1) a) Let R be an integral domain. Prove that ch R (characteristic of R) is 0 or a prime. (8)

Soln.

Defn:- Characteristic of a Ring:-
The characteristic of a ring R is the Least positive integer $n \ni nx=0$ for all x in R. if no such integer exist, we say that R has char. 0.

Note:- An infinite ring can have a nonzero charactristic.

$R = \mathbb{Z}_2[x] = \{a_0 + a_1 x + a_2 x^2 + \cdots \mid a_i \in \mathbb{Z}_2\}$

char $R = 2$

Soln:- $\because$ R is a integral domain i.e. commutative ring with unity.
hence $1 \in R$

If 1 has infinite order, then there is no positive integer $n \ni n \cdot 1 = 0$, so R has char 0.

now suppose 1 has additive order n then $n \cdot 1 = 0$ and n is least positive integer with this property.

we have to show that n is prime.

let $n = st$ where $1 \leq s, t \leq n$

then $0 = n \cdot 1$
$= (s \cdot t) \cdot 1$
$0 = (s \cdot 1)(t \cdot 1)$

So $s \cdot 1 = 0$ or $t \cdot 1 = 0$ --- {R is I.D}

since n is the least positive integer with the property that $n \cdot 1 = 0$,
we must have
$s = n$ or $t = n$
thus n is prime.

Q12) a] Let I and J be ideals in a Ring R. Then prove that the quotient ring $(I+J)/J$ is isomorphic to the quotient ring $I/(I \cap J)$. (10)

proof:- $\because$ I and J are ideals
$\therefore$ $I \cap J$ & $I+J$ is also ideal.
J will be ideal of $I+J$
$\therefore$ $\frac{I+J}{J}$ is well defined. i.e. a quotient ring
~~also~~ let $\phi : I \longrightarrow \frac{I+J}{J}$

$$\phi(x) = x + J$$

we have to prove ϕ is onto homomorphism

$$\phi(x+y) = x+y+J$$
$$= x+J+y+J$$
$$\Rightarrow \phi(x+y) = \phi(x) + \phi(y)$$

$$\phi(x \cdot y) = x \cdot y + J$$
$$= (x+J) \cdot (y+J)$$
$$\phi(x \cdot y) = \phi(x) \cdot \phi(y)$$

$\therefore$ ϕ is homomorphism.
let $x \in \frac{I+J}{J}$ be arbitrary

$\therefore x = b + j$

where $b = i_1 + j_1$ for some $i_1 \in I, j_1 \in J$

$\therefore x = i_1 + j_1 + j$

$x = i_1 + j$ where $i_1 \in I$

$\therefore$ any element of $\frac{I+J}{J}$ is of the form

$x + J$ for $x \in I$

$\therefore \forall\ x + J$ in $\frac{I+J}{J}$ $\exists\ x \in I \ni$

$\phi(x) = x + J$

$\therefore \phi$ is onto.

$\ker \phi = \{ x \in I : \phi(x) = J \}$

$\therefore \quad \phi \quad x + J = J$

$\Rightarrow x \in J$

$\cancel{\text{so}}\ x \in I, x \in J$

$\Rightarrow x \in I \cap J$

$\therefore \ker \phi = I \cap J$

$\therefore$ by theorem of 1^{st} isomorphism

$$\frac{I}{I \cap J} \cong \frac{I+J}{J}$$

Q-3) a) If in the group G, $a^5 = e$, $aba^{-1} = b^2$ for some $a, b \in G$ find order of b. (10)

Sol:- Given $a^5 = e$

$$aba^{-1} = b^2$$

or $b^2 = aba^{-1}$

$$b^4 = b^2 \cdot b^2$$

$$= (aba^{-1})(aba^{-1})$$

$$= ab(a^{-1}a)(ba^{-1})$$

$$= ab\,e\,ba^{-1}$$

$$b^4 = ab^2a^{-1} = a\,aba^{-1}a^{-1} = a^2ba^{-2}$$

$$b^8 = b^4 \cdot b^4 = \cancel{ab^2}\, a^2ba^{-2}a^2ba^{-2}$$

$$= a^2b^2a^{-2}$$

$$= a^2aba^{-1}a^{-2}$$

$$b^8 = a^3ba^{-3}$$

$$b^{16} = b^8 \cdot b^8$$

$$= a^3ba^{-3} \cdot a^3ba^{-3}$$

$$= a^3b^2a^{-3}$$

$$= a^3\,aba^{-1}\,a^{-3}$$

$$b^{16} = a^4ba^{-4}$$

$$b^{32} = b^{16} \cdot b^{16} =$$

$$a^4 \cdot ba^{-4} \cdot a^4 \cdot ba^{-4}$$

$$= a^4 \cdot b^2a^{-4}$$

$$= a^4 \cdot aba^{-1}a^{-4}$$

$$= a^5 \cdot ba^{-5}$$

$$= ebe$$

$$\therefore b^{32} = b$$

$$\therefore b^{31} = e$$

$$\therefore o(b) = 31$$

Q4] Show that the smallest subgroup V of A_4 containing $(1,2)(3,4)$, $(1,3)(2,4)$, and $(1,4)(2,3)$ is isomorphic to the Klein 4-group. (10)

Sol^n:- $K_4 = \{e, a, b, c\}$

$a^2 = b^2 = c^2 = e$

$a \cdot b = c$, $b \cdot c = a$, $a \cdot c = b$

$V_4 = \{e', (12)(34), (13)(24), (14)(23)\}$

∴ defining $\phi: K_4 \to V_4$

$\phi(e) = e'$

$\phi(a) = (12)(34)$

$\phi(b) = (13)(24)$

$\phi(c) = (14)(23)$

To show isomorphism

ϕ is one-one & onto — {above statem

$\phi(a \cdot b) = \phi(c)$

$= (14)(23)$

$\phi(a) \cdot \phi(b) = (12)(34)(13)(24)$

$= (14)(23)$

similarly we can check $\forall x \in K_4$