3

Integration of Irrational Algebraic Fractions

§ 3·1. Integration by rationalisation.

In many problems rationalisation is brought about by multiplying a similar quantity both in numerator and denominator. Sometimes this quantity may differ in sign.

Ex. 1 (a). *Evaluate* $\int \sqrt{\left(\frac{1+x}{1-x}\right)}\, dx$.

(Gorakhpur 1972; Meerut 90 S)

Sol. Multiplying the numerator and the denominator by $\sqrt{(1+x)}$, we have the given integral

$$= \int \frac{\sqrt{(1+x)}}{\sqrt{(1-x)}} \cdot \frac{\sqrt{(1+x)}}{\sqrt{(1+x)}}\, dx = \int \frac{(1+x)}{\sqrt{(1-x^2)}}\, dx$$

$$= \int \frac{1.\, dx}{\sqrt{(1-x^2)}} + \int \frac{x.\, dx}{\sqrt{(1-x^2)}} = \sin^{-1} x + \int \frac{x\, dx}{\sqrt{(1-x^2)}}$$

$$= \sin^{-1} x - \tfrac{1}{2} \int (1-x^2)^{-1/2} (-2x)\, dx$$

$$= (\sin^{-1} x) - \tfrac{1}{2} \{(1-x^2)^{1/2}\}/(1/2), \qquad \text{[By power formula]}$$

$$= \sin^{-1} x - \sqrt{(1-x^2)}.$$

•Ex. 1 (b). *Integrate* $\sqrt{x}/(1+x)$.

Sol. Rationalising the numerator, we have

$$I = \int \frac{\sqrt{x}\, dx}{(1+x)} = \int \frac{x\, dx}{(1+x)\sqrt{x}} = \int \frac{\{(1+x)-1\}}{(1+x)\sqrt{x}}\, dx \qquad \text{(Note)}$$

$$= \int \frac{(1+x)}{(1+x)\sqrt{x}}\, dx - \int \frac{1}{(1+x)\sqrt{x}}\, dx$$

$$= \int \frac{1}{\sqrt{x}}\, dx - \int \frac{1}{(1+x)\sqrt{x}}\, dx$$

$$= \int x^{-1/2}\, dx - \int \frac{1}{(1+x)\sqrt{x}}\, dx = \frac{x^{1/2}}{(1/2)} - \int \frac{1}{(1+x)\sqrt{x}}\, dx.$$

Now let $I_1 = \int \frac{1}{(1+x)\sqrt{x}}\, dx$. Putting $x = t^2$ so that

$dx = 2t\, dt$, we get $I_1 = \int \frac{2t\, dt}{(1+t^2)\, t} = 2 \int \frac{dt}{1+t^2}$

$= 2 \tan^{-1} t = 2 \tan^{-1} \sqrt{x}$.

$\therefore \quad I = 2\sqrt{x} - I_1 = 2\sqrt{x} - 2 \tan^{-1} \sqrt{x}$.

Ex. 2 (a). *Integrate* $\sqrt{[(x-1)/(x+1)]}$.

Sol. Rationalising the numerator by multiplying the numerator and the denominator by $\sqrt{(x-1)}$, we get

$$\int \sqrt{\left(\frac{x-1}{x+1}\right)}\,dx = \int \frac{(x-1)\,dx}{\sqrt{(x^2-1)}} = \int \frac{x\,dx}{\sqrt{(x^2-1)}} - \int \frac{dx}{\sqrt{(x^2-1)}}$$

$$= \tfrac{1}{2}\int (x^2-1)^{-1/2}(2x)\,dx - \cosh^{-1}x$$

$$= \tfrac{1}{2}\{(x^2-1)^{1/2}\}/(1/2) - \cosh^{-1}x = \sqrt{(x^2-1)} - \cosh^{-1}x.$$

Ex. 2 (b). *Evaluate* $\displaystyle\int \sqrt{\left(\frac{x+1}{x-1}\right)}\,dx$.

Sol. Proceed as in Ex. 2 (a).

Ans. $\sqrt{(x^2-1)} + \cosh^{-1}x$.

Ex. 3. *Evaluate* $\displaystyle\int \frac{dx}{\sqrt{(1+x)}+\sqrt{x}}$. **(Vikram 1973)**

Sol. Rationalising the denominator, we have

$$\int \frac{dx}{\sqrt{(1+x)}+\sqrt{x}} = \int \frac{\{\sqrt{(1+x)}-\sqrt{x}\}\,dx}{\{\sqrt{(1+x)}+\sqrt{x}\}\{\sqrt{(1+x)}-\sqrt{x}\}}$$

$$= \int \frac{\{\sqrt{(1+x)}-\sqrt{x}\}}{(1+x)-x}\,dx = \int [\sqrt{(1+x)}-\sqrt{x}]\,dx$$

$$= \tfrac{2}{3}\cdot(1+x)^{3/2} - \tfrac{2}{3}x^{3/2}.$$

Ex. 4. *Integrate* $1/[x+\sqrt{(x^2-1)}]$. **(Ranchi 1975)**

Sol. Rationalising the denominator, we have

$$\int \frac{dx}{[x+\sqrt{(x^2-1)}]} = \int \frac{\{x-\sqrt{(x^2-1)}\}}{x^2-(x^2-1)}\,dx$$

$$= \int [x-\sqrt{(x^2-1)}]\,dx = \int x\,dx - \int \sqrt{(x^2-1)}\,dx$$

$$= \tfrac{1}{2}x^2 - \tfrac{1}{2}x\sqrt{(x^2-1)} + \tfrac{1}{2}\log\{x+\sqrt{(x^2-1)}\}.$$

***Ex. 5.** *Evaluate* $\displaystyle\int \frac{dx}{\sqrt{(x+a)}+\sqrt{(x+b)}}$.

Sol. Rationalising the denominator, we have

$$\int \frac{dx}{\sqrt{(x+a)}+\sqrt{(x+b)}} = \int \frac{\sqrt{(x+b)}-\sqrt{(x+a)}}{(x+b)-(x+a)}\,dx$$

$$= \int \frac{(x+b)^{1/2}-(x+a)^{1/2}}{b-a}\,dx = \frac{1}{b-a}\left[\tfrac{2}{3}(x+b)^{3/2} - \tfrac{2}{3}(x+a)^{3/2}\right]$$

$$= \tfrac{2}{3}\cdot\frac{1}{(b-a)}[(x+b)^{3/2}-(x+a)^{3/2}].$$

Ex. 6 (a). *Evaluate* $\displaystyle\int \frac{1}{x}\sqrt{\left(\frac{x-1}{x+1}\right)}\,dx$.

Sol. $\displaystyle\int \frac{1}{x}\sqrt{\left(\frac{x-1}{x+1}\right)}\,dx = \int \frac{1}{x}\cdot\frac{(x-1)}{\sqrt{(x^2-1)}}\,dx$

$$= \int \frac{x}{x\sqrt{(x^2-1)}}dx - \int \frac{1}{x\sqrt{(x^2-1)}}dx$$

$$= \int \frac{dx}{\sqrt{(x^2-1)}} - \int \frac{dx}{x\sqrt{(x^2-1)}} = \cosh^{-1}x - \sec^{-1}x.$$

Ex. 6 (b). *Evaluate* $\int \frac{dx}{x\sqrt{(1+x^n)}}$. **(Meerut 1982)**

Sol. The given integral

$I = \int \frac{x^{n-1}\,dx}{x^n\sqrt{(1+x^n)}}$, multiplying above and below by x^{n-1}.

Put $1 + x^n = t^2$ so that $nx^{n-1}\,dx = 2t\,dt$.

$$\therefore \quad I = \int \frac{2t\,dt}{n(t^2-1)t} = \frac{2}{n}\int \frac{dt}{t^2-1}$$

$$= \frac{2}{n}\cdot\frac{1}{2}\log\frac{t-1}{t+1} = \frac{1}{n}\log\frac{\sqrt{(1+x^n)}-1}{\sqrt{(1+x^n)}+1}.$$

Ex. 6 (c). *Show that* $\int \frac{dx}{x\sqrt{(1+x^3)}} = \frac{1}{3}\log\frac{\sqrt{(1+x^3)}-1}{\sqrt{(1+x^3)}+1}$.

Sol. Multiply above and below by x^2 and proceed as in Ex. 6 (b).

Ex. 6 (d). *Evaluate* $\int \frac{dx}{x+\sqrt{(x-1)}}$.

Sol. Put $x - 1 = t^2$ so that $dx = 2t\,dt$. Then the given integral

$$I = \int \frac{2t\,dt}{t^2+1+t} = \int \frac{(2t+1)-1}{t^2+t+1}dt$$

$$= \int \frac{2t+1}{t^2+t+1}dt - \int \frac{1}{t^2+t+1}dt$$

$$= \log(t^2+t+1) - \int \frac{1}{(t+\frac{1}{2})^2+(\sqrt{3}/2)^2}dt$$

$$= \log(t^2+t+1) - \frac{2}{\sqrt{3}}\tan^{-1}\frac{t+\frac{1}{2}}{\sqrt{3}/2}$$

$$= \log(t^2+t+1) - \frac{2}{\sqrt{3}}\tan^{-1}\frac{2t+1}{\sqrt{3}}$$

$$= \log[x+\sqrt{(x-1)}] - \frac{2}{\sqrt{3}}\tan^{-1}\frac{2\sqrt{(x-1)}+1}{\sqrt{3}}.$$

Ex. 6 (e). *Evaluate* $\int \sqrt{\left(\frac{x+1}{x+2}\right)}\frac{dx}{x+3}$.

Sol. Put $\frac{x+1}{x+2} = t^2$; $\quad \therefore \quad x = \frac{2t^2-1}{1-t^2} = -2 + \frac{1}{1-t^2}$.

$\therefore \quad x + 3 = \frac{2-t^2}{1-t^2}$ and $dx = \frac{2t\,dt}{(1-t^2)^2}$.

$\therefore$ the given integral

$$I = \int t \cdot \frac{1-t^2}{(2-t^2)} \cdot \frac{2t\,dt}{(1-t^2)^2} = \int \frac{2t^2}{(2-t^2)(1-t^2)}\,dt$$

$$= \int \left[\frac{2}{1-t^2} - \frac{4}{2-t^2}\right] dt, \text{ resolving into partial fractions}$$

$$= \log\frac{1+t}{1-t} - \sqrt{2}\log\frac{\sqrt{2}+t}{\sqrt{2}-t}, \text{ where } t = \sqrt{\left(\frac{x+1}{x+2}\right)}.$$

§ 3·2. Integration of $\dfrac{1}{(ax+b)\sqrt{(cx+d)}}$.

In such problems, put $cx + d = t^2$, so that $c\,dx = 2t\,dt$; then the fraction reduces to a form which can be easily integrated.

Ex. 7 (a). *Integrate* $1/[(x+2)\sqrt{(x+1)}]$. **(Meerut 1970; Ranchi 76)**

Sol. Put $(x+1) = t^2$, so that $dx = 2t\,dt$.

$$\therefore \int \frac{dx}{(x+2)\sqrt{(x+1)}} = \int \frac{2t\,dt}{(t^2+1).t} = 2\int \frac{dt}{t^2+1}$$

$$= 2\tan^{-1} t = 2\tan^{-1}[\sqrt{(x+1)}].$$

Ex. 7 (b). *Evaluate* $\displaystyle\int \frac{dx}{(x+2)\sqrt{(x+3)}}$. **(Meerut 1981)**

Sol. Proceed exactly as in Ex. 7 (a). The answer is

$$\log\frac{\sqrt{(x+3)}-1}{\sqrt{(x+3)}+1}.$$

Ex. 7 (c). *Evaluate* $\displaystyle\int \frac{dx}{(x+2)\sqrt{(x-1)}}$. **(Meerut 1983)**

Sol. Proceed as in Ex. 7 (a). The answer is

$$\frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{\sqrt{(x-1)}}{\sqrt{3}}\right].$$

Ex. 8. *Integrate* $1/[(x-3)\sqrt{(x+2)}]$. **(Meerut 1976)**

Sol. Put $x + 2 = t^2$ so that $dx = 2t\,dt$.

$$\therefore \int \frac{dx}{(x-3)\sqrt{(x+2)}} = \int \frac{2t\,dt}{[(t^2-2)-3].t}$$

$$= 2\int \frac{dt}{t^2-5} = 2\int \frac{dt}{t^2-(\sqrt{5})^2} = 2\cdot\frac{1}{2\sqrt{5}}\log\left\{\frac{t-\sqrt{5}}{t+\sqrt{5}}\right\}$$

$$= \frac{1}{\sqrt{5}}\log\left\{\frac{\sqrt{(x+2)}-\sqrt{5}}{\sqrt{(x+2)}+\sqrt{5}}\right\}.$$

Ex. 9. *Integrate* $1/[(2x+1)\sqrt{(4x+3)}]$.

Sol. Put $4x + 3 = t^2$, so that $4\,dx = 2t\,dt$ and

$$(2x+1) = \frac{2(t^2-3)}{4} + 1 = \frac{t^2-3}{2} + 1 = \frac{t^2-1}{2}.$$

$$\therefore \int \frac{dx}{[(2x+1)\sqrt{(4x+3)}]} = \int \frac{\frac{1}{2}t\,dt}{\frac{1}{2}(t^2-1)\,t} = \int \frac{dt}{(t^2-1)}$$

$$= \tfrac{1}{2}\log\left\{\frac{t-1}{t+1}\right\} = \tfrac{1}{2}\log\frac{\sqrt{(4x+3)}-1}{\sqrt{(4x+3)}+1}.$$

Ex. 10. *Evaluate* $\int \frac{x+1}{(x-1)\sqrt{(x+2)}}\,dx$.

Sol. Put $x+2=t^2$, so that $dx = 2t\,dt$. Also $x = t^2-2$, $x+1 = t^2-2+1 = t^2-1$, and $x-1 = t^2-2-1 = t^2-3$.

$$\therefore \int \frac{(x+1)\,dx}{(x-1)\sqrt{(x+2)}} = \int \frac{(t^2-1)\,.2t\,dt}{(t^2-3)\,.\,t}$$

$$= 2\int \frac{t^2-1}{t^2-3}\,dt = 2\int \frac{(t^2-3)+2}{t^2-3}\,dt \qquad \text{(Note)}$$

$$= 2\int \left(1+\frac{2}{t^2-3}\right)dt = 2\left[t+\frac{2}{2\sqrt{3}}\log\frac{t-\sqrt{3}}{t+\sqrt{3}}\right]$$

$$= 2\left[\sqrt{(x+2)}+\frac{1}{\sqrt{3}}\log\frac{\sqrt{(x+2)}-\sqrt{3}}{\sqrt{(x+2)}+\sqrt{3}}\right].$$

Ex. 11. *Evaluate* $\int \frac{x^2\,dx}{(x-1)\sqrt{(x+2)}}$.

Sol. Put $(x+2)=t^2$, so that $dx = 2t\,dt$. Also $x = t^2-2$.

$$\therefore \int \frac{x^2\,dx}{(x-1)\sqrt{(x+2)}} = \int \frac{(t^2-2)^2.\,2t\,dt}{(t^2-3)\,.\,t} = 2\int \frac{t^4-4t^2+4}{t^2-3}\,dt$$

$= 2\int [t^2-1+\{1/(t^2-3)\}]\,dt$, dividing the numerator by the denominator

$$= 2[\tfrac{1}{3}t^3 - t + \{1/(2\sqrt{3})\}\log\{(t-\sqrt{3})/(t+\sqrt{3})\}]$$

$$= 2\left[\frac{(x+2)^{3/2}}{3} - \sqrt{(x+2)} + \frac{1}{2\sqrt{3}}\log\frac{\sqrt{(x+2)}-\sqrt{3}}{\sqrt{(x+2)}+\sqrt{3}}\right].$$

Ex. 12. *Integrate* $x^3/\{(x-1)\sqrt{(x-2)}\}$. **(Meerut 1983 S, 73)**

Sol. Put $(x-2)=t^2$, so that $dx = 2t\,dt$. Also $x = t^2+2$.

$$\therefore \int \frac{x^3\,dx}{(x-1)\sqrt{(x-2)}} = \int \frac{(t^2+2)^3.\,2t\,dt}{(t^2+2-1)\sqrt{t^2}}$$

$$= \int \frac{2(t^2+2)^3\,dt}{(t^2+1)} = 2\int \frac{(t^6+6t^4+12t^2+8)\,dt}{(t^2+1)}$$

$= 2\int \left[t^4+5t^2-3+\frac{11}{(t^2+1)}\right]dt$, by actual division

$$= 2[\tfrac{1}{5}t^5+\tfrac{5}{3}t^3-3t+11\tan^{-1}t]$$

$$= \tfrac{2}{5}(x-2)^{5/2}+\tfrac{10}{3}(x-2)^{3/2}-6(x-2)^{1/2}+22\tan^{-1}\{\sqrt{(x-2)}\}.$$

§ 3·3. Integration of $1/\{(ax^2 + bx + c)\sqrt{(Ax + B)}\}$.

Such fractions are integrated by putting $Ax + B = t^2$.

***Ex. 13.** *Integrate* $1/\{(x^2 + 1)\sqrt{x}\}$.

Sol. Put $x = t^2$, so that $dx = 2t\,dt$.

$$\therefore \quad I = \int \frac{dx}{(x^2+1)\sqrt{x}} = \int \frac{2t\,dt}{(t^4+1).t} = \int \frac{2\,dt}{1+t^4},$$

[Note the form of the integrand]

$$= \int \frac{(2/t^2)\,dt}{t^2 + (1/t^2)},$$ dividing the numerator and the denominator by t^2

$$= \int \frac{\{1 + (1/t^2)\} - \{1 - (1/t^2)\}}{t^2 + (1/t^2)}\,dt$$

$$= \int \frac{\{1 + (1/t^2)\}\,dt}{t^2 + (1/t^2)} - \int \frac{\{1 - (1/t^2)\}\,dt}{t^2 + (1/t^2)}$$

$$= \int \frac{\{1 + (1/t^2)\}\,dt}{\{t - (1/t)\}^2 + 2} - \int \frac{\{1 - (1/t^2)\}\,dt}{\{t + (1/t)\}^2 - 2}.$$

In the first integral put $t - (1/t) = u$ so that $\{1 + (1/t^2)\}\,dt = du$. In the second integral put $t + (1/t) = v$ so that $\{1 - (1/t^2)\}\,dt = dv$.

$$\text{Then} \quad I = \int \frac{du}{u^2+2} - \int \frac{dv}{v^2-2}$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left[\frac{u}{\sqrt{2}}\right] - \frac{1}{2\sqrt{2}}\log\left[\frac{v-\sqrt{2}}{v+\sqrt{2}}\right]$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\frac{\{t - (1/t)\}}{\sqrt{2}} - \frac{1}{2\sqrt{2}}\log\frac{\{t + (1/t)\} - \sqrt{2}}{\{t + (1/t)\} + \sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\frac{x-1}{\sqrt{(2x)}} - \frac{1}{2\sqrt{2}}\log\left[\frac{x - \sqrt{(2x)} + 1}{x + \sqrt{(2x)} + 1}\right], \quad [\because t^2 = x].$$

Ex. 14. *Integrate* $1/\{x^2\sqrt{(x+1)}\}$.

Sol. Put $(x + 1) = t^2$, so that $dx = 2t\,dt$. Also $x = t^2 - 1$.

$$\therefore \quad \int \frac{dx}{x^2\sqrt{(x+1)}} = \int \frac{2t\,dt}{(t^2-1)^2.t} = 2\int \frac{dt}{(t+1)^2(t-1)^2}$$

$$= \int \frac{1}{2}\left[\frac{1}{(t+1)^2} + \frac{1}{(t+1)} + \frac{1}{(t-1)^2} - \frac{1}{(t-1)}\right]dt,$$

by partial fractions

$$= \frac{1}{2}\int \frac{dt}{(t+1)^2} + \frac{1}{2}\int \frac{dt}{(t+1)} + \frac{1}{2}\int \frac{dt}{(t-1)^2} - \frac{1}{2}\int \frac{dt}{(t-1)}$$

$$= -\tfrac{1}{2}\{1/(t+1)\} + \tfrac{1}{2}\log(t+1) - \tfrac{1}{2}\{1/(t-1)\} - \tfrac{1}{2}\log(t-1)$$

$$= -\tfrac{1}{2}[1/(t+1) + 1/(t-1)] + \tfrac{1}{2}\log\{(t+1)/(t-1)\}$$

$$= -\frac{\sqrt{(x+1)}}{x} + \tfrac{1}{2}\log\left\{\frac{\sqrt{(x+1)}+1}{\sqrt{(x+1)}-1}\right\}.$$

Ex. 15. *Integrate* $1/\{(x^2-4)\sqrt{(x+1)}\}$.

Sol. Put $x+1=t^2$, so that $dx = 2t\,dt$. Also $x^2 = t^2-1$.

$$\therefore \quad \int \frac{dx}{(x^2-4)\sqrt{(x+1)}} = \int \frac{2t\,dt}{\{(t^2-1)^2-2^2\}\,t}$$

$$= 2\int \frac{dt}{(t^2-1+2)(t^2-1-2)} = 2\int \frac{dt}{(t^2+1)(t^2-3)}$$

$$= \tfrac{1}{2}\int \left[\frac{1}{(t^2-3)} - \frac{1}{(t^2+1)}\right] dt, \text{ by partial fractions}$$

$$= \tfrac{1}{2}\int \frac{dt}{t^2-3} - \tfrac{1}{2}\int \frac{dt}{t^2+1}$$

$$= \tfrac{1}{2}\cdot\frac{1}{2\sqrt{3}}\log\left\{\frac{t-\sqrt{3}}{t+\sqrt{3}}\right\} - \tfrac{1}{2}\tan^{-1} t$$

$$= \frac{1}{4\sqrt{3}}\log\left\{\frac{\sqrt{(x+1)}-\sqrt{3}}{\sqrt{(x+1)}+\sqrt{3}}\right\} - \tfrac{1}{2}\tan^{-1}\{\sqrt{(x+1)}\}.$$

Ex. 16. *Integrate* $(x+2)/\{(x^2+3x+3)\sqrt{(x+1)}\}$.

Sol. Put $x+1=t^2$, so that $dx = 2t\,dt$. Also $x = t^2-1$ and $x^2+3x+3 = (t^2-1)^2+3(t^2-1)+3 = t^4+t^2+1$.

$$\therefore \quad I = \int \frac{(x+2)\,dx}{(x^2+3x+3)\sqrt{(x+1)}} = \int \frac{(t^2+1)\,.\,2t\,dt}{(t^4+t^2+1)\,.\,t}$$

$$= 2\int \frac{t^2+1}{t^4+t^2+1}dt, \qquad \text{[Note the form of the integrand]}$$

$$= 2\int \frac{\{1+(1/t^2)\}}{t^2+1+(1/t^2)}dt, \text{ dividing the numerator and denominator by } t^2$$

$$= 2\int \frac{1+(1/t^2)}{\{t-(1/t)\}^2+3}dt.$$

Now put $t-(1/t) = z$ so that $\{1+(1/t^2)\}\,dt = dz$.

$$\therefore \quad I = 2\int \frac{dz}{z^2+3} = \frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{z}{\sqrt{3}}\right)$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left\{\frac{t-(1/t)}{\sqrt{3}}\right\} = \frac{2}{\sqrt{3}}\tan^{-1}\left\{\frac{(t^2-1)}{t\sqrt{3}}\right\}$$

$$= \frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{(x+1)-1}{\sqrt{(x+1)}\,.\,\sqrt{3}}\right] = \frac{2}{\sqrt{3}}\tan^{-1}\left[\frac{x}{\sqrt{\{3(x+1)\}}}\right].$$

§ 3·4. Integration of $1/\sqrt{(ax^2+bx+c)}$.

We can express ax^2+bx+c as $a\,\{x^2+(b/a)x+(c/a)\}$

or $$a\left\{\left(x+\frac{b}{2a}\right)^2+\frac{c}{a}-\frac{b^2}{4a^2}\right\}$$

or $$a\left[\left(x+\frac{b}{2a}\right)^2+\frac{4ac-b^2}{4a^2}\right].$$

This is of the form $a\{(x+\alpha)^2 \pm \beta^2\}$.

Thus the given integral can be reduced to one of the standard forms

$$\int \frac{dx}{\sqrt{(x^2+a^2)}}, \int \frac{dx}{\sqrt{(x^2-a^2)}} \text{ or } \int \frac{dx}{\sqrt{(a^2-x^2)}}.$$

So it can be easily evaluated.

Ex. 17. *Integrate* $1/\sqrt{(2+x-3x^2)}$.

Sol. We have $$\int \frac{dx}{\sqrt{(2+x-3x^2)}} = \frac{1}{\sqrt{3}}\int \frac{dx}{\sqrt{\{\frac{2}{3}-(x^2-\frac{1}{3}x)\}}}$$

$$= \frac{1}{\sqrt{3}}\int \frac{dx}{\{\frac{2}{3}+\frac{1}{36}-(x^2-\frac{1}{3}x+\frac{1}{36})\}} = \frac{1}{\sqrt{3}}\int \frac{dx}{\sqrt{\{\frac{25}{36}-(x-\frac{1}{6})^2\}}}$$

$$= \frac{1}{\sqrt{3}}\int \frac{dx}{\sqrt{\{(\frac{5}{6})^2-(x-\frac{1}{6})^2\}}}$$

$$= \frac{1}{\sqrt{3}}\sin^{-1}\left\{\frac{(x-\frac{1}{6})}{5/6}\right\} = \frac{1}{\sqrt{3}}\sin^{-1}\left(\frac{6x-1}{5}\right).$$

Note. Remember that $$\int \frac{dx}{\sqrt{\{a^2-(x-b)^2\}}} = \sin^{-1}\left(\frac{x-b}{a}\right).$$

Ex. 18. *Integrate* $1/\sqrt{(2x^2-x+2)}$.

Sol. We have $$\int \frac{dx}{\sqrt{(2x^2-x+2)}} = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{(x^2-\frac{1}{2}x+1)}}$$

$$= \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{(x^2-\frac{1}{2}x+\frac{1}{16}+\frac{15}{16})}} = \frac{1}{\sqrt{2}}\int \frac{dx}{\sqrt{\{(x-\frac{1}{4})^2+(\sqrt{15}/4)^2\}}}$$

$$= \frac{1}{\sqrt{2}}\sinh^{-1}\frac{x-\frac{1}{4}}{\sqrt{(15)}/4} = \frac{1}{\sqrt{2}}\sinh^{-1}\left[\frac{4x-1}{\sqrt{15}}\right].$$

Ex. 19. (a). *Integrate* $1/\sqrt{(2x^2+3x+4)}$.

Sol. We have $$\int \frac{dx}{\sqrt{(2x^2+3x+4)}} = \int \frac{dx}{\sqrt{2}.\sqrt{\{x^2+(3/2)x+2\}}}$$

$$= \int \frac{dx}{\sqrt{2}.\sqrt{\{(x+\frac{3}{4})^2+(2-\frac{9}{16})\}}} = \int \frac{dx}{\sqrt{2}.\sqrt{\{(x+\frac{3}{4})^2+\frac{23}{16}\}}}$$

$$= \int \frac{dx}{\sqrt{2}.\sqrt{\{(x+\frac{3}{4})^2+(\sqrt{23}/4)^2\}}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dt}{\sqrt{\{t^2 + (\sqrt{23}/4)^2\}}}, \text{ putting } x + \tfrac{3}{4} = t \text{ so that } dx = dt$$

$$= \frac{1}{\sqrt{2}} \sinh^{-1} \{t/(\sqrt{23}/4)\} = \frac{1}{\sqrt{2}} \sinh^{-1} \{4t/\sqrt{(23)}\}$$

$$= \frac{1}{\sqrt{2}} \sinh^{-1} \left\{\frac{4x+3}{\sqrt{23}}\right\}.$$

Ex. 19 (b). *Integrate* $\dfrac{1}{\sqrt{(x^2 + x + 1)}}$. **(Meerut 19**

Sol. We have $\displaystyle\int \frac{dx}{\sqrt{(x^2 + x + 1)}}$

$$= \int \frac{dx}{\sqrt{\{(x + \frac{1}{2})^2 + \frac{3}{4}\}}} = \int \frac{dx}{\sqrt{\{(x + \frac{1}{2})^2 + (\frac{1}{2}\sqrt{3})^2\}}}$$

$$= \sinh^{-1} \left(\frac{x + \frac{1}{2}}{\frac{1}{2}\sqrt{3}}\right) = \sinh^{-1} \left(\frac{2x+1}{\sqrt{3}}\right).$$

Ex. 20. *Integrate* $1/\sqrt{(4 + 3x - 2x^2)}$.

Sol. We have $\displaystyle\int \frac{dx}{\sqrt{(4 + 3x - 2x^2)}} = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\{2 + (3/2)x - x^2\}}}$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\{2 - (x^2 - \frac{3}{2}x)\}}} = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{(2 + \frac{9}{16} - (x^2 - \frac{3}{2}x + \frac{9}{16}))}}$$

$$= \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\{(41/16) - (x - 3/4)^2\}}} = \frac{1}{\sqrt{2}} \int \frac{dx}{\sqrt{\{(\sqrt{41}/4)^2 - (x - \frac{3}{4})^2\}}}$$

$$= \frac{1}{\sqrt{2}} \sin^{-1} \left\{\frac{x - \frac{3}{4}}{(\sqrt{41}/4)}\right\} = \frac{1}{\sqrt{2}} \sin^{-1} \left(\frac{4x - 3}{\sqrt{41}}\right).$$

Ex. 21. *Evaluate* $\frac{1}{2} \displaystyle\int \frac{dx}{\sqrt{\{x(1-x)\}}}$.

Sol. We have $\frac{1}{2} \displaystyle\int \frac{dx}{\sqrt{\{x(1-x)\}}} = \frac{1}{2} \int \frac{dx}{\sqrt{(x - x^2)}}$

$$= \frac{1}{2} \int \frac{dx}{\sqrt{\{0 - (x^2 - x)\}}}$$

$$= \frac{1}{2} \int \frac{dx}{\sqrt{\{0 - (x - \frac{1}{2})^2 + \frac{1}{4}\}}} = \frac{1}{2} \int \frac{dx}{\sqrt{\{\frac{1}{4} - (x - \frac{1}{2})^2\}}}$$

$$= \frac{1}{2} \sin^{-1} \frac{x - \frac{1}{2}}{1/2} = \frac{1}{2} \sin^{-1} (2x - 1).$$

Ex. 22. *Integrate* $1/\sqrt{(3x - x^2 - 2)}$. **(Kanpur 1980)**

Sol. We have $\displaystyle\int \frac{dx}{\sqrt{(3x - x^2 - 2)}} = \int \frac{dx}{\sqrt{\{-2 - (x^2 - 3x)\}}}$

$$= \int \frac{dx}{\sqrt{\{-2-(x-\frac{3}{2})^2+\frac{9}{4}\}}}$$

$$= \int \frac{dx}{\sqrt{\{\frac{1}{4}-(x-\frac{3}{2})^2\}}} = \sin^{-1}\left\{\frac{x-\frac{3}{2}}{1/2}\right\} = \sin^{-1}(2x-3).$$

Ex. 23. *Integrate* $1/\sqrt{(1-x-x^2)}$.

Sol. We have $\int \frac{dx}{\sqrt{(1-x-x^2)}} = \int \frac{dx}{\sqrt{\{1-(x^2+x)\}}}$

$$= \int \frac{dx}{\sqrt{\{1-(x+\frac{1}{2})^2+\frac{1}{4}\}}}$$

$$= \int \frac{dx}{\sqrt{\{\frac{5}{4}-(x+\frac{1}{2})^2\}}} = \sin^{-1}\left\{\frac{x+\frac{1}{2}}{\frac{1}{2}\sqrt{5}}\right\} = \sin^{-1}\left(\frac{2x+1}{\sqrt{5}}\right).$$

Ex. 24. *Evaluate* $\int \sqrt{(\sec x - 1)}\, dx$. **(Meerut 1986 P, 84 S, 90 P)**

Sol. We have $\int \sqrt{(\sec x - 1)}\, dx$

$$= \int \frac{\sqrt{(1-\cos x)}}{\sqrt{(\cos x)}} dx = \int \frac{\sqrt{2}.\sin\frac{1}{2}x\,.\,dx}{\sqrt{(2\cos^2\frac{1}{2}x-1)}}.$$

Now putting $\sqrt{2}\cos\frac{1}{2}x = t$ so that $-\frac{1}{2}\sqrt{2}\sin\frac{1}{2}x\,dx = dt$,

we get the required integral $= -2\int dt/\sqrt{(t^2-1)} = -2\cosh^{-1} t$

$$= -2\cosh^{-1}(\sqrt{2}\cos\tfrac{1}{2}x).$$

§ 3·5. Integration of $\sqrt{(ax^2+bx+c)}$.

$\sqrt{(ax^2+bx+c)}$ can be integrated by reducing ax^2+bx+c to the form $a\,\{(x+\alpha)^2 \pm \beta^2\}$. The given integral can then be easily evaluated by applying one of the following standard results.

$$\int \sqrt{(x^2+a^2)}\,dx = \tfrac{1}{2}x\sqrt{(x^2+a^2)} + \tfrac{1}{2}a^2\sinh^{-1}(x/a)$$

$$= \tfrac{1}{2}x\sqrt{(x^2+a^2)} + \tfrac{1}{2}a^2\log\{x+\sqrt{(x^2+a^2)}\};$$

$$\int \sqrt{(x^2-a^2)}\,dx = \tfrac{1}{2}x\sqrt{(x^2-a^2)} - \tfrac{1}{2}a^2\cosh^{-1}(x/a)$$

$$= \tfrac{1}{2}x\sqrt{(x^2-a^2)} - \tfrac{1}{2}a^2\log[x+\sqrt{(x^2-a^2)}];$$

and $\int \sqrt{(a^2-x^2)}\,dx = \frac{1}{2}x\sqrt{(a^2-x^2)} + \frac{1}{2}a^2\sin^{-1}(x/a)$.

Note that in each result the sign before a^2 in the second term is the same as the sign before a^2 in the expression under the radical sign.

Ex. 25. *Integrate* $\sqrt{(x^2-x+1)}$.

Sol. We have $\int \sqrt{(x^2-x+1)}\,dx = \int \sqrt{\{(x-\frac{1}{2})^2+\frac{3}{4}\}}\,dx$,

[form $\int \sqrt{(x^2+a^2)}\,dx$]

$$= \tfrac{1}{2}(x-\tfrac{1}{2})\sqrt{\{(x-\tfrac{1}{2})^2+\tfrac{3}{4}\}} + \tfrac{1}{2}\cdot(\tfrac{3}{4})\sinh^{-1}\{(x-\tfrac{1}{2})/(\tfrac{1}{2}\sqrt{3})\}$$

$$= \tfrac{1}{2}(x-\tfrac{1}{2})\sqrt{(x^2-x+1)} + \tfrac{3}{8}\sinh^{-1}\{(2x-1)/\sqrt{3}\}.$$

Ex. 26. *Integrate* $\sqrt{(x^2 + x - 1)}\, dx$.

Sol. We have $\int \sqrt{(x^2 + x - 1)}\, dx = \int \sqrt{\{(x + \frac{1}{2})^2 - (\frac{5}{4})\}}$,

[form $\int \sqrt{(x^2 - a^2)}\, dx$]

$= \frac{1}{2}(x + \frac{1}{2}) \sqrt{\{(x + \frac{1}{2})^2 - (\frac{5}{4})\}} - \frac{1}{2} \cdot \frac{5}{4} \cosh^{-1} \{(x + \frac{1}{2})/(\frac{1}{2}\sqrt{5})\}$

$= \frac{1}{4}(2x + 1) \sqrt{(x^2 + x - 1)} - \frac{5}{8} \cosh^{-1} \{(2x + 1)/\sqrt{5}\}$.

Ex. 27. *Integrate* $\sqrt{(2x^2 + 3x + 4)}$.

Sol. We have $\int \sqrt{(2x^2 + 3x + 4)}\, dx = \sqrt{2} \int \sqrt{(x^2 + \frac{3}{2}x + 2)}\, dx$.

$= \sqrt{2} \int \sqrt{\{(x + \frac{3}{4})^2 + (23/16)\}}\, dx$, [form $\int \sqrt{(x^2 + a^2)}\, dx$]

$= \sqrt{2}\, [\frac{1}{2}(x + \frac{3}{4}) \sqrt{\{(x + \frac{3}{4})^2 + \frac{23}{16}\}} + \frac{1}{2} \cdot \frac{23}{16} \sinh^{-1} \{(x + \frac{3}{4})/(\frac{1}{4}\sqrt{23})\}]$

$= \frac{1}{8}(4x + 3) \sqrt{(2x^2 + 3x + 4)} + (23\sqrt{2}/32) \sinh^{-1} \{(4x + 3)/\sqrt{23}\}$.

Ex. 28. *Integrate* $\sqrt{(4 - 3x - 2x^2)}$.

Sol. We have $\int \sqrt{(4 - 3x - 2x^2)}\, dx = \sqrt{2} \int \sqrt{(2 - \frac{3}{2}x - x^2)}\, dx$

$= \sqrt{2} \int \sqrt{\{2 - (x^2 + \frac{3}{2}x)\}}\, dx = \sqrt{2} \int \sqrt{\{2 + \frac{9}{16} - (x^2 + \frac{3}{2}x + \frac{9}{16})\}}\, dx$

$= \sqrt{2} \int \sqrt{\{\frac{41}{16} - (x + \frac{3}{4})^2\}}\, dx$, [form $\int \sqrt{(a^2 - x^2)}\, dx$]

$= \sqrt{2} \int \sqrt{(\frac{41}{16} - t^2)}\, dt$, putting $x + \frac{3}{4} = t$ so that $dx = dt$

$$= \sqrt{2} \cdot \frac{t}{2} \sqrt{\left(\frac{41}{16} - t^2\right)} + \sqrt{2} \cdot \frac{41}{32} \sin^{-1} \{t/(\sqrt{41}/4)\}$$

$$= \sqrt{2} \cdot \frac{x + \frac{3}{4}}{2} \sqrt{\{\frac{41}{16} - (x + \frac{3}{4})^2\}} + \sqrt{2} \cdot \frac{41}{32} \sin^{-1} \left\{\frac{x + \frac{3}{4}}{\sqrt{41}/4}\right\}$$

$$= \frac{4x + 3}{8} \sqrt{(4 - 3x - 2x^2)} + \frac{41\sqrt{2}}{32} \sin^{-1} \left(\frac{4x + 3}{\sqrt{41}}\right).$$

Ex. 29. *Evaluate* $\int \sqrt{\{(x - 1)(2 - x)\}}\, dx$.

Sol. We have $\int \sqrt{\{(x - 1)(2 - x)\}}\, dx = \int \sqrt{(-x^2 + 3x - 2)}\, dx$

$= \int \sqrt{\{-2 - (x^2 - 3x)\}}\, dx$

$= \int \sqrt{\{-2 - (x - \frac{3}{2})^2 + \frac{9}{4}\}}\, dx = \int \sqrt{\{\frac{1}{4} - (x - \frac{3}{2})^2\}}\, dx$,

[form $\int \sqrt{(a^2 - x^2)}\, dx$]

$= \frac{1}{2}(x - \frac{3}{2}) \sqrt{\{\frac{1}{4} - (x - \frac{3}{2})^2\}} + \frac{1}{2} \cdot \frac{1}{4} \sin^{-1} \{(x - \frac{3}{2})/(1/2)\}$

$= \frac{1}{4}(2x - 3) \sqrt{(3x - x^2 - 2)} + \frac{1}{8} \sin^{-1} (2x - 3)$.

§ 3-6. Integration of $(px + q)/\sqrt{(ax^2 + bx + c)}$.

This integral is evaluated by first breaking the numerator into two parts such that one part is a constant multiple of the differential coefficient of $ax^2 + bx + c$ *i.e.*, of $2ax + b$ and the other part is a constant. Thus we put

$px + q = L \times$ diff. coeff. of $(ax^2 + bx + c) + M$

$= L(2ax + b) + M$, L and M are suitable constants.

On comparing coefficients on both sides, we have

$$L = p/(2a) \quad \text{and} \quad M = q - \{(pb)/(2a)\}.$$

$$\therefore \quad \int \frac{(px+q)\,dx}{\sqrt{(ax^2+bx+c)}}$$

$$= \frac{p}{2a}\int \frac{(2ax+b)\,dx}{\sqrt{(ax^2+bx+c)}} + \int \frac{\{q-(pb)/(2a)\}}{\sqrt{(ax^2+bx+c)}}\,dx$$

$$= \frac{p}{a}\sqrt{(ax^2+bx+c)} + \left(q - \frac{pb}{2a}\right)\int \frac{dx}{\sqrt{(ax^2+bx+c)}},$$

[Applying power formula to evaluate the first integral]

The integral on the right can be evaluated by methods discussed earlier.

Ex. 30 (a). *Integrate* $(2x+3)/\sqrt{(x^2+x+1)}$. **(Meerut 1986 P, 91 P)**

Sol. We have $(d/dx)(x^2+x+1) = 2x+1$.

Now $2x+3 = (2x+1)+2$.

$$\therefore \quad \int \frac{(2x+3)\,dx}{\sqrt{(x^2+x+1)}} = \int \frac{(2x+1)\,dx}{\sqrt{(x^2+x+1)}} + \int \frac{2\,dx}{\sqrt{(x^2+x+1)}}$$

$$= \int (x^2+x+1)^{-1/2}(2x+1)\,dx + \int \frac{2\,dx}{\sqrt{(x^2+x+1)}}$$

$$= 2\sqrt{(x^2+x+1)} + 2\int \frac{dx}{\sqrt{\{(x+\frac{1}{2})^2+1-\frac{1}{4}\}}}$$

$$= 2\sqrt{(x^2+x+1)} + 2\int \frac{dx}{\sqrt{\{(x+\frac{1}{2})^2+(\sqrt{3}/2)^2\}}}$$

$$= 2\sqrt{(x^2+x+1)} + 2\sinh^{-1}\left(\frac{x+\frac{1}{2}}{\sqrt{3}/2}\right)$$

$$= 2\sqrt{(x^2+x+1)} + 2\sinh^{-1}\{(2x+1)/\sqrt{3}\}.$$

Ex. 30 (b). *Evaluate* $\displaystyle\int \frac{x+2}{\sqrt{(x^2+3x+1)}}\,dx$. **(Meerut 1988)**

Sol. Here the given integral $I = \displaystyle\int \frac{\frac{1}{2}(2x+3)+\frac{1}{2}}{\sqrt{(x^2+3x+1)}}\,dx$

$$= \tfrac{1}{2}\int (x^2+3x+1)^{-1/2}(2x+3)\,dx$$

$$+ \tfrac{1}{2}\int \frac{dx}{\sqrt{[\{x+(3/2)\}^2-(\sqrt{5}/2)^2]}}$$

$$= (x^2+3x+1)^{1/2} + \tfrac{1}{2}\cosh^{-1}\left[\frac{x+(3/2)}{\sqrt{5}/2}\right]$$

$$= \sqrt{(x^2+3x+1)} + \tfrac{1}{2}\cosh^{-1}[(2x+3)/\sqrt{5}].$$

Ex. 31 (a). *Integrate* $(2x+5)/\sqrt{(x^2+3x+1)}$

(Vikram 1976; Jiwaji 76)

Sol. We have $(d/dx)(x^2 + 3x + 1) = 2x + 3$.

Now $2x + 5 = (2x + 3) + 2$.

$$\therefore \int \frac{(2x+5)\,dx}{\sqrt{(x^2+3x+1)}} = \int \frac{(2x+3)+2}{\sqrt{(x^2+3x+1)}}\,dx$$

$$= \int \frac{(2x+3)\,dx}{\sqrt{(x^2+3x+1)}} + 2\int \frac{dx}{\sqrt{(x^2+3x+1)}}$$

$$= \int (x^2+3x+1)^{-1/2}(2x+3)\,dx + 2\int \frac{dx}{\sqrt{(x^2+3x+1)}}$$

$$= 2\sqrt{(x^2+3x+1)} + 2\int \frac{dx}{\sqrt{\{(x+\frac{3}{2})^2 - (\sqrt{\frac{5}{4}})^2\}}},$$

[evaluating the first integral by power formula]

$$= 2\sqrt{(x^2+3x+1)} + 2\cosh^{-1}\{(x+\tfrac{3}{2})/\sqrt{(\tfrac{5}{4})}\}$$

$$= 2\sqrt{(x^2+3x+1)} + 2\cosh^{-1}\{(2x+3)/\sqrt{5}\}.$$

Ex. 31 (b). *Evaluate* $\displaystyle\int \frac{(x+2)\,dx}{\sqrt{(5-12x-9x^2)}}$. **(Delhi 1983)**

Sol. We have $\displaystyle\int \frac{(x+2)\,dx}{\sqrt{(5-12x-9x^2)}}$

$$= \frac{1}{3}\int \frac{(x+2)\,dx}{\sqrt{(\frac{5}{9}-\frac{4}{3}x-x^2)}}, \text{ taking 9 common from the denominator}$$

$$= \frac{1}{3}\int \frac{-\frac{1}{3}(-2x-\frac{4}{3})+2-\frac{2}{3}}{\sqrt{(\frac{5}{9}-\frac{4}{3}x-x^2)}}\,dx$$

$$= -\frac{1}{3}\int \left(\frac{5}{9}-\frac{4}{3}x-x^2\right)^{-1/2}\left(-2x-\frac{4}{3}\right)dx$$

$$+ \frac{4}{9}\int \frac{dx}{\sqrt{\{\frac{5}{9}-(x^2+\frac{4}{3}x)\}}}$$

$$= -\frac{1}{3}\frac{(\frac{5}{9}-\frac{4}{3}x-x^2)^{1/2}}{\frac{1}{2}} + \frac{4}{9}\int \frac{dx}{\sqrt{\{\frac{5}{9}-(x+\frac{2}{3})^2+\frac{4}{9}\}}}$$

$$= -\frac{2}{3}\sqrt{\left(\frac{5}{9}-\frac{4}{3}x-x^2\right)} + \frac{4}{9}\int \frac{dx}{\sqrt{\{1-(x+\frac{2}{3})^2\}}}$$

$$= -\frac{2}{3}\sqrt{(5-12x-9x^2)} + \frac{4}{9}\sin^{-1}\left(\frac{x+\frac{2}{3}}{1}\right)$$

$$= -\frac{2}{3}\sqrt{(5-12x-9x^2)} + \frac{4}{9}\sin^{-1}\left(\frac{3x+2}{3}\right).$$

Ex. 31 (c). *Evaluate* $\displaystyle\int \frac{x\,dx}{\sqrt{(x^2+4x+5)}}$ **(Delhi 1982)**

Sol. Proceed exactly as in Ex. 31 (a).

The answer is

$$\sqrt{(x^2 + 4x + 5)} - 2\sinh^{-1}(x + 2).$$

Ex. 32. *Integrate* $(x^2 + 2x + 3)/\sqrt{(x^2 + x + 1)}$. **(Jiwaji 1977)**

Sol. We have $\displaystyle\int \frac{(x^2 + 2x + 3)\,dx}{\sqrt{(x^2 + x + 1)}} = \int \frac{(x^2 + x + 1) + (x + 2)}{\sqrt{(x^2 + x + 1)}}\,dx$

(Note)

$$= \int \sqrt{(x^2 + x + 1)}\,dx + \int \frac{\frac{1}{2}(2x + 1) + 2 - \frac{1}{2}}{\sqrt{(x^2 + x + 1)}}\,dx$$

$$= \int \sqrt{\{(x + \tfrac{1}{2})^2 + \tfrac{3}{4}\}}\,dx + \frac{1}{2}\int \frac{2x + 1}{\sqrt{(x^2 + x + 1)}}\,dx + \frac{3}{2}\int \frac{dx}{\sqrt{\{(x + \frac{1}{2})^2 + \frac{3}{4}\}}}$$

$$= \frac{1}{2}(x + \tfrac{1}{2})\sqrt{\{(x + \tfrac{1}{2})^2 + \tfrac{3}{4}\}} + \frac{1}{2}\cdot\frac{3}{4}\sinh^{-1}\{(x + \tfrac{1}{2})/\sqrt{(\tfrac{3}{4})}\} + \sqrt{(x^2 + x + 1)} + \frac{3}{2}\sinh^{-1}\{(x + \tfrac{1}{2})/\sqrt{(\tfrac{3}{4})}\}$$

$$= \frac{1}{4}(2x + 1)\sqrt{(x^2 + x + 1)} + \frac{15}{8}\sinh^{-1}\{(2x + 1)/\sqrt{3}\} + \sqrt{(x^2 + x + 1)}$$

$$= \frac{1}{4}(2x + 5)\sqrt{(x^2 + x + 1)} + \frac{15}{8}\sinh^{-1}\{(2x + 1)/\sqrt{3}\}.$$

Ex. 33. *Integrate* $(x^2 + x + 1)/\sqrt{(x^2 + 2x + 3)}$.

Sol. We have $\displaystyle\int \frac{(x^2 + x + 1)\,dx}{\sqrt{(x^2 + 2x + 3)}} = \int \frac{(x^2 + 2x + 3) - x - 2}{\sqrt{(x^2 + 2x + 3)}}\,dx$

$$= \int \frac{x^2 + 2x + 3}{\sqrt{(x^2 + 2x + 3)}}\,dx - \int \frac{x + 2}{\sqrt{(x^2 + 2x + 3)}}\,dx$$

$$= \int \frac{(x^2 + 2x + 3)\,dx}{\sqrt{(x^2 + 2x + 3)}} - \int \frac{\frac{1}{2}(2x + 2) + 2 - 1}{\sqrt{(x^2 + 2x + 3)}}\,dx$$

$$= \int \sqrt{(x^2 + 2x + 3)}\,dx - \frac{1}{2}\int \frac{(2x + 2)\,dx}{\sqrt{(x^2 + 2x + 3)}} - \int \frac{dx}{\sqrt{(x^2 + 2x + 3)}}$$

$$= \int \sqrt{\{(x + 1)^2 + 2\}}\,dx - \frac{1}{2}\int \frac{(2x + 2)\,dx}{\sqrt{(x^2 + 2x + 3)}} - \int \frac{dx}{\sqrt{\{(x + 1)^2 + 2\}}}$$

$$= I_1 - \tfrac{1}{2}I_2 - I_3.$$

For the 1st integral, put $(x + 1) = t$ so that $dx = dt$.

$$\therefore \quad I_1 = \sqrt{(t^2 + 2)}\,dt = \tfrac{1}{2}t.\sqrt{(t^2 + 2)} + \tfrac{1}{2}\cdot 2\sinh^{-1}(t/\sqrt{2})$$

$$= \tfrac{1}{2}(x + 1)\sqrt{\{(x + 1)^2 + 2\}} + \sinh^{-1}\{(x + 1)/\sqrt{2}\}.$$

For the 2nd integral,

put $(x^2 + 2x + 3) = t$ so that $(2x + 2)\,dx = dt$.

$\therefore \quad I_2 = \int (1/\sqrt{t})\, dt = \int t^{-1/2} dt = 2\sqrt{t} = 2\sqrt{(x^2 + 2x + 3)}$.

For the 3rd integral, put $(x + 1) = t$ so that $dx = dt$.

$$\therefore \quad I_3 = \int \frac{dt}{\sqrt{(t^2 + 2)}} = \sinh^{-1}\left(\frac{t}{\sqrt{2}}\right) = \sinh^{-1}\left\{\frac{x+1}{\sqrt{2}}\right\}.$$

Hence the required integral

$$= \tfrac{1}{2}(x + 1)\sqrt{\{(x + 1)^2 + 2\}} + \sinh^{-1}\{(x + 1)/\sqrt{2}\}$$
$$- \tfrac{1}{2} \cdot 2\sqrt{(x^2 + 2x + 3)} - \sinh^{-1}\{(x + 1)/\sqrt{2}\}$$
$$= \tfrac{1}{2}(x - 1)\sqrt{(x^2 + 2x + 3)}.$$

Note. The values of the integrals I_1, I_2 and I_3 can also be written directly without making the substitutions.

Ex. 34. *Integrate* $(x^3 + 3)/\sqrt{(x^2 + 1)}$. **(Lucknow 1983)**

Sol. We have $\displaystyle\int \frac{(x^3 + 3)\, dx}{\sqrt{(x^2 + 1)}} = \int \frac{x(x^2 + 1) - x + 3}{\sqrt{(x^2 + 1)}} dx$. **(Note)**

$$= \int \frac{x(x^2 + 1)}{\sqrt{(x^2 + 1)}} dx - \int \frac{x\, dx}{\sqrt{(x^2 + 1)}} + 3\int \frac{dx}{\sqrt{(x^2 + 1)}}$$
$$= \tfrac{1}{2}\int (2x)\sqrt{(x^2 + 1)}\, dx - \tfrac{1}{2}\int \frac{2x\, dx}{\sqrt{(x^2 + 1)}} + 3\int \frac{dx}{\sqrt{(x^2 + 1)}}$$
$$= \tfrac{1}{2}[\tfrac{2}{3}(x^2 + 1)^{3/2}] - \tfrac{1}{2}[2\sqrt{(x^2 + 1)}] + 3\sinh^{-1} x$$
$$= \tfrac{1}{3}(x^2 + 1)^{3/2} - \sqrt{(x^2 + 1)} + 3\sinh^{-1} x.$$

§ 3·7. Integration of $(px + q)\sqrt{(ax^2 + bx + c)}$.

We write $px + q = \dfrac{p}{2a}(2ax + b) + \left(q - \dfrac{pb}{2a}\right)$.

$$\therefore \quad \int (px + q)\sqrt{(ax^2 + bx + c)}\, dx$$
$$= \int (p/2a)(2ax + b)\sqrt{(ax^2 + bx + c)}\, dx$$
$$+ \int \{q - (pb/2a)\}\sqrt{(ax^2 + bx + c)}\, dx$$
$$= \frac{p}{2a} \cdot \frac{2}{3}(ax^2 + bx + c)^{3/2} + \left(q - \frac{pb}{2a}\right)\int \sqrt{(ax^2 + bx + c)}\, dx,$$

the first integral being evaluated by applying the power formula.

The integral left on the right hand side can now be evaluated by methods discussed earlier.

Ex. 35. *Integrate* $(3x - 2)\sqrt{(x^2 + x + 1)}$. **(Meerut 1986)**

Sol. Let $(3x - 2) = A$. diff. coeff. of $(x^2 + x + 1) + B$, where A and B are constants

i.e., $\qquad 3x - 2 = A.(2x + 1) + B.$

On comparing the coefficients, we get $3 = 2A$; $\therefore A = \frac{3}{2}$.

Also $\qquad -2 = A + B; \quad \therefore \quad B = -2 - \frac{3}{2} = -7/2.$

Thus the given integral

$= \int [\frac{3}{2}(2x+1) - \frac{7}{2}] \sqrt{(x^2+x+1)}\, dx$

$= \int \frac{3}{2}(2x+1)\sqrt{(x^2+x+1)}\, dx - \frac{7}{2}\int \sqrt{(x^2+x+1)}\, dx$

$= \frac{3}{2}\int (x^2+x+1)^{1/2}(2x+1)\, dx - \frac{7}{2}\int \sqrt{\{(x+\frac{1}{2})^2+\frac{3}{4}\}}\, dx,$

[Apply power formula], [Form $\int \sqrt{(x^2+a^2)}\, dx$]

$= \frac{3}{2}\cdot\frac{2}{3}(x^2+x+1)^{3/2} - \frac{7}{2}[\frac{1}{2}(x+\frac{1}{2})\sqrt{\{(x+\frac{1}{2})^2+\frac{3}{4}\}}$

$+ \frac{1}{2}\cdot\frac{3}{4}\sinh^{-1}\{(x+\frac{1}{2})/\sqrt{(\frac{3}{4})}\}]$

$= (x^2+x+1)^{3/2} - \frac{7}{4}(x+\frac{1}{2})\sqrt{(x^2+x+1)} - \frac{21}{16}\sinh^{-1}\left(\frac{2x+1}{\sqrt{3}}\right).$

***Ex. 36.** *Integrate* $(x+1)\sqrt{(x^2-x+1)}$. **(Kashmir 1975)**

Sol. We have $x+1 = \frac{1}{2}(2x-1) + 1 + \frac{1}{2} = \frac{1}{2}(2x-1) + \frac{3}{2}$.

$\therefore \quad \int (x+1)\sqrt{(x^2-x+1)}\, dx$

$= \int [\frac{1}{2}(2x-1) + \frac{3}{2}]\sqrt{(x^2-x+1)}\, dx$

$= \frac{1}{2}\int (2x-1)\sqrt{(x^2-x+1)}\, dx + \frac{3}{2}\int \sqrt{(x^2-x+1)}\, dx$

$= \frac{1}{2}\cdot\frac{2}{3}\cdot(x^2-x+1)^{3/2} + \frac{3}{2}\int \sqrt{\{(x-\frac{1}{2})^2+\frac{3}{4}\}}\, dx$

$= \frac{1}{3}(x^2-x+1)^{3/2} + \frac{3}{2}\cdot\frac{1}{2}(x-\frac{1}{2})\sqrt{\{(x-\frac{1}{2})^2+\frac{3}{4}\}}$

$+ \frac{3}{2}\cdot\frac{3}{4}\cdot\frac{1}{2}\sinh^{-1}\left\{\frac{x-\frac{1}{2}}{(\sqrt{\frac{3}{4}})}\right\}$

$= (1/24)(8x^2+10x-1)\sqrt{(x^2-x+1)}$

$+ (9/16)\sinh^{-1}\{(2x-1)/\sqrt{3}\}.$

Ex. 37. *Integrate* $(x+1)\sqrt{(2x^2+3)}$.

Sol. We have $\int (x+1)\sqrt{(2x^2+3)}\, dx$

$= \int x\sqrt{(2x^2+3)}\, dx + \int \sqrt{(2x^2+3)}\, dx$

$= \frac{1}{4}\int (4x)\sqrt{(2x^2+3)}\, dx + \int \sqrt{2}\,.\sqrt{(x^2+\frac{3}{2})}\, dx$

$= \frac{1}{4}\cdot\frac{(2x^2+3)^{3/2}}{\frac{3}{2}} + \sqrt{2}\left\{\frac{1}{2}x\,.\sqrt{(x^2+\frac{3}{2})} + \frac{1}{2}\cdot\frac{3}{2}\sinh^{-1}\left(\frac{x\sqrt{2}}{\sqrt{3}}\right)\right\}$

$= \frac{1}{6}(2x^2+3)^{3/2} + \frac{1}{2}x\sqrt{(2x^2+3)} + \frac{3}{4}\sqrt{2}\,.\sinh^{-1}\{(x\sqrt{2})/\sqrt{3}\}.$

Ex. 38. *Integrate* $(2x-5)\sqrt{(2+3x-x^2)}$.

Sol. We have $(2x-5) = -(3-2x) - 2$.

$\therefore \quad \int (2x-5)\sqrt{(2+3x-x^2)}\, dx$

$= -\int (3-2x)\sqrt{(2+3x-x^2)}\, dx - 2\int \sqrt{(2+3x-x^2)}\, dx$

$= -\dfrac{(2+3x-x^2)^{3/2}}{\frac{3}{2}} - 2\int \sqrt{\left\{\frac{17}{4} - \left(x-\frac{3}{2}\right)^2\right\}}\, dx$

$= -\frac{2}{3}(2+3x-x^2)^{3/2} - 2\left[\frac{1}{2}\left(x-\frac{3}{2}\right)\sqrt{\left\{\frac{17}{4} - \left(x-\frac{3}{2}\right)^2\right\}}\right.$

$$+\frac{1}{2}\cdot\frac{17}{4}\sin^{-1}\left\{\frac{x-\frac{3}{2}}{\sqrt{(\frac{17}{4})}}\right\}$$

$$=-\tfrac{2}{3}(2+3x-x^2)^{3/2}-\tfrac{1}{2}(2x-3)\sqrt{(2+3x-x^2)}$$

$$-\frac{17}{4}\sin^{-1}\left(\frac{2x-3}{\sqrt{17}}\right)$$

§ 3·8. To evaluate $\displaystyle\int\frac{dx}{(px+q)\sqrt{(ax^2+bx+c)}}$.

The integral of this form can be easily evaluated by putting $px+q=\dfrac{1}{t}$ so that $p\,dx=-\dfrac{1}{t^2}dt$. Also $x=\dfrac{1}{p}\left(\dfrac{1}{t}-q\right)$.

Now expressing (ax^2+bx+c) in terms of t, we have

$$ax^2+bx+c=\frac{a}{p^2}\left(\frac{1}{t}-q\right)^2+\frac{b}{p}\left(\frac{1}{t}-q\right)+c$$

$=(At^2+Bt+C)/t^2$, say, where A, B, and C are some constants.

Thus $\int[1/\{(px+q)\sqrt{(ax^2+bx+c)}\}]\,dx$

$$=-\frac{1}{p}\int\frac{(1/t^2)\,dt}{(1/t)\,.\,\sqrt{\{(At^2+Bt+C)/t^2\}}}=-\frac{1}{p}\int\frac{dt}{\sqrt{(At^2+Bt+C)}}$$

This can be integrated by article 3·4 on page 104.

Ex. 39. *Integrate* $1/\{(x+1)\sqrt{(x^2+1)}\}$. **(Meerut 1991)**

Sol. Put $(x+1)=1/t$ so that $dx=(-1/t^2)\,dt$. Also $x=(1/t)-1$.

Hence the required integral

$$=\int\frac{-(1/t^2)\,dt}{(1/t)\,.\,\sqrt{[\{(1/t)-1\}^2+1]}}=\int\frac{-(1/t^2)\,dt}{(1/t)\,.\,\sqrt{[\{(1-t)^2+t^2\}/t^2]}}$$

$$=-\int\frac{dt}{\sqrt{(1-2t+t^2+t^2)}}=-\int\frac{dt}{\sqrt{(1-2t+2t^2)}}$$

$$=-\int\frac{dt}{\sqrt{2}\,.\,\sqrt{(t^2-t+\frac{1}{2})}}=-\frac{1}{\sqrt{2}}\int\frac{dt}{\sqrt{\{(t-\frac{1}{2})^2+\frac{1}{2}-\frac{1}{4}\}}}$$

$$=-\frac{1}{\sqrt{2}}\int\frac{dt}{\sqrt{\{(t-\frac{1}{2})^2+(\frac{1}{2})^2\}}}=-\frac{1}{\sqrt{2}}\sinh^{-1}\left(\frac{t-\frac{1}{2}}{1/2}\right)$$

$$=-\frac{1}{\sqrt{2}}\sinh^{-1}(2t-1)=-\frac{1}{\sqrt{2}}\sinh^{-1}\left\{\frac{2}{x+1}-1\right\}$$

$$=-\frac{1}{\sqrt{2}}\sinh^{-1}\left(\frac{1-x}{1+x}\right).$$

Ex. 40. *Integrate* $1/\{(x-1)\sqrt{(x^2+1)}\}$. **(Agra 1971; Vikram 76)**

Sol. Put $(x-1)=1/t$, so that $dx=-(1/t^2)\,dt$. Also $x=1+(1/t)$.

$$\therefore \quad \int \frac{dx}{(x-1)\sqrt{(x^2+1)}} = \int \frac{-(1/t^2)\,dt}{(1/t)\,.\,\sqrt{[\{1+(1/t)\}^2+1]}}$$

$$= -\int \frac{dt}{\sqrt{\{(1+t)^2+t^2\}}} = -\int \frac{dt}{\sqrt{(1+2t+2t^2)}}$$

$$= -\frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{(t^2+t+\frac{1}{2})}} = -\frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{\{(t+\frac{1}{2})^2+(\frac{1}{2})^2\}}}$$

$$= -\frac{1}{\sqrt{2}}\sinh^{-1}\left(\frac{t+\frac{1}{2}}{1/2}\right) = -\frac{1}{\sqrt{2}}\sinh^{-1}(2t+1)$$

$$= -\frac{1}{\sqrt{2}}\sinh^{-1}\left\{\frac{2}{(x-1)}+1\right\} = -\frac{1}{\sqrt{2}}\sinh^{-1}\left(\frac{x+1}{x-1}\right).$$

****Ex. 41.** *Integrate* $1/\{(x+1)\sqrt{(x^2-1)}\}$.

(Meerut 1983 S; Lucknow '68, 79)

Sol. Put $(x+1) = 1/t$, so that $dx = -(1/t^2)\,dt$.

$$\therefore \quad \int \frac{dx}{(x+1)\sqrt{(x^2-1)}} = \int \frac{-(1/t^2)\,dt}{(1/t)\sqrt{[\{(1/t)-1\}^2-1]}}$$

$$= -\int \frac{dt}{\sqrt{\{(1-t)^2-t^2\}}} = -\int \frac{dt}{\sqrt{(1-2t)}}$$

$$= \tfrac{1}{2}\int (1-2t)^{-1/2}(-2)\,dt = \tfrac{1}{2}\cdot 2(1-2t)^{1/2}$$

$$= (1-2t)^{1/2} = \sqrt{\left\{1-\frac{2}{(x+1)}\right\}} = \sqrt{\left(\frac{x-1}{x+1}\right)}.$$

Ex. 42. *Integrate* $1/\{(1+x)\sqrt{(1-x^2)}\}$. **(Gorakpur 1976)**

Sol. Put $(1+x) = 1/t$, so that $dx = -(1/t^2)\,dt$.

Also $x = (1/t) - 1$.

$$\therefore \quad \int \frac{dx}{(1+x)\sqrt{(1-x^2)}} = \int \frac{-(1/t^2)\,dt}{(1/t)\sqrt{[1-\{(1/t)-1\}^2]}}$$

$$= -\int \frac{dt}{\sqrt{[t^2-(1-t)^2]}} = -\int \frac{dt}{\sqrt{(2t-1)}}$$

$$= -\tfrac{1}{2}\int (2t-1)^{-1/2}\,.\,2\,dt = -\sqrt{(2t-1)}$$

$$= -\sqrt{\left[\frac{2}{1+x}-1\right]} = -\sqrt{\left(\frac{1-x}{1+x}\right)}.$$

Ex. 43. *Integrate* $1/\{(1-x)\sqrt{(1-x^2)}\}$.

Sol. Put $(1-x) = (1/t)$, so that $-dx = -(1/t^2)dt$.

Also $x = 1-(1/t)$.

$$\therefore \quad \int \frac{dx}{(1-x)\sqrt{(1-x^2)}} = \int \frac{(1/t^2)\,dt}{(1/t)\sqrt{[1-\{1-(1/t)\}^2]}}$$

$$= \int \frac{dt}{\sqrt{(2t-1)}} = \tfrac{1}{2}\int (2t-1)^{-1/2}.\,2\,dt$$

$$= \sqrt{(2t-1)} = \sqrt{\left[\frac{2}{(1-x)} - 1\right]} = \sqrt{\left(\frac{1+x}{1-x}\right)}.$$

Ex. 44. *Integrate* $1/\{(x-1)\sqrt{(x^2-1)}\}$.

Sol. Put $(x-1) = (1/t)$, so that $dx = -(1/t^2)\,dt$.

Also $x = 1 + (1/t)$.

$$\therefore \quad \int \frac{dx}{(x-1)\sqrt{(x^2-1)}} = \int \frac{-(1/t^2)\,dt}{(1/t)\sqrt{[\{(1/t)+1\}^2 - 1]}}$$

$$= \int \frac{-dt}{\sqrt{(1+2t)}}$$

$$= -\tfrac{1}{2}\int (2t+1)^{-1/2}.\,2\,dt = -\tfrac{1}{2}(2t+1)^{1/2}.\,2 = -\sqrt{(2t+1)}$$

$$= -\sqrt{\left[\frac{2}{x-1} + 1\right]} = -\sqrt{\left(\frac{x+1}{x-1}\right)}.$$

Ex. 45. *Integrate* $1/\{(x-a)\sqrt{(x^2-a^2)}\}$.

Sol. Put $(x-a) = (1/t)$, so that $dx = -(1/t^2)\,dt$.

Also $x = a + (1/t)$.

$$\therefore \quad \int \frac{dx}{(x-a)\sqrt{(x^2-a^2)}} = \int \frac{-(1/t^2)\,dt}{(1/t)\sqrt{[\{(1/t)+a\}^2 - a^2]}}$$

$$= -\int \frac{dt}{\sqrt{(1+2at)}} = -\int (1+2at)^{-1/2}\,dt = -\frac{(1+2\,at)^{1/2}}{1/2}\cdot\frac{1}{2a}$$

$$= -\frac{1}{a}\sqrt{(1+2at)} = -\frac{1}{a}\sqrt{\left\{1 + \frac{2a}{x-a}\right\}} = -\frac{1}{a}\sqrt{\left(\frac{x+a}{x-a}\right)}.$$

Ex. 46. *Integrate* $1/\{(1+x)\sqrt{(1+2x-x^2)}\}$. **(Meerut 1983 P)**

Sol. Put $1 + x = (1/t)$ *i.e.*, $x = (1/t) - 1$ and $dx = -(1/t^2)\,dt$.

Hence $(1 + 2x - x^2) = 1 + 2\left(\frac{1}{t} - 1\right) - \left(\frac{1}{t} - 1\right)^2 = \frac{4t - 2t^2 - 1}{t^2}$.

$\therefore$ the given integral $\displaystyle\int \frac{1}{(1+x)\sqrt{(1+2x-x^2)}}\,dx$

$$= \int \frac{1\,.\,(-1/t^2)\,dt}{(1/t)\,.\,\{\sqrt{(4t-2t^2-1)}\}.\,(1/t)}$$

$$= -\frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{(2t - t^2 - \frac{1}{2})}} = -\frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{\{-\frac{1}{2} - (t^2 - 2t)\}}}$$

$$= -\frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{\{-\frac{1}{2} - (t-1)^2 + 1\}}} = \frac{1}{\sqrt{2}}\int \frac{dt}{\sqrt{[\frac{1}{2} - (t-1)^2]}}$$

$$= -\frac{1}{\sqrt{2}}\sin^{-1}\left\{\frac{t-1}{1/\sqrt{2}}\right\} = -\frac{1}{\sqrt{2}}\sin^{-1}\left[\left\{\frac{1}{(1+x)} - 1\right\}\cdot\sqrt{2}\right]$$

$$= -\frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{-x\sqrt{2}}{1+x}\right) = \frac{1}{\sqrt{2}}\sin^{-1}\left(\frac{x\sqrt{2}}{1+x}\right),$$

$[\because \sin^{-1}(-x) = -\sin^{-1}x]$.

Ex. 47. *Integrate* $1/\{x\sqrt{(x^2+x+1)}\}$. **(Gorakhpur 1974)**

Sol. Put $x = 1/t$, so that $dx = -(1/t^2)\,dt$.

$$\therefore \quad \int \frac{dx}{x\sqrt{(x^2+x+1)}} = -\int \frac{(1/t^2)\,dt}{(1/t)\sqrt{[(1/t)^2+(1/t)+1]}}$$

$$= -\int \frac{dt}{\sqrt{(t^2+t+1)}} = -\int \frac{dt}{\sqrt{[(t+\frac{1}{2})^2+(\frac{3}{4})]}}$$

$$= -\sinh^{-1}\left\{\frac{t+\frac{1}{2}}{\frac{1}{2}\sqrt{3}}\right\} = -\sinh^{-1}\left(\frac{2t+1}{\sqrt{3}}\right)$$

$$= -\sinh^{-1}\left[\frac{2(1/x)+1}{\sqrt{3}}\right] = -\sinh^{-1}\left[\frac{2+x}{x\sqrt{3}}\right].$$

Ex. 48 (a). *Integrate* $\dfrac{1}{(x-1)\sqrt{(x^2+x+1)}}$, $(x > 1)$. **(Ranchi 1974)**

Sol. Put $(x-1) = 1/t$, so that $dx = -(1/t^2)\,dt$.

Also $x = 1 + (1/t) = (1+t)/t$.

$$\therefore \quad \int \frac{dx}{(x-1)\sqrt{(x^2+x+1)}}$$

$$= \int \frac{-(1/t^2)\,dt}{\frac{1}{t}\sqrt{\left[\left(\frac{1+t}{t}\right)^2+\left(\frac{1+t}{t}\right)+1\right]}}$$

$$= -\int \frac{dt}{\sqrt{(3t^2+3t+1)}} = -\frac{1}{\sqrt{3}}\int \frac{dt}{\sqrt{\{(t+\frac{1}{2})^2+\frac{1}{12}\}}}$$

$$= -\frac{1}{\sqrt{3}}\sinh^{-1}\left[\frac{t+\frac{1}{2}}{1/(2\sqrt{3})}\right]$$

$$= -\frac{1}{\sqrt{3}}\sinh^{-1}\{2\sqrt{3}\,(t+\tfrac{1}{2})\} = -\frac{1}{\sqrt{3}}\sinh^{-1}\left[\frac{x+1}{x-1}\sqrt{3}\right].$$

Ex. 48 (b). *Evaluate* $\displaystyle\int \frac{dx}{(x+2)\sqrt{(x^2+6x+7)}}$. **(Meerut 1981)**

Sol. Let $I = \displaystyle\int \frac{dx}{(x+2)\sqrt{(x^2+6x+7)}}$.

Put $x + 2 = 1/t$, so that $dx = -(1/t^2)\,dt$.

Also $x = \dfrac{1}{t} - 2 = \dfrac{1-2t}{t}$.

$$\therefore \quad I = \int \frac{-(1/t^2)\,dt}{\frac{1}{t}\sqrt{\left[\left(\frac{1-2t}{t}\right)^2+6\left(\frac{1-2t}{t}\right)+7\right]}}$$

$$= -\int \frac{dt}{\sqrt{\{(1-2t)^2 + 6t(1-2t) + 7t^2\}}}$$

$$= -\int \frac{dt}{\sqrt{(1+2t-t^2)}} = -\int \frac{dt}{\sqrt{\{1-(t^2-2t)\}}}$$

$$= -\int \frac{dt}{\sqrt{\{1-(t-1)^2+1\}}} = -\int \frac{dt}{\sqrt{\{2-(t-1)^2\}}}$$

$$= -\sin^{-1}\left(\frac{t-1}{\sqrt{2}}\right) = -\sin^{-1}\left[\frac{\{1/(x+2)\}-1}{\sqrt{2}}\right]$$

$$= -\sin^{-1}\left[\frac{-(x+1)}{\sqrt{2}\,(x+2)}\right] = \sin^{-1}\left[\frac{x+1}{\sqrt{2}\,(x+2)}\right].$$

Ex. 49. *Integrate* $1/\{(2-x)\sqrt{(1-2x+3x^2)}\}$.

Sol. Put $(2-x) = 1/t$ so that $dx = (1/t^2)\,dt$.

Also $x = 2-(1/t) = (2t-1)/t$.

$$\therefore \quad (1-2x+3x^2) = 1 - 2\left(\frac{2t-1}{t}\right) + 3\left(\frac{2t-1}{t}\right)^2$$

$$= \frac{1}{t^2}\{t^2 - 2t(2t-1) + 3(2t-1)^2\} = \frac{1}{t^2}\{9t^2 - 10t + 3\}.$$

$$\therefore \quad \int \frac{dx}{(2-x)\sqrt{(1-2x+3x^2)}} = \int \frac{(1/t^2)\,dt}{(1/t)(1/t)\sqrt{(9t^2-10t+3)}}$$

$$= \int \frac{dt}{\sqrt{(9t^2-10t+3)}} = \tfrac{1}{3}\int \frac{dt}{\sqrt{[t^2-(10/9)t+\frac{1}{3}]}}$$

$$= \tfrac{1}{3}\int \frac{dt}{\sqrt{\{(t-\frac{5}{9})^2+\frac{2}{81}\}}} = \tfrac{1}{3}\sinh^{-1}\left\{\frac{t-\frac{5}{9}}{\sqrt{(2/81)}}\right\}$$

$$= \tfrac{1}{3}\sinh^{-1}\left\{\frac{9(t-\frac{5}{9})}{\sqrt{2}}\right\} = \tfrac{1}{3}\sinh^{-1}\left(\frac{9t-5}{\sqrt{2}}\right)$$

$$= \tfrac{1}{3}\sinh^{-1}\left\{\frac{1}{\sqrt{2}}\left(\frac{9}{2-x}-5\right)\right\} = \tfrac{1}{3}\sinh^{-1}\left\{\frac{1}{\sqrt{2}}\left(\frac{5x-1}{2-x}\right)\right\}.$$

Ex. 50. *Evaluate* $\displaystyle\int \frac{\sqrt{(x+1)}}{(x+2)\sqrt{(x+3)}}\,dx$.

Sol. Multiplying the Nr. and Deno. by $\sqrt{(x+1)}$, we get

$$\int \frac{\sqrt{(x+1)}\,dx}{(x+2)\sqrt{(x+3)}} = \int \frac{(x+1)\,dx}{(x+2)\sqrt{[(x+3)(x+1)]}}$$

$$= \int \frac{\{(x+2)-1\}\,dx}{(x+2)\sqrt{(x^2+4x+3)}}$$

$$= \int \frac{dx}{\sqrt{(x^2+4x+3)}} - \int \frac{dx}{(x+2)\sqrt{(x^2+4x+3)}}$$

$= I_1 - I_2$, say.

Now the Ist integral $I_1 = \int \frac{dx}{\sqrt{(x^2 + 4x + 3)}}$

$$= \int \frac{dx}{\sqrt{[(x+2)^2 - 1]}} = \cosh^{-1}(x+2).$$

And in the 2nd integral I_2 put $x + 2 = 1/t$, so that $dx = -(1/t^2)\,dt$.

Also $x = (1/t) - 2 = (1 - 2t)/t$.

$$\therefore \quad I_2 = \int \frac{-(1/t^2)\,dt}{(1/t)\sqrt{[\{(1-2t)/t\}^2 + 4\{(1-2t)/t\} + 3]}}$$

$$= \int \frac{-dt}{\sqrt{[(1-2t)^2 + 4t(1-2t) + 3t^2]}} = \int \frac{-dt}{\sqrt{(1-t^2)}}$$

$$= -\sin^{-1} t = -\sin^{-1}\{1/(x+2)\}.$$

Hence the required integral $= I_1 - I_2$

$$= \cosh^{-1}(x+2) + \sin^{-1}\{1/(x+2)\}.$$

Ex. 51. *Evaluate* $\int_0^{1/2} \frac{dx}{(x-1)\sqrt{(x-x^2)}}$.

Sol. Put $x - 1 = 1/t$, so that $dx = -(1/t^2)\,dt$. Also $x = 1 + (1/t) = (1+t)/t$.

Also when $x = 0$, $t = -1$; and when $x = \frac{1}{2}$, $t = -2$.

$$\therefore \quad \int_0^{1/2} \frac{dx}{(x-1)\sqrt{(x-x^2)}} = \int_{-1}^{-2} \frac{-(1/t^2)\,dt}{\frac{1}{t}\sqrt{\left[\left(\frac{1+t}{t}\right) - \left(\frac{1+t}{t}\right)^2\right]}}$$

$$= -\int_{-1}^{-2} \frac{dt}{\sqrt{(-t-1)}} = \int_{-1}^{-2} (-t-1)^{-1/2}(-1)\,dt$$

$$= \left[\frac{(-t-1)^{1/2}}{\frac{1}{2}}\right]_{-1}^{-2} = 2\left[(-t-1)^{1/2}\right]_{-1}^{-2} = 2[1-0] = 2.$$

§ 3·9. Integration of $1/\{(Ax^2 + B)\sqrt{(Cx^2 + D)}\}$.

Here we put $x = 1/t$ so that $dx = -(1/t^2)\,dt$.

Then $\int \frac{dx}{(Ax^2 + B)\sqrt{(Cx^2 + D)}} = \int \frac{-(1/t^2)\,dt}{\left(\frac{A}{t^2} + B\right)\sqrt{\left(\frac{C}{t^2} + D\right)}}$

$$= -\int \frac{t\,dt}{(A + Bt^2)\sqrt{(C + Dt^2)}}.$$

Now substituting $C + Dt^2 = u^2$ or $2Dt\,dt = 2u\,du$,

the given integral $= -\int \dfrac{(1/D)\, u\, du}{[A + \{B(u^2 - C)/D\}]\, u}$

$$= -\int \frac{du}{(Bu^2 + AD - BC)}.$$

This integral is of the standard form and so it can be easily evaluated.

**Ex. 52. *Evaluate* $\displaystyle\int \frac{dx}{(1+x^2)\sqrt{(1-x^2)}}$.

(Meerut 1974, 80, 86, 88P, 90, 91P; Lucknow 79)

Sol. Put $x = 1/t$, so that $dx = -(1/t^2)\, dt$.

$$\therefore \quad I = \int \frac{dx}{(1+x^2)\sqrt{(1-x^2)}} = \int \frac{(-1/t^2)\, dt}{\left(1+\dfrac{1}{t^2}\right)\sqrt{\left(1-\dfrac{1}{t^2}\right)}}$$

$$= -\int \frac{t\, dt}{(t^2+1)\sqrt{(t^2-1)}}.$$

Now putting $t^2 - 1 = z^2$ so that $2t\, dt = 2z\, dz$, we get

$$I = -\int \frac{z\, dz}{z(z^2+2)} = -\int \frac{dz}{z^2+2} = -\frac{1}{\sqrt{2}}\left[\tan^{-1}\frac{z}{\sqrt{2}}\right]$$

$$= -\frac{1}{\sqrt{2}}\tan^{-1}\left[\frac{\sqrt{(t^2-1)}}{\sqrt{2}}\right], \qquad \because \; z^2 = (t^2-1)$$

$$= -\frac{1}{\sqrt{2}}\tan^{-1}\left[\frac{\sqrt{\{(1/x^2)-1\}}}{\sqrt{2}}\right] = -\frac{1}{\sqrt{2}}\tan^{-1}\left[\frac{\sqrt{(1-x^2)}}{x\sqrt{2}}\right].$$

**Ex. 53. *Prove that* $\displaystyle\int_0^{1/\sqrt{3}} \frac{dx}{(1+x^2)\sqrt{(1-x^2)}} = \frac{\pi}{4\sqrt{2}}$.

(Lucknow 1982; Kanpur 79; Gorakhpur 75)

Sol. Proceed as in Ex. 52. The limits for t are ∞ to $\sqrt{3}$ and then the limits for z are ∞ to $\sqrt{2}$. Hence

$$\int_0^{1/\sqrt{3}} \frac{dx}{(1+x^2)\sqrt{(1-x^2)}} = -\int_\infty^{\sqrt{2}} \frac{dz}{z^2+2}$$

$$= -\frac{1}{\sqrt{2}}\left[\tan^{-1}\left(\frac{z}{\sqrt{2}}\right)\right]_\infty^{\sqrt{2}}$$

$$= -\frac{1}{\sqrt{2}}[\tan^{-1}1 - \tan^{-1}\infty] = -\frac{1}{\sqrt{2}}\left[\frac{\pi}{4} - \frac{\pi}{2}\right] = \frac{\pi}{4\sqrt{2}}.$$

Ex. 54. *Evaluate* $\displaystyle\int \frac{dx}{(x^2-1)\sqrt{(x^2+1)}}$.

Sol. Put $x = 1/t$, so that $dx = -(1/t^2)\, dt$.

$$\therefore \quad I = \int \frac{dx}{(x^2-1)\sqrt{(x^2+1)}} = \int \frac{-(1/t^2)\, dt}{\{(1/t^2)-1\}\sqrt{\{(1/t^2)+1\}}}$$

$$= -\int \frac{t\,dt}{(1-t^2)\sqrt{(1+t^2)}}.$$

Now put $1 + t^2 = z^2$ so that $t\,dt = z\,dz$. Then

$$I = -\int \frac{z\,dz}{[1-(z^2-1)]z} = -\int \frac{dz}{2-z^2} = \int \frac{dz}{z^2-2}$$

$$= \frac{1}{2\sqrt{2}} \log \frac{z-\sqrt{2}}{z+\sqrt{2}} = \frac{1}{2\sqrt{2}} \log \frac{\sqrt{(1+t^2)}-\sqrt{2}}{\sqrt{(1+t^2)}+\sqrt{2}}, \quad [\because z^2 = 1+t^2]$$

$$= \frac{1}{2\sqrt{2}} \log \frac{\sqrt{\{1+(1/x^2)\}}-\sqrt{2}}{\sqrt{\{1+(1/x^2)\}}+\sqrt{2}}, \quad [\because t = 1/x]$$

$$= \frac{1}{2\sqrt{2}} \log \left[\frac{\sqrt{(1+x^2)}-x\sqrt{2}}{\sqrt{(1+x^2)}+x\sqrt{2}}\right].$$

Ex. 55. *Evaluate* $\displaystyle\int \frac{\sqrt{(1+x^2)}}{(1-x^2)}\,dx.$

Sol. Multiplying the numerator and the denominator by $\sqrt{(1+x^2)}$, we get

$$\int \frac{\sqrt{(1+x^2)}}{1-x^2}\,dx = \int \frac{(1+x^2)\,dx}{(1-x^2)\sqrt{(1+x^2)}}$$

$$= \int \frac{2-(1-x^2)}{(1-x^2)\sqrt{(1+x^2)}}\,dx = 2\int \frac{dx}{(1-x^2)\sqrt{(1+x^2)}} - \int \frac{dx}{\sqrt{(1+x^2)}}$$

$$= -2\int \frac{dx}{(x^2-1)\sqrt{(x^2+1)}} - \sinh^{-1} x$$

$$= -2 \cdot \frac{1}{2\sqrt{2}} \log \frac{\sqrt{(1+x^2)}-x\sqrt{2}}{\sqrt{(1+x^2)}+x\sqrt{2}} - \sinh^{-1} x. \qquad \text{[See Ex. 54]}$$

Ex. 56 (a). *Evaluate* $\displaystyle\int \frac{dx}{(2x^2+3)\sqrt{(x^2-4)}}.$

Sol. Put $x = 1/t$, so that $dx = -(1/t^2)\,dt$.

$$\therefore \text{ Given integral } I = \int \frac{(-1/t^2)\,dt}{[(2/t^2)+3]\sqrt{[(1/t^2)-4]}}$$

$$= -\int \frac{t\,dt}{(2+3t^2)\sqrt{(1-4t^2)}}.$$

Now put $1 - 4t^2 = z^2$, so that $-8t\,dt = 2z\,dz$

and $\quad 2 + 3t^2 = 2 + \frac{3}{4}(1-z^2) = \frac{1}{4}(11-3z^2).$

Then the integral $I = \frac{1}{4}\displaystyle\int \frac{z\,dz}{\frac{1}{4}(11-3z^2)z} = \int \frac{dz}{3(\frac{11}{3}-z^2)}$

$$= \frac{1}{3} \cdot \frac{1}{2}\sqrt{\left(\frac{3}{11}\right)} \cdot \log \frac{\sqrt{(11/3)}+z}{\sqrt{(11/3)}-z}$$

$$= \frac{1}{2\sqrt{(33)}} \log \frac{\sqrt{(11)}+\sqrt{[3(1-4t^2)]}}{\sqrt{(11)}-\sqrt{[3(1-4t^2)]}}$$

$$= \frac{1}{2\sqrt{(33)}} \log \frac{\sqrt{11} + \sqrt{\{3 - (12/x^2)\}}}{\sqrt{11} - \sqrt{\{3 - (12/x^2)\}}}, \qquad \left[\because t = \frac{1}{x}\right]$$

$$= \frac{1}{2\sqrt{(33)}} \log \frac{x\sqrt{(11)} + \sqrt{(3x^2 - 12)}}{x\sqrt{(11)} - \sqrt{(3x^2 - 12)}}.$$

Ex. 56 (b). *Evaluate* $\displaystyle\int \frac{dx}{(x^2 + 1)\sqrt{(x^2 - 1)}}$. **(Lucknow 1981)**

Sol. Put $x = 1/t$, so that $dx = -(1/t^2)\,dt$.

$\therefore$ the given integral $I = \displaystyle\int \frac{-(1/t^2)\,dt}{(1/t^2 + 1)\sqrt{(1/t^2 - 1)}}$

$$= -\int \frac{t\,dt}{(1 + t^2)\sqrt{(1 - t^2)}}.$$

Now put $1 - t^2 = z^2$, so that $-2t\,dt = 2z\,dz$

and $\quad 1 + t^2 = 1 + (1 - z^2) = 2 - z^2$.

Then $\quad I = \displaystyle\int \frac{z\,dz}{(2 - z^2)\,z} = \int \frac{dz}{2 - z^2}$

$$= \frac{1}{2\sqrt{2}} \log \frac{\sqrt{2} + z}{\sqrt{2} - z} = \frac{1}{2\sqrt{2}} \log \frac{\sqrt{2} + \sqrt{(1 - t^2)}}{\sqrt{2} - \sqrt{(1 - t^2)}}$$

$$= \frac{1}{2\sqrt{2}} \log \frac{\sqrt{2} + \sqrt{(1 - 1/x^2)}}{\sqrt{2} - \sqrt{(1 - 1/x^2)}}, \qquad \left[\because t = \frac{1}{x}\right]$$

$$= \frac{1}{2\sqrt{2}} \log \frac{\sqrt{2}x + \sqrt{(x^2 - 1)}}{\sqrt{2}x - \sqrt{(x^2 - 1)}}.$$

Ex. 57. *Evaluate* $\displaystyle\int \frac{1}{(3 + 4x^2)\sqrt{(4 - 3x^2)}}\,dx$.

Sol. Put $x = 1/t$, so that $dx = -(1/t^2)\,dt$.

$$\therefore \quad I = \int \frac{dx}{(3 + 4x^2)\sqrt{(4 - 3x^2)}} = \int \frac{-(1/t^2)\,dt}{\{3 + (4/t^2)\}\sqrt{\{4 - (3/t^2)\}}}$$

$$= -\int \frac{t\,dt}{(3t^2 + 4)\sqrt{(4t^2 - 3)}}.$$

Now putting $4t^2 - 3 = z^2$ so that $8t\,dt = 2z\,dz$, we get

$$I = -\tfrac{1}{4}\int \frac{z\,dz}{[3\,.\,\{(z^2 + 3)/4\} + 4]\,z} = -\int \frac{dz}{3z^2 + 25}$$

$$= -\tfrac{1}{2}\int \frac{dz}{z^2 + (5/\sqrt{3})^2} = -\tfrac{1}{3}\cdot\frac{\sqrt{3}}{5}\tan^{-1}\left(\frac{z}{5/\sqrt{3}}\right)$$

$$= -\frac{1}{5\sqrt{3}}\tan^{-1}\left(\frac{z\sqrt{3}}{5}\right)$$

$$= -\frac{1}{5\sqrt{3}}\tan^{-1}\left\{\frac{\sqrt{3}}{5}\sqrt{(4t^2 - 3)}\right\}, \qquad [\because z^2 = 4t^2 - 3]$$

$$= -\frac{1}{5\sqrt{3}}\tan^{-1}\left[\frac{\sqrt{3}}{5}\sqrt{\{(4/x^2)-3\}}\right]$$

$$= -\frac{1}{5\sqrt{3}}\tan^{-1}\left\{\frac{\sqrt{3}\sqrt{(4-3x^2)}}{5x}\right\}$$

$$= -\frac{1}{5\sqrt{3}}\tan^{-1}\left\{\frac{\sqrt{(12-9x^2)}}{5x}\right\}.$$

Ex. 58. *Evaluate* $\displaystyle\int \frac{x\sqrt{(a^2-x^2)}}{a^2+x^2}\,dx.$

Sol. Put $a^2 - x^2 = t^2$, so that $-x\,dx = t\,dt$.

$$\therefore \quad \int \frac{x\sqrt{(a^2-x^2)}}{a^2+x^2}\,dx = \int \frac{t(-t\,dt)}{a^2+(a^2-t^2)} = \int \frac{-t^2dt}{2a^2-t^2}$$

$$= \int \frac{(2a^2-t^2)-2a^2}{2a^2-t^2}\,dt = \int dt - 2a^2\int \frac{dt}{2a^2-t^2}$$

$$= t - 2a^2\cdot\frac{1}{2.\sqrt{(2a^2)}}\log\frac{a\sqrt{2}+t}{a\sqrt{2}-t}$$

$$= \sqrt{(a^2-x^2)} - \frac{1}{\sqrt{2}}a\log\frac{a\sqrt{2}+\sqrt{(a^2-x^2)}}{a\sqrt{2}-\sqrt{(a^2-x^2)}}.$$

Ex. 59. *Evaluate* $\displaystyle\int_0^{1/2} \frac{dx}{(1-2x^2)\sqrt{(1-x^2)}}.$

Sol. Put $x = 1/t$ so that $dx = -(1/t^2)\,dt$.

Also when $x = 0$, $t = \infty$ and when $x = \frac{1}{2}$, $t = 2$.

$$\therefore \quad I = \int_0^{1/2} \frac{dx}{(1-2x^2)\sqrt{(1-x^2)}}$$

$$= \int_\infty^2 \frac{(-1/t^2)\,dt}{\{1-(2/t^2)\}\sqrt{\{1-(1/t^2)\}}}$$

$$= \int_\infty^2 \frac{-t\,dt}{(t^2-2)\sqrt{(t^2-1)}}.$$

Now put $t^2 - 1 = z^2$ so that $2t\,dt = 2z\,dz$. The limits for z are ∞ to $\sqrt{3}$.

$$\therefore \quad I = -\int_\infty^{\sqrt{3}} \frac{z\,dz}{(z^2-1)z} = -\int_\infty^{\sqrt{3}} \frac{dz}{(z^2-1)} = -\tfrac{1}{3}\left[\log\frac{z-1}{z+1}\right]_\infty^{\sqrt{3}}$$

$$= -\tfrac{1}{2}\left[\log\frac{\sqrt{3}-1}{\sqrt{3}+1} - \lim_{z\to\infty}\log\frac{z-1}{z+1}\right]$$

$$= -\tfrac{1}{2}\log\frac{\sqrt{3}-1}{\sqrt{3}+1} + \lim_{z\to\infty}\log\left\{\frac{1-(1/z)}{1+(1/z)}\right\}$$

$$= \tfrac{1}{2}\log\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right)^{-1} + \log 1 = \tfrac{1}{2}\log\frac{\sqrt{3}+1}{\sqrt{3}-1}, \qquad [\because \log 1 = 0]$$

$$= \tfrac{1}{2}\log\left\{\frac{(\sqrt{3}+1)(\sqrt{3}+1)}{(\sqrt{3}-1)(\sqrt{3}+1)}\right\} = \tfrac{1}{2}\log\left(\frac{4+2\sqrt{3}}{2}\right) = \tfrac{1}{2}\log(2+\sqrt{3}).$$

Ex. 60 (a). *Show that*

$$\int \frac{dx}{(x^2+a^2)\sqrt{(x^2+b^2)}} = \frac{1}{a\sqrt{(b^2-a^2)}}\cos^{-1}\left\{\frac{a}{b}\sqrt{\left(\frac{x^2+b^2}{x^2+a^2}\right)}\right\},$$

$$b^2 > a^2.$$

Sol. Putting $x = 1/t$, so that $dx = -(1/t^2)\,dt$, we have

$$I = \int \frac{dx}{(x^2+a^2)\sqrt{(x^2+b^2)}} = -\int \frac{-(1/t^2)\,dt}{\{(1/t^2)+a^2\}\sqrt{\{(1/t^2)+b^2\}}}$$

$$= -\int \frac{t\,dt}{(1+a^2t^2)\sqrt{(1+t^2b^2)}}.$$

Now putting $1 + t^2b^2 = z^2$ so that $2b^2\,t\,dt = 2z\,dz$, we get

$$I = -\frac{1}{b^2}\int \frac{z\,dz}{[1+a^2\{(z^2-1)/b^2\}].z} = -\int \frac{dz}{(b^2-a^2)+a^2z^2}$$

$$= -\frac{1}{a^2}\int \frac{dz}{z^2+\{(b^2-a^2)/a^2)\}} = \frac{1}{a^2}\frac{a}{\sqrt{(b^2-a^2)}}\cot^{-1}\left\{\frac{az}{\sqrt{(b^2-a^2)}}\right\}$$

$$= \frac{1}{a\sqrt{(b^2-a^2)}}\cot^{-1}\left\{\frac{a}{x}\sqrt{\left(\frac{x^2+b^2}{b^2-a^2}\right)}\right\}.$$

$$\left[\because z = \sqrt{(1+b^2t^2)} = \sqrt{\left(1+\frac{b^2}{x^2}\right)} = \frac{\sqrt{(x^2+b^2)}}{x}\right]$$

Now to bring the result in the required form, let

$$\cot^{-1}\left\{\frac{a}{x}\sqrt{\left(\frac{x^2+b^2}{b^2-a^2}\right)}\right\} = \theta, \text{ so that } \cot\theta = \frac{a}{x}\sqrt{\left(\frac{x^2+b^2}{b^2-a^2}\right)}.$$

$$\therefore \quad \sec^2\theta = 1+\tan^2\theta = 1+\frac{x^2(b^2-a^2)}{a^2(x^2+b^2)} = \frac{b^2(x^2+a^2)}{a^2(x^2+b^2)}.$$

$$\therefore \quad \cos\theta = \frac{a}{b}\sqrt{\left(\frac{x^2+b^2}{x^2+a^2}\right)} \quad \text{or} \quad \theta = \cos^{-1}\left\{\frac{a}{b}\sqrt{\left(\frac{x^2+b^2}{x^2+a^2}\right)}\right\}.$$

Putting this value of θ, we get

$$\int \frac{dx}{(x^2+a^2)\sqrt{(x^2+b^2)}} = \frac{1}{a\sqrt{(b^2-a^2)}}\cos^{-1}\left\{\frac{a}{b}\sqrt{\left(\frac{x^2+b^2}{x^2+a^2}\right)}\right\}.$$

Ex. 60 (b). *Evaluate* $\displaystyle\int \frac{dx}{(x^2+4)\sqrt{(x^2+9)}}$. **(Meerut 1982)**

Sol. Proceed as in Ex. 60 (a). The answer is

$$\frac{1}{2\sqrt{5}}\cot^{-1}\left[\frac{2\sqrt{(x^2+9)}}{x\sqrt{5}}\right].$$

Examples on Algebraic and other substitutions.

***Ex. 61.** *Integrate* $1/\{x\sqrt{(x^6+1)}\}$. **(Meerut 1972, 85 S, 87, 88)**

Sol. Put $x^6 + 1 = t^2$, so that $6x^5\,dx = 2t\,dt$ or $3x^5\,dx = t\,dt$.

$$\therefore \quad \int \frac{dx}{x\sqrt{(x^6+1)}} = \tfrac{1}{3}\int \frac{3x^5\,dx}{x^6\sqrt{(x^6+1)}} = \tfrac{1}{3}\int \frac{t\,dt}{(t^2-1)\,t}$$

$$= \tfrac{1}{3}\int \frac{dt}{(t^2-1)} = \tfrac{1}{3}\cdot\tfrac{1}{2}\log\left(\frac{t-1}{t+1}\right)$$

$$= \tfrac{1}{6}\log\left[\{\sqrt{(x^6+1)}-1\}/\{\sqrt{(x^6+1)}+1\}\right].$$

Ex. 62. *Integrate* $1/\sqrt{(2e^x - 1)}$.

Sol. Let $2e^x - 1 = t^2$, so that $2e^x\,dx = 2t\,dt$.

$$\therefore \quad \int \frac{dx}{\sqrt{(2e^x-1)}} = \int \frac{2e^x\,dx}{2e^x\sqrt{(2e^x-1)}} = \int \frac{2t\,dt}{(t^2+1)\,t}$$

$$= 2\int \frac{dt}{t^2+1} = 2\tan^{-1}(t) = 2\tan^{-1}\sqrt{\{(2e^x-1)\}}.$$

Ex. 63. *Integrate* $1/[x\{6(\log x)^2 + 7(\log x) + 2\}]$.

Sol. Put $\log x = t$, so that $(1/x)\,dx = dt$.

$$\therefore \quad \text{given integral} = \int \frac{dt}{(6t^2+7t+2)} = \int \frac{dt}{(3t+2)(2t+1)}$$

$$= \int \left[\frac{2}{(2t+1)} - \frac{3}{(3t+2)}\right] dt, \qquad \text{by partial fractions}$$

$$= \log(2t+1) - \log(3t+2) = \log\{(2t+1)/(3t+2)\}$$

$$= \log\{(2\log x + 1)/(3\log x + 2)\}.$$

Ex. 64 (a). *Integrate* $1/\{(1+x)^{1/2} - (1+x)^{1/3}\}$.

Sol. Put $1 + x = t^6$, so that $dx = 6t^5\,dt$.

$$\therefore \quad \int \frac{dx}{(1+x)^{1/2}-(1+x)^{1/3}} = \int \frac{6t^5\,dt}{(t^3-t^2)} = 6\int \frac{t^3\,dt}{(t-1)}$$

$$= 6\int \left\{t^2 + t + 1 + \frac{1}{(t-1)}\right\} dt, \qquad \text{by actual division}$$

$$= 6\left[\tfrac{1}{3}t^3 + \tfrac{1}{2}t^2 + t + \log(t-1)\right] = 2t^3 + 3t^2 + 6t + 6\log(t-1)$$

$$= 2(1+x)^{1/2} + 3(1+x)^{1/3} + 6(1+x)^{1/6} + \log\{(1+x)^{1/6} - 1\}.$$

Ex. 64 (b). *Evaluate* $\displaystyle\int \frac{dx}{(x-a)^{3/2}(x+a)^{1/2}}$. **(Meerut 1983)**

Sol. Let $$I = \int \frac{dx}{(x-a)^{3/2}(x+a)^{1/2}}$$

$$= \int \frac{dx}{(x-a)\sqrt{(x^2-a^2)}}.$$

Put $x - a = 1/t$, so that $dx = -(1/t^2)\,dt$.

Also $x = a + 1/t$, so that

$$x^2 - a^2 = \left(a + \frac{1}{t}\right)^2 - a^2 = \frac{2a}{t} + \frac{1}{t^2} = \frac{2at+1}{t^2}.$$

$$\therefore \quad I = \int \frac{-(1/t^2)\,dt}{(1/t)\,[\sqrt{(2at+1)}]\,.\,(1/t)} = -\int (2at+1)^{-1/2}\,dt$$

$$= -\frac{(2at+1)^{1/2}}{2a\,(\frac{1}{2})} = -\frac{1}{a}\sqrt{(2at+1)}$$

$$= -\frac{1}{a}\sqrt{\left(2a\cdot\frac{1}{x-a}+1\right)} = -\frac{1}{a}\sqrt{\left(\frac{x+a}{x-a}\right)}.$$

Ex. 65. *Evaluate* $\int_1^3 \left(x+\frac{1}{x}\right)^{3/2}\left(\frac{x^2-1}{x^2}\right)dx.$

(Meerut 1984 P)

Sol. Given integral $= \int_1^3 \left(x+\frac{1}{x}\right)^{3/2}\left(1-\frac{1}{x^2}\right)dx.$

Now put $x + (1/x) = t$, so that $\{1-(1/x^2)\}\,dx = dt$.
Also when $x = 1$, $t = 2$ and when $x = 3$, $t = 10/3$.
$\therefore$ the given integral

$$= \int_2^{10/3} t^{3/2}\,dt = \left[\frac{2}{5}t^{5/2}\right]_2^{10/3} = \frac{2}{5}\left[t^{5/2}\right]_2^{10/3}$$

$$= \tfrac{2}{5}[(10/3)^{5/2} - (2)^{5/2}] = \tfrac{2}{5}[(2\cdot\tfrac{5}{3})^{5/2} - 2^{5/2}]$$

$$= \frac{2}{5}(2)^{5/2}\left[\left(\frac{5}{3}\right)^{5/2} - 1\right] = \frac{8\sqrt{2}}{5}\left[\frac{25}{9}\left(\frac{5}{3}\right)^{1/2} - 1\right].$$

Integration by Trigonometric substitutions.

****Ex. 66.** *Evaluate* $\int_\alpha^\beta \sqrt{[(x-\alpha)(\beta-x)]}\,dx.$

(Meerut 1982, 89 P; Allahabad 82; Lucknow 82, 80; Gorakhpur 75; Magadh 76; Ranchi 76; Vikram 74)

Sol. Put $x = \alpha\cos^2\theta + \beta\sin^2\theta$.
Then $dx = (-2\alpha\cos\theta\sin\theta + 2\beta\sin\theta\cos\theta)\,d\theta$
$= 2(\beta-\alpha)\sin\theta\cos\theta\,d\theta.$

Also $(x-\alpha) = \alpha\cos^2\theta + \beta\sin^2\theta - \alpha = \beta\sin^2\theta - \alpha(1-\cos^2\theta)$
$= \beta\sin^2\theta - \alpha\sin^2\theta = (\beta-\alpha)\sin^2\theta$ ...(1)

and $(\beta - x) = \beta - (\alpha\cos^2\theta + \beta\sin^2\theta) = \beta(1-\sin^2\theta) - \alpha\cos^2\theta$
$= (\beta-\alpha)\cos^2\theta.$ ...(2)

Now at $x = \alpha$, $x - \alpha = 0$; $\therefore$ from (1) $\sin\theta = 0$ or $\theta = 0$.
Also when $x = \beta$, $\beta - x = 0$; $\therefore$ from (2), $\cos\theta = 0$
or $\theta = \pi/2$.

Thus the given integral $\int_\alpha^\beta \sqrt{\{(x-\alpha)(\beta-x)\}}\,dx$

$$= 2(\beta-\alpha)^2\int_0^{\pi/2}\sin^2\theta\cos^2\theta\,d\theta$$

$$= \tfrac{1}{2}(\beta-\alpha)^2\int_0^{\pi/2}\sin^2 2\theta\,d\theta, \qquad [\because \sin 2\theta = 2\sin\theta\cos\theta]$$

$$= \tfrac{1}{4}(\beta-\alpha)^2 \int_0^{\pi/2} (1-\cos 4\theta)\, d\theta$$

$$= \tfrac{1}{4}(\beta-\alpha)^2 \left[\theta - \tfrac{1}{4}\sin 4\theta\right]_0^{\pi/2} = \frac{\pi}{8}(\beta-\alpha)^2.$$

Alternative solution.

$$\int_\alpha^\beta \sqrt{[(x-\alpha)(\beta-x)]}\, dx = \int_\alpha^\beta \sqrt{[-\alpha\beta - x^2 + x(\alpha+\beta)]}\, dx$$

$$= \int_\alpha^\beta \sqrt{[-\alpha\beta - \{x^2 - x(\alpha+\beta)\}]}\, dx$$

$$= \int_\alpha^\beta \sqrt{[-\alpha\beta - \{x - \tfrac{1}{2}(\alpha+\beta)\}^2 + \tfrac{1}{4}(\alpha+\beta)^2]}\, dx$$

$$= \int_\alpha^\beta \sqrt{[\tfrac{1}{4}(\beta-\alpha)^2 - \{x - \tfrac{1}{2}(\alpha+\beta)\}^2]}\, dx$$

$$= \left[\tfrac{1}{2}\{x - \tfrac{1}{2}(\alpha+\beta)\}\sqrt{[\tfrac{1}{4}(\beta-\alpha)^2 - \{x - \tfrac{1}{2}(\alpha+\beta)\}^2]} + \frac{1}{8}(\beta-\alpha)^2 \sin^{-1}\left\{\frac{x - \tfrac{1}{2}(\alpha+\beta)}{\tfrac{1}{2}(\beta-\alpha)}\right\}\right]_\alpha^\beta$$

$$= \left[\tfrac{1}{2}\{x - \tfrac{1}{2}(\alpha+\beta)\}\sqrt{\{(x-\alpha)(\beta-x)\}} + \frac{1}{8}(\beta-\alpha)^2 \sin^{-1}\left\{\frac{x - \tfrac{1}{2}(\alpha+\beta)}{\tfrac{1}{2}(\beta-\alpha)}\right\}\right]_\alpha^\beta$$

$$= \{0 + \tfrac{1}{8}(\beta-\alpha)^2 \sin^{-1} 1\} - \{0 + \tfrac{1}{8}(\beta-\alpha)^2 \sin^{-1}(-1)\}$$

$$= \tfrac{1}{8}\pi(\beta-\alpha)^2, \qquad [\because \sin^{-1} 1 = \pi/2, \sin^{-1}(-1) = -\pi/2].$$

Ex. 67. *Evaluate* $\displaystyle\int_\alpha^\beta \sqrt{\left(\frac{x-\alpha}{\beta-x}\right)}\, dx.$

Sol. Put $x = \alpha\cos^2\theta + \beta\sin^2\theta$. Then proceeding as in Ex. 66, we have

$$\int_\alpha^\beta \sqrt{\left(\frac{x-\alpha}{\beta-x}\right)}\, dx$$

$$= \int_0^{\pi/2} \sqrt{\left[\frac{(\beta-\alpha)\sin^2\theta}{(\beta-\alpha)\cos^2\theta}\right]}\, 2(\beta-\alpha)\sin\theta\cos\theta\, d\theta$$

$$= (\beta-\alpha)\int_0^{\pi/2} 2\sin^2\theta\, d\theta = (\beta-\alpha)\int_0^{\pi/2} (1-\cos 2\theta)\, d\theta$$

$$= (\beta-\alpha)\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2} = (\beta-\alpha)\cdot\frac{\pi}{2}.$$

Ex. 68. *Evaluate* $\displaystyle\int \frac{dx}{\sqrt{\{(x-\alpha)(\beta-x)\}}}.$

(Meerut 1983 S; Magadh 77)

Sol. Put $x = \alpha\cos^2\theta + \beta\sin^2\theta$

so that $dx = 2(\beta-\alpha)\sin\theta\cos\theta\, d\theta.$

Also $(x-\alpha) = (\beta-\alpha)\sin^2\theta$, and $(\beta - x) = (\beta-\alpha)\cos^2\theta$ (See Ex. 66)

Making these substitutions, the given integral

$$= \int \frac{2(\beta-\alpha)\sin\theta\cos\theta\, d\theta}{\sqrt{\{(\beta-\alpha)\cos^2\theta\,.\,(\beta-\alpha)\sin^2\theta\}}} = \int \frac{2(\beta-\alpha)\sin\theta\cos\theta}{(\beta-\alpha)\cos\theta\sin\theta}\, d\theta$$

$$= 2\int d\theta = 2\theta = \cos^{-1}(\cos 2\theta). \quad ...(1)$$

But $x = \alpha\cos^2\theta + \beta\sin^2\theta$;

$\therefore \quad 2x = \alpha(1+\cos 2\theta) + \beta(1-\cos 2\theta)$

i.e., $(\beta-\alpha)\cos 2\theta = (\alpha+\beta-2x)$

or $\cos 2\theta = (\alpha+\beta-2x)/(\beta-\alpha)$.

$\therefore$ from (1), we get

the given integral $= \cos^{-1}\left(\dfrac{\alpha+\beta-2x}{\beta-\alpha}\right)$.

***Ex. 69 (a).** *Evaluate* $\displaystyle\int_\alpha^\beta \frac{dx}{\sqrt{\{(x-\alpha)(\beta-x)\}}}$.

(Lucknow 1981; Ranchi 74)

Sol. Put $x = \alpha\cos^2\theta + \beta\sin^2\theta$. Then proceeding as in Ex. 66, we have

$$\int_\alpha^\beta \frac{dx}{\sqrt{[(x-\alpha)(\beta-x)]}} = \int_0^{\pi/2} \frac{(\beta-\alpha)\,2\sin\theta\cos\theta\, d\theta}{\sqrt{[(\beta-\alpha)^2\sin^2\theta\cos^2\theta]}}$$

$$= \int_0^{\pi/2} \frac{(\beta-\alpha)\,2\sin\theta\cos\theta\, d\theta}{(\beta-\alpha)\sin\theta\cos\theta} = \int_0^{\pi/2} 2\, d\theta$$

$$= 2\Big[\theta\Big]_0^{\pi/2} = 2\left(\tfrac{1}{2}\pi - 0\right) = \pi.$$

Ex. 69 (b). *Evaluate* $\displaystyle\int_3^4 \frac{dx}{\sqrt{(4-x)(x-3)}}$. **(Meerut 1981 S)**

Sol. Proceed exactly as in Ex. 69 (a).

The answer is π.

***Ex. 70 (a).** *Integrate* $1/\sqrt{\{(x-a)(x-b)\}}$.

(Kanpur 1975; Meerut 1984)

Sol. Put $x = a\sec^2\theta - b\tan^2\theta$. ...(1) **(Note)**

$\therefore \quad dx = [a\,.\,2\sec^2\theta\tan\theta - 2b\tan\theta\sec^2\theta]\, d\theta$

$= 2(a-b)\sec^2\theta\tan\theta\, d\theta$.

Thus $x - a = a(\sec^2\theta - 1) - b\tan^2\theta = (a-b)\tan^2\theta$ ...(2)

and $x - b = a\sec^2\theta - b(1+\tan^2\theta) = (a-b)\sec^2\theta$ ...(3)

$$\therefore \quad \int \frac{dx}{\sqrt{\{(x-a)(x-b)\}}} = \int \frac{2(a-b)\sec^2\theta\tan\theta}{(a-b)\tan\theta\sec\theta}\, d\theta$$

$$= 2\int \sec\theta\, d\theta = 2\log(\sec\theta + \tan\theta).$$

From (2) and (3), $\sec\theta = \sqrt{\left(\frac{x-b}{a-b}\right)}$ and $\tan\theta = \sqrt{\left(\frac{x-a}{a-b}\right)}$.

$\therefore$ the given integral $= 2\log\left[\frac{\sqrt{(x-b)}+\sqrt{(x-a)}}{\sqrt{(a-b)}}\right]$

$= 2\log\{\sqrt{(x-b)}+\sqrt{(x-a)}\}$,

omitting the constant term $-2\log\sqrt{(a-b)}$ which may be added to the constant of integration c.

Alternative solution. We have

$$I = \int \frac{dx}{\sqrt{\{(x-a)(x-b)\}}} = \int \frac{1}{\sqrt{(x-a)}\sqrt{\{(x-a)+(a-b)\}}}\,dx.$$

(Note)

Now put $\sqrt{(x-a)} = t$ so that $\frac{1}{2\sqrt{(x-a)}}dx = dt$.

$$\therefore \quad I = \int \frac{2\,dt}{\sqrt{\{t^2+(a-b)\}}} = 2\log[t+\sqrt{\{t^2+(a-b)\}}]$$

$$= 2\log[\sqrt{(x-a)}+\sqrt{\{(x-a)+(a-b)\}}]$$

$$= 2\log[\sqrt{(x-a)}+\sqrt{(x-b)}].$$

Ex. 70 (b). *If α and β are the roots of $ax^2+bx+c=0$, then prove that*

$$\int \frac{dx}{\sqrt{(ax^2+bx+c)}} = \frac{2}{\sqrt{a}}\log[\sqrt{(x-\alpha)}+\sqrt{(x-\beta)}].$$

Sol. Since α and β are the roots of $ax^2+bx+c=0$, therefore $\alpha+\beta = -\frac{b}{a}$ and $\alpha\beta = \frac{c}{a}$.

Now the given integral $I = \frac{1}{\sqrt{a}}\int \frac{dx}{\sqrt{[x^2+(b/a)x+(c/a)]}}$

$$= \frac{1}{\sqrt{a}}\int \frac{dx}{[x^2-(\alpha+\beta)x+\alpha\beta]} = \frac{1}{\sqrt{a}}\int \frac{dx}{\sqrt{\{(x-\alpha)(x-\beta)\}}}$$

$$= \frac{1}{\sqrt{a}}\cdot 2\log[\sqrt{(x-\alpha)}+\sqrt{(x-\beta)}],$$

proceeding as in the alternative solution of Ex. 70 (a).

Ex. 71. *Evaluate* $\int_0^1 \frac{1-4x+2x^2}{\sqrt{(2x-x^2)}}dx$. **(Lucknow 1981; Kashmir 75)**

Sol. The given integral $= \int_0^1 \frac{[1-2(2x-x^2)]\,dx}{\sqrt{(2x-x^2)}}$ **(Note)**

$$= \int_0^1 \frac{dx}{\sqrt{(2x-x^2)}} - 2\int_0^1 \sqrt{(2x-x^2)}\,dx$$

$$= \int_0^1 \frac{dx}{\sqrt{\{1-(x-1)^2\}}} - 2\int_0^1 \sqrt{\{1-(x-1)^2\}}\,dx \qquad \textbf{(Note)}$$

$$= \int_0^1 \frac{dx}{\sqrt{\{1-(1-x)^2\}}} - 2\int_0^1 \sqrt{\{1-(1-x)^2\}}\, dx,$$

[Note that $0 < x < 1$]

$$= \left[\sin^{-1}(1-x)\right]_0^1 - 2\left[\tfrac{1}{2}(1-x)\sqrt{\{1-(1-x)^2\}} + \tfrac{1}{2}.1.\sin^{-1}(1-x)\right]_0^1$$

$$= [\sin^{-1}0 - \sin^{-1}] - 2[\{0 + \tfrac{1}{2}\sin^{-1}0\} - \{0 + \tfrac{1}{2}\sin^{-1}1\}]$$

$$= -\sin^{-1}1 + \sin^{-1}1 = 0.$$ [Note that $\sin^{-1}0 = 0$]

Ex. 72. *Evaluate* $\displaystyle\int_0^a \frac{x\sqrt{(a^2-x^2)}}{\sqrt{(a^2+x^2)}}\, dx.$ **(Meerut 1985 P)**

Sol. Let $\displaystyle I = \int_0^a \frac{x\sqrt{(a^2-x^2)}}{\sqrt{(a^2+x^2)}}\, dx = \int_0^a \frac{x\,(a^2-x^2)}{\sqrt{(a^4-x^4)}}\, dx,$

multiplying the numerator and the denominator of the integrand by $\sqrt{(a^2-x^2)}$.

Now put $x^2 = a^2\sin\theta$ so that $2x\,dx = a^2\cos\theta\, d\theta$.

Also when $x = 1$, $\sin\theta = 1$ so that $\theta = \frac{1}{2}\pi$;

and when $x = 0$, $\sin\theta = 0$ so that $\theta = 0$.

$$\therefore \quad I = \int_0^{\pi/2} \frac{(a^2 - a^2\sin\theta)}{\sqrt{(a^4 - a^4\sin^2\theta)}} \cdot \tfrac{1}{2}a^2\cos\theta\, d\theta$$

$$= \int_0^{\pi/2} \tfrac{1}{2}a^2(1-\sin\theta)\, d\theta$$

$$= \tfrac{1}{2}a^2\left[\theta + \cos\theta\right]_0^{\pi/2} = \tfrac{1}{2}a^2[\{\tfrac{1}{2}\pi + \cos\tfrac{1}{2}\pi\} - \{0 + \cos 0\}]$$

$$= \tfrac{1}{2}a^2[\tfrac{1}{2}\pi - 1] = \tfrac{1}{4}a^2(\pi - 2).$$

□