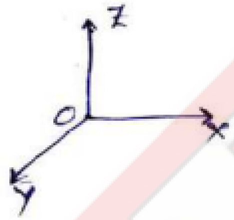
Set – III

The Straight Line

Representation of line:

Consider any two of the co-ordinate planes say YOZ and ZOX, whose equations are $x=0$; $y=0$ respectively.

These two planes intersect in Z-axis.



A point (x,y,z) lies on the Z-axis.

$\Leftrightarrow$ the point (x,y,z) lies on the YOZ plane and the point (x,y,z) lies on the ZOX plane.

$\Leftrightarrow$ $x=0$ and $y=0$.

i.e, the point (x,y,z) lies on the z-axis $\Leftrightarrow$ $x=0$, $y=0$

$\therefore$ $x=0$, $y=0$ are the two eqns of Z-axis

i.e, The common line of intersection is Z-axis.

$\therefore$ Eqns to the Z-axis are $x=0$, $y=0$.

i.e Eqn to the planes passing through the Z-axis.

Similarly, equations to the X-axis are $y=0$, $z=0$

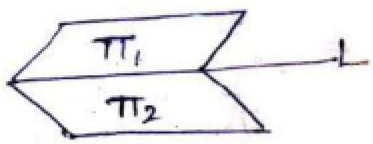
i.e, equation to the planes passing through the X-axis

and equations to the Y-axis are $z=0$, $x=0$

i.e, equation to the planes passing through the Y-axis

General form:

Consider any line L and any two planes Π_1, Π_2 whose line of intersection is L.



$\longrightarrow$ To prove that two general equations of first degree in x, y, z taken together represent a straight line.

Solⁿ: Let $a_1x+b_1y+c_1z+d_1=0$ —— ①

$a_2x+b_2y+c_2z+d_2=0$ —— ②

be two equations of first degree in x, y, z.

But every equation of first degree in x, y, z represents a plane.

The two planes taken together represent a curve of intersection of the planes which is a straight line.

($\because$ a plane cuts another plane in a straight line)

$\therefore$ Equations ① & ② taken together represent a straight line.

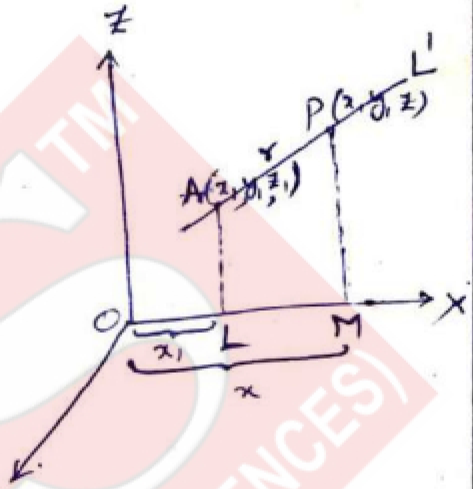
Note: Equations ① & ② taken together are called equations of straight line in the general form.

Equations of a straight line in symmetrical form:

S.T the eqns of the straight line passing through the point (x_1, y_1, z_1) and having direction cosines l, m, n are

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$$

Solⁿ:



Let L' be the line and $A(x_1, y_1, z_1)$ be the point on L'

Let $P(x, y, z)$ be any other point on L' and let $AP=r$

Let L, M be the feet of the perpendiculars from A, P on x-axis.

$\therefore OL=x_1$; $OM=x$.

$LM = OM-OL$
$= x-x_1$ —— ①

Also LM = projection of AP on x-axis

$= AP.\cos\alpha$
$= rl$ $(\because \cos\alpha = l)$ —— ②

from ① & ②, we have

$x-x_1 = lr \Rightarrow r=\frac{x-x_1}{l}$.

Similarly

$y-y_1 = mr$ & $z-z_1 = nr$

$\Rightarrow \frac{y-y_1}{m} = r$ $\Rightarrow \frac{z-z_1}{n} = r$

$\therefore \frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} = r$

which are the required eqns of the line.

Note: A general point on the line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} = r$ is (x_1+lr, y_1+mr, z_1+nr).

Parametric form of eqns of a line:

The eqns $x = x_1 + lr$, $y = y_1 + mr$, $z = z_1 + nr$ are known as the eqns of a line in parametric form, where r is the parameter.

→ The eqns of a straight line passing through (x_1, y_1, z_1) and having d.r's a, b, c are

$$\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

Soln: The d.c's of the line are $\frac{a}{\sqrt{a^2+b^2+c^2}}, \frac{b}{\sqrt{a^2+b^2+c^2}}, \frac{c}{\sqrt{a^2+b^2+c^2}}$

$\therefore$ The required eqns of the line are

$$\frac{x-x_1}{\frac{a}{\sqrt{\Sigma a^2}}} = \frac{y-y_1}{\frac{b}{\sqrt{\Sigma a^2}}} = \frac{z-z_1}{\frac{c}{\sqrt{\Sigma a^2}}} = r$$

where $\Sigma a^2 = a^2+b^2+c^2$

$$\Rightarrow \frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} = \frac{r}{\sqrt{\Sigma a^2}} = r_1 \text{ (say)}$$

$$\Rightarrow \frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c}$$

which are required eqns of the line.

Note: A general point on the line $\frac{x-x_1}{a} = \frac{y-y_1}{b} = \frac{z-z_1}{c} = r_1$ is $(ar_1+x_1, br_1+y_1, cr_1+z_1)$

Two point Form (i.e, the eqns of the line passing through two points):-

→ The eqns of the line passing through the points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}$$

Soln: The d.r's of AB are

$x_2-x_1,\ y_2-y_1,\ z_2-z_1$.

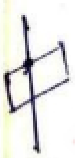
∴ The eqns of a line passing through (x_1, y_1, z_1) and having the d.r's $x_2-x_1,\ y_2-y_1,\ z_2-z_1$ are

$$\frac{x-x_1}{x_2-x_1} = \frac{y-y_1}{y_2-y_1} = \frac{z-z_1}{z_2-z_1}.$$

Problems:

→ Find the co-ordinates of the point of intersection of the line $\frac{x+1}{1} = \frac{y+3}{3} = \frac{z-2}{-2}$ with the plane $3x+4y+5z=5$

Soln: Given line is



$$\frac{x+1}{1} = \frac{y+3}{3} = \frac{z-2}{-2} = r \text{ (say)} \quad \text{—(1)}$$

and given plane is

$$3x+4y+5z-5=0 \quad \text{—(2)}$$

(1) $\equiv (x,y,z) = (r-1,\ 3r-3,\ -2r+2)$ is a point on the line for all values of r.

If this point lies on the plane (2) then

$$3(r-1)+4(3r-3)+5(-2r+2)-5=0$$

$$\Rightarrow \boxed{r=2}$$

∴ (3) $\equiv (x,y,z) = (1, 3, -2)$ which is the reqd point of intersection of (1) & (2)

→ Find the point where the line joining $(2,-3,1), (3,-4,-5)$ cuts the plane $2x+y+z=7$.

Soln: Given points are

$(2,-3,1)$ & $(3,-4,-5)$.

Now the eqns of a straight line passing through the two points are

$$\frac{x-2}{1} = \frac{y+3}{-1} = \frac{z-1}{-6} = r \text{ (say)} \quad \text{—(1)}$$

A general point is

$$(x,y,z) = (r+2,\ -r-3,\ -6r+1) \quad \text{—(2)}$$

and given that $2x+y+z=7$ —(3)

If the point $(x,y,z) = (r+2, -r-3, -6r+1)$ lies on the plane (3) we get $\boxed{r=-1}$

(2) $\equiv (x,y,z) = (1,-2,7)$ which is the required point.

→ Find K so that the lines

$\frac{x-1}{-3} = \frac{y-2}{2k} = \frac{z-3}{2}$;

$\frac{x-1}{3k} = \frac{y-5}{1} = \frac{z-6}{-5}$ may be perpendicular to each other.

Soln: The d.r's of the given lines are $-3, 2K, 2$ and $3K, 1, -5$.

$\therefore$ If these lines are $\perp^r$ to each other then $a_1a_2+b_1b_2+c_1c_2 = 0$

$\Rightarrow (-3)(3K)+(2K)(1)+(2)(-5)=0$

$\Rightarrow \boxed{K = \frac{-10}{7}}$

→ Find the distance of the point $(-1,-5,-10)$ from the point of intersection of the line

$\frac{1}{2}(x-2) = \frac{1}{4}(y+1) = \frac{1}{12}(z-2)$

and the plane $x-y+z=5$.

Soln: Given line is

$\frac{1}{2}(x-2) = \frac{1}{4}(y+1) = \frac{1}{12}(z-2) = r$ (say)

and the plane $x-y+z=5$.

$\therefore$ General point on this line is $(x,y,z) = (2r+2, 4r-1, 12r+2)$ —— ①

If this point lies on the given plane $x-y+z=5$

$\therefore 2r+2-(4r-1)+12r+2=5$

$\Rightarrow 10r+5=5$

$\Rightarrow \boxed{r=0}$

① ≡ $(x,y,z) = (2,-1,2)$.

which is the required point of intersection of line & plane.

Now the distance of the point $(-1,-5,-10)$ from the point $(2,-1,2)$ is $\sqrt{9+16+144}$

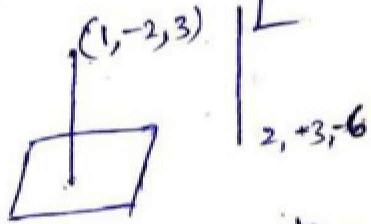
$= \sqrt{169}$

$= \underline{13}$

→ Find the distance of the point $(1,-2,3)$ from the plane $x-y+z=5$ measured parallel to the line

$\frac{1}{2}x = \frac{1}{3}y = -\frac{1}{6}z$.

Soln:



Now the eqns to the line through $(1,-2,3)$ and Parallel to the line whose d.r's $2,3,-6$ are

$\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{-6} = r$ (say)

Now any point on this line is

$(x, y, z) = (2r+1, 3r-2, -6r+3)$ —— ①

If this point lies on the given plane $x-y+z-5=0$

then we have

$2r+1-(3r-2)+(-6r+3)-5=0$

$\Rightarrow \boxed{r = 1/7}$

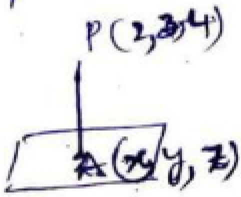
① $\equiv (x, y, z) = \left(\frac{9}{7}, \frac{-11}{7}, \frac{15}{7}\right)$

$\therefore$ The required distance of this point from the given point $(1,-2,3)$ is .

$\sqrt{\left(\frac{9}{7}-1\right)^2 + \left(-\frac{11}{7}+2\right)^2 + \left(\frac{15}{7}-3\right)^2} = 1.$

→ Find the distance of the point $(3,-4,5)$ from the plane $2x+5y-6z=16$ measured along a line with direction cosines proportional to $(2,1,-2)$.

→ Find the foot of the perpendicular from the point $P(2,3,4)$ to the plane $x+y-z+4=0$.



Solⁿ:

The given plane is

$x+y-z+4=0$ —— ①

The d.r's of the normal to the plane are $1, 1, -1$.

The d.r's of line ⊥ to the plane are $1, 1, -1$.

$\therefore$ Eqns of the line through the point $P(2,3,4)$ and having the d.r's $1,1,-1$ (i.e, ⊥ to the plane ①) are $\frac{x-2}{1} = \frac{y-3}{1} = \frac{z-4}{-1} = r$ (say) —— ②

A general point on this line is $A(x, y, z) = (r+2, r+3, -r+4)$

If this point lies on the plane ①, then

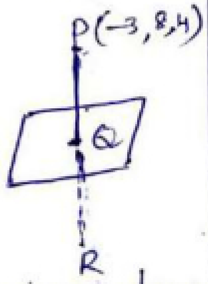
$(r+2)+(r+3)-(-r+4)+4=0$

$\Rightarrow \boxed{r = -5/3}$

$\therefore A(x, y, z) = \left(\frac{1}{3}, \frac{4}{3}, \frac{17}{3}\right)$

which is the required foot of the $\perp^r$ from the point $P(2,3,4)$.

→ Find the image of the point $P(-3,8,4)$ in the plane $6x-3y-2z+1=0$



Sol^n:

Let $R(x_1,y_1,z_1)$ be the image of $P(-3,8,4)$ in the given plane, then the mid point of PR is the foot of the $\perp^r$ from P to the given plane.

Now first of all we have to find out the foot of the $\perp^r$ from the point $P(-3,8,4)$ to the given plane.

The given plane is

$$6x-3y-2z+1=0 \quad \text{①}$$

Now the d.r.'s of the normal to plane are $6,-3,-2$.

The d.r.'s of any line $\perp^r$ to the plane are $6,-3,-2$.

$\therefore$ Equations to the line through $P(-3,8,4)$ and $\perp^r$ to plane ① are

$$\frac{x+3}{6}=\frac{y-8}{-3}=\frac{z-4}{-2}=r \text{ (say)}$$

A general point on this line is $Q(x,y,z)=Q(6r-3,-3r+8,-2r+4)$

If this point lies on the plane ① then $\boxed{r=1}$.

$\therefore Q(x,y,z)=Q(3,5,2)$, which is the foot of the $\perp^r$ from P to the given plane.

Now $R(x_1,y_1,z_1)$ is the image of $P(-3,8,4)$ and $Q(3,5,2)$ is the mid point of PR.

$$\therefore \left(\frac{x_1-3}{2},\frac{y_1+8}{2},\frac{z_1+4}{2}\right)=(3,5,2)$$

$\Rightarrow x_1=9,\ y_1=2$ and $z_1=0$

$\therefore R(x_1,y_1,z_1)=R(9,2,0)$ which is the image of P.

→ Find the image of the point $P(1,3,4)$ in the plane $2x-y+z+3=0$.

→ Find the co-ordinates of the foot of the perpendicular drawn from the origin to the plane $2x+3y-4z+1=0$; also find the co-ordinates of the point which is the image of the origin in the plane.

(Ans: $\left(-\frac{2}{29},-\frac{3}{29},\frac{4}{29}\right)$; $\left(\frac{-4}{29},\frac{-6}{29},\frac{8}{29}\right)$)

→ Find the eqns to the line through the point $(-1,3,2)$ and $\perp^r$ to the plane $x+2y+2z=3$, the length of the $\perp^r$ and the co-ordinates of its foot.

$\left(\text{Ans: } 2;\ \left(\frac{-5}{3}, \frac{5}{3}, \frac{2}{3}\right) \right)$

→ find the eqns to the line through (x_1, y_1, z_1) perpendicular to the plane $ax+by+cz+d=0$ and the co-ordinates of its foot. Deduce the expression for the $\perp^r$ distance of the given point from the given plane.

Ans: $(ar+x_1,\ br+y_1,\ cr+z_1)$

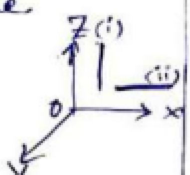
where $r = \dfrac{ax_1+by_1+cz_1+d}{\sqrt{a^2+b^2+c^2}}$

→ Find the eqns of the straight line through (a,b,c) which are (i) parallel to z-axis and (ii) $\perp^r$ to z-axis.

Solⁿ: The eqns of the line through (a,b,c) are

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} \quad \text{—— (1)}$$

where l,m,n are d.c's of the line.



(i) If this line || to z-axis then its d.c's are proportional to $(0,0,1)$ the d.c.'s of the z-axis

$$\therefore (1) \equiv \frac{x-a}{0} = \frac{y-b}{0} = \frac{z-c}{1}$$ which is the required line eqn.

(ii) If the line (1) $\perp^r$ to z-axis i.e, ||lel to xy-plane, then $0l+0m+1\cdot n=0$

$\Rightarrow n=0$

$$\therefore \frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{0}.$$

→ Show that the eqns to the straight line through (a,b,c) ||lel to the x-axis are $y=b$, $z=c$

→ S.T the straight line $m(x-a)=l(y-b)$, $z=c$ is $\perp^r$ to the z-axis.

Solⁿ: Eqns to the line are $m(x-a)=l(y-b)$, $z=c$

$$\Rightarrow \frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{0}. \quad \text{—— (1)}$$

Since the d.c.'s of the z-axis are $0,0,1$.

Since $l_1l_2+m_1m_2+n_1n_2 = l(0)+0(m)+0(1) = 0$

∴ The given line (1) is $\perp^r$ to the z-axis.

→ S.T the line $m(x-a)=l(z-b)$, $y=c$ is $\perp^r$ to the y-axis.

Transformation from the unsymmetrical to the symmetrical form:

To transform the eqns $ax+by+cz+d=0$, $a_1x+b_1y+c_1z+d_1=0$ of a line to the symmetrical form.

Sol^n: The given eqns of the line (in the general form) are

$$\left.\begin{array}{l} ax+by+cz+d=0 \\ a_1x+b_1y+c_1z+d_1=0 \end{array}\right\} \text{---} (1)$$

To transform these eqns to the symmetrical form, we require:

(i) the d.r.'s of the line and
(ii) the co-ordinates of any one point on it.

(i) **To find the d.r.'s of the line:**

Let l, m, n be d.r.'s of the line.

Since line lies in both the planes $ax+by+cz+d=0$ & $a_1x+b_1y+c_1z+d_1=0$, it is $\perp$ to the normals to both of them.

The d.r.'s of the normals to the planes are a, b, c & a_1, b_1, c_1.

Now we have

$$\left.\begin{array}{l} al+bm+cn=0 \\ a_1l+b_1m+c_1n=0 \end{array}\right\} \text{---} (2)$$

Solving by cross multiplication

$$\frac{l}{bc_1-b_1c}=\frac{m}{ca_1-c_1a}=\frac{n}{ab_1-a_1b}$$

$\therefore$ The d.r.'s l, m, n are $bc_1-b_1c,\ ca_1-c_1a,\ ab_1-a_1b$.

(ii) **To find the co-ordinates of any one point:**

putting $z=0$ in (1), we get

$$\left.\begin{array}{l} ax+by+d=0 \\ a_1x+b_1y+d_1=0 \end{array}\right\} \text{---} (3)$$

Solving by cross multiplication

$$\frac{x}{bd_1-b_1d}=\frac{y}{a_1d-ad_1}=\frac{1}{ab_1-a_1b}$$

$$\Rightarrow (x,y,z)=\left(\frac{bd_1-b_1d}{ab_1-a_1b}, \frac{a_1d-ad_1}{ab_1-a_1b}, 0\right)$$

Now the required line in symmetrical form

$$\frac{x-\left(\frac{bd_1-b_1d}{ab_1-a_1b}\right)}{bc_1-b_1c}=\frac{y-\left(\frac{a_1d-ad_1}{ab_1-a_1b}\right)}{ca_1-c_1a}=\frac{z-0}{ab_1-a_1b}$$

Problems:

→ put in symmetrical form, the eqns of the line $3x-y+z+1=0$ and $5x+y+3z=0$. Also find the eqn to a plane through $(2,1,4)$ and $\perp^r$ to the given line.

Soln: The given eqns of the line in the general form are

$$\left.\begin{aligned} 3x-y+z+1&=0 \\ \& \quad 5x+y+3z&=0 \end{aligned}\right\} \text{—} (1)$$

Let l, m, n be the d.r.'s of the line then this line lies on the both the planes given by (1)

it is $\perp^r$ to the normals to both of them.

The d.r.'s of the normals to the planes are $3,-1,1$ & $5,1,3$

we have

$$3l-m+n=0$$
$$5l+m+3n=0$$

$$\Rightarrow \frac{l}{1}=\frac{m}{1}=\frac{n}{-2}$$

∴ The required line d.r.'s l,m,n are $1,1,-2$.

Now suppose the given line intersects the plane $z=0$ at $(x_1,y_1,0)$ then

$$3x_1-y_1+1=0$$
and $$5x_1+y_1=0$$

$$\Rightarrow x_1=-\frac{1}{8},\ y_1=\frac{5}{8}$$

∴ The point on the required line is $\left(-\frac{1}{8},\frac{5}{8},0\right)$

Hence the symmetrical form of the line is

$$\frac{x+\frac{1}{8}}{1}=\frac{y-\frac{5}{8}}{1}=\frac{z}{-2} \text{ —— } (2)$$

Equation of any plane perpendicular to the line is

$$x+y-2z+d=0$$

But this plane also passes through $(2,1,4)$

$$\therefore 2+1-8+d=0$$

$$\Rightarrow \boxed{d=5}$$

∴ The required plane is

$$x+y-2z+5=0$$

→ Find in a symmetrical form, the eqns of the line $x+y+z+1=0$, $4x+y-2z+2=0$ and find its d.c's.

[Ans: $\frac{x+\frac{1}{3}}{1}=\frac{y+\frac{2}{3}}{-2}=\frac{z}{1}$; d.c.s, $\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}$]

→ obtain the symmetrical form of eqns of the line

$x-2y+3z=4,\quad 2x-3y+4z=5$.

(Ans: $x+2=\frac{1}{2}(y+3)=z$)

→ Find the equation of the plane through the point (1,1,1) and $\perp^r$ to the line

$x-2y+z=2,\ 4x+3y-z+1=0$

→ Find the equation of the line through the point (1,2,3) parallel to the line

$x-y+2z=5,\quad 3x+y+z=6$

Soln.

The given eqns of the line in general form are

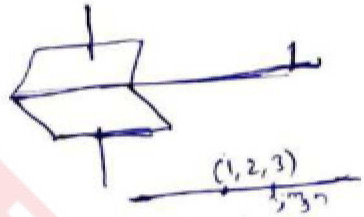
$$\left.\begin{aligned} x-y+2z&=5\\ 3x+y+z&=6\end{aligned}\right\} \text{——} (1)$$

Let l, m, n be the d.r.'s of the required line.

Since it is $\parallel$ to the given line.

$\therefore$ the direction ratios of the given line are also l, m, n.

But the given line is the intersection of the planes given by (1) and hence, lies in both the planes and $\perp^r$ to the normals of these planes.



The d.r's of the normals to the planes are $1,-1,2$ & $3,1,1$.

we have

$l\cdot 1 - m\cdot 1 + n\cdot 2 = 0$

$l\cdot 3 + m\cdot 1 + n\cdot 1 = 0$

$\Rightarrow \frac{l}{-3} = \frac{m}{5} = \frac{n}{4}$

$\therefore$ The reqd line d.r's l, m, n are $-3, 5, 4$.

$\therefore$ The eqns of the line in symmetrical form are

$$\frac{x-1}{-3} = \frac{y-2}{5} = \frac{z-3}{4}$$

→ Find the eqns of the line through the point (1,2,4) parallel to the line

$3x+2y-z=4,\ x-2y-2z=5$

→ prove that eqns to the line through (α,β,γ) at right angles to the lines

$\frac{x}{l_1}=\frac{y}{m_1}=\frac{z}{n_1}$; $\frac{x}{l_2}=\frac{y}{m_2}=\frac{z}{n_2}$

are

$$\frac{x-\alpha}{m_1n_2-m_2n_1} = \frac{y-\beta}{n_1l_2-n_2l_1} = \frac{z-\gamma}{l_1m_2-l_2m_1}$$

→ Find the condition that the line $x = ay + b$, $z = cy + d$ and $x = a'y + b'$, $z = c'y + d'$ may be perpendicular.

(Ans: ~~$aa' + bb' + 1$~~ $ac' + cc' = 0$)

→ Find a, b, c, d so that the line $x = ay + b$, $z = cy + d$ may pass through the points $(3, 1, -3)$, $(4, 2, -4)$

Sol'n: The eqns of line are

$$x = ay + b, \quad z = cy + d \quad \text{(1)}$$

Since (1) passes through $(3, 1, -3)$, $(4, 2, -4)$ we get

$$3 = a + b$$
$$-3 = c + d$$
and
$$4 = 2a + b$$
$$-4 = 2c + d$$

Solving above we get

$$a = 1, \ b = 2, \ c = -1, \ d = -2$$

∴ The line becomes

$$x = y + 2 \ \text{ and } \ z = -y - 2.$$

→ Find the points of intersection of the line $x + y - z + 1 = 0 = 14x + 9y - 7z - 1$ with the xy and yz planes and hence put down the symmetrical form of its equations.

→ Find the angle between the lines in which the planes $3x - 7y - 5z = 1$, $5x - 13y + 3z + 2 = 0$ cut the plane $8x - 11y + z = 0$

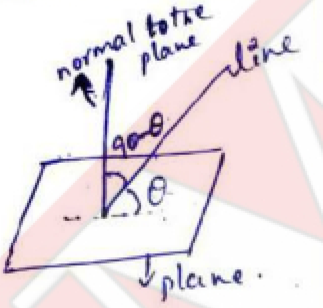
→ Find the angle between the lines

$$3x + 2y + z - 5 = 0 = x + y - 2z - 3$$
$$2x - y - z = 0 = 7x + 10y - 8z$$

Angle between a line and a plane:

To find the angle b/w the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ and the plane $ax+by+cz+d=0$

Sol^n: The angle between a line and plane is the complement of the angle b/w the line and the normal to the plane.



If 'θ' is the angle which the line makes with the plane, the line makes an angle of $(90-\theta)$ with the normal to the plane.

Now the d.r.'s of the line & normal to the plane are l,m,n & a,b,c resply

$$\therefore \cos(90-\theta)=\frac{al+mb+nc}{\sqrt{l^2+m^2+n^2}\sqrt{a^2+b^2+c^2}}$$

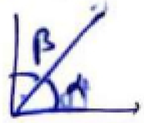
$$\sin\theta=\frac{al+mb+nc}{\sqrt{\Sigma l^2}\sqrt{\Sigma a^2}}$$

[Complementary angles:-

Two angles are complementary if their sum is equal to $90°$

i.e., $\alpha+\beta=90°$

$\therefore \alpha,\beta$ are Complementary angles.]



Note: The straight line is $\parallel$ to the plane

$\Rightarrow \theta=0°$

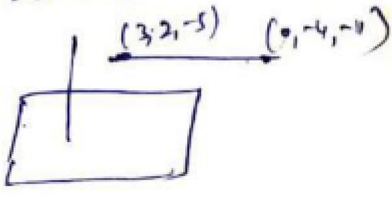
$\therefore al+bm+cn=0.$

The condition is also evident from the fact a line $\parallel$ to a plane

$\Leftrightarrow$ it is $\perp^r$ to the normal to plane

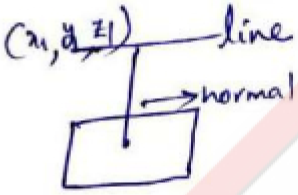
$\rightarrow$ Show that the line $\frac{x-2}{3}=\frac{y-3}{4}=\frac{z-4}{5}$ is $\parallel$ to the plane $2x+y-2z=3$.

$\rightarrow$ S.T the eqn of the plane $\parallel$ to the join of $(3,2,-5)$ and $(0,-4,-11)$ and passing through the points $(-2,1,-3)$ & $(4,3,3)$ is $4x+3y-5z=10$.

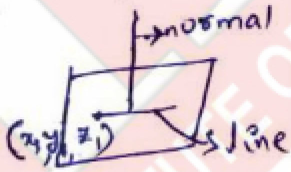


Deductions:

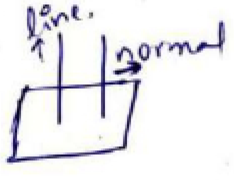
→ The condition that the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ $\parallel$ to the plane $ax+by+cz+d=0$ is $al+bm+cn=0$ & $ax_1+by_1+cz_1+d\neq 0$



→ The condition that the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ lies on the plane $ax+by+cz+d=0$ is $al+bm+cn=0$ and $ax_1+by_1+cz_1+d=0$



→ The condition for the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ to be $\perp^r$ to the plane $ax+by+cz+d=0$ is $\frac{l}{a}=\frac{m}{b}=\frac{n}{c}$.



→ find the equation to the plane which passes through the point (x_1,y_1,z_1) and the line $\frac{x-a}{l}=\frac{y-b}{m}=\frac{z-c}{n}$.

Solⁿ:

The general equation of the plane containing the given line is

$A(x-a)+B(y-b)+C(z-c)=0$ —— (1)

where A, B, C are parameters subjected to the condition

$Al+Bm+Cn=0$ —— (2)

The plane (1) will pass through the point (x_1, y_1, z_1) if $A(x_1-a)+B(y_1-b)+C(z_1-c)=0$ —— (3)

Eliminating A, B, C from (1), (2) and (3), we have

$$\begin{vmatrix} x-a & y-b & z-c \\ l & m & n \\ x_1-a & y_1-b & z_1-c \end{vmatrix}=0$$

which is the required equation.

→ Find the equation to the plane containing the line $\frac{x+1}{-3}=\frac{y+3}{2}=\frac{z+2}{1}$ and the point $(0,7,-7)$ and show that the line $\frac{x}{1}=\frac{y-7}{-3}=\frac{z+7}{2}$ also lies in the same plane.

Sol: Let the required eqn of the plane be

$Ax+By+Cz+d=0$ —— (1)

The given line will lie in it if

$-A+3B-2C+D=0$ —— (2)

and $-3A+2B+C=0$ —— (3)

The plane (1) will pass through $(0,7,-7)$ if

$0A+7B-7C+D=0$ —— (4)

Eliminating A, B, C, D from the eqn (1), (2), (3) and (4), the required plane is

$$\begin{vmatrix} x & y & z & 1 \\ -1 & 3 & -2 & 1 \\ -3 & 2 & 1 & 0 \\ 0 & 7 & -7 & 1 \end{vmatrix}=0$$

$\Rightarrow x+y+z=0$ —— (5)

Now, the line $\frac{x}{1}=\frac{y-7}{-3}=\frac{z+7}{2}$ will lie in the plane (5) if the point $(0,7,-7)$ on the line lies on the plane and the line (d.r.s $1,-3,2$) is perpendicular to the normal to the plane.

i.e, $0+7+(-7)=0$

and $1-3+2=0$

→ Prove that the plane through (α,β,γ) and the line

$x = py+q = rz+s$ is given by

$$\begin{vmatrix} x & py+q & rz+s \\ \alpha & p\beta+q & r\gamma+s \\ 1 & 1 & 1 \end{vmatrix} = 0$$

Sol^n: The given line can be written as

$$\frac{x}{1} = \frac{y+\frac{q}{p}}{1/p} = \frac{z+\frac{s}{r}}{1/r} \quad \text{---(1)}$$

Let the eqn of any plane be

$$Ax+By+Cz+D=0 \quad \text{---(2)}$$

It will pass through the line (1), if

$$A(0)+B(-q/p)+C(-s/r)+D=0 \quad \text{---(3)}$$

$$A(1)+B(\tfrac{1}{p})+C(\tfrac{1}{r})=0 \quad \text{---(4)}$$

and The plane will pass through (α,β,γ) if

$$A(\alpha)+B(\beta)+C(\gamma)+D=0 \quad \text{---(5)}$$

subtracting (3) from (2) & (5).

we get.

$$Ax+B(y+\tfrac{q}{p})+C(z+\tfrac{s}{r})=0 \quad \text{---(6)}$$

$$A\alpha+B(\beta+\tfrac{q}{p})+C(\gamma+\tfrac{s}{r})=0 \quad \text{---(7)}$$

Eliminating A, B, C from (6), (7) and (4)

$$\begin{vmatrix} x & y+q/p & z+s/r \\ \alpha & \beta+q/p & \gamma+s/r \\ 1 & 1/p & 1/r \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} x & py+q & rz+s \\ \alpha & p\beta+q & r\gamma+s \\ 1 & 1 & 1 \end{vmatrix} = 0$$

→ Find the equations to the line through (f,g,h), which is ||lel to the plane $lx+my+nz=0$ and intersects the line $ax+by+cz+d=0=a'x+b'y+c'z+d'$.

Sol^n: Any plane parallel to $lx+my+nz=0$ and through (f,g,h) is $l(x-f)+m(y-g)+n(z-h)=0$ --- (1)

any plane through the given line is

$$(ax+by+cz+d)+\lambda(a'x+b'y+c'z+d')=0$$

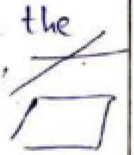
if it passes through (f,g,h), then

$$\lambda = -\frac{af+bg+ch+d}{a'f+b'g+c'h+d'}$$

Hence, the plane becomes

$$\frac{ax+by+cz+d}{af+bg+ch+d} = \frac{a'x+b'y+c'z+d'}{a'f+b'g+c'h+d'} \quad \text{---(2)}$$

∴ The equations (1) and (2) give the required line.



→ show that the line $x+10 = (8-y)/2 = z$ lies in the plane $x+2y+3z=6$ and the line $\frac{1}{3}(x-2) = -(y+2) = \frac{1}{4}(z-3)$ lies in the plane $2x+2y-z+3=0$

→ Find the equation of the plane containing the line $\frac{1}{2}(x+2)+\frac{1}{3}(y+3) = -\frac{1}{2}(z-4)$ and the point $(0,6,0)$

[Ans: $3x+2y+6z-12=0$]

→ $\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$ and $\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$ are two straight lines. find the equation of plane containing the first line and parallel to the second.

[Ans. $\Sigma(x-x_1)(m_1n_2-m_2n_1)=0$]

→ find the equation to the plane containing the line $y/b+z/c=1, x=0$ and parallel to the line $x/a+z/c=1$, $y=0$.

[Ans: $x/a-y/b-z/c+1=0$]

→ find the equation to the plane which passes through the z-axis and is perpendicular to the line $\frac{x-1}{\cos\theta} = \frac{y+2}{\sin\theta} = \frac{z-3}{0}$

[Ans. $x\cos\theta+y\sin\theta=0$].

→ show that the equation of the plane which passes through the line $\frac{x-1}{3} = \frac{y+6}{4} = \frac{z+1}{2}$ and is parallel to the line $\frac{x-2}{2} = \frac{y-1}{-3} = \frac{z+4}{5}$, is $26x-11y-17z-109=0$ and show that the point $(2,1,-4)$ lies on it.

→ Find the equation of the plane containing the line $-\frac{1}{3}(x+1) = \frac{1}{2}(y-3) + (z+2)$ and the point $(0,7,-7)$ and show that the line $x = \frac{1}{3}(7-y) = \frac{1}{2}(z+7)$ lies in the same plane. [Ans: $x+y+z=0$].

→ Find the equation of the plane which contains the line $\frac{(x-1)}{2} = -y-1 = \frac{(z-3)}{4}$ and is perpendicular to the plane $x+2y+z=12$.

[Ans: $9x-2y-5z+4=0$]

Coplanar Lines:

→ To find the condition that the two lines

$$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}; \quad \text{--- (1)}$$

$$\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2} \quad \text{--- (2)}$$

may intersect (or may be coplanar) and find the eqn of the plane in which they lie.

Note: If the lines intersect, they lie in a plane.

Solⁿ:

Method (I):

eqn of any plane containing the line (1) is

$$A(x-x_1) + B(y-y_1) + C(z-z_1) = 0 \quad \text{--- (3)}$$

and $Al_1 + Bm_1 + Cn_1 = 0$ — (4)

The plane (3) will contain the line (2) if

(i) the point (x_2, y_2, z_2) lies on it

$$\Rightarrow A(x_2-x_1) + B(y_2-y_1) + C(z_2-z_1) = 0 \quad (5)$$

(ii) the line is $\perp^r$ to normal to plane

$$\Rightarrow Al_2 + Bm_2 + Cn_2 = 0 \quad \text{--- (6)}$$

The two lines will be coplanar if

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

which is the required condition for the lines to intersect.

Now required eqn of the plane is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

∴ This is the eqn of the plane containing the two lines.

Method-II

Two lines are coplanar iff they intersect (or) are parallel.

Now we first consider the case of intersection.

Any point on the line (1) is

$$(x_1 + l_1 r_1,\ y_1 + m_1 r_1,\ z_1 + n_1 r_1) \quad \text{--- (A)}$$

Any point on the line (2) is

$$(x_2 + l_2 r_2,\ y_2 + m_2 r_2,\ z_2 + n_2 r_2) \quad \text{--- (B)}$$

where r_1, r_2 are any two numbers.

In case the lines ① & ② are intersect, these points should be coincide for some values of r_1, r_2.

$\therefore x_1 + l_1 r_1 = x_2 + l_2 r_2$

$\Rightarrow x_1 - x_2 + l_1 r_1 - l_2 r_2 = 0$

$y_1 + m_1 r_1 = y_2 + m_2 r_2$

$\Rightarrow y_1 - y_2 + m_1 r_1 - m_2 r_2 = 0$

$z_1 + n_1 r_1 = z_2 + n_2 r_2$

$\Rightarrow z_1 - z_2 + n_1 r_1 - n_2 r_2 = 0$

Eliminating r_1, r_2 from these eqns, we get

$$\begin{vmatrix} x_1 - x_2 & l_1 & l_2 \\ y_1 - y_2 & m_1 & m_2 \\ z_1 - z_2 & n_1 & n_2 \end{vmatrix} = 0$$

(or) $$\begin{vmatrix} x_1 - x_2 & y_1 - y_2 & z_1 - z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

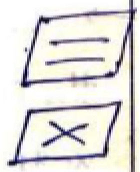
which is required condition for coplanarity.

Note:

$\boxed{1}$ The condition is clearly satisfied if the lines are parallel, for then last two rows are identical
$\therefore$ determinant vanishes.

In general, the eqn

$$\begin{vmatrix} x - x_1 & y - y_1 & z - z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$



represents the plane which passes through the line ① and is $\parallel$ to line ②

and the eqn

$$\begin{vmatrix} x - x_2 & y - y_2 & z - z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$$

represents the plane which passes through the line ② and $\parallel$ to the line ①

$\boxed{2}$. working rule:

(i) write down the co-ordinates of any point on the first & second line.

(ii) Equating the corresponding co-ordinates of two points and find the values of r_1, r_2 from any two eqns. show that these values of r_1, r_2 satisfies third eqn.

(iii) If we put the value of r_1 (or) r_2 in Ⓐ we get the required point of intersection.

→ To find the condition that the lines

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \quad \text{——(1)}$$

& $ax+by+cz+d=0=a'x+b'y+c'z+d'$ ——(2)

may intersect and find the eqn of the plane in which they lie.

Solⁿ: Any plane containing line (2) is

$$(ax+by+cz+d)+K(a'x+b'y+c'z+d')=0 \quad \text{——(3)}$$

The line (1) lies in plane (3) if (i) the point (x_1, y_1, z_1) on the line (1) lies on the plane (3). and

(ii) $al+bm+cn=0$

i.e, $ax_1+by_1+cz_1+d+K(a'x_1+b'y_1+c'z_1+d')=0$ ——(4)

and

$l(a+Ka')+m(b+Kb')+n(c+Kc')=0$ ——(5)

(4) ≡ $K=\dfrac{-(ax_1+by_1+cz_1+d)}{a'x_1+b'y_1+c'z_1+d'}$ ——(6)

(5) ≡ $la+mb+nc+K(a'l+b'm+n'c)=0$

$\Rightarrow K=\dfrac{-(la+mb+nc)}{la'+mb'+nc'}$ ——

Equating two values of K, we have

$$\frac{ax_1+by_1+cz_1+d}{a'x_1+b'y_1+c'z_1+d'}=\frac{la+mb+nc}{la'+mb'+nc'}$$

$$\Rightarrow \frac{ax_1+by_1+cz_1+d}{la+mb+nc}=\frac{a'x+b'y+c'z}{la'+mb'+nc'}$$

which is the required condition

putting the value of K from (7) in (3) we get the required plane.

→ To find the condition that the lines $a_1x+b_1y+c_1z+d_1=0=a_2x+b_2y+c_2z+d_2$ ——(1)

and

$a_3x+b_3y+c_3z+d_3=0=a_4x+b_4y+c_4z+d_4$ ——(2)

may intersect (or be coplanar)

Solⁿ: If the two lines are intersect at (α, β, γ) then (α, β, γ) lies on each of the four planes.

$$\therefore a_1\alpha+b_1\beta+c_1\gamma+d_1=0$$
$$a_2\alpha+b_2\beta+c_2\gamma+d_2=0$$
$$a_3\alpha+b_3\beta+c_3\gamma+d_3=0$$
$$a_4\alpha+b_4\beta+c_4\gamma+d_4=0$$

Now eliminating α, β, γ from the above eqns,

we get,

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = 0$$

which is the required Condition.

→ Prove that the lines

$x-3y+2z+4=0=2x+y+4z+1$;

$3x+2y+5z-1=0=2y+z$

intersect and find the co-ordinates of their point of intersection.

Sol^n: The given lines (in general form)

$x-3y+2z+4=0=2x+y+4z+1$ ——(1)

$3x+2y+5z-1=0=2y+z$ ——(2)

To reduce the line (1) into symmetrical form,

Now let l, m, n be the d.r's of the line (1)

then $l-3m+2n=0$

$2l+m+4n=0$

Now solving:

$$\frac{l}{-14}=\frac{m}{0}=\frac{n}{7}$$

$$\Rightarrow \frac{l}{-2}=\frac{m}{0}=\frac{n}{1}$$

Now putting $z=0$ in (1)

$\therefore x-3y+4=0$

$2x+y+1=0$

Solving: $\frac{x}{-7}=\frac{y}{7}=\frac{1}{7}$

$\Rightarrow x=-1,\ y=1$

$\therefore (x,y,z)=(-1,1,0)$.

$\therefore$ The eqns of line (in symmetrical form) through the point $(-1,1,0)$ and having the d.r's $-2,0,1$

are $\frac{x+1}{-2}=\frac{y-1}{0}=\frac{z}{1}$ ——(3)

Now the eqn of any plane through the line (2) is

$(3x+2y+5z-1)+\lambda(2y+z)=0$ ——(4)

The line (3) lies in the plane (4) if (i) the point $(-1,1,0)$ on the line (3) lies in the plane (4)

and (ii) $al+bm+cn=0$

Now the point $(-1,1,0)$ lies in the plane (4)

if $3(-1)+2(1)+5(0)-1+\lambda[2(1)+0]=0$

$\Rightarrow -2+2\lambda=0$

$\Rightarrow \boxed{\lambda=1}$

(4) $\equiv 3x+2y+5z-1+(2y+z)=0$

$\Rightarrow 3x+4y+6z-1=0$ ——(5)

Now $al+bm+cn = 3(-2)+4(0)+6(1)$
$= 0$

$\therefore$ The given lines are coplanar.

Now the two lines (3) & (2) are intersect in a point where the line (3) meets one of the planes of the line (2)

Now any point on the line (3) is $(-2r-1, 1, r)$ —— (6)

If this lies on the second plane of line (2)

$2(1)+r=0$

$\Rightarrow \boxed{r=-2}$

(6) $\equiv (4-1, 1, -2) = (3, 1, -2)$

which is the required point of intersection of two lines.

→ Show that the lines

$\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}$,

$\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8}$

intersect and find the co-ordinates of the point of intersection.

Soln: The given lines are

$\frac{x-4}{1} = \frac{y+3}{-4} = \frac{z+1}{7}$ —— (1)

$\frac{x-1}{2} = \frac{y+1}{-3} = \frac{z+10}{8}$ —— (2)

Any point on the line (1) is

$(r_1+4, -4r_1-3, 7r_1-1)$ —— (3)

Sly any point on the line (2) is

$(2r_2+1, -3r_2-1, 8r_2-10)$ —— (4)

If the two given lines intersect then for some value of r_1 & r_2 the points (3) & (4) must coincide.

i.e, $r_1+4 = 2r_2+1 \Rightarrow r_1-2r_2-1=0$

$-4r_1-3 = -3r_2-1 \Rightarrow 4r_1-3r_2+2=0$

$7r_1-1 = 8r_2-10 \Rightarrow 7r_1-8r_2+9=0$

Solving the first two of these eqns we get $r_1=1$, $r_2=2$ which also, clearly satisfy third eqn.

Hence, the lines intersect and the point of intersection obtained by putting $r_1=1$ (or $r_2=2$) in (3) (or (4)) is $(5, -7, 6)$

→ Show that the lines

$\frac{x+3}{2} = \frac{y+5}{3} = \frac{z-7}{-3}$.

$\frac{x+1}{4} = \frac{y+1}{5} = \frac{z+1}{-1}$ are coplanar and find the eqn of the plane containing them.

Solⁿ: The given lines are

$\frac{x+3}{2} = \frac{y+5}{3} = \frac{z-7}{-3}$ —— ①

and $\frac{x+1}{4} = \frac{y+1}{5} = \frac{z+1}{-1}$ —— ②

Eqn of the plane through the line ① and || to ② is

$$\begin{vmatrix} x+3 & y+5 & z-7 \\ 2 & 3 & -3 \\ 4 & 5 & -1 \end{vmatrix} = 0$$

$\Rightarrow (x+3)(-3+15) + (y+5)(-12+2) + (z-7)(10-12) = 0$

$\Rightarrow (x+3)(12) + (y+5)(-10) + (z-7)(-2) = 0$

$\Rightarrow 12x + 36 - 10y - 50 - 2z + 14 = 0$

$\Rightarrow 12x - 10y - 2z = 0$

$\Rightarrow 6x - 5y - z = 0$ —— ③

The plane ③ passes through $(-1,-1,-1)$, a point on the line ② if

$6(-1) - 5(-1) - z(-1) = 0$

(or) $0 = 0$

which is true.

∴ The two lines are coplanar and the eqn of the plane containing them is $6x - 5y - z = 0$

→ Show that the lines

$\frac{x+5}{3} = \frac{y+4}{1} = \frac{z-7}{-2}$

$3x+2y+z-2 = 0 = x-3y+2z+13$

are coplanar and find the eqn to the plane in which they lie.

Solⁿ: The given lines are

$\frac{x+5}{3} = \frac{y+4}{1} = \frac{z-7}{-2}$ —— ①

$3x+2y+z-2 = 0 = x-3y+2z+13$ —— ②

Any plane through the line ② is

$(3x+2y+z-2) + \lambda(x-3y+2z+13) = 0$ —— ③

The line ① lies on this plane ③ if

(i) the point $(-5,-4,7)$ on the line ① lies in the plane ③ and

(ii) $al + bm + cn = 0$

Now the point $(-5,-4,7)$ lies on the plane ③ if

$$3(-5)+2(-4)+7-2+\lambda(-5-3(-4)+2(7)-13)=0$$

$$\Rightarrow -\cancel{15}\ +7-2+\lambda(-5+12+14-13)=0$$

$$\Rightarrow -18+\lambda(8)=0$$

$$\Rightarrow \lambda=\frac{18}{8}=\frac{9}{4}$$

$$\therefore \boxed{\lambda=\frac{9}{4}}$$

$\therefore$ ③ $\equiv$

$$(3x+2y+z-2)+\frac{9}{4}(x-3y+2z-13)=0$$

$$\Rightarrow 21x-19y+22z-125=0$$

Now $al+bm+cn=21(3)+(-19)(1)+22(-2)$

$$=63-63=0$$

$\therefore$ The given lines are coplanar and the eqn of the plane in which they lie is $21x-19y+22z-125=0$

→ S.T the lines

$$\frac{x+4}{3}=\frac{y+6}{5}=-\frac{1}{2}(z-1)$$

$$3x-2y+z+5=0=2x+3y+4z-4$$

are coplanar. Find also the co-ordinates of their point of intersection and the eqn of the plane in which they lie.

→ Prove that the lines

$$\frac{x-1}{2}=\frac{y+1}{-3}=\frac{z+10}{8};$$

$$\frac{x-4}{1}=\frac{y+3}{-4}=\frac{z+1}{7}$$ intersect

find also their point of intersection and the plane through them.

(Ans: $(5,-7,6)$; $11x-6y-5z-67=0$)

→ P.T the lines $\frac{x+1}{3}=\frac{y+3}{5}=\frac{z+5}{7}$

$\frac{x-2}{1}=\frac{y-4}{3}=\frac{z-6}{5}$ intersect. Find their point of intersection and the plane in which they lie.

→ $x+2y-z-3=0$, $3x-y+2z-1=0$, $2x-2y+3z-2=0$, $x-y+z+1=0$ are two given pair of planes. Show that the line of intersection of the first pair is coplanar with the line of intersection of the latter.

→ S.T. the line of intersection of the planes $7x-4y+7z+16=0$, $4x+3y-2z+3=0$ is coplanar with the line of intersection of planes $x-3y+4z+6=0$, $x-y+z+1=0$.

Obtain the eqn of the plane through the two lines. (Ans: $3x-7y+9z+13=0$)

Hint: The line of intersection of the first two planes is

$$\left.\begin{array}{l}7x-4y+7z+16=0\\4x+3y-2z+3=0\end{array}\right\} \longrightarrow (1)$$

The line of intersection of last two planes

$$\left.\begin{array}{l}x-3y+4z+6=0\\x-y+z+1=0\end{array}\right\} \longrightarrow (2)$$

Now we p.T two lines (1) & (2) are coplanar.

To reduce the line (1) in symmetrical form.]

→ P.T the lines

$$\frac{x-a}{a'}=\frac{y-b}{b'}=\frac{z-c}{c'} \text{ and }$$

$$\frac{x-a'}{a}=\frac{y-b'}{b}=\frac{z-c'}{c} \text{ intersect}$$

and find the co-ordinates of the point of intersection and the eqn of plane in which they lie.

[Ans: $a+a', b+b', c+c'$;

$\sum x(bc'-b'c)=0$]

→ S.T the two straight lines $x=mz+a$, $y=nz+b$ and $x=m'z+a'$, $y=n'z+b'$. would intersect iff $(a-a')(n-n')=(b-b')(m-m')$.

Sol^n: The given straight lines are $x=mz+a$, $y=nz+b$ — (1)

$x=m'z+a'$, $y=n'z+b'$ — (2)

$(1) \equiv x-a=mz, \ y-b=nz$

$\Rightarrow \frac{x-a}{m}=z, \ \frac{y-b}{n}=z$

$\Rightarrow \frac{x-a}{m}=\frac{y-b}{n}=\frac{z}{1}$ — (3)

Similarly

$(2) \equiv \frac{x-a'}{m'}=\frac{y-b'}{n'}=\frac{z}{1}$ — (4)

$$\begin{vmatrix} a'-a & b'-b & 0 \\ m & n & 1 \\ m' & n' & 1 \end{vmatrix}$$

$=(a'-a)(n-n')-(b'-b)(m-m')$

$=(b-b')(m-m')-(b'-b)(m-m')$

$\left(\because (a'-a)(n-n')=(b-b')(m-m')\right)$

$=0$

∴ The given lines are intersect.

→ S.T the plane containing the two parallel lines

$x-4=-\frac{1}{4}(y-3)=\frac{1}{5}(z-2)$,

$x-3=-\frac{1}{4}(y+2)=\frac{1}{5}z$ is

$11x-y-3z=35$

→ S.T the line $\frac{x+a}{1}=\frac{y+b}{1}=\frac{z+c}{1}$ intersect each of the four lines

(i) $x=0,\ y+z=3a$

(ii) $y=0,\ z+x=3b$

(iii) $z=0,\ x+y=3c$

(iv) $x+y+z=3k,$

$a(a-k)^{-1}x+b(b-k)^{-1}y+c(c-k)^{-1}z=0$

at right angles if $a+b+c=0$

→ S.T. the eqn of the plane which passes through the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ —(1) and is $\perp^r$ to the plane containing the lines

$\frac{x}{m}=\frac{y}{n}=\frac{z}{l}$ —(2) and $\frac{x}{n}=\frac{y}{l}=\frac{z}{m}$ —(3)

is $(m-n)x+(n-l)y+(l-m)z=0$

Sol^n: Now the eqn of any plane containing the two lines (2) & (3) is

$$\begin{vmatrix} x & y & z \\ m & n & l \\ n & l & m \end{vmatrix}=0$$

$\Rightarrow (mn-l^2)x+(nl-m^2)y+(ml-n^2)z=0$ —(4)

Now the eqn of any plane through the line (1) is

$Ax+By+Cz=0$ —(5)

where $Al+Bm+Cn=0$ —(6)

Since the plane (5) is $\perp^r$ to (4)

$\therefore a_1a_2+b_1b_2+c_1c_2=0$

i.e, $A(mn-l^2)+B(nl-m^2)+C(lm-n^2)=0$ —(7)

solving (6) & (7) we get

$$\frac{A}{m(lm-n^2)-n(nl-m^2)}=\frac{B}{n(mn-l^2)-l(lm-n^2)}$$

$$=\frac{C}{l(nl-m^2)-m(mn-l^2)}$$

$$\Rightarrow \frac{A}{lm^2-mn^2-ln^2+m^2n}=\frac{B}{mn^2-nl^2-l^2m+ln^2}$$

$$=\frac{C}{nl^2-lm^2-m^2n+ml^2}$$

$$\Rightarrow \frac{A}{l(m-n)(m+n)+mn(m-n)}=\frac{B}{\quad}=\frac{C}{\quad}$$

$$\Rightarrow \frac{A}{(m-n)[l(m+n)+mn]}=\frac{B}{(n-l)[m(n+l)+nl]}=\frac{C}{(l-m)[n(l+m)+lm]}$$

$$\Rightarrow \frac{A}{m-n}=\frac{B}{n-l}=\frac{B}{l-m}$$

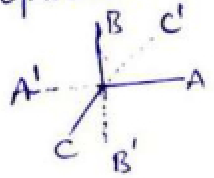
$\therefore$ putting these values of A, B, C in (5) we get

$(m-n)x+(n-l)y+(l-m)z=0$

which is the required plane

→ A, A', B, B', C, C' are points on the axes. Show that the lines of intersection of the planes $A'BC$, $AB'C'$, $B'CA$, $BC'A'$, $C'AB$, $CA'B'$ are coplanar.

Sol:



Let the points

A, A' be $(a,0,0), (a',0,0)$

B, B' be $(0,b,0), (0,b',c)$

and C, C' be $(0,0,c), (0,0,c')$

Now the eqns of the plane $A'BC$ (intercept form) is

$$\frac{x}{a'} + \frac{y}{b} + \frac{z}{c} = 1,$$

eqns of the plane $AB'C'$ is

$$\frac{x}{a} + \frac{y}{b'} + \frac{z}{c'} = 1$$

∴ The line of intersection of the planes $A'BC$ & $AB'C'$ is

$$\frac{x}{a'} + \frac{y}{b} + \frac{z}{c} - 1 = 0 = \frac{x}{a} + \frac{y}{b'} + \frac{z}{c'} - 1 \quad \text{--- (1)}$$

Sly. the line of intersection of the planes $B'CA$ & $BC'A'$

is $$\frac{x}{a} + \frac{y}{b'} + \frac{z}{c} - 1 = 0 = \frac{x}{a'} + \frac{y}{b} + \frac{z}{c'} - 1 \quad \text{--- (2)}$$

and the line of intersection of the planes $C'AB$ & $CA'B'$ is

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c'} - 1 = 0 = \frac{x}{a'} + \frac{y}{b'} + \frac{z}{c} - 1 \quad \text{--- (3)}$$

Now we show that the lines (1) & (2) are coplanar.

Now
$$\begin{vmatrix} \frac{1}{a'} & \frac{1}{b} & \frac{1}{c} & -1 \\ \frac{1}{a} & \frac{1}{b'} & \frac{1}{c'} & -1 \\ \frac{1}{a} & \frac{1}{b'} & \frac{1}{c} & -1 \\ \frac{1}{a'} & \frac{1}{b} & \frac{1}{c'} & -1 \end{vmatrix}$$

$$= \begin{vmatrix} \frac{1}{a}+\frac{1}{a'} & \frac{1}{b}+\frac{1}{b'} & \frac{1}{c}+\frac{1}{c'} & -2 \\ \frac{1}{a} & \frac{1}{b'} & \frac{1}{c'} & -1 \\ \frac{1}{a}+\frac{1}{a'} & \frac{1}{b}+\frac{1}{b'} & \frac{1}{c}+\frac{1}{c'} & -2 \\ \frac{1}{a'} & \frac{1}{b} & \frac{1}{c'} & -1 \end{vmatrix}$$

$R_1 \to R_1 + R_2$
$R_3 \to R_3 + R_4$.

$= 0$ (∵ R_1 & R_3 rows are identical)

∴ The lines (1) & (2) are coplanar.

Sly the line (2) & (3) are coplanar.

∴ the lines of intersection of the given planes are coplanar.

1986 → Find the eqn of the plane passing through the line $\frac{x}{2}=\frac{y}{3}=\frac{z}{-1}$ and $\perp^r$ to the plane containing the lines $\frac{x}{3}=\frac{y}{-1}=\frac{z}{2}$ and $\frac{x}{-1}=\frac{y}{2}=\frac{z}{3}$.

→ To find the general eqns of a straightline involve four arbitrary constants.

(or)

Change the eqns of a line from symmetrical form to general form.

Soln: Let the eqns of a straight-line in a symmetrical form be

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \quad \text{--- (1)}$$

To reduce into general form:

From the first two members of (1), we have

$$\frac{x-x_1}{l}=\frac{y-y_1}{m} \Rightarrow x-x_1=\frac{l}{m}(y-y_1)$$

$$\Rightarrow x = x_1+\frac{l}{m}(y-y_1)$$

$$=\frac{l}{m}y+\left(x_1-\frac{l}{m}y_1\right)$$

$$= ay+b$$

where $a=\frac{l}{m}$; $b=x_1-\frac{l}{m}y_1$

from the last two members of (1), we have

$$\frac{y-y_1}{m}=\frac{z-z_1}{n} \Rightarrow y-y_1=\frac{m}{n}(z-z_1)$$

$$\Rightarrow y=\frac{m}{n}z+y_1-\frac{m}{n}z_1$$

$$\Rightarrow y=cz+d$$

where $c=\frac{m}{n}$; $d=y_1-\frac{m}{n}z_1$

∴ The eqns of straight-line (1) in the general form are

$$x=ay+b, \quad y=cz+d$$

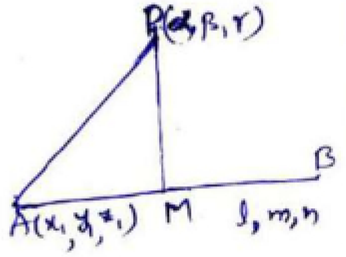
which involves only four arbitrary constants a, b, c, d.

Note: When the eqns of the line in the general form are of the type $x=ay+b$, $y=cz+d$ i.e, each eqn contains only two variables, then the eqns can be put in the symmetrical form by equating the value of the common variable in the two eqns.

Perpendicular distance

Formula:

The perpendicular distance of the point (α,β,γ) from the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$



Working rule:

First Method:

Method of Projection:

Let the point $P(\alpha,\beta,\gamma)$ and the line be $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ —— (1)

(1) Find AP by distance formula.

(2) Find AM = projection of AP on the line AB

$=l(x_1-\alpha)+m(y_1-\beta)+n(z_1-\gamma)$

[By Using $l(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1)$]

(3) Find MP from the relation $MP^2=AP^2-AM^2$

Second Method:

(1) Take any point on the line (1) is $(lr+x_1,\ mr+y_1,\ nr+z_1)$ —— (2)

(2) The d.r's of MP are $lr+x_1-\alpha,\ mr+y_1-\beta,\ nr+z_1-\gamma$

(3) $MP\perp AB$

$\therefore l(lr+x_1-\alpha)+m(mr+y_1-\beta)+n(nr+z_1-\gamma)=0$

which gives r. putting this in (2) we get the co-ordinates of M.

(4) Find the distance MP by the distance formula

(5) we can also find the eqns of the perpendicular MP.

→ Find the eqns of the $\perp^r$ from

(i) $(2,4,-1)$ to $\frac{x+5}{1}=\frac{1}{4}(y+3)=-\frac{1}{9}(z-6)$

(ii) $(-2,2,-3)$ to $x-3=\frac{1}{2}(y+1)=-\frac{1}{4}(z-2)$

(iii) $(0,0,0)$ to $x+2y+3z+4=0=2x+3y+4z+5$.

(iv) $(-2,2,-3)$ to $2x+y+z-7=0=4x+z-14$.

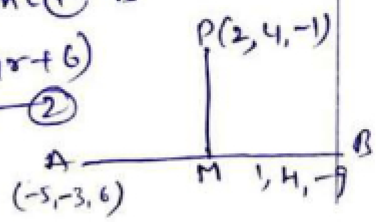
Obtain also the feet of the perpendiculars.

Sol^n:

(i) $(2,4,-1)$;

$x+5=\frac{1}{4}(y+3)=\frac{-1}{9}(z-6)$ —— (1)

Any point on line (1) is $M(r-5,\ 4r-3,\ -9r+6)$ —— (2)



Now the d.r's of MP is

$r-5-2,\ 4r-3-4,\ -9r+6+1$

by using $x_2-x_1, y_2-y_1, z_2-z_1$.

$\Rightarrow r-7,\ 4r-7,\ -9r+7$

Since $MP \perp AB$.

$a_1a_2+b_1b_2+c_1c_2=0$

$1(r-7)+4(4r-7)-9(-9r+7)=0$

$\Rightarrow r-7+16r-28+81r-63=0$

$\Rightarrow 98r=98$

$\Rightarrow \boxed{r=1}$

(2) $\equiv M(-4,1,-3)$ which is required foot of the $\perp^r$.

Now the eqns of the line through $P(2,4,-1)$ & $M(-4,1,-3)$ are $\dfrac{x-2}{6}=\dfrac{y-4}{3}=\dfrac{z+1}{2}$

(iii) The eqns of the line transform into symmetrical form.

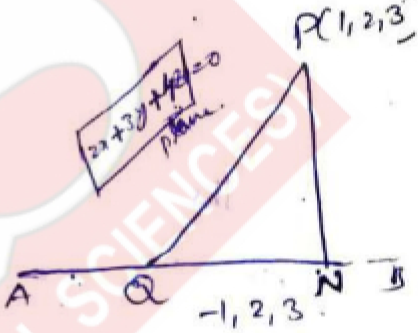
→ Find the eqns of the line which passes through the point $(3,-1,11)$ and is $\perp^r$ to the line $\frac{1}{2}x=\frac{1}{3}(y-2)=\frac{1}{4}(z-3)$.

Obtain also the foot of the $\perp^{lar}$.

$\left(\text{Ans: } (2,5,7)\quad \dfrac{x-3}{1}=\dfrac{y+1}{6}=\dfrac{z-11}{4}\right)$

→ The eqns to AB line are $\dfrac{x}{1}=\dfrac{y}{-2}=\dfrac{z}{3}$ through a point $P(1,2,3)$, PN is drawn $\perp^r$ to AB and PQ is drawn parallel to the plane $2x+3y+4z=0$ to meet AB in Q. Find the eqns of PN and PQ and the co-ordinates of N and Q.

Sol^n:



A Given line AB is

$\dfrac{x}{1}=\dfrac{y}{-2}=\dfrac{z}{3}$ —— (1)

and the point on line (1) is

$N(r,-2r,3r)$ —— (2)

Now the d.r's of the line PN

$r-1,\ -2r-2,\ 3r-3$

Since $PN \perp AB$ line

$(-1)(r-1)-2(-2r-2)+3(3r-3)=0$

$\Rightarrow \boxed{r=\frac{3}{7}}$

(2) $\equiv N\left(\frac{3}{7}, \frac{-6}{7}, \frac{9}{7}\right)$

Now the eqns of PN line through P are

$\dfrac{x-1}{1}=\dfrac{y-2}{5}=\dfrac{z-3}{3}$ —— (3)

Now let point Q given by ① be $(r, -2r, 3r)$

Since the line PQ $\parallel$ to the plane $2x+3y+4z=0$

Now the d.r.'s of PQ are $r-1,\ -2r-2,\ 3r-3$.

Since PQ $\parallel$ to the plane

$\therefore al+bm+cn=0$

$\therefore 2(r-1)+3(-2r-2)+4(3r-3)=0$

$\Rightarrow \boxed{r=5/2}$

$\therefore$ The point Q is $\left(\frac{5}{2}, -5, \frac{15}{2}\right)$

Now the eqns of PQ are

$$\frac{x-1}{3} = \frac{y-2}{-14} = \frac{z-3}{9}.$$

(Two point form)

IFoS 2008

→ Find the distance of the point $(-2,3,-4)$ from the line $\frac{x+2}{3} = \frac{2y+3}{4} = \frac{3z+4}{5}$ measured parallel to the plane $4x+12y-3z+1=0$

Sol^n: The given eqns of the line are $\frac{x+2}{3} = \frac{2y+3}{4} = \frac{3z+4}{5}$

$$\Rightarrow \frac{x+2}{3} = \frac{y+\frac{3}{2}}{2} = \frac{z+\frac{4}{3}}{5/3} \quad ①$$

Any point on the line ① is

$$Q\left(3r-2, 2r-\frac{3}{2}, \frac{5}{3}r-\frac{4}{3}\right) \quad ②$$

Let $P(-2,3,-4)$.

Now the d.r.'s of line PQ are

$3r-2+2,\ 2r-\frac{3}{2}-3,\ \frac{5}{3}r-\frac{4}{3}+4$

$\Rightarrow 3r,\ 2r-\frac{9}{2},\ \frac{5r}{3}+\frac{8}{3}$.

Since the line PQ $\parallel$ to the plane $4x+12y-3z+1=0$

$\therefore$ PQ is $\perp$ to the normal to the plane.

$\therefore al+bm+cn=0$

$4(3r)+12\left(2r-\frac{9}{2}\right)-3\left(\frac{5}{3}r+\frac{8}{3}\right)=0$

$\Rightarrow \boxed{r=2}$

$\therefore$ ② ≡ The point Q $\left(4, \frac{5}{2}, 2\right)$

$\therefore$ The required distance

$$PQ = \sqrt{(4+2)^2+\left(\frac{5}{2}-3\right)^2+(2+4)^2}$$

$$= \sqrt{36+\frac{1}{4}+36}$$

$$= \sqrt{\frac{289}{4}} = \frac{17}{2}$$

→ Find the perpendicular distance of $P(1,2,3)$ from the line $\frac{x-6}{3} = \frac{y-7}{2} = \frac{z-7}{-2}$ (Ans: 7)

2007 → Find the locus of the point which moves so that its distance from the plane $x+y-z=1$ is twice its distance from the line $x=-y=z$.

16(i)

→ Find the eqn of the two straightlines through the origin, each of which intersect the line $\frac{x-3}{2} = y-3 = z$ at angles of $60°$.

Sol^n: Any line through the origin $O(0,0,0)$ is

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \quad \text{--- (1)}$$

The given line is

$$\frac{x-3}{2} = \frac{y-3}{1} = \frac{z}{1} \quad \text{--- (2)}$$

Any point on the line (1) is $P(lr, mr, nr)$.

If the line (1) meets the line (2), let it meet at 'P', so that co-ordinates of P satisfy (2).

$$\therefore \frac{lr-3}{2} = \frac{mr-3}{1} = \frac{nr}{1} \quad \text{--- (3)}$$

From first two members of (3), we have

$$\frac{lr-3}{2} = \frac{mr-3}{1}$$

$$\Rightarrow r(2m-l) = 3 \quad \text{--- (4)}$$

From last two members of (3), we have

$$mr-3 = nr.$$

$$\Rightarrow r(m-n) = 3 \quad \text{--- (5)}$$

Dividing (4) & (5), we have

$$\frac{2m-l}{m-n} = \frac{3}{3} = 1$$

$$\Rightarrow 2m-l = m-n$$

$$\Rightarrow \boxed{l-m-n=0} \quad \text{--- (6)}$$

Given the angle between the line (1) & (2) is $60°$.

$$\therefore \cos 60° = \pm \frac{2l+m+n}{\sqrt{4+1+1}\sqrt{l^2+m^2+n^2}}$$

$$\Rightarrow \frac{1}{2} = \pm\frac{(2l+m+n)}{\sqrt{6(l^2+m^2+n^2)}}$$

$$\Rightarrow \sqrt{6(l^2+m^2+n^2)} = \pm 2(2l+m+n)$$

$$\Rightarrow 6(l^2+m^2+n^2) = 4(2l+m+n)^2$$

$$\Rightarrow 5l^2-m^2-n^2+8l(m+n) + 4mn = 0 \quad \text{--- (7)}$$

The eqns (6) & (7) determine d.c.'s of the required line (1).

(6) ≡ $l = m+n$

putting this in (7) we get

$$5(m+n)^2-m^2-n^2+8(m+n)^2 + 4mn = 0$$

$$\Rightarrow 2m^2+5mn+2n^2=0$$

$$\Rightarrow (2m+n)(m+2n)=0$$

$\Rightarrow$ $2m+n=0$ —(i) (or) $m+2n=0$ —(ii)

from (i) & (6), we have

$0l+2m+n=0$

$l-m-n=0$

$\therefore \frac{l}{1}=\frac{m}{-1}=\frac{n}{2}$ —(8)

from (ii) & (6) we have

$0l+m+2n=0$

$l-m-n=0$

$\therefore \frac{l}{1}=\frac{m}{2}=\frac{n}{-1}$ —(9)

putting these values of l,m,n in (1)

the required lines are

$\frac{x}{1}=\frac{y}{-1}=\frac{z}{2}$ & $\frac{x}{1}=\frac{y}{2}=\frac{z}{-1}$.

2003 → find the eqn of the two straight lines through the point $(1,1,1)$ that intersect the line $x-3=2(y-4)=2(z-1)$ at angle of $60°$.

→ Find the length of the $\perp^r$ from the point $(4,-5,3)$ to the line $\frac{x-5}{3}=\frac{y+2}{-4}=\frac{z-6}{5}$

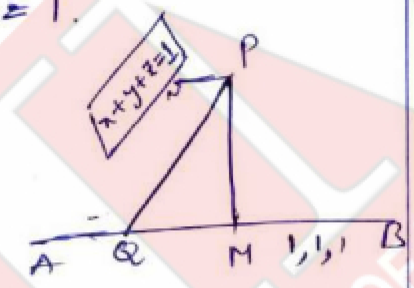
$\left(\text{Ans: } \frac{\sqrt{457}}{5}\right)$

→ Find the length of the $\perp^r$ from the point $P(5,4,-1)$ upon the line.

$\frac{1}{2}(x-1)=\frac{1}{9}y=\frac{1}{5}z$.

→ Find the locus of a point which moves so that its distance from the line $x=y=z$ is twice its distance from the plane $x+y+z=1$.

Soln:



Let $P(x_1,y_1,z_1)$ be any point in the locus.

and the given line $x=y=z$ —— ①

Let PM & PN be the $\perp^{rs}$ from P on the line AB & the given plane $x+y+z=1$ —— ②

Given that $PM=2PN$

$\Rightarrow PM^2=4PN^2$ —— ③

Now $PN=\perp^r$ distance of $P(x_1,y_1,z_1)$ from the plane ②

$=\frac{ax_1+by_1+cz_1}{\sqrt{a^2+b^2+c^2}}$

$=\frac{x_1+y_1+z_1-1}{\sqrt{1+1+1}}=\frac{x_1+y_1+z_1-1}{\sqrt{3}}$ —— ④

Now any point on the line ① is $M(r,r,r)$ —— ⑤

Now the d.r's of PM are

$r-x_1,\ r-y_1,\ r-z_1$.

Since $MP\perp AB$ line.

$\therefore 1(r-x_1)+1(r-y_1)+1(r-z_1)=0$

$\Rightarrow 3r-(x_1+y_1+z_1)=0$

$\Rightarrow r=\frac{x_1+y_1+z_1}{3}$

⑤ $\equiv$ $M\left(\frac{x_1+y_1+z_1}{3},\frac{x_1+y_1+z_1}{3},\frac{x_1+y_1+z_1}{3}\right)$

$PM=\sqrt{\left(\frac{x_1+y_1+z_1}{3}-x_1\right)^2+\left(\frac{x_1+y_1+z_1}{3}-y_1\right)^2+\left(\frac{x_1+y_1+z_1}{3}-z_1\right)^2}$

$PM=\sqrt{\frac{(-2x_1+y_1+z_1)^2}{9}+\frac{(x_1-2y_1+z_1)^2}{9}+\frac{(x_1+y_1-2z_1)^2}{9}}$

$PM^2=\frac{(-2x_1+y_1+z_1)^2}{9}+\frac{(x_1-2y_1+z_1)^2}{9}+\frac{(x_1+y_1-2z_1)^2}{9}$

③ $\equiv$ $\frac{(-2x_1+y_1+z_1)^2}{9}+\frac{(x_1-2y_1+z_1)^2}{9}+\frac{(x_1+y_1-2z_1)^2}{9}$

$=4\frac{(x_1+y_1+z_1-1)^2}{3}$

we get

$x_1^2+y_1^2+z_1^2+5x_1y_1+5y_1z_1+5z_1x_1-4x_1-4y_1-4z_1+2=0$

Now the required locus of $P(x_1,y_1,z_1)$ is

$x^2+y^2+z^2+5xy+5yz+5zx-4x-4y-4z+2=0$

Any line intersecting two given lines:

(Eqns in general form).

→ To Prove that the eqns of any line intersecting the lines $u_1=0, v_1=0$ and $u_2=0, v_2=0$ are $u_1+k_1v_1=0$, $u_2+k_2v_2=0$ where k_1, k_2 are any constants.

Sol^n: The given lines are

$u_1=0,\ v_1=0$ —— ①

and $u_2=0,\ v_2=0$ —— ②

Consider the eqns $u_1+k_1v_1=0$

$u_2+k_2v_2=0$ —— ③

These two eqns taken together represent a straight line.

The line ③ lies in the plane $u_1+k_1v_1=0$ which again contains the line $u_1=0, v_1=0$

∴ The lines ① & ③ are coplanar and hence ③ intersects the line ①.

Similarly the line ③ intersects the line ②.

Problems: Find the eqns of the line which passes through the point $(2,-1,1)$ and intersects the lines $2x+y-4=0=y+2z$

$x+3z=4,\ 2x+5z=8$

Sol^n: Given lines are

$2x+y-4=0=y+2z$ —— ①

$x+3z-4=0=2x+5z-8$ —— ②

Now any line intersecting the given lines ① & ② are

$(2x+y-4)+\lambda_1(y+2z)=0,$

$(x+3z-4)+\lambda_2(2x+5z-8)=0$ —— ③

If this line passes through the point $(2,-1,1)$.

$-1+\lambda_1=0$ and $1+\lambda_2=0$

$\Rightarrow \lambda_1=1,\ \lambda_2=-1$

∴ The required eqns of the line $x+y+z=2$ and $x+2z=4$

→ Find the line drawn from the origin to intersect the lines

$3x+2y+4z-5=0,\ 2x-3y+4z+1=0$

$2x-4y+z+6=0=3x-4y+z-3$

(Ans: $13x-13y+4z=0=8x-12y+3z$)

→ Find the eqns of the line which intersects each of the two lines

$2x+y-1=0=x-2y+3z$;

$3x-y+z+2=0=4x+5y-2z-3$

and is ‖ to the line

$\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$.

Sol: The eqns of the line intersecting the two given lines

$2x+y-1+\lambda_1(x-2y+3z)=0$

$3x-y+z+2+\lambda_2(4x+5y-2z-3)=0$ ——— (1)

This line is ‖ to the line

$\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ ——— (2)

The line (2) is $\perp^r$ to the normals to the first and second plane of line (1).

$\therefore (2+\lambda_1)+2(1-2\lambda_1)+3(3\lambda_1)=0$

$\Rightarrow \lambda_1=-2/3$

and $(3+4\lambda_2)+2(-1+5\lambda_2)+3(1-2\lambda_2)=0$

$\Rightarrow \lambda_2=-1/2$

∴ The required eqns of the line are

$4x+7y-6z-3=0=2x-7y+4z+7$.

→ Find the eqns of the line ‖ to $\frac{x}{2}=\frac{y}{3}=\frac{z}{4}$ and intersecting the lines

$9x+y+z+4=0=5x+y+3z$;

$x+2y-3z-3=0=2x-5y+3z+3$

(Ans: $\frac{x+1}{2}=\frac{y}{3}=\frac{z}{4}$)

<u>Note</u>: To obtain the eqns of a straight-line intersecting two given lines in symmetrical form:

Let the given lines be

$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}=r_1$ (say) ——— (1)

and $\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}=r_2$ (say) ——— (2)

Any point on the line (1) is

$(x_1+l_1r_1,\ y_1+m_1r_1,\ z_1+n_1r_1)$

and any point on the line (2) is $(x_2+l_2r_2,\ y_2+m_2r_2,\ z_2+n_2r_2)$.

The required line is the line which joins these two points for some values r_1 & r_2 which will be obtained from othe given condition.

→ A line with d.c.'s proportional to 2,1,2 meets each of the lines given by the eqns

$x = y + a = z$; $x + a = 2y = 2z$.

find the co-ordinates of each of the points of intersection.

Solⁿ: The given lines are

$\frac{x}{1} = \frac{y+a}{1} = \frac{z}{1} = r_1$ (say) —— (1)

and $\frac{x+a}{2} = \frac{y}{1} = \frac{z}{1} = r_2$ (say) —— (2)

Any point on (1) is $P(r_1, r_1 - a, r_1)$ —— (3)

Any point on (2) is $Q(2r_2 - a, r_2, r_2)$ —— (4)

Now the d.c.'s proportional to the line PQ are 2, 1, 2.

we have

$$\frac{r_1 - 2r_2 + a}{2} = \frac{r_1 - r_2 - a}{1} = \frac{r_1 - r_2}{2}$$

from first two members, we get

$$\frac{r_1 - 2r_2 + a}{2} = r_1 - r_2 - a.$$

$\Rightarrow \boxed{r_1 = 3a}$

from last two members.,

$$\frac{r_1 - r_2 - a}{1} = \frac{r_1 - r_2}{2}$$

$\Rightarrow 2r_1 - 2r_2 - 2a - r_1 + r_2 = 0$

$\Rightarrow r_1 - r_2 - 2a = 0$

$\Rightarrow \boxed{r_2 = a}$

∴ (3) ≡ P $(3a, 2a, 3a)$

(4) ≡ Q (a, a, a)

→ A line with direction cosines proportional (7,4,−1) is drawn to intersect the lines $\frac{x-1}{3} = \frac{y-3}{-1} = \frac{z+2}{1}$,

$\frac{x+3}{-3} = \frac{y-3}{2} = \frac{z-5}{4}$.

Find the points of intersection and the length intercepted on it.

Solⁿ: Proceed as above problem. and find length of PQ.

(2007) → A line with direction ratios 2, 7, −5 is drawn to intersect the lines

$\frac{x}{3} = \frac{y-1}{2} = \frac{z-2}{1}$ and

$\frac{x-11}{3} = \frac{y-5}{1} = \frac{z}{1}$. find the co-ordinates of the points of intersection and the length intercepted on it.

→ Find in symmetrical form the eqn to line which intersects the lines

$x + y + z = 1$, $2x - y - z = 2$;

$x - y - z = 3$, $2x + 4y - z = 4$ and passes through the point (1,1,1)

Solⁿ: Given lines are

$x + y + z - 1 = 0 = 2x - y - z - 2$ —— (1)

and $x - y - z - 3 = 0 = 2x + 4y - z - 4$ —— (2)

Any line intersecting the two lines ① & ② are

$(x+y+z-1)+\lambda_1(2x-y-z-2)=0$
$(x-y-z-3)+\lambda_2(2x+4y-z-4)=0$ ③

Since the line ③ passes through the point $(1,1,1)$.

$\therefore (1+1+1-1)+\lambda_1(2-1-1-2)=0$
& $(1-1-1-3)+\lambda_2(2+4-1-4)=0$
$\Rightarrow \lambda_1=1$; $\lambda_2=4$.

$\therefore$ ③ $\equiv$ $\left.\begin{array}{l} x-1=0 \\ 9x+15y-5z-19=0 \end{array}\right\}$ ④

Now it reduces into symmetrical form:

From the first eqn of ④, we get $x=1$.

putting this in the second eqn of ④, we get

$9+15y-5z-19=0$
$\Rightarrow 15y-5z-10=0$
$\Rightarrow 3y-z-2=0$
$\Rightarrow 3(y-1)-(z-1)=0$
$\Rightarrow 3(y-1)=z-1$
$\Rightarrow \frac{y-1}{1}=\frac{z-1}{3}$ ⑤

Also from first eqn of ④ we have $x-1=0$ ⑥

combining ⑤ & ⑥, we get

$\frac{x-1}{0}=\frac{y-1}{1}=\frac{z-1}{3}$.

which are the eqns of required line in symmetrical form.

$\longrightarrow$ Obtain the line drawn through the point $(1,0,-1)$ and intersecting the lines $x=2y=2z$; $3x+4y=1$, $4x+5z=2$

Sol: Given lines $x=2y=2z$ ①
$3x+4y-1=0=4x+5z-2$ ②

The line ① reduces to the general form:

From first two members of the line, we have
$x=2y \Rightarrow x-2y=0$ ③ and
from last two members of the line, we have
$2y=2z \Rightarrow y-z=0$ ④

$\therefore$ Combine ③ & ④, we have
$x-2y=0=y-z$ ⑤

Any line intersecting the lines ② & ⑤

$\left.\begin{array}{l} (3x+4y-1)+\lambda_1(4x+5z-2)=0 \\ (x-2y)+\lambda_2(y-z)=0 \end{array}\right\}$ ⑥

Since this line passes through the point $(1,0,-2)$.

$\therefore$ ⑥ $\equiv$ $\lambda_1=\frac{2}{3}$, $\lambda_2=-1$

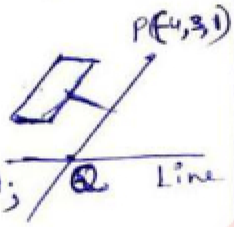
$\therefore$ ⑥ $\equiv$ $(3x+4y-1)+\frac{2}{3}(4x+5z-2)=0$,
$(x-2y)-1(y-z)=0$

$\Rightarrow \left.\begin{array}{l} 17x+12y+10z-7=0 \\ x-3y+z=0 \end{array}\right\}$ ⑦

Now it reduces into symmetrical form which is the required line

→ Find the eqns of the line which passes through the point $(-4,3,1)$ is parallel to the plane $x+2y-z=5$ and intersects the line $-\frac{(x+1)}{3}=\frac{y-3}{2}=\frac{-(z-2)}{1}$. Find also the point of intersection.

IAS 2011



Solⁿ: Let $P(-4,3,1)$ be the given point; the given plane be

$$x+2y-z-5=0 \quad \text{--- (1)}$$

and the given line be

$$-\frac{(x+1)}{3}=\frac{y-3}{2}=\frac{z-2}{-1} \quad \text{--- (2)}$$

The required line passes through the point $P(-4,3,1)$ parallel to the plane (1) and meet the given line (2).

Any point on the line (2) is $Q(-3r-1,\ 2r+3,\ -r+2)$.

Now the d.r.'s of PQ are

$-4+3r+1,\ 3-2r-3,\ 1+r-2$

$\Rightarrow 3r-3,\ -2r,\ r-1$

Since the PQ line is $\parallel$ to the plane (1)

∴ The line PQ is $\perp$ to the normal to the plane.

$\therefore 1(3r-3)+2(-2r)-1(r-1)=0$

$\Rightarrow \boxed{r=-1}$

∴ $Q(2,1,3)$ which is the reqd point of intersection.

∴ The eqns of the line PQ through P & Q are

$$\frac{x+4}{6}=\frac{y-3}{-2}=\frac{z-1}{2}$$

$$\Rightarrow \frac{x+4}{3}=\frac{y-3}{-1}=\frac{z-1}{1}$$

→ Find the eqns of the straight line through the point $(2,3,4)$ $\perp^r$ to the X-axis and intersecting the line $x=y=z$.

Solⁿ: Let the given point be $P(2,3,4)$

Let the eqns of line on x-axis is $y=0=z$ --- (1)

and given line $x=y=z$ --- (2)

(2) ≡ $x-y=0=y-z$ --- (3)

Any line intersecting the lines (1) & (3) are

$$\left.\begin{array}{l} y+\lambda_1 z=0, \\ (x-y)+\lambda_2(y-z)=0 \end{array}\right\} \quad \text{--- (4)}$$

Since this line passes through the point $P(2,3,4)$

$\therefore 3+\lambda_1 4=0 \Rightarrow \lambda_1=-\frac{3}{4}$

$-1+\lambda_2(-1)=0 \Rightarrow \lambda_2=-1$

$$(4) \equiv \left.\begin{array}{l} 4y-3z=0 \\ x-2y+z=0 \end{array}\right\}$$

which is the required line.

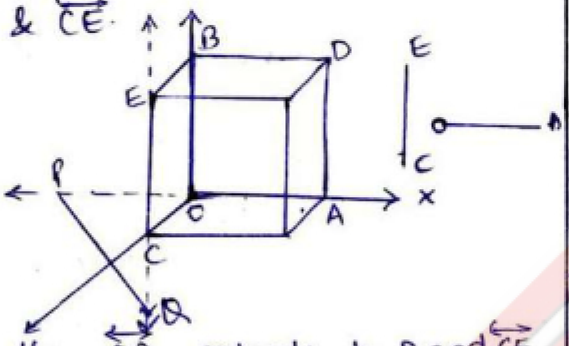
→ Find the eqns of the line through the origin which will intersect the lines

$$\frac{x-1}{2}=\frac{y+3}{4}=\frac{z-5}{3};$$

$$\frac{x-4}{2}=\frac{y+3}{3}=\frac{z-14}{4}$$

* Shortest distance between two lines :-

Let $\overleftrightarrow{OA}$ & $\overleftrightarrow{CE}$ non intersecting lines and $\overrightarrow{OC}$ is $\perp$lar to $\overleftrightarrow{OA}$ & $\overleftrightarrow{CE}$.



Now the $\overleftrightarrow{OA}$ extends to P and $\overleftrightarrow{CE}$ extends to Q. The PQ distance change to OC.

$\therefore PQ = OC$

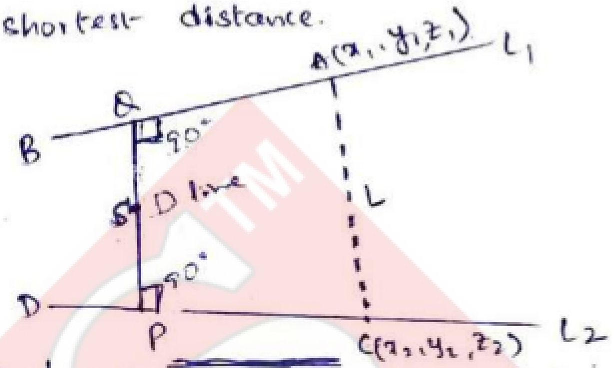
$\therefore$ which is the smallest distance between two non-intersecting lines.

* Skew lines :-

Two straight lines are said to be skew lines if they are neither $\parallel$lel nor intersecting (i.e. the lines which donot lie in the same plane.)

* Shortest distance between the skew lines :-

The straight line which is perpendicular to each of these two non-intersecting lines is called the line of shortest distance and the length of shortest distance intercepted between the skew lines is called the length of the shortest distance.



* Length and equations of the shortest distance :-

Method-1 :- General Coordinates.

Let the equations of the skew lines be $\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$ —— (1)

and $\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$ —— (2)

Any general point on line (1) is

$P(l_1r_1 + x_1, m_1r_1 + y_1, n_1r_1 + z_1)$ —— (3)

and any point on line (2) is

$Q(l_2r_2 + x_2, m_2r_2 + y_2, n_2r_2 + z_2)$ —— (4)

Let P & Q be the points, where the line of shortest distance intersects the given lines (1) & (2).

$\therefore$ The line PQ is $\perp$ to both the given lines (1) & (2).

Now the d.rs of $\overleftrightarrow{PQ}$ are

$l_2r_2 - l_1r_1 + x_2 - x_1 = l$, $m_2r_2 - m_1r_1 + y_2 - y_1 = m$,

$n_2r_2 - n_1r_1 + z_2 - z_1 = n$ (say)

Since PQ is ⊥lar to both, the given lines ①&②.

$$l_1 l + m_1 m + n_1 n = 0 \quad —⑤$$
$$l_2 l + m_2 m + n_2 n = 0 \quad —⑥$$

From these equations we get the values of r_1 & r_2

Now substituting the values of r_1 & r_2 we get the coordinates of P & Q.

∴ SD = distance of PQ and the equations of SD = Equations of line joining P & Q.

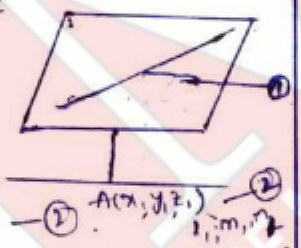
method-2: (method of parallel plane)

(one line in general form & other line in symmetrical form):

Let the given lines be

$$a_1x+b_1y+c_1z+d_1=0=a_2x+b_2y+c_2z+d_2 \quad —①$$

& $$\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1} \quad —②$$



Any plane through the line ① is

$$(a_1x+b_1y+c_1z+d_1)+\lambda(a_2x+b_2y+c_2z+d_2)=0$$

$$\Rightarrow (a_1+a_2\lambda)x+(b_1+b_2\lambda)y+(c_1+c_2\lambda)z+(d_1+d_2\lambda)=0 \quad —③$$

If this plane is ∥el to the line ② then

$$l_1(a_1+a_2\lambda)+m_1(b_1+b_2\lambda)+n_1(c_1+c_2\lambda)=0$$

$$\lambda=\frac{-(l_1a_1+m_1b_1+n_1c_1)}{l_1a_2+m_1b_2+n_1c_2}$$

This value substitute in ③ which gives a plane containing the line ① and ∥el to the line ②.

Let this plane be

$$Ax+By+Cz+D=0 \quad —④$$

SD = length of the ⊥lar from (x_1,y_1,z_1) to the plane ④

Now the equation of the line of S.D. are given by the equations of two planes. namely

(a) the plane containing given line ① and ⊥ to the plane ④.

and

(b) the plane containing the given line ② and ⊥lar to the plane ④.

method-3:- Both the lines in general form.

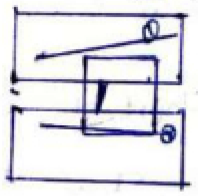
Let the equations of the two given lines are $u_1=0=v_1$ —①

& $u_2=0=v_2$ —②.

The equations of any planes through the lines ① & ② are

$$u_1+\lambda_1v_1=0 \quad —③, \qquad u_2+\lambda_2v_2=0 \quad —④$$

Now determine λ_1,λ_2 with the conditions that the planes ③&④ are parallel.



Then S.D = Distance between these parallel planes ③ & ④

Now the equations of the line of S.D are given by two planes namely

(a) the plane through the line ① and ⊥lar to the plane ③ or ④

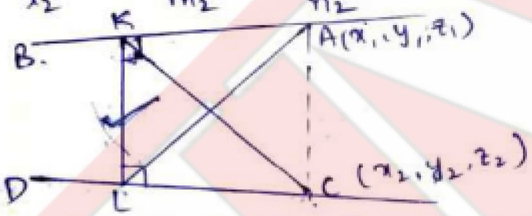
(b) the plane through the line②, and ⊥lar to the plane ③ or ④.

Note :— we can reduce the equations of one or both the straight lines to symmetrical form and then use the methods I or II (, as explained above.

method - 4 :- (Projection method) :-

Let the given lines (in symmetrical form) be $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ —— ①

$$\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2} \quad \text{——②}$$



Let AB, CD denote these lines and A,C are the points (x_1,y_1,z_1) and (x_2,y_2,z_2).

Let KL be the shortest distance between AB & CD.

Now let l,m,n be its d.c's

Since KL ⊥ to AB & CD both.

$\therefore ll_1+mm_1+nn_1=0$

and $ll_2+mm_2+nn_2=0$

solving :

$$\frac{l}{m_1n_2-m_2n_1}=\frac{m}{n_1l_2-l_1n_2}=\frac{n}{l_1m_2-m_1l_2}$$

$\therefore$ the d.r's of KL are

$m_1n_2-m_2n_1,\ n_1l_2-l_1n_2,\ l_1m_2-m_1l_2$ ③

Dividing each by $\sqrt{\Sigma(m_1n_2-n_1m_2)^2}$

Now the actual d.c's of KL are

$$\frac{m_1n_2-m_2n_1}{\sqrt{\Sigma(m_1n_2-n_1m_2)^2}},\ \frac{n_1l_2-l_1n_2}{\sqrt{\Sigma(m_1n_2-n_1m_2)^2}},\ \frac{l_1m_2-m_1l_2}{\sqrt{\Sigma(m_1n_2-n_1m_2)^2}}$$

$\therefore$ length of the shortest distance(SD) = KL

= Projection of AC on KL

$$= \frac{(x_2-x_1)(m_1n_2-m_2n_1)+(y_2-y_1)(n_1l_2-l_1n_2)+(z_2-z_1)(l_1m_2-m_1l_2)}{\sqrt{\Sigma(m_1n_2-n_1m_2)^2}}$$

| using $l(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1)$

Note :- The above expression for length of SD can also be written as

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} \div \sqrt{\Sigma(m_1n_2-m_2n_1)^2}$$

Equations of S.D :-

→ The equations of line of SD is the line of intersection of the planes AKL and CLK.

Now the equation of the plane AlK. i.e. the plane containing the lines AB & KL.

i.e. the plane through AB and ‖lel to KL is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ m_1n_2-n_1m_2 & n_1l_2-l_1n_2 & l_1m_2-m_1l_2 \end{vmatrix}=0 \quad ④$$

and equations of plane CLK (ie the plane through the lines CD and KL is

$$\begin{vmatrix} x-x_2 & y-y_2 & z-z_2 \\ l_2 & m_2 & n_2 \\ m_1n_2-n_1m_2 & n_1l_2-l_1n_2 & l_1m_2-m_1l_2 \end{vmatrix} = 0 \quad \text{——(5)}$$

Equations (4) & (5) taken together represent the equations of the S.D.

Problems :-

→ find the magnitude and the eqn's of the line of shortest distance between the lines.

$$\frac{x-8}{3} = \frac{y+9}{-16} = \frac{z-10}{7} \quad \text{——(1)}$$

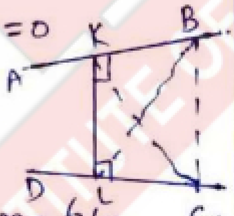
$$\frac{x-15}{3} = \frac{y-29}{8} = \frac{z-5}{-5} \quad \text{——(2)}$$

Projection method :-

Let l, m, n be d.c's of S.D and it is ⊥lar to the lines (1)&(2)

we have $3l - 16m + 7n = 0$

$3l + 8m - 5n = 0$



solving: $\frac{l}{2} = \frac{m}{3} = \frac{n}{6}$

$\therefore l = 2/7, \ m = 3/7, \ n = 6/7$

The magnitude of S.D

$= (x_2-x_1)l + (y_2-y_1)m + (z_2-z_1)n$

$= (15-8)\frac{2}{7} + (29+9)\frac{3}{7} + (5-10)\frac{6}{7}$

$= 7 \times \frac{2}{7} + 38 \times \frac{7}{7} + 5 \times \frac{1}{7} = 14$

The equation of the plane containing the first line and the line of S.D. is

$$\begin{vmatrix} x-8 & y-9 & z-10 \\ 3 & -16 & 7 \\ 2 & 3 & 6 \end{vmatrix} = 0$$

$\Rightarrow 117x + 4y - 41z - 490 = 0$

Also the equations of the plane containing the second line and the line of SD is

$$\begin{vmatrix} x-15 & y-29 & z-5 \\ 3 & 8 & -5 \\ 2 & 3 & +6 \end{vmatrix} = 0$$

$\Rightarrow 9x - 4y - z = 14.$

$\therefore$ the equations of the SD line are $117x + 4y - 41z - 490 = 0 = 9x - 4y - z - 14$

* Coordinates Method :-

Any point on the line (1) is $K(3r_1+8, -16r_1-9, 7r_1+10)$.

and any point on the line (2) is $L(3r_2+15, 8r_2+29, -5r_2+5)$

Now KL is the required line of S.D if it is ⊥lar to both the given lines (1)&(2)

Now the direction ratios of KL are

$3r_2 - 3r_1 - 7, \ 8r_2 + 16r_1 + 38, \ -5r_2 - 7r_1 - 5$

Since KL ⊥lar to the line ①

$\therefore 3(3r_2 - 3r_1 - 7) - 16(8r_2 + 16r_1 + 38) + 7(-5r_2 - 7r_1 - 5) = 0$

$\Rightarrow \boxed{157r_1 + 77r_2 + 311 = 0}$ —— ③

Since KL ⊥ to the line ②

$\therefore 3(3r_2 - 3r_1 - 7) + 8(8r_2 + 16r_1 + 38) - 5(-5r_2 - 7r_1 - 5) = 0$

$\Rightarrow 11r_1 + 7r_2 + 25 = 0$ —— ④

solving ③ & ④ we get

$r_1 = -1, \quad r_2 = -2.$

$\therefore$ The coordinates of K (5,7,3) & L (9,13,15).

$\therefore$ shortest distance of KL = 14

and its equations are

$$\frac{x-5}{2} = \frac{y-7}{3} = \frac{z-3}{6}.$$

→ find the magnitude and the equation of the line of S.D between the two lines.

(i) $\frac{x-3}{2} = \frac{y+15}{-7} = \frac{z-9}{5}$;

$\frac{x+1}{2} = \frac{y-1}{1} = \frac{z-9}{-3}$.

(ii) $\frac{x-3}{-1} = \frac{y-4}{2} = \frac{z+2}{1}$; $\frac{x-1}{1} = \frac{y+7}{3} = \frac{z+2}{2}$

Ans :- (i) $x = y = z$; $4\sqrt{3}$

(ii) $x-4 = \frac{y-2}{2} = -\frac{(z+3)}{5}$; $\sqrt{35}$

→ find the length and equations of the shortest distance line between

$5x - y - z = 0 \quad x - 2y + z + 3 = 0$

$7x - 4y - 2z = 0, \quad x - y + z + 3 = 0.$

→ find the magnitude and the position shortest distance between the lines.

(i) $2x + y - z = 0, \quad x - y + 2z = 0;$

$x + 2y - 3z = 4, \quad 2x - 3y + 4z = 5$

(ii) $\frac{x}{4} = \frac{y+1}{3} = \frac{z-2}{2}$; $5x - 2y - 3z + 6 = 0,$

$x - 3y + 2z - 3 = 0$

→ obtain the coordinates of the points where the shortest distance line between the lines.

$\frac{x-23}{-6} = \frac{y-19}{-4} = \frac{z-25}{3}$, $\frac{x-12}{-9} = \frac{y-1}{4} = \frac{z-5}{2}$

meets them.

→ Find ~~the~~ the shortest distance between the lines.

$\frac{x-1}{2} = \frac{y-2}{3} = \frac{z-3}{4}$; $\frac{x-2}{3} = \frac{y-3}{4} = \frac{z-4}{5}$

show also that the lines are coplanar

→ find the length and equations of the line of shortest distance of the lines $\frac{x+3}{-4} = \frac{y-6}{3} = \frac{z}{2}$;

$\frac{x+2}{-4} = \frac{y}{1} = \frac{z-7}{1}.$

→ show that the shortest distance between the lines

$x + a = 2y = -12z$ and $x = y + 2a = 6z - 6a$

is $2a$.

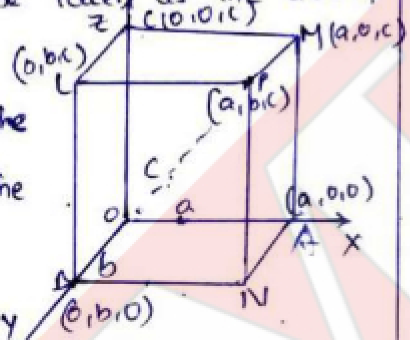
IFoS – 2008

IAS 2007

→ Prove that S.D between the diagonals of rectangular parallelopiped and the edges not meeting it are $\frac{bc}{\sqrt{b^2+c^2}}, \frac{ca}{\sqrt{c^2+a^2}}, \frac{ab}{\sqrt{a^2+b^2}}$

where a, b, c are lengths of the edges.

Sol'n :- Let the continuous edges (coterminus edges) OA, OB, OC be taken as the axes of coordinates.



Now we find the S.D between the diagonal OP & the edge BL which doesnot meet OP.

∴ The equations of OP are

$$\frac{x-0}{a-0}=\frac{y-0}{b-0}=\frac{z-0}{c-0} \Rightarrow \frac{x}{a}=\frac{y}{b}=\frac{z}{c} \quad \text{---(1)}$$

and the eqs of BL are

$$\frac{x-0}{0-0}=\frac{y-b}{b-b}=\frac{z-0}{c-0}$$

$$\Rightarrow \frac{x}{0}=\frac{y-b}{0}=\frac{z}{c} \quad \text{---(2)}$$

By coordinates method we find S.D between (1) & (2)

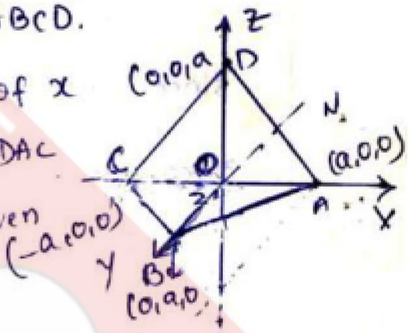
we get $\frac{bc}{\sqrt{b^2+c^2}}$

Similarly S.D between OP & AN; OP & MC are $\frac{ca}{\sqrt{c^2+a^2}}, \frac{ab}{\sqrt{a^2+b^2}}$

2005

→ A square ABCD of diagonal 2a is folded along the diagonal AC, so that the planes DAC, BAC are at right angles. Show that the shortest distance between DC and AB is then $2a/\sqrt{3}$.

Sol'n :- Let 'O' be the centre of the square ABCD.



Take OA as axis of x

Now the planes DAC and BAC are given at right angles.

So that, ⊥ lar at OB and OD as axes of Y and Z respectively.

Then the four vertices of the square are A(a,0,0), B(0,a,0) ∴ C(−a,0,0), D(0,0,a).

Now the equations of DC are

$$\frac{x-0}{-a-0}=\frac{y-0}{0-0}=\frac{z-a}{0-a}$$

$$\Rightarrow \frac{x}{-a}=\frac{y}{0}=\frac{z}{0-a}$$

$$\Rightarrow \frac{x}{1}=\frac{y}{0}=\frac{z-a}{1}$$

and the equations of AB are

$$\frac{x-a}{x-a}=\frac{y-0}{a-0}=\frac{z-0}{0-0}$$

$$\Rightarrow \frac{x}{-1}=\frac{y}{1}=\frac{z}{0} \quad \text{---(2)}$$

Now find the shortest distance between the lines DC & AB ?

2001

→ Find the shortest distance between the axis of z and the line.

$ax+by+cz+d=0,\ a'x+b'y+c'z+d'=0$

(23)

Sol

Now the equations of the line on Z-axis are

$x=0=y$ ——(1)

and $ax+by+cz+d=0=a'x+b'y+c'z+d'$ ——(2)

Now the equations of the planes through the lines (1) and (2) are

$x+\lambda_1 y=0$ ——(3)

and $(ax+by+cz+d)+\lambda_2(a'x+b'y+c'z+d')=0$

$\Rightarrow (a+a'\lambda_2)x+(b+b'\lambda_2)y+(c+c'\lambda_2)z+(d+d'\lambda_2)=0$ ——(4)

If these two planes (3) and (4) are parallel then

$$\frac{a+a'\lambda_2}{1}=\frac{b+b'\lambda_2}{\lambda_1}=\frac{c+c'\lambda_2}{0}$$

$\Rightarrow a+a'\lambda_2=1$ | $b+b'\lambda_2=\lambda_1$ | $c+c'\lambda_2=0$

$\Rightarrow \boxed{\lambda_2=\frac{1-a}{a'}}$ ——(i) | $\lambda_1=b+b'\lambda_2$, $\boxed{\lambda_1=b-\frac{c}{c'}b'}$ ——(ii) | $\boxed{\lambda_2=\frac{-c}{c'}}$ ——(iii)

$\therefore$ from (3), we have

$x+\left(b-\frac{c}{c'}b'\right)y=0$

$\Rightarrow c'x+(bc'-cb')y=0$ ——(5)

from (4), we have

$\left(a-\frac{ca'}{c'}\right)x+\left(b-\frac{cb'}{c'}\right)y+\left(c-\frac{cc'}{c'}\right)z+\left(d-\frac{cd'}{c'}\right)=0$

$\Rightarrow (ac'-a'c)x+(bc'-cb')y+(dc'-cd')=0$ ——(6)

from (i) & (iii), we have

$\frac{1-a}{a'}=\frac{-c}{c'}$

$\Rightarrow c'-ac'=-ca'$

$\Rightarrow \boxed{c'=ac'-ca'}$

$\therefore$ from (5), we have

$(ac'-ca')x+(bc'-cb')y=0$ ——(7)

$\therefore$ S.D = distance between two parallel planes is

$$\frac{|0-(dc'-cd')|}{\sqrt{(ac'-a'c)^2+(bc'-cb')^2}}$$

$$=\frac{|(dc'-cd')|}{\sqrt{(ac'-a'c)^2+(bc'-cb')^2}}$$

→ If the axes are rectangular, show that S.D between the lines

$y = az + b,\ z = \alpha x + \beta;$

$y = a'z + b',\ z = \alpha' x + \beta'$ is

$$\frac{(\alpha-\alpha')(b-b') - (\alpha'\beta - \beta'\alpha)(a-a')}{\left[\alpha^2\alpha'^2(a-a')^2 + (\alpha-\alpha')^2 + (a\alpha'-a'\alpha)^2\right]^{1/2}}$$

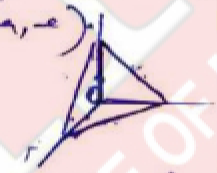
imp → Show that the S.D between any two opposite edges of the tetrahedron formed by the planes

$y+z=0,\ z+x=0,\ x+y=0,$

$x+y+z=a$ is $\frac{2a}{\sqrt{6}}$.

and that three lines of shortest distance intersect in the point $x=y=z=-a$.

i.e $(-a,-a,-a)$.



Sol] The equations to one of the pairs of opposite edges are $y+z=0,\ z+x=0$ —— (1)

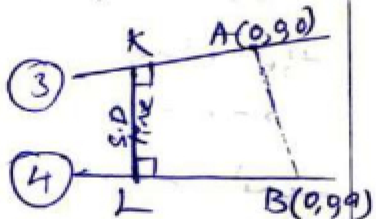
and $x+y=0,\ x+y+z=a$ —— (2)

These equations can be written in symmetrical form as

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{-1} \quad \text{—— (3)}$$

and $$\frac{x}{1} = \frac{y}{-1} = \frac{z-a}{0} \quad \text{—— (4)}$$

Let l, m, n be the d.c.'s of KL, the line of S.D.



Since the S.D is $\perp^r$ to both the lines (3) & (4).

$\therefore\ l+m-n=0$

$l-m+0.n=0$

$$\Rightarrow \frac{l}{0-1} = \frac{m}{-1-0} = \frac{n}{-1-1}$$

$$\Rightarrow \frac{l}{-1} = \frac{m}{-1} = \frac{n}{-2}$$

$$\Rightarrow \frac{l}{1} = \frac{m}{1} = \frac{n}{2}$$

i.e, the d.r's of S.D are $1, 1, 2$.

$\therefore$ The actual d.c's of S.D are $\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}$

Let two points $(0,0,0)$, $(0,0,a)$ on the two lines (3) & (4) be denoted by A, B respectively.

$\therefore$ Length of S.D = projection of AB on KL, the line of S.D.

$$= \frac{1}{\sqrt{6}}(0-0) + \frac{1}{\sqrt{6}}(0-0) + \frac{2}{\sqrt{6}}(a-0)$$

$= \frac{2a}{\sqrt{6}}$.

Similarly in case of other pairs of opposite edges.

The eqns of KL, the line of intersection of the planes AKL and BKL are

$$\begin{vmatrix} x & y & z \\ 1 & 1 & -1 \\ 1 & 1 & 2 \end{vmatrix} = 0 \quad \text{and}$$

$$\begin{vmatrix} x & y & z-a \\ 1 & -1 & 0 \\ 1 & 1 & 2 \end{vmatrix} = 0$$

$\Rightarrow x(2+1) - y(2+1) + z(1-1) = 0$

and $x(-2-0) - y(2-0) + (z-a)(1+1) = 0$

$\Rightarrow 3x - 3y = 0$ &

$-2x - 2y + 2z - 2a = 0$

$\Rightarrow x - y = 0$ and

$x + y - z + a = 0$

which clearly pass through the point $x = y = z = -a$.

i.e, the point $(-a, -a, -a)$.

Similarly it can be shown that the lines of S.D between other opposite edges pass through the same point.

$\therefore$ The three lines of S.D intersect at the point

$x = y = z = -a$

i.e, $(-a, -a, -a)$.

→ Find the length and equation of the lines of S.D. between the two lines $\frac{x-1}{2}=\frac{y-3}{4}=z+2$ & $3x-y-2z+4=0=2x+y+z+1$

Solⁿ: The given lines are

$$\frac{x-1}{2}=\frac{y-3}{4}=\frac{z+2}{1} \quad \text{--- (1)}$$

$$3x-y-2z+4=0=2x+y+z+1 \quad \text{--- (2)}$$

Any plane through line (2) is

$$3x-y-2z+4+\lambda(2x+y+z+1)=0$$

$$\Rightarrow (3+2\lambda)x+(\lambda-1)y+(\lambda-2)z+(4+\lambda)=0 \quad \text{--- (3)}$$

If (3) is || to (1), then

$$2(3+2\lambda)+4(\lambda-1)+1(\lambda-2)=0$$

$$\Rightarrow 9\lambda+6-6=0$$

$$\Rightarrow \boxed{\lambda=0}$$

$$\therefore (3) \equiv 3x-y-2z+4=0 \quad \text{--- (4)}$$

Now, S.D. = length of ⊥ʳ from $(1,3,-2)$ on (4)

$$=\frac{3(1)-3-2(-2)+4}{\sqrt{9+1+4}}$$

$$=\frac{8}{\sqrt{14}}$$

Eqⁿ of line of S.D.

Any plane through (1) is

$$A(x-1)+B(y-3)+C(z+2)=0 \quad \text{--- (5)}$$

where $A(2)+B(4)+C(1)=0$ --- (6)

If (5) is ⊥ʳ to (4), then

$$A(3)+B(-1)+C(-2)=0 \quad \text{--- (7)}$$

Eliminating A, B, C from (5), (6) & (7), by determinant we get

$$\begin{vmatrix} x-1 & y-3 & z+2 \\ 2 & 4 & 1 \\ 3 & -1 & -2 \end{vmatrix}=0$$

$$\Rightarrow x-y+2z+6=0 \quad \text{--- (8)}$$

Again, any plane through (2) is,

$$3x-y-2z+4+\lambda(2x+y+z+1)=0$$

$$\Rightarrow (3+2\lambda)x+(\lambda-1)y+(\lambda-2)z+(\lambda+4)=0 \quad \text{--- (9)}$$

If (9) is ⊥ʳ to (4), then

$$3(3+2\lambda)-1(\lambda-1)-2(\lambda-2)=0$$

$$\Rightarrow \boxed{\lambda=\frac{-14}{3}}$$

(9) ⇒ we get

$$19x+17y+20z+2=0 \quad (10)$$

Hence the eqn of the line of S.D are given by (8) & (10)

i.e.,

$$x-y+2z+6=0=19x+17y+20z+2.$$

→ Find the length of the S.D. between the lines $x=y=z$ and $x+y=2$, $x+z=2$.

(Ans: 0)

→ Find the shortest distance between the z-axis and the line $x+y+2z=3$, $2x+3y+4z+4=0$

(Ans: 2)

→ Find the shortest distance between the lines $x=0$, $\frac{y}{2}+\frac{z}{3}=1$, and $y=0$, $\frac{x}{4}-\frac{z}{3}=1$.

(Ans: $\frac{24}{\sqrt{61}}$)

→ Find the length and equations of shortest distance between the lines $x-y+z=0=2x-3y+4z$ & $x+y+2z-3=0=2x+3y+3z-4$

[Ans: $\frac{13}{\sqrt{66}}$, $3x-y-z=0=x+2y+z-1$]

→ Find the length and eqns of S.D. between $3x-9y+5z=0=x+y-z$, $6x+8y+3z-10=0=x+2y+z-3$.

[Ans: $\frac{13}{\sqrt{342}}$; $10x-29y+16z=0=13x+82y+55z-151$.]

Imp → The lengths of two opposite edges of tetrahedron are a, b; their S.D is equal to 'd'; and the angle between them is 'θ'; prove that the volume is $\frac{1}{6}abd\sin\theta$.

Sol.

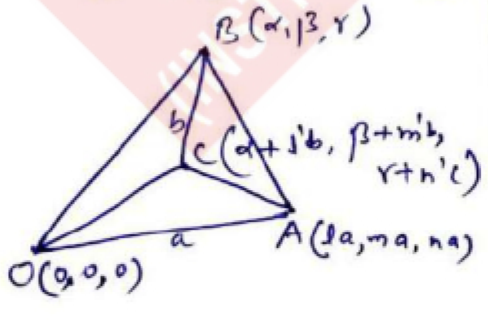
Let OA and BC be two opposite edges of the tetrahedron OABC.

Take one of the vertices to be origin, say 'O'.

Let l, m, n be the actual d.c's of line OA;

Since OA = a (given)

∴ the co-ordinates of A are (la, ma, na). [using (lr, mr, nr)]



Again let B be the point (α, β, γ) and l', m', n' be the d.c's of line BC.

Then the eqns of line BC are

$$\frac{x-\alpha}{l'} = \frac{y-\beta}{m'} = \frac{z-\gamma}{n'} \quad \text{—— ①}$$
$$= b \text{ (say)}.$$

Then the co-ordinates of the point C, distant b from (α, β, γ) are

$(\alpha+l'b, \beta+m'b, \gamma+n'b)$

Now eqns of OA are

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n}.$$

Let L, M, N be the d.c's of the S.D between OA and BC, i.e., the lines ② & ①.

then since S.D is $\perp^r$ to ② & ①

$$Ll + Mm + Nn = 0$$
$$Ll' + Mm' + Nn' = 0$$

$$\frac{L}{mn'-m'n} = \frac{M}{l'n-n'l} = \frac{N}{lm'-l'm}$$

∴ D.c's of S.D are proportional to

$mn'-m'n,\ nl'-n'l,\ lm'-l'm.$

Dividing each by $\sqrt{\Sigma (mn'-m'n)^2} = \sin\theta$,

where θ is the angle between the opposite edges OA, BC

∴ we have actual d.c.'s the line of S.D as

$$\frac{mn'-m'n}{\sin\theta} = \frac{nl'-ln'}{\sin\theta} = \frac{lm'-l'm}{\sin\theta}.$$

∴ d = S.D between OA and BC

= projection of OB on the line of S.D.

$$= \frac{(\alpha-0)(mn'-nm')+(\beta-0)(nl'-ln')+(\gamma-0)(lm'-l'm)}{\sin\theta}$$

$$= \frac{1}{\sin\theta}\begin{vmatrix} \alpha & \beta & \gamma \\ l & m & n \\ l' & m' & n' \end{vmatrix} \longrightarrow (3)$$

Now the volume of tetrahedron OABC

$$= \frac{1}{6}\begin{vmatrix} 0 & 0 & 0 & 1 \\ la & ma & na & 1 \\ \alpha & \beta & \gamma & 1 \\ \alpha+l'b & \beta+m'b & \gamma+n'b & 1 \end{vmatrix}$$

$$= -\frac{1}{6}\begin{vmatrix} la & ma & na \\ \alpha & \beta & \gamma \\ \alpha+l'b & \beta+m'b & \gamma+n'b \end{vmatrix}$$

(on expanding by R_1)

(Apply R_3-R_2)

$$= -\frac{1}{6}\begin{vmatrix} la & ma & na \\ \alpha & \beta & \gamma \\ l'b & m'b & n'b \end{vmatrix}$$

$$= -\frac{1}{6}ab\begin{vmatrix} l & m & n \\ \alpha & \beta & \gamma \\ l' & m' & n' \end{vmatrix}$$

$$= +\frac{1}{6}ab\begin{vmatrix} \alpha & \beta & \gamma \\ l & m & n \\ l' & m' & n' \end{vmatrix}$$

on interchanging R_1 and R_2

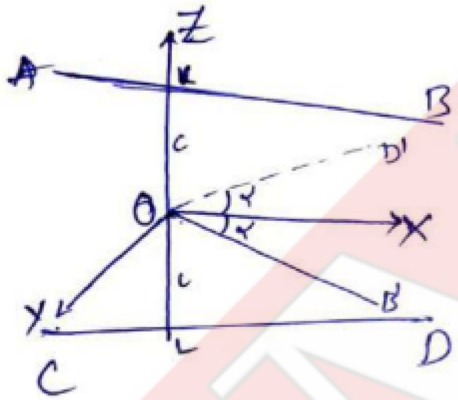
$$= \frac{1}{6}abd\sin\theta. \quad \text{(by (3))}$$

Hence the result.

* **Equations of two lines in simplest form:-**

To prove that by proper choice of axes, the equations of any skew lines can be put in the form
$y = x\tan\alpha,\ z = c$;
$y = -x\tan\alpha,\ z = -c$.

Sol



Let AB, CD be two non-intersecting i.e, skew lines and let $KL = 2c$ be the S.D. between them.

Take 'O' the mid point of KL, as the origin and OK as z-axis.

Through 'O', draw OB' and OD' respectively $\parallel$ to AB and CD.

Take OX, ~~OY~~ the internal and external bisectors of $\angle B'OD'$, as the axis of x and y.

Now, if $\angle B'OD' = 2\alpha$, then $\angle B'OX = \alpha$, $\angle XOD' = -\alpha$.

Since OB' makes angles α, $90-\alpha$, $90°$ with the co-ordinate axes, thus its d.c.'s are
$\cos\alpha, \cos(90°-\alpha), \cos 90$
i.e, $\cos\alpha, \sin\alpha, 0$.

$\therefore$ d.c.'s of AB are also
$\cos\alpha, \sin\alpha, 0$ $[\because AB \parallel OB']$

Further AB passes through the point $K(0,0,c)$.
Hence its eqns are

$$\frac{x-0}{\cos\alpha} = \frac{y-0}{\sin\alpha} = \frac{z-c}{0}. \quad (1)$$

$\Rightarrow y = x\tan\alpha,\ z = c$.

Again the line OD' makes angles $-\alpha$, $90°+\alpha$, $90°$ with the co-ordinate axes.

d.c.'s of OD' and therefore of CD are

$\cos(-\alpha)$, $\cos(90+\alpha)$, $\cos 90°$

$(\because CD \parallel OD')$

$\Rightarrow \cos\alpha, -\sin\alpha, 0.$

Also CD passes through

$L(0,0,-c)$.

$\therefore$ its eqns are

$$\frac{x-0}{\cos\alpha} = \frac{y-0}{-\sin\alpha} = \frac{z+c}{0} \quad (2)$$

$\Rightarrow y = -x\tan\alpha$, $z = -c$.

Hence the eqns of the lines AB and CD can be put in the form

$y = x\tan\alpha$, $z = c$;

$y = -x\tan\alpha$, $z = -c$. (3)

→ To prove that by proper choice of axes the equations of two skew lines can be put in the form

$y = mx$, $z = c$;

$y = -mx$; $z = -c$.

Ans: In equations, (3) if we put $\tan\alpha = m$, the result follows.

→ To prove that the equations of two skew lines can be put in the form

$$\frac{x}{\cos\alpha} = \frac{y}{\sin\alpha} = \frac{z-c}{0};$$

$$\frac{x}{\cos\alpha} = \frac{y}{-\sin\alpha} = \frac{z+c}{0}$$

Ans: The equations (1) and (2) of the above give the required result.

Locus of a line intersecting the three lines:

To find the locus of a intersecting three lines $u_1=0,\ v_1=0;\ u_2=0,\ v_2=0;\ u_3=0,\ v_3=0$.

Sol^n: The given lines are

$u_1=0,\ v_1=0$ ——— (1)

$u_2=0,\ v_2=0$ ——— (2)

and $u_3=0,\ v_3=0$ ——— (3)

Any line intersecting (1) & (2) is $u_1+k_1v_1=0,\ u_2+k_2v_2=0$ ——— (4)

If the line (4) intersects the line (3) also, then at the point of intersection, the eqns (3) and (4) have common values of x, y, z.

$\therefore$ Eliminating x, y, z from (3) and (4), we get a relation in k_1, k_2 which can be written as

$f(k_1, k_2)=0$ ——— (5)

To find the locus, eliminate k_1, k_2 from (4) & (5)

Now from (4),

$$k_1=\frac{u_1}{v_1},\quad k_2=\frac{u_2}{v_2}.$$

$$\therefore (5) \equiv f\left(\frac{u_1}{v_1}, \frac{u_2}{v_2}\right)=0$$

which is the required locus.

Note: If instead of third line, we are given some curve, the method of procedure is the same.

→ Prove that the locus of a variable line which intersects the three lines $y=mx,\ z=c;\ y=-mx,\ z=-c;\ y=z,\ mx=-c$ is the surface $y^2-m^2x^2=z^2-c^2$.

Sol^n: The given lines are

$y-mx=0,\ z-c=0$ ——— (1)

$y+mx=0,\ z+c=0$ ——— (2)

and $y-z=0,\ mx+c=0$ ——— (3)

Any line intersecting (1) & (2) is

$$\left.\begin{array}{l} y-mx+k_1(z-c)=0 \\ y+mx+k_2(z+c)=0 \end{array}\right\} \text{——— (4)}$$

If it intersects (3) also,

we have to eliminate x, y, z from (3) & (4).

Now putting $y = z$ & $mx = -c$ from (3) in (4), we get

(4) ≡
$$(z+c) + k_1(z-c) = 0$$
$$(z-c) + k_2(z+c) = 0$$

$$\Rightarrow \left(\frac{z+c}{z-c}\right) = -k_1 \quad \text{and}$$

$$\left(\frac{z+c}{z-c}\right) = -\frac{1}{k_2}.$$

$$\therefore \quad -k_1 = -\frac{1}{k_2}$$

$$\Rightarrow \boxed{k_1 k_2 = 1} \quad (5)$$

To find the locus, eliminate k_1, k_2 from (4) and (5)

(4) ≡ $k_1 = -\dfrac{(y-mx)}{z-c}$

& $k_2 = -\dfrac{(y+mx)}{z+c}$

putting these values in (5), we get

$$\left(\frac{y-mx}{z-c}\right)\left(\frac{y+mx}{z+c}\right) = 1$$

$$\Rightarrow y^2 - m^2x^2 = z^2 - c^2.$$

which is the required locus.

→ Find the locus of lines which intersect the three lines

$y = b,\ z = -c;\ z = c,\ x = -a;$
$x = a,\ y = -b.$

$\boxed{\text{Ans: } ayz + bzx + cxy + abc = 0}$

(2000) → A variable straight line always intersects the lines

$x = c,\ y = 0;\quad y = c,\ z = 0;$
$z = c,\ x = 0.$

Find the eqns to its locus.

1983 → Show that the locus of the line intersecting the three lines $y - z = 1,\ x = 0;$ $z - x = 1,\ y = 0;\ x - y = 1,\ z = 0$ is $x^2 + y^2 + z^2 - 2xy - 2yz - 2zx = 1.$

→ Show that the eqn to the surface generated by straight-lines intersecting the three lines

$x = 4a,\ y + 2z = 0;$
$x + 4a = 0,\ y = 2z;$
$y = 4a,\ x = 2z$ is

$$x^2 + y^2 - 4z^2 = 16a^2.$$

1991, 2004 → Prove that the locus of a line which meets the lines $y=\pm mx$, $z=\pm c$ and the circle $x^2+y^2=a^2$, $z=0$ is

$$c^2m^2(cy-mxz)^2+c^2(yz-cmx)^2 = a^2m^2(z^2-c^2)^2.$$

Sol: The given lines are

$$y-mx=0,\ z-c=0 \quad \text{--- (1)}$$
$$y+mx=0,\ z+c=0 \quad \text{--- (2)}$$

and the circle is

$$x^2+y^2=a^2;\ z=0 \quad \text{--- (3)}$$

Any line intersecting (1) & (2) is

$$\left.\begin{array}{l} y-mx+k_1(z-c)=0 \\ y+mx+k_2(z+c)=0 \end{array}\right\} \text{--- (4)}$$

If it meets the circle (3), we have to eliminate x,y,z from (3) & (4)

(3) ≡ putting $z=0$ in (4), we get

$$y-mx+k_1(-c)=0$$
$$y+mx+k_2(c)=0$$

Solving:

$$\frac{y}{-mk_2c+mk_1c} = \frac{x}{-ck_1-ck_2} = \frac{1}{m+m}$$

$$\Rightarrow x = \frac{-(k_1+k_2)c}{2m}$$

$$y = \frac{c(k_1-k_2)}{2}$$

putting these values of x,y in (3), we get

$$\frac{c^2(k_1+k_2)^2}{4m^2}+\frac{c^2(k_1-k_2)^2}{4}=a^2$$

$$\Rightarrow c^2(k_1+k_2)^2+c^2m^2(k_1-k_2)^2=4a^2m^2 \quad \text{--- (5)}$$

To find the locus, eliminate k_1,k_2 from (4) & (5)

$$\therefore (4) \equiv k_1 = \frac{-(y-mx)}{z-c} = \frac{mx-y}{z-c}$$

$$k_2 = \frac{-(y+mx)}{z+c}.$$

Substituting these values in (5).

$$\therefore (5) \equiv c^2\left[\left(\frac{mx-y}{z-c}\right)+\left(\frac{-mx-y}{z+c}\right)\right]^2 + c^2m^2\left[\left(\frac{mx-y}{z-c}\right)+\left(\frac{mx+y}{z+c}\right)\right]^2 = 4a^2m^2.$$

On simplification we get

$$c^2m^2(cy-mxz)^2+c^2(yz-mx)^2 = a^2m^2(z^2-c^2)^2$$

which is the required locus

2009
→ A straightline is drawn through a variable point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z=0$ to meet two fixed lines $y=mx,\ z=c$; $y=-mx,\ z=-c$. Find the eqn to the surface generated.

[Ans: $a^2c^2m^2(cy-mxz)^2+b^2c^2(mcx-yz)^2 = a^2b^2m^2(c^2-z^2)^2$]

→ Obtain the locus of the straight line which intersects the circle $x^2+y^2=r^2,\ z=0$ and the two straight lines $x=0=z+a$; $y=0=z-a$.

Ans: $a^2\{x^2(z-a)^2+y^2(z+a)^2=r^2(z-a)^2\}$

→ Prove that the line which intersects the lines $y=mx,\ z=c$; $y=-mx,\ z=-c$ and x-axis lie on the surface $mxz=cy$.

Sol^n: The given lines are

$y-mx=0,\ z-c=0$ —①

$y+mx=0,\ z+c=0$ —②

Any line intersecting ① & ② is

$$\left.\begin{aligned} y-mx+k_1(z-c)=0\\ y+mx+k_2(z+c)=0\end{aligned}\right\}$$ —③

If it intersects x-axis i.e., $y=0=z$ —④ also, we have to eliminate x,y,z from ③ & ④

Now putting $y=0$ & $z=0$ in ③, we get

$-mx+k_1(-c)=0$

and $mx+k_2(c)=0$

$\Rightarrow mx+k_1c=0$

and $mx+k_2c=0$

$\Rightarrow k_1=-\frac{mx}{c}$ & $k_2=\frac{-mx}{c}$

$\therefore k_1=k_2$ —⑤

Now eliminating k_1,k_2 between ①, ② & ⑤, we get

$$-\frac{(y-mx)}{z-c}=-\frac{(y+mx)}{z+c}$$

$\Rightarrow (y-mx)(z+c)=(y+mx)(z-c)$

$\Rightarrow \cancel{yz}+yc-mxz-\cancel{mxc}=\cancel{yz}-yc+mxz-\cancel{mxc}$

$\Rightarrow 2mxz=2yc$

$\Rightarrow \boxed{mxz=cy}$

which is the required locus.

→ P.T the lines, which meet the lines $y=mx,\ z=c$; $y=-mx,\ z=-c$ and the hyperbola $xy=c^2,\ z=0$ generate the surface given by $(cmx-yz)(mxz-cy)+m(c^2-z^2)^2=0$

Intersection of three planes:-

Let the equations of three planes be $u_1 = a_1x+b_1y+c_1z+d_1=0$ ——(1)

$u_2 = a_2x+b_2y+c_2z+d_2=0$ ——(2)

and $u_3 = a_3x+b_3y+c_3z+d_3=0$ ——(3)

No two of these three planes are parallel. we have the following three possibilities in respect of their intersection.

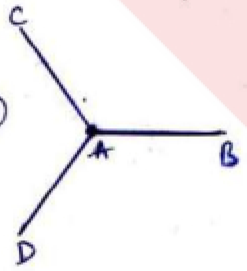
The three planes may

(i) have only one point in common

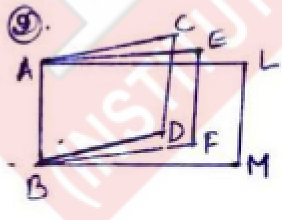
(ii) have a common line of intersection. i.e. three lines of intersection are coincident.

(iii) three lines may be ∥ and then three planes form a triangular prism. i.e. if the three planes form a Δlar prism then the line of intersection of any two planes is ∥lel to third plane.

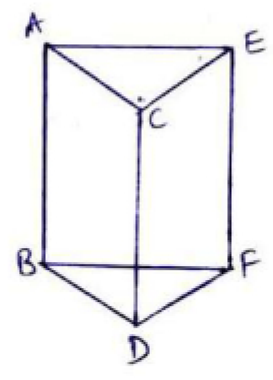
(1)



(2).



(3)



→ to find the line of intersection of (1) & (2) in the symmetrical form:

Given planes are $a_1x+b_1y+c_1z+d_1=0$ ——(1)

and $a_2x+b_2y+c_2z+d_2=0$ ——(2)

Let l, m, n be d.cs of the line of intersection. (1) & (2)

$$\therefore a_1l+b_1m+c_1n=0$$

$$a_2l+b_2m+c_2n=0$$

Solving we get

$$\frac{l}{b_1c_2-c_1b_2} = \frac{m}{c_1a_2-a_1c_2} = \frac{n}{a_1b_2-b_1a_2}$$

Now putting $z=0$ in (1) & (2) we get

$$a_1x+b_1y+d_1=0$$

$$a_2x+b_2y+d_2=0$$

$$\therefore \frac{x}{b_1d_2-d_1b_2} = \frac{y}{d_1a_2-a_1d_2} = \frac{z}{a_1b_2-b_1a_2}$$

$$\therefore x = \frac{b_1d_2-d_1b_2}{a_1b_2-b_1a_2},\ y = \frac{d_1a_2-a_1d_2}{a_1b_2-b_1a_2},\ z=0.$$

∴ one point on the line of intersection (1) & (2) is

$$\left(\frac{b_1d_2-d_1b_2}{a_1b_2-b_1a_2},\ \frac{d_1a_2-a_1d_2}{a_1b_2-b_1a_2},\ 0\right)$$

∴ equations of the line of intersection (1) & (2) in the symmetrical form are

$$\frac{x-\left(\frac{b_1d_2-d_1b_2}{a_1b_2-b_1a_2}\right)}{b_1c_2-c_1b_2} = \frac{y-\left(\frac{a_2d_1-a_1d_2}{a_1b_2-b_1a_2}\right)}{c_1a_2-a_1c_2} = \frac{z-0}{a_1b_2-b_1a_2}$$ ——(4)

* **Condition for the three planes to meet in a point :-**

The three planes ①, ② and ③ will intersect in a point of the line of intersection of ① & ②.

i.e. the line ④ cuts the plane ③ at a point. For this line ④ is not ||el to the plane ③.

i.e. $al+bm+cn \neq 0$

i.e. $a_3(b_1a_2 - c_1b_2) + b_3(c_1a_2 - a_1c_2) + c_3(a_1b_2 - a_2b_1) \neq 0$

(or) $\Delta_4 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \neq 0$ (or)

also solving the equations ①, ② & ③ by the determinants method.

$$\frac{x}{\begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}} = \frac{-y}{\begin{vmatrix} a_1 & c_1 & d_1 \\ a_2 & c_2 & d_2 \\ a_3 & c_3 & d_3 \end{vmatrix}} = \frac{z}{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}} = \frac{-1}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}$$

$$\frac{x}{\Delta_1} = \frac{-y}{\Delta_2} = \frac{z}{\Delta_3} = \frac{-1}{\Delta_4}$$

$$\therefore x = \frac{-\Delta_1}{\Delta_4}, \quad y = \frac{\Delta_2}{\Delta_4}, \quad z = \frac{-\Delta_3}{\Delta_4} \quad \text{—— (A)}$$

If the planes intersect at a point, the coordinates of their point of intersection are finite.

$\Delta_4 \neq 0$

from (A) which is the required condition.

* **Condition for the three planes to form a triangular Prism :-**

If the planes ①, ② and ③ form a triangular prism then the line of intersection of ① & ② i.e. the line ④ should be ||el to the plane ③ for which (i) $al+bm+cn=0$ and (ii) one point on ④ should not lie in plane ③

Now (i) $al+bm+cn=0$ if

$a_3(b_1c_2 - c_1b_2) + b_3(c_1a_2 - a_1c_2) + c_3(a_1b_2 - b_1a_2) = 0.$

i.e. $\Delta_4 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

and (ii) one point

$$\left(\frac{b_1d_2 - d_1b_2}{a_1b_2 - b_1a_2}, \frac{a_2d_1 - d_2a_1}{a_1b_2 - b_1a_2}, 0\right)$$

on the line ④ will not lie on the plane ③ if

$$a_3\left(\frac{b_1d_2 - d_1b_1}{a_1b_2 - b_1a_2}\right) + b_3\left(\frac{a_2d_1 - d_2a_1}{a_1b_2 - b_1a_2}\right) + c_3(0) + d_3 \neq 0$$

i.e. $a_3(b_1d_2 - d_1b_2) + b_3(a_2d_1 - d_2a_1) + d_3(a_1b_2 - b_1a_2) \neq 0.$

i.e. $\Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} \neq 0.$

$\therefore$ Hence the required conditions for the three given planes to form a triangular prism are $\Delta_4 = 0$ and any one of Δ_1, Δ_2, Δ_3 is non-zero.

* **Conditions for the three planes to intersect in a ~~plane~~ line:**–

The three planes ①, ② and ③ will meet in a line if the line of intersection of any two planes say ① and ② lies on the third i.e. ③

Now the line of intersection of ① & ②.

i.e. the line ④ will lie on the plane ③ if (i) $al+bm+cn=0$ and (ii) one point on the line ④ lies on the plane ③

Now (i) $al+bm+cn=0$ if

$$a_3(b_1c_2-c_1b_2)+b_3(c_1a_2-a_1c_2)+c_3(a_1b_2-b_1a_2)=0$$

ie. if $\Delta_4 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

(ii) one point $\left(\frac{b_1d_2-d_1b_2}{a_1b_2-b_1a_2}, \frac{a_2d_1-d_2a_1}{a_1b_2-b_1a_2}, 0\right)$ on the line ④ lies on plane ③ if

$$a_3\left(\frac{b_1d_2-d_1b_2}{a_1b_2-b_1a_2}\right)+b_3\left(\frac{d_1a_2-a_1d_2}{a_1b_2-a_2b_1}\right)+c_3(0)+d_3=0$$

(or) $a_3(b_1d_2-d_1b_2)+b_3(d_1a_2-a_1d_2)+d_3(a_1b_2-b_1a_2)=0.$

(or) $\Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix} = 0$

$\therefore$ the required conditions are $\Delta_4 = 0$ and $\Delta_1 = \Delta_2 = \Delta_3 = 0$.

* **Working rule for finding the intersection three planes:**–

Let the three planes be

$a_1x+b_1y+c_1z+d_1=0$ —— ①

$a_2x+b_2y+c_2z+d_2=0$ —— ②

$a_3x+b_3y+c_3z+d_3=0$ —— ③

Step-I :- write the coefficients in the equations to get the rectangular array.

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{vmatrix}$$

Step-II :- omit the ~~fourth~~ column to get the determinant.

$$\Delta_4 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Now if $\Delta_4 \neq 0$ then the planes are intersect in a point.

Step-III :- If $\Delta_4 = 0$ then omit the third column in Δ to get

$$\Delta_3 = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

if $\Delta_3 \neq 0$, the planes form a triangular Prism.

If $\Delta_3 = 0$, the planes intersect in a line.

Note :- If the three planes intersect in a point, the point of intersection is obtained by solving the given equation for x, y, z.

Problems :-

→ Examine the nature of intersection of the planes $2x-5y+z=3$, $x+y+4z=5$, $x+3y+6z=1$.

Sol^n :- write the coefficients in the equations, to get the rectangular array

$$\Delta = \begin{vmatrix} 2 & -5 & 1 & -3 \\ 1 & 1 & 4 & -5 \\ 1 & 3 & 6 & -1 \end{vmatrix}$$

Now omiting C_4 column, we get

$$\Delta_4 = \begin{vmatrix} 2 & -5 & 1 \\ 1 & 1 & 4 \\ 1 & 3 & 6 \end{vmatrix} = 0$$

and omiting C_3 column i.e. Δ_3

$$\Delta_3 = \begin{vmatrix} 2 & -5 & -3 \\ 1 & 1 & -5 \\ 1 & 3 & -1 \end{vmatrix} \neq 0.$$

∴ the three planes form a prism.

→ show that the following sets of planes intersect in lines.

(i) $4x+3y+2z+7=0$, $2x+y-2z+1=0$, $x-7z-7=0$ (ii) $2x+y+z+4=0$, $y-z+4=0$, $3x+2y+z+8=0$

→ show that the following sets of planes form triangular prisms.

(i) $x+y+z+3=0$, $3x+y-2z+2=0$, $2x+4y+7z-7=0$ (ii) $x-y-z-1=0$, $x+y-2z-3=0$, $x-2y+z-3=0$

→ Examine the nature of intersection of the following sets of planes.

(i) $4x-5y-2z-2=0$, $5x-4y+2z+2=0$, $2x+2y+8z-1=0$ (ii) $2x+3y-z-2=0$, $3x+3y+z-4=0$, $x-y+2z-5=0$

(iii) $5x+3y+7z-4=0$, $3x+26y+2z-9=0$, $7x+2y+10z-5=0$ (iv) $2x+6y+11=0$, $6x+20y-6z+3=0$, $6y-18z+1=0$

→ show that the planes $bx-ay=n$, $cy-bz=l$, $az-cx=m$ will intersect in a line if $al+bm+cn=0$ and the d.r's of the line are a, b, c.

Sol^n :- The equations of the given plane can be written as –

$bx - ay + 0z = n$

$0x - cy - bz = l$

$-cx + 0y + az = m$

Now $\Delta_{12} = \begin{vmatrix} b & -a & 0 \\ 0 & -c & -b \\ -c & 0 & a \end{vmatrix}$

$= -abc + abc = 0.$

Now $\Delta_3 = \begin{vmatrix} b & -a & n \\ 0 & -c & l \\ -c & 0 & m \end{vmatrix}$

$= bcm + c^2n + lca$

Now let L, M, N be the d.c's of the line. Then

$bL - aM + 0.N = 0$

$0L - cM - bN = 0$

$-cL + 0.M + aN = 0$

solving first two equations for L, M, N we get.

$\frac{L}{ab} = \frac{M}{b^2} = \frac{N}{bc}$

These numbers are satisfy the third equation.

$\therefore$ the d.r's of the line are a, b, c.

→ If the planes $x = y + z$, $y = az + x$, $z = x + ay$ pass through one line, find the value of a.

→ Prove that the planes $x = cy + bz$, $y = az + cx$, $z = bx + ay$ pass through one line if $a^2 + b^2 + c^2 + 2abc = 1$. show that the equations of this line are.

$$\frac{x}{\sqrt{1-a^2}} = \frac{y}{\sqrt{1-b^2}} = \frac{z}{\sqrt{1-c^2}}$$

Sol'n:- The three given planes are

$x - cy - bz = 0$ —— ①

$cx - y + az = 0$ —— ②

$bx + ay - z = 0$ —— ③

Let l, m, n d.c's of the line of intersection ① & ②.

$\therefore l - cm - bn = 0$

$cl - m + an = 0$

solving: $\frac{l}{-ac-b} = \frac{m}{-bc-a} = \frac{n}{-1+c^2}$

$\frac{l}{ac+b} = \frac{m}{bc+a} = \frac{n}{1-c^2}$

both the planes ① & ② pass through the origin.

$\therefore$ The line of intersection pass through the origin.

Now the eqn of line of intersection of ① & ② is

$\frac{x}{ac+b} = \frac{y}{bc+a} = \frac{z}{1-c^2}$ —— ④

Now the three planes will intersect in a line if ④ lies in ③ as' on line (4) lies on the plane (3).

Now the point $(0,0,0)$ on the line (4) satisfies plane (3).

and the line (4) lies in (3).

if $al + bm + cn = 0$

i.e, $b(ac+b) + a(bc+a) - (1-c^2) = 0$

$\Rightarrow a^2+b^2+c^2+2abc = 1$ —— (5)

which is the required condition.

Now to find line of intersection (4) in the form

$ac+b = \sqrt{(ac+b)^2}$

$= \sqrt{a^2c^2+b^2+2abc}$

$= \sqrt{a^2c^2+1-a^2-c^2}$ (from (5))

$= \sqrt{(1-a^2)(1-c^2)}$

Similarly $bc+a = \sqrt{(1-b^2)(1-c^2)}$

putting these values in (4) we get

$$\frac{x}{\sqrt{(1-a^2)(1-c^2)}} = \frac{y}{\sqrt{(1-b^2)(1-c^2)}} = \frac{z}{1-c^2}.$$

$$\Rightarrow \frac{x}{\sqrt{1-a^2}} = \frac{y}{\sqrt{1-b^2}} = \frac{z}{\sqrt{1-c^2}}$$

1989 → prove that the planes $ny - mz = \lambda$, $lz - nx = \mu$ and $mx - ly = \nu$ have a common line if $l\lambda + m\mu + n\nu = 0$. Show also that the distance of the line from the origin is $\left(\dfrac{\lambda^2+\mu^2+\nu^2}{l^2+m^2+n^2}\right)^{1/2}$