2013 IFOS

Eliminate the arbitrary function f from the given equation
$f(x^2+y^2+z^2,\ x+y+z)=0$

Solution: Let $u = x^2+y^2+z^2$ and $v = x+y+z$

The given equation $f(x^2+y^2+z^2,\ x+y+z)=0$ becomes
$f(u,v)=0$

Differenting it partially w.r.t x and y respectively, we get $\frac{\partial f}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial f}{\partial v}\cdot\frac{\partial v}{\partial x}=0$ and $\frac{\partial f}{\partial u}\cdot\frac{\partial u}{\partial y}+\frac{\partial f}{\partial v}\frac{\partial v}{\partial y}=0$

$\Rightarrow \frac{\partial f}{\partial u}(2x+2zp)+\frac{\partial f}{\partial v}(1+p)=0$ and $\frac{\partial f}{\partial u}(2y+2zq)+\frac{\partial f}{\partial v}(1+q)=0$

Eliminating $\frac{\partial f}{\partial u}$ and $\frac{\partial f}{\partial v}$ from these equations, we get

$$\begin{vmatrix} 2x+2zp & 1+p \\ 2y+2zq & 1+q \end{vmatrix}=0$$

$\Rightarrow (1+q)(2x+2zp)-(1+p)(2y+2zq)=0$

$\Rightarrow (1+q)(x+zp)-(1+p)(y+zq)=0$

$\Rightarrow x+zp+qx+zpq-y-zq-py-zpq=0$

$\Rightarrow (y-z)p+(z-x)q = x-y.$

2013 Solve the pde:
$xu_x+yu_y+zu_z = xyz$

Sol^n:- We have
$xu_x+yu_y+zu_z = xyz$

Subsidiary equations are
$$\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z}=\frac{du}{xyz}$$

Considering 1st and 2nd ratios,
$$\frac{dx}{x}=\frac{dy}{y}$$

Integrating we get
$\log x = \log y + \log c$

$\Rightarrow \frac{x}{y} = a$, $a=$ constant.

From 2nd and 3rd ratios, we get
$$\frac{dy}{y}=\frac{dz}{z}$$

Integrating, we get
$\frac{y}{z} = b$, $b=$ constant.

contd.

Each ratio $= \dfrac{yz\,dx + zx\,dy + xy\,dz}{xyz + xyz + xyz}$ [choosing (yz, zx, xy) as multipliers]

$= \dfrac{yz\,dx + zx\,dy + xy\,dz}{3xyz}$ — (A)

Comparing (A) with 4th ratio, we get

$$\frac{yz\,dx + zx\,dy + xy\,dz}{3xyz} = \frac{du}{xyz}$$

$\Rightarrow yz\,dx + zx\,dy + xy\,dz = 3du$

$\Rightarrow d(xyz) = 3du$

Integrating, we get

$xyz = 3u + c$, $c =$ constant

$\Rightarrow xyz - 3c = c$.

$\therefore \phi\left(\frac{x}{y}, \frac{y}{z}, xyz - 3c\right) = 0$ is the required solution.

2013 (IFoS)

Rewrite the hyperbolic equation $x^2 u_{xx} - y^2 u_{yy} = 0 \ (x>0, y>0)$ in canonical form.

The given equation can be written as $x^2 r - y^2 t = 0$

Here, $R = x^2$, $S = 0$, $T = -y^2$.

$\therefore S^2 - 4RT = 4x^2y^2 > 0$ for $x \neq 0, y \neq 0$, which is hyperbolic.

The quadratic equation $R\lambda^2 + S\lambda + T = 0$ becomes

$x^2\lambda^2 - y^2 = 0 \Rightarrow \lambda = \pm \frac{y}{x}$

Now, $\frac{dy}{dx} + \lambda_i(x,y) = 0 \Rightarrow \frac{dy}{dx} \pm \frac{y}{x} = 0 \Rightarrow x\,dx \pm y\,dy = 0$

Integrating we get $xy = c_1$ and $\frac{x}{y} = c_2$

Let $U = xy$ and $V = \frac{x}{y}$.

$\therefore p = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial U}\frac{\partial U}{\partial x} + \frac{\partial u}{\partial V}\frac{\partial V}{\partial x} = \frac{\partial u}{\partial U} y + \frac{\partial u}{\partial V}\cdot\frac{1}{y}$

$q = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial U}\frac{\partial U}{\partial y} + \frac{\partial u}{\partial V}\frac{\partial V}{\partial y} = \frac{\partial u}{\partial U} x - \frac{\partial u}{\partial V}\frac{x}{y^2}$

$r = \frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial U} y\right) + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial V}\cdot\frac{1}{y}\right)$

$= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial U}\right) y + \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial V}\right)\frac{1}{y}$

$= \left(\frac{\partial^2 u}{\partial U^2} y + \frac{\partial^2 u}{\partial V \partial U}\frac{1}{y}\right) y + \left(\frac{\partial^2 u}{\partial U \partial V} y + \frac{\partial^2 u}{\partial V^2}\cdot\frac{1}{y}\right)\frac{1}{y}$

$$= y^2 \frac{\partial^2 u}{\partial U^2} + 2\frac{\partial^2 u}{\partial U \partial V} + \frac{1}{y^2}\frac{\partial^2 u}{\partial V^2}$$

$$t = \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial U}x\right) - \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial V}\frac{x}{y^2}\right)$$

$$= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial U}\right)x + \left[\frac{\partial u}{\partial V}\left(\frac{2x}{y^3}\right) - \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial V}\right)\frac{x}{y^2}\right]$$

$$= \left[\frac{\partial^2 u}{\partial U^2}x + \frac{\partial^2 u}{\partial V \partial U}\left(-\frac{x}{y^2}\right)\right]x + \frac{2x}{y^3}\frac{\partial u}{\partial V} - \left[\frac{\partial^2 u}{\partial U \partial V}x + \frac{\partial^2 u}{\partial V^2}\left(-\frac{x}{y^2}\right)\right]\frac{x}{y^2}$$

$$= x^2\frac{\partial^2 u}{\partial U^2} - \frac{2x^2}{y^2}\frac{\partial^2 u}{\partial U \partial V} + \frac{x^2}{y^4}\frac{\partial^2 u}{\partial V^2} + \frac{2x}{y^3}\frac{\partial u}{\partial V}$$

On substituting these values in the given equation, the required canonical form is

$$2U\frac{\partial^2 u}{\partial U \partial V} - \frac{\partial u}{\partial V} = 0$$

Q. Find the solution of the equation $\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = 1$ that passes through the circle. $x^2+y^2=1,\ u=1$.

The given equation can be written as $p^2+q^2=1$, where $p=\frac{\partial u}{\partial x}$, $q=\frac{\partial u}{\partial y}$ $\left[f(p,q)=0 \text{ form}\right.$

Taking $p=a$ in $p^2+q^2=1$, we get $q=\sqrt{1-a^2}$

$\therefore\ du = p\,dx + q\,dy$

$= a\,dx + \sqrt{1-a^2}\,dy$

Integrating, we get

$\therefore\ u = ax + \sqrt{1-a^2}\,y + b$, is the required solution. (1)

Given circle $x^2+y^2=1$, $u=1$

Since the solution (1) passes through $u=1, x^2+y^2=1$ or $u=1, x=\sqrt{1-y^2}$

$$1 = a\sqrt{1-y^2} + \sqrt{1-a^2}\,y + b$$

$\therefore$??

Q. Solve the following heat equation, using the method of separation of variables:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad 0<x<1,\ t>0$$

subject to the conditions

$u=0$ at $x=0$ and $x=1$ for $t>0$

$u=4x(1-x)$, at $t=0$ for $0\le x\le 1$.

Here, $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}, \ 0<x<1,\ t>0$ — (1)

with boundary and initial conditions

$u=0$ at $x=0$ and $x=1$ for $t>0$ — (2)

$u=4x(1-x)$, at $t=0$ for $0\le x\le 1$ — (3)

We assume a separable solution of (1) in the form

$u(x,t) = X(x)\,T(t) \neq 0$ — (4)

$\Rightarrow (1) \Rightarrow \dfrac{X''}{X} = \dfrac{T'}{T}$ — (5)

Since the left hand side depends only on x and the right hand side is a function of t only, so (5) holds if both sides are equal to the same constant λ.

$\therefore\ X''-\lambda X=0$ and $T'-\lambda T=0$ — (6)

Now, $u(0,t) = X(0)T(t) = 0$ for $t>0$

$u(1,t) = X(1)T(t) = 0$ for $t>0$

We take $T(t)\neq 0$, since otherwise $u(x,t)=0$, a trivial solution of (1), contradicts (4).

$X(0)=0=X(1)$ — (7)

Case-I: $\lambda=0$.

$X''=0$

$\Rightarrow X(x) = Ax+B$

Using (7), $0+B=0$ and $A+B=0 \Rightarrow A=0=B$

we get a trivial solution $X(x)=0$, so we do not consider it.

Case-II: Let $\lambda=\alpha^2>0$. Then the solution of $X''-\alpha^2X=0$ is

$\therefore X(x) = Ae^{\alpha x}+Be^{-\alpha x}$,

Using (7), $A+B=0 \Rightarrow B=-A$

$Ae^{\alpha x}+Be^{-\alpha x}=0 \Rightarrow A(e^{\alpha x}-e^{-\alpha x})=0 \Rightarrow A=0 \Rightarrow B=0$

and so $u(x,t)=0$

Again a trivial solution $X(x)=0$ and so $u(x,t)=0$

Case-III: Let $\lambda<0$, say $\lambda=-\alpha^2$.

$\Rightarrow (6)$ becomes

$X'' + \alpha^2 X = 0$, $T' + \alpha^2 T = 0$

$\therefore$ The solutions of these equations are,

$X(x) = A\cos\alpha x + B\sin\alpha x$ and $T(t) = Ce^{-\alpha^2 t}$ —(8),

Using (7), $A = 0$ and $B\sin\alpha = 0$

For non trivial solution, $B \neq 0$. It follows that

$\sin\alpha = 0 \Rightarrow \alpha = n\pi$ —(9)

$\Rightarrow \alpha = \alpha_n = n\pi$; $n = 1, 2, 3, \ldots$

[$n \neq 0$, for otherwise $\alpha = 0$ and so $X(x) = 0$]

Using (9) in (8), we get —(10)

$X_n(x) = B_n \sin(n\pi x)$, $T_n(t) = C_n e^{-(n^2\pi^2 t)}$.

where B_n and C_n are non-zero constants.

$\therefore$ $u_n(x,t) = b_n \sin(n\pi x) e^{-(n^2\pi^2 t)}$ —(11), where $b_n = B_n C_n$ are new constants ; $n = 1, 2, 3, \ldots$

$\therefore$ The most general solution of (1) is —(12)

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) e^{-n^2\pi^2 t}$$

Using the initial condition (3) in (12), we get

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} b_n \sin(n\pi x) , \quad 0 \le x \le 1$$

which is a Fourier-sine series in $[0,1]$, where the Fourier coefficients b_n are

$$b_n = 2\int_0^1 f(x)\sin(n\pi x)\,dx , \quad n = 1, 2, 3, \ldots$$

$$= 2\int_0^1 4x(1-x)\sin(n\pi x)\,dx$$

$$= \begin{cases} 0 , & \text{if } n \text{ is even} \\ \dfrac{32}{n^3\pi^3} , & \text{if } n \text{ is odd} \end{cases}$$

$\therefore$ $b_{2n-1} = \dfrac{32}{(2n-1)^3\pi^3}$, for $n = 1, 2, 3, \ldots$

$\therefore$ (12) implies

$$u(x,t) = \frac{32}{\pi^3}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^3} e^{-(2n-1)^2\pi^2 t} \sin\{(2n-1)\pi x\}$$