Chapter 7

2014

7.1 Section-A

Question-1(a) Show that $u_1 = (1, -1, 0), u_2 = (1, 1, 0)$ and $u_3 = (0, 1, 1)$ form a basis for $\mathbb{R}^3$. Express (5,3,4) in terms of u_1, u_2 and u_3.

[8 Marks]

Solution: Consider

$$xu_1 + yu_2 + zu_3 = 0, \text{ where } x, y, z \in R$$

$$\begin{gathered} x(1, -1, 0) + y(1, 1, 0) + z(0, 1, 1) = (0, 0, 0) \\ (x + y, -x + y + z, z) = (0, 0, 0) \\ \Rightarrow x + y = 0, -x + y + z = 0 \quad z = 0 \\ x = 0, \quad y = 0, \quad z = 0 \end{gathered}$$

Hence, u_1, u_2 and u_3 are linearly independent. Again, let

$$(a, b, c) \in \mathbb{R}^3$$

and

$$\begin{gathered} xu_1 + yu_2 + zu_3 = (a, b, c) \\ \Rightarrow \quad x + y = a, \quad -x + y + z = b, \quad z = c \\ x + y = a, x - y = c - b \\ \therefore x = \frac{a - b + c}{2}, \quad y = \frac{a + b - c}{2}, \quad z = c \\ \therefore (a, b, c) = \left(\frac{a - b + c}{2}\right) u_1 + \left(\frac{a + b - c}{2}\right) u_2 + cu_3 \end{gathered}$$

Taking $(a, b, c) = (5, 3, 4)$

$$\begin{aligned} (5, 3, 4) &= \left(\frac{5 - 3 + 4}{2}\right) u_1 + \left(\frac{5 + 3 - 4}{2}\right) u_2 + 4u_3 \\ &= 3u_1 + 2u_2 + 4u_3 \\ &= 3(1, -1, 0) + 2(1, 1, 0) + 4(0, 1, 1) \end{aligned}$$

Question-1(b) For the matrix $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. **Prove that**

$$A^n = A^{n-2} + A^2 - I, n \geq 3$$

[8 Marks]

Solution: Consider $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)\left(\lambda^2 - 1\right) = 0$$
$$\Rightarrow \lambda^2 - \lambda^3 - 1 + \lambda = 0$$
$$\Rightarrow \lambda^3 - \lambda^2 - \lambda + 1 = 0$$

Cayley-Hamilton theorem states that every square matrix satisfies its own characteristic equation.

$$\therefore A^3 - A^2 - A + I = 0$$
$$\Rightarrow \quad A^3 = A^2 + A - I$$

Hence given relation is true for $n = 3$.
Now, assume that this statement holds true for n.
We have to prove that it also holds true for $n + 1$.
Let $A^n = A^{n-2} + A^2 - I$. Multiply both sides by A,

$$\begin{aligned} A^{n+1} &= A^{n-1} + A^3 - A \\ &= A^{n-1} + \left(A^2 + A - I\right) - A \quad \text{using (1)} \\ &= A^{(n+1)-2} + A^2 - I \end{aligned}$$

Hence given statement is true for $n + 1$ also.
Using principle of Mathematical Induction (PMI), we infer that the statement holds true for all natural numbers greater than or equal to 3.

Question-1(c) Show that the function given by

$$f(x) = \begin{cases} \dfrac{x\left(e^{1/x} - 1\right)}{\left(e^{1/x} + 1\right)}, & x \neq 0 \\ 0 & , x = 0 \end{cases}$$

is continuous but not differentiable at $x = 0$**.**

[8 Marks]

Solution:

$$f(x)=\begin{cases}\frac{x\left(e^{1/x}-1\right)}{\left(e^{1/x}+1\right)}, & x\neq 0\\ 0 & ,x=0\end{cases}$$

$$RHL=\text{Lim}_{x\to 0^+}$$

$$f(x)=\lim_{x\to 0^+}\frac{x\left(e^{1/x}-1\right)}{e^{1/x}+1}$$

$$=\lim_{x\to 0^+}\frac{x\left(1-e^{-1/x}\right)}{1+e^{-1/x}}$$

$$=\frac{0(1-0)}{1+0}=0$$

$$\left(\because \text{ as } x\to 0^+, -\frac{1}{x}\to -\infty \Rightarrow e^{-1/x}+0\right)$$

$$LHL=\lim_{x\to 0^-}f(x)=\lim_{x\to 0^-}\frac{x\left(e^{1/x}-1\right)}{e^{1/x}+1}$$

$$=\lim_{h\to 0^+}\frac{(0-h)\left(e^{\frac{1}{(0-h)}-1}\right)}{\left(e^{\frac{1}{0-h}+1}\right)}$$

$$=\lim_{h\to 0^+}\frac{(-h)\left(e^{-1/h}-1\right)}{e^{-1/h}+1}$$

$$=\frac{0(0-1)}{0+1}=0$$

Hence, $LHL=RHL=f(0)\therefore f$ is continuous at $x=0$. For differentiability,

$$f'(0)=\lim_{h\to 0}\frac{f(h)-f(0)}{h-0}$$

$$=\lim_{h\to 0}\frac{1}{h}\left[\frac{h\left(e^{1/h}-1\right)}{e^{1/h}+1}-0\right]$$

$$=\lim_{h\to 0}\frac{e^{1/h}-1}{e^{1/h}+1}$$

This limit does not exist as

$$RHL=\lim_{h\to 0^+}\frac{e^{1/h}-1}{e^{1/h}+1}=\lim_{h\to 0^+}\frac{1-e^{-1/h}}{1+e^{-1/h}}$$

$$=\frac{1-0}{1+0}=1$$

$$LHL=\lim_{h\to 0^-}\frac{e^{1/h}-1}{e^{1/h}+1}=\frac{0-1}{0+1}=-1$$

Hence, $f(x)$ is not differentiable at $x=0$.

Question-1(d) Evaluate $\iint_{\mathrm{R}} \mathrm{y}\dfrac{\sin \mathrm{x}}{\mathrm{x}}\mathrm{dxdy}$ over R where $\mathrm{R} = \{(\mathrm{x},\mathrm{y}) : \mathrm{y} \le \mathrm{x} \le \pi/2, \quad 0 \le \mathrm{y} \le \pi/2\}$.

[8 Marks]

Solution:

$$
\begin{aligned}
I &= \iint\limits_R y\frac{\sin x}{x}dxdy \\
&= \int_{x=0}^{\pi/2}\int_{y=0}^{x} \frac{\sin x}{x} ydydx \\
&= \int_0^{\pi/2} \frac{\sin x}{x}\left[\frac{y^2}{2}\right]_0^x dx \\
&= \frac{1}{2}\int_0^{\pi/2} \frac{\sin x}{x}\cdot\left(x^2 - 0\right) dx \\
&= \frac{1}{2}\int_0^{\pi/2} x\cdot \sin x dx \\
&= \frac{1}{2}\left[\int_0^{\pi/2} [x(-\cos x)]_0^{\pi/2} + \int_0^{\pi/2} 1\cdot\cos x dx\right] \\
&= \frac{1}{2}\left[(0-0) + [\sin x]_0^{\pi/2}\right] \\
&= \frac{1}{2}(1-0) = \frac{1}{2}
\end{aligned}
$$

Question-1(e) Prove that the locus of a variable line which intersects the three lines:

$$y = mx, z = c; \quad y = -mx, \quad z = -c; \quad y = z, mx = -c$$

is the surface $y^2 - m^2x^2 = z^2 - c^2$.

[8 Marks]

Solution: The given lines are

$$
\begin{aligned}
&y - mx = 0, z - c = 0...(i) \\
&y + mx = 0, z + c = 0...(ii) \\
&y - z = 0, mx + c = 0...(iii)
\end{aligned}
$$

Any line intersecting (i) and (ii) is

$$y - mx - k_1(z - c) = 0, y + mx - k_2(z + c) = 0...(iv)$$

If it intersects (iii) also, we have to eliminate x, y, z from (iii) and (iv).

Now putting $y = z$ and $mx = -c$ from (iii) in (iv), we get

$$z + c - k_1(z - c) = 0$$

$$z - c - k_2(z + c) = 0$$

$$z\,(1 - k_1) + c\,(1 + k_1) = 0$$

$$z\,(1 - k_2) - c\,(1 + k_2) = 0$$

Equating the two values of z, we get

$$\frac{c\,(1 + k_1)}{k_1 - 1} = \frac{c\,(1 + k_2)}{1 - k_2}(= z)$$

$$(1 + k_1)\,(1 - k_2) = (1 + k_2)\,(k_1 - 1)$$

$$1 + k_1 - k_2 - k_1k_2 = k_1 + k_1k_2 - 1 - k_2$$

$$2k_1k_2 - 2 = 0$$

$$k_1k_2 = 1$$

To find the locus, we have to eliminate k_1, k_2 from (iv) and (v) From (iv)

$$k_1 = \frac{y - mx}{z - c}$$

$$k_2 = \frac{y + mx}{z + c}$$

Putting these values in (v), we get

$$\left(\frac{y - mx}{z - c}\right)\left(\frac{y + mx}{z + c}\right) = 1$$

$$\frac{y^2 - m^2x^2}{z^2 - c^2} = 1$$

$$y^2 - m^2x^2 = z^2 - c^2$$

which is the required locus.

Question-2(a) Let $B = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$. Find all eigenvalues and corresponding eigenvectors of B viewed as a matrix over:

(i) the real field R
(ii) the complex field C.

[10 Marks]

Solution:

$$|B - \lambda I| = 0$$

$$\begin{vmatrix} 1 - \lambda & -1 \\ 2 & -1 - \lambda \end{vmatrix} = 0$$

$$(\lambda - 1)(\lambda + 1) + 2 = 0$$
$$\lambda^2 + 1 = 0 \quad \Rightarrow \quad \lambda = i, -i$$

v is the eigenvector

$$Bv = \lambda v$$
$$\Rightarrow \quad (B - \lambda I)v = 0$$
$$\lambda = i$$
$$\Rightarrow \begin{pmatrix} 1-i & -1 \\ 2 & -1-i \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$\begin{matrix} (1-i)x - y = 0 \\ 2x - (1+i)y = 0 \end{matrix} \Rightarrow y = (1-i)x$$
$$v = \begin{bmatrix} x \\ y \end{bmatrix}$$
$$= \begin{bmatrix} x \\ (1-i)x \end{bmatrix}$$
$$= x \begin{bmatrix} 1 \\ 1-i \end{bmatrix}$$
$$\lambda = -i$$
$$\begin{bmatrix} 1+i & -1 \\ 2 & -1+i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$
$$(1+i)x - y = 0$$
$$2x - y(1-i) = 0$$
$$y = (1+i)x$$
$$v = \begin{pmatrix} x \\ y \end{pmatrix}$$
$$= \begin{pmatrix} x \\ (1+i)x \end{pmatrix}$$
$$= \begin{pmatrix} 1 \\ 1+i \end{pmatrix} x$$

When B is viewed as matrix over the complex field, then eigenvectors are

$$\begin{bmatrix} 1 \\ 1-i \end{bmatrix} \text{ and } [1+i]$$

When B is viewed as matrix over the real field the eigenvectros

$$v = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \quad for \quad \lambda = i \quad \text{i.e.} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$
$$a + ib \rightarrow \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$
$$v = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \quad for \quad \lambda = -i \quad \text{i.e.} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Question-2(b) If $xyz = a^3$ then show that the minimum value of $x^2 + y^2 + z^2$ is $3a^2$.

[10 Marks]

Solution: Let

$$f(x, y, z) = x^2 + y^2 + z^2$$
$$g(x, y, z) = xyz - a^3$$

Let

$$f(x, y, z) = x^2 + y^2 + z^2$$
$$g(x, y, z) = xyz - a^3$$

To get maximum value of f, we use Lagrange's multiplier method.
Consider,

$$F(x, y, z) = x^2 + y^2 + z^2 + \lambda\left(xyz - a^3\right)$$

For critical points,

$$dF = 0, \text{ i.e. } F_x = 0, \quad F_y = 0, \quad F_z = 0$$

$$\begin{aligned} 2x + \lambda(yz) = 0 \quad &\Rightarrow 2x^2 + \lambda xyz = 0 \\ 2y + \lambda(xz) = 0 \quad &\Rightarrow 2y^2 + \lambda xyz = 0 \\ 2z + \lambda(xy) = 0 \quad &\Rightarrow 2z^2 + \lambda xyz = 0 \end{aligned}$$

$$\therefore \quad x^2 = y^2 = z^2$$

Also, given $xyz = a^3$ which implies $(x \cdot x + x)a = a^3$ ie. $x^3 = a^3$

$$= \quad x^2 = y^2 = z^2 = a^2$$

Hence, minimum value of $x^2 + y^2 + z^2$ will be

$$a^2 + a^2 + a^2 = 3a^2$$

Question-2(c) Prove that every sphere passing through the circle

$$x^2 + y^2 + 2ax + r^2 = 0, \quad z = 0$$

cut orthogonally every sphere through the circle

$$x^2 + z^2 = r^2 \quad y = 0$$

[10 Marks]

Solution: Equations of two spheres can be taken as

$$\begin{aligned} &S_1 : x^2 + y^2 + z^2 - 2ax + r^2 + \lambda z = 0 \\ &S_2 : x^2 + y^2 + z^2 - r^2 + \mu y = 0 \end{aligned}$$

condition of orthogonality

$$2\left(u_1u_2+v_1v_2+w_1w_2\right)=d_1+d_2$$

$$2\left[a\cdot 0+0\cdot\left(-\frac{\mu}{2}\right)+\left(-\frac{\lambda}{2}\right)\cdot 0\right]=r^2-(r)^2$$

$$2(0+0+0)=0$$

$$0=0$$

which is true for all values of parameters λ and μ. Hence proved.

Question-2(d) Show that the mapping $\mathrm{T}:\mathrm{V}_2(\overline{\mathrm{R}})\rightarrow\mathrm{V}_3(\overline{\mathrm{R}})$ **defined as** $\mathrm{T(a,b)}=\mathrm{(a+b,a-b,b)}$ **is a linear transformation. Find the range, rank and nullity of** $\overline{T}$**.**

[10 Marks]

Solution:

$$\mathrm{T}(a)=(a+b,a-b,b),\forall a,b\in\mathrm{R}$$

Let $\alpha_1=(a_1,b_1)$ and $\alpha_2=(a_2,b_2)$ be any two elements of $\mathrm{V}_2(\mathrm{R})$ then

$$\left.\begin{array}{r}\mathrm{T}\left(\alpha_1\right)=\mathrm{T}\left(a_1,b_1\right)=\left(a_1+b_1,a_1-b_1,b_1\right)\\ \text{and }\mathrm{T}\left(\alpha_2\right)=\mathrm{T}\left(a_2,b_2\right)=\left(a_2+b_2,a_2-b_2,b_2\right)\end{array}\right\}\text{from (i)}$$

Now,

$$a,b\in\mathrm{R}\Rightarrow a\alpha_1+b\alpha_2\in\mathrm{V}_2(\mathrm{R})$$

$$\begin{aligned}\therefore\mathrm{T}\left(a\alpha_1+b\alpha_2\right)&=\mathrm{T}\left[a\left(a_1,b_1\right)+b\left(a_2,b_2\right)\right]\\&=\mathrm{T}\left[aa_1+ba_2,ab_1+bb_2\right]\\&=\left(aa_1+ba_2+ab_1+bb_2,aa_1+ba_2-ab_1-bb_2,ab_1+bb_2\right)\quad(by def.of T)\\&=\left[a\left(a_1+b_1\right)+b\left(a_2+b_2\right),a\left(a_1-b_1\right)+b\left(a_2-b_2\right)\ ab_1+bb_2\right]\\&=a\left(a_1+b_1,a_1-b_1,b_1\right)+b\left(a_2+b_2,a_2-b_2,b_2\right)\\&=a\ \mathrm{T}\left(\alpha_1\right)+b\ \mathrm{T}\left(\alpha_2\right)\end{aligned}$$

which proves that $T:V_2(R)\rightarrow V_3(R)$ as defined in (i) is a linear transformation.
Now, we will calculate the null space of T.
If $\alpha=(a,b)$, then

$$N(T)=\{\alpha\in V_2(R);T(\alpha)=0\in V_3(R)\}$$

Now,

$$T(\alpha)=T(a,b)=(a+b,a-b,b)=(0,0,0)$$

$$\Rightarrow a+b=0,a-b=0,b=0$$

$$\Rightarrow a=0,b=0$$

$$\therefore\alpha=(a,b)=(0,0)\in\mathrm{N(T)}$$

showing that null space consists of only zero vector of $\mathrm{V}_2(\mathrm{R})$ i.e. domain or in other words null space of T is the zero subspace of $\mathrm{V}_2(\mathrm{R})$ i.e. nullity of T = dim[N(T)] = 0.

Range space of T. We have

$$R(T) = \{\beta \in V_3(R) : \beta = T(\alpha), \alpha \in V, (R)\}$$

Now $\{(1,0),(0,1)\}$ is the basis of $V_2(R)$.

Also $T(1,0) = (1+0, 1-0, 0) = (1,1,0)$ and $T(0,1) = (0+1, 0-1, 1) = (1,-1,1)$

Hence the range space of T is a sub-snace ***of*** $V_3(R)$ generated by $(1,1,0)$ and $(1,-1,0)$.

Now

$$\begin{aligned} a(1,1,0) + b(1,-1,1) &= (0,0,0) \forall a, b \in R \\ \Rightarrow (a+b, a-b, b) &= (0,0,0) \\ \Rightarrow a+b = 0, a-b &= 0, b = 0 \\ \Rightarrow a = 0, b &= 0 \end{aligned}$$

Therefore $(1,1,0),(1,-1,1)$; elements of $R(T)$ are L.I. and generates $R(T)$ Hence, $\{(1,1,0),(1,-1,1)\}$ is the basis of R(T)

$$\therefore \dim(\mathrm{R(T)}) = \mathrm{rank(T)} = 2$$

Question-3(a) Examine whether the matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & 6 \\ -1 & -2 & 0 \end{bmatrix}$ is diagonalizable. Find all eigenvalues. Then obtain a matrix P such that $P^{-1}AP$ is a diagonal matrix.

[10 Marks]

Solution: Characterestic Equation of a square matrix A is given by :

$$| A - \lambda I | = 0$$

i.e.

$$\begin{aligned} \lambda^3 - (\text{ trace of A})\lambda^2 + (C_{11} + C_{22} + C_{33})\,\lambda - |A| &= 0 \\ \mathrm{trace}(A) &= -2 + 1 + 0 = -1 \\ C_{11} + C_{22} + C_{33} &= (0-12) + (0-3) + (-2-4) \\ &= -12 - 3 - 6 = -21 \\ |A| &= 45 \end{aligned}$$

$\therefore$ Characterestic Equation: $\lambda^3 + \lambda^2 - 21\lambda - 45 = 0$

$$\Rightarrow \lambda = 5, -3, 3 \qquad \text{(Use Calci)}$$

Now, let us find the corresponding eigen-vectors for each eigen-values.

$$\text{(i) } \lambda = 5$$

$$\therefore (A - \lambda I)X = 0 \Rightarrow (A - 5I)X = 0$$

$$\begin{bmatrix} -7 & 2 & -3 \\ 2 & -4 & -6 \\ -1 & -2 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_1 \leftrightarrow R_3$$

$$\begin{bmatrix} -1 & -2 & -5 \\ 2 & -4 & -6 \\ -7 & 2 & -3 \end{bmatrix} \sim \begin{bmatrix} -1 & -2 & -5 \\ 0 & -8 & -16 \\ 0 & 16 & 32 \end{bmatrix}$$

$$R_2 \to R_2 + 2R_1; \quad R_3 \to R_3 - 7R_1$$

$$R_2 \to R_2 - 2R_1; \quad R_1 \to \frac{R_1}{-8}$$

$$\sim \begin{bmatrix} -1 & -2 & -5 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_1 \to R_1 + 2R_2$$

$$\begin{aligned} -x - z &= 0 \\ y + 2z &= 0 \\ \text{i.e.} \quad x = -z&; \quad y = -2z \\ X = (x, y, z) &= (-z, -2z, z) = z(-1, -2, 0) \end{aligned}$$

$\therefore X_1 = (-1, -2, 0)$ is the given vector corresponding to eigen value $\lambda = 5$

$$\text{(ii)} \quad \lambda = -3, \quad (A + 3I)X = 0$$

$$\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & -6 \\ -1 & -2 & 3 \end{bmatrix} [\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R_2 \to R_2 - 2R_1, R_3 \to R_3 + R_1$$

$$\sim \begin{bmatrix} 1 & 2 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{i.e. } x + 2y - 3z = 0 \quad \text{i.e. } x = -2y + 3z$$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2y + 3z \\ y \\ z \end{bmatrix} = y \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

$$x_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \quad x_3 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

X_2 and X_3 are eigen-vectors, corresponding to eigen value $\lambda = -3$.
We notice that for each eigen-value, algebraic multiplicity (number of same roots) is equal to geometric multiplicity ie. number of independent eigen-vectors.
Hence, A is diagonalizable. Now, for $P^{-1}AP = D$,
Transformation matrix P is obtained by placing eigen-vectors as columns

$$P = [X_1 \quad X_2 \quad X_3] = \begin{bmatrix} -1 & -2 & 3 \\ -2 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

and Diagonal matrix, D consists of eigen-values placed at diagonal positions

$$D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & -3 \end{bmatrix}$$

We can verify that

$$P^{-1} = \frac{1}{8}\begin{bmatrix} -1 & -2 & 3 \\ -2 & 4 & 6 \\ 1 & 2 & 5 \end{bmatrix}$$

$$P^{-1}AP = D$$

Question-3(b) A moving plane passes through a fixed point $(2, 2, 2)$ and meets the coordinate axes at the points A, B, C, all away from the origin O. Find the locus of the centre of the sphere passing through the points O, A, B, C.

[10 Marks]

Solution: Let the eqn of plane be given by:

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

Let the points on coordinate axes through which the plane passes be given by $A(a, 0, 0)$, $B(0, b, 0)$, $C(0, 0, c)$.

$$\Rightarrow \frac{2}{a} + \frac{2}{b} + \frac{2}{c} = 1 \qquad \ldots(1)$$

Let general eqn of sphere

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$$

Origin O(0,0,0) lies on it $\Rightarrow d = 0$
$A(a, 0, 0)$ lies on it $\Rightarrow a^2 + 2ua = 0 \Rightarrow 2u = -a$.
Similarly, $B(0, b, 0)$ gives $2v = b$
$C(0, 0, c)$ gives $2\omega = c$

$$\therefore \quad x^2 + y^2 + z^2 + ax - by - cz = 0$$

Centre

$$x_1 = \frac{a}{2}, \quad y_1 = \frac{b}{2}, \quad z_1 = \frac{c}{2}$$

Using (1),

$$\frac{2}{2x_1} + \frac{2}{2y_1} + \frac{2}{2z_1} = 1$$

$$\Rightarrow \quad \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$$

is the required locus.

Question-3(c) Evaluate the integral

$$I = \int_0^\infty 2^{-ax^2} dx$$

using Gamma function.

[10 Marks]

Solution:

$$\begin{aligned}
I &= \int_0^\infty 2^{-ax^2} dx \\
&= \int_0^\infty e^{-ax^2(\log 2)} dx \quad \left[t = e^{\log t}\right] \\
\text{Let } a(\log 2)x^2 &= y \\
\Rightarrow a(\log 2)2x dx &= dy \\
I &= \int_0^\infty e^{-y} \cdot \frac{dy}{2a(\log 2)x} \\
&= \int_0^\infty e^{-y} \cdot \frac{dy}{2a(\log 2)\sqrt{y}} \times \sqrt{a \log 2} \\
&= \frac{1}{2\sqrt{a \log_2 2}} \int_0^\infty e^{-y} \cdot y^{-1/2} dy \\
&= \frac{1}{2\sqrt{a \log 2}} \cdot T\left(\frac{-1}{2} + 1\right) \\
&= \frac{\sqrt{\pi}}{2\sqrt{a \log 2}}
\end{aligned}$$

Question-3(d) Prove that the equation:

$$4x^2 - y^2 + z^2 + 2xy - 3yz + 2xy + 12x - 11y + 6z + 4 = 0$$

represents a cone with vertex at $(-1, -2, -3)$

[10 Marks]

Solution: Making the given equation homogeneous, we get

$$F(x, y, z, t) \equiv 4x^2 - y^2 + 2z^2 + 2xy - 3yz + 12xt - 11yt + 6zt + 4t^2 = 0$$

$\frac{\partial F}{\partial x} = 0$ gives $8x + 2y + 12t = 0$ or $4x + y + 6t = 0$
$\frac{\partial F}{\partial y} = 0$ gives $-2y + 2x - 3z - 11t = 0$ or $2x - 2y - 3z - 11t = 0$
$\frac{\partial F}{\partial z} = 0$ gives $4z - 3y + 6t = 0$ or $3y - 4z - 6t = 0$
$\frac{\partial F}{\partial t} = 0$ gives $12x - 11y + 6z + 8t = 0$

Putting $t = 1$, these equations become

$$4x + y + 6 = 0 \qquad \ldots(i); \qquad 2x - 2y - 3z - 11 = 0 \ldots(ii)$$

$$3y - 4z - 6 = 0 \quad \ldots(iii); \qquad 12x - 11y + 6z + 8 = 0 \ldots (iv)$$

From (ii), we get $4x - 4y - 6z - 22 = 0$

Subtracting (i) from it, we get $5y + 6z + 28 = 0$, or

$$10y + 12z + 56 = 0$$

Multiplying (iii) by, 3 we get $9y - 12z - 18 = 0 \quad \ldots(vi)$.

Adding (v) and (vi), we get $19y + 38 = 0$ or $y = -2$.

$\therefore$ From (iii), we get $3(-2) - 4z - 6 = 0$ or $z = -3$

From (i), we get $4x + (-2) + 6 = 0$ or $x = -1$.

These values, i.e., $x = -1, y = -2, z = -3$ satisfy (iv) and so the given equation represents a cone and its vertex is (-1,-2,-3).

Question-4(a) Let f be a real valued function defined on [0,1] as follows:

$$f(x) = \begin{cases} \dfrac{1}{a^{r-1}}, & \dfrac{1}{a^r} < x \leq \dfrac{1}{a^{r-1}}, r = 1, 2, 3 \ldots\ldots \\ 0 & x = 0 \end{cases}$$

where a is an integer greater than 2. Show that $\int_0^1 f(x)dx$ exists and is equal to $\dfrac{a}{a+1}$.

[10 Marks]

Solution:

$$f(x) = \begin{cases} \frac{1}{a^{1-1}} = 1, \frac{1}{a} < x \leq 1 \\ \frac{1}{a}, \frac{1}{a^2} < x \leq \frac{1}{a} \\ \frac{1}{a^2}, \frac{1}{a^3} < x \leq \frac{1}{a^2} \\ \frac{1}{a^{r-1}}, \frac{1}{a^r} < x \leq \frac{1}{a^{r-1}} \\ 0, x = 0 \end{cases}$$

Clearly $f(x) \in [0, 1]$ for all $x \in [0, 1] \Rightarrow f$ is bounded on [0,1] as $a > 2$

Also, it is continuous on [0,1] except at points $0, \frac{1}{a}, \frac{1}{a^2}, a^3$

The set of points of discontinuities has only one limit point O and hence, f is integrable on $[0,1]$.

$$\begin{aligned}
\int_{\frac{1}{a^r}}^{1} f(x)dx &= \int_{\frac{1}{a}}^{1} f dx + \int_{\frac{1}{a^2}}^{\frac{1}{a}} f dx + \ldots\ldots + \int_{\frac{1}{a^r}}^{\frac{1}{a^{r-1}}} f dx \\
&= \int_{\frac{1}{a}}^{1} 1.dx + \int_{\frac{1}{a^2}}^{\frac{1}{a}} \frac{1}{a} dx + \int_{\frac{1}{a^3}}^{\frac{1}{a^2}} \frac{1}{a^2} dx + \ldots\ldots + \int_{\frac{1}{a^r}}^{\frac{1}{a^{r-1}}} \frac{1}{a^{r-1}} dx \\
&= \left(1 - \frac{1}{a}\right) + \left(\frac{1}{a} - \frac{1}{a^2}\right)\frac{1}{a} + \left(\frac{1}{a^2} - \frac{1}{a^3}\right)\frac{1}{a^2} + \ldots\ldots + \left(\frac{1}{a^8 - 1} - \frac{1}{a^8}\right) \cdot \frac{1}{a^8 - 1} \\
&= \left(1 - \frac{1}{a}\right)\left[1 + \frac{1}{a^2} + \frac{1}{a^4} + \ldots + \frac{1}{a^2(h-1)}\right] \\
&= \frac{a-1}{a} \times \frac{1\left[1 - \left(\frac{1}{a^2}\right)^r\right]}{1 - \frac{1}{a^2}} \\
S_n &= \frac{a\left(1 - \lambda^n\right)}{1 - r} \\
&= \frac{a}{a+1} \cdot \left(1 - \frac{1}{a^2}\right)
\end{aligned}$$

Taking limit $r \to \infty$

$$\int_0^1 f(x)dx = \frac{a}{a+1}$$

Question-4(b) Prove that the plane $ax+by+cz=0$ cuts the cone $yz+zx+xy=0$ in perpendicular lines if $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$.

[10 Marks]

Solution: Let

$$\frac{x-0}{l} = \frac{y-0}{m} = \frac{z-0}{n}$$

be the line of section.
$\therefore l, m, n$ satisfies equation of cone and plane.

$$\therefore \quad lm + mn + \ln = 0$$

and

$$al + bm + cn = 0 \ldots (i)$$

$$n = \frac{al + bm}{-c}$$

Substituting we get

$$lm - \frac{m}{c}(al + bm) - \frac{l}{c}(al + bm) = 0$$

$$\therefore \quad al^2 + lm(a + b - c) + bm^2 = 0$$

$$\therefore \quad a\left(\frac{l}{m}\right)^2 + (a + b - c)\left(\frac{l}{m}\right) + b = 0$$

which is a quadratic in $\frac{l}{m}$.
Let $\frac{l_1}{m_1}$ and $\frac{l_2}{m_2}$ be the two roots of this equation.
$\therefore$ Product of the roots $= \frac{b}{a}$ i.e.

$$\frac{l_1}{m_1} \cdot \frac{l_2}{m_2} = \frac{b}{a}$$

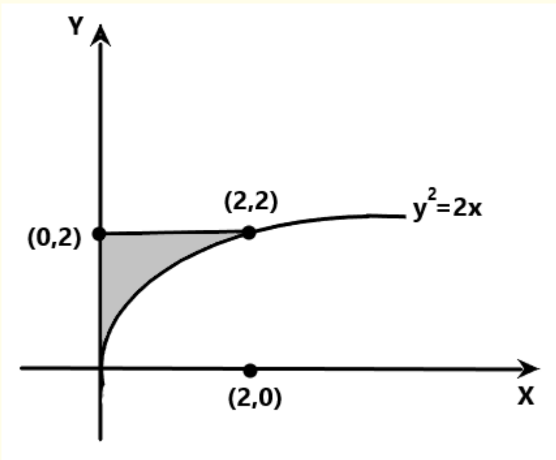
$$\therefore \frac{l_1 l_2}{1/a} = \frac{m_1 m_2}{1/b} = \frac{n_1 n_2}{1/c}$$

by symmetry.
$\therefore$ *perpendicular* if

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$$

Question-4(c) Evaluate the integral $\iint_R \frac{y}{\sqrt{x^2+y^2+1}} dxdy$ **over the region** R **bounded between** $0 \le x \le \frac{y^2}{2}$ **and** $0 \le y \le 2$**.**

[10 Marks]



Solution:

$$
\begin{aligned}
I &= \int_R \frac{y}{\sqrt{x^2+y^2+1}} dxdy \\
&= \int_{x=0}^{2} \int_{y=\sqrt{2x}}^{2} \frac{2y}{2\sqrt{1+x^2+y^2}} dydx \\
&= \int_{x=0}^{2} \left(1+x^2+y^2\right)^{1/2} \Big|_{y=\sqrt{2x}}^{y=2} dx \\
&= \int_0^2 \left(\sqrt{1+4+x^2} - \sqrt{1+x^2+2x}\right) dx \\
x &= 0^2\sqrt{5+x^2} - (1+x)dx \\
&= \int_0^2 \sqrt{5+x^2} - (1+x)dx \\
&= -\frac{\left[\frac{1}{2}x\sqrt{5+x^2} + \frac{5^2}{2}\log\left|x+\sqrt{5+x^2}\right|_0^2\right.}{-\left[x+\frac{x^2}{2}\right]_0^2} \\
&= \left(\frac{2x\sqrt{9}}{2} + \frac{5}{2}\log(2+\sqrt{9}) - \frac{5}{2}\log\sqrt{5}\right) - \left(2+\frac{4}{2}\right) \\
I &= \frac{5}{4}\log 5 - 1
\end{aligned}
$$

Question-4(d) Consider the linear mapping $\mathrm{F} : \mathbb{R}^2 \to \mathbb{R}^2$ given as $\mathrm{F(x, y)} = (3\mathrm{x}+4\mathrm{y}, 2\mathrm{x}-5\mathrm{y})$ with usual basis. Find the matrix associated with the linear transformation relative to the basis $\mathrm{S} = \{\mathrm{u_1}, \mathrm{u_2}\}$ where $u_1 = (1, 2), u_2 = (2, 3)$.

[10 Marks]

Solution: $F : R^2 \to R^2$

$$
\begin{aligned}
F(x, y) &= (3x+4y, 2x-5y) \\
s &= \{(1, 2), (2, 3)\}
\end{aligned}
$$

$$
\begin{aligned}
F(1, 2) &= (3+8, 2-10) = (11, -8) \\
&= -49(1, 2) + 30(2, 3) \text{ [Using calculator]} \\
F(2, 3) &= (6+12, 4-15) = (18, -11) \\
&= -76(1, 2) + 47(2, 3)
\end{aligned}
$$

$\therefore$ Matrix of LT wrt. Basis S

$$
[M] = \begin{bmatrix} -49 & 30 \\ -76 & 47 \end{bmatrix}' = \begin{bmatrix} -49 & -76 \\ 30 & 47 \end{bmatrix}
$$

7.2 Section-B

Question-5(a) Solve the differential equation :

$$y = 2px + p^2 y, p = \frac{dy}{dx}$$

and obtain the non-singular solution.

[8 Marks]

Solution: We have

$$y = 2xp + yp^2 \qquad \ldots (i)$$

$$\Rightarrow \quad 2xp = y - yp^2$$

$$\Rightarrow \quad x = \frac{y}{2p} - \frac{py}{2} \qquad \ldots (ii)$$

Differentiating (ii) w.r.t. y, we get

$$\frac{dx}{dy} = \frac{1}{2}\left(\frac{1}{p} \cdot 1 + y \cdot -\frac{1}{p^2}\frac{dp}{dy}\right) - \frac{1}{2}\left(p \cdot 1 + y\frac{dp}{dy}\right)$$

$$\Rightarrow \frac{1}{p} = \frac{1}{2p} - \frac{y.}{2p^2}\frac{dp}{dy} - \frac{p}{2} - \frac{y}{2}\frac{dp}{dy}$$

$$\Rightarrow \quad \frac{1}{2p} - \frac{p}{2} = \left(\frac{y}{2} - \frac{y}{2p^2}\right)\frac{dp}{dy}$$

$$\Rightarrow \quad -\left(\frac{p}{2} - \frac{1}{2p}\right) = \frac{y}{p}\left(\frac{p}{2} - \frac{1}{2p}\right)\frac{dp}{dy}$$

$$\Rightarrow \quad -1 = \frac{y}{p}\frac{dp}{dy}$$

$$\Rightarrow \quad \frac{dp}{p} + \frac{dy}{y} = 0$$

Integrating,

$$\log p + \log y = \log c$$

$$\Rightarrow py = c$$

$\Rightarrow$

$$p = c/y$$

Putting the value of p in (i), we get

$$y = 2x\left(\frac{c}{y}\right) + y\left(\frac{c}{y}\right)^2$$

$$\Rightarrow \quad y^2 = 2cx + c^2$$

$$\Rightarrow y^2 - 2cx - c^2 = 0$$

Question-5(b) Solve :

$$\frac{d^4y}{dx^4} - 16y = x^4 + \sin x$$

[8 Marks]

Solution: Auxiliary Egn: $D^4 - 16 = 0$ ie. $(D^2 - 4)(D^2 + 4)$

$$D = \pm 2, \pm 2i$$
$$C \cdot F \cdot = c_1 e^{2x} + c_2 e^{-2x} + c_3' \cos 2x + c_4' \sin 2x$$
$$= c_1 e^{2x} + c_2 e^{-2x} + c_3 \cos(2x + (4).$$

$$P.I_1 = \frac{1}{D^4 - 16}\left(x^4 + \sin x\right)$$
$$= \frac{1}{D^4 - 16} x^4 + \frac{1}{D^4 - 16} \sin x$$
$$= \frac{-1}{16}\left(1 - \frac{D^4}{16}\right)^{-1} x^4 + \frac{1}{(-1^2)(-1^2) - 16} \sin x$$
$$\left(D^4 = D^2 \cdot D^2\right)$$
$$= \frac{-1}{16}\left(1 + \frac{D^4}{16}\right) x^4 - \frac{11}{15} \sin x$$
$$= \frac{-1}{16}\left(x^4 + \frac{\cancel{4} \cdot 3 \cdot x \cdot 1}{+6x_2}\right) - \frac{1}{15} \sin x$$

Hence, complete solution,

$$y = C.F. + P.I.$$
$$y = ce^{2x} + c_2 e^{-2x} + C_3 \operatorname{cs}(2x + c_4) \frac{-x^4}{16} - \frac{3}{32} - \frac{1}{15} \sin x$$

Question-5(c) A particle whose mass is m, is acted upon by a force $m\mu\left(x + \frac{a^4}{x^3}\right)$ towards the origin. If it starts from rest at a distance 'a' from the origin, prove that it will arrive at the origin in time $\frac{\pi}{4\sqrt{\mu}}$.

[8 Marks]

Solution: Given

$$\frac{d^2x}{dt^2} = -\mu\left[x + \frac{a^4}{x^3}\right], \quad \ldots (i)$$

the -ve sign being taken because the force is attractive.

Integrating it after multiplying throughout by $2(dx/dt)$, we get

$$\left(\frac{dx}{dt}\right)^2 = \mu\left[-x^2 + \frac{a^4}{x^2}\right] + C$$

When $x = a, dx/dt = 0$, so that $C = 0$

$$\therefore \left(\frac{dx}{dt}\right)^2 = \mu\left[\frac{a^4 - x^4}{x^2}\right]$$

$$\frac{dx}{dt} = -\frac{\sqrt{\mu}\sqrt{(a^4 - x^4)}}{x}$$

the -ve sign is taken because the particle is moving in the direction of x decreasing. If t_1 be the time taken to reach the origin, then integrating (ii), we get

$$\begin{aligned} t_1 &= -\frac{1}{\sqrt{\mu}}\int_a^0 \frac{x}{\sqrt{(a^4 - x^4)}}dx \\ &= \frac{1}{\sqrt{\mu}}\int_0^t \frac{xdx}{\sqrt{(a^4 - x^4)}} \end{aligned}$$

Put $x^2 = a^2 \sin\theta$ so that

$$2xdx = a^2 \cos\theta d\theta$$

When $x = 0, \theta = 0$ and when

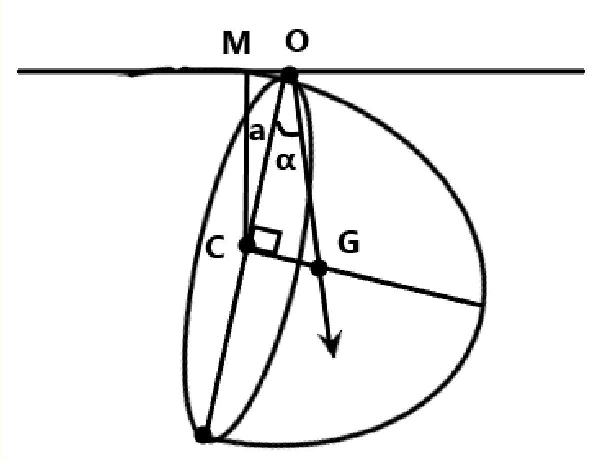
$$x = a, \theta = \frac{\pi}{2}$$

$$\begin{aligned} t_1 &= \frac{1}{\sqrt{\mu}}\int_0^{\pi/2} \frac{\frac{1}{2}a^2\cos\theta d\theta}{a^2\cos\theta} \\ &= \frac{1}{2\sqrt{\mu}}\int_0^{\pi/2} d\theta = \frac{1}{2\sqrt{\mu}}[\theta]_0^{\pi/2} \\ &= \frac{1}{2\sqrt{\mu}} \cdot \frac{\pi}{2} \\ &= \frac{\pi}{4\sqrt{\mu}} \end{aligned}$$

Question-5(d) A hollow weightless hemisphere filled with liquid is suspended from a point on the rim of its base. Show that the ratio of the thrust on the plane base to the weight of the contained liquid is $12 : \sqrt{73}$.

[8 Marks]

Solution: Let a' be the radius of the hemisphere and O the point of rim from which it is suspended.



Let G be the CG (centre of gravity) of the hemisphere, then $CG = \frac{3}{8}a$ and OG must be vertical.
If α be the inclination of the base to the vertical, then

$$\tan\alpha = \frac{3}{8} \qquad \ldots(1)$$

$$\text{The whole pressure (thrust) on the base } = w \cdot \pi a^2 \cdot (a \operatorname{css}\alpha)$$

Here, $w =$ weight per unit volume of liquid.

Depth of the center of gravity of the boy below surface of liquid = CM=$a\cos\alpha$.

$$\text{Weight of the liquid contained } = w \cdot \left(\frac{2}{3}\pi a^3\right)$$

$$\therefore \text{ Required ratio is } = \frac{w \cdot \pi a^2(a\cos\alpha)}{w \cdot \frac{2}{3}\pi a^3}$$

$$= \frac{3}{2} \cdot \frac{8^4}{\sqrt{73}}$$

$$[\text{From (1)}, \ \tan\alpha = \frac{3}{8} \Rightarrow \cos\alpha = \frac{8}{\sqrt{73}}]$$

$$= \frac{12}{\sqrt{73}}$$

Question-5(e) For three vectors show that:

$$\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b}) = 0$$

[8 Marks]

Solution: We know that

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$$
$$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$$

Hence,

$$\vec{a} \times (\vec{b} \times \vec{c}) + \vec{b} \times (\vec{c} \times \vec{a}) + \vec{c} \times (\vec{a} \times \vec{b})$$
$$= [(\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}] + [(\vec{b} \cdot \vec{a})\vec{c} - (\vec{b} \cdot \vec{c})\vec{a}] + [(\vec{c} \cdot \vec{b})\vec{a} - (\vec{c} \cdot \vec{a})\vec{b}]$$

$$= 0$$

Question-6(a) Solve the following differential equation:

$$\frac{dy}{dx} = \frac{2y}{x} + \frac{x^3}{y} + x \tan \frac{y}{x^2}$$

[10 Marks]

Solution: Put $\frac{y}{x^2} = t$ ie, $y = tx^2$

$$\frac{dy}{dx} = 2tx + x^2 \cdot \frac{dt}{dx}$$

Now DE becomes

$$2tx + x^2 \frac{dt}{dx} = \frac{2tx^2}{x} + \frac{x^3}{tx^2} + x + \tan t$$

$$x\frac{dt}{dx} = \frac{1}{t} + \frac{\cos t + t \sin t}{t \cos t}$$

$$\int \frac{t \cos t dt}{\cos t + t \sin t} = \int \frac{dx}{x}$$

$$\log(\cos t + t \sin t) = \log x + \log c$$

$$\Rightarrow \quad \cos t + t \sin t = cx$$

i.e.

$$\cos\left(\frac{y}{x^2}\right) + \frac{y}{x^2} \sin\left(\frac{y}{x^2}\right) = cx$$

which is the required solution.

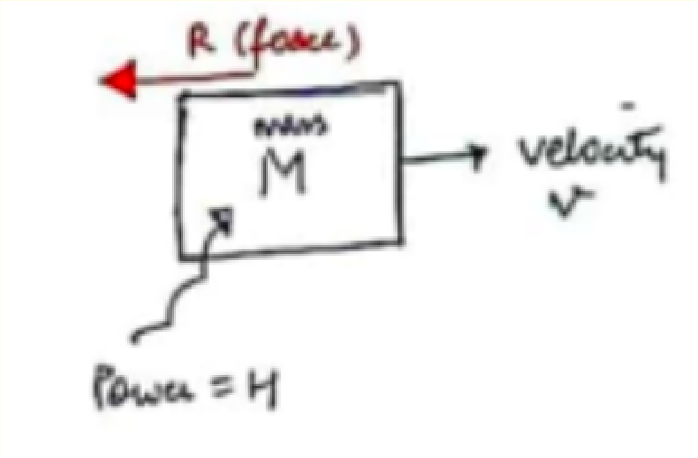
Question-6(b) An engine, working at a constant rate H, draws a load M against a resistance R. Show that the maximum speed is H/R and the time taken to attain half of this speed is $\frac{\mathrm{MH}}{\mathrm{R}^2}\left(\log 2 - \frac{1}{2}\right)$

[10 Marks]

Solution: Energy equation for time dt,

$$\text{Energy supplied} = \mathrm{Hdt}$$

$$\text{Energy lost due to the resistance } = \text{ Force } \times \text{ distance } == Rvdt$$



Assuming no change of PE;

$$\triangle PE = 0$$

$$\sum \text{ energy supplied } - \sum \text{ enegy lost } = \Delta(k.E.)$$

$$Hdt - RVdt = d\cdot\left(\frac{1}{2}\mathrm{mv}^2\right)$$

$$Hdt - Rvdt = mvdv$$

$$H - Rv = mv\frac{dv}{dt}$$

For max. velocity,

$$\frac{dv}{dt} = 0$$

$$\Rightarrow \text{ accelaration } = 0$$

$$H - Rv = 0$$

$$V_{\max} = H/R$$

Now, integrating

$$H - Rv = mvd\frac{d}{dt}$$

$$
\begin{aligned}
dt &= m\left(\frac{vdv}{\beta tRv}\right)\\
&= \frac{m}{R}\left(\frac{kv}{\mu - Rv}\right)dv\\
dt &= \frac{m}{R}\left(\frac{Rv - H + H}{H - Rv}\right)dv\\
&= \frac{m}{R}\left(\frac{H}{H + Rv} - 1\right)dv\\
\int_0^t dt &= \int_0^{\frac{V_{\max}}{2}} \frac{M}{R}\left(\frac{H}{H - Rv} - 1\right)dv\\
t &= \int_0^{H/2R} \frac{M}{R}\left(\frac{M}{H - Rv} - 1\right)dv\\
&= \frac{M}{R}\left[\frac{-H}{R}\log(H - Rv) - v\right]_0^{H/2R}\\
&= \frac{M}{R}\left[\frac{-H}{R}\log\left(H - R\cdot\frac{H}{2R}\right) - \frac{H}{2R} + \frac{H}{R}\log(H - R\cdot 0) - 0\right]\\
&= \frac{M}{R}\left[-\frac{H}{R}\log\frac{H}{2} - \frac{H}{2R} + \frac{H}{R}\log H\right]\\
&= \frac{M}{R}\left[\left(\frac{-H}{R}\log H + \frac{H}{R}\log 2\right) - \frac{H}{2R} + \frac{H}{R}\log H\right]\\
&= \frac{MH}{R^2}\left[(\log 2) - \frac{1}{2}\right]
\end{aligned}
$$

Question-6(c) Solve by the method of variation of parameters:

$$y'' + 3y' + 2y = x + \cos x$$

[10 Marks]

Solution: D.E.

$$\Rightarrow \quad \left(D^2 + 3D + 2\right)y = x + \cos x$$

Auxiliary Eqn:

$$D^2 + 3D + 2 = 0$$

$$(D + 1)(D + 2) = 0$$

$$D = -1, -2$$

$$C\cdot F = C_1' e^{-x} + C_2' e^{-2x}$$

$$y_1 = e^{-x} \quad , \quad y_2 = e^{-2x}$$

$$
\begin{aligned}
W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} &= \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix}\\
&= -2e^{-3x} + e^{-3x} = -e^{-3x} \neq 0
\end{aligned}
$$

$\therefore y_1$ & y_2 are linearly independent.

To get complete solution by variation of parameters, we replace C_1' and C_2' in C.F. by functions A and B.

$$\begin{aligned}
y &= Ae^{-x} + Be^{-2x} \\
&= Ay_1 + By_2 \\
A &= -\int \frac{Ry_2}{w} dx \\
&= -\int \frac{(x+\cos x)e^{-2x}}{-e^{-3x}} dx \\
&= \int e^x (x+\cos x) dx \\
&= \int x \cdot e^x dx + \int e^x \cos x dx \\
&= xe^x - \int e^x dx + \frac{1}{2} e^x (\cos x + \sin x) + c_1 \\
&= xe^x - e^x + \frac{e^x}{2} (\cos x + \sin x) + c_1 \\
\because \int e^{ax} \sin axdx &= \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx), \text{ and} \\
\int e^{ax} \cos bxdx &= \frac{e^{ax}}{a^2+b^2}(a \cos bx + b \sin bx) \\
\text{Also, } B &= \int \frac{y_1 R}{w} dx \\
&= \int \frac{e^{-x}(x+\cos x)}{-e^{-3x}} dx \\
&= \left[\int e^{-2x} xdx + \int e^{-2x} \cos xdx \right] \\
&= -\left[x \cdot \frac{e^{-2x}}{-2} - \int 1 \frac{e^{-2x}}{-2} dx + \frac{e^{-2x}}{4+1}(-2\cos x + \sin x) \right] \\
&= \frac{x}{2} e^{-2x} + \frac{1}{4} e^{-2x} + \frac{e^{-2x}}{5}(-2\cos x + \sin x) + C_2
\end{aligned}$$

Hence, complete solution is given by

$$y = Ay_1 + By_2$$

$$y = \left[xe^x - e^x + \tfrac{e^x}{2}(\cos x + \sin x) + c_1 \right] e^{-x} + \left[\tfrac{x}{2} e^{-2x} + \tfrac{e^{-2x}}{4} + \tfrac{e^{-2x}}{5}(-2\sin x + \sin x) + C_2 \right] e^{-2x}$$

Question-6(d) For the vector $\bar{A} = \dfrac{x\hat{i} + y\hat{j} + z\hat{k}}{x^2+y^2+z^2}$ examine if $\bar{A}$ is an irrotational vector. Then determine ϕ such that $\vec{A} = \nabla\phi$

[10 Marks]

Solution: $\vec{A}$ is irrotational if

$$\nabla \times \vec{A} = 0$$

$$\nabla \times \vec{A} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{x}{r^2} & \frac{y}{r^2} & \frac{z}{r^2} \end{vmatrix} \quad (r^2 = x^2 + y^2 + z^2)$$

$$= i\left(\frac{-2z}{r^3} \cdot 2y + \frac{2y}{r^3} 2z\right) + j\left(\frac{-2x}{r^3} 2z + \frac{2}{x^3} 2x\right) + k\left(\frac{-2y}{r^3} 2x + \frac{2x}{r^3} 2y\right)$$

$$= 0$$

$\therefore \vec{A}$ is irrotational vector.

$$\vec{A} = \nabla\phi = i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

$$\frac{\partial phi}{\partial x} = \frac{x}{x^2 + y^2 + 2^2}$$

$$\frac{\partial\phi}{\partial y} = \frac{y}{x^2 + y^2 + z^2}$$

$$\frac{\partial\phi}{\partial z} = \frac{2}{x^2 + y^2 + 2^2}$$

$$\Rightarrow \phi = \frac{1}{2}\log\left(x^2 + y^2 + z^2\right) + C$$

$\therefore$ Scalar potential ϕ is such that $\vec{A} = \nabla\phi$

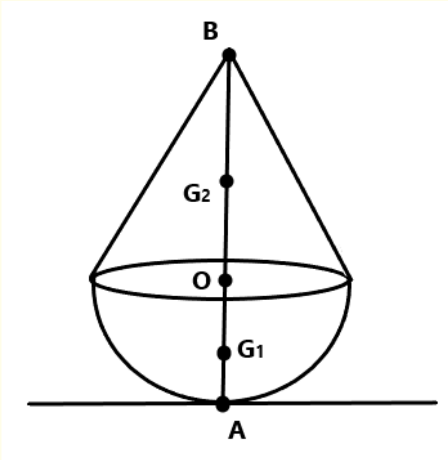
Question-7(a) A solid consisting of a cone and a hemisphere on the same base rests on a rough horizontal table with the hemisphere in contact with the table. Show that the largest height of the cone so that the equilibrium is stable is $\sqrt{3}\times$ radius of hemisphere.

[15 Marks]

Solution: Let us first try to find out the $C \cdot G \cdot$ of the whole body.
As we know, CG of a solid hemisphere is a point on its axis at a distance $3a/8$ from the centre of its base, where $'a'$ is radius of sphere

$$x_1 = AG_1 = a - \frac{3a}{8} = \frac{5a}{8}$$

.



$$\begin{aligned}
\omega_1 &= \text{ weight of hemisphere} \\
&= \frac{2}{3}\pi a^3 \rho q \\
x_2 &= \text{ distance of centre of gravity of cone from table} \\
&= AO + OG_2 \\
&= a + \frac{H}{4} \\
\omega_2 &= \text{ weight of cone} \\
&= \frac{1}{3}\pi a^2 \text{Hpg}
\end{aligned}$$

$$\begin{aligned}
h &= \text{ distance of C.G of combined body from horizontal plane} \\
&= \frac{w_1 x_1 + \omega_2 x_2}{w_1 + w_2} \\
&= \frac{\frac{2}{3}\pi a^3 \text{ pg } \frac{5a}{8} + \frac{1}{3}\pi a^2 \text{ Heg } \left(a + \frac{1}{4}\right)}{\frac{2}{3}\pi a^3 pg + \frac{1}{3} i + a^2 + 1g} \\
&= \frac{\frac{5}{4}a^2 + 1 + \left(a + \frac{11}{4}\right)}{2a + H} \\
&= \frac{5a^2 + 11(4a + 11)}{4(2a + 11)}
\end{aligned}$$

Let $R =$ radius of lower surface $= \infty$ & $r =$ radius of upper furface $= a$

For stable equilibrium,

$$\begin{aligned}
\frac{1}{h} &> \frac{1}{r} + \frac{1}{R} \\
\frac{4(2a + H)}{5a^2 + H(4a + H)} &> \frac{1}{a} + \frac{1}{\infty} \\
a(8a + 4H) &> 5a^2 + 4aH + H^2 \\
3a^2 &> H^2 \\
H &< \sqrt{3}a
\end{aligned}$$

Question-7(b) Evaluate $\iint_S \nabla \times \vec{A} \cdot \hat{n} dS$ for $\vec{A} = (x^2 + y - 4)\hat{i} + 3xy\hat{j} + (2xz + z^2)\hat{k}$ and S is the surface of hemisphere $x^2 + y^2 + z^2 = 16$ above xy plane.

[15 Marks]

Solution: The boundary C of the surface S is the circle $x^2 + y^2 = a^2, z = 0$. Suppose $x = a\cos t$, $y = a\sin t, z = 0, 0 \le t \le 2\pi$ are the parametric equations of C. By Stokes' theorem, we have

$$\begin{aligned}
\int\int_S (\nabla \times \overline{A}) \cdot \bar{n} ds &= \int_C \overline{A} \cdot \bar{dr} \\
&= \int_C [\left(x^2 + y - 4\right)\hat{i} + 3xy\hat{j} + \left(2xz + z^2\right)\hat{k}] \cdot (\hat{i}dx + \hat{j}dy + \hat{k}dz) \\
&= \int_C \left(x^2 + y - 4\right) dx + 3xy \cdot dy + \left(2xz + z^2\right) \cdot dz \\
&= \int_C \left(x^2 + y - 4\right) dx + 3xy \cdot dy \quad [\because z = 0 \therefore dz = 0] \\
&= \int_0^{2\pi} \left(a^2\cos^2 t + a\sin t - 4\right)(-a\sin t) \cdot dt + 3a\cos t \cdot a\sin t(a\cos t) \cdot dt
\end{aligned}$$

$$\begin{aligned}
&= \int_0^{2\pi} \left[2a^3 \cdot \cos^2 t \cdot \sin t - \frac{a^2}{2}(1 - \cos 2t) + 4a\sin t\right] \cdot dt \\
&= \frac{-2a^3}{3}\left(\cos^3 t\right)_0^{2x} - \frac{a^2}{2}[t]_0^{2\pi} + \frac{a^2}{4}[\sin 2t]_0^{2\pi} - 4a[\cos t]_0^{2\pi} \\
&= 0 - a^2 \cdot \pi + 0 - 0 \\
&= -16\pi \quad \left(\because a = 4 \Rightarrow a^2 = 16\right)
\end{aligned}$$

Question-7(c) Solve the D.E.:

$$\frac{d^3y}{dx^3} - 3\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 2y = e^x + \cos x$$

[10 Marks]

Solution:

$$\left(D^3 - 3D^2 + 4D - 2\right)y = e^x + \cos x$$

Auxiliary eqn:

$$D^3 - 3D^2 + 4D - 2 = 0$$

$$(D - 1)\left(D^2 - 2D + 2\right) = 0$$

$$D = 1, \frac{2 \pm \sqrt{4 - 8}}{2}$$

i.e.

$$D = 1, 1 \pm i$$

$$\begin{aligned}CF &= c_1e^x + e^x\left(c_2^1\cos x + c_3'\sin x\right)\\ &= c_1e^x + c_2e^x\,\text{css}\,(x+c_3)\end{aligned}$$

$$\begin{aligned}P.I. &= \frac{1}{D^3-3D^2+4D-2}\left(e^x+\cos x\right)\\ &= x\cdot\frac{1}{3D^2-6D+4}e^x + \frac{1}{D\left(-1^2\right)-3\left(-1^2\right)+4D-2}\cos x\\ &= \frac{x\cdot e^x}{3-6+4} + \frac{1}{(3D+1)}\cdot\frac{3D-1}{3D-1}\cos x\\ &= x'e^x + \frac{1}{9D^2-1}\left((3D-1)-(\cos x)\right.\\ &= xe^x + \frac{1}{9\left(-1^2\right)-1}(-3\sin x-\cos x)\\ &= xe^x + \frac{1}{10}(3\sin x+\cos x)\end{aligned}$$

$\therefore$ Complete solution, $y = CF + PI$

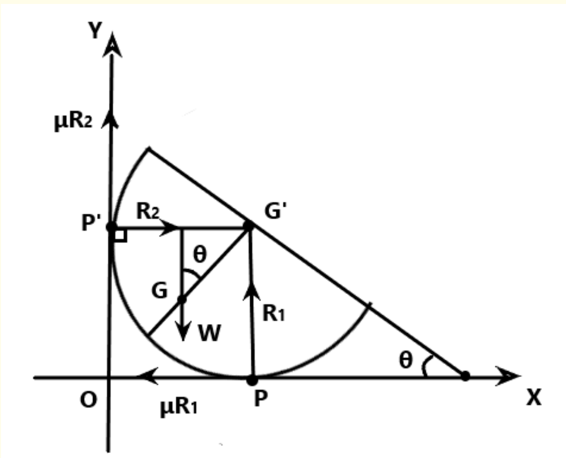
$$y = ge^x + c_2e^x\cos(x+c_3) + xe^x + \frac{1}{10}(3\sin x+\cos x)$$

Question-8(a) A semi circular disc rests in a vertical plane with its curved edge on a rough horizontal and equally rough vertical plane. If the coefficient of friction is μ, prove that the greatest angle that the bounding diameter can make with the horizontal plane is:

$$\sin^{-1}\left(\frac{3\pi}{4}\frac{\mu+\mu^2}{1+\mu^2}\right)$$

[10 Marks]

Solution: Let the disc's diameter makes angle θ with the x-axis (horizontal)
At equilibrium ie. before motion



$$\sum F_x = 0$$

$$\Rightarrow R_2 - \mu R_1 = 0 \quad \ldots \quad (1)$$

$$\sum F_y = 0$$

$$\Rightarrow \quad \mu R_2 + R_1 - W = 0 \quad \ldots \quad (2))$$

Taking moments about G'

$$(\mu R_2)\, r + (\mu R_1)\, r - W\,(GG' \sin\theta) = 0$$

Where r is radius of disc.

$$GG' = \frac{4}{3\pi} r \text{ [A result from chapter on center of gravity]}$$

$$\mu r\,(R_1 + R_2) = w \frac{4r}{3\pi} \sin\theta \quad \ldots \quad (3)$$

Using (1) and (2) in (3), we get

$$R_2 = \mu R_1$$

$$\mu R_2 + R_1 - W = 0$$

$$\Rightarrow \mu^2 R_1 + R_1 - W = 0$$

$$R_1 = \frac{W}{1+\mu^2}$$

$$\therefore \mu \left(\frac{W}{1+\mu^2} + \frac{\mu W}{1+\mu^2} \right) = \frac{4W \sin\theta}{3\pi}$$

$$\frac{\mu}{1+\mu^2}(1+\mu) = \frac{4}{3\pi} \sin\theta$$

$$\Rightarrow \theta = \sin^{-1}\left[\frac{3\pi}{4} \cdot \left(\frac{\mu + \mu^2}{1+\mu^2} \right) \right]$$

Hence, proved.

Question-8(b) A body floating in water has volumes V_1, V_2 and V_3 above the surface when the densities of the surrounding air are ρ_1, ρ_2, ρ_3 respectively. Prove that:

$$\frac{\rho_2 - \rho_3}{\mathrm{V}_1} + \frac{\rho_3 - \rho_1}{\mathrm{V}_2} + \frac{\rho_1 - \rho_2}{\mathrm{V}_3} = 0$$

[10 Marks]

Solution: Let V be the volume and W the weight of the body. Then the volumes immersed in water in the three faces are

$$(V - V_1)\,, (V - V_2) \text{ and } (V - V_3)$$

Let ρ be the density of water.
For equilibrium,
weight of the body = weight of water displaced + weight of air displaced

$$\therefore \quad W = (V - V_1)\,\rho g + V_1 \rho_1 g \text{ or } W - V\rho g = V_1 g\,(\rho_1 - \rho)$$

$$\frac{W - V\rho g}{V_1} = g\left(\rho_1 - \rho\right) \qquad \ldots (1)$$

Similarly,

$$\frac{W - V\rho g}{V_2} = g\left(\rho_2 - \rho\right) \qquad \ldots (2)$$

and

$$\frac{W - V\rho g}{V_3} = g\left(\rho_3 - \rho\right) \qquad \ldots (3)$$

Multiplying (1) by $(\rho_2 = \rho_3)$, (2) by $(\rho_3 - \rho_1)$ and (3) by $(\rho_1 - \rho_2)$ and adding, we get

$$(W - V\rho g)\left[\frac{\rho_2 - \rho_3}{V_1} + \frac{\rho_3 - \rho_1}{V_2} + \frac{\rho_1 - \rho_2}{V_3}\right] = 0$$
$$\frac{\rho_2 - \rho_3}{V_1} + \frac{\rho_3 - \rho_1}{V_2} + \frac{\rho_1 - \rho_2}{V_3} = 0$$

Hence, proved.
Note that the above result can also be put in the form

$$V_2V_3\left(\rho_2 - \rho_3\right) + V_3V_1\left(\rho_3 - \rho_1\right) + V_1V_2\left(\rho_1 - \rho_2\right) = 0$$
$$\rho_1V_1\left(V_2 - V_3\right) + \rho_2V_2\left(V_3 - V_1\right) + \rho_3V_3\left(V_1 - V_2\right) = 0$$

Question-8(c) Verify the divergence theorem for $\bar{A} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ over the region $x^2 + y^2 = 4$, $z = 0$, $z = 3$.

[10 Marks]

Solution: The divergence theorem is

$$\iiint\limits_V \nabla \cdot \bar{A} dv = \iint\limits_S \mathrm{A} \cdot \hat{n} i ds$$

Now, volume integral

$$= \iiint \left\{\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial x}\hat{j} + \frac{\partial}{\partial x}\hat{k}\right\} \cdot \left\{4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}\right\} dxdydz$$
$$= \iiint\limits_V \left(\frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}\left(-2y^2\right) + \frac{\partial}{\partial z}\left(z^2\right)\right) dxdydz$$
$$= \int_{x=-2}^{2} \int_{-y_1}^{y_1} \int_{z=0}^{3} (4 - 4y + 2z) dxdydz$$
$$= \int_{r=-2}^{2} \int_{z=0}^{3} \int_{y=-\sqrt{(4-x^2)}}^{\sqrt{(4-x^2)}} (4 - 4y + 2z) dxdzdy$$
$$= \int_{x=-2}^{2} \int_{z=0}^{3} \left[4y - 2y^2 + 2zy\right]_{-\sqrt{(4-x^2)}}^{\sqrt{(4-x^2)}} dxdy$$
$$= 2\int_{-2}^{2} \int_{0}^{3} [(4 + 2z)y]_0^{\sqrt{(4-x^2)}} dxdz$$

$$= 2\int_{-2}^{2}\int_{0}^{3}\left\{(4+2z)\sqrt{(4-x^2)}\right\}dxdz$$
$$= 4\int_{0}^{2}\int_{0}^{2}(4+2z)\sqrt{(4-x^2)}dxdz$$
$$= 4\left[\left\{4z+z^2\right\}_0^3\left\{\frac{x}{2}\sqrt{(4-x^2)}+2\sin^{-1}\left(\frac{x}{2}\right)\right\}_0^2\right]$$
$$= 4[(12+9)(\pi)]$$
$$= 84\pi$$

Now, we proceed to find the surface integral.
The surface S of the cylinder consists of a base $S_1(z=0)$, the top $S_2(z=3)$ and the convex portion

$$S_3\left(x^2+y^2=4\right)$$

For S_1 : Normal is towards $-\hat{k}$ direction and $z=0$

$$\therefore \iint_{S_1}(\vec{A}\cdot\vec{n})dS = \iint_{S_1}\left(4x\hat{i}-2y^2\hat{j}+z^2\hat{k}\right)\cdot(-\hat{k})dS$$
$$= \iint_{S_1} -z^2 dS = 0$$

For S_2 : Normal is towards $\hat{k}$ direction and $z=3$

$$\therefore \iint_{S_2}(4x\hat{i}-2y^2\hat{j}+9\hat{k})\cdot(\hat{k})ds = \iint_{S_2} 9dS$$
$$= 9(2\pi r^2) \qquad [\because \text{area of } S_2 = 2\pi r^2 = 4\pi]$$
$$= 36\pi$$

For S_3 : Vector normal to S_3 i.e., $x^2+y^2=4$

$$\therefore \hat{n} = \frac{\nabla(x^2+y^2-4)}{|\nabla(x^2+y^2-4)|} = \frac{2x\hat{i}+2y\hat{j}}{4} = \frac{x\hat{i}+y\hat{j}}{2} \qquad [\because x^2+y^2=4 \quad \text{on } S_3]$$
$$\therefore \text{ On } S_3, (\vec{A}\cdot\vec{n}) = (4x\hat{i}-2y^2\hat{j}+z^2\hat{k})\cdot\left(\frac{x\hat{i}+y\hat{j}}{2}\right)$$
$$= 2x^2 - y^3$$

Also, dS = elementary area on the surface S_3
$$= 2d\theta dz \qquad \ldots[\text{Polar Coordinates } dS = rd\theta dz \text{ and } r = z]$$

$$
\begin{aligned}
\therefore \iint\limits_{S_3} (\vec{A}.\vec{n})dS &= \iint\limits_{S_3} (2x^2 - y^3)2d\theta dz \qquad \ldots [x = 2\cos\theta, y = 2\sin\theta] \\
&= \int_{F=0}^{F=3} \int_{\theta=0}^{2\pi} 2(B\cos^2\theta - 8\sin^3\theta)d\theta dF \\
&= 16\int_{\theta=0}^{2\pi} (\cos^2\theta - \sin^3\theta)[F]_0^3 d\theta \\
&= 16.3\int_{\theta=0}^{2\pi} (\cos^2\theta - \sin^3\theta)d\theta \\
&= 48\left[\int_{\theta=0}^{2\pi} \cos^2\theta d\theta - \int_{\theta=0}^{2\pi} \sin^2\theta d\theta\right] \\
&= 48\left[\left(\frac{\theta}{2} + \frac{\sin 2\theta}{4}\right)_0^{2\pi} - 0\right] \qquad \ldots [\sin\theta \text{ is a odd function}] \\
&= 48.\frac{2\pi}{2} = 48\pi
\end{aligned}
$$

$$
\begin{aligned}
\therefore \quad \iint\limits_{S} \vec{A}\cdot\vec{n}dS &= \left(\iint\limits_{S_1} + \iint\limits_{S_2} + \iint\limits_{S_3}\right)(\vec{A}\cdot\vec{n})dS \\
&= 0 + 36\pi + 48\pi \\
&= 84\pi
\end{aligned}
$$

$$
\therefore \iint\limits_{S} (\vec{A}\cdot\vec{n})dS = \iiint\limits_{v} (\nabla\cdot\vec{A})dv
$$

Hence, the divergence theorem is proved.