(y-x)^2+c_1$

putting the above values of $\phi_1(y)$ and $\phi_2(y-x)$ in ② we get

$z = 3y^2 - 4y + c_2 + 8(y-x) - 4(y-x)^2 + c_1$

$\Rightarrow z = -y^2 + 4y - 8x - 4x^2 + 8xy + c_3$ where $c_3 = c_1 + c_2$ — (9)

Equating the values of z from (3) and (9) we get

$4x^2 + y^2 = -y^2 + 4y - 8x - 4x^2 + 8xy + c_3$ where $y = 2x+1$

$$c_3 = 8x^2 + 2y^2 - 4y + 8x - 8xy$$

$$= 8x^2 + 2(2x+1)^2 - 4(2x+1) + 8x - 8x(2x+1)$$

$$= -2$$

Hence from (9) the required surface is

$$4x^2 - 8xy + y^2 + 8x - 4y + z + 2 = 0.$$

5.(c) From the fallowing table, estimate the number of students who obtained marks between 40 and 45 using Newton's forward interpolation formula.

Marks:	30-40	40-50	50-60	60-70	70-80
No. of Sudents:	31	42	51	35	31

Sol^n First we prepare the cumulative frequency table, as fallows :-

Marks less than (x):	40	50	60	70	80
No. of students (y_x):	31	73	124	159	190

Now the difference table is —

x	y_x	Δy_x	$\Delta^2 y_x$	$\Delta^3 y_x$	$\Delta^4 y_x$
40	**31**				
		42			
50	73		**9**		
		51		**-25**	
60	124		**-16**		**37**
		35		**12**	
70	159		**-4**		
		31			
80	190				

We shall find y_{45} i.e. number of students with marks less than 45. Taking $x_0 = 40$, $x = 45$, we have

$$p = \frac{x - x_0}{h} = \frac{5}{10} = 0.5$$

$\therefore$ Using Newton's farward interpolation formula, we get,

$$y_{45} = y_{40} + p\Delta y_{40} + \frac{p(p-1)}{2!}\Delta^2 y_{40}$$

$$+ \frac{p(p-1)(p-2)}{3!}\Delta^3 y_{40}$$

$$= 31 + 0.5 \times 42 + \frac{0.5(-0.5)}{2} \times 9 + \frac{0.5(-0.5)(-1.5)}{6} \cdot \times (-25) + \frac{0.5(-0.5)(-1.5)(-2.5)}{24} \times 37$$

$$= 31 + 21 - 1.125 - 1.5625 - 1.4453$$

$$= 47.87, \text{ on simplification.}$$

The number of students with marks less than 45 is 47.87 ie. 48.

But the number of students with marks less than 40 is 31.

Hence the number of students getting marks between 40 and 45 $= 48 - 31$

$$= 17.$$

5.(d) → Using Gauss Seidel iterative method and the starting solution $x_1 = x_2 = x_3 = 0$ determine the solution of the following system of equations in two iterations., $10x_1 - x_2 - x_3 = 8$; $x_1 + 10x_2 + x_3 = 12$

$x_1 - x_2 + 10x_3 = 10$.

Solution :- The given equation can be rewritten as —

$$x_1 = \frac{8 + x_2 + x_3}{10} = 0.8 + 0.1x_2 + 0.1x_3$$

$$x_2 = \frac{12 - x_1 - x_3}{10} = 1.2 - 0.1x_1 - 0.1x_3$$

$$x_3 = \frac{10 - x_1 + x_2}{10} = 1 - 0.1x_1 + 0.1x_2$$

Also given that $x_1 = x_2 = x_3 = 0$

and we need to find the solution of the system of equation in two iteration.

Iteration - 1

Taking; $x_1 = x_2 = x_3 = 0$

$x_1 = 0.8 + 0.1 \times 0 + 0.1 \times 0 = 0.8$

$x_2 = 1.2 - 0.1 \times 0.8 - 0.1 \times 0 = 1.12$.

$x_3 = 1 - 0.1 \times 0.8 + 0.1 \times 1.12 = 1.032$.

After first Iteration ; $x_1 = 0.8$

$x_2 = 1.12$

$x_3 = 1.032$.

Iteration - II

$x_1 = 0.8$, $x_2 = 1.12$, $x_3 = 1.032$

$x_1 = 0.8 + 0.1 \times 1.12 + 0.1 \times 1.032 = 1.0152$

$x_2 = 1.2 - 0.1 \times 1.0152 - 0.1 \times 1.032$

$= 1.2 - 0.10152 - 0.1032 = 0.9953$

$x_3 = 1 - 0.1 \times 1.0152 + 0.1 \times 0.9953$

$= 1 - 0.10152 + 0.09953 = 0.99801.$

$\therefore$ After 2nd Iteration $\Rightarrow$ $\boxed{\begin{array}{l} x_1 = 1.0152 \\ x_2 = 0.9953 \\ x_3 = 0.99801 \end{array}}$

which is required solution

5(e) Prove that the necessary and sufficient condition that vortex lines may be at right angles to the streamlines are $u, v, w = \mu\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$, where μ and ϕ are functions of x, y, z, t.

Sol^n: The differential equations of streamlines and vortex lines are respectively.

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \quad \text{— (1)}$$

and $$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dz}{\zeta} \quad \text{— (2)}$$

∴ (1) and (2) will intersect orthogonally iff

$$u\xi + v\eta + w\zeta = 0$$

$$\Rightarrow u\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + v\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + w\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

But this is the condition that

$u\,dx + v\,dy + w\,dz$ is perfect differential

$$\Rightarrow u\,dx + v\,dy + w\,dz = \mu\, d\phi$$

$$= \mu\left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right)$$

this $\Rightarrow u, v, w = \mu\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$.

6(a) Find a partial differential equation by eliminating a, b, c from $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

Sol^n: Given that $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ —— (1)

Differentiating (1) w.r.t x and y, we get

$$\frac{2x}{a^2}+\frac{2z}{c^2}\frac{\partial z}{\partial x}=0 \Rightarrow c^2x+a^2z\frac{\partial z}{\partial x}=0 \quad \text{—— (2)}$$

and $$\frac{2y}{b^2}+\frac{2z}{c^2}\frac{\partial z}{\partial y}=0 \Rightarrow c^2y+b^2z\frac{\partial z}{\partial y}=0 \quad \text{—— (3)}$$

Differentiating (2) w.r.t x and (3) w.r.t y, we have

$$c^2+a^2\left(\frac{\partial z}{\partial x}\right)^2+a^2z\frac{\partial^2 z}{\partial x^2}=0 \quad \text{—— (4)}$$

& $$c^2+b^2\left(\frac{\partial z}{\partial y}\right)^2+b^2z\frac{\partial^2 z}{\partial y^2}=0 \quad \text{—— (5)}$$

from (2), $c^2=-\frac{a^2z}{x}\left(\frac{\partial z}{\partial x}\right)$.

putting this value of c^2 in (4) and dividing by a^2, we obtain

$$-\frac{z}{x}\frac{\partial z}{\partial x}+\left(\frac{\partial z}{\partial x}\right)^2+z\frac{\partial^2 z}{\partial x^2} \quad \text{—— (6)}$$

Similarly, from (3) & (5),

$$zy\frac{\partial^2 z}{\partial y^2}+y\left(\frac{\partial z}{\partial y}\right)^2-z\frac{\partial z}{\partial y}=0 \quad \text{—— (7)}$$

Differentiating (2) partially w.r.t y, we get

$$a^2\left\{\left(\frac{\partial z}{\partial y}\right)\left(\frac{\partial z}{\partial x}\right)+z\frac{\partial^2 z}{\partial x\partial y}\right\}=0$$

i.e. $$\frac{\partial z}{\partial x}\frac{\partial z}{\partial y}+z\frac{\partial^2 z}{\partial x\partial y}=0 \quad \text{—— (8)}$$

$\therefore$ (6), (7) and (8) are three possible forms of the required partial differential equations.

6(b) Reduce the equation $yr+(x+y)s+xt=0$ to canonical form and hence find its general solution.

Sol^n: Given $yr+(x+y)s+xt=0 \rightarrow (1)$

Comparing (1) with $Rr+Ss+Tt+f(x,y,z,p,q)=0$

here $R=y, S=x+y$ and $T=x$ so that

$$S^2-4RT=(x+y)^2-4xy=(x-y)^2>0 \text{ for } x\neq y.$$

and so (1) is hyperbolic. The λ-quadratic equation

$R\lambda^2+S\lambda+T=0$ reduces to $y\lambda^2+(x+y)\lambda+x=0$

$$\text{(or)} \quad (y\lambda+x)(\lambda+1)=0$$

So that $\lambda=-1, -x/y$. Then the corresponding characteristic equations are given by

$$\frac{dy}{dx}-1=0 \text{ and } \frac{dy}{dx}-(x/y)=0$$

Integrating these $y-x=c_1$ and $\frac{y^2}{2}-x^2/2=c_2$

In order to reduce one (1) to its canonical form, we choose.

$$u=y-x \text{ and } v=y^2/2-x^2/2 \longrightarrow (2)$$

$$\therefore p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$=-\left(\frac{\partial z}{\partial u}+x\frac{\partial z}{\partial v}\right), \text{ using (2)} \rightarrow (3)$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\frac{\partial z}{\partial u}+y\frac{\partial z}{\partial v}), \text{ using (2)} \rightarrow (4)$$

$$r=\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=-\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)-\frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial v}\right) \text{ using } ③$$

$$=-\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)-\left[x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)+\frac{\partial z}{\partial v}\right]=-\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)-x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)-\frac{\partial z}{\partial v}$$

$$\Rightarrow -\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right)-x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]-\frac{\partial z}{\partial v}$$

$$=-\left(-\frac{\partial^2 z}{\partial u^2}-x\frac{\partial^2 z}{\partial v\partial u}\right)-x\left(-\frac{\partial^2 z}{\partial u\partial v}-x\frac{\partial^2 z}{\partial v^2}\right)-\frac{\partial z}{\partial v} \text{ using } ②$$

$$r=\frac{\partial^2 z}{\partial u^2}+2x\frac{\partial^2 z}{\partial u\partial v}+x^2\frac{\partial^2 z}{\partial v^2}-\frac{\partial z}{\partial v}$$

Now, $t=\dfrac{\partial^2 z}{\partial y^2}=\dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial y}\right)=\dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial u}+y\dfrac{\partial z}{\partial v}\right)$

$$=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)+\frac{\partial}{\partial y}\left(y\frac{\partial z}{\partial v}\right) \text{ by } ④$$

$$=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)+y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)+\frac{\partial z}{\partial v}$$

$$=\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\left(\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}+y\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right\}+\frac{\partial z}{\partial v}$$

$$\Rightarrow \frac{\partial^2 z}{\partial u^2}+y\frac{\partial^2 z}{\partial u\partial v}+y\left(\frac{\partial^2 z}{\partial u\partial v}+y\frac{\partial^2 z}{\partial v^2}\right)+\frac{\partial z}{\partial v}.$$

$$\therefore\ t = \frac{\partial^2 z}{\partial u^2} + 2y\frac{\partial^2 z}{\partial u \partial v} + y^2\frac{\partial^2 z}{\partial v^2} + \frac{\partial z}{\partial v}$$

Also $s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u} + y\frac{\partial z}{\partial v}\right)$ using (4)'

$$= \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \frac{\partial}{\partial}\left(y\frac{\partial z}{\partial v}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x} + y\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x}\right.$$

$$\left. + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right\}$$

$$\Rightarrow -\frac{\partial^2 z}{\partial u^2} - x\frac{\partial^2 z}{\partial u \partial u} - y\frac{\partial^2 z}{\partial u \partial v} - xy\frac{\partial^2 z}{\partial v^2}.\ \text{using (2)}$$

$$\therefore\ s = -\frac{\partial^2 z}{\partial u^2} - (x+y)\frac{\partial^2 z}{\partial u \partial v} - xy\frac{\partial^2 z}{\partial v^2}$$

$\hookrightarrow$ (7)

using (5), (6) and (7) in (1) we get.

$$y\left(\frac{\partial^2 z}{\partial u^2} + 2x\frac{\partial^2 z}{\partial u \partial v} + x^2\frac{\partial^2 z}{\partial v^2} - \frac{\partial z}{\partial v}\right) +$$

$$(x+y)\left\{-\frac{\partial^2 z}{\partial u^2} - (x+y)\frac{\partial^2 z}{\partial u \partial v} - xy\frac{\partial^2 z}{\partial v^2}\right\}$$

$$+ x\left\{\frac{\partial^2 z}{\partial u^2} + 2y\frac{\partial^2 z}{\partial u \partial v} + y^2\frac{\partial^2 z}{\partial v^2} + \frac{\partial z}{\partial v}\right\} = 0$$

(or) $\left\{4xy - (x+y)^2\right\}\frac{\partial^2 z}{\partial u \partial v} - y\frac{\partial z}{\partial v} + x\frac{\partial z}{\partial v} = 0$

(or) $(y-x)^2\frac{\partial^2 z}{\partial u \partial v} + (y-x)\frac{\partial z}{\partial v} = 0$

(or) $u^2\dfrac{\partial^2 z}{\partial u\partial v}+u\dfrac{\partial z}{\partial v}=0$ (or) $u\dfrac{\partial^2 z}{\partial u\partial v}+\dfrac{\partial z}{\partial v}=0 \rightarrow (8)$

[$\because u\neq 0$ and $y-x=u$, by (2)]

(8) is the required Canonical form of (1).

Solution of (8) multiplying both sides of (8) by v we get

$uv\left(\dfrac{\partial^2 z}{\partial u\partial v}\right)+v\left(\partial z/\partial v\right)=0$ (or) $(uvDD'+vD')z=0$ $\rightarrow (9)$

where $D\equiv\dfrac{\partial}{\partial u}$ and $D'\equiv\partial/\partial v$. To reduce (9) into linear equation with constant coefficients, we take new variables x and y as follows

let $u=e^{x}$ and $v=e^{y}$ so that $x=\log u$, $y=\log v$ $\rightarrow (10)$

let $D_1\equiv\partial/\partial x$ and $D_1'\equiv\partial/\partial y$ then (9) reduces to

$(D_1D_1'+D_1')z=0$ (or) $D_1'(D+1)z=0$

Its general solution is

$z=e^{-x}\phi_1(y)+\phi_2(x)=u^{-1}\phi_1(\log v)+\phi_2(\log u)$

(or) $z=u^{-1}\psi_1(v)+\psi_2(u)=(y-x)^{-1}\psi_1(y^2-x^2)+\psi_2(y-x)$,

where ψ_1 and ψ_2 are arbitrary functions.

6.(d) The points of trisection of a string are pulled aside through a distance h on opposite sides of the position of equilibrium, and the string is released from rest. Derive an expression for the string at any subsequent time and show that the middle point of the string always remains at rest.

Solⁿ:

Let the length of the string is $= 3l$.

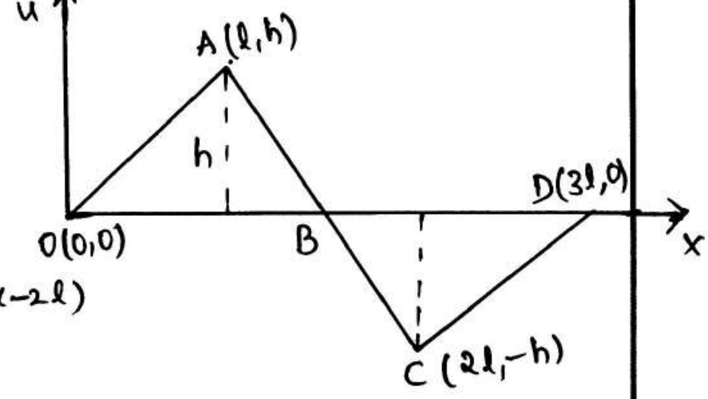
The coordinates of A, C and D are (l, h), $(2l, -h)$, $(3l, 0)$ respectively.

Initial deflection is given by OABCD.

Equation of OA is

$$u - 0 = \frac{h}{l}(x-0) \text{ i.e. } u = \frac{hx}{l}$$

Equations of AC and CD respectively are given by



$$u - h = \frac{-h-h}{2l-l}(x-l)$$

$$\Rightarrow u = \frac{h(3l-2x)}{l}$$

and $$u-(-h) = \frac{0-(-h)}{3l-2l}(x-2l)$$

$$\Rightarrow u = \frac{h(x-3l)}{l}$$

The required deflection is given by

$$u(x,t) = \sum_{n=1}^{\infty} E_n \cos\frac{n\pi ct}{3l} \sin\frac{n\pi x}{3l} \quad \text{—— (I)}$$

where $E_n = \frac{2}{3l} \cdot \int_0^{3l} f(x) \sin\frac{n\pi x}{3l}\, dx$.

The displacement $y(x,t)$ of any point of the string is given by

Boundary Conditions $y(0,t) = y(3l,t) = 0 \quad \forall\ t \geq 0$ —— ①

Initial conditions $y(x,0) = \begin{cases} hx/l & \text{when } 0 \leq x \leq l \\ \dfrac{h(3l-2x)}{l} & \text{when } l \leq x \leq 2l \\ \dfrac{h(x-3l)}{l} & \text{when } 2l \leq x \leq 3l \end{cases}$ —— ②

and $\left(\dfrac{\partial y}{\partial t}\right)_{t=0} = 0$ —— ③

we have

$$y(x,t) = \sum_{n=1}^{\infty} \left\{ E_n \cos\frac{n\pi ct}{3l} + F_n \sin\frac{n\pi ct}{3l} \right\} \sin\frac{n\pi x}{3l} \quad \text{—— ④}$$

Differentiating ③ partially w.r.t t, we get

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} \left\{ -E_n \frac{n\pi c}{3l} \sin\frac{n\pi ct}{3l} + \frac{n\pi c}{3l} F_n \cos\frac{n\pi ct}{3l} \right\} \sin\frac{n\pi x}{3l} \quad \text{—— ⑤}$$

Putting $t=0$ in ④ and ⑤

and using the I.C ② and ③ we get

$$⑤ \equiv \left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum_{n=1}^{\infty} F_n \frac{n\pi c}{3l} \sin\frac{n\pi x}{3l} = 0 \quad (\text{by ③})$$

where $F_n = \dfrac{2}{n\pi c} \displaystyle\int_0^{3l} (0) \sin\frac{n\pi x}{3l}\, dx = 0$

$$④ \equiv y(x,0) = f(x) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{3l}$$

where $E_n = \dfrac{2}{3l} \displaystyle\int_0^{3l} f(x) \sin\frac{n\pi x}{3l}\, dx$ —— ⑥

Now

$$E_n = \frac{2}{3l} \int_0^{3l} f(x) \sin\frac{n\pi x}{3l}\, dx$$

$$= \frac{2}{3l}\left[\int_0^l f(x)\sin\frac{n\pi x}{3l}dx + \int_l^{2l} f(x)\sin\frac{n\pi x}{3l}dx + \int_{2l}^{3l} f(x)\sin\frac{n\pi x}{3l}dx\right.$$

$$= \frac{2}{3l}\left[\int_0^l \frac{hx}{l}\sin\frac{n\pi x}{3l}dx + \int_l^{2l}\frac{h(3l-2x)}{l}\sin\frac{n\pi x}{3l}dx\right.$$

$$+ \int_{2l}^{3l}\frac{h(x-3l)}{l}\sin\frac{n\pi x}{3l}dx$$

Continuing in this way we get

$$E_n = \frac{18h}{n^2\pi^2}\left[\sin\frac{n\pi}{3} - \left\{\sin\left(n\pi - \frac{n\pi}{3}\right)\right\}\right]$$

$$= \frac{18h}{n^2\pi^2}\left[\sin\frac{n\pi}{3} - \left(\sin n\pi\cos\frac{n\pi}{3} - \cos n\pi\sin\frac{n\pi}{3}\right)\right]$$

$$= \frac{18h}{n^2\pi^2}\left[\sin\frac{n\pi}{3} - 0 + \cos n\pi\sin\frac{n\pi}{3}\right] \quad [\because \sin n\pi = 0\ \forall n]$$

$$= \frac{18h}{n^2\pi^2}\left[1+\cos n\pi\right]\sin\frac{n\pi}{3} = \frac{18h}{n^2\pi^2}\left[1+(-1)^n\right]\sin\frac{n\pi}{3}.$$

Thus $E_n = 0$ if n is odd.

if n is even put $n=2m$, $m=1,2,\dots$

$$E_n = \frac{36h}{n^2\pi^2}\sin\frac{n\pi}{3}$$

i.e, $$E_n = \frac{36h}{4m^2\pi^2}\sin\frac{2m\pi}{3}, \quad m=1,2,\dots$$

$$= \frac{9h}{m^2\pi^2}\sin\frac{2m\pi}{3}$$

putting the values of E_n & F_n in (4) the required deflection is given by $y(x,t) = \sum_{m=1}^{\infty}\frac{9h}{m^2\pi^2}\sin\frac{2m\pi}{3}\sin\frac{n\pi x}{3l}\cos\frac{n\pi ct}{3l}$

$$\Rightarrow y(x,t) = \frac{9h}{\pi^2}\sum_{m=1}^{\infty}\sin\frac{2m\pi}{3}\cos\frac{n\pi ct}{3l}\sin\frac{n\pi x}{3l} \quad \text{--- (7)}$$

putting $x = \frac{3l}{2}$ in (7), we find that the displacement of the midpoint of the string i.e. $y\left(\frac{3l}{2},t\right) = 0$.

because $\sin m\pi = 0$, for all integral values of m.

This shows that the mid-point of the string always rest.

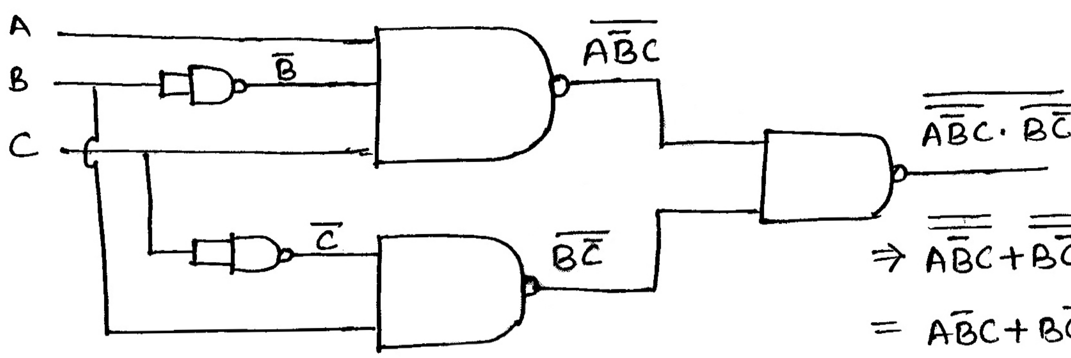
Ques: 7(b)(i) Draw the circuit diagram for $\bar{F} = A\bar{B}C + \bar{C}B$ using NAND to NAND logic long.

Solution:-

$$\bar{F} = A\bar{B}C + \bar{C}B$$

$$\bar{F} = \overline{\overline{A\bar{B}C}} + \overline{\overline{\bar{C}B}}$$

$$\boxed{\bar{F} = \overline{\overline{A\bar{B}C} \cdot \overline{B\bar{C}}}}$$



$$\Rightarrow \overline{\overline{A\bar{B}C}} + \overline{\overline{B\bar{C}}}$$

$$= A\bar{B}C + B\bar{C}$$

$$= \bar{F}$$

Ques: 7(b)(ii) In a Boolean Algebra B, for any a and b prove that $ab' + a'b = 0$ if and only if $a = b$.

Solution: Given $ab' + a'b = 0$

$$\Leftrightarrow \quad ab' = (a'b)' \qquad [-(a'b) = (a'b)']$$

$$ab' = a'' + b'$$

$$ab' = a + b'$$

$$ab' - (b') = a$$

$$b'(a-1) = a$$

$$-(b') = a$$

$$(b')' = a \quad \Rightarrow \quad \boxed{b = a} \text{ Hence proved.}$$

7.(b) (iii) Design a logic circuit having three inputs A, B, C such that output is 1 when A=0 or whenever B=C=1. Also obtain logic circuit using only NAND gates?

Solution

A	B	C	Output = y	m_i (output).
0	0	0	1	$m_0 = \bar{A}\bar{B}\bar{C}$
0	0	1	1	$m_1 = \bar{A}\bar{B}C$
0	1	0	1	$m_2 = \bar{A}B\bar{C}$
0	1	1	1	$m_3 = \bar{A}BC$
1	0	0	0	$m_4 = A\bar{B}\bar{C}$
1	0	1	0	$m_5 = A\bar{B}C$
1	1	0	0	$m_6 = AB\bar{C}$
1	1	1	1	$m_7 = ABC$

$y = 1$ (output

$$y = m_0 + m_1 + m_2 + m_3 + m_7$$

$$y = \bar{A}\bar{B}\bar{C} + \bar{A}\bar{B}C + \bar{A}B\bar{C} + \bar{A}BC + ABC$$

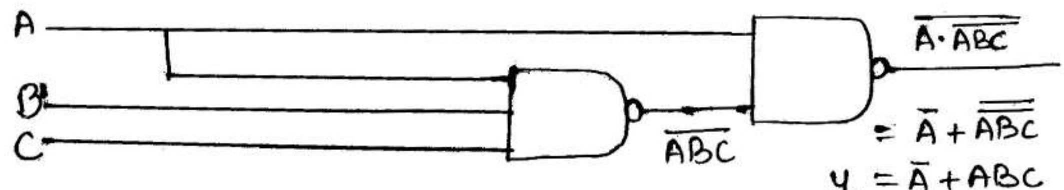
$$y = \bar{A}\left[\bar{B}\bar{C} + \bar{B}C + B\bar{C} + BC\right] + ABC$$

$$y = \bar{A}\left[\bar{B}[\bar{C}+C] + B[\bar{C}+C]\right] + ABC$$

$$[\because \bar{C}+C = 1]$$

$$y = \bar{A}[\bar{B}+B] + ABC \qquad [\because [\bar{B}+B=1]]$$

$$\boxed{y = \bar{A} + ABC}$$



$$= \bar{A} + \overline{\overline{ABC}}$$

$$y = \bar{A} + ABC$$

7(c) Using Runge – Kutta method of fourth order, solve

$\frac{dy}{dx} = \frac{y^2-x^2}{y^2+x^2}$ with $y(0)=1$ at $x=0.2, 0.4$.

Sol'n: we have $f(x,y) = \frac{y^2-x^2}{y^2+x^2}$

To find $y(0.2)$:

Here $x_0=0$, $y_0=1$, $h=0.2$

$k_1 = hf(x_0, y_0) = (0.2) f(0,1) = 0.2$

$k_2 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_1}{2}\right) = (0.2) f(0.1, 1.1) = 0.19672$

$k_3 = hf\left(x_0+\frac{h}{2}, y_0+\frac{k_2}{2}\right) = (0.2) f(0.1, 1.09836) = 0.1967$

$k_4 = hf(x_0+h, y_0+k_3) = (0.2) f(0.2, 1.1967) = 0.1891$

$k = \frac{1}{6}(k_1+2k_2+2k_3+k_4) = 0.19599$

$\therefore y(0.2) = y_0+k = 1.196.$

To find $y(0.4)$:

Here $x_1=0.2$, $y_1=1.196$, $h=0.2$

$k_1 = hf(x_1, y_1) = 0.1891$

$k_2 = hf\left(x_1+\frac{h}{2}, y_1+\frac{k_1}{2}\right) = (0.2) f(0.3, 1.2906) = 0.1795$

$k_3 = hf\left(x_1+\frac{h}{2}, y_1+\frac{k_2}{2}\right) = (0.2) f(0.3, 1.2858) = 0.1793$

$k_4 = hf(x_1+h, y_1+k_3) = (0.2) f(0.4, 1.3753) = 0.1688$

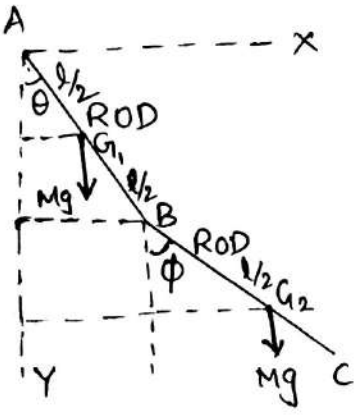
$k = \frac{1}{6}(k_1+2k_2+2k_3+k_4) = 0.1792$

$\therefore y(0.4) = y_1+k = 1.196+0.1792 = 1.3752$

8(a) Two equal rods AB and BC, each of length l smoothly joined at B are suspended from A and oscillate in a vertical plane through A. Show that the periods of normal oscillations are $\frac{2\pi}{n}$, where $n^2 = \left(3 \pm \frac{6}{\sqrt{7}}\right)\frac{g}{l}$.

Sol^n: Let AB and BC be the rods of equal length l and mass M. At time t, let the two rods make angles θ and ϕ to the vertical respectively.

Referred to A as origin horizontal and vertical lines AX and AY as axes the coordinates of C.G. G_1 of rod AB and that of C.G. G_2 of rod BC are given by



$x_{G_1} = \frac{1}{2} l \sin\theta$, $y_{G_1} = \frac{1}{2} l \cos\theta$

$x_{G_2} = l\sin\theta + \frac{1}{2} l \sin\phi$,

$y_{G_2} = l\cos\theta + \frac{1}{2} l \cos\phi$.

$\therefore$ If v_{G_1} and v_{G_2} are velocities of G_1 & G_2, then

$$v_{G_1}^2 = \dot{x}_{G_1}^2 + \dot{y}_{G_1}^2 = \left(\tfrac{1}{2} l\cos\theta\dot{\theta}\right)^2 + \left(-\tfrac{1}{2} l \sin\theta\dot{\theta}\right)^2$$
$$= \tfrac{1}{4} l^2 \dot{\theta}^2$$

$$v_{G_2}^2 = \dot{x}_{G_2}^2 + \dot{y}_{G_2}^2 = \left(l\cos\theta\dot{\theta} + \tfrac{1}{2}\cos\phi\dot{\phi}\right)^2 + \left(-l\sin\theta\dot{\theta} - \tfrac{1}{2} l\sin\phi\dot{\phi}\right)^2$$
$$= l^2\left[\dot{\theta}^2 + \tfrac{1}{4}\dot{\phi}^2 + \dot{\theta}\dot{\phi}\cos(\theta-\phi)\right]$$
$$= l^2\left[\dot{\theta}^2 + \tfrac{1}{4}\dot{\phi}^2 + \dot{\theta}\dot{\phi}\right] \quad (\because \theta, \phi \text{ are small})$$

If T be the total kinetic energy and W the work function of the system, then

$$T = \text{K.E of rod AB} + \text{K.E of rod BC}$$
$$= \left[\tfrac{1}{2} M \cdot \tfrac{1}{3}\left(\tfrac{1}{2} l\right)^2 \dot{\theta}^2 + \tfrac{1}{2} M \cdot v_{G_1}^2\right] + \left[\tfrac{1}{2} m \cdot \tfrac{1}{3}\left(\tfrac{1}{2} l\right)\dot{\phi}^2 + \tfrac{1}{2} M \cdot v_{G_2}^2\right]$$
$$= \tfrac{1}{2} M\left[\tfrac{1}{12} l^2\dot{\theta}^2 + \tfrac{1}{4} l^2\dot{\theta}^2\right] + \tfrac{1}{2} M\left[\tfrac{1}{12} l^2\dot{\phi}^2 + l^2\left(\dot{\theta}^2 + \tfrac{1}{4}\dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\right)\right]$$
$$= \tfrac{1}{2} M l^2 \left(\tfrac{4}{3}\dot{\theta}^2 + \tfrac{1}{3}\dot{\phi}^2 + \dot{\theta}\phi\right)$$

and $W = Mg\,y_{G_1} + Mg\,y_{G_2} + C$

$= Mg\left[\frac{1}{2}l\cos\theta + l\cos\theta + \frac{1}{2}l\cos\phi\right] + C$

$= \frac{1}{2}Mgl\,(3\cos\theta + \cos\phi).$

$\therefore$ Lagrange's θ- equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right) - \frac{\partial T}{\partial\theta} = \frac{\partial W}{\partial\theta}$

i.e $\frac{d}{dt}\left[\frac{1}{2}Ml^2\left(\frac{8}{3}\dot\theta + \dot\phi\right)\right] - 0 = \frac{1}{2}Mgl(-3\sin\theta) = -\frac{3}{2}Mgl\theta$ ($\because \theta$ is small)

$\Rightarrow 8\ddot\theta + 3\ddot\phi = -9c\theta$, (where $c = g/l$) ——— (1)

Equations (1) and (2) can be written as

$(8D^2 + 9c)\theta + 3D^2\phi = 0$ and $3D^2\theta + (2D^2 + 3c)\phi = 0$

Eliminating ϕ between these two equations, we get

$[(2D^2+3c)(8D^2+9c) - 9D^4] = 0$

$\Rightarrow (7D^4 + 42cD^2 + 27c^2)\theta = 0$

If the periods of normal oscillations are $2\pi/n$, then the solution of (3), must be

$\theta = A\cos(nt + B)$

$\therefore D^2\theta = -n^2\theta$ and $D^4\theta = n^4\theta$.

Substituting in (3), we get

$(7n^4 - 42cn^2 + 27c^2)\theta = 0$

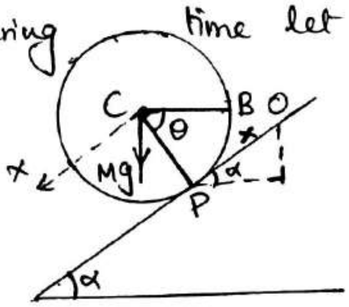
$\Rightarrow 7n^4 - 42cn^2 + 27c^2 = 0 \quad \because \theta \neq 0.$

$$\therefore n^2 = \frac{42 \pm \sqrt{[(42c)^2 - 4.7.27c^2]}}{2.7}$$

$$\Rightarrow n^2 = \left(3 \pm \frac{6}{\sqrt7}\right)c = \left(3 + \frac{6}{\sqrt7}\right)\frac{g}{l} \quad (\because c = g/l)$$

8(b) A sphere of radius a and mass M rolls down a rough plane inclined at an angle α to the horizontal. If x be the distance of the point of contact of the sphere from a fixed point on the plane, find the acceleration by using Hamilton's equation.

Sol'n: Let a sphere of radius a and mass M roll down a rough plane inclined at an angle α starting initially from a fixed point O of the plane. In time t, let the sphere roll down a distance x and during time let it turn through an angle θ.



Since there is no slipping

$\therefore x = OA = \text{arc } AB = a\theta.$

So that $\dot{x} = a\dot{\theta}$

If T and V are the kinetic and potential energies of the sphere, then $T = \frac{1}{2}MK^2\dot{\theta}^2 + \frac{1}{2}M\dot{x}^2 = \frac{1}{2}M\frac{2}{5}a^2\dot{\theta}^2 + \frac{1}{2}M(a\dot{\theta})^2$

$$\Rightarrow T = \frac{7}{10}M\dot{x}^2$$

and $V = -MgOL = -Mgx\sin\alpha$ (Since the sphere moves down the plane)

$$\therefore L = T - V = \frac{7}{10}M\dot{x}^2 + Mgx\sin\alpha$$

Here x is the only generalised coordinate.

$$\therefore p_x = \frac{\partial L}{\partial \dot{x}} = \frac{7}{5}M\dot{x} \quad \text{——— (1)}$$

Since L does not contain t explicitly,

$$\therefore H = T + V = \frac{7}{10}M\dot{x}^2 - Mgx\sin\alpha$$

$$\Rightarrow H = \frac{7}{10}M\left(\frac{5}{7M}p_x\right) - Mgx\sin\alpha$$

$$= \frac{5}{14M}p_x^2 - Mgx\sin\alpha \quad \cdot \text{ from (1)}$$

Hence the two Hamilton's equations are

$$\dot{p}_x = -\frac{\partial H}{\partial x} = Mg\sin\alpha \quad \text{—}(H_1), \quad \dot{x} = \frac{\partial H}{\partial p_x} = \frac{5}{7M}p_x \quad \text{—}(H_2)$$

Differentiating (H_2) and using (H_1), we get

$$\ddot{x} = \frac{5}{7M}\dot{p}_x = \frac{5}{7M}Mg\sin\alpha$$

$$\Rightarrow \boxed{\ddot{x} = \frac{5}{7}g\sin\alpha}$$

which gives the required acceleration.

Ques: 8(c) Show that $\phi = x f(r)$ is a possible form for the velocity potential for an incompressible fluid motion. If the fluid velocity $\vec{q} \to 0$ as $r \to \infty$, find the surfaces of constant speed.

Solution:- given; $\phi = x f(r)$ —— (1)

$$q = -\nabla \phi = -\nabla [x f(r)]$$

$$q = -[f(r)\nabla x + x \nabla f(r)] \quad \text{—— (2)}$$

Now, $r^2 = x^2 + y^2 + z^2$

$$\Rightarrow 2r\left(\frac{\partial r}{\partial x}\right) = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r} \quad \text{—— (3)}$$

Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$ —— (4)

Also; $\nabla x = \left[i\left(\frac{\partial}{\partial x}\right) + j\left(\frac{\partial}{\partial y}\right) + k\left(\frac{\partial}{\partial z}\right)\right] x = i$

and $\nabla f(r) = \left[i\left(\frac{\partial}{\partial x}\right) + j\left(\frac{\partial}{\partial y}\right) + k\left(\frac{\partial}{\partial z}\right)\right] f(r)$

$$\nabla f(r) = i f'(r)\left(\partial r/\partial x\right) + j f'(r)\left(\partial r/\partial y\right) + \hat{k} f'(r)\left(\frac{\partial r}{\partial z}\right)$$

$$\nabla f(r) = \hat{i} f'(r)\cdot\left(\frac{x}{r}\right) + \hat{j} f'(r)\left(y/r\right) + \hat{k} f'(r)\left(\frac{z}{r}\right)$$

$$\nabla f(r) = \frac{1}{r} f'(r)\left(\hat{i}x + \hat{j}y + \hat{k}z\right) = \frac{1}{r}\cdot f'(r)\cdot\vec{r}$$

$$\therefore (2) \Rightarrow q = -f(r)\, i - \left(\frac{x}{r}\right) f'(r)\cdot\vec{r} \quad \text{—— (5)}$$

For a possible motion of an incompressible fluid, we have —

$$\nabla\cdot q = 0 \quad \text{or} \quad \nabla\cdot(-\nabla\phi) = 0$$

$$\Rightarrow \nabla^2\phi = 0$$

$$\Rightarrow \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]\left[x f(r)\right] = 0 \text{ , (using (1))} \quad \text{—— (6)}$$

Now, $\dfrac{\partial^2}{\partial x^2}\left[x f(r)\right] = \dfrac{\partial}{\partial x}\left[\dfrac{\partial}{\partial x}\{x f(r)\}\right]$

$$= \frac{\partial}{\partial x}\left[f(r) + x\frac{\partial f(r)}{\partial x}\right]$$

$$\therefore \frac{\partial^2}{\partial x^2}\left[x f(r)\right] = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial x} + x\frac{\partial^2 f(r)}{\partial x^2} = 2\frac{\partial f}{\partial x} + x\frac{\partial^2 f(r)}{\partial x^2}.$$

Also; $\dfrac{\partial^2}{\partial y^2}\left[x f(r)\right] = x\dfrac{\partial^2 f}{\partial y^2}$

$$\frac{\partial^2}{\partial z^2}\left[x f(r)\right] = x\frac{\partial^2 f}{\partial z^2}$$

$\therefore$ (6) becomes

$$2\frac{\partial f}{\partial x} + x\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) = 0 \quad \text{—— (7)}$$

Now, $\dfrac{\partial f}{\partial x} = \dfrac{\partial f}{\partial r}\cdot\dfrac{\partial r}{\partial x} = f'(x/r)$ —— using (3)

and $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial}{\partial x}\left(f'(x/r)\right) = \dfrac{f'}{r} + x\dfrac{\partial}{\partial x}\left(\dfrac{f'}{r}\right)$ —— (8)

$$= \frac{f'}{r} + x\frac{\partial}{\partial r}\left(\frac{f'}{r}\right)\frac{\partial r}{\partial x} = \frac{f'}{r} + x\left(\frac{\partial f'' - f'}{r^2}\right)\frac{x}{r}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{f'}{r} + \frac{x^2}{r^2}f'' - \frac{x^2}{r^3}f' \quad \text{——— (9)}$$

Similarly,

$$\frac{\partial^2 f}{\partial y^2} = \frac{f'}{r} + \frac{y^2}{r^2}f'' - \frac{y^2}{r^3}f' \quad \text{——— (10)}$$

and $$\frac{\partial^2 f}{\partial z^2} = \frac{f'}{r} + \frac{z^2}{r^2}f'' - \frac{z^2}{r^3}f' \quad \text{——— (11)}$$

Adding (9), (10) and (11), we get.

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{3f'}{r} + \left[\frac{x^2+y^2+z^2}{r^2}\right]f'' - \left(\frac{x^2+y^2+z^2}{r^3}\right)f'$$

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{3f'}{r} + \frac{r^2}{r^2}\cdot f'' - \frac{1}{r}\cdot f'$$

$$\boxed{\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{3f'}{r} + f'' - \frac{1}{r}f' = \frac{2}{r}f' + f''} \text{ — (12)}$$

Using (8) and (12), (7) reduces to

$$\frac{2f'(x)}{r} + x\left(\frac{2f'}{r} + f''\right) = 0$$

$$\Rightarrow f'' + \frac{4f'}{r} = 0$$

$$\Rightarrow \boxed{\frac{f''}{f'} + \frac{4}{r} = 0}$$

Integrating $\log f' + 4\log r = \log c_1$

So that $\Rightarrow \boxed{f' = c_1 r^{-4}}$ ——— (13)

Integrating (13); $f = -\left(c_1/_3\right) \times r^{-3} + c_2$

C_2 being an arbitrary constant

Substituting the values of f' and f from (13) & (14) in (5), we get

$$\vec{q} = -\{(c_1/3r^3) - c_2\}\vec{i} - (c_1 x/r^5)\vec{r} \quad \text{— (15)}$$

Given, that $\vec{q} \to 0$ as $r \to \infty$,

hence (15) shows that $c_2 = 0$

$\therefore$ from (15) ; $\boxed{\vec{q} = \dfrac{c_1}{3r^3}\left(\vec{i} - \dfrac{3x\vec{r}}{r^2}\right)}$ — 16

Now; $q^2 = \vec{q}\cdot\vec{q} = \dfrac{c_1^2}{9r^6}\left(\vec{i} - \dfrac{3x\vec{r}}{r^2}\right)\cdot\left(\vec{i} - \dfrac{3x\vec{r}}{r^2}\right)$

$$= \frac{c_1^2}{9r^6}\left[\vec{i}\cdot\vec{i} - \frac{6x}{r^2}\cdot\vec{r}\cdot\vec{i} + \frac{9x^2r^2}{r^4}\right]$$

as $\vec{r}\cdot\vec{r} = r^2$
& $\vec{r}\cdot\vec{i} = x$

$$= \frac{c_1^2}{9r^6}\left(1 - \frac{6x^2}{r^2} + \frac{9x^2\cancel{r^2}}{r^{\cancel{4}2}}\right)$$

$$= \frac{c_1^2}{9r^6}\left(1 + \frac{3x^2}{r^2}\right)$$

$$\boxed{q^2 = \frac{c_1^2}{9r^8}(r^2 + 3x^2)}$$

Hence, the required surfaces of constant speed are

$$q^2 = \text{constant} \text{ (or) } \left(\frac{c_1^2}{9r^8}\right)(r^2 + 3x^2) = \text{constant}$$

$\boxed{q^2 = (r^2 + 3x^2)r^{-8} = \text{constant}}$ required solution