classmate
Date
Page

IFoS 2019

5(d) Let $\vec{r} = \vec{r}(s)$ represent a space curve. Find $\frac{d^3\vec{r}}{ds^3}$ in terms of $\vec{T}$, $\vec{N}$ and $\vec{B}$, where $\vec{T}$, $\vec{N}$ and $\vec{B}$ represent tangent, principal normal and binormal respectively.
Compute,
$$\frac{d\vec{r}}{ds}\cdot\left(\frac{d^2\vec{r}}{ds^2}\times\frac{d^3\vec{r}}{ds^3}\right)$$
in terms of radius of curvature and the torsion.

$$\vec{T} = \frac{d\vec{r}}{ds}$$

$$\kappa\vec{N} = \frac{d\vec{T}}{ds} = \frac{d}{ds}\left(\frac{d\vec{r}}{ds}\right) = \frac{d^2\vec{r}}{ds^2}$$

ie. $$\frac{d^2\vec{r}}{ds^2} = \kappa\vec{N}$$

$$\Rightarrow \frac{d^3\vec{r}}{ds^3} = \kappa\frac{d\vec{N}}{ds} + \frac{d\kappa}{ds}\vec{N} \quad (*)$$

$$\frac{d^3\vec{r}}{ds^3} = \kappa(\vec{B}\tau - \kappa\vec{T}) + \frac{d}{ds}\left(\left|\frac{d\vec{T}}{ds}\right|\right)\vec{N} \quad (*)$$

(Serret Frenet $\Rightarrow \frac{d\vec{N}}{ds} = \tau\vec{B} - \kappa\vec{T}$)

=

$$\frac{d^2\vec{r}}{ds^2} \times \frac{d^3\vec{r}}{ds^3} = k\vec{N} \times \left[k(\vec{B}\tau - k\vec{T}) + \frac{dk}{ds}\vec{N}\right]$$

using Ⓐ

$$= k^2\tau(\vec{N}\times\vec{B}) - k^3(\vec{N}\times\vec{T})$$

$$= k^2\tau\vec{T} - k^3\vec{B}$$

$$\therefore \frac{d\vec{r}}{ds}\cdot\left(\frac{d^2\vec{r}}{ds^2}\times\frac{d^3\vec{r}}{ds^3}\right)$$

$$= \vec{T}\cdot\left(k^2\tau\vec{T} - k^3\vec{B}\right)$$

$$= k^2\tau \qquad (\because \vec{T}\cdot\vec{B} = 0)$$

Page

5(e) Evaluate $\int_{(0,0)}^{(2,1)} (10x^4 - 2xy^3)dx - 3x^2y^2 dy$

along the path $x^4 - 6xy^3 = 4y^2$.

The integral is of the form

$$\int_C M dx + N dy$$

where $M = 10x^4 - 2xy^3$

$N = -3x^2y^2$

$$\frac{\partial M}{\partial y} = -6xy^2, \quad \frac{\partial N}{\partial x} = -6xy^2$$

Method-1

As $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Hence the given integral is path-independent.

It means we can use any path,

Let the path consists of straight line L_1 : from $(0,0)$ to $(2,0)$ and then L_2 : from $(2,0)$ to $(2,1)$.

Along L_1 : $y = 0 \Rightarrow dy = 0$

Along L_2 : $x = 2 \Rightarrow dx = 0$

$\therefore$ Value of integral $\int_{x=0}^{2} 10x^4 dx + \int_{y=0}^{1} -3(2)^2 y^2 dy$

$$= 2x^5 \Big|_0^2 - 4y^3 \Big|_0^1 = 64 - 4 = 60.$$

Scanned with CamScanner

Method-2 :

As $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$

$\therefore (10x^4 - 2xy^3)\,dx - (3x^2y^2)\,dy$ is an exact differential of $(2x^5 - x^2y^3)$.

$$\therefore \int_{(0,0)}^{(2,1)} (10x^4 - 2xy^3)\,dx - 3x^2y^2\,dy$$

$$= \int_{(0,0)}^{(2,1)} d(2x^5 - x^2y^3),$$

$$= (2x^5 - x^2y^3)\Big|_{(0,0)}^{(2,1)}$$

$$= 64 - 4 = 60.$$

$$=-\tfrac{1}{2}\int_0^{2\pi}(1-\cos 2t)\,dt=-\tfrac{1}{2}\left[t-\frac{\sin 2t}{2}\right]_0^{2\pi}$$
$$=-\pi.$$ ...(1)

Now let us evaluate $\iint_S \text{curl } \mathbf{F}\cdot\mathbf{n}\, dS$. We have curl $\mathbf{F}$

$$=\nabla\times\mathbf{F}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & z & x \end{vmatrix}=-\mathbf{i}-\mathbf{j}-\mathbf{k}.$$

If S_1 is the plane region bounded by the circle C, then by an application of divergence theorem, we have

$$\iint_S \text{curl } \mathbf{F}\cdot\mathbf{n}\, dS=\iint_{S_1} \text{curl } \mathbf{F}\cdot\mathbf{k}\, dS \text{ [See Ex. 36 Page 126]}$$

$$=\iint_{S_1}(-\mathbf{i}-\mathbf{j}-\mathbf{k})\cdot\mathbf{k}\, dS=\iint_{S_1}(-1)\, dS=-\iint_{S_1} dS=-S_1.$$

But S_1=area of a circle of radius $1=\pi\,(1)^2=\pi$.

$$\therefore \iint_S \text{curl } \mathbf{F}\cdot\mathbf{n}\, dS=-\pi.$$ ...(2)

Hence from (1) and (2), the theorem is verified.

IFS 2019

Ex. 8. *Verify Stoke's theorem for* $\mathbf{F}=(2x-y)\,\mathbf{i}-yz^2\,\mathbf{j}-y^2z\mathbf{k}$, *where S is the upper half surface of the sphere* $x^2+y^2+z^2=1$ *and C is its boundary.*

[**Kanpur 1970; Rohilkhand 78; Allahabad 78; Agra 73, 76, 80**]

Solution. The boundary C of S is a circle in the xy plane of radius unity and centre origin. Suppose $x=\cos t$, $y=\sin t$, $z=0$, $0\leqslant t<2\pi$ are parametric equations of C. Then

$$\oint_C \mathbf{F}\cdot d\mathbf{r}=\oint_C[(2x-y)\,\mathbf{i}-yz^2\,\mathbf{j}-y^2z\mathbf{k}]\cdot(dx\mathbf{i}+dy\mathbf{j}+dz\mathbf{k})$$

$$=\oint_C[(2x-y)\,dx-yz^2\,dy-y^2z\,dz]$$

$$=\oint_C(2x-y)\,dx,\text{ since } z=0 \text{ and } dz=0$$

$$=\int_0^{2\pi}(2\cos t-\sin t)\frac{dx}{dt}\,dt=-\int_0^{2\pi}(2\cos t-\sin t)\sin t\,dt$$

Scanned with CamScanner

$$=-\int_0^{2\pi}[\sin 2t-\tfrac{1}{2}(1-\cos 2t)]\,dt=-\left[-\frac{\cos 2t}{2}-\tfrac{1}{2}t+\tfrac{1}{2}\frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$=-[(-\tfrac{1}{2}+\tfrac{1}{2})-\tfrac{1}{2}(\pi-0)+\tfrac{1}{4}(0-0)]=\pi. \qquad \ldots(1)$$

Also $(\nabla\times\mathbf{F})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ 2x-y & -yz^2 & -y^2z\end{vmatrix}$

$$=(-2yz+2yz)\,\mathbf{i}-(0-0)\,\mathbf{j}+(0+1)\,\mathbf{k}=\mathbf{k}.$$

If S_1 is the plane region bounded by the circle C, then

$$\iint_S(\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS=\iint_{S_1}(\nabla\times\mathbf{F})\cdot\mathbf{k}\,dS$$

[by an application of divergence theorem, see Ex. 36, page 126]

$$=\iint_{S_1}\mathbf{k}\cdot\mathbf{k}\,dS=\iint_{S_1}dS=S_1=\pi. \qquad \ldots(2)$$

Hence from (1) and (2), the theorem is verified.

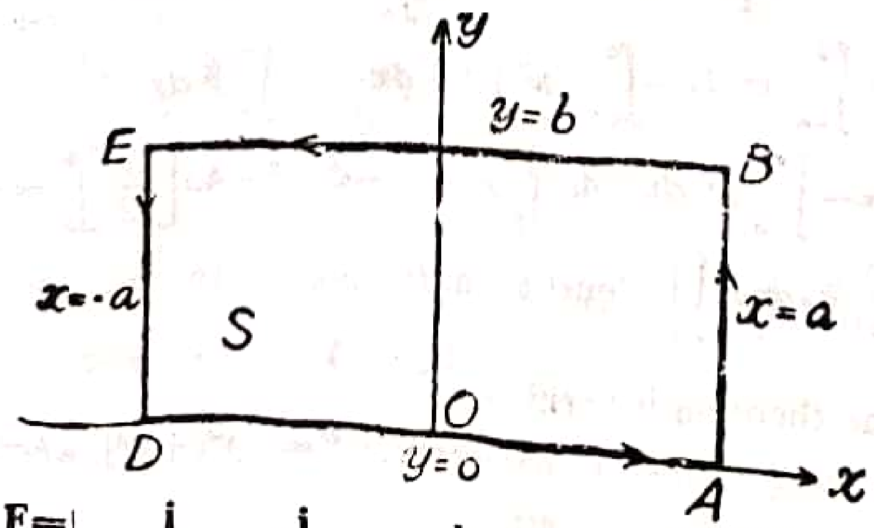
Ex. 9. *Verify Stoke's theorem for*

$$\mathbf{F}=(x^2+y^2)\,\mathbf{i}-2xy\,\mathbf{j}$$

taken round the rectangle bounded by

$$x=\pm a,\ y=0,\ y=b. \qquad \text{[Meerut 1967]}$$

Solution. We have



curl $\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\end{vmatrix}$

2. Serret - Frenet Formulae

Set of relations involving derivatives of fundamental vectors T, N, B ~~the Board~~ is collectively known as

① $\frac{dT}{ds} = kN$ ② $\frac{dB}{ds} = -\tau N$ ③ $\frac{dN}{ds} = \tau B - kT$

where τ is scalar called torsion $\sigma = \frac{1}{\tau}$ is radius of torsion

(i) Principal Normal Vector :-

Any line $\perp$ to tangent to a curve at a pt is normal line at P
Normal line lying in oscillatory plane $\equiv$ Principal Normal
Unit Principal Normal : N

(ii) Plane

- Oscillating Plane: It contains tangent & principal normal at a pt. P
- Normal Plane: thru pt P, $\perp$ to tangent
- Rectifying Plane: thru pt P, $\perp$ to principal normal

(iii) Binormal :

T: unit tangent vector
N: unit principal normal

} B is unit vector $\perp$ to both T & N s.t T, N, B form a right handed system.

$\Downarrow$

$$\boxed{\begin{array}{l} T\cdot T = N\cdot N = B\cdot B = 1 \\ T\times N = B \quad N\times B = T \quad B\times T = N \end{array}}$$

PROOF: (i) $\frac{dT}{ds} = kN$

Let $\vec{r}(t)$ be position vector of point P

$\frac{d\vec{r}}{ds} = T$ is unit tangent vector at P

$|T| = 1$ i.e. T is Constant Magnitude.

$\Rightarrow \boxed{T\cdot\frac{dT}{ds} = 0} \Rightarrow \frac{dT}{ds}$ is $\perp$ to T

W.k.T $\boxed{\frac{dT}{ds}}$ lies in oscillating Plane $\Rightarrow \boxed{\frac{dT}{ds} \text{ is } \parallel \text{ to } N}$

$\frac{dT}{ds} = kN$

Curvature : Rate of change of T w.r.t s i.e. $k = \left|\frac{dT}{ds}\right|$

ρ (radius of curvature) $= \frac{1}{k}$

(ii) $\frac{dB}{ds} = -\tau N$

Since $|B| = 1$ i.e Constant Magnitude vector

$\therefore \boxed{B \cdot \frac{dB}{ds} = 0} \Rightarrow \frac{dB}{ds}$ is $\perp$ to B —(1)

—(2)

W.K.T $\boxed{\frac{dB}{ds}}$ lies in osculating Plane.

$B \cdot T = 0 \Rightarrow B \cdot \frac{dT}{ds} + \frac{dB}{ds} \cdot T = 0$

$B \cdot (KN) + \frac{dB}{ds} \cdot T = 0$

$(B \cdot N) K + \frac{dB}{ds} \cdot T = 0 \Rightarrow \boxed{\frac{dB}{ds} \cdot T = 0}$

$\Rightarrow \frac{dB}{ds}$ is $\perp$ to T —(3)

From (1), (2) & (3) $\frac{dB}{ds}$ is $\parallel$ to N

$\Rightarrow \frac{dB}{ds} = -\tau N$

Torsion : Rate of change of B w.r.t s is called Torsion.

$\left|\frac{dB}{ds}\right| = \tau$ σ (radius of torsion) $= \frac{1}{\tau}$

(iii) $\frac{dN}{ds} = \tau B - KT$

$\boxed{B \times T = N}$

$B \times \frac{dT}{ds} + \frac{dB}{ds} \times T = \frac{dN}{ds}$

$B \times (KN) + (-\tau N) \times T = \frac{dN}{ds}$

$K(-T) + (-\tau)(-B) =$

$\Rightarrow \frac{dN}{ds} = \tau B - KT$

Q K is curvature & τ is torsion of curve $\vec{r}(s)$ then

2009 $K = \left|\frac{d\vec{r}}{ds} \times \frac{d^2\vec{r}}{ds^2}\right|$ $\tau = \frac{\left[\frac{d\vec{r}}{ds} \; \frac{d^2\vec{r}}{ds^2} \; \frac{d^3\vec{r}}{ds^3}\right]}{\left|\frac{dr}{ds} \times \frac{d^2r}{ds^2}\right|^2}$

***

$\boxed{T = \frac{dr}{ds}}$ $\frac{dT}{ds} = \frac{d^2\vec{r}}{ds^2}$

$\Rightarrow \frac{dr}{ds} \times \frac{d^2\vec{r}}{ds^2} = T \times \frac{dT}{ds} = T \times (KN) = KB$

$\Rightarrow K = \left|\frac{dr}{ds} \; \frac{d^2\vec{r}}{ds^2}\right|$

Scanned with CamScanner