IFoS - 2017 → Paper II

5)(d) Evaluate $\int_0^1 e^{-x^2} dx$ using the Composite trapezoidal rule with four decimal precission. i.e, with the absolute value of the error not exceeding 5×10^{-5}.

⇒ we choose ten interval for this problem.

$\therefore f(x) = e^{-x^2}$, $a=0$, $b=1$, $n=10 \Rightarrow h = \frac{1-0}{10} = 0.1$

x_i $i=0$ to 10	y_i $i=0$ to 10	y_i $i=0,10$	y_i $i=1,2,3,4,5,6,7,8,9$
$x_0 = 0$	1.00000	1.00000	—
$x_1 = 0.1$	0.99005	—	0.99005
$x_2 = 0.2$	0.96079	—	0.96079
$x_3 = 0.3$	0.91393	—	0.91393
$x_4 = 0.4$	0.85214	—	0.85214
$x_5 = 0.5$	0.77880	—	0.77880
$x_6 = 0.6$	0.69768	—	0.69768
$x_7 = 0.7$	0.61263	—	0.61263
$x_8 = 0.8$	0.52792	—	0.52792
$x_9 = 0.9$	0.44486	—	0.44486
$x_{10} = 1.0$	0.36788	0.36788	—
		$\Sigma y_i = 1.36788 (=Y_1)$	$\Sigma y_i = 6.77880 (=Y_2)$

Now, by The Trapezoidal rule,

$$\int_0^1 e^{-x^2} dx = \frac{h}{2}\left[y_0 + y_{10} + 2(y_1+y_2+y_3+y_4+y_5+y_6+y_7+y_8+y_9)\right]$$

$$= \frac{h}{2}[Y_2 + 2Y_2] = \frac{0.1}{2} \times 14.92548$$

$$= 0.746274$$

$$\simeq 0.7463 \text{ (Approximate value)}$$

7) (a) Find the real root of the equation, $x^3+x^2+3x+4=0$, correct upto five places of decimal using Newton-Raphson method.

⇒ Let, $f(x) = x^3+x^2+3x+4$, there ~~is~~ no '+ve' real root.

∴ $f(-2)=-6<0$ & $f(-1)=1>0$, $f'(x)=3x^2+2x+3$

Thus $f(x)=0$ has a real root between (-2) and (-1)

Now, we take, $x_0=-1$ & compute successive approx. in a table

n	x_n	$f(x_n)$	$f'(x_n)$	$h_n = -\frac{f(x_n)}{f'(x_n)}$	$x_{n+1}=x_n+h_n$
0	−1	1	4	−0.25	−1.25
1	−1.25	−0.140625	5.1875	0.027108	−1.222892
2	−1.222892	−0.002003	5.040611	0.000397	−1.222495
3	−1.222495	−0.000002	5.038492	0.0000004	−1.2224946

Thus, −1.22249 is the root of the given equation, correct upto four decimal places.

7) (b) A river is 80 metre wide, the depth y, in metre of the river at the distance x from ~~the~~ one bank is given by the following table,

x	0	10	20	30	40	50	60	70	80
y	0	4	7	9	12	15	14	8	3

Find the area of cross-section of the river using Simpson's 1/3rd rule.

⇒ Let A be the area of the cross-section,

then $A = \int_0^{80} (xy)\,dx$

$= \int_0^{80} z\,dx$ where $z = xy$

and $h=10$.

x_i $i=0$ to 8	z_i $i=0$ to 8	z_i $i=0,8$	z_i $i=1,3,5,7$	z_i $i=2,4,6$
0	0	0	—	—
10	40	—	40	—
20	140	—	—	140
30	270	—	270	—
40	480	—	—	480
50	750	—	750	—
60	840	—	—	840
70	560	—	560	—
80	240	240	—	—
		$\sum z_i = 240 (= Y_1)$	$\sum z_i = 1620 (= Y_2)$	$\sum z_i = 1460 (= Y_3)$

Now by Simpson's $1/3$rd rule,

$$A = \frac{h}{3}\left[z_0 + z_8 + 4(z_1 + z_3 + z_5 + z_7) + 2(z_2 + z_4 + z_6)\right]$$

$$= \frac{h}{3}\left[Y_1 + 4Y_2 + 2Y_3\right]$$

$$= \frac{10}{3}\left[240 + (4 \times 1620) + (2 \times 1460)\right]$$

$$= \frac{10}{3}\left[9640\right] = 10711.11 \text{ sq.m.}$$

7) (c) Find y for $x = 0.2$ taking $h = 0.1$ by modified euler method and compute the error, given that,

$$\frac{dy}{dx} = x + y, \quad y(0) = 1.$$

$\Rightarrow$ here, $f(x,y) = x + y$, $x_0 = 0$, $y_0 = 1$, $h = 0.1$

$\therefore f(x_0, y_0) = 0 + 1 = 1$

we have, $y_1^{(1)} = y_0 + h f(x_0, y_0) = 1 + 0.1 = 1.1$

$\therefore f(x_1, y_1^{(1)}) = 0.1 + 1.1 = 1.2$

$\therefore$ 2nd Approximation of y_1 is,

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right] = 1.11$$

$\therefore f(x_1, y_1^{(2)}) = 0.1 + 1.11 = 1.21$

$\therefore y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(2)})]$

$= 1 + \frac{0.1}{2}[1 + 1.21] = 1.1105$

$f(x_1, y_1^{(3)}) = 0.1 + 1.1105 = 1.2105$

$y_1^{(4)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(3)})] = 1.110525$

since, $y_1^{(3)} = y_1^{(4)}$ hence, $\boxed{y_1 = 1.1105}$

Now, $y_2^{(1)} = y_1 + h f(x_1, y_1)$

$= 1.1105 + 0.1\{0.1 + 1.1105\} = 1.23155$

$f(x_2, y_2^{(1)}) = 0.2 + 1.23155 = 1.43155$

Then, $y_2^{(2)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(1)})]$

$= 1.1105 + \frac{0.1}{2}[\{0.1 + 1.1105\} + 1.43155] = 1.2426025$

$f(x_2, y_2^{(2)}) = 1.4426025$

Then, $y_2^{(3)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(2)})]$

$= 1.1105 + \frac{0.1}{2}[\{0.1 + 1.1105\} + 1.4426025]$

$= 1.243155$

Then $y_2^{(4)} = y_1 + \frac{h}{2}[f(x_1, y_1) + f(x_2, y_2^{(3)})]$

$= 1.1105 + \frac{0.1}{2}[\{0.1 + 1.1105\} + \{0.2 + 1.243155\}]$

$= 1.24318275$

$\therefore y_2^{(3)} = y_2^{(4)} = 1.2432$ (Correct upto 4 decimal place)

Hence $y = 1.2432$, is the solution of the given equation at $x = 0.2$.