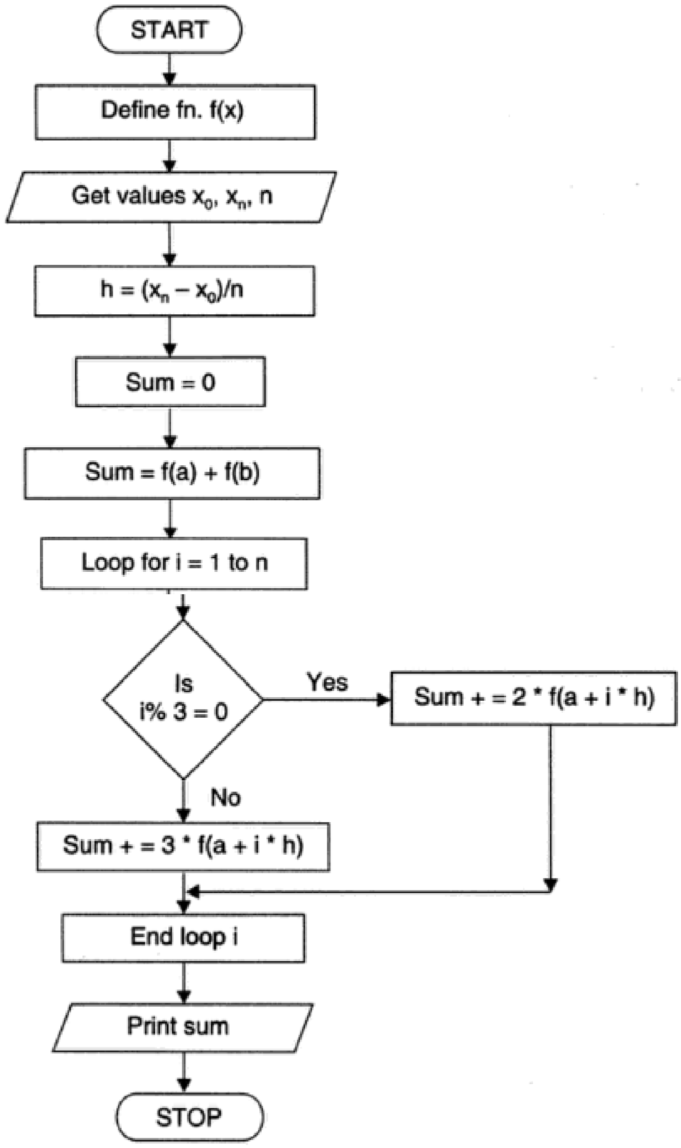
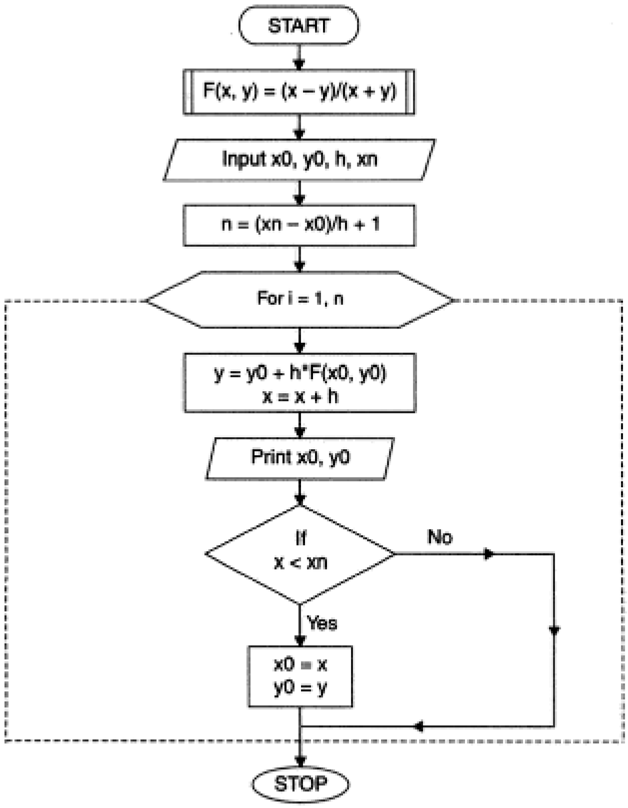
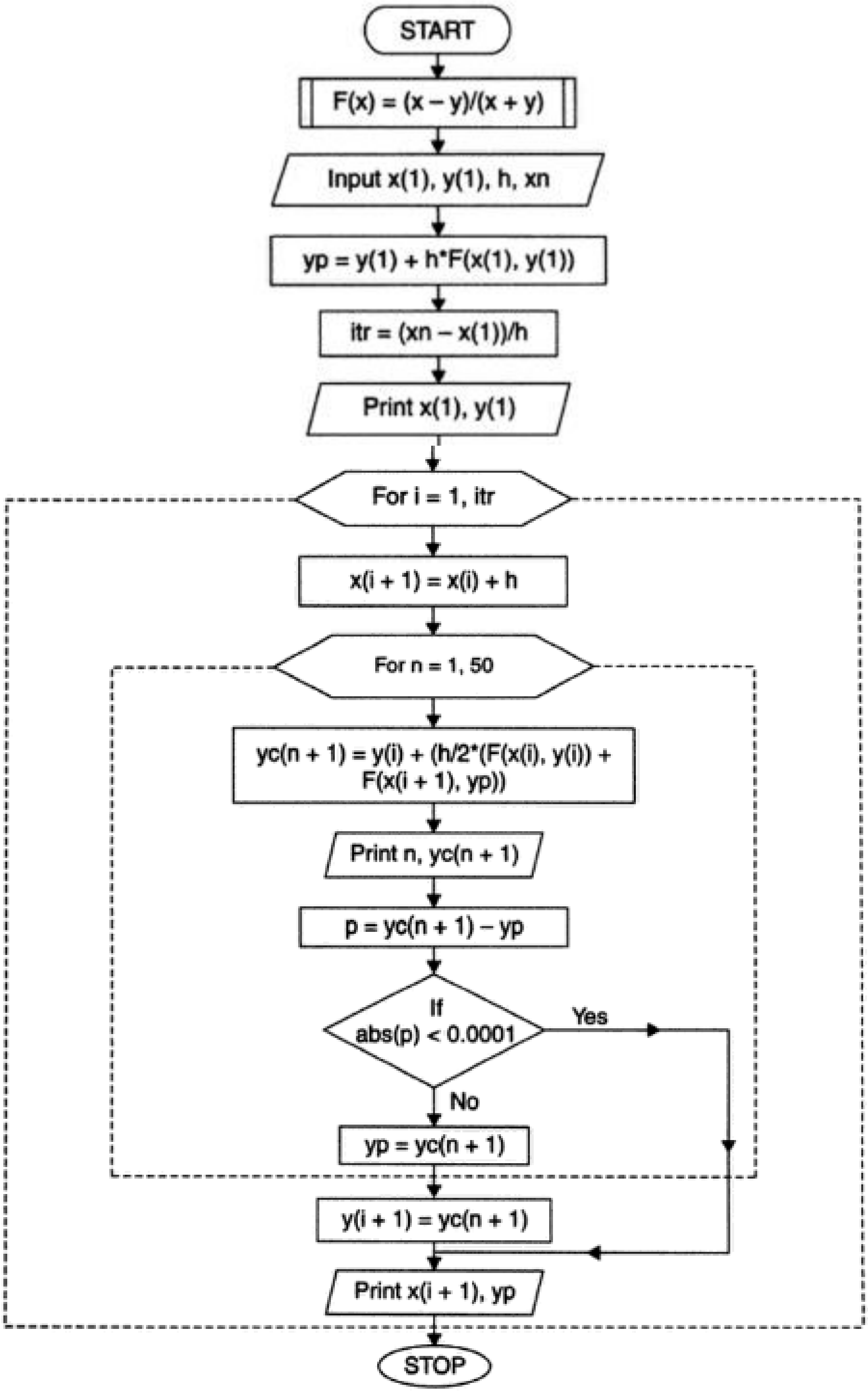

START
Define fn. f(x)
Get values x0, xn, n
h = (xn – x0)/n
Sum = 0
Sum = f(a) + f(b)
Loop for i = 1 to n
Is
i% 3 = 0
Yes
Sum + = 2 * f(a + i * h)
No
Sum + = 3 * f(a + i * h)
End loop i
Print sum
STOP

FLOW-CHART OF EULER'S METHOD



FLOW-CHART OF MODIFIED EULER'S METHOD



ALGORITHM OF SIMPSON'S 1/3rd RULE

Step 01.	Start of the program.
Step 02.	Input Lower limit a
Step 03.	Input Upper limit b
Step 04.	Input number of subintervals n
Step 05.	h=(b–a)/n
Step 06.	sum=0
Step 07.	sum=fun(a)+4*fun(a+h)+fun(b)
Step 08.	for i=3; i<n; i + = 2
Step 09.	sum + = 2*fun(a+(i – 1)*h) + 4*fun(a+i*h)
Step 10.	End of loop i
Step 11.	result=sum*h/3
Step 12.	Print Output result
Step 13.	End of Program
Step 14.	Start of Section fun
Step 15.	temp = 1/(1+(x*x))

ALGORITHM OF SIMPSON'S 3/8th RULE

Step 01.	Start of the program.
Step 02.	Input Lower limit a
Step 03.	Input Upper limit b
Step 04.	Input number of sub itervals n
Step 05.	h = (b – a)/n
Step 06.	sum = 0
Step 07.	sum = fun(a) + fun (b)
Step 08.	for i = 1; i < n; i++
Step 09.	if i%3=0:
Step 10.	sum + = 2*fun(a + i*h)
Step 11.	else:
Step 12.	sum + = 3*fun(a+(i)*h)
Step 13.	End of loop i
Step 14.	result = sum*3*h/8
Step 15.	Print Output result
Step 16.	End of Program
Step 17.	Start of Section fun
Step 18.	temp = 1/(1+(x*x))
Step 19.	Return temp
Step 20.	End of section fun

ALGORITHM OF TRAPEZOIDAL RULE

Step 01. Start of the program.
Step 02. Input Lower limit a
Step 03. Input Upper Limit b
Step 04. Input number of sub intervals n
Step 05. h=(b-a)/n
Step 06. sum=0
Step 07. sum=fun(a)+fun(b)
Step 08. for i=1; i<n; i++
Step 09. sum +=2*fun(a+i)
Step 10. End Loop i
Step 11. result =sum*h/2;
Step 12. Print Output result
Step 13. End of Program
Step 14. Start of Section fun
Step 15. temp = 1/(1+(x*x))
Step 16. Return temp
Step 17. End of Section fun.

Example 16. *A solid of revolution is formed by rotating about x-axis, the lines $x = 0$ and $x = 1$ and a curve through the points with the following coordinates.*

x:	*0*	*0.25*	*0.5*	*0.75 1*	
y:	*1*	*0.9896*	*0.9589*	*0.9089*	*0.8415*

Estimate the volume of the solid formed using Simpson's rule.

Sol. If V is the volume of the solid formed then we know that

$$V = \pi \int_0^1 y^2 \, dx$$

Hence we need the values of y^2 and these are tabulated below correct to four decimal places

0	.25	.5	.75	1
1	.9793	.9195	.8261	.7081

with $h = 0.25$, Simpson's rule gives

$$V = \pi \frac{(0.25)}{3} [(1 + .7081) + 4(.9793 + .8261) + 2(.9195)]$$

$$= 2.8192.$$

Example 6. *Find* $\int_1^{11} f(x)\, dx$, *where f(x) is given by the following table, using a suitable integration formula.*

x:	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*
f(x):	*543*	*512*	*501*	*489*	*453*	*400*	*352*	*310*	*250*	*172*	*95*

Sol. Since the number of subintervals is 10 (even) hence we shall use Simpson's $\frac{1}{3}$rd rule.

$$\int_1^{11} f(x)\, dx = \frac{h}{3} [(y_0 + y_{10}) + 4(y_1 + y_3 + y_5 + y_7 + y_9) + 2(y_2 + y_4 + y_6 + y_8)]$$

$$= \frac{1}{3} [(543 + 95) + 4(512 + 489 + 400 + 310 + 172) + 2(501 + 453 + 352 + 250)]$$

$$= \frac{1}{3} [638 + 7532 + 3112] = 3760.67.$$

Example 12. *A train is moving at the speed of 30 m / sec. Suddenly brakes are applied. The speed of the train per second after t seconds is given by*

Time (t):	*0*	*5*	*10*	*15*	*20*	*25*	*30*	*35*	*40*	*45*
Speed (v):	*30*	*24*	*19*	*16*	*13*	*11*	*10*	*8*	*7*	*5*

Apply Simpson's three-eighth rule to determine the distance moved by the train in 45 seconds.

Sol. If s meters is the distance covered in t seconds, then

$$\frac{ds}{dt} = v \qquad \Rightarrow \qquad \Big[s \Big]_{t=0}^{t=45} = \int_0^{45} v\, dt$$

Since the number of subintervals is **9 (a multiple of 3)** hence by using Simpson's $\left(\frac{3}{8}\right)^{\text{th}}$ rule,

$$\int_0^{45} v\, dt = \frac{3h}{8} [(v_0 + v_9) + 3(v_1 + v_2 + v_4 + v_5 + v_7 + v_8) + 2(v_3 + v_6)]$$

$$= \frac{15}{8} [(30 + 5) + 3(24 + 19 + 13 + 11 + 8 + 7) + 2(16 + 10)]$$

$$= 624.375 \text{ meters.}$$

Hence the distance moved by the train in 45 seconds is **624.375** meters.

Example 13. *Find f ′(4) from the following data:*

x:	*0*	*2*	*5*	*1*
f(x):	*0*	*8*	*125*	*1.*

Sol. Though this problem can be solved by Newton's divided difference formula, we are giving here, as an alternative, Lagrange's method. Lagrange's polynomial, in this case, is given by

$$f(x) = \frac{(x-2)(x-5)(x-1)}{(0-2)(0-5)(0-1)}(0) + \frac{(x-0)(x-5)(x-1)}{(2-0)(2-5)(2-1)}(8)$$

$$+ \frac{(x-0)(x-2)(x-1)}{(5-0)(5-2)(5-1)}(125) + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)}(1)$$

$$= -\frac{4}{3}(x^3 - 6x^2 + 5x) + \frac{25}{12}(x^3 - 3x^2 + 2x) + \frac{1}{4}(x^3 - 7x^2 + 10x)$$

$$= x^3$$

$$\therefore \qquad f'(x) = 3x^2$$

when $x = 4$, $f'(4) = 3(4)^2 = 48$

Example 20. *A reservoir discharging water through sluices at a depth h below the water surface, has a surface area A for various values of h as given below:*

h (in meters):	*10*	*11*	*12*	*13*	*14*
A (in sq. meters):	*950*	*1070*	*1200*	*1350*	*1530*

If t denotes time in minutes, the rate of fall of the surface is given by

$$\frac{dh}{dt} = -\frac{48}{A}\sqrt{h}$$

Estimate the time taken for the water level to fall from 14 to 10 m above the sluices.

Sol. From $\quad \dfrac{dh}{dt} = -\dfrac{48}{A}\sqrt{h}$, we have

$$dt = -\frac{A}{48}\frac{dh}{\sqrt{h}}$$

Integration yields,

$$t = -\frac{1}{48}\int_{14}^{10}\frac{A}{\sqrt{h}}\,dh = \frac{1}{48}\int_{10}^{14}\frac{A}{\sqrt{h}}\,dh$$

Here, $y = \dfrac{A}{\sqrt{h}}$. The table of values is as follows:

h:	10	11	12	13	14
A:	950	1070	1200	1350	1530
$\frac{A}{\sqrt{h}}$:	300.4164	322.6171	346.4102	374.4226	408.9097

Use Trapezoidal or Simpson Rule

Amdahl's Law (I)

1.16 Suppose a computer spends 90 per cent of its time handling a particular type of computation when running a given program, and its manufacturers make a change that improves its performance on that type of computation by a factor of 10.

1. If the program originally took 100s to execute, what will its execution time be after the change?
2. What is the speedup from the old system to the new system?
3. What fraction of its execution time does the new system spend executing the type of computation that was improved?

Solution

1. This is a direct application of Amdahl's Law:

$$\text{Execution Time}_{new} = \text{Execution Time}_{old} \times \left[\text{Frac}_{unused} + \frac{\text{Frac}_{used}}{\text{Speedup}_{used}}\right]$$

Execution $\text{Time}_{old} = 100$ s, $\text{Frac}_{used} = 0.9 > \text{Frac}_{unused} = 0.1 >$ and $\text{Speedup}_{used} = 10$. This gives an Execution Time_{new} of 19 s.

2. Using the definition of speedup, we get a speedup of 5.3. Alternately, we could substitute the values from part 1 into the speedup version of Amdahl's Law to get the same result.
3. Amdahl's Law doesn't give us a direct way to answer this question. The original system spent 90 percent of its time executing the type of computation that was improved, so it spent 90 s of a 100 s program executing that type of computation. Since the computation was improved by a factor of 10, the improved system spends 90/10 = 9 s executing that type of computation. Because 9 s is 47 percent of 19 s, the new execution time, the new system spends 47 percent of its time executing the type of computation that was improved. Alternately, we could have calculated the time that the original system spent executing computations that weren't improved (10 s). Since these computations weren't changed where the improvement was made, the amount of time spent executing them in the new system is the same as the old system. This could then be used to compute the percent of time spent on computations that weren't improved, and the percent of time spent on computations that were improved generated by subtracting that from 100.

Duality

- The dual of a boolean expression is obtained by interchanging boolean sums and products and interchanging 0's and 1's.
- Examples
  - the dual of x(y + 0) is x + (y � 1)
  - the dual of (~x � 1) + (~y + z) is (~x + 0) � (~y � z)
- Construct the dual of the absorption law, x(x + y) = x
  - the dual is x + (xy) = x

```
x  y   xy    x + (xy)
---------------------
0  0   0      0
0  1   0      0
1  0   0      1
1  1   1      1
```

Example 5: The velocity of a train which starts from rest is given by the following table, the time being reckoned in minutes from the start and the speed in km/hour.

t (minutes)	2	4	6	8	10	12	14	16	18	20
v (km/hr)	16	28.8	40	46.4	51.2	32.0	17.6	8	3.2	0

Estimate approximately the total distance run in 20 minutes.

Solution: $$v = \frac{ds}{dt} \quad \Rightarrow \quad ds = v.dt$$

$$\Rightarrow \qquad \int ds = \int v.dt$$

$$s = \int_0^{20} v.dt.$$

The train starts from rest, $\therefore$ the velocity $v = 0$ when $t = 0$.

The given table of velocities can be written

t	0	2	4	6	8	10	12	14	16	18	20
v	0	16	28.8	40	46.4	51.2	32.0	17.6	8	3.2	0
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}

$$h = \frac{2}{60}\text{ hrs} = \frac{1}{30}\text{ hrs.}$$

The Simpson's rule is

$$s = \int_0^{20} v.dt = \frac{h}{3}[(y_{|} + y_{||}) + 4(y_{|} + y_{|} + y_{|} + y_{|}) + 2(y_{|} + y_{|} + y_{|} + y_{|})]$$

$$= \frac{1}{30 \times 3}[(0 + 0)\ 4(16 + 40 + 51.2 + 17.6 + 3.2) + 2(28.8 + 46.4 + 32.0 + 8)]$$

$$= \frac{1}{90}[0\ + 4 \times 128 + 2 \times 115.2] = 8.25\text{ km.}$$

$\therefore$ The distance run by the train in 20 minutes = 8.25 km.

Evaluate $\int_0^6 \frac{dx}{1+x^2}$ *by using (i) trapezoidal rule, (ii) Simpson's 1/3 rule, (iii) Simpson's 3/8 rule.*

[Summer 2014]

Solution

$a = 0,\ b = 6$

Dividing the interval into six equal parts, i.e., $n = 6$,

$$h = \frac{b-a}{n} = \frac{6-0}{6} = 1$$

$$y = f(x) = \frac{1}{1+x^2}$$

x	0	1	2	3	4	5	6
$f(x)$	1	0.5	0.2	0.1	0.0588	0.0385	0.027
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

(i) By the trapezoidal rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0+y_6)+2(y_1+y_2+y_3+y_4+y_5)\right]$$

$$= \frac{1}{2}\left[(1+0.027)+2(0.5+0.2+0.1+0.0588+0.0385)\right]$$

$$= 1.4108$$

(ii) By Simpson's 1/3 rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{h}{3}\left[(y_0+y_6)+4(y_1+y_3+y_5)+2(y_2+y_4)\right]$$

$$= \frac{1}{3}\left[(1+0.027)+4(0.5+0.1+0.0385)+2(0.2+0.0588)\right]$$

$$= 1.3662$$

(iii) By Simpson's 3/8 rule,

$$\int_0^6 \frac{dx}{1+x^2} = \frac{3h}{8}\left[(y_0+y_6)+2(y_3)+3(y_1+y_2+y_4+y_5)\right]$$

$$= \frac{3}{8}\left[(1+0.027)+2(0.1)+3(0.5+0.2+0.0588+0.0385)\right]$$

$$= 1.3571$$

Example 1

Evaluate $\int_0^3 \frac{1}{1+x}$ d *x with n* = 6 *by using Simpson's* 3/8 *rule and, hence, calculate* log 2.

[Summer 2014]

Solution

$$a = 0, \quad b = 3, \quad n = 6$$

$$h = \frac{b-a}{n} = \frac{3-0}{6} = 0.5$$

$$y = f(x) = \frac{1}{1+x}$$

x	0	0.5	1	1.5	2	2.5	3
$f(x)$	1	0.6667	0.5	0.4	0.3333	0.2857	0.25
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

By Simpson's 3/8 rule,

$$\int_0^3 \frac{1}{1+x}\,dx = \frac{3h}{8}\left[(y_0 + y_6) + 2(y_3) + 3(y_1 + y_2 + y_4 + y_5)\right]$$

$$= \frac{3(0.5)}{8}\left[(1+0.25) + 2(0.4) + 3(0.6667 + 0.5 + 0.3333 + 0.2857)\right]$$

$$= 1.3888 \qquad \text{...(1)}$$

By direct integration,

$$\int_0^3 \frac{1}{1+x}\,dx = \left|\log(1+x)\right|_0^3$$

$$= \log 4$$

$$= \log (2)^2$$

$$= 2\log 2 \qquad \text{...(2)}$$

From Eqs (1) and (2),

2 log 2 = 1.3888

log 2 = 0.6944

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Following, e.g. Wikipedia, let us define a *boolean algebra* to be a set A, together with two binary operations $\wedge$ and $\vee$, a unary operation $'$, and two nullary operations 0 and 1, satisfying the following axioms:

$$\begin{array}{rll}
a \vee (b \vee c) = (a \vee b) \vee c, & a \wedge (b \wedge c) = (a \wedge b) \wedge c, & \text{(associativity)} \\
a \vee b = b \vee a, & a \wedge b = b \wedge a, & \text{(commutativity)} \\
a \vee (a \wedge b) = a, & a \wedge (a \vee b) = a, & \text{(absorption)} \\
a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c), & a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c) & \text{(distributivity)} \\
a \vee a' = 1, & a \wedge a' = 0. & \text{(complements)}
\end{array}$$

You want to use these axioms to prove that $(a \wedge b)' = a' \vee b'$ and $(a \vee b)' = a' \wedge b'$.

Lemma 1. $a \wedge 1 = a$ and $a \vee 0 = a$ for all a.

Proof. $a \wedge 1 = a \wedge (a \vee a') = a$, by complements and absorption; likewise, $a \vee 0 = a \vee (a \wedge a') = a$ by complements and absorption. □

Lemma 2. $a \wedge 0 = 0$ and $a \vee 1 = 1$ for all a.

Proof. $a \wedge 0 = a \wedge (a \wedge a') = (a \wedge a) \wedge a' = a \wedge a' = 0$. And $a \vee 1 = a \vee (a \vee a') = (a \vee a) \vee a' = a \vee a' = 1$. □

Lemma 3. If $a \wedge b' = 0$ and $a \vee b' = 1$, then $a = b$.

Proof.

$$\begin{aligned}
b &= b \wedge 1 \\
&= b \wedge (a \vee b') \\
&= (b \wedge a) \vee (b \wedge b') \\
&= (b \wedge a) \vee 0 \\
&= (b \wedge a) \vee (a \wedge b') \\
&= (a \wedge b) \vee (a \wedge b') \\
&= a \wedge (b \vee b') \\
&= a \wedge 1 \\
&= a. \ \square
\end{aligned}$$

Lemma 4. For all a, $(a')' = a$.

Proof. By Lemma 3, it suffices to show that $(a')' \wedge a' = 0$ and $(a')' \vee a' = 1$. But this follows directly by complementation. □

Theorem. $(a \wedge b)' = a' \vee b'$.

By Lemmas 3 and 4, it suffices to show that $(a \wedge b) \wedge (a' \vee b') = 0$ and $(a \wedge b) \vee (a' \vee b') = 1$; for by Lemma 4, this is the same as proving $(a \wedge b) \wedge (a' \vee b')'' = 0$ and $(a \wedge b) \vee (a' \vee b')'' = 1$; by Lemma 3, this gives $(a \wedge b) = (a' \vee b')'$, and applying Lemma 4 again we get $(a \wedge b)' = (a' \vee b')'' = a' \vee b'$, which is what we want.

We have:

$$\begin{aligned}
(a \wedge b) \wedge (a' \vee b') &= \big((a \wedge b) \wedge a'\big) \vee \big((a \wedge b) \wedge b'\big) && \text{(by distributivity)} \\
&= \big((a \wedge a') \wedge b\big) \vee \big(a \wedge (b \wedge b')\big) && \text{(associativity and commutativity)} \\
&= (0 \wedge b) \vee (a \wedge 0) \\
&= 0 \vee 0 \\
&= 0.
\end{aligned}$$

And

$$\begin{aligned}
(a \wedge b) \vee (a' \vee b') &= \big((a \wedge b) \vee a'\big) \vee b' && \text{(by associativity)} \\
&= \big((a \vee a') \wedge (b \vee a')\big) \vee b' && \text{(by distributivity)} \\
&= \big(1 \wedge (b \vee a')\big) \vee b' && \text{(by complements)} \\
&= (b \vee a') \vee b' && \text{(by Lemma 1)} \\
&= (b \vee b') \vee a' && \text{(by commutativity and associativity)} \\
&= 1 \vee a' && \text{(by complements)} \\
&= 1 && \text{(by Lemma 2).}
\end{aligned}$$

Since $(a \wedge b) \wedge (a' \vee b') = 0$ and $(a \wedge b) \vee (a' \vee b') = 1$, the conclusion follows. □

Theorem. $(a \vee b)' = a' \wedge b'$.

Proof. Left as an exercise for the interested reader. □

Example 4.7 Given that $f(0) = 1$, $f(1) = 3$, $f(3) = 55$, find the unique polynomial of degree 2 or less, which fits the given data. Find the bound on the error.

We have $x_0 = 0$, $x_1 = 1$, $x_2 = 3$, $f_0 = 1$, $f_1 = 3$ and $f_2 = 55$. The Lagrange fundamental polynomials are given by

$$l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-1)(x-3)}{(-1)(-3)} = \frac{1}{3}\ (x^2 - 4x + 3)$$

$$l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{x(x-3)}{(1)(-2)} = \frac{1}{2}\ (3x - x^2)$$

$$l_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{x(x-1)}{3(2)} = \frac{1}{6}\ (x^2 - x).$$

Hence, the Lagrange quadratic interpolating polynomial is given by

$$\begin{aligned} P_2(x) &= l_0(x)\, f_0 + l_1(x)\, f_1 + l_2(x)\, f_2 \\ &= \frac{1}{3}\ (x^2 - 4x + 3) + \frac{3}{2}\ (3x - x^2) + \frac{55}{6}\ (x^2 - x) \\ &= 8x^2 - 6x + 1. \end{aligned}$$

We have,

$$\begin{aligned} |E_2(f;\, x)| &\le \frac{1}{6}\ M_3 \left[\max_{0 \le x \le 3} |x(x-1)(x-3)| \right] \\ &= \frac{1}{6}\ (2.1126)\ M_3 = 0.3521\ M_3 \end{aligned}$$

where $M_3 = \max\limits_{0 \le x \le 3} |f'''(x)|$ and since the maximum of $|x(x-1)(x-3)|$ occurs at $x = 2.2152$.

Example 4.8 The following values of the function $f(x) = \sin x + \cos x$, are given

x	10°	20°	30°
$f(x)$	1.1585	1.2817	1.3660

Construct the quadratic interpolating polynomial that fits the data. Hence, find $f(\pi/12)$. Compare with the exact value.

Since the value of f at $\pi/12$ radians is required, we convert the data into radian measure. We have

$$x_0 = 10° = \frac{\pi}{18} = 0.1745, \quad x_1 = 20° = \frac{\pi}{9} = 0.3491,$$

$$x_2 = 30° = \frac{\pi}{6} = 0.5236.$$

The Lagrange fundamental polynomials are given by

$$l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = \frac{(x-0.3491)(x-0.5236)}{(-0.1746)(-0.3491)}$$

$$= 16.4061(x^2 - 0.8727x + 0.1828)$$

$$l_1(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{(x-0.1745)(x-0.5236)}{(0.1746)(-0.1745)}$$

$$= -32.8616(x^2 - 0.6981x + 0.0914)$$

$$l_2(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{(x-0.1745)(x-0.3491)}{(0.3491)(0.1745)}$$

$$= 16.4155(x^2 - 0.5236x + 0.0609).$$

Hence, the Lagrange quadratic interpolating polynomial is given by

$$P_2(x) = 16.4061\ (x^2 - 0.8727\,x + 0.1828)\ (1.1585)$$

$$- 32.8616\ (x^2 - 0.6981\,x + 0.0914)\ (1.2817)$$

$$+ 16.4155\ (x^2 - 0.5236\,x + 0.0609)\ (1.3660)$$

$$= -0.6887x^2 + 1.0751x + 0.9903.$$

Hence, $f(\pi/12) = f(0.2618) = 1.2246$.

The exact value is $f(0.2618) = \sin(0.2618) + \cos(0.2618) = 1.2247$.

Example 4.15 For the following data, calculate the differences and obtain the forward and backward difference polynomials. Interpolate at $x = 0.25$ and $x = 0.35$.

x	0.1	0.2	0.3	0.4	0.5
$f(x)$	1.40	1.56	1.76	2.00	2.28

The difference table is obtained as

0.1	1.40				
		0.16			
0.2	1.56		0.04		
		0.20		0.0	
0.3	1.76		0.04		0.0
		0.24		0.0	
0.4	2.00		0.04		
		0.28			
0.5	2.28				

The forward difference polynomial is given by

$$P(x) = 1.4 + (x - 0.1)\ \frac{0.16}{0.1} + \frac{(x-0.1)(x-0.2)}{2}\frac{0.04}{0.01}$$
$$= 2x^2 + x + 1.28.$$

The backward difference polynomial is obtained as

$$P(x) = 2.28 + (x - 0.5)\ \frac{0.28}{0.1} + \frac{(x-0.5)(x-0.4)}{2}\frac{0.04}{0.01}$$
$$= 2x^2 + x + 1.28.$$

Both the polynomials are identical and we obtain

$f(0.25) = 1.655$ and $f(0.35) = 1.875$.

We can obtain the interpolated values directly also. For $x = 0.25$, we choose $x_0 = 0.2$ and write as

$$u = \frac{x - x_0}{h} = \frac{0.25-0.2}{0.1} = 0.5$$

and $\quad f(0.25) = f(0.2) + (0.5)\ \Delta f(0.2) + \frac{1}{2}(0.5)\ (-0.5)\ \Delta^2 f(0.2)$

$$= 1.56 + (0.5)\ (0.20) - (0.125)\ (0.04) = 1.655.$$

For $x = 0.35$, we choose $x_n = 0.4$ and write in backward differences as

$$u = \frac{x - x_n}{h} = \frac{0.35 - 0.4}{0.1} = -0.5$$

and $\quad f(0.35) = f(0.4) + (-0.5)\ \nabla f(0.4) + \frac{1}{2}\ (-0.5)\ (0.5)\ \nabla^2 f(0.4)$

$$= 2.00 - (0.5)\ (0.24) - (0.125)\ (0.04) = 1.875.$$

If we use only the first differences in (4.45), we get with $x_0 = 0.1$

$$P(x) = f(x_0) + (x - x_0)\ \frac{\Delta f_0}{h}$$

$$P(0.25) = 1.40 + \frac{0.15 \times 0.16}{0.1} = 1.64.$$

Similarly, using only the first difference in (4.50) we get with $x_n = 0.5$

$$P(x) = f(x_n) + (x - x_n)\ \frac{\nabla f_n}{h}$$

$$P(0.35) = 2.28 + \frac{(-0.15)(0.38)}{0.1} = 1.86.$$

However, if we write

$$x = 0.25 = 0.2 + (0.5)(0.1)$$

and use the initial point as $x_0 = 0.2$, $u = 0.5$ and keeping only the first difference in (4.46), we get

$$P(0.25) = 1.56 + (0.5)(0.2) = 1.66.$$

Again, we write

$$x = 0.35 = 0.4 - (0.5)(0.1)$$

and use the initial point as $x_n = 0.4$. Keeping only the first difference in (4.50), we obtain

$$P(0.35) = 2.00 + \frac{(-0.05)(0.24)}{0.1} = 1.88.$$

Thus, when the degree of polynomial is fixed, a judicious choice of the initial point may improve the result considerably.

Example 2.3 Perform five iterations of the bisection method to obtain the smallest positive root of the equation

$$f(x) = x^3 - 5x + 1 = 0.$$

Since $f(0) > 0$ and $f(1) < 0$, the smallest positive root lies in the interval (0, 1). Taking $a_0 = 0$, $b_0 = 1$, we get

$$m_1 = \frac{1}{2}(a_0 + b_0) = \frac{1}{2}(0 + 1) = 0.5$$

$$f(m_1) = -1.375 \text{ and } f(a_0) f(m_1) < 0.$$

Thus, the root lies in the interval (0, 0.5). Taking $a_1 = 0$, $b_1 = 0.5$, we get

$$m_2 = \frac{1}{2}(a_1 + b_1) = \frac{1}{2}(0 + 0.5) = 0.25$$

$$f(m_2) = f(0.25) = -0.234375 \text{ and } f(a_1) f(m_2) < 0.$$

Thus the root lies in the interval (0, 0.25). The sequence of intervals is given in Table 2.4.

Table 2.4 Sequence of Intervals for the Bisection Method

k	a_{k-1}	b_{k-1}	m_k	$f(m_k)\, f(a_{k-1})$
1	0	1	0.5	< 0
2	0	0.5	0.25	< 0
3	0	0.25	0.125	> 0
4	0.125	0.25	0.1875	> 0
5	0.1875	0.25	0.21875	< 0

Hence, the root lies in (0.1875, 0.21875). The approximate root is taken as the midpoint of this interval, that is 0.203125.

Example 2.8 Perform four iterations of the Newton-Raphson method to obtain the approximate value of $(17)^{1/3}$ starting with the initial approximation $x_0 = 2$.

Let $x = (17)^{1/3}$. We obtain $x^3 = 17$ and $f(x) = x^3 - 17 = 0$. Using the Newton-Raphson method

$$x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)}, \ k = 0, 1, \cdots$$

we get

$$x_{k+1} = x_k - \frac{x_k^3 - 17}{3x_k^2} = \frac{2x_k^3 + 17}{3x_k^2}, \ k = 0, 1, \cdots$$

Starting with $x_0 = 2$, we obtain

$$x_1 = \frac{2x_0^3 + 17}{3x_0^2} = 2.75, \qquad x_2 = \frac{2x_1^3 + 17}{3x_1^2} = 2.582645$$

$$x_3 = \frac{2x_2^3 + 17}{3x_2^2} = 2.571332, \quad x_4 = \frac{2x_3^3 + 17}{3x_3^2} = 2.571282.$$

The exact value correct to six decimal places is 2.571282.

Example 2.18 Show that the following two sequences have convergence of the second order with the same limit $\sqrt{a}$.

$$\text{(i)} \ x_{n+1} = \frac{1}{2} x_n \left(1 + \frac{a}{x_n^2}\right), \qquad \text{(ii)} \ x_{n+1} = \frac{1}{2} x_n \left(3 - \frac{x_n^2}{a}\right)$$

If x_n is a suitably close approximation to $\sqrt{a}$, show that the magnitude of the error in the first formula for x_{n+1} is about one-third of that in the second formula, and deduce that the formula

$$\text{(iii)} \ x_{n+1} = \frac{1}{8} x_n \left(6 + \frac{3a}{x_n^2} - \frac{x_n^2}{a}\right)$$

gives a sequence with third-order convergence.

Taking the limits as $n \to \infty$ and noting that $\lim_{n \to \infty} x_n = \xi$, $\lim_{n \to \infty} x_{n+1} = \xi$, where ξ is the exact root, we obtain from all the three methods $\xi^2 = a$. Thus all the three methods determine $\sqrt{a}$, where a is any positive real number.

Substituting $x_n = \xi + \varepsilon_n$, $x_{n+1} = \xi + \varepsilon_{n+1}$ and $a = \xi^2$, we get

(i) $\xi + \varepsilon_{n+1} = \frac{1}{2}(\xi + \varepsilon_n)\left[1 + \frac{\xi^2}{(\xi+\varepsilon_n)^2}\right]$

$$= \frac{1}{2}(\xi + \varepsilon_n)\left[1 + \left(1 + \frac{\varepsilon_n}{\xi}\right)^{-2}\right]$$

$$= \frac{1}{2}(\xi + \varepsilon_n)\left[2 - \frac{2\varepsilon_n}{\xi} + \frac{3\varepsilon_n^2}{\xi^2} - \cdots\right]$$

$$= \frac{1}{2}\left[2\xi_n + (2-2)\varepsilon_n + (3-2)\frac{\varepsilon_n^2}{\xi} + \cdots\right]$$

Therefore,

$$\varepsilon_{n+1} = \frac{1}{2\xi}\ \varepsilon_n^2 + O(\varepsilon_n^3). \tag{2.47}$$

Hence, the method has second order convergence, with the error constant $C = 1/(2\xi)$.

(ii) $\xi + \varepsilon_{n+1} = \frac{1}{2}(\xi + \varepsilon_n)\left[3 - \frac{1}{\xi^2}(\xi+\varepsilon_n)^2\right]$

$$= (\xi + \varepsilon_n)\left(1 - \frac{\varepsilon_n}{\xi} - \frac{\varepsilon_n^2}{2\xi^2}\right)$$

Therefore,

$$\varepsilon_{n+1} = -\frac{3}{2\xi}\ \varepsilon_n^2 + O(\varepsilon_n^3). \tag{2.48}$$

Hence, the method has second order convergence with the error constant $C^* = -3/(2\xi)$.

Therefore, the magnitude of the error in the first formula is about one-third of that in the second formula.

(iii) If we multiply (2.47) by 3 and add to (2.48), we find that

$$\varepsilon_{n+1} = O(\varepsilon_n^3). \tag{2.49}$$

It can be verified that $O(\varepsilon_n^3)$ term in (2.49) does not vanish.

Adding 3 times the first formula to the second formula, we obtain the new formula

$$x_{n+1} = \frac{1}{8} x_n \left(6 + \frac{3a}{x_n} - \frac{x_n^2}{a}\right)$$

which has third order convergence.

Example 37: *From the following data estimate the number of persons having incomes between Rs. 1000 and Rs. 1500.*

Income	Below 400	500-1000	1000-2000	2000-3000	3000-4000
No. of persons	6000	4250	3600	1500	650

Solution: First of all we put all the frequencies in the form of cumulative frequencies and form the following table:

Income less than	500	1000	2000	3000	4000
No. of persons	6000	10250	13850	15350	16000

Now shift the origin to 500; and divide the scale by 500. In this way, we have known $f(x)$ for $x = 0, 1, 3, 5$ and 7.

Now, we shall find number of persons whose income is less than Rs. 1500 *i.e.* $f(x)$ for $x = \frac{1500 - 500}{500} = 2$.

By Lagrange's formula,

$$f(2) = \frac{(2-1)(2-3)(2-5)(2-7)}{(0-1)(0-3)(0-5)(0-7)} \times 6000 + \frac{(2-0)(2-3)(2-5)(2-7)}{(1-0)(1-3)(1-5)(1-7)} \times 10250$$

$$+ \frac{(2-0)(2-1)(2-5)(2-7)}{(3-0)(3-1)(3-5)(3-7)} \times 13850 + \frac{(2-0)(2-1)(2-3)(2-7)}{(5-0)(5-1)(5-3)(5-7)} \times 15350$$

$$+ \frac{(2-0)(2-1)(2-3)(2-5)}{(7-0)(7-1)(7-3)(7-5)} \times 16000$$

$$= -\frac{1}{7} \times 6000 + \frac{5}{8} \times 10250 + \frac{5}{8} \times 13850 - \frac{1}{8} \times 15350 + \frac{1}{56} \times 16000$$

$$= -857.14 + 6406.25 + 8656.25 - 1918.75 + 285.71 = 12572 \text{ approx.}$$

i.e., number of persons having income less than Rs. 1500 = 12572. But number of persons having income less than Rs. 1000 = 10250. Therefore persons having income between Rs. 1000 and Rs. 1500 are 2322.

Prove that a unique polynomial of degree n or less passes through n+1 data points.

If the polynomial is not unique, then at least two polynomials of order n or less pass through the n+1 data points. Assume two polynomials of order n or less, $P_n(x)$ and $Q_n(x)$ go through $(n + 1)$ data points, $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$

Then define

$$R_n(x) = P_n(x) - Q_n(x)$$

Since $P_n(x)$ and $Q_n(x)$ pass through all the $(n + 1)$ data points,

$$P_n(x_i) = Q_n(x_i), \; i = 0, \ldots, n$$

Hence

$$R_n(x_i) = P_n(x_i) - Q_n(x_i) = 0, \; i = 0, \ldots, n$$

The n^{th} order polynomial $R_n(x)$ has $(n + 1)$ zeros. A polynomial of order n can have more than n zeros (in this case n+1) only if it is identical to a zero polynomial, that is,

$$R_n(x) \equiv 0$$

Hence

$$P_n(x) \equiv Q_n(x)$$

How can one show that if a second order polynomial has three zeros, then it is zero everywhere. If $R_2(x) = a_0 + a_1 x + a_2 x^2$, then if it has three zeros at x_1, x_2, and x_3, then

$$R_1(x_1) = a_0 + a_1 x_1 + a_2 x_1^2 = 0$$

$$R_2(x_2) = a_0 + a_1 x_2 + a_2 x_2^2 = 0$$

$$R_3(x_3) = a_0 + a_1 x_3 + a_2 x_3^2 = 0$$

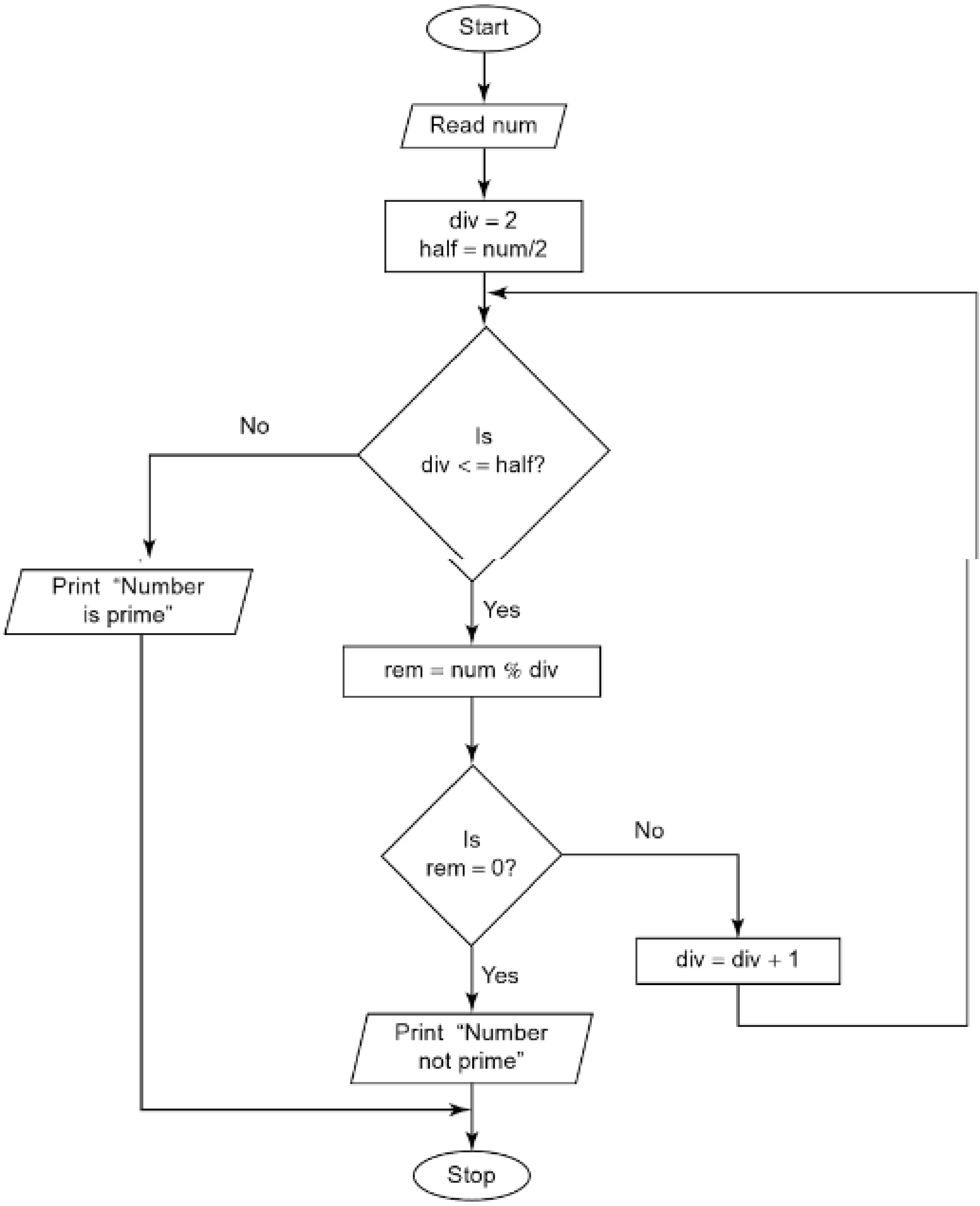
which in matrix form gives

$$\begin{bmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ 1 & x_3 & x_3^2 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The above set of equations has a trivial solution, that is, $a_0 = a_1 = a_2 = 0$.

32. Draw a flowchart to test whether a given number is prime or not.

Ans: The flowchart to check whether a number is prime or not is shown here.



Here, we will discuss the Simpson's 1/3 rule of approximating integrals of the form

$$I = \int_a^b f(x)dx$$

where

$f(x)$ is called the integrand,

$a =$ lower limit of integration

$b =$ upper limit of integration

Simpson's 1/3 Rule

The trapezoidal rule was based on approximating the integrand by a first order polynomial, and then integrating the polynomial over interval of integration. Simpson's 1/3 rule is an extension of Trapezoidal rule where the integrand is approximated by a second order polynomial.

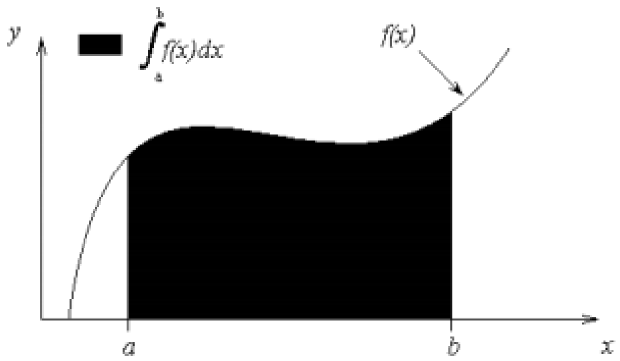


Figure 1 Integration of a function

Method 1:

Hence

$$I = \int_a^b f(x)dx \approx \int_a^b f_2(x)dx$$

where $f_2(x)$ is a second order polynomial given by

$$f_2(x) = a_0 + a_1 x + a_2 x^2.$$

Choose

$$(a, f(a)), \left(\frac{a+b}{2}, f\left(\frac{a+b}{2}\right)\right), \text{ and } (b, f(b))$$

as the three points of the function to evaluate a_0, a_1 and a_2.

$$f(a) = f_2(a) = a_0 + a_1 a + a_2 a^2$$

$$f\left(\frac{a+b}{2}\right) = f_2\left(\frac{a+b}{2}\right) = a_0 + a_1\left(\frac{a+b}{2}\right) + a_2\left(\frac{a+b}{2}\right)^2$$

$$f(b) = f_2(b) = a_0 + a_1 b + a_2 b^2$$

Solving the above three equations for unknowns, a_0, a_1 and a_2 give

$$a_0 = \frac{a^2 f(b) + abf(b) - 4abf\left(\frac{a+b}{2}\right) + abf(a) + b^2 f(a)}{a^2 - 2ab + b^2}$$

$$a_1 = -\frac{af(a) - 4af\left(\frac{a+b}{2}\right) + 3af(b) + 3bf(a) - 4bf\left(\frac{a+b}{2}\right) + bf(b)}{a^2 - 2ab + b^2}$$

$$a_2 = \frac{2\left(f(a) - 2f\left(\frac{a+b}{2}\right) + f(b)\right)}{a^2 - 2ab + b^2}$$

Then

$$I \approx \int_a^b f_2(x)dx$$

$$= \int_a^b \left(a_0 + a_1 x + a_2 x^2\right)dx$$

$$= \left[a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3}\right]_a^b$$

$$= a_0(b-a) + a_1 \frac{b^2 - a^2}{2} + a_2 \frac{b^3 - a^3}{3}$$

Substituting values of a_0, a_1 and a_2 give

$$\int_a^b f_2(x)dx = \frac{b-a}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Since for Simpson 1/3 rule, the interval $[a, b]$ is broken into 2 segments, the segment width

$$h = \frac{b-a}{2}$$

Hence the Simpson's 1/3 rule is given by

$$\int_a^b f(x)dx \approx \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right]$$

Since the above form has 1/3 in its formula, it is called Simpson's 1/3 rule.

Multiple-segment Simpson's 1/3 Rule

Just like in multiple-segment trapezoidal rule, one can subdivide the interval $[a,b]$ into n segments and apply Simpson's 1/3 rule repeatedly over every two segments. Note that n needs to be even. Divide interval $[a,b]$ into n equal segments, so that the segment width is given by

$$h = \frac{b-a}{n}.$$

Now

$$\int_a^b f(x)dx = \int_{x_0}^{x_n} f(x)dx$$

where

$$x_0 = a$$
$$x_n = b$$
$$\int_a^b f(x)dx = \int_{x_0}^{x_2} f(x)dx + \int_{x_2}^{x_4} f(x)dx + ... + \int_{x_{n-4}}^{x_{n-2}} f(x)dx + \int_{x_{n-2}}^{x_n} f(x)dx$$

Apply Simpson's 1/3rd Rule over each interval,

$$\int_a^b f(x)dx \cong (x_2 - x_0)\left[\frac{f(x_0)+4f(x_1)+f(x_2)}{6}\right] + (x_4 - x_2)\left[\frac{f(x_2)+4f(x_3)+f(x_4)}{6}\right] + \ldots$$

$$+ (x_{n-2} - x_{n-4})\left[\frac{f(x_{n-4})+4f(x_{n-3})+f(x_{n-2})}{6}\right] + (x_n - x_{n-2})\left[\frac{f(x_{n-2})+4f(x_{n-1})+f(x_n)}{6}\right]$$

Since

$$x_i - x_{i-2} = 2h$$
$$i = 2, 4, \ldots, n$$

then

$$\int_a^b f(x)dx \cong 2h\left[\frac{f(x_0)+4f(x_1)+f(x_2)}{6}\right] + 2h\left[\frac{f(x_2)+4f(x_3)+f(x_4)}{6}\right] + \ldots$$
$$+ 2h\left[\frac{f(x_{n-4})+4f(x_{n-3})+f(x_{n-2})}{6}\right] + 2h\left[\frac{f(x_{n-2})+4f(x_{n-1})+f(x_n)}{6}\right]$$

$$= \frac{h}{3}\left[f(x_0) + 4\{f(x_1)+f(x_3)+\ldots+f(x_{n-1})\} + 2\{f(x_2)+f(x_4)+\ldots+f(x_{n-2})\} + f(x_n)\right]$$

$$= \frac{h}{3}\left[f(x_0) + 4\sum_{\substack{i=1\\ i=odd}}^{n-1} f(x_i) + 2\sum_{\substack{i=2\\ i=even}}^{n-2} f(x_i) + f(x_n)\right]$$

$$\boxed{\int_a^b f(x)dx \cong \frac{b-a}{3n}\left[f(x_0) + 4\sum_{\substack{i=1\\ i=odd}}^{n-1} f(x_i) + 2\sum_{\substack{i=2\\ i=even}}^{n-2} f(x_i) + f(x_n)\right]}$$

Error in Multiple-segment Simpson's 1/3 rule

The true error in a single application of Simpson's 1/3rd Rule is given[1] by

$$E_t = -\frac{(b-a)^5}{2880} f^{(4)}(\zeta), \quad a < \zeta < b$$

In multiple-segment Simpson's 1/3 rule, the error is the sum of the errors in each application of Simpson's 1/3 rule. The error in the n segments Simpson's 1/3rd Rule is given by

$$E_1 = -\frac{(x_2 - x_0)^5}{2880} f^{(4)}(\zeta_1), \quad x_0 < \zeta_1 < x_2$$

$$= -\frac{h^5}{90} f^{(4)}(\zeta_1)$$

$$E_2 = -\frac{(x_4 - x_2)^5}{2880} f^{(4)}(\zeta_2), \quad x_2 < \zeta_2 < x_4$$

$$= -\frac{h^5}{90} f^{(4)}(\zeta_2)$$

$$\vdots$$

$$E_i = -\frac{(x_{2i} - x_{2(i-1)})^5}{2880} f^{(4)}(\zeta_i), \quad x_{2(i-1)} < \zeta_i < x_{2i}$$

$$= -\frac{h^5}{90} f^{(4)}(\zeta_i)$$

$$E_{\frac{n}{2}-1} = -\frac{(x_{n-2} - x_{n-4})^5}{2880} f^{(4)}\left(\zeta_{\frac{n}{2}-1}\right), \quad x_{n-4} < \zeta_{\frac{n}{2}-1} < x_{n-2}$$

$$= -\frac{h^5}{90} f^{(4)}\left(\zeta_{\frac{n}{2}-1}\right)$$

$$E_{\frac{n}{2}} = -\frac{(x_n - x_{n-2})^5}{2880} f^{(4)}\left(\zeta_{\frac{n}{2}}\right), x_{n-2} < \zeta_{\frac{n}{2}} < x_n$$

Hence, the total error in the multiple-segment Simpson's 1/3 rule is

$$= -\frac{h^5}{90} f^{(4)}\left(\zeta_{\frac{n}{2}}\right)$$

$$E_t = \sum_{i=1}^{\frac{n}{2}} E_i$$

$$= -\frac{h^5}{90} \sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)$$

$$= -\frac{(b-a)^5}{90n^5} \sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)$$

$$= -\frac{(b-a)^5}{90n^4} \frac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{n}$$

The term $\dfrac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{n}$ is an approximate average value of $f^{(4)}(x),\ a < x < b$. Hence

$$E_t = -\frac{(b-a)^5}{90n^4}\overline{f}^{(4)}$$

where

$$\overline{f}^{(4)} = \frac{\sum_{i=1}^{\frac{n}{2}} f^{(4)}(\zeta_i)}{n}$$

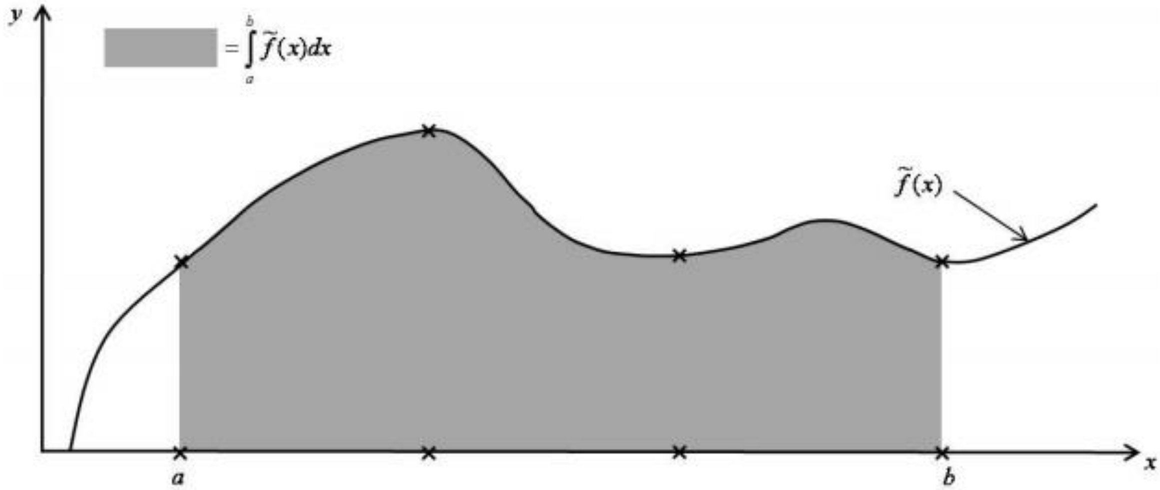


Figure 1 $\tilde{f}(x)$ Cubic function.

In a similar fashion, Simpson 3/8 rule for integration can be derived by approximating the given function $f(x)$ with the 3rd order (cubic) polynomial $f_3(x)$

$$\left.\begin{aligned} f_3(x) &= a_0 + a_1x + a_2x^2 + a_3x^3 \\ &= \{1, x, x^2, x^3\}\times\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} \end{aligned}\right\} \qquad (3)$$

which can also be symbolically represented in Figure 1.

Method 1

The unknown coefficients a_0, a_1, a_2 and a_3 in Equation (3) can be obtained by substituting 4 known coordinate data points $\{x_0, f(x_0)\}, \{x_1, f(x_1)\}, \{x_2, f(x_2)\}$ and $\{x_3, f(x_3)\}$ into Equation (3) as follows.

$$\left.\begin{aligned} f(x_0) &= a_0 + a_1 x_0 + a_2 x_0^2 + a_3 x_0^2 \\ f(x_1) &= a_0 + a_1 x_1 + a_2 x_1^2 + a_3 x_1^2 \\ f(x_2) &= a_0 + a_1 x_2 + a_2 x_2^2 + a_3 x_2^2 \\ f(x_3) &= a_0 + a_1 x_3 + a_2 x_3^2 + a_3 x_3^2 \end{aligned}\right\} \tag{4}$$

Equation (4) can be expressed in matrix notation as

$$\begin{bmatrix} 1 & x_0 & x_0^2 & x_0^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} f(x_0) \\ f(x_1) \\ f(x_2) \\ f(x_3) \end{bmatrix} \tag{5}$$

The above Equation (5) can symbolically be represented as

$$[A]_{4\times 4}\,\vec{a}_{4\times 1} = \vec{f}_{4\times 1} \tag{6}$$

Thus,

$$\vec{a} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} = [A]^{-1} \times \vec{f} \tag{7}$$

Substituting Equation (7) into Equation (3), one gets

$$f_3(x) = \{1, x, x^2, x^3\} \times [A]^{-1} \times \vec{f} \tag{8}$$

As indicated in Figure 1, one has

$$\left.\begin{aligned} x_0 &= a \\ x_1 &= a + h \\ &= a + \frac{b-a}{3} \\ &= \frac{2a+b}{3} \\ x_2 &= a + 2h \\ &= a + \frac{2b-2a}{3} \\ &= \frac{a+2b}{3} \\ x_3 &= a + 3h \\ &= a + \frac{3b-3a}{3} \\ &= b \end{aligned}\right\} \tag{9}$$

Simpsons 3/8 Rule for Integration

Substituting the form of $f_3(x)$ from Method (1) or Method (2),

$$I = \int_a^b f(x)dx$$

$$\approx \int_a^b f_3(x)dx$$

$$= (b-a) \times \frac{\{f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\}}{8} \tag{11}$$

Since

$$h = \frac{b-a}{3}$$

$$b - a = 3h$$

and Equation (11) becomes

$$I \approx \frac{3h}{8} \times \{f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3)\} \tag{12}$$

Note the 3/8 in the formula, and hence the name of method as the Simpson's 3/8 rule.
The true error in Simpson 3/8 rule can be derived as [Ref. 1]

$$E_t = -\frac{(b-a)^5}{6480} \times f''''(\zeta) \text{ , where } a \le \zeta \le b \tag{13}$$

Multiple Segments for Simpson 3/8 Rule

Using n = number of equal segments, the width h can be defined as

$$h = \frac{b-a}{n} \tag{14}$$

The number of segments need to be an integer multiple of 3 as a single application of Simpson 3/8 rule requires 3 segments.
The integral shown in Equation (1) can be expressed as

$$I = \int_a^b f(x)dx$$

$$\approx \int_a^b f_3(x)dx$$

$$\approx \int_{x_0=a}^{x_3} f_3(x)dx + \int_{x_3}^{x_6} f_3(x)dx + \dots\dots + \int_{x_{n-3}}^{x_n=b} f_3(x)dx \tag{15}$$

Using Simpson 3/8 rule (See Equation 12) into Equation (15), one gets

$$I = \frac{3h}{8}\left\{\begin{matrix} f(x_0) + 3f(x_1) + 3f(x_2) + f(x_3) + f(x_3) + 3f(x_4) + 3f(x_5) + f(x_6) \\ + \dots + f(x_{n-3}) + 3f(x_{n-2}) + 3f(x_{n-1}) + f(x_n) \end{matrix}\right\} \tag{16}$$

$$= \frac{3h}{8}\left\{ f(x_0) + 3\sum_{i=1,4,7,..}^{n-2} f(x_i) + 3\sum_{i=2,5,8,..}^{n-1} f(x_i) + 2\sum_{i=3,6,9,..}^{n-3} f(x_i) + f(x_n) \right\} \tag{17}$$

Comparing the truncated error of Simpson 1/3 rule

$$E_t = -\frac{(b-a)^5}{2880} \times f''''(\zeta) \qquad (18)$$

With Simpson 3/8 rule (See Equation 12), it seems to offer slightly more accurate answer than the former. However, the cost associated with Simpson 3/8 rule (using 3rd order polynomial function) is significantly higher than the one associated with Simpson 1/3 rule (using 2nd order polynomial function).

The number of multiple segments that can be used in the conjunction with Simpson 1/3 rule is 2, 4, 6, 8, ... (any even numbers) for

$$I = \int_a^b f(x)dx$$

$$\approx \left(\frac{h}{3}\right)\{f(x_0)+4f(x_1)+f(x_2)+f(x_2)+4f(x_3)+f(x_4)+.....+f(x_{n-2})+4f(x_{n-1})+f(x_n)\}$$

$$= \left(\frac{h}{3}\right)\left\{f(x_0)+4\sum_{i=1,3,...}^{n-1} f(x_i)+2\sum_{i=2,4,6...}^{n-2} f(x_i)+f(x_n)\right\} \qquad (19)$$

However, Simpson 3/8 rule can be used with the number of segments equal to 3,6,9,12,.. (can be certain integers that are multiples of 3).

If the user wishes to use, say 7 segments, then the mixed Simpson 1/3 rule (for the first 4 segments), and Simpson 3/8 rule (for the last 3 segments) would be appropriate.

Gauss Quadrature rule is another method of estimating an integral. The two point Gauss Quadrature Rule is an extension of the Trapezoidal Rule approximation where the integral estimate was based on taking the area under the straight line connecting the function values at the limits of the integration interval, *a* and *b*. However, unlike the Trapezoidal Rule approximation, the two point Gauss Quadrature rule is based on evaluating the area under a straight line connecting two points on the curve that are not predetermined as *a* and *b*, but as unknowns x_1 and x_2. Thus, in the two point Gauss Quadrature Rule, the integral is approximated as

$$I = \int_a^b f(x)dx$$

$$\approx c_1 f(x_1) + c_2 f(x_2)$$

There are now four unknowns that must be evaluated x_1, x_2, c_1 and c_2. These are found by assuming that the formula gives exact results for integrating a general third order polynomial, $f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$.

Hence, $\int_a^b f(x)dx = \int_a^b (a_0 + a_1 x + a_2 x^2 + a_3 x^3)dx$

$$= \left[a_0 x + a_1 \frac{x^2}{2} + a_2 \frac{x^3}{3} + a_3 \frac{x^4}{4}\right]_a^b$$

$$= a_0(b-a) + a_1\left(\frac{b^2-a^2}{2}\right) + a_2\left(\frac{b^3-a^3}{3}\right) + a_3\left(\frac{b^4-a^4}{4}\right)$$

The formula would then give

$$\int_a^b f(x)dx = c_1(a_0 + a_1x_1 + a_2x_1^2 + a_3x_1^3) + c_2(a_0 + a_1x_2 + a_2x_2^2 + a_3x_2^3)$$

Equating the last two equations gives

$$a_0(b-a) + a_1\left(\frac{b^2-a^2}{2}\right) + a_2\left(\frac{b^3-a^3}{3}\right) + a_3\left(\frac{b^4-a^4}{4}\right)$$

$$= c_1(a_0 + a_1x_1 + a_2x_1^2 + a_3x_1^3) + c_2(a_0 + a_1x_2 + a_2x_2^2 + a_3x_2^3)$$

$$= a_0(c_1 + c_2) + a_1(c_1x_1 + c_2x_2) + a_2(c_1x_1^2 + c_2x_2^2) + a_3(c_1x_1^3 + c_2x_2^3)$$

The constatnts a_0, a_1, a_2, and a_3 are arbitrary

$$b - a = c_1 + c_2$$

$$\frac{b^2-a^2}{2} = c_1x_1 + c_2x_2$$

$$\frac{b^3-a^3}{3} = c_1x_1^2 + c_2x_2^2$$

$$\frac{b^4-a^4}{4} = c_1x_1^3 + c_2x_2^3$$

Solving these equations simultaneously, we have

$$c_1 = \frac{b-a}{2}$$

$$c_2 = \frac{b-a}{2}$$

$$x_1 = \left(\frac{b-a}{2}\right)\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}$$

$$x_2 = \left(\frac{b-a}{2}\right)\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}$$

Hence, $\int_a^b f(x)dx \approx c_1 f(x_1) + c_2 f(x_2)$

$$= \frac{b-a}{2} f\left[\left(\frac{b-a}{2}\right)\left(-\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right] + \frac{b-a}{2} f\left[\left(\frac{b-a}{2}\right)\left(\frac{1}{\sqrt{3}}\right) + \frac{b+a}{2}\right]$$

This is called the two-point Gauss Quadrature Rule since two points were chosen. For n-points rules formula, it can be developed using the general form

$$I \cong c_0 f(x_0) + c_1 f(x_1) + \ldots + c_{n-1} f(x_{n-1})$$

Here, we will discuss the trapezoidal rule of approximating integrals of the form

$$I = \int_a^b f(x)dx$$

where

$f(x)$ is called the integrand,

$a =$ lower limit of integration

$b =$ upper limit of integration

What is the trapezoidal rule?

The trapezoidal rule is based on the Newton-Cotes formula that if one approximates the integrand by an n^{th} order polynomial, then the integral of the function is approximated by the integral of that n^{th} order polynomial. Integrating polynomials is simple and is based on the calculus formula.

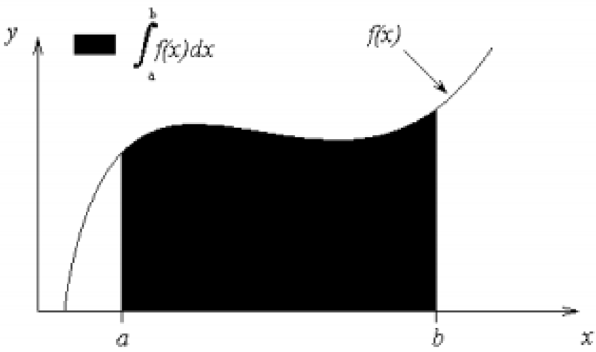


Figure 1 Integration of a function

$$\int_a^b x^n dx = \left(\frac{b^{n+1} - a^{n+1}}{n+1}\right), n \neq -1 \tag{1}$$

So if we want to approximate the integral

$$I = \int_a^b f(x)dx \tag{2}$$

to find the value of the above integral, one assumes

$$f(x) \approx f_n(x) \tag{3}$$

where

$$f_n(x) = a_0 + a_1 x + \ldots + a_{n-1}x^{n-1} + a_n x^n . \tag{4}$$

where $f_n(x)$ is a n^{th} order polynomial. The trapezoidal rule assumes $n = 1$, that is, approximating the integral by a linear polynomial (straight line),

$$\int_a^b f(x)dx \approx \int_a^b f_1(x)dx$$

Method 3: Derived from Geometry

The trapezoidal rule can also be derived from geometry. Look at Figure 2. The area under the curve $f_1(x)$ is the area of a trapezoid. The integral

$$\int_a^b f(x)dx \approx \text{Area of trapezoid}$$

$$= \frac{1}{2}(\text{Sum of length of parallel sides})(\text{Perpendicular distance between parallel sides})$$

$$= \frac{1}{2}(f(b)+f(a))(b-a)$$

$$= (b-a)\left[\frac{f(a)+f(b)}{2}\right] \tag{12}$$

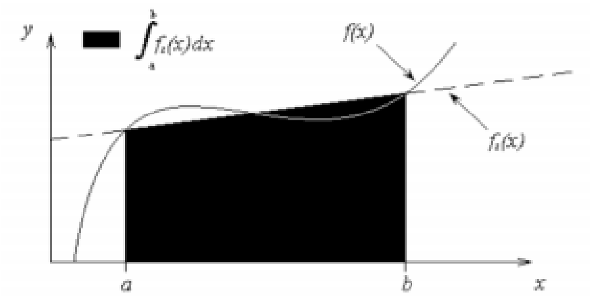


Figure 2 Geometric representation of trapezoidal rule.

Applying trapezoidal rule Equation (27) on each segment gives

$$\int_a^b f(x)dx = [(a+h)-a]\left[\frac{f(a)+f(a+h)}{2}\right]$$

$$+[(a+2h)-(a+h)]\left[\frac{f(a+h)+f(a+2h)}{2}\right]$$

$$+ \ldots\ldots\ldots\ldots + [(a+(n-1)h)-(a+(n-2)h)]\left[\frac{f(a+(n-2)h)+f(a+(n-1)h)}{2}\right]$$

$$+[b-(a+(n-1)h)]\left[\frac{f(a+(n-1)h)+f(b)}{2}\right]$$

$$= h\left[\frac{f(a)+f(a+h)}{2}\right] + h\left[\frac{f(a+h)+f(a+2h)}{2}\right] + \ldots\ldots\ldots\ldots$$

$$+ h\left[\frac{f(a+(n-2)h)+f(a+(n-1)h)}{2}\right] + h\left[\frac{f(a+(n-1)h)+f(b)}{2}\right]$$

$$= h\left[\frac{f(a)+2f(a+h)+2f(a+2h)+\ldots+2f(a+(n-1)h)+f(b)}{2}\right]$$

$$= \frac{h}{2}\left[f(a)+2\left\{\sum_{i=1}^{n-1} f(a+ih)\right\}+f(b)\right]$$

$$= \frac{b-a}{2n}\left[f(a)+2\left\{\sum_{i=1}^{n-1} f(a+ih)\right\}+f(b)\right] \tag{28}$$

Error in Multiple-segment Trapezoidal Rule

The true error for a single segment Trapezoidal rule is given by

$$E_t = -\frac{(b-a)^3}{12} f''(\zeta), \quad a < \zeta < b$$

Where ζ is some point in $[a,b]$.

What is the error then in the multiple-segment trapezoidal rule? It will be simply the sum of the errors from each segment, where the error in each segment is that of the single segment trapezoidal rule. The error in each segment is

$$E_1 = -\frac{[(a+h)-a]^3}{12} f''(\zeta_1), \quad a < \zeta_1 < a+h$$

$$= -\frac{h^3}{12} f''(\zeta_1)$$

$$E_2 = -\frac{[(a+2h)-(a+h)]^3}{12} f''(\zeta_2), \quad a+h < \zeta_2 < a+2h$$

$$= -\frac{h^3}{12} f''(\zeta_2)$$

.

$$E_{n-1} = -\frac{[\{a+(n-1)h\}-\{a+(n-2)h\}]^3}{12} f''(\zeta_{n-1}), \quad a+(n-2)h < \zeta_{n-1} < a+(n-1)h$$

$$= -\frac{h^3}{12} f''(\zeta_{n-1})$$

$$E_n = -\frac{[b-\{a+(n-1)h\}]^3}{12} f''(\zeta_n), \quad a+(n-1)h < \zeta_n < b$$

$$= -\frac{h^3}{12} f''(\zeta_n)$$

Hence the total error in the multiple-segment trapezoidal rule is

$$E_t = \sum_{i=1}^{n} E_i$$

$$= -\frac{h^3}{12} \sum_{i=1}^{n} f''(\zeta_i)$$

$$= -\frac{(b-a)^3}{12n^3} \sum_{i=1}^{n} f''(\zeta_i)$$

$$= -\frac{(b-a)^3}{12n^2} \frac{\sum_{i=1}^{n} f''(\zeta_i)}{n}$$

The term $\frac{\sum_{i=1}^{n} f''(\zeta_i)}{n}$ is an approximate average value of the second derivative $f''(x),\ a < x < b$.

Hence

$$E_t = -\frac{(b-a)^3}{12n^2} \frac{\sum_{i=1}^{n} f''(\zeta_i)}{n}$$

(*vi*) **The symbol $\Rightarrow$.** If p and q are two statements such that the truth of p implies that of q, we write

$p \Rightarrow q$ (one way implication).

It is read as '*p implies q*'. For example, $x=4 \Rightarrow x^2=16$

We shall assume that the statement $p \Rightarrow q$ is false only when p is true and q is false. In other words a true statement can imply only a true statement while a false statement can imply a true statement or a false statement.

(*vii*) **The symbol $\Leftrightarrow$.** It is used to stand for '*implies and is implied by*' or *if and only if*. Sometimes *iff* is also used to stand for '*if and only if*'. If the truth of the statement p implies that of q and also the truth of q implies that of p we write

$p \Leftrightarrow q$ (both way implication).

For example, $x+2=9 \Leftrightarrow x=7$.

The statement $p \Leftrightarrow q$ is true only when p and q are either both true or both false. It is false when one of the statements is true and the other is false.

(*viii*) **Negation of a Statement.** Associated with each statement is another statement called its **negation**. The negation of a statement p is denoted by '$-p$' or '$\sim p$'. For example, if p is the statement "x is 3", then $\sim p$ is the statement "x is not 3."

The negation of a true statement is false and the negation of a false statement is true.

Tautologies. A statement is called a *tautology* if it is always true. We shall give below some examples of tautologies.

Example 1. *The statement* $(p \wedge q) \Rightarrow p$ *is a tautology.*

We shall prepare the **truth table** for the statement $(p \wedge q) \Rightarrow p$.

p	q	$p \wedge q$	$p \wedge q \Rightarrow p$
T	T	T	T
T	F	F	T
F	T	F	T
F	F	F	T

We observe that the statement $(p \wedge q) \Rightarrow p$ has its truth value T for all its entries in the truth table. Thus it is always true. Hence it is a tautology.

Example 2. *The statement* $\sim(p \wedge q) \Leftrightarrow (\sim p \vee \sim q)$ *is a tautology.*

Truth table for $\sim(p \wedge q) \Leftrightarrow (\sim p \vee \sim q)$

p	q	$p \wedge q$	$\sim(p \wedge q)$	$\sim p$	$\sim q$	$\sim p \vee \sim q$
T	T	T	F	F	F	F
T	F	F	T	F	T	T
F	T	F	T	T	F	T
F	F	F	T	T	T	T

Since the corresponding entries under the columns $\sim(p \wedge q)$ and $\sim p \vee \sim q$ are identical, therefore the statements $\sim(p \wedge q) \Rightarrow (\sim p \vee \sim q)$ and $(\sim p \vee \sim q) \Rightarrow \sim(p \wedge q)$ are both tautologies. Hence the statement $\sim(p \wedge q) \Leftrightarrow (\sim p \vee \sim q)$ is a tautology.

Tautologies are very helpful to us whenever we are to deduce some new statement from some given statements. If the statement $p \Rightarrow q$ is a tautology we can safely conclude the truth of q from the truth of p. We shall now give a list of some important tautologies.

(*i*) $p \vee p \Leftrightarrow p$; $p \wedge p \Leftrightarrow p$. (Idempotent laws)

(*ii*) $p \wedge q \Rightarrow p$; $(p \wedge q) \Rightarrow q$. (Laws of simplification)

(*iii*) $p \Rightarrow (p \vee q)$; $q \Rightarrow (p \vee q)$. (Laws of addition)

(*iv*) $(p \vee q) \Leftrightarrow (q \vee p)$; $(p \wedge q) \Leftrightarrow (q \wedge p)$ (Commutative laws)

(*v*) $\left.\begin{array}{l} \sim(p \wedge q) \Leftrightarrow (\sim p \vee \sim q); \\ \sim(p \vee q) \Leftrightarrow (\sim p \wedge \sim q); \end{array}\right\}$ (De-Morgan's laws)

(*vi*) $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge$
$p \vee (q \vee r) \Leftrightarrow (p \vee q) \vee$ (Associative laws)

(*vii*) $\left.\begin{array}{l} p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee (p \wedge r); \\ p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge (p \vee r) \end{array}\right\}$ (Distributive laws)

(*viii*) $p \Leftrightarrow \sim(\sim p)$. (Law of double negation)

(*ix*) $p \vee \sim p$. (Law of the exclusive middle)

(*x*) $(p \Rightarrow q) \Leftrightarrow (\sim q \Rightarrow \sim p)$. (Law of the contrapositive)

(*xi*) $p \wedge (q \wedge r) \Leftrightarrow (p \wedge q) \wedge (p \wedge r)$.