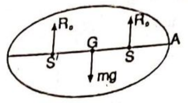
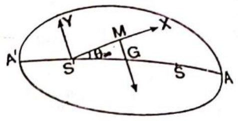




With the focus S' as the origin and horizontal and vertical lines through S' as the axes, the co-ordination of C.G. *i. e.* point G are

$$\left.\begin{aligned} x &= S'G\cos\theta = ae\cos\theta = ae, \\ y &= S'G\sin\theta = ae\sin\theta = ae\,\theta. \end{aligned}\right\} \text{ neglecting squares and higher powers of } \theta.$$

$\therefore \ddot{x} = 0;\ \ddot{y} = ae\,\ddot{\theta}$ for initial values.

Let X and Y be the horizontal and vertical components of the reaction at S'. Equations of motions at the lamina are

$$\left.\begin{aligned} m\ddot{x} &= X \\ m\ddot{y} &= mg - Y \end{aligned}\right\} \text{ these given motion of the C.G. 'G'}$$

and $m.\dfrac{a^2+b^2}{4}.\ddot{\theta} = X\,ae\sin\theta + X\,ae\cos\theta$

(taking moment about the centre of gravity G).

Equations of initial motion are obtained by taking $\sin\theta = \theta$, $\cos\theta = 1$ and putting initial values of $\ddot{x}$ and $\ddot{y}$ in the last three equations.

i. e. $\quad X = 0 \quad$ ...(1) $\qquad mae\,\ddot{\theta} = mg - Y \quad$ ...(2)

and $\quad \dfrac{a^2+b^2}{4}\ddot{\theta} = Y\,ae. \quad$ ...(3)

From (1), (2) and (3) on eliminating $\ddot{\theta}$, we get

$$\frac{4\,ae}{a^2+b^2} = \frac{mg-Y}{ae\,Y} \text{ or } 4\,a^2e^2\,Y = mg\,(a^2+b^2) - (a^2+b^2)\,Y$$

or $[4a^2e^2 + (a^2+b^2)]\,Y = (a^2+b^2)\,mg$ or $Y = \dfrac{a^2+b^2}{4a^2e^2+a^2+b^2}\,mg$

or $Y = \dfrac{1+1-e^2}{4e^2+1+1-e^2}\,mg \quad \left(\because \dfrac{b^2}{a^2} = 1-e^2\right)$ or $\quad Y = \dfrac{2-e^2}{2+3e^2}\,mg \quad$ ...(ii)

Since $X = 0$, the only reaction is Y. If the reaction remains unaltered, then from (i) and (ii), we have $\frac{1}{2}\,mg = \dfrac{2-e^2}{2+3e^2}\,mg$ or $2+3e^2 = 4-2e^2$

or $\quad 5e^2 = 2 \quad$ or $\quad e = \sqrt{\left(\dfrac{2}{5}\right)}$

☆☆☆

7

Lagrange's Equations of Motion Small Oscillations : Normal Co-Ordinates

7·01. Generalised Co-ordinates.

(Meerut 91)

Suppose that a particle or a system of N-particles moves subject to possible constraints, as for example a particle moving along a circular wire or a rigid body moving along an inclined plane, then there will be necessarily a minimum number of independent co-ordinates then needed to specify the motion. These co-ordinates denoted by $q_1, q_2, \ldots, q_n$ are called generalised co-ordinates. These co-ordinates may be distances, angles or quantities relating to them.

7·02. Degrees of freedom.

(Meerut 91)

The number of independent co-ordinates required to specify the position of a system of one or more particles is called the number of degrees of freedom of the system.

Ex. 1. *A particle moving freely in space require 3 co-ordinates, e.g. (x, y, z), to specify its position. Thus the number of degrees of freedom is 3.*

Ex. 2. *A system containing of N-particles moving freely in space require 3N co-ordinates to specify the position. The number of degrees of freedom is 3N.*

A rigid body which can move freely in space has 6 degrees of freedom *i.e.* 6 co-ordinates are required to specify the position.

Let 3 non-collinear points of a rigid body be fixed in space, then the rigid body also fixed in space. Let these points have co-ordinates (x_1, y_1, z_1); (x_2, y_2, z_2); (x_3, y_3, z_3) respectively, a total of 9. Since the body is rigid, we must have

$$(x_1-x_2)^2 + (y_1-y_2)^2 + (z_1-z_2)^2 = \text{constant.}$$
$$(x_2-x_3)^2 + (y_2-y_3)^2 + (z_2-z_3)^2 = \text{constant.}$$
$$(x_3-x_1)^2 + (y_3-y_1)^2 + (z_3-z_1)^2 = \text{constant.}$$

Hence 3 co-ordinates can be expressed in terms of the remaining six. Thus six independent co-ordinates are needed to describe the motion *i.e* there exit six degrees of freedom.

7·03. Transformation equation.

Let $r_\nu = x_\nu\ i + y_\nu\ j + z_\nu\ k$ be the position vector of ν-th particle with respect to xyz co-ordinates system. The relationships of the genaralised co-ordinates $q_1, q_2, ..., q_n$ the position co-ordinates are given by the *transformation equations.*

$$\left.\begin{aligned} x_\nu &= x_\nu\ (q_1, q_2, ..., q_n\ ; t) \\ y_\nu &= y_\nu\ (q_1, q_2, ..., q_n\ ; t) \\ z_\nu &= z_\nu\ (q_1, q_2, ..., q_n\ ; t) \end{aligned}\right\} \quad ...(1)$$

where t denotes the time. In vector (1) can be written as

$$r_\nu = r_\nu\ (q_1, q_2, ... q_n\ ; t) \quad ....(2)$$

where the functions in (1) or (2) are continuous and have continuous derivatives.

7·04. Classification of Mechanical systems.

(a) Scleronomic system.

The mechanical system in which t, the time, does not enter explicity in equations (1) *or* (2) *is called a scleronomic system.*

(b) Rheonomic system.

The mechanical system in which the moving constraints are involved and the time t does enter explicity is called a Rheonomic system.

(Meerut 1989)

(c) Holonomic system and Non Holonomic system.

Let $q_1, q_2, ..., q_n$ denote the generalised co-ordinates describing a system and let t denote the time. If all the constraints of the system can be expressed as equations having the fom $(q_1, q_2, ..., q_n\ ; t) = 0$ or their equivalent, then the system is said to be Holonomic otherwise it is be Non-Holonomic system.

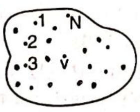
(d) Conservative and non-conservative system.

If the forces acting on the system are derivable from a potential function [or potential energy] V, then the system is called conservative, otherwise it is non-conservative.

7·05. Kinetic energy and generalised velocities.

(T)

The K.E of the system is $T = \frac{1}{2} \sum_{\nu=1}^{n} m_\nu\ \dot{r}_\nu^2$.



The K.E of the system can be written as a quadratic fom in the generalised co-ordinates q_α. If the system is independent of time explicitly i.e. Scleronomic then the quadratic fom has only terms of the type $a_{\alpha\beta}\ q_\alpha q_\beta$. In case the system is Rheonomic, linear terms in q_α are also present.

7·06. Generalised Forces.

(Meerut 1989)

If W is the total work done on a system of particles by forces F_ν acting on the ν-th particle, then

Potential Energy, $V = f(q)$, $T = f(\dot{q})$

$$dW = \sum_{\alpha=1}^{n} \phi_\alpha\ dq_\alpha \text{ where } \phi_\alpha = \sum_{\nu=1}^{N} F_\nu \bullet \frac{\partial r_\nu}{\partial q_\alpha}$$

is called the **generalised force** *associated with generalised co-ordinates* q_α.

Suppose that a system undergoes increments $dq_1, dq_2, ..., dq_n$ of the generalised co-ordinates $q_1, q_2, ..., q_n$, then the ν-th particle undergoes a displacement.

$$dr_\nu = \sum_{\alpha=1}^{n} \frac{\partial r_\nu}{\partial q_\alpha} dq_\alpha \quad ...(4)$$

$\therefore$ Total work done is given by

$$dW = \sum_{\nu=1}^{N} F_\nu \bullet dr_\nu = \sum_{\nu=1}^{N} \left\{ \sum_{\alpha=1}^{n} F_\nu \bullet \frac{\partial r_\nu}{\partial q_\alpha} \right\} dq_\alpha \quad ...(5)$$

Now, let $\phi_\alpha = \sum_{\nu=1}^{N} F_\nu \bullet \frac{\partial r_\nu}{\partial q_\alpha}$

then (5) $dW = \sum_{\alpha=1}^{N} \left(\sum_{\nu=1}^{N} F_\nu \bullet \frac{\partial r_\nu}{\partial q_\alpha} \right) d\, q = \sum_{\alpha=1}^{n} \phi_\alpha d\, q$ ...(6)

We have $dW = \sum_{\alpha=1}^{N} \frac{\partial W}{\partial q_\alpha} dq_\alpha$, $\therefore \frac{\partial W}{\partial q_\alpha} = \phi_\alpha$ ...(7)

Note. **(i)** α varies from (1) to n, the number of degrees of freedom.

(ii) ν aries from 1 to N, the number of particles in the system.

7·07. Lagrange's equations.

(Meerut 1993, 95)

Let F be the net external force acting on the Vth particle of a system, then by Newton"s second law applied to Vth particle, we have

$$m_\nu \ddot{r}_\nu = F_\nu$$

$$\Rightarrow m_\nu \ddot{r}_\nu \bullet \frac{\partial r_\nu}{\partial q_\alpha} = F_\nu \bullet \frac{\partial r_\nu}{\partial q_\alpha} \quad ...(8)$$

$$\Rightarrow \sum_{\nu=1}^{N} m_\nu \ddot{r}_\nu \bullet \frac{\partial r_\nu}{\partial q_\alpha} = \sum_{\nu=1}^{N} F_\nu \bullet \frac{\partial r_\nu}{\partial q_\alpha} \quad ...(9)$$

$$\Rightarrow \frac{d}{dt}\left[\sum_{\nu=1}^{N} m_\nu\ \dot{r}_\nu \bullet \frac{\partial r_\nu}{\partial q_\alpha} \right] - \sum_{\nu=1}^{N} m_\nu \dot{r}_\nu \bullet \frac{d}{dt}\left(\frac{\partial r_\nu}{\partial q_\alpha} \right) = \sum_{\nu=1}^{N} F_\nu \bullet \frac{\partial r_\nu}{\partial q_\alpha}$$

But $r_\nu = r_\nu\ (q_1, q_2, ..., q_n\ ; t)$ ...(10)

$$\therefore \dot{r}_\nu = \frac{\partial r_\nu}{\partial q_1} \dot{q}_1 + \frac{\partial r_\nu}{\partial q_2} \dot{q}_2 + ... + \frac{\partial r_\nu}{\partial q_n} \dot{q}_n + \frac{\partial r_\nu}{\partial t} \quad ...(11)$$

$$\Rightarrow \frac{\partial \dot{r}_\nu}{\partial \dot{q}_\alpha} = \frac{\partial r_\nu}{\partial q_\alpha} \quad \textbf{[Cancellation law of the dots]} \quad ...(12)$$

Also, $\frac{\partial}{\partial q_\alpha}(\dot{r}_\nu) = \frac{\partial}{\partial q_\alpha}\left(\frac{\partial r_\nu}{\partial q_1}\dot{q}_1 + \frac{\partial r_\nu}{\partial q_2}\dot{q}_2 + \dots + \frac{\partial r_\nu}{\partial q_n}\dot{q}_n + \frac{\partial r_\nu}{\partial t}\right)$

$$= \frac{\partial^2 r_\nu}{\partial q_\alpha \partial q_1}\dot{q}_1 + \frac{\partial^2 r_\nu}{\partial q_\alpha \partial q_1}\dot{q}_2 + \dots + \frac{\partial^2 r_\nu}{\partial q_\alpha \partial q_n}\dot{q}_n + \frac{\partial}{\partial q_\alpha}\left(\frac{\partial r_\nu}{\partial t}\right)$$

$$= \frac{\partial}{\partial q_1}\left(\frac{\partial r_\nu}{\partial q_\alpha}\right)\dot{q}_1 + \frac{\partial}{\partial q_2}\left(\frac{\partial r_\nu}{\partial q_\alpha}\right)\dot{q}_2 + \dots + \frac{\partial}{\partial q_\alpha}\left(\frac{\partial r_\nu}{\partial q_n}\right)\dot{q}_n + \frac{\partial}{\partial t}\left(\frac{\partial r_\nu}{\partial q_\alpha}\right)$$

or $\frac{\partial}{\partial q_\alpha}\left(\frac{dr_\nu}{dt}\right) = \frac{d}{dt}\left(\frac{\partial r_\nu}{\partial q_\alpha}\right) \Rightarrow \frac{d}{dt}\left(\frac{\partial}{\partial q_\alpha}\right) \equiv \frac{\partial}{\partial q_\alpha}\left(\frac{d}{dt}\right)$ ...(13)

[interchange law of the order of operators]

Now $\frac{d}{dt}\left\{\sum_{\nu=1}^{N} m_\nu \dot{r}_\nu \cdot \frac{\partial r_\nu}{\partial q_\alpha}\right\} - \sum_{\nu=1}^{N} m_\nu \dot{r}_\nu \cdot \frac{\partial \dot{r}_\nu}{\partial q_\alpha} = \sum_{\nu=1}^{N} F_\nu \cdot \frac{\partial r_\nu}{\partial q_\alpha}$ ...(14)

and $T = \frac{1}{2}\sum_\nu m_\nu \dot{r}_\nu^2 = \frac{1}{2}\sum_\nu m_\nu (\dot{r}_\nu \cdot \dot{r}_\nu)$ ...(15)

$\Rightarrow \frac{\partial T}{\partial q_\alpha} = \sum_\nu m_\nu \dot{r}_\nu \cdot \frac{\partial \dot{r}_\nu}{\partial q_\alpha}$ ...(16)

and $\frac{\partial T}{\partial \dot{q}_\alpha} = \sum_\nu m_\nu \dot{r}_\nu \cdot \frac{\partial \dot{r}_\nu}{\partial \dot{q}_\alpha} = \sum m_\nu \dot{r}_\nu \cdot \frac{\partial r_\nu}{\partial q_\alpha}$ [using (12)] ...(17)

$\therefore$ (14) $\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = \sum_\nu F_\nu \cdot \frac{\partial r_\nu}{\partial q_\alpha}$

or $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = \phi_\alpha = \frac{\partial W}{\partial q_\alpha}$ using (7) ...(18)

Note. The quantity $p_\alpha = \frac{\partial T}{\partial \dot{q}_\alpha}$ is called the generalised momentum associated with the general co-ordinates q_α.

7·08. Lagrangian function.

If the forces are derivable from a potential function V, then

$$\phi_\alpha = \frac{\partial W}{\partial q_\alpha} = -\frac{\partial V}{\partial q_\alpha},$$

since the potential, or potential energy is a function of q's only (and possibly the name t) then, we have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = -\frac{\partial V}{\partial q_\alpha} \Rightarrow \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_\alpha}(T - V)\right] - \left(\frac{\partial T}{\partial q_\alpha} - \frac{\partial V}{\partial q_\alpha}\right) = 0$$

$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = 0$, where $L = T - V$ ...(19)

The function L defined by $L = T - V$ is said to be Lagrangian function.

7·09. Generalised momentum.

We defined $p_\alpha = \frac{\partial T}{\partial \dot{q}_\alpha}$ to be the generalised momentum associated with generalised co-ordinates q_α or the conjugate momentum.

In case the system is conservative, we have

$$T = L + V \Rightarrow (\partial T/\partial \dot{q}_\alpha) = (\partial L/\partial \dot{q}_\alpha) + (\partial V/\partial \dot{q}_\alpha) = (\partial L/\partial \dot{q}_\alpha)$$

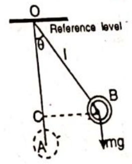
because V, the P. E. of the system does not depend upon $\dot{q}_\alpha$

$\therefore p_\alpha = (\partial L/\partial \dot{q}_\alpha)$.

7·10. Illustrative Examples.

Ex. 1. (i) *Set up the Lagrangian for a simple pendulum, and* (ii) *obtain an equation describing its motion.*

Sol. (i) Choose as generalised coordinates, the angle θ made by the string OB of the pendulum and the vertical OA. Let l be the length of OA, then K. E. is given by



$$T = \frac{1}{2}mv^2 = \frac{1}{2}m(l\dot{\theta})^2 = \frac{1}{2}ml^2\dot{\theta}^2$$

where m is the mass of the bob.

The potential energy of mass m is given by

$$V = mg(OA - OC) = mg(l - l\cos\theta) = mgl(1 - \cos\theta)$$

$$\therefore L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos\theta)$$

(ii) Hence Lagrange's θ equation gives

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \Rightarrow \frac{d}{dt}(ml^2\dot{\theta}) - (-mgl\sin\theta) = 0$$

$$\Rightarrow l\ddot{\theta} = -g\sin\theta \Rightarrow \ddot{\theta} = -\frac{g}{l}\sin\theta,$$

which is the required equation of motion.

7·11. Kinetic energy as a Quadratic function of velocities.

If at time t, the position of the ν^{th} particle (mass m), of a holonomic system is defined by r_ν, then K. E. is given by

$T = \frac{1}{2}\sum_{\nu=1}^{N} m_\nu \dot{r}_\nu^2$, where $r_\nu = r_\nu(q_1 \ldots\ldots q_n; t)$ ...(1)

so that $\dot{r}_\nu = \dot{q}_1 \frac{\partial r_\nu}{\partial q_1} + \dot{q}_2 \frac{\partial r_\nu}{\partial q_2} + \ldots + \dot{q}_n \frac{\partial r_\nu}{\partial q_n} + \frac{\partial r_\nu}{\partial t}$

$$\Rightarrow T = \frac{1}{2}\sum_{\nu=1}^{N} m_\nu \left(\dot{q}_1 \frac{\partial r_\nu}{\partial q_1} + \dot{q}_2 \frac{\partial r_\nu}{\partial q_2} + \ldots + \dot{q}_n \frac{\partial r_\nu}{\partial q_n} + \frac{\partial r_\nu}{\partial t} \right)^2$$

$$= \frac{1}{2}[(a_{11}\dot{q}_1^2 + a_{22}\dot{q}_2^2 + \ldots + a_{nn}\dot{q}_n^2 + 2a_{12}\dot{q}_1\dot{q}_2 + 2a_{1n}\dot{q}_1\dot{q}_n + \ldots)$$

$$+ 2(a_1\dot{q}_1 + a_2\dot{q}_2 + . + a_n\dot{q}_n) + a] \quad \ldots(2)$$

where $5a_{rs} = \sum_{\nu=1}^{N} m_\nu (\partial r_\nu / \partial q_r) \cdot (\partial r_\nu / \partial q_s) \ (s \geq r)$

$$a_{rr} = \sum_{\nu=1}^{N} m_\nu (\partial r_\nu / \partial q_r)^2, a = \sum_{\nu=1}^{N} m_\nu (\partial r_\nu / \partial t)^2, a_r = \sum_{\nu=1}^{N} m_\nu \left(\frac{\partial r_\nu}{\partial q_r} \right) \cdot \left(\frac{\partial r_\nu}{\partial t} \right)$$

From (2), we see that T is a quadratic function of the generalised velocities.

The case t is not explicity involved, is of considerable importance. Hence we have $\frac{\partial r_\nu}{\partial t} = 0$ and therefore (2) implies that

$$T = \frac{1}{2}(a_{11}\dot{q}_1^2 + a_{22}\dot{q}_2^2 + \ldots + a_{nn}\dot{q}_n^2 + 2a_{12}\dot{q}_1\dot{q}_2 + \ldots) \quad \ldots(3)$$

$$= \frac{1}{2}\sum_{s=1}^{n}\sum_{r=1}^{n} a_{rs}\dot{q}_r\dot{q}_s \text{ where } a_{rs} = a_{sr}. \quad \ldots(4)$$

Now using Euler's theorem for homogeneous functions, we get

$$\dot{q}_1 \frac{\partial T}{\partial \dot{q}_1} + \dot{q}_2 \frac{\partial T}{\partial \dot{q}_2} + \ldots + \dot{q}_n \frac{\partial T}{\partial \dot{q}_n} = 2T$$

$$\Rightarrow 2T = \sum_{\alpha=1}^{n} \dot{q}_\alpha \frac{\partial T}{\partial \dot{q}_\alpha} = \sum_{\alpha=1}^{n} p_\alpha \dot{q}_\alpha$$

i.e $2T = p_1\dot{q}_1 + p_1\dot{q}_2 + \ldots + p_n\dot{q}_n$.

7·12. To deduce the principle of energy from The Lagrange's equations (Conservative field).

Lagrange's equations are :

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = -\frac{\partial V}{\partial q_\alpha}; \quad (\alpha = 1, 2, \ldots n) \quad \ldots(1)$$

Also by 7·11. equation we know that

$$T = \frac{1}{2}(a_{11}\dot{q}_1^2 + a_{22}\dot{q}_2^2 + \ldots + a_m\dot{q}_n^2 + 2a_{12}\dot{q}_1\dot{q}_2 + \ldots),$$

that is, T can be expressed as a quadratic expression in generalised velocities. Hence applying Euler's theorem. we get

$$\sum_{\alpha=1}^{n} \dot{q}_\alpha \frac{\partial T}{\partial \dot{q}_\alpha} = 2T \quad \ldots(2)$$

Also, $\frac{dT}{dt} = \sum_{\alpha=1}^{n} \frac{\partial T}{\partial q_\alpha}\dot{q}_\alpha + \sum_{\alpha=1}^{n} \frac{\partial T}{\partial \dot{q}_\alpha}\ddot{q}_\alpha$ ...(3)

Now multiplying the n equations of (1) by $\dot{q}_1, \dot{q}_2, \ldots, \dot{q}_n$ respectively and then adding. we get

$$\left\{\dot{q}_1 \frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_1}\right] + \ldots + \dot{q}_n \frac{d}{dt}\left[\frac{\partial T}{\partial \dot{q}_1}\right]\right\} - \left\{\dot{q}_1 \frac{\partial T}{\partial q_1} + \ldots + \dot{q}_n \frac{\partial T}{\partial q_n}\right\}$$

$$= -\left\{\dot{q}_1 \frac{\partial V}{\partial q_1} + \ldots + \dot{q}_n \frac{\partial V}{\partial q_n}\right\}$$

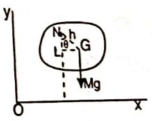
$$\Rightarrow \frac{d}{dt}\left\{\sum_{\alpha=1}^{n} \dot{q}_\alpha \frac{\partial T}{\partial \dot{q}_\alpha}\right\} - \left(\sum_{\alpha=1}^{n} \ddot{q}_\alpha \frac{\partial T}{\partial \dot{q}_\alpha}\right) - \left\{\sum_{\alpha=1}^{n} \dot{q}_n \frac{\partial T}{\partial q_\alpha}\right\} = -\left(\sum_{\alpha=1}^{n} \dot{q}_\alpha \frac{\partial V}{\partial q_\alpha}\right)$$

$$\Rightarrow \frac{d}{dt}(2T) - \frac{dT}{dt} = -\frac{dV}{dt} \Rightarrow \frac{dT}{dt} + \frac{dV}{dt} = 0 \Rightarrow \frac{d}{dt}(T + V) = 0 \Rightarrow T + V =$$

constant.

Ex. 2. *Use Lagrange's equations to find the differantial equation for a compound pendulum which oscillates in a vertical plane about a fixed horizontal axis.*

Sol. Let the plane of oscillation be represented by xy-plane, where N is its intersection with the axis of rotation and G is the centre of gravity. Let the mass of the pendulum be M and let its moment of interia about the axis of rotation be MK^2. Then potential energy relative to the horizontal plane through N is $V = -Mgh\cos\theta$.



Also $T = \frac{1}{2}Mk^2\dot{\theta}^2 \quad \therefore L = T - V = \frac{1}{2}Mk^2\dot{\theta}^2 + Mgh\cos\theta$ ...(1)

$\Rightarrow \frac{\partial L}{\partial \dot{\theta}} = Mk^2\dot{\theta}$ and $\frac{\partial L}{\partial \theta} = -Mgh\sin\theta$ ...(2)

Now Lagrange's θ equation gives

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \Rightarrow \frac{d}{dt}(Mk^2\dot{\theta}) + Mgh\sin\theta = 0$$

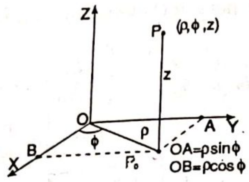
i.e. $Mk^2\ddot{\theta} + Mgh\sin\theta = 0 \Rightarrow \ddot{\theta} = -\frac{gh}{k^2}\sin\theta$

When θ is small, we have $D^2\theta = -\frac{gh}{k^2}\theta$ $\quad (\because \sin\theta = \theta)$

or $\left(D^2+\frac{gh}{k^2}\right)\theta=0.$

This is the differential equation of the pendulum.

Ex. 3. *A particle of mass m moves in a conservative force field. Find* **(a)** *the Lagrangian function,* **(b)** *the equations of motion in cylindrical co-ordinates* (ρ, ϕ, z). **[Meerut 95]**



Sol. we have $OP=OP_0+P_0P=OA+AP_0+P_0P=\vec{\rho}$ (say)

$\therefore \vec{\rho}=\rho\sin\phi\, j+\rho\cos\phi\, i+zk$ where i, j, k are the unit vector along OX, OY and OZ respectively.

Hence the unit vector along the direction of ρ increasing is

given by $\vec{\rho_1}=\frac{\partial\vec{\rho}}{\partial\rho}\Big/\left|\frac{\partial\vec{\rho}}{\partial\rho}\right|=\sin\phi\, j+\cos\phi\, i$

Similarly $\vec{\phi_1}=\frac{\partial\vec{\rho}}{\partial\phi}\Big/\left|\frac{\partial\vec{\rho}}{\partial\phi}\right|$

$=\frac{\rho\cos\phi\, j-\rho\sin\phi\, i}{\rho}=-\sin\phi\, i+\cos\phi\, j$

Now $v=\frac{d\vec{\rho}}{dt}=\frac{d}{dt}(\rho\sin\phi\, j+\rho\cos\phi\, i+zk)$

$=\rho\cos\phi\dot{\phi}\, j+\dot{\rho}\sin\phi\, j-\rho\sin\phi\dot{\phi}\, i+\dot{\rho}\cos\phi\, i+k\dot{z}$

$=\dot{\rho}\cos\phi\, i+\dot{\rho}\sin\phi\, j+\rho\dot{\phi}\,(\cos\phi\, j-\sin\phi\, i)+\dot{z}k$

$=\dot{\rho}\vec{\rho_1}+\rho\dot{\phi}\vec{\phi_1}+\dot{z}k$

$\therefore T=\frac{1}{2}m\,\{\dot{\rho}^2+\rho^2\dot{\phi}^2+\dot{z}^2\}$ and $V=V(\rho,\phi,z)$

(a) Hence the Lagrangian function is

$$L=T-V=\frac{1}{2}m\,[\dot{\rho}^2+\rho^2\dot{\phi}^2+\dot{z}^2]-V(\rho,\phi,z)$$

(b) Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\rho}}\right)-\frac{\partial L}{\partial\rho}=0 \text{ i.e. } \frac{d}{dt}(m\dot{\rho})-\left(m\rho\dot{\phi}^2\frac{\partial V}{\partial\rho}\right)=0$$

i.e. $m\,(\ddot{\rho}-\rho\dot{\phi}^2)=-\frac{\partial V}{\partial\rho}$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right)-\frac{\partial L}{\partial\phi}=0 \text{ i.e. } \frac{d}{dt}(m\rho^2\dot{\phi})+\frac{\partial V}{\partial\phi}=0, \qquad ...(1)$$

or $\frac{d}{dt}\left(\rho^2\dot{\phi}\right)=-\frac{\partial V}{\partial\phi}$

and $\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{z}}\right)-\frac{\partial L}{\partial z}=0$ *i.e.* $\frac{d}{dt}(m\dot{z})+\frac{\partial V}{\partial z}=0$ *or* $m\ddot{z}=-\frac{\partial V}{\partial z}$ ...(2) ...(3)

Ex. 4. *A particle Q moves on a smooth horizontal circular wire of radius a which is free to rotate about a vertical axis through a point O, distance c from the centre C. If the* $\angle QCO=\theta$, *show that*

$$a\ddot{\theta}+\dot{\omega}\,(a-c\cos\theta)=c\,\omega^2\sin\theta.$$

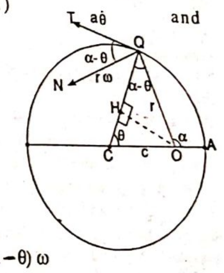
where ω *is the angular velocity of the wire.*

[Meerut 1995,1993,84,86; Garhwal 1982]

Sol. Let $OQ=r$, and $\angle AOQ=\alpha$

$\Rightarrow r^2=a^2+c^2-2ac\cos\theta$ ...(1)

$r\cos(\alpha-\theta)=a-c\cos\theta.$ ...(2) and



The particle Q moves on circle of radius a, **so its velocity along the tangent QT will be $a\dot{\theta}$ but Q revolves about O with angular velocity ω, which causes a velocity $a\omega$ at the right angles to OQ.**

$\Rightarrow v_Q^2=$ (velocity)^2 of the particle at Q

$$=a^2\dot{\theta}^2+r^2\omega^2+2ar\dot{\theta}\cos(\alpha-\theta)\,\omega$$

$\angle AOQ=\alpha$, $\angle CQO=\alpha-\theta$

Now $T=\frac{1}{2}m\,v_Q^{\,2}=\frac{1}{2}m\,[a^2\dot{\theta}^2+r^2\omega^2+2a\,r\omega\dot{\theta}\cos(\alpha-\theta)]$

$\left[\begin{array}{l}\angle NQT=\alpha-\theta, OQ=r\\ HQ=a-c\cos\theta\end{array}\right.$

NOTE :- *If **r** is the position vector of the particle at any time t, then $\partial r/\partial r$ is the vector tangent to the curve $\theta=$ constant i.e. a vector in the direction of **r** (increasing r). A unit vector in this direction is thus given by $r_1=\frac{\partial r}{\partial r}\Big/\left|\frac{\partial r}{\partial r}\right|$.

Similerly, $\partial r/\partial\theta$ is the vector tangant to the curve r=constant, A unit vector in this direction is given by $\vec{\theta_1}=\frac{\partial r}{\partial\theta}\Big/\left|\frac{\partial r}{\partial\theta}\right|$.

$= \frac{1}{2} m \, [a^2 \dot{\theta}^2 + (a^2 + c^2$ $\qquad = r \cos(\alpha - \theta)]$

$- 2ac \cos\theta) \, \omega^2 + 2a\omega \, \dot{\theta} \, (a - c \cos\theta)$

and work function $= 0$ $\quad (\because$ weight does no work)

$\therefore$ Lagrange's θ equation $\Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = 0$.

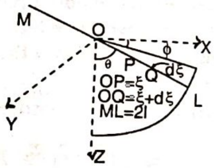
$$\Rightarrow \frac{d}{dt} [a^2 \dot{\theta} + a\omega (a - c\cos\theta)] - ac \, \omega^2 \sin\theta - ac \, \omega \dot{\theta} \sin\theta = 0$$

$$\Rightarrow a^2 \ddot{\theta} + a\dot{\omega}(a - c\cos\theta) + a\omega c \dot{\theta} \sin\theta - ac \, \omega^2 \sin\theta - ac \, \omega \dot{\theta} \sin\theta = 0$$

$$\Rightarrow a\ddot{\theta} + \dot{\omega}(a - c\cos\theta) = c \, \omega^2 \sin\theta .$$

Ex. 5. *A uniform rod, of mass 3m and length 2l, has its middle point fixed and a mass m attached at one extremity. The rod when in horizontal position is set rotating about a vertical axis through its centre with an angular velocity equal to* $\sqrt{\left(\frac{2ng}{l}\right)}$ *show that the heavy end of the rod will fall till the inclination of the rod to the vertical is* $\cos^{-1} [\sqrt{(n^2 + 1)} - n]$ *and will then rise again.* **[Meerut 91,93,95; Agra 1992]**

Sol. The mass m is attached at L. On the rod ML, take a point p such that $OP = \xi$, the element $PQ = d\xi$.



Further at any time t, let the plane through it and the vertical have turned through nn angle ϕ from its initial position and let the rod be inclined at an angle θ to the vertical. Taking O, the mid point of the rod, as the origin and OX, OY (a line perpendicular to the plane of the paper) and OZ as axes of refrence, then co-ordinates of the point P on the rod are:

$$x = \xi \sin\theta \cos\phi, \; y = \xi \sin\theta \sin\phi, \; z = \xi \cos\theta$$

$$\therefore \; \dot{x} = \xi \cos\theta \cos\phi \, \dot{\theta} - \xi \sin\phi \, \dot{\phi} \sin\theta,$$

$$\dot{y} = \xi \cos\theta \sin\phi \, \dot{\theta} + \xi \sin\theta \cos\phi \, \dot{\phi}, \; \dot{z} = -\xi \sin\theta \, \dot{\theta}. \text{ Thus,}$$

$$v_\rho^2 = (\text{velocity})^2 \text{ of } P = \dot{x}^2 + \dot{y}^2 + \dot{z}^2 = \xi^2 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) .$$

$$\therefore \; v_L^2 = l^2 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) = (\text{velocity})^2 \text{ of mass } m ,$$

Now mass of the element $PQ = \frac{3m}{2l} d\xi = dm$, say.

$\therefore$ Its kinetic energy

$$= \frac{1}{2} dm . v_\rho^2 = \frac{1}{2} \cdot \frac{3m}{2l} d\xi \, (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) \, \xi^2$$

$$= \frac{3m}{4l} (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) \, \xi^2 \, d\xi$$

and K.E. of the rod $= \frac{3m}{4l} (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) \int_{-l}^{l} \xi^2 \, d\xi$

$$= \frac{1}{2} m (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) \, l^2 .$$

Again, $(\text{velocity})^2$ of the particle $m = l^2 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta)$.

$\therefore$ Kinetic energy of the particle of mass $m = \frac{1}{2} m l^2 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta)$.

$\therefore$ Total K.E. $= T =$ K.E. of the rod + K.E. of the particle

$$= \frac{1}{2} m l^2 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) + \frac{1}{2} m l^2 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) \qquad \textit{i.e.}$$

$$T = m l^2 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) .$$

Also the work function is given by $W = mgl \cos\theta + C$.

Lagrange's ϕ-equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} = \frac{\partial W}{\partial \phi}$

which gives $\frac{d}{dt} (2ml^2 \dot{\phi} \sin^2\theta) = 0$.

Integrating it, we get $\quad \dot{\phi} \sin^2\theta = K$ (constant).

Initially, $\theta = \frac{\pi}{2}$ and $\dot{\phi} = \sqrt{\left(\frac{2ng}{l}\right)}$. $\therefore K = \sqrt{\left(\frac{2ng}{l}\right)}$.

$$\therefore \; \dot{\phi} \sin^2\theta = \sqrt{\left(\frac{2ng}{l}\right)} . \qquad \text{...(1)}$$

and Lagrange's θ-equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = \frac{\partial W}{\partial \theta}$

i.e. $\quad \frac{d}{dt} (2ml^2 \dot{\theta}) - 2ml^2 \dot{\phi}^2 \sin\theta \cos\theta = -mgl \sin\theta$

$$\text{or } \; 2l\ddot{\theta} - 2l\dot{\phi}^2 \sin\theta \cos\theta = -g \sin\theta . \qquad \text{...(2)}$$

Substituting value of $\dot{\phi}$ from (1) in (2), we have

$$2l\ddot{\theta} - 4ng \cot\theta \operatorname{cosec}^2\theta = -g \sin\theta . \qquad \text{...(2')}$$

Integration provides us $\quad 2l\dot{\theta}^2 + 4ng \cot^2\theta = 2g\cos\theta + k$.

Initially $\theta = \frac{\pi}{2}, \dot{\theta} = 0, \; \therefore k = 0$.

$\therefore\ 2l\dot{\theta}^2 + 4ng\cot^2\theta = 2g\cos\theta$. ...(3)

The rod will fall till $\dot{\theta} = 0$.

i.e. $4ng\cot^2\theta = 2g\cos\theta$ or $2n\cos^2\theta - \cos\theta\sin^2\theta = 0$.

$\therefore$ either $\cos\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$ which gives initial position.

as $2n\cos\theta - \sin^2\theta = 0 \Rightarrow \cos^2\theta + 2n\cos\theta - 1 = 0$.

Solving it, $\cos\theta = \dfrac{-2n \pm \sqrt{(4n^2+4)}}{2} = \left\{-n + \sqrt{(n^2+1)}\right\}$

[the other value being inadmissible because θ can not be obtuse]

or $\theta = \cos^{-1}[-n + \sqrt{(n^2+1)}]$. This proves the required result. If we substitute this value of θ in equation (2′), then we find that $\ddot{\theta}$ comes out to be positive. Hence at that time the rod begins to rise.

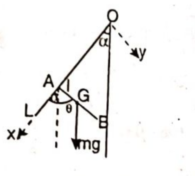
Ex. 6. *A bead, of mass M, slides on a smooth fixed wire, whose inclination to the vertical is α, and has hinged to it a rod, of mass m and length 2l, which can move freely in the vertical plane through the wire. If the system starts from rest with the rod hanging vertically, show that*

$$\{4M + m(1 + 3\cos^2\theta)\}\, l\dot{\theta}^2 = 6(M+m)\, g\sin\alpha(\sin\theta - \sin\alpha)$$

where θ is the angle between the rod and the lower part of the wire.

(Meerut 95, Agra 1993, 91)

Sol. Let OL be the fixed wire. At any time t, let the bead of mass M be at A where $OA = x$, also let θ be the angle which the rod AB makes with the lower part of the fixed wire.



Take O as origin and the fixed wire OL as x axis and a line through O and prep. to OL as y axis; the co-ordinates of G, the C.G. of the rod AB, are $\{x + l\cos\theta,\ l\sin\theta\}$

i.e. $x_G = (x + l\cos\theta)$ and $y_G = l\sin\theta$.

$$\dot{x}_G = (\dot{x} - l\sin\theta\,\dot{\theta})\ ; \qquad \dot{y}_G = l\cos\theta\,\dot{\theta}$$

$\therefore$ (velocity)^2 of $G = v_G^2 = \dot{x}_G^2 + \dot{y}_G^2 = (\dot{x} - l\sin\theta\,\dot{\theta})^2 + (l\cos\theta\,\dot{\theta})^2$.

Now let T be the kinetic energy and W the work function of the system. Then we easily get

Total energy $= T = K.E.$ of the bead $+ K.E.$ of the rod

$$= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\left[\frac{l^2}{3}\dot{\theta}^2 + (\dot{x} - l\sin\theta\,\dot{\theta})^2 + (l\cos\theta\,\dot{\theta})^2\right]$$

$$= \frac{1}{2}(M+m)\dot{x}^2 - ml\dot{x}\,\dot{\theta}\sin\theta + \frac{2}{3}ml^2\dot{\theta}^2.$$

Also, the work function is given by

$$W = Mgx\cos\alpha + mg\{x\cos\alpha + l\cos(\theta-\alpha)\}$$
$$= (M+m)gx\cos\alpha + mgl\cos(\theta-\alpha)$$

$\therefore$ Lagrange's x-equation gives, $\dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{x}}\right) - \dfrac{\partial T}{\partial x} = \dfrac{\partial W}{\partial x}$

i.e. $\dfrac{d}{dt}[(M+m)\dot{x} - ml\dot{\theta}\sin\theta] = (M+m)g\cos\alpha$

or $(M+m)\ddot{x} - ml\ddot{\theta}\sin\theta - ml\dot{\theta}^2\cos\theta = (M+m)g\cos\alpha$. ...(i)

Again Lagrang's θ-equation gives

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = \frac{\partial W}{\partial \theta}$$

i.e $\dfrac{d}{dt}[-ml\dot{x}\sin\theta + \frac{4}{3}ml^2\dot{\theta}] + ml\dot{x}\dot{\theta}\cos\theta$

$= -mgl\sin(\theta-\alpha)$,

or $-\ddot{x}ml\sin\theta - \dot{x}\dot{\theta}ml\cos\theta + \frac{4}{3}ml^2\ddot{\theta} + \dot{x}\dot{\theta}ml\cos\theta$

$= -mgl\sin(\theta-\alpha)$.

or $-\ddot{x}\sin\theta + \frac{4}{3}l\ddot{\theta} = -g\sin(\theta-\alpha)$ ...(ii)

Eliminating $\ddot{x}$ between (i) and (ii), we get

$$\ddot{\theta}[-ml\sin^2\theta + \tfrac{4}{3}(M+m)l] - ml\dot{\theta}^2\sin\theta\cos\theta$$
$$= (M+m)g[\cos\alpha\sin\theta - \sin(\theta-\alpha)]$$

or $l\ddot{\theta}[3M + m + 3m\cos^2\theta] - 3ml\dot{\theta}^2\sin\theta\cos\theta$

$= 3(M+m)g\cos\theta\sin\alpha$.

Whence on integrating, we get

$$l\dot{\theta}^2[4M + m + 3m\cos^2\theta] = 6(M+m)g\sin\alpha\sin\theta + C \qquad ...(iii)$$

When $\theta = \alpha$, $\dot{\theta} = 0$, $\therefore C = -6(M+m)g\sin^2\alpha$.

Putting the value of C in (iii), we get

$$l\dot{\theta}^2(4M + m + 3m\cos^2\theta) = 6(M+m)g\sin\alpha(\sin\theta - \sin\alpha).$$

7·13. Small Oscillations. *To explain how Lagrange's equations are used in case of small oscillations.* **[Meerut 1990]**

(when there exist three generalised co-ordinates).

In order to investigate the theory of small oscillations by the use of Lagrange's equation about the position of equilibrium, the generalised co-ordinates q_1, q_2, q_3 must be chosen in such a way that they vanish in the position of equilibrium.

But the system makes small oscillations about the position of equilibrium, so q_1, q_2, q_3 and their differentials $\dot{q}_1, \dot{q}_2, \dot{q}_3$ will remain small during the whole motion. Hence we shall reject all the powers of small quantites except the lowest one.

Now, let T be kinetic energy, and W the work-function of the system; then we have

$$T = A_{11}\dot{q}_1^2 + A_{22}\dot{q}_2^2 + A_{33}\dot{q}_3^2 + 2A_{12}\dot{q}_2\dot{q}_1 + 2A_{23}\dot{q}_2\dot{q}_3$$
$$+ 2A_{13}\dot{q}_1\dot{q}_3 \text{ [by 7.11]} \quad \text{...(1)}$$
$$W = C + B_1 q_1 + B_2 q_2 + B_3 q_3 + B_{11} q_1^2 + B_{22} q_2^2 + B_{33} q_3^2 \quad \text{...(2)}$$

Now assume that q_1, q_2, q_3 can be expressed in terms of X, Y, Z by the equations of the form

$$q_1 = \lambda_1 X + \lambda_2 Y + \lambda_3 Z; \quad q_2 = \mu_1 X + \mu_2 Y + \mu_3 Z;$$
$$q_3 = \nu_1 X + \nu_2 Y + \nu_3 Z;$$

and further assume $\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3, \nu_1, \nu_2, \nu_3$ in such a way that if we put value of q_1, q_2, q_3, in (1) and (2), then T does not involve $\dot{Y}\dot{Z}, \dot{Z}\dot{X}, \dot{X}\dot{Y}$ and W does not involve YZ, ZX, XY. Then X, Y, Z, are called the **normal or Principal Co-ordinates.**

Thus equation (1) and (2) $\Rightarrow$

$$T = A'_{11}\dot{X}^2 + A'_{22}\dot{Y}^2 + A'_{33}\dot{Z}^2$$
$$W = C' + B'_1 X + B'_2 Y + B'_3 Z + B'_{11} X^2 + B'_{22} Y^2 + B'_{33} Z^2$$

Now, using Lagrange's equations,

$$viz. \; \frac{d}{dt}\left(\frac{dT}{d\dot{X}}\right) - \left(\frac{dT}{dX}\right) = \left(\frac{dW}{dX}\right) \text{ etc., we obtain}$$

$$2A'_{11}\ddot{X} = B'_1 + 2B'_{11}X; \quad 2A'_{22}\ddot{Y} = B'_2 + 2B'_{22}Y;$$

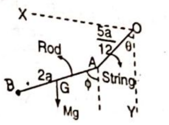
$$2A'_{33}\ddot{Z} = B'_3 + 2B'_{33}Z,$$

which can be put in the forms

$$\ddot{X} = -\omega_1^2 X, \; \ddot{Y} = -\omega_3^2 Y. \; \ddot{Z} = -\omega_3^2 Z \text{ etc.}$$

which represent simple Harmonic Motions, Or in other words they give small oscillations about the position of equilibrium.

Ex. 7. *A uniform rod, of length 2a, which has one end attached to a fixed point by a light inextensible string, of length $\frac{5}{12}a$, performing small oscillations in a vertical plane about its position of equillibrium. Find the position at any time, and show that the period of its principal oscallations are $2\pi\sqrt{\left(\frac{5a}{3g}\right)}$ and $\pi\sqrt{\left(\frac{a}{3g}\right)}$.*



Sol. Figure is self explanatory. At any time t, let the string and the rod be inclined at θ and ϕ to the vertical OY.

Co-ordinates of G are given by

$$x_G = \frac{5}{12}a \sin\theta + a\sin\phi,$$
$$y_G = \frac{5}{12}a\cos\theta + a\cos\phi.$$
$$\dot{x}_G = \frac{5a}{12}a\cos\theta\dot{\theta} + a\cos\phi\,\dot{\phi}.$$
$$\dot{y}_G = -\left(\frac{5a}{12}\sin\theta\dot{\theta} + a\sin\phi\,\dot{\phi}\right)$$

$$\therefore \dot{x}_G^2 + \dot{y}_G^2 = (\text{velocity})^2 \text{ of G}$$
$$= \frac{25a^2}{144}\dot{\theta}^2 + a^2\dot{\phi}^2 + \frac{5a^2}{6}\dot{\theta}\dot{\phi}\cos(\theta+\phi) = \frac{25a^2}{144}\dot{\theta}^2 + a^2\dot{\phi}^2 + \frac{5}{6}a^2\dot{\theta}\dot{\phi}$$

[$\because$ θ and ϕ are small so $\cos(\theta+\phi) = 1$]

Again let T, be the kinetic energy and W, the work function of the system, then we easily get

$$T = \frac{1}{2}m\left[\frac{a^2}{3}\dot{\phi}^2 + \left(\frac{25}{144}a^2\dot{\theta}^2 + a^2\dot{\phi}^2 + \frac{5}{6}a^2\dot{\theta}\dot{\phi}\right)\right]$$
$$= \frac{ma^2}{288}[25\dot{\theta}^2 + 192\dot{\phi}^2 + 120\dot{\theta}\dot{\phi}]$$

and $W = mg\left[\frac{5}{12}a\cos\theta + a\cos\phi\right]$

$\therefore$ Lagrange's θ-equation gives

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right) - \frac{\partial T}{\partial\theta} = \frac{\partial W}{\partial\theta} \Rightarrow \frac{d}{dt}\left\{\frac{ma^2}{144}(25\dot{\theta} + 60\dot{\phi})\right\} = -\frac{5mga}{12}\theta$$

{ $\sin\theta = \theta$ as θ is small }

$$\Rightarrow 5\ddot{\theta} + 12\ddot{\phi} = -\frac{12g}{a}\theta. \quad \text{...(1)}$$

and Lagrange's ϕ-equation gives

$$\frac{d}{dt}\left\{\frac{ma^2}{144}(192\dot{\phi} + 60\dot{\theta})\right\} = -mga\phi \Rightarrow 5\ddot{\theta} + 16\ddot{\phi} = -12\frac{g}{a} \quad \text{...(2)}$$

Equations (1) and (2) $\Rightarrow (5D^2 + 12c)\theta + 12D^2\phi = 0$ ...(3)

and $5D^2\theta + (6D^2 + 12c)\phi = 0$ where $(g/a) = c$. ...(4)

Now eliminating ϕ between these two equations, we get

$$[(5D^2 + 12c)(16D^2 + 12c) - 60D^4]\theta = 0$$

or $(5D^4 + 63cD^2 + 36c^2)\,\theta = 0$. ...(5)

Let $\theta = A\cos(pt+B)$ $\therefore$ $D\theta = -pA\sin(pt+B)$,

$D^2\theta = -p^2A\cos(pt+B) = -p^2\theta$ and $D^4\theta = p^4\theta$.

Substituting these values in (5), we get

$(5p^4 - 63c^2 - 36c^2)\,\theta = 0 \Rightarrow (5p^4 - 63cp^2 + 36c^2) = 0$ $(\because\ \theta \neq 0)$

$\Rightarrow (5p^2 - 3c)(p^2 - 12c) = 0 \Rightarrow \left(5p^2 - \dfrac{3g}{a}\right)\left(p^2 - \dfrac{12g}{a}\right) = 0$

$\therefore\ p_1^2 = \dfrac{3g}{5a}$ and $p_2^2 = \dfrac{12g}{a}$

The periods of oscillations are $\dfrac{2\pi}{p_1}$ and $\dfrac{2\pi}{p_2}$

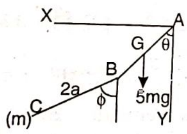
i.e. $2\pi\sqrt{\left(\dfrac{5a}{3g}\right)}$ and $2\pi\sqrt{\left(\dfrac{a}{12g}\right)}$ i.e. $2\pi\sqrt{\left(\dfrac{5a}{3g}\right)}$ and $\pi\sqrt{\left(\dfrac{a}{3g}\right)}$.

Ex. 8. *A uniform rod, of mass 5m and length 2a, turns freely about one end which is fixed, to its other extremity is attached one end of a light string of length 2a, which carries at its other end a particle of mass m, show that the periods of the small oscillations in a vertical plane are the same as those of simple pendulums of length* $\dfrac{2a}{3}$ *and* $\dfrac{20a}{7}$.

[Meerut 74, 81]

Sol. *Let the string BC and the rod AB make angles ϕ and θ with the vertical at any time t. The particle of mass m is tied to the end C of the string.*

Now $x_c = 2a\sin\theta + 2a\sin\phi$,



$\dot{x}_c = 2a(\cos\theta\,\dot{\theta} + \cos\phi\,\dot{\phi})$

$y_c = 2a\cos\theta + 2a\cos\phi$.

$\dot{y}_c = -2a(\sin\theta\,\dot{\theta} + \sin\phi\,\dot{\phi})$

$\therefore$ (velocity)^2 of $m = \dot{x}_c^2 + \dot{y}_c^2$

$= 4a^2(\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi})$.

Again co-ordinates of G are $(a\sin\theta, a\cos\theta)$.

$\therefore$ (velocity)^2 of $G = a^2\dot{\theta}^2$

Now let T be the kinetic energy, and W the work function of the system, then we have

Total K.E. = K.E. of rod + K.E. of particle of mass m.

$T = \frac{1}{2}\,5m\left(\dfrac{a^2}{3}\dot{\theta}^2 + a^2\dot{\theta}^2\right) + \frac{1}{2}m\,.\,4a^2(\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi})$

$= ma^2\left(\frac{16}{3}\dot{\theta}^2 + 2\dot{\phi}^2 + 4\dot{\theta}\dot{\phi}\right)$

and $W = 5mga\cos\theta + mg\,.\,2a(\cos\theta + \cos\phi)$

$= 7mga\cos\theta + 2mga\cos\phi = 7mag\left(1 - \dfrac{\theta^2}{2}\right) + 2mag\left(1 - \dfrac{\phi^2}{2}\right)$

$\therefore$ Lagrange's θ equation is given by

$\dfrac{d}{dt}\left(\dfrac{\partial T}{\partial\dot{\theta}}\right) - \dfrac{\partial T}{\partial\theta} = \dfrac{\partial W}{\partial\theta}$

$\Rightarrow \dfrac{d}{dt}\left(\dfrac{32}{3}\dot{\theta} + 4\dot{\phi}\right) = -\dfrac{7g}{a}\theta \Rightarrow 32\ddot{\theta} + 12\ddot{\phi} = -21\dfrac{g}{a}\theta$ ...(1)

Lagrange's ϕ equation is given by

$\dfrac{d}{dt}(4\dot{\phi} + 4\dot{\theta}) = -\dfrac{2g}{a}\phi \Rightarrow$ i.e. $2\ddot{\theta} + 2\ddot{\phi} = -\dfrac{g}{a}\phi$ ...(2)

$\therefore$ (1) and (2) $\Rightarrow (32D^2 + 21c)\,\theta + 12D^2\phi = 0$

and $2D^2\theta + (2D^2 + c)\,\phi = 0$ where $\dfrac{g}{a} = c$. ...(3)

Now eliminating 'ϕ' between (3) and (4), we get

$[(32D^2 + 21c)\,.\,(2D^2 + c) - 24D^2]\,\theta = 0$

or $[40D^4 + 74cD^2 + 21c^2]\,\theta = 0$

Now let $\theta = A\cos(pt+B) \Rightarrow D\theta = -pA\sin(pt+B)$. ...(5)

$D^2\theta = -p^2A\cos(pt+B) = -p^2\theta$ and $D^4\theta = p^4\theta$.

Substituting these in (5), we get

$(40p^4 - 74cp^2 + 21c^2)\,\theta = 0$ i.e. $40p^4 - 74cp^2 + 21c^2 = 0$ as $\theta \neq 0$

or $(2p^2 - 2c)(20p^2 - 7c) = 0$

i.e. $\left(2p^2 - \dfrac{3g}{a}\right)\left(20p^2 - 7\dfrac{g}{a}\right) = 0$. $\Rightarrow p_1^2 = \dfrac{3g}{2a}$ and $p_2^2 = \dfrac{7g}{20a}$.

*Hence length of equivalent pendulums are

$\dfrac{g}{p_1^2}$ and $\dfrac{g}{p_2^2}$ i.e. $\dfrac{2a}{3}$ and $\dfrac{20}{7}a$.

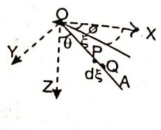
Ex. 9. *A uniform rod, of length 2a, can turn freely about one end, which is fixed. Initially it is inclind at an angle α, to the down-ward drawn vertical and it is set rotating about a vertical axis through its fixed end with angular velocity ω. Show that, during the motion, the rod is always inclined to the vertical at an angle which is > or < α according as ω^2 > or <* $\dfrac{3}{4a\cos\alpha}$ *and that in each case its motion is inclined between the inclination α and*

$$\cos^{-1}[-n + \sqrt{(1 - 2n\cos\alpha + n^2)}], \text{ when } n = \frac{a\omega^2\sin^2\alpha}{3g}$$

If it be slightly disturbed when revolving steadily at a constant angle α, show that the time of a small oscillation is

$$2\pi\sqrt{\left[\frac{4a\cos\alpha}{3g(1 + 3\cos^2\alpha)}\right]}$$

Sol. The rod OA is turning about the end O. Take a point P on the rod such that $OP = \xi$, and the element $PQ = d\xi$.



$\therefore$ mass of element $PQ = \frac{m}{2a} d\xi$,

where m is the mass of the rod

Further at any time t, let the rod be inclined at an angle θ to the vertical and let the plane through the rod and the vertical have turned through angle ϕ from its initial position OX, then co-ordinates of the point P are $x_p = \xi \sin\theta \cos\phi$, $y_p = \xi \sin\theta \sin\phi$, $z = \xi \cos\theta$.

$\therefore v_P^2 =$ (velocity)^2 of $P = \dot{x}_P^2 + \dot{y}_P^2 + \dot{z}_P^2 = \xi^2 (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta)$.

and kinetic energy of the element $PQ = \frac{1}{2}\frac{m}{2a} v_P^2$

$$= \frac{1}{2}\frac{m}{2a} d\xi\, (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta)\, \xi^2$$

Now, let T, *be the K.E. of the rod OA*, then we have

$$T = \frac{1}{2}\frac{m}{2a} (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta) \int_0^{2a} \xi^2\, d\xi = \frac{2ma^2}{3} (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta)$$

or $T = \frac{2ma^2}{3} (\dot{\theta}^2 + \dot{\phi}^2 \sin^2\theta)$

Also the work function, $W = mga \cos\theta + C$.

Lagrange's ϕ-equation gives

$$\frac{d}{dt}\left(\frac{4ma^2}{3} \dot{\phi} \sin^2\theta\right) = 0 \text{ i.e. } \frac{d}{dt} [\dot{\phi} \sin^2\theta] = 0 \quad \text{...(1)}$$

$$\Rightarrow \dot{\phi} \sin^2\theta = K \text{ (constant)}, \quad \text{...(2)}$$

Initially $\theta = \alpha$, $\dot{\phi} = \omega$, $\quad \therefore K = \omega \sin^2\alpha$.

Thus (2) gives $\dot{\phi} \sin^2\theta = \omega \sin^2\alpha$, ...(3)

and Lagrange's θ-equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = 0$.

*When $\theta = A\,(pt + B)$, the period of motion is given by $T = \frac{2\pi}{p}$. If l is the length of the simple equivalent pendulum, we have

$T = 2\pi \sqrt{(l/g)} \Rightarrow l = \frac{g}{p^2}$

$$\Rightarrow \frac{d}{dt}\left(\frac{4ma^2}{3}\dot{\theta}\right) - \frac{2ma^2}{3}\dot{\phi}^2 .2 \sin\theta \cos\theta = -mga \sin\theta.$$

$$\Rightarrow \ddot{\theta} - \dot{\phi}^2 \sin\theta \cos\theta = -\frac{g}{4a} \sin\theta. \quad \text{...(4)}$$

Eliminating $\dot{\phi}$ between (4) and (3), we have

$$\ddot{\theta} - \frac{\omega^2 \sin^4\alpha}{\sin^3\theta} \cos\dot{\theta} = -\frac{3g}{4a} \sin\theta. \quad \text{...(5)}$$

$$\Rightarrow \dot{\theta}^2 + \frac{\omega^2 \sin^4\alpha}{\sin^3\theta} = \frac{3g}{4a} \cos\theta + A. \quad \text{...(6)}$$

Initially $\theta = \alpha$, $\dot{\theta} = 0$, $\quad \therefore A = \omega^2 \sin^2\alpha - \frac{3g}{2a} \cos\alpha$.

Substituting this value of A in (6), we get

$$\dot{\theta}^2 + \frac{\omega^2 \sin^4\alpha}{\sin^2\theta} = \frac{3g}{2a} \cos\theta + \omega^2 \sin^2\alpha - \frac{3g}{2a} \cos\alpha,$$

or $\dot{\theta}^2 = \omega^2 \sin^2\alpha \left(1 - \frac{\sin^2\alpha}{\sin^2\theta}\right) + \frac{3g}{2a} (\cos\theta - \cos\alpha)$

$$= \frac{3ng}{a}\left(1 - \frac{\sin^2\alpha}{\sin^2\theta}\right) + \frac{3g}{2a} (\cos\theta - \cos\alpha) \left[\because n = \frac{a\omega^2 \sin^2\alpha}{3g}\right]$$

$$= \frac{3g}{2a} \cdot \frac{\cos\alpha - \cos\theta}{\sin^2\theta} [2n (\cos\alpha + \cos\theta) - \sin^2\theta]$$

i.e. $\dot{\theta}^2 = \frac{3g}{2a} \cdot \frac{\cos\alpha - \cos\theta}{\sin^2\theta} [(\cos^2\theta + 2n \cos\theta + 2n \cos\alpha - 1)]$ ...(7)

From (7), we see that $\dot{\theta} = 0$, when

$$(\cos\alpha - \cos\theta) [(\cos^2\theta + 2n \cos\theta + 2n \cos\alpha - 1)] = 0$$

i.e. if either $\cos\alpha - \cos\theta$ i.e. $\theta = \alpha$ (the initial position).

or $\cos^2\theta + 2n \cos\theta + 2n \cos\alpha - 1 = 0$

i.e. $\cos\theta = \frac{-2n \pm \sqrt{[4n^2 + 4(1 - 2\cos\alpha)]}}{2}$

or $\cos\theta = -n + \sqrt{(1 - 3n \cos\alpha + n^2)}$ ...(8)

(the other value being inadmissible because that gives value of $\cos\theta$ numerically greater than unity.

Hence the motion is included between $\theta = \alpha$ and $\theta = \theta_1$ where

$$\cos\theta_1 = \{\sqrt{(1 - 2n \cos\alpha + n^2)} - n\}$$

The rod will move above or below its initial position, if $\theta_1 >$ or $< \alpha$

or if $\cos\theta_1 <$ or $> \cos\alpha$

i.e. if $1 - 2n \cos\alpha + n^2 <$ or $> (n + \cos\alpha)^2$

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i.e. if $\frac{3ng}{a\omega^2} <$ or $> 4n \cos\alpha$ i.e. if $\omega^2 >$ or $< \frac{3g}{4a\cos\alpha}$.

2nd part.

Small oscillations about the steady motion :— The motion will be steady if the rod goes round, inclined at the same angle α with the vertical or mathematically if $\theta = \alpha$ (throughout the motion), then $\ddot{\theta} = 0$.

Making these substitutions in (5), we get,

$$-\frac{\omega^2 \sin^4\alpha}{\sin^3\theta}\cos\theta = -\frac{3g}{4a}\sin\theta \text{ i.e. } \omega^2 = \frac{3g}{4a\cos\alpha}$$

When ω^2 has this value and there are small oscillations about the position $\theta = \alpha$, then putting $\theta = \alpha + \psi$ in equation (5) we get

$$\ddot{\psi} = \frac{3g}{4a\cos\alpha}\frac{\sin^4\alpha}{\sin^3(\alpha+\psi)}\cos(\alpha+\psi) - \frac{3g}{4a}\sin(\alpha+\psi)$$

$$= \frac{3g}{4a}\left[\frac{\sin^4\alpha(\cos\alpha\cos\psi - \sin\alpha\sin\psi)}{\cos\alpha(\sin\alpha\cos\psi + \cos\alpha\sin\psi)^3} - (\sin\alpha\cos\psi + \cos\alpha\sin\psi)\right]$$

$$= \frac{3g}{4a}\left[\frac{\sin^4\alpha(\cos\alpha - \psi\sin\alpha)}{\cos\alpha(\sin\alpha + \psi\cos\alpha)^3} - (\sin\alpha + \psi\cos\alpha)\right], \text{ approximately}$$

$$= -\frac{3g\sin\alpha}{4a}[(1-\psi\tan\alpha)(1+\psi\cot\alpha)^{-3} - (1+\psi\cot\alpha)], \text{ approx.}$$

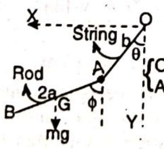
$$= -\frac{3g\sin\alpha}{4}(4\cot\alpha + \tan\alpha)\psi, \text{ app.} = \frac{-3g(1+3\cos^2\alpha)}{4a\cos\alpha}\psi = -\mu\psi \text{ say}$$

$\therefore$ time of small oscillation $= \frac{2\pi}{\sqrt{\mu}} = 2\pi\sqrt{\left(\frac{4a\cos\alpha}{3g(1+3\cos^2\alpha)}\right)}$.

Ex. 10. *A uniform bar of length 2a is hung from a fixed point by a string of length b fastened to one end of the bar. Show that when the system makes small normal oscillations in a vertical plane, the length l of the equivalent simple pendulum is a root of the quadratic,*

$$l^2 - \left(\frac{4}{3}a + b\right)l + \frac{ab}{3} = 0.$$

Sol. Figure is self explanatory.



At any time t, let the string OA and the rod AB make angles θ and ϕ with the vertical.

$x_G = b\sin\theta + a\sin\phi$.

$y_G = b\cos\theta + a\cos\phi$.

$\therefore \dot{x}_G^2 + \dot{y}_G^2 =$ (velocity)^2 of G

$$= b^2\dot{\theta}^2 + a^2\dot{\phi}^2 + 2ab\,\dot{\theta}\dot{\phi}\cos(\theta - \phi)$$

$$= b^2\dot{\theta}^2 + a^2\dot{\phi}^2 + 2ab\,\dot{\theta}\dot{\phi} \qquad [\because \theta \text{ and } \phi \text{ are small}]$$

Now let T be the kinetic energy and W the work function of the system, then we easily obtain

$W = mg[b\cos\theta + a\cos\phi]$

and $T = \frac{1}{2}m\left[\frac{a^2}{3}\dot{\phi}^2 + b^2\dot{\theta}^2 + a^2\dot{\phi}^2 + 2ab\,\dot{\theta}\dot{\phi}\right]$

$$= \frac{1}{2}m\left[\frac{4a^2}{3}\dot{\phi}^2 + b^2\dot{\theta}^2 + 2ab\,\dot{\theta}\dot{\phi}\right]$$

$\therefore$ Lagrange's θ-equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right) - \frac{\partial T}{\partial\theta} = \frac{\partial W}{\partial\theta}$

$\Rightarrow \frac{d}{dt}\{m(b^2\dot{\theta} + ab\dot{\phi})\} = -mgb\theta. \qquad \{\because \sin\theta = 0\}$

$\Rightarrow b\ddot{\theta} + a\ddot{\phi} = -g\theta$ ...(1)

Lagrange's ϕ-equation is given by $\frac{d}{dt}\left\{m\left(\frac{4}{3}a^2\dot{\phi} + ab\dot{\theta}\right)\right\} = -mag\phi$

$\Rightarrow 4a\ddot{\phi} + 3b\ddot{\theta} = -3g\phi$. ...(2)

Equations (1) and (2) again can be written as

$(bD^2 + g)\,\theta + aD^2\,\phi = 0$ ...(3)

and $3bD^2\theta + (4aD^2 + 3g)\,\phi = 0$. ...(4)

Eliminating ϕ between these equations, we obtain

$[(bD^2 + g)(4aD^2 + 3g) - 3abD^4]\theta = 0$

i.e. $[abD^4 + (4a + 3b)gD^2 + 3g^2]\theta = 0$. ...(5)

Now let $\theta = A\cos\left[\sqrt{\left(\frac{g}{l}\right)}t + B\right]$

where l is the length of the simple equivalent pendulum.

Then $D\theta = -\sqrt{\left(\frac{g}{l}\right)}A\sin\left[\sqrt{\left(\frac{g}{l}\right)}t + B\right]$

$D^2\theta = -\frac{g}{l}A\cos\left[\sqrt{\left(\frac{g}{l}\right)}t + B\right] = -\frac{g}{l}\theta$ and $D^4\theta = \frac{g^2}{l^2}\theta$,

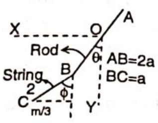
$\therefore (5) \Rightarrow \left[ab\frac{g^2}{l^2} - (4a + 3b)\frac{g^2}{l} + 3g^2\right]\theta = 0$

$\Rightarrow 3l^2 - (4a + 3b)\,l + ab = 0 \qquad \{\because \theta \neq 0\}$

$\Rightarrow l^2 - \left(\frac{4}{3}a + b\right)l + \frac{ab}{3} = 0$.

Ex. 11. *A uniform straight rod of length 2a, is freely movable about its centre and a particle of mass one-third that of the rod is attached by a light inextensible string, of length a, to one end of the rod ; show that one period of principle oscillation is* $(\sqrt{5} + 1)\pi\sqrt{\left(\frac{a}{g}\right)}$.

Sol. Figure is self explanatory.
At time t, let θ and ϕ be the inclinations of the rod and the string to the vertical.



Co-ordinates of C are
$x_C = a \sin\theta + a \sin\phi$ and
$y_C = a \cos\theta + a \cos\phi$.

$\therefore \dot{x}_C = a \cos\theta\,\dot\theta + a \cos\phi\,\dot\phi$. and $\dot{y}_C = -a \sin\theta\,\dot\theta - a \sin\phi\,\dot\phi$

$\Rightarrow \dot{x}_C^2 + \dot{y}_C^2 = a^2\dot\theta^2 + a^2\dot\phi^2 + 2a^2\cos(\theta-\phi)\,\dot\theta\dot\phi$

$= a^2\dot\theta^2 + a^2\dot\phi^2 + 2a^2\dot\theta\dot\phi$
[neglecting higher powers of small quantites]

$\therefore$ (velocity)^2 of the particle C $= v_C^2 = a^2\dot\theta^2 + a^2\dot\phi^2 + 2a^2\dot\theta\dot\phi$.
And velocity of the C.G. of the rod *i.e.* of O, is zero.
Now let T, be the kinetic energy and W, the work function of the system, then we easily get $W = \dfrac{mg}{3}(a\cos\theta + a\cos\phi) + C$

and $T = \dfrac{1}{2}m\dfrac{a^2}{3}\dot\theta^2 + \dfrac{1}{2}\left(\dfrac{m}{3}\right)[a^2\dot\theta^2 + a^2\dot\phi^2 + 2a^2\dot\theta\dot\phi]$

$= \dfrac{ma^2}{6}[2\dot\theta^2 + \dot\phi^2 + 2\dot\theta\dot\phi]$.

$\therefore$ Lagrange's θ-equation is given by

$$\frac{d}{dt}\left(\frac{2ma^2}{3}\dot\theta + \frac{ma^2}{3}\dot\phi\right) = -\frac{mga}{3}\theta \Rightarrow 2\ddot\theta + \ddot\phi = -\frac{g}{a}\theta \quad ...(1)$$

While Lagrange's ϕ-equation gives $\dfrac{d}{dt}\left[\dfrac{ma^2}{3}\dot\phi + \dfrac{ma^2}{3}\dot\theta\right] = -\dfrac{mga}{3}\phi$

i.e $\ddot\theta + \ddot\phi = -\dfrac{g}{a}\phi$. ...(2)

Equations (1) and (2) again give
$(2D^2 + c)\theta + D^2\phi = 0$ ...(3) and $D^2\theta + (D^2 + c)\phi = 0$ ...(4)

where $c = \dfrac{g}{a}$.

Eliminating ϕ in between (3) and (4), we get

$\{(D^2 + c)(2D^2 + c) - D^4\}\theta = 0$ *i.e.* $[D^4 + 3cD^2 + c^2]\theta = 0$. ...(5)

To solve (5), let $\theta = A\cos(pt + B)$ $\therefore D\theta = -pA\sin(pt + B)$,
$D^2\theta = -p^2A\cos(pt + B) = -p^2\theta$ and $D^4\theta = p^4\theta$.
With these substitutions, (5) gives
$(p^4 - 3cp^2 + c^2)\theta = 0 \Rightarrow p^4 - 3cp^2 + c^2 = 0$ $(\because \theta \neq 0)$

$\therefore p^2 = \dfrac{3c \pm \sqrt{(9c^2 - 4c^2)}}{2} = \left(\dfrac{3 \pm \sqrt{5}}{2}\right)c = \dfrac{(3 \pm \sqrt{5})}{2}\cdot\dfrac{g}{a}$

$\therefore p_1^2 = \dfrac{3 - \sqrt{5}}{2}\cdot\dfrac{g}{a}$ and $p_2^2 = \dfrac{3 + \sqrt{5}}{2}\cdot\dfrac{g}{a}$.

$\therefore$ one period of principal oscillations corresponding to p_1, is given by

$\dfrac{2\pi}{p_1} = 2\pi\sqrt{\left(\dfrac{2}{3 - \sqrt{5}}\cdot\dfrac{a}{g}\right)} = 2\pi\sqrt{\left(\dfrac{a}{g}\right)}\sqrt{\left\{\dfrac{2(3 + \sqrt{5})}{9 - 5}\right\}}$

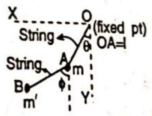
$= 2\pi\sqrt{\left(\dfrac{a}{g}\right)}\sqrt{\left(\dfrac{6 + 2\sqrt{5}}{4}\right)} = 2\pi\sqrt{\left(\dfrac{a}{g}\right)}\sqrt{\left\{\dfrac{(\sqrt{5} + 1)^2}{4}\right\}}$

$= (\sqrt{5} + 1)\pi\sqrt{\left(\dfrac{a}{g}\right)}$.

Ex. 12. *A mass m hangs from a fixed point by a light string of length l and a mass m' hangs from m by a second string of length l'. For oscillations in a vertical plane, show that the periods of the principal oscillations are the values of $\dfrac{2\pi}{n}$ where n is given by the equation*

$$n^4 - gn^2\frac{m + m'}{m}\left(\frac{1}{l} + \frac{1}{l'}\right) + g^2\frac{m + m'}{ml'l} = 0.$$

Sol. At any time t, let the strings be inclined at angles θ and ϕ to the vertical. Co-ordinates of m are $(l\sin\theta, l\cos\theta)$.



$\therefore$ (velocity)^2 of $m = l^2\dot\theta^2$
while co-ordinates of m' are
$x_B = l\sin\theta + l'\sin\phi$, $\Rightarrow$

$\dot{x}_B = l\cos\theta\,\dot\theta + l'\cos\phi\,\dot\phi$
$y_B = l\cos\theta + l'\cos\phi$ $\Rightarrow$

$\dot{y}_B = -(l\sin\theta\,\dot\theta + l'\sin\phi\,\dot\phi)$

$\therefore$ (velocity)^2 of $m' = \dot{x}_B^2 + \dot{y}_B^2 = l^2\dot\theta^2 + l'^2\dot\phi^2 + 2ll'\dot\theta\dot\phi$

[$\because$ θ and ϕ are small]

Now let T, be the kinetic energy and, W the work function, of the system, then we have

$W = mgl\cos\theta + m'g(l\cos\theta + l'\cos\phi)$
$= gl(m + m')\cos\theta + m'gl'\cos\phi$

and $T = \dfrac{1}{2}ml^2\dot\theta^2 + \dfrac{1}{2}m'[l^2\dot\theta^2 + l'^2\dot\theta^2 + 2ll'\dot\theta\dot\phi]$

$= \dfrac{1}{2}[(m + m')l^2\dot\theta^2 + m'l'^2\dot\theta^2 + 2m'll'\dot\theta\dot\phi]$

Lagrange's θ-equation is given by $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\theta}\right) - \frac{\partial T}{\partial \theta} = \frac{\partial W}{\partial \theta}$

$$\Rightarrow \frac{d}{dt}[(m+m')l^2\dot\theta + m'll'\dot\phi] = -gl(m+m')\theta.$$

$$\Rightarrow (m+m')l\ddot\theta + m'l'\ddot\phi = -g(m+m')\theta \quad ...(1)$$

While Lagrange's ϕ-equation gives

$$\frac{d}{dt}[m'l'^2\dot\phi + m'll'\dot\theta] = -m'gl'\phi$$

$$\Rightarrow l'\ddot\phi + l\ddot\theta = -g\phi \quad ...(2)$$

Equation (1) and (2) again give

$$(m+m')(lD^2+g)\theta + m'l'D^2\phi = 0 \quad ...(3)$$

$$lD^2\theta + (l'D^2+g)\phi = 0 \quad ...(4)$$

Eliminating ϕ, we get $[(m+m')(lD^2+g)(l'D^2+g) - m'll'D^4]\theta = 0$

i.e. $[mll'D^4 + (m+m')(l+l')gD^2 + (m+m')g^2]\theta = 0$ ...(5)

Now let $\theta = A\cos(nt+B)$; $\therefore D\theta = -nA\sin(nt+B)$

$D^2\theta = -n^2A\cos(nt+B) = -n^2\theta$ and $D^4\theta = n^4\theta$ ...(6)

$\therefore$ (5) and (6) give $mll'n^4 - (m+m')(l+l')gn^2 + (m+m')g^2 = 0$

or $$n^4 - \frac{m+m'}{m}\left(\frac{1}{l}+\frac{1}{l'}\right)gn^2 + \frac{(m+m')g^2}{mll'} = 0. \quad ...(7)$$

Ex. 13. (a) *A mass M hangs from a fixed point at the end of a very long string whose length l is a, to M is suspended a mass m by a string whose length l is small compound with a ; prove that the time of a small oscillation of m is* $2\pi\sqrt{\left(\frac{M}{M+m}\cdot\frac{l}{g}\right)}$

Sol. Here, we have $m = M$, $m' = m$, $l = a$, $l' = l$.

$$\therefore n^4 - \frac{M+m}{M}\left(\frac{l}{a}+\frac{1}{l}\right)gn^2 + \frac{(M+m)g^2}{Mal} = 0$$

i.e. $$n^4 - \frac{M+m}{M}\left(\frac{l}{a}+1\right)\frac{g}{l}n^2 + \frac{(M+m)g^2}{Ml^2}\cdot\frac{l}{a} = 0 \quad ...(8)$$

But a is large compared to l $\therefore \frac{l}{a} \to 0$

Hence the equation (8), gives

$$n^4 - \frac{M+m}{M}\cdot\frac{g}{l}\cdot n^2 = 0 \text{ i.e } n^2 = \frac{M+m}{M}\cdot\frac{g}{l}$$

$\therefore$ Time of a small oscillation $= \frac{2\pi}{n} = 2\pi\sqrt{\left\{\frac{M}{M+m}\cdot\frac{l}{g}\right\}}$.

Ex. 13. (b) *At the lowest point of a smooth circular tube, of mass M and radius a, is placed a particle of mass M', the tube hangs in a vertical plane from its highest point, which is fixed, and can turn freely in its own plane*

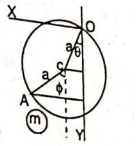
about this point. If the system be slightly displaced, show that the periods of the two independent oscillations of the system are

$$2\pi\sqrt{\left(\frac{2a}{g}\right)} \text{ and } 2\pi\sqrt{\left(\frac{Mag^{-1}}{M+M'}\right)}.$$

(Meerut 1983)

And that for one principal mode of oscillations, the particle remains at rest relative to the tube and for the other, the centre of gravity of the particle and the tube remain at rest.

Sol. Let C be the centre of the tube and A the position of the particle M' at time t when OC and CA make angle 3 θ and ϕ with the vertical



$\therefore x_A = a\sin\theta + a\sin\phi$,

$y_A = a\cos\theta + a\cos\phi$.

and

(velocity)^2 of $A = \dot x_A^2 + \dot y_A^2$

$$= (a\cos\theta\dot\theta + a\cos\phi\dot\phi)^2 + (-a\sin\theta\dot\theta - a\sin\phi\dot\phi)^2$$

$$= a^2\dot\theta^2 + a^2\dot\phi^2 + 2a^2\dot\theta\dot\phi\cos(\theta-\phi) = a^2\dot\theta^2 + a^2\dot\phi^2 + 2a^2\dot\theta\dot\phi$$

[neglecting small quantities of the higher order].

Also $C \equiv (a\sin\theta, a\cos\theta)$

(velocity)^2 of $C = (a\cos\theta\dot\theta)^2 + (-a\sin\theta\dot\theta)^2 = a^2\dot\theta^2$

Now let T, be the kinetic energy and W the work function of the system

then we readily obtain $W = Mga\cos\theta + M'g(a\cos\theta + a\cos\phi) + K$
$= (M+M')ga\cos\theta + M'ga\cos\phi + K$

$T =$ K.E. of circular tube + K.E. of particle

$$= \tfrac{1}{2}M(a^2\dot\theta^2 + a^2\dot\theta^2) + \tfrac{1}{2}M(a^2\dot\theta^2 + a^2\dot\phi^2 + 2a^2\dot\theta\dot\phi)$$

$$= \frac{2M+M'}{2}a^2\dot\theta^2 + \tfrac{1}{2}M'a^2\dot\phi^2 + M'a^2\dot\theta\dot\phi. \quad ...(2)$$

$\therefore$ Lagrange's θ-equation gives

$$\frac{d}{dt}[(2M+M')a^2\dot\theta + M'a^2\dot\phi] = -(M+M')ga\theta.$$

$$\Rightarrow (2M+M')\ddot\theta + M'\ddot\phi = -(M+M')\frac{g}{a}\theta. \quad ...(3)$$

Also Lagrange's ϕ-equation gives

$$\frac{d}{dt}[M'a^2\dot\phi + M'a^2\dot\theta] = -M'ga\phi \Rightarrow \ddot\phi + \ddot\theta = -\frac{g}{a}\phi \quad ...(4)$$

Equations (3) and (4) can be re-written as

$$[(2M+M')D^2+(M+M')c]\theta+M'D^2\phi=0$$

and $D^2\theta+(D^2+c)\phi=0$ where $c=\dfrac{g}{a}$. ...(5)

...(6)

Eliminating ϕ between these two equations, we get

$$[\{(2M+M')D^2+(M+M')c\}(D^2+c)-M'D^4]\theta=0$$

i.e. $[2MD^4+c(3M+2M')D^2+c^2(M+M')]\theta=0.$

To solve (7), ...(7)

let $\theta=A\cos(pt+B)$; $D\theta=-pA\sin(pt+B)$

$D^2\theta=-p^2A\cos(pt+B)=-p^2\theta$ and $D^4\theta=p^4\theta$ ...(8)

$\therefore$ (7) and (8) give $[2Mp^4-c(3M+2M')p^2+c^2(M+M')]\theta=0$

i.e. $2Mp^4-c(3M+2M')p+c^2(M+M')=0.$ $[\because \theta\neq 0]$

which again gives $(2p^2-c)[Mp^2-c(M+M')]=0$

$\therefore$ $p_1^2=\dfrac{c}{2}$ and $p_2^2=\dfrac{c(M+M')}{M}$

i.e. $p_1^2=\dfrac{g}{2a}$ and $p_2^2=\dfrac{M+M'}{M}\dfrac{g}{a}$. $\left(\because c=\dfrac{g}{a}\right)$

Hence periods of oscillations are given by

$$\frac{2\pi}{p_1} \text{ and } \frac{2\pi}{p_2} \quad i.e. \text{ by } 2\pi\sqrt{(2a/g)} \text{ and } 2\pi\sqrt{\left(\frac{M}{(M+M')}\frac{a}{g}\right)}.$$

Multiplying (6) by λ and adding to (5), we have

$D^2\{(2M+M'+\lambda)\theta+(M'+\lambda)\phi\}=-\{(M+M')\theta+\lambda\phi\}$ ....(9)

Now chose λ such that

$$\frac{2M+M'+\lambda}{M'+\lambda}=\frac{M+M'}{\lambda}\Rightarrow\lambda=M' \text{ and } \lambda=-(M+M').$$

taking $\lambda=M'$, equation (9) reduces to

$$D^2\{(M+M')\theta+M'\phi\}=-\tfrac{1}{2}c\{(M+M')\theta+M'\phi\}$$

and when $\lambda=-(M+M')$, equation (9) reduces to

$$D^2(\theta-\phi)=-\frac{M+M'}{M}c(\theta-\phi)$$

$\therefore$ Principal co-ordinates are $\theta-\phi$ and $\{(M+M')\theta+M'\phi\}$.

For the first mode, $\theta-\phi=0$. *i.e.* $\theta=\phi$. This shows that the particle is at rest relative to the tube. For the second mode we have $(M+M')\theta+M'\phi=0.$

Further, the x-coordinates of C.G. of the particle and the tube

$$=\frac{Ma\sin\theta+M'(a\sin\theta+a\sin\phi)}{M+M'}$$

$$=\frac{a}{M+M'}\{M\theta+M'(\theta+\phi)\}$$ (since θ and ϕ are small, $\therefore$ $\sin\theta=\theta$ and $\sin\phi=\phi$)

$$=\frac{a}{M+M'}\{(M+M')\theta+M'\phi\}=0$$ [using above results]

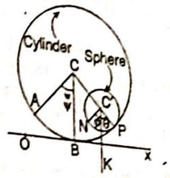
$\Rightarrow$ The common C.G. of the particle and the tube remains at rest.

Ex 14. *A perfectly rough sphere lying inside a hollow cylinder, which rests on a perfectly rough plane, is slightly displaced from its position of equilibrium. Show that the time of a small oscillation is*

$$2\pi\sqrt{\left(\frac{a-b}{g}\cdot\frac{4M}{10M+7m}\right)}$$

where a is the radius of the cylinder, b that of the sphere, and M, m are the masses of the cylinder and the sphere.

Sol. Let CA be a line fixed in the cylinder and $C'N$, a line fixed in the sphere, which were initially vertical. Let these lines, after a time t, make angles ψ and ϕ with the vertical. Further let θ be the angle which the line joining centre makes with the vertical.



Initially O was the point of contact with the horizontal plane, which is regarded as origin. There is no slipping ; between the cylinder, horizontal plane and between the cylinder, sphere ; therefore, we get $OB=$ arc $AB=a\psi$ and arc $AP=$ arc NP

i.e. $a(\theta+\psi)=b(\theta+\phi)$, *i.e.* $b\dot{\phi}=(a-b)\dot{\theta}+a\dot{\psi}$.

Referred to the horizontal and vertical through O as co-ordinate axes, we have

$x_C'=a\psi+a(a-b)\sin\theta$, $y_C'=a-(a-b)\cos\theta$.

(co-ordinates of C')

$\therefore$ (velocity)^2 of C'

$$\dot{x}_C'^2+\dot{y}_C'^2=\{a\dot{\psi}+(a-b)\cos\theta\dot{\theta}\}^2+ man\{(a-b)\sin\theta\dot{\theta}\}^2$$

$$=a^2\dot{\psi}^2+(a-b)^2\dot{\theta}^2+2a(a-b)\dot{\psi}\dot{\theta}\cos\theta$$

$$=a^2\dot{\psi}^2+(a-b)^2\dot{\theta}^2+2a(a-b)\dot{\psi}\dot{\theta}.$$ ...(1)

Also co-ordinates of C are $(a\psi, a)$.

$\therefore$ (velocity)^2 of $C=(a\dot{\psi})^2=a^2\dot{\psi}^2$. ...(2)

Now let T be the kinetic energy and $\dot{W}$, the work function. Then we have ...(3)

$$W=(a-b)mg\cos\theta+C$$

and $T=\tfrac{1}{2}M\left(v_C^2+a^2\dot{\psi}^2\right)+\tfrac{1}{2}m\left(v_{C'}^2+\dfrac{2b^2}{5}\dot{\phi}^2\right)$

$\Rightarrow T=\tfrac{1}{2}M(a^2\dot{\psi}^2+a^2\dot{\psi}^2)$

$$+\frac{1}{2}m\left[\frac{2b^2}{5}\dot{\phi}^2+a^2\dot{\psi}^2+(a-b)^2\dot{\theta}^2+2a(a-b)\dot{\psi}\dot{\theta}\right]$$

$$= Ma^2\dot{\psi}^2+\tfrac{1}{2}m\ [\tfrac{2}{5}\{(a-b\dot{\theta}+a\dot{\psi}\}^2+a^2\dot{\psi}^2$$

$$+(a-b)^2\dot{\theta}^2+2a(a-b)\dot{\psi}\dot{\theta}]$$

$$=\frac{10M+7m}{10}a^2\dot{\psi}^2+\frac{7m}{10}[(a-b)^2\,\theta^2+2a(a-b)\dot{\theta}\dot{\psi}] \quad ...(4)$$

$\therefore$ Lagrange's θ-equation gives

$$\frac{d}{dt}\left[\frac{7m}{5}\{(a-b)^2\dot{\theta}+a(a-b).\dot{\psi}\}\right]=-(a-b)\,mg\sin\theta$$

$$\Rightarrow 7(a-b)\ddot{\theta}+5a\ddot{\psi}=-5g\theta \text{ (as } \theta \text{ is small)}. \quad ...(5)$$

Also Lagrange's ψ-equation gives

$$\frac{d}{dt}\left[\frac{10M+7m}{5}a^2\dot{\psi}+\frac{7m}{5}a(a-b)\ \theta\right]=0$$

$$\Rightarrow (10M+7m)\,a\ddot{\psi}+7m(a-b)\ddot{\theta}=0$$

$$\Rightarrow 7m(a-b)\ddot{\theta}+(10M+7m)\,a\ddot{\psi}=0. \quad ...(6)$$

Eliminating $\ddot{\psi}$ between (5) and (6), we get

$$[7(10M+7m)-49m](a-b)\ddot{\theta}=-5(10M+7m)\,g\theta$$

or $\quad 70M(a-b)\ddot{\theta}=-5(10M+7m)\,g\theta$

or $\quad \ddot{\theta}=-\dfrac{(10M+7m)}{14M}\cdot\dfrac{g}{a-b}\theta=-\mu\theta$ say,

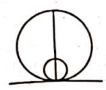
$\therefore$ time of a small oscillation $=\dfrac{2\pi}{\sqrt{\mu}}=2\pi\sqrt{\left(\dfrac{a-b}{g}\cdot\dfrac{14M}{10M+7m}\right)}$.

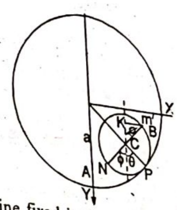
Ex. 15. *A perfectly rough sphere, of mass m and radius b, rests at the lowest point of a fixed spherical cavity of radius a. To the highest point of the movable sphere is attached a particle of mass m and the system is disturbed. Show that the oscillations are the same as those of a simple pendulum of length*

$$(a-b)\frac{4m'+\frac{2}{5}m}{m+m'\left(2-\dfrac{a}{b}\right)}$$

(Agra 1995, 89)

Sol. O is the centre of the fixed cavity, C the centre of the sphere and at B, a particle of m' is attached. Initially NB was vertical, and N conicided with A.





After the time t, let BN or CN, a line fixed in the sphere make an angle ϕ with the vertical. But there is no slipping between the cylinder and the sphere.

$\therefore$ arc $AP=$ arc NP, *i.e.* $a\theta=b(\theta+\phi)$

i.e. $\quad b\phi=(a-b)\theta\Rightarrow b\dot{\phi}=(a-b)\dot{\theta}$

Now co-ordinates of B (i.e. of m') are

$x_B=(a-b)\sin\theta+b\sin\phi,\ y_B=(a-b)\cos\theta-b\cos\phi,$

and co-ordinates of the centre C are

$\{(a-b)\sin\theta,(a-b)\cos\theta\}$

$\therefore$ (velocity)^2 of $B=\dot{x}_B^{\,2}+\dot{y}_B^{\,2}$

$$=\{(a-b)\cos\theta\dot{\theta}+b\cos\phi\dot{\phi}\}^2+\{-(a-b)\sin\theta\dot{\theta}+b\sin\phi\dot{\phi}\}^2$$

$$=(a-b)^2\dot{\theta}^2+b^2\dot{\phi}^2+2b(a-b)\dot{\theta}\dot{\phi}$$

[neglecting small quantities of second and higher orders]

and (velocity)^2 of $C=\{(a-b)\cos\theta\dot{\theta}\}^2+\{-(a-b)\sin\theta\dot{\theta}\}^2$

$$=(a-b)^2\dot{\theta}^2$$

Now let T, be the kinetic energy and W the work function of the system, then we readily obtain

$$W=mg(a-b)\cos\theta+m'g\{(a-b)\cos\theta-b\cos\phi\}+D$$

$$=(a-b)(m+m')g\cos\theta-m'gb\cos\phi+D$$

$$=(a-b)(m+m')g\cos\left(\frac{b}{a-b}\phi\right)-m'gb\cos\phi+D$$

$$\{\because\ (a-b)\theta=b.\phi\}$$

and $\quad T=\dfrac{1}{2}m\left[\dfrac{2b^2\dot{\phi}^2}{5}+(a-b)^2\dot{\theta}^2\right]+\dfrac{1}{2}m'\{(a-b)^2\dot{\theta}^2+b^2\dot{\phi}^2$

$$+2b(a-b)\dot{\theta}\dot{\phi}\}$$

$$=\frac{1}{2}m\left[\frac{2b^2\dot{\phi}^2}{5}+b^2\dot{\phi}^2\right]+\frac{1}{2}m'[b^2\dot{\phi}^2+b^2\dot{\phi}^2+2b^2\dot{\phi}^2]$$

$$[\because \quad (a-b)\,\dot{\theta} = b\dot{\phi}\,]$$

$$= \frac{b^2}{10}(7m + 20m')\dot{\phi}^2.$$

Further Lagrange's ϕ-equation gives $\frac{d}{dt}\left[\frac{b^2}{5}(7m + 20m')\,\dot{\phi}\right]$

$$= -(a-b)(m+m')g\frac{b}{a-b}\sin\left(\frac{b}{a-b}\phi\right) + m' g b \sin\phi$$

$$= -(m+m')bg\sin\left(\frac{b\phi}{a-b}\right) + m' g b \sin\phi$$

$$\Rightarrow \frac{b^2}{5}(7m + 20m')\,\ddot{\phi} = -(m+m')\,bg.\frac{b}{a-b}\phi + m' g b\phi \text{ app.}$$

$$\Rightarrow \left(4m' + \frac{7m}{5}\right)\ddot{\phi} = -\frac{g}{a-b}\left[m + m'\left(2 - \frac{a}{b}\right)\right]\phi, \text{ app.}$$

$$\Rightarrow \ddot{\phi} = -\frac{g}{a-b}\cdot\frac{m + m'\left(2-\frac{a}{b}\right)}{4m' + \frac{7}{5}m}\phi = -\mu\phi \qquad [\text{ form } \ddot{x} = -\mu x]$$

$\therefore$ Length of the simple equivalent pendulum $= \frac{g}{\mu}$

$$= (a-b)\frac{4m' + \frac{7}{5}m}{m + m'\left(2 - \frac{a}{b}\right)}.$$

Ex. 16. *A perfectly rough sphere rests at the lowest point of a fixed spherical cavity of double its own radius. To the highest point of the movable sphere is attached a particle of mass $\frac{7}{20}$ times that of the sphere and the system is disturbed. Show that the oscillations are the same as those of a simple pendulum of length $\frac{14}{5}$ times the radius of the sphere.* **[Agra 91]**

Sol. Just like EX. 15.

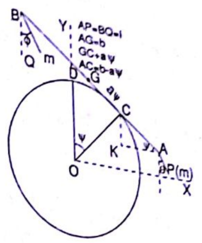
Put $m' = \frac{7m}{20}$ in Ex. 15 and proceed similarly to get the required result.

Ex. 17. *A plank, of mass M, radius of gyration k and length 2b, can swing like a sea-saw across a perfectly rough fixed cylinder of radius a. At its ends hang two particles each of mass m, by strings of length l. Show that, as the system swings, the lengths of its simple equivalent pendulum are* l and $\frac{Mk^2 + 2mb^2}{(M + 2m)\,a}$.

Sol. Let the strings be AP and BQ such that their free extremities are having particles of mass m. Initially G was at D and AP, BQ were vertical.

After a time t, when the plank has turned through an angle ψ with the horizontal. let the strings be inclined at angle θ and ϕ to the vertical.

But there is no slipping between the plank and the cylinder, so we have $CG = \text{arc } DC = a\psi$



Now $x_P = a\sin\psi + (b - a\psi)\cos\psi + l\sin\theta = b + l\theta - \frac{1}{2}b\psi^2$ ...(1)

$$y_P = a\cos\psi - (b - a\psi)\sin\psi - l\cos\theta$$

$$= a\left(1 - \frac{1}{2}\psi^2\right) - (b - a\psi)\psi - l\left(1 - \frac{1}{2}\theta^2\right)$$

neglecting higher powers of θ and ψ.

$$= a - l - b\psi + \frac{1}{2}a\psi^2 - \frac{1}{2}l\theta^2 \qquad \text{...(2)}$$

$\therefore \dot{x}_P = l\dot{\theta} - b\psi\dot{\psi}$ and $\dot{y}_P = -b\dot{\psi} + a\psi\dot{\psi} + l\theta\dot{\theta}$

$\therefore$ (velocity)^2 of $P = \dot{x}_P^2 + \dot{y}_P^2 = l^2\dot{\theta}^2 + b^2\dot{\psi}^2$

Thus kinetic energy of m at $P = \frac{1}{2}m\,(l^2\dot{\theta}^2 + b^2\dot{\psi}^2)$

*Smilarly kinetic energy of m at $Q = \frac{1}{2}m\,(l^2\dot{\phi}^2 + b^2\dot{\psi}^2)$

Also y-co-ordinate of $Q = a - l + b\psi + \frac{1}{2}a\psi^2 + \frac{1}{2}l\phi^2 = y_Q$ *say*

[put $-\psi$ for ψ and ϕ for θ in (2)]

the co-ordinates of G are $\{a\sin\psi - a\psi\cos\psi, a\cos\psi + a\psi\sin\psi\}$

(neglecting higher powers of ψ as it is small)

i.e. $\left\{\frac{a\psi^2}{2}, a + \frac{a\psi^2}{2}\right\}$.

$\therefore$ (velocity)^2 of $G = a^2\psi^2\dot{\psi}^2$ [$\because \dot{x} = a\dot{\psi}\psi, \dot{y} = a\psi\dot{\psi}$]

(neglecting smaller, quantities). Thus kinetic energy of the plank

$$= \frac{1}{2}M\,(k^2\dot{\psi}^2 + a^2\psi^2\dot{\psi}^2) = \frac{1}{2}Mk^2\dot{\psi}^2\text{, approximately}$$

*To obtain the corresponding quantities for the point Q, we shall write $-\psi$ for ψ and ϕ for θ.

Now let T be the kinetic energy and W, the work function of the system, then we obtain $W = -mg\left(a - l - b\psi + \frac{a\psi^2}{2} + l\frac{\theta^2}{2}\right)$

$$-mg\left(a - l + b\psi + \frac{a\psi^2}{3} + l\frac{\phi^2}{2}\right) - mg\left(a + \frac{a\psi^2}{2}\right)$$

$$= K - \tfrac{1}{2}a\,(M + 2m)\,g\psi^2 - \tfrac{1}{2}mgl\,(\theta^2 - \phi^2)$$

and $T = \frac{1}{2}m\,(l^2\dot{\theta}^2 + b^2\dot{\psi}^2) + \frac{1}{2}m\,(l^2\dot{\phi}^2 + b^2\dot{\psi}^2) + \frac{1}{2}MK^2\dot{\psi}^2$

$$= \tfrac{1}{2}(MK^2 + 2mbq^2)\,\dot{\psi}^2 + \tfrac{1}{2}Ml^2\dot{\theta}^2 + \tfrac{1}{2}ml^2\dot{\phi}^2$$

Lagrange's θ-equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right) - \frac{\partial T}{\partial\theta} = \frac{\partial W}{\partial\theta}$

$$\Rightarrow \frac{d}{dt}(ml^2\dot{\theta}^2) = -mgl\theta \Rightarrow \ddot{\theta} = -\frac{g}{l}\theta,$$

$\therefore$ length of the simple equivalent pendulum $= l$

Also Lagrange's $\psi -$ eq^n gives

$$\frac{d}{dt}\{(Mk^2 + 2mb^2)\,\dot{\psi}\} = -a\,(M + 2m)\,g\psi \Rightarrow \ddot{\psi} = -ag\frac{M + 2m}{Mk^2 + 2mb^2}\psi$$

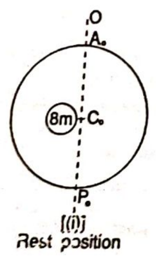
$\therefore$ length of the other equivalent pendulum $= \frac{Mk^2 + 2mb^2}{(M + 2m)\,a}$.

Ex. 18. *A smooth circular wire, of mass 8m and radius a, swings in a vertical plane, being suspended by an inxtensible string of length a attached to one point of it, a particle of mass m can slide on the wire. Prove that the periods of small oscillations are*

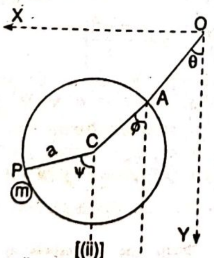
$$2\pi\sqrt{\left(\frac{8a}{3g}\right)},\ 2\pi\sqrt{\left(\frac{a}{3g}\right)},\ 2\pi\sqrt{\left(\frac{8a}{9g}\right)}.$$ **(Agra 91)**

Sol. At any time t, let the string OA, and the radius AC be inclined at angles θ and ϕ with the vertical and further let the radius to the particle (m) be inclined at an angle ψ with the vertical.

Now co-ordinates of C are



[(i)] Rest position



[(ii)] displaced position

$(a\sin\theta + a\sin\phi,\ a\cos\theta + a\cos\phi)$.

$\therefore$ (velocity)^2 of $C = a^2\dot{\theta}^2 + a^2\dot{\phi}^2 + 2a^2\dot{\theta}\dot{\phi}\cos(\theta - \phi)$

$= a^2\dot{\theta}^2 + a^2\dot{\phi}^2 + 2a^2\dot{\theta}\dot{\phi}$ approximately

$$\left[\because \begin{array}{l} \dot{x}_C = a\cos\theta\,\dot{\theta} + a\cos\phi\,\dot{\phi} \\ \dot{y}_C = -a\,(\sin\theta\,\dot{\theta} + \sin\phi\,\dot{\phi}) \end{array}\right]$$

Also co-ordinates of the particle m (*i.e.* of the pt. P) are

$$x_P = a\,(\sin\theta + \sin\phi + \sin\psi),$$
$$y_P = a\,(\cos\theta + \cos\phi + \cos\psi).$$

$\because$ (velocity)^2 of $m = a^2(\dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2 + 2\dot{\theta}\dot{\phi} + 2\dot{\phi}\dot{\psi} + 2\dot{\psi}\dot{\theta})$ app.

Let T, be the kinetic energy and W, the work function of the system, then we readily get

$$W = 8mg\,(a\cos\theta + a\cos\phi) + mg\,(a\cos\theta + a\cos\phi + a\cos\psi)$$
$$= mga\,[9\cos\theta + 9\cos\phi + \cos\psi] \qquad ...(1)$$

and $T = \frac{1}{2}\cdot 8m\,[a^2\dot{\phi}^2 + (a^2\dot{\theta}^2 + a^2\dot{\phi}^2 + 2a^2\dot{\theta}\dot{\phi})]$

$$+ \tfrac{1}{2}ma^2\,[\dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2 + 2\dot{\theta}\dot{\phi} + 2\dot{\phi}\dot{\psi} + 2\dot{\psi}\dot{\theta}]$$

i.e. $T = \frac{1}{2}m\,[9\dot{\theta}^2 + 17\dot{\phi}^2 + \dot{\psi}^2 + 18\dot{\theta}\dot{\phi} + 2\dot{\phi}\dot{\psi} + 2\dot{\psi}\dot{\theta}]$ ...(2)

$\therefore$ Lagrange's θ, and ψ equations give

$9\ddot{\theta} + 9\ddot{\phi} + \ddot{\psi} = -9\frac{g}{a}\theta$ ...(3) $9\ddot{\theta} + 17\ddot{\phi} + \ddot{\psi} = -9\frac{g}{a}\phi$ ...(4)

and $\ddot{\theta} + \ddot{\phi} + \ddot{\psi} = -\frac{g}{a}\psi,$ ...(5)

which can be rewritten as

$$(9D^2 + 9c)\,\theta + 9D^2\phi + D^2\psi = 0 \qquad ...(6)$$
$$9D^2\theta + (17D^2 + 9c)\,\phi + D^2\psi = 0 \qquad ...(7)$$

and $D^2\theta + D^2\phi + (D^2 + c)\,\psi = 0$ ...(8)

Eliminating ϕ and ψ in (6), (7) and (8), we get

$$\begin{vmatrix} 9D^2 + 9c & 9D^2 & D^2 \\ 9D^2 & 17D^2 + 9c & D^2 \\ D^2 & D^2 & D^2 + c \end{vmatrix}\theta = 0$$

i.e. $(8D^2 + 9c)\,[9c\,(2D^2 + c) + D^2(8D^2 + 9c)]\,\theta = 0$

i.e. $(8D^2 + 9c)\,[8D^4 + 27cD^2 + 9c^2]\,\theta = 0$

i.e. $[(8D^2+9c)(8D^2+3c)(D^2+3c)]\,\theta=0$ ...(9)

Now let $\theta = A\cos(pt+B)$, then

$D\theta = -pA\sin(pt+B),\ D^2\theta = -p^2A\cos(pt+B) = -p^2\theta$ ...(10)

$\therefore$ (9) gives $(8p^2-9c)(8p^2-3c)(p^2-3c)=0$ $[\because\ \theta \neq \theta]$

$$\Rightarrow \left(8p^2-\frac{9g}{a}\right)\left(8p^2-\frac{3g}{a}\right)\left(p^2-\frac{3g}{a}\right)=0.$$

i.e. $p_1^2=\dfrac{9g}{8a},\ p_2^2=\dfrac{3g}{8a},\ p_3^2=\dfrac{3g}{a}$

Thus periods of small oscillations are $\dfrac{2\pi}{p_1}, \dfrac{2\pi}{p_2}, \dfrac{2\pi}{p_3}$

i.e. $2\pi\sqrt{\left(\dfrac{8a}{9g}\right)},\ 2\pi\sqrt{\left(\dfrac{8a}{3g}\right)},\ 2\pi\sqrt{\left(\dfrac{a}{9g}\right)}$

Ex. 19. *A plank, 2a feet long, is placed symmetrically across a light cylinder, of radius a, which rests and is free to roll on a perfectly rough horizontal plane. A heavy particle whose mass is n times that of plank is embedded in the cylinder at its lowest point. If the system is sligthly displaced, show that its periods of oscillation are the values of*

$$\frac{2\pi}{p}\sqrt{\left(\frac{a}{g}\right)}$$

given by the equation $4p^4-(n+12)p^2+3(n-1)=0.$

Sol. C is the centre of the cylinder whose radius is a.

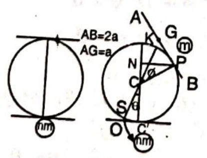
Let the mass of the plank be m, therefore mass of the particle embedded at the lowest point of the cylinder is nm.

Suppose θ and ϕ are the angles through which the cylinder and the plank have turned in time t, from their initial positions.

Initially G was coinciding with K which was at the highest point of the cylinder.

There is no slipping between the plank and the cylinder, hence

Further there is no sliding between the cylinder and the horizontal plane,



$$PG = \text{arc } KP = a(\phi-\theta)$$

hence $OC_1 = \text{arc } C_1S = a\theta$, where O was the initial point of contact of the cylinder and the horizontal plane.

Taking O as origin and the horizontal and vertical through O as co-ordinates axes, we obtain $x_G = a\theta + a\sin\phi - a(\phi-\theta)\cos\phi$

$= a\theta + a\phi - a(\phi-\theta) = 2a\theta$

and $y_G = a + a\cos\phi + a(\phi-\theta)\sin\phi = 2a - a\theta\phi + a\dfrac{\phi^2}{2}$

up to first approximations.

$\therefore \dot{x}_G^2+\dot{y}_G^2=(2a\dot\theta)^2+(-a\dot\theta\phi-a\phi\dot\theta+a\phi\dot\phi)^2=4a^2\dot\theta^2$

Further co-ordinates of the particle nm (*i.e.* of the point S)

$\text{arc} \equiv \{a\theta - a\sin\theta,\ a - a\cos\theta\}$

$\therefore$ (velocity)^2 of $(nm) = (a\dot\theta - a\cos\theta\dot\theta)^2 + (a\sin\theta\dot\theta)^2,$

$= (a^2+a^2-2a^2\cos\theta)\,\dot\theta^2.$

Now let T, be the kinetic energy and W, the work function of the system, then we readily obtain

$$W = -nmg(a-a\cos\theta) - mg\left(2a - a\theta\phi + a\frac{\phi^2}{2}\right)$$

$$= -nmga\frac{\theta^2}{2} - mga\left(\frac{\phi^2}{2}-\theta\phi\right) - 2mga$$

and $T = \dfrac{1}{2}m\left[\dfrac{a^2}{3}\dot\phi^2 + 4a^2\dot\theta^2\right] + \dfrac{1}{2}nm\,[2a^2-2a^2\cos\theta]\,\dot\theta^2$

$= \dfrac{1}{2}m\left(\dfrac{a^2}{3}\dot\phi^2 + 4a^2\dot\theta^2\right) + \dfrac{1}{2}nm.\,2a^2\dot\theta^2\dfrac{\theta^2}{2}$

$= \dfrac{1}{2}m\left[\dfrac{a^2}{3}\dot\phi^2 + 4a^2\dot\theta^2\right]$, approximately,

(neglecting the second part)

$\therefore$ Lagrange's θ-equation is given by $\dfrac{d}{dt}\left(\dfrac{\partial T}{\partial\dot\theta}\right) - \dfrac{\partial T}{\partial\theta} = \dfrac{\partial W}{\partial\theta}$

$\Rightarrow \dfrac{d}{dt}(4ma^2\dot\theta) = -nmga\theta + mga\phi \Rightarrow 4\ddot\theta = -\dfrac{g}{a}(n\theta-\phi)$ ...(1)

Further Lagrange's ϕ-equation gives

$\dfrac{d}{dt}\left(\dfrac{ma^2}{3}\dot\phi\right) = -mg(\phi-\theta) \Rightarrow \ddot\phi = -\dfrac{3g}{a}(\phi-\theta)$ ...(2)

Equations (1) and (2) can be re-written a $(4D^2+nc)\theta - c\phi = 0$ ...(3)

and $3c\theta - (D^2+3c)\phi = 0$ where $c=(g/a)$ ...(4)

Eliminating ϕ in between (3) and (4), we get

$[(4D^2+nc)(D^2+3c)-3c^2]\,\theta=0$ ...(5)

$\Rightarrow [4D^2 + c(n+12)D^2 + 3(n-1)c^2]\,\theta = 0$

Now if the periods of oscillation are the values of

$\frac{2\pi}{p}\sqrt{\left(\frac{a}{g}\right)}$ *i.e.* values of $\frac{2\pi}{\sqrt{\left(\frac{g}{a}p^2\right)}}$,

then the solution of the above equation must be of the form

$$\theta = A\cos\left[\sqrt{\left(\frac{g}{a}\right)}pt + B\right]$$

$$\therefore D\theta = -\sqrt{\left(\frac{g}{a}\right)}pA\sin\left[\sqrt{\left(\frac{g}{a}\right)}pt + B\right]$$

$$D^2\theta = -\left(\frac{g}{a}\right)p^2A\cos\left[\sqrt{\left(\frac{g}{a}\right)}pt + B\right]$$

$$= -\left(\frac{g}{a}\right)p^2\theta = -cp^2\theta \text{ and } D^4\theta = c^2p^4\theta.$$

Substituting these values in (5) we get

$$[4c^2p^4 - c^2(n+12)p^2 + 3(n-1)c^2]\theta = 0$$

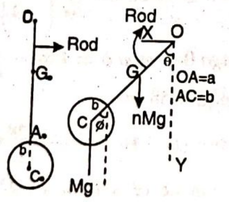
$$\Rightarrow 4p^4 - (n+12)p^2 + 3(n-1) = 0 \qquad (\because \theta \neq 0)$$

Ex. 20. *To a point of a solid homogeneous sphere, of mass M, is freely hinged one end of a homogeneous rod, of mass nM ; and the other end is freely hinged to a fixed point. If the system makes small oscillations under gravity about the position of equilibrium, the centre of a sphere and the rod being always in a vertical plane passing through the fixed point, show that the periods of the principal oscillations are the values of* $\frac{2\pi}{p}$ *given by the equation*

$$2ab(6+7n)p^4 - p^2g\{10a(3+n) + 21b(2+n)\} + 15g^2(2+n) = 0,$$

where a is the length of the rod and b is the radius of the sphere.

Sol. At any time t, let the rod and the sphere have turned through



angles θ and ϕ to vertical, where $OA = a$, $AC = b$.
Now co-ordinates of C are

$$x_C = a\sin\theta + b\sin\phi,\ y_C = a\cos\theta + b\cos\phi$$

$$\therefore (\text{velocity})^2 \text{ of } C = \dot{x}_C^2 + \dot{y}_C^2 = a^2\dot{\theta}^2 + b^2\dot{\phi}^2 + 2ab\dot{\theta}\dot{\phi}\cos(\theta - \phi)$$

$$= a^2\dot{\theta}^2 + b^2\dot{\phi}^2 + 2ab\dot{\theta}\dot{\phi} \quad [\because \theta \text{ and } \phi \text{ are small}]$$

Also the co-ordinates of G are $\left(\frac{a}{2}\sin\theta, \frac{a}{2}\cos\theta\right)$

$\therefore (\text{velocity})^2$ of $G = \frac{a^2}{4}\dot{\theta}^2$

Now let T be, the kinetic energy and W, the work function of the system, then we readily obtain $W = n\,Mg\frac{a}{2}\cos\theta + Mg(a\cos\theta + b\cos\phi)$

$$= \tfrac{1}{2}a(n+2)Mg\cos\theta + bMg\cos\phi$$

and $T = \frac{1}{2}M\left[\frac{2b^2}{5}\dot{\phi}^2 + (a^2\dot{\theta}^2 + b^2\dot{\theta}^2 + 2ab\dot{\theta}\dot{\phi})\right] + \frac{1}{2}n.M\left[\frac{a^2}{12}\dot{\theta}^2 + \frac{a^2}{4}\dot{\theta}^2\right]$

$$= \frac{1}{6}a^2(n+3)M\dot{\theta}^2 + \frac{7b^2}{10}M\dot{\phi}^2 + abM\dot{\theta}\dot{\phi}$$

$\therefore$ Lagrange's θ-equation is given by $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right) - \frac{\partial T}{\partial\theta} = \frac{\partial W}{\partial\theta}$

$$\Rightarrow \frac{d}{dt}\{\tfrac{1}{3}a^2(n+3)M\dot{\theta} + abM\dot{\phi}\} = -\tfrac{1}{2}a(n+2)Mg\theta \text{ app.}$$

$$\Rightarrow 2a(n+3)\ddot{\theta} + 6b\ddot{\phi} = -3(n+2)g\theta \qquad \text{...(1)}$$

While Lagrange's ϕ-equation gives

$$\frac{d}{dt}\left\{\frac{7b^2}{5}M\dot{\phi} + abM\dot{\theta}\right\} = -bMg\phi$$

$$\Rightarrow 7b\ddot{\phi} + 5a\ddot{\theta} = -5g\phi \qquad \text{...(2)}$$

These can be re-written as

$$\{2a(n+3)D^2 + 3(n+2)g\}\theta + 6bD^2\phi = 0 \qquad \text{...(3)}$$

and $5aD^2\theta + (7bD^2 + 5g)\phi = 0$ ...(4)

Eliminating ϕ between (3) and (4), we get

$$[2ab(7n+6)D^4 + \{10a(n+3) + 21b(n+2)\}gD^2 + 15(n+2)g^2]\theta = 0 \qquad \text{...(5)}$$

To solve (5), let us put

$$\theta = A\cos(pt + B), \qquad \therefore D\theta = -pA\sin(pt + B).$$

$$D^2\theta = -p^2A\cos(pt + B) = -p^2\theta \text{ and } D^4\theta = p^4\theta.$$

Substituting these values in (5), we have

$$2ab(7n+6)p^4 - \{10(n+3)a + 21b(n+2)\}p^2g + 15(n+2)g^2 = 0.$$

Ex. 21. *A hollow cylindrical garden roller is fitted with a counterpoise which can turn on the axis of the cylinder; the system is placed on the*

rough horizontal plane and oscillates under gravity, if $\frac{2\pi}{p}$ be the time of a small ascillation, show that p is given by the equation
$p^2\,[(2M+M')\,k^2 - M'\,h^2] = (2M+M')\,gh,$
where M and M′ are the masses of the roller and counterpoise, k is the radius of gyration of M′ about the axis of the cylinder and h is the distance of its centre of mass from the axis.

Sol. Suppose O' is the centre of the cylinderical roller and G is the centre of gravity of the counterpoise, whose mass is M'. Obviously the line $O'K$ is a line fixed in the roller and the line $O'G$ is fixed in the counterpoise which were initially vertical. Suppose they make angles θ and ϕ with the vertical at the time t. Initially K was at O.

There is no slipping between the cylinder and the horizontal plane, hence $OB =$ arc $KB = a\,\theta$. Now assuming O as origin and the horizontal and vertical through O as co-ordinates axes, we have co-ordinates of G :-

$$x_G = a\,\theta + h\sin\phi\,,\ y_G = a - h\cos\phi,$$

$\therefore$ (velocity)^2 of $G = \dot{x}_G^2 + \dot{y}_G^2 = (a\,\dot\theta + h\cos\phi\,\dot\phi)^2 + (h\sin\phi\,\dot\phi)^2$

$$= a^2\dot\theta^2 + h^2\dot\phi^2 + 2\,a\,h\,\dot\theta\,\dot\phi \qquad \text{[neglecting higher powers of } \phi]$$

But radius of gyration of the counterpoise about O' is k, where $O'G = h$. Hence the square of radius of gyration about G is $(k^2 - h^2)$.

Also (velocity)^2 of $O' = (a\,\dot\theta)^2 = a^2\,\dot\theta^2$.

[$\therefore$ co-ordinates of O' are $(a\,\theta,\ a)$]

Now let T, be the kinetic energy and W, the work function of the system then we get

$$T = \tfrac{1}{2}M\,(a^2\dot\theta^2 + a^2\dot\theta^2) + \tfrac{1}{2}M'\,[(k^2 - h^2)\,\dot\phi^2 + a^2\dot\theta^2 + h^2\dot\phi^2 + 2ah\,\dot\theta\,\dot h]$$

$$= \frac{M+M'}{2}\,a^2\,\dot\theta^2 + \tfrac{1}{2}M'\,k^2\dot\phi^2 + M'\,ah\dot\theta\,\dot\phi.$$

$$W = M'\,gh\cos\phi + D.$$

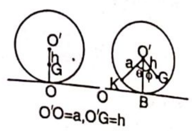
$\therefore$ Lagrange's θ-equation gives $\frac{d}{dt}\,[(2M+M')\,a^2\dot\theta + M'\,ah\,\dot\phi] = 0$

$$\Rightarrow (2M+M')\,a\,\ddot\theta + M'\,h\,\ddot\phi = 0 \qquad ...(1)$$

While Lagrange's ϕ-equation gives

$$\frac{d}{dt}\,[M'k^2\dot\phi + M'\,ah\dot\theta] = -M'\,gh\,\phi \Rightarrow k^2\ddot\phi + ah\,\ddot\theta = -gh\phi$$

$$\Rightarrow ah\ddot\theta + k^2\,\ddot\phi = -\,gh\,\phi, \qquad ...(2)$$



$$O'O = a, \quad O'G = h$$

Now eliminating $\ddot\theta$ between (1) and (2), we get

$$[M\,h^2 - (2M+M')k^2]\,\ddot\phi = (2M+M')\,gh\phi.$$

or $\quad [(2M+M')\,k^2 - M'\,h^2]\,\ddot\phi = -\,(2M+M')\,gh\,\phi \qquad ...(3)$

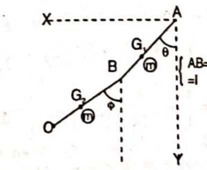
But, if $\frac{2\pi}{p}$ is the time of small oscillation, then we assume $\phi = A\cos(pt+b)$

$\therefore\ \dot\phi = -\,pA\sin(pt+B)$ and $\ddot\phi = -\,p^2A\cos(pt+B) = -\,p^2\phi$

$\therefore$ Equation (3) gives

$$-\,p^2\,[(2M+M')\,k^2 - M'h^2] = -\,(2M+M')\,gh \qquad [\because\ \phi \neq 0]$$

Ex. 22. *Two equal rods AB and BC; each of length L, smoothly jointed at B, are suspended from A and oscillate in a vertical plane through A. Show that the periods of normal oscillations are* $\frac{2\pi}{n}$ *where* $n^2 = \left(3 \pm \frac{6}{\sqrt{7}}\right)\frac{g}{l}$.



Sol. At any time t, let the rod AB and BC be inclined at angles θ and ϕ to the vertical.

$\therefore$ Co-ordinates of G_1 are $\left(\frac{l}{2}\sin\theta,\ \frac{l}{2}\cos\theta\right)$

$$\therefore\ v_{G_1}^2 = \left(\frac{l}{2}\cos\theta\,\dot\theta\right)^2 + \left(-\frac{l}{2}\sin\theta\,\dot\theta\right)^2 = \frac{l^2}{4}\,\dot\theta^2$$

$$X_{G_2} = l\sin\theta + \frac{l}{2}\sin\phi,\ y_{G_2} = l\cos\theta + \frac{l}{2}\cos\phi$$

$$\therefore\ v_{G_2}^2 = l^2\dot\theta^2 + \tfrac{1}{2}l^2\dot\phi^2 + l^2\dot\theta\,\dot\phi\cos(\theta - \phi) = l^2\,\dot\theta^2 + \tfrac{1}{4}\,l^2\dot\phi^2 + l^2\dot\theta\,\dot\phi$$

$[\because\ \theta$ and ϕ are small$]$

now, let T, be the kinetic energy and W, the work function, of the system, then we have $W = mg\frac{l}{2}\cos\theta + mg\left(l\cos\theta + \frac{l}{2}\cos\phi\right)$

$$= \frac{1}{2} mgl(3\cos\theta + \cos\phi)$$

and $T = \frac{1}{2} m\,[\frac{1}{12} l^2\dot{\theta}^2 + \frac{1}{2} l^2\dot{\theta}^2] + \frac{1}{2} m\,\{\frac{1}{12} l^2\dot{\phi}^2 + l^2(\dot{\theta}^2 + \frac{1}{4}\dot{\phi}^2 + \dot{\theta}\dot{\phi})\}$

$$= \frac{1}{2} ml^2 (\tfrac{4}{3}\dot{\theta}^2 + \tfrac{1}{3}\dot{\phi}^2 + \dot{\theta}\dot{\phi})$$

$\therefore$ Lagrange's θ-equations is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = \frac{\partial W}{\partial \theta} \Rightarrow \frac{d}{dt}\left(\frac{4}{3} l\dot{\theta} + \frac{1}{2} l\dot{\phi}\right) = -\frac{3}{2} g\theta$$

$$\Rightarrow 8\ddot{\theta} + 3\ddot{\phi} = -\frac{9g}{l}\theta \qquad ...(1)$$

While Lagrange's ϕ equation gives

$$\frac{d}{dt}(\tfrac{1}{3} l\dot{\phi} + \tfrac{1}{2} l\dot{\theta}) = -\tfrac{1}{2} g\phi \Rightarrow 3\ddot{\theta} + 2\ddot{\phi} = -\frac{3g}{l}\phi \qquad ...(2)$$

Equations (1) and (2) can be written as

$$(8D^2 + 9c)\theta + 3D^2\phi = 0 \qquad ...(3)$$

and $3D^2\theta + (2D^2 + 3c)\phi = 0$ where $c = \left(\frac{g}{l}\right)$ ...(4)

Eliminating ϕ between (3) and (4), we get

$$[(8D^2 + 9c)(2D^2 + 3c) - 9D^4]\theta = 0$$

i.e $(7D^2 + 42\,cD^2 + 27\,c^2)\theta = 0$ ...(5)

Let us assume, $\theta = A\cos(nt + B) \Rightarrow D\theta = nA\sin(nt + B)$

$$D^2\theta = -n^2 A\cos(nt + B) = -n^2\theta \text{ and } D^4\theta = n^4\theta$$

Substituting these values in (5), we get

$(7n^4 - 42cn^2 + 27c^2)\theta = 0$ i.e. $7n^4 - 42cn^2 - 27c^2 = 0$ $[\because \theta \neq 0]$

i.e. $n^2 = \frac{42c \pm \sqrt{[\{(42c)^2 - 4(7)(27c^2)\}]}}{14}$

$$= \left[3 \pm \frac{6}{14}\sqrt{(28)}\right]c = \left(3 \pm \frac{6}{\sqrt{7}}\right)c = \left\{3 \pm \frac{6}{\sqrt{7}}\right\}\frac{g}{l}.$$

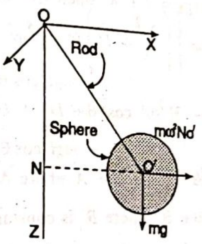
Ex. 23. *A solid uniform sphere has a light rod rigidly attached to it which passes through its centre. This rod is so jointed to a fixed vertical axis that the angle θ between the rod and the axis may alter but the rod must turn with the axis. If the vertical axis be forced to revolve constantly with uniform angular velocity, show that the equation of motion is of the form*

$$\dot{\theta}^2 = n^2(\cos\theta - \cos\beta)(\cos\alpha - \cos\theta)$$

Show also that the total energy imparted to the sphere as θ increases from θ_1 to θ_2 varies as $\cos^2\theta_1 - \cos^2\theta_2$. **[Meerut 1989, Agra 86]**

Sol. First of all, we reduce the revolving axis OZ to rest by introducing an additional force $m\omega^2 NO'$, along NO' i.e. a force $m\omega^2 l\sin\theta$ along NO'.

After the interval of time t. let the rod make an angle θ with the vertical ONZ (taken as Z-axis. (Further let ϕ be the angle through which the sphere turn in the horizontal plane, then the co-ordinates of C.G. of the sphere, are given by



$$x_{0'} = l\sin\theta\cos\phi.\ y_{0'} = l\sin\theta\sin\phi,\ z_{0'} = l\cos\theta$$

$$\therefore \dot{x}_0' = l\dot{\theta}\cos\theta\cos\phi - l\dot{\phi}\sin\theta\sin\phi,$$

$$\dot{y}_0' = l\dot{\theta}\cos\theta\cos\phi + l\dot{\phi}\sin\theta\cos\phi,\quad \dot{z}_0' = -l\dot{\theta}\sin\phi$$

$$\Rightarrow \dot{x}_0'^2 + \dot{y}_0'^2 + \dot{z}_0'^2 = l^2\dot{\theta}^2 + l^2\sin^2\theta\dot{\phi}^2$$

$$\therefore T = \frac{1}{2}m\left[\frac{2a^2}{5}\dot{\theta}^2 + \dot{x}_{0'}^2 + \dot{y}_{0'}^2 + \dot{z}_{0'}^2\right]$$

$$= \frac{1}{2}m\left[\frac{2a^2}{5}\dot{\theta}^2 + l^2\dot{\theta}^2 + l^2\dot{\phi}^2\sin^2\theta\right]$$

$$= \frac{1}{2}m\left[\left(\frac{2a^2}{5} l^2\right)\dot{\theta}^2 + l^2\omega^2\sin^2\theta\right] \qquad (\because \dot{\phi} \neq \omega \text{ given})$$

and $W^* = mgl\cos\theta + m\omega^2 l\sin\theta.\, l\sin\theta = mgl\cos\theta + m\omega^2 l^2\sin^2\theta$ ...(2)

$\because$ Lagrange's θ-equation gives

$$\frac{d}{dt}\left\{m\left(\frac{2a^2}{5} + l^2\right)\dot{\theta}\right\} - \frac{1}{2}ml^2\omega^2.2\sin\theta\cos\theta$$

$$= -mgl\sin\theta + 2m\omega^2 l^2\sin\theta\cos\theta$$

$$\Rightarrow \left(\frac{2a^2}{5} + l^2\right)\ddot{\theta} = -mgl\sin\theta + 3m\,\omega^2 l^2\sin\theta\cos\theta$$

Multiplying both sides by $2\dot{\theta}$ and integrating, we get

$$\left(\tfrac{2}{5}a^2 + l^2\right)\dot{\theta}^2 = 2gl\cos\theta - 3l^2\omega^2\cos^2\theta + D$$

$$\Rightarrow \dot{\theta}^2 = n^2(\cos\theta - \cos\beta)(\cos\alpha - \cos\theta)$$

where n is constant and α, β are the values of θ for which $\dot{\theta}$ vanish.

SECOND PART.

**K.E+ Potential Energy $= T + K$, when P.E. is K and is given by

$$K = T - W + C = \tfrac{1}{2}m\left[\left(\frac{2a^2}{5} + l^2\right)\dot{\theta}^2 + l^2\omega^2\sin^2\theta\right] - mgl\cos\theta - m\omega^2 l^2\sin^2\theta + C$$

$$= \tfrac{1}{2}m\,[\{2gl\cos\theta - 3l^2\omega^2\cos^2\theta + D\} + l^2\omega^2\sin^2\theta] - mgl\cos\theta - ml^2\omega^2\sin^2\theta + C$$

$$= -\tfrac{1}{2}m\omega^2l^2\cos^2\theta - \tfrac{1}{2}m\omega^2l^2 + A \text{ where } A \text{ is constant.}$$

$$= -\tfrac{1}{2}m\omega^2l^2\cos^2\theta + B \text{ where } B \text{ is constant}$$

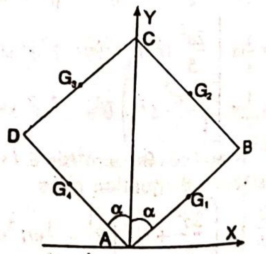
$\therefore$ Total Energy imparted

$$= \left[-\tfrac{1}{2}m\omega^2l^2\cos^2\theta + B\right]_{\theta_1}^{\theta_2} = \tfrac{1}{2}m\omega^2l^2(\cos^2\theta_1 - \cos^2\theta_2)$$

which varies as $(\cos^2\theta_1 - \cos^2\theta_2)$.

Ex. 24. *Four uniform rods, each of length $2a$, are hinged at their ends so as to form a rhombus ABCD. The angles B and D are connected by an elastic string and the lowest end A rests on a horizontal plane while the end C slides on a smooth vertical wire passing through A; in the position of equlibrium the string is strected to twice its natural length and the angle BAD is 2α. Show that the time of a small oscillation about this position*

is $2\pi\left\{\dfrac{2a(1 + 3\sin^2\alpha)}{3g\cos 2\alpha}\cos\alpha\right\}^{1/2}$.



Sol. In the position of equilibrium, rods are making angles α with the vertical. When the system is slightly displaced from the position of equilibrium, let the rods make angle $(\alpha + \theta)$ with the vertical θ being a small displacement. Now assuming the fixed end A as origin and the horizontal and vertical lines through it as co-ordinate axes, the co-ordinates of G_2 are $\{a\sin(\alpha + \theta), 3a\cos(\alpha + \theta)\}$

$\therefore$ (velocity)^2 of $G_2 = \{a\cos(\alpha + \theta)\dot{\theta}\}^2 + \{-3a\sin(\alpha + \theta)\dot{\theta}\}^2$

$$= a^2[(1 + 8\sin^2(\alpha + \theta)]\dot{\theta}^2.$$

Co-ordinates of G_1 are $\{a\sin(\alpha + \theta), a\cos(\alpha + \theta)\}$

$\therefore$ (velocity)^2 of $G_1 = a^2\dot{\theta}^2$

$\therefore$ Kinetic energy of the four rods taken together is

$$T = 2\cdot\tfrac{1}{2}m\cdot\left[\frac{a^2}{3}\dot{\theta}^2 + a^2\dot{\theta}^2\right] + 2\cdot\tfrac{1}{2}m\left[\frac{a^2}{3}\dot{\theta}^2 + a^2\{1 + 8\sin^2(\alpha + \theta)\}\dot{\theta}^2\right]$$

$$= \frac{8ma^2}{3}\left[1 + \sin^2(\alpha + \theta)\right]\dot{\theta}^2 \quad (\because v_{G_1} = v_{G_4} \text{ and } v_{G_2} = v_{G_3})$$

The work function W is given by $W = 2\{-mga\cos(\alpha + \theta)\}$

$$+ 2\{-mg\,3a\cos(\alpha + \theta)\} - 2\int_0^{2a\sin(\theta + \alpha)}\lambda\left(\frac{y - c}{c}\right)dy$$

$$= -8mga\cos(\alpha + \theta) - \frac{\lambda}{c}\{2a\sin(\alpha + \theta) - c\}^2$$

Lagrange's θ equation gives

$$\frac{d}{dt}\left[\frac{16ma^2}{3}\{1 + 3\sin^2(\alpha + \theta)\}\dot{\theta}\right] - 16ma^2\sin(\alpha + \theta)\cos(\alpha + \theta)\dot{\theta}^2$$

$$= 8mga\sin(\alpha + \theta) - \frac{4\lambda}{c}a\cos(\alpha + \theta)\{2a\sin(\alpha + \theta) - c\}$$

$$\Rightarrow \frac{16ma^2}{3}(1 + 3\sin^2(\alpha + \theta)\}\ddot{\theta}$$

$$= 8mga\sin(\alpha + \theta) - \frac{4\lambda a}{c}\cos(\alpha + \theta)\{2a\sin(\alpha + \theta) - c\} \quad ...(1)$$

Initially when $\theta = 0$, $\dot{\theta} = 0$, $\ddot{\theta} = 0$, $c = a\sin\alpha$, hence (1) gives

$$\lambda = \frac{2mgc}{a\cos\alpha}.$$

Putting this value of λ in equation (1), we get

$$\frac{16ma^2}{3}\{1 + 3\sin^2(\alpha + \theta)\}\ddot{\theta}$$

$$= mga\,8\sin(\alpha + \theta) - \frac{8mg\cos(\alpha + \theta)}{\cos\alpha}\{2a\sin(\alpha + \theta) - c\}$$

---

*The force $m\omega^2 l\sin\theta$ also contributes to W. The distance of the point of applicat ion at O' of this force from the vertical OZ is equal to $l\sin\theta$, hence the contribution $m\omega^2l^2\sin^2\theta$ to W is as given in (2).

** If W is the work function of the system, then P.E. $= C - W$.

i.e. $\frac{16ma^2}{3}\{1+3\sin^2\alpha\}\ddot{\theta} = 8mga(\sin\alpha+\theta\cos\alpha)$

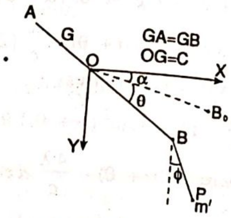
$$-\frac{8mg}{\cos\alpha}(\cos\alpha-\theta\sin\alpha)\{2a(\sin\alpha+\theta\cos\alpha)-a\sin\alpha\}.$$

$$= -\frac{8mga\cos 2\alpha}{\cos\alpha}\theta \text{ app.} \Rightarrow \ddot{\theta} = -\frac{3g\cos 2\alpha}{2a\cos\alpha(1+3\sin^2\alpha)}\theta \text{ app.}$$

$\therefore$ Time of a small oscillation about the position of equilibrium is given by $2\pi\sqrt{\left\{\frac{2a\cos\alpha(1+3\sin^2\alpha)}{3g\cos 2\alpha}\right\}}$.

Ex. 25. *A uniform rod AB, of length 2a can turn freely about a point distance c from its centre, and is at rest at an angle α to the horizon when a particle is hung by a light string of length l from one end. If the particle be displaced slightly in the vertical plane of the rod show that it will oscillate in the same time as a simple pendulum of length* $l\frac{a^2+3ac\cos^2\alpha+3c^2\sin^2\alpha}{a^2+3ac}$. **[Meerut 95]**

Sol. Let M be the mass and G the centre of gravity of the rod. Further let O be the point about which it can turn where $OG = c$. Let BP be a string of length l and m the mass of the particle tied to the end P of the string BP. Initially rod was making an angle α with the horizontal and the string was vertical.



Let B_0 be the initial position of B so that $\angle B_0OX = \alpha$. After a time t let the rod make an angle θ with OB_0 i.e. an angle $(\alpha+\theta)$ with the horizontal and let the string be inclined at an angle ϕ with the vertical.

Initially the system was at rest, hence taking moments about O, (in this position), we get $Mc = m(a-c)$.

Assuming O as origin and the horizontal and the vertical through O as co-ordinates axes, we obtain the following results

$x_p = (a-c)\cos(\alpha+\theta) + l\sin\phi$. $y_p = (a-c)\sin(\alpha+\theta) + l\cos\phi$.

$\therefore$ (velocity)^2 of $P = \{-(a-c)\sin(\alpha+\theta)\dot{\theta} + l\cos\phi\dot{\phi}\}^2$

$$+\{(a-c)\cos(\alpha+\theta)\dot{\theta} - l\sin\phi\dot{\phi}\}^2$$

$$= (a-c)^2\dot{\theta}^2 + l^2\dot{\phi}^2 - 2l(a-c)\dot{\theta}\dot{\phi}\sin(\alpha+\theta+\phi)$$

$$= (a-c)^2\dot{\theta}^2 + l^2\dot{\phi}^2 - 2l(a-c)\dot{\theta}\dot{\phi}\sin\alpha$$

(neglecting small quantities of the higher order)

Also co-ordinates of G (the C.G. of the rod) are $\{-c\cos(\alpha+\theta), -c\sin(\alpha+\theta)\}$

$\therefore$ (velocity)^2 of $G = \{c\sin(\alpha+\theta)\dot{\theta}\}^2 + \{-c\cos(\alpha+\theta)\dot{\theta}\}^2 = c^2\dot{\theta}^2$.

Now let T, be the kinetic energy and W, the work function of the system, then we easily obtain

$$W = mg[(a-c)\sin(\alpha+\theta) + l\cos\phi] - Mgc\sin(\alpha+\theta) + D$$

$$= mgl\cos\phi + D$$

[the first and the last terms cancel, because $m(a-c) = Mc$]

and $T = \frac{1}{2}M\left[\frac{a^2}{3}\dot{\theta}^2 + c^2\dot{\theta}^2\right] + \frac{1}{2}m[(a-c)^2\dot{\theta}^2 + l^2\dot{\phi}^2 - 2l(a-c)\dot{\theta}\dot{\phi}\sin\alpha]$

$\therefore$ Lagrange's θ-equation gives

$$\frac{d}{dt}\left[\left\{M\left(\frac{a^2}{3}+c^2\right)+m(a-c)^2\right\}\dot{\theta} - ml(a-c)\dot{\phi}\sin\alpha\right] = 0$$

$$\Rightarrow \left\{M\left(\frac{a^2}{3}+c^2\right)+m(a-c)^2\right\}\ddot{\theta} - ml(a-c)\ddot{\phi}\sin\alpha = 0.$$

$$\Rightarrow \left\{M\left(\frac{a^2}{3}+c^2\right)+Mc(a-c)\right\}\ddot{\theta} - Mcl\ddot{\phi}\sin\alpha = 0 \quad [\because m(a-c) = Mc]$$

$\Rightarrow (a^2+3ac)\ddot{\theta} - 3lc\sin\alpha\ddot{\phi} = 0$. While Lagrange's ϕ equation gives

$$\frac{d}{dt}\left\{ml^2\dot{\phi} - ml(a-c)\dot{\theta}\sin\alpha\right\} = -mgl\phi$$

$$\Rightarrow l\ddot{\phi} - (a-c)\ddot{\theta}\sin\alpha = -g\phi \qquad ...(2)$$

Eliminating θ in (1) and (2), we have $l\ddot{\phi} - (a-c)\sin\alpha\frac{3lc\sin\alpha}{a^2+3ac}\ddot{\phi} = -g\phi$

i.e. $\Rightarrow \frac{a^2+3ac-3c(a-c)\sin^2\alpha}{a^2+3ac}l\ddot{\phi} = -g\phi$

$$\Rightarrow \frac{a^2+3ac\cos^2\alpha+3c\sin^2\alpha}{a^2+3ac}l\ddot{\phi} = -g\phi$$

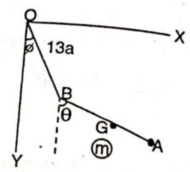
$$\Rightarrow \ddot{\phi} = -\frac{g}{l(a^2+3ac\cos^2\alpha+3c^2\sin^2\alpha)}\phi = -\mu\phi \text{ (say)}$$

$\therefore$ Length of the simple equivalent pendulum $= \frac{g}{\mu}$

$$= \frac{a^2+3ac\cos^2\alpha+3c^2\sin^2\alpha}{a^2+3ac}l.$$

Ex. 26. *A uniform rod AB of length 8a is suspended from a fixed point O by means of light inextensible string, of length 13a, attached to B. If the system is slightly displaced in a vertical plane, show that $(\theta+3\phi)$ and $(12\theta-13\phi)$ are principal co-ordinates where θ and ϕ are the angles which the rod and string respectively make with the vertical. Also show that periods of small oscillations are*

$$2\pi\sqrt{\left(\frac{a}{g}\right)} \text{ and } 2\pi\sqrt{\left(\frac{52a}{3g}\right)}.$$



Sol. We have

$x_G = 13a\sin\phi + 4a\sin\theta$

and $y_G = 13a\cos\phi + 4a\cos\theta$

$\therefore \dot{x}_G^2 + \dot{y}_G^2 = 169a^2\dot{\phi}^2 + 16a^2\dot{\theta}^2$

$+ 104a^2\dot{\theta}\dot{\phi}\cos(\theta-\phi)$

Thus, $T = \frac{1}{2}m[k^2\dot{\theta}^2 + (\dot{x}_G^2 + \dot{y}_G^2)] = \frac{1}{2}ma^2\left[\frac{64}{3}\dot{\theta}^2 + 169\dot{\phi}^2 + 104\dot{\theta}\dot{\phi}\right]$

and the work function

$W = mg(13a\cos\phi + 4a\cos\theta)$

$\therefore$ Lagrange's θ-equation gives

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right) - \frac{\partial T}{\partial\theta} \Rightarrow 61\ddot{\theta} + 39\ddot{\phi} = -\frac{3g}{a}\theta \qquad ...(1)$$

While Lagrange's ϕ-equation gives $4\ddot{\theta} + 13\ddot{\phi} = -\frac{g}{a}\phi$ ...(2)

Equations (1) and (2) $\Rightarrow D^2(\theta+3\phi) = -\frac{3g}{52a}(\theta+3\phi)$

and $D^2(12\theta-13\phi) = -\frac{g}{a}(12\theta-13\phi)$

Now putting $\theta+3\phi = X$ and $12\theta-13\phi = Y$ in these equations, we get $D^2X = -\frac{3g}{52a}X$ and $D^2Y = -\frac{g}{a}.Y$;

which obviously represent two independent simple harmonic motions. Hence X and Y are principal co-ordinates, that is $(\theta+3\phi)$ and $(12\theta-13\phi)$ are principal co-ordinates.

Also, periods of small oscillations are given by

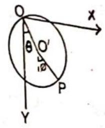
$$2\pi\sqrt{\left(\frac{52a}{3g}\right)} \text{ and } 2\pi\sqrt{\left(\frac{a}{g}\right)}$$

Ex. 27. *A ring slides on a smooth circular hoop of equal mass and of radius a which can turn in a vertical plane about a fixed point O in its circumference. If θ and ϕ be the inclinations to the vertical of the radius through O and of the radius through the ring, prove that principal co-ordinates are $(2\theta+\phi)$ and $(\phi-\theta)$, and the periods of small oscillations are*

$$2\pi\sqrt{\left(\frac{a}{2g}\right)} \text{ and } 2\pi\sqrt{\left(\frac{2a}{g}\right)}.$$

[Agra 1994]

Sol. Let the mass of the hoop and the ring be m. In a displaced position, let O' be the centre of the hoop and P the position of the ring which slides on the hoop.



$\therefore$ Co-ordiantes of the point P are given by

$x_P = a(\sin\theta + \sin\phi)$

$y_P = a(\cos\theta + \cos\phi)$

$\Rightarrow \dot{x}_P^2 + \dot{y}_P^2 = a^2(\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi})$

when θ and ϕ are small.

Also co-ordinates of O' are $(a\sin\theta, a\cos\theta)$

$\therefore$ (velocity)^2 of $O' = a^2\dot{\theta}^2$

Thus T= Energy of circular hoop+ Energy of ring

$= \frac{1}{2}m(a^2\dot{\theta}^2 + a^2\dot{\theta}^2) + \frac{1}{2}ma^2(\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}) = \frac{1}{2}ma^2(3\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi})$

Also the work function

$W = mga\cos\theta + mg(a\cos\theta + a\cos\phi) = mga(2\cos\theta + \cos\phi)$.

$\therefore$ Lagrange's θ-equation gives $3\ddot{\theta} + \ddot{\phi} = -\frac{2g}{a}\theta$ ...(1)

Lagrange's ϕ-equation gives $\ddot{\theta} + \ddot{\phi} = -\frac{g}{a}\phi$ ...(2)

Now adding (1) and (2) and multiplying (2) by 2 and subtracting (1) from that, we get $D^2(2\theta+\phi) = -\frac{g}{2a}(2\theta+\phi)$

and $D^2(\phi-\theta) = -\frac{2g}{a}(\phi-\theta)$

Taking $2\theta+\phi = X$ and $\phi-\theta = Y$, we get

$D^2X = -\frac{g}{2a}X$ and $D^2Y = -\frac{2g}{a}Y$;

which obviously represent two independent simple harmonic motions. Thus X and Y are principal co-ordinates, that is $(2\theta+\phi)$ and $(\phi-\theta)$ are principal co-ordinates.

Also, periods of small oscillations are given by

$$2\pi\sqrt{\left(\frac{2a}{g}\right)} \text{ and } 2\pi\sqrt{\left(\frac{a}{2g}\right)}.$$

[Meerut 1995]

7.14. Lagrange's Equations with impulsive forces

When the forces are finite, we have by 7.07

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = \phi_\alpha \text{ where } \phi_\alpha = \sum_\nu F_\nu \bullet \frac{\partial r_\nu}{\partial q_\alpha}$$

Integrating both sides of (1) w.r.t "t" form $t = 0$ to $t = \tau$, we get

$$\left[\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)\right]_{t=0}^{t=\tau} - \int_0^\tau \frac{\partial T}{\partial q_\alpha}\,dt = \int_{t=0}^{\tau} \phi_\alpha\,dt = \sum_\nu \left\{\left(\int_{t=0}^{\tau} F_\nu\,dt\right) \bullet \frac{\partial r_\nu}{\partial q_\alpha}\right\}$$

$$\Rightarrow \left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)_{t=\tau} - \left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)_{t=0} - \int_0^\tau \frac{\partial T}{\partial q_\alpha}\,dt = \sum_\nu \left\{\left(\int_0^\tau F_\nu\,dt\right) \bullet \frac{\partial r_\nu}{\partial q_\alpha}\right\} \qquad \text{...(2)}$$

Taking the limit as $\tau \to 0$, we get

$$\lim_{\tau \to 0}\left\{\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)_{t=\tau} - \left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)_{t=0}\right\} - \lim_{\tau \to 0}\int_0^\tau \frac{\partial T}{\partial q_\alpha}\,dt$$

$$= \sum_\nu \left\{\left(\lim_{\tau \to 0}\int_0^\tau F_\nu\,dt\right) \bullet \frac{\partial r_\nu}{\partial q_\alpha}\right\}$$

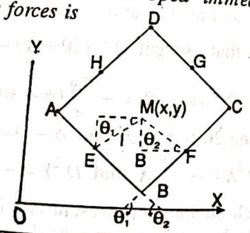
$$\Rightarrow \left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)_1 - \left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)_0 = \Sigma^* I_\nu \bullet \frac{\partial r_\nu}{\partial q_\alpha} = {}^{**}P_\alpha \text{ (say)} \qquad \text{...(3)}$$

where subscripts 0 and 1 denote respectively quantities before and after the application of the implusive force.

These equations are known as Lagrange's equations under implusive forces.

Ex. 28. *A square ABCD formed by four rods each of length 2l and mass hinged at their ends, rests on a horizontal frictionless table. An implusive of magnitube I is applied to the vertex A in the direction AD Find the equations of motion, and prove that tha K.E. devloped immediately after the application of the impulsive forces is*

$$T = (l^2/2m)$$



Sol. When the square is struck, its shape will in general be a rhombus. Suppose that at any time t, the angles made by sides AD (or BC) and BA (or CD) with the x-axis are θ_1 and θ_2 respectively, while the co-ordinates of the centre M are (x, y).

*Suppose that the force $\mathbf{F}_\nu$ acting on a system are such that $\lim_{\tau \to 0}\int_0^\tau \mathbf{F}_\nu\,dt = \mathbf{I}_\nu$ where τ represents a time interval, Then we call $\mathbf{F}_\nu$ impulsive forces and $\mathbf{I}_\nu$ are called Impulsse.

** If we call P_α to be the generalised impules, (3) can be written as

Hence x, y, θ_1, θ_2 are the generalised co-ordinates.

From the adjoining diagram, we see that the position vectors of the centre E, F, G, H of the rods are given respectively by

$$\mathbf{r}_E = (x - l\cos\theta_1)\,\mathbf{i} + (y - l\sin\theta_1)\,\mathbf{j},$$
$$\mathbf{r}_F = (x + l\cos\theta_2)\,\mathbf{i} + (y - l\sin\theta_2)\,\mathbf{j},$$
$$\mathbf{r}_G = (x + l\cos\theta_1)\,\mathbf{i} + (y + l\sin\theta_1)\,\mathbf{j},$$

and $\mathbf{r}_H = (x - l\cos\theta_2)\,\mathbf{i} + (y + l\sin\theta_1)\,\mathbf{j}.$

$$\therefore \quad \mathbf{v}_E = \dot{\mathbf{r}}_E = (\dot{x} + l\sin\theta_1\,\dot{\theta}_1)\,\mathbf{i} + (\dot{y} - l\cos\theta_1\,\dot{\theta}_1)\,\mathbf{j},$$

$$\mathbf{v}_F = \dot{\mathbf{r}}_F = (\dot{x} - l\sin\theta_2\,\dot{\theta}_2)\,\mathbf{i} + (\dot{y} - l\cos\theta_2\,\dot{\theta}_2)\,\mathbf{j},$$

$$\mathbf{v}_G = \dot{\mathbf{r}}_G = (\dot{x} - l\sin\theta_1\,\dot{\theta}_1)\,\mathbf{i} + (\dot{y} + l\cos\theta_1\,\dot{\theta}_1)\,\mathbf{j},$$

$$\mathbf{v}_H = \dot{\mathbf{r}}_H = (\dot{x} + l\sin\theta_2\,\dot{\theta}_2)\,\mathbf{i} + (\dot{y} + l\cos\theta_2\,\dot{\theta}_2)\,\mathbf{j}.$$

Now, K.E. of the rod $AB = \frac{1}{2}m\,\dot{\mathbf{r}}_E^2 + \frac{1}{3}m\,l^2\dot{\theta}_2^2 \cdot \frac{1}{2} = T_{AB}$ say

K.E. of the rod $CB = \frac{1}{2}m\,\dot{\mathbf{r}}_F^2 + \frac{1}{3}m\,l^2\dot{\theta}_1^2 \cdot \frac{1}{2} = T_{BC}$ say

K.E. of the rod $CD = \frac{1}{2}m\,\dot{\mathbf{r}}_G^2 + \frac{1}{3}m\,l^2\dot{\theta}_2^2 \cdot \frac{1}{2} = T_{CD}$ say

K.E. of the rod $DA = \frac{1}{2}m\,\dot{\mathbf{r}}_H^2 + \frac{1}{3}m\,l^2\dot{\theta}_1^2 \cdot \frac{1}{2} = T_{DA}$ say

$\therefore$ K.E. of system$= \frac{1}{2}m\,(\dot{\mathbf{r}}_E^2 + \dot{\mathbf{r}}_F^2 + \dot{\mathbf{r}}_G^2 + \dot{\mathbf{r}}_H^2) + \frac{2}{3}ml^2(\dot{\theta}_1^2 + \dot{\theta}_2^2) \cdot \frac{1}{2} = T$ say

or $T = \frac{1}{2}m\,(4\dot{x}^2 + 4\dot{y}^2 + 2l^2\dot{\theta}_1^2 + 2l^2\dot{\theta}_2^2) + \frac{ml^2}{3}(\dot{\theta}_1^2 + \dot{\theta}_2^2).$

$$= 2m\,(\dot{x}^2 + \dot{y}^2) + \frac{4}{3}ml^2(\dot{\theta}_1^2 + \dot{\theta}_2^2). \qquad \text{...(1)}$$

Let us now assume that initially the rhombus is a square at rest with its sides parallel to the co-ordinate axes and its centre located at the origin.

Then, we get $x = 0 = y,\ \theta_1 = \frac{\pi}{2},\ \theta_2 = 0, \dot{x} = 0,\ \dot{y} = 0, \dot{\theta}_1 = 0, \dot{\theta}_2 = 0.$

Now if we use the notation, $(\ )_1$ and $(\ )_2$ to denote quantities before and after the impulse is applied, we have

$$\left(\frac{\partial T}{\partial \dot{x}}\right)_1 = (4m\dot{x})_1 = 0 \left(\frac{\partial T}{\partial \dot{y}}\right)_1 = (4m\dot{y})_1 = 0,$$

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$$\left(\frac{\partial T}{\partial \dot\theta}\right)_1 = \left(\frac{8}{3} ml^2 \dot\theta_1\right) = 0, \left(\frac{\partial T}{\partial \dot\theta_2}\right)_1 = \left(\frac{8}{3} ml^2 \dot\theta_2\right) = 0$$

and
$$\left(\frac{\partial T}{\partial \dot x}\right)_2 = (4m\dot x)_2 = 4m\dot x, \left(\frac{\partial T}{\partial \dot y}\right)_2 = (4m\dot y)_2 = 4m,$$

$$\left(\frac{\partial T}{\partial \dot\theta_1}\right)_2 = \left(\frac{8}{3} ml^2 \dot\theta_1\right)_2 = \frac{8}{3} ml^2 \dot\theta_1, \left(\frac{\partial T}{\partial \dot\theta_2}\right)_2 = \frac{8}{3} ml^2 \dot\theta_2 .$$

Hence Lagrange's equations are

$$\left(\frac{\partial T}{\partial \dot x}\right)_2 - \left(\frac{\partial T}{\partial \dot x}\right)_1 = P_x \Rightarrow 4m\dot x = P_x \quad ...(2)$$

$$\left(\frac{\partial T}{\partial \dot y}\right)_2 - \left(\frac{\partial T}{\partial \dot y}\right)_1 = P_y \Rightarrow 4m\dot y = P_y \quad ...(3)$$

$$\left(\frac{\partial T}{\partial \dot\theta_1}\right)_2 - \left(\frac{\partial T}{\partial \dot\theta_1}\right)_1 = P_{\theta_1} \Rightarrow \frac{8}{3} ml^2 \dot\theta_1 = P_{\theta_1}. \quad ...(4)$$

$$\left(\frac{\partial T}{\partial \dot\theta_1}\right)_2 - \left(\frac{\partial T}{\partial \dot\theta_2}\right)_1 = P_{\theta_1} \Rightarrow \frac{8}{3} ml^2 \dot\theta_2 = P_{\theta_2}. \quad ...(5)$$

Now we shall find the value of P_x, P_y, P_{θ_1} and P_{θ_2}

We have $P_\alpha = \sum_v \mathbf{I}_v \cdot \frac{\partial \mathbf{r}_v}{\partial q_\alpha}$ where $\mathbf{I}_v$ are the impulsive forces.

$$\therefore P_x = \mathbf{I}_A \cdot \frac{\partial \mathbf{r_A}}{\partial x} + \mathbf{I}_B \cdot \frac{\partial \mathbf{r_B}}{\partial x} + \mathbf{I_C} \cdot \frac{\partial \mathbf{r_C}}{\partial x} + \mathbf{I_D} \cdot \frac{\partial \mathbf{r_D}}{\partial x} \quad ...(6)$$

$$P_y = \mathbf{I}_A \cdot \frac{\partial \mathbf{r_A}}{\partial y} + \mathbf{I}_B \cdot \frac{\partial \mathbf{r_B}}{\partial y} + \mathbf{I_C} \cdot \frac{\partial \mathbf{r_C}}{\partial y} + \mathbf{I_D} \cdot \frac{\partial \mathbf{r_D}}{\partial y} \quad ...(7)$$

$$P_{\theta_1} = \mathbf{I}_A \cdot \frac{\partial \mathbf{r_A}}{\partial \theta_1} + \mathbf{I}_B \cdot \frac{\partial \mathbf{r_B}}{\partial \theta_1} + \mathbf{I_C} \cdot \frac{\partial \mathbf{r_C}}{\partial \theta_1} + \mathbf{I_D} \cdot \frac{\partial \mathbf{r_D}}{\partial \theta_1} \quad ...(8)$$

$$P_{\theta_2} = \mathbf{I}_A \cdot \frac{\partial \mathbf{r_A}}{\partial \theta_2} + \mathbf{I}_B \cdot \frac{\partial \mathbf{r_B}}{\partial \theta_2} + \mathbf{I_C} \cdot \frac{\partial \mathbf{r_C}}{\partial \theta_2} + \mathbf{I_D} \cdot \frac{\partial \mathbf{r_D}}{\partial \theta_2} \quad ...(9)$$

where

$$\mathbf{r}_A = (x - l\cos\theta_1 - l\cos\theta_2)\,\mathbf{i} + (y - l\sin\theta_1 + l\sin\theta_2)\,\mathbf{j} \quad ...(10)$$

$$\mathbf{r}_B = (x - l\cos\theta_1 + l\cos\theta_2)\,\mathbf{i} + (y - l\sin\theta_1 - l\sin\theta_2)\,\mathbf{j} \quad ...(11)$$

$$\mathbf{r}_C = (x + l\cos\theta_1 + l\cos\theta_2)\,\mathbf{i} + (y + l\sin\theta_1 - l\sin\theta_2)\,\mathbf{j} \quad ...(12)$$

$$\mathbf{r}_D = (x + l\cos\theta_1 - l\cos\theta_2)\,\mathbf{i} + (y + l\sin\theta_1 + l\sin\theta_2)\,\mathbf{j} \quad ...(13)$$

But initially the impulsive force at A is in the direction of the positive y-axis, we have

$$\mathbf{I}_A = I\,\mathbf{j}.$$

$\therefore$ Equation (6), (7), (8) and (9) $\Rightarrow$

$$P_x = 0, P_y = \mathrm{I}, P_{\theta_1} = -\mathrm{I}l\cos\theta_1 \text{ and } P_{\theta_2} = \mathrm{I}l\cos\theta_2. \quad ...(14)$$

Thus equations (2) and (3), (4) and (5) give ...(15)

$$4m\dot x = 0, 4m\dot y = 1, \tfrac{8}{3}ml^2\dot\theta_1 = -\mathrm{I}l\cos\theta_1, \tfrac{8}{3}ml^2\dot\theta_2 = \mathrm{I}l\cos\theta_2. \quad ...(16)$$

Second Part.

We have $\dot x = 0$, $\dot y = \frac{1}{4m}$, $\dot\theta_1 = -\frac{3\mathrm{I}}{8ml}\cos\theta_1$ and $\dot\theta_2 = \frac{3\mathrm{I}}{8ml}\cos\theta_2$.

$$\therefore T = 2m\,(\dot x^2 + \dot y^2) + \tfrac{4}{3}ml^2\,(\dot\theta_1^2 + \dot\theta_2^2)$$

$$= 2m\left(0 + \frac{\mathrm{I}^2}{16m^2}\right) + \tfrac{4}{3}ml^2\left[\frac{9\mathrm{I}^2}{64m^2l^2}\cos^2\theta_1 + \frac{9\mathrm{I}^2}{64m^2l^2}\cos^2\theta_2\right] \quad ...(17)$$

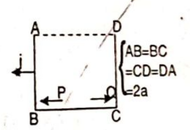
But immediately after the appiication of the implusive forces,

$\theta_1 = \frac{1}{2}\pi$ and $\theta_2 = 0$ approximately, so (17) gives $T = \frac{I^2}{2m}$.

Ex. 29. *Three equal uniform rods, AB, BC, CD are freely joined at B and C and the ends A and B are fastened to smooth, fixed pivots whose distance apart is equal to the length of either rod. The frame being at rest in the form of the square, a blow J is given perpendicular to AB at its middle point and in the plane of square. Show that the energy set up is $\frac{3J^2}{40m}$ where m is the mass of each rod.*

Find also the blows at the joints B and C.

Sol. The blow J is given at G_1 the C.G. of AB, at right angles to AB



The rods AB and CD will turn through the same angle θ but the rod **BC** will remain parallel to AD.

Now let T be the K.E. of the system, then we have

$$T = 2.\tfrac{1}{2}m\left[\frac{a^2}{3}\dot\theta^2 + a^2\dot\theta^2\right] + \tfrac{1}{2}m\,(2a\dot\theta)^2 = \frac{10ma^2}{3}\dot\theta^2.$$

$\therefore$ Lagrange's θ-equation gives

$$\left(\frac{\partial T}{\partial \dot\theta}\right) - \left(\frac{\partial T}{\partial \dot\theta}\right)_1 = J \Rightarrow \frac{20ma^2}{3}\dot\theta = Ja$$

i.e. $\dot\theta = \frac{3J}{20ma}$ $\quad\therefore T = \frac{10ma^2}{3} \cdot \frac{9J^2}{400m^2a^2} = \frac{3J^2}{40m}$.

SECOND PART.

Let P and Q be the impulses at B and C respectively. with directions, as shown in the figure, then considering the motion of AB and taking moments about A, we get

Change in the angular momentum about the axis through A

= moments of the impulses about this axis

i.e. $m\frac{4a^2}{3}\dot{\theta} = Ja - P\,2a \Rightarrow P = \frac{1}{2}J - \frac{2ma}{3}$

$$\therefore\ P = \frac{1}{2}J - \frac{2ma}{3}\cdot\frac{3J}{20ma} = \frac{2J}{5}$$

Again considering the motion of CD and taking moments about D, we obtain

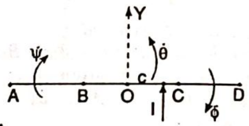
$$m\cdot\frac{4a^2}{3}\dot{\theta} = Q\,2a \Rightarrow Q = \frac{2ma}{3}\dot{\theta} = \frac{2ma}{3}\cdot\frac{3J}{20ma} = \frac{1}{10}J.$$

Ex. 30. *Three equal uniform rods AB, BC, CD, each of mass m and length $2a$, are at rest in a straight line smoothly jointed at B and C. A blow I is given to the middle rod at a distance c from the centre O in a direction perpendicular to it, show that the initial velocity of O is $\frac{2I}{3T}$, and that the initial angular velocities of the rods are*

$$\frac{(5a+9c)\,I}{10ma^2},\ \frac{6cI}{5ma^2} \text{ and } \frac{(5a-9c)\,I}{10ma^2}$$

(Nagpur 1993)

Sol. Just after the blow let $\dot{y}$ be the linear velocity of O, and let θ, ϕ, ψ be the inclination of BC, CD and AB to their initial positions. Now let T be K.E. of the system, then we have



$$T = \frac{1}{2}\left(\frac{a^2}{3}\dot{\theta}^2 + \dot{y}^2\right) + \frac{1}{2}m\left\{\frac{a^2}{3}\dot{\psi}^2 + (\dot{y} - a\dot{\theta} + a\dot{\psi})^2\right\}$$

$$+ \frac{1}{2}m\left\{\frac{a^2}{3}\dot{\phi}^2 + (\dot{y} + a\dot{\theta} + a\dot{\phi})^2\right\}$$

$$= \frac{1}{2}m\left[3\dot{y}^2 + \frac{7a^2}{3}\dot{\theta}^2 + \frac{4a^2}{3}\dot{\psi}^2 + \frac{4}{3}a^2\dot{\phi}^2 + 2a\dot{y}\dot{\psi} + 2a\dot{y}\dot{\phi} + 2a\dot{\theta}\dot{\phi} - 2a\dot{\theta}\dot{\psi}\right]$$

Also just before the action of the impluse, we have $T = 0$

$\therefore$ Lagrange's equation for blows gives

$$\left(\frac{\partial T}{\partial \dot{y}}\right)_2 - \left(\frac{\partial T}{\partial \dot{y}}\right)_1 = I \Rightarrow 3m\dot{y} + ma\dot{\phi} + ma\dot{\psi} = I \Rightarrow 3\dot{y} + a\dot{\theta} + a\dot{\psi} = \frac{I}{m}.$$

Similarly Lagrange's θ, ϕ, ψ equations give

$$\frac{7}{3}a\dot{\theta} + a\dot{\phi} - a\dot{\psi} = \frac{Ic}{ma},\ \dot{y} + \frac{4}{3}a\dot{\phi} + a\dot{\theta} = 0,\ \dot{y} - a\dot{\theta} + \frac{4}{3}a\dot{\psi} = 0.$$

Last two equations $\Rightarrow 3\dot{y} + 2a\dot{\phi} + 2a\dot{\psi} = 0$ $3\dot{y} = \frac{2I}{m}$, or $\dot{y} = \frac{2I}{3m}$

Also $2a\dot{\theta} = \frac{4a}{3}(\dot{\psi} - \dot{\phi})$, i.e. $\frac{3}{2}a\dot{\theta} + a\dot{\phi} - a\dot{\psi} = 0$

$$\therefore\ \frac{5a}{6}\dot{\theta} = \frac{Ic}{ma} \text{ i.e. } \dot{\theta} = \frac{6Ic}{5ma^2}.$$

Substituting now the values of $\dot{y}$ and $\dot{\theta}$ in the equation

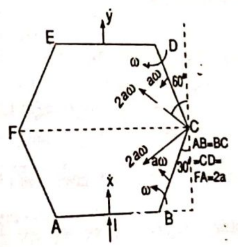
$$\dot{y} + \frac{4}{3}a\dot{\phi} + a\dot{\theta} = 0 \quad \text{we get } \frac{2I}{3m} + \frac{4}{3}a\dot{\phi} + \frac{6Ic}{5ma} = 0 \text{ or } \dot{\phi} = \frac{(5a-9c)}{10ma^2}.$$

Also substituting values of $\dot{y}$ and $\dot{\theta}$ in equation $\dot{y} - a\dot{\theta} + \frac{4}{3}a\dot{\psi} = 0$,

we get $\frac{I}{3m} - \frac{6Ic}{5ma} + \frac{4}{3}a\dot{\psi} = 0$ or $\dot{\psi} = \frac{(5a-9c)\,I}{10ma^2}$.

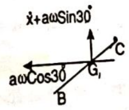
Ex. 31. *Six equal uniform rods form a regular hexagon, loosely jointed at the angular points, and rest on a smooth table; a blow is given perpendicular to one of them at its middle point, find the resulting motion and show that the opposite rod begins to move with one tenth of the velocity of the rod that is struck.* **(Meerut 94; Rajasthan 1983; Agra 81, 89)**

Sol. The impluse is given at the middle point of AB in the direction as shown in the figure so the ensuing motion of BC, AF and CD, EF will be symmertrical.



Therefore they have after impluse same angular velocity say ω.

Also the rods AB and DE will begin to move at right angles to themselves. Now let $\dot{x}$ and $\dot{y}$ be their velocities just after the action of impluse.



$\therefore$ Velocity of C in the direction of $\dot{x}$

= Velocity of B + velocity of C relative to B

$$= \dot{x} + 2a\omega \sin 30° = \dot{x} + a\omega \quad \text{...(1)}$$

and velocity of C in the same direction (with respect to the rod CD)

= Velocity of D + velocity of C relative to D,

$$= \dot{y} - 2a\omega \sin 30^\circ = \dot{y} - a\omega \quad \text{...(2)}$$

$\therefore$ (1) and (2) $\Rightarrow \dot{x} + a\omega = \dot{y} - a\omega \Rightarrow a\omega = \frac{1}{2}(\dot{y} - \dot{x})$

Now actual velocity of the C.G of the rod BC is $a\omega \cos 30^\circ$ parallel to AB and $\dot{x} + a\omega \sin 30^\circ$ at right angles to AB

$\therefore$ (velocity)^2 of $G_1 = (a\omega \cos 30^\circ)^2 + (\dot{x} + a\omega \sin 30^\circ)^2$

Thus K.E. of BC

$$= \frac{1}{2}m\left[\frac{1}{3}a^2\omega^2 + \{(a\omega \cos 30)^2 + (\dot{x} + a\omega \sin 30)^2\}\right]$$

$$= \frac{1}{2}m\left(\frac{4}{3}a^2\omega^2 + \dot{x}^2 + a\omega\dot{x}\right) = \frac{1}{12}m\,[5\dot{x}^2 + 2\dot{y}^2 - \dot{x}\dot{y}] = T_1 \text{ say} \left[\because a\omega = \frac{1}{2}(\dot{y} - \dot{x})\right]$$

Also Kinetic energy of CD

$$= \frac{1}{2}m\left[\frac{1}{3}a^2\omega^2 + \{(a\omega \cos 30)^2 + (\dot{y} - a\omega \sin 30)^2\}\right]$$

$$= \frac{1}{2}m\left[\frac{4}{3}a^2\omega^2 + \dot{y}^2 - a\omega\dot{y}\right] = \frac{1}{12}m\,[2\dot{x}^2 + 5\dot{y}^2 - \dot{x}\dot{y}) = T_2 \text{ say}$$

$\therefore$ Total Kinetic Energy T is given by $T = 2T_1 + 2T_2 + T_3 + T_4$ where T_3 and T_4 are K. E.'s of AB and DE respectively.

$$\therefore T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\dot{y}^2 + 2.(m/12)\,[5\dot{x}^2 + 2\dot{y}^2 - \dot{x}\dot{y}]$$

$$+ 2.(m/12)\,[5\dot{y}^2 + 2\dot{x}^2 - \dot{x}\dot{y}]$$

or $T = \frac{1}{3}m\,(5\dot{x}^2 + 5\dot{y}^2 - \dot{x}\dot{y})$ $\therefore$ Lagrange's x-equation is

$$\left(\frac{\partial T}{\partial \dot{x}}\right)_2 - \left(\frac{\partial T}{\partial \dot{x}}\right)_1 = I \Rightarrow \frac{1}{3}m\,(10\dot{x} - \dot{y}) = I$$

Lagrange's y-equation gives $\frac{1}{3}m\,(10\dot{y} - \dot{x}) = 0$

Solving (1) and (2), we get $\dot{x} = \frac{10}{33}\frac{I}{m}$ and $\dot{y} = \frac{I}{33m}$

Also $\dot{y} = (x/10)$ and $a\omega = \frac{1}{2}(\dot{y} - \dot{x}) \Rightarrow \omega = -\frac{3I}{22m}a$

Revision at a Glance

(1) Generalised Co-ordinates

Suppose that a particle or a system of N-particles moves subject to possible constraints, as for example a pa.ticle moving along a circular wire or a rigid body moving along an inclined plane, then there will be necessarily a minimum number of independent co-ordinates thcn needed to specify the motion. These co-ordinates denoted by $q_1, q_2, \ldots, q_n$ are called generalised

co-ordinates. These co-ordinates may be distances, angles or quantities to them.

(2) Degrees of freedom.

The number of co-ordinates required to specify the position of a system of one or more particles is called the number of degrees of freedom of the system.

(3) Transformation equations.

Let $r_\nu = x_\nu i + y_\nu j + z_\nu k$ be the position vector of νth particle with respect to xyz co-ordinate system. The relationships of the generalised co-ordinates $q_1, q_2 \ldots, q_n$ to the position co-ordinates are given by the transformation equations.

$$\left.\begin{aligned} x_\nu &= x_\nu(q_1, q_2, \ldots, q_n, t) \\ y_\nu &= y_\nu(q_1, q_2, \ldots, q_n, t) \\ z_\nu &= z_\nu(q_1, q_2, \ldots, q_n\ t) \end{aligned}\right\} \quad \text{...(1)}$$

where t denotes the time. In vector, (1) can be written as

$$r_\nu = r_\nu(q_1, q_2, \ldots, t) \quad \text{...(2)}$$

where the functions in (1) or (2) are continuous and have continuous derivatives.

4. Classification of Mechanical systems.

(a) Scleronmic system.

The mechanical system in which t, the time, does not enter explicitly in equations (1) or (2) is called a sceleronomic system.

(b) Rhenomic system

The mechanical system in which the moving constraints are involved and the time t does enter explicitly is called a Rhenomic system.

(c) Holonomic system and Non Holonomic system.

(Meerut 1983 (P))

Let $q_1, q_2, \ldots, q_n$ denote the generalised co-ordinates describing a system and let t denote the time. If all the constraints of the systemm can be expressed as equations having the form $f(q_1, q_2, \ldots, q_n\, t) = 0$ or their equivalent, then the system is siad to be Holonomic otherwise it is be Non-Holonomic system.

(d) Conservative and non conservative system.

If the forces acting on the system are dervable from a potential function [or potential energy V]. then the system is called conservative, otherwise it is non-conservative.

5. Kinetic energy and genralised velocities

The K.E. of the system is $T = \frac{1}{2}\sum_{\nu=1}^{n} m_\nu \dot{r}_\nu^2$.

The K.E. of the system can be written as a quadratic form in the generalised $\dot{q}_\alpha$ If the system is independent of time explicitly i.e. Sclemomic then the system is Rheonomic, linear terms in $\dot{q}_\alpha$ are also present.

6. Generalised Forces.

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If W is the total work done on a system of particles by forces $\mathbf{F}_\nu$ *acting on the vth particle, then*

$$dW = \sum_{\alpha=1}^{n} \phi_\alpha \, dq_\alpha \quad \text{where } \phi_\alpha = \sum_{\nu=1}^{N} \mathbf{F}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha}$$

is called the generalised force associated with generalised coordinates q_α

7. Lagrange's equations.

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = \phi_1 = \frac{\partial W}{\partial q_\alpha}$$

For $\alpha = 1, 2, \ldots, n$, *we have n different equations which are called Lagrange's equations.*

Note. The quantity $q_\alpha = \dfrac{\partial T}{\partial \dot{q}_\alpha}$ is called the generalised momentum associated with the genera! co-ordinate q_α.

8. If the forces are derivable from a potential V, then

$$\phi_\alpha = \frac{\partial W}{\partial q_\alpha} = -\frac{\partial V}{\partial q_\alpha},$$

since the potential, or potential energy is a function of q's only (and possible the time t) then, we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = -\frac{\partial V}{\partial q_\alpha} \Rightarrow \frac{d}{dt}\left[\frac{\partial}{\partial \dot{q}_\alpha}(T - V)\right] - \left(\frac{\partial T}{\partial q_\alpha} - \frac{\partial V}{\partial q_\alpha}\right) = 0$$

$$\Rightarrow \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = 0, \text{ where } L = T - V \qquad \ldots(19)$$

The function L defined by $L = T - V$ *is said to be Lagrangian function.*

9. Generalising momentum.

We defined $p_\alpha = \dfrac{\partial T}{\partial \dot{q}_\alpha}$ *to be the generalised momentum associated with generalised co-ordinate* q_α. *We usually refer* p_α *as the momentum conjugate to* q_α *or the conjugate momentum.*

In case the system is conservative, we have

$$T = L + V \Rightarrow (\partial T/\partial \dot{q}_\alpha) = (\partial L/\partial \dot{q}_\alpha) + (\partial V/\partial \dot{q}_\alpha) = (\partial L/\partial \dot{q}_\alpha)$$

because V, the P.E. system does not depend upon q_α

$$\therefore \; q_\alpha = (\partial L/\partial \dot{q}_\alpha).$$

10. Lagrange's Equation with Impulsive forces.

$$\Rightarrow \left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)_2 - \left(\frac{\partial T}{\partial \dot{q}_\alpha}\right)_1 = \Sigma\, \mathbf{I}_\nu \cdot \frac{\partial \mathbf{r}_\nu}{\partial q_\alpha} = P_\alpha \quad \text{(say)}$$

where subscripts 1 and 2 denote respective quantities before and after the application of the i plusive forces.

These equations are known as Lagrange's equatious under impuisive forces.

SUPPLEMENTARY PROBLEMS

1. A hollow circular roller of radius a and mass **M** has attached to the centre of its axis a particle of mass **m** by a light a rod of length **b**. The particle is free to swing sbout the axis a plane perpendicular to the axis. The roller is at rest on a horizontal plane rough enough to prevent slipping, with rod held at an angle α with the downward vertical. If the rod is then released prove that the centre of the roller will oscillate through a distance

$$\frac{2mba^2 \sin\alpha}{M(a^2 + k^2) + ma^2},$$

where **k** is the radius of gyration of the roller about the axis.

2. A homogeneous rod **OA**, of mass m_1 and length $2a$, is freely at **O** to a fixed point ; as its other end is freely attached another homogeneous rod **AB**, of mass m_2 and length $2b$; the system moves under gravity, find equations to determine motion. **(Meerut 1994)**

3. A thin circular ring of a radius **a** and **M** lies on a smooth horizontal plane and two light elastic strings are attached to it at ends of a diameter, the other ends of the string being fastened fixed points in the diameter produced. Show that for small oscillations in the piane of the ring the periods are the values of $2\pi/p$ given by

$$\frac{Mlp^2}{2T} - 1 = 0 \text{ or } \frac{b}{l-b} \text{ or } \frac{l}{a}$$

where **b** is the natural length, l the equilibrium length, and **T** the equilibrium tension of each string. **(Agra 1994)**

4. A heavy uniform rod of mass m and length $2a$ rotating in a vertical plane falls and strikes a smooth horizontal plane. If u and ω be its linear and angular velocities and θ the inclination to the vertical just before impact, prove that the impulse **J** is given by

$$(1 + 3\sin^2\theta)\, J = m\,(u + a\omega \sin\theta).$$

5. A rhombus of smoothly jointed equal uniform rods of length $2a$ lies on a smooth horizontal table. It is struck by a blow **J** perpendicular to one of its sides at a point distant **c** from the middle point, show that the angular velocity of the side instantly becomes $\dfrac{3Jc}{8ma^2}$, where **m** is the mass of each side.

6. A square **ABCD** formed of four freely jointed uniform rods, is rotating about its centre of gravity on smooth horizontal table with angular velocity ω when the joint **A** is suddenly fixed. Show that the initial angular velocities of the rods are equal and that 3/5 of the kinetic energy immediately lost.

7. A uniform straight rod of length $2a$, is freely movable about; its on the and a particle of mass one fourth that of the rod is attached by a light inextensible string of length $3a$, to one end of the rod, find the period of principle oscillations. **(Meerut 1990)**

8. Obtain Lagrange's equations of motion for a system under finite forces and hence deduce the same under implusive forces. **(Meerut 1989)**

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