IMS

MATHS

BOOK-07

Set – IV

INTERPOLATION

→ The interpolation has been defined as the art of reading between the lines of a table, and in elementary mathematics the term usually denotes the process of computing intermediate values of a function from a set of given values of that function.

For example:

Consider the table that lists the population of Delhi. The population census is taken every ten years and the table gives poulation for the years 1901, 1911, 1961, 1971, 1981 and 1991 in Delhi.

we would like to know whether this table could be used to estimate the poupulation of Delhi in 1936 say or even in 1996. Such estimates of population can be made using a function that fits the given data.

→ Let $y=f(x)$ be a real valued function defined on the interval $[a,b]$ and we denote $f(x_k)$ by f_k or y_k.

Suppose that the values of the function $f(x)$ are given to be $f_0, f_1, f_2, \dots f_n$ when $x = x_0, x_1, \dots x_n$ respectively, where $x_0 < x_1 < \dots < x_{n-1} < x_n$ lying in the interval $[a,b]$.

The function $f(x)$ may not be known to us. This technique of determining the value $f(x)$

for a non-tabular value of x which lies in the interval $[a, b]$ is called 'interpolation'.

The process of determining the value of $f(x)$ for a value of 'x' lying outside the interval $[a, b]$ is called extrapolation.

It may be noted that if the function $f(x)$ is known, the value of $y = f(x)$ corresponding to any x can be readily computed to the desired accuracy. But, in practice, it may be difficult or sometimes impossible to know the function $y = f(x)$ in its exact form. In such cases the function $f(x)$ is replaced by a polynomial $\phi(x)$ of degree $\leq n$ which agrees with the values of $f(x)$ at the given $(n+1)$ distinct points, called nodes or abscissas. In other words, we can find a polynomial $\phi(x)$ such that $\phi(x_j) = f_j$, $j = 0, 1, 2, \ldots n$. Such a polynomial $\phi(x)$ is called the interpolating polynomial of $f(x)$.

In general, for interpolation of a tabulated function, the concept of finite differences is important. The knowledge about various finite difference operators and their symbolic relations are very much needed to establish various interpolation formulae.

Interpolation with Equal Intervals:-

Finite Difference Operators:-

Assume that we have a table of values (x_k, y_k), $K = 0, 1, 2 \ldots n$ of any function $y = f(x)$, the values of x being equally spaced i.e., $x_k = x_0 + kh$; $K = 0, 1, 2 \ldots n$.

Forward Differences :

If $y_0, y_1, y_2 \dots y_n$ denote a set of values of y, then $y_1 - y_0, y_2 - y_1, y_3 - y_2, \dots y_n - y_{n-1}$ are called the differences of y.

Denoting these differences by $\Delta y_0, \Delta y_1, \dots \Delta y_{n-1}$ respectively, we have

$$\Delta y_0 = y_1 - y_0$$
$$\Delta y_1 = y_2 - y_1$$
$$\vdots$$
$$\Delta y_{n-1} = y_n - y_{n-1}$$

The symbol Δ is called forward difference operator and $\Delta y_0, \Delta y_1 \dots$ are called first forward differences.

— The differences of the first forward differences are called second forward differences and are denoted by $\Delta^2 y_0, \Delta^2 y_1, \dots \Delta^2 y_{n-1}$.

we have

$$\Delta^2 y_0 = \Delta[\Delta y_0]$$
$$= \Delta[y_1 - y_0]$$
$$= \Delta y_1 - \Delta y_0$$
$$= (y_2 - y_1) - (y_1 - y_0)$$
$$\boxed{\Delta^2 y_0 = y_2 - 2y_1 + y_0}$$

$$\Delta^2 y_1 = \Delta y_2 - \Delta y_1$$
$$= (y_3 - y_2) - (y_2 - y_1)$$
$$\boxed{\Delta^2 y_1 = y_3 - 2y_2 + y_1} \text{ etc.}$$

Thus, in general

$$\boxed{\Delta^2 y_{n-1} = \Delta y_n - \Delta y_{n-1}}$$

The symbol Δ^2 is called second forward difference operator.

Similarly $\Delta^3 y_0 = \Delta^2 y_1 - \Delta^2 y_0$

$$= y_3 - 2y_2 + y_1 - [y_2 - 2y_1 + y_0].$$

$$\therefore \boxed{\Delta^3 y_0 = y_3 - 3y_2 + 3y_1 - y_0}$$

Thus, continuing, we can define, rth difference of y as

$$\boxed{\Delta^r y_{n-1} = \Delta^{r-1} y_n - \Delta^{r-1} y_{n-1}}.$$

By defining a difference table as a convenient device for displaying various differences, the above defined differences can be written down systematically by constructing a difference table for values (x_k, y_k), $k = 0, 1, \ldots 6$ as shown below:

Forward Difference Table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$	$\Delta^6 y$
x_0	y_0						
		Δy_0					
x_1	y_1		$\Delta^2 y_0$				
		Δy_1		$\Delta^3 y_0$			
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$		
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$	
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$		$\Delta^6 y_0$
		Δy_3		$\Delta^3 y_2$		$\Delta^5 y_1$	
x_4	y_4		$\Delta^2 y_3$		$\Delta^4 y_2$		
		Δy_4		$\Delta^3 y_3$			
x_5	y_5		$\Delta^2 y_4$				
		Δy_5					
x_6	y_6						

This difference table is called forward difference table or diagonal difference table. Here, each difference is located in its appropriate column, midway between the elements of the previous column. It can be noted that the subscript remains constant along each diagonal of the table. The first term in the table, that is y_0 is called the leading term, while the differences $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0, \ldots$ are called leading differences.

→ Construct a forward difference table for the following values of x and y.

x	0.1	0.3	0.5	0.7	0.9	1.1	1.3
y	0.003	0.067	0.148	0.248	0.370	0.518	0.697

Sol:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0.1	0.003					
		0.064				
0.3	0.067		0.017			
		0.081		0.002		
0.5	0.148		0.019		0.001	
		0.100		0.003		0.000
0.7	0.248		0.022		0.001	
		0.122		0.004		0.000
0.9	0.370		0.026		0.001	
		0.148		0.005		
1.1	0.518		0.031			
		0.179				
1.3	0.697					

* Backward Differences :-

Let $y=f(x)$ be a function given by the values $y_0, y_1, y_2, \ldots\ldots y_n$ which it takes for the equally spaced abscissas $x_0, x_1, x_2 \ldots\ldots x_n$ of the independent variable x. Then y_1-y_0, y_2-y_1, $\ldots\ldots y_n-y_{n-1}$ are called the first backward differences of $y=f(x)$

Denoting these differences by $\nabla y_1, \nabla y_2 \ldots \nabla y_n$ respectively. we have

$$\nabla y_1 = y_1 - y_0$$

$$\nabla y_2 = y_2 - y_1$$

$$\vdots$$

$$\nabla y_n = y_n - y_{n-1}.$$

The differences of these differences are called second differences and they are denoted by $\nabla^2 y_2, \nabla^2 y_3 \ldots\ldots \nabla^2 y_n$. That is,

$$\nabla^2 y_2 = \nabla y_2 - \nabla y_1$$

$$\nabla^2 y_3 = \nabla y_3 - \nabla y_2$$

$$\vdots$$

$$\nabla^2 y_n = \nabla y_n - \nabla y_{n-1}$$

Thus, in general, the second backward differences are

$$\nabla^2 y_n = \nabla y_n - \nabla y_{n-1}.$$

while the K-th backward differences are given as $\nabla^k y_n = \nabla^{k-1} y_n - \nabla^{k-1} y_{n-1}$.

These backward differences can be systematically arranged for a table of values of (x_k, y_k), $k = 0, 1, 2, \ldots 6$ as indicated below:

Backward Difference Table

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$	$\nabla^6 y$
x_0	y_0						
		∇y_1					
x_1	y_1		$\nabla^2 y_2$				
		∇y_2		$\nabla^3 y_3$			
x_2	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$		
		∇y_3		$\nabla^3 y_4$		$\nabla^5 y_5$	
x_3	y_3		$\nabla^2 y_4$		$\nabla^4 y_5$		$\nabla^6 y_6$
		∇y_1		$\nabla^3 y_5$		$\nabla^5 y_6$	
x_4	y_4		$\nabla^2 y_5$		$\nabla^4 y_6$		
		∇y_5		$\nabla^2 y_6$			
x_5	y_5		$\nabla^2 y_6$				
		∇y_6					
x_6	y_6						

From this table, it can be observed that the subscript remains constant along every backward diagonal.

Central Differences:-

In some applications, central difference notation is found to be more convenient to represent the successive differences of a function. Here, we use the symbol δ to represent central difference operator and the subscript of δy for any difference as the average of the subscripts of the two members of the difference.

Thus, we write

$$\delta y_{1/2} = y_1 - y_0$$

$$\delta y_{3/2} = y_2 - y_1 \quad , \text{etc.}$$

In general

$$\delta y_n = y_{n+\frac{1}{2}} - y_{n-\frac{1}{2}}$$

Higher differences are defined as follows:

$$\delta^2 y_n = \delta y_{n+\frac{1}{2}} - \delta y_{n-\frac{1}{2}}$$

$$\delta^k y_n = \delta^{k-1} y_{n+\frac{1}{2}} - \delta^{k-1} y_{n-\frac{1}{2}}$$

These central differences can be systematically arranged as shown below.

Central Difference Table.

x	y	δy	$\delta^2 y$	$\delta^3 y$	$\delta^4 y$	$\delta^5 y$	$\delta^6 y$
x_0	y_0						
		$\delta y_{1/2}$					
x_1	y_1		$\delta^2 y_1$				
		$\delta y_{3/2}$		$\delta^3 y_{3/2}$			
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$		
		$\delta y_{5/2}$		$\delta^3 y_{5/2}$		$\delta^5 y_{5/2}$	
x_3	y_3		$\delta^2 y_3$		$\delta^4 y_3$		$\delta^6 y_3$
		$\delta y_{7/2}$		$\delta^3 y_{7/2}$		$\delta^5 y_{7/2}$	
x_4	y_4		$\delta^2 y_4$		$\delta^4 y_4$		
		$\delta y_{9/2}$		$\delta^3 y_{9/2}$			
x_5	y_5		$\delta^2 y_5$				
		$\delta y_{11/2}$					
x_6	y_6						

Thus we observe that all the odd differences have a fractional suffix and all the even differences with the same subscript lie horizontally.

The following alternative notation may also be adopted to introduce finite difference operators. Let $y = f(x)$ be a functional relation between x and y, which is also denoted by y_x.

Suppose, we are given consecutive values of x differing by h say $x, x+h, x+2h, x+3h$, etc. The corresponding values of y are $y_x, y_{x+h}, y_{x+2h}, y_{x+3h}$ etc.

As before, we can form the differences of these values.

Thus, $\Delta y_x = y_{x+h} - y_x$
$$= f(x+h) - f(x)$$
$$\Delta^2 y_x = \Delta y_{x+h} - \Delta y_x$$

Similarly,
$$\nabla y_x = y_x - y_{x-h}$$
$$= f(x) - f(x-h)$$

and
$$\delta y_x = y_{x+\frac{h}{2}} - y_{x-\frac{h}{2}}$$
$$= f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)$$

Note: we should note that it is only the notation which changes and not the differences.

i.e, it is clear from the three tables that in a definite numerical case, the same numbers occur in the same positions whether we use forward, backward, (or) central differences.

Thus we obtain

$$y_1 - y_0 = \Delta y_0 = \nabla y_1 = \delta y_{1/2} ;$$

$$\Delta^3 y_2 = \nabla^3 y_5 = \delta^3 y_{7/2} \quad etc.$$

* Symbolic Relations and seperation of symbols:

Shift operator (E):-

Let $y = f(x)$ be a function of x and $x, x+h, x+2h, x+3h, \ldots$ etc be the consecutive values of x, then the operator E is defined as

$$E f(x) = f(x+h)$$

Thus, when E operates on $f(x)$, the result is the next value of the function. Here, E is called the shift operator. If we apply the operator E twice on $f(x)$,

we get

$$E^2 f(x) = E[Ef(x)]$$
$$= E[f(x+h)]$$
$$= f(x+2h)$$

Thus, in general, if we apply the operator E n times on $f(x)$, we arrive at

$$E^n f(x) = f(x+nh)$$

In terms of new notation, we can write

$$E^n y_x = y_{x+nh}$$

(or)

$$E^n f(x) = f(x+nh).$$

for all real values of n. Also, if $y_0, y_1, y_2, \ldots$ are the consecutive values of the function y_x, then we can also write

$$E y_0 = y_1,\quad E^2 y_0 = y_2,\quad E^4 y_0 = y_4, \ldots\ E^2 y_2 = y_4$$

and so on.

The inverse operator E^{-1} is defined as

$$\boxed{E^{-1} f(x) = f(x-h)}$$

and similarly

$$\boxed{E^{-n} f(x) = f(x-nh)}$$

Average operator (μ):-

The average operator μ is defined as

$$\mu f(x) = \frac{1}{2}\left[f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)\right]$$

$$= \frac{1}{2}\left[y_{x+\left(\frac{h}{2}\right)} + y_{x-\left(\frac{h}{2}\right)}\right].$$

or

$$= \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right].$$

Differential operator (D):-

The differential operator is defined as

$$D f(x) = \frac{d}{dx} f(x) = f'(x)$$

$$D^2 f(x) = \frac{d^2}{dx^2} f(x) = f''(x) \quad \text{etc.}$$

* Relation b/w operators Δ, ∇, δ, E and μ:-

[I] We know that $\Delta y_x = y_{x+h} - y_x$

$$= E y_x - y_x$$

$$\Delta y_x = (E-1)y_x$$

$$\therefore \boxed{\Delta = E - 1}$$

$$\boxed{E = 1+\Delta}$$

[2] we know that

$$\nabla y_x = y_x - y_{x-h}$$

$$= y_x - E^{-1} y_x$$

$$= (1-E^{-1}) y_x$$

$$\therefore \nabla y_x = (1-E^{-1}) y_x$$

$$\therefore \boxed{\nabla = 1 - \frac{1}{E}}$$

$$\boxed{E = (1-\nabla)^{-1}}$$

[3]. The definition of operators δ and E gives

$$\delta y_x = y_{x+(\frac{h}{2})} - y_{x-(\frac{h}{2})}$$

$$= E^{1/2} y_x - E^{-1/2} y_x$$

$$= (E^{1/2} - E^{-1/2}) y_x$$

Hence $\boxed{\delta = E^{1/2} - E^{-1/2}}$

[4] The definition of operations μ and E gives

$$\mu y_x = \frac{1}{2}\left[y_{x+\frac{h}{2}} + y_{x-h/2}\right]$$

$$= \frac{1}{2}\left[E^{1/2} + E^{-1/2}\right] y_x$$

$$\therefore \boxed{\mu = 1/2 \left(E^{1/2} + E^{-1/2}\right)}$$

5. We know that $E y_x = y_{x+h} = f(x+h)$ (7)

Using Taylor's series expansion, we have

$$E y_x = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \cdots$$

$$= f(x) + hD f(x) + \frac{h^2}{2!} D^2 f(x) + \cdots$$

$$= \left[1 + \frac{hD}{1!} + \frac{h^2}{2!} D^2 + \frac{h^3 D^3}{3!} + \cdots\right] f(x)$$

$$\therefore E y_x = e^{hD} y_x$$

$$\boxed{E = e^{hD}}$$

$$\boxed{hD = \log E}$$

Hence, all the operators can be expressed in terms of operators E.

* The properties of operator Δ:

(1) If c is a constant then $\Delta C = 0$

Sol^n: Let $f(x) = c$

then $f(x+h) = c$

where h is the interval of differencing.

$$\therefore \Delta f(x) = f(x+h) - f(x)$$

$$= c - c = 0$$

$$\Rightarrow \boxed{\Delta C = 0}$$

(2) $\Delta[f(x) + g(x)] = \Delta f(x) + \Delta g(x)$

(3) If c is a constant then $\Delta[c\, f(x)] = c\, \Delta f(x)$.

(iv) If m and n are +ve integers then

$$\Delta^m \Delta^n f(x) = \Delta^{m+n} f(x).$$

(v) $\Delta[f_1(x) + f_2(x) + \cdots + f_n(x)] = \Delta f_1(x) + \Delta f_2(x) + \cdots + \Delta f_n(x).$

(vi) $\Delta[f(x)\, g(x)] = f(x)\, \Delta g(x) + g(x)\, \Delta f(x).$

(vii) $$\Delta\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\, \Delta f(x) - f(x)\, \Delta g(x)}{g(x)\, g(x+h)}.$$

Note:

(1). From the properties (2) & (3) it is clear that Δ is a linear operator.

(2). If n is a +ve integer $\Delta^n [\Delta^{-n} f(x)] = f(x)$.

and in particular when $n = 1$,

$$\Delta [\Delta^{-1} f(x)] = f(x).$$

Example: Find (a) Δe^{ax} (b) $\Delta^2 e^{ax}$ (c) $\Delta \sin x$ (d) $\Delta \log x$ (e) $\Delta \tan^{-1} x$.

Sol^n: (a)

$$\Delta e^{ax} = e^{a(x+h)} - e^{ax}$$
$$= e^{ax}[e^{ah} - 1]$$

(b)

$$\Delta^2 e^{ax} = \Delta(\Delta e^{ax})$$
$$= \Delta[e^{ax}(e^{ah} - 1)]$$
$$= (e^{ah} - 1)\, \Delta e^{ax}$$
$$= (e^{ah} - 1)[e^{ah} - 1]\, e^{ax}$$
$$= (e^{ah} - 1)^2 e^{ax}.$$

(c) $\Delta \sin x = \sin(x+h) - \sin x$

$$= 2\cos\left(\frac{x+h+x}{2}\right) \sin\left(\frac{x+h-x}{2}\right)$$

$$= 2\cos\left(x+\frac{h}{2}\right) \sin\frac{h}{2}$$

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(d) $\Delta \log x = \log(x+h) - \log x$

$$= \log\left(\frac{x+h}{x}\right)$$

$$= \log\left(1+\frac{h}{x}\right)$$

(e) $\Delta \tan^{-1} x = \tan^{-1}(x+h) - \tan^{-1} x$

$$= \tan^{-1}\left[\frac{x+h-x}{1+(x+h)x}\right] \quad \left(\because \tan^{-1}x - \tan^{-1}y = \tan^{-1}\left(\frac{x-y}{1+xy}\right)\right)$$

$$= \tan^{-1}\left[\frac{h}{1+hx+x^2}\right]$$

→ prove that $hD = \log(1+\Delta) = -\log(1-\nabla) = \sinh^{-1}(\mu\delta)$

Sol^n: we know that

$$e^{hD} = E = 1+\Delta$$

$$\Rightarrow hD = \log(1+\Delta)$$

$$= -\log E^{-1}$$

$$= -\log(1-\nabla) \quad (\because E^{-1} = 1-\nabla)$$

$$\therefore hD = \log(1+\Delta) = -\log(1-\nabla) \quad \text{——(1)}$$

we have $\mu = \frac{1}{2}\left(E^{1/2}+E^{-1/2}\right)$

and $\delta = E^{1/2} - E^{-1/2}$

$$\therefore \mu\delta = \frac{1}{2}\left(E^{1/2}+E^{-1/2}\right)\left(E^{1/2}-E^{-1/2}\right)$$

$$= \frac{1}{2}(E - E^{-1})$$

$$= \frac{1}{2}(e^{hD} - e^{-hD})$$

$$\mu\delta = \sinh D$$

$$\Rightarrow hD = \sinh^{-1}(\mu\delta) \text{ --- } (2)$$

$\therefore$ from (1) & (2)

$$hD = \log(1+\Delta) = -\log(1-\nabla) = \sinh^{-1}(\mu\delta)$$

$\longrightarrow$ Show that

(i) $1+\delta^2\mu^2 = \left(1+\frac{\delta^2}{2}\right)^2$ (ii) $E^{1/2} = \mu + \frac{\delta}{2}$

(iii) $\Delta = \frac{\delta^2}{2} + \delta\sqrt{1+(\delta^2/4)}$ (iv) $\mu\delta = \frac{\Delta E^{-1}}{2} + \frac{\Delta}{2}$

(v) $\mu\delta = \frac{\Delta+\nabla}{2}$

Solⁿ: (i) we have

$$\mu\delta = \frac{1}{2}(E^{1/2}+E^{-1/2})(E^{1/2}-E^{-1/2})$$

$$= \frac{1}{2}(E-E^{-1})$$

$$\Rightarrow 1+\mu^2\delta^2 = 1+\frac{1}{4}(E-E^{-1})^2$$

$$= 1+\frac{1}{4}(E^2-2+E^{-2})$$

$$= \frac{1}{4}(E^2+2+E^{-2})$$

$$= \frac{1}{4}(E+E^{-1})^2 \text{ --- } (1)$$

Also

$$1+\frac{\delta^2}{2} = 1+\frac{1}{2}(E^{1/2}-E^{-1/2})^2$$

$$= 1+\frac{1}{2}(E-2+E^{-1})$$

$$= \frac{1}{2}(E+E^{-1})$$

$$\Rightarrow \left(1+\frac{\delta^2}{2}\right)^2 = \frac{1}{4}(E+E^{-1})^2 \text{ --- } (2)$$

from (1) & (2) $1+\delta^2\mu^2 = \left(1+\frac{\delta^2}{2}\right)^2$

(19)

(ii) $\mu + \frac{\delta}{2} = \frac{1}{2}(E^{1/2} + E^{-1/2}) + \frac{1}{2}(E^{1/2} - E^{-1/2})$

$= \frac{1}{2}(2E^{1/2})$

$= E^{1/2}$.

(iii) $\frac{1}{2}\delta^2 + \delta\sqrt{1+(\delta^2/4)} = \left(\frac{E^{1/2} - E^{-1/2}}{2}\right)^2 + (E^{1/2} - E^{-1/2})\sqrt{1+\frac{1}{4}(E^{1/2} - E^{-1/2})^2}$

$= \frac{E-2+E^{-1}}{2} + \frac{1}{2}(E^{1/2} - E^{-1/2})\sqrt{(4+E-2+E^{-1})}$

$= \frac{E-2+E^{-1}}{2} + \frac{1}{2}(E^{1/2} - E^{-1/2})\sqrt{(E^{1/2} + E^{-1/2})^2}$

$= \frac{E-2+E^{-1}}{2} + \frac{1}{2}(E^{1/2} - E^{-1/2})(E^{1/2} + E^{-1/2})$

$= \frac{E-2+E^{-1}}{2} + \frac{1}{2}(E - E^{-1})$

$= \frac{1}{2}(E-2+E^{-1}+E-E^{-1})$

$= \frac{1}{2}(2E-2)$

$= E-1$

$= \Delta \quad (\because \Delta = E-1)$.

(iv) $\mu\delta = \frac{1}{2}(E^{1/2} + E^{-1/2})(E^{1/2} - E^{-1/2})$

$= \frac{1}{2}(E - E^{-1}) \quad (\because E = 1+\Delta)$

$= \frac{1}{2}(1+\Delta - E^{-1})$

$= \frac{\Delta}{2} + \frac{1}{2}(1-E^{-1})$

$= \frac{\Delta}{2} + \frac{1}{2}\left(1-\frac{1}{E}\right)$

$= \frac{\Delta}{2} + \frac{1}{2}\left(\frac{E-1}{E}\right) \quad (\because E-1 = \Delta)$

$= \frac{\Delta}{2} + \frac{1}{2}\frac{\Delta}{E} = \frac{\Delta}{2} + \frac{\Delta E^{-1}}{2}$

(v) $\mu\delta = \frac{1}{2}(E^{1/2}+E^{-1/2})(E^{1/2}-E^{-1/2})$

$= \frac{1}{2}(E-E^{-1})$

$= \frac{1}{2}[1+\Delta-(1-\nabla)]$

$= \frac{1}{2}(\Delta+\nabla)$ $\quad \left(\because E-1=\Delta \text{ and } 1-E^{-1}=\nabla\right)$

→ Construct the forward difference table for the following data.

x	0	10	20	30
y	0	0.174	0.347	0.518

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	0			
		0.174		
10	0.174		-0.001	
		0.173		-0.001
20	0.347		-0.002	
		0.171		
30	0.518			

→ Construct a difference table for $y=f(x) = x^3+2x+1$ for $x=1,2,3,4,5$

Sol:

x	$y=f(x)$	Δy	$\Delta^2 y$	$\Delta^3 y$
1	4			
		9		
2	13		12	
		21		6
3	34		18	
		39		6
4	73		24	
		63		
5	136			

Differences of a polynomial:

The n^{th} differences of a polynomial of the n^{th} degree are constant and all higher order differences are zero when the values of independent variable are at equal intervals.

proof: Let the polynomial of the n^{th} degree in x, be

$$f(x) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n.$$

where $a_0, a_1, a_2, \cdots a_n$ are constants and $a_0 \neq 0$.

$$\therefore f(x+h) = a_0 (x+h)^n + a_1 (x+h)^{n-1} + \cdots + a_{n-1}(x+h) + a_n$$

where h is the interval of differencing.

$$\therefore \Delta f(x) = f(x+h) - f(x)$$

$$= a_0 (x+h)^n + a_1 (x+h)^{n-1} + \cdots + a_{n-1}(x+h) + a_n - a_0 x^n - a_1 x^{n-1} - \cdots - a_{n-1} x - a_n$$

$$= a_0 \left[(x+h)^n - x^n\right] + a_1\left[(x+h)^{n-1} - x^{n-1}\right] + \cdots + a_{n-1}\left[(x+h) - x\right]$$

$$= a_0 \left[\cancel{x^n} + nC_1 x^{n-1} h + nC_2 x^{n-2} h^2 + \cdots + nC_{n-1} h^{n-1} + h^n - \cancel{x^n}\right] + a_1\left[\cancel{x^{n-1}} + {}^{n-1}C_1 x^{n-2} h + {}^{n-1}C_2 x^{n-3} h^2 + \cdots + h^{n-1} - \cancel{x^{n-1}}\right] + \cdots + a_{n-1} h$$

$$= a_0 n h x^{n-1} + \left[a_0 h^2\, nC_2 + a_1 h (n-1)C_1\right] x^{n-2} + \cdots + a_{n-1} h.$$

$= a_0 n h x^{n-1} + b_2 x^{n-2} + b_3' x^{n-3} + \cdots + b_{n-1} x + b_n$ ——— (1)

where $b_2, b_3, \ldots \ldots b_n$ are constants.

From (1), it is clear that the first difference of $f(x)$ is a polynomial of $(n-1)^{th}$ degree.

Similarly, $\Delta^2 f(x) = \Delta(\Delta f(x))$

$= \Delta[f(x+h) - f(x)]$

$= \Delta f(x+h) - \Delta f(x)$.

$= a_0 n h[(x+h)^{n-1} - x^{n-1}] + b_2[(x+h)^{n-2} - x^{n-2}] + \cdots$

$\cdots\cdots + b_{n-1}[x+h-x]$

$= a_0 n(n-1) h^2 x^{n-2} + c_3 x^{n-3} + c_4 x^{n-4} + \cdots + c_{n-1} x + c_n$

where $c_3, c_4, \ldots\ldots c_{n-1}$ are constants.

Therefore the second differences of $f(x)$ reduces to a polynomial of $(n-2)^{th}$ degree. proceeding as above and differencing for n times we get

$\Delta^n f(x) = a_0 n(n-1)\ldots\ldots 3\times 2\times 1 . h^n x^{n-n}$

$= a_0 n! h^n$.

which is a constant

and $\Delta^{n+1} f(x) = \Delta(\Delta^n f(x))$.

$= a_0 n! h^n - a_0 n! h^n = 0$

i.e., the $(n+1)^{th}$ and higher order differences of a polynomial of n^{th} degree will be zero.

Note: The converse of the above theorem is true. i.e., if the n^{th} differences of a tabulated function and the values of the independent variable are equally spaced then the function is a polynomial of degree n.

Effect of an Error on a difference table :

Difference tables can be used to check errors in tabular values. Suppose that there is an error ϵ in the entry y_5 of a table. As higher differences are formed, this error spreads out and is considerably magnified.

Let us see, how it effects the difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
x_0	y_0				
		Δy_0			
x_1	y_1		$\Delta^2 y_0$		
		Δy_1		$\Delta^3 y_0$	
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$
		Δy_2		$\Delta^3 y_1$	
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1+\epsilon$
		Δy_3		$\Delta^3 y_2+\epsilon$	
x_4	y_4		$\Delta^2 y_3+\epsilon$		$\Delta^4 y_2-4\epsilon$
		$\Delta y_4+\epsilon$		$\Delta^3 y_3-3\epsilon$	
x_5	$y_5+\epsilon$		$\Delta^2 y_4-2\epsilon$		$\Delta^4 y_3+6\epsilon$
		$\Delta y_5-\epsilon$		$\Delta^3 y_4+3\epsilon$	
x_6	y_6		$\Delta^2 y_5+\epsilon$		$\Delta^4 y_4-4\epsilon$
		Δy_6		$\Delta^3 y_5-\epsilon$	
x_7	y_7		$\Delta^2 y_6$		$\Delta^4 y_4+\epsilon$
		Δy_7		$\Delta^3 y_6$	
x_8	y_8		$\Delta^2 y_7$		
		Δy_8			
x_9	y_9				

The above table shows that :

(i) The error increases with the order of differences.

(ii) The coefficients of ϵ's in any column are binomial coefficients of $(1-\epsilon)^n$. Thus the errors in the fourth difference column are $\epsilon, -4\epsilon, 6\epsilon, -4\epsilon, \epsilon$.

(iii) The algebraic sum of the errors in any difference column is zero.

(iv) The maximum error in each column, occurs opposite to the entry containing the error i.e, y_s

The above facts enable us to detect errors in a difference table.

→ One entry in the following table is incorrect and y is a cubic polynomial in x. Use the difference table to locate and correct the error.

x	0	1	2	3	4	5	6	7
y	25	21	18	18	27	45	76	123

Soln:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0	25			
		-4		
1	21		1	
		-3		2
2	18		3	
		0		6
3	18		9	
		9		0
4	27		9	
		18		4
5	45		13	
		31		3
6	76		16	
		47		
7	123			

y being a polynomial of the 3rd degree, $\Delta^3 y$ must be constant. i.e, the same.

The sum of the third differences being 15, each entry under $\Delta^3 y$ must be $15/5$ i.e, 3.

Thus the four entries under $\Delta^3 y$ are in error which can be written as

$2 = 3+(-1)$, $6 = 3-3(-1)$, $0 = 3+3(-1)$,
$4 = 3-(-1)$.

Taking $\epsilon = -1$, we find that the entry corresponding to $x=3$ is in error.

$\therefore y+\epsilon = 18$

$\Rightarrow y = 18-\epsilon$

$\Rightarrow y = 18-(-1) = 19$.

Thus the true value of $y = 19$ at $x=3$.

→ The following is a table of values of a polynomial of degree 5. It is given that $f(3)$ is in error. Correct the error.

x	0	1	2	3	4	5	6
y	1	2	33	254	1054	3126	7777

Sol^n: It is given that

$y = f(x)$ is a polynomial of degree 5.

$\therefore \Delta^5 y$ must be constant.

Also given $f(3)$ is in error.

Let $254+\epsilon$ be the true value,

now we form the difference table.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0	1					
		1				
1	2		30			
		31		$160+\epsilon$		
2	33		$190+\epsilon$		$200-4\epsilon$	
		$221+\epsilon$		$360-3\epsilon$		$220+10\epsilon$
3	$254+\epsilon$		$550-2\epsilon$		$420+6\epsilon$	
		$771-\epsilon$		$780+3\epsilon$		$20-10\epsilon$
4	1025		$1330+\epsilon$		$440-4\epsilon$	
		2101		$1220-\epsilon$		
5	3126		2550			
		4651				
6	7777					

Since the fifth differences of y are constant

$$220+10\epsilon = 20-10\epsilon$$

$$\Rightarrow 20\epsilon = -200 \Rightarrow \boxed{\epsilon = -10}$$

$$\therefore f(3) = 254+\epsilon = 254-10 = 244.$$

Missing values: Let a function $y = f(x)$ be given for equally spaced values $x_0, x_1, x_2, \ldots, x_n$ of the argument and $y_0, y_1, y_2, \ldots, y_n$ denote the corresponding values of the function. If one or more values of $y = f(x)$ are missing we can find the missing values by using the construction of difference table or using the relation between the operators E and Δ.

→ By constructing a difference table and taking the second order differences as a constant, find the sixth term of the series

$8, 12, 19, 29, 42, \underline{\ ?\ }$

Sol: Let k be the sixth term of the series

The difference table is.

x	y	Δy	$\Delta^2 y$
1	8		
		4	
2	12		3
		7	
3	19		3
		10	
4	29		3
		13	
5	42		$K-55$
		$K-42$	
6	K		

Since the second order differences are constant

$\therefore k - 55 = 3$

$\Rightarrow \boxed{k = 58}$

$\therefore$ The sixth ~~term~~ of the series is 58.

Assuming that the following values of y belong to a polynomial degree 4, compute the next three values.

x	0	1	2	3	4	5	6	7
y	1	-1	1	-1	1	—	—	—

Sol^n: we construct the difference table from the given data.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0	$y_0 = 1$				
		-2			
1	$y_1 = -1$		4		
				-8	
2	$y_2 = 1$	2	-4		$16 = \Delta^4 y_0$
		-2		8	
3	$y_3 = -1$		4		$16 = \Delta^4 y_1$
		2		$\Delta^3 y_2$	
4	$y_4 = 1$		$\Delta^2 y_3$		16
		Δy_4		$\Delta^3 y_3$	
5	y_5		$\Delta^2 y_4$		16
		Δy_5		$\Delta^3 y_4$	
6	y_6		$\Delta^2 y_5$		
		Δy_6			
7	y_7				

Since the values of y belong to a polynomial of degree 4, the fourth differences must be constant. But $\Delta^4 y_0 = 16$

$\therefore$ The other fourth order differences must also 16.

Thus $\Delta^4 y_1 = 16 = \Delta^3 y_2 - \Delta^3 y_1$

i.e., $\Delta^3 y_2 = \Delta^4 y_1 + \Delta^3 y_1$

$\Delta^3 y_2 = 16 + 8 = 24$

Now $\Delta^3 y_2 = \Delta^2 y_3 - \Delta^2 y_2$

$24 = \Delta^2 y_3 - 4$

$\Rightarrow \boxed{\Delta^2 y_3 = 28}$

$\Delta^2 y_3 = \Delta y_4 - \Delta y_3$

$\Rightarrow 28 = \Delta y_4 - 2$

$\Rightarrow \boxed{\Delta y_4 = 30}$

$\Delta y_4 = y_5 - y_4$

$30 = y_5 - 1 \Rightarrow \boxed{y_5 = 31}$

Similarly starting with $\Delta^4 y_2 = 16$

we get $\Delta^3 y_3 = 40$, $\Delta^2 y_4 = 68$,

$\Delta y_5 = 98$, $y_6 = 129$

Starting with $\Delta^4 y_3 = 16$;

we obtain $\Delta^3 y_4 = 56$, $\Delta^2 y_5 = 124$

$\Delta y_6 = 222$, $y_7 = 351$

4.3 Newton's binomial expansion formula

Let $y_0, y_1, y_2, \dots, y_n$ denote the values of the function $y = f(x)$ corresponding to the values $x_0, x_0 + h, x_0 + 2h, \dots, x_0 + nh$ of x and let one of the values of y be missing since n values of the functions are known. We have

$$\Delta^n y_0 = 0$$

$$\Rightarrow (E - 1)^n y_0 = 0$$

$$\Rightarrow [E^n - {}^nc_1 E^{n-1} + {}^nc_2 E^{n-2} - \dots + (-1)^n] y_0 = 0$$

$$\Rightarrow E^n y_0 - nE^{n-1} y_0 + \frac{n(n-1)}{1 \times 2} E^{n-2} y_0 + \dots + (-1)^n y_0 = 0$$

$$\Rightarrow y_n - n y_{n-1} + \frac{n(n-1)}{2} y_{n-2} + \dots + (-1)^n y_0 = 0$$

The above formula is called Newton's binomial expansion formula and is useful in finding the missing values without constructing the difference table.

Example 1 : Find the missing entry in the following table

x	0	1	2	3	4
y_x	1	3	9	–	81

Solution : Given $y_0 = 1, y_1 = 3, y_2 = 9, \dots, y_3 = ?, y_4 = 81$ four values of y are given. Let y be polynomial of degree 3

$$\therefore \Delta^4 y_0 = 0$$

$$(E - 1)^4 y_0 = 0$$

$$\Rightarrow (E^4 - 4E^3 + 6E^2 - 4E + 1) y_0 = 0$$

$$\Rightarrow E^4 y_0 - 4E^3 y_0 + 6E^2 y_0 - 4E y_0 + y_0 = 0$$

$$y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0$$

$$\therefore 81 - 4y_3 + 6 \times 9 - 4 \times 3 + 1 = 0$$

$$y_3 = 31.$$

Example 2 : Following are the population of a district

Year (x)	1881	1891	1901	1911	1921	1931
Population (y)	363	391	421	?	467	501

Find the population of the year 1911.

Solution : We have

$$y_0 = 363$$
$$y_1 = 391$$
$$y_2 = 421$$
$$y_3 = ?$$

$$y_4 = 467$$
$$y_5 = 501$$

Five values of y are given. Let us assume that y is a polynomial in x of degree 4.

$$\Delta^5 y_0 = 0$$
$$\Rightarrow (E - 1)^5 y_0 = 0$$
$$(E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1)\, y_0 = 0$$
$$y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0 = 0$$
$$501 - 5 \times 461 + 10y_3 - 10 \times 421 + 5 \times 391 - 363 = 0$$
$$10\, y_3 - 4452 = 0$$
$$y_3 = 445.2$$

The population of the district in 1911 is 445.2 lakhs.

Example 3 : Interpolate the missing entries

x	0	1	2	3	4	5
$y = f(x)$	0	–	8	15	–	35

Solution : Given $y_0 = 0, y_1 = ?, y_2 = 8, y_3 = 15, y_4 = ?, y_5 = 35$. Three values are known. Let us assume that $y = f(x)$ is a polynomial of degree 3.

$$\Delta^4 y_0 = 0$$
$$\Rightarrow (E - 1)^4 y_0 = 0$$
$$(E^4 - 4E^3 + 6E^2 - 4E + 1)\, y_0 = 0$$
$$\therefore y_4 - 4y_3 + 6y_2 - 4y_1 + y_0 = 0$$
$$\therefore y_4 - 4 \times 15 + 6 \times 8 - 4y_1 - 0 = 0$$

and
$$\therefore y_4 - 4y_1 = 12 \qquad (1)$$

$$\Delta^5 y_0 = 0$$
$$\Rightarrow (E - 1)^5 y_0 = 0$$
$$\Rightarrow (E^5 - 5E^4 + 10E^3 - 10E^2 + 5E - 1)\, y_0 = 0$$
$$y_5 - 5y_4 + 10y_3 - 10y_2 + 5y_1 - y_0 = 0$$
$$35 - 5y_4 + 10 \times 15 - 10 \times 8 + 5y_1 - 0 = 0$$
$$y_4 - y_1 = 21 \qquad (2)$$

solving (1) and (2) we get $y_1 = 3$, $y_4 = 24$.

INTERPOLATING POLYNOMIALS USING FINITE DIFFERENCES

4.4 Newton's forward interpolation formula (Gregory)

Let $y = f(x)$ be a function which takes the values $y_0, y_1, y_2, \ldots, y_n$ corresponding to the $(n + 1)$ values $x_0, x_1, x_2, \ldots, x_n$ of the independent variable x. Let the values x be equally spaced i.e.,

$$x_r = x_0 + rh;\ r = 0, 1, 2, \ldots, n)$$

(15)

where h is the interval of differencing. Let $\phi(x)$ be a polynomial of the nth degree in x taking the same values as y corresponding to $x = x_0, x_1, \ldots, x_n$ i.e., $\phi(x)$ represents the continuous function $y = f(x)$ such that $f(x_r) = \phi(x_r)$, $r = 0, 1, 2, \ldots, n$ and at all other points $f(x) = \phi(x) + R(x)$ where $R(x)$ is called the error term (Remainder term) of the interpolation formula. Ignoring the error term let us assume

$$f(x) \approx \phi(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + \ldots + a_n(x - x_0)(x - x_1)\ldots(x - x_{n-1}) \qquad (3)$$

the constants $a_0, a_1, a_2, \ldots a_n$ can be determine as follows.

Putting $x = x_0$ in (3) we get

$$f(x_0) = \phi(x_0) = a_0$$
$$\Rightarrow y_0 = a_0$$

(34)

putting $x = x_1$ in (3) we get

$$f(x_1) = \phi(x_1) = a_0 + a_1(x_1 - x_0) = y_0 + a_1 h$$
$$\therefore y_1 = y_0 + a_1 h$$
$$\Rightarrow a_1 = \frac{y_1 - y_0}{h} = \frac{\Delta y_0}{h}$$

Putting $x = x_2$ in (3) we get

$$f(x_2) = \phi(x_2) = a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1)$$
$$\therefore y_2 = y_0 + \frac{\Delta y_0}{h}(2h) + a_2(2h)(h)$$
$$\Rightarrow y_2 = y_0 + 2(y_1 - y_0) + a_2(2h^2)$$
$$\Rightarrow a_2 = \frac{y_2 - 2y_1 + y_0}{2h^2} = \frac{\Delta^2 y_0}{2!h^2}$$

Similarly by putting $x = x_3, x = x_4, \ldots, x = x_n$ in (3) we get

$$a_3 = \frac{\Delta^3 y_0}{3!h^3}, \quad a_4 = \frac{\Delta^4 y_0}{4!h^4}, \quad \ldots, \quad a_n = \frac{\Delta^n y_0}{n!h^n}$$

putting the values of $a_0, a_1, \ldots a_n$ in (3) we get

$$f(x) \approx \phi(x) = y_0 + \frac{\Delta y_0}{h}(x - x_0) + \frac{\Delta^2 y_0}{2!h^2}(x - x_0)(x - x_1) + \frac{\Delta^3 y_0}{3!h^3}(x - x_0)(x - x_1)(x - x_2) + \ldots + \frac{\Delta^n y_0}{n!h^n}(x - x_0)(x - x_1)\ldots(x - x_{n-1}) \qquad (4)$$

writing $u = \frac{x - x_0}{h}$, we get

$$x - x_0 = uh$$

$$x - x_1 = x - x_0 + x_0 - x_1$$

$$= (x - x_0) - (x_1 - x_0)$$

$$= uh - h = (u-1)h$$

similarly

$$x - x_2 = (u-2)h$$

$$x - x_3 = (u-3)h$$

$$\ldots$$

$$x - x_{n-1} = (u - (n+1))h$$

Equation (4) can be written as

$$\phi(x) = y_0 + u\frac{\Delta y_0}{1!} + \frac{u(u-1)}{2!}\Delta^2 y_0 + \ldots + \frac{u(u-1)\ldots(u-n+1)}{n!}\Delta^n y_0.$$

The above formula is called Newton's forward interpolation formula.

Note :

1. Newton forward interpolation formula is used to interpolate the values of y near the beginning of a set of tabular values.

2. y_0 may be taken as any point of the table, but the formula contains only those values of y which come after the value chosen as y_0.

Example 4 : Given that

$\sqrt{12500} = 111.8034$, $\sqrt{12510} = 111.8481$

$\sqrt{12520} = 111.8928$, $\sqrt{12530} = 111.9375$

find the value of $\sqrt{12516}$.

Solution : The difference table is

x	$y = \sqrt{x}$	Δy	$\Delta^2 y$
12500 x_0	111.8034 y_0		
		0.0447 Δy_0	
12510	111.8481		0 $\Delta^2 y_0$
		0.0447	
12520	111.8928		0
		0.0447	
12530	111.9375		

We have $x_0 = 12500$, $h = 10$ and $x = 12516$

$$u = \frac{x-x_0}{h} = \frac{12516-12500}{10} = 1.6$$

from Newton's forward interpolation formula,

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \ldots$$

$$\Rightarrow f(12516) = 111.8034 + 1.6 \times 0.0447 + 0 + \ldots$$

$$= 111.8034 + 0.07152$$

$$= 111.87492$$

$$\therefore \sqrt{12516} = 111.87492.$$

Example 5 : Evaluate $y = e^{2x}$ for $x = 0.05$ using the following table

x	0.00	0.10	0.20	0.30	0.40
$y = e^{2x}$	1.000	1.2214	1.4918	1.8221	2.255

Solution : The difference table is

x	$y = e^{2x}$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.000	1.0000				
		0.2214			
0.10	1.2214		0.0490		
		0.2704		0.0109	
0.20	1.4918		0.0599		0.0023
		0.3303		0.0132	
0.30	1.8221		0.0731		
		0.4034			
0.40	2.2255				

We have $x_0 = 0.00$, $x = 0.05$, $h = 0.1$.

$$\therefore u = \frac{x - x_0}{h} = \frac{0.05 - 0.00}{0.1} 0.5$$

Using Newton's forward formula

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$+ \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \ldots$$

$$f(0.05) = 1.0000 + 0.5 \times 0.2214 + \frac{0.5(0.5-1)}{2}(0.0490)$$

$$+ \frac{0.5(0.5-1)(0.5-2)}{6}(0.0109)$$

$$+ \frac{0.5(0.5-1)(0.5-2)(0.5-3)}{24}(0.0023)$$

$$= 1.000 - 0.006125 + 0.000681 - 0.000090$$

$$= 1.105166$$

$$\therefore f(0.05) = 1.052.$$

Example 6 : The values of $\sin x$ are given below for different values of x. Find the value fo $\sin 32°$

x	30°	35°	40°	45°	50°
$y = \sin x$	0.5000	0.5736	0.6428	0.7071	0.7660

Solution : $x = 32°$ is very near to the starting value $x_0 = 30°$. By using Newton's forward interpolation formula the difference table is

x	$y = \sin x$	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
30°	0.5000				
		0.0736			
35°	0.5736		−0.0044		
		0.0692		−0.005	
40°	0.6428		−0.0049		0
		0.0643		−0.005	
45°	0.7071		−0.0054		
		0.0589			
50°	0.7660				

$$u = \frac{x - x_0}{h} = \frac{32^0 - 30^0}{5} = 0.4$$

We have $y_0 = 0.5000$, $\Delta y_0 = 0.0736$, $\Delta^2 y_0 = -0.0044$, $\Delta^3 y_0 = -0.005$

putting these values in Newton's forward interpolation formula we get

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \dots$$

$$\Rightarrow f(32°) = 0.5000 + 0.4 \times 0.0736 + \frac{(0.4)(0.4-1)}{2}(-0.0044)$$

$$+ \frac{(0.4)(0.4-1)(0.4-2)}{6}(-0.0005)$$

$$f(32°) = 0.5000 + 0.02944 + 0.000528 - 0.00032 = 0.529936 = 0.5299$$

0.5298

Example 7 : In an examination the number of candidates who obtained marks between certain limits were as follows

Marks	30–40	40–50	50–60	60–70	70–80
No. of Students	31	42	51	35	31

Find the number of candidates whose scores lie between 45 and 50.

Solution : First of all we construct a cumulative frequency table for the given data.

Upper limits of the class intervals	40	50	60	70	80
Cumulative frequency	31	73	124	159	190

The difference table is

x marks	y cumulative frequencies	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
40	31				
		42			
50	73		9		
		51		−25	
60	124		−16		37
		35		12	
70	159		−4		
		31			
80	190				

we have $x_0 = 40$, $x = 45$, $h = 10$

$$u = \frac{x - x_0}{h} = \frac{45 - 40}{10} = 0.5$$

and $y_0 = 31$, $\Delta y_0 = 42$ $\Delta^2 y_0 = 9$, $\Delta^3 y_0 = -25$, $\Delta^4 y_0 = 37$.

From Newton's forward interpolation formula

$$f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$+ \frac{u(u-1)(u-2)(u-3)}{4!}\Delta^4 y_0 + \ldots$$

$$\therefore f(45) = 31 + (0.5)(42) + \frac{(0.5)(-0.5)}{2} \times 9$$

$$+ \frac{(0.5)(0.5-1)(0.5-2)}{6}(-25)$$

$$+ \frac{(0.5)(0.5-1)(0.5-2)(0.5-3)}{24} \times (37)$$

$$= 31 + 21 - 1.125 - 1.5625 - 1.4452$$
$$= 47.8673$$
$$= 48 \text{ (approximately)}$$

$\therefore$ The number of students who obtained mark less than 45 = 48, and the number of students who scored marks between 45 and 50 = 73 − 48 = 25.

Example: 8 : A second degree polynomial passes through the points (1, −1), (2, −1), (3, 1), (4, 5). Find the polynomial.

Solution : We construct difference table with the given values of x and y

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
1	−1			
		0		
2	−1		2	
		2		0
3	1		2	
		4		
4	5			

We have $x_0 = 1,\ h = 1,\ y_0 = -1,\ \Delta y_0 = 0,\ \Delta^2 y_0 = 2,$

$$u = \frac{x - x_0}{h} = (x - 1).$$

From Newton's forward interpolation we get

$$y = f(x) = y_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \ldots$$

$$\Rightarrow f(x) = -1 + (x-1)\cdot 0 + \frac{(x-1)(x-1-1)}{2}\cdot 2$$

$$\therefore f(x) = x^2 - 3x + 1.$$

Note : There may be polynomials of higher degree which also fit the data, but Newton's formula gives us the polynomial of least degree which fits the data.

[illegible] Newton - Gregory backward interpolation formula

Newton's forward interpolation formula cannot be used for interpolating a value of y near the end of a table of values. For this purpose, we use another formula known as Newton - Gregory backward interpolation formula. It can be derived as follows.

Let $y = f(x)$ be a function which takes the values $y_0, y_1, y_2, \ldots, y_n$ corresponding to the values $x_0, x_1, x_2, \ldots, x_n$ of the independent variable x. Let the values of x be equally spaced with h as the interval of differencing, i.e., Let

$$x_r = x_0 + rh,\ r = 0, 1, 2, \ldots, n$$

Let $\phi(x)$ be a polynomial of the nth degree in x taking the same values as y corresponding to $x = x_0, x_1, \ldots x_n$ i.e., $\phi(x)$ represents $y = f(x)$ such that $f(x_r) = \phi(x_r),\ r = 0, 1, 2, \ldots, n$ we may write $\phi(x)$ as

$$f(x) \approx \phi(x) = a_0 + a_1(x - x_n) + a_2(x - x_n)(x - x_{n-1}) + \ldots + a_n(x - x_n)(x - x_{n-1}) \ldots (x - x_1) \qquad (5)$$

putting $x = x_n$ in (5) we get

$$f(x_n) \approx \phi(x_n) = a_0$$

$$\Rightarrow y_n = a_0.$$

Putting $x = x_{n-1}$ in (5) we get

$$f(x_{n-1}) \approx \phi(x_{n-1}) = a_0 + a_1(x_{n-1} - x_n)$$

$$\Rightarrow y_{n-1} = y_n + a_1(-h)$$

$$\Rightarrow a_1 h = y_n - y_{n-1} = \nabla y_n$$

$$\Rightarrow a_1 = \frac{\nabla y_n}{1!h}$$

Putting $x = x_{n-2}$, we get

$$f(x_{n-2}) = \phi(x_{n-2}) = a_0 + a_1(x_{n-2} - x_n) + a_2(x_{n-2} - x_n)(x_{n-2} - x_{n-1})$$

$$\Rightarrow y_{n-2} = y_n + \left(\frac{y_n - y_{n-1}}{h}\right)(-2h) + a_2(-2h)(-h)$$

$$\Rightarrow y_{n-2} = y_n - 2y_n + 2y_{n-1} + 2h^2 a_2$$

$$\Rightarrow a_2 = \frac{y_n - 2y_{n-1} + y_{n-2}}{2h^2} = \frac{\nabla^2 y_n}{2!h^2}$$

similarly putting $x = x_{n-3}, x = x_{n-4}, \ldots, x = x_0$ we get

$$a_3 = \frac{\nabla^3 y_n}{3!h^3},\ a_4 = \frac{\nabla^4 y_n}{4!h^4}, \ldots, a_n = \frac{\nabla^n y_n}{n!h^n}$$

substituting these values in (5)

$$f(x) \approx \phi(n) = y_n + \frac{\nabla y_n}{h}(x - x_n) + \frac{\nabla^2 y_n}{2!h^2}(x - x_n)(x - x_{n-1})$$

$$+ \frac{\nabla^3 y_n}{3!h^3}(x - x_n)(x - x_{n-1})(x - x_{n-2})$$

$$+ \ldots + \frac{\nabla^n y_n}{n!h^n}(x - x_n)(x - x_{n-1}) \ldots (x - x_1) \qquad (6)$$

writing $u = \dfrac{x - x_n}{h}$ we get

$$x - x_n = uh.$$

$\therefore x - x_{n-1} = x - x_n + x_n - x_{n-1} = (uh) + h = (u+1)h$

$\Rightarrow x - x_{n-2} = (u+2)h, \ldots, (x - x_1) = (u + n - 1)h$

$\therefore$ The equation (6) may be written as

$$f(x) \approx \phi(x) = y_n + \frac{u \nabla y_n}{1!} + \frac{u(u+1)}{2!}\nabla^2 y_n + \frac{u(u+1)(u+2)}{3!}\nabla^3 y_n$$

$$+ \ldots + \frac{u(u+1)(u+2)\ldots(u+n-1)}{n!}\nabla^n y_n .$$

The above formula is known as Newton's backward interpolation formula.

Example 9 : The following data gives the melting point of an alloy of lead and zinc, where t is the temperature in degrees c and P is the percentage of lead in the alloy.

P	40	50	60	70	80	90
t	184	204	226	250	276	304

Find the melting point of the alloy containing 84 percent lead.

Solution : The value of 84 is near the end of the table therefore we use the Newton's backward interpolation formula. The difference table is

P	t	∇	∇^2	∇^3	∇^4	∇^5
40	184					
		20				
50	204		2			
		22		0		
60	226		2		0	
		24		0		0
70	250		2		0	
		26		0		
80	276		2			
		28				
90	304					

We have $x_n = 90$, $x = 84$, $h = 10$, $t_n = y_n = 304$, $\nabla t_n = \nabla y_n = 28$, $\nabla^2 y_n = 2$, and

$$\nabla^3 y_n = \nabla^4 y_n = \nabla^5 y_n = 0,$$

$$u = \frac{x - x_n}{h} = \frac{84 - 90}{10} = -0.6$$

From Newton backward formula

$$f(84) = t_n + u\nabla t_n + \frac{u(u+1)}{2}\nabla^2 t_n + \ldots$$

$$f(84)=304-0.6\times 28+\frac{(-0.6)(-0.6+1)}{2}2$$

$$=304-16.8-0.24$$

$$=286.96$$

Example 10 : Calculate the value of $f(7.5)$ for the table

x	1	2	3	4	5	6	7	8
$f(x)$	1	8	27	64	125	216	343	512

Solution : 7.5 is near to the end of the table, we use numbers backward formula to find $f(7.5)$.

x	y	∇y	$\nabla^2 y$	$\nabla^3 y$	$\nabla^4 y$	$\nabla^5 y$
1	1					
		7				
2	8		12			
		19		6		
3	27		18		0	
		37		6		0
4	64		24		0	
		61		6		0
5	125		30		0	
		91		6		0
6	216		36		0	
		127		6		
7	343		42			
		169				
8	512					

We have

$x_n=8,\ x=7.5, h=1, y_n=512;\ \nabla y_n=169,\ \nabla^2 y_n=42,\ \nabla^3 y_n=6,\ \nabla^4 y_n=\nabla^5 y_n=...=0$

$$u=\frac{x-x_n}{h}=\frac{7.5-8}{1}=-0.5$$

$\therefore$ We get

$$f(x)=y_n+u\nabla y_n+\frac{u(u+1)}{2!}\nabla^2 y_n+\frac{u(u+1)(u+2)}{3!}\nabla^3 y_n$$

$$f(7.5)=512+(-0.5)(169)+\frac{(-0.5)(-0.5+1)}{2}(42)$$

$$+\frac{(-0.5)(-0.5+1)(-0.5+2)}{6}(6)$$

$$=512-84.5-5.25-0.375$$

$$=421.87.$$

6 Error in the interpolation formula

Let the function $f(x)$ be continuous and possess continuous derivatives of all orders with in the interval $[x_0, x_n]$ and let $\phi(x)$ denote the interpolating polynomial. Define the auxiliary function $F(t)$ as given below,

$$F(t) = f(t) - \phi(t) - \{f(x) - \phi(x)\}\frac{(t-x_0)(t-x_1)\dots(t-x_n)}{(x-x_0)(x-x_1)\dots(x-x_n)}$$

The function $F(t)$ is continuous in $[x_0, x_n]$. $F(t)$ possesses continuous derivatives of all orders in $[x_0, x_n]$ and variables for the values $t = x, x_0, \dots, x_n$. Therefore $F(t)$ satisfies all the conditions of Rollers Theorem in each of the subintervals $(x_0, x_1), (x_1, x_2) \dots (x_{n-1}, x_n)$. Hence $F'(t)$ vanishes at least once in each of the sub intervals. Therefore $f'(t)$ vanishes at least $(n+1)$ times in (x_0, x_n), $f''(t)$ vanishes at least n times in the interval (x_0, x_n), ..., $F^{n+1}(t)$ vanishes at least once in (x_0, x_n) say at ζ, where $x_0 < \zeta < x_n$.

The expression $(t - x_0)(t - x_1) \dots (t - x_n)$ is a polynomial of degree $(n+1)$ in t and the coefficient of $t = 1$.

$\therefore$ The $(n+1)$th derivative of the polynomial is $(n+1)!$

$$\therefore F^{n+1}(\xi) = f^{n+1}(\xi) - \{f(x) - \phi(x)\}\frac{(n+1)!}{(x-x_0)(x-x_1)\dots(x-x_n)} = 0$$

$$\Rightarrow f(x) - \phi(x) - \frac{f^{n+1}(\xi)}{(n+1)!}(x-x_0)(x-x_1)\dots(x-x_n)$$

If $R(x)$ denotes the error in the formula then $R(x) = f(x) - \phi(x)$

$$\therefore R(x) = \frac{f^{n+1}(\xi)}{(n+1)!}(x-x_0)(x-x_1)\dots(x-x_n)$$

But $x - x_0 = uh \Rightarrow x - x_1 = (u-1)h, \dots (x - x_n) = (u-n)h$ where h is the interval of differencing therefore we can write

$$\therefore \text{Error } R(x) = \frac{h^{n+1}f^{n+1}(\xi)}{(n+1)!}u(u-1)(u-2)\dots(u-n).$$

Using the relation $D = \frac{1}{h}\Delta$

we get

$$D^{n+1} \approx \frac{1}{h^{n+1}}\Delta^{n+1}$$

$$\Rightarrow f^{n+1}(\xi) \approx \frac{\Delta^{n+1} f(x_0)}{n+1}$$

The error in the forward interpolation formula is

$$\boxed{R(x) = \frac{\Delta^{n+1} y_0}{(n+1)!} u(u-1)(u-2)\ldots(u-n)}$$

Similarly by taking the auxiliary function $F(t)$ in the form

$$F(t) = f(t) - \phi(t) - \{f(x) - \phi(x)\} \frac{(t-x_n)(t-x_{n-1})\ldots(t-x_0)}{(x-x_n)(x-x_{n-1})\ldots(x-x_0)},$$

and proceeding as above we get the error in the Newton backward interpolation formula as

$$\boxed{R(x) = \frac{\nabla^{n+1} y_n}{(n+1)!} u(u+1)\ldots(u+n) \quad \text{where } uh = x - x_n}$$

Example 11 : Use Newton's forward interpolation formula and find the value of sin 52° from the following data. Estimate the error.

x	45°	50°	55°	60°
$y = \sin x$	0.7071	0.7660	0.8192	0.8660

Solution : The difference table is

x	$y = \sin x$	Δy	$\Delta^2 y$	$\Delta^3 y$
45°	0.7071			
		0.0589		
50°	0.7660		−0.0057	
		0.0532		−0.0007
55°	0.8192		−0.0064	
		0.0468		
60°	0.8660			

$\therefore$ We have $x_0 = 45°$, $x_1 = 52°$, $y_0 = 0.7071$, $\Delta y_0 = 0.0589$, $\Delta^2 y_0 = -0.0057$ and $\Delta^3 y_0 = -0.0007$,

$$u = \frac{x - x_0}{h} = \frac{52° - 45°}{5°} = 1.4 .$$

From Newton's formula,

$$y = u_0 + u\Delta y_0 + \frac{u(u-1)}{2!}\Delta^2 y_0 + \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0 + \ldots$$

$$\therefore f(52) = 0.7071 + 1.4 \times 0.0589 + \frac{(1.4)(1.4-1)}{2} \times (-0.0057)$$

$$+ \frac{(1.4)(1.4-1)(1.4-2)}{6}(-0.0007)$$

$$= 0.7071 + 0.8246 - 0.001596 + 0.0000392$$
$$= 0.7880032$$

$\therefore \sin 52^\circ = 0.7880032$

$$\text{Error} = \frac{u(u-1)(u-2)\ldots(u-n)}{(n+1)!}\Delta^{n+1}y_0$$

taking $n = 2$ we get

$$\text{Error} = \frac{u(u-1)(u-2)}{3!}\Delta^3 y_0$$

$$= \frac{(1.4)(1.4-1)(1.4-2)}{6}(-0.0007) = 0.0000392$$

Exercise 4.1

1. Find the missing term in the following table

x	1	2	3	4	5	6	7
$f(x)$	2	4	8	–	32	64	128

2. Estimate the production of cotton in the year 1985 from the data given below

year (x)	1981	1982	1983	1984	1985	1986	1987
production (y)	17.1	13.0	14.0	9.6	–	12.4	18.2

3. Find the missing figure in the frequency table

x	15–19	20–24	25–29	30–34	35–39	40–44
f	7	21	35	?	57	58

4. Find the missing figures in the following table

x	0	5	10	15	20	25
y	7	11	–	18	–	32

5. Complete the table

x	2	3	4	5	6	7	8
$f(x)$	0.135	–	0.111	0.100	–	0.082	0.074

6. Find $f(1.1)$ from the table

x	1	2	3	4	5
$f(x)$	7	12	29	64	123

7. The following are data from the steam table

Temperature °C	140	150	160	17	180
Pressure kgt/cm2	3.685	4.84	6.302	8.076	10.225

Using the Newton's formula, find the pressure of the steam for a temperature of 142°C.

8. The area A of circle of diameter d is given for the following values

A	80	85	90	95	100
d	5026	5674	6362	708	7854

Find approximate values for the areas of circles of diameter 82 and 91 respectively.

9. Compute (i) $f(1.38)$ from the table

x	1.1	1.2	1.3	1.4
$f(x)$	7.831	8.728	9.627	10.744

10. Find the value of y when $x = 0.37$, using the given values

x	0.000	0.10	0.20	0.30	0.40
$y = e^{2x}$	1.000	1.2214	1.4918	1.8221	2.2255

11. Find the value of $\log_{10} 2.91$, using table given below

x	2.0	2.2	2.4	2.6	2.8	3.0
$y = \log_{10} x$	0.30103	0.34242	0.38021	0.41497	0.44716	0.47721

12. Find $f(2.8)$ from the following table

x	0	1	2	3
$f(x)$	1	2	11	34

13. Find the polynomial which takes on the following values

x	0	1	2	3	4	5
$f(x)$	41	43	47	53	61	71

14. Find a polynomial y which satisfies the following table

x	0	1	2	3	4	5
y	0	5	34	111	260	505

15. Given the following table find $f(x)$ and hence find $f(4.5)$

x	0	2	4	6	8
$f(x)$	−1	13	43	89	151

16. A second degree polynomial passes through (0,1) (1,3 (2, 7) (3, 13), find the polynomial.

17. Find a cubic polynomial which takes the following values

x	0	1	2	3
$f(x)$	1	0	1	10

18. $u_0 = 560, u_1 = 556, u_2 = 520, u_4 = 385$ show that $u_3 = 465$.

19. In an examination the number of candidates who secured marks between certain limit were as follows :

Marks	0 – 19	20 – 39	40 – 59	60 – 79	80 – 99
No. of candidates	41	62	65	50	17

Estimate the number of candidates whose marks are less than 70.

20. Given the following score distribution of statistics

Marks	30 – 40	40 – 50	50 – 60	60 – 70
No of students	52	36	21	14

Find

(i) the number of students who secured below 35

(ii) the number of students who secured above 65

(iii) the number of students who secured between 35–45

Answers

1. 17 2. 6.60 3. 48 4. 23.5, 14.25 5. $f(3) = 0.123$, $f(6) = 0.090$ 6. 7.13
7. 3.899 8. 5281, 6504 9. 10.963 10. 2.0959 11. 0.46389 12. 27.992
13. $x^2 + x + 41$ 14. $4x^3 + x$ 15. $2x^3 + 3x - 1$ 16. $x^2 + x + 1$ 17. $x^3 - 2x^2 + 1$
19. 197 20. (i) 26 (ii) 7 (iii) 46

→ The table gives the distance in nautical miles of the visible horizon for the given heights in feet above the earth's surface.

x: height	100	150	200	250	300	350	400
y: distance	10.63	13.03	15.04	16.81	18.42	19.90	21.27

Find the values of y when

(i) $x = 218$ ft

(ii) $y = 410$ ft.

Sol: The difference table is :—

x	y	Δ	Δ^2	Δ^3	Δ^4
100	10.63				
		2.40			
150	13.03		−0.39		
		2.01		0.15	
200	15.04		−0.24		−0.07
		1.77		0.08	
250	16.81		−0.16		−0.05
		1.61		0.03	
300	18.42		−0.13		−0.01
		1.48		0.02	
350	19.90		−0.11		
		1.37			
400	21.27				

(1) If we take $x_0 = 200$, then $y_0 = 15.04$,

$\Delta y_0 = 1.77$, $\Delta^2 y_0 = -0.16$, $\Delta^3 = 0.03$ etc.

Since $x = 218$ and $h = 50$.

$$\therefore p = \frac{x - x_0}{h} = \frac{18}{50} = 0.36.$$

$\therefore$ Using newton's forward interpolation formula, we get

$$f(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \cdots$$

$$f(218) = 15.04 + 0.36(1.77) + \frac{0.36(-0.64)}{2}(-0.16) + \frac{0.36(-0.64)(-1.64)}{6}(0.03) + \frac{(0.36)(-0.64)(-1.64)(-2.64)}{24}(-0.01)$$

$$= 15.04 + 0.637 + 0.018 + 0.002 + 0.0004$$

$$= 15.697$$

$$= 15.7 \text{ nautical miles.}$$

(ii) Since $x = 410$ is near the end of the table we use newton's backward interpolation formula.

$\therefore$ Taking $x_n = 400$, $p = \frac{x - x_n}{h} = \frac{10}{50} = 0.2$

using the line of backward difference

$$y_n = 21.27, \quad \nabla y_n = 1.37, \quad \nabla^2 y_n = -0.11$$

$$\nabla^3 y_n = 0.02 \text{ etc.}$$

$\therefore$ newton's backward formula gives

$$y(410) = y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n$$

$$+ \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n$$

$$= 21.27 + (0.2)(1.37) + \frac{(0.2)(1.2)}{2!}(-0.11)$$

$$+ \frac{(0.2)(1.2)(2.2)}{3!}(0.02) + \frac{(0.2)(1.2)(2.2)(3.2)}{4!}(-0.01)$$

$$= 21.27 + 0.274 - 0.0132 + 0.0018 - 0.0007$$

$= 21.53$ nautical miles.

*Interpolation With unequal Intervals:

The various interpolation formulae derived so far possess the disadvantage of being applicable only to equally spaced values of the argument. It is, therefore, desirable to develop interpolation formulae for unequally spaced values of x. Now we shall study two such formulae.

(i) Lagrange's Interpolation formula.

(ii) Newton's - general Interpolation formula with divided differences

* Lagrange's Interpolation Formula :-

Let $x_0, x_1, x_2, ----x_n$ be $n+1$ distinct points on the real line and let $f(x)$ be a real valued function defined on some interval $I=[a,b]$ containing these points. Then, there exists exactly one polynomial $P_n(x)$ of degree $\leq n$, which interpolates $f(x)$ at $x_0, x_1, ---x_n$ that is $P_n(x_i) = f(x_i) = f_i$ ——— (A)
$i = 0, 1, 2, 3 ------ n.$

Proof:

First of all, we discuss the uniqueness of the interpolating polynomial.

Let $P_n(x)$ and $Q_n(x)$ be two distinct interpolating polynomials of degree $\leq n$, which interpolate $f(x)$ at $(n+1)$ distinct points $x_0, x_1, x_2, ---x_n$.

and also satisfy $P_n(x_i) = f(x_i)$ and $Q_n(x_i) = f(x_i)$; $i = 0, 1, 2, ---n$.

Now let us consider the polynomial

$$h(x) = P_n(x) - Q_n(x)$$

Since $P_n(x)$ and $Q_n(x)$, are both polynomials of degree $\leq n$, then $h(x)$ is also a polynomial of degree $\leq n$ and satisfying the condition.

$$h(x_i) = P_n(x_i) - Q_n(x_i); \quad i = 0, 1, 2 \cdots n$$

$$= f(x_i) - f(x_i) = 0.$$

$$\therefore h(x_i) = 0; \quad i = 0, 1, 2, \cdots n$$

i.e. $h(x)$ has $(n+1)$ distinct zero's. But $h(x)$ is of degree $\leq n$

This implies that $h(x) = 0$.

because, a polynomial $h(x)$ of degree 'n' has exactly 'n' roots real or complex.

$$\therefore P_n(x) = Q_n(x).$$

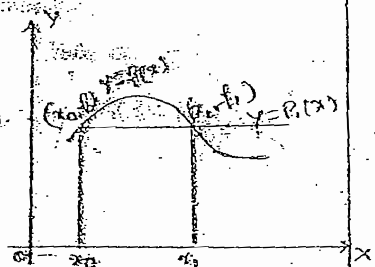
This shows that the uniqueness of the polynomial.

Let the data be given at the points

$(x_0, f_0), (x_1, f_1), (x_2, f_2), \cdots (x_n, f_n)$

Before deriving the general formula, we first consider a simpler case namely, the equation of the straight line (a linear polynomial) passing through two points (x_0, f_0) and (x_1, f_1).

Such a polynomial say $P_1(x)$.



$$P_1(x) = \frac{x - x_1}{(x_0 - x_1)} f_0 + \frac{x - x_0}{x_1 - x_0} f_1$$

$$\left[\because \underset{x_1 \;\; y_1}{(x_0, f_0)}, \underset{x_2 \;\; y_2}{(x_1, f_1)} \right.$$

$$\Rightarrow (y - f_0)(x_1 - x_0) = (f_1 - f_0)(x - x_0)$$

$$\Rightarrow y = \frac{x - x_1}{(x_0 - x_1)} f_0 + \frac{x - x_0}{x_1 - x_0} f_1$$

$$\left. \Rightarrow P_1(x) = \frac{x - x_1}{x_0 - x_1} f_0 + \frac{x - x_0}{x_1 - x_0} f_1 \right]$$

$$= l_0(x) f_0 + l_1(x) f_1$$

$$\boxed{P_1(x) = \sum_{i=0}^{1} l_i(x) f_i} \quad \text{———} \; (1)$$

where $\left.\begin{aligned} l_0(x) &= \frac{x-x_1}{x_0-x_1} \\ l_1(x) &= \frac{x-x_0}{x_1-x_0} \end{aligned}\right]$ ——— ②

Putting $x = x_0$ in ②, we get

$l_0(x_0) = 1$

$l_1(x_0) = 0$

Putting $x = x_1$ in ②, we get

$l_0(x_1) = 0$

$l_1(x_1) = 1$

∴ The relations can be expressed in a more convenient form as

$$l_i(x_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \quad \text{——— ③}$$

from ②, we have

$$l_0(x) + l_1(x) = \frac{x-x_1}{x_0-x_1} + \frac{x-x_0}{x_1-x_0}$$

$$= 1$$

$$\boxed{\sum_{i=0}^{1} l_i(x) = l_0(x) + l_1(x) = 1} \quad \text{——— ④}$$

Equation ① is the Lagrange Polynomial of degree one passing through two points $(x_0, f_0), (x_1, f_1)$.

Similarly, the lagrange polynomial of degree two passing through three points $(x_0, f_0), (x_1, f_1)$ & (x_2, f_2) is written as.

$$P_2(x) = \sum_{i=0}^{2} l_i(x) f_i$$

$$= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f_2 \quad \text{——— ⑤}$$

where the $l_i(x)$ satisfy the conditions given in ③ & ④.

To derive the general formula,

let $P_n(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n$ ——— ⑥

be the required polynomial of the nth degree such that conditions (5) (called the interpolatory conditions) are satisfied.

Substituting these conditions in (6)

we get the equations

$$\left.\begin{aligned} f_0 &= a_0 + a_1 x_0 + a_2 x_0^2 + \cdots + a_n x_0^n \\ f_1 &= a_0 + a_1 x_1 + a_2 x_1^2 + \cdots + a_n x_1^n \\ f_2 &= a_0 + a_1 x_2 + a_2 x_2^2 + \cdots + a_n x_2^n \\ &\cdots\cdots \\ f_n &= a_0 + a_1 x_n + a_2 x_n^2 + \cdots + a_n x_n^n \end{aligned}\right\} \quad (7)$$

The set of equations (7) will have a solution if

$$\begin{vmatrix} 1 & x_0 & x_0^2 & x_0^3 & \cdots & x_0^n \\ 1 & x_1 & x_1^2 & x_1^3 & \cdots & x_1^n \\ 1 & x_2 & x_2^2 & x_2^3 & \cdots & x_2^n \\ \cdots & & & & & \\ 1 & x_n & x_n^2 & x_n^3 & \cdots & x_n^n \end{vmatrix} \neq 0$$

The value of this determinant, called Vandermonde's determinant, is $(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)(x_1-x_2)\cdots(x_1-x_n)$

$(x_2-x_3)(x_2-x_4)\cdots(x_2-x_n)\cdots(x_{n-1}-x_n)$

Eliminating $a_0, a_1, a_2, \ldots a_n$ from equations (6) and (7), we obtain

$$\begin{vmatrix} P_n(x) & 1 & x & x^2 & \cdots & x^n \\ f_0 & 1 & x_0 & x_0^2 & \cdots & x_0^n \\ f_1 & 1 & x_1 & x_1^2 & \cdots & x_1^n \\ \cdots & & & & & \\ f_n & 1 & x_n & x_n^2 & \cdots & x_n^n \end{vmatrix}_{(n+2)\times(n+2)} = 0 \quad (8)$$

which shows that $P_n(x)$ is a linear combination of $f_0, f_1, f_2, \ldots f_n$

Hence write $P_n(x) = \sum_{i=0}^{n} l_i(x) f_i$ ——— (9)

where $l_i(x)$ are polynomials in x of degree 'n'.

since $P_n(x_j) = f_j$ for $j = 0, 1, 2 \ldots n$.

the equation (4) gives

$$l_i(x_j) = 0 \text{ if } i \neq j$$
$$l_i(x_j) = 1 \text{ for all } i = j.$$

which are the same as (3). Hence $l_i(x)$ may be written as

$$l_i(x) = \frac{(x-x_0)(x-x_1)\ldots(x-x_{i-1})(x-x_{i+1})\ldots(x-x_n)}{(x_i-x_0)(x_i-x_1)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_n)} \quad \text{---} (10)$$

which obviously satisfies the condition (3)

If we now set

$$\pi_{n+1}(x) = (x-x_0)(x-x_1)\ldots(x-x_{i-1})(x-x_i)(x-x_{i+1})\ldots(x-x_n) \quad \text{---} (11)$$

then $\pi'_{n+1}(x_i) = \dfrac{d}{dx}\Big[\pi_{n+1}(x)\Big]_{x=x_i}$

$$= (x_i-x_0)(x_i-x_1)\ldots(x_i-x_{i-1})(x_i-x_{i+1})\ldots(x_i-x_n) \quad \text{---} (12)$$

so that (10) becomes

$$l_i(x) = \frac{\pi_{n+1}(x)}{(x-x_i)\,\pi'_{n+1}(x_i)} \quad \text{---} (13)$$

Hence (9) gives

$$P_n(x) = \sum_{i=0}^{n} \frac{\pi_{n+1}(x)}{(x-x_i)\,\pi'_{n+1}(x_i)} f_i \quad \text{---} (14)$$

which is called Lagrange's interpolation formula.

– The coefficients $l_i(x)$, defined in (10) are called Lagranges Interpolation Coefficients.

* Inverse Interpolation Formula :–

In inverse interpolation, in a table of values of x and $y = f(x)$ one is given a number $\bar{y}$ and wishes to find the point $\bar{x}$ so that $f(\bar{x}) = \bar{y}$, where $f(x)$ is the tabulated function. This problem can always be solved if $f(x)$ is continuous and strictly increasing (or) decreasing (i.e. the inverse of 'f' exists). This is done by considering the table of values $x_i, f(x_i)$; $i = 0,1,2,3,\dots n$ to be a table of values $y_i, g(y_i)$, $i = 0,1,2 - - - n$ for the inverse function $g(y) = f^{-1}(y) = x$ by taking $y_i = f(x_i)$, $g(y_i) = x_i$, $i = 0,1,2 - - - n$. Then we can interpolate for the unknown value $g(\bar{y})$ in this table.

$$P_n(y) = \sum_{i=1}^{n} \frac{\pi_{n+1}(y)}{(y-y_i)\,\pi'_{n+1}(y_i)}\, x_i \quad \text{—— (15)}$$

and $\bar{x} \simeq P_n(\bar{y})$.

This process is called Inverse Interpolation.

Note (1): The Lagrange form [equation (9)] of interpolating polynomial makes it easy to show the existence of an interpolating polynomial. But its evaluation at a point x_i involves a lot of computation.

Note (2): Moreover the lagrange form of interpolating polynomial can be determined for equally spaced or unequally spaced nodes.

Note (3): The Lagrangian coefficients in (13) can conveniently be computed in practice by the following scheme. We first compute the differences, row wise, as given below.

$$\begin{array}{ccccc}
x_0-x_0 & x_0-x_1 & x_0-x_2 & \cdots & x_0-x_n \\
x_1-x_0 & x_1-x_1 & x_1-x_2 & \cdots & x_1-x_n \\
x_2-x_0 & x_2-x_1 & x_2-x_2 & \cdots & x_2-x_n \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
x_n-x_0 & x_n-x_1 & x_n-x_2 & \cdots & x_n-x_n
\end{array}$$

we note that the product of the elts along the diagonal line is $\Pi_{n+1}(x)$.

Similarly, the product of the elements of the first row $(x-x_0)\,\Pi'_{n+1}(x_0)$, of the second row is $(x-x_1)\,\Pi'_{n+1}(x_1)$ and of the $(n+1)^{th}$ row is $(x-x_n)\,\Pi'_{n+1}(x_n)$.

The Lagrangian coefficients can then be computed by using formula (13).

— The Lagrange's Interpolation formula (14) can be written as

$$P_n(x) = \sum_{i=0}^{n} \frac{\Pi_{n+1}(x)}{(x-x_i)\,\Pi'_{n+1}(x_i)} f_i$$

$$= \frac{\Pi_{n+1}(x)}{(x-x_0)\,\Pi'_{n+1}(x_0)} f_0 + \frac{\Pi_{n+1}(x)}{(x-x_1)\,\Pi'_{n+1}(x_1)} f_1 + \cdots + \frac{\Pi_{n+1}(x)}{(x-x_n)\,\Pi'_{n+1}(x_n)} f_n$$

$$\Rightarrow P_n(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)} f_0 + \frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)} f_1 + \cdots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})} f_n \qquad (16)$$

— The Lagrange's inverse interpolation formula (15) can be written as

$$x = P_n(f) = \frac{(f-f_1)(f-f_2)\cdots(f-f_n)}{(f_0-f_1)(f_0-f_2)\cdots(f_0-f_n)} x_0 + \frac{(f-f_0)(f-f_2)\cdots(f-f_n)}{(f_1-f_0)(f_1-f_2)\cdots(f_1-f_n)} x_1 + \cdots + \frac{(f-f_0)(f-f_1)\cdots(f-f_{n-1})}{(f_n-f_0)(f_n-f_1)\cdots(f_n-f_{n-1})} x_n \qquad (17)$$

problems

① Certain corresponding values of x and $\log_{10} x$ are $(300, 2.4771)$, $(304, 2.4829)$, $(305, 2.4843)$ and $(307, 2.4871)$ find $\log_{10} 301$.

Sol: we have

$x_0 = 300, \; x_1 = 304, \; x_2 = 305, \; x_3 = 307$

and $f_0 = 2.4771, \; f_1 = 2.4829, \; f_2 = 2.4843$,

$f_3 = 2.4871$.

Now the Lagrange's interpolation formula with 4 points is

$$f(x) = P_3(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} f_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} f_1$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} f_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} f_3$$

$$\log_{10} 301 = \frac{(-3)(-4)(-6)}{(-4)(-5)(-7)}(2.4771) + \frac{(1)(-4)(-6)}{(4)(-1)(-3)}(2.4829)$$
$$+ \frac{(1)(-3)(-6)}{(5)(1)(-2)}(2.4843) + \frac{(1)(-3)(-4)}{(7)(3)(2)}(2.4871)$$

$= 1.2739 + 4.9658 - 4.4717 + 0.7106$

$= 2.4786$

② If $f_1 = 4$, $f_3 = 12$, $f_4 = 19$ and $f_x = 7$ find x. by using Lagrange's inverse interpolation formula.

Sol: we have $f_1 = 4$, $f_3 = 12$, $f_4 = 19$

$f_x = 7$ as

i.e the given points are $(1, 4), (3, 12), (4, 19)$ and $(x, 7)$
$\quad x_0\ y_0 \quad x_1\ y_1 \quad x_2\ y_2 \quad x\ y$

now we find x by using the Lagrange's inverse interpolation formula.

$$x = \frac{(y-y_1)(y-y_2)}{(y_0-y_1)(y_0-y_2)} x_0 + \frac{(y-y_0)(y-y_2)}{(y_1-y_0)(y_1-y_2)} x_1 + \frac{(y-y_0)(y-y_1)}{(y_2-y_0)(y_2-y_1)} x_2$$

$$\Rightarrow x = \frac{(7-12)(7-19)}{(4-12)(4-19)}(1) + \frac{(7-4)(7-19)}{(12-4)(12-19)}(3) + \frac{(7-4)(7-12)}{(19-4)(19-12)}(4)$$

$$= \frac{(-5)(-12)}{(-8)(-15)}(1) + \frac{(3)(-12)}{(8)(-7)}(3) + \frac{(3)(-5)}{(15)(7)}(4)$$

$$= \frac{1}{2} + \frac{27}{14} - \frac{4}{7}$$

$$= \underline{\underline{1.86}}$$

→ The function $y = \sin x$ is obtained below:

x	0	$\pi/4$	$\pi/2$
$y = \sin x$	0	0.70711	1.0

Using Lagrange's interpolation formula, find the value of $\sin(\pi/6)$.

sol we have

$$\sin \pi/6 = \frac{\left(\frac{\pi}{6}-0\right)\left(\pi/6-\pi/2\right)}{\left(\frac{\pi}{4}-0\right)\left(\frac{\pi}{4}-\pi/2\right)}(0.70711) + \frac{\left(\frac{\pi}{6}-0\right)\left(\pi/6-\pi/4\right)}{\left(\pi/2-0\right)\left(\pi/2-\pi/4\right)}(1)$$

$$= \frac{8}{9}(0.70711) - \frac{1}{9}$$

$$= \underline{0.51743}$$

→ Using Lagrange's interpolation formula, find the form of the function $y(x)$ from the following table

x	0	1	3	4
y	-12	0	12	24

sol Since $y=0$ when $x=1$

$\therefore (x-1)$ is a factor

Let $y(x) = (x-1)R(x)$ —— (1)

then $R(x) = \dfrac{y(x)}{x-1}$

we now tabulate the values of $x, R(x)$

x	0	3	4
$R(x)$	12	6	8

Applying Lagrange's formula to the above table, we find

$$R(x) = \frac{(x-3)(x-4)}{(-3)(-4)}(12) + \frac{(x-0)(x-4)}{(3-0)(3-4)}(6)$$

$$+ \frac{(x-0)(x-3)}{(4-0)(4-3)}(8)$$

$$= (x-3)(x-4) - 2x(x-4) + 2x(x-3)$$

$$= x^2 - 5x + 12$$

Hence the reqd polynomial approximation to $y(x)$ is given by

$$\boxed{y(x) = (x-1)(x^2 - 5x + 12)}$$

→ 2002 10M — Find the cubic polynomial which takes the following values:— $y(0) = 1$, $y(1) = 0$, $y(2) = 1$ and $y(3) = 10$. Hence, or otherwise, obtain $y(4)$.

→ 2005 15M — Find the unique polynomial $p(x)$ of degree 2 or less such that $p(1) = 1$, $p(3) = 27$, $p(4) = 64$. Using the Lagrange's interpolation formula.

→ If $y(1) = -3$, $y(3) = 9$, $y(4) = 30$ and $y(6) = 132$, find the four-point Lagrange interpolation polynomial that takes the same values as the function y at the given points.

→ Given table of values

x	150	152	154	156
$y = \sqrt{x}$	12.247	12.329	12.410	12.490

Evaluate $\sqrt{155}$ using Lagrange's interpolation formula.

→ Show that $l_0(x) + l_1(x) + l_2(x) + l_3(x) = 1$ for all x

→ Applying Lagrange's formula, find a cubic polynomial which approximates the following data

x	-2	-1	2	3
$y(x)$	-12	-8	3	5

Note: Lagrange's interpolation formula can be used to express a rational function as a sum of partial fractions in the following way.

Let the given rational function be

$$\frac{3x^2+x+1}{(x-1)(x-2)(x-3)}.$$

We consider the numerator $f(x) = 3x^2+x+1$ and tabulate its values for $x = 1, 2, 3$. We get

x	1	2	3
$f(x) = 3x^2+x+1$	5	15	31

Using Lagrange's interpolation formula, we get

$$f(x) = \frac{(x-2)(x-3)}{(1-2)(1-3)}(5) + \frac{(x-1)(x-3)}{(2-1)(2-3)}(15) + \frac{(x-1)(x-2)}{(3-1)(3-2)}(31)$$

$$= \frac{5}{2}(x-2)(x-3) - 15(x-1)(x-3) + \frac{31}{2}(x-1)(x-2)$$

Hence

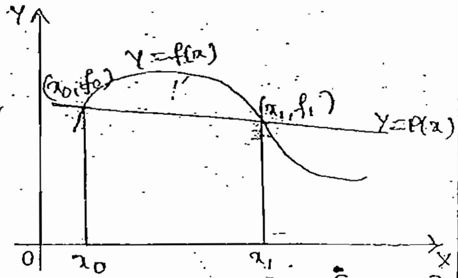
$$\frac{3x^2+x+1}{(x-1)(x-2)(x-3)} = \frac{5}{2(x-1)} - \frac{15}{(x-2)} + \frac{31}{2(x-3)}$$

→ Using Lagrange's interpolation formula, express the function $\frac{x^2+x-3}{x^3-2x^2-x+2}$ as sums of partial fractions.

→ Express the functions $\frac{x^2+6x+1}{(x-1)(x+1)(x-4)(x-6)}$ as sums of partial fractions.

Truncation Error Bounds:

The polynomial $P(x)$ coincides with the $f(x)$ at x_0 and x_1, and it deviates at all other points, in the interval (x_0, x_1) as shown in the figure.



This deviation is called the truncation error and may be written as $E_1(f; x) = f(x) - P(x)$ —— ①

We will now derive an expression for $E_1(f; x)$ for $x \in [x_0, x_1]$

we use the result ROLLE Theorem.

If $g(x)$ is continuous function $[a,b]$ and differentiable on (a,b) and if $g(a) = g(b) = 0$.

then there exist at least one point $\xi \in (a,b)$ such that $g'(\xi) = 0$

- If $x = x_0$ (or) $x = x_1$ then $E_1(f; x) = 0$
- If $x \in [a,b]$, $x \neq x_0, x_1$ be fixed then for this x, we define a function $g(t)$ as

$$g(t) = f(t) - P(t) - [f(x) - P(x)] \frac{(t-x_0)(t-x_1)}{(x-x_0)(x-x_1)} \quad \text{—— ②}$$

clearly $g(t) = 0$ at the three distinct points $t = x_0$, $t = x_1$ and $t = x$.

Differentiating ② twice with respect to 't', we get

$$g''(t) = f''(t) - \frac{2(f(x) - P(x))}{(x-x_0)(x-x_1)} \quad \text{—— ③}$$

Now by Rolle's Theorem,

$$g''(\xi) = 0$$

solving ③ for $f(x)$, we get

$$f(x) = P(x) + \tfrac{1}{2}(x-x_0)(x-x_1) f''(\xi) \quad \text{—— ④}$$

where $\text{Min}(x_0, x_1, x) < \xi < \text{Max}(x_0, x_1, x)$.

$$\Rightarrow f(x) - P(x) = \tfrac{1}{2}(x-x_0)(x-x_1) f''(\xi)$$

$$\boxed{E_1(f;x) = \tfrac{1}{2}(x-x_0)(x-x_1) f''(\xi)} \quad \text{—— (5)}$$

which is known as the truncation error in linear Interpolation.

If we determine a bound for $f''(x)$ in $[x_0, x_1]$

i.e. $|f''(x)| < M_2 \quad x \in [x_0, x_1]$

then $|f(x) - P(x)| = \left|\tfrac{1}{2}(x-x_0)(x-x_1) f''(\xi)\right|$

$$\leq \tfrac{1}{2} \max_{x_0 \leq x \leq x_1} \left|(x-x_0)(x-x_1) f''(\xi)\right|$$

$$\leq \tfrac{1}{2} \max_{x_0 \leq x \leq x_1} \left|(x-x_0)(x-x_1)\right| M_2 \quad \text{—— (6)}$$

The maximum value of $|(x-x_0)(x-x_1)|$ occurs at $x = \tfrac{1}{2}(x_0+x_1)$

and (6) becomes

$$\boxed{|f(x) - P(x)| \leq \tfrac{1}{8}(x_1 - x_0)^2 M_2} \quad \text{—— (7)}$$

Further the equation (7) may also be used to construct a table of values for a function $f(x)$ ~~for a function f(x)~~ for equally spaced nodal points $x_i = a + ih$, $i = 0, 1, 2, \ldots n$.

$h = \frac{b-a}{n}$;

So that the maximum truncation error using the linear interpolating polynomial $P(x)$ is less than given $\epsilon > 0$, we have

$$\frac{h^2}{8} \max_{a \leq x \leq b} |f''(x)| \leq \epsilon.$$

→ Using $\sin(0.1) = 0.09983$ and $\sin(0.2) = 0.19867$, find an approximate value of $\sin(0.15)$ by Lagrange interpolation. Obtain a bound on the truncation error.

Sol^n: We have

$$P_1(0.15) = \frac{(0.15-0.2)}{(0.1-0.2)}(0.09983) + \frac{(0.15-0.1)}{(0.2-0.1)}(0.19867)$$

$$=(0.5)(0.09983)+(0.5)(0.19867)$$

$$= 0.14925$$

The truncation error is

$$E_1(f;x) = \frac{(x-0.1)(x-0.2)}{2}(-\sin\xi)$$

[Let $f(x)=\sin x$, $f''(x)=-\sin x$]

where $0.1 < \xi < 0.2$

The maximum value of $|-\sin\xi|$, $\xi \in [0.1, 0.2]$ is $\sin(0.2) = 0.19867$.

$$\therefore |E_1(f;x)| \le \left|\frac{(0.15-0.1)(0.15-0.2)}{2}\right| (0.19867)$$

$$= (0.19867)(0.00125)$$

$$\approx 0.00025$$

$f''(\xi) = |-\sin\xi| = 0.19867$, $\xi \in [0.1, 0.2]$.

→ Determine the stepsize h to be used in tabulation of $f(x) = \sin x$ in the interval $[1,3]$ so that the linear interpolation will be correct to four decimal places.

Solⁿ: $f(x) = \sin x$

$\Rightarrow f'(x) = \cos x$

$\Rightarrow f''(x) = -\sin x$

Also $\max\limits_{1\le x\le 3} |-\sin x| = 1$

Hence we obtain

$$\frac{h^2}{8}(1) \le 5\times 10^{-5}$$

This gives

$$h \le 0.02$$

* Truncation error for Lagrange interpolating polynomial of degree n :-

we know that the Lagrange interpolating polynomial of degree n is

$$P(x) = \sum_{i=0}^{n} \frac{\pi_{n+1}(x)}{(x-x_i)\,\pi'_{n+1}(x_i)} \quad \text{—— (1)}$$

where $\Pi_{n+1}(x) = (x-x_0)(x-x_1) \cdots (x-x_n)$

The truncation error in the lagrange interpolation is given by

$E_n(f;x) = f(x) - P(x)$

Since $E_n(f;x) = 0$ at $x = x_i$, $i = 0, 1, 2 \cdots n$, then for $x \in [a,b]$ and $x \neq x_i$,

we define a function $g(t)$ as

$$g(t) = f(t) - P(t) - [f(x) - P(x)] \frac{(t-x_0)(t-x_1) \cdots (t-x_n)}{(x-x_0)(x-x_1) \cdots (x-x_n)} \quad \text{——— (2)}$$

clearly $g(t) = 0$ at $t = x$ and $t = x_i$, $i = 0, 1, 2 \cdots n$

Differentiating (2) $n+1$ times with respect to t, we get

$$g^{(n+1)}(t) = f^{(n+1)}(t) - \frac{(n+1)!\,[f(x) - P(x)]}{(x-x_0)(x-x_1) \cdots (x-x_n)} \quad \text{——— (3)}$$

By using the Rolle's theorem, $g^{(n+1)}(\xi) = 0$

Solving (3) for $f(x)$, we get

$$f(x) = P(x) + \frac{\Pi_{n+1}(x)}{(n+1)!} f^{(n+1)}(\xi)$$

where 'ξ' is some point in $[x_0, x_1, x_2 \cdots x_n, x]$.

Hence the truncation error in lagranges interpolation is

$$E_n(f;x) = \frac{\Pi_{n+1}(x)}{(n+1)!} f^{n+1}(\xi) \quad \text{——— (4)}$$

Note: The error formula (equation (4)) is an important theoretical result because lagrange interpolating polynomials are extensively used in deriving important formulae for numerical differentiation and numerical integration.

→ The following table gives the values of $f(x) = e^x$. If we fit an interpolating polynomial of degree four to the data, find the magnitude of the maximum possible error in the computed value of $f(x)$ when $x = 1.25$

x	1.2	1.3	1.4	1.5	1.6
$f(x)$	3.3201	3.6692	4.0552	4.4817	4.9530

Sol^n: The magnitude of the error associated with the 4^{th} degree polynomial approximation is given by

$$E_4(x) = (x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)\frac{f^5(\xi)}{5!}$$

$$|E_4(x)| = \left|(x-1.2)(x-1.3)(x-1.4)(x-1.5)(x-1.6)\frac{f^5(\xi)}{5!}\right| \quad \text{--- (1)}$$

Since $f(x) = e^x \Rightarrow f^{(5)}(x) = e^x$

$$\max_{1.2 \le x \le 1.6} |f^{(5)}(x)| = e^{1.6} = 4.9530 \quad \text{--- (2)}$$

Substituting (2) in (1) and putting $x = 1.25$, the upperbound on the magnitude of the error

$$= |(0.05)(-0.05)(-0.15)(-0.25)(-0.35)|\left(\frac{4.9530}{120}\right)$$

$$= 0.00000135.$$

→ Find the lagrange interpolating polynomial of degree 2 approximating the function $y = \log_e x$ defined by the following table of values. Hence determine the value of 2.7 and also obtain a bound on the error.

x	2	2.5	3.0
$y = \log_e x$	0.69315	0.91629	1.09861

Sol^n: Using the Lagrange's interpolation formula we have

$$P_2(x) = \frac{(x-2.5)(x-3)}{(-0.5)(-1.0)}(0.69315) + \frac{(x-2)(x-3)}{(0.5)(-0.5)}(0.91629) + \frac{(x-2)(x-2.5)}{(1)(0.5)}$$

$$= (2x^2-11x+15)(0.69315) - (4x^2-20x+24)(0.91629)$$
$$+(2x^2-9x+10)(1.09861)$$

$$= -0.08164x^2 + 0.81866x - 0.60761 \quad \text{——} \quad (i)$$

which is the required quadratic polynomial

putting $x = 2.7$ in the equation (i)

$$\log_e(2.7) = 0.9941164.$$

Since $y = \log_e x$

$$\Rightarrow y' = \frac{1}{x}$$

$$y'' = \frac{-1}{x^2} \text{ and } y''' = \frac{2}{x^3}$$

The magnitude of the error associated with 2nd degree polynomial approximation is given by

$$|E_2(x)| = \left| (x-x_0)(x-x_1)(x-x_2) \frac{f^{(3)}(\xi)}{3!} \right|$$

$$|E_2(x)| = \left| (2.7-2)(2.7-2.5)(2.7-3.0) \frac{f^{(3)}(\xi)}{3!} \right| \quad \text{——} \quad (2)$$

$$\max_{2 \le x \le 3} |f^{(3)}(x)| = \max_{2 \le x \le 3} \left| \frac{2}{x^3} \right|$$

$$= \frac{1}{4} \quad \text{——} \quad (3)$$

Substituting (3) in (2),

the magnitude of the error

$$|E_2(x)| \le \left| (0.7)(0.2)(0.3) \frac{1}{4} \times \frac{1}{6} \right|$$

$$\approx 0.00175$$

Note: Actual value $\log_e 2.7 = 0.9932518$

so that $|\text{error}| = |0.9941164 - 0.9932518|$

$$= 0.0008646 \quad (\text{Actual Error})$$

and we have $|E_2(x)| \le 0.00175$ (truncation error)

which agrees with the actual error.

Example : Using Lagrange's interpolation formula find a polynomial which passes the points (0, −12), (1, 0), (3, 6), (4, 12).

Solution : We have

$x_0 = 0,\ x_1 = 1,\ x_2 = 3,\ x_3 = 4,\ y_0 = f(x_0) = -12,\ y_1 = f(x_1) = 0,\ y_2 = f(x_2) = 6,$
$y_3 = f(x_3) = 12.$

Using Lagrange's interpolation formula we can write

$$f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)...(x_0-x_3)}f(x_0)+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}f(x_1)$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)...(x_2-x_3)}f(x_2)+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)...(x_3-x_2)}f(x_3).$$

$$f(x)=\frac{(x-1)(x-3)(x-4)}{(0-1)(0-3)(0-4)}\times(-12)+\frac{(x-0)(x-3)(x-4)}{(1-0)(1-3)(1-4)}\times 0$$

$$+\frac{(x-0)(x-1)(x-4)}{(3-0)(3-1)(3-4)}\times(6)+\frac{(x-0)(x-1)(x-3)}{(4-0)(4-1)(4-3)}\times(12)$$

$$=\frac{(x^3-8x^2+19x-12)}{12}\times 12+\frac{(x^3-5x^2+4x)}{(-6)}\times(6)+\frac{(x^3-4x^2+3x)}{(12)}\times 12$$

$\therefore f(x) = x^3 - 7x^2 + 18x - 12$

is the required polynomial.

Example : Using Lagrange's interpolation formula, find the value of y corresponding to $x = 10$ form the following table

x	5	6	9	11
$y = f(x)$	12	13	14	16

Solution : We have

$x_0 = 5,\ x_1 = 6,\ x_2 = 9,\ x_3 = 11,\ y_0 = 5,\ y_1 = 13,\ y_2 = 14,\ y_3 = 16.$

Using Lagrange's Interpolation formula we can write

$$y=f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3.$$

Substituting we get

$$f(10)=\frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)}\times(12)+\frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)}\times(13)$$

$$+\frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)}\times(14)+\frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)}\times(16)$$

$$=2-\frac{13}{3}+\frac{35}{3}+\frac{16}{3}=\frac{44}{3}$$

Exercise :

1. Find by Lagrange's formula the interpolation polynomial which corresponds to the following data

x	-1	0	2	5
$f(x)$	9	5	3	15

2. Find the polynomial degree three relevant to the following data

x	-1	0	1	2
$f(x)$	1	1	1	-3

3. Find the polynomial of the least degree which attains the prescribed values at the given points

x	-2	1	2	4
$f(x)$	25	-8	-15	-25

4. Compute $f(0.4)$ using the table

x	0.3	0.5	0.6
$f(x)$	0.61	0.69	0.72

5. The following table gives the sales of a concern for the five years. Estimate the sales for the years (a) 1986 (b) 1992

Year	1985	1987	1989	1991	1993
Sales (in thousands)	40	43	48	52	57

6. Compute $\sin 39^\circ$ form the table

x°	0	10	20	30	40
$\sin x^\circ$	0	0.1736	0.3420	0.5000	0.6428

7. Find $\log 5.15$ from the table

x	5.1	5.2	5.3	5.4	5.5
$\log_{10} x$	0.7076	0.7160	0.7243	0.7324	0.7404

8. Use Lagrange's interpolation formula and find $f(0.35)$

x	0.0	0.1	0.2	0.3	0.4
$f(x)$	1.0000	1.1052	1.2214	1.3499	1.4918

9. Use Lagrange's formula and compute

9. Use Lagrange's formula and compute

x	0.20	0.22	0.24	0.26	0.28	0.30
$f(x)$	1.6596	1.6698	1.6804	1.6912	1.7024	1.7139

10. Use Lagrange's interpolation formula to find the value of $f(x)$ for $x = 0$ given the following table

x	-1	-2	2	4
$f(x)$	-1	-9	11	69

Answers

1. $x^2 - 3x + 5$ 2. $-x^3 + x + 1$ 3. $x^2 - 10x + 1$ 4. 0.65 5. (a) 41.02 (b) 54.46
6. 0.6293 7. 0.7118 8. 1.4191 9. 1.6751 10. 1

Example : The following table gives the value of the elliptical integral

$$F(\phi)=\int_0^{\phi}\frac{d\phi}{\sqrt{1-\frac{1}{2}\sin^2\phi}}$$

for certain values of ϕ. Find the values of ϕ if $F(\phi)=0.3887$

ϕ	21°	23°	25°
$F(f)$	0.3706	0.4068	0.4433

Solution : We have

$\phi_0=21°, \phi_1=23°, \phi_2, 25°, F=0.3887, F_0=0.3706, F_1=0.4068$ and $F_2=0.4433$

Using the inverse interpolation formula we can write

$$\phi=\frac{(F-F_1)(F-F_2)}{(F_0-F_1)(F_0-F_2)}\phi_0+\frac{(F-F_0)(F-F_2)}{(F_1-F_0)(F_1-F_2)}\phi_1+\frac{(F-F_0)(F-F_1)}{(F_2-F_0)(F_2-F_1)}\phi_2,$$

$$\Rightarrow \phi=\frac{(0.3887-0.4068)(0.3887-0.4433)}{(0.3706-0.4068)(0.3706-0.4433)}\times 21$$

$$+\frac{(0.3887-0.3706)(0.3887-0.4433)}{(0.4068-0.3706)(0.4068-0.4433)}\times 23$$

$$+\frac{(0.3887-0.3706)(0.3887-0.4068)}{(0.4433-0.3706)(0.4433-0.4068)}\times 25$$

$$=7.884+17.20-3.087$$

$$=21.999\ 22$$

$$\therefore \phi=22°.$$

Example : Find the value of x when $y=0.3$ by applying Lagrange's formula inversely

x	0.4	0.6	0.8
y	0.3683	0.3332	0.2897

Solution : From Lagrange's inverse interpolation formula we get

$$x=\frac{(y-y_1)(y-y_2)}{(y_0-y_1)(y_0-y_2)}x_0+\frac{(y-y_0)(y-y_2)}{(y_1-y_0)(y_1-y_2)}x_1+\frac{(y-y_0)(y-y_1)}{(y_2-y_0)(y_2-y_1)}x_2.$$

Substituting $x_0=0.4$, $x_1=0.6$, $x_2=0.8$, $y_0=0.3683$, $y_1=0.3332$, $y_2=0.2899$ in the above formula, we get

$$x=\frac{(0.3-0.3332)(0.3-0.2897)}{(0.3683-0.3332)(0.3683-0.2897)}\times(0.4)$$

$$+\frac{(0.3-0.3683)(0.3-0.2897)}{(0.3332-0.3683)(0.3332-0.2897)}\times(0.6)$$

$$+\frac{(0.3-0.3683)(0.3-0.3332)}{(0.2897-0.3683)(0.2897-0.3332)}\times(0.8)$$

$$=0.757358.$$

Example : The following table gives the values of the probability integral $y=\frac{2}{\sqrt{\pi}}\int_0^{\pi} e^{-x^2}dx$ corresponding to certain values of x. For what value of x is this integral equal to $\frac{1}{2}$?

x	0.46	0.47	0.48	0.49
$y=\frac{2}{\sqrt{\pi}}\int_0^{\pi} e^{-x^2}dx$	0.484655	0.4937452	0.5027498	0.5116683

Solution : Here $x_0 = 0.46$, $x_1 = 0.47$, $x_2 = 0.48$, $x_3 = 0.48$, $y_0 = 0.484655$, $y_1 = 0.4937452$, $y_2 = 0.5027498$, $y_3 = 0.5116683$ and $y = \frac{1}{2}$.

From Lagrange's inverse interpolation formula,

$$x=\frac{(y-y_1)(y-y_2)(y-y_3)}{(y_0-y_1)(y_0-y_2)(y_0-y_3)}x_0+\frac{(y-y_0)(y-y_2)(y-y_3)}{(y_1-y_0)(y_1-y_2)(y_1-y_3)}x_1$$

$$+\frac{(y-y_0)(y-y_1)(y-y_3)}{(y_2-y_0)(y_2-y_1)(y_2-y_3)}x_2+\frac{(y-y_0)(y-y_1)(y-y_2)}{(y_3-y_0)(y_3-y_1)(y_3-y_2)}x_3.$$

$$\therefore\ x=\frac{(0.5-0.4937452)(0.5-0.5027498)(0.5-0.5116683)}{(0.4846555-0.4937452)(0.484655-0.5027498)(0.4846555-0.5116683)}\times 0.46$$

$$+\frac{(0.5-0.4846555)(0.5-0.5027498)(0.5-0.5116683)}{(0.4937452-0.4846555)(0.4937452-0.5027498)(0.4937452-0.5116683)}\times 0.47$$

$$+\frac{(0.5-0.4846555)(0.5-0.4937452)(0.5-0.5116683)}{(0.5027498-0.484655)(0.5027498-0.4937452)(0.5027498-0.5116683)}\times 0.48$$

$$+\frac{(0.5-0.484655)(0.5-0.4937452)(0.5-0.5027498)}{(0.5116683-0.484655)(0.5116683-0.4937452)(0.5-0.5027498)}\times 0.49$$

$$=-0.0207787+0.157737+0.369928-0.0299495=0.476937.$$

Example : Show that Lagrange's interpolation formula can be written in the form

$$f(x)=\sum_{r=0}^{r=n}\frac{\phi(x_r)}{(x-x_r)\phi^1(x_r)}$$

where $\phi(x)=(x-x_0)(x-x_1)\ldots(x-x_n)$

and $\phi^1(x_r)=\frac{d}{dx}[\phi(x)]$ at $x=x_r$

Solution : we have

$$\phi(x)=(x-x_0)(x-x_1)\ldots(x-x_n)$$
$$\phi'(x)=(x-x_1)(x-x_2)\ldots(x-x_n)$$
$$+(x-x_0)(x-x_2)\ldots(x-x_n)$$
$$+\ldots+(x-x_0)(x-x_1)\ldots(x-x_{r-1})(x-x_{r+1})\ldots(x-x_n)$$
$$+\ldots+(x-x_0)(x-x_1)\ldots(x-x_{n-1})$$

$$\therefore\ \phi'(x_r)=(x_r-x_0)(x_r-x_1)\ldots(x_r-x_{r-1})(x_r-x_{r+1})\ldots(x_r-x_n)$$

$$\therefore\ f(x)=\sum_{r=0}^{r=n}\frac{\phi(x)}{(x-x_r)\phi^1(x_r)}f(x_r)$$

Example : By means of lagrange's formula prove that

(i) $y_1=y_3-0.3[y_5-y_{-3}]+0.2[y_{-3}-y_{-5}]$

(ii) $y_o=\frac{1}{2}(y_1+y_{-1})-\frac{1}{8}\left[\frac{1}{2}(y_3-y_1)-\frac{1}{2}(y_{-1}-y_{-3})\right]$

Solution : (i) y_{-5}, y_{-3}, y_3 and y_5 are given, therefore the values of the arguments are -5, -3, 3, and 5, y_1 is to be obtained. By lagrange's formula

$$y_x=\frac{[x-(-3)](x-3)(x-5)}{[-5-(-3)][-5-3](-5-5)}y_{-5}+\frac{[x-(-5)](x-3)(x-5)}{[-3-(-5)](-3-3)(-3-5)}y_{-3}$$
$$+\frac{[x-(-5)][x-(-3)](x-5)}{[3-(-5)][3-(-3)](3-5)}y_3+\frac{[x-(-5)][x-(-3)](x-3)}{[5-(-5)][5-(-3)](5-3)}y_5$$

Taking $x=1$, we get

$$y_1=\frac{(1+3)(1-3)(1-5)}{(-5+3)(-5-3)(-5-5)}y_{-5}+\frac{(1+5)(1-3)(1-5)}{(-3+5)(-3-3)(-3-5)}y_{-3}$$
$$+\frac{(1+5)(1+3)(1-5)}{(3+5)(3+3)(3-5)}y_3+\frac{(1+5)(1+3)(1-3)}{(5+5)(5+3)(5-3)}y_5$$

$$\Rightarrow y_1 = \frac{(4)(-2)(-4)}{(-2)(-8)(-10)}y_{-5} + \frac{(6)(-2)(-4)}{(2)(-6)(-8)}y_{-3} + \frac{(6)(4)(-4)}{(8)(6)(-2)}y_3 + \frac{(6)(4)(-2)}{(10)(8)(2)}y_5$$

$$= \frac{y_{-5}}{5} + \frac{y_{-3}}{2} + y_3 - \frac{3}{10}y_5$$

$$= y_3 + 0.2y_{-5} + 0.5y_{-3} - 0.3y_5$$

$$= y_3 + 0.2y_{-5} + 0.2y_{-3} + 0.3y_{-3} - 0.3y_5$$

$$y_1 = y_3 - 0.3(y_5 - y_{-3}) + 0.2(y_{-3} - y_{-5})$$

(ii) y_{-3}, y_{-1}, y_1, and y_3 are given, y_0 is to be obtained. By Lagrange's formula

$$y_0 = \frac{(0+1)(0-1)(0-3)}{(-3+1)(-3-1)(-3-3)}y_{-3} + \frac{(0+3)(0-1)(0-3)}{(-1+3)(-1-1)(-1-3)}y_{-1}$$

$$+ \frac{(0+3)(0+1)(0-3)}{(1+3)(1+1)(1-3)}y_1 + \frac{(0+3)(0+1)(0-1)}{(3+3)(3+1)(3-1)}y_3$$

$$= -\frac{1}{16}y_{-3} + \frac{9}{16}y_{-1} + \frac{9}{16}y_1 - \frac{1}{16}y_3$$

$$= \frac{1}{2}(y_1 + y_{-1}) - \frac{1}{16}\left[(y_3 - y_1) - (y_{-1} - y_{-3})\right]$$

$$\therefore y_0 = \frac{1}{2}(y_1 + y_{-1}) - \frac{1}{8}\left[\frac{1}{2}(y_3 - y_1) - \frac{1}{2}(y_{-1} - y_{-3})\right]$$

Example : The values of $f(x)$ are given at a, b, and c. Show that the maximum is obtained by

$$x = \frac{f(a)\cdot(b^2 - c^2) + f(b)\cdot(c^2 - a^2) + f(c)\cdot(a^2 - b^2)}{2[f(a)\cdot(b-c) + f(b)\cdot(c-a) + f(c)\cdot(a-b)]}$$

Solution : By lagrange's formula $f(x)$ for the arguments a, b, and c is given by-

$$f(x) = \frac{(x-b)(x-c)}{(a-b)(a-c)}f(a) + \frac{(x-a)(x-c)}{(b-a)(b-c)}f(b)$$

$$+ \frac{(x-a)(x-b)}{(c-a)(c-b)}f(c)$$

$$= \frac{x^2 - (b+c)x + bc}{(a-b)(a-c)}f(a) + \frac{x^2 - (c+a)x + ca}{(b-c)(b-a)}f(b) + \frac{x^2 - (a+b)x + ab}{(c-a)(c-b)}f(c)$$

for maximum or minimum we have $f'(x)=0$

$$\therefore \frac{2x-(b+c)}{(a-b)(c-a)}f(a)-\frac{2x-(a+c)}{(a-b)(b-c)}f(b)-\frac{2x-(a+b)}{(b-c)(c-a)}f(c)=0$$

$$\Rightarrow 2x[(b-c)f(a)+(c-a)f(b)+(a-b)f(c)]$$

$$-[(b^2-c^2)f(a)+(c^2-a^2)f(b)+(a^2-b^2)f(c)]=0$$

$$\therefore x=\frac{(b^2-c^2)f(a)+(c^2-a^2)f(b)+(a^2-b^2)f(c)}{2[(b-c)f(a)+(c-a)f(b)+(a-b)f(c)]}$$

Example : Given $\log_{10}654 = 2.8156$, $\log_{10}658 = 2.8182$, $\log_{10}659 = 2.8189$, $\log_{10}661 = 2.8202$, find $\log_{10}656$.

Solution : Here $x_0 = 654$, $x_1 = 658$, $x_2 = 659$, $x_3 = 661$ and $f(x) = \log_{10}656$.

By Lagrange's formula we have

$$f(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0+\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$

$$+\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2+\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$\therefore \log_{10}656=\frac{(656-658)(656-659)(656-661)}{(654-658)(654-659)(654-661)}\times(2.8156)$$

$$+\frac{(656-654)(656-659)(656-661)}{(658-654)(658-659)(658-661)}\times(2.8182)$$

$$+\frac{(656-654)(656-658)(656-661)}{(659-654)(659-658)(659-661)}\times(2.8189)$$

$$+\frac{(656-654)(656-658)(656-659)}{(661-654)(661-658)(661-659)}\times(2.8202)$$

$$=\frac{3}{14}(2.8156)+\frac{5}{2}(2.8182)-2(2.8189)+\frac{2}{7}(2.8202)$$

$$=0.6033+7.045-5.6378+0.8057$$

$$=2.8170$$

Example : Write down the Lagrange's polynomial passing through (x_0, f_0), (x_1, f_1) and (x_2, f_2). Hence express $\dfrac{3x^2+x+1}{(x-1)(x-2)(x-3)}$ as sum of partial fractions.

Solution : The Lagrange's polynomial through the points (x_0, f_0), (x_1, f_1) and (x_2, f_2) is given by

$$f(x)=\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}\times f_0+\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}\times f_1+\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}\times f_2$$

Consider the numerator $3x^2+x+1$.

Let

$$f(x) = 3x^2+x+1,$$

tabulating the values of $f(x)$ for $x = 1, 2, 3$ we get

x	1	2	3
$f(x)$	5	15	31

Using Lagrange's formula, we get

$$f(x)=\frac{(x-2)(x-3)}{(1-2)(1-3)}\times 5+\frac{(x-1)(x-3)}{(2-1)(2-3)}\times 15+\frac{(x-1)(x-2)}{(3-1)(3-2)}\times 31$$

$$=\frac{5}{2}(x-2)(x-3)-15(x-1)(x-3)+\frac{31}{2}(x-1)(x-2)$$

$$\therefore \quad \frac{3x^2+x+1}{(x-1)(x-2)(x-3)}=\frac{5}{2}(x-2)(x-3)-15(x-1)(x-3)+\frac{31}{2}(x-1)(x-2).$$

Exercise

1. Show that the sum of coefficients of y_i s' in the Lagrange's interpolation formula is unity.

2. Given u_{-1}, y_0, u_1, and u_2. Using Lagrange's formula show that

$$u_x = yu_0 + xu_1 + \frac{y(y^2-1)}{3!}\Delta^2 u_{-1} + \frac{x(x^2-1)}{3!}\Delta^2 u_0 \quad \text{where } x+y=1.$$

3. If $y_0, y_1, y_2, \ldots, y_6$ are the consecutive terms of a series then prove that

$$y_3 = 0.05(y_0+y_6) - 0.3(y_1+y_5) + 0.75(y_2+y_4).$$

4. If all terms except y_5 of the sequence $y_1, y_2, y_3, ..., y_9$ be given, show that the value of y_5 is

$$\left[\frac{156(y_4+y_6)-28(y_3+y_7)+8(y_2+y_8)-(y_1+y_9)}{70}\right]$$

5. The following table is given.

x	0	1	2	5
$f(x)$	2	3	12	147

show that the form of $f(x)$ is $x^3 + x^2 - x + 2$.

6. Using Lagrange's interpolation formula, express the function

$$\frac{x^2+x-3}{x^3-2x^2-x+2}$$ as sum of partial fractions.

7. Express the function $\dfrac{x^2+6x+1}{(x-1)(x+1)(x-4)(x-6)}$ as sum of partial functions.

8. Using Lagrange's formula show that

a. $$\frac{x^3-10x+13}{(x-1)(x-2)(x-3)}=\frac{2}{(x-1)}+\frac{3}{(x-2)}-\frac{4}{(x-3)}$$

b. $$\frac{x^2+6x+1}{(x^2-1)(x-4)(x+6)}=\frac{-2}{25(x+1)}-\frac{4}{21(x-1)}+\frac{41}{150(x-4)}-\frac{1}{350(x+6)}$$

9. The following values of the function $f(x)$ for values of x are given: $f(1) = 4$, $f(2) = 5$, $f(7) = 5$, $f(8) = 4$. Find the values of $f(6)$ and also the value of x for which $f(x)$ is maximum or minimum.

Answers

6. $$\frac{1}{2(x-1)}-\frac{1}{(x-2)}-\frac{1}{2(x+1)}$$

7. $$\frac{2}{35(x+1)}+\frac{4}{15(x-1)}-\frac{41}{30(x-4)}+\frac{73}{70(x-6)}$$

9. $f(6) = 5.66$, maximum at $x = 4.5$

△ △ △

7.13. DIVIDED DIFFERENCES

The Lagrange's formula has the drawback that if another interpolation value were inserted, then the interpolation coefficients are required to be recalculated. This labour of recomputing the interpolation coefficients is saved by using Newton's general interpolation formula which employs what are called 'divided differences'. Before deriving this formula, we shall first define these differences.

If $(x_0, y_0), (x_1, y_1), (x_2, y_2), \ldots$ be given points, then the *first divided difference* for the arguments x_0, x_1 is defined by the relation $[x_0, x_1] = \dfrac{y_1 - y_0}{x_1 - x_0}$.

Similarly $[x_1, x_2] = \dfrac{y_2 - y_1}{x_2 - x_1}$ and $[x_2, x_3] = \dfrac{y_3 - y_2}{x_3 - x_2}$ etc.

The *second divided difference* for x_0, x_1, x_2 is defined as $[x_0, x_1, x_2] = \dfrac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}$.

The *third divided difference* for x_0, x_1, x_2, x_3 is defined as

$$[x_0, x_1, x_2, x_3] = \frac{[x_1, x_2, x_3] - [x_0, x_1, x_2]}{x_3 - x_0} \text{ and so on.}$$

Divided difference (d.d) Table.

		first d.d	second d.d	third d.d
x_0	y_0			
		$[x_0, x_1]$		
x_1	y_1		$[x_0, x_1, x_2]$	
		$[x_1, x_2]$		$[x_0, x_1, x_2, x_3]$
x_2	y_2		$[x_1, x_2, x_3]$	
		$[x_2, x_3]$		
x_3	y_3			

Obs. 1. *The divided differences are symmetrical in their arguments i.e. independent of the order of the arguments.* For it is easy to write $[x_0, x_1] = \dfrac{y_0}{x_0 - x_1} + \dfrac{y_1}{x_1 - x_0} = [x_1, x_0]$, $[x_0, x_1, x_2]$

$$= \frac{y_0}{(x_0 - x_1)(x_0 - x_2)} + \frac{y_1}{(x_1 - x_0)(x_1 - x_2)} + \frac{y_2}{(x_2 - x_0)(x_2 - x_1)}$$

$= [x_1, x_2, x_0]$ or $[x_2, x_0, x_1]$ and so on.

Obs. 2. *The nth divided differences of a polynomial of the nth degree are constant.*

Let the arguments be equally spaced so that $x_1 - x_0 = x_2 - x_1 = \ldots = x_n - x_{n-1} = h$. Then

$$[x_0, x_1] = \frac{y_1 - y_0}{x_1 - x_0} = \frac{\Delta y_0}{h}$$

$$[x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0 - x_1]}{x_2 - x_0} = \frac{1}{2h}\left\{\frac{\Delta y_1}{h} - \frac{\Delta y_0}{h}\right\}$$

$$= \frac{1}{2!\,h^2}\,\Delta^2 y_0 \text{ and in general, } [x_0, x_1, \ldots, x_n] = \frac{1}{n!\,h^n}\,\Delta^n y_0.$$

If the tabulated function is a nth degree polynomial, then $\Delta^n y_0$ will be constant. Hence the nth divided differences will also be constant.

7.14. NEWTON'S DIVIDED DIFFERENCE FORMULA

Let $y_0, y_1, \ldots, y_n$ be the values of $y = f(x)$ corresponding to the arguments $x_0, x_1, \ldots, x_n$. Then from the definition of divided differences, we have

$$[x, x_0] = \frac{y - y_0}{x - x_0}$$

so that
$$y = y_0 + (x - x_0)[x, x_0] \qquad \ldots(1)$$

Again
$$[x, x_0, x_1] = \frac{[x, x_0] - [x_0, x_1]}{x - x_1}$$

which gives
$$[x, x_0] = [x_0, x_1] + (x - x_1)[x, x_0, x_1]$$

Substituting this value of $[x, x_0]$ in (1), we get

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x, x_0, x_1] \qquad \ldots(2)$$

Also
$$[x, x_0, x_1, x_2] = \frac{[x, x_0, x_1] - [x_0\, x_1, x_2]}{x - x_2}$$

which gives
$$[x, x_0, x_1] = [x_0, x_1, x_2] + (x - x_2)[x, x_0, x_1, x_2]$$

Substituting this value of $[x, x_0, x_1]$ in (2), we obtain

$$y = y_0 + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + (x - x_0)(x - x_1)(x - x_2)[x, x_0, x_1, x_2]$$

Proceeding in this manner, we get

$$\begin{aligned} y = f(x) = y_0 &+ (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] \\ &+ (x - x_0)(x - x_1)(x - x_2)[x_0, x_1, x_2, x_3] + \ldots \\ &+ (x - x_0)(x - x_1)\ldots(x - x_n)[x, x_0, x_1, \ldots x_n] \end{aligned} \qquad \ldots(3)$$

which is called *Newton's general interpolation formula with divided differences.*

Example 7.12. *Given the values*

x :	5	7	11	13	17
$f(x)$:	150	392	1452	2366	5202,

evaluate f(9), using (i) Lagrange's formula

(ii) Newton's divided difference formula. (*P.T.U., B. Tech., 2005*)

Sol. (*i*) Here $x_0 = 5$, $x_1 = 7$, $x_2 = 11$, $x_3 = 13$, $x_4 = 17$

and $y_0 = 150, y_1 = 392, y_2 = 1452, y_3 = 2366, y_4 = 5202$.

Putting $x = 9$ and substituting the above values in Lagrange's formula, we get

$$f(9) = \frac{(9-7)(9-11)(9-13)(9-17)}{(5-7)(5-11)(5-13)(5-17)} \times 150 + \frac{(9-5)(9-11)(9-13)(9-17)}{(7-5)(7-11)(7-13)(7-17)} \times 392$$

$$+ \frac{(9-5)(9-7)(9-13)(9-17)}{(11-5)(11-7)(11-13)(11-17)} \times 1452$$

$$+ \frac{(9-5)(9-7)(9-11)(9-17)}{(13-5)(13-7)(13-11)(13-17)} \times 2366$$

$$+ \frac{(9-5)(9-7)(9-11)(9-13)}{(17-5)(17-7)(17-11)(17-13)} \times 5202$$

$$= -\frac{50}{3} + \frac{3136}{15} + \frac{3872}{3} - \frac{2366}{3} + \frac{578}{5} = 810$$

(*ii*) The divided differences table is

x	y	*1st divided differences*	*2nd divided differences*	*3rd divided differences*
5	150			
		$\frac{392-150}{7-5} = 121$		
7	392		$\frac{265-121}{11-5} = 24$	
		$\frac{1452-392}{11-7} = 265$		$\frac{32-24}{13-5} = 1$
11	1452		$\frac{457-265}{13-7} = 32$	
		$\frac{2366-1452}{13-11} = 457$		$\frac{42-32}{17-7} = 1$
13	2366		$\frac{709-457}{17-11} = 42$	
		$\frac{5202-2366}{17-13} = 709$		
17	5202			

Taking $x = 9$ in the Newton's divided difference formula, we obtain

$$f(9) = 150 + (9-5) \times 121 + (9-5)(9-7) \times 24 + (9-5)(9-7)(9-11) \times 1$$

$$= 150 + 484 + 192 - 16 = 810.$$

Example 7.13. *Using Newton's divided differences formula, evaluate f(8) and f(15) given :*

x :	4	5	7	10	11	13
$f(x)$:	48	100	294	900	1210	2028

(V.T.U., B. Tech., 2008)

Sol. The divided differences table is

x	$f(x)$	1st divided differences	2nd divided differences	3rd divided differences	4th divided differences
4	48				
		52			
5	100		15		
		97		1	
7	294		21		0
		202		1	
10	900		27		0
		310		1	
11	1210		33		
		409			
13	2028				

Taking $x = 8$ in the Newton's divided difference formula, we obtain

$$f(8) = 48 + (8-4)\,52 + (8-4)(8-5)\,15 + (8-4)(8-5)(8-7)\,1$$
$$= 448.$$

Similarly $f(15) = 3150.$

Example 7.14. *Determine f(x) as a polynomial in x for the following data :*

x :	-4	-1	0	2	5
$f(x)$:	1245	33	5	9	1335.

(V.T.U., B. Tech., 2007)

Sol. The divided differences table is

x	$f(x)$	1st divided differences	2nd divided differences	3rd divided differences	4th divided differences
-4	1245				
		-404			
-1	33		94		
		-28		-14	
0	5		10		3
		2		13	
2	9		88		
		442			
5	1335				

Applying Newton's divided difference formula

$$f(x) = f(x_0) + (x - x_0)[x_0, x_1] + (x - x_0)(x - x_1)[x_0, x_1, x_2] + \ldots$$
$$= 1245 + (x+4)(-404) + (x+4)(x+1)(94)$$
$$+ (x+4)(x+1)(x-0)(-14) + (x+4)(x+1)x(x-2)(3)$$
$$= 3x^4 - 5x^3 + 6x^2 - 14x + 5.$$

Numerical Integration

Consider a function of single variable $y=f(x)$. If the function is known and simple, we can easily evaluate its definite integral. However, if we do not know the function as such or the function is complicated such as $f(x)=e^{-x^2}$, $f(x)=\frac{\sin x}{x}$ which have no anti-derivatives expressible interms of elementary functions and is given in a tabular form at a set of points $x_0, x_1, \ldots x_n$, we use only numerical methods for integration of the function.

To evaluate the integral, we fit up a suitable interpolation polynomial to the given set of values of $f(x)$ and then integrate it within the definite limits. Here we integrate an approximate interpolation formula instead of $f(x)$. When this technique is applied on a function of single variable, the process is called Quadrature.

We have studied several interpolation formulas which fits the given data (x_k, f_k) $k=0,1,2\ldots n$. So, the different integration formulae can be obtained depending upon the type of the interpolation formula we used.

Now we derive a general quadrature formula for numerical integration using Newton's forward difference formula.

Newton-cotes formula (A general quadrature formula for equidistant ordinates):-

Consider the integral $I = \int_a^b f(x)\,dx.$ —— (1)

where $f(x)$ takes the values

$f(x_0) = y_0$, $f(x_0+h) = y_1$, $f(x_0+2h) = y_2$, ----

$f(x_0+nh) = y_n$ when $x = x_0$, $x = x_0+h$,

$x = x_0+2h$, ---- $x = x_0+nh$ respectively.

To evaluate I, we replace $f(x)$ by a suitable interpolation formula.

Let the interval $[a,b]$ be divided into n sub-intervals of width h so that $x_0 = a$, $x_1 = x_0+h$, $x_2 = x_0+2h$, ---- ---- $x_n = x_0+nh = b$.

Approximating $f(x)$ by Newton's forward interpolation formula we can write the integral (1) as

$$I = \int_{x_0}^{x_n} f(x)\,dx.$$

$$= \int_{x_0}^{x_0+nh} f(x)\,dx$$

$$= \int_{x_0}^{x_0+nh} \left[y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots \right] dx \quad \text{--- (2)}$$

Since $p = \frac{x - x_0}{h}$

$\Rightarrow x = x_0 + ph$

$\Rightarrow dx = h\,dp$

when $x = x_0$, $p = 0$

and when $x = x_0 + nh$, $p = n$.

Equation (2) can be written as

$$I = h\int_0^n \left[y_0 + p\Delta y_0 + \frac{p^2 - p}{2!}\Delta^2 y_0 + \frac{p^3 - 3p^2 + 2p}{3!}\Delta^3 y_0 + \cdots \right] dp$$

$$= h\left[py_0 + \frac{p^2}{2}\Delta y_0 + \frac{\frac{p^3}{3} - \frac{p^2}{2}}{2!}\Delta^2 y_0 + \frac{1}{6}\left(\frac{p^4}{4} - p^3 + p^2\right)\Delta^3 y_0 + \cdots \right]_0^n$$

$$= h\left[ny_0 + \frac{n^2}{2}\Delta y_0 + \frac{1}{2}\left(\frac{n^3}{3} - \frac{n^2}{2}\right)\Delta^2 y_0 + \frac{1}{6}\left(\frac{n^4}{4} - n^3 + n^2\right)\Delta^3 y_0 + \cdots \right].$$

This is known as Newton-cotes quadrature formula. From this general formula, we can obtain different integration formulae by putting $n = 1, 2, 3, \ldots$ etc.

Trapezoidal Rule :-

putting $n=1$, in the quadrature formula, and taking the curve through (x_0, y_0) and (x_1, y_1) as a straight line i.e, a polynomial of first order so that differences of order higher than first become zero.

we get
$$\int_{x_0}^{x_0+h} f(x)\,dx = h\left[y_0 + \tfrac{1}{2}\Delta y_0\right]$$
$$= h\left[y_0 + \tfrac{1}{2}(y_1 - y_0)\right]$$
$$= \frac{h}{2}(y_0 + y_1)$$

for the next interval $[x_1, x_2]$, we deduce

Similarly
$$\int_{x_1}^{x_2} f(x)\,dx = \int_{x_0+h}^{x_0+2h} f(x)\,dx$$
$$= \frac{h}{2}\left(y_1 + \tfrac{1}{2}\Delta y_1\right)$$
$$= \frac{h}{2}(y_1 + y_2)$$

and so on.

for the last interval $[x_{n-1}, x_n]$,

we have
$$\int_{x_{n-1}}^{x_n} f(x)\,dx = \int_{x_0+(n-1)h}^{x_0+nh} f(x)\,dx$$
$$= \frac{h}{2}(y_{n-1} + y_n)$$

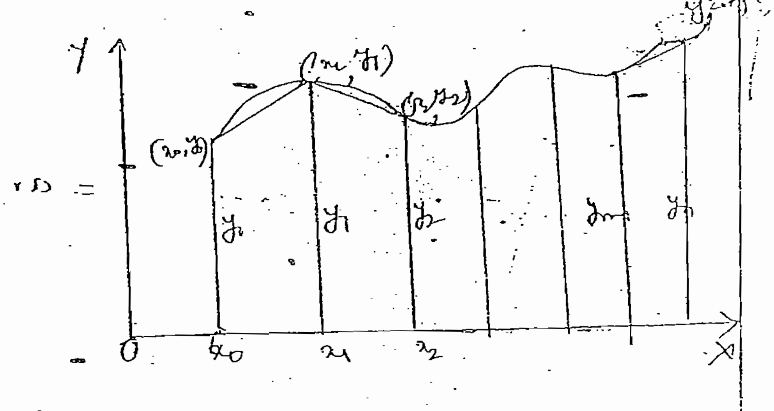
Adding these n integrals, we obtain

$$\int_{x_0}^{x_0+nh} f(x)\,dx = \frac{h}{2}\left[(y_0+y_n)+2(y_1+y_2+\cdots y_{n-1})\right]$$

which is known as the trapezoidal rule.

Geometrical interpretation:

The geometrical significance of this rule is that the curve $y=f(x)$ is replaced by n straight lines joining the points (x_0, y_0), and (x_1, y_1); (x_1, y_1) and (x_2, y_2); ----; (x_{n-1}, y_{n-1}) and (x_n, y_n). The area bounded by the curve $y=f(x)$, the ordinates $x=x_0$ and $x=x_n$, and the x-axis is then approximately equivalent to the sum of the areas of the n trapeziums obtained.



The error of the trapezoidal formula can be obtained in the following way:

- Let $f(x)$ be continuous, well-behaved and possess continuous derivatives in $[x_0, x_n]$.

Expanding y in a Taylor's series around $x=x_0$,

$$y(x) = y_0 + \frac{(x-x_0)}{1!} y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \cdots \quad \text{--- (1)}$$

- where $y_0' = [y'(x)]_{x=x_0}$.

$$\int_{x_0}^{x_1} y\,dx = \int_{x_0}^{x_1} \left[y_0 + \frac{(x-x_0)}{1!} y_0' + \frac{(x-x_0)^2}{2!} y_0'' + \cdots \right] dx$$

$$= \left[y_0 x + \frac{(x-x_0)^2}{2!} y_0' + \frac{(x-x_0)^3}{3!} y_0'' + \cdots \right]_{x_0}^{x_1}$$

$$= y_0 (x_1 - x_0) + \frac{(x_1-x_0)^2}{2!} y_0' + \frac{(x_1-x_0)^3}{3!} y_0'' + \cdots$$

$$= h y_0 + \frac{h^2}{2!} y_0' + \frac{h^3}{3!} y_0'' + \cdots \quad \text{--- (2)}$$

where h is the equal interval length.

Also $\int_{x_0}^{x_1} y\,dx = \frac{h}{2}(y_0 + y_1)$

= area of the first trapezium

$= A_0$. ——— (3)

Putting $x = x_1$ in (1), we get

$$y(x_1) = y_1 = y_0 + \frac{(x_1-x_0)}{1!} y_0' + \frac{(x_1-x_0)^2}{2!} y_0'' + \cdots$$

$$\Rightarrow y_1 = y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \cdots \quad \text{--- (4)}$$

$\therefore$ From (3) & (4), we have

$$A_0 = \frac{h}{2}\left[y_0 + y_0 + \frac{h}{1!} y_0' + \frac{h^2}{2!} y_0'' + \cdots\right]$$

$$= h y_0 + \frac{h^2}{2} y_0' + \frac{h^3}{2\times 2!} y_0'' + \cdots$$

Subtracting A_0 value from (2)

$$\int_{x_0}^{x_1} y\,dx - A_0 = h^3 y_0''\left[\frac{1}{3!} - \frac{1}{2\times 2!}\right] + \cdots$$

$$= -\frac{1}{12} h^3 y_0'' + \cdots$$

which is the error in the first interval $[x_0, x_1]$ (neglecting the higher powers of h)

proceeding in a similar manner we obtain the errors in the remaining subintervals, viz., $[x_1, x_2]$, $[x_2, x_3]$, ... and $[x_{n-1}, x_n]$.

Thus we have

$$E = -\frac{1}{12} h^3 \left[y_0'' + y_1'' + \cdots + y_{n-1}''\right]$$

where E is the total error.

Assuming that $y''(\bar{x})$ is the largest value of the n quantities y_0'', y_1'', ... y_{n-1}''

we obtain

$$E = -\frac{1}{12} h^3 n y''(\bar{x})$$

$$= -\frac{(b-a)}{12} h^2 y''(\bar{x}) \quad (\because nh = b-a)$$

Hence, the error in the trapezoidal rule is of the order h^2.

Simpson's one-third Rule :-

putting $n=2$ in the quadrature formula and taking the curve through (x_0, y_0), (x_1, y_1) and (x_2, y_2) as a parabola. i.e, a polynomial of second order so that differences of order higher than second vanish, we get

$$\int_{x_0}^{x_2} f(x)\,dx = \int_{x_0}^{x_0+2h} f(x)\,dx$$

$$= h\left[2y_0 + \frac{4}{2}\Delta y_0 + \frac{1}{2}\left(\frac{8}{3} - \frac{4}{2}\right)\Delta^2 y_0\right]$$

$$= h\left[2y_0 + 2(y_1 - y_0) + \frac{1}{3}\Delta^2 y_0\right]$$

$$= h\left[2y_1 + \frac{1}{3}(y_2 - 2y_1 + y_0)\right]$$

$$= \frac{h}{3}\left[y_0 + 4y_1 + y_2\right] \qquad \left(\because \Delta^2 y_0 = \Delta y_1 - \Delta y_0 = y_2 - y_1 - (y_1 - y_0) = y_2 - 2y_1 + y_0\right)$$

Similarly

$$\int_{x_2}^{x_4} f(x)\,dx = \frac{h}{3}\left[y_2 + 4y_3 + y_4\right]$$

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$$\int_{x_{n-2}}^{x_n} f(x)\,dx = \frac{h}{3}\left[y_{n-2} + 4y_{n-1} + y_n\right], \text{ n being even.}$$

Adding all these integrals, if n is even +ve integer. i.e, the number of ordinates $y_0, y_1, y_2, \ldots y_n$ is odd, we have

$$\int_{x_0}^{x_n} f(x)\,dx = \int_{x_0}^{x_0+nh} f(x)\,dx$$

$$= \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \cdots + \int_{x_{n-2}}^{x_n} f(x)\,dx.$$

$$= \frac{h}{3}\left[(y_0+4y_1+y_2)+(y_2+4y_3+y_4)+(y_4+4y_5+y_6)\right.$$

$$\left.+\cdots\cdots+(y_{n-2}+4y_{n-1}+y_n)\right]$$

$$= \frac{h}{3}\left[(y_0+y_n)+2(y_2+y_4+\cdots y_{n-2})+4(y_1+y_3+\cdots+y_{n-1})\right]$$

$= \frac{h}{3}$ [sum of the first and last ordinates + 4 (sum of even ordinates) + 2 (sum of remaining odd ordinates)]

which is known as Simpson's $\frac{1}{3}$ rule. Simply Simpson's rule.

Note:

1. Though y_2 has suffix even, it is the third ordinate (odd).
2. This rule requires the division of the whole range into an even number of subintervals of width 'h'.

Error in Simpson's formula:

Following the method outlined as in case of Trapezoidal rule, it can be shown that the error in Simpson's rule is given by

$$\boxed{E = \frac{-nh^5}{90}\, y^{iv}(\bar{x})}$$

where $y^{iv}(\bar{x})$ is the largest value of the fourth derivatives.

$$\boxed{E = \frac{-(b-a)}{180}\, h^4\, y^{iv}(\bar{x})}$$

$\left(\because (2n)h = b-a \Rightarrow nh = \frac{b-a}{2}\right)$

Hence, the error in Simpson's one-third rule is of the order h^4.

Simpson's 3/8 - Rule :-

putting $n=3$ in the quadrature formula and taking the curve through (x_i, y_i) ; $i = 0, 1, 2, 3$. as a polynomial of third order so that differences above the third order vanish.

we get

$$\int_{x_0}^{x_3} f(x)dx = \int_{x_0}^{x_0+3h} f(x)dx$$

$$= 3h\left(y_0 + \frac{3}{2}\Delta y_0 + \frac{3}{4}\Delta^2 y_0 + \frac{1}{8}\Delta^3 y_0\right).$$

$$= \frac{3h}{8}\left(y_0 + 3y_1 + 3y_2 + y_3\right).$$

Similarly,

$$\int_{x_0+3h}^{x_0+6h} f(x)dx = \frac{3h}{8}\left(y_3 + 3y_4 + 3y_5 + y_6\right)$$

and so on.

Adding all these integrals from x_0 to x_0+nh where 'n' is a multiple of 3.

we obtain

$$\int_{x_0}^{x_0+nh} f(x)dx = \int_{x_0}^{x_0+3h} f(x)dx + \int_{x_0+3h}^{x_0+6h} f(x)dx + \cdots$$

$$+ \int_{x_0+(n-3)h}^{x_0+nh} f(x)dx.$$

$$= \frac{3h}{8}\Big[\left(y_0 + 3y_1 + 3y_2 + y_3\right) + \left(y_3 + 3y_4 + 3y_5 + y_6\right)$$
$$+ \cdots + \left(y_{n-3} + 3y_{n-2} + 3y_{n-1} + y_n\right)\Big]$$

$$= \frac{3h}{8}\Big[(y_0+y_n)+3(y_1+y_2+y_4+y_5+\cdots+y_{n-2}+y_{n-1}) + 2(y_3+y_6+\cdots\cdots+y_{n-3})\Big]$$

which is known as Simpson's $3/8$ - rule.

Note: [1]. While applying Simpson's $3/8$ - rule, the number of sub-intervals should be taken as multiple of 3.

The error in Simpson's $3/8$ - rule is given by

$$\boxed{E = -\frac{3h^5}{80}\, y^{iv}(\bar{x})}$$

where $y^{iv}(\bar{x})$ is the largest value of the fourth order derivative

Note: It may be noted that the errors in Simpson's $1/3$ and $3/8$ - rules are of the same order. However, if we consider the magnitudes of the error terms, Simpson's $1/3$ rule is superior to Simpson's $3/8$ rule.

Hence, Simpson's $3/8$ - rule, is not so accurate as Simpson's rule, the dominant term in the error of this formula being $-\frac{3}{80}h^5 y^{iv}(\bar{x})$.

Example 1 : Calculate the value $\int_0^1 \frac{x}{1+x}dx$ correct upto three significant figures taking six intervals by Trapezoidal rule.

Solution : Here we have

$$f(x)=\frac{x}{1+x},$$

$$a=0,\ b=1 \text{ and } n=6.$$

$$\therefore\ h=\frac{b-a}{n}=\frac{1-0}{6}=\frac{1}{6}$$

x	0	1/6	2/6	3/6	4/6	5/6	6/6=1
$y=f(x)$	0.00000	0.14286	0.25000	0.33333	0.40000	0.45454	0.50000
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

The Trapezoidal rule can be written as

$$I=\frac{h}{2}\left[\left(y_0+y_6\right)+2\left(y_1+y_2+y_3+y_4+y_5\right)\right]$$

$$=\frac{1}{12}\left[(0.00000+0.50000)+2(0.14286+0.25000+0.3333+0.40000+0.45454)\right]$$

$$=0.30512.$$

$\therefore\ I=0.0305$, correct to three significant figures.

Example 2 : Find the value of $\int_0^1 \frac{1}{1+x^2}dx$ taking 5 subinterval by Trapezoidal rule, correct to five significant figures. Also compare it with its exact value.

Solution : Here

$$f(x)=\frac{1}{1+x^2},$$

$$a=0,\ b=1, \text{ and } n=5.$$

$$\therefore\ h=\frac{1-0}{5}=\frac{1}{5}=0.2.$$

x	0.0	0.2	0.4	0.6	0.8	1
$y=f(x)$	1.000000	0.961538	0.832069	0.735294	0.609756	0.500000
	y_0	y_1	y_2	y_3	y_4	y_5

Using trapezoidal rule we get

$$I=\int_0^1\frac{dx}{1+x^2}=\frac{h}{2}\left[\left(y_0+y_5\right)+2\left(y_1+y_2+y_3+y_4\right)\right]$$

$$=\frac{0.2}{2}\left[(1.000000+.500000)+2(0.961538+0.862069+0.735294+0.609756)\right]$$

$$=0.7837314,$$

$\therefore I=0.78373$, correct to five significant figures.

The exact value

$$=\int_0^1 \frac{1}{1+x^2}dx=\left[\tan^{-1}x\right]_0^1$$

$$=\tan^{-1}1-\tan^{-1}0$$

$$=\frac{\pi}{4}=0.7853981$$

$$\int_0^1 \frac{1}{1+x^2}dx=0.78540,$$

correct to five significant figures.

$\therefore$ The error is $=0.78540-0.78373$

$=0.00167$

$\therefore$ Absolute error $=0.00167$.

Example 3: Find the value of $\int_1^5 \log_{10}x\,dx$, taking 8 subintervals correct to four decimal places by Trapezoidal rule.

Solution : Here

$$f(x)=\log_{10}x,$$
$$a=1,\ b=5,\ n=8,$$

$$\therefore\ h=\frac{b-a}{n}=\frac{5-1}{8}=0.5.$$

x	1.0	1.5	2.0	2.5	3.0	3.5	4.0	4.5	5.0
$f(x)$	0.00000	0.17609	0.30103	0.39794	0.47712	0.54407	0.60206	0.65321	0.69897
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8

Using Trapezoidal rule we can write

$$I=\frac{h}{2}\left[(y_0+y_8)+2(y_1+y_2+y_3+y_4+y_5+y_6+y_7)\right]$$

$$=\frac{0.5}{2}\left[(0.00000+0.69897)+2(0.17609+0.30103+0.39794)\right]$$

(8)

$$+\frac{0.5}{2}\left[2(0.47712+0.54407+0.60206+0.65321)\right]$$

$$= 1.7505025$$

$$\therefore I=\int_1^5 \log_{10} x\,dx = 1.75050\,.$$

Example 4. Find the value $\int_0^{0.6} e^x dx$, taking $n = 6$, correct to five significant figures by Simpson's one-third rule.

Solution : We have

$$f(x) = e^x,$$

$$a = 0,\ b = 0.6,\ n = 6.$$

$$\therefore h=\frac{b-a}{n}=\frac{0.6-0}{6}=0.1.$$

x	0.0	0.1	0.2	0.3	0.4	0.5	0.6
$y=f(x)$	1.0000	1.10517	1.22140	1.34986	1.49182	1.64872	1.82212
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

The Simpson's rule is

$$I=\frac{h}{3}\left[\left(y_0+y_6\right)+4\left(y_1+y_3+y_5\right)+2\left(y_2+y_4\right)\right]$$

$$=\frac{0.1}{3}\left[(1.00000+1.82212)+4(1.10517+1.34986+1.64872)+2(1.22140+1.49182)\right]$$

$$=\frac{0.1}{3}\left[(2.82212)+4(4.10375)+2(2.71322)\right]$$

$$= 0.8221186 \simeq 0.82212$$

$$\therefore I = 0.82212.$$

Example 5. The velocity of a train which starts from rest is given by the following table, the time being reckoned in minutes from the start and the speed in km/hour.

t (minutes)	2	4	6	8	10	12	14	16	18	20
v (km/hr)	16	28.8	40	46.4	51.2	32.0	17.6	8	3.2	0

Estimate approximately the total distance run in 20 minutes.

Solution :

$$v = \frac{ds}{dt} \Rightarrow ds = v.dt$$

$$\Rightarrow \int ds = \int v.dt$$

$$s = \int_0^{20} v \cdot dt .$$

The train starts from rest, ∴ the velocity $v = 0$ when $t = 0$.

The given table of velocities can be written

t	0	2	4	6	8	10	12	14	16	18	20
v	0	16	28.8	40	46.4	51.2	32.0	17.6	8	3.2	0
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}

$$h = \frac{2}{60}\text{ hrs} = \frac{1}{30}\text{ hrs}.$$

The Simpson's rule is

$$s = \int_0^{20} v \cdot dt = \frac{h}{3}\left[(y_0 + y_{10}) + 4(y_1 + y_3 + y_5 + y_7 + y_9) + 2(y_2 + y_4 + y_6 + y_8)\right]$$

$$= \frac{1}{30 \times 3}\left[(0+0) + 4(16 + 40 + 51.2 + 17.6 + 3.2) + 2(28.8 + 46.4 + 32.0 + 8)\right]$$

$$= \frac{1}{90}[0 + 4 \times 128 + 2 \times 115.2]$$

$$= 8.25 \text{ km.}$$

∴ The distance run by the train in 20 minutes = 8.25 kms.

Example 6 : A tank in discharging water through an orifice at a depth of x meter below the surface of the water whose area is Am^2. The following are the values of x for the corresponding values of A.

A	1.257	1.39	1.52	1.65	1.809	1.962	2.123	2.295	2.462	2.650	2.827
x	1.50	1.65	1.80	1.95	2.10	2.25	2.40	2.55	2.70	2.85	3.00

Using the formula $(0.018)T = \int_{1.5}^{3.0} \frac{A}{\sqrt{x}} dx$, calculate T, the time in seconds for the level of the water to drop from 3.0 m to 15 m above the orifice.

Solution : We have $h = 0.15$,

the table of values of x and the corresponding values of $\frac{A}{\sqrt{x}}$ is

x	1.50	1.65	1.80	1.95	2.10	2.25	2.40	2.55	2.70	2.85	3.00
$y=\frac{A}{\sqrt{x}}$	1.025	1.081	1.132	1.182	1.249	1.308	1.375	1.438	1.498	1.571	1.632

Using Simpson's rule, we get

$$\int_{1.5}^{3.0}\frac{A}{\sqrt{x}}dx=\frac{h}{3}\left[(y_0+y_{10})+4(y_1+y_3+y_5+y_7+y_9)+2(y_2+y_4+y_6+y_8)\right]$$

$$=\frac{0.15}{3}\left[(1.025+1.632)+4(1.081+1.182+1.308+1.438+1.571)\right]$$

$$+\frac{0.15}{3}\left[2(1.132+1.249+1.375+1.498)\right]$$

$$=1.9743$$

$$\therefore \int_{1.5}^{3}\frac{A}{\sqrt{x}}dx=1.9743.$$

Using the formula

$$(0.018)T=\int\frac{A}{\sqrt{x}}dx,$$

we get

$$(0.018)T=1.9743$$

$$\Rightarrow T=\frac{1.9743}{0.018}=110 \text{ sec (approximately)}$$

$$\therefore T=110 \text{ sec.}$$

Example 7 : Evaluate $\int_0^1\frac{1}{1+x^2}dx$, by taking seven ordinates.

Solution : We have

$$n+1=7\Rightarrow n=6$$

The points of division are

$$0,\ \frac{1}{6},\ \frac{2}{6},\ \frac{3}{6},\ \frac{4}{6},\ \frac{5}{6},\ 1.$$

The table of values is

x	0	1/6	2/6	3/6	4/6	5/6	1
$y=\frac{1}{1+x^2}$	1.0000000	0.9729730	0.9000000	0.8000000	0.6923077	0.5901639	0.5000000

Here $h=\frac{1}{6}$

the Simpson's three–eighth rule is

$$I=\frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2(y_3)\right]$$

$$=\frac{3}{6\times 8}\left[(1+0.5000000)+3(0.9729730+0.9000000)\right]$$

$$+\frac{3}{6\times 8}\left[3(0.6923077+0.5901639)+2(0.800000)\right]$$

$$=\frac{1}{16}\left[1.5000000+9.4663338+1.6000000\right]$$

$$=0.7853959.$$

Simpson's 3/8 Rule

Example 8 Calculate $\int_0^{\frac{\pi}{2}} e^{\sin x}\,dx$, correct to four decimal places.

Solution : We divide the range in three equal points with the division points

$$x_0=0,\ x_1=\frac{\pi}{6},\ x_2=\frac{\pi}{3},\ x_3=\frac{\pi}{2}$$

where $h=\frac{\pi}{6}$.

The table of values of the function is

x	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$
$y=e^{\sin x}$	1	1.64872	2.36320	2.71828
	y_0	y_1	y_2	y_3

By Simpson's three–eighth rule we get

$$I=\int_0^{\frac{\pi}{2}} e^{\sin x}\,dx=\frac{3h}{8}\left[(y_0+y_3)+3(y_1+y_2)\right.$$

$$= \frac{3}{8}\frac{\pi}{6}\left[(1+2.71828)+3(1.64872+2.36320)\right]$$

$$= \frac{\pi}{16}\left[(3.71828+12.03576)\right]$$

$$\doteq 0.091111$$

$$I = \int_0^{\frac{\pi}{2}} e^{\sin x}\, dx = 0.091111.$$

Example 9 : Compute the integral $\int_0^{\frac{\pi}{2}} \sqrt{1-0.162\sin^2\phi}\, d\phi$ by Weddle's rule.

Solution : Here we have

$$y = f(\phi) = \sqrt{1-0.162\sin^2\phi},$$

$$a = 0,\ b = \frac{\pi}{2},$$

taking $n = 12$ we get

$$h = \frac{b-a}{n} = \frac{\frac{\pi}{2}-0}{12} = \frac{\pi}{24}$$

f	$y = f(f)$		f	$y = f(f)$	
0	1.000000	y_0	$\frac{6\pi}{24}$	0.958645	y_6
$\frac{\pi}{24}$	0.998619	y_1	$\frac{7\pi}{24}$	0.947647	y_7
$\frac{2\pi}{24}$	0.994559	y_2	$\frac{8\pi}{24}$	0.937283	y_8
$\frac{3\pi}{24}$	0.988067	y_3	$\frac{9\pi}{24}$	0.928291	y_9
$\frac{4\pi}{24}$	0.979541	y_4	$\frac{10\pi}{24}$	0.921332	y_{10}
$\frac{5\pi}{24}$	0.969518	y_5	$\frac{11\pi}{24}$	0.916930	y_{11}
			$\frac{12\pi}{24} = \frac{\pi}{2}$	0.915423	y_{12}

By Weddle's rule we have

$$I=\int_0^{\frac{\pi}{2}}\sqrt{-0.162\sin^2\phi}\ d\phi$$

$$=\frac{3h}{10}\left[(y_0+y_{12})+5(y_1+y_5+y_7+y_{11})\right]$$

$$+\frac{3h}{10}\left[(y_2+y_4+y_8+y_{10})+6(y_3+y_9)+2y_6\right]$$

$$=\frac{3\pi}{240}\left[(11.000000+0.915423)+5(0.998619+0.969518+0.947647+0.916930)\right]$$

$$+\frac{3\pi}{240}\left[(0.994559+0.979541+0.937283+0.9213322)\right]$$

$$+\frac{3\pi}{240}\left[6(0.988067+0.928291)+2(0.958645)\right]$$

$\therefore\ I=1.505103504$.

Example 10 : Find the value of $\int_4^{5.2}\log_e x\,dx$ by Weddle's rule.

Solution : Here $f(x)=\log_e n$, $a=x_0=4$, $b=x_n=5.2$ taking $n=6$ (a multiple of six) we have

$$h=\frac{5.2-4}{6}=0.2,$$

x	4.0	4.2	4.4	4.6	4.8	5.0	5.2
$y=f(x)$	1.3863	1.4351	1.4816	1.5261	1.5686	1.6094	1.6457

Weddle's rule is

$$I=\int_4^{5.2}\log_e x\,dx=\frac{3h}{10}\left[y_0+5y_1+y_2+6y_3+y_4+5y_5+y_6\right]$$

$$=\frac{3\times(0.2)}{10}\left[1.3863+7.1755+1.4816+9.1566+1.5686+8.0470+1.6487\right]$$

$$=0.06[30.4643]$$

$$=1.827858$$

$$\int_4^{5.2}\log_e x=1.827858.$$

Exercise 9.1

1. Evaluate $\int_0^1 x^3\,dx$ by Trapezoidal rule.

2. Evaluate $\int_0^1 (4x-3x^2)\,dx$ taking 10 intervals by Trapezoidal rule.

3. Given that $e^0 = 1$, $e^1 = 2.72$, $e^2 = 7.39$, $e^3 = 20.09$, $e^4 = 54.60$, find an approximation value of $\int_0^4 e^x\,dx$ by Trapezoidal rule.

4. Evaluate $\int_0^1 \sqrt{1-x^3}\,dx$ by (i) Simpson's rule and (ii) Trapezoidal rule, taking six interval correct to two decimal places.

5. Evaluate $\int_0^{\frac{\pi}{2}} \sqrt{\sin x}\,dx$ taking $x = 6$, correct to four significant figures by (i) Simpson's one–third rule and (ii) Trapezoidal rule.

6. Evaluate $\int_1^2 \frac{dx}{x}$ taking 4 subintervals, correct to five decimal places (i) Simpson's one third rule (ii) Trapezoidal rule.

7. Compute by Simpson's one–third rule, the integral $\int_0^1 x^2(1-x)\,dx$ correct to three places of decimal, taking step length equal to 0.1.

8. Evaluate $\int_0^1 \sin x^2\,dx$ by (i) Trapezoidal rule and (ii) Simpson's one–third rule, correct to four decimals taking $x = 10$.

9. Calculate approximate value of $\int_{-3}^{3} \sin x^4\,dx$ by using (i) Trapezoidal rule and (ii) Simpson's rule, taking $n = 6$.

10. Find the value of $\int_0^{\frac{\pi}{2}} \sqrt{\cos x}\,dx$ by (ii) Trapezoidal rule and (ii) Simpson's one–third rule taking $x = 6$.

11. Compute $\int_1^{1.5} e^x\,dx$ by (i) Trapezoidal rule and (ii) Simpson's one–third rule taking $x = 10$.

12. Evaluate $\int_0^{0.5} \frac{x}{\cos x}\,dx$ taking $n = 10$, by (i) Trapezoidal rule and (ii) Simpson's one–third rule.

13. Evaluate $\int_0^{0.4} \cos x\,dx$ taking four–equal intervals by (i) Trapezoidal rule and (ii) Simpson's one–third rule.

14. Evaluate $\int_0^{\frac{\pi}{2}} \sqrt{\cos x}\,dx$ by Weddle's rule taking $n = 6$.

15. Evaluate $\int_0^1 \frac{x^2+2}{x^2+1}\,dx$ by Weddle's rule, correct to four decimals taking $n = 12$.

16. Evaluate $\int_0^2 \frac{1}{1+x^2}\,dx$ by using Weddle's rule taking twelve–intervals.

17. Evaluate $\int_{0.4}^{1.6} \frac{x}{\sinh x}\,dx$ taking thirteen–ordinates by Weddle's rule correct to five decimals.

18. Using Simpson's rule evaluate $\int_0^{\frac{\pi}{2}} \sqrt{2+\sin x}\,dx$ with seven ordinates.

19. Using Simpson's rule evaluate $\int_1^2 \sqrt{x-1/x}\,dx$ with five ordinates.

20. Using Simpson's rule evaluate $\int_2^6 \frac{1}{\log_e x} dx$ taking $n = 4$.

21. A river is 80 unit wide. The depth at a distance x unit from one bank d is given by the following table

x	0	10	20	30	40	50	60	70	80
d	0	4	7	9	12	15	14	8	3

22. Find the approximate value of $\int_0^{\frac{\pi}{2}} \sqrt{\cos\theta}\, d\theta$ using Simpson's rule with six intervals.

Answers

1. 0.260 2. 0.995 3. 58.00 4. 0.83 5. 1.187, 1.170 6. 0.69326, 0.69702
7. 0.083 8. 0.3112, 0.3103 9. 115, 98 10. 1.170, 1.187 11. 1.764, 1.763
12. 0.133494, 0.133400 13. 0.3891, 0.3894 14. 1.18916 15. 1.7854 16. 1.1071
17. 1.1020 18. 2.545 19. 1.007 20. 3.1832 21. 710 Sq units 22. 1.1872

9.7 Newton - Cotes formula

Consider Lagrange's interpolation formula

$$f(x) = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)} f(x_0)$$

$$+ \frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)} f(x_1)$$

$$+ \ldots + \frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\ldots(x_n-x_{n-1})} f(x_n),$$

Integrating between the limits x_0 and x_0+nh we get

$$\int_{x_0}^{x_0+h} f(x)dx = H_0 f(x_0) + H_1 f(x_1) + \ldots + H_r f(x_r) + \ldots + H_n f(x_n) \qquad (5)$$

Expression (5) is known as Newton – Cotes formula. Taking

$$x_{r+1} - x_r = h$$

for all r such that

$$x_r = x_0 + rh$$

Numerical Solution of ordinary Differential Equations

Set-VI

In the field of Science and technology, a number of problems can be formulated into differential equations. Such problems in this field are reduced to the problem of solving differential equations satisfying certain given conditions. The analytical methods of solving differential equations are applicable only to a limited class of equations. Quite often differential equations appearing in physical problems do not belong to any of these familiar types and one is obliged to resort to numerical methods.

These methods are of even greater importance when we realise that computing machines are now readily available which reduce numerical work considerably.

The solution of an ordinary differential equation means finding an explicit expression for y in terms of a finite number of elementary functions of x. Such a solution of a differential equation is known as the closed or finite form of solution. In the absence of such a solution, we have recourse to numerical methods of solution.

To describe various numerical methods for the solution of ordinary differential equations, Let us consider the first order differential equation

$$\frac{dy}{dx} = f(x, y), \quad \text{---} \quad (1)$$

with the initial condition

$$y(x_0) = y_0.$$

The methods so developed can, in general be applied to the solution of systems of first order equations, and will yield the solution of one of the two forms.

(i) A series for y in terms of powers of x, from which the value of y can be obtained by direct substitution.

(ii) A set of tabulated values of x and y.

The methods of Taylor and picard belong to class (i). In these methods, y in (1) approximated by a truncated series, each term of which is a function of x. The information about the curve at one point is utilized and the solution is not iterated. As such, these are referred to as single step methods.

The methods of Euler, Runge-Kutta, Milne, Adams-Bashforth etc belong to the latter class solutions (i.e, class (ii)). In these methods, the next point on the curve is evaluated in short steps ahead, by performing iterations till sufficient accuracy is achieved. As such, these methods are called step by step methods (or) multistep methods.

Euler and Runge - Kutta methods are used for computing y over a limit range of x values whereas Milne and Adams methods may be applied for finding y over a wider range of x-values.

The methods Milne and Adams require starting values which are found by picard's Taylor's series or Runge-Kutta methods.

Initial and boundary conditions:

An ordinary differential equation of the n^{th} order is of the form

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots \frac{d^ny}{dx^n}\right) = 0 \quad (2)$$

Its general solution contains 'n' arbitrary constants and is of the form

$$\phi(x, y, c_1, c_2, \ldots c_n) = 0 \quad (3).$$

To obtain its particular solution, n conditions must be given so that the constants $c_1, c_2, \ldots c_n$ can be determined.

If these conditions are prescribed at one point only (say: x_0), then the differential equation together with the conditions constitute an <u>initial value problem</u> of the n^{th} order.

If the conditions are prescribed at two or more points, then the problem is termed as <u>boundary value problem</u>.

We shall first describe methods for solving initial value problems of the type

$$\frac{dy}{dx} = f(x, y) \; ; \quad y(x_0) = y_0 .$$

and at the end of the chapter we will outline methods for solving boundary value problems for second order differential equations.

Solution by Taylor's series method.

The Taylor series method provides a solution of the equation.

$$\frac{dy}{dx} = f(x,y) \; ; \; y(x_0) = y_0 \qquad \text{——— ①}$$

- we assume that $f(x,y)$ is sufficiently differentiable with respect to x and y.
- If $y(x)$ is the exact solution of ①, then the Taylor's series for $y(x)$ around $x = x_0$ in power of $(x-x_0)$ is given by

$$y(x) = y_0 + (x-x_0)\,y_0' + \frac{(x-x_0)^2}{2!}\,y_0'' + \cdots$$

(or) $y(x_0+h) = y_0 + h y_0' + \frac{h^2}{2!} y_0'' + \cdots$ where $h = x - x_0 \Rightarrow x = x_0 + h$

Since, the solution is not known, the derivatives in the above expansion are not known explicitly. However, f is assumed to be sufficiently differentiable and therefore, the derivatives can be obtained directly from the given differential equation itself. Noting that f is an implicit function of y, we have.

$$y' = \frac{dy}{dx} = f(x,y) = f$$

$$y'' = \frac{d}{dx}(y') = \frac{\partial f}{\partial x}\frac{dx}{dx} + \frac{\partial f}{\partial y}\frac{dy}{dx}$$

$$= f_x + f_y f$$

$$y''' = \frac{d}{dx}(y'')$$

$$= \frac{d}{dx}\left[f_x + f_y f\right]$$

$$= \frac{\partial}{\partial x} f_x \frac{dx}{dx} + \frac{\partial f_x}{\partial y} \frac{dy}{dx} + f\left[\frac{\partial f_y}{\partial x}\frac{dx}{dx} + \frac{\partial f_y}{\partial y}\frac{dy}{dx}\right]$$

$$+ f_y\left[f_x + f_y f\right]$$

$$= f_{xx} + \underline{f_{yx} f} + f\left[\underline{f_{xy}} + f_{yy} f\right] + f_y\left[f_x + f_y f\right]$$

$$= f_{xx} + 2ff_{xy} + f^2 f_{yy} + f_y(f_x + f_y f)$$

$$y^{iv} = f_{xxx} + 3ff_{xxy} + 3f^2 f_{xyy} + f_y(f_{xx} + 2ff_{xy} + f^2 f_{yy})$$

$$+ 3(f_x + ff_y)(f_{xy} + ff_{yy}) + f_y^2(f_x + ff_y)$$

etc.

By continuing in this manner, we can express the derivative of y in terms of $f(x, y)$ and its partial derivatives.

The method is best understood by the following example.

→ From the Taylor series for $y(x)$, find $y(0.1)$ correct to four decimal places if $y(x)$ satisfies $y' = x - y^2$ and $y(0) = 1$.

Sol^n: The Taylor series for $y(x)$ is given by

$$y(x) = 1 + xy_0' + \frac{x^2}{2} y_0'' + \frac{x^3}{6} y_0''' + \frac{x^4}{24} y_0^{iv} + \frac{x^5}{120} y_0^{v} + \text{-----}$$

The derivatives y_0', y_0'', ---- etc are obtained thus:

$y'(x) = x - y^2 \Rightarrow y_0' = -1$

$y''(x) = 1 - 2yy' \Rightarrow y_0'' = 3$

$y'''(x) = -2yy'' - 2y'^2 \Rightarrow y_0''' = -8$

$y^{iv}(x) = -2yy''' - 6y'y'' \Rightarrow y_0^{iv} = 34$

$y^{v}(x) = -2yy^{iv} - 8y'y''' - 6y''^2 \Rightarrow y_0^{(v)} = -186.$

- Using these values the Taylor series becomes

$$y(x) = 1 - x + \frac{3}{2}x^2 - \frac{4}{3}x^3 + \frac{17}{12}x^4 - \frac{31}{20}x^5 + \cdots$$

To obtain the value of $y(0.1)$ correct to four decimal places, it is found that the terms upto x^4 should be considered, and we have $\boxed{y(0.1) = 0.9138}$

Suppose that we wish to find the range of values of 'x' for which the above series, truncated after the term containing x^4, can be used to compute the values of y correct to four decimal places.

we need only to write

$$\frac{31}{20}x^5 \le 0.00005.$$

so that $x \le 0.126$

→ find $y(1.1)$, given $y' = 2x - y$ and $y(1) = 3$.

Sol^n: Given $y' = 2x - y$, $x_0 = 1$, $y_0 = 3$, $x_1 = 1.1$,

$h = 0.1$

Taylor series is given by

$$y_1 = y_0 + h y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \dots \quad (1)$$

$$y' = 2x - y \Rightarrow y_0' = 2x_0 - y_0$$
$$= 2(1) - 3$$
$$= -1$$

$$y'' = 2 - y' \Rightarrow y_0'' = 2 - y_0'$$
$$= 2 - (-1)$$
$$= 3$$

$$y''' = -y'' \Rightarrow y_0''' = -y_0''$$
$$= -3$$

Equation (1) becomes

$$y_1 = 3 + (0.1)(-1) + \frac{(0.1)^2}{2}(3) + \frac{(0.1)^3}{6}(-3) + \dots$$

$$= 3 - 0.1 + 0.015 - 0.0005 + \dots$$

$$= 2.9145$$

(5)

Example 8.1 Using Taylor's series method, find the solution of the initial value problem

$$\frac{dy}{dt} = t + y, \qquad y(1) = 0$$

at $t = 1.2$, with $h = 0.1$ and compare the result with the closed form solution.

Solution Let us compute the first few derivatives from the given differential equation as follows:

$$y' = t + y, \qquad y'' = 1 + y', \qquad y''' = y'', \qquad y^{IV} = y''', \qquad y^{V} = y^{IV} \tag{1}$$

Prescribing the initial condition, that is, at $t_0 = 1$, $y_0 = y(t_0) = 0$, we have

$$y_0' = 1, \qquad y_0'' = 2, \qquad y_0''' = y_0^{IV} = y_0^{V} = 2$$

Now, using Taylor's series method, we have

$$y(t) = y_0 + (t - t_0)y_0' + \frac{(t-t_0)^2}{2}y_0'' + \frac{(t-t_0)^3}{6}y_0'''$$

$$+ \frac{(t-t_0)^4}{24}y_0^{IV} + \frac{(t-t_0)^5}{120}y_0^{V} + \cdots \tag{2}$$

Substituting the above values of the derivatives, and the initial condition, into (2), we obtain

$$y(1.1) = 0 + (0.1)(1) + \frac{0.01}{2}(2) + \frac{0.001}{6}(2) + \frac{0.0001}{24}(2) + \frac{0.00001}{120}(2) + \cdots$$

$$= 0.1 + 0.01 + \frac{0.001}{3} + \frac{0.0001}{12} + \frac{0.00001}{60} + \cdots$$

$$= 0.1 + 0.01 + 0.000333 + 0.0000083 + 0.0000001 + \cdots$$

$$\cong 0.1103414.$$

Therefore,

$$y(1.1) \cong y_1 \cong 0.1103414 \cong 0.1103$$

Taking $y_1 = 0.1103$ at $t = 1.1$, the values of the derivatives as computed from Eq. (1) are

$$y_1' = 1.1 + 0.1103 = 1.2103$$

$$y_1'' = 1 + 1.2103 = 2.2103$$

$$y_1''' = y_1^{IV} = y_1^{V} = 2.2103$$

Substituting the value of y_1 and its derivatives into Taylor's series expansion (2) we get, after retaining terms up to fifth derivative only

$$y(1.2) = y_1 + (t - t_1)y_1' + \frac{(t-t_1)^2}{2}y_1'' + \frac{(t-t_1)^3}{6}y_1''' + \frac{(t-t_1)^4}{24}y_1^{IV} + \frac{(t-t_1)^5}{120}y_1^{V}$$

$$= 0.1103 + 0.12103 + 0.0110515 + 0.0003683 + 0.000184 + 0.0000003$$

$$= 0.2429341$$

Hence,

$$y(1.2) = 0.2429341 \cong 0.2429 \qquad (3)$$

To obtain the closed form solution, we rewrite the given IVP as

$$\frac{dy}{dt} - y = x \quad \text{or} \quad d(ye^{-t}) = te^{-t}$$

On integration, we get

$$y = -e^t(te^{-t} + e^{-t}) + ce^t = ce^t - t - 1$$

Using the initial condition, we get

$$0 = ce - 2 \quad \text{or} \quad c = \frac{2}{e}$$

Therefore, the closed form solution is

$$y = -t - 1 + 2e^{t-1}$$

when $t = 1.2$, the closed form solution becomes

$$y(1.2) = -1.2 - 1 + 2(1.2214028) = -2.2 + 2.4428056 = 0.2428 \qquad (4)$$

Comparing the results (3) and (4), obtained numerically and in closed form, we observe that they agree up to three decimals.

EXAMPLE 7.2 Given $\frac{dy}{dx} = 3x + \frac{y}{2}$ and $y(0) = 1$. Find the values of $y(0.1)$ and $y(0.2)$, using the Taylor series method.

Solution

Given

$$y' = 3x + y/2 \text{ and } x_0 = 0,\ y_0 = 1$$

Taylor series is given by

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{iv} + \ldots \qquad \text{(i)}$$

To find y(0.1):

$$y' = 3x + \frac{y}{2} \Rightarrow y_0' = 3x_0 + \frac{y_0}{2}$$
$$= 3(0) + \frac{1}{2}$$
$$= 0.5$$

$$y'' = 3 + \frac{y'}{2} \Rightarrow y_0'' = 3 + \frac{y_0'}{2}$$
$$= 3 + \frac{0.5}{2}$$
$$= 3.25$$

$$y''' = \frac{y''}{2} \Rightarrow y_0''' = \frac{y_0''}{2}$$
$$= \frac{3.25}{2}$$
$$= 1.625$$

$$y^{iv} = \frac{y'''}{2} \Rightarrow y_0^{iv} = \frac{y_0'''}{2}$$
$$= \frac{1.625}{2}$$
$$= 0.8125$$

$\therefore$ Equation (i) $\Rightarrow$

$$y_1 = 1 + (0.1)(0.5) + \frac{(0.1)^2}{2}(3.25) + \frac{(0.1)^3}{6}(1.625) + \frac{(0.1)^4}{24}(0.8125) + \ldots$$
$$= 1.0665$$
$$y_1 = y(0.1)$$
$$= 1.0665$$

).1)

To find y(0.2):

Taylors series formula for $y(0.2)$ is

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!}y_1'' + \frac{h^3}{3!}y_1''' + \frac{h^4}{4!}y_1^{iv} + \ldots \quad \text{(ii)}$$

$$y' = 3x + \frac{y}{2} \Rightarrow y_1' = 3x_1 + \frac{y_1}{2}$$

(i)

$$= 3(0.1) + \frac{1.0665}{2}$$

$$= 0.83325$$

$$y'' = 3 + \frac{y'}{2} \Rightarrow y_1'' = 3 + \frac{y_1'}{2}$$

$$= 3 + \frac{0.83325}{2}$$

$$= 3.416625$$

$$y''' = \frac{y''}{2} \Rightarrow y_1''' = \frac{y_1''}{2}$$

$$= \frac{3.416625}{2}$$

$$= 1.7083125$$

$$y^{iv} = \frac{y'''}{2} \Rightarrow y_1^{iv} = \frac{y_1'''}{2}$$

$$= \frac{1.7083125}{2}$$

Equation (ii) $\Rightarrow$

$$= 0.85415625$$

$$y_2 = 1.0665 + (0.1)(0.83325) + \frac{(0.1)^2}{2}(3.416625)$$

$$+ \frac{(0.1)^3}{6}(1.7083125) + \frac{(0.1)^4}{24}(0.85415625) + \ldots$$

$$= 1.167196.$$

$$y_2 = y(0.2)$$

$$= 1.167196.$$

EXAMPLE 7.3 Obtain $y(4.2)$ and $y(4.4)$, given

$$\frac{dy}{dx} = \frac{1}{x^2 + y}, \; y(4) = 4, \text{ taking } h = 0.2$$

Solution

Given $\quad y' = \dfrac{1}{x^2 + y}, \; x_0 = 4, y_0 = 4, x_1 = 4.2, x_2 = 4.4, h = 0.2.$

Taylor series is given by

$$y_1 = y_0 + h y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \ldots \qquad \text{(i)}$$

To find y(4.2):

$$y' = \frac{1}{x^2 + y} \Rightarrow y_0' = \frac{1}{x^2 + y_0}$$

$$= \frac{1}{4^2 + 4}$$

$$= 0.05$$

$$y'' = -1\,(x^2 + y)^{-2}\, y' \Rightarrow y_0'' = \frac{-y_0'}{(x_0^2 + y_0)^2}$$

$$= \frac{-0.05}{(20)^2}$$

$$= -0.000125$$

$$y''' = 2(x^2 + y)^{-3} (y')^2 + (-1)(x^2 + y)^{-2} y''$$

$$y_0''' = \frac{2(y_0')^2}{(x_0^2 + y_0)^3} - \frac{y_0''}{(x_0^2 + y_0)^2}$$

$$= \frac{2(0.05)^2}{(20)^3} - \frac{0.000125}{(20)^2}$$

$$= 0.0000009375$$

$\therefore$ Equation (i) $\Rightarrow$

$$y_1 = 4 + (0.2)(0.05) + \frac{(0.2)^2}{2}(-0.000125) + \frac{(0.2)^3}{6}(0.0000009375) + \ldots$$

$$= 4 + 0.01 - 0.0000025 + 0.00000000125$$

$$= 4.0099975$$

$$y_1 = y(4.2)$$

$$= 4.009998$$

To find y(4.4):

Taylor series is given by

$$y_2 = y_1 + h y_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \ldots \qquad \text{(ii)}$$

$$y' = \frac{1}{x^2 + y} \quad \Rightarrow \quad y_1' = \frac{1}{x_1^2 + y_1}$$

$$= \frac{1}{(4.2)^2 + 4.009998}$$

$$= 0.046189$$

(i)

$$y_1'' = \frac{-y_1'}{(x_1^2 + y_1)^2} = \frac{-0.046189}{[(4.2)^2 + 4.009998]^2}$$

$$= -0.000098542$$

$$y_1''' = \frac{2(y_1')^2}{(x_1^2 + y_1)^3} - \frac{y_1''}{(x_1^2 + y_1)^2}$$

$$= \frac{2(0.046189)^2}{[(4.2)^2 + 4.009998]^3} + \frac{0.000098542}{[(4.2)^2 + 4.009998]^2}$$

$$= 0.000000420469 + 0.00000021024$$

$$= 0.000000630704$$

$\therefore$ Equation (ii) $\Rightarrow$

$$y_2 = 4.009998 + (0.2)(0.046189) + \frac{(0.2)^2}{2}(-0.000098542)$$

$$+ \frac{(0.2)^3}{6}(0.000000630704)$$

$$= 4.009998 + 0.0092378 - 0.00000197084 + 0.00000000094094$$

$$= 4.019234$$

EXAMPLE 7.4 Find $y(0.1)$, given $y' = x^2 y - 1$, $y(0) = 1$.

Solution

Given $y' = x^2 y - 1$, $x_0 = 0$, $y_0 = 1$, $x_1 = 0.1$, $h = 0.1$

Taylor series is given by

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \ldots \qquad \text{(i)}$$

$$y' = x^2 y - 1 \quad \Rightarrow \quad y_0' = x_0^2 y_0 - 1$$

$$= (0)(1) - 1$$

$$= -1$$

$$y'' = x^2 y' + 2xy \quad \Rightarrow \quad y_0'' = x_0^2 y_0' + 2x_0 y_0$$

$$= 0$$

(ii)

$$y''' = x^2 (y')^2 + 2xy' + 2xy' + 2y$$

$$y_0''' = x_0^2 (y_0')^2 + 2x_0\, y_0' + 2x_0 y_0' + 2y_0$$
$$= 2$$

$\therefore$ Equation (i) $\Rightarrow$

$$y_1 = 1 + (0.1)(-1) + \frac{(0.1)^2}{2!}(0) + \frac{(0.1)^3}{6}(2) + \ldots$$
$$= 1 - 0.1 + 0.0003333 + \ldots$$
$$= 0.900333$$
$$y_1 = y(0.1)$$
$$= 0.900333$$

7.1.1 Taylor Series Method for Simultaneous First-order Differential Equations

The simultaneous first-order differential equations of the form

$$\frac{dy}{dx} = f_1(x, y, z)$$

and

$$\frac{dz}{dx} = f_2(x, y, z)$$

with initial values $y(x_0) = y_0$ and $z(x_0) = z_0$

To solve this system of equations at an interval h, the increments in y and z are obtained by using the formulae

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \ldots$$

and

$$z_1 = z_0 + hz_0' + \frac{h^2}{2!} z_0'' + \frac{h^3}{3!} z_0''' + \ldots$$

EXAMPLE 7.5 Solve the differential equations using the Taylor series

$$\frac{dy}{dx} = 1 + xz, \quad \frac{dz}{dx} = -xy, \quad \text{for } x = 0.3$$

given that $x = 0$, $y = 0$, $z = 1$.

Solution

$$\frac{dy}{dx} = 1 + xz$$

$$\frac{dz}{dx} = -xy$$

$$x = 0,\ y = 0,\ z = 1,\ h = 0.3$$

Taylors series for y' is given by

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \ldots \quad \text{(i)}$$

$$y' = 1 + xz \Rightarrow y_0' = 1 + x_0 z_0$$
$$= 1 + (0)(1)$$
$$= 1$$

$$y'' = xz' + z \Rightarrow y_0'' = x_0 z_0' + z_0$$
$$= x_0(-x_0 y_0) + z_0$$
$$= 0 + 1$$
$$= 1$$

$$y''' = x(z')^2 + z' + z' \Rightarrow y_0''' = x_0 (z_0')^2 + 2z_0'$$
$$= x_0(-x_0 y_0)^2 + 2(x_0 y_0)$$
$$= 0$$

$\therefore$ Equation (i) $\Rightarrow$

$$y_1 = 0 + (0.3)(1) + \frac{(0.3)^2}{2}(1) + \frac{(0.3)^3}{6}(0) + \ldots$$
$$= 0.3 + 0.045$$
$$= 0.345$$
$$y_1 = y(0.3)$$
$$= 0.345$$

Taylor series for z' is given by

$$z_1 = z_0 + hz_0' + \frac{h^2}{2!} z_0'' + \frac{h^3}{3!} z_0''' + \ldots \quad \text{(ii)}$$

$$z' = -xy \Rightarrow z_0' = -x_0 y_0 = 0$$
$$z'' = -(xy' + y) \Rightarrow z_0'' = -x_0 y_0' + y_0$$
$$= -(0)(1) + 0$$
$$= 0$$
$$z''' = -[xy'' + y' + y'] \Rightarrow z_0''' = -x_0 y_0'' + 2y_0'$$
$$= -(0)(1) + 2(1)$$
$$= 2.$$

$\therefore$ Equation (ii) $\Rightarrow$

$$z_1 = 1 + (0.3)(0) + \frac{(0.3)^2}{2}(0) + \frac{(0.3)^3}{6}(2) + \ldots$$

$$= 1 + 0.009$$
$$= 1.009$$
$$z_1 = z(0.3)$$
$$= 1.009$$

EXAMPLE 7.6 Find $y(0.1)$, $y(0.2)$, $z(0.1)$, $z(0.2)$, given

$$\frac{dy}{dx} = x + z, \frac{dz}{dx} = x - y^2 \text{ and } y(0) = 2, z(0) = 1$$

Solution

Given

$$y' = x + z, z' = x - y^2, x_0 = 0, y_0 = 2, z_0 = 1, h = 0.1$$

To find y(0.1):

Taylor series is given by

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \ldots \quad \text{(i)}$$

$$y' = x + z, \quad z' = x - y^2$$
$$y'' = 1 + z', \quad z'' = 1 - 2yy'$$
$$y''' = z''$$
$$z''' = -[2y\cancel{y''} + 2(y')^2]$$

$$\therefore \quad y' = x + z \Rightarrow y_0' = x_0 + z_0$$
$$= 0 + 1$$
$$= 1$$
$$y'' = 1 + z' = 1 + x - y^2 \Rightarrow y_0'' = 1 + x_0 - y_0^2$$
$$= 1 - 2^2$$
$$= -3$$

$\therefore$ Equation (i) $\Rightarrow$

$$y_1 = 2 + (0.1)(1) + \frac{(0.1)^2}{2}(-3) + \frac{(0.1)^3}{6}(-3) + \ldots$$
$$= 2 + 0.1 - 0.015 - 0.0005 \ldots$$
$$= 2.0845$$

To find z(0.1):

Taylor series is given by

$$z_1 = z_0 + hz_0' + \frac{h^2}{2!} z_0'' + \frac{h^3}{3!} z_0''' + \ldots \quad \text{(ii)}$$

$$z_0' = x_0 - y_0^2$$
$$= 0 - 2^2$$
$$= -9$$

$$z_0'' = 1 - 2y_0 y_0'$$
$$= 1 - 2(2)(1)$$
$$= -3$$

$$z_0''' = -[2y_0 y_0'' + 2(y_0')^2]$$
$$= -[2(2)(-3) + 2(1)^2]$$
$$= -[-12 + 2]$$
$$= 10$$

$\therefore$ Equation (ii) $\Rightarrow$

$$z_1 = 1 + (0.1)(-4) + \frac{(0.1)^2}{2}(-3) + \frac{(0.1)^3}{6}(10) + \ldots$$
$$= 1 \quad 0.4 + (-0.015) + 0.0016667$$
$$= 1 - 0.4 - 0.015 + 0.001667$$
$$= 0.586667$$

(i)

To find y(0.2):
Taylor series is given by

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \ldots \quad \text{(iii)}$$

$$y_1' = x_1 + z_1$$
$$= 0.1 + 0.586667$$
$$= 0.686667$$

$$y_1'' = 1 + z_1'$$
$$= 1 + x_1 - y_1^2$$
$$= 1 + 0.1 - 2.0845^2$$
$$= -3.24514$$

$$y_1''' = z_1''$$
$$= 1 - 2y_1 y_1'$$
$$= 1 - 2(2.0845)(0.686667)$$
$$= -1.862415$$

$\therefore$ Equation (iii) $\Rightarrow$

(ii)

$$y_2 = 2.0845 + (0.1)(0.686667) + \frac{(0.1)^2}{2}(-3.24514) + \frac{(0.1)^3}{6}(-1.862715) + \ldots$$

(10)

$$= 2.0845 + 0.0686667 - 0.01623 - 0.00031045$$
$$= 2.13663$$

To find z(0.2):

Taylor series is given by

$$z_2 = z_1 + hz_1' + \frac{h^2}{2!}z_1'' + \frac{h^3}{3!}z_1''' + \ldots$$

$$z_1' = x_1 - y_1^2$$
$$= 0.1 - 2.0845^2$$
$$= -4.24514$$

$$z_1'' = 1 - 2y_1y_1'$$
$$= 1 - 2(2.0845)(0.686667)$$
$$= -1.862715$$

$$z_1''' = -[2y_1\,y_1'' + 2(y_1')^2]$$
$$= -[2(2.0845)(-3.24514) + 2(0.686664)^2]$$
$$= -[-13.52899 + 0.943023]$$
$$= 12.585967$$

$\therefore$ Equation $\Rightarrow$

$$z_2 = (0.1) + (0.1)(-4.24514) + \frac{(0.1)^2}{2}(-1.862715)$$
$$+ \frac{(0.1)^3}{6}(12.585967) + \ldots$$
$$= 0.1 - 0.424514 - 0.009313575 + 0.00209766$$
$$= -0.33173$$

7.1.2 Taylor Series Method for Second-order Differential Equations

The differential equation of the second-order can be solved by reducing it to a lower-order differential equation. A second-order differential equation can be reduced to a first-order differential equation by transformation $y' = z$.

Suppose $$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right)$$

i.e., $$y'' = f(x, y, y') \qquad (7.2)$$

is the differential equation together with initial conditions.

$$y(x_0) = y_0 \quad \text{and} \quad y'(x_0) = y_0' \qquad (7.3)$$

where y_0, y_0' are known values.

Setting $p = y'$, we get $y'' = p'$

The equation (7.2) becomes

$$p' = f(x, y, p) \text{ with initial conditions}$$

$$y(x_0) = y_0 \quad (7.4)$$

$$y'(x_0) = y_0' \quad (7.5)$$

where y_0, y_0' are known values.

By putting $y' = p, y'' = p'$, Eq. (7.2) becomes

$$p' = f(x, y, p)$$

with initial conditions

$$y(x_0) = y_0$$

and $$y'(x_0) = y_0' \Rightarrow p(x_0) = p_0$$

Solving p' by using Eqs. (7.4) and (7.5), we get

$$p_1 = p_0 + hp_0' + \frac{h^2}{2!} p_0'' + \frac{h^3}{3!} p_0''' + \ldots \quad (7.6)$$

where $h = x_1 - x_0$

Since $p = y'$, we get Eq. (7.6) as

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \ldots$$

Similarly, proceeding in similar manner, we get

$$y_2 = y_1 + hy_1' + \frac{h^2}{2!} y_1'' + \frac{h^3}{3!} y_1''' + \ldots$$

$\therefore$ We calculate $y_1, y_2 \ldots$

EXAMPLE 7.7 Solve $y'' = y + xy'$, given $y(0) = 1$, $y'(0) = 0$ and calculate $y(0.1)$.

Solution

$$x_0 = 0$$

$$y_0 = 1$$

$$y_0' = 0$$

$$y'' = y + xy'$$

$$\Rightarrow \quad y_0'' = y_0 + x_0 y_0'$$

$$= 1 + (0)(0)$$

$$= 1$$

Differentiating with respect to x

$$y''' = y' + y' + xy'' = 2y' + xy''$$

$$\Rightarrow \qquad y_0''' = 2y_0' + x_0 y_0''$$

$$= 2(0) + 0(1) = 0$$

$$y^{iv} = 2y'' + y'' + xy'''$$

$$= 3y'' + xy'''$$

$$\Rightarrow \qquad y_0^{iv} = 3y_0'' + x_0 y_0'''$$

$$= 3(1) + (0)(0)$$

$$= 3$$

$$y^{v} = 4y''' + xy^{iv}$$

$$y^{v} = 4y_0''' + x_0 y_0^{iv}$$

$$= 4(0) + 0(3)$$

$$= 0$$

We know that Taylor series is given by

$$y(x) = y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \frac{h^4}{4!} y_0^{iv} + \ldots$$

$$= 1 + (0.1)(0) + \frac{(0.1)^2}{2}(1) + \frac{(0.1)^3}{6}(0) + \frac{(0.1)^4}{24}(3) + \ldots$$

$$= 1 + \frac{(0.1)^2}{2} + \frac{(0.1)^4}{24}(3) + \ldots$$

$$= 1.0050125$$

EXAMPLE 7.8 Find $y(0.2)$, given $y'' + y = 0$, $y(0) = 1$, $y'(0) = 0$.

Solution

$$y'' = -y,\ x_0 = 0,\ y_0 = 1,\ y_0' = 0,\ h = 0.2.$$

To find y(0.2)

We know that Taylor series is given by

$$y_1 = y_0 + hy_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \ldots \qquad \text{(i)}$$

$$\begin{aligned}
y'' &= -y & \Rightarrow \quad & y_0'' = -y_0 = -1 \\
y''' &= -y' & \Rightarrow \quad & y_0''' = -y_0' = 0 \\
y^{iv} &= -y'' & \Rightarrow \quad & y_0^{iv} = -y_0'' = -(-1) = 1 \\
y^{v} &= -y''' & \Rightarrow \quad & y_0^{v} = -y_0''' = 0
\end{aligned}$$

$\therefore$ Equation (i) $\Rightarrow$

$$y_1 = 1 + (0.2)(0) + \frac{(0.2)^2}{2}(-1) + \frac{(0.2)^3}{6}(6) + \frac{(0.2)^4}{24}(1) + \ldots$$

$$= 1 + \frac{(0.2)^2}{2}(-1) + \frac{(0.2)^4}{24}(1) + \ldots$$

$= 1 - 0.02 + 0.00006667$

$= 0.9800667$

EXERCISES

7.1 Solve $\frac{dy}{dx} = x + y$, given $y(1) = 0$ and get $y(1.1)$, $y(1.2)$ by Taylor series method.

[*Ans.* $y(1.1) = 0.110342$, $y(1.2) = 0.24281$]

7.2 Using the Taylor series method, find correct to four decimal places, the value of $y(0.1)$, given

$$\frac{dy}{dx} = x^2 + y^2 \text{ and } y(0) = 1.$$

[*Ans.* $y(0.1) = 1.11145$]

7.3 Using the Taylor method, compute $y(0.2)$ and $y(0.4)$ correct to four decimal places, given

$$\frac{dy}{dx} = 1 - 2xy \text{ and } y(0) = 0.$$

[*Ans.* $y(0.2) = 0.19475$, $y(0.4) = 0.359884$]

7.4 Given $\frac{dy}{dx} = 3x + \frac{y}{2}$ and $y(0) = 1$. Find the values of $y(0.1)$ and $y(0.2)$, using the Taylor series method.

[*Ans.* $y(0.1) = 1.0665$, $y(0.2) = 1.167196$]

7.5 Solve by the Taylor series method of third-order the problem

$$\frac{dy}{dx} = (x^3 + xy^2)e^{-x}, \; y(0) = 1$$

to find y, for $x = 0.1, 0.2, 0.3$.

[*Ans.* $y(0.1) = 1.0047$, $y(0.2) = 1.01812$, $y(0.3) = 1.03995$]

7.6 Solve by the Taylor series method (of fourth-order)

$$\frac{dy}{dx} = xy^2 + 1, \quad y(0) = 1 \quad \text{at } x = 0.2, 0.4.$$

[*Ans.* $y(0.2) = 1.226$, $y(0.4) = 1.54205$]

7.7 Using the Taylor series method, solve

$$\frac{dy}{dx} = x^2 - y, \; y(0) = 1 \text{ at } x = 0.1, 0.2, 0.3 \text{ and } 0.4.$$

[*Ans.* $y(0.1) = 0.9052$, $y(0.2) = 0.8213$,
$y(0.3) = 0.7492$, $y(0.4) = 0.6897$]

7.8 Find $y(0.1)$, given $\frac{dy}{dx} = x + y$, $y(0) = 1$.

[*Ans.* $y(0.1) = 1.1103$]

7.9 Find $y(0.1)$, $y(0.2)$, $y(0.3)$, given

$$y' = \frac{x^3 + xy^2}{e^x},\ y(0) = 1.$$

[*Ans.* $y(0.1) = 1.0047$, $y(0.2) = 1.01812$, $y(0.3) = 1.03995$]

7.10 Solve $\frac{dy}{dx} = y + x^3$, for $x = 1.1, 1.2, 1.3$, given $y(1) = 1$.

[*Ans.* $y(1.1) = 1.225$, $y(1.2) = 1.512$, $y(1.3) = 1.874$]

7.11 Find $y(0.1)$, $y(0.2)$, $z(0.1)$, $z(0.2)$, given

$$\frac{dy}{dx} = x + z,\ \frac{dz}{dx} = x - y^2 \text{ and } y(0) = 2,\ z(0) = 1.$$

[*Ans.* 2.0845, 2.1367, 0.5867, 0.1550]

7.12 Evaluate $x(0.1)$, $y(0.1)$, $x(0.2)$, $y(0.2)$, given

$$\frac{dx}{dt} = t_y + 1,\ \frac{dy}{dt} = -t_x$$

given $x = 0$, $y = 1$ at $t = 0$.

[*Ans.* $x(0.1) = 0.105$, $y(0.1) = 0.9987$, $x(0.2) = 0.21998$, $y(0.2) = 0.9972$]

7.13 Find $y(0.3)$, $z(0.3)$, given—

$$\frac{dz}{dx} = -xy,\ \frac{dy}{dx} = 1 + xz$$

where $y(0) = 0$, $z(0) = 1$.

[*Ans.* $y(0.3) = 0.3448$, $z(0.3) = 0.991$]

7.14 Solve for x and y

$$\frac{dx}{dt} = x + y + t,\quad \frac{dy}{dt} = 2x - t$$

given $x = 0$, $y = 1$ at $t = 1$.

$$\left[\textit{Ans. } x = 2t + t^2 + \frac{5}{6}t^3 + \ldots \quad y = 1 - t + \frac{3}{2}t^2 + \frac{2}{3}t^3 + \ldots\right]$$

7.15 Solve numerically, using the Taylor series method find approximate values of y and z corresponding to $x = 0.1, 0.2$ given that

$$y(0) = 2,\ z(0) = 1 \text{ and } \frac{dy}{dx} = x + z,\ \frac{dz}{dx} = x - y^2$$

[*Ans.* $y(0.1) = 2.0845$, $z(0.1) = 0.5867$
$y(0.2) = 2.1367$, $z(0.2) = 0.15497$]

7.16 Find the value of $y(1.1)$ and $y(1.2)$ from $\dfrac{d^2y}{dx^2} + y^2\dfrac{dy}{dx} = x^3$, $y(1) = 1$, $y'(1) = 1$ by using the Taylor series method.

[*Ans.* $y(1.1) = 1.1002$, $y(1.2) = 1.2015$]

7.17 Given $\dfrac{d^2y}{dx^2} - x\left(\dfrac{dy}{dx}\right)^2 + y^2 = 0$ with $y(0) = 1$, $y'(0) = 0$, obtain the values of $y(0.1)$ and $y(0.2)$, correct to 3 decimal places, using the Taylor series method.

[*Ans.* $y(0.1) = 0.995$, $y(0.2) = 0.981$]

7.18 Using the Taylor series method, find $y(0.1)$, $y(0.2)$, given $y'' + xy = 0$ and $y(0) = 1$, $y'(0) = 0.5$.

[*Ans.* $y(0.1) = 1.0498$, $y(0.2) = 1.0986$]

Euler's Method:

we have so far discussed the methods which yield the solution of a differential equation in the form of a power series. we will now describe the methods which give the solution in the form ~~of~~ a set of tabulated values.

Consider the differential equation of first order

$$\frac{dy}{dx} = f(x, y) \quad \text{——— (1)}$$

with the initial condition $y(x_0) = y_0$.

Suppose we want to find ~~the~~ approximate value of y say y_n when $x = x_n$.

we divide the interval $[x_0, x_n]$ into n- subintervals of equal length say h, with the division points $x_0, x_1, \ldots\ldots x_n$, where $x_r = x_0 + rh$ $(r = 1, 2, \ldots n)$

Let us assume that

$$f(x, y) \approx f(x_{r-1}, y_{r-1}) \quad \text{——— (2)}$$

in $[x_{r-1}, x_r]$.

Integrating equation (1) in $[x_{r-1}, x_r]$,

we get

$$\int_{x_{r-1}}^{x_r} dy = \int_{x_{r-1}}^{x_r} f(x, y)\, dx$$

$$\Rightarrow [y_r - y_{r-1}] = \int_{x_{r-1}}^{x_r} f(x, y)\, dx$$

$\Rightarrow y_r = y_{r-1} + f(x_{r-1}, y_{r-1}) \int_{x_{r-1}}^{x_r} dx$

$\Rightarrow$ $\left[\because f(x,y) \simeq f(x_{r-1}, y_{r-1}) \text{ in } [x_{r-1}, x_r]\right]$

$\Rightarrow y_r \simeq y_{r-1} + f(x_{r-1}, y_{r-1})(x_r - x_{r-1})$

$\therefore y_r \simeq y_{r-1} + h f(x_{r-1}, y_{r-1})$ ——— (2)(i)

which is known as Euler's iteration formula.

Taking $r = 0, 1, 2 \ldots n$ in (2)(i), we get the successive approximate values of y as follows

$$y_1 = y(x_1) = y_0 + h f(x_0, y_0)$$
$$y_2 = y(x_2) = y_1 + h f(x_1, y_1)$$
$$\vdots$$
$$y_n = y(x_n) = y_{n-1} + h f(x_{n-1}, y_{n-1}).$$

Note. Euler's method has limited usage because of the large errors that is accumulated as the process proceeds. The process is very slow and to obtain reasonable accuracy with Euler's method we have to take a smaller value of h. Further the method should not be used for a larger range of 'x' as the values found by this method go on becoming farther and farther away from the true values. So the method is unsuitable for practical use and a modification of it, known as the modified Euler method, which gives more accurate results.

Modified Euler's Method:

From Euler's formula we know that

$$y_r \approx y_{r-1} + h f(x_{r-1}, y_{r-1}) \quad \text{——— (1)}$$

Let $y(x_r) = y_r$ denote the initial value. An approximate value of $y_r^{(0)}$ can be using (1) an approximate value of can be calculated as

$$\bar{y}_r^{(0)} = y_{r-1} + \int_{x_{r-1}}^{x_r} f(x, y)\, dx.$$

$$\Rightarrow y_r^{(0)} \approx y_{r-1} + h f(x_{r-1}, y_{r-1})$$

Replacing $f(x, y)$ by $f(x_{r-1}, y_{r-1})$ in $x_{r-1} \le x < x_r$.

Using Trapezoidal rule in $[x_{r-1}, x_r]$, we can write

$$y_r^{(0)} = y_{r-1} + \frac{h}{2}\left[f(x_{r-1}, y_{r-1}) + f(x_r, y_r)\right]$$

Replacing $f(x_r, y_r)$ by its approximate value $f(x_r, \bar{y}_r^{(0)})$ at the end point of the interval $[x_{r-1}, x_r]$, we get

$$y_r^{(1)} = y_{r-1} + \frac{h}{2}\left[f(x_{r-1}, y_{r-1}) + f(x_r, y_r^{(0)})\right]$$

where $y_r^{(1)}$ is the first approximation to $y_r = y(x_r)$ proceeding as above we get the iteration formula

$$y_r^{(n)} = y_{r-1} + \frac{h}{2}\left[f(x_{r-1}, y_{r-1}) + f(x_r, y_r^{(n-1)})\right]$$

where $y_n^{(k+1)}$ denotes the k^{th} approximation to y_n.

we have

$$y_n = y_n^{(k+1)} = y_{n-1} + \frac{h}{2}\left[f(x_{n-1}, y_{n-1}) + f(x_n, y_n^{(k)})\right]$$

Example : Solve the equation $\frac{dy}{dx}=1-y$, with the initial condition $x = 0$, $y = 0$, using Euler's algorithm and tabulate the solutions at $x = 0.1, 0.2, 0.3$.

Solution : Given $\frac{dy}{dx}=1-y$, with the initial condition $x = 0, y = 0$

$$\therefore f(x, y) = 1 - y$$

we have

$$h = 0.1$$

$$\therefore x_0 = 0, y_0 = 0$$

$$x_1 = x_0 + h = 0 + 0.1 = 0.1$$

$$x_2 = 0.2, x_3 = 0.3$$

Taking $n = 0$ in

$$y_{n+1} = y_n + hf(x_n, y_n)$$

we get

$$y_1 = y_0 + hf(x_0, y_0)$$

$$= 0 + (0.1)(1 - 0) = 0.1$$

$$\therefore y_1 = 0.1 \text{ i.e. } y(0.1) = 0.1,$$

$$y_2 = y_1 + hf(x_1, y_1)$$

$$y_2 = 0.1 + (0.1)(1 - y1)$$

$$= 0.1 + (0.1)(1 - 0.1)$$

$$= 0.19$$

$$\therefore y_2 = y(0.2) = 0.19,$$

$$y_3 = y_2 + hf(x_2, y_2)$$

$$\therefore y_3 = 0.19 + (0.1)(1 - y_2)$$

$$= 0.19 + (0.1)(1 - 0.19)$$

$$= 0.19 + (0.1)(0.81)$$

$$= 0.271$$

$$\therefore y_3 = y(0.3) = 0.271.$$

x	Solution by Euler's method
0	0
0.1	0.1
0.2	0.19
0.3	0.271

Example : Given $\frac{dy}{dx} = x^3 + y$, $y(0) = 1$, compute $y(0.2)$ by Euler's method taking $h = 0.01$.

Solution : Given

$$\frac{dy}{dx} = x^3 + y,$$

with the initial condition $y(0) = 1$.

∴ We have

$$f(x, y) = x^3 + y$$

$$x_0 = 0,\ y_0 = 1,\ h = 0.01$$

$$x_1 = x_0 + h = 0 + 0.01 = 0.01,$$

$$x_2 = x_0 + 2h = 0 + 2(0.01) = 0.02.$$

Applying Euler's formula we get

$$y_1 = y_0 + hf(x_0, y_0)$$

$$\therefore\ y_1 = 1 + (0.01)(x_0^3 + y_0)$$

$$= 1 + (0.01)(0^3 + 1)$$

$$= 1.01$$

$$\therefore\ y_1 = y(0.01) = 1.01,$$

$$y_2 = y_1 + hf(x_1, y_1)$$

$$= 1.01 + (0.01)[x_1^3 + y_1]$$

$$= 1.01 + (0.01)[(0.01)^3 + 1.01] = 1.0201$$

$$\therefore\ y_2 = y(0.02) = 1.0201.$$

Example : Solve by Euler's method the following differential equation $x = 0.1$ correct to four decimal places $\frac{dy}{dx} = \frac{y-x}{y+x}$ with the initial condition $y(0) = 1$.

Solution : Here

$$\frac{dy}{dx} = \frac{y-x}{y+x}$$

$$\Rightarrow f(x,y) = \frac{y-x}{y+x},$$

the initial condition is $y(0) = 1$.

[illegible] $= 0.02$, we get [illegible]

$$x_1 = 0.02,$$
$$x_2 = 0.04,$$
$$x_3 = 0.06,$$
$$x_4 = 0.08,$$
$$x_5 = 0.1.$$

[illegible]

$$y_1 = y(0.02) = y_0 + hf(x_0, y_0)$$

$$= y_0 + h\left(\frac{y_0 - x_0}{y_0 + x_0}\right)$$

$$= 1 + (0.02)\left(\frac{1-0}{1+0}\right)$$

$$= 1.0200$$

$$\therefore y(0.02) = 1.0200,$$

$$y_2 = y(0.04) = y_1 + hf(x_1, y_1)$$

$$= y_1 + h\left(\frac{y_1 - x_1}{y_1 + x_1}\right)$$

$$= 1.0200 + (0.02)\left(\frac{1.0200 - 0.02}{1.0200 + 0.02}\right)$$

$$= 1.0392$$

$$\therefore y_2 = y(0.04) = 1.0392,$$

$y_3 =$ [illegible] $\left(\frac{y_2 - x_2}{y_2 + x_2}\right)$

$=$ [illegible] $\left[\frac{\text{[illegible]} - 0.04}{\text{[illegible]} + 0.04}\right]$

$\therefore y_3 =$ [illegible]

$y_4 =$ [illegible]

$= y_3 + h$ [illegible]

$=$ [illegible] $\left[\frac{\text{[illegible]} - 0.06}{\text{[illegible]} + 0.06}\right]$

$=$ [illegible]

$\therefore y_4 =$ [illegible]

$$y_5 = y(0.1) = y_4 + hf(x_4, y_4)$$

$$= y_4 + h\left(\frac{y_4 - x_4}{y_4 + x_4}\right)$$

$$= 1.0756 + (0.02)\left[\frac{1.0756 - 0.08}{1.0756 + 0.08}\right]$$

$$= 1.0928$$

$\therefore y(0.1) = 1.0928.$

Example – : Solve the Euler's modified method the following differential equation for $x = 0.02$ by taking $h = 0.01$ $\frac{dy}{dx} = x^2 + y$, $y = 1$, when $x = 0$.

Solution : Here we have

$$f(x, y) = x^2 + y$$

$$h = 0.01,\ x_0 = 0,\ y_0 = y(0) = 1$$

$$x_1 = 0.01,\ x_2 = 0.02$$

we get

$$\therefore y_1^{(0)} = y_0 + hf(x_0, y_0)$$

$$= 1 + (0.01)(x_0^2 + y_0)$$

$$= 1 + (0.01)(0^2 + 1) = 1.01$$

$\therefore y_1^{(0)} = 1.01.$

Applying Euler's modified formula we get

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(0)}\right)\right]$$

$$= 1 + \frac{0.01}{2}\left[0^2 + 1 + (0.01)^2 + 1.01\right]$$

$$= 1.01005$$

$\therefore y_1^{(1)} = 1.01005,$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)\right]$$

$$= 1 + \frac{0.01}{2}\left[0^2 + 1 + (0.01)^2 + 1.1005\right]$$

$$= 1.01005$$

$y_1^{(2)} = 1.01005,$

$\therefore y_1^{(1)} = y_1^{(2)} = 1.01005,$

$\therefore y_2^{(0)} = y_1 + hf(x_1, y_1)$

$= 1.01005 + (0.01)(x_1^2 + y_1)$

$= 1.01005 + (0.01)((0.01)^2 + 1.01005)$

$= 1.02015,$

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right]$$

$$= 1.01005 + \frac{0.01}{2}\left[(0.01)^2 + (1.01005) + (0.02)^2 + (1.02015)\right]$$

$= 1.020204,$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right]$$

$$= 1.01 + \frac{0.01}{2}\left[(0.01)^2 + (1.01005) + (0.02)^2 + (1.020204)\right]$$

$\therefore y_2 = 1.020204$

$\therefore y_2 = y(0.02) = 1.020204$

Exercises

1. Given $\frac{dy}{dx} = \frac{1}{x^2 + y}$, $y(4) = 4$. Find y[illegible] by Taylor's series method, taking $h = 0.1$.

2. Given that $\frac{dy}{dx} = x + y^2$, $y(0) = 1$, find y[illegible].

3. Solve $\frac{dy}{dx} = 3x + y^2$, $y = 1$, when $x = 0$ numerically upto $x = 0.1$ by Taylor's series method.

4. Apply Taylor's algorithm to $y' = x^2 + y^2$, $y(0) = 1$. Take $h = 0.5$ and determine approximations to $y(0.5)$. Carry the calculations upto 3 decimals.

5. Find $y(1)$ by Euler's method from the differential equation $\frac{dy}{dx} = \frac{-y}{1+x}$, when $y(0.3) = 2$. Convert upto four decimal places taking step length $h = 0.1$.

6. Find $y(4.4)$, by Euler's modified method taking $h = 0.2$ from the differential equation $\frac{dy}{dx} = \frac{2-y^2}{5x}$, $y = 1$, when $x = 4$.

7. Given $\frac{dy}{dx} = x^2 + y$, with $y(0) = 1$, evaluate $y(0.02)$, $y(0.04)$ by Euler's method.

8. Given $\frac{dy}{dx} = y - x$, where $y(0) = 2$, find $y(0.1)$ and $y(0.2)$ by Euler's method upto two decimal places.

9. Given $\frac{dy}{dx} = -\frac{y-x}{1+x}$, with boundary condition $y(0) = 1$, find approximately y for $x = 0.1$, by Euler's method (five steps).

10. Use modified Euler's method with one step to find the value of y at $x = 0.1$ to five significant figures, where $\frac{dy}{dx} = x^2 + y$, $y = 0.94$, when $x = 0$.

11. Solve $y' = x - y^2$, by Euler's method for $x = 0.2$ to 0.6 with $h = 0.2$ initially $x = 0$, $y = 1$.

12. Determine $y(0.02)$, $y(0.04)$ and $y(0.06)$ using Euler's modified method.

Answers

1. 4.0098 2. 1.2375 3. 1.12725 4. 1.052 5. 1.2632 6. 1.01871 7. 1.0202, 1.0408, 1.0619 8. 2.42, 2.89 9. 1.0928 10. 1.039474 11. $y(0.2) = 0.8512$, $y(0.4) = 0.7798$, $y(0.6) = 0.7260$ 12. $y(0.02) = 1.0202$, $y(0.04) = 1.0408$, $y(0.06) = 1.0619$

Runge-Kutta method:

As already, we considered the initial value problem $y' = f(x,y)$; $y(x_0) = y_0$ — (1) and developed Taylor series method and Euler's method for its solution.

As mentioned earlier, Euler's method being a first order method, requires a very small step size for reasonable accuracy and therefore may require lot of computations.

Higher order Taylor series methods require evaluation of higher order derivatives either manually or computationally. For complicated functions, finding second, third and higher order total derivatives is tedious. Hence Taylor series methods of higher order are not of much practical use in finding the solution of initial value problems of the form given by equation (1).

However there is a class of methods known as Runge-kutta methods which do not require the calculations of higher order derivatives and give greater accuracy. The Runge-kutta formulae possess the advantage of requiring only the function values at some selected points. These methods agree with Taylor's series solution upto the term in h^r where 'r' differs from method to method and is called the order of that method.

(i) First order R-K method:

we have seen that Euler's method gives

$$y_1 = y_0 + h f(x_0, y_0)$$
$$= y_0 + h y_0' \qquad (\because y' = f(x,y))$$

Expanding y_1 by Taylor's series

$$y_1 = y(x_0+h) = y_0 + h y_0' + \frac{h^2}{2!} y_0'' + \cdots$$

It follows that the Euler's method agrees with the Taylor's series solution upto the term in h.

Hence, Euler's method is the Runge-Kutta method of the first order.

(ii) Second order R-K method:

The modified Euler's method gives

$$y_1 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_0+h, y_1^{(1)})\right] \quad \text{--- (1)}$$

where $y_1^{(1)} = y_0 + h f(x_0, y_0)$.

Substituting $y_1^{(1)} = y_0 + h f(x_0, y_0)$ on the R.H.S of (1), we obtain

$$y_1 = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_0+h, y_0 + h f(x_0, y_0))\right]$$
$$= y_0 + \frac{h}{2}\left[f_0 + f(x_0+h, y_0 + h f_0)\right] \quad \text{--- (2)}$$

where $f_0 = f(x_0, y_0)$.

Expanding LHS by Taylor's series we get

$$y_1 = y(x_0+h) = y_0 + h y_0' + \frac{h^2}{2!} y_0'' + \frac{h^3}{3!} y_0''' + \cdots \quad (3)$$

Expanding $f(x_0+h,\ y_0+hf_0)$ by Taylor's series for a function of two variables, eqn (2) gives

$$y_1 = y_0 + \frac{h}{2}\left[f_0 + \left\{ f(x_0,y_0) + h\left(\frac{\partial f}{\partial x}\right)_0 + hf_0\left(\frac{\partial f}{\partial y}\right)_0 + O(h^2)\right.\right.$$

where $O(h^2)$ — terms containing second and higher powers of h and is read as order of h^2.

$$\left[\because f(x,y) = f(x_0,y_0) + \left(\frac{\partial f}{\partial x}\right)_0 (x-x_0) + \left(\frac{\partial f}{\partial y}\right)_0 (y-y_0) + \cdots\right.$$

$$= f(x_0,y_0) + \left(\frac{\partial f}{\partial x}\right)_0 h + \left(\frac{\partial f}{\partial y}\right)_0 h + \cdots$$

$$\left.\text{putting } x = x_0+h,\ y = y_0+h\right]$$

$$= y_0 + \frac{1}{2}\left[hf_0 + hf_0 + h^2\left\{\left(\frac{\partial f}{\partial x}\right)_0 + f_0\left(\frac{\partial f}{\partial y}\right)_0\right\} + O(h^3)\right]$$

$$= y_0 + \frac{1}{2}\left[2hf_0 + h^2 f_0' + O(h^3)\right]$$

$$\left[\because \frac{d}{dx} f(x,y) = \frac{\partial f}{\partial x} + f\frac{\partial f}{\partial y}\right.$$

$$= y_0 + hf_0 + \frac{h^2}{2} f_0' + O(h^3) \qquad \left[\frac{d}{dx} f(x,y)\right]_{(x_0,y_0)} = \left(\frac{\partial f}{\partial x}\right)_0 + f_0\left(\frac{\partial f}{\partial y}\right)_0$$

$$= y_0 + h y_0' + \frac{h^2}{2} y_0'' + O(h^3) \quad (4)$$

$$\left[\because y' = f(x,y)\right]$$

Comparing (3) & (4) it follows that the modified Euler's method agrees with the Taylor's series solution up to the term in h^2.

Hence the modified Euler's method is the Runge-Kutta method of the second order.

The second order Runge-Kutta formula is

$$y_1 = y_0 + \frac{1}{2}(k_1 + k_2)$$

where $k_1 = hf(x_0, y_0)$

and $k_2 = hf(x_0 + h, y_0 + k_1)$

The third order Runge-Kutta formula is

$$y_1 = y_0 + \frac{1}{6}(k_1 + 4k_2 + k_3)$$

where $k_1 = hf(x_0, y_0)$,

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$$k_3 = hf(x_0 + h, y_0 + k')$$

where $k' = hf(x_0 + h, y_0 + k_1)$

Fourth order R-K method:

This method is most commonly used and is often referred to as Runge-Kutta method only.

Working rule:- for finding the increment k of y corresponding to an increment h of x by Runge-Kutta method

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

is as follows.

Calculate successively

$$k_1 = hf(x_0, y_0)$$

$$k_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$k_3 = h\,f(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2})$

and $k_4 = h\,f(x_0 + h, y_0 + k_3)$

Finally compute $k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

which gives the required approximate value as $y_1 = y_0 + k$.

Note: One of the advantages of these methods is that the operation is identical whether the differential equation is linear or non-linear.

→ Apply Runge-kutta fourth order method to find an approximate value of y when $x = 0.2$ given that $\frac{dy}{dx} = x + y$ and $y = 1$ when $x = 0$.

Soln:

we have

$x_0 = 0,\ y_0 = 1$

$f(x, y) = x + y$ and $h = 0.2$

$f(x_0, y_0) = x_0 + y_0$

$= 0 + 1 = 1$

$k_1 = h f(x_0, y_0)$

$= 0.2 \times 1 = 0.2$

$k_2 = h\,f(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2})$

$= (0.2)\,f(0.1, 1 + 0.1)$

$= (0.2)\,f(0.1, 1.1)$

$= (0.2)(1.2)$

$= 0.24$

$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$

$= (0.2) f(0.1, 1 + 0.12)$

$= (0.2) f(0.1, 1.12)$

$= (0.2)(1.22) = 0.2440$

$k_4 = h f(x_0 + h, y_0 + k_3)$

$= (0.2) f(0.2, 1 + 0.244)$

$= (0.2) f(0.2, 1.244)$

$= (0.2)(1.444) = 0.2888$

$\therefore k = \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$= \frac{1}{6}(0.2 + 2(0.24) + 2(0.244) + 0.2888)$

$= \frac{1}{6}(1.4568)$

$= 0.2428$

Hence the required approximate value of y is $= y_0 + k$

$= 1 + 0.2428$

$= 1.2428$

Example : Use Runge – Kutta method to approximate y when $x = 0.1$, given that $y = 1$, when

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$$x = 0 \text{ and } \frac{dy}{dx} = x + y .$$

Solution : We have

$$x = 0 , y_0 = 1$$

$$f(x, y) = x + y, \text{ and } h = 0.1.$$

$$\therefore f(x_0, y_0) = x_0 + y_0 = 0 + 1 = 1,$$

we get

$$k_1 = h f(x_0, y_0)$$
$$= 0.1 \times 1$$
$$= 0.1,$$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$
$$= (0.1)(f(0 + 0.05), 1 + 0.05)$$
$$= (0.1) f(0.05, 1.05)$$
$$= (0.1)(0.05 + 1.05)$$
$$= 0.11,$$

$$k_3 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$= (0.1)(f(0 + 0.05), 1 + 0.055)$

$= (0.1)(0.05 + 1.055)$

$= (0.1)(1.105)$

$= 0.1105,$

$k_4 = hf(x_0 + h, y_0 + k_3)$

$= (0.1)f(0 + 0.01, 1 + 0.1105)$

$= (0.1)f(0.1, 1.1105)$

$= (0.1)(1.2105)$

$= 0.12105,$

$$\therefore \Delta y = \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$= \frac{1}{6}\left(0.1 + 0.22 + 0.2210 + 0.12105\right)$$

$= 0.11034.$

Hence

$x_1 = x_0 + h = 0 + 0.1 = 0.1$

$y_1 = y_0 + \Delta y = 1 + 0.11034$

$= 1.11034.$

Example : Using Runge–Kutta method, find an approximate value of y for $x = 0.2$, if $\frac{dy}{dx} = x + y^2$, given that $y = 1$ when $x = 0$.

Solution: Taking step–length $h = 0.1$, we have

$$x_0 = 0,\ y_0 = 1,\ \frac{dy}{dx} = f(x,y) = x + y^2.$$

Now

$k_1 = hf(x_0, y_0) = (0.1)(0 + 1) - 0.1,$

$$k_2 = h f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}\right)$$

$= (0.1)(0.05 + 1.1025)$

$= (0.1)(1.1525)$

$= 0.11525,$

$$k_3 = h\, f\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}\right)$$

$$= (0.1)(0.05 + 1.1185)$$

$$= (0.1)(1.1685)$$

$$= 0.11685$$

$$k_4 = h\, f(x_0 + h,\ y_0 + k_3)$$

$$= (0.1)(0.01 + 1.2474)$$

$$= (0.1)(1.3474)$$

$$= 0.13474,$$

$$\therefore\ K = \Delta y = \frac{1}{6}\left(k_1 + 2k_2 + 2k_3 + k_4\right)$$

$$= \frac{1}{6}\left(0.1 + 2(0.11525) + 2(0.11685) + 0.13474\right)$$

$$= \frac{1}{6}\left(0.6991\right) = 0.1165\,.$$

We get

$$y_1 = y_0 + \Delta y = 1 + 0.1165$$

$$\therefore y(0.1) = 1.1165.$$

For the second step, we have

$$x_0 = 0.1,\ y_0 = 1.1165,$$

$$k_1 = (0.1)(0.1 + 1.2466) = 0.1347,$$

$$k_2 = (0.1)(0.15 + 1.4014) = (0.1)(1.55514)$$

$$= 0.1551,$$

$$k_3 = (0.1)(0.15 + 1.4259) = (0.1)(1.5759)$$

$$= 0.1576,$$

$$k_4 = (0.1)(0.2 + 1.6233) = (0.1)(1.8233)$$

$$= 0.1823,$$

$$K = \Delta y = \frac{1}{6}\left(0.9424\right) = 0.1571\,,$$

$$\therefore y(0.2) = 1.1165 + 0.1571 = 1.2736.$$

$$\therefore y(0.1) = 1.1165 \text{ and } y(0.2) = 1.2736.$$

Exercise

1. Solve the equation $\frac{dy}{dx} = x - y^2$, $y(0) = 1$ for $x = 0.2$ and $x = 0.4$ to 3 decimal places by Runge – Kutta fourth order method.

2. Use the Runge–Kutta method to approximate y at $x = 0.1$ and $x = 0.2$ for the equation $\frac{dy}{dx} = x + y$, $y(0) = 1$.

3. For the equation $\frac{dy}{dx} = 3x + \frac{y}{2}$, $y(0) = 1$. Find y at the following points with the given step – lengths.

4. Use Runge – Kutta method to solve $y' = xy$ for $x = 1.4$, initially $x = 1$, $y = 2$ (by taking step – length $h = 0.2$).

5. $\frac{dy}{dx} = \frac{y^2 - 2x}{y^2 + x}$, use Runge – Kutta method to find y at $x = 0.1, 0.2, 0.3$ and 0.4, given that $y = 1$ when $x = 0$.

6. Use Runge – Kutta method to obtain y when $x = 1.1$ given that $y = 1.2$ when $x = 1$ and y satisfies the equation $\frac{dy}{dx} = 3x + y^2$.

Answers

1. 0.851, 0.780 2. 1.1103, 1.2428

3.

x	h	y
0.1	0.1	1.0665242
0.2	0.2	1.1672208
0.4	0.4	1.4782

4. 2.99485866 5. $y(0.1) = 1.0874$, $y(0.2) = 1.1557$, $y(0.3) = 1.2104$, $y(0.4) = 1.2544$

6. $y(1.1) = 1.7271$

Runge-Kutta Method for a System of Equations

The fourth-order Runge-Kutta method can be extended to numerically solve the higher order ordinary differential equations - linear or non-linear.

Let us consider a second order ordinary differential equation of the form

$$\frac{d^2y}{dx^2} = f\left(x, y, \frac{dy}{dx}\right) \quad \text{with the initial condition } y(x_0) = y_0 \ \& \ y'(x_0) = y_0' \quad \text{--- (1)}$$

By writing $\frac{dy}{dx} = p$, it can be reduced to two first order differential equations as given below.

$$\frac{dy}{dx} = p = f(x, y, p), \quad \frac{dp}{dx} = f_2(x, y, p)$$

with the initial conditions

$$y(x_0) = y_0$$

$$\text{and } y'(x_0) = y_0' \Rightarrow p(x_0) = p_0$$

Now, the Runge-Kutta method is applied as follows starting at (x_0, y_0, p_0) and taking the step-sizes for x, y, z to be h, k, l respectively,

then we define,

$$k_1 = hf_1(x_0, y_0, p_0) \quad ; \quad l_1 = hf_2(x_0, y_0, p_0)$$

$$k_2 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, p_0 + \frac{l_1}{2}\right) \quad ; \quad l_2 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, p_0 + \frac{l_1}{2}\right)$$

$$k_3 = hf_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, p_0 + \frac{l_2}{2}\right) \quad ; \quad l_3 = hf_2\left(x_0 + \frac{h}{2}, y_0 + \frac{k_2}{2}, p_0 + \frac{l_2}{2}\right)$$

$$k_4 = hf_1(x_0 + h, y_0 + k_3, p_0 + l_3) \quad ; \quad l_4 = hf_2(x_0 + h, y_0 + k_3, p_0 + l_3),$$

Hence $y_1 = y_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

$p_1 = p_0 + \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

To compute y_2 and p_2, we simply replace x_0, y_0, p_0 by x_1, y_1, p_1 in the above formulae.

→ Using Runge-kutta method, solve $y'' = xy'^2 - y^2$ for $x = 0.2$ correct to 4 decimal places. Initial conditions are $x = 0$, $y = 1$, $y' = 0$.

Sol:- Let $\frac{dy}{dx} = p = f_1(x, y, p)$

Then $\frac{dp}{dx} = xp^2 - y^2 = f_2(x, y, p)$

we have $x_0 = 0$, $y_0 = 1$, $p_0 = 0$, $h = 0.2$

∴ Runge-kutta formulae become

$k_1 = h f_1(x_0, y_0, p_0)$
$= (0.2)(0) = 0$

$l_1 = h f_2(x_0, y_0, p_0)$
$= (0.2)(x_0 p_0^2 - y_0^2)$
$= (0.2)[(0)(0) - 1^2]$
$= (0.2)(-1) = -0.2$

$k_2 = h f_1\left(x_0 + \frac{h}{2}, y_0 + \frac{k_1}{2}, p_0 + \frac{l_1}{2}\right)$
$= (0.2) f_1(0.1, 1, -0.1)$
$= (0.2)(-0.1) = -0.02$

$\therefore\; K = \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)$

$= \frac{1}{6}\left[0.00 + 2(-0.02) + 2(-0.01998) - 0.0392\right]$

$= \frac{1}{6}(-0.11916)$

$= -0.01986 = -0.0199$

$l = \frac{1}{6}(l_1 + 2l_2 + 2l_3 + l_4)$

$= \frac{1}{6}\left[-0.2 + 2(-0.1998) + 2(-0.1958) - 0.1905\right]$

$= \frac{1}{6}(-1.1817) = -0.19695$

$= -0.1970$

Hence at $x = 0.2$

$y = y_0 + K = 1 - 0.0199$
$= 0.9801$

and $y' = p = p_0 + l = 0 - 0.1970$
$= -0.1970.$

→ Using Runge-kutta method of order 4, solve $y'' = y + xy'$, $y(0) = 1$, $y'(0) = 0$ to find $y(0.2)$ and $y'(0.2)$.

→ Solve the differential equations $\frac{dy}{dx} = 1 + xp$, $\frac{dp}{dx} = -xy$ for $x = 0.3$, using fourth order Runge-kutta method. Initial values are $x = 0$, $y = 0$, $p = 1$.

$$l_2 = h f_2 \left(x_0 + \frac{h}{2},\ y_0 + \frac{k_1}{2},\ p_0 + \frac{l_1}{2}\right)$$
$$= (0.2)\,(0.1,\ 1,\ -0.1)$$
$$= (0.2)\left[(0.1)(-0.1)^2 - 1^2\right]$$
$$= (0.2)\left[0.001 - 1\right]$$
$$= (0.2)\,(-0.999) = -0.1998$$

$$k_3 = h f_1 \left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2},\ p_0 + \frac{l_2}{2}\right)$$
$$= (0.2)\, f_1 (0.1,\ 0.99,\ -0.0999)$$
$$= (0.2)\,(-0.0999)$$
$$= -0.01998$$

$$l_3 = h f_2 \left(x_0 + \frac{h}{2},\ y_0 + \frac{k_2}{2},\ p_0 + \frac{l_2}{2}\right)$$
$$= (0.2)\, f_2 (0.1,\ 0.99,\ -0.0999)$$
$$= (0.2)\left[(0.1)(-0.0999)^2 - (0.99)^2\right]$$
$$= (0.2)\,(-0.9791)$$
$$= -0.1958$$

$$k_4 = h f_1 (x_0 + h,\ y_0 + k_3,\ p_0 + l_3)$$
$$= (0.2)\, f_1 (0.2,\ 0.98002,\ -0.1958)$$
$$= (0.2)\,(-0.1958)$$
$$= -0.03916 = -0.0392$$

$$l_4 = h f_2 (x_0 + h,\ y_0 + k_3,\ p_0 + l_3)$$
$$= (0.2)\, f_2 (0.2,\ 0.98002,\ -0.1958)$$
$$= (0.2)\left[(0.2)(-0.1958)^2 - (0.98002)^2\right]$$
$$= (0.2)\,(-0.9527) = -0.1905$$

Number Systems and Codes

Introduction: –

- We all are familiar with the number system in which an ordered set of ten symbols — 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9, known as digits — are used to specify any number.
- This number system is popularly known as the decimal number system.
- The radix or base of this number system is 10 (number of distinct digits).
- Any number is a collection of these digits.

For example: 1980.0918 signifies a number with an integer part equal to 1980 and a fractional part equal to 0.0918, separated from the integer part with a radix point also known as decimal point.

- There are some other systems also, used to represent numbers.
- Some of the other commonly used number systems are: binary, octal and hexadecimal number systems.

These number systems are widely used in digital systems like microprocessors, logic circuits, computers etc. and therefore, the knowledge of these number systems is very essential for understanding, analysing and designing digital systems.

— computers and other digital circuits use binary signals but are required to handle data which may be numeric, alphabets or special characters. Therefore, the information available in any other form is required to be converted into suitable binary form before it can be processed by digital circuits. This means that the information available in the form of numeral, alphabets and special characters or in any combination of these must be converted into binary format.

To achieve this, a process of coding is employed where by each numeral, alphabet or special character is coded in a unique combination of 0's and 1's using a coding scheme, known as a code.

— The process of coding is known as encoding.

— There can be a variety of coding schemes (codes) to serve different purposes, such as arithmetic operations, data entry, error detection and correction etc. In digital systems, a large number of codes are in use. Selection of a particular code depends on its suitability for the purpose. In one digital system, different codes may be used for different operations and it may be necessary to convert data from one code to another code.

In general, in any number system there is an ordered set of symbols known as *digits* with rules defined for performing arithmetic operations like addition, multiplication, etc.

A collection of these digits makes a number which in general has two parts - *integer* and *fractional*, set apart by a radix point (.), that is

$$(N)_b = \underbrace{d_{n-1}\, d_{n-2}\, d_{n-3} \cdots d_i \cdots d_2\, d_1\, d_0}_{\text{integer portion}} \cdot \underbrace{d_{-1}\, d_{-2} \cdots d_{-f} \cdots d_{-m}}_{\text{fractional portion}}$$

Radix point

where

N = a number

b = radix or base of the number system

n = number of digits in integer portion $(n-1, n-2, \cdots i \cdots 1, 0)$

m = number of digits in fractional portion

d_{n-1} = most significant digit (msd)

d_{-m} = least significant digit (lsd)

and $0 \leq (d_i \text{ or } d_{-f}) \leq b-1$.

— The digits in a number are placed side by side and each position in the number is assigned a *weight* or *index of importance* by some predesigned rule.

* commonly used number systems: -

The following table shows the used number systems:

Table - 1

Number System	Base or radix(b)	Symbols used (d_i or d_{-f})	weight assigned to position i	weight assigned to position $-f$	Example
Binary	2	0, 1	2^i	2^{-f}	1011.11
Octal	8	0, 1, 2, 3, 4, 5, 6, 7	8^i	8^{-f}	3567.25
Decimal	10	0, 1, 2, 3, 4, 5, 6, 7, 8, 9	10^i	10^{-f}	3974.57
Hexadecimal	16	0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F	16^i	16^{-f}	3FA9.56

Note: (i) The base is usually denoted as a suffix of the number. If no suffix is mentioned, the base is assumed to be 10.

(2) 10, 11, 12, 13, 14, 15 are denoted by A, B, C, D, E, F respectively.

(3) value of the digit: we know that value of the number is based on each digit in it, for this, weights are attached to each digit position in arriving at the value of the digit.

Basically, there are two types of values as follows:
(i) Face value and (ii) place value.

Face value: Face value of the number is simply the number.

place value: The place value of a number relates to weights attached to the position of the number.

The number system that is normally used in decimal system. while assigning weights to each digit of a number, rightmost digit gets the weight of unity and the successive digits to its left usually have weights $10^1, 10^2, 10^3$ and so on. Each digit is multiplied by its weight and the sum of product provides value of the number.

Example (1):

Decimal number 7536_{10} may be expressed as

$$7 \times 10^3 = 7000$$
$$5 \times 10^2 = 500$$
$$3 \times 10^1 = 30$$
$$6 \times 10^0 = 6$$

Thus, $7 \times 10^3 + 5 \times 10^2 + 3 \times 10^1 + 6 \times 10^0 = 7536_{10}$

It should be noted that the digit 7 has a face value and a place value $10^3 = 1000$. Thus the digit represents the product 7×10^3.

Example (2): Decimal number 65355_{10} may be expressed as

$$6 \times 10^4 = 60000$$
$$5 \times 10^3 = 5000$$
$$3 \times 10^2 = 300$$
$$5 \times 10^1 = 50$$
$$5 \times 10^0 = 5$$

Thus, $6 \times 10^4 + 5 \times 10^3 + 3 \times 10^2 + 5 \times 10^1 + 5 \times 10^0 = 65355_{10}$.

* Binary System:

The binary system may be defined as a number system, having two symbols 0 and 1. The base (or radix) of a binary number system is 2. and the symbols 0 and 1 are termed as binary digits or bits.

Example (1): 1101_2 is a binary number, to find the decimal value of the binary number, powers of two (2) are used as weights in a binary system, and is as follows:

$1 \times 2^3 = 8$

$1 \times 2^2 = 4$

$0 \times 2^1 = 0$

$1 \times 2^0 = 1$

Thus the decimal value of 1101_2 is

$1 \times 2^3 + 1 \times 2^2 + 0 \times 2^1 + 1 \times 2^0 = 13.$

Note: (1) while assigning weights to each digit of a number, rightmost digit gets the weight of unity and the successive digits to its left usually have weights $2^1, 2^2, 2^3$ and so on.

- Each digit is multiplied by its weight and the sum of product provides the decimal value of the binary number.

1101_2 → least significant bit.

most significant bit

- The rightmost bit is called the least significant bit

bit.

(2) Any number of 0's can be added to the left of the number without changing the value of the number.

(3) In the binary number system, a group of four bits is known as a nibble and a group of eight bits is known as a byte.

(4) Just as power of '10' are important in the decimal system of enumeration (countable), powers of '2' are important in binary systems.

We thus give in Table powers of 2 and their decimal equivalents.

The abbreviation K in table stands for 1024 which is approximately 1000, a kilo.

- Thus the notation 16 K means $16 \times 1024 = 16384$
- The abbreviation M (Mega) stands for $1024 \times 1024 = 1048576$.
- The abbreviation G (Giga) is used to represent $1024 \times 1024 \times 1024$ which is nearly a billion.
- The abbreviation T (Tera) is used to represent $1024 \times 1024 \times 1024 \times 1024$, which is nearly a trillion.

powers of 2	Decimal equivalent	Abbreviation
2^0	1	
2^1	2	
2^2	4	
2^3	8	
2^4	16	
2^5	32	
2^6	64	
2^7	128	
2^8	256	
2^9	512	
2^{10}	1024	1 K
2^{11}	2048	2 K
2^{12}	4096	4 K
2^{13}	8192	8 K
2^{14}	16384	16 K
2^{15}	32768	32 K
2^{16}	65536	64 K
2^{17}	131072	128 K
2^{18}	262144	256 K
2^{19}	524288	512 K
2^{20}	1048576	1 M
2^{30}	1073741824	1 G

* Octal system:

The octal system is the number system consisting of eight symbols 0, 1, 2, 3, 4, 5, 6, and 7. The base of an octal number system is '8'. Thus the weight assigned to each digit in octal sytem is power of 8.

Example (1): The decimal value of the octal number 176 can be obtained in the following manner.

$1 \times 8^2 = 64$

$7 \times 8^1 = 56$

$6 \times 8^0 = 6$

Thus the derived decimal value of 176 is $1 \times 8^2 + 7 \times 8^1 + 6 \times 8^0 = 126$.

Example (2): The decimal value of the octal number 1116 can be obtained in the following manner:

$1 \times 8^3 = 512$

$1 \times 8^2 = 64$

$1 \times 8^1 = 8$

$6 \times 8^0 = 6$

Thus the derived decimal value of 1116 is $1 \times 8^3 + 1 \times 8^2 + 1 \times 8^2 + 1 \times 8^1 + 6 \times 8^0 = 590$.

*** Hexadecimal system:**

The number system having symbols such as 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F is called hexadecimal system. It has 16 as the basis.

Example(1): The decimal value of hexadecimal number 84A8 is obtained as follows:

$8 \times 16^3 = 32768$

$4 \times 16^2 = 1024$

$A \times 16^1 = 160$

$8 \times 16^0 = 8$

Thus the decimal value of 84A8 is $8 \times 16^3 + 4 \times 16^2 + A \times 16^1 + 8 \times 16^0 = 33960$.

Example(2): The decimal value of hexadecimal number 21E3 is obtained as follows:

$2 \times 16^3 = 8192$

$1 \times 16^2 = 256$

$E \times 16^1 = 224$

$3 \times 16^0 = 3$

Thus the decimal value of 21E3 is

$2 \times 16^3 + 1 \times 16^2 + E \times 16^1 + 3 \times 16^0 = 8675$.

Example (3):

The decimal value of hexadecimal number 12AF is obtained as follows:

$1 \times 16^3 = 4096$

$2 \times 16^2 = 512$

$A \times 16^1 = 160$

$F \times 16^0 = 15$

sum 4783

Thus the decimal value of $(12AF)_{16}$ is $(4783)_{10}$

i.e $(12AF)_{16} = 1 \times 16^3 + 2 \times 16^2 + A \times 16^1 + F \times 16^0$

$= (4783)_{10}$

* Binary to Decimal conversion:

Method 1:

To convert a binary system number to a decimal system, the digits are obtained first. The sum of the products of the value of the digit in the decimal system and the digit gives the decimal equivalent.

Example(1):

convert 101101_2 to decimal.

Sol

Binary number	value of the digit (in decimal system)
1	$2^5 = 32$
0	$2^4 = 16$
1	$2^3 = 8$
1	$2^2 = 4$
0	$2^1 = 2$
1	$2^0 = 1$

Thus the decimal value of the binary number 101101_2 is

$1\times2^5 + 0\times2^4 + 1\times2^3 + 1\times2^2 + 0\times2^1 + 1\times2^0 = 45.$

HW → convert the binary number 111111_2 into decimal number.

Method(2)

The procedure to convert a binary number to decimal is called dibble-dabble method.

i.e.
- We start with the left hand bit.
- Multiply this value by 2 and add the next bit.
- Again multiply by 2 and add the next bit.
- Stop when the bit on extreme right hand side is reached.

Example(1) convert 100101_2 to decimal.

Sol Left hand bit 1

Multiply by 2 and add next bit $2\times1+0 = 2$

Multiply by 2 and add next bit $2\times2+0=4$

Multiply by 2 and add next bit $2\times4+1 = 9$

Multiply by 2 and add next bit $2\times9+0 = 18$

Multiply by 2 and add next bit $2\times18+1 = 37$

$\therefore 100101_2 = 37_{10}$.

Method (3)

To convert binary number to decimal number is as under:

(1) write the binary number.

(2) write the weights $2^0, 2^1, 2^2, 2^3$ etc, under the binary digits starting with the bit on right hand side.

(3) Cross out weights under zeros.

(4) Add the remaining weights.

Example (1): ~~convert~~ 1101_2 into equivalent decimal number.

Sol

1	1	0	1	Binary number
8	4	2	1	write weights
8	4	$\cancel{2}$	1	cross out weights under zeros.
$8+4+0+1 = 13$				add weights

$\therefore 1101_2 = 13_{10}$.

HW → Convert 110011011001_2 into equivalent decimal number

Ans: 3289_{10}.

Decimal to Binary conversion:

A systematic way to convert a decimal number into equivalent binary is known as double-dabble method. This method involves successive division by 2 and recording the remainder (the remainder will be always 0 or 1). The division is stopped when we get a quotient of '0' with a remainder of 1. The remainders when read upwards give the equivalent binary number.

Example (1) convert decimal number 747 into its equivalent binary number.

Sol

Division	Quotient	Remainder
747/2	373	1
373/2	186	1
186/2	93	0
93/2	46	1
46/2	23	0
23/2	11	1
11/2	5	1

5/2	2	
2/2	1	0
1/2	0	1

Thus the binary equivalent of 747 is 1011101011_2

H.W. → Convert decimal number 101 into its equivalent binary number.

→ Convert 3289_{10} into binary.

Sol^n

2	3289	
2	1644	remainder 1
2	822	remainder 0
2	411	remainder 0
2	205	remainder 1
2	102	remainder 1
2	51	remainder 0
2	25	remainder 1
2	12	remainder 1
2	6	remainder 0
2	3	remainder 0
2	1	remainder 1
	0	remainder 1

$\therefore 3289_{10} = 110011011001_2$

HW → convert 10_{10} into binary.

HW → convert 25_{10} into binary

Binary fractions:

So far we have discussed only whole numbers. However, to represent fractions is also important.

The decimal number 2568 is represented as $2568 = 2000+500+60+8$

$= 2\times10^3 + 5\times10^2 + 6\times10^1 + 8\times10^0$

Similarly ① 25.68 can be represented as

$25.68 = 20+5+0.6+0.08$

$= 2\times10^1 + 5\times10^0 + 6\times10^{-1} + 8\times10^{-2}$

② $0.238 = 0.2 + 0.03 + 0.008$

$= 2\times10^{-1} + 3\times10^{-2} + 8\times10^{-3}$

* Conversion of Binary to Decimal:

In the binary system, the weights of the binary bits after the binary point, can be written as

$0.1011 = 0.1 + 0.00 + 0.001 + 0.0001$

$= 1\times2^{-1} + 0\times2^{-2} + 1\times2^{-3} + 1\times2^{-4}$

$= 1\times\frac{1}{2} + 0\times\frac{1}{2^2} + 1\times\frac{1}{2^3} + 1\times\frac{1}{2^4}$

$= 0.5 + 0 + 0.125 + 0.0625$

$= 0.6875$ (decimal).

→ Determine the decimal numbers represented by the following binary numbers.

(1) 111011.101 (2) 101101.10101 (3) 1100.1011

(4) 1001.0101 (5) 0.10101 (6) 11000.0011

Soln:

(1) $(111011.101)_2 = 1\times2^5+1\times2^4+1\times2^3+0\times2^2+1\times2^1+1\times2^0+1\times2^{-1}+0\times2^{-2}+1\times2^{-3}$

$= 32+16+8+0+2+1+\frac{1}{2}+0+\frac{1}{8}$

$= 59+0.5+0.125$

$= (59.625)_{10}$.

(3) $(1100.1011)_2 = 1\times2^3+1\times2^2+0\times2+0\times2^0+1\times2^{-1}+0\times2^{-2}+1\times2^{-3}+1\times2^{-4}$

$= 8+4+0+0+\frac{1}{2}+0+\frac{1}{8}+\frac{1}{32}$

$= 8+4+0.5+0.125+0.0625$

$(12.6875)_{10}$.

Conversion of decimal to binary :

To convert a decimal to binary, a method of successive multiplication by 2 is used. After each multiplication, the integer part is noted seperately and the fraction is again multiplied by 2 till the remainder becomes zero.

Sometimes it is possible that the remainder doesnot become zero even after many stages. In such case, an approximation is made and the result is taken upto a certain number of bits after the binary point

A similar procedure is adopted for a number having both integer and fractions.

Binary fractions can be added, subtracted etc. as the decimal numbers.

→ Example:
express the number 0.6875 into Binary equivalent.

Soln

Fraction	Fraction × 2	Remainder new fraction	Integer
0.6875	1.375	0.375	1 (MSB)
0.375	0.75	0.75	0
0.75	1.5	0.5	1
0.5	1	0	1 (LSB)

The binary equivalent is 0.1011

→ Convert the decimal number 0.634 into its binary equivalent.

Soln.

Fraction	fraction × 2	Remainder new fraction	Integer
0.634	1.268	0.268	1 (MSB)
0.268	0.536	0.536	0
0.536	1.072	0.072	1
0.072	0.144	0.144	0
0.144	0.288	0.288	0
0.288	0.576	0.576	0
0.576	1.152	0.152	1 (LSB)

It is seen that it is not possible to get a zero as remainder even after 7 stages.
The process can be continued further or an

terminated here.

The binary equivalent is 0.1010001.

It is important to note that the decimal equivalent of 0.1010001 (binary) is 0.6328125 (decimal)

→ Convert the decimal number 39.12 into binary

Soln: Taking the integer part first:

2	39	
2	19	remainder 1 (LSB)
2	9	remainder 1
2	4	remainder 1
2	2	remainder 0
2	1	remainder 0
		remainder 1 (MSB)

$(39)_{10} = (111001)_2$

Taking the fractional part:

fraction	fraction × 2	Remainder new fraction	Integer
0.12	0.24	0.24	0 (MSB)
0.24	0.48	0.48	0
0.48	0.96	0.96	0
0.96	1.92	0.92	1
0.92	1.84	0.84	1
0.84	1.68	0.68	1
0.68	1.36	0.36	1 (LSB)

This fraction cannot be converted into exact binary number. The process can be terminated here

The result is 0.0001111.

Adding the binary equivalent of 39 and 0.12

$$
\begin{array}{r}
100111.0000000 \\
0.0001111 \\
\hline
100111.0001111
\end{array}
$$

$\therefore (39.12)_{10} = (100111.0001111)_2$

→ Express the following decimal numbers in the binary form.

(a) 25.5 (b) 10.625 (c) 0.6875

* Binary Arithmetic :-

We all are familiar with the arithmetic operations such as addition, subtraction, multiplication and division of decimal numbers.

Similar operations can be performed on binary numbers: in fact, binary arithmetic is much simpler than decimal arithmetic because here only two digits 0 and 1 are involved.

Binary addition :-

The rules of binary addition are given in table:

Augend	Addend	Sum	Carry	Result
0	0	0	0	0
0	1	1	0	1
1	0	1	0	1
1	1	0	1	10

that is, carry = 0; whereas in the fourth row a carry is produced (since the largest digit possible is 1) i.e, carry = 1, and similar to decimal addition it is added to the next higher binary position.

→ Add the binary numbers:

(i) 1011 and 1100 (ii) 0101 and 1111

Sol^n:

$$\begin{array}{r} 1011 \\ (+)\ 1100 \\ \hline 10111 \end{array}$$

↑ carry

Sol^n:

(1) (1) (1) ← carry

$$\begin{array}{r} 0101 \\ +\ 1111 \\ \hline 10100 \end{array}$$

↑ carry

→ Add the binary numbers:

0 1 1 0 1 0 1 0
0 0 0 0 1 0 0 0
1 0 0 0 0 0 0 1
1 1 1 1 1 1 1 1

Sol^n:

Two pair of 1's in the previous column.

one pair of 1's in the previous column.

(1) (1) (1) $\binom{1}{1}$ (1) (1) (1)

$$\begin{array}{r} 0\ 1\ 1\ 0\ 1\ 0\ 1\ 0 \\ 0\ 0\ 0\ 0\ 1\ 0\ 0\ 0 \\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 1 \\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1 \\ \hline 1\ 1\ 1\ 1\ 1\ 0\ 0\ 1\ 0 \end{array}$$

Even number of 1's in the column.

carry

Odd number of 1's in the column.

∴ The sum = 111110010.

From the above example, we observe the following:

(i) if the number of 1's to be added in a column is even then the sum bit is 0, and if the number of 1's to be added in a column is odd then the sum bit is 1.

(ii) Every pair of 1's in a column produces a carry (1) to be added to the next higher bit column.

Binary subtraction:

The rules of binary subtraction are given in table:

Minuend	Subtrahend	Difference	Borrow
0	0	0	0
0	1	1	1
1	0	1	0
1	1	0	0

Except in the second row above, the borrow = 0. When the borrow = 1, as in the second row, this is subtracted from the next higher binary bit as it is done in decimal subtraction.

→ Subtract 0111 from 1010.

Sol:

$$\begin{array}{r} 1010 \\ -\,0111 \\ \hline 0011 \\ \hline \end{array}$$

is 0. So we borrow 1 from next digit and subtract 1 to give 1.

Now in the second column we have 0. So we again borrow 1 from the next higher column and subtract 1 to give 1. In the third column we borrow 1 from the next higher column and 1 − 1 gives '0'.

In the fourth column, 0 (after lending) − 0 give

(1)
$$\begin{array}{r} 11001 \\ -\ 1110 \\ \hline 1011 \end{array}$$

(2)
$$\begin{array}{r} 1011 \\ -\ 0110 \\ \hline 0101 \end{array}$$

(3)
$$\begin{array}{r} 100 \\ -\ 001 \\ \hline 011 \end{array}$$

Binary Multiplication:

The four basic rules for binary multiplication are:

$0 \times 0 = 0$

$0 \times 1 = 0$

$1 \times 0 = 0$

$1 \times 1 = 1$

The method of binary multiplication is similar to decimal multiplication.

In binary, each partial product is either zero (multiplication by 0) or exactly same as the multiplicand (multiplication by 1).

Examples

(1) Multiply 1001 by 1101

Sol^n:

```
     1001   ← multiplicand
   × 1101   ← multiplier
   ------
     1001  ┐
    0000   │  partial products
   1001    │
  1001     ┘
  -------
  1110101   ← final product.
```

(2) (i) Multiply 1011_2 by 101_2

(ii) Convert 1011_2 and 101_2 into decimal numbers. Multiply them, convert the result into binary and compare with the result of (i).

Sol^n (i)

```
    1011
   × 101
  ------
    1011
   0000
  1011
  ------
  110111
```

(ii)

1	0	1	1	Binary number
8	4	2	1	write weights
8	~~4~~	2	1	cross out weights under zero

$8+2+1 = \underline{\underline{11}}$ Add weights.

1	0	1	Binary number
4	2	1	write weights
4	~~2~~	1	cross out weights under zero.

$4+1 = \underline{\underline{5}}$ Add weights.

Now $11 \times 5 = 55$.

2	27	remainder 1
2	13	remainder 1
2	6	remainder 1
2	3	remainder 0
2	1	remainder 1
	0	remainder 1.

$\therefore (55)_{10} = (110111)_2$.

$\therefore$ The result is same as in part (i)

H.W. → Multiply 1110_2 by 10001_2

Binary Division:

Binary division is obtained using the same procedure as decimal division.

Examples

→ (1) Divide 110101 by 1001

Soln:

```
1001) 110101 (1101
      1001
      ----
       1011
       1001
      ------
      001001
        1001
      ------
        0000
```

Ans: 1101

→ (2) Divide 10100111 by 1001

Soln

```
1001) 10100111 (1001
      1001
      -------
      0001011
         1001
      -------
        00101
```

remainder 101
quotient 1001.

(i) Divide 110110 by 101

(ii) Convert 110110 and 101 into equivalent decimal number obtain division, convert results into binary and compare the results with those in part (i).

Sol: (i)

```
101) 110110 (101  quotient
     101
     -----
     00111
       101
     -----
       100  remainder
```

(ii)

1	1	0	1	1	0	Binary number
32	16	8	4	2	1	write weights
32	16	$\cancel{8}$	4	2	$\cancel{1}$	cross out weights under zero

$32+16+4+2 = \underline{54}$ Add weights

and

1	0	1	Binary number
4	2	1	write weights
4	$\cancel{2}$	1	cross out weights under zero.

$4+1=5$ Add weights.

Now

```
5) 54 (10 → quotient
   50
   --
    4 → remainder.
```

Quotient

```
2 | 10
2 | 5   remainder 0
2 | 2   remainder 1
2 | 1   remainder 0
  | 0   remainder 1
```

Remainder.

```
2 | 4
2 | 2   remainder 0
2 | 1   remainder 0
  | 0   remainder 1
```

∴ The quotient is 1010 and the remainder is 100.

These are the same as in part (i).

In the decimal number system a plus (+) sign is used to denote a positive number and a minus (−) sign for denoting a negative number. The plus sign is usually dropped, and the absence of any sign means that the number has positive value. This representation of numbers is known as signed number.

As is well known, digital circuits can understand only two symbols, 0 and 1; therefore, we must use the same symbols to indicate the sign of the number also.

Normally, an additional bit is used as the sign bit and it is placed as the most significant bit. A '0' is used to represent a positive number and a 1 to represent a negative number.

for example:

An 8-bit signed number 01000100 represents a positive number and its value (magnitude) is $(1000100)_2 = (68)_{10}$

The left most '0' (MSB) indicates that the number is positive.

On the other hand, in the signed binary form, 11000100 represents a negative number with magnitude $(1000100)_2 = (68)_{10}$

The 1 in the left most position (MSB) indicates that the number is negative and the other seven bits give its magnitude.

This kind of representation for signed numbers is known as sign-magnitude representation.

→ Find the decimal equivalent of the following binary numbers assuming sign-magnitude representation of the binary numbers.

(a) 101100 (b) 001000 (c) 0111 (d) 1111 :

Soln :

(a) 101100.

Sign bit is 1, which means the number is negative.

$$\text{Magnitude} = 01100$$
$$= 0\times2^4 + 1\times2^3 + 1\times2^2 + 0\times2^1 + 0\times2^0$$
$$= 0 + 8 + 4 + 0 + 0$$
$$= (12)_{10}$$

$$\therefore (101100)_2 = (-12)_{10}$$

(b) 001000

Sign bit is 0, which means the number is positive.

$$\text{Magnitude} = 01000$$
$$= 0\times2^4 + 1\times2^3 + 0\times2^2 + 0\times2^1 + 0\times2^0$$
$$= (8)_{10}$$

$$\therefore (001000)_2 = (+8)_{10}$$

(a) +8, (b) −8 (c) +165 (d) −165.

Sol^n: (a)

2	8
2	4 remainder 0
2	2 remainder 0
2	1 remainder 0
	0 remainder 1.

The binary number is 1000.

For the 16 bit system, we use 16 bits, 0 (which stands for +) in the left most position 1000 in the last 4 bits and 0 in the remaining 11 position.

∴ The signed 16 bit binary number is

+8 = 0000 0000 0000 1000

(b) Using 1 in the left most position (to represent the − sign). The rest of the representation is the same as in part.

∴ −8 = 1000 0000 0000 1000.

→ Represent $(-17)_{10}$ in sign − magnitude.

Sol^n:

2	17
2	8 remainder 1
2	4 remainder 0
2	2 remainder 0
2	1 remainder 0
	0 remainder 1

Using 1 in the left most position (for − sign).

∴ $(-17)_{10} = (110001)_2$

One's Complement Representation:

In a binary number, if each '1' is replaced by '0' and each '0' by '1', the resulting number is known as the one's complement of the first number. In fact, both the numbers are complement of each other.

If one of these numbers is positive, then the other number is negative with the same magnitude and vice-versa.

For example:

$(0101)_2$ represents $(+5)_{10}$, whereas $(1010)_2$ represents $(-5)_{10}$ in this representation. This method is widely used for representing signed numbers.

In this representation also, MSB is '0' for positive numbers and 1 for negative numbers.

→ Find the one's complement of the following binary numbers.

(a) 0100111001 (b) 11011010.

Sol^n: (a) one's complement of the binary number 0100111001 is 1011000110

(b) One's complement of the binary number 11011010 is 00100101.

→ Represent the following numbers in one's complement form.

(a) +7 and −7 (b) +8 and −8 (c) +15 and −15.

Sol: In one's comp

(a) $(+7)_{10} = (0111)_2$

and $(-7)_{10} = (1000)_2$

(b) $(+8)_{10} = (01000)_2$

and $(-8)_{10} = (10111)_2$

(c) $(+15)_{10} = (01111)_2$

$(-15)_{10} = (10000)_2$

from the above examples, it can be observed that for an n-bit number, the maximum positive number which can be represented in 1's complement representation is $(2^{n-1}-1)$ and the maximum negative number is $-(2^{n-1}-1)$

Two's complement Representation:

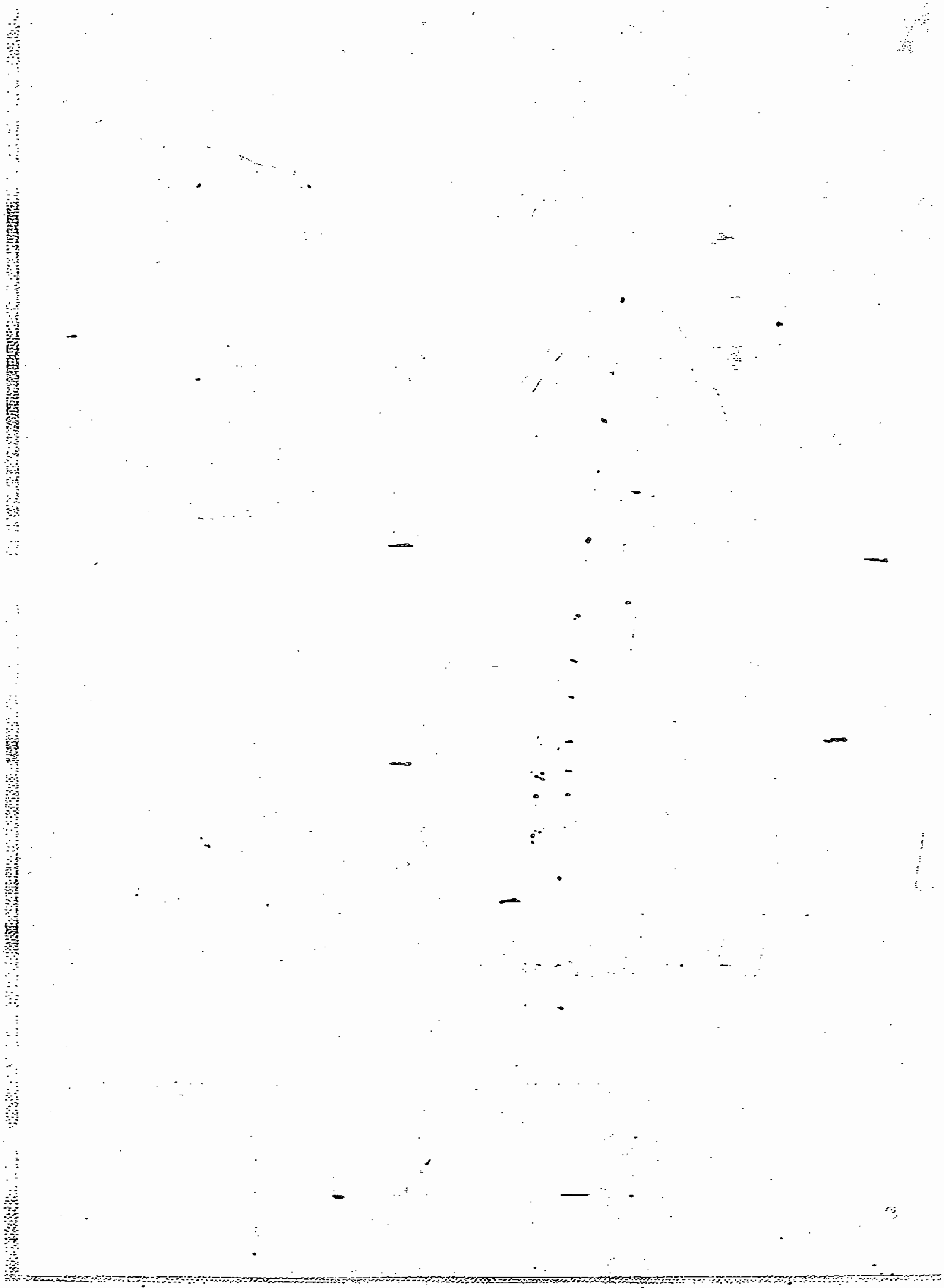
If 1 is added to 1's complement of a binary number, the resulting number is known as the two's complement of the binary number.

For example:

2's complement of 0101 is 1011.

Since 0101 represents $(+5)_{10}$; therefore, 1011 represents $(-5)_{10}$ in 2's complement representati

In this representation also, if the MSB is '0' the number is positive, whereas if the MSB is '1' the number is negative.



For an n-bit number, the maximum positive number which can be represented in 2's complement form is $(2^{n-1}-1)$ and the maximum negative number is -2^{n-1}.

The below table gives sign-magnitude, 1's and 2's complement numbers represented by 4-bit binary numbers:

Sign-magnitude, 1's and 2's complement representation using four bits:

Decimal number	Binary number		
	Sign-magnitude	One's complement	Two's complement
0	0000	0000	0000
1	0001	0001	0001
2	0010	0010	0010
3	0011	0011	0011
4	0100	0100	0100
5	0101	0101	0101
6	0110	0110	0110
7	0111	0111	0111
−8	—	—	1000
−7	1111	1000	1001
−6	1110	1001	1010
−5	1101	1010	1011
−4	1100	1011	1100
−3	1011	1100	1101
−2	1010	1101	1110
−1	1001	1110	1111
−0	1000	1111	—

- From the table, it is observed that the maximum positive number is 0111 = +7, where as the maximum negative number is 1000 = −8 using four bits in 2's complement format.

- It is also observed that the 2's complement of the 2's complement of a number is the number itself.

<u>For example</u>

Let the binary number 0101 = A (say)

NOW, Number A = 0101

1's complement $\bar{A}$ = 1010

Add 1   1

2's complement A' = 1011

1's complement of A' $\overline{A'}$ = 0100

2's complement of A' is A'' = 0101

$$\left(\because \begin{array}{r} 0100 \\ +\quad 1 \\ \hline 0101 \end{array}\right)$$

∴ The 2's complement of the 2's complement of a number is the number itself.

→ Find the 2's complement of the numbers.

(i) 01001110 (ii) 00110101

<u>Sol:</u> (i) Number 01001110

1's complement 10110001

Add 1   1

10110010

(iii) Number 00110101

1's complement 11001010

Add 1   1

11001011

From the above example, we observe the following.

(1) If the LSB of the number is 1, its 2's complement is obtained by changing each '0' to '1' and '1' to '0' except the least significant bit

(2) If the LSB of the number is '0', its 2's complement is obtained by scanning the number from the LSB to MSB bit by bit and retaining the bits as they are up to and including the occurence of the first 1 and complement all other bits.

→ Find two's complement of the numbers. (i) 01100100 (ii) 10010010 (iii) 11011000 (iv) 01100111.

Sol: Using the rules of conversion given above, we obtain

(i) Number → 01100100

2's complement → 10011100

(ii) Number → 10010010

2's complement → 01101110

(iii) Number → 11011000

2's complement → 00101000

(iv) Number → 01100111

2's complement → 10011001

(i) Sign-magnitude
(ii) One's complement
(iii) two's complement

Sol^n:

2	17
2	8 remainder 1
2	4 remainder 0
2	2 remainder 0
2	1 remainder 0
	0 remainder 1

$$(17)_{10} = (10001)_2$$

Using 0 in the left most position for '+' sign.

$$(+17)_{10} = (010001)_2$$

∴ The minimum number of bits required to represent $(+17)_{10}$ in signed number format is six.

— Therefore, $(-17)_{10}$ is represented by

(i) 110001 in sign-magnitude representation
(ii) 101110 in 1's complement representation
(iii) 101111 in 2's complement representation

⟹ Find the largest positive and negative numbers which can be stored with 8 bits.

Sol^n: The largest positive number is

0111 1111 = +127.

$(2^{8-1}-1)=12$

The largest negative number is

1000 0000 = -128.

Thus, with 8 bits we can store numbers between -128 and $+127$.

2's complement Addition, Subtraction:

The use of 2's complement representation has simplified the computer hardware for arithmetic operations. When A and B are to be added, the B bits are not inverted so that we get,

$$S = A + B \quad \text{———} \; (1)$$

When B is to be subtracted from A, the computer hardware forms the 2's complement of B and then adds it to A.

Thus $S = A + B'$

$$= A + (-B) = A - B \quad \text{———} \; (2)$$

Equations (1) and (2) represent algebraic addition and subtraction. A and B may represent either positive or negative numbers.

Example: If $A = -24$ and $B = +16$, (a) represent A & B in 8-bit 2's complement.
(b) find A+B (c) find A−B.

Sol: (a)

2	24	
2	12	remainder 0
2	6	remainder 0
2	3	remainder 0
2	1	remainder 1
	0	remainder 1

$= (0001\ 1000)_2$ which is ~~in~~ in 8 bit for

$24 = 0001\,1000$

$\overline{24} = 1110\,0111$ (1's complement)

$\quad + \qquad\qquad 1$

$-24 = 1110\,1000$ (2's complement)

NOW

2	16	
2	8	remainder 0
2	4	remainder 0
2	2	remainder 0
2	1	remainder 0
	0	remainder 1

$B = +16 = (10000)_2$

$= (0001\,0000)_2$ (8-bit form)

(b)

$$\begin{aligned} -24 &= 1110\,1000 \\ +16 &= 0001\,0000 \\ \hline -8 &= 1111\,1000 \end{aligned}$$

(c)

$B = 0001\,0000$

$\overline{B} = 1110\,1111$ (1's complement)

$\quad + \qquad\qquad 1$

$B' = 1111\,0000$ (2's complement)

NOW

$$\begin{aligned} -24 &= 1110\,1000 \\ +(-16) &= 1111\,0000 \\ \hline -40 &= \overline{1}1011\,1000 \end{aligned}$$

Note : For the above problem.

2's Complement : The signed binary numbers required too much electronic circuitry for addition and subtraction.

Therefore, positive decimal numbers are expressed in sign-magnitude form but negative decimal numbers are expressed in 2's complement.

Double precision numbers :-

→ Most present day computers are 16 bit. In these computers the numbers from +32,767 to -32,768 can be stored in each register.

→ To store numbers greater than these numbers, double precision system is used.

→ In this method two storage locations are used to represent each number.

The format is

First word	S	High order bits
Second word	0	Low order bits

Where 'S' is the sign bit and '0' is a zero. Thus numbers with 31 bit length can be represented in 16 bit registers.

→ For still bigger numbers triple precision can be used. In triple precision 3 word lengths (each 16 bit) is used to represent each number.

A 16-bit computer cannot store a positive number larger than 32767. What if we want to handle a fractional number like 35.7812 or a large number like 987654321?

Such numbers are stored and processed in what is known as exponential form. These number have an embedded decimal point and are called floating point numbers or Real numbers.

For example:

35.7812 can be expressed 0.357812×10^2.

Similarly, the number 987654321 can be expressed as 0.987654×10^9.

By writing a large number in exponential form, we lose some digits.

If 'x' is a Real number, its floating point for representation

$$x = f \times 10^E.$$

Where the number f is called mantissa and E is the exponent.

Floating point numbers are stored differently. The entire memory location is divided into three fields (or) parts as shown below.

31	30 – 24	23 – 0
SIGN	EXPONENT	MANTISSA
1 bit	7 bits	24 bits

Floating point representation.

The first part (1 bit) is reserved for the sign, the second part (7 bits) for the exponent of the number and the third (24 bits) for the mantissa of the number.

Typically, floating numbers use a field width of 32 bits where 24 bits are used for the mantissa and 7 bits for the exponent.

Thus, we can represent very small fractions or very large numbers within the computer using the floating point representation.

<u>for Example:</u>

Here the mantissa has a 10 bit length and exponent has 6 bit length.

Mantissa	Exponent
0111001101	100111

Floating point format. fig (1)

→ The left most bit of mantissa is sign bit.

→ The binary point is to the right of this sign bit.

→ The 6-bit exponent has a base of 2. The exponent can represent 0 to 63.

→ To express negative exponents the number $(32)_{10}$ (ie $(100000)_2$) has been added to the exponent. It is known as excess -32 format ~~as~~ notation and is a common floating point format.

Actual exponent	Binary Representation in excess-32 format
-32	000000
-1	011111
0	100000
+7	100111
+15	101111
+31	111111

The number represented in fig (1) is

Mantissa $+0.111001101$

Exponent 100111.

Subtracting 100 000 from exponent, we get

000111.

The number is 0.111001101×2^{7}

$= (1110011.01)_2$

$= (115.25)_{10}$.

→ What does the floating point number

0110100000|010101 represent

Sol: Mantissa is $+0.110100000$

Exponent is 010101.

Subtracting 100000 from exponent, we get

110101.

The given number is $+0.110100000 \times 2^{-11}$

$= (0.00000000000110100000)_2$

$= (+0.000396728)_{10}$

Octal - ...

Any octal number can be converted into its equivalent decimal number by using the weights assigned to each octal digit position as given in Table-1. (Pg. 2 after).

→ For example:

Convert the octal number 1054 into decimal number.

Sol^n:

1 0 5 4 octal number

$8^3\ 8^2\ 8^1\ 8^0$ weights

$\therefore (1054)_8 = 1\times8^3 + 0\times8^2 + 5\times8 + 4\times8^0$

$= 512 + 0 + 40 + 4$

$= (556)_{10}$

H.W → Convert the octal number $(1502)_8$ into decimal numb

→ Convert $(6327.4051)_8$ into its equivalent decimal number.

Sol^n: Using the weights given in Table-1, we obtain

$(6327.4051)_8 = 6\times8^3 + 3\times8^2 + 2\times8^1 + 7\times8^0 + 4\times8^{-1} + 0\times8^{-2} + 5\times8^{-3} + 1\times8^{-4}$

$= 3072 + 192 + 16 + 7 + \frac{4}{8} + 0 + \frac{5}{512} + \frac{1}{4096}$

$= (3287.5100098)_{10}$

$\therefore (6327.4051)_8 = (3287.5100098)_{10}$

Decimal-to-octal conversion:

The conversion from decimal to octal (base-10 to base-8) is similar to the conversion procedure for base-10 to base-2 conversion.

The only difference is that the number 8 is used in place of 2 for division in the case of integers and for multiplication in the case of fractional numbers.

Examples:

→ Convert decimal number 574 into octal.

Soln:

$$\begin{array}{c|l} 8 & 574 \\ \hline 8 & 71 \text{ remainder } 6 \\ \hline 8 & 8 \text{ remainder } 7 \\ \hline 8 & 1 \text{ remainder } 0 \\ \hline & 0 \text{ remainder } 1 \end{array}$$

$\therefore (574)_{10} = (1076)_8$

→ (a) convert $(247)_{10}$ into octal

(b) Convert $(0.6875)_{10}$ into octal

(c) Convert $(468)_{2}$ into octal.

(d) Convert $(3287.5100098)_{10}$ into octal.

Soln:

(b)

fraction	fraction×8	Remainder new fraction	Integer
0.6875	5.5000	0.5000	5
0.5000	4.0000	0.0000	4

$\therefore (0.6875)_{10} = (0.54)_8$

(d) Integer part:

$$\begin{array}{c|l} 8 & 3287 \\ \hline 8 & 410 \text{ remainder } 7 \\ \hline 8 & 51 \text{ remainder } 2 \\ \hline 8 & 6 \text{ remainder } 3 \\ \hline & 0 \text{ remainder } 6 \end{array}$$

$\therefore (3287)_{10} = (6327)_8$.

Fraction	fraction x 8	Remainder new fraction	Integer
0.5100098	4.0800784	0.0800784	4
0.0800784	0.6406272	0.6406272	0
0.6406272	5.1250176	0.1250176	5
0.1250176	1.0001408	0.0001408	1

Thus $(0.5100098)_{10} \cong (0.4051)_8$

$\therefore (3287.5100098)_{10} = (6327.4051)_8$

Note: Conversion for fractional numbers may not be exact.

In general, an approximate equivalent can be determined by terminating the process of multiplication by eight at the desired point.

Octal-to-Binary Conversion: octal numbers can be converted into equivalent binary numbers by replacing each octal digit by its 3-bit equivalent binary.

For example: Convert $(71)_8$ into an equivalent binary number.

7	1	octal number
↓	↓	
111	001	Binary number

i.e. binary equivalent of 7 and 1 are 111 and 001 respectively

$\therefore (71)_8 = (111\,001)_2$

→ Convert $(736)_8$ into an equivalent binary number.

Sol:

7	3	6	octal number
↓	↓	↓	
111	011	110	Binary number

i.e. The binary equivalents of 7, 3 and 6 are 111, 011 and 110 respectively

$\therefore (736)_8 = (111\,011\,110)_2$

The following table gives octal number and their binary equivalents for decimal numbers 0 to 15.

Binary and decimal equivalents of octal numbers

Octal	Decimal	Binary
0	0	000
1	1	001
2	2	010
3	3	011
4	4	100
5	5	101
6	6	110
7	7	110
10	8	001000
11	9	001001
12	10	001010
13	11	001011
14	12	001100
15	13	001101
16	14	001110
17	15	001111

Binary-to-octal conversion:

- Binary numbers can be converted into equivalent octal numbers by making groups of three bits starting from LSB and moving towards MSB for integer part of the number and then replacing each group of three bits by its octal representation.

For fractional part, the groupings of three bits are made starting from the binary point.

Sol

$$(1001110)_2 = (\underbrace{001}_{1}\ \underbrace{001}_{1}\ \underbrace{110}_{6})_2$$

$$= (1\ 1\ 6)_8$$

$$= (116)_8$$

→ convert the following binary numbers into octal numbers.

(i) 110101 (ii) 0.1010110 (iii) 111000101 (iv) 1001101.1011

(v) 11001110001.000101111001

(vi) 101101111O.1100101OO11

(vii) 1111100O1.1001100110l.

sol^n:

(i)
$$(110101)_2 = (\underbrace{110}_{6}\ \underbrace{101}_{5})_2$$

$$= (65)_8$$

(ii)
$$(0.1010110)_2 = (0.\underbrace{101}_{5}\ \underbrace{001}_{1}\ \underbrace{100}_{4})_2$$

$$= (0.514)_8$$

(iv)
$$(1001101.1011)_2 = (\underbrace{001}_{1}\ \underbrace{001}_{1}\ \underbrace{101}_{5}.\underbrace{101}_{5}\ \underbrace{100}_{4})_2$$

$$= (115.54)_8$$

From the above examples we observe that in forming the 3-bit groupings, 0's may be required to complete the first (most significant digit) group in the integer part and the last (least significant digit) group in the fractional part.

- One method to convert a hexadecimal number into its decimal equivalent is to first convert hexadecimal to binary and then convert binary to decimal.
- A direct conversion of hexadecimal into decimal is also possible.

Since the base of a hexadecimal is 16, the weights of different bits are $16^0, 16^1, 16^2$ etc. starting with the bit on the extreme right.

The decimal equivalent of a hexadecimal number equals the sum of all digits multiplied by their weights.

For example:

E	7	F	6	Hexadecimal
16^3	16^2	16^1	16^0	weights

$$E7F6 = (E \times 16^3) + (7 \times 16^2) + (F \times 16^1) + (6 \times 16^0)$$
$$= (14 \times 4096) + (7 \times 256) + (15 \times 16) + (6 \times 1)$$
$$= 57344 + 1792 + 240 + 6$$
$$= (59382)_{10}.$$

→ obtain decimal equivalent of hexadecimal number $(3A.2F)_{16}$.

Sol: $(3A.2F)_{16} = (3 \times 16^1) + (A \times 16^0) + (2 \times 16^{-1}) + (F \times 16^{-2})$
$$= 48 + (10 \times 1) + \frac{2}{16} + \frac{15}{16^2}$$
$$= (58.1836)_{10}.$$

Note: The fractional part may not be an exact equivalent, may give a small error.

→ Convert the hexadecimal number $8A3_{10}$ into decimal equivalent (a) by first converting it into binary (b) directly.

→ Decimal - to - Hexadecimal Conversion:

- One method is to convert the decimal to binary and then convert binary to hexadecimal.

- The direct method is successive division by 16 and to write the hexadecimal equivalents of remainder.

For example:

→ Convert decimal number 5390 into hexadecimal

16	5390	
16	336	remainder E
16	21	remainder 0
16	1	remainder 5
	0	remainder 1

The hexadecimal equivalent is 150E.

→ Convert the following decimal numbers into hexadecimal numbers —

(a) 95.5 (b) 675.625 (c) 268 (d) 5741

Sol: (a) Integral part:

16	95	
16	5	remainder 15 = F
	0	remainder 5

i.e., $(95)_{10} = (5F)_{16}$

Fractional part:

Fraction	Fraction × 16	Remainder new fraction	Integer
0.5	8.0	0.0	8
0.0	0.0	0.0	0

i.e. $(0.5)_{10} = (0.8)_{16}$

$\therefore (95.5)_{10} = (5F.8)_{16}$

numbers:

Hexadecimal	Decimal	Binary
0	0	0000
1	1	0001
2	2	0010
3	3	0011
4	4	0100
5	5	0101
6	6	0110
7	7	0111
8	8	1000
9	9	1001
A	10	1010
B	11	1011
C	12	1100
D	13	1101
E	14	1110
F	15	1111

Hexadecimal to Binary conversion:-

Hexadecimal numbers can be converted into equivalent binary numbers by replacing each hex digit by its equivalent 4-bit binary number.

Example: Convert $(2F9A)_{16}$ to equivalent binary number.

Sol: Using the above table, write the binary equivalent of each hex digit.

$(2F9A)_{16} = (0010\ 1111\ 1001\ 1010)_2$

$= (0010111110011010)_2$

Binary-to-hexadecimal Conversion:-

Binary numbers can be converted into the equivalent hexadecimal numbers by making groups of four bits starting from LSB and moving towards MSB for integer part and then replacing each group of four bits by its hexadecimal representation.

For the fractional part, the above procedure is repeated starting from the bit next to the binary point and moving towards the right.

→ Convert the following binary numbers to their equivalent hex numbers.

(a) 10100110101111 (b) 0.00011110101101

(c) 1100111110001.000101111001 (d) 1011011110.110010100111

(a) $(10100110101111)_2 = \underbrace{0010}_{2}\ \underbrace{1001}_{9}\ \underbrace{1010}_{A}\ \underbrace{1111}_{F}$

$= (29AF)_{16}$.

(b) $(0.00011110101101)_2 = 0.\underbrace{0001}_{1}\ \underbrace{1110}_{E}\ \underbrace{1011}_{B}\ \underbrace{0100}_{4}$

$= (0.1EB4)_{16}$

(c) $(11001110001.000101111001)_2$

$= \underbrace{0110}_{6}\ \underbrace{0111}_{7}\ \underbrace{0001}_{1}\ .\ \underbrace{0001}_{1}\ \underbrace{0111}_{7}\ \underbrace{1001}_{9}$

$= (671.179)_{16}$

From the above examples, we observe that in forming 4-bit groupings 0's may be required to complete the first (MSB) group in the integer part and the (last) (LSB) group in the fractional part

Hexadecimal numbers can be converted to equivalent octal numbers and octal numbers can be converted to equivalent hex numbers by converting the hex/octal number to equivalent binary and then to octal/hex, respectively.

→ Convert the following hex numbers to octal numbers.

(a) A72E (b) 0.BF85

Soln: (a) $(A72E)_{16} = (1010\ 0111\ 0010\ 1110)_2$

$= (\underbrace{001}_{1}\ \underbrace{010}_{2}\ \underbrace{011}_{3}\ \underbrace{100}_{4}\ \underbrace{101}_{5}\ \underbrace{110}_{6})_2$

$= (123456)_8$

(b) $(0.BF85)_{16} = (0.1011\ 1111\ 1000\ 0101)_2$

$= (0.\underbrace{101}_{5}\ \underbrace{111}_{7}\ \underbrace{111}_{7}\ \underbrace{000}_{0}\ \underbrace{010}_{2}\ \underbrace{100}_{4})_2$

$= (0.577024)_8$

→ Convert $(247.36)_8$ to equivalent hex number.

Soln: $(247.36)_8 = (010\ 100\ 111 . 011\ 110)_2$

$= (0\ \underbrace{1010}\ \underbrace{0111} . \underbrace{0111}\ \underbrace{1000})_2$

$= (A7.78)_{16}$