Date : 7/07/19

A CONSOLIDATED QUESTION PAPER-CUM-ANSWER BOOKLET

IMS™
(INSTITUTE OF MATHEMATICAL SCIENCES)

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MAINS TEST SERIES-2019

(JUNE-2019 *to* SEPT.-2019)

Under the guidance *of* K. Venkanna

206/250

MATHEMATICS

PAPER - I : ODE, STATICS & DYNAMICS AND VECTOR ANALYSIS

TEST CODE: TEST-3: IAS(M)/23-JUNE.-2019

Time: 3 Hours

Maximum Marks: 250

INSTRUCTIONS

1. This question paper-cum-answer booklet has 46 pages and has 32 PART/SUBPART questions. Please ensure that the copy of the question paper-cum-answer booklet you have received contains all the questions.
2. Write your Name, Roll Number, Name of the Test Centre and Medium in the appropriate space provided on the right side.
3. A consolidated Question Paper-cum-Answer Booklet, having space below each part/sub part of a question shall be provided to them for writing the answers. Candidates shall be required to attempt answer to the part/sub-part of a question strictly within the pre-defined space. Any attempt outside the pre-defined space shall not be evaluated."
4. Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
5. Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any THREE of the remaining questions selecting at least ONE question from each Section.
6. The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
7. Symbols/notations carry their usual meanings, unless otherwise indicated.
8. All questions carry equal marks.
9. All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
10. All rough work should be done in the space provided and scored out finally.
11. The candidate should respect the instructions given by the invigilator.
12. The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name: Gurwinder Singh

Roll No. 9855731040

Test Centre

Medium

Do not write your Roll Number or Name anywhere else in this Question Paper-cum-Answer Booklet.

I have read all the instructions and shall abide by them

Gurwinder Singh

Signature of the Candidate

I have verified the information filled by the candidate above

Signature of the invigilator

IMPORTANT NOTE:
Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed. Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

P.T.O.

DO NOT WRITE ON THIS SPACE

INDEX TABLE

QUESTION	No.	PAGE NO.	MAX. MARKS	MARKS OBTAINED	
1	(a)			08	36
	(b)			05	
	(c)			08	
	(d)			07	
	(e)			08	
2	(a)			12	39
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3	(a)				
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4	(a)			13	46
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5	(a)			07	38
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	~~(d)~~				
8	(a)				
	(b)				
	(c)				
	(d)				
Total Marks					

$\frac{206}{250}$

P.T.O.

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SECTION – A

1. (a) Solve $(x^2 + y^2)(1 + p)^2 - 2(x + y)(1 + p)(x + yp) + (x + yp)^2 = 0$ — ① **[10]**

Let $u = x+y$, $v = x^2+y^2$ $\Rightarrow$ $\frac{du}{dx} = 1 + \frac{dy}{dx}$

$\frac{dv}{dx} = 2(x + y\frac{dy}{dx})$

$P = \frac{dv}{du} = \frac{2(x+yp)}{1+p}$

$P + Pp = 2x + 2yp \Rightarrow P - 2x = (2y - P)p$

$\Rightarrow \boxed{p = \frac{P-2x}{2y-P}}$

① ≡ $x^2+y^2\left(1+\frac{P-2x}{2y-P}\right)^2 - 2(x+y)\left(1+\frac{P-2x}{2y-P}\right)\left[x+y\frac{(P-2x)}{2y-P}\right] + \left[x+y\left(\frac{P-2x}{2y-P}\right)\right]^2 = 0$

$(x^2+y^2)\left(\frac{2y-\cancel{P}+\cancel{P}-2x}{2y-P}\right)^2 - 2(x+y)\left(\frac{2y-\cancel{P}+\cancel{P}-2x}{2y-P}\right)\left(\frac{\cancel{2xy}-Px+Py-\cancel{2xy}}{2y-P}\right) + \left(\frac{\cancel{2xy}+Px+yP-\cancel{2x}}{2y-P}\right)^2 = 0$

$(x^2+y^2)\left(\frac{2(y-x)}{2y-P}\right)^2 - 2(x+y)\left(\frac{2y-2x}{2y-P}\right)\left(\frac{P(y-x)}{2y-P}\right) + \left(\frac{P(y-x)}{2y-P}\right)^2 = 0$

$(x^2+y^2)4(y-x)^2 - 4P(x+y)(y-x)^2 + P^2(y-x)^2 = 0$

$4(x^2+y^2) - 4P(x+y) + P^2 = 0$.

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$4v - 4uP + P^2 = 0$

$4v = 4uP - P^2$

$v = uP - \frac{P^2}{4}$ Clairaut's form

$\boxed{v = uc - \frac{c^2}{4}}$ is the solution

$\boxed{(x^2+y^2) = (x+y)c - \frac{c^2}{4}}$

1. (b) (i) If $L^{-1}\left\{\dfrac{e^{-1/p}}{p^{1/2}}\right\}=\dfrac{\cos 2\sqrt{t}}{\sqrt{(\pi t)}}$, find $L^{-1}\left\{\dfrac{e^{-a/p}}{p^{1/2}}\right\}$ where a > 0.

(ii) Find $L^{-1}\left\{\log\left(1+\dfrac{1}{p^2}\right)\right\}$. **[10]**

(i) $\cancel{L^{-1}}$ $L^{-1}\left[\dfrac{e^{-1/p}}{p^{1/2}}\right]=\dfrac{\cos 2\sqrt{t}}{\sqrt{\pi t}}$

we know that $F(ap)=\frac{1}{a}f\left(\frac{t}{a}\right)$.

$\cancel{L^{-1}\left[\dfrac{e^{-a/p}}{p^{1/2}}\right]}$ $L^{-1}\left[\dfrac{e^{-1/(p/a)}}{\left(\frac{p}{a}\right)^{1/2}}\right]=a\dfrac{\cos 2\sqrt{at}}{\sqrt{\pi a t}}$

$L^{-1}\left[a^{1/2}\dfrac{e^{-a/p}}{p^{1/2}}\right]=a\dfrac{\cos 2\sqrt{at}}{\sqrt{\pi a t}}$

$\boxed{L^{-1}\left[\dfrac{e^{-a/p}}{p^{1/2}}\right]=a^{1/2}\dfrac{\cos 2\sqrt{at}}{\sqrt{\pi a t}}}$.

(ii) $L^{-1}\left[\log\left(\dfrac{s^2+1}{s^2}\right)\right]$

let $F(s)=\log(s^2+1)-\log s^2$

$F'(s)=\dfrac{2s}{s^2+1}-\dfrac{2s}{s^2}$

05

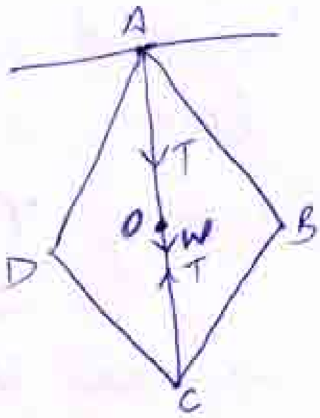
$L^{-1}[F'(s)]=\sin t-t$

$(-1)^1 t^1 L^{-1}[F(s)]=\sin t-t$

$\boxed{L^{-1}\left[\log\left(\dfrac{s^2+1}{s^2}\right)\right]=1-\dfrac{\sin t}{t}}$

1. (c) Four uniform rods are freely jointed at their extremities and form a parallelogram ABCD, which is suspended by the joint A, and is kept in shape by a string AC. Prove that the tension of the string is equal to half the weight of all the four rods. [10]

Let ABCD be given framework of Parallelogram shown in diagram.



Let T be Tension of string

& W be weight of all four rods

By Principle of virtual work,

$$-T\delta(AC) + W\delta(AO) = 0$$

Let $AO = x \Rightarrow AC = 2x$

$$\Rightarrow -T\delta(2x) + W(\delta(x)) = 0$$

$$-T \times 2\delta x + W\delta x = 0$$

$$(-2T + W)\delta x = 0$$

$$-2T + W = 0 \qquad \delta x \neq 0$$

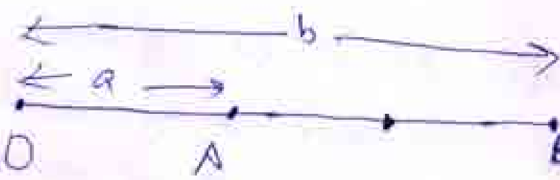
08

$$\boxed{T = \frac{W}{2}}$$

Tension of string = $\frac{\text{weight of four rods}}{2}$

1. (d) A body moving in a straight line OAB with S.H.M. has zero velocity when at the points A and B whose distances from O are a and b respectively, and has velocity v when half way between them. show that the complete period is $\pi(b-a)/v$.**[10]**

Time period of SHM is given by, $T = \frac{2\pi}{\sqrt{\mu}}$



we know that velocity at the center of SHM is $\sqrt{\mu}$ times the amplitude

i.e. $v = \sqrt{\mu}\, A$ where A is Amplitude

So, $v = \sqrt{\mu}\left(\frac{b-a}{2}\right)$.

$\Rightarrow \sqrt{\mu} = \frac{2v}{b-a}$.

Now $T = \frac{2\pi}{\sqrt{\mu}}$

$= \frac{2\pi}{\left(\frac{2v}{b-a}\right)}$

$$\boxed{T = \frac{\pi(b-a)}{v}}$$

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1. (e) If $\nabla^2 f(r) = 0$, show that $f(r) = c_1 \log r + c_2$

where $r^2 = x^2 + y^2$ and c_1, c_2 are arbitrary constants. **[10]**

$$\nabla^2 f(r) = \frac{\partial^2 f(r)}{\partial x^2} + \frac{\partial^2 f(r)}{\partial y^2} = 0$$

$$\nabla f(r) = \frac{\partial}{\partial x} f(r)\, \vec{i} + \frac{\partial}{\partial y} f(r)\, \vec{j} + \frac{\partial}{\partial z} f(r)\, \vec{k}$$

$$\nabla f(r) = \frac{f'(r)}{r} \vec{r}$$

$$\nabla^2 f(r) = \frac{\partial}{\partial x}\left(f'(r)\frac{\vec{r}}{r}\right)\vec{i} + \frac{\partial}{\partial y}\left(\frac{\vec{r} f'(r)}{r}\right)\vec{j} + \frac{\partial}{\partial z}\left(\frac{\vec{r} f'(r)}{r}\right)\vec{k}$$

$$= \left(f''(r)\frac{\vec{r}}{r}\frac{\partial r}{\partial x} + \frac{f'(r)}{r} - \frac{1}{r^2} f'(r)\, \vec{r}\frac{\partial r}{\partial x}\right)\vec{i}$$

$$+ \left(f''(r)\frac{\vec{r}}{r}\frac{\partial r}{\partial y} + \frac{f'(r)}{r} - \frac{1}{r^2} f'(r)\, \vec{r}\frac{\partial r}{\partial y}\right)\vec{j}$$

$$+ \left(f''(r)\frac{\vec{r}}{r}\frac{\partial r}{\partial z} + \frac{f'(r)}{r} - \frac{1}{r^2} f'(r)\, \vec{r}\frac{\partial r}{\partial z}\right)\vec{k}$$

Adding $\nabla^2 f(r) = 0$

we get, $r f''(r) + f'(r) = 0$

Integrate $\int \frac{f''(r)}{f'(r)} = \int -\frac{1}{r} \Rightarrow \log f'(r) = -\log r + \log c_1$

08

$\Rightarrow \log f'(r) = \log \frac{c_1}{r} \Rightarrow f'(r) = \frac{c_1}{r}$

Integrate $\int f'(r) = \int \frac{c_1}{r} \Rightarrow \boxed{f(r) = c_1 \log r + c_2}$

2. (a) Use the method of variation of parameters to find the general solution of $x^2y'' - 4xy' + 6y = -x^4 \sin x$. **[15]**

$x^2y'' - 4xy' + 6y = -x^4 \sin x$

$y'' - \frac{4}{x}y' + \frac{6}{x^2}y = -x^2 \sin x$

where $R = -x^2 \sin x$

$x^2y'' - 4xy' + 6y = 0.$

Let $x = e^z \Rightarrow \log x = z$

$(D_1(D_1 - 1) - 4D_1 + 6)y = 0.$

$D_1^2 - D_1 - 4D_1 + 6 = 0 \Rightarrow D_1^2 - 5D_1 + 6 = 0$

$(m-2)(m-3) = 0 \Rightarrow m = 2, 3$

So, $C.F. = c_1 e^{2z} + c_2 e^{3z}$

$y_c = c_1 x^2 + c_2 x^3$

let $u = x^2$, $v = x^3$ $\quad W = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix}$

$W = 3x^4 - 2x^4 = x^4 \neq 0.$

$y_p = uf(x) + vg(x)$

$f(x) = -\int \frac{vR}{W} dx$, $\quad g(x) = \int \frac{uR}{W} dx$

Now $y'' + Py' + Q = R$

$P = \frac{-4}{x}$, $Q = \frac{6}{x^2}$, $R = -x^2 \sin x$

$f(x) = -\int \frac{x^3(-x^2 \sin x)}{x^4} dx = \int x \sin x \, dx$

$= -x\cos x - \int (-\cos x) dx$

$f(x) = -x\cos x + \sin x$

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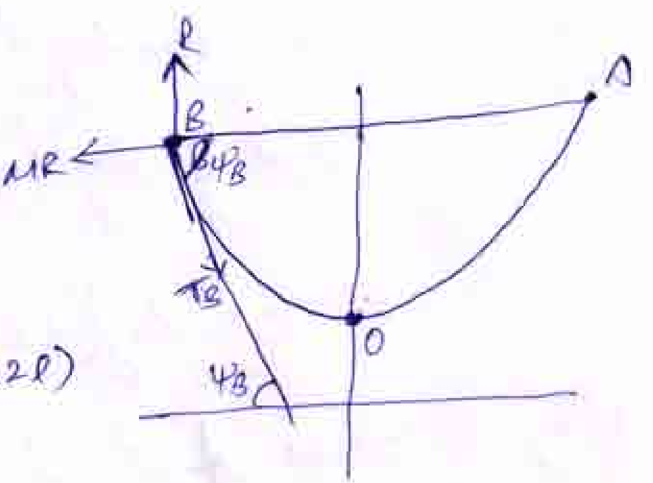
$g(x) = \int \frac{x^2(-x^2 \sin x)}{x^4} dx = -\int \sin x \, dx = \cos x$

So, $y_p = x^2(\sin x - x\cos x) + x^3(\cos x)$ simplify it here

So, $\boxed{y = y_c + y_p = c_1 x^2 + c_2 x^3 + x^2(\sin x - x\cos x) + x^3 \cos x}$

2. (b) A heavy chain, of length $2l$, has one end tied at A and the other is attached to a small heavy ring which can slide on a rough horizontal rod which passes through A. If the weight of the ring be n times the weight of the chain, show that its greatest possible distance from A is $\frac{2l}{\lambda}\log\left\{\lambda+\sqrt{(1+\lambda^2)}\right\}$, $1/\lambda = \mu\,(2n+1)$ and μ is the coefficient of friction. **[15]**

Let w be weight of chain per unit length.



weight of Ring $= n(w \times 2l)$
$= 2nlw$

$T_B \cos\psi_B = wc$ — (1)

$T_B \sin\psi_B = w\cancel{s} = wl$ — (2)

from (1) & (2) $\tan\psi_B = \frac{l}{c}$ — (3)

Greatest possible distance $= 2c\log(\sec\psi_B + \tan\psi_B)$

Now $R = T_B\sin\psi_B + 2nlw$

$R = wl + 2nlw$

$R = (2n+1)wl$ — (4)

Now $\mu R = T_B\cos\psi_B$

$\mu R = wc$

$\mu(2n+1)\cancel{w}l = \cancel{w}c$

$c = \mu(2n+1)l = \frac{l}{\lambda}$ where $\lambda = \frac{1}{\mu(2n+1)}$

(3) $\Rightarrow$ $\tan\psi_B = \frac{l}{c} = \frac{l}{l/\lambda} = \lambda \Rightarrow \tan\psi_B = \lambda$

Now Greatest distance from A

$$= 2c \log(\tan\psi_B + \sec\psi_B)$$

$$= 2\frac{l}{\lambda} \log\left(\lambda^2 + \sqrt{1+\tan^2\psi_B}\right)$$

$$= \frac{2l}{\lambda} \log\left(\lambda^2 + \sqrt{1+\lambda^2}\right)$$

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when $\lambda = \frac{1}{u(2n+1)}$

2. (c) Show that for the curve

$x = a(3u - u^3)$, $y = 3au^2$, $z = a(3u + u^3)$

$\kappa = \tau = \frac{1}{3a(1+u^2)^2}$.

[13]

$x = a(3u-u^3)$, $y = 3au^2$, $z = a(3u+u^3)$

Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\vec{r} = a(3u-u^3)\hat{i} + 3au^2\hat{j} + a(3u+u^3)\hat{k}$

$$\frac{d\vec{r}}{du} = a(3-3u^2)\hat{i} + 6au\hat{j} + a(3+3u^2)\hat{k}$$

$$\left|\frac{d\vec{r}}{du}\right| = \sqrt{9a^2(1-u^2)^2 + 36a^2u^2 + 9a^2(1+u^2)^2}$$

$$= \sqrt{9a^2(1+u^4-\cancel{2u^2}) + 36a^2u^2 + 9a^2(1+u^4+\cancel{2u^2})}$$

$$= \sqrt{18a^2(1+u^4) + 36a^2u^2}$$

$$\left|\frac{dr}{du}\right| = 3\sqrt{2}\,a(1+u^2)$$

$$\frac{d^2r}{du^2} = a(-6u)\hat{i} + 6a\hat{j} + a(6u)\hat{k}$$
$$= -6au\,\hat{i} + 6a\hat{j} + 6au\,\hat{k}$$

$$\frac{dr}{du}\times\frac{d^2r}{du^2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a(3-3u^2) & 6au & a(3+3u^2) \\ -6au & 6a & 6au \end{vmatrix}$$

$$= \hat{i}(36a^2u^2 - 18a^2 - 18a^2u^2) - \hat{j}(\cancel{18a^2u - 18a^2u^3}\ \ 18a^2u + 18a^2u^3)$$
$$+ \hat{k}(18a^2 - 18a^2u^2 + 36a^2u^2)$$

$$= (18a^2u^2 - 18a^2)\hat{i} + \hat{j}(-36a^2u) + \hat{k}(18a^2u^2 + 18a^2)$$

$$K = \frac{\left|\frac{dr}{du}\times\frac{d^2r}{du^2}\right|}{\left|\frac{dr}{du}\right|^3} = \frac{\sqrt{(18a^2u^2-18a^2)^2 + (36a^2u)^2 + (18a^2u^2+18a^2)^2}}{(3\sqrt{2}\,a(1+u^2))^3}$$

$$\boxed{K = \frac{1}{3a(1+u^2)^2}}$$

$$\tau = \frac{\left[\frac{dr}{du}\ \frac{d^2r}{du^2}\ \frac{d^3r}{du^3}\right]}{\left|\frac{dr}{du}\times\frac{d^2r}{du^2}\right|^2}$$

Now $\frac{d^3r}{du^3} = -6a\hat{i} + 0\hat{j} + 6a\hat{k}$

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$$\left[\frac{dr}{du}\ \frac{d^2r}{du^2}\ \frac{d^3r}{du^3}\right] = (-6au\,\hat{i} + 6a\hat{j} + 6au\,\hat{k})\cdot(-6a\hat{i} + 6a\hat{k})$$
$$= 36a^2u + 36a^2u = 72a^2u \quad 216a^3$$

$$\tau = \frac{72a^2u \quad 216a^3}{\sqrt{(18a^2(1-u^2))^2 + (36a^2u)^2 + (18a^2(1+u^2))^2}^{\,2}} = \frac{1}{3a(1+u^2)^2}$$

2. (d) Find the angle between the surfaces $x^2 + y^2 + z^2 = 9$, and $z = x^2 + y^2 - 3$ at the point (2, −1, 2). **[07]**

Let $f_1 = x^2+y^2+z^2-9$

$f_2 = x^2+y^2-z-3$

$\nabla f_1 = 2x\hat{i}+2y\hat{j}+2z\hat{k}$

$\nabla f_1 \Big|_{(2,-1,2)} = 4\hat{i}-2\hat{j}+4\hat{k}$

$\nabla f_2 = 2x\hat{i}+2y\hat{j}-\hat{k}$

$\nabla f_2 \Big|_{(2,-1,2)} = 4\hat{i}-2\hat{j}-\hat{k}$

Let Angle between f_1 & f_2 be θ

$$\cos\theta = \frac{\nabla f_1 \cdot \nabla f_2}{|\nabla f_1|\,|\nabla f_2|}$$

$$= \frac{(4\hat{i}-2\hat{j}+4\hat{k})\cdot(4\hat{i}-2\hat{j}-\hat{k})}{\sqrt{16+16+4}\,\sqrt{16+4+1}}$$

$$= \frac{16+\cancel{4}-\cancel{4}}{\sqrt{36}\,\sqrt{21}}$$

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$$= \frac{16}{6\sqrt{21}} = \frac{8}{3\sqrt{21}}$$

$$\boxed{\theta = \cos^{-1}\frac{8}{3\sqrt{21}}}$$

4. (a) By using Laplace transform method, Solve $(D^2 + m^2)\, y = a \cos nt$, $t > 0$, if $y = 0$, $Dy = 0$ when $t = 0$. **[15]**

$(D^2+m^2)y = a\cos nt$

Take Laplace on Both sides

$L[D^2y] + L[m^2y] = a\,L[\cos nt]$

$s^2 L[y] - s\,y(0) - y'(0) + m^2 L[y] = a\dfrac{s}{s^2+n^2}$

$L[y](s^2+m^2) - s(0) - 0 \cancel{+m^2} = \dfrac{as}{s^2+n^2}$

$L[y] = \dfrac{as}{(s^2+m^2)(s^2+n^2)}$

$= \dfrac{as}{m^2-n^2}\left[\dfrac{1}{s^2+n^2} - \dfrac{1}{s^2+m^2}\right]$

$y = \dfrac{a}{m^2-n^2} L^{-1}\left[\dfrac{s}{s^2+n^2} - \dfrac{s}{s^2+m^2}\right]$

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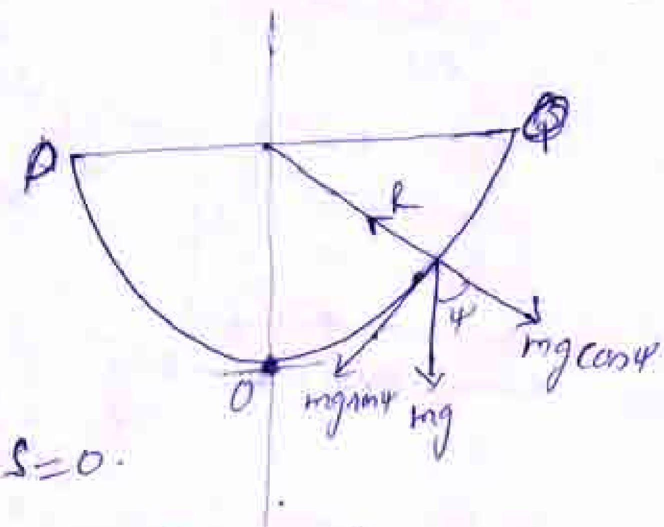
$y = \dfrac{a}{m^2-n^2}(\cos nt - \cos mt)$

$$\boxed{y = \frac{a\cos nt}{m^2-n^2} - \frac{a\cos mt}{m^2-n^2}}$$

4. (b) A particle slides down the arc of a smooth cycloid whose axis is vertical and vertex lowest, starting at rest from the cusp. Prove that the time occupied in falling down the first half of the vertical height is equal to the time of falling down the second half. **[20]**

Let $s = 4a \sin\psi$ be Equation of cycloid..



[illegible]

At Q, $s = 4a$
and At vertex O, $s = 0$.

Now, $m\dfrac{d^2s}{dt^2} = -mg\sin\psi$ — (1)

$R - mg\cos\psi = \dfrac{mv^2}{\rho}$ — (2)

Use $S = 4a\sin\psi$ in ① $\equiv \frac{d^2s}{dt^2} = -\frac{gS}{4a}$

Multiply Both side with $2\frac{ds}{dt}$ & integrate,

we get, $\left(\frac{ds}{dt}\right)^2 = \frac{-gs^2}{4a} + A$

Initially, at Q, $S = 4a$, $\frac{ds}{dt} = 0$

So, $\left(\frac{ds}{dt}\right)^2 = 0 = \frac{-g(4a)^2}{4a} + A \Rightarrow \boxed{A = 4ag}$

$\left(\frac{ds}{dt}\right)^2 = \frac{g}{4a}(16a^2 - s^2) \Rightarrow \frac{ds}{dt} = -\frac{1}{2}\sqrt{\frac{g}{a}}\sqrt{16a^2 - s^2}$

(-ve sign for s decreasing)

Now $s^2 = 8ay$

for first Half, $y = a$, $S = 2\sqrt{2}a$

$$-2\sqrt{\frac{a}{g}}\int_{4a}^{2\sqrt{2}a}\frac{1}{\sqrt{16a^2 - s^2}}\,ds = \int_0^{t_1} dt$$

$$t_1 = +2\sqrt{\frac{a}{g}}\left[\cos^{-1}\frac{s}{4a}\right]_{4a}^{2\sqrt{2}a} = 2\sqrt{\frac{a}{g}}\left[\frac{\pi}{4} - 0\right]$$

$$\boxed{t_1 = \frac{\pi}{2}\sqrt{\frac{a}{g}}}$$

For second Half, $-2\sqrt{\frac{a}{g}}\int_{2\sqrt{2}a}^{0}\frac{1}{\sqrt{16a^2 - s^2}}\,ds = \int_0^{t_2} dt$

$$t_2 = 2\sqrt{\frac{a}{g}}\left[\cos^{-1}\frac{s}{4a}\right]_{2\sqrt{2}a}^{0} = 2\sqrt{\frac{a}{g}}\left[\frac{\pi}{4}\right] = \frac{\pi}{2}\sqrt{\frac{a}{g}}$$

$\boxed{t_2 = \frac{\pi}{2}\sqrt{\frac{a}{g}}}$ Hence $\boxed{t_1 = t_2}$

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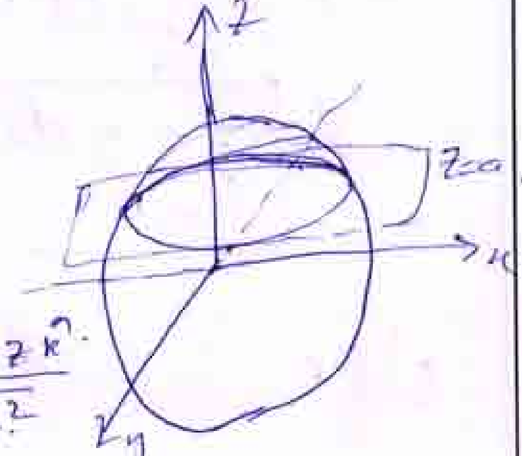
P.T.O.

4. (c) Let S be the spherical cap $x^2 + y^2 + z^2 = 2a^2$, $z \geq a$, together with its base $x^2 + y^2 \leq a^2$, $z = a$. find the flux of $\mathbf{F} = xz\mathbf{i} - yz\mathbf{j} + y^2\mathbf{k}$ outward through S (i) by evaluating $\iint_S \mathbf{F}\cdot\mathbf{n}\, d\sigma$ directly, (ii) by applying the divergence theorem. **[15]**

(i) let $S = S_1 + S_2$

$S_1 \rightarrow$ curved surface of cap

$S_2 \rightarrow x^2+y^2 = a^2,\ z = a$



$\iint_{S_1} F\cdot\hat{n}\, dS_1$ Now $\hat{n} = \dfrac{2x\hat{i} + 2y\hat{j} + 2z\hat{k}}{\sqrt{4x^2+4y^2+4z^2}}$

$\hat{n} = \dfrac{x}{\sqrt{2}a}\hat{i} + \dfrac{y}{\sqrt{2}a}\hat{j} + \dfrac{z}{\sqrt{2}a}\hat{k}$.

$\iint_{S_1} (xz\hat{i} - yz\hat{j} + y^2\hat{k})\cdot\left(\dfrac{x}{\sqrt{2}a}\hat{i} + \dfrac{y}{\sqrt{2}a}\hat{j} + \dfrac{z}{\sqrt{2}a}\hat{k}\right) dS$

$= \iint_{S_1} \dfrac{1}{\sqrt{2}a}(x^2z - \cancel{y^2z} + \cancel{y^2z})\, dS = \dfrac{1}{\sqrt{2}a}\iint x^2z\, \dfrac{dx\,dy}{|n\cdot\hat{k}|}$

$= \iint \dfrac{1}{\sqrt{2}a}\, x^2z\, \dfrac{dx\,dy}{\frac{z}{\sqrt{2}a}} = \iint x^2\, dx\,dy$

$= \dfrac{\pi a^4}{4}$

$\iint_{S_2} (xz\hat{i} - yz\hat{j} + y^2\hat{k})\cdot(-\hat{k})\, dS = \iint_{S_2} -y^2\, dS$

$= -2\int_{-a}^{a}\int_{0}^{\sqrt{a^2-x^2}} y^2\, dx\,dy = -\dfrac{2}{3}\int_{-a}^{a} (a^2-x^2)^{3/2} dx$

$= -\dfrac{4}{3}\int_0^a (a^2-x^2)^{3/2} dx = -\dfrac{4}{3}\int_0^{\pi/2} a^4\cos^4\theta\, d\theta = -\dfrac{4}{3}a^4\cdot\dfrac{3}{4}\cdot\dfrac{1}{2}\cdot\dfrac{\pi}{2} = -\dfrac{\pi a^4}{4}$

$\iint_S F\cdot n\, dS = \dfrac{\pi a^4}{4} - \dfrac{\pi a^4}{4} = 0$

(ii) $\iiint_V \nabla\cdot F\, dv = \iint_S F\cdot n\, dS$

$\nabla \cdot F = z - z + 0 = 0$

So, $\iiint_V (\nabla \cdot F)\, dv = \iiint 0\, dv = 0$

So $\boxed{\iint_S F \cdot n\, ds = 0}$

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SECTION – B

5. (a) Solve $(D-1)^2 (D^2+1)^2 y = \sin^2 (x/2) + e^x + x$. **[10]**

$(D-1)^2 (D^2+1)^2 y = \sin^2 \frac{x}{2} + e^x + x$

Auxilliary eqn is $(m-1)^2 = 0$ and $(m^2+1)^2 = 0$

$(m-1)^2 = 0 \Rightarrow m = 1, 1 \Rightarrow C.F = C_1 e^x + C_2 x e^x$

$(m^2+1)^2 = 0 \Rightarrow m = \pm i, \pm i \Rightarrow C.F = C_3 \cos x + C_4 \sin x$
$+ C_5 x \cos x + C_6 x \sin x$

$P.I. = \frac{1}{(D-1)^2 (D^2+1)^2} \left[\sin^2 \frac{x}{2} + e^x + x \right]$

$\frac{1}{(D-1)^2 (D^2+1)^2} \sin^2 \frac{x}{2} = \frac{1}{(D-1)^2 (D^2+1)^2} \left(\frac{1 - \cos x}{2} \right)$

$= \frac{1}{2} - \frac{1}{2}\frac{1}{(D-1)^2(D^2+1)^2}(\cos x) = \frac{1}{2} - \frac{1}{8}\frac{1}{(D-1)^2}\cos x$

$\Rightarrow \frac{1}{2} - \frac{1}{8}\frac{x}{2D}\cos x = \frac{1}{2} - \frac{1}{16}x\sin x$]

Now, $\frac{1}{(D-1)^2(D^2+1)^2}e^x = \frac{1}{4}\frac{1}{(D-1)^2}e^x = \frac{1}{4}\frac{x^2}{\lfloor 2}e^x = \frac{x^2}{8}e^x$]

and $\frac{1}{(D-1)^2(D^2+1)^2}x = \frac{1}{(D-1)^2}(1+D^2)^{-2}x = \frac{1}{(D-1)^2}x = (1-D)^{-2}x$
$= x+2$]

So, $y = y_c + y_p$

$= (c_1 + c_2 x)e^x + (c_3 + c_5 x)\cos x + (c_4 + c_6 x)\sin x$

$+ \frac{x^2 e^x}{8} - \frac{x\sin x}{16} + x + \frac{5}{2}$ Ans

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$-\frac{x^2 \sin x}{32}$

5. (b) Find the orthogonal trajectories of cardioids r = a (1 − cos θ), a being parameter. **[10]**

$r = a(1-\cos\theta)$

$\log r = \log a + \log(1-\cos\theta)$

$\frac{1}{r}\frac{dr}{d\theta} = 0 + \frac{1}{1-\cos\theta}\sin\theta$

For orthogonal trajectory,

Put $\frac{dr}{d\theta} \to -r^2\frac{d\theta}{dr}$

$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \frac{\sin\theta}{1-\cos\theta} \Rightarrow -r\frac{d\theta}{dr} = \frac{\sin\theta}{1-\cos\theta}$

$$\frac{1-\cos\theta}{\sin\theta}\,d\theta = -\frac{dr}{r} \Rightarrow \int(\text{cosec}\,\theta - \cot\theta)\,d\theta = \int -\frac{1}{r}\,dr$$

Integrate, $-\log(\text{cosec}\,\theta + \cot\theta) - \log\sin\theta = -\log r + \log b$

$$-\log(1+\cos\theta) = -\log r + \log b$$

$$\log\left(\frac{r}{1+\cos\theta}\right) = \log b$$

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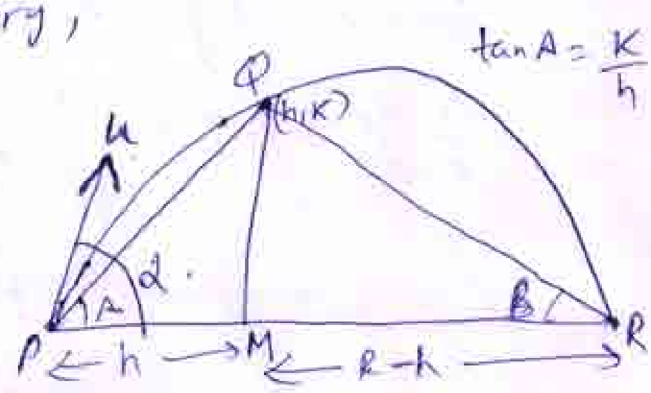
$$\boxed{r = b(1+\cos\theta)}$$

5. (c) A particle is thrown over a triangle from one end of a horizontal base and grazing over the vertex falls on the other end of the base. If A, B be the base angles of the triangle and α the the angle of projection, prove that

$\tan\alpha = \tan A + \tan B.$ **[10]**

Equation of trajectory,

$$y = x\tan\alpha - \frac{g}{2}\frac{x^2}{u^2\cos^2\alpha} \quad \text{—} (1)$$



It passes through (h, K)

$$K = h\tan\alpha - \frac{g}{2}\frac{h^2}{u^2\cos^2\alpha} \quad \text{—} (2)$$

we know $R = \frac{u^2 \sin 2\alpha}{g}$ — (3)

Use (3) in (2) : $\Rightarrow$ $\frac{k}{h} = \tan\alpha - \frac{g}{2}\frac{h}{\cos^2\alpha} \cdot \frac{\sin 2\alpha}{Rg}$

$\tan A = \tan\alpha - \frac{1}{g}\frac{h}{\cos^2\alpha}\frac{2\sin\alpha\cos\alpha}{R}$

$\tan A = \tan\alpha - \frac{h\tan\alpha}{R} = \left(\frac{R-h}{R}\right)\tan\alpha$

From tringle, $\boxed{\frac{h}{R-h} = \frac{\tan B}{\tan A}}$

$\Rightarrow \tan\alpha = \frac{R}{R-h}\tan A$

$\Rightarrow \tan\alpha = \left(\frac{R-h+h}{R-h}\right)\tan A$

$\Rightarrow \tan\alpha = \tan A + \frac{h}{R-h}\tan A$

$\Rightarrow \boxed{\tan\alpha = \tan A + \tan B}$.

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5. (d) Find the value of **r** satisfying the equation
$\frac{d^2\mathbf{r}}{dt^2} = 6t\mathbf{i} - 24t^2\mathbf{j} + 4\sin t\mathbf{k}$,

given that $\mathbf{r} = 2\mathbf{i} + \mathbf{j}$ and $\frac{d\mathbf{r}}{dt} = -\mathbf{i} - 3\mathbf{k}$ at $t = 0$.

[10]

$\int \frac{d^2r}{dt^2} = \int 6t\hat{i} - \int 24t^2\hat{j} + \int 4\sin t\,\hat{k}$

Integrate, $\frac{dr}{dt} = 6\frac{t^2}{2}\hat{i} - 24\frac{t^3}{3}\hat{j} - 4\cos t\,\hat{k} + C_1$

given $-\hat{i} - 3\hat{k} = 0 + 0 - 4\cos 0\,\hat{k} + C_1$

$C_1 = -\hat{i} + \hat{k}$

So, $\frac{d\vec{r}}{dt} = (3t^2-1)\hat{i} - 8t^3\hat{j} + (1-4\cos t)\hat{k}$

Integrating, $\vec{r} = \left(\frac{3t^3}{3} - t\right)\hat{i} - \frac{8t^4}{4}\hat{j} + (t - 4\sin t)\hat{k} + C_2$

given $2\hat{i} + \hat{j} = 0 + 0 + (0 - 0) + C_2$

$\boxed{C_2 = 2\hat{i} + \hat{j}}$

So, $\boxed{\vec{r} = (t^3 - t + 2)\hat{i} + (1 - 2t^4)\hat{j} + (t - 4\sin t)\hat{k}}$

08

5. (e) (i) If $\mathbf{u} = (1/r)\,\mathbf{r}$, show that $\nabla \times \mathbf{u} = 0$. **[10]**

(ii) If $\mathbf{u} = \left(\frac{1}{r}\right)\mathbf{r}$ find grad (div $\mathbf{u}$)

(i) $u = \left(\frac{1}{r}\right)\vec{r}$

$\nabla \times u = \nabla \times \left(\frac{1}{r}\vec{r}\right)$

we know, $\nabla \times \phi A = \nabla\phi \times \nabla A + \phi \nabla \times A$.

So, $\nabla \times u = \nabla\left(\frac{1}{r}\right) \times \nabla\vec{r} + \frac{1}{r}\nabla \times \vec{r}$

$= -\frac{1}{r^2}(\nabla\vec{r} \times \nabla\vec{r}) + \frac{1}{r}(0)$.

$= -\frac{1}{r^2}(0) + 0 = 0$

$[\because \nabla \times \vec{r} = 0]$

So, $\boxed{\nabla \times u = 0.}$

(ii) $u = \frac{1}{r}\vec{r}$, grad div u $= \nabla(\nabla \cdot u)$

Now, $\nabla \cdot u = \nabla \cdot \frac{1}{r}\vec{r}$ $\quad \left[\nabla \cdot \phi A = \phi \nabla \cdot A + \nabla\phi \cdot A\right]$

$= \frac{1}{r}\nabla \cdot \vec{r} + \nabla\left(\frac{1}{r}\right) \cdot \vec{r}$

$= \frac{1}{r}(3) + \frac{-1}{r^2}\nabla r \cdot \vec{r}$

$= \frac{3}{r} - \frac{1}{r^2}\left(\frac{\vec{r}}{r} \cdot \vec{r}\right)$

07

$= \frac{3}{r} - \frac{1}{r^3}(r^2) = \frac{3}{r} - \frac{1}{r}$

$\boxed{\nabla \cdot u = \frac{2}{r}}$ Ans $\Rightarrow$ grad$(\nabla \cdot u)$

??

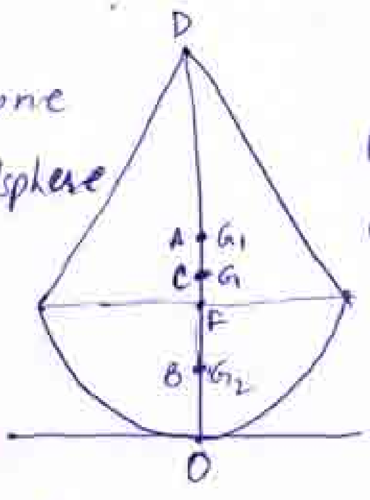
6. (a) (i) Solve $(2xy^4 e^y + 2xy^3 + y)\,dx + (x^2y^4 e^y - x^2 y^2 - 3x)\,dy = 0$

(ii) Solve $(1 + y^2)\,dx + (x - e^{-\tan^{-1} y})\,dy = 0,\ y(1) = 0$ **[14]**

6. (d) Prove that $L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\frac{1}{p}$ and hence find $L\left\{\frac{\sin at}{t}\right\}$. Does the Laplace transform of $\frac{\cos at}{t}$ exist ? **[10]**

7. (a) A body, consisting of a cone and a hemisphere on the same base, rests on a rough horizontal table the hemisphere being in contact with the table show that the greatest height of the cone so that the equilibrium may be stable, is $\sqrt{3}$ times the radius of the hemisphere. **[16]**

Let H be height of cone
r be radius of Hemisphere



$FA = \frac{H}{4}$

$FB = \frac{3r}{8}$.

To prove :- $H < \sqrt{3}r$

let h be height of C.G. of combined body from O.

$$h = \frac{\frac{1}{3}\pi r^2 H\left(\frac{H}{4}+r\right) + \frac{2}{3}\pi r^3\left(r - \frac{3r}{8}\right)}{\frac{1}{3}\pi r^2 H + \frac{2}{3}\pi r^3}$$

$$h = \frac{H\left(\frac{H}{4}+r\right) + 2r\left(\frac{5r}{8}\right)}{H+2r} \quad \text{—(1)}$$

~~h =~~ For stable Equilibrium, $\frac{1}{h} > \frac{1}{\rho_1} + \frac{1}{\rho_2}$

$\frac{1}{h} > \frac{1}{r} + \frac{1}{\infty} \quad \Rightarrow \quad h < r \quad \text{—(2)}$

Use (1) in (2) $\equiv$ $H\left(\frac{H}{4}+r\right) + \frac{5r^2}{4} < (H+2r)r$.

$\cancel{H^2 + Hr}$ $H(H+4r) + 5r^2 < 4r(H+2r)$.

$H^2 + \cancel{4Hr} + 5r^2 < \cancel{4Hr} + 8r^2$.

$H^2 < 3r^2$

$\boxed{H < \sqrt{3}\, r}$.

So, Greatest height of cone for stable Equilibrium is $\sqrt{3}$ times radius of Hemisphere.

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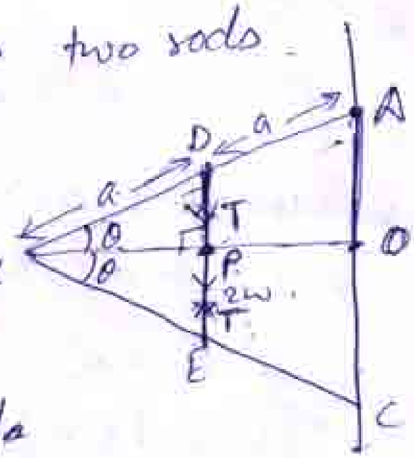
7. (b) One end of a uniform rod AB, of length 2a and weight W, is attached by a frictionless joint to a smooth vertical wall, and the other end B is smoothly jointed to an equal rod BC. The middle points of the rods are jointed by an elastic string, of natural length a and modulus of elasticity 4W. Prove that the system can rest in equilibrium in a vertical plane with C in contact with the wall below A, and the angle between the rods is 2 $\sin^{-1}(3/4)$. [16]

$AB = 2a$, $AD = a$, $BD = a$

Let 2θ be angle between two rods.

Let T be the tension in the string.



$w =$ weight of rods

By Principle of virtual work,

$$-T\delta(DE) + 2w\,\delta(AD) = 0$$

$$-T\,\delta(2a\sin\theta) + 2w\,\delta(2a\sin\theta) = 0$$

$$\boxed{T = 2w}$$

Now Modulus of elasticity $= 4w$

Natural length of string $= a$

$$So,\ T = 4w\ \frac{2a\sin\theta - a}{a}.$$

$$2w = 4w\left(\frac{2a\sin\theta - a}{a}\right)$$

$$2a\sin\theta - a = \frac{a}{2}$$

$$2a\sin\theta = \frac{3a}{2}$$

$$\sin\theta = \frac{3}{4}$$

$$\theta = \sin^{-1}\frac{3}{4}$$

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$$\boxed{2\theta = 2\sin^{-1}\frac{3}{4} = \text{Angle b/w rods.}}$$

7. (c) A particle moves under a force
$m\mu\{3au^4 - 2(a^2 - b^2)u^5\}$, $a > b$
and is projected from an apse at a distance $(a + b)$ with velocity $\sqrt{\mu/(a+b)}$. Show that the equation of its path is $r = a + b\cos\theta$. **[18]**

Differential Eqn of Apse is

$$\boxed{h^2\left(u + \frac{d^2u}{d\theta^2}\right) = \frac{P}{u^2}} \quad \text{——} (1)$$

given $P = \mu[3au^4 - 2(a^2-b^2)u^5]$

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \mu[3au^2 - 2(a^2-b^2)u^3]$$

Multiply Both sides with $2\frac{du}{d\theta}$ & Integrate

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu[2au^3 - (a^2-b^2)u^4] + A$$

given $r = a+b \Rightarrow u = \frac{1}{a+b}$, $v = \frac{\sqrt{\mu}}{a+b}$, $\frac{du}{d\theta} = 0$

$$\frac{\mu}{(a+b)^2} = h^2\left[\frac{1}{(a+b)^2} + 0\right] = \mu\left[\frac{2a}{(a+b)^3} - \frac{a^2-b^2}{(a+b)^4}\right] + A.$$

$$\frac{\mu}{(a+b)^2} = \frac{h^2}{(a+b)^2} = \frac{\mu}{(a+b)^2} + A \Rightarrow \boxed{A=0}$$

$$\boxed{h^2=\mu}$$

Now $u^2 + \left(\frac{du}{d\theta}\right)^2 = 2au^3 - (a^2-b^2)u^4$ — (2)

put $u = \frac{1}{r} \Rightarrow \frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$ in (2)

(2) $\equiv$ $$\frac{1}{r^2} + \left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 = \frac{2a}{r^3} - \frac{a^2-b^2}{r^4}$$

$$\cancel{\frac{1}{r^2}}\left[1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right] = \cancel{\frac{1}{r^2}}\left[\frac{2a}{r} - \frac{a^2-b^2}{r^2}\right]$$

$$\frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2 = \frac{2a}{r} - \frac{a^2-b^2}{r^2} - 1$$

$$\left(\frac{dr}{d\theta}\right)^2 = b^2 - (r-a)^2 \Rightarrow \int \frac{dr}{\sqrt{b^2-(r-a)^2}} = \int d\theta \Rightarrow \theta + B = \sin^{-1}\frac{r-a}{b}$$

Initially, $r = a+b$, $\theta = 0 \Rightarrow 0 + B = \sin^{-1}\frac{a+b-a}{b} = \frac{\pi}{2}$

So $\theta + \frac{\pi}{2} = \sin^{-1}\frac{r-a}{b} \Rightarrow \sin\left(\frac{\pi}{2}+\theta\right) = \frac{r-a}{b}$

$$\frac{r-a}{b} = \cos\theta \Rightarrow \boxed{r = a + b\cos\theta}$$