CSE-2016

Q1 Find P.I. of $\frac{d^2y}{dx^2} + y = e^{x/2} \sin \frac{x\sqrt{3}}{2}$

sol.

$$P.I = \frac{1}{D^2+1} e^{x/2} \sin \frac{x\sqrt{3}}{2}$$

$$= e^{x/2} \frac{1}{(D+\frac{1}{2})^2+1} \sin \frac{x\sqrt{3}}{2} = e^{x/2} \frac{1}{D^2+D+\frac{5}{4}} \sin\left(\frac{x\sqrt{3}}{2}\right)$$

$$= e^{x/2} \frac{1}{-(\frac{3}{4})+D+\frac{5}{4}} \sin\left(\frac{x\sqrt{3}}{2}\right) = e^{x/2} \frac{1}{(D+\frac{1}{2})} \sin\left(\frac{x\sqrt{3}}{2}\right)$$

$$= e^{x/2} \frac{(D-\frac{1}{2})}{D^2-\frac{1}{4}} \sin\left(\frac{x\sqrt{3}}{2}\right) = e^{x/2} \frac{\left[D\left(\sin \frac{x\sqrt{3}}{2}\right) - \frac{1}{2} \sin \frac{x\sqrt{3}}{2}\right]}{-\frac{3}{4}-\frac{1}{4}}$$

$$= -e^{x/2}\left[\frac{\sqrt{3}}{2} \cos \frac{x\sqrt{3}}{2} - \frac{1}{2} \sin \frac{x\sqrt{3}}{2}\right]$$

$$\boxed{P.I. = \frac{-e^{x/2}}{2}\left(\sqrt{3} \cos \frac{x\sqrt{3}}{2} - \sin \frac{x\sqrt{3}}{2}\right)}$$

Q2 Solve $\frac{dy}{dx} = \frac{1}{1+x^2}\left(e^{\tan^{-1}x} - y\right)$

sol.

$$\frac{dy}{dx} = \frac{e^{\tan^{-1}x}}{1+x^2} - \frac{y}{1+x^2}$$

$$\Rightarrow \cancel{\frac{dy}{dx} + \frac{y}{1+x^2}x} \quad \frac{dy}{dx} + \frac{y}{1+x^2} = \frac{e^{\tan^{-1}x}}{1+x^2}$$ (Linear form)

$$I.F. = e^{\int \frac{1}{1+x^2}dx} = e^{\tan^{-1}x}$$

$\therefore$ solution is

$$y\left(e^{\tan^{-1}x}\right) = \int \frac{e^{\tan^{-1}x}}{1+x^2}\left(e^{\tan^{-1}x}\right) dx$$

$$y\, e^{\tan^{-1}x} = \int e^{2\tan^{-1}x}\, d(\tan^{-1}x) \quad \left[\frac{1}{1+x^2}dx = d(\tan^{-1}x)\right]$$

$$y\left(e^{\tan^{-1}x}\right) = \frac{\left(e^{\tan^{-1}x}\right)^3}{3} + C$$

$$\boxed{y = \frac{\left(e^{\tan^{-1}x}\right)^2}{3} + C e^{-\tan^{-1}x}}$$ is the solution.

Q Show that family of parabolas $y^2 = 4cx + 4c^2$ is self-orthogonal.

sol. $y^2 = 4cx + 4c^2$ — (1)

$\therefore 2yy' = 4c$

$\therefore c = \frac{yy'}{2}$, Putting in original equation (1)

$y^2 = 4\left(\frac{yy'}{2}\right)x + 4\left(\frac{yy'}{2}\right)^2$

$y^2 = 2xyy' + y^2y'^2$ — (2)

Replacing y' by $-1/y'$ to find orthogonal trajectory

$y^2 = 2xy\left(-\frac{1}{y'}\right) + y^2\left(-\frac{1}{y'}\right)^2$

$y^2 = -\frac{2xy}{y'} + \frac{y^2}{y'^2}$

$\Rightarrow y^2y'^2 = -2xyy' + y^2$

$\Rightarrow \boxed{y^2y'^2 + 2xyy' = y^2}$ — (3)

As both (2) & (3) are identical

$\therefore y^2 = 4cx + 4c^2$ is self-orthogonal.

Q solve $\{y(1 - x\tan x) + x^2\cos x\}dx - xdy = 0$

Method I:

$\{y(1 - x\tan x) + x^2\cos x\}dx = xdy$

$\Rightarrow x\frac{dy}{dx} - y(1 - x\tan x) = x^2\cos x$

$\Rightarrow \frac{dy}{dx} + y\left(\tan x - \frac{1}{x}\right) = x\cos x$ [Linear form]

$I.F. = e^{\int(\tan x - \frac{1}{x})dx} = e^{\ln|\sec x| - \ln x} = e^{\ln\left|\frac{\sec x}{x}\right|} = \frac{\sec x}{x}$

$\therefore$ solution is $y\left(\frac{\sec x}{x}\right) = \int(x\cos x)\left(\frac{\sec x}{x}\right)dx$

$y\left(\frac{\sec x}{x}\right) = \int 1 dx = x + C$

$\therefore \boxed{y = x^2\cos x + Cx\cos x}$ is the solution.

Q Use method of variation of parameters, solve the DE

$$(D^2+2D+1)y = e^{x}\log x \ ; \ \left[D = \frac{d}{dx}\right]$$

sol: Auxiliary eq^n is

$$m^2+2m+1 = 0$$

$$(m+1)^2 = 0$$

$$m = -1, -1$$

$$\therefore \ C.F. = (C_1 + C_2 x)e^{-x}$$

$$\therefore \quad u = e^{-x} \qquad v = xe^{-x}$$

$$W = \begin{vmatrix} u & u' \\ v & v' \end{vmatrix} = \begin{vmatrix} e^{-x} & -e^{-x} \\ xe^{-x} & (1-x)e^{-x} \end{vmatrix} = (1-x)e^{-2x} + xe^{-2x}$$

$$= e^{-2x} \neq 0$$

$\therefore$ u & v are independent

<u>Using variation of parameters.</u>

Let A and B be solutions of y such that

$$y_p = Au + Bv \quad , \quad R = e^{x}\log x$$

then $A = -\int \frac{vR}{W} = -\int \frac{xe^{-x}(e^{-x}\log x)}{e^{-2x}} dx = -\int x \log x \, dx$

$= -\left[\log x\left(\frac{x^2}{2}\right) - \frac{x^2}{4}\right] = \frac{x^2}{2}\left[\frac{1}{2} - \log x\right]$

similarly $B = \int \frac{uR}{W} = \int \frac{e^{-x}(e^{-x}\log x)}{e^{-2x}} = \int \log x \, dx$

$= [\log x(x) - x] = x(\log x - 1)$

$\therefore y_p = Au + Bv = \left(\frac{x^2}{2}\left[\frac{1}{2} - \log x\right]\right)(e^{-x}) + (x\log x - x)(xe^{-x})$

$= \left(\frac{x^2}{4} - \frac{x^2}{2}\log x\right)e^{-x} + (x^2\log x - x^2)e^{-x}$

$= \left(\frac{x^2}{2}\log x - \frac{3x^2}{4}\right)e^{-x} = \frac{x^2 e^{-x}}{2}\left(\log x - \frac{3}{2}\right)$

The complete solution

$y = \text{C.F.} + y_p$

$$\boxed{y = (C_1 + C_2 x)e^{-x} + \frac{x^2 e^{-x}}{2}\left(\log x - \frac{3}{2}\right)}$$

Q Find the general solution of $x^2\frac{d^3y}{dx^3} - 4x\frac{d^2y}{dx^2} + 6\frac{dy}{dx} = 4$

sol. Multiplying eq^n by x,

$x^3\frac{d^3y}{dx^3} - 4x^2\frac{d^2y}{dx^2} + 6x\frac{dy}{dx} = 4x$

$\therefore (x^3D^3 - 4x^2D^2 + 6xD)y = 4x$ where $D = \frac{d}{dx}$

Let $x = e^z$ and $\theta = \frac{d}{dz}$

then $xD = \theta$

$x^2D = \theta(\theta - 1)$

$x^3D = \theta(\theta-1)(\theta-2)$, Putting these in equation.

$\theta(\theta-1)(\theta-2) - 4(\theta(\theta-1)) + 6\theta = 4e^z$

$\Rightarrow \theta^3 - 7\theta^2 + 12\theta = 4e^z$

Auxiliary eqⁿ is

$$m^3 - 7m^2 + 12m = 0$$
$$m(m^2 - 7m + 12) = 0$$
$$m(m^2 - 4m - 3m + 12) = 0$$
$$m(m-4)(m-3) = 0$$
$$\therefore \quad m = 0, 3, 4$$

Hence, $C.F. = C_1 e^{0z} + C_2 e^{3z} + C_3 e^{4z}$

$C.F. = C_1 + C_2 x^3 + C_3 x^4 \qquad (\because e^z = x)$

Now.
$$P.I. = \frac{1}{\theta^3 - 7\theta^2 + 12\theta} e^z.$$
$$= e^z \frac{1}{1^3 - 7(1)^2 + 12(1)} = \frac{e^z}{6} = \frac{x}{6}$$

$$\therefore \quad y = C.F. + P.I$$

$\boxed{y = (C_1 + C_2 x^3 + C_3 x^4) + \frac{x}{6}}$ is the solution.

Q Using laplace transform, solve the following
$$y'' - 2y' - 8y = 0, \quad y(0) = 3, \; y'(0) = 6$$

sol. Taking Laplace transform

$$L(y'' - 2y' - 8y) = L(0)$$
$$\Rightarrow L(y'') - 2L(y') - 8L(y) = 0$$
$$\Rightarrow s^2 L(y) - sy(0) - y'(0) - 2(sL(y) - y(0)) - 8L(y) = 0$$
$$\Rightarrow s^2 L(y) - s(3) - 6 - 2(sL(y) - 3) - 8L(y) = 0$$
$$\Rightarrow s^2 L(y) - 3s - 6 - 2sL(y) + 6 - 8L(y) = 0$$
$$\Rightarrow (s^2 - 2s - 8) L(y) - 3s = 0$$
$$\Rightarrow (s^2 - 2s - 8) L(y) = 3s$$

$\therefore \quad L(y) = \frac{3s}{s^2-2s-8} = \frac{3s}{s^2-4s+2s-8} = \frac{3s}{(s-4)(s+2)}$

Using Partial fractions

$$\frac{3s}{(s-4)(s+2)} = \frac{A}{s-4} + \frac{B}{s+2}$$

$\Rightarrow \quad 3s = A(s+2) + B(s-4)$

If $s=4$

$$12 = A(6) + B(0)$$

$$\therefore \quad A = 2$$

If $s = -2$

$$-6 = A(0) + B(-6)$$

$$\therefore \quad B = 1$$

Hence.

$$\frac{3s}{(s-4)(s+2)} = \frac{2}{(s-4)} + \frac{1}{(s+2)}$$

$\Rightarrow \quad L(y) = \frac{2}{s-4} + \frac{1}{s+2}$

$$y = L^{-1}\left[\frac{2}{s-4} + \frac{1}{s+2}\right] = L^{-1}\left[\frac{2}{s-4}\right] + L^{-1}\left[\frac{1}{s+2}\right]$$

$$= 2e^{+4x} + e^{-2x}$$

$\therefore \quad \boxed{y = 2e^{4x} + e^{-2x}}$ is the solution.