2012

44 (5c)

(c) A particle is projected vertically upwards from the earth's surface with a velocity just sufficient to carry it to infinity. Prove that the time it takes to reach a height h is

$$\frac{1}{3}\sqrt{\left(\frac{2a}{g}\right)}\left[\left(1+\frac{h}{a}\right)^{3/2}-1\right].$$ 8

45 (5d)

(d) A triangle ABC is immersed in a liquid with the vertex C in the surface and the sides AC, BC equally inclined to the surface. Show that the vertical C divides the triangle into two others, the fluid pressures on which are as $b^3+3ab^2 : a^3+3a^2b$ where a and b are the sides BC & AC respectively. 8

46 (6c)

(c) A particle is projected with a velocity u and strikes at right angle on a plane through the plane of projection inclined at an angle β to the horizon. Show that the time of flight is

$$\frac{2u}{g\sqrt{(1+3\sin^2\beta)}},$$

range on the plane is $\dfrac{2u^2}{g}\cdot\dfrac{\sin\beta}{1+3\sin^2\beta}$

and the vertical height of the point struck is

$\dfrac{2u^2\sin^2\beta}{g(1+3\sin^2\beta)}$ above the point of projection.

47 (7a)

7. (a) A particle is moving with central acceleration $\mu\left[r^5 - c^4 r\right]$ being projected from an apse at a distance c with velocity $\sqrt{\left(\frac{2\mu}{3}\right)c^3}$, show that its path is a curve, $x^4 + y^4 = c^4$. 13

48 (7b)

(b) A thin equilateral rectangular plate of uniform thickness and density rests with one end of its base on a rough horizontal plane and the other against a small vertical wall. Show that the least angle, its base can make with the horizontal plane is given by

$$\cot\theta = 2\mu + \frac{1}{\sqrt{3}}$$

μ, being the coefficient of friction. 14

49 (7c)

(c) A semicircular area of radius a is immersed vertically with its diameter horizontal at a depth b. If the circumference be below the centre, prove that the depth of centre of pressure is

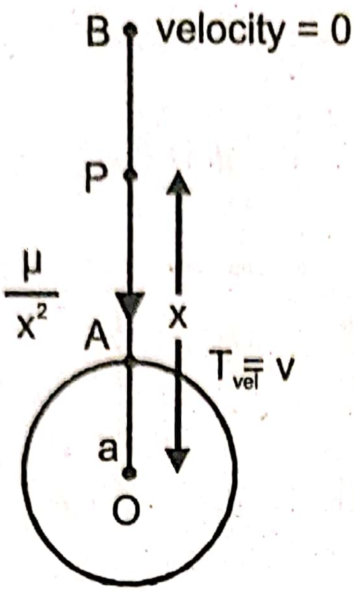
$$\frac{1}{4}\frac{3\pi\left(a^2+4b^2\right)+32ab}{4a+3\pi b}.$$ 13.

50 (8c)

(c) A heavy elastic string, whose natural length is $2\pi a$, is placed round a smooth cone whose axis is vertical and whose semi vertical angle is α. If W be the weight and λ the modulus of elasticity of the string, prove that it will be in equilibrium when in the form of a circle whose radius is

$$a\left(1+\frac{W}{2\pi\lambda}\cot\alpha\right).$$ 10

5. (c)



Let O be the centre of the earth and A be the point of projection on the earth's surface.

If P be the position of the particle at any time t, such that OP = x, then the acceleration at P = $\frac{\mu}{x^2}$ directed towards O.

∴ The equation of motion of the particle at P is

$$\frac{d^2x}{dt^2} = \frac{-\mu}{x^2}$$

(Negative sign indicates that acceleration acts in the direction of x decreasing.)

But at the point A, on the surface of the earth,

$$x = a \text{ and } \frac{d^2x}{dt^2} = -g$$

$$\therefore \quad -g = \frac{-\mu}{a^2} \quad \text{or } \mu = a^2 g$$

$$\therefore \quad \frac{d^2x}{dt^2} = \frac{-a^2 g}{x^2}$$

Multiplying by $2\left(\frac{dx}{dt}\right)$ and integrating with respect to (t) we get

$$\left(\frac{dx}{dt}\right)^2 = \frac{2a^2 g}{x} + C$$

where C is a constant

But when $x \to \infty$, $\frac{dx}{dt}$(velocity) $\to 0$

$\therefore$ $C = 0$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2 = \frac{2a^2 g}{x}$$

(Here +ve sign is taken because the particle is moving in the direction of x increasing)

$$\Rightarrow \quad \frac{dx}{dt} = a\sqrt{\frac{2g}{x}}$$

Separating the variables, we have

$$dt = \frac{1}{a\sqrt{2g}}\sqrt{x}\,dx$$

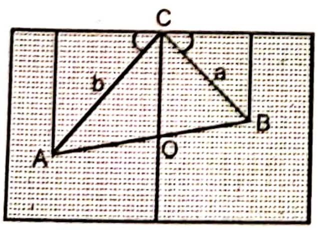
Integrating between the limits $x = a$ to $x = a + h$, the required time t to reach height h is given by

$$t = \frac{1}{a\sqrt{2g}}\int_a^{a+h}\sqrt{x}\,dx = \frac{1}{a\sqrt{2g}}\left[\frac{2}{3}x^{3/2}\right]_a^{a+h}$$

$$= \frac{1}{3a}\sqrt{\frac{2}{g}}\left[(a+h)^{3/2} - a^{3/2}\right]$$

$$= \frac{1}{3}\sqrt{\frac{2a}{g}}\left[\left(1+\frac{h}{a}\right)^{3/2} - 1\right].$$

5. (d)



Let the vertical through C meets AB at O.

then $\angle ACO = \angle BCO = \frac{1}{2}\angle C$

Area of $\Delta AOC = \frac{1}{2}$ AC . OC sin $\angle ACO$

Area of $\Delta BOC = \frac{1}{2}$ BC . OC sin $\angle BCO$

The depth of the centre of gravity (C.G.) of ΔAOC below the surface of the liquid

$$= \frac{1}{3}(AC\cos\angle ACO + OC)$$

and the depth of the C.G of ΔBOC below the surface of the liquid

$$= \frac{1}{3}(BC\cos\angle BCO + OC)$$

$$\therefore \quad \frac{\text{Pressure on } \Delta AOC}{\text{Pressure on } \Delta BOC}$$

$$= \frac{\frac{1}{2}AC\cdot OC\sin\angle ACO\cdot\frac{1}{3}(AC\cos\angle ACO + OC)\cdot w}{\frac{1}{2}BC\cdot OC\sin\angle BCO\cdot\frac{1}{3}(BC\cos\angle BCO + OC)\cdot w}$$

$$= \frac{\left(\frac{1}{2}b\,OC\sin\frac{C}{2}\right)\left(\frac{1}{3}\left(b\cos\frac{C}{2}+OC\right)\right)}{\left(\frac{1}{2}a\,OC\sin\frac{C}{2}\right)\left(\frac{1}{3}\left(a\cos\frac{C}{2}+OC\right)\right)}$$

$$= \frac{b\left(b\cos\frac{C}{2}+OC\right)}{a\left(a\cos\frac{C}{2}+OC\right)}$$

From Δ's BCO and ACO, we have

$$\frac{CO}{\sin B} = \frac{OB}{\sin\frac{C}{2}} \text{ and } \frac{CO}{\sin A} = \frac{AO}{\sin\frac{C}{2}} \quad ...(1)$$

$$\text{Also } \frac{AO}{b} = \frac{OB}{a} = \frac{AO+OB}{b+a} = \frac{c}{b+a} \quad ...(2)$$

$\therefore$ The required ratio

$$= \frac{b\left(b\cos\frac{C}{2}+\frac{OB\sin B}{\sin\frac{C}{2}}\right)}{a\left(a\cos\frac{C}{2}+\frac{AO\sin A}{\sin\frac{C}{2}}\right)} \quad \text{[using (1)]}$$

$$= \frac{b(b\sin C + 2OB\sin B)}{a(a\sin C + 2OA\sin A)}$$

$$\left(\because \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}\right)$$

$$= \frac{b\left(b\sin C + 2OB\frac{b\sin C}{c}\right)}{a\left(a\sin C + 2OA\, a\frac{\sin C}{c}\right)}$$

$$= \frac{b^2}{a^2}\cdot\left(\frac{c+2OB}{c+2OA}\right) = \frac{b^2}{a^2}\cdot\frac{\left(c+\frac{2ac}{b+a}\right)}{\left(c+\frac{2bc}{b+a}\right)}$$

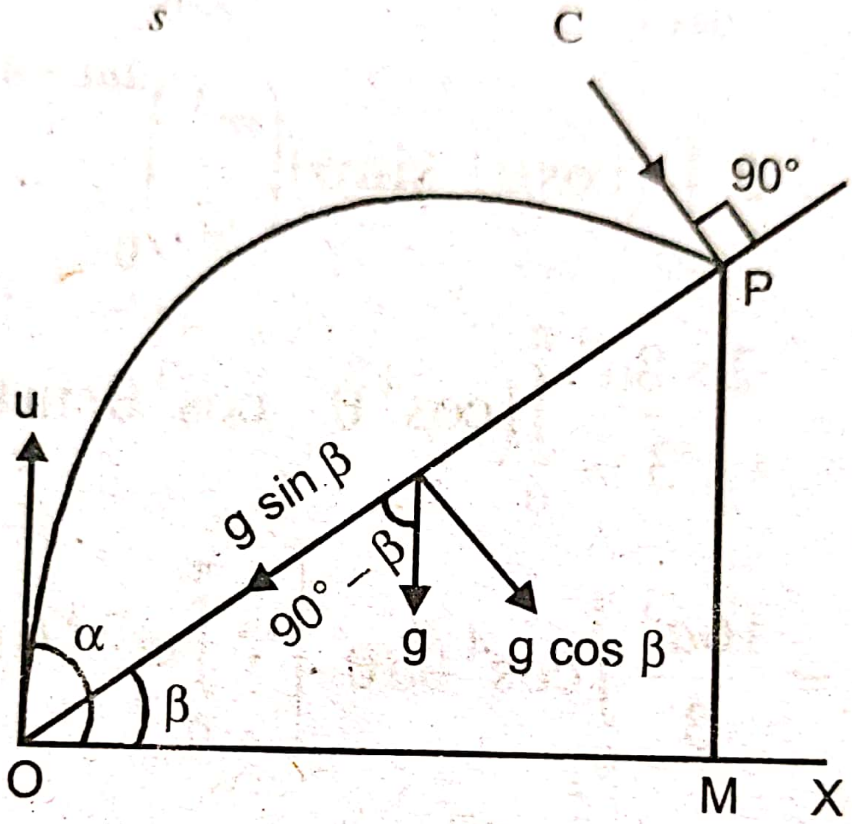
[using (2)]

$$= \frac{b^2}{a^2}\cdot\left[\frac{c(a+b)+2ac}{c(a+b)+2bc}\right]$$

$$= \frac{b^2(3a+b)}{a^2(a+3b)} = \frac{b^3+3ab^2}{a^3+3a^2b}.$$

5. (a) Given $u = x + y + z$, $v = x^2 + y^2 + z^2$

6. (c)



Let O be the point of projection, u be the velocity of projection, α be the angle of projection and P be the point where the particle strikes the plane at right angles. Let T be the time of flight from O to P. Then by the formula for the time of flight on an inclined plane, we have

$$T = \frac{2u \sin(\alpha - \beta)}{g \cos\beta} \qquad ...(1)$$

Since the particle strikes the inclined plane at right angles at P, therefore the velocity of the particle at P along the inclined plane is zero.

Also the resolved part of the velocity of the particle at O along the inclined plane is $u\cos(\alpha-\beta)$ upwards and the resolved part of the acceleration g along the inclined plane is $g \sin\beta$ downwards.

So, considering the motion of the particle from O to P along the inclined plane and using the formula

$v = u + at$, we have

$0 = u\cos(\alpha-\beta) - g\sin\beta T$

$$\text{or, } T = \frac{u\cos(\alpha-\beta)}{g\sin\beta} \quad ...(2)$$

Equating the values of T from (1) and (2), we have

$$\frac{2u\sin(\alpha-\beta)}{g\cos\beta} = \frac{u\cos(\alpha-\beta)}{g\sin\beta}$$

$$\text{or, } \tan(\alpha-\beta) = \frac{1}{2}\cot\beta \quad ...(3)$$

The condition for striking the plane at right angles.

(*i*) To prove $T = \dfrac{2u}{g\sqrt{1+3\sin^2\beta}}$

Proof: From (2) we have

$$T = \frac{u}{g\sin\beta}\cos(\alpha-\beta) = \frac{u}{g\sin\beta\sec(\alpha-\beta)}$$

$$= \frac{u}{g\sin\beta\sqrt{1+\tan^2(\alpha-\beta)}}$$

$$= \frac{u}{g\sin\beta\sqrt{1+\frac{1}{4}\cot^2\beta}}$$

[substituting value from (3)]

$$= \frac{2u\sin\beta}{g\sin\beta\sqrt{4\sin^2\beta+\cos^2\beta}}$$

$$= \frac{2u}{g\sqrt{\sin^2\beta+\cos^2\beta+3\sin^2\beta}}$$

$$\therefore T = \frac{2u}{g\sqrt{1+3\sin^2\beta}}.$$

(*ii*) Range, R on the plane $= \dfrac{2u^2}{g}\dfrac{\sin\beta}{1+3\sin^2\beta}$

Proof: Let R be the range on the inclined plane then R = OP considering the motion from O to P along the inclined plane and using the formula $v^2 = u^2 + 2as$, we have

$0 = u^2\cos^2(\alpha-\beta) - 2g\sin\beta R$

$$\text{or, } R = \frac{u^2\cos^2(\alpha-\beta)}{2g\sin\beta} = \frac{u^2}{2g\sin\beta\sec^2(\alpha-\beta)}$$

$$= \frac{u^2}{2g\sin\beta\left[1+\tan^2(\alpha-\beta)\right]}$$

$$= \frac{u^2}{2g\sin\beta\left[1+\frac{1}{4}\cot^2\beta\right]} \quad \text{[From (3)]}$$

$$= \frac{4u^2\sin^2\beta}{2g\sin\beta(4\sin^2\beta+\cos^2\beta)}$$

$$\text{Hence, Range, } R = \frac{2u^2\sin\beta}{g(1+3\sin^2\beta)}.$$

(*iii*) The vertical height of the point struck is

$$\frac{2u^2\sin^2\beta}{g(1+3\sin^2\beta)}$$

Proof: The vertical height of P above O = PM

$$= OP\sin\beta = R\sin\beta = \frac{2u^2\sin^2\beta}{g(1+3\sin^2\beta)}.$$

6. (d) Solve $\dfrac{d^4y}{dx^4} + 2\dfrac{d^2y}{dx^2} + y = x^2\cos x$

Let $D \equiv \dfrac{d}{dx}$, then the given differential equation becomes

$(D^4 + 2D^2 + 1)y = x^2\cos x$

This equation is the differential equation of first order with constant coefficients. It is solved by the following method.

The auxiliary equation is

required solution.

7. (a) Here, the central acceleration,

$$p = \mu[r^5 - c^4 r] = \mu\left[\frac{1}{u^5} - \frac{c^4}{u}\right] \quad \left(\because r = \frac{1}{u}\right)$$

∴ The differential equation of the path is

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{p}{u^2} = \frac{\mu}{u^2}\left[\frac{1}{u^5} - \frac{c^4}{u}\right]$$

$$\Rightarrow u^2 = h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{p}{u^2} = \mu\left[\frac{1}{u^7} - \frac{c^4}{u^3}\right]$$

Multiplying both sides by $2\left(\frac{du}{d\theta}\right)$, we get

$$h^2\left[2\left(\frac{du}{d\theta}\right)u + 2\left(\frac{du}{d\theta}\right)\frac{d^2u}{d\theta^2}\right] = \frac{2p}{u^2}\left(\frac{du}{d\theta}\right)$$

$$\frac{h^2 d}{d\theta}\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \frac{2p}{u^2}\left(\frac{du}{d\theta}\right)$$

Now, integrating above equation with respect to 'θ' we have

$$h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = 2\int \frac{p}{u^2}\,du + A$$

where A is a constant

$$\text{or, } v^2 = h\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = 2\mu\int\left(\frac{1}{u^7} - \frac{c^4}{u^3}\right) + A$$

or, $v^2 = h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]$

$= \mu\left(\frac{-1}{3u^6}+\frac{c^4}{u^2}\right)+\text{A}$ ...(1)

But initially when $r = c$ *i.e.*,

$$u = \frac{1}{c},\quad \frac{du}{d\theta} = 0 \qquad \text{(at apse)}$$

and $\quad v = c^3\sqrt{\frac{2\mu}{3}}$.

∴ From (1) we have

$$\frac{2\mu c^6}{3} = h^2\cdot\frac{1}{c^2} = \mu\left[\frac{-c^6}{3}+c^6\right]+\text{A}$$

$\therefore\ h^2 = \frac{2}{3}\mu c^8,\ \text{A} = 0$

Substituting the values of h^2 and A, in (1) we have

$$\frac{2}{3}\mu c^8\left[u^2+\left(\frac{du}{d\theta}\right)^2\right] = \mu\left[\frac{-1}{3u^6}+\frac{c^4}{u^2}\right]$$

or, $c^8\left(\frac{du}{d\theta}\right)^2 = \frac{-1}{2u^6}+\frac{3c^4}{2u^2}-c^8u^2$

$$= \frac{1}{u^6}\left[\frac{-1}{2}+\frac{3}{2}c^4u^4-c^8u^8\right]$$

$$\Rightarrow c^8\left(\frac{du}{d\theta}\right)^2 = \frac{1}{u^6}\left[\frac{-1}{2}-\left(c^8u^8-\frac{3}{2}c^4u^4\right)\right]$$

$$= \frac{1}{u^6}\left[\frac{-1}{2}-\left(c^4u^4-\frac{3}{4}\right)^2+\frac{9}{16}\right]$$

$$c^8\left(\frac{du}{d\theta}\right)^2 = \frac{1}{u^6}\left[\left(\frac{1}{4}\right)^2-\left(c^4u^4-\frac{3}{4}\right)^2\right]$$

$$\therefore\ c^4u^3\frac{du}{d\theta} = \sqrt{\left(\frac{1}{4}\right)^2-\left(c^4u^4-\frac{3}{4}\right)^2}$$

$$\text{or, } d\theta = \frac{c^4u^3du}{\sqrt{\left(\frac{1}{4}\right)^2-\left(c^4u^4-\frac{3}{4}\right)^2}}$$

Putting $c^4u^4-\frac{3}{4} = z$, so that $4c^4u^3du = dz$, we have

$$4d\theta = \frac{dz}{\sqrt{\left(\frac{1}{4}\right)^2-z^2}}$$

Integrating, $4\theta + \text{B} = \sin^{-1}\left(\frac{z}{1/4}\right)$

$\Rightarrow\ 4\theta + \text{B} = \sin^{-1}(4z)$

where B is a constant

$\Rightarrow\ 4\theta + \text{B} = \sin^{-1}(4c^4u^4 - 3)$

But initially when $u = \frac{1}{c}$, $\theta = 0$

$\therefore\quad \text{B} = \sin^{-1}(1)$

$\Rightarrow\quad \text{B} = \frac{\pi}{2}$

$\therefore\quad 4\theta+\frac{\pi}{2} = \sin^{-1}(4c^4u^4-3)$

$\Rightarrow\ \sin\left(\frac{\pi}{2}+4\theta\right) = 4c^4u^4 - 3$

$\Rightarrow\ \cos 4\theta = 4c^4u^4 - 3$

$\Rightarrow\ 4c^4u^4 = 3 + \cos 4\theta$

$\Rightarrow\ \frac{4c^4}{r^4} = 3 + \cos 4\theta$

$\Rightarrow\ 4c^4 = r^4\,[3 + 2\cos^2 2\theta - 1]$

$= 2r^4\,[1 + \cos^2 2\theta]$

$= 2r^4\left[(\cos^2\theta+\sin^2\theta)^2 + (\cos^2\theta-\sin^2\theta)^2\right]$

$= 4r^4\,(\cos^4\theta + \sin^4\theta)$

$\therefore\quad c^4 = r^4\,(\cos^4\theta + \sin^4\theta)$

$\Rightarrow\quad c^4 = (r\cos\theta)^4 + (r\sin\theta)^4$

$\Rightarrow\quad c^4 = x^4 + y^4$

($\because\ x = r\cos\theta$ and $y = r\sin\theta$)

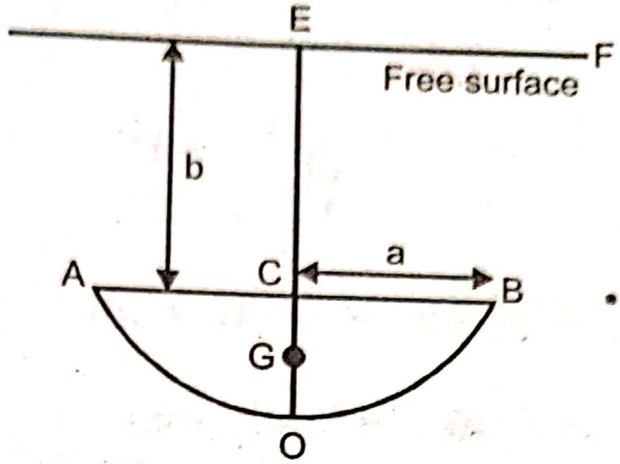
Hence, $x^4 + y^4 = c^4$ is the equation of path.

7. (c) Depth of the centre of pressure of the semi-circular area $= \frac{k^2}{h}$

where k is the radius of gyration about the line EF on the free surface and

h = depth of the CG. of the lemina below EF = EG

k^2 = "k^2" about paralleloxi through G + $(EG)^2$



Now, $CG = \dfrac{4a}{3\pi}$ and hence $EG = b + \dfrac{4a}{3\pi}$

$$\Rightarrow EG = h = \frac{4a + 3b\pi}{3\pi} \quad ...(1)$$

$\therefore$ k^2 = "k^2" about AB $- (CG)^2 + (EG)^2$

$$= \frac{a^2}{4} - \left(\frac{4a}{3\pi}\right)^2 + \left(\frac{4a + 3b\pi}{3\pi}\right)^2$$

$$= \frac{9\pi^2 a^2 + 36b^2\pi^2 + 96ab\pi}{36\pi^2}$$

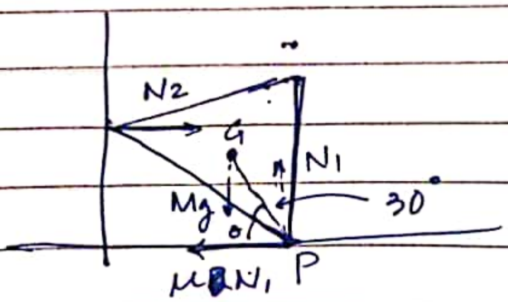
$$\therefore k^2 = \frac{3\pi(a^2 + 4b^2) + 32ab}{12\pi} \quad ...(2)$$

From (1) and (2) we get

Depth of the centre of pressure

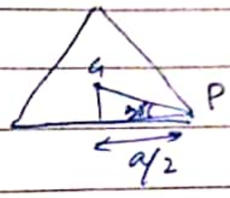
$$= \frac{k^2}{h} = \left(\frac{3\pi(a^2 + 4b^2) + 32ab}{12\pi}\right) \Big/ \left(\frac{4a + 3b\pi}{3\pi}\right)$$

$$= \frac{1}{4}\left(\frac{3\pi(a^2 + 4b^2) + 32ab}{4a + 3\pi b}\right)$$



$N_1 = mg$

$N_2 = \mu N_1$



Taking moment about P.

$$- Mg\, GP \cos(30+\theta) + N_2\, a \sin\theta = 0.$$

$$\cancel{a} N_2 \sin\theta = Mg \frac{\cancel{a}}{\sqrt{3}}\left(\frac{\sqrt{3}}{2}\cos\theta - \frac{1}{2}\sin\theta\right)$$

$\cos 30 \cos\theta$
$- \sin 30 \sin\theta$

$$\mu \cancel{Mg} \sin\theta = \frac{\cancel{Mg}}{\sqrt{3}} \frac{(\sqrt{3}\, c_\theta - \sin\theta)}{2}.$$

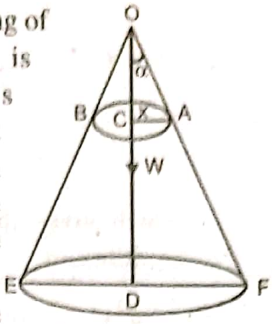
$$2\sqrt{3}\mu\, s_\theta = \sqrt{3}\, c_\theta - s_\theta.$$

$$2\sqrt{3}\mu = \sqrt{3}\cot\theta - 1.$$

$$\cot\theta = \frac{2\sqrt{3}\mu + 1}{\sqrt{3}} = \boxed{2\mu + \frac{1}{\sqrt{3}}}$$

8(c) OEF is a smooth fixed cone of semi-vertical angle α, axis OD of cone being vertical

A heavy elastic string of natural length $2\pi a$ is placed round this cone and suppose it rests in the form of a circle whose centre is C and whose radius CA is x.



The weight W of the string acts at its centre of gravity C.
Let T be the tension in this string.
Give the string a small displacement in which x changes to $x + \delta x$. The point O remains fixed, the point C is slightly displaced. $\angle\alpha$ is fixed and the length of the string slight changed.
We have the length of the string AB in the form of a circle of radius x is $2\pi x$ and so the work done by the tension T of this string is $-T\delta(2\pi x)$.
Also, the depth of the point of application C of the weight W below the fixed point O $= OC = AC \cot\alpha = x \cot\alpha$
and so the work done by the weight W during this small displacement $= W\delta\,(x \cot\alpha)$
Since the reactions at the various points of contact do work, we have by the **Principle of virtual work,**

$$-T\delta(2\pi x) + W\delta(x \cot\alpha) = 0$$

$$\Rightarrow -2\pi\, T\delta x + W \cot\alpha\, \delta x = 0$$

or $\quad (-2\pi T + W \cot\alpha)\delta x = 0$

$$\Rightarrow \quad -2\pi T + W \cot\alpha = 0 \; (\because \delta x \neq 0)$$

or $\quad T = \dfrac{W \cot\alpha}{2\pi}$

2010

Now, by Hooke's law the tension T in the elastic string AB is given by

$$T = \lambda \frac{(2\pi x - 2\pi a)}{2\pi a}$$

$$T = \lambda \frac{x - a}{a}$$

Equating the two values of T, we get

$$\frac{W \cot\alpha}{2\pi} = \lambda \frac{(x - a)}{a}$$

$$\Rightarrow \quad x = a\left(1 + \frac{W}{2\pi\lambda} \cot\alpha\right)$$