APPENDIX II

Cantor's Theory of Real Numbers

In chapter 1, we followed Dedekind's theory to extend the rational number system to the real number system. We now introduce another method, which is less algebraic but more sophisticated, due to G. Cantor. The construction of real numbers by Cantor's theory depends on the existence of an equivalence relation in the set of all Cauchy sequences of rational numbers. Defining each equivalence class as a real number and by suitably defining addition and multiplication, and order, this set **R** of Cantor's real numbers will be made into an ordered field which will be an extension of the ordered field **Q** or rational numbers. It will then be shown that this ordered field **R** is also *complete* (in the sense defined earlier) i.e., the field **R** is *order complete*.

Since the concept of *Cauchy sequences of rational numbers* and their limits are basic to Cantor's Theory, we start the discussion by a brief introduction to these sequences.

1. SEQUENCES OF RATIONAL NUMBERS

Definitions

1. A function S on the set **N** of natural numbers into the set **Q** of rational numbers is called a *rational sequence* or a *sequence of rational numbers* and is symbolically denoted as $\mathbf{S} : \mathbf{N} \to \mathbf{Q}$.

2. A rational sequence $\{S_n\}$ is said to be *bounded* if there exists a rational numbers $K > 0$ such that

$$|S_n| \leq K, \quad \forall\ n.$$

3. A sequence $\{S_n\}$ of rational numbers is called a *Cauchy sequence* or a *fundamental sequence*, if for each rational number $\varepsilon > 0$ there exists a positive integer m_0, such that

$$|S_n - S_m| < \varepsilon, \quad \forall\ m, n \geq m_0$$

or

$$|S_{n+p} - S_n| < \varepsilon, \quad \forall\ n \geq m_0, \text{ integer } p \geq 1.$$

4. A rational sequence $\{S_n\}$ is said to *converge* to a rational number l (or we have the number l as its *limit*) if for each rational number $\varepsilon > 0$, there exists a positive integer m_0, (depending on ε), such that

$$|S_n - l| < \varepsilon \text{ for all } n \geq m_0, \text{ and we then write}$$

$$\lim_{n\to\infty} S_n = l \text{ or } S_n \to l$$

It may be easily shown that

(*i*) every Cauchy sequence is bounded,

(*ii*) every convergent sequence is bounded, and

(*iii*) every convergent sequence is a Cauchy sequence, i.e., a necessary condition for convergence of a rational sequence is that it is a Cauchy sequence.

Thus *a necessary condition for convergence* of a rational sequence is that for any rational number $\varepsilon > 0$, there exists a positive integer m_0 such that

$$|S_{n+p} - S_n| < \varepsilon, \quad \forall\, n \geq m_0,\ p \geq 1 \qquad \ldots(1)$$

Example. Show that the sequence $\{S_n\}$, where

$$S_{n+1} = \frac{2 + S_n}{1 + S_n}, \quad n \geq 1$$

$$S_1 = 1$$

is not convergent in the field or rational numbers.

If the sequence converges to a rational number, say q, then

$$\lim S_{n+1} = q = \lim S_n$$

$$\therefore \qquad q = \lim \frac{2 + S_n}{1 + S_n} = \frac{2 + q}{1 + q}$$

or $$q^2 = 2$$

But no rational number exists whose square is equal to 2. Hence, the sequence is not convergent in the field of rational numbers.

Note: $\{S_n\}$ is the rational sequence, $1, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \ldots$

Thus by considering the sequence $\{S_n\}$, where $S_{n+1} = \dfrac{2 + S_n}{1 + S_n}$, for $n \geq 1$, $S_1 = 1$, it may be shown that (1) is not a sufficient condition for convergence of a rational sequence. Thus every Cauchy sequence is not convergent in the field of rationals (compare § 6.1, Ch. 3).

Ex. 1. Show that the sequence $\{S_n\}$, where $S_{n+1} = \dfrac{l + 2 + kS_n}{k + lS_n}$, $S_1 = 1$, l, k are finite non-zero numbers, is not convergent in the field of rationals.

Ex. 2. If $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences of rational numbers, then

(*i*) the sequences $\{a_n + b_n\}$, $\{a_n b_n\}$ are also Cauchy sequences,

(*ii*) the sequences $\left\{\dfrac{1}{b_n}\right\}$ and $\left\{\dfrac{a_n}{b_n}\right\}$ are also Cauchy sequences provided $\{b_n\}$ does not converge to zero, and $b_n \neq 0$ for any n.

2. CANTOR REAL NUMBER

We shall use F_Q to denote the set of all Cauchy sequences of rational numbers.

Definition. *A sequence $\{a_n\} \in F_Q$ is said to be equivalent to $\{b_n\} \in F_Q$ whenever $\{a_n - b_n\}$ converges to zero.* Expressed symbolically,

$$\{a_n\} \sim \{b_n\} \text{ iff } \lim (a_n - b_n) = 0 .$$

Theorem 1. *The relation $\sim$ in the set of all Cauchy sequences of rational numbers, defined by*

$$\{a_n\} \sim \{b_n\} \text{ iff } \lim (a_n - b_n) = 0$$

is an equivalence relation.

(1) For any Cauchy sequence $\{a_n\} \in F_Q$

$$\lim (a_n - a_n) = 0$$

$$\therefore \quad \{a_n\} \sim \{a_n\}$$

and so the relation $\sim$ is reflexive.

(2) Let $\{a_n\}, \{b_n\}$ be sequences in F_Q such that $\lim (a_n - b_n) = 0$.

Then

$$\lim (b_n - a_n) = \lim [-(a_n - b_n)] = 0$$

$$\therefore \quad \{a_n\} \sim \{b_n\} \Rightarrow \{b_n\} \sim \{a_n\}$$

and so the relation $\sim$ is symmetric.

(3) Let $\{a_n\}, \{b_n\}, \{c_n\}$ be sequences in F_Q, such that

$$\{a_n\} \sim \{b_n\} \text{ and } \{b_n\} \sim \{c_n\}$$

i.e.,

$$\lim (a_n - b_n) = 0 \text{ and } \lim (b_n - c_n) = 0$$

Then,

$$\lim (a_n - c_n) = \lim (a_n - b_n + b_n - c_n)$$

$$= \lim (a_n - b_n) + \lim (b_n - c_n) = 0$$

$$\therefore \quad \{a_n\} \sim \{b_n\} \wedge \{b_n\} \sim \{c_n\} \Rightarrow \{a_n\} \sim \{c_n\}$$

Hence, the relation $\sim$ is transitive.

Hence from (1), (2) and (3) it follows that the relation $\sim$ is an equivalence relation in F_Q.

Notation. Let $[a_n]$ denote the equivalence class containing the sequence $\{a_n\}$, i.e., the set of rational Cauchy sequences equivalent to $\{a_n\}$. Thus,

$$[a_n] = \{\{x_n\} \in F_Q \mid \{x_n\} \sim \{a_n\}\}.$$

Ex. 1. If $\{a_n\} \in F_Q$, then $\lim (a_n) = a$ iff $\{a_n\} \sim \{a\}$.

Ex. 2. Two equivalence classes $[a_n]$, and $[a_n']$ are equal iff $\{a_n\} \sim \{a_n'\}$.

Definition. A Cantor *real number* is an equivalence class $[a_n]$ with respect to the equivalence relation $\sim$ in F_Q defined by the condition

$$\{a_n\} \sim \{b_n\} \text{ iff } \lim (a_n - b_n) = 0$$

Thus if ξ is a real number $[a_n]$, then

$$\xi = \{\{x_n\} \in F_Q \mid \{x_n\} \sim \{a_n\}\}$$

We shall denote by **R** the set of all real numbers and use $\xi, \eta, \ldots$ to denote the real numbers.

3. ADDITION AND MULTIPLICATION IN R

We shall define two binary operations (+ and .) in **R**, to be called addition and multiplication and discuss some of their properties. But we shall need the following theorems before we can do so.

Theorem 2. *If $\{x_n\}$, $\{y_n\}$, $\{a_n\}$, and $\{b_n\}$ belong to F_Q such that $\{x_n\} \sim \{a_n\}$ and $\{y_n\} \sim \{b_n\}$, then $\{x_n + y_n\}$ and $\{a_n + b_n\}$ also belong to F_Q and $\{x_n + y_n\} \sim \{a_n + b_n\}$.*

Since $\{x_n\}$, $\{y_n\}$, $\{a_n\}$ and $\{b_n\}$ all belong to F_Q therefore as in Ex. 2 § 1, $\{x_n + y_n\}$ and $\{a_n + b_n\}$ also belong to F_Q. Also

$$\{x_n\} \sim \{a_n\} \Rightarrow \lim (x_n - a_n) = 0$$

and

$$\{y_n\} \sim \{b_n\} \Rightarrow \lim (y_n - b_n) = 0$$

$$\therefore \quad \lim (x_n + y_n - a_n - b_n) = \lim (x_n - a_n) + \lim (y_n - b_n) = 0$$

$$\Rightarrow \quad \{x_n + y_n\} \sim \{a_n + b_n\}.$$

Theorem 3. *If $\{x_n\}$, $\{y_n\}$, $\{a_n\}$ and $\{b_n\}$ belong to F_Q such that $\{x_n\} \sim \{a_n\}$ and $\{y_n\} \sim \{b_n\}$, then $\{x_n y_n\}$ and $\{a_n b_n\}$ also belong to F_Q and $\{x_n y_n\} \sim \{a_n b_n\}$.*

Since $\{x_n\}$, $\{y_n\}$, $\{a_n\}$ and $\{b_n\}$ all belong to F_Q, therefore $\{x_n y_n\}$ and $\{a_n b_n\}$ also belong to F_Q.

Again since $\{x_n\}$, $\{b_n\}$ are rational Cauchy sequences, they are bounded and therefore $\exists$ positive rational numbers k_1, k_2 such that

$$|x_n| < k_1, |b_n| < k_2, \quad \forall\, n \in \mathbf{N}$$

Also since $\{x_n\} \sim \{a_n\}$ and $\{y_n\} \sim \{b_n\}$,

$$\therefore \quad \lim (x_n - a_n) = 0 \text{ and } \lim (y_n - b_n) = 0$$

so that for rational $\varepsilon > 0$, $\exists$ positive integers m_1, m_2 that

$$|x_n - a_n| < \varepsilon/2k_2, \quad \text{for } n \geq m_1$$

and

$$|y_n - b_n| < \varepsilon/2k_1, \quad \text{for } n \geq m_2$$

Let $m = \max(m_1, m_2)$.

Hence for $n \geq m$, we have

$$|x_n y_n - a_n b_n| = |x_n(y_n - b_n) + b_n(x_n - a_n)|$$

$$\leq |x_n||y_n - b_n| + |b_n||x_n - a_n|$$

$$< k_1 \frac{\varepsilon}{2k_1} + k_2 \frac{\varepsilon}{2k_2} = \varepsilon$$

$$\Rightarrow \qquad \lim (x_n y_n - a_n b_n) = 0$$

$$\therefore \qquad \{x_n y_n\} \sim \{a_n b_n\}$$

Theorem 4. (*Addition in* **R**). *There is a binary operation* $f: \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ *such that for every pair of real numbers* $\xi = [a_n]$, $\eta = [b_n]$,

$$f(\xi, \eta) = \zeta$$

where $\zeta = [a_n + b_n]$

Let (ξ, η) be an arbitrary element of $\mathbf{R} \times \mathbf{R}$ and let $\xi = [a_n]$, $\eta = [b_n]$, for some sequences $\{a_n\}$ and $\{b_n\}$ in F_Q.

Since $\{a_n + b_n\} \in F_Q$, therefore $[a_n + b_n] \in \mathbf{R}$.

Let $f(\xi, \eta) = \zeta$, where $\zeta = [a_n + b_n]$.

We now show that mapping f is well defined.

Let, if possible, $f(\xi, \eta) = \zeta'$, where $\zeta' = [a_n' + b_n']$, $\xi = [a_n']$, and $\eta = [b_n']$.

Now

$$[a_n] = \xi = [a'_n] \text{ and } \eta = [b_n] = [b_n']$$

$$\therefore \qquad \{a_n\} \sim \{a_n'\} \text{ and } \{b_n\} \sim \{b_n'\}$$

$$\Rightarrow \qquad \{a_n + b_n\} \sim \{a_n' + b_n'\}$$

$$\Rightarrow \qquad [a_n + b_n] = [a_n' + b_n']$$

$$\Rightarrow \qquad \zeta = \zeta'$$

Thus f is a binary operation on **R**.

Definition. The binary operation f on **R** is called *addition* in **R** and is denoted by +. Thus

For every pair ξ, η *of real numbers where* $\xi = [a_n]$ *and* $\eta = [b_n]$, *for some sequences* $\{a_n\}$ *and* $\{b_n\}$ *in* F_Q, *the real number* $\zeta = [a_n + b_n]$ *is called the sum of* ξ *and* η *and is denoted by* $\xi + \eta$.

Ex. For real numbers $\xi, \eta, \zeta, \ldots$, prove that

(*i*) $\xi + \eta = \eta + \xi$

(*ii*) $\xi + (\eta + \zeta) = (\xi + \eta) + \zeta$

(*iii*) There is a unique real number 0 such that

$$\xi + 0 = 0 + \xi = \xi$$

[*Hint*: Take $0 = [0_n]$, where $0_n = 0$ for all n]

(*iv*) For every $\xi \in \mathbf{R}$, there exists a unique $\eta \in \mathbf{R}$ such that

$$\xi + \eta = 0 = \eta + \xi.$$

Theorem 5. (*Multiplication in* **R**). *There is a binary operation* $g: \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ *such that for every pair of real numbers* $\xi = [a_n]$, $\eta = [b_n]$,

$$g(\xi, \eta) = \zeta,$$

where $\zeta = [a_n b_n]$.

Let (ξ, η) be an arbitrary element of $\mathbf{R} \times \mathbf{R}$ and let $\xi = [a_n]$, $\eta = [b_n]$, for some sequences $\{a_n\}$ and $\{b_n\}$ in F_Q.

Since $\{a_n b_n\} \in F_Q$, therefore $[a_n b_n] \in \mathbf{R}$.

Let $g(\xi, \eta) = \zeta$, where $\zeta = [a_n b_n]$.

We now show that the mapping g is well defined.

Let, if possible, $g(\xi, \eta) = \zeta'$, where $\zeta' = [a_n' b_n']$, and $\xi = [a_n']$, $\eta = [b_n']$.

Now

$$[a_n] = [a_n'], \text{ and } [b_n] = [b_n']$$

$$\therefore \quad \{a_n\} \sim \{a_n'\} \text{ and } \{b_n\} \sim \{b_n'\}$$

$$\Rightarrow \quad \{a_n b_n\} \sim \{a_n' b_n'\}$$

$$\Rightarrow \quad [a_n b_n] = [a_n' b_n']$$

$$\Rightarrow \quad \zeta = \zeta'$$

Thus g is a binary operation on **R**.

Definition. The binary operation g on **R** is called *multiplication* in **R** and is denoted by (.). Thus for every pair ξ, η or real numbers where $\xi = [a_n]$ and $\eta = [b_n]$ for some sequences $\{a_n\}$ and $\{b_n\}$ in F_Q, the real number $\zeta = [a_n b_n]$ is called the product of ξ and η and is denoted by $\xi \cdot \eta$.

Ex. For real numbers $\xi, \eta, \zeta, \ldots$, show that

(*i*) $\xi \cdot \eta = \eta \cdot \xi$

(*ii*) $\xi \cdot (\eta \cdot \zeta) = (\xi \cdot \eta) \cdot \zeta$,

(*iii*) There exists a real number 1, called the multiplicative identity such that

$$\xi \cdot 1 = 1 \cdot \xi = \xi$$

[*Hint*: Take $1 = [I_n]$, where $I_n = 1$ for all n.]

(*iv*) The multiplicative identity is unique.

(v) To each real number $\xi \neq 0$, there corresponds a unique real number η such that

$$\xi \cdot \eta = 1 = \eta \cdot \xi$$

Let $\xi = [a_n]$. Since $\xi \neq 0$, the sequence $\{a_n\}$ can have at the most a finite number of terms equal to zero. Let $m \in \mathbf{N}$ be such that $a_n \neq 0$ for $n > m$.

Let us define a sequence $\{b_n\}$, where $b_n = \dfrac{1}{a_n}$, for $n > m$, and $b_n = 1$ for $n \leq m$.

The real number $\eta = [b_n]$ has the desired property.

The real number η is called the *multiplicative inverse* of ξ and is denoted by ξ^{-1}.

(vi) The multiplicative inverse ξ^{-1} is unique.

Theorem 6. *Prove that* $(R, +, .)$ *is a field.*

The proof is left to the reader.

4. ORDER IN R

We shall now give an *order structure* to the field of real numbers. To do so we shall first define the set of positive elements of **R**.

4.1 *Definition* (*A positive sequence of rational numbers*). *A rational sequence* $\{a_n\}$ *is called a positive sequence if there exists a positive rational number e and a positive integer m such that*

$a_n > e$, *for all* $n \geq m$

From the above definition it may be easily shown that *if* $\{a_n\}$ *and* $\{b_n\}$ *are positive sequences of rational numbers, then* $\{a_n + b_n\}$ *and* $\{a_n b_n\}$ *are also positive sequences of rational numbers*.

Theorem 7. *If* $\{a_n\} \sim \{a_n'\}$, *and* $\{a_n\}$ *is a positive rational sequence, then* $\{a_n'\}$ *is also a positive rational sequence*.

Since $\{a_n\}$ is a positive rational sequence, therefore $\exists\, e > 0$ in $\mathbf{Q}^+$ and $m_1 \in \mathbf{N}$ such that

$$a_n > e, \text{ for } n \geq m_1 \qquad \ldots(1)$$

Again since $\{a_n\} \sim \{a_n'\}$, therefore $\exists\, m_2 \in \mathbf{N}$ such that

$$|a_n' - a_n| < \frac{e}{2}, \text{ for all } n \geq m_2$$

or

$$-\frac{e}{2} < a_n' - a_n < \frac{e}{2}, \text{ for } n \geq m_2 \qquad \ldots(2)$$

Let $m_0 = \max(m_1, m_2)$.

$$\therefore \quad a_n' = (a_n' - a_n) + a_n$$

$$> -\frac{e}{2} + e, \text{ for } n \geq m_0 \qquad \text{[from (1) and (2)]}$$

$$= \frac{e}{2} > 0, \text{ for } all\ n \geq m_0$$

Hence $\{a_n'\}$ is a positive rational sequence.

Corollary. If ξ is a real number and $\{a_n\} \in \xi$ be a positive sequence in Q then every sequence $\{a_n'\} \in \xi$ is also a positive sequence in **Q**.

4.2 *Definition. A real number ξ is positive if every sequence in ξ is a positive sequence.*

In view of Theorem 7.1, it follows that a *real number ξ is positive if and only if there exists a positive rational sequence in ξ.*

We shall denote the set of positive real numbers by $\mathbf{R}^+$. Thus

$$\mathbf{R}^+ = \begin{cases} \{\xi \in \mathbf{R} \mid \xi \text{ is positive}\} \\ \{\xi \in \mathbf{R} \mid \text{for some } \{a_n\} \in \xi, \{a_n\} \text{ is positive}\} \end{cases}$$

From definition, the following result may be easily proved.

If ξ, η *are positive real numbers then so also are* $\xi + \eta$ and $\xi \cdot \eta$.

Theorem 8. *If $\xi \in \mathbf{R}$, then one and only one of the following statements is true:*

(*i*) $\xi = 0$, (*ii*) $\xi \in \mathbf{R}^+$, (*iii*) $-\xi \in \mathbf{R}^+$.

We first show that at *least one* of the three statements is true.

Let $\xi = [a_n]$, so that $\{a_n\} \in \xi$, and $\{a_n\} \in F_Q$.

When $\xi = 0$, there is nothing to prove, for then (*i*) holds.

Let $\xi \neq 0$, so that $\{a_n\}$ is not equivalent to $\{0_n\}$, where $0_n = 0$ for all n, consequently $\{a_n\}$ does not converge to 0. Hence $\exists\ e \in Q^+$ and $m_1 \in \mathbf{N}$ such that

$$|a_n| > e \quad \forall\ n \geq m_1 \qquad \ldots(1)$$

Again since $\{a_n\}$ is a Cauchy sequence and $e > 0$, therefore $\exists\ m_2 \in \mathbf{N}$ such that

$$|a_{n+p} - a_n| < \frac{e}{2}, \text{ for } n \geq m_2,\ p \geq 1 \qquad \ldots(2)$$

Let $m = \max(m_1, m_2)$, then from (2),

$$a_m - \frac{e}{2} < a_{m+p} < a_m + \frac{e}{2},\ p \geq 1 \qquad \ldots(3)$$

From (1) either $a_m > e$ or $a_m < -e$.

If $a_m > e$, then from (3),

$$a_{m+p} > e - \frac{e}{2} = \frac{e}{2} > 0,\ p \geq 1$$

Therefore $\{a_n\}$ is a positive sequence and hence $\xi \in \mathbf{R}^+$.

And if $a_m < -e$, then again from (3),

$$a_{m+p} < -e + \frac{e}{2} = -\frac{e}{2}, \; p \geq 1$$

i.e.,
$$-a_{m+p} > \frac{e}{2} > 0, \; p \geq 1$$

Therefore $\{-a_n\}$ is a positive sequence and hence $-\xi \in \mathbf{R}^+$.

Thus we have shown that at least one of the three statements is true.

We now proceed to show that *not more than one* of the three statements is true.

If $\xi = 0$, then $\{a_n\} \sim \{0_n\}$. Hence for rational $e > 0$, $\exists\, m' \in \mathbf{N}$ such that

$$|a_n| < e, \text{ for } n \geq m'$$

Hence there is no $e \in \mathbf{Q}^+$ such that for some $m_0 \in \mathbf{N}$, either

$$a_n \geq e \quad \forall\, n \geq m_0$$

or
$$-a_n \geq e \quad \forall\, n \geq m_0$$

Thus if $\xi = 0$, then neither $\xi \in \mathbf{R}^+$ nor $-\xi \in \mathbf{R}^+$.

Now, if possible, $\xi \in \mathbf{R}^+$ and $-\xi \in \mathbf{R}^+$.

Then for some e, e' $(e < e'$, say$) \in \mathbf{Q}^+$ and $n_1, n_2 \in \mathbf{N}$,

$$a_n \geq e \quad \forall\, n \geq n_1$$
$$-a_n \geq e' \quad \forall\, n \geq n_2$$

Hence for $n = \max(n_1, n_2)$,

$$0 < e' \leq -a_n \leq -e < 0$$

which is impossible,

$\therefore$
$$\xi \in \mathbf{R}^+ \wedge -\xi \in \mathbf{R}^+ \text{ is false}$$

Hence the theorem.

4.3 *Thus* $\mathbf{R}^+$ *is a set of positive elements of* $\mathbf{R}$ *such that*

(*a*) *If* $\xi, \eta \in \mathbf{R}^+$, *then* $\xi + \eta \in \mathbf{R}^+$ *and* $\xi . \eta \in \mathbf{R}^+$.

(*b*) *For each* $\xi \in \mathbf{R}$, *one and only one of the following is true*:

(*i*) $\xi \in \mathbf{R}^+$ (*ii*) $\xi = 0$ (*iii*) $-\xi \in \mathbf{R}^+$.

4.4 *Definition. A real number* ξ *is said to be greater than a real number* η if $\xi - \eta \in \mathbf{R}^+$.

Using the symbol $>$ to denote '*greater than*', we write

$$\xi > \eta \text{ iff } \xi - \eta \in \mathbf{R}^+$$

The same thing can also be expressed by saying that η is '*less than*' ξ and write $\eta < \xi$.

Ex. For real numbers $\xi, \eta, \zeta, \ldots$, prove that

1. Exactly one of the following holds:

   (i) $\xi > \eta$. (ii) $\xi = \eta$. (iii) $\xi < \eta$.

2. $\xi > \eta$ and $\eta > \zeta \Rightarrow \xi > \zeta$.

3. $\xi > \eta \Rightarrow \xi + \zeta > \eta + \zeta$.

4. For $\zeta \in \mathbf{R}^+$, $\xi > \eta \Rightarrow \xi \cdot \zeta > \eta \cdot \zeta$.

Theorem 9. *Prove that* $(\mathbf{R}, +, \cdot, >)$ *is an ordered field.*

The proof is left to the reader.

5. REAL RATIONAL AND IRRATIONAL NUMBERS

If a Cauchy sequence of rational numbers converges in the field of rationals then the real number determined by it is called a *real rational number*. Thus if a rational Cauchy sequence $\{a_n\}$ converges to a rational number α then $[a_n]$ is the real rational number α.

Theorem 10. *If α is a rational number then there exists a rational Cauchy sequence converging to α.*

Let $\{a_n\}$, where $a_n = \alpha$ for all n, be a rational sequence, which clearly converges to α. Also it is a Cauchy sequence, since $|a_{n+p} - a_n| = 0$ for all n, p.

Theorem 11. *If $\{a_n\}$ and $\{b_n\}$ are two rational Cauchy sequences converging to the same limit α, then*

$$\{a_n\} \sim \{b_n\}$$

$\because$
$$\lim a_n = \alpha = \lim b_n$$

$\therefore$
$$\lim (a_n - b_n) = 0$$

$\therefore$
$$\{a_n\} \sim \{b_n\}$$

Theorem 12. *To every rational number α there corresponds a unique real rational number.*

Let $\{a_n\}$ be a rational Cauchy sequence that converges to α, so that $\{a_n\}$ is a real rational number that corresponds to α.

If $[a_n']$ be another real rational number that corresponds to α, then the rational Cauchy sequence $\{a_n'\}$ also converges to α. Hence by Theorem 11,

$$\{a_n\} \sim \{a_n'\}$$

$\therefore$
$$[a_n] = [a_n'].$$

Notation. The set of all real rational numbers will be denoted by $\mathbf{R}^*$, where $\mathbf{R}^* = \{\alpha \mid \alpha$ is a real rational number$\}$.

If $\xi \in \mathbf{R}$ but $\xi \notin \mathbf{R}^*$, then ξ is called *real irrational* number and the set of all real irrational numbers is $\mathbf{R} - \mathbf{R}^*$,

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6. SOME PROPERTIES OF REAL NUMBERS

Theorem 13. *If ξ is a real number and $\{x_n\}$ a Cauchy sequence in* **Q** *such that $\{x_n\} \in \xi$, then $\lim x_n = \xi$* in **R**.

Let $\varepsilon > 0$ be a real number and e a rational number such that $0 < e < \varepsilon$.

Since $\{x_n\}$ is a Cauchy sequence in **Q**, therefore for $e > 0$, $\exists\, m_0 \in \mathbf{N}$ such that

$$|x_n - x_m| < \frac{e}{2} \quad \forall\, m, n \geq m_0$$

Hence for all $m, n \geq m_0$, $\quad e - |x_n - x_m| > \dfrac{e}{2}$

and so for each $n \geq m_0$ the sequence $\{y_m\}$, where $y_m = e - |x_n - x_m|$, is a positive sequence in **R**.

$$\therefore \qquad [y_m] > 0 \text{ in } \mathbf{R}$$

or
$$e - [|x_n - x_m|] > 0 \text{ in } \mathbf{R}$$

Thus for $n \geq m_0$

$$\begin{aligned} |x_n - \xi| &= |x_n - [x_m]| \\ &= |[x_n] - [x_m]| \\ &= |[x_n - x_m]| = [|x_n - x_m|] < e < \varepsilon \end{aligned}$$

$$\therefore \qquad \lim x_n = \xi, \text{ and } \xi \in \mathbf{R}.$$

Corollary 1. *If $\xi \in \mathbf{R}$ and $\varepsilon > 0$ in* **R** *then there is an $x \in \mathbf{Q}$ such that $|\xi - x| < \varepsilon$* in **R**.

Let $\{x_n\}$ be a Cauchy sequence in **Q** such that $\{x_n\} \in \xi$.

$$\therefore \qquad \lim x_n = \xi \text{ in } \mathbf{R}$$

Hence for every $\varepsilon > 0$ in **R**, $\exists$ an $m \in \mathbf{N}$ such that

$$|\xi - x_n| < \varepsilon \text{ in } \mathbf{R} \quad \forall\, n \geq m$$

In particular for $x = x_m \in \mathbf{Q}$,

$$|\xi - x| < \varepsilon$$

Corollary 2. *If $\xi < \eta$ in* **R**, *then there is an $x \in \mathbf{Q}$ such that $\xi < x < \eta$.*

We know that every ordered field is dense. Therefore, there is a real number ζ such that $\xi < \zeta < \eta$. If $\varepsilon = \min(\zeta - \xi, \eta - \zeta)$, then by Cor. 1. There is a rational number x such that

$$\xi < \zeta - \varepsilon < x < \zeta + \varepsilon < \eta.$$

Corollary 3. **R** *is Archimedean*

or

For each pair of positive real numbers ξ, η, there exists a positive integer n such that $n\,\xi > \eta$.

For $0 < \xi < \eta$ in **R**, let x, y be rational numbers such that $0 < x < \xi < \eta < y$ in **R**. Since the field **Q** is Archimedean, therefore $\exists$ an $n \in \mathbf{N}$ such that $nx > y$.

$\therefore$
$$n\xi > nx > y > \eta$$

Hence **R** is Archimedean.

Theorem 14. *Every Cauchy sequence of Cantor real numbers converges in* **R**.

Let $\{\xi_n\}$ be a Cauchy sequence in **R**. Then for $\varepsilon > 0$ in **R**, $\exists\, m_1 \in \mathbf{N}$ such that
$$|\xi_n - \xi_m| < \varepsilon/3 \qquad \forall\, m, n \geq m_1 \qquad \text{...(1)}$$

By Cor. 1, for each $n \in \mathbf{N}$, there exists a rational number x_n such that $|\xi_n - x_n| < 1/n$.

For above $\varepsilon > 0$, we can choose $m_2 \in \mathbf{N}$ such that
$$1/n < \tfrac{1}{3}\varepsilon \text{ and } |\xi_n - x_n| < 1/n < \tfrac{1}{3}\varepsilon \qquad \forall\, n \geq m_2 \qquad \text{...(2)}$$

Let $m_0 = \max(m_1, m_2)$, from (1) and (2),
$$\begin{aligned}|x_n - x_m| &= |x_n - \xi_n + \xi_n - \xi_m + \xi_m - x_m| \\ &\leq |x_n - \xi_n| + |\xi_n - \xi_m| + |\xi_m - x_m| \\ &< \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon + \tfrac{1}{3}\varepsilon = \varepsilon, \quad \forall\, m, n \geq m_0\end{aligned}$$

$\therefore$ $\{x_n\}$ is a Cauchy sequence in **Q**.

Hence $[x_n]$ is a real number ξ, say,

$\therefore$ $\lim x_n = \xi$ in **R** [Theorem 13]

Again, since $\{x_n\}$ converges to ξ in **R**, for $\varepsilon > 0$ in **R**, $\exists$ an $m_3 \in \mathbf{N}$ such that
$$|x_n - \xi| < \frac{2}{3}\varepsilon, \quad \forall n \geq m_3 \qquad \forall\, n \geq m_3 \qquad \text{...(3)}$$

Hence for $n \geq m' = \max(m_2, m_3)$, we have
$$\begin{aligned}|\xi_n - \xi| &= |\xi_n - x_n + x_n - \xi| \\ &\leq |\xi_n - x_n| + |x_n - \xi| \\ &< \tfrac{1}{3}\varepsilon + \tfrac{2}{3}\varepsilon = \varepsilon \qquad \text{[using (2), (3)]}\end{aligned}$$

$\therefore$
$$\lim \xi_n = \xi \text{ in } \mathbf{R}.$$

7. COMPLETENESS IN R

Theorem 15. ***Order-completeness property.*** *Every non-empty subset of real numbers which is bounded above has the supremum in* **R**.

(*i*) Suppose A is a non-empty subset of **R**, b is an upper bound of A, and $\alpha \in A$ so that $\alpha \leq b$.

Since **R** is Archimedean, therefore, for each $n \in \mathbf{N}$, there exists an $\overline{m} \in \mathbf{N}$ such that $\alpha + (\overline{m}/n) \geq b$ in A, and therefore $\alpha + (\overline{m}/n)$ is an upper bound of A. Hence for each $n \in \mathbf{N}$, the set
$$B_n = \left\{ m \mid \alpha + \frac{m}{n} \text{ is an upper bound of } A, m \in \mathbf{N} \right\}$$

is a non-empty subset of **N**.

But, since every non-empty subset of natural numbers has a least element, let m_n be the least element of B_n. Thus for each $n \in \mathbf{N}$,

(1) $y_n = \alpha + \dfrac{m_n}{n}$ is an upper bound of A, and

(2) $x_n = y_n - \dfrac{1}{n} = \alpha + \dfrac{m_n - 1}{n}$ is not an upper bound of A.

$$\therefore \qquad x_m < y_n, \quad \forall\, m, n \in \mathbf{N}$$

Now

$$x_n - x_m < y_m - x_m = \frac{1}{m}$$

and

$$x_m - x_n < y_n - x_n = \frac{1}{n}$$

$$\therefore \qquad |x_n - x_m| = \max\,(x_n - x_m, x_m - x_n)$$

$$< \max\left(\frac{1}{m}, \frac{1}{n}\right), \quad \forall\, m, n \in \mathbf{N}$$

$$< \frac{1}{n_0}, \text{ for } \; m, n \geq n_0$$

Hence, $\{x_n\}$ is a Cauchy sequence in $\mathbf{R}$ which by Cauchy property of $\mathbf{R}$ (Theorem 14, § 6) converges in $\mathbf{R}$.

$$\therefore \qquad \lim x_n = \xi, \text{ where } \xi \in \mathbf{R}\,.$$

(*ii*) We shall now show that $\xi = \sup A$.

Let, if possible, ξ be not an upper bound of A.

Hence $\xi < x$, for some x in A.

Since $\lim x_n = \xi$ and $\lim 1/n = 0$, there is some $n \in \mathbf{N}$ such that

$$\frac{1}{n} < \frac{x - \xi}{2}$$

and

$$x_n - \xi \leq |x_n - \xi| < \frac{x - \xi}{2}$$

Then by equations (1) and (2), we have

$$y_n = x_n + \frac{1}{n} < \left(\xi + \frac{x - \xi}{2}\right) + \frac{x - \xi}{2} = x \text{ in } \mathbf{R}$$

But this is impossible by (1), since $x \in A$.

Hence ξ is an upper bound of A.

If ξ is not the least upper bound, let $\eta < \xi$ be the least upper bound of A.

Let $\xi - \eta = \delta > 0$.

Since $\lim x_n = \xi$, therefore for $\delta > 0, \exists\, n \in \mathbf{N}$ such that

$$\xi - x_n \le |\xi - x_n| < \delta = \xi - \eta$$

$$\therefore \qquad \eta < x_n \le x, \text{ for some } x \in A$$

which is a contradiction.

Hence $\xi \le \eta$, so that ξ is the least upper bound of A.

Thus, the non-empty set A of real numbers has the supremum in $\mathbf{R}$.