5(c) IAS-2014

A particle is performing a Simple harmonic motion of period T about a Centre O and it passes through a point P where $OP = b$ with velocity v in the direction OP; Prove that the time which elapses before it returns to P is $\frac{T}{\pi}\cos^{-1}\left(\frac{b}{a}\right)$.

Sol^n: Let the equation of the S.H.M with centre O as origin be $\frac{d^2x}{dt^2} = -\mu x$

The time period $T = 2\pi/\sqrt{\mu}$

Let the amplitude be a. Then $\left(\frac{dx}{dt}\right)^2 = \mu(a^2 - x^2)$ —— (1)

when the particle passes through P its velocity is given to be v in the direction OP. Also $OP = b$. So putting $x = b$ and $dx/dt = v$ in (1), we get

$$v^2 = \mu(a^2 - b^2) \quad \text{—— (2)}$$

Let A be an extremity of the motion. From P the particle comes to instantaneous rest at A and then returns back to P.

In S.H.M the time from P to A is equal to the time from A to P

$\therefore$ the required time $= 2$. time from A to P.

Now for the motion from A to P, we have

$$\frac{dx}{dt} = -\sqrt{\mu}\sqrt{(a^2 - x^2)} \Rightarrow dt = \frac{-1}{\sqrt{\mu}}\frac{dx}{\sqrt{a^2 - x^2}}$$

Let t_1 be the time from A to P. Then at $t = 0$, $x = a$ and at P, $t = t_1$ and $x = b$. Therefore integrating (3) we get

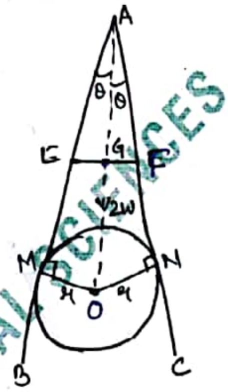
$$\int_0^{t_1} dt = \frac{1}{\sqrt{\mu}}\int_a^b \frac{-dx}{\sqrt{a^2 - x^2}} \Rightarrow t_1 = \frac{1}{\sqrt{\mu}}\left[\cos^{-1}\frac{x}{a}\right]_a^b$$

$$= \frac{1}{\sqrt{\mu}}\left[\cos^{-1}b/a - \cos^{-1}1\right] = \frac{1}{\sqrt{\mu}}\cos^{-1}b/a$$

Hence the required time $= 2t_1 = \frac{2}{\sqrt{\mu}}\cos^{-1}b/a$

$\left[\because T = \frac{2\pi}{\sqrt{\mu}} \text{ so that } \sqrt{\mu} = \frac{T}{2\pi}\right]$ $\quad = \frac{T}{\pi}\cos^{-1}b/a$

5(d)
Ques. Two equal uniform rods AB and AC, each of length 'l' are freely jointed at 'A' and rest on a smooth fixed verticle circle of radius 'r'. If 2θ is the angle b/w the rods, then find the relation between l, r and θ, by using principle of virtual work.



Sol: Let 'O' be the centre of the given fixed circle and 'W' be the weight each of the rods AB and AC. If 'E' and 'F' are the middle pts of AB and AC, then the total weight '2W' of the two rods can be taken as acting at 'G', middle point of EF. The line AO is vertical.

we have;

$$\angle BAO = \angle CAO = \theta$$

Also, $AB = l$, $AE = l/2$. If the rod AB touches the circle at M; then $\angle OMA = 90°$ and '$OM = r$' the radius of circle.

Give the rods a small symmetrical displacement in which 'θ' changes to $\theta + \delta\theta$. The point O, remains fixed and the point 'G' is slightly displaced.

The $\angle AMO$ remains $90°$, we have.

the height of G above the fixed point 'O'

$$\Rightarrow OG = OA - GA = OM \operatorname{cosec}\theta - AE\cos\theta$$

$$OG = r\operatorname{cosec}\theta - \frac{l}{2}\cos\theta$$

Equation of Virtual work is –

$$\Rightarrow -2W\,\delta(OG) = 0$$

or $\Rightarrow$ $\delta(r\operatorname{cosec}\theta - l/2\cos\theta) = 0$

$$= (-r\operatorname{cosec}\theta\cot\theta + l/2\sin\theta)\,\delta\theta = 0$$

$$r\operatorname{cosec}\theta\cot\theta = l/2\sin\theta$$

$$\frac{2r.1}{\sin\theta}\cdot\frac{\cos\theta}{\sin\theta} = l\sin\theta.$$

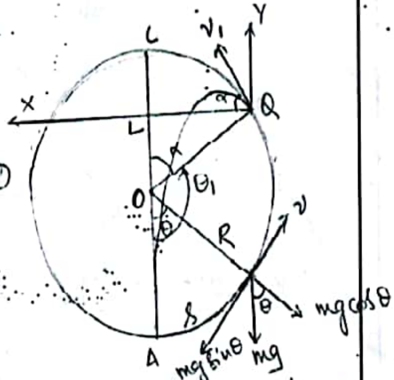
$$\boxed{2r.\cos\theta = l\sin^3\theta}$$

7(b) IAS-2014 DY

A heavy particle hanging vertically from a fixed point by a light inextensible cord of length l is struck by a horizontal blow which imparts it a velocity $2\sqrt{gl}$; Prove that the cord becomes slack when the particle has risen to a height $\frac{2}{3}l$ above the fixed point.

Sol^n: Take $R = T$ (i.e. the tension in the string)

Let a particle tied to a cord OA of length l be struck by a horizontal blow which imparts it a velocity $2\sqrt{(gl)}$. If P is the position of the particle at time t such that $\angle AOP = \theta$, then the equations of motion are.



$$m\frac{d^2s}{dt^2} = -mg\sin\theta \quad \text{—— ①}$$

and $$m\frac{v^2}{l} = T - mg\cos\theta \quad \text{—— ②}$$

Also $s = l\theta$

From ① & ③, we have

$$l\frac{d^2\theta}{dt^2} = -g\sin\theta$$

Multiplying bothsides by $2l\frac{d\theta}{dt}$ and integrating, we have

$$v^2 = \left(l\frac{d\theta}{dt}\right)^2 = 2lg\cos\theta + A$$

But at the point A, $\theta = 0$ and $v = 2\sqrt{gl}$

$$\therefore 4gl = 2lg + A \quad \text{so that } A = 2gl$$

$$\therefore v^2 = 2lg(\cos\theta + 1) \quad \text{—— ④}$$

from ② and ④, we have

$$T = \frac{m}{l}(v^2 + gl\cos\theta) = mg(3\cos\theta + 2) \quad \text{—— ⑤}$$

If the cord becomes slack at the point Q, where $\theta = \theta_1$, then from ⑤, we have

$$T = 0 - mg(3\cos\theta_1 + 2)$$

giving as $\cos\theta_1 = -2/3$.

If $\angle COQ = \alpha$, then $\alpha = \pi - \theta$, and $\cos\alpha = 2/3$

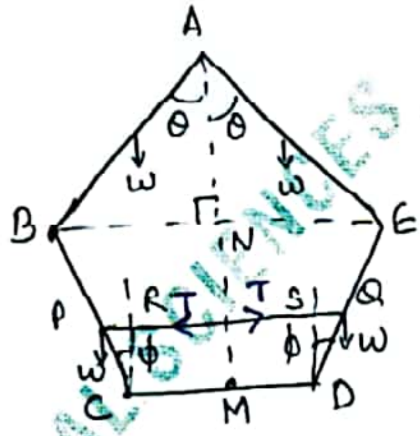
If v_1 is the velocity of the particle at Q, then $v = v_1$ where $\theta = \theta_1$. Therefore from ④ we have

$$v_1^2 = 2lg(1 + \cos\theta_1) = 2lg(1 - 2/3) = 2lg/3$$

Now $OL = l\cos\alpha = \frac{2}{3}l$

Thus the particle leaves the circular path at the point Q at a height $\frac{2}{3}l$ above the fixed point O with velocity $v_1 = \sqrt{(2lg/3)}$ at an angle α to the horizontal and subsequently it describes a parabolic path.

7(c)
Que) A regular pentagon ABCDE. formed of equal heavy uniform bars jointed together, is suspended from point 'A', and is maintained in form by a light rod joining the middle points of BC and DE. find the stress in this rod.



Sol: ABCDE is a pentagon formed of five equal rods each of weight 'w' and length '2a'.
It is suspended from 'A' and midpoints of BC and ED is jointed by a weightless (light) rod PQ.
Let ~~EA~~, The thrust in the rod PQ. The line AM joining 'A' to the middle point 'M' of CD is verticle and PQ is horizontal. The weights of the rods AB, BC, CD, DE & EA act at their respective middle points. In the portion of equilibrium the pentagon is a regular one so that each of the interior angles is $108°$ or $\frac{3\pi}{5}$ radians.

let, 'θ' be the angles which the upper slat rods AB and AE make with the verticle and φ be the angle which the lower slant rods CB and DE make with the verticle.
Replace the rod PQ by two equal and opposite forces T as shown in figure.
Give the system a small displacement about the vertical AM in which 'θ' changes to θ + δθ and φ changes to φ+δφ.

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The point A remains fixed. The lengths of the rods AB, BC etc remains fixed, the length BE changes and the middle point of the rods AB, BC etc are slightly displaced. The $\angle ANB$ remains 90°.

we have.

$$PQ = PR + PS + SQ = PR + CD + SQ = a\sin\phi + 2a + a\sin\phi$$

$$PQ = 2a(1+\sin\phi).$$

The depth of the middle pnt. of AB or AE below A $= a\cos\theta$

The depth of the middle pt. of BC or ED below A $= 2a\cos\theta + a\cos\phi$

and depth of the middle pt M of CD below A $= 2a(\cos\phi + \cos\theta)$

The equation of virtual work is –

$$T[\delta(2a + 2a\sin\phi)] + 2w\delta(a\cos\theta) + 2w\delta(2a\cos\theta + a\cos\phi)$$
$$+ w\delta(2a\cos\theta + 2a\cos\phi) = 0$$

$$\Rightarrow 2T\cos\phi\,\delta\phi - 2w\sin\theta\,\delta\theta - 4w\sin\theta\,\delta\theta - 2w\sin\phi\,\delta\phi$$
$$- 2w\sin\theta\,\delta\theta - 2w\sin\phi\,\delta\phi = 0$$

$$\Rightarrow [T\cos\phi - 2w\sin\phi]\delta\phi = 4w\sin\theta\,\delta\theta \quad \text{—— (1)}$$

from the figure, finding the length of BE in two ways; i.e from the upper portion AE and from lower portion BCDE,

we have. $\quad 4a\sin\theta = 2a + 4a\sin\phi$

Differentiating, we get $\quad 4a\cos\theta\,\delta\theta = 4a\cos\phi\,\delta\phi$

or $\quad \cos\theta\,\delta\theta = \cos\phi\,\delta\phi \quad$ —— (2)

Diving (1) by (2), we get.

$$\Rightarrow \frac{T\cos\phi - 2w\sin\phi}{\cos\phi} = \frac{4w\sin\theta}{\cos\theta} \Rightarrow \underline{T = 2W(\tan\phi + 2\tan\theta)}$$

But in the position of equilibrium

$$\theta = \frac{1}{2}\cdot\frac{3\pi}{5} = \frac{3\pi}{10} \;;\; \phi = \frac{3\pi}{5} - \frac{\pi}{2} = \frac{\pi}{10}$$

$$\therefore T = 2W(\tan\pi/10 + 2\tan 3\pi/10) = 2W(\tan\pi/10 + 2\cot 2\pi/10)$$

$$T = 2W\left[\tan\pi/10 + 2\cdot\left(\frac{1 - \tan^2(\pi/10)}{2\tan\pi/10}\right)\right] = 2W\cot(\pi/10)$$

8(b) A particle is acted on by a force parallel to the axis of y whose acceleration (always towards the axis of x) is μy^{-2} and when $y=a$, it is projected parallel to the axis of x with velocity $\sqrt{\frac{2\mu}{a}}$. Find the parametric equation of the path of the particle. Here μ is a constant.

Sol^n: Here we are given that

$$\frac{d^2y}{dt^2} = -\mu y^{-2} \quad \text{——— (1)}$$

the negative sign has been taken because the force is in the direction of y increasing. the force is in the direction of y increasing. the force is parallel to the axis of x.

Also there is no force parallel

Therefore $\frac{d^2x}{dt^2} = 0$ ——— (2)

Multiplying both sides of (1) by $2\frac{dy}{dt}$ and then integrating w.r.t. t, we have

$\left(\frac{dy}{dt}\right)^2 = \frac{2\mu}{y} + A$, where A is a constant.

Initially, when $y=a$, $\frac{dy}{dt}=0$ (Note that initially there is no velocity $\parallel$ to y-axis)

$$\therefore A = -\frac{2\mu}{a}$$

$$\therefore \left(\frac{dy}{dt}\right)^2 = \frac{2\mu}{y} - \frac{2\mu}{a} = 2\mu\left(\frac{1}{y} - \frac{1}{a}\right) = \frac{2\mu}{a}\left(\frac{a-y}{y}\right)$$

$$\Rightarrow \frac{dy}{dt} = -\sqrt{\left(\frac{2\mu}{a}\right)} \cdot \sqrt{\frac{a-y}{y}} \quad \text{——— (3)}$$

(–ve sign has been taken because the particle is moving in the direction of y decreasing).

Now integrating ②, we have

$\frac{dx}{dt} = B$, where B is a constant.

Initially, when $y = a$, $\frac{dx}{dt} = \sqrt{\frac{2\mu}{a}}$.

so that $B = \sqrt{\frac{2\mu}{a}}$.

$\therefore \frac{dx}{dt} = \sqrt{\frac{2\mu}{a}}$ ——— ④

Dividing ③ by ④, we have

$$\frac{dy}{dx} = -\sqrt{\frac{a-y}{y}}$$

$$\Rightarrow dx = -\sqrt{\frac{y}{a-y}}\,dy$$

Integrating,

$$\int dx = -\int\sqrt{\frac{y}{a-y}}\,dy$$

$$= 2a\int \frac{\cos\theta}{\sin\theta}\cdot\cos\theta\sin\theta\,d\theta + C.$$

(putting $y = a\cos^2\theta$, so that $dy = -2a\sin\theta\cos\theta\,d\theta$)

$$= a\int(1+\cos 2\theta)\,d\theta + C$$

$$= a\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + C$$

$$= \frac{a}{2}(2\theta + \sin 2\theta) + C.$$

Let us take $x = 0$, when $y = a$

So, when $a\cos^2\theta = a \Rightarrow \cos\theta = 1 \Rightarrow \theta = 0$

Then $0 = \frac{1}{2}a(0+0) + C \Rightarrow C = 0$

$\therefore x = \frac{1}{2}a(2\theta + \sin 2\theta)$ ——— ⑤

Also $y = a\cos^2\theta = \frac{a}{2}(1+\cos 2\theta)$ ——— ⑥

The equations ⑤ & ⑥ give us the path of the particle.

But these are the parametric equations of a cycloid.