CHAPTER 17

Partial Differential Equations

17.1 INTRODUCTION

The reader has, already been introduced to the notion of partial differential equations. Here, we shall begin by studying the ways in which partial differential equations are formed. Then we shall investigate the solutions of special types of partial differential equations of the first and higher orders.

In what follows x and y will, usually be taken as the independent variables and z, the dependent variable so that $z = f(x, y)$ and we shall employ the following notation :

$$\frac{\partial z}{\partial x} = p,\ \frac{\partial z}{\partial y} = q,\ \frac{\partial^2 z}{\partial x^2} = r,\ \frac{\partial^2 z}{\partial x \partial y} = s,\ \frac{\partial^2 z}{\partial y^2} = t.$$

17.2 FORMATION OF PARTIAL DIFFERENTIAL EQUATIONS

Unlike the case of ordinary differential equations which arise from the elimination of arbitrary constants; *the partial differential equations can be formed either by the elimination of arbitrary constants or by the elimination of arbitrary functions from a relation involving three or more variables.* The method is best illustrated by the following examples :

Example 17.1. *Derive a partial differential equation (by eliminating the constants) from the equation*

$$2z = \frac{x^2}{a^2} + \frac{y^2}{b^2}. \qquad ...(i)$$

Solution. Differentiating (*i*) partially with respect to x and y, we get

$$2\frac{\partial z}{\partial x} = \frac{2x}{a^2} \quad \text{or} \quad \frac{1}{a^2} = \frac{1}{x}\frac{\partial z}{\partial x} = \frac{p}{x}$$

and

$$\frac{2\partial z}{\partial y} = \frac{2y}{b^2} \quad \text{or} \quad \frac{1}{b^2} = \frac{1}{y}\frac{\partial z}{\partial y} = \frac{q}{y}$$

Substituting these values of $1/a^2$ and $1/b^2$ in (*i*), we get

$$2z = xp + yq$$

as the desired partial differential equation of the first order.

Example 17.2. *Form the partial differential equations (by eliminating the arbitrary functions) from*
(*a*) $z = (x + y)\,\phi\,(x^2 - y^2)$ (*P.T.U., 2009*)
(*b*) $z = f(x + at) + g(x - at)$ (*V.T.U., 2009*) (*c*) $f(x^2 + y^2, z - xy) = 0$ (*S.V.T.U., 2007*)

Solution. (*a*) We have $z = (x + y)\,\phi\,(x^2 - y^2)$

Differentiating z partially with respect to x and y,

$$p = \frac{\partial z}{\partial x} = (x + y)\,\phi'\,(x^2 - y^2)\,.\,2x + \phi\,(x^2 - y^2), \qquad ...(i)$$

$$q = \frac{\partial z}{\partial y} = (x + y)\,\phi'\,(x^2 - y^2)\,.\,(-2y) + \phi\,(x^2 - y^2) \qquad ...(ii)$$

From (*i*), $$p - \frac{z}{x + y} = 2x\,(x + y)\,\phi'\,(x^2 - y^2)$$

From (*ii*), $$q - \frac{z}{x + y} = -2y\,(x + y)\,\phi'\,(x^2 - y^2)$$

Division gives $$\frac{p - z/(x + y)}{q - z/(x + y)} = -\frac{x}{y}$$

i.e., $[p(x + y) - z]y + [q\,(x + y) - z]x$

i.e., $(x + y)\,(py + qx) - z(x + y) = 0$

Hence $py + qz = z$ is required equation.

(*b*) We have $z = f\,(x + at) + g(x - at)$ ...(*i*)

Differentiating z partially with respect to x and t,

$$\frac{\partial z}{\partial x} = f'\,(x + at) + g'(x - at),\ \frac{\partial^2 z}{\partial x^2} = f''\,(x + at) + g''(x - at) \qquad ...(ii)$$

$$\frac{\partial z}{\partial t} = af'\,(x + at) - ag'(x - at),\ \frac{\partial^2 z}{\partial t^2} = a^2 f''\,(x + at) + a^2 g''(x - at) = a^2\frac{\partial^2 z}{\partial x^2} \qquad \text{[By (ii)]}$$

Thus the desired partial differential equation is $\dfrac{\partial^2 z}{\partial t^2} = a^2\,\dfrac{\partial^2 z}{\partial x^2}$

which is an equation of the second order and (*i*) is its solution.

(*c*) Let $x^2 + y^2 = u$ and $z - xy = v$ so that $f\,(u, v) = 0$.

Differentiating partially w.r.t. x and y, we have

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}p\right) = 0$$

or $$\frac{\partial f}{\partial u}(2x) + \frac{\partial f}{\partial v}(-y + p) = 0 \qquad ...(i)$$

and $$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}q\right) = 0 \quad \text{or} \quad \frac{\partial f}{\partial u}(2y) + \frac{\partial f}{\partial v}(-x + q) = 0 \qquad ...(ii)$$

Eliminating $\dfrac{\partial f}{\partial u}$ and $\dfrac{\partial f}{\partial v}$ from (*i*) and (*ii*), we get

$$\begin{vmatrix} 2x & -y + p \\ 2y & -x + q \end{vmatrix} = 0 \qquad \text{or} \qquad xq - yp = x^2 - y^2.$$

Example 17.3. *Find the differential equation of all planes which are at a constant distance a from the origin.* (*V.T.U., 2009 S ; Kurukshetra, 2006*)

Solution. The equation of the plane in 'normal form' is

$$lx + my + nz = a \qquad ...(i)$$

where l, m, n are the d.c.s of the normal from the origin to the plane.

Then $\quad l^2 + m^2 + n^2 = 1$ or $n = \sqrt{(1 - l^2 - m^2)}$

$\therefore$ (*i*) becomes $\quad lx + my + \sqrt{(1 - l^2 - m^2)}\ z = a \quad ...(ii)$

Differentiating partially w.r.t. x, we get

$$l + \sqrt{(1 - l^2 - m^2)}\ .\ p = 0 \quad ...(iii)$$

Differentiating partially w.r.t. y, we get

$$m + \sqrt{(1 - l^2 - m^2)}\ .\ q = 0 \quad ...(iv)$$

Now we have to eliminate l, m from (*ii*), (*iii*) and (*iv*).

From (*iii*), $l = -\sqrt{(1 - l^2 - m^2)}\ .\ p$ and $m = -\sqrt{(1 - l^2 - m^2)}\ .\ q$

Squaring and adding, $l^2 + m^2 = (1 - l^2 - m^2)(p^2 + q^2)$

or $\quad (l^2 + m^2)(1 + p^2 + q^2) = p^2 + q^2$ or $1 - l^2 - m^2 = 1 - \dfrac{p^2 + q^2}{1 + p^2 + q^2} = \dfrac{1}{1 + p^2 + q^2}$

Also $\quad l = -\dfrac{p}{\sqrt{(1 + p^2 + q^2)}}$ and $m = -\dfrac{q}{\sqrt{(1 + p^2 + q^2)}}$

Substituting the values of l, m and $1 - l^2 - m^2$ in (*ii*), we obtain

$$\frac{-px}{\sqrt{(1 + p^2 + q^2)}} - \frac{qy}{\sqrt{(1 + p^2 + q^2)}} + \frac{1}{\sqrt{(1 + p^2 + q^2)}}\ z = a$$

or $\quad z = px + qy + a\sqrt{(1 + p^2 + q^2)}$ which is the required partial differential equation.

PROBLEMS 17.1

From the partial differential equation (by eliminating the arbitrary constants from :

1. $z = ax + by + a^2 + b^2$.
2. $(x - a)^2 + (y - b)^2 + z^2 = c^2$. (*Kottayam, 2005*)
3. $(x - a)^2 + (y - b)^2 = z^2 \cot^2 \alpha$ (*Anna, 2009*)
4. $z = a \log \left\{\dfrac{b(y-1)}{1-x}\right\}$ (*J.N.T.U., 2002 S*)
5. Find the differential equation of all spheres of fixed radius having their centres in the xy-plane. (*Madras 2000 S*)
6. Find the differential equation of all spheres whose centres lie on the z-axis. (*Kerala, 2005*)

Form the partial differential equations (by eliminating the arbitrary functions) from :

7. $z = f(x^2 - y^2)$ (*S.V.T.U., 2008*)
8. $z = f(x^2 + y^2) + x + y$ (*Anna, 2009*)
9. $z = yf(x) + xg(y)$. (*V.T.U., 2004*)
10. $z = x^2 f(y) + y^2 g(x)$. (*Anna, 2003*)
11. $z = f(x) + e^y g(x)$.
12. $xyz = \phi(x + y + z)$.
13. $z = f_1(x) f_2(y)$.
14. $z = e^{my} \phi(x - y)$. (*P.T.U., 2002*)
15. $z = y^2 + 2f\left(\dfrac{1}{x} + \log y\right)$. (*V.T.U., 2010 ; J.N.T.U., 2010 ; Madras, 2000*)
16. $z = f_1(y + 2x) + f_2(y - 3x)$. (*Kurukshetra, 2005*)
17. $v = \dfrac{1}{r}[f(r - at) + F(r + at)]$.
18. $z = xf_1(x + t) + f_2(x + t)$.
19. $F(xy + z^2, x + y + z) = 0$. (*V.T.U., 2006*)
20. $F(x + y + z, x^2 + y^2 + z^2) = 0$. (*S.V.T.U., 2007*)
21. If $u = f(x^2 + 2yz, y^2 + 2zx)$, prove that $(y^2 - zx)\dfrac{\partial u}{\partial x} + (x^2 - yz)\dfrac{\partial u}{\partial y} + (z^2 - xy)\dfrac{\partial u}{\partial z} = 0$.

17.3 SOLUTIONS OF A PARTIAL DIFFERENTIAL EQUATION

It is clear from the above examples that a partial differential equation can result both from elimination of arbitrary constants and from the elimination of arbitrary functions.

The solution $\quad f(x, y, z, a, b) = 0 \quad ...(1)$

of a first order partial differential equation which contains two arbitrary constants is called a *complete integral*.

A solution obtained from the complete integral by assigning particular values to the arbitrary constants is called a particular integral.

If we put $b = \phi(a)$ in (1) and find the envelope of the family of surfaces $f[x, y, z, \phi(a)] = 0$, then we get a solution containing an arbitrary function ϕ, which is called the *general integral*.

The envelope of the family of surfaces (1), with parameters a and b, if it exists, is called a *singular integral*. The singular integral differs from the particular integral in that it is not obtained from the complete integral by giving particular values to the constants.

17.4 EQUATIONS SOLVABLE BY DIRECT INTEGRATION

We now consider such partial differential equations which can be solved by direct integration. In place of the usual constants of integration, we must, however, use arbitrary functions of the variable held fixed.

Example 17.4. *Solve* $\dfrac{\partial^3 z}{\partial x^2 \partial y} + 18xy^2 + \sin(2x - y) = 0.$ *(V.T.U., 2010)*

Solution. Integrating twice with respect to x (keeping y fixed),

$$\frac{\partial^2 z}{\partial x \partial y} + 9x^2y^2 - \frac{1}{2}\cos(2x - y) = f(y)$$

$$\frac{\partial z}{\partial y} + 3x^3y^2 - \frac{1}{4}\sin(2x - y) = xf(y) + g(y).$$

Now integrating with respect to y (keeping x fixed)

$$z + x^3y^3 - \frac{1}{4}\cos(2x - y) = x\int f(y)dy + \int g(y)dy + w(x)$$

The result may be simplified by writing

$$\int f(y)dy = u(y) \text{ and } \int g(y)dy = v(y).$$

Thus $z = \dfrac{1}{4}\cos(2x - y) - x^3y^3 + xu(y) + v(y) + w(x)$ where u, v, w are arbitrary functions.

Example 17.5. *Solve* $\dfrac{\partial^2 z}{\partial x^2} + z = 0$, *given that when* $x = 0$, $z = e^y$ *and* $\dfrac{\partial z}{\partial x} = 1.$

Solution. If z were function of x alone, the solution would have been $z = A\sin x + B\cos x$, where A and B are constants. Since z is a function of x and y, A and B can be arbitrary functions of y. Hence the solution of the given equation is $z = f(y)\sin x + \phi(y)\cos x$

$$\therefore \qquad \frac{\partial z}{\partial x} = f(y)\cos x - \phi(y)\sin x$$

When $x = 0$; $z = e^y$, $\quad \therefore e^y = \phi(y).$ $\quad$ When $x = 0$, $\dfrac{\partial z}{\partial x} = 1$, $\quad \therefore \; 1 = f(y).$

Hence the desired solution is $z = \sin x + e^y \cos x$.

Example 17.6. *Solve* $\dfrac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$, *for which* $\dfrac{\partial z}{\partial y} = -2\sin y$ *when* $x = 0$ *and* $z = 0$ *when* y *is an odd multiple of* $\pi/2$. *(V.T.U., 2010 S)*

Solution. Given equation is $\dfrac{\partial^2 z}{\partial x \partial y} = \sin x \sin y$

Integrating w.r.t. x, keeping y constant, we get

$$\frac{\partial z}{\partial y} = -\cos x \sin y + f(y) \qquad \text{...}(i)$$

When $x = 0$, $\frac{\partial z}{\partial y} = -2 \sin y$, $\quad \therefore \quad -2 \sin y = -\sin y + f(y)$ or $f(y) = -\sin y$

$\therefore$ (*i*) becomes $\frac{\partial z}{\partial y} = -\cos x \sin y - \sin y$

Now integrating w.r.t. y, keeping x constant, we get

$$z = \cos x \cos y + \cos y + g(x) \quad \text{...}(ii)$$

When y is an odd multiple of $\pi/2$, $z = 0$.

$\therefore \quad 0 = 0 + 0 + g(x)$ or $g(x) = 0$ $\quad [\because \cos (2n + 1) \pi/2 = 0]$

Hence from (*ii*), the complete solution is $z = (1 + \cos x) \cos y$.

PROBLEMS 17.2

Solve the following equations :

1. $\frac{\partial^2 z}{\partial x \partial y} = \frac{x}{y} + a.$
2. $\frac{\partial^2 z}{\partial x^2} = xy.$
3. $\frac{\partial^2 u}{\partial x \partial t} = e^{-t} \cos x.$
4. $\frac{\partial^3 z}{\partial x^2 \partial y} = \cos (2x + 3y).$
5. $\frac{\partial^2 z}{\partial y^2} = z$, gives that when $y = 0$, $z = e^x$ and $\frac{\partial z}{\partial y} = e^{-x}$
6. $\frac{\partial^2 z}{\partial x^2} = a^2 z$ given that when $x = 0$, $\frac{\partial z}{\partial x} = a \sin y$ and $\frac{\partial z}{\partial y} = 0.$

17.5 LINEAR EQUATIONS OF THE FIRST ORDER

A linear partial differential equation of the first order, commonly known as *Lagrange's Linear equation**, is of the form

$$Pp + Qq = R \quad \text{...(1)}$$

where P, Q and R are functions of x, y, z. This equation is called a *quasi-linear equation*. When P, Q and R are independent of z it is known as *linear equation*.

Such an equation is obtained by eliminating an arbitrary function ϕ from $\phi(u, v) = 0$ ...(2)

where u, v are some functions of x, y, z.

Differentiating (2) partially with respect to x and y.

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} p\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} p\right) = 0 \text{ and } \frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} q\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} q\right) = 0.$$

Eliminating $\frac{\partial \phi}{\partial u}$ and $\frac{\partial \phi}{\partial v}$, we get $\begin{vmatrix} \frac{\partial u}{\partial x} + \frac{\partial u}{\partial z} p & \frac{\partial v}{\partial x} + \frac{\partial v}{\partial z} p \\ \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} q & \frac{\partial v}{\partial y} + \frac{\partial v}{\partial z} q \end{vmatrix} = 0$

which simplifies to $\left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}\right) p + \left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\right) q = \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)$ ...(3)

This is of the same form as (1).

Now suppose $u = a$ and $v = b$, where a, b are constants, so that

$$\frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz = du = 0$$

$$\frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy + \frac{\partial v}{\partial z} dz = dv = 0.$$

*See footnote p. 142.

By cross-multiplication, we have

$$\frac{dx}{\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}}=\frac{dy}{\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}}=\frac{dz}{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}}.$$

or $$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$$ ...(4) [By virtue of (1) and (3)]

The solutions of these equations are $u = a$ and $v = b$.

$\therefore$ $\phi(u, v) = 0$ is the required solution of (1).

Thus to solve the equation $Pp + Qq = R$.

(*i*) *form the subsidiary equations* $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$.

(*ii*) *solve these simultaneous equations by the method of § 16.10 giving $u = a$ and $v = b$ as its solutions.*

(*iii*) *write the complete solution as $\phi(u, v) = 0$* or $u = f(v)$.

Example 17.7. *Solve* $\frac{y^2z}{x}p + xzq = y^2$. *(Kottayam, 2005)*

Solution. Rewriting the given equation as

$$y^2zp + x^2zq = y^2x,$$

The subsidiary equations are $\frac{dx}{y^2z}=\frac{dy}{x^2z}=\frac{dz}{y^2x}$

The first two fractions give $x^2dx = y^2dy$.

Integrating, we get $x^3 - y^3 = a$ ...(*i*)

Again the first and third fractions give $xdx = zdz$

Integrating, we get $x^2 - z^2 = b$ ...(*ii*)

Hence from (*i*) and (*ii*), the complete solution is

$$x^3 - y^3 = f(x^2 - z^2).$$

Example 17.8. *Solve* $(mz - ny)\frac{\partial z}{\partial x} + (nx - lz)\frac{\partial z}{\partial y} = ly - mx$. *(V.T.U., 2010 ; S.V.T.U., 2009)*

Solution. Here the subsidiary equations are $\frac{dx}{mz-ny}=\frac{dy}{mx-lz}=\frac{dz}{ly-mx}$

Using multipliers x, y, and z, we get each fraction $= \frac{xdx+ydy+zdz}{0}$

$\therefore$ $xdx + ydy + zdz = 0$ which on integration gives $x^2 + y^2 + z^2 = a$ ...(*i*)

Again using multipliers l, m and n, we get each fraction $= \frac{ldx+mdy+ndz}{0}$

$\therefore$ $ldx + mdy + ndx = 0$ which on integration gives $lx + my + nz = b$ ...(*ii*)

Hence from (*i*) and (*ii*), the required solution is $x^2 + y^2 + z^2 = f(lx + my + nz)$.

Example 17.9. *Solve* $(x^2 - y^2 - z^2)p + 2xyq = 2xz$. *(V.T.U., 2010 ; Anna, 2009 ; S.V.T.U., 2008)*

Solution. Here the subsidiary equations are $\frac{dx}{x^2-y^2-z^2}=\frac{dy}{2xy}=\frac{dz}{2xz}$

From the last two fractions, we have $\frac{dy}{y}=\frac{dz}{z}$

which on integration gives $\log y = \log z + \log a$ or $y/z = a$ ...(*i*)

Using multipliers x, y and z, we have

each fraction $= \frac{xdx+ydy+zdz}{x(x^2+y^2+z^2)}$ $\therefore$ $\frac{2xdx+2ydy+2zdz}{x^2+y^2+z^2}=\frac{dz}{z}$

which on integration gives $\log (x^2 + y^2 + z^2) = \log z + \log b$

or $$\frac{x^2 + y^2 + z^2}{z} = b \quad \text{...(ii)}$$

Hence from (*i*) and (*ii*), the required solution is $x^2 + y^2 + z^2 = zf(y/z)$.

Example 17.10. *Solve* $x^2 (y - z) p + y^2 (z - x) q = z^2 (x - y)$. *(P.T.U., 2009 ; Bhopal, 2008 ; S.V.T.U. 2007)*

Solution. Here the subsidiary equations are

$$\frac{dx}{x^2 (y - z)} = \frac{dy}{y^2 (z - x)} = \frac{dz}{z^2 (x - y)}$$

Using the multipliers $1/x$, $1/y$ and $1/z$, we have

$$\text{each fraction} = \frac{\frac{1}{x} dx + \frac{1}{y} dy + \frac{1}{z} dz}{0}$$

$$\therefore \quad \frac{dx}{x} + \frac{dy}{y} + \frac{dz}{z} = 0 \text{ which on integration gives}$$

$$\log x + \log y + \log z = \log a \quad \text{or} \quad xyz = a \quad \text{...(i)}$$

Using the multipliers $\frac{1}{x^2}, \frac{1}{y^2}$ and $\frac{1}{z^2}$, we get

$$\text{each fraction} = \frac{\frac{1}{x^2} dx + \frac{1}{y^2} dy + \frac{1}{z^2} dz}{0}$$

$$\therefore \quad \frac{dx}{x^2} + \frac{dy}{y^2} + \frac{dz}{z^2} = 0, \text{ which on integrating gives}$$

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0 \quad \text{...(ii)}$$

Hence from (*i*) and (*ii*), the complete solution is

$$xyz = f\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right).$$

Example 17.11. *Solve* $(x^2 - yz) p + (y^2 - zx) q = z^2 - xy$. *(Bhopal, 2008 ; V.T.U., 2006 ; Madras, 2000)*

Solution. Here the subsidiary equations are

$$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy} \quad \text{...(i)}$$

$$\text{Each of these equations} = \frac{dx - dy}{x^2 - y^2 - (y - x)z} = \frac{dy - dz}{y^2 - z^2 - x(z - y)}$$

i.e., $$\frac{d(x - y)}{(x - y)(x + y + z)} = \frac{d(y - z)}{(y - z)(x + y + z)} \quad \text{or} \quad \frac{d(x - y)}{x - y} = \frac{d(y - z)}{y - z}$$

Integrating, $$\log (x - y) = \log (y - z) + \log c \quad \text{or} \quad \frac{x - y}{y - z} = c \quad \text{...(ii)}$$

$$\text{Each of the subsidiary equations } (i) = \frac{xdx + ydy + zdz}{x^3 + y^3 + z^3 - 3xyz}$$

$$= \frac{xdx + ydy + zdz}{(x + y + z)(x^2 + y^2 + z^2 - yz - zx - xy)} \quad \text{...(iii)}$$

$$\text{Also each of the subsidiary equations} = \frac{dx + dy + dz}{x^2 + y^2 + z^2 - yz - zx - xy} \quad \text{...(iv)}$$

Equating (*iii*) and (*iv*) and cancelling the common factor, we get

$$\frac{xdx + ydy + zdz}{x + y + z} = dx + dy + dz$$

or $$\int (xdx + ydy + zdz) = \int (x + y + z) d\,(x + y + z) + c'$$

or $$x^2 + y^2 + z^2 = (x + y + z)^2 + 2c' \quad \text{or} \quad xy + yz + zx + c' = 0 \qquad ...(v)$$

Combining (*ii*) and (*v*), the general solution is

$$\frac{x - y}{y - z} = f\,(xy + yz + zx).$$

PROBLEMS 17.3

Solve the following equations :

1. $xp + yq = 3z$.
2. $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$.
3. $(z - y)p + (x - z)q = y - x$.
4. $p \cos (x + y) + q \sin (x + y) = z$.
5. $pyz + qzx = xy$.
6. $p \tan x + q \tan y = \tan z$.
7. $p - q = \log (x + y)$.
8. $xp - yq = y^2 - x^2$ (J.N.T.U., 2002 S)
9. $(y + z)p - (z + x)\,q = x - y$.
10. $x(y - z)p + y(z - x)q = z(x - y)$. (Bhopal, 2007)
11. $x(y^2 - z^2)p + y(z^2 - x^2)q - z(x^2 - y^2) = 0$. (V.T.U., 2010 ; Anna, 2008)
12. $y^2p - xyq = x(z - 2y)$. (S.V.T.U., 2008)
13. $(y^2 + z^2)p - xyq + zx = 0$. (P.T.U., 2009 ; V.T.U., 2009)
14. $(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx$. (Kerala, 2005)
15. $px(z - 2y^2) = (z - qy)(z - y^2 - 2x^3)$.

17.6 NON-LINEAR EQUATIONS OF THE FIRST ORDER

Those equations in which p and q occur other than in the first degree are called *non-linear partial differential equations of the first order*. The *complete solution* of such an equation contains only two arbitrary constants (*i.e.*, equal to the number of independent variables involved) and the particular integral is obtained by giving particular values to the constants.]

Here we shall discuss four standard forms of these equations.

Form I. f(p, q) = 0, *i.e., equations containing p and q only.*

Its complete solution is $z = ax + by + c$ ...(1)

where a and b are connected by the relation $f(a, b) = 0$ ...(2)

[Since from (1), $p = \frac{\partial z}{\partial x} = a$ and $q = \frac{\partial z}{\partial y} = b$, which when substituted in (2) give $f(p, q) = 0$].

Expressing (2) as $b = \phi(a)$ and substituting this value of b in (1), we get the required solution as $z = ax + \phi(a)y + c$ in which a and c are arbitrary constants.

Example 17.12. *Solve $p - q = 1$.* (Anna, 2009)

Solution. The complete solution is $z = ax + by + c$ where $a - b = 1$

Hence $z = ax + a - 1y + c$ is the desired solution.

Example 17.13. *Solve $x^2p^2 + y^2q^2 = z^2$.* (Anna, 2008 ; Bhopal, 2008 ; Kerala, 2005 ; Kurukshetra, 2005)

Solution. Given equation can be reduced to the above form by writing it as

$$\left(\frac{x}{z}\cdot\frac{\partial z}{\partial x}\right)^2 + \left(\frac{y}{z}\cdot\frac{\partial z}{\partial y}\right)^2 = 1 \qquad ...(i)$$

and setting $$\frac{dx}{x} = du,\ \frac{dy}{y} = dv,\ \frac{dz}{z} = dw \text{ so that } u = \log x,\ v = \log y,\ w = \log z.$$

Then (*i*) becomes $$\left(\frac{\partial w}{\partial u}\right)^2 + \left(\frac{\partial w}{\partial v}\right)^2 = 1$$

i.e., $$P^2 + Q^2 = 1 \quad \text{where} \quad P = \frac{\partial w}{\partial u} \quad \text{and} \quad Q = \frac{\partial w}{\partial v}.$$

Its complete solution is $w = au + bv + c$ ...(*ii*)

where $a^2 + b^2 = 1$ or $b = \sqrt{(1-a^2)}$.

$\therefore$ (*ii*) becomes $w = au + \sqrt{(1-a^2)}\, v + c$

or $$\log z = a \log x + \sqrt{(1-a^2)} \log y + c \text{ which is the required solution.}$$

Form II. f(z, p, q) = 0, *i.e., equations not containing x and y.*

As a trial solution, assume that z is a function of $u = x + ay$, where a is an arbitrary constant.

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} = \frac{dz}{du} \qquad q = \frac{\partial z}{\partial y} = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} = a\frac{dz}{du}$$

Substituting the values of p and q in $f(z, p, q) = 0$, we get

$$f\left(z, \frac{\partial z}{\partial u}, a\frac{dz}{du}\right) = 0 \text{ which is an ordinary differential equation of the first order.}$$

Rewiting it as $\dfrac{dz}{du} = \phi(z, a)$ it can be easily integrated giving

$$F(z, a) = u + b, \text{ or } x + ay + b = F(z, a) \text{ which is the desired complete solution.}$$

Thus to solve $f(z, p, q) = 0$,

(*i*) *assume* $u = x + ay$ *and substitute* $p = dz/du$, $q = a\, dz/du$ *in the given equation;*

(*ii*) *solve the resulting ordinary differential equation in z and u;*

(*iii*) *replace u by x + ay.*

Example 17.14. *Solve* $p(1 + q) = qz$. (*Madras, 2000 S*)

Solution. Let $u = x + ay$, so that $p = dz/du$ and $q = a\, dz/du$.

Substituting these values of p and q in the given equation, we have

$$\frac{dz}{du}\left(1 + a\frac{dz}{du}\right) = az\frac{dz}{du} \text{ or } a\frac{dz}{du} = az - 1 \quad \text{or} \quad \int \frac{a\,dz}{az-1} = \int du + b$$

or $$\log(az - 1) = u + b \text{ or } \log(az - 1) = x + ay + b$$

which is the required complete solution.

Example 17.15. *Solve* $q^2 = z^2p^2(1 - p^2)$. (*J.NT.U., 2005 ; Kerala, 2005*)

Solution. Setting $u = y + ax$ and $z = f(u)$, we get

$$p = \frac{\partial z}{\partial x} = \frac{dz}{du} \cdot \frac{\partial u}{\partial x} = a\frac{dz}{du} \text{ and } q = \frac{dz}{du} \cdot \frac{\partial u}{\partial y} = \frac{dz}{du}$$

$\therefore$ The given equation becomes $\left(\dfrac{dz}{du}\right)^2 = a^2z^2\left(\dfrac{dz}{du}\right)^2 \left\{1 - a^2\left(\dfrac{dz}{du}\right)^2\right\}$ ...(*i*)

or $$a^4z^2\left(\frac{dz}{du}\right)^2 = a^2z^2 - 1 \quad \text{or} \quad \frac{dz}{du} = \frac{\sqrt{(a^2z^2-1)}}{a^2z}$$

Integrating, $$\int \frac{a^2z}{\sqrt{(a^2z^2-1)}} dz = \int du + c \quad \text{or} \quad (a^2z^2 - 1)^{1/2} = u + c$$

i.e., $$a^2z^2 = (y + ax + c)^2 + 1 \qquad [\because u = y + ax]$$

The second factor in (*i*) is $dz/du = 0$. Its solution is $z = c'$.

Example 17.16. *Solve* $z^2(p^2x^2 + q^2) = 1$. (*Bhopal, 2008 S*)

Solution. Given equation can be reduced to the above form by writing it as

$$z^2\left[\left(x\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]=1 \qquad ...(i)$$

Putting $X = \log x$, so that $x\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial X}$, (i) takes the standard form

$$z^2\left[\left(\frac{\partial z}{\partial X}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2\right]=1 \qquad ...(ii)$$

Let $u = X + ay$ and put $\dfrac{\partial z}{\partial X}=\dfrac{dz}{du}$ and $\dfrac{\partial z}{\partial y}=a\dfrac{dz}{du}$ in (ii), so that

$$z^2\left[\left(\frac{dz}{du}\right)^2+a^2\left(\frac{dz}{du}\right)^2\right]=1 \quad \text{or} \quad \sqrt{(1+a^2)}\,zdz=\pm\, du$$

Integrating, $\sqrt{(1+a^2)}\,z^2 = \pm\, 2u + b = \pm\, 2(X+ay)+b$

or $z^2\sqrt{(1+a^2)} = \pm\, 2\,(\log x + ay) + b$

which is the complete solution required.

Form III. f(x, p) = F(y, q), *i.e., equations in which z is absent and the terms containing x and p can be separated from those containing y and q.*

As a trial solution assume that $f(x, p) = F(y, q) = a$, say

Then solving for p, we get $p = \phi(x)$

and solving for q, we get $q = \psi(y)$

Since $dz = \dfrac{\partial z}{\partial x}dx+\dfrac{\partial z}{\partial y}dy = pdx + qdy$

$\therefore$ $dz = \phi(x)dx + \psi(y)dy$

Integrating, $z = \int \phi(x)dx + \int \psi(y)dy + b$

which is the desired complete solution containing two constants a and b.

Example 17.17. *Solve* $p^2 + q^2 = x + y$. *(Bhopal, 2006 ; Madras, 2003)*

Solution. Given equation is $p^2 - x = y - q^2 = a$, say

$\therefore$ $p^2 - x = a$ gives $p = \sqrt{(a+x)}$

and $y - q^2 = a$ gives $q = \sqrt{(y-a)}$

Substituting these values of p and q in $dz = pdx + qdy$, we get

$$dz = \sqrt{(a+x)}\,dx + \sqrt{(y-a)}\,dy$$

$\therefore$ integrating gives, $z = \dfrac{2}{3}(a+x)^{3/2} + \dfrac{2}{3}(y-a)^{3/2} + b$

which is the required complete solution.

Example 17.18. *Solve* $z^2(p^2 + q^2) = x^2 + y^2$. *(Bhopal, 2008)*

Solution. The equation can be reduced to the above form by writing it as

$$\left(z\frac{\partial z}{\partial x}\right)^2+\left(z\frac{\partial z}{\partial y}\right)^2=x^2+y^2 \qquad ...(i)$$

and putting $zdz = dZ$, *i.e.*, $Z = \dfrac{1}{2}z^2$

$$\therefore \quad \frac{\partial Z}{\partial x}=\frac{\partial Z}{\partial z}\cdot\frac{\partial z}{\partial x}=z\frac{\partial z}{\partial x}=P$$

and

$$\frac{\partial Z}{\partial y}=\frac{\partial Z}{\partial z}\cdot\frac{\partial z}{\partial y}=z\frac{\partial z}{\partial y}=Q$$

$\therefore$ (*i*) becomes $\quad P^2 + Q^2 = x^2 + y^2$

or $\quad P^2 - x^2 = y^2 - Q^2 = a$, say.

$$\therefore \quad P = \sqrt{(x^2 + a)} \text{ and } Q = \sqrt{(y^2 - a)}.$$

$\therefore \quad dZ = Pdx + Qdy$ gives

$$dZ = \sqrt{(x^2 + a)}\, dx + \sqrt{(y^2 - a)}\, dy$$

Integrating, we have

$$Z = \frac{1}{2} x \sqrt{(x^2 + a)} + \frac{1}{2} a \log [x + \sqrt{(x^2 + a)}] + \frac{1}{2} y \sqrt{(y^2 - a)} - \frac{1}{2} a \log [y + \sqrt{(y^2 - a)}] + b$$

or

$$z^2 = x\sqrt{(x^2 + a)} + y\sqrt{(y^2 - a} + a \log \frac{x + \sqrt{(x^2 + a)}}{y + \sqrt{(y^2 - a)}} + 2b$$

which is the required complete solution.

Example 17.19. *Solve* $(x + y)(p + q)^2 + (x - y)(p - q)^2 = 1$.

(*Bhopal, 2006 ; Rajasthan, 2006 ; V.T.U., 2003*)

Solution. This equation can be reduced to the form $f(x, q) = F(y, q)$ by putting $u = x + y$, $v = x - y$ and taking $z = z(u, v)$.

Then $\quad p = \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial u} \cdot \dfrac{\partial u}{\partial x} + \dfrac{\partial z}{\partial v} \cdot \dfrac{\partial v}{\partial x} = P + Q$

and $\quad q = \dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial u} \cdot \dfrac{\partial u}{\partial y} + \dfrac{\partial z}{\partial v} \cdot \dfrac{\partial v}{\partial y} = P - Q$, where $P = \dfrac{\partial z}{\partial u}, Q = \dfrac{\partial z}{\partial v}$

Substituting these, the given equation reduces to

$$u(2P)^2 + v(2Q)^2 = 1 \quad \text{or} \quad 4P^2u = 1 - 4Q^2v = a \text{ (say)}$$

$$P = \pm \frac{1}{2}\sqrt{\frac{a}{u}}, Q = \pm \frac{1}{2}\sqrt{\frac{1-a}{v}}$$

$$\therefore \quad dz = \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv = Pdu + Qdv$$

$$= \pm \frac{\sqrt{a}}{2} \frac{du}{\sqrt{u}} \pm \frac{\sqrt{1-a}}{2} \frac{dv}{\sqrt{v}}$$

Integrating, we have $\quad z = \pm \sqrt{a}\sqrt{u} \pm \sqrt{1-a}\sqrt{v} + b$

or $\quad z = \pm \sqrt{a(x + y)} \pm \sqrt{(1 - a)(x - y)} + b$

which is the required complete solution.

Form IV. $z = px + qy + f(p, q)$: *an equation analogous to the Clairaut's equation (§ 11.14).*

Its complete solution is $z = ax + by + f(a, b)$ which is obtained by writing a for p and b for q in the given equation.

Example 17.20. *Solve* $z = px + qy + \sqrt{(1 + p^2 + q^2)}$. (*Anna, 2009*)

Solution. Given equation is of the form $z = px + qy + f(p, q)$ where $f(p, q) = \sqrt{(1 + p^2 + q^2)}$

$\therefore$ Its complete solution is $z = ax + by + \sqrt{(1 + a^2 + b^2)}$.

PROBLEMS 17.4

Obtain the complete solution of the following equations :

1. $pq + p + q = 0$.
2. $p^2 + q^2 = 1$. (*Osmania, 2000*)
3. $z = p^2 + q^2$. (*Anna, 2005 S ; J.N.T.U., 2002 S*)
4. $p(1 - q^2) = q(1 - z)$ (*Anna, 2006*)
5. $yp + xq + pq = 0$.
6. $p + q = \sin x + \sin y$.

7. $p^2 - q^2 = x - y$.
8. $\sqrt{p} + \sqrt{q} = x + y$.
9. $p^2 + q^2 = x^2 + y^2$. *(Osmania, 2003)*
10. $z = px + qy + \sin(x + y)$.
11. $\sqrt{p} + \sqrt{q} = 2x$. *(J.N.T.U., 2006)*
12. $z = px + qy - 2\sqrt{(pq)}$.
13. $(x - y)(px - qy) = (p - q)^2$. **[Hint.** Use $x + y = u, xy = v$]

17.7 CHARPIT'S METHOD*

We now explain a general method for finding the complete integral of a non-linear partial differential equation which is due to Charpit.

Consider the equation

$$f(x, y\ z, p, q) = 0 \quad \text{...(1)}$$

Since z depends on x and y, we have

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = pdx + qdy \quad \text{...(2)}$$

Now if we can find another relation involving x, y, z, p, q such as $\phi(x, y, z, p, q) = 0$ ...(3)

then we can solve (1) and (3) for p and q and substitute in (2). This will give the solution provided (2) is integrable.

To determine ϕ, we differentiate (1) and (3) with respect to x and y giving

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} p + \frac{\partial f}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial x} = 0 \quad \text{...(4)}$$

$$\frac{\partial \phi}{\partial x} + \frac{\partial \phi}{\partial z} p + \frac{\partial \phi}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial \phi}{\partial q}\frac{\partial q}{\partial x} = 0 \quad \text{...(5)}$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} q + \frac{\partial f}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial y} = 0 \quad \text{...(6)}$$

$$\frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial z} q + \frac{\partial \phi}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial \phi}{\partial q}\frac{\partial q}{\partial y} = 0 \quad \text{...(7)}$$

Eliminating $\dfrac{\partial p}{\partial x}$ between the equations (4) and (5), we get

$$\left(\frac{\partial f}{\partial x}\frac{\partial \phi}{\partial p} - \frac{\partial \phi}{\partial x}\frac{\partial f}{\partial p}\right) + \left(\frac{\partial f}{\partial z}\frac{\partial \phi}{\partial p} - \frac{\partial \phi}{\partial z}\frac{\partial f}{\partial p}\right) p + \left(\frac{\partial f}{\partial q}\frac{\partial \phi}{\partial p} - \frac{\partial \phi}{\partial q}\frac{\partial f}{\partial p}\right)\frac{\partial q}{\partial x} = 0 \quad \text{...(8)}$$

Also eliminating $\dfrac{\partial q}{\partial y}$ between the equations (6) and (7), we obtain

$$\left(\frac{\partial f}{\partial y}\frac{\partial \phi}{\partial q} - \frac{\partial \phi}{\partial y}\frac{\partial f}{\partial q}\right) + \left(\frac{\partial f}{\partial z}\frac{\partial \phi}{\partial q} - \frac{\partial \phi}{\partial z}\frac{\partial f}{\partial q}\right) q + \left(\frac{\partial f}{\partial p}\frac{\partial \phi}{\partial q} - \frac{\partial \phi}{\partial p}\frac{\partial f}{\partial q}\right)\frac{\partial p}{\partial y} = 0 \quad \text{...(9)}$$

Adding (8) and (9) and using $\dfrac{\partial q}{\partial x} = \dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial p}{\partial y}$,

we find that the last terms in both cancel and the other terms, on rearrangement, give

$$\left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right)\frac{\partial \phi}{\partial p} + \left(\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}\right)\frac{\partial \phi}{\partial q} + \left(-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}\right)\frac{\partial \phi}{\partial z} + \left(-\frac{\partial f}{\partial p}\right)\frac{\partial \phi}{\partial x} + \left(-\frac{\partial f}{\partial q}\right)\frac{\partial \phi}{\partial y} = 0 \quad \text{...(10)}$$

i.e.,
$$\left(-\frac{\partial f}{\partial p}\right)\frac{\partial \phi}{\partial x} + \left(-\frac{\partial f}{\partial q}\right)\frac{\partial \phi}{\partial y} + \left(-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}\right)\frac{\partial \phi}{\partial z} + \left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right)\frac{\partial \phi}{\partial p} + \left(\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}\right)\frac{\partial \phi}{\partial q} = 0 \quad \text{...(11)}$$

This is Lagrange's linear equation (§ 17.5) with x, y, z, p, q as independent variables and ϕ as the dependent variable. Its solution will depend on the solution of the subsidiary equations

*Charpit's memoir containing this method was presented to the Paris Academy of Sciences in 1784.

$$\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{\partial \phi}{0}$$

An integral of these equations involving p or q or both, can be taken as the required relation (3), which alongwith (1) will give the values of p and q to make (2) integrable. Of course, we should take the simplest of the integrals so that it may be easier to solve for p and q.

Example 17.21. *Solve* $(p^2 + q^2)y = qz$. *(V.T.U., 2007 ; Hissar, 2005)*

Solution. Let $f(x, y, z, p, q) = (p^2 + q^2)y - qz = 0$ ...(*i*)

Charpit's subsidiary equations are

$$\frac{dx}{-2py} = \frac{dy}{z - 2qy} = \frac{dz}{-qz} = \frac{dp}{-pq} = \frac{dq}{p^2}$$

The last two of these give $pdp + qdq = 0$

Integrating, $p^2 + q^2 = c^2$ ...(*ii*)

Now to solve (*i*) and (*ii*), put $p^2 + q^2 = c^2$ in (*i*), so that $q = c^2y/z$

Substituting this value of q in (*ii*), we get $p = c\sqrt{(z^2 - c^2y^2)}/z$

Hence $$dz = pdx + qdy = \frac{c}{z}\sqrt{(z^2 - c^2y^2)}\,dx + \frac{c^2y}{z}dy$$

or $$zdz - c^2y\,dy = c\sqrt{(z^2 - c^2y^2)}\,dx \quad \text{or} \quad \frac{\frac{1}{2}d(z^2 - c^2y^2)}{\sqrt{(z^2 - c^2y^2)}} = c\,dx$$

Integrating, we get $\sqrt{(z^2 - c^2y^2)} = cx + a$ or $z^2 = (a + cx)^2 + c^2y^2$ which is the required complete integral.

Example 17.22. *Solve* $2xz - px^2 - 2qxy + pq = 0$. *(Rajasthan, 2006)*

Solution. Let $f(x, y, z, p, q) = 2xz - px^2 - 2qxy + pq = 0$ ...(*i*)

Charpit's subsidiary equations are

$$\frac{dx}{x^2 - q} = \frac{dy}{2xy - p} = \frac{dz}{px^2 - 2pq + 2qxy} = \frac{dp}{2z - 2qy} = \frac{dq}{0}$$

$\therefore$ $dq = 0$ or $q = a$.

Putting $q = a$ in (*i*), we get $p = \dfrac{2x(z - ay)}{x^2 - a}$

$$\therefore \quad dz = pdx + qdy = \frac{2x(z - ay)}{x^2 - a}dx + ady \quad \text{or} \quad \frac{dz - ady}{z - ay} = \frac{2x}{x^2 - a}dx$$

Integrating, $\log(z - ay) = \log(x^2 - a) + \log b$

or $z - ay = b(x^2 - a)$ or $z = ay + b(x^2 - a)$

which is the required complete solution.

Example 17.23. *Solve* $2z + p^2 + qy + 2y^2 = 0$. *(J.N.T.U., 2005 ; Kurukshetra, 2005)*

Solution. Let $f(x, y, z, p, q) = 2z + p^2 + qy + 2y^2$

Charpit's subsidiary equations are

$$\frac{dx}{-2p} = \frac{dy}{-y} = \frac{dz}{-(2p^2 + qy)} = \frac{dp}{2p} = \frac{dq}{4y + 3q}$$

From first and fourth ratios,

$$dp = -dx \quad \text{or} \quad p = -x + a$$

Substituting $p = a - x$ in the given equation, we get

$$q = \frac{1}{y}[-2z - 2y^2 - (a - x)^2]$$

$\therefore \qquad dz = pdx + qdy = (a - x)dx - \dfrac{1}{y}[2z + 2y^2 + (a - x)^2]dy$

Multiplying both sides by $2y^2$,

$$2y^2dz + 4yz\,dy = 2y^2(a - x)dx - 4y^3dy - 2y(a - x)^2dy$$

Integrating $\qquad 2zy^2 = -[y^2(a - x)^2 + y^4] + b$

or $\qquad y^2[(x - a)^2 + 2z + y^2] = b$, which is the desired solution.

PROBLEMS 17.5

Solve the following equations :

1. $z = p^2x + q^2x$.
2. $z^2 = pq\,xy$. *(Anna, 2009 ; V.T.U., 2004)*
3. $1 + p^2 = qz$.
4. $pxy + pq + qy = yz$. *(J.N.T.U., 2006 ; Kurukshetra, 2006)*
5. $p(p^2 + 1) + (b - z)q = 0$.
6. $q + xp = p^2$. *(Osmania, 2003)*

17.8 HOMOGENEOUS LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS

An equation of the form

$$\frac{\partial^n z}{\partial x^n} + k_1\frac{\partial^n z}{\partial x^{n-1}\partial y} + \dots + k_n\frac{\partial^n z}{\partial y^n} = F(x, y) \qquad \text{...(1)}$$

in which k's are constants, is called a *homogeneous linear partial differential equation of the nth order with constant coefficients*. It is called homogeneous because all terms contain derivatives of the same order.

On writing, $\dfrac{\partial^r}{\partial x^r} = D^r$ and $\dfrac{\partial^r}{\partial y^r} = D'^r$. (1) becomes $(D^n + k_1D^{n-1}D'^r + D' + \dots + k_nD'^n)z = F(x, y)$

or briefly $\qquad f(D, D')z = F(x, y) \qquad$ ...(2)

As in the case of ordinary linear equations with constant coefficients the complete solution of (1) consists of two parts, namely : the *complementary function* and *the particular integral.*

The complementary function is the complete solution of the equation $f(D, D')z = 0$, which must contain n arbitrary functions. The particular integral is the particular solution of equation (2).

17.9 RULES FOR FINDING THE COMPLEMENTARY FUNCTION

Consider the equation $\dfrac{\partial^2 z}{\partial x^2} + k_1\dfrac{\partial^2 z}{\partial x\partial y} + k_2\dfrac{\partial^2 z}{\partial y^2} = 0 \qquad$ ...(1)

which in *symbolic form is* $(D^2 + k_1DD' + k_2D'^2)z = 0 \qquad$ ...(2)

Its symbolic operator equated to zero, *i.e.*, $D^2 + k_1DD' + k_2D'^2 = 0$ is called the *auxiliary equation* (A.E.)

Let its root be $\qquad D/D' = m_1, m_2$.

Case I. *If the roots be real and distinct* then (2) is equivalent to

$$(D - m_1D')(D - m_2D')z = 0 \qquad \text{...(3)}$$

It will be satisfied by the solution of

$$(D - m_2D')z = 0, \text{ i.e., } p - m_2q = 0.$$

This is a Lagrange's linear and the subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m_2} = \frac{dz}{0}, \text{ whence } y + m_2x = a \text{ and } z = b.$$

$\therefore$ its solution is $z = \phi(y + m_2x)$.

Similarly (3) will also be satisfied by the solution of

$$(D - m_1D')z = 0, \quad i.e., \quad \text{by } z = f(y + m_1x)$$

Hence the complete solution of (1) is $z = f(y + m_1x) + \phi(y + m_2x)$.

Case II. *If the roots be equal* (*i.e.*, $m_1 = m_2$), then (2) is equivalent to

$$(D - m_1D')^2z = 0 \quad ...(4)$$

Putting $(D - m_1D')z = u$, it becomes $(D - m_1D')u = 0$ which gives

$$u = \phi(y + m_1x)$$

$\therefore$ (4) takes the form $(D - m_1'D)z = \phi(y + m_1x)$ or $p - m_1q) = \phi(y + m_1x)$

This is again Lagrange's linear and the subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m_1} = \frac{dz}{\phi(y + m_1x)}$$

giving $y + m_1x = a$ and $dz = \phi(a)\,dx$, *i.e.*, $z = \phi(a)x + b$

Thus the complete solution of (1) is

$$z - x\phi(y + m_1x) = f(y + m_1x). \quad i.e., \quad z = f(y + m_1x) + x\phi(y + m_1x).$$

Example 17.24. *Solve* $2\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0.$

Solution. Given equation in *symbolic form* is $(2D^2 + 5DD' + 2D'^2)z = 0$.

Its *auxiliary equation* is $2m^2 + 5m + 2 = 0$, where $m = D/D'$.

which gives $m = -2, -1/2$.

Here the complete solution is $z = f_1(y - 2x) + f_2(y - \frac{1}{2}x)$

which may be written as $z = f_1(y - 2x) + f_2(2y - x)$.

Example 17.25. *Solve* $4r + 12s + 9t = 0$. *(P.T.U., 2010)*

Solution. Given equation in *symbolic form* is $(4D^2 + 12DD' + 9D'^2)z = 0$

for $r = \frac{\partial^2 z}{\partial x^2} = D^2z,\ s = \frac{\partial^2 z}{\partial x \partial y} = DD'z$ and $t = \frac{\partial^2 z}{\partial y^2} = D'^2z.$

$\therefore$ Its *auxiliary equation* is $4m^2 + 12m + 9 = 0$, whence $m = -3/2, -3/2$

Hence the complete solution is $z = f_1(y - 1.5x) + xf_2(y - 1.5x)$.

17.10 RULES FOR FINDING THE PARTICULAR INTEGRAL

Consider the equation $(D^2 + k_1DD' + k_2D'^2)z = F(x, y)$ *i.e.*, $f(D, D')z = F(x, y)$.

$$\therefore \quad \text{P.I.} = \frac{1}{f(D, D')}F(x, y)$$

Case I. *When* $F(x, y) = e^{ax + by}$

Since $De^{ax + by} = ae^{ax + by}\,;\ D'e^{ax + by} = be^{ax + by}$

$\therefore$ $D^2e^{ax + by} = a^2e^{ax + by}\,;\ DD'e^{ax + by} = abe^{ax + by}$

and $D'^2e^{ax + by} = b^2e^{ax + by}$

$\therefore$ $(D^2 + k_1DD' + k_2D'^2)e^{ax + by} = (a^2 + k_1ab + k_2b^2)\,e^{ax + by}$

i.e., $f(D, D')e^{ax + by} = f(a, b)\,e^{ax + by}$

Operating both sides by $1/f(D, D')$, we get

$$\text{P.I.} = \frac{1}{f(D, D')}e^{ax + by} = \frac{1}{f(a, b)}e^{ax + by}$$

Case II. *When* $F(x, y) = \sin(mx + ny)$ or $\cos(mx + ny)$

Since $D^2 \sin(mx + ny) = -m^2 \sin(mx + ny)$

$DD' \sin(mx + ny) = -mn \sin(mx + ny)$

and $D'^2 \sin(mx + ny) = -n^2 \sin(mx + ny)$.

$\therefore$ $f(D^2, DD', D'^2) \sin(mx + ny) = f(-m^2, -mn, -n^2) \sin(mx + ny)$

Operating both sides by $1/f(D^2, DD', D'^2)$, we get

$$\text{P.I.} = \frac{1}{f(D^2, DD', D'^2)} \sin(mx + ny) = \frac{1}{f(-m^2 - mn, -n^2)} \sin(mx + ny)$$

Similarly about the P.I. for $\cos(mx + ny)$.

Case III. *When* $F(x, y) = x^m y^n$, m and n being constants.

$$\therefore \quad \text{P.I.} = \frac{1}{f(D, D')} x^m y^n = [f(D, D')]^{-1} x^m y^n.$$

To evaluate it, we expand $[f(D, D')]^{-1}$ in ascending powers of D or D' by Binomial theorem and then operate on $x^m y^n$ term by term.

Case IV. *When* $F(x, y)$ *is any function of* x *and* y.

$$\therefore \quad \text{P.I.} = \frac{1}{f(D, D')} F(x, y)$$

To evaluate it, we resolve $1/f(D, D')$ into partial fractions treating $f(D, D')$ as a function of D alone and operate each partial fraction on $F(x, y)$ remembering that

$$\frac{1}{D - mD'} F(x, y) = \int F(x, c - mx)\, dx$$

where c is replaced by $y + mx$ after integration.

17.11 WORKING PROCEDURE TO SOLVE THE EQUATION

$$\frac{\partial^n z}{\partial x^n} + k_1 \frac{\partial^n z}{\partial x^{n-1} \partial y} + \dots + k^n \frac{\partial^n z}{\partial y^n} = F(x, y).$$

Its *symbolic form* is $\quad (D^n + k_1 D^{n-1} D' + \dots + k_n D'^n) z = F(x, y)$

or briefly $\quad f(D, D') z = F(x, y)$

Step I. To find the C.F.

(*i*) *Write the A.E.*

i.e., $\quad m^n + k_1 m^{n-1} + \dots + k_n = 0$ *and solve it for m.*

(*ii*) *Write the C.F. as follows*

Roots of A.E.	*C.F.*
1. $m_1, m_2, m_3 \dots$ (distinct roots)	$f_1(y + m_1 x) + f_2(y + m_2 x) + f_3(y + m_3 x) + \dots$
2. $m_1, m_1, m_3 \dots$ (two equal roots)	$f_1(y + m_1 x) + x f_2(y + m_1 x) + f_3(y + m_3 x) + \dots$
3. $m_1, m_1, m_1 \dots$ (three equal roots)	$f_1(y + m_1 x) + x f_2(y + m_1 x) + x^2 f_3(y + m_1 x) + \dots$

Step II. To find the P.I.

From the *symbolic form*, $\text{P.I.} = \dfrac{1}{f(D, D')} F(x, y)$.

(*i*) *When* $F(x, y) = e^{ax + by}$ $\text{P.I.} = \dfrac{1}{f(D, D')} e^{ax + by}$ [Put $D = a$ and $D' = b$]

(*ii*) *When* $F(x, y) = \sin(mx + ny)$ *or* $\cos(mx + ny)$

$$\text{P.I.} = \frac{1}{f(D^2, DD', D'^2)} \sin \text{ or } \cos(mx + ny) \qquad [\text{Put } D^2 = -m^2, DD' = -mn, D'^2 = -n^2]$$

(*iii*) *When* $F(x, y) = x^m y^n$, $\text{P.I.} = \dfrac{1}{f(D, D')} x^m y^n = [f(D, D')]^{-1} x^m y^n$.

Expand $[f(D, D')]^{-1}$ *in ascending powers of* D *or* D' *and operate on* $x^m y^n$ *term by term.*

(*iv*) *When* $F(x, y)$ *is any function of* x *and* y $\text{P.I.} = \dfrac{1}{f(D, D')} F(x, y)$.

Resolve $1/f(D, D')$ into partial fractions considering $f(D, D')$ as a function of D alone and operate each partial fraction on $F(x, y)$ remembering that

$$\frac{1}{D - mD'} F(x, y) = \int F(x, c - mx)\, dx \text{ where } c \text{ is replaced by } y + mx \text{ after integration.}$$

Example 17.26. *Solve* $(D^2 + 4DD' - 5D'^2)z = \sin(2x + 3y)$. *(Madras, 2006)*

Solution. A.E. of the given equation is $m^2 + 4m - 5 = 0$ *i.e.*, $m = 1, -5$

$\therefore$ C.F. $= f_1(y + x) + f_2(y - 5x)$

$$\text{P.I.} = \frac{1}{D^2 + 4DD' - 5D'^2}\sin(2x + 3y) \quad [\text{Put } D^2 = -2^2, DD' = -2 \times 3, D'^2 = -3^2$$

$$= \frac{1}{-4 + 4(-6) - 5(-9)}\sin(2x + 3y) = \frac{1}{17}\sin(2x + 3y).$$

Hence the C.S. is $z = f_1(y + x) + f_2(y - 5x) + \frac{1}{17}\sin(2x + 3y)$.

Example 17.27. *Solve* $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} = \cos x \cos 2y$. *(Bhopal, 2008 S)*

Solution. Given equation in *symbolic form* is $(D^2 - DD')z = \cos x \cos 2y$.

Its A.E. is $m^2 - m = 0$, whence $m = 0, 1$.

$\therefore$ C.F. $= f_1(y) + f_2(y + x)$

$$\text{P.I.} = \frac{1}{D^2 - DD'}\cos x \cos 2y = \frac{1}{2}\frac{1}{D^2 - DD'}[\cos(x + 2y) + \cos(x - 2y)]$$

$$= \frac{1}{2}\left[\frac{1}{D^2 - DD'}\cos(x + 2y)\right. \qquad [\text{Put } D^2 = -1, DD' = -2]$$

$$\left. + \frac{1}{D^2 - DD'}\cos(x - 2y)\right] \qquad [\text{Put } D^2 = -1, DD' = 2]$$

$$= \frac{1}{2}\left[\frac{1}{-1 + 2}\cos(x + 2y) + \frac{1}{-1 - 2}\cos(x - 2y)\right] = \frac{1}{2}\cos(x + 2y) - \frac{1}{6}\cos(x - 2y)$$

Hence the C.S. is $z = f_1(y) + f_2(y + x) + \frac{1}{2}\cos(x + 2y) - \frac{1}{6}\cos(x - 2y)$.

Example 17.28. *Solve* $\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial x^2 \partial y} = 2e^{2x} + 3x^2 y$. *(S.V.T.U., 2007)*

Solution. Given equation in *symbolic form* is

$$(D^3 - 2D^2D')z = 2e^{2x} + 3x^2y$$

Its A.E. is $m^3 - 2m^2 = 0$, whence $m = 0, 0, 2$.

$\therefore$ C.F. $= f_1(y) + xf_2(y) + f_3(y + 2x)$

$$\text{P.I.} = \frac{1}{D^3 - 2D^2D'}(2e^{2x} + 3x^2y) = 2\frac{1}{D^3 - 2D^2D'}e^{2x} + 3\frac{1}{D^3(1 - 2D'/D)}x^2y$$

$$= 2\frac{1}{2^3 - 2 \cdot 2^2(0)}e^{2x} + \frac{3}{D^3}(1 - 2D'/D)^{-1}x^2y = \frac{1}{4}e^{2x} + \frac{3}{D^3}\left(1 + \frac{2D'}{D} + \frac{4D'^2}{D^2} + \dots\right)x^2y$$

$$= \frac{1}{4}e^{2x} + \frac{3}{D^3}\left(x^2y + \frac{2}{D}x^2 \cdot 1\right) = \frac{1}{4}e^{2x} + \frac{3}{D^3}\left(x^2y + \frac{2}{3}x^3\right) \qquad \left[\because \frac{1}{D}f(x) = \int f(x)\,dx\right]$$

$$= \frac{1}{4}e^{2x} + 3y\frac{x^5}{3 \cdot 4 \cdot 5} + 2 \cdot \frac{x^6}{4 \cdot 5 \cdot 6} \qquad \left[\because \frac{1}{D^3}f(x) = \int\left[\int\left(\int f(x)\,dx\right)dx\right]dx\right]$$

$$= \frac{e^{2x}}{4} + \frac{x^5 y}{20} + \frac{x^6}{60}$$

Hence the C.S. is $z = f_1(y) + xf_2(y) + f_3(y + 2x) + \frac{1}{60}(15e^{2x} + 3x^5 y + x^6)$.

Example 17.29. *Solve* $r - 4s + 4t = e^{2x+y}$.

Solution. Given equation is $\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + 4\frac{\partial^2 z}{\partial y^2} = e^{2x+y}$.

i.e., in *symbolic form* $(D^2 - 4DD' + 4D'^2)z = e^{2x+y}$.

Its A.E. is $(m-2)^2 = 0$, whence $m = 2, 2$.

$\therefore$ $\quad$ C.F. $= f_1(y + 2x) + xf_2(y + 2x)$

$$\text{P.I.} = \frac{1}{(D - 2D')^2} e^{2x+y}$$

The usual rule fails because $(D - 2D')^2 = 0$ for $D = 2$ and $D' = 1$.

$\therefore$ to obtain the P.I., we find from $(D - 2D')u = e^{2x+y}$, the solution

$$u = \int F(x, c - mx)\, dx = \int e^{2x + (c - 2x)}\, dx = xe^c = xe^{2x+y} \qquad [\because\ y = c - mx = c - 2x]$$

and from $(D - 2D')z = u = xe^{2x+y}$, the solution

$$z = \int xe^{2x + (c-2x)} dy = \frac{1}{2}x^2 e^c = \frac{1}{2}x^2 e^{2x+y} \qquad [\because\ y = c - mx = c - 2x]$$

Hence the C.S. is $z = f_1(y + 2x) + xf_2(y + 2x) + \frac{1}{2}x^2 e^{2x+y}$.

Example 17.30. *Solve* $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = \cos(2x + y)$. *(P.T.U., 2010 ; S.V.T.U., 2009)*

Solution. Given equation in *symbolic form* is $(D^2 + DD' - 6D'^2)z = \cos(2x + y)$

Its A.E. is $m^2 + m - 6 = 0$ whence $m = -3, 2$.

$\therefore$ $\quad$ C.F. $= f_1(y - 3x) + f_2(y + 2x)$.

Since $D^2 + DD' - 6D'^2 = -2^2 - (2)(1) - 6(-1)^2 = 0$

$\therefore$ It is a case of failure and we have to apply the general method.

$$\text{P.I.} = \frac{1}{D^2 + DD' - 6D'^2}\cos(2x + y) = \frac{1}{(D + 3D')(D - 2D')}\cos(2x + y)$$

$$= \frac{1}{D + 3D'}\left[\int \cos(2x + \overline{c - 2x})dx\right]_{c \to y+2x} = \frac{1}{D + 3D'}\left[\int \cos c\, dx\right]_{c \to y+2x}$$

$$[\because\ y = c - mx = c - 2x$$

$$= \frac{1}{D + 3D'}x\cos(y + 2x) = \left[\int x\cos(\overline{c + 3x} + 2x)dx\right]_{c \to y-3x} = \left[\int x\cos(5x + c)dx\right]_{c \to y-3x}$$

$$= \left[\frac{x\sin(5x + c)}{5} + \frac{\cos(5x + c)}{25}\right]_{c \to y-3x} \qquad \text{[Integrating by parts]}$$

$$= \frac{x}{5}\sin(5x + \overline{y - 3x}) + \frac{1}{25}\cos(5x + \overline{y - 3x}) = \frac{x}{5}\sin(2x + y) + \frac{1}{25}\cos(2x + y)$$

Hence the C.S. is

$$z = f_1(y - 3x) + f_2(y + 2x) + \frac{x}{5}\sin(2x + y) + \frac{1}{25}\cos(2x + y)$$

$$z = f_1(y - 3x) + f_2(y + 2x) + \frac{x}{5}\sin(2x + y) + \frac{1}{25}\cos(2x + y).$$

Example 17.31. *Solve* $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} - 6\frac{\partial^2 z}{\partial y^2} = y \cos x.$ *(Anna, 2005 S ; U.P.T.U., 2003)*

or $r + s - 6t = y \cos x.$ *(Bhopal, 2008 ; S.V.T.U., 2008)*

Solution. Its *symbolic form* is $(D^2 + DD' - 6D'^2)z = y \cos x$
and the A.E. is $m^2 + m - 6 = 0$, whence $m = -3, 2$.

$\therefore$ C.F. $= f_1(y - 3x) + f_2(y + 2x)$

$$\text{P.I.} = \frac{1}{(D - 2D')(D + 3D')} y \cos x = \frac{1}{D - 2D'}\left[\int (c + 3x) \cos x \, dx\right]_{c \to y - 3x}$$

$[\because \; y = c - mx = c + 3x]$

$$= \frac{1}{D - 2D'}[(c + 3x) \sin x + 3 \cos x]_{c \to y - 3x} \qquad \text{[Integrating by parts]}$$

$$= \frac{1}{D - 2D'}(y \sin x + 3 \cos x) = \left[\int \{(c - 2x) \sin x \; 3 \cos x\} \, dx\right]_{c \to y - 2x}$$

$$= [(c - 2x)(-\cos x) - (-2)(-\sin x) + 3 \sin x]_{c \to y + 2x}$$

$$= -y \cos x + \sin x$$

Hence the C.S. is $z = f_1(y - 3x) + f_2(y + 2x) + \sin x - y \cos x.$

Example 17.32. *Solve* $4\frac{\partial^2 z}{\partial x^2} - 4\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 16 \log (x + 2y).$

Solution. Its *symbolic form* is $4D^2 - 4DD' + D'^2 = 16 \log (x + 2y)$
and the A.E. is $4m^2 - 4m + 1 = 0$, $m = 1/2, 1/2$.

$$\therefore \quad \text{C.F.} = f_1\left(y + \frac{1}{2}x\right) + x f_2\left(y + \frac{1}{2}x\right)$$

$$\text{P.I.} = \frac{1}{(2D - D')^2} 16 \log (x + 2y) = 4 \frac{1}{\left(D - \frac{1}{2}D'\right)} \left\{\frac{1}{D - \frac{1}{2}D'} \log (x + 2y)\right\}$$

$$= 4 \frac{1}{D - \frac{1}{2}D'}\left[\int \log \left\{x + 2\left(c - \frac{x}{2}\right)\right\} dx\right]_{c \to y + x/2} \qquad [\because \; y = c - mx = c - x/2]$$

$$= 4 \frac{1}{D - \frac{1}{2}D'}\left[\int \log (2c) \, dx\right]_{c \to y + x/2} = 4 \frac{1}{D - \frac{1}{2}D'}[x \log (x + 2y)]$$

$$= 4\left[\int \left\{x \log \left[x + 2\left(c - \frac{x}{2}\right)\right]\right\} dx\right]_{c \to y + x/2} = 4\left[\log 2c \int x \, dx\right]_{c \to y + x/2} = 2x^2 \log (x + 2y)$$

Hence the C.S. is $z = f_1\left(y + \frac{x}{2}\right) + x f_2\left(y + \frac{x}{2}\right) + 2x^2 \log (x + 2y).$

PROBLEMS 17.6

Solve the following equations :

1. $\frac{\partial^3 z}{\partial x^3} - 4\frac{\partial^3 x}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial x \partial y^2} = 0.$

2. $\frac{\partial^3 z}{\partial x^3} - 3\frac{\partial^3 z}{\partial x^2 \partial y} + 4\frac{\partial^3 z}{\partial y^3} = e^{x + 2y}.$ *(Burdwan, 2003)*

3. $(D^2 - 2DD' + D'^2)z = e^{x + y}.$ *(Bhopal, 2007)*

4. $\frac{\partial^3 z}{\partial x^3} - 4\frac{\partial^3 z}{\partial x^2 \partial y} + 5\frac{\partial^3 z}{\partial x \partial y^2} - 2\frac{\partial^3 z}{\partial y^3} = e^{2x + y}.$ *(Bhopal, 2008)*

5. $\dfrac{\partial^2 z}{\partial x^2} - 2\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial^2 z}{\partial y^2} = \sin x$. *(P.T.U., 2009 S)*

6. $\dfrac{\partial^2 y}{\partial t^2} - a^2 \dfrac{\partial^2 y}{\partial x^2} = E \sin pt$.

7. $\dfrac{\partial^3 z}{\partial x^3} - \dfrac{4\partial^3 z}{\partial z^2 \partial y} + 4\dfrac{\partial^3 z}{\partial x \partial y^2} = 2 \sin (3x + 2y)$. *(S.V.T.U., 2007)*

8. $(D^3 - 7DD'^2 - 6D'^3)z = \cos (x + 2y) + 4$. *(Anna, 2008)*

9. $\dfrac{\partial^2 z}{\partial x^2} - 3\dfrac{\partial^2 z}{\partial x \partial y} + 2\dfrac{\partial^2 z}{\partial y^2} = e^{2x-y} + e^{x+y} + \cos (x + 2y)$. *(U.P.T.U., 2006)*

10. $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial x \partial y} = \sin x \cos 2y$. *(U.P.T.U., 2003)*

11. $(D^2 - DD')z = \cos 2y (\sin x + \cos x)$.

12. $(D^2 - D'^2)z = e^{x-y} \sin (x + 2y)$. *(Anna, 2009)*

13. $(D^2 + 3DD' + 2D'^2)z = 24xy$.

14. $\dfrac{\partial^2 z}{\partial x^2} + 2\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial^2 z}{\partial y^2} = x^2 + xy + y^2$.

15. $(D^2 - DD' - 2D'^2)z = (y - 1)\, e^x$. *(Bhopal, 2006)*

16. $(D^3 + D^2D' - DD'^2 - D'^3)z = e^x \cos 2y$.

17. $(D^2 + 2DD' + D'^2)z = 2 \cos y - x \sin y$. *(P.T.U., 2005)*

17.12 NON-HOMOGENEOUS LINEAR EQUATIONS

If in the equation $\qquad f(D, D')z = F(x, y) \qquad$ ...(1)

the polynomial expression $f(D, D')$ is not homogeneous, then (1) is a non-homogeneous linear partial differential equation. As in the case of homogeneous linear partial differential equations, its complete solution = C.F. + P.I.

The methods to find P.I. are the same as those for homogeneous linear equations.

To find the C.F., we factorize $f(D, D')$ into factors of the form $D - mD' - c$. To find the solution of $(D - mD' - c)z = 0$, we write it as $p - mq = cz$ ...(2)

The subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{cz}$$

Its integrals are $y + mx = a$ and $z = be^{cx}$.

Taking $b = \phi(a)$, we get $z = e^{cx}\, \phi(y + mx)$

as the solution of (2). The solution corresponding to various factors added up, give the C.F. of (1).

Example 17.32. *Solve $(D^2 + 2DD' + D'^2 - 2D - 2D')z = \sin (x + 2y)$.* *(U.P.T.U., 2004)*

Solution. Here $\qquad f(D, D') = (D + D')(D + D' - 2)$

Since the solution corresponding to the factor $D - mD' - c$ is known to be

$$z = e^{cx}\, \phi(y + mx)$$

$$\therefore \qquad \text{C.F.} = \phi_1(y - x) + e^{2x} f_2(y - x)$$

$$\therefore \qquad \text{P.I.} = \frac{1}{D^2 + 2DD' + D'^2 - 2D - 2D'} \sin (x + 2y)$$

$$= \frac{1}{-1 + 2(-2) + (-4) - 2D - 2D'} \sin (x + 2y)$$

$$= -\frac{1}{2(D + D') + 9} \sin (x + 2y) = -\frac{2(D + D') - 9}{4(D^2 + 2DD' + D'^2) - 81} \sin (x + 2y)$$

$$= \frac{1}{39}[2 \cos (x + 2y) - 3 \sin (x + 2y)]$$

Hence the complete solution is

$$z = \phi_1(y - x) + e^{2x}\phi_2(y - x) + \frac{1}{39}[2 \cos (x + 2y) - 3 \sin (x + 2y)].$$

PROBLEMS 17.7

Solve the following equations :

1. $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial y} - z = e^{-x}$.
2. $(D - D' - 1)(D - D' - 2)z = e^{2x - y}$.
3. $(D + D' - 1)(D + 2D' - 3)z = 4 + 3x + 6y$.
4. $\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial z}{\partial y} = x^2 + y^2$. *(Madras, 2000 S)*
5. $(D^2 + DD' + D' - 1)z = \sin(x + 2y)$. *(S.V.T.U., 2009)*
6. $(2DD' + D'^2 - 3D')z = 3\cos(3x - 2y)$.

17.13 NON-LINEAR EQUATIONS OF THE SECOND ORDER

We now give a method due to *Monge**, for integrating the equation $Rr + Ss + Tt = V$ ...(1)
in which R, S, T, V are functions of x, y, z, p and q.

Since $dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy = rdx + tdy$, and $dq = sdx + tdy$,

we have $r = (dp - tdy)/dx$ and $t = (dq - sdx)/dy$.

Substituting these values of r and t in (1), and rearranging the terms, we get

$$(Rdpdy + Tdqdx - Vdxdy) - s(Rdy^2 - Sdydx + Tdx^2) = 0 \quad ...(2)$$

Let us consider the equations

$$Rdy^2 - Sdydx + Tdx^2 = 0 \quad ...(3)$$

$$Rdpdy + Tdqdx - Vdxdy = 0 \quad ...(4)$$

which are known as *Monge's equations*.

Since (3) can be factorised, we obtain its integral first. In case the factors are different, we may get two distinct integrals of (3). Either of these together with (4) will give an integral of (4). If need be, we may also use the relation $dz = pdx + qdy$ while solving (3) and (4).

Let $u(x, y, z, p, q) = a$ and $v(x, y, z, p, q) = b$ be the integrals of (3) and (4) respectively. Then $u = a$, $v = b$ evidently constitute a solution of (2) and therefore, of (1) also. Taking $b = \phi(a)$, we find a general solution of (1) to be $v = \phi(u)$, which should be further integrated by methods of first order equations.

Example 17.34. *Solve* $(x - y)(xr - xs - ys + yt) = (x + y)(p - q)$. *(S.V.T.U., 2007)*

Solution. Monge's equations are

$$xdy^2 + (x + y)dy\,dx + ydx^2 = 0 \quad ...(i)$$

$$xdpdy + ydqdx - \frac{x + y}{x - y}(p - q)\,dydx = 0 \quad ...(ii)$$

(*i*) may be factorised as $(xdy + ydx)(dx + dy) = 0$ whose integrals are $xy = c$ and $x + y = c$.

Taking $xy = c$ and dividing each term of (*ii*) by xdy or its equivalent $-ydx$, we get

$$dp - dq - \frac{dx - dy}{x - y}(p - q) = 0 \quad \text{or} \quad \frac{d(p - q)}{p - q} - \frac{d(x - y)}{x - y} = 0$$

This gives on integration $(p - q)/(x - y) = c$.

Hence a first integral of the given equation is $p - q = (x - y)\,\phi(xy)$ which is a Lagrange's linear equation. Its subsidiary equations are

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{(x - y)\phi(xy)}$$

From the first two equations, we have $x + y = a$

Using this, we have

$$dz = -\phi(ax - x^2)\,.\,(a - 2x)\,dx \quad \text{which gives } z = \phi_1(ax - x^2) + b$$

Writing $b = \phi_2(a)$ and $a = x + y$, we get

$$z = \phi_1(xy) + \phi_2(x + y).$$

* Named after *Gaspard Monge* (1746–1818), Professor at Paris.

Obs. Had we started with the integral $x + y = c$ and divided each term of (*ii*) by dx or $-dy$, we would have arrived at the same solution.

Example 17.35. *Solve* $y^2r - 2ys + t = p + 6y$. *(Osmania, 2002)*

Solution. Monge's equations are $y^2dy^2 + 2ydydx + dx^2 = 0$ ...(*i*)

and $$y^2dpdy + dqdx - (p + 6y)dydx = 0 \quad ...(ii)$$

(*i*) gives $$(ydy + dx)^2 = 0 \text{ i.e. } y^2 + 2x = c \quad ...(iii)$$

Putting $ydy = -dx$ in (*ii*), we get

$$ydp - dq + (p + 6y)dy = 0 \quad \text{or} \quad (ydp + pdy) - dq + 6ydy = 0$$

whose integral is $$py - q + 3y^2 = a$$

Combining this with (*iii*), we get the integral $py - q + 3y^2 = \phi(y^2 + 2x)$

The subsidiary equations for this Lagrange's linear equation are

$$\frac{dx}{y} = \frac{dy}{-1} = \frac{dz}{\phi(y^2 + 2x) - 3y^2}$$

From the first two equations, we have $y^2 + 2x = c$

Using this, we have $dz + [\phi(c) - 3y^2]\, dy = 0$

whose solution is $$z + y\phi(c) - y^3 = b.$$

Hence the required solution is $z = y^3 - y\phi(y^2 + 2x) + \psi(y^2 + 2x)$.

PROBLEMS 17.8

Solve :

1. $(q + 1)s = (p + 1)t$.
2. $r - t\cos^2 x + p\tan x = 0$.
3. $2x^2r - 5xys + 2y^2t + 2(px + qy) = 0$. *(J.N.T.U., 2006)*
4. $xy(t - r) + (x^2 - y^2)(s - 2) = py - qx$.
5. $q^2r - 2pqs + p^2t = pq^2$.
6. $(1 + q)^2r - 2(1 + p + q + pq)s + (1 + p)^2t = 0$.

17.14 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 17.9

Fill up the blanks or choose the correct answer in each of the following problems :

1. The equation $\frac{\partial^2 z}{\partial x^2} + 2xy\left(\frac{\partial z}{\partial x}\right)^2 + \frac{\partial z}{\partial y} = 5$ is of order .... and degree ....
2. The complementary function of $(D^2 - 4DD' + 4D'^2)z = x + y$ is ......
3. The solution of $\frac{\partial^2 z}{\partial y^2} = \sin(xy)$ is ......
4. A solution of $(y - z)p + (z - x)q = x - y$ is ....
5. The particular integral of $(D^2 + DD')z = \sin(x + y)$ is ....
6. The partial differential equation obtained from $z = ax + by + ab$ by eliminating a and b is ......
7. Solution of $\sqrt{p} + \sqrt{q} = 1$ is ......
8. Solution of $p\sqrt{x} + q\sqrt{y} = \sqrt{z}$ is ......
9. Solution of $p - q = \log(x + y)$.
10. The order of the partial differential equation obtained by eliminating f from $z = f(x^2 + y^2)$, is ......
11. The solution of $x\,\partial z/\partial x = 2x + y$ is
12. By eliminating a and b from $z = a(x + y) + b$, the p.d.e. formed is ......
13. The solution of $[D^3 - 3D^2D' + 2DD'^2]z = 0$ is ......
14. By eliminating the arbitrary constants from $z = a^2x + ay^2 + b$, the partial differential equation formed is ...... *(Anna, 2008)*
15. A solution of $u_{xy} = 0$ is of the form ....
16. If $u = x^2 + t^2$ is a solution of $c^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$, then c = ...

17. The general solution of $u_{xx} = xy$ is ....

18. The complementary function of $r - 7s + 6t = e^{x+y}$ is ....

19. The solution of $xp + yq = z$ is

(i) $f(x^2, y^2) = 0$ (ii) $f(xy, yz)$ (iii) $f(x, y) = 0$ (iv) $f\left(\frac{x}{y}, \frac{y}{z}\right) = 0$.

20. The solution of $(y - z)p + (z - x)q = x - y$, is

(i) $f(x^2 + y^2 + z^2) = xyz$ (ii) $f(x + y + z) = xyz$

(iii) $f(x + y + z) = x^2 + y^2 + z^2$ (iv) $f(x^2 + y^2 + z^2, xyz) = 0$.

21. The partial differential equation from $z = (c + x)^2 + y$ is

(i) $z = \left(\frac{\partial z}{\partial x}\right)^2 + y$ (ii) $z = \left(\frac{\partial z}{\partial y}\right)^2 + y$ (iii) $z = \frac{1}{4}\left(\frac{\partial z}{\partial x}\right)^2 + y$ (iv) $z = \frac{1}{4}\left(\frac{\partial z}{\partial y}\right)^2 + y$.

22. The solution of $p + q = z$ is

(i) $f(xy, y \log z) = 0$ (ii) $f(x + y, y + \log z) = 0$

(iii) $f(x - y, y - \log z) = 0$ (iv) None of these.

23. Particular integral of $(2D^2 - 3DD' + D'^2)z = e^{x+2y}$ is

(i) $\frac{1}{2}e^{x+2y}$ (ii) $-\frac{x}{2}e^{x+2y}$ (iii) xe^{x+2y} (iv) x^2e^{x+2y}.

24. The solution of $\frac{\partial^3 z}{\partial x^3} = 0$ is

(i) $z = (1 + x + x^2) f(y)$ (ii) $z = (1 + y + y^2) f(x)$

(iii) $z = f_1(x) + y f_2(x) + y^2 f_3(x)$ (iv) $z = f_1(y) + x f_2(y) + x^2 f_3(y)$.

25. Particular integral of $(D^2 - D'^2)z = \cos(x + y)$ is

(i) $x \cos(x + y)$ (ii) $\frac{x}{2}\cos(x + y)$ (iii) $x \sin(x + y)$ (iv) $\frac{x}{2}\sin(x + y)$

26. The solution of $\partial^2 z/\partial x^2 = \partial^2 z/\partial y^2$ is

(i) $z = f_1(y + x) + f_2(y - x)$ (ii) $z = f_1(y + x) + f_1(y - x)$

(iii) $z = f(x^2 - y^2)$ (iv) $z = f(x^2 + y^2)$.

27. $xu_x + yu_y = u^2$ is a non-linear partial differential equation. (True or False)

28. $xu_x + u_{xx} = 0$ is a non-linear partial differential equation. (True or False)

29. $u = x^2 - y^2$ is a solution of $u_{xx} + u_{yy} = 0$. (True or False)

30. $u = e^{-t} \sin x$ is a solution of $\frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} = 0$. (True or False)

31. $x\frac{\partial u}{\partial x} + t\frac{\partial u}{\partial t} = 2u$ is an ordinary differential equation. (True or False)

CHAPTER 18

Applications of Partial Differential Equations

18.1 INTRODUCTION

In physical problems, we always seek a solution of the differential equation which satisfies some specified conditions known as the boundary conditions. The differential equation together with these boundary conditions, constitute a *boundary value problem.*

In problems involving ordinary differential equations, we may first find the general solution and then determine the arbitrary constants from the initial values. But the same process is not applicable to problems involving partial differential equations for the general solution of a partial differential equation contains arbitrary functions which are difficult to adjust so as to satisfy the given boundary conditions. Most of the boundary value problems involving linear partial differential equations can be solved by the following method.

18.2 METHOD OF SEPARATION OF VARIABLES

It involves a solution which breaks up into a product of functions each of which contains only one of the variables. The following example explains this method :

Example 18.1. *Solve (by the method of separation of variables) :*

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0.$$

(P.T.U., 2009 S ; Bhopal 2008 ; U.P.T.U., 2005)

Solution. Assume the trial solution $z = X(x)Y(y)$ ...(*i*)

where X is a function of x alone and Y that of y alone.

Substituting this value of z in the given equation, we have

$$X''Y - 2X'Y + XY' = 0 \qquad \text{where } X' = \frac{dX}{dx},\ Y' = \frac{dY}{dy} \text{ etc.}$$

Separating the variables, we get $\dfrac{X'' - 2X'}{X} = -\dfrac{Y'}{Y}$ ...(*ii*)

Since x and y are independent variables, therefore, (*ii*) can only be true if each side is equal to the same constant, a (say).

$\therefore \quad \dfrac{X'' - 2X'}{X} = a$, i.e. $X'' - 2X' - aX = 0$ ...(iii)

and $\quad -Y'/Y = a$, i.e., $Y' + aY = 0$ ...(iv)

To solve the ordinary linear equation (iii), the auxiliary equation is

$$m^2 - 2m - a = 0, \quad \text{whence } m = 1 \sqrt{(1 + a)}.$$

$\therefore$ the solution of (iii) is $X = c_1 e^{[1 + \sqrt{(1+a)}]x} + c_2 e^{[1 - \sqrt{(1+a)}]x}$

and the solution of (iv) is $\quad Y = c_3 e^{-ay}$.

Substituting these values of X and Y in (i), we get

$$z = \{c_1 e^{[1+\sqrt{(1+a)}]x} + c_2 e^{[1-\sqrt{(1+a)}]x}\} \cdot c_3 e^{-ay}$$

i.e., $$z = \{k_1 e^{[1+\sqrt{(1+a)}]x} + k_2 e^{[1-\sqrt{(1+a)}]x}\} e^{-ay}$$

which is the required complete solution.

Obs. In practical problems, the unknown constants a, k_1, k_2 are determined from the given boundary conditions.

Example 18.2. *Using the method of separation of variables, solve* $\dfrac{\partial u}{\partial x} = 2\dfrac{\partial u}{\partial t} + u$ *where* $u(x, 0) = 6e^{-3x}$.

(V.T.U., 2009 ; Kurukshetra, 2006 ; Kerala, 2005)

Solution. Assume the solution $u(x, t) = X(x)T(t)$

Substituting in the given equation, we have

$$X'T = 2XT' + XT \quad \text{or} \quad (X' - X)\, T = 2XT'$$

or $$\frac{X' - X}{2X} = \frac{T'}{T} = k \text{ (say)}$$

$\therefore \quad X' - X - 2kX = 0$ or $\dfrac{X'}{X} = 1 + 2k$ ...(i) and $\dfrac{T'}{T} = k$ ...(ii)

Solving (i), $\quad \log X = (1 + 2k)x + \log c$ or $X = ce^{(1+2k)x}$

From (ii), $\quad \log T = kt + \log c'$ or $T = c'e^{kt}$

Thus $\quad u(x, t) = XT = cc'\, e^{(1+2k)x}\, e^{kt}$ ...(iii)

Now $\quad 6e^{-3x} = u(x, 0) = cc'\, e^{(1+2k)x}$

$\therefore \quad cc' = 6$ and $1 + 2k = -3$ or $k = -2$

Substituting these values in (iii), we get

$$u = 6e^{-3x} e^{-2t} \quad \text{i.e.,} \quad u = 6e^{-(3x+2t)}$$ which is the required solution.

PROBLEMS 18.1

Solve the following equations by the method of *separation of variables* :

1. $py^3 + qx^2 = 0$. (V.T.U., 2011 ; S.V.T.U., 2008)
2. $x^2 \dfrac{\partial u}{\partial x} + y^2 \dfrac{\partial u}{\partial y} = 0$. (V.T.U., 2008)
3. $\dfrac{\partial u}{\partial x} = 4\dfrac{\partial u}{\partial y}$, given that $u(0, y) = 8e^{-3y}$. (J.N.T.U., 2006)
4. $4\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = 3u$, given $u = 3e^{-y} - e^{-5y}$ when $x = 0$. (S.V.T.U., 2008)
5. $3\dfrac{\partial u}{\partial x} + 2\dfrac{\partial u}{\partial y} = 0$, $u(x, 0) = 4e^{-x}$. (V.T.U., 2008 S)
6. Find a solution of the equation $\dfrac{\partial^2 u}{\partial x^2} = \dfrac{\partial u}{\partial y} + 2u$ in the form $u = f(x)\, g(y)$. Solve the equation subject to the conditions $u = 0$ and $\partial u/\partial x = 1 + e^{-3y}$, when $x = 0$ for all values of y. (Andhra, 2000)

18.3 PARTIAL DIFFERENTIAL EQUATIONS OF ENGINEERING

A number of problems in engineering give rise to the following well-known partial differential equations :

(*i*) *Wave equation :* $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$.

(*ii*) *One dimensional heat flow equation :* $\dfrac{\partial u}{\partial t} = c^2 \dfrac{\partial^2 u}{\partial x^2}$.

(*iii*) *Two dimensional heat flow equation* which in steady state becomes the two dimensional *Laplace's equation :* $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

(*iv*) *Transmission line equations.*

(*v*) *Vibrating membrane.* Two dimensional wave equation.

(*vi*) *Laplace's equation* in three dimensions.

Besides these, the partial differential equations frequently occur in the theory of Elasticity and Hydraulics.

Starting with the method of separation of variables, we find their solutions subject to specific boundary conditions and the combination of such solution gives the desired solution. Quite often a certain condition is not applicable. In such cases, the most general solution is written as the sum of the particular solutions already found and the constants are determined using Fourier series so as to satisfy the remaining conditions.

18.4 VIBRATIONS OF A STRETCHED STRING—WAVE EQUATION

Consider a tightly stretched elastic string of length l and fixed ends A and B and subjected to constant tension T (Fig. 18.1). The tension T will be considered to be large as compared to the weight of the string so that the effects of gravity are negligible.

Let the string be released from rest and allowed to vibrate. We shall study the subsequent motion of the string, with no external forces acting on it, assuming that each point of the string makes small vibrations at right angles to the equilibrium position AB, of the string entirely in one plane.

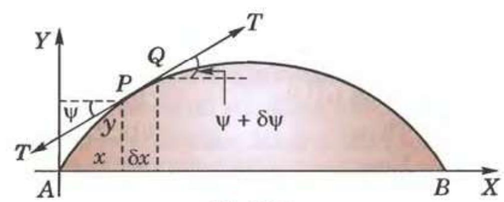


Fig. 18.1

Taking the end A as the origin, AB as the x-axis and AY perpendicular to it as the y-axis ; so that the motion takes place entirely in the xy-plane. Figure 18.1 shows the string in the position APB at time t. Consider the motion of the element PQ of the string between its points $P(x, y)$ and $Q(x + \delta x, y + \delta y)$, where the tangents make angles ψ and $\psi + \delta\psi$ with the x-axis. Clearly the element is moving upwards with the acceleration $\partial^2 y/\partial t^2$. Also the vertical component of the force acting on this element.

$$= T \sin(\psi + \delta\psi) - T \sin\psi = T[\sin(\psi + \delta\psi) - \sin\psi]$$

$$= T\,[\tan(\psi + \delta\psi) - \tan\psi], \text{ since } \psi \text{ is small} = T\left[\left\{\frac{\partial y}{\partial x}\right\}_{x+\delta x} - \left\{\frac{\partial y}{\partial x}\right\}_{x}\right]$$

If m be the mass per unit length of the string, then by Newton's second law of motion, we have

$$m\delta x \,.\, \frac{\partial^2 y}{\partial t^2} = T\left[\left\{\frac{\partial y}{\partial x}\right\}_{x+\delta x} - \left\{\frac{\partial y}{\partial x}\right\}_{x}\right] \quad i.e., \quad \frac{\partial^2 y}{\partial t^2} = \frac{T}{m}\left[\frac{\left\{\frac{\partial y}{\partial x}\right\}_{x+\delta x} - \left\{\frac{\partial y}{\partial x}\right\}_{x}}{\delta x}\right]$$

Taking limits as $Q \to P$ *i.e.*, $dx \to 0$, we have $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$, where $c^2 = \dfrac{T}{m}$ ...(1)

This is the partial differential equation giving the transverse vibrations of the string. It is also called the one dimensional *wave equation*.

(2) Solution of the wave equation. Assume that a solution of (1) is of the form

$z = X(x)T(t)$ where X is a function of x and T is a function of t only.

Then $$\frac{\partial^2 y}{\partial t^2} = X \,.\, T'' \text{ and } \frac{\partial^2 y}{\partial x^2} = X'' \,.\, T$$

Substituting these in (1), we get $XT'' = c^2X''T$ *i.e.*, $\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T}$ ...(2)

Clearly the left side of (2) is a function of x only and the right side is a function of t only. Since x and t are independent variables, (2) can hold good if each side is equal to a constant k (say). Then (2) leads to the ordinary differential equations :

$$\frac{d^2X}{dx^2} - kX = 0 \quad ...(3) \qquad \text{and} \qquad \frac{d^2T}{dt^2} - kc^2T = 0 \quad ...(4)$$

Solving (3) and (4), we get

(*i*) *When k is positive* and $= p^2$, say $X = c_1e^{px} + c_2e^{-px}$; $T = c_3e^{cpt} + c_4e^{-cpt}$.

(*ii*) *When k is negative* and $= -p^2$ say $X = c_5 \cos px + c_6 \sin px$; $T = c_7 \cos cpt + c_8 \sin cpt$.

(*iii*) *When k is zero*. $X = c_9x + c_{10}$; $T = c_{11}t + c_{12}$.

Thus the various possible solutions of wave-equation (1) are

$$y = (c_1e^{px} + c_2e^{-px})(c_3e^{cpt} + c_4e^{-cpt}) \quad ...(5)$$

$$y = (c_5 \cos px + c_6 \sin px)(c_7 \cos cpt + c_8 \sin cpt) \quad ...(6)$$

$$y = (c_9x + c_{10})(c_{11}t + c_{12}) \quad ...(7)$$

Of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem. As we will be dealing with problems on vibrations, y must be a periodic function of x and t. Hence their solution must involve trigonometric terms. Accordingly the solution given by (6), *i.e.*, of the form

$$y = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt) \quad ...(8)$$

is the only suitable solution of the *wave equation*. (*Bhopal, 2008*)

Example 18.3. *A string is stretched and fastened to two points l apart. Motion is started by displacing the string in the form $y = a \sin (\pi x/l)$ from which it is released at time $t = 0$. Show that the displacement of any point at a distance x from one end at time t is given by*

$y(x, t) = a \sin (\pi x/l) \cos (\pi ct/l)$. (*V.T.U., 2010 ; S.V.T.U., 2008 ; Kerala, 2005 ; U.P.T.U., 2004*)

Solution. The vibration of the string is given by $\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$ ...(*i*)

As the end points of the string are fixed, for all time,

$$y(0, t) = 0 \quad ...(ii) \qquad \text{and} \qquad y(l, t) = 0 \quad ...(iii)$$

Since the initial transverse velocity of any point of the string is zero,

therefore, $$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \quad ...(iv)$$

Also $$y(x, 0) = a \sin (\pi x/l) \quad ...(v)$$

Now we have to solve (*i*) subject to the *boundary conditions* (*ii*) and (*iii*) and *initial conditions* (*iv*) and (*v*). Since the vibration of the string is periodic, therefore, the solution of (*i*) is of the form

$$y(x, t) = (C_1 \cos px + C_2 \sin px)(C_3 \cos cpt + C_4 \sin cpt) \quad ...(vi)$$

By (*ii*), $$y(0, t) = C_1(C_3 \cos cpt + C_4 \sin cpt) = 0$$

For this to be true for all time, $C_1 = 0$.

Hence $$y(x, t) = C_2 \sin px (C_3 \cos cpt + C_4 \sin cpt) \quad ...(vii)$$

and $$\frac{\partial y}{\partial t} = C_2 \sin px \,\{C_3(-cp \,.\, \sin cpt) + C_4(cp \,.\, \cos cpt)\}$$

$\therefore$ By (*iv*), $$\left(\frac{\partial y}{\partial t}\right)_{t=0} = C_2 \sin px \,.\, (C_4\, cp) = 0, \text{ whence } C_2C_4cp = 0.$$

If $C_2 = 0$, (*vii*) will lead to the trivial solution $y(x, t) = 0$,

$\therefore$ the only possibility is that $C_4 = 0$.

Thus (*vii*) becomes $y(x, t) = C_2C_3 \sin px \cos cpt$ ...(*viii*)

$\therefore$ By (*iii*), $\quad y(l, t) = C_2C_3 \sin pl \cos cpt = 0$ for all t.

Since C_2 and $C_3 \neq 0$, we have $\sin pl = 0$. $\therefore$ $pl = n\pi$, *i.e.*, $p = n\pi/l$, where n is an integer.

Hence (*i*) reduces to $\quad y(x, t) = C_2C_3 \sin \dfrac{n\pi x}{l} \cos \dfrac{n\pi ct}{l}$.

[These are the solutions of (*i*) satisfying the boundary conditions. These functions are called the **eigen functions** corresponding to the **eigen values** $\lambda_n = cn\pi/l$ of the vibrating string. The set of values $\lambda_1, \lambda_2, \lambda_3, \ldots$ is called its **spectrum.**]

Finally, imposing the last condition (*v*), we have $y(x, 0) = C_2C_3 \sin \dfrac{n\pi x}{l} = a \sin \dfrac{\pi x}{l}$

which will be satisfied by taking $C_2C_3 = a$ and $n = 1$.

Hence the required solution is $y(x, t) = a \sin \dfrac{\pi x}{l} \cos \dfrac{\pi ct}{l}$ ...(*ix*)

Obs. We have from (*ix*) $\dfrac{\partial^2 y}{\partial t^2} = -a\left(\dfrac{\pi c}{l}\right)^2 \sin \dfrac{\pi x}{l} \cos \dfrac{\pi ct}{l} = -\left(\dfrac{\pi c}{l}\right)^2 y$.

This shows that the motion of each point $y(x, t)$ of the string is simple harmonic with period $= 2\pi/(\pi c/l)$, *i.e.*, $2l/c$.

Thus we can look upon (*ix*) as a sine wave $y = y_0 \sin (\pi x/l)$ of wave length l, wave-velocity c and amplitude $y_0 = a \cos (\pi ct/l)$ which varies harmonically with time t. Whatever t may be, $y = 0$ when $x = 0, l, 2l, 3l$ etc. and these points called *nodes*, remain undisturbed during wave motion. Thus (*ix*) represents a *stationary sine wave* of varying amplitudes whose frequency is $c/2l$. Such waves often occur in electrical and mechanical vibratory systems.

Example 18.4. *A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by $y = y_0 \sin^3 (\pi x/l)$. If it is released from rest from this position, find the displacement $y(x, t)$.*

(Rajasthan, 2006 ; V.T.U., 2003 ; J.N.T.U., 2002)

Solution. The equation of the vibrating string is $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$ ...(*i*)

The boundary conditions are $\quad y(0, t) = 0, y(l, t) = 0$ ...(*ii*)

Also the initial conditions are $y(x, 0) = y_0 \sin^3\left(\dfrac{\pi x}{l}\right)$ ...(*iii*)

and $\quad \left(\dfrac{\partial y}{\partial t}\right)_{t=0} = 0$ ...(*iv*)

Since the vibration of the string is periodic, therefore, the solution of (*i*) is of the form

$$y(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$$

By (*ii*), $\quad y(0, t) = c_1(c_3 \cos cpt + c_4 \sin cpt) = 0$

For this to be true for all time, $c_1 = 0$.

$\therefore$ $\quad y(x, t) = c_2 \sin px\,(c_3 \cos cpt + c_4 \sin cpt)$

Also by (*ii*), $\quad y(l, t) = c_2 \sin pl\,(c_3 \cos cpt + c_4 \sin cpt) = 0$ for all t.

This gives $pl = n\pi$ or $p = n\pi/l$, n being an integer.

Thus $\quad y(x, t) = c_2 \sin \dfrac{n\pi x}{l}\left(c_3 \cos \dfrac{cn\pi t}{l} + c_4 \sin \dfrac{cn\pi t}{l}\right)$ ...(*v*)

$$\frac{\partial y}{\partial t} = \left(c_2 \sin \frac{n\pi x}{l}\right)\frac{cn\pi}{l}\left(-c_3 \sin \frac{cn\pi t}{l} + c_4 \cos \frac{cn\pi t}{l}\right)$$

By (*iv*), $\quad \left(\dfrac{\partial y}{\partial t}\right)_{t=0} = \left(c_2 \sin \dfrac{n\pi x}{l}\right)\dfrac{cn\pi}{l} \cdot c_4 = 0$, *i.e.* $c_4 = 0$.

Thus (*v*) becomes $y(x, t) = c_2c_3 \sin \dfrac{n\pi x}{l} \cos \dfrac{n\pi ct}{l} = b_n \sin \dfrac{n\pi x}{l} \cos \dfrac{n\pi ct}{l}$

Adding all such solutions the general solution of (*i*) is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \qquad \ldots(vi)$$

$\therefore$ from (*iii*), $$y_0 \sin^3 \frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

or $$y_0 \left\{ \frac{3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l}}{4} \right\} = b_1 \sin \frac{\pi x}{l} + b_2 \sin \frac{2\pi x}{l} + b_3 \sin \frac{3\pi x}{l} + \ldots$$

Comparing both sides, we have

$$b_1 = 3\, y_0/4,\ b_2 = 0,\ b_3 = -\, y_0/4,\ b_4 = b_5 = \ldots = 0.$$

Hence from (*vi*), the desired solution is

$$y(x, t) = \frac{3y_0}{4} \sin \frac{\pi x}{l} \cos \frac{\pi ct}{l} - \frac{y_0}{4} \sin \frac{3\pi x}{l} \cos \frac{3\pi ct}{l}.$$

Example 18.5. *A tightly stretched flexible string has its ends fixed at $x = 0$ and $x = l$. At time $t = 0$, the string is given a shape defined by $F(x) = \mu x(l - x)$, where μ is a constant, and then released. Find the displacement of any point x of the string at any time $t > 0$.*

(*Bhopal, 2008 ; Madras, 2006 ; J.N.T.U., 2005 ; P.T.U., 2005*)

Solution. The equation of the string is $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$ ...(*i*)

The boundary conditions are $y(0, t) = 0, y(l, t) = 0$ ...(*ii*)

Also the initial conditions are $y(x, 0) = \mu x(l - x)$ ...(*iii*)

and $$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \qquad \ldots(iv)$$

The solution of (*i*) is of the form

$$y(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$$

By (*ii*), $y(0, t) = c_1(c_3 \cos cpt + c_4 \sin cpt) = 0$

For this to be true for all time, $c_1 = 0$.

$\therefore$ $y(x, t) = c_2 \sin px\,(c_3 \cos cpt + c_4 \sin cpt)$

Also by (*ii*) $y(l, t) = c_2 \sin pl\,(c_3 \cos cpt + c_4 \sin cpt) = 0$ for all t.

This gives $pl = n\pi$ or $p = n\pi/l$, n being an integer.

Thus $$y(x, t) = c_2 \sin \frac{n\pi x}{l}\left(c_3 \cos \frac{n\pi ct}{l} + c_4 \sin \frac{n\pi ct}{l}\right) \qquad \ldots(v)$$

$$\frac{\partial y}{\partial t} = \left(c_2 \sin \frac{n\pi x}{l}\right)\frac{n\pi c}{l}\left(-\,c_3 \sin \frac{n\pi ct}{l} + c_4 \cos \frac{n\pi ct}{l}\right)$$

$\therefore$ by (*iv*) $$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \left(c_2 \sin \frac{n\pi x}{l}\right)\frac{n\pi c}{l}\,.\,c_4 = 0$$

Thus (*v*) becomes $$y(x, t) = c_2 c_3 \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} = b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}$$

Adding all such solutions, the general solution of (*i*) is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \qquad \ldots(vi)$$

From (*iii*), $$\mu(lx - x^2) = y(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

where $$b_n = \frac{2}{l}\int_0^l \mu(lx - x^2) \sin \frac{n\pi x}{l}\, dx, \text{ by Fourier half-range sine series}$$

$$= \frac{2\mu}{l}\left\{\left|(lx - x^2)\left(-\frac{\cos n\pi x/l}{n\pi/l}\right)\right|_0^l - \int_0^l (l - 2x)\left(-\frac{\cos n\pi x/l}{n\pi/l}\right) dx\right\}$$

$$= \frac{2\mu}{l} \cdot \frac{1}{n\pi} \left\{ \int_0^l (l - 2x) \frac{\cos n\pi x}{l} dx \right\} = \frac{2\mu}{n\pi} \left\{ \left| (l - 2x) \frac{\sin n\pi x/l}{n\pi/l} \right|_0^l - \int_0^l (-2) \frac{\sin n\pi x/l}{n\pi/l} dx \right\}$$

$$= \frac{2\mu}{n\pi} \cdot \frac{2l}{n\pi} \int_0^l \sin \frac{n\pi x}{l} dx = \frac{4\mu l}{n^2\pi^2} \left| \frac{-\cos n\pi x/l}{n\pi/l} \right|_0^l = \frac{4\mu l^2}{n^3\pi^3} \{1 - (-1)^n\}$$

Hence from (*vi*), the desired solution is

$$y(x, t) = \frac{4\mu l^2}{\pi^3} \sum_{n=1}^{\infty} \frac{1 - (-1)^n}{n^3} \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l}$$

$$= \frac{8\mu l^2}{\pi^3} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^3} \sin \frac{(2m-1)\pi}{l} x \cos \frac{(2m-1)\pi ct}{l}.$$

Example 18.6. *A tightly stretched string of length l with fixed ends is initially in equilibrium position. It is set vibrating by giving each point a velocity $v_0 \sin^3 \pi x/l$. Find the displacement y(x, t).*

(S.V.T.U., 2008 ; V.T.U., 2008 ; U.P.T.U., 2006)

Solution. The equation of the vibrating string is $\dfrac{\partial^2 y}{\partial t^2} = c^2 \dfrac{\partial^2 y}{\partial x^2}$ ...(*i*)

The boundary conditions are $y(0, t) = 0$, $y(l, t) = 0$ ...(*ii*)

Also the initial conditions are $y(x, 0) = 0$ ...(*iii*)

and $$\left(\frac{\partial y}{\partial t}\right)_{t=0} = v_0 \sin^3 \frac{\pi x}{l}$$ ...(*iv*)

Since the vibration of the string is periodic, therefore, the solution of (*i*) is of the form

$$y(x, t) = (c_1 \cos px + c_2 \sin px)(c_3 \cos cpt + c_4 \sin cpt)$$

By (*ii*), $y(0, t) = c_1(c_3 \cos cpt + c_4 \sin cpt) = 0$

For this to be true for all time $c_1 = 0$.

$\therefore$ $y(x, t) = c_2 \sin px\,(c_3 \cos cpt + c_4 \sin cpt)$

Also $y(l, t) = c_2 \sin pl\,(c_3 \cos cpt + c_4 \sin cpt) = 0$ for all t.

This gives $pl = n\pi$ or $p = \dfrac{n\pi}{l}$, n being an integer.

Thus $$y(x, t) = c_2 \sin \frac{n\pi x}{l} \left(c_3 \cos \frac{cn\pi}{l} t + c_4 \sin \frac{cn\pi}{l} t \right)$$

By (*iii*), $0 = c_2 c_3 \sin \dfrac{n\pi x}{l}$ for all x *i.e.*, $c_2 c_3 = 0$

$\therefore$ $y(x, t) = b_n \sin \dfrac{n\pi x}{l} \sin \dfrac{cn\pi t}{l}$ where $b_n = c_2 c_4$

Adding all such solutions, the general solution of (*i*) is

$$y(x, t) = \sum b_n \sin \frac{n\pi x}{l} \sin \frac{cn\pi t}{l}$$ ...(*v*)

Now $$\frac{\partial y}{\partial t} = \sum b_n \sin \frac{n\pi x}{l} \cdot \frac{cn\pi}{l} \cos \frac{cn\pi t}{l}$$

By (*iv*), $$v_0 \sin^3 \frac{\pi x}{l} = \left(\frac{\partial y}{\partial t}\right)_{t=0} = \sum \frac{cn\pi}{l} b_n \sin \frac{n\pi x}{l}$$

or $$\frac{v_0}{4}\left(3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l}\right) = \sum \frac{cn\pi}{l} b_n \sin \frac{n\pi x}{l}$$ [$\because$ $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$]

$$= \frac{c\pi}{l} b_1 \sin \frac{\pi x}{l} + \frac{2c\pi}{l} b_2 \sin \frac{2\pi x}{l} + \frac{3c\pi}{l} b_3 \sin \frac{3\pi x}{l} + \ldots$$

Equating coefficients from both sides, we get

$$\frac{3v_0}{4} = \frac{c\pi}{l} b_1, \quad 0 = \frac{2c\pi}{l} b_2, \quad -\frac{v_0}{4} = \frac{3c\pi}{l} b_3, \ldots$$

$$\therefore \qquad b_1 = \frac{3lv_0}{4c\pi}, \quad b_3 = -\frac{lv_0}{12c\pi}, \quad b_2 = b_4 = b_3 = \ldots = 0$$

Substituting in (v), the desired solution is

$$y = \frac{lv_0}{12c\pi}\left(9 \sin \frac{\pi x}{l} \sin \frac{c\pi t}{l} - \sin \frac{3\pi x}{l} \sin \frac{3c\pi t}{l}\right).$$

Example 18.7. *A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially at rest in its equilibrium position. If it is vibrating by giving to each of its points a velocity $\lambda x(l-x)$, find the displacement of the string at any distance x from one end at any time t.* (Anna, 2009 ; U.P.T.U., 2002)

Solution. The equation of the vibrating string is $\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$ ...(i)

The boundary conditions are $\quad y(0, t) = 0, y(l, t) = 0$ ...(ii)

Also the initial conditions are $\quad y(x, 0) = 0$ ...(iii)

and $$\left(\frac{\partial y}{\partial t}\right)_{t=0} = \lambda x(l-x) \qquad \ldots(iv)$$

As in example 18.6, the general solution of (i) satisfying the conditions (ii) and (iii) is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cdot \sin \frac{n\pi ct}{l} \qquad \ldots(v)$$

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cdot \cos \frac{n\pi ct}{l} \cdot \left(\frac{n\pi c}{l}\right)$$

By (iv), $$\lambda x(l-x) = \left(\frac{\partial y}{\partial t}\right)_{t=0} = \frac{\pi c}{l} \sum_{n=1}^{\infty} nb_n \sin \frac{n\pi x}{l}$$

$$\therefore \qquad \frac{\pi c}{l} nb_n = \frac{2}{l} \int_0^l \lambda x(l-x) \sin \frac{n\pi x}{l}\, dx$$

$$= \frac{2\lambda}{l}\left| (lx - x^2)\left(-\frac{l}{n\pi} \cos \frac{n\pi x}{l}\right) - (l-2x)\left(-\frac{l^2}{n^2\pi^2} \sin \frac{n\pi x}{l}\right) + (-2)\left(\frac{l^3}{n^3\pi^3} \cos \frac{n\pi x}{l}\right)\right|_0^l$$

$$= \frac{4\lambda l^2}{n^3\pi^3}(1 - \cos n\pi) = \frac{4\lambda l^2}{n^3\pi^3}\,[1 - (-1)^n]$$

or $$b_n = \frac{4\lambda l^3}{c\pi^4 n^4}[1 - (-1)^n] = \frac{8\lambda l^3}{c\pi^4 (2m-1)^4} \text{ taking } n = 2m - 1.$$

Hence, from (v), the desired solution is

$$y = \frac{8\lambda l^3}{c\pi^4} \sum_{m=1}^{\infty} \frac{1}{(2m-1)^4} \sin \frac{(2m-1)\pi x}{l} \sin \frac{(2m-1)\pi ct}{l}.$$

Example 18.8. *The points of trisection of a string are pulled aside through the same distance on opposite sides of the position of equilibrium and the string is released from rest. Derive an expression for the displacement of the string at subsequent time and show that the mid-point of the string always remains at rest.* (Kerala, 2005)

Solution. Let B and C be the points of the trisection of the string $OA (= l)$ (Fig. 18.2). Initially the string is held in the form $OB'C'A$, where $BB' = CC' = a$(say).

The displacement $y(x, t)$ of any point of the string is given by

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad ...(i)$$

and the boundary conditions are

$$y(0, t) = 0 \quad ...(ii)$$

$$y(l, t) = 0 \quad ...(iii)$$

$$\left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \quad ...(iv)$$

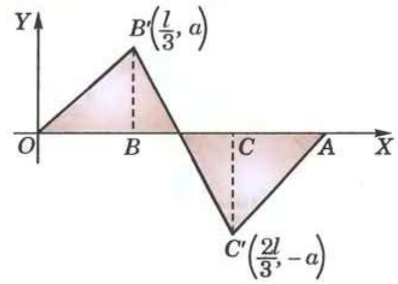


Fig. 18.2

The remaining condition is that at $t = 0$, the string rests in the form of the broken line $OB'C'A$. The equation of OB' is $y = (3a/l)\,x$;

the equation of $B'C'$ is $\quad y - a = \dfrac{-2a}{(l/3)}\left(x - \dfrac{l}{3}\right)$, i.e.. $y = \dfrac{3a}{l}(l - 2x)$

and the equation of $C'A$ is $\quad y = \dfrac{3a}{l}(x - l)$

Hence the fourth boundary condition is

$$\left.\begin{aligned} y(x, 0) &= \frac{3a}{l}x, \; 0 \le x \le \frac{l}{3} \\ &= \frac{3a}{l}(l - 2x), \; \frac{l}{3} \le x \le \frac{2l}{3} \\ &= \frac{3a}{l}(x - l), \; \frac{2l}{3} \le x \le l \end{aligned}\right\} \quad ...(v)$$

As in example 18.6, the solution of (i) satisfying the boundary conditions (ii), (iii) and (iv), is

$$y(x, t) = b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \qquad [\text{Where } b_n = C_2C_3]$$

Adding all such solutions, the most general solution of (i) is

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \quad ...(vi)$$

Putting $t = 0$, we have $y(x, 0) = \displaystyle\sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$ $\quad ...(vii)$

In order that the condition (v) may be satisfied, (v) and (vii) must be same. This requires the expansion of $y(x, 0)$ into a Fourier half-range sine series in the interval $(0, l)$.

$\therefore$ by (1) of § 10.7,

$$b_n = \frac{2}{l}\left[\int_0^{l/3} \frac{3ax}{l} \sin \frac{n\pi x}{l}\,dx + \int_{l/3}^{2l/3} \frac{3a}{l}(l - 2x) \sin \frac{n\pi x}{l}\,dx + \int_{2l/3}^{l} \frac{3a}{l}(x - l) \sin \frac{n\pi x}{l}\,dx\right]$$

$$= \frac{6a}{l^2}\left[\left| x\left\{-\frac{\cos(n\pi x/l)}{(n\pi/l)}\right\} - 1\left\{-\frac{\sin(n\pi x/l)}{(n\pi/l)^2}\right\}\right|_0^{l/3}\right.$$

$$+ \left|(l - 2x)\left\{-\frac{\cos(n\pi x/l)}{(n\pi/l)}\right\} - (-2)\left\{\frac{\sin(n\pi x/l)}{(n\pi/l)^2}\right\}\right|_{l/3}^{2l/3}$$

$$\left. + \left|(x - l)\left\{-\frac{\cos(n\pi x/l)}{(n\pi/l)}\right\} - (1).\left\{-\frac{\sin(n\pi x/l)}{(n\pi/l)^2}\right\}\right|_{2l/3}^{l}\right]$$

$$= \frac{6a}{l^2}\left[\left(-\frac{l^2}{3n\pi}\cos\frac{n\pi}{3} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{3}\right) + \frac{l^2}{3n\pi}\cos\frac{2n\pi}{3} - \frac{2l^2}{n^2\pi^2}\sin\frac{2n\pi}{3} + \frac{l^2}{3n\pi}\cos\frac{n\pi}{3}\right.$$

$$\left. + \frac{2l^2}{n^2\pi^2}\sin\frac{n\pi}{3} - \left(\frac{l^2}{3n\pi}\cos\frac{2n\pi}{3} + \frac{l^2}{n^2\pi^2}\sin\frac{2n\pi}{3}\right)\right]$$

$$= \frac{6a}{l^2} \cdot \frac{3l^2}{n^2\pi^2}\left(\sin\frac{n\pi}{3} - \sin\frac{2n\pi}{3}\right)$$

$$= \frac{18a}{n^2\pi^2}\sin\frac{n\pi}{3}[1 + (-1)^n] \qquad \left[\because \sin\frac{2n\pi}{3} = \sin\left(n\pi - \frac{n\pi}{3}\right) = -(-1)^n \sin\frac{n\pi}{3}\right]$$

Thus $b_n = 0$, when n is odd.

$$= \frac{36a}{n^2\pi^2}\sin\frac{n\pi}{3}, \text{ when } n \text{ is even.}$$

Hence (vi) gives

$$y(x, t) = \sum_{n=2,4,\ldots}^{\infty} \frac{36a}{n^2\pi^2}\sin\frac{n\pi}{3}\sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \qquad [\text{Take } n = 2m]$$

$$= \frac{9a}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\sin\frac{2m\pi}{3}\sin\frac{2m\pi x}{l}\cos\frac{2m\pi ct}{l} \qquad \ldots(vii)$$

Putting $x = l/2$ in (vii), we find that the displacement of the mid-point of the string, *i.e.* $y(l/2, t) = 0$, because $\sin m\pi = 0$ for all integral values of m.

This shows that the mid-point of the string is always at rest.

(3) D'Alembert's solution of the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2} \qquad \ldots(1)$$

Let us introduce the new independent variables $u = x + ct$, $v = x - ct$ so that y becomes a function of u and v.

Then $\frac{\partial y}{\partial x} = \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}$

and
$$\frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right) = \frac{\partial}{\partial u}\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{\partial y}{\partial u} + \frac{\partial y}{\partial v}\right) = \frac{\partial^2 y}{\partial u^2} + 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}$$

Similarly,
$$\frac{\partial^2 y}{\partial t^2} = c^2\left(\frac{\partial^2 y}{\partial u^2} - 2\frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2}\right)$$

Substituting in (1), we get $\frac{\partial^2 y}{\partial u \partial v} = 0$ ...(2)

Integrating (2) w.r.t. v, we get $\frac{\partial y}{\partial u} = f(u)$ ...(3)

where $f(u)$ is an arbitrary function of u. Now integrating (3) w.r.t. u, we obtain

$$y = \int f(u)\,du + \psi(v)$$

where $\psi(v)$ is an arbitrary function of v. Since the integral is a function of u alone, we may denote it by $\phi(u)$. Thus

$$y = \phi(u) + \psi(v)$$

i.e.
$$y(x, t) = \phi(x + ct) + \psi(x - ct) \qquad \ldots(4)$$

This is the *general solution of the wave equation* (1).

Now to determine ϕ and ψ, suppose initially $u(x, 0) = f(x)$ and $\partial y(x, 0)/\partial t = 0$.

Differentiating (4) w.r.t. t, we get $\frac{\partial y}{\partial t} = c\phi'(x + ct) - c\psi'(x - ct)$

At $t = 0$, $\qquad \phi'(x) = \psi'(x)$ ...(5)

and $\qquad y(x, 0) = \phi(x) + \psi(x) = f(x)$ ...(6)

(5) gives, $\qquad \phi(x) = \psi(x) + k$

$\therefore$ (6) becomes $2\psi(x) + k = f(x)$

or
$$\psi(x) = \frac{1}{2}[f(x) - k] \text{ and } \phi(x) = \frac{1}{2}[f(x) + k]$$

Hence the solution of (4) takes the form

$$y(x, t) = \frac{1}{2}[f(x + ct) + k] + \frac{1}{2}[f(x - ct) - k] = f(x + ct) + f(x - ct) \quad \text{...(7)}$$

which is the *d'Alembert's solution* of the wave equation* (1) *(V.T.U., 2011 S)*

Obs. The above solution gives a very useful method of solving partial differential equations by change of variables.

Example 18.9. *Find the deflection of a vibrating string of unit length having fixed ends with initial velocity zero and initial deflection f(x) = k(sin x – sin 2x).* *(V.T.U., 2011)*

Solution. By d'Alembert's method, the solution is

$$y(x, t) = \frac{1}{2}[f(x + ct) + f(x - ct)]$$

$$= \frac{1}{2}[k\{\sin(x + ct) - \sin 2(x + ct)\} + k\{\sin(x - ct) - \sin 2(x - ct)\}]$$

$$= k[\sin x \cos ct - \sin 2x \cos 2ct]$$

Also $\quad y(x, 0) = k(\sin x - \sin 2x) = f(x)$

and $\quad \partial y(x, 0)/\partial t = k(-c \sin x \sin ct + 2c \sin 2x \sin 2ct)_{t=0} = 0$

i.e., the given boundary conditions are satisfied.

PROBLEMS 18.2

1. Solve completely the equation $\partial^2 y/\partial t^2 = c^2 \partial^2 y/\partial x^2$, representing the vibrations of a string of length l, fixed at both ends, given that $y(0, t) = 0$; $y(l, t) = 0$; $y(x, 0) = f(x)$ and $\partial y(x, 0)/\partial t = 0$, $0 < x < l$. *(Bhopal, 2007 S ; U.P.T.U., 2005)*
2. Solve the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ under the conditions $u(0, t) = 0$, $u(l, t) = 0$ for all t ; $u(x, 0) = f(x)$ and $\left(\frac{\partial u}{\partial t}\right)_{t=0} = g(x)$, $0 < x < l$.
3. Find the solution of the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$, corresponding to the triangular initial deflection
$$f(x) = \frac{2k}{l}x \text{ when } 0 < x < \frac{l}{2}. = \frac{2k}{l}(l - x) \text{ when } \frac{l}{2} < x < l,$$
and initial velocity zero. *(Bhopal, 2006 ; Kerala, M.E., 2005)*
4. A tightly stretched string of length l has its ends fastened at $x = 0$, $x = l$. The mid-point of the string is then taken to height h and then released from rest in that position. Find the lateral displacement of a point of the string at time t from the instant of release. *(Anna, 2005)*
5. A tightly stretched string with fixed end points at $x = 0$ and $x = 1$, is initially in a position given by
$$f(x) = \begin{cases} x, 0 \le x \le \frac{1}{2} \\ 1 - x, \frac{1}{2} \le x \le 1 \end{cases}$$
If it is released from this position with velocity a, perpendicular to the x-axis, show that the displacement $u(x, t)$ at any point x of the string at any time $t > 0$, is given by
$$u(x, t) = \frac{4\sqrt{2}}{\pi^2}\left[\sum_{n=1}^{\infty}\left\{\frac{\sin[(4pi - 3)\pi x]\cos[(4pi - 3)\pi at - \pi/4]}{(4n - 3)^2} - \frac{\sin[(4pi - 1)\pi x]\cos[(4pi - 1)\pi at - \pi/4]}{(4n - 1)^2}\right\}\right]$$
6. If a string of length l is initially at rest in equilibrium position and each of its points is given a velocity v such that $v = cx$ for $< x < l/2$
$c(l . x)$ for $l/2 < x < l$, determine the displacement $y(x, t)$ at anytime t. *(Anna, 2008)*
7. Using d'Alembert's method, find the deflection of a vibrating string of unit length having fixed ends, with initial velocity zero and initial deflection :
(*i*) $f(x) = a(x - x^2)$ *(Kerala, M. Tech., 2005)* (*ii*) $f(x) = a \sin^2 \pi x$.

*See footnote of p. 373.

18.5 (1) ONE-DIMENSIONAL HEAT FLOW

Consider a homogeneous bar of uniform cross-section α(cm^2). Suppose that the sides are covered with a material impervious to heat so that the stream lines of heat-flow are all parallel and perpendicular to the area α. Take one end of the bar as the origin and the direction of flow as the positive x-axis (Fig. 18.3). Let ρ be the density (gr/cm^3), s the specific heat (cal./gr. deg.) and k the thermal conductivity (cal./cm. deg. sec.).

Let $u(x, -t)$ be the temperature at a distance x from O. If δu be the temperature change in a slab of thickness δx of the bar, then by § 12.7 (*ii*) p. 466, the quantity of heat in this slab = $s\rho\alpha\ \delta x \delta u$. Hence the rate of increase of heat in this slab, *i.e.*, $s\rho\alpha\delta x \dfrac{\partial u}{\partial t} = R_1 - R_2$, where R_1 and R_2 are respectively the rate (cal./sec.) of inflow and outflow of heat.

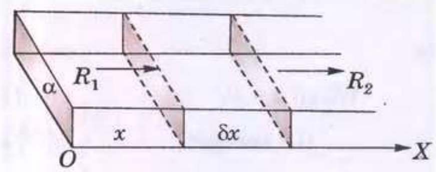


Fig. 18.3

Now by (*A*) of p. 466, $R_1 = -k\alpha\left(\dfrac{\partial u}{\partial x}\right)_x$ and $R_2 = -k\alpha\left(\dfrac{\partial u}{\partial x}\right)_{x+\delta x}$

the negative sign appearing as a result of (*i*) on p. 466.

Hence $s\rho\alpha\delta x \dfrac{\partial u}{\partial t} = -k\alpha\left(\dfrac{\partial u}{\partial x}\right)_x + k\alpha\left(\dfrac{\partial u}{\partial x}\right)_{x+\delta x}$ *i.e.*, $\dfrac{\partial u}{\partial t} = \dfrac{k}{s\rho}\left\{\dfrac{(\partial u/\partial x)_{x+\delta x} - (\partial u/\partial x)_x}{\delta x}\right\}$

Writing $k/s\rho = c^2$, called the *diffusivity* of the substance (cm^2/sec.), and taking the limit as $\delta x \to 0$, we get

$$\frac{\partial \mathbf{u}}{\partial \mathbf{t}} = \mathbf{c}^2 \frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} \quad \text{...(1)}$$

This is the *one-dimensional heat-flow equation.* (*V.T.U., 2011*)

(2) Solution of the heat equation. Assume that a solution of (1) is of the form

$$u(x, t) = X(x) \cdot T(t)$$

where X is a function of x alone and T is a function of t only.

Substituting this in (1), we get

$$XT' = c^2 X''T, \ i.e., X''/X = T'/c^2T \quad \text{...(2)}$$

Clearly the left side of (2) is a function of x only and the right side is a function of t alone. Since x and t are independent variables, (2) can hold good if each side is equal to a constant k (say). Then (2) leads to the ordinary differential equations

$$\frac{d^2X}{dx^2} - kX = 0 \quad \text{...(3)} \qquad \text{and} \qquad \frac{dT}{dt} - kc^2T = 0 \quad \text{...(4)}$$

Solving (3) and (4), we get

(*i*) *When k is positive* and $= p^2$, say :

$$X = c_1e^{px} + c_2e^{-px},\ T = c_3\, e^{c^2p^2t};$$

(*ii*) *When k is negative* and $= -p^2$, say :

$$X = c_4 \cos px + c_5 \sin px,\ T = c_6\, e^{-c^2p^2t};$$

(*iii*) *When k is zero :*

$$X = c_7x + c_8,\ T = c_9.$$

Thus the various possible solutions of the heat-equation (1) are

$$u = (c_1e^{px} + c_2e^{-px})\,c_3e^{c^2p^2t} \quad \text{...(5)}$$

$$u = (c_4 \cos px + c_5 \sin px)\,c_6e^{-c^2p^2t} \quad \text{...(6)}$$

$$u = (c_7x + c_8)c_9 \quad \text{...(7)}$$

Of these three solutions, we have to choose that solution which is consistent with the physical nature of the problem. As we are dealing with problems on heat conduction, it must be a transient solution, *i.e.*, u is to decrease with the increase of time t. Accordingly, the solution given by (6), *i.e.*, of the form

$$u = (C_1 \cos px + C_2 \sin px)\,e^{-c^2p^2t} \quad \text{...(8)}$$

is the only suitable solution of the heat equation.

Example 18.10. *Solve the equation* $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ *with boundary conditions* $u(x, 0) = 3 \sin n\pi x$, $u(0, t) = 0$ *and* $u(1, t) = 0$, *where* $0 < x < 1$, $t > 0$.

Solution. The solution of the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$ ...(*i*)

is $$u(x, t) = (c_1 \cos px + c_2 \sin px)\, e^{-p^2 t} \quad ...(ii)$$

When $x = 0$, $u(0, t) = c_1 e^{-p^2 t} = 0$ *i.e.*, $c_1 = 0$.

$\therefore$ (*ii*) becomes $$u(x, t) = c_2 \sin px e^{-p^2 t} \quad ...(iii)$$

When $x = 1$, $u(1, t) = c_2 \sin p \,.\, e^{-p^2 t} = 0$ or $\sin p = 0$

i.e., $p = n\pi$.

$\therefore$ (*iii*) reduces to $u(x, t) = b_n e^{-(n\pi)^2 t} \sin n\pi x$ where $b_n = c_2$

Thus the general solution of (*i*) is $u(x, t) = \sum b_n e^{-n^2\pi^2 t} \sin n\pi x$ ...(*iv*)

When $t = 0$, $3 \sin n\pi x = u(0, t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t} \sin n\pi x$

Comparing both sides, $b_n = 3$

Hence from (*iv*), the desired solution is

$$u(x, t) = 3 \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} \sin n\pi x.$$

Example 18.11. *Solve the differential equation* $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$ *for the conduction of heat along a rod without radiation, subject to the following conditions :*

(*i*) *u is not infinite for* $t \to \infty$, (*ii*) $\frac{\partial u}{\partial x} = 0$ *for* $x = 0$ *and* $x = l$,

(*ii*) $u = lx - x^2$ *for* $t = 0$, *between* $x = 0$ *and* $x = l$. (*P.T.U., 2007*)

Solution. Substituting $u = X(x)T(t)$ in the given equation, we get

$$XT' = \alpha^2 X''T \quad i.e., \quad X''/X = \frac{T'}{\alpha^2 T} = -k^2 \text{ (say)}$$

$$\therefore \quad \frac{d^2X}{dx^2} + k^2 X = 0 \text{ and } \frac{dT}{dt} + k^2\alpha^2 T = 0 \quad ...(1)$$

Their solutions are $$X = c_1 \cos kx + c_2 \sin kx,\ T = c_3 e^{-k^2\alpha^2 t} \quad ...(2)$$

If k^2 is changed to $-k^2$, the solutions are

$$X = c_4 e^{kx} + c_5 e^{-kx},\ T = c_6 e^{k^2\alpha^2 t} \quad ...(3)$$

If $k^2 = 0$, the solutions are $$X = c_7 x + c_8,\ T = c_9 \quad ...(4)$$

In (3), $T \to \infty$ for $t \to \infty$ therefore, u also $\to \infty$ *i.e.*, the given condition (*i*) is not satisfied. So we reject the solutions (3) while (2) and (4), satisfy this condition.

Applying the condition (*ii*) to (4), we get $c_7 = 0$.

$$\therefore \quad u = XT = c_8 c_9 = a_0 \quad \text{(say)} \quad ...(5)$$

From (2), $$\frac{\partial u}{\partial x} = (-c_1 \sin kx + c_2 \cos kx)\, k c_3 e^{-k^2\alpha^2 t}$$

Applying the condition (*ii*), we get $c_2 = 0$ and $-c_1 \sin kl + c_2 \cos kl = 0$

i.e., $c_2 = 0$ and $kl = n\pi$ (n an integer)

$$\therefore \quad u = c_1 \cos kx \,.\, c_3 e^{-k^2\alpha^2 t} = a_n \cos\left(\frac{n\pi x}{l}\right) \frac{e^{-n^2\pi^2\alpha^2 t}}{l^2} \quad ...(6)$$

Thus the general solution being the sum of (5) and (6), is

$$u = a_0 + \Sigma a_n \cos(n\pi x/l)\, e^{-n^2\pi^2\alpha^2 t/l^2} \quad ...(7)$$

Now using the condition (*iii*), we get

$$lx - x^2 = a_0 + \Sigma a_n \cos(n\pi x/l)$$

This being the expansion of $lx - x^2$ as a half-range cosine series in $(0, l)$, we get

$$a_0 = \frac{1}{l}\int_0^l (lx - x^2)\,dx = \frac{1}{l}\left|\frac{lx^2}{2} - \frac{x^3}{3}\right|_0^l = \frac{l^2}{6}$$

and

$$a_n = \frac{2}{l}\int_0^l (lx - x^2)\cos\frac{n\pi x}{l}\,dx = \frac{2}{l}\left|(lx - x^2)\left(\frac{l}{n\pi}\sin\frac{n\pi x}{l}\right)\right.$$

$$\left. - (l - 2x)\left(-\frac{l^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right) + (-2)\left(-\frac{l^3}{n^3\pi^3}\sin\frac{n\pi x}{l}\right)\right|_0^l$$

$$= \frac{2}{l}\left\{0 - \frac{l^3}{n^2\pi^2}(\cos n\pi + 1) + 0\right\} = -\frac{4l^2}{n^2\pi^2} \text{ when } n \text{ is even, otherwise } 0.$$

Hence taking $n = 2m$, the required solution is

$$u = \frac{l^2}{6} - \frac{l^2}{\pi^2}\sum_{m=1}^{\infty}\frac{1}{m^2}\cos\left(\frac{2m\pi x}{l}\right)e^{-4m^2\pi^2\alpha^2 t/l^2}.$$

Example 18.12. (*a*) *An insulated rod of length l has its ends A and B maintained at 0°C and 100°C respectively until steady state conditions prevail. If B is suddenly reduced to 0°C and maintained at 0°C, find the temperature at a distance x from A at time t.* (U.P.T.U., 2005)

(*b*) *Solve the above problem if the change consists of raising the temperature of A to 20°C and reducing that of B to 80°C.* (Madras, 2000 S)

Solution. (*a*) Let the equation for the conduction of heat be

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2} \quad ...(i)$$

Prior to the temperature change at the end B, when $t = 0$, the heat flow was independent of time (*steady state condition*). When u depends only on x, (*i*) reduces to $\partial^2 u/\partial x^2 = 0$.

Its general solution is $u = ax + b$ ...(*ii*)

Since $u = 0$ for $x = 0$ and $u = 100$ for $x = l$, therefore, (*ii*) gives $b = 0$ and $a = 100/l$.

Thus the *initial condition* is expressed by $u(x, 0) = \dfrac{100}{l}x$ ...(*iii*)

Also the *boundary conditions* for the subsequent flow are

$$u(0, t) = 0 \text{ for all values of } t \quad ...(iv)$$

and

$$u(l, t) = 0 \text{ for all values of } t \quad ...(v)$$

Thus we have to find a temperature function $u(x, t)$ satisfying the differential equation (*i*) subject to the initial condition (*iii*) and the boundary conditions (*iv*) and (*v*).

Now the solution of (*i*) is of the form

$$u(x, t) = (C_1\cos px + C_2\sin px)\,e^{-c^2p^2t} \quad ...(vi)$$

By (*iv*), $u(0, t) = C_1 e^{-c^2p^2t} = 0$, for all values of t.

Hence $C_1 = 0$ and (*vi*) reduces to $u(x, t) = C_2 \sin px \,.\, e^{-c^2p^2t}$ ...(*vii*)

Applying (*v*), (*vii*) gives $u(l, t) = C_2 \sin pl \,.\, e^{-c^2p^2t} = 0$, for all values of t.

This requires $\sin pl = 0$ *i.e.*, $pl = n\pi$ as $C_2 \neq 0$. $\therefore\ p = n\pi/l$, where n is any integer.

Hence (*vii*) reduces to $u(x, t) = b_n \sin\dfrac{n\pi x}{l}\cdot e^{-c^2n^2\pi^2t/l^2}$, where $b_n = C_2$.

[These are the solutions of (*i*) satisfying the boundary conditions (*iv*) and (*v*). These are the **eigen functions** corresponding to the **eigen values** $\lambda_n = cn\pi/l$, of the problem.]

Adding all such solutions, the most general solution of (*i*) satisfying the boundary conditions (*iv*) and (*v*) is

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cdot e^{-c^2n^2\pi^2t/l^2} \qquad ...(viii)$$

Putting $t = 0$,
$$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \qquad ...(ix)$$

In order that the condition (*iii*) may be satisfied, (*iii*) and (*ix*) must be same. This requires the expansion of $100x/l$ as a half-range Fourier sine series in $(0, l)$. Thus

$$\frac{100x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \text{ where } b_n = \frac{2}{l}\int_0^l \frac{100x}{l} \cdot \sin \frac{n\pi x}{l}\,dx$$

$$= \frac{200}{l^2}\left[x\left\{-\frac{\cos(n\pi x/l)}{(n\pi/l)}\right\} - (1)\left\{-\frac{\sin(n\pi x/l)}{(n\pi/l)^2}\right\}\right]_0^l = \frac{200}{l^2}\left(-\frac{l^2}{n\pi}\cos n\pi\right) = \frac{200}{n\pi}(-1)^{n+1}$$

Hence (*viii*) gives $u(x, t) = \dfrac{200}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin\frac{n\pi x}{l} \cdot e^{-(cn\pi/l)^2 t}$

(*b*) Here the initial condition remains the same as (*iii*) above, and the boundary conditions are

$$u(0, t) = 20 \text{ for all values of } t \qquad ...(x)$$
$$u(l, t) = 80 \text{ for all values of } t \qquad ...(xi)$$

In part (*a*), the boundary values (*i.e.*, the temperature at the ends) being zero, we were able to find the desired solution easily. Now the boundary values being non-zero, we have to modify the procedure.

We split up the temperature function $u(x, t)$ into two parts as

$$u(x, t) = u_s(x) + u_t(x, t) \qquad ...(xii)$$

where $u_s(x)$ is a solution of (*i*) involving x only and satisfying the boundary conditions (*x*) and (*xi*); $u_t(x, t)$ is then a function defined by (*xii*). Thus $u_s(x)$ is a steady state solution of the form (*ii*) and $u_t(x, t)$ may be regarded as a transient part of the solution which decreases with increase of t.

Since $u_s(0) = 20$ and $u_s(l) = 80$, therefore, using (*ii*) we get

$$u_s(x) = 20 + (60/l)x \qquad ...(xiii)$$

Putting $x = 0$ in (*xii*), we have by (*x*),

$$u_t(0, t) = u(0, t) - u_s(0) = 20 - 20 = 0 \qquad ...(xiv)$$

Putting $x = l$ in (*xii*), we have by (*xi*),

$$u_t(l, t) = u(l, t) - u_s(l) = 80 - 80 = 0 \qquad ...(xv)$$

Also
$$u_t(x, 0) = u(x, 0) - u_s(x) = \frac{100x}{l} - \left(\frac{60x}{l} + 20\right) \qquad \text{[by (iii) and (xiii)]}$$

$$= \frac{40x}{l} - 20 \qquad ...(xvi)$$

Hence (*xiv*) and (*xv*) give the boundary conditions and (*xvi*) gives the initial condition relative to the transient solution. Since the boundary values given by (*xiv*) and (*xv*) are both zero, therefore, as in part (*a*), we have $u_t(x, t) = (C_1 \cos px + C_2 \sin px)\, e^{-c^2p^2t}$

By (*xiv*), $\quad u_t(0, t) = C_1 e^{-c^2p^2t} = 0$, for all values of t.

Hence $C_1 = 0$ and $\quad u_t(x, t) = C_2 \sin px \,.\, e^{-c^2p^2t} \qquad ...(xvii)$

Applying (*xv*), it gives $\quad u_t(l, t) = C_2 \sin pl e^{-c^2p^2t} = 0$ for all values of t.

This requires $\quad \sin pl = 0$, *i.e.* $pl = n\pi$ as $C_2 \neq 0$. $p = n\pi/l$, when n is any integer.

Hence (*xvii*) reduces to $u_t(x, t) = b_n \sin \dfrac{n\pi x}{l} e^{-c^2n^2\pi^2t/l^2}$ where $b_n = C_2$.

Adding all such solutions, the most general solution of (*xvii*) satisfying the boundary conditions (*xiv*) and (*xv*) is

$$u_t(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} e^{-c^2n^2\pi^2 t/l^2} \qquad \text{...(xviii)}$$

Putting $t = 0$, we have $u_t(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$ ...(*xix*)

In order that the condition (*xvi*) may be satisfied, (*xvi*) and (*xix*) must be same. This requires the expansion of $(40/l)\, x - 20$ as a half-range Fourier sine series in $(0, l)$. Thus

$$\frac{40x}{l} - 20 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \quad \text{where} \quad b_n = \frac{2}{l}\int_0^l \left(\frac{40x}{l} - 20\right) \sin \frac{n\pi x}{l}\, dx = -\frac{40}{n\pi}\,(1 + \cos n\pi)$$

i.e., $b_n = 0$, when n is odd ; $= -80/n\pi$, when n is even

Hence (*xviii*) becomes $u_t(x, t) = \sum_{n=2,4...}^{\infty} \left(\frac{-80}{n\pi}\right) \sin \frac{n\pi x}{l} \cdot e^{-c^2n^2\pi^2 t/l^2}$ [Take $n = 2m$]

$$= -\frac{40}{\pi} \sum_{m=1}^{\infty} \frac{1}{m} \sin \frac{2m\pi x}{l} \cdot e^{-4c^2m^2\pi^2 t/l^2} \qquad \text{...(xx)}$$

Finally combining (*xiii*) and (*xx*), the required solution is

$$u(x, t) = \frac{40x}{l} + 20 - \frac{40}{\pi} \sum_{m=1}^{\infty} \frac{1}{m} \sin \frac{2m\pi x}{l} \cdot e^{-4c^2m^2\pi^2 t/l^2}.$$

Example 18.13. *The ends A and B of a rod 20 cm long have the temperature at 30°C and 80°C until steady-state prevails. The temperature of the ends are changed to 40°C and 60°C respectively. Find the temperature distribution in the rod at time t.*

Solution. Let the heat equation be $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ ...(*i*)

In steady state condition, u is independent of time and depends on x only, (*i*) reduces to

$$\partial^2 u / \partial x^2 = 0. \qquad \text{...(ii)}$$

Its solution is $u = a + bx$

Since $u = 30$ for $x = 0$ and $u = 80$ for $x = 20$, therefore $a = 30$, $b = (80 - 30)/20 = 5/2$

Thus the initial conditions are expressed by

$$u\,(x, 0) = 30 + \frac{5}{2}x \qquad \text{...(iii)}$$

The boundary conditions are $u\,(0, t) = 40$, $u\,(20, t) = 60$

Using (*ii*), the steady state temperature is

$$u\,(x, 0) = 40 + \frac{60 - 40}{20}\, x = 40 + x \qquad \text{...(iv)}$$

To find the temperature u in the intermediate period,

$$u\,(x, t) = u_s\,(x) + u_t\,(x, t)$$

where $u_s\,(x)$ is the steady state temperature distribution of the form (*iv*) and $u_t\,(x, t)$ is the transient temperature distribution which decreases to zero as t increases.

Since $u_t\,(x, t)$ satisfies one dimensional heat equation

$$\therefore \quad u(x, t) = 40 + x + \sum_{n=1}^{\infty} (a_n \cos px + b_n \sin px)\, e^{-p^2 t} \qquad \text{...(v)}$$

$$u(0, t) = 40 = 40 + \sum_{n=1}^{\infty} a_n e^{-p^2 t} \quad \text{whence } a_n = 0.$$

$\therefore$ (v) reduces to $$u(x, t) = 40 + x + \sum_{n=1}^{\infty} b_n \sin pxe^{-p^2 t} \quad ...(vi)$$

Also $$u(20, t) = 60 = 40 + 20 + \sum_{n=1}^{\infty} b_n \sin 20\, pe^{-p^2 t}$$

or $$\sum_{n=1}^{\infty} b_n \sin 20\, pe^{-p^2 t} = 0 \text{ i.e., } \sin 20p = 0 \text{ i.e., } p = n\pi/20$$

Thus (vi) becomes $$u(x, t) = 40 + x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20} e^{-n\pi t/20} \quad ...(vii)$$

Using (iii), $$30 + \frac{5}{2}x = u(0, t) = 40 + x + \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20}$$

or $$\frac{3x}{2} - 10 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{20}$$

where $b_n = \frac{2}{20}\int_0^{20}\left(\frac{3x}{2} - 10\right)\sin\frac{n\pi x}{20}\,dx = -\frac{20}{n\pi}(1 + 2\cos n\pi)$

Hence from (vii), the desired solution is

$$u = 40 + x - \frac{20}{\pi}\sum_{n=1}^{\infty}\frac{1 + 2\cos n\pi}{n}\sin\frac{n\pi x}{20}e^{-(n\pi/20)^2 t}.$$

Example 18.14. Bar with insulated ends. *A bar 100 cm long, with insulated sides, has its ends kept at 0°C and 100°C until steady state conditions prevail. The two ends are then suddenly insulated and kept so. Find the temperature distribution.*

Solution. The temperature $u(x, t)$ along the bar satisfies the equation

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad ...(i)$$

By law of heat conduction, the rate of heat flow is proportional to the gradient of the temperature. Thus, if the ends $x = 0$ and $x = l$ (= 100 cm) of the bar are insulated (Fig. 18.4) so that no heat can flow through the ends, the boundary conditions are

$$\frac{\partial u(0, t)}{\partial x} = 0, \frac{\partial u(l, t)}{\partial x} = 0 \text{ for all } t \quad ...(ii)$$

Initially, under steady state conditions, $\frac{\partial^2 u}{\partial x^2} = 0$. Its solution is $u = ax + b$.

Since $u = 0$ for $x = 0$ and $u = 100$ for $x = l$ $\therefore$ $b = 0$ and $a = 1$.

Thus the initial condition is $u(x, 0) = x \quad 0 < x < l$. ...(iii)

Now the solution of (i) is of the form $u(x, t) = (c_1 \cos px + c_2 \sin px)e^{-c^2 p^2 t}$ ...(iv)

Differentiating partially w.r.t. x, we get

$$\frac{\partial u}{\partial x} = (-c_1 p \sin px + c_2 p \cos px)e^{-c^2 p^2 t} \quad ...(v)$$

Putting $x = 0$, $\left(\frac{\partial u}{\partial x}\right)_0 = c_2 pe^{-c^2 p^2 t} = 0$ for all t. [By (ii)]

$\therefore$ $c_2 = 0$

Putting $x = l$ in (v), $\left(\frac{\partial u}{\partial x}\right)_l = -c_1 p \sin ple^{-c^2 p^2 t}$ for all t. [By (ii)]

$\therefore$ $c_1 p \sin pl = 0$ i.e., p being $\neq 0$, either $c_1 = 0$ or $\sin pl = 0$.

When $c_1 = 0$, (iv) gives $u(x, t) = 0$ which is a trivial solution, therefore $\sin pl = 0$.

or $$pl = n\pi \quad \text{or} \quad p = n\pi/l, \quad n = 0, 1, 2, ...$$

Hence (*iv*) becomes $u(x, t) = c_1 \cos \dfrac{n\pi x}{l} e^{-c^2 n^2 \pi^2 t/l^2}$.

$\therefore$ the most general solution of (*i*) satisfying the boundary conditions (*ii*) is

$$u(x, t) = \sum_{n=0}^{\infty} A_n \cos \frac{n\pi x}{l} e^{-c^2 n^2 \pi^2 t/l^2} = A_0 + \sum_{n=1}^{\infty} A_n \cos \frac{n\pi x}{l} e^{-c^2 n^2 \pi^2 t/l^2} \qquad \text{(where } A_n = c_1) \quad ...(vi)$$

Putting $t = 0$, $u(x, 0) = A_0 + \sum_{n=1}^{\infty} A_n \cos \dfrac{n\pi x}{l} = x$ [by (*iii*)]

This requires the expansion of x into a half range cosine series in $(0, l)$.

Thus $$x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x/l \qquad \text{where } a_0 = \frac{2}{l}\int_0^l x\,dx = l$$

and $$a_n = \frac{2}{l}\int_0^l x \cos \frac{n\pi x}{l} dx = \frac{2l}{n^2\pi^2}(\cos n\pi - 1)$$

$= 0$, where n is even ; $= -4l/n^2\pi^2$, when n is odd.

$\therefore$ $A_0 = \dfrac{a_0}{2} = l/2$, and $A_n = a_n = 0$ for n even ; $= -4l/n^2\pi^2$ for n odd.

Hence (*vi*) takes the form

$$u(x, t) = \frac{l}{2} + \sum_{n=1,3,\ldots}^{\infty} \frac{4l}{n^2\pi^2} \cos \frac{n\pi x}{l} e^{-c^2 n^2 \pi^2 t/l^2}$$

$$= \frac{l}{2} - \frac{4l}{\pi^2} \sum_{1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi x}{l} e^{-c^2 (2n-1)^2 \pi^2 t/l^2} \qquad ...(vii)$$

This is the required temperature at a point P_1 distant x from end A at any time t.

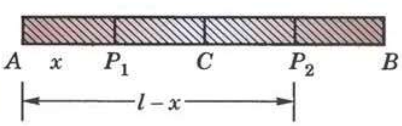


Fig. 18.4

Obs. *The sum of the temperatures at any two points equidistant from the centre is always 100°C, a constant.*

Let P_1, P_2 be two points equidistant from the centre C of the bar so that $CP_1 = CP_2$ (Fig. 18.4).

If $AP_1 = BP_2 = x$ (say), then $AP_2 = l - x$.

$\therefore$ Replacing x by $l - x$ in (*vii*), we get the temperature at P_2 as

$$u(l - x, t) = \frac{l}{2} - \frac{4l}{\pi^2} \sum_{1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi(l-x)}{l} e^{\frac{-c^2(2n-1)^2\pi^2 t}{l^2}}$$

$$= \frac{l}{2} + \frac{4l}{\pi^2} \sum_{1}^{\infty} \frac{1}{(2n-1)^2} \cos \frac{(2n-1)\pi x}{l} e^{\frac{-c^2(2n-1)^2\pi^2 t}{l^2}} \qquad ...(viii)$$

$$\left\{\because \cos \frac{(2n-1)\pi(l-x)}{l} = \cos\left[2n\pi - \pi - \frac{(2n-1)\pi x}{l}\right] = -\cos \frac{(2n-1)\pi x}{l}\right.$$

Adding (*vii*) and (*viii*), we get $u(x, t) + u(l - x, t) = l = 100°C$.

PROBLEMS 18.3

1. A homogeneous rod of conducting material of length 100 cm has its ends kept at zero temperature and the temperature initially is

$$u(x, 0) = x, \qquad 0 \le x \le 50$$
$$= 100 - x, \qquad 50 \le x \le 100.$$

Find the temperature $u(x, t)$ at any time. (*Bhopal, 2007 ; S.V.T.U., 2007 ; Kurukshetra, 2006*)

2. Find the temperature $u(x, t)$ in a homogeneous bar of heat conducting material of length l, whose ends are kept at temperature 0°C and whose initial temperature in (°C) is given by $ax(l-x)/l^2$. *(P.T.U., 2009)*

3. A rod 30 cm. long, has its ends A and B kept at 20° and 80°C respectively until steady state conditions prevail. The temperature at each end is then suddenly reduced to 0°C and kept so. Find the resulting temperature function $u(x, t)$ taking $x = 0$ at A. *(Anna, 2008)*

4. A bar of 10 cm long, with insulated sides has its ends A and B maintained at temperatures 50°C and 100°C respectively, until steady-state conditions prevail. The temperature A is suddenly raised to 90°C and at the same time that at B is lowered to 60°C. Find the temperature distribution in the bar at time t. *(P.T.U., 2010)*

   Show that the temperature at the middle point of the bar remains unaltered for all time, regardless of the material of the bar.

5. Solve the following boundary value problem :
$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < l, \qquad \frac{\partial u(0,t)}{\partial x} = 0, \frac{\partial u(l,t)}{\partial x} = 0, u(x, 0) = x. \qquad \textit{(S.V.T.U., 2008)}$$

6. The temperatures at one end of a bar, 50 cm long with insulated sides, is kept at 0°C and that the other end is kept at 100°C until steady-state conditions prevail. The two ends are then suddenly insulated, so that the temperature gradient is zero at each end thereafter. Find the temperature distribution.

7. Find the solution of $\frac{\partial \theta}{\partial t} = k \frac{\partial^2 \theta}{\partial x^2}$, such that

   (*i*) θ is not infinite when $t \to +\infty$; (*ii*) $\left.\begin{matrix} \frac{\partial \theta}{\partial x} = 0 & \text{when } x = 0 \\ \theta = 0, & \text{when } x = l \end{matrix}\right\}$ for all values of t ;

   (*iii*) $\theta = \theta_0$, when $t = 0$, for all values of x between 0 and l. *(S.V.T.U., 2008)*

8. Find the solution of $\frac{\partial V}{\partial t} = k \frac{\partial^2 V}{\partial x^2}$ having given that $V = V_0 \sin nt$ when $x = 0$ for all values of t and $V = 0$ when x is very large.

18.6 TWO-DIMENSIONAL HEAT FLOW

Consider the flow of heat in a metal plate of uniform thickness α (cm), density ρ (gr/cm³), specific heat s (cal/gr deg) and thermal conductivity k (cal/cm sec deg). Let XOY plane be taken in one face of the plate (Fig. 18.5). If the temperature at any point is independent of the z-coordinate and depends only on x, y and time t, then the flow is said to be two-dimensional. In this case, the heat flow is in the XY-plane only and is zero along the normal to the XY-plane.

Consider a rectangular element $ABCD$ of the plane with sides δx and δy. By (*A*) on p. 466, the amount of heat entering the element in 1 sec. from the side AB

$$= -k\alpha\delta x \left(\frac{\partial u}{\partial y}\right)_y$$

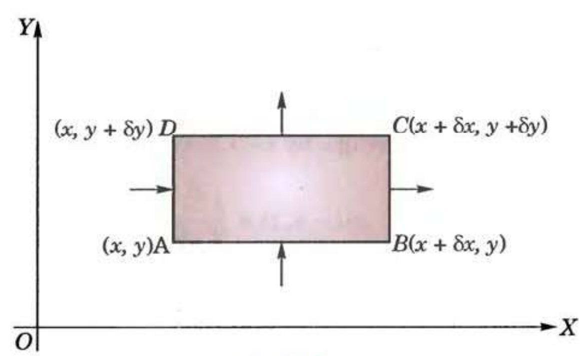


Fig. 18.5

and the amount of heat entering the element in 1 second from the side $AD = -k\alpha\delta y \left(\frac{\partial u}{\partial x}\right)_x$

The quantity of heat flowing out through the side CD per sec. $= -k\alpha\delta x \left(\frac{\partial u}{\partial y}\right)_{y+\delta y}$

and the quantity of heat flowing out through the side BC per second $= -k\alpha\delta y \left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$

Hence the total gain of heat by the rectangular element $ABCD$ per second

$$= -k\alpha\delta x \left(\frac{\partial u}{\partial y}\right)_y - k\alpha\delta y \left(\frac{\partial u}{\partial x}\right)_x + k\alpha\delta x \left(\frac{\partial u}{\partial y}\right)_{y+\delta y} + k\alpha\delta y \left(\frac{\partial u}{\partial x}\right)_{x+\delta x}$$

$$= k\alpha\delta x\left[\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} - \left(\frac{\partial u}{\partial y}\right)_y\right] + k\alpha\delta y\left[\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right]$$

$$= k\alpha\delta x\delta y\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\partial x} + \frac{\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} - \left(\frac{\partial u}{\partial y}\right)_y}{\partial y}\right] \quad ...(1)$$

Also the rate of gain of heat by the element

$$= \rho\delta x\delta y\alpha s\,\frac{\partial u}{\partial t} \quad ...(2)$$

Thus equating (1) and (2),

$$k\alpha\delta x\delta y\left[\frac{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x}{\delta x} + \frac{\left(\frac{\partial u}{\partial y}\right)_{y+\delta y} - \left(\frac{\partial u}{\partial y}\right)_y}{\delta y}\right] = \rho\delta x\delta y\alpha s\,\frac{\partial u}{\partial t}.$$

Dividing both sides by $\alpha\delta x\delta y$ and taking limits as $\delta x \to 0$, $\delta y \to 0$, we get

$$k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = \rho s\frac{\partial u}{\partial t}$$

i.e., $$\frac{\partial \mathbf{u}}{\partial \mathbf{t}} = \mathbf{c}^2\left(\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} + \frac{\partial^2 \mathbf{u}}{\partial \mathbf{y}^2}\right) \text{ where } c^2 = k/\rho s \text{ is the } \textit{diffusivity}. \quad ...(3)$$

Hence the equation (3) gives the temperature distribution of the plane in the *transient state.*

Cor. In the *steady state*, u is independent of t, so that $\partial u/\partial t = 0$ and the above equation reduces to,

$$\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} + \frac{\partial^2 \mathbf{u}}{\partial \mathbf{y}^2} = \mathbf{0}$$

which is the well known **Laplace's equation** *in two dimensions.*

Obs. When the stream lines are curves in space, *i.e.*, the heat flow is three dimensional, we shall similarly arrive at the equation

$$\frac{\partial u}{\partial t} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

In a *steady state*, it reduces to $\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} + \frac{\partial^2 \mathbf{u}}{\partial \mathbf{y}^2} + \frac{\partial^2 \mathbf{u}}{\partial \mathbf{z}^2} = \mathbf{0}$

which is the *three dimensional* **Laplace's equation.**

18.7 SOLUTION OF LAPLACE'S EQUATION

$$\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} + \frac{\partial^2 \mathbf{u}}{\partial \mathbf{y}^2} = \mathbf{0} \quad ...(1)$$

Let $u = X(x)Y(y)$ be a solution of (1).

Substituting it in (1), we get $\frac{d^2X}{dx^2}Y + X\frac{\partial^2 Y}{\partial y^2} = 0$

or separating the variables, $$\frac{1}{X}\frac{d^2X}{dx^2} = -\frac{1}{Y}\frac{d^2Y}{dy^2} \quad ...(2)$$

Since x and y are independent variables, (2) can hold good only if each side of (2) is equal to a constant k (say). Then (2) leads to the ordinary differential equations

$$\frac{d^2X}{dx^2} - kX = 0 \text{ and } \frac{d^2Y}{dy^2} + kY = 0.$$

Solving these equations, we get

(*i*) *When k is positive* and is equal to p^2, say

$$X = c_1e^{px} + c_2e^{-px}, \; Y = c_3 \cos py + c_4 \sin py$$

(*ii*) *When k is negative*, and is equal to $-p^2$, say

$$X = c_5 \cos px + c_6 \sin px, \; Y = c_7e^{py} + c_8e^{-py}$$

(*iii*) *When k is zero* ; $X = c_9x + c_{10}$, $Y = c_{11}y + c_{12}$.

Thus the various possible solutions of (1) are

$$u = (c_1e^{px} + c_2e^{-px})(c_3 \cos py + c_4 \sin py) \qquad ...(3)$$

$$u = (c_5 \cos px + c_6 \sin px)(c_7e^{py} + c_8e^{-py}) \qquad ...(4)$$

$$u = (c_9x + c_{10})(c_{11}y + c_{12}) \qquad ...(5)$$

Of these we take that solution which is consistent with the given boundary conditions.

(*V.T.U., 2011 S ; Kerala, 2005*)

Temperature distribution in long plates

Example 18.15. *An infinitely long plane uniform plate is bounded by two parallel edges and an end at right angles to them. The breadth is π; this end is maintained at a temperature u_0 at all points and other edges are at zero temperature. Determine the temperature at any point of the plate in the steady-state.*

(*P.T.U., 2005 ; J.N.T.U., 2002 S*)

Solution. In the steady state (Fig. 18.6), the temperature $u(x, y)$ at any point $P(x, y)$ satisfies the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \qquad ...(i)$$

The boundary conditions are $u(0, y) = 0$ for all values of y ...(*ii*)

$u(\pi, y) = 0$ for all values of y ...(*iii*)

$u(x, \infty) = 0$ in $0 < x < \pi$ ...(*iv*)

$u(x, 0) = u_0$ in $0 < x < \pi$ ...(*v*)

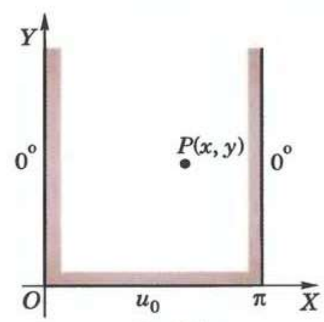


Fig. 18.6

Now the three possible solutions of (*i*) are

$$u = (c_1e^{px} + c_2e^{-px})(c_3 \cos py + c_4 \sin py) \qquad ...(vi)$$

$$u = (c_5 \cos px + c_6 \sin px)(c_7 e^{py} + c_8e^{-py}) \qquad ...(vii)$$

$$u = (c_9x + c_{10})(c_{11}y + c_{12}) \qquad ...(viii)$$

Of these, we have to choose that solution which is consistent with the physical nature of the problem. The solution (*vi*) cannot satisfy the condition (*ii*) for $u \neq 0$ for $x = 0$, for all values of y. The solution (*viii*) cannot satisfy the condition (*iv*). Thus the only possible solution is (*vii*), *i.e.* of the form

$$u(x, y) = (C_1 \cos px + C_2 \sin px)(C_3 e^{py} + C_4 e^{-py}) \qquad ...(ix)$$

By (*ii*), $u(0, y) = C_1(C_3e^{py} + C_4e^{-py}) = 0$ for all y.

Hence $C_1 = 0$ and (*ix*) reduces to

$$u(x, y) = C_2 \sin px \,(C_3e^{py} + C_4e^{-py}) \qquad ...(x)$$

By (*iii*), $u(\pi, y) = C_2 \sin p\pi(C_3e^{py} + C_4e^{-py}) = 0$, for all y.

This requires $\sin p\pi = 0$, *i.e.* $p\pi = n\pi$ as $C_2 \neq 0$. $\therefore\; p = n$, an integer.

Also to satisfy the condition (*iv*), *i.e.*, $u = 0$ as $y \to \infty$, $C_3 = 0$.

Hence (*x*) takes the form $u(x, y) = b_n \sin nx \,.\, e^{-ny}$, where $b_n = C_2C_4$.

$\therefore$ the most general solution satisfying (*ii*), (*iii*) and (*iv*) is of the form

$$u(x, y) = \sum_{n=1}^{\infty} b_n \sin nx \,.\, e^{-ny} \qquad ...(xi)$$

Putting $y = 0$,

$$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin nx \qquad ...(xii)$$

In order that the condition (*v*) may be satisfied, (*v*) and (*xii*) must be same. This requires the expansion of u as a half-range Fourier sine series in $(0, \pi)$. Thus

$$u = \sum_{n=1}^{\infty} b_n \sin nx \text{ where } b_n = \frac{2}{\pi}\int_0^{\pi} u_0 \sin nx\, dx = \frac{2u_0}{n\pi}[1-(-1)^n]$$

i.e., $b_n = 0$, if n is even ; $= 4u_0/n\pi$, if n is odd.

Hence (*xi*) becomes $u(x, y) = \dfrac{4u_0}{\pi}\left[e^{-y}\sin x + \dfrac{1}{3}e^{-3y}\sin 3x + \dfrac{1}{5}e^{-5y}\sin 5x + \ldots\right]$.

Temperature distribution in finite plates

Example 18.16. *Solve the Laplace equation* $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ *subject to the conditions* $u(0, y) = u(l, y) = u(x, 0) = 0$ *and* $u(x, a) = \sin n\pi x/l$. *(V.T.U., 2011 ; J.N.T.U., 2006 ; Kerala M. Tech., 2005, U.P.T.U., 2004)*

Solution. The three possible solutions of

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \ldots(i)$$

are

$$u = (c_1e^{px} + c_2e^{-px})(c_3\cos py + c_4\sin py) \quad \ldots(ii)$$

$$u = (c_5\cos px + c_6\sin px)(c_7e^{py} + c_8e^{-py}) \quad \ldots(iii)$$

$$u = (c_9x + c_{10})(c_{11}y + c_{12}) \quad \ldots(iv)$$

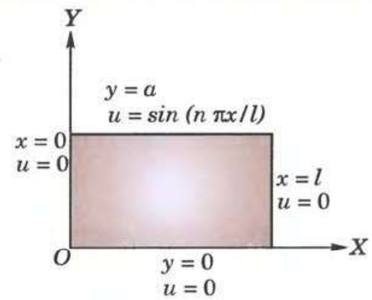


Fig. 18.7

We have to solve (*i*) satisfying the following boundary conditions

$$u(0, y) = 0 \quad \ldots(v) \qquad u(l, y) = 0 \quad \ldots(vi)$$

$$u(x, 0) = 0 \quad \ldots(vii) \qquad u(x, a) = \sin n\pi x/l \quad \ldots(viii)$$

Using (*v*) and (*vi*) in (*ii*), we get

$$c_1 + c_2 = 0, \quad \text{and} \quad c_1e^{pl} + c_2e^{-pl} = 0$$

Solving these equations, we get $c_1 = c_2 = 0$ which lead to trivial solution. Similarly, we get a trivial solution by using (*v*) and (*vi*) in (*iv*). Hence the suitable solution for the present problem is solution (*iii*). Using (*v*) in (*iii*), we have $c_5(c_7e^{py} + c_8e^{-py}) = 0$ *i.e.*, $c_5 = 0$

$\therefore$ (*iii*) becomes $u = c_6 \sin px(c_7e^{py} + c_8e^{-py})$ ...(*ix*)

Using (*vi*), we have $c_6 \sin pl\,(c_7e^{py} + c_8e^{-py}) = 0$

$\therefore$ either $c_6 = 0$ or $\sin pl = 0$

If we take $c_6 = 0$, we get a trivial solution.

Thus $\sin pl = 0$ whence $pl = n\pi$ or $p = n\pi/l$ where $n = 0, 1, 2, \ldots\ldots$

$\therefore$ (*ix*) becomes $u = c_6 \sin(n\pi x/l)(c_7e^{n\pi y/l} + c_8e^{-n\pi y/l})$ ...(*x*)

Using (*vii*), we have $0 = c_6 \sin n\pi x/l \,.\, (c_7 + c_8)$ *i.e.*, $c_8 = -c_7$.

Thus the solution suitable for this problem is

$$u(x, y) = b_n \sin\frac{n\pi x}{l}(e^{n\pi y/l} - e^{-n\pi y/l}) \quad \text{where} \quad b_n = c_6c_7$$

Now using the condition (*viii*), we have

$$u(x, a) = \sin\frac{n\pi x}{l} = b_n \sin\frac{n\pi x}{l}(e^{n\pi a/l} - e^{-n\pi a/l}),$$

we get

$$b_n = \frac{1}{(e^{n\pi a/l} - e^{-n\pi a/l})}$$

Hence the required solution is

$$u(x, y) = \frac{e^{n\pi y/l} - e^{-n\pi y/l}}{e^{n\pi a/l} - e^{-n\pi a/l}}\sin\frac{n\pi x}{l} = \frac{\sinh(n\pi y/l)}{\sinh(n\pi a/l)}\sin\frac{n\pi x}{l}.$$

Example 18.17. *The function* $v(x, y)$ *satisfies the Laplace's equation in rectangular coordinates* (x, y) *and for points within the rectangle* $x = 0, x = a, y = 0, y = b$, *it satisfies the conditions* $v(0, y) = v(a, y) = v(x, b) = 0$ *and* $v(x, 0) = x(a - x)$, $0 < x < a$. *Show that* $v(x, y)$ *is given by*

$$v(x, y) = \frac{8a^2}{\pi^3}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\pi x/a}{(2n+1)^3}\frac{\sinh(2n+1)\pi(b-y)/a}{\sinh(2n+1)\pi b/a} \quad \textit{(Madras, 2003)}$$

Solution. The only possible solution of

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0 \quad \text{...}(i)$$

is of the form $v(x, y) = (c_1 \cos px + c_2 \sin px)(c_3 e^{py} + c_4 e^{-py})$ ...(*ii*)

The boundary conditions are

$$v(0, y) = 0; \quad v(a, y) = 0 \quad \text{...}(iii)$$

$$v(x, b) = 0 \quad \text{...}(iv)$$

$$v(x, 0) = x(a - x), \; 0 < x < a. \quad \text{...}(v)$$

Using (*iii*) $v(0, y) = c_1(c_3 e^{py} + c_4 e^{-py}) = 0$ *i.e.*, $c_1 = 0$.

$\therefore$ (*ii*) becomes $v(x, y) = c_2 \sin px\,(c_3 e^{py} + c_4 e^{-py})$ ...(*vi*)

Again using (*iii*), $v(a, y) = c_2 \sin pa\,(c_3 e^{py} + c_4 e^{-py}) = 0$.

i.e., $\sin pa = 0$, *i.e.* $pa = n\pi$ or $p = n\pi/a$

$\therefore$ (*vi*) becomes $v(x, y) = c_2 \sin \dfrac{n\pi x}{a}\left(c_3 e^{\frac{n\pi y}{a}} + c_4 e^{-\frac{n\pi y}{a}}\right)$

or $v(x, y) = \sin \dfrac{n\pi x}{a}(Ae^{n\pi y/a} + Be^{-n\pi y/a})$ where $A = c_2 c_3, B = c_2 c_4$ ...(*vii*)

Now using (*iv*), $v(x, b) = \sin \dfrac{n\pi x}{a}\left(A e^{\frac{n\pi b}{a}} + B e^{-\frac{n\pi b}{a}}\right) = 0$

i.e., $Ae^{n\pi b/a} + Be^{-n\pi b/a} = 0$ or $Ae^{n\pi b/a} - Be^{-n\pi b/a} = -\dfrac{1}{2}b_n$ (say)

Thus (*vii*) becomes $v(x, y) = \sin \dfrac{n\pi x}{a} \cdot \dfrac{1}{2}b_n \left\{e^{n\pi(b-y)/a} - e^{-n\pi(b-y)/a}\right\}$

$$= b_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi(b-y)}{a}$$

$\therefore$ the most general solution of (*i*) is

$$v(x, y) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi(b-y)}{a} \quad \text{...}(viii)$$

Using the condition (*v*), we have

$$x(a - x) = v(x, 0) = \sum_{n=1}^{\infty} b_n \sinh \frac{n\pi b}{a} \sin \frac{n\pi x}{a}$$

where $b_n \sinh \dfrac{n\pi b}{a} = \dfrac{2}{a}\displaystyle\int_0^a x(a - x) \sin \dfrac{n\pi x}{a}\, dx$

$$= \frac{2}{a}\left|(ax - x^2)\left(\frac{-\cos n\pi x/a}{n\pi/a}\right) - (a - 2x)\left(-\frac{\sin n\pi x/a}{(n\pi/a)^2}\right) + (-2)\left\{\frac{\cos n\pi x/a}{(n\pi/a)^3}\right\}\right|_0^a$$

$$= 0 - 0 + \frac{4a^2}{n^3\pi^3}(1 - \cos n\pi)$$

$$= \frac{8a^2}{n^3\pi^3} \text{ when } n \text{ is odd, otherwise zero when } n \text{ is even.}$$

Hence from (*viii*), the required solution is

$$v(x, y) = \frac{8a^2}{\pi^3} \sum_{n=1,3,5,\ldots}^{\infty} \frac{\sinh n\pi(b-y)/a}{n^3 \sinh n\pi b/a} \sin \frac{n\pi x}{a}$$

or
$$v(x, y) = \frac{8a^2}{\pi^3} \sum_{n=0}^{\infty} \frac{\sinh (2n+1)\pi(b-y)/a}{(2n+1)^3 \sinh (2n+1)\pi b/a} \sin \frac{(2n+1)\pi x}{a}.$$

PROBLEMS 18.4

1. A long rectangular plate of width a cm. with insulated surface has its temperature v equal to zero on both the long sides and one of the short sides so that $v(0, y) = 0$, $v(a, y) = 0$, $v(x, \infty) = 0$, $v(x, 0) = kx$. Show that the steady-state temperature within the plate is

$$v(x, y) = \frac{2ak}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \cdot e^{-n\pi y/a} \sin \frac{n\pi x}{a}. \qquad \text{(J.N.T.U., 2005)}$$

2. A rectangular plate with insulated surface is 8 cm. wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature along one short edge $y = 0$ is given by

$$u(x, 0) = 100 \sin (\pi x/8), \quad 0 < x < 8;$$

while the two long edges $x = 0$ and $x = 8$ as well as the other short edge are kept at 0°C, show that the steady-state temperature at any point of the plane is given by

$$u(x, y) = 100e^{-\pi y/8} \sin (\pi x/8).$$

3. A rectangular plate with insulated surface is 10 cm. wide and so long compared to its width that it may be considered infinite in length without introducing an appreciable error. If the temperature of the short edge $y = 0$ is given by

$$u = 20\,x \quad \text{for } 0 \le x \le 5$$

and

$$u = 20(10 - x) \quad \text{for } 5 \le x \le 10$$

and the two long edges $x = 0$, $x = 10$ as well as the other short edge are kept at 0°C, prove that the temperature u at any point (x, y) is given by

$$u = \frac{40}{\pi^2} \sum_{1}^{\infty} \frac{(-1)^{n+1}}{(2n-1)^2} \sin \frac{(2n-1)\pi x}{10} \cdot e^{-(2n-1)\pi y/10} \qquad \text{(Anna, 2009)}$$

4. Solve $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$ for $0 < x < \pi$, $0 < y < \pi$, with conditions given : $u(0, y) = u(\pi, y) = u(x, \pi) = 0$, $u(x, 0) = \sin^2 x$.

5. A square plate is bounded by the lines $x = 0$, $y = 0$, $x = 20$ and $y = 20$. Its faces are insulated. The temperature along the upper horizontal edge is given by

$$u(x, 20) = x(20 - x), \text{ when } 0 < x < 20,$$

while other three edges are kept at 0°C. Find the steady state temperature in the plate. *(Madras, 2003)*

6. The temperature u is maintained at 0° along three edges of a square plate of length 100 cm. and the fourth edge is maintained at 100° until steady-state conditions prevail. Find an expression for the temperature u at any point (x, y). Hence show that the temperature at the centre of the plate

$$= \frac{200}{\pi}\left[\frac{1}{\cosh \pi/2} - \frac{1}{3 \cosh 3\pi/2} + \frac{1}{5 \cosh 5\pi/2} - \ldots\right].$$

7. A square thin metal plate of side a is bounded by the lines $x = 0$, $x = a$, $y = 0$, $y = a$. The edges $x = 0$, $y = a$ are kept at zero temperature, the edge $y = 0$ is insulated and the edge $x = a$ is kept at constant temperature T_0. Show that in the steady state conditions, the temperature $u(x, y)$ at the point (x, y) is given by

$$u(x, y) = \frac{4T_0}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n-1} \sinh \frac{(2n-1)\pi x}{2a} \cos \frac{(2n-1)\pi y}{2a}}{(2n-1) \sinh \frac{(2n-1)\pi}{2}}.$$

8. A rectangular plate has sides a and b. Taking the side of length a as OX and that of length b as OY and other sides to be $x = a$ and $y = b$, the sides $x = 0$, $x = a$, $y = b$ are insulated and the edge $y = 0$ is kept at temperature $u_0 \cos \dfrac{\pi x}{a}$. Find the temperature $u(x, y)$ in the steady-state.

18.8 (1) LAPLACE'S EQUATION IN POLAR COORDINATES

In the study of steady-state temperature distribution in a rectangular plate, it is usually convenient to employ Cartesian coordinates as hitherto done. Sometimes Polar coordinates (r, θ) are found to be more useful and the Cartesian form of Laplace's equation is replaced by its polar form :

$$r^2 \frac{\partial^2 u}{\partial r^2} + r \frac{\partial u}{\partial r} + \frac{\partial^2 u}{\partial \theta^2} = 0$$

(See Ex. 5.24, p. 213-214)

(2) Solution of Laplace's equation

$$r^2\frac{\partial^2 u}{\partial r^2}+r\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial \theta^2}=0 \quad ...(1)$$

Assume that a solution of (1) is of the form $u = R(r) \,.\, \phi(\theta)$ where R is a function of r alone and ϕ is a function of θ only.

Substituting it in (1), we get $r^2R''\phi + rR'\phi + R\phi'' = 0$ or $\phi(r^2R'' + rR') + R\phi'' = 0$.

Separating the variables $\dfrac{r^2R''+rR'}{R}=-\dfrac{\phi''}{\phi}$ ...(2)

Clearly the left side of (2) is a function of r only and the right side is a function of θ alone. Since r and θ are independent variables, (2) can hold good only if each side is equal to a constant k (say). Then (2) leads to the ordinary differential equations

$$r^2\frac{d^2R}{dr^2}+r\frac{dR}{dr}-kR=0 \quad ...(3) \qquad \text{and} \qquad \frac{d^2\phi}{d\theta^2}+k\phi=0 \quad ...(4)$$

Putting $r = e^z$, (3) reduces to $\dfrac{d^2R}{dz^2}-kR=0$ ...(5)

Solving (5) and (4), we get

(*i*) *When k is positive* and $= p^2$, say :

$$R = c_1e^{pz} + c_2e^{-pz} = c_1r^p + c_2r^{-p},\ \phi = c_3 \cos p\theta + c_4 \sin p\theta$$

(*ii*) *When k is negative* and $= -p^2$, say

$$R = c_5 \cos pz + c_6 \sin pz = c_5 \cos (p \log r) + c_6 \sin (p \log r),\ \phi = c_7e^{p\theta} + c_8e^{-p\theta}$$

(*iii*) *When k is zero* :

$$R = c_9z + c_{10} = c_9 \log r + c_{10},\ \phi = c_{11}\theta + c_{12}$$

Thus the three possible solutions of (1) are

$$u = (c_1r^p + c_2r^{-p})(c_3 \cos p\theta + c_4 \sin p\theta) \quad ...(6)$$

$$u = [c_5 \cos (p \log r) + c_6 \sin (p \log r)](c_7e^{p\theta} + c_8e^{-p\theta}) \quad ...(7)$$

$$u = (c_9 \log r + c_{10})(c_{11}\theta + c_{12}) \quad ...(8)$$

Of these solutions, we have to take that solution which is consistent with the physical nature of the problem. The general solution will consist of a sum of terms of type (6), (7) or (8). (*S.V.T.U., 2008*)

Example 18.18. *The diameter of a semi-circular plate of radius a is kept at 0°C and the temperature at the semi-circular boundary is T°C. Show that the steady state temperature in the plate is given by*

$$u(r,\theta)=\frac{4T}{\pi}\sum_{n=1}^{\infty}\frac{1}{2n-1}\left(\frac{r}{a}\right)^{2n-1}\sin(2n-1)\theta. \qquad (\textit{Kerala M. Tech., 2005})$$

Solution. Take the centre of the circle as the pole and bounding diameter as the initial line as in Fig. 18.8. Let the steady state temperature at any point $P(r, \theta)$ be $u(r, \theta)$, so that u satisfies the equation

$$r^2\frac{\partial^2 u}{\partial r^2}+r\frac{\partial u}{\partial r}+\frac{\partial^2 u}{\partial \theta^2}=0 \quad ...(i)$$

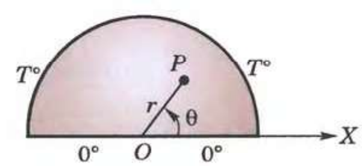


Fig. 18.8

The boundary conditions are :

$$u(r, 0) = 0 \quad \text{in } 0 \le r \le a \quad ...(ii)$$

$$u(r, \pi) = 0 \quad \text{in } 0 \le r \le a \quad ...(iii)$$

and $$u(a, \theta) = T \quad ...(iv)$$

The three possible solutions of (*i*) are

$$u = (c_1r^p + c_2r^{-p})(c_3 \cos p\theta + c_4 \sin p\theta) \quad ...(v)$$

$$u = [c_5 \cos (p \log r) + c_6 \sin (p \log r)](c_7e^{p\theta} + c_8e^{-p\theta}) \quad ...(vi)$$

$$u = (c_9 \log r + c_{10})(c_{11}\theta + c_{12}) \quad ...(vii)$$

From (*ii*) and (*iii*), $u = 0$ when $r = 0$ *i.e.*, u must be finite at the origin. Thus the solutions (*vi*) and (*vii*) are to be rejected. Hence the only suitable solution is (*v*).

By (ii), $u(r, \theta) = (c_1 r^p + c_2 r^{-p})\, c_3 = 0$

Hence $c_3 = 0$ and (v) becomes $u(r, \theta) = (c_1 r^p + c_2 r^{-p})\, c_4 \sin p\theta$ ...$(viii)$

By (iii), $u(r, \pi) = (c_1 r^p + c_2 r^{-p})\, c_4 \sin p\pi = 0$.

As $c_4 \neq 0$, $\sin p\pi = 0$, *i.e.*, $p = n$, where n is any integer.

Hence $(viii)$ reduces to $u(r, \theta) = (c_1 r^n + c_2 r^{-n})\, c_4 \sin n\theta$ ...(ix)

Since $u = 0$, when $r = 0$, $\therefore$ $c_2 = 0$ and (ix) becomes

$u(r, \theta) = b_n r^n \sin n\theta$, where $b_n = c_1 c_4$.

$\therefore$ the most general solution of (i) is of the form

$$u(r, \theta) = \sum_{n=1}^{\infty} b_n r^n \sin n\theta \qquad ...(x)$$

Putting $r = a$, $$u(a, \theta) = \sum_{n=1}^{\infty} b_n a^n \sin n\theta. \qquad ...(xi)$$

In order that (iv) may be satisfied, (iv) and (xi) must be same. This requires the expansion of T as a half-range Fourier sine series in $(0, \pi)$. Thus

$$T = \sum_{n=1}^{\infty} B_n \sin n\theta \quad \text{where} \quad B_n = \frac{2}{\pi}\int_0^{\pi} T \sin n\theta \, d\theta = \frac{2T}{n\pi}(1 - \cos n\pi) \quad \text{and } B_n = b_n a^n$$

$$\therefore \quad b_n = \frac{B_n}{a^n} = \frac{2T}{n\pi a^n}(1 - \cos n\pi)$$

i.e., $b_n = 0$, if n is even

$$= \frac{4T}{n\pi a^n}, \text{ if } n \text{ is odd.}$$

Hence (x) gives $$u(r, \theta) = \frac{4T}{\pi}\left\{\frac{(r/a)}{1}\sin\theta + \frac{(r/a)^3}{3}\sin 3\theta + \frac{(r/a)^5}{5}\sin 5\theta + \ldots\right\}$$

Example 18.19. *The bounding diameter of a semi-circular plate of radius a cm is kept at 0°C and the temperature along the semi-circular boundary is given by*

$$u(a, \theta) = \begin{cases} 50\theta, \text{ when } 0 < \theta \le \pi/2 \\ 50(\pi - \theta) \text{ when } \pi/2 < \theta < \pi \end{cases}$$

Find the steady-state temperature function u (r, θ). (Madras, 2003)

Solution. We know that $u(r, \theta)$ satisfies the equation

$$r^2 \frac{\partial^2 u}{\partial \theta^2} + r\frac{\partial u}{\partial \theta} + \frac{\partial^2 u}{\partial \theta^2} = 0 \qquad ...(i)$$

The boundary conditions are $u(r, \theta) = 0$, $u(r, \pi) = 0$ ...(ii)

and $u(a, \theta) = 50\theta$ for $0 \le \theta \le \pi/2$; $u(a, \theta) = 50(\pi - \theta)$ for $\pi/2 \le \theta \le \pi$ ...(iii)

As in example 18.18, the most general solution of (i) satisfying the boundary conditions (ii) is of the form

$$u(r, \theta) = \sum_{n=1}^{\infty} b_n r^n \sin n\theta \qquad ...(iv)$$

Putting $r = a$, $$r(a, \theta) = \sum_{n=1}^{\infty} b_n a^n \sin n\theta$$

In order that the boundary condition (iii) is satisfied, we have $r(a, \theta) = \sum_{n=1}^{\infty} B_n \sin n\theta$

where $$b_n a^n = B_n = \frac{2}{\pi}\left\{\int_0^{\pi/2} 50\theta \sin n\theta \, d\theta + \int_{\pi/2}^{\pi} 50(\pi - \theta)\sin n\theta \, d\theta\right\} \qquad ...(v)$$

$$= \frac{100}{\pi}\left\{\left| \theta\left(\frac{-\cos n\theta}{\theta}\right) - (1)\left(\frac{-\sin n\theta}{n^2}\right)\right|_0^{\pi/2} + \left|(\pi-\theta)\left(\frac{-\cos n\theta}{n}\right) - (-1)\left(\frac{-\sin n\theta}{n^2}\right)\right|_{\pi/2}^{\pi}\right\}$$

$$= \frac{100}{\pi}\left\{-\frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{\sin n\pi/2}{n^2} + \frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{\sin n\pi/2}{n^2}\right\} = \frac{200}{\pi n^2}\sin n\pi/2.$$

When n is even $B_n = 0$, so taking $n = 1, 3, 5$ etc, (iv) gives

$$u(r,\theta) = \sum_{n=1,3,5,\ldots}^{\infty}\left(\frac{200}{\pi n^2}\sin\frac{n\pi}{2}\right)\frac{1}{a^n}\cdot r^n \sin n\theta$$

$$= \frac{200}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{(2m-1)^2}\left(\frac{r}{a}\right)^{2m-1}\sin(2m-1)\theta.$$

[Taking $n = 2m - 1$, $n = 1, 3, 5, \ldots$; gives $m = 1, 2, 3$, $\sin n\pi/2 = \sin(2m-1)\pi/2 = (-1)^{m-1}$. This gives the required temperature function.

PROBLEMS 18.5

1. A semi-circular plate of radius a has its circumference kept at temperature $u(a,\theta), = k\theta(\pi-\theta)$ while the boundary diameter is kept at zero temperature. Find the steady state temperature distribution $u(r,\theta)$ of the plate assuming the lateral surfaces of the plate to be insulated.
2. A semi-circular plate of radius 10 cm has insulated faces and heat flows in plane curves. The bounding diameter is kept at 0°C and on the circumference the temperature distribution maintained is $u(10,\theta) = (400/\pi)(\pi\theta - \theta^2)$, $0 \le \theta \le \pi$. Determine the temperature distribution $u(r,\theta)$ at any point on the plate.
3. A plate in the shape of truncated quadrant of a circle, is bounded by $r = a$, $r = b$ and $\theta = 0$, $\theta = \pi/2$. It has its faces insulated and heat flows in plane curves. It is kept at temperature 0°C along three of the edges while along the edge $r = a$, it is kept at temperature $\theta(\pi/2 - \theta)$. Determine the temperature distribution.
4. Determine the steady state temperature at the points on the sector $0 \le \theta \le \pi/4$, $0 \le r \le a$ of a circular plate, if the temperature is maintained at 0°C along the side edges and at a constant temperature k°C along the curved edges.
5. Find the steady-state temperature in a circular plate of radius a which has one-half of its circumference at 0°C and the other half at 60°C.
6. If the radii of the inner and outer boundaries of a circular annulus area 10 cm and 20 cm and
$$u(10,\theta) = 15\cos\theta,\ u(20,\theta) = 30\sin\theta,$$
find the value of $u(r,\theta)$ in the annulus. [$u(r,\theta)$ satisfies Laplace equation in the interior of the annulus.]
7. A plate in the form of a ring is bounded by the lines $r = 2$ and $r = 4$. Its surfaces are insulated and the temperature along the boundaries are
$$u(2,\theta) = 10\sin\theta + 6\cos\theta,\ u(4,\theta) = 17\sin\theta + 15\cos\theta$$
Find the steady-state temperature $u(r,\theta)$ in the ring.

18.9 (1) VIBRATING MEMBRANE—TWO DIMENSIONAL WAVE EQUATION

We shall now derive the equation for the vibrations of a tightly stretched membrane, such as the membrane of a drum. We shall assume that the membrane is uniform and the tension T in it per unit length is the same in all directions at every point.

Consider the forces acting on an element $\delta x\delta y$ of the membrane (Fig. 18.9). Forces $T\delta x$ and $T\delta y$ act on the edges along the tangent to the membrane. Let u be its small displacement perpendicular to the xy-plane, so that the forces $T\delta y$ on its opposite edges of length δy make angles α and β to the horizontal. So their vertical component

$= T\delta y\sin\beta - T\delta y\sin\alpha$

$= T\delta y(\tan\beta - \tan\alpha)$ approximately, since α and β are small

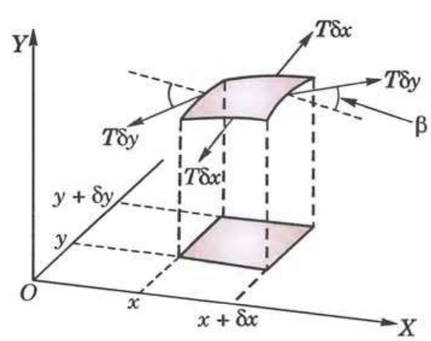


Fig. 18.9

$$= T\delta y\left\{\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} - \left(\frac{\partial u}{\partial x}\right)_x\right\} = T\delta y \frac{\partial^2 u}{\partial x^2}\delta x, \text{ up to a first order of approximation.}$$

Similarly, the vertical component of the force $T\delta x$ acting on the edges of length δx

$$= T\delta x \frac{\partial^2 u}{\partial y^2}\delta y$$

If m be the mass per unit area of the membrane, then the equation of motion of the element is

$$m\delta x\delta y \frac{\partial^2 u}{\partial t^2} = T\left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right]\delta x\delta y \quad \text{or} \quad \frac{\partial^2 \mathbf{u}}{\partial \mathbf{t}^2} = \mathbf{c}^2\left[\frac{\partial^2 \mathbf{u}}{\partial \mathbf{x}^2} + \frac{\partial^2 \mathbf{u}}{\partial \mathbf{y}^2}\right] \quad \text{where} \quad c^2 = T/m \qquad ...(1)$$

This is the wave equation in two dimensions.

(2) Solution of the two-dimensional wave equation - Rectangular membrane. Assume that a solution of (1) is of the form $u = X(x)Y(y)T(t)$

Substituting this in (1) and dividing by XYT, we get

$$\frac{1}{c^2 T}\frac{d^2T}{dt^2} = \frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2}$$

This can hold good if each member is a constant. Choosing the constants suitably, we have

$$\frac{d^2X}{dx^2} + k^2X = 0, \quad \frac{d^2Y}{dy^2} + l^2Y = 0 \quad \text{and} \quad \frac{d^2T}{dt^2} + (k^2 + l^2)c^2T = 0$$

Hence a solution of (1) is

$$u = (c_1 \cos kx + c_2 \sin kx)(c_3 \cos ly + c_4 \sin ly) \times [c_5 \cos\sqrt{(k^2 + l^2)}ct + c_6 \sin\sqrt{(k^2 + l^2)}ct] \qquad ...(2)$$

Now suppose the membrane is rectangular and is stretched between the lines $x = 0, x = a, y = 0, y = b$. Then the condition $u = 0$ when $x = 0$ gives

$$0 = c_1(c_3 \cos ly + c_4 \sin ly)[c_5 \cos\sqrt{(k^2 + l^2)}ct + c_6 \sin\sqrt{(k^2 + l^2)}ct] \quad i.e., \quad c_1 = 0.$$

Then putting $c_1 = 0$ in (2) and applying the condition $u = 0$ when $x = a$, we get $\sin ka = 0$ or $k = m\pi/a$. (m being an integer)

Similarly, applying the conditions $u = 0$, when $y = 0$ and $y = b$, we obtain

$$c_3 = 0 \quad \text{and} \quad l = n\pi/b \qquad (n \text{ being an integer})$$

Thus the solution (2) becomes

$$u(x, y, t) = c_2c_4 \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}(c_5 \cos p_{mn}t + c_6 \sin p_{mn}t)$$

where $p_{mn} = \pi c\sqrt{[(m/a)^2 + (n/b)^2]}$ ...(3)

[These are the solutions of the wave equation (1) which are zero on the boundary of the rectangular membrane. These functions are called **eigen functions** and the numbers p_{mn} are the **eigen values** of the vibrating membrane.]

Choosing the constants c_2 and c_4 so that $c_2c_4 = 1$, we can write the general solution of the equation (1) as

$$u = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}(A_{mn} \cos pt + B_{mn} \sin pt) \qquad ...(4)$$

If the membrane starts from rest from the initial position $u = f(x, y)$, *i.e.*, $\frac{\partial u}{\partial t} = 0$ when $t = 0$, then (3) gives $B_{mn} = 0$.

Also using the condition $u = f(x, y)$ when $t = 0$, we get

$$f(x, y) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b}$$

This is *double Fourier series*. Multiplying both sides by $\sin(m\pi x/a)\sin(n\pi y/b)$ and integrating from $x = 0$ to $x = a$ and $y = 0$ to $y = b$, every term on the right except one, becomes zero. Hence we obtain

$$\int_0^a \int_0^b f(x, y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\, dydx = \frac{ab}{4} A_{mn} \qquad ...(5)$$

which gives the coefficients in the solution and is called the **generalised Euler's formula.**

Rectangular Membranes

Example 18.20. *Find the deflection $u(x, y, t)$ of the square membrane with $a = b = 1$ and $c = 1$, if the initial velocity is zero and the initial deflection is $f(x, y) = A \sin \pi x \sin 2\pi y$.*

Solution. Taking $a = b = 1$ and $f(x, y) = A \sin \pi x \sin 2\pi y$, in (5), we get

$$A_{mn} = 4 \int_0^1 \int_0^1 A \sin \pi x \sin 2\pi y \sin m\pi x \sin n\pi y\, dydx$$

$$= 4A \int_0^1 \sin \pi x \sin m\pi x\, dx \left(\int_0^1 \sin 2\pi y \sin n\pi y\, dy\right) = 0, \quad \text{for } m \neq 1$$

$$= 4A\left(\frac{1}{2}\right) \int_0^1 \sin 2\pi y \sin n\pi y\, dy, \text{ for } m = 1 \qquad \left[\because \int_0^1 \sin \pi x \sin \pi x\, dx = \tfrac{1}{2}\right]$$

i.e.,
$$A_{mn} = 2A \int_0^1 \sin 2\pi y \sin n\pi y\, dy = 0, \text{ for } n \neq 2$$

$$= 2A\left(\frac{1}{2}\right), \text{ for } n = 2.$$

$\therefore$ $A_{12} = A$. Also from (3), $p_{mn} = \pi\sqrt{(m^2 + n^2)}$ $\qquad [\because a = b = 1 = c]$

$\therefore$ $p_{12} = \pi\sqrt{(1^2 + 2^2)} = \sqrt{5}\,\pi$.

Hence from (4), the required solution is $u(x, y, t) = A \sin \pi x \sin 2\pi y \cos (\sqrt{5}\,\pi t)$.

Example 18.21. *Find the vibration $u(x, y, t)$ of a rectangular membrane $(0 < x < a, 0 < y < b)$ whose boundary is fixed given that it starts from rest and $u(x, y, 0) = hxy(a - x)(b - y)$.*

Solution. Proceeding as in § 18.9 (**2**), we have from (4),

$$u = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} (A_{mn} \cos pt + B_{mn} \sin pt) \text{ where } p = \pi c\sqrt{[(m/a)^2 + (n/b)^2}$$

Since the membrane starts from rest $\partial u/\partial t = 0$ when $t = 0$,

$$\therefore \quad \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} (-A_{mn}\, p \sin pt + pB_{mn} \cos pt) = 0 \text{ when } t = 0$$

This gives $B_{mn} = 0$

$$\therefore \quad u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos pt \qquad ...(i)$$

Then $\quad hxy(a - x)(b - y) = u(x, y, 0) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}$

where $A_{mn} = \frac{2}{a} \cdot \frac{2}{b} \int_0^a \int_0^b hxy(a - x)(b - y) \,.\, \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\, dydx$

$$= \frac{4h}{ab} \left\{\int_0^a x(a - x) \sin \frac{m\pi x}{a} dx\right\} \left\{\int_0^b y(b - y) \sin \frac{n\pi y}{b} dy\right\}$$

$$= \frac{4h}{ab} \left| (ax - x^2)\left(\frac{-\cos m\pi x/a}{m\pi/a}\right) - (a - 2x)\left\{\frac{-\sin \frac{m\pi x}{a}}{(m\pi/a)^2}\right\} + (-2)\frac{\cos m\pi x/a}{(m\pi/a)^3} \right|_0^a$$

$$\times \left| (by - y^2)\left(\frac{-\cos n\pi y/b}{n\pi/b}\right) - (b - 2y)\left\{\frac{-\sin n\pi/b}{(n\pi/b)^2}\right\} + (-2)\frac{\cos n\pi y/b}{(n\pi/b)^3} \right|_0^b$$

$$= \frac{4h}{ab} \frac{2a^3}{m^3\pi^3} \cdot \frac{2b^3}{n^3\pi^3} (1 - \cos m\pi)(1 - \cos n\pi)$$

Hence from (*i*), we get

$$u(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} A_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos pt$$

where $A_{mn} = \dfrac{16 h a^2 b^2}{m^3 n^3 \pi^6} (1 - \cos m\pi)(1 - \cos n\pi)$ and $p = \pi c \sqrt{[(m/a)^2 + (n/b)^2]}$

Circular Membranes*

Example 18.22. *A circular membrane of unit radius fixed along its boundary starts vibrating from rest and has initial deflection $u(r, 0) = f(r)$. Show that the deflection $u(r, t)$ of the membrane at any instant is given by*

$$u(r, t) = \sum_{m=1}^{\infty} A_m \cos (c\alpha_m t) \cdot J_o(\alpha_m r) \text{ where } A_m = \frac{2}{J_1^2(\alpha_m)} \int_0^1 r f(r) J_0(\alpha_m r)\, dr,$$

and $\alpha_m (m = 1, 2, ...)$ are the positive roots of the Bessel function $J_0(k) = 0$.

Solution. The vibrations of a plane circular membrane are governed by 2-dimensional wave equation in polar coordinates *i.e.*,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} \right)$$

For a radially symmetric membrane (in which u does not depend on θ) the above equation reduces to

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) \quad ...(i)$$

For the given membrane fixed along its boundary, the boundary condition is

$$u(1, t) = 0 \quad \text{for all } t \geq 0 \quad ...(ii)$$

For solutions not depending on θ,

initial deflection $u(r, 0) = f(r)$ ...(*iii*)

and initial velocity $\left(\dfrac{\partial u}{\partial t} \right)_{t=0} = 0$ ...(*iv*)

which are the initial conditions. We find the solutions $u(r, t) = R(r)T(t)$ ...(*v*)

satisfying the boundary condition (*ii*).

Differentiating and substituting (*v*) in (*i*), we get

$$\frac{\partial^2 T/\partial t^2}{c^2 T} = \frac{1}{R} \left(\frac{\partial^2 R}{\partial r^2} + \frac{1}{r} \frac{\partial R}{\partial r} \right) = -k^2 \text{ (say)}$$

This leads to $\dfrac{\partial^2 T}{\partial t^2} + p^2 T = 0$ where $p = ck$ ...(*vi*)

and $$\frac{\partial^2 R}{\partial r^2} + \frac{1}{r} \frac{\partial R}{\partial r} + k^2 R = 0 \quad ...(vii)$$

Now putting $s = kr$, (*vii*) transforms to $\dfrac{d^2 R}{ds^2} + \dfrac{1}{s} \dfrac{dR}{ds} + R = 0$ which is Bessel's equation. Its general solution $R = aJ_0(s) + bY_0(s)$ where J_0 and Y_0 are Bessel's functions of the first and second kind of order zero.

Since the deflection of the membrane is always finite, we must have $b = 0$. Then taking $a = 1$, we get

$$R(r) = J_0(s) = J_0(kr)$$

On the boundary of the circular membrane, we must have $J_0(k) = 0$, which is satisfied for

$$k = \alpha_m,\ m = 1, 2, ...$$

*Drums, telephones and microphones provide examples of circular membrane and as such are quite useful in engineering.

Thus the solutions of (*vii*) are $R(r) = J_0(\alpha_m r)$, $m = 1, 2, \ldots$ and the corresponding solutions of (*vi*) are $T(t) = A_m \cos p_m t + B_m \sin p_m t$, where $p_m = ck_m = c\alpha_m$.

Hence the general solution of (*i*) satisfying (*ii*) are

$$u(r, t) = (A_m \cos p_m t + B_m \sin p_m t)\, J_0(\alpha_m r)$$

which are the *eigen functions* of the problem and the corresponding *eigen values* are p_m.

To find that solution which also satisfies the initial conditions (*iii*) and (*iv*), consider the series

$$u(r, t) = \sum_{m=1}^{\infty} (A_m \cos p_m t + B_m \sin p_m t)\, J_0(\alpha_m r)$$

Putting $t = 0$ and using (*iii*), we get $u(r, 0) = \sum_{m=1}^{\infty} A_m J_0(\alpha_m r) = f(r)$

Here, the constants A_m must be the coefficients of Fourier-Bessel series [(8) page 560] with $m = 0$, *i.e.*,

$$A_m = \frac{2}{J_1^2(\alpha_m)} \int_0^1 r\, f(r)\, J_0(\alpha_m r)\, dr$$

Using (*iv*), we get $B_m = 0$. Hence the result.

PROBLEMS 18.6

1. A tightly stretched unit square membrane starts vibrating from rest and its initial displacement is $k \sin 2\pi x \sin \pi y$. Show that the deflection at any instant is $k \sin 2\pi x \sin \pi y \cos(\sqrt{5}\, \pi ct)$.
2. Find the deflection $u(r, t)$ of the circular membrane of unit radius if $c = 1$, the initial velocity is zero and the initial deflection is $0.25\,(1 - r^2)$.

18.10 TRANSMISSION LINE

Consider a cable l km in length, carrying an electric current with resistance R ohms/km, inductance L henries/km ; capacitance C farads/km and leakance G mhos/km (Fig. 18.10).

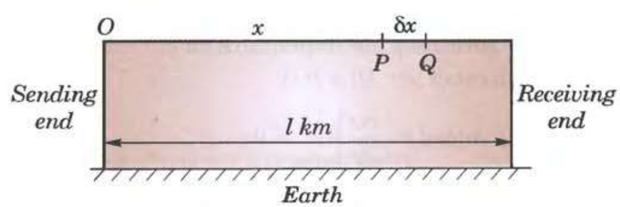


Fig. 18.10

Let the instantaneous voltage and current at any point P, distant x km from the sending end O, and at time t sec be $v(x, t)$ and $i(x, t)$ respectively. Consider a small length $PQ (= \delta x)$ of the cable.

Now since the voltage drop across the segment δx

= voltage drop due to resistance + voltage drop due to inductance

$$\therefore \qquad -\delta v = iR\delta x + L\delta x \cdot \frac{\partial i}{\partial t}$$

and dividing by δx and taking limits as $\delta x \to 0$, we get

$$-\frac{\partial v}{\partial x} = Ri + L\frac{\partial i}{\partial t} \qquad \ldots(1)$$

Similarly the current loss between P and Q

= current lost due to capacitance and leakance

$$\therefore \qquad -\delta i = C\frac{\partial v}{\partial t}\delta x + Gv\delta x \text{ from which as before, we get}$$

$$-\frac{\partial i}{\partial x} = C\frac{\partial v}{\partial t} + Gv \qquad \ldots(2)$$

Rewriting the simultaneous partial differential equations (1) and (2) as

$$\left(R + L\frac{\partial}{\partial t}\right) i + \frac{\partial v}{\partial x} = 0 \qquad \ldots(3)$$

and $$\frac{\partial i}{\partial x}+\left(C\frac{\partial}{\partial t}+G\right)v=0, \qquad ...(4)$$

we shall eliminate i and v in turn.

$\therefore$ operating (3) by $\frac{\partial}{\partial x}$ and (4) by $\left(R+L\frac{\partial}{\partial t}\right)$ and subtracting

$$\frac{\partial^2 v}{\partial x^2}-\left(R+L\frac{\partial}{\partial t}\right)\left(C\frac{\partial}{\partial t}+G\right)v=0$$

or $$\frac{\partial^2 v}{\partial x^2}=LC\frac{\partial^2 v}{\partial t^2}+(LG+RC)\frac{\partial v}{\partial t}+RGv \qquad ...(5)$$

Again operating (3) by $\left(C\frac{\partial}{\partial t}+G\right)$ and (4) by $\frac{\partial}{\partial x}$ and subtracting

$$\left(C\frac{\partial}{\partial t}+G\right)\left(R+L\frac{\partial}{\partial t}\right)i-\frac{\partial^2 i}{\partial x^2}=0$$

or $$\frac{\partial^2 i}{\partial x^2}=LC\frac{\partial^2 i}{\partial t^2}+(LG+RC)\frac{\partial i}{\partial t}+RGi \qquad ...(6)$$

which is (5) with v replaced by i. The equations (5) and (6) are called the *telephone equations.*

Cor. 1. *If* $L = G = 0$, the equations (5) and (6) become

$$\frac{\partial^2 v}{\partial x^2}=RC\frac{\partial v}{\partial t} \qquad ...(7) \qquad\qquad \frac{\partial^2 i}{\partial x^2}=RC\frac{\partial i}{\partial t} \qquad ...(8)$$

which are known as the *telegraph equations.*

Rewriting (7) as $\frac{\partial v}{\partial t}=\frac{1}{RC}\frac{\partial^2 v}{\partial x^2}$, we observe that it is similar to the heat equation [(1) p. 611].

Cor. 2. *If* $R = G = 0$, the equations (5) and (6) become

$$\frac{\partial^2 v}{\partial x^2}=LC\frac{\partial^2 v}{\partial t^2} \qquad ...(9) \qquad\qquad \frac{\partial^2 i}{\partial x^2}=LC\frac{\partial^2 i}{\partial t^2} \qquad ...(10)$$

which are called the *radio equations.*

Rewriting (9) as $\frac{\partial^2 v}{\partial t^2}=k^2\frac{\partial^2 v}{\partial x^2}$ where $k^2=\frac{1}{LC}$,

its general solution is $v(x, t) = f_1(x + kt) + f_2(x - kt)$. [See (4) p. 609]

Similarly from (10), $i(x, t) = F_1(x + kt) + F_2(x - kt)$.

Thus the voltage $v(x, t)$ for the current $i(x, t)$ at any point along the lossless transmission line can be obtained by the superposition of a progressive wave and a receding wave travelling with equal velocities (k). This is the case of oscillations of $v(x, t)$ and $i(x, t)$ at high frequencies.

Cor. 3. *If* $L = C = 0$, *e.g.*, in the case of a submarine cable, then (5) gives

$$\frac{\partial^2 v}{\partial x^2}=GRv, \text{ i.e. } (D^2-GR)v=0$$

$$\therefore \qquad v(x)=A\cosh(\sqrt{GR}\,.\,x)+B\sinh(\sqrt{GR}\,.\,x) \qquad ...(11)$$

Since by (1), $$Ri=-\frac{\partial v}{\partial x}=-\sqrt{GR}\,[A\sinh(\sqrt{GR}\,.\,x)+B\cosh(\sqrt{GR}\,.\,x)]$$

$$\therefore \qquad i(x)=-\sqrt{G/R}\,[A\sinh(\sqrt{GR}\,.\,x)+B\cosh(\sqrt{GR}\,.\,x)] \qquad ...(12)$$

If $v(0) = v_0$ and $i(0) = i_0$, then $v_0 = A$ and $i_0 = -\sqrt{(G/R)}B$.

Hence writing $\sqrt{(GR)} = \gamma$ and $\sqrt{(R/G)} = z_0$, (11) and (12) give

$$v(x)=v_0\cosh\gamma x-i_0z_0\sinh\gamma x \qquad ...(13)$$

and $$i(x)=i_0\cosh\gamma x-\frac{v_0}{z_0}\sinh\gamma x. \qquad ...(14)$$

Obs. Steady-state solutions. We have so far considered the transient state solutions only. The steady-state solutions of transmission lines are however, obtained by assuming $v = Ve^{i\omega t}$ and $i = Ie^{i\omega t}$, where V and I are complex functions of x only. Substituting these in (5) and (6), we get two ordinary linear differential equations of the second order which can be solved at once.

Example 18.23. *Neglecting R and G, find the e.m.f. v(x, t) in a line of length l, t seconds after the ends were suddenly grounded, given that* $i(x, 0) = i_0$ *and* $v(x, 0) = e_1 \sin \frac{\pi x}{l} + e_5 \sin \frac{5\pi x}{l}$. *(S.V.T.U., 2008)*

Solution. Since R and G are negligible, we use the *Radio equation* $\frac{\partial^2 v}{\partial x^2} = LC\frac{\partial^2 v}{\partial t^2}$ ...(*i*)

Since the ends are suddenly grounded, we have the boundary conditions

$$v(0, t) = 0,\ v(l, t) = 0 \quad ...(ii)$$

Also the initial conditions are $i(x, 0) = i_0$

and
$$v(x, 0) = e_1 \sin \pi x/l + e_5 \sin 5\pi x/l \quad ...(iii)$$

$$\therefore \quad \frac{\partial i}{\partial x} = -c\frac{\partial v}{\partial t} \quad \text{gives} \quad \frac{\partial v}{\partial t}(x, 0) = 0 \quad ...(iv)$$

Let $v = X(x)T(t)$ be the solution of (*i*).

$\therefore$ (*i*) gives
$$X''T = LCXT''$$

$$\frac{X''}{X} = LC\frac{T''}{T} = -k^2 \text{ (say)}$$

$$\therefore \quad X'' + k^2X = 0 \quad \text{and} \quad T'' + (k^2/LC)T = 0$$

Solving these equations, we get

$$v = (c_1 \cos kx + c_2 \sin kx)\left(c_3 \cos \frac{k}{\sqrt{LC}} . t + c_4 \sin \frac{k}{\sqrt{LC}} t\right)$$

Using the boundary conditions (*ii*), we get

$$c_1 = 0 \quad \text{and} \quad k = n\pi/l.$$

$$\therefore \quad v = \sin \frac{n\pi x}{l}\left(a_n \cos \frac{n\pi}{l\sqrt{LC}} t + b_n \sin \frac{n\pi}{l\sqrt{LC}} t\right)$$

Using the initial condition (*iv*), we get $b_n = 0$

$$\therefore \quad v = a_n \sin \frac{n\pi x}{l} \cos \frac{n\pi}{l\sqrt{LC}} t$$

Thus the most general solution of (*i*) is

$$v = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{l} \cos \frac{n\pi t}{l\sqrt{LC}}$$

Finally by the initial condition (*iii*), we have

$$e_1 \sin \frac{\pi x}{l} + e_5 \sin \frac{5\pi x}{l} = \sum a_n \sin \frac{n\pi x}{l}$$

$$\therefore \quad a_1 = e_1 \quad \text{and} \quad a_5 = e_5$$ while all other a's are zero.

Hence
$$v = e_1 \sin \frac{\pi x}{l} \cos \frac{\pi t}{l\sqrt{LC}} + e_5 \sin \frac{5\pi x}{l} \cos \frac{5\pi t}{l\sqrt{LC}}$$

which is the required solution.

Example 18.24. *A telephone line 3000 km. long has a resistance of 4 ohms/km. and a capacitance of* 5×10^{-7} *farad/km. Initially both the ends are grounded so that the line is uncharged. At time t = 0, a constant e.m.f. E is applied to one end, while the other end is left grounded. Assuming the inductance and leakance to be negligible, show that the steady state current of the grounded end at the end of 1 sec. is 5.3%.*

Solution. Since $L = 0$, $G = 0$, we use the telegraph equation

$$\frac{\partial^2 v}{\partial x^2} = RC\frac{\partial v}{\partial t}$$

Let $v = X(x)T(t)$ be its solution so that

$$TX'' = RCXT' \quad \text{or} \quad \frac{X''}{X} = RC\frac{T'}{T} = -k^2 \text{ (say)}$$

$\therefore$ $$X'' + k^2X = 0 \text{ and } T' + (k^2/RC)T = 0$$

Solving these equations, we get

$$X = c_1 \cos kx + c_2 \sin kx,\ T = c_3 e^{-k^2t/RC}$$

giving $$v = (c_1 \cos kx + c_2 \sin kx)\, c_3 e^{-k^2t/RC} \qquad ...(i)$$

When $t = 0$; $v = 0$ at $x = 0$ and $v = 0$ at $x = l$

$\therefore$ $$0 = c_1c_3 \,;\, 0 = (c_1 \cos kl + c_2 \sin kl)c_3$$

i.e., $$c_1c_3 = 0 \text{ and } kl = n\pi \ (n \text{ an integer})$$

Putting these in (i) and adding a linear term, we have

$$v = a_0x + b_0 + \sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\, e^{-n^2\pi^2t/RCl^2} \qquad ...(ii)$$

The end conditions of the problem are

$$v = 0 \text{ at } x = 0 \text{ and } v = E \text{ at } x = l$$

Using these, (ii) gives $b_0 = 0$ and $a_0 = E/l$

Then (ii) becomes $$v = \frac{E}{l}x + \sum b_n \sin\frac{n\pi x}{l}\, e^{-n^2\pi^2t/RCl^2}$$

Also $v = 0$ when $t = 0$, we get $-Ex/l = \Sigma b_n \sin n\pi x/l$

This requires the expansion of $(-Ex/l)$ as a half-range sine series in $(0, l)$.

$\therefore$ $$b_n = \frac{2}{l}\int_0^l \left(\frac{-Ex}{l}\right)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\left[\left(\frac{-Ex}{l}\right)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - \left(\frac{-E}{l}\right)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right]_0^l = \frac{2}{l}\left(\frac{El}{n\pi}\cos n\pi\right) = \frac{2E}{n\pi}(-1)^n.$$

Thus $$v = \frac{Ex}{l} + \frac{2E}{\pi}\sum_{n=1}^{\infty} \sin\frac{n\pi x}{l}\, e^{-n^2\pi^2t/RCl^2} \qquad ...(iii)$$

Also when $L = 0$, $\dfrac{-\partial v}{\partial x} = Ri$

i.e., $$i = -\frac{1}{R}\frac{\partial v}{\partial x} = -\frac{E}{lR} - \frac{2E}{lR}\sum_{n=1}^{\infty}(-1)^n \cos\frac{n\pi x}{l}\, e^{-n^2\pi^2t/RCl^2}$$

At the grounded end $(x = 0)$, the current is

$$i = -\frac{E}{lR} - \frac{2E}{lR}\sum_{n=1}^{\infty}(-1)^n e^{-n^2\pi^2t/RCl^2}$$

When $t = 1$ sec, $$i = -\frac{E}{lR}\left(1 - 2e^{\pi^2/RCl^2} + 2e^{-4\pi^2/RCl^2} - ...\right) \qquad ...(iv)$$

Since $$\frac{\pi^2}{RCl^2} = \frac{(3.14)^2}{4(5\times10^{-7})(3000)^2} = 0.548$$

$\therefore$ $$e^{-\pi^2/RCl^2} = e^{-0.548} = 0.578$$

When $t \to \infty, i \to -E/lR$

Hence from (*iv*), we have

$$i = -\frac{E}{lR}\{1 - 2(0.578) + 2(0.578)^4 - 2(0.578)^9 + \ldots\}$$
$$= -\frac{E}{lR}\{1 - 1.156 + 0.223 - 0.014 + \ldots\}$$
$$= i_\infty(0.053) = 5.3\% \text{ of } i_\infty.$$

Example 18.25. *A transmission line 1000 kilometers long is initially under steady-state conditions with potential 1300 volts at the sending end (x = 0) and 1200 volts at the receiving end (x = 1000). The terminal end of the line is suddenly grounded, but the potential at the source is kept at 1300 volts. Assuming the inductance and leakance to be negligible, find the potential v(x, t).* (Andhra, 2000)

Solution. The equation of the telegraph line is

$$\frac{\partial^2 v}{\partial x^2} = RC\frac{\partial v}{\partial t} \quad \text{or} \quad \frac{\partial v}{\partial t} = \frac{1}{RC}\frac{\partial^2 v}{\partial x^2} \qquad \ldots(i)$$

$$v_s = \text{initial steady voltage satisfying } \frac{\partial^2 v}{\partial x^2} = 0 = 1300 - x/10 = v(x, 0) \qquad \ldots(ii)$$

v_s' = steady voltage (after grounding the terminal end) when steady conditions are ultimately reached = $1300 - 1.3x$

$\therefore \quad v(x, t) = v_s' + v_t(x, t)$ where $v_t(x, t)$ is the transient part

$$= 1300 - 1.3x + \sum_{n=1}^{\infty} b_n e^{-(n^2\pi^2 t)/(l^2 RC)} \sin\frac{n\pi x}{l} \qquad \text{[By (viii), p. 614]} \qquad \ldots(iii)$$

where $l = 1000$ kilometers.

Putting $t = 0$, we have from (*ii*) and (*iii*)

$$1300 - 0.1x = v(x, 0) = 1300 - 1.3x + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

i.e. $$1.2x = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l} \text{ where } b_n = \frac{2}{l}\int_0^1 1.2 \sin\frac{n\pi x}{l}\,dx = \frac{2400}{\pi}\cdot\frac{(-1)^{n+1}}{n}$$

Hence $$v(x, t) = 1300 - 1.3x + \frac{2400}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n} e^{-(n^2\pi^2 t)/(l^2 RC)} \sin\frac{n\pi x}{1000}.$$

PROBLEMS 18.7

1. Find the current i and voltage e in a line of length l, t seconds after the ends are suddenly grounded, given that $i(x, 0) = i_0$, $e(x, 0) = e_0 \sin(\pi x/l)$.
   Also R and G are negligible.
2. Show that a transmission line with negligible resistance and leakage propagates waves of current and potential with a velocity equal to $l/\sqrt{(LC)}$, where L is the self-inductance and C is the capacitance.
3. A steady voltage distribution of 20 volts at the sending end and 12 volts at the receiving end is maintained in a telephone wire of length l. At time $t = 0$, the receiving end is grounded. Find the voltage and current t sec later. Neglect leakance and inductance.
4. Obtain the solution of the *radio equation*
$$\frac{\partial^2 v}{\partial x^2} = LC\frac{\partial^2 v}{\partial t^2}$$
   appropriate to the case when a periodic e.m.f. V_0 cost pt is applied at the end $x = 0$ of the line.

18.11 LAPLACE'S EQUATION IN THREE DIMENSIONS

We have seen that the three dimensional heat flow equation in steady state reduces to

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0 \qquad ...(1)$$

which is the *Laplace's equation in three dimensions.*

Laplace's equation also arises in the study of gravitational potential at (x, y, z) of a particle of mass m situated at (ξ, η, ζ) given by

$$\frac{Gm}{r} \text{ where } r = \sqrt{[(x-\zeta)^2 + (y-\eta)^2 + (z-\zeta)^2]}$$

This function is called the *potential of the gravitational field* and satisfies the Laplace's equation.

If a mass of density ρ at (ξ, η, ζ) is distributed throughout a region R, then the gravitational potential u at an external point (x, y, z) is given by

$$u(x, y, z) = G \iiint_R \frac{\rho}{r}\, d\xi\, d\eta\, d\zeta \qquad ...(2)$$

Since $\nabla^2(1/r) = 0$ and ρ is independent of x, y and z, we get

$$\nabla^2 u = \iiint_R \rho\, \nabla^2 (1/r)\, d\xi\, d\eta\, d\zeta = 0.$$

This shows that the gravitational potential defined by (2) also obeys Laplace's equation.

Thus Laplace's equation (1) is one of the most important equations arising in connection with numerous problems of physics and engineering. *The theory of its solutions is called the* **potential theory** *and its solutions are called the* **harmonic functions.**

In most of the problems leading to Laplace's equation, it is required to solve the equation subject to certain boundary conditions. A proper choice of coordinate system makes the solution of the problem simpler. Now we proceed to take up the solutions of (1) in its other forms.

18.12 SOLUTIONS OF THREE DIMENSIONAL LAPLACE'S EQUATION

(1) **Cartesian form of $\nabla^2 u = 0$** is $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$ ...(1)

Let $u = X(x)Y(y)Z(z)$ ...(2)

be a solution of (1). Substituting (2) in (1) and dividing by XYZ, we obtain

$$\frac{1}{X}\frac{d^2X}{dx^2} + \frac{1}{Y}\frac{d^2Y}{dy^2} + \frac{1}{Z}\cdot\frac{d^2Z}{dz^2} = 0 \qquad ...(3)$$

which is of the form $F_1(x) + F_2(y) + F_3(z) = 0$.

As x, y, z are independent, this will hold good only if F_1, F_2, F_3 are constants. Assuming these constants to be $k^2, l^2, -(k^2 + l^2)$ respectively, (3) leads to the equations

$$\frac{d^2X}{dx^2} - k^2X = 0,\ \frac{d^2Y}{dy^2} - l^2Y = 0,\ \frac{d^2Z}{dz^2} + (k^2 + l^2)Z = 0$$

Their solutions are $X = c_1e^{kx} + c_2e^{-kx},\ Y = c_3e^{ly} + c_4e^{-ly}$

$$Z = c_5 \cos \sqrt{(k^2 + l^2)}z + c_6 \sin \sqrt{(k^2 + l^2)}z$$

Thus a possible solution of (1) is

$$u = (c_1e^{kx} + c_2e^{-kx})(c_3e^{ly} + c_4e^{-ly})\{c_5 \cos \sqrt{(k^2 + l^2)}z + c_6 \sin \sqrt{(k^2 + l^2)}z\}.$$

Since the three constants could have been taken as $-k^2, -l^2$ and $k^2 + l^2$, an alternative solution of (1) will be

$$u = (c_1 \cos kx + c_2 \sin kx)(c_3 \cos ly + c_4 \sin ly)\{c_5e^{\sqrt{(k^2+l^2)}z} + c_6e^{-\sqrt{(k^2+l^2)}z}\}.$$

(2) **Cylindrical form of $\nabla^2 u = 0$** is $\dfrac{\partial^2 u}{\partial \rho^2} + \dfrac{1}{\rho}\dfrac{\partial u}{\partial \rho} + \dfrac{1}{\rho^2}\dfrac{\partial^2 u}{\partial \phi^2} + \dfrac{\partial^2 u}{\partial z^2} = 0$ ...(1)

Let $$u = R(\rho)\, H(\phi)\, Z(z)$$ [(iv) p. 359]

be a solution of (1). Substituting it in (1), and dividing by RHZ, we get

$$\frac{1}{R}\left(\frac{d^2R}{d\rho^2}+\frac{1}{\rho}\frac{dR}{d\rho}\right)+\frac{1}{\rho^2 H}\frac{d^2H}{d\phi^2}+\frac{1}{Z}\frac{d^2Z}{dz^2}=0 \qquad ...(2)$$

Assuming that $\dfrac{d^2H}{d\phi^2} = -n^2H$ and $\dfrac{d^2Z}{dz^2} = k^2Z$, ..(3)

(2) reduces to $$\frac{1}{R}\left(\frac{d^2R}{d\rho^2}+\frac{1}{\rho}\frac{dR}{d\rho}\right)-\frac{n^2}{\rho^2}+k^2=0$$

or $$\rho^2\frac{d^2R}{d\rho^2}+\rho\frac{dR}{d\rho}+(k^2\rho^2-n^2)R=0.$$

This is Bessel's equation [§ 16.10 (1)] and its solution is $R = c_1J_n(k\rho) + c_2Y_n(k\rho)$.

Also the solutions of equations (3) are

$$H = c_3 \cos n\phi + c_4 \sin n\phi,\ Z = c_5e^{kz} + c_6e^{-kz}$$

Thus a solution of (1) is

$$u = [c_1J_n(k\rho) + c_2Y_n(k\rho)]\,[c_3 \cos n\phi + c_4 \sin n\phi]\,[c_5 e^{kz} + c_6e^{-kz}]$$

which is known as a *cylindrical harmonic.* (*Assam, 1999*)

(3) Spherical form of $\nabla^2 u = 0$ is

$$\frac{\partial^2u}{\partial r^2}+\frac{2}{r}\frac{\partial u}{\partial r}+\frac{1}{r^2}\frac{\partial^2u}{\partial\theta^2}+\frac{\cot\theta}{r^2}\frac{\partial u}{\partial\theta}+\frac{1}{r^2\sin^2\theta}\frac{\partial^2u}{\partial\phi^2}=0 \qquad ...(1)\quad [(iv)\text{ p. }361]$$

Let $u = R(r)\, G(\theta)\, H(\phi)$ be a solution of (1).

Then $$\frac{1}{R}\left(r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}\right)+\frac{1}{G}\left(\frac{d^2G}{d\theta^2}+\cot\theta\frac{dG}{d\theta}\right)+\frac{1}{H\sin^2\theta}\frac{d^2H}{d\phi^2}=0$$

Putting $$\frac{1}{R}\left(r^2\frac{d^2R}{dr^2}+2r\frac{dR}{dr}\right)=n(n+1) \qquad ...(2) \qquad \text{and } \frac{1}{H}\frac{d^2H}{d\phi^2}=-m^2, \qquad ...(3)$$

the above equation takes the form

$$\frac{d^2G}{d\theta^2}+\cot\theta\frac{dG}{d\theta}+[n(n+1)-m^2\operatorname{cosec}^2\theta]G=0 \qquad ...(4)$$

Now differentiating the *Legendre's equation* (§ 16.13)

$$(1-x^2)y''-2xy'+n(n+1)y=0,$$

m times with respect to x and writing $u = d^my/dx^m$, we get

$$(1-x^2)u''-2(m+1)xu'+(n-m)(n+m+1)u=0 \qquad ...(5)$$

Now putting $G = (1 - x^2)^{m/2}\, u$ in (5), we get

$$(1-x^2)\frac{d^2G}{dx^2}-2x\frac{dG}{dx}+\left[n(n+1)-\frac{m^2}{1-x^2}\right]G=0 \qquad ...(6)$$

Now putting $x = \cos\theta$ in (6), it reduces to (4) and its solution is

$$G = c_1P_n^m(\cos\theta) + c_2Q_n^m(\cos\theta)$$

The solution of (3) is $H = c_3 \cos m\phi + c_4 \sin m\phi$

To solve (2), write $R = r^k$, so that $k(k - 1) + 2k = n(n + 1)$ which gives $k = n$ or $-(n + 1)$

Thus $$R = c_5r^n + c_6r^{-n-1}$$

Hence the general solution of (1) is

$$u=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\{c_1P_n^m(\cos\theta)+c_2Q_n^m(\cos\theta)\}(c_3\cos m\phi+c_4\sin m\phi)\times(c_5r^n+c_6r^{-n-1})$$

Any solution of (1) is known as a *spherical harmonic.*

Example 18.26. *Find the potential in the interior of a sphere of unit radius when the potential on the surface is $f(\theta) = \cos^2 \theta$.*

Solution. Take the origin at the centre of the given sphere S. Since the potential is independent of ϕ on S, so also is the potential at any point. Therefore, the Laplace's equation in spherical co-ordinates reduces to

$$\frac{\partial^2 u}{\partial r^2} + \frac{2}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\cot\theta}{r^2}\frac{\partial u}{\partial \theta} = 0 \qquad ...(i)$$

Putting $u(r, \theta) = R(r)\, G(\theta)$ in (i) and proceeding as in § 18.12 (3), we obtain the equations

$$\frac{\partial^2 G}{d\theta^2} + \cot\theta \frac{dG}{d\theta} + n(n+1)G = 0 \qquad ...(ii)$$

and
$$\frac{1}{R}\left(r^2 \frac{d^2 R}{dr^2} + 2r\frac{dR}{dr}\right) = n(n+1) \qquad ...(iii)$$

Putting $\cot\theta = v$, (ii) takes the form

$$(1 - v^2)\frac{d^2 G}{dv^2} - 2v\frac{dG}{dv} + n(n+1)G = 0$$

which is Legendre's equation. Its solutions are

$$G = P_n(v) = P_n(\cos\theta) \text{ for } n = 0, 1, 2, ...$$

The solutions of (iii) are $R_n(r) = r^n$, $\overline{R_n}(r) = 1/r^{n+1}$.

Hence the equation (i) has the following two sets of solutions

$$u_n(r, \theta) = c_n r^n P_n(\cos\theta) \text{ and } \bar{u}_n(r, \theta) = c_n P_n(\cos\theta)/r^{n+1}, \text{ where } n = 0, 1, 2, ...$$

For points inside S, we have the general equation $u(r, \theta) = \sum_{n=0}^{\infty} c_n r^n P_n(\cos\theta)$ ...(iv)

On the boundary of S, $u(1, \theta) = f(\theta)$ $\quad\therefore\quad f(\theta) = \sum_{n=0}^{\infty} c_n P_n(\cos\theta)$

which is Fourier-Legendre expansion of $f(\theta)$. Hence by (5) p. 560, we have

$$c_n = \left(n + \frac{1}{2}\right)\int_{-1}^{1} f(\theta)\, P_n(x)\, dx \text{ where } x = \cos\theta.$$

$$= \left(n + \frac{1}{2}\right)\int_{-1}^{1} x^2 P_n(x) dx \qquad [\because\ f(\theta) = \cos^2\theta]$$

$$= \left(n + \frac{1}{2}\right)\int_{-1}^{1}\left[\frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)\right]P_n(x)\, dx \qquad [\because\ P_2(x) = \tfrac{1}{2}(3x^2 - 1)]$$

Using the orthogonality of Legendre polynomials, we get

$c_n = 0$, except for $n = 0, 2$. Hence

$$c_0 = \frac{1}{2}\cdot\frac{1}{3}\int_{-1}^{1} P_0^2(x)\, dx = \frac{1}{3},\ c_2 = \frac{5}{2}\cdot\frac{2}{3}\int_{-1}^{0} P_2^2(x)\, dx = \frac{2}{3}.$$

Substituting in (iv), we get $u(r, \theta) = \frac{1}{3} + \frac{2}{3}r^2 P_2(\cos\theta)$ or $u(r, \theta) = \frac{1}{3} + r^2(\cos^2\theta - \frac{1}{3})$.

PROBLEMS 18.8

1. Show that a solution of Laplace's equation in cylindrical co-ordinates, which remains finite at $r = 0$, may be expressed in the form

$$u = \sum_{n=0}^{\infty} J_n(kr)\{e^{kz}(A_n \cos n\theta + B_n \sin n\theta) + e^{-kz}(C_n \cos n\theta + D_n \sin n\theta)\}.$$

2. The potential on the surface of a unit sphere is $f(\theta) = \cos 2\theta$. Show that the potential at all points of space is given by

$$u(r, \theta) = 2r^2(\cos^2\theta - 1/3) - \frac{1}{3} \text{ for } r < 1,$$

and $$u(r, \theta) = 2r^{-3}(\cos^2\theta - 1/3) - r^{-1}/3 \text{ for } r > 1.$$

3. Show that in spherical polar coordinates (r, θ, ϕ), Laplace's equation possesses solutions of the form

$$(Ar^n + B/r^{n+1})P_n(\mu)e^{\pm im\phi},$$

where $\mu = \cos\theta$, A, B, m, n are constants and $P_n(\mu)$ satisfies Legendre's equation

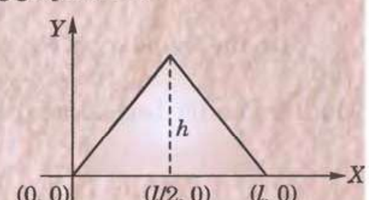
$$(1-\mu^2)\frac{d^2P_n}{d\mu^2} - 2\mu\frac{dP_n}{d\mu} + \left\{n(n+1) - \frac{m^2}{1-\mu^2}\right\}P_n = 0.$$

18.13 OBJECTIVE TYPE OF QUESTIONS

PROBLEMS 18.9

Fill up the blanks in each of the following questions :

1. The radio equations for the potential and current are ....
2. The partial differential equation representing variable heat flow in three dimensions, is ....
3. Temperature gradient is defined as ....
4. The differential equation $f_{xx} + 2f_{xy} + 4f_{yy} = 0$ is classified as ....
5. The partial differential equation of the transverse vibrations of a string is ....
6. The steady state temperature of a rod of length l whose ends are kept at 30° and 40° is ....
7. The equation $u_t = c^2u_{xx}$ is classified as ....
8. The two dimensional steady state heat flow equation in polar coordinates is ....
9. The mathematical function of the initial form of the string given by the following graph is ....



10. When a vibrating string fastened to two points l apart, has an initial velocity u_0, its initial conditions are ....
11. In two dimensional heat flow, the temperature along the normal to the xy-plane is ....
12. If a square plate has its faces and the edge $y = 0$ insulated, its edges $x = 0$ and $x = a$ are kept at zero temperature and the fourth edge is kept at temperature u, then the boundary conditions for this problem are ....
13. If the ends $x = 0$ and $x = l$ are insulated in one dimensional heat flow problems, then the boundary conditions are ....
14. D'Alembert's solution of the wave equation is ....
15. The partial differential equation of 2-dimensional heat flow in ....
16. A rod 50 cm long with insulated sides has its end A and B kept at 20° and 70°C respectively. The steady state temperature distribution of the rod is .... *(Anna, 2008)*
17. The three possible solutions of Laplace equation in polar coordinates are ....
18. Solution of $\frac{\partial u}{\partial x} = 4\frac{\partial u}{\partial y}$, given $u(0, y) = 8e^{-3y}$, is ....
19. Solution of $\frac{\partial z}{\partial x} + 4z = \frac{\partial z}{\partial t}$, given $z(x, 0) = 4e^{-3x}$, is ....
20. In the equation $\frac{\partial u}{\partial t} = \alpha^2\frac{\partial^2 u}{\partial x^2}$, α^2 represents ....
21. The telegraph equations for potential and current are ......
22. The general solution of one-dimensional heat flow equation when both ends of the bar are kept at zero temperature, is of the form ....
23. The three possible solutions of Laplace equation $\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$ are ....