INTEGRAL CALCULUS

1. INTEGRAL AS A LIMIT OF SUM, SUMMATION OF SERIES

1. 1d 2018

Find the limit $\lim\limits_{n \to \infty} \frac{1}{n^2} \sum\limits_{r=0}^{n-1} \sqrt{n^2 - r^2}$.

2. 3b 2012

(b) Define a sequence s_n of real numbers by

$$s_n = \sum_{i=1}^{n} \frac{(\log(n+i) - \log n)^2}{n+i}$$

Does $\lim\limits_{n \to \infty} s_n$ exist? If so, compute the value of this limit and justify your answer. 20

2. DEFINITE INTEGRALS

1. 4b 2020 P-2

Show that $\displaystyle\int_0^{\pi/2} \frac{\sin^2 x}{\sin x + \cos x}\,dx = \frac{1}{\sqrt{2}}\log_e(1+\sqrt{2})$ 15

2. 2a 2020

Evaluate $\int_0^1 \tan^{-1}\left(1-\frac{1}{x}\right)dx.$ 15

3. 1d 2015

Evaluate the following integral :

$$\int_{\pi/6}^{\pi/3} \frac{\sqrt[3]{\sin x}}{\sqrt[3]{\sin x}+\sqrt[3]{\cos x}}\,dx.$$ 10

4. 1d 2014

Evaluate : 10

$$\int_0^1 \frac{\log_e(1+x)}{1+x^2}\,dx$$

5. 1d 2015 IFoS

(d) For $x>0$, let $f(x)=\int_1^x \frac{\ln t}{1+t}\,dt$. Evaluate $f(e)+f\left(\frac{1}{e}\right)$. 8

6. 1c 2013

Evaluate $\int_0^1 \left(2x \sin\frac{1}{x} - \cos\frac{1}{x}\right) dx$.

7. 4a 2013 IFoS

) Evaluate

$$\int_0^{\pi/2} \frac{x \sin x \cos x \, dx}{\sin^4 x + \cos^4 x}.$$ 10

8. 3a(ii) 2011

(ii) $\int_0^1 \ell n \, x \, dx$. (8, 12)

3. CONVERGENCE OF IMPROPER AND INFINITE INTEGRALS

1. 2019 P-2 4c

Discuss the convergence of $\int_1^2 \frac{\sqrt{x}}{l_n x} dx$. 15

2. 2b 2019 P-2

(b) Show that the integral $\int_0^{\pi/2} \log \sin x \, dx$ is convergent and hence evaluate it. 15

3. 2018 P-2 3d

(d) Show that the improper integral $\int_0^1 \frac{\sin \frac{1}{x}}{\sqrt{x}} dx$ is convergent. 10

4. 4c 2017

Examine if the improper integral $\int_0^3 \frac{2x dx}{(1-x^2)^{2/3}}$ exists. 10

5. 2017 P-2 3c

Prove that $\int_0^\infty \frac{\sin x}{x} dx$ is convergent but not absolutely convergent. 12

6. 2016 P-2 4a

4.(a) Evaluate the integral $\int_0^\infty \frac{dx}{\sqrt{x}(1+x)}$. 8

7. 2014 P-2 1b

Test the convergence of the improper integral $\int_1^\infty \frac{dx}{x^2(1+e^{-x})}$. 10

8. 3b 2011 IFoS

Test for convergence the integral $\int_0^\infty \sqrt{x\,e^{-x}}\,dx$. 8

9. 2011 P-2 3a

3. (a) Examine the convergence of

$$\int_0^\infty \frac{dx}{(1+x)\sqrt{x}}$$

and evaluate, if possible. 13

10. 1d 2010

(d) Does the integral $\int_{-1}^{1} \sqrt{\frac{1+x}{1-x}}\,dx$ exist?

If so, find its value. 12

11. 3a 2010

Discuss the convergence of the integral

$$\int_0^\infty \frac{dx}{1+x^4\sin^2 x}$$ 10

4. BETA GAMMA FUNCTIONS

1. 4c(ii) 2020 IFoS

(ii) Evaluate the following integral :

$$\int_{-\infty}^{\infty} xe^{-x^2}\,dx$$

6+9=15

2. 2b 2017 IFoS

Show that

$$\int_0^{\pi/2} \sin^p\theta\cos^q\theta\,d\theta = \frac{1}{2}\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)},\ p, q > -1.$$

Hence evaluate the following integrals :

(i) $\int_0^{\pi/2} \sin^4 x\cos^5 x\,dx$

(ii) $\int_0^{1} x^3(1-x^2)^{5/2}\,dx$

(iii) $\int_0^{1} x^4(1-x)^3\,dx$ *10*

3. 1c 2016

Evaluate : 10

$$I = \int_0^1 \sqrt[3]{x\log\left(\frac{1}{x}\right)}\,dx$$

4. 4b 2016 IFoS

Show that the integral $\int_{0}^{\infty} e^{-x} x^{\alpha-1} dx, \quad \alpha > 0$ exists, by separately taking the cases for $\alpha \geq 1$ and $0 < \alpha < 1$. 10

5. 4c 2016 IFoS

Prove that $\Gamma(2z) = \frac{2^{2z-1}}{\sqrt{\pi}} \Gamma(z) \Gamma\left(z + \frac{1}{2}\right)$. 10

6. 3c 2014 IFoS

Q. 3(c) Evaluate the integral

$$I = \int_{0}^{\infty} 2^{-ax^2} dx$$

using Gamma function

7. 2012 P-1 3c

(c) Find all the real values of p and q so that the integral $\int_0^1 x^p (\log \frac{1}{x})^q \, dx$ converges. 20

8. 4d 2012 IFoS

Evaluate the following in terms of Gamma function :

$$\int_0^a \sqrt{\left(\frac{x^3}{a^3 - x^3}\right)} dx.$$ 10

5. DOUBLE INTEGRALS

1. 1c 2020 IFoS P-2

Evaluate the integral $\iint_R (x-y)^2 \cos^2(x+y)\, dx\, dy$, where R is the rhombus with successive vertices at $(\pi, 0)$, $(2\pi, \pi)$, $(\pi, 2\pi)$ and $(0, \pi)$. 8

2. 2018 4b

Evaluate the integral $\int_0^a \int_{x/a}^{x} \frac{x\, dy\, dx}{x^2+y^2}$.

3. 2018 3d IFoS

) Evaluate $\iint_R (x^2+xy)\, dx\, dy$ over the region R bounded by $xy=1$, $y=0$, $y=x$ and $x=2$. 10

4. IFoS 2018 4a P-2

Show that

$$\iint_R x^{m-1} y^{n-1} (1-x-y)^{l-1} dx\, dy = \frac{\Gamma(l)\Gamma(m)\Gamma(n)}{\Gamma(l+m+n)}; \quad l, m, n > 0$$

taken over R : the triangle bounded by $x=0$, $y=0$, $x+y=1$. 10

5. 2017 4d

Prove that $\frac{\pi}{3} \le \iint_D \frac{dxdy}{\sqrt{x^2+(y-2)^2}} \le \pi$ where D is the unit disc. 10

6. 2017 3c IFoS

Evaluate the integral $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$, by changing to polar coordinates. Hence show that $\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}$. 10

7. IFoS 2017 4c P-2

4.(c) Evaluate $\int_{x=0}^{\infty} \int_{y=0}^{x} x\, e^{-x^2/y}\, dy\, dx$ 8

8. 2016 4c

Evaluate $\iint_R f(x, y)\, dx\, dy$ over the rectangle $R = [0, 1; 0, 1]$ where

$$f(x, y) = \begin{cases} x + y, & \text{if } x^2 < y < 2x^2 \\ 0, & \text{elsewhere} \end{cases}$$

15

9. 2016 2a IFoS

After changing the order of integration of $\int_0^\infty \int_0^\infty e^{-xy} \sin nx\, dx\, dy$, show that $\int_0^\infty \frac{\sin nx}{x}\, dx = \frac{\pi}{2}$. 10

10. IFoS 2016 3d P-2

Evaluate the integral $\int_0^2 \int_0^{y^2/2} \frac{y}{\left(x^2+y^2+1\right)^{\frac{1}{2}}} \, dx \, dy$. 12

11. 2015 3d

Evaluate the integral

$$\iint_R (x-y)^2 \cos^2(x+y) \, dx \, dy$$

where R is the rhombus with successive vertices as $(\pi, 0)$ $(2\pi, \pi)$ $(\pi, 2\pi)$ $(0, \pi)$. 12

12. 2015 4a

Evaluate $\iint_R \sqrt{|y-x^2|} \, dx \, dy$

where $R = [-1, 1 ; 0, 2]$. 13

13. 2014 2c

By using the transformation $x + y = u$, $y = uv$, evaluate the integral $\iint \{xy\,(1 - x - y)\}^{1/2} \, dx \, dy$ taken over the area enclosed by the straight lines $x = 0$, $y = 0$ and $x + y = 1$. *15*

14. 2014 1d IFoS

Evaluate $\iint_R y \frac{\sin x}{x} \, dx \, dy$ over R where $R = \{(x, y) : y \leq x \leq \pi/2,\ 0 \leq y \leq \pi/2\}$. 8

15. 2014 4c IFoS

Evaluate the integral $\iint\limits_R \frac{y}{\sqrt{x^2+y^2+1}}\,dx\,dy$ over the region R bounded between $0 \le x \le \frac{y^2}{2}$ and $0 \le y \le 2$. 10

16. IFoS 2014 3b P-2

(b) Change the order of integration and evaluate $\int_{-2}^{1}\int_{y^2}^{2-y} dx\,dy$. 15

17. 2013 3c

f_{xy} and f_{yx} at (0, 0).

Evaluate $\iint\limits_D xy\; dA$, where D is the region bounded by the line $y = x - 1$ and the parabola $y^2 = 2x + 6$. 15

18. 2013 1c

Evaluate the integral $\int_0^{\infty}\int_0^{x} x\,e^{-x^2/y}\,dy\,dx$ by changing the order of integration. 8

19. 2011 4a P-2

4. (a) Evaluate

$$\iint \sqrt{4x^2 - y^2}\; dx\, dy$$

over the triangle formed by the straight lines $y = 0,\ x = 1,\ y = x$. 13

20. 2010 3d

(d) Evaluate $\iint_D (x + 2y)\, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$. 10

21. 2010 3b P-2

(b) Evaluate

$$\iint_R (x - y + 1)\, dx\, dy$$

where R is the region inside the unit square in which $x + y \geq \frac{1}{2}$. 13

22. 2009 4b

(b) Evaluate

$$I = \iint_S x\, dy\, dz + dz\, dx + xz^2\, dx\, dy$$

where S is the outer side of the part of the sphere $x^2 + y^2 + z^2 = 1$ in the first octant. 20

6. TRIPLE INTEGRALS

1. 2015 2c

(c) Consider the three-dimensional region R bounded by $x + y + z = 1$, $y = 0$, $z = 0$. Evaluate $\iiint\limits_R (x^2 + y^2 + z^2)\,dx\,dy\,dz$. 10

2. 2010 3b

(b) Let D be the region determined by the inequalities $x > 0$, $y > 0$, $z < 8$ and $z > x^2 + y^2$. Compute

$$\iiint\limits_D 2x\,dx\,dy\,dz$$ 20

7. SURFACE AREAS AND VOLUMES

1. 2019 1c

(c) Find the volume lying inside the cylinder $x^2+y^2-2x=0$ and outside the paraboloid $x^2+y^2=2z$, while bounded by xy-plane. 8

2. 2018 2c

The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ revolves about the x-axis. Find the volume of the solid of revolution. 13

3. 2017 2a

Find the volume of the solid above the xy-plane and directly below the portion of the elliptic paraboloid $x^2+\frac{y^2}{4}=z$ which is cut off by the plane $z=9$. 15

4. IFoS 2017 3d P-2

Find the volume of the region common to the cylinders $x^2+y^2=a^2$ and $x^2+z^2=a^2$. 8

5. 3c 2016 IFoS

(c) Obtain the area between the curve $r = 3(\sec\theta + \cos\theta)$ and its asymptote $x = 3$. *10*

6. 2015 2d IFoS

(d) Find the area enclosed by the curve in which the plane $z=2$ cuts the ellipsoid

$$\frac{x^2}{25}+y^2+\frac{z^2}{5}=1$$ 10

7. IFoS 2015 4a P-2

(a) Compute the double integral which will give the area of the region between the y-axis, the circle $(x-2)^2 + (y-4)^2 = z^2$ and the parabola $2y = x^2$. Compute the integral and find the area. *15*

8. 2013 4a P-2 IFoS

(a) Find the area of the region between the x-axis and $y = (x-1)^3$ from $x = 0$ to $x = 2$. *13*

9. 2012 4a

4. *(a)* Compute the volume of the solid enclosed between the surfaces $x^2 + y^2 = 9$ and $x^2 + z^2 = 9$. 20

10. 2012 4a IFoS

Find by triple Integration the volume cut off from the cylinder $x^2 + y^2 = ax$ by the planes $z = mx$ and $z = nx$. 10

11. 2012 3a P-2

3. (a) Find the volume of the solid bounded above by the parabolic cylinder $z = 4 - y^2$ and bounded below by the elliptic paraboloid $z = x^2 + 3y^2$. 13

12. 2011 3c

(c) Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$ above the xy-plane and inside the cylinder $x^2 + y^2 = 2x$. 20

13. 2011 3c

Show that the area of the surface of the sphere $x^2 + y^2 + z^2 = a^2$ cut off by $x^2 + y^2 = ax$ is $2(\pi - 2)a^2$. *12*

*MISCELLANEOUS

1. 4b 2019 IFoS

(b) Find the centroid of the solid generated by revolving the upper half of the cardioid $r = a(1 + \cos\theta)$ bounded by the line $\theta = 0$ about the initial line. Take the density of the solid as uniform. 10

2. 1b 2012

(b) Let p and q be positive real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$. Show that for real numbers $a, b \geq 0$

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$$

12

3. 3a 2011 IFoS

(a) Show that the equation $3^x + 4^x = 5^x$ has exactly one root. 8

4. 1d 2009 P-2

(d) If f is the derivative of some function defined on $[a, b]$, prove that there exists a number $\eta \in [a, b]$, such that

$$\int_a^b f(t)\,dt = f(\eta)(b - a)$$

12