1(a) find the inverse of the Matrix:

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$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & -1 & 7 \\ 3 & 2 & -1 \end{bmatrix}$$

By using elementary row operations. Hence solve the system of linear equation.

$$x+3y+z=10$$
$$2x-y+7z=21$$
$$3x+2y-z=4$$

Soln

$$A = \begin{bmatrix} 1 & 3 & 1 \\ 2 & -1 & 7 \\ 3 & 2 & -1 \end{bmatrix}$$

$$[A \,\vdots\, I] = \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 2 & -1 & 7 & 0 & 1 & 0 \\ 3 & 2 & -1 & 0 & 0 & 1 \end{array}\right]$$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$

$$= \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -7 & 5 & -2 & 1 & 0 \\ 0 & -7 & -4 & -3 & 0 & 1 \end{array}\right]$$

$R_3 \to R_3 - R_2$

$$= \left[\begin{array}{ccc|ccc} 1 & 3 & 1 & 1 & 0 & 0 \\ 0 & -7 & 5 & -2 & 1 & 0 \\ 0 & 0 & -9 & -1 & -1 & 1 \end{array}\right]$$

$R_1 \to 7R_1 + 3R_2$

$$\sim \left[\begin{array}{ccc|ccc} 7 & 0 & 22 & 1 & 3 & 0 \\ 0 & -7 & 5 & -2 & 1 & 0 \\ 0 & 0 & -9 & -1 & -1 & 1 \end{array}\right]$$

$R_1 \to R_1/7$, $R_2 \to -R_2/7$, $R_3 \to -R_3/9$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 22/7 & 1/7 & 3/7 & 0 \\ 0 & 1 & -5/7 & 2/7 & -1/7 & 0 \\ 0 & 0 & 1 & 1/9 & 1/9 & -1/9 \end{array}\right]$$

$R_1 \to R_1 - 22/7\, R_3$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -13/63 & 5/63 & 22/63 \\ 0 & 1 & -5/7 & 2/7 & -1/7 & 0 \\ 0 & 0 & 1 & 1/9 & 1/9 & -1/9 \end{array}\right]$$

$R_2 \to R_2 + 5/7\, R_3$

$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -13/63 & 5/63 & 22/63 \\ 23/63 & -4/63 & -5/63 \\ 1/9 & 1/9 & -1/9 \end{bmatrix}$$

$$\therefore A^{-1} = \begin{bmatrix} -13/63 & 5/63 & 22/63 \\ 23/63 & -4/63 & -5/63 \\ 1/9 & 1/9 & -1/9 \end{bmatrix}$$

then.
$$\begin{bmatrix} 1 & 3 & 1 \\ 2 & -1 & 7 \\ 3 & 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 21 \\ 4 \end{bmatrix} \quad (\because AX = B \quad X = A^{-1}B)$$

$$x \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 \\ 2 & -1 & 7 \\ 3 & 2 & -1 \end{bmatrix}^{-1} \begin{bmatrix} 10 \\ 21 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} -13/63 & 5/63 & 22/63 \\ 23/63 & -4/13 & -5/63 \\ 1/9 & 1/9 & -1/9 \end{bmatrix} \begin{bmatrix} 10 \\ 21 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 63/63 \\ 126/63 \\ 27/9 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\Rightarrow x = 1, \; y = 2, \; z = 3$$

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1(b) IAS -2013

Let A be a square matrix and A^* be its adjoint, show that the eigen values of matrices AA^* and A^*A are real. further show that trace (AA^*) = trace (A^*A).

Soln Define an inner product.

$$\langle x, y\rangle = \langle \overline{y, x}\rangle$$

$$\langle Ax, y\rangle = \langle x, A^*y\rangle$$

where A^* is adjoint of A. Let V be the eigenvector of matrix AA^*. λ be the corresponding eigenvalue.

$$\Rightarrow \langle AA^*V, V\rangle = \langle \lambda V, V\rangle$$

$$\Rightarrow \langle AA^*V, V\rangle = \lambda\langle V, V\rangle \quad \text{—— (1)}$$

Now, $\langle V, AA^*V\rangle = \langle V, \lambda V\rangle$

$$\langle V, AA^*V\rangle = \bar{\lambda}\langle V, V\rangle \quad \text{—— (2)}$$

Now, $\langle V, AA^*V\rangle = \langle A^*V, A^*V\rangle$

$$= \langle (A^*)^* A^*V, V\rangle$$

$$= \langle AA^*V, V\rangle$$

$$\langle V, \lambda V\rangle = \langle \lambda V, V\rangle \quad \text{using eqn (1) + (2)}$$

$$\Rightarrow \bar{\lambda}\langle V, V\rangle = \lambda\langle V, V\rangle$$

$$\Rightarrow \bar{\lambda} = \lambda$$

$\Rightarrow \lambda$ is real.

Continue in this way.

1(c) Evaluate $\int_0^1 \left(2x\sin\frac{1}{x} - \cos\frac{1}{x}\right)dx$.

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Sol^n The function

$$f(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x}, & x \in]0,1[\\ 0, & x=0 \end{cases}$$

is not continuous on $[0,1]$ (it is discontinuous at $x=0$), but it is bounded and continuous on $]0,1]$ and thus Riemann-integrable on $[0,1]$.

The function $g(x) = \begin{cases} x^2\sin\frac{1}{x}, & x \in]0,1] \\ 0, & x=0 \end{cases}$

is differentiable on $[0,1]$ and satisfies

$$g'(x) = f(x), \ \forall x \in [0,1]$$

$$\therefore \int_0^1 \left(2x\sin\frac{1}{x} - \cos\frac{1}{x}\right)dx = g(1) - g(0) = \sin 1.$$

1(d) IAS-2013 P-I

Find the Equation of the plane which passes through the points (0,1,1) and (2,0,-1) and is parallel to the line joining the points (-1,1,-2), (3,-2,4). Find also the distance b/w the line and the plane.

1(d) IAS 2013 P-I.

Equation of plane through (0,1,1) is

$$a(x-0)+b(y-1)+c(z-1)=0$$

$$\Rightarrow ax+by+cz-b-c=0 \longrightarrow (1)$$

this also passes through (2,0,-1), then

$$a(2)+b(0)+c(-1)-b-c=0$$

$$\Rightarrow \quad 2a-b-2c=0 \longrightarrow (2)$$

Given plane is parallel to line joining (-1,1,-2) and (3,-2,4)

Now dr's of line joining (-1,1,-2) and (3,-2,4)
(4,-3,6)

as plane is parallel to this line, its normal will be perpendicular to it, Then

$$4a-3b+6c=0 \longrightarrow (3)$$

from (2) and (3)

$$\frac{a}{-6-6}=\frac{b}{-8-12}=\frac{c}{-6+4}$$

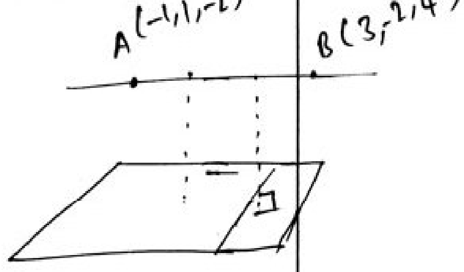
(or) $$\frac{a}{-12}=\frac{b}{-20}=\frac{c}{-2}$$

(or) $$\frac{a}{6}=\frac{b}{10}=\frac{c}{1}$$

Thus, Equation of plane is

$$P \equiv 6x+10y+z-11=0$$

and Equation of line is



$$\frac{x+1}{4} = \frac{y-1}{-3} = \frac{z+2}{6}$$

from figure, let M be the foot of the perpendicular from point B (3, -2, 4) on the plane p.

Equation of BM $\equiv \dfrac{x-3}{\frac{6}{\sqrt{137}}} = \dfrac{y}{\frac{10}{\sqrt{137}}} = \dfrac{z-4}{\frac{1}{\sqrt{137}}} = r.$

where $r = |BM|$

then $\left(3 + \frac{6r}{\sqrt{137}}, -2 + \frac{10r}{\sqrt{137}}, 4 + \frac{r}{\sqrt{137}}\right)$ lies on the plane p.

thus $6\left(3 + \frac{6r}{\sqrt{137}}\right) + 10\left(-2 + \frac{10r}{\sqrt{137}}\right) + \left(4 + \frac{r}{\sqrt{137}}\right) - 11 = 0.$

(or) $\frac{137r}{\sqrt{137}} + 18 - 20 + 4 - 11 = 0$

$\Rightarrow \sqrt{137}\, r = 9$

(or) $r = \frac{9}{\sqrt{137}}.$

1(e) A Sphere S has points $(0,1,0)$, $(3,-5,2)$ at opposite ends of a diameter. find the equation of the sphere having the intersection of the sphere S with the plane $5x-2y+4z+7=0$ as a great circle.

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Soln:- Equation of sphere S is

$$(x-0)(x-3)+(y-1)(y+5)+(z-0)(z-2)=0$$

$$\Rightarrow x^2-3x+y^2+4y-5+z^2-2z=0$$

$$S \equiv x^2+y^2+z^2-3x+4y-2z-5=0$$

Equation of family of circles through the Intersection of S and plane $P \equiv 5x-2y+4z+7=0$ is given by

$$S+\lambda P=0$$

(or) $S' \equiv x^2+y^2+z^2+(5\lambda-3)x+(-2\lambda+4)y+(4\lambda-2)z$
$+(7\lambda-5)=0$

for the intersection of S and P to be great circle of S' its centre must lie on P.

Centre of $S' \equiv \left(-\frac{5\lambda-5}{2}, -\frac{-2\lambda+4}{2}, -\frac{4\lambda-2}{2}\right)$

$$5\left(\mp\frac{5\lambda-3}{2}\right)-2\left(\mp\frac{-2\lambda+4}{2}\right)+4\left(\mp\frac{4\lambda-2}{2}\right)\pm 7=0.$$

$$25\lambda-15+4\lambda-8+16\lambda-8-14=0$$

$$45\lambda=45 \Rightarrow \lambda=1$$

So, $S' \equiv x^2+y^2+z^2+2x+2y+2z+2=0.$

distance

f_{min} = 5units and f_{max} = 45 units.

Q(a)(i) Let P_n denote the vectorspace of all real polynomials of degree atmost n and $T: P_2 \to P_3$ be a linear transformation given by $T(p(x)) = \int_0^x p(t)\,dt$, $p(x) \in P_2$

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find the matrix of T w.r.to the bases $\{1, x, x^2\}$ and $\{1, x, 1+x^2, 1+x^3\}$ of P_2 and P_3 respectively. Also, find the null space of T.

(ii) Let V be an n-dimensional vectorspace and $T: V \to V$ be an invertible linear operator. If $\beta = \{x_1, x_2, \ldots, x_n\}$ is a basis of V, show that $B' = \{Tx_1, Tx_2, \ldots, Tx_n\}$ is also a basis of V.

Sol^n: (i) Given $T(p(x)) = \int_0^x p(t)\,dt$, $p(x) \in P_2$

basis for P_2 is $\{1, x, x^2\}$ and

basis for P_3 is $\{1, x, 1+x^2, 1+x^3\}$

Now

$$T(1) = \int_0^x 1\,dt = x = 0.1 + 1.x + 0(1+x^2) + 0(1+x^3)$$

$$T(x) = \int_0^x t\,dt = \frac{x^2}{2} = -\tfrac{1}{2}.1 + 0.x + \tfrac{1}{2}(1+x^2) + 0(1+x^3)$$

$$T(x^2) = \int_0^x t^2\,dt = \frac{x^3}{3} = -\tfrac{1}{3}.1 + 0.x + 0.(1+x^2) + \tfrac{1}{3}(1+x^3)$$

Matrix of T w.r.t bases β_1 and β_2 is

$$[T : \beta_1, \beta_2] = \begin{bmatrix} 0 & -\frac{1}{2} & -\frac{1}{3} \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{3} \end{bmatrix}$$

Null Space of T will be given by

$\int_0^2 p(t)dt = 0$ i.e. if $p(x) = a_0 + a_1x + a_2x^2$.

$$\begin{bmatrix} 0 & -\frac{1}{2} & -\frac{1}{3} \\ 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow a_0 = 0$; $a_1 = 0$; $a_2 = 0$

$\therefore a_0 = a_1 = a_2 = 0$

$\therefore p(x) = 0$

$\therefore$ Null Space of T Contains only single element $\{0\}$

2(a)(ii) Since it is given that $T: V \to V$ is invertible. So, T must also be one-one and onto. Also T is linear

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So, $T(\alpha) = 0 \Leftrightarrow \alpha = 0$ —— ①

Again V is a vectorspace of dimension 'n', so any Linearly independent set of dimension 'n' can form its basis. —— ②

Consider set $B' = \{Tx_1, Tx_2, \dots Tx_n\}$

Let $\alpha_1, \alpha_2 \dots \alpha_n$ be n scalars such that

$$\alpha_1 Tx_1 + \alpha_2 Tx_2 + \dots \alpha_n Tx_n = 0 \quad \text{—— ③}$$

by property of linear transformation ③ becomes

$$T(\alpha_1 x_1 + \alpha_2 x_2 + \dots \alpha_n x_n) = 0$$

from ① $\alpha_1 x_1 + \alpha_2 x_2 + \dots \alpha_n x_n = 0$.

$B(x_1, x_2 \dots x_n)$ forms a basis of V. so must be LI

$\alpha_1 = \alpha_2 = \dots = \alpha_n = 0$

$Tx_1, Tx_2 \dots Tx_n$ are LI.

So, from ② set B' forms a basis for V.

2(b)i → Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix}$ where $\omega (\neq 1)$ is a

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cube root of unity. If $\lambda_1, \lambda_2, \lambda_3$ denote the eigenvalues of A^2, show that

$$|\lambda_1| + |\lambda_2| + |\lambda_3| \leq 9.$$

Sol^n → ω is a cube root of unity.

$\Rightarrow \omega^2 = 1$ and $1 + \omega + \omega^2 = 0$ —— (1)

$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix}$ is given

$$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{bmatrix} = \begin{bmatrix} 3 & 1+\omega^2+\omega & 1+\omega+\omega^2 \\ 1+\omega^2+\omega & 1+\omega^4+\omega^2 & 1+\omega^3+\omega^3 \\ 1+\omega+\omega^2 & 1+\omega^3+\omega^3 & 1+\omega^2+\omega^4 \end{bmatrix}$$

$$\Rightarrow A^2 = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 3 \\ 0 & 3 & 0 \end{bmatrix} \text{ using equation (1)}$$

Let λ be the eigenvalue of A^2, then charateristic matrix is

$$[A^2 - \lambda I] = \begin{bmatrix} 3-\lambda & 0 & 0 \\ 0 & -\lambda & 3 \\ 0 & 3 & -\lambda \end{bmatrix}$$

Characteristic polynomial is -

$$(3-\lambda)(\lambda^2 - 9) = 0$$

$\Rightarrow (3-\lambda)(\lambda+3)(\lambda-3) = 0$

$\Rightarrow \lambda_1 = -3, \lambda_2 = 3, \lambda_3 = 3$

$\Rightarrow |\lambda_1| + |\lambda_2| + |\lambda_3| = 3+3+3 = 9 \leq 9$

$\Rightarrow |\lambda_1| + |\lambda_2| + |\lambda_3| \leq 9$

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2(b) ii → find the rank of the matrix

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$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 8 & 12 \\ 3 & 5 & 8 & 12 & 17 \\ 5 & 8 & 12 & 17 & 23 \\ 8 & 12 & 17 & 23 & 30 \end{bmatrix}$$

Solⁿ → Given $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 5 & 8 & 12 \\ 3 & 5 & 8 & 12 & 17 \\ 5 & 8 & 12 & 17 & 23 \\ 8 & 12 & 17 & 23 & 30 \end{bmatrix}$

$R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$, $R_4 \to R_4 - 5R_1$

$R_5 \to R_5 - 8R_1$

$$\Rightarrow A \sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -1 & -1 & 0 & 2 \\ 0 & -1 & -1 & 0 & 2 \\ 0 & -2 & -3 & -3 & -2 \\ 0 & -4 & -7 & -9 & -10 \end{bmatrix}$$

$R_3 \to R_3 - R_3$, $R_4 \to R_4 - 2R_2$, $R_5 \to R_5 - 4R_2$

$$\Rightarrow A \sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -3 & -6 \\ 0 & 0 & -3 & -9 & -18 \end{bmatrix}$$

$R_5 \to R_5 - 3R_4$ then.

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & -3 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$R_3 \leftrightarrow R_4$

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & -1 & -1 & 0 & 2 \\ 0 & 0 & -1 & -3 & -6 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

this is echelon form of Matrix A and since there are 3 non-zero rows hence

$\rho(A) =$ no. of non-zero rows.

$\Rightarrow \boxed{\rho(A) = 3}$

2(c) i → Let A be the Hermetian matrix having all distinct eigenvalues $\lambda_1, \lambda_2, \cdots \lambda_n$. If $X_1, X_2, \cdots X_n$ are corresponding eigen vectors then show that the $n \times n$ matrix C whose k^{th} coloumn consist of the vector X_k is non singular.

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Sol^n → A be a Hermetian matrix.

given A have $\lambda_1, \lambda_2, \cdots, \lambda_n$ all distinct eigenvalues.

Corresponding eigenvectors are $X_1, X_2, \cdots, X_n$

consider a matrix C whose coloumn are $X_1, X_2, \cdots, X_n$.

$$\Rightarrow \quad C = [X_1, X_2, \cdots, X_n]_{n \times n}$$

To show that C is invertible, it would be enough to show $X_1, \cdots, X_n$ are L.I.

Let us consider $X_1, X_2, \cdots X_n$ are not L.I.

$\Rightarrow \exists X_i$ s.t. X_i can be written as unique linear combination of remaining vectors.

W.L.O.G. take $X_i = X_1 \neq 0$

$\Rightarrow \exists \alpha_2, \alpha_3 \cdots \alpha_n$

s.t. $$X_1 = \sum_{k=2}^{n} \alpha_k X_k \quad \text{—— (1)}$$

($\alpha_1, \ldots, \alpha_n$ are unique)

$\because \lambda_i$ is eigen value of A and corresponding eigen vector is X_i

$\Rightarrow A X_i = \lambda_i X_i \quad \forall\ i = 1, 2, \ldots n$ —— (2)

from equation (1) multiply by A, we get

$$A X_1 = A \sum_{k=2}^{n} \alpha_k X_k \Rightarrow A X_1 = \sum_{k=2}^{n} \alpha_k A X_k$$

(using eqn (2))

$$\Rightarrow \lambda_1 X_1 = \sum_{k=2}^{n} \alpha_k \lambda_k X_k$$

$$\Rightarrow X_1 = \sum_{k=2}^{n} \alpha_k \frac{\lambda_k}{\lambda_1} X_k$$

$\lambda_1 \neq 0$ (if $\lambda_1 = 0$, $\Rightarrow X_1 = 0$ which is contradiction)

$$\Rightarrow X_1 = \sum_{k=2}^{n} \alpha_k \frac{\lambda_k}{\lambda} X_k \quad \text{—— (3)}$$

$\because X_1$ can be written as unique linear combination of $X_2 \ldots X_n$.

Hence from (1) & (3)

$$\alpha_k \frac{\lambda_k}{\lambda_1} = \alpha_k$$

$$\Rightarrow \lambda_1 = \lambda_k$$

$\Rightarrow$ all eigenvalues are not distinct, which is a contradiction.

Hence our assumption $X_1, X_2, \ldots X_n$ are not L.I. is wrong.

$\Rightarrow x_1, x_2, \ldots, x_n$ are linearly Independent.

$\Rightarrow [C]_{n\times n} = [x_1, \ldots x_n]_{n\times n}$ has no zero column.

$\Rightarrow \quad \rho(c) = n$

$\Rightarrow \quad C$ is invertible.

$\Rightarrow \quad C$ is non-singular.

2(c) ii → Show that the vectors $X_1 = (1, 1+i, i)$, $X_2 = (i, -i, 1-i)$ and $X_3 = (0, 1-2i, 2-i)$ in $\mathbb{C}^3$ are linearly independent over the field of real numbers but are linearly dependent over the field of complex numbers.

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Sol^n → Given $X_1 = (1, 1+i, i)$, $X_2 = (i, -i, 1-i)$

$X_3 = (0, 1-2i, 2-i)$

consider $\exists$ $\alpha_1, \alpha_2, \alpha_3$ st.

$$\alpha_1 X_1 + \alpha_2 X_2 + \alpha_3 X_3 = 0$$

$\Rightarrow \alpha_1(1, 1+i, i) + \alpha_2(-i, -i, 1-i) + \alpha_3(0, 1-2i, 2-i) = (0,0,0)$

$\Rightarrow \alpha_1 + i\alpha_2 = 0$ —— (1)

$(1+i)\alpha_1 - i\alpha_2 + (1-2i)\alpha_3 = 0$ —— (2)

$i\alpha_1 + (1-i)\alpha_2 + (2-i)\alpha_3 = 0$ —— (3)

Case (1) if $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$

from (1) $\alpha_1 + i\alpha_2 = 0$

$\Leftrightarrow \alpha_1 = 0, \alpha_2 = 0$

putting values of α_1, α_2 in equation (2).

we get $\alpha_3 = 0$

$\Rightarrow \alpha_1 = 0 = \alpha_2 = \alpha_3$

$\Rightarrow X_1, X_2, X_3$ are L.I.

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Case (2) if field is complex numbers $\mathbb{C}$

i.e. if $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}$

from eqn (1) $\alpha_1 = -i\alpha_2$

putting the value of α_1 in eqn (2) we get

$$(1+i)(-i\alpha_2) - i\alpha_2 + (1-2i)\alpha_3 = 0$$

$$\Rightarrow (1-2i)(\alpha_2+\alpha_3) = 0$$

$$\Rightarrow \alpha_2 = -\alpha_3$$

Hence $(\alpha_1, \alpha_2, \alpha_3) = (-i\alpha_2, \alpha_2, -\alpha_2)$

put $\alpha_2 = 1$

we get $(\alpha_1, \alpha_2, \alpha_3) = (-i, 1, -1)$

Hence all $\alpha_1, \alpha_2, \alpha_3$ are non-zero.

$\Rightarrow X_1, X_2, X_3$ are linearly dependent.

8(a) 2013 P-I using Lagrange's multiplier method, find the shortest distance b/w the line $y = 10 - 2x$ and the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

Sol^n:- let $p(x,y)$ be any point on the given ellipse.

Then, length of perpendicular from $p(x,y)$ to the line $y = 10 - 2x$ is

$$\left| \frac{2x + y - 10}{\sqrt{5}} \right| \longrightarrow (\star)$$

let $f(x,y) = \frac{(2x+y-10)^2}{5}$

we are to maximize this function $f(x,y)$ subject to the constraint $g(x,y) = 0$, where

$$g(x,y) = \frac{x^2}{4} + \frac{y^2}{9} - 1$$

let $F(x,y) = f(x,y) + \lambda g(x,y)$

$$= \frac{(2x+y-10)^2}{5} + \lambda \left(\frac{x^2}{4} + \frac{y^2}{9} - 1 \right)$$

Now, $F_x(x,y) = \frac{4(2x+y-10)}{5} + \frac{\lambda x}{2} = 0 \rightarrow (1)$

$F_y(x,y) = \frac{2(2x+y-10)}{5} + \frac{2\lambda y}{9} = 0 \rightarrow (2)$

from (1) and (2), solve for x, y and λ. This will give you critical points, substituting which into $(\star)$ will give you the shortest

3(b) → Compute $f_{xy}(0,0)$ and $f_{yx}(0,0)$ for the function.

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$$f(x,y)=\begin{cases}\dfrac{xy^3}{x+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

Also discuss the continuity of f_{xy} and f_{yx} at $(0,0)$.

Sol^n →

$$f(x,y)=\begin{cases}\dfrac{xy^3}{x+y^2}, & (x,y)\neq(0,0)\\ 0, & (x,y)=(0,0)\end{cases}$$

(i) $f_{xy}(0,0)=\lim\limits_{x\to 0}\dfrac{f_y(0+h,0)-f_y(0,0)}{h}$

$f_y(h,0)=\lim\limits_{x\to 0}\dfrac{f(h,0+k)-f(h,0)}{h}$

$=\lim\limits_{x\to 0}\dfrac{\frac{hx^3}{h+x^2}-0}{k}=0.$

$f_{xy}(0,0)=\lim\limits_{x\to 0}\dfrac{0-0}{x}=0$

(ii) $f_{yx}(0,0)=\lim\limits_{x\to 0}\dfrac{f_x(0,0+k)-f_x(0,0)}{k}$

$f_x(0,h)=\lim\limits_{h\to 0}\dfrac{f(0+h,x)-f_x(0,k)}{h}$

$\lim\limits_{x\to 0}\dfrac{\frac{hx^3}{h+x^2}-0}{h}=K.$

$f_{yx}(0,0)=\lim\limits_{x\to 0}\dfrac{K-0}{K}=1.$

Clearly. f_{xy} and f_{yx} are not equal at $(0,0)$

$\therefore$ f_{xy} and f_{yx} are not continuous at $(0,0)$.

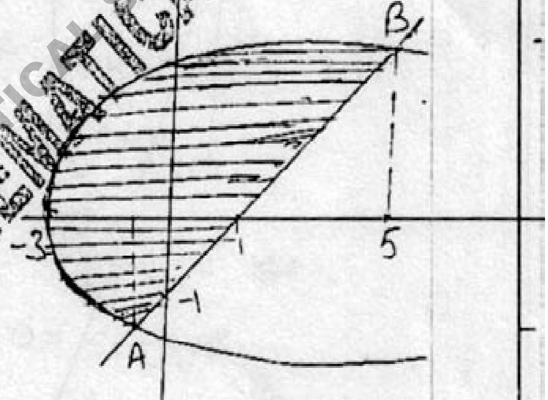
3(c). IAS-2013. Evaluate $\iint_D xy\,dA$, where D is the region bounded by the line $y=x-1$ and the parabola $y^2=2x+6$.

Sol'n: solving the line and parabola we get

A(−1, −2) and B(5, 4)

$\therefore \iint_D xy\,dA$ where D is the region given by

$$D=\left\{(x,y)\in\mathbb{R}^2 : y\in[-2,4],\ \frac{y^2}{2}-3\le x\le y+1\right\}$$



$$\therefore \iint_D xy\,dA=\int_{-2}^{4}\int_{\frac{y^2}{2}-3}^{y+1} xy\,dx\,dy$$

$$=\int_{-2}^{4}\left[\frac{x^2}{2}\cdot y\right]_{\frac{y^2}{2}-3}^{y+1}dy$$

$$=\frac{1}{2}\int_{-2}^{4} y\left[(y+1)^2-\left(\frac{y^2}{2}-3\right)^2\right]dy$$

$$=\frac{1}{2}\int_{-2}^{4} y\left[-\frac{y^4}{4}+4y^3+2y^2-8y\right]dy$$

$$=\frac{1}{2}\left[-\frac{y^6}{24}+y^4+\frac{2y^3}{3}-4y^2\right]_{-2}^{4}$$

$$=36.$$

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4(a) Show that three mutually perpendicular tangent lines can be drawn to the sphere $x^2+y^2+z^2=r^2$ from any point on the sphere $2(x^2+y^2+z^2)=3r^2$.

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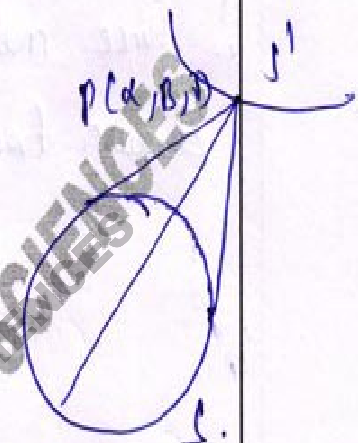
Sol^n

let S': $x^2+y^2+z^2=\frac{3r^2}{2}$ and



$P(\alpha,\beta,\gamma)$ be any point on S'

$$\alpha^2+\beta^2+\gamma^2=\frac{3r^2}{2} \rightarrow (1)$$

Enveloping cone of S with P as vertex

S: $x^2+y^2+z^2=r^2$ is given by $SS_1=T^2$

i.e $(x^2+y^2+z^2-r^2)(\alpha^2+\beta^2+\gamma^2-r^2)$

$$=(\alpha x+\beta y+\gamma z-r^2)^2 \rightarrow (2)$$

this cone has 3 mutually ⊥er generators (which are the tangent lines to the sphere) iff Coeff of x^2 + Coeff of y^2 + coeff of $z^2=0$.

$\therefore$ from (2) $(\alpha^2+\beta^2+\gamma^2-r^2-\alpha^2)$

$$+(\alpha^2+\beta^2+\gamma^2-r^2-\beta^2)+(\alpha^2+\beta^2+\gamma^2-r^2-\gamma^2)=0$$

$\Rightarrow 2(\alpha^2+\beta^2+\gamma^2)-3r^2=0$

$2\left(\frac{3r^2}{2}\right)-3r^2=0 \Rightarrow 0=0$ Satisfied

$\therefore$ Three mutually perpendicular lines can be drawn from any point on S' onto S.

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4(b) → A cone has for its guiding curve the circle $x^2+y^2+2ax+2by=0$, $z=0$ and passes through a fixed point $(0,0,c)$. If the section of the cone by the plane $y=0$ is a rectangular hyperbola, prove that the vertex lies on the fixed circle

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$$x^2+y^2+z^2+2ax+2by=0$$

$$2ax+2by+cz=0$$

Sol^n → let (α,β,γ) be the vertex of the cone. Any line through (α,β,γ) is

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \rightarrow (1)$$

It meets the plane $z=0$ in $\left(\alpha-\frac{l\gamma}{n}, \beta-\frac{m\gamma}{n}, 0\right)$

and if this point lies on the given conic, we have

$$\left(\alpha-\frac{l\gamma}{n}\right)^2+\left(\beta-\frac{m\gamma}{n}\right)^2+2a\left(\alpha-\frac{l\gamma}{n}\right)+2b\left(\beta-\frac{m\gamma}{n}\right)=0 \rightarrow (2)$$

Eliminating l, m, n between (i) and (ii), the equation of the cone is

$$\left[\alpha - \left(\frac{x-\alpha}{z-\gamma}\right)\gamma\right]^2 + \left[\beta - \left(\frac{y-\beta}{z-\gamma}\right)\gamma\right]^2 +$$

$$2a\left[\alpha - \left(\frac{x-\alpha}{z-\gamma}\right)\gamma\right] + 2b\left[\beta - \left(\frac{y-\beta}{z-\gamma}\right)\gamma\right] = 0$$

(or)

$$(\alpha z - x\gamma)^2 + (\beta z - y\gamma)^2 + 2a(\alpha z - x\gamma)(z-\gamma) + 2b(\beta z - y\gamma)(z-\gamma) = 0$$

If this cone passes through $(0, 0, c)$ then

$$(\alpha c)^2 + (\beta c)^2 + 2a(\alpha c)(c-\gamma) + 2b(\beta c)(c-\gamma) = 0 \quad \to (3)$$

Again the section of cone by zx-plane i.e. $y=0$. is

$$(\alpha z - \gamma x)^2 + (\beta z)^2 + 2a(\alpha z - x\gamma)(z-\gamma) + 2b(\beta z)(z-\gamma) = 0.$$

and if this section is a rectangular hyperbola in the zx plane then the sum of the coefficients of x^2 and z^2 should be zero.

$$\gamma^2 + (\alpha^2 + \beta^2 + 2a\alpha + 2b\beta) = 0. \longrightarrow \text{(iv)}.$$

$\therefore$ The locus of (α, β, γ) from (iii) and (iv)

$$c(x^2+y^2) + 2ax(c-z) + 2by(c-z) = 0 \longrightarrow \text{(v)}.$$

$$x^2+y^2+z^2+2ax+2by = 0. \longrightarrow \text{(vi)}$$

Multiplying (vi) by c and subtracting (v) from the result so obtained we get

$$cz^2 + 2azx + 2byz = 0 \text{ (or)}$$

$$2ax + 2by + cz = 0. \longrightarrow \text{(vii)}$$

which is ~~the~~ equation of a plane.

4(c) → A variable generator meets two generators of the system through the extremities B and B' of the minor axis of the principal elliptic section of the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-z^2/c^2=1$ in P and P'. Prove that $BP \cdot B'P' = a^2+c^2$.

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Sol^n → We have the hyperboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

Generating lines of hyperboloid can be written in the standard form:

$$\frac{x-a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{-b\cos\theta}=\frac{z}{c} \quad \text{—①}$$

and $$\frac{x-a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{-b\cos\theta}=\frac{z}{-c} \quad \text{—②}$$

from ①, we can obtained the equations of two generators passing through minor axis by putting $\theta=90°$ and $\theta=-90°$

$$\frac{x}{a}=\frac{y-b}{0}=\frac{z}{c} \quad \text{—③}$$

and $$\frac{x}{-a}=\frac{y+b}{0}=\frac{z}{c} \quad \text{—④}$$

According to the question

$B = (0, b, 0)$

$B' = (0, -b, 0)$

further, P is the intersection of (2) and (3)

$$P = \left(\frac{a\cos\theta}{1+\sin\theta}, b, \frac{\cos\theta}{c(1+\sin\theta)}\right)$$

and P' is the intersection of (2) & (4):

$$P' = \left(\frac{-a\cos\theta}{\sin\theta - 1}, -b, \frac{\cos\theta}{c(\sin\theta-1)}\right)$$

Now, we just apply distance formula.

$BP \cdot B'P'$ – then we get

$$BP \cdot B'P' = a^2 + c^2$$

Hence, proved.

5(a). y is a function of x, such that the differential coefficient $\frac{dy}{dx}$ is equal to $\cos(x + y)$ + $\sin(x + y)$. Find out a relation between x and y, which is free from any derivative/ differential.

SOLUTION

Given $\frac{dy}{dx} = \cos(x+y) + \sin(x+y)$ .....(1)

take x+y = t $\Rightarrow 1 + \frac{dy}{dx} = \frac{dt}{dx}$

$$\frac{dt}{dx} - 1 = \cos t + \sin t$$

$$\frac{dt}{dx} = 1 + \cos t + \sin t$$

$$\frac{dt}{1+\cos t+\sin t} = dx$$

$$\frac{dt}{2\cos^2 t/2 + 2\sin t/2\, \cos t/2} = dx$$

Integrating both sides

$$\int \frac{\frac{1}{2}\sec^2\frac{t}{2}\,dt}{1+\tan\frac{t}{2}} = \int dx$$

$$\log\left(1+\tan\frac{t}{2}\right) = x + c$$

$$\boxed{\log\left(1+\tan\left(\frac{x+y}{2}\right)\right) = x + c}$$

Required solution.

5(b). Obtain the equation of the orthogonal trajectory of the family of curves represented by $r^n = a \sin n\theta$, (r, θ) being the plane polar coordinates.

SOLUTION

Given $$r^n = a \sin n\theta \quad \text{.....(1)}$$

applying log on both sides

$$n \log r = \log a + \log \sin n\theta$$

differenting w.r.t. θ.

$$\frac{n}{r}\frac{dr}{d\theta} = n \cot n\theta$$

for finding orthogonal trajectories replace

$\dfrac{dr}{d\theta}$ by $\dfrac{-r^2 d\theta}{dr}$

$$\therefore \quad \frac{n}{r}\left(\frac{-r^2 d\theta}{dr}\right) = n \cot n\theta$$

$$\frac{d\theta}{\cot n\theta} = \frac{-dr}{r}$$

integerating both sides

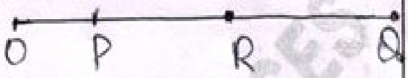
$$\frac{\log(\sec n\theta)}{n} = -\log r + c$$

∴ Simplifying

$\boxed{r^n = K \cos n\theta}$ K being arbitrary constant

5(c) IAS-2013 P-I → A body is performing S.H.M in a straight line OPQ. Its velocity is zero at points P and Q whose distances from O are x and y respectively and its velocity is v at the mid-point between P and Q. Find the time of one complete oscillation.

Sol^n: from figure P and Q are the positions of instantaneous rest in a S.H.M. Let R be the middle point of PQ. Then R is the Centre of the motion.



Also it is given that OP = a, OQ = b.

The amplitude of the motion $= \frac{1}{2} PQ = \frac{1}{2}(OQ - OP)$
$= \frac{1}{2}(y - x)$

Now in a S.H.M the velocity at the Centre $= \sqrt{\mu} \times$ amplitude. Since in this case the velocity at the Centre is given to be v.

$$\therefore\ v = \frac{1}{2}(y-x).\sqrt{\mu} \Rightarrow \sqrt{\mu} = 2v/(y-x)$$

Hence time period $T = \frac{2\pi}{\sqrt{\mu}} = 2\pi\left[(y-x)/2v\right]$

$$= \frac{\pi(y-x)}{v}$$

5(d) (1)

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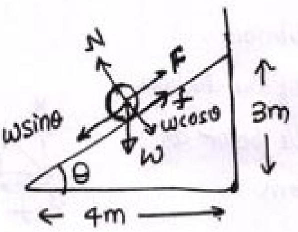
P-1



$\tan\theta = \frac{3}{4}$

$\sin\theta = \frac{3}{5}$

$\cos\theta = \frac{4}{5}$

where

F : force applied parallel to inclined plane

f : friction force

N : Normal reaction

w : weight of the body.

θ : inclination angle

$\Rightarrow \quad w\sin\theta = F + f \quad \text{———} (1)$

$w\cos\theta = N \quad \text{———} (2)$

$f = \mu N = \mu w\cos\theta$ where μ is coefficient of friction.

(1) ⇒

$w\sin\theta = F + \mu w\cos\theta$

$$\therefore \mu = \frac{w\sin\theta - F}{w\cos\theta} = \tan\theta - \frac{F}{w\cos\theta}$$

$$= \frac{3}{4} - \frac{8 \times 5}{20 \times 4}$$

$$= \frac{3}{4} - \frac{1}{2} = \frac{1}{4}.$$

$$\therefore \boxed{\mu = \frac{1}{4}}$$

5(e) Show that the curve

$$\vec{x}(t) = t\hat{i} + \left(\frac{1+t}{t}\right)\hat{j} + \left(\frac{1-t^2}{t}\right)\hat{k}$$

lies in a plane.

SOLUTION

$$\vec{r}(t) = t\hat{i} + \left(\frac{1+t}{t}\right)\hat{j} - \left(\frac{1-t^2}{t}\right)\hat{k}$$

$$\frac{d\vec{r}}{dt} = \left((i) - \frac{1}{t^2}\right)\hat{j} - \left(-1 - \frac{1}{t^2}\right)\hat{k}$$

$$\frac{d^2\vec{r}}{dt^2} = 0\hat{i} + \frac{2}{t^3}\hat{j} - \left(\frac{2}{t^3}\right)\hat{k}$$

$$\frac{d^3r}{dt^3} = \frac{-6}{t^4}\hat{j} + \frac{6}{t^4}\hat{j}$$

$$\left[\frac{d\vec{r}}{dt}\,\frac{d^2r}{dt^2}\,\frac{d^3r}{dt^3}\right] = \begin{vmatrix} 1 & -\frac{1}{t^2} & \left(1+\frac{1}{t^2}\right) \\ 0 & \frac{2}{t^3} & \frac{-2}{t^3} \\ 0 & \frac{-6}{t^4} & \frac{6}{t^4} \end{vmatrix}$$

$$1\left[\frac{12}{t^7} - \left[\frac{12}{t^7}\right]\right] = 0$$

$\therefore$ $z = 0$ so the curve lies in the plane

6(a). Solve the differential equation

$(5x^3 + 12x^2 + 6y^2)\, dx + 6xy\, dy = 0.$

SOLUTION

Given that,

$(5x^3 + 12x^2 + 6y^2)\, dx + 6xy\, dy = 0$....(1)

Comparing with Mdx + Ndy = 0

$$M = 5x^3 + 12x^2 + 6y^2$$
$$N = 6xy$$
$$\frac{\partial M}{\partial y} = 12y$$
$$\frac{\partial N}{\partial x} = 6y$$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$ given equation is not exact

To find I.F.

$$\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{6y}{6xy} = \frac{1}{x} = f(x)$$

$\therefore$ $$\text{I.F.} = e^{\int f(x)dx} = e^{\int \frac{1}{x}dx} = x$$

$\therefore$ Multiplying (1) with I.f.

We have $(5x^4 + 12x^3 + 6y^2x)\, dx + (6x^2y)\, dy = 0$

$\therefore$ Solution is given by $\int Mdx + \int Ndy = 0$

y constant y terms not containing

$\int \left(5x^4 + 12x^3 + 6y^2x\right)dx + 0 = C$

$\boxed{x^5 + 3x^4 + 3x^2y^2 = C}$

6(b). Using the method of variation of parameters, solve the differential equation $\frac{d^2y}{dx^2} + a^2y = \sec ax.$

SOLUTION

Given that,

$$\frac{d^2y}{dx^2} + a^2y = \sec ax \qquad .....(1)$$

$$(1) \equiv \left(D^2 + a^2\right)y = \sec ax$$

where $\qquad D = \frac{d}{dx}$

Auxiliary equation given by

$m^2 + a^2 = 1 \Rightarrow m = \pm ia$

$\therefore$ C. F $= C_1 \cos ax + C_2 \sin ax$

Assuming $\qquad y_p = Au + Bv$

where $\qquad u = \cos ax;\ v = \sin ax$

u.v needs to be linearly independent

$$w(u.v) = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a(\cos^2 ax + \sin^2 ax) = a(1)$$

$$a \neq 0$$

$$\therefore \qquad A(x) = \int \frac{-vR}{w}dx$$

$$= \int \frac{-\sin ax.\sec ax}{a}dx$$

$$A = \frac{-\log(\sec ax)}{a^2}$$

$$B = \int \frac{uR}{w}dx$$

$$= \int \frac{\cos ax.\sec ax}{a}dx$$

$$= \frac{x}{a}$$

$$\therefore \qquad y_p = \frac{-\log(\sec ax)}{a^2}\cdot\cos ax + \frac{x}{a}\sin ax$$

$$\therefore \qquad y = y_c + y_p$$

$$\boxed{\begin{array}{l} y = c_1\cos ax + c_2\sin ax + \frac{x}{a}\sin ax \\ \quad + \frac{\log(\cos ax)\cos ax}{a^2} \end{array}}$$

Required Solution.

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6(c). Find the general solution of the equation

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=\ln x\sin(\ln x).$$

SOLUTION

Given that,

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y=\ln x\sin(\ln x) \quad .....(1)$$

Put $x=e^z, z=\log x$

transforming D^2, D,

$$x^2D^2y=D_1(D_1-1)y$$
$$xDy=D_1y$$

where $D=\frac{d}{dx}$

$$D_1=\frac{d}{dz}$$

$\therefore$ given equation transform to

$(D_1(D_1-1)+D_1+1)y=z\sin z$

$$\left(D_1^2+1\right)y=z\sin z \quad .....(2)$$

Auxiliary equation of (2) is given by

$$m^2+1=0\Rightarrow m=i,-i$$

$\therefore$ $y_c=C_1\cos z+C_2\sin z$

$$y_p=\frac{1}{D_1^2+1}(z\sin z)$$

$$=\text{Imaginary part } \text{Im}\left[\frac{1}{D_1^{\,2}+1}\left(ze^{iz}\right)\right]$$

$$=\text{Im.P.}\left[e^{iz}\frac{z}{(D_1+i)^2+1}\right]$$

$$=\text{Im.P.}\left[e^{iz}\frac{z}{D_1^{\,2}+2D_1i}\right]$$

$$=\text{Im.P.}\left[e^{iz}\frac{1}{2D_1i}\left(1-\frac{D_1}{2i}+\left(\frac{D_1}{zi}\right)^2.....\right)z\right]$$

$$=\text{Im.P.}\left[e^{iz}\cdot\left(\frac{-z^2i}{4}+\frac{z}{4}\right)\right]$$

$$=\frac{-z^2}{4}\cos z+\frac{z}{4}\sin z$$

$$y=y_c+y_p$$

$$y=C_1\cos z+C_2\sin z+\frac{z}{4}\sin z-\frac{z^2}{4}\cos z$$

$$\boxed{y=C_1\cos(\log x)+C_2\sin(\log x)+\frac{(\log x)}{4}\sin(\log x)-\frac{(\log x)^2}{4}\cos(\log x)}$$

Required solution

6(d). By using Laplace transform method, solve the differential equation $(D^2 + n^2)\, x = a \sin(nt + \alpha)$, $D^2 = \frac{d^2}{dt^2}$ subject to the initial conditions $x = 0$ and $\frac{dx}{dt} = 0$, at $t = 0$, in which a, n and α are constants.

SOLUTION

Given $(D^2 + n^2)\, x = a \sin(nt + \alpha)$ .....(1)

Applying Laplace transform on both sides

$S^2\, \mathcal{L}(x) - S.\, x(0) - x'(0) + n^2\, \mathcal{L}(x) = \mathcal{L}[a \sin nt \cos\alpha + a \cos nt.\, \sin\alpha]$

$$(S^2 + n^2)\, \mathcal{L}(x) = a\cos\alpha \cdot \frac{n}{S^2+n^2} + a\sin\alpha \cdot \frac{S}{S^2+n^2}$$

$$\mathcal{L}(n) = a\cos\alpha \frac{n}{\left(S^2+n^2\right)} + a\sin\alpha \frac{S}{\left(S^2+n^2\right)}$$

$$X(t) = a\cos\alpha\, \mathcal{L}^{-1}\left(\frac{n}{\left(S^2+n^2\right)^2}\right) + a\sin\alpha\, \mathcal{L}^{-1}\left(\frac{S}{\left(S^2+n^2\right)^2}\right) \quad .....(2)$$

we know

$$\mathcal{L}^{-1}\frac{n}{\left(S^2+n^2\right)} = \sin(nt)$$

$$\mathcal{L}^{-1}\left(\frac{d}{dS}\left(\frac{n}{S^2+n^2}\right)\right) = (-1)t.\sin(nt)$$

$$\therefore \quad \mathcal{L}^{-1}\left(\frac{S}{\left(S^2+n^2\right)^2}\right) = \int_0^t \sin nt \quad .....(3)$$

again,

$$\mathcal{L}^{-1}\left(\frac{1}{S}\cdot\frac{S}{(S^2+n^2)^2}\right) = \int_0^t \frac{t}{2n}\cdot \sin nt\, dt$$

$$\mathcal{L}^{-1}\left(\frac{S}{(S^2+n^2)^2}\right) = \frac{1}{2n}\left[t\cdot\frac{(-\cos nt)}{n} - \frac{(-\sin nt)}{n^2}\right]$$

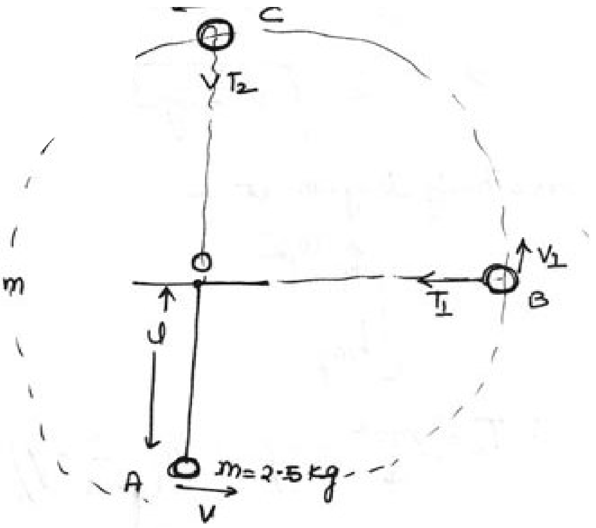
$$\mathcal{L}^{-1}\left(\frac{1}{(S^2+n^2)^2}\right) = \frac{1}{2n^3}\left[-nt\cos nt + \sin nt\right] \quad .....(4)$$

using (3) & (4) in (2), we obtain.

$$\boxed{\begin{array}{l} X(t) = \dfrac{a}{2n^2}\left(\sin nt - nt\cos nt\right) \\ \cos\alpha + \dfrac{at}{2n}\left(\sin nt\right)\cdot \sin\alpha \end{array}}$$

$$\boxed{X(t) = \frac{a}{2n^2}\sin nt - \frac{at}{2n}\cos\left(nt+\alpha\right)}$$

Required solution.



7(a)
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l = length of the string = 0.9 m

$m = 2.5\,kg$

$V = 8\,m/sec$

$g = 9.8\,m/sec^2$

Let $V_1, T_1 \rightarrow$ velocity and tension in the string at B

$V_2, T_2 \rightarrow$ Velocity and tension in the string at C

$\Rightarrow$ Applying conservation of Total mechanical energy at A & B, taking Potential energy at A = 0.

$\therefore$ Potential Energy at A (P.E) = 0

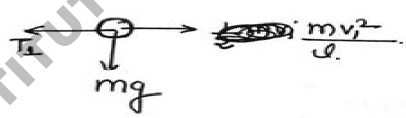
Kinetic energy at A $= \frac{1}{2}mV^2$

Sly P.E at B = mgl

K.E at B $= \frac{1}{2}mV_1^2$

$\therefore\ 0 + \frac{1}{2}mV^2 = mgl + \frac{1}{2}mV_1^2 \Rightarrow V_1 = \sqrt{2\left(\frac{V^2}{2} - gl\right)} = \sqrt{V^2 - 2gl}$
$= 6.81\,m/sec$ upward

Free body diagram at B.



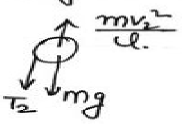
$\therefore\ T_1 = \frac{mV_1^2}{l} = 128.8\,N\ (\leftarrow)$

Sly applying conservation of Total Mechanical energy b/w A & C, we get

$0 + \frac{1}{2}mV^2 = mg\cdot 2l + \frac{1}{2}mV_2^2$

$\Rightarrow\ V_2 = \sqrt{V^2 - 4gl} = 5.36\,m/sec$ leftward.

Free body diagram at C



$\therefore\ T_2 = \frac{mV_2^2}{l} - mg = m\left(\frac{V_2^2}{l} - g\right)$

$= 55.28\,N$ (downward)

7(b)

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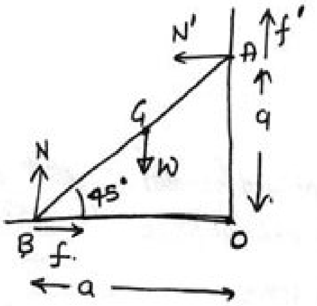
FBD under equilibrium:-
W is the weight of the ladder acting at middle point as ladder is uniform.



$AO = OB = a$

$N, f \rightarrow$ is the Normal reaction and friction force exerted by ground at point B.

$N', f' \rightarrow$ is the normal rxn and friction force exerted by wall at point A.

$\Rightarrow$ max^m value of $f = \mu N$.

max^m value of $f' = \mu' N'$

$\Rightarrow$ When force is applied so as to move the ladder, it will be min^M when it is applied at G.

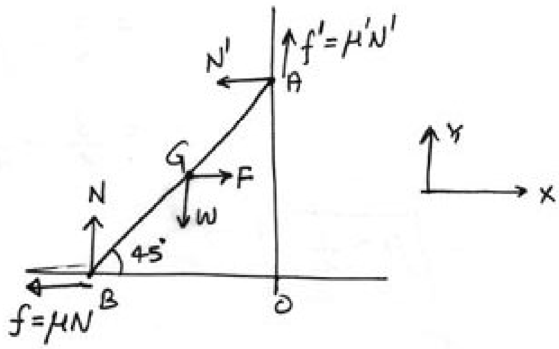
New Free body diagram with force 'F'

Balancing force in X direction

$F = \mu N + N'$ —— (1)

in Y direction:-

$W = N + \mu' N'$ —— (2)



(3) Σ Moment about $G = 0$

$\Rightarrow N' \cdot \frac{a}{2} + (\mu' N') \cdot \frac{a}{2} = N \cdot \frac{a}{2} + \mu N \cdot \frac{a}{2}$

$\Rightarrow \frac{N'}{N} = \frac{(1+\mu)}{1+\mu'}$ —— (3)

(2) $\equiv$
$W = N\left[1 + \frac{\mu'(1+\mu)}{1+\mu'}\right]$ —— (4) $\Rightarrow N = \frac{W}{\left[1 + \frac{\mu'(1+\mu)}{1+\mu'}\right]}$

Cont.

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Cont'

(1) ≡ $F = \cancel{\mu} N\left(\mu + \frac{1+\mu}{1+\mu'}\right)$

$$= \frac{W}{\frac{1+\mu'(1+\mu)}{(1+\mu')}} * \left(\mu + \frac{1+\mu}{1+\mu'}\right)$$

$$= \frac{W}{(1+\mu') + \mu'(1+\mu)} * \left[\mu(1+\mu') + (1+\mu)\right]$$

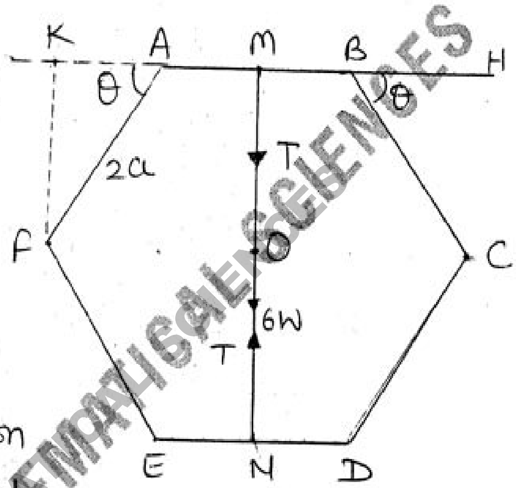
$$\boxed{F = W * \frac{[2\mu + \mu'\mu + 1]}{[2\mu' + \mu\mu' + 1]}}$$

7(c) IAS-2013 P-I

Six equal rods AB, BC, CD, DE, EF and FA each of weight W and are freely joined at their extremities so as to form a hexagon. the rod AB is fixed in a horizontal position and the middle of AB and DE are joined by a string; prove that its tension is 3W.

Sol^n

ABCDEF is a hexagon formed of six equal rods each of weight W and say of length 2a. The rod AB is fixed in a horizontal position and the middle points M and N of AB and DE are joined by a string



Let T be the tension in the string MN. The total weight 6W of all the six rods AB, BC etc. can be taken acting at O, the middle point of MN. Let $\angle FAK = \theta = \angle CBH$

Give the system a small symmetrical displacement about the vertical line MN in which θ changes to $\theta + \delta\theta$. The line AB remains fixed. The length of the rods AB, BC etc. remain fixed, the length MN changes and the point O also changes.

We have $MN = 2MO = 2KF = 2AF\sin\theta$

$= 4a\sin\theta$

Also the depth of O below the fixed line AB

$= MO = 2a\sin\theta$.

By the principle of virtual work, we have.

$$-T\delta(4a\sin\theta) + 6W\delta(2a\sin\theta) = 0$$

or $\quad -4aT\cos\theta\,\delta\theta + 12aW\cos\theta\,\delta\theta = 0$

or, $\quad 4a[-T+3W]\cos\theta\, d\theta = 0$

or, $\quad -T+3W = 0$

or, $\quad T = 3W \qquad (\because \delta\theta \neq 0$ and $\cos\theta \neq 0)$

8(a). Calculate $\nabla^2(r^n)$ and find its expression in terms of *r* and *n*, *r* being the distance of any point (*x,y,z*) from the origin, n being a constant and ∇^2 being the Laplace operator.

SOLUTION

$$\nabla^2 r^n = \nabla\cdot\nabla r^n$$

$$\nabla r^n = \nabla\cdot\left(\frac{\partial}{\partial u}\hat{i}+\frac{\partial}{\partial y}\hat{j}+\frac{\partial}{\partial z}\hat{k}\right)r^n$$

$$= nr^{n-1}\left(\frac{\partial r}{\partial u}\hat{i}+\frac{\partial r}{\partial y}\hat{j}+\frac{\partial r}{\partial z}\hat{k}\right)$$

We know $r^2 = x^2 + y^2 + z^2$

$$\therefore \quad \frac{dr}{dx}=\frac{x}{r},\frac{\partial r}{\partial y}=\frac{y}{r},\frac{\partial r}{\partial z}=\frac{z}{r}$$

$$= nr^{n-1}\left(\frac{x}{r}\hat{i}+\frac{y}{r}\hat{j}+\frac{z}{r}\hat{k}\right)$$

$$\nabla r^n = nr^{n-2}\vec{r}$$

$$\nabla\cdot\nabla r^n = \nabla\cdot\left(nr^{r-2}\vec{r}\right)$$

Let $\phi = nr^{n-2}$

$$\therefore \quad \nabla(\phi\vec{r}) = \nabla\phi\cdot\vec{r}+\phi\nabla\cdot\vec{r}$$

$$= \nabla(nr^{n-2})\cdot\vec{r}+nr^{n-2}(\nabla\cdot\vec{r})$$

$$= n(n-2)r^{n-4}(\vec{r}\cdot\vec{r})+3nr^{n-2}$$

$$= n(n-2)r^{n-2}+3nr^{n-2}$$

$$\therefore \quad \nabla^2 r^n = r^{n-2}(n^2-2n+3n)$$

$$= n(n+1)\ r^{n-2}$$

IAS-2013 P-1

8(a) A curve in space is defined by the vector equation $\bar{r} = t^2\hat{i} + 2t\hat{j} - t^3\hat{k}$. Determine the angle between the tangents to this curve at the points $t=+1$ and $t=-1$. By using Divergence theorem of Gauss, evaluate the surface integral.

Sol^n: Given that $\vec{r} = t^2\hat{i} + 2t\hat{j} - t^3\hat{k}$

Tangent vector is given by

$$\frac{d\vec{r}}{dt} = 2t\hat{i} + 2\hat{j} - 3t^2\hat{k}$$

At $t=1$, $\frac{d\vec{r}}{dt} = 2\hat{i} + 2\hat{j} - 3\hat{k} = T_1$ (say)

and at $t=-1$, $\frac{d\vec{r}}{dt} = -2\hat{i} + 2\hat{j} - 3\hat{k} = T_2$ (say)

Angle between the tangents T_1 and T_2 is given by

$$\cos\theta = \frac{T_1 \cdot T_2}{|T_1||T_2|}$$

$$= \frac{(2\hat{i} + 2\hat{j} - 3k)(-2\hat{i} + 2\hat{j} - 3k)}{\sqrt{4+4+9}\,\sqrt{4+4+9}}$$

$$\cos\theta = \frac{-4+4+9}{\sqrt{17}\,\sqrt{17}} = 9/17$$

$$\Rightarrow \theta = \cos^{-1}\left(9/17\right)$$

8(b) Find the curvature (κ) and torsion (τ) for the space curve $x = t - t^3/3$, $y = t^2$, $z = t + t^3/3$.

Sol^n: Ans $\kappa = \tau = \frac{1}{(1+t^2)^2}$

8(c). By using Divergence theorem of Gauss, evaluate the surface integral.

IAS-2013 P-I

$\iint (a^2x^2+b^2y^2+c^2z^2)^{-1/2} ds$, where S is the surface of the ellipsoid $ax^2+by^2+cz^2=1$, a, b & c being all +ve constants.

Sol'n: Let us first put the integral

$$\iint_S (a^2x^2+b^2y^2+c^2z^2)^{-1/2} ds \text{ in the form}$$

$$\iint_S F.n\,ds,$$

where n is a unit normal vector to the closed surface S whose equation is $ax^2+by^2+cz^2=1$.

The normal vector to $\phi(x,y,z) \equiv ax^2+by^2+cz^2-1=0$ is

$$= \nabla\phi = 2ax\hat{i}+2by\hat{j}+2cz\hat{k}$$

$$\therefore n = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2ax\hat{i}+2by\hat{j}+2cz\hat{k}}{\sqrt{(4a^2x^2+4b^2y^2+4c^2z^2)}} = \frac{ax\hat{i}+by\hat{j}+cz\hat{k}}{\sqrt{(a^2x^2+b^2y^2+c^2z^2)}}$$

Here we are to choose F such that

$$F.n = \frac{1}{\sqrt{a^2x^2+b^2y^2+c^2z^2}} \text{ on } S.$$

Obviously $F = x\hat{i}+y\hat{j}+z\hat{k}$, because then

$$F.n = \frac{ax^2+by^2+cz^2}{\sqrt{a^2x^2+b^2y^2+c^2z^2}} = \frac{1}{\sqrt{(a^2x^2+b^2y^2+c^2z^2)}} \text{ on } S.$$

Note that on S, $ax^2+by^2+cz^2=1$

Now $$\iint_S \frac{1}{\sqrt{(a^2x^2+b^2y^2+c^2z^2)}}\,ds$$

$$= \iint_S F\cdot n\, ds, \text{ where } F = x\hat{i} + y\hat{j} + z\hat{k}$$

$$= \iiint_V (\nabla\cdot F)\, dv$$ by divergence theorem; V is the volume enclosed by surface S.

$$= \iiint_V \left[\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)\right] dV$$

$$= \iiint_V 3\, dV$$

$$= 3V = 3\cdot\frac{4}{3}\pi\cdot\frac{1}{\sqrt{a}}\cdot\frac{1}{\sqrt{b}}\cdot\frac{1}{\sqrt{c}}$$

$$= \frac{4\pi}{\sqrt{abc}}$$