MATHEMATICS

Under the guidance of K. Venkanna

TEST CODE: TEST-7: IAS(M)/21-JULY-2019

PAPER - I : FULL SYLLABUS

Time: 3 Hours	Maximum Marks: 250

INSTRUCTIONS

Each question is printed only in English.

Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the cover of the answer-book in the space provided for the purpose. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.

Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any **THREE** of the remaining questions selecting at least **ONE** question from each Section.

The number of marks carried by each question is indicated at the end of the question.

Assume suitable data if considered necessary and indicate the same clearly.

Symbols/notations carry their usual meanings, unless otherwise indicated.

All questions carry equal marks.

Important Note: Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed.

Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

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SECTION – A

1. (a) (i) The rank of a product of two matrices cannot exceed the rank of either matrix.

(ii) Prove : The zero vector **0** = (0, 0, ..., 0) is a solution (the zero solution) of any homogeneous system AX = 0. **[10]**

1. (b) Define a basis of a vector space over a field F and the dimension of vector space. What is the dimension of complex vector space over the field of complex numbers ? Give an example of a vector space the dimension of which is not finite. **[10]**

1. (c) If $z = (x + y) + (x + y)\,\phi\,(y/x)$, prove that

$$x\left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y \partial x}\right) = y\left(\frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x \partial y}\right)$$ **[10]**

1. (d) Show that the right circular cylinder of the given surface and maximum volume is such that its height is equal to the diameter of its base. **[10]**

1. (e) Prove that the plane $x + 2y - z = 4$ cuts the sphere $x^2 + y^2 + z^2 - x + z + 2 = 0$ in a circle of radius unity and find the equations of the sphere which has this circle for one of its great circles. **[10]**

2. (a) Let W be the solution space of the homogeneous system

$$x + 2y - 3z + 2s - 4t = 0$$
$$2x + 4y - 5z + s - 6t = 0$$
$$5x + 10y - 13z + 4s - 16t = 0$$

Find the dimension and a basis for W. **[15]**

2. (b) Show that the function

$$f(x,y) = \begin{cases} x^2y/(x^2+y^2), & \text{when } x^2+y^2 \neq 0 \\ 0, & \text{when } x^2+y^2 = 0 \end{cases}$$

is continuous but not differentiable at (0, 0). **[10]**

2. (d) Evaluate $\int_0^{\pi/2} \frac{\sin^2 x\,dx}{1+\sin x \cos x}$ **[10]**

2. (c) (i) The plane $lx + my = 0$ is rotated through an angle α about its line of intersection with the plane $z = 0$. Prove that equation to the plane in its new position is $lx + my \pm z\sqrt{(l^2 + m^2)}\tan\alpha = 0$.

(ii) Find the distance of the point (3, 8, 2) from the line $\frac{1}{2}(x-1) = \frac{1}{4}(y-3) = \frac{1}{3}(z-2)$

measured parallel to the plane $3x + 2y - 2z + 15 = 0$ **[15]**

3. (a) (i) Consider the bases $B = \{(1, 2), (3, -1)$ and $B' = \{1, 0), (0, 1)\}$ of $\mathbf{R}^2$. If $\mathbf{u}$ is a vector such that $\mathbf{u}_B = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, find $\mathbf{u}_{B'}$.

(ii) Consider the bases $B = \{(1, 2), (3, -1)$ and $B' = \{(3, 1), (5, 2)\}$ of $\mathbf{R}^2$. Find the transition matrix, from B to B'. If $\mathbf{u}$ is a vector such that $\mathbf{u}_B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, find $\mathbf{u}_{B'}$.

(iii) Prove that similar matrices have the same eigenvalues but not a converse. **[5+5+10=20]**

3. (b) A space probe in the shape of the ellipsoid

$4x^2 + y^2 + 4z^2 = 16$

enters the earth's atmosphere and its surface begins to heat. After one hour, the temperature at the point (x, y, z) on the probe's surface is

$T(x, y, z) = 8x^2 + 4yz - 16z + 600.$

Find the hottest point on the probe's surface. **[15]**

3. (c) Show that the locus of points from which three mutually perpendicular tangents can be drawn to the paraboloid $ax^2 + by^2 = 2z$ is given by

$$ab\left(x^2 + y^2\right) - 2(a + b)z - 1 = 0$$

[15]

4. (a) Let A be an $n \times n$ matrix.

(i) If A has n linearly independent eigenvectors it is diagonalizable. The matrix C whose columns consist of n linearly independent eigenvectors can be used in a similarity transformation $C^{-1} AC$ to give a diagonal matrix D. The diagonal elements of D will be the eigenvalues of A.

(ii) If A is diagonalizable then it has n linearly independent eigenvectors.

[18]

4. (b) The volume bounded by the elliptic paraboloids $z = x^2 + 9y^2$ and $z = 18 - x^2 - 9y^2$.

[15]

4. (c) Prove that in general two generators of the hyperboloid $(x^2/a^2) + (y^2/b^2) - (z^2/c^2) = 1$ can be drawn to cut a given generator at right angles. Also show that if they meet the plane $z = 0$ in P and Q, PQ touches the ellipse $(x^2/a^6) + (y^2/b^6) = c^4/(a^4b^4)$.

[17]

SECTION – B

5. (a) Solve $(D^4 - 4D^2 - 5)\,y = e^x(x + \cos x)$. **[10]**

5. (b) Solve the differential equation $(D^2 - 2D + 2)\,y = e^x \tan x$, $D \equiv d/dx$ by method of variation of parameters. **[10]**

5. (c) Two equal rods, AB and AC, each of length 2b, are freely jointed at A and rest on a smooth vertical circle of radius a. Show that if 2θ be the angle between them, then $b \sin^3\theta = a\cos\theta$. **[10]**

5. (d) Show that the Frenet-Serret formulae can be written in the form $\frac{dT}{ds} = \omega \times T, \frac{dN}{ds} = \omega \times N, \frac{dB}{ds} = \omega \times B$ and determine ω.

[10]

5. (e) Show that

$(y^2z^3 \cos x - 4x^3z)\,dx + 2z^3y \sin x\,dy + (3y^2z^2 \sin x - x^4)\,dz$ is an exact differential of some function ϕ and find this function. **[10]**

6. (a) Solve $\left(3y^2 - 7x^2 + 7\right)dx + \left(7y^2 - 3x^2 + 3\right)dy = 0$ **[13]**

6. (b) A heavy hemispherical shell of radius r has a particle attached to a point on the rim, and rests with the curved surface in contact with a rough sphere of radius R at the highest point. Prove that if $R/r > \sqrt{5} - 1$, the equilibrium is stable, whatever be the weight of the particle. **[15]**

6. (c) (i) Find the most general differentiable function f(r) so that $f(r)\,\mathbf{r}$ is solenoidal.

(ii) show that $E = \mathbf{r}/r^2$ is irrotational. Find ϕ such that $\mathbf{E} = -\nabla\phi$ and such that $\phi(a) = 0$ where $a > 0$. **[14]**

6. (d) Prove that
grad $(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{B}\cdot\nabla)\mathbf{A} + (\mathbf{A}\cdot\nabla)\mathbf{B} + \mathbf{B}\times \text{curl}\,\mathbf{A} + \mathbf{A}\times\text{curl}\mathbf{B}$.

[08]

7. (a) (i) Solve $6\cos^2 x\,(dy/dx) - y\sin x + 2y^4\sin^3 x = 0$

(ii) Reduce the equation $x^2p^2 + py(2x + y) + y^2 = 0$ where $p = dy/dx$ to Clairaut's form and find its complete primitive and its singular solution. **[6+14=20]**

7. (b) A particle is free to move on a smooth vertical circular wire of radius a. It is projected from the lowest point with velocity just sufficient to carry it to the highest point. show that the reaction between the particle and the wire is zero after a time

$$\sqrt{(a/g)}.\log\left(\sqrt{5}+\sqrt{6}\right).$$

[17]

7. (c) Verify Green's theorem in the plane for $\oint_C \left(2x - y^3\right)dx - xy\,dy$,

where C is the boundary of the region enclosed by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$.

[13]

8. (a) By using Laplace transform method solve

$(D^2 + n^2)\,y = a\sin(nt + \alpha)$, $y = Dy = 0$ when $t = 0$.

[17]

8. (b) Assuming that a particle falling freely under gravity can penetrate the earth without meeting any resistance, show that a particle falling from rest at a distance $b(b > a)$ from the centre of the earth would on reaching the centre acquire a velocity $\sqrt{\left[ga\left(3b-2a\right)/b\right]}$ and the time to travel from the surface to the centre of the eath is $\sqrt{\left(\frac{a}{b}\right)}\sin^{-1}\sqrt{\left[\frac{b}{(3b-2a)}\right]}$, where a is the radius of the earth and g is the acceleration due to gravity on the earth's surface.

[18]

8. (c)

8. (c) Evaluate $\iint_S (\nabla \times F) \cdot \mathbf{n}\, dS$, where

$F = (x^2 + y - 4)\,\mathbf{i} + 3xy\,\mathbf{j} + (2xy + z^2)\,\mathbf{k}$ and S is the surface of the paraboloid $z = 4 - (x^2 + y^2)$ above the xy-plane.

[15]