(c) A particle is undergoing simple harmonic motion of period T about a centre O and it passes through the position P (OP = b) with velocity v in the direction OP. Prove that the time that elapses before it returns to P is $\frac{T}{\pi} \tan^{-1}\left(\frac{vT}{2\pi b}\right)$. 8

Q.5(c) $x = a \sin \omega t$ $\qquad v = a\omega \cos \omega t$

$\omega = \frac{2\pi}{T}$

At $x = b$, $\quad b = a \sin \omega t_b \qquad v = a\omega \cos \omega t_b$

$\Rightarrow t_b = \frac{1}{\omega} \sin^{-1} \frac{b}{a} \qquad \Rightarrow v = a\omega \left(1 - \sin^2 \omega t_b\right)^{1/2}$

$= a\omega \left(1 - \left(\frac{b}{a}\right)^2\right)^{1/2}$

$\Rightarrow t_b = \frac{T}{2\pi} \sin^{-1} \frac{b}{a}$

$= \omega \sqrt{a^2 - b^2}$

$\Rightarrow a^2 = \left(\frac{v}{\omega}\right)^2 + b^2$

$\cancel{t_a = \frac{1}{\omega} \sin^{-1} \frac{a}{a} =}$

$= \left(\frac{vT}{2\pi}\right)^2 + b^2$

$t_a = \frac{T}{4} = \frac{2\pi}{\omega} \times \frac{1}{4}$

Time required $t = 2(t_a - t_b) = 2\left(\frac{T}{4} - \frac{T}{2\pi} \sin^{-1} \frac{b}{a}\right)$

$\Rightarrow t = \frac{T}{\pi}\left(\frac{\pi}{2} - \sin^{-1} \frac{b}{a}\right) = \frac{T}{\pi} \cos^{-1} \frac{b}{a}$

$= \frac{T}{\pi} \cos^{-1} \frac{b}{\left(\left(Tv/2\pi\right)^2 + b^2\right)^{1/2}} = \frac{T}{\pi} \tan^{-1} \frac{Tv/2\pi}{b}$

$\therefore t = \frac{T}{\pi} \tan^{-1} \frac{Tv}{2\pi b}$

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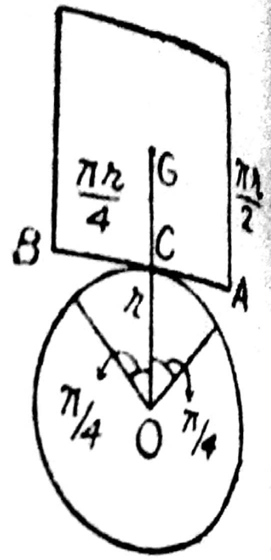
(d) A heavy uniform cube balances on the highest point of a sphere whose radius is r. If the sphere is rough enough to prevent sliding and if the side of the cube be $\frac{\pi r}{2}$, then prove that the total angle through which the cube can swing without falling is 90°. 8

Sol. A heavy uniform cube balances on the highest point C of a sphere whose centre is O and radius r. The length of a side of the cube is $\pi r/2$. If G is the C.G. of the cube, then for equilibrium the line OCG must be vertical. In the figure we have shown a cross section of the bodies by a vertical plane through the point of contact C.



First we shall show that the equilibrium of the cube is stable.

Here ρ_1 = the radius of curvature of the upper body at the point of contact $C=\infty$,

and ρ_2 = the radius of curvature of the lower body at the point of contact $=r$.

Also h = the height of the centre of gravity G of the upper body above the point of contact C = half the edge of the cube $=\pi r/4$.

The equilibrium will be stable if

$$\frac{1}{h} > \frac{1}{\rho_1} + \frac{1}{\rho_2} \quad i.e., \quad \frac{1}{\pi r/4} > \frac{1}{\infty} + \frac{1}{r}$$

i.e., $$\frac{4}{\pi r} > \frac{1}{r} \quad i.e., \quad \frac{4}{\pi} > 1 \quad i.e., \quad 4 > \pi$$

which is so because the value of π lies between 3 and 4.

Hence the equilibrium is stable. So if the cube is slightly displaced, it will tend to come back to its original position of equilibrium. During a swing to the right, the cube will not fall down till the right hand corner A of the lowest edge comes in contact with the sphere.

If θ is the angle through which the cube turns when the right hand corner A of the lowest edge comes in contact with the sphere, we have

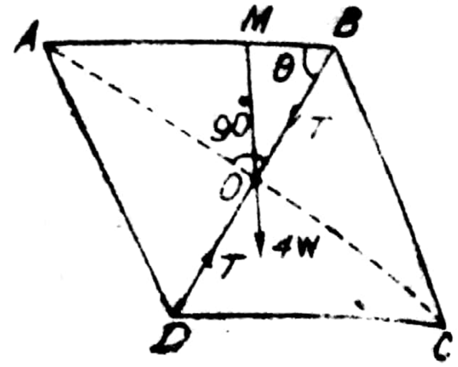
$$r\theta = \text{half the edge of the cube} = \pi r/4,$$

so that $$\theta = \pi/4.$$

Similarly the cube can turn through an angle $\pi/4$ to the left side on the sphere. Hence the total angle through which the cube can swing (or rock) without falling is $2 \cdot \frac{1}{4}\pi$ *i.e.*, $\frac{1}{2}\pi$.

(b) A string of length a, forms the shorter diagonal of a rhombus formed of four uniform rods, each of length b and weight W, which are hinged together. If one of the rods is supported in a horizontal position, then prove that the tension of the string is $\dfrac{2W(2b^2 - a^2)}{b\sqrt{4b^2 - a^2}}$.

Sol. $ABCD$ is a framework in the shape of a rhombus formed of four equal uniform rods each of length b and weight W. The rod AB is fixed in a horizontal position and B and D are joined by a string of length a forming the shorter diagonal of the rhombus.



Let T be the tension in the string BD. The total weight $4W$ of the rods AB, BC, CD and DA can be taken as acting at the point of intersection O of the diagonals AC and BD. We have
$\angle AOB=90°$.

Let $\angle ABO=\theta$. Draw OM perpendicular to AB.

Give the system a small symmetrical displacement in which θ changes to $\theta+\delta\theta$. The line AB remains fixed. The points O, C and D change. The lengths of the rods AB, AC, CD and DA do not change while the length BD changes. The $\angle AOB$ will remain $90°$.

We have $BD=2BO=2AB\cos\theta=2b\cos\theta$.

[Note that in the position of equilibrium $BD=a$. But during the displacement BD changes and so we have found BD in terms of θ.]

The depth of O below the fixed line $AB=MO$.

$=BO\sin\theta=(AB\cos\theta)\sin\theta=b\sin\theta\cos\theta$.

By the principle of virtual work, we have

$$-T\delta(2b\cos\theta)+4W\delta(b\sin\theta\cos\theta)=0$$

or $\quad 2bT\sin\theta\,\delta\theta+4bW(\cos^2\theta-\sin^2\theta)\,\delta\theta=0$

or $\quad 2b[T\sin\theta-2W(\sin^2\theta-\cos^2\theta)]\,\delta\theta=0$

or $\quad T\sin\theta-2W(\sin^2\theta-\cos^2\theta)=0 \qquad [\because \delta\theta\neq 0]$

or $$T=\frac{2W(\sin^2\theta-\cos^2\theta)}{\sin\theta}=\frac{2W(1-2\cos^2\theta)}{\sqrt{(1-\cos^2\theta)}}.$$

In the position of equilibrium, $BD=a$ or $BO=\frac{1}{2}a$. So in the position of equilibrium, $\cos\theta=\dfrac{BO}{AB}=\dfrac{\frac{1}{2}a}{b}=\dfrac{a}{2b}$.

$$\therefore\quad T=\frac{2W\{1-2(a^2/4b^2)\}}{\sqrt{\{1-(a^2/4b^2)\}}}=\frac{2W(2b^2-a^2)}{b\sqrt{(4b^2-a^2)}}.$$

Solve $\frac{d^2y}{dx^2} + 4y = \tan 2x$ by using the method of variation of parameter. 1(

A particle moves in a straight line, its acceleration directed towards a fixed point O in the line and is always equal to $\mu\left(\frac{a^5}{x^2}\right)^{\frac{1}{3}}$ when it is at a distance x from O. If it starts from rest at a distance a from O, then prove that it will arrive at O with a velocity $a\sqrt{6\mu}$ after time $\frac{8}{15}\sqrt{\frac{6}{\mu}}$.

$f = \frac{d^2x}{dt^2} = -\mu \frac{a^{5/3}}{x^{2/3}}$

$\Rightarrow \left(\frac{dx}{dt}\right)^2 \Big|_0^{v_0} = -2\mu a^{5/3} \frac{x^{1/3}}{1/3}\Big|_a^0 = -6\mu a^{5/3} x^{1/3}\Big|_a^0$

$\Rightarrow v_0^2 = 6\mu a^{5/3} a^{1/3} = 6\mu a^2$

$\Rightarrow v_0 = \underline{\underline{a\sqrt{6\mu}}}$.

Also, $\left(\frac{dx}{dt}\right)^2\Big|_0^{dx/dt} = -6\mu a^{5/3} x^{1/3}\Big|_a^x$

$\Rightarrow \left(\frac{dx}{dt}\right)^2 = 6\mu a^{5/3}\left(a^{1/3} - x^{1/3}\right)$

$\Rightarrow \frac{dx}{\sqrt{a^{1/3} - x^{1/3}}} = -\sqrt{6\mu a^{5/3}}\, dt$

$\Rightarrow \int_a^0 \frac{dx}{(a^{1/3} - x^{1/3})^{1/2}} = -\int_0^{t_0} \sqrt{6\mu a^{5/3}}\, dt$

$\Rightarrow \cancel{x^{1/3}}-$

Let $x^{1/3} = a^{1/3}\sin^2\theta \Rightarrow x = a\sin^6\theta \Rightarrow dx = 6a\sin^5\theta\cos\theta\, d\theta$

$x = a \to \theta = \cancel{0}\,\pi/2$

$x = 0 \to \theta = 0$

$\Rightarrow \int_{\pi/2}^0 \frac{6a\sin^5\theta\cos\theta}{a^{1/6}(1-\sin^2\theta)^{1/2}}\, d\theta = -\int_0^{\cancel{\pi/2}\ t_0} \sqrt{6\mu a^{5/3}}\, dt$

$\Rightarrow \sqrt{6\mu a^{5/3}}\, t_0 = \int_0^{\pi/2} 6a^{5/6}\sin^5\theta\, d\theta$

$\Rightarrow \sqrt{6\mu}\, a^{5/6}\, t_0 = 6a^{5/6} \cdot \frac{4\cdot 2}{1\cdot 3\cdot 5} = \frac{16}{5}a^{5/6}$

$\Rightarrow t_0 = \frac{16}{5\sqrt{6\mu}} = \underline{\underline{\frac{8}{15}\sqrt{\frac{6}{\mu}}}}$

(b) A planet is describing an ellipse about the Sun as a focus. Show that its velocity away from the Sun is the greatest when the radius vector to the planet is at a right angle to the major axis of path and that the velocity then is $\frac{2\pi ae}{T\sqrt{1-e^2}}$, where 2a is the major axis, e is the eccentricity and T is the periodic time. 1