3.2.3. Let T be the linear operator on $\mathbb{R}^3$ defined by

$$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ x_1 - x_2 \\ 2x_1 + x_2 + x_3 \end{bmatrix}$$

Is T invertible? If so, find a rule which defines T^{-1} like the one which defines T.

We do this with the aid of matrices. The linear operator T can be rephrased using matrices as follows: $T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

So, letting $A = \begin{bmatrix} 3 & 0 & 0 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \end{bmatrix}$, we see that $T(v) = Av$ for all vectors $v \in \mathbb{R}^3$. To show that T is invertible, it suffices to show that the matrix A is invertible. Indeed, in that case the linear transformation given by $S(v) = A^{-1}v$ will be inverse to T, since $T(S(v)) = AA^{-1}v = Iv = v$, and $S(T(v)) = A^{-1}Av = Iv = v$.

So we just need to find A^{-1}, which we do using row-reduction. I will skip the steps

3.1.13. Let V be a vector space and T a linear transformation from V to V. Prove that the following two statements about T are equivalent:

a) The intersection of the null space of T (a.k.a. $\ker(T)$) and the range of T (a.k.a. $\operatorname{im}(T)$) is the 0 subspace.

b) If $T(T(\alpha)) = 0$, then $T(\alpha) = 0$.

Proof: We must show that a) implies b) and b) implies a).

a) $\implies$ b): Suppose $\ker(T) \cap \operatorname{im}(T) = \{0\}$. We want to show that if $T(T(\alpha)) = 0$, then $T(\alpha) = 0$. So let $\alpha \in V$ and suppose $T(T(\alpha)) = 0$. We want to show that $T(\alpha) = 0$.

Indeed, since $T(T(\alpha)) = 0$, it follows that $T(\alpha) \in \ker(T)$. But also, by definition of image, $T(\alpha) \in \operatorname{im}(T)$. Thus $T(\alpha) \in \ker(T) \cap \operatorname{im}(T)$, which by our assumption is $\{0\}$. Thus $T(\alpha) = 0$, as desired.

b) $\implies$ a): Suppose $T(T(\alpha)) = 0$ implies $T(\alpha) = 0$. We want to show $\ker(T) \cap \operatorname{im}(T) = \{0\}$. Of course, $\{0\} \subset \ker(T) \cap \operatorname{im}(T)$, since kernel and image contain 0. So it suffices to show $\ker(T) \cap \operatorname{im}(T) \subset \{0\}$. So let $v \in \ker(T) \cap \operatorname{im}(T)$. We want to show $v = 0$.

Indeed, since $v \in \operatorname{im}(T)$, we can write $v = T(w)$ for some $w \in V$. On the other hand, since $v \in \ker(T)$, we see that $T(T(w)) = T(v) = 0$. But by our assumption, this implies $T(w) = 0$. So $v = T(w) = 0$, as desired.

EXAMPLE 3 Consider the linear transformation $T: \mathbf{R}^3 \to \mathbf{R}^2$, defined by $T(x, y, z) = (x + y, 2z)$. Find the matrix of T with respect to the bases $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ and $\{\mathbf{u}'_1, \mathbf{u}'_2\}$ of $\mathbf{R}^3$ and $\mathbf{R}^2$, where

$$\mathbf{u}_1 = (1, 1, 0), \quad \mathbf{u}_2 = (0, 1, 4), \quad \mathbf{u}_3 = (1, 2, 3) \quad \text{and} \quad \mathbf{u}'_1 = (1, 0), \quad \mathbf{u}'_2 = (0, 2)$$

Use this matrix to find the image of the vector $\mathbf{u} = (2, 3, 5)$.

SOLUTION

We find the effect of T on the basis vectors of $\mathbf{R}^3$.

$$\begin{aligned}
T(\mathbf{u}_1) &= T(1, 1, 0) = (2, 0) = 2(1, 0) + 0(0, 2) = 2\mathbf{u}'_1 + 0\mathbf{u}'_2 \\
T(\mathbf{u}_2) &= T(0, 1, 4) = (1, 8) = 1(1, 0) + 4(0, 2) = 1\mathbf{u}'_1 + 4\mathbf{u}'_2 \\
T(\mathbf{u}_3) &= T(1, 2, 3) = (3, 6) = 3(1, 0) + 3(0, 2) = 3\mathbf{u}'_1 + 3\mathbf{u}'_2
\end{aligned}$$

The coordinate vectors of $T(\mathbf{u}_1)$, $T(\mathbf{u}_2)$, and $T(\mathbf{u}_3)$ are thus $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $\begin{bmatrix} 1 \\ 4 \end{bmatrix}$, and $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$. These vectors form the columns of the matrix of T.

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 4 & 3 \end{bmatrix}$$

Let us now use A to find the image of the vector $\mathbf{u} = (2, 3, 5)$. We determine the coordinate vector of $\mathbf{u}$. It can be shown that

$$\mathbf{u} = (2, 3, 5) = 3(1, 1, 0) + 2(0, 1, 4) - (1, 2, 3) = 3\mathbf{u}_1 + 2\mathbf{u}_2 + (-1)\mathbf{u}_3$$

The coordinate vector of $\mathbf{u}$ is thus $\mathbf{a} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$. The coordinate vector of $T(\mathbf{u})$ is

$$\mathbf{b} = A\mathbf{a} = \begin{bmatrix} 2 & 1 & 3 \\ 0 & 4 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

Therefore, $T(\mathbf{u}) = 5\mathbf{u}'_1 + 5\mathbf{u}'_2 = 5(1, 0) + 5(0, 2) = (5, 10)$.

We can check this result directly using the definition $T(x, y, z) = (x + y, 2z)$. For $\mathbf{u} = (2, 3, 5)$ this gives

$$T(\mathbf{u}) = T(2, 3, 5) = (5, 10)$$

1. T: $P_3 \to R$ where $T(a_3x^3 + a_2x^2 + a_1x + a_0) = a_0$.

Ker(T): To find the kernel, we want to find all the polynomials that get mapped to the zero polynomial. So we set $T(a_3x^3 + a_2x^2 + a_1x + a_0) = 0$. But this means that $a_0 = 0$, and a_3, a_2, and a_1 can be anything we want. So $\ker(T) = \{a_3x^3 + a_2x^2 + a_1x \mid a_3, a_2, a_1 \in \boldsymbol{R}\}$. A basis for ker(T) would be $\{x, x^2, x^3\}$. So the nullity (i.e., the dimension of the kernel) is 3, because there are 3 basis vectors in a basis for ker(T).

R(T): To find the range, we want to find what values of $\boldsymbol{R}$ get hit by all the polynomials in P_3. Since we can choose a_0 to be anything we want, we can hit all of $\boldsymbol{R}$, which means $R(T) = \boldsymbol{R}$. A basis for R(T) is {1}, so the rank (the dimension of the range) is 1.

2. T: $P_3 \to P_2$ where $T(a_3x^3 + a_2x^2 + a_1x + a_0) = 4a_3x^2 - 2a_2x + 2a_1 - a_0$.

Ker(T): To find the kernel, we once again set $T(a_3x^3 + a_2x^2 + a_1x + a_0) = 0$ and see what restrictions are placed on a_3, a_2, a_1, and a_0 so that the polynomial gets mapped to 0. In this case, we get $4a_3x^2 - 2a_2x + 2a_1 - a_0 = 0$. Then $a_3 = 0$, $a_2 = 0$, and $2a_1 - a_0 = 0$. From the last statement, we get $a_0 = 2a_1$. Thus, all the polynomials that get mapped to 0 are of the form $0x^3 + 0x^2 + a_1x + 2a_1 = a_1(x + 2)$. So all of the polynomials that get mapped to 0 are multiples of x + 2, so {x + 2} is a basis for ker(T). The nullity is 1, because that's how many vectors are in a basis for ker(T).

R(T): To find the rank, we set $T(a_3x^3 + a_2x^2 + a_1x + a_0) = b_2x^2 + b_1x + b_0$, a generic vector in P_2. This means $4a_3x^2 - 2a_2x + 2a_1 - a_0 = b_2x^2 + b_1x + b_0$, which implies $4a_3 = b_2$, $-2a_2 = b_1$, and $2a_1 - a_0 = b_0$. Note that we can choose a_3, a_2, a_1, and a_0 in such as a way so as to produce any b_2, b_1, and b_0 we desire. Thus, there are no restrictions on b_2, b_1, and b_0, so $R(T) = P_2$. A basis for R(T) is $\{1, x, x^2\}$. So the rank is 3.

3. T: $P_2 \to M_{2,2}$ where $T(a_2x^2 + a_1x + a_0) = \begin{bmatrix} a_2 & 0 \\ a_1 & a_0 \end{bmatrix}$.

Ker(T): Setting $T(a_2x^2 + a_1x + a_0) = 0$, we get $\begin{bmatrix} a_2 & 0 \\ a_1 & a_0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. So a_2, a_1 and a_0 must all be 0. Thus, ker(T) = {0}, and the nullity is 0.

R(T): Setting $T(a_2x^2 + a_1x + a_0) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, a generic vector in $M_{2,2}$, we get $\begin{bmatrix} a_2 & 0 \\ a_1 & a_0 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, which implies $a_2 = a$, $0 = b$, $a_1 = c$, $a_0 = d$. So we can make a, c, and d anything we want by carefully selecting a_2, a_1 and a_0, but b must be 0. So our range is the set $\left\{ \begin{bmatrix} a & 0 \\ c & d \end{bmatrix} \middle| a, c, d \in R \right\}$. A basis for this subspace would be $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

4. T: $M_{2,2} \to M_{2,2}$ where $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} a & b-c \\ 0 & d \end{bmatrix}$.

Ker(T): Setting $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, we get a = 0, b - c = 0, d = 0. The second equation gives us b = c, so the matrices that get mapped to zero are of the form $\begin{bmatrix} 0 & b \\ b & 0 \end{bmatrix} = b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Since every matrix that gets mapped to zero is a multiple of $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, so this single matrix forms a basis for ker(T). The nullity is 1.

R(T): Setting $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} e & f \\ g & h \end{bmatrix}$, we get a = e, b - c = f, 0 = g, d = h. We can produce any e, f, and h we want by carefully choosing a, b, c and d. But no matter how we choose a, b, c and d, g must always be 0. So R(T) = $\left\{\begin{bmatrix} e & f \\ 0 & h \end{bmatrix} \middle| \, e,f,g \in R\right\}$, which has dimension 3. So the rank is 3.

5. $T: R^4 \to R^3$ where T is multiplication by $A = \begin{bmatrix} 1 & -1 & 1 & 4 \\ 5 & 2 & 12 & -1 \\ -3 & -2 & -8 & 0 \end{bmatrix}$.

Ker(T): The null space of A.

R(T): the column space of A.

Ex. 5. *Describe explicitly a linear transformation from V_3 (**R**) into V_3 (**R**) which has its range the subspace spanned by* $(1, 0, -1)$ *and* $(1, 2, 2)$. **(Kanpur 1996)**

Sol. The set $B = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is a basis for V_3 (**R**).

Also $\{(1, 0, -1), (1, 2, 2), (0, 0, 0)\}$ is a subset of V_3 (**R**). It should be noted that in this subset the number of vectors has been taken the same as is the number of vectors in the set B.

There exists a unique linear transformation T from V_3 (**R**) into V_3 (**R**) such that

$$\left.\begin{array}{ll} & T(1, 0, 0) = (1, 0, -1), \\ & T(0, 1, 0) = (1, 2, 2), \\ \text{and} & T(0, 0, 1) = (0, 0, 0). \end{array}\right\} \qquad \ldots(1)$$

Now the vectors $T(1, 0, 0), T(0, 1, 0), T(0, 0, 1)$ span the range of T. In other words the vectors

$$(1, 0, -1), (1, 2, 2), (0, 0, 0)$$

span the range of T. Thus the range of T is the subspace of V_3 (**R**) spanned by the set $\{(1, 0, -1), (1, 2, 2)\}$ because the zero vector $(0, 0, 0)$ can be omitted from the spanning set. Therefore T defined in (1) is the required transformation.

Now let us find an explicit expression for T. Let (a, b, c) be any element of V_3 (**R**). Then we can write

$$(a, b, c) = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1).$$

$$\begin{aligned} \therefore \quad T(a, b, c) &= aT(1, 0, 0) + bT(0, 1, 0) + cT(0, 0, 1) \\ &= a(1, 0, -1) + b(1, 2, 2) + c(0, 0, 0) \qquad \text{[from (1)]} \\ &= (a + b, 2b, 2b - a). \end{aligned}$$

Ex. 6. *Describe explicitly a linear transformation from V_3 (**R**) into V_4 (**R**) which has its range the subspace spanned by the vectors* $(1, 2, 0, -4), (2, 0, -1, -3)$.

Sol. The set $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ is a basis for V_3 (**R**).

Also $\{(1, 2, 0, -4), (2, 0, -1, -3), (0, 0, 0, 0)\}$ is a subset of V_4 (**R**).

There exists a unique linear transformation T from V_3 (**R**) into V_4 (**R**) such that

$$T(1, 0, 0) = (1, 2, 0, -4),\ T(0, 1, 0) = (2, 0, -1, -3),$$
$$T(0, 0, 1) = (0, 0, 0, 0).$$

Now the vectors $T(1, 0, 0), T(0, 1, 0), T(0, 0, 1)$ span the range of T *i.e.*, $(1, 2, 0, -4), (2, 0, -1, -3), (0, 0, 0, 0)$ span the range of T. Thus the range of T is the subspace of V_4 (**R**) spanned by $(1, 2, 0, -4), (2, 0, -1, -3)$. Hence T defined above is the required transformation. Let us find an explicit expression for T. Let $(a, b, c) \in V_3$ (**R**). Then we have

$$(a, b, c) = a(1, 0, 0) + b(0, 1, 0) + c(0, 0, 1).$$

$$\begin{aligned} \therefore \quad T(a, b, c) &= aT(1, 0, 0) + bT(0, 1, 0) + cT(0, 0, 1) \\ &= a(1, 2, 0, -4) + b(2, 0, -1, -3) + c(0, 0, 0, 0) \\ &= (a + 2b, 2a, -b, -4a - 3b). \end{aligned}$$

Ex. 8. *Let F be any field and let T be a linear operator on F^2 defined by $T(a, b) = (a + b, a)$.*

Show that T is invertible and find a rule for T^{-1} like the one which defines T.

Sol. The null space of T is the set of all $(a, b) \in F^2$ such that

$$T(a, b) = (0, 0) \quad i.e., \quad (a + b, a) = (0, 0)$$

i.e., $$a + b = 0, a = 0 \quad i.e., \quad a = 0, b = 0.$$

Thus $N(T) = \{\mathbf{0}\}$. Hence T is non-singular and so it is invertible.

If $T(a, b) = (p, q)$ then $T^{-1}(p, q) = (a, b)$.

Now $T(a, b) = (p, q) \Rightarrow (a + b, a) = (p, q)$

$\Rightarrow$ $a + b = p,\ a = q \Rightarrow a = q,\ b = p - q.$

$\therefore$ $T^{-1}(p, q) = (q, p - q) \ \forall\ (p, q) \in F^2$ is the rule which defines T^{-1}.

Ex. 9. *Let T be a linear operator on $V_3(\mathbf{R})$ defined by*

$$T(a, b, c) = (3a, a - b, 2a + b + c) \ \forall\ (a, b, c) \in V_3(\mathbf{R}).$$

Is T invertible ? If so, find a rule for T^{-1} like the one which defines T.

Sol. Let us see that T is one-one or not.

Let $\alpha = (a_1, b_1, c_1), \beta = (a_2, b_2, c_2) \in V_3(\mathbf{R})$.

Then $T(\alpha) = T(\beta)$

$\Rightarrow$ $T(a_1, b_1, c_1) = T(a_2, b_2, c_2)$

$\Rightarrow$ $(3a_1, a_1 - b_1, 2a_1 + b_1 + c_1) = (3a_2, a_2 - b_2, 2a_2 + b_2 + c_2)$

$\Rightarrow$ $3a_1 = 3a_2, a_1 - b_1 = a_2 - b_2, 2a_1 + b_1 + c_1 = 2a_2 + b_2 + c_2$

$\Rightarrow$ $a_1 = a_2, b_1 = b_2, c_1 = c_2$.

$\therefore$ T is one-one.

Now T is a linear transformation on a finite dimensional vector space $V_3(\mathbf{R})$ whose dimension is 3. Since T is one-one, therefore T must be onto also and thus T is invertible.

If $T(a, b, c) = (p, q, r)$, then $T^{-1}(p, q, r) = (a, b, c)$.

Now $T(a, b, c) = (p, q, r)$

$\Rightarrow$ $(3a, a - b, 2a + b + c) = (p, q, r)$

$\Rightarrow$ $p = 3a, q = a - b, r = 2a + b + c$

$\Rightarrow$ $a = \dfrac{p}{3}, b = \dfrac{p}{3} - q, c = r - 2a - b = r - \dfrac{2p}{3} - \dfrac{p}{3} + q = r - p + q.$

$\therefore$ $$T^{-1}(p, q, r) = \left(\frac{p}{3}, \frac{p}{3} - q, r - p + q\right) \ \forall\ (p, q, r) \in V_3(\mathbf{R})$$

is the rule which defines T^{-1}.

Ex. 11. *For the linear operator T of Ex. 9, prove that $(T^2 - I)(T - 3I) = \hat{\mathbf{0}}$.*

(Kanpur 1999)

Sol. We have

$$(T - 3I)(a, b, c) = T(a, b, c) - 3I(a, b, c)$$
$$= (3a, a - b, 2a + b + c) - 3(a, b, c), \text{ by def. of } T \text{ and } I$$
$$= (0, a - 4b, 2a + b - 2c). \qquad \ldots(1)$$

$\therefore$
$$(T^2 - I)(T - 3I)(a, b, c) = (T^2 - I)[(T - 3I)(a, b, c)]$$
$$= (T^2 - I)(0, a - 4b, 2a + b - 2c), \text{ by (1)}$$
$$= T^2(A, B, C) - I(A, B, C), \qquad \ldots(2)$$

where $A = 0, B = a - 4b, C = 2a + b - 2c$.

Now $T(A, B, C) = (3A, A - B, 2A + B + C)$, by def. of T.

$\therefore$
$$T^2(A, B, C) = T(3A, A - B, 2A + B + C)$$
$$= T(l, m, n), \text{ say}$$
$$= (3l, l - m, 2l + m + n), \text{ by def. of } T$$
$$= (9A, 3A - A + B, 6A + A - B + 2A + B + C),$$

on putting the values of l, m, n

$$= (9A, 2A + B, 9A + C)$$
$$= (0, a - 4b, 2a + b - 2c).$$

Also $I(A, B, C) = (A, B, C) = (0, a - 4b, 2a + b - 2c)$.

Hence from (2), we have

$$(T^2 - I)(T - 3I)(a, b, c) = T^2(A, B, C) - I(A, B, C)$$
$$= (0, a - 4b, 2a + b - 2c) - (0, a - 4b, 2a + b - 2c)$$
$$= (0, 0, 0) = \hat{\mathbf{0}}(a, b, c) \ \forall\ (a, b, c) \in V_3(\mathbf{R}).$$

Therefore, by def. of zero transformation, we have

$$(T^2 - I)(T - 3I) = \hat{\mathbf{0}}.$$

Ex. 12. *A linear transformation T is defined on $V_2(\mathbf{C})$ by*

$$T(a, b) = (\alpha a + \beta b, \gamma a + \delta b),$$

where $\alpha, \beta, \gamma, \delta$ are fixed elements of $\mathbf{C}$. *Prove that T is invertible if and only if* $\alpha\delta - \beta\gamma \neq 0$.

Sol. The vector space $V_2(\mathbf{C})$ is of dimension 2. Therefore T is a linear transformation on a finite-dimensional vector space. T will be invertible if and only if the null space of T consists of zero vector alone. The zero vector of the space $V_2(\mathbf{C})$ is the ordered pair $(0, 0)$. Thus T is invertible

iff $T(x, y) = (0, 0) \Rightarrow x = 0, y = 0$

i.e., iff $(\alpha x + \beta y, \gamma x + \delta y) = (0, 0) \Rightarrow x = 0, y = 0$

i.e., iff $\alpha x + \beta y = 0, \gamma x + \delta y = 0 \Rightarrow x = 0, y = 0$.

Now the necessary and sufficient condition for the equations $\alpha x + \beta y = 0$, $\gamma x + \delta y = 0$ to have the only solution $x = 0, y = 0$ is that

$$\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \neq 0.$$

Hence T is invertible iff $\alpha\delta - \beta\gamma \neq 0$.

Ex. 13. *Let T and U be the linear operators on $\mathbf{R}^2$ defined by $T(a, b) = (b, a)$ and $U(a, b) = (a, 0)$. Give rules like the one defining T and U for each of the linear transformation $(U + T)$, UT, TU, T^2, U^2.*

Sol. $(U + T)(a, b) = U(a, b) + T(a, b) = (a, 0) + (b, a) = (a + b, a)$;

$(UT)(a, b) = U[T(a, b)] = U(b, a) = (b, 0)$;

$(TU)(a, b) = T[U(a, b)] = T(a, 0) = (0, a)$;

$T^2(a, b) = T[T(a, b)] = T(b, a) = (a, b)$;

$U^2(a, b) = U[U(a, b)] = U(a, 0) = (a, 0)$.

Ex. 14. *Let T be the (unique) linear operator on $\mathbf{C}^3$ for which*

$$T(1, 0, 0) = (1, 0, i),\ T(0, 1, 0) = (0, 1, 1),\ T(0, 0, 1) = (i, 1, 0).$$

Show that T is not invertible.

Sol. The vectors $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$, $e_3 = (0, 0, 1)$, form a basis for $\mathbf{C}^3$. Show that $T(e_1)$, $T(e_2)$, $T(e_3)$ are linearly dependent vectors. Consequently they do not form a basis for $\mathbf{C}^3$. Since T does not map a basis of $\mathbf{C}^3$ onto a basis of $\mathbf{C}^3$, therefore T is not invertible.

Ex. 1. *Describe explicitly the linear transformation T from F^2 to F^2 such that $T(e_1) = (a, b)$, $T(e_2) = (c, d)$ where $e_1 = (1, 0)$, $e_2 = (0, 1)$.*

Sol. Let (x_1, x_2) be any member of F^2. Then we are to find a formula for $T(x_1, x_2)$ under the given conditions that $T(1, 0) = (a, b)$, $T(0, 1) = (c, d)$.

We know that the set $\{e_1, e_2\}$ is a basis for the vector space F^2. Therefore any vector $(x_1, x_2) \in F^2$ can be expressed as a linear combination of the elements of this basis set.

Obviously $(x_1, x_2) = x_1 (1, 0) + x_2 (0, 1) = x_1 e_1 + x_2 e_2$.

$\therefore\ T(x_1, x_2) = T(x_1 e_1 + x_2 e_2) = x_1 T(e_1) + x_2 T(e_2)$, by linearity of T

$= x_1 (a, b) + x_2 (c, d) = (x_1 a, x_1 b) + (x_2 c, x_2 d)$

$= (x_1 a + x_2 c, x_1 b + x_2 d)$.

Ex. 2. *Describe explicitly the linear transformation* $T : \mathbf{R}^2 \to \mathbf{R}^2$ *such that* $T(2, 3) = (4, 5)$ *and* $T(1, 0) = (0, 0)$. **(Kanpur 1998, 99)**

Sol. First we shall show that the set $\{(2, 3), (1, 0)\}$ is a basis of $\mathbf{R}^2$. For linear independence of this set let

$$a(2, 3) + b(1, 0) = (0, 0), \text{ where } a, b \in \mathbf{R}.$$

Then $(2a + b, 3a) = (0, 0)$

$\Rightarrow$ $2a + b = 0, 3a = 0$

$\Rightarrow$ $a = 0, b = 0.$

Hence the set $\{(2, 3), (1, 0)\}$ is linearly independent.

Now we shall show that the set $\{(2, 3), (1, 0)\}$ spans $\mathbf{R}^2$. Let $(x_1, x_2) \in \mathbf{R}^2$ and let $(x_1, x_2) = a(2, 3) + b(1, 0) = (2a + b, 3a)$.

Then $2a + b = x_1, 3a = x_2$. Therefore

$$a = \frac{x_2}{3}, b = \frac{3x_1 - 2x_2}{3}.$$

$$\therefore \quad (x_1, x_2) = \frac{x_2}{3}(2, 3) + \frac{3x_1 - 2x_2}{3}(1, 0). \qquad \ldots(1)$$

From the relation (1) we see that the set $\{(2, 3), (1, 0)\}$ spans $\mathbf{R}^2$. Hence this set is a basis for $\mathbf{R}^2$.

Now let (x_1, x_2) be any member of $\mathbf{R}^2$. Then we are to find a formula for $T(x_1, x_2)$ under the conditions that $T(2, 3) = (4, 5)$, $T(1, 0) = (0, 0)$. We have

$$T(x_1, x_2) = T\left[\frac{x_2}{3}(2, 3) + \frac{3x_1 - 2x_2}{3}(1, 0)\right], \text{ by (1)}$$

$$= \frac{x_2}{3} T(2, 3) + \frac{3x_1 - 2x_2}{3} T(1, 0), \text{ by linearity of } T$$

$$= \frac{x_2}{3}(4, 5) + \frac{3x_1 - 2x_2}{3}(0, 0) = \left(\frac{4x_2}{3}, \frac{5x_2}{3}\right).$$

Theorem 10. *Let U and V be finite dimensional vector spaces over the field F such that $\dim U = \dim V$. If T is a linear transformation from U into V, the following are equivalent.*

(i) T is invertible.

(ii) T is non-singular.

(iii) The range of T is V.

(iv) If $\{\alpha_1, \dots, \alpha_n\}$ is any basis for U, then

$\{T(\alpha_1), \dots, T(\alpha_n)\}$ is a basis for V.

(v) There is some basis $\{\alpha_1, \dots, \alpha_n\}$ for U such that

$\{T(\alpha_1), \dots, T(\alpha_n)\}$ is a basis for V.

Proof. (i) $\Rightarrow$ (ii).

If T is invertible, then T is one-one. Therefore T is non-singular.

(ii) $\Rightarrow$ (iii).

Let T be non-singular. Let $\{\alpha_1, \dots, \alpha_n\}$ be a basis for U. Then $\{\alpha_1, \dots, \alpha_n\}$ is a linearly independent subset of U. Since T is non-singular therefore $\{T(\alpha_1), \dots, T(\alpha_n)\}$ is a linearly independent subset of V and it contains n vectors. Since $\dim V$ is also n, therefore this set of vectors is a basis for V. Now let β be any vector in V. Then there exist scalars $a_1, \dots, a_n \in F$ such that

$$\beta = a_1 T(\alpha_1) + \dots + a_n T(\alpha_n) = T(a_1 \alpha_1 + \dots + a_n \alpha_n)$$

which shows that β is in the range of T because

$$a_1\alpha_1 + \dots + a_n \alpha_n \in U.$$

Thus every vector in V is in the range of T. Hence range of T is V.

(iii) $\Rightarrow$ (iv).

Now suppose that range of T is V *i.e.*, T is onto. If $\{\alpha_1, \dots, \alpha_n\}$ is any basis for U, then the vectors $T(\alpha_1), \dots, T(\alpha_n)$ span the range of T which is equal to V. Thus the vectors $T(\alpha_1), \dots, T(\alpha_n)$ which are n in number span V whose dimension is also n. Therefore $\{T(\alpha_1), \dots, T(\alpha_n)\}$ must be a basis set for V.

(iv) $\Rightarrow$ (v).

Suppose there is some basis $\{\alpha_1, \dots, \alpha_n\}$ for U such that $\{T(\alpha_1), \dots, T(\alpha_n)\}$ is a basis for V. The vectors $\{T(\alpha_1), \dots, T(\alpha_n)\}$ span the range of T. Also they span V. Therefore the range of T must be all of V *i.e.*, T is onto.

If $\alpha = c_1 \alpha_1 + \dots + c_n \alpha_n$ is in the null space of T, then

$$T(c_1 \alpha_1 + \dots + c_n \alpha_n) = \mathbf{0} \quad \Rightarrow \quad c_1 T(\alpha_1) + \dots + c_n T(\alpha_n) = \mathbf{0}$$

$\Rightarrow$ $c_i = 0,\ 1 \le i \le n$ because $T(\alpha_1), \dots, T(\alpha_n)$ are linearly independent

$\Rightarrow$ $\alpha = \mathbf{0}$.

$\therefore$ T is non-singular and consequently T is one-one. Hence T is invertible.

pexercise 1. Which of the following maps T from $\mathbb{R}^2$ into $\mathbb{R}^2$ are linear transformations?

(a) $T(x_1, x_2) = (1 + x_1, x_2)$; No, because $T(0,0) \neq (0,0)$.

(b) $T(x_1, x_2) = (x_2, x_1)$; Yes, because x_2 and x_1 are linear homogeneous functions of x_1, x_2.

(c) $T(x_1, x_2) = (x_1^2, x_2)$; No, because, say, $2T(1,0) \neq T(2,0)$.

(d) $T(x_1, x_2) = (\sin x_1, x_2)$; No, since $2T(\frac{\pi}{2}, 0) = (2,0) \neq (0,0) = T(\pi, 0)$.

(e) $T(x_1, x_2) = (x_1 - x_2, 0)$. Yes, because $x_1 - x_2$ and 0 are linear homogeneous functions of x_1, x_2.

exercise 3. Find the range, rank, null space, and nullity for th e differentiation transformation D on the space of polynomials of degree $\leq k$:

$$D(f) = f'.$$

Do the same for the integration transformation T:

$$T(f) = \int_0^x f(t)dt.$$

solution: The range of D consists of all polynomials of degree strictly less than k, since any polynomial $p(x) = a_n x^n + \ldots + a_0$ is the derivative of the polynomial $\int p(x) = \frac{a_n}{n+1} x^{n+1} + \ldots + a_0 x$. The null space of D consists of all constants. Hence, the rank of D is k, and the nullity is 1.

The range of T consists of all continuous functions f such that f has *continuous* first derivative and $f(0) = 0$. The null space of T is trivial, because if a function is not identically zero then so is its integral. Hence, the rank of T is infinite, and the nullity is 0.

exercise 7. Let F be a subfield of the complex numbers and let T be the function from F^3 into F^3 defined by

$$T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3, 2x_1 + x_2, -x_1 - 2x_2 + 2x_3)$$

(a) Verify that T is a linear transformation.

(b) If (a, b, c) is a vector in F^3, what are the conditions on a, b and c that the vector be in the range of T? What is the rank of T?

(c) What are the conditions on a, b, and c that (a, b, c) be in the null space of T? What is the nullity of T?

solution:

(a) The coordinate functions of T are given by homogeneous polynomials of degree 1.

(b) Let

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 2 & 1 & 0 \\ -1 & -2 & 2 \end{pmatrix}$$

that is, A is the matrix which represents T with respect to the canonical basis of F^3. If we row reduce A^T we obtain

$$\tilde{A} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}$$

therefore, a basis for the image of T is $(1, 0, 1), (0, 1, -1)$, i.e. (a, b, c) is in the range of T if and only if there are scalars $s, t \in F$ such that

$$(a, b, c) = s(1, 0, 1) + t(0, 1, -1)$$

i.e. The rank of T is 2

(c) The conditions for (a, b, c) to be in the kernel are

$$a = -\frac{2}{3}c, \quad b = \frac{4}{3}c$$

The nullity of T is 1 by the dimension formula.

EXERCISE 5. Let $\mathbb{C}^{2\times 2}$ be the complex vector space of 2×2 matrices with complex entries. Let

$$B=\begin{pmatrix}1 & -1\\ -4 & 4\end{pmatrix}$$

and let T be the linear operator on $\mathbb{C}^{2\times 2}$ defined by $T(A)=BA$. What is the rank of T? can you describe T^2?

SOLUTION: Let

$$\mathcal{B}=\left\{e_1=\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}, e_2=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix}, e_3=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}, e_4=\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix}\right\}$$

$\mathcal{B}$ is a basis for $\mathbb{C}^3$. Since $T(e_1)=-T(e_3)$ and $T(e_2)=-T(e_4)$ we conclude that the rank of T is less than or equal to 2. Since $T(e_1)$ and $T(e_2)$ are linearly independent. The rank is 2. Notice that $B^2=5B$ therefore $T^2(A)=B^2A=5BA=5T(A)$

EXERCISE 7. Find two linear operators T and U on $\mathbb{R}^2$ such that $TU=0$ but $UT\neq 0$.

SOLUTION: Take $T(x_1,x_2)=(x_2,0)$ and $U(x_1,x_2)=(0,x_2)$

P.86

EXERCISE 2. Let V be a vector space over the field of complex numbers, and suppose there is an isomorphism T of V onto $\mathbb{C}^3$. Let $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ be vectors in V such that

$$T\alpha_1=(1,0,i),\quad T\alpha_2=(-2,1+i,0),$$
$$T\alpha_3=(-1,1,1),\quad T\alpha_4=(\sqrt{2},i,3).$$

(a) Is α_1 in the subspace spanned by α_2 and α_3?

(b) Let W_1 be the subspace spanned by α_1 and α_2, and let W_2 be the subspace spanned by α_3 and α_4. What is the intersection of W_1 and W_2?

(c) Find a basis for the subspace of V spanned by the four vectors α_j.

SOLUTION: (a) Note that since T is an isomorphism, it is one-to-one. So α_1 is in the subspace spanned by α_2 and α_3 if and only if $T\alpha_1$ is in the subspace spanned by $T\alpha_2$ and $T\alpha_3$. It is easy to find that $T\alpha_1=(1,0,i)=-\frac{1+i}{2}(-2,1+i,0)+i(-1,1,1)=-\frac{1+i}{2}T\alpha_2+iT\alpha_3$. Hence, $T\alpha_1$ belongs to the subspace spanned by $T\alpha_2$ and $T\alpha_3$.

(b) The intersection of W_1 and W_2 is the image of the intersection TW_1 and TW_2 under the action of T^{-1}. So first, find $TW_1\cap TW_2$. Since we already know from the part (a) that $T\alpha_1+\frac{1+i}{2}T\alpha_2=-iT\alpha_3$, we get that $T\alpha_3$ does belong to TW_1. On the other hand, it is easy to check that $T\alpha_4$ does not. Indeed, if $a(1,0,i)+b(-2,1+i,0)=(\sqrt{2},i,3)$, then $ai=3$ and $b(1+i)=i$, but then $a-2b\neq\sqrt{2}$. Hence, the intersection $TW_1\cap TW_2$ is spanned by $T\alpha_3$, and the intersection $W_1\cap W_2$ is spanned by α_3.

(c) From parts (a) and (b) we know that α_3 lies in the subspace spanned by α_1,α_2, but α_4 does not. Hence, vectors α_1,α_2,a_4 are linearly independent and span any of the four vectors α_j. So they form a basis for the subspace of V spanned by the four vectors α_j. Note that this subspace coincides with V itself, since they both have dimension 3.

Problem 1:
A matrix A is idempotent if $A^2 = A$. Show that the only possible eigenvalues of an idempotent matrix are $\lambda = 0$and $\lambda = 1$. Then give an example of a matrix that is idempotent and has both of these two values as eigenvalues.

Solution:
Suppose that λ is an eigenvalue of A. Then there is an eigenvector x, such that $Ax = \lambda x$. We have

$$\begin{aligned}
\lambda x &= A\,x \\
&= A^2\,x && \text{as A is idempotent} \\
&= A(A\,x) \\
&= A(\lambda x) && \text{as x is eigenvector of A} \\
&= \lambda(A\,x) \\
&= \lambda(\lambda\,x) && \text{as x is eigenvector of A} \\
&= \lambda^2\,x
\end{aligned}$$

From this we get:

$$\begin{aligned}
0 &= \lambda^2\,x - \lambda\,x \\
&= (\lambda^2 - \lambda)\,x
\end{aligned}$$

Since x is an eigenvector, it is nonzero, we get the conclusion that $\lambda^2 - \lambda = 0$, and the solutions to this quadratic polynomial equation for λ are $\lambda = 0$ and $\lambda = 1$. The matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

is idempotent (check this!) and since it is a diagonal matrix, its eigenvalues are the diagonal entries, $\lambda = 0$ and $\lambda = 1$, so each of these possible values for an eigenvalue of an idempotent matrix actually occurs as an eigenvalue of some idempotent matrix.

Problem 2:
The Matrix A_t is given:

$$\begin{bmatrix} -\frac{1}{3} & t \\ -2 & 1 \end{bmatrix}$$

a) For which $t \in \Re$ there do not exist any real eigenvalues for A_t?

b) For which $t \in \Re$ there does exist exactly one real eigenvalue for A_t? Give the eigenvalue!

c) For which $t \in \Re$ there do exist two different real eigenvalues for A_t? Give the eigenvalues in dependence on t!

d) Give the second real eigenvalue for c) if -1 is the first real eigenvalue!

Solution:

$$
\begin{array}{rrcl}
 & (-\frac{1}{3}-\lambda)(1-\lambda)+2\,t & = & 0 \\
\Leftrightarrow & -\frac{1}{3}-\frac{2}{3}\lambda+\lambda^2+2\,t & = & 0 \\
\Leftrightarrow & (\lambda-\frac{1}{3})^2-\frac{4}{9}+2t & = & 0
\end{array}
$$

a)no real eigenvalues:

$$-\frac{4}{9}+2\,t>0 \ \rightarrow t>\frac{2}{9}$$

The Matrix A_t has no real eigenvalues for all $t>\frac{2}{9}$

b)exactly one real eigenvalue:

$$-\frac{4}{9}+2\,t=0 \ \rightarrow t=\frac{2}{9}$$

The Matrix A_t has one real eigenvalue for $t=\frac{2}{9}$ which is $\lambda=\frac{1}{3}$

c)two real eigenvalues:

$$-\frac{4}{9}+2\,t<0 \ \rightarrow t<\frac{2}{9}$$

The Matrix A_t has two real eigenvalues for $t<\frac{2}{9}$, which are:
$\lambda=\frac{1}{3}+\sqrt{\frac{4}{9}-2\,t}$ and $\lambda=\frac{1}{3}-\sqrt{\frac{4}{9}-2\,t}$

d)First we have to find the t for which one real eigenvalue is -1:

$$
\begin{array}{rrl}
 & -1 & =\frac{1}{3}-\sqrt{\frac{4}{9}-2\,t} \\
\Leftrightarrow & \frac{4}{3} & =-\sqrt{\frac{4}{9}-2\,t} \\
\Leftrightarrow & \frac{16}{9} & =\frac{4}{9}-2\,t \\
\Leftrightarrow & \frac{12}{9} & =-2\,t \\
\Leftrightarrow & -\frac{2}{3} & =t
\end{array}
$$

Then we can calculate the second real eigenvalue with the given t:

$$
\begin{array}{rl}
 & \lambda=\frac{1}{3}+\sqrt{\frac{4}{9}-2\,(-\frac{2}{3})} \\
\Leftrightarrow & \lambda=\frac{1}{3}+\sqrt{\frac{4}{9}+\frac{4}{3}} \\
\Leftrightarrow & \lambda=\frac{1}{3}+\sqrt{\frac{16}{9}} \\
\Leftrightarrow & \lambda=\frac{5}{3}
\end{array}
$$

So finally the second real eigenvalue is $\frac{5}{3}$.

1. Let $\mathcal{P}_2$ be the space of polynomials of degree at most 2, and define the linear transformation

$$T : \mathcal{P}_2 \to \mathbb{R}^2$$
$$T(p(x)) = \begin{bmatrix} p(0) \\ p(1) \end{bmatrix}$$

For example $T(x^2+1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

(a) Using the basis $\{1, x, x^2\}$ for $\mathcal{P}_2$, and the standard basis for $\mathbb{R}^2$, find the matrix representation of T.

(b) Find a basis for the kernel of T, writing your answer as polynomials.

Solution.

(a) The matrix representation is $A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$, since $T(1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, T(x) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, T(x^2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(b) The nullspace of A is spanned by $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$, which corresponds to the polynomial $x - x^2$.

2.

(a) Find the coordinate vector of the element $1 + 3x - 6x^2$ in $\mathcal{P}_2$, relative to the basis

$$B = \{1 - x^2, x - x^2, 2 - x + x^2\}.$$

(b) In the space $\mathcal{P}_3$ of polynomials of degree at most 3, are the vectors $\{1 + 2x^3, 2 + x - 3x^2, -x + 2x^2 - x^3\}$ linearly independent?

(c) Do the vectors in part (b) form a basis for $\mathcal{P}_3$?

Solution.

(a) The coordinate vector is $\begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$. To see this, solve the system

$$x_1 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -6 \end{bmatrix}$$

using row reduction.

(b) Yes, they are. Row reduce the matrix

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & -3 & 2 \\ 2 & 0 & -1 \end{bmatrix}$$

to see this. You should get a pivot in each column.

(c) No, they do not form a basis because they do not span $\mathcal{P}_3$. (You would need 4 vectors to span $\mathcal{P}_3$.)

3. Let $M_{2\times 2}$ be the vector space of 2×2 matrices with the usual operations of addition and scalar multiplication. Define the linear transformation

$$T : M_{2\times 2} \to M_{2\times 2}$$
$$T(A) = A + A^T,$$

where A^T is the transpose of A.

(a) Find the matrix representation of T relative to the basis $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

(b) Find the dimension of the kernel of T.

Solution.

(a) The matrix representation is $\begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$.

(b) After row reducing the matrix from part (a) there is one free variable. So the dimension of the kernel is 1.

1. Find a basis for and the dimension of the subspaces defined by the following conditions:

(a) $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$ such that

$$x_1 + x_4 = 0\,,$$
$$3x_1 + x_2 + x_4 = 0\,.$$

We solve the equations. First we have $x_1 = -x_4$. That implies $2x_1 = -x_2$. So the subspace may be represented by $(x_1, -2x_1, x_3, -x_1)$ and the dimension is the number of independent variables or 2. We can write the basis as $(1, -2, 0, -1)$ and $(0, 0, 1, 0)$.

(b) $\{ f \in \text{Span}\{e^x, e^{2x}, e^{3x}\} |\, f(0) = f'(0) = 0\}$
Let $g \in f$, then $g = ae^x + be^{2x} + ce^{3x}$. If we apply the conditions, we have

$$a + b + c = 0\,,$$
$$a + 2b + 3c = 0\,.$$

Solving, we get $b = -2c$ and $a = c$. So may write the basis as $(1, -2, 1)$ and the subspace is 1-dimensional.

2. For each of the following matrices, defining a linear transformation between vector spaces of the appropriate dimensions, find bases for Ker(T) and Im(T).

(a)

$$\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$$

If we solve $Tx = 0$, we get the equations

$$x + 2y = 0\,,$$
$$2x + 2x = 0\,.$$

The only solution is $(x, y) = (0, 0)$. So Ker$(T) = \{0\}$ and a basis for the image of T is formed by the column vectors $(1, 2)$ and $(2, 2)$. Since the image is all of $\mathbb{R}^2$ any two independent vectors would work.

(b)

$$\begin{pmatrix} -1 & 2 & 2 \\ 2 & -4 & 2 \end{pmatrix}$$

If we solve $Tx = 0$, we get the equations,

$$-x + 2y + 2z = 0\,,$$
$$2x - 4y + 2z = 0\,.$$

Solving, we find $z = 0$ and $x = 2y$. A basis for the kernel is $(1, 2, 0)$. Since the kernel is a one-dimensional subspace, the dimension of the image must be 2. We just take two independent vectors from the column space, for example $(-1, 2)$ and $(2, 2)$. Again the image is all of $\mathbb{R}^2$, so any two independent vectors would do.

3. For each of the following linear transformations, determine if it is an isomorphism and if so find its inverse.

(a) $T : P_2((R)) \to \mathbb{R}^2$ given by $T(p(x)) = (p(0), p(1), p(2))$.
We pick a basis for $P_2(x)$ as $\{1, x, x^2\}$. We look at how T acts on a basis.

$$T(1) = (1, 1, 1)\,,$$
$$T(x) = (0, 1, 2)\,,$$
$$T(x^2) = (0, 1, 4)\,.$$

We can represent T as the following matrix using this basis

$$\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 2 & 4 \end{pmatrix}.$$

We can find its inverse

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right),$$
$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 2 & 4 & -1 & 0 & 1 \end{array}\right),$$
$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 2 & 1 & -2 & 1 \end{array}\right),$$
$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & \frac{1}{2} & -1 & \frac{1}{2} \end{array}\right),$$
$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -\frac{3}{2} & 2 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{2} & -1 & \frac{1}{2} \end{array}\right),$$

So the inverse of T using the basis we have chosen is represented by the matrix

$$\begin{pmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 2 & -\frac{1}{2} \\ \frac{1}{2} & -1 & \frac{1}{2} \end{pmatrix}$$

(b) $T : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ given by $T(p(x)) = x\frac{dp}{dx}$.
We note that all constant polynomials are mapped to 0. Since the kernel is nontrivial, T can't be an isomorphism.

(c) $T : V \to V$, $\dim(V) = 4$, and with respect to the basis $\{v_1, v_2, v_3, v_4\}$, T is defined by $T(v_1) = v_2$, $T(v_2) = v_1$, $T(v_3) = v_4$ and $T(v_4) = v_3$.
We can write T with respect to the given basis as the matrix,

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

If you think about it you should see why T is its own inverse, but you should check by squaring the matrix representation.

Note: $P_n(\mathbb{R})$ is the space of polynomials of degree at most n.

4. Let $V = P_3(\mathbb{R})$ and $W = P_4(\mathbb{R})$. Let $D : W \to V$ be the derivative mapping $D(p) = p'$ and $\mathrm{Int} : V \to W$ be the integration mapping $\mathrm{Int}(p) = \int_0^x p(t)\,dt$. Let $\alpha = \{1, x, x^2, x^3\}$ and $\beta = \{1, x, x^2, x^3, x^4\}$ be the bases for V and W.

(a) Compute $[D]_\beta^\alpha$ and $[\mathrm{Int}]_\alpha^\beta$ the matrix representations with respect to given bases.
First we evaluate the action of the operators on the basis.

$$\begin{aligned} D(1) &= 0\,, \\ D(x) &= 1\,, \\ D(x^2) &= 2x\,, \\ D(x^3) &= 3x^2\,, \\ D(x^4) &= 4x^3\,, \\ \mathrm{Int}(1) &= x\,, \\ \mathrm{Int}(x) &= \frac{x^2}{2}\,, \\ \mathrm{Int}(x^2) &= \frac{x^3}{3}\,, \\ \mathrm{Int}(x^3) &= \frac{x^4}{4}\,, \end{aligned}$$

So we can represent the transformations with the following matrices:

$$[D]_\beta^\alpha = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix},$$

$$[\mathrm{Int}]_\alpha^\beta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix}.$$

(b) Compute $[D\text{Int}]_\alpha^\alpha$ and $[\text{Int}D]_\beta^\beta$ using the appropriate matrix products. What theorem of calculus is reflected in these results? What are the Ker(DInt) and Ker(Int$D$)?
A simple matrix multiplication gives us,

$$[D\text{Int}]_\alpha^\alpha = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$[\text{Int}D]_\beta^\beta = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

It is clear that D and Int are almost inverses. This is a reflection of the Fundamental Theorem of Calculus. It should be clear from the matrix representation that Ker(DInt) = $\{0\}$ and Ker(Int$D$) = span$\{1\}$

EXAMPLE 2.69

Find a non-singular matrix P such that P^TAP is a diagonal matrix, where

$$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}.$$

Find the quadratic form and its rank.

Solution. Write $A = IAI$, that is,

$$\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Using congruent operations, we shall reduce A to diagonal form. Performing congruent operations $R_2 \to R_2 + \frac{1}{3}R_1, C_2 \to C_2 + \frac{1}{3}C_1$ and $R_3 \to R_3 - \frac{1}{3}R_1, C_3 \to C_3 - \frac{1}{3}C_1$, we have

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & \frac{7}{3} & -\frac{1}{3} \\ 0 & -\frac{1}{3} & \frac{7}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ -\frac{1}{3} & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & \frac{1}{3} & -\frac{1}{3} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now performing congruent operation $R_3 \to R_3 + \frac{1}{7}R_2, C_3 \to C_3 + \frac{1}{7}C_2$, we have

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & \frac{7}{3} & 0 \\ 0 & 0 & \frac{16}{7} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{3} & 1 & 0 \\ -\frac{2}{7} & \frac{1}{7} & 1 \end{bmatrix} A \begin{bmatrix} 1 & \frac{1}{3} & -\frac{2}{7} \\ 0 & 1 & \frac{1}{7} \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus,

$$diag\begin{bmatrix} 6 & \frac{7}{3} & \frac{16}{7} \end{bmatrix} = P^{-1}\,AP,$$

where

$$P = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{2}{7} \\ 0 & 1 & \frac{1}{7} \\ 0 & 0 & 1 \end{bmatrix}.$$

The quadratic form corresponding to the matrix A is

$$X^T AX = 6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + 4x_3x_1. \tag{48}$$

The non-singular transformation $X = PY$ corresponding to the matrix P is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{3} & -\frac{2}{7} \\ 0 & 1 & \frac{1}{7} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix},$$

which yields

$$x_1 = y_1 + \frac{1}{3}y_2 - \frac{2}{7}y_3$$

$$x_2 = y_2 + \frac{1}{7}y_3$$

$$x_3 = y_3.$$

Substituting these values in (48), we get

$$(PY)^T A(PY) = 6y_1^2 + \frac{7}{3}y_2^2 + \frac{16}{7}y_3^2.$$

It contains a sum of *three* squares. Thus, the rank of the quadratic form is 3.

Example 7.41 Given two real symmetric matrices A and B by

$$A = \begin{bmatrix} 6 & 4 & -2 \\ 4 & 12 & -4 \\ -2 & -4 & 13 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \tag{7.141}$$

determine a nonsingular matrix P such that $P^TAP = I$ and P^TBP is diagonal.

Solution The characteristic polynomial of A is $\phi(\lambda) = \det(\lambda I - A) = \lambda^3 - 31\lambda^2 + 270\lambda - 648 = (\lambda - 4)(\lambda - 9)(\lambda - 18)$. The eigenvalues are therefore 4, 9, 18, and the associated orthogonal eigenvectors are, respectively,

$$\mathbf{x}_1 = [-2\ 1\ 0]^T, \qquad \mathbf{x}_2 = [2\ 4\ 5]^T, \qquad \mathbf{x}_3 = [1\ 2\ -2]^T$$

We now normalize these vectors (see Definition 5.12) and construct an orthogonal matrix with these normalized vectors as columns to get

$$U = \frac{1}{3\sqrt{5}} \begin{bmatrix} -6 & 2 & \sqrt{5} \\ 3 & 4 & 2\sqrt{5} \\ 0 & 5 & -2\sqrt{5} \end{bmatrix}$$

It can be checked that $U^TAU = \text{diag}\,[4, 9, 18]$. We now define $Q^{-1} = \text{diag}\,[2, 3, 3\sqrt{2}]$ so that

$$(UQ)^TA(UQ) = I$$

We now compute the matrix C defined by $C = (UQ)^TA(UQ)$ to get

$$C = \frac{1}{180} \begin{bmatrix} 81 & -18 & 9\sqrt{10} \\ -18 & 4 & -2\sqrt{10} \\ 9\sqrt{10} & -2\sqrt{10} & 10 \end{bmatrix}$$

The characteristic equation of C is $\det(\lambda I - C) = \lambda^2(\lambda - 19/36)$. Hence its eigenvalues are $\lambda_1 = 0$, $\lambda_2 = 0$, $\lambda_3 = 19/36$, and the associated eigenvectors are, respectively,

$$[-2, -4, \sqrt{10}]^T, \qquad [1, -1/2, -\sqrt{10}]^T, \qquad [9, -2, \sqrt{10}]^T$$

By Theorem 7.30 we orthogonalize the first two eigenvectors associated with eigenvalues 0 and then normalize all the three eigenvectors to get

$$\mathbf{y}_1 = (1/\sqrt{30})[-2, -4, \sqrt{10}]^T, \qquad \mathbf{y}_2 = (1/\sqrt{285})[2, -11, -4\sqrt{10}]^T,$$

$$\mathbf{y}_3 = (1/\sqrt{95})[9, -2, \sqrt{10}]^T$$

as a set of three orthonormal eigenvectors. If we now define the orthogonal matrix R by $R := [\mathbf{y}_1\mathbf{y}_2\mathbf{y}_3]$, then it can be easily checked that $R^TCR = (UQR)^TB(UQR) = \text{diag}\,[0, 0, 19/36]$. Hence the required nonsingular transformation is

$$P = UQR = \frac{1}{90\sqrt{57}} \begin{bmatrix} 15\sqrt{38} & -120 & -160\sqrt{3} \\ -15\sqrt{38} & -150 & 85\sqrt{3} \\ -30\sqrt{38} & -30 & -40\sqrt{3} \end{bmatrix} \tag{7.142}$$

The matrix P is not unique. This is because another set of eigenvectors $\mathbf{x}_i$'s and $\mathbf{y}_i$'s would give the same result.

We have already observed that a characteristic feature of a Hermitian matrix is that it is a simple matrix. In other words, an n-square Hermitian matrix has always an orthogonal set of n eigenvectors as established in Theorem 7.31. Because of this property, a Hermitian matrix can always be reduced to a diagonal matrix by a unitary transformation. Now an important question is: What is the most general class of nth order matrices A which have an orthogonal set of n eigenvectors? The answer is that this would depend on whether or not the matrices are normal which is now given in the following definition.

Corollary 7.14 A normal matrix is unitarily similar to a diagonal matrix, i.e., for a normal matrix over $\mathbb{C}$ there is a unitary matrix U such that U^*AU is diagonal.

Example 7.42 The matrix

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$

is normal as it satisfies $A^*A = AA^*$. The characteristic polynomial of A is $\phi(\lambda) = \det(\lambda I - A) = \lambda^3 - 5\lambda^2 + 9\lambda - 9$, and hence its eigenvalues and eigenvectors are, respectively, 3, $1 + \sqrt{2}i$, $1 - \sqrt{2}i$ and $\mathbf{x}_1 = [0\ 1\ 1]^T$, $\mathbf{x}_2 = [1, -i/\sqrt{2}, i/\sqrt{2}]^T$, $\mathbf{x}_3 = [1, i/\sqrt{2}, -i/\sqrt{2}]^T$. These eigenvectors are orthogonal having norms $\|\mathbf{x}_i\| = \sqrt{2}$ $(i = 1, 2, 3)$. Let us therefore choose

$$U = (\sqrt{2})^{-1}[\mathbf{x}_1\ \mathbf{x}_2\ \mathbf{x}_3]$$

Then it is easy to check that $UU^* = U^*U = I$ and that $U^*AU = \operatorname{diag}[1, 1 + \sqrt{2}i, 1 - \sqrt{2}i]$.

Example 2 A quadratic form in 3 variables :

$$Q(\boldsymbol{x}) = Q(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_1z_3 + 2x_2x_3$$

(Note that the quadratic form $Q(\boldsymbol{x})$ is not a linear function on $\mathbb{R}^3$)

Using $\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, we can write $Q(\boldsymbol{x}) =$

$$Q(x_1, x_2, x_3) \quad = \boldsymbol{x}^T \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \boldsymbol{x} = x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3,$$

as you can check directly. How was the matrix A created? The entries a_{ii} are the coefficients (possibly 0) of the terms x_i^2 in $Q(\boldsymbol{x})$; the coefficient of a "cross-product" term x_ix_j in $Q(\boldsymbol{x})$ **is** "split in half" to form the two entries a_{ij} and a_{ji} in A. From Example 1,

$$A = \begin{bmatrix} -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} = U\,DU^T$$

A change of coordinates now lets us understand the quadratic form much better: the columns of U are an orthonormal basis $\mathcal{B} = \{\boldsymbol{u_1}, \boldsymbol{u_2}, \boldsymbol{u_3}\}$ for $\mathbb{R}^3$. For notational convenience, we will use $\boldsymbol{y}$ to describe $\mathcal{B}$-coordinates: $[\boldsymbol{x}]_{\mathcal{B}} = \boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$. The change of coordinate matrix $U_{\mathcal{B}} = U$ relates the old and new coordinates: $\boldsymbol{x} = U_{\mathcal{B}}[\boldsymbol{x}]_{\mathcal{B}}) = U\boldsymbol{y}$. When we make this change of coordinates we get $Q(\boldsymbol{x}) \quad = x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3$

$$= \boldsymbol{x}^T A\boldsymbol{x} = (U\boldsymbol{y})^T A(U\boldsymbol{y}) = \boldsymbol{y}^T U^T AU\boldsymbol{y} = \boldsymbol{y}^T D\boldsymbol{y}$$

$$= [y_1 \ y_2 \ y_3] \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = 0y_1^2 + 0y_2^2 + 3y_3^2 = 3y_3^2$$

In the new coordinates, all the cross-product terms like x_1x_2 in the quadratic form have disappeared and only a linear combination of "pure" terms y_1^2, y_2^2, y_3^2 remains. (*In this particular example, y_1^2 and y_2^2 also drop out because $\lambda_1 = \lambda_2 = 0$, as read from the diagonal of D.*)

You can imagine that we took the standard $x_1x_2x_3$ axes and repositioned them (still orthogonal to each other) in $\mathbb{R}^3$ to set up a new $y_1y_2y_3$ coordinate system. Consider a geometric point P in $\mathbb{R}^3$ with no coordinates assigned for P. What is the value of the quadratic form Q at this point? A formula to find the value of Q at the point P depends on the coordinates we choose:

In standard coordinates: the "coordinate name" of P is (x_1, x_2, x_3), and

$$Q(P) = Q(x_1, x_2, x_3) = x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3$$

In $\mathcal{B}$-coordinates: the "coordinate name" of that point P now is (y_1, y_2, y_3) and

$$Q(P) = 3y_3^2$$

Be sure you understand that the numeric value of Q at P does not change. $x_1^2 + x_2^2 + x_3^2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3 = 3y_3^2$ because of how the coordinate systems are related. It is the formula to evaluate Q at P that changes when the coordinate system changes.

One coordinate system may give us better insight. For example, switching into the new $\boldsymbol{y}$ coordinates makes it clear that there are infinitely many points where the quadratic form has value 0 – for example, at every point P with $\boldsymbol{y}$ coordinates $(y_1, y_2, y_3) = (c, d, 0)$. In $\boldsymbol{y}$ coordinates, the formula also makes it clear that this quadratic form never has a negative value.

Example 8.3

The real symmetrix matrix

$$A = \begin{bmatrix} 5 & 2 & 2 \\ 2 & 2 & 1 \\ 2 & 1 & 2 \end{bmatrix}$$

has the characteristic polynomial $d(s) = (s-1)^2(s-7)$. We observe that the eigenvalues are real.

Two linearly independent eigenvectors associated with the multiple eigenvalue $\lambda_1 = 1$ can be found by solving

$$(A - \lambda_1 I)\mathbf{v} = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix} \mathbf{v} = \mathbf{0}$$

as

$$\mathbf{v}_{11} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{v}_{22} = \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

Applying the Gram-Schmidt process to $\{\mathbf{v}_{11}, \mathbf{v}_{12}\}$, and normalizing the orthogonal eigenvector generated by the process, we obtain two orthonormal eigenvectors associated with $\lambda_1 = 1$ as

$$\mathbf{u}_{11} = \frac{1}{\sqrt{5}} \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{u}_{12} = \frac{1}{\sqrt{30}} \begin{bmatrix} -2 \\ -1 \\ 5 \end{bmatrix}$$

An eigenvector associated with $\lambda_2 = 7$ is found as

$$\mathbf{v}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

Like the eigenvectors of a unitary matrix, eigenvectors of a Hermitian matrix associated with distinct eigenvalues are also orthogonal (see Exercise 8.11). Therefore, we need not specifically look for an eigenvector $\mathbf{v}_2$ that is orthogonal to $\mathbf{v}_{11}$ and $\mathbf{v}_{12}$. After normalizing $\mathbf{v}_2$, we obtain a unit eigenvector associated with $\lambda_2 = 7$ as

$$\mathbf{u}_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

The reader can verify that the modal matrix

$$P = [\, \mathbf{u}_{11} \;\; \mathbf{u}_{12} \;\; \mathbf{u}_2 \,] = \begin{bmatrix} -\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{30}} & \frac{2}{\sqrt{6}} \\ \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{30}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{5}{\sqrt{30}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

is orthogonal and that

$$P^t A P = \operatorname{diag}[1, 1, 7]$$

"One-to-one" and "onto" are properties of functions in general, not just linear transformations.

Definition. Let $f\colon X \to Y$ be a function.

- f is **one-to-one** if and only if for every $y \in Y$ there is *at most* one $x \in X$ such that $f(x) = y$; equivalently, if and only if $f(x_1) = f(x_2)$ implies $x_1 = x_2$.
- f is **onto** (or **onto** Y, if the codomain is not clear from context) if and only if for every $y \in Y$ there *at least* one $x \in X$ such that $f(x) = y$.

This definition applies to linear transformations as well, and in particular for linear transformations $T\colon \mathbb{R}^n \to \mathbb{R}^m$, and by extension to matrices, since an $m \times n$ matrix A can be identified with the linear transformation $L_A\colon \mathbb{R}^n \to \mathbb{R}^m$ given by $L_A(\mathbf{x}) = A\mathbf{x}$.

So, the definitions are for *any* functions. But when our sets X and Y have more structure to them, and the functions are not arbitrary, but special kinds of functions, we can often obtain *other* ways of characterizing a function as one-to-one or onto which is easier/better/more useful/more conceptual/has interesting applications. This is indeed the case when we have such a rich structure as linear transformations and vector spaces.

One-to-one is probably the easiest; this is because whether a function is one-to-one depends *only* on its domain, and not on its codomain. By contrast, whether a function is onto depends on *both* on the domain and the codomain (so, for instance, $f(x) = x^2$ is onto if we think of it as a function $f\colon \mathbb{R} \to [0, \infty)$, but not if we think of it as a function $f\colon \mathbb{R} \to \mathbb{R}$, or $f\colon [2, \infty) \to [0, \infty)$).

Theorem. *Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. The following are equivalent:*

1. *T is one-to-one.*
2. *$T(\mathbf{x}) = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$.*
3. *If A is the standard matrix of T, then the columns of A are linearly independent.*

Proof. The equivalence of (1) and (2) is basic in linear algebra, so let's deal with that:

(1)⇒(2): If T is one-to-one, then for all $\mathbf{x}$, since $T(\mathbf{0}) = \mathbf{0}$ (being linear), then $T(\mathbf{x}) = \mathbf{0} = T(\mathbf{0})$ implies $\mathbf{x} = \mathbf{0}$; this proves (2).

(2)⇒(1): Suppose $T(\mathbf{x}_1) = T(\mathbf{x}_2)$. Then

$$\mathbf{0} = T(\mathbf{x}_1) - T(\mathbf{x}_2) = T(\mathbf{x}_1 - \mathbf{x}_2),$$

since T is linear; because we are assuming (2), $T(\mathbf{x}_1 - \mathbf{x}_2) = \mathbf{0}$ implies that $\mathbf{x}_1 - \mathbf{x}_2 = \mathbf{0}$, so $\mathbf{x}_1 = \mathbf{x}_2$, which proves that T is indeed one-to-one.

The key to the connection with (3) (and eventually to your confusion) is that multiplying a matrix by a vector can be seen as an operation on columns. If

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix},$$

then let the *columns of* A, $A_1, A_2, \dots, A_n$ be:

$$A_1 = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{pmatrix}, \quad A_2 = \begin{pmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{pmatrix}, \quad \dots, A_n = \begin{pmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{pmatrix}.$$

Then we have the following:

$$A \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1 A_1 + a_2 A_2 + \cdots + a_n A_n.$$

That is, multiplying A by $\mathbf{x}$ gives a linear combination of the columns of A. This gives the direct connection we need between conditions (1) and (2), and condition (3).

(2)⇒(3): To show that the columns of A are linearly independent, we need to show that if $\alpha_1 A_1 + \cdots + \alpha_n A_n = \mathbf{0}$, then $\alpha_1 = \cdots = \alpha_n = 0$. So suppose $\alpha_1 A_1 + \cdots + \alpha_n A_n = \mathbf{0}$. Then

$$T(\alpha) = A\alpha = \alpha_1 A_1 + \cdots + \alpha_n A_n = \mathbf{0}, \qquad \text{where } \alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix}.$$

Because we are assuming (2), that means that from $T(\alpha) = \mathbf{0}$ we can conclude taht $\alpha = \mathbf{0}$; therefore, $\alpha_1 = \cdots = \alpha_n = 0$. This proves that $A_1, \ldots, A_n$ are linearly independent.

(3)⇒(2): Suppose the columns of A are linearly independent, and

$$\mathbf{0} = T(\mathbf{x}) = A\mathbf{x} \quad \text{where } \mathbf{x} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}.$$

This means that $a_1A_1 + \cdots a_nA_n = \mathbf{0}$; since the columns of A are assumed to be linearly independent, we conclude that $a_1 = \cdots = a_n = 0$, so $\mathbf{x} = \mathbf{0}$, proving (2). **QED**

What about onto? There are two things here. One is a theorem similar to the one above; the other is the Rank-Nullity Theorem.

Theorem. *Let $T\colon \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. The following are equivalent:*

1. *T is onto.*
2. *The equation $T(\mathbf{x}) = \mathbf{b}$ has solutions for every $\mathbf{b} \in \mathbb{R}^m$.*
3. *If A is the standard matrix of T, then the columns of A span $\mathbb{R}^m$.* That is: every $\mathbf{b} \in \mathbb{R}^m$ is a linear combination of the columns of A.

Proof. (1)⇔(2) is essentially the definition, only cast in terms of equations for the sake of similarity to the previous theorem.

(2)⇒(3) Let $\mathbf{b} \in \mathbb{R}^m$. Then by (2) there exists an $\mathbf{a} \in \mathbb{R}^n$ such that $T(\mathbf{a}) = \mathbf{b}$. We have:

$$\mathbf{b} = T(\mathbf{a}) = A\mathbf{a} = A\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} = a_1A_1 + a_2A_2 + \cdots + a_nA_n.$$

That is, we can express $\mathbf{b}$ as a linear combination of the columns of A. Since $\mathbf{b}$ is arbitrary, every vector in $\mathbb{R}^m$ can be expressed as a linear combination of the columns of A, so the columns of A span $\mathbb{R}^m$; this proves (3).

(3)⇒(2) Suppose the columns of A span $\mathbb{R}^m$ and let $\mathbf{b} \in \mathbb{R}^m$. We want to show that $T(\mathbf{x}) = \mathbf{b}$ has at least one solution.

Since the columns of A span $\mathbb{R}^m$, there exist scalars $\alpha_1, \ldots, \alpha_n$ such that

$$\mathbf{b} = \alpha_1A_1 + \cdots + \alpha_nA_n = A\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = T(\alpha).$$

So α), where

$$\alpha = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix},$$

is a solution to $T(\mathbf{x}) = \mathbf{b}$. This establishes (2). **QED**

So: "one-to-one"-ness is related to linear independence; "onto"-ness is related to spanning properties. Note that linear independence is an *intrinsic* property (it depends only on the set of vectors), whereas spanning is an *extrinsic* properties (it depends also on the space we are considering; it is *contextual*). This matches the fact that whether a function is one-to-one or not depends *only* on the domain, but whether it is onto depends on *both* the domain and the codomain of the function.

But there is a deep connection between the two. Remember the following:

Definition. Let A be an $m \times n$ matrix. The *nullity of* A, $\mathbf{nullity}(A)$, is the dimension of the kernel of A, that is, of the subspace of $\mathbb{R}^n$ given by

$$\ker(A) = \left\{ \mathbf{x} \in \mathbb{R}^n \;\middle|\; A\mathbf{x} = \mathbf{0} \right\}.$$

The *rank of* A, $\mathbf{rank}(A)$ is the dimension of the image of A; that is, of the subspace of $\mathbb{R}^m$ given by

$$\begin{aligned} \mathrm{Im}(A) &= \left\{ \mathbf{b} \in \mathbb{R}^m \;\middle|\; A\mathbf{x} = \mathbf{b} \text{ has at least one solution} \right\} \\ &= \left\{ A(\mathbf{x}) \;\middle|\; \mathbf{x} \in \mathbb{R}^n \right\}. \end{aligned}$$

The deep connection between them is given by the Rank-Nullity Theorem:

Rank-Nullity Theorem. Let A be an $m \times n$ matrix. Then

$$\mathbf{rank}(A) + \mathbf{nullity}(A) = n.$$

Now we get two more equivalences for one-to-one and onto:

Theorem. *Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. The following are equivalent:*

1. T *is one-to-one.*
2. *The equation $T(\mathbf{x}) = \mathbf{0}$ *has only the trivial solution* $\mathbf{x} = \mathbf{0}$.
3. *If A is the standard matrix of T, then the columns of A are linearly independent.*
4. $\ker(A) = \{\mathbf{0}\}$.
5. $\mathbf{nullity}(A) = 0$.
6. $\mathbf{rank}(A) = n$.

Proof. The equivalence of (4) and (5) follows because only the trivial subspace has dimension $\mathbf{0}$; the equivalence of (4) and (2) follows by definition of the kernel. The equivalence of (5) and (6) follows from the Rank-Nullity Theorem, since $n = \mathbf{nullity}(A) + \mathbf{rank}(A)$, so $\mathbf{nullity}(A) = \mathbf{0}$ if and only if $\mathbf{rank}(A) = n$. Since we already know (1), (2), and (3) are equivalent, the result follows. **QED**

Theorem. *Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. The following are equivalent:*

1. *T is onto.*
2. *The equation $T(\mathbf{x}) = \mathbf{b}$ has solutions for every $\mathbf{b} \in \mathbb{R}^m$.*
3. *If A is the standard matrix of T, then the columns of A span $\mathbb{R}^m$.* That is: every $\mathbf{b} \in \mathbb{R}^m$ is a linear combination of the columns of A.
4. $\operatorname{Im}(A) = \mathbb{R}^m$.
5. $\operatorname{rank}(A) = m$.
6. $\operatorname{nullity}(A) = n - m$.

Proof. We already know that (1), (2), and (3) are equivalent. The equivalence of (4) and (2) follows by the definition of the image. The equivalence of (4) and (5) follows because the only subspace of $\mathbb{R}^m$ that has dimension m is the whole space. Finally, the equivalence of (5) and (6) follows from the rank nullity theorem: since $n = \operatorname{rank}(A) + \operatorname{nullity}(A)$, then $\operatorname{nullity}(A) = n - \operatorname{rank}(A)$. So the rank equals m if and only if the nullity equals $n - m$. **QED**

Exercise 2. Let

$$\alpha_1 = (1, 1, -2, 1), \quad \alpha_2 = (3, 0, 4, -1), \quad \alpha_3 = (-1, 2, 5, 2).$$

Let

$$\alpha = (4, -5, 9, -7), \quad \beta = (3, 1, -4, 4), \quad \gamma = (-1, 1, 0, 1).$$

(a) Which of the vectors α, β, γ are in the subspace of $\mathbb{R}^4$ spanned by the α_i?
(b) Which of the vectors α, β, γ are in the subspace of $\mathbb{C}^4$ spanned by the α_i?
(c) Does this suggest a theorem?

Solution: (a)(b) This problem is analogous to part (b) of Exercise 4 (p.55) above. Namely, consider the 4×6 matrix whose columns are vectors $\alpha_1, \alpha_2, \alpha_3, \alpha, \beta, \gamma$. Row reduce this matrix to see if the systems $x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3 = \alpha (= \beta, = \gamma)$ are consistent.

(c) Theorem: Let V be a complex vector space with some basis, and $v_1, \ldots, v_n$ and v vectors with *real* coordinates with respect to this basis. Then if v is a linear combination of $v_1, \ldots, v_n$ with *complex* coefficients, then v can also be represented as a linear combination of $v_1, \ldots, v_n$ with *real* coefficients.

Exercise 2. Let

$$\alpha_1 = (1, 1, -2, 1), \quad \alpha_2 = (3, 0, 4, -1), \quad \alpha_3 = (-1, 2, 5, 2).$$

Let

$$\alpha = (4, -5, 9, -7), \quad \beta = (3, 1, -4, 4), \quad \gamma = (-1, 1, 0, 1).$$

(a) Which of the vectors α, β, γ are in the subspace of $\mathbb{R}^4$ spanned by the α_i?
(b) Which of the vectors α, β, γ are in the subspace of $\mathbb{C}^4$ spanned by the α_i?
(c) Does this suggest a theorem?

Solution: (a)(b) This problem is analogous to part (b) of Exercise 4 (p.55) above. Namely, consider the 4×6 matrix whose columns are vectors $\alpha_1, \alpha_2, \alpha_3, \alpha, \beta, \gamma$. Row reduce this matrix to see if the systems $x_1\alpha_1 + x_2\alpha_2 + x_3\alpha_3 = \alpha(= \beta, = \gamma)$ are consistent.

(c) Theorem: Let V be a complex vector space with some basis, and $v_1, \dots, v_n$ and v vectors with *real* coordinates with respect to this basis. Then if v is a linear combination of $v_1, \dots, v_n$ with *complex* coefficients, then v can also be represented as a linear combination of $v_1, \dots, v_n$ with *real* coefficients.

Exercise 3. Consider the vectors in $\mathbb{R}^4$ defined by $\alpha_1 = (-1, 0, 1, 2), \alpha_2 = (3, 4, -2, 5), \alpha_3 = (1, 4, 0, 9)$ Find a system of homogeneous linear equations for which the space of solutions is exactly the subspace of $\mathbb{R}^4$ spanned by the three given vectors.

Solution Row reducing the matrix whose rows are the α_i's we get

$$\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & \frac{1}{4} & \frac{11}{4} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore we need to find a matrix whose kernel is the space V spanned by $\{\alpha_1 = (1, 0, -1, -2), \alpha_2 = (0, 1, \frac{1}{4}, \frac{11}{4})\}$. The set $\mathcal{B} = \{\alpha_1, \alpha_2, (0, 0, 1, 0), (0, 0, 0, 1)\}$ is a basis for $\mathbb{R}^4$. The kernel of the matrix

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is exactly the space generated by e_1 and e_2. Thus, the matrix which maps the canonical basis to the basis $\mathcal{B}$ will map the kernel of S to the space spanned by α_1 and α_2. Let P be the change of basis, then the previous assertion expressed in terms of matrices is $P(\ker(S)) = V$ therefore $\ker(S) = P^{-1}(V)$. This means that if we apply S to any vector in $P^{-1}(V)$ we get 0, that is, if we apply P^{-1} to any vector in V and then we apply S, we get zero. But this is exactly what we want, to find a transformation whose kernel is V, such a transformation is given by SP^{-1}. Explicitly a transformations whose kernel is V is

$$SP^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & \frac{1}{4} & 1 & 0 \\ -2 & -\frac{11}{4} & 0 & 1 \end{pmatrix}$$

Exercise 6. Let V be the real vector space spanned by the rows of the matrix

$$A = \begin{pmatrix} 3 & 21 & 0 & 9 & 0 \\ 1 & 7 & -1 & -2 & -1 \\ 2 & 14 & 0 & 6 & 1 \\ 6 & 42 & -1 & 13 & 0 \end{pmatrix}.$$

(a) Find a basis for V.

(b) Tell which vectors $v = (x_1, x_2, x_3, x_4, x_5)$ are elements of V.

(c) If $v = (x_1, x_2, x_3, x_4, x_5)$ is in V what are its coordinates in the basis chosen in part (a)?

Solution:(a)(c) Row reduce the matrix A. The nonzero rows of the row reduced matrix $\tilde{A}$ give the basis in V. It is easy to check that the coordinates of v relative to this basis form an ordered subset of coordinates x_1, x_2, x_3, x_4, x_5. This subset consists of all x_i such that i coincides with the number of a column of $\tilde{A}$ that contains the leading coefficient of a nonzero row.

(b) (The same as part (b) of Exercise 4 (p.55) above.) Take the matrix A whose columns are the basis from part (a) and the vector v. Row reduce it to write the condition on v for the system with augmented matrix A to be consistent.

Section 5.4 p244 Problem 1. In each part, explain why the given vectors do not form a basis for the indicated vector space. (Solve the problem by inspection.)

(a) $\boldsymbol{u}_1 = (1, 2),\ \boldsymbol{u}_2 = (0, 3),\ \boldsymbol{u}_3 = (2, 7)$ for $\boldsymbol{R}^2$

(b) $\boldsymbol{u}_1 = (-1, 3, 2),\ \boldsymbol{u}_2 = (6, 1, 1)$ for $\boldsymbol{R}^3$

(c) $\boldsymbol{p}_1 = 1 + x + x^2,\ \boldsymbol{p}_2 = x - 1$ for P_2.

(d) $A = \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix},\ B = \begin{bmatrix} 6 & 0 \\ -1 & 4 \end{bmatrix},\ C = \begin{bmatrix} 3 & 0 \\ 1 & 7 \end{bmatrix},\ D = \begin{bmatrix} 5 & 1 \\ 4 & 2 \end{bmatrix},\ E = \begin{bmatrix} 7 & 1 \\ 2 & 9 \end{bmatrix},$ for M_{22}

Solution.

(a) Too many vectors: 3 vectors in the 2-dimensional space $\boldsymbol{R}^2$ must form a dependent set.

(b) Too few vectors: 2 vectors in the 3-dimensional space $\boldsymbol{R}^3$ cannot span $\boldsymbol{R}^3$.

(c) Too few: 2 vectors in the 3-dimensional space P_2 cannot span P_2.

(d) Too many: 5 vectors in the 4-dimensional space M_{22} must form a dependent set.

Section 5.4 p244 Problem 3b. Do the vectors $(3,1,-4),(2,5,6),(1,4,8)$ form a basis for $\boldsymbol{R}^3$?

Solution. Since we have the correct count (3 vectors for a 3-dimensional space) there is certainly a chance. If these 3 vectors form an independent set, then one of the theorems in 5.4 tells us that they'll form a basis. If not, they can't form a basis. So we need only test for independence. So: does the equation $k_1(3,1,-4)+k_2(2,5,6)+k_3(1,4,8)=(0,0,0)$ have nontrivial solutions or not? Equivalently, does the system

$$\begin{bmatrix} 3 & 2 & 1 \\ 1 & 5 & 4 \\ -4 & 6 & 8 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

have nontrivial solutions? According to the Long Theorem, the answer depends on whether the indicated coefficient matrix has determinant 0 or not. Since

$$\det \begin{bmatrix} 3 & 2 & 1 \\ 1 & 5 & 4 \\ -4 & 6 & 8 \end{bmatrix} = 42 \neq 0$$

there are no nontrivial solutions. The given vectors form an independent set, hence a basis for $\boldsymbol{R}^3$.

Section 5.4 p244 Problem 4b. Do the vectors/functions $4+6x+x^2, -1+4x+2x^2, 5+2x-x^2$ form a basis for P_2?

Solution. Once again we have the correct count, since P_2 has dimension 3. We need only test for independence. So consider the condition

$$k_1(4+6x+x^2)+k_2(-1+4x+2x^2)+k_3(5+2x-x^2)=0$$

for every x. This is equivalent to the system

$$\begin{bmatrix} 4 & -1 & 5 \\ 6 & 4 & 2 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This time we find

$$\det \begin{bmatrix} 4 & -1 & 5 \\ 6 & 4 & 2 \\ 1 & 2 & -1 \end{bmatrix} = 0$$

According to the Long Theorem, we have a homogeneous system with nontrivial solutions. Therefore the given three vectors/functions form a dependent set, hence do not form a basis.

Section 5.4 p244 Problem 5. Show that the set $\{A, B, C, D\}$ is a basis for M_{22}.

$$A = \begin{bmatrix} 3 & 6 \\ 3 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & -8 \\ -12 & -4 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$$

Solution. (or outline of solution). It suffices to show that the set $\{A, B, C, D\}$ is independent. (Why?) To this end, consider the equation $k_1A + k_2B + k_3C + k_4D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. This equation is equivalent to the system

$$\begin{bmatrix} 3 & 0 & 0 & 1 \\ 6 & -1 & -8 & 0 \\ 3 & -1 & -12 & -1 \\ -6 & 0 & -4 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

One can show that this system has no nontrivial solutions. (Do this.) Therefore the only values of $k_1, \ldots k_4$ that will give us

$$k_1A + k_2B + k_3C + k_4D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

are

$$k_1 = k_2 = k_3 = k_4 = 0.$$

This finishes the proof. (Why?)

Section 5.4 p244 Problem 12. Find a basis for and the dimension of the solution space for the homogeneous system

$$3x_1 + x_2 + x_3 + x_4 = 0$$

$$5x_1 - x_2 + x_3 - x_4 = 0$$

Solution. First solve the given system, using row reduction. The augmented matrix for the given system is

$$\left[\begin{array}{cccc|c} 3 & 1 & 1 & 1 & 0 \\ 5 & -1 & 1 & -1 & 0 \end{array}\right]$$

which reduces to

$$\left[\begin{array}{cccc|c} 1 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 1 & \frac{1}{4} & 1 & 0 \end{array}\right].$$

The nonleading variables are x_3 and x_4. Setting $x_3 = s$ and $x_4 = t$, and solving for the leading variables in terms of the nonleading ones, we find a parametric description for the solutions. The solutions are of the form

$$(x_1, x_2, x_3, x_4) = s(-\frac{1}{4}, -\frac{1}{4}, 1, 0,) + t(0, -1, 0, 1).$$

We can read off from this description that the vectors

$$(-1, -1, 4, 0), \quad (0, -1, 0, 1)$$

span the solution space. By inspection, we can see that they form an independent set (why?). Therefore the set

$$\{(-1, -1, 4, 0), (0, -1, 0, 1)\}$$

is a basis for the solution space, which has dimension 2. Other correct answers are possible.

Section 5.4 p244 Problem 17. Find bases for the following subspaces of $\boldsymbol{R}^3$.

(a) The plane $3x - 2y + 5z = 0$

(b) The plane $x - y = 0$

(c) The line $x = 2t, y = -t, z = 4t$

(d) The set of all vectors of the form (a, b, c) where $b = a + c$

Solution. (a) Find a basis for the solution space to the equation $3x - 2y + 5z = 0$. First find a parametric description for the solutions. We can do this by putting $y = s, z = t$, and solving for x. The solutions are $(x, y, z) = s(\frac{2}{3}, 1, 0) + t(-\frac{5}{3}, 0, 1)$. Therefore the vectors

$$(2, 3, 0), \ (-5, 0, 3)$$

span the solution space. They also form an independent set, so they form a basis. Since there are exactly two vectors in the basis, the dimension of the solution space is 2. In other words, as we knew all along, the dimension of the specified plane is 2.

(b) One method is to follow the same procedure as for (a). The solutions for the equation $x-y=0$ can be described by

$$(x,y,z) = s(1,1,0) + t(0,0,1).$$

The vectors $(1,1,0)$ and $(0,0,1)$ span the solution set for $x-y=0$ and they form an independent set. Hence they form a basis for the plane $x-y=0$, a 2-dimensional subspace of $\boldsymbol{R}^3$.

Here's another method. The points on the given plane are exactly the points of the form (x,x,z) where x and z are real numbers. In other words, they're the points of the form $x(1,1,0)+z(0,0,1)$. Now read off that the set

$$\{(1,1,0),\ \ (0,0,1)\}$$

is a basis for the plane in question.

(c) Points on this line are of the form $t(2,-1,4)$ where t can be any real number. Thus the vector $(2,-1,4)$ spans this line. Moreover, the singleton set

$$\{(2,-1,4)\}$$

is independent. Thus it forms a basis for the line in question, which therefore (as we anticipated) has dimension 1.

(d) The vectors in question can also be written in the form $(a,a+c,c)$, hence in the form $a(1,1,0)+c(0,1,1)$. Together, the vectors

$$(1,1,0), \quad (0,1,1)$$

span the space in question. They form an independent set, hence a basis. The set in question has dimension 2.

Section 5.4 p244 Problem 18. Find the dimensions of the following subspaces of $\boldsymbol{R}^4$.

(a) The set of all vectors of the form $(a,b,c,0)$.

(b) The set of all vectors of the form (a,b,c,d) where $d=a+b$ and $c=a-b$.

(c) The set of all vectors of the form (a,b,c,d) where $a=b=c=d$.

Solution. (a) Let W be the subspace in question. The vectors in W can be written in the form $a(1,0,0,0)+b(0,1,0,0)+c(0,0,1,0)$. By inspection, we see that $\{(1,0,0,0),(0,1,0,0),(0,0,1,0)\}$ forms a basis for W. Therefore W has dimension 3.

(b) Denote the given subspace by W. Vectors in W can be written in the form $(a,b,a-b,a+b)$, hence in the form $a(1,1,1,0)+b(0,1,-1,1)$. This time we see that the set $\{(1,1,1,0),(0,1,-1,1)\}$ is a basis for W, which therefore has dimension 2.

(c) Denote the subspace by W. Vectors in W are those of the form (a,a,a,a), hence of the form $a(1,1,1,1)$. The singleton set $\{(1,1,1,1)\}$ forms a basis for W, which is therefore a 1-dimensional subspace of $\boldsymbol{R}^4$.

Section 5.4 p244 Problem 21. Find standard basis vectors that can be added to the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2\}$ to produce a basis for $\boldsymbol{R}^4$, given that

$$\boldsymbol{v}_1 = (1, -4, 2, -3), \quad \boldsymbol{v}_2 = (-3, 8, -4, 6)$$

Solution. Here's one solution. The standard basis vector $(0, 0, 0, 1)$, which I'll call $\boldsymbol{u}$, is not a linear combination of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ (verify this). According to the Plus/Minus Theorem, the set $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{u}\}$ is independent. Inspecting the second and third entries in $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ it looks like we could use either $(0, 1, 0, 0)$ or $(0, 0, 1, 0)$ for the fourth vector. Pick one (either one will work) and test the new collection of four vectors to confirm independence. Other correct answers are possible. (See the text for a list of all possible correct answers.)

Section 5.4 p244 Problem 28ab. The figure given in the text shows a rectangular xy-coordinate system determined by the unit basis vectors $\boldsymbol{i}$ and $\boldsymbol{j}$ and an $x'y'$-coordinate system determined by unit basis vectors $\boldsymbol{u}_1$ and $\boldsymbol{u}_2$. Find the $x'y'$-coordinates of the points whose xy-coordinates are given.

In the figure, $\boldsymbol{u}_2 = \boldsymbol{j}$ and $\boldsymbol{u}_1$ is a unit vector that makes an angle of $30°$ with the positive x-axis. Thus $\boldsymbol{u}_2 = (0, 1)$ and $\boldsymbol{u}_1 = (\cos 30°, \sin 30°) = (\frac{\sqrt{3}}{2}, \frac{1}{2})$.

(a) $(\sqrt{3}, 1)$ (b) $(1, 0)$

Solution. (a) By inspection, $(\sqrt{3}, 1) = 2\boldsymbol{u}_1 + 0\boldsymbol{u}_2$. The $x'y'$-coordinate vector is $(2, 0)$.

(b) Solve the system $(1, 0) = a(\frac{\sqrt{3}}{2}, \frac{1}{2}) + b(0, 1)$ to find $a = \frac{2}{\sqrt{3}}, b = -\frac{a}{2} = -\frac{1}{\sqrt{3}}$. Thus $(1, 0) = \frac{2}{\sqrt{3}}(\frac{\sqrt{3}}{2}, \frac{1}{2}) - \frac{1}{\sqrt{3}}(0, 1) +$. The $x'y'$-coordinate vector is $(\frac{2}{\sqrt{3}}, -\frac{1}{\sqrt{3}})$.

Section 5.4 p244 Problem 31. Find the dimension for each of the following vector spaces.

(a) The vector space of all diagonal $n \times n$ matrices.

(b) The vector space of all symmetric $n \times n$ matrices.

(c) The vector space of all upper triangular $n \times n$ matrices.

Solution. (a) Let D_{nn} denote the specified vector space of diagonal matrices. One basis for D_{nn} consists of the n different $n \times n$ matrices, each of which has exactly one of the diagonal entries equal to 1 and all other entries in the matrix equal to 0. [For D_{33}, for instance, this basis would use the 3 matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

as basis vectors.
Thus D_{nn} has dimension n.

(b) Let S_{nn} denote the specified vector space of symmetric matrices. Let's look at a special case first, just to get oriented. Consider S_{33} for instance. For a particular 3×3 matrix to belong to S_{nn} the entries on the main diagonal can be anything, but the entries off the main diagonal must be paired up to match symmetrically. A typical matrix in S_{33} would look like

$$\begin{bmatrix} a & x & y \\ x & b & z \\ y & z & c \end{bmatrix}$$

One basis for S_{33} would use the following 6 matrices as the basis vectors:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

So S_{33} would have dimension 6.

Theorem. *Let U, V be two subspaces of a finite-dimensional vector space W. Then*

$$\dim U + \dim V = \dim(U \cap V) + \dim(U + V). \tag{2}$$

This result is illustrated by the following example (whose method is often used as a proof).

Example. Let $W = \mathbb{R}^5$. Consider the two subspaces

$$U = \mathscr{L}\{(x_1, x_2, x_3, x_4, x_5) : 2x_1 - x_2 - x_3 = 0 = x_4 - 3x_5\},$$
$$V = \mathscr{L}\{(x_1, x_2, x_3, x_4, x_5) : x_3 + x_4 = 0\}.$$

We are required to find a basis of $\mathbb{R}^5$ that contains *both* a basis of U *and* a basis of V. The trick is to start by finding a basis of $U \cap V$. It is easy to see that $\dim U = 3$ and $\dim V = 4$; this is because the homogeneous linear systems have rank 2 and 1. Now, a vector $\mathbf{v} \in \mathbb{R}^5$ belongs to $U \cap V$ iff it satisfes *all three* equations. Since the associated matrix

$$A = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} \boxed{1} & -\frac{1}{2} & 0 & 0 & \frac{3}{2} \\ 0 & 0 & \boxed{1} & 0 & 3 \\ 0 & 0 & 0 & \boxed{1} & -3 \end{pmatrix}$$

has rank 3, we deduce that $\dim(U \cap V) = 5 - 3 = 2$. Indeed, we may take $x_2 = s$ and $x_5 = t$ to be free variables and obtain (as a row) the general solution

$$\mathbf{v} = \left(\tfrac{1}{2}s - \tfrac{3}{2}t,\ s,\ -3t,\ 3t,\ t\right).$$

A basis of $U \cap V$ consists of

$$\mathbf{w}_1 = (\tfrac{1}{2}, 1, 0, 0, 0), \qquad \mathbf{w}_2 = (-\tfrac{3}{2}, 0, -3, 3, 1)$$

(take first $s = 1, t = 0$ and second $s = 0, t = 1$). Extend this basis in any way to

a basis $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ of U, and

a basis $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4, \mathbf{w}_5\}$ of V, and

Then $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4, \mathbf{w}_5\}$ will *always* be LI and thus a basis of $\mathbb{R}^5$. There are lots of choices in this example, but we could take

$\mathbf{w}_3 = (0, -1, 1, 0, 0)$ (this works since $\mathbf{w}_3 \in U$ but $\mathbf{w}_3 \notin \mathscr{L}\{\mathbf{w}_1, \mathbf{w}_2\}$),

$\mathbf{w}_4 = (0, 0, 1, -1, 0)$, $\mathbf{w}_5 = (0, 0, 0, 0, 1)$ (note that $\mathbf{w}_5 \notin \mathscr{L}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4\}$).

In conclusion, $U + V = \mathbb{R}^5$, and the required basis is

$$\overbrace{\mathbf{w}_5 \quad \underbrace{\mathbf{w}_1 \quad \mathbf{w}_2}_{}}^{U} \! \quad \underbrace{\phantom{\mathbf{w}_1 \quad \mathbf{w}_2} \quad \mathbf{w}_3 \quad \mathbf{w}_4}_{V}$$

5. (v.1) The set $\mathcal{B} = \{1+t, 1+t^2, t+t^2\}$ is a basis for $\mathbb{P}_2$. Find the $\mathcal{B}$-coordinate vector of $p(t) = 6 + 3t - t^2$.

The coordinate vector of $p(t)$ with respect to the basis $\mathcal{B}$ is the vector

$$[p(t)]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \qquad \textit{such that} \qquad p(t) = c_1(1+t) + c_2(1+t^2) + c_3(t+t^2).$$

In our case, that means

$$6 + 3t - t^2 = c_1(1+t) + c_2(1+t^2) + c_3(t+t^2)$$

There are a few ways of seeing this, but the net result must be that

$$c_1 + c_2 = 6 \qquad c_1 + c_3 = 3 \qquad c_2 + c_3 = -1.$$

This can be rewritten

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 3 \\ -1 \end{bmatrix}$$

Whichever way you attack this problem, the correct answer is

$$[p(t)]_{\mathcal{B}} = \begin{bmatrix} 5 \\ 1 \\ -2 \end{bmatrix}$$

5. (v.2) The set $\mathcal{B} = \{1+t, 1+t^2, t+t^2\}$ is a basis for $\mathbb{P}_2$. Find the $\mathcal{B}$-coordinate vector of $p(t) = 6 - t + t^2$.

The coordinate vector of $p(t)$ with respect to the basis $\mathcal{B}$ is the vector

$$[p(t)]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \qquad \textit{such that} \qquad p(t) = c_1(1+t) + c_2(1+t^2) + c_3(t+t^2).$$

In our case, that means

$$6 - t + t^2 = c_1(1+t) + c_2(1+t^2) + c_3(t+t^2)$$

There are a few ways of seeing this, but the net result must be that

$$c_1 + c_2 = 6 \qquad c_1 + c_3 = -1 \qquad c_2 + c_3 = 1.$$

This can be rewritten

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 6 \\ -1 \\ 1 \end{bmatrix}$$

Whichever way you attack this problem, the correct answer is

$$[p(t)]_{\mathcal{B}} = \begin{bmatrix} 2 \\ 4 \\ -3 \end{bmatrix}$$

6. Define a linear transformation $T : \mathbb{P}_3 \to \mathbb{R}^3$ by

$$T(p(t)) = \begin{bmatrix} p(0) \\ p(0) \\ p(0) \end{bmatrix}$$

Find a basis for the kernel of T, and describe the range of T.

This is almost identical to a homework problem we had. A polynomial $p \in \mathbb{P}_2$ has the form

$$p(t) = a_0 + a_1 t + a_2 t^2 + a_3 t^3$$

where the coefficients a_i are real numbers. Thus

$$p(0) = a_0 + a_1(0) + a_2(0)^2 + a_3(t)^3 = a_0.$$

We see that an equivalent definition of the transformation T would be

$$T(p(t)) = \begin{bmatrix} a_0 \\ a_0 \\ a_0 \end{bmatrix} = a_0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad (\textit{since } a_0 \textit{ is a scalar}).$$

Now we can begin to discuss the kernel and the range of T.

$$\begin{aligned} kernel(T) &= \{p \in \mathbb{P}_3 : T(p) = 0\} \\ &= \left\{ a_0 + a_1 t + a_2 t^2 + a_3 t^3 : a_0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\} \end{aligned}$$

Clearly we see that a_0 must be zero for $a_0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Thus

$$kernel(T) = \{a_1 t + a_2 t^2 + a_3 t^3 : a_i \in \mathbb{R}\}.$$

A basis of this set would be $\{t, t^2, t^3\}$, as these are three linearly independent polynomials and they span the kernel. As for the range,

$$\begin{aligned} range(T) &= \{v \in \mathbb{R}^3 : T(p) = v \textit{ for some } p \in \mathbb{P}_3\} \\ &= \left\{ \begin{bmatrix} p(0) \\ p(0) \\ p(0) \end{bmatrix} : p \in \mathbb{P}_3 \right\} \\ &= \left\{ a_0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} : a_0 \in \mathbb{R} \right\} \\ &= span \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \end{aligned}$$

Since the range of T is the span of a single vector, it is actually a line in $\mathbb{R}^3$. ∎

Exercise 7. Let V be the (real) vector space of all polynomial functions from $\mathbb{R}$ into $\mathbb{R}$ of degree 2 or less, i.e. the space of functions of the form

$$f(x) = c_0 + c_1 x + c_2 x^2.$$

Let t be a fixed real number and define

$$g_1(x) = 1, \quad g_2(x) = x + t, \quad g_3(x) = (x+t)^2.$$

Prove that $\mathcal{B} = \{g_1, g_2, g_3\}$ is a basis for V. If

$$f(x) = c_0 + c_1 x + c_2 x^2$$

what are the coordinates of f in the ordered basis $\mathcal{B}$?

Solution: Since

$$x^2 = ((x+t)-t)^2 = (x+t)^2 - 2t(x+t) + t^2, \quad x = x + t - t$$

we get that

$$f(x) = c_2(x+t)^2 + (c_1 - 2tc_2)(x+t) + (c_0 - tc_1 + t^2 c_2).$$

Thus, $\mathcal{B}$ span the space V, so $\mathcal{B}$ is a basis. The coordinates of f relative to $\mathcal{B}$ are $(c_2, c_1 - 2tc_2, c_0 - tc_1 + t^2 c_2)$, respectively.

2. Let V be the vector space of all polynomials of degree at most three.

   (a) Determine whether

   $$p_1(t) := 1 - t, p_2(t) := -1 + 2t + 3t^2 + t^3, p_3(t) := t^2 + 5t^3, p_4(t) = -1 + 3t + 7t^2 + 7t^3$$

   are linearly independent or not.
   Solution: Pick an ordered basis for V, e.g. $B = \{1; t; t^2; t^3\}$. Then we only have to check whether

   $$c_B(p_1(t)) = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix}, c_B(p_2(t)) = \begin{pmatrix} -1 \\ 2 \\ 3 \\ 1 \end{pmatrix}, c_B(p_3(t)) = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 5 \end{pmatrix}, c_B(p_4(t)) = \begin{pmatrix} -1 \\ 3 \\ 7 \\ 7 \end{pmatrix}$$

   are linearly independent. We consider the big 4×4–matrix with columns given by the above vectors, after computing the row echelon form you will see that the last column is a non–leading column. So the vectors are linearly dependent.

   (b) Find $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ such that

   $$\lambda_1 p_1(t) + \lambda_2 p_2(t) + \lambda_3 p_3(t) + \lambda_4 p_4(t) = -3 + 8t + 18t^2 + 20t^3.$$

   (you do NOT have to find ALL such λ's).
   Solution: Again using the ordered basis $B = \{1; t; t^2; t^3\}$ this boils down to finding a solution to

   $$\left(\begin{array}{cccc|c} 1 & -1 & 0 & -1 & -3 \\ -1 & 2 & 0 & 3 & 8 \\ 0 & 3 & 1 & 7 & 18 \\ 0 & 1 & 5 & 7 & 20 \end{array}\right)$$

>Let F be a field and let n be a positive integer (n>=2). Let V be the vector
>space of all n x n matrices over F. Which of the following sets of matrices
>A in V are subspaces of V?
>
>1) all invertible A;
>2) all non-invertible A;
>3) all A such that AB=BA,where B is some fixed matrix in V;
>4) all A such that A^2 = A

Remember H is a subspace of V if
a) the zerovector is an element of H
b) H is closed under addition: if a,b in H, then a+b in H
c) H is closed under scalarmultiplication: if x in H and c a scalar then cx in H

1) H={A: A invertible} is not a subspace, because the zerovector which is the zero-matrix is not invertible. So the zerovector is not an element of H, so H can't be a vectorspace.

2) H={A: A non-inversible} is not a subspace, because it is not closed under addition.
Take for instance in the space of 2x2 matrices the matrix A=
[1 0]
[0 0]
and the matrix B=
[0 0]
[0 1]
Then are both A,B non inversible, thus A,B in H, but A+B is the identity-matrix, which is invertible.

3) H={A: AB=BA for a fixed matrix B in V} is a subspace, because
a) 0, the zeromatrix is in H, because 0B=B0=0
b) H is closed under addition: Let A,C in H, thus AB=BA and CB=BC, then,
(A+C)B=AB+CB=BA+BC=B(A+C), thus A+C in H
c) let c be a scalar and A in H, than (cA)B=c(AB)=c(BA)=(cB)A=(Bc)A=B(cA), thus cA in H

4) H={A: A^2=A} is not a subspace of V because it isn't closed under scalar multiplication.
Take for instance the identity-matrix I.
Then I in H, but (2I)^2=(2^2)I^2=4I. Thus 2I is not an element of H.

Example: coordinates in polynomial space

find a basis for $W = Sp\{v_1,v_2,v_3,v_4\} < P_3$ where

$v_1 = t^3 - 2t^2 + 4t + 1$

$v_2 = 2t^3 - 3t^2 + 9t + 1$

$v_3 = t^3 + 6t - 5$

$v_4 = 2t^3 - 5t^2 + 7t + 5$

the coordinates of these polynomials with respect to the monomial basis $\{t^3, t^2, t, 1\}$ are

$[v_1] = (1, -2,4,1)$

$[v_2] = (2, -3,9,1)$

$[v_3] = (1, 0,6,-5)$

$[v_4] = (2,-5,7,5)$ ....

4.3.10 Find a basis for the null space of the following matrix: $A = \begin{bmatrix} 1 & 0 & -5 & 1 & 4 \\ -2 & 1 & 6 & -2 & -2 \\ 0 & 2 & -8 & 1 & 9 \end{bmatrix}$

We need to find a basis for the solutions to the equation $A\mathbf{x} = \mathbf{0}$. To do this we first put A in row reduced echelon form. The result (according to the computer) is: $\begin{bmatrix} 1 & 0 & -5 & 0 & 7 \\ 0 & 1 & -4 & 0 & 6 \\ 0 & 0 & 0 & 1 & -3 \end{bmatrix}$.

From this we can read the general solution, $\mathbf{x} = \begin{bmatrix} 5x_3 - 7x_5 \\ 4x_3 - 6x_5 \\ x_3 \\ 3x_5 \\ x_5 \end{bmatrix}$. We can also write this as $\mathbf{x} = x_3 \begin{bmatrix} 5 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -7 \\ -6 \\ 0 \\ 3 \\ 1 \end{bmatrix}$, or Span$\left\{ \begin{bmatrix} 5 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ -6 \\ 0 \\ 3 \\ 1 \end{bmatrix} \right\}$. Because these two vectors are clearly not multiples of one another, they also give a basis. So a basis for null(A) is $\left\{ \begin{bmatrix} 5 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ -6 \\ 0 \\ 3 \\ 1 \end{bmatrix} \right\}$.

3.3.28 Let $\mathbf{v}_1 = \begin{bmatrix} 7 \\ 4 \\ -9 \\ -5 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ -7 \\ 2 \\ 5 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ -5 \\ 3 \\ 4 \end{bmatrix}$. It can be verified that $\mathbf{v}_1 - 3\mathbf{v}_2 + 5\mathbf{v}_3 = \mathbf{0}$. Use this information to find a basis for $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

By the Spanning Set Theorem, some subset of the $\mathbf{v}_i$ is a basis for H. In class we showed how to find this subset. We simply remove any of the vectors involved in a non-trivial linear relation. So I choose to remove $\mathbf{v}_3$ (I could have

removed any of the $\mathbf{v}_i$ because they each occur with a non-zero coefficient in the dependency relation $\mathbf{v}_1 - 3\mathbf{v}_2 + 5\mathbf{v}_3 = \mathbf{0}$). The remaining vectors then give a basis for H. We know they span by the Spanning Set Theorem. They are also linearly independent, because they are not multiples of one another.

4.4.14 The set $\mathcal{B} = \{1 - t^2, t - t^2, 2 - 2t + t^2\}$ is a basis for $\mathbb{P}_2$. Find the coordinate vector of $\mathbf{p}(t) = 3 + t - 6t^2$ relative to $\mathcal{B}$.

We need to write $\mathbf{p}$ in terms of the basis $\mathcal{B}$, that is, find $x_1, x_2, x_3 \in \mathbb{R}$ such that $x_1(1 - t^2) + x_2(t - t^2) + x_3(2 - 2t + t^2) = 3 + t - 6t^2$. Multiplying things out, we get $(x_1 + 2x_3) + (x_2 - 2x_3)t + (-x_1 - x_2 + x_3)t^2 = 3 + t - 6t^2$. Thus we have to solve the three linear equations:

$$\begin{aligned} x_1 + 2x_3 &= 3 \\ x_2 - 2x_3 &= 1 \\ -x_1 - x_2 + x_3 &= -6 \end{aligned}$$

We form the augmented matrix for this system,

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -2 & 1 \\ -1 & -1 & 1 & -6 \end{bmatrix}.$$

In row reduced echelon form, this is the matrix $\begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$. So we see that $x_1 = 7$, $x_2 = -3$, and $x_3 = -2$.

This implies that $7(1 - t^2) + (-3)(t - t^2) + (-2)(2 - 2t + t^2) = 3 + t - 6t^2$, so

$$[3 + t - 6t^2]_{\mathcal{B}} = \begin{bmatrix} 7 \\ -3 \\ -2 \end{bmatrix} \in \mathbb{R}^3.$$

4.5.22 The first four Laguerre polynomials are 1, $1-t$, $2-4t+t^2$, and $6-18t+9t^3-t^3$. Show that these polynomials form a basis of $\mathbb{P}_3$.

Consider the basis $\mathcal{B} = \{1, t, t^2, t^3\}$ of $\mathbb{P}_3$. Utilizing the same arguments as we did in question 4.4.32, we know that it is enough to show that the images of these polynomials form a basis of $\mathbb{R}^4$ under to coordinate mapping. Their images under this mapping are: $[1]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $[1-t]_{\mathcal{B}} = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$, $[2-4t+t^2]_{\mathcal{B}} = \begin{bmatrix} 2 \\ -4 \\ 1 \\ 0 \end{bmatrix}$,

and $[6-18t+9t^3-t^3]_{\mathcal{B}} = \begin{bmatrix} 6 \\ -18 \\ 9 \\ -1 \end{bmatrix}$.

By IMT, these will form a basis if and only if the matrix $\begin{bmatrix} 1 & 1 & 2 & 6 \\ 0 & -1 & -4 & -18 \\ 0 & 0 & 1 & 9 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ is invertible. This matrix will be invertible if and only if its determinate is non-zero (theorem 4 page 194). The computer tells me that the determinate of this matrix is equal to 1. So we conclude that the given polynomials do form a basis.

Section 5.4 p244 Problem 31. Find the dimension for each of the following vector spaces.

(a) The vector space of all diagonal $n \times n$ matrices.

(b) The vector space of all symmetric $n \times n$ matrices.

(c) The vector space of all upper triangular $n \times n$ matrices.

Solution. (a) Let D_{nn} denote the specified vector space of diagonal matrices. One basis for D_{nn} consists of the n different $n \times n$ matrices, each of which has exactly one of the diagonal entries equal to 1 and all other entries in the matrix equal to 0. [For D_{33}, for instance, this basis would use the 3 matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

as basis vectors.
Thus D_{nn} has dimension n.

(b) Let S_{nn} denote the specified vector space of symmetric matrices. Let's look at a special case first, just to get oriented. Consider S_{33} for instance. For a particular 3×3 matrix to belong to S_{nn} the entries on the main diagonal can be anything, but the entries off the main diagonal must be paired up to match symmetrically. A typical matrix in S_{33} would look like

$$\begin{bmatrix} a & x & y \\ x & b & z \\ y & z & c \end{bmatrix}$$

One basis for S_{33} would use the following 6 matrices as the basis vectors:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

So S_{33} would have dimension 6.

More generally to create a basis for S_{nn} we could use the following:

- n diagonal matrices, each having a single 1 on the diagonal and all other diagonal entries 0
- $n-1$ matrices, each having exactly one first-row entry above the diagonal equalling 1, the symmetric entry equalling 1, and all other entries equalling 0,
- $n-2$ matrices, each having exactly one second-row entry above the diagonal equalling 1, the symmetric entry equalling 1, and all other entries equalling 0,
- and so forth, continuing through
- 1 matrix having the very last entry of the next to last row equalling 1, the symmetric entry in the last row equalling 1, and all other entries equalling 0.

This gives

$$n + (n-1) + (n-2) + \ldots + (n-(n-1))$$

matrices in the basis. The total number of basis vectors is therefore

$$n^2 - (1 + 2 + \ldots + (n-1)) = n^2 - \frac{n(n-1)}{2} = \frac{n^2+n}{2} = \frac{n(n+1)}{2}.$$

The combinatorialists among you may well find a faster way to count these. In any case,

$$\dim(S_{nn}) = \frac{n(n+1)}{2}.$$

(c) A variation on the reasoning used in (b) gives the dimension to be $\frac{n(n+1)}{2}$. Just leave out the bit about making the relevant symmetric entry equal to 1.

Section 5.4 p244 Problem 4b. Do the vectors/functions $4+6x+x^2, -1+4x+2x^2, 5+2x-x^2$ form a basis for P_2?

Solution. Once again we have the correct count, since P_2 has dimension 3. We need only test for independence. So consider the condition

$$k_1(4+6x+x^2) + k_2(-1+4x+2x^2) + k_3(5+2x-x^2) = 0$$

for every x. This is equivalent to the system

$$\begin{bmatrix} 4 & -1 & 5 \\ 6 & 4 & 2 \\ 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

This time we find

$$\det \begin{bmatrix} 4 & -1 & 5 \\ 6 & 4 & 2 \\ 1 & 2 & -1 \end{bmatrix} = 0$$

According to the Long Theorem, we have a homogeneous system with nontrivial solutions. Therefore the given three vectors/functions form a dependent set, hence do not form a basis.

Section 5.4 p244 Problem 12. Find a basis for and the dimension of the solution space for the homogeneous system

$$3x_1 + x_2 + x_3 + x_4 = 0$$

$$5x_1 - x_2 + x_3 - x_4 = 0$$

Solution. First solve the given system, using row reduction. The augmented matrix for the given system is

$$\left[\begin{array}{cccc|c} 3 & 1 & 1 & 1 & 0 \\ 5 & -1 & 1 & -1 & 0 \end{array}\right]$$

which reduces to

$$\left[\begin{array}{cccc|c} 1 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 1 & \frac{1}{4} & 1 & 0 \end{array}\right].$$

The nonleading variables are x_3 and x_4. Setting $x_3 = s$ and $x_4 = t$, and solving for the leading variables in terms of the nonleading ones, we find a parametric description for the solutions. The solutions are of the form

$$(x_1, x_2, x_3, x_4) = s(-\frac{1}{4}, -\frac{1}{4}, 1, 0,) + t(0, -1, 0, 1).$$

We can read off from this description that the vectors

$$(-1, -1, 4, 0), \quad (0, -1, 0, 1)$$

span the solution space. By inspection, we can see that they form an independent set (why?). Therefore the set

$$\{(-1, -1, 4, 0), (0, -1, 0, 1)\}$$

is a basis for the solution space, which has dimension 2. Other correct answers are possible.

Let T be a linear transformation from a vector space V over reals into V such that $T - T^2 = I$. Show that T is invertible

$T - T^2 = I$, so $(I - T)T = I \implies ST = I$ with $S = (I - T)$ which means T is left invertible.

Similarly, $T - T^2 = I$, so $T(I - T) = I \implies TU = I$ with $U = (I - T)$ which means T is right invertible, and hence invertible where $T^{-1} = I - T$.

Problem 76. *If* $\mathbf{H}$ *is a Hermitian matrix, what kind of matrix is* $e^{i\mathbf{H}}$*?*

(*Rohilkhand, 1992; Meerut, 1969*)

Hint: $\mathbf{H}$ is Hermitian $\Rightarrow \mathbf{H}^{\Theta} = \mathbf{H}$

Consider $(e^{i\mathbf{H}})^{\Theta} \,.\, e^{i\mathbf{H}} = e^{-i}\mathbf{H}^{\Theta}.\, e^{i\mathbf{H}} = e^{-i\mathbf{H}}\, e^{i\mathbf{H}} = e^{\circ} = \mathbf{I} \Rightarrow e^{i\mathbf{H}}$ is unitary.

Problem 77. *Show that product of two orthogonal matrices is also orthogonal.*

Hint: $\mathbf{A}$, $\mathbf{B}$ are orthogonal $\Rightarrow \mathbf{AA}' = \mathbf{I} = \mathbf{A}'\mathbf{A}$ and $\mathbf{BB}' = \mathbf{I} = \mathbf{B}'\mathbf{B}$.

$\therefore \quad (\mathbf{AB})'(\mathbf{AB}) = \mathbf{B}'\mathbf{A}'\mathbf{AB} = \mathbf{B}'\mathbf{IB} = \mathbf{B}'\mathbf{B} = \mathbf{I} \Rightarrow \mathbf{AB}$ is orthogonal.

- **Problem 1**

Let V be the subspace of R^4 generated by the vectors $v_1 = (1,1,0,0), v_2 = (0,1,1,0), v_3 = (0,0,1,1)$ and W generated by the vectors $w_1 = (1,0,1,0), w_2 = (0,2,1,1), w_3 = (1,2,1,2)$ in $\mathbb{R}^4$.

(a) Determine the dimensions of V and W.

Solution. $\dim(V) = 3$ since $V =$ span$\{v_1, v_2, v_3\}$ and the vectors v_1, v_2, v_3 are lin. independent. Indeed, $a_1v_1 + a_2v_2 + a_3v_3 = 0 \Rightarrow a_1(1,1,0,0) + a_2(0,1,1,0) + a_3(0,0,1,1) = (0,0,0,0) \Rightarrow (a_1, a_1 + a_2, a_2 + a_3, a_3) = (0,0,0,0) \Rightarrow a_1 = 0, a_2 = 0, a_3 = 0$.

Same argument for W: $\dim(W) = 3$ since w_1, w_2, w_3 are lin. ind. (a similar argument is needed!)

(b) Find a basis for the sum $V + W$.

Solution. Since $\{v_1, v_2, v_3\}$ is lin. ind. and $w_1 \notin$ span$\{v_1, v_2, v_3\}$ (This needs proof!) then, by Theorem 1.7, $\{v_1, v_2, v_3, w_1\}$ is lin. ind. in $\mathbb{R}^4$. Hence span$\{v_1, v_2, v_3, w_1\} = \mathbb{R}^4$. Now $\{v_1, v_2, v_3, w_1\} \subset V + W$ means that $V + W = \mathbb{R}^4$. We conclude that $\{v_1, v_2, v_3, w_1\}$ is a basis for $V + W$.

(c) Find a basis for the intersection $V \cap W$. Verify that $\dim(V + W) + \dim(V \cap W) = \dim(V) + \dim(W)$.

Solution. For a vector v to belong to $V \cap W$, it is necessary and sufficient to have v as a linear combination of both bases $\{v_1, v_2, v_3\}$ (for V) and $\{w_1, w_2, w_3\}$ for W:

$$v = a_1v_1 + a_2v_2 + a_3v_3 = b_1w_1 + b_2w_2 + b_3w_3, \qquad \text{for } a_i, b_i \in \mathbb{R}, i = 1,2,3.$$

Hence a's and b's are solutions to the underdetermined linear system

$$a_1\begin{pmatrix}1\\1\\0\\0\end{pmatrix} + a_2\begin{pmatrix}0\\1\\1\\0\end{pmatrix} + a_3\begin{pmatrix}0\\0\\1\\1\end{pmatrix} = b_1\begin{pmatrix}1\\0\\1\\0\end{pmatrix} + b_2\begin{pmatrix}0\\2\\1\\1\end{pmatrix} + b_3\begin{pmatrix}1\\2\\1\\2\end{pmatrix}$$

Performing reduced row echelon form on the matrix $\begin{pmatrix}1&0&0&1&0&1\\1&1&0&0&2&2\\0&1&1&1&1&1\\0&0&1&0&1&2\end{pmatrix}$ yields: $\begin{pmatrix}1&0&0&0&1&2\\0&1&0&0&1&0\\0&0&1&0&1&2\\0&0&0&1&-1&-1\end{pmatrix}$

hence the system above is equivalent to

$$a_1\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + a_2\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + a_3\begin{pmatrix}0\\0\\1\\0\end{pmatrix} = b_1\begin{pmatrix}0\\0\\0\\1\end{pmatrix} + b_2\begin{pmatrix}1\\1\\1\\-1\end{pmatrix} + b_3\begin{pmatrix}2\\0\\2\\-1\end{pmatrix},$$

or $a_1 = b_2 + 2b_3, a_2 = b_2, a_3 = b_2 + 2b_3, 0 = b_1 - b_2 - b_3$. As a consequence, we can choose arbitrarily $b_2 = s$ and $b_3 = t$, then $a_1 = s + 2t, a_2 = s, a_3 = s + 2t, b_1 = s + t$.

Then every $v \in V \cap W$ can be expressed as $v = (s+2t)v_1 + sv_2 + (s+2t)v_3 = (s+t)w_1 + sw_2 + tw_3$. or $v = s(v_1 + v_2 + v_3) + t(2v_1 + 2v_3) = s(w_1 + w_2) + t(w_1 + w_3)$. It means that a basis for $V \cap W$ consists of the two vectors $v_1 + v_2 + v_3 = w_1 + w_2 = (1,2,2,1)$ and $2v_1 + 2v_3 = w_1 + w_3 = (2,2,2,2)$.

One verifies that $\dim(V + W) + \dim(V \cap W) = 4 + 2 = 6 = 3 + 3 = \dim(V) + \dim(W)$. Q.E.D.

- **Problem 5**

Given a 2×2 matrix $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, define the map $T : \mathcal{M}_{2\times 2}(\mathbb{R}) \to \mathcal{M}_{2\times 2}(\mathbb{R})$ given by left multiplication by P:

$$T(A) = PA, \quad \text{for } A \in \mathcal{M}_{2\times 2}(\mathbb{R}).$$

(a) Show that T is linear.

Solution. We easily verify $T(A_1 + A_2) = P(A_1 + A_2) = PA_1 + PA_2 = T(A_1) + T(A_2)$ and $T(cA) = PcA = cPA = cT(A)$ for $c \in \mathbb{R}$, by using the properties of matrix addition and multiplication.

(b) Determine the matrix representation of T in the standard ordered basis for $\mathcal{M}_{2\times 2}(\mathbb{R})$

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

Solution. We compute $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix}$ etc and therefore obtain

$$[T] = \begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \end{pmatrix}.$$

Example. Let $W = \mathbb{R}^5$. Consider the two subspaces

$$\begin{aligned} U &= \mathscr{L}\{(x_1, x_2, x_3, x_4, x_5) : 2x_1 - x_2 - x_3 = 0 = x_4 - 3x_5\}, \\ V &= \mathscr{L}\{(x_1, x_2, x_3, x_4, x_5) : x_3 + x_4 = 0\}. \end{aligned}$$

We are required to find a basis of $\mathbb{R}^5$ that contains *both* a basis of U *and* a basis of V. The trick is to start by finding a basis of $U \cap V$. It is easy to see that $\dim U = 3$ and $\dim V = 4$; this is because the homogeneous linear systems have rank 2 and 1. Now, a vector $\mathbf{v} \in \mathbb{R}^5$ belongs to $U \cap V$ iff it satisfes *all three* equations. Since the associated matrix

$$A = \begin{pmatrix} 2 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix} \sim \begin{pmatrix} \boxed{1} & -\frac{1}{2} & 0 & 0 & \frac{3}{2} \\ 0 & 0 & \boxed{1} & 0 & 3 \\ 0 & 0 & 0 & \boxed{1} & -3 \end{pmatrix}$$

has rank 3, we deduce that $\dim(U \cap V) = 5 - 3 = 2$. Indeed, we may take $x_2 = s$ and $x_5 = t$ to be free variables and obtain (as a row) the general solution

$$\mathbf{v} = \left(\tfrac{1}{2}s - \tfrac{3}{2}t,\ s,\ -3t,\ 3t,\ t\right).$$

A basis of $U \cap V$ consists of

$$\mathbf{w}_1 = (\tfrac{1}{2}, 1, 0, 0, 0), \qquad \mathbf{w}_2 = (-\tfrac{3}{2}, 0, -3, 3, 1)$$

(take first $s = 1$, $t = 0$ and second $s = 0$, $t = 1$). Extend this basis in any way to

a basis $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ of U, and

a basis $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4, \mathbf{w}_5\}$ of V, and

Then $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3, \mathbf{w}_4, \mathbf{w}_5\}$ will *always* be LI and thus a basis of $\mathbb{R}^5$. There are lots of choices in this example, but we could take

$\mathbf{w}_3 = (0, -1, 1, 0, 0)$ (this works since $\mathbf{w}_3 \in U$ but $\mathbf{w}_3 \notin \mathscr{L}\{\mathbf{w}_1, \mathbf{w}_2\}$),

$\mathbf{w}_4 = (0, 0, 1, -1, 0)$, $\mathbf{w}_5 = (0, 0, 0, 0, 1)$ (note that $\mathbf{w}_5 \notin \mathscr{L}\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_4\}$).

In conclusion, $U + V = \mathbb{R}^5$, and the required basis is

$$\rlap{$\overbrace{\phantom{\mathbf{w}_5 \quad \mathbf{w}_1 \quad \mathbf{w}_2}}^{U}$}\mathbf{w}_5 \quad \underbrace{\mathbf{w}_1 \quad \mathbf{w}_2 \quad \mathbf{w}_3 \quad \mathbf{w}_4}_{V}$$

4.4.14 The set $\mathcal{B} = \{1 - t^2, t - t^2, 2 - 2t + t^2\}$ is a basis for $\mathbb{P}_2$. Find the coordinate vector of $\mathbf{p}(t) = 3 + t - 6t^2$ relative to $\mathcal{B}$.

We need to write $\mathbf{p}$ in terms of the basis $\mathcal{B}$, that is, find $x_1, x_2, x_3 \in \mathbb{R}$ such that $x_1(1 - t^2) + x_2(t - t^2) + x_3(2 - 2t + t^2) = 3 + t - 6t^2$. Multiplying things out, we get $(x_1 + 2x_3) + (x_2 - 2x_3)t + (-x_1 - x_2 + x_3)t^2 = 3 + t - 6t^2$. Thus we have to solve the three linear equations:

$$\begin{aligned} x_1 + 2x_3 &= 3 \\ x_2 - 2x_3 &= 1 \\ -x_1 - x_2 + x_3 &= -6 \end{aligned}$$

We form the augmented matrix for this system,

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -2 & 1 \\ -1 & -1 & 1 & -6 \end{bmatrix}.$$

In row reduced echelon form, this is the matrix $\begin{bmatrix} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$. So we see that $x_1 = 7$, $x_2 = -3$, and $x_3 = -2$.

This implies that $7(1 - t^2) + (-3)(t - t^2) + (-2)(2 - 2t + t^2) = 3 + t - 6t^2$, so

$$[3 + t - 6t^2]_{\mathcal{B}} = \begin{bmatrix} 7 \\ -3 \\ -2 \end{bmatrix} \in \mathbb{R}^3.$$

Example 3. Let $V = \mathbb{R}^2$ and $W = \mathbb{R}^3$. Define $L\colon\ V \to W$ by $L(x_1, x_2) = (x_1 - x_2, x_1, x_2)$. Let $F = \{(1,1), (-1,1)\}$, and let $G = \{(1,0,1),\ (0,1,1),\ (1,1,0)\}$.

a. Find the matrix representation of L using the standard bases in both V and W.

b. Find the matrix representation of L using the standard basis in V and the basis G in W.

c. Find the matrix representation of L using the basis F in $\mathbb{R}^2$ and the standard basis in $\mathbb{R}^3$.

d. Find the matrix representation of L using the bases F and G.

Solution.

a. $L(\boldsymbol{e}_1) = L(1,0) = (1,1,0) = \boldsymbol{e}_1 + \boldsymbol{e}_2$
$L(\boldsymbol{e}_2) = L(0,1) = (-1,0,1) = -\boldsymbol{e}_1 + \boldsymbol{e}_3$

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

b. $L(\boldsymbol{e}_1) = (1,1,0) = 0(1,0,1) + 0(0,1,1) + (1,1,0) = 0\boldsymbol{g}_1 + 0\boldsymbol{g}_2 + \boldsymbol{g}_3$
$L(\boldsymbol{e}_2) = (-1,0,1) = 0(1,0,1) + (0,1,1) - (1,1,0) = 0\boldsymbol{g}_1 + \boldsymbol{g}_2 - \boldsymbol{g}_3$

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}$$

c. $L(\boldsymbol{f}_1) = L(1,1) = (0,1,1) = \boldsymbol{e}_2 + \boldsymbol{e}_3$
$L(\boldsymbol{f}_2) = L(-1,1) = (-2,-1,1) = -2\boldsymbol{e}_1 - \boldsymbol{e}_2 + \boldsymbol{e}_3$

$$A = \begin{bmatrix} 0 & -2 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}$$

d. $L(\boldsymbol{f}_1) = 0\boldsymbol{g}_1 + \boldsymbol{g}_2 + 0\boldsymbol{g}_3$
$L(\boldsymbol{f}_2) = 0\boldsymbol{g}_1 + \boldsymbol{g}_2 - 2\boldsymbol{g}_3$

$$A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & -2 \end{bmatrix} \qquad \square$$

Example 5. Let $F = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$. Thus F is the standard basis of M_{22}. Let $B = \begin{bmatrix} -2 & 1 \\ 3 & 4 \end{bmatrix}$. Define $L\colon\ M_{22} \to M_{22}$ by $L(\boldsymbol{x}) = B\boldsymbol{x}$. Find the matrix representation of L with respect to the standard basis F of M_{22}.

$$\begin{aligned}
L(\boldsymbol{f}_1) = B\boldsymbol{f}_1 &= \begin{bmatrix} -2 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 3 & 0 \end{bmatrix} \\
&= -2\boldsymbol{f}_1 + 3\boldsymbol{f}_3 \\
L(\boldsymbol{f}_2) = B\boldsymbol{f}_2 &= \begin{bmatrix} -2 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 0 & 3 \end{bmatrix} \\
&= -2\boldsymbol{f}_2 + 3\boldsymbol{f}_4 \\
L(\boldsymbol{f}_3) = B\boldsymbol{f}_3 &= \begin{bmatrix} -2 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4 & 0 \end{bmatrix} \\
&= \boldsymbol{f}_1 + 4\boldsymbol{f}_3 \\
L(\boldsymbol{f}_4) = B\boldsymbol{f}_4 &= \begin{bmatrix} -2 & 1 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 4 \end{bmatrix} \\
&= \boldsymbol{f}_2 + 4\boldsymbol{f}_4
\end{aligned}$$

Thus, the matrix representation of L is

$$\begin{bmatrix} -2 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \\ 3 & 0 & 4 & 0 \\ 0 & 3 & 0 & 4 \end{bmatrix}$$

$\square$

Cor. 2. *If A is an n-square non-singular matrix, there exist non-singular matrices, P and Q such that PAQ =* I_n.

Proof. Since every matrix is equivalent to its normal form

$\therefore$ $$A \sim I_n.$$

$\therefore$ there exist non-singular matrices P and Q (as in the above theorem) such that

$$PAQ = I_n.$$

Cor. 3. *Every non-singular matrix can be expressed as a product of elementary matrices.*

Proof. Let A be an n-square non-singular matrix.

$\therefore$ there exist non-singular matrices P, Q which are products of elementary matrices such that

$$PAQ = I_n$$

9. If $A = \begin{bmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{bmatrix}$, find A^{-1}. Also find two non-singular matrices P and Q such that PAQ $= \mathbb{I}$, where $\mathbb{I}$ is the unit matrix and verify that $A^{-1} = QP$.

Solution

The combined matrix [A I] is given by

$$\begin{bmatrix} 3 & -3 & 4 & 1 & 0 & 0 \\ 2 & -3 & 4 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

By further row operations,

$$\sim R_1 \mapsto R_1 - R_2 \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 0 \\ 2 & -3 & 4 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{bmatrix} \sim R_2 \mapsto 2R_2 - R_1 \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & -3 & 4 & -2 & 3 & 0 \\ 0 & -1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

$$\sim R_3 \mapsto R_2 - 3R_3 \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & -3 & 4 & -2 & 3 & 0 \\ 0 & 0 & 1 & -2 & 3 & -3 \end{bmatrix} \sim R_2 \mapsto R_2 - 4R_3;\, R_2 \mapsto \frac{R_2}{-3} \begin{bmatrix} 1 & 0 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -2 & 3 & -4 \\ 0 & 0 & 1 & -2 & 3 & -3 \end{bmatrix}$$

So finally $A^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ -2 & 3 & -4 \\ -2 & 3 & -3 \end{bmatrix}$

Now second part of the question is to find Q and P such that QAP $= \mathbb{I}$. The most straight forward answer is to choose either P or Q to be A^{-1} and to choose the other as $\mathbb{I}$.So one of the choice will be

$$P = A^{-1} \qquad \text{and} \qquad Q = \mathbb{I} \tag{0.0.1}$$

Then

$$PAQ = A^{-1}A\mathbb{I} = \mathbb{I} \times \mathbb{I} = \mathbb{I} \tag{0.0.2}$$

It also satisfies the condition

$$PQ = A^{-1}\mathbb{I} = A^{-1} \tag{0.0.3}$$

We also note that there are infinitely many choice for P and Q which satisfies the condition in the question.

Example 2. Find a non-singular matrices P and Q such that PAQ is in the normal form for the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}$$

Let $\quad A = I_3 A I_4$

$$\begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} C_2 - 2C_1 \\ C_3 - 3C_1 \\ C_4 + 2C_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -6 & -5 & 7 \\ 3 & -6 & -5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -5 & 7 \\ 0 & -6 & -5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} R_3 - R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -5 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} -1/6\,C_2 \\ -1/5\,C_3 \\ 1/7\,C_4 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/3 & 3/5 & 2/7 \\ 0 & -1/6 & 0 & 0 \\ 0 & 0 & -1/5 & 0 \\ 0 & 0 & 0 & 1/7 \end{bmatrix} \begin{matrix} C_3 - C_2 \\ C_4 - C_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/3 & 4/15 & -1/21 \\ 0 & -1/6 & 1/6 & 1/6 \\ 0 & 0 & -1/5 & 0 \\ 0 & 0 & 0 & 1/7 \end{bmatrix}$$

$$\therefore \qquad P = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 1/3 & 4/15 & -1/21 \\ 0 & -1/6 & 1/6 & 1/6 \\ 0 & 0 & -1/5 & 0 \\ 0 & 0 & 0 & 1/7 \end{bmatrix}$$

where both P and Q are non-singular.

Problem #7 Determine a non-singular matrix P that is $|P| \neq 0$ such that $P'AP$ is a Diagonal Matrix where $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & -3 \end{bmatrix}$ interpret the result in terms of Quadratic Form?

Solution: Let A = IAI $\Rightarrow \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Reduce matrix A to Diagonal Matrix by congruent operations on R.H.S IAI, Elementary row pre-factor on I and Elementary column post-factor on I we get as below :

$R_2 \to R_2 + (1/3)R_1$; $C_2 \to C_2 + (1/3)C_1$; and $R_3 \to R_3 - (1/3)R_1$; $C_3 \to C_3 - (1/3)C_1 \Rightarrow$

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 7/3 & -1/3 \\ 0 & -1/3 & 7/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ -1/3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/3 & -1/3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 + (1/7)R_2$; $C_3 \to C_3 + (1/7)C_2 \Rightarrow$

$$\begin{bmatrix} 6 & 0 & 0 \\ 0 & 7/3 & 0 \\ 0 & 0 & 16/3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/3 & 1 & 0 \\ -2/7 & 1/7 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/3 & -2/7 \\ 0 & 1 & 1/7 \\ 0 & 0 & 1 \end{bmatrix}$$

On observation we obtain a non-singular matrix $P = \begin{bmatrix} 1 & 1/3 & -2/7 \\ 0 & 1 & 1/7 \\ 0 & 0 & 1 \end{bmatrix}$ such that $P'AP = D$

Where D = Diag. [6 7/3 16/3].

Quadratic form corresponding to matrix A = X′AX

$= 6x_1^2 + 3x_2^2 + 3x_3^2 - 4x_1x_2 - 2x_2x_3 + x_3x_1$

.... equation (1)

Therefore, the on-singular Transformation corresponding to P is X = PY i.e.,

$$X = PY \quad \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 1/3 & -2/7 \\ 0 & 1 & 1/7 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

Equivalent to write in the form of linear system of equations

$$\boxed{x_1 = y_1 + \frac{1}{3}y_2 - \frac{2}{7}y_3; \quad x_2 = \quad y_2 + \frac{1}{7}y_3; \quad x_3 = \quad y_3 \text{............ equation (2)}}$$

the transformation (2) reduce to Quadratic Form (1) to Diag. Form

$$\boxed{Y'P'AP = 6y_1^2 + 7/3\, y_2^2 + 16/7\, y_3^2}$$

Rank of Quadratic Form = ρ (Q.F) = X′AX = 3 = No. of positive square terms in Q.F.
Signature = sign = No. Of positive terms – No. Of negative terms = 3 – 0 = 3.
Hence the solution.

Example 2. Find a non-singular matrices P and Q such that PAQ is in the normal form for the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix}$$

Let $\qquad A = I_3 A I_4$

$$\begin{bmatrix} 1 & 2 & 3 & -2 \\ 2 & -2 & 1 & 3 \\ 3 & 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} C_2 - 2C_1 \\ C_3 - 3C_1 \\ C_4 + 2C_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & -6 & -5 & 7 \\ 3 & -6 & -5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} R_2 - 2R_1 \\ R_3 - 3R_1 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -5 & 7 \\ 0 & -6 & -5 & 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} R_3 - R_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -6 & -5 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & -2 & -3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{matrix} -1/6\, C_2 \\ -1/5\, C_3 \\ 1/7\, C_4 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/3 & 3/5 & 2/7 \\ 0 & -1/6 & 0 & 0 \\ 0 & 0 & -1/5 & 0 \\ 0 & 0 & 0 & 1/7 \end{bmatrix} \begin{matrix} C_3 - C_2 \\ C_4 - C_2 \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 1/3 & 4/15 & -1/21 \\ 0 & -1/6 & 1/6 & 1/6 \\ 0 & 0 & -1/5 & 0 \\ 0 & 0 & 0 & 1/7 \end{bmatrix}$$

$$\therefore \qquad P = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & -1 & 1 \end{bmatrix}, \; Q = \begin{bmatrix} 1 & 1/3 & 4/15 & -1/21 \\ 0 & -1/6 & 1/6 & 1/6 \\ 0 & 0 & -1/5 & 0 \\ 0 & 0 & 0 & 1/7 \end{bmatrix}$$

where both P and Q are non-singular.

Example: We considered the matrix

$$A = \begin{pmatrix} 3 & -1 & 1 \\ 0 & 2 & 0 \\ 1 & -1 & 3 \end{pmatrix}$$

above, and we saw that it has only two eigenvalues, 4 and 2. If we want to diagonalise it, we need to find three linearly independent eigenvectors. We found that an eigenvector corresponding to $\lambda = 4$ is

$$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix},$$

and that, for $\lambda = 2$, the eigenvectors are given by the non-zero-vector solutions to the system consisting of just the single equation $x_1 - x_2 + x_3 = 0$. Above, we simply wanted to find **an** eigenvector, but now we want to find two which, together with the eigenvector for $\lambda = 4$, form a linearly independent set. Now, the system for the eigenvectors corresponding to $\lambda = 2$ has just one equation and is therefore of rank 1; it follows that the solution set is two-dimensional. Let's see exactly what the general solution looks like. We have $x_1 = x_2 - x_3$, and x_2, x_3 can be chosen independently of each other. Setting $x_3 = r$ and $x_2 = s$, we see that the general solution is

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} s - r \\ s \\ r \end{pmatrix} = s \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + r \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}, \quad (r, s \in \mathbf{R}).$$

This shows that the solution space (the **eigenspace**, as it is called in this instance) is spanned by the two linearly independent vectors

$$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Now, each of these is an eigenvector corresponding to eigenvalue 2 and, together with our eigenvector for $\lambda = 4$, the three form a linearly independent set. So there are

three linearly independent eigenvectors, even though two of them correspond to the same eigenvalue. The matrix is therefore diagonalisable. We may take

$$P = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}.$$

Then (Check!) $P^{-1}AP = D = \mathrm{diag}(4, 2, 2)$.

Example: Let

$$A = \begin{pmatrix} 7 & 0 & 9 \\ 0 & 2 & 0 \\ 9 & 0 & 7 \end{pmatrix}.$$

Note that A is symmetric. We find an orthogonal matrix P such that $P^T AP$ is a diagonal matrix. The characteristic polynomial of A is

$$\begin{aligned} |A - \lambda I| &= \begin{vmatrix} 7-\lambda & 0 & 9 \\ 0 & 2-\lambda & 0 \\ 9 & 0 & 7-\lambda \end{vmatrix} \\ &= (2-\lambda)[(7-\lambda)(7-\lambda) - 81] \\ &= (2-\lambda)(\lambda^2 - 14\lambda - 32) \\ &= (2-\lambda)(\lambda - 16)(\lambda + 2), \end{aligned}$$

where we have expanded the determinant using the middle row. So the eigenvalues are $2, 16, -2$. An eigenvector for $\lambda = 2$ is given by

$$5x + 9z = 0, \quad 9x + 5z = 0.$$

This means $x = z = 0$. So we may take $(0, 1, 0)^T$. This already has length 1 so there is no need to normalise it. (Recall that we need three eigenvectors which are of length 1.) For $\lambda = -2$ we find that an eigenvector is $(-1, 0, 1)^T$ (or some multiple of this). To normalise (that is, to make of length 1), we divide by its length, which is $\sqrt{2}$, obtaining $(1/\sqrt{2})(-1, 0, 1)^T$. For $\lambda = 16$, we find a normalised eigenvector is $(1/\sqrt{2})(1, 0, 1)$. It follows that if we let

$$P = \begin{pmatrix} 0 & -1/\sqrt{2} & 1/\sqrt{2} \\ 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix},$$

then P is orthogonal and $P^T AP = D = \text{diag}(2, -2, 16)$. Check this!

Example 9.11. Find the minimal polynomial $m(t)$ of $A = \begin{bmatrix} 2 & 2 & -5 \\ 3 & 7 & -15 \\ 1 & 2 & -4 \end{bmatrix}$.

First find the characteristic polynomial $\Delta(t)$ of A. We have

$$\text{tr}(A) = 5, \qquad A_{11} + A_{22} + A_{33} = 2 - 3 + 8 = 7 \qquad \text{and} \qquad |A| = 3$$

Hence

$$\Delta(t) = t^3 - 5t^2 + 7t - 3 = (t-1)^2(t-3)$$

The minimal polynomial $m(t)$ must divide $\Delta(t)$. Also, each irreducible factor of $\Delta(t)$, that is, $t - 1$ and $t - 3$, must also be a factor of $m(t)$. Thus $m(t)$ is exactly one of the following:

$$f(t) = (t-3)(t-1) \qquad \text{or} \qquad g(t) = (t-3)(t-1)^2$$

We know, by the Cayley–Hamilton Theorem, that $g(A) = \Delta(A) = 0$. Hence we need only test $f(t)$. We have

$$f(A) = (A - I)(A - 3I) = \begin{bmatrix} 1 & 2 & -5 \\ 3 & 6 & -15 \\ 1 & 2 & -5 \end{bmatrix} \begin{bmatrix} -1 & 2 & -5 \\ 3 & 4 & -15 \\ 1 & 2 & -7 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus $f(t) = m(t) = (t-1)(t-3) = t^2 - 4t + 3$ is the minimal polynomial of A.

Example 9.14. Find the characteristic polynomal $\Delta(t)$ and the minimal polynomial $m(t)$ of the block diagonal matrix

$$M = \left[\begin{array}{cc|cc|c} 2 & 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ \hline 0 & 0 & 4 & 2 & 0 \\ 0 & 0 & 3 & 5 & 0 \\ \hline 0 & 0 & 0 & 0 & 7 \end{array}\right] = \text{diag}(A_1, A_2, A_3), \text{ where } A_1 = \begin{bmatrix} 2 & 5 \\ 0 & 2 \end{bmatrix}, A_2 = \begin{bmatrix} 4 & 2 \\ 3 & 5 \end{bmatrix}, A_3 = [7]$$

Then $\Delta(t)$ is the product of the characterization polynomials $\Delta_1(t)$, $\Delta_2(t)$, $\Delta_3(t)$ of A_1, A_2, A_3, respectively. One can show that

$$\Delta_1(t) = (t-2)^2, \qquad \Delta_2(t) = (t-2)(t-7), \qquad \Delta_3(t) = t - 7$$

Thus $\Delta(t) = (t-2)^3(t-7)^2$. [As expected, $\deg \Delta(t) = 5$.]

The minimal polynomials $m_1(t)$, $m_2(t)$, $m_3(t)$ of the diagonal blocks A_1, A_2, A_3, respectively, are equal to the characteristic polynomials, that is,

$$m_1(t) = (t-2)^2, \qquad m_2(t) = (t-2)(t-7), \qquad m_3(t) = t - 7$$

But $m(t)$ is equal to the least common multiple of $m_1(t), m_2(t), m_3(t)$. Thus $m(t) = (t-2)^2(t-7)$.

Problem 2.4.28. Construct a matrix with $(1, 0, 1)$ and $(1, 2, 0)$ as a basis for its row space and its column space. Why can't this be a basis for the row space and nullspace?

Solution: Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -4 & 2 \\ 1 & 2 & 0 \end{bmatrix}.$$

Notice, first of all, that A is symmetric, so a basis for its column space will also be a basis for its row space (and *vice versa*). Notice that the second column is twice the first column minus twice the third column, so we can ignore the second column for the purposes of finding a basis for the column space.

Also, the vectors $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ are certainly linearly independent since they aren't multiples of each other. Therefore, these two columns do indeed give a basis for the column space of A (and, by symmetry, for the row space of A).

Now, I claim that this cannot be a basis for the row space and the nullspace of any 3×3 matrix A. To see why, remember that the dimension of the row space of A is equal to the rank of A, whereas the dimension of the nullspace is equal to the number of columns minus the rank, namely

$$3 - \text{rank}(A).$$

If these two vectors are a basis for both the row space and the nullspace, then both the row space and the nullspace must be two-dimensional, meaning that

$$\text{rank}(A) = 2 \quad \text{and} \quad 3 - \text{rank}(A) = 2,$$

meaning that the rank of A must be both 2 and 1 simultaneously. This is clearly impossible, so we see that the row space and the nullspace can't both be two-dimensional, so no two vectors can be a basis for both the row space and the nullspace of any 3×3 matrix.

Example 6. Show that there are two values of λ for which the equations

$$\begin{aligned} 3x - \lambda y - 8 &= 0, \\ 2\lambda x + 3y - 5 &= 0, \\ 5x + 4\lambda y - 2 &= 0 \end{aligned}$$

are consistent, and find these values. Find also the corresponding values of x and y.

$$\Delta = \begin{vmatrix} 3 & -\lambda & -8 \\ 2\lambda & 3 & -5 \\ 5 & 4\lambda & -2 \end{vmatrix} = \begin{vmatrix} -17 & -17\lambda & 0 \\ 2\lambda - \frac{25}{2} & 3-10\lambda & 0 \\ 5 & 4\lambda & -2 \end{vmatrix} = 3\lambda \begin{vmatrix} 1 & \lambda \\ 24 - \frac{25}{2} & 3-10\lambda \end{vmatrix}$$

$$= 34(3 - 10\lambda + \tfrac{25}{2}\lambda - 2\lambda^2) = -17(4\lambda^2 - 5\lambda - 6).$$

For consistency

$$4\lambda^2 - 5\lambda - 6 = (4\lambda+3)(\lambda-2) = 0.$$

When $\lambda = 2$, $3x - 2y - 8 = 0$, $4x + 3y - 5 = 0$, $5x + 8y - 2 = 0$.

In this case $x = 2$, $y = -1$.

When $\lambda = -\frac{3}{4}$, $12x + 3y - 32 = 0$, $-3x + 6y - 10 = 0$, $5x - 3y - 2 = 0$.

In this case $x = 2$, $y = \frac{8}{3}$.

In two-dimenensional analytical geometry an equation of the first degree represents a straight line, so that, for an arbitrary value of λ, the above three equations represent three straight lines. When Δ vanishes these lines are concurrent, the coordinates of their meeting point being given by the solution for x and y in each case.

For a strictly complex matrix A,

1) Can we comment on determinant of A^* (conjugate of entries of A) , A^T (transpose of A) and A^H (hermitian of A). I know that for real matrices, $\det(A) = \det(A^T)$. Does it carry over to complex matrices, i.e. does $\det(A) = \det(A^T)$ in general? I understand $\det(A) = \det(A^H)$ (from Schur triangularization).

2) The same question as first, now about eigenvalues of A. I would like to know about special cases, for instance what if A is hermitian or positive definite and so on.

Since complex conjugation satisfies $\overline{xy} = \overline{x} \cdot \overline{y}$ and $\overline{x+y} = \overline{x} + \overline{y}$, you can see with the Leibniz formula quickly that $\det[A^*] = \overline{\det[A]}$.

For complex matrices $\det[A] = \det[A^T]$ still holds and doesn't require any changes to the proof for real matrices.

Together this means that $\det[A] = \overline{\det[A^H]}$.

This applies to the eigenvalues as well: the characteristic polynomial of A^* is given by $\det[tI - A^*] = \det[(\bar{t}I - A)^*] = \overline{\det[\bar{t}I - A]}$ and the eigenvalues of A^* are exactly the complex conjugates of those of A.

In particular if A is hermitian, $A = A^*$ and so all eigenvalues are equal to their complex conjugates - in other words, they're real.

2. Let W_1 and W_2 be subspaces of a vector space V. The *sum* of W_1 and W_2 is the subset of V defined by

$$W_1 + W_2 = \{\mathbf{w}_1 + \mathbf{w}_2 \in V \mid \mathbf{w}_1 \in W_1, \mathbf{w}_2 \in W_2\}.$$

(a) Prove that $W_1 + W_2$ is a subspace of V.

SOLUTION. Note that $\mathbf{0} = \mathbf{0} + \mathbf{0} \in W_1 + W_2$ and so it is nonempty. Let $\alpha, \beta \in \mathbb{F}$. Let $\mathbf{w}_1 + \mathbf{w}_2, \mathbf{w}_1' + \mathbf{w}_2' \in W_1 + W_2$ where $\mathbf{w}_1, \mathbf{w}_1' \in W_1, \mathbf{w}_2, \mathbf{w}_2' \in W_2$. Since W_1 and W_2 are subspaces, $\alpha\mathbf{w}_1 + \beta\mathbf{w}_1' \in W_1$ and $\alpha\mathbf{w}_2 + \beta\mathbf{w}_2' \in W_2$. Hence

$$\alpha(\mathbf{w}_1 + \mathbf{w}_2) + \beta(\mathbf{w}_1' + \mathbf{w}_2') = (\alpha\mathbf{w}_1 + \beta\mathbf{w}_1') + (\alpha\mathbf{w}_2 + \beta\mathbf{w}_2') \in W_1 + W_2$$

and $W_1 + W_2$ is a subspace by Theorem **1.8**.

Proposition 1. *Let V be a vector space and let $W \subset V$ be a subset. Then W is a subspace if and only if it is closed under addition and scalar multiplication and $0 \in W$.*

Proof. First suppose that W is a subspace of V. Then by definition, $0 \in W$ and, for any $w_1, w_2 \in W$ and $a, b \in \mathbf{R}$ we have $aw_1 + bw_2 \in W$.

Now suppose that $W \subset V$ contains 0 and is closed under addition and scalar multiplication. Then W inherits all the algebraic properties of V, such as the associative and distributive laws. Thus it is a straigtforward exercise to check that W satisfies all the conditions of being a vector space. □

Proposition 2. *Let W_1 and W_2 be subspaces of a vector space V. Then $W_1 \cap W_2$ is also a subspace.*

Proof. We only need to show that $W_1 \cap W_2$ contains 0 and is closed under linear combinations. Indeed, $0 \in W_1$ and $0 \in W_2$, so $0 \in W_1 \cap W_2$. Let $u, v \in W_1 \cap W_2$ and let $a, b \in \mathbf{R}$. Then $au + bv \in W_1$ and $au + bv \in W_2$, so $au + bv \in W_1 \cap W_2$. □

Example: It is not always true that the union of two subspaces is a subspace. Let $V = \mathbf{R}^2$, let $W_1 = \{(x, 0) : x \in \mathbf{R}\}$, and let $W_2 = \{(0, y) : y \in \mathbf{R}\}$. Then W_1 and W_2 are subspaces of V, but $W_1 \cup W_2$ is not. Indeed, $(1, 0) \in W_1$ and $(0, 1) \in W_2$, but $(1, 1) = (1, 0) + (0, 1) \notin W_1 \cup W_2$. Thus $W_1 \cup W_2$ is not closed under vector addition, so it cannot be a subspace.

Exercise: Prove that $W_1 \cup W_2$ is a subspace if and only if either $W_1 \subset W_2$ or $W_2 \subset W_1$.

Definition 21. *Let V be a vector space, and let $W_1, W_2 \subset V$ be subspace. Then*

$$W_1 + W_2 = \{w_1 + w_2 : w_1 \in W_1, w_2 \in W_2\}.$$

Proposition 49. *The sum $W_1 + W_2$ is a subspace of V. In fact, it is the smallest subspace containing both W_1 and W_2.*

Proof. We need first to verify that if $w, \tilde{w} \in W_1 + W_2$ and $a, \tilde{a} \in \mathbf{R}$ then $aw + \tilde{a}\tilde{w} \in W_1 + W_2$. We have $w = w_1 + w_2$, where $w_1 \in W_1$ and $w_2 \in W_2$. Similarly, $\tilde{w} = \tilde{w}_1 + \tilde{w}_2$, where $\tilde{w}_1 \in W_1$ and $\tilde{w}_2 \in W_2$. Then

$$aw + \tilde{a}\tilde{w} = a(w_1 + w_2) + \tilde{a}(\tilde{w}_1 + \tilde{w}_2) = (aw_1 + \tilde{a}\tilde{w}_1) + (aw_2 + \tilde{a}\tilde{w}_2),$$

and we have now written $aw + \tilde{a}\tilde{w}$ as the sum of a vector in W_1 and a vector in W_2. Thus $W_1 + W_2$ is closed under linear combinations, so it must be a subspace.

Now suppose U is another subspace of V containing both W_1 and W_2, and choose $w = w_1 + w_2 \in W_1 + W_2$. However, w_1 and w_2 must both be elements of U, and U is closed under addition, so $w_1 + w_2 \in U$. We have just shown that $W_1 + W_2 \subset U$. □

Ex. 25. *If the product of two non-zero square matrices is a zero matrix, show that both of them must be singular matrices.*

(Delhi 1959; Sagar 66)

Solution. Let **A** and **B** be two non-zero square matrices each of the type $n \times n$. It is given that **AB**=a null matrix *i.e.*, **AB**=**O**.

Let $|\mathbf{B}|$ be not equal to zero. Then $\mathbf{B}^{-1}$ exists. So post-multiplying both sides of **AB**=**O** by $\mathbf{B}^{-1}$, we get

$$\mathbf{ABB}^{-1}=\mathbf{O} \text{ or } \mathbf{AI}_n=\mathbf{O} \text{ or } \mathbf{A}=\mathbf{O}.$$

But **A** is not a zero matrix. Hence $|\mathbf{B}|$ must be equal to zero.

Now suppose $|\mathbf{A}|$ is not equal to zero. Then $\mathbf{A}^{-1}$ exists. So pre-multiplying both sides of **AB**=**O** by $\mathbf{A}^{-1}$, we get

$$\mathbf{A}^{-1}\mathbf{AB}=\mathbf{O} \text{ or } \mathbf{I}_n\mathbf{B}=\mathbf{O} \text{ or } \mathbf{B}=\mathbf{O}.$$

But **B** is not a null matrix. Hence $|\mathbf{A}|$ must be equal to zero.

Exercise 3. Consider the vectors in $\mathbb{R}^4$ defined by $\alpha_1 = (-1, 0, 1, 2), \alpha_2 = (3, 4, -2, 5), \alpha_3 = (1, 4, 0, 9)$ Find a system of homogeneous linear equations for which the space of solutions is exactly the subspace of $\mathbb{R}^4$ spanned by the three given vectors.

Solution Row reducing the matrix whose rows are the α_i's we get

$$\begin{pmatrix} 1 & 0 & -1 & -2 \\ 0 & 1 & \frac{1}{4} & \frac{11}{4} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Therefore we need to find a matrix whose kernel is the space V spanned by $\{\alpha_1 = (1, 0, -1, -2), \alpha_2 = (0, 1, \frac{1}{4}, \frac{11}{4})\}$. The set $\mathcal{B} = \{\alpha_1, \alpha_2, (0, 0, 1, 0), (0, 0, 0, 1)\}$ is a basis for $\mathbb{R}^4$. The kernel of the matrix

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is exactly the space generated by e_1 and e_2. Thus, the matrix which maps the canonical basis to the basis $\mathcal{B}$ will map the kernel of S to the space spanned by α_1 and α_2. Let P be the change of basis, then the previous assertion expressed in terms of matrices is $P(\ker(S)) = V$ therefore $\ker(S) = P^{-1}(V)$. This means that if we apply S to any vector in $P^{-1}(V)$ we get 0, that is, if we apply P^{-1} to any vector in V and then we apply S, we get zero. But this is exactly what we want, to find a transformation whose kernel is V, such a transformation is given by SP^{-1}. Explicitly a transformations whose kernel is V is

$$SP^{-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & \frac{1}{4} & 1 & 0 \\ -2 & -\frac{11}{4} & 0 & 1 \end{pmatrix}$$

Theorem VI. *Non-zero eigen vectors belonging to distinct values are linearly independent.*

(G.N.D.U. 1996 ; P.U. 1989 ; Pbi. U. 1986)

Proof. Let $v_1, v_2, \ldots\ldots, v_n$ be n non-zero eigen vectors of a linear operator T : V→V belonging to distinct eigen values $\lambda_1, \lambda_2, \ldots\ldots, \lambda_n$ respectively.

We prove the result by induction.

When $n = 1$. Here v_1 is L.I. because $v_1 \neq 0$.

Thus the result is true when $n = 1$.

Let us assume that the result is true when the number of vectors $< n$.

Let $\alpha_1 v_1 + \alpha_2 v_2 + \ldots\ldots + \alpha_n v_n = 0$ ...(1),

where $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n \in F$

$\Rightarrow \quad \alpha_1 T(v_1) + \alpha_2 T(v_2) + \ldots\ldots + \alpha_n T(v_n) = T(0)$ [*Applying T*]

$\Rightarrow \quad \alpha_1 T(v_1) + \alpha_2 T(v_2) + \ldots\ldots + \alpha_n T(v_n) = 0$

Since $T(v_i) = \lambda_i v_i$ for $i = 1, 2, \ldots\ldots, n$

$\therefore \quad \alpha_1 \lambda_1 v_1 + \alpha_2 \lambda_2 v_2 + \ldots\ldots + a_n \lambda_n v_n = 0$ ...(2)

Multiplying (1) by λ_n, we get

$\alpha_1 \lambda_n v_1 + \alpha_2 \lambda_n v_2 + \ldots\ldots + a_n \lambda_n v_n = 0$ ...(3)

Subtracting (3) from (2), we get

$\alpha_1 (\lambda_1 - \lambda_n) v_1 + \alpha_2 (\lambda_2 - \lambda_n) v_2 + \ldots\ldots + \alpha_{n-1} (\lambda_{n-1} - \lambda_n) v_{n-1} = 0$

$\Rightarrow \quad \alpha_1 (\lambda_1 - \lambda_n) = 0, \ \alpha_2 (\lambda_2 - \lambda_n) = 0, \ \ldots\ldots, \ \alpha_{n-1} (\lambda_{n-1} - \lambda_n) = 0$ ...(4)

[$\because$ $v_1, v_2, \ldots\ldots, v_{n-1}$ *are L.I. (assumed)*]

But since λ_i are distinct,

$\therefore \quad \lambda_1 - \lambda_n \neq 0, \lambda_2 - \lambda_n \neq 0, \ldots\ldots, \lambda_{n-1} - \lambda_n \neq 0.$

$\therefore$ (4) $\Rightarrow \alpha_1 = 0, \ \alpha_2 = 0, \ldots\ldots, \alpha_{n-1} = 0.$

Putting in (1), $\quad \alpha_n v_n = 0$

$\Rightarrow \quad \alpha_n = 0.$ [$\because$ $v_n \neq 0$]

Thus $\alpha_1 = 0, \ \alpha_2 = 0, \ldots\ldots, \alpha_n = 0.$

Hence the vectors $v_1, v_2, \ldots\ldots, v_n$ are L.I.

Theorem VI. *Let* $T : V \to W$ *be a linear transformation and suppose* $x_1, x_2, \ldots, x_n \in V$ *have the property that their images* $T(x_1), T(x_2), \ldots, T(x_n)$ *are L.I. Show that the vectors* $x_1, x_2, \ldots, x_n$ *are L.I.*

(G.N.D.U. 1986 ; P.U. 1985 S)

Proof. Let there exist scalars $\alpha_1, \alpha_2, \ldots\ldots, \alpha_n \in F$ such that

$\alpha_1 x_1 + \alpha_2 x_2 + \ldots\ldots + \alpha_n x_n = 0$

Now $T(\alpha_1 x_1 + \alpha_2 x_2 + \ldots\ldots + \alpha_n x_n) = T(0) = 0$

$\Rightarrow \quad \alpha_1 T(x_1) + \alpha_2 T(x_2) + \ldots\ldots + \alpha_n T(x_n) = 0$ [$\because$ *T is L.T.*]

$\Rightarrow \quad \alpha_1 = 0, \alpha_2 = 0, \ldots\ldots, \alpha_n = 0$ [$\because$ *$T(x_1), T(x_2), \ldots\ldots, T(x_n)$ are L.I.*]

Hence $x_1, x_2, \ldots\ldots, x_n$ are also L.I.

Example 1. *If W_1 and W_2 are sub-spaces of $V(F)$, prove that*

$$W_1 + W_2 = \{ w_1 + w_2 \mid w_1 \in W_1, w_2 \in W_2 \}$$

is a sub-space of V. **(P.U. 1985 S)**

Sol. If $x_1, x_2 \in W_1$; $y_1, y_2 \in W_2$

so that $x = x_1 + y_1, y = x_2 + y_2 \in W_1 + W_2$.

Now $\alpha, \beta \in F$ and $x_1, x_2 \in W_1$

$\therefore$ $\alpha x_1 + \beta x_2 \in W_1$ [$\because$ *W_1 is a sub-space*]

and $\alpha, \beta \in F$ and $y_1, y_2 \in W_2$

$\Rightarrow$ $\alpha y_1 + \beta y_2 \in W_2$ [$\because$ *W_2 is a sub-space*]

Thus $(\alpha x_1 + \beta x_2) + (\alpha y_1 + \beta y_2) \in W_1 + W_2$ ...(1)

Also for $\alpha, \beta \in F, x, y \in W_1 + W_2$

$\Rightarrow$ $\alpha x + \beta y = \alpha (x_1 + y_1) + \beta (x_2 + y_2) = \alpha x_1 + \alpha y_1 + \beta x_2 + \beta y_2 = (\alpha x_1 + \beta x_2) + (\alpha y_1 + \beta y_2)$

$\in W_1 + W_2$. [*Using (1)*]

Hence $W_1 + W_2$ is a sub-space of V.

Example 18. *If $V(\mathbf{R})$ be a vector space of 2×3 matrices over $\mathbf{R}$, then show that the matrices*

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 3 & -2 & 4 \end{bmatrix};\ B = \begin{bmatrix} 1 & 1 & -3 \\ -2 & 0 & 5 \end{bmatrix};\ C = \begin{bmatrix} 4 & -1 & 2 \\ 1 & -2 & 3 \end{bmatrix}$$

in $V(\mathbf{R})$ are linearly independent.

Sol. Let α, β, γ be the scalars in $\mathbf{R}$ such that

$$\alpha \begin{bmatrix} 2 & 1 & -1 \\ 3 & -2 & 4 \end{bmatrix} + \beta \begin{bmatrix} 1 & 1 & -3 \\ -2 & 0 & 5 \end{bmatrix} + \gamma \begin{bmatrix} 4 & -1 & 2 \\ 1 & -2 & 3 \end{bmatrix} = O \quad \textit{(zero matrix)}$$

$$\Rightarrow \begin{bmatrix} 2\alpha + \beta + 4\gamma & \alpha + \beta - \gamma & -\alpha - 3\beta + 2\gamma \\ 3\alpha - 2\beta + \gamma & -2\alpha - 2\gamma & 4\alpha + 5\beta + 3\gamma \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad ...(1)$$

Then $2\alpha + \beta + 4\gamma = 0,\ \alpha + \beta - \gamma = 0,\ -\alpha - 3\beta + 2\gamma = 0$

and $3\alpha - 2\beta + \gamma = 0,\ -2\alpha - 2\gamma = 0,\ 4\alpha + 5\beta + 3\gamma = 0$.

By second and fifth, we get :

$$2\alpha + \beta = 0.$$

Then by first, we get :

$$\gamma = 0$$

and so by fifth, we get

$$\alpha = 0,\ \beta = 0.$$

Thus (1) is true only if $\alpha = \beta = \gamma = 0$ and so the matrices A, B, C in $V(\mathbf{R})$ are linearly independent.