(b) Evaluate $\iint_S \nabla \times \bar{A} \cdot \bar{n}\, dS$ for

$\bar{A} = (x^2 + y - 4)\hat{i} + 3xy\,\hat{j} + (2xz + z^2)\hat{k}$ and S is the surface of hemisphere $x^2 + y^2 + z^2 = 16$ above xy plane. **15**

(c) Solve the D.E. :

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2y = e^x + \cos x. \quad \textbf{10}$$

8. (a) A semi circular disc rests in a vertical plane with its curved edge on a rough horizontal and equally rough vertical plane. If the coeff. of friction is μ, prove that the greatest angle that the bounding diameter can make with the horizontal plane is:

$$\sin^{-1}\left(\frac{3\pi}{4}\,\frac{\mu + \mu^2}{1 + \mu^2}\right). \quad \textbf{15}$$

(b) A body floating in water has volumes V_1, V_2 and V_3 above the surface when the densities of the surrounding air are ρ_1, ρ_2, ρ_3 respectively. Prove that:

$$\frac{\rho_2 - \rho_3}{V_1} + \frac{\rho_3 - \rho_1}{V_2} + \frac{\rho_1 - \rho_2}{V_3} = 0. \quad \textbf{10}$$

(c) Verify the divergence theorem for $\bar{A} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ over the region $x^2 + y^2 = 4$, $z = 0$, $z = 3$. **15**

PAPER-II

IFS 2014

Instructions: *Candidates should attempt Question Nos. 1 and 5 which are compulsory, and any THREE of the remaining questions, selecting at least ONE question from each Section. All questions carry equal marks. The number of marks carried by each part of a question is indicated against each. Answers must be written in ENGLISH only. Assume suitable data, if considered necessary, and indicate the same clearly. Symbols and notations have their usual meanings, unless indicated otherwise.*

Section-A

1. Answer the following:

(a) If G is a group in which $(a \cdot b)^4 = a^4 \cdot b^4$, $(a \cdot b)^5 = a^5 \cdot b^5$ and $(a \cdot b)^6 = a^6 \cdot b^6$, for all $a, b \in G$, then prove that G is Abelian. **8**

(b) Let f be defined on [0, 1] as

$$f(x) = \begin{cases} \sqrt{1 - x^2}, & \text{if } x \text{ is rational} \\ 1 - x, & \text{if } x \text{ is irrational} \end{cases}$$

Find the upper and lower Riemann integrals of f over [0, 1]. **8**

(c) Using Cauchy integral formula, evaluate

$$\int_C \frac{z+2}{(z+1)^2(z-2)}\,dz$$

where C is the circle $|z - i| = 2$. **8**

(d) Obtain the initial basic feasible solution for the transportation problem by North-West corner rule: **8**

Retail Shop

		R_1	R_2	R_3	R_4	R_5	Supply
Factory	F_1	1	9	13	36	51	50
	F_2	24	12	16	20	1	100
	F_3	14	35	1	23	26	150
		100	70	50	40	40	

(e) Find the constants a, b, c such that the function

$f(z) = 2x^2 - 2xy - y^2 + i(ax^2 - bxy + cy^2)$

is analytic for all z $(= x + iy)$ and express $f(z)$ in terms of z. **8**

2. (a) Let J_n be the set of integers mod n. Then prove that J_n is a ring under the operations of addition and multiplication mod n. Under what conditions on n, J_n is a field? Justify your answer. **10**

(*b*) Show that the function $f(x) = \sin\frac{1}{x}$ is continuous but not uniformly continuous on $(0, \pi)$. **15**

(*c*) Evaluate:

$$\int_{|z|=1} \frac{z}{z^4 - 6z^2 + 1} dz$$ **15**

3. (*a*) Let R be an integral domain with unity. Prove that the units of R and R[*x*] are same. **10**

(*b*) Change the order of integration and evaluate $\int_{-2}^{1}\int_{y^2}^{2-y} dx\, dy$. **15**

(*c*) Find the bilinear transformations which map the points $-1, \infty, i$ into the points—

(*i*) $i, 1, 1+i$

(*ii*) $\infty, i, 1$

(*iii*) $0, \infty, 1$ **15**

4. (*a*) Show that the function $f(x) = \sin x$ is Riemann integrable in any interval $[0, t]$ by taking the partition $P = \left\{0, \frac{t}{n}, \frac{2t}{n}, \frac{3t}{n}, ..., \frac{nt}{n}\right\}$ and $\int_0^t \sin x\, dx = 1 - \cos t$. **10**

(*b*) Find the Laurent series expansion at $z = 0$ for the function

$f(z) = \frac{1}{z^2(z^2 + 2z - 3)}$ in the regions

(*i*) $1 < |z| < 3$ and (*ii*) $|z| > 3$. **15**

(*c*) Solve the following LPP graphically:

Maximize $Z = 3x_1 + 4x_2$

subject to $x_1 + x_2 \le 6$

$x_1 - x_2 \le 2$

$x_2 \le 4$

$x_1, x_2 \ge 0$

Write the dual problem of the above and obtain the optimal value of the objective function of the dual without actually solving it. **15**

Section-B

5. (*a*) Use Lagrange's formula to find the form of $f(x)$ from the following table: **8**

x	0	2	3	6
$f(x)$	648	704	729	792

(*b*) Write a program in BASIC to integrate

$\int_0^1 e^{-2x} \sin x\, dx$

by Simpson's $\frac{1}{3}$rd rule with 20 sub-intervals. **8**

(*c*) Show that the general solution of the pde

$$\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 z}{\partial t^2}$$

is of the form $Z(x, y) = F(x + ct) + G(x - ct)$, where F and G are arbitrary functions. **8**

(*d*) Prove that the vorticity vector $\vec{\Omega}$ of an incompressible viscous fluid moving in the absence of an external force satisfies the differential equation

$\frac{D\vec{\Omega}}{Dt} = (\vec{\Omega}\cdot\nabla)\vec{q} + \nu\nabla^2\vec{\Omega}$ where $\vec{q}$ is the velocity vector with $\vec{\Omega} = \nabla \times \vec{q}$. **8**

(*e*) Find the condition that $f(x, y, \lambda) = 0$ should be a possible system of streamlines for steady irrotational motion in two dimensions, where λ is a variable parameter. **8**

6. (*a*) Verify that the differential equation

$$(y^2 + yz)dx + (xz + z^2)dy + (y^2 - xy)dz = 0$$

is integrable and find its primitive. **10**

(*b*) Show that the moment of inertia of a uniform rectangular mass M and sides $2a$ and $2b$ about a diagonal is $\frac{2Ma^2b^2}{3(a^2 + b^2)}$ **10**

(*c*) The values of $f(x)$ for different values of x are given as $f(1) = 4$, $f(2) = 5$, $f(7) = 5$ and $f(8) = 4$. Using Lagrange's interpolation formula, find the value of $f(6)$. Also find the value of x for which $f(x)$ is optimum. **10** (*d*) Write a BASIC program to sum the series $S = 1 + x + x^2 + \ldots + x^n$, for $n = 30, 60$ and 90 for the values of $x = 0.1\ (0.1)\ 0.3$. **10**

7. (*a*) Solve:

$$(D - 3D' - 2)^2 z = 2e^{2x}\cot(y + 3x)$$ **10**

(*b*) Solve the following system of equations:

$$2x_1 + x_2 + x_3 - 2x_4 = -10$$
$$4x_1 + 2x_3 + x_4 = 8$$
$$3x_1 + 2x_2 + 2x_3 = 7$$
$$x_1 + 3x_2 + 2x_3 - x_4 = -5$$

(*c*) A uniform rod OA of length $2a$ is free to turn about its end O, revolves with uniform angular velocity ω about a vertical axis OZ through O and is inclined at a constant angle α to OZ. Show that the value of α is either zero or $\cos^{-1}\left(\frac{3g}{4a\omega^2}\right)$ **15**

8. (*a*) Using Runge-Kutta 4th order method, find y from

$$\frac{dy}{dx} = \frac{y^2 - x^2}{y^2 + x^2}$$

with $y(0) = 1$ at $x = 0.2, 0.4$. **10**

(*b*) A plank of mass M is initially at rest along a straight line of greatest slope of a smooth plane inclined at an angle α to the horizon and a man of mass M' starting from the upper end walks down the plank so that it does not move. Show that he gets to the other end in time

$$\sqrt{\frac{2M'a}{(M+M')g\sin\alpha}}$$

where α is the length of the plank. **15**

(*c*) Prove that

$$\frac{x^2}{a^2}\tan^2 t + \frac{y^2}{b^2}\cot^2 t = 1$$

is a possible form for the bounding surface of a liquid and find the velocity components. **15**

ANSWERS

PAPER-I

Section-A

1. (*a*) $x \cdot u_1 + y \cdot u_2 + z \cdot u_3 = (5, 3, 4)$

$x(1, -1, 0) + y(1, 1, 0) + z(0, 1, 1) = (5, 3, 4)$

$x + y = 5, \quad -x + y + z = 3$ and $z = 4$.

$\therefore\ x + y = 5$ and $-x + y + 4 = 3$

so, $x - y = 1$

From, $x + y = 5$ and $x - y = 1$

$x = 3, y = 2$ and $z = 4$

$\therefore\ u_1 = 3(1, -1, 0) = (3, -3, 0)$

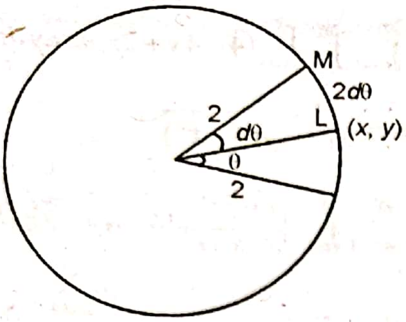
$u_2 = 2(1, 1, 0) = (2, 2, 0)$

$u_3 = 4(0, 1, 1) = (0, 4, 4)$

1. (*b*)
$$A^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^2 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

$$A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^3 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$A^4 = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}^4 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}$$



$2d\theta\, dz = ds_3$ and $\iint A_z \cdot n_3 ds_3$

$$= \iint_{S_3} \left[\{4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}\} \cdot \left\{ \frac{2x\hat{i} + 2y\hat{j}}{\sqrt{16}} \right\} \right] ds_3$$

$$= \iint_{S_3} \left\{ \frac{8x^2 - 4y^2}{4} \right\} ds_3 = \iint_{S_3} (2x^2 - y^3)(2d\theta\, dz)$$

$$= \int_{\theta=0}^{2x} \int_{z=0}^{2} \{2(2\cos\theta)^2 - 2^3 \sin^2\theta\} (2d\theta\, dz)$$

$$= 48 \int_{\theta=0}^{2x} (\cos^2\theta - \sin^3\theta) d\theta = 48 \int_{\theta=0}^{2\pi} \cos^2\theta\, d\theta$$

$$= 48.4 \int_{\theta=0}^{\frac{x}{2}} \cos^2\theta\, d\theta = 48.4 \cdot \frac{\pi}{4} = 48\pi. \quad ...(iii)$$

Adding (*i*), (*ii*), (*iii*), we get

$$\iint A \cdot \hat{n} ds = 36\pi + 48\pi = 84\pi.$$

Thus the divergence theorem is verified.

Paper-II

1. (*a*) Fix $a, b \in G$. Then $(ab)(ab)(ab)(ab)(ab) = (ab)^5 = a^5b^5$ and cancellation of the end-terms, or multiplication by inverses, implies that $(ba)^4 = b(ab)(ab)(ab)a = a^4b^4$. Likewise, $(ab)(ab)(ab) = (ab)^3 = a^3b^3$ which implies that

$$(ba)^2 = a^2b^2 \quad .....(i)$$

again by cancellation. But $(ba)^4 = (ba)^2(ba)^2 = a^2b^2a^2b^2$ so that $a^4b^4 = a^2b^2a^2b^2$.

Cancellation again implies $a^2b^2 = b^2a^2$ which is certainly getting us closer. Now, using Equation (*i*), but switching the roles of *a* and *b*, we have $(ab)^2 = b^2a^2$, so that $a^2b^2 = b^2a^2 = (ab)^2 = (ab)(ab)$. Cancellation of the end-terms one last time yields $ab = ba$ which proves that G is Abelian.

1. (*b*) $f(x) = \sqrt{1-x^2}$, if x is rational

$= 1 - x$, if x is irrational

Here, $\sqrt{1-x^2} - (1-x)$

$$= \sqrt{1-x}\left(\sqrt{1+x} - \sqrt{1-x}\right)$$

$f(x)$ is increasing from

0 to $\frac{1}{\sqrt{2}}$ and decreasing from $\frac{1}{\sqrt{2}}$ to 1.

The upper Riemann integral

$$= \int_0^1 f(x).dx$$

$$= \int_0^{\frac{1}{\sqrt{2}}} \sqrt{1-x^2}.dx + \int_{\frac{1}{\sqrt{2}}}^{1} (1-x).dx$$

$$= \left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x\right]_0^{\frac{1}{\sqrt{2}}} + \left[x - \frac{x^2}{2}\right]_{\frac{1}{\sqrt{2}}}^{1}$$

$$= \frac{1}{4} + \frac{\pi}{8} + \left(1 - \frac{1}{\sqrt{2}}\right) - \frac{1}{2}\left(1 - \frac{1}{2}\right)$$

$$= \frac{\sqrt{2}-1}{\sqrt{2}} + \frac{\pi}{8}.$$

and lower Riemann integral

$$= \int_0^1 f(x).dx$$

$$= \int_0^{\frac{1}{\sqrt{2}}} (1-x).dx + \int_{\frac{1}{\sqrt{2}}}^{1} \sqrt{1-x^2}.dx$$

$$= \left[x - \frac{x^2}{2}\right]_0^{\frac{1}{\sqrt{2}}} + \left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}.\sin^{-1} x\right]_{\frac{1}{\sqrt{2}}}^{1}$$

$$= \frac{1}{\sqrt{2}} - \frac{1}{4} - \frac{1}{4} + \frac{\pi}{4} - \frac{\pi}{8}$$

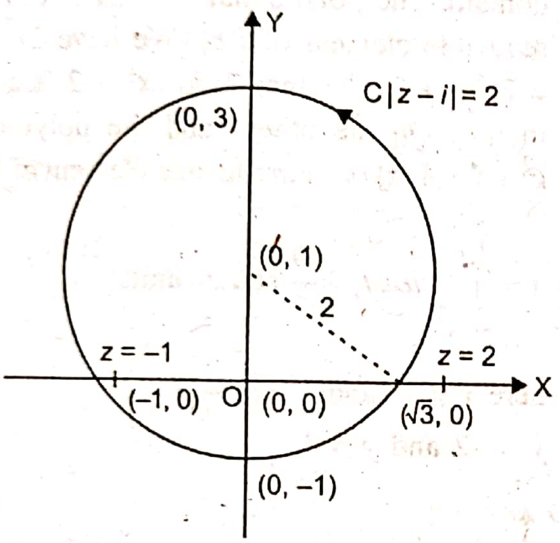
$$= \frac{\sqrt{2}-1}{2} + \frac{\pi}{8}.$$

1. (*c*) Poles of the integrand are given by

$$(z+1)^2 (z-2) = 0$$

$$\Rightarrow \qquad z = -1, 2$$

$z = -1$ is a double pole while $z = 2$ is a simple pole.



The given curve C is a circle with centre at (0, 1) and radius 2.

Clearly, the pole $z = -1$ lies inside the given circle while the pole $z = 2$ lies outside it.

Hence,

$$\oint_C \frac{z+2}{(z+1)^2(z-2)}\,dz = \oint_C \frac{\left(\frac{z+2}{z-2}\right)}{(z+1)^2}\,dz$$

$$= \frac{2\pi i}{1!}\left\{\frac{d}{dz}\left(\frac{z+2}{z-2}\right)\right\}_{z=-1}$$

By Cauchy's Integral formula for derivative

$$= 2\pi i\left\{\frac{-4}{(z-2)^2}\right\}_{z=-1} = -\frac{8\pi i}{9}.$$

1. (*d*) We first check whether the problem is balanced or not. It is, so we proceed with the North-West Corner cell (1, 1).

	R_1		R_2		R_3		R_4		R_5		
F_1	50	1		9		13		36		51	50/0
F_2	50	24	50	12		16		20		1	100/50/0
F_3		14	20	35	50	1	40	23	40	26	150/130/80/40/0
	100/50/0		70/20/0		50/0		40/0		40/0		

Since 50 < 100, we ship 50 by cell (1, 1); we slash the row 1 with a zero (/ 0) since b_1 has been reduced to zero. In addition, a_1 is reduced to 50. The remaining matrix (after neglecting first row), has the cell (2, 1) (corresponding to $F_2 - R_1$) as its north-west corner cell. Again $a_1 = 50 < b_2 = 100$. So we ship 50 by the cell (2, 1) and now $b_2 = 50$ and $a_1 = 0$. So we neglect column 1 and proceed. The cell (2, 2) is now the north-west corner cell for the new table and $b_2 = 50$, $a_2 = 70$ $(b_2 < a_2)$. So we ship 50 by cell (2, 2) and now $a_2 = 20$, $b_2 = 0$. So we neglect row 2 and proceed.

Finally, the initial basic feasible solution is:

$x_{11} = 50$, $x_{21} = 50$, $x_{22} = 50$, $x_{32} = 20$, $x_{33} = 50$, $x_{34} = 40$, $x_{35} = 40$.

The corresponding transportation cost is given by

C = 50 (1) + 50 (24) + 50 (12) + 20 (35) + 50 (1) + 40 (23) + 40 (26) = ₹ 4560.

2. (*b*) A function that is not continuous at a point p_0 is said to be discontinuous there. The term discontinuity is used in two ways. The first refers to a point at which the function is defined but is not continuous. For example, consider the function *f* described by

$$f(p) = f(x, y) = \begin{cases} x^2 + y^2 & \text{when } |p| \le 1 \\ 0 & \text{when } |p| > 1 \end{cases}$$

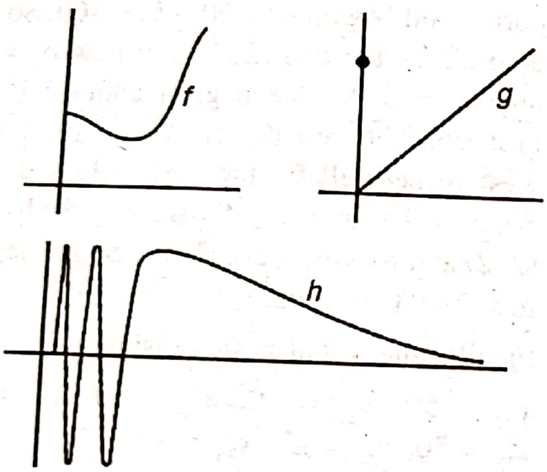
This function is defined in the whole plane and is continuous there except at the points p with $|p| = 1$; each point of this circle is thus a discontinuity for *f*.

In its second usage, the term discontinuity is also applied to points where a function is not defined. For example, the function described by $f(x) = 1/x$ might be said to be discontinuous (or to have a discontinuity) at $x = 0$. For example,

$$f(x) = x^x \qquad \text{when } x > 0$$

$$g(x) = \begin{cases} x & \text{when } x > 0 \\ 2 & \text{when } x = 0 \end{cases}$$

$$h(x) = \sin\left(\frac{1}{x}\right) \qquad \text{when } x > 0$$



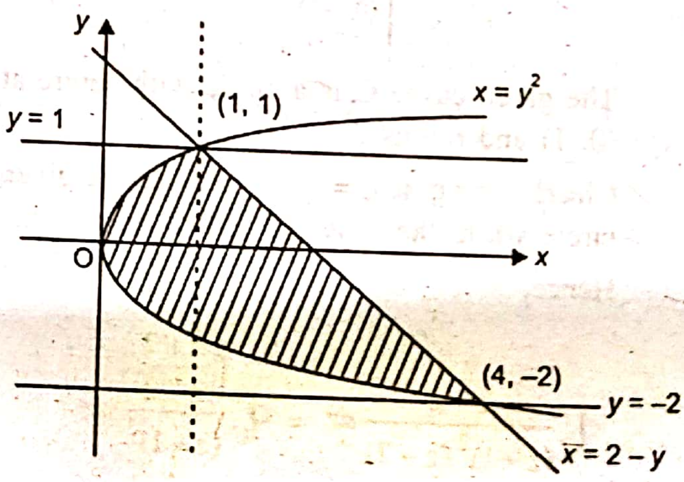
All are continuous on the open interval $0 < x$, and can be said to have discontinuities at the origin. However, this is a removable discontinuity for f and g, and an essential discontinuity for h (see fig.). To explain this, we observe that f is not defined for $x = 0$, but that it may be shown that $\lim_{x\to 0} f(x) = 1$; if we set $f(0) = 1$, the extended function is now continuous on the closed interval $0 \le x$. The function g is defined at the origin, but $g(0) = 2 \neq 0 = \lim_{x\to 0} g(x)$. If we alter g there so that $g(0) = 0$, then g too is continuous for $0 \le x$. The third function, h is not defined at the origin, nor does $\lim_{x\to 0} h(x)$ exist, and no choice for $h(0)$ will make h continuous there.

3. (a) Polynomial rings over unique factorization domains. Let R be a unique factorization domain. Since R is an integral domain with unity, therefore R[x] is also an integral domain with unity. Also any unit, (inversible element) in R[x] must already be a unit in R. Thus the only units in R[x] are the units of R. A polynomial $p(x) \in$ R [x] is *irreducible* over R *i.e.*, irreducible as an element of R[x] if whenever $p(x) = a(x)\, b(x)$ with $a(x), b(x) \in$ R[x], then one of $a(x)$ or $b(x)$ is a unit in R[x] *i.e.*, a unit in R. For example, if I is the ring of integers, then I is a unique factorization domain. The polynomial $2x^2 + 4 \in$ I[x] is a reducible element of I[x]. We have $2x^2 + 4 = 2\,(x^2 + 2)$. Neither 2 nor $x^2 + 2$ is a unit in I[x]. On the other hand the polynomial $x^2 + 1 \in$ I [x] is an irreducible element of I [x].

3. (b) $$\mathrm{I} = \int_{-2}^{1}\int_{y^2}^{2-y} dx.dy = \frac{9}{2} \text{ square units}$$

here $x = y^2$ and $x = 2 - y$

$y = -2$ and $y = 1$



By changing the order of integration,

$y = \sqrt{x}$ and $y = 2 - x$

$x = 0$ to $x = 4$

$$\mathrm{I} = 2\times\int_0^1\int_0^{\sqrt{x}} dy.dx + \int_1^4\int_{-\sqrt{x}}^{2-x} dy.dx$$

$$= 2\int_0^1 \sqrt{x}.dx + \int_1^4 (2 - x - \sqrt{x}).dx$$

$$= 2\frac{2}{3}\left[(x)^{3/2}\right]_0^1 + \left[2x - \frac{x^2}{2} + \frac{2}{3}x^{3/2}\right]_1^4$$

$$= \frac{4}{3} + 2(4-1) - \frac{1}{2}(4^2 - 1^2) + \frac{2}{3}(2^3 - 1)$$

$$= \frac{4}{3} + 6 - \frac{15}{2} + \frac{14}{3} = \frac{9}{2} \text{ square units.}$$

3. (*c*) Here, $z_1 = -1$, $z_2 = \infty$, $z_3 = i$

(*i*) $w_1 = i$, $w_2 = 1$, $w_3 = 1 + i$

$$\frac{(z - z_1)(z_3 - z_2)}{(z - z_2)(z_3 - z_1)} = \frac{(w - w_1)(w_3 - w_2)}{(w - w_2)(w_3 - w_1)}$$

$$\frac{(z+1)\left(\frac{z_3}{z_2} - 1\right)}{\left(\frac{z}{z_2} - 1\right)(z_3 - z_1)} = \frac{(w-i)(i)}{(w-1)}$$

$$\frac{(z+1)}{(i+1)} = \frac{i(w-i)}{w-1}$$

$$(z + 1) = \frac{(w-i)}{(w-1)} i(1+i)$$

$$\Rightarrow \qquad z = \frac{i(1+w)}{(w-1)} - 2$$

(*ii*) $w_1 = \infty$, $w_2 = i$, $w_3 = 1$

$$\frac{z+1}{(1+i)} = \frac{\left(\frac{w}{w_1} - 1\right)(1-i)}{(w-i)\left(\frac{w_3}{w_1} - 1\right)}$$

$$\frac{(z+1)}{(1+i)} = \frac{(1-i)}{(w-i)}$$

$$z + 1 = \frac{2}{w-i}$$

$$z = \frac{2 - w + i}{w - i}$$

(*iii*) $w_1 = 0$, $w_2 = \infty$, $w_3 = 1$

$$\frac{z+1}{1+i} = \frac{(w-0)\left(\frac{w_3}{w_2} - 1\right)}{\left(\frac{w}{w_2} - 1\right)(1-0)}$$

$$w = \frac{z+1}{1+i}.$$

5. (*a*) Here, $x_0 = 0$, $x_1 = 2$, $x_2 = 3$, $x_3 = 6$;

$y_0 = 648$, $y_1 = 704$, $y_2 = 729$, $y_3 = 792$

Using Lagrange's formula,

$$f(x) = y_0 \frac{(x - x_1)(x - x_2)(x - x_3)}{(x_0 - x_1)(x_0 - x_2)(x_0 - x_3)}$$

$$+ y_1 \frac{(x - x_0)(x - x_2)(x - x_3)}{(x_1 - x_0)(x_1 - x_2)(x_1 - x_3)}$$

$$+ y_2 \frac{(x - x_0)(x - x_1)(x - x_3)}{(x_2 - x_0)(x_2 - x_1)(x_2 - x_3)}$$

$$+ y_3 \frac{(x - x_0)(x - x_1)(x - x_2)}{(x_3 - x_0)(x_3 - x_1)(x_3 - x_2)}$$

$$= (648)\frac{(x-2)(x-3)(x-6)}{(0-2)(0-3)(0-6)}$$

$$+ (704)\frac{(x-0)(x-3)(x-6)}{(2-0)(2-3)(2-6)}$$

$$+ (729)\frac{(x-0)(x-2)(x-6)}{(3-0)(3-2)(3-6)}$$

$$+ (792)\frac{(x-0)(x-2)(x-3)}{(6-0)(6-2)(6-3)}$$

$$= -18\,(x^3 - 11x^2 + 36x - 36)$$
$$+ 88\,(x^3 - 9x^2 + 18x).$$
$$- 81\,(x^3 - 8x^2 + 12x)$$
$$+ 11\,(x^3 - 5x^2 + 6x)$$
$$= 0.$$

5. (c) We will solve the equation for a uniaxial displacement u:

$$\frac{\partial^2 u}{\partial t^2} = C_0^2 \frac{\partial^2 u}{\partial x^2}$$

This is a linear, homogeneous, second-order partial differential equation.

Partial differential equations of the form

$$A\frac{\partial^2}{\partial x^2}+B\frac{\partial^2}{\partial x\,\partial t}+C\frac{\partial^2}{\partial t^2}+\ldots = 0$$

are classified into

$B^2 - 4AC > 0$ (hyperbolic)

$B^2 - 4AC = 0$ (parabolic)

$B^2 - 4AC < 0$ (elliptic)

In our case,

$$C_0^2 \frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = 0$$

$$A = C_0^2 \quad B = 0 \quad C = -1$$

$$B^2 - 4AC = 4C_0^2 > 0$$

Thus, this is a hyperbolic equation, and there are two principal methods of solution: separation of variables and transformation. The method of separation of variables is more appropriate for standing waves, whereas the method of transformations is better suited for travelling waves. The transformation method (also called the method of characteristics), originally developed by Riemann and Hadamard, transforms the equation to a new set of variables. This transformation is often obtained by experience, although it can also be formally obtained. For the standing-wave solution, the separation of variables for the harmonic wave can be expressed as

$$u(x, t) = u_0\left(\sin\frac{n\pi x}{l}\cos\frac{n\pi C_0 t}{l}\right)$$

where l is a characteristic length and C_0 is the wave velocity. We can express the trigonometric function as

$$\sin\frac{n\pi x}{l}\cos\frac{n\pi C_0 t}{l}$$

$$= \frac{1}{2}\left[\sin\frac{n\pi}{l}(x-C_0 t)+\sin\frac{n\pi}{l}(x+C_0 t)\right]$$

Thus, the displacement u can be expressed as

$u(x, t)$

$$= \frac{u_0}{2}\sin\frac{n\pi}{l}(x-C_0 t)+\frac{u_0}{2}\sin\frac{n\pi}{l}(x+C_0 t)$$

Generalizing the above equation, assuming nonharmonic functions F and G, one would have

$u(x, t) = F(x - C_0 t) + G(x + C_0 t)$.

5. (d) Navier-Stokes equation for incompressible viscous fluid with constant viscosity is

$$\partial q/\partial t + (\mathbf{q} \cdot \nabla)\mathbf{q} = \mathbf{B} - (1/\rho) \times \nabla p + \nu\, \nabla^2 \mathbf{q}. \quad \ldots(i)$$

Let the forces be conservative. Then there exists a force potential V such that $B = -\nabla V$.

Again, by vector calculus

$$\nabla \mathbf{q}^2 = \nabla(\mathbf{q}.\, \mathbf{q})$$

$$= 2[(\mathbf{q}.\, \nabla)\, \mathbf{q} + \mathbf{q} \times \text{curl}\, \mathbf{q}.]$$

or $\quad (\mathbf{q} \cdot \nabla)\mathbf{q} = \nabla(\mathbf{q}^2/2) - \mathbf{q} \times \text{curl}\, \mathbf{q}$

or $\quad (\mathbf{q} \cdot \nabla)\mathbf{q} = \nabla(\mathbf{q}^2/2) - 2\mathbf{q} \times \Omega$

[Taking $\Omega = (1/2) \times \text{curl}\, \mathbf{q}$]

Then (i) reduces to

$$\partial \mathbf{q}/\partial t + \nabla(\mathbf{q}^2/2) - 2\mathbf{q} \times \Omega = -\nabla V - (1/\rho) \times \nabla p + \nu\, \nabla^2 \mathbf{q}$$

or $\partial \mathbf{q}/\partial t - 2\mathbf{q} \times \Omega$

$$= -\nabla(V + p/\rho + \mathbf{q}^2/2) + \nu\nabla^2\mathbf{q}$$

Taking curl of both sides and using the results curl grad = 0 and

$$\text{curl}\, (\partial \mathbf{q}/\partial t) = \partial(\text{curl}\, \mathbf{q})/\partial t$$

$$= 2(\partial\Omega/\partial t) \text{ and}$$

$$\text{curl}\, \nabla^2\mathbf{q} = \nabla^2\, \text{curl}\, \mathbf{q} = 2\nabla^2\Omega,$$

we obtain

$\partial\Omega/\partial t$ – curl $(\mathbf{q} \times \Omega)$

$$= \nu\, \nabla^2\Omega$$

or $\partial\Omega/\partial t - [\mathbf{q}\ \text{div}\ \Omega - \Omega\ \text{div}\ \mathbf{q} + (\Omega\,.\,\nabla)\mathbf{q} - (\mathbf{q}\,.\,\nabla)\Omega] = \nu\nabla^2\Omega$

or $\partial\Omega/\partial t + (\mathbf{q}.\,\nabla)\Omega = (\Omega\,.\,\nabla)\mathbf{q} + \nu\nabla^2\Omega$

[$\because$ Equation of continuity is div $\mathbf{q} = 0$
Also div Ω = div curl $\mathbf{q} = 0$]

or $\qquad D\Omega/Dt = (\Omega\,.\,\nabla)\mathbf{q} + \nu\nabla^2\Omega \qquad ...(ii)$

which is known as vorticity equation or vorticity transport equation.

5. (*e*) If ψ is the stream function, then streamlines are given by

$$\psi = C \text{ (constant)} \qquad ...(i)$$

Given that $f(x, y, \lambda) = 0 \qquad ...(ii)$

represents a system of streamlines, λ being parameter. Then for $\lambda = \lambda'$ (say), (*ii*) must give a streamline which corresponds with (*i*) for $C = C'$. Hence ψ is a function of λ alone. Moreover λ is a function of x and y from (*ii*). Hence, we obtain

$$\frac{\partial\psi}{\partial x} = \frac{d\psi}{d\lambda}\frac{\partial\lambda}{\partial x}$$

and

$$\frac{\partial\psi}{\partial y} = \frac{d\psi}{d\lambda}\frac{\partial\lambda}{\partial y}$$

Again,

$$\frac{\partial^2\psi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d\psi}{d\lambda}\cdot\frac{\partial\lambda}{\partial x}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{d\psi}{d\lambda}\right)\frac{\partial\lambda}{\partial x} + \frac{d\psi}{d\lambda}\frac{\partial}{\partial x}\left(\frac{\partial\lambda}{\partial x}\right)$$

so that

$$\frac{\partial^2\psi}{\partial x^2} = \left\{\frac{d}{d\lambda}\left(\frac{d\psi}{d\lambda}\right)\right\}\frac{\partial\lambda}{\partial x}\frac{\partial\lambda}{\partial x} + \frac{d\psi}{d\lambda}\frac{\partial^2\lambda}{\partial x^2}$$

Thus,

$$\frac{\partial^2\psi}{\partial x^2} = \frac{d^2\psi}{d\lambda^2}\left(\frac{\partial\lambda}{\partial x}\right)^2 + \frac{d\psi}{d\lambda}\frac{\partial^2\lambda}{\partial x^2} \qquad ...(iii)$$

Similarly,

$$\frac{\partial^2\psi}{\partial y^2} = \frac{d^2\psi}{d\lambda^2}\left(\frac{\partial\lambda}{\partial y}\right)^2 + \frac{d\psi}{d\lambda}\frac{\partial^2\lambda}{\partial y^2} \qquad ...(iv)$$

For the irrotational motion,

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \qquad ...(v)$$

Adding (*iii*) and (*iv*) and using (*v*), we get

$$\frac{d^2\psi}{d\lambda^2}\left[\left(\frac{\partial\lambda}{\partial x}\right)^2 + \left(\frac{\partial\lambda}{\partial y}\right)^2\right] + \frac{d\psi}{d\lambda}\left(\frac{\partial^2\lambda}{\partial x^2} + \frac{\partial^2\lambda}{\partial y^2}\right) = 0$$

or

$$\frac{\left[\left(\frac{\partial\lambda}{\partial x}\right)^2 + \left(\frac{\partial\lambda}{\partial y}\right)^2\right]}{\left[\frac{\partial^2\lambda}{\partial x^2} + \frac{\partial^2\lambda}{\partial y^2}\right]} = -\frac{\frac{d\psi}{d\lambda}}{\frac{d^2\psi}{d\lambda^2}} \qquad ...(vi)$$

Since the R.H.S. of (*vi*) is a function of λ alone, the required condition is that the L.H.S. of (*vi*) should be a function of λ alone.

7. (*b*)
$$2x_1 + x_2 + x_3 - 2x_4 = -10 \qquad ...(i)$$
$$4x_1 + 2x_3 + x_4 = 8 \qquad ...(ii)$$
$$3x_1 + 2x_2 + 2x_3 = 7 \qquad ...(iii)$$
$$x_1 + 3x_2 + 2x_3 - x_4 = -5 \qquad ...(iv)$$

From (*ii*) and (*iv*), we get

$$5x_1 + 3x_2 + 4x_3 = 3 \qquad ...(v)$$

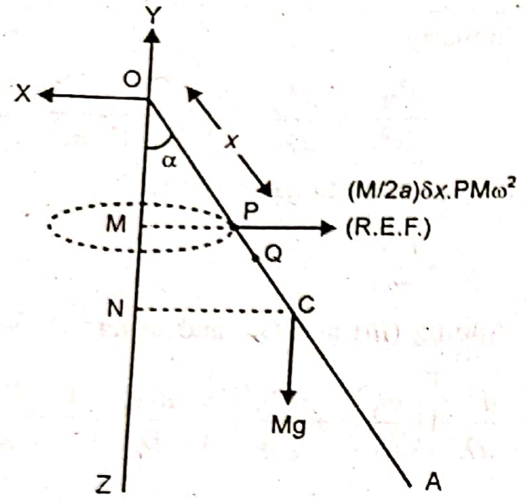
From (*i*) and (*iv*), we get

$$5x_2 + 3x_3 = 0 \qquad ...(vi)$$

From (*iii*), (*v*) and (*vi*), we get

$x_1 = 5,\ x_2 = 6,\ x_3 = -10,\ x_4 = 8.$

7. (*c*) Let the rod OA of length $2a$ and mass M revolve with uniform angular velocity ω about the vertical OZ through O, inclined at a constant angle α to OZ. Let PQ = δx be an element angle of the rod at a distance x from O. The mass of the element PQ is $\frac{M}{2a}\delta x$.



The element PQ will make a circle in the horizontal plane with radius PM (= $x \sin \alpha$) and the centre at M. Since the rod revolves with uniform angular velocity, the only effective force on this element is

$$\frac{M}{2a}\delta x.\,PM\,\omega^2 \text{ along PM.}$$

Thus, the reversed effective force on the element PQ is

$$\frac{M}{2a}\delta x.\,x \sin\alpha.\,\omega^2 \text{ along MP.}$$

Now by D'Alembert's principle all the reversed effective forces acting at different points of the rod, and the external forces, weight Mg and reaction at O, are in equilibrium.

To avoid reaction at O, taking moments about O, we get

$$\sum\left[\left(\frac{M}{2a}\delta x.\omega^2.x\sin\alpha\right).OM\right] - Mg.NG = 0$$

or

$$\int_0^{2a}\frac{M}{2a}\omega^2 x^2 \sin\alpha\cos\alpha\,dx - Mg.a\sin\alpha = 0$$

$$(\because OM = x\cos\alpha)$$

or

$$\frac{M}{2a}\omega^2.\left\{\frac{1}{3}(2a)^3\right\}.\sin\alpha\cos\alpha - Mg\,a\sin\alpha = 0$$

or

$$Mg\,a\sin\alpha\left(\frac{4a}{3g}\omega^2\cos\alpha - 1\right) = 0.$$

$\therefore$ either $\sin\alpha = 0$ *i.e.*, $\alpha = 0$

or $\quad \frac{4a}{3g}\omega^2\cos\alpha - 1 = 0$

i.e., $\quad \cos\alpha = \frac{3g}{4a\omega^2}$

Hence, the rod is inclined at an angle zero or $\cos^{-1}\left(\frac{3g}{4a\omega^2}\right)$.

Note. If $\omega^2 < \frac{3g}{4a}$, then $\cos\alpha > 1$, therefore in this case $\cos\alpha = \frac{3g}{4a\omega^2}$ gives an impossible value of α *i.e.*, when $\omega^2 < \frac{3g}{4a}$, then $\alpha = 0$ is the only possible value of α.