FORCES ACTING AT A POINT (Concurrent Forces)

CHAPTER 1

1.1. RESULTANT AND COMPONENTS

If two or more forces act upon a rigid body and if a single force can be found such that the effect of it upon the body is the same as that of all the forces taken together, then the single force is called the **resultant** *of the forces and the given forces themselves are called the* **components** *of this resultant force.*

Forces are said to be in *equilibrium* if there is no resultant force.

Note :

When we say that forces are acting on a particle, it is meant that the forces are acting on a point.

1.2. PARALLELOGRAM LAW OF FORCES

If two forces, acting at a point, be represented in magnitude and direction by the two adjacent sides of a parallelogram through the point of application, their resultant will be completely represented by the diagonal of the parallelogram through that point.

Thus if two forces P and Q acting at a point O be represented in magnitude and direction by the sides OA and OB of the parallelogram OACB, then their resultant R is completely represented by the diagonal OC.

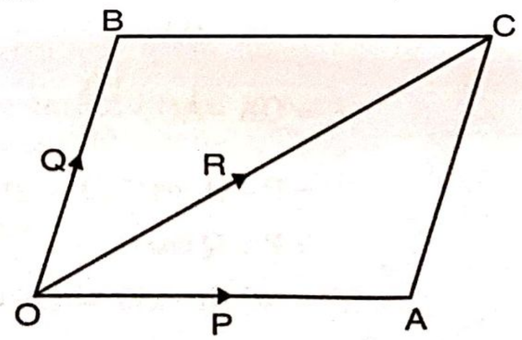


Fig. 1.1

In vector notation, this law may be written as :

$$\vec{OA} + \vec{OB} = \vec{OC}$$

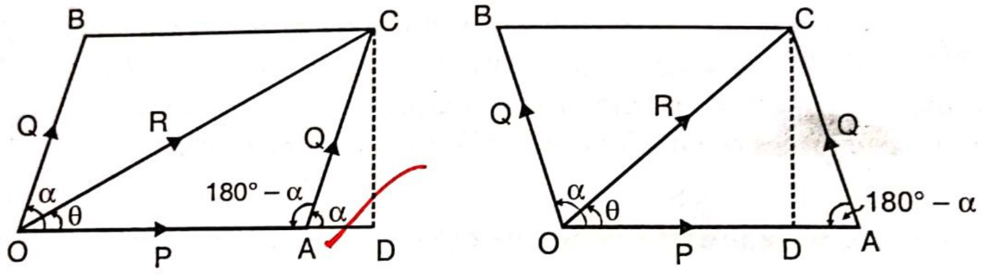
Remark :

If the diagonals of the parallelogram meet in M, then M is the middle point of OC. The resultant R is represented by 2OM. In vector notation it can be written as :

$$\vec{OA} + \vec{OB} = 2\vec{OM}.$$

1.3. MAGNITUDE AND DIRECTION OF THE RESULTANT

To find the magnitude and direction of the resultant of two forces acting at a point.



(*i*) *Fig. 1.2* (*ii*)

Let the two given forces P and Q acting at an angle α be represented in magnitude and direction by OA and OB respectively. Complete the parallelogram OACB, then the resultant R is represented in magnitude and direction by the diagonal OC.

Draw CD ⊥ OA as in fig 1.2 (*ii*) or produced if necessary as in fig. 1.2 (*i*).

Since AC and OB are parallel and equal, therefore AC will also represent the force Q.

Also since $\angle AOB = \alpha$, thus $\angle CAD = \alpha$ and $180° - \alpha$ in fig. 1.2 (*i*) and 1.2 (*ii*) respectively.

Then in fig. 1.2 (*i*), $OD = OA + AD$

$$= OA + AC \cdot \frac{AD}{AC} = P + Q\cos\alpha$$

and $CD = AC \sin\alpha = Q\sin\alpha.$

In fig. 1.2 (*ii*), $OD = OA - DA$

$$= OA - AC \cdot \frac{DA}{AC}$$

$$= P - Q\cos(180° - \alpha)$$

$$= P + Q\cos\alpha$$

and $CD = AC\sin(180° - \alpha) = Q\sin\alpha$

Hence in each case, $OD = P + Q\cos\alpha$ and $CD = Q\sin\alpha$

Now in right angled ΔOCD, we have

$$OC^2 = OD^2 + CD^2$$

$$\therefore \quad R^2 = (P + Q\cos\alpha)^2 + (Q\sin\alpha)^2$$
$$= P^2 + Q^2(\cos^2\alpha + \sin^2\alpha) + 2PQ\cos\alpha$$
$$= P^2 + Q^2 + 2PQ\cos\alpha$$

$$\therefore \quad \mathbf{R = \sqrt{P^2 + Q^2 + 2PQ\cos\alpha}} \qquad ...(1)$$

If θ be the angle which the resultant makes with OA, then

$$\tan\theta = \frac{CD}{OD} = \frac{Q\sin\alpha}{P + Q\cos\alpha} \Rightarrow \theta = \tan^{-1}\frac{Q\sin\alpha}{P + Q\cos\alpha} \qquad ...(2)$$

Equations (1) and (2) give the *magnitude and the direction of the resultant.*

Note :

Since $\tan(180° + \theta) = \tan\theta$, the equation

$\tan\theta = \dfrac{Q\sin\alpha}{P + Q\cos\alpha}$ will, in general, have two values of θ lying between 0 and 2π.

We take that value of θ which satisfies two equations,

$$\sin\theta = \frac{CD}{OC} = \frac{Q\sin\alpha}{R}$$

$$\cos\theta = \frac{OD}{OC} = \frac{P + Q\cos\alpha}{R}.$$

Cor. 1. If ϕ be the angle which the resultant makes with OB, then

$$\tan\phi = \frac{P\sin\alpha}{Q + P\cos\alpha}$$

Cor. 2. If the two forces P and Q are perpendicular to one another *i.e.*, if $\alpha = \frac{\pi}{2}$, then

$$R = \sqrt{P^2 + Q^2 + 2PQ\cos\frac{\pi}{2}}$$

$$\therefore \quad R = \sqrt{\mathbf{P^2 + Q^2}} \quad \text{and} \quad \tan\theta = \frac{Q\sin\frac{\pi}{2}}{P + Q\cos\frac{\pi}{2}} = \frac{\mathbf{Q}}{\mathbf{P}}.$$

Geometrically, when forces are perpendicular to each other, the parallelogram OACB reduces to a rectangle. Then in right angled ΔOAC, we have

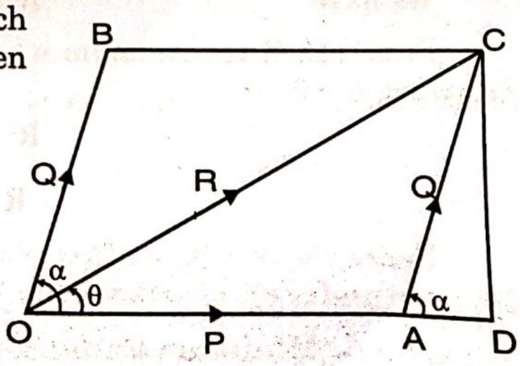


Fig. 1.3

$$OC^2 = OA^2 + AC^2$$

$$\therefore \quad R^2 = P^2 + Q^2$$

or $$R = \sqrt{P^2 + Q^2}$$

and $$\tan\theta = \frac{AC}{OA} = \frac{Q}{P}.$$

Cor 3. If P = Q *i.e.*, if two forces are equal, then

$$R = \sqrt{P^2 + P^2 + 2P^2 \cos\alpha}$$

$$= \sqrt{2P^2(1+\cos\alpha)} = \sqrt{2P^2\left(2\cos^2\frac{\alpha}{2}\right)}$$

$$\therefore \qquad \mathbf{R = 2P \cos\frac{\alpha}{2}}. \qquad \text{[K.U. 2010; M.D.U. 2010]}$$

and

$$\tan\theta = \frac{P\sin\alpha}{P + P\cos\alpha} = \frac{\sin\alpha}{1+\cos\alpha}$$

$$= \frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\cos^2\frac{\alpha}{2}} = \tan\frac{\alpha}{2}$$

$$\therefore \qquad \theta = \frac{\alpha}{2}.$$

Thus the resultant of two equal forces, each equal to P, acting at an angle α is 2P cos $\frac{\alpha}{2}$ and the direction of the resultant bisects the angle between the forces.

Cor 4. If P > Q, then P + Q cos α > Q + Q cos α

i.e.,

$$\frac{P + Q\cos\alpha}{Q\sin\alpha} > \frac{Q + Q\cos\alpha}{Q\sin\alpha} \qquad \text{[Dividing throughout by } Q\sin\alpha]$$

i.e.,

$$\frac{Q\sin\alpha}{P + Q\cos\alpha} < \frac{\sin\alpha}{1+\cos\alpha} \Rightarrow \tan\theta < \frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{2\cos^2\frac{\alpha}{2}}$$

$$\Rightarrow \qquad \tan\theta < \tan\frac{\alpha}{2} \Rightarrow \theta < \frac{\alpha}{2}$$

Thus the line of action of the resultant is always inclined more towards the line of action of the greater force.

Cor 5. Maximum Value of the Resultant

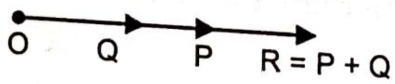
We have

$$R^2 = P^2 + Q^2 + 2\,PQ\cos\alpha \qquad ...(1)$$

From (1), R is maximum when cos α is maximum. But maximum value of cos α = 1, *i.e.*, when α = 0

$$\therefore \qquad R^2 = P^2 + Q^2 + 2\,PQ = (P+Q)^2$$

$$\therefore \qquad R = P + Q.$$



Hence, *the resultant of two forces acting at a point is maximum when they act in the same direction and is equal to their sum.*

Cor. 6. Minimum Value of the Resultant

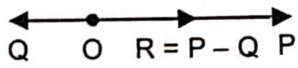
We have

$$R^2 = P^2 + Q^2 + 2PQ\cos\alpha \qquad ...(1)$$

From (1), R is minimum when $\cos \alpha$ is minimum. But minimum value of $\cos \alpha = -1$, *i.e.*, $\alpha = 180°$.

$$\therefore \quad R^2 = P^2 + Q^2 + 2\,PQ\,(-1)$$

or $$R^2 = (P - Q)^2$$



$$\therefore \quad R = P - Q$$

Hence, *the resultant of two forces acting at a point is minimum when they act in opposite directions and is equal to their difference acting in the direction of the greater force.*

SOLVED EXAMPLES

Example 1.

Find the magnitude and direction of the resultant of two forces of magnitudes 12 N and 14 N, acting at a point and inclined to each other at an angle of 45°.

Solution. Let P = 12 N and Q = 14 N and let R be their resultant acting at O making an angle θ with P (= 12 N).

Also, $\alpha = 45°$

$$\therefore \quad R = \sqrt{P^2 + Q^2 + 2PQ\cos\alpha}$$

$$= \sqrt{(12)^2 + (14)^2 + 2 \times 12 \times 14 \times \cos 45°}$$

$$= \sqrt{144 + 196 + 168\sqrt{2}}$$

$$= \sqrt{144 + 196 + 168 \times 1{\cdot}414}$$

$$= \sqrt{577{\cdot}552} = \mathbf{24{\cdot}03\ N}$$

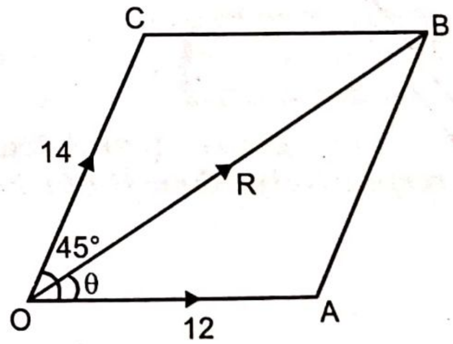


Fig. 1.4

Also, $$\tan\theta = \frac{Q\sin\alpha}{P + Q\sin\alpha} = \frac{14\sin 45°}{12 + 14\sin 45°}$$

$$= \frac{14 \times \frac{1}{\sqrt{2}}}{12 + 14 \times \frac{1}{\sqrt{2}}} = \frac{14}{12\sqrt{2} + 14} = 0{\cdot}45$$

$$\therefore \quad \theta = \mathbf{\tan^{-1}(0{\cdot}45).}$$

Thus the resultant is of magnitude 24·03 N and makes an angle of $\tan^{-1}(0{\cdot}45)$ with the direction of force 12 N.

Example 2.

Two forces P and 2P act on a particle. If the first be doubled and the second be increased by 10 kg.wt., the direction of the resultant is unaltered. Find the value of P.

[*M.D.U. 2018*]

Solution. Let α be the angle between the forces P and 2P and θ be the angle which the resultant makes with P.

$$\therefore \qquad \tan\theta = \frac{2P\sin\alpha}{P + 2P\cos\alpha} \qquad ...(1)$$

When P is doubled and 2P is increased by 10 kg.wt., then θ remains unchanged.

$$\therefore \qquad \tan\theta = \frac{(2P+10)\sin\alpha}{2P + (2P+10)\cos\alpha} \qquad ...(2)$$

From (1) and (2), we have

$$\frac{2P\sin\alpha}{P+2P\cos\alpha} = \frac{(2P+10)\sin\alpha}{2P+(2P+10)\cos\alpha}$$

or
$$\frac{2P}{P+2P\cos\alpha} = \frac{2P+10}{2P+(2P+10)\cos\alpha}$$

or
$$\frac{2}{1+2\cos\alpha} = \frac{P+5}{P+(P+5)\cos\alpha}$$

or
$$2P + (2P+10)\cos\alpha = (P+5) + (2P+10)\cos\alpha$$

or
$$2P = P + 5$$

$$\therefore \qquad P = \mathbf{5\ kg.\ wt.}$$

Example 3.

The greatest and least resultants of two forces are of magnitude P and Q respectively. Show that when they act at an angle θ, their resultant is of magnitude

$$\sqrt{P^2\cos^2\frac{\theta}{2} + Q^2\sin^2\frac{\theta}{2}}$$

Solution. Let F_1 and F_2 ($F_1 > F_2$) be the two given forces.

$$\therefore \qquad \text{Greatest resultant} = F_1 + F_2 = P \qquad ...(1)$$

and
$$\text{Least resultant} = F_1 - F_2 = Q \qquad ...(2)$$

Adding (1) and (2), we have $\quad F_1 = \dfrac{P+Q}{2}$

Subtracting (1) and (2), we have $\quad F_2 = \dfrac{P-Q}{2}$

When θ is the angle between the forces F_1 and F_2, let R be their resultant.

$$\therefore \qquad R^2 = F_1^2 + F_2^2 + 2F_1F_2\cos\theta$$

$$= \left(\frac{P+Q}{2}\right)^2 + \left(\frac{P-Q}{2}\right)^2 + 2\left(\frac{P+Q}{2}\right)\left(\frac{P-Q}{2}\right)\cos\theta$$

$$= \frac{1}{4}[(P+Q)^2 + (P-Q)^2] + \frac{1}{2}(P^2 - Q^2)\cos\theta$$

$$= \frac{1}{4}[2P^2 + 2Q^2] + \frac{1}{2}(P^2 - Q^2)\cos\theta$$

$$= \frac{1}{2}[P^2(1+\cos\theta) + Q^2(1-\cos\theta)]$$

$$= \frac{1}{2}\left[P^2 . 2\cos^2\frac{\theta}{2} + Q^2 . 2\sin^2\frac{\theta}{2}\right]$$

$$= P^2\cos^2\frac{\theta}{2} + Q^2\sin^2\frac{\theta}{2}$$

$$\therefore \quad R = \sqrt{P^2\cos^2\frac{\theta}{2} + Q^2\sin^2\frac{\theta}{2}}.$$

Example 4.

The resultant of forces P and Q is R. If Q be doubled, R is doubled; if Q be reversed, R is again doubled. Show that

$$P^2 : Q^2 : R^2 = 2 : 3 : 2 \quad \text{or} \quad P : Q : R = \sqrt{2} : \sqrt{3} : \sqrt{2}$$

Solution. Let α be the angle between the forces P and Q

$$\therefore \quad R^2 = P^2 + Q^2 + 2PQ\cos\alpha \quad \text{...(1)}$$

When Q changes to 2Q, R changes to 2R

$$\therefore \quad (2R)^2 = P^2 + (2Q)^2 + 2P\,.\,(2Q)\cos\alpha$$

or
$$4R^2 = P^2 + 4Q^2 + 4PQ\cos\alpha \quad \text{...(2)}$$

Again when Q changes to – Q, R changes to 2R

$$\therefore \quad (2R)^2 = P^2 + (-Q)^2 + 2P(-Q)\cos\alpha$$

or
$$4R^2 = P^2 + Q^2 - 2PQ\cos\alpha \quad \text{...(3)}$$

Adding (1) and (3), we get

$$5R^2 = 2P^2 + 2Q^2$$

or
$$2P^2 + 2Q^2 - 5R^2 = 0 \quad \text{...(4)}$$

Multiplying (3) by 2 and adding to (2), we get

$$12R^2 = 3P^2 + 6Q^2$$

or
$$P^2 + 2Q^2 - 4R^2 = 0 \quad \text{...(5)}$$

Solving (4) and (5) by cross-multiplication, we get

$$\frac{P^2}{-8+10} = \frac{Q^2}{-5+8} = \frac{R^2}{4-2}$$

or
$$\frac{P^2}{2} = \frac{Q^2}{3} = \frac{R^2}{2}$$

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$\therefore$ $$P^2 : Q^2 : R^2 = 2 : 3 : 2$$

Hence $$\mathbf{P : Q : R = \sqrt{2} : \sqrt{3} : \sqrt{2}}.$$

Example 5.

The resultant of two forces P and Q is of magnitude Q. Show that if the force Q be doubled, P remaining the same, the new resultant will be at right angle to P and its magnitude will be $\sqrt{4Q^2 - P^2}$.

[C.D.L.U. 2012]

Solution. Let α be the angle between the forces P and Q whose resultant is Q.

$\therefore$ $$Q^2 = P^2 + Q^2 + 2PQ \cos \alpha$$

or $$0 = P^2 + 2PQ \cos \alpha$$

$\therefore$ $$P(P + 2Q \cos \alpha) = 0 \Rightarrow P + 2Q \cos \alpha = 0$$

[$\because P \neq 0$]

$\therefore$ $$\cos \alpha = \frac{-P}{2Q} \qquad ...(1)$$

When Q changes to 2Q, let the new resultant make an angle θ with the force P

$\therefore$ $$\tan \theta = \frac{2Q \sin \alpha}{P + 2Q \cos \alpha}$$

$$= \frac{2Q \sin \alpha}{P + 2Q\left(\frac{-P}{2Q}\right)} \qquad \text{[Using (1)]}$$

$\therefore$ $$\tan \theta = \frac{2Q \sin \alpha}{0} = \infty \Rightarrow \theta = \frac{\pi}{2}$$

$\therefore$ The new resultant is at right angle to P

Again, $$R'^2 = P^2 + (2Q)^2 + 2P(2Q) \cos \alpha$$

$$= P^2 + 4Q^2 + 4PQ\left(\frac{-P}{2Q}\right) \qquad \text{[Using (1)]}$$

$$= P^2 + 4Q^2 - 2P^2 = 4Q^2 - P^2$$

$\therefore$ $$\mathbf{R' = \sqrt{4Q^2 - P^2}}.$$

Example 6.

Two forces P + Q and P – Q make an angle 2α with one another and their resultant makes an angle θ with the bisector of the angle between them. Show that

$$\boldsymbol{P \tan \theta = Q \tan \alpha}$$

[K.U. 2015]

Solution. Let the given forces P + Q and P − Q be represented by OA and OB respectively such that $\angle AOB = 2\alpha$.

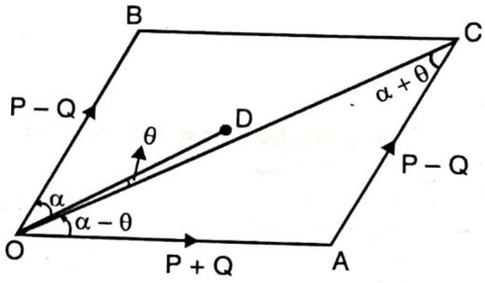


Fig. 1.5

Complete the parallelogram OACB. Then the resultant is represented by OC. Let OD be the bisector of $\angle AOB$

Since OD makes an angle θ with OC.

$\therefore$
$$\angle AOC = \alpha - \theta$$

and
$$\angle OCA = \angle COB \quad [\textit{Alternate angles}]$$
$$= \angle COD + \angle DOB$$
$$= \theta + \alpha = \alpha + \theta.$$

From ΔOAC, by sine formula,

$$\frac{OA}{\sin(\alpha+\theta)} = \frac{AC}{\sin(\alpha-\theta)}$$

i.e.,
$$\frac{P+Q}{\sin(\alpha+\theta)} = \frac{P-Q}{\sin(\alpha-\theta)}$$

or
$$\frac{P+Q}{P-Q} = \frac{\sin(\alpha+\theta)}{\sin(\alpha-\theta)}$$

By componendo and dividendo, we have

$$\frac{(P+Q)+(P-Q)}{(P+Q)-(P-Q)} = \frac{\sin(\alpha+\theta)+\sin(\alpha-\theta)}{\sin(\alpha+\theta)-\sin(\alpha-\theta)}$$

or
$$\frac{2P}{2Q} = \frac{2\sin\alpha\cos\theta}{2\cos\alpha\sin\theta}$$

or
$$\frac{P}{Q} = \tan\alpha\cot\theta = \frac{\tan\alpha}{\tan\theta}$$

$\therefore$
$$\mathbf{P\tan\theta = Q\tan\alpha.}$$

Example 7.

If the greatest possible resultant of two forces P and Q is n times the least, show that the angle between them when their resultant is half of their sum is $\cos^{-1}\left(-\dfrac{n^2+2}{2(n^2-1)}\right)$.

[K.U. 2012, M.D.U. 2012,11, 02]

Solution. Greatest possible resultant of P and Q = P + Q

Least resultant of P and Q = P − Q

It is given that
$$P + Q = n\,(P - Q)$$

$\therefore$
$$(n-1)\,P = (n+1)\,Q$$

or $$P = \frac{n+1}{n-1} Q \qquad ...(1)$$

Let α be the angle between the forces when their resultant is given by

$$R = \frac{1}{2}[P + Q]$$

$$= \frac{1}{2}\left[\frac{n+1}{n-1} Q + Q\right] \qquad [From (1)]$$

$$= \frac{1}{2}\left[\frac{2n}{n-1} Q\right] = \frac{n}{n-1} Q$$

Now, $$R^2 = P^2 + Q^2 + 2PQ \cos\alpha$$

$\therefore$ $$\frac{n^2}{(n-1)^2} Q^2 = \frac{(n+1)^2}{(n-1)^2} Q^2 + Q^2 + 2\left(\frac{n+1}{n-1}\right) Q.Q \cos\alpha$$

or $$n^2 = (n+1)^2 + (n-1)^2 + 2(n+1)(n-1)\cos\alpha$$

or $$n^2 = 2n^2 + 2 + 2(n^2 - 1)\cos\alpha$$

or $$2(n^2 - 1)\cos\alpha = -n^2 - 2$$

$\therefore$ $$\cos\alpha = -\frac{n^2+2}{2(n^2-1)} \Rightarrow \alpha = \cos^{-1}\left[-\frac{n^2+2}{2(n^2-1)}\right].$$

Example 8.

The resultant of two forces P and Q trisects the angle between them. Show that if $P > Q$, then the angle between them is $3\cos^{-1}\left(\frac{P}{2Q}\right)$ and the resultant is $\frac{P^2 - Q^2}{Q}$.

[C.D.L.U. 2016; M.D.U. 2013; K.U. 2011, 01]

Solution. Let the two forces P and Q be represented in magnitude and direction by the sides OA and OB of the parallelogram OACB. The diagonal OC represents the resultant R of the forces.

Let $\angle AOB = 3\alpha$

Since $P > Q$,

$\therefore$ $\angle AOC = \alpha$ and $\angle COB = 2\alpha$

In ΔOAC, $\angle ACO = \angle COB = 2\alpha$

$\angle OAC = 180° - 3\alpha$

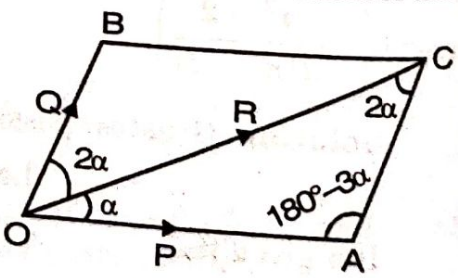


Fig. 1.6

Then by sine formula,

$$\frac{OA}{\sin 2\alpha} = \frac{AC}{\sin \alpha} = \frac{OC}{\sin (180° - 3\alpha)}$$

or
$$\frac{P}{\sin 2\alpha} = \frac{Q}{\sin \alpha} = \frac{R}{\sin 3\alpha}$$

From first two members, we have

$$\frac{P}{2 \sin \alpha \cos \alpha} = \frac{Q}{\sin \alpha} \Rightarrow \cos \alpha = \frac{P}{2Q}$$

i.e.,
$$\alpha = \cos^{-1} \frac{P}{2Q}$$

$\therefore$ Angle between the forces $= 3\alpha = 3 \cos^{-1} \left(\frac{P}{2Q}\right)$

Again, from last two members, we have

$$\frac{Q}{\sin \alpha} = \frac{R}{3 \sin \alpha - 4 \sin^3 \alpha}$$

i.e.,
$$R = Q\,(3 - 4 \sin^2 \alpha) = Q\,[3 - 4\,(1 - \cos^2 \alpha)]$$

$$= Q\,[4 \cos^2 \alpha - 1] = Q \left[4.\frac{P^2}{4Q^2} - 1\right]$$

$\therefore$
$$\text{Resultant} = \frac{\mathbf{P^2 - Q^2}}{\mathbf{Q}}.$$

Example 9.

If forces P and Q acting at an angle θ be interchanged in position, show that their resultant turns through an angle ϕ such that,

[*K.U. 2015*]

$$\tan \frac{\phi}{2} = \frac{P - Q}{P + Q} \tan \frac{\theta}{2}$$

Solution. Let the forces P and Q be represented by OA and OB such that $\angle AOB = \theta$. Then the diagonal OC of the parallelogram OACB represents the resultant.

When P and Q are interchanged in position, let P be represented by OA′, Q by OB′ and the resultant is turned through an angle ϕ.

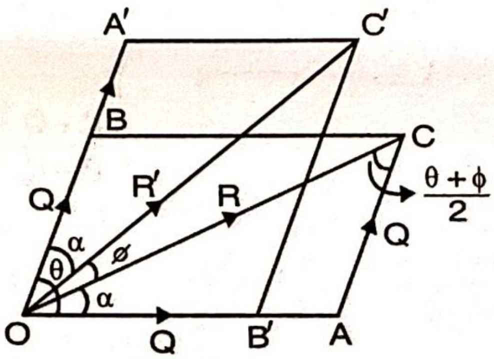


Fig. 1.7

$\therefore$
$$COC' = \phi$$

In Δ's OAC and OC′A′, $\quad OA = OA'$

$$OB = OB'$$

$$OC = OC'$$

$$\therefore \quad \Delta OAC \equiv \Delta OC'A'$$

$$\therefore \quad \angle AOC = \angle A'OC' = \alpha \text{ (say)}$$

$$\angle AOB = \theta$$

$$\therefore \quad \alpha + \phi + \alpha = \theta$$

or
$$2\alpha = \theta - \phi \Rightarrow \alpha = \frac{\theta - \phi}{2} \qquad ...(1)$$

Also,
$$\angle ACO = \angle COA \qquad [\textit{Alternate } \angle s]$$

$$= \angle COC' + \angle COA'$$

$$= \phi + \alpha = \phi + \frac{\theta - \phi}{2} \qquad [\textit{Using (1)}]$$

$$= \frac{\theta + \phi}{2} \qquad ...(2)$$

Now, from ΔOAC, by sine formula, we have

$$\frac{OA}{\sin(\alpha + \phi)} = \frac{AC}{\sin \alpha}$$

or
$$\frac{P}{\sin(\alpha + \phi)} = \frac{Q}{\sin \alpha} \qquad [\because AC = OB]$$

$$\therefore \quad \frac{P}{Q} = \frac{\sin(\alpha + \phi)}{\sin \alpha}$$

or
$$\frac{P}{Q} = \frac{\sin \frac{\theta + \phi}{2}}{\sin \frac{\theta - \phi}{2}} \qquad [\textit{From (1) and (2)}]$$

Applying componendo and dividendo, we have

$$\frac{P + Q}{P - Q} = \frac{\sin \frac{\theta + \phi}{2} + \sin \frac{\theta - \phi}{2}}{\sin \frac{\theta + \phi}{2} - \sin \frac{\theta - \phi}{2}}$$

$$= \frac{2 \sin \frac{\theta}{2} \cos \frac{\phi}{2}}{2 \cos \frac{\theta}{2} \sin \frac{\phi}{2}} = \tan \frac{\theta}{2} \cot \frac{\phi}{2}$$

$$\therefore \qquad \frac{P+Q}{P-Q} = \frac{\tan\frac{\theta}{2}}{\tan\frac{\phi}{2}}$$

Hence, $$\tan\frac{\phi}{2} = \frac{P-Q}{P+Q}\tan\frac{\theta}{2}.$$

EXERCISE 1.1

1. Two forces of magnitudes 8 N and 6 N act at a point and the angle between them is 60°. Find the magnitude and direction of their resultant. [*M.D.U. 2014; C.D.L.U. 2013*]

2. When two equal forces are inclined at an angle 2α, their resultant is twice as great as when they are inclined at an angle 2β. Prove that $\cos\alpha = 2\cos\beta$.

3. The resultant of two forces P and Q is R. If one of the forces be reversed in direction, the resultant is S. Show that $R^2 + S^2 = 2(P^2 + Q^2)$.

4. Two forces act at an angle of 120°. The greater is represented by 80 kg. wt. and the resultant is at right angles to lesser. Find the latter.

5. Find the angle between two equal forces P, P when the square of their resultant is equal to $(2-\sqrt{3})$ times their product

6. The resultant of two forces P and Q act at right angles to P. Show that the angle between the forces is $\cos^{-1}\left(-\frac{P}{Q}\right)$. Also, show that $R^2 = Q^2 - P^2$. [*M.D.U. 2014; K.U. 2011*]

7. Two forces acting at a point are such that if the direction of one is reversed, the direction of the resultant is turned through a right angle. Prove that the forces must be equal in magnitude. ?

90 − θ

8. Two forces P and Q acting along two intersecting lines, have a resultant R. If the force Q is reversed in direction and changed into $\frac{R^2 - P^2}{Q}$, show that the resultant is still of magnitude R.

9. The resultant of two forces P and Q acting at an angle θ is equal to $(2m+1)\sqrt{P^2+Q^2}$ and when they act at angle $90° - \theta$, the resultant is $(2m-1)\sqrt{P^2+Q^2}$. Prove that $\tan\theta = \frac{m-1}{m+1}$.

10. The resultant of two forces P and Q acting along the lines OA and OB respectively is at right angles to OA and the resultant of forces P' and Q' acting respectively along the same straight lines is at right angles to OB. Show that $PP' = QQ'$.

11. The resultant of two forces P and Q is perpendicular to P; the resultant of P and Q' acting at the same angle is perpendicular to Q'. Prove that P is the geometric mean between Q and Q'

12. Two forces P and Q have a resultant R. If force P be increased, then the new resultant bisects the angle between R and P. Find the increase in P.

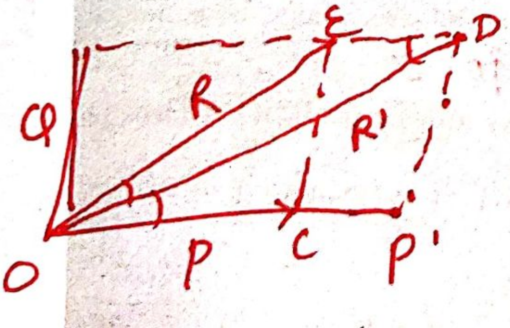
13. The resultant R of forces P and Q makes an angle 2θ with the line of action of P. P is now replaced by P + R, Q remaining unchanged. Show that the resultant makes an angle θ with P.

14. If the greatest possible resultant of two forces, P and Q, is n times the least, show that the angle α between P and Q when their resultant is twice the square root of their product is given by $\tan^2 \dfrac{\alpha}{2} = \dfrac{1}{n^2 - 2}$. [M.D.U. 2017]

15. Two forces P and Q act at an angle α and have a resultant R. If each force is increased by R, prove that the new resultant makes with R an angle

$$\tan^{-1}\left[\frac{(P - Q)\sin\alpha}{P + Q + R + (P + Q)\cos\alpha}\right].$$ [K.U. 2015]

16. Two forces P, Q act at a point along two straight lines making an angle α with each other and R is their resultant. Two other forces P', Q' act along the same two lines and have a resultant R'. Prove that the angle between the lines of action of the resultant is



$$\cos^{-1}\left(\frac{PP' + QQ' + (PQ' + P'Q)\cos\alpha}{RR'}\right).$$

ANSWERS

1. $2\sqrt{37}$ N; $\tan^{-1}\left(\dfrac{3\sqrt{3}}{11}\right)$

4. 40 kg. wt.

5. 150°

12. R.

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1.4. RESOLUTION OF A GIVEN FORCE IN TWO GIVEN DIRECTIONS

Resolution of forces is the converse of the *composition of forces*. When a force is given, we are to find the *component forces* in given directions. Given a resultant force in magnitude and direction, we can resolve it into two components in an infinite number of ways; since on a given line as diagonal an *infinite* number of parallelograms can be constructed.

1.5. COMPONENTS OF A GIVEN FORCE IN TWO GIVEN DIRECTIONS

Let F be the given force represented in magnitude and direction by OC and let OX, OY be the given directions making angles α, β respectively with OC along which components are to be found.

Through C, draw CA and CB parallel to OY and OX respectively to meet OX and OY in A and B. Then OACB is a parallelogram and OA, OB are the required components of the force F in magnitude and direction.

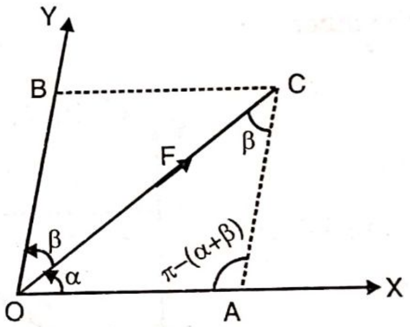


Fig. 1.8

Now, AC is equal to and parallel to OB.

Thus, both represent the same force in magnitude and direction.

Now, in Δ OAC, $\angle AOC = \alpha$, $\angle OCA = \angle COB = \beta$ [*Alternate angles*]

and $\angle OAC = \pi - (\alpha + \beta)$

From ΔOAC, by using sine formula, we have

$$\frac{OA}{\sin \angle OCA} = \frac{AC}{\sin \angle AOC} = \frac{OC}{\sin \angle CAO}$$

or
$$\frac{OA}{\sin \beta} = \frac{OB}{\sin \alpha} = \frac{OC}{\sin [\pi - (\alpha + \beta)]}$$

or
$$\frac{OA}{\sin \beta} = \frac{OB}{\sin \alpha} = \frac{F}{\sin (\alpha + \beta)}$$

Hence, $OA = \dfrac{F \sin \beta}{\sin (\alpha + \beta)}$ and $OB = \dfrac{F \sin \alpha}{\sin (\alpha + \beta)}$

In words : ***Components in any direction*** $= \dfrac{\textbf{\textit{F} × sine (other angle)}}{\textbf{sine (sum of angles)}}$.

Note :

By other angle we mean the angle which the other direction makes with the given force.

1.6. RESOLVED PARTS OF A GIVEN FORCE

Definition. *If a force be resolved into two components, which are at right angles to each other, then these components are called the resolved parts of the force.*

To determine the resolved part of a given force in any direction

Let F be the given force represented in magnitude and direction by OC and OX be the given direction, making an angle α (acute as in fig. 1.9 (*i*) or obtuse as in fig. 1.9 (*ii*) with OC. Draw $OY \perp OX$. Then OX is the direction of one of the resolved parts and OY is the direction of the other.

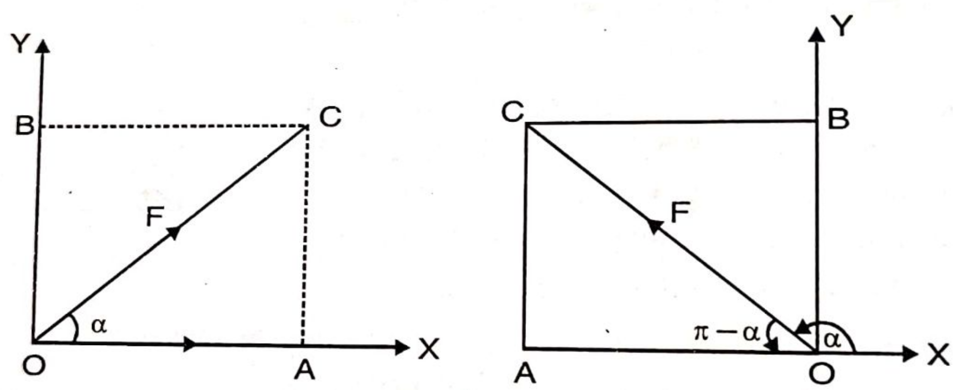


Fig. 1.9

Through C, draw CA and CB parallel to OY and OX respectively to meet OX and OY in A and B. Then in each figure OACB is a rectangle.

In fig. 1.9 (*i*); OA , OB are the required resolved parts and in fig. 1.9 (*ii*); AO, OB are the required resolved parts.

Since AC is equal to and parallel to OB, therefore both represent the same force in magnitude and direction.

In fig. 1.9 (*i*), the resolved part of F along OX

$$= OA = OC \cos \alpha = F \cos \alpha$$

The resolved part of F along OY

$$= OB = AC = OC \sin \alpha = F \sin \alpha$$

In fig. 1.9 (*ii*), the resolved part of F along OX

$$= AO = -OA = -OC\cos(\pi - \alpha)$$

$$= OC\cos\alpha = F\cos\alpha$$

The resolved part of F along OY

$$= OB = AC$$

$$= OC\sin(\pi - \alpha) = F\sin\alpha$$

$\therefore$ ***Resolved part of a force in any direction***

= Force × cosine of the angle which the force makes with that direction.

Cor 1. The resolved part of a force F in its own direction

$$= F\cos 0^\circ = F \qquad [\because \alpha = 0^\circ]$$

Cor 2. The resolved part of a force F in a direction perpendicular to it

$$= F\cos 90^\circ = 0 \qquad [\because \alpha = 90^\circ]$$

Hence a force cannot produce any effect in a direction perpendicular to its line of action.

Cor 3. The resolved part of a given force F in a given direction represents the *whole effect* of the force in that direction.

SOLVED EXAMPLES

Example 1.

If a force F be resolved into component forces and if one component be at right angles to F and equal to $\sqrt{3}$ F in magnitude, find the direction and magnitude of the other component. [*C.D.L.U. 2013*]

Solution. Let the other component P of the resultant F make an angle θ with it.

Component $\sqrt{3}$ F is perpendicular to force F.

$$\therefore \qquad \angle AOB = 90^\circ$$

$$\therefore \qquad \sqrt{3}\,F = \frac{F\sin\theta}{\sin(90^\circ + \theta)}$$

or

$$\sqrt{3} = \frac{\sin\theta}{\cos\theta}$$

or

$$\tan\theta = \sqrt{3} \Rightarrow \theta = \mathbf{60^\circ}.$$

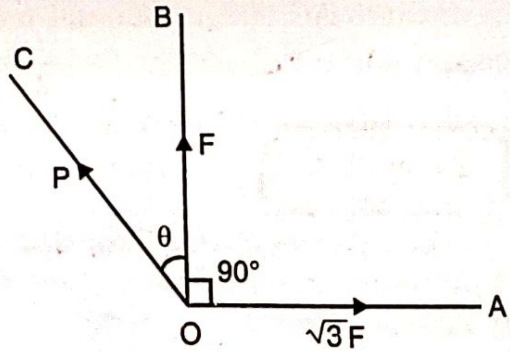


Fig. 1.10

The other component P is given by

$$P = \frac{F \sin 90^\circ}{\sin (90^\circ + 60^\circ)} = \frac{F}{\cos 60^\circ}$$

$$\therefore \quad P = \frac{F}{\frac{1}{2}} = 2F$$

Hence the other component 2F makes an angle of 60° with F.

Example 2.

The resultant of two forces P and Q is R. The resolved part of R in the direction of P is of magnitude Q. Show that the angle between the forces is

$$2 \sin^{-1} \sqrt{\frac{P}{2Q}}.$$

[K.U. 2015]

Solution. Let the given forces P and Q, represented by OA and OB be inclined at an angle α and θ be the angle which the resultant R makes with OA. Complete the parallelogram OACB and from C draw CD ⊥ OA (produced). Diagonal OC represents the force R.

Resolved part of R along OA

$$= R \cos \theta$$

$$= OC \cdot \frac{OD}{OC}$$

$$= OD = OA + AD$$

$$= P + Q \cos \alpha$$

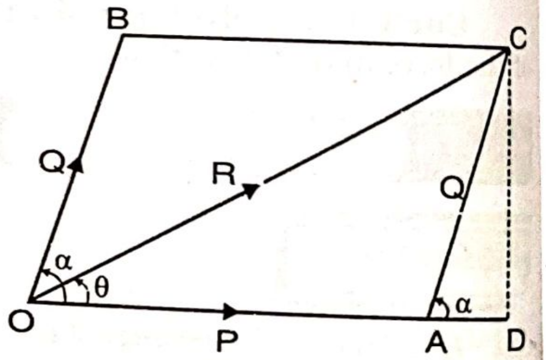


Fig. 1.11

But this is given to be equal to Q

$$\therefore \quad P + Q \cos \alpha = Q$$

or

$$P = Q(1 - \cos \alpha) = 2Q \cdot \sin^2 \frac{\alpha}{2}$$

$$\therefore \quad \sin \frac{\alpha}{2} = \sqrt{\frac{P}{2Q}}$$

or

$$\frac{\alpha}{2} = \sin^{-1} \sqrt{\frac{P}{2Q}} \Rightarrow \alpha = 2 \sin^{-1} \sqrt{\frac{P}{2Q}}.$$

Example 3.

Two forces P and Q, acting on a particle are inclined at the angle θ. If the sum of their resolved parts in a certain direction be X and that along a perpendicular direction by Y, prove that

$$\theta = \cos^{-1} \frac{X^2 + Y^2 - P^2 - Q^2}{2PQ}.$$

[M.D.U. 2018]

Solution. Let the given forces P, Q act at a point O and let OX, the given direction make an angle α with P.

$\therefore$ Q makes an angle $(\alpha + \theta)$ with OX.

Let $OY \perp OX$

Now, resolving the forces P and Q along OX and OY, we have

$$X = P\cos\alpha + Q\cos(\alpha+\theta) \quad ...(1)$$

$$Y = P\sin\alpha + Q\sin(\alpha+\theta) \quad ...(2)$$

Squaring (1) and (2) and then adding, we have

$$X^2 + Y^2 = P^2\cos^2\alpha + Q^2\cos^2(\alpha+\theta) + 2PQ\,.\cos\alpha\cos(\alpha+\theta) + P^2\sin^2\alpha + Q^2\sin^2(\alpha+\theta) + 2PQ\sin\alpha\sin(\alpha+\theta)$$

$$= P^2(\cos^2\alpha + \sin^2\alpha) + Q^2[\cos^2(\alpha+\theta) + \sin^2(\alpha+\theta)] + 2PQ[\cos\alpha\cos(\alpha+\theta) + \sin\alpha\sin(\alpha+\theta)]$$

$$= P^2 + Q^2 + 2PQ\cos(\alpha - \alpha - \theta)$$

$$= P^2 + Q^2 + 2PQ\cos(-\theta)$$

$$= P^2 + Q^2 + 2PQ\cos\theta$$

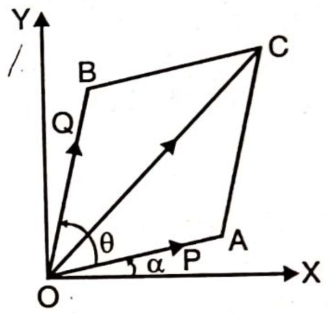


Fig. 1.12

$\therefore$ $$2PQ\cos\theta = X^2 + Y^2 - P^2 - Q^2$$

or $$\cos\theta = \frac{X^2 + Y^2 - P^2 - Q^2}{2PQ}$$

Hence, $$\theta = \cos^{-1}\frac{X^2 + Y^2 - P^2 - Q^2}{2PQ}.$$

EXERCISE 1.2

1. A force of magnitude 24 N is inclined at an angle of 30° to the horizontal. Find its resolved parts in horizontal and vertical directions.
2. Find the components of a force P along two directions making angles 45° and 30° with it on opposite sides. [*C.D.L.U. 2012*]
3. A given force P is resolved into two components inclined at 45° and α. If the latter component is $\frac{\sqrt{2}}{\sqrt{3}}$P, find α and the other components.
4. The resolved part of a force F in a direction is $\frac{\sqrt{3}}{2}$F. Find its inclination with the force. Also, find the other resolved part.

5. Find the resolved part of a force equal to 60 kg wt. in a direction making an angle equal to $\tan^{-1} 3/4$ with its direction. [K.U. 2016; M.D.U. 2011]

6. The resolved part of force along a direction inclined at an angle α with it is of magnitude 30 N. If $\alpha = \tan^{-1}\left(\frac{4}{3}\right)$, find the magnitude of the force. Also, find its other resolved part.

7. A force F is resolved into two components. If one of the components is at right angles to the force and equal to it in magnitude, find the direction and magnitude of the other component.

8. AD is the altitude of ΔABC. Show that the force AD has components along AB and AC equal to

$$\frac{a^2+b^2-c^2}{2a^2}\,AB \quad \text{and} \quad \frac{c^2+a^2-b^2}{2a^2}\,AC.$$ [M.D.U. 2012]

ANSWERS

1. $12\sqrt{3}$ N; 12 N
2. $\frac{\sqrt{3}-1}{\sqrt{2}}$ P; $(\sqrt{3}-1)$ P
3. 15° or 75°; $\frac{\sqrt{3}-1}{\sqrt{6}}$ P or $\frac{\sqrt{3}+1}{\sqrt{6}}$ P
4. 30°; $\frac{1}{2}$ F
5. 48 kg. wt.
6. 50 N; 40 N
7. 45° with F; $\sqrt{2}$ F.

1.7. TRIANGLE LAW OF FORCES

If three forces, acting at a point, be represented in magnitude and direction by the sides of a triangle, (taken in order), then the forces are in equilibrium.

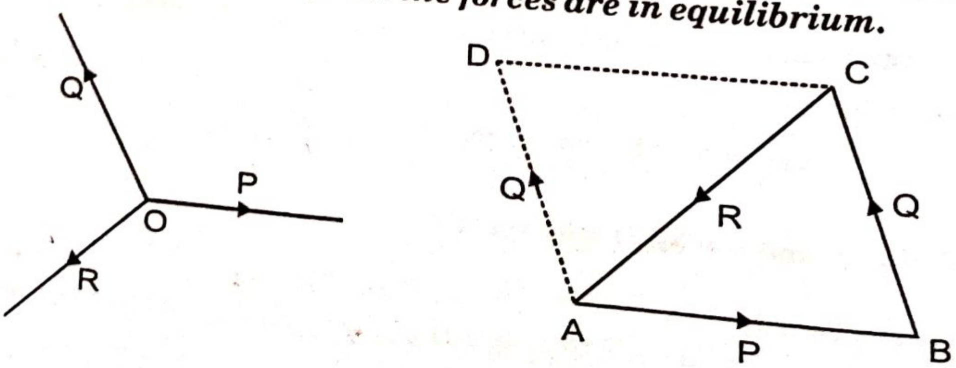


Fig. 1.13.

Let P, Q and R be the three forces acting at the point O and represented in magnitude and direction by the sides AB, BC and CA respectively of a ΔABC.

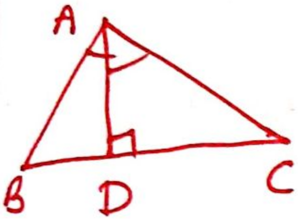
Complete the parallelogram ABCD.

Since AD is equal and parallel to BC, therefore, it represents the force Q.

By parallelogram law of forces, the resultant of the forces represented by AB and AD is represented by AC.

i.e., $$\vec{AB} + \vec{AD} = \vec{AC} \qquad ...(1)$$

Now, the resultant of the forces P, Q and R



= resultant of forces represented by $\vec{AB}$, $\vec{BC}$, $\vec{CA}$

= resultant of forces represented by $\vec{AB}$, $\vec{AD}$, $\vec{CA}$

= resultant of forces represented by $\vec{AC}$, $\vec{CA}$ [*Using (1)*]

= 0 [$\because$ $\vec{AC}$, $\vec{CA}$ *being equal and opposite, neutralize each other*]

Hence the forces P, Q, R are in equilibrium.

Cor. Since $\vec{AB} + \vec{BC} + \vec{CA} = 0$

$\therefore$ $$\vec{AB} + \vec{BC} = -\vec{CA}$$

or $$\vec{AB} + \vec{BC} = \vec{AC}$$

Hence the triangle law of forces can also be stated as :

If two forces, acting at a point, be represented in magnitude and direction by two sides of a triangle taken in order, their resultant is represented in magnitude and direction by the third side taken in the opposite order.

Note :

It should be noted that the forces must act at a point and they are represented by the sides of the triangle in magnitude and direction only and not in the line of action.

1.8. CONVERSE OF THE TRIANGLE LAW OF FORCES

If three forces acting at a point be in equilibrium, they can be represented by the sides (taken in order) of any triangle which is drawn so as to have its sides respectively parallel to the directions of the forces.

Let the three forces P, Q, R acting at a point O be in equilibrium.

Let P, Q and R be represented by OA, OB and OD on some scale

Complete the parallelogram OACB.

Since the forces P, Q, R acting at O are in equilibrium

$$\therefore \quad \vec{OA} + \vec{OB} + \vec{OD} = 0 \qquad ...(1)$$

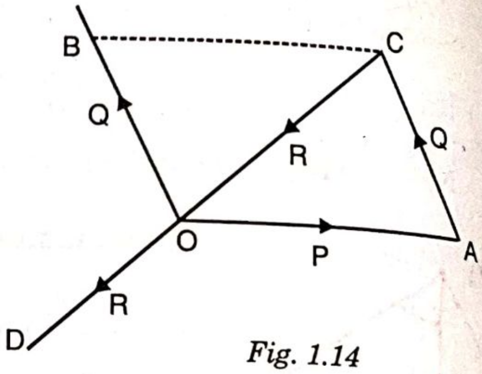


Fig. 1.14

But by parallelogram law of forces,

$$\vec{OA} + \vec{OB} = \vec{OC}$$

$\therefore$ From (1), we have

$$\vec{OC} + \vec{OD} = 0$$

or
$$\vec{OD} = -\vec{OC} = \vec{CO}$$

Hence R is represented by $\vec{CO}$ which was represented by $\vec{OD}$.

Also, AC is equal and parallel to OB and hence represents the force Q in magnitude and direction.

Thus the sides OA, AC, CO taken in order of the triangle OAC represent the forces P, Q, R respectively.

$$\therefore \quad \frac{P}{OA} = \frac{Q}{AC} = \frac{R}{CO} \qquad ...(2)$$

Let O′A′C′ be any other triangle whose sides are respectively parallel to those of OAC and therefore parallel to the directions of the forces P, Q and R.

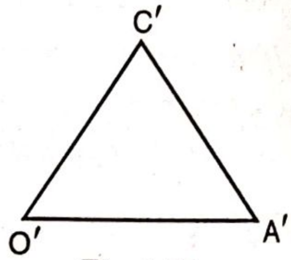


Fig. 1.15

$\therefore$ Δ's OAC and O′A′C′ are equiangular and hence similar.

$$\therefore \quad \frac{O'A'}{OA} = \frac{A'C'}{AC} = \frac{C'O'}{CO}$$

or
$$\frac{P}{O'A'} = \frac{Q}{A'C'} = \frac{R}{C'O'} \qquad \text{[From (2)]}$$

Hence the sides O′A′, A′C′, C′O′ taken in order of the $\Delta O'A'C'$ represent the forces P, Q, R in magnitude and direction.

Hence the theorem.

Note :

Any such triangle like OAC of the above article is called the "*triangle of forces*".

1.9. OTHER FORM OF TRIANGLE OF FORCES

If the triangle O′A′C′ of the above article be rotated through a constant angle in its own plane, it takes the form of a new triangle whose sides make the same constant angle with the directions of the forces P, Q, R in the same sense and represent these forces in magnitude only. Thus the sides of the ΔO′A′C′ will also be proportional to the forces.

An important particular case of such a triangle of forces occurs when the sides of the ΔO′A′C′ are respectively perpendicular to the directions of the forces P, Q and R. Such a triangle is called the **Perpendicular Triangle of Forces,** which states :

If three forces acting at a point be such that their magnitudes are proportional to the sides of a triangle and their directions are perpendicular to the corresponding sides, all inwards, or all outwards, the forces will be in equilibrium.

The converse of this theorem is also true and may be stated as :

If three forces acting at a point be in equilibrium, and if the forces act perpendicularly to the sides of any triangle, all inwards or all outwards, they are proportional to the sides of the triangle.

1.10. $\lambda - \mu$ THEOREM

The resultant of two forces acting at a point O in direction OA and OB and represented in magnitude by λ.OA and μ.OB is represented by $(\lambda + \mu)$.OC, where C is a point in AB such that λ.CA = μ.CB. [K.U. 2000]

Proof. Let the two given forces λ.OA and μ.OB act at the point O in the direction OA and OB respectively.

Take a point C on AB such that λ.CA = μ.CB.

From the ΔOAC, by the triangle law of forces, using vector notation, we have

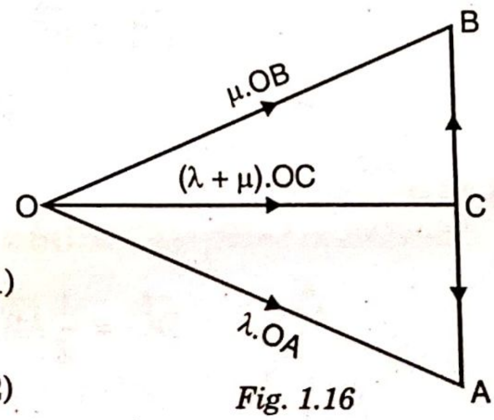


Fig. 1.16

$$\vec{OA} = \vec{OC} + \vec{CA}$$

$$\therefore \quad \lambda.\vec{OA} = \lambda.\vec{OC} + \lambda.\vec{CA} \quad \quad ...(1)$$

Similarly, from ΔOCB,

$$\therefore \quad \mu.\vec{OB} = \mu.\vec{OC} + \mu.\vec{CB} \quad \quad ...(2)$$

Adding (1) and (2), we have

$$\lambda.\vec{OA} + \mu.\vec{OB} = (\lambda+\mu).\vec{OC} + \lambda.\vec{CA} + \mu.\vec{CB}$$

$$= (\lambda+\mu).\vec{OC} + \lambda.\vec{CA} - \mu.\vec{BC}$$

$$= (\lambda+\mu).\vec{OC} \quad \quad [\because \lambda.\vec{CA} = \mu.\vec{BC}]$$

Hence the result.

Cor. 1. *The resultant of two forces $\vec{OA}$ and $\vec{OB}$ is a force $2.\vec{OC}$ where C is the middle point of AB.*

If $\lambda = \mu = 1$, then we have

$$1.\vec{OA} + 1.\vec{OB} = (1+1).\vec{OC}$$

or $$\vec{OA} + \vec{OB} = 2\vec{OC}, \text{ where C is the mid point of AB.}$$

Cor. 2. The resultant of $\lambda.\vec{AO}$ and $\mu.\vec{BO}$ is $(\lambda+\mu).\vec{CO}$

Cor. 3. If $\lambda = -\mu$ *i.e.*, $\lambda + \mu = 0$

Then $$\lambda.\vec{OA} + \mu.\vec{OB} = \lambda(\vec{OA} - \vec{OB})$$

$$= \lambda(\vec{OA} + \vec{BO}) = \lambda\vec{BA}.$$

SOLVED EXAMPLES

Example 1.

If D and E are the middle points of the sides AB and AC of a triangle ABC, prove that the resultant of the forces represented by BE and DC is represented in magnitude and direction by $\frac{3}{2}BC$.

Solution. In Δ BCE,

$$\vec{BE} = \vec{BC} + \vec{CE} \quad \text{[Triangle law of forces]}$$

$$\therefore \quad \vec{BE} = \vec{BC} + \frac{1}{2}\vec{CA} \quad ...(1)$$

[*Since E is the mid point of CA*]

Again in ΔBCD,

$$\vec{DC} = \vec{DB} + \vec{BC} \quad \text{[Triangle law of forces]}$$

$$\therefore \quad \vec{DC} = \frac{1}{2}\vec{AB} + \vec{BC} \quad ...(2)$$

[*Since D is the mid point of AB*]

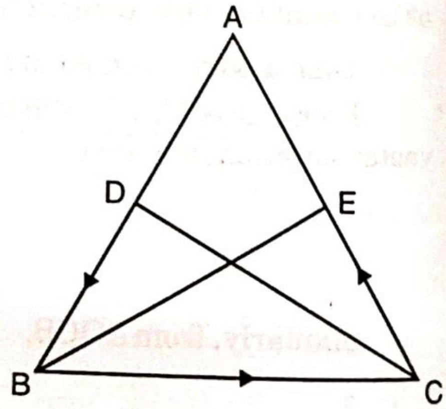


Fig. 1.17

Adding (1) and (2), we get

$$\vec{BE} + \vec{DC} = 2\vec{BC} + \frac{1}{2}(\vec{CA} + \vec{AB})$$

$$= 2\vec{BC} + \frac{1}{2}\vec{CB} \quad \text{[Triangle law of forces]}$$

$$= 2\vec{BC} - \frac{1}{2}\vec{BC}$$

Hence, $$\vec{BE} + \vec{DC} = \frac{3}{2}\vec{BC}.$$

Example 2.

Find a point O inside a quadrilateral ABCD such that if a particle placed at it be acted upon by forces represented by OA, OB, OC, OD, it will be in equilibrium.

Solution. Let O be the given point inside the quadrilateral ABCD. Let E and F be the mid-points of sides AB and CD respectively. Let G be the mid point of EF.

Since E is the mid point of AB

$\therefore$ $$\vec{OA} + \vec{OB} = 2\vec{OE}$$

Similarly, $$\vec{OC} + \vec{OD} = 2\vec{OF}$$

$\therefore$ $$\vec{OA} + \vec{OB} + \vec{OC} + \vec{OD} = 2(\vec{OE} + \vec{OF})$$

$$= 2(2\vec{OG}) = 4\vec{OG}$$

[Since G is mid-point of EF]

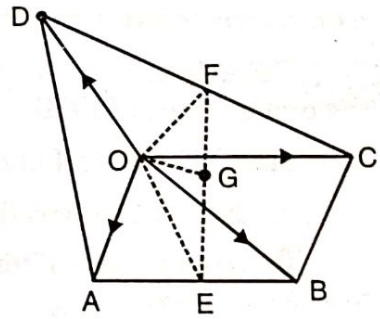


Fig. 1.18

The particle will be in equilibrium if the resultant vanishes

$\therefore$ $$4\vec{OG} = 0$$

Thus, O coincides with G.

Hence the required point O is the mid-point of the join of the mid-points of opposite sides of the quadrilateral.

Example 3.

If G is the centroid of ΔABC, prove that the forces acting at G and represented by GA, 2GB, 3GC have a resultant force represented by 3GH, where H is a point in BC such that BH = 2HC

Solution. Let D be the mid-point of BC

$\therefore$ G, the centroid of the ΔABC divides AD in the ratio 2 : 1.

$\therefore$ $$AG = 2GD \qquad ...(1)$$

Now, $$\vec{GA} + 2\vec{GB} + 3\vec{GC} = (\vec{GA} + \vec{GB} + \vec{GC}) + (\vec{GB} + 2\vec{GC})$$

$$= (\vec{GA} + 2\vec{GD}) + (\vec{GB} + 2\vec{GC})$$

[Since D is mid point of BC]

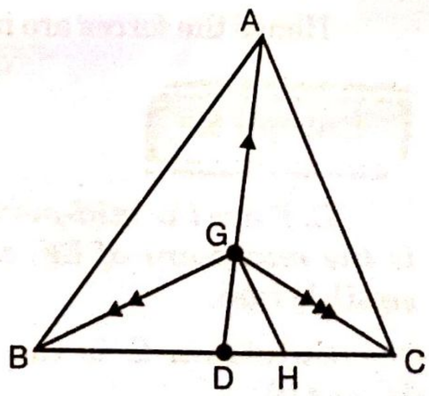


Fig. 1.19

$$= (\vec{GA} + \vec{AG}) + (\vec{GB} + 2\vec{GC}) \qquad \text{[From (1)]}$$

$$= 0 + \vec{GB} + 2.\vec{GC}$$

$$= 1.\vec{GB} + 2.\vec{GC}$$

$$= (1 + 2)\vec{GH} = 3\vec{GH} \qquad [By\ \lambda - \mu\ theorem]$$

where H is a point on BC such that 1.BH = 2.HC.

Example 4.

ABCD is a quadrilateral and E, F, G, H are the middle points of the sides AB, BC, CD, DA respectively. Show that the forces represented by EG, FH and DB in magnitude and direction acting on a particle will keep it in equilibrium. [M.D.U. 2012]

Solution. E, F, G, H are the mid-points of the sides AB, BC, CD and DA respectively of the quadrilateral ABCD.

Join EF, FG, GH and HE.

$\therefore$ EFGH is a parallelogram.

By triangle law of forces,

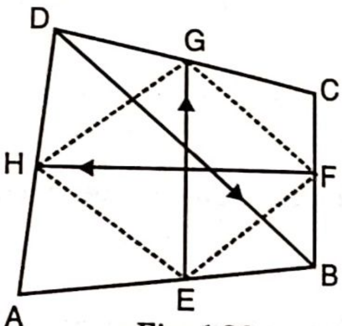


Fig. 1.20

For ΔEFG, $\quad \vec{EG} = \vec{EF} + \vec{FG}$ ...(1)

For ΔEFH, $\quad \vec{FH} = \vec{FE} + \vec{EH}$ ...(2)

$\vec{DB} = 2\vec{HE} = 2\vec{GF}$ ...(3)

Adding (1), (2), (3), we have

$$\vec{EG} + \vec{FH} + \vec{DB} = \vec{EF} + \vec{FG} + \vec{FE} + \vec{EH} + 2\vec{GF}$$

$$= \vec{EF} + \vec{FG} - \vec{EF} + \vec{FG} + 2\vec{GF}$$

$$= 2(\vec{FG} + \vec{GF}) = 0 \qquad [Since\ in\ ||gm\ EFGH,\ \vec{EH} = \vec{FG}]$$

Hence the forces are in equilibrium.

Example 5.

E, F are the mid-points of the diagonals AC and BD of a quadrilateral ABCD. If G is the mid-point of EF, show that the forces represented by GA, GB, GC, GD are in equilibrium.

Solution. G is the mid-point EF, where E and F are mid-points of diagonals AC and BD.

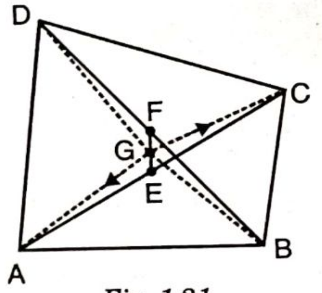


Fig. 1.21

$\therefore$ By $(\lambda - \mu)$ theorem,

$$\vec{GA} + \vec{GC} = 2\vec{GE} \quad ...(1)$$

and
$$\vec{GB} + \vec{GD} = 2\vec{GF} \quad ...(2)$$

Adding (1) and (2), we get

$$\vec{GA} + \vec{GC} + \vec{GB} + \vec{GD} = 2(\vec{GE} + \vec{GF})$$

i.e.,
$$\vec{GA} + \vec{GB} + \vec{GC} + \vec{GD} = 2(0) = 0$$

[*Since $\vec{GE}$ and $\vec{GF}$ are equal and opposite forces*]

Hence forces represented by GA, GB, GC and GD are in equilibrium.

Example 6.

A transversal cuts the lines of action of three concurrent forces P, Q, R in L, M, N respectively. If R is the resultant of P and Q, show that

$$\frac{P}{OL} + \frac{Q}{OM} = \frac{R}{ON}$$

where O is the point of concurrence of the forces. [*K.U. 2017, 01*]

Solution. Let $\frac{P}{OL} = \lambda$ and $\frac{Q}{OM} = \mu$.

$\therefore \quad P = \lambda.OL$ and $Q = \mu.OM$

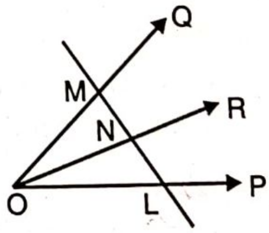


Fig. 1.22

Now, forces along OL and OM are λ.OL and μ.OM respectively.

$\therefore \quad R$ = resultant of forces P and Q along OL and OM respectively.

= resultant of forces λ.OL along OL and μ.OM along OM

$\therefore \quad R = (\lambda + \mu)$ ON along ON [*$\lambda - \mu$ theorem*]

$$\Rightarrow \quad \frac{R}{ON} = \lambda + \mu$$

$$\Rightarrow \quad \frac{R}{ON} = \frac{P}{OL} + \frac{Q}{OM} \quad \text{[From 1]}$$

Hence $\frac{P}{OL} + \frac{Q}{OM} = \frac{R}{ON}$.

EXERCISE 1.3

1. Show that the three forces directed from the vertices and represented by the three medians of a triangle are in equilibrium.
2. ABC is a triangle and D, E, F are the middle points of the sides. Forces represented by AD, $\frac{2}{3}$BE and $\frac{1}{3}$CF act on a particle at the point where AD and BE meet. Show that the resultant is represented in magnitude and direction by $\frac{1}{2}$AC and that its line of action divides BC in the ratio 2 : 1.
3. Find a point O within a triangle ABC so that the forces represented by OA, OB and OC in magnitude and direction and acting at O, may be in equilibrium.

   [Ans. Centroid of Δ ABC]
4. If G is the centroid of a triangle ABC, show that
   (*i*) Forces acting on a particle and represented by GA, GB, GC are in equilibrium.
   (*ii*) Forces represented by GD, GE, GF are in equilibrium, where D, E, F are mid points of sides BC, AC and AB respectively.
5. P is any point on Δ ABC and D, E, F are the middle points of its sides. Prove that forces represented by AP, BP, CP, PD, PE, PF are in equilibrium.
6. If P is any point in the plane of the triangle ABC, show that forces PA, PB, PC are equal to 3PG, where G is the centroid of the triangle.
7. ABCD is a quadrilateral. Forces represented by BC, BA, DC, DA are acting at a point. Show that the resultant will be represented by four times the line joining the mid-points of the diagonals.
8. ABCD is a quadrilateral and E is the point of intersection of the lines joining the middle points of opposite sides. Prove that the resultant of forces represented by OA, OB, OC, OD is given by 4OE wherever the point O may be. When will these forces be in equilibrium ?

   [Ans. When O coincides with E]
9. If P, Q, R, S are the middle points of the sides of a quadrilateral ABCD and O is any point, show that the forces represented by OA, OB, OC, OD have the same resultant as the forces represented by OP, OQ, OR, OS.
10. ABCD is a parallelogram and P is any point in its plane. A particle P is attracted towards A and C by forces proportional to PA and PC respectively and repelled from B and D by forces proportional to BP and DP respectively (the constant of proportionality being the same in both the cases). Show that P is in equilibrium in all positions.
11. Show that the resultant of the forces OA tan A and OB tan B acting along the sides OA and OB of a triangle OAB is AB tan A tan B and acts in the direction of the perpendicular from O on AB.

12. O is a point in the plane of triangle ABC. AO meets BC in D. Show that the components along AB and AC respectively of the forces represented by AD are

(i) $\frac{1}{2}$ AB and $\frac{1}{2}$ AC if O is the centroid

(ii) $\frac{b}{b+c}$ AB and $\frac{c}{b+c}$ AC if O is the incentre.

(iii) $\frac{\tan B}{\tan B + \tan C}$. AB and $\frac{\tan C}{\tan B + \tan C}$. AC if O is the orthocentre of the triangle.

[*M.D.U. 1998*]

1.11. LAMI'S THEOREM

Statement. ***If three coplanar forces acting at a point are in equilibrium, then each is proportional to the sine of the angle between the other two.*** [*M.D.U. 2018, 13, 10*]

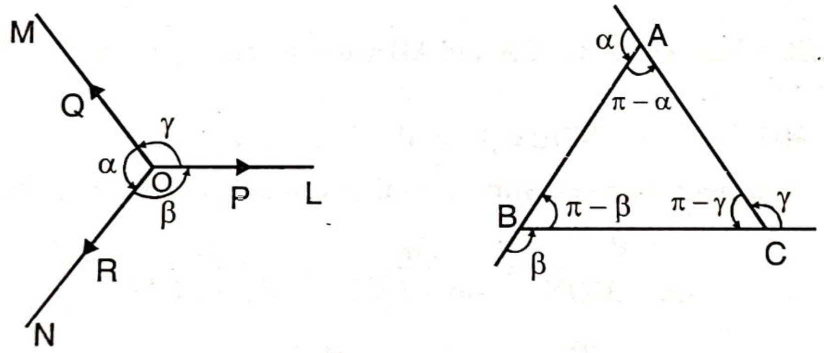


Fig. 1.23

Proof. Let the three coplanar forces P, Q, R acting at the point O, along OL, OM and ON respectively, be in equilibrium. Draw a triangle ABC whose sides BC, CA and AB are respectively parallel to the directions of forces P, Q and R

Then by the converse of triangle law of forces, the sides of the ΔABC will be proportional to the forces in magnitude

$$\therefore \qquad \frac{P}{BC} = \frac{Q}{CA} = \frac{R}{AB} \qquad ...(1)$$

Let $\angle MON = \alpha$, $\angle NOL = \beta$ and $\angle LOM = \gamma$

Again from Δ ABC, by sine formula

$$\frac{BC}{\sin A} = \frac{CA}{\sin B} = \frac{AB}{\sin C}$$

i.e.,
$$\frac{BC}{\sin(\pi-\alpha)}=\frac{CA}{\sin(\pi-\beta)}=\frac{AB}{\sin(\pi-\gamma)}$$

or
$$\frac{BC}{\sin\alpha}=\frac{CA}{\sin\beta}=\frac{AB}{\sin\gamma} \quad ...(2)$$

From (1) and (2), we have

$$\frac{P}{\sin\alpha}=\frac{Q}{\sin\beta}=\frac{R}{\sin\gamma}.$$

Hence the theorem.

1.12. CONVERSE OF LAMI'S THEOREM

If three coplanar forces acting at a point be such that each is proportional to the sine of the angle between the directions of the other two, the forces are in equilibrium.

[M.D.U. 2012]

Proof. Let the three coplanar forces P, Q, R act at a point O along OL, OM and ON respectively.

Draw Δ ABC whose sides BC, CA and AB are respectively parallel to the directions of forces P, Q and R.

Let $\angle MON=\alpha$, $\angle NOL=\beta$ and $\angle LOM=\gamma$

It is given that each force is proportional to the sine of the angle between the other two [Refer fig. 1.23]

$\therefore$
$$\frac{P}{\sin\angle MON}=\frac{Q}{\sin\angle NOL}=\frac{R}{\sin\angle LOM}$$

i.e.,
$$\frac{P}{\sin\alpha}=\frac{Q}{\sin\beta}=\frac{R}{\sin\gamma}$$

i.e.,
$$\frac{P}{\sin(\pi-\alpha)}=\frac{Q}{\sin(\pi-\beta)}=\frac{R}{\sin(\pi-\gamma)}$$

or
$$\frac{P}{\sin A}=\frac{Q}{\sin B}=\frac{R}{\sin C} \quad ...(1)$$

Again, from Δ ABC, by sine formula, we have

$$\frac{BC}{\sin A}=\frac{CA}{\sin B}=\frac{AB}{\sin C} \quad ...(2)$$

$\therefore$ From (1) and (2), we get

$$\frac{P}{BC}=\frac{Q}{CA}=\frac{R}{AB} \quad ...(3)$$

Result (3) shows that the sides of the triangle ABC represent on some scale the forces P, Q and R in magnitude and direction.

$\therefore$ By triangle law of forces, the forces P, Q, R are in equilibrium.

Note :

The proof of the converse of Lami's theorem is based on the reversion of the steps of proof of Lami's theorem.

SOLVED EXAMPLES

Example 1.

Three coplanar forces acting on a particle are in equilibrium. The angle between the first and the second is 60° and that between the second and the third is 150°. Find the ratio of the magnitudes of the forces. [K.U. 2016, 14]

Solution. Let the forces be P, Q, R and act at the point O.

Now, angle between P and Q = 60° and angle between Q and R = 150°

$\therefore$ Angle between R and P = 360° – (60° + 150°)

= 150°

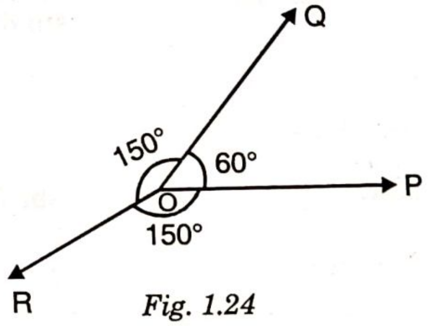


Fig. 1.24

The forces P, Q, R acting at point O are in equilibrium.

$\therefore$ By Lami's theorem,

$$\frac{P}{\sin 150^\circ} = \frac{Q}{\sin 150^\circ} = \frac{R}{\sin 150^\circ}$$

$$\Rightarrow \quad \frac{P}{\frac{1}{2}} = \frac{Q}{\frac{1}{2}} = \frac{R}{\frac{1}{2}} \quad \Rightarrow \quad \frac{P}{1} = \frac{Q}{1} = \frac{R}{\sqrt{3}}$$

Hence the ratio of the magnitude of the forces is as $\mathbf{1 : 1 : \sqrt{3}}$.

Example 2.

Find the greatest weight which can be supported by two light strings making angles 60° and 45° with the vertical; it being known that either string will break under the tension of W kg. wt.

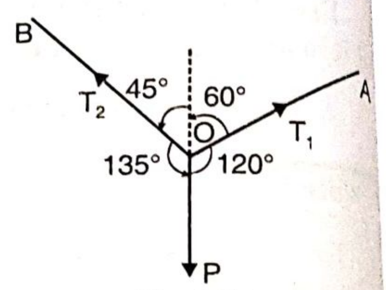


Fig. 1.25

Solution. Let P be the greatest weight supported by two strings OA and OB. Let T_1 and T_2 be the tensions along the strings OA and OB respectively.

Since the forces T_1, T_2 and P acting at a point O are in equilibrium, therefore, by Lami's theorem,

$$\frac{T_1}{\sin 135°} = \frac{T_2}{\sin 120°} = \frac{P}{\sin 105°}$$

or
$$\frac{T_1}{\sin (180° - 45°)} = \frac{T_2}{\sin (180° - 60°)} = \frac{P}{\sin (180° - 75°)}$$

or
$$\frac{T_1}{\sin 45°} = \frac{T_2}{\sin 60°} = \frac{P}{\sin 75°} \quad ...(1)$$

Since $\sin 60° > \sin 45°$, therefore $T_2 > T_1$

When the string OB having tension T_2 breaks, then

$$T_2 = W$$

$\therefore$ From (1),
$$\frac{W}{\sin 60°} = \frac{P}{\sin 75°}$$

or
$$P = \frac{\sin 75°}{\sin 60°} W \quad ...(2)$$

Now,
$$\sin 75° = \sin (45° + 30°)$$
$$= \sin 45° \cos 30° + \cos 45° \sin 30°$$
$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{\sqrt{3}+1}{2\sqrt{2}}$$

$\therefore$ From (2),
$$P = \frac{\sqrt{3}+1}{2\sqrt{2}} \times \frac{2}{\sqrt{3}} .W. \text{ kg. wt.}$$

Hence,
$$\mathbf{P = \frac{W(\sqrt{3}+1)}{\sqrt{6}} \text{ kg.wt}}$$

Example 3.

Three forces P, Q, R acting at a point O are in equilibrium and the angle between P and Q is double the angle between P and R. Show that $R^2 = Q(Q - P)$. [*K.U.* 2014, 10]

Solution. Let α be the angle between P and R. Then angle between P and Q is 2α.

$\therefore$ The angle between R and Q is $360° - 3\alpha$

Now, the forces P, Q and R acting at O are in equilibrium

$\therefore$ By Lami's theorem, we have

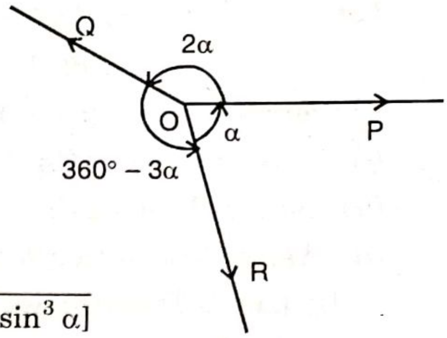


Fig. 1.26

$$\frac{Q}{\sin\alpha} = \frac{R}{\sin 2\alpha} = \frac{P}{\sin(360° - 3\alpha)}$$

or
$$\frac{Q}{\sin\alpha} = \frac{R}{\sin 2\alpha} = \frac{P}{-\sin 3\alpha}$$

or
$$\frac{Q}{\sin\alpha} = \frac{R}{2\sin\alpha\cos\alpha} = \frac{P}{-[3\sin\alpha - 4\sin^3\alpha]}$$

or
$$\frac{Q}{1} = \frac{R}{2\cos\alpha} = \frac{P}{4\sin^2\alpha - 3}$$ [*Dividing by sin α*]

From first and second members,

$$\cos\alpha = \frac{R}{2Q} \qquad ...(1)$$

From first and third members,

$$\frac{P}{Q} = 4\sin^2\alpha - 3$$
$$= 4(1 - \cos^2\alpha) - 3 = 1 - 4\cos^2\alpha$$

$\therefore$
$$\frac{P}{Q} = 1 - 4\left(\frac{R}{2Q}\right)^2 = 1 - \frac{R^2}{Q^2}$$ [*Using (1)*]

i.e.,
$$\frac{P}{Q} = \frac{Q^2 - R^2}{Q^2}$$

or
$$PQ = Q^2 - R^2$$

$\therefore$
$$R^2 = Q^2 - PQ = \mathbf{Q(Q - P)}.$$

Example 4.

AB and AC are two strings 9 m. and 12 m. long attached to pegs B and C at a horizontal distance 15 m apart. Find the tensions in the strings when a weight of 10 kg is suspended from A.

[C.D.L.U. 2013]

Solution. Let T_1 and T_2 be the tensions in the string AB and AC respectively when a wt. of 10 kg is suspended at A.

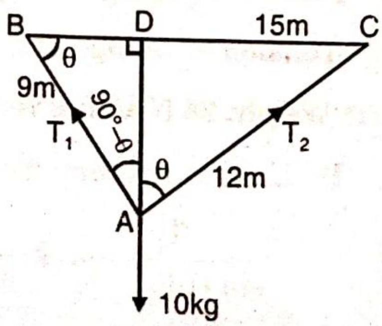


Fig. 1.27

Now, AB = 9 m., AC = 12 m. and BC = 15 m.

Since
$$(15)^2 = (9)^2 + (12)^2$$

$\therefore$
$$BC^2 = AB^2 + AC^2$$

$\therefore$
$$\angle BAC = 90°$$

Let $\angle CAD = \theta$

$\therefore$ $\angle BAD = (90° - \theta)$

The forces acting at point A in equilibrium are

(*i*) Tension T_1 along CA

(*ii*) Tension T_2 along CB

(*iii*) Weight 10 kg acting vertically downwards

$\therefore$ By Lami's Theorem, we have

$$\frac{T_1}{\sin(180° - \theta)} = \frac{T_2}{\sin(90° + \theta)} = \frac{10}{\sin 90°}$$

or
$$\frac{T_1}{\sin\theta} = \frac{T_2}{\cos\theta} = 10$$

$\therefore$ $T_1 = 10 \sin\theta$ and $T_2 = 10\cos\theta$ ...(1)

In the right angled ΔBAC, $\angle ABC = \theta$

$\therefore$ $\sin\theta = \frac{12}{15}$ and $\cos\theta = \frac{9}{15}$

$\therefore$ From (1), $T_1 = 10 \times \frac{12}{15} = \mathbf{8\ kg}$ and $T_2 = 10 \times \frac{9}{15} = \mathbf{6\ kg.}$

Example 5.

A weight of 26 N is suspended by two light inelastic strings of length 5 m and 12 m from two points at the same level and 14 m apart. Find the tensions in the strings.

Solution. Let the weight 26 N be suspended from C and let T_1 and T_2 be the tensions in the strings AC and CB respectively.

Let $\angle ACD = \theta$ and $\angle BCD = \phi$

The forces acting at the point C in equilibrium are :

(*i*) Tension T_1 along CA

(*ii*) Tension T_2 along CB

(*iii*) Weight 26 N acting vertically downwards

$\therefore$ By Lami's theorem, we have

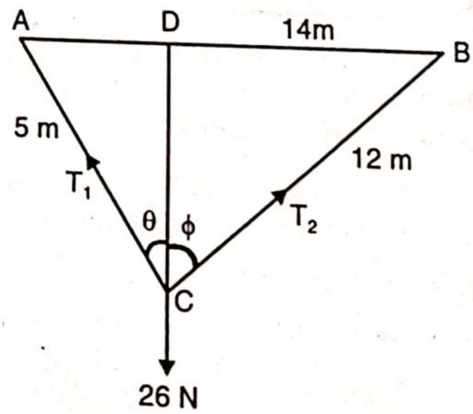


Fig. 1.28

$$\frac{T_1}{\sin(180° - \phi)} = \frac{T_2}{\sin(180° - \theta)} = \frac{26}{\sin(\theta + \phi)}$$

$$\Rightarrow \quad \frac{T_1}{\sin \phi} = \frac{T_2}{\sin \theta} = \frac{26}{\sin(\theta + \phi)}$$

$$\Rightarrow \quad \frac{T_1}{\sin(90° - B)} = \frac{T_2}{\sin(90° - A)} = \frac{26}{\sin[180° - (A + B)]}$$

$$\Rightarrow \quad \frac{T_1}{\cos B} = \frac{T_2}{\cos A} = \frac{26}{\sin(A + B)} \qquad \text{...(1)}$$

$$\Rightarrow \quad T_1 = 26\left(\frac{\cos B}{\sin(A + B)}\right)$$

and

$$T_2 = 26\left(\frac{\cos A}{\sin(A + B)}\right) \qquad \text{...(2)}$$

Now, by cosine formula, we have

$$\cos A = \frac{(5)^2 + (14)^2 - (12)^2}{2(5)(14)} = \frac{11}{20}$$

and

$$\cos B = \frac{(12)^2 + (14)^2 - (5)^2}{2(12)(14)} = \frac{15}{16}$$

$$\therefore \quad \sin A = \sqrt{1 - \left(\frac{11}{20}\right)^2} = \frac{\sqrt{279}}{20}$$

and

$$\sin B = \sqrt{1 - \left(\frac{15}{16}\right)^2} = \frac{\sqrt{31}}{16}$$

$$\therefore \quad \sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$= \frac{\sqrt{279}}{20} \cdot \frac{15}{16} + \frac{11}{20} \cdot \frac{\sqrt{31}}{16} = \frac{45\sqrt{31} + 11\sqrt{31}}{320} = \frac{56\sqrt{31}}{320} \quad [\textit{Using } (1)]$$

$$\therefore \quad T_1 = 26 \cdot \left(\frac{\cos B}{\sin(A + B)}\right) = 26 \cdot \left(\frac{\frac{15}{16}}{\frac{56\sqrt{31}}{320}}\right) = \frac{7800}{56\sqrt{31}} = \frac{975}{7\sqrt{31}}\,\text{N}$$

and

$$T_2 = 26 \cdot \left(\frac{\cos A}{\sin(A + B)}\right) = 26 \cdot \left(\frac{\frac{11}{20}}{\frac{56\sqrt{31}}{320}}\right) = \frac{572}{7\sqrt{31}}\,\text{N}.$$

Example 6.

A uniform lamina in the form of a rhombus one of whose angles is 120° is supported by two forces P, Q applied at the centre in the directions of diagonals so that one side is horizontal. Show that if P > Q, then $P^2 = 3Q^2$.

Solution. Let ABCD be the rhombus and W be its weight which acts vertically downwards through O, the point where diagonals AC and BD bisect each other.

$\angle BAD = 120°$. The other angles are shown in fig. 1.29

Since P > Q , the resultant always acts nearer the greater force P.

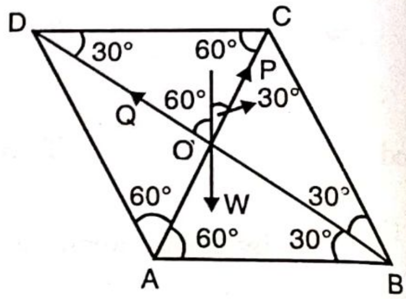


Fig. 1.29

∴ Forces P and Q act along OC and OD.

The forces P, Q and W acting at O are in equilibrium.

∴ By Lami's theorem, we get

$$\frac{P}{\sin(180° - 60°)} = \frac{Q}{\sin(180° - 30°)} = \frac{W}{\sin 90°}$$

i.e.,

$$\frac{P}{\sin 60°} = \frac{Q}{\sin 30°} = \frac{W}{1}$$

From 1st two members, we have

$$\frac{P}{\frac{\sqrt{3}}{2}} = \frac{Q}{\frac{1}{2}}$$

$$\therefore \qquad P = \sqrt{3}Q \quad \Rightarrow \quad \mathbf{P^2 = 3Q^2}.$$

Example 7.

ABC is a triangle and forces P, Q, R acting at a point O along the lines OA, OB, OC are in equilibrium. Prove that

(a) If O is the incentre of △ ABC, then $\dfrac{P}{\cos\frac{A}{2}} = \dfrac{Q}{\cos\frac{B}{2}} = \dfrac{R}{\cos\frac{C}{2}}$

(b) If O is the orthocentre of the △ ABC, then $P : Q : R = a : b : c$

(c) If O is the centroid of △ABC, then $\dfrac{P}{OA} = \dfrac{Q}{OB} = \dfrac{R}{OC}$.

Solution. (*a*) Here O is the incentre of the △ ABC.

∴ OA, OB and OC are the internal bisectors of the angles A, B and C respectively.

$$\therefore \quad \angle OBC = \frac{B}{2} \quad \text{and} \quad \angle BCO = \frac{C}{2}$$

$$\therefore \quad \angle BOC = 180° - \left(\frac{B}{2} + \frac{C}{2}\right)$$

$$= 180° - \left(\frac{B + C}{2}\right)$$

$$= 180° - \left(90° - \frac{A}{2}\right) \quad [\because A + B + C = 180°]$$

$$= 90° + \frac{A}{2}.$$

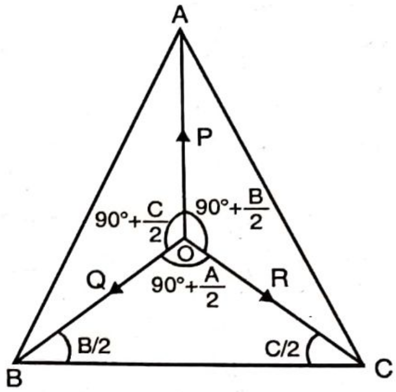


Fig. 1.30

Similarly, $\angle COA = 90° + \frac{B}{2}$ and $\angle AOB = 90° + \frac{C}{2}$

Since forces P, Q, R acting at O along OA, OB, OC respectively, are in equilibrium

∴ By Lami's theorem,

$$\frac{P}{\sin\left(90° + \frac{A}{2}\right)} = \frac{Q}{\sin\left(90° + \frac{B}{2}\right)} = \frac{R}{\sin\left(90° + \frac{C}{2}\right)}$$

i.e.,

$$\frac{\mathbf{P}}{\cos\frac{\mathbf{A}}{\mathbf{2}}} = \frac{\mathbf{Q}}{\cos\frac{\mathbf{B}}{\mathbf{2}}} = \frac{\mathbf{R}}{\cos\frac{\mathbf{C}}{\mathbf{2}}}.$$

(*b*) Let AL, BM and CN be the altitudes of Δ ABC. Let O be their point of intersection which is the orthocentre of Δ ABC.

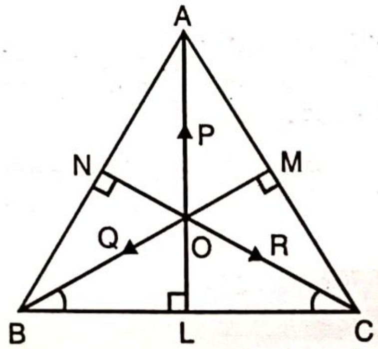


Fig. 1.31

In right angled Δ BMC, $\angle MBC = 90° - C$

In right angled Δ BNC,

$$\angle NCB = 90° - B$$

$$\therefore \text{ In } \Delta BOC, \quad \angle BOC = 180° - [(90° - B) + (90° - C)]$$

$$= B + C = 180° - A$$

Similarly, $\angle COA = 180° - B$

and $\angle AOB = 180° - C$

Since the forces P, Q, R acting at O along OA, OB, OC respectively are in equilibrium.

∴ By Lami's theorem, we have

$$\frac{P}{\sin(180° - A)} = \frac{Q}{\sin(180° - B)} = \frac{R}{\sin(180° - C)}$$

i.e., $$\frac{P}{\sin A} = \frac{Q}{\sin B} = \frac{R}{\sin C}$$

or $$\frac{P}{a} = \frac{Q}{b} = \frac{R}{c}$$ [*By sine formula*]

Hence, $\mathbf{P : Q : R = \mathit{a} : \mathit{b} : \mathit{c}}$.

(*c*) Here O is the centroid of the Δ ABC where the medians of the triangle meet.

$\therefore$ From Geometry, $\Delta OBC = \Delta OCA = \Delta OAB$

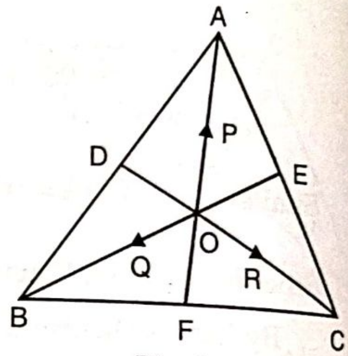


Fig. 1.32

$$\therefore \quad \frac{1}{2} OB.OC. \sin \angle BOC = \frac{1}{2} OC.OA \angle \sin COA$$

$$= \frac{1}{2} OA.OB. \angle \sin AOB$$

$$[\because \Delta ABC = \frac{1}{2} ab \sin C \text{ etc.}]$$

Dividing throughout by $\frac{1}{2}$OA.OB.OC, we have

$$\frac{\sin \angle BOC}{OA} = \frac{\sin \angle COA}{OB} = \frac{\sin \angle AOB}{OC} \quad ...(1)$$

Since the forces acting at O along OA, OB, OC respectively are in equilibrium

$\therefore$ By Lami's theorem, we get

$$\frac{P}{\sin \angle BOC} = \frac{Q}{\sin \angle COA} = \frac{R}{\sin \angle AOB}$$

$\therefore$ From (1) and (2), we have $\mathbf{\dfrac{P}{OA} = \dfrac{Q}{OB} = \dfrac{R}{OC}}$.

EXERCISE 1.4

1. A weight of 100 kg is supported by two ropes which are tied to the weight and are inclined at angles 45° and 30° to the vertical. Find the tensions in the string.
2. Two forces acting on a particle are at right angles and are balanced by a third force making an angle 150° with one of them. The greater of the two is 3 kg. What are the values of the other two ?
3. A weight W hangs by a string and is drawn aside by a horizontal force until the string makes an angle of 60° with the vertical. Find the horizontal force and tension in the string.

4. Three forces P, Q and R act on a particle and keep it in equilibrium. If the angle between the forces P and Q and between Q and R each be 120°, prove that P = Q = R. [M.D.U. 2013]

5. Three equal forces acting at a point are in equilibrium. Show that they are equally inclined to one another and conversely.

6. Two fine light strings support a weight W kg and are inclined to the vertical at angles 30° and 60°. Find the tensions in the strings.

7. The ends of a string 23 cms long, are attached to two points A and B in a horizontal line. A weight of 51 gms. is fastened to a point C, of the string. Find the tensions in the strings, if AB = 17 cms and AC = 8 cms when the weight is at rest. [M.D.U. 2015, 13]

8. Two strings of length 4 m and 5 m are fastened to a particle of mass W kg., their ends being fastened to points at the same level 6 m apart. Find the tensions in the string.

9. A body of mass 26 N is suspended by two strings 5 m and 12 m long, their other ends being fastened to the extremities of a rod 13 m long. If the rod be so held that the body hangs immediately below the middle point, find the tensions of the string.

10. A, B are two fixed points in a horizontal line at a distance c apart. Two light strings AC and BC of length b and a respectively support a mass at C. Show that the tensions of the strings are in the ratio $b(c^2 + a^2 - b^2) : a(b^2 + c^2 - a^2)$. [K.U. 1993]

11. Forces P, Q, R acting along OA, OB, OC in the plane of the Δ ABC are in equilibrium. Prove that, if O is the circumcentre of Δ ABC, then

(a) $\dfrac{P}{\sin 2A} = \dfrac{Q}{\sin 2B} = \dfrac{R}{\sin 2C}$ [K.U. 2016; 2000]

(b) $\dfrac{P}{a^2(b^2 + c^2 - a^2)} = \dfrac{Q}{b^2(c^2 + a^2 - b^2)} = \dfrac{R}{c^2(a^2 + b^2 - c^2)}$.

ANSWERS

1. $\dfrac{100\sqrt{2}}{\sqrt{3}+1}$ kg wt.; $\dfrac{200}{\sqrt{3}+1}$ kg wt.

2. $\sqrt{3}$ kg; $2\sqrt{3}$ kg

3. Horizontal force = $\sqrt{3}$ W; Tension = 2W.

6. $\dfrac{\sqrt{3}}{2}$ W kg; $\dfrac{W}{2}$ kg

7. 45 gms. wt.; 24 gms. wt.

8. $\dfrac{2}{\sqrt{7}}$ W kg; $\dfrac{3}{2\sqrt{7}}$ W kg

9. 10 N; 24 N.

1.13. POLYGON LAW OF FORCES

Statement. *If any number of forces, acting at a point, be represented in magnitude and direction by the sides of a polygon, taken in order, the forces will be in equilibrium.*

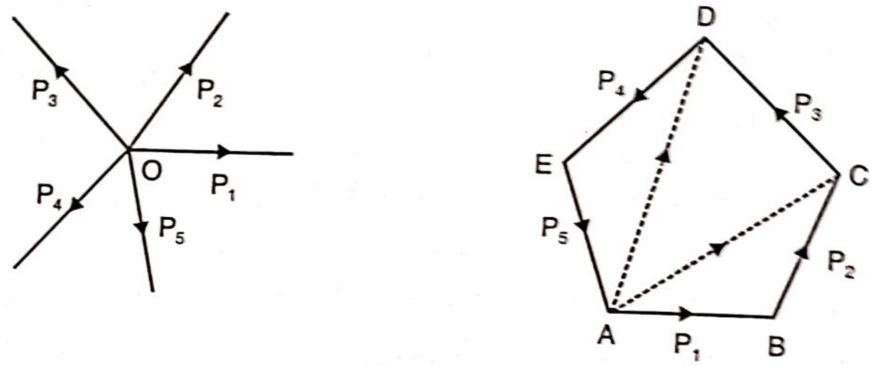


Fig. 1.33

Let the forces P_1, P_2, P_3, P_4 and P_5 acting at a point O, be represented in magnitude and direction by the sides AB, BC, CD, DE and EA respectively of the polygon ABCDE. Join AC and AD.

Using vector notations, by triangle law of forces, we have

$$\vec{AB} + \vec{BC} = \vec{AC}$$

Again
$$\vec{AB} + \vec{BC} + \vec{CD} = \vec{AC} + \vec{CD} = \vec{AD}$$

and
$$\vec{AB} + \vec{BC} + \vec{CD} + \vec{DE} + \vec{EA} = \vec{AD} + \vec{DE} + \vec{EA}$$
$$= \vec{AE} + \vec{EA} = 0$$

$\therefore$
$$P_1 + P_2 + P_3 + P_4 + P_5 = 0$$

Hence the forces P_1, P_2, P_3, P_4 and P_5 are in equilibrium.

Note :

We have taken only five forces. A similar method of proof will apply for any number of forces acting at a point.

Cor. Since $\vec{AB} + \vec{BC} + \vec{CD} + \vec{DE} + \vec{EA} = 0$

$\therefore$
$$\vec{AB} + \vec{BC} + \vec{CD} + \vec{DE} = -\vec{EA} = \vec{AE}$$

Hence the Polygon Law of forces can also be stated as :

If any number of forces, acting at a point, can be represented in magnitude and direction by the sides, taken in order, of an open polygon, their resultant is represented in magnitude and direction by the closing side of the polygon in the opposite order.

Note :

The converse of the polygon law of forces is not true in the same sense as that of the triangle law of forces; because the ratios of the sides of a polygon are not known when the directions of the sides are known.

1.14. THEOREM OF RESOLVED PARTS

Statement. ***The algebraic sum of the resolved parts of two concurrent forces in any direction in their plane is equal to the resolved part of their resultant in the same direction.***

Proof. Let P and Q be the two given forces represented by OA and OB respectively. Complete the parallelogram OACB. Their resultant R is represented by the diagonal OC.

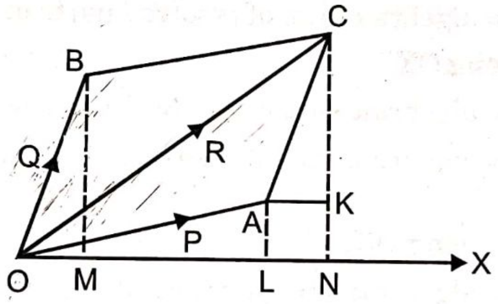


Fig. 1.34

Let OX be parallel to the direction in which the forces are to be resolved.

Draw AL, BM and CN ⊥'s to OX and AK ⊥ CN. Clearly Δ's OMB and AKC are congruent.

$\therefore$ OM = AK

$\Rightarrow$ OM = LN ...(1) [$\because AK = LN$]

Now, resolved part of P along OX = P cos ∠AOL

and resolved part of Q along OX = Q cos ∠BOM

$\therefore$ The algebraic sum of the resolved parts of P and Q along OX

$$= P \cos \angle AOL + Q \cos \angle BOM$$

$$= OA \cdot \frac{OL}{OA} + OB \cdot \frac{OM}{OB}$$

$$= OL + OM = OL + LN \qquad [Using\ (1)]$$

$$= ON = OC \cdot \frac{ON}{OC}$$

$$= R \cdot \cos \angle CON = \text{Resolved part of R along OX}$$

Hence the theorem.

1.15. GENERALIZATION OF THEOREM OF RESOLVED PARTS

Statement. ***The algebraic sum of the resolved parts of any number of concurrent forces in any direction is equal to the resolved part of their resultant in the same direction.***

Proof. Let $P_1, P_2, P_3, \ldots P_n$ be n concurrent forces acting at a point O and OX be the given direction.

Let R_1 be the resultant of forces P_1, P_2

R_2 be the resultant of forces R_1 and P_3 (*i.e.*, P_1, P_2, P_3) and so on.

Finally, let R_{n-1} be the resultant of R_{n-2} and P_n so that R_{n-1} is the resultant of $P_1, P_2, P_3, \ldots P_n$.

Now, resolved part of R_1 along OX

= algebraic sum of resolved parts of P_1 and P_2 along OX

Resolved part of R_2 along OX

= algebraic sum of resolved parts of R_1 and P_3 along OX

= algebraic sum of resolved parts of P_1, P_2 and P_3 along OX

and so on.

Resolved part of R_{n-1} along OX

= algebraic sum of resolved parts of R_{n-2} and P_n along OX

= algebraic sum of resolved parts of $P_1, P_2, P_3, \ldots P_n$ along OX

Hence the theorem.

1.16. RESULTANT OF ANY NUMBER OF CONCURRENT AND COPLANAR FORCES

[K.U. 1995]

Let $P_1, P_2, P_3, \ldots P_n$ be n coplanar forces acting at a point O in directions making angles $\alpha_1, \alpha_2, \alpha_3, \ldots \alpha_n$ respectively with a fixed straight line OX lying in the plane of the forces.

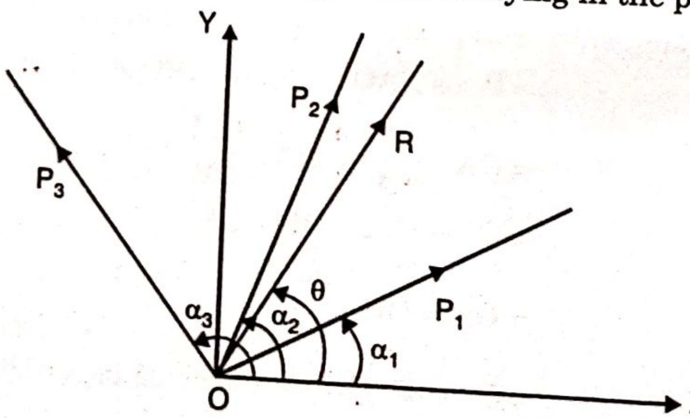


Fig. 1.35

Through O draw OY $\perp$ OX.

Let their resultant R make an angle θ with OX. Now by the Generalised Theorem of Resolved parts :

Resolving along OX, we have

$$R\cos\theta = P_1\cos\alpha_1 + P_2\cos\alpha_2 + P_3\cos\alpha_3 + \ldots + P_n\cos\alpha_n \quad \ldots(1)$$
$$= X \text{ (say)}$$

Resolving along OY, we have

$$R\sin\theta = P_1\sin\alpha_1 + P_2\sin\alpha_2 + P_3\sin\alpha_3 + \ldots + P_n\sin\alpha_n \quad \ldots(2)$$
$$= Y \text{ (say)}$$

Squaring (1) and (2) and adding, we have

$$R^2 = X^2 + Y^2$$

$\therefore$ $R = \sqrt{X^2 + Y^2}$, which is the magnitude of the resultant.

Further, dividing (2) by (1), we get

$$\tan\theta = \frac{Y}{X} \Rightarrow \theta = \tan^{-1}\frac{Y}{X}$$, which gives the direction of the resultant.

Remarks :

1. Here X stands for the algebraic sum of the resolved parts of all the forces along OX and Y stands for the algebraic sum of the resolved parts of all the forces along OY.
2. $\tan\theta = \frac{Y}{X}$, where θ is chosen so that it may satisfy the equations, $\cos\theta = \frac{X}{R}$ and $\sin\theta = \frac{Y}{R}$.

SOLVED EXAMPLES

Example 1. ***Forces P – Q, P, P + Q act at a point in direction parallel to the sides of an equilateral triangle, taken in order. Find their resultant.***

Solution. Let ABC be an equilateral triangle. Let the forces P, P + Q, P – Q act at O along OL, OM and ON which are parallel to the sides BC, CA and AB respectively. Let R be their resultant making an angle θ with OL

$\therefore$ $\angle LOM = 120° = \angle MON$

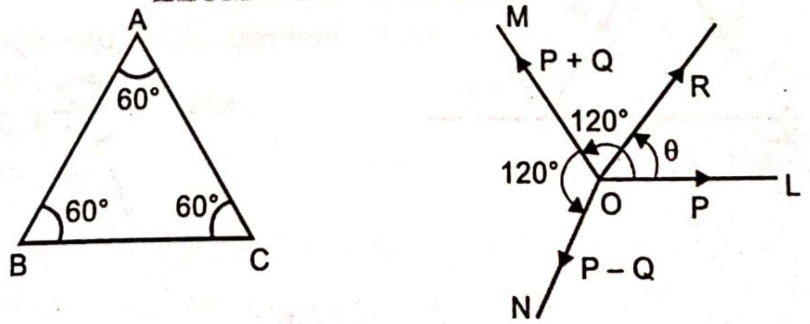


Fig. 1.36

Resolving the forces along OL and perpendicular to OL, we have

$$R\cos\theta = P\cos 0^\circ + (P+Q)\cos 120^\circ + (P-Q)\cos 240^\circ$$

$$= P + (P+Q)\left(-\frac{1}{2}\right) + (P-Q)\left(-\frac{1}{2}\right) \quad \left[\because \cos 120^\circ = \cos 240^\circ = -\frac{1}{2}\right]$$

$$= P - \frac{1}{2}P - \frac{1}{2}P = P - P = 0 \qquad \ldots(1)$$

and

$$R\sin\theta = P\sin 0^\circ + (P+Q)\sin 120^\circ + (P-Q)\sin 240^\circ$$

$$= 0 + (P+Q)\sin(180^\circ - 60^\circ) + (P-Q)\sin(180^\circ + 60^\circ)$$

$$= (P+Q)\sin 60^\circ - (P-Q)\sin 60^\circ$$

$$= [(P+Q)-(P-Q)]\,\frac{\sqrt{3}}{2} = \sqrt{3}\,Q \qquad \ldots(2)$$

Squaring (1) and (2) and adding, we have

$$R^2 = 3Q^2 \;\Rightarrow\; R = \sqrt{3}\,Q$$

Dividing (2) by (1), we have

$$\tan\theta = \frac{\sqrt{3}Q}{0} = \infty = \tan 90^\circ$$

$$\therefore \quad \theta = 90^\circ$$

Hence the resultant $\sqrt{3}\,Q$ acts along perpendicular to the direction of P.

Example 2.

Forces each equal to P act at a point parallel to the sides of a triangle ABC. Show that their resultant is given by

$$P\sqrt{3 - 2\cos A - 2\cos B - 2\cos C}$$

[*M.D.U. 2016, 14, 11*]

Solution. Let the three forces each equal to P, act at O, along OL, OM and ON which are parallel to the sides BC, CA and AB of Δ ABC respectively. Let R be their resultant making an angle θ with OL.

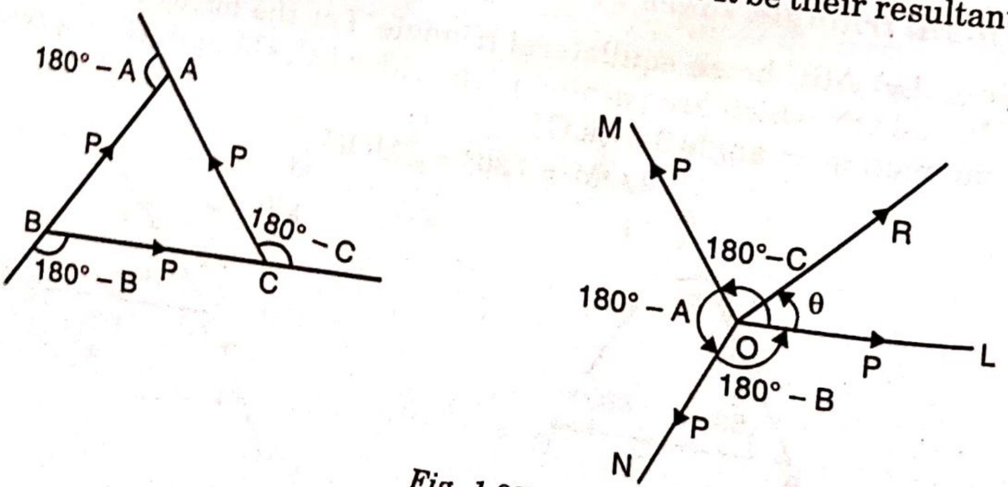


Fig. 1.37

$\therefore \quad \angle LOM = 180° - C; \ \angle MON = 180° - A; \ \angle NOL = 180° - B$

Resolving the forces along and perpendicular to OL, we get

$$\begin{aligned}R\cos\theta &= P\cos 0° + P\cos(180° - C) + P\cos(360° - \overline{C + A})\\ &= P - P\cos C + P\cos(C + A)\\ &= P - P\cos C + P\cos(180° - B)\\ &= P - P\cos C - P\cos B\\ &= P(1 - \cos B - \cos C) \qquad ...(1)\end{aligned}$$

$$\begin{aligned}R\sin\theta &= P\sin 0° + P\sin(180° - C) + P\sin(360° - \overline{C + A})\\ &= P\sin C - P\sin(C + A)\\ &= P\sin C - P\sin(180° - B)\\ &= P(\sin C - \sin B) \qquad ...(2)\end{aligned}$$

Squaring (1) and (2) and adding, we have

$$\begin{aligned}R^2 &= P^2[(1 - \cos B - \cos C)^2 + (\sin C - \sin B)^2]\\ &= P^2[1 + \cos^2 B + \cos^2 C - 2\cos B + 2\cos B\cos C\\ &\qquad - 2\cos C + \sin^2 C + \sin^2 B - 2\sin B\sin C]\end{aligned}$$

$$\begin{aligned}\Rightarrow \quad R^2 &= P^2[1 + (\cos^2 B + \sin^2 B) + (\cos^2 C + \sin^2 C)\\ &\qquad - 2\cos B - 2\cos C + 2(\cos B\cos C - \sin B\sin C]\\ &= P^2[1 + 1 + 1 - 2\cos B - 2\cos C + 2\cos(B + C)]\\ &= P^2[3 - 2\cos B - 2\cos C + 2\cos(180° - A)]\end{aligned}$$

Hence, $\quad R = P\sqrt{3 - 2\cos A - 2\cos B - 2\cos C}$.

Example 3.

ABCDEF is a regular hexagon. Forces of magnitude 4, $8\sqrt{3}$, 16, $4\sqrt{3}$ and 8 Newtons act at A in the directions AB, AC, AD, AE and AF respectively. Find the resultant of the forces.

Solution. ABCDEF is a regular hexagon in which each interior angle = 120°

$\therefore \angle BAF = 120°$

The lines AC, AD and AE, divide the angle BAF into four equal parts.

Hence each part = 30°

$\therefore$ AE is perpendicular to AB

Take AB and AE as axes of x and y respectively.

Let R be the resultant of the forces making an angle θ with AX.

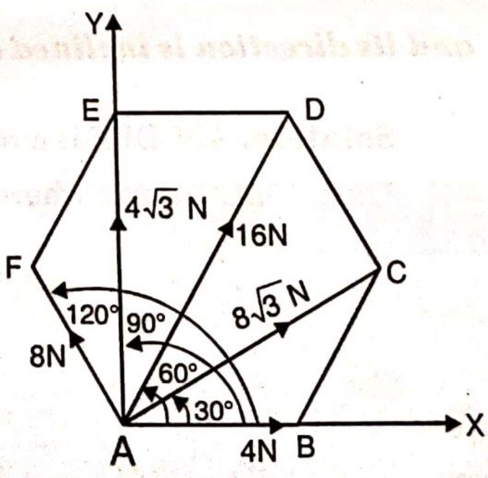


Fig. 1.38

The directions AB, AC, AD, AE, AF of the forces make angles 0°, 30°, 60°, 90°, 120° respectively with the X-axis.

Resolving the forces along AX and AY, we have

$$R\cos\theta = 4\cos 0° + 8\sqrt{3}\cos 30° + 16\cos 60° + 4\sqrt{3}\cos 90° + 8\cos 120°$$

$$= 4 + 8\sqrt{3}.\frac{\sqrt{3}}{2} + 16.\frac{1}{2} + 0 + 8\left(-\frac{1}{2}\right)$$

$$= 4 + 12 + 8 - 4 = 20 \quad ...(1)$$

$$R\sin\theta = 4\sin 0° + 8\sqrt{3}\sin 30° + 16\sin 60° + 4\sqrt{3}\sin 90° + 8\sin 120°$$

$$= 0 + 8\sqrt{3}.\frac{1}{2} + 16.\frac{\sqrt{3}}{2} + 4\sqrt{3} + 8.\frac{\sqrt{3}}{2}$$

$$= 4\sqrt{3} + 8\sqrt{3} + 4\sqrt{3} + 4\sqrt{3} = 20\sqrt{3} \quad ...(2)$$

Squaring (1) and (2) and adding, we have

$$R^2 = (20)^2 + (20\sqrt{3})^2$$

$$\therefore \quad R = 20 \times 2 = \textbf{40 Newtons.}$$

Again dividing (2) by (1), we get

$$\tan\theta = \frac{20\sqrt{3}}{20} = \sqrt{3} \quad \Rightarrow \quad \theta = 60°$$

Hence the resultant acts along AD.

Example 4.

ABCDEF is a regular hexagon. The forces represented in magnitude and direction by AB, 2AC, 3AD, 4AE, 5AF act at A. Show that the magnitude of the resultant is $\sqrt{351}$ AB and its direction is inclined at an angle $\tan^{-1}\frac{7}{\sqrt{3}}$ to AB.

Solution. ABCDEF is a regular hexagon. Let each side = a

Then $\quad AC = 2\,AL$, where $BL \perp AC$

$$= 2\,AB\cos 30° = 2a.\frac{\sqrt{3}}{2} = \sqrt{3}\,a$$

Also, $\quad AE = AC = \sqrt{3}\,a$

and $\quad AD = \sqrt{(AC)^2 + (CD)^2} = \sqrt{3a^2 + a^2} = 2a$

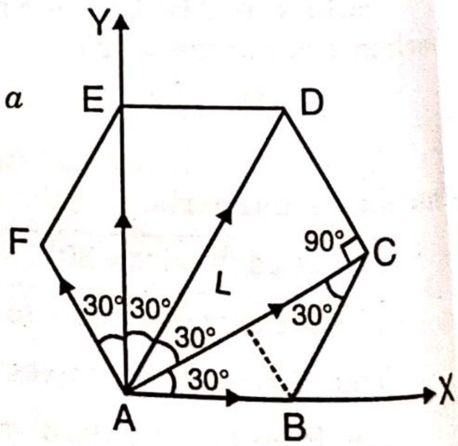


Fig. 1.39

Thus the given forces are :

(*i*) $\overrightarrow{AB} = a$ along AB,

(*ii*) $2\overrightarrow{AC} = 2\sqrt{3}.a$ along AC

(*iii*) $3\overrightarrow{AD} = 6a$ along AD

(*iv*) $4\overrightarrow{AE} = 4\sqrt{3}\,a$ along AE

(*v*) $5\overrightarrow{AF} = 5a$ along AF.

Let R be the resultant making angle θ with AB. Resolving the forces along AX and AY, we have

$$R\cos\theta = a\cos 0° + 2\sqrt{3}\,a\cos 30° + 6a\cos 60° + 4\sqrt{3}\,a\cos 30° + 5a\cos 120°$$

$$= a + 2\sqrt{3}a.\frac{\sqrt{3}}{2} + 6a.\frac{1}{2} + 4\sqrt{3}a.0 + 5a\left(-\frac{1}{2}\right)$$

$$= a + 3a + 3a - \frac{5a}{2} = \frac{9a}{2} \qquad ...(1)$$

and

$$R\sin\theta = a\sin 0° + 2\sqrt{3}\,a\sin 30° + 6a\sin 60° + 4\sqrt{3}\,a\sin 90° + 5a\sin 120°$$

$$= 2\sqrt{3}\,a.\frac{1}{2} + 6a\frac{\sqrt{3}}{2} + 4\sqrt{3}\,a(1) + 5a.\frac{\sqrt{3}}{2}$$

$$= \sqrt{3}\,a + 3\sqrt{3}\,a + 4\sqrt{3}\,a + \frac{5\sqrt{3}}{2}a = \frac{21\sqrt{3}}{2}a \qquad ...(2)$$

Squaring (1) and (2) and adding, we have

$$R^2 = \frac{81a^2}{4} + \frac{1323a^2}{4} = \frac{1404}{4}a^2 = 351\,a^2$$

$$\therefore \qquad R = \sqrt{351}\,a = \sqrt{351}\,\mathbf{AB}.$$

Dividing (2) by (1), we get

$$\tan\theta = \frac{21\sqrt{3}}{9} = \frac{7}{\sqrt{3}} \quad\Rightarrow\quad \theta = \tan^{-1}\frac{7}{\sqrt{3}}.$$

EXERCISE 1.5

1. Three coplanar forces of 5 kg. each act on a particle. If their lines of action make equal angles with each other, determine their resultant.
2. Forces 2P, 3P, 4P act at a point in directions parallel to the sides of an equilateral triangle taken in order. Find the magnitude and the line of action of resultant.

3. Forces equal to P each act at a point in directions parallel to the sides of lengths 3, 4, 5 m of a triangle taken in order. Find their resultant, in magnitude and direction.

4. If the forces of magnitude P, Q, R act at a point parallel to the sides BC, CA, AB respectively of a triangle ABC, prove that the magnitude of the resultant is

$$\sqrt{P^2+Q^2+R^2-2QR\cos A-2RP\cos B-2PQ\cos C}\,.\qquad \text{[M.D.U. 1992]}$$

5. Three forces acting at a point are parallel to the sides of a triangle ABC taken in order and are proportional to the cosine of the opposite angles. Show that their resultant is proportional to

$$\sqrt{1-8\cos A\cos B\cos C}\qquad \text{[K.U. 2013; M.D.U. 2010]}$$

6. Five forces acting at a point O are in equilibrium. Four of them whose magnitudes are 3, 4, 4, 3 kg. wt., act along the coplanar lines OA, OB, OC, OD respectively, such that

$$\angle AOB = 15°,\ \ \angle BOC = 60°,\ \ \angle COD = 15°;$$

Find the magnitude and direction of the fifth force.

7. ABCD is a quadrilateral with AB = 2 cms, BC = 3 cms, CD = 5 cms, DA = 7 cms. Forces of 6, 9, 15, 20 gms.wt. act on a particle parallel to the respective sides of the quadrilateral. Show that their resultant is a force of 1 gm wt. parallel to AD.

[**Hint.** Introduce two equal and opposite forces each = 1 gm wt. along DA and AD. Now, forces become respectively proportional to the sides and hence are in equilibrium. A force of 1 gm wt. is left along AD etc.]

8. Forces of 6, 8, 12 kg. wt. act along BC, CA, AB of the sides of a triangle of lengths 3, 4, 5 cm respectively. Show that their resultant is a force of 2 kg. wt. acting parallel to AB. [M.D.U. 1999]

9. ABCDEF is a regular hexagon. Forces of magnitude $1, 2\sqrt{3}, 2, 4\sqrt{3}, 5$ N respectively act at one of the angular points of the hexagon towards the five others taken in order. Find the magnitude and the direction of the resultant of the forces.

10. ABCDEF is a regular hexagon. Find the direction and magnitude of resultant of forces equal to 7, 18, 5, 9 and 19 kg wt. acting at A respectively along AB, CA, AD, AE, AF.

ANSWERS

1. 0

2. $\sqrt{3}P$; act at an angle 210° with 2P

3. $\dfrac{P}{\sqrt{5}}$; angle = $\left[\pi+\tan^{-1}\left(\dfrac{1}{2}\right)\right]$ with side of 5 m or $\theta=\pi-\tan^{-1}2$ with side 4m.

6. 11·17 kg wt. acting at ∠225° with OA

9. $\sqrt{223}$ N ; $\theta=\tan^{-1}\left(\dfrac{17}{5}.\sqrt{3}\right)$

10. $15\sqrt{3}$ kg .wt.; $\theta=\tan^{-1}\left(-\dfrac{4}{3}\right)$, lying in 2nd quadrant.

1.17. CONDITIONS OF EQUILIBRIUM OF CONCURRENT FORCES

To find the necessary and sufficient conditions of equilibrium of a number of forces acting at a point.

Let the given forces be acting at O

Let X and Y be the algebraic sums of the resolved parts of the forces in the directions of two perpendicular lines in the plane of the forces.

(*i*) **The Conditions are Necessary :**

Let R be the resultant of the forces

$$\therefore \qquad R^2 = X^2 + Y^2$$

The system is in equilibrium

$$\therefore \qquad R = 0$$

i.e.,

$$X^2 + Y^2 = 0$$

This is possible only if $X = 0$, $Y = 0$, which are the required conditions of equilibrium.

(*ii*) **The Conditions are Sufficient :**

If $X = 0$ and $Y = 0$, then

$$R^2 = X^2 + Y^2 = 0$$

$$\therefore \qquad R = 0$$

i.e., the resultant is zero.

Thus the forces are in equilibrium.

Hence, the necessary and sufficient conditions of equilibrium of a number of forces acting a point are :

The algebraic sum of their resolved parts along each of two perpendicular directions must vanish separately.

1.17.1. Working rule for solving problems on equilibrium of a number of concurrent forces :

1. Choose two convenient directions at right angles to each other in the figure as axes, usually horizontal and vertical.
2. Resolve the forces along each of these directions and equate them separately equal to zero.
3. Solve these two equations and obtain the required result.

Note :

For problems where there are only three forces acting on a particle, use **Lami's Theorem** to get the desired result

SOLVED EXAMPLES

Example 1.

A string ABCD is suspended from two fixed points A and D. It carries weights of 30 kg and W kg respectively at two points B and C in it. The inclination to the vertical of AB is 30° and that of CD is 60°, the angle BCD being 120°. Find W and the tension in the different parts of the string.

Solution. Let T_1, T_2 and T_3 be the tensions in the portions AB, BC and CD respectively of the string ABCD.

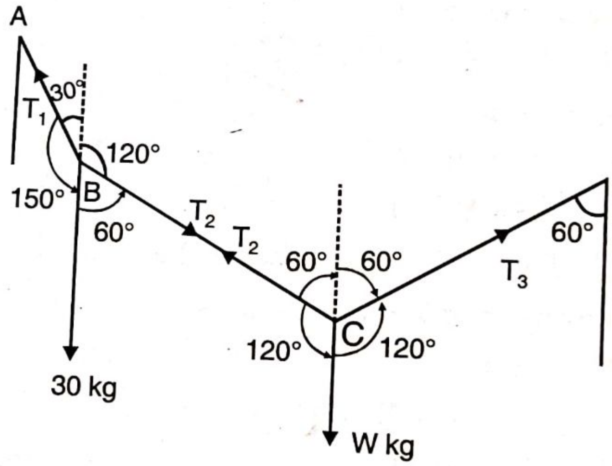


Fig. 1.40

Forces acting in equilibrium at B are

(i) Tension T_1 in the string BA

(ii) Tension T_2 in the string BC

(iii) Weight 30 kg acting vertically downwards.

∴ By Lami's Theorem,

$$\frac{T_1}{\sin 60°} = \frac{T_2}{\sin 150°} = \frac{30}{\sin 150°}$$

i.e.,

$$\frac{T_1}{\frac{\sqrt{3}}{2}} = \frac{T_2}{\sin 30°} = \frac{30}{\sin 30°}$$

or

$$\frac{T_1}{\frac{\sqrt{3}}{2}} = \frac{T_2}{\frac{1}{2}} = \frac{30}{\frac{1}{2}}$$

∴ $T_1 = 30 \times \sqrt{3} = 30\sqrt{3}$ kg and $T_2 = 30$ kg

Again, forces acting in the equilibrium at C are

(*i*) Tension T_2 in the string CB

(*ii*) Tension T_3 in the string CD

(*iii*) Weight W kg acting vertically downwards

$\therefore$ By Lami's Theorem,

$$\frac{T_2}{\sin 120°} = \frac{T_3}{\sin 120°} = \frac{W}{\sin 120°}$$

i.e., $T_2 = T_3 = W =$ **30 kg.** $[\because T_2 = 30\ kg]$

Example 2.

A string of length l is fastened to two points A, B at the same level and at a distance 'a' apart. A ring of weight W can slide on the string and a horizontal force P is applied to it such that it is in equilibrium vertically below B. Show that

$$P = \frac{aW}{l} \text{ and tension of the string is } \frac{W(l^2 + a^2)}{2l^2}.$$ [M.D.U. 2014]

Solution. In equilibrium position, let the ring of weight W be at C where the horizontal force P is applied to it towards right so that it is vertically below B.

Since the string passes through the smooth ring, the tensions in CA and CB, the two portions of the string are the same. Let it be T. Let $\angle BCA = \theta$.

The ring is in equilibrium under the action of the following forces :

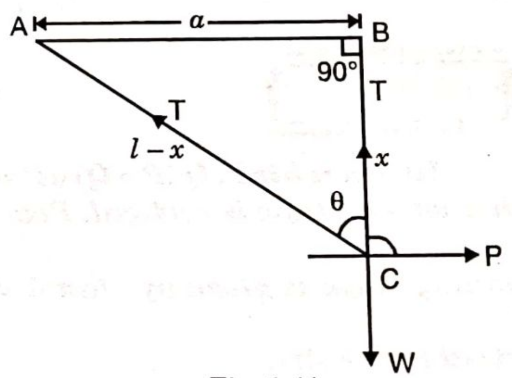


Fig. 1.41

(*i*) Tension T along CA

(*ii*) Tension T along CB

(*iii*) Weight W acting vertically downwards

(*iv*) Horizontal force P towards the right.

Resolving the forces horizontally and vertically, we have

$$P + T\cos(90° + \theta) = 0 \quad i.e., \quad P = T\sin\theta \quad ...(1)$$

and

$$T + T\cos\theta = W \quad i.e., \quad T(1 + \cos\theta) = W \quad ...(2)$$

Now, $AB = a$, $AC + CB = l$

Let $CB = x$, then $AC = l - x$

In right angled ΔABC,

$$AB^2 + BC^2 = CA^2$$

i.e.,

$$a^2 + x^2 = (l - x)^2$$

or $$a^2 = l^2 - 2lx \Rightarrow x = \frac{l^2 - a^2}{2l}$$

$\therefore$ $$l - x = l - \frac{l^2 - a^2}{2l} = \frac{l^2 + a^2}{2l}$$

Also, $$\cos\theta = \frac{x}{l - x} \quad \text{and} \quad \sin\theta = \frac{a}{l - x}$$

From (2), $$T\left(1 + \frac{x}{l - x}\right) = W$$

or $$T.\frac{l}{l - x} = W$$

$\therefore$ $$T = \frac{l - x}{l}.W = \frac{(l^2 + a^2)}{2l^2}.W$$

and $$P = T\sin\theta = \frac{l - x}{l}.W.\frac{a}{l - x}$$

i.e., $$P = \frac{aW}{l}.$$

Example 3.

Two weights P, Q (P > Q) attached to the ends of a string rest on a smooth circula[r] disc whose plane is vertical. Prove that the inclination θ to the horizontal of the lin[e] joining them is given by, $\tan\theta = \frac{P - Q}{P + Q}\tan\alpha$, ***where 2α is the angle subtended b[y] PQ at the centre.***

Solution. Let the weights P and Q be at rest at A and B on the circular disc wit[h] centre O

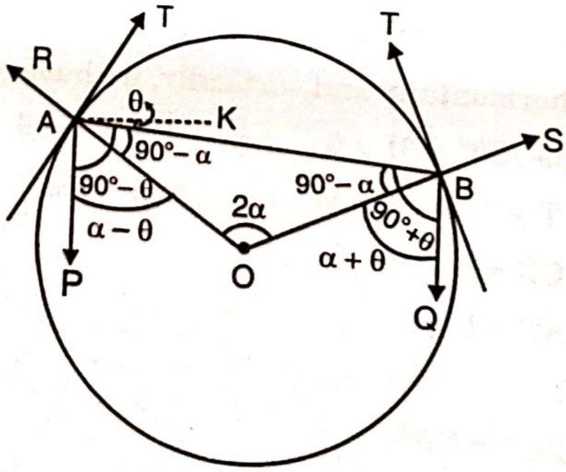


Fig. 1.42

The string ACB lies along the circumference of the disc.

The disc being smooth, the tensions at A and B will be along the tangents at A and B.

Let R and S be the reactions at A and B

Since AB is inclined at an angle θ to the horizontal, so it makes an angle $90° - \theta$ with the vertical

$\therefore \quad \angle PAB = 90° - \theta$ and $\angle QBA = 90° + \theta$.

Since $\quad \angle AOB = 2\alpha$ (given)

$$\therefore \quad \angle OAB = \angle OBA = \frac{180° - 2\alpha}{2} = 90° - \alpha.$$

$\angle PAK = 90°; \quad \angle BAK = \theta$

$\therefore \quad \angle PAO = 90° - (\theta + 90° - \alpha) = \alpha - \theta$

Similarly, $\quad \angle OBQ = \alpha + \theta$

Since the forces P, T and R, at A are in equilibrium

$\therefore$ By Lami's Theorem, we get

$$\frac{P}{\sin 90°} = \frac{T}{\sin[180° - (\alpha - \theta)]}$$

$$\therefore \quad P = \frac{T}{\sin(\alpha - \theta)} \qquad ...(1)$$

Again for equilibrium at B, by Lami's Theorem

$$\frac{Q}{\sin 90°} = \frac{T}{\sin[180° - (\alpha + \theta)]}$$

$$\therefore \quad Q = \frac{T}{\sin(\alpha + \theta)} \qquad ...(2)$$

$\therefore$ From (1) and (2), $\quad \dfrac{P}{Q} = \dfrac{\sin(\alpha + \theta)}{\sin(\alpha - \theta)}$

Applying componendo and dividendo, we have

$$\frac{P+Q}{P-Q} = \frac{\sin(\alpha+\theta) + \sin(\alpha-\theta)}{\sin(\alpha+\theta) - \sin(\alpha-\theta)}$$

$$= \frac{2\sin\alpha\cos\theta}{2\cos\alpha\sin\theta} = \frac{\tan\alpha}{\tan\theta}$$

Hence, $\quad \tan\theta = \dfrac{\mathbf{P-Q}}{\mathbf{P+Q}}\,\mathbf{\tan}\,\alpha.$

Example 4.

Three equal strings of no sensible weight are knotted together to form an equilateral triangle ABC and a weight W is suspended from A. If the triangle and the weight be supported with BC horizontal by means of two strings at B and C each at an angle of 135° with BC, show that the tension in BC is $\frac{W}{6}(3-\sqrt{3})$.

Solution. Since ABC is an equilateral triangle and line of action of W bisects $\angle BAC$, the tensions in AB and AC are equal, say T_1

$\therefore$ W is to be equal in magnitude to resultant of tensions along AB and AC.

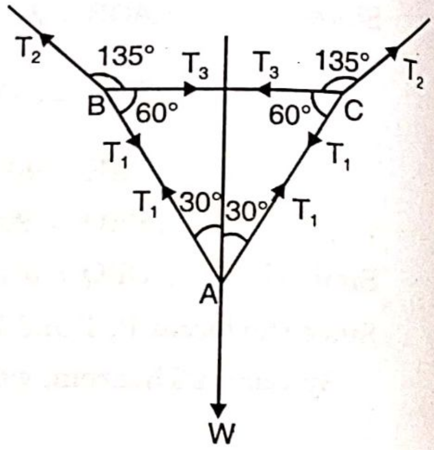


Fig. 1.43

$$\therefore \quad W = 2T_1 \cos\frac{60°}{2}$$

$$= 2T_1\cos 30° = \sqrt{3}\,T_1 \quad \Rightarrow \quad T_1 = \frac{W}{\sqrt{3}} \quad ...(1)$$

The other tensions in the strings are as shown in fig. 1.43. For the equilibrium at B, by Lami's theorem, we have

$$\frac{T_3}{\sin 165°} = \frac{T_1}{\sin 135°} \quad \Rightarrow \quad T_3 = T_1\frac{\sin 165°}{\sin 135°} = \frac{W}{\sqrt{3}}\frac{\sin 15°}{\sin 45°} \qquad \text{[From (1)]}$$

$$= \frac{W}{\sqrt{3}}\cdot\frac{\frac{\sqrt{3}-1}{2\sqrt{2}}}{\frac{1}{\sqrt{2}}} = \frac{W(\sqrt{3}-1)}{2\sqrt{3}}$$

Hence, $\quad T_3 = \frac{W}{6}(3-\sqrt{3})$.

Note :

Result (1) can also be obtained by applying Lami's theorem at A as follows :
Since the forces T_1, T_1, W acting at A are in equilibrium, therefore by Lami's theorem

$$\frac{W}{\sin 60°} = \frac{T_1}{\sin 150°} = \frac{T_1}{\sin 150°} \Rightarrow W = \frac{T_1 \sin 60°}{\sin 150°} = \frac{\sqrt{3}}{2}\times 2T_1 \Rightarrow T_1 = \frac{W}{\sqrt{3}}.$$

Example 5.

Two weights P and Q are suspended from a fixed point O by string OA, OB and are kept apart by a light rod AB. If the string makes angles α and β with the rod, show that the angle θ which the rod makes with the vertical is given by

$$\tan\theta = \frac{P+Q}{P\cot\alpha - Q\cot\beta}.$$

[K.U. 2014]

Solution. Let T_1, T_2 be the tensions in the string AO and BO respectively. The thrust S of the rod at A and B will be equal and opposite in directions as shown in fig. 1.44

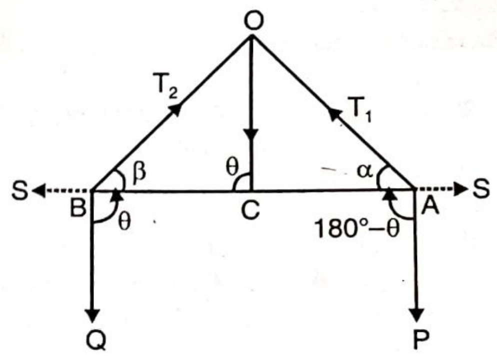


Fig. 1.44

Let the rod BA make angle θ with the vertical OC.

Also, $\angle BAO = \alpha$ and $\angle OBA = \beta$

Since the forces T_1, S and P acting at A are in equilibrium

∴ By Lami's theorem, we have

$$\frac{T_1}{\sin\theta} = \frac{S}{\sin(180° - \theta + \alpha)} = \frac{P}{\sin(180° - \alpha)}$$

∴ $$S = \frac{P\sin(\theta - \alpha)}{\sin\alpha} \quad ...(1)$$

Similarly, at B, by Lami's theorem, we get

$$\frac{T_2}{\sin(180° - \theta)} = \frac{S}{\sin(\theta + \beta)} = \frac{Q}{\sin(180° - \beta)}$$

or $$\frac{T_2}{\sin\theta} = \frac{S}{\sin(\theta+\beta)} = \frac{Q}{\sin\beta}$$

∴ $$S = \frac{Q\sin(\theta+\beta)}{\sin\beta} \quad ...(2)$$

∴ From (1) and (2), we have

$$\frac{P\sin(\theta-\alpha)}{\sin\alpha} = \frac{Q\sin(\theta+\beta)}{\sin\beta}$$

or $$P\left[\frac{\sin\theta\cos\alpha - \cos\theta\sin\alpha}{\sin\alpha}\right] = Q\left[\frac{\sin\theta\cos\beta + \cos\beta\sin\theta}{\sin\beta}\right]$$

or $$P[\sin\theta\cot\alpha - \cos\theta] = Q[\sin\theta\cot\beta + \cos\theta]$$

Dividing both sides by cos θ, we have

$$P[\tan\theta\cot\alpha - 1] = Q[\tan\theta\cot\beta + 1]$$

or $$\tan\theta[P\cot\alpha - Q\cot\beta] = P + Q$$

∴ $$\tan\theta = \frac{P+Q}{P\cot\alpha - Q\cot\beta}.$$

EXERCISE 1.6

1. ABCDEF is a regular hexagon. Forces of 4, P, 10, Q and 6 N acting along AB, CA, AD, AE and FA respectively are in equilibrium. Find the magnitude of P and Q. [M.D.U. 1991]

2. ABCD is a string suspended from points A and D and it carries a weight 20 kg at B and a weight W kg at C. The portion AB is horizontal and BC and CD make angles of 60° and 30° with the vertical. Find W and the tensions in the three portions of the string.

3. A string ABCD attached to two fixed points A and D, has two equal weights, W each, knotted to it at B and C and rests with the portions AB and CD inclined at angles 30° and 60° respectively to the vertical. Find the tensions of the portions of the string and the inclination of BC to the vertical.

4. Two equal weights, each of W kg are attached to the ends of a thin string which passes over three pegs in a wall arranged in the form of an isosceles triangle with the base horizontal and vertical angle 150°. Find the magnitude and direction of pressure of each peg.

5. A string of length 2 m is attached to two points A and B at the same level and at a distance 1 m apart. A ring of 10 kg slung on the string is acted on by a horizontal force P which holds it in equilibrium vertically below B. Find the tension in the string and magnitude of P. [K.U. 1997]

6. The ends of an inelastic and weightless string 21 cms long are attached to two points 15 cms apart in the same horizontal line. A weight of 21 kg is attached to the string 9 cms from one end. Find the tension in each portion of the string.

7. A string is tied to two pcints in a horizontal plane; a ring of weight W can slide freely along the string and is pulled by a horizontal force P. If in the position of equilibrium the portions of the strings be inclined at angle 45° and 60° to the horizontal, find the value of P.

ANSWERS

1. $P = 8\sqrt{3}$ N ; $Q = 2\sqrt{3}$ N

2. $T_1 = 20\sqrt{3}$ kg ; $T_2 = W = 40$ kg; $T_3 = 40\sqrt{3}$ kg

3. $W\sqrt{3}$; W; W; 60°.

4. Pressure on each base peg = $2W \cos 52\frac{1}{2}^\circ$ inclined at $37\frac{1}{2}^\circ$ with the base ;

   Pressure on 3rd peg = $\frac{W}{2}(\sqrt{6}-\sqrt{2})$ along downward vertical.

5. $T = 6\frac{1}{4}$ kg . wt.; $P = 5$ kg wt.

6. $16\frac{4}{5}$ kg ; $12\frac{3}{5}$ kg

7. $\frac{\sqrt{2}-1}{\sqrt{3}+\sqrt{2}}W$.

1.18. EQUILIBRIUM OF BODIES PLACED ON A SMOOTH INCLINED PLANE

***(a)** A body of weight W is placed on a smooth inclined plane of inclination α and is supported by a force acting horizontally. To find the force and the reaction of the plane.*

The plane being smooth, the normal reaction R of the plane on the body placed at O is along the perpendicular to the inclined plane of inclination α. Let P be the horizontal force supporting the body.

Now, the body is in equilibrium under the action of the following forces acting at O :

(*i*) W, the weight of the body, acting vertically downwards

(*ii*) Force P acting horizontally along OD

(*iii*) Normal reaction R along OC.

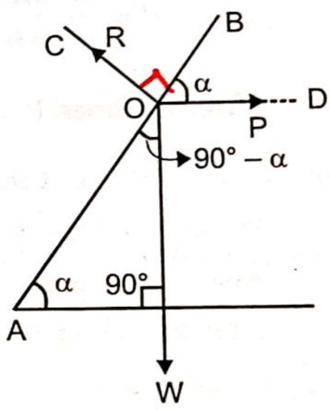


Fig. 1.45

Resolving the forces horizontally along OD, we have

$$P + R\cos(90° + \alpha) + W\cos 90° = 0$$

i.e., $$P - R\sin\alpha = 0$$

$\therefore$ $$P = R\sin\alpha \qquad ...(1)$$

Resolving the forces perpendicular to OD, we have

$$P\cos 90° + R\sin(90° + \alpha) - W = 0$$

$\therefore$ $$R\cos\alpha = W \quad \Rightarrow \quad R = \frac{W}{\cos\alpha} = W\sec\alpha$$

Putting the value of R in (1), we get

$$P = \frac{W}{\cos\alpha}(\sin\alpha) = W\tan\alpha.$$

Remark :

The above result can also be obtained by applying Lami's theorem at O.

Applying Lami's theorem at O, we have

$$\frac{R}{\sin 90°} = \frac{P}{\sin(180° - \alpha)} = \frac{W}{\sin(90° + \alpha)} \Rightarrow \frac{R}{1} = \frac{P}{\sin\alpha} = \frac{W}{\cos\alpha}$$

From first and third members, we have $R = W\sec\alpha$

From second and third members, we have $P = W\tan\alpha$.

***(b)** A body of weight W is placed on a smooth inclined plane of inclination α and is kept in equilibrium by a force P which acts in a vertical plane in a direction making an angle θ with the plane i.e., with the line of greatest slope through the body. To find the magnitude of P and the normal reaction.*

Let the force P act along OC making an angle θ with the line of greatest slope. The plane being smooth, the reaction R of the plane of the body at O is along the normal to the inclined plane.

Now the following three forces acting at O are in equilibrium :

(*i*) The force P along OC.

(*ii*) Normal reaction R along OD

(*iii*) W, the weight of the body, acting vertically downwards.

Hence, by Lami's theorem at O, we have

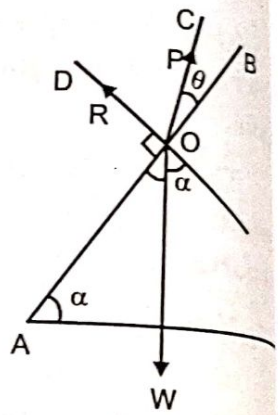


Fig. 1.46

$$\frac{P}{\sin(180° - \alpha)} = \frac{R}{\sin(90° + \alpha + \theta)} = \frac{W}{\sin(90° - \theta)}$$

or
$$\frac{P}{\sin\alpha} = \frac{R}{\cos(\alpha + \theta)} = \frac{W}{\cos\theta}$$

$$\therefore \quad P = \frac{W\sin\alpha}{\cos\theta} \quad \text{and} \quad R = \frac{W\cos(\alpha+\theta)}{\cos\theta}$$

Cor. 1. Since $P = \dfrac{W\sin\alpha}{\cos\theta}$, therefore P is least if cos θ is greatest *i.e.*, when cos θ = 1 or when θ = 0, and then P = W sin α.

Hence the least value of the force is W sin α and acts along the plane.

Also in this case R = W cos α.

Cor. 2. When θ = – α *i.e.*, when the force act horizontally

$$P = \frac{W\sin\alpha}{\cos(-\alpha)} = \frac{W\sin\alpha}{\cos\alpha} = W\tan\alpha.$$

[*Ref. Art. 1.18(a)*]

SOLVED EXAMPLES

Example 1. ***Two forces P and Q acting parallel to the length and the base of a smooth inclined plane would, each of them, singly support a weight W on the plane. Prove that***

$$\frac{1}{P^2} - \frac{1}{Q^2} = \frac{1}{W^2}.$$

Solution. Let R be the normal reaction acting at O and P be the force applied along AB. In the second case, let S be the normal reaction and the force Q acts horizontally making an angle α with AB.

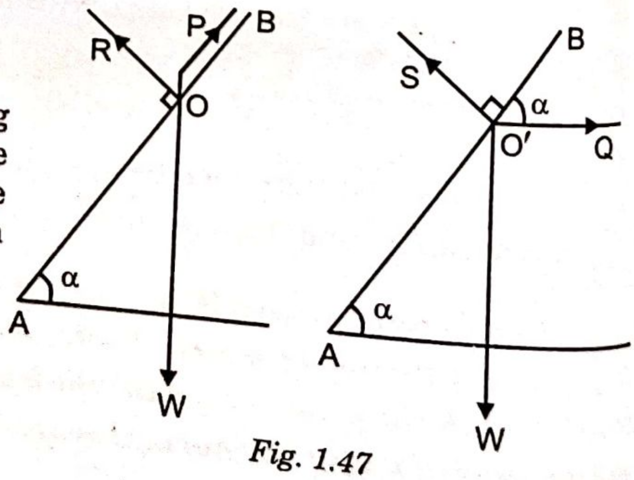


Fig. 1.47

Case I. By Lami's theorem at O, we have

$$\frac{P}{\sin(180° - \alpha)} = \frac{W}{\sin 90°} = \frac{R}{\sin(90° + \alpha)}$$

From first two members,

$$\frac{P}{\sin \alpha} = W \Rightarrow \operatorname{cosec} \alpha = \frac{W}{P} \quad ...(1)$$

Case II. By Lami's theorem at O, we get

$$\frac{Q}{\sin (180° - \alpha)} = \frac{W}{\sin (90° + \alpha)} = \frac{S}{\sin 90°}$$

or
$$\frac{Q}{\sin \alpha} = \frac{W}{\cos \alpha} \Rightarrow Q = \frac{W \sin \alpha}{\cos \alpha}$$

or
$$\cot \alpha = \frac{W}{Q} \quad ...(2)$$

Squaring (1) and (2) and subtracting, we have

$$\operatorname{cosec}^2 \alpha - \cot^2 \alpha = \frac{W^2}{P^2} - \frac{W^2}{Q^2} \quad \text{or} \quad \frac{1}{P^2} - \frac{1}{Q^2} = \frac{1}{W^2}.$$

Example 2.

Two weights P and Q rest on each of two smooth planes placed back to back, of inclination α and β, being connected by a string which runs horizontally from one to the other. Show that P tan α = Q tan β.

If the string passes over a smooth pulley at the top of inclined planes, show that P sin α = Q sin β.

Solution. (*i*) Let AB and AC be the planes inclined at angles α and β to the horizontal. Let the weights P and Q be at O and O′ respectively.

Let R and R′ be the normal reactions at O and O′ and T be the tension in the string.

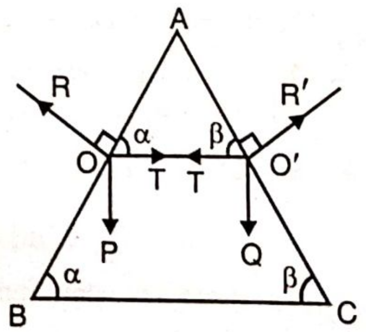


Fig. 1.48

The forces P, R and T, acting at O, are in equilibrium

∴ By Lami's theorem, we have

$$\frac{P}{\sin (90° + \alpha)} = \frac{T}{\sin (180° - \alpha)}$$

or
$$\frac{P}{\cos \alpha} = \frac{T}{\sin \alpha} \Rightarrow T = P \tan \alpha \quad ...(1)$$

Similarly, by Lami's theorem at O′, we have

$$\frac{Q}{\sin (90° + \beta)} = \frac{T}{\sin (180° - \beta)}$$

or
$$\frac{Q}{\cos \beta} = \frac{T}{\sin \beta} \Rightarrow T = \frac{Q \sin \beta}{\cos \beta} = Q \tan \beta \quad ...(2)$$

$\therefore$ From (1) and (2), we have $P \tan \alpha = Q \tan \beta$

(*ii*) In the 2nd case, let S and S′ be the normal reactions. Since the pulley is smooth, the tension in the string is the same throughout and let it be T′.

By Lami's theorem at O, we have

$$\frac{P}{\sin 90°} = \frac{T'}{\sin (180° - \alpha)}$$

$$\therefore \quad T' = P \sin \alpha \qquad ...(3)$$

Similarly, by Lami's theorem at O′,

$$\frac{Q}{\sin 90°} = \frac{T'}{\sin (180° - \beta)}$$

$$\Rightarrow \quad T' = Q \sin \beta \qquad ...(4)$$

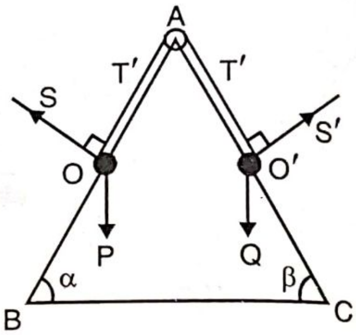


Fig. 1.49

$\therefore$ From (3) and (4), we have

$$P \sin \alpha = Q \sin \beta.$$

EXERCISE 1.7

1. Two forces $\frac{P}{\sqrt{2}}$ and Q, acting parallel to the length and base of a smooth inclined plane respectively would, each of them, singly support a weight W on the plane. Prove that P^2 is the harmonic mean between Q^2 and W^2
2. A force P when applied along the smooth inclined plane supports a weight W_1 and when applied horizontally balances a weight W_2 on the same plane. Prove that
$$P^2 = W_1^2 - W_2^2$$
3. Show that the smallest force which will keep a body in equilibrium on a smooth inclined plane must act along the plane.
4. A horizontal force of 15 kg. wt. can support a weight 20 kg on a smooth inclined plane. What force acting parallel to the plane will support the same weight ?
5. A particle of weight 50 kg is placed on a smooth plane inclined to the horizontal at an angle $\sin^{-1}\frac{3}{5}$. Find the magnitude of the force (*i*) acting parallel to the plane (*ii*) acting horizontally, required to keep the body in equilibrium.
6. A body is sustained on a smooth plane by two forces each equal to half of its weight, one acting horizontally and the other along the plane. Find the inclination of the plane.

ANSWERS

4. 12 kg. wt. **5.** (*i*) 30 kg. wt. (*ii*) 37·5 kg.wt. **6.** $2 \tan^{-1}\frac{1}{2}$.