Mains Test Series - 2019

Test - 2 (Paper - II)

Answer Key

Modern Algebra, Real Analysis, Complex Analysis & LPP

SECTION - A

Q.1.(a) Are the following groups cyclic groups? Give reasons.

(i) The Klein 4-group

(ii) The dihedral group D_4.

(iii) The group of all roots (real or complex) of the equation $x^n - 1 = 0$

(iv) The group Q^* of non-zero rationals, for multiplication.

Solution:

(i) Klein 4-group

It is not a cyclic group because order of each and every element of Klein 4-group is 2 except identity element.

Thus, no element can be a generator of entire group.

$\because$ A cyclic group of order 'n' has an element of order 'n'.

(ii) Dihedral group D_4.

It is not a cyclic group because order of D_4 is 8 whereas the order of every element is less than or equal to 4. Thus, no element can be a generator of entire group.

$\because$ A cyclic group of order 'n' has an element of order 'n'.

(iii) The group of all roots (real or complex) of the equation $x^n - 1 = 0$.

$\because x^n - 1 = 0 \Rightarrow x = 1^{1/n}$
$= e^{\frac{2\pi i}{n} l}$, $l = 0, 1, 2, \ldots, n-1$

Clearly this group is cyclic with the generator $e^{\frac{2\pi i}{n}}$.

(iv) The group Q^* of non-zero rationals, for multiplication.

$Q^* = \{ Q \setminus \{0\} \} = \{ p/q \mid p, q \neq 0,\ p, q \in \mathbb{R} \}$

is not a cyclic group.

Justification: Suppose it is generated by 'a', then $a^n = -1$ for some $n \in \mathbb{Z}$.

$\Rightarrow a = -1$

But $\langle -1 \rangle = \{-1, 1\} \neq Q^*$

Hence the result.

Q.1.(b) Let R be a commutative ring with unit element whose only ideals are (0) and R itself. Then R is a field.

<u>Solution</u>:

Let $0 \neq a \in R$.

Now, we first prove that $I = \{ax / x \in R\}$ is an ideal of R.

$\because I = \{ax / x \in R\}$

(i) Let $x = 1$

$\therefore a = a \cdot 1$

$\Rightarrow a \in I$

$\Rightarrow I \neq \phi$; Also by def^n. of I, $I \subseteq R$

(ii) Now, let $v, w \in I$

$\therefore v = ax, \; w = ay$ for some $x, y \in R$

$v - w = ax - ay$; $a, x, y \in R$

$= a(x-y)$

$= a \cdot z$ for some $z \in R$.

$\Rightarrow v - w \in I$

(iii) Let $v \in I$ (arbitrary) and $r \in R$ (arbitrary)

$v = a \cdot x$ for some $x \in R$

$v \cdot r = (a \cdot x) r = a(xr) = aw$ $\because$ ('$\cdot$' is associative)

$\therefore vw \in I$

Now,

$$\begin{aligned} r\cdot v &= r\cdot(a\cdot x) \\ &= (r\cdot a)\cdot x \quad \text{—— } \because \cdot \text{ is associative} \\ &= (a\cdot r)\,x \quad \text{—— } \because \cdot \text{ is commutative} \\ &= a\cdot(r\cdot x) \quad \text{—— } \because \cdot \text{ is associative} \\ &= a\cdot z \end{aligned}$$

$\therefore\ r\cdot v \in I.$

from (i), (ii) and (iii), we conclude that
$I = \{ax / x \in R\}$ is an ideal of R.

Also, $I \neq \{0\}$ $\left[\because a = a\cdot 1 \Rightarrow a \in I \ (a\neq 0)\right]$

But (0) and R itself are only ideals of R.
(i.e. R has no proper ideal)

$\Rightarrow I = R$ —— ($\because$ R is a ring with unity)

$\therefore\ 1 \in I$

$\Rightarrow 1 = a\cdot x$ for some $x \in R$

[$\because$ R is commutative ring]

$\therefore\ x\cdot a = 1.$

$\Rightarrow x$ is a multiplicative inverse of a $(a\neq 0)$.

$\Rightarrow R$ is a field.

Hence, the result.

Q.1.(c) For $u_1 > 0$, the sequence u_n defined by $u_{n+1} = 1 + \frac{1}{u_n}$ $\forall n$, converges to $\left(\frac{\sqrt{5}+1}{2}\right)$

Solution:

Here, for any $u_1 > 0$,

$$2 \geq u_n \geq 3/2 \quad \forall n \geq 3,$$

as $2 \geq u_n \geq 3/2$,

$$\Rightarrow 2 > 5/3 \geq u_{n+1} = 1 + \frac{1}{u_n} \geq 3/2 \quad \forall n \geq 3.$$

So that $\forall n \geq 3$,

$$|u_{n+1} - u_n| = \frac{|u_n - u_{n-1}|}{u_n \cdot u_{n-1}} = \dots =$$

$$\frac{|u_4 - u_3|}{u_n u_{n-1}^2 \cdots u_3} \leq \frac{|u_4 - u_3|}{(3/2)^{2(n-3)}}$$

$$\Rightarrow |u_{n+p} - u_n| \leq |u_{n+p} - u_{n+p-1}| +$$

$$|u_{n+p-1} - u_{n+p-2}| + \dots + |u_{n+1} - u_n|$$

$$\leq |u_4 - u_3| \left(\frac{2}{3}\right)^{2(n-3)} \left\{1 + \left(\frac{2}{3}\right)^2 + \dots + \left(\frac{2}{3}\right)^{2p}\right\}$$

$$< 2|u_4 - u_3| \left(\frac{2}{3}\right)^{2(n-3)} \longrightarrow 0 \text{ as } n \to \infty$$

$\Rightarrow u_n$ converges, say to l, then

$$u_{n+1} = 1 + \frac{1}{u_n}, \quad \text{as } n \to \infty$$

$\Rightarrow l^2 - l - 1 = 0$ i.e. $l = \frac{\sqrt{5}+1}{2}$, as $l \geq \frac{3}{2}$

$\Rightarrow u_n$ converges to $\frac{\sqrt{5}+1}{2}$.

Hence, the result.

<u>Alternatively,</u>

Here, for any $u_1 > 0$, $2 \geq u_n \geq \frac{3}{2}$ $\forall n \geq 3$

and $$\left| u_n - \frac{\sqrt{5}+1}{2} \right| = \left| u_{n-1} - \frac{\sqrt{5}+1}{2} \right| \cdot \frac{1}{u_{n-1}}$$

$$\leq \left| u_3 - \frac{\sqrt{5}+1}{2} \right| \cdot \frac{1}{u_{n-1} \cdot u_{n-2} \cdots u_3}$$

$$\leq \left| u_3 - \frac{\sqrt{5}+1}{2} \right| \cdot \left(\frac{2}{3}\right)^{n-3} \to 0 \quad \text{as } n \to \infty$$

$\Rightarrow u_n$ converges to $\frac{\sqrt{5}+1}{2}$

Hence, the result.

Q. 1. (d) Prove that the function $u = e^{-x}(x\cos y + y\sin y)$ is harmonic and find the corresponding analytic function.

<u>Solution</u> :

Let $u = e^{-x}(x\cos y + y\sin y)$

Then $\frac{\partial u}{\partial x} = e^{-x}(\cos y) + (x\cos y + y\sin y)(-e^{-x})$

$= e^{-x}[\cos y - x\cos y - y\sin y]$

$\frac{\partial u}{\partial y} = e^{-x}(-x\sin y + y\cos y + \sin y)$

Also, $\frac{\partial^2 u}{\partial x^2} = e^{-x}[-\cos y] + (\cos y - x\cos y - y\sin y)(-e^{-x})$

$= e^{-x}[-\cos y - \cos y + x\cos y + y\sin y]$

$= e^{-x}[x\cos y + y\sin y - 2\cos y]$

$\frac{\partial^2 u}{\partial y^2} = e^{-x}[-x\cos y + \cos y - y\sin y + \cos y]$

$= e^{-x}[-x\cos y - y\sin y + 2\cos y]$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-x}[\cancel{x\cos y} + \cancel{y\sin y} - \cancel{2\cos y} - \cancel{x\cos y} - \cancel{y\sin y} + \cancel{2\cos y}]$

$= 0$

$\Rightarrow$ u is harmonic. ——— (i)

Let v be the harmonic conjugate of u, then $f(z) = u + iv$ be analytic function.

$$\therefore f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \quad \left[\text{By C-R Equations, } \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}\right]$$

$$\therefore f'(z) = e^{-x}\left[\cos y - x\cos y - y\sin y\right] - i\cdot e^{-x}\left[-x\sin y + y\cos y + \sin y\right] \quad \text{——— (A)}$$

Now, by applying Milne-Thomson's method, replacing x by z and y by 0, in Eq^n (A), we get,

$$f'(z) = e^{-z}\left[1 - z - 0\right] - ie^{-z}[0]$$

$$f'(z) = e^{-z}\left[1 - z\right]$$

$$\Rightarrow f(z) = (1-z)(-e^{-z}) - \int -e^{-z}(-1)\,dz + \text{constant}$$

[On integrating with respect to z]

$$= -(1-z)e^{-z} - \int e^{-z}dz + \text{constant}$$

$$= -(1-z)e^{-z} - (-e^{-z}) + \text{constant}$$

$$= e^{-z}(-1 + z + 1) + \text{constant}$$

$$= e^{-x-iy}(x+iy) + \text{constant}$$

$$= e^{-x}(\cos y - i\sin y)(x+iy) + \text{constant}$$

$$= e^{-x}(x\cos y + y\sin y) + ie^{-x}(y\cos y - x\sin y) + ic$$

$$f(z) = u + i\left[e^{-x}(y\cos y - x\sin y) + c\right] \quad \text{(B)}$$

$$\therefore\ v = e^{-x}(y\cos y - x\sin y) + c$$

where Eq^n (B) represents the required analytic function.

Hence, the result.

Q.1.(c) Write the dual of the problem.

Minimize $Z = 2x_2 + 5x_3$

Subject to
$$x_1 + x_2 \geq 2$$
$$2x_1 + x_2 + 6x_3 \leq 6$$
$$x_1 - x_2 + 3x_3 = 4$$
$$x_1, x_2, x_3 \geq 0.$$

<u>Solution:</u>

First of all, we shall write the given problem in the standard primal form as follows :-

i) Since it is a minimization problem, all

the constraints must involve the sign $\geq$.
$\therefore$ We multiply the second constraint by -1, and get $-2x_1 - x_2 - 6x_3 \geq -6$

(ii) The third constraint is an equality, so we replace it by two constraints.

$$x_1 - x_2 + 3x_3 \leq 4$$

and $$x_1 - x_2 + 3x_3 \geq 4$$

The first on multiplying by -1, reduce to

$$-x_1 + x_2 - 3x_3 \geq -4.$$

Thus, the given problem in the standard primal form is as follows:

Minimize $Z = 0x_1 + 2x_2 + 5x_3$

subject to constraints

$$x_1 + x_2 + 0x_3 \geq 2$$
$$-2x_1 - x_2 - 6x_3 \geq -6$$
$$-x_1 + x_2 - 3x_3 \geq -4$$
$$x_1 - x_2 + 3x_3 \geq 4$$
$$x_1, x_2, x_3 \geq 0$$

The dual of the given problem is given by

Maximize $Z^* = 2y_1 - 6y_2 - 4y_3' - 4y_3''$

subject to constraints

$$y_1 - 2y_2 - (y_3' - y_3'') \le 0$$

$$y_1 - y_2 + (y_3' - y_3'') \le 2$$

$$0 \cdot y_1 - 6y_2 - 3(y_3' - y_3'') \le 5$$

$$y_1, y_2, y_3', y_3'' \ge 0.$$

Now, writing $y_3' - y_3'' = y_3$, the dual problem is given as:

Maximize $Z^* = 2y_1 - 6y_2 - 4y_3$

subject to the contraints,

$$y_1 - 2y_2 - y_3 \le 0$$

$$y_1 - y_2 + y_3 \le 2$$

$$-6y_2 - 3y_3 \le 5$$

$y_1, y_2 \ge 0$, y_3 unrestricted in sign.

Hence, the result.

Q.2. (a)

(i) Show that every subgroup of an abelian group is normal.

(ii) Is the converse of Problem (i) true? If yes, prove it, if no, give an example of a non-abelian group all of whose subgroups are normal.

Solution :

(i) Let G be an abelian group and H be subgroup of G.

$\because$ Every subgroup of an abelian group is abelian

$\therefore$ H is an abelian group.

We have, $h \in H$ and $a \in G$

$$ha = ah$$

$\Rightarrow aha^{-1} \in H$

$\Rightarrow$ H is normal subgroup of G.

$\{$ $\because$ By definition, Let G be a group and $H \leq G$ be a subgroup. H is said to be normal in G, denoted by $H \triangleleft G$, if for every $g \in G$ and $h \in H$ we have $ghg^{-1} \in H$, i.e. if for every $g \in G$, we have $gHg^{-1} \subseteq H$. $\}$

(ii)
By the converse of Problem (i), it means that any subgroup N of a group G is normal then G must be abelian.

This is definitely not true, and it is a reason why normal subgroups are interesting and important.

a) The simplest example of a normal subgroup in a non-abelian group is the subgroup $N = \langle(1\,2\,3)\rangle \subset S_3$.

b) The smallest example is the quaternion group of order 8, $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$. The subgroup of order 4 is normal because it has index 2; the only subgroup of order 2 is $\{\pm 1\}$ which is normal because it equals the center of the group.

Hence, the required counter-example(s) is/are provided.

Q.2 (b) Suppose that N and M are two normal subgroups of G and that $N \cap M = \{e\}$. show that for any $n \in N$, $m \in M$, $nm = mn$.

Solution:

Consider, $nmn^{-1}m^{-1}$ in two ways:-

Firstly,

$(nmn^{-1})m^{-1} \in M$ as $nmn^{-1} \in M$ — [M is normal subgroup of G]

and $m^{-1} \in M$. (I)

Secondly,

$n(mn^{-1}m^{-1}) \in N$ as $n \in N$ and $mn^{-1}m^{-1} \in N$ — [$\because n \in N \Rightarrow n^{-1} \in N$ and N is a normal subgroup of G] (II)

Hence,
from (I) and (II),

$nmn^{-1}m^{-1} \in N \cap M = e$

$\Rightarrow nmn^{-1}m^{-1} = e$

$\Rightarrow nmn^{-1} = em$ — [Multiplying m on the right hand of each expression]

$\Rightarrow nm = emn$ — [Post-Multiplying n on both the sides]

$\Rightarrow nm = mn$ — [$e \cdot a = a$] $\forall a \in G$.

i.e. $nm = mn$
$\forall n \in N$ and $m \in M$.

Hence, the result.

Q.2. (c)(i) Test the convergence of $\int_0^1 \frac{dx}{\sqrt{1-x^3}}$

(ii) Prove that the integral $\int_0^\infty x^{m-1}e^{-x}dx$ is convergent if and only if $m>0$.

Solution: (i)

Let $f(x) = \frac{1}{\sqrt{1-x^3}} = \frac{1}{(1-x)^{1/2}} \cdot \frac{1}{(1+x+x^2)^{1/2}}$

Clearly, $\frac{1}{(1+x+x^2)^{1/2}}$ is a bounded function

and let M be its upper bound.

$\therefore f(x) \le \frac{M}{(1-x)^{1/2}}$

Also, $\int_0^1 \frac{dx}{(1-x)^{1/2}}$ is convergent.

$\therefore$ By Comparison test, $\int_0^1 \frac{dx}{\sqrt{1-x^3}}$ is convergent.

Hence, the result.

(ii)

Let $f(x) = x^{m-1}e^{-x} = \frac{e^{-x}}{x^{1-m}}$

The integrand f has infinite discontinuity at 0 if $m<1$.

So, we have to examine convergence at 0 and ∞ both.

Putting $\int_0^\infty x^{m-1}e^{-x}dx = \int_0^1 x^{m-1}e^{-x}dx + \int_1^\infty x^{m-1}e^{-x}dx$,

we test the two integrals on the right for convergence at 0 and ∞ respectively.

Convergence at 0, $m<1$.

Let $g(x) = \frac{1}{x^{1-m}}$ so that $\frac{f(x)}{g(x)} = e^{-x} \to 1$ as $x \to 0$

Also, $\int_0^1 g\,dx = \int_0^1 \frac{dx}{x^{1-m}}$, converges iff $1-m<1$ i.e. $m>0$

Hence, $\int_0^1 x^{m-1}e^{-x}dx$ convergences iff $m>0$.

Convergence at ∞

Let $g(x) = \frac{1}{x^2}$, so that $\frac{f(x)}{g(x)} = \frac{x^{m+1}}{e^x} \to 0$ as $x \to \infty$, for all m.

As $\int_1^\infty \frac{dx}{x^2}$ converges, therefore $\int_1^\infty x^{m-1}e^{-x}$

also converges for all m.

Hence, $\int_0^\infty x^{m-1} e^{-x}\,dx$ is convergent iff $m>0$.

This integral, $\int_0^\infty x^{m-1} e^{-x}\,dx$, $m>0$ called Gamma function, is denoted by $\Gamma(m)$.

Hence, the result.

Q.2.(d) Apply the method of contour integration to prove that $\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\,d\theta = \frac{\pi}{6}$.

Solution :

Let $I = \int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\,d\theta$ = real part of $\int_0^{2\pi} \frac{e^{2i\theta}}{5+4\cos\theta}\,d\theta$

$$= \text{real part of } \int_C \frac{z^2}{5+2\left(z+\frac{1}{z}\right)}\cdot\frac{dz}{iz}$$

$$\left[\begin{array}{l}\text{As, } z=e^{i\theta} \text{ then } \cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right) \\ \text{and } \cos 2\theta = \frac{e^{2i\theta}+e^{-2i\theta}}{2} = \frac{z^2+1/z^2}{2}.\end{array}\right]$$

[Also Since, $z = e^{i\theta} \Rightarrow \frac{dz}{d\theta} = ie^{i\theta}$

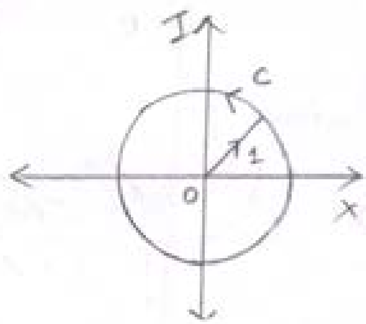
$\Rightarrow \boxed{d\theta = \frac{dz}{iz}}$]

$$= \text{real part of } \frac{1}{i}\int_C \frac{z^2}{2z^2+5z+2}\,dz$$

$$= \text{real part of } \frac{1}{2i}\int_C \frac{z^2}{\left(z+\frac{1}{2}\right)(z+2)}\,dz$$

$$= \text{real part of } \int_C f(z)\,dz, \text{ (say)},$$

where $f(z) = \frac{z^2}{2i\left(z+\frac{1}{2}\right)(z+2)}$ and C is the unit circle.



Now, $z = -1/2$ and $z = -2$ are simple poles of $f(z)$

Out of these only $z = -1/2$ lies inside C.

Residue at $z = -1/2$ of order 1 is

$$\lim_{z\to -1/2}\left[\left(z+1/2\right)f(z)\right] = \lim_{z\to -1/2} \cancel{\left(z+1/2\right)}\cdot\frac{z^2}{2i\cancel{\left(z+1/2\right)}(z+2)}$$

$$= \lim_{z \to -1/2} \frac{z^2}{2i(z+2)}$$

$$= \frac{1}{12i}$$

$$\therefore \int_C f(z)\,dz = 2\pi i \times \text{Residue at } z = -1/2$$

$$= \cancel{2}\pi \cancel{i} \times \frac{1}{\underset{6}{\cancel{12}}\cancel{i}}$$

$$= \frac{\pi}{6}.$$

Hence, $\boxed{\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\,d\theta = \text{real part of } \int_C f(z)\,dz = \frac{\pi}{6}.}$

Hence, proved.

Q.3. (a). Prove, by an example, that we can find three groups $E \subset F \subset G$, where E is normal in F. F is normal in G, but E is not normal in G.

Solution:

Let $G = S_4$.

Now, we select E and F such that

(i) $E \subset F \subset G = S_4$ and

(ii) E is normal in F,
F is normal in G but
E is not normal in G. (can be easily proved).

$\therefore$ Let E be the subgroup generated by one pair of disjoint transpositions.

i.e. $E = \langle (1\,2)(3\,4) \rangle$

and let F be the subgroup generated by all pairs of disjoint transpositions.

i.e. $F = \langle (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3) \rangle$

Hence, $\boxed{G = S_4,\ E = \langle (1\,2)(3\,4) \rangle,}$
$\boxed{F = \langle (1\,2)(3\,4), (1\,3)(2\,4), (1\,4)(2\,3) \rangle}$ is the required example.

Q.3.(b) Give an example of an integral domain which has an infinite number of elements, yet is of finite characteristic.

Solution:

Consider the ring $\mathbb{Z}_n[x]$ of polynomials in one variable x with co-efficients in $\mathbb{Z}_n$, where n is prime.

It is an infinite ring, since $x^m \in \mathbb{Z}_n[x]$ for all +ve integers m, and

$$x^{m_1} \neq x^{m_2} \text{ for } m_1 \neq m_2$$

But the characteristic of $\mathbb{Z}_n[x]$ is clearly 'n'.

Hence, the required example.

Q.3.(c) Suppose that f is defined on $I = \{x : 0 \le x \le 1\}$ by the formula

$$f(x) = \begin{cases} \frac{1}{2^n} & \text{if } x = \frac{j}{2^n} \text{ where } j \text{ is an odd integer and } 0 < j < 2^n, \quad n = 1, 2, \cdots \\ 0 & \text{otherwise.} \end{cases}$$

Determine whether or not f is integrable and prove your result.

Solution:

$$f(x) = \frac{1}{2^n} \quad \text{when } x = \frac{j}{2^n} \text{ where } j \text{ is an odd integer}$$

and $0 < j < 2^n$, $n = 1, 2, \ldots$

$= \frac{1}{2}$ when $x = \frac{1}{2}$

$= \frac{1}{2^2}$ when $x = \frac{1}{2^2}, \frac{3}{2^2}$

$= \frac{1}{2^3}$ when $x = \frac{1}{2^3}, \frac{3}{2^3}, \frac{5}{2^3}, \frac{7}{2^3}$

$= \frac{1}{2^4}$ when $x = \frac{1}{2^4}, \frac{3}{2^4}, \frac{5}{2^4}, \frac{7}{2^4}, \frac{9}{2^4}, \frac{11}{2^4}, \frac{13}{2^4}, \frac{15}{2^4}$

$\vdots$

$= 0$ otherwise

$\Rightarrow f$ is bounded and continuous on $[0, 1]$ except at the points $1/2$, $1/2^2$, $3/2^2$, $1/2^3$, $3/2^3$, $5/2^3$, $7/2^3$, $1/2^4$, ...

The set of points of discontinuity of f on $[0,1]$ is $\left\{\frac{1}{2}, \frac{1}{2^2}, \frac{3}{2^2}, \frac{1}{2^3}, \frac{3}{2^3}, \frac{5}{2^3}, \frac{7}{2^3}, \frac{1}{2^4}, \ldots\right\}$ which has only one limit point '0'.

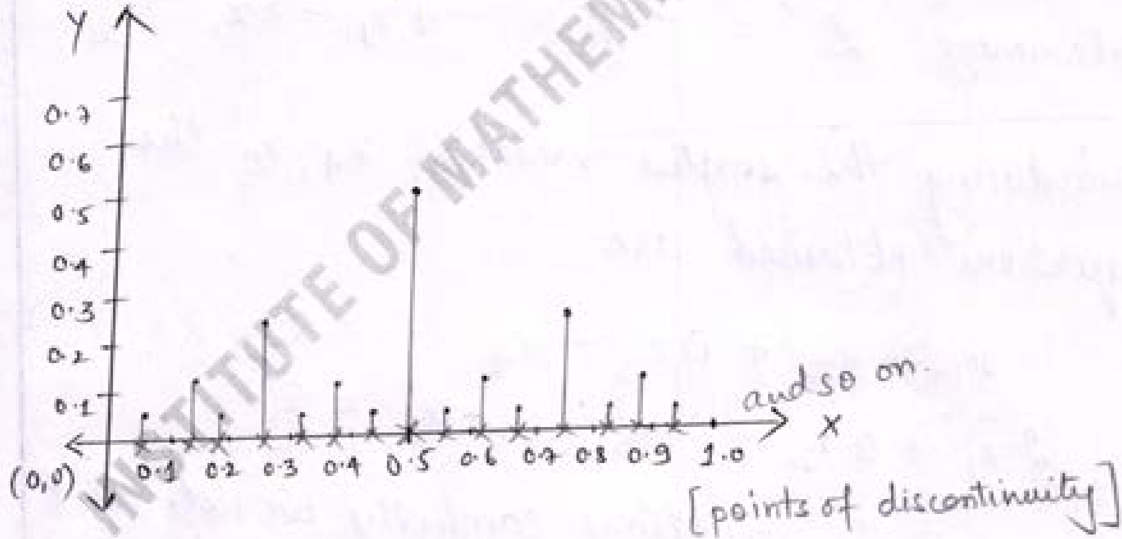
Since the set of points of discontinuity of 'f' on $[0,1]$ has a finite number of limit points.

$\therefore$ f is integrable on $[0,1]$.

Hence, the result.

Additional input :-

Refer to the graph of $f(x)$:-



Q.3.(d). Using simplex method, solve the L.P. problem.

Minimize $Z = 4x_1 + 8x_2 + 3x_3$

Subject to constraints

$$x_1 + x_2 \geq 2$$
$$2x_1 + x_3 \geq 5$$
$$x_1, x_2, x_3 \geq 0.$$

Solution:

First, we convert the problem of minimization to maximization problem by taking the objective function as Z'.

Maximize $Z' = -Z = -4x_1 - 8x_2 - 3x_3$

Introducing the surplus variables x_4, x_5 the equations obtained are

$$x_1 + x_2 + 0x_3 - x_4 = 2$$
$$2x_1 + 0 \cdot x_2 + x_3 - x_5 = 5$$

Examining the equations carefully we note that the columns of the coefficient of x_2 and x_3 form a unit matrix.

Therefore there is no need to introduce the artificial variables.

Taking $x_1 = 0$, $x_4 = 0$, $x_5 = 0$, we have $x_2 = 2$, $x_3 = 5$ which is the starting B.F.S.

B	C_B	c_j / X_B	-4 / Y_1	-8 / Y_2	-3 / Y_3	0 / Y_4	0 / Y_5	Mini-Ratio X_B/Y_1
Y_2	-8	2	$\boxed{1}$	1	0	-1	0	2 (minimum) →
Y_3	-3	5	2	0	1	0	-1	$5/2$
$Z' = C_B X_B = -31$		Δ_j	10 ↑	0	0	-8	-3	X_B/Y_4
Y_1	-4	2	1	1	0	-1	0	(−ve)
Y_3	-3	1	0	-2	1	$\boxed{2}$	1	$1/2$ (minimum) →
$Z' = C_B X_B = -11$		Δ_j	0	-10	0	2 ↑	-3	
Y_1	-4	$5/2$	1	0	$1/2$	0	$-1/2$	
Y_4	0	$1/2$	0	-1	$1/2$	1	$-1/2$	
$Z' = C_B X_B = -10$		Δ_j	0	-8	-1	0	-2	

Here, no $\Delta_j > 0$, so this solution is optimal.

$\therefore$ Optimal solution is

$x_1 = 5/2$, $x_2 = 0$, $x_3 = 0$ and

Minimum $Z = -Z' = 10$.

Hence, the result.

Q.4.(b) The function $f(x) = 1/x$ is continuous on $(0,1)$ but not uniformly continuous.

Solution:

Since x is continuous on $(0,1)$ and $x \neq 0$ on $(0,1)$,

$\Rightarrow \frac{1}{x}$ is continuous on $(0,1)$.

Now,
for any $\delta > 0$, $\exists$ $m \in \mathbb{N}$ such that $\frac{1}{n} < \delta$ $\forall n \geq m$.

Let $x_1 = \frac{1}{m}$, $x_2 = \frac{1}{2m}$

Then $x_1, x_2 \in (0,1)$ and $|x_1 - x_2| < \delta$

but $|f(x_1) - f(x_2)| = m$ which cannot be smaller than every $\varepsilon > 0$.

Thus, $f(x) = 1/x$ is not uniformly continuous on $(0,1)$.

Hence, the result.

Q.4.(c)
(i) Specify the nature of singularity at $z=-2$ of $f(z) = (z-3)\sin\frac{1}{z+2}$

(ii) Find zeros and poles of $\left(\frac{z+1}{z^2+1}\right)^2$.

Solution:

(i) Zeros of $f(z)$ are given by $f(z)=0$

i.e. $(z-3)\sin\frac{1}{z+2} = 0 \Rightarrow z=3, \sin\frac{1}{z+2}=0$

Now,

$$\sin\left(\frac{1}{z+2}\right) = 0 = \sin n\pi$$

$$\Rightarrow \frac{1}{z+2} = n\pi$$

$$\Rightarrow z+2 = \frac{1}{n\pi} \Rightarrow z = -2+\frac{1}{n\pi} \;;\; n=0,1,2,\dots$$

Limit point of zeros is $z=-2$.

$\therefore$ $z=-2$ is an isolated essential singularity.

(ii)

We have $f(z) = \left(\frac{z+1}{z^2+1}\right)^2 = \frac{(z+1)^2}{(z^2+1)^2}$

Zeros of $f(z)$ are given by $f(z)=0$

$$\Rightarrow (z+1)^2 = 0 \Rightarrow z=-1,-1.$$

$\therefore$ $z=-1$ is a zero of order 2.

Poles of $f(z)$ are given by.

$$(z^2+1)^2=0$$

$$\Rightarrow \left[(z+i)^2(z-i)^2\right]=0$$

$$\Rightarrow z=-i,-i,i,i$$

$\therefore$ $z=i$ and $z=-i$ are poles each of order 2.

Hence, the result.

Q. 4. (d) An automobile dealer wishes to put four repairmen to four different jobs. The repairmen have somewhat different kinds of skills and they exhibit different levels of efficiency from one job to another. The dealer has estimated the number of manhours that would be required for each job-man combination. This is given in the matrix form in adjacent table. Find the optimum assignment that will result in minimum manhours needed.

Man \ Job	A	B	C	D
1	5	3	2	8
2	7	9	2	6
3	6	4	5	7
4	5	7	7	8

Solution:

Step 1: Subtracting the smallest element of each row from all the elements of that row and then in the second matrix subtracting the smallest element of each column from all the elements of that columns the initial feasible solution determined by zeros is obtained.

Man \ Job	A	B	C	D
1	3	1	0	3
2	5	7	0	1
3	2	0	1	0
4	0	2	2	0

Table (I)

Step 2: We now examine each row successively for a single zero Enrectangling these zeros and crossing (x) all the remaining zeros in the respective columns and then repeating same procedure for each column, we

Man \ Job	A	B	C	D
1	3	1	$\boxed{0}$	3
2	5	7	$\cancel{0}$	1
3	2	$\boxed{0}$	1	$\cancel{0}$
4	$\boxed{0}$	2	2	$\cancel{0}$

Table (II)

Step 3: We now draw the minimum number of horizontal and vertical lines necessary to cover all zeros atleast once.

Man \ Job	A	B	C	D
1	3	1	$\boxed{0}$	3
2	5	7	$\cancel{0}$	1
3	2	$\boxed{0}$	1	$\cancel{0}$
4	$\boxed{0}$	2	2	$\cancel{0}$

L_2

L_3

Table (III)

L_1

Select the smallest element not covered by the lines (i.e. 1) and subtract it from all the elements not covered by the lines and add the same to the elements of the intersection of the lines.

We obtain Table (IV), providing second feasible solution to the problem.

Man \ Job	A	B	C	D
1	2	0	0	2
2	4	6	0	0
3	2	0	2	0
4	0	2	3	0

Table (IV)

Step 4:

Repeating Step 2, we make the 6 zero-assignments as shown in Table 'V' and Table 'VI'.

Man \ Job	A	B	C	D
1	2	$\boxed{0}$	$\cancel{0}$	2
2	4	6	$\boxed{0}$	$\cancel{0}$
3	2	$\cancel{0}$	2	$\boxed{0}$
4	$\boxed{0}$	2	3	$\cancel{0}$

Table(V)

Optimum Solution 1.

Man	Job	Man hours.
1.	B	3
2	C	2
3	D	7
4	A	5

17 hours

It may be noted that an assignment problem can have more than one optimum solution. The other solution is shown in Table (VI).

Man \ Job	A	B	C	D
1	2	$\cancel{0}$	$\boxed{0}$	2
2	4	6	$\cancel{0}$	$\boxed{0}$
3	2	$\boxed{0}$	2	$\cancel{0}$
4	$\boxed{0}$	2	3	$\cancel{0}$

Optimum Solution-2.

Man	Job	Man hours
1	C	2
2	D	6
3	B	4
4	A	5

17 hours

Hence, the result.

SECTION – B.

Q.5. (a) Give an example of a group which is not a cyclic group, but every proper subgroup of which is cyclic.

Solution:

Let us consider Klein 4-group,

$\therefore\ G = \{\pm 1, \pm i, \pm j, \pm k\}$ where $i^2 = j^2 = k^2 = 1$

and $ij = k,\ jk = i,\ ki = j$.

Also, G is not a cyclic group since there is no element in G of order 4 which can generate all other elements of Group G.

Klein-4-group has four proper subgroups, namely,

$H_1 = \{\pm 1\} = \langle -1 \rangle$

$H_2 = \{\pm 1, \pm i\} = \langle i \rangle$

$H_3 = \{\pm 1, \pm j\} = \langle j \rangle$

$H_4 = \{\pm 1, \pm k\} = \langle k \rangle$

Thus, every proper subgroup of G is cyclic.

Hence, Klein 4-group is the required example.

Alternatively, The group $U(8) = \{1, 3, 5, 7\}$ is not a cyclic group since $1 = 3^2 = 5^2 = 7^2 = 1$.

[i.e. no generator in $U(8)$]
The only proper subgroups are $\{1\}$, $\{1,3\}$, $\{1,5\}$ and $\{1,7\}$ which are obviously cyclic.

Q.5(b) Consider the following rings R and R' with four elements $R = \{a,b,c,d\}$ with $+$ and $\cdot$ defined by

$+$	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	a	b
c	a	a	c	c
d	a	b	c	d

and $R' = \{x, y, z, t\}$ with $+$ and $\cdot$ defined by

$+$	x	y	z	t
x	x	y	z	t
y	y	z	t	x
z	z	t	x	y
t	t	x	y	z

$\cdot$	x	y	z	t
x	x	x	x	x
y	x	y	z	t
z	x	z	x	z
t	x	t	z	y

Investigate whether R and R' are isomorphic.

Solution: (if possible)
Let $f: R \to R'$ be an isomorphism defined by

$$f(a) = x \quad \text{and} \quad f(d) = y$$

as a and x are additive identities of R and R'

respectively, whereas d and y are multiplicative identities of R and R'.

$\because$ f is one-one, we have following two cases for $f(b)$ and $f(c)$.

Case i:

$f(b) = z,\ f(c) = t$

$\therefore f(b \cdot c) = f(a) = x$

$\because f(b)\cdot f(c) = z \cdot t = z$

Thus, $f(b\cdot c) \neq f(b)\cdot f(c)$

$\Rightarrow f$ is not a homomorphism

$\Rightarrow f$ is not an isomorphism.

Case ii:

$f(b) = t\ ,\ f(c) = z$

$\therefore f(b\cdot c) = f(a) = x$

$\therefore f(b)\cdot f(c) = t \cdot z = z$

Again, $f(b\cdot c) \neq f(b)\cdot f(c)$

$\Rightarrow f$ is not a homomorphism

$\Rightarrow f$ is not an isomorphism.

Thus, R and R' are not isomorphic to each other.

Hence, the result.

Q.5.(c) The sequence $nx/(1+n^2x^2)$ is not uniformly convergent over $\mathbb{R}$ but it is uniformly convergent on $\{x : |x| > k > 0\}$.

Solution:

We know that $nx/(1+n^2x^2)$ converges to '0' for any $x \in \mathbb{R}$.

Let the convergence be uniform on $\mathbb{R}$, then for $\epsilon = \frac{1}{20}$ $\exists$ $m \in \mathbb{N}$ such that

$$\left| \frac{2nx}{1-4n^2x^2} - \frac{nx}{1+n^2x^2} \right| < \frac{1}{20} \quad \forall\, n \geq m$$
and $x \in \mathbb{R}$.

(For $p = n$ by Cauchy's principle).

On putting $x = 1/m$, $n = m$, we get,

$$\left| \frac{2}{5} - \frac{1}{2} \right| = \frac{1}{10} < \frac{1}{20}$$

which is a contradiction.

This contradiction arises due to our wrong assumption.

Therefore, $nx/(1+n^2x^2)$ is <u>not uniformly convergent on $\mathbb{R}$.</u>

Further, on $\{x : |x| > k > 0\}$ for $\varepsilon > 0$

$$\left| \frac{nx}{1+n^2x^2} - 0 \right| < \frac{1}{|nx|} < \frac{1}{nx} < \varepsilon$$

$$\forall\ n \geq \left[\frac{1}{k\varepsilon}\right]$$

Hence, $\frac{nx}{(1+n^2x^2)}$ converges uniformly on $\{x : |x| > k > 0\}$.

Hence, the result.

Q.5.(d) If $f(z) = \frac{x^3y(y-ix)}{x^6+y^2}$, $z \neq 0$ and $f(0) = 0$, show that $\frac{f(z)-f(0)}{z} \to 0$ as $z \to 0$ along any radius vector but not as $z \to 0$ in any manner.

Solution :

We have, $\frac{f(z)-f(0)}{z} = \frac{f(z)-0}{z} = \frac{f(z)}{z}$

$$= \frac{x^3y(y-ix)}{(x^6+y^2)(x+iy)}$$

$$= \frac{-i x^3 y \cancel{(x+iy)}}{(x^6+y^2)\cancel{(x+iy)}}$$

$$= \frac{-i x^3 y}{x^6+y^2}$$

Let $z \to 0$ along the radius vector $y = mx$.
Then we have,

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{x\to 0} \frac{-i x^3.(mx)}{x^6+(mx)^2}$$

$$= \lim_{x\to 0} \frac{-i m x^2}{x^4+m^2}$$

$$= 0$$

i.e. $\frac{f(z)-f(0)}{z} \to 0$ as $z \to 0$ along any radius vector. —— (i).

Now, let $z \to 0$ along $y = x^3$, then

$$\lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{z\to 0} \frac{-i x^3(x^3)}{x^6+(x^3)^2} = \frac{-i}{2} \neq 0$$

i.e. $\frac{f(z)-f(0)}{z} \not\to 0$ as $z \to 0$ along any other manner. Hence, the result.

Q.5. (e) A firm plans to purchase atleast 200 quintals of scrap containing high quality metal X and low quality metal Y. It decides that the scrap to be purchased must contain atleast 100 quintals of X-metal and not more than 30 quintals of Y-metal. The firm can purchase the scrap from two suppliers (A and B) in unlimited quantities. The percentage of X and Y metals in terms of weight in the scraps supplied by A and B is given below:

Metals	Supplier A	Supplier B
X	25%	75%
Y	10%	20%

The price of A's scrap is ₹200 per quintal and that of B's scrap is ₹400 per quintal. Formulate this problem as LP model and solve it graphically to determine the quantities that the firm should buy from the two suppliers, so as to minimize total purchase cost.

Solution:

The formulation of the given problem is:

Minimize (total cost) $Z = 200x_1 + 400x_2$

subject to the constraints:

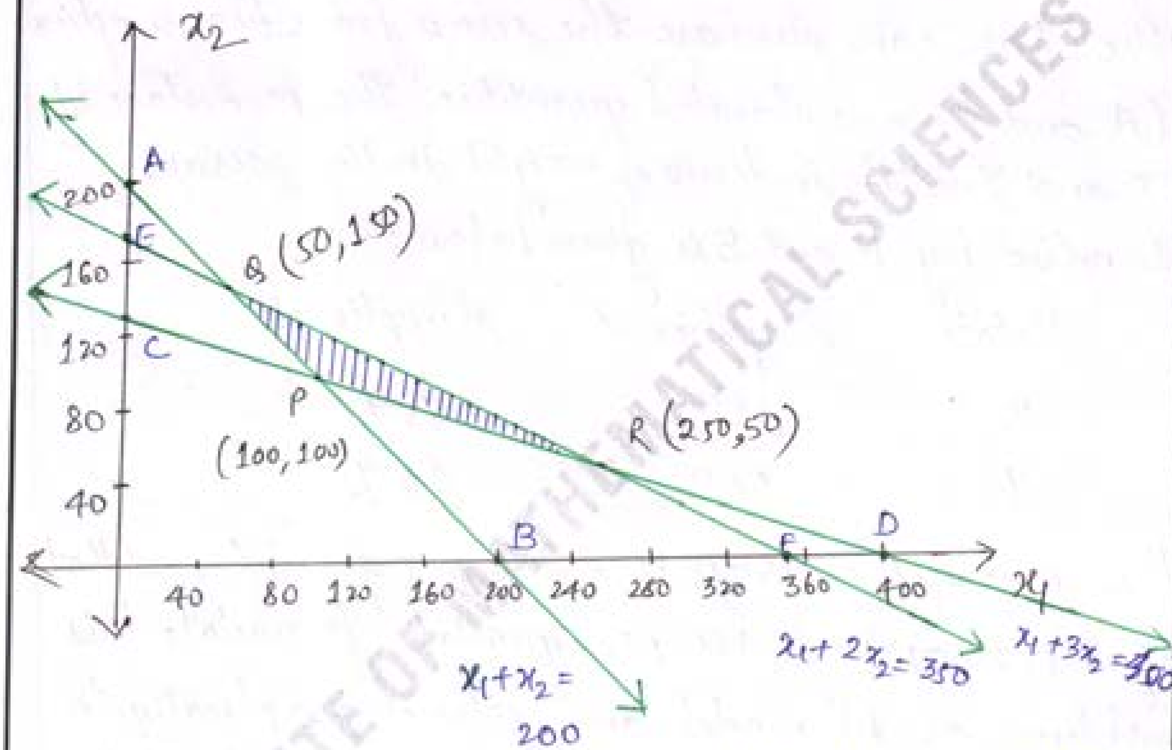
$$x_1 + x_2 \geq 200,$$

$$\frac{1}{4}x_1 + \frac{3}{4}x_2 \geq 100,$$

$\frac{1}{10}x_1 + \frac{1}{5}x_2 \leq 35,$

$x_1, x_2 \geq 0$

where x_1, x_2 represent the number of quintals of scrap from two suppliers A and B respectively.



The feasible region is the shaded area PQR which is obtained by drawing the graph of the given constraints.

The co-ordinates of the corner points of the feasible region are $P(100,100)$, $Q(50,150)$, $R(250,50)$.

Thus, Z has minimum value at the point $P(100,100)$.

Hence, the required answer is $\boxed{x_1 = 100, x_2 = 100}$

$\boxed{\text{minimum } Z = ₹\ 60{,}000}$

Q.6.(a)

(i) Let G be the group of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where a, b, c, d are integers modulo p, p is a prime number, such that $ad-bc \neq 0$. G forms a group relative to matrix multiplication. What is $o(G)$?

(ii) Let H be the subgroup of G of part (i) defined by $H = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \;/\; ad-bc = 1 \right\}$.

What is $o(H)$?

Solution:

(i)
For the first row (a, b) of a matrix in G, a and b could be anything in $\mathbb{Z}_p$, but we must exclude the case $a=0$ and $b=0$.
$[\because ad-bc \neq 0]$

Hence, $(p\times p)-1$ possibilities for the first row.

Now, for the second row, we note that it should NOT be a multiple of the first row.

Hence, for second row, we have $(p\times p)-p$ possibilities.

$\Rightarrow$ The number of elements in G is
$(p^2-1)\cdot(p^2-p)$.

i.e. $\boxed{o(G) = (p^2-1)(p^2-p)}$ —— (i)

(ii)
The set of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $ad-bc=1$

forms a subgroup H of G.

Moreover for any $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ either

$ad-bc = \det(g) = 1$ or $\det(g) = -1$
$[\equiv (p-1)(\bmod p)]$

On the other hand, for any,
$Hg_1, Hg_2 \in G$, $Hg_1 = Hg_2$ iff
$\det(g_1) = \det(g_2)$.

Hence, the above subgroup has index 2 in G.

i.e. $\boxed{o(H) = \frac{(p^2-1)(p^2-p)}{2}}$ —— (ii)

Hence, the result.

Q.6.(b) Investigate whether the following ideals are (i) prime ideals, (ii) maximal ideals in the ring R.

(i) $R = \mathbb{Z}_6$ and $I = \{\bar{0}, \bar{2}, \bar{4}\}$.

(ii) $R = 2\mathbb{Z}$ and $I = 4\mathbb{Z}$

(iii) $R = C[0,1]$ the ring of continuous functions on $[0,1]$ and $I = \{x \mid x(1/2) = 0\}$.

Solution :

(i) I is a maximal ideal of R as

$R/I \cong \mathbb{Z}$ (field).

Hence, it also a prime ideal.

(ii) I is a maximal ideal of R

If $J = n\mathbb{Z} \supset I$, 2 divides n which divides 4.

Hence, $n = 2$ or $n = 4$.

i.e. $J = R$ or $J = I$.

It is also a prime ideal.

(iii) I is a prime ideal of R.

Consider the map $\theta : C[0,1] \to F$ [F reals]

defined by $\theta(x) = x(1/2)$

It is a ring homomorphism with

$\text{Ker}\,\theta = I$.

Hence, $\frac{C[0,1]}{I} \simeq Im(\theta)$

Also, θ is 'onto' because given any value λ of F, there exists a continuous map x on $[0,1]$ with $x(1/2) = \lambda$.

Since F is a field, I is a maximal ideal and also a prime ideal.

Q-6-(c) Prove that every integral domain can be imbedded in a field.

Solution:

Let D be an integral domain with atleast two elements.

Let us consider $S = \{(a,b) / a, b \in D : b \neq 0\}$ then $S \neq \phi$ and $S \subset D \times D$.

$\forall (a,b), (c,d) \in S$

Define a relation '$\sim$' on 'S' as

$(a,b) \sim (c,d) \Leftrightarrow ad = bc$

We now prove that $\sim$ is an equivalence relation on S.

(i) For each $(a,b) \in S$

We have $ab = ba$

$\Rightarrow (a,b) \sim (a,b)$

(ii) for $(a,b), (c,d) \in S$

We have $(a,b) \sim (c,d)$

$\Rightarrow ad = bc$

$\Rightarrow cb = da$

$\Rightarrow (c,d) \sim (a,b)$

(iii) for $(a,b), (c,d), (e,f) \in S$

$(a,b) \sim (c,d)$, $(c,d) \sim (e,f)$

$\Rightarrow ad = bc$, $cf = de$

$\Rightarrow (ad)f = (bc)f$, $cf = de$

$\Rightarrow (af)d = b(de)$

$\Rightarrow (af)d = b(ed)$

$\Rightarrow (af)d = (be)d$

$\Rightarrow af = be \quad (\because d \neq 0)$

$\Rightarrow (a,b) \sim (e,f)$

$\therefore$ '$\sim$' is an equivalence relation on 'S'.

The equivalence relation '$\sim$' partitions the set 'S' into equivalence classes which are identical or disjoint.

For $(a,b) \in S$,

let $\frac{a}{b}$ denote equivalence class of (a,b). then

$$\frac{a}{b} = \{(x,y) \in S / (x,y) \sim (a,b)\}$$

i.e. $\frac{a}{b} = \{(x,y) \in S / (x,y) \sim (a,b) \Leftrightarrow xb = ya\}$

If $\frac{a}{b}, \frac{c}{d}$ are the equivalence classes of (a,b), $(c,d) \in S$, then either $\frac{a}{b} = \frac{c}{d}$ or $\frac{a}{b} \cap \frac{c}{d} = \phi$

It is evident that $\frac{a}{b} = \frac{c}{d} \Leftrightarrow ad = bc$.

Let F denote the set of all the equivalence classes or the set of quotients then

$$F = \left\{ \frac{a}{b} / (a,b) \in S \right\}$$

Since D has atleast two elements say $0, a \in D$

We have quotients $\frac{0}{a}, \frac{a}{a} \in F$ and $\frac{0}{a} \neq \frac{a}{a}$.

$\therefore$ the set F has atleast two elements.

for $\frac{a}{b}, \frac{c}{d} \in F$, define addition $(+)$ and multiplication $(\cdot)$ as

(i) $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$ and

(ii) $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$

Since D is without zero divisors,

$$\underbrace{0 \neq b, d \neq 0}_{\in D} \Rightarrow bd \neq 0.$$

so $\frac{ad+bc}{bd}, \frac{ac}{bd} \in F$.

Now, we prove that the $+^n$ and $\times^n$ defined above are well-defined.

Let $\frac{a}{b} = \frac{a'}{b'}$ and $\frac{c}{d} = \frac{c'}{d'}$

then $ab' = a'b$ and $cd' = c'd$ — (I)

Now, (I) $\Rightarrow ab'dd' = a'bdd'$ and

$$bb'cd' = bb'c'd$$

$\Rightarrow ab'dd' + bb'cd' = a'bdd' + bb'c'd$

$\Rightarrow (ad+bc)b'd' = (a'd'+b'c')bd$

$\Rightarrow \dfrac{ad+bc}{bd} = \dfrac{a'd'+b'c'}{b'd'}$

Also (I) $\Rightarrow ab'cd' = a'bc'd$

$\Rightarrow (ac)(b'd') = (a'c')(bd)$

$\Rightarrow \dfrac{ac}{bd} = \dfrac{a'c'}{b'd'}$

$\therefore$ $+^n$ and $\times^n$ of quotients are well-defined binary operations on F.

We now prove that $(F, +, \cdot)$ is a field:

(i) for $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in F$;

$$\left(\frac{a}{b}+\frac{c}{d}\right)+\frac{e}{f} = \frac{ad+bc}{bd}+\frac{e}{f}$$

$$= \frac{(ad+bc)f+(bd)e}{(bd)f}$$

$$= \frac{a(df)+(cf+de)b}{b(df)}$$

$$= \frac{a}{b}+\frac{cf+de}{df}$$

$$= \frac{a}{b}+\left(\frac{c}{d}+\frac{e}{f}\right)$$

$\therefore$ addition is associative.

(ii) for $\frac{a}{b}, \frac{c}{d} \in F$; $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$

$$= \frac{bc+ad}{db}$$

$$= \frac{c}{d} + \frac{a}{b}$$

$\therefore$ addition is commutative.

(iii) for $u \neq 0 \in D$ we have $\frac{0}{u} \in F$ such that

$$\frac{0}{u} + \frac{a}{b} = \frac{0b+ua}{ub} = \frac{ua}{ub} = \frac{a}{b} \quad \forall \frac{a}{b} \in F$$

$\therefore$ $\frac{0}{u} \in F$ is the zero element.

(iv) Let $\frac{a}{b} \in F$ then $-\frac{a}{b} \in F$ such that

$$\frac{a}{b} + \left(-\frac{a}{b}\right) = \frac{ab+(-a)b}{b^2} = \frac{0}{b^2} = \frac{0}{u}$$

$$(\because 0u = 0b^2)$$

$\therefore$ every element in F has additive inverse.

(v) for $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in F$;

$$\left(\frac{a}{b}\cdot\frac{c}{d}\right)\cdot\frac{e}{f} = \frac{ac}{bd}\cdot\frac{e}{f}$$

$$= \frac{(ac)e}{(bd)f}$$

$$= \frac{a(ce)}{b(df)} = \frac{a}{b}\cdot\frac{ce}{df}$$

$= \frac{a}{b} \cdot \left(\frac{c}{d} \cdot \frac{e}{f}\right)$

$\therefore$ multiplication is associative.

(vi) for $\frac{a}{b}, \frac{c}{d} \in F$;

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} = \frac{ca}{db} = \frac{c}{d} \cdot \frac{a}{b}$$

$\therefore$ multiplication is commutative.

(vii) for $u \neq 0 \in D$, we have $\frac{u}{u} \in F$ such that $\frac{a}{b} \cdot \frac{u}{u} = \frac{au}{bu} = \frac{a}{b}$ $\forall \frac{a}{b} \in F$.

$\therefore$ $\frac{u}{u} \in F$ is the unity element.

(viii) Let $\frac{a}{b} \in F$ and $\frac{a}{b} \neq \frac{0}{u}$

Then $au \neq 0 \Rightarrow a \neq 0$ and already $u \neq 0$

$\therefore$ $b \neq 0$ and $a \neq 0 \Rightarrow \frac{b}{a} \in F$

$\therefore$ for $\frac{a}{b}\left(\neq \frac{0}{u}\right) \in F$ there exists $\frac{b}{a} \in F$

such that $\frac{a}{b} \cdot \frac{b}{a} = \frac{ab}{ba} = \frac{u}{u}$ $[\because (ab)u = (ba)u]$

$\therefore$ Every non-zero element in F has multiplicative inverse.

(ix) for $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in F$;

$$\frac{a}{b}\cdot\left(\frac{c}{d}+\frac{e}{f}\right) = \frac{a}{b}\cdot\frac{cf+de}{df}$$

$$= \frac{a(cf+de)}{b(df)}$$

$$= \frac{(acf+ade)(bdf)}{(bdf)(bdf)}$$

$$\left[\because \frac{bdf}{bdf} = \frac{u}{u}\right]$$

$$= \frac{acf\,bdf + ade\,bdf}{(bdf)(bdf)}$$

$$= \frac{acf}{bdf} + \frac{ade}{bdf}$$

$$= \frac{ac}{bd} + \frac{ae}{bf}$$

$$= \frac{a}{b}\cdot\frac{c}{d} + \frac{a}{b}\cdot\frac{e}{f}$$

Similarly, we can prove that,

$$\left(\frac{c}{d}+\frac{e}{f}\right)\cdot\frac{a}{b} = \frac{c}{d}\cdot\frac{a}{b} + \frac{e}{f}\cdot\frac{a}{b}$$

$\therefore$ multiplication is distributive over addition.

Thus, in view of (i) to (ix) results,

$(F, +, \cdot)$ is a field.

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Now, we have to prove that D is embedded in the field F, i.e. we have to show that there exists an isomorphism of D into F.

Define the mapping $\phi : D \to F$ by
$$\phi(a) = \frac{ax}{x} \quad \forall\, a \in D \text{ and } x(\neq 0) \in D.$$

$a, b \in D$ and $\phi(a) = \phi(b)$

$$\Rightarrow \frac{ax}{x} = \frac{bx}{x}$$
$$\Rightarrow (ax)x = (bx)x$$
$$\Rightarrow (a-b)x^2 = 0$$
$$\Rightarrow a-b=0 \quad \text{since } x^2 \neq 0.$$
$$\Rightarrow a=b.$$

$\therefore$ ϕ is one-one.

for $a, b \in D$;

$$\phi(a+b) = \frac{(a+b)x}{x} = \frac{(a+b)xx}{xx} = \frac{axx+bxx}{xx}$$

$$= \frac{axx}{xx} + \frac{bxx}{xx} = \frac{ax}{x} + \frac{bx}{x} = \phi(a) + \phi(b) \quad \text{—(i)}$$

$$\phi(ab) = \frac{(ab)x}{x} = \frac{(ab)xx}{xx} = \frac{ax}{x} \cdot \frac{bx}{x}$$

$= \phi(a)\cdot\phi(b)$ ——— (ii)

from (i) & (ii),

ϕ is a homomorphism.

Hence, ϕ is an isomorphism of D into F.

$\therefore$ The Integral Domain D is embedded in the field F.

Hence, the theorem.

Q.7.(a) Prove that the following sets are bounded.

$\{n^{1/n} : n\in\mathbb{N}\}$, $\{(1+\frac{1}{n})^n : n\in\mathbb{N}\}$, $\{a^{1/n} : a>0$ and $n\in\mathbb{N}\}$. Give supremum and infimum of each of these sets.

Solution:

(i) $\{n^{1/n} : n\in\mathbb{N}\} = \{1, 2^{1/2}, 3^{1/3}, 4^{1/4}, \dots\}$

as, we know that $\lim_{n\to\infty} n^{1/n} = 1$.

$\Rightarrow$ The set is bounded. [$\because$ Every cgt seq. is bounded.]

Here, the infimum is 1 and the supremum is $3^{1/3}$.

(ii) $\left\{\left(1+\frac{1}{n}\right)^n : n \in \mathbb{N}\right\}$

Also, we know that $\lim_{n\to\infty} \left\{\left(1+\frac{1}{n}\right)^n : n\in\mathbb{N}\right\} = e$

$\Rightarrow$ It is bounded.

Here, the infimum is 2 and supremum is e.

Additional steps to show limit :-

$$e^{\ln(1+1/n)^n} = e^{n\ln(1+1/n)}$$

$$\lim_{n\to\infty}\left(1+1/n\right)^n = \lim_{n\to\infty} e^{n\ln(1+1/n)}$$

$$= \lim_{n\to\infty} n\ln(1+1/n)$$

$$= e$$

$$= e^{\lim_{n\to\infty} \frac{\ln(1+1/n)}{1/n}}$$

Applying L'Hopital's rule,

$$= e^{\lim_{n\to\infty} \frac{\frac{1}{1+1/n}\cdot\cancel{\left(-\frac{1}{n^2}\right)}}{\cancel{\left(-1/n^2\right)}}}$$

$\therefore \lim_{n\to\infty}\left(1+1/n\right)^n = e$

Similarly, $\lim_{n\to\infty} n^{1/n} = 1$ can be proved.

(iii) $\{a^{1/n} : a>0 \text{ and } n \in \mathbb{N}\}$

Let $L = \lim_{n\to\infty} a^{1/n}$

Taking log on both sides, we have

$$\log L = \lim_{n\to\infty} \frac{1}{n} \log a$$

$$\Rightarrow \underline{L = 1}.$$

i.e. for any $a>0$, $(a^{1/n})$ converges to 1.
$\Rightarrow$ It is bounded.

Here infimum is and supremum is .

Hence, the result.

Q.7.(b) Show that $\prod_{n=1}^{\infty}\left(1-\frac{1}{4n^2}\right)$ converges and its limit lies between $1/2$ and 1.

__Solution:__

The given infinite product is $\prod_{n=1}^{\infty}\left(1-\frac{1}{4n^2}\right)$

$$= \prod_{n=1}^{\infty}(1-b_n), \text{ where } b_n = \frac{1}{4n^2}$$

and $n \geq 1$

so that $0 < b_n < 1$.

$\therefore$ The product $\prod_{n=1}^{\infty}(1-b_n)$ and the series $\sum_{n=1}^{\infty} b_n$ converge or diverge together.

But the series $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{4n^2}$ is convergent

[By P-Test]

$\therefore$ The given product $\prod_{n=1}^{\infty}\left(1-\frac{1}{4n^2}\right)$ is convergent.

Now,

$\because\ 0 < b_n < 1$

$\Rightarrow -1 < -b_n < 0$

$\Rightarrow 1-1 < 1-b_n < 1$

$\Rightarrow\ 0 < 1-b_n < 1.$

$\Rightarrow \prod_{n=1}^{\infty}(0) < \prod_{n=1}^{\infty}(1-b_n) < \prod_{n=1}^{\infty}(1)$

$\Rightarrow\ 0 < \prod_{n=1}^{\infty}\left(1-\frac{1}{4n^2}\right) < 1$ —— (*)

Also, $1-\frac{1}{4n^2}$ is an increasing function with increasing n.

$\left[\because \text{ Let } f(x) = 1-\frac{1}{4n^2}, \ f'(x) = \frac{1}{2n^3} > 0 \ \forall n > 0\right]$

$$\Rightarrow \prod_{n=1}^{\infty}(1-b_n) = \prod_{n=1}^{\infty}\left(1-\frac{1}{4n^2}\right) = \prod_{n=1}^{\infty}\left(\frac{4n^2-1}{4n^2}\right)$$

$$= \prod_{n=1}^{\infty}\left(\frac{2n+1}{2n}\right)\left(\frac{2n-1}{2n}\right) = \prod_{n=1}^{\infty}\left(1+\frac{1}{2n}\right)\left(1-\frac{1}{2n}\right)$$

$$> \left(1+\frac{1}{2}\right)\left(1-\frac{1}{2}\right)$$

$$> 1-\frac{1}{2}$$

$$= \frac{1}{2}$$

i.e. $\prod_{n=1}^{\infty}\left(1-\frac{1}{4n^2}\right) > \frac{1}{2}$ ——— (**)

From (*) and (**), we conclude that, the limit of $\prod_{n=1}^{\infty}\left(1-\frac{1}{4n^2}\right)$ lies between $\frac{1}{2}$ and 1.

Hence, the result.

Q.7.(c) for $x > -1$, test for convergence

$$1 + \frac{2^2}{3.4}x + \frac{2^2.4^2}{3.4.5.6}x^2 + \frac{2^2.4^2.6^2}{3.4.5.6.7.8}x^3 + \cdots$$

Solution:

Leaving the first term,

$$u_n = \frac{2^2.4^2.\cdots\cdots(2n)^2}{3.4.5.6.\cdots(2n+2)}x^n$$

$$u_{n+1} = \frac{2^2.4^2.\cdots\cdots(2n)^2(2n+2)^2}{3.4.5.6\cdots\cdots(2n+2)(2n+3)(2n+4)}.x^{n+1}$$

$$\Rightarrow \frac{u_n}{u_{n+1}} = \frac{(2n+4)(2n+3)}{(2n+2)^2}.\frac{1}{x}$$

$$= \frac{(2+4/n)(2+3/n)}{(2+2/n)^2}.\frac{1}{x}$$

$$\therefore \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \frac{1}{x}$$

$\therefore$ By D'Alembert's Ratio test,

$\sum u_n$ converges if $1/x > 1$ i.e. $x < 1$

and $\sum u_n$ diverges if $1/x < 1$ i.e. $x > 1$.

If $x=1$, then the ratio test fails.

when $x=1$,

$$\frac{u_n}{u_{n+1}} = \frac{(2n+4)(2n+3)}{(2n+2)^2}$$

$$= \frac{(1+4/2n)(1+3/2n)}{(1+2/2n)^2}$$

$$= \frac{\left[1+7/2n+\cancel{12}^{3}/\cancel{4}n^2\right]}{\left[1+\frac{2\cdot\cancel{2}}{\cancel{2}n}+\frac{\cancel{4}}{\cancel{4}n^2}\right]}$$

$$= \left[1+\frac{7}{2n}+\frac{3}{n^2}\right]\left[1+\left(\frac{2}{n}+\frac{1}{n^2}\right)\right]^{-1}$$

$$= \left(1+\frac{7}{2n}+\frac{3}{n^2}\right)\left(1-\frac{2}{n}-\frac{1}{n^2}+\frac{4}{n^2}+\frac{1}{n^4}-\frac{8}{n^3} - \frac{1}{n^6}+\cdots\right)$$

$$= \left(1+\frac{7}{2n}+\frac{3}{n^2}\right)\left(1-\frac{2}{n}+\frac{3}{n^2}-\frac{8}{n^3}+\frac{1}{n^4}+\cdots\right)$$

$$= \left(1-\frac{2}{n}+\frac{3}{n^2}-\frac{8}{n^3}+\cdots\right)+\left(\frac{7}{2n}-\frac{14}{2n^2}+\frac{21}{2n^3}-\cdots\right)+\left(\frac{3}{n^2}-\frac{6}{n^3}+\frac{9}{n^4}-\cdots\right)$$

$$= 1+\frac{3}{2n}-\frac{1}{n^2}+\cdots$$

$$= 1 + \left(\frac{3}{2}\right)\left(\frac{1}{n}\right) + O\left(\frac{1}{n^2}\right)$$

Comparing it with

$$\frac{u_n}{u_{n+1}} = 1 + \frac{\lambda}{n} + O\left(\frac{1}{n^2}\right)$$

where $\lambda = \frac{3}{2} > 1$.

$\therefore$ By Gauss Test,

$\sum u_n$ is convergent at $x = 1$.

$\therefore$ The given series converges if $x \leq 1$ and diverges if $x > 1$.

Hence, the result.

Q.7.(d) Let a function $f : \mathbb{R} \to \mathbb{R}$ satisfy the equation $f(x+y) = f(x) + f(y)$, $\forall x, y \in \mathbb{R}$.

Show that

(i) If f is continuous at the point $x = a$, then it is continuous for all $x \in \mathbb{R}$.

(ii) If f is continuous then $f(x) = kx$, for some constant k.

Solution:

(ii)

Let f be continuous at the point $x=a$.

We have $f(a+0) = \lim_{h\to 0} f(a+h)$

$= \lim_{h\to 0} [f(a)+f(h)]$ — (by def^n. of f)

$= \lim_{h\to 0} f(a) + \lim_{h\to 0} f(h)$

$= f(a) + \lim_{h\to 0} f(h)$

$\because$ f is continuous at a, we have,

$f(a) = f(a+0)$

$\Rightarrow f(a) = f(a) + \lim_{h\to 0} f(h)$

$\Rightarrow \lim_{h\to 0} f(h) = 0$ —— (1)

Similarly, $f(a) = f(a-0) = \lim_{h\to 0} f(-h) = 0$ —— (2)

Now, let α be any real number.

To show that: f is continuous at α.

We have, $f(\alpha+0) = \lim_{h\to 0} f(\alpha+h)$

$= \lim_{h\to 0} [f(\alpha) + f(h)]$ — (def^n. of f)

$$= \lim_{h\to 0} f(\alpha) + \lim_{h\to 0} f(h)$$

$$= f(\alpha) \quad \text{— using (1)}$$

Similarly,
using (2), we have $f(\alpha-0) = f(\alpha)$

Thus, $f(\alpha+0) = f(\alpha-0) = f(\alpha)$.

Hence, f is continuous at $x=\alpha$.
Since α is arbitrary, so f is continuous for all $x \in \mathbb{R}$. — (i) Hence, proved

(ii)
Here, we consider the following cases:-

Case (I): Let $x=0$

$\because f(x+x) = f(x)+f(x)$, we have

$f(0) = f(0+0) = f(0)+f(0)$ and so

$f(0)=0$.

Hence, $f(x) = kx$ for every constant k in this case.

Case (II): Let x be any +ve integer, then

$f(x) = f(1+1+\cdots \; x \text{ times}) = \underbrace{f(1)+f(1)+\cdots}_{x \text{ times}}$

$= x \cdot f(1) = k \cdot x$, where $k = f(1)$ i.e. constant

Case (III): Let x be any $-$ve integer, then Put $x = -y$ so that y is a +ve integer.

Now $0 = f(0)$ — [by case (I)]

$= f(y - y)$

$= f(y) + f(-y)$ (by defn of f)

$\therefore f(-y) = -f(y)$

Hence, $f(x) = -f(y) = -y \cdot f(1)$ — (by Case II)

$= k \cdot x$ where $k = -f(1)$ i.e. constant

Case (IV): Let x be any rational number. Put $x = p/q$ where q is a +ve integer and p is any integer, +ve, $-$ve or zero.

Now, $f(q \cdot p/q) = f(\underbrace{p/q + p/q + \cdots}_{q \text{ times}})$

$= \underbrace{f(p/q) + f(p/q) + \cdots}_{q \text{ times}}$

$\therefore f(p) = q \cdot f(p/q)$

But $f(p) = kp$ by previous cases.

Hence, $q \cdot f(p/q) = kp$

$\Rightarrow f(p/q) = k \cdot p/q$

$\Rightarrow f(x) = k \cdot x$, in this case also.

<u>Case (V)</u>: finally,

Let x be any real number.

Let $\langle x_n \rangle$ be a sequence of rational numbers converging to x.

Since, f is continuous at x, the sequence $\langle f(x_n) \rangle$ converges to $f(x)$.

Thus, we have,

$\lim\limits_{n\to\infty} x_n = x$ and $\lim\limits_{n\to\infty} f(x_n) = f(x)$

$\because$ x_n is a rational number, we have by case (IV), $f(x_n) = k \cdot x_n$

$\therefore \lim\limits_{n\to\infty} f(x_n) = \lim\limits_{n\to\infty} k \cdot x_n = k \cdot \lim\limits_{n\to\infty} x_n = k \cdot x$

$\Rightarrow f(x) = k \cdot x$.

Hence, $f(x) = kx \ \forall \ x \in \mathbb{R}$ —— (ii)

Hence, proved.

Q. 8. (b) Find the Laurent expansion of $\dfrac{z}{(z+1)(z+2)}$ about the singularity $z=-2$.

Specify the region of convergence.

Solution:

We have $f(z) = \dfrac{z}{(z+1)(z+2)} = \dfrac{2}{z+2} - \dfrac{1}{z+1}$

$$\Rightarrow f(z) = \frac{2}{z+2} - \frac{1}{z+1} \quad \text{———} \quad (1)$$

To find Laurent expansion for $\phi(z) = \dfrac{1}{z+1}$ about $z=-2$, we write

$$\phi(z) = \sum_{n=0}^{\infty} a_n (z+2)^n \quad \text{———} \quad (2)$$

where $a_n = \dfrac{\phi^{(n)}(-2)}{n!}$.

But $\phi^{(n)}(z) = \dfrac{(-1)^n n!}{(z+1)^{n+1}}$

$$\therefore \frac{\phi^{(n)}(-2)}{n!} = \frac{(-1)^n}{(-2+1)^{n+1}} = \frac{(-1)^n}{(-1)^{n+1}} = -1.$$

$$\text{or } a_n = \frac{\phi^{(n)}(-2)}{n!} = -1. \quad \text{———} \quad (3)$$

Put ③ in ②, we get,

$$\frac{1}{z+1} = \phi(z) = \sum_{n=0}^{\infty} (-1)(z+2)^n$$

$$\Rightarrow -\frac{1}{z+1} = \sum_{n=0}^{\infty} (z+2)^n$$

Now, ① reduces to $\boxed{f(z) = \frac{2}{2+z} + \sum_{n=0}^{\infty} (z+2)^n}$ —— ④

which is the required expansion.

Let $\sum_{n=0}^{\infty} (z+2)^n = \sum u_n$

Then $\left|\frac{u_{n+1}}{u_n}\right| = \left|\frac{(z+2)^{n+1}}{(z+2)^n}\right| = |z+2|$

series will be convergent if $\left|\frac{u_{n+1}}{u_n}\right| < 1$

i.e. if $|z+2| < 1$

$\therefore$ Radius of convergence $= 1$.

Series is convergent $\forall z$, inside the circle whose centre is $z = -2$ and radius $= 1$ unit.

Hence, the result.

Q. 8.(c). By integrating $e^{iz}/(z-ai)$, $(a>0)$ round a suitable contour, prove that

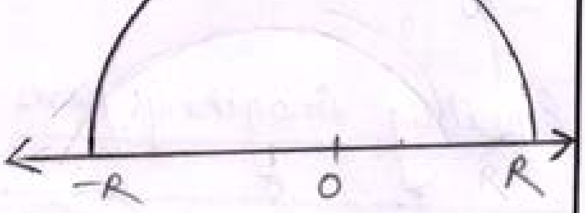
$$\int_{-\infty}^{\infty} \frac{a\cos x + x\sin x}{x^2+a^2}\,dx = 2\pi e^{-a}.$$

Solution :

Consider the integral, $\int_C \frac{e^{iz}}{(z-ai)}\,dz = \int_C f(z)\,dz$,

where C is the contour consisting of a large semi-circle T' of radius R containing all the poles of the integrand in the upper half plane and the part of real axis from $-R$ to R.

$z = ia$ is the simple pole of $f(z)$ and it lies inside C.



$\therefore$ Residue at $z = ia = \lim_{z\to ia} (z-ia) f(z)$

$$= \lim_{z\to ia} e^{iz} = \boxed{e^{-a}}$$

By Cauchy's Residue theorem, we have,

$$\int_C f(z)\,dz = \int_{-R}^{R} \frac{e^{iz}}{z-ia}\,dz + \int_{T'} \frac{e^{iz}}{z-ia}\,dz = 2\pi i \Sigma R$$

By Jordan's lemma, $\lim_{R\to\infty} \int_{\Gamma} \frac{e^{iz}}{z-ia}\, dz = 0$

$$\therefore \lim_{R\to\infty} \int_{-R}^{R} \frac{e^{iz}}{z-ia}\, dz = 2\pi i \sum R^+$$

$$\text{or} \int_{-\infty}^{\infty} \frac{e^{iz}}{z-ia}\, dz = 2\pi i \sum R^+$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{(\cos x + i\sin x)(x+ia)}{(x-ia)(x+ia)}\, dx = 2\pi i e^{-a}$$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{(x\cos x - a\sin x) + i(x\sin x + a\cos x)}{x^2+a^2}\, dx = i2\pi e^{-a}$$

Equating imaginary parts on both sides, we get,

$$\boxed{\int_{-\infty}^{\infty} \frac{x\sin x + a\cos x}{x^2+a^2}\, dx = 2\pi e^{-a}}$$

Hence, proved.

Q. 8. (d) Solve the following transportation problem.

From \ To				Available
	2	7	4	5
	3	3	1	8
	5	4	7	7
	1	6	2	14
Required	7	9	18	34

Solution:

By Vogel's approximation, the initial basic feasible solution having the transportation cost of ₹80 is obtained.

i.e. Initial basic feasible solution is $\{3, 2, 0, 0, 0, 8, 0, 7, 0, 4, 0, 10\}$

	I	II	III
A	2 (3)	7 (2)	4 ×
B	3 ×	3 ×	1 (8)
C	5 ×	4 (7)	7 ×
D	1 (4)	6 ×	2 (10)

Also, no of basic cells $= 6 = m+n-1$.

$\therefore$ no degeneracy.

Now, using Mod-I method (u-v method) to find optimal basic feasible solution,

	I	II	III	
A	2 3 $+\varepsilon$ (0)	7 2 $-\varepsilon$ (0)	4 (−1)	$u_1(0)$
B	3 (−3)	3 ε (2)	1 8 $-\varepsilon$ (0)	$u_2(-2)$
C	5 (−6)	4 7 (0)	7 (−7)	$u_3(-3)$
D	1 4 $-\varepsilon$ (0)	6 (0)	2 10 $+\varepsilon$ (0)	$u_4(-1)$
	v_1 (2)	v_2 (7)	v_3 (3)	

Let $u_1 = 0$

$u_1 + v_1 = 2 \Rightarrow v_1 = 2$

$u_1 + v_2 = 7 \Rightarrow v_2 = 7$

$u_4 + v_1 = 1 \Rightarrow u_4 = -1$

$u_4 + v_3 = 2 \Rightarrow v_3 = 3$

$u_2 + v_3 = 1 \Rightarrow u_2 = -2$

$u_3 + v_2 = 4 \Rightarrow u_3 = -3$

$\Delta_{13} = u_1 + v_3 - a_{13} = -1$

$\Delta_{22} = u_2 + v_2 - a_{22} = 2 > 0.$

only $\Delta_{22} > 0$.

Hence, (2, 2) enters basis.

$\varepsilon = 2$, leaving cell : (2, 1)

	I	II	III	
A	2 5 (0)	7 (−2)	4 (−1)	$u_1(2)$
B	3 (−3)	3 2 (0)	1 6 (0)	$u_2(0)$
C	6 (−4)	4 7 (0)	7 (−5)	$u_3(1)$
D	1 2 (0)	6 (−2)	2 12 (0)	$u_4(1)$
	v_1 (0)	v_2 (3)	v_3 (1)	

Let $u_2 = 0$.

Here, all the net evaluations are ≤ 0.
Hence, optimality is obtained.
At optimality,

A transports 5 units to I,
B transports 6 units to III, 2 units to II.
C transports 7 units to II.
D transports 2 units to I, 12 units to III.

Also, optimal cost of transportation =
$5(2) + 2(3) + 6(1) + 7(4) + 2(1) + 12(2)$
$= \boxed{₹\,76}$

Hence, the result.