§ 1. Definitions.

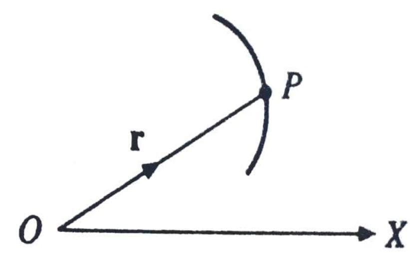
1. **Central force.** *A force whose line of action always passes through a fixed point is called a* **central force**. *The fixed point is known as the* **centre of force**.

2. **Central orbit.** *A* **central orbit** *is the path described by a particle moving under the action of a central force.* The motion of a planet about the sun is an important example of a central orbit.

Theorem. *A central orbit is always a plane curve.*

[Meerut 87, 94; Allahabad 79]

Proof. Take the centre of force O as the origin of vectors. Let P be the position of a particle moving in a central orbit at any time t and let $\vec{OP} = \mathbf{r}$. Then $\dfrac{d^2\mathbf{r}}{dt^2}$ is the expression for the acceleration vector of the particle at the point P. Since the particle moves under the action of a central force with centre at O, therefore the only force acting on the particle at P is along the line OP or PO. So the acceleration vector of P is parallel to the vector $\vec{OP}$.



$$\therefore \frac{d^2\mathbf{r}}{dt^2} \text{ is parallel to } \mathbf{r} \Rightarrow \frac{d^2\mathbf{r}}{dt^2} \times \mathbf{r} = \mathbf{0}$$

$$\Rightarrow \frac{d^2\mathbf{r}}{dt^2} \times \mathbf{r} + \frac{d\mathbf{r}}{dt} \times \frac{d\mathbf{r}}{dt} = \mathbf{0} \qquad \left[\because \frac{d\mathbf{r}}{dt} \times \frac{d\mathbf{r}}{dt} = \mathbf{0}\right]$$

$$\Rightarrow \frac{d}{dt}\left(\frac{d\mathbf{r}}{dt} \times \mathbf{r}\right) = \mathbf{0}$$

$$\Rightarrow \frac{d\mathbf{r}}{dt} \times \mathbf{r} = \text{a constant vector} = \mathbf{h}, \text{ say.} \qquad \text{...(1)}$$

Taking dot product of both sides of (1) with the vector $\mathbf{r}$, we get

$$\mathbf{r} \cdot \left(\frac{d\mathbf{r}}{dt} \times \mathbf{r}\right) = \mathbf{r} \cdot \mathbf{h}.$$

But the left hand member is a scalar triple product involving two equal vectors, and so it vanishes.

$$\therefore \mathbf{r} \cdot \mathbf{h} = 0,$$

which shows that **r** is always perpendicular to a constant vector **h**.

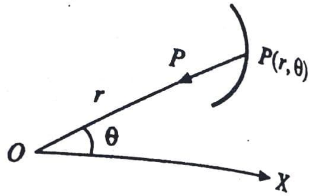
Thus the radius vector OP is always perpendicular to a fixed direction and hence lies in a plane. Therefore the path of P is a plane curve.

§ 2. Differential equation of a central orbit.

A particle moves in a plane with an acceleration which is always directed to a fixed point O in the plane ; to obtain the differential equation of the path.

[Allahabad 78; Agra 85, 88; Meerut 77S, 78, 79, 79S, 82S, 83S, 85P, 86, 87, 89S, 91, 91P; Rohilkhand 85, 86]

Let a particle move in a plane with an acceleration P which is always directed to a fixed point O in the plane. Take the centre of force O as the pole. Let OX be the initial line and (r, θ) the polar co-ordinates of the position P of the moving particle at any instant t.



Since the acceleration of the particle is always directed towards the pole O, therefore the particle has only the radial acceleration and the transverse component of the acceleration of the particle is always zero. So the equations of motion of the particle at the point P are

the radial acceleration *i.e.*, $$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -P, \quad \text{...(1)}$$

(the-ive sign has been taken because the radial acceleration P is in the direction of r decreasing)

and the transverse acceleration *i.e.*, $$\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0. \quad \text{...(2)}$$

From (2), we have $\dfrac{d}{dt}\left(r^2\dfrac{d\theta}{dt}\right) = 0.$

Integrating, we get $$\mathbf{r^2\frac{d\theta}{dt} = \text{constant} = h, \text{ say.}} \quad \text{...(3)}$$

Let $r = 1/u$.

Now from (3), we have

$$d\theta/dt = h/r^2 = hu^2.$$

Also $$\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\cdot\frac{d\theta}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\cdot u^2h = -h\frac{du}{d\theta},$$

and $$\frac{d^2r}{dt^2} = -h\frac{d^2u}{d\theta^2}\cdot\frac{d\theta}{dt} = -h\frac{d^2u}{d\theta^2}(u^2h) = -h^2u^2\frac{d^2u}{d\theta^2}.$$

Substituting in (1), we have

$$-h^2u^2\frac{d^2u}{d\theta^2} - \frac{1}{u}\cdot(u^2h)^2 = -P \quad \text{or} \quad h^2u^2\frac{d^2u}{d\theta^2} + h^2u^3 = P$$

or $$\mathbf{\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2}}, \quad \text{...(4)}$$

which is the differential equation of a central orbit in **polar form** referred to the centre of force as the pole.

Pedal form. If p is the length of the perpendicular drawn from the origin upon the tangent at the point P, we have

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2.$$

But $u = \dfrac{1}{r}$. Therefore $\dfrac{du}{d\theta} = -\dfrac{1}{r^2}\dfrac{dr}{d\theta}$

i.e., $$\left(\frac{du}{d\theta}\right)^2 = \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2.$$

So $$\mathbf{\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2}. \quad \text{...(5)}$$

Differentiating both sides of (5) w.r.t. 'θ', we have

$$-\frac{2}{p^3}\frac{dp}{d\theta} = 2u\frac{du}{d\theta} + 2\frac{du}{d\theta}\cdot\frac{d^2u}{d\theta^2} = 2\frac{du}{d\theta}\left(u + \frac{d^2u}{d\theta^2}\right)$$

or $$-\frac{1}{p^3}\frac{dp}{d\theta} = \frac{du}{d\theta}\cdot\frac{P}{h^2u^2} \quad \text{[From (4)]}$$

or $$-\frac{1}{p^3}\cdot\frac{dp}{dr}\cdot\frac{dr}{d\theta} = \left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)\left(\frac{P}{h^2u^2}\right), \quad \left\{\because \frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}\right\}$$

or $$\frac{1}{p^3}\frac{dp}{dr} = \frac{1}{r^2}\cdot\frac{P}{h^2u^2} = u^2\cdot\frac{P}{h^2u^2} = \frac{P}{h^2}$$

or $$\mathbf{P = \frac{h^2}{p^3}\frac{dp}{dr}}, \quad \text{...(6)}$$

[Rohilkhand 79; Meerut 79S, 83S, 92S, 93S, 95BP]

which is the differential equation of a central orbit in **pedal form.**

Angular momentum or moment of momentum. The expression $r^2(d\theta/dt)$ is called the angular momentum or the moment of momentum about the pole O of a particle of unit mass moving in a plane curve. Since in a central orbit $r^2(d\theta/dt)$ =constant, therefore *in a central orbit the angular momentum is conserved.*

[Meerut 77; Allahabad 79]

§ 3. Rate of description of the sectorial area.

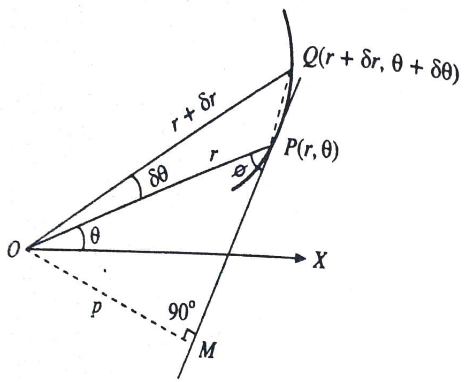
In every central orbit, the sectorial area traced out by the radius vector to the centre of force increases uniformly per unit of time, and the linear velocity varies inversely as the perpendicular from the centre upon the tangent to the path.

[Rohilkhand 77]

Take the centre of force O as the pole and OX as the initial line. Let $P(r, \theta)$ and $Q(r + \delta r, \theta + \delta\theta)$ be the positions of a particle moving in a central orbit at times t and $t + \delta t$ respectively.

Sectorial area OPQ described by the particle in time δt

= area of the ΔOPQ



[$\because$ the point Q is very close to P and ultimately we have to take limit as $Q \to P$]

$$= \tfrac{1}{2} OP.OQ \sin \angle POQ = \tfrac{1}{2} r(r + \delta r) \sin \delta\theta.$$

$\therefore$ rate of description of the sectorial area

$$= \lim_{\delta t \to 0} \frac{\text{sectorial area } OPQ}{\delta t} = \lim_{\delta t \to 0} \frac{\tfrac{1}{2}(r + \delta r) \sin \delta\theta}{\delta t}$$

$$= \lim_{\delta t \to 0} \tfrac{1}{2} r(r + \delta r) \cdot \frac{\sin \delta\theta}{\delta\theta} \cdot \frac{\delta\theta}{\delta t} = \tfrac{1}{2} r^2 \frac{d\theta}{dt} = \tfrac{1}{2} h. \quad \text{...(1)}$$

[$\because r^2 (d\theta/dt) = h$]

Thus the rate of description of the sectorial area is constant and is equal to $h/2$.

[Kanpur 76]

The rate of description of the sectorial area is also called the *areal velocity* of the particle about the fixed point O.

Again for a central orbit, we have $r^2 \dfrac{d\theta}{dt} = h$.

$$\therefore r^2 \frac{d\theta}{ds} \frac{ds}{dt} = h \text{ or } r^2 \frac{d\theta}{ds} \cdot v = h. \quad \text{...(2)}$$

[$\because ds/dt = v$ (*i.e.*, the linear velocity)]

But from differential calculus, we have $r \dfrac{d\theta}{ds} = \sin \phi$, where ϕ is the angle between the radius vector and the tangent.

$\therefore r^2 \dfrac{d\theta}{ds} = r \sin \phi = p$, where p is the length of the perpendicular drawn from the pole O on the tangent at P.

Putting $r^2 (d\theta/ds) = p$ in (2), we get $vp = h$.

$$\mathbf{v} = \frac{\mathbf{h}}{\mathbf{p}}. \quad \text{...(3)}$$

[Rohilkhand 87]

or $\therefore v \propto 1/p$

i.e., the linear velocity at P varies inversely as the perpendicular from the fixed point upon the tangent to the path.

From (3), we have $v^2 = \dfrac{h^2}{p^2}$.

But
$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4} \left(\frac{dr}{d\theta}\right)^2 = u^2 + \left(\frac{du}{d\theta}\right)^2.$$

$$\therefore \mathbf{v^2 = h^2 \left[u^2 + \left(\frac{du}{d\theta}\right)^2\right]}. \quad \text{...(4)}$$

[Meerut 79, 88, 89S, 93; Agra 86; Rohilkhand 81]

The equation (4) gives the linear velocity at any point of the path of a central orbit.

§ 4. Elliptic orbit (Focus as the centre of force).

A particle moves in an ellipse under a force which is always directed towards its focus: to find

(i) the law of force, **[Meerut 76S]**

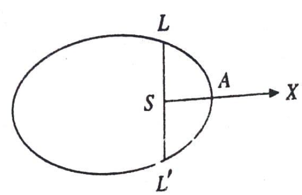
(ii) the velocity at any point of its path **[Meerut 76S, 79]**

and (iii) the periodic time. **[Meerut 81S]**

We know that the polar equation of an ellipse referred to its focus S as pole is

$$\frac{l}{r} = 1 + e \cos \theta$$

or
$$u = \frac{1}{l} + \frac{e}{l} \cos \theta, \quad \text{...(1)}$$



where $u = 1/r$.

Differentiating, we have

$$\frac{du}{d\theta} = -\frac{e}{l} \sin \theta \text{ and } \frac{d^2u}{d\theta^2} = -\frac{e}{l} \cos \theta.$$

(i) Law of force. We know that the differential equation of a central orbit referred to the centre of force as pole is

$$\frac{P}{h^2u^2} = u + \frac{d^2u}{d\theta^2}$$

where P is the central acceleration assumed to be attractive.

Now here $P = h^2u^2 \left[u + \dfrac{d^2u}{d\theta^2}\right]$

$$= h^2u^2 \left[\frac{1}{l} + \frac{e}{l}\cos\theta - \frac{e}{l}\cos\theta\right],$$

substituting for u and $d^2u/d\theta^2$

$$= \frac{h^2u^2}{l} = \frac{h^2/l}{r^2} = \frac{\mu}{r^2},$$

where $\mu = h^2/l$ or $\mathbf{h^2 = \mu l}$. ...(2)

$$\therefore P \propto \frac{1}{r^2}. \qquad ...(3)$$

Hence the acceleration varies inversely as the square of the distance of the particle from the focus. Also the force is attractive because the value of P is positive.

(ii) **Velocity.** We know that the velocity in a central orbit is given by

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right].$$

$\therefore$ here, $v^2 = h^2\left[\left(\dfrac{1}{l} + \dfrac{e}{l}\cos\theta\right)^2 + \left(-\dfrac{e}{l}\sin\theta\right)^2\right]$

$$= h^2\left[\frac{1}{l^2} + \frac{2e}{l^2}\cos\theta + \frac{e^2}{l^2}\right] = \frac{h^2}{l}\left[\frac{1+e^2}{l} + 2\frac{e\cos\theta}{l}\right]$$

$$= \mu\left[\frac{1+e^2}{l} + 2\left(u - \frac{1}{l}\right)\right] \qquad \text{[from (1) and (3)]}$$

$$= \mu\left[2u - \frac{1-e^2}{l}\right] = \mu\left[\frac{2}{r} - \frac{1-e^2}{l}\right]. \qquad \textbf{[Meerut 79]}$$

If $2a$ and $2b$ are the lengths of the major and the minor axes of the ellipse, we have

$$l = \text{the semi latus rectum} = \frac{b^2}{a} = \frac{a^2(1-e^2)}{a} = a(1-e^2).$$

$$\therefore \frac{1-e^2}{l} = \frac{1}{a}. \quad \therefore \mathbf{v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right)}, \qquad ...(4)$$

which gives the velocity of the particle at any point of its path.

Equation (4) *shows that the magnitude of the velocity at any point of the path depends only on the distance from the focus and that it is independent of the direction of the motion. Also* $v^2 < 2\mu/r$.

(iii) **Periodic time.** We know that in a central orbit the rate of description of the sectorial area is constant and is equal to $h/2$. Let T be the time period for one complete revolution *i.e.*, the time taken by the particle in describing the whole of the ellipse. The sectorial area traced in describing the whole arc of the ellipse is equal to the whole area of the ellipse.

$\therefore T(h/2) =$ the whole area of the ellipse $= \pi ab$.

$$\therefore T = \frac{2\pi ab}{h} = \frac{2\pi ab}{\sqrt{(\mu l)}} \qquad [\because h^2 = \mu l]$$

or

$$T = \frac{2\pi ab}{\sqrt{\{\mu(b^2/a)\}}} \qquad [\because l = b^2/a]$$

or

$$\mathbf{T = \frac{2\pi a^{3/2}}{\sqrt{\mu}}}, \qquad ...(5)$$

i.e., the time period for one complete revolution is proportional to $a^{3/2}$, *a being semi-major axis.* **[Meerut 81S ; Allahabad 79]**

§ 5. Hyperbolic and parabolic orbits. (Centre of force being the focus).

(i) **Hyperbolic orbit.** In the case of hyperbola, we have $e > 1$.

Also $l = \dfrac{b^2}{a} = \dfrac{a^2(e^2-1)}{a} = a(e^2-1)$.

Proceeding as in § 4, we have $P = \mu/r^2$, where $h^2 = \mu l$.

[Note that this result does not depend upon the value of e].

Also proceeding as in establishing the result (4) of § 4, we have here

$$v^2 = \mu\left[\frac{2}{r} + \frac{e^2-1}{l}\right] \qquad [\because e > 1]$$

or

$$\mathbf{v^2 = \mu\left[\frac{2}{r} + \frac{1}{a}\right]}. \text{ Note that here } v^2 > 2\mu/r.$$

(ii) **Parabolic orbit.** In this case $e = 1$.

Proceeding as in § 4, we have here $P = \mu/r^2$ and $v^2 = 2\mu/r$.

§ 6. Velocity from infinity.

In connection with the central orbits by the phrase '*velocity from infinity at any point*' we mean the velocity that a particle would acquire if it moved from rest at infinity in a straight line to that point under the action of an attractive force in accordance with the law associated with the orbit.

Suppose a particle falls from rest from infinity in a straight line under the action of a central attractive acceleration P directed towards the centre of force O.

Let Q be the position of the particle at any time t, where $OQ = r$.

Suppose v is the velocity of the particle at Q. The expression for acceleration at the point Q is $v\,(dv/dr)$.

The equation of motion of the particle at the point Q is

$v\dfrac{dv}{dr} = -P$, [–ive sign has been taken because the acceleration P is in the direction of r decreasing] ...(1)

or $\quad v\,dv = -P dr$.

Let V be the velocity acquired in falling from rest at infinity to a point distant a from the centre of force O. Then integrating (1) from infinity to the point $r = a$, we get

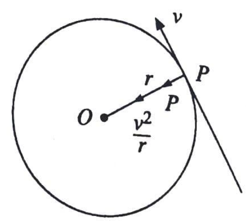
$$\int_0^V v\,dv = -\int_\infty^a P\,dr$$

or $\quad \frac{1}{2}V^2 = -\int_\infty^a P\,dr$ or $V^2 = -2\int_\infty^a P\,dr$, **(Remember)**

which gives the velocity from infinity at a distance a from the centre of force while moving under the central acceleration P associated with the orbit.

§ 7. Velocity in a circle.

The phrase '*velocity in a circle*' at any point of a central orbit means the velocity necessary to describe a circle, passing through that point and with centre at the centre of force, while moving under the action of the prescribed force associated with the orbit.



Take the centre of force O as the pole. Let P be the central acceleration, directed towords O, at any point P of the orbit where $OP = r$. Suppose v is the velocity in a circle at P. Then v is the velocity at the point P of a particle which moves, under the same central acceleration P, in a circle with centre at O. But for a circle with centre at the pole O, the radius vector OP is also normal to the circle at P. Therefore, the central radial acceleration P

= the inward normal acceleration v^2/ρ

i.e., $\quad P = v^2/r$. [$\because$ for the circle, $\rho = r$]

$\therefore \; v^2 = rP$. **(Remember)**

Thus while moving under a central attractive acceleration P, the velocity V in a circle at a distance a from the centre of force is given by

$$V^2 = a\,.\left[P\right]_{r=a}.$$

§ 8. Given the central orbit, to find the law of force.

Case I. *The equation of the orbit being given in the polar form* (r, θ).

We know that referred to the centre of force as pole, the differential equation of a central orbit is

$$\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2}, \qquad ...(1)$$

where P is the central acceleration assumed to be attractive.

From the given equation of the orbit we can easily calculate u and $d^2u/d\theta^2$ and substituting their values in (1) we can determine P. Thus we find the law of force. If the value of P is positive, the force is attractive and if the value of P is negative, the force is repulsive.

Case II. *The equation of the orbit being given in the pedal form* (p, r).

The differential equation of a central orbit in (p, r) form is

$$\frac{h^2}{p^3}\frac{dp}{dr} = P. \qquad ...(2)$$

From the given equation of the orbit in (p, r) form, we can find out dp/dr and then substituting its value in (2) we can determine P.

Solved Examples

Ex. 1. *Find the law of force towards the pole under which the following curves are described.*

(*i*) $au = e^{n\theta}$ *and* (*ii*) $r = ae^{\theta \cot \alpha}$. **(Meerut 1986P, 97)**

Sol. (i) We have $\quad au = e^{n\theta}$ ...(1)

Differentiating w.r.t. 'θ', we have

$$\frac{du}{d\theta} = \frac{n}{a}e^{n\theta} = nu \quad \text{and} \quad \frac{d^2u}{d\theta^2} = n\frac{du}{d\theta} = n\,.\,nu = n^2u.$$

Referred to the centre of force as pole, the differential equation of a central orbit is

$$\frac{P}{h^2u^2} = u + \frac{d^2u}{d\theta^2},$$

where P is the central acceleration assumed to be attractive.

$$\therefore \quad P = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right) = h^2u^2\,(u + n^2u) = h^2\,(1 + n^2)\,u^3$$

$$= \frac{h^2(1+n^2)}{r^3}. \qquad [\because u = 1/r]$$

$\therefore$ $P \propto 1/r^3$ *i.e.*, the force varies inversely as the cube of the distance from the pole. Also the positive value of P indicates that the force is attractive *i.e.*, is directed towards the pole.

(ii) We have $\quad r = ae^{\theta \cot \alpha}$

or $\quad \dfrac{1}{u} = ae^{\theta \cot \alpha}$, $\quad [\because r = 1/u]$.

$\therefore \quad u = \dfrac{1}{a} e^{-\theta \cot \alpha}$.

Differentiating w.r.t. 'θ', we have

$$\frac{du}{d\theta} = -\frac{\cot \alpha}{a} e^{-\theta \cot \alpha} = -u \cot \alpha$$

and $$\frac{d^2u}{d\theta^2} = -\frac{du}{d\theta} \cot \alpha = -(-u \cot \alpha) \cot \alpha = u \cot^2 \alpha.$$

The differential equation of the central orbit is

$$\frac{P}{h^2u^2} = u + \frac{d^2u}{d\theta^2}.$$

$$\therefore \quad P = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right) = h^2u^2(u + u\cot^2\alpha) = h^2(1+\cot^2\alpha)u^3$$

$$= \frac{h^2 \operatorname{cosec}^2 \alpha}{r^3}.$$

$\therefore$ $P \propto 1/r^3$ *i.e.*, the force varies inversely as the cube of the distance from the pole. Also the positive value of P indicates that the force is attractive.

Ex. 2. *A paritcle describes the curve $r^n = a^n \cos n\theta$ under a force to the pole. Find the law of force.* **[Meerut 83, 85S, 95BP, L.F.S. 77]**

Hence obtain the law of force under which a cardioid can be described.

Sol. The equation of the curve is $r^n = a^n \cos n\theta$.

Putting $r = 1/u$, we have

$$\frac{1}{u^n} = a^n \cos n\theta \quad \text{or} \quad a^n u^n = \sec n\theta. \qquad ...(1)$$

Taking logarithm of both sides of (1), we have

$$n \log a + n \log u = \log \sec n\theta.$$

Differentiating w.r.t. 'θ', we have

$$\frac{n}{u}\frac{du}{d\theta} = \frac{1}{\sec n\theta} n \sec n\theta \tan n\theta \quad \text{or} \quad \frac{du}{d\theta} = u \tan n\theta.$$

Differentiating again w.r.t.. 'θ', we have

$$\frac{d^2u}{d\theta^2} = \frac{du}{d\theta} \tan n\theta + u(\sec^2 n\theta) \cdot n$$

$$= u \tan n\theta \cdot \tan n\theta + un \sec^2 n\theta \qquad [\because du/d\theta = u \tan n\theta]$$

$$= u \tan^2 n\theta + un \sec^2 n\theta. \qquad ...(2)$$

The differential equation of the central orbit is

$$\frac{P}{h^2u^2} = u + \frac{d^2u}{d\theta^2}.$$

$$\therefore \quad P = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right) = h^2u^2(u + u\tan^2 n\theta + un \sec^2 n\theta)$$

[putting the value of $d^2u/d\theta^2$ from (2)]

$$= h^2u^3(\sec^2 n\theta + n\sec^2 n\theta) = h^2u^3(1+n)\sec^2 n\theta$$

[substituting for $\sec n\theta$ from (1)]

$$= h^2(1+n)u^3 \cdot (a^nu^n)^2$$

$$= h^2a^{2n}(1+n)u^{2n+3} = \frac{h^2a^{2n}(1+n)}{r^{2n+3}}.$$

$\therefore$ $P \propto 1/r^{2n+3}$ *i.e.*, the force varies inversely as the $(2n+3)$th power of the distance from the pole.

Second part. Putting $n = 1/2$ in the equation of the path, we get

$$r^{1/2} = a^{1/2} \cos \tfrac{1}{2}\theta$$

or $$r = a \cos^2 \tfrac{1}{2}\theta \qquad \text{[squaring both sides]}$$

or $r = \frac{1}{2}a \cdot 2\cos^2 \frac{1}{2}\theta = \frac{1}{2}a(1 + \cos\theta)$, which is the equation of a cardioid.

Now putting $n = \frac{1}{2}$ in the value of P, we get

$$P \propto \frac{1}{r^{1+3}} \quad i.e., \quad P \propto \frac{1}{r^4}.$$

Ex. 3. *A particle describes the curve $r^2 = a^2 \cos 2\theta$ under a force to the pole. Find the law of force.* **(Agra 1975)**

Sol. The equation of the curve is $\quad r^2 = a^2 \cos 2\theta$.

Proceed as in Ex. 2. Replacing n by 2 in the preceding exercise 2, we have

$$P = \frac{3h^2a^4}{r^7}. \quad \text{Therefore } P \propto \frac{1}{r^7}$$

i.e., the force varies inversely as the seventh power of the distance from the pole.

Ex. 4. *Find the law of force towards the pole under which the curve $r^n \cos n\theta = a^n$ is described.* **(Rohilkhand 1981)**

Sol. The equation of the curve is $r^n \cos n\theta = a^n$.

Replacing r by $1/u$, we have

$$\frac{1}{u^n}\cos n\theta = a^n$$

or
$$a^n u^n = \cos n\theta.$$

Taking logarithm of both sides of (1), we have

$$n\log a + n\log u = \log\cos n\theta. \qquad ...(1)$$

Differentiating w.r.t. 'θ', we have

$$\frac{n\,du}{u\,d\theta} = \frac{1}{\cos n\theta}\cdot(-n\sin n\theta)$$

or
$$\frac{du}{d\theta} = -u\tan n\theta. \qquad ...(2)$$

Differentiating again w.r.t., 'θ', we have

$$\frac{d^2u}{d\theta^2} = -\frac{du}{d\theta}\tan n\theta - un\sec^2 n\theta = u\tan^2 n\theta - un\sec^2 n\theta$$

[Substituting for $du/d\theta$ from (2)]

The differential equation of the central orbit is

$$\frac{P}{h^2u^2} = u + \frac{d^2u}{d\theta^2}\,.$$

$$\therefore\quad P = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right) = h^2u^2(u + u\tan^2 n\theta - un\sec^2 n\theta)$$

$$= h^2u^3(\sec^2 n\theta - n\sec^2 n\theta) = h^2u^3(1-n)\sec^2 n\theta$$

$$= h^2u^3(1-n)\cdot\left(\frac{1}{a^nu^n}\right)^2 = \frac{h^2(1-n)}{a^{2n}\,u^{2n-3}} = \frac{h^2(1-n)}{a^{2n}}\cdot r^{2n-3}.$$

$\therefore$ $P \propto r^{2n-3}$ *i.e.*, the force is proportional to the $(2n-3)$th power of the distance from the pole.

Ex. 5. *A particle describes the curve $r^n = A\cos n\theta + B\sin n\theta$ under a force to the pole. Find the law of force.*

Sol. Here $r^n = A\cos n\theta + B\sin n\theta$.

Let $A = k\cos\alpha$ and $B = k\sin\alpha$, where k and α are constants.

Re lacing r by $1/u$, we have

$$r^n = u^{-n} = k\cos(n\theta-\alpha).$$

$$\therefore\ -n\log u = \log k + \log\cos(n\theta-\alpha). \qquad ...(1)$$

Differentiating both sides w.r.t. 'θ', we have

$$\frac{-n\,du}{u\,d\theta} = -n\tan(n\theta-\alpha) \quad\text{or}\quad \frac{du}{d\theta} = u\tan(n\theta-\alpha).$$

$$\therefore\ \frac{d^2u}{d\theta^2} = \frac{du}{d\theta}\cdot\tan(n\theta-\alpha) + un\sec^2(n\theta-\alpha)$$

$$= u\tan^2(n\theta-\alpha) + un\sec^2(n\theta-\alpha).$$

The differential equation of the path is

$$\frac{P}{h^2u^2} = u + \frac{d^2u}{d\theta^2}\,.$$

$$\therefore\ P = h^2u^2[u + u\tan^2(n\theta-\alpha) + un\sec^2(n\theta-\alpha)]$$

$$= h^2u^3[\sec^2(n\theta-\alpha) + n\sec^2(n\theta-\alpha)]$$

$$= (1+n)h^2u^3\sec^2(n\theta-\alpha)$$

$$= (1+n)h^2u^3(ku^n)^2 \qquad [\because \text{ from (1), } \sec(n\theta-\alpha) = k\,u^n]$$

$$= \frac{(1+n)h^2k^2}{r^{2n+3}}\,.$$

Thus $P \propto \dfrac{1}{r^{2n+3}}$ *i.e.*, the force is inversely proportional to the $(2n+3)$th power of the distance from the pole.

Ex. 6. *A particle describes a circle, pole on its circumference, under a force P to the pole. Find the law of force.* **(Meerut 1975, 81, 82, 83, 86)**

Or

A particle describes the curve $r = 2a\cos\theta$ under the force P to the pole. Find the law of force. **(Meerut 1980, 81)**

Sol. Let a be the radius of the circle. If we take pole on the circumference of the circle and the diameter through the pole as the initial line, the equation of the circle is

$$r = 2a\cos\theta \qquad ...(1)$$

or
$$1/u = 2a\cos\theta.$$

$$\therefore\quad -\log u = \log(2a) + \log\cos\theta.$$

Differentiating w.r.t. 'θ', we have

$$-\frac{1}{u}\frac{du}{d\theta} = -\tan\theta \quad\text{or}\quad \frac{du}{d\theta} = u\tan\theta,$$

and
$$\frac{d^2u}{d\theta^2} = u\,.\sec^2\theta + \frac{du}{d\theta}\tan\theta$$

$$= u\sec^2\theta + u\tan\theta\,.\tan\theta = u\sec^2\theta + u\tan^2\theta.$$

The differential equation of the path is

$$\frac{P}{h^2u^2} = u + \frac{d^2u}{d\theta^2}\,.$$

$$\therefore\ P = h^2u^2[u + u\sec^2\theta + u\tan^2\theta] = h^2u^3[(1+\tan^2\theta) + \sec^2\theta]$$

$$= 2h^2u^3\sec^2\theta$$

$$= 2h^2u^3(2au)^2 \qquad [\text{substituting for } \sec\theta \text{ from (1)}]$$

$$= \frac{8a^2h^2}{r^5}\,.$$

$\therefore$ $P \propto 1/r^5$ *i.e.*, the force varies inversely as the fifth power of the distance from the pole. Also the positive value of P indicates that the force is attractive.

Ex. 7. *Find the law of force towards the pole under which the following curves are described.*

(*i*) $a = r\cosh n\theta$ and (*ii*) $a = r\tanh(\theta/\sqrt{2})$.

Sol. (i) The equation of the curve is

$$a = r\cosh n\theta = (1/u)\cosh n\theta$$

or $$u = (1/a)\cosh n\theta.$$

Differentiating, $\dfrac{du}{d\theta} = \dfrac{n}{a}\sinh n\theta$ and $\dfrac{d^2u}{d\theta^2} = \dfrac{n^2}{a}\cosh n\theta.$ ...(1)

The differential equation of the central orbit is

$$\frac{P}{h^2u^2} = u + \frac{d^2u}{d\theta^2}.$$

$$\therefore\quad P = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right) = h^2u^2\left(u + \frac{n^2}{a}\cosh n\theta\right) = h^2u^2(u + n^2u)$$

[substituting for $\cosh n\theta$ from (1)]

$$= h^2(1+n^2)u^3 = \frac{h^2(1+n^2)}{r^3}.$$

$\therefore$ $P \propto 1/r^3$ *i.e.*, the force varies inversely as the cube of the distance from the pole.

(ii) The equation of the curve is

$$a = r\tanh(\theta/\sqrt{2}) = (1/u)\tanh(\theta/\sqrt{2})$$

or $$u = (1/a)\tanh(\theta/\sqrt{2}).$$ ...(1)

Differentiating, $\dfrac{du}{d\theta} = \dfrac{1}{a\sqrt{2}}\operatorname{sech}^2(\theta/\sqrt{2})$

and $$\frac{d^2u}{d\theta^2} = \frac{1}{a\sqrt{2}}\cdot 2\operatorname{sech}(\theta/\sqrt{2}).\left\{-\frac{1}{\sqrt{2}}\operatorname{sech}(\theta/\sqrt{2})\tanh(\theta/\sqrt{2})\right\}$$

$$= -\frac{1}{a}\operatorname{sech}^2(\theta/\sqrt{2})\tanh(\theta/\sqrt{2}) = -u\operatorname{sech}^2(\theta/\sqrt{2}).$$

The differential equation of the central orbit is

$$\frac{P}{h^2u^2} = u + \frac{d^2u}{d\theta^2}.$$

$$\therefore\quad P = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right)$$

$$= h^2u^2[u - u\operatorname{sech}^2(\theta/\sqrt{2})] = h^2u^3[1 - \operatorname{sech}^2(\theta/\sqrt{2})]$$

$$= h^2u^3\tanh^2(\theta/\sqrt{2}) \qquad [\because \operatorname{sech}^2\theta = 1 - \tanh^2\theta]$$

$$= h^2u^3(au)^2 \qquad \text{[from (1)]}$$

$$= h^2a^2u^5 = \frac{h^2a^2}{r^5}.$$

$\therefore$ $P \propto 1/r^5$ *i.e.*, the force varies inversely as the 5*th* power of the distance from the pole.

Ex. 8. *A particle describes the curve $r = a\sin n\theta$ under a force P to the pole. Find the law of force.* **(Meerut 1975, 91S)**

Sol. The equation of the curve is

$$r = a\sin n\theta$$

or $$u = \frac{1}{r} = \frac{1}{a}\operatorname{cosec} n\theta.$$ ...(1)

Differentiating, $\dfrac{du}{d\theta} = -\dfrac{n}{a}\operatorname{cosec} n\theta\cot n\theta = -nu\cot n\theta,$

and $$\frac{d^2u}{d\theta^2} = n^2u\operatorname{cosec}^2 n\theta - n\frac{du}{d\theta}\cot n\theta$$

$$= n^2u\operatorname{cosec}^2 n\theta - n.(-nu\cot n\theta)\cot n\theta$$

$$= n^2u^2\operatorname{cosec}^2 n\theta + n^2u\cot^2 n\theta.$$

The differential equation of the central orbit is

$$\frac{P}{h^2u^2} = u + \frac{d^2u}{d\theta^2}.$$

$$\therefore\quad P = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right) = h^2u^2(u + n^2u\operatorname{cosec}^2 n\theta + n^2u\cot^2 n\theta)$$

$$= h^2u^3[1 + n^2\operatorname{cosec}^2 n\theta + n^2(\operatorname{cosec}^2 n\theta - 1)]$$

$$= h^2u^3[2n^2\operatorname{cosec}^2 n\theta - (n^2-1)]$$

$$= h^2u^3[2n^2(au)^2 - (n^2-1)]$$

[substituting for $\operatorname{cosec} n\theta$ from (1)]

$$= h^2[2n^2a^2u^5 - (n^2-1)u^3]$$

$$= h^2\left[\frac{2n^2a^2}{r^5} - \frac{(n^2-1)}{r^3}\right].$$

$$\therefore\quad P \propto \left[\frac{2n^2a^2}{r^5} - \frac{(n^2-1)}{r^3}\right].$$

Ex. 9. *Find the law of force towards the pole under which the following curves are described.*

(*i*) $r^2 = 2ap$, (*ii*) $p^2 = ar$ and (*iii*) $b^2/p^2 = (2a/r) - 1$.

Sol. (i) The equation of the curve is $r^2 = 2ap$.

$$\therefore\quad \frac{1}{p} = \frac{2a}{r^2} \quad\text{or}\quad \frac{1}{p^2} = \frac{4a^2}{r^4}.$$ **[Note]**

Differentiating w.r.t. 'r', we have

$$-\frac{2}{p^3}\frac{dp}{dr} = -\frac{16a^2}{r^5}.$$

$$\therefore\quad \frac{h^2}{p^3}\frac{dp}{dr} = \frac{8a^2h^2}{r^5}$$ ...(1)

Now from the pedal equation of a central orbit, we have

$$P = \frac{h^2}{p^3}\frac{dp}{dr} = \frac{8a^2h^2}{r^5}. \qquad \text{[from (1)]}$$

$\therefore$ $P \propto 1/r^5$ *i.e.*, the force varies inversely as the fifth power of the distance from the pole.

(ii) The equation of the curve is $p^2 = ar$, which is the pedal equation of a parabola referred to the focus as the pole.

$$\therefore \quad \frac{1}{p^2} = \frac{1}{a}\frac{1}{r}.$$

Differentiating w.r.t. 'r', we get

$$-\frac{2}{p^3}\frac{dp}{dr} = -\frac{1}{a}\frac{1}{r^2}.$$

$$\therefore \quad \frac{h^2}{p^3}\frac{dp}{dr} = \frac{h^2}{2a}\frac{1}{r^2}. \qquad \text{...(1)}$$

From the pedal equation of a central orbit, we have

$$P = \frac{h^2}{p^3}\frac{dp}{dr} = \frac{h^2}{2a}\frac{1}{r^2} \qquad \text{[from (1)]}$$

$\therefore$ $P \propto 1/r^2$ *i.e.*, the force varies inversely as the square of the distance from the pole.

(iii) The equation of the given central orbit is

$$\frac{b^2}{p^2} = \frac{2a}{r} - 1. \qquad \text{...(1)}$$

(i) is the pedal equation of an ellipse referred to the focus as pole. Differentiating both sides of (1) w.r.t. 'r', we get

$$-\frac{2b^2}{p^3}\frac{dp}{dr} = -\frac{2a}{r^2}, \text{ or } \frac{h^2}{p^3}\frac{dp}{dr} = \frac{a}{b^2}\frac{h^2}{r^2}.$$

$$\therefore \quad P = \frac{h^2}{p^3}\frac{dp}{dr} = \frac{ah^2}{b^2}\frac{1}{r^2}.$$

Thus $P \propto 1/r^2$ *i.e.*, the acceleration varies inversely as the square of the distance from the focus of the ellipse.

Ex. 10. *A particle describes the curve $r^2 = a^2 \cos^2\theta + b^2 \sin^2\theta$ under an attraction to the origin, prove that the attraction at a distance r is*

$$h^2\,[2\,(a^2 + b^2)\,r^2 - 3a^2b^2]\,.\,r^{-7}.$$

Sol. The equation of the given curve is

$$r^2 = a^2 \cos^2\theta + b^2 \sin^2\theta$$

or $$\frac{1}{u^2} = \frac{a^2}{2}(1 + \cos 2\theta) + \frac{b^2}{2}(1 - \cos 2\theta)$$

or $$\frac{1}{u^2} = \tfrac{1}{2}(a^2 + b^2) + \tfrac{1}{2}(a^2 - b^2)\cos 2\theta. \qquad \text{...(1)}$$

Differentiating w.r.t. 'θ', we have

$$-\frac{2}{u^3}\frac{du}{d\theta} = -(a^2 - b^2)\sin 2\theta$$

$$\frac{du}{d\theta} = \tfrac{1}{2}(a^2 - b^2)\,u^3 \sin 2\theta.$$

Differentiating again w.r.t. 'θ', we have

$$\frac{d^2u}{d\theta^2} = \frac{3}{2}(a^2 - b^2)\,u^2 \cdot \frac{du}{d\theta}\sin 2\theta + (a^2 - b^2)\,u^3 \cos 2\theta$$

$$= \tfrac{3}{2}(a^2 - b^2)\,u^2 \,.\, \tfrac{1}{2}(a^2 - b^2)\,u^3 \sin 2\theta \,.\, \sin 2\theta + (a^2 - b^2)\,u^3 \cos 2\theta$$

$$= \tfrac{3}{4}u^5\,(a^2 - b^2)^2 \sin^2 2\theta + (a^2 - b^2)\,u^3 \cos 2\theta$$

$$= \tfrac{3}{4}u^5\,(a^2 - b^2)^2\,(1 - \cos^2 2\theta) + u^3\,(a^2 - b^2) \cos 2\theta$$

$$= \tfrac{3}{4}u^5\,(a^2 - b^2)^2 - \tfrac{3}{4}u^5 \,.\, \{(a^2 - b^2)\cos 2\theta\}^2 + u^3\,(a^2 - b^2)\cos 2\theta.$$

Now from (1), $(a^2 - b^2)\cos 2\theta = \dfrac{2}{u^2} - (a^2 + b^2)$.

$$\therefore \frac{d^2u}{d\theta^2} = \tfrac{3}{4}u^5\,(a^2 - b^2)^2 - \tfrac{3}{4}u^5 \left\{\frac{2}{u^2} - (a^2 + b^2)\right\}^2 + u^3 \,.\, \left\{\frac{2}{u^2} - (a^2 + b^2)\right\}$$

$$= \tfrac{3}{4}u^5\,(a^2 - b^2)^2 - \tfrac{3}{4}u^5 \left\{\frac{4}{u^4} - \frac{4}{u^2}(a^2 + b^2) + (a^2 + b^2)^2\right\} + 2u - (a^2 + b^2)\,u^3$$

$$= \tfrac{3}{4}u^5\,(a^2 - b^2)^2 - 3u + 3u^3\,(a^2 + b^2) - \tfrac{3}{4}u^5\,(a^2 + b^2)^2 + 2u - (a^2 + b^2)\,u^3$$

$$= \tfrac{3}{4}u^5\,\{(a^2 - b^2)^2 - (a^2 + b^2)^2\} + 2u^3\,(a^2 + b^2) - u$$

$$= 2\,(a^2 + b^2)\,u^3 - 3a^2b^2u^5 - u.$$

The differential equation of the central orbit is

$$\frac{P}{h^2u^2} = u + \frac{d^2u}{d\theta^2}.$$

$$\therefore \quad P = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right) = h^2u^2\,[u + 2\,(a^2 + b^2)\,u^3 - 3a^2b^2u^5 - u]$$

$$= h^2u^7\,[2\,(a^2 + b^2)\,r^2 - 3a^2b^2] = h^2r^{-7}\,[2\,(a^2 + b^2)\,r^2 - 3a^2b^2].$$

Ex. 11. *Show that the only law for a central attraction for which the velocity in a circle at any distance is equal to the velocity acquired in falling from infinity to the distance is that of inverse cube.*

Sol. Let the central acceleration P be given by

$$P = f'(r). \qquad \text{...(1)}$$

[Note the form we have assumed for P]

The equation of motion of the particle falling from infinity under the cental acceleration given by (1) is

$$v\frac{dv}{dr} = -P = -f'(r)$$

[Refer § 6 of this chapter on page ...

or $$2v\,dv = -2f'(r)\,dr.$$

Integrating, $$v^2 = -2\int f'(r)\,dr + A,$$

where A is constant of integration

or $$v^2 = -2f(r) + A. \quad ...(2)$$

Thus the velocity v at a distance r acquired in falling from infinity is given by (2). Again let V be the velocity of the particle moving in a circle under the same central acceleration P at a distance r from the centre of the circle. For a circle with centre at the centre of force pole we have

the central radial attractive acceleration P = the inward normal acceleration V^2/ρ

$$\therefore \quad P = V^2/r \qquad [\because \text{ for the circle, } \rho = r]$$

or $$V^2 = rP = rf'(r).$$

But according to the question

$$V = v \quad \text{or} \quad V^2 = v^2.$$

$$\therefore \quad rf'(r) = -2f(r) + A$$

or $$r^2 f'(r) + 2rf(r) = Ar$$

or $$\frac{d}{dr}\{r^2 f(r)\} = Ar.$$

Integrating both sides w.r.t. 'r', we have

$$r^2 f(r) = \tfrac{1}{2}Ar^2 + B, \text{ where } B \text{ is a constant}$$

or $$f(r) = \frac{A}{2} + \frac{B}{r^2}.$$

Differentiating both sides w.r.t. 'r', we have

$$f'(r) = \frac{-2B}{r^3}$$

so that $$P = -\frac{2B}{r^3}. \qquad [\because P = f'(r)]$$

$\therefore$ $P \propto 1/r^3$ *i.e.*, the law of force is that of inverse cube.

Ex. 12. *In a central orbit described under a force to a centre, the velocity at any point is inversely proportional to the distance of the point from the centre of force. Show that the path is an equiangular spiral.*

Sol. If v is the velocity of the particle at any point at a distance r from the centre of force, then according to the question

$$v \propto \frac{1}{r} \quad \text{or} \quad v = \frac{k}{r}, \quad ...(1)$$

where k is a constant.

But in a central orbit $$v = h/p, \quad ...(2)$$

where p is the length of the perpendicular from the pole on the tangent at any point of the path.

From (1) and (2), we have

$$\frac{k}{r} = \frac{h}{p} \quad \text{or} \quad p = \frac{h}{k}r$$

or $p = ar$, where $a = h/k$ = a constant.

This is the pedal equation of an equiangular spiral. Hence the path is an equiangular spiral.

Ex. 13. *The velocity at any point of a central orbit is $(1/n)$th of what it would be for a circular orbit at the same distance. Show that the central force varies as* $\dfrac{1}{r^{(2n^2+1)}}$ *and that the equation of the orbit is*

$$r^{n^2-1} = a^{n^2-1}\cos(n^2-1)\theta. \qquad \textbf{[Meerut 75]}$$

Sol. Under the same central force P, let v and V be the velocities at a distance r from the centre of force in the central orbit and the circular orbit respectively. Then according to the question, we have $v = V/n$

or $$v^2 = V^2/n^2 \quad ...(1)$$

But $$V^2/r = P \qquad [\text{See § 7, page 8}]$$

or $$V^2 = Pr = P/u. \quad ...(2)$$

$\therefore$ from (1) and (2), we have

$$v^2 = \frac{P}{n^2u} \quad \text{or} \quad h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \frac{P}{n^2u} \quad ...(3)$$

$$[\because \text{ for a central orbit, } v^2 = h^2\{u^2 + (du/d\theta)^2\}].$$

Differentiating both sides of (3) w.r.t. 'θ', we have

$$h^2\left[2u\frac{du}{d\theta} + 2\frac{du}{d\theta}\frac{d^2u}{d\theta^2}\right] = \frac{1}{n^2}\left[\frac{1}{u}\frac{dP}{d\theta} - \frac{P}{u^2}\frac{du}{d\theta}\right]$$

$$= \frac{1}{n^2}\left[\frac{1}{u}\frac{dP}{du}\frac{du}{d\theta} - \frac{P}{u^2}\frac{du}{d\theta}\right].$$

$$\therefore \quad 2h^2\frac{du}{d\theta}\cdot\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{1}{n^2}\frac{du}{d\theta}\left[\frac{1}{u}\frac{dP}{du} - \frac{P}{u^2}\right].$$

Dividing out by $du/d\theta$, we get

$$2h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{1}{n^2}\left[\frac{1}{u}\frac{dP}{du} - \frac{P}{u^2}\right]$$

or $$2\cdot\frac{P}{u^2} = \frac{1}{n^2}\left[\frac{1}{u}\frac{dP}{du} - \frac{P}{u^2}\right] \qquad \left[\because \frac{P}{h^2u^2} = u + \frac{d^2u}{d\theta^2}\right]$$

or $$2n^2 \cdot \frac{P}{u^2} = \left[\frac{1}{u}\frac{dP}{du} - \frac{P}{u^2}\right] \quad \text{or} \quad (2n^2+1)\frac{P}{u^2} = \frac{1}{u}\frac{dP}{du}$$

or $$\frac{dP}{P} = (2n^2+1) \cdot \frac{du}{u}.$$

Integrating, $\log P = (2n^2+1)\log u + \log A$.

$$\therefore \quad P = Au^{2n^2+1} = \frac{A}{r^{2n^2+1}}.$$

$\therefore \quad P \propto \dfrac{1}{r^{2n^2+1}}$, which proves the first result.

Substituting $P = Au^{2n^2+1}$ in (3), we have

$$h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \frac{Au^{2n^2+1}}{n^2u} = \frac{A}{n^2}u^{2n^2}.$$

Putting $u = \dfrac{1}{r}$ so that $\dfrac{du}{d\theta} = -\dfrac{1}{r^2}\dfrac{dr}{d\theta}$, we have

$$\frac{1}{r^2} + \left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 = \frac{A}{n^2h^2r^{2n^2}}$$

or $$r^{2n^2-2} + r^{2n^2-4}\left(\frac{dr}{d\theta}\right)^2 = \frac{A}{n^2h^2}$$

or $$r^{2n^2-4}\left(\frac{dr}{d\theta}\right)^2 = \frac{A}{n^2h^2} - r^{2n^2-2}$$

or $$\left(r^{n^2-2}\right)^2\left(\frac{dr}{d\theta}\right)^2 = a^{2n^2-2} - r^{2n^2-2},$$

setting $A/n^2h^2 = a^{2n^2-2}$ to get the required form of the answer.

$$\therefore \quad \frac{dr}{d\theta} = \frac{\sqrt{\{a^{2n^2-2} - r^{2n^2-2}\}}}{r^{n^2-2}}$$

or $$\frac{r^{n^2-2}\,dr}{\sqrt{\left\{\left(a^{n^2-1}\right)^2 - \left(r^{n^2-1}\right)^2\right\}}} = d\theta.$$

Putting $r^{n^2-1} = z$ so that $(n^2-1)\,r^{n^2-2}\,dr = dz$, we have

$$\frac{dz}{\sqrt{\left\{\left(a^{n^2-1}\right)^2 - z^2\right\}}} = (n^2-1)\,d\theta.$$

Integrating, $\sin^{-1}\left(\dfrac{z}{a^{n^2-1}}\right) = (n^2-1)\,\theta + B$

or $$\sin^{-1}\left(\frac{r^{n^2-1}}{a^{n^2-1}}\right) = (n^2-1)\,\theta + B.$$

Initially when $\theta = 0$, let $r = a$. Then $B = \sin^{-1} 1 = \pi/2$.

$$\therefore \quad \sin^{-1}\left(\frac{r^{n^2-1}}{a^{n^2-1}}\right) = (n^2-1)\,\theta + \frac{1}{2}\pi$$

or $$\frac{r^{n^2-1}}{a^{n^2-1}} = \sin\{(n^2-1)\,\theta + \tfrac{1}{2}\pi\} = \cos(n^2-1)\,\theta$$

or $$r^{n^2-1} = a^{(n^2-1)}\cos(n^2-1)\,\theta,$$

which is the required equation of the orbit.

Ex. 14. *A particle moves with a central acceleration $\mu/(\text{distance})^2$; it is projected with velocity V at a distance R. Show that its path is a rectangular hyperbola if the angle of projection is*

$$\sin^{-1}\left[\frac{\mu}{VR\left(V^2 - \dfrac{2\mu}{R}\right)^{1/2}}\right].$$

[Meerut 93S]

Sol. If the particle describes a hyperbola under the central acceleration $\mu/(\text{distance})^2$, then the velocity v of the particle at a distance r from the centre of force is given by

$$v^2 = \mu\left(\frac{2}{r} + \frac{1}{a}\right), \quad \text{...(1)}$$

where $2a$ is the transverse axis of the hyperbola.

Since the particle is projected with velocity V at a distance R, therefore from (1), we have

$$V^2 = \mu\left(\frac{2}{R} + \frac{1}{a}\right) \quad \text{or} \quad \frac{\mu}{a} = V^2 - \frac{2\mu}{R}. \quad \text{...(2)}$$

If α is the required angle of projection to describe a rectangular hyperbola, then at the point of projection from the relation $h = vp$, we have

$$h = Vp = VR\sin\alpha \quad \text{...(3)}$$

[$\because p = r\sin\phi$ and initially $r = R, \phi = \alpha$]

Also $$h = \sqrt{(\mu l)} = \sqrt{\{\mu \cdot (b^2/a)\}} = \sqrt{(\mu a)} \quad \text{...(4)}$$

[$\because b = a$ for a rectangular hyperbola]

From (3) and (4), we have

$$VR\sin\alpha = \sqrt{(\mu a)}$$

or $\sin\alpha = \dfrac{\sqrt{(\mu a)}}{VR} = \dfrac{\mu\sqrt{a}}{VR\sqrt{\mu}} = \dfrac{\mu}{VR\sqrt{(\mu/a)}}$.

Substituting for μ/a from (2), we have

$$\sin\alpha = \mu/\{VR\sqrt{(V^2 - 2\mu/R)}\}$$

or $\alpha = \sin^{-1}[\mu/\{VR\sqrt{(V^2-2\mu/R)}\}]$,

which is the required angle of projection.

Ex. 15. *A particle of unit mass describes an equiangular spiral of angle α, under a force which is always in the direction perpendicular to the straight line joining the particle to the pole of the spiral; show that the force is $\mu r^{2\sec^2\alpha - 3}$ and that the rate of description of sectorial area about the pole is*

$$\tfrac{1}{2}\sqrt{(\mu \sin\alpha\cos\alpha)}\,.\,r^{\sec^2\alpha}.$$

Sol. Here the particle is moving under a force which is always in the direction perpendicular to the straight line joining the particle to the pole of the spiral.

$\therefore$ the central radial acceleration $= \ddot{r} - r\dot{\theta}^2 = 0$ ...(1)

If F is the force on the particle of unit mass, perpendicular to the line joining the particle to the pole, then

F = transverse acceleration

i.e., $F = \dfrac{1}{r}\dfrac{d}{dt}(r^2\dot{\theta})$. ...(2)

The equation of the equiangular spiral is

$r = ae^{\theta\cot\alpha}$. ...(3)

Differentiating (3) w.r.t. 't', we have

$\dot{r} = ae^{\theta\cot\alpha}\dot{\theta}\cot\alpha = r\dot{\theta}\cot\alpha$

or $\dot{\theta} = \dfrac{\dot{r}}{r}\tan\alpha$. ...(4)

$\therefore$ from (1) and (4), we have

$$\ddot{r} = r\left(\frac{\dot{r}}{r}\tan\alpha\right)^2$$

or $\dfrac{\ddot{r}}{\dot{r}} = \dfrac{\dot{r}}{r}\tan^2\alpha$.

Integrating, we have

$$\log\dot{r} = (\tan^2\alpha)\log r + \log A,$$

where A is a constant of integration

or $\log\dot{r} = \log\left(A r^{\tan^2\alpha}\right)$

or $\dot{r} = A r^{\tan^2\alpha}$. ...(5)

Substituting the value of $\dot{r}$ from (5) in (4), we have

$$\dot{\theta} = \frac{1}{r}\tan\alpha\,.\,A r^{\tan^2\alpha}$$

or $\dot{\theta} = A\tan\alpha\,.\,r^{\tan^2\alpha - 1}$ ...(6)

$\therefore$ from (2), we have

$F = \dfrac{1}{r}\dfrac{d}{dt}\left(r^2 A\tan\alpha\,.\,r^{\tan^2\alpha - 1}\right) = \dfrac{A\tan\alpha}{r}\dfrac{d}{dt}\left(r^{\tan^2\alpha + 1}\right)$

$= \dfrac{A\tan\alpha}{r}\dfrac{d}{dt}\left(r^{\sec^2\alpha}\right) = \dfrac{A\tan\alpha}{r}\cdot\sec^2\alpha\, r^{\sec^2\alpha - 1}\dot{r}$

$= A\tan\alpha\sec^2\alpha\,.\,r^{\sec^2\alpha - 2}\,.\,A r^{\tan^2\alpha}$ [substituting from (5)]

$= A^2\tan\alpha\sec^2\alpha\, r^{\sec^2\alpha - 2 + \tan^2\alpha}$

$= \mu r^{\sec^2\alpha - 2 + \sec^2\alpha - 1}$, where $\mu = A^2\tan\alpha\sec^2\alpha$. ...(7)

Thus $F = \mu r^{2\sec^2\alpha - 3}$, which proves the first part.

Second Part. The rate of description of the sectorial area

$= \tfrac{1}{2}r^2\dot{\theta}$

$= \tfrac{1}{2}r^2 A\tan\alpha\, r^{\tan^2\alpha - 1}$ [substituting from (6)]

$= \tfrac{1}{2}A\tan\alpha\, r^{2+\tan^2\alpha - 1}$

$= \tfrac{1}{2}\sqrt{(\mu\cot\alpha\cos^2\alpha)}\tan\alpha\, r^{\tan^2\alpha + 1}$

[substituting $A = \sqrt{(\mu\cot\alpha\cos^2\alpha)}$, from (7)]

$= \tfrac{1}{2}\sqrt{(\mu\cot\alpha\cos^2\alpha\tan^2\alpha)}\, r^{\sec^2\alpha} = \tfrac{1}{2}\sqrt{(\mu\sin\alpha\cos\alpha)}\, r^{\sec^2\alpha}$.

§ 9. Apse and Apsidal distance.

1. Apse. Definition. *An apse is a point on the central orbit at which the radius vector from the centre of force to the point has a maximum or minimum value.*

2. Apsidal distance. *The length of the radius vector at an apse is called an apsidal distance.*

3. Apsidal angle. *The angle between two consecutive apsidal distances is called an apsidal angle.*

Theorem. *At an apse the radius vector is perpendicular to the tangent i.e., at an apse the particle moves at right angles to the radius vector.* **[Meerut 90, 90S]**

From the definition of an apse, r is maximum or minimum at an apse *i.e.,* $u = 1/r$ is minimum or maximum at an apse.

$\therefore$ at an apse, $du/d\theta = 0$.

But we know that $\dfrac{1}{p^2} = u^2 + \left(\dfrac{du}{d\theta}\right)^2$.

$\therefore$ at an apse, $\dfrac{1}{p^2} = u^2 = \dfrac{1}{r^2}$

or $p = r$ or $r\sin\phi = r$

or $\sin\phi = 1$ or $\phi = 90°$.

This proves that at an apse the radius vector is perpendic... the tangent or in other words at an apse the particle moves at r... angles to the radius vector.

Remember. At an apse $dr/d\theta = 0$, $du/d\theta = 0$, $\phi = 90°$, $p = r$... the direction of motion is at right angles to the radius vector.

§ 10. Property of the apse-line. Theorem.

If the central acceleration P is a single valued function of the dist... every apse-line divides the orbit into equal and symmetrical portions... thus there can only be two apsidal distances.

Proof. Since the central acceleration P is a single valued func... of r, therefore the acceleration of the particle is the same at the sa... distance r.

The differential equation of a central orbit is

$$\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2} \quad \text{or} \quad h^2\left[\frac{d^2u}{d\theta^2} + u\right] = \frac{P}{u^2}.$$

Multiplying both sides by $2\,(du/d\theta)$ and integrating w.r.t. 'θ', w... have

$$v^2 = h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] = 2\int \frac{P}{u^2}\,du + C,$$

or

$$v^2 = C - 2\int P\,dr. \qquad \left\{\because \frac{1}{u} = r \Rightarrow -\frac{1}{u^2}\,du = dr\right\} \qquad \ldots(1)$$

The equation (1) shows that if P is a single valued function of the distance r, then the velocity of the particle is the same at the same distance r and is independent of the direction of motion.

Thus we observe that both velocity and acceleration are the same at the same distance from the centre. Therefore if at an apse the direction of velocity is reversed, the particle will describe symmetrica... orbit on both sides of the apse-line.

Now when the particle comes to a second apse, the path for the same reasons, is symmetrical about this second apsidal distance also. But this is possible only when the next (third) apsidal distance is equal to the one (first) before it and the angle between the first and the second apsidal distances is the same as the angle between the second and the third apsidal distances. Therefore if the central acceleration is a single valued function of the distance r, there are only two differen... apsidal distances. Also the angle between any two consecutive apsidal distances always remains the same and is called the *apsidal angle*.

§ 11. *To prove analytically that when the central acceleration varies as some integral power of the distance, there are at most two apsidal distances.* **[Meerut 951**

Let the central acceleration P be given by

$$P = \mu r^n, \text{ where } n \text{ is an integer.}$$

Thus $P = \mu u^{-n}$ because $r = 1/u = u^{-1}$.

$\therefore$ the differential equation of the path is

$$h^2\left\{u + \frac{d^2u}{d\theta^2}\right\} = \frac{P}{u^2} = \frac{\mu u^{-n}}{u^2} = \mu u^{-(n+2)}.$$

Multiplying both sides by $2\,(du/d\theta)$ and then integrating, we have

$$h^2\left\{u^2 + \left(\frac{du}{d\theta}\right)^2\right\} = \frac{\mu u^{-(n+1)}}{-(n+1)} + A. \qquad \ldots(1)$$

But at an apse $du/d\theta = 0$. So putting $du/d\theta = 0$ in (1), we have

$$h^2u^2 = -\frac{\mu}{n+1}u^{-(n+1)} + A$$

or

$$r^{n+3} - \frac{(n+1)}{\mu}Ar^2 + \frac{(n+1)}{\mu}h^2 = 0.$$

Whatever be the values of n or A this equation cannot have more than two changes of sign. Therefore by Descarte's rule of signs it cannot have more than two positive roots. Hence there are at most two positive values of r *i.e.*, at most two apsidal distances.

§ 12. Given the law of force, to find the orbit.

This problem is converse to that given in § 8 on page 9. For solving such a problem we substitute the given expression for P in the differential equation

$$h^2\left[\frac{d^2u}{d\theta^2} + u\right] = \frac{P}{u^2} \qquad \ldots(1)$$

or

$$\frac{h^2}{p^3}\frac{dp}{dr} = P, \qquad \ldots(2)$$

whichever is convenient. In case the force is repulsive, we take the value of P with negative sign.

Then integrating the resulting differential equation of the central orbit with the help of the given initial conditions, we get the (r, θ) or (p, r) equation of the orbit.

Illustrative Examples

Ex. 16 (a). *A particle moves with a central acceleration $\mu(r + a^4/r^3)$ being projected from an apse at a distance 'a' with a velocity $2a\sqrt{\mu}$. Prove that it describes the curve $r^2(2 + \cos\sqrt{3}\theta) = 3a^2$.* **(Agr 1978)**

Sol. Here, the central acceleration,

$P = \mu\,(r + a^4/r^3) = \mu\,\{(1/u) + a^4u^3\}$, where $u = 1/r$.

$\therefore$ the differential equation of the path is

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{\mu}{u^2}\left(\frac{1}{u} + a^4u^3\right)$$

or
$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \mu\left(\frac{1}{u^3} + a^4u\right).$$

Multiplying both sides by $2\,(du/d\theta)$ and integrating w.r.t. 'θ', we have

$$h^2\left[2\cdot\frac{u^2}{2} + \left(\frac{du}{d\theta}\right)^2\right] = 2\mu\left(-\frac{1}{2u^2} + \frac{a^4u^2}{2}\right) + A$$

or
$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\left(-\frac{1}{u^2} + a^4u^2\right) + A,$$

where A is a constant. ...(1)

Now initially the particle has been projected from an apse (say, the point A) at a distance a with velocity $2\sqrt{\mu}\,a$. Therefore when $r = a$ *i.e.*, $u = 1/a$, $du/d\theta = 0$ (at an apse) and $v = 2\sqrt{\mu}\,a$.

$\therefore$ from (1), we have

$$4\mu a^2 = h^2\left[\frac{1}{a^2}\right] = \mu\left(-a^2 + a^4\cdot\frac{1}{a^2}\right) + A.$$

(i) (ii) (iii)

From (i) and (ii), we have $h^2 = 4\mu a^4$ and from (i) and (iii), we have

$$4\mu a^2 = 0 + A \quad i.e., \quad A = 4\mu a^2.$$

Substituting the values of h^2 and A in (1), we have

$$4\mu a^4\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\left(-\frac{1}{u^2} + a^4u^2\right) + 4\mu a^2$$

or
$$4a^4\left(\frac{du}{d\theta}\right)^2 = -4a^4u^2 - \frac{1}{u^2} + a^4u^2 + 4a^2$$

or
$$4a^4u^2\left(\frac{du}{d\theta}\right)^2 = (-1 - 3a^4u^4 + 4a^2u^2)$$

or
$$2a^2u\,\frac{du}{d\theta} = \sqrt{[-1 - 3a^4u^4 + 4a^2u^2]} \qquad \text{[taking square root]} \qquad ...(2)$$

or
$$d\theta = \frac{2a^2u\,du}{\sqrt{[-1 - 3a^4u^4 + 4a^2u^2]}}$$
$$= \frac{2a^2u\,du}{\sqrt{3}\,.\sqrt{[-\frac{1}{3} - (a^4u^4 - \frac{4}{3}a^2u^2)]}}$$

$$= \frac{2a^2u\,du}{\sqrt{3}\,.\sqrt{[-\frac{1}{3} - (a^2u^2 - \frac{2}{3})^2 + \frac{4}{9}]}}$$
$$= \frac{2a^2u\,du}{\sqrt{3}\,.\sqrt{[(\frac{1}{3})^2 - (a^2u^2 - \frac{2}{3})^2]}}$$

or
$$\sqrt{3}\,d\theta = \frac{2a^2u\,du}{\sqrt{[(\frac{1}{3})^2 - (a^2u^2 - \frac{2}{3})^2]}}.$$

Substituting $a^2u^2 - \frac{2}{3} = z$, so that $2a^2u\,du = dz$, we have

$$\sqrt{3}\,d\theta = \frac{dz}{\sqrt{[(\frac{1}{3})^2 - z^2]}}.$$

Integrating, $\sqrt{3}\theta + B = \sin^{-1}(3z)$ where B is a constant

or
$$\sqrt{3}\theta + B = \sin^{-1}(3a^2u^2 - 2). \qquad ...(3)$$

Now take the aspe-line OA as the initial line. Then initially

$r = a, u = 1/a$ and $\theta = 0$.

$\therefore$ from (3), $0 + B = \sin^{-1} 1$ or $B = \frac{1}{2}\pi$.

Putting $B = \frac{1}{2}\pi$ in (3), we have

$$\sqrt{3}\theta + \tfrac{1}{2}\pi = \sin^{-1}(3a^2u^2 - 2)$$

or
$$3a^2u^2 - 2 = \sin(\tfrac{1}{2}\pi + \sqrt{3}\theta) = \cos(\sqrt{3}\theta)$$

or
$$\frac{3a^2}{r^2} - 2 = \cos(\sqrt{3}\theta) \quad \text{or} \quad 3a^2 - 2r^2 = r^2\cos(\sqrt{3}\theta).$$

$\therefore$ $3a^2 = r^2\,[2 + \cos(\sqrt{3}\theta)]$,

which is the equation of the required curve.

Remarks. We know that a central orbit is symmetrical about an apse- line. So if we take an apse-line as the initial line, then while extracting the square root of the equation (2) we can keep either the positive sign or the negative sign. In both the cases we shall get the same result. The students can verify it by solving the above problem while keeping the negative sign on extracting the square root of (2).

After extracting the square root of the equation (2) and then separating the variables, we should first try to integrate with respect to u. If we find any difficulty in integrating w.r.t. 'u', we should change u to r by putting $u = 1/r$.

Ex. 16 (b). *A particle subject to the central acceleration $(\mu/r^3) + f$ is projected from an apse at a distance 'a' with the velocity $\sqrt{\mu/a}$; prove that at any subsequent time t, $r = a - \frac{1}{2}ft^2$.* **(Meerut 1977)**

Sol. Here the central acceleration

$$P = \frac{\mu}{r^3} + f = \mu u^3 + f, \quad \text{where } \frac{1}{r} = u.$$

$\therefore$ the differential equation of the path is

$$h^2\left[u+\frac{d^2u}{d\theta^2}\right]=\frac{P}{u^2}=\frac{1}{u^2}(\mu u^3+f)$$

or $$h^2\left[u+\frac{d^2u}{d\theta^2}\right]=\mu u+\frac{f}{u^2}.$$

Multiplying both sides by $2\,(du/d\theta)$ and integrating, we have

$$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu u^2-\frac{2f}{u}+A,$$

where A is a constant.

But initially when $r=a$ i.e., $u=1/a$, $du/d\theta=0$ (at an apse) and $v=\sqrt{\mu}/a$.

$\therefore$ from (1), we have $\dfrac{\mu}{a^2}=h^2\left(\dfrac{1}{a^2}\right)=\dfrac{\mu}{a^2}-2fa+A.$

$\therefore$ $h^2=\mu$ and $A=2fa$.

Substituting the values of h^2 and A in (1), we have

$$\mu\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu u^2-\frac{2f}{u}+2fa$$

or $$\mu\left(\frac{du}{d\theta}\right)^2=2fa-\frac{2f}{u}. \quad ...(2)$$

Now $u=1/r$, so that $\dfrac{du}{d\theta}=-\dfrac{1}{r^2}\dfrac{dr}{d\theta}$. Therefore, from (2), we have

$$\mu\left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)^2=2fa-2fr=2f(a-r)$$

or $$\left(\frac{dr}{d\theta}\right)^2=\frac{2fr^4}{\mu}(a-r) \quad \text{or} \quad \frac{dr}{d\theta}=-\sqrt{(2f/\mu)}\,.\,r^2\sqrt{(a-r)}$$

Also $$h=r^2\frac{d\theta}{dt}=r^2\frac{d\theta}{dr}\cdot\frac{dr}{dt}.$$

$$\therefore \sqrt{\mu}=r^2\,.\sqrt{\left(\frac{\mu}{2f}\right)}\cdot\frac{(-1)}{r^2\sqrt{(a-r)}}\cdot\frac{dr}{dt}$$

[substituting for h and $dr/d\theta$]

or $$dt=\frac{-1}{\sqrt{(2f)}}\cdot(a-r)^{-1/2}\,dr.$$

Integrating, $$t=\frac{1}{\sqrt{(2f)}}\cdot 2\,(a-r)^{1/2}+B,$$ where B is a constant.

But initially when $t=0, r=a$; $\therefore$ $B=0$.

$\therefore$ $t=\sqrt{(2/f)}\,.\,(a-r)^{1/2}$

or $t^2=(2/f)(a-r)$

or $a-r=\frac{1}{2}ft^2$. $\therefore$ $r=a-\frac{1}{2}ft^2$.

Ex. 17. *A particle moves under a force $m\mu\,\{3au^4-2\,(a^2-b^2)\,u^5\}$, $a>b$ and is projected from an apse at a distance $(a+b)$ with velocity $\sqrt{\mu}/(a+b)$. Show that the equation of its path is $r=a+b\cos\theta$.*

(Meerut 1982, 88S; Agra 77, 79, 85; Rohilkhand 86)

Sol. Here the central acceleration

$$P=\mu\,\{3au^4-2\,(a^2-b^2)\,u^5\}.$$

$\therefore$ the differential equation of the path is

$$h^2\left[u+\frac{d^2u}{d\theta^2}\right]=\frac{P}{u^2}=\frac{\mu}{u^2}\{3au^4-2\,(a^2-b^2)\,u^5\}$$

or $$h^2\left[u+\frac{d^2u}{d\theta^2}\right]=\mu\,\{3au^2-2\,(a^2-b^2)\,u^3\}.$$

Multiplying both sides by $2\,(du/d\theta)$ and integrating, we have

$$h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=2\mu\left\{au^3-2\,(a^2-b^2)\frac{u^4}{4}\right\}+A$$

or $$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu\,\{2au^3-(a^2-b^2)\,u^4\}+A, \quad ...(1)$$

where A is a constant.

But initially at an apse, $r=a+b, u=1/(a+b), du/d\theta=0$ and $v=\sqrt{\mu}/(a+b)$.

$\therefore$ from (1), we have

$$\frac{\mu}{(a+b)^2}=h^2\left[\frac{1}{(a+b)^2}\right]=\mu\left[\frac{2a}{(a+b)^3}-\frac{(a^2-b^2)}{(a+b)^4}\right]+A.$$

$\therefore$ $h^2=\mu$ and $A=0$.

Substituting the values of h^2 and A in (1), we have

$$\mu\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu\,\{2au^3-(a^2-b^2)\,u^4\}$$

or $$\left(\frac{du}{d\theta}\right)^2=-u^2+2au^3-(a^2-b^2)\,u^4. \quad ...(2)$$

But $u=\dfrac{1}{r}$, so that $\dfrac{du}{d\theta}=-\dfrac{1}{r^2}\dfrac{dr}{d\theta}$.

Substituting in (2), we have

$$\left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)^2=-\frac{1}{r^2}+\frac{2a}{r^3}-\frac{(a^2-b^2)}{r^4}$$

or $$\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2=\frac{1}{r^4}[-r^2+2ar-(a^2-b^2)]$$

or $$\left(\frac{dr}{d\theta}\right)^2=-r^2+2ar-a^2+b^2=b^2-(r^2-2ar+a^2)$$

$$= b^2 - (r-a)^2.$$

$\therefore \quad \dfrac{dr}{d\theta} = \sqrt{\{b^2 - (r-a)^2\}}$ or $d\theta = \dfrac{dr}{\sqrt{\{b^2-(r-a)^2\}}}$.

Integrating, $\quad \theta + B = \sin^{-1}\left(\dfrac{r-a}{b}\right)$.

But initially when $r = a + b$, let us take $\theta = 0$. Then from (3)

$$B = \sin^{-1}(1) = \pi/2.$$

Substituting in (3), we have

$$\theta + \tfrac{1}{2}\pi = \sin^{-1}\left(\frac{r-a}{b}\right) \quad \text{or} \quad r - a = b\sin(\tfrac{1}{2}\pi + \theta)$$

or $\quad r = a + b\cos\theta$, which is the required equation of the p

Ex. 18. *A particle moves under a repulsive force $m\mu/(\text{distance})^3$ and is projected from an apse at a distance a with a velocity V; show that the equation to the path is $r\cos p\theta = a$, and that the angle θ described in time t is $(1/p)\tan^{-1}(pVt/a)$, where*

$$p^2 = (\mu + a^2V^2)/(a^2V^2).$$

(Meerut 1976, 88, 92S, 93; Rohilkhand 85,

Sol. Since the particle moves under a **repulsive** force

$$\frac{m\mu}{(\text{distane})^3} = \frac{m\mu}{r^3},$$

$\therefore$ the central acceleration $P = -\dfrac{\mu}{r^3} = -\mu u^3$.

$\therefore$ the differential equation of the path is

$$h^2\left[u^2 + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{-\mu u^3}{u^2} = -\mu u.$$

Multiplying both sides by $2\,(du/d\theta)$ and integrating, we have

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = -\mu u^2 + A, \quad \text{...(1)}$$

where A is a constant.

But initially at an apse, $r = a, u = 1/a, du/d\theta = 0$ and $v = V$.

$\therefore$ from (1), we have

$$V^2 = h^2\left[\frac{1}{a^2}\right] = -\frac{\mu}{a^2} + A.$$

$\therefore \quad h^2 = a^2V^2 \quad$ and $\quad A = V^2 + (\mu/a^2)$. ...(2)

Substituting the values of h^2 and A in (1), we have

$$a^2V^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = -\mu u^2 + V^2 + \frac{\mu}{a^2}$$

or $\quad a^2V^2\left(\dfrac{du}{d\theta}\right)^2 = -(a^2V^2 + \mu)u^2 + \dfrac{(a^2V^2+\mu)}{a^2}$

$$a^2\left(\frac{du}{d\theta}\right)^2 = \frac{(a^2V^2+\mu)}{a^2V^2}(1 - a^2u^2)$$

or $\quad a^2\left(\dfrac{du}{d\theta}\right)^2 = p^2(1 - a^2u^2), \quad$ where $p^2 = \dfrac{\mu + a^2V^2}{a^2V^2}$

or $\quad a\dfrac{du}{d\theta} = p\sqrt{(1-a^2u^2)} \quad$ or $\quad p\,d\theta = \dfrac{a\,du}{\sqrt{(1-a^2u^2)}}$.

or

Integrating, $p\theta + B = \sin^{-1}(au)$, where B is a constant.

But initially when $u = 1/a$, let $\theta = 0$. Then $B = \sin^{-1}1 = \tfrac{1}{2}\pi$.

$$p\theta + \tfrac{1}{2}\pi = \sin^{-1}(au)$$

$\therefore \quad au = \sin(\tfrac{1}{2}\pi + p\theta)$

or $\quad a/r = \cos p\theta$

or $\quad r\cos p\theta = a$, ...(3)

or

which is the equation of the path.

Second part. We have

$h = r^2\dfrac{d\theta}{dt}$ **[Note** that in a central orbit for finding the time, we use this formula]

$aV = a^2\sec^2 p\theta\,\dfrac{d\theta}{dt}$, [substituting for h from (2) and for r from (3)]

or $\quad dt = (a/V)\sec^2 p\theta\,d\theta$.

or

Integrating, $\quad t + C = \dfrac{a}{pV}\tan p\theta$.

But initially $t = 0$ and $\theta = 0$. Therefore $C = 0$.

$\therefore \quad t = \dfrac{a}{pV}\tan p\theta \quad$ or $\quad \tan p\theta = pVt/a$.

$\therefore \quad \theta = (1/p)\tan^{-1}(pVt/a)$,

which gives the angle θ described in time t.

Ex. 19. *A particle moves under a central force*

$$m\lambda\,(3a^3u^4 + 8au^2).$$

It is projected from an apse at a distance a from the centre of force with velocity $\sqrt{(10\lambda)}$. Show that the second apsidal distance is half of the first and that the equation to the path is

$$2r = a\,[1 + \text{sech}\,(\theta/\sqrt{5})].$$

(Rohilkhand 1985)

Sol. Here the particle moves under the central force $m\lambda\,(3a^3u^4 + 8au^2)$. Therefore the central acceleration P is given by

$$P = \lambda\,(3a^3u^4 + 8au^2).$$

$\therefore$ the differential equation of the path is

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{\lambda}{u^2}(3a^3u^4 + 8au^2)$$

or $$h^2\left[u+\frac{d^2u}{d\theta^2}\right]=\lambda\,(3a^3u^2+8a).$$

Multiplying both sides by $2\,(du/d\theta)$ and integrating, we have

$$h^2\left[2.\frac{u^2}{2}+\left(\frac{du}{d\theta}\right)^2\right]=2\lambda\,.\,(a^3u^3+8au)+A$$

or $$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\lambda\,(2a^3u^3+16au)+A, \quad ...(1)$$

where A is a constant.

But initially at an apse, $r=a, u=1/a, du/d\theta=0$ and $v=\sqrt{(10\lambda)}$.

$\therefore$ from (1), we have

$$10\lambda=h^2\left[\frac{1}{a^2}\right]=\lambda\left(2a^3.\frac{1}{a^3}+16a.\frac{1}{a}\right)+A.$$

$\therefore \quad h^2=10a^2\lambda \quad$ and $\quad A=10\lambda-18\lambda=-8\lambda.$

Substituting the values of h^2 and A in (1), we have

$$10a^2\lambda\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\lambda\,(2a^3u^3+16au)-8\lambda$$

or $$10a^2\left(\frac{du}{d\theta}\right)^2=2a^3u^3-10a^2u^2+16au-8$$

or $$5a^2\left(\frac{du}{d\theta}\right)^2=[a^3u^3-5a^2u^2+8au-4]$$
$$=a^2u^2\,(au-1)-4au\,(au-1)+4\,(au-1)$$
$$=(au-1)\,(a^2u^2-4au+4)$$
$$=(au-1)\,(au-2)^2. \quad ...(2)$$

To find the second apsidal distance. At an apse, we have $du/d\theta=0$.

$\therefore$ from (2), $\quad 0=(au-1)\,(au-2)^2$

or $\quad u=1/a \quad$ and $\quad 2/a \quad$ or $\quad r=a$ and $a/2$.

But $r=a$ is the first apsidal distance. Therefore the second apsidal distance is $a/2$ which is half of the first.

To find the equation of the path. From equation (2), we have

$$\sqrt{5}a\,\frac{du}{d\theta}=-\,(au-2)\,\sqrt{(au-1)}.$$

$$\therefore \quad \frac{d\theta}{\sqrt{5}}=\frac{-\,a\,du}{(au-2)\,\sqrt{au-1}}.$$

Substituting $au-1=z^2$, so that $adu=2z\,dz$, we have

$$\frac{d\theta}{\sqrt{5}}=\frac{-\,2z\,dz}{(z^2-1)\,z}$$

or $$\frac{d\theta}{2\sqrt{5}}=\frac{dz}{1-z^2}.$$

Integrating, $\dfrac{\theta}{2\sqrt{5}}+B=\tanh^{-1}z$, where B is a constant

or $$\frac{\theta}{2\sqrt{5}}+B=\tanh^{-1}\sqrt{(au-1)}. \quad ...(3)$$

But initially, when $u=1/a, \theta=0$.

$\therefore$ from (3), $B=0$.

Putting $B=0$ in (3), we get

$$\frac{\theta}{2\sqrt{5}}=\tanh^{-1}\sqrt{(au-1)}$$

or $$\tanh\left(\frac{\theta}{2\sqrt{5}}\right)=\sqrt{(au-1)}.$$

Now $$\cosh 2A=\frac{1+\tanh^2 A}{1-\tanh^2 A}\cdot \textbf{(Remember)}$$

$$\therefore \quad \cosh\left(\frac{\theta}{\sqrt{5}}\right)=\frac{1+\tanh^2(\theta/2\sqrt{5})}{1-\tanh^2(\theta/2\sqrt{5})}=\frac{1+(au-1)}{1-(au-1)}=\frac{au}{2-au}$$

or $$2-au=au\,\text{sech}\,(\theta/\sqrt{5})$$

or $$2=au\,[1+\text{sech}\,(\theta/\sqrt{5})]=(a/r)\,[1+\text{sech}\,(\theta/\sqrt{5})]$$

or $$2r=a\,[1+\text{sech}\,(\theta/\sqrt{5})],$$

which is the required equation of the path.

Ex. 20. *A particle subject to a central force per unit of mass equal to $\mu\,\{2\,(a^2+b^2)\,u^5-3a^2b^2u^7\}$ is projected at the distance a with velocity $\sqrt{\mu}/a$ in a direction at right angles to the initial distance; show that the path is the curve*

$$r^2=a^2\cos^2\theta+b^2\sin^2\theta. \quad \textbf{(Meerut 1975, 81)}$$

Sol. Here, the central acceleration

$$P=\mu\,\{2\,(a^2+b^2)\,u^5-3a^2b^2u^7\}.$$

$\therefore$ the differential equation of the path is

$$h^2\left[u+\frac{d^2u}{d\theta^2}\right]=\frac{P}{u^2}=\frac{\mu}{u^2}\{2\,(a^2+b^2)\,u^5-3a^2b^2u^7\}$$

or $$h^2\left[u+\frac{d^2u}{d\theta^2}\right]=\mu\,\{2\,(a^2+b^2)\,u^3-3a^2b^2u^5\}.$$

Multiplying both sides by $2\,(du/d\theta)$ and integrating, we have

$$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu\,\{(a^2+b^2)\,u^4-a^2b^2u^6\}+A, \quad ...(1)$$

where A is a constant.

Now at the point of projection the direction of velocity is perpendicular to the radius vector. So the point of projection is an apse. Therefore initially when $r=a, u=1/a, \ du/d\theta=0$ and $v=\sqrt{\mu}/a$.

$\therefore$ from (1), we have

$$\frac{\mu}{a^2} = h^2 \left[\frac{1}{a^2}\right] = \mu \left\{\frac{(a^2+b^2)}{a^4} - \frac{a^2b^2}{a^6}\right\} + A.$$

$\therefore \quad h^2 = \mu \quad$ and $\quad A = 0.$

Substituting the values of h^2 and A in (1), we have

$$\mu \left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu \left\{(a^2+b^2)u^4 - a^2b^2u^6\right\}$$

or $$\left(\frac{du}{d\theta}\right)^2 = -u^2 + (a^2+b^2)u^4 - a^2b^2u^6.$$

Putting $u = \frac{1}{r}$, so that $\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$, we have

$$\left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 = -\frac{1}{r^2} + (a^2+b^2)\frac{1}{r^4} - a^2b^2\frac{1}{r^6}$$

or $$\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{1}{r^6}\left\{-r^4 + (a^2+b^2)r^2 - a^2b^2\right]$$

or $$\left(\frac{dr}{d\theta}\right)^2 = \frac{1}{r^2}\left[-a^2b^2 - \{r^4 - (a^2+b^2)r^2\}\right]$$

$$= \frac{1}{r^2}\left[-a^2b^2 - \{r^2 - \tfrac{1}{2}(a^2+b^2)\}^2 + \tfrac{1}{4}(a^2+b^2)^2\right]$$

$$= \frac{1}{r^2}\left[\tfrac{1}{4}(a^2-b^2)^2 - \{r^2 - \tfrac{1}{2}(a^2+b^2)\}^2\right].$$

$$\therefore \quad \frac{dr}{d\theta} = -\frac{1}{r}\sqrt{\left[\tfrac{1}{4}(a^2-b^2)^2 - \{r^2 - \tfrac{1}{2}(a^2+b^2)\}^2\right]}$$

or $$d\theta = \frac{-r\,dr}{\sqrt{[\tfrac{1}{4}(a^2-b^2)^2 - \{r^2 - \tfrac{1}{2}(a^2+b^2)\}^2]}}.$$

Putting $r^2 - \frac{1}{2}(a^2+b^2) = z$, so that $2r\,dr = dz$, we have

$$d\theta = \frac{-\tfrac{1}{2}dz}{\sqrt{[\tfrac{1}{4}(a^2-b^2)^2 - z^2]}}.$$

Integrating, we get

$$\theta + B = \tfrac{1}{2}\cos^{-1}\left\{\frac{z}{\tfrac{1}{2}(a^2-b^2)}\right\} = \tfrac{1}{2}\cos^{-1}\left\{\frac{r^2 - \tfrac{1}{2}(a^2+b^2)}{\tfrac{1}{2}(a^2-b^2)}\right\},$$

where B is a constant.

Initially when $r = a, \theta = 0.$

$$\therefore \quad B = \tfrac{1}{2}\cos^{-1}\left\{\frac{a^2 - \tfrac{1}{2}(a^2+b^2)}{\tfrac{1}{2}(a^2-b^2)}\right\} = \tfrac{1}{2}\cos^{-1}\left\{\frac{\tfrac{1}{2}(a^2-b^2)}{\tfrac{1}{2}(a^2-b^2)}\right\}$$

$$= \tfrac{1}{2}\cos^{-1}1 = 0.$$

$$\text{Hence } \theta = \tfrac{1}{2}\cos^{-1}\left\{\frac{r^2 - \tfrac{1}{2}(a^2+b^2)}{\tfrac{1}{2}(a^2-b^2)}\right\}$$

or $$2\theta = \cos^{-1}\left\{\frac{r^2 - \tfrac{1}{2}(a^2+b^2)}{\tfrac{1}{2}(a^2-b^2)}\right\}$$

or $$\cos 2\theta = \frac{r^2 - \tfrac{1}{2}(a^2+b^2)}{\tfrac{1}{2}(a^2-b^2)}$$

or $$r^2 - \tfrac{1}{2}(a^2+b^2) = \tfrac{1}{2}(a^2-b^2)\cos 2\theta$$

or $$r^2 = \tfrac{1}{2}(a^2+b^2) + \tfrac{1}{2}(a^2-b^2)\cos 2\theta$$

$$= \tfrac{1}{2}a^2(1+\cos 2\theta) + \tfrac{1}{2}b^2(1-\cos 2\theta)$$

or $$r^2 = a^2\cos^2\theta + b^2\sin^2\theta,$$

which is the required equation of the path.

Ex. 21. *A particle moves with a central acceleration $\lambda^2(8au^2 + a^4u^5)$; it is projected with velocity 9λ from an apse at a distance $a/3$ from the origin; show that the equation to its path is*

$$\frac{1}{\sqrt{3}}\sqrt{\left(\frac{au+5}{au-3}\right)} = \cot(\theta/\sqrt{6}).$$

Sol. Here the central acceleration $P = \lambda^2(8au^2 + a^4u^5)$.

$\therefore$ the differential equation of the path is

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{\lambda^2}{u^2}(8au^2 + a^4u^5)$$

or $$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \lambda^2(8a + a^4u^3).$$

Multiplying both sides by $2(du/d\theta)$ and integrating, we have

$$h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = 2\lambda^2\left(8au + \frac{a^4u^4}{4}\right) + A$$

or $$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \lambda^2\left(16au + \frac{a^4u^4}{2}\right) + A, \quad \text{...(1)}$$

where A is a constant.

But initially when $r = a/3$ *i.e.*, $u = 3/a$, $du/d\theta = 0$ (at an apse) and $v = 9\lambda$.

$\therefore$ from (1), we have

$$81\lambda^2 = h^2\left[\frac{9}{a^2}\right] = \lambda^2\left(16a\cdot\frac{3}{a} + \frac{a^4}{2}\cdot\frac{81}{a^4}\right) + A.$$

$\therefore \quad h^2 = 9a^2\lambda^2 \quad$ and $\quad A = \frac{-15}{2}\lambda^2.$

Substituting the values of h^2 and A in (1), we have

$$9a^2\lambda^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\lambda^2\left(16au+\frac{a^4u^4}{2}\right)-\frac{15}{2}\lambda^2$$

or $$9a^2\left(\frac{du}{d\theta}\right)^2=\frac{a^4u^4}{2}-9a^2u^2+16au-\frac{15}{2}$$

or $$18a^2\left(\frac{du}{d\theta}\right)^2=a^4u^4-18a^2u^2+32au-15$$
$$=a^3u^3(au-1)+a^2u^2(au-1)-17au(au-1)+15(au-1)$$
$$=(au-1)(a^3u^3+a^2u^2-17au+15)$$
$$=(au-1)\{a^2u^2(au-1)+2au(au-1)-15(au-1)\}$$
$$=(au-1)^2(a^2u^2+2au-15)$$
$$=(au-1)^2(au-3)(au+5).$$

∴ $$3\sqrt{2}a\frac{du}{d\theta}=(au-1)\sqrt{[(au-3)(au+5)]}$$ [Note]

or $$\frac{d\theta}{3\sqrt{2}}=\frac{a\,du}{(au-1)\sqrt{[(au-3)(au+5)]}}.$$

Substituting $au+5=(au-3)z^2$, so that $au=\dfrac{3z^2+5}{z^2-1}$

and $$a\,du=\frac{6z(z^2-1)-(3z^2+5).2z}{(z^2-1)^2}dz=\frac{-16z\,dz}{(z^2-1)^2},$$ we have

$$\frac{d\theta}{3\sqrt{2}}=\frac{-\dfrac{16z\,dz}{(z^2-1)^2}}{\left(\dfrac{3z^2+5}{z^2-1}-1\right)\left(\dfrac{3z^2+5}{z^2-1}-3\right).z}$$

or $$\frac{d\theta}{3\sqrt{2}}=-\frac{dz}{z^2+3}.$$

Integrating, $\dfrac{\theta}{3\sqrt{2}}+B=\dfrac{1}{\sqrt{3}}\cot^{-1}(z/\sqrt{3})$, where B is a constant

or $$\frac{\theta}{3\sqrt{2}}+B=\frac{1}{\sqrt{3}}\cot^{-1}\left\{\sqrt{\left(\frac{au+5}{au-3}\right)}\cdot\frac{1}{\sqrt{3}}\right\}.$$

But initially, $u=3/a$ and $\theta=0$.

∴ $0+B=\dfrac{1}{\sqrt{3}}\cot^{-1}\infty=0$ or $B=0$.

∴ $$\frac{\theta}{3\sqrt{2}}=\frac{1}{\sqrt{3}}\cot^{-1}\left\{\frac{1}{\sqrt{3}}\sqrt{\left(\frac{au+5}{au-3}\right)}\right\}$$

or $$\cot^{-1}\left\{\frac{1}{\sqrt{3}}\sqrt{\left(\frac{au+5}{au-3}\right)}\right\}=\frac{\theta}{\sqrt{6}}$$

or $$\frac{1}{\sqrt{3}}\sqrt{\left(\frac{au+5}{au-3}\right)}=\cot(\theta/\sqrt{6}),$$

which is the required equation of the path.

Ex. 22. *A particle moving with a central acceleration $\mu/(\text{distance})^3$ is projected from an apse at a distance a with a velocity V; show that the path is*

$$r\cosh\left\{\frac{\sqrt{(\mu-a^2V^2)}}{aV}\theta\right\}=a \quad or \quad r\cos\left\{\frac{\sqrt{(a^2V^2-\mu)}}{aV}\theta\right\}=a$$

according as V is < or > the velocity from infinity. **[Meerut 88]**

Sol. Here, the central acceleration P

$$=\frac{\mu}{(\text{distance})^3}=\frac{\mu}{r^3}=\mu u^3.$$

The differential equation of the path is

$$h^2\left[u+\frac{d^2u}{d\theta^2}\right]=\frac{P}{u^2}=\frac{\mu u^3}{u^2}=\mu u.$$

Multiplying both sides by $2\,(du/d\theta)$ and integrating, we have

$$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu u^2+A, \quad ...(1)$$

where A is a constant.

But initially when $r=a$ i.e., $u=\dfrac{1}{a}$, $\dfrac{du}{d\theta}=0$ (at an apse) and $v=V$.

∴ from (1), $V^2=h^2\left[\dfrac{1}{a^2}\right]=\dfrac{\mu}{a^2}+A$.

∴ $h^2=a^2V^2$ and $A=V^2-\dfrac{\mu}{a^2}=\dfrac{(V^2a^2-\mu)}{a^2}$.

Substituting the values of h^2 and A in (1), we have

$$a^2V^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu u^2+\frac{(V^2a^2-\mu)}{a^2}$$

or $$a^2V^2\left(\frac{du}{d\theta}\right)^2=-a^2V^2u^2+\mu u^2+\frac{(V^2a^2-\mu)}{a^2}$$
$$=-(a^2V^2-\mu)u^2+(a^2V^2-\mu)/a^2$$
$$=(a^2V^2-\mu)(-u^2+1/a^2)$$

or $$a^4V^2\left(\frac{du}{d\theta}\right)^2=(a^2V^2-\mu)(1-a^2u^2). \quad ...(2)$$

If V_1 is the velocity acquired by the particle in falling form infinity to the distance a, then

$$V_1^2=-2\int_\infty^a P\,dr=-2\int_\infty^a\frac{\mu}{r^3}dr=-2\left[-\frac{\mu}{2r^2}\right]_\infty^a=\frac{\mu}{a^2}.$$

Case I. When $V < V_1$ (velocity from infinity), we have
$V^2 < V_1^2$ or $V^2 < \mu/a^2$ or $a^2V^2 < \mu$ or $\mu - a^2V^2 > 0$.

$\therefore$ from (2), we have

$$a^4V^2\left(\frac{du}{d\theta}\right)^2 = (\mu - a^2V^2)(a^2u^2 - 1)$$

or $$a^2V\frac{du}{d\theta} = \sqrt{(\mu - a^2V^2)}\,.\,\sqrt{(a^2u^2 - 1)}$$

or $$\frac{\sqrt{(\mu - a^2V^2)}}{aV}\,d\theta = \frac{adu}{\sqrt{(a^2u^2 - 1)}}.$$

Substituting $au = z$, so that $adu = dz$, we have

$$\frac{\sqrt{(\mu - a^2V^2)}}{aV}\,d\theta = \frac{dz}{\sqrt{(z^2 - 1)}}.$$

Integrating, $$\frac{\sqrt{(\mu - a^2V^2)}}{aV}\,\theta + B = \cosh^{-1} z$$

or $$\frac{\sqrt{(\mu - a^2V^2)}}{aV}\,\theta + B = \cosh^{-1}(au).$$

But initially when $u = 1/a, \theta = 0$.

$$\therefore\ 0 + B = \cosh^{-1} 1 = 0 \quad \text{or} \quad B = 0.$$

$$\therefore\ \frac{\sqrt{(\mu - a^2V^2)}}{aV}\,\theta = \cosh^{-1}(au)$$

or $$au = \frac{a}{r} = \cosh\left\{\frac{\sqrt{(\mu - a^2V^2)}}{aV}\,\theta\right\}$$

or $$r\cosh\left\{\frac{\sqrt{(\mu - a^2V^2)}}{aV}\,\theta\right\} = a.$$

Case II. When $V > V_1$ (velocity from infinity), we have
$V^2 > V_1^2$ or $V^2 > \mu/a^2$ or $a^2V^2 - \mu > 0$.

$\therefore$ from (2), we have

$$a^4V^2\left(\frac{du}{d\theta}\right)^2 = (a^2V^2 - \mu)(1 - a^2u^2)$$

or $$a^2V\left(\frac{du}{d\theta}\right) = \sqrt{(a^2V^2 - \mu)}\,.\,\sqrt{(1 - a^2u^2)}$$

or $$\frac{\sqrt{(a^2V^2 - \mu)}}{aV}\,d\theta = \frac{adu}{\sqrt{(1 - a^2u^2)}}.$$

Integrating, $$\frac{\sqrt{(a^2V^2 - \mu)}}{aV}\,\theta + C = \sin^{-1}(au).$$

But initially when $u = 1/a, \theta = 0$.

$$\therefore\ 0 + C = \sin^{-1} 1 \quad \text{or} \quad C = \pi/2.$$

$$\therefore\ \frac{\sqrt{(a^2V^2 - \mu)}}{aV}\,\theta + \frac{\pi}{2} = \sin^{-1}(au)$$

$$\therefore\ au = \frac{a}{r} = \sin\left\{\frac{\sqrt{(a^2V^2 - \mu)}}{aV}\,\theta + \frac{\pi}{2}\right\}$$

or $$a = r\cos\left\{\frac{\sqrt{(a^2V^2 - \mu)}}{aV}\,\theta\right\}.$$

or

Ex. 23. *A particle, acted on by a repulsive central force $\mu r/(r^2 - 9c^2)^2$, is projected from an apse at a distance c with velocity $\sqrt{(\mu/8c^2)}$. Find the equation of its path and show that the time to the cusp is $\frac{4}{3}\pi c^2\sqrt{(2/\mu)}$.*

Sol. Considering the particle of unit mass, the central acceleration

$$P = \frac{-\mu r}{(r^2 - 9c^2)^2}$$

(Negative sign is taken because the force is repulsive).

The differential equation of the path in **pedal form** is

$$\frac{h^2}{p^3}\frac{dp}{dr} = P = -\frac{\mu r}{(r^2 - 9c^2)^2}$$

or $$-\frac{2h^2}{p^3}\,dp = \frac{2\mu r\,dr}{(r^2 - 9c^2)^2} = 2\mu r\,(r^2 - 9c^2)^{-2}\,dr.$$

Integrating, $$v^2 = \frac{h^2}{p^2} = -\frac{\mu}{(r^2 - 9c^2)} + A, \qquad \text{...(1)}$$

where A is a constant.

But the particle is projected from an apse at a distance c. Also at an apse, $p = r$. Therefore initially $p = r = c$ and $v = \sqrt{(\mu/8c^2)}$.

$\therefore$ from (1), we have

$$\frac{\mu}{8c^2} = \frac{h^2}{c^2} = -\frac{\mu}{(c^2 - 9c^2)} + A.$$

$$\therefore\ h^2 = \mu/8 \quad \text{and} \quad A = \frac{\mu}{8c^2} - \frac{\mu}{8c^2} = 0.$$

Substituting the values of h^2 and A in (1), we have

$$\frac{\mu}{8p^2} = -\frac{\mu}{(r^2 - 9c^2)} \quad \text{or} \quad 8p^2 = 9c^2 - r^2, \qquad \text{...(2)}$$

which is the pedal equation of the path and is a three-cusped hypocycloid.

Second part. Now we are to find the time to reach the cusp. At the cusp we have $p = 0$. So it is required to find the time from $p = c$ to $p = 0$.

We know that in a central orbit

$$v=\frac{ds}{dt}=\frac{h}{p}.$$

$\therefore \quad h\,dt=p\,ds \qquad$ or $\qquad h\,dt=p\,\dfrac{ds}{dr}\cdot dr.$

But $dr/ds=\cos\phi$.

$$\therefore \quad h\,dt=p\cdot\frac{1}{\cos\phi}\,dr=\frac{p\,dr}{\sqrt{(1-\sin^2\phi)}}=\frac{p\,dr}{\sqrt{\{1-(p^2/r^2)\}}}$$

$$=\frac{pr\,dr}{\sqrt{(r^2-p^2)}}=\frac{p\,(-8p)\,dp}{\sqrt{(9c^2-8p^2-p^2)}}$$

[$\because$ from (2), $-r\,dr=8p\,dp$]

$$=\frac{-8p^2\,dp}{3\sqrt{(c^2-p^2)}}.$$

Let t_1 be the required time to the cusp. Then integrating from $p=c$ to $p=0$, we get

$$ht_1=-\tfrac{1}{3}\int_c^0\frac{8p^2\,dp}{\sqrt{(c^2-p^2)}}=\frac{8}{3}\int_0^c\frac{p^2\,dp}{\sqrt{(c^2-p^2)}}$$

$$=\frac{8}{3}\int_0^{\pi/2}\frac{c^2\sin^2 z}{c\cos z}\,c\cos z\,dz$$

[putting $p=c\sin z$, so that $dp=c=\cos z\,dz$]

$$=\frac{8}{3}c^2\int_0^{\pi/2}\sin^2 z\,dz=\frac{8}{3}c^2\cdot\tfrac{1}{2}\times\frac{\pi}{2}=\frac{2\pi c^2}{3}\,.$$

$$\therefore \quad t_1=\frac{2\pi c^2}{3h}=\frac{2\pi c^2}{3}\cdot\sqrt{\left(\frac{8}{\mu}\right)} \qquad [\because h^2=\mu/8]$$

$$=\frac{4\pi c^2}{3}\sqrt{\left(\frac{2}{\mu}\right)}.$$

Ex. 24. *A particle is moving with central acceleration $\mu(r^5-c^4r)$ being projected from an apse at a distance c with velocity $c^3\sqrt{(2\mu/3)}$, show that its path is the curve $x^4+y^4=c^4$.*

(Meerut 1980, 90, 94; Rohilkhand 78, 79; Agra 84, 86, 88)

Sol. Here the central acceleration

$$P=\mu(r^5-c^4r)=\mu\left(\frac{1}{u^5}-\frac{c^4}{u}\right).$$

$\therefore$ The differential equation of the path is

$$h^2\left[u+\frac{d^2u}{d\theta^2}\right]=\frac{P}{u^2}=\frac{\mu}{u^2}\left(\frac{1}{u^5}-\frac{c^4}{u}\right)=\mu\left(\frac{1}{u^7}-\frac{c^4}{u^3}\right).$$

Multiplying both sides by $2\,(du/d\theta)$ and then integrating, we have

$$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu\left(-\frac{1}{3u^6}+\frac{c^4}{u^2}\right)+A, \qquad ...(1)$$

where A is a constant.

But initially, when $r=c$ i.e., $u=1/c$, $du/d\theta=0$ (at an apse) and $v=c^3\sqrt{(2\mu/3)}$.

$\therefore$ from (1), we have $\quad \dfrac{2\mu c^6}{3}=h^2\cdot\dfrac{1}{c^2}=\mu\left(-\dfrac{c^6}{3}+c^6\right)+A.$

$\therefore \quad h^2=\tfrac{2}{3}\mu c^8,\ A=0.$

Substituting the values of h^2 and A in (1), we have

$$\tfrac{2}{3}\mu c^8\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu\left(-\frac{1}{3u^6}+\frac{c^4}{u^2}\right)$$

or $$c^8\left(\frac{du}{d\theta}\right)^2=-\frac{1}{2u^6}+\frac{3c^4}{2u^2}-c^8u^2=\frac{1}{u^6}\left[-\tfrac{1}{2}+\tfrac{3}{2}c^4u^4-c^8u^8\right]$$

$$=\frac{1}{u^6}\left[-\tfrac{1}{2}-(c^8u^8-\tfrac{3}{2}c^4u^4)\right]=\frac{1}{u^6}\left[-\tfrac{1}{2}-(c^4u^4-\tfrac{3}{4})^2+\tfrac{9}{16}\right]$$

$$=\frac{1}{u^6}\left[(\tfrac{1}{4})^2-(c^4u^4-\tfrac{3}{4})^2\right].$$

$$\therefore \quad c^4u^3\frac{du}{d\theta}=\sqrt{\left[(\tfrac{1}{4})^2-(c^4u^4-\tfrac{3}{4})^2\right]}$$

or $$d\theta=\frac{c^4u^3\,du}{\sqrt{\left[(\tfrac{1}{4})^2-(c^4u^4-\tfrac{3}{4})^2\right]}}.$$

Putting $c^4u^4-\frac{3}{4}=z$, so that $4c^4u^3\,du=dz$, we have

$$4\,d\theta=\frac{dz}{\sqrt{\left[(\tfrac{1}{4})^2-z^2\right]}}.$$

Integrating, $4\theta+B=\sin^{-1}\left(\dfrac{z}{\frac{1}{4}}\right)=\sin^{-1}(4z),$

where B is a constant

or $\quad 4\theta+B=\sin^{-1}(4c^4u^4-3).$

But initially when $u=1/c,\theta=0$. $\quad\therefore\quad B=\sin^{-1}1=\pi/2.$

$\therefore \quad 4\theta+\tfrac{1}{2}\pi=\sin^{-1}(4c^4u^4-3)$

or $\quad \sin(\tfrac{1}{2}\pi+4\theta)=4c^4u^4-3$

or $\quad \cos 4\theta=4c^4u^4-3$

or $\quad 4c^4u^4=3+\cos 4\theta$

or $\quad 4c^4/r^4=[3+\cos 4\theta]$

or $\quad 4c^4=r^4[3+(2\cos^2 2\theta-1)]=2r^4[1+\cos^2 2\theta]$

$\quad =2r^4[(\cos^2\theta+\sin^2\theta)^2+(\cos^2\theta-\sin^2\theta)^2]$

$= 4r^4 (\cos^4\theta + \sin^4\theta).$

$\therefore \quad c^4 = (r\cos\theta)^4 + (r\sin\theta)^4$

or $\quad c^4 = x^4 + y^4, \quad [\because x = r\cos\theta$ and $y = r\sin\theta]$

which is the required equation of the path.

Ex. 25. *If the law of force be $\mu\,(u^4 - \frac{10}{9}au^5)$ and the particle be projected from an apse at a distance 5a with a velocity equal to $\sqrt{(5/7)}$ of that in a circle at the same distance, show that the orbit is the limacon $r = a\,(3 + 2\cos\theta)$.*

Sol. Here the central acceleration

$$P = \mu\left(u^4 - \frac{10}{9}au^5\right) = \mu\left(\frac{1}{r^4} - \frac{10a}{9r^5}\right).$$

If V is the velocity for a circle at a distance $5a$, then

$$\frac{V^2}{5a} = [P]_{r=5a} = \mu\left[\frac{1}{(5a)^4} - \frac{10a}{9\,(5a)^5}\right] = \frac{7\mu}{9\,(5a)^4}.$$

$$\therefore \quad V = \sqrt{\left[\frac{7\mu}{9\,.\,(5a)^3}\right]}.$$

If v_1 is the velocity of projection of the particle, then

$$v_1 = \sqrt{\left(\frac{5}{7}\right)}\cdot V = \sqrt{\left(\frac{5}{7}\right)}\cdot\sqrt{\left(\frac{7\mu}{9\,(5a)^3}\right)} = \sqrt{\left(\frac{\mu}{225a^3}\right)}.$$

The differential equation of the path is

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{\mu}{u^2}\left(u^4 - \frac{10}{9}au^5\right) = \mu\left(u^2 - \frac{10}{9}au^3\right).$$

Multiplying both sides by $2\,(du/d\theta)$ and then integrating, we have

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\,(\tfrac{2}{3}u^3 - \tfrac{5}{9}au^4) + A, \qquad ...(1)$$

where A is a constant.

But initially, when $r = 5a$ i.e., $u = \dfrac{1}{5a}$, $\dfrac{du}{d\theta} = 0$ and $v^2 = \dfrac{\mu}{225a^3}$.

$\therefore$ from (1), we have

$$\frac{\mu}{225a^3} = h^2\left(\frac{1}{5a}\right)^2 = \mu\left[\frac{2}{3}\left(\frac{1}{5a}\right)^3 - \frac{5a}{9}\left(\frac{1}{5a}\right)^4\right] + A.$$

$$\therefore \quad h^2 = \frac{\mu}{9a}, \quad A = 0.$$

Substituting the values of h^2 and A in (1), we have

$$\frac{\mu}{9a}\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\left(\frac{2}{3}u^3 - \frac{5a}{9}u^4\right)$$

or $$\left(\frac{du}{d\theta}\right)^2 = 6au^3 - 5a^2u^4 - u^2.$$

Putting $u = \dfrac{1}{r}$, so that $\dfrac{du}{d\theta} = -\dfrac{1}{r^2}\dfrac{dr}{d\theta}$, we have

$$\left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 = \frac{6a}{r^3} - \frac{5a^2}{r^4} - \frac{1}{r^2}$$

or $$\left(\frac{dr}{d\theta}\right)^2 = 6ar - 5a^2 - r^2 = -5a^2 - (r^2 - 6ar)$$

$$= -5a^2 - (r - 3a)^2 + 9a^2 = 4a^2 - (r - 3a)^2.$$

$$\therefore \quad \frac{dr}{d\theta} = \sqrt{[(2a)^2 - (r - 3a)^2]}$$

or $$d\theta = \frac{dr}{\sqrt{[(2a)^2 - (r - 3a)^2]}}.$$

Integrating, $\theta + B = \sin^{-1}\left(\dfrac{r - 3a}{2a}\right)$, where B is a constant.

But initially when $r = 5a, \theta = 0$. $\quad \therefore \quad B = \sin^{-1}1 = \pi/2$.

$$\therefore \quad \theta + \tfrac{1}{2}\pi = \sin^{-1}\left(\frac{r - 3a}{2a}\right) \quad \text{or} \quad \sin(\tfrac{1}{2}\pi + \theta) = \frac{r - 3a}{2a}$$

or $\quad r - 3a = 2a\cos\theta \quad$ or $\quad r = a\,(3 + 2\cos\theta)$,

which is the required equation of the orbit.

Ex. 26. *A particle is projected from an apse at a distance a with the velocity from infinity under the action of a central acceleration μ/r^{2n+3}. Prove that the equation of the path is $r^n = a^n\cos n\theta$.*

(Meerut 1974, 87, 91, 95)

Sol. Here, the central acceleration $P = \mu/r^{2n+3} = \mu u^{2n+3}$.

If V is the velocity of the particle at a distance a acquired in falling from rest from infinity under the same acceleration, then as in § 6, page 8,

$$V^2 = -2\int_\infty^a P\,dr = -2\int_\infty^a \frac{\mu}{r^{2n+3}}\,dr = -2\int_\infty^a \mu r^{-2n-3}\,dr$$

$$= -2\mu\left[\frac{r^{-2n-2}}{-2n-2}\right]_\infty^a = \frac{\mu}{(n+1)}\left[\frac{1}{r^{2n+2}}\right]_\infty^a = \frac{\mu}{(n+1)\,a^{2n+2}}.$$

The differential equation of the path is

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{1}{u^2}\cdot\mu u^{2n+3} = \mu u^{2n+1}.$$

Multiplying both sides by $2\,(du/d\theta)$ and integrating, we get

$$h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \frac{2\mu u^{2n+2}}{2\,(n+1)} + A, \text{ where } A \text{ is a constant}$$

or $$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \frac{\mu}{(n+1)}\cdot u^{2n+2} + A. \qquad ...(1)$$

But initially when $r = a$, *i.e.*, $u = 1/a$, $du/d\theta = 0$ (at an apse) and $v = V = \sqrt{\mu}/\sqrt{[(n+1)\,a^{2n+2}]}$.

$\therefore$ from (1) we have

$$\frac{\mu}{(n+1)\,a^{2n+2}} = h^2\left[\frac{1}{a^2}\right] = \frac{\mu}{(n+1)} \cdot \frac{1}{a^{2n+2}} + A.$$

$$\therefore \quad h^2 = \frac{\mu}{(n+1)\,a^{2n}} \text{ and } A = 0.$$

Substituting the values of h^2 and A in (1), we have

$$\frac{\mu}{(n+1)\,a^{2n}} \cdot \left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \frac{\mu}{(n+1)} u^{2n+2}$$

or $$\left(\frac{du}{d\theta}\right)^2 = a^{2n} \,.\, u^{2n+2} - u^2.$$

Putting $u = \dfrac{1}{r}$, so that $\dfrac{du}{d\theta} = -\dfrac{1}{r^2}\dfrac{dr}{d\theta}$, we have

$$\left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 = \frac{a^{2n}}{r^{2n+2}} - \frac{1}{r^2}$$

or $$\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{a^{2n}}{r^{2n+2}} - \frac{1}{r^2} = \frac{a^{2n} - r^{2n}}{r^{2n+2}}$$

or $$\left(\frac{dr}{d\theta}\right)^2 = \frac{a^{2n} - r^{2n}}{r^{2n-2}} \quad \text{or} \quad \frac{dr}{d\theta} = \frac{\sqrt{(a^{2n} - r^{2n})}}{r^{n-1}}$$

or $$d\theta = \frac{r^{n-1}\,dr}{\sqrt{[(a^{2n} - r^{2n})}}.$$

Substituting $r^n = z$, so that $nr^{n-1}\,dr = dz$, we have

$$n\,d\theta = \frac{dz}{\sqrt{[(a^n)^2 - z^2]}}.$$

Integrating, $n\theta + B = \sin^{-1}(z/a^n)$, where B is a constant

or $$n\theta + B = \sin^{-1}(r^n/a^n).$$

But initially when $r = a, \theta = 0$. $\quad \therefore \quad B = \sin^{-1}(1) = \pi/2$.

$$\therefore \quad n\theta + \tfrac{1}{2}\pi = \sin^{-1}(r^n/a^n)$$

or $$r^n/a^n = \sin(\tfrac{1}{2}\pi + n\theta) = \cos n\theta \quad \text{or} \quad r^n = a^n \cos n\theta,$$

which is the required equation of the path.

Ex. 27. (a) *A particle is projected from an apse at a distance a with the velocity from infinity, the acceleration being μu^7; show that the equation to its path is*

$$r^2 = a^2 \cos 2\theta. \quad \textbf{(Meerut 96)}$$

Sol. Proceed as in Ex. 26. Here $n = 2$.

(b) *A particle is projected from an apse at a distance a with velocity of projection $\sqrt{\mu}/(a^2\sqrt{2})$ under the action of a central force μu^5. Prove that the path is the circle $r = a\cos\theta$.* **(Meerut 1973, 77, 78, 82P, 82S, 86)**

Sol. Proceed as in Ex. 26. Here $n = 1$.

(c) *If the central force varies as the cube of the distance from a fixed point then find the orbit.* **(Meerut 96, 97)**

Sol. We know that referred to the centre of force as pole the differential equation of a central orbit in pedal form is

$$\frac{h^2}{p^3}\frac{dp}{dr} = P, \qquad \text{...(1)}$$

where P is the central acceleration assumed to be attractive.

Here $P = \mu r^3$. Putting $P = \mu r^3$ in (1), we get

$$\frac{h^2}{p^3}\frac{dp}{dr} = \mu r^3$$

or $$\frac{h^2}{p^3}\,dp = \mu r^3\,dr$$

or $$-2\frac{h^2}{p^3}\,dp = -2\mu r^3\,dr.$$

Integrating both sides, we get

$$v^2 = \frac{h^2}{p^2} = -\frac{\mu r^4}{2} + C \qquad \text{...(2)}$$

Let $v = v_0$ when $r = r_0$.

Then $$v_0^2 = -\frac{\mu r_0^4}{2} + C$$

or $$C = v_0^2 + \frac{\mu r_0^4}{2}.$$

Putting this value of C in (2), the pedal equation of the central orbit is

$$\frac{h^2}{p^2} = -\frac{\mu r^4}{2} + v_0^2 + \frac{\mu r_0^4}{2}.$$

Ex. 28. *A particle moves with a central acceleration which varies inversely as the cube of the distance. If it be projected from an apse at a distance a from the origin with a velocity which is $\sqrt{2}$ times the velocity for a circle of radius a, show that the equation to its path is $r\cos(\theta/\sqrt{2}) = a$.* **[Rohilkhand 77, 81; Allahabad 78; Meerut 78; Agra 86]**

Sol. Here the central acceleration varies inversely as the cube of the distance *i.e.*, $P = \mu/r^3 = \mu u^3$, where μ is a constant.

If V is the velocity for a circle of radius a, then

$$\frac{V^2}{a} = [P]_{r=a} = \frac{\mu}{a^3}$$

or $$V = \sqrt{(\mu/a^2)}.$$

$\therefore$ the velocity of projection $v_1 = \sqrt{2}V = \sqrt{(2\mu/a^2)}$.

The differential equation of the path is

$$h^2\left[u+\frac{d^2u}{d\theta^2}\right]=\frac{P}{u^2}=\frac{\mu u^3}{u^2}=\mu u.$$

Multiplying both sides by $2\,(du/d\theta)$ and integrating, we have

$$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu u^2+A,$$

where A is a constant. ...(1)

But initially when $r=a$ *i.e.*, $u=1/a$, $du/d\theta=0$ (at an apse), and $v=v_1=\sqrt{(2\mu/a^2)}$.

$\therefore$ from (1), we have

$$\frac{2\mu}{a^2}=h^2\left[\frac{1}{a^2}\right]=\frac{\mu}{a^2}+A.$$

$\therefore$ $h^2=2\mu$ and $A=\mu/a^2$.

Substituting the values of h^2 and A in (1), we have

$$2\mu\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu u^2+\frac{\mu}{a^2}$$

or $$2\left(\frac{du}{d\theta}\right)^2=\frac{1}{a^2}+u^2-2u^2=\frac{1-a^2u^2}{a^2}$$

or $$\sqrt{2}\,a\frac{du}{d\theta}=\sqrt{(1-a^2u^2)}\quad\text{or}\quad\frac{d\theta}{\sqrt{2}}=\frac{a\,du}{\sqrt{(1-a^2u^2)}}.$$

Integrating, $(\theta/\sqrt{2})+B=\sin^{-1}(au)$, where B is a constant.

But initially, when $u=1/a,\theta=0$. $\therefore$ $B=\sin^{-1}1=\frac{1}{2}\pi$.

$\therefore$ $(\theta/\sqrt{2})+\frac{1}{2}\pi=\frac{1}{2}\sin^{-1}(au)$ or $au=a/r=\sin\{\frac{1}{2}\pi+(\theta/\sqrt{2})\}$

or $a=r\cos(\theta/\sqrt{2})$, which is the required equation of the path.

Ex. 29. *A particle moving under a constant force from a centre is projected at a distance a from the centre in a direction perpendicular to the radius vector with velocity acquired in falling to the point of projection from the centre, show that its path is* $(a/r)^3=\cos^2(\frac{3}{2}\theta)$.

(Meerut 1976, 85)

Also show that the particle will ultimately move in a straight line through the origin in the same way as if its path had always been this line.

If the velocity of projection be double that in the previous case show that the path is

$$\frac{\theta}{2}=\tan^{-1}\sqrt{\left(\frac{r-a}{a}\right)}-\frac{1}{\sqrt{3}}\tan^{-1}\sqrt{\left(\frac{r-a}{3a}\right)}.$$

Sol. Since the particle moves under a constant force directed away from a centre, therefore the central acceleration $P=-f$, where f is a constant.

While falling in a straight line from the centre of force to the point of projection, if v is the velocity of the particle at a distance r from the centre of force, then

$$v\frac{dv}{dr}=f\quad\text{or}\quad v\,dv=f\,dr.$$

Let V be the velocity of the particle acquired in falling from the centre to a distance a. Then

$$\int_0^V v\,dv=\int_0^a f\,dr\quad\text{or}\quad\frac{V^2}{2}=af\quad\text{or}\quad V=\sqrt{(2af)}.$$

Therefore the particle is projected from a distance a with velocity $\sqrt{(2af)}$ in a direction perpendicular to the radius vector.

The differential equation of the path is

$$h^2\left[u+\frac{d^2u}{d\theta^2}\right]=\frac{P}{u^2}=-\frac{f}{u^2}.$$

Multiplying both sides by $2\,(du/d\theta)$ and integrating, we have

$$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\frac{2f}{u}+A, \qquad ...(1)$$

where A is a constant.

But initially, when $r=a$ *i.e.*, $u=1/a$, $du/d\theta=0$ (since the particle is projected perpendicular to the radius vector), and $v=V=\sqrt{(2af)}$.

$\therefore$ from (1), $2af=h^2\left[\frac{1}{a^2}\right]=2fa+A$.

$\therefore$ $h^2=2fa^3$ and $A=0$.

Substituting the values of h^2 and A in (1), we have

$$2fa^3\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\frac{2f}{u}$$

or $$a^3\left(\frac{du}{d\theta}\right)^2=-a^3u^2+\frac{1}{u}=\frac{1-a^3u^3}{u}$$

or $$a^{3/2}\frac{du}{d\theta}=\frac{\sqrt{(1-a^3u^3)}}{u^{1/2}}$$

or $$d\theta=\frac{a^{3/2}u^{1/2}\,du}{\sqrt{(1-a^3u^3)}}.$$

Substituting $a^{3/2}u^{3/2}=z$, so that $\frac{3}{2}a^{3/2}u^{1/2}\,du=dz$, we have

$$\tfrac{3}{2}\,d\theta=\frac{dz}{\sqrt{(1-z^2)}}.$$

Integrating $\frac{3}{2}\theta + B = \sin^{-1}(z) = \sin^{-1}(a^{3/2}u^{3/2})$,

where B is a constant.

But initially when $u = 1/a, \theta = 0$. $\therefore\ B = \sin^{-1} 1 = \frac{1}{2}\pi$.

$$\therefore \quad \tfrac{3}{2}\theta + \tfrac{1}{2}\pi = \sin^{-1}(a^{3/2}u^{3/2})$$

or $$a^{3/2}u^{3/2} = \sin(\tfrac{1}{2}\pi + \tfrac{3}{2}\theta) = \cos\tfrac{3}{2}\theta$$

or $$a^{3/2}/r^{3/2} = \cos(\tfrac{3}{2}\theta)$$

or $$(a/r)^3 = \cos^2(\tfrac{3}{2}\theta).$$

This is the required equation of the path. ...(2)

Second part. Now as $r \to \infty$, $\cos(\frac{3}{2}\theta) \to 0$ *i.e.*, $\frac{3}{2}\theta \to \frac{1}{2}\pi$

i.e., $\theta \to \pi/3$.

Hence the particle ultimately moves in a straight line through the origin, inclined at an angle $\theta = \pi/3$, in the same way as if its path had always been this line.

Third part. If the velocity of projection of the particle is double of that in the previous case, then the initial conditions are

$r = a, u = 1/a, du/d\theta = 0$ and $v = 2V = 2\sqrt{(2af)}$.

$\therefore$ from (1), we have $8af = h^2\left[\dfrac{1}{a^2}\right] = 2fa + A$.

$\therefore\ h^2 = 8a^3f$ and $A = 6af$.

Substituting these values of h^2 and A in (1), we have

$$8a^3f\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \frac{2f}{u} + 6af$$

or $$4a^3\left(\frac{du}{d\theta}\right)^2 = -4a^3u^2 + \frac{1}{u} + 3a.$$

Putting $u = \dfrac{1}{r}$, so that $\dfrac{du}{d\theta} = -\dfrac{1}{r^2}\dfrac{dr}{d\theta}$, we have

$$4a^3\left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 = -\frac{4a^3}{r^2} + r + 3a$$

or $$4a^3(dr/d\theta)^2 = r^5 + 3ar^4 - 4a^3r^2 = r^2(r^3 + 3ar^2 - 4a^3)$$
$$= r^2[r^2(r-a) + 4ar(r-a) + 4a^2(r-a)]$$
$$= r^2(r-a)(r^2 + 4ar + 4a^2) = r^2(r-a)(r+2a)^2$$

or $$2a^{3/2}(dr/d\theta) = r(r+2a)\sqrt{(r-a)}$$

or $$\frac{d\theta}{2} = \frac{a^{3/2}\,dr}{r(r+2a)\sqrt{(r-a)}}.$$

Substituting $r - a = z^2$, so that $dr = 2z\,dz$, we have

$$\frac{d\theta}{2} = \frac{2a^{3/2}z\,dz}{(z^2+a)(z^2+3a)\cdot z}$$

or $$\frac{d\theta}{2} = \sqrt{a}\cdot\left[\frac{1}{z^2+a} - \frac{1}{z^2+3a}\right]dz.$$

Integrating,

$$\frac{\theta}{2} + B = \sqrt{a}\cdot\left[\frac{1}{\sqrt{a}}\tan^{-1}\frac{z}{\sqrt{a}} - \frac{1}{\sqrt{(3a)}}\tan^{-1}\frac{z}{\sqrt{(3a)}}\right],$$

where B is a constant

or $$\frac{\theta}{2} + B = \tan^{-1}\sqrt{\left(\frac{r-a}{a}\right)} - \frac{1}{\sqrt{3}}\tan^{-1}\sqrt{\left(\frac{r-a}{3a}\right)}.$$

But initially when $r = a, \theta = 0$. $\therefore\ B = 0$.

$$\therefore\ \frac{\theta}{2} = \tan^{-1}\sqrt{\left(\frac{r-a}{a}\right)} - \frac{1}{\sqrt{3}}\tan^{-1}\sqrt{\left(\frac{r-a}{3a}\right)},$$

which is the required equation of the path.

Ex. 30. *A particle moves with a central acceleration $\mu/(\text{distance})^5$ and projected from the apse at a distance a with a velocity equal to n times that which would be acquired in falling from infinity; show that the other apsidal distance is $a/\sqrt{(n^2-1)}$.*

If $n = 1$ and particle be projected in any direction, show that the path is a circle passing through the centre of force. **(Rohilkhand 1988)**

Sol. Here, the central acceleration

$$P = \frac{\mu}{(\text{distance})^5} = \frac{\mu}{r^5} = \mu u^5.$$

Let V be the velocity from infinity to a distance a from the centre under the same acceleration. Then as in § 6 of this chapter on page 7,

$$V^2 = -2\int_\infty^a P\,dr = -2\int_\infty^a \frac{\mu}{r^5}dr = -2\left[\frac{\mu}{-4r^4}\right]_\infty^a = \frac{\mu}{2a^4}.$$

$\therefore\ V = \sqrt{(\mu/2a^4)}$.

The differential equation of the path is

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{\mu u^5}{u^2} = \mu u^3.$$

Multiplying both sides by $2(du/d\theta)$ and integrating, we have

$$h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \frac{2\mu u^4}{4} + A, \text{ where } A \text{ is a constant}$$

or $$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \frac{\mu u^4}{2} + A. \quad ...(1)$$

But initially, when $r = a$ *i.e.*, $u = 1/a, du/d\theta = 0$ (at an apse) and $v = nV = n\sqrt{(\mu/2a^4)}$.

$\therefore$ from (1), we have $\dfrac{n^2\mu}{2a^4} = h^2\left[\dfrac{1}{a^2}\right] = \dfrac{\mu}{2a^4} + A.$

$$\therefore \quad h^2 = \frac{n^2 \mu}{2a^2} \quad \text{and} \quad A = \frac{(n^2-1)\mu}{2a^4}.$$

Substituting the values of h^2 and A in (1), we have

$$\frac{n^2\mu}{2a^2}\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \frac{\mu u^4}{2} + \frac{(n^2-1)\mu}{2a^4}$$

or $$\left(\frac{du}{d\theta}\right)^2 = \frac{1}{n^2a^2}\left[a^4u^4 - a^2n^2u^2 + (n^2-1)\right].$$

At an apse, we have $du/d\theta = 0$. Therefore the apsidal distances are given by

$$0 = (1/n^2a^2)\,[a^4u^4 - a^2n^2u^2 + (n^2-1)]$$

or $$a^4u^4 - a^2n^2u^2 + (n^2-1) = 0$$

or $$\frac{a^4}{r^4} - \frac{a^2n^2}{r^2} + (n^2-1) = 0 \qquad \left[\because u = \frac{1}{r}\right]$$

or $$(n^2-1)\,r^4 - a^2n^2r^2 + a^4 = 0,$$

which is a quadratic equation in r^2.

If r_1^2 and r_2^2 are its roots, then $r_1^2 r_2^2 = a^4/(n^2-1)$.

or $$r_1 r_2 = a^2/\sqrt{(n^2-1)}. \qquad \text{...(2)}$$

But the first apsidal distance, say r_1, is a.

$\therefore$ from (2), $ar_2 = a^2/\sqrt{(n^2-1)}$

i.e., the second apsidal distance $r_2 = a/\sqrt{(n^2-1)}$.

Second part. When $n = 1$ and the particle is projected in any direction, say at an angle α to the radius vector, then at the point of projection, we have $\phi = \alpha$, $p = r \sin\phi = a \sin\alpha$

and so $$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2 = \frac{1}{(a\sin\alpha)^2}.$$

Thus in this case initially when $r = a$ *i.e.*, $u = 1/a$, we have

$$v = V = \sqrt{(\mu/2a^4)} \quad \text{and} \quad u^2 + (du/d\theta)^2 = 1/(a^2\sin^2\alpha).$$

$\therefore$ from (1), we have $$\frac{\mu}{2a^4} = \frac{h^2}{(a^2\sin^2\alpha)} = \frac{\mu}{2a^4} + A.$$

$$\therefore \quad h^2 = (\mu \sin^2\alpha)/(2a^2) \quad \text{and} \quad A = 0.$$

Substituting the values of h^2 and A in (1), we have

$$\frac{(\mu\sin^2\alpha)}{2a^2}\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \frac{\mu u^4}{2}$$

or $$u^2 + \left(\frac{du}{d\theta}\right)^2 = \frac{a^2u^4}{\sin^2\alpha} \quad \text{or} \quad \left(\frac{du}{d\theta}\right)^2 = \frac{a^2u^4}{\sin^2\alpha} - u^2.$$

Putting $u = \dfrac{1}{r}$, so that $\dfrac{du}{d\theta} = -\dfrac{1}{r^2}\dfrac{dr}{d\theta}$, we have

$$\left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 = \frac{a^2}{r^4\sin^2\alpha} - \frac{1}{r^2}$$

or $$\left(\frac{dr}{d\theta}\right)^2 = a^2\,\text{cosec}^2\,\alpha - r^2 \quad \text{or} \quad \frac{dr}{d\theta} = \sqrt{(a^2\,\text{cosec}^2\,\alpha - r^2)}$$

or $$d\theta = \frac{dr}{\sqrt{(a^2\,\text{cosec}^2\,\alpha - r^2)}}.$$

Integrating, $\theta + B = \sin^{-1}\left(\dfrac{r}{a\,\text{cosec}\,\alpha}\right)$, where B is a constant.

Initially when $r = a$, let $\theta = 0$. Then $B = \sin^{-1}(\sin\alpha) = \alpha$.

$$\therefore \quad \theta + \alpha = \sin^{-1}\{r/(a\,\text{cosec}\,\alpha)\}$$

$$r = (a\,\text{cosec}\,\alpha)\sin(\theta+\alpha)$$

or $$r = (a\,\text{cosec}\,\alpha)\cos\{\tfrac{1}{2}\pi - (\theta+\alpha)\}$$

or $$r = (a\,\text{cosec}\,\alpha)\cos\{(\theta+\alpha) - \tfrac{1}{2}\pi\}$$

or $$r = (a\,\text{cosec}\,\alpha)\cos\{\theta - (\tfrac{1}{2}\pi - \alpha)\}$$

or $$r = (a\,\text{cosec}\,\alpha)\cos(\theta - \beta), \text{ where } \beta = \tfrac{1}{2}\pi - \alpha.$$

This represents a circle of diameter $a\,\text{cosec}\,\alpha$ and pole on its circumference. Hence the path of the particle is a circle through the centre of force.

Ex. 31. *If the acceleration at a distance r is μ/r^5 and the particle is projected at a distance a from the centre of force with velocity $\sqrt{(\mu/2a^4)}$, prove that the orbit is a circle through O of diameter $a\,\text{cosec}\,\alpha$, where α is the inclination of the direction of projection to the radius vector.* **(Meerut 1979)**

Sol. This is Ex. 30, part II. Do yourself.

Ex. 32. *A particle describes an orbit with a central acceleration $\mu u^3 - \lambda u^5$ being projected from an apse at a distance a with velocity equal to that from infinity. Show that its path is $r = a\cosh(\theta/n)$, where $n^2 + 1 = 2\mu a^2/\lambda$.* **(Rohilkhand 1980)**

Prove also that it will be at a distance r at the end of time

$$\sqrt{\left(\frac{a^2}{2\lambda}\right)}\left[a^2 \log\left\{\frac{r + \sqrt{(r^2-a^2)}}{a}\right\} + r\sqrt{(r^2-a^2)}\right].$$

Sol. Here, the central acceleration

$$P = \mu u^3 - \lambda u^5 = \frac{\mu}{r^3} - \frac{\lambda}{r^5}.$$

Let V be the velocity from infinity at the distance a under the same acceleration. Then

$$V^2 = -2\int_\infty^a P\,dr = -2\int_\infty^a \left(\frac{\mu}{r^3} - \frac{\lambda}{r^5}\right)dr$$

$$= -2\left[-\frac{\mu}{2r^2} + \frac{\lambda}{4r^4}\right]_\infty^a = \frac{\mu}{a^2} - \frac{\lambda}{2a^4}$$

$$= \frac{\lambda}{2a^4}\left(\frac{2\mu a^2}{\lambda} - 1\right) = \frac{\lambda n^2}{2a^4} \qquad \left[\because\ n^2 + 1 = \frac{2\mu a^2}{\lambda}\right].$$

$\therefore\quad V = (n/a^2)\sqrt{(\lambda/2)}$.

The differential equation of the path is

$$h^2\left\{u + \frac{d^2u}{d\theta^2}\right\} = \frac{P}{u^2} = \frac{\mu u^3 - \lambda u^5}{u^2} = \mu u - \lambda u^3.$$

Multiplying both sides by $2\,(du/d\theta)$ and integrating, we have

$$h^2\left\{u^2 + \left(\frac{du}{d\theta}\right)^2\right\} = 2\left(\frac{\mu u^2}{2} - \frac{\lambda u^4}{4}\right) + A, \quad \text{where } A \text{ is a constant}$$

or
$$v^2 = h^2\left\{u^2 + \left(\frac{du}{d\theta}\right)^2\right\} = \mu u^2 - \frac{\lambda u^4}{2} + A. \qquad \text{...(1)}$$

But initially when $r = a$ *i.e.*, $u = 1/a$, $du/d\theta = 0$ (at an apse) and $v = V = (n/a^2)\sqrt{(\lambda/2)}$. Therefore from (1), we have

$$\frac{\lambda n^2}{2a^4} = h^2\left\{\frac{1}{a^2}\right\} = \frac{\mu}{a^2} - \frac{\lambda}{2a^4} + A.$$

$$\therefore\quad h^2 = \frac{\lambda n^2}{2a^2} \text{ and } A = \frac{\lambda n^2}{2a^4} - \left(\frac{\mu}{a^2} - \frac{\lambda}{2a^4}\right) = \frac{\lambda}{2a^4}(n^2 + 1) - \frac{\mu}{a^2}$$

$$= \frac{\lambda}{2a^4}\cdot\left(\frac{2\mu a^2}{\lambda}\right) - \frac{\mu}{a^2} = 0. \qquad \left[\because\ n^2 + 1 = \frac{2\mu a^2}{\lambda}\right]$$

Substituting the values of h^2 and A in (1), we have

$$\frac{\lambda n^2}{2a^2}\left\{u^2 + \left(\frac{du}{d\theta}\right)^2\right\} = \mu u^2 - \frac{\lambda u^4}{2}$$

$$= \frac{\lambda}{2a^2}(n^2 + 1)\,u^2 - \frac{\lambda u^4}{2} \qquad \left[\because\ n^2 + 1 = \frac{2\mu a^2}{\lambda}\right]$$

or
$$n^2u^2 + n^2\left(\frac{du}{d\theta}\right)^2 = (n^2 + 1)\,u^2 - a^2u^4$$

or
$$n^2\left(\frac{du}{d\theta}\right)^2 = u^2 - a^2u^4.$$

Putting $u = \dfrac{1}{r}$ so that $\dfrac{du}{d\theta} = -\dfrac{1}{r^2}\dfrac{dr}{d\theta}$, we have

$$n^2\left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 = \frac{1}{r^2} - \frac{a^2}{r^4} \quad \text{or} \quad n^2\left(\frac{dr}{d\theta}\right)^2 = r^2 - a^2$$

or
$$\frac{dr}{d\theta} = \frac{\sqrt{(r^2 - a^2)}}{n}$$

or
$$\frac{d\theta}{n} = \frac{dr}{\sqrt{(r^2 - a^2)}}. \qquad \text{...(2)}$$

Integrating, $\theta/n + B = \cosh^{-1}(r/a)$, where B is a constant.

But initially when $r = a, \theta = 0$ (say). Then $B = \cosh^{-1}(1) = 0$.

$$\therefore\quad \theta/n = \cosh^{-1}(r/a) \quad \text{or} \quad r = a\cosh(\theta/n),$$

which is the required equation of the path.

Second part. We know that

$$h = r^2\frac{d\theta}{dt}$$

or
$$h = r^2\frac{d\theta}{dr}\cdot\frac{dr}{dt}.$$

Substituting for h and $dr/d\theta$, we have

$$\frac{n}{a}\sqrt{\left(\frac{\lambda}{2}\right)} = r^2\cdot\frac{n}{\sqrt{(r^2 - a^2)}}\frac{dr}{dt}$$

or
$$dt = a\sqrt{\left(\frac{2}{\lambda}\right)}\frac{r^2\,dr}{\sqrt{(r^2 - a^2)}}.$$

Integrating, the time t from the distance a to the distance r is given by

$$t = a\sqrt{(2/\lambda)}\int_{r=a}^{r}\frac{r^2\,dr}{\sqrt{(r^2 - a^2)}} = a\sqrt{(2/\lambda)}\int_a^r\frac{(r^2 - a^2) + a^2}{\sqrt{(r^2 - a^2)}}\,dr$$

$$= a\sqrt{(2/\lambda)}\int_a^r\left\{\sqrt{(r^2 - a^2)} + \frac{a^2}{\sqrt{(r^2 - a^2)}}\right\}dr$$

$$= a\sqrt{(2/\lambda)}\left[\frac{r}{2}\sqrt{(r^2 - a^2)} - \frac{a^2}{2}\log\{r + \sqrt{(r^2 - a^2)}\}\right.$$
$$\left. + a^2\log\{r + \sqrt{(r^2 - a^2)}\}\right]$$

$$= a\sqrt{(2/\lambda)}\left[\frac{r}{2}\sqrt{(r^2 - a^2)} + \frac{a^2}{2}\log\{r + \sqrt{(r^2 - a^2)}\}\right]_a^r$$

$$= a\sqrt{(2/\lambda)}\left[\frac{r}{2}\sqrt{(r^2 - a^2)} + \frac{a^2}{2}\log\{r + \sqrt{(r^2 - a^2)}\} - \frac{a^2}{2}\log a\right]$$

$$= a\sqrt{(2/\lambda)}\left[\frac{r}{2}\sqrt{(r^2 - a^2)} + \frac{a^2}{2}\log\left\{\frac{r + \sqrt{(r^2 - a^2)}}{a}\right\}\right]$$

$$= \sqrt{(a^2/2\lambda)}\left[r\sqrt{(r^2 - a^2)} + a^2\log\left\{\frac{r + \sqrt{(r^2 - a^2)}}{a}\right\}\right].$$

Ex. 33. *A particle is acted on by a central repulsive force which varies as the nth power of the distance. If the velocity at any point be equal to that which would be acquired in falling from the centre to the point, show that the equation to the path is of the form*

$$r^{(n+3)/2}\cos\tfrac{1}{2}(n + 3)\,\theta = \text{constant}.$$

Sol. Since the particle is acted on by a central repulsive force which varies as the n^{th} power of the distance, therefore the central acceleration

$$P = -\mu\,(\text{distance})^n = -\mu r^n = -\mu/u^n.$$

While falling in a straight line from rest from the centre of force if v is the velocity of the particle at a distance x from the centre, then

$$v\frac{dv}{dx} = \mu x^n \quad \text{or} \quad v\,dv = \mu x^n\,dx.$$

Let V be the velocity of the particle acquired in falling from the centre to a distance r. Then

$$\int_0^V v\,dv = \int_0^r \mu x^n\,dx$$

or $$\tfrac{1}{2}V^2 = \mu\left[\frac{x^{n+1}}{n+1}\right]_0^r = \frac{\mu}{n+1}r^{n+1}$$

or $$V^2 = \{2\mu/(n+1)\}\,r^{n+1}. \qquad \text{...(1)}$$

The differential equation of the central orbit is

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = -\frac{\mu/u^n}{u^2} = -\mu u^{-n-2}.$$

Multiplying both sides by 2 $(du/d\theta)$ and integrating, we have

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \frac{-2\mu u^{-n-1}}{(-n-1)} + A = \frac{2\mu}{(n+1)\,u^{n+1}} + A, \qquad \text{...(2)}$$

where A is a constant and v is the velocity of the particle in the orbit at a distance r from the centre.

But according to the question, we have

$$v^2 = V^2 \quad i.e., \quad \frac{2\mu}{(n+1)\,u^{n+1}} + A = \frac{2\mu}{n+1}r^{n+1}.$$

$\therefore\ A = 0.\qquad [\because\ u = 1/r].$

Substituting the value of A in (2), we have

$$h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \frac{2\mu}{(n+1)\,u^{n+1}}$$

or $$u^2 + \left(\frac{du}{d\theta}\right)^2 = \frac{2\mu}{(n+1)\,h^2u^{n+1}} = \frac{\lambda^2}{u^{n+1}},$$

where $\lambda^2 = \dfrac{2\mu}{(n+1)\,h^2}$ = constant

or $$\left(\frac{du}{d\theta}\right)^2 = \frac{\lambda^2}{u^{n+1}} - u^2 = \frac{\lambda^2 - u^{n+3}}{u^{n+1}}$$

or $$\frac{du}{d\theta} = -\frac{\sqrt{(\lambda^2 - u^{n+3})}}{u^{(n+1)/2}} \quad \text{or} \quad d\theta = \frac{-u^{(n+1)/2}\,du}{\sqrt{(\lambda^2 - u^{n+3})}}.$$

Substituting $u^{(n+3)/2} = z$, so that $\frac{1}{2}(n+3)\,u^{(n+1)/2}\,du = dz$,

we have $$d\theta = \frac{-2\,dz}{(n+3)\sqrt{(\lambda^2 - z^2)}}$$

or $$\tfrac{1}{2}(n+3)\,d\theta = -\frac{dz}{\sqrt{(\lambda^2 - z^2)}}.$$

Integrating, $\frac{1}{2}(n+3)\,\theta + B = \cos^{-1}(z/\lambda) = \cos^{-1}\{u^{(n+3)/2}/\lambda\}.$ ...(3)

Now choose λ such that when $u = 1/a, \theta = 0$, $(1/\lambda)(1/a)^{(n+3)/2} = 1$.

Then from (3), $0 + B = \cos^{-1}1 = 0$. Therefore $B = 0$.

Putting $B = 0$ in (3), we have

$$\tfrac{1}{2}(n+3)\,\theta = \cos^{-1}\{u^{(n+3)/2}/\lambda\}$$

or $$u^{(n+3)/2} = \lambda\cos\{\tfrac{1}{2}(n+3)\,\theta\}$$

or $$r^{(n+3)/2}\cos\{\tfrac{1}{2}(n+3)\,\theta\} = 1/\lambda = \text{constant}.$$

This gives the required equation to the path.

Ex. 34. *A particle subject to a force producing an acceleration $\mu(r+2a)/r^5$ towards the origin is projected from the point $(a, 0)$ with a velocity equal to the velocity from infinity at an angle $\cot^{-1}2$ with the initial line; show that the equation to the path is*

$$r = a\,(1 + 2\sin\theta).$$ **[Meerut 80S, 90S, 91S]**

Sol. Here, the central acceleration

$$P = \frac{\mu\,(r+2a)}{r^5} = \mu\left(\frac{1}{r^4} + \frac{2a}{r^5}\right) = \mu\,(u^4 + 2au^5).$$

Let V be the velocity of the particle acquired in falling from rest from infinity under the same acceleration to the point of projection which is at a distance a from the centre. Then

$$V^2 = -2\int_\infty^a P\,dr = -2\int_\infty^a \mu\left(\frac{1}{r^4} + \frac{2a}{r^5}\right)dr$$

$$= -2\mu\left[-\frac{1}{3r^3} - \frac{2a}{4r^4}\right]_\infty^a = 2\mu\left[\frac{1}{3a^3} + \frac{1}{2a^3}\right] = \frac{5\mu}{3a^3}$$

or $V = \sqrt{(5\mu/3a^3)}$.

According to the question the velocity of projection of the particle is equal to V i.e., $\sqrt{(5\mu/3a^3)}$.

Now the differential equation of the path is

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{\mu}{u^2}(u^4 + 2au^5) = \mu\,(u^2 + 2au^3).$$

Multiplying both sides by 2 $(du/d\theta)$ and integrating, we have

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\left(\frac{2u^3}{3} + au^4\right) + A,$$

where A is a constant.

Initially when $r = a$ i.e., $u = 1/a$, $v = \sqrt{(5\mu/3a^3)}$.

Also initially $\phi = \cot^{-1} 2$ or $\cot\phi = 2$ or $\sin\phi = 1/\sqrt{5}$.

But $p = r\sin\phi$. Therefore initially $p = a\,(1/\sqrt{5}) = a/\sqrt{5}$

or $\quad 1/p^2 = 5/a^2$.

But $1/p^2 = u^2 + (du/d\theta)^2$. Therefore initially, when $r = a$, we have $u^2 + (du/d\theta)^2 = 5/a^2$.

Applying the above initial conditions in (1), we have

$$\frac{5\mu}{3a^3} = h^2\frac{5}{a^2} = \mu\left(\frac{2}{3a^3} + \frac{a}{a^4}\right) + A. \ \therefore\ h^2 = \mu/3a, A = 0.$$

Substituting the values of h^2 and A in (1), we have

$$\frac{\mu}{3a}\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\left(\frac{2}{3}u^3 + au^4\right)$$

or $$\left(\frac{du}{d\theta}\right)^2 = 2au^3 + 3a^2u^4 - u^2.$$

Putting $u = \dfrac{1}{r}$, so that $\dfrac{du}{d\theta} = -\dfrac{1}{r^2}\dfrac{dr}{d\theta}$, we have

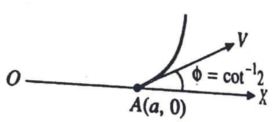
$$\left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 = \frac{2a}{r^3} + \frac{3a^2}{r^4} - \frac{1}{r^2}$$

or $\quad (dr/d\theta)^2 = 2ar + 3a^2 - r^2 = 3a^2 - (r^2 - 2ar)$

$\quad = 3a^2 - (r-a)^2 + a^2 = 4a^2 - (r-a)^2$

or $\quad dr/d\theta = \sqrt{[(2a)^2 - (r-a)^2]}$

[Note that as the particle starts moving from A, r increases as θ increases. So we have taken $dr/d\theta$ with +ive sign.]



or $$d\theta = \frac{dr}{\sqrt{[(2a)^2 - (r-a)^2]}}.$$

Integrating, $\theta + B = \sin^{-1}\left(\dfrac{r-a}{2a}\right)$.

But initially when $r = a, \theta = 0$. $\therefore B = \sin^{-1} 0 = 0$.

$\therefore \quad \theta = \sin^{-1}\left(\dfrac{r-a}{2a}\right)$ or $\sin\theta = \dfrac{r-a}{2a}$

or $\quad r = a\,(1 + 2\sin\theta)$, which is the required equation of the path.

Ex. 35. *In a central orbit the force is $\mu u^3(3 + 2a^2u^2)$; if the particle be projected at a distance a with a velocity $\sqrt{(5\mu/a^2)}$ in a direction making an angle $\tan^{-1}(\frac{1}{2})$ with the radius vector, show that the equation to the path is $r = a\tan\theta$.*

(Meerut 1970, 86P, 88P, 92; Rohilkhand 86; I.F.S. 75)

Sol. Here, the central acceleration $P = \mu u^3(3 + 2a^2u^2)$.

The differential equation of the path is

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{\mu u^3}{u^2}(3 + 2a^2u^2) = \mu(3u + 2a^2u^3).$$

Multiplying both sides by $2\,(du/d\theta)$ and integrating, we have

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = 2\mu\left(\frac{3u^2}{2} + \frac{2a^2u^4}{4}\right) + A,$$

where A is a constant

or $$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu(3u^2 + a^2u^4) + A. \quad ...(1)$$

But initially when $r = a$ i.e., $u = 1/a$, $v = \sqrt{(5\mu/a^2)}$, $\phi = \tan^{-1}(1/2)$

or $\quad \tan\phi = 1/2$ or $\sin\phi = 1/\sqrt{5}$

or $\quad p = r\sin\phi = a/\sqrt{5}$

or $\quad 1/p^2 = u^2 + (du/d\theta)^2 = 5/a^2$.

$\therefore$ from (1), we have $\dfrac{5\mu}{a^2} = h^2\cdot\dfrac{5}{a^2} = \mu\left(\dfrac{3}{a^2} + \dfrac{a^2}{a^4}\right) + A$.

$\therefore \quad h^2 = \mu \quad$ and $\quad A = \mu/a^2$.

Substituting the values of h^2 and A in (1), we have

$$\mu\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu(3u^2 + a^2u^4) + \frac{\mu}{a^2}$$

or $$\left(\frac{du}{d\theta}\right)^2 = 2u^2 + a^2u^4 + \frac{1}{a^2} = \frac{1}{a^2}(2a^2u^2 + a^4u^4 + 1).$$

Putting $u = \dfrac{1}{r}$, so that $\dfrac{du}{d\theta} = -\dfrac{1}{r^2}\dfrac{dr}{d\theta}$, we have

$$\left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 = \frac{1}{a^2}\left(\frac{2a^2}{r^2} + \frac{a^4}{r^4} + 1\right)$$

or $$\left(\frac{dr}{d\theta}\right)^2 = \frac{1}{a^2}(2a^2r^2 + a^4 + r^4) = \frac{1}{a^2}(a^2 + r^2)^2$$

or $\quad \dfrac{dr}{d\theta} = \dfrac{1}{a}(r^2 + a^2)$ or $d\theta = \dfrac{a\,dr}{(r^2 + a^2)}$.

Integrating, $\quad \theta + B = \tan^{-1}(r/a), \quad ...(2)$

where B is a constant.

But initially when $r = a$, let $\theta = \pi/4$.

Then $\frac{1}{4}\pi + B = \tan^{-1} 1 = \frac{1}{4}\pi$, so that $B = 0$.

Putting $B = 0$ in (2), we get $\theta = \tan^{-1}(r/a)$

or $r = a \tan\theta$, is the required equation of the path.

Ex. 36. *A particle moves with a central acceleration $\mu(u^5 - \frac{1}{8}a^2u^7)$; it is projected at a distance a with a velocity $\sqrt{(25/7)}$ times the velocity for a circle at that distance and at an inclination $\tan^{-1}(4/3)$ to the radius vector, show that its path is the curve*

$$4r^2 - a^2 = 3a^2/(1-\theta)^2.$$

(Meerut 1990 P)

Sol. Here, the central acceleration

$$P = \mu(u^5 - \tfrac{1}{8}a^2u^7) = \mu\left(\frac{1}{r^5} - \frac{a^2}{8r^7}\right).$$

If V is the velocity for a circle at a distance a under the same acceleration, then

$$\frac{V^2}{a} = \Big[P\Big]_{r=a} = \mu\left(\frac{1}{a^5} - \frac{a^2}{8a^7}\right) = \frac{7\mu}{8a^5}.$$

$\therefore$ $V^2 = 7\mu/8a^4$ or $V = \sqrt{(7\mu/8a^4)}$.

$\therefore$ velocity of projection of the particle

$= \sqrt{(25/7)}\,.\,V = \sqrt{(25/7)}\sqrt{(7\mu/8a^4)} = \sqrt{(25\mu/8a^4)}$.

The differential equation of the path is

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{\mu}{u^2}(u^5 - \tfrac{1}{8}a^2u^7)$$

or
$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \mu(u^3 - \tfrac{1}{8}a^2u^5).$$

Multiplying both sides by $2\,(du/d\theta)$ and integrating, we have

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\left(\frac{u^4}{2} - \frac{a^2u^6}{24}\right) + A. \qquad \text{...(1)}$$

But initially when $r = a$ i.e., $u = 1/a$, $v = \sqrt{(25\mu/8a^4)}$,

$\phi = \tan^{-1}(4/3)$ or $\tan\phi = 4/3$ or $\sin\phi = 4/5$

or $p = r\sin\phi = 4a/5$ or $1/p^2 = u^2 + (du/d\theta)^2 = 25/(16a^2)$.

Substituting the above initial conditions in (1), we get

$$\frac{25\mu}{8a^4} = h^2 . \frac{25}{16a^2} = \mu\left(\frac{1}{2a^4} - \frac{a^2}{24a^6}\right) + A.$$

$$\therefore \quad h^2 = \frac{2\mu}{a^2} \text{ and } A = \frac{25\mu}{8a^4} - \frac{11\mu}{24a^4} = \frac{8\mu}{3a^4}.$$

Substituting the values of h^2 and A in (1), we have

$$\frac{2\mu}{a^2}\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\left(\frac{u^4}{2} - \frac{1}{24}a^2u^6\right) + \frac{8\mu}{3a^4}$$

or
$$u^2 + \left(\frac{du}{d\theta}\right)^2 = \frac{a^2u^4}{4} - \frac{a^4u^6}{48} + \frac{4}{3a^2}$$

or
$$\left(\frac{du}{d\theta}\right)^2 = \frac{a^2u^4}{4} - \frac{a^4u^6}{48} + \frac{4}{3a^2} - u^2$$

$$= \frac{1}{48a^2}(64 - 48a^2u^2 + 12a^4u^4 - a^6u^6) = \frac{1}{48a^2}(4 - a^2u^2)^3.$$

Putting $u = \dfrac{1}{r}$, so that $\dfrac{du}{d\theta} = -\dfrac{1}{r^2}\dfrac{dr}{d\theta}$, we get

$$\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{1}{48a^2}\left(4 - \frac{a^2}{r^2}\right)^3 = \frac{1}{48a^2r^6}(4r^2 - a^2)^3$$

or
$$\left(\frac{dr}{d\theta}\right)^2 = \frac{1}{48a^2r^2}(4r^2 - a^2)^3 \text{ or } \frac{dr}{d\theta} = \frac{1}{4\sqrt{3}ar}(4r^2 - a^2)^{3/2}$$

or
$$d\theta = \frac{4\sqrt{3}ar\,dr}{(4r^2 - a^2)^{3/2}} = \left(\frac{\sqrt{3}}{2}a\right)(4r^2 - a^2)^{-3/2}(8r)\,dr.$$

Integrating, $\theta + B = \left(\dfrac{\sqrt{3}}{2}a\right)\dfrac{(4r^2 - a^2)^{-1/2}}{-1/2}$

or
$$\theta + B = \frac{-a\sqrt{3}}{\sqrt{(4r^2 - a^2)}}. \qquad \text{...(2)}$$

But initially when $r = a$, let $\theta = 0$. Then

$0 + B = -1$ or $B = -1$.

Putting $B = -1$ in (2), we get

$$\theta - 1 = \frac{-a\sqrt{3}}{\sqrt{(4r^2 - a^2)}} \text{ or } 1 - \theta = \frac{a\sqrt{3}}{\sqrt{(4r^2 - a^2)}}$$

or
$$\sqrt{(4r^2 - a^2)} = \frac{a\sqrt{3}}{1-\theta} \text{ or } 4r^2 - a^2 = \frac{3a^2}{(1-\theta)^2},$$

which is the required path.

Ex. 37. *A particle of mass m moves under a central force $m\mu/(\text{distance})^3$ and is projected at a distance a from the centre of force with the velocity which at an angle α to the radius would be acquired by a fall from rest at infinity to the point of projection; prove that the orbit is an equiangular spiral.*

Sol. Here, the central acceleration

$$P = \frac{\mu}{(\text{distance})^3} = \frac{\mu}{r^3} = \mu u^3. \qquad \left[\because u = \frac{1}{r}\right]$$

If V is the velocity of the particle at distance a acquired in falling from rest from infinity under the same acceleration, then

$$V^2 = -2\int_{\infty}^{a} P\,dr = -2\int_{\infty}^{a}\frac{\mu}{r^3}dr = -2\mu\left[-\frac{1}{2r^2}\right]_{\infty}^{a} = \frac{\mu}{a^2}$$

or $V = \sqrt{\mu}/a$.

The differential equation of the path is

$$h^2\left[u+\frac{d^2u}{d\theta^2}\right]=\frac{P}{u^2}=\frac{\mu u^3}{u^2}=\mu u.$$

Multiplying both sides by $2\,(du/d\theta)$ and integrating, we have

$$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu u^2+A. \quad \text{...(1)}$$

But initially, when $r=a$ *i.e.*, $u=1/a,\ v=V=\sqrt{\mu}/a,\ \phi=\alpha$

$p=r\sin\phi=a\sin\alpha$ so that $\dfrac{1}{p^2}=u^2+\left(\dfrac{du}{d\theta}\right)^2=\dfrac{1}{a^2\sin^2\alpha}$.

$\therefore$ from (1), we have $\dfrac{\mu}{a^2}=h^2\cdot\dfrac{1}{a^2\sin^2\alpha}=\dfrac{\mu}{a^2}+A.$

$\therefore\ h^2=\mu\sin^2\alpha$ and $A=0.$

Substituting the values of h^2 and A in (1), we get

$$\mu\sin^2\alpha\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu u^2$$

or $\left(\dfrac{du}{d\theta}\right)^2=u^2\operatorname{cosec}^2\alpha-u^2=u^2(\operatorname{cosec}^2\alpha-1)=u^2\cot^2\alpha.$

Putting $u=\dfrac{1}{r}$, so that $\dfrac{du}{d\theta}=-\dfrac{1}{r^2}\dfrac{dr}{d\theta}$, we have

$\dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2=\dfrac{1}{r^2}\cot^2\alpha$ or $\left(\dfrac{dr}{d\theta}\right)^2=r^2\cot^2\alpha$

or $\dfrac{dr}{d\theta}=r\cot\alpha$ or $\dfrac{dr}{r}=(\cot\alpha)\,d\theta.$

Integrating, $\log r=\theta\cot\alpha+B.$

But initially when $r=a$, let $\theta=0$. Then $B=\log a$.

$\therefore\ \log r=\theta\cot\alpha+\log a$

or $\log r-\log a=\theta\cot\alpha$ or $\log(r/a)=\theta\cot\alpha$

or $r=ae^{\theta\cot\alpha}$, which is an equiangular spiral.

Ex. 38. *A particle acted on by a central attractive force μu^3 is projecte with a velocity $\sqrt{\mu}/a$ at an angle $\pi/4$ with its initial distance a from the centre of force. Show that its orbit is the equiangular spiral $r=ae^{-\theta}$.*

(Meerut 1973, 74; Allahabad 77; Agra 84)

Sol. Here, the central acceleration $P=\mu u^3$.

The differential equation of the central orbit is

$$h^2\left[u+\frac{d^2u}{d\theta^2}\right]=\frac{P}{u^2}=\frac{\mu u^3}{u^2}=\mu u.$$

Multiplying both sides by $2\,(du/d\theta)$ and integrating, we have

$$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu u^2+A. \quad \text{...(1)}$$

Initially when $r=a,\ u=1/a,\ v=\sqrt{\mu}/a,\ \phi=\pi/4,$

$p=r\sin\phi=a\sin\frac{1}{4}\pi=a/\sqrt{2}$ so that $1/p^2=u^2+(du/d\theta)^2=2/a^2.$

$\therefore$ from (1), we have $\dfrac{\mu}{a^2}=h^2\left(\dfrac{2}{a^2}\right)=\dfrac{\mu}{a^2}+A.$

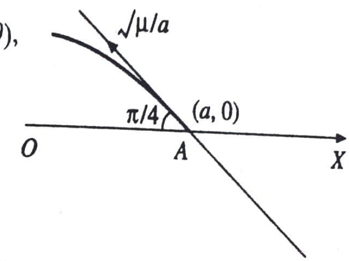
$\therefore\ h^2=\mu/2,\quad A=0.$

Substituting the values of h^2 and A in (1), we have

$$\frac{\mu}{2}\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu u^2$$

or $(du/d\theta)^2=u^2.$

Putting $u=1/r$, so that $du/d\theta=(-1/r^2)(dr/d\theta)$,



we have $\dfrac{1}{r^4}\left(\dfrac{dr}{d\theta}\right)^2=\dfrac{1}{r^2}$

or $\left(\dfrac{dr}{d\theta}\right)^2=r^2$

or $\dfrac{dr}{d\theta}=-r$

[the –ive sign has been taken because r decreases while θ increases as the particle starts moving from the point A]

or $\dfrac{dr}{r}=-d\theta.$

Integrating, $\log r=-\theta+B.$

Initially when $r=a$, let $\theta=0$. Then $B=\log a$.

$\therefore\ \log r=-\theta+\log a$ or $\log(r/a)=-\theta$

or $r=ae^{-\theta}$, which is the required equation of the path and is an equiangular spiral.

Ex. 39. *A particle moves with a central acceleration $\mu(3u^3+a^2u^5)$ being projected from a distance a at an angle 45° with a velocity equal to that in a circle at the same distance. Prove that the time to the centre of force is*

$$\frac{a^2}{\sqrt{(2\mu)}}\left(2-\tfrac{1}{2}\pi\right).$$

Sol. Here the central acceleration

$$P=\mu(3u^3+a^2u^5)=\mu\left(\frac{3}{r^3}+\frac{a^2}{r^5}\right). \qquad [\because u=1/r]$$

If V is the velocity in a circle at a distance a under the same acceleration, then

$$\frac{V^2}{a} = \left[P\right]_{r=a} = \mu\left(\frac{3}{a^3} + \frac{a^2}{a^5}\right)$$

or $V^2 = \frac{4\mu}{a^2}$ or $V = 2\sqrt{\mu/a}$.

The differential equation of the path is

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{\mu}{u^2}(3u^3 + a^2u^5) = \mu(3u + a^2u^3).$$

Multiplying both sides by $2(du/d\theta)$ and integrating, we have

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\left(3u^2 + \frac{a^2}{2}u^4\right) + A,$$

where A is a constant.

But initially when $r = a$ i.e., $u = 1/a$, $v = 2\sqrt{\mu/a}$, $\phi = 45°$, $p = r\sin\phi = a\sin\frac{1}{4}\pi = a/\sqrt{2}$ so that $1/p^2 = u^2 + (du/d\theta)^2 = 2/a^2$.

$\therefore$ from (1), we have $\frac{4\mu}{a^2} = h^2 . \frac{2}{a^2} = \mu\left(\frac{3}{a^2} + \frac{a^2}{2a^4}\right) + A$.

$\therefore$ $h^2 = 2\mu$ and $A = \mu/2a^2$.

Substituting the values of h^2 and A in (1), we have

$$2\mu\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\left(3u^2 + \frac{a^2}{2}u^4\right) + \frac{\mu}{2a^2}$$

or
$$2\left(\frac{du}{d\theta}\right)^2 = u^2 + \frac{a^2}{2}u^4 + \frac{1}{2a^2}.$$

Putting $u = \frac{1}{r}$, so that $\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$, we have

$$\frac{2}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{1}{r^2} + \frac{a^2}{2r^4} + \frac{1}{2a^2}$$

or
$$4a^2\left(\frac{dr}{d\theta}\right)^2 = 2a^2r^2 + a^4 + r^4 = (r^2 + a^2)^2$$

or
$$\frac{dr}{d\theta} = -\frac{r^2 + a^2}{2a}. \qquad ...(2)$$

[Negative sign is taken because r decreases when θ increases. See figure in Ex. 38].

We have

$$h = r^2\frac{d\theta}{dt} = r^2\frac{d\theta}{dr} . \frac{dr}{dt}$$

or
$$\sqrt{(2\mu)} = -r^2 . \frac{2a}{(r^2 + a^2)} . \frac{dr}{dt} \quad \text{[substituting for } h \text{ and } dr/d\theta]$$

or
$$dt = -\frac{2a}{\sqrt{(2\mu)}} . \frac{r^2\,dr}{(r^2 + a^2)}.$$

Integrating between the limits $r = a$ to $r = 0$, the required time t_1 from the distance a to the centre of force is given by

$$t_1 = -\frac{2a}{\sqrt{(2\mu)}}\int_{r=a}^{0}\frac{r^2\,dr}{r^2 + a^2} = -\frac{2a}{\sqrt{(2\mu)}} . \int_a^0\left(1 - \frac{a^2}{r^2 + a^2}\right)dr$$

$$= -\frac{2a}{\sqrt{(2\mu)}} . \left[r - a\tan^{-1}\left(\frac{r}{a}\right)\right]_a^0$$

$$= -\frac{2a}{\sqrt{(2\mu)}}[\{0 - a\tan^{-1}0\} - \{a - a\tan^{-1}(a/a)\}]$$

$$= \frac{2a}{\sqrt{(2\mu)}}\left[a - a . \tfrac{1}{4}\pi\right] = \frac{a^2}{\sqrt{(2\mu)}}\left[2 - \tfrac{1}{2}\pi\right].$$

Ex. 40. *A particle moves with central acceleration* $\mu\left(r + \frac{2a^3}{r^2}\right)$ *being projected from an apse at a distance a with twice the velocity for a circle at that distance; find the apsidal distance and show that equation to the path is*

$$\frac{\theta}{2} = \tan^{-1}(t\sqrt{3}) - \left(\frac{1}{\sqrt{5}}\right)\tan^{-1}\left(\sqrt{(5/3)}\,t\right), \text{where } t^2 = \frac{(r-a)}{(3a-r)}.$$

Sol. Here, the central acceleration

$$P = \mu\left(r + \frac{2a^3}{r^2}\right) = \mu\left(\frac{1}{u} + 2a^3u^2\right).$$

If V is the velocity for a circle at a distance a under the same acceleration, we have

$$\frac{V^2}{a} = \left[P\right]_{r=a} = \mu\left(a + \frac{2a^3}{a^2}\right) = \mu . 3a.$$

$\therefore$ $V^2 = 3\mu a^2$, or $V = a\sqrt{(3\mu)}$.

$\therefore$ the velocity of projection of the particle $= 2V = 2a\sqrt{(3\mu)}$.

The differential equation of the path is

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{\mu}{u^2}\left(\frac{1}{u} + 2a^3u^2\right) = \mu\left(\frac{1}{u^3} + 2a^3\right).$$

Multiplying both sides by $2(du/d\theta)$ and integrating, we have

$$h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = 2\mu\left(-\frac{1}{2u^2} + 2a^3u\right) + A,$$

where A is a constant

or
$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\left(-\frac{1}{u^2} + 4a^3u\right) + A. \qquad ...(1)$$

But initially when $r = a$, i.e., $u = 1/a$, $du/d\theta = 0$ (at an apse) and $v = 2a\sqrt{(3\mu)}$.

$\therefore$ from (1), we have

$$12\mu a^2 = h^2\left[\frac{1}{a^2}\right] = \mu\left(-a^2+\frac{4a^3}{a}\right)+A.$$

$\therefore\quad h^2 = 12\mu a^4 \quad$ and $\quad A = 9\mu a^2.$

Substituting the values of h^2 and A in (1), we have

$$12\mu a^4\left[u^2+\left(\frac{du}{d\theta}\right)^2\right] = \mu\left(-\frac{1}{u^2}+4a^3u\right)+9\mu a^2$$

or $$12a^4\left(\frac{du}{d\theta}\right)^2 = -12a^4u^2-\frac{1}{u^2}+4a^3u+9a^2.$$

Putting $\quad u=\frac{1}{r},\quad$ so that $\quad \frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta},\quad$ we have

$$12a^4\cdot\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = -\frac{12a^4}{r^2}-r^2+\frac{4a^3}{r}+9a^2$$

or $$12a^4\left(\frac{dr}{d\theta}\right)^2 = -12a^4r^2-r^6+4a^3r^3+9a^2r^4$$

$$= r^2[-r^4+9a^2r^2+4a^3r-12a^4]$$

$$= r^2[-r^3(r-a)-r^2a(r-a)+8a^2r(r-a)+12a^3(r-a)]$$

$$= r^2(r-a)(-r^3-r^2a+8a^2r+12a^3)$$

$$= r^2(r-a)[-r^2(r-3a)-4ar(r-3a)-4a^2(r-3a)]$$

$$= r^2(r-a)(3a-r)(r^2+4ar+4a^2) = r^2(r-a)(3a-r)(r+2a)^2.$$

$$\therefore\quad 2\sqrt{3}a^2\frac{dr}{d\theta} = r(r+2a)\sqrt{\{(r-a)(3a-r)\}}. \qquad \text{...(2)}$$

At an apse, we have $\quad du/d\theta = 0 \quad$ or $\quad dr/d\theta = 0.$

Putting $dr/d\theta = 0$ in (2), we have

$$0 = r(r+2a)\sqrt{\{(r-a)(3a-r)\}}.$$

The positive roots of r are $r = a, r = 3a$.

But $r = a$ is the first apsidal distance. Therefore the second apsidal distance is $3a$.

Now to find the equation of the path from the equation (2), we have

$$\frac{d\theta}{2} = \frac{a^2\sqrt{3}\,dr}{r(r+2a)\sqrt{\{(r-a)(3a-r)\}}}. \qquad \text{...(3)}$$

Put $\quad (r-a) = (3a-r)t^2$

or $\quad r = \frac{a(3t^2+1)}{1+t^2},\quad$ so that $\quad dr = \frac{4at\,dt}{(1+t^2)^2}.$

Then from (3), we have

$$\frac{d\theta}{2} = \frac{a^2\sqrt{3}\,\frac{4at\,dt}{(1+t^2)^2}}{\frac{a(3t^2+1)}{(1+t^2)}\cdot\left\{\frac{a(3t^2+1)}{1+t^2}+2a\right\}\left\{3a-\frac{a(3t^2+1)}{1+t^2}\right\}t}$$

$$\frac{d\theta}{2} = \frac{2\sqrt{3}(t^2+1)}{(3t^2+1)(5t^2+3)}dt$$

$$\frac{d\theta}{2} = \sqrt{3}\left[\frac{1}{3t^2+1}-\frac{1}{5t^2+3}\right]dt.$$

Integrating, $$\frac{\theta}{2}+B = \tan^{-1}(t\sqrt{3})-\sqrt{3}\cdot\frac{1}{\sqrt{3}\sqrt{5}}\tan^{-1}\left(\frac{\sqrt{5}t}{\sqrt{3}}\right),$$

where B is a constant

or $$\frac{\theta}{2}+B = \tan^{-1}(t\sqrt{3})-\frac{1}{\sqrt{5}}\tan^{-1}[\sqrt{(5/3)}\,.\,t].$$

But initially when $r = a$, let $\theta = 0$.

Also initially $$t = \sqrt{\left(\frac{a-a}{3a-a}\right)} = 0.$$

$\therefore\quad B = 0.$

$$\therefore\quad \frac{\theta}{2} = \tan^{-1}(t\sqrt{3})-\frac{1}{\sqrt{5}}\tan^{-1}[\sqrt{(5/3)}\,.\,t],$$

where $$t^2 = \frac{(r-a)}{(3a-r)}.$$

This is the required equation of the path.

Ex. 41. *A particle is projected with velocity $\sqrt{(2\mu/3c^3)}$ from a point P in a field of attractive force μ/r^4 to a point O distant c from P, where r denotes the distance from O.*

If the direction of projection makes an angle 45° with PO, prove that the orbit is a cardioid and the particle will arrive at O after a time $(\frac{3}{4}\pi-2)\cdot\sqrt{(3c^5/\mu)}$.

Sol. Here, the central acceleration

$$P = \mu/r^4 = \mu u^4. \qquad [\because\ u = 1/r]$$

The differential equation of the path is

$$h^2\left[u+\frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{\mu u^4}{u^2} = \mu u^2.$$

Multiplying both sides by $2\,(du/d\theta)$ and integrating, we have

$$v^2 = h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right] = \frac{2\mu}{3}u^3+A. \qquad \text{...(1)}$$

But initially at P, $r = c$, $u = 1/c$, $v = \sqrt{(2\mu/3c^3)}$, $\phi = 45°$, $p = r\sin\phi = c\sin 45° = c/\sqrt{2}$ so that $1/p^2 = u^2+(du/d\theta)^2 = 2/c^2$.

$\therefore$ from (1), we have

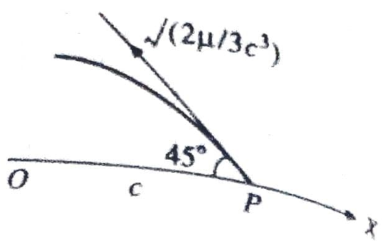
$$\frac{2\mu}{3c^3} = h^2\left[\frac{2}{c^2}\right] = \frac{2\mu}{3c^3}+A.$$

$\therefore\quad h^2 = \mu/(3c) \quad$ and $\quad A = 0.$

Substituting the values of h^2 and A in (1), we have

$$\frac{\mu}{3c}\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\frac{2\mu}{3}u^3$$

or $$u^2+\left(\frac{du}{d\theta}\right)^2=2cu^3.$$



Putting $u=\dfrac{1}{r}$ so that

$$\frac{du}{d\theta}=-\frac{1}{r^2}\frac{dr}{d\theta},\text{ we have}$$

$$\frac{1}{r^2}+\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2=\frac{2c}{r^3}$$

or $$\left(\frac{dr}{d\theta}\right)^2=2cr-r^2=c^2-(r-c)^2$$

or $$\frac{dr}{d\theta}=-\sqrt{\{c^2-(r-c)^2\}}, \qquad \text{...(2)}$$ the $-$ ive sign has been taken because r decreases and θ increases as the particle starts moving from P.

$$\therefore\quad d\theta=\frac{-dr}{\sqrt{\{c^2-(r-c)^2\}}}.$$

Integrating, $\theta+B=\cos^{-1}\left(\dfrac{r-c}{c}\right)$.

But initially when $r=c$, let $\theta=0$. Then $B=\cos^{-1}0=\frac{1}{2}\pi$.

$$\therefore\quad \theta+\tfrac{1}{2}\pi=\cos^{-1}\left(\frac{r-c}{c}\right)$$

or $$\frac{r-c}{c}=\cos\left(\tfrac{1}{2}\pi+\theta\right)=-\sin\theta$$

or $$r-c=-c\sin\theta \quad\text{or}\quad r=c\,(1-\sin\theta),$$

which is a cardioid.

Second part. We have $h=r^2\,(d\theta/dt)$.

But $h=\sqrt{(\mu/3c)}$ and $r=c\,(1-\sin\theta)$, as found above.

$$\therefore\quad dt=\frac{r^2}{h}\,d\theta=\frac{c^2\,(1-\sin\theta)^2}{\sqrt{(\mu/3c)}}\,d\theta=\sqrt{(3c^5/\mu)}\,(1-\sin\theta)^2\,d\theta. \qquad\text{...(3)}$$

At the starting point P, $\theta=0$. Also at the point O, $r=0$.

Putting $r=0$ in the equation of the path $r=c\,(1-\sin\theta)$, we get

$1-\sin\theta=0$ or $\sin\theta=1$ or $\theta=\frac{1}{2}\pi$.

So at the point O, $\theta=\frac{1}{2}\pi$. Let t_1 be the time from P to O. Then integrating (3) from P to O, we get

$$t_1=\int_0^{\pi/2}\sqrt{(3c^5/\mu)}\,(1-2\sin\theta+\sin^2\theta)\,d\theta$$

$$=\sqrt{\left(\frac{3c^5}{\mu}\right)}\left[\tfrac{1}{2}\pi-2\cdot1+\tfrac{1}{2}\cdot\tfrac{1}{2}\pi\right]\quad\left[\text{Note that }\int_0^{\pi/2}\sin\theta\,d\theta=1\text{ and }\int_0^{\pi/2}\sin^2\theta\,d\theta=\tfrac{1}{2}\cdot\tfrac{1}{2}\pi\right]$$

$$=\sqrt{(3c^5/\mu)}\,\left[\tfrac{3}{4}\pi-2\right].$$

Ex. 42. *A particle of mass m moves under a central attractive force $m\mu\,(5/r^3+8c^2/r^5)$, and is projected from an apse at a distance c with velocity $3\sqrt{\mu}/c$, prove that the orbit is $r=c\cos(2\theta/3)$, and that it will arrive at the origin after a time $\pi c^2/(8\sqrt{\mu})$.* **[Meerut 84, 89, 91P]**

Sol. Here, the central acceleration

$$P=\mu\,(5/r^3+8c^2/r^5)=\mu\,(5u^3+8c^2u^5)$$

The differential equation of the central orbit is

$$h^2\left[u+\frac{d^2u}{d\theta^2}\right]=\frac{P}{u^2}=\frac{\mu}{u^2}(5u^3+8c^2u^5)=\mu\,(5u+8c^2u^3).$$

Multiplying both sides by $2\,(du/d\theta)$ and integrating, we have

$$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu\,(5u^2+4c^2u^4)+A. \qquad\text{...(1)}$$

But initially at an apse, $r=c$, $u=1/c$, $du/d\theta=0$, $v=3\sqrt{\mu}/c$.

$\therefore$ from (1), we have

$$\frac{9\mu}{c^2}=h^2\cdot\frac{1}{c^2}=\mu\left(\frac{5}{c^2}+\frac{4}{c^2}\right)+A.$$

$$\therefore\quad h^2=9\mu,\ A=0.$$

Substituting the values of h^2 and A in (1), we have

$$9\mu\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu\,(5u^2+4c^2u^4)$$

or $$9\,(du/d\theta)^2=4c^2u^4-4u^2.$$

Putting $u=\dfrac{1}{r}$, so that $\dfrac{du}{d\theta}=-\dfrac{1}{r^2}\dfrac{dr}{d\theta}$, we have

$$9\,\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2=4\left(\frac{c^2}{r^4}-\frac{1}{r^2}\right)$$

or $$9\,(dr/d\theta)^2=4\,(c^2-r^2)\quad\text{or}\quad dr/d\theta=-\tfrac{2}{3}\sqrt{(c^2-r^2)}$$

or $$\frac{2}{3}\,d\theta=\frac{-dr}{\sqrt{(c^2-r^2)}}.$$

Integrating, $2\theta/3 + B = \cos^{-1}(r/c)$, where B is a constant.

Initially when $r = c$, let $\theta = 0$. Then $B = \cos^{-1} 1 = 0$.

$\therefore \quad 2\theta/3 = \cos^{-1}(r/c)$ or $r/c = \cos(2\theta/3)$

or $\quad r = c\cos(2\theta/3)$, which is the required equation of the path.

Second Part. We have $h = r^2 (d\theta/dt)$.

But $h = 3\sqrt{\mu}$ and $r = c\cos(2\theta/3)$, as found above.

$$\therefore \quad dt = \frac{r^2}{h}\, d\theta = \frac{c^2\cos^2(2\theta/3)}{3\sqrt{\mu}}\, d\theta.$$

At the point of projection, we have taken $\theta = 0$. Also at the pole $O, r = 0$. Putting $r = 0$ in the equation of the path, we get $0 = c\cos(2\theta/3)$ giving $2\theta/3 = \frac{1}{2}\pi$ or $\theta = 3\pi/4$. So at $O, \theta = 3\pi/4$.

Let t_1 be the time from the point of projection to the point O. Then integrating (2), we have

$$t_1 = \int_0^{3\pi/4} \frac{c^2}{3\sqrt{\mu}} \cos^2(2\theta/3)\, d\theta.$$

Put $2\theta/3 = z$, so that $(2/3)\, d\theta = dz$. When $\theta = 0, z = 0$ and when $\theta = 3\pi/4, z = \frac{1}{2}\pi$.

$$\therefore \quad t_1 = \int_0^{\pi/2} \frac{c^2}{3\sqrt{\mu}} (\cos^2 z) \,.\, (3/2)\, dz$$

$$= \frac{c^2}{2\sqrt{\mu}} \int_0^{\pi/2} \cos^2 z\, dz = \frac{c^2}{2\sqrt{\mu}} \cdot \frac{1}{2} \cdot \frac{1}{2}\pi = \frac{\pi c^2}{8\sqrt{\mu}}.$$

Ex. 43. *A particle moves in a curve under a central acceleration so that its velocity at any point is equal to that in a circle at the same distance and under the same attraction. Show that the law of force is that of inverse cube, and the path is an equiangular spiral.*

Sol. Let the central acceleration $P = u^2\phi'(u)$. [Note]

The differential equation of the central orbit is

$$h^2\left\{u + \frac{d^2u}{d\theta^2}\right\} = \frac{P}{u^2} = \phi'(u).$$

Multiplying both sides by $2\,(du/d\theta)$ and integrating, we have

$$v^2 = h^2\left\{u^2 + \left(\frac{du}{d\theta}\right)^2\right\} = 2\phi(u) + A. \qquad ...(1)$$

But according to the question the velocity of the particle at any point in the orbit is equal to that in a circle at the same distance under the same acceleration.

$$\therefore \quad \frac{v^2}{r} = P \quad \text{or} \quad v^2 = rP = ru^2\phi'(u) = u\phi'(u). \qquad ...(2)$$

Substituting the value of v^2 from (2) in (1), we have

$$u\phi'(u) = 2\phi(u) + A$$

or $\quad \dfrac{\phi'(u)}{u^2} - \dfrac{2\phi(u)}{u^3} = \dfrac{A}{u^3}$ [dividing both sides by u^3]

or $\quad \dfrac{d}{du}\left[\dfrac{1}{u^2}\phi(u)\right] = \dfrac{A}{u^3}.$

Integrating, $\quad \dfrac{1}{u^2}\phi(u) = -\dfrac{A}{2u^2} + B$

or $\quad \phi(u) = -\dfrac{A}{2} + Bu^2.$

Differentiating w.r.t. 'u', we have

$$\phi'(u) = 2Bu.$$

$$\therefore \quad P = u^2\phi'(u) = u^2 \,.\, 2Bu = 2Bu^3 = 2B/r^3 \qquad ...(3)$$

or $\quad P \propto 1/r^3$ *i.e.*, the acceleration varies inversely as the cube of the distance from the centre.

To find the equation of the path. The differential equation of the path in pedal form is

$$\frac{h^2}{p^3}\frac{dp}{dr} = P.$$

$$\therefore \quad \frac{h^2}{p^3}\frac{dp}{dr} = \frac{2B}{r^3} \qquad \left[\because \text{ from (3), } P = \frac{2B}{r^3}\right]$$

or $\quad -\dfrac{2h^2}{p^3}\, dp = -\dfrac{4B}{r^3}\, dr.$ [Multiplying both sides by -2]

Integrating, $\quad \dfrac{h^2}{p^2} = \dfrac{2B}{r^2} + C.$

If $p \to \infty$ when $r \to \infty$, we have $C = 0$.

$$\therefore \quad \frac{h^2}{p^2} = \frac{2B}{r^2} \quad \text{or} \quad p^2 = \frac{h^2}{2B} r^2$$

or $\quad p = ar$, where a is a constant.

This is the pedal equation of an equiangular spiral and is the required path.

Ex. 44. *A particle moves in a plane under a central force which varies inversely as the square of the distance from the fixed point, find the orbit.* **(Meerut 85 P)**

Sol. We know that referred to the centre of force as pole the differential equation of a central orbit in pedal form is

$$\frac{h^2}{p^3}\frac{dp}{dr} = P, \qquad ...(1)$$

where P is the central acceleration assumed to be attractive.

Here $P = \mu/r^2$. Putting $P = \mu/r^2$ in (1), we get

$$\frac{h^2}{p^3}\frac{dp}{dr} = \frac{\mu}{r^2}$$

or
$$\frac{h^2}{p^3}dp = \frac{\mu}{r^2}dr$$

or
$$-2\frac{h^2}{p^3}dp = -\frac{2\mu}{r^2}dr.$$

Integrating both sides, we get

$$v^2 = \frac{h^2}{p^2} = \frac{2\mu}{r} + C. \quad ...(2)$$

Let $v = v_0$ when $r = r_0$.

Then $v_0^2 = \frac{2\mu}{r_0} + C$ or $C = v_0^2 - \frac{2\mu}{r_0}$.

Putting this value of C in (2), the pedal equation of the central orbit is

$$\frac{h^2}{p^2} = \frac{2\mu}{r} + v_0^2 - \frac{2\mu}{r_0}. \quad ...(3)$$

Case I. Let $v_0^2 = \frac{2\mu}{r_0}$. Then the equation (3) becomes $\frac{h^2}{p^2} = \frac{2\mu}{r}$ which is of the form $p^2 = ar$.

This is the pedal equation of a parabola referred to focus as pole. Hence in this case the orbit is a parabola with centre of force at the focus.

Case II. Let $v_0^2 < \frac{2\mu}{r_0}$. In this case the equation (3) reduces to the form

$$\frac{b^2}{p^2} = \frac{2a}{r} - 1.$$

This is the pedal equation of an ellipse referred to a focus as pole. Hence in this case the orbit is an ellipse with centre of force at its focus.

Case III. Let $v_0^2 > \frac{2\mu}{r_0}$. In this case the equation (3) reduces to the form

$$\frac{b^2}{p^2} = \frac{2a}{r} + 1.$$

This is the pedal equation of a hyperbola referred to a focus as pole. It represents that branch of the hyperbola which is nearer to the focus taken as pole.

Hence we conclude that under inverse square law the central orbit is always a conic with centre of force at the focus.

Ex. 45. *If the central force varies inversely as the cube of the distance from a fixed point, find the orbit.* **(Meerut 88 P, 92)**

Sol. We know that referred to the centre of force as pole the differential equation of a central orbit in pedal form is

$$\frac{h^2}{p^3}\frac{dp}{dr} = P, \quad ...(1)$$

where P is the central acceleration assumed to be attractive.

Here $P = \mu/r^3$. Putting $P = \mu/r^3$ in (1), we get

$$\frac{h^2}{p^3}\frac{dp}{dr} = \frac{\mu}{r^3} \quad \text{or} \quad \frac{h^2}{p^3}dp = \frac{\mu}{r^3}dr$$

or
$$-\frac{2h^2}{p^3}dp = -\frac{2\mu}{r^3}dr.$$

Integrating both sides, we get

$$\frac{h^2}{p^2} = \frac{\mu}{r^2} + C. \quad ...(2)$$

Let $p \to \infty$ as $r \to \infty$. Then $0 = 0 + C$ or $C = 0$.

Putting $C = 0$ in (2), the pedal equation of the orbit is

$$\frac{h^2}{p^2} = \frac{\mu}{r^2} \quad \text{or} \quad p^2 = \frac{h^2}{\mu}r^2$$

or $p = ar$ where a is some constant.

This is the pedal equation of an equiangular spiral. [Note that the pedal equation of the equiangular spiral $r = ae^{\theta \cot \alpha}$ is $p = r \sin \alpha$].

Hence under inverse cube law the central orbit is an equiangular spiral.

Ex. 46. *A particle moves with central acceleration $(\mu u^2 + \lambda u^3)$ and the velocity of projection at distance R is V; show that the particle will ultimately go off to infinity if $V^2 > \frac{2\mu}{R} + \frac{\lambda}{R^2}$.*

Sol. Here, the central acceleration $P = \mu u^2 + \lambda u^3$.

The differential equation of the path is

$$h^2\left\{u + \frac{d^2u}{d\theta^2}\right\} = \frac{P}{u^2} = \frac{1}{u^2}(\mu u^2 + \lambda u^3) = \mu + \lambda u.$$

Multiplying both sides by $2\,(du/d\theta)$ and integrating, we have

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = 2\mu u + \lambda u^2 + A. \quad ...(1)$$

But initially when $r = R$ i.e., $u = 1/R$, $v = V$.

$\therefore$ from (1), we have

$$V^2 = \frac{2\mu}{R} + \frac{\lambda}{R^2} + A \quad \text{or} \quad A = V^2 - \frac{2\mu}{R} - \frac{\lambda}{R^2}. \quad \text{...(2)}$$

Hence the equation (1) is

$$h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = 2\mu u + \lambda u^2 + A,$$

where A is given by the equation (2)

or $$h^2\left(\frac{du}{d\theta}\right)^2 = (\lambda - h^2)\,u^2 + 2\mu u + A$$

$$= (\lambda - h^2)\left\{u^2 + \frac{2\mu}{(\lambda - h^2)}u + \frac{A}{(\lambda - h^2)}\right\}$$

$$= (\lambda - h^2)\left\{\left(u + \frac{\mu}{(\lambda - h^2)}\right)^2 + \left(\frac{A}{(\lambda - h^2)} - \frac{\mu^2}{(\lambda - h^2)^2}\right)\right\}$$

or $$h\left(\frac{du}{d\theta}\right) = \sqrt{(\lambda - h^2)}\cdot\left\{\left(u + \frac{\mu}{(\lambda - h^2)}\right)^2 + \left(\frac{A}{(\lambda - h^2)} - \frac{\mu^2}{(\lambda - h^2)^2}\right)\right\}^{1/2}$$

or $$d\theta = \frac{h}{\sqrt{(\lambda - h^2)}}\cdot\frac{du}{\left\{\left(u + \frac{\mu}{(\lambda - h^2)}\right)^2 + \left(\frac{A}{(\lambda - h^2)} - \frac{\mu^2}{(\lambda - h^2)^2}\right)\right\}^{1/2}}$$

Integrating,

$$\theta + B = \frac{h}{\sqrt{(\lambda - h^2)}}\cdot\log\left[\left(u + \frac{\mu}{(\lambda - h^2)}\right) + \left\{\left(u + \frac{\mu}{(\lambda - h^2)}\right)^2 + \left(\frac{A}{(\lambda - h^2)} - \frac{\mu^2}{(\lambda - h^2)^2}\right)\right\}^{1/2}\right]. \quad \text{...(3)}$$

Ultimately means when $r \to \infty$. So the particle will ultimately go off to infinity if θ is real when $r \to \infty$ *i.e.*, when $u \to 0$.

Now when $u = 0$, the equation (3) becomes

$$\theta + B = \frac{h}{(\lambda - h^2)}\cdot\log\left[\frac{\mu}{(\lambda - h^2)} + \left(\frac{A}{\lambda - h^2}\right)^{1/2}\right]. \quad \text{...(4)}$$

Assuming $\lambda > h^2$, we see that the equation (4) always gives a real value of θ provided A is positive. Therefore the particle will ultimately go off to infinity if $A > 0$

i.e., if $V^2 - \frac{2\mu}{R} - \frac{\lambda}{R^2} > 0$ [using (2)]

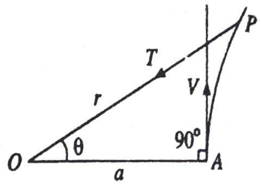
i.e., if $V^2 > \frac{2\mu}{R} + \frac{\lambda}{R^2}$.

Ex. 47. *A particle of mass m is attached to a fixed point by an elastic string of natural length a, the coefficient of elasticity being nmg; it*

is projected from an apse at a distance a with velocity $\sqrt{(2pgh)}$; show that the other apsidal distance is given by the equation

$$nr^2(r - a) - 2pha\,(r + a) = 0.$$

Sol. Let a particle of mass m be attached to a fixed point O by an elastic string of natural length a. Initially the particle is at A such that $OA = a$ and is projected perpendicular to OA with velocity $V = \sqrt{(2pgh)}$. Let P be the position of the particle at any time t, where $OP = r$ and $\angle AOP = \theta$. The only force acting on the particle at P in the plane of motion is the tension T in the string OP and is always directed towards the fixed centre O. So the path of the particle is a central orbit. By Hooke's law, the tension T in the string OP is given by $T = \lambda\frac{OP - a}{a} = nmg\frac{r - a}{a}$.



[$\because \lambda = nmg$]

$\therefore$ P = the central acceleration of the particle at P

$$= \frac{T}{m} \qquad \left[\because \text{acceleration} = \frac{\text{force}}{\text{mass}}\right]$$

$$= ng\left(\frac{r - a}{a}\right) = \frac{ng}{a}\left(\frac{1}{u} - a\right). \qquad \left[\because r = \frac{1}{u}\right]$$

The differential equation of the path of the particle is

$$H^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{1}{u^2}\cdot\frac{ng}{a}\left(\frac{1}{u} - a\right) = \frac{ng}{a}\left(\frac{1}{u^3} - \frac{a}{u^2}\right).$$

Here the letter H has been used because the letter h is given in the problem.

Multiplying both sides by $2\,(du/d\theta)$ and integrating, we have

$$v^2 = H^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \frac{ng}{a}\left(-\frac{1}{u^2} + \frac{2a}{u}\right) + A, \quad \text{...(1)}$$

where A is a constant.

But initially at the apse A, $r = a, u = 1/a, du/d\theta = 0, v = \sqrt{(2pgh)}$.

Applying these initial conditions to (1), we have

$$2pgh = \frac{H^2}{a^2} = \frac{ng}{a}(-a^2 + 2a^2) + A.$$

$\therefore$ $H^2 = 2pgha^2$ and $A = 2pgh - nga$.

Substituting the values of H^2 and A in (1), we have

$$2pgha^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \frac{ng}{a}\left(-\frac{1}{u^2} + \frac{2a}{u}\right) + 2pgh - nga. \quad \text{...(2)}$$

Now at an apse, $du/d\theta = 0$. Therefore putting $du/d\theta = 0$ in (2), the apsidal distances are given by the equation

$$2pgha^2u^2 = \frac{ng}{a}\left(-\frac{1}{u^2}+\frac{2a}{u}\right)+2pgh-nga$$

or $\frac{2pha^2}{r^2} = \frac{n}{a}(-r^2+2ar)+2ph-na \qquad \left[\because \frac{1}{u}=r\right]$

or $2pha^3 = nr^2(-r^2+2ar)+(2ph-na)ar^2$

or $2pha^3 - 2phar^2 + nr^2(r^2-2ar)+na^2r^2 = 0$

or $2pha(a^2-r^2)+nr^2(r^2-2ar+a^2)=0$

or $nr^2(r-a)^2 - 2pha(r-a)(r+a)=0$

or $(r-a)\{nr^2(r-a)-2pha(r+a)\}=0.$

But $r-a=0$ gives the first apsidal distance $r=a$. Therefore the other apsidal distance is given by the equation

$$nr^2(r-a)-2pha(r+a)=0.$$

Ex. 48. *A particle is attached to a fixed point on a smooth horizontal plane by an elastic string of natural length a. Initially the particle is at rest on the plane with the string just taut and it is projected horizontally in a direction perpendicular to the string with a kinetic energy equal to the potential energy of the string when its extension is $3a/\sqrt{2}$. Prove that the second apsidal distance is equal to 3a.*

Sol. By Hooke's law, the tension in the string when its extension is $3a/\sqrt{2}$

$$= \lambda\cdot\frac{3a/\sqrt{2}}{a} = \frac{3\lambda}{\sqrt{2}},$$

where λ is the modulus of elasticity of the string.

We know that the potential energy of an elastic string in any stretched position $= \frac{1}{2}$ (initial tension + final tension) × extension.

$\therefore$ the potential energy of the string when its extension is $3a/\sqrt{2}$

$$= \frac{1}{2}\left[0+\frac{3\lambda}{\sqrt{2}}\right]\times\frac{3a}{\sqrt{2}} = \frac{9a\lambda}{4}. \qquad \left[\begin{array}{l}\text{Note that the initial}\\ \text{tension is zero}\end{array}\right]$$

If V is the velocity of projection of the particle, then its kinetic energy at that time $= \frac{1}{2}mV^2$.

According to the question,

$$\frac{1}{2}mV^2 = \frac{9a\lambda}{4} \quad\text{or}\quad V^2 = \frac{9a\lambda}{2m} \quad\text{or}\quad V=\sqrt{\left(\frac{9a\lambda}{2m}\right)}.$$

Now suppose the particle is initially at A, where $OA = a =$ natural length of the string. [Refer figure of Ex. 47.]

The particle is projected from A perpendicular to OA with velocity $V = \sqrt{(9a\lambda/2m)}$. Let P be the position of the particle at any time t, where $OP = r$. The only force acting on the particle at P in the plane of motion is the tension T in the string OP and is always directed towards the fixed centre O. By Hooke's law,

$$T = \lambda\frac{OP-a}{a} = \lambda\frac{r-a}{a}.$$

$\therefore P =$ the central acceleration of the particle at the point P

$$= \frac{T}{m} = \frac{\lambda}{am}(r-a) = \frac{\lambda}{am}\left(\frac{1}{u}-a\right).$$

The differential equation of the particle is

$$h^2\left[u+\frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{\lambda}{am}\left(\frac{1}{u^3}-\frac{a}{u^2}\right).$$

Multiplying both sides by $2\,(du/d\theta)$ and integrating, we have

$$v^2 = h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right] = \frac{\lambda}{am}\left(-\frac{1}{u^2}+\frac{2a}{u}\right)+A. \qquad \text{...(1)}$$

Now the point A is an apse. So initially at A, $r=a$, $u=1/a$, $du/d\theta = 0$, $v=\sqrt{(9a\lambda/2m)}$.

$\therefore$ from (1), we have

$$\frac{9a\lambda}{2m} = h^2\cdot\frac{1}{a^2} = \frac{\lambda}{am}(-a^2+2a^2)+A.$$

$$\therefore h^2 = \frac{9a^3\lambda}{2m}, A = \frac{7a\lambda}{2m}.$$

Substituting the values of h^2 and A in (1), we get

$$\frac{9a^3\lambda}{2m}\left\{u^2+\left(\frac{du}{d\theta}\right)^2\right\} = \frac{\lambda}{am}\left(-\frac{1}{u^2}+\frac{2a}{u}\right)+\frac{7a\lambda}{2m}.$$

Putting $du/d\theta = 0$, the apsidal distances are given by

$$\frac{9}{2}a^3u^2 = -\frac{1}{au^2}+\frac{2}{u}+\frac{7a}{2}, \quad\text{or}\quad \frac{9a^3}{2r^2} = -\frac{r^2}{a}+2r+\frac{7a}{2}$$

or $9a^4 - 7a^2r^2 - 4ar^3 + 2r^4 = 0$ or $2r^4 - 4ar^3 - 7a^2r^2 + 9a^4 = 0$

or $(r-a)(r-3a)(2r^2+4ar+3a^2)=0.$

Here $r=a, r=3a$ are +ive real roots. But $r=a$ is the given apsidal distance. Therefore $r=3a$ is the other apsidal distance.

Ex. 49. *The attraction to a fixed point being μ/r^5, a particle is projected in a direction making an angle $\tan^{-1}(2\sqrt{2}/3)$ with the initial distance c from the centre of force with velocity $\sqrt{(17\mu)}/\sqrt{2}\,c^2$. Prove that the orbit is*

$$\frac{2r}{c} = \frac{3e^{\theta\sqrt{2}}+1}{3e^{\theta\sqrt{2}}-1}.$$

(Calcutta 1990)

Sol. Here, the central acceleration $P = \mu/r^5 = \mu u^5$. The differential equation of the central orbit is

$$h^2\left[u+\frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{\mu u^5}{u^2} = \mu u^3.$$

Multiplying both sides by $2\,(du/d\theta)$ and integrating, we have

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \frac{\mu u^4}{2} + A. \qquad \ldots(1)$$

Initially when $r = c, u = 1/c, v = \sqrt{(17\mu)}/\sqrt{2}\,c^2, \phi = \tan^{-1}(2\sqrt{2}/3)$, $\tan\phi = 2\sqrt{2}/3, \sin\phi = 2\sqrt{2}/\sqrt{(17)}, p = r\sin\phi = c.[2\sqrt{2}/\sqrt{(17)}]$, so that $\quad 1/p^2 = v^2 + (du/d\theta)^2 = 17/(8c^2)$.

$\therefore$ from (1), we have $\dfrac{17\mu}{2c^4} = h^2.\left(\dfrac{17}{8c^2}\right) = \dfrac{\mu}{2c^4} + A.$

$\therefore \quad h^2 = 4\mu/c^2, A = 8\mu/c^4.$

Substituting the values of h^2 and A in (1), we have

$$\frac{4\mu}{c^2}\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \frac{\mu u^4}{2} + \frac{8\mu}{c^4}$$

or $$u^2 + \left(\frac{du}{d\theta}\right)^2 = \frac{c^2u^4}{8} + \frac{2}{c^2}$$

or $$\left(\frac{du}{d\theta}\right)^2 = \frac{c^2u^4}{8} + \frac{2}{c^2} - u^2.$$

Putting $u = 1/r$, so that $du/d\theta = (-1/r^2)(dr/d\theta)$, we have

$$\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{c^2}{8r^4} + \frac{2}{c^2} - \frac{1}{r^2}$$

or $$\left(\frac{dr}{d\theta}\right)^2 = \frac{c^2}{8} + \frac{2r^4}{c^2} - r^2 = \frac{16r^4 - 8c^2r^2 + c^4}{8c^2} = \frac{(4r^2 - c^2)^2}{8c^2}$$

or $\dfrac{dr}{d\theta} = -\dfrac{4r^2 - c^2}{2\sqrt{2}\,c}$, the negative sign has been taken because r decreases while θ increases as the particle starts moving from the point $r = c$. [Refer figure of Ex. 38]

or $$\frac{2c\,dr}{4r^2 - c^2} = -\frac{d\theta}{\sqrt{2}}.$$

Integrating, $$2c \cdot \frac{1}{2c} \cdot \frac{1}{2}\log\frac{2r - c}{2r + c} = -\frac{\theta}{\sqrt{2}} + B$$

or $$\frac{1}{2}\log\frac{2r - c}{2r + c} = -\frac{\theta}{\sqrt{2}} + B.$$

Initially when $r = c$, let $\theta = 0$. Then $B = \frac{1}{2}\log\frac{1}{3}$.

$\therefore \quad \dfrac{1}{2}\log\dfrac{2r - c}{2r + c} = -\dfrac{\theta}{\sqrt{2}} + \dfrac{1}{2}\log\dfrac{1}{3} \quad$ or $\quad \log\left[3\left(\dfrac{2r - c}{2r + c}\right)\right] = -\theta\sqrt{2}$

or $\quad 3\cdot\dfrac{2r - c}{2r + c} = e^{-\theta\sqrt{2}} \quad$ or $\quad \dfrac{2r + c}{2r - c} = 3e^{\theta\sqrt{2}}$

or $\quad 2r + c = (2r - c)\,3e^{\theta\sqrt{2}} \quad$ or $\quad 2r(3e^{\theta\sqrt{2}} - 1) = c(3e^{\theta\sqrt{2}} + 1)$

or $\quad \dfrac{2r}{c} = \dfrac{3e^{\theta\sqrt{2}} + 1}{3e^{\theta\sqrt{2}} - 1}$, which is the required equation of the orbit.

□□□