IMS

MATHS

BOOK-06

Vector Analysis

Set-I

Paper - I

Section - B

Syllabus:

Scalar and vector fields, differentiation of vector field of scalar variable; Gradient divergence and curl in cartesian and cylindrical co-ordinates; Higher order derivatives; vector identities and vector equations.

Application to geometry; curves in space, curvature and torsion; Serret - Frenet's formulae.

Gauss and Stoke's theorems, Greens identities.

Some basic Concepts:

→ Many physical quantities can be divided into broadly two types (i). Scalars (ii) vectors.

— A scalar is a quantity that is determined by its magnitude, ~~its number~~ of units measured on a suitable scale.

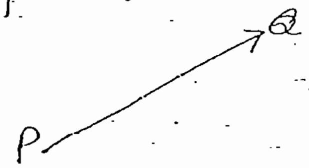
For example: length, temperature and voltage are scalars.

— A vector is a quantity that is determined by both its magnitude and its direction.

For example: displacement, velocity, acceleration and force are vectors.

→ A vector is represented by a directed line segment.-

i.e, $\overrightarrow{PQ}$ represents a vector whose magnitude is the length PQ and direction is from P to Q.



The point P is called initial (tail) point of vector $\overrightarrow{PQ}$ and Q is called the terminal point (or) head (or) tip.

→ vectors are generally denoted by $\vec{a}, \vec{b}, \vec{c}$ etc

→ The magnitude of a vector $\vec{a}$ is the +ve number which is the measure of its length and is denoted by $|\vec{a}|$ or a.

→ A vector of unit magnitude is called unit vector.

→ A vector of zero magnitude (which can have no direction) is called zero (null) vector.

→ The vector $\overrightarrow{QP}$ represents the negative of $\overrightarrow{PQ}$

→ Two vectors $\vec{a}$ & $\vec{b}$ having the same magnitude and same (or parallel) directions said to be equal and we write $\vec{a} = \vec{b}$.

Product of two vectors:

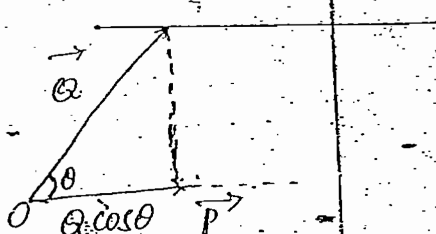
As a vector quantity involves magnitude and direction both, therefore, it is difficult to assign a definite meaning to the product of two vectors, whether the product is a scalar or a vector quantity. Accordingly, there are two types of products of two vectors.

(i). Scalar product (or) dot product:

~~The~~ scalar product or dot product of any two vectors is the product of the magnitude of the first vector and the component of the second vector in the direction of the first vector.

(or)

The scalar product or dot product of any two vectors is the product of the magnitude of the two vectors and the cosine of the angle between them.



If $\vec{P}$ and $\vec{Q}$ are any two vectors, then the scalar product or dot product of these vectors is given by

$$\vec{P}\cdot\vec{Q} = |\vec{P}||\vec{Q}|\cos\theta.$$
$$= PQ\cos\theta.$$

Note: By convention we take 'θ' to be the angle smaller than or equal to π so that $0 \leq \theta \leq \pi$.

The dot product of two vectors is a scalar quantity

<u>Properties of Scalar product:</u>

If $\vec{P}, \vec{Q}$ and $\vec{R}$ are any three vectors and K is a scalar quantity, then from

$$\vec{P}\cdot\vec{Q} = PQ\cos\theta$$

(1) $\vec{P}\cdot\vec{Q} = \vec{Q}\cdot\vec{P}$, i.e, the scalar product is commutative.

(2) $\vec{P}\cdot(\vec{Q}+\vec{R}) = \vec{P}\cdot\vec{Q} + \vec{P}\cdot\vec{R}$.
i.e, the scalar product is distributive over addition.

(3) $(K\vec{P})\cdot\vec{Q} = K(\vec{P}\cdot\vec{Q}) = \vec{P}\cdot(K\vec{Q})$

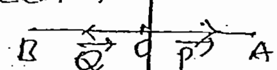
(4) If $\vec{P}\cdot\vec{Q} = 0$ and $\vec{P}$ & $\vec{Q}$ are not zero vectors then $\vec{P}$ is $\perp$lar to $\vec{Q}$.

(5) $\vec{P}\cdot\vec{P} = |\vec{P}||\vec{P}|\cos 0^\circ = |\vec{P}|^2$ $(\because \cos 0^\circ = 1)$

(6) $\vec{P}\cdot\vec{P} > 0$ for any non-zero vector $\vec{P}$.

(7) $\vec{P}\cdot\vec{P} = 0$ only if $\vec{P} = 0$.

(8) $\vec{P}, \vec{Q}$ are vectors in opposite direction



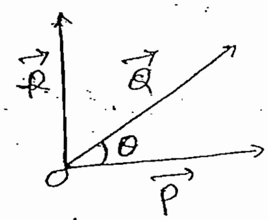
$(\vec{P},\vec{Q}) = 180^\circ \Rightarrow \cos(\vec{P},\vec{Q}) = -1$
$\Rightarrow \vec{P}\cdot\vec{Q} = -|\vec{P}||\vec{Q}|.$

(3)

(ii) Vector product or Cross product of two vectors:

→ The vector product of any two vectors is another vector which is ⊥lar to the plane formed by these two vectors. Its magnitude is equal to the product of the magnitudes of the two vectors and the sine of the angle between them.

→ If $\vec{P}$ and $\vec{Q}$ are two vectors then the cross product of $\vec{P}$ and $\vec{Q}$ is given by



$$\vec{P} \times \vec{Q} = PQ \sin\theta \, \hat{n} \; ; \quad 0 \leq \theta \leq \pi$$
$$= \vec{R}$$

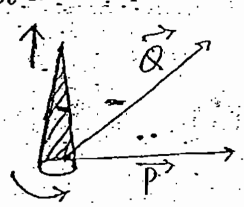
where P and Q are magnitudes of $\vec{P}$ & $\vec{Q}$,

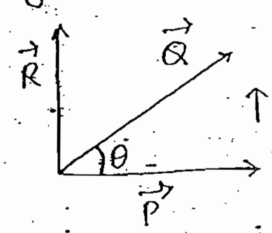
θ is the angle between $\vec{P}$ & $\vec{Q}$;

$\hat{n}$ is a unit vector ⊥lar to the plane containing $\vec{P}$ & $\vec{Q}$.

→ $|\vec{P} \times \vec{Q}| = PQ \sin\theta.$

→ The direction of the vector $\vec{R}$ can be obtained from the right-handed screw.





The axis of the screw is ⊥lar to the plane containing the vectors $\vec{P}$ & $\vec{Q}$ be turned from $\vec{P}$ to $\vec{Q}$ through an angle θ - between them, the direction of the advancement of the screw gives the direction of the vector. i.e, $\vec{P}\times\vec{Q}$.

Properties of vector product or Cross Product:-

i) $\vec{P}$, $\vec{Q}$ and $\vec{R}$ are any three vectors and K is a scalar quantity, then $|\vec{P}\times\vec{Q}| = PQ\sin\theta$.

(1) $\vec{P}\times\vec{Q}$ is a vector

(2) $\vec{P}\times\vec{Q} = -\vec{Q}\times\vec{P}$

(3) If $\vec{P}$ and $\vec{Q}$ are non-zero vectors, and $\vec{P}\times\vec{Q} = 0$, then $\vec{P}$ is parallel to $\vec{Q}$.

(4) $\vec{P}\times\vec{P} = 0$, for any vector $\vec{P}$.

i.e, $\vec{P}\times\vec{P} = P.P\sin 0^\circ = 0$

(5) $\vec{P}\times(\vec{Q}+\vec{R}) = (\vec{P}\times\vec{Q}) + (\vec{P}\times\vec{R})$

(6) $(\vec{P}+\vec{Q})\times\vec{R} = (\vec{P}\times\vec{R}) + (\vec{Q}\times\vec{R})$

i.e, the vector product is distributive over addition.

(7) $(K\vec{P})\times\vec{Q} = K(\vec{P}\times\vec{Q}) = \vec{P}\times(K\vec{Q})$

Hence a position vector of a point $P(x,y,z)$ (5)

in space is $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Now $OP^2 = OC'^2 + C'P^2$

$= (OB^2 + BC'^2) + C'P^2$

$= (OB^2 + OA^2) + OC^2$

$= OA^2 + OB^2 + OC^2$

$OP^2 = x^2 + y^2 + z^2$

$\therefore \quad OP = \sqrt{x^2 + y^2 + z^2}$

$\Rightarrow |\vec{r}| = \sqrt{x^2 + y^2 + z^2}$.

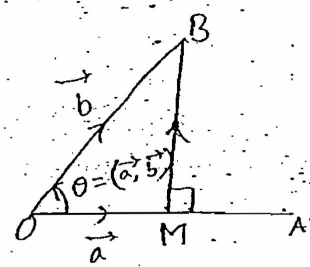
$\therefore$ If a point in space has coordinates (x,y,z), then its position vector $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$. The vectors $x\hat{i}, y\hat{j}, z\hat{k}$ are known as the component vectors of $\vec{r}$ along x, y and z axes respectively.

Projection of a vector along another vector :-

Let $\vec{a} \neq \vec{0}$, $\vec{b} \neq \vec{0}$. choose points O, A and B s.t. $\vec{OA} = \vec{a}$ and $\vec{OB} = \vec{b}$.

Let $\angle AOB = \theta$

From B draw BM perpendicular to $\overleftrightarrow{OA}$. Then OM is the projection of $\vec{b}$ on $\vec{a}$.



From definition,

$$\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$$
$$= |\vec{a}||\vec{OB}|\cos\theta$$
$$= |\vec{a}|\cdot OB\cos\theta$$
$$= |\vec{a}|\cdot OM$$
$$= |\vec{a}|\,(\text{projection of } \vec{b} \text{ on } \vec{a}).$$

Similarly $\vec{b}\cdot\vec{a} = |\vec{b}|$ (projection of $\vec{a}$ on $\vec{b}$)

$\Rightarrow$ projection of $\vec{b}$ on $\vec{a}$ $= \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}|}$

$$= \frac{\vec{a}}{|\vec{a}|}\cdot\vec{b}$$
$$= \hat{a}\cdot\vec{b}.$$

Similarly projection of $\vec{a}$ on $\vec{b}$ $= \dfrac{\vec{a}\cdot\vec{b}}{|\vec{b}|}$

$(\because \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a})$

$$= \vec{a}\cdot\hat{b}$$

$\rightarrow$ If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are non-zero vectors and $\vec{a}\cdot\vec{c} = \vec{b}\cdot\vec{c}$ then $\vec{a} = \vec{b}$ (or) $(\vec{a}-\vec{b})$ is perpendicular to $\vec{c}$.

Sol^n: $\vec{a}\cdot\vec{c} = \vec{b}\cdot\vec{c}$

$\Rightarrow (\vec{a}\cdot\vec{c}) - (\vec{b}\cdot\vec{c}) = 0$

~~$\Rightarrow (\vec{a}\cdot\vec{c}) + (-\vec{b}\cdot\vec{c}) = 0$~~

$\Rightarrow (\vec{a}-\vec{b})\cdot\vec{c} = 0$

$\Rightarrow (\vec{a}-\vec{b}) = 0$ (or) $(\vec{a}-\vec{b}) \perp \vec{c}$

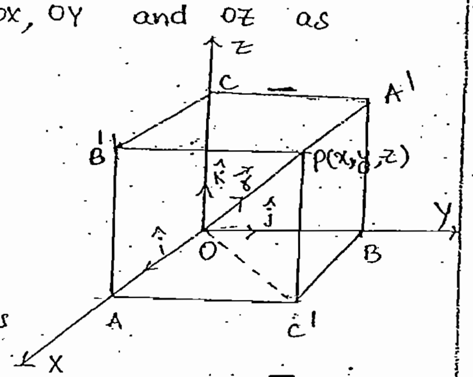
$\Rightarrow \vec{a} = \vec{b}$ (or) $(\vec{a}-\vec{b}) \perp \vec{c}$

whose direction is along OX (or) OX' according as x is positive (or) negative.

(iv) The component of $\overrightarrow{OP}$ along y-axis is a vector $y\hat{j}$ whose magnitude is $|y|$ and whose direction is along OY (or) OY' according as y is +ve (or) −ve.

Components of a vector in three Dimensions :—

Let $P(x, y, z)$ be a point in three dimensional space with reference to OX, OY and OZ as coordinate axes.



Then $OA = x$, $OB = y$ and $OC = z$

Let $\hat{i}, \hat{j}, \hat{k}$ be unit vectors along OX, OY and OZ respectively.

Then $\overrightarrow{OA} = x\hat{i}$; $\overrightarrow{OB} = y\hat{j}$ and $\overrightarrow{OC} = z\hat{k}$.

we have

$$\overrightarrow{BC'} = \overrightarrow{OA} = x\hat{i}$$

$$\overrightarrow{C'P} = \overrightarrow{OC} = z\hat{k}$$

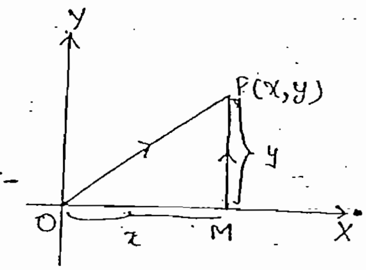
$$\begin{aligned}\overrightarrow{OP} &= \overrightarrow{OC'} + \overrightarrow{C'P} \\ &= \overrightarrow{OB} + \overrightarrow{BC'} + \overrightarrow{C'P} \\ &= \overrightarrow{OB} + \overrightarrow{OA} + \overrightarrow{OC} \\ &= x\hat{i} + y\hat{j} + z\hat{k}.\end{aligned}$$

Let $\overrightarrow{OP} = \vec{r}$. Then $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Components of Vectors :—

→ Let $P(x,y)$ be a point in a plane with reference OX & OY as the coordinate axes.

Then $OM = x$ & $MP = y$



→ Let $\hat{i}, \hat{j}$ be unit vectors along ox and oy respectively.

Then $\vec{OM} = x\hat{i}$ and $\vec{MP} = y\hat{j}$.

→ vectors $\vec{OM}$ and $\vec{MP}$ are known as the components of $\vec{OP}$ along x-axis & y-axis respectively.

Now $\vec{OP} = \vec{OM} + \vec{MP}$ [By triangle law of addition]

$$\Rightarrow \vec{OP} = x\hat{i} + y\hat{j}$$

→ Let $\vec{OP} = \vec{r}$ then $\vec{r} = x\hat{i} + y\hat{j}$

Now $OP^2 = OM^2 + MP^2$

$$\Rightarrow OP^2 = x^2 + y^2$$

$$\Rightarrow OP = \sqrt{x^2+y^2}$$

$$\Rightarrow |\vec{r}| = \sqrt{x^2+y^2}$$

Hence if a point 'P' in a plane has coordinates (x,y) then

(i) $\vec{OP} = x\hat{i} + y\hat{j}$

(ii) $|\vec{OP}| = \sqrt{x^2+y^2}$

(iii) The component of $\vec{OP}$ along x-axis is a vector $x\hat{i}$, whose magnitude is $|x|$ and

→ We know that $\vec{a}\cdot\vec{b} = ab\cos\theta$.
where 'θ' be the angle between the vectors $\vec{a}$ & $\vec{b}$.

For two mutually perpendicular vectors
$\vec{a}\cdot\vec{b} = ab\cos\pi/2 = 0$;

therefore if $\hat{i}, \hat{j}, \hat{k}$ are unit vectors along three mutually ⊥lar axes,
then $\hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$

obviously two given vectors are orthogonal iff $\vec{a}\cdot\vec{b} = 0$.

→ The scalar product of two equal vectors is given by $\vec{a}\cdot\vec{a} = |\vec{a}||\vec{a}|\cos 0$
$= a\cdot a\cos 0 = a^2$ $(\because |\vec{a}| = a)$

i.e, the square of any vector is equal to the square of its modulus,
thus, if $\hat{i}, \hat{j}, \hat{k}$ are unit vectors
$\hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1$.

→ If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ and $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$
then $\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3$.
$(\because \hat{i}\cdot\hat{j} = \hat{j}\cdot\hat{k} = \hat{k}\cdot\hat{i} = 0$
$\& \ \hat{i}\cdot\hat{i} = \hat{j}\cdot\hat{j} = \hat{k}\cdot\hat{k} = 1)$

* **Angle between two non-zero vectors $\vec{a}$ and $\vec{b}$:**

Let $(\vec{a}, \vec{b}) = \theta$, then $\vec{a}\cdot\vec{b} = |\vec{a}||\vec{b}|\cos\theta$
$\Rightarrow \cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$.

$$\Rightarrow \theta = \cos^{-1}\left(\frac{\vec{a}\cdot\vec{b}}{|\vec{a}|\cdot|\vec{b}|}\right)$$

→ If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$; $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ are non-zero vectors and $(\vec{a}, \vec{b}) = \theta$

then $\cos\theta = \dfrac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$

$$= \frac{a_1b_1 + a_2b_2 + a_3b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2}\sqrt{b_1^2 + b_2^2 + b_3^2}}$$

Note:

(1) If $\vec{a}$ and $\vec{b}$ are perpendicular then $a_1b_1 + a_2b_2 + a_3b_3 = 0$

(2) If $\vec{a}$ & $\vec{b}$ are parallel, then $\exists$ a scalar 't' such that $a_1 = tb_1$, $a_2 = tb_2$, & $a_3 = tb_3$

→ For any two vectors $\vec{a}$ and $\vec{b}$, the following are true

(i) $|\vec{a} + \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 + 2\,\vec{a}\cdot\vec{b}$

(ii) $|\vec{a} - \vec{b}|^2 = |\vec{a}|^2 + |\vec{b}|^2 - 2\,\vec{a}\cdot\vec{b}$

(iii) $(\vec{a} + \vec{b})\cdot(\vec{a} - \vec{b}) = |\vec{a}|^2 - |\vec{b}|^2$

(iv) $|\vec{a}\cdot\vec{b}| \le |\vec{a}|\,|\vec{b}|$

Sol: (iv) $|\vec{a}\cdot\vec{b}| = \left|\,|\vec{a}|\,|\vec{b}|\cos(\vec{a},\vec{b})\right|$

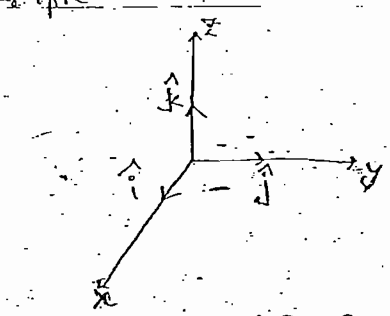
$= |\vec{a}||\vec{b}|\,|\cos(\vec{a},\vec{b})|$

$\le |\vec{a}|\,|\vec{b}|$ $\quad (\because |\cos\theta| \le 1)$

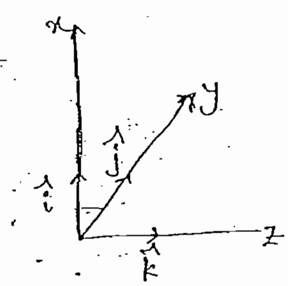
Vector product in component form:

Let us express the vector product in component form w.r.t a cartesian Coordinate system. But before we do so we would like to note that there are two types of such systems. Depending on the orientation of axes, they are termed as right-handed (or) left-handed.

By definition, a cartesian Co-ordinate system is called right handed if the unit vectors $\hat{i}, \hat{j}, \hat{k}$ in the directions of positive x, y, z axes form a right handed triple. A Cartesian Co-ordinate system is called left-handed if $\hat{i}, \hat{j}, \hat{k}$ form a left-handed left-handed triple.



Right-handed Cartesian co-ordinate System.
fig (i)



Left-handed cartesian Co-ordinate System.
fig (ii)

Now let two vectors $\vec{a}$ and $\vec{b}$ be given as $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$; $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$.

where a_1, a_2, a_3 and b_1, b_2, b_3 are their Components w.r.t a Right handed cartesian coordinate System.

Then

$$\vec{a}\times\vec{b} = (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \times (b_1\hat{i} + b_2\hat{j} + b_3\hat{k})$$
$$= a_1b_1\,\hat{i}\times\hat{i} + a_1b_2\,\hat{i}\times\hat{j} + a_1b_3\,\hat{i}\times\hat{k}$$
$$+ a_2b_1\,\hat{j}\times\hat{i} + a_2b_2\,\hat{j}\times\hat{j} + a_2b_3\,\hat{j}\times\hat{k}$$
$$+ a_3b_1\,\hat{k}\times\hat{i} + a_3b_2\,\hat{k}\times\hat{j} + a_3b_3\,\hat{k}\times\hat{k}.$$

Now we have to know the cross product of the unit vectors $\hat{i}, \hat{j}, \hat{k}$ with themselves and each other to determine $\vec{a}\times\vec{b}$.

Let us consider the products $\hat{i}\times\hat{i}$ and $\hat{i}\times\hat{j}$.

From the definition of cross product,

$$|\hat{i}\times\hat{i}| = 1.1(\sin 0^\circ) = 0$$

so $\hat{i}\times\hat{i} = \vec{0}$

And $|\hat{i}\times\hat{j}| = 1.1(\sin 90^\circ) = 1$

According to the right hand rule the direction of $\hat{i}\times\hat{j}$ is along $\hat{k}$ (fig (i))

So that $\hat{i}\times\hat{j} = \hat{k}$.

Similarly; $\hat{j}\times\hat{j} = 0$ and $\hat{k}\times\hat{k} = 0$;

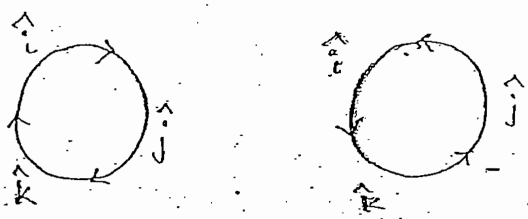
$\hat{j}\times\hat{k} = \hat{i}$, $\hat{k}\times\hat{i} = \hat{j}$;

$\hat{j}\times\hat{i} = -\hat{k}$, $\hat{k}\times\hat{j} = -\hat{i}$, $\hat{i}\times\hat{k} = -\hat{j}$

From above we can see a cyclic pattern in the cross-products $\hat{i}\times\hat{j}$, $\hat{j}\times\hat{k}$, $\hat{k}\times\hat{i}$ etc. It is a good way to remember these cross products.

If we go around the circle clockwise all vector products are positive, i.e, $\hat{i}\times\hat{j}=\hat{k}$ and so on.

If we go in an anti-clockwise direction the cross products are negative. i.e, $\hat{j}\times\hat{i}=-\hat{k}$ and so on.



Using the above results we can write the vector product in its component form.

$$\vec{a}\times\vec{b}=(a_2b_3-a_3b_2)\,\hat{i}+(a_3b_1-a_1b_3)\,\hat{j}+(a_1b_2-a_2b_1)\,\hat{k}.$$

$$=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix}$$

$\rightarrow$ $|\vec{a}\times\vec{b}|^2+|\vec{a}\cdot\vec{b}|^2=|\vec{a}|^2\,|\vec{b}|^2.$

Sol :- $|\vec{a}\times\vec{b}|^2+|\vec{a}\cdot\vec{b}|^2$

$$=\left|\,|\vec{a}|\,|\vec{b}|\sin\theta\,\hat{n}\right|^2+\left|\,|\vec{a}|\,|\vec{b}|\cos\theta\right|^2$$

$$=|\vec{a}|^2|\vec{b}|^2\sin^2\theta+|\vec{a}|^2|\vec{b}|^2\cos^2\theta$$

$$=|\vec{a}|^2|\vec{b}|^2(\sin^2\theta+\cos^2\theta)$$

$$=|\vec{a}|^2|\vec{b}|^2$$

which is known as Lagranges Identity.

→ $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}| \sin\theta$ where $\theta = (\vec{a}, \vec{b})$

$$\Rightarrow \sin\theta = \frac{|\vec{a} \times \vec{b}|}{|\vec{a}||\vec{b}|}$$

Scalar Tripple Product

→ Let $\vec{a}, \vec{b}$ and $\vec{c}$ be three vectors. Then we call $(\vec{a} \times \vec{b}) \cdot \vec{c}$, the scalar triple product of $\vec{a}, \vec{b}$ & $\vec{c}$. This is a real number.

→ If $(\vec{a} \times \vec{b}) \cdot \vec{c} = 0$, the following cases arise

(i) Atleast one of the vectors $\vec{a}, \vec{b}$ and $\vec{c}$ should be zero vector.

(ii) $\vec{a} \neq \vec{0}$, $\vec{b} \neq \vec{0}$, $\vec{c} \neq \vec{0}$ and $\vec{c}$ is $\perp$lar to $\vec{a} \times \vec{b}$.

→ If $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$, $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$
$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$ then

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Note :— 1. We write the scalar triple product $(\vec{a} \times \vec{b}) \cdot \vec{c}$ as $[\vec{a}\ \vec{b}\ \vec{c}]$ (or) $(\vec{a}, \vec{b}, \vec{c})$.

Properties of Scalar Tripple Product :-

→ If $\vec{a}, \vec{b}, \vec{c}$ are cyclically permuted the value of scalar triple product remains same.

i.e. $(\vec{a} \times \vec{b}) \cdot \vec{c} = (\vec{b} \times \vec{c}) \cdot \vec{a} = (\vec{c} \times \vec{a}) \cdot \vec{b}$

(or)

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a}) = \vec{c} \cdot (\vec{a} \times \vec{b})$$

(or)

$$[\vec{a}\ \vec{b}\ \vec{c}] = [\vec{b}\ \vec{c}\ \vec{a}] = [\vec{c}\ \vec{a}\ \vec{b}].$$

→ The change of cyclic order of vectors in scalar triple product changes the sign of the scalar triple product but not in magnitude.

i.e. $[\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{b}\ \vec{a}\ \vec{c}] = -[\vec{c}\ \vec{b}\ \vec{a}]$
$= -[\vec{a}\ \vec{c}\ \vec{b}]$

→ In scalar triple product the positions of dot and cross can be interchanged provided that the cyclic order of the vectors remains same.

i.e. $(\vec{a} \times \vec{b}) \cdot \vec{c} = \vec{a} \cdot (\vec{b} \times \vec{c})$

→ The scalar triple product of three vectors is zero if any two of them are equal.

i.e. if two of the vectors $\vec{a}, \vec{b}, \vec{c}$ are equal then $[\vec{a}\ \vec{b}\ \vec{c}] = 0$.

→ For any three vectors $\vec{a}, \vec{b}, \vec{c}$ and scalar λ,
$[\lambda\vec{a}\ \vec{b}\ \vec{c}] = \lambda[\vec{a}\ \vec{b}\ \vec{c}]$.

→ The scalar triple product of three vectors is zero if any two of them are parallel or collinear.

i.e. If two of the vectors $\vec{a}, \vec{b}, \vec{c}$ are parallel (or) collinear then $[\vec{a}\ \vec{b}\ \vec{c}] = 0$.

→ The necessary and sufficient condition for three non-zero non collinear vectors $\vec{a}, \vec{b}, \vec{c}$ to be coplanar is that $[\vec{a}\ \vec{b}\ \vec{c}] = 0$.

i.e. $\vec{a}, \vec{b}, \vec{c}$ are coplanar $\Leftrightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 0$.

→ For points with position vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ will be coplanar if

$$[\vec{d}\ \vec{b}\ \vec{c}] + [\vec{d}\ \vec{c}\ \vec{a}] + [\vec{d}\ \vec{a}\ \vec{b}] = [\vec{a}\ \vec{b}\ \vec{c}].$$

→ If $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are four vectors, then
$$[\vec{a}+\vec{b}\ \ \vec{c}\ \ \vec{d}] = [\vec{a}\ \vec{c}\ \vec{d}] + [\vec{b}\ \vec{c}\ \vec{d}]$$

→ If $\vec{a}, \vec{b}, \vec{c}$ are not coplanar and they are adjacent sides of a parallelopiped, then the volume of parallelopiped is $[\vec{a}\ \vec{b}\ \vec{c}]$.

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→ If $A = (x_1, y_1, z_1)$; $B = (x_2, y_2, z_2)$; $C = (x_3, y_3, z_3)$ and $D = (x_4, y_4, z_4)$ are the vertices of a tetrahedron, then the volume of the ~~tetrahedron~~

$$= \frac{1}{6}\left|\left[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}\right]\right|$$

$$= \frac{1}{6}\left|\left[(x_2-x_1,\ y_2-y_1,\ z_2-z_1)\ (x_3-x_1,\ y_3-y_1,\ z_3-z_1)\ (x_4-x_1,\ y_4-y_1,\ z_4-z_1)\right]\right|$$

$$= \frac{1}{6}\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \\ x_4-x_1 & y_4-y_1 & z_4-z_1 \end{vmatrix}$$

Vector Triple Product

→ Let $\vec{a}, \vec{b}, \vec{c}$ be any three vectors, then the vectors $\vec{a} \times (\vec{b} \times \vec{c})$ and $(\vec{a} \times \vec{b}) \times \vec{c}$ are called vector triple products of $\vec{a}, \vec{b}, \vec{c}$.

Note :— (i) If any one of $\vec{a}, \vec{b}$ and $\vec{c}$ is the zero vector, then $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \times \vec{b}) \times \vec{c} = 0$

(ii) If $\vec{a}$ is parallel to $\vec{b}$, then $(\vec{a} \times \vec{b}) \times \vec{c} = \vec{0}$.

$(\because \vec{a} \times \vec{b} = \vec{0})$

(iii) If $\vec{c}$ is a perpendicular to both $\vec{a}$ & $\vec{b}$, then $(\vec{a} \times \vec{b}) \times \vec{c} = \vec{0}$

→ For any vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$

$$(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{b}\cdot\vec{c})\vec{a}.$$

Note :— 1. $\vec{a}\times(\vec{b}\times\vec{c}) = -((\vec{b}\times\vec{c})\times\vec{a})$

$$= -((\vec{b}\cdot\vec{a})\vec{c} - (\vec{c}\cdot\vec{a})\vec{b})$$

$$= (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$$

2. $(\vec{a}\times\vec{b})\times\vec{c} \neq \vec{a}\times(\vec{b}\times\vec{c})$.

→ For any vectors $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$

$$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})$$

$$= \begin{vmatrix} \vec{a}\cdot\vec{c} & \vec{a}\cdot\vec{d} \\ \vec{b}\cdot\vec{c} & \vec{b}\cdot\vec{d} \end{vmatrix}$$

Sol :— $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = (\vec{a}\times\vec{b})\cdot\vec{r}$ where $\vec{r} = \vec{c}\times\vec{d}$

$$= \vec{a}\cdot(\vec{b}\times\vec{r})$$

$$= \vec{a}\cdot[\vec{b}\times(\vec{c}\times\vec{d})]$$

$$= \vec{a}\cdot[(\vec{b}\cdot\vec{d})\vec{c} - (\vec{b}\cdot\vec{c})\vec{d}]$$

$$= (\vec{a}\cdot\vec{c})(\vec{b}\cdot\vec{d}) - (\vec{a}\cdot\vec{d})(\vec{b}\cdot\vec{c})$$

$$= \begin{vmatrix} \vec{a}\cdot\vec{c} & \vec{a}\cdot\vec{d} \\ \vec{b}\cdot\vec{c} & \vec{b}\cdot\vec{d} \end{vmatrix}$$

→ For any vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$,

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a}\ \vec{c}\ \vec{d}]\vec{b} - [\vec{b}\ \vec{c}\ \vec{d}]\vec{a}$$

$$= [\vec{a}\ \vec{b}\ \vec{d}]\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\vec{d}$$

Sol: $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = (\vec{a}\times\vec{b})\times\vec{m}$

where $\vec{m} = \vec{c}\times\vec{d}$

$$= (\vec{a}\cdot\vec{m})\vec{b} - (\vec{b}\cdot\vec{m})\vec{a}$$

$$= (\vec{a}\cdot(\vec{c}\times\vec{d}))\vec{b} - (\vec{b}\cdot(\vec{c}\times\vec{d}))\vec{a}$$

$$= [\vec{a}\ \vec{c}\ \vec{d}]\vec{b} - [\vec{b}\ \vec{c}\ \vec{d}]\vec{a} \quad \text{——— (1)}$$

Now $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = -(\vec{c}\times\vec{d})\times(\vec{a}\times\vec{b})$

$$= -\{[\vec{c}\ \vec{a}\ \vec{b}]\vec{d} - [\vec{d}\ \vec{a}\ \vec{b}]\vec{c}\}$$

$$= [\vec{d}\ \vec{a}\ \vec{b}]\vec{c} - [\vec{c}\ \vec{a}\ \vec{b}]\vec{d}$$

$$= [\vec{a}\ \vec{b}\ \vec{d}]\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\vec{d} \quad \text{——— (2)}$$

From (1) & (2) we have

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a}\ \vec{c}\ \vec{d}]\vec{b} - [\vec{b}\ \vec{c}\ \vec{d}]\vec{a}$$

$$= [\vec{a}\ \vec{b}\ \vec{d}]\vec{c} - [\vec{a}\ \vec{b}\ \vec{c}]\vec{d}$$

→ **Reciprocal system of vectors:**

The sets of vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{a'}, \vec{b'}, \vec{c'}$ are called reciprocal sets (or) system of vectors if $\vec{a}\cdot\vec{a'} = \vec{b}\cdot\vec{b'} = \vec{c}\cdot\vec{c'} = 1$

$\vec{a'}\cdot\vec{b} = \vec{a'}\cdot\vec{c} = \vec{b'}\cdot\vec{a} = \vec{c'}\cdot\vec{a} = \vec{c'}\cdot\vec{b} = \vec{b'}\cdot\vec{c} = 0$

The sets $\vec{a}, \vec{b}, \vec{c}$ and $\vec{a'}, \vec{b'}, \vec{c'}$ are reciprocal of vectors iff

$$\vec{a'} = \frac{\vec{b}\times\vec{c}}{\vec{a}\cdot(\vec{b}\times\vec{c})}, \quad \vec{b'} = \frac{\vec{c}\times\vec{a}}{\vec{a}\cdot(\vec{b}\times\vec{c})}$$

$$\vec{c'} = \frac{\vec{a}\times\vec{b}}{\vec{a}\cdot(\vec{b}\times\vec{c})} \quad ; \text{where } \vec{a}\cdot(\vec{b}\times\vec{c}) \neq 0$$

→ Prove that $(\vec{a}\times\vec{b})\cdot(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a}) = (\vec{a}\cdot\vec{b}\times\vec{c})^2$

Sol^n: W.K.T $\vec{x}\times(\vec{c}\times\vec{a}) = \vec{c}(\vec{x}\cdot\vec{a}) - \vec{a}(\vec{x}\cdot\vec{c})$.

Let $\vec{x} = \vec{b}\times\vec{c}$

then $(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a}) = \vec{c}(\vec{b}\times\vec{c}\cdot\vec{a}) - \vec{a}(\vec{b}\times\vec{c}\cdot\vec{c})$

$= \vec{c}(\vec{a}\cdot\vec{b}\times\vec{c}) - \vec{a}(\vec{b}\times\vec{c}\times\vec{c})$

$[\because (\vec{b}\times\vec{c})\cdot\vec{c} = \vec{b}\cdot(\vec{c}\times\vec{c})]$

$= \vec{c}(\vec{a}\cdot\vec{b}\times\vec{c})$

$(\because \vec{c}\times\vec{c} = 0)$

$\therefore (\vec{a}\times\vec{b})\cdot(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a}) = (\vec{a}\times\vec{b})\cdot\vec{c}(\vec{a}\cdot\vec{b}\times\vec{c})$

$= (\vec{a}\times\vec{b}\cdot\vec{c})(\vec{a}\cdot\vec{b}\times\vec{c})$

$= (\vec{a}\cdot\vec{b}\times\vec{c})(\vec{a}\cdot\vec{b}\times\vec{c})$

$(\because \vec{a}\times\vec{b}\cdot\vec{c} = \vec{a}\cdot(\vec{b}\times\vec{c}))$

$= (\vec{a}\cdot\vec{b}\times\vec{c})^2$

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→ Given the vectors $\vec{a}' = \dfrac{\vec{b}\times\vec{c}}{\vec{a}\cdot\vec{b}\times\vec{c}}$, $\vec{b}' = \dfrac{\vec{c}\times\vec{a}}{\vec{a}\cdot\vec{b}\times\vec{c}}$ and (12)

$\vec{c}' = \dfrac{\vec{a}\times\vec{b}}{\vec{a}\cdot\vec{b}\times\vec{c}}$, show that if $\vec{a}\cdot\vec{b}\times\vec{c} \neq 0$

(i) $\vec{a}'\cdot\vec{a} = \vec{b}'\cdot\vec{b} = \vec{c}'\cdot\vec{c} = 1$,

(ii) $\vec{a}'\cdot\vec{b} = \vec{a}'\cdot\vec{c} = 0$, $\vec{b}'\cdot\vec{a} = \vec{b}'\cdot\vec{c} = 0$, $\vec{c}'\cdot\vec{a} = \vec{c}'\cdot\vec{b} = 0$,

(iii) if $\vec{a}\cdot\vec{b}\times\vec{c} = V$ then $\vec{a}'\cdot\vec{b}'\times\vec{c}' = 1/V$,

(iv) $\vec{a}'$, $\vec{b}'$ and $\vec{c}'$ are non-coplanar if $\vec{a}, \vec{b}$ and $\vec{c}$ are non-coplanar.

(i) $\vec{a}'\cdot\vec{a} = \vec{a}\cdot\vec{a}' = \vec{a}\cdot\dfrac{\vec{b}\times\vec{c}}{\vec{a}\cdot\vec{b}\times\vec{c}} = \dfrac{\vec{a}\cdot\vec{b}\times\vec{c}}{\vec{a}\cdot\vec{b}\times\vec{c}} = 1$

$\vec{b}'\cdot\vec{b} = \vec{b}\cdot\vec{b}' = \vec{b}\cdot\dfrac{\vec{c}\times\vec{a}}{\vec{a}\cdot\vec{b}\times\vec{c}} = \dfrac{\vec{b}\cdot\vec{c}\times\vec{a}}{\vec{a}\cdot\vec{b}\times\vec{c}} = \dfrac{\vec{a}\cdot\vec{b}\times\vec{c}}{\vec{a}\cdot\vec{b}\times\vec{c}} = 1$

$\vec{c}'\cdot\vec{c} = \vec{c}\cdot\vec{c}' = \vec{c}\cdot\dfrac{\vec{a}\times\vec{b}}{\vec{a}\cdot\vec{b}\times\vec{c}} = \dfrac{\vec{c}\cdot\vec{a}\times\vec{b}}{\vec{a}\cdot\vec{b}\times\vec{c}} = \dfrac{\vec{a}\cdot\vec{b}\times\vec{c}}{\vec{a}\cdot\vec{b}\times\vec{c}} = 1$

(ii) $\vec{a}'\cdot\vec{b} = \vec{b}\cdot\vec{a}' = \vec{b}\cdot\dfrac{\vec{b}\times\vec{c}}{\vec{a}\cdot\vec{b}\times\vec{c}} = \dfrac{\vec{b}\cdot\vec{b}\times\vec{c}}{\vec{a}\cdot\vec{b}\times\vec{c}} = \dfrac{\vec{b}\times\vec{b}\cdot\vec{c}}{\vec{a}\cdot\vec{b}\times\vec{c}} = 0$

Similarly $\vec{a}'\cdot\vec{c} = \vec{c}\cdot\vec{a}' = 0$

$\vec{b}'\cdot\vec{a} = \vec{a}\cdot\vec{b}' = 0$, $\vec{b}'\cdot\vec{c} = \vec{c}\cdot\vec{b}' = 0$

$\vec{c}'\cdot\vec{a} = \vec{a}\cdot\vec{c}' = 0$, $\vec{c}'\cdot\vec{b} = \vec{b}\cdot\vec{c}' = 0$

(iii) $\vec{a}' = \dfrac{\vec{b}\times\vec{c}}{V}$, $\vec{b}' = \dfrac{\vec{c}\times\vec{a}}{V}$, $\vec{c}' = \dfrac{\vec{a}\times\vec{b}}{V}$

Then $\vec{a}'\cdot\vec{b}'\times\vec{c}' = \dfrac{(\vec{b}\times\vec{c})\cdot(\vec{c}\times\vec{a})\times(\vec{a}\times\vec{b})}{V^3}$

$= \dfrac{(\vec{a}\times\vec{b})\cdot(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})}{V^3}$

$$= \frac{(\vec{a}\cdot\vec{b}\times\vec{c})^2}{V^3} = \frac{V^2}{V^3} = \frac{1}{V}$$

(iv) If $\vec{a}, \vec{b}$, and $\vec{c}$ are non-coplanar $\vec{a}\cdot\vec{b}\times\vec{c}\neq 0$ i.e. $[\vec{a}\ \vec{b}\ \vec{c}]\neq 0$. then from part (iii) it follows that $\vec{a}'\cdot\vec{b}'\times\vec{c}'\neq 0$, so that $\vec{a}', \vec{b}'$ and $\vec{c}'$ are also non-coplanar.

(IAS 2003) Show that if $\vec{a}', \vec{b}'$ and $\vec{c}'$ are the reciprocals of the non-coplanar vectors $\vec{a}, \vec{b}$ and $\vec{c}$, then any vector $\vec{r}$ may be expressed as

$$\vec{r} = (\vec{r}\cdot\vec{a}')\vec{a} + (\vec{r}\cdot\vec{b}')\vec{b} + (\vec{r}\cdot\vec{c}')\vec{c}.$$

Sol) We know that

$$(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = \vec{b}(\vec{a}\cdot\vec{c}\times\vec{d}) - \vec{a}(\vec{b}\cdot\vec{c}\times\vec{d})$$
$$= \vec{c}(\vec{a}\cdot\vec{b}\times\vec{d}) - \vec{d}(\vec{a}\cdot\vec{b}\times\vec{c}).$$

we take $\vec{b}(\vec{a}\cdot\vec{c}\times\vec{d}) - \vec{a}(\vec{b}\cdot\vec{c}\times\vec{d}) = \vec{c}(\vec{a}\cdot\vec{b}\times\vec{d}) - \vec{d}(\vec{a}\cdot\vec{b}\times\vec{c})$.

$$\Rightarrow \vec{d}(\vec{a}\cdot\vec{b}\times\vec{c}) = \vec{a}(\vec{b}\cdot\vec{c}\times\vec{d}) - \vec{b}(\vec{a}\cdot\vec{c}\times\vec{d}) + \vec{c}(\vec{a}\cdot\vec{b}\times\vec{d}).$$

$$\Rightarrow \vec{d} = \frac{\vec{a}(\vec{b}\cdot\vec{c}\times\vec{d})}{\vec{a}\cdot\vec{b}\times\vec{c}} - \frac{\vec{b}(\vec{a}\cdot\vec{c}\times\vec{d})}{\vec{a}\cdot\vec{b}\times\vec{c}} + \frac{\vec{c}(\vec{a}\cdot\vec{b}\times\vec{d})}{\vec{a}\cdot\vec{b}\times\vec{c}}$$

$\left(\because \vec{a}\cdot\vec{b}\times\vec{c} = [\vec{a}\ \vec{b}\ \vec{c}]\neq 0\right)$.

Let $\vec{d} = \vec{r}$ then

$$\vec{r} = \frac{\vec{r}\cdot\vec{b}\times\vec{c}}{\vec{a}\cdot\vec{b}\times\vec{c}}\vec{a} + \frac{\vec{r}\cdot\vec{c}\times\vec{a}}{\vec{a}\cdot\vec{b}\times\vec{c}}\vec{b} + \frac{\vec{r}\cdot\vec{a}\times\vec{b}}{\vec{a}\cdot\vec{b}\times\vec{c}}\vec{c}$$

$$= \vec{r}\cdot\left(\frac{\vec{b}\times\vec{c}}{\vec{a}\cdot\vec{b}\times\vec{c}}\right)\vec{a} + \vec{r}\cdot\left(\frac{\vec{c}\times\vec{a}}{\vec{a}\cdot\vec{b}\times\vec{c}}\right)\vec{b} + \vec{r}\cdot\left(\frac{\vec{a}\times\vec{b}}{\vec{a}\cdot\vec{b}\times\vec{c}}\right)\vec{c}$$

$$= (\vec{r}\cdot\vec{a}')\vec{a} + (\vec{r}\cdot\vec{b}')\vec{b} + (\vec{r}\cdot\vec{c}')\vec{c}$$

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Simplify $(A+B)\cdot(B+C)\times(C+A)$.

Sol $(A+B)\cdot(B+C)\times(C+A) = (A+B)\cdot\left[(B+C)\times(C+A)\right]$

$= (A+B)\cdot\left[(B+C)\times C + (B+C)\times A\right]$

$= (A+B)\cdot\left[B\times C + C\times C + B\times A + C\times A\right]$

$= (A+B)\cdot\left[B\times C + B\times A + C\times A\right]$ $(\because C\times C = 0)$

$= A\cdot(B\times C) + A\cdot(B\times A) + A\cdot(C\times A) +$
$B\cdot(B\times C) + B\cdot(B\times A) + B\cdot(C\times A)$

$= A\cdot(B\times C) + B\cdot(A\times A) + C\cdot(A\times A) + C\cdot(B\times B)$
$+ A\cdot(B\times B) + A\cdot(B\times C)$.

$= 2A\cdot(B\times C)$

$= 2A\cdot B\times C$.

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Vector Differential Calculus

This is the beginning of vector calculus, which involves two kinds of functions, vector functions and scalar functions.

Scalar function:

Let $S \subset \mathbb{R}$, if for each scalar (element) $t \in S$, $\exists$ a unique real number $f(t)$, then $f(t)$ is said to scalar function of the scalar variable 't'. Here 'S' is called domain of $f(t)$ and $f(t)$ is a scalar quantity, so f is scalar function.

Vector function of a Scalar variable:

Let $S \subset \mathbb{R}$, if for each scalar $t \in S$, $\exists$ a unique vector $\overrightarrow{f(t)}$ then $\overrightarrow{f(t)}$ is said

to be vector function of the scalar variable 't'.
Here $\vec{f(t)}$ is a vector quantity so $\vec{f}$ is a vector function.

— Let $\vec{i}, \vec{j}, \vec{k}$ be the three mutually perpendicular unit vectors in three dimensional space then the vector function $\vec{f}(t)$ may be expressed in the form

$$\vec{f}(t) = f_1(t)\vec{i} + f_2(t)\vec{j} + f_3(t)\vec{k}.$$

Here $f_1(t), f_2(t), f_3(t)$ are real valued functions and are called the components of $\vec{f}(t)$.

Scalar field:

If to each point (x, y, z) of a region R in space there corresponds a unique number or scalar $\phi(x, y, z)$, then ϕ is called a scalar function of position or scalar point function and we say that a scalar field ϕ has been defined in R.

Ex: 1) The temperature at any point ~~within or on the~~ earth's surface at a certain time defines a scalar field.

2) $\phi(x, y, z) = x^3y - z^2$ defines a scalar field.

vector field: If to each point (x, y, z) of a region R in space there corresponds a vector $\vec{V}(x, y, z)$ then $\vec{V}$ is called a vector function of position or vector point function and we say that a vector field $\vec{V}$ has been defined in R

Examples: 1- If the velocity at any point (x, y, z) within a moving fluid is known at a certain time, then a vector field is defined.

(2) $\vec{V}(x, y, z) = xy^2\vec{i} - 2yz^3\vec{j} + x^2z\vec{k}$ defines a vector field.

Limit of a vector function:—

A vector function $\vec{f}(t)$ said to have limit 'L' when t tends to t_0, if given $\epsilon > 0$, $\exists$ a $\delta > 0$ (depending on ϵ) such that

$|\vec{f}(t) - L| < \epsilon$ whenever $0 < |t - t_0| < \delta$.

i.e, $\vec{f}(t) \rightarrow L$ as $t \rightarrow t_0$

i.e, $\underset{t \to t_0}{Lt} \vec{f}(t) = L$

Note: Let $\underset{t \to t_0}{Lt} \vec{f}(t) = L$ and $\underset{t \to t_0}{Lt} \vec{g}(t) = M$ and λ is a constant

then (1) $\underset{t \to t_0}{Lt} \left[\vec{f}(t) \pm \vec{g}(t)\right] = L \pm M$ (3) $\underset{t \to t_0}{Lt} \left(\vec{f}(t) \times \vec{g}(t)\right) = L \times M$

(2) $\underset{t \to t_0}{Lt} \left[\vec{f}(t) \cdot \vec{g}(t)\right] = LM$ (4) $\underset{t \to t_0}{Lt} \left[\lambda \vec{f}(t)\right] = \lambda L$

Continuity of vector function:

A vector function $\vec{f}(t)$ is said to be continuous at $t = t_0$ if (i) $f(t_0)$ is defined.

(ii) given any $\epsilon > 0$ (however small) $\exists$ a $\delta > 0$ (depending on ϵ) such that $|\vec{f}(t) - \vec{f}(t_0)| < \epsilon$ whenever $|t - t_0| < \delta$.

i.e, $\vec{f}(t) \rightarrow \vec{f}(t_0)$ as $t \rightarrow t_0$.

i.e, $\underset{t \to t_0}{Lt} \vec{f}(t) = \vec{f}(t_0)$.

Note (1) The function f is said to be continuous on I if f is continuous at each point of I.

(2). If f and g are continuous then $f \pm g$, $f \cdot g$ and $f \times g$ are also continuous.

Derivative of a vector function with respect to a scalar:

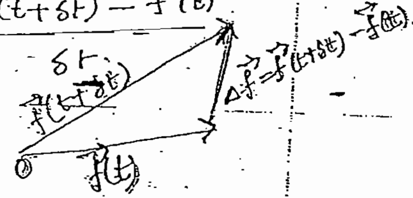
Let $\vec{f}(t)$ be a vector function on an interval I and $t_0 \in I$ then $\underset{t\to t_0}{Lt} \frac{\vec{f}(t)-\vec{f}(t_0)}{t-t_0}$ if it exists, it is called the derivative of $\vec{f}(t)$ at t_0. and is denoted by $\vec{f}'(t_0)$ or $\left(\frac{d\vec{f}}{dt}\right)_{t=t_0}$

Also it is said that $\vec{f}(t)$ is differentiable at $t=t_0$

Note: 1. If $\vec{f}(t)$ is differentiable on I and $t_0 \in I$ then the derivative of $\vec{f}(t)$ at t is denoted by $\frac{d\vec{f}}{dt}$.

2. If the changes in t, $\vec{f}(t)$ are denoted by δt and $\delta\vec{f}(t)$ respectively then we have

$$\frac{d\vec{f}}{dt} = \underset{\delta t\to 0}{Lt} \frac{\delta\vec{f}}{\delta t} = \underset{\delta t\to 0}{Lt} \frac{\vec{f}(t+\delta t)-\vec{f}(t)}{\delta t}$$



Higher order Derivatives:

Let $\vec{f}(t)$ be differentiable on an interval I and $\vec{f}' = \frac{d\vec{f}}{dt}$ be the derivative of $\vec{f}$.

If $\underset{t\to t_0}{Lt} \frac{\vec{f}'(t)-\vec{f}'(t_0)}{t-t_0}$ exists for each $t \in I_1 \subset I$

Then $\vec{f}'$ is said to be differentiable on I_1

Also $\vec{f}(t)$ is said to possess second derivative on I_1 and is denoted by $\vec{f}''(t)$ or $\frac{d^2\vec{f}}{dt^2}$

Similarly the derivative of $\frac{d^2\vec{f}}{dt^2}$ is denoted by $\frac{d^3\vec{f}}{dt^3}$ and is called the third derivative of $\vec{f}(t)$ and so on.

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→ $\frac{d\vec{f}}{dt}, \frac{d^2\vec{f}}{dt^2}, \ldots$ are also denoted by $\dot{\vec{f}}, \ddot{\vec{f}}, \ldots$ respectively. (16)

→ Let $\vec{A}$, $\vec{B}$ and $\vec{C}$ be three differentiable vector functions of scalar variable 't' and ϕ is a differentiable scalar function of the same variable t, then

(1) $\frac{d}{dt}(\vec{A} \pm \vec{B}) = \frac{d\vec{A}}{dt} \pm \frac{d\vec{B}}{dt}$

(2) $\frac{d}{dt}(\vec{A} \cdot \vec{B}) = \vec{A} \cdot \frac{d\vec{B}}{dt} + \frac{d\vec{A}}{dt} \cdot \vec{B}$

(3) $\frac{d}{dt}(\vec{A} \times \vec{B}) = \vec{A} \times \frac{d\vec{B}}{dt} + \frac{d\vec{A}}{dt} \times \vec{B}$

(4) $\frac{d}{dt}(\phi \vec{A}) = \phi \frac{d\vec{A}}{dt} + \vec{A}\frac{d\phi}{dt}$

(5) $\frac{d}{dt}[\vec{A}\,\vec{B}\,\vec{C}] = \left[\frac{d\vec{A}}{dt}\,\vec{B}\,\vec{C}\right] + \left[\vec{A}\,\frac{d\vec{B}}{dt}\,\vec{C}\right] + \left[\vec{A}\,\vec{B}\,\frac{d\vec{C}}{dt}\right]$

(6) $\frac{d}{dt}\{\vec{A} \times (\vec{B} \times \vec{C})\} = \frac{d\vec{A}}{dt} \times (\vec{B} \times \vec{C}) + \vec{A} \times \left(\frac{d\vec{B}}{dt} \times \vec{C}\right) + \vec{A} \times \left(\vec{B} \times \frac{d\vec{C}}{dt}\right)$

Derivative of a constant vector:

A vector is said to be constant only if both its magnitude and direction are fixed. If either of these changes, then the vector will change and thus it will not be constant.

→ Let $\vec{A}$ be a constant vector function in the interval I and $t_0 \in I$ then $f'(t_0) = 0$

Sol: Let $\vec{f}(t) = C$, where C is a constant vector.

then $\lim_{t \to t_0} \frac{\vec{f}(t) - \vec{f}(t_0)}{t - t_0} = \lim_{t \to t_0} \frac{C - C}{t - t_0}$

$= \lim_{t \to t_0} (0) = 0$

$\therefore \left(\frac{d\vec{f}}{dt}\right)_{t=t_0} = \vec{f}'(t_0) = 0$

→ Derivative of a vector function in terms of its Components.

Let $\vec{r}$ be a vector function of the scalar variable t.

Let $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ where the components x, y, z are scalar functions of the scalar variable t and $\vec{i}, \vec{j}, \vec{k}$ are fixed unit vectors.

we have $\quad \vec{r} + \delta\vec{r} = (x+\delta x)\vec{i} + (y+\delta y)\vec{j} + (z+\delta z)\vec{k}$

$$\therefore \delta\vec{r} = (\vec{r} + \delta\vec{r}) - \vec{r} = \delta x\,\vec{i} + \delta y\,\vec{j} + \delta z\,\vec{k}.$$

$$\therefore \frac{\delta\vec{r}}{\delta t} = \frac{\delta x}{\delta t}\vec{i} + \frac{\delta y}{\delta t}\vec{j} + \frac{\delta z}{\delta t}\vec{k}.$$

$$\therefore \lim_{\delta t \to 0} \frac{\delta\vec{r}}{\delta t} = \lim_{\delta t \to 0} \left\{ \frac{\delta x}{\delta t}\vec{i} + \frac{\delta y}{\delta t}\vec{j} + \frac{\delta z}{\delta t}\vec{k} \right\}.$$

$$\therefore \frac{d\vec{r}}{dt} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}.$$

Thus in order to differentiate a vector we should differentiate its components.

<u>Note</u>: if $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, then sometimes we also write it as $\vec{r} = (x, y, z)$.

In this notation

$$\frac{d\vec{r}}{dt} = \left(\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right)$$

$$\frac{d^2\vec{r}}{dt^2} = \left(\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \frac{d^2z}{dt^2}\right) \text{, and so on.}$$

<u>Alternative method</u>:

we have $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$, where $\vec{i}, \vec{j}, \vec{k}$ are constant vectors and so their derivatives will be zero.

Now $\frac{d\vec{r}}{dt} = \frac{d}{dt}(x\vec{i} + y\vec{j} + z\vec{k})$

$= \frac{d}{dt}(x\vec{i}) + \frac{d}{dt}(y\vec{j}) + \frac{d}{dt}(z\vec{k})$ (7)

$= \frac{dx}{dt}\vec{i} + x\frac{d\vec{i}}{dt} + \frac{dy}{dt}\vec{j} + y\frac{d\vec{j}}{dt} + \frac{dz}{dt}\vec{k} + z\frac{d\vec{k}}{dt}$

$= \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k}$.

$\left(\because \frac{d\vec{i}}{dt} = \frac{d\vec{j}}{dt} = \frac{d\vec{k}}{dt} = 0\right)$

Note:— If f_1, f_2, f_3 are constant functions then $\vec{f} = f_1 i + f_2 j + f_3 k$ is called a constant vector function.

* Some important results:

⟶ A vector function $\vec{f}$ is constant iff $\frac{d\vec{f}}{dt} = 0$.

proof: suppose $\vec{f}$ is constant. Then $\frac{d\vec{f}}{dt} = 0$.

conversely suppose that $\frac{d\vec{f}}{dt} = 0$.

Let $\vec{f} = f_1 i + f_2 j + f_3 k$

since $\frac{d\vec{f}}{dt} = 0 \Rightarrow \frac{df_1}{dt} i + \frac{df_2}{dt} j + \frac{df_3}{dt} k = 0$

$\Rightarrow \frac{df_1}{dt} = 0, \frac{df_2}{dt} = 0, \frac{df_3}{dt} = 0$.

$\Rightarrow f_1, f_2, f_3$ are constants.

$\Rightarrow \vec{f}$ is a constant vector function.

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⟶ A vector function $\vec{f}$ is of constant magnitude iff $\vec{f} \cdot \frac{d\vec{f}}{dt} = 0$

proof Let $\vec{f}$ be a vector of constant magnitude.

Then $\vec{f} \cdot \vec{f} = |\vec{f}|^2 = $ constant.

Now diff. w.r.t. 't', we get $\vec{f} \cdot \frac{d\vec{f}}{dt} + \frac{d\vec{f}}{dt} \cdot \vec{f} = 0$

$\Rightarrow 2\left(\vec{f} \cdot \frac{d\vec{f}}{dt}\right) = 0 \quad (\because \vec{a}\cdot\vec{b} = \vec{b}\cdot\vec{a})$.

$$\Rightarrow \vec{f}\cdot\frac{d\vec{f}}{dt} = 0.$$

Conversely suppose that

$$\vec{f}\cdot\frac{d\vec{f}}{dt} = 0.$$

then $\frac{1}{2}\frac{d}{dt}(\vec{f}\cdot\vec{f}) = 0$

$$\Rightarrow \vec{f}\cdot\vec{f} = \text{const.}$$

$$\Rightarrow |\vec{f}|^2 = \text{const.}$$

$$\Rightarrow |\vec{f}| = \text{const.}$$

→ If $\vec{a}$ is a differentiable vector function of the scalar variable 't' and if $|\vec{a}| = a$, then

(i) $\frac{d}{dt}(\vec{a}^2) = 2a\frac{da}{dt}$ and (ii) $\vec{a}\cdot\frac{d\vec{a}}{dt} = a\frac{da}{dt}$.

Sol (i) we have $\vec{a}^2 = \vec{a}\cdot\vec{a}$

$$= (a)(a)\cos 0$$

$$= a^2$$

$$\therefore \frac{d}{dt}(\vec{a}^2) = \frac{d}{dt}(a^2) = 2a\frac{da}{dt}.$$

(ii) we have $\frac{d}{dt}(\vec{a}^2) = \frac{d}{dt}(\vec{a}\cdot\vec{a})$

$$= \frac{d\vec{a}}{dt}\cdot\vec{a} + \vec{a}\cdot\frac{d\vec{a}}{dt}$$

$$= 2\vec{a}\cdot\frac{d\vec{a}}{dt} \quad \text{--- (1)}$$

Also $\frac{d}{dt}(\vec{a}^2) = \frac{d}{dt}(a^2)$

$$= 2a\frac{da}{dt} \quad \text{--- (2)}$$

$\therefore$ from (1) & (2), we have

$$2\vec{a}\cdot\frac{d\vec{a}}{dt} = 2a\frac{da}{dt}$$

$$\Rightarrow \boxed{\vec{a}\cdot\frac{d\vec{a}}{dt} = a\frac{da}{dt}.}$$

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→ If $\vec{a}$ has constant length (fixed magnitude) then $\vec{a}$ and $\frac{d\vec{a}}{dt}$ are perpendicular provided $\left|\frac{d\vec{a}}{dt}\right| \neq 0$. (18)

sol. Let $|\vec{a}| = a$ (constant).

Then $\vec{a}\cdot\vec{a} = a^2$ (constant)

$\therefore \frac{d}{dt}(\vec{a}\cdot\vec{a}) = 0$

$\Rightarrow \frac{d\vec{a}}{dt}\cdot\vec{a} + \vec{a}\cdot\frac{d\vec{a}}{dt} = 0$

$\Rightarrow 2\vec{a}\cdot\frac{d\vec{a}}{dt} = 0$

$\Rightarrow \vec{a}\cdot\frac{d\vec{a}}{dt} = 0$.

$\therefore$ The scalar product of two vectors $\vec{a}$ and $\frac{d\vec{a}}{dt}$ is zero.

$\therefore \vec{a}$ is $\perp^{lar}$ to $\frac{d\vec{a}}{dt}$, provided $\frac{d\vec{a}}{dt}$ is not null vector

i.e provided $\left|\frac{d\vec{a}}{dt}\right| \neq 0$.

Thus the derivative of a vector of constant length is perpendicular to the vector itself provided the vector itself is not constant.

→ If $\vec{a}$ is a differentiable vector function of the scalar variable t, then

$$\frac{d}{dt}\left(\vec{a}\times\frac{d\vec{a}}{dt}\right) = \vec{a}\times\frac{d^2\vec{a}}{dt^2}.$$

sol. We have $\frac{d}{dt}\left(\vec{a}\times\frac{d\vec{a}}{dt}\right) = \frac{d\vec{a}}{dt}\times\frac{d\vec{a}}{dt} + \vec{a}\times\frac{d^2\vec{a}}{dt^2}$

$$= 0 + \vec{a}\times\frac{d^2\vec{a}}{dt^2}$$

$$= \vec{a}\times\frac{d^2\vec{a}}{dt^2}$$

→ A vector function $\vec{a}(t)$ has constant direction iff $\vec{a} \times \frac{d\vec{a}}{dt} = 0$.

proof Let $\vec{a}(t) = a(t)A(t)$

where $a(t) = |\vec{a}(t)|$

and $A(t)$ is a vector function with unit magnitude, for every t in the domain of $\vec{a}(t)$.

$$\therefore \frac{d\vec{a}}{dt} = \frac{d}{dt}(aA)$$

$$= a\frac{dA}{dt} + \frac{da}{dt}A$$

Now $\vec{a} \times \frac{d\vec{a}}{dt} = aA \times \left(a\frac{dA}{dt} + \frac{da}{dt}A\right)$

$$= \left(aA \times a\frac{dA}{dt}\right) +$$

$$aA \times \frac{da}{dt}A$$

$$= a^2\left(A \times \frac{dA}{dt}\right) + 0$$

$$= a^2\left(A \times \frac{dA}{dt}\right) \quad \text{--- (1)}$$

Suppose $\vec{a}(t)$ has constant direction.

Then A is constant

$$\Rightarrow \frac{dA}{dt} = 0$$

$$\Rightarrow A \times \frac{dA}{dt} = 0 \quad \text{--- (2)}.$$

$$\therefore \vec{a} \times \frac{d\vec{a}}{dt} = a^2\left(A \times \frac{dA}{dt}\right) \quad (\text{by } (1))$$

$$= a^2(0) \quad (\text{by } (2))$$

$$= 0.$$

$$\therefore \boxed{\vec{a} \times \frac{d\vec{a}}{dt} = 0}$$

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Conversely suppose that

$$\vec{a} \times \frac{d\vec{a}}{dt} = 0$$

Then $a^2 \left(A \times \frac{dA}{dt}\right) = 0$

$$\Rightarrow A \times \frac{dA}{dt} = 0.$$

Since A is of unit length

$$\therefore A \cdot \frac{dA}{dt} = 0$$

Now $A \times \frac{dA}{dt} = 0$, $A \cdot \frac{dA}{dt} = 0$

$$\Rightarrow \frac{dA}{dt} = 0$$

$\Rightarrow A$ is constant.

$\Rightarrow A$ has constant direction.

Ex: If $\vec{r}$ is a vector function of a scalar t and $\vec{a}$ is a constant vector, m a constant, differentiate the following w.r.t t:

(i) $\vec{r} \cdot \vec{a}$ (ii) $\vec{r} \times \vec{a}$ (iii) $\vec{r} \times \frac{d\vec{r}}{dt}$ (iv) $\vec{r} \cdot \frac{d\vec{r}}{dt}$.

(v) $\vec{r}^2 + \frac{1}{\vec{r}^2}$ (vi) $m\left(\frac{d\vec{r}}{dt}\right)^2$ (vii) $\frac{\vec{r}+\vec{a}}{\vec{r}^2+\vec{a}^2}$ (viii) $\frac{\vec{r}\times\vec{a}}{\vec{r}\cdot\vec{a}}$.

Sol^n: (i) Let $R = \vec{r}\cdot\vec{a}$

Then $\frac{dR}{dt} = \frac{d}{dt}(\vec{r}\cdot\vec{a})$

$$= \frac{d\vec{r}}{dt}\cdot\vec{a} + \vec{r}\cdot\frac{d\vec{a}}{dt}$$

$$= \frac{d\vec{r}}{dt}\cdot\vec{a} + \vec{r}\cdot 0 \quad \left(\because \frac{d\vec{a}}{dt} = 0, \text{ as } \vec{a} \text{ is constant vector}\right)$$

$$= \frac{d\vec{r}}{dt}\cdot\vec{a}$$

(iv) Let $R = \vec{r}\cdot\frac{d\vec{r}}{dt}$. Then $\frac{dR}{dt} = \frac{d}{dt}\left[\vec{r}\cdot\frac{d\vec{r}}{dt}\right]$

$$= \vec{r}\cdot\frac{d^2\vec{r}}{dt^2} + \frac{d\vec{r}}{dt}\cdot\frac{d\vec{r}}{dt}$$

$$= \vec{r}\cdot\frac{d^2\vec{r}}{dt^2} + \left(\frac{d\vec{r}}{dt}\right)^2$$

(v) Let $R = \vec{r}^2 + \frac{1}{\vec{r}^2}$. Then $\frac{dR}{dt} = \frac{d}{dt}\left\{\vec{r}^2 + \frac{1}{\vec{r}^2}\right\}$

$$= \frac{d}{dt}\left\{r^2 + \frac{1}{r^2}\right\} \quad \left(\because \vec{r} = x\vec{i} + y\vec{j} + z\vec{k} \right.$$

$$\Rightarrow r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$$

$$\Rightarrow r^2 = x^2+y^2+z^2 = \vec{r}\cdot\vec{r} = (\vec{r})^2 \Big)$$

$$= 2r\frac{dr}{dt} - \frac{2}{r^3}\frac{dr}{dt}$$

$$= 2\left(r - \frac{1}{r^3}\right)\frac{dr}{dt}$$

(vi) Let $R = m\left(\frac{d\vec{r}}{dt}\right)^2$. Then $\frac{dR}{dt} = m\frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)^2$

$$= m\frac{d}{dt}\left[\frac{d\vec{r}}{dt}\cdot\frac{d\vec{r}}{dt}\right]$$

$$= m\left[\frac{d^2\vec{r}}{dt^2}\cdot\frac{d\vec{r}}{dt} + \frac{d\vec{r}}{dt}\cdot\frac{d^2\vec{r}}{dt^2}\right]$$

$$= m\left[2\frac{d\vec{r}}{dt}\cdot\frac{d^2\vec{r}}{dt^2}\right]$$

$$= 2m\,\frac{d\vec{r}}{dt}\cdot\frac{d^2\vec{r}}{dt^2}$$

(vii) Let $R = \frac{\vec{r}+\vec{a}}{\vec{r}^2+\vec{a}^2}$. Then $\frac{dR}{dt} = \frac{d}{dt}\left[\frac{\vec{r}+\vec{a}}{\vec{r}^2+\vec{a}^2}\right]$

$$= \frac{1}{\vec{r}^2+\vec{a}^2}\left(\frac{d\vec{r}}{dt} + \frac{d\vec{a}}{dt}\right) + \left\{\frac{d}{dt}\left(\frac{1}{\vec{r}^2+\vec{a}^2}\right)\right\}(\vec{r}+\vec{a})$$

$$= \frac{1}{\vec{r}^2+\vec{a}^2}\left(\frac{d\vec{r}}{dt}\right) + \left\{\frac{(-1)}{(\vec{r}^2+\vec{a}^2)^2}\frac{d}{dt}(\vec{r}^2+\vec{a}^2)\right\}(\vec{r}+\vec{a})$$

$$= \frac{1}{\vec{r}^2+\vec{a}^2}\left(\frac{d\vec{r}}{dt}\right) + \left\{\frac{(-1)}{(\vec{r}^2+\vec{a}^2)^2}\,2\vec{r}\cdot\frac{d\vec{r}}{dt}\right\}(\vec{r}+\vec{a})$$

$$= \frac{1}{\vec{r}^2+\vec{a}^2}\left(\frac{d\vec{r}}{dt}\right) - \frac{2\vec{r}\cdot\frac{d\vec{r}}{dt}}{\vec{r}^2+\vec{a}^2}(\vec{r}+\vec{a})$$

(viii) Let $R = \frac{\vec{r}\times\vec{a}}{\vec{r}\cdot\vec{a}}$. Then $\frac{dR}{dt} = \frac{d}{dt}\left[\frac{\vec{r}\times\vec{a}}{\vec{r}\cdot\vec{a}}\right]$

$$= \frac{1}{\vec{r}\cdot\vec{a}}\frac{d}{dt}(\vec{r}\times\vec{a}) + \left\{\frac{d}{dt}\left(\frac{1}{\vec{r}\cdot\vec{a}}\right)\right\}(\vec{r}\times\vec{a})$$

$$= \frac{1}{\vec{r}\cdot\vec{a}}\left(\frac{d\vec{r}}{dt}\times\vec{a}\right) + \left[\frac{-1}{(\vec{r}\cdot\vec{a})^2}\left(\frac{d\vec{r}}{dt}\cdot\vec{a}\right)\right](\vec{r}\times\vec{a})$$

$$= \frac{\frac{d\vec{r}}{dt}\times\vec{a}}{\vec{r}\cdot\vec{a}} - \left\{\frac{\frac{d\vec{r}}{dt}\cdot\vec{a}}{(\vec{r}\cdot\vec{a})^2}(\vec{r}\times\vec{a})\right\}$$

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Problems :-

(28)

→ If $\vec{r} = \sin t\,\vec{i} + \cos t\,\vec{j} + t\vec{k}$, find

(i) $\frac{d\vec{r}}{dt}$ (ii) $\frac{d^2\vec{r}}{dt^2}$ (iii) $\left|\frac{d\vec{r}}{dt}\right|$ (iv) $\left|\frac{d^2\vec{r}}{dt^2}\right|$

(i) Given $\vec{r} = \sin t\,\vec{i} + \cos t\,\vec{j} + t\vec{k}$

$$\frac{d\vec{r}}{dt} = \frac{d}{dt}\left(\sin t\,\vec{i} + \cos t\,\vec{j} + t\vec{k}\right)$$
$$= \frac{d}{dt}(\sin t)\vec{i} + \frac{d}{dt}(\cos t)\vec{j} + \frac{d}{dt}(t)\vec{k}$$
$$= \cos t\,\vec{i} - \sin t\,\vec{j} + \vec{k}.$$

(ii) $$\frac{d^2\vec{r}}{dt^2} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right) = \frac{d}{dt}\left(\cos t\,\vec{i} - \sin t\,\vec{j} + \vec{k}\right)$$
$$= \frac{d}{dt}(\cos t)\vec{i} - \frac{d}{dt}(\sin t)\vec{j} + \frac{d}{dt}(1)\vec{k}$$
$$= -\sin t\,\vec{i} - \cos t\,\vec{j} + 0$$
$$= -\sin t\,\vec{i} - \cos t\,\vec{j}$$

(iii) $$\left|\frac{d\vec{r}}{dt}\right| = \sqrt{(\cos t)^2 + (-\sin t)^2 + (1)^2} = \sqrt{\cos^2 t + \sin^2 t + 1}$$
$$= \sqrt{1+1} = \sqrt{2}.$$

(iv) $$\left|\frac{d^2\vec{r}}{dt^2}\right| = \sqrt{(-\sin t)^2 + (-\cos t)^2}$$
$$= \sqrt{\sin^2 t + \cos^2 t}$$
$$= 1.$$

→ If $\vec{a}, \vec{b}$ are constant vectors, ω is a constant and $\vec{r}$ is a vector function of the scalar variable t given by $\vec{r} = \cos\omega t\,\vec{a} + \sin\omega t\,\vec{a}$.
Show that (i) $\frac{d^2\vec{r}}{dt^2} + \omega^2\vec{r} = 0$ (ii) $\vec{r} \times \frac{d\vec{r}}{dt} = \omega\,\vec{a}\times\vec{b}$.

Sol^n: Since $\vec{a}, \vec{b}$ are constant vectors.

$$\therefore \frac{d\vec{a}}{dt} = 0, \quad \frac{d\vec{b}}{dt} = 0$$

(i) $$\frac{d\vec{r}}{dt} = \frac{d}{dt}\left(\cos\omega t\,\vec{a} + \sin\omega t\,\vec{b}\right)$$

$$= \frac{d}{dt}(\cos\omega t)\vec{a} + \frac{d}{dt}(\sin\omega t)\vec{b}$$

$$\frac{d\vec{r}}{dt} = -\omega\sin\omega t\,\vec{a} + \omega\cos\omega t\,\vec{b}$$

$$\therefore \frac{d^2\vec{r}}{dt^2} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)$$

$$= \frac{d}{dt}\left(-\omega\sin\omega t\,\vec{a} + \omega\cos\omega t\,\vec{b}\right)$$

$$= \frac{d}{dt}(-\omega\sin\omega t)\vec{a} + \frac{d}{dt}(\omega\cos\omega t)\vec{b}$$

$$= -\omega\frac{d}{dt}(\sin\omega t)\vec{a} + \omega\frac{d}{dt}(\cos\omega t)\vec{b}$$

$$= -\omega(\omega\cos\omega t)\vec{a} + \omega(-\omega\sin\omega t)\vec{b}$$

$$= -\omega^2\cos\omega t\,\vec{a} - \omega^2\sin\omega t\,\vec{b}$$

$$= -\omega^2(\cos\omega t\,\vec{a} + \sin\omega t\,\vec{b})$$

$$\frac{d^2\vec{r}}{dt^2} = -\omega^2\vec{r} \qquad (\because \vec{r} = \cos\omega t\,\vec{a} + \sin\omega t\,\vec{b})$$

$$\therefore \frac{d^2\vec{r}}{dt^2} + \omega^2\vec{r} = 0.$$

(ii) $$\vec{r}\times\frac{d\vec{r}}{dt} = (\cos\omega t\,\vec{a} + \sin\omega t\,\vec{b})\times(-\omega\sin\omega t\,\vec{a} + \omega\cos\omega t\,\vec{b})$$

$$= -\omega\cos\omega t\sin\omega t\,\vec{a}\times\vec{a} + \omega\cos^2\omega t\,\vec{a}\times\vec{b} - \omega\sin^2\omega t\,\vec{b}\times\vec{a} + \omega\sin\omega t\cos\omega t\,\vec{b}\times\vec{b}$$

$$= \omega\cos^2\omega t\,\vec{a}\times\vec{b} - \omega\sin^2\omega t\,\vec{b}\times\vec{a} \qquad \left(\because \vec{a}\times\vec{a} = 0 \text{ \& } \vec{b}\times\vec{b} = 0\right)$$

$$= \omega\cos^2\omega t\,\vec{a}\times\vec{b} + \omega\sin^2\omega t\,\vec{a}\times\vec{b}$$

$$\because \vec{b}\times\vec{a} = -(\vec{a}\times\vec{b})$$

$$= \omega(\cos^2\omega t + \sin^2\omega t)\vec{a}\times\vec{b}$$

$$= \omega\,\vec{a}\times\vec{b}.$$

$$\therefore \vec{r}\times\frac{d\vec{r}}{dt} = \omega\,\vec{a}\times\vec{b}.$$

41 $\to$ If $\vec{r} = a\cos t\,\vec{i} + a\sin t\,\vec{j} + at\tan\alpha\,\vec{k}$, find $\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right|$ and $\left[\frac{d\vec{r}}{dt}, \frac{d^2\vec{r}}{dt^2}, \frac{d^3\vec{r}}{dt^3}\right]$ (21)

Sol^n: Given $\vec{r} = a\cos t\,\vec{i} + a\sin t\,\vec{j} + at\tan\alpha\,\vec{k}$

$$\frac{d\vec{r}}{dt} = -a\sin t\,\vec{i} + a\cos t\,\vec{j} + a\tan\alpha\,\vec{k}$$

$$\frac{d^2\vec{r}}{dt^2} = -a\cos t\,\vec{i} - a\sin t\,\vec{j}$$

$$\frac{d^3\vec{r}}{dt^3} = a\sin t\,\vec{i} - a\cos t\,\vec{j}$$

$$\therefore \frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2} = \begin{vmatrix} i & j & k \\ -a\sin t & a\cos t & a\tan\alpha \\ -a\cos t & -a\sin t & 0 \end{vmatrix}$$

$$= a^2\sin t\tan\alpha\,\vec{i} - a^2\cos t\tan\alpha\,\vec{j} + a^2\vec{k}$$

$$\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right| = \sqrt{(a^2\sin t\tan\alpha)^2 + (-a^2\cos t\tan\alpha)^2 + (a^2)^2}$$

$$= \sqrt{a^4\sin^2 t\tan^2\alpha + a^4\cos^2 t\tan^2\alpha + a^4}$$

$$= \sqrt{a^4\tan^2\alpha(\sin^2 t + \cos^2 t) + a^4}$$

$$= \sqrt{a^4\tan^2\alpha + a^4}$$

$$= \sqrt{a^4(1+\tan^2\alpha)} = \sqrt{a^4(\sec^2\alpha)}$$

$$= a^2\sec\alpha.$$

Also $\left[\frac{d\vec{r}}{dt}, \frac{d^2\vec{r}}{dt^2}, \frac{d^3\vec{r}}{dt^3}\right] = \left(\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right) \cdot \frac{d^3\vec{r}}{dt^3}$

$\left\{\because [\vec{a}\ \vec{b}\ \vec{c}] = (\vec{a}\times\vec{b})\cdot\vec{c}\right\}$

$$= \left(a^2\sin t\tan\alpha\,\vec{i} - a^2\cos t\tan\alpha\,\vec{j} + a^2\vec{k}\right)\cdot\left(a\sin t\,\vec{i} - a\cos t\,\vec{j}\right)$$

$$= a^3\sin^2 t\tan\alpha\,\vec{i}\cdot\vec{i} + a^3\cos^2 t\tan\alpha\,\vec{j}\cdot\vec{j}$$

$\left[\because \vec{i}\cdot\vec{j} = \vec{j}\cdot\vec{i} = \vec{k}\cdot\vec{i} \ldots\right]$

$= a^3 \sin^2 t \tan\alpha + a^3 \cos^2 t \tan\alpha$

$(\because \vec{i}\cdot\vec{i} = \vec{j}\cdot\vec{j} = 1)$

$= a^3 \tan\alpha (\sin^2 t + \cos^2 t)$

$= a^3 \tan\alpha$

$$\therefore \left[\frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\ \frac{d^3\vec{r}}{dt^3}\right] = a^3 \tan\alpha$$

→ If $\frac{d\vec{u}}{dt} = \vec{w}\times\vec{u}$, $\frac{d\vec{v}}{dt} = \vec{w}\times\vec{v}$, show that

$$\frac{d}{dt}(\vec{u}\times\vec{v}) = \vec{w}\times(\vec{u}\times\vec{v})$$

Sol^n: WKT

$$\frac{d}{dt}(\vec{u}\times\vec{v}) = \frac{d\vec{u}}{dt}\times\vec{v} + \vec{u}\times\frac{d\vec{v}}{dt}$$

$$= (\vec{w}\times\vec{u})\times\vec{v} + \vec{u}\times(\vec{w}\times\vec{v})$$

$\left[\because \frac{d\vec{u}}{dt} = \vec{w}\times\vec{u}\ \&\ \frac{d\vec{v}}{dt} = \vec{w}\times\vec{v}\right]$

$$= (\vec{v}\cdot\vec{w})\vec{u} - (\vec{v}\cdot\vec{u})\vec{w} + (\vec{u}\cdot\vec{v})\vec{w} - (\vec{u}\cdot\vec{w})\vec{v}$$

$$= (\vec{v}\cdot\vec{w})\vec{u} - (\vec{u}\cdot\vec{w})\vec{v} \quad (\because \vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u})$$

$$= \vec{w}\times(\vec{u}\times\vec{v})$$

2002 → If $\vec{R}$ be a unit vector in the direction of $\vec{r}$, prove that $\vec{R}\times\frac{d\vec{R}}{dt} = \frac{1}{r^2}\vec{r}\times\frac{d\vec{r}}{dt}$, where $r = |\vec{r}|$

Sol^n: We have $\vec{r} = r\vec{R}$;

$$\Rightarrow \vec{R} = \frac{1}{r}\vec{r}$$

$$\therefore \frac{d\vec{R}}{dt} = \frac{1}{r}\frac{d\vec{r}}{dt} - \frac{1}{r^2}\frac{dr}{dt}\vec{r}$$

Hence $\vec{R}\times\frac{d\vec{R}}{dt} = \frac{1}{r}\vec{r}\times\left(\frac{1}{r}\frac{d\vec{r}}{dt} - \frac{1}{r^2}\frac{dr}{dt}\vec{r}\right)$

$$= \frac{1}{r^2}\vec{r}\times\frac{d\vec{r}}{dt} - \frac{1}{r^3}\frac{dr}{dt}\vec{r}\times\vec{r}$$

$$= \frac{1}{r^2}\vec{r}\times\frac{d\vec{r}}{dt} \quad (\because \vec{r}\times\vec{r} = 0)$$

$$\therefore \vec{R}\times\frac{d\vec{R}}{dt} = \frac{1}{r^2}\vec{r}\times\frac{d\vec{r}}{dt}$$

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HW → If $\vec{A} = 5t^2\vec{i} + t\vec{j} - t^3\vec{k}$ and $\vec{B} = \sin t\,\vec{i} - \cos t\,\vec{j}$, find (a) $\frac{d}{dt}(\vec{A}\cdot\vec{B})$, (b) $\frac{d}{dt}(\vec{A}\times\vec{B})$, (c) $\frac{d}{dt}(\vec{A}\cdot\vec{A})$.

HW → If $\vec{A} = t^2\vec{i} - t\vec{j} + (2t+1)\vec{k}$ and $\vec{B} = (2t-3)\vec{i} + \vec{j} - t\vec{k}$, find (a) $\frac{d}{dt}(\vec{A}\cdot\vec{B})$ (b) $\frac{d}{dt}(\vec{A}\times\vec{B})$ (c) $\frac{d}{dt}|A+B|$, (d) $\frac{d}{dt}\left(\vec{A}\times\frac{d\vec{B}}{dt}\right)$ at $t=1$

HW → If $\vec{r} = e^{-t}\vec{i} + \log(t^2+1)\vec{j} - \tan t\,\vec{k}$ then find $\frac{d\vec{r}}{dt}$, $\frac{d^2\vec{r}}{dt^2}$, $\left|\frac{d\vec{r}}{dt}\right|$, $\left|\frac{d^2\vec{r}}{dt^2}\right|$ at $t=0$.

HW → If $\vec{r} = a\cos t\,\vec{i} + a\sin t\,\vec{j} + bt\vec{k}$ then find $\left|\frac{d\vec{r}}{dt}\right|$

HW → If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $x = 2\sin 3t$, $y = 2\cos 3t$, $z = 8t$ then show that $|\vec{r}'| = 10$ and $|\vec{r}''| = 18$

HW → If $\vec{A} = \sin t\,\vec{i} + \cos t\,\vec{j} + t\vec{k}$, $\vec{B} = \cos t\,\vec{i} - \sin t\,\vec{j} - 3\vec{k}$ and $\vec{C} = 2\vec{i} + 3\vec{j} - \vec{k}$ then find $\frac{d}{dt}\left[\vec{A}\times(\vec{B}\times\vec{C})\right]$ at $t=0$

HW → If $\vec{A} = 3t^2\vec{i} - (t+4)\vec{j} + (t^2 - 2t\vec{k})$ and $\vec{B} = \sin t\,\vec{i} + 3e^{-t}\vec{j} - 3\cos t\,\vec{k}$, find $\frac{d^2}{dt^2}(\vec{A}\times\vec{B})$ at $t=0$,

→ If $\vec{r} = e^t(c\cos 2t + d\sin 2t)$ where c and d are constant vectors, then show that $\frac{d^2\vec{r}}{dt^2} - 2\frac{d\vec{r}}{dt} + 5\vec{r} = 0$

→ If $\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$ and $x = \frac{2t+1}{t-1}$, $y = \frac{t^2}{t-1}$, $z = t+2$ then find $[\vec{r}'\ \vec{r}''\ \vec{r}''']$

→ If $\vec{r} = 2t\vec{i} + t^2\vec{j} + \frac{t^3}{3}\vec{k}$ then show that

$$\frac{[\vec{r}'\ \vec{r}''\ \vec{r}''']}{(\vec{r}'\times\vec{r}'')^2} = \frac{|\vec{r}'\times\vec{r}''|}{|\vec{r}'|^3} \text{ at } t=1$$

Differential Geometry

Differential Geometry involves a study of space curves and surfaces. It differs essentially from algebraic geometry which deals with a much narrower and restricted class of curves and surfaces and employs algebra as its principal tool.

For example:

The theory of conic sections or quadric surface come under the purview of algebraic geometry whereas the study of curvature of a general curve or the tangent plane to a general surface pertains to differential geometry.

→ parametric representation of space curves:

There are two ways of representing a curve analytically.

One of them is to represent it as the intersection of two surfaces represented by two equations of the form -

$$F_1(x,y,z)=0,\ F_2(x,y,z)=0. \quad (1)$$

The other one is the parametric representation of the form

$$x=f_1(t),\ y=f_2(t),\ z=f_3(t) \quad (2)$$

where x, y, z are scalar functions of the scalar 't', also represents a curve in three dimensional space.

Here (x, y, z) are co-ordinates of a current point of the curve.
The scalar variable 't' may range over a set of values $a \le t \le b$.

In vector notation, an equation of the form $\vec{r} = \vec{f}(t)$, represents a curve in three dimensional space if $\vec{r}$ is the position vector of a current point on the curve. As 't' changes, $\vec{r}$ will give position vectors of different points on the curve.

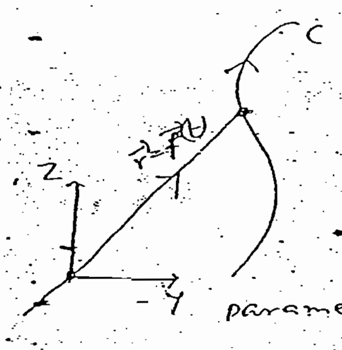
The vector $\vec{f}(t)$ can be expressed as $f_1(t)i + f_2(t)j + f_3(t)k$.
Also if (x, y, z) are the coordinates of a current point on the curve whose position vector is $\vec{r}$, then $\vec{r} = xi + yj + zk$.

- Therefore the single vector equation

$$\vec{r} = \vec{f}(t)$$

i.e $xi + yj + zk = f_1(t)i + f_2(t)j + f_3(t)k$

is equivalent to the three parametric equations $x = f_1(t)$, $y = f_2(t)$, $z = f_3(t)$.
Thus a curve in space may be defined as the locus of a point whose co-ordinates may be expressed as a function of a single parameter.



parametric representation curve.

* Typical examples

Kinds of curves

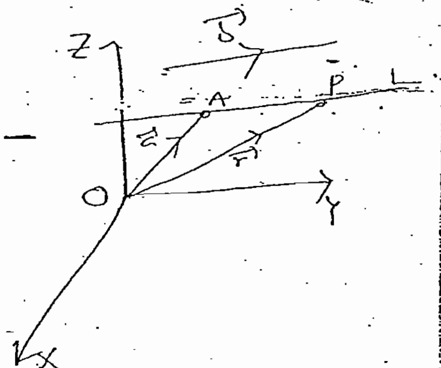
→ Straight line:

A straight line 'L' through a point with position vector $\vec{a}$ in the direction of a constant vector $\vec{b}$ can be represented in the form $\vec{r} = \vec{a} + t\vec{b}$

$$= [a_1 + tb_1, a_2 + tb_2, a_3 + tb_3]$$

where $\vec{a} = a_1 i + a_2 j + a_3 k$

$\vec{b} = b_1 i + b_2 j + b_3 k$; $t \in \mathbb{R}$.



Sol

Let O be the origin and $\vec{OA} = \vec{a}$.

Let 'L' be the straight line parallel to the given vector $\vec{b}$ passing through 'A'.

Let 'P' be the point on L and $\vec{OP} = \vec{r}$.

Then $\vec{AP} \parallel \vec{b}$.

$\therefore \vec{AP} = t\vec{b}$; $t \in \mathbb{R}$ — (1)

Since $\vec{AP} = \vec{OP} - \vec{OA}$

$\therefore \vec{OP} = \vec{AP} + \vec{OA}$

$\vec{OP} = \vec{a} + t\vec{b}$.

$\therefore \boxed{\vec{r} = \vec{a} + t\vec{b}}$; $t \in \mathbb{R}$

This is the required vector equation of the straight line.

→ Ellipse, circle:

The vector function $\vec{r}(t) = [a\cos t, b\sin t, 0]$

$= a\cos t\,\hat{i} + b\sin t\,\hat{j}$ —— (1)

represents an ellipse in the xy-plane with centre at the origin and principal axes in the direction of the x and y axes.

since $\cos^2 t + \sin^2 t = 1$

$\therefore$ (1) ≡ we have $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $z = 0$

If $b = a$, then (1) represents a circle of radius 'a'.

→ A plane curve is a curve that lies in a plane in space.

→ A curve that is not plane is called a twisted curve.

For example:

Circular helix

The twisted curve C represented by the vector

$\vec{r}(t) = [a\cos t, a\sin t, ct]$

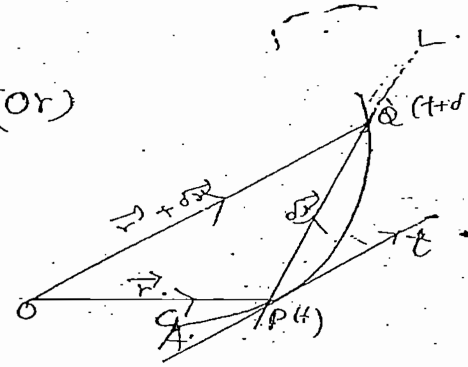
$= a\cos t\,\hat{i} + a\sin t\,\hat{j} + ct\hat{k}$; $c \neq 0$

is called a circular helix.

Tangent to a curve:

The tangent to a curve 'C' at a point 'P' of C is the limiting position of a straight line 'L' through P and a point Q of curve 'C' approaches P along 'C'.

(Or)



Let $\vec r = \vec f(t)$ be a curve 'C' and 'P' be a point on the curve.

If Q ($\neq$P) is a point on the curve then $\overleftrightarrow{PQ}$ is called a secant line. If the secant line $\overleftrightarrow{PQ}$ approaches the same limiting position as Q moves along the curve and approaches to 'P' from either sides, then the limiting position is called a tangent line to the curve at 'P'.

A vector parallel to the tangent at 'P' is called a tangent vector to the curve at 'P'.

Theorem → If $\vec r = \vec f(t)$ be a differentiable vector function represents a curve 'C' and 'P' is a point on the curve then the tangent vector to the curve at 'P' is $\frac{d\vec r}{dt}$.

proof —

Let $\vec{r}$, $\vec{r}+\delta\vec{r}$ be the position vectors of two neighbouring points P and Q on the curve 'c'.

Thus we have

$$\vec{OP} = \vec{r} = \vec{f}(t)$$

and
$$\vec{OQ} = \vec{r}+\delta\vec{r} = \vec{f}(t+\delta t).$$

$$\therefore \vec{PQ} = \vec{OQ} - \vec{OP}$$
$$= (\vec{r}+\delta\vec{r}) - \vec{r}$$
$$= \delta\vec{r}$$

$$\therefore \delta\vec{r} = \vec{OQ} - \vec{OP}$$

$$\boxed{\delta\vec{r} = \vec{f}(t+\delta t) - \vec{f}(t)}.$$

$$\Rightarrow \frac{\delta\vec{r}}{\delta t} = \frac{\vec{f}(t+\delta t) - \vec{f}(t)}{\delta t}$$

Thus, $\frac{\delta\vec{r}}{\delta t}$ is a vector parallel to the chord PQ.

As Q → P

i.e. δt → 0, chord PQ → tangent at P to the curve

$\therefore \lim\limits_{\delta t\to 0} \frac{\delta\vec{r}}{\delta t} = \frac{d\vec{r}}{dt}$ is a vector parallel to the tangent at 'P' to the curve $\vec{r} = \vec{f}(t)$.

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Unit tangent vector:

(26)

Suppose in place of the scalar parameter 't', we take the parameter as 's' where 's' denotes the arc length measured along the curve from some fixed point A on the curve.

Thus arc $AP = s$ and arc $AQ = s + \delta s$.

Then $\frac{d\vec{r}}{ds}$ will be a vector along the tangent at P to the curve and in the direction of s increasing.

Also

$$\left|\frac{d\vec{r}}{ds}\right| = \lim_{\delta s \to 0} \left|\frac{\delta \vec{r}}{\delta s}\right|$$

$$= \lim_{Q \to P} \frac{|\delta \vec{r}|}{\text{arc } PQ}$$

$$= \lim_{Q \to P} \frac{\text{Chord } PQ}{\text{arc } PQ}$$

$$= 1.$$

Thus $\frac{d\vec{r}}{ds}$ is a unit vector along the tangent at P in the direction of s increasing. We denote it by 'T'. i.e. $\boxed{T = \frac{d\vec{r}}{ds}}$

If x, y, z are cartesian co-ordinates of P we have

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}.$$

$$\Rightarrow \frac{d\vec{r}}{ds} = \frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} + \frac{dz}{ds}\hat{k}.$$

$$\Rightarrow T = \frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} + \frac{dz}{ds}\hat{k} \qquad \left(\because T = \frac{d\vec{r}}{ds}\right)$$

$$\Rightarrow |T| = \left|\frac{dx}{ds}\hat{i} + \frac{dy}{ds}\hat{j} + \frac{dz}{ds}\hat{k}\right|$$

$$\Rightarrow 1 = \sqrt{\left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2} \qquad (\because |T| = 1)$$

$$\Rightarrow 1 = \left(\frac{dx}{ds}\right)^2 + \left(\frac{dy}{ds}\right)^2 + \left(\frac{dz}{ds}\right)^2$$

$$\Rightarrow 1 = \left(\frac{dx}{dt}\right)^2\left(\frac{dt}{ds}\right)^2 + \left(\frac{dy}{dt}\right)^2\left(\frac{dt}{ds}\right)^2 + \left(\frac{dz}{dt}\right)^2\left(\frac{dt}{ds}\right)^2$$

where 't' is any parameter.

$$\Rightarrow \left(\frac{ds}{dt}\right)^2 = \left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2$$

$$\Rightarrow \left(\frac{ds}{dt}\right)^2 = \left|\frac{d\vec{r}}{dt}\right|^2$$

$$\Rightarrow \boxed{\frac{ds}{dt} = \left|\frac{d\vec{r}}{dt}\right|}$$

Note: we shall denote differentiation w.r.t arc length 's' by using primes (ie, dashes) and differentiation w.r.t any other parameter 't' with dots.

i.e, r' for $\frac{dr}{ds}$, r'' for $\frac{d^2r}{ds^2}$

and $\dot{r}$ for $\frac{dr}{dt}$, $\ddot{r}$ for $\frac{d^2r}{dt^2}$ etc.

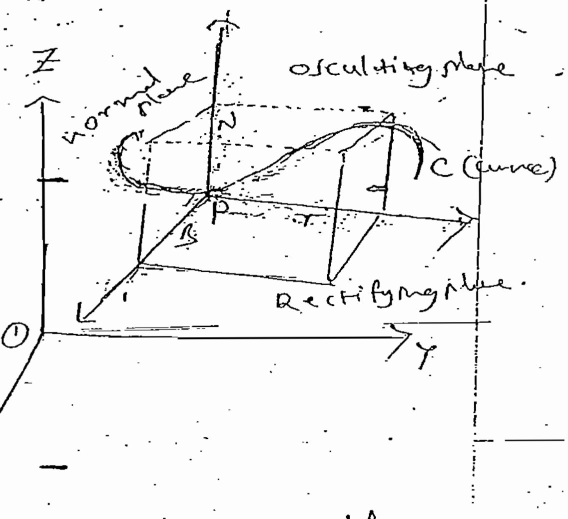
* Serret – Frenet formulae :-

The set of relations involving the derivatives of the fundamental vectors T, N, B is known collectively as the serret-frenet formulae given by

(1) $\frac{dT}{ds} = kN$; (2) $\frac{dB}{ds} = -\tau N$ - (3) $\frac{dN}{ds} = \tau B - kT$.

where τ is a scalar called the torsion.
The quantity $\sigma = \frac{1}{\tau}$ is called the radius of the torsion.

* principal Normal vector :-



Any line perpendicular to the tangent to a curve at a point 'p' is called a normal line at 'p'.
The normal line lying in the osculating plane is called the principal normal at 'p'.
– The unit principal normal is denoted by N.
– The osculating plane to a curve at a point is the plane containing the tangent and principal normal at 'p'.

Normal plane :-
The plane through the point 'p' perpendicular to the tangent at 'p' is called the normal plane at 'p'.

Rectifying plane :-

The plane through the point 'p' perpendicular to the principal normal is called rectifying plane.

Binormal !

Let T be the unit tangent vector, N be the unit principal normal vector to the curve at a point 'p'.
If B is the unit vector perpendicular to both T and N such that T, N, B form a right handed system then B is called binormal vector to the curve at 'p'.
Thus the binormal is the perpendicular to the osculating plane.

* Right-handed system of T, N, B !-

The vectors T, N, B form a right handed system of ~~unit~~ vectors

$\therefore\ T\cdot T = 1,\ N\cdot N = 1,\ B\cdot B = 1\ ;\ T\cdot N = N\cdot B = B\cdot T = 0.$

and $T\times N = B,\ N\times B = T,\ B\times T = N,$

$T\times T = N\times N = B\times B = 0.$

proof of Serret-Frenet formulae:

(1) $\frac{dT}{ds} = kN.$

Let $\vec{r}(t)$ be the position vector of the point 'p' on the curve $(\vec{r}(t) = \vec{r}(s))$, then the unit vector T at p is given by $\frac{d\vec{r}}{ds} = T.$

Since $|T| = 1$ i.e T is of constant magnitude

we have $T\cdot\frac{dT}{ds} = 0.$

$\therefore \frac{dT}{ds}$ is perpendicular to T.

But we know that $\frac{dT}{ds}$ lies in the osculating plane.

$\therefore \frac{dT}{ds}$ is parallel to $N \Rightarrow \frac{dT}{ds} = k N$ for some scalar k.

By convention, we take +ve sign.

$\Rightarrow \frac{dT}{ds} = kN$. for some scalar 'k'. —— (1)

<u>curvature</u>:- If T is the unit tangent vector to the curve $\bar{r}(s)$ at a point 'p' then the rate of change of 'T' w.r.t 's' is called curvature of the curve at 'p'. It is denoted by k.

The reciprocal of k is called radius of curvature of the curve at 'p'. It is denoted by ρ i.e $\rho = \frac{1}{k}$.

<u>Note</u>: $\left|\frac{dT}{ds}\right| = k$. (from (1), $|N|=1$).

(2) $\frac{dB}{ds} = -\tau N$.

since $|B|=1$

i.e B is of constant magnitude.

$\therefore B \cdot \frac{dB}{ds} = 0$.

$\therefore \frac{dB}{ds}$ is perpendicular to B

we know that $\frac{dB}{ds}$ lies in the osculating plane

Now we have

$B \cdot T = 0$

$\Rightarrow B \cdot \frac{dT}{ds} + \frac{dB}{ds} \cdot T = 0$ (by diff w.r.t s).

$\Rightarrow B \cdot (kN) + \frac{dB}{ds} \cdot T = 0$. ($\because \frac{dT}{ds} = kN$).

$\Rightarrow \frac{dB}{ds} \cdot T + (B \cdot N) k = 0$

$\Rightarrow \frac{dB}{ds} \cdot T = 0 \quad (\because B \cdot N = 0)$

$\Rightarrow T \cdot \frac{dB}{ds} = 0$

$\Rightarrow \frac{dB}{ds}$ is $\perp$ to T.

Since $\frac{dB}{ds}$ lies in the osculating plane

so it must be parallel to N.

$\therefore \frac{dB}{ds} = \pm \tau N$.

$\therefore$ By convention, $\frac{dB}{ds} = -\tau N$. (ii)

Torsion:- If B is the binormal vector to the curve $\vec{r}(s)$ at a point 'p' then the rate of change of B w.r.t. 's' is called torsion of the curve at 'p'. It is denoted by τ.

The reciprocal of τ is called the radius of torsion and is denoted by σ

i.e. $\sigma = \frac{1}{\tau}$.

Note:- $\left| \frac{dB}{ds} \right| = \tau \quad$ (from (ii); $|N| = 1$).

(3) $\frac{dN}{ds} = \tau B - kT$

Now we have $B \times T = N$.

$\Rightarrow B \times \frac{dT}{ds} + \frac{dB}{ds} \times T = \frac{dN}{ds}$ (by diff. w.r.t. s)

$\Rightarrow B \times (kN) + (-\tau N) \times T = \frac{dN}{ds}$

$\Rightarrow \frac{dN}{ds} = k(B \times N) - \tau (N \times T)$

$= k(-T) - \tau(-B)$

$= \tau B - kT$

→ If k is the curvature and τ is the torsion of a curve $\vec{r}(s)$ then $k = \left|\frac{d\vec{r}}{ds} \times \frac{d^2\vec{r}}{ds^2}\right|$ and

$$\tau = \left[\frac{d\vec{r}}{ds}\ \frac{d^2\vec{r}}{ds^2}\ \frac{d^3\vec{r}}{ds^3}\right] \Big/ \left|\frac{d\vec{r}}{ds} \times \frac{d^2\vec{r}}{ds^2}\right|^2$$

Sol^n: We know that $T = \frac{d\vec{r}}{ds}$

and $kN = \frac{d^2\vec{r}}{ds^2}$

$$\text{Now } \frac{d\vec{r}}{ds} \times \frac{d^2\vec{r}}{ds^2} = T \times kN$$

$$= k(T \times N)$$

$$= kB \qquad (\because T \times N = B)$$

$$\therefore k = \left|\frac{d\vec{r}}{ds} \times \frac{d^2\vec{r}}{ds^2}\right| \quad \text{--- (1)}$$

$$\frac{d^3\vec{r}}{ds^3} = \frac{d}{ds}\left(\frac{d^2\vec{r}}{ds^2}\right) = \frac{d}{ds}(kN)$$

$$= k\frac{dN}{ds} + \frac{dk}{ds}N$$

$$= k(\tau B - kT) + \frac{dk}{ds}N$$

$$\left(\because \frac{dN}{ds} = \tau B - kT\right)$$

$$= k\tau B - k^2 T + \frac{dk}{ds}N$$

$$\left[\frac{d\vec{r}}{ds}\ \frac{d^2\vec{r}}{ds^2}\ \frac{d^3\vec{r}}{ds^3}\right] = \left(\frac{d\vec{r}}{ds} \times \frac{d^2\vec{r}}{ds^2}\right) \cdot \frac{d^3\vec{r}}{ds^3} \qquad \left(\because [\vec{a}\ \vec{b}\ \vec{c}] = (\vec{a} \times \vec{b}) \cdot \vec{c}\right)$$

$$= kB \cdot \left(k\tau B - k^2 T + \frac{dk}{ds}N\right)$$

$$= k^2\tau\, B \cdot B - k^3 (B \cdot T) + k\frac{dk}{ds}(B \cdot N)$$

$$= k^2\tau \qquad (\because B \cdot B = 1;\ B \cdot T = 0 \text{ and } B \cdot N = 0)$$

$$\Rightarrow \tau = \frac{\left[\frac{d\vec{r}}{ds}\ \frac{d^2\vec{r}}{ds^2}\ \frac{d^3\vec{r}}{ds^3}\right]}{k^2}$$

$$\therefore \tau = \frac{\left[\frac{d\vec{r}}{ds}\ \frac{d^2\vec{r}}{ds^2}\ \frac{d^3\vec{r}}{ds^3}\right]}{\left|\frac{d\vec{r}}{ds} \times \frac{d^2\vec{r}}{ds^2}\right|^2} \qquad \text{(by ①)}$$

→ If κ is the curvature and τ is the torsion of a curve $\vec{r}(t)$ then $k = \left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right| \Big/ \left|\frac{d\vec{r}}{dt}\right|^3$

and $\tau = \left[\frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\ \frac{d^3\vec{r}}{dt^3}\right] \Big/ \left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right|^2$

Sol^n: W.K.T $\left|\frac{d\vec{r}}{dt}\right| = \frac{ds}{dt}$

$$\frac{d\vec{r}}{ds} = \frac{d\vec{r}}{dt}\frac{dt}{ds} \quad \text{—— ①}$$

$$\frac{d^2\vec{r}}{ds^2} = \frac{d}{ds}\left(\frac{d\vec{r}}{ds}\right)$$

$$= \frac{d}{ds}\left(\frac{d\vec{r}}{dt}\frac{dt}{ds}\right)$$

$$= \frac{d\vec{r}}{dt}\frac{d^2t}{ds^2} + \frac{d^2\vec{r}}{dt^2}\left(\frac{dt}{ds}\right)^2$$

$$\frac{d^3\vec{r}}{ds^3} = \frac{d\vec{r}}{dt}\frac{d^3t}{ds^3} + \frac{d^2\vec{r}}{dt^2}\frac{dt}{ds}\frac{d^2t}{ds^2} + 2\frac{d^2\vec{r}}{dt^2}\frac{dt}{ds}\frac{d^2t}{ds^2}$$

$$+ \frac{d^3\vec{r}}{dt^3}\left(\frac{dt}{ds}\right)^3$$

$$= \frac{d\vec{r}}{dt}\frac{d^3t}{ds^3} + 3\frac{d^2\vec{r}}{dt^2}\frac{dt}{ds}\frac{d^2t}{ds^2} + \frac{d^3\vec{r}}{dt^3}\left(\frac{dt}{ds}\right)^3$$

Now $\frac{d\vec{r}}{ds} \times \frac{d^2\vec{r}}{ds^2} = \frac{d\vec{r}}{dt}\frac{dt}{ds} \times \left[\frac{d\vec{r}}{dt}\frac{d^2t}{ds^2} + \frac{d^2\vec{r}}{dt^2}\left(\frac{dt}{ds}\right)^2\right]$

$$= \left(\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right)\left(\frac{dt}{ds}\right)^3$$

$$\therefore k = \left|\frac{d\vec{r}}{ds} \times \frac{d^2\vec{r}}{ds^2}\right|$$

$$= \left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right| \Big/ \left(\frac{ds}{dt}\right)^3$$

$$= \left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right| \Big/ \left|\frac{d\vec{r}}{dt}\right|^3$$

$$\left[\frac{d\vec{r}}{ds}\ \frac{d^2\vec{r}}{ds^2}\ \frac{d^3\vec{r}}{ds^3}\right] = \left(\frac{d\vec{r}}{ds} \times \frac{d^2\vec{r}}{ds^2}\right)\cdot\frac{d^3\vec{r}}{ds^3}$$

$$= \left(\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right)\left(\frac{dt}{ds}\right)^3 \cdot \left[\frac{d\vec{r}}{dt}\frac{d^3t}{ds^3} + 3\frac{d^2\vec{r}}{dt^2}\frac{dt}{ds}\frac{d^2t}{ds^2} + \frac{d^3\vec{r}}{dt^3}\left(\frac{dt}{ds}\right)^3\right]$$

$$= \left[\left(\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right)\cdot\frac{d^3\vec{r}}{dt^3}\right]\left(\frac{dt}{ds}\right)^6$$

$$= \left[\frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\ \frac{d^3\vec{r}}{dt^3}\right]\cdot\left(\frac{dt}{ds}\right)^6$$

$$\tau = \frac{\left[\frac{d\vec{r}}{ds}\ \frac{d^2\vec{r}}{ds^2}\ \frac{d^3\vec{r}}{ds^3}\right]}{\left|\frac{d\vec{r}}{ds} \times \frac{d^2\vec{r}}{ds^2}\right|^2}$$

$$= \frac{\left[\frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\ \frac{d^3\vec{r}}{dt^3}\right]\left(\frac{dt}{ds}\right)^6}{\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right|^2\left(\frac{dt}{ds}\right)^6}$$

$$= \frac{\left[\frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\ \frac{d^3\vec{r}}{dt^3}\right]}{\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right|^2}$$

2004 → Show that the Frenet-Serret formulae can be written in the form $\frac{d\vec{T}}{ds} = \vec{\omega} \times \vec{T}$, $\frac{d\vec{N}}{ds} = \vec{\omega} \times \vec{N}$ and $\frac{d\vec{B}}{ds} = \vec{\omega} \times \vec{B}$

where $\vec{\omega} = \tau \vec{T} + k\vec{B}$.

Sol^n: $\vec{\omega} \times \vec{T} = (\tau \vec{T} + k\vec{B}) \times \vec{T}$

$= \tau \vec{T} \times \vec{T} + k\vec{B} \times \vec{T}$

$= 0 + k\vec{N} \quad (\because \vec{B} \times \vec{T} = \vec{N})$

$= k\vec{N}$

$= \frac{d\vec{T}}{ds}$

$\vec{\omega} \times \vec{N} = (\tau \vec{T} + k\vec{B}) \times \vec{N}$

$= \tau \vec{T} \times \vec{N} + k\vec{B} \times \vec{N}$

$= \tau \vec{B} - k\vec{T}$

$= \frac{d\vec{N}}{ds}$

TNB

$\vec{\omega} \times \vec{N} = (\tau \vec{T} + k\vec{B}) \times \vec{B}$

$= \tau \vec{T} \times \vec{B} + k\vec{B} \times \vec{B}$

$= -\tau \vec{B} + 0$

$= -\tau B$

61 : problems

⟶ for the curve $x = 3\cos t,\ y = 3\sin t,\ z = 4t$,
find T, N, B and k, τ.

Sol. The position vector for any point on the curve is $\vec{r} = 3\cos t\,i + 3\sin t\,j + (4t)k$.

$$\therefore \frac{d\vec{r}}{dt} = -3\sin t\,i + 3\cos t\,j + 4k \quad \text{--- (1)}$$

Now $\frac{ds}{dt} = \left|\frac{d\vec{r}}{dt}\right|$

$$= \sqrt{9\sin^2 t + 9\cos^2 t + 16}$$

$$= \sqrt{9(1) + 16}$$

$$= \underline{5}$$

$\frac{d\vec{r}}{ds} = \frac{d\vec{r}/dt}{ds/dt}$

Now $T = \frac{d\vec{r}}{ds} = \frac{d\vec{r}/dt}{ds/dt} = \frac{1}{5}\left(-3\sin t\,i + 3\cos t\,j + 4k\right)$.

$\frac{dT}{dt}$

$$\therefore \frac{dT}{dt} = \frac{1}{5}\left(-3\cos t\,i - 3\sin t\,j\right)$$

$$\therefore \frac{dT}{ds} = \frac{dT/dt}{ds/dt} = \frac{1}{25}\left(-3\cos t\,i - 3\sin t\,j\right).$$

$\therefore$ curvature $k = \left|\frac{dT}{ds}\right|$

$$= \sqrt{\left(\frac{-3\cos t}{25}\right)^2 + \left(\frac{-3}{25}\sin t\right)^2}$$

$$\boxed{k = \frac{3}{25}}$$

we know that

$\frac{dT}{ds} = kN$

$\Longrightarrow N = \frac{1}{k}\frac{dT}{ds}$

$$= \frac{25}{3}\left(\frac{1}{25}\right)\left(-3\cos t\,i - 3\sin t\,j\right)$$

$$= \frac{1}{3}\left(-3\cos t\,i - 3\sin t\,j\right).$$

$= -\cos t\,\hat{i} - \sin t\,\hat{j}$

$B = T \times N$

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -\frac{3}{5}\sin t & \frac{3}{5}\cos t & \frac{4}{5} \\ -\cos t & -\sin t & 0 \end{vmatrix} = \frac{1}{5}\left(4\sin t\,\hat{i} - 4\cos t\,\hat{j} + 3\hat{k}\right)$$

$$\therefore \frac{dB}{dt} = \frac{1}{5}\left(4\cos t\,\hat{i} + 4\sin t\,\hat{j}\right)$$

Now $\frac{dB}{ds} = \frac{dB/dt}{ds/dt} = \frac{1}{25}\left(4\cos t\,\hat{i} + 4\sin t\,\hat{j}\right)$

$= \frac{4}{25}(\cos t\,\hat{i} + \sin t\,\hat{j})$.

we know that

$$\frac{dB}{ds} = -\tau N.$$

$$\Rightarrow \frac{4}{25}(\cos t\,\hat{i} + \sin t\,\hat{j}) = -\tau\left(-\cos t\,\hat{i} - \sin t\,\hat{j}\right)$$

$$\Rightarrow \boxed{\frac{4}{25} = \tau}$$

Note :- whenever only finding k and τ, you can better to use the formulas $k = \frac{\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right|}{\left|\frac{d\vec{r}}{dt}\right|^3}$ &

$$\tau = \frac{\left[\frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\ \frac{d^3\vec{r}}{dt^3}\right]}{\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right|^2}$$

63. HW → For the curve
IPS-2005 $x = a\cos t,\ y = a\sin t,\ z = bt$. Show that

$$K = \frac{a}{a^2+b^2},\quad \tau = \frac{b}{a^2+b^2}.$$

→ Show that the torsion τ for the space curve $x = \frac{2t+1}{t-1},\ y = \frac{t^2}{t-1},\ z = t+2$ is zero.

→ Find the curvature K and torsion τ for the space curve $x = t - \sin t$,
$y = 1 - \cos t$,
$z = t$.

→ Find the curvature K and torsion τ for the space curve $\vec{r} = e^t\,\hat{i} - e^{-t}\,\hat{j} + \sqrt{2}\,t\,\hat{k}$.

2002 → Find the curvature K for the space curve: $\cancel{x = a\cos\theta}$, $y = a\sin\theta$, $\cancel{z = a\theta\tan\alpha}$.

IPS 2005 → Find the curvature and the torsion of the space curve $\cancel{x = a(3u - u^3)}$,
$y = 3au^2$,
$z = a(3u + u^3)$.

2007 → Find the curvature and torsion at any point of the curve $x = a\cos 2t,\ y = a\sin 2t$,
$z = 2a\sin t$.

→ Find T, N, B, Curvature K; torsion τ for the space curve
(i): $x = t,\ y = t^2,\ z = \frac{2}{3}t^3$ at $t = 1$.

IPS 2007 (ii) $x = a\cos\theta,\ y = a\sin\theta,\ z = a\theta\cot\alpha$.

(32)

→ Find the Curvature κ and τ for the Space Curve.

(i) $r = 3ti + 3t^2 j + 2t^3 k$ (ii) $r = (3t - t^3)i + 3t^2 j + (3t + t^3)k$

(iii) $r = ti + t^2 j + t^3 k$ at $t = 1$. (iv) $x = \frac{t}{2}$, $y = \frac{t^2}{4}$, $z = \frac{t^3}{12}$

(v) $x = t - \frac{t^3}{3}$, $y = t^2$, $z = t + \frac{t^3}{3}$

(vi) $x = \theta - \sin\theta$, $y = 1 - \cos\theta$, $z = 4\sin\theta/2$

(vii) $x = t$, $y = t^2$, $z = 2t^3/3$ at $(\sqrt{3}, 3, 2\sqrt{3})$.

→ Find the unit tangent vector at any point on the curve $x = t^2 + 2$, $y = 4t - 5$, $z = 2t^2 - 6t$. Also determine the unit tangent vector at $t = 2$.

BPS-2002
→ for the space curve $x = t - \frac{1}{3}t^3$, $y = t^2$, $z = t + \frac{1}{3}t^3$ show that

$$\kappa = \tau = \frac{1}{(1+t^2)^2}.$$

→ for the curve $\vec{r} = e^u(i - j) + \sqrt{2}\,uk$, show that $\kappa = -\tau = \frac{\sqrt{2}}{(e^u + e^{-u})^2}$.

BPS-2001
BAS-2008
→ for the space curve $x = t$, $y = t^2$, $z = \frac{2}{3}t^3$ find the values of κ and τ at $t = 1$.

Set II

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Gradient, Divergence and Curl

§ 1. Partial Derivatives of Vectors.

Suppose $\mathbf{r}$ is a vector depending on more than one scalar variable. Let $\mathbf{r}=\mathbf{f}(x, y, z)$ i.e. let $\mathbf{r}$ be a function of three scalar variables x, y and z. The partial derivative of $\mathbf{r}$ with respect to x is defined as

$$\frac{\partial \mathbf{r}}{\partial x}=\lim_{\delta x \to 0} \frac{\mathbf{f}(x+\delta x, y, z)-\mathbf{f}(x, y, z)}{\delta x}$$

if this limit exists. Thus $\partial \mathbf{r}/\partial x$ is nothing ~~but~~ the ordinary derivative of $\mathbf{r}$ with respect to x provided the other variables y and z are regarded as constants. Similarly we may define the partial derivatives $\dfrac{\partial \mathbf{r}}{\partial y}$ and $\dfrac{\partial \mathbf{r}}{\partial z}$.

Higher partial derivatives can also be defined as in Scalar Calculus. Thus, for example,

$$\frac{\partial^2 \mathbf{r}}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial \mathbf{r}}{\partial x}\right), \frac{\partial^2 \mathbf{r}}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial \mathbf{r}}{\partial y}\right), \frac{\partial^2 \mathbf{r}}{\partial z^2}=\frac{\partial}{\partial z}\left(\frac{\partial \mathbf{r}}{\partial z}\right),$$

$$\frac{\partial^2 \mathbf{r}}{\partial x\, \partial y}=\frac{\partial}{\partial x}\left(\frac{\partial \mathbf{r}}{\partial y}\right), \frac{\partial^2 \mathbf{r}}{\partial y\, \partial x}=\frac{\partial}{\partial y}\left(\frac{\partial \mathbf{r}}{\partial x}\right).$$

If $\mathbf{r}$ has continuous partial derivatives of the second order at least, then, $\dfrac{\partial^2 \mathbf{r}}{\partial x\, \partial y}=\dfrac{\partial^2 \mathbf{r}}{\partial y\, \partial x}$ i.e. the order of differentiation is immaterial. If $\mathbf{r}=\mathbf{f}(x, y, z)$ the total differential $d\mathbf{r}$ of $\mathbf{r}$ is given by

$$d\mathbf{r}=\frac{\partial \mathbf{r}}{\partial x} dx+\frac{\partial \mathbf{r}}{\partial y} dy+\frac{\partial \mathbf{r}}{\partial z} dz.$$

§ 2. The Vector Differential Operator Del. (∇).

The vector differential operator ∇ (read as *del* or *nabla*) is defined as

$$\nabla \equiv \frac{\partial}{\partial x} \mathbf{i}+\frac{\partial}{\partial y} \mathbf{j}+\frac{\partial}{\partial z} \mathbf{k} \equiv \mathbf{i} \frac{\partial}{\partial x}+\mathbf{j} \frac{\partial}{\partial y}+\mathbf{k} \frac{\partial}{\partial z}$$

and operates distributively.

The vector operator ∇ can generally be treated to behave as an ordinary vector. It possesses properties like ordinary vectors. The symbols $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ can be treated as its components along **i, j, k**.

§ 3. Gradient of a scalar Field. Definition.

Let $f(x, y, z)$ be defined and differentiable at each point (x, y, z) in a certain region of space (i.e., defines a differentiable scalar field). Then the gradient of f, written as ∇f or grad f, is defined as

$$\nabla f = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k} \right) f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}.$$

[Kerala 1975; Allahabad 79]

It should be noted that ∇f is a vector whose three successive components are $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$. Thus the gradient of a scalar field defines a vector field. *If f a scalar point function, then ∇f is a vector point function.*

§ 4. Formulas involving gradient.

Theorem 1. Gradient of the sum of two scalar point functions. *If f and g are two scalar point functions, then*

$$grad\,(f+g) = grad\, f + grad\, g$$

or

$$\nabla (f+g) = \nabla f + \nabla g.$$

Proof. We have $\nabla (f+g) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \right)(f+g)$

$$= \mathbf{i}\frac{\partial}{\partial x}(f+g) + \mathbf{j}\frac{\partial}{\partial y}(f+g) + \mathbf{k}\frac{\partial}{\partial z}(f+g)$$

$$= \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{i}\frac{\partial g}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{j}\frac{\partial g}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z} + \mathbf{k}\frac{\partial g}{\partial z}$$

$$= \left(\mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z} \right) + \left(\mathbf{i}\frac{\partial g}{\partial x} + \mathbf{j}\frac{\partial g}{\partial y} + \mathbf{k}\frac{\partial g}{\partial z} \right)$$

$$= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \right) f + \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} \right) g$$

$$= \nabla f + \nabla g = \text{grad } f + \text{grad } g.$$

Similarly, we can prove that $\nabla (f-g) = \nabla f - \nabla g$.

Theorem 2. Gradient of a constant. *The necessary and sufficient condition for a scalar point function to be constant is that*

$$\nabla f = 0.$$

Proof. If $f(x, y, z)$ is constant, then

$$\frac{\partial f}{\partial x}=0, \frac{\partial f}{\partial y}=0, \frac{\partial f}{\partial z}=0.$$

Therefore grad $f=\mathbf{i}\frac{\partial f}{\partial x}+\mathbf{j}\frac{\partial f}{\partial y}+\mathbf{k}\frac{\partial f}{\partial z}=0\mathbf{i}+0\mathbf{j}+0\mathbf{k}=\mathbf{0}$.

Hence the condition is necessary.

Conversely, let grad $f=\mathbf{0}$. Then $\mathbf{i}\frac{\partial f}{\partial x}+\mathbf{j}\frac{\partial f}{\partial y}+\mathbf{k}\frac{\partial f}{\partial z}=\mathbf{0}$.

Therefore $\frac{\partial f}{\partial x}=0, \frac{\partial f}{\partial y}=0, \frac{\partial f}{\partial z}=0.$

$\therefore$ f must be independent of x, y and z.

$\therefore$ f must be a constant. Hence the condition is sufficient.

Theorem 3. Gradient of the product of two scalar point functions. *If f and g are scalar point functions, then*

$$\text{grad}\,(fg)=f\,\text{grad}\,g+g\,\text{grad}\,f$$

or

$$\nabla(fg)=f\nabla g+g\nabla f.$$

[Meerut 1972; Bombay 69]

Proof. We have $\nabla(fg)=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)(fg)$

$$=\mathbf{i}\frac{\partial}{\partial x}(fg)+\mathbf{j}\frac{\partial}{\partial y}(fg)+\mathbf{k}\frac{\partial}{\partial z}(fg)$$

$$=\mathbf{i}\left(f\frac{\partial g}{\partial x}+g\frac{\partial f}{\partial x}\right)+\mathbf{j}\left(f\frac{\partial g}{\partial y}+g\frac{\partial f}{\partial y}\right)+\mathbf{k}\left(f\frac{\partial g}{\partial z}+g\frac{\partial f}{\partial z}\right)$$

$$=f\left(\mathbf{i}\frac{\partial g}{\partial x}+\mathbf{j}\frac{\partial g}{\partial y}+\mathbf{k}\frac{\partial g}{\partial z}\right)+g\left(\mathbf{i}\frac{\partial f}{\partial x}+\mathbf{j}\frac{\partial f}{\partial y}+\mathbf{k}\frac{\partial f}{\partial z}\right)$$

$$=f\nabla g+g\nabla f=f\,\text{grad}\,g+g\,\text{grad}\,f.$$

In particular if c is a constant, then

$$\nabla(cf)=c\nabla f+f\nabla c=c\nabla f+\mathbf{0}=c\nabla f.$$

Theorem 4. Gradient of the Quotient of two scalar functions. *If f and g are two scalar point functions, then*

$$\nabla\left(\frac{f}{g}\right)=\frac{g\nabla f-f\nabla g}{g^2}$$

[Jiwaji 1982]

Proof. We have $\nabla\left(\frac{f}{g}\right)=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\left(\frac{f}{g}\right)$

$$=\mathbf{i}\frac{\partial}{\partial x}\left(\frac{f}{g}\right)+\mathbf{j}\frac{\partial}{\partial y}\left(\frac{f}{g}\right)+\mathbf{k}\frac{\partial}{\partial z}\left(\frac{f}{g}\right).$$

But $\frac{\partial}{\partial x}\left(\frac{f}{g}\right)=\frac{g\frac{\partial f}{\partial x}-f\frac{\partial g}{\partial x}}{g^2}, \frac{\partial}{\partial y}\left(\frac{f}{g}\right)=\frac{g\frac{\partial f}{\partial y}-f\frac{\partial g}{\partial y}}{g^2},$

and $\dfrac{\partial}{\partial z}\left(\dfrac{f}{g}\right)=\dfrac{g\dfrac{\partial f}{\partial z}-f\dfrac{\partial g}{\partial z}}{g^2}$.

$$\therefore \quad \nabla\left(\frac{f}{g}\right)=\frac{1}{g^2}\left\{\mathbf{i}\left(g\frac{\partial f}{\partial x}-f\frac{\partial g}{\partial x}\right)+\mathbf{j}\left(g\frac{\partial f}{\partial y}-f\frac{\partial g}{\partial y}\right)+\mathbf{k}\left(g\frac{\partial f}{\partial y}-f\frac{\partial g}{\partial z}\right)\right\}$$

$$=\frac{1}{g^2}\left\{g\left(\mathbf{i}\frac{\partial f}{\partial x}+\mathbf{j}\frac{\partial f}{\partial y}+\mathbf{k}\frac{\partial f}{\partial z}\right)-f\left(\mathbf{i}\frac{\partial g}{\partial x}+\mathbf{j}\frac{\partial g}{\partial y}+\mathbf{k}\frac{\partial g}{\partial z}\right)\right\}$$

$$=\frac{1}{g^2}\left\{g\,\nabla f-f\,\nabla g\right\}.$$

Solved Examples

Ex. 1. *If* $\mathbf{F}=e^{xy}\,\mathbf{i}+(x-2y)\,\mathbf{j}+x\sin y\,\mathbf{k}$, *calculate*

(*i*) $\dfrac{\partial \mathbf{F}}{\partial x}$, (*ii*) $\dfrac{\partial \mathbf{F}}{\partial y}$, (*iii*) $\dfrac{\partial^2 \mathbf{F}}{\partial x^2}$, (*iv*) $\dfrac{\partial^2 \mathbf{F}}{\partial x\,\partial y}$, (*v*) $\dfrac{\partial^2 \mathbf{F}}{\partial y^2}$.

Sol. (i) $\dfrac{\partial \mathbf{F}}{\partial x}=\left[\dfrac{\partial}{\partial x}(e^{xy})\right]\mathbf{i}+\left[\dfrac{\partial}{\partial x}(x-2y)\right]\mathbf{j}+\left[\dfrac{\partial}{\partial x}(x\sin y)\right]\mathbf{k}$

$=(ye^{xy})\,\mathbf{i}+(1)\,\mathbf{j}+(\sin y)\,\mathbf{k}=ye^{xy}\,\mathbf{i}+\mathbf{j}+\sin y\,\mathbf{k}$.

(ii) $\dfrac{\partial \mathbf{F}}{\partial y}=\left[\dfrac{\partial}{\partial y}(e^{xy})\right]\mathbf{i}+\left[\dfrac{\partial}{\partial y}(x-2y)\right]\mathbf{j}+\left[\dfrac{\partial}{\partial y}(x\sin y)\right]\mathbf{k}$

$=xe^{xy}\,\mathbf{i}-2\mathbf{j}+x\cos y\,\mathbf{k}$.

(iii) $\dfrac{\partial^2 \mathbf{F}}{\partial x^2}=\dfrac{\partial}{\partial x}\left(\dfrac{\partial \mathbf{F}}{\partial x}\right)=\dfrac{\partial}{\partial x}[ye^{xy}\,\mathbf{i}+\mathbf{j}+\sin y\,\mathbf{k}]$

$=\left[\dfrac{\partial}{\partial x}(ye^{xy})\right]\mathbf{i}+\dfrac{\partial}{\partial x}\mathbf{j}+\left[\dfrac{\partial}{\partial x}(\sin y)\right]\mathbf{k}$

$=y^2e^{xy}\,\mathbf{i}+0+0\,\mathbf{k}=y^2e^{xy}\,\mathbf{i}$.

(iv) $\dfrac{\partial^2 \mathbf{F}}{\partial x\,\partial y}=\dfrac{\partial}{\partial y}\left(\dfrac{\partial \mathbf{F}}{\partial x}\right)=\dfrac{\partial}{\partial y}[ye^{xy}\,\mathbf{i}+\mathbf{j}+\sin y\,\mathbf{k}]$

$=\left[\dfrac{\partial}{\partial y}(ye^{xy})\right]\mathbf{i}+\dfrac{\partial}{\partial y}\mathbf{j}+\left(\dfrac{\partial}{\partial y}\sin y\right)\mathbf{k}$

$=(e^{xy}+xye^{xy})\,\mathbf{i}+0+\cos y\,\mathbf{k}$

$=e^{xy}(xy+1)\,\mathbf{i}+\cos y\,\mathbf{k}$.

(v) $\dfrac{\partial^2 \mathbf{F}}{\partial y^2}=\dfrac{\partial}{\partial y}\left(\dfrac{\partial \mathbf{F}}{\partial y}\right)=\dfrac{\partial}{\partial y}[xe^{xy}\,\mathbf{i}-2\mathbf{j}+x\cos y\,\mathbf{k}]$

$=\left[\dfrac{\partial}{\partial y}(xe^{xy})\right]\mathbf{i}+\dfrac{\partial}{\partial y}(-2\mathbf{j})+\left[\dfrac{\partial}{\partial y}(x\cos y)\right]\mathbf{k}$

$=x^2e^{xy}\,\mathbf{i}+0-x\sin y\,\mathbf{k}=x^2e^{xy}\,\mathbf{i}-x\sin y\,\mathbf{k}$.

Ex. 2. *If* $\phi(x, y, z)=xy^2z$ *and* $\mathbf{f}=xz\,\mathbf{i}-xy\,\mathbf{j}+yz^2\,\mathbf{k}$, *show that* $\dfrac{\partial^3}{\partial x^2\,\partial z}(\phi\,\mathbf{f})$ *at* $(2, -1, 1)$ *is* $4\,\mathbf{i}+2\,\mathbf{j}$. [Garhwal 1985]

Sol. We have $\phi\,\mathbf{f}=xy^2z(xz\,\mathbf{i}-xy\,\mathbf{j}+yz^2\,\mathbf{k})$

$$=x^2y^2z^2\,\mathbf{i}-x^2y^3z\,\mathbf{j}+xy^3z^3\,\mathbf{k}.$$

$$\therefore\ \frac{\partial}{\partial x}(\phi\mathbf{f})=2xy^2z^2\,\mathbf{i}-2xy^3z\,\mathbf{j}+y^3z^3\,\mathbf{k},$$

$$\frac{\partial^2}{\partial x^2}(\phi\mathbf{f})=\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(\phi\mathbf{f})\right]=\frac{\partial}{\partial x}[2xy^2z^2\,\mathbf{i}-2xy^3z\,\mathbf{j}+y^3z^3\,\mathbf{k}]$$

$$=2y^2z^2\,\mathbf{i}-2y^3z\,\mathbf{j}+0\mathbf{k}=2y^2z^2\,\mathbf{i}-2y^3z\,\mathbf{j}$$

and $$\frac{\partial^3}{\partial x^2\,\partial z}(\phi\mathbf{f})=\frac{\partial}{\partial z}\left[\frac{\partial^2}{\partial x^2}(\phi\mathbf{f})\right]=\frac{\partial}{\partial z}(2y^2z^2\,\mathbf{i}-2y^3z\,\mathbf{j})$$

$$=4y^2z\,\mathbf{i}-2y^3\,\mathbf{j}.$$

$\therefore$ at the point $(2, -1, 1)$, $$\frac{\partial^3}{\partial x^2\,\partial z}(\phi\mathbf{f})=4.(-1)^2.1\,\mathbf{i}-2.(-1)^3\,\mathbf{j}=4\,\mathbf{i}+2\,\mathbf{j}.$$

Ex. 3. *If* $\mathbf{f}=(2x^2y-x^4)\,\mathbf{i}+(e^{xy}-y\sin x)\,\mathbf{j}+x^2\cos y\,\mathbf{k}$, *verify that*

$$\frac{\partial^2\mathbf{f}}{\partial y\,\partial x}=\frac{\partial^2\mathbf{f}}{\partial x\,\partial y}.$$ [Agra 1981]

Sol. We have $$\frac{\partial\mathbf{f}}{\partial x}=\left[\frac{\partial}{\partial x}(2x^2y-x^4)\right]\mathbf{i}+\left[\frac{\partial}{\partial x}(e^{xy}-y\sin x)\right]\mathbf{j}+\left[\frac{\partial}{\partial x}(x^2\cos y)\right]\mathbf{k}$$

$$=(4xy-4x^3)\,\mathbf{i}+(ye^{xy}-y\cos x)\,\mathbf{j}+2x\cos y\,\mathbf{k}.$$

$$\therefore\ \frac{\partial^2\mathbf{f}}{\partial x\,\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial\mathbf{f}}{\partial x}\right)=\frac{\partial}{\partial y}[(4xy-4x^3)\,\mathbf{i}+(ye^{xy}-y\cos x)\,\mathbf{j}+2x\cos y\,\mathbf{k}]$$

$$=\left[\frac{\partial}{\partial y}(4xy-4x^3)\right]\mathbf{i}+\left[\frac{\partial}{\partial y}(ye^{xy}-y\cos x)\right]\mathbf{j}+\left[\frac{\partial}{\partial y}(2x\cos y)\right]\mathbf{k}$$

$$=(4x)\,\mathbf{i}+(e^{xy}+xy\,e^{xy}-\cos x)\,\mathbf{j}+(-2x\sin y)\,\mathbf{k}$$

$$=4x\,\mathbf{i}+(e^{xy}+xy\,e^{xy}-\cos x)\,\mathbf{j}-2x\sin y\,\mathbf{k}. \quad \ldots(1)$$

Again $$\frac{\partial\mathbf{f}}{\partial y}=\left[\frac{\partial}{\partial y}(2x^2y-x^4)\right]\mathbf{i}+\left[\frac{\partial}{\partial y}(e^{xy}-y\sin x)\right]\mathbf{j}+\left[\frac{\partial}{\partial y}(x^2\cos y)\right]\mathbf{k}$$

$$=2x^2\,\mathbf{i}+(xe^{xy}-\sin x)\,\mathbf{j}-x^2\sin y\,\mathbf{k}.$$

$$\therefore \frac{\partial^2 \mathbf{f}}{\partial y \partial x}=\frac{\partial}{\partial x}\left(\frac{\partial \mathbf{f}}{\partial y}\right)=\left[\frac{\partial}{\partial x}(2x^2)\right]\mathbf{i}+\left[\frac{\partial}{\partial x}(xe^{xy}-\sin x)\right]\mathbf{j}-\left[\frac{\partial}{\partial x}(x^2\sin y)\right]\mathbf{k}$$

$$=4x\,\mathbf{i}+(e^{xy}+xye^{xy}-\cos x)\,\mathbf{j}-2x\sin y\,\mathbf{k}. \qquad \ldots(2)$$

From (1) and (2), we have $\dfrac{\partial^2 \mathbf{f}}{\partial x \partial y}=\dfrac{\partial^2 \mathbf{f}}{\partial y \partial x}$.

Ex. 4. *If* $\mathbf{u}=xyz\,\mathbf{i}+xz^2\,\mathbf{j}-y^3\,\mathbf{k}$ *and* $\mathbf{v}=x^3\,\mathbf{i}-xyz\,\mathbf{j}+x^2z\,\mathbf{k}$, *calculate* $\dfrac{\partial^2 \mathbf{u}}{\partial y^2}\times\dfrac{\partial^2 \mathbf{v}}{\partial x^2}$ *at the point* (1, 1, 0).

Sol. We have $\dfrac{\partial \mathbf{u}}{\partial y}=\left[\dfrac{\partial}{\partial y}(xyz)\right]\mathbf{i}+\left[\dfrac{\partial}{\partial y}(xz^2)\right]\mathbf{j}-\left[\dfrac{\partial}{\partial y}(y^3)\right]\mathbf{k}$

$$=xz\,\mathbf{i}+0\,\mathbf{j}-3y^2\,\mathbf{k}=xz\,\mathbf{i}-3y^2\,\mathbf{k}$$

and $\dfrac{\partial^2 \mathbf{u}}{\partial y^2}=\dfrac{\partial}{\partial y}\left(\dfrac{\partial \mathbf{u}}{\partial y}\right)=0\,\mathbf{i}-6y\,\mathbf{k}=-6y\,\mathbf{k}.$

Again $\dfrac{\partial \mathbf{v}}{\partial x}=\left[\dfrac{\partial}{\partial x}(x^3)\right]\mathbf{i}-\left[\dfrac{\partial}{\partial x}(xyz)\right]\mathbf{j}+\left[\dfrac{\partial}{\partial x}(x^2z)\right]\mathbf{k}$

$$=3x^2\,\mathbf{i}-yz\,\mathbf{j}+2xz\,\mathbf{k}$$

and $\dfrac{\partial^2 \mathbf{v}}{\partial x^2}=\dfrac{\partial}{\partial x}\left(\dfrac{\partial \mathbf{v}}{\partial x}\right)=6x\,\mathbf{i}-0\,\mathbf{j}+2z\,\mathbf{k}=6x\,\mathbf{i}+2z\,\mathbf{k}.$

$$\therefore \frac{\partial^2 \mathbf{u}}{\partial y^2}\times\frac{\partial^2 \mathbf{v}}{\partial x^2}=(-6y\mathbf{k})\times(6x\mathbf{i}+2z\mathbf{k})$$

$$=-36xy\,\mathbf{k}\times\mathbf{i}-12yz\,\mathbf{k}\times\mathbf{k}$$

$$=-36xy\mathbf{j} \qquad [\because \mathbf{k}\times\mathbf{i}=\mathbf{j} \text{ and } \mathbf{k}\times\mathbf{k}=\mathbf{0}]$$

$\therefore$ at the point (1, 1, 0), we have $\dfrac{\partial^2 \mathbf{u}}{\partial y^2}\times\dfrac{\partial^2 \mathbf{v}}{\partial x^2}=-36\mathbf{j}.$

Ex. 5. *If* $\mathbf{A}=x^2yz\,\mathbf{i}-2xz^3\,\mathbf{j}+xz^2\,\mathbf{k}$, $\mathbf{B}=2z\,\mathbf{i}+y\,\mathbf{j}-x^2\,\mathbf{k}$, *find the value of* $\dfrac{\partial^2}{\partial x \partial y}(\mathbf{A}\times\mathbf{B})$ *at* $(1, 0, -2)$. **[Kanpur 1985, 81]**

Sol. We have $\mathbf{A}\times\mathbf{B}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x^2yz & -2xz^3 & xz^2 \\ 2z & y & -x^2 \end{vmatrix}$

$$=(2x^3z^3-xyz^2)\,\mathbf{i}+(2xz^3+x^4yz)\,\mathbf{j}+(x^2y^2z+4xz^4)\,\mathbf{k}.$$

$$\therefore \frac{\partial}{\partial y}(\mathbf{A}\times\mathbf{B})=-xz^2\,\mathbf{i}+x^4z\,\mathbf{j}+2x^2yz\,\mathbf{k}.$$

Again $\dfrac{\partial^2}{\partial x \partial y}(\mathbf{A}\times\mathbf{B})=\dfrac{\partial}{\partial x}\left\{\dfrac{\partial}{\partial y}(\mathbf{A}\times\mathbf{B})\right\}$

$$=-z^2\,\mathbf{i}+4x^3z\,\mathbf{j}+4xyz\,\mathbf{k}. \qquad \ldots(1)$$

Putting $x=1, y=0$ and $z=-2$ in (1), we get the required derivative at the point $(1, 0, -2)=-4\mathbf{i}-8\mathbf{j}$.

Ex. 6. *If $f(x, y, z)=3x^2y-y^3z^2$, find grad f at the point* $(1, -2, -1)$. [Agra 1978; Rohilkhand 83]

Sol. We have

$$\text{grad } f=\nabla f=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)(3x^2y-y^3z^2)$$

$$=\mathbf{i}\frac{\partial}{\partial x}(3x^2y-y^3z^2)+\mathbf{j}\frac{\partial}{\partial y}(3x^2y-y^3z^2)+\mathbf{k}\frac{\partial}{\partial z}(3x^2y-y^3z^2)$$

$$=\mathbf{i}(6xy)+\mathbf{j}(3x^2-3y^2z^2)+\mathbf{k}(-2y^3z)$$

$$=6xy\,\mathbf{i}+(3x^2-3y^2z^2)\,\mathbf{j}-2y^3z\mathbf{k}.$$

Putting $x=1, y=-2, z=-1$, we get

$$\nabla f=6(1)(-2)\,\mathbf{i}+\{3(1)^2-3(-2)^2(-1)^2\}\,\mathbf{j}-2(-2)^3(-1)\,\mathbf{k}$$

$$=-12\mathbf{i}-9\mathbf{j}-16\mathbf{k}.$$

Ex. 7. *If $r=|\mathbf{r}|$ where $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, prove that*

(i) $\nabla f(r)=f'(r)\nabla r$, (ii) $\nabla r=\dfrac{1}{r}\mathbf{r}$, [Rohilkhand 1984]

(iii) $\nabla f(r)\times\mathbf{r}=0$, (iv) $\nabla\left(\dfrac{1}{r}\right)=-\dfrac{\mathbf{r}}{r^3}$, [Meerut 1991]

(v) $\nabla \log|\mathbf{r}|=\dfrac{\mathbf{r}}{r^2}$, 2007 [Meerut 1991; Kanpur 88]

(vi) $\nabla r^n=nr^{n-2}\mathbf{r}$. [Kanpur 1986; Agra 86; Rohilkhand 90; Garhwal 84]

Sol. If $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, then $r=|\mathbf{r}|=\sqrt{(x^2+y^2+z^2)}$.

$$\therefore\quad r^2=x^2+y^2+z^2.$$

(i) $\nabla f(r)=\left(\mathbf{i}\dfrac{\partial}{\partial x}+\mathbf{j}\dfrac{\partial}{\partial y}+\mathbf{k}\dfrac{\partial}{\partial z}\right)f(r)$

$$=\mathbf{i}\frac{\partial}{\partial x}f(r)+\mathbf{j}\frac{\partial}{\partial y}f(r)+\mathbf{k}\frac{\partial}{\partial z}f(r)$$

$$=\mathbf{i}f'(r)\frac{\partial r}{\partial x}+\mathbf{j}f'(r)\frac{\partial r}{\partial y}+\mathbf{k}f'(r)\frac{\partial r}{\partial z}$$

$$=f'(r)\left(\mathbf{i}\frac{\partial r}{\partial x}+\mathbf{j}\frac{\partial r}{\partial y}+\mathbf{k}\frac{\partial r}{\partial z}\right)=f'(r)\nabla r.$$

(ii) We have $\nabla r=\mathbf{i}\dfrac{\partial r}{\partial x}+\mathbf{j}\dfrac{\partial r}{\partial y}+\mathbf{k}\dfrac{\partial r}{\partial z}$.

Now $r^2=x^2+y^2+z^2$; $\therefore\ 2r\dfrac{\partial r}{\partial x}=2x$ i.e., $\dfrac{\partial r}{\partial x}=\dfrac{x}{r}$.

Similarly $\frac{\partial r}{\partial y}=\frac{y}{r}$ and $\frac{\partial r}{\partial z}=\frac{z}{r}$.

$\therefore\quad \nabla r=\frac{x}{r}\,\mathbf{i}+\frac{y}{r}\,\mathbf{j}+\frac{z}{r}\,\mathbf{k}=\frac{1}{r}\,(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})=\frac{1}{r}\,\mathbf{r}=\hat{\mathbf{r}}.$

(iii) We have as in part (i), $\nabla f(r)=f'(r)\,\nabla r.$

But as in part (ii) $\nabla r=\frac{1}{r}\,\mathbf{r}.$

$$\therefore\quad \nabla f(r)=f'(r)\,\frac{1}{r}\,\mathbf{r}.$$

$\therefore\ \nabla f(r)\times\mathbf{r}=\left\{f'(r)\,\frac{1}{r}\,\mathbf{r}\right\}\times\mathbf{r}=\left\{\frac{1}{r}\,f'(r)\right\}(\mathbf{r}\times\mathbf{r})$

$=\mathbf{0}$, since $\mathbf{r}\times\mathbf{r}=\mathbf{0}.$

(iv) We have $\nabla\left(\frac{1}{r}\right)=\mathbf{i}\,\frac{\partial}{\partial x}\left(\frac{1}{r}\right)+\mathbf{j}\,\frac{\partial}{\partial y}\left(\frac{1}{r}\right)+\mathbf{k}\,\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$

$=\mathbf{i}\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right)+\mathbf{j}\left(-\frac{1}{r^2}\frac{\partial r}{\partial y}\right)+\mathbf{k}\left(-\frac{1}{r^2}\frac{\partial r}{\partial z}\right)$

$=-\frac{1}{r^2}\left(\frac{\partial r}{\partial x}\,\mathbf{i}+\frac{\partial r}{\partial y}\,\mathbf{j}+\frac{\partial r}{\partial z}\,\mathbf{k}\right)$

$=-\frac{1}{r^2}\left(\frac{x}{r}\,\mathbf{i}+\frac{y}{r}\,\mathbf{j}+\frac{z}{r}\,\mathbf{k}\right)$ [see part (ii)]

$=-\frac{1}{r^3}\,(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})=-\frac{1}{r^3}\,\mathbf{r}.$

(v) We have $\nabla\log|\mathbf{r}|=\nabla\log r$

$=\mathbf{i}\,\frac{\partial}{\partial x}\log r+\mathbf{j}\,\frac{\partial}{\partial y}\log r+\mathbf{k}\,\frac{\partial}{\partial z}\log r$

$=\frac{1}{r}\frac{\partial r}{\partial x}\,\mathbf{i}+\frac{1}{r}\frac{\partial r}{\partial y}\,\mathbf{j}+\frac{1}{r}\frac{\partial r}{\partial z}\,\mathbf{k}=\frac{1}{r}\left(\frac{x}{r}\,\mathbf{i}+\frac{y}{r}\,\mathbf{j}+\frac{z}{r}\,\mathbf{k}\right)$

$=\frac{1}{r^2}\,(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})=\frac{1}{r^2}\,\mathbf{r}.$

(vi) We have $\nabla r^n=\mathbf{i}\,\frac{\partial}{\partial x}\,r^n+\mathbf{j}\,\frac{\partial}{\partial y}\,r^n+\mathbf{k}\,\frac{\partial}{\partial z}\,r^n$

$=\mathbf{i}\;nr^{n-1}\,\frac{\partial r}{\partial x}+\mathbf{j}\,nr^{n-1}\,\frac{\partial r}{\partial y}+\mathbf{k}\;nr^{n-1}\,\frac{\partial r}{\partial z}=nr^{n-1}\left(\mathbf{i}\,\frac{\partial r}{\partial x}+\mathbf{j}\,\frac{\partial r}{\partial y}+\mathbf{k}\,\frac{\partial r}{\partial z}\right)$

$=nr^{n-1}\,\nabla r$

$=nr^{n-1}\,\frac{1}{r}\,\mathbf{r}$ $\left[\because\ \nabla r=\frac{\mathbf{r}}{r}\text{ as in part (ii)}\right]$

$=nr^{n-2}\,\mathbf{r}.$

Ex. 8. *Prove that* $f(u)\,\nabla u=\nabla\int f(u)\,du.$

Sol. We have $\nabla \int f'(u)\, du$

$$=\Sigma\, \mathbf{i} \frac{\partial}{\partial x}\left\{\int f(u)\, du\right\} \quad \text{[by def. of gradient]}$$

$$=\Sigma\, \mathbf{i}\left\{\frac{d}{du}\int f(u)\, du\right\}\frac{\partial u}{\partial x}=\Sigma \mathbf{i} f(u)\frac{\partial u}{\partial x}=f(u)\,\Sigma\, \mathbf{i}\frac{\partial u}{\partial x}=f(u)\,\nabla u.$$

Ex. 9. *Show that*

(*i*) *grad* $(\mathbf{r}\cdot\mathbf{a})=\mathbf{a}$, (*ii*) *grad* $[\mathbf{r}, \mathbf{a}, \mathbf{b}]=\mathbf{a}\times\mathbf{b}$,

where $\mathbf{a}$ *and* $\mathbf{b}$ *are constant vectors.* **[Rohilkhand 1981; Kanpur 87]**

Sol. (i) Let $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$. Then a_1, a_2, a_3 are constants. Also $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$.

$\therefore\ \mathbf{r}\cdot\mathbf{a}=a_1x+a_2y+a_3z.$

$\therefore$ grad $(\mathbf{r}\cdot\mathbf{a})=\nabla(\mathbf{r}\cdot\mathbf{a})=\nabla(a_1x+a_2y+a_3z)$

$$=\mathbf{i}\frac{\partial}{\partial x}(a_1x+a_2y+a_3z)+\mathbf{j}\frac{\partial}{\partial y}(a_1x+a_2y+a_3z)+\mathbf{k}\frac{\partial}{\partial z}(a_1x+a_2y+a_3z)$$

$=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}=\mathbf{a}.$

(ii) grad $[\mathbf{r}, \mathbf{a}, \mathbf{b}]=$ grad $\{\mathbf{r}\cdot(\mathbf{a}\times\mathbf{b})\}$, where $\mathbf{a}\times\mathbf{b}$ is a constant vector

$=\mathbf{a}\times\mathbf{b}$ as in part (i).

Ex. 10. *If* $\phi(x, y, z)=x^2y+y^2x+z^2$, *find* $\nabla\phi$ *at the point* $(1, 1, 1)$. **[Agra 1979]**

Sol. We have $\nabla\phi=\dfrac{\partial\phi}{\partial x}\mathbf{i}+\dfrac{\partial\phi}{\partial y}\mathbf{j}+\dfrac{\partial\phi}{\partial z}\mathbf{k}$

$$=\left[\frac{\partial}{\partial x}(x^2y+y^2x+z^2)\right]\mathbf{i}+\left[\frac{\partial}{\partial y}(x^2y+y^2x+z^2)\right]\mathbf{j}+\left[\frac{\partial}{\partial z}(x^2y+y^2x+z^2)\right]\mathbf{k}$$

$=(2xy+y^2)\mathbf{i}+(x^2+2xy)\mathbf{j}+2z\mathbf{k}.$

Putting $x=1, y=1, z=1$, we get

$\nabla\phi$ at the point $(1, 1, 1)=3\mathbf{i}+3\mathbf{j}+2\mathbf{k}.$

Ex. 11. *Find grad* f, *where* f *is given by*

$f=x^3-y^3+xz^2$, *at the point* $(1, -1, 2)$. **[Agra 1977]**

Sol. We have grad $f=\nabla f=\dfrac{\partial f}{\partial x}\mathbf{i}+\dfrac{\partial f}{\partial y}\mathbf{j}+\dfrac{\partial f}{\partial z}\mathbf{k}$

$$=\left[\frac{\partial}{\partial x}(x^3-y^3+xz^2)\right]\mathbf{i}+\left[\frac{\partial}{\partial y}(x^3-y^3+xz^2)\right]\mathbf{j}+\left[\frac{\partial}{\partial z}(x^3-y^3+xz^2)\right]\mathbf{k}$$

$=(3x^2+z^2)\mathbf{i}+(-3y^2)\mathbf{j}+2xz\mathbf{k}.$

Putting $x=1, y=-1, z=2$, we get

∇f at the point $(1, -1, 2) = 7\,\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$.

Ex. 12 *If* $u = x + y + z$, $v = x^2 + y^2 + z^2$, $w = yz + zx + xy$, *prove that*

$(\text{grad } u) \cdot [(\text{grad } v) \times (\text{grad } w)] = 0$. [Kohlapur 1978]

Sol. We have $\text{grad } u = \dfrac{\partial u}{\partial x}\mathbf{i} + \dfrac{\partial u}{\partial y}\mathbf{j} + \dfrac{\partial u}{\partial z}\mathbf{k}$

$$= 1\,\mathbf{i} + 1\,\mathbf{j} + 1\,\mathbf{k} = \mathbf{i} + \mathbf{j} + \mathbf{k},$$

$$\text{grad } v = \frac{\partial v}{\partial x}\mathbf{i} + \frac{\partial v}{\partial y}\mathbf{j} + \frac{\partial v}{\partial z}\mathbf{k} = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}$$

and $\text{grad } w = \dfrac{\partial w}{\partial x}\mathbf{i} + \dfrac{\partial w}{\partial y}\mathbf{j} + \dfrac{\partial w}{\partial z}\mathbf{k}$

$$= (y+z)\,\mathbf{i} + (z+x)\,\mathbf{j} + (x+y)\,\mathbf{k}.$$

$\therefore$ $\text{grad } u \cdot [(\text{grad } v) \times (\text{grad } w)]$ = scalar triple product of the vectors grad u, grad v and grad w

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & z+x & x+y \end{vmatrix} = 2 \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ y+z & z+x & x+y \end{vmatrix}$$

$$= 2 \begin{vmatrix} 1 & 1 & 1 \\ x+y+z & x+y+z & x+y+z \\ y+z & z+x & x+y \end{vmatrix}, \text{ by } R_2 + R_3$$

$$= 2\,(x+y+z) \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ y+z & z+x & x+y \end{vmatrix}$$

$= 2\,(x+y+z).0$, the first two rows of the determinant being identical

$= 0$.

Ex. 13. *If* $\mathbf{F} = \left(y\dfrac{\partial f}{\partial z} - z\dfrac{\partial f}{\partial y}\right)\mathbf{i} + \left(z\dfrac{\partial f}{\partial x} - x\dfrac{\partial f}{\partial z}\right)\mathbf{j}$

$$+ \left(x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x}\right)\mathbf{k},$$

prove that

(*i*) $\mathbf{F} = \mathbf{r} \times \nabla f$, (*ii*) $\mathbf{F} \cdot \mathbf{r} = 0$, (*iii*) $\mathbf{F} \cdot \nabla f = 0$.

Sol. We have $\nabla f = \dfrac{\partial f}{\partial x}\mathbf{i} + \dfrac{\partial f}{\partial y}\mathbf{j} + \dfrac{\partial f}{\partial z}\mathbf{k}$ and $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$

(i) $\mathbf{r}\times\nabla f=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ \partial f/\partial x & \partial f/\partial y & \partial f/\partial z \end{vmatrix}$

$$=\left(y\frac{\partial f}{\partial z}-z\frac{\partial f}{\partial y}\right)\mathbf{i}+\left(z\frac{\partial f}{\partial x}-x\frac{\partial f}{\partial z}\right)\mathbf{j}+\left(x\frac{\partial f}{\partial y}-y\frac{\partial f}{\partial x}\right)\mathbf{k}=\mathbf{F}$$

(ii) $\mathbf{F}\cdot\mathbf{r}=(\mathbf{r}\times\nabla f)\cdot\mathbf{r}$ [$\because$ $\mathbf{F}=\mathbf{r}\times\nabla f$]

$=0$, because the value of a scalar triple product having two vectors equal is zero.

(iii) $\mathbf{F}\cdot\nabla f=(\mathbf{r}\times\nabla f)\cdot\nabla f=[\mathbf{r}, \nabla f, \nabla f]$

$=0$, because the value of a scalar triple product having two vectors equal is zero.

Ex. 14. *Prove that* $\mathbf{A}\cdot\left(\nabla\frac{1}{r}\right)=-\frac{\mathbf{A}\cdot\mathbf{r}}{r^3}$.

Sol. First prove that $\nabla\frac{1}{r}=-\frac{\mathbf{r}}{r^3}$.

[For its complete solution see Ex. 7, part (iv)]

$$\therefore\quad \mathbf{A}\cdot\nabla\left(\frac{1}{r}\right)=\mathbf{A}\cdot\left(-\frac{\mathbf{r}}{r^3}\right)=-\frac{\mathbf{A}\cdot\mathbf{r}}{r^3}.$$

Ex. 15. *Prove that* $\nabla r^{-3}=-3r^{-5}\mathbf{r}$. [Agra 1974]

Sol. We have

$$\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k} \text{ and } r=|\mathbf{r}|=\sqrt{(x^2+y^2+z^2)}$$

so that $r^2=x^2+y^2+z^2$.

Now $\nabla r^{-3}=\left(\frac{\partial}{\partial x}r^{-3}\right)\mathbf{i}+\left(\frac{\partial}{\partial y}r^{-3}\right)\mathbf{j}+\left(\frac{\partial}{\partial z}r^{-3}\right)\mathbf{k}$

$$=-3r^{-4}\frac{\partial r}{\partial x}\mathbf{i}-3r^{-4}\frac{\partial r}{\partial y}\mathbf{j}-3r^{-4}\frac{\partial r}{\partial z}\mathbf{k}$$

$$=-3r^{-4}\left(\frac{\partial r}{\partial x}\mathbf{i}+\frac{\partial r}{\partial y}\mathbf{j}+\frac{\partial r}{\partial z}\mathbf{k}\right). \qquad \ldots(1)$$

Differentiating both sides of $r^2=x^2+y^2+z^2$ partially w.r.t. x, we have

$$2r\frac{\partial r}{\partial x}=2x \text{ or } \frac{\partial r}{\partial x}=\frac{x}{r}.$$

Similarly $\frac{\partial r}{\partial y}=\frac{y}{r}$ and $\frac{\partial r}{\partial z}=\frac{z}{r}$.

$\therefore$ from (1), we have

$$\nabla r^{-3}=-3r^{-4}\left(\frac{x}{r}\,\mathbf{i}+\frac{y}{r}\,\mathbf{j}+\frac{z}{r}\,\mathbf{k}\right)$$

$$=-3r^{-5}\,(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})=-3r^{-5}\,\mathbf{r}.$$

Ex. 16. *Prove that* $\nabla\phi\cdot d\mathbf{r}=d\phi$.

Sol. We have $\nabla\phi=\dfrac{\partial\phi}{\partial x}\,\mathbf{i}+\dfrac{\partial\phi}{\partial y}\,\mathbf{j}+\dfrac{\partial\phi}{\partial z}\,\mathbf{k}$. ...(1)

Also $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k} \Rightarrow d\mathbf{r}=dx\,\mathbf{i}+dy\,\mathbf{j}+dz\,\mathbf{k}$. ...(2)

From (1) and (2), $\nabla\phi\cdot d\mathbf{r}$

$$=\left(\frac{\partial\phi}{\partial x}\,\mathbf{i}+\frac{\partial\phi}{\partial y}\,\mathbf{j}+\frac{\partial\phi}{\partial z}\,\mathbf{k}\right)\cdot(dx\,\mathbf{i}+dy\,\mathbf{j}+dz\,\mathbf{k})$$

$$=\frac{\partial\phi}{\partial x}\,dx+\frac{\partial\phi}{\partial y}\,dy+\frac{\partial\phi}{\partial z}\,dz=d\phi.$$

Ex. 17. *Show that*

$$\frac{d\phi}{ds}=\nabla\phi\cdot\frac{d\mathbf{r}}{ds},\ \textit{where}\ \mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$$

and ϕ *is a function of* x, y *and* z.

Sol. We have $\nabla\phi=\dfrac{\partial\phi}{\partial x}\,\mathbf{i}+\dfrac{\partial\phi}{\partial y}\,\mathbf{j}+\dfrac{\partial\phi}{\partial z}\,\mathbf{k}$ ...(1)

Also $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k} \Rightarrow \dfrac{d\mathbf{r}}{ds}=\dfrac{dx}{ds}\,\mathbf{i}+\dfrac{dy}{ds}\,\mathbf{j}+\dfrac{dz}{ds}\,\mathbf{k}$. ...(2)

From (1) and (2), we have

$$\nabla\phi\cdot\frac{d\mathbf{r}}{ds}=\left(\frac{\partial\phi}{\partial x}\,\mathbf{i}+\frac{\partial\phi}{\partial y}\,\mathbf{j}+\frac{\partial\phi}{\partial z}\,\mathbf{k}\right)\cdot\left(\frac{dx}{ds}\,\mathbf{i}+\frac{dy}{ds}\,\mathbf{j}+\frac{dz}{ds}\,\mathbf{k}\right)$$

$$=\frac{\partial\phi}{\partial x}\frac{dx}{ds}+\frac{\partial\phi}{\partial y}\frac{dy}{ds}+\frac{\partial\phi}{\partial z}\frac{dz}{ds}=\frac{d\phi}{ds}.$$

Ex. 18. ρ *and* p *are two scalar point functions such that* ρ *is a function of* p; *show that*

$$\nabla\rho=\frac{d\rho}{dp}\,\nabla p.$$

Sol. We have $\nabla\rho=\dfrac{\partial\rho}{\partial x}\,\mathbf{i}+\dfrac{\partial\rho}{\partial y}\,\mathbf{j}+\dfrac{\partial\rho}{\partial z}\,\mathbf{k}$. ...(1)

Since ρ is a function of p, therefore

$$\frac{\partial\rho}{\partial x}=\frac{d\rho}{dp}\frac{\partial p}{\partial x},\ \frac{\partial\rho}{\partial y}=\frac{d\rho}{dp}\frac{\partial p}{\partial y},\ \frac{\partial\rho}{\partial z}=\frac{d\rho}{dp}\frac{\partial p}{\partial z}.$$

$\therefore$ from (1), we have

$$\nabla\rho=\frac{d\rho}{dp}\frac{\partial p}{\partial x}\,\mathbf{i}+\frac{d\rho}{dp}\frac{\partial p}{\partial y}\,\mathbf{j}+\frac{d\rho}{dp}\frac{\partial p}{\partial z}\,\mathbf{k}$$

$$=\frac{d\rho}{dp}\left(\frac{\partial p}{\partial x}\,\mathbf{i}+\frac{\partial p}{\partial y}\,\mathbf{j}+\frac{\partial p}{\partial z}\,\mathbf{k}\right)=\frac{d\rho}{dp}\,\nabla p.$$

Ex. 19. *If* $\phi=(3r^2-4r^{1/2}+6r^{-1/3})$, *show that*

$$\nabla\phi=2\,(3-r^{-3/2}-r^{-7/3})\,\mathbf{r}.$$

Sol. We have $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ and $r=|\,\mathbf{r}\,|=\sqrt{(x^2+y^2+z^2)}$ so that $r^2=x^2+y^2+z^2$.

Now ϕ is a function of r.

$$\therefore\quad \nabla\phi=\frac{d\phi}{dr}\,\nabla r \qquad \text{[See Ex. 7 part (i)]}$$

$$=[6r-4.\tfrac{1}{2}r^{-1/2}+6.(-\tfrac{1}{3})\,r^{-4/3}]\,\nabla r$$

$$=(6r-2r^{-1/2}-2r^{-4/3})\,\frac{1}{r}\,\mathbf{r}\left[\because\quad \nabla r=\frac{1}{r}\,\mathbf{r}\right]$$

$$=(6-2r^{-3/2}-2r^{-7/3})\,\mathbf{r}=2\,(3-r^{-3/2}-r^{-7/3})\,\mathbf{r}.$$

Ex. 20. (*i*) *Interpret the symbol* $\mathbf{a}\cdot\nabla$.

(*ii*) *Show that* $(\mathbf{a}\cdot\nabla)\,\phi=\mathbf{a}\cdot\nabla\,\phi$.

(*iii*) *Show that* $(\mathbf{a}\cdot\nabla)\,\mathbf{r}=\mathbf{a}$.

Sol. (i) Let $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$. Then

$$\mathbf{a}\cdot\nabla=(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\cdot\left(\mathbf{i}\,\frac{\partial}{\partial x}+\mathbf{j}\,\frac{\partial}{\partial y}+\mathbf{k}\,\frac{\partial}{\partial z}\right)$$

$$=a_1\,\frac{\partial}{\partial x}+a_2\,\frac{\partial}{\partial y}+a_3\,\frac{\partial}{\partial z}.$$

Thus the symbol $\mathbf{a}\cdot\nabla$ stands for the operator

$$a_1\,\frac{\partial}{\partial x}+a_2\,\frac{\partial}{\partial y}+a_3\,\frac{\partial}{\partial z}.$$

(ii) $$(\mathbf{a}\cdot\nabla)\,\phi=\left(a_1\,\frac{\partial}{\partial x}+a_2\,\frac{\partial}{\partial y}+a_3\,\frac{\partial}{\partial z}\right)\phi.$$

Also $$\mathbf{a}\cdot\nabla\phi=(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\cdot\left(\frac{\partial\phi}{\partial x}\,\mathbf{i}+\frac{\partial\phi}{\partial y}\,\mathbf{j}+\frac{\partial\phi}{\partial z}\,\mathbf{k}\right)$$

$$=a_1\,\frac{\partial\phi}{\partial x}+a_2\,\frac{\partial\phi}{\partial y}+a_3\,\frac{\partial\phi}{\partial z}.$$

Hence $(\mathbf{a}\cdot\nabla)\,\phi=\mathbf{a}\cdot\nabla\phi$.

(iii) $$(\mathbf{a}\cdot\nabla)\,\mathbf{r}=\left(a_1\,\frac{\partial}{\partial x}+a_2\,\frac{\partial}{\partial y}+a_3\,\frac{\partial}{\partial z}\right)\mathbf{r}$$

$$=a_1\,\frac{\partial\mathbf{r}}{\partial x}+a_2\,\frac{\partial\mathbf{r}}{\partial y}+a_3\,\frac{\partial\mathbf{r}}{\partial z}.$$

But $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$. $\therefore\ \dfrac{\partial\mathbf{r}}{\partial x}=\mathbf{i},\ \dfrac{\partial\mathbf{r}}{\partial y}=\mathbf{j},\ \dfrac{\partial\mathbf{r}}{\partial z}=\mathbf{k}$.

$$\therefore\ (\mathbf{a}\cdot\nabla)\,\mathbf{r}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}=\mathbf{a}.$$

§ 8. **Divergence of a vector point function.**

Definition. *Let* V *be any given differentiable vector point function. Then the divergence of* V, *written as,*

$$\nabla \cdot \mathbf{V} \quad \textit{or} \quad \textit{div } \mathbf{V},$$

is defined as $\text{div } \mathbf{V} = \nabla \cdot \mathbf{V} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot \mathbf{V}$

$$= \mathbf{i} \cdot \frac{\partial \mathbf{V}}{\partial x} + \mathbf{j} \cdot \frac{\partial \mathbf{V}}{\partial y} + \mathbf{k} \cdot \frac{\partial \mathbf{V}}{\partial z} = \Sigma \mathbf{i} \cdot \frac{\partial \mathbf{V}}{\partial x}.$$

[Sagar 1983; Kerala 74; Bombay 70]

It should be noted that div V is a scalar quantity. Thus *the divergence of a vector point function is a scalar point function.*

Theorem. *If* $\mathbf{V} = V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}$ *is a differentiable vector point function, then* $\text{div } \mathbf{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}.$

Proof. We have by definition

$$\text{div } \mathbf{V} = \nabla \cdot \mathbf{V} = \mathbf{i} \cdot \frac{\partial \mathbf{V}}{\partial x} + \mathbf{j} \cdot \frac{\partial \mathbf{V}}{\partial y} + \mathbf{k} \cdot \frac{\partial \mathbf{V}}{\partial z}.$$

Now $\mathbf{V} = V_1\mathbf{i} + V_2\mathbf{j} + V_3\mathbf{k}$; $\therefore \quad \frac{\partial \mathbf{V}}{\partial x} = \frac{\partial V_1}{\partial x}\mathbf{i} + \frac{\partial V_2}{\partial x}\mathbf{j} + \frac{\partial V_3}{\partial x}\mathbf{k}.$

$$\therefore \quad \mathbf{i} \cdot \frac{\partial \mathbf{V}}{\partial x} = \mathbf{i} \cdot \left(\frac{\partial V_1}{\partial x}\mathbf{i} + \frac{\partial V_2}{\partial x}\mathbf{j} + \frac{\partial V_3}{\partial x}\mathbf{k}\right) = \frac{\partial V_1}{\partial x}.$$

Similarly $\mathbf{j} \cdot \frac{\partial \mathbf{V}}{\partial y} = \frac{\partial V_2}{\partial y}$ and $\mathbf{k} \cdot \frac{\partial \mathbf{V}}{\partial z} = \frac{\partial V_3}{\partial z}.$

Hence $\quad \text{div } \mathbf{V} = \frac{\partial V_1}{\partial x} + \frac{\partial V_2}{\partial y} + \frac{\partial V_3}{\partial z}.$

Solenoidal Vector. Definition. *A vector* V *is said to be solenoidal if div* $\mathbf{V} = 0$. [Meerut 1991 S; Calcutta 75]

§ 9. **Curl of a vector point function. Definition.**

Let f *be any given differentiable vector point function. Then the curl or rotation of* f, *written as* $\nabla \times \mathbf{f}$, *curl* f *or rot* f *is defined as*

$$\text{curl } \mathbf{f}=\nabla\times\mathbf{f}=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\times\mathbf{f}$$

$$=\mathbf{i}\times\frac{\partial\mathbf{f}}{\partial x}+\mathbf{j}\times\frac{\partial\mathbf{f}}{\partial y}+\mathbf{k}\times\frac{\partial\mathbf{f}}{\partial z}=\Sigma\,\mathbf{i}\times\frac{\partial\mathbf{f}}{\partial x}.$$

[Sagar 1983; Bombay 86; Punjab 88]

It should be noted that curl **f** is a vector quantity. *Thus the curl of a vector point function is a vector point function.*

Theorem. *If* $\mathbf{f}=f_1\,\mathbf{i}+f_2\,\mathbf{j}+f_3\,\mathbf{k}$ *is a differentiable vector point function, then*

$$\text{curl } \mathbf{f}=\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right)\mathbf{i}+\left(\frac{\partial f_1}{\partial z}-\frac{\partial f_3}{\partial x}\right)\mathbf{j}+\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right)\mathbf{k}.$$

Proof. We have by definition

$$\text{curl } \mathbf{f}=\nabla\times\mathbf{f}=\mathbf{i}\times\frac{\partial\mathbf{f}}{\partial x}+\mathbf{j}\times\frac{\partial\mathbf{f}}{\partial y}+\mathbf{k}\times\frac{\partial\mathbf{f}}{\partial z}$$

$$=\mathbf{i}\times\frac{\partial}{\partial x}(f_1\,\mathbf{i}+f_2\,\mathbf{j}+f_3\,\mathbf{k})+\mathbf{j}\times\frac{\partial}{\partial y}(f_1\,\mathbf{i}+f_2\,\mathbf{j}+f_3\,\mathbf{k})$$
$$+\mathbf{k}\times\frac{\partial}{\partial z}(f_1\,\mathbf{i}+f_2\,\mathbf{j}+f_3\,\mathbf{k})$$

$$=\mathbf{i}\times\left(\frac{\partial f_1}{\partial x}\,\mathbf{i}+\frac{\partial f_2}{\partial x}\,\mathbf{j}+\frac{\partial f_3}{\partial x}\,\mathbf{k}\right)+\mathbf{j}\times\left(\frac{\partial f_1}{\partial y}\,\mathbf{i}+\frac{\partial f_2}{\partial y}\,\mathbf{j}+\frac{\partial f_3}{\partial y}\,\mathbf{k}\right)$$
$$+\mathbf{k}\times\left(\frac{\partial f_1}{\partial z}\,\mathbf{i}+\frac{\partial f_2}{\partial z}\,\mathbf{j}+\frac{\partial f_3}{\partial z}\,\mathbf{k}\right)$$

$$=\left(\frac{\partial f_2}{\partial x}\,\mathbf{k}-\frac{\partial f_3}{\partial x}\,\mathbf{j}\right)+\left(-\frac{\partial f_1}{\partial y}\,\mathbf{k}+\frac{\partial f_3}{\partial y}\,\mathbf{i}\right)+\left(\frac{\partial f_1}{\partial z}\,\mathbf{j}-\frac{\partial f_2}{\partial z}\,\mathbf{i}\right)$$

$$=\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right)\mathbf{i}+\left(\frac{\partial f_1}{\partial z}-\frac{\partial f_3}{\partial x}\right)\mathbf{j}+\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right)\mathbf{k}.$$

Note. It should be noted that the expression for curl **f** can be written immediately if we treat the operator ∇ as a vector quantity. Thus

$$\text{Curl } \mathbf{f}=\nabla\times\mathbf{f}=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\times(f_1\,\mathbf{i}+f_2\,\mathbf{j}+f_3\,\mathbf{k})$$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ f_1 & f_2 & f_3\end{vmatrix}$$

$$=\begin{vmatrix}\frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ f_2 & f_3\end{vmatrix}\mathbf{i}-\begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial z}\\ f_1 & f_3\end{vmatrix}\mathbf{j}+\begin{vmatrix}\frac{\partial}{\partial x} & \frac{\partial}{\partial y}\\ f_1 & f_2\end{vmatrix}\mathbf{k}$$

$$=\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right)\mathbf{i}+\left(\frac{\partial f_1}{\partial z}-\frac{\partial f_3}{\partial x}\right)\mathbf{j}+\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right)\mathbf{k}.$$

But we must take care that in the expansion of the determinant the operators $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ must *precede* the functions f_1, f_2, f_3.

Irrotational vector. Definition. *A vector* $\mathbf{f}$ *is said to be irrotational if* $\nabla\times\mathbf{f}=\mathbf{0}$. **[Meerut 1991 S]**

§ 10. The Laplacian operator ∇^2.

The Laplacian operator ∇^2 *is defined as*

$$\nabla^2=\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}.$$

If f is a scalar point function, then

$$\nabla^2 f=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}.$$

It should be noted that $\nabla^2 f$ is also a scalar quantity.

If $\mathbf{f}$ is a vector point function, then

$$\nabla^2\mathbf{f}=\frac{\partial^2\mathbf{f}}{\partial x^2}+\frac{\partial^2\mathbf{f}}{\partial y^2}+\frac{\partial^2\mathbf{f}}{\partial z^2}.$$

It should be noted that $\nabla^2\mathbf{f}$ is also a vector quantity.

Laplace's equation. *The equation* $\nabla^2 f=0$ *is called Laplace's equation.* A function which satisfies Laplace's equation is called **a harmonic function.**

Solved Examples

Ex. 1. *Prove that div* $\mathbf{r}=3$.

[Agra 1978; Rohilkhand 81; Kanpur 75; Gorakhpur 88]

Sol. We have $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$.

By definition, div $\mathbf{r}=\nabla\cdot\mathbf{r}=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\cdot\mathbf{r}$

$$=\mathbf{i}\cdot\frac{\partial\mathbf{r}}{\partial x}+\mathbf{j}\cdot\frac{\partial\mathbf{r}}{\partial y}+\mathbf{k}\cdot\frac{\partial\mathbf{r}}{\partial z}$$

$$=\mathbf{i}\cdot\mathbf{i}+\mathbf{j}\cdot\mathbf{j}+\mathbf{k}\cdot\mathbf{k}\quad\left[\because\ \frac{\partial\mathbf{r}}{\partial x}=\mathbf{i},\ \frac{\partial\mathbf{r}}{\partial y}=\mathbf{j},\ \frac{\partial\mathbf{r}}{\partial z}=\mathbf{k}\right]$$

$$=1+1+1=3.$$

Ex. 2. *Prove that curl* $\mathbf{r}=\mathbf{0}$.

[Agra 1968; Kanpur 75, 79; Rohilkhand 76; Gorakhpur 88]

Sol. We have by definition

$$\text{Curl } \mathbf{r} = \nabla \times \mathbf{r} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \times \mathbf{r}$$

$$= \mathbf{i} \times \frac{\partial \mathbf{r}}{\partial x} + \mathbf{j} \times \frac{\partial \mathbf{r}}{\partial y} + \mathbf{k} \times \frac{\partial \mathbf{r}}{\partial z}$$

Now $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. $\therefore$ $\dfrac{\partial \mathbf{r}}{\partial x} = \mathbf{i}, \dfrac{\partial \mathbf{r}}{\partial y} = \mathbf{j}, \dfrac{\partial \mathbf{r}}{\partial z} = \mathbf{k}$.

$\therefore$ $\text{Curl } \mathbf{r} = \mathbf{i} \times \mathbf{i} + \mathbf{j} \times \mathbf{j} + \mathbf{k} \times \mathbf{k} = \mathbf{0} + \mathbf{0} + \mathbf{0} = \mathbf{0}$.

Ex. 3. *If* $\mathbf{f} = x^2 y\, \mathbf{i} - 2xz\, \mathbf{j} + 2yz\, \mathbf{k}$, *find*

(*i*) *div* $\mathbf{f}$, (*ii*) *curl* $\mathbf{f}$, (*iii*) *curl curl* $\mathbf{f}$. [Agra 1986]

Sol. (i) We have

$$\text{div } \mathbf{f} = \nabla \cdot \mathbf{f} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot (x^2 y\, \mathbf{i} - 2xz\, \mathbf{j} + 2yz\, \mathbf{k})$$

$$= \frac{\partial}{\partial x}(x^2 y) + \frac{\partial}{\partial y}(-2xz) + \frac{\partial}{\partial z}(2yz) = 2xy + 0 + 2y = 2y\,(x+1).$$

(ii) We have $\text{curl } \mathbf{f} = \nabla \times \mathbf{f} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 y & -2xz & 2yz \end{vmatrix}$

$$= \left[\frac{\partial}{\partial y}(2yz) - \frac{\partial}{\partial z}(-2xz)\right] \mathbf{i} - \left[\frac{\partial}{\partial x}(2yz) - \frac{\partial}{\partial z}(x^2 y)\right] \mathbf{j} + \left[\frac{\partial}{\partial x}(-2xz) - \frac{\partial}{\partial y}(x^2 y)\right] \mathbf{k}$$

$$= (2z + 2x)\, \mathbf{i} - 0\, \mathbf{j} + (-2z - x^2)\, \mathbf{k} = (2x + 2z)\, \mathbf{i} - (x^2 + 2z)\, \mathbf{k}.$$

(iii) We have $\text{curl curl } \mathbf{f} = \nabla \times (\nabla \times \mathbf{f})$

$$= \nabla \times [(2x + 2z)\, \mathbf{i} - (x^2 + 2z)\, \mathbf{k}]$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x + 2z & 0 & -x^2 - 2z \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(-x^2 - 2z)\right] \mathbf{i} - \left[\frac{\partial}{\partial x}(-x^2 - 2z) - \frac{\partial}{\partial z}(2x + 2z)\right] \mathbf{j} + \left[0 - \frac{\partial}{\partial y}(2x + 2z)\right] \mathbf{k}$$

$$= 0\, \mathbf{i} - (-2x - 2)\, \mathbf{j} + (0 - 0)\, \mathbf{k} = (2x + 2)\, \mathbf{j}.$$

Ex. 4. *Find the divergence and curl of the vector*
$\mathbf{f}=(x^2-y^2)\,\mathbf{i}+2xy\,\mathbf{j}+(y^2-xy)\,\mathbf{k}$. [Agra 1982]

Sol. We have div $\mathbf{f}=\nabla\cdot\mathbf{f}$

$$=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\cdot[(x^2-y^2)\,\mathbf{i}+2xy\,\mathbf{j}+(y^2-xy)\,\mathbf{k}]$$

$$=\frac{\partial}{\partial x}(x^2-y^2)+\frac{\partial}{\partial y}(2xy)+\frac{\partial}{\partial z}(y^2-xy)$$

$$=2x+2x+0=4x.$$

Also curl $\mathbf{f}=\nabla\times\mathbf{f}$

$$=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\times[(x^2-y^2)\,\mathbf{i}+2xy\,\mathbf{j}+(y^2-xy)\,\mathbf{k}]$$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ x^2-y^2 & 2xy & y^2-xy\end{vmatrix}$$

$$=\left[\frac{\partial}{\partial y}(y^2-xy)-\frac{\partial}{\partial z}(2xy)\right]\mathbf{i}+\left[\frac{\partial}{\partial z}(x^2-y^2)-\frac{\partial}{\partial x}(y^2-xy)\right]\mathbf{j}$$
$$+\left[\frac{\partial}{\partial x}(2xy)-\frac{\partial}{\partial y}(x^2-y^2)\right]\mathbf{k}$$

$$=[(2y-x)-0]\,\mathbf{i}+(0+y)\,\mathbf{j}+(2y+2y)\,\mathbf{k}$$
$$=(2y-x)\,\mathbf{i}+y\mathbf{j}+4y\,\mathbf{k}.$$

Ex. 5. *Find div* $\mathbf{F}$ *and curl* $\mathbf{F}$ *where*
$\mathbf{F}=grad\,(x^3+y^3+z^3-3xyz)$.

Sol. We have $\mathbf{F}=\text{grad}\,(x^3+y^3+z^3-3xyz)$

$$=\mathbf{i}\frac{\partial}{\partial x}(x^3+y^3+z^3-3xyz)+\mathbf{j}\frac{\partial}{\partial y}(x^3+y^3+z^3-3xyz)$$
$$+\mathbf{k}\frac{\partial}{\partial z}(x^3+y^3+z^3-3xyz)$$

$$=(3x^2-3yz)\,\mathbf{i}+(3y^2-3xz)\,\mathbf{j}+(3z^2-3xy)\,\mathbf{k}.$$

Now div $\mathbf{F}=\nabla\cdot\mathbf{F}$

$$=\frac{\partial}{\partial x}(3x^2-3yz)+\frac{\partial}{\partial y}(3y^2-3xz)+\frac{\partial}{\partial z}(3z^2-3xy)$$

$$=6x+6y+6z=6\,(x+y+z).$$

Also curl $\mathbf{F}=\nabla\times\mathbf{F}$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy\end{vmatrix}$$

$$=\left[\frac{\partial}{\partial y}(3z^2-3xy)-\frac{\partial}{\partial z}(3y^2-3xz)\right]\mathbf{i}$$
$$+\left[\frac{\partial}{\partial z}(3x^2-3yz)-\frac{\partial}{\partial x}(3z^2-3xy)\right]\mathbf{j}$$
$$+\left[\frac{\partial}{\partial x}(3y^2-3xz)-\frac{\partial}{\partial y}(3x^2-3yz)\right]\mathbf{k}$$
$$=(-3x+3x)\,\mathbf{i}+(-3y+3y)\,\mathbf{j}+(-3z+3z)\,\mathbf{k}$$
$$=0\mathbf{i}+0\mathbf{j}+0\mathbf{k}=\mathbf{0}.$$

Ex. 6. *Given $\phi=2x^3y^2z^4$, find div (grad ϕ).*

Sol. We have grad $\phi=$ grad $(2x^3y^2z^4)$

$$=\mathbf{i}\frac{\partial}{\partial x}(2x^3y^2z^4)+\mathbf{j}\frac{\partial}{\partial y}(2x^3y^2z^4)+\mathbf{k}\frac{\partial}{\partial z}(2x^3y^2z^4)$$
$$=6x^2y^2z^4\,\mathbf{i}+4x^3yz^4\,\mathbf{j}+8x^3y^2z^3\,\mathbf{k}.$$

Now div (grad ϕ) $=\nabla\cdot$(grad ϕ) $=\nabla\cdot(6x^2y^2z^4\,\mathbf{i}+4x^3yz^4\,\mathbf{j}+8x^3y^2z^3\,\mathbf{k})$

$$=\frac{\partial}{\partial x}(6x^2y^2z^4)+\frac{\partial}{\partial y}(4x^3yz^4)+\frac{\partial}{\partial z}(8x^3y^2z^3)$$
$$=12xy^2z^4+4x^3z^4+24x^3y^2z^2.$$

Ex. 7. *If $\mathbf{f}=xy^2\,\mathbf{i}+2x^2yz\,\mathbf{j}-3yz^2\,\mathbf{k}$, find div $\mathbf{f}$ and curl $\mathbf{f}$. What are their values at the point $(1,-1,1)$?*

[Rohilkhand 1982]

Sol. We have div $\mathbf{f}=\nabla\cdot\mathbf{f}$

$$=\frac{\partial}{\partial x}(xy^2)+\frac{\partial}{\partial y}(2x^2yz)+\frac{\partial}{\partial z}(-3yz^2)$$
$$=y^2+2x^2z-6yz.$$

$\therefore$ div $\mathbf{f}$ at $(1,-1,1)=(-1)^2+2.1^2.1-6.(-1).1=1+2+6=9.$

Also curl $\mathbf{f}=\nabla\times\mathbf{f}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ xy^2 & 2x^2yz & -3yz^2\end{vmatrix}$

$$=\left[\frac{\partial}{\partial y}(-3yz^2)-\frac{\partial}{\partial z}(2x^2yz)\right]\mathbf{i}-\left[\frac{\partial}{\partial x}(-3yz^2)-\frac{\partial}{\partial z}(xy^2)\right]\mathbf{j}$$
$$+\left[\frac{\partial}{\partial x}(2x^2yz)-\frac{\partial}{\partial y}(xy^2)\right]\mathbf{k}$$
$$=(-3z^2-2x^2y)\,\mathbf{i}-(0-0)\,\mathbf{j}+(4xyz-2xy)\,\mathbf{k}$$
$$=-(3z^2+2x^2y)\,\mathbf{i}+(4xyz-2xy)\,\mathbf{k}.$$

$\therefore$ curl $\mathbf{f}$ at $(1,-1,1)=-[3\cdot1^2+2.1^2.(-1)]\,\mathbf{i}$
$$+[4.1.(-1).1-2.1.(-1)]\,\mathbf{k}$$

$$=-\mathbf{i}-2\mathbf{k}.$$

Ex. 8. *If* $\mathbf{F}=x^2z\,\mathbf{i}-2y^3z^2\,\mathbf{j}+xy^2z\,\mathbf{k}$, *find div* $\mathbf{F}$, *curl* $\mathbf{F}$ *at* $(1, -1, 1)$. [Garhwal 1979; Madras 78]

Sol. We have div $\mathbf{F}=\nabla\cdot\mathbf{F}$

$$=\frac{\partial}{\partial x}(x^2z)+\frac{\partial}{\partial y}(-2y^3z^2)+\frac{\partial}{\partial z}(xy^2z)$$

$$=2xz-6y^2z^2+xy^2.$$

$\therefore$ div $\mathbf{F}$ at $(1, -1, 1)=2.1.1-6.(-1)^2.1^2+1.(-1)^2$

$$=2-6+1=-3.$$

Also curl $\mathbf{F}=\nabla\times\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ x^2z & -2y^3z^2 & xy^2z\end{vmatrix}$

$$=\left[\frac{\partial}{\partial y}(xy^2z)-\frac{\partial}{\partial z}(-2y^3z^2)\right]\mathbf{i}+\left[\frac{\partial}{\partial z}(x^2z)-\frac{\partial}{\partial x}(xy^2z)\right]\mathbf{j}$$

$$+\left[\frac{\partial}{\partial x}(-2y^3z^2)-\frac{\partial}{\partial y}(x^2z)\right]\mathbf{k}$$

$$=(2xyz+4y^3z)\,\mathbf{i}+(x^2-y^2z)\,\mathbf{j}+(0-0)\,\mathbf{k}$$

$$=2\,(xyz+2y^3z)\,\mathbf{i}+(x^2-y^2z)\,\mathbf{j}.$$

$\therefore$ curl $\mathbf{F}$ at $(1, -1, 1)=2\,[1.(-1).1+2.(-1)^3.1]\,\mathbf{i}$

$$+[1^2-(-1)^2.1]\,\mathbf{j}$$

$$=2(-1-2)\,\mathbf{i}+(1-1)\,\mathbf{j}=-6\mathbf{i}+0\mathbf{j}$$

$$=-6\mathbf{i}.$$

Ex. 9. *If* $\mathbf{F}=(y^2+z^2-x^2)\,\mathbf{i}+(z^2+x^2-y^2)\,\mathbf{j}+(x^2+y^2-z^2)\,\mathbf{k}$, *find div* $\mathbf{f}$ *and curl* $\mathbf{f}$.

Sol. We have div $\mathbf{f}=\nabla\cdot\mathbf{f}$

$$=\frac{\partial}{\partial x}(y^2+z^2-x^2)+\frac{\partial}{\partial y}(z^2+x^2-y^2)+\frac{\partial}{\partial z}(x^2+y^2-z^2)$$

$$=-2x-2y-2z=-2\,(x+y+z).$$

Also curl $\mathbf{f}=\nabla\times\mathbf{f}$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ y^2+z^2-x^2 & z^2+x^2-y^2 & x^2+y^2-z^2\end{vmatrix}$$

$$=\left[\frac{\partial}{\partial y}(x^2+y^2-z^2)-\frac{\partial}{\partial z}(z^2+x^2-y^2)\right]\mathbf{i}$$

$$+\left[\frac{\partial}{\partial z}(y^2+z^2-x^2)-\frac{\partial}{\partial x}(x^2+y^2-z^2)\right]\mathbf{j}$$

$$+\left[\frac{\partial}{\partial x}(z^2+x^2-y^2)-\frac{\partial}{\partial y}(y^2+z^2-x^2)\right]\mathbf{k}$$

$$=(2y-2z)\,\mathbf{i}+(2z-2x)\,\mathbf{j}+(2x-2y)\,\mathbf{k}$$

$$=2\,(y-z)\,\mathbf{i}+2\,(z-x)\,\mathbf{j}+2\,(x-y)\,\mathbf{k}.$$

Ex. 10. *If* $\mathbf{f}=(x+y+1)\,\mathbf{i}+\mathbf{j}+(-x-y)\,\mathbf{k}$, *prove that* $\mathbf{f}\cdot \textit{curl}\ \mathbf{f}=0.$ [Kanpur 1988; Agra 86]

Sol. We have curl $\mathbf{f}=\nabla\times\mathbf{f}$

$$=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x+y+1 & 1 & -x-y \end{vmatrix}$$

$$=\left[\frac{\partial}{\partial y}(-x-y)-\frac{\partial}{\partial z}(1)\right]\mathbf{i}+\left[\frac{\partial}{\partial z}(x+y+1)-\frac{\partial}{\partial x}(-x-y)\right]\mathbf{j}$$

$$+\left[\frac{\partial}{\partial x}(1)-\frac{\partial}{\partial y}(x+y+1)\right]\mathbf{k}$$

$$=(-1-0)\,\mathbf{i}+(0+1)\,\mathbf{j}+(0-1)\,\mathbf{k}=-\mathbf{i}+\mathbf{j}-\mathbf{k}.$$

$$\therefore\quad \mathbf{f}\cdot\text{curl } \mathbf{f}=[(x+y+1)\,\mathbf{i}+\mathbf{j}+(-x-y)\,\mathbf{k}]\cdot(-\mathbf{i}+\mathbf{j}-\mathbf{k})$$

$$=(x+y+1)\cdot(-1)+1\cdot 1+(-x-y)\cdot(-1)$$

$$=-x-y-1+1+x+y=0.$$

Ex. 11. *If* $u=3x^2y$ *and* $v=xz^2-2y$, *then find* *grad* $[(\textit{grad}\ u)\cdot(\textit{grad}\ v)]$.

Sol. We have grad $u=\mathbf{i}\dfrac{\partial}{\partial x}(3x^2y)+\mathbf{j}\dfrac{\partial}{\partial y}(3x^2y)+\mathbf{k}\dfrac{\partial}{\partial z}(3x^2y)$

$$=6xy\,\mathbf{i}+3x^2\,\mathbf{j}+0\,\mathbf{k}=6xy\,\mathbf{i}+3x^3\,\mathbf{j}.$$

Also grad $v=\mathbf{i}\dfrac{\partial}{\partial x}(xz^2-2y)+\mathbf{j}\dfrac{\partial}{\partial y}(xz^2-2y)+\mathbf{k}\dfrac{\partial}{\partial z}(xz^2-2y)$

$$=z^2\,\mathbf{i}-2\mathbf{j}+2xz\,\mathbf{k}.$$

$$\therefore\quad (\text{grad } u)\cdot(\text{grad } v)=(6xy\,\mathbf{i}+3x^2\,\mathbf{j})\cdot(z^2\,\mathbf{i}-2\mathbf{j}+2xz\,\mathbf{k})$$

$$=6xyz^2-6x^2.$$

Hence grad $[(\text{grad } u)\cdot(\text{grad } v)]=\text{grad }(6xyz^2-6x^2)$

$$=\mathbf{i}\frac{\partial}{\partial x}(6xyz^2-6x^2)+\frac{\partial}{\partial y}(6xyz^2-6x^2)+\mathbf{k}\frac{\partial}{\partial z}(6xyz^2-6x^2)$$

$$=(6yz^2-12x)\,\mathbf{i}+(0xz^2)\,\mathbf{j}+(12xyz)\,\mathbf{k}.$$

Ex. 12. *If* $u=x^2-y^2+4z$, *show that* $\nabla^2 u=0.$

Sol. We have $\nabla^2 u=\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^3}+\dfrac{\partial^2}{\partial z^2}\right)u$

$$=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$$

Now $\frac{\partial u}{\partial x}=\frac{\partial}{\partial x}(x^2-y^2+4z)=2x.$

$\therefore \quad \frac{\partial^2 u}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)=\frac{\partial}{\partial x}(2x)=2.$

Again $\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}(x^2-y^2+4z)=-2y.$

$\therefore \quad \frac{\partial^2 u}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right)=\frac{\partial}{\partial y}(-2y)=-2.$

Finally $\frac{\partial u}{\partial z}=\frac{\partial}{\partial z}(x^2-y^2+4z)=4.$

$\therefore \quad \frac{\partial^2 u}{\partial z^2}=\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right)=\frac{\partial}{\partial z}(4)=0.$

$\therefore \quad \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}=2-2+0=0.$

Hence $\nabla^2 u=0.$

Ex. 13. *If* $\mathbf{f}=f_1\,\mathbf{i}+f_2\,\mathbf{j}+f_3\,\mathbf{k}$, *show that* $\nabla\cdot\mathbf{f}=\nabla f_1\cdot\mathbf{i}+\nabla f_2\cdot\mathbf{j}+\nabla f_3\cdot\mathbf{k}.$

Sol. We have $\nabla f_1=\frac{\partial f_1}{\partial x}\,\mathbf{i}+\frac{\partial f_1}{\partial y}\,\mathbf{j}+\frac{\partial f_1}{\partial z}\,\mathbf{k}.$

$\therefore \quad \nabla f_1\cdot\mathbf{i}=\left(\frac{\partial f_1}{\partial x}\,\mathbf{i}+\frac{\partial f_1}{\partial y}\,\mathbf{j}+\frac{\partial f_1}{\partial z}\,\mathbf{k}\right)\cdot\mathbf{i}=\frac{\partial f_1}{\partial x}.$

Similarly $\nabla f_2\cdot\mathbf{j}=\frac{\partial f_2}{\partial y}$ and $\nabla f_3\cdot\mathbf{k}=\frac{\partial f_3}{\partial z}.$

$\therefore \quad \nabla f_1\cdot\mathbf{i}+\nabla f_2\cdot\mathbf{j}+\nabla f_3\cdot\mathbf{k}$

$$=\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}=\text{div}\,\mathbf{f}=\nabla\cdot\mathbf{f}.$$

Ex. 14. *Prove that* $\nabla\cdot(r^3\mathbf{r})=6r^3.$

Sol. We have $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}.$

$\therefore \quad r^3\mathbf{r}=r^3(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})=r^3x\mathbf{i}+r^3y\mathbf{j}+r^3z\mathbf{k}.$

$\therefore \quad \nabla\cdot(r^3\mathbf{r})=\text{div}\,(r^3\mathbf{r})=\frac{\partial}{\partial x}(r^3x)+\frac{\partial}{\partial y}(r^3y)+\frac{\partial}{\partial z}(r^3z)$

$$=r^3+3r^2x\frac{\partial r}{\partial x}+r^3+3r^2y\frac{\partial r}{\partial y}+r^3+3r^2z\frac{\partial r}{\partial z}$$

$$=3r^3+3r^2\left(x\frac{\partial r}{\partial x}+y\frac{\partial r}{\partial y}+z\frac{\partial r}{\partial z}\right) \qquad \ldots(1)$$

Now $r^2=x^2+y^2+z^2.$

$\therefore \quad 2r\frac{\partial r}{\partial x}=2x$ or $\frac{\partial r}{\partial x}=\frac{x}{r}.$

Similarly $\frac{\partial r}{\partial y}=\frac{y}{r}$ and $\frac{\partial r}{\partial z}=\frac{z}{r}$.

$\therefore$ from (1), $\nabla\cdot(r^3\mathbf{r})=3r^3+3r^2\left(x.\frac{x}{r}+y.\frac{y}{r}+z.\frac{z}{r}\right)$

$$=3r^3+3r^2\left(\frac{x^2+y^2+z^2}{r}\right)=3r^3+3r^2.\frac{r^2}{r}$$

$$=3r^3+3r^3=6r^3.$$

Ex. 15. *Find the constants a, b, c so that the vector* $\mathbf{F}=(x+2y+az)\,\mathbf{i}+(bx-3y-z)\,\mathbf{j}+(4x+cy+2z)\,\mathbf{k}$ *is irrotational.*

Sol. The vector $\mathbf{F}$ is irrotational if curl $\mathbf{F}=\mathbf{0}$.

We have curl $\mathbf{F}=\nabla\times\mathbf{F}$

$$=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$$

$$=\left[\frac{\partial}{\partial y}(4x+cy+z)-\frac{\partial}{\partial z}(bx-3y-z)\right]\mathbf{i}$$

$$+\left[\frac{\partial}{\partial z}(x+2y+az)-\frac{\partial}{\partial x}(4x+cy+2z)\right]\mathbf{j}$$

$$+\left[\frac{\partial}{\partial x}(bx-3y-z)-\frac{\partial}{\partial y}(x+2y+az)\right]\mathbf{k}$$

$$=(c+1)\,\mathbf{i}+(a-4)\,\mathbf{j}+(b-2)\,\mathbf{k}.$$

$\therefore$ curl $\mathbf{F}=\mathbf{0} \Rightarrow (c+1)\,\mathbf{i}+(a-4)\,\mathbf{j}+(b-2)\,\mathbf{k}=\mathbf{0}$

$$\Rightarrow c+1=0,\ a-4=0,\ b-2=0$$

$$\Rightarrow c=-1,\ a=4,\ b=2.$$

Hence the vector $\mathbf{F}$ is irrotational if $a=4$, $b=2$, $c=-1$.

Ex. 16. *Determine the constant a so that the vector* $\mathbf{V}=(x+3y)\,\mathbf{i}+(y-2z)\,\mathbf{j}+(x+az)\,\mathbf{k}$ *is solenoidal.* **[Kanpur 1978]**

Sol. A vector $\mathbf{V}$ is said to be solenoidal if div $\mathbf{V}=\mathbf{0}$.

We have div $\mathbf{V}=\nabla\cdot\mathbf{V}=\frac{\partial}{\partial x}(x+3y)+\frac{\partial}{\partial y}(y-2z)+\frac{\partial}{\partial z}(x+az)$

$$=1+1+a=2+a.$$

Now div $\mathbf{V}=0$ if $2+a=0$ *i.e.* if $a=-2$.

Ex. 17. *Show that the vector* $\mathbf{V}=(\sin y+z)\,\mathbf{i}+(x\cos y-z)\,\mathbf{j}+(x-y)\,\mathbf{k}$ *is irrotational.*

Sol. A vector $\mathbf{V}$ is said to be irrotational if curl $\mathbf{V}=\mathbf{0}$.

We have curl $\mathbf{V}=\nabla\times\mathbf{V}$

$$=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin y+z & x\cos y-z & x-y \end{vmatrix}$$

$$=\left[\frac{\partial}{\partial y}(x-y)-\frac{\partial}{\partial z}(x\cos y-z)\right]\mathbf{i}-\left[\frac{\partial}{\partial x}(x-y)-\frac{\partial}{\partial z}(\sin y+z)\right]\mathbf{j}$$
$$+\left[\frac{\partial}{\partial x}(x\cos y-z)-\frac{\partial}{\partial y}(\sin y+z)\right]\mathbf{k}$$

$$=(-1+1)\,\mathbf{i}-(1-1)\,\mathbf{j}+(\cos y-\cos y)\,\mathbf{k}=\mathbf{0}.$$

$\therefore$ **V** is irrotational.

Ex. 18. *If* **V** *is a constant vector, show that*

(*i*) *div* $\mathbf{V}=0$, (*ii*) *curl* $\mathbf{V}=\mathbf{0}$.

Sol. (i). We have $\text{div } \mathbf{V}=\mathbf{i}\cdot\frac{\partial \mathbf{V}}{\partial x}+\mathbf{j}\cdot\frac{\partial \mathbf{V}}{\partial y}+\mathbf{k}\cdot\frac{\partial \mathbf{V}}{\partial z}$

$$=\mathbf{i}\cdot\mathbf{0}+\mathbf{j}\cdot\mathbf{0}+\mathbf{k}\cdot\mathbf{0}=0.$$

(ii) We have $\text{curl } \mathbf{V}=\mathbf{i}\times\frac{\partial \mathbf{V}}{\partial x}+\mathbf{j}\times\frac{\partial \mathbf{V}}{\partial y}+\mathbf{k}\times\frac{\partial \mathbf{V}}{\partial z}$

$$=\mathbf{i}\times\mathbf{0}+\mathbf{j}\times\mathbf{0}+\mathbf{k}\times\mathbf{0}=\mathbf{0}.$$

Ex. 19. *If* **a** *is a constant vector, find*

(*i*) *div* $(\mathbf{r}\times\mathbf{a})$, **[Rohilkhand 1981, 84]**

2007 (*ii*) *curl* $(\mathbf{r}\times\mathbf{a})$. **[Rohilkhand 1984; Indore 83]**

Sol. We have $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$.

Let $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$. Then the scalars a_1, a_2, a_3 are all constants.

We have $\mathbf{r}\times\mathbf{a}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ a_1 & a_2 & a_3 \end{vmatrix}$

$$=(a_3y-a_2z)\,\mathbf{i}+(a_1z-a_3x)\,\mathbf{j}+(a_2x-a_1y)\,\mathbf{k}.$$

(i) $\text{div}\,(\mathbf{r}\times\mathbf{a})=\frac{\partial}{\partial x}(a_3y-a_2z)+\frac{\partial}{\partial y}(a_1z-a_3x)+\frac{\partial}{\partial z}(a_2x-a_1y)$

$$=0+0+0=0.$$

(ii) $\text{curl}\,(\mathbf{r}\times\mathbf{a})=\nabla\times(\mathbf{r}\times\mathbf{a})$

$$=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_3y-a_2z & a_1z-a_3x & a_2x-a_1y \end{vmatrix}$$

$$=\left[\frac{\partial}{\partial y}(a_2x-a_1y)-\frac{\partial}{\partial z}(a_1z-a_3x)\right]\mathbf{i}-\left[\frac{\partial}{\partial x}(a_2x-a_1y)\right.$$

$$\left.-\frac{\partial}{\partial z}(a_3y-a_2z)\right]\mathbf{j}+\left[\frac{\partial}{\partial x}(a_1z-a_3x)-\frac{\partial}{\partial y}(a_3y-a_2z)\right]\mathbf{k}$$

$$=-2a_1\mathbf{i}-2a_2\mathbf{j}-2a_3\mathbf{k}=-2(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})=-2\mathbf{a}.$$

Ex. 20. *If* $\mathbf{V}=e^{xyz}(\mathbf{i}+\mathbf{j}+\mathbf{k})$, *find curl* $\mathbf{V}$.

[Meerut 1991 P; Kanpur 87; Agra 83]

Sol. We have curl $\mathbf{V}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^{xyz} & e^{xyz} & e^{xyz} \end{vmatrix}$

$$=\left[\frac{\partial}{\partial y}(e^{xyz})-\frac{\partial}{\partial z}(e^{xyz})\right]\mathbf{i}+\left[\frac{\partial}{\partial z}(e^{xyz})-\frac{\partial}{\partial x}(e^{xyz})\right]\mathbf{j}$$

$$+\left[\frac{\partial}{\partial x}(e^{xyz})-\frac{\partial}{\partial y}(e^{xyz})\right]\mathbf{k}$$

$$=e^{xyz}(xz-xy)\mathbf{i}+e^{xyz}(xy-yz)\mathbf{j}+e^{xyz}(yz-xz)\mathbf{k}.$$

Ex. 21. *Evaluate div* $\mathbf{f}$ *where*

$$\mathbf{f}=2x^2z\mathbf{i}-xy^2z\mathbf{j}+3y^2x\mathbf{k}.$$ [Kanpur 1988]

Sol. We have

$$\text{div}\,\mathbf{f}=\nabla\cdot\mathbf{f}=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\cdot(2x^2z\mathbf{i}-xy^2z\mathbf{j}+3y^2x\,\mathbf{k})$$

$$=\frac{\partial}{\partial x}(2x^2z)+\frac{\partial}{\partial y}(-xy^2z)+\frac{\partial}{\partial z}(3y^2x)$$

$$=4xz-2xyz+0=2xz(2-y).$$

Ex. 22. *Show that* $\nabla^2(x/r^3)=0$.

[Meerut 1991 P; Rohilkhand 92]

Sol. $\nabla^2\left(\dfrac{x}{r^3}\right)=\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}+\dfrac{\partial^2}{\partial z^2}\right)\left(\dfrac{x}{r^3}\right).$

Now $\dfrac{\partial^2}{\partial x^2}\left(\dfrac{x}{r^3}\right)=\dfrac{\partial}{\partial x}\left\{\dfrac{\partial}{\partial x}\left(\dfrac{x}{r^3}\right)\right\}=\dfrac{\partial}{\partial x}\left\{\dfrac{1}{r^3}-\dfrac{3x}{r^4}\dfrac{\partial r}{\partial x}\right\}$

$$=\frac{\partial}{\partial x}\left\{\frac{1}{r^3}-\frac{3x}{r^4}\frac{x}{r}\right\}\left[\because\ r^2=x^2+y^2+z^2 \text{ gives } \frac{\partial r}{\partial x}=\frac{x}{r}\right]$$

$$=\frac{\partial}{\partial x}\left\{\frac{1}{r^3}-\frac{3x^2}{r^5}\right\}=-\frac{3}{r^4}\frac{\partial r}{\partial x}-\frac{6x}{r^5}+\frac{15x^2}{r^6}\frac{\partial r}{\partial x}$$

$$=-\frac{3}{r^4}\frac{x}{r}-\frac{6x}{r^5}+\frac{15x^2}{r^6}\frac{x}{r}=-\frac{9x}{r^5}+\frac{15x^3}{r^7}.$$

Again $\dfrac{\partial^2}{\partial y^2}\left(\dfrac{x}{r^3}\right)=\dfrac{\partial}{\partial y}\left\{\dfrac{\partial}{\partial y}\left(\dfrac{x}{r^3}\right)\right\}=\dfrac{\partial}{\partial y}\left\{-\dfrac{3x}{r^4}\dfrac{\partial r}{\partial y}\right\}$

$$=\frac{\partial}{\partial y}\left\{-\frac{3x}{r^4}\frac{y}{r}\right\}\qquad\left[\because\ \frac{\partial r}{\partial y}=\frac{y}{r}\right]$$

$$=\frac{\partial}{\partial y}\left(-\frac{3xy}{r^5}\right)=-\frac{3x}{r^5}+\frac{15xy}{r^6}\frac{\partial r}{\partial y}=-\frac{3x}{r^5}+\frac{15xy^2}{r^7}.$$

Similarly $\dfrac{\partial^2}{\partial z^2}\left(\dfrac{x}{r^3}\right)=-\dfrac{3x}{r^5}+\dfrac{15xz^2}{r^7}.$

Therefore adding we get

$$\nabla^2\left(\frac{x}{r^3}\right)=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\left(\frac{x}{r^3}\right)$$

$$=-\frac{9x}{r^5}+\frac{15x^3}{r^7}-\frac{3x}{r^5}+\frac{15xy^2}{r^7}-\frac{3x}{r^5}+\frac{15xz^2}{r^7}$$

$$=-\frac{15x}{r^5}+\frac{15x}{r^7}(x^2+y^2+z^2)=-\frac{15x}{r^5}+\frac{15x}{r^7}r^2=0.$$

§ 11. Important Vector Identities.

1. *Prove that div* $(\mathbf{A}+\mathbf{B})=$ *div* $\mathbf{A}+$ *div* $\mathbf{B}$

or $\nabla\cdot(\mathbf{A}+\mathbf{B})=\nabla\cdot\mathbf{A}+\nabla\cdot\mathbf{B}$. [Meerut 1992]

Proof. We have

$$\text{div}\,(\mathbf{A}+\mathbf{B})=\nabla\cdot(\mathbf{A}+\mathbf{B})=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\cdot(\mathbf{A}+\mathbf{B})$$

$$=\mathbf{i}\cdot\frac{\partial}{\partial x}(\mathbf{A}+\mathbf{B})+\mathbf{j}\cdot\frac{\partial}{\partial y}(\mathbf{A}+\mathbf{B})+\mathbf{k}\cdot\frac{\partial}{\partial z}(\mathbf{A}+\mathbf{B})$$

$$=\mathbf{i}\cdot\left(\frac{\partial\mathbf{A}}{\partial x}+\frac{\partial\mathbf{B}}{\partial x}\right)+\mathbf{j}\cdot\left(\frac{\partial\mathbf{A}}{\partial y}+\frac{\partial\mathbf{B}}{\partial y}\right)+\mathbf{k}\cdot\left(\frac{\partial\mathbf{A}}{\partial z}+\frac{\partial\mathbf{B}}{\partial z}\right)$$

$$=\left(\mathbf{i}\cdot\frac{\partial\mathbf{A}}{\partial x}+\mathbf{j}\cdot\frac{\partial\mathbf{A}}{\partial y}+\mathbf{k}\cdot\frac{\partial\mathbf{A}}{\partial z}\right)+\left(\mathbf{i}\cdot\frac{\partial\mathbf{B}}{\partial x}+\mathbf{j}\cdot\frac{\partial\mathbf{B}}{\partial y}+\mathbf{k}\cdot\frac{\partial\mathbf{B}}{\partial z}\right)$$

$$=\nabla\cdot\mathbf{A}+\nabla\cdot\mathbf{B}=\text{div}\,\mathbf{A}+\text{div}\,\mathbf{B}.$$

2. *Prove that curl* $(\mathbf{A}+\mathbf{B})=$ *curl* $\mathbf{A}+$ *curl* $\mathbf{B}$

or $\nabla\times(\mathbf{A}+\mathbf{B})=\nabla\times\mathbf{A}+\nabla\times\mathbf{B}$.

Proof. We have curl $(\mathbf{A}+\mathbf{B})=\nabla\times(\mathbf{A}+\mathbf{B})$

$$=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\times(\mathbf{A}+\mathbf{B})=\Sigma\,\mathbf{i}\times\frac{\partial}{\partial x}(\mathbf{A}+\mathbf{B})=\Sigma\,\mathbf{i}\times\left(\frac{\partial\mathbf{A}}{\partial x}+\frac{\partial\mathbf{B}}{\partial x}\right)$$

$$=\Sigma \mathbf{i}\times\frac{\partial \mathbf{A}}{\partial x}+\Sigma \mathbf{i}\times\frac{\partial \mathbf{B}}{\partial x}=\text{curl }\mathbf{A}+\text{curl }\mathbf{B}.$$

3. *If* A *is a differentiable vector function and* ϕ *is a differentiable scalar function, then*

$$\textit{div}\,(\phi\mathbf{A})=(\textit{grad}\,\phi)\cdot\mathbf{A}+\phi\ \textit{div}\ \mathbf{A}$$

or $$\nabla\cdot(\phi\mathbf{A})=(\nabla\phi)\cdot\mathbf{A}+\phi\,(\nabla\cdot\mathbf{A}).$$

[Meerut B. Sc. Physics 1983; Gorakhpur 85; Garhwal 84; Rohilkhand 82; Agra 81; Bombay 86]

Proof. We have

$$\text{div}\,(\phi\mathbf{A})=\nabla\cdot(\phi\mathbf{A})=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\cdot(\phi\mathbf{A})$$

$$=\mathbf{i}\cdot\frac{\partial}{\partial x}(\phi\mathbf{A})+\mathbf{j}\cdot\frac{\partial}{\partial y}(\phi\mathbf{A})+\mathbf{k}\cdot\frac{\partial}{\partial z}(\phi\mathbf{A})$$

$$=\Sigma\left\{\mathbf{i}\cdot\left(\frac{\partial}{\partial x}(\phi\mathbf{A})\right)\right\}=\Sigma\left\{\mathbf{i}\cdot\left(\frac{\partial\phi}{\partial x}\mathbf{A}+\phi\frac{\partial\mathbf{A}}{\partial x}\right)\right\}$$

$$=\Sigma\left\{\mathbf{i}\cdot\left(\frac{\partial\phi}{\partial x}\mathbf{A}\right)\right\}+\Sigma\left\{\mathbf{i}\cdot\left(\phi\frac{\partial\mathbf{A}}{\partial x}\right)\right\}$$

$$=\Sigma\left\{\left(\frac{\partial\phi}{\partial x}\mathbf{i}\right)\cdot\mathbf{A}\right\}+\Sigma\left\{\phi\left(\mathbf{i}\cdot\frac{\partial\mathbf{A}}{\partial x}\right)\right\}$$

[Note $\mathbf{a}\cdot(m\mathbf{b})=(m\mathbf{a})\cdot\mathbf{b}=m\,(\mathbf{a}\cdot\mathbf{b})$]

$$=\left\{\Sigma\frac{\partial\phi}{\partial x}\mathbf{i}\right\}\cdot\mathbf{A}+\phi\Sigma\left(\mathbf{i}\cdot\frac{\partial\mathbf{A}}{\partial x}\right)=(\nabla\phi)\cdot\mathbf{A}+\phi\,(\nabla\cdot\mathbf{A}).$$

4. *Prove that curl* $(\phi\mathbf{A})=(\textit{grad}\,\phi)\times\mathbf{A}+\phi$ *curl* A

or $$\nabla\times(\phi\mathbf{A})=(\nabla\phi)\times\mathbf{A}+\phi\,(\nabla\times\mathbf{A}).$$

[Agra 1968; Meerut 67, 68, 72; Bombay 68; Kanpur 76; Punjab 63]

Proof. We have

$$\text{curl}\,(\phi\mathbf{A})=\nabla\times(\phi\mathbf{A})=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\times(\phi\mathbf{A})$$

$$=\Sigma\left\{\mathbf{i}\times\frac{\partial}{\partial x}(\phi\mathbf{A})\right\}=\Sigma\left\{\mathbf{i}\times\left(\frac{\partial\phi}{\partial x}\mathbf{A}+\phi\frac{\partial\mathbf{A}}{\partial x}\right)\right\}$$

$$=\Sigma\left\{\mathbf{i}\times\left(\frac{\partial\phi}{\partial x}\mathbf{A}\right)\right\}+\Sigma\left\{\mathbf{i}\times\left(\phi\frac{\partial\mathbf{A}}{\partial x}\right)\right\}$$

$$=\Sigma\left\{\left(\frac{\partial\phi}{\partial x}\mathbf{i}\right)\times\mathbf{A}\right\}+\Sigma\left\{\phi\left(\mathbf{i}\times\frac{\partial\mathbf{A}}{\partial x}\right)\right\}$$

[Note that $\mathbf{a}\times(m\mathbf{b})=(m\mathbf{a})\times\mathbf{b}=m\,(\mathbf{a}\times\mathbf{b})$]

$$=\left\{\Sigma\left(\frac{\partial\phi}{\partial x}\mathbf{i}\right)\right\}\times\mathbf{A}+\phi\Sigma\left(\mathbf{i}\times\frac{\partial\mathbf{A}}{\partial x}\right)=(\nabla\phi)\times\mathbf{A}+\phi\,(\nabla\times\mathbf{A}).$$

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5. *Prove that* $div\,(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot curl\,\mathbf{A}-\mathbf{A}\cdot curl\,\mathbf{B}$

or $\nabla\cdot(\mathbf{A}\times\mathbf{B})=\mathbf{B}(\nabla\times\mathbf{A})-\mathbf{A}\cdot(\nabla\times\mathbf{B})$.

[Agra 1984, 85; Kanpur 87; Calicut 74; Allahabad 79; Gorakhpur 88]

Proof. We have

$$\text{div}\,(\mathbf{A}\times\mathbf{B})=\Sigma\left\{\mathbf{i}\cdot\frac{\partial}{\partial x}(\mathbf{A}\times\mathbf{B})\right\}=\Sigma\left\{\mathbf{i}\cdot\left(\frac{\partial\mathbf{A}}{\partial x}\times\mathbf{B}+\mathbf{A}\times\frac{\partial\mathbf{B}}{\partial x}\right)\right\}$$

$$=\Sigma\left\{\mathbf{i}\cdot\left(\frac{\partial\mathbf{A}}{\partial x}\times\mathbf{B}\right)\right\}+\Sigma\left\{\mathbf{i}\cdot\left(\mathbf{A}\times\frac{\partial\mathbf{B}}{\partial x}\right)\right\}$$

$$=\Sigma\left\{\left(\mathbf{i}\times\frac{\partial\mathbf{A}}{\partial x}\right)\cdot\mathbf{B}\right\}-\Sigma\left\{\mathbf{i}\cdot\left(\frac{\partial\mathbf{B}}{\partial x}\times\mathbf{A}\right)\right\}$$

[Note $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$ and $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=-\mathbf{a}\cdot(\mathbf{c}\times\mathbf{b})$]

$$=\left\{\Sigma\left(\mathbf{i}\times\frac{\partial\mathbf{A}}{\partial x}\right)\right\}\cdot\mathbf{B}-\Sigma\left\{\left(\mathbf{i}\times\frac{\partial\mathbf{B}}{\partial x}\right)\cdot\mathbf{A}\right\}=(\text{curl}\,\mathbf{A})\cdot\mathbf{B}-\left\{\Sigma\left(\mathbf{i}\times\frac{\partial\mathbf{B}}{\partial x}\right)\right\}\cdot\mathbf{A}$$

$$=(\text{curl}\,\mathbf{A})\cdot\mathbf{B}-(\text{curl}\,\mathbf{B})\cdot\mathbf{A}=\mathbf{B}\cdot\text{curl}\,\mathbf{A}-\mathbf{A}\cdot\text{curl}\,\mathbf{B}.$$

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6. *Prove that*

$curl\,(\mathbf{A}\times\mathbf{B})=(\mathbf{B}\cdot\nabla)\,\mathbf{A}-\mathbf{B}\,div\,\mathbf{A}-(\mathbf{A}\cdot\nabla)\,\mathbf{B}+\mathbf{A}\,div\,\mathbf{B}.$

[Meerut 1991; Agra 74; Allahabad 77; Ravishankar 82]

Proof. We have $\text{curl}\,(\mathbf{A}\times\mathbf{B})=\nabla\times(\mathbf{A}\times\mathbf{B})$

$$=\Sigma\left\{\mathbf{i}\times\frac{\partial}{\partial x}(\mathbf{A}\times\mathbf{B})\right\}=\Sigma\left\{\mathbf{i}\times\left(\mathbf{A}\times\frac{\partial\mathbf{B}}{\partial x}+\frac{\partial\mathbf{A}}{\partial x}\times\mathbf{B}\right)\right\}$$

$$=\Sigma\left\{\mathbf{i}\times\left(\mathbf{A}\times\frac{\partial\mathbf{B}}{\partial x}\right)\right\}+\Sigma\left\{\mathbf{i}\times\left(\frac{\partial\mathbf{A}}{\partial x}\times\mathbf{B}\right)\right\}$$

$$=\Sigma\left\{\left(\mathbf{i}\cdot\frac{\partial\mathbf{B}}{\partial x}\right)\mathbf{A}-(\mathbf{i}\cdot\mathbf{A})\frac{\partial\mathbf{B}}{\partial x}\right\}+\Sigma\left\{(\mathbf{i}\cdot\mathbf{B})\frac{\partial\mathbf{A}}{\partial x}-\left(\mathbf{i}\cdot\frac{\partial\mathbf{A}}{\partial x}\right)\mathbf{B}\right\}$$

$$=\Sigma\left\{\left(\mathbf{i}\cdot\frac{\partial\mathbf{B}}{\partial x}\right)\mathbf{A}\right\}-\Sigma\left\{(\mathbf{A}\cdot\mathbf{i})\frac{\partial\mathbf{B}}{\partial x}\right\}+\Sigma\left\{(\mathbf{B}\cdot\mathbf{i})\frac{\partial\mathbf{A}}{\partial x}\right\}-\Sigma\left\{\left(\mathbf{i}\cdot\frac{\partial\mathbf{A}}{\partial x}\right)\mathbf{B}\right\}$$

$$=\left\{\Sigma\left(\mathbf{i}\cdot\frac{\partial\mathbf{B}}{\partial x}\right)\right\}\mathbf{A}-\left\{\mathbf{A}\cdot\Sigma\mathbf{i}\frac{\partial}{\partial x}\right\}\mathbf{B}+\left\{\mathbf{B}\cdot\Sigma\mathbf{i}\frac{\partial}{\partial x}\right\}\mathbf{A}-\left\{\Sigma\left(\mathbf{i}\cdot\frac{\partial\mathbf{A}}{\partial x}\right)\right\}\mathbf{B}$$

$$=(\text{div}\,\mathbf{B})\,\mathbf{A}-(\mathbf{A}\cdot\nabla)\,\mathbf{B}+(\mathbf{B}\cdot\nabla)\,\mathbf{A}-(\text{div}\,\mathbf{A})\,\mathbf{B}.$$

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7. *Prove that*

$grad\,(\mathbf{A}\cdot\mathbf{B})=(\mathbf{B}\cdot\nabla)\,\mathbf{A}+(\mathbf{A}\cdot\nabla)\,\mathbf{B}+\mathbf{B}\times curl\,\mathbf{A}+\mathbf{A}\times curl\,\mathbf{B}.$

~~[Allahabad 1980, 82; Rohilkhand 78; Jiwaji 81]~~

Proof. We have

$$\text{grad}\,(\mathbf{A}\cdot\mathbf{B})=\nabla\,(\mathbf{A}\cdot\mathbf{B})=\Sigma\mathbf{i}\frac{\partial}{\partial x}(\mathbf{A}\cdot\mathbf{B})=\Sigma\mathbf{i}\left(\mathbf{A}\cdot\frac{\partial\mathbf{B}}{\partial x}+\frac{\partial\mathbf{A}}{\partial x}\cdot\mathbf{B}\right)$$

$$=\Sigma\left\{\left(\mathbf{A}\cdot\frac{\partial\mathbf{B}}{\partial x}\right)\mathbf{i}\right\}+\Sigma\left\{\left(\mathbf{B}\cdot\frac{\partial\mathbf{A}}{\partial x}\right)\mathbf{i}\right\}. \quad ...(1)$$

Now we know that $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\,\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}$.

$$\therefore\quad (\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}=(\mathbf{a}\cdot\mathbf{c})\,\mathbf{b}-\mathbf{a}\times(\mathbf{b}\times\mathbf{c}).$$

$$\therefore\quad \left(\mathbf{A}\cdot\frac{\partial\mathbf{B}}{\partial x}\right)\mathbf{i}=(\mathbf{A}\cdot\mathbf{i})\,\frac{\partial\mathbf{B}}{\partial x}-\mathbf{A}\times\left(\frac{\partial\mathbf{B}}{\partial x}\times\mathbf{i}\right)$$

$$=(\mathbf{A}\cdot\mathbf{i})\,\frac{\partial\mathbf{B}}{\partial x}+\mathbf{A}\times\left(\mathbf{i}\times\frac{\partial\mathbf{B}}{\partial x}\right).$$

Thus $\Sigma\left\{\left(\mathbf{A}\cdot\frac{\partial\mathbf{B}}{\partial x}\right)\mathbf{i}\right\}=\Sigma\left\{(\mathbf{A}\cdot\mathbf{i})\,\frac{\partial\mathbf{B}}{\partial x}\right\}+\Sigma\left\{\mathbf{A}\times\left(\mathbf{i}\times\frac{\partial\mathbf{B}}{\partial x}\right)\right\}$

$$=\left\{\mathbf{A}\cdot\Sigma\mathbf{i}\,\frac{\partial}{\partial x}\right\}\mathbf{B}+\mathbf{A}\times\Sigma\left(\mathbf{i}\times\frac{\partial\mathbf{B}}{\partial x}\right)$$

$$=(\mathbf{A}\cdot\nabla)\,\mathbf{B}+\mathbf{A}\times(\nabla\times\mathbf{B}). \qquad \ldots(2)$$

Similarly $\Sigma\left\{\left(\mathbf{B}\cdot\frac{\partial\mathbf{A}}{\partial x}\right)\mathbf{i}\right\}=(\mathbf{B}\cdot\nabla)\,\mathbf{A}+\mathbf{B}\times(\nabla\times\mathbf{A}). \qquad \ldots(3)$

Putting the values from (2) and (3) in (1), we get

$$\text{grad}\,(\mathbf{A}\cdot\mathbf{B})=(\mathbf{A}\cdot\nabla)\,\mathbf{B}+\mathbf{A}\times(\nabla\times\mathbf{B})+(\mathbf{B}\cdot\nabla)\,\mathbf{A}+\mathbf{B}\times(\nabla\times\mathbf{A}).$$

Note. If we put $\mathbf{A}$ in place of $\mathbf{B}$, then

$$\text{grad}\,(\mathbf{A}\cdot\mathbf{A})=2\,(\mathbf{A}\cdot\nabla)\,\mathbf{A}+2\mathbf{A}\times(\nabla\times\mathbf{A})$$

or $\quad \tfrac{1}{2}\,\text{grad}\,A^2=(\mathbf{A}\cdot\nabla)\,\mathbf{A}+\mathbf{A}\times\text{curl}\,\mathbf{A}.$

2003 **8.** *Prove that div grad* $\phi=\nabla^2\phi$

i.e. $\quad \nabla\cdot(\nabla\phi)=\nabla^2\phi.$ **[Rohilkhand 1981; Garhwal 85]**

Proof. We have

$$\nabla\cdot(\nabla\phi)=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\cdot\left(\frac{\partial\phi}{\partial x}\mathbf{i}+\frac{\partial\phi}{\partial y}\mathbf{j}+\frac{\partial\phi}{\partial z}\mathbf{k}\right)$$

$$=\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial z}\right)$$

$$=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\phi=\nabla^2\phi.$$

9. *Prove that curl of the gradient of* ϕ *is zero*

i.e. $\quad \nabla\times(\nabla\phi)=\mathbf{0},\quad$ *i.e. curl grad* $\phi=\mathbf{0}$.

[Meerut 1991S; Rohilkhand 81; Agra 74; Garhwal 82; Kerala 74; Jiwaji 83]

Proof. We have grad $\phi=\dfrac{\partial\phi}{\partial x}\mathbf{i}+\dfrac{\partial\phi}{\partial y}\mathbf{j}+\dfrac{\partial\phi}{\partial z}\mathbf{k}.$

$$\therefore\quad \text{curl grad}\,\phi=\nabla\times\text{grad}\,\phi$$

$$=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\times\left(\frac{\partial\phi}{\partial x}\mathbf{i}+\frac{\partial\phi}{\partial y}\mathbf{j}+\frac{\partial\phi}{\partial z}\mathbf{k}\right)$$

$$=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial\phi}{\partial x} & \dfrac{\partial\phi}{\partial y} & \dfrac{\partial\phi}{\partial z} \end{vmatrix}$$

$$=\left(\frac{\partial^2\phi}{\partial y\,\partial z}-\frac{\partial^2\phi}{\partial z\,\partial y}\right)\mathbf{i}+\left(\frac{\partial^2\phi}{\partial z\,\partial x}-\frac{\partial^2\phi}{\partial x\,\partial z}\right)\mathbf{j}+\left(\frac{\partial^2\phi}{\partial x\,\partial y}-\frac{\partial^2\phi}{\partial y\,\partial x}\right)\mathbf{k}$$

$$=0\mathbf{i}+0\mathbf{j}+0\mathbf{k}=\mathbf{0},$$

provided we suppose that ϕ has continuous second partial derivatives so that the order of differentiation is immaterial.

10 *Prove that div curl* $\mathbf{A}=0$, *i.e.*, $\nabla\cdot(\nabla\times\mathbf{A})=0$.

[Meerut 1992; Kanpur 89; Agra 82; Rohilkhand 90]

Proof. Let $\mathbf{A}=A_1\mathbf{i}+A_2\mathbf{j}+A_3\mathbf{k}$.

Then curl $\mathbf{A}=\nabla\times\mathbf{A}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$

$$=\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)\mathbf{i}+\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)\mathbf{j}+\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)\mathbf{k}.$$

Now div curl $\mathbf{A}=\nabla\cdot(\nabla\times\mathbf{A})$

$$=\frac{\partial}{\partial x}\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)$$

$$=\frac{\partial^2 A_3}{\partial x\,\partial y}-\frac{\partial^2 A_2}{\partial x\,\partial z}+\frac{\partial^2 A_1}{\partial y\,\partial z}-\frac{\partial^2 A_3}{\partial y\,\partial x}+\frac{\partial^2 A_2}{\partial z\,\partial x}-\frac{\partial^2 A_1}{\partial z\,\partial y}$$

$=0$, assuming that $\mathbf{A}$ has continuous second partial derivatives.

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11. *Prove that*

$\nabla\times(\nabla\times\mathbf{A})=\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}$. [Meerut B.Sc. Physics 1983; Kanpur 86; Allahabad 81; Rohilkhand 90]

Proof. Let $\mathbf{A}=A_1\mathbf{i}+A_2\mathbf{j}+A_3\mathbf{k}$.

Then $\nabla\times\mathbf{A}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$

$$=\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)\mathbf{i}+\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)\mathbf{j}+\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)\mathbf{k}.$$

$$\therefore\ \nabla\times(\nabla\times\mathbf{A})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ \dfrac{\partial A_3}{\partial y}-\dfrac{\partial A_2}{\partial z} & \dfrac{\partial A_1}{\partial z}-\dfrac{\partial A_3}{\partial x} & \dfrac{\partial A_2}{\partial x}-\dfrac{\partial A_1}{\partial y}\end{vmatrix}$$

$$=\Sigma\left[\left\{\frac{\partial}{\partial y}\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)-\frac{\partial}{\partial z}\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)\right\}\mathbf{i}\right]$$

$$=\Sigma\left[\left\{\left(\frac{\partial^2 A_2}{\partial y\,\partial x}+\frac{\partial^2 A_3}{\partial z\,\partial x}\right)-\left(\frac{\partial^2 A_1}{\partial y^2}+\frac{\partial^2 A_1}{\partial z^2}\right)\right\}\mathbf{i}\right]$$

$$=\Sigma\left[\left\{\frac{\partial}{\partial x}\left(\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}\right)-\left(\frac{\partial^2 A_1}{\partial y^2}+\frac{\partial^2 A_1}{\partial z^2}\right)\right\}\mathbf{i}\right]$$

$$=\Sigma\left[\left\{\frac{\partial}{\partial x}\left(\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}\right)-\left(\frac{\partial^2 A_1}{\partial x^2}+\frac{\partial^2 A_1}{\partial y^2}+\frac{\partial^2 A_1}{\partial z^2}\right)\right\}\mathbf{i}\right]$$

$$=\Sigma\left[\left\{\frac{\partial}{\partial x}(\nabla\cdot\mathbf{A})-(\nabla^2 A_1)\right\}\mathbf{i}\right]$$

$$=\Sigma\left[\left\{\frac{\partial}{\partial x}(\nabla\cdot\mathbf{A})\right\}\mathbf{i}\right]-\nabla^2\Sigma A_1\mathbf{i}=\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}.$$

Solved Examples

Ex. 1. *Prove that grad $f(u)=f'(u)$ grad u.*

Sol. We have

$$\text{grad } f(u)=\mathbf{i}\frac{\partial}{\partial x}f(u)+\mathbf{j}\frac{\partial}{\partial y}f(u)+\mathbf{k}\frac{\partial}{\partial z}f(u)$$

$$=\mathbf{i}\,f'(u)\frac{\partial u}{\partial x}+\mathbf{j}\,f'(u)\frac{\partial u}{\partial y}+\mathbf{k}\,f'(u)\frac{\partial u}{\partial z}$$

$$=f'(u)\left[\mathbf{i}\frac{\partial u}{\partial x}+\mathbf{j}\frac{\partial u}{\partial y}+\mathbf{k}\frac{\partial u}{\partial z}\right]=f'(u)\text{ grad } u.$$

Ex. 2. *Taking* $\mathbf{F}=x^2y\,\mathbf{i}+xz\,\mathbf{j}+2yz\,\mathbf{k}$, *verify that div curl* $\mathbf{F}=0$. [Agra 1986, Rohilkhand 85]

Sol. We have Curl $\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ x^2y & xz & 2yz\end{vmatrix}$

$$=\left[\frac{\partial}{\partial y}(2yz)-\frac{\partial}{\partial z}(xz)\right]\mathbf{i}-\left[\frac{\partial}{\partial x}(2yz)-\frac{\partial}{\partial z}(x^2y)\right]\mathbf{j}$$

$$+\left[\frac{\partial}{\partial x}(xz)-\frac{\partial}{\partial y}(x^2y)\right]\mathbf{k}$$

$$=(2z-x)\,\mathbf{i}-0\,\mathbf{j}+(z-x^2)\,\mathbf{k}=(2z-x)\,\mathbf{i}+(z-x^2)\,\mathbf{k}.$$

Now div curl $\mathbf{F}$=div $[(2z-x)\,\mathbf{i}+(z-x^2)\,\mathbf{k}]$

$$=\frac{\partial}{\partial x}(2z-x)+\frac{\partial}{\partial z}(z-x^2)=-1+1=0.$$

Ex. 2. *Verify that curl grad $f=\mathbf{0}$, where*

$$f=x^2y+2xy+z^2.$$

[Agra 1973]

Sol. We have grad $f=(\partial f/\partial x)\,\mathbf{i}+(\partial f/\partial y)\,\mathbf{j}+(\partial f/\partial z)\,\mathbf{k}$

$$=(2xy+2y)\,\mathbf{i}+(x^2+2x)\,\mathbf{j}+2z\mathbf{k}.$$

$\therefore$ curl grad $f=\nabla\times[(2xy+2y)\,\mathbf{i}+(x^2+2x)\,\mathbf{j}+2z\mathbf{k}]$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2xy+2y & x^2+2x & 2z\end{vmatrix}$$

$$=\left[\frac{\partial}{\partial y}(2z)-\frac{\partial}{\partial z}(x^2+2x)\right]\mathbf{i}+\left[\frac{\partial}{\partial z}(2xy+2y)-\frac{\partial}{\partial x}(2z)\right]\mathbf{j}$$

$$+\left[\frac{\partial}{\partial x}(x^2+2x)-\frac{\partial}{\partial y}(2xy+2y)\right]\mathbf{k}$$

$$=(0-0)\,\mathbf{i}+(0-0)\,\mathbf{j}+(2x+2-2x-2)\,\mathbf{k}$$

$$=0\mathbf{i}+0\mathbf{j}+0\mathbf{k}=\mathbf{0}.$$

Ex. 4. *Prove that*

curl $(\psi\nabla\phi)=\nabla\psi\times\nabla\phi=-$*curl* $(\phi\nabla\psi)$. [Bombay 1986]

Sol. We know that curl $(\phi\mathbf{A})=(\nabla\phi)\times\mathbf{A}+\phi$ curl $\mathbf{A}$.

In the above formula replacing ϕ by ψ and $\mathbf{A}$ by $\nabla\phi$, we have

curl $(\psi\nabla\phi)=(\nabla\psi)\times\nabla\phi+\psi$ curl $\nabla\phi$

$$=\nabla\psi\times\nabla\phi+\mathbf{0}$$

[$\because$ curl $\nabla\phi$=curl grad $\phi=\mathbf{0}$]

$$=\nabla\psi\times\nabla\phi. \quad \ldots(1)$$

Similarly curl $(\phi\nabla\psi)=(\nabla\phi)\times\nabla\psi+\phi$ curl $\nabla\psi$

$$=\nabla\phi\times\nabla\psi+\mathbf{0}$$

$$=\nabla\phi\times\nabla\psi$$

$$=-\nabla\psi\times\nabla\phi. \quad \ldots(2)$$

From (1) and (2) we have

curl $(\psi\nabla\phi)=\nabla\psi\times\nabla\phi=-$curl $(\phi\nabla\psi)$.

Ex. 5. *Show that curl* $(\mathbf{a}\cdot\mathbf{r})\,\mathbf{a}=0$, *where* $\mathbf{a}$ *is a constant vector.*

Sol. We know that curl $(\phi\mathbf{A})=(\nabla\phi)\times\mathbf{A}+\phi$ curl $\mathbf{A}$.

Replacing ϕ by $\mathbf{a}\cdot\mathbf{r}$ and $\mathbf{A}$ by $\mathbf{a}$ in the above formula, we have

$$\text{curl}\,(\mathbf{a}\cdot\mathbf{r})\,\mathbf{a}=[\nabla(\mathbf{a}\cdot\mathbf{r})]\times\mathbf{a}+(\mathbf{a}\cdot\mathbf{r})\,\text{curl}\,\mathbf{a}. \quad ...(1)$$

But if $\mathbf{a}$ is a constant vector, then curl $\mathbf{a}=0$ and $\nabla(\mathbf{a}\cdot\mathbf{r})=\mathbf{a}$.

$\therefore$ from (1), we have

$$\text{curl}\,(\mathbf{a}\cdot\mathbf{r})\,\mathbf{a}=\mathbf{a}\times\mathbf{a}+0=0+0=0.$$

Ex. 6. *Find* $\nabla\phi$ *and* $|\nabla\phi|$ *when*

$$\phi=(x^2+y^2+z^2)\,e^{-(x^2+y^2+z^2)^{1/2}}.$$

Sol. Let $r^2=x^2+y^2+z^2$. Then we can write $\phi=r^2e^{-r}$.

Now $\quad \nabla\phi=\dfrac{\partial\phi}{\partial x}\mathbf{i}+\dfrac{\partial\phi}{\partial y}\mathbf{j}+\dfrac{\partial\phi}{\partial z}\mathbf{k}.$

We have $\quad \dfrac{\partial\phi}{\partial x}=\dfrac{\partial\phi}{\partial r}\dfrac{\partial r}{\partial x}=[2re^{-r}-r^2e^{-r}]\dfrac{\partial r}{\partial x}.$

But $\quad r^2=x^2+y^2+z^2.$

Therefore $2r\dfrac{\partial r}{\partial x}=2x$ or $\dfrac{\partial r}{\partial x}=\dfrac{x}{r}.$

So $\quad \dfrac{\partial\phi}{\partial x}=re^{-r}(2-r)\dfrac{x}{r}=(2-r)\,e^{-r}x.$

Similarly, $\dfrac{\partial\phi}{\partial y}=(2-r)\,e^{-r}y$ and $\dfrac{\partial\phi}{\partial z}=(2-r)\,e^{-r}z.$

Therefore $\nabla\phi=(2-r)\,e^{-r}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})=(2-r)\,e^{-r}\mathbf{r}.$

Also $|\nabla\phi|=|(2-r)\,e^{-r}\mathbf{r}|=(2-r)\,e^{-r}|\mathbf{r}|=(2-r)\,e^{-r}r.$

Ex. 7. *Prove that* div $(r^n\mathbf{r})=(n+3)\,r^n$.

[Gorakhpur 1985; Rohilkhand 78; Kanpur 87]

Sol. We have

$$\text{div}\,(\phi\mathbf{A})=\phi\,(\text{div}\,\mathbf{A})+\mathbf{A}\cdot\text{grad}\,\phi.$$

Putting $\mathbf{A}=\mathbf{r}$ and $\phi=r^n$ in this identity, we get

$$\text{div}\,(r^n\mathbf{r})=r^n\,\text{div}\,\mathbf{r}+\mathbf{r}\cdot\text{grad}\,r^n$$

$$=3r^n+\mathbf{r}\cdot(nr^{n-1}\,\text{grad}\,r)$$

[$\because$ div $\mathbf{r}=3$ and grad $f(u)=f'(u)$ grad u]

$$=3r^n+\mathbf{r}\cdot\left[nr^{n-1}\frac{1}{r}\mathbf{r}\right] \qquad \left[\because\ \text{grad}\,r=\hat{\mathbf{r}}=\frac{1}{r}\mathbf{r}\right]$$

$$=3r^n+nr^{n-2}(\mathbf{r}\cdot\mathbf{r})=3r^n+nr^{n-2}r^2=(n+3)\,r^n.$$

Ex. 8. *Prove that* $\nabla^2(r^n\mathbf{r})=n(n+3)\,r^{n-2}\mathbf{r}$. **[Kanpur 1988]**

Sol. We have $\nabla^2(r^n\mathbf{r})=\nabla[\nabla\cdot(r^n\mathbf{r})]=\text{grad}\,[\text{div}\,(r^n\mathbf{r})]$

$$=\text{grad}\,[(\text{grad}\,r^n)\cdot\mathbf{r}+r^n\,\text{div}\,\mathbf{r}]$$

$$=\text{grad}\,[(nr^{n-2}\,\mathbf{r})\cdot\mathbf{r}+3r^n]=\text{grad}\,[nr^{n-2}\,r^2+3r^n]$$
$$=\text{grad}\,[nr^{n-2}\,r^2+3r^n]=\text{grad}\,[(n+3)\,r^n]$$
$$=(n+3)\,\text{grad}\,r^n=(n+3)\,nr^{n-2}\,\mathbf{r}=n\,(n+3)\,r^{n-2}\,\mathbf{r}.$$

Ex. 9. *Prove that* div $\left(\dfrac{\mathbf{r}}{r^3}\right)=0.$ **[Banaras 1978]**

Sol. We have $\text{div}\left(\dfrac{1}{r^3}\,\mathbf{r}\right)=\text{div}\,(r^{-3}\,\mathbf{r})$

$$=r^{-3}\,\text{div}\,\mathbf{r}+\mathbf{r}\cdot\text{grad}\,r^{-3}=3r^{-3}+\mathbf{r}\cdot(-3r^{-4}\,\text{grad}\,r)$$
$$=3r^{-3}+\mathbf{r}\cdot\left(-3r^{-4}\,\frac{1}{r}\,\mathbf{r}\right)$$
$$=3r^{-3}-3r^{-5}\,(\mathbf{r}\cdot\mathbf{r})=3r^{-3}-3r^{-5}\,r^2=3r^{-3}-3r^{-3}=0.$$

$\therefore$ the vector $r^{-3}\,\mathbf{r}$ is solenoidal.

Ex. 10. *Prove that* div $\hat{\mathbf{r}}=2/r.$ **[Kanpur 1979]**

Sol. $\text{div}\,(\hat{\mathbf{r}})=\text{div}\left(\dfrac{1}{r}\,\mathbf{r}\right).$ Now proceed as in Ex. 7.

Alternative Method.

$$\text{div}\,\hat{\mathbf{r}}=\text{div}\left(\frac{1}{r}\,\mathbf{r}\right)=\text{div}\left[\frac{1}{r}\,(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\right]$$
$$=\text{div}\left(\frac{x}{r}\,\mathbf{i}+\frac{y}{r}\,\mathbf{j}+\frac{z}{r}\,\mathbf{k}\right)=\frac{\partial}{\partial x}\left(\frac{x}{r}\right)+\frac{\partial}{\partial y}\left(\frac{y}{r}\right)+\frac{\partial}{\partial z}\left(\frac{z}{r}\right)$$
$$=\left(\frac{1}{r}-\frac{x}{r^2}\,\frac{\partial r}{\partial x}\right)+\left(\frac{1}{r}-\frac{y}{r^2}\,\frac{\partial r}{\partial y}\right)+\left(\frac{1}{r}-\frac{z}{r^2}\,\frac{\partial r}{\partial z}\right)$$

Now $r^2=x^2+y^2+z^2$. $\therefore$ $2r\,\dfrac{\partial r}{\partial x}=2x$ *i.e.* $\dfrac{\partial r}{\partial x}=\dfrac{x}{r}.$

Similarly $\dfrac{\partial r}{\partial y}=\dfrac{y}{r}$ and $\dfrac{\partial r}{\partial z}=\dfrac{z}{r}.$

$$\therefore\quad \text{div}\,\hat{\mathbf{r}}=\frac{3}{r}-\left(\frac{x}{r^2}\,\frac{x}{r}+\frac{y}{r^2}\,\frac{y}{r}+\frac{z}{r^2}\,\frac{z}{r}\right)$$
$$=\frac{3}{r}-\frac{x^2+y^2+z^2}{r^3}=\frac{3}{r}-\frac{r^2}{r^3}=\frac{3}{r}-\frac{1}{r}=\frac{2}{r}.$$

Ex. 11. *Prove that vector* $f(r)\,\mathbf{r}$ *is irrotational.* **[Agra 1974; Kanpur 75]**

Sol. The vector $f(r)\,\mathbf{r}$ will be irrotational if

$$\text{curl}\,[f(r)\,\mathbf{r}]=\mathbf{0}.$$

We know that $\text{Curl}\,(\phi\mathbf{A})=(\text{grad}\,\phi)\times\mathbf{A}+\phi\,\text{curl}\,\mathbf{A}.$

Putting $\phi=f(r)$ and $\mathbf{A}=\mathbf{r}$ in this identity, we get

$$\text{Curl}\,[f(r)\,\mathbf{r}]=[\text{grad}\,f(r)]\times\mathbf{r}+f(r)\,\text{curl}\,\mathbf{r}$$
$$=[f'(r)\,\text{grad}\,r]\times\mathbf{r}+f(r)\,\mathbf{0}\qquad[\because\ \text{curl}\,\mathbf{r}=\mathbf{0}]$$

$$=\left[f'(r)\,\frac{1}{r}\,\mathbf{r}\right]\times\mathbf{r}=f'(r)\,\frac{1}{r}\,(\mathbf{r}\times\mathbf{r})=0,\ \text{since } \mathbf{r}\times\mathbf{r}=0.$$

$\therefore$ The vector $f(r)\,\mathbf{r}$ is irrotational.

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Ex. 12. *Prove that* $\nabla^2 f(r)=f''(r)+\frac{2}{r}f'(r)$. [Agra 1977]

Sol. We know that if ϕ is a scalar function then

$$\nabla^2\phi=\nabla\cdot(\nabla\phi).$$

$$\therefore\quad \nabla^2 f(r)=\nabla\cdot\{\nabla f(r)\}=\text{div}\,\{\text{grad}\,f(r)\}$$

$$=\text{div}\,\{f'(r)\,\text{grad}\,r\}=\text{div}\left\{\frac{1}{r}\,f'(r)\,\mathbf{r}\right\}$$

$$=\frac{1}{r}\,f'(r)\,\text{div}\,\mathbf{r}+\mathbf{r}\cdot\text{grad}\left\{\frac{1}{r}\,f'(r)\right\}$$

$$=\frac{3}{r}\,f'(r)+\mathbf{r}\cdot\left[\frac{d}{dr}\left\{\frac{1}{r}\,f'(r)\right\}\text{grad}\,r\right]$$

$$=\frac{3}{r}\,f'(r)+\mathbf{r}\cdot\left[\left\{-\frac{1}{r^2}f'(r)+\frac{1}{r}f''(r)\right\}\frac{1}{r}\,\mathbf{r}\right]$$

$$=\frac{3}{r}\,f'(r)+\left[\frac{1}{r}\left\{-\frac{1}{r^2}f'(r)+\frac{1}{r}\,f''(r)\right\}\right](\mathbf{r}\cdot\mathbf{r})$$

$$=\frac{3}{r}\,f'(r)+\left[\frac{1}{r}\left\{-\frac{1}{r^2}f'(r)+\frac{1}{r}\,f''(r)\right\}\right]r^2$$

$$=\frac{3}{r}\,f'(r)-\frac{1}{r}\,f'(r)+f''(r)=f''(r)+\frac{2}{r}\,f'(r).$$

Ex. 13. *If* $\nabla^2 f(r)=0$, *show that*

$$f(r)=\frac{c_1}{r}+c_2,$$

where $r^2=x^2+y^2+z^2$ *and* c_1, c_2 *are arbitrary constants.* [Bombay 1989]

Sol. As shown in the preceding example, if

$r^2=x^2+y^2+z^2$, then $\nabla^2 f(r)=f''(r)+\frac{2}{r}\,f'(r)$.

$\therefore$ if $\nabla^2 f(r)=0$, then

$$f''(r)+\frac{2}{r}\,f'(r)=0\quad\text{or}\quad\frac{f''(r)}{f'(r)}=-\frac{2}{r}.$$

Integrating with respect to r, we get

$$\log f'(r)=-2\log r+\log c,\ \text{where } c \text{ is a constant}$$

$$=\log\frac{c}{r^2}.$$

$$\therefore\quad f'(r)=\frac{c}{r^2}.$$

Again integrating,

$$f(r)=-\frac{c}{r}+c_2 \text{ where } c_2 \text{ is a constant}$$

$$=\frac{c_1}{r}+c_2, \text{ replacing } -c \text{ by } c_1.$$

Ex. 14. *Prove that* $\nabla^2\left(\frac{1}{r}\right)=0$ *or* $div\left(grad\frac{1}{r}\right)=0.$

[Meerut 1991S; Agra 84; Rohilkhand 81; Kanpur 79]

Sol. We have

$$\nabla^2\left(\frac{1}{r}\right)=\nabla\cdot\left(\nabla\frac{1}{r}\right)=\text{div}\left(\text{grad }\frac{1}{r}\right)$$

$$=\text{div}\left(-\frac{1}{r^2}\text{ grad } r\right)=\text{div}\left(-\frac{1}{r^2}\frac{1}{r}\mathbf{r}\right)=\text{div}\left(-\frac{1}{r^3}\mathbf{r}\right)$$

$$=\left(-\frac{1}{r^3}\right)\text{div }\mathbf{r}+\mathbf{r}\cdot\text{grad}\left(-\frac{1}{r^3}\right)=-\frac{3}{r^3}+\mathbf{r}\cdot\left[\frac{d}{dr}\left(-\frac{1}{r^3}\right)\text{ grad } r\right]$$

$$=-\frac{3}{r^3}+\mathbf{r}\cdot\left(\frac{3}{r^4}\frac{1}{r}\mathbf{r}\right)=-\frac{3}{r^3}+\frac{3}{r^5}(\mathbf{r}\cdot\mathbf{r})=-\frac{3}{r^3}+\frac{3}{r^5}r^2=0.$$

$\therefore$ $1/r$ is a solution of Laplace's equation.

Ex. 15. *Prove that* $div\ grad\ r^n=n(n+1)r^{n-2}$,

i.e. $\nabla^2 r^n=n(n+1)r^{n-2}$.

[Kanpur 1978, 80; Rohilkhand 81; Agra 84; Jiwaji 83]

Sol. We have $\nabla^2 r^n=\nabla\cdot(\nabla r^n)=\text{div (grad } r^n)$

$$=\text{div }(nr^{n-1}\text{ grad } r)=\text{div}\left(nr^{n-1}\frac{1}{r}\mathbf{r}\right)=\text{div }(nr^{n-2}\mathbf{r})$$

$$=(nr^{n-2})\text{ div }\mathbf{r}+\mathbf{r}\cdot(\text{grad } nr^{n-2})$$

$$=3nr^{n-2}+\mathbf{r}\cdot[n(n-2)\,r^{n-3}\text{ grad } r]$$

$$=3nr^{n-2}+\mathbf{r}\cdot\left[n(n-2)\,r^{n-3}\frac{1}{r}\mathbf{r}\right]$$

$$=3nr^{n-2}+\mathbf{r}\cdot[n(n-2)\,r^{n-4}\mathbf{r}]=3nr^{n-2}+n(n-2)\,r^{n-4}(\mathbf{r}\cdot\mathbf{r})$$

$$=3nr^{n-2}+n(n-2)\,r^{n-4}\,r^2=nr^{n-2}(3+n-2)=n(n+1)\,r^{n-2}.$$

Note. If $n=-1$, then $\nabla^2(r^{-1})=(-1)(-1+1)\,r^{-3}=0.$

Ex. 16. *Prove that curl grad* $r^n=0$.

[Rohilkhand 1992; Garhwal 81]

Sol. Let $r^n=\phi$. Now proceed as in identity 9 of § 11.

Ex. 17. *If* $\mathbf{r}$ *is the position vector of the point* (x, y, z) *show that curl* $(r^n\mathbf{r})=0$, *where* r *is the module of* $\mathbf{r}$. [Kanpur 1978]

Sol. We know that curl $(\phi\mathbf{A})=(\nabla\phi)\times\mathbf{A}+\phi$ curl $\mathbf{A}$. Putting $\phi=r^n$ and $\mathbf{A}=\mathbf{r}$ in this identity, we get

$$\text{curl }(r^n\mathbf{r})=(\nabla r^n)\times\mathbf{r}+r^n\text{ curl }\mathbf{r}$$

$$=(nr^{n-1}\nabla r)\times\mathbf{r}+r^n\,\mathbf{0}$$

[$\because$ $\nabla f(r)=f'(r)\,\nabla r$ and curl $\mathbf{r}$=curl $(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})=\mathbf{0}$]

$$=\left(nr^{n-1}\frac{1}{r}\,\mathbf{r}\right)\times\mathbf{r} \qquad \left[\because\ \nabla r=\frac{1}{r}\,\mathbf{r}\right]$$

$$=nr^{n-2}\,(\mathbf{r}\times\mathbf{r})=nr^{n-2}\,\mathbf{0} \qquad [\because\ \mathbf{r}\times\mathbf{r}=\mathbf{0}]$$

$$=\mathbf{0}.$$

2006 **Ex. 18.** *Prove that $r^n\,\mathbf{r}$ is an irrotational vector for any value of n but is solenoidal only if $n+3=0$.* **[Agra 1976; Rohilkhand 78]**

Sol. Let $\mathbf{F}=r^n\,\mathbf{r}$.

The vector $\mathbf{F}$ is irrotational if curl $\mathbf{F}=\mathbf{0}$. Proceeding as in Ex. 17 show that curl $(r^n\,\mathbf{r})=\mathbf{0}$ for any value of n.

$\therefore$ $r^n\,\mathbf{r}$ is an irrotational vector for any value of n.

The vector $\mathbf{F}$ is solenoidal if div $\mathbf{F}=0$. Proceeding as in Ex. 7, show that div $(r^n\,\mathbf{r})=(n+3)\,r^n$.

$\therefore$ the vector $r^n\,\mathbf{r}$ is solenoidal only if $(n+3)\,r^n=0$ *i.e.*, only if $n+3=0$ *i.e.*, only if $n=-3$.

2007 **Ex. 19.** *If* $\mathbf{u}=(1/r)\,\mathbf{r}$, *show that* $\nabla\times\mathbf{u}=\mathbf{0}$. **[Kanpur 1979]**

Sol. We have $\nabla\times\mathbf{u}$=curl $\mathbf{u}$=curl $[(1/r)\,\mathbf{r}]$.

We know that curl $(\phi\mathbf{A})=(\nabla\phi)\times\mathbf{A}+\phi$ curl $\mathbf{A}$.

Replacing ϕ by $1/r$ and $\mathbf{A}$ by $\mathbf{r}$ in this identity, we have

curl $[(1/r)\,\mathbf{r}]=[\nabla(1/r)]\times\mathbf{r}+(1/r)$ curl $\mathbf{r}$

$$=\left[\left(-\frac{1}{r^2}\right)\nabla r\right]\times\mathbf{r}+(1/r)\,\mathbf{0}$$

[$\because$ curl $\mathbf{r}=\mathbf{0}$ and $\nabla f(r)=f'(r)\,\nabla r$]

$$=\left[-\frac{1}{r^2}\frac{1}{r}\,\mathbf{r}\right]\times\mathbf{r} \qquad \left[\because\ \nabla r=\frac{1}{r}\,\mathbf{r}\right]$$

$$=-\frac{1}{r^3}\,(\mathbf{r}\times\mathbf{r})=-\frac{1}{r^3}\,\mathbf{0}=\mathbf{0}.$$

Hence $\nabla\times\mathbf{u}=\mathbf{0}$ if $\mathbf{u}=(1/r)\,\mathbf{r}$.

Ex. 20. *If* $\mathbf{u}=(1/r)\,\mathbf{r}$ *find grad (div* $\mathbf{u}$*).* **[Kanpur 1976]**

Sol. Proceeding as in Ex. 10, first show that div $\mathbf{u}$=div $[(1/r)\,\mathbf{r}]=2/r$.

$\therefore$ grad (div $\mathbf{u}$)=grad $(2/r)=(-2/r^2)$ grad r

[$\because$ grad $f(r)=f'(r)$ grad r]

$$=-\frac{2}{r^2}\left(\frac{1}{r}\,\mathbf{r}\right)=-\frac{2}{r^3}\,\mathbf{r}.$$

Ex. 21. *If* $\nabla^2 f(r)=0$ *show that* $f(r)=c_1\log r+c_2$ *where* $r^2=x^2+y^2$ *and* c_1, c_2 *are arbitrary constants.* **[Poona 1970]**

Sol. We have $\nabla^2 f(r)=\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}\right)f(r)=\dfrac{\partial^2}{\partial x^2}f(r)+\dfrac{\partial^2}{\partial y^2}f(r)$

Now $\frac{\partial}{\partial x} f(r)=f'(r) \cdot \frac{\partial r}{\partial x}$.

But from $r^2=x^2+y^2$, we have $2r \frac{\partial r}{\partial x}=2x$ or $\frac{\partial r}{\partial x}=\frac{x}{r}$.

$\therefore \quad \frac{\partial}{\partial x} f(r)=f'(r) \cdot \frac{x}{r}=\frac{x}{r} f'(r)$.

$$\therefore \quad \frac{\partial^2}{\partial x^2} f(r)=\frac{\partial}{\partial x}\left[\frac{x}{r} f'(r)\right]$$

$$=\frac{1}{r} f'(r)-\frac{x}{r^2} \frac{\partial r}{\partial x} f'(r)+\frac{x}{r} f''(r) \cdot \frac{\partial r}{\partial x}$$

$$=\frac{1}{r} f'(r)-\frac{x}{r^2} \cdot \frac{x}{r} f'(r)+\frac{x}{r} \cdot \frac{x}{r} f''(r)$$

$$=\frac{1}{r} f'(r)-\frac{x^2}{r^3} f'(r)+\frac{x^2}{r^2} f''(r).$$

Similarly, by symmetry,

$$\frac{\partial^2}{\partial y^2} f(r)=\frac{1}{r} f'(r)-\frac{y^2}{r^3} f'(r)+\frac{y^2}{r^2} f''(r).$$

$$\therefore \quad \nabla^2 f(r)=\frac{2}{r} f'(r)-\frac{x^2+y^2}{r^3} f'(r)+\frac{x^2+y^2}{r^2} f''(r)$$

$$=\frac{2}{r} f'(r)-\frac{r^2}{r^3} f'(r)+\frac{r^2}{r^2} f''(r)$$

$$=\frac{1}{r} f'(r)+f''(r).$$

$\therefore$ if $\nabla^2 f(r)=0$, then

$$f''(r)+\frac{1}{r} f'(r)=0 \text{ or } \frac{f''(r)}{f'(r)}=-\frac{1}{r}.$$

Integrating with respect to r, we get

$\log f'(r)=-\log r+\log c_1$, where c_1 is a constant

$\qquad =\log (c_1/r)$.

$\therefore \ f'(r)=c_1/r$.

Again integrating,

$f(r)=c_1 \log r+c_2$, where c_2 is a constant.

Hence $f(r)=c_1 \log r+c_2$, where c_1, c_2 are arbitrary constants.

2003 **Ex. 22.** *Prove that* $\frac{1}{2}\nabla \mathbf{a}^2=(\mathbf{a} \cdot \nabla)\, \mathbf{a}+\mathbf{a} \times$ *curl* $\mathbf{a}$.

[Kanpur 1986]

Sol. Proceed as in identity 7 by taking $\mathbf{A}=\mathbf{a}$ and $\mathbf{B}=\mathbf{a}$.

We have $\frac{1}{2} \nabla \mathbf{a}^2=\frac{1}{2} \nabla (\mathbf{a} \cdot \mathbf{a})$

$$=\frac{1}{2} \Sigma \mathbf{i} \frac{\partial}{\partial x} (\mathbf{a} \cdot \mathbf{a})=\frac{1}{2} \Sigma \mathbf{i} \left(\mathbf{a} \cdot \frac{\partial \mathbf{a}}{\partial x}+\frac{\partial \mathbf{a}}{\partial x} \cdot \mathbf{a} \right)$$

$$=\frac{1}{2}\Sigma 2\mathbf{i}\left(\mathbf{a}\cdot\frac{\partial \mathbf{a}}{\partial x}\right)=\Sigma\left(\mathbf{a}\cdot\frac{\partial \mathbf{a}}{\partial x}\right)\mathbf{i}.$$

We know that $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=(\mathbf{A}\cdot\mathbf{C})\mathbf{B}-(\mathbf{A}\cdot\mathbf{B})\mathbf{C}$.

$\therefore$ $(\mathbf{A}\cdot\mathbf{B})\mathbf{C}=(\mathbf{A}\cdot\mathbf{C})\mathbf{B}-\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$.

$$\therefore\quad \left(\mathbf{a}\cdot\frac{\partial \mathbf{a}}{\partial x}\right)\mathbf{i}=(\mathbf{a}\cdot\mathbf{i})\frac{\partial \mathbf{a}}{\partial x}-\mathbf{a}\times\left(\frac{\partial \mathbf{a}}{\partial x}\times\mathbf{i}\right)=(\mathbf{a}\cdot\mathbf{i})\frac{\partial \mathbf{a}}{\partial x}+\mathbf{a}\times\left(\mathbf{i}\times\frac{\partial \mathbf{a}}{\partial x}\right).$$

$$\text{Thus } \Sigma\left\{\left(\mathbf{a}\cdot\frac{\partial \mathbf{a}}{\partial x}\right)\mathbf{i}\right\}=\Sigma\left\{(\mathbf{a}\cdot\mathbf{i})\frac{\partial \mathbf{a}}{\partial x}\right\}+\Sigma\left\{\mathbf{a}\times\left(\mathbf{i}\times\frac{\partial \mathbf{a}}{\partial x}\right)\right\}$$

$$=\left\{\mathbf{a}\cdot\Sigma\mathbf{i}\frac{\partial}{\partial x}\right\}\mathbf{a}+\mathbf{a}\times\Sigma\left(\mathbf{i}\times\frac{\partial \mathbf{a}}{\partial x}\right)=(\mathbf{a}\cdot\nabla)\mathbf{a}+\mathbf{a}\times(\nabla\times\mathbf{a})$$

$$=(\mathbf{a}\cdot\nabla)\mathbf{a}+\mathbf{a}\times\text{curl }\mathbf{a}.$$

Hence $\frac{1}{2}\nabla\, \mathbf{a}^2=(\mathbf{a}\cdot\nabla)\mathbf{a}+\mathbf{a}\times\text{curl }\mathbf{a}$.

Ex. 23. *Show that* curl $\mathbf{a}\,\phi(r)=\frac{1}{r}\phi'(r)\,\mathbf{r}\times\mathbf{a}$, *where* $\mathbf{a}$ *is a constant vector.* [Kanpur 1982]

Sol. We know that curl $(\phi\mathbf{A})=(\nabla\phi)\times\mathbf{A}+\phi$ curl $\mathbf{A}$. Replacing ϕ by $\phi(r)$ and $\mathbf{A}$ by $\mathbf{a}$ in this identity, we have

$$\text{curl}\,[\mathbf{a}\,\phi(r)]=[\nabla\phi(r)]\times\mathbf{a}+\phi(r)\text{ curl }\mathbf{a}$$

$$=[\phi'(r)\nabla r]\times\mathbf{a}+\phi(r).0$$

[$\because$ $\mathbf{a}$ is a constant vector $\Rightarrow$ curl $\mathbf{a}=0$]

$$=\left[\phi'(r)\frac{1}{r}\mathbf{r}\right]\times\mathbf{a} \qquad \left[\because\ \nabla r=\frac{1}{r}\mathbf{r}\right]$$

$$=\frac{1}{r}\phi'(r)\,\mathbf{r}\times\mathbf{a}.$$

Ex. 24. *Prove that* $\nabla^2(\phi\psi)=\phi\nabla^2\psi+2\nabla\phi\cdot\nabla\psi+\psi\nabla^2\phi$. [Meerut 1972; Bombay 86]

Sol. We have $\nabla^2(\phi\psi)=\nabla\cdot[\nabla(\phi\psi)]$

$$=\nabla\cdot[\phi(\nabla\psi)+\psi(\nabla\phi)]=\nabla\cdot[\phi(\nabla\psi)]+\nabla\cdot[\psi(\nabla\phi)]$$

$$=\phi\nabla\cdot(\nabla\psi)+(\nabla\psi)\cdot(\nabla\phi)+\psi\nabla\cdot(\nabla\phi)+(\nabla\psi)\cdot(\nabla\phi)$$

$$=\phi\nabla^2\psi+2\nabla\phi\cdot\nabla\psi+\psi\nabla^2\phi.$$

Ex. 25. *Prove that* div $(\nabla\phi\times\nabla\psi)=0$.

Sol. We know that

div $(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot\text{curl }\mathbf{A}-\mathbf{A}\cdot\text{Curl }\mathbf{B}$.

$\therefore$ div $(\nabla\phi\times\nabla\psi)=(\nabla\psi)\cdot(\text{Curl }\nabla\phi)-(\nabla\phi)\cdot(\text{Curl }\nabla\psi)$

$=(\nabla\psi)\cdot 0-(\nabla\phi)\cdot 0$ [$\because$ curl grad $\phi=0$]

$=0$.

Ex. 26. *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are irrotational, prove that* $\mathbf{A}\times\mathbf{B}$ *is solenoidal.* [Bombay 1988; Kanpur 77, 79]

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Sol. If $\mathbf{A}$ and $\mathbf{B}$ are irrotational, then

$$\text{curl } \mathbf{A}=0, \text{ curl } \mathbf{B}=0.$$

Now div $(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot\text{curl } \mathbf{A}-\mathbf{A}\cdot\text{curl } \mathbf{B}=\mathbf{B}\cdot 0-\mathbf{A}\cdot 0=0.$

Since div $(\mathbf{A}\times\mathbf{B})$ is zero, therefore $\mathbf{A}\times\mathbf{B}$ is solenoidal.

Ex. 27. *Prove that curl* $(\phi \text{ grad } \phi)=0$.

Sol. We know that

$$\text{curl } (\phi\mathbf{A})=\text{grad } \phi\times\mathbf{A}+\phi \text{ curl } \mathbf{A}.$$

Putting grad ϕ in place of $\mathbf{A}$, we get

$$\text{curl } (\phi \text{ grad } \phi)=\text{grad } \phi\times\text{grad } \phi+\phi \text{ curl grad } \phi$$
$$=0+\phi 0.$$

Here grad $\phi\times$ grad $\phi=0$, since it is the cross product of two equal vectors. Also curl grad $\phi=0$.

$$\therefore \quad \text{curl } (\phi \text{ grad } \phi)=0+0=0.$$

Ex. 28. *If* f *and* g *are two scalar point functions, prove that* $\text{div } (f\nabla g)=f\nabla^2 g+\nabla f\cdot\nabla g$. [Meerut 1972]

Sol. We have $\nabla g=\dfrac{\partial g}{\partial x}\mathbf{i}+\dfrac{\partial g}{\partial y}\mathbf{j}+\dfrac{\partial g}{\partial z}\mathbf{k}.$

Therefore $f\nabla g=f\dfrac{\partial g}{\partial x}\mathbf{i}+f\dfrac{\partial g}{\partial y}\mathbf{j}+f\dfrac{\partial g}{\partial z}\mathbf{k}.$

So $\text{div } (f\nabla g)=\dfrac{\partial}{\partial x}\left(f\dfrac{\partial g}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(f\dfrac{\partial g}{\partial y}\right)+\dfrac{\partial}{\partial z}\left(f\dfrac{\partial g}{\partial z}\right)$

$$=f\left(\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial y^2}+\frac{\partial^2 g}{\partial z^2}\right)+\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial x}+\frac{\partial f}{\partial y}\frac{\partial g}{\partial y}+\frac{\partial f}{\partial z}\frac{\partial g}{\partial z}\right)$$

$$=f\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)$$
$$+\left(\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}\right)\cdot\left(\frac{\partial g}{\partial x}\mathbf{i}+\frac{\partial g}{\partial y}\mathbf{j}+\frac{\partial g}{\partial z}\mathbf{k}\right)$$

$$=f\nabla^2 g+\nabla f\cdot\nabla g.$$

Ex. 29. *A vector function* $\mathbf{f}$ *is the product of a scalar function and the gradient of a scalar function. Show that*

$$\mathbf{f}\cdot\textit{curl } \mathbf{f}=0.$$ [Kerala 1975]

Sol. Let $\mathbf{f}=\psi \text{ grad } \phi$, where ψ and ϕ are scalar functions. We have curl $\mathbf{f}=$ curl $(\psi \text{ grad } \phi)$.

We know that curl $(\phi\mathbf{A})=(\text{grad } \phi)\times\mathbf{A}+\phi \text{ curl } \mathbf{A}.$

$$\therefore \quad \text{curl } (\psi \text{ grad } \phi)=(\text{grad } \psi)\times(\text{grad } \phi)+\psi\,(\text{curl grad } \phi)$$
$$=(\text{grad } \psi)\times(\text{grad } \phi) \quad [\because \text{ curl grad } \phi=0]$$

Now $\mathbf{f}\cdot\text{curl } \mathbf{f}=(\psi \text{ grad } \phi)\cdot\{(\text{grad } \psi)\times(\text{grad } \phi)\}$

$$=[\psi \text{ grad } \phi, \text{ grad } \psi, \text{ grad } \phi]=\psi\,[\text{grad } \phi, \text{ grad } \psi, \text{ grad } \phi]$$

$=0$, since the value of a scalar triple product is zero if two vectors are equal.

Ex. 30. *Given that* $\rho\,\mathbf{F}=\nabla p$, *where* ρ, p, $\mathbf{F}$ *are point functions, prove that* $\mathbf{F}\cdot curl\,\mathbf{F}=0$. [Kerala 1975]

Sol. We have $\mathbf{F}=(1/\rho)\,\nabla p$, where $1/\rho$ and p are scalar functions. Now proceed as in Ex. 29.

Ex. 31. *Prove that* $\nabla\cdot\left\{r\nabla\left(\dfrac{1}{r^3}\right)\right\}=\dfrac{3}{r^4}$

or, $div\,[r\,grad\,r^{-3}]=3r^{-4}$. [Meerut 1991P]

Sol. We have $\nabla\left(\dfrac{1}{r^3}\right)=\text{grad}\,r^{-3}$

$$=\frac{\partial}{\partial x}(r^{-3})\,\mathbf{i}+\frac{\partial}{\partial y}(r^{-3})\,\mathbf{j}+\frac{\partial}{\partial z}(r^{-3})\,\mathbf{k}.$$

Now $\dfrac{\partial}{\partial x}(r^{-3})=-3r^{-4}\dfrac{\partial r}{\partial x}$. But $r^2=x^2+y^2+z^2$.

Therefore $2r\dfrac{\partial r}{\partial x}=2x$ or $\dfrac{\partial r}{\partial x}=\dfrac{x}{r}$.

So $\dfrac{\partial}{\partial x}(r^{-3})=-3r^{-4}\dfrac{x}{r}=-3r^{-5}\,x$.

Similarly $\dfrac{\partial}{\partial y}(r^{-3})=-3r^{-5}\,y$ and $\dfrac{\partial}{\partial z}(r^{-3})=-3r^{-5}\,z$.

Therefore $\nabla\left(\dfrac{1}{r^3}\right)=-3r^{-5}\,(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$.

$$\therefore\quad r\nabla\left(\frac{1}{r^3}\right)=-3r^{-4}\,(x\mathbf{i}+y\mathbf{j}+z\mathbf{k}).$$

$$\therefore\quad \nabla\cdot\left(r\nabla\frac{1}{r^3}\right)=\frac{\partial}{\partial x}(-3r^{-4}x)+\frac{\partial}{\partial y}(-3r^{-4}y)+\frac{\partial}{\partial z}(-3r^{-4}z).$$

Now $\dfrac{\partial}{\partial x}(-3r^{-4}x)=12\,r^{-5}\dfrac{\partial r}{\partial x}x-3r^{-4}$

$$=12\,r^{-5}\frac{x}{r}x-3r^{-4}=12r^{-6}x^2-3r^{-4}.$$

Similarly $\dfrac{\partial}{\partial y}(-3r^{-4}y)=12r^{-6}y^2-3r^{-4}$

and $\dfrac{\partial}{\partial z}(-3r^{-4}z)=12r^{-6}z^2-3r^{-4}$.

Hence $\nabla\cdot\left(r\nabla\dfrac{1}{r^3}\right)=12r^{-6}(x^2+y^2+z^2)-9r^{-4}$

$$=12r^{-6}r^2-9r^{-4}=12r^{-4}-9r^{-4}=3r^{-4}.$$

Ex. 32. *Prove that* $\mathbf{a}\cdot\left(\nabla\dfrac{1}{r}\right)=-\dfrac{\mathbf{a}\cdot\mathbf{r}}{r^3}$.

Sol. We have

$$\text{grad } \frac{1}{r} = -\frac{1}{r^2} \text{ grad } r = -\frac{1}{r^2} \frac{1}{r} \mathbf{r} = -\frac{1}{r^3} \mathbf{r}.$$

$$\therefore \quad \mathbf{a} \cdot \left(\nabla \frac{1}{r}\right) = \mathbf{a} \cdot \left(-\frac{1}{r^3} \mathbf{r}\right) = -\frac{\mathbf{a} \cdot \mathbf{r}}{r^3}.$$

Ex. 33. *Prove that*

$$\mathbf{b} \cdot \nabla \left(\mathbf{a} \cdot \nabla \frac{1}{r}\right) = \frac{3\,(\mathbf{a} \cdot \mathbf{r})(\mathbf{b} \cdot \mathbf{r})}{r^5} - \frac{\mathbf{a} \cdot \mathbf{b}}{r^3}$$

where **a** *and* **b** *are constant vectors.*

Sol. As shown in the last example, we have

$$\mathbf{a} \cdot \nabla \frac{1}{r} = -\frac{\mathbf{a} \cdot \mathbf{r}}{r^3}.$$

$$\therefore \quad \mathbf{b} \cdot \nabla \left(\mathbf{a} \cdot \nabla \frac{1}{r}\right) = \mathbf{b} \cdot \nabla \left(-\frac{\mathbf{a} \cdot \mathbf{r}}{r^3}\right) = \mathbf{b} \cdot \Sigma \mathbf{i} \frac{\partial}{\partial x}\left(-\frac{\mathbf{a} \cdot \mathbf{r}}{r^3}\right)$$

$$= \mathbf{b} \cdot \Sigma \mathbf{i} \left\{-\frac{1}{r^3} \frac{\partial}{\partial x} (\mathbf{a} \cdot \mathbf{r}) + (\mathbf{a} \cdot \mathbf{r}) \frac{\partial}{\partial x}\left(-\frac{1}{r^3}\right)\right\}$$

$$= \mathbf{b} \cdot \Sigma \mathbf{i} \left\{-\frac{1}{r^3}\left(\mathbf{a} \cdot \frac{\partial \mathbf{r}}{\partial x}\right) + 3\,(\mathbf{a} \cdot \mathbf{r})\, r^{-4} \frac{\partial r}{\partial x}\right\}$$

[$\because$ **a** is a constant vector]

$$= \mathbf{b} \cdot \Sigma \mathbf{i} \left\{-\frac{\mathbf{a} \cdot \mathbf{i}}{r^3} + \frac{3x}{r^5} (\mathbf{a} \cdot \mathbf{r})\right\} \qquad \left[\because \frac{\partial \mathbf{r}}{\partial x} = \mathbf{i} \text{ and } \frac{\partial r}{\partial x} = \frac{x}{r}\right]$$

$$= \mathbf{b} \cdot \Sigma \left\{-\frac{1}{r^3} (\mathbf{a} \cdot \mathbf{i})\, \mathbf{i} + \frac{3}{r^5} (\mathbf{a} \cdot \mathbf{r})\, x\mathbf{i}\right\}$$

$$= \mathbf{b} \cdot \left\{-\frac{1}{r^3} \mathbf{a} + \frac{3}{r^5} (\mathbf{a} \cdot \mathbf{r})\, \mathbf{r}\right\}$$

[$\because$ $\Sigma(\mathbf{a} \cdot \mathbf{i})\, \mathbf{i} = \mathbf{a}$, and $\Sigma x\mathbf{i} = \mathbf{r}$]

$$= -\frac{\mathbf{a} \cdot \mathbf{b}}{r^3} + \frac{3\,(\mathbf{a} \cdot \mathbf{r})\,(\mathbf{b} \cdot \mathbf{r})}{r^5}.$$

Ex. 34. *Prove that div* $(\mathbf{A} \times \mathbf{r}) = \mathbf{r} \cdot$ *curl* **A**. [Rohilkhand 1979]

Sol. We know that

$$\text{div } (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \text{curl } \mathbf{A} - \mathbf{A} \cdot \text{curl } \mathbf{B}.$$

$$\therefore \quad \text{div } (\mathbf{A} \times \mathbf{r}) = \mathbf{r} \cdot \text{curl } \mathbf{A} - \mathbf{A} \cdot \text{curl } \mathbf{r}$$

$$= \mathbf{r} \cdot \text{curl } \mathbf{A} - \mathbf{A} \cdot \mathbf{0} \qquad [\because \text{curl } \mathbf{r} = \mathbf{0}]$$

$$= \mathbf{r} \cdot \text{curl } \mathbf{A}.$$

Ex. 35. *If* **a** *is a constant vector, prove that*

$\text{div } \{r^n (\mathbf{a} \times \mathbf{r})\} = 0.$ [Allahabad 1980; Rohilkhand 77]

Sol. We have

$$\text{div } (\phi \mathbf{A}) = \phi \text{ div } \mathbf{A} + \mathbf{A} \cdot \text{grad } \phi.$$

$$\therefore \quad \text{div } \{r^n (\mathbf{a} \times \mathbf{r})\} = r^n \text{ div } (\mathbf{a} \times \mathbf{r}) + (\mathbf{a} \times \mathbf{r}) \cdot \text{grad } r^n$$

$$= r^n \text{ div } (\mathbf{a} \times \mathbf{r}) + (\mathbf{a} \times \mathbf{r}) \cdot (n r^{n-1} \text{ grad } r)$$

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$=r^n\,(\mathbf{r}\cdot\text{curl }\mathbf{a}-\mathbf{a}\cdot\text{curl }\mathbf{r})+(\mathbf{a}\times\mathbf{r})\cdot\left(nr^{n-1}\dfrac{1}{r}\,\mathbf{r}\right)$

$=r^n\,(\mathbf{r}\cdot\mathbf{0}-\mathbf{a}\cdot\mathbf{0})+nr^{n-2}\,(\mathbf{a}\times\mathbf{r})\cdot\mathbf{r}$

[$\because$ curl of constant vector is zero and curl $\mathbf{r}=\mathbf{0}$]

$=nr^{n-2}\,[\mathbf{a},\,\mathbf{r},\,\mathbf{r}]$

$=0$, since a scalar triple product having two equal vectors is zero.

Ex. 36. *Prove that*

$$\nabla\cdot(U\nabla V-V\nabla U)=U\nabla^2V-V\nabla U.$$

[Meerut 1969; Bombay 89; Agra 70]

Sol. We have $\nabla\cdot(U\nabla V-V\nabla U)$

$$=\nabla\cdot(U\nabla V)-\nabla\cdot(V\nabla U).$$

Now $\nabla\cdot(U\nabla V)=U\,\{\nabla\cdot(\nabla V)\}+(\nabla U)\cdot(\nabla V)$

$$=U\nabla^2V+(\nabla U)\cdot(\nabla V).$$

Interchanging U and V, we get

$$\nabla\cdot(V\nabla U)=V\nabla^2U+(\nabla V)\cdot(\nabla U).$$

$\therefore\quad \nabla\cdot(U\nabla V-V\nabla U)$

$=[U\nabla^2V+(\nabla U)\cdot(\nabla V)]-[V\nabla^2U+(\nabla V)\cdot(\nabla U)]$

$=U\nabla^2V-V\nabla^2U.$

Ex. 37. *If* $\mathbf{a}$ *and* $\mathbf{b}$ *are constant vectors, prove that*

(*i*) *div* $[(\mathbf{r}\times\mathbf{a})\times\mathbf{b}]=-2\mathbf{b}\cdot\mathbf{a}.$ [Rohilkhand 1979]

(*ii*) *curl* $[(\mathbf{r}\times\mathbf{a})\times\mathbf{b}]=\mathbf{b}\times\mathbf{a}.$ [Rohilkhand 1979; Gorakhpur 87]

Sol. (i) We have $(\mathbf{r}\times\mathbf{a})\times\mathbf{b}=(\mathbf{b}\cdot\mathbf{r})\,\mathbf{a}-(\mathbf{b}\cdot\mathbf{a})\,\mathbf{r}.$

$\therefore\quad$ div $[(\mathbf{r}\times\mathbf{a})\times\mathbf{b}]=$ div $[(\mathbf{b}\cdot\mathbf{r})\,\mathbf{a}-(\mathbf{b}\cdot\mathbf{a})\,\mathbf{r}]$

$=$ div $[(\mathbf{b}\cdot\mathbf{r})\,\mathbf{a}]-$ div $[(\mathbf{b}\cdot\mathbf{a})\,\mathbf{r}]$ ...(1)

But div $(\phi\mathbf{A})=\phi$ div $\mathbf{A}+\mathbf{A}\cdot$ grad ϕ.

Taking $\phi=\mathbf{b}\cdot\mathbf{r}$ and $\mathbf{A}=\mathbf{a}$, we get

div $[(\mathbf{b}\cdot\mathbf{r})\,\mathbf{a}]=(\mathbf{b}\cdot\mathbf{r})$ div $\mathbf{a}+\mathbf{a}\cdot$ grad $(\mathbf{b}\cdot\mathbf{r})$.

Since $\mathbf{a}$ is a constant vector, therefore div $\mathbf{a}=0$.

Also let $\quad \mathbf{b}=b_1\,\mathbf{i}+b_2\,\mathbf{j}+b_3\,\mathbf{k}.$

Then $\mathbf{b}\cdot\mathbf{r}=(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})\cdot(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$

$=b_1x+b_2y+b_3z$ where $b_1,\,b_2,\,b_3$ are constants.

$\therefore\quad$ grad $(\mathbf{b}\cdot\mathbf{r})=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}=\mathbf{b}.$

$\therefore\quad$ div $[(\mathbf{b}\cdot\mathbf{r})\,\mathbf{a}]=\mathbf{a}\cdot\mathbf{b}.$ ...(2)

Again div $[(\mathbf{b}\cdot\mathbf{a})]\,\mathbf{r}=(\mathbf{b}\cdot\mathbf{a})$ div $\mathbf{r}+\mathbf{r}\cdot$ grad $(\mathbf{b}\cdot\mathbf{a})$.

But div $\mathbf{r}=3$. Also grad $(\mathbf{b}\cdot\mathbf{a})=\mathbf{0}$ because $\mathbf{b}\cdot\mathbf{a}$ is constant.

$\therefore\quad$ div $[(\mathbf{b}\cdot\mathbf{a})\,\mathbf{r}]=3\,(\mathbf{b}\cdot\mathbf{a}).$ ...(3)

Substituting the values from (2) and (3) in (1), we get

$$\text{div }[(\mathbf{r}\times\mathbf{a})\times\mathbf{b}]=(\mathbf{a}\cdot\mathbf{b})-3\,(\mathbf{b}\cdot\mathbf{a})=-2\mathbf{b}\cdot\mathbf{a}.$$

(ii) $\text{Curl}\,[(\mathbf{r}\times\mathbf{a})\times\mathbf{b}]=\text{curl}\,[(\mathbf{b}\cdot\mathbf{r})\,\mathbf{a}-(\mathbf{b}\cdot\mathbf{a})\,\mathbf{r}]$

$$=\text{curl}\,[(\mathbf{b}\cdot\mathbf{r})\,\mathbf{a}]-\text{curl}\,[(\mathbf{b}\cdot\mathbf{a})\,\mathbf{r}].$$

But $\quad \text{curl}\,(\phi\mathbf{A})=\text{grad}\,\phi\times\mathbf{A}+\phi\,\text{curl}\,\mathbf{A}.$

$\therefore\quad \text{curl}\,[(\mathbf{b}\cdot\mathbf{r})\,\mathbf{a}]=[\text{grad}\,(\mathbf{b}\cdot\mathbf{r})]\times\mathbf{a}+(\mathbf{b}\cdot\mathbf{r})\,\text{curl}\,\mathbf{a}$

$$=\mathbf{b}\times\mathbf{a}\;[\because\;\text{curl}\,\mathbf{a}=\mathbf{0}\text{ and grad }(\mathbf{b}\cdot\mathbf{r})=\mathbf{b}]$$

Also $\quad \text{curl}\,[(\mathbf{b}\cdot\mathbf{a})\,\mathbf{r}]=[\text{grad}\,(\mathbf{b}\cdot\mathbf{a})]\times\mathbf{r}+(\mathbf{b}\cdot\mathbf{a})\,\text{curl}\,\mathbf{r}$

$$=\mathbf{0}\;[\because\;\text{grad}\,(\mathbf{b}\cdot\mathbf{a})=\mathbf{0}\text{ and curl }\mathbf{r}=\mathbf{0}]$$

$\therefore\quad \text{curl}\,[(\mathbf{r}\times\mathbf{a})\times\mathbf{b}]=\mathbf{b}\times\mathbf{a}-\mathbf{0}=\mathbf{b}\times\mathbf{a}.$

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Ex. 38. *If* **a** *is a constant vector, prove that*

$$\textit{curl}\,\frac{\mathbf{a}\times\mathbf{r}}{r^3}=-\frac{\mathbf{a}}{r^3}+\frac{3\mathbf{r}}{r^5}\,(\mathbf{a}\cdot\mathbf{r}).$$

[Rajasthan 1981]

Sol. We have

$$\text{curl}\,\frac{\mathbf{a}\times\mathbf{r}}{r^3}=\nabla\times\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right)=\Sigma\left\{\mathbf{i}\times\frac{\partial}{\partial x}\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right)\right\}.$$

Now $\displaystyle\frac{\partial}{\partial x}\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right)=-\frac{3}{r^4}\frac{\partial r}{\partial x}(\mathbf{a}\times\mathbf{r})+\frac{1}{r^3}\left(\mathbf{a}\times\frac{\partial\mathbf{r}}{\partial x}\right)+\frac{1}{r^3}\left(\frac{\partial\mathbf{a}}{\partial x}\times\mathbf{r}\right)$ ...(1)

Now $\dfrac{\partial\mathbf{a}}{\partial x}=\mathbf{0}$ because **a** is a constant vector.

Also $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}.\qquad\therefore\quad \dfrac{\partial\mathbf{r}}{\partial x}=\mathbf{i}.$

Further $\dfrac{\partial r}{\partial x}=\dfrac{x}{r}.$

$\therefore$ (1) becomes

$$\frac{\partial}{\partial x}\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right)=-\frac{3}{r^4}\frac{x}{r}(\mathbf{a}\times\mathbf{r})+\frac{1}{r^3}(\mathbf{a}\times\mathbf{i})$$

$$=-\frac{3x}{r^5}(\mathbf{a}\times\mathbf{r})+\frac{1}{r^3}(\mathbf{a}\times\mathbf{i}).$$

$$\therefore\quad \mathbf{i}\times\frac{\partial}{\partial x}\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right)=-\frac{3x}{r^5}\,\mathbf{i}\times(\mathbf{a}\times\mathbf{r})+\frac{1}{r^3}\,\mathbf{i}\times(\mathbf{a}\times\mathbf{i})$$

$$=-\frac{3x}{r^5}\Big[(\mathbf{i}\cdot\mathbf{r})\,\mathbf{a}-(\mathbf{i}\cdot\mathbf{a})\,\mathbf{r}\Big]+\frac{1}{r^3}\,[(\mathbf{i}\cdot\mathbf{i})\,\mathbf{a}-(\mathbf{i}\cdot\mathbf{a})\,\mathbf{i}]$$

$$=-\frac{3x}{r^5}\,x\mathbf{a}+\frac{3x}{r^5}\,a_1\mathbf{r}+\frac{1}{r^3}\,\mathbf{a}-\frac{1}{r^3}\,a_1\mathbf{i}$$

$$[\because\;\mathbf{i}\cdot\mathbf{r}=x\text{ and }\mathbf{i}\cdot\mathbf{a}=a_1\text{ if }\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}]$$

$$=-\frac{3x^2}{r^5}\,\mathbf{a}+\frac{3}{r^5}\,a_1x\,\mathbf{r}+\frac{1}{r^3}\,\mathbf{a}-\frac{1}{r^3}\,a_1\mathbf{i}.$$

$$\therefore\quad \Sigma\left\{\mathbf{i}\times\frac{\partial}{\partial x}\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right)\right\}$$

$$=\left\{-\frac{3}{r^5}\Sigma x^2\right\}\mathbf{a}+\left\{\frac{3}{r^5}\Sigma a_1 x\right\}\mathbf{r}+\frac{3}{r^3}\mathbf{a}-\frac{1}{r^3}\Sigma a_1\mathbf{i}$$

$$=-\frac{3}{r^5}r^2\mathbf{a}+\frac{3}{r^5}(\mathbf{r}\cdot\mathbf{a})\mathbf{r}+\frac{3}{r^3}\mathbf{a}-\frac{1}{r^3}\mathbf{a}$$

[$\because$ $\Sigma x^2=r^2$, $\Sigma a_1 x=\mathbf{r}\cdot\mathbf{a}$, $\Sigma a_1\mathbf{i}=\mathbf{a}$]

$$=-\frac{\mathbf{a}}{r^3}+\frac{3}{r^5}(\mathbf{a}\cdot\mathbf{r})\mathbf{r}.$$

Ex. 39. *Prove that* $\text{div}\left\{\frac{f(r)\,\mathbf{r}}{r}\right\}=\frac{1}{r^2}\frac{d}{dr}(r^2 f)$. [Agra 1981]

Sol. We have

$$\text{div}\left\{\frac{f(r)\,\mathbf{r}}{r}\right\}=\text{div}\left\{\frac{f(r)}{r}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\right\}$$

$$=\frac{\partial}{\partial x}\left\{\frac{f(r)}{r}x\right\}+\frac{\partial}{\partial y}\left\{\frac{f(r)}{r}y\right\}+\frac{\partial}{\partial z}\left\{\frac{f(r)}{r}z\right\} \quad \ldots(1)$$

Now $\frac{\partial}{\partial x}\left\{\frac{f(r)}{r}x\right\}=\frac{f(r)}{r}+x\frac{d}{dr}\left\{\frac{f(r)}{r}\right\}\frac{\partial r}{\partial x}$

$$=\frac{f(r)}{r}+x\left\{\frac{f'(r)}{r}-\frac{1}{r^2}f(r)\right\}\frac{x}{r}=\frac{f(r)}{r}+\frac{x^2}{r^2}f'(r)-\frac{x^2}{r^3}f(r).$$

Similarly $\frac{\partial}{\partial y}\left\{\frac{f(r)}{r}y\right\}=\frac{f(r)}{r}+\frac{y^2}{r^2}f'(r)-\frac{y^2}{r^3}f(r)$

and $\frac{\partial}{\partial z}\left\{\frac{f(r)}{r}z\right\}=\frac{f(r)}{r}+\frac{z^2}{r^2}f'(r)-\frac{z^2}{r^3}f(r).$

Putting these values in (1), we get

$$\text{div}\left\{\frac{f(r)\,\mathbf{r}}{r}\right\}=\frac{3}{r}f(r)+\frac{r^2}{r^2}f'(r)-\frac{r^2}{r^3}f(r)$$

$$=\frac{2}{r}f(r)+f'(r)=\frac{1}{r^2}\left[2rf(r)+r^2 f'(r)\right]=\frac{1}{r^2}\frac{d}{dr}\left[r^2 f(r)\right].$$

Ex. 40. *Evaluate* $\text{div}\,\{\mathbf{a}\times(\mathbf{r}\times\mathbf{a})$, *where* $\mathbf{a}$ *is a constant vector.* [Kanpur 1976]

Sol. We have $\text{div}\,\{\mathbf{a}\times(\mathbf{r}\times\mathbf{a})\}$

$=\text{div}\,(\mathbf{a}\times\mathbf{b})$, where $\mathbf{b}=\mathbf{r}\times\mathbf{a}$

$=\mathbf{b}\cdot(\nabla\times\mathbf{a})-\mathbf{a}\cdot(\nabla\times\mathbf{b})$ [See identity 5 of § 11]

$=-\mathbf{a}\cdot(\nabla\times\mathbf{b})$ [$\because$ $\mathbf{a}$ is a constant vector $\Rightarrow \nabla\times\mathbf{a}=\mathbf{0}$]

$=-\mathbf{a}\cdot[\nabla\times(\mathbf{r}\times\mathbf{a})]$. ...(1)

Now $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$.

Let $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$. Then the scalars a_1, a_2, a_3 are all constants.

Proceeding as in Ex. 19 part (ii) after § 10, we have

$\text{curl}\,(\mathbf{r}\times\mathbf{a})=\nabla\times(\mathbf{r}\times\mathbf{a})=-2\mathbf{a}$, [Do it here]

Hence from (1), we have

$$\text{div}\,\{\mathbf{a}\times(\mathbf{r}\times\mathbf{a})\}=-\mathbf{a}\cdot(-2\mathbf{a})=2\mathbf{a}\cdot\mathbf{a}=2a^2.$$

Ex. 41. *If* a *and* b *are constant vectors, then show that*

$$\nabla\cdot(\mathbf{a}\cdot\mathbf{b}\mathbf{r})=3\mathbf{a}\cdot\mathbf{b}.$$

Sol. We know that $\nabla\cdot(\phi\mathbf{A})=(\nabla\phi)\cdot\mathbf{A}+\phi(\nabla\cdot\mathbf{A})$

[See identity 3 of § 11]

Replacing ϕ by $\mathbf{a}\cdot\mathbf{b}$ and $\mathbf{A}$ by $\mathbf{r}$ in the above identity, we get

$$\nabla\cdot(\mathbf{a}\cdot\mathbf{b}\mathbf{r})=[\nabla(\mathbf{a}\cdot\mathbf{b})]\cdot\mathbf{r}+(\mathbf{a}\cdot\mathbf{b})(\nabla\cdot\mathbf{r}). \quad ...(1)$$

Since a and b are constant vectors, therefore $\mathbf{a}\cdot\mathbf{b}$ is a constant scalar.

$$\therefore\quad \nabla(\mathbf{a}\cdot\mathbf{b})=\mathbf{0}.$$

Also $\nabla\cdot\mathbf{r}=\nabla\cdot(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$

$$=\frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(z)=1+1+1=3.$$

Substituting these values in (1), we get

$$\nabla\cdot(\mathbf{a}\cdot\mathbf{b}\mathbf{r})=\mathbf{0}\cdot\mathbf{r}+3(\mathbf{a}\cdot\mathbf{b})=3\mathbf{a}\cdot\mathbf{b}.$$

Ex. 42. *Prove that* $\nabla^2\left[\nabla\cdot\left(\frac{\mathbf{r}}{r^2}\right)\right]=2r^{-4}$.

Sol. We know that $\nabla\cdot(\phi\mathbf{A})=\phi(\nabla\cdot\mathbf{A})+\mathbf{A}\cdot(\nabla\phi)$. ...(1)

Putting $\mathbf{A}=\mathbf{r}$ and $\phi=1/r^2$ in this identity, we get

$$\nabla\cdot\left(\frac{\mathbf{r}}{r^2}\right)=\frac{1}{r^2}(\nabla\cdot\mathbf{r})+\mathbf{r}\cdot\left(\nabla\frac{1}{r^2}\right)$$

$$=\frac{3}{r^2}+\mathbf{r}\cdot\left[-\frac{2}{r^3}\nabla r\right]$$

[$\because$ $\nabla\cdot\mathbf{r}=3$ and $\nabla f(r)=f'(r)\,\nabla r$]

$$=\frac{3}{r^2}+\mathbf{r}\cdot\left(-\frac{2}{r^3}\frac{1}{r}\mathbf{r}\right) \qquad \left[\because\ \nabla r=\frac{1}{r}\mathbf{r}\right]$$

$$=\frac{3}{r^2}-\frac{2}{r^4}(\mathbf{r}\cdot\mathbf{r})=\frac{3}{r^2}-\frac{2}{r^4}r^2=\frac{1}{r^2}.$$

$$\therefore\quad \nabla^2\left[\nabla\cdot\left(\frac{\mathbf{r}}{r^2}\right)\right]=\nabla^2\left(\frac{1}{r^2}\right)=\nabla\cdot\left(\nabla\frac{1}{r^2}\right)$$

$$=\nabla\cdot\left(-\frac{2}{r^3}\nabla r\right)=\nabla\cdot\left(-\frac{2}{r^3}\frac{1}{r}\mathbf{r}\right)$$

$$=\nabla\cdot\left(-\frac{2}{r^4}\mathbf{r}\right)$$

$$=\left(-\frac{2}{r^4}\right)(\nabla\cdot\mathbf{r})+\mathbf{r}\cdot\left[\nabla\left(\frac{-2}{r^4}\right)\right], \text{ using the identity (1)}$$

$$=-\frac{2}{r^4}\cdot 3+\mathbf{r}\cdot\left[\frac{8}{r^5}\nabla r\right]$$

$$=-\frac{6}{r^4}+\mathbf{r}\cdot\left(\frac{8}{r^5}\frac{1}{r}\mathbf{r}\right)=-\frac{6}{r^4}+\frac{8}{r^6}\mathbf{r}\cdot\mathbf{r}$$

$$=-\frac{6}{r^4}+\frac{8}{r^6}r^2=-\frac{6}{r^4}+\frac{8}{r^4}=\frac{2}{r^4}=2r^{-4}.$$

Ex. 43. *Prove that curl* $[\mathbf{r}\times(\mathbf{a}\times\mathbf{r})]=3\mathbf{r}\times\mathbf{a}$, *where* a *is a constant vector.* [Gorakhpur 1983]

Sol. Curl $[\mathbf{r}\times(\mathbf{a}\times\mathbf{r})]$

$=\nabla\times[(\mathbf{r}\cdot\mathbf{r})\,\mathbf{a}-(\mathbf{r}\cdot\mathbf{a})\,\mathbf{r}]$ $\quad[\because\ \mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\,\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}]$

$=\nabla\times[r^2\mathbf{a}-(\mathbf{r}\cdot\mathbf{a})\,\mathbf{r}]$ $\quad[\because\ \mathbf{r}\cdot\mathbf{r}=\mathbf{r}^2=r^2]$

$=\nabla\times(r^2\mathbf{a})-\nabla\times[(\mathbf{r}\cdot\mathbf{a})\,\mathbf{r}]$

$[\because\ \nabla\times(\mathbf{A}+\mathbf{B})=\nabla\times\mathbf{A}+\nabla\times\mathbf{B}]$

$=(\nabla\,r^2)\times\mathbf{a}+r^2\,(\nabla\times\mathbf{a})-[\nabla\,(\mathbf{r}\cdot\mathbf{a})]\times\mathbf{r}-(\mathbf{r}\cdot\mathbf{a})\,(\nabla\times\mathbf{r})$

$[\because\ \nabla\times(\phi\mathbf{A})=(\nabla\phi)\times\mathbf{A}+\phi(\nabla\times\mathbf{A})]$

$=(2r\,\nabla r)\times\mathbf{a}+r^2\mathbf{0}-[\nabla\,(\mathbf{r}\cdot\mathbf{a})]\times\mathbf{r}-(\mathbf{r}\cdot\mathbf{a})\,\mathbf{0}$

$[\because\ \nabla f(r)=f'(r)\,\nabla r;\ \nabla\times\mathbf{a}=\mathbf{0}$, a being a constant vector; and $\nabla\times\mathbf{r}=\mathbf{0}]$

$=\left(2r\,\frac{1}{r}\mathbf{r}\right)\times\mathbf{a}-[\nabla\,(\mathbf{r}\cdot\mathbf{a})]\times\mathbf{r}$

$=2\mathbf{r}\times\mathbf{a}-\mathbf{a}\times\mathbf{r}$ $\quad[\because\ \nabla\,(\mathbf{r}\cdot\mathbf{a})=\mathbf{a}$, if a is a constant vector. See Ex. 9 after § 4. Do it here]

$=2\mathbf{r}\times\mathbf{a}+\mathbf{r}\times\mathbf{a}=3\mathbf{r}\times\mathbf{a}.$

Ex. 44. *Prove that* $\nabla\times(\mathbf{F}\times\mathbf{r})=2\mathbf{F}-(\nabla\cdot\mathbf{F})\,\mathbf{r}+(\mathbf{r}\cdot\nabla)\,\mathbf{F}.$ [Allahabad 1980]

Sol. We know that

$\nabla\times(\mathbf{A}\times\mathbf{B})=\mathbf{A}\,(\nabla\cdot\mathbf{B})-\mathbf{B}\,(\nabla\cdot\mathbf{A})+(\mathbf{B}\cdot\nabla)\,\mathbf{A}-(\mathbf{A}\cdot\nabla)\,\mathbf{B}.$

[See identity 6 after § 11]

Putting $\mathbf{A}=\mathbf{F}$ and $\mathbf{B}=\mathbf{r}$ in this identity, we get

$\nabla\times(\mathbf{F}\times\mathbf{r})=\mathbf{F}\,(\nabla\cdot\mathbf{r})-\mathbf{r}\,(\nabla\cdot\mathbf{F})+(\mathbf{r}\cdot\nabla)\,\mathbf{F}-(\mathbf{F}\cdot\nabla)\,\mathbf{r}.$ ...(1)

Now $\nabla\cdot\mathbf{r}=\nabla\cdot(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$

$$=\frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(z)=1+1+1=3. \quad ...(2)$$

If $\mathbf{F}=F_1\mathbf{i}+F_2\mathbf{j}+F_3\mathbf{k}$, then

$$\mathbf{F}\cdot\nabla=(F_1\mathbf{i}+F_2\mathbf{j}+F_3\mathbf{k})\cdot\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)$$

$$=F_1\frac{\partial}{\partial x}+F_2\frac{\partial}{\partial y}+F_3\frac{\partial}{\partial z}.$$

$$\therefore\ (\mathbf{F}\cdot\nabla)\,\mathbf{r}=\left(F_1\frac{\partial}{\partial x}+F_2\frac{\partial}{\partial y}+F_3\frac{\partial}{\partial z}\right)(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$$

$$=F_1\frac{\partial}{\partial x}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})+\ldots+$$

$$=F_1\mathbf{i}+F_2\mathbf{j}+F_3\mathbf{k}=\mathbf{F}. \quad \ldots(3)$$

$\therefore$ from (1), (2) and (3), we get

$$\nabla\times(\mathbf{F}\times\mathbf{r})=3\mathbf{F}-(\nabla\cdot\mathbf{F})\,\mathbf{r}+(\mathbf{r}\cdot\nabla)\,\mathbf{F}-\mathbf{F}$$

$$=2\mathbf{F}-(\nabla\cdot\mathbf{F})\,\mathbf{r}+(\mathbf{r}\cdot\nabla)\,\mathbf{F}.$$

Ex. 43. *If* **a** *and* **b** *are constant vectors, prove that*

$$\textit{grad}\,[(\mathbf{r}\times\mathbf{a})\cdot(\mathbf{r}\times\mathbf{b})]=(\mathbf{b}\times\mathbf{r})\times\mathbf{a}+(\mathbf{a}\times\mathbf{r})\times\mathbf{b}.$$

[Kanpur 1977]

Sol. We have $(\mathbf{r}\times\mathbf{a})\cdot(\mathbf{r}\times\mathbf{b})$

$$=\begin{vmatrix}\mathbf{r}\cdot\mathbf{r} & \mathbf{r}\cdot\mathbf{b}\\ \mathbf{a}\cdot\mathbf{r} & \mathbf{a}\cdot\mathbf{b}\end{vmatrix}, \text{ by Lagrange's identity}$$

$$=(\mathbf{a}\cdot\mathbf{b})\,(\mathbf{r}\cdot\mathbf{r})-(\mathbf{r}\cdot\mathbf{b})\,(\mathbf{a}\cdot\mathbf{r}).$$

$\therefore$ grad $[(\mathbf{r}\times\mathbf{a})\cdot(\mathbf{r}\times\mathbf{b})]$

$$=\text{grad}\,[(\mathbf{a}\cdot\mathbf{b})\,(\mathbf{r}\cdot\mathbf{r})-(\mathbf{r}\cdot\mathbf{b})\,(\mathbf{a}\cdot\mathbf{r})]$$

$$=\text{grad}\,[(\mathbf{a}\cdot\mathbf{b})\,(\mathbf{r}\cdot\mathbf{r})]-\text{grad}\,[(\mathbf{r}\cdot\mathbf{b})\,(\mathbf{a}\cdot\mathbf{r})]$$

$$=(\mathbf{a}\cdot\mathbf{b})\,\text{grad}\,(\mathbf{r}\cdot\mathbf{r})+(\mathbf{r}\cdot\mathbf{r})\,\text{grad}\,(\mathbf{a}\cdot\mathbf{b})$$

$$-(\mathbf{r}\cdot\mathbf{b})\,\text{grad}\,(\mathbf{a}\cdot\mathbf{r})-(\mathbf{a}\cdot\mathbf{r})\,\text{grad}\,(\mathbf{r}\cdot\mathbf{b}) \quad \ldots(1)$$

$[\because\ \text{grad}\,(\phi\psi)=\phi\,\text{grad}\,\psi+\psi \text{ and } \phi]$

Now grad $(\mathbf{r}\cdot\mathbf{r})=$ grad $\mathbf{r}^2=$ grad $(x^2+y^2+z^2)$

$$=2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}=(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})=2\mathbf{r}.$$

Also if **a** and **b** are constant vectors, then $\mathbf{a}\cdot\mathbf{b}$ is a constant scalar and so grad $(\mathbf{a}\cdot\mathbf{b})=\mathbf{0}$.

Further if **a** is a constant vector, then grad $(\mathbf{r}\cdot\mathbf{a})=\mathbf{a}$. Similarly **b** is a constant vector implies grad $(\mathbf{r}\cdot\mathbf{b})=\mathbf{b}$.

Putting the above values in (1), we have

$$\text{grad}\,[(\mathbf{r}\times\mathbf{a})\cdot(\mathbf{r}\times\mathbf{b})]=(\mathbf{a}\cdot\mathbf{b})\,(2\mathbf{r})+(\mathbf{r}\cdot\mathbf{r})\,\mathbf{0}-(\mathbf{r}\cdot\mathbf{b})\,\mathbf{a}-(\mathbf{a}\cdot\mathbf{r})\,\mathbf{b}$$

$$=[(\mathbf{a}\cdot\mathbf{b})\,\mathbf{r}-(\mathbf{r}\cdot\mathbf{a})\,\mathbf{b}]+[(\mathbf{a}\cdot\mathbf{b})\,\mathbf{r}-(\mathbf{r}\cdot\mathbf{b})\,\mathbf{a}]$$

$$=(\mathbf{b}\times\mathbf{r})\times\mathbf{a}+(\mathbf{a}\times\mathbf{r})\times\mathbf{b}.$$

Ex. 46. *Prove that curl* $[r^n(\mathbf{a}\times\mathbf{r})]=(n+2)\,r^n\mathbf{a}-nr^{n-2}\,(\mathbf{r}\cdot\mathbf{a})\,\mathbf{r}$, *where* **a** *is a constant vector.*

[Rohilkhand 1977]

Sol. We know that curl $(\phi \mathbf{A}) = (\text{grad } \phi) \times \mathbf{A} + \phi \text{ curl } \mathbf{A}$.

Putting $\phi = r^n$ and $\mathbf{A} = \mathbf{a} \times \mathbf{r}$ in this identity, we have

$$\text{curl } [r^n (\mathbf{a} \times \mathbf{r})] = (\nabla r^n) \times (\mathbf{a} \times \mathbf{r}) + r^n \text{ curl } (\mathbf{a} \times \mathbf{r}). \quad \ldots(1)$$

Now $\nabla r^n = n r^{n-1} \nabla r = n r^{n-1} (1/r) \mathbf{r} = n r^{n-2} \mathbf{r}$.

$$\therefore \quad (\nabla r^n) \times (\mathbf{a} \times \mathbf{r}) = (n r^{n-2} \mathbf{r}) \times (\mathbf{a} \times \mathbf{r})$$

$$= n r^{n-2} \mathbf{r} \times (\mathbf{a} \times \mathbf{r}) = n r^{n-2} [(\mathbf{r} \cdot \mathbf{r}) \mathbf{a} - (\mathbf{r} \cdot \mathbf{a}) \mathbf{r}]$$

$$= n r^{n-2} [r^2 \mathbf{a} - (\mathbf{r} \cdot \mathbf{a}) \mathbf{r}]$$

$$= n r^n \mathbf{a} - n r^{n-2} (\mathbf{r} \cdot \mathbf{a}) \mathbf{r}. \quad \ldots(2)$$

Also $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

Let $\mathbf{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}$, where the scalars a_1, a_2, a_3 are all constants.

Then $\mathbf{a} \times \mathbf{r} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ x & y & z \end{vmatrix}$

$$= (a_2 z - a_3 y) \mathbf{i} + (a_3 x - a_1 z) \mathbf{j} + (a_1 y - a_2 x) \mathbf{k}.$$

$$\therefore \quad \text{curl } (\mathbf{a} \times \mathbf{r}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_2 z - a_3 y & a_3 x - a_1 z & a_1 y - a_2 x \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y} (a_1 y - a_2 x) - \frac{\partial}{\partial z} (a_3 x - a_1 z) \right] \mathbf{i} + \left[\frac{\partial}{\partial z} (a_2 z - a_3 y) - \frac{\partial}{\partial x} (a_1 y - a_2 x) \right] \mathbf{j} + \left[\frac{\partial}{\partial x} (a_3 x - a_1 z) - \frac{\partial}{\partial y} (a_2 z - a_3 y) \right] \mathbf{k}$$

$$= (a_1 + a_1) \mathbf{i} + (a_2 + a_2) \mathbf{j} + (a_3 + a_3) \mathbf{k} = 2 (a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}) = 2\mathbf{a} \quad \ldots(3)$$

Substituting from (2) and (3) in (1), we get

$$\text{curl } [r^n (\mathbf{a} \times \mathbf{r})] = n r^n \mathbf{a} - n r^{n-2} (\mathbf{r} \cdot \mathbf{a}) \mathbf{r} + r^n (2\mathbf{a})$$

$$= (n+2) r^n \mathbf{a} - n r^{n-2} (\mathbf{r} \cdot \mathbf{a}) \mathbf{r}.$$

Ex. 47. *Prove that* $\mathbf{a} \cdot \{\nabla (\mathbf{v} \cdot \mathbf{a}) - \nabla \times (\mathbf{v} \times \mathbf{a})\} = \text{div } \mathbf{v}$, *where* $\mathbf{a}$ *is a constant unit vector.*

Sol. We know that

$$\nabla (\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla) \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}).$$

$$\therefore \quad \nabla (\mathbf{v} \cdot \mathbf{a}) = (\mathbf{v} \cdot \nabla) \mathbf{a} + (\mathbf{a} \cdot \nabla) \mathbf{v} + \mathbf{v} \times (\nabla \times \mathbf{a}) + \mathbf{a} \times (\nabla \times \mathbf{v}) \quad \ldots(1)$$

Let $\quad \mathbf{v}=v_1\,\mathbf{i}+v_2\,\mathbf{j}+v_3\,\mathbf{k}$.

Then $\quad \mathbf{v}\cdot\nabla=(v_1\,\mathbf{i}+v_2\,\mathbf{j}+v_3\,\mathbf{k})\cdot\left(\mathbf{i}\dfrac{\partial}{\partial x}+\mathbf{j}\dfrac{\partial}{\partial y}+\mathbf{k}\dfrac{\partial}{\partial z}\right)$

$$=v_1\frac{\partial}{\partial x}+v_2\frac{\partial}{\partial y}+v_3\frac{\partial}{\partial z}.$$

$$\therefore\ (\mathbf{v}\cdot\nabla)\,\mathbf{a}=\left(v_1\frac{\partial}{\partial x}+v_2\frac{\partial}{\partial y}+v_3\frac{\partial}{\partial z}\right)\mathbf{a}$$

$$=v_1\frac{\partial\mathbf{a}}{\partial x}+v_2\frac{\partial\mathbf{a}}{\partial y}+v_3\frac{\partial\mathbf{a}}{\partial z}$$

$=0$, because $\mathbf{a}$ is a constant vector.

Also $\nabla\times\mathbf{a}=0$, $\mathbf{a}$ being a constant vector.

$\therefore$ from (1), we have

$$\nabla(\mathbf{v}\cdot\mathbf{a})=(\mathbf{a}\cdot\nabla)\,\mathbf{v}+\mathbf{a}\times(\nabla\times\mathbf{v}) \qquad \ldots(2)$$

Also we know that

$$\nabla\times(\mathbf{A}\times\mathbf{B})=(\nabla\cdot\mathbf{B})\,\mathbf{A}-(\nabla\cdot\mathbf{A})\,\mathbf{B}+(\mathbf{B}\cdot\nabla)\,\mathbf{A}-(\mathbf{A}\cdot\nabla)\,\mathbf{B}.$$

$$\therefore\ \nabla\times(\mathbf{v}\times\mathbf{a})=(\nabla\cdot\mathbf{a})\,\mathbf{v}-(\nabla\cdot\mathbf{v})\,\mathbf{a}+(\mathbf{a}\cdot\nabla)\,\mathbf{v}-(\mathbf{v}\cdot\nabla)\,\mathbf{a}$$

$$=-(\nabla\cdot\mathbf{v})\,\mathbf{a}+(\mathbf{a}\cdot\nabla)\,\mathbf{v} \qquad \ldots(3)$$

[$\because\ \nabla\cdot\mathbf{a}=0$ and $(\mathbf{v}\cdot\nabla)\,\mathbf{a}=0$]

Subtracting (3) from (2), we get

$$\nabla(\mathbf{v}\cdot\mathbf{a})-\nabla\times(\mathbf{v}\times\mathbf{a})=\mathbf{a}\times(\nabla\times\mathbf{v})+(\nabla\cdot\mathbf{v})\,\mathbf{a} \qquad \ldots(4)$$

Multiplying both sides of (4) scalarly by $\mathbf{a}$, we get

$$\mathbf{a}\cdot[\nabla(\mathbf{v}\cdot\mathbf{a})-\nabla\times(\mathbf{v}\times\mathbf{a})]=\mathbf{a}\cdot[\mathbf{a}\times(\nabla\times\mathbf{v})]+\mathbf{a}\cdot[(\nabla\cdot\mathbf{v})\,\mathbf{a}]$$

$$=[\mathbf{a},\mathbf{a},\nabla\times\mathbf{v}]+(\nabla\cdot\mathbf{v})\,(\mathbf{a}\cdot\mathbf{a})$$

$=0+(\nabla\cdot\mathbf{v})\,\mathbf{a}^2$, since the scalar triple product

$[\mathbf{a},\mathbf{a},\nabla\times\mathbf{v}]=0$

$=\nabla\cdot\mathbf{v}$ $\qquad$ [$\because\ \mathbf{a}^2=|\mathbf{a}|^2=1$, $\mathbf{a}$ being a unit vector]

$=\text{div}\,\mathbf{v}$.

Ex. 48. *If $\mathbf{a}$ is a constant vector, then prove that*

(i) $\nabla(\mathbf{a}\cdot\mathbf{u})=(\mathbf{a}\cdot\nabla)\,\mathbf{u}+\mathbf{a}\times\textit{curl}\,\mathbf{u}$,

(ii) $\nabla\cdot(\mathbf{a}\times\mathbf{u})=-\mathbf{a}\cdot\textit{curl}\,\mathbf{u}$,

(iii) $\nabla\times(\mathbf{a}\times\mathbf{u})=\mathbf{a}\,\textit{div}\,\mathbf{u}-(\mathbf{a}\cdot\nabla)\,\mathbf{u}$.

Sol. (i) Proceed exactly as in Ex. 47.

We have $\nabla(\mathbf{a}\cdot\mathbf{u})=(\mathbf{a}\cdot\nabla)\,\mathbf{u}+(\mathbf{u}\cdot\nabla)\,\mathbf{a}+\mathbf{a}\times(\nabla\times\mathbf{u})$

$+\mathbf{u}\times(\nabla\times\mathbf{a})$.

Since a is a constant vector, therefore

$$(\mathbf{u}\cdot\nabla)\,\mathbf{a}=\mathbf{0} \text{ and } \nabla\times\mathbf{a}=\mathbf{0}.$$

$\therefore\quad \nabla(\mathbf{a}\cdot\mathbf{u})=(\mathbf{a}\cdot\nabla)\,\mathbf{u}+\mathbf{a}\times\operatorname{curl}\mathbf{u}.$

(ii) We know that $\nabla\cdot(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot(\nabla\times\mathbf{A})-\mathbf{A}\cdot(\nabla\times\mathbf{B})$.

$\therefore\quad \nabla\cdot(\mathbf{a}\times\mathbf{u})=\mathbf{u}\cdot(\nabla\times\mathbf{a})-\mathbf{a}\cdot(\nabla\times\mathbf{u})$

$=\mathbf{u}\cdot\mathbf{0}-\mathbf{a}\cdot(\nabla\times\mathbf{u})$, since $\nabla\times\mathbf{a}=\mathbf{0}$,

a being a constant vector

$=0-\mathbf{a}\cdot\operatorname{curl}\mathbf{u}=-\mathbf{a}\cdot\operatorname{curl}\mathbf{u}.$

(iii) Proceed exactly as in Ex. 47, using the identity for $\nabla\times(\mathbf{A}\times\mathbf{B})$.

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11. Velocity and Acceleration.

If the scalar variable t be the time and $\mathbf{r}$ be the position vector of a moving particle P with respect to the origin O, then $\delta\mathbf{r}$ is the displacement of the particle in time δt.

The vector $\frac{\delta \mathbf{r}}{\delta t}$ is the average velocity of the particle during the interval δt. If $\mathbf{v}$ represents the velocity vector of the particle at P, then $\mathbf{v} = \lim_{\delta t \to 0} \frac{\delta \mathbf{r}}{\delta t} = \frac{d\mathbf{r}}{dt}$.

Since $\frac{d\mathbf{r}}{dt}$ is a vector along the tangent at P to the curve in which the particle is moving, therefore the direction of velocity is along the tangent.

If $\delta\mathbf{v}$ be the change in the velocity $\mathbf{v}$ during the time δt, then $\frac{\delta \mathbf{v}}{\delta t}$ is the average acceleration during that interval. If $\mathbf{a}$ represents the acceleration of the particle at time t, then

$$\mathbf{a} = \lim_{\delta t \to 0} \frac{\delta \mathbf{v}}{\delta t} = \frac{d\mathbf{v}}{dt} = \frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right) = \frac{d^2\mathbf{r}}{dt^2}.$$

Solved Examples

Ex. 1. *If* $\mathbf{r} = (t+1)\,\mathbf{i} + (t^2+t+1)\,\mathbf{j} + (t^3+t^2+t+1)\,\mathbf{k}$ *find* $\frac{d\mathbf{r}}{dt}$ *and* $\frac{d^2\mathbf{r}}{dt^2}$.

Sol. Since $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are constant vectors, therefore

$$\frac{d\mathbf{i}}{dt} = 0 = \frac{d\mathbf{j}}{dt} = \frac{d\mathbf{k}}{dt}.$$

$$\therefore \quad \frac{d\mathbf{r}}{dt}=\frac{d}{dt}(t+1)\,\mathbf{i}+\frac{d}{dt}(t^2+t+1)\,\mathbf{j}+\frac{d}{dt}(t^3+t^2+t+1)\,\mathbf{k}$$

$$=\mathbf{i}+(2t+1)\,\mathbf{j}+(3t^2+2t+1)\,\mathbf{k}.$$

Again, $\dfrac{d^2\mathbf{r}}{dt^2}=\dfrac{d}{dt}\left(\dfrac{d\mathbf{r}}{dt}\right)=\dfrac{d\mathbf{i}}{dt}+\dfrac{d}{dt}(2t+1)\,\mathbf{j}+\dfrac{d}{dt}(3t^2+2t+1)\,\mathbf{k}$

$$=0+2\mathbf{j}+(6t+2)\,\mathbf{k}=2\mathbf{j}+(6t+2)\,\mathbf{k}.$$

Ex. 2. *If* $\mathbf{r}=\sin t\,\mathbf{i}+\cos t\,\mathbf{j}+t\,\mathbf{k}$, *find*

(*i*) $\dfrac{d\mathbf{r}}{dt}$, (*ii*) $\dfrac{d^2\mathbf{r}}{dt^2}$, (*iii*) $\left|\dfrac{d\mathbf{r}}{dt}\right|$, (*iv*) $\left|\dfrac{d^2\mathbf{r}}{dt^2}\right|$. [Agra 1978]

Sol. Since $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are constant vectors, therefore $\dfrac{d\mathbf{i}}{dt}=\mathbf{0}$ etc.

Therefore

(i) $\dfrac{d\mathbf{r}}{dt}=\dfrac{d}{dt}(\sin t)\,\mathbf{i}+\dfrac{d}{dt}(\cos t)\,\mathbf{j}+\dfrac{d}{dt}(t)\,\mathbf{k}=\cos t\,\mathbf{i}-\sin t\,\mathbf{j}+\mathbf{k}.$

(ii) $\dfrac{d^2\mathbf{r}}{dt^2}=\dfrac{d}{dt}\left(\dfrac{d\mathbf{r}}{dt}\right)=\dfrac{d}{dt}(\cos t)\,\mathbf{i}-\dfrac{d}{dt}(\sin t)\,\mathbf{j}+\dfrac{d\mathbf{k}}{dt}$

$$=-\sin t\,\mathbf{i}-\cos t\,\mathbf{j}+\mathbf{0}=-\sin t\,\mathbf{i}-\cos t\,\mathbf{j}.$$

(iii) $\left|\dfrac{d\mathbf{r}}{dt}\right|=\sqrt{[(\cos t)^2+(-\sin t)^2+(1)^2]}=\sqrt{2}.$

(iv) $\left|\dfrac{d^2\mathbf{r}}{dt^2}\right|=\sqrt{[(-\sin t)^2+(-\cos t)^2]}=1.$

Ex. 3. *If* $\mathbf{r}=(\cos nt)\,\mathbf{i}+(\sin nt)\,\mathbf{j}$, *where* n *is a constant and* t *varies, show that* $\mathbf{r}\times\dfrac{d\mathbf{r}}{dt}=n\mathbf{k}$

Sol. We have

$$\frac{d\mathbf{r}}{dt}=\frac{d}{dt}(\cos nt)\,\mathbf{i}+\frac{d}{dt}(\sin nt)\,\mathbf{j}=-n\sin nt\,\mathbf{i}+n\cos nt\,\mathbf{j}.$$

$$\therefore \quad \mathbf{r}\times\frac{d\mathbf{r}}{dt}=(\cos nt\,\mathbf{i}+\sin nt\,\mathbf{j})\times(-n\sin nt\,\mathbf{i}+n\cos nt\,\mathbf{j})$$

$$=-n\cos nt\sin nt\,\mathbf{i}\times\mathbf{i}+n\cos^2 nt\,\mathbf{i}\times\mathbf{j}$$

$$-n\sin^2 nt\,\mathbf{j}\times\mathbf{i}+n\cos nt\sin nt\,\mathbf{j}\times\mathbf{j}$$

$$=n\cos^2 nt\,\mathbf{k}+n\sin^2 nt\,\mathbf{k}$$

$$[\because\ \mathbf{i}\times\mathbf{i}=\mathbf{0},\ \mathbf{j}\times\mathbf{j}=\mathbf{0},\ \mathbf{i}\times\mathbf{j}=\mathbf{k},\ \mathbf{j}\times\mathbf{i}=-\mathbf{k}]$$

$$=n(\cos^2 nt+\sin^2 nt)\,\mathbf{k}=n\mathbf{k}.$$

Ex. 4. *If* $\mathbf{a}$, $\mathbf{b}$ *are constant vectors,* ω *is a constant, and* $\mathbf{r}$ *is a vector function of the scalar variable* t *given by*

$$\mathbf{r}=\cos\omega t\,\mathbf{a}+\sin\omega t\,\mathbf{b},$$

show that

(i) $\frac{d^2\mathbf{r}}{dt^2}+\omega^2\mathbf{r}=\mathbf{0}$, *and* (ii) $\mathbf{r}\times\frac{d\mathbf{r}}{dt}=\omega\mathbf{a}\times\mathbf{b}$.

[Rohilkhand 1984; Agra 81; Kumayun 82; Madras 83]

Sol. Since $\mathbf{a}$, $\mathbf{b}$ are constant vectors, therefore

$$\frac{d\mathbf{a}}{dt}=\mathbf{0},\ \frac{d\mathbf{b}}{dt}=\mathbf{0}.$$

(i) $\frac{d\mathbf{r}}{dt}=\frac{d}{dt}(\cos\omega t)\,\mathbf{a}+\frac{d}{dt}(\sin\omega t)\,\mathbf{b}$

$=-\omega\sin\omega t\,\mathbf{a}+\omega\cos\omega t\,\mathbf{b}$.

$\therefore\ \frac{d^2\mathbf{r}}{dt^2}=-\omega^2\cos\omega t\,\mathbf{a}-\omega^2\sin\omega t\,\mathbf{b}$

$=-\omega^2(\cos\omega t\,\mathbf{a}+\sin\omega t\,\mathbf{b})=-\omega^2\mathbf{r}$.

$\therefore\ \frac{d^2\mathbf{r}}{dt^2}+\omega^2\mathbf{r}=\mathbf{0}$.

(ii) $\mathbf{r}\times\frac{d\mathbf{r}}{dt}=(\cos\omega t\,\mathbf{a}+\sin\omega t\,\mathbf{b})\times(-\omega\sin\omega t\,\mathbf{a}+\omega\cos\omega t\,\mathbf{b})$

$=\omega\cos^2\omega t\,\mathbf{a}\times\mathbf{b}-\omega\sin^2\omega t\,\mathbf{b}\times\mathbf{a}$ $\quad[\because\ \mathbf{a}\times\mathbf{a}=\mathbf{0},\ \mathbf{b}\times\mathbf{b}=\mathbf{0}]$

$=\omega\cos^2\omega t\,\mathbf{a}\times\mathbf{b}+\omega\sin^2\omega t\,\mathbf{a}\times\mathbf{b}$

$=\omega(\cos^2\omega t+\sin^2\omega t)\,\mathbf{a}\times\mathbf{b}=\omega\mathbf{a}\times\mathbf{b}$.

Ex. 5. *If* $\mathbf{r}=(\sinh t)\,\mathbf{a}+(\cosh t)\,\mathbf{b}$, *where* $\mathbf{a}$ *and* $\mathbf{b}$ *are constant vectors, then show that* $\frac{d^2\mathbf{r}}{dt^2}=\mathbf{r}$.

Sol. Since $\mathbf{a}$, $\mathbf{b}$ are constant vectors, therefore

$$\frac{d\mathbf{a}}{dt}=\mathbf{0},\ \frac{d\mathbf{b}}{dt}=\mathbf{0}.$$

$\therefore\ \frac{d\mathbf{r}}{dt}=\frac{d}{dt}(\sinh t)\,\mathbf{a}+\frac{d}{dt}(\cosh t)\,\mathbf{b}$.

$=(\cosh t)\,\mathbf{a}+(\sinh t)\,\mathbf{b}$.

$\therefore\ \frac{d^2\mathbf{r}}{dt^2}=(\sinh t)\,\mathbf{a}+(\cosh t)\,\mathbf{b}=\mathbf{r}$.

Ex. 6. *If* $\mathbf{r}=t^3\,\mathbf{i}+\left(2t^3-\frac{1}{5t^2}\right)\mathbf{j}$, *show that* $\mathbf{r}\times\frac{d\mathbf{r}}{dt}=\mathbf{k}$.

[Utkal 1973]

Sol. We have $\mathbf{r}=t^3\,\mathbf{i}+\left(2t^3-\frac{1}{5t^2}\right)\mathbf{j}$.

$\therefore\ \frac{d\mathbf{r}}{dt}=3t^2\,\mathbf{i}+\left(6t^2+\frac{2}{5t^3}\right)\mathbf{j}$.

$\therefore\ \mathbf{r}\times\frac{d\mathbf{r}}{dt}=\left[t^3\,\mathbf{i}+\left(2t^3-\frac{1}{5t^2}\right)\mathbf{j}\right]\times\left[3t^2\,\mathbf{i}+\left(6t^2+\frac{2}{5t^3}\right)\mathbf{j}\right]$

$$=t^3\left(6t^2+\frac{2}{5t^3}\right)\mathbf{i}\times\mathbf{j}+3t^2\left(2t^3-\frac{1}{5t^2}\right)\mathbf{j}\times\mathbf{i}$$

$$[\because \quad \mathbf{i}\times\mathbf{i}=\mathbf{0},\ \mathbf{j}\times\mathbf{j}=\mathbf{0}]$$

$$=\left(6t^5+\frac{2}{5}\right)\mathbf{k}+\left(6t^5-\frac{3}{5}\right)(-\mathbf{k})$$

$$[\because \quad \mathbf{i}\times\mathbf{j}=\mathbf{k},\ \mathbf{j}\times\mathbf{i}=-\mathbf{k}]$$

$$=\left(6t^5+\frac{2}{5}-6t^5+\frac{3}{5}\right)\mathbf{k}=\mathbf{k}.$$

Ex. 7. *If* $\mathbf{r}=e^{nt}\,\mathbf{a}+e^{-nt}\,\mathbf{b}$, *where* $\mathbf{a}$, $\mathbf{b}$ *are constant vectors, show that* $\dfrac{d^2\mathbf{r}}{dt^2}-n^2\mathbf{r}=\mathbf{0}$. [Agra 1976]

Sol. Given $\mathbf{r}=e^{nt}\,\mathbf{a}+e^{-nt}\,\mathbf{b}$, ...(1)

where $\mathbf{a}$, $\mathbf{b}$ are constant vectors.

$$\therefore \quad \frac{d\mathbf{r}}{dt}=\left[\frac{d}{dt}(e^{nt})\right]\mathbf{a}+e^{nt}\frac{d\mathbf{a}}{dt}+\left[\frac{d}{dt}(e^{-nt})\right]\mathbf{b}+e^{-nt}\frac{d\mathbf{b}}{dt}$$

$$=ne^{nt}\,\mathbf{a}-ne^{-nt}\,\mathbf{b}. \quad \left[\because \ \frac{d\mathbf{a}}{dt}=\mathbf{0}=\frac{d\mathbf{b}}{dt},\ \mathbf{a} \text{ and } \mathbf{b} \text{ being constant vectors}\right]$$

Again differentiating with respect to t, we get

$$\frac{d^2\mathbf{r}}{dt^2}=n^2e^{nt}\,\mathbf{a}+n^2e^{-nt}\,\mathbf{b}=n^2(e^{nt}\mathbf{a}+e^{-nt}\mathbf{b})=n^2\mathbf{r}, \text{ from (1).}$$

$$\therefore \quad \frac{d^2\mathbf{r}}{dt^2}-n^2\mathbf{r}=\mathbf{0}.$$

Ex. 8. *If* $\mathbf{r}=\mathbf{a}\sin\omega t+\mathbf{b}\cos\omega t+\dfrac{\mathbf{c}t}{\omega^2}\sin\omega t$, *prove that*

$$\frac{d^2\mathbf{r}}{dt^2}+\omega^2\mathbf{r}=\frac{2\mathbf{c}}{\omega}\cos\omega t,$$

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *are constant vectors and* ω *is a constant scalar.*

[Meerut 1991; Marathwada 74]

Sol. Given $\mathbf{r}=\mathbf{a}\sin\omega t+\mathbf{b}\cos\omega t+\dfrac{\mathbf{c}t}{\omega^2}\sin\omega t$, ...(1)

where $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are constant vectors and ω is a constant scalar.

$$\therefore \quad \frac{d\mathbf{r}}{dt}=\mathbf{a}\,\omega\cos\omega t-\mathbf{b}\,\omega\sin\omega t+\frac{\mathbf{c}}{\omega^2}\sin\omega t+\frac{\mathbf{c}}{\omega^2}t\omega\cos\omega t$$

and $$\frac{d^2\mathbf{r}}{dt^2}=-\mathbf{a}\omega^2\sin\omega t-\mathbf{b}\,\omega^2\cos\omega t+\frac{\mathbf{c}}{\omega^2}\omega\cos\omega t$$

$$+\frac{\mathbf{c}}{\omega^2}\omega\cos\omega t-\frac{\mathbf{c}}{\omega^2}t\,\omega^2\sin\omega t$$

$$=-\omega^2\left(\mathbf{a}\sin\omega t+\mathbf{b}\cos\omega t+\frac{\mathbf{c}t}{\omega^2}\sin\omega t\right)+\frac{2\mathbf{c}}{\omega}\cos\omega t$$

$$=-\omega^2\,\mathbf{r}+\frac{2\mathbf{c}}{\omega}\cos\omega t, \text{ from (1).}$$

$$\therefore\ \frac{d^2\mathbf{r}}{dt^2}+\omega^2\,\mathbf{r}=\frac{2\mathbf{c}}{\omega}\cos\omega t.$$

Ex. 9. *Show that* $\mathbf{r}=\mathbf{a}\,e^{mt}+\mathbf{b}\,e^{nt}$, *where* $\mathbf{a}$ *and* $\mathbf{b}$ *are the constant vectors, is the solution of the differential equation*

$$\frac{d^2\mathbf{r}}{dt^2}-(m+n)\frac{d\mathbf{r}}{dt}+mn\,\mathbf{r}=0.$$

Hence solve the equation

$$\frac{d^2\mathbf{r}}{dt^2}-\frac{d\mathbf{r}}{dt}-2\mathbf{r}=0, \text{ where}$$

$\mathbf{r}=\mathbf{i}$ *and* $\frac{d\mathbf{r}}{dt}=\mathbf{j}$ *for* $t=0$. [Kanpur 1977]

Sol. We have $\mathbf{r}=\mathbf{a}\,e^{mt}+\mathbf{b}\,e^{nt}$, ...(1)

where $\mathbf{a}$ and $\mathbf{b}$ are constant vectors.

$$\therefore\ \frac{d\mathbf{r}}{dt}=\mathbf{a}\,me^{mt}+\mathbf{b}\,ne^{nt} \qquad ...(2)$$

and $$\frac{d^2\mathbf{r}}{dt^2}=\mathbf{a}\,m^2\,e^{mt}+\mathbf{b}\,n^2\,e^{nt} \qquad ...(3)$$

From (1), (2) and (3), we get

$$\frac{d^2\mathbf{r}}{dt^2}-(m+n)\frac{d\mathbf{r}}{dt}+mn\,\mathbf{r}$$

$$=\mathbf{a}m^2\,e^{mt}+\mathbf{b}\,n^2\,e^{nt}-(m+n)\,[\mathbf{a}\,me^{mt}+\mathbf{b}\,ne^{nt}]$$

$$=e^{mt}\,(m^2-m^2-mn+mn)\,\mathbf{a}+e^{nt}\,(n^2-mn-n^2+mn)\,\mathbf{b}$$

$$=0\mathbf{a}+0\mathbf{b}=\mathbf{0}+\mathbf{0}=\mathbf{0}.$$

Hence $\mathbf{r}=\mathbf{a}\,e^{mt}+\mathbf{b}\,e^{nt}$ is the solution of the differential equation

$$\frac{d^2\mathbf{r}}{dt^2}-(m+n)\frac{d\mathbf{r}}{dt}+mn\,\mathbf{r}=\mathbf{0}. \qquad ...(4)$$

Putting $m=2$ and $n=-1$ in (1) and (4), we see that the general solution of the differential equation

$$\frac{d^2\mathbf{r}}{dt^2}-\frac{d\mathbf{r}}{dt}-2\mathbf{r}=\mathbf{0} \qquad ...(5)$$

is $$\mathbf{r}=\mathbf{a}\,e^{2t}+\mathbf{b}\,e^{-t}, \qquad ...(6)$$

where $\mathbf{a}$ and $\mathbf{b}$ are arbitrary constant vectors.

From (6), $$\frac{d\mathbf{r}}{dt}=\mathbf{a}\cdot 2e^{2t}-\mathbf{b}\,e^{-t}. \qquad ...(7)$$

But it is given that for $t=0$, $\mathbf{r}=\mathbf{i}$ and $\frac{d\mathbf{r}}{dt}=\mathbf{j}$.

$\therefore$ from (6) and (7), we have

$$\mathbf{a}+\mathbf{b}=\mathbf{i} \qquad ...(8)$$

and $\quad 2\mathbf{a}-\mathbf{b}=\mathbf{j}$. ...(9)

Adding (8) and (9), we get $3\mathbf{a}=\mathbf{i}+\mathbf{j}$ or $\mathbf{a}=\frac{1}{3}(\mathbf{i}+\mathbf{j})$.

Now from (8), we have $\mathbf{b}=\mathbf{i}-\mathbf{a}=\mathbf{i}-\frac{1}{3}(\mathbf{i}+\mathbf{j})=\frac{2}{3}\mathbf{i}-\frac{1}{3}\mathbf{j}$.

Putting $\mathbf{a}=\frac{1}{3}(\mathbf{i}+\mathbf{j})$ and $\mathbf{b}=\frac{2}{3}\mathbf{i}-\frac{1}{3}\mathbf{j}$ in (6), the required solution of the differential equation (5) under the given conditions is

$$\mathbf{r}=\frac{1}{3}(\mathbf{i}+\mathbf{j})e^{2t}+\left(\frac{2}{3}\mathbf{i}-\frac{1}{3}\mathbf{j}\right)e^{-t}$$

or $$\mathbf{r}=\frac{1}{3}(e^{2t}+2e^{-t})\mathbf{i}+\frac{1}{3}(e^{2t}-e^{-t})\mathbf{j}.$$

Ex. 10. *Prove the following :*

(*i*) $\frac{d}{dt}\left[\mathbf{a}\cdot\frac{d\mathbf{b}}{dt}-\frac{d\mathbf{a}}{dt}\cdot\mathbf{b}\right]=\mathbf{a}\cdot\frac{d^2\mathbf{b}}{dt^2}-\frac{d^2\mathbf{a}}{dt^2}\cdot\mathbf{b}$.

(*ii*) $\frac{d}{dt}\left[\mathbf{a}\times\frac{d\mathbf{b}}{dt}-\frac{d\mathbf{a}}{dt}\times\mathbf{b}\right]=\mathbf{a}\times\frac{d^2\mathbf{b}}{dt^2}-\frac{d^2\mathbf{a}}{dt^2}\times\mathbf{b}$.

Sol. (i) We have $\frac{d}{dt}\left[\mathbf{a}\cdot\frac{d\mathbf{b}}{dt}-\frac{d\mathbf{a}}{dt}\cdot\mathbf{b}\right]$

$$=\frac{d}{dt}\left(\mathbf{a}\cdot\frac{d\mathbf{b}}{dt}\right)-\frac{d}{dt}\left(\frac{d\mathbf{a}}{dt}\cdot\mathbf{b}\right)$$

$$=\frac{d\mathbf{a}}{dt}\cdot\frac{d\mathbf{b}}{dt}+\mathbf{a}\cdot\frac{d^2\mathbf{b}}{dt^2}-\frac{d^2\mathbf{a}}{dt^2}\cdot\mathbf{b}-\frac{d\mathbf{a}}{dt}\cdot\frac{d\mathbf{b}}{dt}$$

$$=\mathbf{a}\cdot\frac{d^2\mathbf{b}}{dt^2}-\frac{d^2\mathbf{a}}{dt^2}\cdot\mathbf{b}.$$

(ii) $\frac{d}{dt}\left[\mathbf{a}\times\frac{d\mathbf{b}}{dt}-\frac{d\mathbf{a}}{dt}\times\mathbf{b}\right]$

$$=\frac{d}{dt}\left(\mathbf{a}\times\frac{d\mathbf{b}}{dt}\right)-\frac{d}{dt}\left(\frac{d\mathbf{a}}{dt}\times\mathbf{b}\right)$$

$$=\frac{d\mathbf{a}}{dt}\times\frac{d\mathbf{b}}{dt}+\mathbf{a}\times\frac{d}{dt}\left(\frac{d\mathbf{b}}{dt}\right)-\left[\frac{d}{dt}\left(\frac{d\mathbf{a}}{dt}\right)\right]\times\mathbf{b}-\frac{d\mathbf{a}}{dt}\times\frac{d\mathbf{b}}{dt}$$

$$=\mathbf{a}\times\frac{d^2\mathbf{b}}{dt^2}-\frac{d^2\mathbf{a}}{dt^2}\times\mathbf{b}.$$

Ex. 11. *If* $\mathbf{r}=t^2\mathbf{i}-t\mathbf{j}+(2t+1)\mathbf{k}$, *find at* $t=0$, *the values of*

$$\frac{d\mathbf{r}}{dt},\ \frac{d^2\mathbf{r}}{dt^2},\ \left|\frac{d\mathbf{r}}{dt}\right|,\ \left|\frac{d^2\mathbf{r}}{dt^2}\right|.$$

Sol. $\mathbf{r}=t^2\mathbf{i}-t\mathbf{j}+(2t+1)\mathbf{k}$.

$\therefore \quad \frac{d\mathbf{r}}{dt}=2t\mathbf{i}-\mathbf{j}+2\mathbf{k}$ ...(1)

and $\quad \frac{d^2\mathbf{r}}{dt^2}=2\mathbf{i}$ ...(2)

4

From (1) and (2), we have

$$\left|\frac{d\mathbf{r}}{dt}\right|=\sqrt{[(2t)^2+(-1)^2+2^2]}=\sqrt{(4t^2+5)} \quad \text{...(3)}$$

and $\left|\frac{d^2\mathbf{r}}{dt^2}\right|=|2\mathbf{i}|=2.$ ...(4)

Putting $t=0$ in (1), (2), (3) and (4), we have at $t=0$,

$$\frac{d\mathbf{r}}{dt}=-\mathbf{j}+2\mathbf{k},\ \frac{d^2\mathbf{r}}{dt^2}=2\mathbf{i},\ \left|\frac{d\mathbf{r}}{dt}\right|=\sqrt{5},\ \left|\frac{d^2\mathbf{r}}{dt^2}\right|=2.$$

Ex. 12. *If* $\mathbf{u}=t^2\,\mathbf{i}-t\,\mathbf{j}+(2t+1)\,\mathbf{k}$ *and* $\mathbf{v}=(2t-3)\,\mathbf{i}+\mathbf{j}-t\mathbf{k}$, *find* $\frac{d}{dt}(\mathbf{u}\cdot\mathbf{v})$, *when* $t=1$.

[Kanpur 1982]

Sol. $\frac{d\mathbf{u}}{dt}=\frac{d}{dt}\left[t^2\,\mathbf{i}-t\,\mathbf{j}+(2t+1)\,\mathbf{k}\right]=2t\mathbf{i}-\mathbf{j}+2\mathbf{k}$

and $\frac{d\mathbf{v}}{dt}=\frac{d}{dt}[(2t-3)\,\mathbf{i}+\mathbf{j}-t\,\mathbf{k}]=2\mathbf{i}+0\,\mathbf{j}-\mathbf{k}.$

$$\therefore \quad \frac{d}{dt}(\mathbf{u}\cdot\mathbf{v})=\mathbf{u}\cdot\frac{d\mathbf{v}}{dt}+\mathbf{v}\cdot\frac{d\mathbf{u}}{dt}$$

$$=[t^2\,\mathbf{i}-t\,\mathbf{j}+(2t+1)\,\mathbf{k}]\cdot(2\mathbf{i}-\mathbf{k})$$
$$+[(2t-3)\,\mathbf{i}+\mathbf{j}-t\,\mathbf{k}]\cdot(2t\,\mathbf{i}-\mathbf{j}+2\mathbf{k})$$
$$=2t^2-(2t+1)+2t\,(2t-3)-1-2t$$
$$=2t^2-2t-1+4t^2-6t-1-2t$$
$$=6t^2-10t-2$$
$$=-6,\ \text{when } t=1.$$

Ex. 13. *If* $\mathbf{A}=5t^2\,\mathbf{i}+t\,\mathbf{j}-t^3\,\mathbf{k}$ *and* $\mathbf{B}=\sin t\,\mathbf{i}-\cos t\,\mathbf{j}$, *find*

(*i*) $\frac{d}{dt}(\mathbf{A}\cdot\mathbf{B})$; (*ii*) $\frac{d}{dt}(\mathbf{A}\times\mathbf{B})$; (*iii*) $\frac{d}{dt}(\mathbf{A}\cdot\mathbf{A})$.

Sol. We have $\frac{d\mathbf{A}}{dt}=10t\,\mathbf{i}+\mathbf{j}-3t^2\,\mathbf{k}$ and $\frac{d\mathbf{B}}{dt}=\cos t\,\mathbf{i}+\sin t\,\mathbf{j}.$

(i) $\frac{d}{dt}(\mathbf{A}\cdot\mathbf{B})=\mathbf{A}\cdot\frac{d\mathbf{B}}{dt}+\frac{d\mathbf{A}}{dt}\cdot\mathbf{B}$

$$=(5t^2\,\mathbf{i}+t\,\mathbf{j}-t^3\,\mathbf{k})\cdot(\cos t\,\mathbf{i}+\sin t\mathbf{j})$$
$$+(10t\,\mathbf{i}+\mathbf{j}-3t^2\,\mathbf{k})\cdot(\sin t\,\mathbf{i}-\cos t\,\mathbf{j})$$
$$=5t^2\cos t+t\sin t+10t\sin t-\cos t$$
$$=(5t^2-1)\cos t+11t\sin t.$$

(ii) We have $\mathbf{A}\times\mathbf{B}=(5t^2\,\mathbf{i}+t\,\mathbf{j}-t^3\,\mathbf{k})\times(\sin t\,\mathbf{i}-\cos t\,\mathbf{j})$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 5t^2 & t & -t^3\\ \sin t & -\cos t & 0\end{vmatrix}$$

$$=-t^3\cos t\,\mathbf{i}-(0+t^3\sin t)\,\mathbf{j}+(-5t^2\cos t-t\sin t)\,\mathbf{k}$$

$=-t^3 \cos t\,\mathbf{i}-t^3 \sin t\,\mathbf{j}-(5t^2 \cos t+t \sin t)\,\mathbf{k}.$

$\therefore\ \dfrac{d}{dt}(\mathbf{A}\times\mathbf{B})=(t^3 \sin t-3t^2 \cos t)\,\mathbf{i}-(t^3 \cos t+3t^2 \sin t)\,\mathbf{j}$

$\qquad -(10t \cos t-5t^2 \sin t+\sin t+t \cos t)\,\mathbf{k}$

$=t^2(t \sin t-3 \cos t)\,\mathbf{i}-t^2(t \cos t+3 \sin t)\,\mathbf{j}$

$\qquad -(11t \cos t-5t^2 \sin t+\sin t)\,\mathbf{k}.$

(iii) $\dfrac{d}{dt}(\mathbf{A}\cdot\mathbf{A})=\dfrac{d\mathbf{A}}{dt}\cdot\mathbf{A}+\mathbf{A}\cdot\dfrac{d\mathbf{A}}{dt}=2\mathbf{A}\cdot\dfrac{d\mathbf{A}}{dt}$

$=2\,(5t^2\,\mathbf{i}+t\,\mathbf{j}-t^3\,\mathbf{k})\cdot(10t\,\mathbf{i}+\mathbf{j}-3t^2\,\mathbf{k})$

$=2\,[50\,t^3+t+3t^5]=100t^3+2t+6t^5.$

Ex. 14. *If* $\mathbf{r}=a \cos t\,\mathbf{i}+a \sin t\,\mathbf{j}+at \tan\alpha\,\mathbf{k}$, *find*

$$\left|\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right| \text{ and } \left[\frac{d\mathbf{r}}{dt}, \frac{d^2\mathbf{r}}{dt^2}, \frac{d^3\mathbf{r}}{dt^3}\right].$$

[Meerut 1991 P, 92; Agra 82, 88; Kanpur 88]

Sol. We have

$\dfrac{d\mathbf{r}}{dt}=-a \sin t\,\mathbf{i}+a \cos t\,\mathbf{j}+a \tan\alpha\,\mathbf{k}$

$\dfrac{d^2\mathbf{r}}{dt^2}=-a \cos t\,\mathbf{i}-a \sin t\,\mathbf{j},\quad \left[\because\ \dfrac{d\mathbf{k}}{dt}=0\right]$

$\dfrac{d^3\mathbf{r}}{dt^3}=a \sin t\,\mathbf{i}-a \cos t\,\mathbf{j}.$

$$\therefore\ \frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a \sin t & a \cos t & a \tan\alpha \\ -a \cos t & -a \sin t & 0 \end{vmatrix}$$

$=a^2 \sin t \tan\alpha\,\mathbf{i}-a^2 \cos t \tan\alpha\,\mathbf{j}+a^2\mathbf{k}.$

$\therefore\ \left|\dfrac{d\mathbf{r}}{dt}\times\dfrac{d^2\mathbf{r}}{dt^2}\right|=\sqrt{\left(a^4 \sin^2 t \tan^2\alpha+a^4 \cos^2 t \tan^2\alpha+a^4\right)}$

$=a^2 \sec\alpha.$

Also $\left[\dfrac{d\mathbf{r}}{dt}, \dfrac{d^2\mathbf{r}}{dt^2}, \dfrac{d^3\mathbf{r}}{dt^3}\right]=\left(\dfrac{d\mathbf{r}}{dt}\times\dfrac{d^2\mathbf{r}}{dt^2}\right)\cdot\dfrac{d^3\mathbf{r}}{dt^3}$

$=(a^2 \sin t \tan\alpha\,\mathbf{i}-a^2 \cos t \tan\alpha\,\mathbf{j}+a^2\mathbf{k})\cdot(a \sin t\,\mathbf{i}-a \cos t\,\mathbf{j})$

$=a^3 \sin^2 t \tan\alpha\,\mathbf{i}\cdot\mathbf{i}+a^3 \cos^2 t \tan\alpha\,\mathbf{j}\cdot\mathbf{j}\qquad [\because\ \mathbf{i}\cdot\mathbf{j}=0 \text{ etc.}]$

$=a^3 \tan\alpha\,(\sin^2 t+\cos^2 t)\quad [\because\ \mathbf{i}\cdot\mathbf{i}=1=\mathbf{j}\cdot\mathbf{j}]$

$=a^3 \tan\alpha.$

Ex. 15. *If* $\dfrac{d\mathbf{u}}{dt}=\mathbf{w}\times\mathbf{u},\ \dfrac{d\mathbf{v}}{dt}=\mathbf{w}\times\mathbf{v}$, *show that*

$$\frac{d}{dt}(\mathbf{u}\times\mathbf{v})=\mathbf{w}\times(\mathbf{u}\times\mathbf{v}).$$

[Meerut 1991S, Kanpur 88]

Sol. We have

$$\frac{d}{dt}(\mathbf{u}\times\mathbf{v})=\frac{d\mathbf{u}}{dt}\times\mathbf{v}+\mathbf{u}\times\frac{d\mathbf{v}}{dt}=(\mathbf{w}\times\mathbf{u})\times\mathbf{v}+\mathbf{u}\times(\mathbf{w}\times\mathbf{v})$$

$$=(\mathbf{v}\cdot\mathbf{w})\,\mathbf{u}-(\mathbf{v}\cdot\mathbf{u})\,\mathbf{w}+(\mathbf{u}\cdot\mathbf{v})\,\mathbf{w}-(\mathbf{u}\cdot\mathbf{w})\,\mathbf{v}$$

$$=(\mathbf{v}\cdot\mathbf{w})\,\mathbf{u}-(\mathbf{u}\cdot\mathbf{w})\,\mathbf{v}\quad[\because\ \mathbf{u}\cdot\mathbf{v}=\mathbf{v}\cdot\mathbf{u}]$$

$$=(\mathbf{w}\cdot\mathbf{v})\,\mathbf{u}-(\mathbf{w}\cdot\mathbf{u})\,\mathbf{v}=\mathbf{w}\times(\mathbf{u}\times\mathbf{v}).$$

Ex. 16. *If* $\mathbf{R}$ *be a unit vector in the direction of* $\mathbf{r}$, *prove that*

$$\mathbf{R}\times\frac{d\mathbf{R}}{dt}=\frac{1}{r^2}\,\mathbf{r}\times\frac{d\mathbf{r}}{dt};\ \textit{where } r=|\,\mathbf{r}\,|.$$

[Kanpur 1987; Agra 83; Garhwal 86]

Sol. We have $\mathbf{r}=r\mathbf{R}$; so that $\mathbf{R}=\frac{1}{r}\,\mathbf{r}$.

$$\therefore\quad \frac{d\mathbf{R}}{dt}=\frac{1}{r}\frac{d\mathbf{r}}{dt}-\frac{1}{r^2}\frac{dr}{dt}\,\mathbf{r}.$$

Hence $\mathbf{R}\times\frac{d\mathbf{R}}{dt}=\frac{1}{r}\,\mathbf{r}\times\left(\frac{1}{r}\frac{d\mathbf{r}}{dt}-\frac{1}{r^2}\frac{dr}{dt}\,\mathbf{r}\right)$

$$=\frac{1}{r^2}\,\mathbf{r}\times\frac{d\mathbf{r}}{dt}-\frac{1}{r^3}\frac{dr}{dt}\,\mathbf{r}\times\mathbf{r}$$

$$=\frac{1}{r^2}\,\mathbf{r}\times\frac{d\mathbf{r}}{dt}.\qquad[\because\ \mathbf{r}\times\mathbf{r}=\mathbf{0}]$$

Ex. 17. *Show that* $\hat{\mathbf{r}}\times d\hat{\mathbf{r}}=(\mathbf{r}\times d\mathbf{r})/r^2$, *where* $\mathbf{r}=r\,\hat{\mathbf{r}}$.

[Rohilkhand 1991, Agra 83]

Sol. We have $\hat{\mathbf{r}}=\frac{1}{r}\,\mathbf{r}$.

$$\therefore\quad d\hat{\mathbf{r}}=d\left(\frac{1}{r}\mathbf{r}\right)=\frac{1}{r}\,d\mathbf{r}+\left(-\frac{1}{r^2}\,dr\right)\mathbf{r}.$$

Hence $\hat{\mathbf{r}}\times d\hat{\mathbf{r}}=\left(\frac{1}{r}\,\mathbf{r}\right)\times\left[\frac{1}{r}\,d\mathbf{r}-\left(\frac{1}{r^2}\,dr\right)\mathbf{r}\right]$

$$=\frac{1}{r^2}\,\mathbf{r}\times d\mathbf{r}-\left(\frac{1}{r^3}\,dr\right)\mathbf{r}\times\mathbf{r}$$

$$=\frac{\mathbf{r}\times d\mathbf{r}}{r^2};\ \text{since } \mathbf{r}\times\mathbf{r}=\mathbf{0}.$$

Ex. 18. *If* $\mathbf{r}$ *is the position vector of a moving point and* r *is the modulus of* $\mathbf{r}$, *show that*

$$\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}=r\,\frac{dr}{dt}.$$

Interpret the relations $\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}=0$ *and* $\mathbf{r}\times\frac{d\mathbf{r}}{dt}=\mathbf{0}$.

[Rohilkhand 1980]

Sol. We have $\mathbf{r}\cdot\mathbf{r}=|\,\mathbf{r}\,|^2=r^2$.

$$\therefore \quad \frac{d}{dt}(\mathbf{r}\cdot\mathbf{r})=\frac{d}{dt}(r^2)$$

or $$\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}+\frac{d\mathbf{r}}{dt}\cdot\mathbf{r}=2r\frac{dr}{dt}$$

or $$2\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}=2r\frac{dr}{dt} \qquad \left[\because \quad \mathbf{r}\cdot\frac{d\mathbf{r}}{dt}=\frac{d\mathbf{r}}{dt}\cdot\mathbf{r}\right]$$

or $$\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}=r\frac{dr}{dt}.$$

Geometrical interpretation of $\mathbf{r}\cdot\dfrac{d\mathbf{r}}{dt}=0$ and $\mathbf{r}\times\dfrac{d\mathbf{r}}{dt}=0$.

$\mathbf{r}\cdot\dfrac{d\mathbf{r}}{dt}=0$ is a necessary and sufficient condition for the vector $\mathbf{r}(t)$ to have constant modulus while $\mathbf{r}\times\dfrac{d\mathbf{r}}{dt}=0$ is a necessary and sufficient condition for the vector $\mathbf{r}(t)$ to have constant direction.

Ex. 19. *If the direction of a differentiable vector function* $\mathbf{r}(t)$ *is constant, show that* $\mathbf{r}\times(d\mathbf{r}/dt)=0$. **[Kanpur 1982; Rohilkhand 79]**

Or

If $\mathbf{r}(t)$ *is a vector of constant direction, show that its derivative is collinear with it.* **[Allahabad 1981]**

Sol. Let $\mathbf{r}$ be a vector function of the scalar variable t having a constant direction.

If $\mathbf{R}$ be a unit vector in the direction of $\mathbf{r}$, then $\mathbf{R}$ is a constant vector because it has constant direction as well as constant modulus.

If r be the modulus of $\mathbf{r}$, then $\mathbf{r}=r\mathbf{R}$.

$$\therefore \quad \frac{d\mathbf{r}}{dt}=\frac{d}{dt}(r\mathbf{R})=\frac{dr}{dt}\mathbf{R}+r\frac{d\mathbf{R}}{dt}$$

$$=\frac{dr}{dt}\mathbf{R}\left[\because \quad \frac{d\mathbf{R}}{dt}=0, \ \mathbf{R} \text{ being a constant vector}\right]$$

$$\therefore \quad \mathbf{r}\times\frac{d\mathbf{r}}{dt}=(r\mathbf{R})\times\left(\frac{dr}{dt}\mathbf{R}\right)$$

$$=r\frac{dr}{dt}(\mathbf{R}\times\mathbf{R})=0$$

$$[\because \quad \mathbf{R}\times\mathbf{R}=0]$$

Now $\mathbf{r}\times(d\mathbf{r}/dt)=0$ implies that the vector $d\mathbf{r}/dt$ is collinear with $\mathbf{r}$.

Ex. 20. If $\mathbf{r}\times d\mathbf{r}=0$, *show that* $\hat{\mathbf{r}}=$ *constant.*

[Kanpur 1987; Rohilkhand 80]

Sol. Let r be the modulus of the vector $\mathbf{r}$.

Then $\mathbf{r}=r\hat{\mathbf{r}}$.

$\therefore \quad d\mathbf{r}=d(r\hat{\mathbf{r}})=dr\,\hat{\mathbf{r}}+r\,d\hat{\mathbf{r}}$.

$\therefore \quad \mathbf{r}\times d\mathbf{r}=(r\hat{\mathbf{r}})\times(dr\,\hat{\mathbf{r}}+r\,d\hat{\mathbf{r}})$

$=(r\,dr)\,\hat{\mathbf{r}}\times\hat{\mathbf{r}}+r^2\,\hat{\mathbf{r}}\times d\hat{\mathbf{r}}$

$=r^2\,\hat{\mathbf{r}}\times d\hat{\mathbf{r}} \qquad [\because \ \hat{\mathbf{r}}\times\hat{\mathbf{r}}=\mathbf{0}]$

$\therefore \quad \mathbf{r}\times d\mathbf{r}=\mathbf{0} \Rightarrow r^2\,\hat{\mathbf{r}}\times d\hat{\mathbf{r}}=\mathbf{0}$

$\Rightarrow \hat{\mathbf{r}}\times d\hat{\mathbf{r}}=\mathbf{0}$ ...(1)

Since $\hat{\mathbf{r}}$ is of constant modulus, therefore

$\hat{\mathbf{r}}\cdot d\hat{\mathbf{r}}=0.$ ...(2)

From (1) and (2), we get $d\hat{\mathbf{r}}=\mathbf{0}$.

Hence $\hat{\mathbf{r}}$ is a constant vector.

Alternative method.

Let $\mathbf{r}=x\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}$, then $d\mathbf{r}=dx\,\mathbf{i}+dy\,\mathbf{j}+dz\,\mathbf{k}$.

$$\therefore \quad \mathbf{r}\times d\mathbf{r}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ dx & dy & dz \end{vmatrix}$$

$=(y\,dz-z\,dy)\,\mathbf{i}+(z\,dx-x\,dz)\,\mathbf{j}+(x\,dy-y\,dx)\,\mathbf{k}$.

$\therefore \quad \mathbf{r}\times d\mathbf{r}=\mathbf{0} \Rightarrow (y\,dz-z\,dy)\,\mathbf{i}+(z\,dx-x\,dz)\,\mathbf{j}+(x\,dy-y\,dx)\,\mathbf{k}=\mathbf{0}$

$\Rightarrow \quad y\,dz-z\,dy=0,\ z\,dx-x\,dz=0,\ x\,dy-y\,dx=0$.

$$\Rightarrow \quad \frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z}$$

If $\dfrac{dx}{x}=\dfrac{dy}{y}$, then $\log x=\log y+\log c_1$

or $\qquad x=c_1 y.$ ...(1)

If $\dfrac{dy}{y}=\dfrac{dz}{z}$, then $\log y+\log c_2=\log z$

or $\qquad z=c_2 y.$ ...(2)

Now $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}=(c_1 y)\,\mathbf{i}+y\,\mathbf{j}+(c_2 y)\,\mathbf{k}$

$=y\,(c_1\,\mathbf{i}+\mathbf{j}+c_2\,\mathbf{k})$.

$$\therefore \quad \hat{\mathbf{r}}=\frac{\mathbf{r}}{|\mathbf{r}|}=\frac{y\,(c_1\,\mathbf{i}+\mathbf{j}+c_2\,\mathbf{k})}{\sqrt{[(c_1 y)^2+y^2+(c_2 y)^2]}}$$

$=\dfrac{c_1\,\mathbf{i}+\mathbf{j}+c_2\,\mathbf{k}}{\sqrt{(c_1^2+1+c_2^2)}}$, which is a constant vector because it is independent of x, y, z.

Ex. 21. *If* r *is a unit vector, then prove that*

$$\left|\mathbf{r}\times\frac{d\mathbf{r}}{dt}\right|=\left|\frac{d\mathbf{r}}{dt}\right|.$$

[Rajasthan 1974]

Sol. Since r is a unit vector, therefore $|\mathbf{r}|$ is constant and so r is perpendicular to its derivative $d\mathbf{r}/dt$.

Now by the definition of the cross product of two vectors, we have

$$\left|\mathbf{r}\times\frac{d\mathbf{r}}{dt}\right|=|\mathbf{r}|\,.\left|\frac{d\mathbf{r}}{dt}\right|.\sin 90^\circ=1.\left|\frac{d\mathbf{r}}{dt}\right|.\,1=\left|\frac{d\mathbf{r}}{dt}\right|.$$

Ex. 22. *If* e *is the unit vector making an angle* θ *with x-axis, show that* $d\mathbf{e}/d\theta$ *is a unit vector obtained by rotating* e *through a right angle in the direction of* θ *increasing.* [Allahabad 1979]

Sol. Since e is the unit vector which makes an angle θ with x-axis, therefore

$$\mathbf{e}=\cos\theta\,\mathbf{i}+\sin\theta\,\mathbf{j}.\qquad\text{[Draw figure yourself]}$$

$$\therefore\ \frac{d\mathbf{e}}{d\theta}=-\sin\theta\,\mathbf{i}+\cos\theta\,\mathbf{j}=\cos\left(\frac{1}{2}\pi+\theta\right)\mathbf{i}+\sin\left(\frac{1}{2}\pi+\theta\right)\mathbf{j},$$

which is a unit vector which makes an angle $\frac{1}{2}\pi+\theta$ with x-axis.

Thus $d\mathbf{e}/d\theta$ is a unit vector perpendicular to the vector e in the direction of θ increasing.

Hence $d\mathbf{e}/d\theta$ is a unit vector obtained by rotating e through a right angle in the direction of θ increasing.

Ex. 23. *If* r *is a vector function of a scalar t and* a *is a constant vector, m a constant, differentiate the following with respect to t;*

(i) $\mathbf{r}\cdot\mathbf{a}$, (ii) $\mathbf{r}\times\mathbf{a}$, (iii) $\mathbf{r}\times\dfrac{d\mathbf{r}}{dt}$, (iv) $\mathbf{r}\cdot\dfrac{d\mathbf{r}}{dt}$,

(v) $\mathbf{r}^2+\dfrac{1}{\mathbf{r}^2}$, (vi) $m\left(\dfrac{d\mathbf{r}}{dt}\right)^2$, (vi) $\dfrac{\mathbf{r}+\mathbf{a}}{\mathbf{r}^2+\mathbf{a}^2}$, (viii) $\dfrac{\mathbf{r}\times\mathbf{a}}{\mathbf{r}\cdot\mathbf{a}}$.

Sol. (i) Let $R=\mathbf{r}\cdot\mathbf{a}$. [Note $\mathbf{r}\cdot\mathbf{a}$ is a scalar]

$$\text{Then }\frac{dR}{dt}=\frac{d\mathbf{r}}{dt}\cdot\mathbf{a}+\mathbf{r}\cdot\frac{d\mathbf{a}}{dt}$$

$$=\frac{d\mathbf{r}}{dt}\cdot\mathbf{a}+\mathbf{r}\cdot\mathbf{0}\qquad\left[\because\ \frac{d\mathbf{a}}{dt}=\mathbf{0},\text{ as a is constant}\right]$$

$$=\frac{d\mathbf{r}}{dt}\cdot\mathbf{a}+0=\frac{d\mathbf{r}}{dt}\cdot\mathbf{a}.$$

(ii) Let $\mathbf{R}=\mathbf{r}\times\mathbf{a}$.

$$\text{Then }\frac{d\mathbf{R}}{dt}=\frac{d\mathbf{r}}{dt}\times\mathbf{a}+\mathbf{r}\times\frac{d\mathbf{a}}{dt}$$

$$=\frac{d\mathbf{r}}{dt}\times\mathbf{a}+\mathbf{r}\times 0\left[\because\ \frac{d\mathbf{a}}{dt}=0\right]$$

$$=\frac{d\mathbf{r}}{dt}\times\mathbf{a}+0=\frac{d\mathbf{r}}{dt}\times\mathbf{a}.$$

(iii) Let $\mathbf{R}=\mathbf{r}\times\frac{d\mathbf{r}}{dt}$.

Then $\frac{d\mathbf{R}}{dt}=\frac{d\mathbf{r}}{dt}\times\frac{d\mathbf{r}}{dt}+\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}$

$$=0+\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}\quad\left[\because\ \frac{d\mathbf{r}}{dt}\times\frac{d\mathbf{r}}{dt}=0\right]$$

$$=\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}.$$

(iv) Let $R=\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}$.

Then $\frac{dR}{dt}=\frac{d\mathbf{r}}{dt}\cdot\frac{d\mathbf{r}}{dt}+\mathbf{r}\cdot\frac{d^2\mathbf{r}}{dt^2}=\left(\frac{d\mathbf{r}}{dt}\right)^2+\mathbf{r}\cdot\frac{d^2\mathbf{r}}{dt^2}.$

(v) Let $R=\mathbf{r}^2+\frac{1}{\mathbf{r}^2}$.

Then $\frac{dR}{dt}=\frac{d}{dt}(\mathbf{r}^2)+\frac{d}{dt}\left(\frac{1}{\mathbf{r}^2}\right)$

$$=\frac{d}{dt}(r^2)+\frac{d}{dt}\left(\frac{1}{r^2}\right),\text{ where } r=|\mathbf{r}|$$

$$=2r\frac{dr}{dt}-\frac{2}{r^3}\frac{dr}{dt}.$$

(vi) Let $R=m\left(\frac{d\mathbf{r}}{dt}\right)^2$.

Then $\frac{dR}{dt}=m\frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right)^2$

$$=2m\frac{d\mathbf{r}}{dt}\cdot\frac{d^2\mathbf{r}}{dt^2}\qquad\left[\text{Note }\frac{d\mathbf{r}^2}{dt}=2\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}\right]$$

$$=2m\frac{d\mathbf{r}}{dt}\cdot\frac{d^2\mathbf{r}}{dt^2}.$$

(vii) Let $\mathbf{R}=\frac{\mathbf{r}+\mathbf{a}}{\mathbf{r}^2+\mathbf{a}^2}$.

Then $\frac{d\mathbf{R}}{dt}=\frac{1}{(\mathbf{r}^2+\mathbf{a}^2)}\frac{d}{dt}(\mathbf{r}+\mathbf{a})+\left\{\frac{d}{dt}\left(\frac{1}{\mathbf{r}^2+\mathbf{a}^2}\right)\right\}(\mathbf{r}+\mathbf{a})$

[Note that $\mathbf{r}^2+\mathbf{a}^2$ is a scalar]

$$=\frac{1}{\mathbf{r}^2+\mathbf{a}^2}\left(\frac{d\mathbf{r}}{dt}+\frac{d\mathbf{a}}{dt}\right)-\left\{\frac{1}{(\mathbf{r}^2+\mathbf{a}^2)^2}\frac{d}{dt}(\mathbf{r}^2+\mathbf{a}^2)\right\}(\mathbf{r}+\mathbf{a})$$

$$=\frac{1}{(\mathbf{r}^2+\mathbf{a}^2)}\frac{d\mathbf{r}}{dt}-\frac{2\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}}{(\mathbf{r}^2+\mathbf{a}^2)^2}(\mathbf{r}+\mathbf{a}).$$

$$\left[\because \frac{d\mathbf{a}}{dt}=0,\ \frac{d}{dt}\mathbf{r}^2=2\mathbf{r}\cdot\frac{d\mathbf{r}}{dt},\ \frac{d}{dt}\mathbf{a}^2=0\right]$$

(viii) Let $\mathbf{R}=\dfrac{\mathbf{r}\times\mathbf{a}}{\mathbf{r}\cdot\mathbf{a}}$.

Then $\dfrac{d\mathbf{R}}{dt}=\dfrac{1}{\mathbf{r}\cdot\mathbf{a}}\dfrac{d}{dt}(\mathbf{r}\times\mathbf{a})+\left\{\dfrac{d}{dt}\left(\dfrac{1}{\mathbf{r}\cdot\mathbf{a}}\right)\right\}(\mathbf{r}\times\mathbf{a})$

[Note that $\mathbf{r}\cdot\mathbf{a}$ is a scalar quantity.]

$$=\frac{1}{\mathbf{r}\cdot\mathbf{a}}\left(\frac{d\mathbf{r}}{dt}\times\mathbf{a}+\mathbf{r}\times\frac{d\mathbf{a}}{dt}\right)-\left\{\frac{1}{(\mathbf{r}\cdot\mathbf{a})^2}\frac{d}{dt}(\mathbf{r}\cdot\mathbf{a})\right\}(\mathbf{r}\times\mathbf{a})$$

$$=\frac{\frac{d\mathbf{r}}{dt}\times\mathbf{a}}{\mathbf{r}\cdot\mathbf{a}}-\left\{\frac{1}{(\mathbf{r}\cdot\mathbf{a})^2}\left(\frac{d\mathbf{r}}{dt}\cdot\mathbf{a}+\mathbf{r}\cdot\frac{d\mathbf{a}}{dt}\right)\right\}(\mathbf{r}\times\mathbf{a})$$

$$=\frac{\frac{d\mathbf{r}}{dt}\times\mathbf{a}}{\mathbf{r}\cdot\mathbf{a}}-\frac{\frac{d\mathbf{r}}{dt}\cdot\mathbf{a}}{(\mathbf{r}\cdot\mathbf{a})^2}(\mathbf{r}\times\mathbf{a})\qquad\left[\because \frac{d\mathbf{a}}{dt}=0\right]$$

Ex. 24. *If* $\mathbf{r}$ *is a vector function of a scalar* t, r *its module, and* $\mathbf{a}$, $\mathbf{b}$ *are constant vectors, differentiate the following with respect to* t :

(*i*) $r^3\mathbf{r}+\mathbf{a}\times\dfrac{d\mathbf{r}}{dt}$, (*ii*) $r^2\mathbf{r}+(\mathbf{a}\cdot\mathbf{r})\,\mathbf{b}$, (*iii*) $r^n\mathbf{r}$, (*iv*) $(a\mathbf{r}+r\mathbf{b})^2$.

Sol. (i) Let $\mathbf{R}=r^3\mathbf{r}+\mathbf{a}\times\dfrac{d\mathbf{r}}{dt}$.

Then $\dfrac{d\mathbf{R}}{dt}=\dfrac{d}{dt}(r^3\mathbf{r})+\dfrac{d}{dt}\left\{\mathbf{a}\times\dfrac{d\mathbf{r}}{dt}\right\}$

$$=3r^2\frac{dr}{dt}\mathbf{r}+r^3\frac{d\mathbf{r}}{dt}+\frac{d\mathbf{a}}{dt}\times\frac{d\mathbf{r}}{dt}+\mathbf{a}\times\frac{d^2\mathbf{r}}{dt^2}$$

$$=3r^2\frac{dr}{dt}\mathbf{r}+r^3\frac{d\mathbf{r}}{dt}+\mathbf{a}\times\frac{d^2\mathbf{r}}{dt^2}\qquad\left[\because \frac{d\mathbf{a}}{dt}=0\right]$$

(ii) Let $\mathbf{R}=r^2\mathbf{r}+(\mathbf{a}\cdot\mathbf{r})\,\mathbf{b}$.

Then $\dfrac{d\mathbf{R}}{dt}=\dfrac{d}{dt}(r^2\mathbf{r})+\left\{\dfrac{d}{dt}(\mathbf{a}\cdot\mathbf{r})\right\}\mathbf{b}+(\mathbf{a}\cdot\mathbf{r})\dfrac{d\mathbf{b}}{dt}$

$$=2r\frac{dr}{dt}\mathbf{r}+r^2\frac{d\mathbf{r}}{dt}+\left(\frac{d\mathbf{a}}{dt}\cdot\mathbf{r}+\mathbf{a}\cdot\frac{d\mathbf{r}}{dt}\right)\mathbf{b}\qquad\left[\because \frac{d\mathbf{b}}{dt}=0\right]$$

$$=2r\frac{dr}{dt}\mathbf{r}+r^2\frac{d\mathbf{r}}{dt}+\left(\mathbf{a}\cdot\frac{d\mathbf{r}}{dt}\right)\mathbf{b}\qquad\left[\because \frac{d\mathbf{a}}{dt}=0\right]$$

(iii) Let $\mathbf{R}=r^n\mathbf{r}$.

Then $\dfrac{dR}{dt}=\left(\dfrac{d}{dt}r^n\right)\mathbf{r}+r^n\dfrac{d\mathbf{r}}{dt}=\left(nr^{n-1}\dfrac{dr}{dt}\right)\mathbf{r}+r^n\dfrac{d\mathbf{r}}{dt}.$

(iv) Let $R=(a\mathbf{r}+r\mathbf{b})^2$. Then

$$\frac{dR}{dt}=2\,(a\mathbf{r}+r\mathbf{b})\cdot\frac{d}{dt}(a\mathbf{r}+r\mathbf{b}) \qquad \left[\text{Note } \frac{d}{dt}\mathbf{r}^2=2\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}\right]$$

$$=2\,(a\mathbf{r}+r\mathbf{b})\cdot\left(\frac{da}{dt}\mathbf{r}+a\frac{d\mathbf{r}}{dt}+\frac{dr}{dt}\mathbf{b}+r\frac{d\mathbf{b}}{dt}\right)$$

$$=2\,(a\mathbf{r}+r\mathbf{b})\cdot\left(a\frac{d\mathbf{r}}{dt}+\frac{dr}{dt}\mathbf{b}\right) \qquad \left[\because\ \frac{da}{dt}=0,\ \frac{d\mathbf{b}}{dt}=0\right]$$

Ex. 25. *Find*

(i) $\dfrac{d}{dt}\left[\mathbf{r},\dfrac{d\mathbf{r}}{dt},\dfrac{d^2\mathbf{r}}{dt^2}\right]$; (ii) $\dfrac{d^2}{dt^2}\left[\mathbf{r},\dfrac{d\mathbf{r}}{dt},\dfrac{d^2\mathbf{r}}{dt^2}\right]$;

(iii) $\dfrac{d}{dt}\left[\mathbf{r}\times\left(\dfrac{d\mathbf{r}}{dt}\times\dfrac{d^2\mathbf{r}}{dt^2}\right)\right]$.

Sol. (i) Let $R=\left[\mathbf{r},\dfrac{d\mathbf{r}}{dt},\dfrac{d^2\mathbf{r}}{dt^2}\right]$. Then R is the scalar triple product of three vectors $\mathbf{r}$, $\dfrac{d\mathbf{r}}{dt}$ and $\dfrac{d^2\mathbf{r}}{dt^2}$. Therefore using the rule for finding the derivative of a scalar triple product, we have

$$\frac{dR}{dt}=\left[\frac{d\mathbf{r}}{dt},\frac{d\mathbf{r}}{dt},\frac{d^2\mathbf{r}}{dt^2}\right]+\left[\mathbf{r},\frac{d^2\mathbf{r}}{dt^2},\frac{d^2\mathbf{r}}{dt^2}\right]+\left[\mathbf{r},\frac{d\mathbf{r}}{dt},\frac{d^3\mathbf{r}}{dt^3}\right]$$

$=\left[\mathbf{r},\dfrac{d\mathbf{r}}{dt},\dfrac{d^3\mathbf{r}}{dt^3}\right]$, since scalar triple products having two equal vectors vanish.

(ii) Let $R=\left[\mathbf{r},\dfrac{d\mathbf{r}}{dt},\dfrac{d^2\mathbf{r}}{dt^2}\right]$. Then as in part (i)

$$\frac{dR}{dt}=\left[\mathbf{r},\frac{d\mathbf{r}}{dt},\frac{d^3\mathbf{r}}{dt^3}\right].$$

Differentiating again, we get

$$\frac{d^2R}{dt^2}=\left[\frac{d\mathbf{r}}{dt},\frac{d\mathbf{r}}{dt},\frac{d^3\mathbf{r}}{dt^3}\right]+\left[\mathbf{r},\frac{d^2\mathbf{r}}{dt^2},\frac{d^3\mathbf{r}}{dt^3}\right]+\left[\mathbf{r},\frac{d\mathbf{r}}{dt},\frac{d^4\mathbf{r}}{dt^4}\right]$$

$$=\left[\mathbf{r},\frac{d^2\mathbf{r}}{dt^2},\frac{d^3\mathbf{r}}{dt^3}\right]+\left[\mathbf{r},\frac{d\mathbf{r}}{dt},\frac{d^4\mathbf{r}}{dt^4}\right].$$

(iii) Let $\mathbf{R}=\mathbf{r}\times\left(\dfrac{d\mathbf{r}}{dt}\times\dfrac{d^2\mathbf{r}}{dt^2}\right)$. Then $\mathbf{R}$ is the vector triple product of three vectors. Therefore using the rule for finding the derivative of a vector triple product, we have

$$\frac{d\mathbf{R}}{dt}=\frac{d\mathbf{r}}{dt}\times\left(\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right)+\mathbf{r}\times\left(\frac{d^2\mathbf{r}}{dt^2}\times\frac{d^2\mathbf{r}}{dt^2}\right)+\mathbf{r}\times\left(\frac{d\mathbf{r}}{dt}\times\frac{d^3\mathbf{r}}{dt^3}\right)$$

$$=\frac{d\mathbf{r}}{dt}\times\left(\frac{d\mathbf{r}}{dt}\times\frac{d^2\mathbf{r}}{dt^2}\right)+\mathbf{r}\times\left(\frac{d\mathbf{r}}{dt}\times\frac{d^3\mathbf{r}}{dt^3}\right),$$

since $\frac{d^2\mathbf{r}}{dt^2}\times\frac{d^2\mathbf{r}}{dt^2}=\mathbf{0}$, being vector product of two equal vectors.

Ex. 26. *If* $\mathbf{a}=\sin\theta\,\mathbf{i}+\cos\theta\,\mathbf{j}+\theta\mathbf{k}$, $\mathbf{b}=\cos\theta\,\mathbf{i}-\sin\theta\,\mathbf{j}-3\mathbf{k}$, *and* $\mathbf{c}=2\mathbf{i}+3\mathbf{j}-3\mathbf{k}$, *find* $\frac{d}{d\theta}\{\mathbf{a}\times(\mathbf{b}\times\mathbf{c})\}$ *at* $\theta=\frac{\pi}{2}$.

[Kanpur 1987; Rohilkhand 79]

Sol. We have

$$\mathbf{b}\times\mathbf{c}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \cos\theta & -\sin\theta & -3\\ 2 & 3 & -3\end{vmatrix}=(3\sin\theta+9)\,\mathbf{i}+(3\cos\theta-6)\,\mathbf{j}+(3\cos\theta+2\sin\theta)\,\mathbf{k}.$$

$$\therefore\quad \mathbf{a}\times(\mathbf{b}\times\mathbf{c})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \sin\theta & \cos\theta & \theta\\ 3\sin\theta+9 & 3\cos\theta-6 & 3\cos\theta+2\sin\theta\end{vmatrix}$$

$$=(3\cos^2\theta+2\sin\theta\cos\theta-3\theta\cos\theta+6\theta)\,\mathbf{i}+(3\theta\sin\theta+9\theta-3\sin\theta\cos\theta-2\sin^2\theta)\,\mathbf{j}+(-6\sin\theta-9\cos\theta)\,\mathbf{k}.$$

$$\therefore\quad \frac{d}{d\theta}\{\mathbf{a}\times(\mathbf{b}\times\mathbf{c})\}$$

$$=(-6\cos\theta\sin\theta+2\cos^2\theta-2\sin^2\theta-3\cos\theta+3\theta\sin\theta+6)\,\mathbf{i}+(3\sin\theta+3\theta\cos\theta+9-3\cos^2\theta+3\sin^2\theta-4\sin\theta\cos\theta)\,\mathbf{j}+(-6\cos\theta+9\sin\theta)\,\mathbf{k}.$$

Putting $\theta=\pi/2$, we get the required derivative

$$=(4+\tfrac{3}{2}\pi)\,\mathbf{i}+15\mathbf{j}+9\mathbf{k}.$$

Ex. 27. *Show that if* $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ *are constant vectors, then* $\mathbf{r}=\mathbf{a}\,t^2+\mathbf{b}\,t+\mathbf{c}$ *is the path of a particle moving with constant acceleration.*

Sol. The velocity of the particle $=\frac{d\mathbf{r}}{dt}=2t\mathbf{a}+\mathbf{b}$.

The acceleration of the particle $=\frac{d^2\mathbf{r}}{dt^2}=2\mathbf{a}$.

Thus the point whose path is $\mathbf{r}=\mathbf{a}\,t^2+\mathbf{b}\,t+\mathbf{c}$ is moving with constant acceleration.

Ex. 28. *A particle moves along the curve* $x=4\cos t$, $y=4\sin t$, $z=6t$. *Find the velocity and acceleration at time* $t=0$ *and* $t=\frac{1}{2}\pi$. *Find also the magnitudes of the velocity and acceleration at any time* t.

[Kanpur 1980; Agra 81]

Sol. Let $\mathbf{r}$ be the position vector of the particle at time t.

Then $\mathbf{r}=x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}=4\cos t\,\mathbf{i}+4\sin t\,\mathbf{j}+6t\,\mathbf{k}$. If $\mathbf{v}$ is the velocity of the particle at time t and $\mathbf{a}$ its acceleration at that time then $\mathbf{v}=\dfrac{d\mathbf{r}}{dt}=-4\sin t\,\mathbf{i}+4\cos t\,\mathbf{j}+6\mathbf{k}$,

$$\mathbf{a}=\frac{d^2\mathbf{r}}{dt^2}=-4\cos t\,\mathbf{i}-4\sin t\,\mathbf{j}.$$

Magnitude of the velocity at time $t=|\mathbf{v}|$

$$=\sqrt{(16\sin^2 t+16\cos^2 t+36)}=\sqrt{(52)}=2\sqrt{(13)}.$$

Magnitude of the acceleration

$$=|\mathbf{a}|=\sqrt{(16\cos^2 t+16\sin^2 t)}=4.$$

At $t=0$, $\mathbf{v}=4\mathbf{j}+6\mathbf{k}$, $\mathbf{a}=-4\mathbf{i}$

At $t=\frac{1}{2}\pi$, $\mathbf{v}=-4\,\mathbf{i}+6\mathbf{k}$, $\mathbf{a}=-4\,\mathbf{j}$.

Ex. 29. *A particle moves along the curve $x=e^{-t}$, $y=2\cos 3t$, $z=2\sin 3t$. Determine the velocity and acceleration at any time t and their magnitudes at $t=0$.* [Gorakhpur 1985]

Sol. Let $\mathbf{r}$ be the position vector of the particle at time t.

Then $\mathbf{r}=x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}=e^{-t}\,\mathbf{i}+2\cos 3t\,\mathbf{j}+2\sin 3t\,\mathbf{k}$.

If $\mathbf{v}$ is the velocity of the particle at time t and $\mathbf{a}$ its acceleration at that time, then

$$\mathbf{v}=\frac{d\mathbf{r}}{dt}=-e^{-t}\,\mathbf{i}-6\sin 3t\,\mathbf{j}+6\cos 3t\,\mathbf{k},$$

$$\mathbf{a}=\frac{d^2\mathbf{r}}{dt^2}=e^{-t}\,\mathbf{i}-18\cos 3t\,\mathbf{j}-18\sin 3t\,\mathbf{k}.$$

Putting $t=0$ in the above relations, the velocity at $t=0$ is $-\mathbf{i}+6\mathbf{k}$,

and the acceleration at $t=0$ is $\mathbf{i}-18\mathbf{j}$.

Hence at $t=0$,

the magnitude of velocity $=|-\mathbf{i}+6\mathbf{k}|=\sqrt{[(-1)^2+6^2]}=\sqrt{37}$,

and the magnitude of acceleration

$$=|\mathbf{i}-18\mathbf{j}|=\sqrt{[1^2+(-18)^2]}=\sqrt{(325)}.$$

Ex. 30. *A particle moves along the curve $x=t^3+1$, $y=t^2$, $z=2t+5$, where t is the time. Find the components of its velocity and acceleration at $t=1$ in the direction $\mathbf{i}+\mathbf{j}+3\mathbf{k}$.*

[Agra 1979; Rohilkhand 81]

Sol. If $\mathbf{r}$ is the position vector of any point (x, y, z) on the given curve, then

$$\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}=(t^3+1)\,\mathbf{i}+t^2\,\mathbf{j}+(2t+5)\mathbf{k}.$$

Velocity $=\mathbf{v}=\dfrac{d\mathbf{r}}{dt}=3t^2\,\mathbf{i}+2t\mathbf{j}+2\mathbf{k}=3\mathbf{i}+2\mathbf{j}+2\mathbf{k}$ at $t=1$.

$$\text{Acceleration}=\mathbf{a}=\frac{d^2\mathbf{r}}{dt^2}=\frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right)=6t\mathbf{i}+2\mathbf{j}=6\mathbf{i}+2\mathbf{j} \text{ at } t=1.$$

Now the unit vector in the given direction $\mathbf{i}+\mathbf{j}+3\mathbf{k}$

$$=\frac{\mathbf{i}+\mathbf{j}+3\mathbf{k}}{|\mathbf{i}+\mathbf{j}+3\mathbf{k}|}=\frac{\mathbf{i}+\mathbf{j}+3\mathbf{k}}{\sqrt{(11)}}=\mathbf{b}, \text{ say.}$$

$\therefore$ the component of velocity in the given direction

$$=\mathbf{v}\cdot\mathbf{b}=\frac{(3\mathbf{i}+2\mathbf{j}+2\mathbf{k})\cdot(\mathbf{i}+\mathbf{j}+3\mathbf{k})}{\sqrt{(11)}}=\frac{11}{\sqrt{(11)}}=\sqrt{(11)};$$

and the component of acceleration in the given direction

$$=\mathbf{a}\cdot\mathbf{b}=\frac{(6\mathbf{i}+2\mathbf{j})\cdot(\mathbf{i}+\mathbf{j}+3\mathbf{k})}{\sqrt{(11)}}=\frac{8}{\sqrt{(11)}}.$$

Ex. 31. *A particle moves so that its position vector is given by* $\mathbf{r}=\cos\omega t\,\mathbf{i}+\sin\omega t\,\mathbf{j}$ *where* ω *is a constant; show that (i) the velocity of the particle is perpendicular to* $\mathbf{r}$, *(ii) the acceleration is directed towards the origin and has magnitude proportional to the distance from the origin, (iii)* $\mathbf{r}\times\dfrac{d\mathbf{r}}{dt}$ *is a constant vector.*

Sol. (i) Velocity $\mathbf{v}=\dfrac{d\mathbf{r}}{dt}=-\omega\sin\omega t\,\mathbf{i}+\omega\cos\omega t\,\mathbf{j}$.

We have $\mathbf{r}\cdot\dfrac{d\mathbf{r}}{dt}=(\cos\omega t\,\mathbf{i}+\sin\omega t\,\mathbf{j})\cdot(-\omega\sin\omega t\,\mathbf{i}+\omega\cos\omega t\,\mathbf{j})$

$$=-\omega\cos\omega t\sin\omega t+\omega\sin\omega t\cos\omega t=0.$$

Therefore the velocity is perpendicular to $\mathbf{r}$.

(ii) Acceleration of the particle

$$=\mathbf{a}=\frac{d^2\mathbf{r}}{dt^2}=-\omega^2\cos\omega t\,\mathbf{i}-\omega^2\sin\omega t\,\mathbf{j}$$

$$=-\omega^2(\cos\omega t\,\mathbf{i}+\sin\omega t\,\mathbf{j})=-\omega^2\mathbf{r}.$$

$\therefore$ Acceleration is a vector opposite to the direction of $\mathbf{r}$ *i.e.* acceleration is directed towards the origin. Also magnitude of acceleration $=|\mathbf{a}|=|-\omega^2\mathbf{r}|=\omega^2 r$ which is proportional to r *i.e.*, the distance of the particle from the origin.

(iii) $\mathbf{r}\times\dfrac{d\mathbf{r}}{dt}=(\cos\omega t\,\mathbf{i}+\sin\omega t\,\mathbf{j})\times(-\omega\sin\omega t\,\mathbf{i}+\omega\cos\omega t\,\mathbf{j})$

$$=\omega\cos^2\omega t\,\mathbf{i}\times\mathbf{j}-\omega\sin^2\omega t\,\mathbf{j}\times\mathbf{i}\quad[\because \mathbf{i}\times\mathbf{i}=0,\ \mathbf{j}\times\mathbf{j}=0]$$

$$=\omega\cos^2\omega t\,\mathbf{k}+\omega\sin^2\omega t\,\mathbf{k}\quad[\because \mathbf{i}\times\mathbf{j}=\mathbf{k}=-\mathbf{j}\times\mathbf{i}]$$

$$=\omega(\cos^2\omega t+\sin^2\omega t)\,\mathbf{k}=\omega\mathbf{k}, \text{ a constant vector.}$$

Ex. 32. *Find the unit tangent vector to any point on the curve* $x=a\cos t,\ y=a\sin t,\ z=bt.$

Sol. If $\mathbf{r}$ is the position vector of any point (x, y, z) on the given curve, then

$$\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}=a \cos t\ \mathbf{i}+a \sin t\ \mathbf{j}+bt\ \mathbf{k}.$$

The vector $\dfrac{d\mathbf{r}}{dt}$ is also the tangent at the point (x, y, z) to the given curve.

We have $\dfrac{d\mathbf{r}}{dt}=-a \sin t\ \mathbf{i}+a \cos t\ \mathbf{j}+b\ \mathbf{k}.$

$$\therefore \left|\frac{d\mathbf{r}}{dt}\right|=\sqrt{(a^2 \sin^2 t+a^2 \cos^2 t+b^2)}=\sqrt{(a^2+b^2)}.$$

Hence the unit tangent vector $\mathbf{t}$

$$=\frac{d\mathbf{r}/dt}{|d\mathbf{r}/dt|}=\frac{-a \sin t\ \mathbf{i}+a \cos t\ \mathbf{j}+b\mathbf{k}}{\sqrt{(a^2+b^2)}}$$

$$=\frac{1}{\sqrt{(a^2+b^2)}}(-a \sin t\ \mathbf{i}+a \cos t\ \mathbf{j}+b\mathbf{k}).$$

§ 12. Integration of Vector Functions.

We shall define *integration as the reverse process of differentiation*. Let $\mathbf{f}(t)$ and $\mathbf{F}(t)$ be two vector functions of the scalar t such that $\dfrac{d}{dt}\mathbf{F}(t)=\mathbf{f}(t)$.

Then $\mathbf{F}(t)$ is called the *indefinite integral* of $\mathbf{f}(t)$ with respect to t and symbolically we write $\int \mathbf{f}(t)\, dt=\mathbf{F}(t)$. ...(1)

The function $\mathbf{f}(t)$ to be integrated is called the *integrand*.

If $\mathbf{c}$ is any *arbitrary constant vector* independent of t, then

$$\frac{d}{dt}\left\{\mathbf{F}(t)+\mathbf{c}\right\}=\mathbf{f}(t).$$

This is equivalent to $\int \mathbf{f}(t)\, dt=\mathbf{F}(t)+\mathbf{c}$. ...(2)

From (2) it is obvious that the integral $\mathbf{F}(t)$ of $\mathbf{f}(t)$ is indefinite to the extent of an additive arbitrary constant $\mathbf{c}$. Therefore $\mathbf{F}(t)$ is called the indefinite integral of $\mathbf{f}(t)$. The constant vector $\mathbf{c}$ is called the *constant of integration*. It can be determined if we are given some initial conditions.

If $\dfrac{d}{dt}\mathbf{F}(t)=\mathbf{f}(t)$ for all t in the interval $[a, b]$, then the *definite integral* between the limits $t=a$ and $t=b$ can in such case be written

$$\int_a^b \mathbf{f}(t)\, dt=\int_a^b \left\{\frac{d}{dt}\mathbf{F}(t)\right\} dt$$

$$=\Big[\mathbf{F}(t)+\mathbf{c}\Big]_a^b=\mathbf{F}(b)-\mathbf{F}(a).$$

Theorem. *If* $\mathbf{f}(t)=f_1(t)\,\mathbf{i}+f_2(t)\,\mathbf{j}+f_3(t)\,\mathbf{k}$, *then*

$$\int \mathbf{f}(t)\,dt=\mathbf{i}\int f_1(t)\,dt+\mathbf{j}\int f_2(t)\,dt+\mathbf{k}\int f_3(t)\,dt.$$

Proof. Let $\dfrac{d}{dt}\mathbf{F}(t)=\mathbf{f}(t)$. ...(1)

Then $\int \mathbf{f}(t)\,dt=\mathbf{F}(t)$. ...(2)

Let $\mathbf{F}(t)=F_1(t)\,\mathbf{i}+F_2(t)\,\mathbf{j}+F_3(t)\,\mathbf{k}$.

Then from (1), we have

$$\frac{d}{dt}\{F_1(t)\,\mathbf{i}+F_2(t)\,\mathbf{j}+F_3(t)\,\mathbf{k}\}=\mathbf{f}(t)$$

or

$$\left\{\frac{d}{dt}F_1(t)\right\}\mathbf{i}+\left\{\frac{d}{dt}F_2(t)\right\}\mathbf{j}+\left\{\frac{d}{dt}F_3(t)\right\}\mathbf{k}$$
$$=f_1(t)\,\mathbf{i}+f_2(t)\,\mathbf{j}+f_3(t)\,\mathbf{k}.$$

Equating the coefficients of $\mathbf{i}, \mathbf{j}, \mathbf{k}$, we get

$$\frac{d}{dt}F_1(t)=f_1(t),\ \frac{d}{dt}F_2(t)=f_2(t),\ \frac{d}{dt}F_3(t)=f_3(t).$$

$$\therefore\ F_1(t)=\int f_1(t)\,dt,\ F_2(t)=\int f_2(t)\,dt,\ F_3(t)=\int f_3(t)\,dt.$$

$$\therefore\ \mathbf{F}(t)=\left\{\int f_1(t)\,dt\right\}\mathbf{i}+\left\{\int f_2(t)\,dt\right\}\mathbf{j}+\left\{\int f_3(t)\,dt\right\}\mathbf{k}.$$

So from (2), we get

$$\int \mathbf{f}(t)\,dt=\mathbf{i}\int f_1(t)\,dt+\mathbf{j}\int f_2(t)\,dt+\mathbf{k}\int f_3(t)\,dt.$$

Note. From this theorem we conclude that the definition of the integral of a vector function implies the definition of integrals of three scalar functions which are the components of that vector function. Thus in order to integrate a vector function we should integrate its components.

§ 13. Some Standard Results.

We have already obtained some standard results for differentiation. With the help of these results we can obtain some standard results for integration.

1. We have $\dfrac{d}{dt}(\mathbf{r}\cdot\mathbf{s})=\dfrac{d\mathbf{r}}{dt}\cdot\mathbf{s}+\mathbf{r}\cdot\dfrac{d\mathbf{s}}{dt}$.

Therefore $\displaystyle\int\left(\frac{d\mathbf{r}}{dt}\cdot\mathbf{s}+\mathbf{r}\cdot\frac{d\mathbf{s}}{dt}\right)dt=\mathbf{r}\cdot\mathbf{s}+c,$

where c is the constant of integration. It should be noted that c is here a scalar quantity since the integrand is also scalar.

2. We have $\dfrac{d}{dt}(\mathbf{r}^2)=2\mathbf{r}\cdot\dfrac{d\mathbf{r}}{dt}$.

Therefore $\displaystyle\int\left(2\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}\right)dt=\mathbf{r}^2+c.$

Here the constant of integration c is a scalar quantity.

3. We have $\frac{d}{dt}\left(\frac{d\mathbf{r}}{dt}\right)^2=2\,\frac{d\mathbf{r}}{dt}\cdot\frac{d^2\mathbf{r}}{dt^2}$.

Therefore we have

$$\int\left(2\,\frac{d\mathbf{r}}{dt}\cdot\frac{d^2\mathbf{r}}{dt^2}\right)\,dt=\left(\frac{d\mathbf{r}}{dt}\right)^2+c.$$

Here the constant of integration c is a scalar quantity.

Also $$\left(\frac{d\mathbf{r}}{dt}\right)^2=\frac{d\mathbf{r}}{dt}\cdot\frac{d\mathbf{r}}{dt}.$$

4. We have $\frac{d}{dt}\left(\mathbf{r}\times\frac{d\mathbf{r}}{dt}\right)=\frac{d\mathbf{r}}{dt}\times\frac{d\mathbf{r}}{dt}+\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}=\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}$.

$$\therefore\int\left(\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}\right)dt=\mathbf{r}\times\frac{d\mathbf{r}}{dt}+\mathbf{c}.$$

Here the constant of integration $\mathbf{c}$ is a vector quantity since the integrand $\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}$ is also a vector quantity.

5. If $\mathbf{a}$ is a constant vector, we have

$$\frac{d}{dt}\,(\mathbf{a}\times\mathbf{r})=\frac{d\mathbf{a}}{dt}\times\mathbf{r}+\mathbf{a}\times\frac{d\mathbf{r}}{dt}=\mathbf{a}\times\frac{d\mathbf{r}}{dt}.$$

Therefore $\int\left(\mathbf{a}\times\frac{d\mathbf{r}}{dt}\right)dt=\mathbf{a}\times\mathbf{r}+\mathbf{c}.$

Here the constant of integration $\mathbf{c}$ is a vector quantity.

6. If $r=|\mathbf{r}|$ and $\hat{\mathbf{r}}$ is a unit vector in the direction of $\mathbf{r}$ then

$$\frac{d}{dt}\,(\hat{\mathbf{r}})=\frac{d}{dt}\left(\frac{1}{r}\,\mathbf{r}\right)=\frac{1}{r}\,\frac{d\mathbf{r}}{dt}-\frac{1}{r^2}\,\frac{dr}{dt}\,\mathbf{r}.$$

Therefore $\int\left(\frac{1}{r}\,\frac{d\mathbf{r}}{dt}-\frac{1}{r^2}\,\frac{dr}{dt}\,\mathbf{r}\right)\,dt=\hat{\mathbf{r}}+\mathbf{c}.$

7. If c is a constant scalar and $\mathbf{r}$ a vector function of a scalar t, then obviously $\int c\mathbf{r}\,dt=c\int\mathbf{r}\,dt.$

8. If $\mathbf{r}$ and $\mathbf{s}$ are two vector functions of the scalar t, then obviously $\int(\mathbf{r}+\mathbf{s})\,dt=\int\mathbf{r}\,dt+\int\mathbf{s}\,dt.$

Solved Examples

Ex. 1. *If* $\mathbf{f}(t)=(t-t^2)\,\mathbf{i}+2t^3\,\mathbf{j}-3\mathbf{k}$, *find*

(i) $\int\mathbf{f}(t)\,dt$ *and (ii)* $\int_1^2\mathbf{f}(t)\,dt.$

Sol. (i) $\int\mathbf{f}(t)\,dt=\int\{(t-t^2)\,\mathbf{i}+2t^3\,\mathbf{j}-3\mathbf{k}\}\,dt$

$$=\mathbf{i}\int (t-t^2)\,dt+\mathbf{j}\int 2t^3\,dt+\mathbf{k}\int -3\,dt$$

$$=\mathbf{i}\left(\frac{t^2}{2}-\frac{t^3}{3}\right)+\mathbf{j}\left(2\cdot\frac{t^4}{4}\right)+\mathbf{k}\,(-3t)+\mathbf{c},$$

where $\mathbf{c}$ is an arbitrary constant vector

$$=\left(\frac{t^2}{2}-\frac{t^3}{3}\right)\mathbf{i}+\frac{t^4}{2}\mathbf{j}-3t\mathbf{k}+\mathbf{c}.$$

(ii) $\int_1^2 \mathbf{f}(t)\,dt=\int_1^2 \{(t-t^2)\,\mathbf{i}+2t^3\,\mathbf{j}-3\mathbf{k}\}\,dt$

$$=\mathbf{i}\int_1^2 (t-t^2)\,dt+\mathbf{j}\int_1^2 2t^3\,dt-\mathbf{k}\int_1^2 3dt$$

$$=\mathbf{i}\left[\frac{t^2}{2}-\frac{t^3}{3}\right]_1^2+\mathbf{j}\left[2\cdot\frac{t^4}{4}\right]_1^2-3\mathbf{k}\left[\,t\,\right]_1^2=-\frac{5}{6}\,\mathbf{i}+\frac{15}{2}\,\mathbf{j}-3\mathbf{k}.$$

Ex. 2. *Evaluate* $\int_0^1 (e^t\,\mathbf{i}+e^{-2t}\,\mathbf{j}+t\mathbf{k})\,dt.$

Sol. $\int_0^1 (e^t\,\mathbf{i}+e^{-2t}\,\mathbf{j}+t\mathbf{k})\,dt$

$$=\mathbf{i}\int_0^1 e^t\,dt+\mathbf{j}\int_0^1 e^{-2t}\,dt+\mathbf{k}\int_0^1 t\,dt$$

$$=\mathbf{i}\left[e^t\right]_0^1+\mathbf{j}\left[-\tfrac{1}{2}\cdot e^{-2t}\right]_0^1+\mathbf{k}\left[\tfrac{1}{2}t^2\right]_0^1$$

$$=(e-1)\,\mathbf{i}-\tfrac{1}{2}\,(e^{-2}-1)\,\mathbf{j}+\tfrac{1}{2}\mathbf{k}.$$

Ex. 3. *If* $\mathbf{f}(t)=t\,\mathbf{i}+(t^2-2t)\,\mathbf{j}+(3t^2+3t^3)\,\mathbf{k}$, *find*

$$\int_0^1 \mathbf{f}(t)\,dt.$$

[Agra 1977]

Sol. $\int_0^1 \mathbf{f}(t)\,dt=\int_0^1 [t\,\mathbf{i}+(t^2-2t)\,\mathbf{j}+(3t^2+3t^3)\,\mathbf{k}]\,dt$

$$=\mathbf{i}\int_0^1 t\,dt+\mathbf{j}\int_0^1 (t^2-2t)\,dt+\mathbf{k}\int_0^1 (3t^2+3t^3)\,dt$$

$$=\mathbf{i}\left[\tfrac{1}{2}t^2\right]_0^1+\mathbf{j}\left[\frac{t^3}{3}-t^2\right]_0^1+\mathbf{k}\left[t^3+\frac{3t^4}{4}\right]_0^1$$

$$=\tfrac{1}{2}\,\mathbf{i}+(\tfrac{1}{3}-1)\,\mathbf{j}+(1+\tfrac{3}{4})\,\mathbf{k}=\tfrac{1}{2}\,\mathbf{i}-\tfrac{2}{3}\,\mathbf{j}+\tfrac{7}{4}\mathbf{k}.$$

Ex. 4. *If* $\mathbf{r}=t\,\mathbf{i}-t^2\,\mathbf{j}+(t-1)\,\mathbf{k}$ *and* $\mathbf{s}=2t^2\,\mathbf{i}+6t\,\mathbf{k}$, *evaluate*

(*i*) $\int_0^2 \mathbf{r}\cdot\mathbf{s}\,dt,$ (*ii*) $\int_0^2 \mathbf{r}\times\mathbf{s}\,dt$

[Meerut 1992]

Sol. (i) We have $\mathbf{r}\cdot\mathbf{s}=[t\,\mathbf{i}-t^2\,\mathbf{j}+(t-1)\,\mathbf{k}]\cdot(2t^2\,\mathbf{i}+6t\,\mathbf{k})$

$$=2t^3+6t\,(t-1)=2t^3+6t^2-6t.$$

$$\therefore \int_0^2 \mathbf{r}\cdot\mathbf{s}\,dt=\int_0^2 (2t^3+6t^2-6t)\,dt$$

$$=\left[\frac{t^4}{2}+2t^3-3t^2\right]_0^2=8+16-12=12.$$

(ii) We have $\mathbf{r}\times\mathbf{s}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & -t^2 & t-1 \\ 2t^2 & 0 & 6t \end{vmatrix}$

$$=-6t^3\,\mathbf{i}-[6t^2-2t^2(t-1)]\,\mathbf{j}+2t^4\,\mathbf{k}$$

$$=-6t^3\,\mathbf{i}-(8t^2-2t^3)\,\mathbf{j}+2t^4\,\mathbf{k}.$$

$$\therefore \int_0^2 \mathbf{r}\times\mathbf{s}\,dt=\int_0^2 [-6t^3\,\mathbf{i}-(8t^2-2t^3)\,\mathbf{j}+2t^4\,\mathbf{k}]\,dt$$

$$=\mathbf{i}\int_0^2 -6t^3\,dt-\mathbf{j}\int_0^2 (8t^2-2t^3)\,dt+\mathbf{k}\int_0^2 2t^4\,dt$$

$$=\mathbf{i}\left[-\frac{3}{2}t^4\right]_0^2-\mathbf{j}\left[\frac{8t^3}{3}-\frac{t^4}{2}\right]_0^2+\mathbf{k}\left[\frac{2t^5}{5}\right]_0^2$$

$$=-24\,\mathbf{i}-\left(\frac{64}{3}-8\right)\mathbf{j}+\frac{64}{5}\mathbf{k}$$

$$=-24\,\mathbf{i}-\frac{40}{3}\mathbf{j}+\frac{64}{5}\mathbf{k}.$$

Ex. 5. *Evaluate* $\int_1^2 (\mathbf{a}\cdot\mathbf{b}\times\mathbf{c})\,dt$, *where*

$\mathbf{a}=t\,\mathbf{i}-3\mathbf{j}+2t\,\mathbf{k}$, $\mathbf{b}=\mathbf{i}-2\,\mathbf{j}+2\mathbf{k}$, $\mathbf{c}=3\,\mathbf{i}+t\,\mathbf{j}-\mathbf{k}$.

[Garhwal 1977]

Sol. We have $\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}=[\mathbf{a}\ \mathbf{b}\ \mathbf{c}]$

$$=\begin{vmatrix} t & -3 & 2t \\ 1 & -2 & 2 \\ 3 & t & -1 \end{vmatrix}$$

$$=\begin{vmatrix} t & -3+2t & 0 \\ 1 & 0 & 0 \\ 3 & t+6 & -7 \end{vmatrix}, \text{ by } C_2+2C_1 \text{ and } C_3-2C_1$$

$$=-1.\{-7(2t-3)-0\},$$

expanding the determinant along R_2

$$=7(2t-3).$$

$$\therefore \int_1^2 (\mathbf{a} \cdot \mathbf{b} \times \mathbf{c})\, dt = \int_1^2 7(2t-3)\, dt = 7 \int_1^2 (2t-3)\, dt$$

$$= 7\left[t^2 - 3t\right]_1^2 = 7\{(4-6)-(1-3)\} = 7.(-2+2) = 0.$$

Ex. 6. *Evaluate* $\int_1^2 \mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} dt$, *where* $\mathbf{r} = 2t^2\, \mathbf{i} + t\, \mathbf{j} - 3t^3\, \mathbf{k}$.

[Kanpur 1975]

Sol. Given $\mathbf{r} = 2t^2\, \mathbf{i} + t\, \mathbf{j} - 3t^3\, \mathbf{k}$.

$$\therefore \frac{d\mathbf{r}}{dt} = 4t\, \mathbf{i} + \mathbf{j} - 9t^2\, \mathbf{k} \text{ and } \frac{d^2\mathbf{r}}{dt^2} = 4\, \mathbf{i} + 0\, \mathbf{j} - 18\, t\, \mathbf{k}.$$

$$\therefore \mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} = (2t^2\, \mathbf{i} + t\, \mathbf{j} - 3t^3\, \mathbf{k}) \times (4\, \mathbf{i} + 0\, \mathbf{j} - 18\, t\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2t^2 & t & -3t^3 \\ 4 & 0 & -18t \end{vmatrix}$$

$$= -18t^2\, \mathbf{i} - (-36\, t^3 + 12\, t^3)\, \mathbf{j} - 4t\, \mathbf{k}$$

$$= -18t^2\, \mathbf{i} + 24\, t^3\, \mathbf{j} - 4t\, \mathbf{k}.$$

$$\therefore \int_1^2 \mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} dt = \int_1^2 (-18\, t^2\, \mathbf{i} + 24\, t^3\, \mathbf{j} - 4t\, \mathbf{k})\, dt$$

$$= -18\, \mathbf{i} \int_1^2 t^2\, dt + 24\, \mathbf{j} \int_1^2 t^3\, dt - 4\, \mathbf{k} \int_1^2 t\, dt$$

$$= -18\, \mathbf{i}\left[\frac{t^3}{3}\right]_1^2 + 24\, \mathbf{j}\left[\frac{t^4}{4}\right]_1^2 - 4\, \mathbf{k}\left[\frac{t^2}{2}\right]_1^2$$

$$= -6\,(8-1)\, \mathbf{i} + 6(16-1)\, \mathbf{j} - 2\,(4-1)\, \mathbf{k}$$

$$= -42\, \mathbf{i} + 90\, \mathbf{j} - 6\, \mathbf{k}.$$

Ex. 7. *Find the value of* $\mathbf{r}$ *satisfying the equation* $\frac{d^2\mathbf{r}}{dt^2} = \mathbf{a}$, *where* $\mathbf{a}$ *is a constant vector. Also it is given that when* $t = 0$, $\mathbf{r} = \mathbf{0}$ *and* $\frac{d\mathbf{r}}{dt} = \mathbf{u}$.

[Agra 1981]

Sol. Integrating the equation $\frac{d^2\mathbf{r}}{dt^2} = \mathbf{a}$, we get

$\frac{d\mathbf{r}}{dt} = t\mathbf{a} + \mathbf{b}$, where $\mathbf{b}$ is an arbitrary constant vector.

But it is given that when $t = 0$, $\frac{d\mathbf{r}}{dt} = \mathbf{u}$.

$$\therefore \quad \mathbf{u}=0\mathbf{a}+\mathbf{b} \quad \text{or} \quad \mathbf{b}=\mathbf{a}.$$

$$\therefore \quad \frac{d\mathbf{r}}{dt}=t\mathbf{a}+\mathbf{u}.$$

Integrating again with respect to t, we get

$$\mathbf{r}=\tfrac{1}{2}t^2\,\mathbf{a}+t\mathbf{u}+\mathbf{c}, \text{ where } \mathbf{c} \text{ is constant.}$$

But when $t=0$, $\mathbf{r}=0$.

$$\therefore \quad 0=0+0+\mathbf{c} \quad \text{or} \quad \mathbf{c}=0.$$

$$\therefore \quad \mathbf{r}=\tfrac{1}{2}t^2\,\mathbf{a}+t\mathbf{u}.$$

Ex. 8. *Solve the equation* $\dfrac{d^2\mathbf{r}}{dt^2}=\mathbf{a}$ *where* $\mathbf{a}$ *is a constant vector; given that* $\mathbf{r}=0$ *and* $\dfrac{d\mathbf{r}}{dt}=0$ *when* $t=0$.

Sol. Proceed as in Ex. 7. Ans. $\mathbf{r}=\frac{1}{2}t^2\mathbf{a}$.

Ex. 9. *Find the value of* $\mathbf{r}$ *satisfying the equation* $\dfrac{d^2\mathbf{r}}{dt^2}=t\mathbf{a}+\mathbf{b}$, *where* $\mathbf{a}$ *and* $\mathbf{b}$ *are constant vectors.* [Agra 1979; Rohilkhand 83]

Sol. Integrating the equation $\dfrac{d^2\mathbf{r}}{dt^2}=t\mathbf{a}+\mathbf{b}$, we get

$$\frac{d\mathbf{r}}{dt}=\tfrac{1}{2}t^2\,\mathbf{a}+t\mathbf{b}+\mathbf{c}, \text{ where } \mathbf{c} \text{ is constant.}$$

Again integrating, we get

$$\mathbf{r}=\tfrac{1}{6}t^3\,\mathbf{a}+\tfrac{1}{2}t^2\,\mathbf{b}+t\,\mathbf{c}+\mathbf{d}, \text{ where } \mathbf{d} \text{ is constant.}$$

Ex. 10. *Integrate* $\dfrac{d^2\mathbf{r}}{dt^2}=-n^2\mathbf{r}$.

Sol. We have $\dfrac{d^2\mathbf{r}}{dt^2}=-n^2\mathbf{r}$. ...(1)

Forming the scalar product of each side of (1) with the vector $2\dfrac{d\mathbf{r}}{dt}$, we get $2\,\dfrac{d\mathbf{r}}{dt}\cdot\dfrac{d^2\mathbf{r}}{dt^2}=-2n^2\mathbf{r}\cdot\dfrac{d\mathbf{r}}{dt}$.

Now integrating we get

$$\left(\frac{d\mathbf{r}}{dt}\right)^2=-n^2\mathbf{r}^2+c, \text{ where } c \text{ is constant.}$$

Ex. 11. *If* $\mathbf{r}\cdot d\mathbf{r}=0$, *show that* $|\,\mathbf{r}\,|=$ *constant.*

[Agra 1975; Rohilkhand 79]

Sol. We have $\mathbf{r}\cdot d\mathbf{r}=0 \Rightarrow 2\mathbf{r}\cdot d\mathbf{r}=0 \Rightarrow d(\mathbf{r}\cdot\mathbf{r})=0$

$\Rightarrow d(\mathbf{r}^2)=0 \Rightarrow \mathbf{r}^2=\text{constant} \Rightarrow |\mathbf{r}|^2=\text{constant}$

$\Rightarrow |\mathbf{r}|=\text{constant}.$

Ex. 12. *Find the value of* $\mathbf{r}$ *satisfying the equation*

$$\frac{d^2\mathbf{r}}{dt^2}=6t\mathbf{i}-24t^2\mathbf{j}+4\sin t\mathbf{k},$$

given that $\mathbf{r}=2\mathbf{i}+\mathbf{j}$ *and* $d\mathbf{r}/dt=-\mathbf{i}-3\mathbf{k}$ *at* $t=0$.

Sol. Integrating the equation $\dfrac{d^2\mathbf{r}}{dt^2}=6t\mathbf{i}-24t^2\mathbf{j}+4\sin t\mathbf{k}$, we get

$\dfrac{d\mathbf{r}}{dt}=3t^2\mathbf{i}-8t^3\mathbf{j}-4\cos t\mathbf{k}+\mathbf{b}$, where $\mathbf{b}$ is an arbitrary constant vector.

But it is given that when $t=0$, $d\mathbf{r}/dt=-\mathbf{i}-3\mathbf{k}$.

$\therefore\quad -\mathbf{i}-3\mathbf{k}=-4\mathbf{k}+\mathbf{b}$ or $\mathbf{b}=-\mathbf{i}+\mathbf{k}$.

$$\therefore\quad \frac{d\mathbf{r}}{dt}=3t^2\mathbf{i}-8t^3\mathbf{j}-4\cos t\mathbf{k}-\mathbf{i}+\mathbf{k}$$

$$=(3t^2-1)\,\mathbf{i}-8t^3\mathbf{j}+(1-4\cos t)\,\mathbf{k}.$$

Integrating again w.r.t. t, we get

$$\mathbf{r}=(t^3-t)\,\mathbf{i}-2t^4\mathbf{j}+(t-4\sin t)\,\mathbf{k}+\mathbf{c},$$

where $\mathbf{c}$ is an arbitrary constant vector.

But it is given that when $t=0$, $\mathbf{r}=2\mathbf{i}+\mathbf{j}$.

$\therefore\quad 2\mathbf{i}+\mathbf{j}=0+\mathbf{c}=\mathbf{c}.$

$\therefore\quad \mathbf{r}=(t^3-t)\,\mathbf{i}-2t^4\mathbf{j}+(t-4\sin t)\,\mathbf{k}+2\mathbf{i}+\mathbf{j}.$

or $\quad \mathbf{r}=(t^3-t+2)\,\mathbf{i}+(1-2t^4)\,\mathbf{j}+(t-4\sin t)\,\mathbf{k}$

is the required solution of the given differential equation.

Ex. 13. *Show that* $\displaystyle\int\left(\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}\right)dt=\mathbf{r}\times\frac{d\mathbf{r}}{dt}+\mathbf{c}$, *where* $\mathbf{c}$ *is an arbitrary constant vector.*

Sol. We have $\dfrac{d}{dt}\left(\mathbf{r}\times\dfrac{d\mathbf{r}}{dt}\right)=\mathbf{r}\times\dfrac{d^2\mathbf{r}}{dt^2}+\dfrac{d\mathbf{r}}{dt}\times\dfrac{d\mathbf{r}}{dt}$

$$=\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2},\ \text{since}\ \frac{d\mathbf{r}}{dt}\times\frac{d\mathbf{r}}{dt}=\mathbf{0}.$$

Integrating both sides with respect to t, we get

$\displaystyle\int\left(\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}\right)dt=\left(\mathbf{r}\times\frac{d\mathbf{r}}{dt}\right)+\mathbf{c}$, where $\mathbf{c}$ is an arbitrary constant vector.

Ex. 14. *Integrate* $\mathbf{a}\times\dfrac{d^2\mathbf{r}}{dt^2}=\mathbf{b}$, *where* $\mathbf{a}$ *and* $\mathbf{b}$ *are constant vectors.*

Sol. We have $\dfrac{d}{dt}\left\{\mathbf{a}\times\dfrac{d\mathbf{r}}{dt}\right\}=\mathbf{a}\times\dfrac{d^2\mathbf{r}}{dt^2}$.

Therefore integrating $\mathbf{a}\times\dfrac{d^2\mathbf{r}}{dt^2}=\mathbf{b}$, we get

$$\mathbf{a}\times\frac{d\mathbf{r}}{dt}=t\mathbf{b}+\mathbf{c}, \text{ where } \mathbf{c} \text{ is constant.}$$

Again integrating, we get

$$\mathbf{a}\times\mathbf{r}=\tfrac{1}{2}t^2\mathbf{b}+t\mathbf{c}+\mathbf{d}, \text{ where } \mathbf{d} \text{ is constant.}$$

Ex. 15. *If* $\mathbf{r}(t)=5t^2\,\mathbf{i}+t\,\mathbf{j}-t^3\,\mathbf{k}$, *prove that*

$$\int_1^2\left(\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}\right)dt=-14\mathbf{i}+75\mathbf{j}-15\mathbf{k}.$$

[Meerut 1991, Kanpur 87; Agra 82, 86; Rohilkhand 85]

Sol. We have $\displaystyle\int\left(\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}\right)dt=\mathbf{r}\times\frac{d\mathbf{r}}{dt}+\mathbf{c}$.

$$\therefore\quad \int_1^2\left(\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}\right)dt=\left[\mathbf{r}\times\frac{d\mathbf{r}}{dt}\right]_1^2.$$

Let us now find $\mathbf{r}\times\dfrac{d\mathbf{r}}{dt}$. We have $\dfrac{d\mathbf{r}}{dt}=10t\mathbf{i}+\mathbf{j}-3t^2\mathbf{k}$.

$$\therefore\quad \mathbf{r}\times\frac{d\mathbf{r}}{dt}=(5t^2\,\mathbf{i}+t\,\mathbf{j}-t^3\,\mathbf{k})\times(10t\,\mathbf{i}+\mathbf{j}-3t^2\,\mathbf{k})$$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 5t^2 & t & -t^3\\ 10t & 1 & -3t^2\end{vmatrix}=-2t^3\,\mathbf{i}+5t^4\,\mathbf{j}-5t^2\,\mathbf{k}.$$

$$\therefore\quad \int_1^2\left(\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}\right)dt=\left[-2t^3\,\mathbf{i}+5t^4\,\mathbf{j}-5t^2\,\mathbf{k}\right]_1^2$$

$$=\left[-2t^3\right]_1^2\mathbf{i}+\left[5t^4\right]_1^2\mathbf{j}-\left[5t^2\right]_1^2\mathbf{k}=-14\mathbf{i}+75\mathbf{j}-15\mathbf{k}.$$

Ex. 16. *Given that*

$$\mathbf{r}(t)=2\mathbf{i}-\mathbf{j}+2\mathbf{k}, \textit{ when } t=2$$
$$=4\mathbf{i}-2\mathbf{j}+3\mathbf{k}, \textit{ when } t=3,$$

show that $\displaystyle\int_2^3\left(\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}\right)dt=10.$

[Kanpur 1986; Rohilkhand 84; Agra 83, 87]

Sol. We have $\displaystyle\int\left(\mathbf{r}\cdot\frac{d\mathbf{r}}{dt}\right)dt=\tfrac{1}{2}\mathbf{r}^2+c.$

$$\therefore \int_2^3 \left(\mathbf{r} \cdot \frac{d\mathbf{r}}{dt} \right) dt = \left[\tfrac{1}{2} \mathbf{r}^2 \right]_2^3.$$

When $t=3$, $\mathbf{r}=4\mathbf{i}-2\mathbf{j}+3\mathbf{k}$.

$\therefore$ when $t=3$, $\mathbf{r}^2=(4\mathbf{i}-2\mathbf{j}+3\mathbf{k})\cdot(4\mathbf{i}-2\mathbf{j}+3\mathbf{k})=16+4+9=29$.

When $t=2$, $\mathbf{r}=2\mathbf{i}-\mathbf{j}+2\mathbf{k}$.

$\therefore$ When $t=2$, $\mathbf{r}^2=4+1+4=9$.

$$\therefore \int_2^3 \left(\mathbf{r} \cdot \frac{d\mathbf{r}}{dt} \right) dt = \tfrac{1}{2}\,[29-9]=10.$$

Ex. 17. *The acceleration of a particle at any time $t \geqslant 0$ is given by*

$$\mathbf{a}=\frac{d\mathbf{v}}{dt}=12 \cos 2t\ \mathbf{i}-8 \sin 2t\ \mathbf{j}+16t\ \mathbf{k}.$$

If the velocity $\mathbf{v}$ and displacement $\mathbf{r}$ are zero at $t=0$, find $\mathbf{v}$ and $\mathbf{r}$ at any time.

[Meerut 1991 P; Kerala 74]

Sol. We have $\dfrac{d\mathbf{v}}{dt}=12 \cos 2t\ \mathbf{i}-8 \sin 2t\ \mathbf{j}+16t\ \mathbf{k}$.

Integrating, we get

$$\mathbf{v}=\mathbf{i}\int 12 \cos 2t\ dt+\mathbf{j}\int -8 \sin 2t\ dt+\mathbf{k}\int 16t\ dt$$

or $\mathbf{v}=6 \sin 2t\ \mathbf{i}+4 \cos 2t\ \mathbf{j}+8t^2\ \mathbf{k}+\mathbf{c}$.

When $t=0$, $\mathbf{v}=\mathbf{0}$.

$$\therefore \quad \mathbf{0}=0\mathbf{i}+4\mathbf{j}+0\mathbf{k}+\mathbf{c}$$

or
$$\mathbf{c}=-4\mathbf{j}.$$

$$\therefore \quad \mathbf{v}=\frac{d\mathbf{r}}{dt}=6 \sin 2t\ \mathbf{i}+(4 \cos 2t-4)\ \mathbf{j}+8t^2\mathbf{k}.$$

Integrating, we get

$$\mathbf{r}=\mathbf{i}\int 6 \sin 2t\ dt+\mathbf{j}\int (4 \cos 2t-4)\ dt+\mathbf{k}\int 8t^2\ dt$$

$=-3 \cos 2t\ \mathbf{i}+(2 \sin 2t-4t)\ \mathbf{j}+\frac{8}{3}t^3\ \mathbf{k}+\mathbf{d}$, where $\mathbf{d}$ is constant.

When $t=0$, $\mathbf{r}=\mathbf{0}$.

$\therefore \quad \mathbf{0}=-3\mathbf{i}+0\mathbf{j}+0\mathbf{k}+\mathbf{d}$. $\qquad \therefore \quad \mathbf{d}=3\mathbf{i}$.

$\therefore \quad \mathbf{r}=-3 \cos 2t\ \mathbf{i}+(2 \sin 2t-4t)\ \mathbf{j}+\frac{8}{3}t^3\ \mathbf{k}+3\mathbf{i}$

$\qquad =(3-3 \cos 2t)\ \mathbf{i}+(2 \sin 2t-4t)\ \mathbf{j}+\frac{8}{3}t^3\mathbf{k}$.

Ex. 18. *The acceleration of a particle at any time t is $e^t\ \mathbf{i}+e^{2t}\ \mathbf{j}+\mathbf{k}$. Find $\mathbf{v}$, given that $\mathbf{v}=\mathbf{i}+\mathbf{j}$ at $t=0$.*

[Meerut 1991 S; Kanpur 88]

Sol. Given $\frac{d\mathbf{v}}{dt}=e^t \mathbf{i}+e^{2t}\mathbf{j}+\mathbf{k}$, where $\mathbf{v}$ is the velocity vector of the particle at any time t. Integrating with respect to t, we get

$$\mathbf{v}=e^t \mathbf{i}+\tfrac{1}{2}e^{2t}\mathbf{j}+t\mathbf{k}+\mathbf{c},$$

where $\mathbf{c}$ is an arbitrary constant vector.

But at $t=0$, it is given that $\mathbf{v}=\mathbf{i}+\mathbf{j}$.

$\therefore \quad \mathbf{i}+\mathbf{j}=\mathbf{i}+\tfrac{1}{2}\mathbf{j}+\mathbf{c}$ or $\mathbf{c}=\tfrac{1}{2}\mathbf{j}$

$\therefore \quad \mathbf{v}=e^t \mathbf{i}+\tfrac{1}{2}e^{2t}\mathbf{j}+t\mathbf{k}+\tfrac{1}{2}\mathbf{j}$

$\quad\quad =e^t \mathbf{i}+\tfrac{1}{2}(e^{2t}+1)\mathbf{j}+t\mathbf{k}.$

1rd. conti

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§ 5. Level Surfaces.

Let $f(x, y, z)$ be a scalar field over a region R. The points satisfying an equation of the type

$$f(x, y, z)=c, \text{ (arbitrary constant)}$$

constitute a family of surfaces in three dimensional space. The surfaces of this family are called *level surfaces*. Any surface of this family is such that the value of the function f at any point of it is the same. Therefore these surfaces are also called *iso-f-surfaces*.

Theorem 1. *Let $f(x, y, z)$ be a scalar field over a region R. Then through any point of R there passes one and only one level surface.*

Proof. Let (x_1, y_1, z_1) be any point of the region R. Then the level surface $f(x, y, z)=f(x_1, y_1, z_1)$ passes through this point.

Now suppose the level surfaces $f(x, y, z)=c_1$ and $f(x, y, z)=c_2$ pass through the point (x_1, y_1, z_1). Then

$$f(x_1, y_1, z_1)=c_1 \text{ and } f(x_1, y_1, z_1)=c_2.$$

Since $f(x, y, z)$ has a unique value at (x_1, y_1, z_1) therefore we have $\qquad c_1=c_2.$

Hence only one level surface passes through the point

$$(x_1, y_1, z_1).$$

Theorem 2. *∇f is a vector normal to the surface $f(x, y, z)=c$ where c is a constant.* [Agra 1968; Kerala 75]

Proof. Let $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ be the position vector of any point $P(x, y, z)$ on the level surface $f(x, y, z)=c$. Let

$$Q(x+\delta x, y+\delta y, z+\delta z)$$

be a neighbouring point on this surface. Then the position vector of $Q=\mathbf{r}+\delta\mathbf{r}=(x+\delta x)\,\mathbf{i}+(y+\delta y)\,\mathbf{j}+(z+\delta z)\,\mathbf{k}$.

$$\therefore \overrightarrow{PQ}=(\mathbf{r}+\delta\mathbf{r})-\mathbf{r}=\delta\mathbf{r}=\delta x\,\mathbf{i}+\delta y\,\mathbf{j}+\delta z\,\mathbf{k}.$$

As $Q\to P$, the line PQ tends to tangent at P to the level surface. Therefore $d\mathbf{r}=dx\mathbf{i}+dy\mathbf{j}+dz\mathbf{k}$ lies in the tangent plane to the surface at P.

From the differential calculus, we have

$$df=\frac{\partial f}{\partial x}\,dx+\frac{\partial f}{\partial y}\,dy+\frac{\partial f}{\partial z}\,dz$$

$$=\left(\mathbf{i}\frac{\partial f}{\partial x}+\mathbf{j}\frac{\partial f}{\partial y}+\mathbf{k}\frac{\partial f}{\partial z}\right)\cdot(dx\mathbf{i}+dy\mathbf{j}+dz\mathbf{k})=\nabla f\cdot d\mathbf{r}.$$

Since $f(x, y, z)=$ constant, therefore $df=0$.

$\therefore$ $\nabla f\cdot d\mathbf{r}=0$ so that ∇f is a vector perpendicular to $d\mathbf{r}$ and therefore to the tangent plane at P to the surface

$$f(x, y, z)=c.$$

Hence ∇f is a vector normal to the surface $f(x, y, z)=c$.

Thus if $f(x, y, z)$ is a scalar field defined over a region R, then ∇f at any point (x, y, z) is a vector in the direction of normal at that point to the level surface $f(x, y, z)=c$ passing through that point.

§ 6. Directional Derivative of a scalar point function.

[Agra 1972; Kolhapur 73; Bombay 70]

Definition. *Let $f(x, y, z)$ define a scalar field in a region R and let P be any point in this region. Suppose Q is a point in this region in the neighbourhood of P in the direction of a given unit vector $\hat{\mathbf{a}}$.*

Then $\lim_{Q\to P}\dfrac{f(Q)-f(P)}{PQ}$, *if it exists, is called the directional derivative of f at P in the direction of $\hat{\mathbf{a}}$.*

Interpretation of directional derivative. Let P be the point (x, y, z) and let Q be the point $(x+\delta x, y+\delta y, z+\delta z)$. Suppose $PQ=\delta s$. Then δs is a small element at P in the direction of $\hat{\mathbf{a}}$. If $\delta f=f(x+\delta x, y+\delta y, z+\delta z)-f(x, y, z)=f(Q)-f(P)$, then $\dfrac{\delta f}{\delta s}$ represents the average rate of change of f per unit distance in the direction of $\mathbf{a}$. Now the directional derivative of f at P in the direction of $\hat{\mathbf{a}}$ is $\lim_{Q\to P}\dfrac{f(Q)-f(P)}{PQ}=\lim_{\delta s\to 0}\dfrac{\delta f}{\delta s}=\dfrac{df}{ds}$. It represents the rate of change of f with respect to distance at point P in the direction of unit vector $\hat{\mathbf{a}}$.

Theorem 1. *The directional derivative of a scalar field f at a point $P(x, y, z)$ in the direction of a unit vector $\hat{\mathbf{a}}$ is given by*

$$\frac{df}{ds}=\nabla f\cdot\hat{\mathbf{a}}.$$

[Allahabad 1982; Poona 70]

Proof. Let $f(x, y, z)$ define a scalar field in the region R. Let $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ denote the position vector of any point $P(x, y, z)$ in this region. If s denotes the distance of P from some fixed point A in the direction of $\hat{\mathbf{a}}$, then δs denotes small element at P in the direction of $\hat{\mathbf{a}}$. Therefore $\frac{d\mathbf{r}}{ds}$ is a unit vector at P in this direction *i.e.* $\frac{d\mathbf{r}}{ds}=\hat{\mathbf{a}}$.

But $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$. $\therefore$ $\frac{d\mathbf{r}}{ds}=\frac{dx}{ds}\mathbf{i}+\frac{dy}{ds}\mathbf{j}+\frac{dz}{ds}\mathbf{k}=\hat{\mathbf{a}}$.

$$\text{Now } \nabla f\cdot\hat{\mathbf{a}}=\left(\mathbf{i}\frac{\partial f}{\partial x}+\mathbf{j}\frac{\partial f}{\partial y}+\mathbf{k}\frac{\partial f}{\partial z}\right)\cdot\left(\frac{dx}{ds}\mathbf{i}+\frac{dy}{ds}\mathbf{j}+\frac{dz}{ds}\mathbf{k}\right)$$

$$=\frac{\partial f}{\partial x}\frac{dx}{ds}+\frac{\partial f}{\partial y}\frac{dy}{ds}+\frac{\partial f}{\partial z}\frac{dz}{ds}$$

$=\frac{df}{ds}=$ directional derivative of f at P in the direction of $\hat{\mathbf{a}}$.

Alternative Proof. Let Q be a point in the neighbourhood of P in the direction of the given unit vector $\hat{\mathbf{a}}$. If l, m, n are the direction cosines of the line PQ, then $l\mathbf{i}+m\mathbf{j}+n\mathbf{k}=$ the unit vector in the direction of $PQ=\hat{\mathbf{a}}$. Further if $PQ=\delta s$, then the co-ordinates of Q are $(x+l\delta s, y+m\delta s, z+n\delta s)$. Now the directional derivative of f at P in the direction of $\hat{\mathbf{a}}$ is

$$=\lim_{Q\to P}\frac{f(Q)-f(P)}{PQ}$$

$$=\lim_{\delta s\to 0}\frac{f(x+l\delta s, y+m\delta s, z+n\delta s)-f(x, y, z)}{\delta s}$$

$$\lim_{\delta s\to 0}\frac{f(x, y, z)+\left(l\delta s\frac{\partial f}{\partial x}+m\delta s\frac{\partial f}{\partial y}+n\delta s\frac{\partial f}{\partial z}\right)+\ldots-f(x, y, z)}{\delta s},$$

on expanding by Taylor's theorem

$$=l\frac{\partial f}{\partial x}+m\frac{\partial f}{\partial y}+n\frac{\partial f}{\partial z}$$

$$=\left(\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}\right)\cdot(l\mathbf{i}+m\mathbf{j}+n\mathbf{k})=\nabla f\cdot\hat{\mathbf{a}}.$$

Theorem 2. *If* $\hat{\mathbf{n}}$ *be a unit vector normal to the level surface*

$f(x, y, z)=c$ at a point $P(x, y, z)$ and n be the distance of P from some fixed point A in the direction of $\hat{\mathbf{n}}$ so that δn represents element of normal at P in the direction of $\hat{\mathbf{n}}$, then

$$\text{grad } f=\frac{df}{dn}\,\hat{\mathbf{n}}.$$

[Agra 1971]

Proof. We have $\text{grad } f=\nabla f=\dfrac{\partial f}{\partial x}\mathbf{i}+\dfrac{\partial f}{\partial y}\mathbf{j}+\dfrac{\partial f}{\partial z}\mathbf{k}$.

Also grad f is a vector normal to the surface $f(x, y, z)=c$. Since $\hat{\mathbf{n}}$ is a unit vector normal to the surface $f(x, y, z)=c$, therefore let grad $f=A\,\hat{\mathbf{n}}$, where A is some scalar to be determined.

Now $\dfrac{df}{dn}=$ directional derivative of f in the direction of $\hat{\mathbf{n}}$

$$=\nabla f\cdot\hat{\mathbf{n}}$$

$$=A\hat{\mathbf{n}}\cdot\hat{\mathbf{n}} \quad [\because \nabla f=\text{grad } f=A\hat{\mathbf{n}}]$$

$$=A.$$

$$\therefore \text{grad } f=\nabla f=\frac{df}{dn}\,\hat{\mathbf{n}}.$$

Note. If the vector $\hat{\mathbf{n}}$ is in the direction of f increasing, then $\dfrac{df}{dn}$ is positive. Therefore ∇f is a vector normal to the surface $f(x, y, z)=c$ in the direction of f increasing.

Theorem 3. *Grad f is a vector in the direction of which the maximum value of the directional derivative of f i.e. $\dfrac{df}{ds}$ occurs.*

[Agra 1971]

Proof. The directional derivative of f in the direction of $\hat{\mathbf{a}}$ is given by $\dfrac{df}{ds}=\nabla f\cdot\hat{\mathbf{a}}$

$$=\left(\frac{df}{dn}\hat{\mathbf{n}}\right)\cdot\hat{\mathbf{a}} \qquad \left[\because \nabla f=\frac{df}{dn}\hat{\mathbf{n}}\right]$$

$$=\frac{df}{dn}(\hat{\mathbf{n}}\cdot\hat{\mathbf{a}})$$

$$=\frac{df}{dn}\cos\theta, \text{ where } \theta \text{ is the angle between } \hat{\mathbf{a}} \text{ and } \hat{\mathbf{n}}.$$

Now $\frac{df}{dn}$ is fixed. Therefore $\frac{df}{dn}\cos\theta$ is maximum when $\cos\theta$ is maximum *i.e.*, when $\cos\theta=1$. But $\cos\theta$ will be 1 when the angle between $\hat{\mathbf{a}}$ and $\hat{\mathbf{n}}$ is 0 *i.e.* when $\hat{\mathbf{a}}$ is along the unit normal vector $\hat{\mathbf{n}}$.

Therefore the directional derivative is maximum along the normal to the surface. Its maximum value is

$$=\frac{df}{dn}=|\operatorname{grad} f|.$$

§ 7. Tangent plane and Normal to a level surface.

To find the equations of the tangent plane and normal to the surface $f(x, y, z)=c$.

Let $f(x, y, z)=c$ be the equation of a level surface. Let $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ be the position vector of any point $P(x, y, z)$ on this surface.

Then $\nabla f=\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}$ is a vector along the normal to the surface at P *i.e.* ∇f is perpendicular to the tangent plane at P.

Tangent plane at P. Let $\mathbf{R}=X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k}$ be the position vector of current point $Q(X, Y, Z)$ on the tangent plane at P to the surface. The vector

$$\overrightarrow{PQ}=\mathbf{R}-\mathbf{r}=(X-x)\,\mathbf{i}+(Y-y)\,\mathbf{j}+(Z-z)\,\mathbf{k}$$

lies in the tangent plane at P. Therefore it is perpendicular to the vector ∇f.

$$\therefore\quad (\mathbf{R}-\mathbf{r})\cdot\nabla f=0$$

or $$[(X-x)\,\mathbf{i}+(Y-y)\,\mathbf{j}+(Z-z)\,\mathbf{k}]\cdot\left(\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}\right)=0$$

or $$(X-x)\frac{\partial f}{\partial x}+(Y-y)\frac{\partial f}{\partial y}+(Z-z)\frac{\partial f}{\partial z}=0, \quad \text{...(1)}$$

is the equation of the tangent plane at P.

Normal at P. Let $\mathbf{R}=X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k}$ be the position vector of any current point $Q(X, Y, Z)$ on the normal at P to the surface. The vector $\overrightarrow{PQ}=\mathbf{R}-\mathbf{r}=(X-x)\,\mathbf{i}+(Y-y)\,\mathbf{j}+(Z-z)\,\mathbf{k}$ lies along the normal at P to the surface. Therefore it is parallel to the vector ∇f.

$$\therefore \quad (\mathbf{R}-\mathbf{r})\times\nabla f=\mathbf{0} \qquad \ldots(2)$$

is the vector equation of the normal at P to the given surface.

Cartesian form. The vectors

$$(X-x)\,\mathbf{i}+(Y-y)\,\mathbf{j}+(Z-z)\,\mathbf{k} \text{ and } \nabla f=\frac{\partial f}{\partial x}\,\mathbf{i}+\frac{\partial f}{\partial y}\,\mathbf{j}+\frac{\partial f}{\partial z}\,\mathbf{k}$$

will be parallel if

$$(X-x)\,\mathbf{i}+(Y-y)\,\mathbf{j}+(Z-z)\,\mathbf{k}=p\left(\frac{\partial f}{\partial x}\,\mathbf{i}+\frac{\partial f}{\partial y}\,\mathbf{j}+\frac{\partial f}{\partial z}\,\mathbf{k}\right),$$

where p is some scalar.

Equating the coefficients of $\mathbf{i}, \mathbf{j}, \mathbf{k}$, we get

$$X-x=p\,\frac{\partial f}{\partial x},\; Y-y=p\,\frac{\partial f}{\partial y},\; Z-z=p\,\frac{\partial f}{\partial z}$$

or

$$\frac{X-x}{\dfrac{\partial f}{\partial x}}=\frac{Y-y}{\dfrac{\partial f}{\partial y}}=\frac{Z-z}{\dfrac{\partial f}{\partial z}}$$

are the equations of the normal at P.

Solved Examples

Ex. 1. *Find a unit normal vector to the level surface $x^2y+2xz=4$ at the point $(2, -2, 3)$.*

Sol. The equation of the level surface is

$$f(x, y, z)\equiv x^2y+2xz-4.$$

The vector grad f is along the normal to the surface at the point (x, y, z).

We have grad $f=\nabla(x^2y+2xz)=(2xy+2z)\,\mathbf{i}+x^2\,\mathbf{j}+2x\,\mathbf{k}$.

$\therefore$ at the point $(2, -2, 3)$, grad $f=-2\mathbf{i}+4\mathbf{j}+4\mathbf{k}$.

$\therefore$ $-2\mathbf{i}+4\mathbf{j}+4\mathbf{k}$ is a vector along the normal to the given surface at the point $(2, -2, 3)$.

Hence a unit normal vector to the surface at this point

$$=\frac{-2\mathbf{i}+4\mathbf{j}+4\mathbf{k}}{|-2\mathbf{i}+4\mathbf{j}+4\mathbf{k}|}=\frac{-2\mathbf{i}+4\mathbf{j}+4\mathbf{k}}{\sqrt{(4+16+16)}}=-\tfrac{1}{3}\mathbf{i}+\tfrac{2}{3}\mathbf{j}+\tfrac{2}{3}\mathbf{k}.$$

The vector $-(-\tfrac{1}{3}\mathbf{i}+\tfrac{2}{3}\mathbf{j}+\tfrac{2}{3}\mathbf{k})$ *i.e.*, $\tfrac{1}{3}\mathbf{i}-\tfrac{2}{3}\mathbf{j}-\tfrac{2}{3}\mathbf{k}$ is also a unit normal vector to the given surface at the point $(2, -2, 3)$.

Ex. 2. *Find the unit normal to the surface $z=x^2+y^2$ at the point $(-1, -2, 5)$.* [Kanpur 1986]

Sol. The equation of the given surface is

The vector grad f is along the normal to the surface at the point (x, y, z).

We have grad $f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k} = 2x\,\mathbf{i} + 2y\,\mathbf{j} - \mathbf{k}$.

$\therefore$ at the point $(-1, -2, 5)$, grad $f = -2\mathbf{i} - 4\,\mathbf{j} - \mathbf{k}$.

$\therefore$ $-(2\,\mathbf{i} + 4\,\mathbf{j} + \mathbf{k})$ is a vector along the normal to the given surface at the point $(-1, -2, 5)$.

Hence the required unit normal vector to the surface at this point

$$= \frac{\text{grad } f}{|\text{grad } f|} = \frac{-(2\,\mathbf{i}+4\,\mathbf{j}+\mathbf{k})}{|-(2\,\mathbf{i}+4\,\mathbf{j}+\mathbf{k})|} = \frac{-(2\,\mathbf{i}+4\,\mathbf{j}+\mathbf{k})}{\sqrt{(2^2+4^2+1^2)}}$$

$$= \frac{-(2\,\mathbf{i}+4\,\mathbf{j}+\mathbf{k})}{\sqrt{(21)}}.$$

The vector $\frac{2\,\mathbf{i}+4\,\mathbf{j}+\mathbf{k}}{\sqrt{(21)}}$ is also a unit normal vector to the given surface at the point $(-1, -2, 5)$.

Ex. 3. *Find the unit normal to the surface*

$$x^4 - 3xyz + z^2 + 1 = 0$$

at the point $(1, 1, 1)$. [Gorakhpur 1988; Allahabad 79]

Sol. The given surface is

$$f(x, y, z) \equiv x^4 - 3xyz + z^2 + 1 = 0. \qquad \ldots(1)$$

We have grad $f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$

$$= (4x^3 - 3yz)\,\mathbf{i} + (-3xz)\,\mathbf{j} + (-3xy + 2z)\,\mathbf{k}.$$

Now a vector normal to the surface (1) at the point $(1, 1, 1)$

$= \text{grad } f$ at the point $(1, 1, 1) = (4-3)\,\mathbf{i} + (-3)\,\mathbf{j} + (-3+2)\,\mathbf{k}$

$$= \mathbf{i} - 3\,\mathbf{j} - \mathbf{k}.$$

$\therefore$ the required unit normal vector $= \frac{\text{grad } f}{|\text{grad } f|}$

$$= \frac{\mathbf{i}-3\mathbf{j}-\mathbf{k}}{|\mathbf{i}-3\mathbf{j}-\mathbf{k}|} = \frac{\mathbf{i}-3\mathbf{j}-\mathbf{k}}{\sqrt{[1^2+(-3)^2+(-1)^2]}} = \frac{\mathbf{i}-3\mathbf{j}-\mathbf{k}}{\sqrt{(11)}}.$$

Ex. 4. *Find the unit vector normal to the surface* $x^2 - y^2 + z = 2$ *at the point* $(1, -1, 2)$. [Ravi Shankar 1981]

Sol. The given surface is

$$f(x, y, z) \equiv x^2 - y^2 + z - 2 = 0. \qquad \ldots(1)$$

We have grad $f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$

$$= 2x\,\mathbf{i} - 2y\,\mathbf{j} + \mathbf{k}.$$

Now a vector normal to the surface (1) at the point $(1, -1, 2)$
$=$grad f at the point $(1, -1, 2)=2\,\mathbf{i}+2\mathbf{j}+\mathbf{k}$.

Hence the required unit vector normal to the surface (1) at the point $(1, -1, 2)$

$$=\frac{\text{grad } f}{|\text{grad } f|}=\frac{2\mathbf{i}+2\mathbf{j}+\mathbf{k}}{|2\mathbf{i}+2\mathbf{j}+\mathbf{k}|}=\frac{2\mathbf{i}+2\mathbf{j}+\mathbf{k}}{\sqrt{(4+4+1)}}=\tfrac{1}{3}(2\mathbf{i}+2\mathbf{j}+\mathbf{k}).$$

Ex. 5. *Find the gradient and the unit normal to the level surface $x^2+y-z=4$ at the point $(2, 0, 0)$.*

Sol. The given surface is

$$f(x, y, z)\equiv x^2+y-z-4=0. \quad \ldots(1)$$

We have grad $f=\dfrac{\partial f}{\partial x}\mathbf{i}+\dfrac{\partial f}{\partial y}\mathbf{j}+\dfrac{\partial f}{\partial z}\mathbf{k}=2x\,\mathbf{i}+\mathbf{j}-\mathbf{k}$.

$\therefore$ at the point $(2, 0, 0)$, grad $f=4\,\mathbf{i}+\mathbf{j}-\mathbf{k}$.

Now a vector along the normal to the surface (1) at the point $(2, 0, 0)$

$=$grad f at the point $(2, 0, 0)=4\,\mathbf{i}+\mathbf{j}-\mathbf{k}$.

Hence the required unit normal to the surface (1) at the point $(2, 0, 0)$

$$=\frac{4\mathbf{i}+\mathbf{j}-\mathbf{k}}{|4\mathbf{i}+\mathbf{j}-\mathbf{k}|}=\frac{4\,\mathbf{i}+\mathbf{j}-\mathbf{k}}{\sqrt{(16+1+1)}}=\frac{1}{3\sqrt{2}}(4\,\mathbf{i}+\mathbf{j}-\mathbf{k}).$$

Ex. 6. *Find the directional derivatives of a scalar point function f in the direction of coordinate axes.*

Sol. The grad f at any point (x, y, z) is the vector

$$\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}.$$

The directional derivative of f in the direction of $\mathbf{i}$

$$=\text{grad } f\cdot\mathbf{i}=\left(\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}\right)\cdot\mathbf{i}=\frac{\partial f}{\partial x}.$$

Similarly the directional derivatives of f in the directions of $\mathbf{j}$ are $\mathbf{k}$ are $\dfrac{\partial f}{\partial y}$ and $\dfrac{\partial f}{\partial z}$.

Ex. 7. *Find the directional derivative of $f(x, y, z)=x^2yz+4xz^2$ at the point $(1, -2, -1)$ in the direction of the vector $2\mathbf{i}-\mathbf{j}-2\mathbf{k}$.*

[Allahabad 1978]

Sol. We have $f(x, y, z)=x^2yz+4xz^2$.

$\therefore$ grad $f=(2xyz+4z^2)\,\mathbf{i}+x^2z\,\mathbf{j}+(x^2y+8xz)\,\mathbf{k}$

If $\hat{a}$ be the unit vector in the direction of the vector $2i-j-2k$,

then $\hat{a}=\dfrac{2i-j-2k}{\sqrt{(4+1+4)}}=\dfrac{2}{3}i-\dfrac{1}{3}j-\dfrac{2}{3}k.$

Therefore the required directional derivative is

$$\frac{df}{ds}=\text{grad } f\cdot\hat{a}=(8i-j-10k)\cdot(\tfrac{2}{3}i-\tfrac{1}{3}j-\tfrac{2}{3}k)=\frac{16}{3}+\frac{1}{3}+\frac{20}{3}=\frac{37}{3}.$$

Since this is positive, f is increasing in this direction.

Ex. 8. *Find the directional derivative of*

$$f(x, y, z)=x^2-2y^2+4z^2.$$

at the point $(1, 1, -1)$ *in the direction of* $2i+j-k$. [Agra 1979]

Ans. $8/\sqrt{6}$.

Ex. 9. *Find the directional derivative of the function* $f=x^2-y^2+2z^2$ *at the point* $P(1, 2, 3)$ *in the direction of the line* PQ *where* Q *is the point* $(5, 0, 4)$. [Agra 1987]

Sol. Here $\text{grad } f=\dfrac{\partial f}{\partial x}i+\dfrac{\partial f}{\partial y}j+\dfrac{\partial f}{\partial z}k$

$=2x\, i-2y\, j+4z\, k=2i-4j+12k$ at the point $(1, 2, 3)$.

Also $\overrightarrow{PQ}$ = position vector of Q − position vector of P

$=(5i+0j+4k)-(i+2j+3k)=4i-2j+k.$

If $\hat{a}$ be the unit vector in the direction of the vector $\overrightarrow{PQ}$,

then $\hat{a}=\dfrac{4i-2j+k}{\sqrt{(16+4+1)}}=\dfrac{4i-2j+k}{\sqrt{(21)}}.$

$\therefore$ the required directional derivative

$=(\text{grad } f)\cdot\hat{a}=(2i-4j+12k)\cdot\left\{\dfrac{4i-2j+k}{\sqrt{(21)}}\right\}$

$=\dfrac{28}{\sqrt{(21)}}=\dfrac{28}{21}\sqrt{(21)}=\dfrac{4}{3}\sqrt{(21)}.$

Ex. 10. *Find the directional derivatives of the function*

$$f=xy+yz+zx$$

in the direction of the vector $2i+3j+6k$ *at the point* $(3, 1, 2)$.

[Rohilkhand 1980, 81; Agra 75]

Sol. We have $f(x, y, z)=xy+yz+zx$.

$\therefore$ $\text{grad } f=(\partial f/\partial x)\, i+(\partial f/\partial y)\, j+(\partial f/\partial z)\, k$

$=(y+z)\, i+(z+x)\, j+(x+y)\, k$

$=(1+2)\, i+(2+3)\, j+(3+1)\, k$ at the point $(3, 1, 2)$

$=3i+5j+4k.$

If $\hat{a}$ be the unit vector in the direction of the vector $2i+3j+6k$, then

$$\hat{a}=\frac{2i+3j+6k}{|2i+3j+6k|}=\frac{2i+3j+6k}{\sqrt{(4+9+36)}}=\frac{1}{7}(2i+3j+6k).$$

$\therefore$ the required directional derivative

$$=(\text{grad } f)\cdot\hat{a}=(3i+5j+4k)\cdot\frac{1}{7}(2i+3j+6k)$$

$$=\frac{1}{7}(3i+5j+4k)\cdot(2i+3j+6k)$$

$$=\frac{1}{7}(6+15+24)=\frac{45}{7}.$$

Ex. 11. *Find the directional derivative of $\phi=xy+yz+zx$ in the direction of the vector $i+2j+2k$ at $(1, 2, 0)$.* [Agra 1982]

Sol. We have $\phi(x, y, z)=xy+yz+zx$.

$\therefore$ grad $\phi=(\partial\phi/\partial x)\, i+(\partial\phi/\partial y)\, j+(\partial\phi/\partial z)\, k$

$$=(y+z)\, i+(z+x)\, j+(x+y)\, k$$

$$=(2+0)\, i+(0+1)\, j+(1+2)\, k$$

$$=2i+j+3k \qquad \text{at the point } (1, 2, 0)$$

If $\hat{a}$ be the unit vector in the direction of the given vector $i+2j+2k$, then

$$\hat{a}=\frac{i+2j+2k}{|i+2j+2k|}=\frac{i+2j+2k}{\sqrt{(1+4+4)}}=\frac{1}{3}(i+2j+2k).$$

$\therefore$ the required directional derivative

$$=(\text{grad } \phi)\cdot\hat{a}=(2i+j+3k)\cdot\tfrac{1}{3}(i+2j+2k)$$

$$=\tfrac{1}{3}(2+2+6)=\frac{10}{3}.$$

Ex. 12. *Find the directional derivative of*

$$\phi(x, y, z)=x^2yz+4xz^2$$

at the point $(1, -2, 1)$ in the direction of $2i-j-2k$.

Sol. Do yourself. [Indore 1983]

Ans. $-13/3$.

Ex. 13. *Obtain the directional derivative of $\phi=xy^2+yz^2$ at the point $(2, -1, 1)$ in the direction of the vector $i+2j+2k$.*

[Allahabad 1975]

Sol. Do yourself.

Ans. -3.

Ex. 14. *Find the directional derivatives of $\phi=xyz$ at the point $(2, 2, 2)$, in the directions*

(i) i, (ii) j, (iii) $i+j+k$. [Agra 1981]

Sol. We have $\phi(x, y, z) = xyz$.

$\therefore$ grad $\phi = (\partial\phi/\partial x)\,\mathbf{i} + (\partial\phi/\partial y)\,\mathbf{j} + (\partial\phi/\partial z)\,\mathbf{k} = yz\,\mathbf{i} + zx\,\mathbf{j} + xy\,\mathbf{k}$

$= 4\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$ at the point (2, 2, 2).

(i) Directional derivative of ϕ at the point (2, 2, 2) in the direction of the unit vector $\mathbf{i}$

$$= \text{grad}\,\phi \cdot \mathbf{i} = (4\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}) \cdot \mathbf{i} = 4.$$

(ii) Directional derivative of ϕ at the point (2, 2, 2) in the direction of the unit vector $\mathbf{j}$

$$= \text{grad}\,\phi \cdot \mathbf{j} = (4\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}) \cdot \mathbf{j} = 4.$$

(iii) Unit vector $\hat{\mathbf{a}}$ in the direction of the vector $\mathbf{i} + \mathbf{j} + \mathbf{k}$

$$= \frac{\mathbf{i} + \mathbf{j} + \mathbf{k}}{|\mathbf{i} + \mathbf{j} + \mathbf{k}|} = \frac{\mathbf{i} + \mathbf{j} + \mathbf{k}}{\sqrt{3}}.$$

$\therefore$ directional derivative of ϕ at the point (2, 2, 2) in the direction of the vector $\mathbf{i} + \mathbf{j} + \mathbf{k}$

$$= \text{grad}\,\phi \cdot \hat{\mathbf{a}} = (4\mathbf{i} + 4\mathbf{j} + 4\mathbf{k}) \cdot \frac{\mathbf{i} + \mathbf{j} + \mathbf{k}}{\sqrt{3}} = \frac{4 + 4 + 4}{\sqrt{3}} = \frac{12}{\sqrt{3}} = 4\sqrt{3}.$$

Ex. 15. *Find the directional derivative of $\phi = xyz$ at (1, 2, 3) in the direction of the vector* $\mathbf{i}$.

Sol. Do yourself. **Ans.** 6.

Ex. 16. *Find the directional derivative of the function $xy^2 + yz^2 + zx^2$ along the tangent to the curve $x = t,\ y = t^2,\ z = t^3$ at the point* (1, 1, 1).

Sol. Let $\phi(x, y, z) = xy^2 + yz^2 + zx^2$.

Then grad $\phi = (\partial\phi/\partial x)\,\mathbf{i} + (\partial\phi/\partial y)\,\mathbf{j} + (\partial\phi/\partial z)\,\mathbf{k}$

$= (y^2 + 2zx)\,\mathbf{i} + (z^2 + 2xy)\,\mathbf{j} + (x^2 + 2yz)\,\mathbf{k}$

$= 3\mathbf{i} + 3\mathbf{j} + 3\mathbf{k}$, at the point (1, 1, 1)

$= 3(\mathbf{i} + \mathbf{j} + \mathbf{k})$.

Also for the curve $x = t,\ y = t^2,\ z = t^3$, we have

$$dx/dt = 1,\ dy/dt = 2t,\ dz/dt = 3t^2.$$

At the point (1, 1, 1) on the curve $x = t,\ y = t^2,\ z = t^3$, we have $t = 1$.

Now a vector along the tangent to the above curve at the point (x, y, z)

$= (dx/dt)\,\mathbf{i} + (dy/dt)\,\mathbf{j} + (dz/dt)\,\mathbf{k}$

$= \mathbf{i} + 2t\,\mathbf{j} + 3t^2\,\mathbf{k}$.

Putting $t = 1$, a vector along the tangent to the curve at the point (1, 1, 1) $= \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

If $\hat{\mathbf{a}}$ be the unit vector in the direction of this tangent, then

$$\hat{\mathbf{a}}=\frac{\mathbf{i}+2\mathbf{j}+3\mathbf{k}}{|\mathbf{i}+2\mathbf{j}+3\mathbf{k}|}=\frac{\mathbf{i}+2\mathbf{j}+3\mathbf{k}}{\sqrt{}}$$

$\therefore$ the required directional derivative

$$=\hat{\mathbf{a}}\cdot \text{grad } \phi \text{ at } (1, 1, 1)$$
$$=\frac{\mathbf{i}+2\mathbf{j}+3\mathbf{k}}{\sqrt{(14)}}\cdot 3\,(\mathbf{i}+\mathbf{j}+\mathbf{k})$$
$$=\frac{3}{\sqrt{(14)}}\,(1+2+3)=\frac{18}{\sqrt{(14)}}.$$

Ex. 17. *In what direction the directional derivative of $\phi=x^2y^2z$ from (1, 1, 2) will be maximum and what is its magnitude? Also find a unit normal vector to the surface $x^2y^2z=2$ at the point (1, 1, 2).*

Sol. We know that the directional derivative of ϕ at the point (x, y, z) is maximum in the direction of the normal to the surface $\phi=$constant *i.e.*, in the direction of the vector grad ϕ.

Now grad $\phi=(\partial\phi/\partial x)\ \mathbf{i}+(\partial\phi/\partial y)\ \mathbf{j}+(\partial\phi/\partial z)\ \mathbf{k}$

$$=2xy^2z\ \mathbf{i}+2x^2yz\ \mathbf{j}+x^2y^2\ \mathbf{k}$$
$$=4\mathbf{i}+4\mathbf{j}+\mathbf{k}, \text{ at the point } (1, 1, 2).$$

Hence the directional derivative of ϕ at the point (1, 1, 2) will be maximum in the direction of the vector $4\mathbf{i}+4\mathbf{j}+\mathbf{k}$.

Also the magnitude of this maximum directional derivative

$$=\text{modulus of grad } \phi \text{ at } (1, 1, 2)$$
$$=|4\mathbf{i}+4\mathbf{j}+\mathbf{k}|=\sqrt{(16+16+1)}=\sqrt{(33)}.$$

The unit vector along the normal to the surface $x^2y^2z=2$ at the point (1, 1, 2)

$$=\frac{\text{grad } \phi}{|\text{grad } \phi|}, \text{ at } (1, 1, 2)$$
$$=\frac{4\mathbf{i}+4\mathbf{j}+\mathbf{k}}{|4\mathbf{i}+4\mathbf{j}+\mathbf{k}|}=\frac{4\mathbf{i}+4\mathbf{j}+\mathbf{k}}{\sqrt{(33)}}.$$

Ex. 18. *Find the greatest value of the directional derivative of the function $2x^2-y-z^4$ at the point $(2, -1, 1)$.*

Sol. Let $\phi(x, y, z)=2x^2-y-z^4$.

Then grad $\phi=(\partial\phi/\partial x)\ \mathbf{i}+(\partial\phi/\partial y)\ \mathbf{j}+(\partial\phi/\partial z)\ \mathbf{k}$

$$=4x\ \mathbf{i}-\mathbf{j}-4z^3\ \mathbf{k}$$
$$=8\mathbf{i}-\mathbf{j}-4\mathbf{k}, \text{ at the point } (2, -1, 1).$$

Now the greatest value of the directional derivative of ϕ at the

$=$modulus of grad ϕ at the point $(2, -1, 1)$

$= |\, 8\mathbf{i}-\mathbf{j}-4\mathbf{k}\,| = \sqrt{[8^2+(-1)^2+(-4)^2]} = \sqrt{(64+1+16)} = 9.$

Ex. 19. *Find the maximum value of the directional derivative of $\phi = x^2\, yz$ at the point $(1, 4, 1)$.* [Bombay 1970]

Sol. We have grad $\phi = (\partial\phi/\partial x)\,\mathbf{i} + (\partial\phi/\partial y)\,\mathbf{j} + (\partial\phi/\partial z)\,\mathbf{k}$

$= 2xyz\,\mathbf{i} + x^2 z\,\mathbf{j} + x^2 y\,\mathbf{k}$

$= 8\mathbf{i}+\mathbf{j}+4\mathbf{k}$, at the point $(1, 4, 1)$.

Now the maximum value of the directional derivative of $\phi = x^2 yz$ at the point $(1, 4, 1)$

$= |\,$grad ϕ at the point $(1, 4, 1)\,|$

$= |\, 8\mathbf{i}+\mathbf{j}+4\mathbf{k}\,| = \sqrt{(8^2+1^2+4^2)} = \sqrt{(81)} = 9.$

Ex. 20. *Calculate the maximum rate of change and the corresponding direction for the function $\phi = x^2\, y^3\, z^4$ at the point $2\mathbf{i}+3\mathbf{j}-\mathbf{k}$.* [Allahabad 1982]

Sol. The coordinates of the point whose position vector is $2\mathbf{i}+3\mathbf{j}-\mathbf{k}$ are $(2, 3, -1)$.

We have grad $\phi = (\partial\phi/\partial x)\,\mathbf{i} + (\partial\phi/\partial y)\,\mathbf{j} + (\partial\phi/\partial z)\,\mathbf{k}$

$= 2xy^3z^4\,\mathbf{i} + 3x^2y^2z^4\,\mathbf{j} + 4x^2y^3z^3\,\mathbf{k}$

$= 108\mathbf{i}+108\mathbf{j}-432\,\mathbf{k}$, at the point $(2, 3, -1)$

$= 108\,(\mathbf{i}+\mathbf{j}-4\,\mathbf{k})$.

Now the rate of change of ϕ (*i.e.*, the directional derivative of ϕ) at the point $(2, 3, -1)$ is maximum in the direction of the vector grad ϕ at this point *i.e.*, in the direction of the vector $108\,(\mathbf{i}+\mathbf{j}-4\mathbf{k})$.

Also the magnitude of the maximum rate of change $=$ modulus of grad ϕ at the point $(2, 3, -1)$

$= |\, 108\,(\mathbf{i}+\mathbf{j}-4\mathbf{k})\,| = 108\,|\,\mathbf{i}+\mathbf{j}-4\mathbf{k}\,|$.

$= 108\sqrt{[1^2+1^2+(-4)^2]} = 108\sqrt{(18)} = 324\sqrt{2}.$

Ex. 21. *Find the values of the constants a, b, c so that the directional derivative of $\phi = ax^2+by^2+cz^2$ at $(1, 1, 2)$ has a maximum magnitude 4 in the direction parallel to y-axis.*

Sol. We have grad $\phi = (\partial\phi/\partial x)\,\mathbf{i} + (\partial\phi/\partial y)\,\mathbf{j} + (\partial\phi/\partial z)\,\mathbf{k}$

$= 2ax\,\mathbf{i} + 2by\,\mathbf{j} + 2cz\,\mathbf{k}$

$= 2a\,\mathbf{i} + 2b\mathbf{j} + 4c\,\mathbf{k}$, at the point $(1, 1, 2)$.

Now the directional derivative of ϕ at the point $(1, 1, 2)$ is maximum in the direction of the vector grad ϕ at this point. According to the question this directional derivative is maximum in the direction parallel to y-axis *i.e.*, in the direction parallel to the vector $\mathbf{j}$. So if the direction of the vector $2a\mathbf{i}+2b\mathbf{j}+4c\,\mathbf{k}$

is parallel to the vector $\mathbf{j}$, we must have $2a=0$, $4c=0$ *i.e.*, $a=0$ and $c=0$.

Then grad ϕ at $(1, 1, 2)=2b\mathbf{j}$.

Also the maximum value of directional derivative

$$=|\text{grad } \phi|.$$

$\therefore$ $4=|2b\mathbf{j}|$, since according to the question the maximum value of directional derivative is 4.

$\therefore$ $2b=4$ or $b=2$.

Hence $a=0$, $b=2$, $c=0$.

Ex. 22. *In what direction from the point* $(1, 1, -1)$ *is the directional derivative of* $f=x^2-2y^2+4z^2$ *a maximum? Also find the value of this maximum directional derivative.*

Sol. We have grad $f=2x\mathbf{i}-4y\mathbf{j}+8z\mathbf{k}$

$$=2\mathbf{i}-4\mathbf{j}-8\mathbf{k} \text{ at the point } (1, 1, -1).$$

The directional derivative of f is maximum in the direction of grad $f=2\mathbf{i}-4\mathbf{j}-8\mathbf{k}$.

The maximum value of this directional derivative

$=|\text{grad } f|=|2\mathbf{i}-4\mathbf{j}-8\mathbf{k}|=\sqrt{(4+16+64)}=\sqrt{(84)}=2\sqrt{(21)}$.

Ex. 23. *For the function* $f=y/(x^2+y^2)$, *find the value of the directional derivative making an angle* $30°$ *with the positive x-axis at the point* $(0, 1)$. [Agra 1981]

Sol. We have grad $f=\dfrac{\partial f}{\partial x}\mathbf{i}+\dfrac{\partial f}{\partial y}\mathbf{j}$

$$=\frac{-2xy}{(x^2+y^2)^2}\mathbf{i}+\frac{x^2-y^2}{(x^2+y^2)^2}\mathbf{j}=-\mathbf{j} \text{ at the point } (0, 1).$$

If $\hat{\mathbf{a}}$ is a unit vector along the line which makes an angle $30°$ with the positive x-axis, then

$$\hat{\mathbf{a}}=\cos 30°\,\mathbf{i}+\sin 30°\,\mathbf{j}=\frac{\sqrt{3}}{2}\mathbf{i}+\tfrac{1}{2}\mathbf{j}.$$

$\therefore$ the required directional derivative is

$$=\text{grad } f\cdot\hat{\mathbf{a}}=(-\mathbf{j})\cdot\left(\frac{\sqrt{3}}{2}\mathbf{i}+\frac{1}{2}\mathbf{j}\right)=-\tfrac{1}{2}.$$

Ex. 24. *What is the greatest rate of increase of* $u=xyz^2$ *at the point* $(1, 0, 3)$? [Agra 1968]

Sol. We have $\nabla u=yz^2\,\mathbf{i}+xz^2\,\mathbf{j}+2xyz\,\mathbf{k}$.

$\therefore$ at the point $(1, 0, 3)$, we have

$$\nabla u=0\,\mathbf{i}+9\,\mathbf{j}+0\,\mathbf{k}=9\,\mathbf{j}.$$

The greatest rate of increase of u at the point $(1, 0, 3)$

$=$the maximum value of $\dfrac{du}{ds}$ at the point $(1, 0, 3)$

$=|\nabla u|$, at the point $(1, 0, 3)$

$=|9\mathbf{j}|=9.$

Ex. 25. *Show that the directional derivative of a scalar point function at any point along any tangent line to the level surface at the point is zero.*

Sol. Let $f(x, y, z)$ be a scalar point function and let $\mathbf{a}$ be a unit vector along a tangent line to the level surface $f(x, y, z)=c$.

We know that ∇f is a normal vector at any point of the surface $f(x, y, z)=c$. Therefore the vectors ∇f and $\mathbf{a}$ are perpendicular.

Now the directional derivative of f in the direction of $\mathbf{a}$

$$=\mathbf{a}\cdot\nabla f=0.$$

Ex. 26. *Find the equations of the tangent plane and normal to the surface* $2xz^2-3xy-4x=7$ *at the point* $(1, -1, 2)$.

Sol. The equation of the surface is

$$f(x, y, z)\equiv 2xz^2-3xy-4x=7.$$

We have grad $f=(2z^2-3y+4)\,\mathbf{i}-3x\,\mathbf{j}+4xz\,\mathbf{k}$

$$=7\mathbf{i}-3\mathbf{j}+8\mathbf{k},\text{ at the point }(1, -1, 2).$$

$\therefore$ $7\mathbf{i}-3\mathbf{j}+8\mathbf{k}$ is a vector along the normal to the surface at the point $(1, -1, 2)$.

The position vector of the point $(1, -1, 2)$ is $=\mathbf{r}=\mathbf{i}-\mathbf{j}+2\mathbf{k}$.

If $\mathbf{R}=X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k}$ is the position vector of any current point (X, Y, Z) on the tangent plane at $(1, -1, 2)$, then the vector $\mathbf{R}-\mathbf{r}$ is perpendicular to the vector grad f.

$\therefore$ the equation of the tangent plane is

$$(\mathbf{R}-\mathbf{r})\cdot\text{grad } f=0,$$

i.e. $\{(X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k})-(\mathbf{i}-\mathbf{j}+2\mathbf{k})\}\cdot(7\mathbf{i}-3\mathbf{j}+8\mathbf{k})=0,$

i.e. $\{(X-1)\,\mathbf{i}+(Y+1)\,\mathbf{j}+(Z-2)\,\mathbf{k}\}\cdot(7\mathbf{i}-3\mathbf{j}+8\mathbf{k})=0,$

i.e. $7(X-1)-3(Y+1)+8(Z-2)=0.$

The equations of the normal to the surface at the point $(1, -1, 2)$ are

$$\frac{X-1}{\dfrac{\partial f}{\partial x}}=\frac{Y+1}{\dfrac{\partial f}{\partial y}}=\frac{Z-2}{\dfrac{\partial f}{\partial z}},\ i.e.\ \frac{X-1}{7}=\frac{Y+1}{-3}=\frac{Z-2}{8}.$$

Ex. 27. *Find the equations of the tangent plane and normal to the surface* $xyz=4$ *at the point* $(1, 2, 2)$. **[Meerut 1991 P, Agra 82]**

Sol. The equation of the surface is

$$f(x, y, z)\equiv xyz-4=0.$$

We have grad $f=yz\mathbf{i}+xz\mathbf{j}+xy\mathbf{k}$

$=4\mathbf{i}+2\mathbf{j}+2\mathbf{k}$, at the point (1, 2, 2).

$\therefore$ $4\mathbf{i}+2\mathbf{j}+2\mathbf{k}$ is a vector along the normal to the surface at the point (1, 2, 2).

The position vector of the point (1, 2, 2) is $=\mathbf{r}=\mathbf{i}+2\mathbf{j}+2\mathbf{k}$.

If $\mathbf{R}=X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k}$ is the position vector of any current point (X, Y, Z) on the tangent plane at (1, 2, 2), the equation of the tangent plane is

$$(\mathbf{R}-\mathbf{r})\cdot \text{grad } f=0,$$

i.e. $\{(X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k})-(\mathbf{i}+2\mathbf{j}+2\mathbf{k})\}\cdot(4\mathbf{i}+2\mathbf{j}+2\mathbf{k})=0,$

i.e. $\{(X-1)\,\mathbf{i}+(Y-2)\,\mathbf{j}+(Z-2)\,\mathbf{k}\}\cdot(4\mathbf{i}+2\mathbf{j}+2\mathbf{k})=0,$

i.e. $4(X-1)+2(Y-2)+2(Z-2)=0,$

i.e. $4X+2Y+2Z=12$, *i.e.* $2X+Y+Z=6$.

The equations of the normal to the surface at the point (1, 2, 2) are

$$\frac{X-1}{\dfrac{\partial f}{\partial x}}=\frac{Y-2}{\dfrac{\partial f}{\partial y}}=\frac{Z-2}{\dfrac{\partial f}{\partial z}},$$

i.e. $\dfrac{X-1}{4}=\dfrac{Y-2}{2}=\dfrac{Z-2}{2}$, *i.e.*, $\dfrac{X-1}{2}=\dfrac{Y-2}{1}=\dfrac{Z-2}{1}$.

Ex. 28. *Find the equation of the tangent plane to the surface* $yz-zx+xy+5=0$, *at the point* $(1, -1, 2)$.

Sol. The equation of the given surface is

$$f(x, y, z)\equiv yz-zx+xy+5=0.$$

We have $\text{grad } f=(\partial f/\partial x)\,\mathbf{i}+(\partial f/\partial y)\,\mathbf{j}+(\partial f/\partial z)\,\mathbf{k}$

$=(-z+y)\,\mathbf{i}+(z+x)\,\mathbf{j}+(y-x)\,\mathbf{k}$

$=-3\mathbf{i}+3\mathbf{j}-2\mathbf{k}$, at the point $(1, -1, 2)$.

$\therefore$ $-3\mathbf{i}+3\mathbf{j}-2\mathbf{k}$ is a vector along the normal to the surface $f(x, y, z)=0$ at the point $(1, -1, 2)$.

The position vector of the point $(1, -1, 2)$

$=\mathbf{i}-\mathbf{j}+2\mathbf{k}=\mathbf{r}$, say.

If $\mathbf{R}=X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k}$ is the position vector of any current point (X, Y, Z) on the tangent plane at $(1, -1, 2)$, the equation of the tangent plane is

$$(\mathbf{R}-\mathbf{r})\cdot \text{grad } f=0$$

i.e., $\{(X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k})-(\mathbf{i}-\mathbf{j}+2\mathbf{k})\}\cdot(-3\mathbf{i}+3\mathbf{j}-2\mathbf{k})=0$

or $\{(X-1)\,\mathbf{i}+(Y+1)\,\mathbf{j}+(Z-2)\,\mathbf{k}\}\cdot(-3\mathbf{i}+3\mathbf{j}-2\mathbf{k})=0$

or $-3(X-1)+3(Y+1)-2(Z-2)=0$

or $-3X+3Y-2Z+3+3+4=0$

or $3X-3Y+2Z=10$.

Ex. 29. *Find the equations of the tangent plane and normal to the surface* $z=x^2+y^2$ *at the point* $(2, -1, 5)$.

Sol. The equation of the given surface is

$f(x, y, z) \equiv x^2+y^2-z=0.$ ...(1)

We have grad $f=(\partial f/\partial x)\,\mathbf{i}+(\partial f/\partial y)\,\mathbf{j}+(\partial f/\partial z)\,\mathbf{k}$

$=2x\mathbf{i}+2y\mathbf{j}-\mathbf{k}$

$=4\mathbf{i}-2\mathbf{j}-\mathbf{k}$, at the point $(2, -1, 5)$.

$\therefore$ $4\mathbf{i}-2\mathbf{j}-\mathbf{k}$ is a vector along the normal to the surface (1) at the point $(2, -1, 5)$ *i.e.*, perpendicular to the tangent plane to the surface (1) at the point $(2, -1, 5)$.

Hence the equation of the tangent plane to the surface (1) at the point $(2, -1, 5)$ is

$\{(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})-(2\mathbf{i}-\mathbf{j}+5\mathbf{k})\}\cdot(4\mathbf{i}-2\mathbf{j}-\mathbf{k})=0$

or $\{(x-2)\,\mathbf{i}+(y+1)\,\mathbf{j}+(z-5)\,\mathbf{k}\}\cdot(4\mathbf{i}-2\mathbf{j}-\mathbf{k})=0$

or $4\,(x-2)-2\,(y+1)-(z-5)=0$

or $4x-2y-z=5.$

The equations of the normal to the surface (1) at the point $(2, -1, 5)$ are

$$\frac{x-2}{\partial f/\partial x}=\frac{y+1}{\partial f/\partial y}=\frac{z-5}{\partial f/\partial z}$$

i.e., $\dfrac{x-2}{4}=\dfrac{y+1}{2}=\dfrac{z-5}{-1}.$

Ex. 30. *Find the equations of the tangent plane and normal to the surface* $x^2+y^2+z^2=25$ *at the point* $(4, 0, 3)$.

Sol. The given surface is $f(x, y, z) \equiv x^2+y^2+z^2-25=0$ ...(1)

We have grad $f=2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}$

$=8\mathbf{i}+0\mathbf{j}+6\mathbf{k}$, at the point $(4, 0, 3)$.

$\therefore$ the direction cosines of the normal to the surface (1) at the point $(4, 0, 3)$ are proportional to 8, 0, 6.

Hence the equation of the tangent plane to the surface (1) at the point $(4, 0, 3)$ is

$8\,(x-4)+0\,(y-0)+6\,(z-3)=0$ or $4x+3z=25.$

The equations of the normal to the surface (1) at the point $(4, 0, 3)$ are

$$\frac{x-4}{8}=\frac{y-0}{0}=\frac{z-3}{6}\ \textit{i.e.},\ \frac{x-4}{4}=\frac{y}{0}=\frac{z-3}{3}.$$

Ex. 31. *Given the curve* $x^2+y^2+z^2=1$, $x+y+z=1$ *(intersection of two surfaces), find the equations of the tangent line at the point* $(1, 0, 0)$. **[Agra 1983]**

Sol. A normal to $x^2+y^2+z^2=1$ at $(1, 0, 0)$ is

$$\text{grad } f_1=\text{grad }(x^2+y^2+z^2)=2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}=2\mathbf{i}.$$

A normal to $x+y+z=1$ at $(1, 0, 0)$ is

$$\text{grad } f_2=\text{grad }(x+y+z)=1\mathbf{i}+1\mathbf{j}+1\mathbf{k}=\mathbf{i}+\mathbf{j}+\mathbf{k}.$$

The tangent line at the point $(1, 0, 0)$ is perpendicular to both these normals. Therefore it is parallel to the vector

$$(\text{grad } f_1)\times(\text{grad } f_2).$$

Now $(\text{grad } f_1)\times(\text{grad } f_2)=2\mathbf{i}\times(\mathbf{i}+\mathbf{j}+\mathbf{k})$

$$=2\mathbf{i}\times\mathbf{j}+2\mathbf{i}\times\mathbf{k}=2\mathbf{k}-2\mathbf{j}=0\mathbf{i}-2\mathbf{j}+2\mathbf{k}.$$

Now to find the equations of the line through the point $(1, 0, 0)$ and parallel to the vector $0\mathbf{i}-2\mathbf{j}+2\mathbf{k}$.

The required equations are

$$\frac{X-1}{0}=\frac{Y-0}{-2}=\frac{Z-0}{2},$$

i.e., $$X=1,\ \frac{Y}{-1}=\frac{Z}{1}.$$

Ex. 32. *Find the angle between the surfaces $x^2+y^2+z^2=9$, and $z=x^2+y^2-3$ at the point $(2, -1, 2)$.*

[Meerut 1991S; Kanpur 78, 80]

Sol. Angle between two surfaces at a point is the angle between the normals to the surfaces at the point.

Let $f_1=x^2+y^2+z^2$ and $f_2=x^2+y^2-z$.

Then $\text{grad } f_1=2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}$ and $\text{grad } f_2=2x\mathbf{i}+2y\mathbf{j}-\mathbf{k}$.

Let $\mathbf{n}_1=\text{grad } f_1$ at the point $(2, -1, 2)$ and $\mathbf{n}_2=\text{grad } f_2$ at the point $(2, -1, 2)$. Then

$$\mathbf{n}_1=4\mathbf{i}-2\mathbf{j}+4\mathbf{k} \text{ and } \mathbf{n}_2=4\mathbf{i}-2\mathbf{j}-\mathbf{k}.$$

The vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are along normals to the two surfaces at the point $(2, -1, 2)$. If θ is the angle between these vectors then

$$\mathbf{n}_1\cdot\mathbf{n}_2=|\mathbf{n}_1|\,|\mathbf{n}_2|\cos\theta$$

or $$16+4-4=\sqrt{(16+4+16)}\sqrt{(16+4+1)}\cos\theta.$$

$$\therefore \cos\theta=\frac{16}{6\sqrt{(21)}} \text{ or } \theta=\cos^{-1}\frac{8}{3\sqrt{(21)}}.$$

Ex. 33. *Find the angle of intersection at $(4, -3, 2)$ of spheres $x^2+y^2+z^2=29$ and $x^2+y^2+z^2+4x-6y-8z-47=0$.*

Sol. Let $f_1=x^2+y^2+z^2-29$ and $f_2=x^2+y^2+z^2+4x-6y-8z-47$.

Then $\text{grad } f_1=2x\,\mathbf{i}+2y\,\mathbf{j}+2z\,\mathbf{k}$

and $\text{grad } f_2=(2x+4)\,\mathbf{i}+(2y-6)\,\mathbf{j}+(2z-8)\,\mathbf{k}.$

Let $\mathbf{n}_1=\text{grad } f_1$ at the point $(4, -3, 2)$

and $\mathbf{n}_2=\text{grad } f_2$ at the point $(4, -3, 2)$. Then

$$\mathbf{n}_1=8\mathbf{i}-6\mathbf{j}+4\mathbf{k}=2\,(4\mathbf{i}-3\mathbf{j}+2\mathbf{k})$$

and $$\mathbf{n}_2=12\mathbf{i}-12\mathbf{j}-4\mathbf{k}=4\,(3\mathbf{i}-3\mathbf{j}-\mathbf{k}).$$

$\therefore$ the component of $\mathbf{F}$ along the normal

$$=\left\{\mathbf{F}\cdot\frac{\nabla f}{|\nabla f|}\right\}\frac{\nabla f}{|\nabla f|}=\frac{(\mathbf{F}\cdot\nabla f)}{|\nabla f|^2}\nabla f=\frac{(\mathbf{F}\cdot\nabla f)}{(\nabla f)^2}\nabla f.$$

Consequently the tangential component of $\mathbf{F}$ is

$$=\mathbf{F}-\frac{(\mathbf{F}\cdot\nabla f)}{(\nabla f)^2}\nabla f=\frac{(\nabla f\cdot\nabla f)\,\mathbf{F}-(\mathbf{F}\cdot\nabla f)\,\nabla f}{(\nabla f)^2}$$

$$=\frac{\nabla f\times(\mathbf{F}\times\nabla f)}{(\nabla f)^2}\quad[\because\ \mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\,\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}].$$

The vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are along normals to the two spheres at the point (4, −3, 2) and the angle θ between these two vectors is the angle of intersection of the two spheres at the point (4, −3, 2).

We have $\cos\theta = \dfrac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|} = \dfrac{8\,(12+9-2)}{2\sqrt{(16+9+4)}.4\sqrt{(9+9+1)}}$

$$= \frac{19}{\sqrt{(29)}.\sqrt{(19)}}$$

or $\theta = \cos^{-1}\sqrt{(19/29)}$.

Ex. 34. *Find the constants a and b so that the surface $ax^2 - byz = (a+2)\,x$ will be orthogonal to the surface $4x^2y + z^3 = 4$ at the point* (1, −1, 2).

Sol. The given surfaces are

$$f_1 \equiv ax^2 - byz - (a+2)\,x = 0 \qquad \ldots(1)$$

and $$f_2 \equiv 4x^2y + z^3 - 4 = 0 \qquad \ldots(2)$$

The point (1, −1, 2) obviously lies on the surface (2). It will also lie on the surface (1) if

$$a + 2b - (a+2) = 0 \quad \text{or} \quad 2b - 2 = 0 \quad \text{or} \quad b = 1.$$

Now grad $f_1 = [2ax - (a+2)]\,\mathbf{i} - bz\,\mathbf{j} - by\,\mathbf{k}$

and grad $f_2 = 8xy\,\mathbf{i} + 4x^2\,\mathbf{j} + 3z^2\,\mathbf{k}$.

Then $\mathbf{n}_1 =$ grad f_1 at the point $(1, -1, 2) = (a-2)\,\mathbf{i} - 2b\mathbf{j} + b\mathbf{k}$

and $\mathbf{n}_2 =$ grad f_2 at the point $(1, -1, 2) = -8\mathbf{i} + 4\mathbf{j} + 12\mathbf{k}$.

The vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are along the normals to the surfaces (1) and (2) at the point (1, −1, 2). These surfaces will intersect orthogonally at the point (1, −1, 2) if the vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are perpendicular *i.e.*, if $\mathbf{n}_1 \cdot \mathbf{n}_2 = 0$.

or $$-8\,(a-2) - 8b + 12b = 0 \quad \text{or} \quad b - 2a + 4 = 0 \qquad \ldots(3)$$

But $b = 1$, as already found.

Putting $b = 1$ in (3), we get $a = 5/2$. **Ans.** $a = 5/2,\ b = 1$.

Ex. 35. *If* $\mathbf{F}$ *and* f *are point functions, show that the components of the former tangential and normal to the level surface*

$$f = 0 \text{ are } \frac{\nabla f \times (\mathbf{F} \times \nabla f)}{(\nabla f)^2} \text{ and } \frac{(\mathbf{F} \cdot \nabla f)\,\nabla f}{(\nabla f)^2}.$$

Sol. The unit normal vector to the surface $f = 0$ is

$$= \frac{\nabla f}{|\nabla f|}$$

∴ The magnitude of the component of $\mathbf{F}$ along the normal

$$= \mathbf{F} \cdot \frac{\nabla f}{|\nabla f|}$$

§ 12. Invariance.

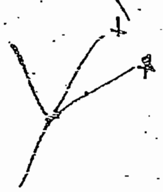
Theorem 1. *Show that under a rotation of rectangular axes, the origin remaining the same, the vector differential operator ∇ remains invariant.*

Proof. Let O be the fixed origin. Let Ox, Oy, Oz be one system of rectangular axes and Ox', Oy', Oz' be the other system of rectangular axes. Take **i**, **j**, **k** as unit vectors along Ox, Oy, Oz and **i**$'$, **j**$'$, **k**$'$ as unit vectors along Ox', Oy', Oz'. Let P be any point in space whose co-ordinates are (x, y, z) or (x', y', z') with respect to the two systems of axes. Let l_1, m_1, n_1 ; l_2, m_2, n_2 ; l_3, m_3, n_3 be the direction cosines of the lines Ox', Oy', Oz' with respect to the co-ordinate axes Ox, Oy, Oz.

The scheme of transformation will be as follows :

$$\left.\begin{aligned} x' &= l_1 x + m_1 y + n_1 z \\ y' &= l_2 x + m_2 y + n_2 z \\ z' &= l_3 x + m_3 y + n_3 z \end{aligned}\right\} \qquad \ldots(1)$$

Also we know that if l, m, n are the direction cosines of a line, then a unit vector along that line is $l\mathbf{i}+m\mathbf{j}+n\mathbf{k}$, where **i**, **j**, **k** are unit vectors along co-ordinate axes. Therefore



$$\left.\begin{aligned} \mathbf{i}' &= l_1\mathbf{i}+m_1\mathbf{j}+n_1\mathbf{k} \\ \mathbf{j}' &= l_2\mathbf{i}+m_2\mathbf{j}+n_2\mathbf{k} \\ \mathbf{k}' &= l_3\mathbf{i}+m_3\mathbf{j}+n_3\mathbf{k} \end{aligned}\right\} \qquad \ldots(2)$$

If V is any function (vector or scalar) of x, y, z, then

$$\frac{\partial V}{\partial x}=\frac{\partial V}{\partial x'}\frac{\partial x'}{\partial x}+\frac{\partial V}{\partial y'}\frac{\partial y'}{\partial x}+\frac{\partial V}{\partial z'}\frac{\partial z'}{\partial x}$$

$$\therefore \quad \frac{\partial}{\partial x}\equiv\frac{\partial x'}{\partial x}\frac{\partial}{\partial x'}+\frac{\partial y'}{\partial x}\frac{\partial}{\partial y'}+\frac{\partial z'}{\partial x}\frac{\partial}{\partial z'}.$$

But from (1), $\dfrac{\partial x'}{\partial x}=l_1$, $\dfrac{\partial y'}{\partial x}=l_2$, $\dfrac{\partial z'}{\partial x}=l_3$.

$$\left.\begin{aligned} \therefore \quad \frac{\partial}{\partial x} &\equiv l_1\frac{\partial}{\partial x'}+l_2\frac{\partial}{\partial y'}+l_3\frac{\partial}{\partial z'} \\ \text{Similarly} \quad \frac{\partial}{\partial y} &\equiv m_1\frac{\partial}{\partial x'}+m_2\frac{\partial}{\partial y'}+m_3\frac{\partial}{\partial z'} \\ \frac{\partial}{\partial z} &\equiv n_1\frac{\partial}{\partial x'}+n_2\frac{\partial}{\partial y'}+n_3\frac{\partial}{\partial z'}. \end{aligned}\right\} \qquad \ldots(3)$$

Multiplying the equations (3) by **i**, **j**, **k** respectively, adding and using the results (2), we get

$$\mathbf{i}\,\frac{\partial}{\partial x}+\mathbf{j}\,\frac{\partial}{\partial y}+\mathbf{k}\,\frac{\partial}{\partial z}\equiv\mathbf{i}'\,\frac{\partial}{\partial x'}+\mathbf{j}'\,\frac{\partial}{\partial y'}+\mathbf{k}'\,\frac{\partial}{\partial z'}.$$

Theorem 2. *If $\phi\,(x, y, z)$ is a scalar invariant with respect to a rotation of axes, then grad ϕ is a vector invariant under this transformation.*

Proof. First proceed exactly in the same manner as in theorem 1 and obtain the equations (1) and (2).

Now suppose the function $\phi\,(x, y, z)$ becomes $\phi'\,(x', y', z')$ after rotation of axes. Then by hypothesis $\phi\,(x, y, z)=\phi'\,(x', y', z')$.

By chain rule of differentiation, we have

$$\frac{\partial \phi}{\partial x}=\frac{\partial \phi'}{\partial x'}\,\frac{\partial x'}{\partial x}+\frac{\partial \phi'}{\partial y'}\,\frac{\partial y'}{\partial x}+\frac{\partial \phi'}{\partial z'}\,\frac{\partial z'}{\partial x}.$$

But from (1), $\dfrac{\partial x'}{\partial x}=l_1,\ \dfrac{\partial y'}{\partial x}=l_2,\ \dfrac{\partial z'}{\partial x}=l_3.$

$$\left.\begin{aligned}\therefore\quad \frac{\partial \phi}{\partial x}&=l_1\,\frac{\partial \phi'}{\partial x'}+l_2\,\frac{\partial \phi'}{\partial y'}+l_3\,\frac{\partial \phi'}{\partial z'}\\ \text{Similarly}\quad \frac{\partial \phi}{\partial y}&=m_1\,\frac{\partial \phi'}{\partial x'}+m_2\,\frac{\partial \phi'}{\partial y'}+m_3\,\frac{\partial \phi'}{\partial z'}\\ \text{and}\quad \frac{\partial \phi}{\partial z}&=n_1\,\frac{\partial \phi'}{\partial x'}+n_2\,\frac{\partial \phi'}{\partial y'}+n_3\,\frac{\partial \phi'}{\partial z'}\end{aligned}\right\}\quad\ldots(3)$$

Multiplying these equations by $\mathbf{i}, \mathbf{j}, \mathbf{k}$ respectively, adding and using the results (2), we get

$$\mathbf{i}\,\frac{\partial \phi}{\partial x}+\mathbf{j}\,\frac{\partial \phi}{\partial y}+\mathbf{k}\,\frac{\partial \phi}{\partial z}=\mathbf{i}'\,\frac{\partial \phi'}{\partial x'}+\mathbf{j}'\,\frac{\partial \phi'}{\partial y'}+\mathbf{k}'\,\frac{\partial \phi'}{\partial z'}$$

or $$\text{grad}\ \phi=\text{grad}\ \phi'.$$

Theorem 3. *If $\mathbf{V}\,(x, y, z)$ is a vector function invariant with respect to a rotation of axes, then div $\mathbf{V}$ is a scalar invariant under this transformation.*

Proof. First proceed exactly in the same manner as in theorems 1 and 2.

Now suppose the function $\mathbf{V}\,(x, y, z)$ becomes $\mathbf{V}'\,(x', y', z')$ after rotation of axes. Then by hypothesis

$$\mathbf{V}\,(x, y, z)=\mathbf{V}'\,(x', y', z').$$

By chain rule of differentiation, we have

$$\frac{\partial \mathbf{V}}{\partial x}=\frac{\partial \mathbf{V}'}{\partial x'}\,\frac{\partial x'}{\partial x}+\frac{\partial \mathbf{V}'}{\partial y'}\,\frac{\partial y'}{\partial x}+\frac{\partial \mathbf{V}'}{\partial z'}\,\frac{\partial z'}{\partial x}.$$

But from (1), $\dfrac{\partial x'}{\partial x}=l_1,\ \dfrac{\partial y'}{\partial x}=l_2,\ \dfrac{\partial z'}{\partial x}=l_3.$

$$\left.\begin{aligned} \therefore\quad \frac{\partial \mathbf{V}}{\partial x} &= l_1 \frac{\partial \mathbf{V}'}{\partial x'} + l_2 \frac{\partial \mathbf{V}'}{\partial y'} + l_3 \frac{\partial \mathbf{V}'}{\partial z'} \\ \text{Similarly}\quad \frac{\partial \mathbf{V}}{\partial y} &= m_1 \frac{\partial \mathbf{V}'}{\partial x'} + m_2 \frac{\partial \mathbf{V}'}{\partial y'} + m_3 \frac{\partial \mathbf{V}'}{\partial z'} \\ \text{and}\quad \frac{\partial \mathbf{V}}{\partial z} &= n_1 \frac{\partial \mathbf{V}'}{\partial x'} + n_2 \frac{\partial \mathbf{V}'}{\partial y'} + n_3 \frac{\partial \mathbf{V}'}{\partial z'} \end{aligned}\right\} \quad \ldots(3)$$

Taking dot product of these three equations by **i**, **j**, **k** respectively, adding and using the results (2), we get

$$\mathbf{i}\cdot\frac{\partial \mathbf{V}}{\partial x} + \mathbf{j}\cdot\frac{\partial \mathbf{V}}{\partial y} + \mathbf{k}\cdot\frac{\partial \mathbf{V}}{\partial z} = \mathbf{i}'\cdot\frac{\partial \mathbf{V}'}{\partial x'} + \mathbf{j}'\cdot\frac{\partial \mathbf{V}'}{\partial y'} + \mathbf{k}'\cdot\frac{\partial \mathbf{V}'}{\partial z'}$$

or $$\text{div } \mathbf{V} = \text{div } \mathbf{V}'.$$

Theorem 4. *If* $\mathbf{V}\ (x, y, z)$ *is a vector function invariant under a rotation of axes, then curl* $\mathbf{V}$ *is a vector invariant under this rotation.*

Proof. Proceed exactly in the same manner as in theorem 3. In place of taking dot product of equations (3), take cross product. We shall get

$$\mathbf{i}\times\frac{\partial \mathbf{V}}{\partial x} + \mathbf{j}\times\frac{\partial \mathbf{V}}{\partial y} + \mathbf{k}\times\frac{\partial \mathbf{V}}{\partial z} = \mathbf{i}'\times\frac{\partial \mathbf{V}'}{\partial x'} + \mathbf{j}'\times\frac{\partial \mathbf{V}'}{\partial y'} + \mathbf{k}'\times\frac{\partial \mathbf{V}'}{\partial z'}$$

or $$\text{curl } \mathbf{V} = \text{curl } \mathbf{V}'.$$

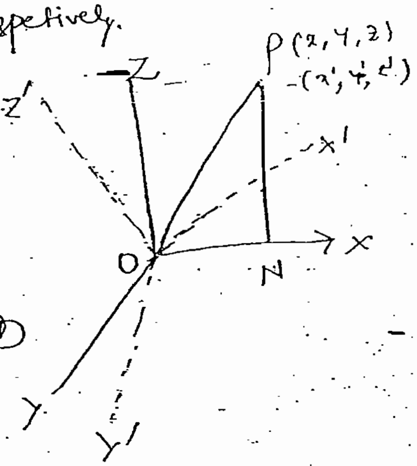
To change the direction of of co-ordinate axes without changing the origin

Let OX, OY, OZ and OX', OY', OZ' be two sets of co-ordinate axes through the common origin O.

Let the direction cosines of OX', OY' and OZ' be l_1, m_1, n_1, l_2, m_2, n_2 and l_3, m_3, n_3 respectively referred to OX, OY and OZ.

Then the direction cosines of OX, OY, OZ referred to OX', OY', OZ' are evidently $l_1\ l_2\ l_3$; $m_1\ m_2, m_3$ and n_1, n_2, n_3 respectively.

Let the co-ordinates of P be (x, y, z) and (x', y', z') referred to the original axes OX, OY, OZ and the new axes OX', OY', OZ' respetively.



From P draw PN perpendicular to OX.

Then $x = ON$ = projection of OP on OX ——— (1)

Now d.c.s of OX referred to the new axes are l_1, l_2, l_3 and the co-ordinates of P referred to the new axes are (x', y', z').

∴ from (1), we have

$$x = l_1(x' - 0) + l_2(y' - 0) + l_3(z' - 0).$$

$\Rightarrow x = l_1 x' + l_2 y' + l_3 z'$

Similarly, $y = m_1 x' + m_2 y' + m_3 z'$

$$\left.\begin{aligned} x &= l_1 x' + l_2 y' + l_3 z' \\ y &= m_1 x' + m_2 y' + m_3 z' \\ z &= n_1 x' + n_2 y' + n_3 z' \end{aligned}\right\} \longleftarrow (2)$$

Multiplying these relations by l_1, m_1, n_1 respectively and adding we get

$$l_1 x + m_1 y + n_1 z = x' \Sigma l_1^2 + y' \Sigma l_1 l_2 + z' \Sigma l_1 l_3$$

$$= x'(1) + y'(0) + z'(0)$$

$$= x' \quad (\because \Sigma l_1^2 = 1, \ \Sigma l_1 l_2 = 0 = \Sigma l_1 l_3)$$

i.e., $x' = l_1 x + m_1 y + n_1 z$

Similarly

$$\left.\begin{aligned} x' &= l_1 x + m_1 y + n_1 z \\ y' &= l_2 x + m_2 y + n_2 z \\ z' &= l_3 x + m_3 y + n_3 z \end{aligned}\right\} \longleftarrow (3)$$

The relations (2) express the old co-ordinates x, y, z in terms of the new co-ordinates x', y', z' and the relations (3) express x', y', z' in terms of x, y, z.

The relations are also written conveniently with the help of adjoining table.

	x	y	z
x'	l_1	m_1	n_1
y'	l_2	m_2	n_2
z'	l_3	m_3	n_3

In this table the horizontal and vertical lines denote the direction cosines of mutually ~~r~~ lar axes.

SET-IV

* VECTOR INTEGRAL *

Introduction :— Integration is the inverse operation of differentiation.

Let $\vec{F}(t)$ be a differentiable vector function of a scalar variable 't' and let $\frac{d}{dt}\vec{F}(t) = \vec{f}(t)$

then $\int \vec{f}(t)\,dt = \vec{F}(t)$ — (1) $\vec{F}(t)$ is called the indefinite integral of $f(t)$ with respect to 't'.

- The function $\vec{f}(t)$ to be integrated is called the integrand.
- If $\vec{c}$ is any arbitrary constant vector independent of 't', then $\frac{d}{dt}\{\vec{F}(t) + \vec{c}\} = \vec{f}(t)$

This is equivalent to $\int \vec{f}(t)\,dt = \vec{F}(t) + \vec{c}$ — (2)

from (3), it is obvious that the integral $\vec{F}(t)$ of $\vec{f}(t)$ is indefinite to the extent of an additive arbitrary constant 'c'. Therefore $\vec{F}(t)$ is called the indefinite integral of $\vec{f}(t)$. The constant vector 'c' is called the constant of integration. It can be determined if we are given some initial conditions.

Note: If $\vec{f}(t) = f_1(t)\,\hat{i} + f_2(t)\,\hat{j} + f_3(t)\,\hat{k}$,

then $\int \vec{f}(t)\,dt = \hat{i}\int f_1(t)\,dt + \hat{j}\int f_2(t)\,dt + \hat{k}\int f_3(t)\,dt + C$.

If $\frac{d}{dt}\vec{F}(t) = \vec{f}(t)$ for all t in the interval $[a, b]$, then the definite integral between the limits $t=a$ and $t=b$ can be written as

$$\int_a^b \vec{f}(t)\,dt = \int_a^b \left\{\frac{d}{dt}\vec{F}(t)\right\} dt$$

$$= \left[\vec{F}(t) + c\right]_a^b$$

$$= F(b) - F(a)$$

Some Standard Results:

→ we have $\frac{d}{dt}(\vec{r}.\vec{s}) = \frac{d\vec{r}}{dt}.\vec{s} + \vec{r}.\frac{d\vec{s}}{dt}$

$$\therefore \int\left(\frac{d\vec{r}}{dt}.\vec{s} + \vec{r}.\frac{d\vec{s}}{dt}\right)dt = \vec{r}.\vec{s} + c.$$

Hence c is a scalar constant; since the integrand is a scalar.

→ we have $\frac{d}{dt}(\vec{r}^2) = 2\vec{r}.\frac{d\vec{r}}{dt}$.

$$\therefore \int\left(2\vec{r}.\frac{d\vec{r}}{dt}\right)dt = \vec{r}^2 + c.$$ where c is a scalar constant

→ $\frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)^2 = 2\frac{d\vec{r}}{dt}.\frac{d^2\vec{r}}{dt^2}$.

$$\therefore \int\left(2\frac{d\vec{r}}{dt}.\frac{d^2\vec{r}}{dt^2}\right)dt = \left(\frac{d\vec{r}}{dt}\right)^2 + c$$

where c is a scalar constant

→ we have $\frac{d}{dt}(\vec{r}\times\vec{s}) = \frac{d\vec{r}}{dt}\times\vec{s} + \vec{r}\times\frac{d\vec{s}}{dt}$.

$$\therefore \int\left(\frac{d\vec{r}}{dt}\times\vec{s} + \vec{r}\times\frac{d\vec{s}}{dt}\right)dt = \vec{r}\times\vec{s} + c.$$

Here the constant c is a vector quantity since the integrand is also a vector quantity.

1(i)

→ If $\vec{a}$ is constant vector, we have

$$\frac{d}{dt}(\vec{a}\times\vec{r}) = \vec{a}\times\frac{d\vec{r}}{dt}$$

$$\therefore \int\left(\vec{a}\times\frac{d\vec{r}}{dt}\right)dt = \vec{a}\times\int\frac{d\vec{r}}{dt}\,dt$$

$$= \vec{a}\times\vec{r}+c.$$

→ Now $\frac{d}{dt}\left(\vec{r}\times\frac{d\vec{r}}{dt}\right) = \frac{d\vec{r}}{dt}\times\frac{d\vec{r}}{dt}+\vec{r}\times\frac{d^2\vec{r}}{dt^2}$

$$= \vec{r}\times\frac{d^2\vec{r}}{dt^2}.$$

$$\therefore \int\left(\vec{r}\times\frac{d^2\vec{r}}{dt^2}\right)dt = \vec{r}\times\frac{d\vec{r}}{dt}+c.$$

→ If $r=|\vec{r}|$, then $\frac{d}{dt}\left(\frac{\vec{r}}{r}\right) = \frac{1}{r}\frac{d\vec{r}}{dt}-\frac{1}{r^2}\frac{dr}{dt}\vec{r}$

$$\therefore \int\left(\frac{1}{r}\frac{d\vec{r}}{dt}-\frac{1}{r^2}\frac{dr}{dt}\vec{r}\right)dt = \frac{\vec{r}}{r}+c.$$

→ If c is a scalar constant then $\int c\vec{r}\,dt = c\int\vec{r}\,dt$

→ If $\vec{r}$ and $\vec{s}$ are any two vector functions of a scalar t, then $\int(\vec{r}+\vec{s})\,dt = \int\vec{r}\,dt+\int\vec{s}\,dt$

→ Evaluate $\int_0^1\left(e^t\hat{i}+e^{-2t}\hat{j}+t\hat{k}\right)dt$

Sol^n: $\int_0^1\left(e^t\hat{i}+e^{-2t}\hat{j}+t\hat{k}\right)dt$

$$= \hat{i}\int_0^1 e^t\,dt+\hat{j}\int_0^1 e^{-2t}\,dt+\hat{k}\int_0^1 t\,dt$$

$$= \hat{i}\left[e^t\right]_0^1+\hat{j}\left[-\frac{1}{2}e^{-2t}\right]_0^1+\hat{k}\left[\frac{t^2}{2}\right]_0^1$$

$$= (e-1)\hat{i}-\frac{1}{2}(e^{-2}-1)\hat{j}+\frac{1}{2}\hat{k}.$$

→ Evaluate $\int_2^3 \vec{f}\cdot\frac{d\vec{f}}{dt}\,dt$ if $\vec{f}(2)=2\hat{i}-\hat{j}+2\hat{k}$ and $\vec{f}(3)=4\hat{i}-2\hat{j}+3\hat{k}$.

Sol: We know that $\int\left(2\vec{f}\cdot\frac{d\vec{f}}{dt}\right)dt=\vec{f}^2+c$.

$$\therefore \int_2^3\left(\vec{f}\cdot\frac{d\vec{f}}{dt}\right)dt=\frac{1}{2}\left[\vec{f}^2\right]_2^3$$

$$=\frac{1}{2}\left[\vec{f}^2(3)-\vec{f}^2(2)\right]$$

$$=\frac{1}{2}\left[(4\hat{i}-2\hat{j}+3\hat{k})^2-(2\hat{i}-\hat{j}+2\hat{k})^2\right]$$

$$=\frac{1}{2}\left[(16+4+9)-(4+1+4)\right]$$

$$=\frac{1}{2}(20)=10$$

→ Find the value of $\frac{d\vec{r}}{dt}$ by integrating $\frac{d^2\vec{r}}{dt^2}=-n^2\vec{r}$.

Sol: The given equation is $\frac{d^2\vec{r}}{dt^2}=-n^2\vec{r}$.

Taking the dot product with $2\frac{d\vec{r}}{dt}$ both sides and integrating, we have

$$\int\left(2\frac{d\vec{r}}{dt}\cdot\frac{d^2\vec{r}}{dt^2}\right)dt=-n^2\int 2\vec{r}\cdot\frac{d\vec{r}}{dt}$$

$$\left(\frac{d\vec{r}}{dt}\right)^2=n^2\vec{r}^2+c.$$

where c is any constant vector

→ If $\vec{f}(t)=5t^2\hat{i}+t\hat{j}-t^3\hat{k}$, find $\int_1^2\left(\vec{f}\times\frac{d^2\vec{f}}{dt^2}\right)dt$.

Sol: $\int_1^2\left(\vec{f}\times\frac{d^2\vec{f}}{dt^2}\right)dt=\left[\vec{f}\times\frac{d\vec{f}}{dt}\right]_1^2$

Given $\vec{f}(t) = 5t^2\hat{i} + t\hat{j} - t^3\hat{k}$ 1(ii)

$$\therefore \frac{d\vec{f}(t)}{dt} = 10t\hat{i} + \hat{j} - 3t^2\hat{k}$$

$$\vec{f} \times \frac{d\vec{f}}{dt} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 5t^2 & t & -t^3 \\ 10t & 1 & -3t^2 \end{vmatrix}$$

$$= -2t^3\hat{i} + 5t^4\hat{j} - 5t^2\hat{k}$$

$$\therefore \left[\vec{f} \times \frac{d\vec{f}}{dt}\right]_1^2 = \left[-2t^3\hat{i} + 5t^4\hat{j} - 5t^2\hat{k}\right]_1^2$$

$$= -14\hat{i} + 75\hat{j} - 15\hat{k}$$

$$\therefore \int_1^2 \left(\vec{f} \times \frac{d^2\vec{f}}{dt^2}\right) dt = -14\hat{i} + 75\hat{j} - 15\hat{k}$$

→ If $\vec{F}(t) = (t - t^2)\hat{i} + 2t^3\hat{j} - 3\hat{k}$ find $\int_1^2 F(t)\,dt$

→ If $\vec{F}(t) = t\hat{i} + (t^2 - 2t)\hat{j} + (3t^2 + 3t^3)\hat{k}$ find $\int_0^1 f(t)\,dt$

→ If $\vec{A} = t\hat{i} - t^2\hat{j} + (t-1)\hat{k}$ and $\vec{B} = 2t^2\hat{i} + 6t\hat{k}$ find (i) $\int_1^2 (\vec{A}\cdot\vec{B})\,dt$ (ii) $\int_0^2 (\vec{A}\times\vec{B})\,dt$.

→ If $\vec{A} = t\hat{i} - 3\hat{j} + 2t\hat{k}$, $\vec{B} = \hat{i} - 2\hat{j} + 2\hat{k}$; $\vec{C} = 3\hat{i} + t\hat{j}$ find (i) $\int_1^2 [ABC]\,dt$ (ii) $\int_1^2 [\vec{A} \times (\vec{B} \times \vec{C})]\,dt$

→ If $\frac{d^2\vec{r}}{dt^2} = 6t\hat{i} - 24t^2\hat{j} + 4\sin t\,\hat{k}$ find $\vec{r}$. given that $\vec{r} = 2\hat{i} + \hat{j}$ and $\frac{d\vec{r}}{dt} = -\hat{i} - 3\hat{k}$ at $t = 0$.

→ Find $\vec{F}(t)$, given $\frac{d\vec{F}}{dt} = 12\cos 2t\,\hat{i} - 8\sin 2t\,\hat{j} + 16t\,\hat{k}$ and $\vec{F}(0) = 0$

→ If $\vec{a}, \vec{b}$ and n are constants and

$$\vec{r} = \vec{a}\cos nt + \vec{b}\sin nt.$$

Show that $\frac{d^2\vec{r}}{dt^2} + n^2\vec{r} = 0.$

→ Given $\frac{d^2\vec{r}}{dt^2} = -k^2\vec{r}$, Show that $\left(\frac{d\vec{r}}{dt}\right)^2 = c - k^2\vec{r}^2$

→ Find the value of $\vec{r}$ satisfying the equation $\frac{d^2\vec{r}}{dt^2} = \vec{a}t + \vec{b}$, where $\vec{a}, \vec{b}$ are constant vectors.

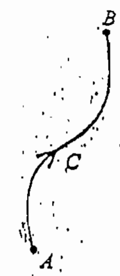
1 (१११)

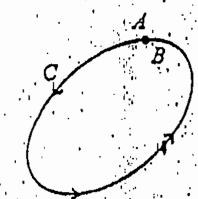
Set – IV

Green's, Gauss's and Stoke's Theorems

§ 1. Some preliminary concepts.

Oriented curve. Suppose C is a curve in space. Let us orient C by taking one of the two directions along C as the *positive direction*; the opposite direction along C is then called the *negative direction*. Suppose A is the initial point and B the terminal point of C under the chosen orientation. In case these two points coincide, the curve C is called a *closed curve*.

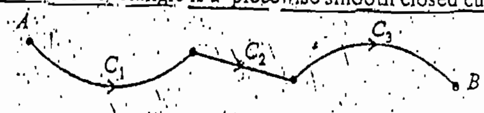




Oriented closed curve

Smooth curve. Let $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$, where $\mathbf{r}(t)$ is the position vector of (x, y, z), be the parametric representation of a curve C joining the points A and B, where $t=t_1$ and $t=t_2$ respectively. We know that $\frac{d\mathbf{r}}{dt}$ is a tangent vector to this curve at the point $\mathbf{r}$. Suppose the function $\mathbf{r}'(t)$ is continuous and has a continuous first derivative not equal to zero vector for all values of t under consideration. Then the curve C possesses a unique tangent at each of its points. A curve satisfying these assumptions is called a smooth curve.

A curve C is said to be piecewise smooth if it is composed of a finite number of smooth curves. The curve C in the adjoining figure is piecewise smooth as it is composed of three smooth curves C_1, C_2 and C_3. The circle is a smooth closed curve while the curve consisting of the four sides of a rectangle is a piecewise smooth closed curve.

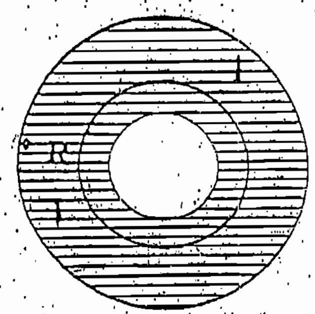


Piecewise smooth curve

Smooth surface. Suppose S is a surface which has a unique normal at each of its points and the direction of this normal depends continuously on the points of S. Then S is called a smooth surface.

If a surface S is not smooth but can be subdivided into a finite number of smooth srufaces, then it is called a piecewise smooth surface. The surface of a sphere is smooth while the surface of a cube is piecewise smooth.

Classification of regions. A region R in which every closed curve can be contracted to a point without passing out of the region is called a simply connected region. Otherwise the region R is multiply-connected. The region interior to a circle is a simply-connected plane region. The region interior to a sphere is a simply-connected region in space. The region between two concentric circles lying in the same plane is a multiply connected plane region.



If we take a closed curve in this region surrounding the inner circle, then it cannot be contracted to a point without passing out of the region. Therefore the region is not simply-connected. However the region between two concentric shperes is a simply-connected region in space. The region between two infinitely long coaxial cylinders is a multiply-connected region in space.

§ 2. Line Integrals. *Any integral which is to be evaluated along a curve is called a line integral.*

Suppose $\mathbf{r}(t) = x(t)\,\mathbf{i}+y(t)\,\mathbf{j}+z(t)\,\mathbf{k}$, where $\mathbf{r}(t)$ is the position vector of (x, y, z) *i.e.*, $\mathbf{r}(t) = x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}$, defines a piecewise smooth curve joining two points A and B. Let $t=t_1$ at A and $t=t_2$ at B. Suppose $\mathbf{F}(x, y, z) = F_1\,\mathbf{i}+F_2\,\mathbf{j}+F_3\,\mathbf{k}$ is a vector point function defined and continuous along C. If s denotes the arc length of the curve C, then $\frac{d\mathbf{r}}{ds} = \mathbf{t}$ is a unit vector along the tangent to the curve C at the point $\mathbf{r}$.

The component of the vector $\mathbf{F}$ along this tangent is $\mathbf{F}\cdot\frac{d\mathbf{r}}{ds}$. The integral of $\mathbf{F}\cdot\frac{d\mathbf{r}}{ds}$ along C from A to B written as

$$\int_A^B \left[\mathbf{F}\cdot\frac{d\mathbf{r}}{ds}\right] ds = \int_A^B \mathbf{F}\cdot d\mathbf{r} = \int_C \mathbf{F}\cdot d\mathbf{r}$$

is an example of a *line integral*. It is called *the tangent line integral* of $\mathbf{F}$ along C.

Since $\mathbf{r} = x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}$, therefore, $d\mathbf{r} = dx\,\mathbf{i}+dy\,\mathbf{j}+dz\,\mathbf{k}$.

$$\therefore\ \mathbf{F}\cdot d\mathbf{r} = (F_1\mathbf{i}+F_2\mathbf{j}+F_3\mathbf{k})\cdot(dx\,\mathbf{i}+dy\,\mathbf{j}+dz\,\mathbf{k})$$
$$= F_1\,dx+F_2\,dy+F_3\,dz.$$

Therefore in components form the above line integral is written as

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (F_1\,dx+F_2\,dy+F_3\,dz).$$

The parametric equations of the curve C are $x=x(t), y=y(t)$ and $z=z(t)$.

Therefore we may write

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{t_1}^{t_2}\left[F_1\frac{dx}{dt}+F_2\frac{dy}{dt}+F_3\frac{dz}{dt}\right] dt.$$

Circulation. *If C is a simple closed curve (i.e. a curve which does not intersect itself anywhere), then the tangent line integral of* $\mathbf{F}$ *around C is called the circulation of* $\mathbf{F}$ *about C. It is often denoted by*

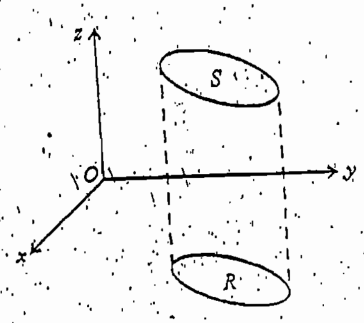
$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C (F_1\,dx+F_2\,dy+F_3\,dz).$$

Work done by a Force. Suppose a force $\mathbf{F}$ acts upon a particle. Let the particle be displaced along a given path C in space. If $\mathbf{r}$ denotes the position vector of a point on C, then $\frac{d\mathbf{r}}{ds}$ is a unit vector along the tangent to C at the point $\mathbf{r}$ in the direction of s increasing. The component of

Note 2. Important. In order to evaluate surface integrals it is convenient to express them as double integrals taken over the orthogonal projection of the surface S on one of the coordinate planes. But this is possible only if any line perpendicular to the co-ordinate plane chosen meets the surface S in no more than one point. If the surface S does not satisfy this condition, then it can be sub-divided into surfaces which do satisfy this condition.

Suppose the surface S is such that any line perpendicular to the xy-plane meets S in no more than one point. Then the equation of the surface S can be written in the form

$$z = h(x, y).$$



Let R be the orthogonal projection of S on the xy-plane. If γ is the acute angle which the undirected normal $\mathbf{n}$ at $P(x, y, z)$ to the surface S makes with z-axis, then it can be shown that

$$\cos\gamma \, dS = dx\, dy,$$

where dS is the small element of area of surface S at the point P.

Therefore $dS = \dfrac{dx\,dy}{\cos\gamma} = \dfrac{dx\,dy}{|\mathbf{n}\cdot\mathbf{k}|}$, where $\mathbf{k}$ is the unit vector along z-axis.

Hence $\displaystyle\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iint_R \mathbf{F}\cdot\mathbf{n}\, \frac{dx\,dy}{|\mathbf{n}\cdot\mathbf{k}|}$.

Thus the surface integral on S can be evaluated with the help of a double integral integrated over R.

§ 4. Volume Integrals.

Suppose V is a volume bounded by a surface S. Suppose $f(x, y, z)$ is a single valued function of position defined over V. Subdivide the volume V into n elements of volumes $\delta V_1, \delta V_2, \ldots, \delta V_n$. In each part δV_k we choose an arbitrary point P_k whose co-ordinates are (x_k, y_k, z_k). We define $f(P_k) = f(x_k, y_k, z_k)$.

Form the sum

$$\sum_{k=1}^{n} f(P_k)\, \delta V_k.$$

Now take the limit of this sum as $n \to \infty$ in such a way that the largest of the volumes δV_k approaches zero. This limit, if it exists, is called the volume integral of $f(x, y, z)$ over V and is denoted by

$$\iiint_V f(x, y, z)\, dV.$$

It can be shown that if the surface is piecewise smooth and the function $f(x, y, z)$ is continuous over V, then the above limit exists *i.e.*, is independent of the choice of sub-divisions and points P_k.

If we subdivide the volume V into small cuboids by drawing lines parallel to the three co-ordinates axes, then $dV = dx\, dy\, dz$ and the above volume integral becomes

$$\iiint_V f(x, y, z)\, dx\, dy\, dz.$$

If $\mathbf{F}(x, y, z)$ is a vector function, then

$$\iiint_V \mathbf{F}\, dV$$

is also an example of a volume integral.

Solved Examples

Ex. 1. *Evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$, *where* $\mathbf{F} = x^2\mathbf{i} + y^3\mathbf{j}$ *and curve* C *is the arc of the parabola* $y = x^2$ *in the* x-y *plane from* $(0, 0)$ *to* $(1, 1)$.

Sol. We shall illustrate two methods for the solution of such a problem.

Method 1. The curve C is the parabola $y = x^2$ from $(0, 0)$ to $(1, 1)$.

Let $x = t$; then $y = t^2$. If $\mathbf{r}$ is the position vector of any point (x, y) on C, then

$$\therefore \quad \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\,\mathbf{j}.$$

Also in terms of t, $\mathbf{F} = t^2\mathbf{i} + t^6\mathbf{j}$.

At the point $(0, 0)$, $t = x = 0$. At the point $(1, 1)$, $t = 1$.

$$\therefore \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \left(\mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\right) dt = \int_0^1 (t^2\mathbf{i} + t^6\mathbf{j})\cdot(\mathbf{i} + 2t\,\mathbf{j})\, dt$$

$$= \int_0^1 (t^2 + 2t^7)\, dt = \left[\frac{t^3}{3} + \frac{2t^8}{8}\right]_0^1 = \frac{1}{3} + \frac{1}{4} = \frac{7}{12}.$$

Method 2. In the xy-plane we have $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$.

$$\therefore \quad d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j}.$$

Therefore $\mathbf{F}\cdot d\mathbf{r} = (x^2\mathbf{i} + y^3\mathbf{j})\cdot(dx\,\mathbf{i} + dy\,\mathbf{j}) = x^2\, dx + y^3\, dy$.

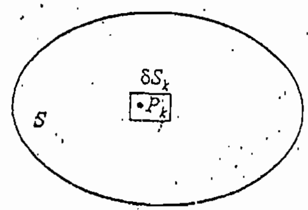
force $\mathbf{F}$ along tangent to C is $\mathbf{F}\cdot\frac{d\mathbf{r}}{ds}$. Therefore the work done by $\mathbf{F}$ during a small displacement ds of the particle along C is $\left[\mathbf{F}\cdot\frac{d\mathbf{r}}{ds}\right]ds$ *i.e.*, $\mathbf{F}\cdot d\mathbf{r}$. The total work W done by $\mathbf{F}$ in this displacement along C, is given by the line integral

$$W=\int_C \mathbf{F}\cdot d\mathbf{r},$$

the integration being taken in the sense of the displacement.

§ 3. Surface Integrals.

Any integral which is to be evaluated over a surface is called a surface integral.



Suppose S is a surface of finite area. Suppose $f(x, y, z)$ is a single valued function of position defined over S. Subdivide the area S into n elements of areas $\delta S_1, \delta S_2, \ldots, \delta S_n$. In each part δS_k we choose an arbitrary point P_k whose coordinates are (x_k, y_k, z_k).

We define

$$f(P_k)=f(x_k, y_k, z_k).$$ Form the sum

$$\sum_{k=1}^{n} f(P_k)\,\delta S_k.$$

Now take the limit of this sum as $n\to\infty$ in such a way that the largest of the areas δS_k approaches zero. This limit if it exists, is called the *surface integral* of $f(x, y, z)$ over S and is denoted by

$$\iint_S f(x, y, z)\, dS.$$

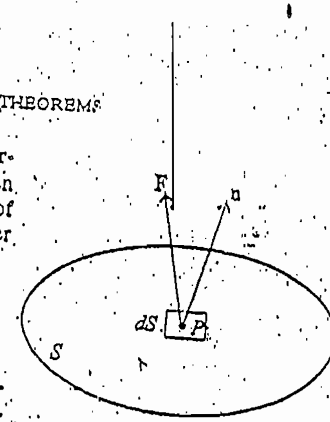
It can be shown that if the surface S is piecewise smooth and the function $f(x, y, z)$ is continuous over S, then the above limit exists *i.e.*, is independent of the choice of sub-division and points P_k.

Flux. Suppose S is a piecewise smooth surface and

$$\mathbf{F}(x, y, z)$$

is a vector function of position defined and continuous over S. Let P be any point on the surface S and let $\mathbf{n}$ be the unit vector at P in the

direction of outward drawn normal to the surface S at P. Then $\mathbf{F}\cdot\mathbf{n}$ is the normal component of $\mathbf{F}$ at P. The integral of $\mathbf{F}\cdot\mathbf{n}$ over S is



$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS.$$

It is called the *flux* of $\mathbf{F}$ over S.

Let us associate with the differential of surface area dS a vector $d\mathbf{S}$ (called *vector area*) whose magnitude is dS and whose direction is that of $\mathbf{n}$. Then $d\mathbf{S}=\mathbf{n}\,dS$. Therefore we can write

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS=\iint_S \mathbf{F}\cdot d\mathbf{S}.$$

Suppose the outward drawn normal to the surface S at P makes angles α, β, γ with the positive directions of x, y and z axes respectively. If l, m, n are the direction cosines of the outward drawn normal, then

$$l=\cos\alpha,\ m=\cos\beta,\ n=\cos\gamma.$$

Also $\mathbf{n}=\cos\alpha\,\mathbf{i}+\cos\beta\,\mathbf{j}+\cos\gamma\,\mathbf{k}=l\,\mathbf{i}+m\,\mathbf{j}+n\,\mathbf{k}.$

Let $\mathbf{F}(x, y, z)=F_1\mathbf{i}+F_2\mathbf{j}+F_3\mathbf{k}$. Then

$$\mathbf{F}\cdot\mathbf{n}=F_1\cos\alpha+F_2\cos\beta+F_3\cos\gamma=F_1 l+F_2 m+F_3 n.$$

Therefore we can write

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS=\iint_S (F_1\cos\alpha+F_2\cos\beta+F_3\cos\gamma)\, dS$$

$$=\iint_S (F_1\, dy\, dz+F_2\, dz\, dx+F_3\, dx\, dy),$$ if we define

$$\iint_S F_1\cos\alpha\, dS=\iint_S F_1\, dy\, dz,$$

$$\iint_S F_2\cos\beta\, dS=\iint_S F_2\, dz\, dx,$$

$$\iint_S F_3\cos\gamma\, dS=\iint_S F_3\, dx\, dy.$$

Note 1. Other examples of surface integrals are

$$\iint_S f\,\mathbf{n}\, dS,\quad \iint_S \mathbf{F}\times d\mathbf{S}$$

where $f(x, y, z)$ is a scalar function of position.

$\therefore \quad \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (x^2\,dx + y^3\,dy)$.

Now along the curve C, $y = x^2$. Therefore $dy = 2x\,dx$.

$$\therefore \quad \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{x=0}^{1} [x^2\,dx + x^6\,(2x)\,dx]$$

$$= \int_0^1 (x^2 + 2x^7)\,dx = \left[\frac{x^3}{3} + \frac{2x^8}{8}\right]_0^1 = \frac{7}{12}.$$

Ex. 2. *Evaluate* $\int_C \mathbf{F} \cdot d\mathbf{r}$, *where* $\mathbf{F} = (x^2 - y^2)\,\mathbf{i} + xy\,\mathbf{j}$ *and curve C is the arc of the curve* $y = x^3$ *from* $(0, 0)$ *to* $(2, 8)$.

Sol. The curve C is the curve $y = x^3$ from $(0, 0)$ to $(2, 8)$. Let $x = t$, then $y = t^3$. If $\mathbf{r}$ is the position vector of any point (x, y) on C, then

$$\mathbf{r}(t) = x\,\mathbf{i} + y\,\mathbf{j} = t\,\mathbf{i} + t^3\,\mathbf{j},$$

$$\therefore \quad \frac{d\mathbf{r}}{dt} = \mathbf{i} + 3t^2\,\mathbf{j}.$$

Also in terms of t, $\mathbf{F} = (t^2 - t^6)\,\mathbf{i} + t^4\,\mathbf{j}$.

At the point $(0, 0)$ $t = x = 0$. At the point $(2, 8)$, $t = 2$.

$$\therefore \quad \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \left(\mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\right) dt = \int_0^2 [(t^2 - t^6)\,\mathbf{i} + t^4\,\mathbf{j}] \cdot (\mathbf{i} + 3t^2\,\mathbf{j})\,dt$$

$$= \int_0^2 [(t^2 - t^6) + 3t^6]\,dt = \int_0^2 [t^2 + 2t^6]\,dt$$

$$= \left[\frac{t^3}{3} + \frac{2t^7}{7}\right]_0^2 = \left[\frac{8}{3} + \frac{256}{7}\right] = \frac{824}{21}.$$

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Ex. 3. *If* $\mathbf{F} = 3xy\,\mathbf{i} - y^2\,\mathbf{j}$, *evaluate* $\int_C \mathbf{F} \cdot d\mathbf{r}$, *where C is the curve in the xy-plane,* $y = 2x^2$, *from* $(0, 0)$ *to* $(1, 2)$.

[Calicut 1983 ; Kanpur 78 ; Agra 76 ; Garhwal 85]

Sol. The parametric equations of the parabola $y = 2x^2$ can be taken as

$$x = t,\; y = 2t^2.$$

At the point $(0, 0)$, $x = 0$ and so $t = 0$. Again at the point $(1, 2)$, $x = 1$ and so $t = 1$.

Now $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (3xy\,\mathbf{i} - y^2\,\mathbf{j}) \cdot (dx\,\mathbf{i} + dy\,\mathbf{j})$

[$\because \mathbf{r} = x\mathbf{i} + y\mathbf{j}$, so that $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j}$]

$$= \int_C (3xy\,dx - y^2\,dy) = \int_{t=0}^{1} \left(3xy\frac{dx}{dt} - y^2\frac{dy}{dt}\right) dt$$

$$= \int_0^1 (3 \cdot t \cdot 2t^2 \cdot 1 - 4t^4 \cdot 4t)\,dt$$

[$\because x = t,\; y = 2t^2$ so that $dx/dt = 1$ and $dy/dt = 4t$]

$$= \int_0^1 (6t^3 - 16t^5)\,dt = \left[6 \cdot \frac{t^4}{4} - 16 \cdot \frac{t^6}{6}\right]_0^1$$

$$= \frac{6}{4} - \frac{16}{6} = \frac{3}{2} - \frac{8}{3} = -\frac{7}{6}.$$

Ex. 4. *Evaluate* $\int_C \mathbf{F} \cdot d\mathbf{r}$ *where* $\mathbf{F}$ *is* $x^2y^2\,\mathbf{i} + y\,\mathbf{j}$ *and C is* $y^2 = 4x$ *in the xy-plane from* $(0, 0)$ *to* $(4, 4)$. (Agra 1986, Kanpur 77)

Sol. In the xy-plane, we have $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$ so that $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j}$.

$$\therefore \quad \mathbf{F} \cdot d\mathbf{r} = (x^2y^2\,\mathbf{i} + y\,\mathbf{j}) \cdot (dx\,\mathbf{i} + dy\,\mathbf{j})$$

$$= x^2y^2\,dx + y\,dy. \qquad [\because \mathbf{i} \cdot \mathbf{i} = 1,\; \mathbf{i} \cdot \mathbf{j} = 0 \text{ etc.}]$$

$\therefore \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (x^2y^2\,dx + y\,dy)$, where C is the given curve $y^2 = 4x$ from $(0, 0)$ to $(4, 4)$

$$= \int_C x^2y^2\,dx + \int_C y\,dy = \int_{x=0}^{4} x^2(4x)\,dx + \int_{y=0}^{4} y\,dy \qquad [\because y^2 = 4x]$$

$$= 4\left[\frac{1}{4}x^4\right]_0^4 + \left[\frac{1}{2}y^2\right]_0^4 = 256 + 8 = 264.$$

Ex. 5. *Integrate the function* $\mathbf{F} = x^2\,\mathbf{i} - xy\,\mathbf{j}$ *from the point* $(0, 0)$ *to* $(1, 1)$ *along parabola* $y^2 = x$.

[Rohilkhand 1978]

Sol. Here the parabola $y^2 = x$ lies in the xy-plane. If $\mathbf{r}$ is the position vector of any point (x, y) on this plane, then

$$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} \text{ so that } d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j}.$$

Let C be the curve $y^2 = x$ from $(0, 0)$ to $(1, 1)$. The parametric equaions of $y^2 = x$ can be taken as $x = t^2$, $y = t$. At the point $(0, 0)$ we have $t = 0$ and at the point $(1, 1)$ we have $t = 1$.

$$\therefore \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (x^2\,\mathbf{i} - xy\,\mathbf{j}) \cdot (dx\,\mathbf{i} + dy\,\mathbf{j})$$

$$= \int_C (x^2\,dx - xy\,dy), \text{ where } x=t^2,\ y=t$$

$$= \int_{t=0}^1 (t^4 . 2t\,dt - t^2 . t\,dt) = \int_0^1 (2t^5 - t^3)\,dt$$

$$= 2\left[\tfrac{1}{6}t^6\right]_0^1 - \left[\tfrac{1}{4}t^4\right]_0^1 = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}.$$

Ex. 6. *Evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$ *where* $\mathbf{F} = (x^2+y^2)\,\mathbf{i} + xy\,\mathbf{j}$ *and the curve C is the arc of the parabola* $y=x^2$ *from* (0, 0) *to* (3, 9) *in the xy-plane.* (Kanpur 1981)

Sol. We have $\int_C \mathbf{F}\cdot d\mathbf{r}$

$$= \int_C [(x^2+y^2)\,\mathbf{i} + xy\,\mathbf{j}] \cdot (dx\,\mathbf{i} + dy\,\mathbf{j}) \quad [\because\ \mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}]$$

$$= \int_C [(x^2+y^2)\,dx + xy\,dy] = \int_C (x^2+y^2)\,dx + \int_C xy\,dy$$

$$= \int_{x=0}^3 (x^2+x^4)\,dx + \int_{y=0}^9 y^{1/2} . y\,dy \quad [\because\ y^2=x \text{ and}$$

for the curve C, x varies form 0 to 3 and y varies from 0 to 9]

$$= \left[\tfrac{1}{3}x^3 + \tfrac{1}{5}x^5\right]_0^3 + \tfrac{2}{5}\left[y^{5/2}\right]_0^9$$

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$$= 9 + \frac{243}{5} + \frac{2}{5}\cdot 243 = \frac{1}{5}[45+243+486]] = \frac{774}{5}.$$

Ex. 7. *Evaluate* $\int \mathbf{F}\cdot d\mathbf{r}$ *along the curve* $x^2+y^2=1$, $z=1$ *in the positive direction from* (0, 1, 1) *to* (1, 0, 1) *where*

$$\mathbf{F} = (2x+yz)\,\mathbf{i} + xz\,\mathbf{j} + (xy+2z)\,\mathbf{k}.$$

Sol. Let the given curve be denoted by C and let A and B be points (0, 1, 1) and (1, 0, 1) respectively.

Along the given curve C, we have $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$.

$\therefore\quad d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$.

$$\therefore \int_C \mathbf{F}\cdot d\mathbf{r} = \int_A^B [(2x+yz)\,\mathbf{i} + xz\,\mathbf{j} + (xy+2z)\,\mathbf{k}] \cdot (dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \int_A^B [(2x+yz)\,dx + xz\,dy + (xy+2z)\,dz]. \quad \ldots(1)$$

In moving from A to B, x varies from 0 to 1, y varies from 1 to 0 and z remains constant. We have $z=1$, and so $dz=0$.

Hence from (1)

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 (2x+y)\,dx + \int_1^0 x\,dy + 0.$$

$$= \int_0^1 [2x + \sqrt{(1-x^2)}]\,dx - \int_0^1 \sqrt{(1-y^2)}\,dy$$

$$= \left[x^2\right]_0^1 = 1,$$

the last two integrals cancel by a property of definite integrals.

Ex. 8. *Find the work done when a force*

$$\mathbf{F} = (x^2-y^2+x)\,\mathbf{i} - (2xy+y)\,\mathbf{j}$$

moves a particle in xy-plane from (0, 0) *to* (1, 1) *along the parabola* $y^2=x$. (Kanpur 1980)

Sol. Let C denote the arc of the parabola $y^2=x$ from the point (0, 0) to the point (1, 1). The parametric equations of the parabola $y^2=x$ can be taken as $x=t^2$, $y=t$. At the point (0, 0), $t=0$ and at the point (1, 1), $t=1$. The required work done

$$= \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \{(x^2-y^2+x)\,\mathbf{i} - (2xy+y)\mathbf{j}\} \cdot (dx\,\mathbf{i} + dy\,\mathbf{j})$$

$$= \int_C [(x^2-y^2+x)\,dx - (2xy+y)\,dy]$$

$$= \int_{t=0}^1 \left[(x^2-y^2+x)\frac{dx}{dt} - (2xy+y)\frac{dy}{dt}\right] dt$$

$$= \int_0^1 [(t^4-t^2+t^2) . 2t - (2t^3+t) . 1]\,dt$$

$$= \int_0^1 [2t^5 - 2t^3 - t]\,dt = \left[2\cdot\frac{t^6}{6} - 2\cdot\frac{t^4}{4} - \frac{t^2}{2}\right]_0^1$$

$$= \tfrac{1}{3} - \tfrac{1}{2} - \tfrac{1}{2} = -\tfrac{2}{3}.$$

Ex. 9. *Find the work done in moving a particle in a force field*

$$\mathbf{F} = 3x^2\,\mathbf{i} + (2xz-y)\,\mathbf{j} + z\,\mathbf{k}$$

along the line joining (0, 0, 0) *to* (2, 1, 3).

Sol. Let C be the straight line joining (0, 0, 0) to (2, 1, 3). The parametric equations of this straight line are

$$\frac{x-0}{2-0}=\frac{y-0}{1-0}=\frac{z-0}{3-0}=t \text{ say}$$

or $\quad x=2t,\ y=t,\ z=3t.$

At the point (0, 0, 0), we have $t=0$ and at the point (2, 1, 3), we have $t=1$.

The required work done

$$=\int_C \mathbf{F}\cdot d\mathbf{r}=\int_C [3x^2\mathbf{i}+(2xz-y)\mathbf{j}+3\mathbf{k}]\cdot(dx\,\mathbf{i}+dy\,\mathbf{j}+dz\,\mathbf{k})$$

$$=\int_C [3x^2\,dx+(2xz-y)\,dy+z\,dz]$$

$$=\int_C [3\,(2t)^2 2\,dt+\{2(2t)\,(3t)-t\}\,dt+(3t)\,3\,dt]$$

$$=\int_{t=0}^{1}(24t^2+12t^2-t+9t)\,dt=\int_0^1 (36t^2+8t)\,dt$$

$$=36\left[\frac{1}{3}t^3\right]_0^1+8\left[\frac{1}{2}t^2\right]_0^1=12+4=16.$$

Ex. 10. *Evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$ *where*

$$\mathbf{F}=c\,[-3a\sin^2 t\cos t\,\mathbf{i}+a\,(2\sin t-3\sin^3 t)\,\mathbf{j}+b\sin 2t\,\mathbf{k}]$$

and C is given by $\mathbf{r}=a\cos t\,\mathbf{i}+a\sin t\,\mathbf{j}+bt\,\mathbf{k}$ *from* $t=\pi/4$ *to* $\pi/2$.

Sol. We have $\frac{d\mathbf{r}}{dt}=-a\sin t\,\mathbf{i}+a\cos t\,\mathbf{j}+b\,\mathbf{k}$.

$$\therefore\ \int_C \mathbf{F}\cdot d\mathbf{r}=\int_{t=\pi/4}^{\pi/2}\left(\mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\right)dt$$

$$=\int_{\pi/4}^{\pi/2} c\,[(-3a\sin^2 t\cos t)\,(-a\sin t)+a(2\sin t-3\sin^3 t)\,(a\cos t)+(b\sin 2t)\,(b)]\,dt$$

$$=c\int_{\pi/4}^{\pi/2}[3a^2\sin^3 t\cos t+a^2\,(2\sin t\cos t-3\sin^3 t\cos t)+b^2\sin 2t]\,dt$$

$$=c\int_{\pi/4}^{\pi/2}(a^2+b^2)\sin 2t\,dt=c\,(a^2+b^2)\left[-\frac{1}{2}\cos 2t\right]_{\pi/4}^{\pi/2}$$

$$=c\,(a^2+b^2)\left[-\tfrac{1}{2}\{(-1)-0\}\right]=\tfrac{1}{2}c\,(a^2+b^2).$$

Ex. 11. *Find* $\int_C \mathbf{t}\cdot d\mathbf{r}$ *where* $\mathbf{t}$ *is the unit tangent vector and C is the unit circle, in xy-plane, with centre at the origin.*

Sol. For any curve, $\frac{d\mathbf{r}}{ds}=$ unit tangent vector $=\mathbf{t}$.

$$\therefore\ \int_C \mathbf{t}\cdot d\mathbf{r}=\int_C\left[\mathbf{t}\cdot\left(\frac{d\mathbf{r}}{ds}\right)\right]ds=\int_C(\mathbf{t}\cdot\mathbf{t})\,ds$$

$$=\int_C ds \qquad [\because\ \mathbf{t}\cdot\mathbf{t}=1,\ \mathbf{t}\text{ being a unit vector}]$$

$$=\int_{s=0}^{2\pi} ds, \text{ since along the unit circle } C,\ s \text{ goes from 0 to } 2\pi$$

$$=\Big[s\Big]_0^{2\pi}=2\pi.$$

Ex. 12. *Evaluate* $\int (x\,dy-y\,dx)$ *around the circle* $x^2+y^2=1$.

Sol. Let C denote the circle $x^2+y^2=1$. The parametric equations of this circle are $x=\cos t,\ y=\sin t$.

To integrate around the circle C, we should vary t from 0 to 2π.

$$\therefore\ \oint_C (x\,dy-y\,dx)=\int_0^{2\pi}\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)dt$$

$$=\int_0^{2\pi}(\cos^2 t+\sin^2 t)\,dt=\int_0^{2\pi}dt=2\pi.$$

Ex. 13. *Evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$,

where $\qquad \mathbf{F}=\mathbf{i}\cos y-\mathbf{j}\,x\sin y$

and C is the curve $y=\sqrt{(1-x^2)}$ *in the xy-plane from (1, 0) to (0, 1).*

Sol. We have $\int_C \mathbf{F}\cdot d\mathbf{r}$

$$=\int_C(\mathbf{i}\cos y-\mathbf{j}\,x\sin y)\cdot(\mathbf{i}\,dx+\mathbf{j}\,dy)$$

$$=\int_C(\cos y\,dx-x\sin y\,dy)$$

$$=\int_C d\,(x\cos y)=\Big[x\cos y\Big]_{(1,0)}^{(0,1)}=0-1=-1.$$

Ex. 14. *Evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$, *where* $\mathbf{F}=xy\,\mathbf{i}+(x^2+y^2)\,\mathbf{j}$ *and curve C is the arc of* $y=x^2-4$ *from (2, 0) to (4, 12).* (Garhwal 1981)

Sol. We have $\int_C \mathbf{F}\cdot d\mathbf{r}$

$$= \int_C [xy\,\mathbf{i}+(x^2+y^2)\,\mathbf{j}]\cdot(dx\,\mathbf{i}+dy\,\mathbf{j})$$

$$= \int_C [xy\,dx+(x^2+y^2)\,dy] = \int_C xy\,dx+\int_C (x^2+y^2)\,dy.$$

Along C, $y=x^2-4$ and $x^2=y+4$.

$$\therefore \int_C \mathbf{F}\cdot d\mathbf{r} = \int_{x=2}^{4} x\,(x^2-4)\,dx+\int_{y=0}^{12} (y+4+y^2)\,dy$$

$$= \left[\frac{x^4}{4}-2x^2\right]_2^4+\left[\frac{y^2}{2}+4y+\frac{y^3}{3}\right]_0^{12} = 732.$$

Ex. 15. *Evaluate* $\int_C xy^3\,ds$, *where* C *is the segment of the line* $y=2x$ *in the xy-plane from* $(-1,-2)$ *to* $(1,2)$.

Sol. The parametric form of the curve C can be taken as

$$\mathbf{r}(t) = t\,\mathbf{i}+2t\,\mathbf{j} \quad (-1\le t\le 1).$$

We have $\dfrac{d\mathbf{r}}{dt} = \mathbf{i}+2\mathbf{j}$.

Now $\dfrac{d\mathbf{r}}{dt} = \dfrac{d\mathbf{r}}{ds}\dfrac{ds}{dt}$

$$\therefore \left|\frac{d\mathbf{r}}{dt}\right| = \left|\frac{d\mathbf{r}}{ds}\right|\frac{ds}{dt} = \frac{ds}{dt}, \text{ because } \frac{d\mathbf{r}}{ds} \text{ is unit vector.}$$

$$\therefore \frac{ds}{dt} = |\mathbf{i}+2\mathbf{j}| = \sqrt{5}.$$

$$\therefore \int_C xy^3\,ds = \int_C \left(xy^3\frac{ds}{dt}\right)dt = \int_{-1}^{1} t\,(2t)^3\sqrt{5}\,dt$$

$$= 8\sqrt{5}\int_{-1}^{1} t^4\,dt = \frac{16}{\sqrt{5}}.$$

Ex. 16. *Evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$,

where $\mathbf{F} = xy\,\mathbf{i}+yz\,\mathbf{j}+zx\,\mathbf{k}$ *and curve* C *is* $\mathbf{r} = t\,\mathbf{i}+t^2\,\mathbf{j}+t^3\,\mathbf{k}$, t *varying from* -1 *to* $+1$.

(Tirupati 1989, Rohilkhand 92)

Sol. Along the curve C,

$$\mathbf{r} = x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k} = t\,\mathbf{i}+t^2\,\mathbf{j}+t^3\,\mathbf{k}.$$

$$\therefore x=t,\ y=t^2,\ z=t^3 \text{ and } \frac{d\mathbf{r}}{dt} = \mathbf{i}+2t\,\mathbf{j}+3t^2\,\mathbf{k}.$$

$\therefore$ Along the curve C, we have

$$\mathbf{F} = (t\times t^2)\,\mathbf{i}+(t^2\times t^3)\,\mathbf{j}+(t^3\times t)\,\mathbf{k} = t^3\mathbf{i}+t^5\,\mathbf{j}+t^4\,\mathbf{k}.$$

Hence $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{-1}^{1}\left(\mathbf{F}\cdot\dfrac{d\mathbf{r}}{dt}\right)dt$

$$= \int_{-1}^{1} (t^3\,\mathbf{i}+t^5\,\mathbf{j}+t^4\,\mathbf{k})\cdot(\mathbf{i}+2t\,\mathbf{j}+3t^2\,\mathbf{k}) = \int_{-1}^{1} (t^3+2t^6+3t^6)\,dt$$

$$= \int_{-1}^{1} (t^3+5t^6)\,dt = \int_{-1}^{1} t^3\,dt+5\int_{-1}^{1} t^6\,dt$$

$$= 0+5(2)\int_0^1 t^6\,dt = 10\left[\frac{t^7}{7}\right]_0^1 = \frac{10}{7}.$$

Ex. 17. *If* $\mathbf{F} = (3x^2+6y)\,\mathbf{i}-14yz\,\mathbf{j}+20xz^2\,\mathbf{k}$, *then evaluate* $\int \mathbf{F}\cdot d\mathbf{r}$ *from* $(0,0,0)$ *to* $(1,1,1)$ *along the curve*

$$x=t,\ y=t^2,\ z=t^3.$$

Sol. Along the given curve C, we have

$$\mathbf{r} = x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k} = t\,\mathbf{i}+t^2\,\mathbf{j}+t^3\,\mathbf{k}.$$

$$\therefore \frac{d\mathbf{r}}{dt} = \mathbf{i}+2t\,\mathbf{j}+3t^2\,\mathbf{k} \qquad \ldots(1)$$

Also from the equations of the given curve we find that the points $(0,0,0)$ and $(1,1,1)$ correspond to $t=0$ and $t=1$ respectively.

$$\therefore \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \left(\mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\right)dt$$

$$= \int_C [(3x^2+6y)\,\mathbf{i}-14yz\,\mathbf{j}+20xz^2\,\mathbf{k}]\cdot(\mathbf{i}+2t\,\mathbf{j}+3t^2\,\mathbf{k})\,dt,$$

from (1)

$$= \int_{t=0}^{1} [(3x^2+6y)-28yzt+60xz^2t^2]\,dt$$

$$= \int_0^1 [(3t^2+6t^2)-28t^6+60t^9]\,dt,$$

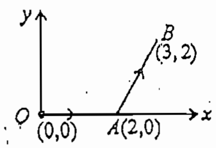
putting $x=t$, $y=t^2$, $z=t^3$

$$= \int_0^1 (9t^2-28t^6+60t^9)\,dt = \left[3t^3-4t^7+6t^{10}\right]_0^1$$

$$= 3-4+6 = 5.$$

Ex. 18. *If* $F = (2x+y)\,i + (3y-x)\,j$, *evaluate* $\int_C F \cdot dr$ *where C is the curve in the xy-plane consisting of the straight lines from* (0, 0) *to* (2, 0) *and then to* (3, 2).

Sol. The path of integration C has been shown in the figure. It consists of the straight lines OA and AB.



We have $\int_C F \cdot dr$

$$= \int_C [(2x+y)\,i + (3y-x)\,j] \cdot (dx\,i + dy\,j)$$

$$= \int_C [(2x+y)\,dx + (3y-x)\,dy].$$

Now along the straight line OA, $y=0$, $dy=0$ and x varies from 0 to 2. The equation of the straight line AB is

$$y-0 = \frac{2-0}{3-2}(x-2) \text{ i.e., } y = 2x-4.$$

$\therefore$ along AB, $y=2x-4$, $dy=2\,dx$ and x varies from 2 to 3.

$$\therefore \int_C F \cdot dr = \int_0^2 [(2x+0)\,dx + 0] + \int_2^3 [(2x+2x-4)\,dx + (6x-12-x)\,2dx]$$

$$= \left[x^2\right]_0^2 + \int_2^3 (14x-28)\,dx = 4 + 14\int_2^3 (x-2)\,dx$$

$$= 4 + 14\left[\frac{(x-2)^2}{2}\right]_2^3 = 4+7 = 11.$$

Ex.19. *If* $F = (3x^2+6y)\,i - 14yz\,j + 20xz^2\,k$, *evaluate* $\int_C F \cdot dr$ *where C is the curve consisting of the straight lines from* (0, 0, 0) *to* (1, 0, 0) *then to* (1, 1, 0) *and then to* (1, 1, 1).

Sol. We have $F \cdot dr$

$$= [(3x^2+6y)\,i - 14yz\,j + 20xz^2\,k] \cdot (dx\,i + dy\,j + dz\,k)$$

$$= (3x^2+6y)\,dx - 14yz\,dy + 20xz^2\,dz.$$

Let C_1 denote the straight line joining (0, 0, 0) to (1, 0, 0). Then along C_1, $y=0$, $z=0$ and x goes from 0 to 1. Obviously along C_1, $dy=0$ and $dz=0$.

Let C_2 denote the straight line joining (1, 0, 0) to (1, 1, 0). Then along C_2, $x=1$, $z=0$ and y varies from 0 to 1. Obviously along C_2, $dx=0$ and $dz=0$.

Again let C_3 denote the straight line joining (1, 1, 0) to (1, 1, 1). Along C_3, we have

$x=1$, $y=1$ so that $dx=0$, $dy=0$. Obviously along C_3, z varies from 0 to 1.

Now $\int_C F \cdot dr = \int_{C_1} F \cdot dr + \int_{C_2} F \cdot dr + \int_{C_3} F \cdot dr.$

We have $\int_{C_1} F \cdot dr = \int_{C_1} [(3x^2+6y)\,dx - 14yz\,dy + 20xz^2\,dz]$

$$= \int_{x=0}^{1} 3x^2\,dx$$

[$\because$ along C_1, $y=0$, $z=0$, $dy=0$, $dz=0$, and x varies from 0 to 1]

$$= 3\left[\frac{1}{3}x^3\right]_0^1 = 1.$$

Again $\int_{C_2} F \cdot dr = \int_{y=0}^{1} -14yz\,dy$

[$\because$ along C_2, $dx=0$, $dz=0$ and y varies from 0 to 1]

$$= \int_{y=0}^{1} -14y \cdot 0\,dy \qquad [\because \text{ along } C_2,\ z=0]$$

$$= 0.$$

Finally $\int_{C_3} F \cdot dr = \int_{z=0}^{1} 20xz^2\,dz$

[$\because$ along C_3, $dx=0$, $dy=0$ and z varies from 0 to 1]

$$= 20\left[\frac{1}{3}z^3\right]_0^1 = \frac{20}{3}.$$

Hence $\int_C F \cdot dr = 1 + 0 + \frac{20}{3} = \frac{23}{3}.$

Ex. 20. *If* $F = (2y+3)\,i + xz\,j + (yz-x)\,k$, *evaluate* $\int_C F \cdot dr$ *where C is the path consisting of the straight lines from* (0, 0, 0) *to* (0, 0, 1) *then to* (0, 1, 1) *and then to* (2, 1, 1).

Sol. We have $F \cdot dr$

$$= [(2y+3)\,\mathbf{i} + xz\,\mathbf{j} + (yz-x)\,\mathbf{k}] \cdot (dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$
$$= (2y+3)\,dx + xz\,dy + (yz-x)\,dz.$$

Let C_1 denote the straight line joining $(0, 0, 0)$ to $(0, 0, 1)$, C_2 denote the straight line joining $(0, 0, 1)$ to $(0, 1, 1)$ and C_3 denote the straight line joining $(0, 1, 1)$ to $(2, 1, 1)$.

Along C_1, $x=0$, $y=0$ so that $dx=0$, $dy=0$.
Also along C_1, z varies from 0 to 1.
Along C_2, $x=0$, $z=1$ so that $dx=0$, $dz=0$.
Also along C_2, y varies from 0 to 1.
Along C_3, $y=1$, $z=1$ so that $dy=0$, $dz=0$.
Also along C_3, x varies from 0 to 2.

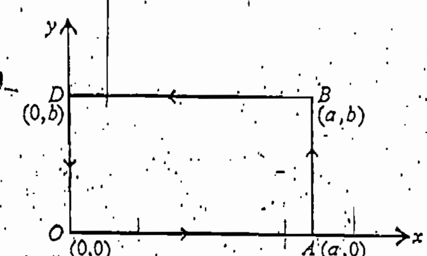
$$\therefore \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} + \int_{C_3} \mathbf{F} \cdot d\mathbf{r}$$
$$= \int_{z=0}^{1} (0 \cdot z - 0)\,dz + \int_{y=0}^{1} (0 \cdot 1)\,dy + \int_{x=0}^{2} (2 \cdot 1 + 3)\,dx$$
$$= 0 + 0 + 5\left[x\right]_0^2 = 10.$$

Ex. 21. *Evaluate* $\int_C \mathbf{F} \cdot d\mathbf{r}$ *where* $\mathbf{F} = (x^2+y^2)\,\mathbf{i} - 2xy\,\mathbf{j}$, *curve C is the rectangle in the xy-plane bounded by* $y=0$, $x=a$, $y=b$, $x=0$.
(Andhra 1992 ; Meerut 81 ; Kanpur 79)

Sol. In the xy-plane $z=0$. Therefore
$\therefore$ $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$ and $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j}$.

The path of integration C has been shown in the figure. It consists of the straight lines OA, AB, BD and DO.

We have $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C [(x^2+y^2)\,\mathbf{i} - 2xy\,\mathbf{j}] \cdot (dx\,\mathbf{i} + dy\,\mathbf{j})$



$$= \int_C [(x^2+y^2)\,dx - 2xy\,dy].$$

Now on OA, $y=0$, $dy=0$ and x varies from 0 to a,
on AB, $x=a$, $dx=0$ and y varies from 0 to b,
on BD, $y=b$, $dy=0$ and x varies from a to 0,
on DO, $x=0$, $dx=0$ and y varies from b to 0.

$$\therefore \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^a x^2\,dx - \int_0^b 2ay\,dy + \int_a^0 (x^2+b^2)\,dx + \int_b^0 0\,dy$$
$$= \left[\frac{x^3}{3}\right]_0^a - 2a\left[\frac{y^2}{2}\right]_0^b + \left[\frac{x^3}{3} + b^2x\right]_a^0 + 0 = -2ab^2.$$

Ex. 22. *Find the total work done in moving a particle in a force field given by* $\mathbf{F} = 3xy\,\mathbf{i} - 5z\,\mathbf{j} + 10x\,\mathbf{k}$ *along the curve* $x=t^2+1$, $y=2t^2$, $z=t^3$ *from* $t=1$ *to* $t=2$.
[Tirupati 1984; Madras 83; Kanpur 78]

Sol. Let C denote the arc of the given curve from $t=1$ to $t=2$. Then the total work done

$$= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (3xy\,\mathbf{i} - 5z\,\mathbf{j} + 10x\,\mathbf{k}) \cdot (dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$
$$= \int_C (3xy\,dx - 5z\,dy + 10x\,dz)$$
$$= \int_1^2 \left(3xy\frac{dx}{dt} - 5z\frac{dy}{dt} + 10x\frac{dz}{dt}\right) dt$$
$$= \int_1^2 [3\,(t^2+1)\,(2t^2)(2t) - (5t^3)(4t) + 10\,(t^2+1)(3t^2)]\,dt$$
$$= \int_1^2 (12t^5 + 12t^3 - 20t^4 + 30t^4 + 30t^2)\,dt$$
$$= \int_1^2 (12t^5 + 10t^4 + 12t^3 + 30t^2)\,dt = 303.$$

Ex. 23. *Find the work done in moving a particle once around a circle C in the xy-plane, if the circle has centre at the origin and radius 2 and if the force field* $\mathbf{F}$ *is given by*
$$\mathbf{F} = (2x-y+2z)\,\mathbf{i} + (x+y-z)\,\mathbf{j} + (3x-2y-5z)\,\mathbf{k}.$$
(Kanpur 1979)

Sol. In the xy-plane, we have $z=0$. Therefore
$$\mathbf{F} = (2x-y)\,\mathbf{i} + (x+y)\,\mathbf{j} + (3x-2y)\,\mathbf{k}.$$
The circle C is given by $x^2+y^2=4$ or $x=2\cos t$, $y=2\sin t$.
$$\therefore \quad \mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}.$$

$\therefore \quad \dfrac{d\mathbf{r}}{dt} = -2 \sin t\, \mathbf{i} + 2 \cos t\, \mathbf{j}.$

Also $\mathbf{F} = (4 \cos t - 2 \sin t)\, \mathbf{i} + (2 \cos t + 2 \sin t)\, \mathbf{j} + (6 \cos t - 4 \sin t)\, \mathbf{k}.$

In moving round the circle once t will vary from 0 to 2π.

The required work done is $= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F} \cdot \dfrac{d\mathbf{r}}{dt}\, dt$

$$= \int_0^{2\pi} [-2 \sin t\, (4 \cos t - 2 \sin t) + 2 \cos t\, (2 \cos t + 2 \sin t)]\, dt$$

$$= \int_0^{2\pi} [4 (\sin^2 t + \cos^2 t) - 4 \sin t \cos t]\, dt$$

$$= \int_0^{2\pi} (4 - 4 \sin t \cos t)\, dt = \Big[4t - 2 \sin 2t\Big]_0^{2\pi} = 8\pi.$$

Ex. 24. *If* $\mathbf{F} = (3x^2 + 6y)\, \mathbf{i} - 14yz\, \mathbf{j} + 20xz^2\, \mathbf{k}$, *evaluate* $\int_C \mathbf{F} \cdot d\mathbf{r}$ *where C is a straight line joining* (0, 0, 0) *to* (1, 1, 1).

(Meerut 1983 ; Bundelkhand 79)

Sol. The equations of the straight line joining (0, 0, 0) and (1, 1, 1) are

$$\frac{x-0}{1-0} = \frac{y-0}{1-0} = \frac{z-0}{1-0} = t \text{ (say)}.$$

Then along C, $x = t$, $y = t$, $z = t$.

Also $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$. $\therefore d\mathbf{r} = (\mathbf{i} + \mathbf{j} + \mathbf{k})\, dt.$

Also along C, $\mathbf{F} = (3t^2 + 6t)\, \mathbf{i} - 14t^2\, \mathbf{j} + 20t^3\, \mathbf{k}.$

At (0, 0, 0), $t = 0$ and at (1, 1, 1), $t = 1$.

$$\therefore \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{t=0}^{1} [(3t^2 + 6t) - 14t^2 + 20t^3]\, dt = \frac{13}{3}.$$

Ex. 25. *If* $\mathbf{F} = y\,\mathbf{i} - x\,\mathbf{j}$, *evaluate* $\int_C \mathbf{F} \cdot d\mathbf{r}$ *from* (0, 0) *to* (1, 1) *along the following paths C :*

(*a*) *the parabola* $y = x^2$, (Agra 1973)

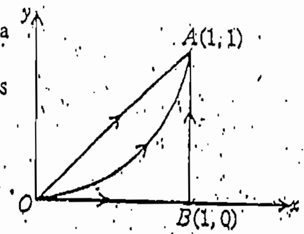
(*b*) *the straight lines from* (0, 0) *to* (1, 0) *and then to* (1, 1).

(*c*) *the straight line joining* (0, 0) *and* (1, 1).

Sol. The three paths of integration have been shown in the figure. We have

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C (y\,\mathbf{i} - x\,\mathbf{j}) \cdot (dx\,\mathbf{i} + dy\,\mathbf{j})$$

$$= \int_C (y\, dx - x\, dy).$$



(a) C is the arc of parabola $y = x^2$ from (0, 0) to (1, 1).

Here $dy = 2x\, dx$ and x varies from 0 to 1.

$$\therefore \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 [x^2\, dx - x\, (2x)\, dx]$$

$$= \int_0^1 -x^2\, dx = -\frac{1}{3}.$$

(b) C is the curve consisting of straight lines OB and BA.

Along OB, $y = 0$, $dy = 0$ and x varies from 0 to 1.

Along BA, $x = 1$, $dx = 0$ and x varies from 0 to 1.

$$\therefore \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 0\, dx + \int_0^1 -1\, dy = -1.$$

(c) C is the straight line OA. The equation of OA is

$$y - 0 = \frac{1-0}{1-0}(x - 0) \text{ i.e., } y = x.$$

$\therefore dy = dx$ and x varies from 0 to 1.

$$\therefore \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (x\, dx - x\, dx) = 0.$$

Ex. 26. *Calculate* $\int_C [(x^2 + y^2)\, \mathbf{i} + (x^2 - y^2)\, \mathbf{j}] \cdot d\mathbf{r}$ *where C is the curve :*

(*i*) $y^2 = x$ *joining* (0, 0) *to* (1, 1).

(*ii*) $x^2 = y$ *joining* (0, 0) *to* (1, 1).

(*iii*) *consisting of two straight lines joining* (0, 0) *to* (1, 0) *and* (1, 0) *to* (1, 1).

(*iv*) *consisting of three straight lines joining* (0, 0) *to* (2, −2), (2, −2) *to* (0, −1) *and* (0, −1) *to* (1, 1).

Sol. Here $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$ so that $d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j}$.

$$\therefore \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C [(x^2 + y^2)\, dx + (x^2 - y^2)\, dy] \quad \text{...(1)}$$

(i) Let C be the curve $y^2 = x$ from (0, 0) to (1, 1). Then along C, $y^2 = x$, x varies from 0 to 1 and y varies from 0 to 1.

$\therefore$ from (1), $\int_C \mathbf{F} \cdot d\mathbf{r}$

$$= \int_C (x^2+y^2)\,dx + \int_C (x^2-y^2)\,dy$$

$$= \int_{x=0}^{1} (x^2+x)\,dx + \int_{y=0}^{1} (y^4-y^2)\,dy$$

$$= \left[\frac{1}{3}x^3+\frac{1}{2}x^2\right]_0^1 + \left[\frac{1}{5}y^5-\frac{1}{3}y^3\right]_0^1$$

$$= \left(\frac{1}{3}+\frac{1}{2}\right)+\left(\frac{1}{5}-\frac{1}{3}\right) = \frac{5}{6}-\frac{2}{15} = \frac{21}{30} = \frac{7}{10}.$$

(ii) Let C be the curve $x^7=y$ from (0, 0) to (1, 1). Then along C, $y=x^7$, $x=y^{1/7}$, x varies from 0 to 1 and y varies from 0 to 1.

$\therefore$ from (1), $\int_C \mathbf{F}\cdot d\mathbf{r}$

$$= \int_{x=0}^{1} (x^2+x^{14})\,dx + \int_{y=0}^{1} (y^{2/7}-y^2)\,dy$$

$$= \left[\frac{1}{3}x^3+\frac{1}{15}x^{15}\right]_0^1 + \left[\frac{7}{9}y^{9/7}-\frac{1}{3}y^3\right]_0^1$$

$$= \left(\frac{1}{3}+\frac{1}{15}\right)+\left(\frac{7}{9}-\frac{1}{3}\right) = \frac{2}{5}+\frac{4}{9} = \frac{38}{45}.$$

(iii) Let C_1 be the straight line joining (0, 0) to (1, 0) and C_2 be the straight line joining (1, 0) to (1, 1).

Along C_1, $y=0$ so that $dy=0$ and x varies from 0 to 1.

Along C_2, $x=1$, $dx=0$ and x varies from 0 to 1.

$\therefore$ from (1), $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} + \int_{C_2} \mathbf{F}\cdot d\mathbf{r}$

$$= \int_{C_1} \{(x^2+y^2)\,dx+(x^2-y^2)\,dy\} + \int_{C_2} \{(x^2+y^2)\,dx+(x^2-y^2)\,dy\}$$

$$= \int_{x=0}^{1} (x^2+0)\,dx + \int_{y=0}^{1} (1-y^2)\,dy$$

$$= \frac{1}{3}\left[x^3\right]_0^1 + \left[y-\frac{1}{3}y^3\right]_0^1 = \frac{1}{3}+\left(1-\frac{1}{3}\right) = 1.$$

(iv) Let C_1 be the straight line joining (0, 0) to (2, −2), C_2 be the straight line joining (2, −2) to (0, −1) and C_3 be the straight line joining (0, −1) to (1, 1).

The equation of the straight line joining (0, 0) and (2, −2) is

$$y = -\frac{2}{2}x \text{ or } y=-x.$$

$\therefore$ along C_1, $y=-x$, $dy=-dx$ and x varies from 0 to 2.

$\therefore$ from (1), $\int_{C_1} \mathbf{F}\cdot d\mathbf{r} = \int_{x=0}^{2} [(x^2+x^2)\,dx+(x^2-x^2)(-dx)]$

$$= 2\int_0^2 x^2\,dx = 2\left[\frac{1}{3}x^3\right]_0^2 = \frac{16}{3}.$$

The equation of the straight line joining (2, −2) and (0, −1) is

$$y+1 = \frac{-2-(-1)}{2-0}(x-0) \quad\text{or}\quad y+1 = -\frac{1}{2}x \quad\text{or}\quad y=-1-\frac{1}{2}x$$

or $\quad y = -\frac{1}{2}(x+2)$.

$\therefore$ along C_2, $y=-\frac{1}{2}(x+2)$, $dy = -\frac{1}{2}dx$ and x varies from 2 to 0.

$\therefore$ from (1),

$$\int_{C_2} \mathbf{F}\cdot d\mathbf{r} = \int_{x=2}^{0} \left[\left\{x^2+\frac{1}{4}(x+2)^2\right\}dx+\left\{x^2-\frac{1}{4}(x+2)^2\right\}\left(-\frac{1}{2}dx\right)\right]$$

$$= \int_2^0 \left[\left(1+\frac{1}{4}-\frac{1}{2}+\frac{1}{8}\right)x^2+\left(1+\frac{1}{2}\right)x+\left(1+\frac{1}{2}\right)\right]dx$$

$$= \int_2^0 \left[\frac{7}{8}x^2+\frac{3}{2}x+\frac{3}{2}\right]dx = \left[\frac{7}{8}\cdot\frac{1}{3}x^3+\frac{3}{4}x^2+\frac{3}{2}x\right]_2^0$$

$$= -\frac{7}{3}-3-3 = -\frac{25}{3}.$$

The equation of the straight line joining (0, −1) and (1, 1) is

$$y+1 = \frac{1+1}{1-0}(x-0) \quad\text{or}\quad y+1 = 2x \text{ or } y = 2x-1.$$

$\therefore$ along C_3, $y=2x-1$, $dy=2\,dx$ and x varies from 0 to 1.

$\therefore$ from (1),

$$\int_{C_3} \mathbf{F}\cdot d\mathbf{r} = \int_{x=0}^{1} \left[\left\{x^2+(2x-1)^2\right\}dx+\left\{x^2-(2x-1)^2\right\}(2\,dx)\right]$$

$$= \int_0^1 \left[-x^2+4x-1\right]dx$$

$$= \left[-\frac{1}{3}x^3+2x^2-x\right]_0^1 = -\frac{1}{3}+2-1 = \frac{2}{3}.$$

Hence $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} + \int_{C_2} \mathbf{F}\cdot d\mathbf{r} + \int_{C_3} \mathbf{F}\cdot d\mathbf{r}$

$$= \frac{16}{3} - \frac{25}{3} + \frac{2}{3} = -\frac{7}{3}.$$

Ex. 27. *Evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$, *where* $\mathbf{F} = yz\,\mathbf{i} + zx\,\mathbf{j} + xy\,\mathbf{k}$ *and C is the portion of the curve* $\mathbf{r} = a\cos t\,\mathbf{i} + b\sin t\,\mathbf{j} + ct\,\mathbf{k}$, *from* $t=0$ *to* $t=\pi/2$.

Sol. Along the curve C,

$$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} = a\cos t\,\mathbf{i} + b\sin t\,\mathbf{j} + ct\,\mathbf{k}.$$

$$\therefore \quad x = a\cos t,\ y = b\sin t,\ z = ct.$$

Now $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (yz\,\mathbf{i} + zx\,\mathbf{j} + xy\,\mathbf{k})\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$

$$= \int_C (yz\,dx + zx\,dy + xy\,dz) = \int_C d\,(xyz)$$

$$= \Big[xyz\Big]_{t=0}^{t=\pi/2} = \Big[(a\cos t)\cdot(b\sin t)\cdot(ct)\Big]_0^{\pi/2}$$

$$= abc\,\Big[t\cos t\sin t\Big]_0^{\pi/2} = abc\,(0-0) = 0.$$

Ex. 28. *Evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$ *where* $\mathbf{F} = z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$ *and C is the arc of the curve* $\mathbf{r} = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k}$ *from* $t=0$ *to* $t=2\pi$.

(Agra 1977 ; Garhwal 86)

Sol. The vector equation of the given curve is

$$\mathbf{r} = (\cos t)\,\mathbf{i} + (\sin t)\,\mathbf{j} + t\,\mathbf{k} \qquad \ldots(1)$$

$\therefore$ the parametric equations of (1) are

$$x = \cos t,\ y = \sin t,\ z = t. \qquad \ldots(2)$$

From (1), $\frac{d\mathbf{r}}{dt} = (-\sin t)\,\mathbf{i} + (\cos t)\,\mathbf{j} + \mathbf{k}$.

$$\therefore \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \left(\mathbf{F}\cdot\frac{d\mathbf{r}}{dt}\right) dt$$

$$= \int_C (z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k})\cdot[(-\sin t)\,\mathbf{i} + (\cos t)\,\mathbf{j} + \mathbf{k}]\,dt$$

$$= \int_C (-z\sin t + x\cos t + y)\,dt$$

$$= \int_{t=0}^{2\pi} (-t\sin t + \cos^2 t + \sin t)\,dt,$$

putting for x, y, z from (2)

$$= -\int_0^{2\pi} t\sin t\,dt + \tfrac{1}{2}\int_0^{2\pi} (1+\cos 2t)\,dt + \int_0^{2\pi} \sin t\,dt$$

$$= -\left[\{t(-\cos t)\}_0^{2\pi} + \int_0^{2\pi} \cos t\,dt\right] + \tfrac{1}{2}\Big[t + \tfrac{1}{2}\sin 2t\Big]_0^{2\pi} + \Big[-\cos t\Big]_0^{2\pi}$$

$$= -\left[\{2\pi(-1)\} + \{\sin t\}_0^{2\pi}\right] + \tfrac{1}{2}[2\pi] + [-\cos 2\pi + \cos 0]$$

$$= 2\pi + \pi = 3\pi.$$

Ex. 29. *Evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$, *where* $\mathbf{F} = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$ *and C is the arc of the curve* $\mathbf{r} = (a\cos\theta)\,\mathbf{i} + (a\sin\theta)\,\mathbf{j} + a\theta\,\mathbf{k}$ *from* $\theta=0$ *to* $\theta=\frac{1}{2}\pi$.

Sol. The parametric equations of the given curve are

$$x = a\cos\theta,\ y = a\sin\theta,\ z = a\theta. \qquad \ldots(1)$$

Also $\frac{d\mathbf{r}}{d\theta} = (-a\sin\theta)\,\mathbf{i} + (a\cos\theta)\,\mathbf{j} + a\,\mathbf{k}$.

$$\therefore \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k})\cdot[(-a\sin\theta)\,\mathbf{i} + (a\cos\theta)\,\mathbf{j} + a\,\mathbf{k}]\,d\theta$$

$$= \int_C (-axy\sin\theta + ayz\cos\theta + azx)\,d\theta$$

$$= \int_{\theta=0}^{\pi/2} (-a^3\cos\theta\sin^2\theta + a^3\theta\sin\theta\cos\theta + a^3\theta\cos\theta)\,d\theta,$$

putting for x, y, z from (1)

$$= -a^3\int_0^{\pi/2} \sin^2\theta\cos\theta\,d\theta + \tfrac{1}{2}a^3\int_0^{\pi/2} \theta\sin 2\theta\,d\theta$$

$$+ a^3\int_0^{\pi/2} \theta\cos\theta\,d\theta$$

$$= -a^3\left[\tfrac{1}{3}\sin^3\theta\right]_0^{\pi/2} + \tfrac{1}{2}a^3\left[\left(-\theta\,\frac{\cos 2\theta}{2}\right)_0^{\pi/2} + \tfrac{1}{2}\int_0^{\pi/2}\cos 2\theta\,d\theta\right]$$

$$+ a^3\left[(\theta\sin\theta)_0^{\pi/2} - \int_0^{\pi/2}\sin\theta\,d\theta\right]$$

$$= -\tfrac{1}{3}a^3 + \tfrac{1}{2}a^3\left[\tfrac{1}{4}\pi + \tfrac{1}{2}\left(\tfrac{1}{2}\sin 2\theta\right)_0^{\pi/2}\right] + a^3\left[\tfrac{1}{2}\pi + (\cos\theta)_0^{\pi/2}\right]$$

$$= -\tfrac{1}{3}a^3 + \tfrac{1}{2}a^3\left[\tfrac{1}{4}\pi\right] + a^3\left[\tfrac{1}{2}\pi - 1\right]$$

$$= a^3\left[-\tfrac{1}{3} + \tfrac{1}{8}\pi + \tfrac{1}{2}\pi - 1\right] = a^3\left[\tfrac{5}{8}\pi - \tfrac{4}{3}\right].$$

Ex. 30. *If* $\mathbf{F} = yz\,\mathbf{i} + zx\,\mathbf{j} - xy\,\mathbf{k}$, *find* $\int_C \mathbf{F}\cdot d\mathbf{r}$ *where C is given by* $x=t,\ y=t^2,\ z=t^3$ *from* $P(0,0,0)$ *to* $Q(2,4,8)$. [Madurai 1985]

Sol. Along the given curve C, we have

$$\mathbf{r} = x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k} = t\,\mathbf{i}+t^2\,\mathbf{j}+t^3\,\mathbf{k}.$$

$$\therefore \quad \frac{d\mathbf{r}}{dt} = \mathbf{i}+2t\,\mathbf{j}+3t^2\,\mathbf{k}.$$

Also along the given curve C, we have

$$\mathbf{F} = (t^2 \cdot t^3)\,\mathbf{i}+(t^3 \cdot t)\,\mathbf{j}-(t \cdot t^2)\,\mathbf{k} = t^5\,\mathbf{i}+t^4\,\mathbf{j}-t^3\,\mathbf{k}.$$

At the point P (0, 0, 0), we have $t=0$ and at the point Q (2, 4, 8), we have $t=2$.

$$\therefore \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \left[\mathbf{F}\cdot\left(\frac{d\mathbf{r}}{dt}\right)\right] dt$$

$$= \int_{t=0}^{2} [(t^5\,\mathbf{i}+t^4\,\mathbf{j}-t^3\,\mathbf{k})\cdot(\mathbf{i}+2t\,\mathbf{j}+3t^2\,\mathbf{k})]\,dt$$

$$= \int_0^2 (t^5+2t^5-3t^5)\,dt = \int_0^2 0\,dt = 0.$$

Ex. 31. *Evaluate* $\int_C x^{-1}(y+z)\,ds$, *where C is the arc of the circle $x^2+y^2=4$ in the xy-plane from A (2, 0, 0) to B ($\sqrt{2}$, $\sqrt{2}$, 0).*

Sol. Let $x=2\cos t$, $y=2\sin t$, $z=0$ be the parametric equations of the circle $x^2+y^2=4$, $z=0$.

For the point A, $x=2$, $y=0$, $z=0$ and so $t=0$ and for the point B, $x=\sqrt{2}$, $y=\sqrt{2}$, $z=0$ and so $t=\pi/4$.

If $\mathbf{r}$ is the position vector of any point (x, y, z) on the circle C, then

$$\mathbf{r} = x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k} = 2\cos t\,\mathbf{i}+2\sin t\,\mathbf{j}+0\,\mathbf{k}.$$

$$\therefore \quad \frac{d\mathbf{r}}{dt} = -2\sin t\,\mathbf{i}+2\cos t\,\mathbf{j}.$$

$$\therefore \quad \left(\frac{d\mathbf{r}}{dt}\right)^2 = 4\sin^2 t+4\cos^2 t = 4.$$

But $\left(\dfrac{d\mathbf{r}}{dt}\right)^2 = \left(\dfrac{d\mathbf{r}}{ds}\dfrac{ds}{dt}\right)^2 = \left(\dfrac{ds}{dt}\right)^2 \mathbf{t}^2$, where $\mathbf{t} = \dfrac{d\mathbf{r}}{ds}$ is unit tangent vector

$$= \left(\frac{ds}{dt}\right)^2 \qquad [\because \ \mathbf{t}^2 = \mathbf{t}\cdot\mathbf{t} = 1, \ \mathbf{t} \text{ being unit vector}]$$

$$\therefore \left(\frac{ds}{dt}\right)^2 = 4 \text{ or } \frac{ds}{dt} = 2.$$

$$\therefore \int_C x^{-1}(y+z)\,ds = \int_{t=0}^{\pi/4} \frac{y+z}{x}\frac{ds}{dt}\,dt$$

$$= \int_0^{\pi/4} \frac{2\sin t+0}{2\cos t}\cdot 2\,dt = 2\int_0^{\pi/4} \tan t\,dt$$

$$= 2\left[\log \sec t\right]_0^{\pi/4} = 2\log\sqrt{2}$$

$$= 2\cdot\tfrac{1}{2}\log 2 = \log 2.$$

Ex. 32. *Evaluate* $\int_C (x^2+y^2+z^2)^2\,ds$, *where C is the arc of the circular helix*

$$\mathbf{r}(t) = \cos t\,\mathbf{i}+\sin t\,\mathbf{j}+3t\,\mathbf{k}$$

from A (1, 0, 0) to B (1, 0, 6π).

Sol. The equation of the curve C is

$$\mathbf{r}(t) = \cos t\,\mathbf{i}+\sin t\,\mathbf{j}+3t\,\mathbf{k}.$$

$$\therefore \quad \frac{d\mathbf{r}}{dt} = -\sin t\,\mathbf{i}+\cos t\,\mathbf{j}+3\,\mathbf{k}.$$

$$\therefore \left(\frac{d\mathbf{r}}{dt}\right)^2 = \sin^2 t+\cos^2 t+9 = 10.$$

$\therefore \left(\dfrac{d\mathbf{r}}{ds}\dfrac{ds}{dt}\right)^2 = 10$ or $\left(\dfrac{ds}{dt}\right)^2 \mathbf{t}^2 = 10$, where $\mathbf{t} = \dfrac{d\mathbf{r}}{ds}$ is unit tangent vector to C at the point 't'

$$\text{or} \quad \left(\frac{ds}{dt}\right)^2 = 10 \qquad [\because \ \mathbf{t}^2 = \mathbf{t}\cdot\mathbf{t} = 1, \ \mathbf{t} \text{ being unit vector}]$$

$$\text{or} \quad \frac{ds}{dt} = \sqrt{(10)}.$$

Parametric equations of C are

$$x = \cos t, \ y = \sin t, \ z = 3t.$$

At the point A, $x=1$, $y=0$, $z=0$ and so $t=0$ and at the point B, $x=1$, $y=0$, $z=6\pi$ and so $t=2\pi$.

$$\therefore \int_C (x^2+y^2+z^2)^2\,ds = \int_{t=0}^{2\pi} (\cos^2 t+\sin^2 t+9t^2)^2 \frac{ds}{dt}\,dt$$

$$= \int_0^{2\pi} (1+9t^2)^2\cdot\sqrt{(10)}\,dt = \sqrt{(10)}\int_0^{2\pi} (1+18t^2+81t^4)\,dt$$

$$= \sqrt{(10)}\left[t+6t^3+\frac{81}{5}t^5\right]_0^{2\pi} = \sqrt{(10)}\left[2\pi+6(2\pi)^3+\frac{81}{5}(2\pi)^5\right].$$

Ex. 33. *Find the circulation of* $\mathbf{F}$ *round the curve C, where*

$$\mathbf{F} = e^x \sin y\,\mathbf{i}+e^x \cos y\,\mathbf{j}$$

and C is the rectangle whose vertices are $(0, 0), (1, 0), (1, \frac{1}{2}\pi), (0, \frac{1}{2}\pi)$.

Sol. The rectangle C lies in the xy-plane. If $\mathbf{r}$ is the position vector of any point (x, y) on this plane, then

$$\mathbf{r} = x\,\mathbf{i}+y\,\mathbf{j} \text{ so that } d\mathbf{r} = dx\,\mathbf{i}+dy\,\mathbf{j}.$$

By definition, the circulation of $\mathbf{F}$ round the curve C

$$= \oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C (e^x \sin y\,\mathbf{i}+e^x \cos y\,\mathbf{j})\cdot(dx\,\mathbf{i}+dy\,\mathbf{j})$$

$$= \oint_C (e^x \sin y\,dx+e^x \cos y\,dy). \qquad \text{...(1)}$$

Draw figure as in solved example 21.

Let O be the point $(0, 0)$, A be the point $(1, 0)$, B be the point $(1, \frac{1}{2}\pi)$ and D be the point $(0, \frac{1}{2}\pi)$.

Now on OA, $y=0$, $dy=0$ and x varies from 0 to 1,
on AB, $x=1$, $dx=0$ and y varies from 0 to $\frac{1}{2}\pi$,
on BD, $y=\frac{1}{2}\pi$, $dy=0$ and x varies from 1 to 0,
on DO, $x=0$, $dx=0$ and y varies from $\frac{1}{2}\pi$ to 0.

$\therefore$ from (1), $\oint_C \mathbf{F}\cdot d\mathbf{r} = \int_{x=0}^{1} e^x \cdot 0\,dx+\int_{y=0}^{\pi/2} e \cos y\,dy$

$$+\int_{x=1}^{0} e^x \sin \tfrac{1}{2}\pi\,dx + \int_{y=\pi/2}^{0} \cos y\,dy$$

$$= e\left[\sin y\right]_0^{\pi/2}+\left[e^x\right]_1^0+\left[\sin y\right]_{\pi/2}^0$$

$$= e+1-e+(0-1) = 0.$$

Ex. 34. *If* $\mathbf{F} = \dfrac{-y\,\mathbf{i}+x\,\mathbf{j}}{(x^2+y^2)}$, *evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$ *around any closed path C in the xy-plane.*

Sol. In the xy-plane, $\mathbf{r} = x\,\mathbf{i}+y\,\mathbf{j}$ so that $d\mathbf{r} = dx\,\mathbf{i}+dy\,\mathbf{j}$.

$$\therefore \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \left(\frac{-y\,\mathbf{i}+x\,\mathbf{j}}{x^2+y^2}\right)\cdot(dx\,\mathbf{i}+dy\,\mathbf{j})$$

$$= \int_C \frac{-y\,dx+x\,dy}{x^2+y^2}$$

We change to polar coordinates by putting $x=r\cos\theta$, $y=r\sin\theta$.

$\therefore$ $dx = -r\sin\theta\,d\theta+\cos\theta\,dr$
and $dy = r\cos\theta\,d\theta+\sin\theta\,dr$.

$$\therefore \frac{-y\,dx+x\,dy}{x^2+y^2} = \frac{1}{r^2}\Big[-r\sin\theta(-r\sin\theta\,d\theta+\cos\theta\,dr)$$
$$+r\cos\theta(r\cos\theta\,d\theta+\sin\theta\,dr)\Big]$$
$$= \frac{r^2(\cos^2\theta+\sin^2\theta)}{r^2}\,d\theta = d\theta.$$

$$\therefore \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C d\theta \qquad \text{...(1)}$$

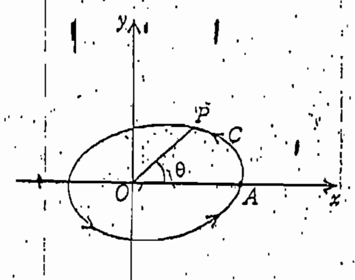


Fig (i)

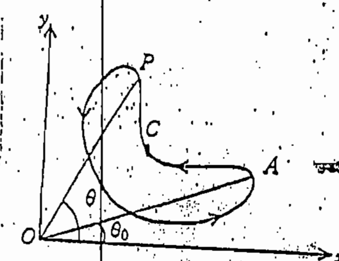


Fig (ii)

Case I. If the origin O lies inside the closed curve C as in fig. (i), then for the curve C at the point A, we have $\theta=0$ and when after a complete round we come back to A, then at A, $\theta=2\pi$. So from (1)

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{\theta=0}^{2\pi} d\theta = 2\pi.$$

Case II. If the origin O lies outside the closed curve C as in fig. (ii), then for the curve C at the point A, we have $\theta=\theta_0$ and when after a complete round along C we come back to A, then also at A, $\theta=\theta_0$. So from (1)

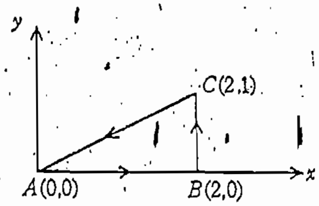
$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{\theta=\theta_0}^{\theta_0} d\theta = \left[\theta\right]_{\theta_0}^{\theta_0} = 0.$$

Ex. 35. *If* $\mathbf{F} = (2x^2+y^2)\,\mathbf{i}+(3y-4x)\,\mathbf{j}$, *evaluate* $\int \mathbf{F}\cdot d\mathbf{r}$ *around the triangle ABC whose vertices are* $A\ (0, 0)$, $B\ (2, 0)$ *and* $C\ (2, 1)$.

Sol. Let C_1 denote the curve consisting of the straight lines AB, BC and CA. Then

$\int \mathbf{F}\cdot d\mathbf{r}$ around the triangle

$$ABC = \int_{C_1} [(2x^2+y^2)\,\mathbf{i}+(3y-4x)\,\mathbf{j}]\cdot(dx\,\mathbf{i}+dy\,\mathbf{j})$$



$$= \int_{C_1} [(2x^2+y^2)\,dx+(3y-4x)\,dy]$$

$$= \int_{AB} [(2x^2+y^2)\,dx+(3y-4x)\,dy] + \int_{BC} [(2x^2+y^2)\,dx+(3y-4x)\,dy] + \int_{CA} [(2x^2+y^2)\,dx+(3y-4x)\,dy] \quad ...(1)$$

Now along the straight line AB, $y = 0$, $dy = 0$ and x varies from 0 to 2;

along the straight line BC, $x = 2$, $dx = 0$ and y varies from 0 to 1;

and along the straight line CA, $y-0 = \frac{1-0}{2-0}x$

i.e., $y=\frac{1}{2}x$ or $x=2y$ so that $dx=2\,dy$ and y varies from 1 to 0.

$\therefore$ from (1), $\int_{C_1} \mathbf{F}\cdot d\mathbf{r}$

$$= \int_{x=0}^{2} 2x^2\,dx + \int_{y=0}^{1} (3y-8)\,dy + \int_{y=1}^{0} [\{2\cdot(2y)^2+y^2\}(2\,dy) + (3y-8y)\,dy]$$

$$= 2\left[\frac{1}{3}x^3\right]_0^2 + \left[\frac{3y^2}{2}-8y\right]_0^1 + \int_1^0 (18y^2-5y)\,dy$$

$$= \frac{16}{3}+\frac{3}{2}-8+\left[6y^3-\frac{5}{2}y^2\right]_1^0 = \frac{16}{3}+\frac{3}{2}-8-6+\frac{5}{2}$$

$$= \frac{16}{3}-10 = -\frac{14}{3}.$$

Ex. 36. *Evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$ *where* $\mathbf{F} = xy\,\mathbf{i}+(x^2+y^2)\,\mathbf{j}$ *and C is the x-axis from* $x=2$ *to* $x=4$ *and the straight line* $x=4$ *from* $y=0$ *to* $y=12$.

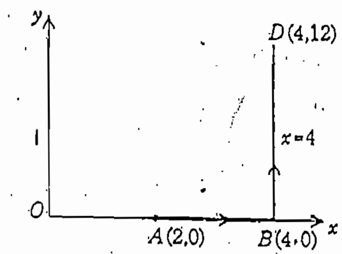
Sol. Here the curve C consists of the straight lines AB and BD where A, B and D are the points $(2, 0)$, $(4, 0)$ and $(4, 12)$ respectively.

We have $\int_C \mathbf{F}\cdot d\mathbf{r}$

$$= \int_C [xy\,\mathbf{i}+(x^2+y^2)\,\mathbf{j}]\cdot(dx\,\mathbf{i}+dy\,\mathbf{j})$$

$$= \int_C [xy\,dx+(x^2+y^2)\,dy]$$

$$= \int_{AB} [xy\,dx+(x^2+y^2)\,dy] + \int_{BD} [xy\,dx+(x^2+y^2)\,dy]. \quad ...(1)$$



Along the straight line AB, $y=0$, $dy=0$ and x varies from 2 to 4.

Along the straight line BD, $x=4$, $dx=0$ and y varies from 0 to 12.

$\therefore$ from (1), $\int_C \mathbf{F}\cdot d\mathbf{r}$

$$= \int_{x=2}^{4} (x\cdot 0)\,dx + \int_{y=0}^{12} (16+y^2)\,dy$$

$$= 0+\left[16y+\frac{1}{3}y^3\right]_0^{12} = 192+576 = 768.$$

Ex. 37. *Evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$, *where* $\mathbf{F} = xy\,\mathbf{i}+(x^2+y^2)\,\mathbf{j}$ *and C is the rectangle in the xy-plane bounded by the lines* $y=2, x=4, y=10$ *and* $x=1$.

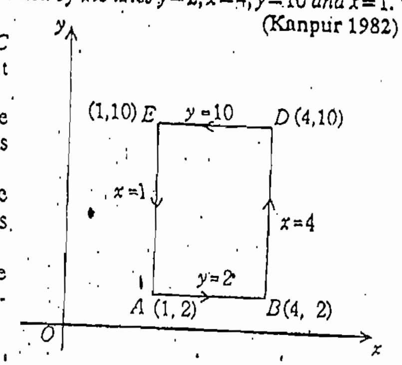
(Kanpur 1982)

Sol. Here the curve C consists of the four straight lines AB, BD, DE and EA.

Along the line AB, $y=2$, $dy=0$ and x varies from 1 to 4.

Along the line BD, $x=4$, $dx=0$ and y varies from 2 to 10.

Along the line DE, $y=10$, $dy=0$ and x varies from 4 to 1.



Along the line EA, $x=1$, $dx=0$ and y varies from 10 to 2.

We have $\mathbf{F}\cdot d\mathbf{r} = [xy\,\mathbf{i}+(x^2+y^2)\,\mathbf{j}]\cdot(dx\,\mathbf{i}+dy\,\mathbf{j})$

$= xy\,dx+(x^2+y^2)\,dy.$

Now $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{AB} \mathbf{F}\cdot d\mathbf{r}+\int_{BD} \mathbf{F}\cdot d\mathbf{r}+\int_{DE} \mathbf{F}\cdot d\mathbf{r}+\int_{EA} \mathbf{F}\cdot d\mathbf{r}$

$$= \int_{x=1}^{4} 2x\,dx+\int_{y=2}^{10}(16+y^2)\,dy+\int_{x=4}^{1}10\,x\,dx+\int_{y=10}^{2}(1+y^2)\,dy$$

$$= \left[x^2\right]_1^4+\left[16y+\frac{1}{3}y^3\right]_2^{10}+\left[5x^2\right]_4^1+\left[y+\frac{1}{3}y^3\right]_{10}^{2}$$

$$= 15+\left[160+\frac{1000}{3}-32-\frac{8}{3}\right]-75+\left[2+\frac{8}{3}-10-\frac{1000}{3}\right]$$

$$= 15+160-32-75+2-10 = 60.$$

Ex. 38. *Evaluate* $\int_C \dfrac{-y^3\,\mathbf{i}+x^3\,\mathbf{j}}{(x^2+y^2)^2}\cdot d\mathbf{r}$, *where C is the boundary of the square $x=\pm a$, $y=\pm a$ in the counter clockwise sense.*

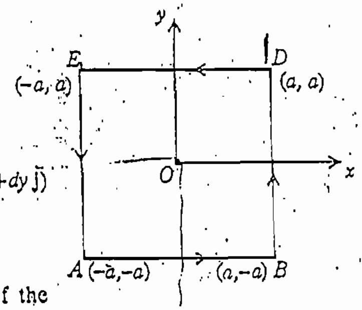
Sol. Let $\mathbf{F} = \dfrac{-y^3\,\mathbf{i}+x^3\,\mathbf{j}}{(x^2+y^2)^2}$.

We have

$$\mathbf{F}\cdot d\mathbf{r}= \frac{-y^3\,\mathbf{i}+x^3\,\mathbf{j}}{(x^2+y^2)^2}\cdot d\mathbf{r}$$

$$= \frac{-y^3\,\mathbf{i}+x^3\,\mathbf{j}}{(x^2+y^2)^2}\cdot(dx\,\mathbf{i}+dy\,\mathbf{j})$$

$$= \frac{-y^3\,dx+x^3\,dy}{(x^2+y^2)^2}.$$



The curve C consists of the four straight lines AB, BD, DE and EA.

Along the line AB, $y=-a$, $dy=0$ and x varies from $-a$ to a.
Along the line BD, $x=a$, $dx=0$ and y varies from $-a$ to a.
Along the line DE, $y=a$, $dy=0$ and x varies from a to $-a$.
Along the line EA, $x=-a$, $dx=0$ and y varies from a to $-a$.

We have $\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{AB} \mathbf{F}\cdot d\mathbf{r}+\int_{BD} \mathbf{F}\cdot d\mathbf{r}+\int_{DE} \mathbf{F}\cdot d\mathbf{r}+\int_{EA} \mathbf{F}\cdot d\mathbf{r}$

$$= \int_{x=-a}^{a} \frac{a^3\,dx}{(x^2+a^2)^2}+\int_{y=-a}^{a} \frac{a^3\,dy}{(y^2+a^2)^2}+\int_{x=a}^{-a} \frac{-a^3\,dx}{(x^2+a^2)^2}+\int_{y=a}^{-a} \frac{-a^3\,dy}{(y^2+a^2)^2}$$

$$= 4\int_{-a}^{a} \frac{a^3\,dx}{(x^2+a^2)^2}$$

$$\left[\because \int_a^b f(x)\,dx = \int_a^b f(y)\,dy = -\int_b^a f(x)\,dx\right].$$

$$= 8a^3\int_0^a \frac{dx}{(x^2+a^2)^2}$$

$$= 8a^3\int_0^{\pi/4} \frac{a\sec^2\theta\,d\theta}{a^4\sec^4\theta}, \text{ putting } x=a\tan\theta \text{ so that}$$

$$dx = a\sec^2\theta\,d\theta$$

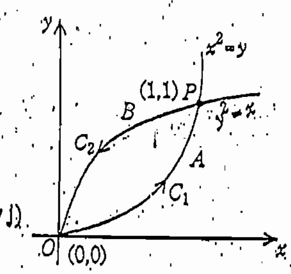
$$= 8\int_0^{\pi/4} \cos^2\theta\,d\theta = 4\int_0^{\pi/4} (1+\cos 2\theta)\,d\theta$$

$$= 4\left[\theta+\frac{1}{2}\sin 2\theta\right]_0^{\pi/4} = 4\left[\frac{\pi}{4}+\frac{1}{2}\right] = \pi+2.$$

Ex. 39. *Find the circulation of* $\mathbf{F}$ *round the curve C where* $\mathbf{F} = (2x+y^2)\,\mathbf{i}+(3y-4x)\,\mathbf{j}$ *and C is the curve $y=x^2$ from* $(0,0)$ *to* $(1,1)$ *and the curve $y^2=x$ from* $(1,1)$ *to* $(0,0)$.

Sol. Here the closed curve C consists of arcs OAP and PBO.

Let C_1 denote the arc OAP and C_2 denote the arc PBO.



Along C_1, we have $y=x^2$ so that $dy=2x\,dx$ and x varies from 0 to 1.

Along C_2, we have $x=y^2$ so that $dx=2y\,dy$ and y varies from 1 to 0.

Also

$\mathbf{F}\cdot d\mathbf{r} = [(2x+y^2)\,\mathbf{i}+(3y-4x)\,\mathbf{j}]\cdot(dx\,\mathbf{i}+dy\,\mathbf{j})$

$= (2x+y^2)\,dx+(3y-4x)\,dy.$

Now circulation of $\mathbf{F}$ round C

$$= \oint_C \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} + \int_{C_2} \mathbf{F}\cdot d\mathbf{r}$$

$$= \int_{C_1} [(2x+y^2)\,dx + (3y-4x)\,dy] + \int_{C_2} [(2x+y^2)\,dx + (3y-4x)\,dy]$$

$$= \int_{x=0}^{1} [(2x+x^4)\,dx + (3x^2-4x)\,2x\,dx]$$

$$+ \int_{y=1}^{0} [(2y^2+y^2)\,2y\,dy + (3y-4y^2)\,dy]$$

$$= \int_0^1 (2x-8x^2+6x^3+x^4)\,dx + \int_1^0 (3y-4y^2+6y^3)\,dy$$

$$= \left[x^2-\frac{8}{3}x^3+\frac{3}{2}x^4+\frac{1}{5}x^5\right]_0^1 + \left[\frac{3}{2}y^2-\frac{4}{3}y^3+\frac{3}{2}y^4\right]_1^0$$

$$= 1-\frac{8}{3}+\frac{3}{2}+\frac{1}{5}-\frac{3}{2}+\frac{4}{3}-\frac{3}{2}$$

$$= 1-\frac{4}{3}+\frac{1}{5}-\frac{3}{2} = \frac{30-40+6-45}{30} = -\frac{49}{30}.$$

Ex. 40. (a) *Evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$ *where*

$$\mathbf{F} = (2x+y)\,\mathbf{i}+(3y-x)\,\mathbf{j}+yz\,\mathbf{k}$$

and C is the curve $x=2t^2$, $y=t$, $z=t^3$ *from* $t=0$ *to* $t=1$.

(b) *If* $\mathbf{A}=(2y+3)\,\mathbf{i}+xz\,\mathbf{j}+(yz-x)\,\mathbf{k}$, *evaluate* $\int_C \mathbf{A}\cdot d\mathbf{r}$ *along the curve C :*

$$x=2t^2,\ y=t,\ z=t^3 \text{ from } t=0 \text{ to } t=1. \quad \text{(Kakatiya 1992)}$$

(c) *Evaluate* $\int_C \mathbf{A}\cdot d\mathbf{r}$ *where C is the line joining* (0, 0, 0) *and* (2, 1, 1) *and* $\mathbf{A} = (2y+3)\,\mathbf{i}+xz\,\mathbf{j}+(yz-x)\,\mathbf{k}$. (Nagarjuna 1991)

Sol. (a) Along the given curve C, we have

$$\mathbf{r} = 2t^2\,\mathbf{i}+t\,\mathbf{j}+t^3\,\mathbf{k},\quad \frac{d\mathbf{r}}{dt} = 4t\,\mathbf{i}+\mathbf{j}+3t^2\,\mathbf{k}$$

and $\mathbf{F} = (4t^2+t)\,\mathbf{i}+(3t-2t^2)\,\mathbf{j}+t^4\,\mathbf{k}$.

$$\therefore \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C \left[\mathbf{F}\cdot\left(\frac{d\mathbf{r}}{dt}\right)\right] dt$$

$$= \int_{t=0}^{1} [4t\,(4t^2+t)+(3t-2t^2)+3t^6]\,dt = \int_0^1 (3t^6+16t^3+2t^2+3t)\,dt$$

$$= \left[\frac{3}{7}t^7+4t^4+\frac{2}{3}t^3+\frac{3}{2}t^2\right]_0^1 = \frac{3}{7}+4+\frac{2}{3}+\frac{3}{2} = \frac{277}{42}.$$

(b) Along the given curve C, we have

$$\mathbf{r} = x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k} = 2t^2\,\mathbf{i}+t\,\mathbf{j}+t^3\,\mathbf{k},$$

on putting the values of x, y, z in terms of t.

$$\therefore \quad \frac{d\mathbf{r}}{dt} = 4t\,\mathbf{i}+\mathbf{j}+3t^2\,\mathbf{k}.$$

Also $\mathbf{A} = (2t+3)\,\mathbf{i}+2t^5\,\mathbf{j}+(t^4-2t^2)\,\mathbf{k}$.

$$\therefore \int_C \mathbf{A}\cdot d\mathbf{r} = \int_C \left[\mathbf{A}\cdot\left(\frac{d\mathbf{r}}{dt}\right)\right] dt$$

$$= \int_{t=0}^{1} [4t\,(2t+3)+1\cdot 2t^5+3t^2\cdot(t^4-2t^2)]\,dt$$

$$= \int_0^1 (8t^2+12t+2t^5+3t^6-6t^4)\,dt$$

$$= \left[\frac{8}{3}t^3+6t^2+\frac{1}{3}t^6+\frac{3}{7}t^7-\frac{6}{5}t^5\right]_0^1$$

$$= \frac{8}{3}+6+\frac{1}{3}+\frac{3}{7}-\frac{6}{5} = \frac{280+630+35+45-126}{105} = \frac{864}{105} = \frac{288}{35}.$$

(c) The equations of the straight line joining (0, 0, 0) and (2, 1, 1) are

$$\frac{x-0}{2-0} = \frac{y-0}{1-0} = \frac{z-0}{1-0} = t \text{ (say)}.$$

Then along C, $x=2t, y=t, z=t$.
At the point (0, 0, 0), $t=0$ and at the point (2, 1, 1), $t=1$.
Along the curve C, we have
$\mathbf{r} = x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k} = 2t\,\mathbf{i}+t\,\mathbf{j}+t\,\mathbf{k}$ and so $d\mathbf{r} = 2\,\mathbf{i}+\mathbf{j}+\mathbf{k}$.
Also along C, $\mathbf{A} = (2t+3)\,\mathbf{i}+2t^2\,\mathbf{j}+(t^2-2t)\,\mathbf{k}$.

$$\therefore \int_C \mathbf{A}\cdot d\mathbf{r} = \int_{t=0}^{1} [2\,(2t+3)+1.2t^2+1.(t^2-2t)]\,dt$$

$$= \int_0^1 [4t+6+2t^2+t^2-2t]\,dt = \int_0^1 [3t^2+2t+6]\,dt.$$

$$= \left[t^3+t^2+6t\right]_0^1 = 1+1+6 = 8.$$

Ex. 41. Evaluate

$$\int_C \{(2xy^3-y^2\cos x)\,dx+(1-2y\sin x+3x^2y^2)\,dy\}$$

where C is the arc of the parabola $2x = \pi y^2$ from $(0, 0)$ to $(\frac{1}{2}\pi, 1)$.
(Meerut 1977)

Sol. We know that $M dx+N dy$ is an exact differential if $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$.

Here $M = 2xy^3-y^2\cos x$; $\quad \therefore \dfrac{\partial M}{\partial y} = 6xy^2-2y\cos x$.

Also $N = 1-2y\sin x+3x^2y^2$; $\quad \therefore \dfrac{\partial N}{\partial x} = -2y\cos x+6xy^2$.

Thus $\dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x}$. Therefore $M dx+N dy$ is an exact differential.

Let $\phi(x, y)$ be such that

$$d\phi = (2xy^3-y^2\cos x)\,dx+(1-2y\sin x+3x^2y^2)\,dy.$$

Then $\dfrac{\partial \phi}{\partial x}dx+\dfrac{\partial \phi}{\partial y}dy = (2xy^3-y^2\cos x)\,dx+(1-2y\sin x+3x^2y^2)\,dy$.

$\therefore \dfrac{\partial \phi}{\partial x} = (2xy^3-y^2\cos x)$ which gives $\phi = x^2y^3-y^2\sin x+f_1(y)$ ...(1)

Also $\dfrac{\partial \phi}{\partial y} = 1-2y\sin x+3x^2y^2$ which gives $\phi = y-y^2\sin x$
$+x^2y^3+f_2(x)$. ...(2)

The values of ϕ given by (1) and (2) agree if we take $f_1(y)=y$ and $f_2(x)=0$. Then $\phi = y-y^2\sin x+x^2y^3$.

$\therefore$ The given integral

$$= \int_C d\phi = \int_C d\,(y-y^2\sin x+x^2y^3)$$

$$= \Big[y-y^2\sin x+x^2y^3\Big]_{(0,0)}^{(\pi/2,1)}$$

$$= \left[\left\{1-1\times\sin\frac{\pi}{2}+\frac{\pi^2}{4}\times 1\right\}-0\right] = \frac{\pi^2}{4}.$$

Ex. 42. *Find the circulation of* F *round the curve C where*

$$\mathbf{F} = y\,\mathbf{i}+z\,\mathbf{j}+x\,\mathbf{k}$$

and C is the circle $x^2+y^2 = 1$, $z=0$.

Sol. By definition, the circulation of F along the curve C is

$$= \oint_C \mathbf{F}\cdot d\mathbf{r}, \text{ where } \mathbf{r} = x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}$$

$$= \oint_C (y\,\mathbf{i}+z\,\mathbf{j}+x\,\mathbf{k})\cdot(dx\,\mathbf{i}+dy\,\mathbf{j}+dz\,\mathbf{k}) = \oint_C (y\,dx+z\,dy+x\,dz)$$

$$= \oint_C y\,dx \qquad [\because \text{ on } C,\ z=0 \text{ and } dz=0]$$

$$= \int_0^{2\pi} \sin\theta\,(-\sin\theta)\,d\theta \qquad [\because \text{ on } C,\ x=\cos\theta,\ y=\sin\theta]$$

$$= -\int_0^{2\pi} \sin^2\theta\,d\theta = -\int_0^{2\pi} \tfrac{1}{2}(1-\cos 2\theta)\,d\theta$$

$$= -\tfrac{1}{2}\left[\theta-\frac{\sin 2\theta}{2}\right]_0^{2\pi} = -\pi.$$

Ex. 43. *Find the circulation of* F *round the curve C, where* $\mathbf{F} = (x-y)\,\mathbf{i}+(x+y)\,\mathbf{j}$ *and C is the circle* $x^2+y^2 = 4$, $z=0$.

Sol. The parametric equations of the circle $x^2+y^2 = 4$, $z=0$ are $x=2\cos t$, $y=2\sin t$, $z=0$.

By definition, the circulation of F along the curve C is

$$= \oint_C \mathbf{F}\cdot d\mathbf{r}, \text{ where } \mathbf{r} = x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}$$

$$= \oint_C [(x-y)\,\mathbf{i}+(x+y)\,\mathbf{j}]\cdot(dx\,\mathbf{i}+dy\,\mathbf{j}+dz\,\mathbf{k})$$

$$= \oint_C [(x-y)\,dx+(x+y)\,dy]$$

$$= \int_{t=0}^{2\pi}\left[(x-y)\frac{dx}{dt}+(x+y)\frac{dy}{dt}\right]dt$$

$$= \int_0^{2\pi} [(2\cos t-2\sin t)\cdot(-2\sin t)+(2\cos t+2\sin t)\cdot 2\cos t]\,dt$$

$$= \int_0^{2\pi} 4\,(\sin^2 t+\cos^2 t)\,dt = 4\int_0^{2\pi} dt = 4\Big[\,t\,\Big]_0^{2\pi} = 8\pi.$$

Ex. 44. *Show that*

$$\int_C \left[-\frac{y}{x^2+y^2}\,\mathbf{i}+\frac{x}{x^2+y^2}\,\mathbf{j}\right]\cdot d\mathbf{r} = 2\pi,$$

where C is the circle $x^2+y^2 = 1$ in the xy-plane described in counter-clockwise sense.

Sol. The parametric equations of the circle are $x=\cos t,\ y=\sin t,\ z=0$ and along the circle t varies from 0 to 2π.

Along the circle C, we have

$$\mathbf{r}=x\,\mathbf{i}+y\,\mathbf{j}=\cos t\,\mathbf{i}+\sin t\,\mathbf{j}.$$

$$\therefore \quad \frac{d\mathbf{r}}{dt}=-\sin t\,\mathbf{i}+\cos t\,\mathbf{j}.$$

Hence the given integral

$$=\int_{t=0}^{2\pi}[(-\sin t\,\mathbf{i}+\cos t\,\mathbf{j})\cdot(-\sin t\,\mathbf{i}+\cos t\,\mathbf{j})]\,dt$$

$$=\int_0^{2\pi}(\sin^2 t+\cos^2 t)\,dt$$

$$=\int_0^{2\pi}dt=\Big[t\Big]_0^{2\pi}=2\pi.$$

Ex. 45. *If $\phi=2xyz^2$, and C is the curve $x=t^2, y=2t, z=t^3$ from $t=0$ to $t=1$, evaluate $\int_C \phi\, d\mathbf{r}$.* (Kanpur 1974)

Sol. We have $\mathbf{r}=x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}$ so that $d\mathbf{r}=dx\,\mathbf{i}+dy\,\mathbf{j}+dz\,\mathbf{k}$.

Now $$\int_C \phi\, d\mathbf{r}=\int_C 2xyz^2(dx\,\mathbf{i}+dy\,\mathbf{j}+dz\,\mathbf{k})$$

$$=2\mathbf{i}\int_C xyz^2\,dx+2\mathbf{j}\int_C xyz^2\,dy+2\mathbf{k}\int_C xyz^2\,dz$$

$$=2\mathbf{i}\int_{t=0}^{1}(t^2)(2t)(t^3)^2\,2t\,dt+2\mathbf{j}\int_{t=0}^{1}(t^2)(2t)(t^3)^2\,2\,dt$$
$$+2\mathbf{k}\int_{t=0}^{1}(t^2)(2t)(t^3)^2\,3t^2\,dt$$

$$[\because x=t^2, y=2t, z=t^3]$$

$$=8\mathbf{i}\int_0^1 t^{10}\,dt+8\mathbf{j}\int_0^1 t^9\,dt+12\mathbf{k}\int_0^1 t^{11}\,dt$$

$$=\frac{8}{11}\mathbf{i}+\frac{4}{5}\mathbf{j}+\mathbf{k}.$$

Ex. 46. *If $\mathbf{F}=xy\,\mathbf{i}-z\,\mathbf{j}+x^2\,\mathbf{k}$ and C is the curve $x=t^2, y=2t, z=t^3$ from $t=0$ to $t=1$, evaluate $\int_C \mathbf{F}\times d\mathbf{r}$.* (Kanpur 1974)

Sol. We have $\mathbf{r}=x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}$ so that

$$d\mathbf{r}=dx\,\mathbf{i}+dy\,\mathbf{j}+dz\,\mathbf{k}.$$

$$\therefore \quad \mathbf{F}\times d\mathbf{r}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ xy & -z & x^2\\ dx & dy & dz\end{vmatrix}$$

$$=-\mathbf{i}\,(z\,dz+x^2\,dy)-\mathbf{j}\,(xy\,dz-x^2\,dx)+\mathbf{k}\,(xy\,dy+z\,dx).$$

Changing in terms of t with the help of parametric equations $x=t^2, y=2t, z=t^3$, we have

$$\mathbf{F}\times d\mathbf{r}=-\mathbf{i}\,(t^3\cdot 3t^2\,dt+t^4\cdot 2\,dt)-\mathbf{j}\,(2t^3\cdot 3t^2\,dt-t^4\cdot 2t\,dt)$$
$$+\mathbf{k}\,(2t^3\cdot 2\,dt+t^3\cdot 2t\,dt)$$

$$=-\mathbf{i}\,(3t^5+2t^4)\,dt-\mathbf{j}\,(4t^5)\,dt+\mathbf{k}\,(4t^3+2t^4)\,dt.$$

$$\therefore \int_C \mathbf{F}\times d\mathbf{r}=-\mathbf{i}\int_0^1(3t^5+2t^4)\,dt-\mathbf{j}\int_0^1 4t^5\,dt+\mathbf{k}\int_0^1(4t^3+2t^4)\,dt$$

$$=-\mathbf{i}\left[\frac{3}{6}+\frac{2}{5}\right]-\mathbf{j}\left(\frac{4}{6}\right)+\mathbf{k}\left[\frac{4}{4}+\frac{2}{5}\right]=-\frac{9}{10}\mathbf{i}-\frac{2}{3}\mathbf{j}+\frac{7}{5}\mathbf{k}.$$

Ex. 47. *Let $\mathbf{r}$ denote the position vector of any point (x,y,z) measured from an origin O and let $r=|\mathbf{r}|$.*

Evaluate $\iint_S \frac{\mathbf{r}}{r^3}\cdot d\mathbf{S}$ where S denotes the sphere of radius a with centre at the origin.

Sol. The equation to the sphere S is $x^2+y^2+z^2=a^2$.

A vector normal to the sphere S at the point (x,y,z) is given by

$$\nabla(x^2+y^2+z^2)=2x\,\mathbf{i}+2y\,\mathbf{j}+2z\,\mathbf{k}.$$

If $\mathbf{n}$ denotes the unit vector along the outward drawn normal to the sphere S at the point (x,y,z), then

$$\mathbf{n}=\frac{2x\,\mathbf{i}+2y\,\mathbf{j}+2z\,\mathbf{k}}{\sqrt{(4x^2+4y^2+4z^2)}}=\frac{x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}}{a},$$

since $x^2+y^2+z^2=a^2$ on the sphere S.

Again $\mathbf{r}=x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}$

and $r=|\mathbf{r}|=\sqrt{(x^2+y^2+z^2)}$.

Let $\mathbf{F}=\frac{\mathbf{r}}{r^3}$.

Then $\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_S \mathbf{F}\cdot\mathbf{n}\, dS$

$= \iint_S \frac{\mathbf{r}}{r^3}\cdot\left(\frac{x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}}{a}\right) dS$

$= \iint_S \frac{(x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k})\cdot(x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k})}{a\,(x^2+y^2+z^2)^{3/2}}\, dS$

$= \iint_S \frac{x^2+y^2+z^2}{a\,(x^2+y^2+z^2)^{3/2}}\, dS = \iint_S \frac{1}{a\,(x^2+y^2+z^2)^{1/2}}\, dS$

$= \iint_S \frac{1}{a^2}\, dS$ $\quad$ [$\because$ on the sphere S, $x^2+y^2+z^2=a^2$]

$= \frac{1}{a^2}\iint_S dS = \frac{1}{a^2}\cdot 4\pi a^2$ $\quad$ [$\because$ surface of the sphere $S = 4\pi a^2$]

$= 4\pi.$

Ex. 48. (a) *Evaluate* $\iint_S \mathbf{F}\cdot\mathbf{n}\, dS$, *where* $\mathbf{F} = yz\,\mathbf{i}+zx\,\mathbf{j}+xy\,\mathbf{k}$ *and S is that part of the surface of the sphere* $x^2+y^2+z^2=1$ *which lies in the first octant.* (Meerut 1984; Agra 74; Kanpur 79)

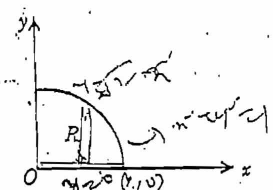
(b) *If* $\mathbf{F} = y\,\mathbf{i}+(x-2xz)\,\mathbf{j}-xy\,\mathbf{k}$ *and S is the surface of the sphere*

$$x^2+y^2+z^2=a^2,\ 0 \le x, y, z \le a,$$

evaluate $\iint_S \mathbf{F}\cdot\mathbf{n}\, dS$, *where* $\mathbf{n}$ *is a unit vector along the outward drawn normal to the surface S.* (Osmania 1989)

Sol. (a). A vector normal to the surface S is given by

$\nabla(x^2+y^2+z^2) = 2x\,\mathbf{i}+2y\,\mathbf{j}+2z\,\mathbf{k}.$



Therefore $\mathbf{n}$ = a unit normal to any point (x, y, z) of S

$= \frac{2x\,\mathbf{i}+2y\,\mathbf{j}+2z\,\mathbf{k}}{\sqrt{(4x^2+4y^2+4z^2)}} = x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}$,

since $x^2+y^2+z^2=1$ on the surface S.

$J = \frac{\partial(x,y)}{\partial(r,\theta)}$

We have $\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iint_R \mathbf{F}\cdot\mathbf{n}\,\frac{dx\,dy}{|\mathbf{n}\cdot\mathbf{k}|}$, where R is the projection of S on the xy-plane. The region R is bounded by x-axis, y-axis and the circle $x^2+y^2=1$, $z=0$.

We have $\mathbf{F}\cdot\mathbf{n} = (yz\,\mathbf{i}+zx\,\mathbf{j}+xy\,\mathbf{k})\cdot(x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k}) = 3xyz.$

Also $\mathbf{n}\cdot\mathbf{k} = (x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k})\cdot\mathbf{k} = z.$ $\quad\therefore\ |\mathbf{n}\cdot\mathbf{k}| = z.$

Hence $\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iint_R \frac{3xyz}{z}\, dx\, dy = 3\iint_R xy\, dx\, dy$

$= 3\int_{\theta=0}^{\pi/2}\int_{r=0}^{1} (r\cos\theta)(r\sin\theta)\, r\, d\theta\, dr$, on changing to polars

$= 3\int_0^{\pi/2}\left[\frac{r^4}{4}\right]_0^1 \cos\theta\sin\theta\, d\theta = \frac{3}{4}\left(\frac{1}{2}\right) = \frac{3}{8}.$

(b) A vector normal to the surface S is given by

$\nabla(x^2+y^2+z^2) = 2x\,\mathbf{i}+2y\,\mathbf{j}+2z\,\mathbf{k}.$

$\therefore$ $\mathbf{n}$ = a unit normal to any point (x, y, z) of S

$= \frac{2x\,\mathbf{i}+2y\,\mathbf{j}+2z\,\mathbf{k}}{\sqrt{(4x^2+4y^2+4z^2)}} = \frac{1}{a}(x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k})$,

since $x^2+y^2+z^2=a^2$ on the surface S.

Since $0 \le x, y, z \le a$, therefore the surface S is that part of the sphere $x^2+y^2+z^2=a^2$ which lies in the positive octant.

We have $\mathbf{F}\cdot\mathbf{n} = [y\,\mathbf{i}+(x-2xz)\,\mathbf{j}-xy\,\mathbf{k}]\cdot\frac{1}{a}(x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k})$

$= \frac{1}{a}[xy + y(x-2xz) - xyz] = \frac{1}{a}(2xy - 3xyz)$

$= \frac{y}{a}(2x - 3xz).$

Now $\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iint_R \mathbf{F}\cdot\mathbf{n}\,\frac{dx\,dz}{|\mathbf{n}\cdot\mathbf{j}|}$, where R is the projection of S on the zx-plane. The region R is bounded by x-axis, z-axis and the circle $x^2+z^2=a^2$, $y=0$.

We have $\mathbf{n}\cdot\mathbf{j} = \frac{1}{a}(x\,\mathbf{i}+y\,\mathbf{j}+z\,\mathbf{k})\cdot\mathbf{j} = \frac{y}{a}.$ $\quad\therefore\ |\mathbf{n}\cdot\mathbf{j}| = \frac{y}{a}.$

Hence $\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iint_R \frac{y}{a}(2x - 3xz)\,\frac{dx\,dz}{y/a}$

$$= \iint_R (2x - 3xz)\, dx\, dz$$

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} [2r\cos\theta - 3(r\cos\theta)(r\sin\theta)]\, r\, d\theta\, dr,$$

on changing to polars

$$= 2\int_{\theta=0}^{\pi/2} \left[\int_{r=0}^{a} r^2\, dr\right] \cos\theta\, d\theta - 3\int_{\theta=0}^{\pi/2} \left[\int_{r=0}^{a} r^3\, dr\right] \cos\theta \sin\theta\, d\theta$$

$$= \frac{2}{3}a^3 \int_0^{\pi/2} \cos\theta\, d\theta - \frac{3}{4}a^4 \int_0^{\pi/2} \cos\theta \sin\theta\, d\theta$$

$$= \frac{2}{3}a^3 - \frac{3}{4}a^4 \cdot \frac{1}{2} = \frac{2}{3}a^3 - \frac{3}{8}a^4.$$

Ex. 49. *If* $\mathbf{F} = y\,\mathbf{i} + (x - 2xz)\,\mathbf{j} - xy\,\mathbf{k}$, *evaluate* $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$ *where S is the surface of the sphere* $x^2 + y^2 + z^2 = a^2$ *above the xy-plane.* (Kanpur 1980; Bundelkhand 79)

Sol. let $\mathbf{f} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x - 2xz & -xy \end{vmatrix}$

$$= \mathbf{i}\left[\frac{\partial}{\partial y}(-xy) - \frac{\partial}{\partial z}(x - 2xz)\right] - \mathbf{j}\left[\frac{\partial}{\partial x}(-xy) - \frac{\partial}{\partial z}(y)\right] + \mathbf{k}\left[\frac{\partial}{\partial x}(x - 2xz) - \frac{\partial}{\partial y}(y)\right]$$

$$= x\,\mathbf{i} + y\,\mathbf{j} - 2z\,\mathbf{k}.$$

A vector normal to the sphere S at the point (x, y, z) is given by

$$\nabla(x^2 + y^2 + z^2) = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}.$$

If $\mathbf{n}$ denotes the unit vector along the outward drawn normal to the sphere S at the point (x, y, z), then

$$\mathbf{n} = \frac{2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}}{\sqrt{(4x^2 + 4y^2 + 4z^2)}} = \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{a},$$

since $x^2 + y^2 + z^2 = a^2$ on the sphere S.

Now $(\nabla \times \mathbf{F}) \cdot \mathbf{n} = \mathbf{f} \cdot \mathbf{n} = \dfrac{(x\,\mathbf{i} + y\,\mathbf{j} - 2z\,\mathbf{k}) \cdot (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})}{a}$

$$= \frac{x^2 + y^2 - 2z^2}{a}.$$

We know that $\iint_S \mathbf{f} \cdot \mathbf{n}\, dS = \iint_R \mathbf{f} \cdot \mathbf{n} \dfrac{dx\, dy}{\mathbf{n} \cdot \mathbf{k}}$, where R is the projection of the surface S on the xy-plane. Obviously the region R is the area bounded by the circle $x^2 + y^2 = a^2, z = 0$ in the xy-plane.

We have $\mathbf{n} \cdot \mathbf{k} = \dfrac{1}{a}(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}) \cdot \mathbf{k} = \dfrac{z}{a}$.

$$\therefore \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_S \mathbf{f} \cdot \mathbf{n}\, dS$$

$$= \iint_R (\mathbf{f} \cdot \mathbf{n}) \frac{dx\, dy}{z/a} = \iint_R \frac{x^2 + y^2 - 2z^2}{a} \frac{dx\, dy}{z/a},$$

where $z^2 = a^2 - x^2 - y^2$

$$= \iint_R \frac{x^2 + y^2 - 2(a^2 - x^2 - y^2)}{\sqrt{(a^2 - x^2 - y^2)}}\, dx\, dy$$

$$= \iint_R \frac{3(x^2 + y^2) - 2a^2}{\sqrt{[a^2 - (x^2 + y^2)]}}\, dx\, dy$$

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{a} \frac{3r^2 - 2a^2}{\sqrt{(a^2 - r^2)}}\, r\, d\theta\, dr, \text{ changing to polar coordinates}$$

$$= 2\pi \int_{r=0}^{a} \frac{r(3r^2 - 2a^2)}{\sqrt{(a^2 - r^2)}}\, dr, \text{ first integrating with respect to } \theta$$

$$= 2\pi \int_0^{\pi/2} \frac{a \sin t\,(3a^2 \sin^2 t - 2a^2)}{a \cos t}\, a \cos t\, dt,$$

putting $r = a \sin t$ so that $dr = a \cos t\, dt$

$$= 2\pi a^3 \left[3 \int_0^{\pi/2} \sin^3 t\, dt - 2 \int_0^{\pi/2} \sin t\, dt\right],$$

$$= 2\pi a^3 \left[3 \cdot \frac{2}{3 \cdot 1} - 2 \cdot 1\right] = 2\pi a^3 (2 - 2) = 0.$$

Ex. 50. *If* $\mathbf{F} = y\,\mathbf{i} + (x - 2xz)\,\mathbf{j} - xy\,\mathbf{k}$, *evaluate* $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$ *where S is the surface of the sphere* $x^2 + y^2 + z^2 = a^2$ *which lies in the first octant.* (Kanpur 1974)

Sol. Proceed as in solved example 49. Here R *i.e.*, the projection of the surface S on the xy-plane is the area in the xy-plane bounded by

x-axis, y-axis and the circle $x^2 + y^2 = a^2, z = 0$. This area is in the form of a quadrant of a circle.

So here $\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$

$$= \int_{\theta=0}^{\pi/2} \int_{r=0}^{a} \frac{3r^2 - 2a^2}{\sqrt{(a^2 - r^2)}}\, r\, d\theta\, dr$$

$= 0$, proceeding as in solved example 49.

Ex. 51. *Evaluate $\iint_S \phi \mathbf{n}\, dS$, where $\phi = \frac{3}{8} xyz$ and S is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between $z = 0$ and $z = 5$.*

Sol. A vector normal to the surface S *i.e.*, the cylinder $x^2 + y^2 = 16$ is given by $\nabla (x^2 + y^2) = 2x\,\mathbf{i} + 2y\,\mathbf{j}$.

Therefore $\mathbf{n}$ = a unit normal to any point of S

$$= \frac{2x\,\mathbf{i} + 2y\,\mathbf{j}}{\sqrt{(4x^2 + 4y^2)}} = \frac{x\,\mathbf{i} + y\,\mathbf{j}}{4}, \text{ since } x^2 + y^2 = 16, \text{ on the surface } S.$$

We have $\iint_S \phi \mathbf{n}\, dS = \iint_R \phi \mathbf{n} \dfrac{dx\, dz}{\mathbf{n} \cdot \mathbf{j}}$,

where R is the projection of S on the xz-plane.

We have $\mathbf{n} \cdot \mathbf{j} = \frac{1}{4}(x\,\mathbf{i} + y\,\mathbf{j}) \cdot \mathbf{j} = \frac{1}{4} y$.

$$\therefore \iint_S \phi \mathbf{n}\, dS = \iint_R \left(\frac{3}{8} xyz\right) \left(\frac{x\,\mathbf{i} + y\,\mathbf{j}}{4}\right) \frac{dx\, dz}{y/4}$$

$$= \frac{3}{8} \int_{z=0}^{5} \int_{x=0}^{4} \left[x^2 z\,\mathbf{i} + xz\sqrt{(16 - x^2)}\,\mathbf{j}\right] dx\, dz,$$

since $y = \sqrt{(16 - x^2)}$ on S

$$= \frac{3}{8} \int_{x=0}^{4} [x^2\,\mathbf{i} + x\sqrt{(16 - x^2)}] \left[\frac{z^2}{2}\right]_0^5 dx,$$

first integrating with respect to z

$$= \frac{3}{8} \cdot \frac{25}{2} \int_0^4 \left[x^2\,\mathbf{i} + \left(-\frac{1}{2}\right)(16 - x^2)^{1/2}(-2x)\,\mathbf{j}\right] dx$$

$$= \frac{75}{16} \left[\frac{x^3}{3}\,\mathbf{i} - \frac{1}{2} \cdot \frac{2}{3}(16 - x^2)^{3/2}\,\mathbf{j}\right]_0^4$$

$$= \frac{75}{16} \left[\frac{64}{3}\,\mathbf{i} - \frac{1}{3}(0 - 64)\,\mathbf{j}\right]$$

$$= \frac{75}{16} \cdot \frac{64}{3} (\mathbf{i} + \mathbf{j}) = 100\,(\mathbf{i} + \mathbf{j}).$$

Ex. 52. *Evaluate $\int_V (2x + y)\, dV$, where V is the closed region bounded by the cylinder $z = 4 - x^2$ and the planes $x = 0, y = 0, y = 2$ and $z = 0$.*

(Kanpur 1981)

Sol. The cylinder $z = 4 - x^2$ meets the x-axis (*i.e.*, $y = 0$, $z = 0$) at $x^2 = 4$ or $x = 2$ on the positive side *i.e.*, at the point $(2, 0, 0)$. It meets z-axis (*i.e.*, $x = 0, y = 0$) at $z = 4$ *i.e.*, at the point $(0, 0, 4)$. Therefore the limits of integration for z are from 0 to $4 - x^2$, for x from 0 to 2 and for y from 0 to 2.

Also $dV = dx\, dy\, dz$.

$$\therefore \int_V (2x + y)\, dV = \int_{x=0}^{2} \int_{y=0}^{2} \int_{z=0}^{4-x^2} (2x + y)\, dx\, dy\, dz$$

$$= \int_{x=0}^{2} \int_{y=0}^{2} (2x + y) \left[z\right]_{z=0}^{4-x^2} dx\, dy$$

$$= \int_{x=0}^{2} \int_{y=0}^{2} (2x + y)(4 - x^2)\, dx\, dy$$

$$= \int_{x=0}^{2} (4 - x^2) \left[2xy + \frac{y^2}{2}\right]_{y=0}^{2} dx$$

$$= \int_0^2 (4 - x^2)[4x + 2]\, dx = 2\int_0^2 (4 - x^2)(2x + 1)\, dx$$

$$= 2\int_0^2 (4 + 8x - x^2 - 2x^3)\, dx$$

$$= 2\left[4x + 4x^2 - \frac{x^3}{3} - \frac{x^4}{2}\right]_0^2 = 2\left[8 + 16 - \frac{8}{3} - 8\right] = \frac{80}{3}.$$

Ex. 53. *Evaluate $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$, where $\mathbf{F} = z\,\mathbf{i} + x\,\mathbf{j} - 3y^2 z\,\mathbf{k}$ and S is the surface of the cylinder $x^2 + y^2 = 16$ included in the first octant between $z = 0$ and $z = 5$.*

(Tirupati 1993)

Sol. A vector normal to the surface S is given by

$$\nabla (x^2 + y^2 = 16) = 2x\,\mathbf{i} + 2y\,\mathbf{j}.$$

Therefore $\mathbf{n}$ = a unit normal to any point of S

$$= \frac{2x\,\mathbf{i} + 2y\,\mathbf{j}}{\sqrt{(4x^2 + 4y^2)}} = \frac{x\,\mathbf{i} + y\,\mathbf{j}}{4}, \text{ since } x^2 + y^2 = 16, \text{ on the surface } S.$$

We have $\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iint_R \mathbf{F}\cdot\mathbf{n} \dfrac{dx\, dz}{|\mathbf{n}\cdot\mathbf{j}|}$, where R is the projection of S on the x-z plane. It should be noted that in this case we cannot take the projection of S on the x-y plane as the surface S is perpendicular to the x-y plane.

Now $\mathbf{F}\cdot\mathbf{n} = (z\mathbf{i} + x\mathbf{j} - 3y^2 z\mathbf{k})\cdot\left(\dfrac{x\mathbf{i}+y\mathbf{j}}{4}\right) = \frac{1}{4}(xz + xy)$,

$$\mathbf{n}\cdot\mathbf{j} = \left(\frac{x\mathbf{i}+y\mathbf{j}}{4}\right)\cdot\mathbf{j} = \frac{y}{4}.$$

Therefore the required surface integral is

$$= \iint_R \frac{xz + xy}{4}\, \frac{dx\, dz}{y/4}$$

$$= \int_{z=0}^{5}\int_{x=0}^{4}\left(\frac{xz}{\sqrt{(16-x^2)}} + x\right) dx\, dz, \text{ since } y = \sqrt{(16-x^2)} \text{ on } S$$

$$= \int_0^5 (4z+8)\, dz = 90.$$

Ex. 54. *If* $\mathbf{F} = (2x^2 - 3z)\mathbf{i} - 2xy\mathbf{j} - 4x\mathbf{k}$, *then evaluate* $\iiint_V \nabla\cdot\mathbf{F}\, dV$ *where* V *is the closed region bounded by the planes* $x = 0,\ y = 0,\ z = 0$ *and* $2x + 2y + z = 4$.

(Osmania 1989, 90; Kanpur 76, 78)

Also Evaluate $\iiint_V \nabla\times\mathbf{F}\, dV$.

Sol. We have $\mathbf{F} = (2x^2 - 3z)\mathbf{i} - 2xy\mathbf{j} - 4x\mathbf{k}$.

$$\therefore \nabla\cdot\mathbf{F} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\cdot[(2x^2 - 3z)\mathbf{i} - 2xy\mathbf{j} - 4x\mathbf{k}]$$

$$= \frac{\partial}{\partial x}(2x^2 - 3z) + \frac{\partial}{\partial y}(-2xy) + \frac{\partial}{\partial z}(-4x)$$

$$= 4x - 2x = 2x.$$

$$\therefore \iiint_V \nabla\cdot\mathbf{F}\, dV = \iiint_V 2x\, dx\, dy\, dz \qquad [\because dV = dx\, dy\, dz]$$

$$= 2\int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} x\, dx\, dy\, dz.$$

[Note that we have taken a thin column parallel to z-axis as the elementary volume. It cuts the boundary at $z = 0$ and $z = 4 - 2x - 2y$. Also the projection of the plane $2x + 2y + z = 4$ on the xy-plane is

bounded by the axes $y = 0$, $x = 0$ and the line $x + y = 2$. Hence the limits for y are from 0 to $2 - x$ and those for x are from 0 to 2]

$$\therefore \iiint_V \nabla\cdot\mathbf{F}\, dV = 2\int_{x=0}^{2}\int_{y=0}^{2-x} x\left[z\right]_{z=0}^{4-2x-2y} dx\, dy$$

$$= 2\int_{x=0}^{2}\int_{y=0}^{2-x} x(4 - 2x - 2y)\, dx\, dy$$

$$= 2\int_{x=0}^{2}\left[4xy - 2x^2 y - xy^2\right]_{y=0}^{2-x} dx$$

$$= 2\int_0^2 [4x(2-x) - 2x^2(2-x) - x(2-x)^2]\, dx$$

$$= 2\int_0^2 [x^3 - 4x^2 + 4x]\, dx, \text{ on simplifying}$$

$$= 2\left[\frac{1}{4}x^4 - \frac{4}{3}x^3 + 2x^2\right]_0^2 = 2\left[4 - \frac{32}{3} + 8\right] = \frac{8}{3}.$$

Second part. We have

$$\nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x^2 - 3z & -2xy & -4x \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(-4x) - \frac{\partial}{\partial z}(-2xy)\right]\mathbf{i} - \left[\frac{\partial}{\partial x}(-4x) - \frac{\partial}{\partial z}(2x^2 - 3z)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(-2xy) - \frac{\partial}{\partial y}(2x^2 - 3z)\right]\mathbf{k}$$

$$= 0\,\mathbf{i} - (-4 + 3)\mathbf{j} + (-2y)\mathbf{k} = \mathbf{j} - 2y\mathbf{k}.$$

$$\therefore \iiint_V \nabla\times\mathbf{F}\, dV = \iiint_V (\mathbf{j} - 2y\mathbf{k})\, dx\, dy\, dz$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} (\mathbf{j} - 2y\mathbf{k})\, dx\, dy\, dz$$

$$= \int_{x=0}^{2}\int_{y=0}^{2-x} (\mathbf{j} - 2y\mathbf{k})(4 - 2x - 2y)\, dx\, dy$$

$$= \int_{x=0}^{2}\left[\mathbf{j}(4y - 2xy - y^2) - 2\mathbf{k}(2y^2 - xy^2 - \tfrac{2}{3}y^3)\right]_{y=0}^{2-x} dx$$

$$= \int_{x=0}^{2}\left[\mathbf{j}(2 - x)(4 - 2x - 2 + x)\right.$$

$$-2\,\mathbf{k}(2-x)^2\left\{2-x-\tfrac{2}{3}(2-x)\right\}\Big]\,dx$$

$$=\int_0^2\left[(2-x)^2\,\mathbf{j}-\tfrac{2}{3}(2-x)^3\,\mathbf{k}\right]dx$$

$$=\int_0^2\left[(x-2)^2\,\mathbf{j}+\tfrac{2}{3}(x-2)^3\,\mathbf{k}\right]dx$$

$$=\left[\frac{(x-2)^3}{3}\right]_0^2\mathbf{j}+\left[\frac{2}{3}\,\frac{(x-2)^4}{4}\right]_0^2\mathbf{k}$$

$$=\frac{8}{3}\,\mathbf{j}-\frac{8}{3}\,\mathbf{k}=\frac{8}{3}(\mathbf{j}-\mathbf{k}).$$

Ex. 55. *Evaluate* $\iiint_V \phi\, dV$, *where* $\phi = 45x^2y$ *and V is the closed region bounded by the planes* $4x+2y+z=8$, $x=0$, $y=0$, $z=0$.

Sol. We have

$$\iiint_V \phi\, dv=\int_{x=0}^{2}\int_{y=0}^{4-2x}\int_{z=0}^{8-4x-2y}45x^2y\,dx\,dy\,dz$$

$$=45\int_{x=0}^{2}\int_{y=0}^{4-2x}x^2y\,\Big[z\Big]_0^{8-4x-2y}dx\,dy$$

$$=45\int_{x=0}^{2}\int_{y=0}^{4-2x}x^2y\,(8-4x-2y)\,dx\,dy$$

$$=45\int_{x=0}^{2}\left[x^2(8-4x)\frac{y^2}{2}-2x^2\frac{y^3}{3}\right]_0^{4-2x}dx$$

$$=45\int_0^2\frac{x^2}{3}(4-2x)^3\,dx=128.$$

Ex. 56. *Evaluate* $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$,

where $\mathbf{F}=(x+y^2)\,\mathbf{i}-2x\,\mathbf{j}+2yz\,\mathbf{k}$ *and S is the surface of the plane* $2x+y+2z=6$ *in the first octant.* (Kanpur 1970)

Sol. A vector normal to the surface S is given by

$$\nabla(2x+y+2z)=2\,\mathbf{i}+\mathbf{j}+2\,\mathbf{k}.$$

$\therefore$ $\mathbf{n}$ = a unit normal vector at any point (x, y, z) of S

$$=\frac{2\,\mathbf{i}+\mathbf{j}+2\,\mathbf{k}}{\sqrt{(4+1+4)}}=\left(\tfrac{2}{3}\,\mathbf{i}+\tfrac{1}{3}\,\mathbf{j}+\tfrac{2}{3}\,\mathbf{k}\right).$$

We have $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS=\iint_R \mathbf{F}\cdot\mathbf{n}\,\dfrac{dx\,dy}{|\mathbf{n}\cdot\mathbf{k}|}$, where R is the projection of S on the xy-plane. The region R is bounded by x-axis, y-axis and the straight line $2x+y=6$, $z=0$.

We have $\mathbf{F}\cdot\mathbf{n}=[(x+y^2)\,\mathbf{i}-2x\,\mathbf{j}+2yz\,\mathbf{k}]\cdot(\tfrac{2}{3}\,\mathbf{i}+\tfrac{1}{3}\,\mathbf{j}+\tfrac{2}{3}\,\mathbf{k})$

$$=\tfrac{2}{3}(x+y^2)-\tfrac{2}{3}x+\tfrac{4}{3}yz=\tfrac{2}{3}y^2+\tfrac{4}{3}yz.$$

Also $\quad \mathbf{n}\cdot\mathbf{k}=(\tfrac{2}{3}\,\mathbf{i}+\tfrac{1}{3}\,\mathbf{j}+\tfrac{2}{3}\,\mathbf{k})\cdot\mathbf{k}=\tfrac{2}{3}$.

Hence $\quad \iint_S \mathbf{F}\cdot\mathbf{n}\,dS=\iint_R\left[\tfrac{2}{3}y^2+\tfrac{4}{3}yz\right]\cdot\tfrac{3}{2}\,dx\,dy$

$$=\iint_R (y^2+2yz)\,dx\,dy$$

$$=\iint_R\left[y^2+2y\left(\frac{6-2x-y}{2}\right)\right]dx\,dy,$$ using the fact that $z=\dfrac{6-2x-y}{2}$ from the equation of S,

$$=\iint_R (y^2+6y-2xy-y^2)\,dx\,dy=2\iint_R y(3-x)\,dx\,dy$$

$$=2\int_{y=0}^{6}\int_{x=0}^{(6-y)/2}y(3-x)\,dx\,dy.$$

[Note that R is bounded by x-axis, y-axis and the straight line $2x+y=6$, $z=0$. To evaluate the double integral over R, keep y fixed and integrate with respect to x from $x=0$ to $x=\dfrac{6-y}{2}$; then integrate with respect to y from $y=0$ to $y=6$. In this way R is completely covered].

$$=2\int_{y=0}^{6}y\left[3x-\frac{x^2}{2}\right]_{x=0}^{(6-y)/2}dy$$

$$=2\int_0^6 y\left[\frac{3(6-y)}{2}-\frac{(6-y)^2}{8}\right]dy$$

$$=2\int_0^6 y\left[9-\frac{3y}{2}-\frac{36}{8}+\frac{12y}{8}-\frac{y^2}{8}\right]dy$$

$$=2\int_0^6 y\left[\frac{36}{8}-\frac{y^2}{8}\right]dy=\int_0^6\left[9y-\frac{y^3}{4}\right]dy$$

$$= \left[9\frac{y^2}{2} - \frac{y^4}{16}\right]_0^6 = \left[9 \cdot \frac{36}{2} - \frac{36 \times 36}{16}\right] = [162 - 81] = 81.$$

Ex. 57. *Evaluate* $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$, *where* $\mathbf{F} = y\,\mathbf{i} + 2x\,\mathbf{j} - z\,\mathbf{k}$ *and S is the surface of the plane* $2x + y = 6$ *in the first octant cut off by the plane* $z = 4$.

Sol. A vector normal to the surface S is given by

$$\nabla (2x + y) = 2\mathbf{i} + \mathbf{j}.$$

Therefore $\mathbf{n}$ = a unit normal vector at any point (x, y, z) of S

$$= \frac{2\mathbf{i} + \mathbf{j}}{\sqrt{(4+1)}} = \frac{1}{\sqrt{5}}(2\mathbf{i} + \mathbf{j}).$$

We have $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_R \mathbf{F} \cdot \mathbf{n} \dfrac{dx\,dz}{|\mathbf{n} \cdot \mathbf{j}|}$, where R is the projection of S on the xz-plane. It should be noted that in this case we cannot take the projection on the xy-plane because the surface S is perpendicular to xy-plane.

Now $\mathbf{F} \cdot \mathbf{n} = (y\,\mathbf{i} + 2x\,\mathbf{j} - z\,\mathbf{k}) \cdot \left(\dfrac{2}{\sqrt{5}}\mathbf{i} + \dfrac{1}{\sqrt{5}}\mathbf{j}\right) = \dfrac{2}{\sqrt{5}}y + \dfrac{2}{\sqrt{5}}x.$

Also $\quad \mathbf{n} \cdot \mathbf{j} = \dfrac{1}{\sqrt{5}}(2\mathbf{i} + \mathbf{j}) \cdot \mathbf{j} = \dfrac{1}{\sqrt{5}}.$

$\therefore$ The required surface integral is

$$= \iint_R \left(\frac{2}{\sqrt{5}}y + \frac{2}{\sqrt{5}}x\right) \cdot \sqrt{5}\, dx\, dz = \iint_R 2(y + x)\, dx\, dz$$

$$= 2\iint_R [6 - 2x + x]\, dx\, dz, \text{ since } y = 6 - 2x \text{ on } S$$

$$= 2\iint_R (6 - x)\, dx\, dz = 2\int_{z=0}^{4}\int_{x=0}^{3} (6 - x)\, dx\, dz$$

$$= 2\int_{x=0}^{3} (6 - x)\left[z\right]_0^4 dx = 8\left[6x - \frac{x^2}{2}\right]_0^3 = 8\left[18 - \frac{9}{2}\right] = 108.$$

Ex. 58. *Evaluate* $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$, *where* $\mathbf{F} = 18z\,\mathbf{i} - 12\,\mathbf{j} + 3y\,\mathbf{k}$ *and S is the surface of the plane* $2x + 3y + 6z = 12$ *in the first octant.*

Sol. A vector normal to the surface S *i.e.*, the plane $2x + 3y + 6z = 12$ is given by

$$\nabla (2x + 3y + 6z) = 2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}.$$

$\therefore$ $\mathbf{n}$ = a unit normal vector at any point (x, y, z) of S

$$= \frac{2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}}{\sqrt{(4 + 9 + 36)}} = \frac{2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}}{7}.$$

We have $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_R \mathbf{F} \cdot \mathbf{n} \dfrac{dx\,dy}{\mathbf{n} \cdot \mathbf{k}}$, where R is the projection of S on the xy-plane. The region R is bounded by x-axis, y-axis and the straight line $2x + 3y = 12$, $z = 0$.

We have $\mathbf{F} \cdot \mathbf{n} = (18z\,\mathbf{i} - 12\,\mathbf{j} + 3y\,\mathbf{k}) \cdot \left(\dfrac{2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}}{7}\right)$

$$= \frac{1}{7}(36z - 36 + 18y) = \frac{18}{7}(2z + y - 2).$$

Also $\mathbf{n} \cdot \mathbf{k} = \dfrac{1}{7}(2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}) \cdot \mathbf{k} = \dfrac{6}{7}.$

Hence $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_R \dfrac{18}{7}(2z + y - 2)\dfrac{dx\,dy}{6/7}$

$$= \iint_R (6z + 3y - 6)\, dx\, dy$$

$$= \iint_R [(12 - 2x - 3y) + 3y - 6]\, dx\, dy,$$

since $6z = 12 - 2x - 3y$ from the equation of S

$$= \iint_R (6 - 2x)\, dx\, dy = 2\iint_R (3 - x)\, dx\, dy$$

$$= 2\int_{x=0}^{6}\int_{y=0}^{(12-2x)/3} (3 - x)\, dx\, dy$$

$$= 2\int_{x=0}^{6} (3 - x)\left[y\right]_{y=0}^{(12-2x)/3} dx$$

$$= 2\int_0^6 (3 - x) \cdot \frac{1}{3}(12 - 2x)\, dx = \frac{4}{3}\int_0^6 (3 - x)(6 - x)\, dx$$

$$= \frac{4}{3}\int_0^6 (18 - 9x + x^2)\, dx = \frac{4}{3}\left[18x - \frac{9}{2}x^2 + \frac{1}{3}x^3\right]_0^6$$

$$= \frac{4}{3}[108 - 162 + 72] = \frac{4}{3} \cdot 18 = 24.$$

Ex. 59. *If* $\mathbf{F} = 2y\,\mathbf{i} - 3\,\mathbf{j} + x^2\,\mathbf{k}$ *and S is the surface of the parabolic cylinder* $y^2 = 8x$ *in the first octant bounded by the planes* $y = 4$ *and* $z = 6$, *then evaluate* $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$.

Sol. A vector normal to the surface S *i.e.*, the cylinder $8x - y^2 = 0$ is given by $\nabla (8x - y^2) = 8\,\mathbf{i} - 2y\,\mathbf{j}$.

$\therefore$ n = a unit normal vector at any point (x, y, z) of S

$$= \frac{8\mathbf{i} - 2y\mathbf{j}}{\sqrt{(64 + 4y^2)}} = \frac{4\mathbf{i} - y\mathbf{j}}{\sqrt{(16 + y^2)}}.$$

We have $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_R \mathbf{F} \cdot \mathbf{n} \frac{dy\, dz}{\mathbf{n} \cdot \mathbf{i}}$,

where R is the projection of S on the yz-plane.

We have $\mathbf{F} \cdot \mathbf{n} = (2y\mathbf{i} - 3\mathbf{j} + x^2\mathbf{k}) \cdot \left[\frac{4\mathbf{i} - y\mathbf{j}}{\sqrt{(16 + y^2)}}\right]$

$$= \frac{8y + 3y}{\sqrt{(16 + y^2)}} = \frac{11y}{\sqrt{(16 + y^2)}}.$$

Also $\mathbf{n} \cdot \mathbf{i} = \left[\frac{4\mathbf{i} - y\mathbf{j}}{\sqrt{(16 + y^2)}}\right] \cdot \mathbf{i} = \frac{4}{\sqrt{(16 + y^2)}}.$

Hence $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_R \frac{11y}{\sqrt{(16 + y^2)}} \cdot \frac{dy\, dz}{4/\sqrt{(16 + y^2)}}$

$$= \frac{11}{4} \iint_R y\, dy\, dz = \frac{11}{4} \int_{y=0}^{4} \int_{z=0}^{6} y\, dy\, dz$$

$$= \frac{11}{4} \int_{y=0}^{4} y \left[z\right]_{z=0}^{6} dy = \frac{11}{4} \cdot 6 \int_0^4 y\, dy$$

$$= \frac{11}{4} \cdot 6 \cdot \left[\frac{y^2}{2}\right]_0^4 = \frac{11}{4} \cdot 6 \cdot 8 = 132.$$

Ex. 60. *Evaluate* $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$ *over the surface S of the cylinder $x^2 + y^2 = 9$ included in the first octant between $z = 0$ and $z = 4$ where* $\mathbf{F} = z\mathbf{i} + x\mathbf{j} - yz\mathbf{k}$.

Sol. A vector normal to the surface S *i.e.*, the cylinder $x^2 + y^2 = 9$ is given by $\nabla(x^2 + y^2) = 2x\mathbf{i} + 2y\mathbf{j}$.

$\therefore$ n = a unit normal vector at any point of S

$$= \frac{2x\mathbf{i} + 2y\mathbf{j}}{\sqrt{(4x^2 + 4y^2)}} = \frac{x\mathbf{i} + y\mathbf{j}}{3}, \text{ since } x^2 + y^2 = 9 \text{ on the surface } S.$$

$$\therefore \mathbf{F} \cdot \mathbf{n} = (z\mathbf{i} + x\mathbf{j} - yz\mathbf{k}) \cdot \left[\tfrac{1}{3}(x\mathbf{i} + y\mathbf{j})\right]$$

$$= \tfrac{1}{3}(xz + xy) = \tfrac{1}{3}x(z + y).$$

Also dS = elementary area on the surface of the cylinder $= 3\, d\theta\, dz$, using cylindrical coordinates r, θ, z.

$$\therefore \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iint_S \frac{1}{3} x(z + y)\, 3\, d\theta\, dz,$$

where $x = 3\cos\theta$, $y = 3\sin\theta$

$$= \int_{z=0}^{4} \int_{\theta=0}^{\pi/2} 3\cos\theta\, (z + 3\sin\theta)\, d\theta\, dz$$

$$= \int_{\theta=0}^{\pi/2} 3\cos\theta \left[\frac{1}{2}z^2 + z \cdot 3\sin\theta\right]_{z=0}^{4} d\theta$$

$$= 3\int_0^{\pi/2} \cos\theta\, [8 + 12\sin\theta]\, d\theta$$

$$= 12\left[2\int_0^{\pi/2} \cos\theta\, d\theta + 3\int_0^{\pi/2} \sin\theta\cos\theta\, d\theta\right]$$

$$= 12\left[2 \cdot 1 + 3 \cdot \frac{1}{2}\right] = 42.$$

Ex. 61. *Evaluate* $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$, *where* $\mathbf{F} = 4x\mathbf{i} - 2y^2\mathbf{j} + z^2\mathbf{k}$ *and S is the closed surface consisting of the cylinder $x^2 + y^2 = 4$ and the circular discs $z = 0$ and $z = 3$.*

Sol. Here the surface S consists of three surfaces: (i) the surface S_1 of the base *i.e.*, the plane face $z = 0$ of the cylinder (ii) the surface S_2 of the top *i.e.*, the plane face $z = 3$ of the cylinder and (iii) the surface S_3 of the convex portion of the cylinder.

For the surface S_1 *i.e.*, $z = 0$, $\mathbf{F} = 4x\mathbf{i} - 2y^2\mathbf{j}$, putting $z = 0$ in $\mathbf{F}$.

A unit vector n along the outward drawn normal to S_1 is obviously $-\mathbf{k}$.

$$\therefore \iint_{S_1} \mathbf{F} \cdot \mathbf{n}\, dS = \iint_{S_1} (4x\mathbf{i} - 2y^2\mathbf{j}) \cdot (-\mathbf{k})\, dS$$

$$= \iint_{S_1} 0\, dS = 0.$$

For the surface S_2 *i.e.*, $z = 3$, $\mathbf{F} = 4x\mathbf{i} - 2y^2\mathbf{j} + 9\mathbf{k}$, putting $z = 3$ in $\mathbf{F}$.

A unit vector n along the outward drawn normal to S_2 is obviously $\mathbf{k}$.

$\therefore \iint_{S_2} \mathbf{F}\cdot\mathbf{n}\, dS = \iint_{S_2} (4x\,\mathbf{i} - 2y^2\,\mathbf{j} + 9\,\mathbf{k})\cdot\mathbf{k}\, dS$

$= \iint_{S_2} 9\, dS = 9 \iint_{S_2} dS = 9\cdot 2\,\pi . 2 = 36\pi$

[$\because$ area of the plane face S_2 of the cylinder $= 2\pi r = 2\pi . 2$]

For the convex portion S_3 i.e., $x^2 + y^2 = 4$, a vector normal to S_3 is given by $\nabla (x^2 + y^2) = 2x\,\mathbf{i} + 2y\,\mathbf{j}$.

$\therefore$ $\mathbf{n}$ = a unit vector along outward drawn normal at any point of S_3

$$= \frac{2x\,\mathbf{i} + 2y\,\mathbf{j}}{\sqrt{(4x^2 + 4y^2)}} = \frac{x\,\mathbf{i} + y\,\mathbf{j}}{2}, \text{ since } x^2 + y^2 = 4 \text{ on } S_3.$$

$\therefore$ on S_3, $\mathbf{F}\cdot\mathbf{n} = (4x\,\mathbf{i} - 2y^2\,\mathbf{j} + z^2\,\mathbf{k})\cdot\left\{\frac{1}{2}(x\,\mathbf{i} + y\,\mathbf{j})\right\}$

$= 2x^2 - y^3.$

Also dS = elementary area on the surface S_3

$= 2\, d\theta\, dz$, using cylindrical coordinates r, θ, z.

$$\therefore \iint_{S_3} \mathbf{F}\cdot\mathbf{n}\, dS = \iint_{S_3} (2x^2 - y^3)\, 2d\theta\, dz,$$

where $x = 2\cos\theta, y = 2\sin\theta$

$$= \int_{z=0}^{3} \int_{\theta=0}^{2\pi} (8\cos^2\theta - 8\sin^3\theta)\, 2\, d\theta\, dz$$

$$= 2\int_{\theta=0}^{2\pi} 8\,(\cos^2\theta - \sin^3\theta)\left[z\right]_{z=0}^{3} d\theta$$

$$= 48\left[\int_0^{2\pi} \cos^2\theta\, d\theta - \int_0^{2\pi} \sin^3\theta\, d\theta\right]$$

$$= 48\left[4\int_0^{\pi/2} \cos^2\theta\, d\theta - 0\right] \quad [\because \sin^3(2\pi - \theta) = -\sin^3\theta]$$

$$= 192\cdot\frac{1}{2}\cdot\frac{\pi}{2} = 48\pi.$$

Hence the required surface integral

$$= \iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iint_{S_1} \mathbf{F}\cdot\mathbf{n}\, dS + \iint_{S_2} \mathbf{F}\cdot\mathbf{n}\, dS + \iint_{S_3} \mathbf{F}\cdot\mathbf{n}\, dS$$

$$= 0 + 36\pi + 48\pi = 84\pi.$$

Ex. 62. *Evaluate* $\iint_S (y^2z^2\,\mathbf{i} + z^2x^2\,\mathbf{j} + x^2y^2\,\mathbf{k})\cdot\mathbf{n}\, dS$ *where* S *is the surface of the sphere* $x^2 + y^2 + z^2 = 1$ *above the xy-plane.*

Sol. Let $\mathbf{F} = y^2z^2\,\mathbf{i} + z^2x^2\,\mathbf{j} + x^2y^2\,\mathbf{k}$.

A vector normal to the surface S i.e., $x^2 + y^2 + z^2 = 1$ is given by $\nabla (x^2 + y^2 + z^2) = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}$.

Therefore $\mathbf{n}$ = a unit normal vector at any point (x, y, z) of S

$$= \frac{2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}}{\sqrt{(4x^2 + 4y^2 + 4z^2)}} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k},$$

since $x^2 + y^2 + z^2 = 1$ on the surface S.

We have $\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iint_R \mathbf{F}\cdot\mathbf{n}\,\frac{dx\,dy}{\mathbf{n}\cdot\mathbf{k}}$, where R is the projection of S on the xy-plane. Obviously the region R is the area of the circle $x^2 + y^2 = 1$, $z = 0$ in the xy-plane.

We have $\mathbf{F}\cdot\mathbf{n} = (y^2z^2\,\mathbf{i} + z^2x^2\,\mathbf{j} + x^2y^2\,\mathbf{k})\cdot(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})$

$= xy^2z^2 + yz^2x^2 + zx^2y^2.$

Also $\mathbf{n}\cdot\mathbf{k} = (x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k})\cdot\mathbf{k} = z.$

Hence $\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iint_R \mathbf{F}\cdot\mathbf{n}\,\frac{dx\,dy}{z}$

$$= \iint_R xyz\,(yz + zx + xy)\,\frac{dx\,dy}{z}$$

$$= \iint_R xy\,[xy + z\,(x + y)]\, dx\,dy$$

$$= \iint_R [x^2y^2 + (x^2y + xy^2)\sqrt{\{1 - (x^2 + y^2)\}}]\, dx\,dy,$$

since $z = \sqrt{(1 - x^2 - y^2)}$ on S

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} [r^4\cos^2\theta\sin^2\theta + (r^3\cos^2\theta\sin\theta + r^3\cos\theta\sin^2\theta)\,\sqrt{(1 - r^2)}]\, r\, d\theta\, dr,$$

on changing to polars

$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{1} r^5\cos^2\theta\sin^2\theta\, d\theta\, dr$$

$$+ \int_{\theta=0}^{2\pi} \int_{r=0}^{1} r^4\sqrt{(1 - r^2)}\cos^2\theta\sin\theta\, d\theta\, dr$$

$$+ \int_{\theta=0}^{2\pi} \int_{r=0}^{1} r^4\sqrt{(1 - r^2)}\cos\theta\sin^2\theta\, d\theta\, dr$$

$$= 4\int_{\theta=0}^{\pi/2}\int_{r=0}^{1} r^5 \cos^2\theta \sin^2\theta \, d\theta \, dr$$

$$\left[\because \int_{\theta=0}^{2\pi} \cos^2\theta \sin\theta \, d\theta = 0 \text{ and } \int_{\theta=0}^{2\pi} \cos\theta \sin^2\theta \, d\theta = 0\right]$$

$$= 4\int_{\theta=0}^{\pi/2} \cos^2\theta \sin^2\theta \left[\frac{r^6}{6}\right]_{r=0}^{1} d\theta$$

$$= \frac{4}{6}\int_0^{\pi/2} \cos^2\theta \sin^2\theta \, d\theta = \frac{2}{3}\cdot\frac{1\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2} = \frac{\pi}{24}.$$

Ex. 63. *Evaluate* $\iint_S \mathbf{r}\cdot d\mathbf{S}$ *where S is the part of the surface of the ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

lying above the plane $z = 0$, *the normal at any point being directed outwards.*

Sol. A normal vector to the surface $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}-1=0$ is given by $\nabla\left(\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}\right)=\dfrac{2x}{a^2}\mathbf{i}+\dfrac{2y}{b^2}\mathbf{j}+\dfrac{2z}{c^2}\mathbf{k}$.

Hence the unit vector $\mathbf{n}$ along the outward drawn normal at any point (x, y, z) of the surface S is given by

$$\mathbf{n} = \frac{(2x/a^2)\,\mathbf{i} + (2y/b^2)\,\mathbf{j} + (2z/c^2)\,\mathbf{k}}{\sqrt{[(4x^2/a^4)+(4y^2/b^4)+(4z^2/c^4)]}}$$

$$= \frac{(x/a^2)\,\mathbf{i} + (y/b^2)\,\mathbf{j} + (z/c^2)\,\mathbf{k}}{\sqrt{[\Sigma(x^2/a^4)]}}.$$

Now $\iint_S \mathbf{r}\cdot d\mathbf{S} = \iint_S \mathbf{r}\cdot\mathbf{n}\, dS = \iint_R \mathbf{r}\cdot\mathbf{n}\,\dfrac{dx\,dy}{\mathbf{n}\cdot\mathbf{k}}$,

where R is the projection of S on the xy-plane. Obviously the region R is the area of the ellipse $x^2/a^2+y^2/b^2=1$, $z=0$ lying in the xy-plane.

Now $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$.

$$\therefore \quad \mathbf{r}\cdot\mathbf{n} = \frac{\Sigma(x^2/a^2)}{\sqrt{[\Sigma(x^2/a^4)]}} = \frac{1}{\sqrt{[\Sigma(x^2/a^4)]}}, \quad \text{on the surface } S.$$

Also $\mathbf{n}\cdot\mathbf{k} = \dfrac{z/c^2}{\sqrt{[\Sigma(x^2/a^4)]}} = \dfrac{z}{c^2\sqrt{[\Sigma(x^2/a^4)]}}$.

$$\therefore \iint_S \mathbf{r}\cdot\mathbf{n}\, dS = \iint_R \frac{1}{\sqrt{[\Sigma(x^2/a^4)]}}\cdot\frac{dx\,dy}{z/[c^2\sqrt{\{\Sigma(x^2/a^4)\}}]}$$

$$= c^2\iint_R \frac{dx\,dy}{z} = c^2\iint_R \frac{dx\,dy}{c\sqrt{[1-(x^2/a^2)-(y^2/b^2)]}},$$

since on S, $z = c\sqrt{[1-(x^2/a^2)-(y^2/b^2)]}$

$$= c\int_{x=-a}^{a}\int_{y=-b\sqrt{(1-x^2/a^2)}}^{b\sqrt{(1-x^2/a^2)}} \frac{b\,dx\,dy}{\sqrt{[b^2\{1-(x^2/a^2)\}-y^2]}}$$

$$= 4bc\int_{x=0}^{a}\int_{y=0}^{b\sqrt{(1-x^2/a^2)}} \frac{dx\,dy}{\sqrt{[b^2\{1-(x^2/a^2)\}-y^2]}}$$

$$= 4bc\int_0^a \left[\sin^{-1}\frac{y}{b\sqrt{(1-x^2/a^2)}}\right]_{y=0}^{b\sqrt{(1-x^2/a^2)}} dx$$

$$= 4bc\int_0^a \frac{\pi}{2}\,dx = 4bc\cdot\frac{\pi}{2}\cdot\left[x\right]_0^a = 2\pi abc.$$

Ex. 64. *Evaluate* $\iint_S \mathbf{A}\cdot\mathbf{n}\, dS$, *where* $\mathbf{A} = xy\,\mathbf{i} - x^2\mathbf{j} + (x+z)\,\mathbf{k}$, *S is the portion of the plane* $2x+2y+z=6$ *included in the first octant and* $\mathbf{n}$ *is a unit normal to S.* (Meerut 1974)

Sol. A vector normal to the surface S i.e., the plane $2x+2y+z=6$ is given by

$$\nabla(2x+2y+z) = 2\mathbf{i}+2\mathbf{j}+\mathbf{k}.$$

$\therefore$ $\mathbf{n}$ = a unit normal vector at any point (x, y, z) of S

$$= \frac{2\mathbf{i}+2\mathbf{j}+\mathbf{k}}{\sqrt{(4+4+1)}} = \frac{2\mathbf{i}+2\mathbf{j}+\mathbf{k}}{3}.$$

We have $\iint_S \mathbf{A}\cdot\mathbf{n}\, dS = \iint_R \mathbf{A}\cdot\mathbf{n}\,\dfrac{dx\,dy}{\mathbf{n}\cdot\mathbf{k}}$, where R is the projection of S on the xy-plane. The region R is the area of the triangle in xy-plane bounded by x-axis, y-axis and the straight line $x+y=3$, $z=0$.

We have $\mathbf{A}\cdot\mathbf{n} = [xy\,\mathbf{i} - x^2\mathbf{j} + (x+z)\,\mathbf{k}]\cdot[\frac{1}{3}(2\mathbf{i}+2\mathbf{j}+\mathbf{k})]$

$$= \tfrac{2}{3}xy - \tfrac{2}{3}x^2 + \tfrac{1}{3}(x+z).$$

Also $\mathbf{n}\cdot\mathbf{k} = \frac{1}{3}(2\mathbf{i}+2\mathbf{j}+\mathbf{k})\cdot\mathbf{k} = \frac{1}{3}$.

Hence $\iint_S \mathbf{A}\cdot\mathbf{n}\,dS = \iint_R \frac{1}{3}[2xy - 2x^2 + x + z]\,\frac{dx\,dy}{1/3}$

$= \iint_R (2xy - 2x^2 + x + z)\,dx\,dy$

$= \iint_R (2xy - 2x^2 + x + 6 - 2x - 2y)\,dx\,dy$, since $z = 6 - 2x - 2y$ from the equation of S

$= \int_{x=0}^{3}\int_{y=0}^{3-x} (2xy - 2x^2 - x - 2y + 6)\,dx\,dy$

$= \int_{x=0}^{3}\left[xy^2 - 2x^2y - xy - y^2 + 6y\right]_{y=0}^{3-x} dx$

$= \int_{x=0}^{3}[x(3-x)^2 - 2x^2(3-x) - x(3-x) - (3-x)^2 + 6(3-x)]\,dx$

$= \int_0^3 (3x^3 - 12x^2 + 6x + 9)\,dx = \left[\frac{3}{4}x^4 - 4x^3 + 3x^2 + 9x\right]_0^3$

$= \frac{243}{4} - 108 + 27 + 27 = \frac{243}{4} - 54 = \frac{27}{4}$.

Ex. 65. *Evaluate* $\iiint_V \mathbf{F}\,dV$ *where* $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ *and* V *is the region bounded by the surfaces* $x = 0, x = 2, y = 0, y = 6, z = 4$ *and* $z = x^2$. (Andhra 1992)

Sol. Here the limits of integration for the region V are $z = x^2$ to $z = 4$, $y = 0$ to $y = 6$ and $x = 0$ to $x = 2$.

We have $\iiint_V \mathbf{F}\,dV = \iiint_V (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})\,dx\,dy\,dz$

$= I_1\mathbf{i} + I_2\mathbf{j} + I_3\mathbf{k}$, say.

Now $I_1 = \int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} x\,dx\,dy\,dz = \int_{x=0}^{2}\int_{y=0}^{6}\left[xz\right]_{z=x^2}^{4} dx\,dy$

$= \int_{x=0}^{2}\int_{y=0}^{6} (4x - x^3)\,dx\,dy = \int_{x=0}^{2}\left[4xy - x^3y\right]_{y=0}^{6} dx$

$= \int_0^2 (24x - 6x^3)\,dx = \left[12x^2 - \frac{3}{2}x^4\right]_0^2 = 48 - 24 = 24$,

$I_2 = \int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} y\,dx\,dy\,dz = \int_{x=0}^{2}\int_{y=0}^{6}\left[yz\right]_{z=x^2}^{4} dx\,dy$

$= \int_{x=0}^{2}\int_{y=0}^{6} (4y - yx^2)\,dx\,dy = \int_{x=0}^{2}\left[2y^2 - \frac{1}{2}y^2x^2\right]_{y=0}^{6} dx$

$= \int_0^2 (72 - 18x^2)\,dx = \left[72x - 6x^3\right]_0^2 = 96$

and $I_3 = \int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} z\,dx\,dy\,dz = \int_{x=0}^{2}\int_{y=0}^{6}\left[\frac{z^2}{2}\right]_{z=x^2}^{4} dx\,dy$

$= \int_{x=0}^{2}\int_{y=0}^{6}\left[8 - \frac{x^4}{2}\right] dx\,dy = \int_{x=0}^{2}\left[8y - \frac{x^4}{2}y\right]_{y=0}^{6} dx$

$= \int_0^2 (48 - 3x^4)\,dx = \left[48x - \frac{3}{5}x^5\right]_0^2 = 96 - \frac{96}{5} = \frac{384}{5}$.

$\therefore \iiint_V \mathbf{F}\,dV = I_1\mathbf{i} + I_2\mathbf{j} + I_3\mathbf{k} = 24\mathbf{i} + 96\mathbf{j} + \frac{384}{5}\mathbf{k}$.

Ex. 66. *Evaluate* $\int_V \mathbf{F}\,dV$ *for* $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ *where* V *is the region bounded by the surfaces* $x = 0, y = 0, y = 6, z = 4$ *and* $z = x^2$.

Sol. If we put $z = 4$ in $z = x^2$, we get $x^2 = 4$ or $x = \pm 2$.

$\therefore$ the limits of integration for the region V are $z = x^2$ to $z = 4$, $y = 0$ to $y = 6$ and $x = 0$ to $x = 2$.

Now proceed as in Ex. 65.

* Green's Theorem in the plane :-

— we will now see a way of evaluating the line integral of a smooth vector field around a simple closed curve. A vector field $\vec{F}(x,y) = P(x,y)\,i + Q(x,y)\,j$ is smooth if its component functions $P(x,y)$ and $Q(x,y)$ are smooth.

— we will use Green's theorem (sometimes called Green's Theorem in the plane) to relate the line integral around a closed curve with double integral over the region inside the curve.

* Statement :

Let R be a region in xy-plane whose boundary is simple closed curve 'C' which is piecewise smooth curve. Let $\vec{f}(x,y) = P(x,y)\,i + Q(x,y)\,j$ be a smooth vector field defined on both R and C. Then

$$\oint_C \vec{f}\cdot d\vec{r} = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

i.e

$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

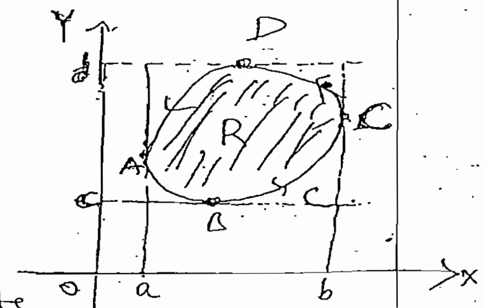
where C is traversed so that R is always on the left side of C.

(or)

If R is a closed region of the xy-plane bounded by a simple closed curve 'C' and if P and Q are continuous functions of x and y having continuous derivatives in R then $\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$

where 'C' is traversed in the positive (counterclockwise) direction.

Note :- Unless otherwise stated we shall always assume $\oint$ to mean ~~that~~ the integral is described in the positive sense.



proof

If C is closed curve which has the property that any straight line parallel to the co-ordinate axes cuts the ~~the~~ curve 'C' in at most two points.

Let the equations of the curves ABC and ADC be $y = Y_1(x)$ and $y = Y_2(x)$ respectively.

If R is the ~~region~~ region bounded by C, we have

$$\iint_R \frac{\partial P}{\partial y}\,dx\,dy = \int_{x=a}^{b} \left[\int_{y=Y_1(x)}^{Y_2(x)} \frac{\partial P}{\partial y}\,dy\right] dx$$

$$= \int_{x=a}^{b} \left[P(x,y)\right]_{y=Y_1(x)}^{Y_2(x)} dx$$

$$= \int_a^b \left[P(x, Y_2) - P(x, Y_1)\right] dx$$

$$= \int_a^b P(x, Y_2)\,dx - \int_a^b P(x, Y_1)\,dx$$

$$= -\int_a^b P(x, y_1)\,dx - \int_b^a P(x, y_2)\,dx$$

$$= -\int_{ABC} P\,dx - \int_{CDA} P\,dx$$

$$= -\int_{ABCD} P\,dx$$

$$= -\oint_C P\,dx \quad \text{——— (1)}$$

Similarly let the equations of curves DAB and BCD be $x = x_1(y)$ and $x = x_2(y)$ respectively.

$$\iint_R \frac{\partial Q}{\partial x}\,dx\,dy = \int_{y=c}^{d}\left[\int_{x=x_1(y)}^{x_2(y)} \frac{\partial Q}{\partial x}\,dx\right] dy$$

$$= \int_{y=c}^{d}\left[Q(x, y)\right]_{x=x_1(y)}^{x_2(y)} dy$$

$$= \int_{y=c}^{d}\left[Q(x_2(y), y)\right] dy - \int_{y=c}^{d}\left[Q(x_1(y), y)\right] dy$$

$$= \int_{y=c}^{d} Q(x_2(y), y)\,dy + \int_{y=d}^{c} Q(x_1(y), y)\,dy$$

$$= \int_{BCD} Q\,dy + \int_{DAB} Q\,dy$$

$$= \int_{ABCD} Q\,dy = \oint_C Q\,dy \quad \text{——— (2)}$$

Now $\iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy = \oint_C Q\,dy - \left[-\oint_C P\,dx\right]$

$$= \oint_C (P\,dx + Q\,dy).$$

Green's theorem in the plane in vector notation:

we have $\vec{r} = x\hat{i} + y\hat{j}$ so that $d\vec{r} = dx\,i + dy\,j$.

Let $\vec{F} = P\hat{i} + Q\hat{j}$

Then $P\,dx + Q\,dy = (Pi + Qj)\cdot(dx\,i + dy\,j)$

$= \vec{F}\cdot d\vec{r}$

Also $\text{curl}\,\vec{F} = \nabla \times \vec{F}$

$$= \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & 0 \end{vmatrix}$$

$$= -\frac{\partial Q}{\partial z} i + \frac{\partial P}{\partial z} j + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) k.$$

$\therefore (\nabla \times \vec{F})\cdot k = \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$.

Hence Green's theorem in plane can be written as

$$\iint_R (\nabla \times \vec{F})\cdot k\, dR = \oint_C \vec{F}\cdot d\vec{r}.$$

where $dR = dx\,dy$ and k is ~~unit~~ unit vector perpendicular to the xy-plane

If s denotes the arc length of C and t is the unit tangent vector to C, then

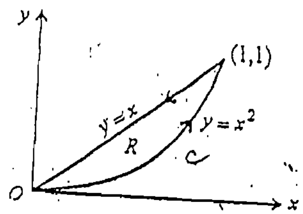
$d\vec{r} = \frac{d\vec{r}}{ds}\,ds = t\,ds$

$\therefore$ The above result can also be written as

$$\iint_R (\nabla \times \vec{F})\cdot k\, dR = \oint_C F\cdot t\, ds.$$

curve of the region bounded by $y=x$ and $y=x^2$.

Sol. By Green's theorem in plane, we have



$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \oint_C (M\,dx + N\,dy).$$

Here $M = xy + y^2$, $N = x^2$.

The curves $y = x$ and $y = x^2$ intersect at $(0, 0)$ and $(1, 1)$. The positive direction in traversing C is as shown in the figure.

We have $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$$= \iint_R \left[\frac{\partial}{\partial x}(x^2) - \frac{\partial}{\partial y}(xy + y^2)\right] dx\,dy$$

$$= \iint_R (2x - x - 2y)\,dx\,dy = \iint_R (x - 2y)\,dx\,dy$$

$$= \int_{x=0}^{1}\int_{y=x^2}^{x} (x - 2y)\,dy\,dx = \int_{x=0}^{1} \left[xy - y^2\right]_{y=x^2}^{x} dx$$

$$= \int_0^1 [x^2 - x^2 - x^3 + x^4]\,dx = \int_0^1 (x^4 - x^3)\,dx$$

$$= \left[\frac{x^5}{5} - \frac{x^4}{4}\right]_0^1 = \frac{1}{5} - \frac{1}{4} = -\frac{1}{20}.$$

Now let us evaluate the line integral along C. Along $y = x^2$, $dy = 2x\,dx$. Therefore along $y = x^2$, the line integral equals

$$\int_0^1 [\{(x)(x^2) + x^4\}\,dx + x^2(2x)\,dx]$$

$$= \int_0^1 (3x^3 + x^4)\,dx = \frac{19}{20}.$$

Along $y = x$, $dy = dx$. Therefore along $y = x$, the line integral equals

$$\therefore \int_1^0 [\{(x)(x) + x^2\}\,dx + x^2\,dx] = \int_1^0 3x^4\,dx = -1.$$

Therefore the required line integral $= \frac{19}{20} - 1 = -\frac{1}{20}$. Hence the theorem is verified.

Ex. 2. *Verify Green's theorem in a plane for*

$$\oint_C [(x^2 - 2xy)\,dx + (x^2y + 3)\,dy]$$

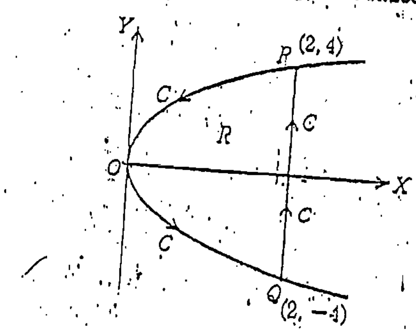
where C is the boundary of the region defined by $y^2 = 8x$ and $x = 2$. (Osmania 1991)

Sol. By Green's theorem in plane, we have

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \oint_C (M\,dx + N\,dy).$$

Here $M = x^2 - 2xy$, $N = x^2y + 3$.

The parabola $y^2 = 8x$ and the straight line $x = 2$ intersect at the points $P(2, 4)$ and $Q(2, -4)$. The positive direction in traversing C is as shown in the figure and R is the region bounded by the curve C.



We have $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$$= \iint_R \left[\frac{\partial}{\partial x}(x^2y + 3) - \frac{\partial}{\partial y}(x^2 - 2xy)\right] dx\,dy$$

$$= \iint_R (2xy + 2x)\,dx\,dy$$

$$= \int_{x=0}^{2}\int_{y=-\sqrt{(8x)}}^{\sqrt{(8x)}} (2xy + 2x)\,dx\,dy$$

[$\because$ for the region R, x varies from 0 to 2 and y varies from $-\sqrt{(8x)}$ to $\sqrt{(8x)}$]

$$= \int_0^2 2x \left[\frac{y^2}{2} + y\right]_{y=-\sqrt{(8x)}}^{\sqrt{(8x)}} dx, \text{ integrating with respect to } y \text{ regarding } x \text{ as constant}$$

$$= \int_0^2 2x \left[0 + 2\sqrt{(8x)}\right] dx$$

$$= 4\sqrt{8} \int_0^2 x^{3/2}\, dx = 8\sqrt{2} \cdot \frac{2}{5} \left[x^{5/2}\right]_0^2 = \frac{16}{5}\sqrt{2} \cdot 2^{5/2} = \frac{128}{5}. \quad \dots (1)$$

Now let us evaluate the line integral along C. Along $y^2 = 8x$, we have $x = y^2/8$, $dx = \frac{1}{4} y\, dy$ and y varies from 4 to -4. Therefore along $y^2 = 8x$, the line integral equals

$$\int_{y=4}^{-4} \left[\left(\frac{y^4}{64} - 2 \cdot \frac{y^2}{8} \cdot y\right) \cdot \frac{1}{4} y\, dy + \left(\frac{y^4}{64} \cdot y + 3\right) dy\right]$$

$$= \int_4^{-4} \left[\frac{y^5}{256} + \frac{y^5}{64} - \frac{1}{16} y^4 + 3\right] dy$$

$$= -\int_{-4}^{4} \left[\frac{y^5}{256} + \frac{y^5}{64} - \frac{1}{16} y^4 + 3\right] dy$$

$$= -2 \int_0^4 \left[-\frac{1}{16} y^4 + 3\right] dy$$

$$\left[\because \int_{-a}^{a} f(x)\, dx = 0 \text{ or } 2\int_0^a f(x)\, dx \text{ according as } f(x) = -f(x) \text{ or } f(-x) = f(x)\right]$$

$$= -2 \left[-\frac{1}{16.5} y^5 + 3y\right]_0^4 = -2\left[-\frac{1}{80} \cdot 4^5 + 3 \cdot 4\right]$$

$$= \frac{128}{5} - 24.$$

Along the st. line $x = 2$, we have $dx = 0$ and y varies from -4 to 4. Therefore along $x = 2$, the line integral equals

$$\int_{y=-4}^{4} \left[0 + (2^2 y + 3)\, dy\right] = \int_{-4}^{4} (4y + 3)\, dy$$

$$= 3 \int_{-4}^{4} dy \qquad \left[\because \int_{-4}^{4} 4y\, dy = 0\right]$$

$$= 6 \int_0^4 dy = 6 \left[y\right]_0^4 = 6.4 = 24.$$

Therefore the total line integral along the curve C i.e.,

$$\oint_C (M\, dx + N\, dy) = \frac{128}{5} - 24 + 24 = \frac{128}{5}. \quad \dots (2)$$

From (1) and (2), we see that

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \oint_C (M\, dx + N\, dy);$$

which verifies Green's theorem in plane.

Ex. 3. *Verify Green's theorem in a plane for*

$$\oint_C [(3x^2 - 8y^2)\, dx + (4y - 6xy)\, dy]$$

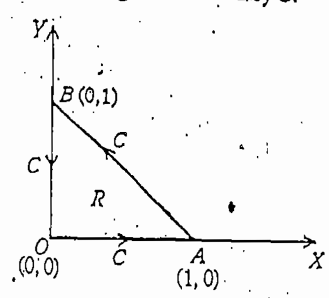
where C is the boundary of the region defined by $x = 0$, $y = 0$ and $x + y = 1$. (Osmania 1992)

Sol. By Green's theorem in plane, we have

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \oint_C (M\, dx + N\, dy).$$

Here $M = 3x^2 - 8y^2$, $N = 4y - 6xy$.

The closed curve C consists of the st. line OA, the st. line AB and the straight line BO. The positive direction in traversing C is as shown in the figure and R is the region bounded by C.



We have $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$= \iint_R \left[\frac{\partial}{\partial x}(4y - 6xy) - \frac{\partial}{\partial y}(3x^2 - 8y^2)\right] dx\,dy$

$= \iint_R [-6y + 16y]\,dx\,dy = 10 \iint_R y\,dx\,dy$

$= 10 \int_{x=0}^{1} \int_{y=0}^{1-x} y\,dx\,dy$ [$\because$ for the region R, x varies from 0 to 1 and y varies from 0 to $1-x$]

$= 10 \int_0^1 \left[\frac{y^2}{2}\right]_{y=0}^{1-x} dx$, integrating with respect to y regarding x as constant

$= 5 \int_0^1 (1-x)^2\,dx = 5 \int_0^1 (x-1)^2\,dx = \frac{5}{3}\left[(x-1)^3\right]_0^1$

$= \frac{5}{3}[0 - (-1)^3] = \frac{5}{3}$. ... (1)

Now let us evaluate the line integral along the curve C.

Along the st. line OA, we have $y = 0$, $dy = 0$ and x varies from 0 to 1.

$\therefore$ line integral along $OA = \int_0^1 3x^2\,dx = \left[x^3\right]_0^1 = 1.$

Along the st. line AB, we have $x = 1 - y$, $dx = -dy$ and y varies from 0 to 1.

$\therefore$ line integral along AB

$= \int_0^1 [\{3(1-y)^2 - 8y^2\}(-dy) + \{4y - 6y(1-y)\}\,dy]$

$= \int_0^1 [-3(1 - 2y + y^2) + 8y^2 + 4y - 6y + 6y^2]\,dy$

$= \int_0^1 (11y^2 + 4y - 3)\,dy = \left[\frac{11}{3}y^3 + 2y^2 - 3y\right]_0^1$

$= \frac{11}{3} + 2 - 3 = \frac{8}{3}$.

Along the st. line BO, we have $x = 0$, $dx = 0$ and y varies from 1 to 0.

$\therefore$ line integral along $BO = \int_1^0 4y\,dy = 2\left[y^2\right]_1^0 = -2.$

$\therefore$ total line integral along the closed curve C

$= 1 + \frac{8}{3} - 2 = \frac{5}{3}$. ... (2)

From (1) and (2), we see that Green's theorem is verified.

Ex. 4. *Verify Green's theorem in the plane for*

$$\oint_C [(3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy],$$

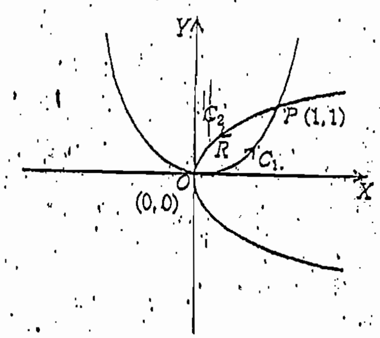
where C is the boundary of the region defined by $y = \sqrt{x}$, $y = x^2$. (Kakatiya 1990)

Sol. By Green's theorem in plane, we have

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \oint_C (M\,dx + N\,dy).$$

Here $M = 3x^2 - 8y^2$, $N = 4y - 6xy$.

The parabola $y = \sqrt{x}$ i.e., $y^2 = x$ and the parabola $y = x^2$ intersect at the points (0, 0) and (1, 1). The closed curve C consists of the arc C_1 of the parabola $y = x^2$ and the arc C_2 of the parabola $y = \sqrt{x}$. Also R is the region bounded by the closed curve C.



We have $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$= \iint_R \left[\frac{\partial}{\partial x}(4y - 6xy) - \frac{\partial}{\partial y}(3x^2 - 8y^2)\right] dx\,dy$

$= \iint_R (-6y + 16y)\,dx\,dy = \iint_R 10y\,dx\,dy$

$= \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} 10y\, dx\, dy$ [$\because$ for the region R, x varies from 0 to 1 and y varies from x^2 to $\sqrt{x}$]

$= \int_0^1 5 \left[y^2\right]_{y=x^2}^{\sqrt{x}} dx = 5 \int_0^1 [x - x^4]\, dx$

$= 5 \left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = 5 \left[\frac{1}{2} - \frac{1}{5}\right] = \frac{15}{10} = \frac{3}{2}.$ ... (1)

Now the line integral along the closed curve C

$= \oint_C (M\, dx + N\, dy) = \int_{C_1} (M\, dx + N\, dy) + \int_{C_2} (M\, dx + N\, dy).$

Along C_1, $x^2 = y$, $dy = 2x\, dx$ and x varies from 0 to 1.

$\therefore$ line integral along $C_1 = \int_0^1 [(3x^2 - 8x^4)\, dx + (4x^2 - 6x^3)\, 2x\, dx]$

$= \int_0^1 (3x^2 + 8x^3 - 20x^4)\, dx = \left[x^3 + 2x^4 - 4x^5\right]_0^1 = 1 + 2 - 4 = -1.$

Along C_2, $y^2 = x$, $\therefore$ $dx = 2y\, dy$ and limits for y are 1 to 0.

$\therefore$ line integral along C_2

$= \int_1^0 [(3y^4 - 8y^2)\, 2y\, dy + (4y - 6y^3)\, dy]$

$= \int_1^0 (6y^5 - 22y^3 + 4y)\, dy = \left[y^6 - \frac{11}{2} y^4 + 2y^2\right]_1^0$

$= -1 + \frac{11}{2} - 2 = \frac{5}{2}.$

$\therefore$ total line integral along the closed curve C

$= -1 + \frac{5}{2} = \frac{3}{2}.$ ... (2)

From (1) and (2), we see that Green's theorem is verified.

Ex. 5. *Verify Green's theorem in the plane for*

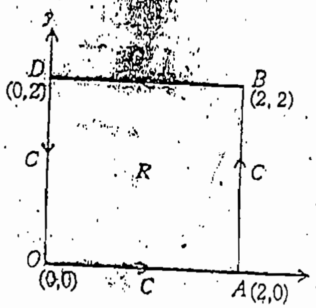
$$\int_C (x^2 - xy^3)\, dx + (y^2 - 2xy)\, dy,$$

where C is the square with vertices $(0, 0)$, $(2, 0)$, $(2, 2)$, $(0, 2)$. (Meerut 1974)

Sol. By Green's theorem in plane, we have

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \oint_C (M\, dx + N\, dy).$$

Here $M = x^2 - xy^3$, $N = y^2 - 2xy$.



The closed curve C consists of the straight lines OA, AB, BD and DO. The positive direction in traversing C is as shown in the figure and R is the region bounded by C.

We have

$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$

$= \iint_R \left[\frac{\partial}{\partial x} (y^2 - 2xy) - \frac{\partial}{\partial y} (x^2 - xy^3)\right] dx\, dy$

$= \iint_R (-2y + 3xy^2)\, dx\, dy = \int_{x=0}^{2} \int_{y=0}^{2} (-2y + 3xy^2)\, dx\, dy$

$= \int_{x=0}^{2} \left[-y^2 + xy^3\right]_{y=0}^{2} dx = \int_0^2 (-4 + 8x)\, dx$

$= \left[-4x + 4x^2\right]_0^2 = -8 + 16 = 8$ ... (1)

Now let us evaluate the line integral along the closed curve C.

Along OA, $y = 0$, $dy = 0$ and x varies from 0 to 2;

along AB, $x = 2$, $dx = 0$ and y varies from 0 to 2;

along BD, $y = 2$, $dy = 0$ and x varies from 2 to 0;

and along DO, $x = 0$, $dx = 0$ and y varies from 2 to 0.

We have

$\int_C (M\, dx + N\, dy) = \int_{OA} (M\, dx + N\, dy) + \int_{AB} (M\, dx + N\, dy) + \int_{BD} (M\, dx + N\, dy) + \int_{DO} (M\, dx + N\, dy)$

$= \int_{x=0}^{2} x^2\, dx + \int_{y=0}^{2} (y^2 - 4y)\, dy + \int_{x=2}^{0} (x^2 - 8x)\, dx + \int_{y=2}^{0} y^2\, dy$

$= \left[\frac{x^3}{3}\right]_0^2 + \left[\frac{y^3}{3} - 2y^2\right]_0^2 + \left[\frac{x^3}{3} - 4x^2\right]_2^0 + \left[\frac{y^3}{3}\right]_2^0$

$= \frac{8}{3} + \frac{8}{3} - 8 - \frac{8}{3} + 16 - \frac{8}{3} = 8$ ... (2)

From (1) and (2), we see that

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \oint_C (M\, dx + N\, dy).$$

This verifies Green's theorem.

Ex. 6. *Verify Green's theorem in the plane for*

$$\int_C [(2xy - x^2)\, dx + (x^2 + y^2)\, dy],$$

where C is the boundary of the region enclosed by $y = x^2$ *and* $y^2 = x$ *described in the positive sense.* (Meerut 1973)

Sol. By Green's theorem in plane, we have

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \oint_C (M\, dx + N\, dy).$$

Here $M = 2xy - x^2$, $N = x^2 + y^2$.

The parabolas $y^2 = x$ and $x^2 = y$ intersect at the points (0, 0) and (1, 1). [See figure of solved example 4]. The closed curve C consists of the arc C_1 of the parabola $y = x^2$ and the arc C_2 of the parabola $y^2 = x$. Also R is the region bounded by the closed curve C.

We have $\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$

$$= \iint_R \left[\frac{\partial}{\partial x}(x^2 + y^2) - \frac{\partial}{\partial y}(2xy - x^2)\right] dx\, dy$$

$$= \iint_R (2x - 2x)\, dx\, dy = \iint_R 0\, dx\, dy = 0. \qquad \dots (1)$$

Now the line integral along the closed curve C

$$= \oint_C (M\, dx + N\, dy) = \int_{C_1} (M\, dx + N\, dy) + \int_{C_2} (M\, dx + N\, dy).$$

Along C_1, $x^2 = y$, $dy = 2x\, dx$ and x varies from 0 to 1.

$\therefore$ the line integral along C_1

$$= \int_{x=0}^{1} [(2x^3 - x^2)\, dx + (x^2 + x^4)\, 2x\, dx]$$

$$= \int_0^1 (4x^3 - x^2 + 2x^5)\, dx = \left[x^4 - \frac{x^3}{3} + \frac{x^6}{3}\right]_0^1$$

$$= 1 - \frac{1}{3} + \frac{1}{3} = 1.$$

Along C_2, $y^2 = x$, $dx = 2y\, dy$ and y varies from 1 to 0.

$\therefore$ the line integral along $C_2 = \int_{y=1}^{0} [(2y^3 - y^4)\, 2y\, dy + (y^4 + y^2)\, dy]$

$$= \int_1^0 (5y^4 - 2y^5 + y^2)\, dy = \left[y^5 - \frac{y^6}{3} + \frac{y^3}{3}\right]_1^0$$

$$= -1 + \frac{1}{3} - \frac{1}{3} = -1.$$

$\therefore$ total line integral along the closed curve C

$= 1 - 1 = 0.$ ... (2)

From (1) and (2), we see that the two integrals are equal and hence Green's theorem is verified.

Ex. 7. *Evaluate by Green's theorem*

$$\oint_C (x^2 - \cosh y)\, dx + (y + \sin x)\, dy,$$

where C is the rectangle with vertices (0, 0), $(\pi, 0)$, $(\pi, 1)$, (0, 1).

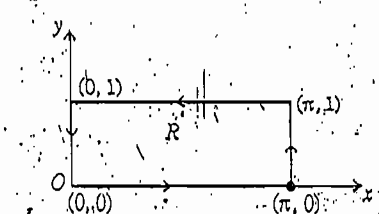
Sol. By Green's theorem in plane, we have

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \oint_C (M\, dx + N\, dy).$$

Here $M = x^2 - \cosh y$, $N = y + \sin x$.

$$\therefore \frac{\partial N}{\partial x} = \cos x, \quad \frac{\partial M}{\partial y} = -\sinh y.$$

Hence the given line integral is equal to



$$\iint_R (\cos x + \sinh y)\, dx\, dy = \int_{x=0}^{\pi} \int_{y=0}^{1} (\cos x + \sinh y)\, dy\, dx$$

$$= \int_{x=0}^{\pi} \left[y \cos x + \cosh y\right]_{y=0}^{1} dx = \int_{x=0}^{\pi} [\cos x + \cosh 1 - 1]\, dx$$

$= \left[\sin x + x\cosh 1 - x\right]_0^{\pi} = (\cosh 1 - 1).$

Ex. 8. *Evaluate by Green's theorem in plane*
$\int_C (e^{-x} \sin y\, dx + e^{-x} \cos y\, dy)$, *where C is the rectangle with vertices* $(0,0), (\pi, 0), \left(\pi, \frac{1}{2}\pi\right), \left(0, \frac{1}{2}\pi\right)$.

Sol. Draw figure as in solved example 7. By Green's theorem in plane

$$\oint_C (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

Here $M = e^{-x} \sin y$, $N = e^{-x} \cos y$.

$\therefore \frac{\partial N}{\partial x} = -e^{-x} \cos y$, $\frac{\partial M}{\partial y} = e^{-x} \cos y$.

Hence the given line integral

$$= \iint_R (-e^{-x} \cos y - e^{-x} \cos y)\, dx\,dy,$$

where R is the region enclosed by the rectangle C

$$= \int_{x=0}^{\pi} \int_{y=0}^{\pi/2} -2e^{-x} \cos y\, dx\,dy$$

$$= \int_{x=0}^{\pi} -2e^{-x} \left[\sin y\right]_{y=0}^{\pi/2} dx = \int_0^{\pi} -2e^{-x}\, dx$$

$$= 2\left[e^{-x}\right]_0^{\pi} = 2(e^{-\pi} - 1).$$

Ex. 9. *If* $\mathbf{F} = (x^2 - y^2)\,\mathbf{i} + 2xy\,\mathbf{j}$ *and* $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$, *find the value of* $\int \mathbf{F} \cdot d\mathbf{r}$ *around the rectangular boundary* $x = 0$, $x = a$, $y = 0$, $y = b$.

(Gauhati 1973)

Sol. Here the four vertices of the rectangle taken in order are $(0,0)$, $(a,0)$, (a,b) and $(0,b)$. Draw figure as in solved example 7.

Let C be the closed curve traversed in positive direction by the boundary of the rectangle and R be the region bounded by this curve C.

We have $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C [(x^2 - y^2)\,\mathbf{i} + 2xy\,\mathbf{j}] \cdot (dx\,\mathbf{i} + dy\,\mathbf{j})$

$$= \int_C [(x^2 - y^2)\, dx + 2xy\, dy] = \int_C M\,dx + N\,dy,$$

where $M = x^2 - y^2$, $N = 2xy$

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy, \quad \text{by Green's theorem}$$

$$= \iint_R \left[\frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(x^2 - y^2)\right] dx\,dy$$

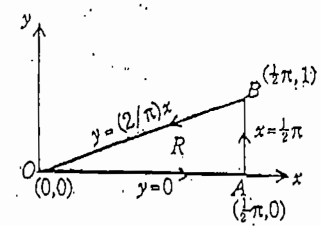
$$= \iint_R (2y + 2y)\, dx\,dy = 4\int_{x=0}^{a} \int_{y=0}^{b} y\, dx\,dy$$

$$= 4\int_{x=0}^{a} \left[\frac{y^2}{2}\right]_{y=0}^{b} dx = 2b^2 \int_0^a dx = 2ab^2.$$

Ex. 10. *Apply Green's theorem in the plane to evaluate*
$\int_C \{(y - \sin x)\, dx + \cos x\, dy\}$,
where C is the triangle enclosed by the lines $y = 0$, $x = 2\pi$, $\pi y = 2x$.

(Agra 1973)

Sol. Here C is the closed curve traversed in positive direction by ΔOAB and R is the region bounded by this curve C.



We have, $\int_C \{(y - \sin x)\, dx + \cos x\, dy\}$

$$= \int_C (M\,dx + N\,dy), \text{ where } M = y - \sin x,\ N = \cos x$$

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy, \text{ by Green's theorem}$$

$$= \iint_R \left[\frac{\partial}{\partial x}(\cos x) - \frac{\partial}{\partial y}(y - \sin x)\right] dx\,dy$$

$$= \int_{x=0}^{\pi/2} \int_{y=0}^{(2/\pi)x} (-\sin x - 1)\, dx\,dy$$

[$\because$ for the region R, y varies from 0 to $(2/\pi)x$ and x varies from 0 to $\pi/2$]

$$= \int_{x=0}^{\pi/2} \left[-y\sin x - y\right]_{y=0}^{(2/\pi)x} dx$$

$$= \int_0^{\pi/2} \left[-\frac{2}{\pi}x\sin x - \frac{2}{\pi}x\right] dx$$

$$= -\frac{2}{\pi}\int_0^{\pi/2} (x + x\sin x)\, dx$$

$$= -\frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi/2} - \frac{2}{\pi}\left[x(-\cos x)\right]_0^{\pi/2} - \frac{2}{\pi}\int_0^{\pi/2} \cos x\, dx$$

$$= -\frac{2}{\pi}\cdot\frac{\pi^2}{8} - \frac{2}{\pi}\cdot 1 = -\frac{\pi}{4} - \frac{2}{\pi}.$$

Ex. 11. *Evaluate by Green's theorem*

$\oint_C (\cos x \sin y - xy)\, dx + \sin x \cos y\, dy$, *where C is the circle* $x^2+y^2 = 1$.

Sol. By Green's theorem in plane, we have

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy = \oint_C (M dx + N dy).$$

Here $M = \cos x \sin y - xy$, $N = \sin x \cos y$.

$$\therefore \frac{\partial M}{\partial y} = \cos x \cos y - x, \quad \frac{\partial N}{\partial x} = \cos x \cos y.$$

Hence the given line integral is equal to

$$\iint_R x\, dx\, dy = \int_{\theta=0}^{2\pi}\int_{r=0}^{1} r\cos\theta\, r\, d\theta\, dr, \text{ changing to polars}$$

$$= \int_{\theta=0}^{2\pi}\left[\frac{r^3}{3}\right]_0^1 \cos\theta\, d\theta = \frac{1}{3}\left[\sin\theta\right]_0^{2\pi} = \tfrac{1}{3}(0) = 0.$$

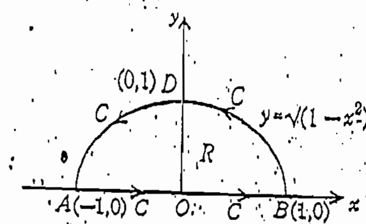
Ex. 12. *Apply Green's theorem in the plane to evaluate*

$$\int_C \left[(2x^2-y^2)\, dx + (x^2+y^2)\, dy\right],$$

where C is the boundary of the surface enclosed by the x-axis and the semi-circle $y = (1-x^2)^{1/2}$.

Sol. Here C is the closed curve traversed in the positive direction by the straight line AOB and the semi-circle BDA. Also R is the region bounded by this curve C.

We have $\int_C \left[(2x^2-y^2)\, dx + (x^2+y^2)\, dy\right]$



$$= \int_C M\, dx + N\, dy, \text{ where } M = 2x^2-y^2,\ N = x^2+y^2$$

$$= \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy, \text{ by Green's theorem}$$

$$= \iint_R \left[\frac{\partial}{\partial x}(x^2+y^2) - \frac{\partial}{\partial y}(2x^2-y^2)\right] dx\, dy$$

$$= \iint_R (2x+2y)\, dx\, dy$$

$$= \int_{x=-1}^{1}\int_{y=0}^{\sqrt{(1-x^2)}} 2(x+y)\, dx\, dy, \text{ since for the region } R, y \text{ varies}$$

from 0 to $\sqrt{(1-x^2)}$ and x varies from -1 to 1

$$= 2\int_{x=-1}^{1}\left[xy + \frac{y^2}{2}\right]_{y=0}^{\sqrt{(1-x^2)}} dx$$

$$= 2\int_{-1}^{1}\left[x\sqrt{(1-x^2)} + \frac{1}{2}(1-x^2)\right] dx$$

$$= 2\int_0^1 (1-x^2)\, dx \quad \left[\because \int_{-1}^{1} x\sqrt{(1-x^2)}\, dx = 0 \text{ and } \int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx \text{ if } f(-x) = f(x)\right]$$

$$= 2\left[x - \frac{x^3}{3}\right]_0^1 = 2\left(1-\frac{1}{3}\right) = \frac{4}{3}.$$

Ex. 13. *If C is the simple closed curve in the xy-plane not enclosing the origin, show that*

$$\int_C \mathbf{F}\cdot d\mathbf{r} = 0, \text{ where } \mathbf{F} = \frac{-\mathbf{i}y+\mathbf{j}x}{x^2+y^2}.$$

Sol. Let R be the region enclosed by the closed curve C.

We have $\int_C \mathbf{F}\cdot d\mathbf{r}$

$$= \int_C \left(\frac{-\mathbf{i}y+\mathbf{j}x}{x^2+y^2}\right)\cdot(dx\,\mathbf{i}+dy\,\mathbf{j})$$

$$= \int_C \left[-\frac{y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy\right]$$

$$= \int_C (M\,dx+N\,dy), \text{ where } M = -\frac{y}{x^2+y^2},\ N = \frac{x}{x^2+y^2}.$$

Since the closed curve C does not enclose origin, therefore both the functions M and N are defined at the origin. So by Green's theorem, we have

$$\int_C (M\,dx+N\,dy) = \iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy$$

$$= \iint_R \left[\frac{1(x^2+y^2)-x(2x)}{(x^2+y^2)^2}+\frac{1(x^2+y^2)-y(2y)}{(x^2+y^2)^2}\right]dx\,dy$$

$$= \iint_R \frac{2(x^2+y^2)-2(x^2+y^2)}{(x^2+y^2)^2}\,dx\,dy = \iint_R 0\,dx\,dy = 0.$$

Hence $\int_C \mathbf{F}\cdot d\mathbf{r} = 0.$

Ex. 14. *Show that the area bounded by a simple closed curve C is given by $\frac{1}{2}\oint_C (x\,dy-y\,dx)$. Hence find the area of the ellipse $x=a\cos\theta,\ y=b\sin\theta$.* (Tirupati 1993; Agra 74)

Sol. By Green's theorem in plane, if R is a plane region bounded by a simple closed curve C, then

$$\iint_R \left(\frac{\partial N}{\partial x}-\frac{\partial M}{\partial y}\right)dx\,dy = \oint_C M\,dx+N\,dy.$$

Putting $M=-y,\ N=x$, we get

$$\oint_C (x\,dy-y\,dx) = \iint_R \left[\frac{\partial}{\partial x}(x)-\frac{\partial}{\partial y}(-y)\right]dx\,dy$$

$$= 2\iint_R dx\,dy = 2A, \text{ where } A \text{ is the area bounded by } C.$$

Hence $\qquad A = \frac{1}{2}\oint_C (x\,dy-y\,dx).$

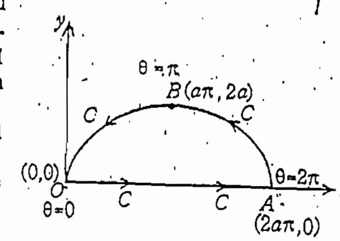
The area of the ellipse $= \frac{1}{2}\oint_C (x\,dy-y\,dx)$

$$= \frac{1}{2}\int_{\theta=0}^{2\pi}\left(a\cos\theta\,\frac{dy}{d\theta}-b\sin\theta\,\frac{dx}{d\theta}\right)d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}(ab\cos^2\theta+ab\sin^2\theta)\,d\theta = \frac{1}{2}ab\int_0^{2\pi}d\theta = \pi ab.$$

Ex. 15. *Find the area bounded by one arch of the cycloid $x=a(\theta-\sin\theta),\ y=a(1-\cos\theta),\ a>0$ and the x-axis.*

Sol. Let C be the closed curve traversed in positive direction by the straight line OA and the arch ABO of the given cycloid.

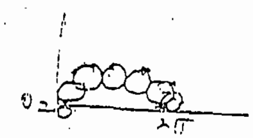


At the point O, $\theta=0$ and at A, $\theta=2\pi$.

The area bounded by one arch of the given cycloid and the x-axis

= the area enclosed by the simple closed curve C,

$$= \frac{1}{2}\oint_C (x\,dy-y\,dx), \text{ by Green's theorem}$$



$$= \frac{1}{2}\int_{OA}(x\,dy-y\,dx)+\frac{1}{2}\int_{\text{arch }ABO}(x\,dy-y\,dx).$$

Now along the straight line OA, we have $y=0,\ dy=0$ and x varies from 0 to $2a\pi$.

$$\therefore\ \frac{1}{2}\int_{OA}(x\,dy-y\,dx) = 0.$$

Hence the required area $= \frac{1}{2}\int_{\text{arch }ABO}(x\,dy-y\,dx)$

$$= \frac{1}{2}\int_{\theta=2\pi}^{0}\left(x\frac{dy}{d\theta} - y\frac{dx}{d\theta}\right)d\theta$$

$$= \frac{1}{2}\int_{2\pi}^{0} [a(\theta-\sin\theta) \,.\, a\sin\theta - a(1-\cos\theta) \,.\, a(1-\cos\theta)]d\theta$$

$$= \frac{a^2}{2}\int_{2\pi}^{0} [\theta\sin\theta - \sin^2\theta - 1 + 2\cos\theta - \cos^2\theta]d\theta$$

$$= \frac{a^2}{2}\int_{2\pi}^{0} (\theta\sin\theta - 2 + 2\cos\theta)d\theta$$

$$= \frac{a^2}{2}\int_{0}^{2\pi} (2 - 2\cos\theta - \theta\sin\theta)d\theta$$

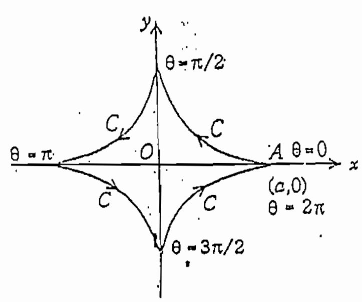
$$= \frac{a^2}{2}\left[2\theta - 2\sin\theta + \theta\cos\theta - \sin\theta\right]_0^{2\pi}$$

$$= \frac{a^2}{2}[4\pi + 2\pi] = 3\pi a^2.$$

Ex. 16. *Use Green's theorem to find the area bounded by the hypocycloid $x^{2/3}+y^{2/3} = a^{2/3}$, $a>0$.*

Sol. The parametric equations of the given curve $x^{2/3}+y^{2/3} = a^{2/3}$ can be taken as

$$x = a\cos^3\theta,\ y = a\sin^3\theta.$$



Here C is the simple closed curve traversed in positive direction by the whole arc of the given hypocycloid.

At the point A, $\theta=0$ and when after one complete round in anti-clockwise sense along the curve C we come back to A, then at A, $\theta=2\pi$.

The area bounded by the given hypocycloid

= the area enclosed by the simple closed curve C

$$= \frac{1}{2}\oint_C (x\,dy - y\,dx), \text{ by Green's theorem}$$

$$= \frac{1}{2}\int_{\theta=0}^{2\pi}\left(x\frac{dy}{d\theta} - y\frac{dx}{d\theta}\right)d\theta, \text{ where } x = a\cos^3\theta, y = a\sin^3\theta$$

$$= \frac{1}{2}\int_0^{2\pi} [a\cos^3\theta \,.\, 3a\sin^2\theta\cos\theta - a\sin^3\theta \,.\, (-3a\cos^2\theta\sin\theta)]d\theta$$

$$= \frac{3a^2}{2}\int_0^{2\pi} (\cos^4\theta\sin^2\theta + \sin^4\theta\cos^2\theta)\,d\theta$$

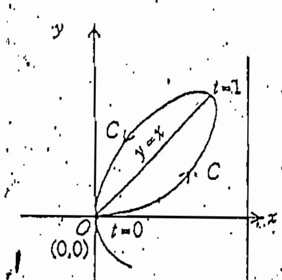
$$= 2\cdot\frac{3a^2}{2}\int_0^{\pi} (\cos^4\theta\sin^2\theta + \sin^4\theta\cos^2\theta)d\theta$$

$$= 4\cdot\frac{3a^2}{2}\int_0^{\pi/2} (\cos^4\theta\sin^2\theta + \sin^4\theta\cos^2\theta)d\theta$$

$$= 6a^2\left[\frac{3.1.1}{6.4.2}\cdot\frac{\pi}{2} + \frac{3.1.1}{6.4.2}\cdot\frac{\pi}{2}\right] = 6a^2\cdot\frac{\pi}{16} = \frac{3\pi a^2}{8}.$$

Ex. 17. *Find the area of the loop of the folium $x^3+y^3 = 3axy$, $a>0$.*

Sol. Let C be the simple closed curve formed by the loop of the given curve.



By Green's theorem the area of the loop

= the area enclosed by the simple closed curve C

$$= \frac{1}{2}\oint_C (x\,dy - y\,dx), \text{ by Green's theorem}$$

$$= \frac{1}{2}\int_C x^2\left[\frac{x\,dy - y\,dx}{x^2}\right] = \frac{1}{2}\int_C x^2\, d\left(\frac{y}{x}\right)$$

$$= \frac{1}{2}\int_C x^2\, dt, \qquad \ldots(1)$$

putting $y = tx$.

Putting $y = tx$ in the given equation of the folium, we have

$$x^3 + t^3x^3 = 3ax\,tx$$

or $\quad x^3(1+t^3) = 3atx^2$

or $\quad x = \dfrac{3at}{1+t^3}.$

Also for half the loop t varies from 0 to 1.

$\therefore$ from (1), the required area of the loop

$$= \frac{1}{2}\int_C \left(\frac{3at}{1+t^3}\right)^2 dt$$

$$= 2\cdot\frac{1}{2}\int_{t=0}^{1}\frac{9a^2t^2}{(1+t^3)^2}\,dt \quad \text{(since the loop is symmetrical about the line } y = x\text{)}$$

$$= 3a^2\int_0^1 (1+t^3)^{-2}(3t^2)\,dt$$

$$= 3a^2\left[\frac{(1+t^3)^{-1}}{-1}\right]_0^1, \text{ by power formula}$$

$$= 3a^2\left[-\frac{1}{1+t^3}\right]_0^1 = 3a^2\left[-\frac{1}{2}+1\right] = \frac{3a^2}{2}.$$

Ex. 18. *Introducing* $A = N\,i - M\,j$, *show that the formula in Green's theorem may be written as*

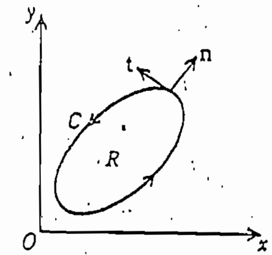
$$\iint_R \operatorname{div} A\, dx\, dy = \oint_C A\cdot n\, ds,$$

where n *is the outward unit normal vector to C and s is the arc length of C.*

Sol. We have $A = N\,i - M\,j$.

$$\therefore \operatorname{div} A = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}.$$

$$\therefore \iint_R \operatorname{div} A\, dx\, dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\, dy$$



$$= \oint_C (M\,dx + N\,dy), \text{ by Green's theorem.}$$

Now $M\,dx + N\,dy = (M\,i + N\,j)\cdot(dx\,i + dy\,j) = (M\,i + N\,j)\cdot d\mathbf{r}$

$$= \left\{(M\,i + N\,j)\cdot\frac{d\mathbf{r}}{ds}\right\} ds.$$

Now if t is a unit tangent vector to C, then $t = \dfrac{d\mathbf{r}}{ds}$. Also if k is a unit vector perpendicular to xy-plane, then $t = k \times n$.

$$\therefore M\,dx + N\,dy = [(M\,i + N\,j)\cdot t]\,ds = [(M\,i + N\,j)\cdot(k\times n)]\,ds$$

$$= [(M\,i + N\,j)\times k]\cdot n\, ds = (M\,i\times k + N\,j\times k)\cdot n\, ds$$

$$= (N\,i - M\,j)\cdot n\, ds = A\cdot n\, ds.$$

Hence the result.

Note. Putting $A = \nabla\phi$ in the above result, we get

$$\iint_R \operatorname{div}(\nabla\phi)\, dx\, dy = \oint_C (\nabla\phi)\cdot n\, ds$$

or

$$\iint \nabla^2\phi\, dx\, dy = \oint \frac{\partial\phi}{\partial n}\, ds, \text{ since } \nabla\phi = \frac{\partial\phi}{\partial n}\, n.$$

Gauss's Divergence Theorem:

We shall now show that triple integrals can be transformed into surface integrals over the boundary surface of a region in space and conversely.

This is of practical interest, since in many cases, one of the two kinds of integral is simpler than the other.

The transformation is done by the so-called divergence theorem, which involves the divergence of a vector function F,

$$\operatorname{div} F = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

Theorem

Divergence theorem of Gauss:-

(transformation b/w volume integrals and surface integrals)

Statement:

Suppose V is the volume bounded by a closed piece-wise smooth surface S. Suppose $\vec{F}(x,y,z)$ is a vector function of position which is continuous and has continuous first partial derivatives in V. Then

$$\iiint_V \nabla\cdot\vec{F}\,dV = \iint_S \vec{F}\cdot\hat{n}\,dS$$

where $\hat{n}$ is the outwards drawn unit normal vector to S.

Note: Since $\vec{F}\cdot\hat{n}$ is the normal component of vector $\vec{F}$, therefore divergence theorem may also be stated as follows:

The surface integral of the normal component of a vector $\vec{F}$ taken over a closed surface is equal to the integral of the divergence of $\vec{F}$ taken over the volume enclosed by the surface.

* Cartesian form:

Let $\vec{F} = F_1 i + F_2 j + F_3 k$

then $\nabla \cdot \vec{F} = \operatorname{div} \vec{F}$

$$= \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}.$$

Let α, β, γ be the angles which outward drawn unit normal $\hat{n}$ makes with positive directions of x, y, z-axes, then $\cos\alpha, \cos\beta, \cos\gamma$ are direction cosines of $\hat{n}$ and

we have $\hat{n} = \cos\alpha\, i + \cos\beta\, j + \cos\gamma\, k$.

$\therefore \vec{F} \cdot \hat{n} = F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma$.

Hence the divergence theorem can be written as

$$\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx\, dy\, dz = \iint_S \left(F_1 \cos\alpha + F_2 \cos\beta + F_3 \cos\gamma \right) ds \quad \text{(1)}$$

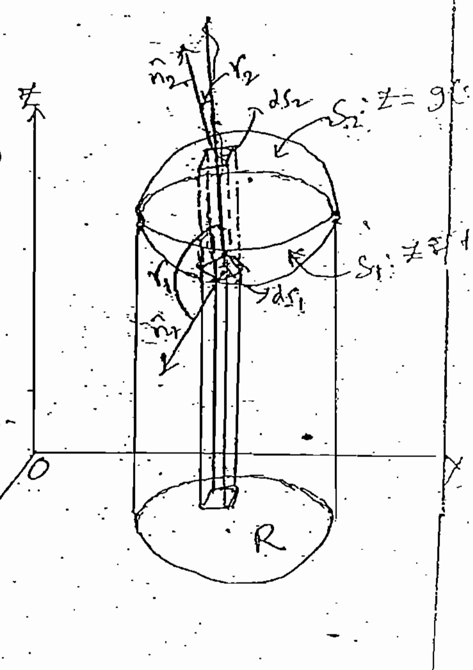
Now the projections of 'S' on xy, yz, zx-planes are $ds = \frac{dx\,dy}{|\hat{n} \cdot k|} = \frac{dx\,dy}{\cos\gamma}$;

$$ds = \frac{dy\,dz}{|\hat{n} \cdot i|} = \frac{dy\,dz}{\cos\alpha} \quad \text{and}$$

$$ds = \frac{dz \cdot dx}{|\hat{n} \cdot j|} = \frac{dz \cdot dx}{\cos\beta}.$$

$$\therefore \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \right) dx\, dy\, dz = \iint_S \left(F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy \right)$$

proof of the divergence theorem



Now we prove the theorem for a special region V which is bounded by a piece-wise smooth closed surface S and has the property that any straight line parallel to any one of the co-ordinate axes and intersecting V has only one segment (or a single point) in common with V.

If R is the orthogonal projection of 'S' on the xy-plane, then V can be represented in the form $f(x,y) \le z \le g(x,y)$. where (x,y) varies in R.

Obviously $z = g(x,y)$ represents the upper portion S_2 of S, $z = f(x,y)$ represents the lower portion S_1 of S.

$$\text{Now } \iiint_V \frac{\partial F_3}{\partial z} dv = \iiint_V \frac{\partial F_3}{\partial z} dx\,dy\,dz = \iint_R \left[\int_{z=f(x,y)}^{g(x,y)} \frac{\partial F_3}{\partial z} dz\right] dx\,dy$$

$$= \iint_R \Big[F_3(x,y,z)\Big]_{z=f(x,y)}^{g(x,y)} dx\,dy$$

$$= \iint_R \left[F_3(x,y,g) - F_3(x,y,f)\right] dx\,dy. \quad \text{——} (A)$$

For the upper portion S_2.

$$dy\,dx = \cos\gamma_2\, ds_2$$
$$= k \cdot \hat{n}_2\, ds_2$$

Since the normal $\hat{n}_2$ to S_2 makes an acute angle γ_2 with k

$$\therefore \boxed{\iint_R F_3(x,y,g)\,dx\,dy = \iint_{S_2} F_3\, \hat{n}_2 \cdot k\, ds_2}$$

For the lower curve:

$$dy\,dx = -\cos\gamma_1\, ds_1$$
$$= -k \cdot \hat{n}_1\, ds_1$$

Since the normal n_1 to S_1 makes an obtuse angle γ_1 with k.

$$\therefore \boxed{\iint_R F_3\,(x,y,f)\,dx\,dy = -\iint_{S_1} F_3\, \hat{n}_1 \cdot k\, ds_1}$$

from (A),

$$\iint_R F_3(x,y,g)\,dx\,dy - \iint_R F(x,y,f)\,dx\,dy =$$
$$\iint_{S_2} F_3\, \hat{n}_2 \cdot k\, ds_2 + \iint_{S_1} F_3\, \hat{n}_1 \cdot k\, ds_1$$

$$= \iint_S F_3\, \hat{n} \cdot k\, ds$$

$$\therefore \iiint_V \frac{\partial F_3}{\partial z}\, dv = \iint_S F_3\, \hat{n} \cdot k\, ds = \iint_S F_3\, k \cdot \hat{n}\, ds \quad \text{(B)}$$

Similarly, by projecting S on other co-ordinate planes

$$\iiint_V \frac{\partial F_1}{\partial x}\, dv = \iint_S F_1\, \hat{n} \cdot i\, ds = \iint F_1\, i \cdot \hat{n}\, ds \quad \text{(C)}$$

and $$\iiint_V \frac{\partial F_2}{\partial y}\, dv = \iint_S F_2\, \hat{n} \cdot j\, ds = \iint_S F_2\, j \cdot \hat{n}\, ds \quad \text{(D)}$$

Adding (B), (C) and (D), we get

$$\iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dv = \iint_S (F_1 i + F_2 j + F_3 k) \cdot \hat{n}\, ds$$

$$\Rightarrow \iiint (\nabla \cdot F)\, dv = \iint F \cdot \hat{n}\, ds.$$

§ 8. Some deductions from divergence theorem.

1. Green's theorem. *Let ϕ and ψ be scalar point functions which together with their derivatives in any direction are uniform and continuous within the region V bounded by a closed surface S, then*

$$\iiint_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\,dV = \iint_S (\phi\nabla\psi - \psi\nabla\phi)\cdot\mathbf{n}\,dS.$$

(Meerut 1979; Agra 71; Gauhati 92; Indore 79)

Proof. By divergence theorem, we have

$$\iiint_V \nabla\cdot\mathbf{F}\,dV = \iint_S \mathbf{F}\cdot\mathbf{n}\,dS.$$

Putting $\mathbf{F} = \phi\nabla\psi$, we get

$$\nabla\cdot\mathbf{F} = \nabla\cdot(\phi\nabla\psi)$$

$$= \phi(\nabla\cdot\nabla\psi) + (\nabla\phi)\cdot(\nabla\psi) = \phi\nabla^2\psi + (\nabla\phi)\cdot(\nabla\psi).$$

Also $\mathbf{F}\cdot\mathbf{n} = (\phi\nabla\psi)\cdot\mathbf{n}$.

$\therefore$ divergence theorem gives

$$\iiint_V [\phi\nabla^2\psi + (\nabla\phi)\cdot(\nabla\psi)]\,dV$$

$$= \iint_S (\phi\nabla\psi)\cdot\mathbf{n}\,dS \qquad \ldots(1)$$

(Osmania 1989; Meerut 70)

This is called *Green's first identity or theorem.*

Interchanging ϕ and ψ in (1), we get

$$\iiint_V [\psi\nabla^2\phi + (\nabla\psi)\cdot(\nabla\phi)]\,dV$$

$$= \iint_S [\psi\nabla\phi]\cdot\mathbf{n}\,dS \qquad \ldots(2)$$

Subtracting (2) from (1), we get

$$\iiint_V (\phi\nabla^2\psi - \psi\nabla^2\phi)\,dV = \iint_S (\phi\nabla\psi - \psi\nabla\phi)\cdot\mathbf{n}\,dS \qquad \ldots(3)$$

This is called *Green's second identity* or *Green's theorem in symmetrical form.*

Since $\nabla\psi = \dfrac{\partial\psi}{\partial n}\mathbf{n}$ and $\nabla\phi = \dfrac{\partial\phi}{\partial n}\mathbf{n}$, therefore,

$$(\phi\nabla\psi - \psi\nabla\phi)\cdot\mathbf{n} = \left(\phi\frac{\partial\psi}{\partial n}\mathbf{n} - \psi\frac{\partial\phi}{\partial n}\mathbf{n}\right)\cdot\mathbf{n}$$

$$= \phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}.$$

Hence (3) can also be written as

$$\iiint_V (\phi\nabla^2\psi-\psi\nabla^2\phi)\,dV=\iint_S\left(\phi\frac{\partial\psi}{\partial n}-\psi\frac{\partial\phi}{\partial n}\right)dS.$$

(Meerut 1980)

Note. Harmonic function. *If a scalar point function ϕ satisfies Laplace's equation $\nabla^2\phi=0$, then ϕ is called harmonic function.* If ϕ and ψ are both harmonic functions, then $\nabla^2\phi=0$, $\nabla^2\psi=0$.

Hence from Green's second identity, we get

$$\iint_S\left(\phi\frac{\partial\psi}{\partial n}-\psi\frac{\partial\phi}{\partial n}\right)dS=0.$$

2. *Prove that* $\iiint_V \nabla\phi\,dV=\iint_S \phi\mathbf{n}\,dS.$

(Agra 1972 ; Allahabad 77)

Proof. By divergence theorem, we have

$$\iiint_V \nabla\cdot\mathbf{F}\,dV=\iint_S \mathbf{F}\cdot\mathbf{n}\,dS.$$

Taking $\mathbf{F}=\phi\mathbf{C}$ where $\mathbf{C}$ is an arbitrary constant non-zero vector, we get

$$\iiint_V \nabla\cdot(\phi\mathbf{C})\,dV=\iint_S (\phi\mathbf{C})\cdot\mathbf{n}\,dS. \quad ...(1)$$

Now $\nabla\cdot(\phi\mathbf{C})=(\nabla\phi)\cdot\mathbf{C}+\phi(\nabla\cdot\mathbf{C})$

$=(\nabla\phi)\cdot\mathbf{C}$, since $\nabla\cdot\mathbf{C}=0$.

Also $(\phi\mathbf{C})\cdot\mathbf{n}=\mathbf{C}\cdot(\phi\mathbf{n})$.

$\therefore$ (1) becomes

$$\iiint_V \mathbf{C}\cdot(\nabla\phi)\,dV=\iint_S \mathbf{C}\cdot(\phi\mathbf{n})\,dS$$

or $$\mathbf{C}\cdot\iiint_V \nabla\phi\,dV=\mathbf{C}\cdot\iint_S (\phi\mathbf{n})\,dS$$

or $$\mathbf{C}\cdot\left[\iiint_V \nabla\phi\,dV-\iint_S \phi\mathbf{n}\,dS\right]=0.$$

Since $\mathbf{C}$ is an arbitrary vector, therefore we must have

$$\iiint_V \nabla\phi\,dV=\iint_S \phi\mathbf{n}\,dS.$$

3. *Prove that* $\iiint_V \nabla\times\mathbf{B}\,dV=\iint_S \mathbf{n}\times\mathbf{B}\,dS.$

(Gauhati 1974)

Proof. In divergence theorem taking $\mathbf{F}=\mathbf{B}\times\mathbf{C}$, where $\mathbf{C}$ is an arbitrary constant vector, we get

$$\iiint_V \nabla\cdot(\mathbf{B}\times\mathbf{C})\,dV=\iint_S (\mathbf{B}\times\mathbf{C})\cdot\mathbf{n}\,dS. \quad ...(1)$$

Now $\nabla\cdot(\mathbf{B}\times\mathbf{C})=\mathbf{C}\cdot\text{curl }\mathbf{B}-\mathbf{B}\cdot\text{curl }\mathbf{C}$

$=\mathbf{C}\cdot\text{curl }\mathbf{B}$, since curl $\mathbf{C}=0$.

Also $(\mathbf{B}\times\mathbf{C})\cdot\mathbf{n}=[\mathbf{B},\mathbf{C},\mathbf{n}]=[\mathbf{C},\mathbf{n},\mathbf{B}]=\mathbf{C}\cdot(\mathbf{n}\times\mathbf{B})$.

$\therefore$ (1) becomes

$$\iiint_V (\mathbf{C}\cdot\text{curl }\mathbf{B})\,dV=\iint_S \mathbf{C}\cdot(\mathbf{n}\times\mathbf{B})\,dS$$

or $$\mathbf{C}\cdot\iiint_V (\nabla\times\mathbf{B})\,dV=\mathbf{C}\cdot\iint_S (\mathbf{n}\times\mathbf{B})\,dS$$

or $$\mathbf{C}\cdot\left[\iiint_V (\nabla\times\mathbf{B})\,dV-\iint_S (\mathbf{n}\times\mathbf{B})\,dS\right]=0.$$

Since $\mathbf{C}$ is an arbitrary vector therefore we can take $\mathbf{C}$ as a non-zero vector which is not perpendicular to the vector

$$\iiint_V (\nabla\times\mathbf{B})\,dV-\iint_S (\mathbf{n}\times\mathbf{B})\,dS.$$

Hence we must have

$$\iiint_V (\nabla\times\mathbf{B})\,dV-\iint_S (\mathbf{n}\times\mathbf{B})\,dS=0$$

or $$\iiint_V (\nabla\times\mathbf{B})\,dV=\iint_S (\mathbf{n}\times\mathbf{B})\,dS.$$

Solved Examples

Ex. 1. *For any closed surface S, prove that*

$$\iint_S \text{curl }\mathbf{F}\cdot\mathbf{n}\,dS=0.$$

Sol. By divergence theorem, we have

$$\iint_S \text{curl }\mathbf{F}\cdot\mathbf{n}\,dS=\iiint_V (\text{div curl }\mathbf{F})\,dV, \text{ where } V \text{ is the volume enclosed by } S$$

$=0$, since div curl $\mathbf{F}=0$.

Ex. 2. *Evaluate* $\iint_S \mathbf{r}\cdot\mathbf{n}\,dS$, *where S is a closed surface.*

(Andhra 1992, Madras 83, Rohilkhand 76, Allahabad 75)

Sol. By the divergence theorem, we have

$$\iint_S \mathbf{r}\cdot\mathbf{n}\,dS=\iiint_V \nabla\cdot\mathbf{r}\,dV$$

$$=\iiint_V 3\,dV, \text{ since } \nabla\cdot\mathbf{r}=\text{div }\mathbf{r}=3$$

$=3V$, where V is the volume enclosed by S.

Ex. 3. *If* $\mathbf{F} = ax\,\mathbf{i}+by\,\mathbf{j}+cz\,\mathbf{k}$, *a, b, c are constants show that* $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \frac{4}{3}\pi(a+b+c)$, *where S is the surface of a unit sphere.*

(Rohilkhand 1992, Allahabad 82, Agra 1980)

Sol. By the divergence theorem, we have

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_V (\nabla\cdot\mathbf{F})\,dV,$$

where V is the volume enclosed by S.

$$= \iiint_V [\nabla\cdot(ax\,\mathbf{i}+by\,\mathbf{j}+cz\,\mathbf{k})]\,dV$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(ax)+\frac{\partial}{\partial y}(by)+\frac{\partial}{\partial z}(cz)\right]dV$$

$$= \iiint_V (a+b+c)\,dV = (a+b+c)\,V = (a+b+c)\,\frac{4}{3}\pi,$$

since the volume V enclosed by a sphere of unit radius is equal to $\frac{4}{3}\pi(1)^3$ i.e., $\frac{4}{3}\pi$.

Ex. 4. *If* $\mathbf{n}$ *is the unit outward drawn normal to any closed surface S, show that* $\iiint_V \operatorname{div}\mathbf{n}\,dV = S$.

(Andhra 1989)

Sol. We have by the divergence theorem,

$$\iiint_V \operatorname{div}\mathbf{n}\,dV = \iint_S \mathbf{n}\cdot\mathbf{n}\,dS = \iint_S dS = S.$$

Ex. 5. *Prove that*

$$\iiint_V \nabla\phi\cdot\mathbf{A}\,dV = \iint_S \phi\mathbf{A}\cdot\mathbf{n}\,dS - \iiint_V \phi\nabla\cdot\mathbf{A}\,dV.$$

Sol. By divergence theorem, we have

$$\iiint_V \nabla\cdot(\phi\mathbf{A})\,dV = \iint_S (\phi\mathbf{A})\cdot\mathbf{n}\,dS \qquad \text{...(1)}$$

Now $\nabla\cdot(\phi\mathbf{A}) = (\nabla\phi)\cdot\mathbf{A}+\phi(\nabla\cdot\mathbf{A})$.

Also $(\phi\mathbf{A})\cdot\mathbf{n} = \phi(\mathbf{A}\cdot\mathbf{n})$.

Hence (1) gives

$$\iiint_V [(\nabla\phi)\cdot\mathbf{A}+\phi(\nabla\cdot\mathbf{A})]\,dV = \iint_S \phi\mathbf{A}\cdot\mathbf{n}\,dS$$

or $$\iiint_V (\nabla\phi)\cdot\mathbf{A}\,dV = \iint_S \phi\mathbf{A}\cdot\mathbf{n}\,dS - \iiint_V \phi\nabla\cdot\mathbf{A}\,dV.$$

Ex. 6. *Prove that* $\int_S \nabla\phi\times\nabla\psi\cdot d\mathbf{S} = 0.$

Sol. We have $\iint_S \nabla\phi\times\nabla\psi\cdot d\mathbf{S} = \int_S (\nabla\phi\times\nabla\psi)\cdot\mathbf{n}\,dS$

$= \int_V \nabla\cdot(\nabla\phi\times\nabla\psi)\,dV$, by divergence throrem

$= 0.$ $\qquad [\because \nabla\cdot(\nabla\phi\times\nabla\psi) = 0].$

Ex. 7. *Prove that*

$$\int_V \nabla\phi\cdot\text{curl}\,\mathbf{F}\,dV = \int_S (\mathbf{F}\times\nabla\phi)\cdot d\mathbf{S}.$$

Sol. We have $\int_S (\mathbf{F}\times\nabla\phi)\cdot d\mathbf{S} = \int_S (\mathbf{F}\times\nabla\phi)\cdot\mathbf{n}\,dS$

$= \int_V \nabla\cdot(\mathbf{F}\times\nabla\phi)\,dV$, by divergence theorem applied to the vector function $\mathbf{F}\times\nabla\phi$

$= \int_V (\nabla\phi\cdot\text{curl}\,\mathbf{F} - \mathbf{F}\cdot\text{curl}\,\nabla\phi)\,dV$

$[\because \nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot\text{curl}\,\mathbf{A} - \mathbf{A}\cdot\text{curl}\,\mathbf{B}]$

$= \int_V \nabla\phi\cdot\text{curl}\,\mathbf{F}\,dV.$ $\qquad [\because \text{curl}\,\nabla\phi = 0]$

Ex. 8. *Prove that* $\iiint_V \frac{dV}{r^2} = \iint_S \frac{\mathbf{r}\cdot\mathbf{n}}{r^2}\,dS.$

Sol. $\iint_S \frac{\mathbf{r}\cdot\mathbf{n}}{r^2}\,dS = \iint_S \left(\frac{\mathbf{r}}{r^2}\right)\cdot\mathbf{n}\,dS$

$= \iiint_V \nabla\cdot\left(\frac{\mathbf{r}}{r^2}\right)dV$, by divergence theorem.

Now $\nabla\cdot\left(\frac{\mathbf{r}}{r^2}\right) = \frac{1}{r^2}(\nabla\cdot\mathbf{r}) + \mathbf{r}\cdot\nabla\left(\frac{1}{r^2}\right)$

$= \frac{3}{r^2}+\mathbf{r}\cdot\left(-\frac{2}{r^3}\nabla r\right) = \frac{3}{r^2}-\frac{2}{r^3}\left(\mathbf{r}\cdot\frac{\mathbf{r}}{r}\right) = \frac{3}{r^2}-\frac{2}{r^4}r^2 = \frac{1}{r^2}.$

Hence $\iint_S \frac{\mathbf{r}\cdot\mathbf{n}}{r^2}\,dS = \iiint_V \frac{dV}{r^2}.$

Ex. 9. *If* $\mathbf{F} = \nabla\phi$ *and* $\nabla^2\phi = 0$, *show that for a closed surface S*

$$\iiint_V F^2\,dV = \iint_S \phi\mathbf{F}\cdot\mathbf{n}\,dS. \quad \text{(Rohilkhand 1978, 79)}$$

Sol. By divergence theorem, we have

$$\iint_S \phi \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_V [\nabla \cdot (\phi \mathbf{F})]\, dV.$$

Now $\nabla \cdot (\phi \mathbf{F}) = (\nabla\phi \cdot \mathbf{F}) + \phi (\nabla \cdot \mathbf{F}) = \mathbf{F} \cdot \mathbf{F} + \phi (\nabla \cdot \nabla \phi)$

$$= F^2 + \phi \nabla^2 \phi = F^2, \text{ since } \nabla^2 \phi = 0.$$

Hence $\iint_S \phi \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_V F^2\, dV.$

Ex. 10. *If* $\mathbf{F} = \nabla \phi, \nabla^2 \phi = -4\pi\rho$, *show that*

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = -4\pi \iiint_V \rho\, dV.$$

Sol. By divergence theorem, we have

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_V (\nabla \cdot \mathbf{F})\, dV.$$

Now $\nabla \cdot \mathbf{F} = \nabla \cdot (\nabla \phi) = \nabla^2 \phi = -4\pi\rho.$

$$\therefore \iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_V (-4\pi\rho)\, dV = -4\pi \iiint_V \rho\, dV.$$

Ex. 11. *If* $\mathbf{C} = \frac{1}{2} \nabla \times \mathbf{B}, \mathbf{B} = \nabla \times \mathbf{A}$, *show that*

$$\tfrac{1}{2} \iiint_V B^2\, dV = \tfrac{1}{2} \iint_S \mathbf{A} \times \mathbf{B} \cdot \mathbf{n}\, dS + \iiint_V \mathbf{A} \cdot \mathbf{C}\, dV.$$

Sol. We have by divergence theorem

$$\tfrac{1}{2} \iint_S (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{n}\, dS = \tfrac{1}{2} \iiint_V \nabla \cdot (\mathbf{A} \times \mathbf{B})\, dV.$$

Now $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot \text{curl}\, \mathbf{A} - \mathbf{A} \cdot \text{curl}\, \mathbf{B}$

$= \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}) = \mathbf{B} \cdot \mathbf{B} - \mathbf{A} \cdot (2\mathbf{C}) = B^2 - 2 (\mathbf{A} \cdot \mathbf{C}).$

Hence $\frac{1}{2} \iint_S (\mathbf{A} \times \mathbf{B}) \cdot \mathbf{n}\, dS = \frac{1}{2} \iiint_V [B^2 - 2 (\mathbf{A} \cdot \mathbf{C})]\, dV$

$$= \tfrac{1}{2} \iiint_V B^2\, dV - \iiint_V \mathbf{A} \cdot \mathbf{C}\, dV$$

or $\quad \frac{1}{2} \iiint_V B^2\, dV = \frac{1}{2} \iint_S \mathbf{A} \times \mathbf{B} \cdot \mathbf{n}\, dS + \iiint_V \mathbf{A} \cdot \mathbf{C}\, dV.$

Ex. 12. *If* ϕ *is harmonic in* V, *then*

$$\iint_S \frac{\partial \phi}{\partial n}\, dS = 0$$

where S *is the surface enclosing* V. (Meerut 1972)

Sol. We have $\iint_S \frac{\partial \phi}{\partial n}\, dS = \iint_S \left(\frac{\partial \phi}{\partial n} \mathbf{n}\right) \cdot \mathbf{n}\, dS$

$$= \iint_S (\nabla \phi) \cdot \mathbf{n}\, dS$$

$$= \iiint_V \nabla \cdot (\nabla \phi)\, dV, \text{ by divergence theorem}$$

$$= \iiint_V \nabla^2 \phi\, dV$$

$= 0$, since $\nabla^2 \phi = 0$ in V because ϕ is harmonic in V.

Ex. 13. *If* ϕ *is harmonic in* V, *then*

$$\iint_S \phi \frac{\partial \phi}{\partial n}\, dS = \iiint_V |\nabla \phi|^2\, dV.$$

(Meerut 1979 ; Agra 70)

Sol. We have

$$\iint_S \phi \frac{\partial \phi}{\partial n}\, dS = \iint_S \left(\phi \frac{\partial \phi}{\partial n} \mathbf{n}\right) \cdot \mathbf{n}\, dS = \iint_S (\phi \nabla \phi) \cdot \mathbf{n}\, dS$$

$$= \iiint_V \nabla \cdot (\phi \nabla \phi)\, dV, \text{ by divergence theorem}$$

$$= \iiint_V [(\nabla \phi \cdot \nabla \phi) + \phi (\nabla \cdot \nabla \phi)]\, dV$$

$$= \iiint_V [(\nabla \phi)^2 + \phi \nabla^2 \phi]\, dV$$

$$= \iiint_V |\nabla \phi|^2\, dV, \text{ since } \nabla^2 \phi = 0 \text{ and } (\nabla \phi)^2 = |\nabla \phi|^2.$$

Ex. 14. *If* ϕ *is harmonic in* V *and* $\frac{\partial \phi}{\partial n} = 0$ *on* S, *then* ϕ *is constant in* V.

Sol. Since ϕ is harmonic in V, therefore as in exercise 13, we have

$$\iint_S \phi \frac{\partial \phi}{\partial n}\, dS = \iiint_V |\nabla \phi|^2\, dV.$$

But $\frac{\partial \phi}{\partial n} = 0$ on S. Therefore $\iint_S \phi \frac{\partial \phi}{\partial n}\, dS = 0.$

$\therefore \iiint_V |\nabla \phi|^2\, dV = 0.$

$\therefore |\nabla \phi|^2 = 0$ in V.

$\therefore \nabla \phi = 0$ in V.

$\therefore \phi =$ constant in V.

Ex. 15. *If* ϕ *and* ψ *are harmonic in* V *and* $\frac{\partial \phi}{\partial n} = \frac{\partial \psi}{\partial n}$ *on* S, *then* $\phi = \psi + c$ *in* V, *where* c *is a constant.*

Sol. We have, $\nabla^2\phi = 0, \nabla^2\psi = 0$ in V.

$\therefore \quad \nabla^2(\phi - \psi) = \nabla^2\phi - \nabla^2\psi = 0$ in V.

Therefore $\phi - \psi$ is harmonic in V.

Again on S, $\dfrac{\partial}{\partial n}(\phi - \psi) = \dfrac{\partial \phi}{\partial n} - \dfrac{\partial \psi}{\partial n} = 0.$

Thus $\phi - \psi$ is harmonic in V and on S we have

$$\frac{\partial}{\partial n}(\phi - \psi) = 0.$$

Hence as in exercise 14, we have

$$\phi - \psi = c, \text{ where } c \text{ is a constant}$$

or $\quad \phi = \psi + c.$

Ex. 16. *If div* **F** *denotes the divergence of a vector field* **F** *at a point P, show that*

$$\text{div }\mathbf{F} = \lim_{\delta V \to 0} \frac{\iint_{\delta S} \mathbf{F}\cdot\mathbf{n}\, dS}{\delta V}$$

where δV is the volume enclosed by the surface δS and the limit is obtained by shrinking δV to the point P.

Sol. We have by the divergence theorem,

$$\iiint_{\delta V} \text{div }\mathbf{F}\, dV = \iint_{\delta S} \mathbf{F}\cdot\mathbf{n}\, dS. \qquad \ldots(1)$$

By the mean value theorem of integral calculus, the left hand side can be written as

$$\overline{\text{div }\mathbf{F}} \iiint_{\delta V} dV = \overline{\text{div }\mathbf{F}}\, \delta V,$$

where $\overline{\text{div }\mathbf{F}}$ is some value intermediate between the maximum and minimum of div **F** throughout δV. Therefore (1) gives

$$\overline{\text{div }\mathbf{F}}\, \delta V = \iint_{\delta S} \mathbf{F}\cdot\mathbf{n}\, dS$$

or $$\overline{\text{div }\mathbf{F}} = \frac{\iint_{\delta S} \mathbf{F}\cdot\mathbf{n}\, dS}{\delta V}.$$

Taking the limit as $\delta V \to 0$ such that P is always interior to δV, $\overline{\text{div }\mathbf{F}}$ approaches the value div **F** at point P. Hence, we get

$$\text{div }\mathbf{F} = \lim_{\delta V \to 0} \frac{\iint_{\delta S} \mathbf{F}\cdot\mathbf{n}\, dS}{\delta V}.$$

Ex. 17. *Show that* $\iint_S \mathbf{n}\, dS = 0$ *for any closed surface S.*

Solution. Let **C** be any arbitrary constant vector. Then

$$\mathbf{C}\cdot \iint_S \mathbf{n}\, dS = \iint_S \mathbf{C}\cdot\mathbf{n}\, dS$$

$$= \iiint_V (\nabla\cdot\mathbf{C})\, dV, \text{ by divergence theorem}$$

$$= 0, \text{ since div } \mathbf{C} = 0.$$

Thus $\mathbf{C}\cdot \iint_S \mathbf{n}\, dS = 0$, where **C** is an arbitrary vector.

Therefore we must have $\iint_S \mathbf{n}\, dS = 0.$

Ex. 18. *Prove that* $\iint_S \mathbf{r}\times\mathbf{n}\, dS = 0$ *for any closed surface S.*

Sol. Let **C** be any arbitrary constant vector. Then

$$\mathbf{C}\cdot \iint_S \mathbf{r}\times\mathbf{n}\, dS = \iint_S \mathbf{C}\cdot[(\mathbf{r}\times\mathbf{n})]\, dS$$

$$= \iint_S (\mathbf{C}\times\mathbf{r})\cdot\mathbf{n}\, dS$$

$$= \iiint_V [\nabla\cdot(\mathbf{C}\times\mathbf{r})]\, dV, \text{ by divergence theorem}$$

$$= \iiint_V [\mathbf{r}\cdot\text{curl }\mathbf{C} - \mathbf{C}\cdot\text{curl }\mathbf{r}]\, dV$$

$$= 0, \text{ since curl } \mathbf{C} = 0 \text{ and curl } \mathbf{r} = 0.$$

Thus $\mathbf{C}\cdot \iint_S \mathbf{r}\times\mathbf{n}\, dS = 0$, where **C** is an arbitrary vector.

Therefore, we must have $\iint_S \mathbf{r}\times\mathbf{n}\, dS = 0.$

Ex. 19. *Prove that* $\iint_S (\nabla\phi)\times\mathbf{n}\, dS = 0$ *for a closed surface S.*

Sol. Let **C** be an arbitrary constant vector. Then

$$\mathbf{C}\cdot \iint_S (\nabla\phi)\times\mathbf{n}\, dS = \iint_S \mathbf{C}\cdot[(\nabla\phi)\times\mathbf{n}]\, dS$$

$$= \iint_S [\mathbf{C}\times\nabla\phi]\cdot\mathbf{n}\, dS$$

$$= \iiint_V [\nabla\cdot(\mathbf{C}\times\nabla\phi)]\, dV, \text{ by div. theorem}$$

$$= \iiint_V [\nabla\phi\cdot\text{curl }\mathbf{C} - \mathbf{C}\cdot\text{curl }\nabla\phi]\, dV$$

$$= 0, \text{ since curl } \mathbf{C} = 0 \text{ and curl } \nabla\phi = 0.$$

Thus $\mathbf{C}\cdot \iint_S (\nabla\phi)\times\mathbf{n}\, dS = 0$, where **C** is an arbitrary vector.

Hence we must have $\iint_S (\nabla\phi) \times \mathbf{n}\, dS = 0.$

Ex. 20. *Prove that* $\iint_S \mathbf{n} \times (\mathbf{a} \times \mathbf{r})\, dS = 2V\mathbf{a},$

where $\mathbf{a}$ *is a constant vector and* V *is the volume enclosed by the closed surface* S.

Sol. We know that

$$\iiint_V \nabla \times \mathbf{B}\, dV = \iint_S \mathbf{n} \times \mathbf{B}\, dS.$$

[See § 8, part 3 page 232]

Putting $\mathbf{B} = \mathbf{a} \times \mathbf{r}$, we get

$$\iint_S \mathbf{n} \times (\mathbf{a} \times \mathbf{r})\, dS = \iiint_V \nabla \times (\mathbf{a} \times \mathbf{r})\, dV$$

$$= \iiint_V \text{curl}\,(\mathbf{a} \times \mathbf{r})\, dV$$

$$= \iiint_V 2\mathbf{a}\, dV, \text{ since curl}\,(\mathbf{a} \times \mathbf{r}) = 2\mathbf{a}$$

$$= 2\mathbf{a} \iiint_V dV = 2\mathbf{a}V.$$

Ex. 21. *A vector* $\mathbf{B}$ *is always normal to a given closed surface* S. *Show that* $\iiint_V \text{curl}\,\mathbf{B}\, dV = 0$, *where* V *is the region bounded by* S.

Sol. We know that

$$\iiint_V \text{curl}\,\mathbf{B}\, dV = \iint_S \mathbf{n} \times \mathbf{B}\, dS.$$

Since $\mathbf{B}$ is normal to S, therefore $\mathbf{B}$ is parallel to $\mathbf{n}$. Therefore $\mathbf{n} \times \mathbf{B} = 0$.

$$\therefore \iint_S \mathbf{n} \times \mathbf{B}\, dS = 0.$$

$$\therefore \iiint_V \text{curl}\,\mathbf{B}\, dV = 0.$$

Ex. 22. *Express* $\int_V \{(\text{grad}\,\rho) \cdot \mathbf{v} + \rho\, \text{div}\,\mathbf{v}\}\, dV$, *as a surface integral.* (Gauhati 1977)

Sol. From a vector identity we know that

$$\text{div}\,(\rho\mathbf{v}) = (\text{grad}\,\rho) \cdot \mathbf{v} + \rho\, \text{div}\,\mathbf{v}.$$

$$\therefore \int_V \{(\text{grad}\,\rho) \cdot \mathbf{v} + \rho\, \text{div}\,\mathbf{v}\}\, dV = \int_V \text{div}\,(\rho\mathbf{v})\, dV$$

$$= \int_V \nabla \cdot (\rho\mathbf{v})\, dV$$

$$= \int_S (\rho\mathbf{v}) \cdot \mathbf{n}\, dS, \text{ by Gauss divergence theorem}$$

$$= \int_S \rho\, (\mathbf{v} \cdot \mathbf{n})\, dS.$$

Ex. 23. *Using the divergence theorem, show that the volume* V *of a region* T *bounded by a surface* S *is*

$$V = \iint_S x\, dy\, dz = \iint_S y\, dz\, dx = \iint_S z\, dx\, dy$$

$$= \tfrac{1}{3} \iint_S (x\, dy\, dz + y\, dz\, dx + z\, dx\, dy).$$

Sol. By divergence theorem, we have

$$\iint_S x\, dy\, dz = \iiint_V \left(\frac{\partial}{\partial x}(x)\right) dV = \iiint_V dV = V$$

$$\iint_S y\, dz\, dx = \iiint_V \left(\frac{\partial}{\partial y}(y)\right) dV = \iiint_V dV = V$$

$$\iint_S z\, dx\, dy = \iiint_V \left(\frac{\partial}{\partial z}(z)\right) dV = \iiint_V dV = V.$$

Adding these results, we get

$$3V = \iint_S (x\, dy\, dz + y\, dz\, dx + z\, dx\, dy)$$

or

$$V = \tfrac{1}{3} \iint_S (x\, dy\, dz + y\, dz\, dx + z\, dx\, dy).$$

Ex. 24 (a). *Verify divergence theorem for*

$$\mathbf{F} = (x^2 - yz)\,\mathbf{i} + (y^2 - zx)\,\mathbf{j} + (z^2 - xy)\,\mathbf{k}$$

taken over the rectangular parallelopiped

$$0 \le x \le a,\ 0 \le y \le b,\ 0 \le z \le c.$$

(Andhra 1990, Meerut 76)

Sol. We have $\text{div}\,\mathbf{F} = \nabla \cdot \mathbf{F}$

$$= \frac{\partial}{\partial x}(x^2 - yz) + \frac{\partial}{\partial y}(y^2 - zx) + \frac{\partial}{\partial z}(z^2 - xy) = 2x + 2y + 2z.$$

$$\therefore \text{volume integral} = \iiint_V \nabla \cdot \mathbf{F}\, dV = \iiint_V 2(x + y + z)\, dV$$

$$= 2 \int_{z=0}^{c} \int_{y=0}^{b} \int_{x=0}^{a} (x + y + z)\, dx\, dy\, dz$$

$$= 2 \int_{z=0}^{c} \int_{y=0}^{b} \left[\frac{x^2}{2} + yx + zx\right]_{x=0}^{a} dy\, dz$$

$$= 2\int_{z=0}^{c}\int_{y=0}^{b}\left[\frac{a^2}{2} + ay + az\right] dy\, dz$$

$$= 2\int_{z=0}^{c}\left[\frac{a^2}{2}y + a\frac{y^2}{2} + azy\right]_{y=0}^{b} dz$$

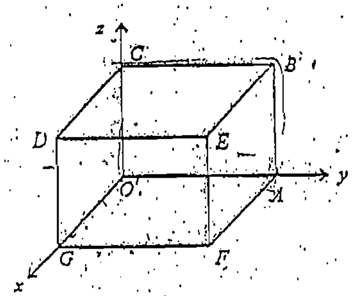
$$= 2\int_{z=0}^{c}\left[\frac{a^2b}{2} + \frac{ab^2}{2} + abz\right] dz = 2\left[\frac{a^2b}{2}z + \frac{ab^2}{2}z + ab\frac{z^2}{2}\right]_0^c$$

$$= [a^2bc + ab^2c + abc^2] = abc\,(a + b + c).$$

Surface Integral. We shall now calculate

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS$$

over the six faces of the rectangular parallelopiped.



Over the face $DEFG$,

$\mathbf{n} = \mathbf{i}$, $x = a$.

Therefore,

$$\iint_{DEFG} \mathbf{F}\cdot\mathbf{n}\, dS$$

$$= \int_{z=0}^{c}\int_{y=0}^{b} [(a^2 - yz)\,\mathbf{i} + (y^2 - za)\,\mathbf{j} + (z^2 - ay)\,\mathbf{k}]\cdot\mathbf{i}\, dy\, dz$$

$$= \int_{z=0}^{c}\int_{y=0}^{b} (a^2 - yz)\, dy\, dz = \int_{z=0}^{c}\left[a^2y - z\frac{y^2}{2}\right]_{y=0}^{b} dz$$

$$= \int_{z=0}^{c}\left[a^2b - \frac{zb^2}{2}\right] dz = \left[a^2bz - \frac{z^2}{4}b^2\right]_0^c$$

$$= a^2bc - \frac{c^2b^2}{4}.$$

Over the face $ABCO$, $\mathbf{n} = -\mathbf{i}$, $x = 0$. Therefore

$$= \iint_{ABCO} \mathbf{F}\cdot\mathbf{n}\, dS = \iint [(0 - yz)\,\mathbf{i} + \ldots + \ldots]\cdot(-\mathbf{i})\, dy\, dz.$$

$$= \int_{z=0}^{c}\int_{y=0}^{b} yz\, dy\, dz = \int_{z=0}^{c}\left[\frac{y^2}{2}z\right]_{y=0}^{b} dz$$

$$= \int_{z=0}^{c}\frac{b^2}{2}z\, dz = \frac{b^2c^2}{4}.$$

Over the face $ABEF$, $\mathbf{n} = \mathbf{j}$, $y = b$. Therefore

$$\iint_{ABEF} \mathbf{F}\cdot\mathbf{n}\, dS = \int_{z=0}^{c}\int_{x=0}^{a} [(x^2 - bz)\,\mathbf{i} + (b^2 - zx)\,\mathbf{j}$$

$$+ (z^2 - bx)\,\mathbf{k}]\cdot\mathbf{j}\, dx\, dz$$

$$= \int_{z=0}^{c}\int_{x=0}^{a} (b^2 - zx)\, dx\, dz = b^2ca - \frac{a^2c^2}{4}.$$

Over the face $OGDC$, $\mathbf{n} = -\mathbf{j}$, $y = 0$. Therefore

$$\iint_{OGDC} \mathbf{F}\cdot\mathbf{n}\, dS = \int_{z=0}^{c}\int_{x=0}^{a} zx\, dx\, dz = \frac{c^2a^2}{4}.$$

Over the face $BCDE$, $\mathbf{n} = \mathbf{k}$, $z = c$. Therefore

$$\iint_{BCDE} \mathbf{F}\cdot\mathbf{n}\, dS = \int_{y=0}^{b}\int_{x=0}^{a} (c^2 - xy)\, dx\, dy = c^2ab - \frac{a^2b^2}{4}.$$

Over the face $AFGO$, $\mathbf{n} = -\mathbf{k}$, $z = 0$. Therefore

$$\iint_{AFGO} \mathbf{F}\cdot\mathbf{n}\, dS = \int_{y=0}^{b}\int_{x=0}^{a} xy\, dx\, dy = \frac{a^2b^2}{4}.$$

Adding the six surface integrals, we get

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \left(a^2bc - \frac{c^2b^2}{4} + \frac{c^2b^2}{4}\right) + \left(b^2ca - \frac{a^2c^2}{4} + \frac{a^2c^2}{4}\right)$$

$$+ \left(c^2ab - \frac{a^2b^2}{4} + \frac{a^2b^2}{4}\right)$$

$= abc\,(a + b + c)$.

Hence the theorem is verified.

Ex. 24. (b). *If* $\mathbf{F} = 4xz\,\mathbf{i} - y^2\mathbf{j} + yz\mathbf{k}$ *and S is the surface bounded by* $x = 0,\ y = 0,\ z = 0,\ x = 1,\ y = 1,\ z = 1$, *evaluate*

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS. \qquad \text{(Osmania 1990)}$$

Sol. By Gauss divergence theorem,

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_V \nabla\cdot\mathbf{F}\,dV,$$

where V is the volume enclosed by the surface S

$$= \iiint_V \left[\frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)\right] dV$$

$$= \iiint_V (4z - 2y + y)\,dV = \iiint_V (4z - y)\,dx\,dy\,dz$$

$$= \int_{x=0}^{1}\int_{y=0}^{1}\int_{z=0}^{1} (4z - y)\,dx\,dy\,dz$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} \left[2z^2 - yz\right]_{z=0}^{1} dx\,dy = \int_{x=0}^{1}\int_{y=0}^{1} (2 - y)\,dx\,dy$$

$$= \int_{x=0}^{1} \left[2y - \frac{y^2}{2}\right]_{y=0}^{1} dx = \int_0^1 \left[2 - \frac{1}{2}\right] dx = \frac{3}{2}\int_0^1 dx = \frac{3}{2}.$$

Ex. 24. (c). *Evaluate* $\iint_S [4xz\,dy\,dz - y^2\,dz\,dx + yz\,dx\,dy]$ *where S is the surface of the cube bounded by the planes* $x = 0,\ y = 0,\ z = 0,\ x = 1,\ y = 1$ *and* $z = 1$. (Osmania 1992)

Sol. By Gauss divergence theorem, the given surface integral is equal to the volume integral

$$= \iiint_V \left[\frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)\right] dV,$$

where V is the volume enclosed by the surface S.

Ex. 24. (d). *Evaluate* $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$ *if* $\mathbf{F} = xy\,\mathbf{i} + z^2\mathbf{j} + 2yz\,\mathbf{k}$ *over the tetrahedron bounded by the planes* $x = 0,\ y = 0,\ z = 0$ *and* $x + y + z = 1$.

Sol. By Gauss divergence theorem,

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_V \nabla\cdot\mathbf{F}\,dV,$$ where V is the volume enclosed by

the surface S of the tetrahedron

$$= \iiint_V \left[\frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(z^2) + \frac{\partial}{\partial z}(2yz)\right] dV$$

$$= \iiint_V (y + 0 + 2y)\,dV$$

$$= 3\int_{x=0}^{1}\int_{y=0}^{1-x}\int_{z=0}^{1-x-y} y\,dx\,dy\,dz = 3\int_{x=0}^{1}\int_{y=0}^{1-x} y\left[z\right]_{z=0}^{1-x-y} dx\,dy$$

$$= 3\int_{x=0}^{1}\int_{y=0}^{1-x} y\,[1 - x - y]\,dx\,dy$$

$$= 3\int_{x=0}^{1}\int_{y=0}^{1-x} [y - xy - y^2]\,dx\,dy$$

$$= 3\int_0^1 \left[\frac{y^2}{2} - x\frac{y^2}{2} - \frac{y^3}{3}\right]_{y=0}^{1-x} dx$$

$$= 3\int_0^1 \left[\tfrac{1}{2}(1-x)^2 - \tfrac{1}{2}x(1-x)^2 - \tfrac{1}{3}(1-x)^3\right] dx$$

$$= 3\int_0^1 \frac{1}{6}(1-x)^3\,dx = \frac{3}{6}\left[\frac{-(1-x)^4}{4}\right]_0^1 = \frac{3}{24} = \frac{1}{8}.$$

Ex. 24. (e). *Find the value of* $\iint_S (\mathbf{F}\times\nabla\phi)\cdot\mathbf{n}\,dS$ *where* $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$, $\phi = xy + yz + zx$,

S:— $x = \pm 1,\ y = \pm 1,\ z = \pm 1$. (Nagarjuna 1991)

Sol. By Gauss divergence theorem,

$$\iint_S (\mathbf{F}\times\nabla\phi)\cdot\mathbf{n}\,dS = \iiint_V \nabla\cdot(\mathbf{F}\times\nabla\phi)\,dV,$$

where V is the volume enclosed by the surface S.

Now from a vector identity, we know that

$\nabla\cdot(\mathbf{A}\times\mathbf{B}) = \mathbf{B}\cdot(\nabla\times\mathbf{A}) - \mathbf{A}\cdot(\nabla\times\mathbf{B})$.

$\therefore\ \nabla\cdot(\mathbf{F}\times\nabla\phi) = \nabla\phi\cdot(\nabla\times\mathbf{F}) - \mathbf{F}\cdot(\nabla\times(\nabla\phi))$

$= \nabla\phi\cdot(\nabla\times\mathbf{F})$. $\quad [\because\ \nabla\times(\nabla\phi) = \text{curl grad}\,\phi = 0]$

Now $\nabla\times\mathbf{F} = \nabla\times(x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k})$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(z^2) - \frac{\partial}{\partial z}(y^2)\right]\mathbf{i} + \left[\frac{\partial}{\partial z}(x^2) - \frac{\partial}{\partial x}(z^2)\right]\mathbf{j} + \left[\frac{\partial}{\partial x}(y^2) - \frac{\partial}{\partial y}(x^2)\right]\mathbf{k}$$

$= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = 0.$

$\therefore \nabla \cdot (\mathbf{F} \times \nabla\phi) = (\nabla\phi) \cdot 0 = 0.$

Hence the given integral

$$= \iiint_V 0\, dV = 0.$$

Ex. 25. *Verify divergence theorem for* $\mathbf{F} = (2x - z)\mathbf{i} + x^2y\mathbf{j} - xz^2\mathbf{k}$ *taken over the region bounded by* $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1.$ (Rohilkhand 1989; Agra 85)

Sol. By divergence theorem, we have

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_V (\nabla \cdot \mathbf{F})\, dV,$$

where S is the surface of the cube bounded by the planes $x = 0$, $x = 1$, $y = 0$, $y = 1$, $z = 0$, $z = 1$ and V is the volume enclosed by the surface S.

We have $\nabla \cdot \mathbf{F} = \nabla \cdot [(2x - z)\mathbf{i} + x^2y\mathbf{j} - xz^2\mathbf{k}]$

$$= \frac{\partial}{\partial x}(2x - z) + \frac{\partial}{\partial y}(x^2y) + \frac{\partial}{\partial z}(-xz^2)$$

$= 2 + x^2 - 2xz.$

$$\therefore \iiint_V (\nabla \cdot \mathbf{F})\, dV = \iiint_V (2 + x^2 - 2xz)\, dV$$

$$= \int_{z=0}^1 \int_{y=0}^1 \int_{x=0}^1 (2 + x^2 - 2xz)\, dx\, dy\, dz$$

$$= \int_{z=0}^1 \int_{y=0}^1 \left[2x + \frac{x^3}{3} - x^2 z\right]_{x=0}^1 dy\, dz$$

$$= \int_{z=0}^1 \int_{y=0}^1 \left(2 + \frac{1}{3} - z\right) dy\, dz = \int_{z=0}^1 \int_{y=0}^1 \left(\frac{7}{3} - z\right) dy\, dz$$

$$= \int_{z=0}^1 \left[\frac{7}{3}y - zy\right]_{y=0}^1 dz = \int_0^1 \left(\frac{7}{3} - z\right) dz$$

$$= \left[\frac{7}{3}z - \frac{1}{2}z^2\right]_0^1 = \frac{7}{3} - \frac{1}{2} = \frac{11}{6}. \quad \ldots(1)$$

We shall now calculate $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$ over the six faces of the cube.

Draw figure as in solved example 24 (a). Over the face $OABC$ which lies in the yz-plane, $\mathbf{n} = -\mathbf{i}$, $x = 0$.

Therefore $\iint_{OABC} \mathbf{F} \cdot \mathbf{n}\, dS = \int_{z=0}^1 \int_{y=0}^1 (-z\mathbf{i}) \cdot (-\mathbf{i})\, dy\, dz$

$$= \int_{z=0}^1 \int_{y=0}^1 z\, dy\, dz = \int_{z=0}^1 z \left[y\right]_{y=0}^1 dz = \int_0^1 z\, dz = \left[\frac{z^2}{2}\right]_0^1 = \frac{1}{2}.$$

Over the opposite face $DEFG$, $x = 1$, $\mathbf{n} = \mathbf{i}$.

Therefore

$$\iint_{DEFG} \mathbf{F} \cdot \mathbf{n}\, dS = \int_{z=0}^1 \int_{y=0}^1 [(2 - z)\mathbf{i} + y\mathbf{j} - z^2\mathbf{k}] \cdot \mathbf{i}\, dy\, dz$$

$$= \int_{z=0}^1 \int_{y=0}^1 (2 - z)\, dy\, dz = \int_{z=0}^1 (2 - z) \left[y\right]_{y=0}^1 dz$$

$$= \int_0^1 (2 - z)\, dz = \left[2z - \frac{z^2}{2}\right]_0^1 = 2 - \frac{1}{2} = \frac{3}{2}.$$

Over the face $OGDC$ which lies in the zx-plane, $y = 0$, $\mathbf{n} = -\mathbf{j}$.

$$\therefore \iint_{OGDC} \mathbf{F} \cdot \mathbf{n}\, dS = \int_{z=0}^1 \int_{x=0}^1 \left[(2x - z)\mathbf{i} - xz^2\mathbf{k}\right] \cdot (-\mathbf{j})\, dz\, dx$$

$$= \int_{z=0}^1 \int_{x=0}^1 0\, dz\, dx = 0.$$

Over the opposite face $ABEF$, $y = 1$, $\mathbf{n} = \mathbf{j}$.

$$\therefore \iint_{ABEF} \mathbf{F} \cdot \mathbf{n}\, dS = \int_{z=0}^1 \int_{x=0}^1 [(2x - z)\mathbf{i} + x^2\mathbf{j} - xz^2\mathbf{k}] \cdot \mathbf{j}\, dz\, dx$$

$$= \int_{z=0}^1 \int_{x=0}^1 x^2\, dz\, dx = \int_{z=0}^1 \left[\frac{x^3}{3}\right]_{x=0}^1 dz = \int_0^1 \frac{1}{3} dz$$

$$= \frac{1}{3}\left[z\right]_0^1 = \frac{1}{3}.$$

Over the face $OGFA$ which lies in the xy-plane, $z = 0$, $\mathbf{n} = -\mathbf{k}$.

$$\therefore \iint_{OGFA} \mathbf{F} \cdot \mathbf{n}\, dS = \int_{x=0}^1 \int_{y=0}^1 (2x\mathbf{i} + x^2y\mathbf{j}) \cdot (-\mathbf{k})\, dx\, dy$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} 0\, dx\, dy = 0.$$

Over the opposite face $BCDE$, $z = 1$, $\mathbf{n} = \mathbf{k}$.

$$\therefore \iint_{BCDE} \mathbf{F}\cdot\mathbf{n}\, dS = \int_{x=0}^{1}\int_{y=0}^{1} [(2x-1)\,\mathbf{i} + x^2 y\,\mathbf{j} - x\,\mathbf{k}]\cdot\mathbf{k}\, dx\, dy$$

$$= \int_{x=0}^{1}\int_{y=0}^{1} -x\, dx\, dy = \int_{x=0}^{1} -x\,\Big[y\Big]_{y=0}^{1} dx = \int_0^1 -x\, dx$$

$$= -\left[\frac{x^2}{2}\right]_0^1 = -\frac{1}{2}.$$

Adding the six surface integrals, we get

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \frac{1}{2} + \frac{3}{2} + 0 + \frac{1}{3} + 0 - \frac{1}{2} = \frac{11}{6}. \qquad \ldots(2)$$

From (1) and (2), we see that

$$\iiint_V \operatorname{div}\mathbf{F}\, dV = \iint_S \mathbf{F}\cdot\mathbf{n}\, dS.$$

This verifies Gauss divergence theorem.

Ex. 26. *Verify divergence theorem for* $\mathbf{F} = 4xz\,\mathbf{i} - y^2\,\mathbf{j} + yz\,\mathbf{k}$ *taken over the cube bounded by* $x = 0, x = 1, y = 0, y = 1, z = 0, z = 1.$

Sol. Proceed as in solved example 25. Here we shall have

$$\iiint_V (\nabla\cdot\mathbf{F})\, dV = \iiint_V \left[\frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)\right] dV$$

$$= \int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{1} (4z - 2y + y)\, dx\, dy\, dz$$

$$= \int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{1} (4z - y)\, dx\, dy\, dz = \frac{3}{2}.$$

The six surface integrals will come out to be $2, 0, -1, 0, \frac{1}{2}$ and 0. Their sum $= \frac{3}{2}$.

Hence the theorem is verified.

Ex. 27. *Evaluate*

$$\iint_S x^2\, dy\, dz + y^2\, dz\, dx + 2z\,(xy - x - y)\, dx\, dy$$

where S *is the surface of the cube*

$0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le z \le 1.$ (Meerut 1986)

Sol. By divergence theorem, the given surface integral is equal to the volume integral

$$\iiint_V \left[\frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(y^2) + \frac{\partial}{\partial z}\{2z\,(xy - x - y)\}\right] dV$$

$$= \int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{1} [2x + 2y + 2xy - 2x - 2y]\, dx\, dy\, dz$$

$$= 2\int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{1} xy\, dx\, dy\, dz = 2\int_{z=0}^{1}\int_{y=0}^{1} \left[\frac{x^2}{2}y\right]_{x=0}^{1} dy\, dz$$

$$= 2\int_{z=0}^{1}\int_{y=0}^{1} \frac{y}{2}\, dy\, dz = \int_{z=0}^{1} \left[\frac{y^2}{2}\right]_{y=0}^{1} dz$$

$$= \int_{z=0}^{1} \tfrac{1}{2}\, dz = \tfrac{1}{2}\Big[z\Big]_0^1 = \tfrac{1}{2}.$$

Ex. 28. *Evaluate, by Green's theorem in space (i.e., Gauss divergence theorem), the integral*

$$\iint_S 4xz\, dy\, dz - y^2\, dz\, dx + yz\, dx\, dy,$$

where S *is the surface of the cube bounded by the planes* $x = 0,\ y = 0,\ z = 0,\ x = 1,\ y = 1,\ z = 1.$ (Meerut 1974, Kanpur 77)

Sol. Let V be the volume enclosed by the surface S. Then by Gauss divergence theorem, we have

$$\iint_S 4xz\, dy\, dz - y^2\, dz\, dx + yz\, dx\, dy$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(yz)\right] dV$$

$= \frac{3}{2}$, as in solved example 26.

Ex. 29. *Use Gauss divergence theorem to show that*

$$\iint_S \{(x^3 - yz)\,\mathbf{i} - 2x^2 y\,\mathbf{j} + 2\,\mathbf{k}\}\cdot\mathbf{n}\, dS = \frac{1}{3}a^5,$$

where S *denotes the surface of the cube bounded by the planes* $x = 0,\ x = a,\ y = 0,\ y = a,\ z = 0,\ z = a.$ (Rohilkhand 1979, Agra 77)

Sol. Let V be the volume enclosed by the surface S of the given cube. Then by Gauss divergence theorem, we have

$$\iint_S \{(x^3 - yz)\,\mathbf{i} - 2x^2 y\,\mathbf{j} + 2\,\mathbf{k}\}\cdot\mathbf{n}\, dS$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(x^3 - yz) + \frac{\partial}{\partial y}(-2x^2 y) + \frac{\partial}{\partial z}(2)\right] dV$$

$$= \iiint_V (3x^2 - 2x^2 + 0)\, dV$$

$$= \int_{z=0}^{a}\int_{y=0}^{a}\int_{x=0}^{a} x^2\, dx\, dy\, dz$$

$$= \int_{z=0}^{a}\int_{y=0}^{a}\left[\frac{x^3}{3}\right]_{x=0}^{a} dy\, dz = \int_{z=0}^{a}\int_{y=0}^{a}\frac{a^3}{3}\, dy\, dz$$

$$= \frac{a^3}{3}\int_{z=0}^{a}\left[y\right]_{y=0}^{a} dz = \frac{a^3}{3}\int_{0}^{a} a\, dz = \frac{a^4}{3}\left[z\right]_0^a = \frac{a^5}{3}.$$

Ex. 30. *Evaluate* $\iint_S (x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot\mathbf{n}\, dS$ *where S denotes the surface of the cube bounded by the planes* $x=0, y=0, z=0, x=a, y=a, z=a$ *by the application of Gauss divergence theorem. Verify your answer by evaluating the integral directly.* (Agra 1979)

Sol. Let V be the volume bounded by the surface S of the given cube. Then by Gauss divergence theorem, we have

$$\iint_S (x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot\mathbf{n}\, dS = \iiint_V [\text{div}\,(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})]\, dV$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)\right] dV = \iiint_V 3\, dV$$

$$= 3\iiint_V dV = 3V = 3a^3,$$ as $V = a^3$ = the volume of the cube whose each edge is of length a.

To verify our answer we shall evaluate $\iint_S (x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot\mathbf{n}\, dS$ over the six faces of the cube.

Draw figure as in solved example 24 (a).

Over the face $OABC$ which lies in the yz-plane, $\mathbf{n} = -\mathbf{i}$, $x = 0$.

$$\therefore \iint_{OABC} (x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot\mathbf{n}\, dS = \iint_{OABC} (y\mathbf{j}+z\mathbf{k})\cdot(-\mathbf{i})\, dS = 0.$$

Over the opposite face $DEFG$, $x = a$, $\mathbf{n} = \mathbf{i}$.

$$\therefore \iint_{DEFG} (x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot\mathbf{n}\, dS = \iint_{DEFG} (a\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot\mathbf{i}\, dS$$

$$= \iint_{DEFG} a\, dS = a\iint_{DEFG} dS = a \cdot \text{area of the face } DEFG$$

$$= a \cdot a^2 = a^3.$$

Similarly calculate the other four surface integrals.

Adding the six surface integrals, we get

$$\iint_S (x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot\mathbf{n}\, dS = 0 + a^3 + 0 + a^3 + 0 + a^3 = 3a^3.$$

Since $\iint_S (x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\cdot\mathbf{n}\, dS = \iiint_V [\text{div}\,(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})]\, dV$,

therefore our answer is verified.

Ex. 31. (a). *By transforming to a triple integral evaluate*

$$I = \iint_S (x^3\, dy\, dz + x^2 y\, dz\, dx + x^2 z\, dx\, dy).$$

where S is the closed surface bounded by the planes $z = 0$, $z = b$ *and the cylinder* $x^2 + y^2 = a^2$. (Meerut 1980)

Solution. By divergence theorem, the required surface integral I is equal to the volume integral

$$\iiint_V \left[\frac{\partial}{\partial x}(x^3) + \frac{\partial}{\partial y}(x^2 y) + \frac{\partial}{\partial z}(x^2 z)\right] dV$$

$$= \int_{z=0}^{b}\int_{y=-a}^{a}\int_{x=-\sqrt{(a^2-y^2)}}^{\sqrt{(a^2-y^2)}} (3x^2 + x^2 + x^2)\, dx\, dy\, dz$$

$$= 4 \times 5\int_{z=0}^{b}\int_{y=0}^{a}\int_{x=0}^{\sqrt{(a^2-y^2)}} x^2\, dx\, dy\, dz$$

$$= 20\int_{z=0}^{b}\int_{y=0}^{a}\left[\frac{x^3}{3}\right]_{x=0}^{\sqrt{(a^2-y^2)}} dy\, dz$$

$$= \frac{20}{3}\int_{z=0}^{b}\int_{y=0}^{a} (a^2-y^2)^{3/2}\, dy\, dz$$

$$= \frac{20}{3}\int_{y=0}^{a}\left[(a^2-y^2)^{3/2} z\right]_{z=0}^{b} dy = \frac{20}{3}\int_{y=0}^{a} b\,(a^2-y^2)^{3/2}\, dy.$$

Put $y = a\sin t$ so that $dy = a\cos t\, dt$.

$$\therefore I = \frac{20}{3} b \int_0^{\pi/2} a^3\cos^3 t\,(a\cos t)\, dt$$

$$= \frac{20}{3} a^4 b \int_0^{\pi/2} \cos^4 t\, dt = \frac{20}{3} a^4 b\, \frac{3}{4}\frac{1}{2}\frac{\pi}{2} = \frac{5}{4}\pi a^4 b.$$

Ex. 31. (b). *Evaluate* $\iint_S (zx^2\, dx\, dy + x^3\, dy\, dz + yx^2\, dz\, dx)$ *where S is the closed surface consisting of the cylinder* $x^2 + y^2 = 4$ *and the circular discs* $z = 0$ *and* $z = 3$. (Osmania 1990)

Sol. Proceed as in Ex. 31 (a). (Ans. 60π)

Ex. 32. *Apply Gauss's divergence theorem to evaluate*

$$\iint_S [(x^3 - yz)\,dy\,dz - 2x^2 y\,dz\,dx + z\,dx\,dy]$$

over the surface of a cube bounded by the coordinate planes and the planes $x = y = z = a$.

Sol. By divergence theorem, we have

$$\iint_S (F_1\,dy\,dz + F_2\,dz\,dx + F_3\,dx\,dy) = \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\,dy\,dz.$$

Here $F_1 = x^3 - yz$, $F_2 = -2x^2 y$, $F_3 = z$.

$$\therefore \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 3x^2 - 2x^2 + 1 = x^2 + 1.$$

$\therefore$ the given surface integral is equal to the volume integral

$$\int_{z=0}^{a} \int_{y=0}^{a} \int_{x=0}^{a} (x^2 + 1)\,dx\,dy\,dz$$

$$= \int_{z=0}^{a} \int_{y=0}^{a} \left[\frac{x^3}{3} + x\right]_{x=0}^{a} dy\,dz$$

$$= \int_{z=0}^{a} \int_{y=0}^{a} \left(\frac{a^3}{3} + a\right) dy\,dz = a^2 \left(\frac{a^3}{3} + a\right).$$

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Ex. 33. *If* $\mathbf{F} = x\mathbf{i} - y\mathbf{j} + (z^2 - 1)\mathbf{k}$, *find the value of* $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$ *where S is the closed surface bounded by the planes* $z = 0$, $z = 1$ *and the cylinder* $x^2 + y^2 = 4$. (Kanpur 1980)

Sol. By divergence theorem, we have

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_V \text{div}\,\mathbf{F}\,dV.$$

Here $\text{div}\,\mathbf{F} = \frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(-y) + \frac{\partial}{\partial z}(z^2 - 1)$

$= 1 - 1 + 2z = 2z.$

$$\therefore \iiint_V \text{div}\,\mathbf{F}\,dV = \int_{z=0}^{1} \int_{y=-2}^{2} \int_{x=-\sqrt{(4-y^2)}}^{\sqrt{(4-y^2)}} 2z\,dx\,dy\,dz$$

$$= \int_{z=0}^{1} \int_{y=-2}^{2} \left[2zx\right]_{x=-\sqrt{(4-y^2)}}^{\sqrt{(4-y^2)}} dy\,dz$$

$$= \int_{z=0}^{1} \int_{y=-2}^{2} 4z\sqrt{(4-y^2)}\,dy\,dz = \int_{y=-2}^{2} \left[4\frac{z^2}{2}\sqrt{(4-y^2)}\right]_{z=0}^{1} dy$$

$$= 2\int_{y=-2}^{2} \sqrt{(4-y^2)}\,dy = 4\int_0^2 \sqrt{(4-y^2)}\,dy$$

$$= 4\left[\frac{y}{2}\sqrt{(4-y^2)} + 2\sin^{-1}\frac{y}{2}\right]_0^2 = 4\,[2\sin^{-1} 1] = 4\,(2)\,\frac{\pi}{2} = 4\pi.$$

Ex. 34. *Find* $\iint_S \mathbf{A}\cdot\mathbf{n}\,dS$,

where $\mathbf{A} = (2x + 3z)\,\mathbf{i} - (xz + y)\,\mathbf{j} + (y^2 + 2z)\,\mathbf{k}$

and S is the surface of the sphere having centre at $(3, -1, 2)$ *and radius* 3. (Kakatiya 1990, Meerut 74)

Sol. Let V be the volume enclosed by the surface S. Then by Gauss divergence theorem, we have

$$\iint_S \mathbf{A}\cdot\mathbf{n}\,dS = \iiint_V \text{div}\,\mathbf{A}\,dV.$$

Now $\text{div}\,\mathbf{A} = \frac{\partial}{\partial x}(2x + 3z) + \frac{\partial}{\partial y}\{-(xz + y)\} + \frac{\partial}{\partial z}(y^2 + 2z)$

$= 2 - 1 + 2 = 3.$

$$\therefore \iint_S \mathbf{A}\cdot\mathbf{n}\,dS = \iiint_V 3\,dV = 3\iiint_V dV = 3V.$$

But V is the volume of a sphere of radius 3. Therefore $V = \frac{4}{3}\pi\,(3)^3 = 36\pi$.

$$\therefore \iint_S \mathbf{A}\cdot\mathbf{n}\,dS = 3V = 3 \times 36\pi = 108\pi.$$

Ex. 35. (a). *Apply divergence theorem to evaluate*

$$\iint_S [(x + z)\,dy\,dz + (y + z)\,dz\,dx + (x + y)\,dx\,dy]$$

where S is the surface of the sphere $x^2 + y^2 + z^2 = 4$. (Andhra 1989)

Sol. By divergence theorem, the given surface integral is equal to the volume integral

$$\iiint_V \left[\frac{\partial}{\partial x}(x + z) + \frac{\partial}{\partial y}(y + z) + \frac{\partial}{\partial z}(x + y)\right] dV$$

$= \iiint_V 2\,dV = 2\iiint_V dV = 2V$, where V is the volume of the sphere $x^2 + y^2 + z^2 = 4$

$$= 2\left[\frac{4}{3}\pi\,(2)^3\right] = \frac{64}{3}\pi.$$

Ex. 35. (b). *By using the Gauss divergence theorem evaluate*

$\iint_S (x\,dy\,dz + y\,dz\,dx + z\,dx\,dy)$ *where S is the surface of the sphere* $x^2 + y^2 + z^2 = 4$. (Osmania 1991)

Sol. By Gauss divergence theorem, the given surface integral is equal to the volume integral

$$\iiint_V \left[\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)\right] dV, \text{ where } V \text{ is the volume enclosed by the sphere } x^2 + y^2 + z^2 = 4$$

$$= \iiint_V (1 + 1 + 1)\, dV = 3 \iiint_V dV = 3V = 3 \cdot \left[\frac{4}{3}\pi (2)^3\right]$$

$= 32\pi$.

Ex. 36. *If S is any closed surface enclosing a volume V and* $\mathbf{F} = x\mathbf{i} + 2y\mathbf{j} + 3z\mathbf{k}$, *prove that*

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = 6V.$$

(Rohilkhand 1980, Kanpur 79, Agra 78)

Sol. By divergence theorem, we have

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_V \operatorname{div} \mathbf{F}\, dV = \iiint_V \operatorname{div}(x\mathbf{i} + 2y\mathbf{j} + 3z\mathbf{k})\, dV$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(2y) + \frac{\partial}{\partial z}(3z)\right] dV$$

$$= \iiint_V (1 + 2 + 3)\, dV = 6 \iiint_V dV = 6V.$$

Ex. 37. *Evaluate*

$$\iint_S (y^2z^2\mathbf{i} + z^2x^2\mathbf{j} + z^2y^2\mathbf{k}) \cdot \mathbf{n}\, dS$$

where S is the part of the sphere $x^2 + y^2 + z^2 = 1$ *above the xy-plane and bounded by this plane.* (Agra 1969, Bombay 66)

Sol. By divergence theorem, we have

$$\iint_S (y^2z^2\mathbf{i} + z^2x^2\mathbf{j} + z^2y^2\mathbf{k}) \cdot \mathbf{n}\, dS$$

$$= \iiint_V \operatorname{div}(y^2z^2\mathbf{i} + z^2x^2\mathbf{j} + z^2y^2\mathbf{k})\, dV,$$

where V is the volume enclosed by S

$$= \iiint_V \left[\frac{\partial}{\partial x}(y^2z^2) + \frac{\partial}{\partial y}(z^2x^2) + \frac{\partial}{\partial z}(z^2y^2)\right] dV$$

$$= \iiint_V 2zy^2\, dV = 2 \iiint_V zy^2\, dV.$$

We shall use spherical polar coordinates (r, θ, ϕ) to evaluate this triple integral. In polars $dV = (dr)(r\,d\theta)(r \sin\theta\, d\phi) = r^2 \sin\theta\, dr\, d\theta\, d\phi$. Also $z = r\cos\theta$, $y = r\sin\theta\sin\phi$. To cover V the limits of r will be 0 to 1, those of θ will be 0 to $\frac{\pi}{2}$ and those of ϕ will be 0 to 2π. The triple integral is

$$= 2 \int_{r=0}^{1} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} (r\cos\theta)(r^2 \sin^2\theta \sin^2\phi)\, r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= 2 \int_{r=0}^{1} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} r^5 \sin^3\theta \cos\theta \sin^2\phi\, dr\, d\theta\, d\phi$$

$$= 2 \cdot \frac{1}{6} \int_{\theta=0}^{\pi/2} \int_{\phi=0}^{2\pi} \sin^3\theta \cos\theta \sin^2\phi\, d\theta\, d\phi,$$

on integrating with respect to r.

[Note that the order of integration is immaterial because the limits of r, θ and ϕ are all constants].

$$= \frac{1}{3} \cdot \frac{2}{4 \cdot 2} \int_0^{2\pi} \sin^2\phi\, d\phi, \text{ on integrating with respect to } \theta$$

$$= \frac{1}{12} \cdot 4 \int_0^{\pi/2} \sin^2\phi\, d\phi = \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{12}.$$

Ex. 38. *By converting the surface integral into a volume integral evaluate*

$$\iint_S (x^3\, dy\, dz + y^3\, dz\, dx + z^3\, dx\, dy),$$

where S is the surface of the sphere $x^2 + y^2 + z^2 = 1$. (Bombay 1970)

Sol. By divergence theorem, we have

$$\iint_S (F_1\, dy\, dz + F_2\, dz\, dx + F_3\, dx\, dy)$$

$$= \iiint_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx\, dy\, dz,$$

where V is the volume enclosed by S.

Here $F_1 = x^3$, $F_2 = y^3$, $F_3 = z^3$.

$$\therefore \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} = 3(x^2 + y^2 + z^2)$$

$\therefore$ the given surface integral

$$= \iiint_V 3(x^2 + y^2 + z^2)\, dx\, dy\, dz$$

$$= 3\int_{r=0}^{1}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} r^2 \sin\theta\, dr\, d\theta\, d\phi, \quad \text{changing to polar spherical coordinates}$$

$$= 3 \times 2\pi \times 2 \times \frac{1}{5} = \frac{12\pi}{5}.$$

Ex. 39. *Evaluate by divergence theorem the integral*

$$\iint_S xz^2\, dy\, dz + (x^2 y - z^3)\, dz\, dx + (2xy + y^2 z)\, dx\, dy,$$

where S is the entire surface of the hemispherical region bounded by $z = \sqrt{(a^2 - x^2 - y^2)}$ *and* $z = 0$. (Meerut 1974)

Sol. Here S is the part of the sphere $x^2 + y^2 + z^2 = a^2$ above the xy-plane and bounded by this plane.

Let V be the volume enclosed by S. By Gauss divergence theorem, we have

$$\iint_S xz^2\, dy\, dz + (x^2 y - z^3)\, dz\, dx + (2xy + y^2 z)\, dx\, dy$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(xz^2) + \frac{\partial}{\partial y}(x^2 y - z^3) + \frac{\partial}{\partial z}(2xy + y^2 z)\right] dV$$

$$= \iiint_V (z^2 + x^2 + y^2)\, dV. \quad \ldots(1)$$

We shall use spherical polar coordinates (r, θ, ϕ) to evaluate the triple integral (1). In polars, $z = r\cos\theta$, $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $dV = (dr)(r\, d\theta)(r\sin\theta\, d\phi) = r^2 \sin\theta\, dr\, d\theta\, d\phi$.

Also $x^2 + y^2 + z^2 = r^2$. To cover V the limits of r will be 0 to a, those of θ will be 0 to $\pi/2$ and those of ϕ will be 0 to 2π. Hence the triple integral (1)

$$= \int_{r=0}^{a}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} r^2 . r^2 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \int_{r=0}^{a}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} r^4 \sin\theta\, dr\, d\theta\, d\phi$$

$$= \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi}\left[\frac{r^5}{5}\right]_{r=0}^{a} \sin\theta\, d\theta\, d\phi$$

$$= \frac{a^5}{5}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} \sin\theta\, d\theta\, d\phi = \frac{a^5}{5}\int_{\theta=0}^{\pi/2} \sin\theta . \left[\phi\right]_{\phi=0}^{2\pi} d\theta$$

$$= \frac{2\pi a^5}{5}\int_0^{\pi/2} \sin\theta\, d\theta = \frac{2\pi a^5}{5} . 1 = \frac{2\pi a^5}{5}.$$

Ex. 40. *By using Gauss divergence theorem, evaluate*

$$\iint_S (x\mathbf{i} + y\mathbf{j} + z^2 \mathbf{k}) \cdot \mathbf{n}\, dS$$

where S is the closed surface bounded by the cone $x^2 + y^2 = z^2$ *and the plane* $z = 1$. (Agra 1973)

Sol. Let V the volume enclosed by the closed surface S. Then by Gauss divergence theorem, we have

$$\iint_S (x\mathbf{i} + y\mathbf{j} + z^2 \mathbf{k}) \cdot \mathbf{n}\, dS$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z^2)\right] dV$$

$$= \iiint_V (2 + 2z)\, dV, \text{ where } V \text{ is the region bounded by the surfaces } z = 0,\ z = 1,\ z^2 = x^2 + y^2$$

$$= 2\int_{z=0}^{1}\int_{y=-z}^{z}\int_{x=-\sqrt{(z^2-y^2)}}^{\sqrt{(z^2-y^2)}} (1+z)\, dx\, dy\, dz$$

$$= 2\int_{z=0}^{1}\int_{y=-z}^{z} (1+z)\left[x\right]_{x=-\sqrt{(z^2-y^2)}}^{\sqrt{(z^2-y^2)}} dy\, dz$$

$$= 2\int_{z=0}^{1}\int_{y=-z}^{z} (1+z) \cdot 2\sqrt{(z^2-y^2)}\, dy\, dz$$

$$= 8\int_{z=0}^{1}\int_{y=0}^{z} (1+z)\sqrt{(z^2-y^2)}\, dy\, dz$$

$$= 8\int_{z=0}^{1} (1+z)\left[\frac{y}{2}\sqrt{(z^2-y^2)} + \frac{z^2}{2}\sin^{-1}\left(\frac{y}{z}\right)\right]_{y=0}^{z} dz$$

$$= 8\int_0^1 (1+z)\left[\frac{z^2}{2} . \frac{\pi}{2}\right] dz = 2\pi\int_0^1 (z^2 + z^3)\, dz$$

$$= 2\pi\left[\frac{z^3}{3} + \frac{z^4}{4}\right]_0^1 = 2\pi\left[\frac{1}{3} + \frac{1}{4}\right] = 2\pi . \frac{7}{12} = \frac{7\pi}{6}.$$

Ex. 41. *Evaluate* $\iint_S \mathbf{F} \cdot \mathbf{n}\, dS$ *over the entire surface of the region above the xy-plane bounded by the cone* $z^2 = x^2 + y^2$ *and the plane* $z = 4$, *if*

$$\mathbf{F} = 4xz\mathbf{i} + xyz^2\mathbf{j} + 3z\mathbf{k}.$$

Sol. By divergence theorem, we have

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS = \iiint_V \operatorname{div} \mathbf{F}\, dV,$$

where V is the volume enclosed by S.

Here $\operatorname{div} \mathbf{F} = \frac{\partial}{\partial x}(4xz) + \frac{\partial}{\partial y}(xyz^2) + \frac{\partial}{\partial z}(3z) = 4z + xz^2 + 3.$

Also V is the region bounded by the surfaces

$$z = 0,\ z = 4,\ z^2 = x^2 + y^2.$$

Therefore, $\iiint_V \operatorname{div} \mathbf{F}\, dV = \iiint_V (4z + xz^2 + 3)\, dx\, dy\, dz$

$$= \int_{z=0}^{4} \int_{y=-z}^{z} \int_{x=-\sqrt{(z^2-y^2)}}^{\sqrt{(z^2-y^2)}} (4z + xz^2 + 3)\, dx\, dy\, dz$$

$$= 2 \int_{z=0}^{4} \int_{y=-z}^{z} \int_{x=0}^{\sqrt{(z^2-y^2)}} (4z + 3)\, dx\, dy\, dz,$$

$$\text{since } \int_{x=-\sqrt{(z^2-y^2)}}^{\sqrt{(z^2-y^2)}} x\, dx = 0$$

$$= 2 \int_{z=0}^{4} \int_{y=-z}^{z} (4z + 3)\sqrt{(z^2 - y^2)}\, dy\, dz,$$

on integrating with respect to x

$$= 4 \int_{z=0}^{4} \int_{y=0}^{z} (4z + 3)\sqrt{(z^2 - y^2)}\, dy\, dz$$

$$= 4 \int_{z=0}^{4} (4z + 3) \left[\frac{y}{2}\sqrt{(z^2 - y^2)} + \frac{z^2}{2} \sin^{-1} \frac{y}{z} \right]_0^z dz$$

$$= 4 \int_0^4 (4z + 3) \left[\frac{z^2}{2} \sin^{-1} 1 \right] dz = \pi \int_0^4 (4z^3 + 3z^2)\, dz$$

$$= \pi \left[z^4 + z^3 \right]_0^4 = \pi (256 + 64) = 320\pi.$$

Ex. 42. *Show that* $\iint_S (x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}) \cdot \mathbf{n}\, dS$ *vanishes where S denotes the surface of the ellipsoid*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Solution. We have by divergence theorem

$$\iint_S (x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}) \cdot \mathbf{n}\, dS$$

$$= \iiint_V \operatorname{div} (x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k})\, dV, \text{ where } V \text{ is the volume enclosed by } S$$

$$= \iiint_V (2x + 2y + 2z)\, dx\, dy\, dz$$

$$= 2 \int_{z=-c}^{c} \int_{y=-b\sqrt{(1-(z^2/c^2))}}^{b\sqrt{(1-(z^2/c^2))}} \int_{x=-a\sqrt{(1-y^2/b^2-z^2/c^2)}}^{a\sqrt{(1-(y^2/b^2)-(z^2/c^2))}} (x + y + z)\, dx\, dy\, dz$$

$$= 4 \int_{z=-c}^{c} \int_{y=-b\sqrt{(1-(z^2/c^2))}}^{b\sqrt{(1-(z^2/c^2))}} (y + z) \sqrt{\left(1 - \frac{y^2}{b^2} - \frac{z^2}{c^2}\right)}\, dy\, dz,$$

on integrating with respect to x

$$\left[\text{Note that } \int_{-a}^{a} f(x)\, dx = 0 \text{ if } f(-x) = -f(x) \text{ and } \int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx \text{ if } f(-x) = f(x)\right]$$

$$= 8 \int_{z=-c}^{c} \int_{y=0}^{b\sqrt{(1-(z^2/c^2))}} z \sqrt{\left(1 - \frac{z^2}{c^2} - \frac{y^2}{b^2}\right)}\, dy\, dz$$

$$= 8 \int_{z=-c}^{c} \int_{y=0}^{b\sqrt{(1-(z^2/c^2))}} \frac{z}{b} \sqrt{\left\{b^2\left(1 - \frac{z^2}{c^2}\right) - y^2\right\}}\, dy\, dz$$

$$= \frac{8}{b} \int_{z=-c}^{c} z \left[\frac{y}{2}\sqrt{\left\{b^2\left(1 - \frac{z^2}{c^2}\right) - y^2\right\}} + \frac{b^2}{2}\left(1 - \frac{z^2}{c^2}\right) \sin^{-1} \frac{y}{b\sqrt{\{1 - (z^2/c^2)\}}} \right]_{y=0}^{b\sqrt{(1-(z^2/c^2))}} dz$$

$$= \frac{8}{b} \int_{z=-c}^{c} z \left[\frac{b^2}{2}\left(1 - \frac{z^2}{c^2}\right) \sin^{-1} 1 \right] dz$$

$$= \frac{8}{b} \int_{z=-c}^{c} z \frac{b^2}{2} \left(1 - \frac{z^2}{c^2}\right) \frac{\pi}{2}\, dz = 0.$$

Ex. 43. *Use divergence theorem to evaluate*

$$\iint_S [x\, dy\, dz + y\, dz\, dx + z\, dx\, dy],$$

where S is the surface $x^2 + y^2 + z^2 = 1$.

Sol. Let V be the volume bounded by the surface S of the sphere $x^2 + y^2 + z^2 = 1$. Then by Gauss divergence theorem, we have

$$\iint_S [x\, dy\, dz + y\, dz\, dx + z\, dx\, dy]$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)\right] dV$$

$= 3 \iiint_V dV = 3V$, where V is the volume enclosed by the sphere $x^2 + y^2 + z^2 = 1$ whose radius is 1

$$= 3 \cdot \tfrac{4}{3}\pi \cdot 1^3 = 4\pi.$$

Ex. 44. *Use divergence theorem to find* $\iint_S \mathbf{F}\cdot\mathbf{n}\, dS$ *for the vector* $\mathbf{F} = x\mathbf{i} - y\mathbf{j} + 2z\mathbf{k}$ *over the sphere* $x^2 + y^2 + (z-1)^2 = 1$.

Sol. Here S is the surface of the sphere $x^2 + y^2 + (z-1)^2 = 1$ whose centre is the point $(0, 0, 1)$ and radius 1. Let V be the volume enclosed by S. Then by Gauss divergence theorem, we have

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iiint_V (\text{div } \mathbf{F})\, dV$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(-y) + \frac{\partial}{\partial z}(2z)\right] dV$$

$$= \iiint_V (1 - 1 + 2)\, dV = 2 \iiint_V dV$$

$= 2V$, where V is the volume of the sphere $x^2 + y^2 + (z-1)^2 = 1$ whose radius is 1

$$= 2 \cdot \frac{4}{3}\pi . 1^3 = \frac{8}{3}\pi.$$

Ex. 45. *If* $\mathbf{F} = ax\mathbf{i} + by\mathbf{j} + cz\mathbf{k}$, *where a,b,c are constants, show that*

$$\iint_S (\mathbf{n}\cdot\mathbf{F})\, dS = \frac{4\pi}{3}(a + b + c),$$

S being the surface of the sphere $(x-1)^2 + (y-2)^2 + (z-3)^2 = 1$. (Gauhati 1971)

Sol. By the divergence theorem, we have

$\iint_S (\mathbf{n}\cdot\mathbf{F})\, dS = \iiint_V (\nabla\cdot\mathbf{F})\, dV$, where V is the volume enclosed by the sphere S whose radius is 1

$$= \iiint_V \left[\frac{\partial}{\partial x}(ax) + \frac{\partial}{\partial y}(by) + \frac{\partial}{\partial z}(cz)\right] dV$$

$$= \iiint_V (a + b + c)\, dV = (a + b + c)\, V = (a + b + c)\, \frac{4}{3}\pi,$$

since the volume V enclosed by a sphere of unit radius is $\frac{4}{3}\pi(1)^3$ *i.e.*, $\frac{4}{3}\pi$

$$= \frac{4}{3}\pi\,(a + b + c).$$

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Ex. 46. *Verify the divergence theorem for*

$$\mathbf{F} = 4x\mathbf{i} - 2y^2\mathbf{j} + z^2\mathbf{k}$$

taken over the region bounded by the surfaces $x^2 + y^2 = 4$, $z = 0$, $z = 3$. (Allahabad 1978)

Sol. Let S denote the closed surface bounded by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$, $z = 3$. Also let V be the volume bounded by the surface S. By Gauss divergence theorem, we have

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS = \iiint_V \text{div } \mathbf{F}\, dV.$$

We have $\iiint_V \text{div } \mathbf{F}\, dV = \iiint_V \nabla\cdot\mathbf{F}\, dV$

$$= \iiint_V \left[\frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2)\right] dV$$

$$= \iiint_V (4 - 4y + 2z)\, dV$$

$$= 2\int_{z=0}^{3}\int_{x=-2}^{2}\int_{y=-\sqrt{(4-x^2)}}^{\sqrt{(4-x^2)}} (2 - 2y + z)\, dz\, dx\, dy$$

$$= 4\int_{z=0}^{3}\int_{x=-2}^{2}\int_{y=0}^{\sqrt{(4-x^2)}} (2 + z)\, dz\, dx\, dy$$

[$\because$ $2y$ is an odd function of y]

$$= 4\int_{z=0}^{3}\int_{x=-2}^{2} \left[2y + zy\right]_{y=0}^{\sqrt{(4-x^2)}} dz\, dx$$

$$= 4\int_{z=0}^{3}\int_{x=-2}^{2} [2\sqrt{(4-x^2)} + z\sqrt{(4-x^2)}]\, dz\, dx$$

$$= 4\int_{x=-2}^{2} \left[2z\sqrt{(4-x^2)} + \frac{z^2}{2}\sqrt{(4-x^2)}\right]_{z=0}^{3} dx$$

$$= 4\int_{-2}^{2} \frac{21}{2}\sqrt{(4-x^2)}\, dx = 4 \cdot \frac{21}{2} \cdot 2\int_{0}^{2} \sqrt{(4-x^2)}\, dx$$

$$= 84\left[\frac{x}{2}\sqrt{(4-x^2)} + 2\sin^{-1}\frac{x}{2}\right]_0^2 = 84\left[2\cdot\frac{\pi}{2}\right] = 84\pi.$$

Now we shall evaluate the surface integral

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS,$$

For evaluating this surface integral give complete solution of solved example 61 on page 197.

Thus $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = 84\pi.$

We see that $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_V \operatorname{div}\mathbf{F}\,dV.$

This completes the verification of Gauss divergence theorem.

Ex. 47. *Use Gauss divergence theorem to find* $\iint_S \mathbf{F}\cdot\mathbf{n}\,dS$, *where* $\mathbf{F} = 2x^2y\mathbf{i} - y^2\mathbf{j} + 4xz^2\mathbf{k}$ *and S is the closed surface in the first octant bounded by* $y^2 + z^2 = 9$ *and* $x = 2$. (Kanpur 1976)

Sol. Let V be the volume enclosed by the closed surface S. Then by Gauss divergence theorem, we have

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_V \operatorname{div}\mathbf{F}\,dV$$

$$= \iiint_V \nabla\cdot(2x^2y\mathbf{i} - y^2\mathbf{j} + 4xz^2)\,dV$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(2x^2y) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(4xz^2)\right] dV$$

$= \iiint_V (4xy - 2y + 8xz)\,dV$, where V is the volume in the first octant bounded by the cylinder $y^2 + z^2 = 9$ and the planes $x = 0$, $x = 2$

$$= 2\int_{x=0}^{2}\int_{z=0}^{3}\int_{y=0}^{\sqrt{(9-z^2)}} (2xy - y + 4xz)\,dx\,dz\,dy$$

$$= 2\int_{x=0}^{2}\int_{z=0}^{3}\left[xy^2 - \tfrac{1}{2}y^2 + 4xzy\right]_{y=0}^{\sqrt{(9-z^2)}} dx\,dz$$

$$= 2\int_{x=0}^{2}\int_{z=0}^{3}\left[x(9-z^2) - \tfrac{1}{2}(9-z^2) + 4xz\sqrt{(9-z^2)}\right] dx\,dz$$

$$= 2\int_{z=0}^{3}\left[\frac{x^2}{2}(9-z^2) - \frac{x}{2}(9-z^2) + 2x^2z\sqrt{(9-z^2)}\right]_{x=0}^{2} dz$$

$$= 2\int_0^3 \left[2(9-z^2) - (9-z^2) + 8z\sqrt{(9-z^2)}\right] dz$$

$$= 2\int_0^3 \left[9 - z^2 - 4(-2z)(9-z^2)^{1/2}\right] dz$$

$$= 2\left[9z - \frac{z^3}{3} - 4\cdot\frac{2}{3}(9-z^2)^{3/2}\right]_0^3$$

$$= 2[27 - 9 + \tfrac{4}{3}\cdot 27] = 2(18 + 72) = 180.$$

Ex. 48. *Verify divergence theorem for the function* $\mathbf{F} = y\mathbf{i} + x\mathbf{j} + z^2\mathbf{k}$ *over the cylindrical region bounded by* $x^2 + y^2 = a^2$, $z = 0$ *and* $z = h$. (Kanpur 1975; Allahabad 79)

Sol. Let S denote the closed surface bounded by the cylinder $x^2 + y^2 = a^2$ and the planes $z = 0$, $z = h$. Also let V be the volume bounded by the surface S. By Gauss divergence theorem, we have

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_V \operatorname{div}\mathbf{F}\,dV.$$

We have $\iiint_V \operatorname{div}\mathbf{F}\,dV = \iiint_V [\operatorname{div}(y\mathbf{i} + x\mathbf{j} + z^2\mathbf{k})]\,dV$

$$= \iiint_V \left[\frac{\partial}{\partial x}(y) + \frac{\partial}{\partial y}(x) + \frac{\partial}{\partial z}(z^2)\right] dV = \iiint_V 2z\,dV$$

$$= 2\int_{z=0}^{h}\int_{x=-a}^{a}\int_{y=-\sqrt{(a^2-x^2)}}^{\sqrt{(a^2-x^2)}} z\,dz\,dx\,dy$$

$$= 4\int_{z=0}^{h}\int_{x=-a}^{a}\int_{y=0}^{\sqrt{(a^2-x^2)}} z\,dz\,dx\,dy$$

$$= 4\int_{z=0}^{h}\int_{x=-a}^{a} z\left[y\right]_{y=0}^{\sqrt{(a^2-x^2)}} dz\,dx$$

$$= 4\int_{z=0}^{h}\int_{x=-a}^{a} z\sqrt{(a^2-x^2)}\,dz\,dx$$

$$= 8\int_{z=0}^{h}\int_{x=0}^{a} z\sqrt{(a^2-x^2)}\,dz\,dx$$

$$= 8\int_{x=0}^{a}\sqrt{(a^2-x^2)}\left[\frac{z^2}{2}\right]_{z=0}^{h} dx$$

$$= 8\int_0^a \frac{h^2}{2}\sqrt{(a^2-x^2)}\,dx = 4h^2\int_0^a \sqrt{(a^2-x^2)}\,dx$$

$$= 4h^2\left[\frac{x}{2}\sqrt{(a^2-x^2)}+\frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a = 4h^2\left[\frac{a^2}{2}\cdot\frac{\pi}{2}\right]$$

$$= \pi a^2h^2. \qquad \ldots (1)$$

Now we shall evaluate the surface integral

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS.$$

The surface S consists of three surfaces: (i) the surface S_1 of the base of the cylinder *i.e.*, the plane face $z = 0$, (ii) the surface S_2 of the top face of the cylinder *i.e.*, the plane face $z = h$ and (iii) the surface S_3 of the convex portion of the cylinder.

For the surface S_1 *i.e.*, $z = 0$; $\mathbf{F} = y\,\mathbf{i} + x\,\mathbf{j}$, putting $z = 0$ in $\mathbf{F}$.

A unit vector $\mathbf{n}$ along the outward drawn normal to S_1 is obviously $-\mathbf{k}$.

$$\therefore \iint_{S_1} \mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_1} (y\,\mathbf{i} + x\,\mathbf{j})\cdot(-\mathbf{k})\,dS = 0.$$

For the surface S_2 *i.e.*, $z = h$, $\mathbf{F} = y\,\mathbf{i} + x\,\mathbf{j} + h^2\mathbf{k}$, putting $z = h$ in $\mathbf{F}$.

A unit vector $\mathbf{n}$ along the outward drawn normal to S_2 is given by $\mathbf{n} = \mathbf{k}$.

$$\therefore \iint_{S_2} \mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_2} (y\,\mathbf{i} + x\,\mathbf{j} + h^2\mathbf{k})\cdot\mathbf{k}\,dS$$

$$= \iint_{S_2} h^2\,dS = h^2\iint_{S_2} dS = h^2 .\text{ area of the plane face } S_2 \text{ of the cylinder}$$

$$= h^2.\pi a^2 = \pi a^2h^2.$$

For the convex portion S_3 *i.e.*, $x^2 + y^2 = a^2$, a vector normal to S_3 is given by

$$\nabla(x^2 + y^2) = 2x\,\mathbf{i} + 2y\,\mathbf{j}.$$

$\therefore$ $\mathbf{n}$ = a unit vector along outward drawn normal at any point of S_3

$$= \frac{2x\,\mathbf{i} + 2y\,\mathbf{j}}{\sqrt{(4x^2+4y^2)}} = \frac{x\,\mathbf{i} + y\,\mathbf{j}}{a}, \text{ since } x^2 + y^2 = a^2 \text{ on } S_3.$$

$$\therefore \text{ on } S_3,\ \mathbf{F}\cdot\mathbf{n} = (y\,\mathbf{i} + x\,\mathbf{j} + z^2\mathbf{k})\cdot\left[\frac{1}{a}(x\,\mathbf{i} + y\,\mathbf{j})\right]$$

$$= \frac{1}{a}xy + \frac{1}{a}xy = \frac{2}{a}xy.$$

Also dS = elementary area on the surface S_3

$= a\,d\theta\,dz$, using cylindrical coordinates r, θ, z.

$$\therefore \iint_{S_3} \mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_3} \frac{2}{a}xy\,a\,d\theta\,dz, \text{ where } x = a\cos\theta,\ y = a\sin\theta$$

$$= \int_{z=0}^{h}\int_{\theta=0}^{2\pi} 2a\cos\theta\,a\sin\theta\,d\theta\,dz$$

$$= 2a^2\int_{\theta=0}^{2\pi}\cos\theta\sin\theta\left[z\right]_{z=0}^{h} d\theta$$

$$= 2a^2h\int_0^{2\pi}\cos\theta\sin\theta\,d\theta = a^2h\int_0^{2\pi}\sin 2\theta\,d\theta$$

$$= a^2h\left[-\frac{\cos 2\theta}{2}\right]_0^{2\pi} = -\frac{a^2h}{2}[\cos 4\pi - \cos 0] = 0.$$

Hence the total surface integral

$$\iint_S \mathbf{F}\cdot\mathbf{n}\,dS = 0 + \pi a^2h^2 + 0 = \pi a^2h^2. \qquad \ldots (2)$$

From (1) and (2), we see that

$$\iiint_V \operatorname{div}\mathbf{F}\,dV = \iint_S \mathbf{F}\cdot\mathbf{n}\,dS.$$

This verifies divergence theorem.

Ex. 49. *If* $\mathbf{F} = (x^2 + y - 4)\,\mathbf{i} + 3xy\,\mathbf{j} + (2xz + z^2)\,\mathbf{k}$, *evaluate* $\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS$ *where S is the surface of the sphere* $x^2 + y^2 + z^2 = 16$ *above the xy-plane.*

Sol. The surface $x^2+y^2+z^2=16$ meets the plane $z=0$ in a circle C given by $x^2+y^2=16, z=0$. Let S_1 be the plane region bounded by the circle C. If Sprime and S_1, then S' is a closed surface. Let V be the region bounded by S'.

If n denotes the outward drawn (drawn outside the region V) unit normal vector to S', then on the plane surface S_1, we have $\mathbf{n}=-\mathbf{k}$. Note that $\mathbf{k}$ is a unit vector normal to S_1 drawn into the region V.

Now by an application of Gauss divergence theorem, we have

$$\iint_{S'} \text{curl}\,\mathbf{F}\cdot\mathbf{n}\,dS=0 \qquad \text{[See Ex. 1 page 233]}$$

or $$\iint_S \text{curl}\,\mathbf{F}\cdot\mathbf{n}\,dS+\iint_{S_1}\text{curl}\,\mathbf{F}\cdot\mathbf{n}\,dS=0$$

[$\because$ S' consists of S and S_1]

or $$\iint_S \text{curl}\,\mathbf{F}\cdot\mathbf{n}\,dS-\iint_{S_1}\text{curl}\,\mathbf{F}\cdot\mathbf{k}\,dS=0$$

[$\because$ on S_1, $\mathbf{n}=-\mathbf{k}$]

or $$\iint_S \text{curl}\,\mathbf{F}\cdot\mathbf{n}\,dS=\iint_{S_1}\text{curl}\,\mathbf{F}\cdot\mathbf{k}\,dS.$$

Now $$\text{curl}\,\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^2+y-4 & 3xy & 2xz+z^2\end{vmatrix}$$

$$=0\,\mathbf{i}-z\,\mathbf{j}+(3y-1)\,\mathbf{k}=-z\,\mathbf{j}+(3y-1)\,\mathbf{k}.$$

$\therefore$ $\text{curl}\,\mathbf{F}\cdot\mathbf{k}=\{-z\,\mathbf{j}+(3y-1)\,\mathbf{k}\}\cdot\mathbf{k}=3y-1.$

$$\therefore \iint_S \text{curl}\,\mathbf{F}\cdot\mathbf{n}\,dS=\iint_{S_1}(3y-1)\,dS$$

$$=\int_{\theta=0}^{2\pi}\int_{r=0}^{4}(3r\sin\theta-1)\,r\,d\theta\,dr, \quad \text{changing to polars}$$

[Note that S_1 is a circle in xy-plane with centre origin and radius 4]

$$=\int_{\theta=0}^{2\pi}\int_{r=0}^{4}3r^2\sin\theta\,d\theta\,dr-\int_{\theta=0}^{2\pi}\int_{r=0}^{4}r\,d\theta\,dr$$

$$=0-\int_{\theta=0}^{2\pi}\left[\frac{r^2}{2}\right]_0^4 d\theta \qquad \left[\because \int_{\theta=0}^{2\pi}\sin\theta\,d\theta=0\right]$$

$$=-8\left[\theta\right]_0^{2\pi}=-16\pi.$$

Ex. 50. *If* $\mathbf{F}=y\,\mathbf{i}+(x-2xz)\,\mathbf{j}-xy\,\mathbf{k}$, *evaluate* $\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS$ *where S is the surface of the sphere* $x^2+y^2+z^2=a^2$ *above the xy-plane.* (Kanpur 1980)

Sol. The surface $x^2+y^2+z^2=a^2$ meets the plane $z=0$ in a circle C given by $x^2+y^2=a^2, z=0$. Let S_1 be the plane region bounded by the circle C. If S' is the surface consisting of the surfaces S and S_1, then S' is a closed surface. Let V be the volume bounded by S'.

If n denotes the outward drawn (drawn outside the region V) unit normal vector to S', then on the plane surface S_1, we have $\mathbf{n}=-\mathbf{k}$. Note that $\mathbf{k}$ is a unit vector normal to S_1 drawn into the region V.

By Gauss divergence theorem, we have

$$\iint_{S'}(\text{curl}\,\mathbf{F})\cdot\mathbf{n}\,dS=\iiint_V[\text{div}\,(\text{curl}\,\mathbf{F})]\,dV$$

$$=0. \qquad [\because \text{div}\,(\text{curl}\,\mathbf{F})=0]$$

$$\therefore \iint_S(\text{curl}\,\mathbf{F})\cdot\mathbf{n}\,dS+\iint_{S_1}(\text{curl}\,\mathbf{F})\cdot\mathbf{n}\,dS=0$$

[$\because$ S' consists of S and S_1]

or $$\iint_S(\text{curl}\,\mathbf{F})\cdot\mathbf{n}\,dS-\iint_{S_1}(\text{curl}\,\mathbf{F})\cdot\mathbf{k}\,dS=0$$

[$\because$ on S_1, $\mathbf{n}=-\mathbf{k}$]

or $$\iint_S(\text{curl}\,\mathbf{F})\cdot\mathbf{n}\,dS=\iint_{S_1}(\text{curl}\,\mathbf{F})\cdot\mathbf{k}\,dS.$$

Now $$\text{curl}\,\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y & x-2xz & -xy\end{vmatrix}$$

$$=\mathbf{i}\left[\frac{\partial}{\partial y}(-xy)-\frac{\partial}{\partial z}(x-2xz)\right]-\mathbf{j}\left[\frac{\partial}{\partial x}(-xy)-\frac{\partial}{\partial z}(y)\right]$$

$$+\mathbf{k}\left[\frac{\partial}{\partial x}(x-2xz)-\frac{\partial}{\partial y}(y)\right]$$

$= x\mathbf{i}+y\mathbf{j}-2z\mathbf{k}.$

$\therefore$ (curl F)$\cdot\mathbf{k}=(x\mathbf{i}+y\mathbf{j}-2z\mathbf{k})\cdot\mathbf{k}$

$=-2z=0$ over the surface S_1 bounded by the circle $x^2+y^2=a^2, z=0.$

Hence $\iint_S$ (curl F)$\cdot\mathbf{n}\,dS=\iint_{S_1} 0\,dS=0.$

Ex. 51. *Evaluate* $\iint_S (\nabla\times\mathbf{A})\cdot\mathbf{n}\,dS$, *where*

$\mathbf{A}=[xye^z+\log(z+1)-\sin x]\,\mathbf{k}$ *and S is the surface of the sphere* $x^2+y^2+z^2=a^2$ *above the xy-plane.*

Sol. Proceed as in solved example 50.

Here curl $\mathbf{A}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 0 & 0 & xye^z+\log(z+1)-\sin x\end{vmatrix}$

$$=\mathbf{i}\frac{\partial}{\partial y}\left[xye^z+\log(z+1)-\sin x\right]$$
$$-\mathbf{j}\frac{\partial}{\partial x}\left[xye^z+\log(z+1)-\sin x\right]$$

$=xe^z\mathbf{i}-(ye^z-\cos x)\mathbf{j}.$

$\therefore$ (curl A)$\cdot\mathbf{k}=[xe^z\mathbf{i}-(ye^z-\cos x)\mathbf{j}]\cdot\mathbf{k}=0.$

$\therefore$ (curl A)$\cdot\mathbf{k}=0$ over the surface S_1.

Hence $\iint_S$ (curl A)$\cdot\mathbf{n}\,dS=\iint_{S_1} 0\,dS=0.$

Ex. 52. *Evaluate* $\iint_S (\nabla\times\mathbf{A})\cdot\mathbf{n}\,dS$, *where*

$\mathbf{A}=(x-z)\mathbf{i}+(x^3+yz)\mathbf{j}-3xy^2\mathbf{k}$ *and S is the surface of the cone* $z=2-\sqrt{(x^2+y^2)}$ *above the xy-plane.* (Meerut 1974)

Solution. The surface $z=2-\sqrt{(x^2+y^2)}$ meets the xy-plane in a circle C given by $x^2+y^2=4$, $z=0$. Let S_1 be the plane region bounded by the circle C. If S' is the surface consisting of the surfaces

S and S_1, then S' is a closed surface. By application of divergence theorem, we have

$$\iint_{S'} \text{curl}\,\mathbf{A}\cdot\mathbf{n}\,dS=0 \qquad \text{[See Ex. 1 page233]}$$

or $\iint_S \text{curl}\,\mathbf{A}\cdot\mathbf{n}\,dS+\iint_{S_1}\text{curl}\,\mathbf{A}\cdot\mathbf{n}\,dS=0$

or $\iint_S \text{curl}\,\mathbf{A}\cdot\mathbf{n}\,dS=\iint_{S_1}\text{curl}\,\mathbf{A}\cdot\mathbf{k}\,dS \quad [\because \text{ on } S_1, \mathbf{n}=-\mathbf{k}]$

Now curl $\mathbf{A}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x-z & x^3+yz & -3xy^2\end{vmatrix}$

$=\mathbf{i}(-6xy-y)+\mathbf{j}(-1+3y^2)+\mathbf{k}(3x^2-0).$

$\therefore$ curl $\mathbf{A}\cdot\mathbf{k}=3x^2.$

$\therefore \iint_S \text{curl}\,\mathbf{A}\cdot\mathbf{n}\,dS=\iint_{S_1} 3x^2\,dS$

$=\int_{\theta=0}^{2\pi}\int_{r=0}^{2} 3r^2\cos^2\theta\, r\,d\theta\,dr$, changing to polars

$=3\int_{\theta=0}^{2\pi}\int_{r=0}^{2} r^3\cos^2\theta\,d\theta\,dr=3\int_{\theta=0}^{2\pi}\left[\frac{r^4}{4}\right]_0^2\cos^2\theta\,d\theta$

$=12\int_0^{2\pi}\cos^2\theta\,d\theta$

$=12\times 4\int_0^{\pi/2}\cos^2\theta\,d\theta=48\times\frac{1}{2}\times\frac{\pi}{2}=12\pi.$

Ex. 53. *Evaluate* $\iint_S (\nabla\times\mathbf{F})\cdot\mathbf{n}\,dS$, *where*

$\mathbf{F}=(x^2+y-4)\mathbf{i}+3xy\mathbf{j}+(2xz+z^2)\mathbf{k}$ *and S is the surface of the paraboloid* $z=4-(x^2+y^2)$ *above the xy-plane.*

Sol. The surface $z=4-(x^2+y^2)$ meets the plane $z=0$ in a circle C given by $x^2+y^2=4$, $z=0$. Let S_1 be the plane region bounded by the circle C. If S' is the surface consisting of the surfaces S and S_1, then S' is a closed surface. Let V be the volume bounded by S'.

If n denotes the outward drawn (drawn outside the region V) unit normal vector to S', then on the plane surface S_1, we have $\mathbf{n} = -\mathbf{k}$. Note that k is a unit vector normal to S_1 drawn into the region V.

By Gauss divergence theorem, we have

$$\iint_{S'} (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = \iiint_V (\text{div curl } \mathbf{F})\, dV = 0,$$

since div curl F = 0.

$$\therefore \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS + \iint_{S_1} (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = 0$$

[$\because$ S' consists of S and S_1]

or $$\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS - \iint_{S_1} (\text{curl } \mathbf{F}) \cdot \mathbf{k}\, dS = 0$$

[$\because$ on S_1, $\mathbf{n} = -\mathbf{k}$]

or $$\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_{S_1} (\text{curl } \mathbf{F}) \cdot \mathbf{k}\, dS.$$

Now $$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 + y - 4 & 3xy & 2xz + z^2 \end{vmatrix}$$

$$= \mathbf{i}\left[\frac{\partial}{\partial y}(2xz + z^2) - \frac{\partial}{\partial z}(3xy)\right] - \mathbf{j}\left[\frac{\partial}{\partial x}(2xz + z^2) - \frac{\partial}{\partial z}(x^2 + y - 4)\right] + \mathbf{k}\left[\frac{\partial}{\partial x}(3xy) - \frac{\partial}{\partial y}(x^2 + y - 4)\right]$$

$$= 0\mathbf{i} - 2z\mathbf{j} + (3y - 1)\mathbf{k}.$$

$$\therefore (\text{curl } \mathbf{F}) \cdot \mathbf{k} = [-2z\mathbf{j} + (3y - 1)\mathbf{k}] \cdot \mathbf{k}$$

$= 3y - 1$ over the surface S_1 bounded by the circle

$$x^2 + y^2 = 4,\ z = 0.$$

Hence, $$\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_{S_1} (3y - 1)\, dS$$

$$= \int_{x=-2}^{2} \int_{y=-\sqrt{(4-x^2)}}^{\sqrt{(4-x^2)}} (3y - 1)\, dx\, dy$$

$$= 2\int_{x=-2}^{2} \int_{y=0}^{\sqrt{(4-x^2)}} (-1)\, dx\, dy \quad [\because\ 3y \text{ is an odd function of } y]$$

$$= -2\int_{x=-2}^{2} \left[y\right]_{y=0}^{\sqrt{(4-x^2)}} dx$$

$$= -2\int_{-2}^{2} \sqrt{(4 - x^2)}\, dx = -4\int_0^2 \sqrt{(4 - x^2)}\, dx$$

$$= -4\left[\frac{x}{2}\sqrt{(4 - x^2)} + \frac{1}{2}\cdot 2\sin^{-1}\frac{x}{2}\right]_0^2 = -4\left[2 \cdot \frac{\pi}{2}\right] = -4\pi.$$

Ex. 54. *Evaluate* $\iint_S (ax^2 + by^2 + cz^2)\, dS$ *over the sphere* $x^2 + y^2 + z^2 = 1$ *using the divergence theorem.*

Solution. Let us first put the integral

$$\iint_S (ax^2 + by^2 + cz^2)\, dS \text{ in the form}$$

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, dS,$$

where n is unit normal vector to S.

The normal vector to $\phi(x, y, z) \equiv x^2 + y^2 + z^2 - 1 = 0$ is

$$= \nabla\phi = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}.$$

$$\therefore \quad \mathbf{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{\sqrt{[4(x^2 + y^2 + z^2)]}}$$

$$= x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \quad [\because\ x^2 + y^2 + z^2 = 1, \text{ on } S]$$

Now we are to choose F such that

$$\mathbf{F} \cdot \mathbf{n} = \mathbf{F} \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = ax^2 + by^2 + cz^2.$$

Obviously $\mathbf{F} = ax\mathbf{i} + by\mathbf{j} + cz\mathbf{k}$.

Now $$\iint_S (ax^2 + by^2 + cz^2)\, dS$$

$$= \iint_S \mathbf{F} \cdot \mathbf{n}\, dS, \text{ where } \mathbf{F} = ax\mathbf{i} + by\mathbf{j} + cz\mathbf{k}$$

$$= \iiint_V \text{div } \mathbf{F}\, dV, \text{ by divergence theorem}$$

$$= \iiint_V (a + b + c)\, dV \quad [\because\ \text{div } \mathbf{F} = a + b + c]$$

$$= (a + b + c) \iiint_V dV = (a + b + c) V$$

$= (a + b + c)\frac{4}{3}\pi$, since the volume V enclosed by the sphere S of unit radius is $\frac{4}{3}\pi$.

Ex. 55. *Compute*

(i) $\iint_S (a^2x^2 + b^2y^2 + c^2z^2)^{1/2}\, dS$, *and*

(ii) $\iint_S (a^2x^2 + b^2y^2 + c^2z^2)^{-1/2}\, dS$

over the ellipsoid $ax^2 + by^2 + cz^2 = 1$.

Sol. (i). Let us first put the integral

$$\iint_S (a^2x^2 + b^2y^2 + c^2z^2)^{1/2}\, dS \text{ in the form}$$

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS,$$

where $\mathbf{n}$ is a unit normal vector to the closed surface S whose equation is $ax^2 + by^2 + cz^2 = 1$.

The normal vector to $\phi(x, y, z) \equiv ax^2 + by^2 + cz^2 - 1 = 0$ is $= \nabla\phi = 2ax\,\mathbf{i} + 2by\,\mathbf{j} + 2cz\,\mathbf{k}$.

$$\therefore \mathbf{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2ax\,\mathbf{i} + 2by\,\mathbf{j} + 2cz\,\mathbf{k}}{\sqrt{(4a^2x^2 + 4b^2y^2 + 4c^2z^2)}} = \frac{ax\,\mathbf{i} + by\,\mathbf{j} + cz\,\mathbf{k}}{\sqrt{(a^2x^2 + b^2y^2 + c^2z^2)}}.$$

Now we are to choose $\mathbf{F}$ such that $\mathbf{F}\cdot\mathbf{n} = \sqrt{(a^2x^2 + b^2y^2 + c^2z^2)}$.

Obviously $\mathbf{F} = ax\,\mathbf{i} + by\,\mathbf{j} + cz\,\mathbf{k}$.

Now $\iint_S (a^2x^2 + b^2y^2 + c^2z^2)^{1/2}\, dS$

$= \iint_S \mathbf{F}\cdot\mathbf{n}\, dS$, where $\mathbf{F} = ax\,\mathbf{i} + by\,\mathbf{j} + cz\,\mathbf{k}$

$= \iiint_V \text{div}\,\mathbf{F}\, dV$, by divergence theorem; V is the volume enclosed by the closed surface S

$$= \iiint_V \left[\frac{\partial}{\partial x}(ax) + \frac{\partial}{\partial y}(by) + \frac{\partial}{\partial z}(cz)\right] dV$$

$$= (a + b + c) \iiint_V dV = (a + b + c)\, V$$

$$= (a + b + c)\cdot\frac{4}{3}\pi\cdot\left(\frac{1}{\sqrt{a}}\cdot\frac{1}{\sqrt{b}}\cdot\frac{1}{\sqrt{c}}\right) = \frac{4\pi}{3}\frac{(a + b + c)}{\sqrt{(abc)}}.$$

Note that the equation of the ellipsoid $ax^2 + by^2 + cz^2 = 1$ can be written as $\frac{x^2}{1/a} + \frac{y^2}{1/b} + \frac{z^2}{1/c} = 1$ and the volume of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is $\frac{4}{3}\pi abc$.

(ii) Proceed as in part (i) of this question.

Here we are to choose $\mathbf{F}$ such that

$\mathbf{F}\cdot\mathbf{n} = 1/\sqrt{(a^2x^2 + b^2y^2 + c^2z^2)}$ on S.

Obviously $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$, because then

$$\mathbf{F}\cdot\mathbf{n} = \frac{ax^2 + by^2 + cz^2}{\sqrt{(a^2x^2 + b^2y^2 + c^2z^2)}} = \frac{1}{\sqrt{(a^2x^2 + b^2y^2 + c^2z^2)}} \text{ on } S.$$

Note that on S, $ax^2 + by^2 + cz^2 = 1$.

Now $\iint_S \frac{1}{\sqrt{(a^2x^2 + b^2y^2 + c^2z^2)}}\, dS$

$= \iint_S \mathbf{F}\cdot\mathbf{n}\, dS$, where $\mathbf{F} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$

$= \iiint_V (\nabla\cdot\mathbf{F})\, dV$, by divergence theorem; V is the volume enclosed by the surface S

$$= \iiint_V \left[\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial y}(y) + \frac{\partial}{\partial z}(z)\right] dV = \iiint_V 3\, dV$$

$$= 3V = 3\cdot\frac{4}{3}\pi\cdot\frac{1}{\sqrt{a}}\cdot\frac{1}{\sqrt{b}}\cdot\frac{1}{\sqrt{c}} = \frac{4\pi}{\sqrt{(abc)}}.$$

Ex. 56. *Evaluate* $\iint_S (x^2 + y^2)\, dS$, *where S is the surface of the cone $z^2 = 3(x^2 + y^2)$ bounded by $z = 0$ and $z = 3$.*

Sol. Let S be the surface of the cone $z^2 = 3(x^2 + y^2)$ bounded by the planes $z = 0$ and $z = 3$. The plane $z = 3$ cuts the surface $z^2 = 3(x^2 + y^2)$ in the circle $x^2 + y^2 = 3$, $z = 3$. Let S_1 be the plane region bounded by this circle. Let S' be the closed surface consisting of the surfaces S and S_1.

Let us first put the integral

$$\iint_{S'} (x^2 + y^2)\, dS \text{ in the form}$$

$$\iint_S \mathbf{F}\cdot\mathbf{n}\, dS,$$

where $\mathbf{n}$ is a unit vector along the outward drawn normal to the surface S whose equation is $\phi(x, y, z) \equiv 3(x^2 + y^2) - z^2 = 0$,

We have $\mathbf{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{6x\,\mathbf{i} + 6y\,\mathbf{j} - 2z\,\mathbf{k}}{\sqrt{(36x^2 + 36y^2 + 4z^2)}}$

$$= \frac{3x\mathbf{i}+3y\mathbf{j}-z\mathbf{k}}{\sqrt{[9(x^2+y^2)+z^2]}} = \frac{3x\mathbf{i}+3y\mathbf{j}-z\mathbf{k}}{\sqrt{(3z^2+z^2)}}, \quad \text{since on } S,\ 3(x^2+y^2)=z^2$$

$$= \frac{3x\mathbf{i}+3y\mathbf{j}-z\mathbf{k}}{2}.$$

Now take $\mathbf{F} = \frac{2z}{3}(x\mathbf{i}+y\mathbf{j})$. Then on S, $\mathbf{F}\cdot\mathbf{n} = x^2+y^2$.

By Gauss divergence theorem, we have

$$\iint_{S'} \mathbf{F}\cdot\mathbf{n}\,dS = \iiint_V \text{div}\,\mathbf{F}\,dV, \qquad \dots(1)$$

where V is the volume enclosed by the closed surface S'.

We have $\text{div}\,\mathbf{F} = \text{div}\left(\frac{2}{3}zx\mathbf{i}+\frac{2}{3}zy\mathbf{j}\right)$

$$= \frac{\partial}{\partial x}\left(\frac{2}{3}zx\right)+\frac{\partial}{\partial y}\left(\frac{2}{3}zy\right) = \frac{2}{3}z+\frac{2}{3}z = \frac{4}{3}z.$$

$\therefore \iiint_V \text{div}\,\mathbf{F}\,dV = \iiint_V \frac{4}{3}z\,dV$, where V is the volume bounded by $z=0$, $z=3$ and $x^2+y^2=z^2/3$

$$= \frac{4}{3}\int_{z=0}^{3}\int_{y=-z/\sqrt{3}}^{z/\sqrt{3}}\int_{x=-\sqrt{((z^2/3)-y^2)}}^{\sqrt{((z^2/3)-y^2)}} z\,dz\,dy\,dx$$

$$= \frac{4}{3}\cdot 2\int_{z=0}^{3}\int_{y=-z/\sqrt{3}}^{z/\sqrt{3}}\int_{x=0}^{\sqrt{((z^2/3)-y^2)}} z\,dz\,dy\,dx$$

$$= \frac{8}{3}\int_{z=0}^{3}\int_{y=-z/\sqrt{3}}^{z/\sqrt{3}} z\,[x]_{x=0}^{\sqrt{((z^2/3)-y^2)}}\,dz\,dy$$

$$= \frac{8}{3}\int_{z=0}^{3}\int_{y=-z/\sqrt{3}}^{z/\sqrt{3}} z\sqrt{\{(z^2/3)-y^2\}}\,dz\,dy$$

$$= 2\cdot\frac{8}{3}\int_{z=0}^{3}\int_{y=0}^{z/\sqrt{3}} z\sqrt{\left(\frac{z^2}{3}-y^2\right)}\,dz\,dy$$

$$= \frac{16}{3}\int_{z=0}^{3} z\left[\frac{y}{2}\sqrt{\left(\frac{z^2}{3}-y^2\right)}+\frac{z^2}{6}\sin^{-1}\left(\frac{y}{z/\sqrt{3}}\right)\right]_{y=0}^{z/\sqrt{3}} dz$$

$$= \frac{16}{3}\int_0^3 z\left[0+\frac{z^2}{6}\cdot\sin^{-1}1\right]dz = \frac{16}{3}\cdot\frac{1}{6}\cdot\frac{\pi}{2}\int_0^3 z^3\,dz$$

$$= \frac{16}{3}\cdot\frac{1}{6}\cdot\frac{\pi}{2}\left[\frac{z^4}{4}\right]_0^3 = \frac{16}{3}\cdot\frac{1}{6}\cdot\frac{\pi}{2}\cdot\frac{81}{4} = 9\pi. \qquad \dots(2)$$

Also $\iint_{S'} (\mathbf{F}\cdot\mathbf{n})\,dS = \iint_S \mathbf{F}\cdot\mathbf{n}\,dS + \iint_{S_1} \mathbf{F}\cdot\mathbf{n}\,dS$

[$\because$ S' consists of S and S_1]

$$= \iint_S (x^2+y^2)\,dS + \iint_{S_1} \frac{2\cdot 3}{3}(x\mathbf{i}+y\mathbf{j})\cdot\mathbf{k}\,dS,$$

since on S_1, $\mathbf{n}=\mathbf{k}$, $z=3$

$$= \iint_S (x^2+y^2)\,dS + 0 = \iint_S (x^2+y^2)\,dS \qquad \dots(3)$$

From (1), (2) and (3), we have

$$\iint_S (x^2+y^2)\,dS = 9\pi.$$

Ex. 57. *Prove that*

$$\int_V \mathbf{f}\cdot\text{curl}\,\mathbf{F}\,dV = \int_S \mathbf{F}\times\mathbf{f}\cdot d\mathbf{S} + \int_V \mathbf{F}\cdot\text{curl}\,\mathbf{f}\,dV.$$

Sol. We have $\int_S \mathbf{F}\times\mathbf{f}\cdot d\mathbf{S} = \int_S (\mathbf{F}\times\mathbf{f})\cdot\mathbf{n}\,dS$,

where $\mathbf{n}$ is a unit normal vector to the surface S

$$= \int_V [\nabla\cdot(\mathbf{F}\times\mathbf{f})]\,dV, \text{ by Gauss divergence theorem}$$

$$= \int_V (\mathbf{f}\cdot\text{curl}\,\mathbf{F} - \mathbf{F}\cdot\text{curl}\,\mathbf{f})\,dV$$

$$= \int_V \mathbf{f}\cdot\text{curl}\,\mathbf{F}\,dV - \int_V \mathbf{F}\cdot\text{curl}\,\mathbf{f}\,dV.$$

$$\therefore \int_V \mathbf{f}\cdot\text{curl}\,\mathbf{F}\,dV = \int_S \mathbf{F}\times\mathbf{f}\cdot d\mathbf{S} + \int_V \mathbf{F}\cdot\text{curl}\,\mathbf{f}\,dV.$$

Stoke's theorem: Let S be a piecewise smooth op. surface bounded by a piecewise smooth simple cl. curve C. Let F (x, y, z) be a continuous vector fun. which has continuous first partial derivatives in a region of space which contains S in its interior. Then $\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_S (\text{curl}\, \mathbf{F}) \cdot d\mathbf{S}$

where C is traversed in the positive direction. The direction of C is called positive if an observer, walking on the boundary of S in this direction, with his head pointing in the direction of outward drawn normal n *to S, has the surface on the left.*

(Meerut 85; Rohilkhand 90; Osmania 89; Kakatiya 90, 92; Tirupati 89, 93)

Note. $\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C \left(\mathbf{F} \cdot \frac{d\mathbf{r}}{ds}\right) ds = \oint_C (\mathbf{F} \cdot \mathbf{t})\, ds$, where t is unit tangent vector to C. Therefore $\mathbf{F} \cdot \mathbf{t}$ is the component of F in the direction of the tangent vector of C. Also $(\nabla \times \mathbf{F}) \cdot \mathbf{n}$ is the component of curl F in the direction of outward drawn normal vector n of S. Therefore in words Stoke's theorem may be stated as follows:

The line integral of the tangential component of vector F *taken around a simple closed curve C is equal to the surface integral of the normal component of the curl of* F *taken over any surface S having C as its boundary.*

Cartesian equivalent of Stoke's theorem.

Let $\mathbf{F} = F_1 \mathbf{i} + F_2 \mathbf{j} + F_3 \mathbf{k}$. Let outward drawn normal vector n of S make angles α, β, γ with positive directions of x, y, z axes.

Then $\mathbf{n} = \cos\alpha\, \mathbf{i} + \cos\beta\, \mathbf{j} + \cos\gamma\, \mathbf{k}$.

Also $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$

$$= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) \mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) \mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \mathbf{k}$$

$$\therefore (\nabla \times \mathbf{F}) \cdot \mathbf{n} = \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) \cos\alpha + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right) \cos\beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) \cos\gamma.$$

Also $\mathbf{F} \cdot d\mathbf{r} = (F_1 \mathbf{i} + F_2 \mathbf{j} + F_3 \mathbf{k}) \cdot (dx\, \mathbf{i} + dy\, \mathbf{j} + dz\, \mathbf{k})$

$= F_1\, dx + F_2\, dy + F_3\, dz.$

$\therefore$ Stoke's theorem can be written as

$$\oint_C F_1\,dx + F_2\,dy + F_3\,dz = \iint_S \left[\left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\cos\alpha + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\cos\beta + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\cos\gamma\right] dS.$$

Proof of Stoke's theorem. Let S be a surface which is such that its projections on the xy, yz and zx planes are regions bounded by simple closed curves. Suppose S can be represented simultaneously in the forms

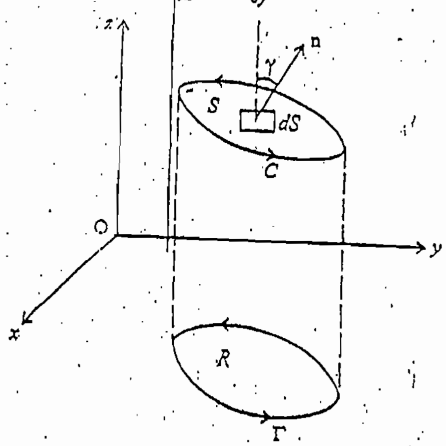
$$z = f(x,y),\; y = g(x,z),\; x = h(z,y)$$

where f, g, h are continuous functions and have continuous first partial derivatives.

Consider the integral $\iint_S [\nabla \times (F_1 \mathbf{i})] \cdot \mathbf{n}\, dS$.

We have $\nabla \times (F_1 \mathbf{i}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & 0 & 0 \end{vmatrix}$

$$= \frac{\partial F_1}{\partial z}\mathbf{j} - \frac{\partial F_1}{\partial y}\mathbf{k}$$



$$\therefore [\nabla \times (F_1 \mathbf{i})] \cdot \mathbf{n} = \left(\frac{\partial F_1}{\partial z}\mathbf{j}\cdot\mathbf{n} - \frac{\partial F_1}{\partial y}\mathbf{k}\cdot\mathbf{n}\right) = \frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma.$$

$$\therefore \iint_S [\nabla \times (F_1 \mathbf{i})] \cdot \mathbf{n}\, dS = \iint_S \left(\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right) dS.$$

We shall prove that

$$\iint_S \left(\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right) dS = \oint_C F_1\, dx.$$

Let R be the orthogonal projection of S on the xy-plane and let Γ be its boundary which is oriented as shown in the figure. Using the representation $z = f(x,y)$ of S, we may write the line integral over C as a line integral over Γ. Thus

$$\oint_C F_1(x,y,z)\, dx = \oint_\Gamma F_1[x,y,f(x,y)]\, dx$$

$$= \oint_\Gamma \{F_1[x,y,f(x,y)]\, dx + 0\, dy\}$$

$$= -\iint_R \frac{\partial F_1}{\partial y}\, dx\, dy, \quad \text{by Green's theorem in plane for the region } R.$$

But
$$\frac{\partial F_1[x,y,f(x,y)]}{\partial y} = \frac{\partial F_1(x,y,z)}{\partial y} + \frac{\partial F_1(x,y,z)}{\partial z}\frac{\partial f}{\partial y} \qquad [\because z = f(x,y)]$$

$$\therefore \oint_C F_1(x,y,z)\, dx = -\iint_R \left(\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial f}{\partial y}\right) dx\, dy \quad \ldots (1)$$

Now the equation $z = f(x,y)$ of the surface S can be written as

$$\phi(x,y,z) \equiv z - f(x,y) = 0.$$

We have $\operatorname{grad}\phi = -\frac{\partial f}{\partial x}\mathbf{i} - \frac{\partial f}{\partial y}\mathbf{j} + \mathbf{k}$.

Let $|\operatorname{grad}\phi| = a$.

Since $\operatorname{grad}\phi$ is normal to S, therefore, we get

$$\mathbf{n} = \pm\frac{\operatorname{grad}\phi}{a}.$$

But the components of both $\mathbf{n}$ and $\operatorname{grad}\phi$ in positive direction of z-axis are positive. Therefore

$$\mathbf{n} = +\frac{\operatorname{grad}\phi}{a}$$

or $\cos\alpha\, \mathbf{i} + \cos\beta\, \mathbf{j} + \cos\gamma\, \mathbf{k} = -\frac{1}{a}\frac{\partial f}{\partial x}\mathbf{i} - \frac{1}{a}\frac{\partial f}{\partial y}\mathbf{j} + \frac{1}{a}\mathbf{k}.$

$\therefore \cos\alpha = -\frac{1}{a}\frac{\partial f}{\partial x},\ \cos\beta = -\frac{1}{a}\frac{\partial f}{\partial y},\ \cos\gamma = \frac{1}{a}.$

Now $dS = \frac{dx\,dy}{\cos\gamma} = a\,dx\,dy.$

$$\therefore \iint_S \left(\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right) dS$$

$$= \iint_R \left[\frac{\partial F_1}{\partial z}\left(-\frac{1}{a}\frac{\partial f}{\partial y}\right) - \frac{\partial F_1}{\partial y}\frac{1}{a}\right] a\,dx\,dy$$

$$= -\iint_R \left(\frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}\frac{\partial f}{\partial y}\right) dx\,dy. \qquad \dots (2)$$

From (1) and (2), we get

$$\oint_C F_1\,dx = \iint_S \left(\frac{\partial F_1}{\partial z}\cos\beta - \frac{\partial F_1}{\partial y}\cos\gamma\right) dS$$

$$= \iint_S [\nabla \times (F_1\mathbf{i})]\cdot\mathbf{n}\,dS \qquad \dots (3)$$

Similarly, by projections on the other coordinate planes, we get

$$\oint_C F_2\,dy = \iint_S [\nabla \times (F_2\mathbf{j})]\cdot\mathbf{n}\,dS \qquad \dots (4)$$

$$\oint_C F_3\,dz = \iint_S [\nabla \times (F_3\mathbf{k})]\cdot\mathbf{n}\,dS \qquad \dots (5)$$

Adding (3), (4), (5), we get

$$\oint_C (F_1\,dx + F_2\,dy + F_3\,dz) = \iint_S [\nabla \times (F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k})]\cdot\mathbf{n}\,dS$$

or $$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F})\cdot\mathbf{n}\,dS.$$

If the surface S does not satisfy the restrictions imposed above, even then Stoke's theorem will be true provided S can be subdivided into surfaces $S_1, S_2, \dots, S_k$ with boundaries $C_1, C_2, \dots, C_k$ which do satisfy the restrictions. Stoke's theorem holds for each such surface. The sum of surface integrals over $S_1, S_2, \dots, S_k$ will give us surface integral over S while the sum of the integrals over $C_1, C_2, \dots, C_k$ will give us line integral over C.

Note. Green's theorem in plane is a special case of Stoke's theorem. If R is a region in the xy-plane bounded by a closed curve C, then in vector form Green's theorem in plane can be written as

$$\iint_R (\nabla \times \mathbf{F})\cdot\mathbf{k}\,dR = \oint_C \mathbf{F}\cdot d\mathbf{r}.$$

This is nothing but a special case of Stoke's theorem because here $\mathbf{k} = \mathbf{n}$ = outward drawn unit normal to the surface of region R.

Solved Examples

Ex. 1. *Prove that* $\oint_C \mathbf{r}\cdot d\mathbf{r} = 0.$ (Andhra 1989)

Sol. By Stoke's theorem

$$\oint_C \mathbf{r}\cdot d\mathbf{r} = \iint_S (\text{curl}\,\mathbf{r})\cdot\mathbf{n}\,dS = 0, \text{ since curl } \mathbf{r} = 0.$$

Ex. 2. *Prove that* $\oint_C \phi\nabla\psi\cdot d\mathbf{r} = -\oint_C \psi\nabla\phi\cdot d\mathbf{r}.$

Sol. By Stoke's theorem, we have

$$\oint_C \nabla(\phi\psi)\cdot d\mathbf{r} = \iint_S [\text{curl grad}\,(\phi\psi)]\cdot\mathbf{n}\,dS$$

$$= 0, \text{ since curl grad }(\phi\psi) = 0.$$

But $\nabla(\phi\psi) = \phi\nabla\psi + \psi\nabla\phi.$

$$\therefore \oint_C (\phi\nabla\psi + \psi\nabla\phi)\cdot d\mathbf{r} = 0$$

or $$\oint_C \phi\nabla\psi\cdot d\mathbf{r} = -\oint_C \psi\nabla\phi\cdot d\mathbf{r}.$$

Ex. 3. (a) *Prove that* $\oint_C \phi\nabla\psi\cdot d\mathbf{r} = \iint_S [\nabla\phi \times \nabla\psi]\cdot\mathbf{n}\,dS.$

Sol. By Stoke's theorem, we have

$$\oint_C \phi\nabla\psi\cdot d\mathbf{r} = \iint_S [\nabla \times (\phi\nabla\psi)]\cdot\mathbf{n}\,dS$$

$$= \iint_S [\nabla\phi \times \nabla\psi + \phi\,\text{curl grad}\,\psi]\cdot\mathbf{n}\,dS$$

$$= \iint_S [\nabla\phi \times \nabla\psi]\cdot\mathbf{n}\,dS, \text{ since curl grad } \psi = 0.$$

Ex. 3 (b). *Show that* $\int_C \phi\nabla\phi\cdot d\mathbf{r} = 0$, *C being a closed curve.*

Sol. Applying Stoke's theorem to the vector function $\phi\nabla\phi$, we have

$$\int_C (\phi\nabla\phi)\cdot d\mathbf{r} = \iint_S [\text{curl}\,(\phi\nabla\phi)]\cdot\mathbf{n}\,dS$$

$$= \iint_S [\phi \operatorname{curl} \nabla\phi + \nabla\phi \times \nabla\phi] \cdot \mathbf{n}\, dS$$

$$= \iint_S 0 \cdot \mathbf{n}\, dS \qquad [\because \operatorname{curl} \nabla\phi = 0 \text{ and } \nabla\phi \times \nabla\phi = 0]$$

$= 0.$

Ex. 4. *Prove that* $\oint_C \phi\, d\mathbf{r} = \iint_S d\mathbf{S} \times \nabla\phi.$ (Kanpur 1977)

Sol. Let A be any arbitrary constant vector. Let $\mathbf{F} = \phi\mathbf{A}$. Applying Stoke's theorem for F, we get

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S [\nabla \times (\phi\mathbf{A})] \cdot \mathbf{n}\, dS = \iint_S [\nabla\phi \times \mathbf{A} + \phi \operatorname{curl} \mathbf{A}] \cdot d\mathbf{S}$$

$$= \iint_S (\nabla\phi \times \mathbf{A}) \cdot d\mathbf{S}, \qquad \text{since curl } \mathbf{A} = 0.$$

$$\therefore \oint_C (\phi\mathbf{A}) \cdot d\mathbf{r} = \iint_S \mathbf{A} \cdot (d\mathbf{S} \times \nabla\phi)$$

or $\mathbf{A} \cdot \oint_C \phi\, d\mathbf{r} = \mathbf{A} \cdot \iint_S d\mathbf{S} \times \nabla\phi$ or $\mathbf{A} \cdot \left[\oint_C \phi\, d\mathbf{r} - \iint_S d\mathbf{S} \times \nabla\phi\right] = 0.$

Since A is an arbitrary vector, therefore we must have

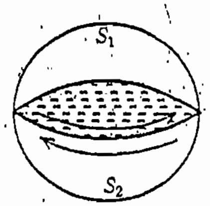
$$\oint_C \phi\, d\mathbf{r} = \iint_S d\mathbf{S} \times \nabla\phi.$$

Ex. 5. *By Stoke's theorem prove that div curl* $\mathbf{F} = 0.$

Sol. Let V be any volume enclosed by a closed surface. Then by divergence theorem

$$\iiint_V \nabla \cdot (\operatorname{curl} \mathbf{F})\, dV$$

$$= \iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n}\, dS.$$



Divide the surface S into two portions and S_2 by a closed curve C. Then

$$\iint_S (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n}\, dS$$

$$= \iint_{S_1} (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n}\, dS_1 + \iint_{S_2} (\operatorname{curl} \mathbf{F}) \cdot \mathbf{n}\, dS_2. \qquad \ldots(1)$$

By Stoke's theorem right hand side of (1) is

$$= \oint_C \mathbf{F} \cdot d\mathbf{r} - \oint_C \mathbf{F} \cdot d\mathbf{r} = 0.$$

Negative sign has been taken in the second integral because the positive directions about the boundaries of the two surfaces are opposite.

$$\therefore \iiint_V \nabla \cdot (\operatorname{curl} \mathbf{F})\, dV = 0.$$

Now this equation is true for all volume elements V. Therefore we have $\nabla \cdot (\operatorname{curl} \mathbf{F}) = 0$

or $\quad$ div curl $\mathbf{F} = 0.$

Ex. 6. *By Stoke's theorem prove that curl grad* $\phi = 0.$

Solution. Let S be any surface enclosed by a simple closed curve C. Then by Stoke's theorem, we have

$$\iint_S (\operatorname{curl} \operatorname{grad} \phi) \cdot \mathbf{n}\, dS = \oint_C \operatorname{grad} \phi \cdot d\mathbf{r}.$$

Now $\operatorname{grad} \phi \cdot d\mathbf{r} = \left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\right) \cdot (dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$

$$= \frac{\partial\phi}{\partial x} dx + \frac{\partial\phi}{\partial y} dy + \frac{\partial\phi}{\partial z} dz = d\phi.$$

$\therefore \oint_C \operatorname{grad} \phi \cdot d\mathbf{r} = \oint_C d\phi = \left[\phi\right]_A^A$, where A is any point on C

$= 0.$

Therefore we have $\iint_S (\operatorname{curl} \operatorname{grad} \phi) \cdot \mathbf{n}\, dS = 0.$

Now this equation is true for all surface elements S.

Therefore we have, curl grad $\phi = 0.$

Ex. 7. (a). *Verify Stoke's theorem for* $\mathbf{F} = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$ *where S is the upper half surface of the sphere* $x^2 + y^2 + z^2 = 1$ *and C is its boundary.* (Meerut 1981; Rohilkhand 91; Agra 70; Andhra 92)

Sol. The boundary C of S is a circle in the xy-plane of radius unity and centre origin. The equations of the curve C are $x^2 + y^2 = 1$, $z = 0$. Suppose $x = \cos t$, $y = \sin t$, $z = 0$, $0 \le t < 2\pi$ are parametric equation of C. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C (y\mathbf{i} + z\mathbf{j} + x\mathbf{k}) \cdot (dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \oint_C (y\,dx + z\,dy + x\,dz) = \oint_C y\,dx, \text{ since on } C,\ z = 0 \text{ and } dz = 0$$

$$= \int_0^{2\pi} \sin t \frac{dx}{dt}\, dt = \int_0^{2\pi} -\sin^2 t\, dt$$

$$= -\frac{1}{2}\int_0^{2\pi}(1-\cos 2t)\,dt = -\frac{1}{2}\left[t - \frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$= -\pi. \quad \ldots (1)$$

Now let us evaluate $\iint_S$ curl $\mathbf{F}\cdot\mathbf{n}\,dS$. We have curl $\mathbf{F}$

$$= \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} = -\mathbf{i} - \mathbf{j} - \mathbf{k}.$$

If S_1 is the plane region bounded by the circle C, then by an application of divergence theorem, we have

$$\iint_S \text{curl}\,\mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_1} \text{curl}\,\mathbf{F}\cdot\mathbf{k}\,dS \quad \text{[See Ex. 50 page 267]}$$

$$= \iint_{S_1}(-\mathbf{i}-\mathbf{j}-\mathbf{k})\cdot\mathbf{k}\,dS = \iint_{S_1}(-1)\,dS = -\iint_{S_1} dS = -S_1.$$

But S_1 = area of a circle of radius $1 = \pi(1)^2 = \pi$.

$$\therefore \iint_S \text{curl}\,\mathbf{F}\cdot\mathbf{n}\,dS = -\pi. \quad \ldots (2)$$

Hence from (1) and (2), the theorem is verified.

Ex. 7 (b). *Verify Stoke's theorem for*

$$\mathbf{F} = (x^2+y-4)\,\mathbf{i} + 3xy\,\mathbf{j} + (2xz+z^2)\,\mathbf{k}$$

where S is the upper half of the sphere $x^2+y^2+z^2=16$ and C is its boundary. (Osmania 1991)

Sol. The boundary C of S is the circle $x^2+y^2=16, z=0$ lying in the xy-plane. Suppose $x=4\cos t$, $y=4\sin t$, $z=0$, $0\le t<2\pi$ are parametric equations of C. Then

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C \{(x^2+y-4)\,\mathbf{i} + 3xy\,\mathbf{j} + (2xz+z^2)\,\mathbf{k}\}\cdot(dx\,\mathbf{i}+dy\,\mathbf{j}+dz\,\mathbf{k})$$

$$= \oint_C \left[(x^2+y-4)\,dx + 3xy\,dy + (2xz+z^2)\,dz\right]$$

$$= \oint_C \left[(x^2+y-4)\,dx + 3xy\,dy\right], \text{ since on } C,\ z=0 \text{ and } dz=0$$

$$= \int_0^{2\pi}\left[(x^2+y-4)\frac{dx}{dt} + 3xy\frac{dy}{dt}\right]dt$$

$$= \int_0^{2\pi}\left[(16\cos^2 t + 4\sin t - 4)(-4\sin t) + 3\cdot 16\sin t\cos t\cdot 4\cos t\right]dt$$

$$= 128\int_0^{2\pi}\cos^2 t\sin t\,dt - 16\int_0^{2\pi}\sin^2 t\,dt + 16\int_0^{2\pi}\sin t\,dt$$

$$= 128.0 - 16.4\int_0^{\pi/2}\sin^2 t\,dt + 16.0 = -64\cdot\frac{1}{2}\cdot\frac{\pi}{2} = -16\pi \quad \ldots (1)$$

$\left[\text{Note that } \int_0^{2a} f(x)\,dx = 0, \text{ if } f(2a-x) = -f(x)\right.$

$\left.\text{and } = 2\int_0^a f(x)\,dx, \text{ if } f(2a-x) = f(x)\right]$

Now let us evaluate $\iint_S$ (curl $\mathbf{F}$)$\cdot\mathbf{n}\,dS$. We have curl $\mathbf{F}$

$$= \nabla\times\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y-4 & 3xy & 2xz+z^2 \end{vmatrix}$$

$$= 0\,\mathbf{i} - z\,\mathbf{j} + (3y-1)\,\mathbf{k} = -z\,\mathbf{j} + (3y-1)\,\mathbf{k}.$$

If S_1 is the plane region bounded by the circle C, then by an application of Gauss divergence theorem, we have

$$\iint_S \text{curl}\,\mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_1}\text{curl}\,\mathbf{F}\cdot\mathbf{k}\,dS \quad \text{[See Ex. 50 page 267]}$$

$$= \iint_{S_1}\{-z\,\mathbf{j} + (3y-1)\,\mathbf{k}\}\cdot\mathbf{k}\,dS = \iint_{S_1}(3y-1)\,dS$$

$$= \int_{\theta=0}^{2\pi}\int_{r=0}^{4}(3r\sin\theta - 1)\,r\,d\theta\,dr, \text{ changing to polars}$$

[Note that S_1 is a circle in xy-plane with centre origin and radius 4]

$$= -16\pi. \quad \ldots (2)$$

Hence from (1) and (2), Stoke's theorem is verified. [See Ex. 49 pag 265]

Ex. 8. *Verify Stoke's theorem for* $\mathbf{F} = (2x-y)\,\mathbf{i} - yz^2\,\mathbf{j} - y^2z\,\mathbf{k}$, *where S is the upper half surface of the sphere $x^2+y^2+z^2=1$ and C is its boundary.*

(Agra 1960; Rohilkhand 78; Allahabad 78; Kanpur 70; Osmania 89, 91)

Sol. The boundary C of S is a circle in the xy-plane of radius unity and centre origin. Suppose $x = \cos t$, $y = \sin t$, $z = 0$, $0 \le t < 2\pi$ are parametric equations of C. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C [(2x - y)\mathbf{i} - yz^2 \mathbf{j} - y^2 z \mathbf{k}] \cdot (dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \oint_C [(2x - y)\,dx - yz^2\,dy - y^2 z\,dz]$$

$$= \oint_C (2x - y)\,dx, \text{ since } z = 0 \text{ and } dz = 0$$

$$= \int_0^{2\pi} (2\cos t - \sin t)\frac{dx}{dt}\,dt = -\int_0^{2\pi} (2\cos t - \sin t)\sin t\,dt$$

$$= -\int_0^{2\pi} [\sin 2t - \tfrac{1}{2}(1 - \cos 2t)]\,dt = -\left[-\frac{\cos 2t}{2} - \frac{1}{2}t + \frac{1}{2}\frac{\sin 2t}{2}\right]_0^{2\pi}$$

$$= -[(-\tfrac{1}{2} + \tfrac{1}{2}) - \tfrac{1}{2}(\pi - 0) + \tfrac{1}{4}(0 - 0)] = \pi. \quad \ldots (1)$$

And
$$(\nabla \times \mathbf{F}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix}$$

$$= (-2yz + 2yz)\,\mathbf{i} - (0 - 0)\,\mathbf{j} + (0 + 1)\,\mathbf{k} = \mathbf{k}.$$

Let S_1 be the plane region bounded by the circle C. If S' is the surface consisting of the surfaces S and S_1, then S' is a closed surface.

$\therefore$ by an application of Gauss divergence theorem, we have

$$\iint_{S'} \text{curl } \mathbf{F} \cdot \mathbf{n}\,dS = 0 \qquad \text{[See Ex. 1 page 233]}$$

or
$$\iint_S \text{curl } \mathbf{F} \cdot \mathbf{n}\,dS + \iint_{S_1} \text{curl } \mathbf{F} \cdot \mathbf{n}\,dS = 0$$

$[\because\ S' \text{ consists of } S \text{ and } S_1]$

or
$$\iint_S \text{curl } \mathbf{F} \cdot \mathbf{n}\,dS - \iint_S \text{curl } \mathbf{F} \cdot \mathbf{k}\,dS = 0$$

$[\because\ \text{on } S_1,\ \mathbf{n} = -\mathbf{k}]$

or
$$\iint_S \text{curl } \mathbf{F} \cdot \mathbf{n}\,dS = \iint_S \text{curl } \mathbf{F} \cdot \mathbf{k}\,dS.$$

$$\therefore \iint_S \text{curl } \mathbf{F} \cdot \mathbf{n}\,dS = \iint_{S_1} \text{curl } \mathbf{F} \cdot \mathbf{k}\,dS$$

$$= \iint_{S_1} \mathbf{k} \cdot \mathbf{k}\,dS = \iint_{S_1} dS = S_1 = \pi. \quad \ldots (2)$$

Note that S_1 = area of a circle of radius 1

$$= \pi(1)^2 = \pi.$$

Hence from (1) and (2) Stoke's theorem is verified.

Ex. 9. *Verify Stoke's theorem for the function*

$$\mathbf{F} = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$$

where curve is the unit circle in the xy-plane bounding the hemisphere $z = \sqrt{(1 - x^2 - y^2)}$. (Rohilkhand 1981; Kanpur 78; Agra 75)

Sol. Here the surface S is the surface of the sphere $x^2 + y^2 + z^2 = 1$ lying above the xy-plane. The curve C is the boundary of the surface S and is a circle in the xy-plane of radius 1 and centre origin.

The equations of the curve C are $x^2 + y^2 = 1$, $z = 0$. Suppose $x = \cos t$, $y = \sin t$, $z = 0$, $0 \le t < 2\pi$ are parametric equations of C. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \oint_C (z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}) \cdot (dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \oint_C (z\,dx + x\,dy + y\,dz) = \oint_C x\,dy, \text{ since on } C,\ z = 0 \text{ and } dz = 0$$

$$= \int_0^{2\pi} \cos t\,\frac{dy}{dt}\,dt = \int_0^{2\pi} \cos t \cdot \cos t\,dt = \int_0^{2\pi} \cos^2 t\,dt$$

$$= \tfrac{1}{2}\int_0^{2\pi} (1 + \cos 2t)\,dt = \tfrac{1}{2}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi} = \tfrac{1}{2} \cdot 2\pi = \pi. \quad \ldots (1)$$

Now let us evaluate $\iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\,dS$.

We have
$$\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix}$$

$$= \mathbf{i}\left[\frac{\partial}{\partial y}(y) - \frac{\partial}{\partial x}(z)\right] - \mathbf{j}\left[\frac{\partial}{\partial x}(y) - \frac{\partial}{\partial z}(z)\right] + \mathbf{k}\left[\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(z)\right]$$

$$= \mathbf{i} + \mathbf{j} + \mathbf{k}.$$

Let S_1 be the plane region bounded by the circle C. If S' is the surface consisting of the surfaces S and S_1, then S' is a closed surface.

By Gauss divergence theorem,

$$\iint_{S'} (\text{curl } \mathbf{F})\cdot\mathbf{n}\, dS = \iiint_V (\text{div curl } \mathbf{F})\, dV, \quad \text{where } V \text{ is the volume enclosed by } S'$$

$$= 0, \text{ since div curl } \mathbf{F} = 0.$$

$$\therefore \quad \iint_S (\text{curl } \mathbf{F})\cdot\mathbf{n}\, dS + \iint_{S_1} (\text{curl } \mathbf{F})\cdot\mathbf{n}\, dS = 0$$

[$\because$ S' consists of S and S_1]

$$\text{or} \quad \iint_S (\text{curl } \mathbf{F})\cdot\mathbf{n}\, dS - \iint_{S_1} (\text{curl } \mathbf{F})\cdot\mathbf{k}\, dS = 0$$

[$\because$ on S_1, $\mathbf{n} = -\mathbf{k}$]

$$\text{or} \quad \iint_S (\text{curl } \mathbf{F})\cdot\mathbf{n}\, dS = \iint_{S_1} (\text{curl } \mathbf{F})\cdot\mathbf{k}\, dS.$$

$$\therefore \quad \iint_S (\text{curl } \mathbf{F})\cdot\mathbf{n}\, dS = \iint_{S_1} (\text{curl } \mathbf{F})\cdot\mathbf{k}\, dS$$

$$= \iint_{S_1} (\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot\mathbf{k}\, dS = \iint_{S_1} dS$$

$= S_1$, where S_1 is the area of the circle $x^2 + y^2 = 1, z = 0$

$$= \pi \cdot 1^2 = \pi. \qquad \ldots (2)$$

From (1) and (2), we see that

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F})\cdot\mathbf{n}\, dS.$$

This verifies Stoke's theorem.

Ex. 10. *Verify Stoke's theorem for* $\mathbf{A} = 2y\,\mathbf{i} + 3x\,\mathbf{j} - z^2\,\mathbf{k}$, *where S is the upper half surface of the sphere* $x^2 + y^2 + z^2 = 9$ *and C is its boundary.* (Meerut 1975)

Sol. Proceed as in solved example 9.

Here the parametric equations of the circle are $x = 3\cos t$, $y = 3 \sin t$, $z = 0$, $0 \le t < 2\pi$.

We have $\oint_C \mathbf{A}\cdot d\mathbf{r} = \oint_C (2y\,\mathbf{i} + 3x\,\mathbf{j} - z^2\mathbf{k})\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$

$$= \oint_C (2y\, dx + 3x\, dy - z^2\, dz) = \oint_C (2y\, dx + 3x\, dy),$$

since on C, $z = 0$ and $dz = 0$

$$= \int_{t=0}^{2\pi} \left(2y \frac{dx}{dt} + 3x \frac{dy}{dt}\right) dt = \int_0^{2\pi} [6 \sin t \cdot (-3 \sin t) + 9 \cos t \cdot (3 \cos t)]\, dt$$

$$= \int_0^{2\pi} (27 \cos^2 t - 18 \sin^2 t)\, dt$$

$$= 27.4 \int_0^{\pi/2} \cos^2 t\, dt - 18.4 \int_0^{\pi/2} \sin^2 t\, dt$$

$$= 108 \cdot \frac{1}{2} \cdot \frac{\pi}{2} - 72 \cdot \frac{1}{2} \cdot \frac{\pi}{2} = 9\pi.$$

Again here curl $\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & 3x & -z^2 \end{vmatrix}$

$$= \mathbf{i}\left[\frac{\partial}{\partial y}(-z^2) - \frac{\partial}{\partial z}(3x)\right] - \mathbf{j}\left[\frac{\partial}{\partial x}(-z^2) - \frac{\partial}{\partial z}(2y)\right] + \mathbf{k}\left[\frac{\partial}{\partial x}(3x) - \frac{\partial}{\partial y}(2y)\right]$$

$$= 0\,\mathbf{i} - 0\,\mathbf{j} + \mathbf{k} = \mathbf{k}.$$

$$\iint_S (\text{curl } \mathbf{A})\cdot\mathbf{n}\, dS = \iint_{S_1} (\text{curl } \mathbf{A})\cdot\mathbf{k}\, dS$$

$= \iint_{S_1} \mathbf{k}\cdot\mathbf{k}\, dS = \iint_{S_1} dS = S_1$, where S_1 is the area of the circle $x^2 + y^2 = 9, z = 0$

$= \pi \cdot 3^2 = 9\pi$, since radius of the circle is 3.

We see that $\oint_C \mathbf{A}\cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{A})\cdot\mathbf{n}\, dS.$

This verifies Stoke's theorem.

Ex. 11. *Verify Stoke's theorem for the vector* $\mathbf{F} = z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k}$ *taken over the half of the sphere* $x^2 + y^2 + z^2 = a^2$ *lying above the xy-plane.* (Gauhati 1973)

Sol. Here let S be the surface of the sphere $x^2 + y^2 + z^2 = a^2$ lying above the xy-plane and let the curve C be the boundary of this surface. Obviously the curve C is a circle in the xy-plane of radius a and centre origin and its equations are $x^2 + y^2 = a^2$, $z = 0$. Suppose $x = a \cos t$, $y = a \sin t$, $z = 0$, $0 \le t < 2\pi$ are parametric equations of C.

By Stoke's theorem, we have

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F})\cdot \mathbf{n}\, dS. \qquad \dots(1)$$

Let us verify (1).

We have $\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C (z\,\mathbf{i} + x\,\mathbf{j} + y\,\mathbf{k})\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$.

$$= \oint_C (z\,dx + x\,dy + y\,dz) = \oint_C x\,dy, \text{ since on } C,\ z = 0 \text{ and } dz = 0$$

$$= \int_0^{2\pi} a\cos t \cdot \frac{dy}{dt}\,dt = \int_0^{2\pi} a\cos t \,.\, a\cos t\, dt = a^2 \int_0^{2\pi} \cos^2 t\, dt$$

$$= \frac{a^2}{2}\int_0^{2\pi} (1 + \cos 2t)\,dt = \frac{a^2}{2}\left[t + \frac{\sin 2t}{2}\right]_0^{2\pi} = \frac{a^2}{2}\cdot 2\pi = \pi a^2. \qquad \dots(2)$$

Now let us find $\iint_S (\text{curl } \mathbf{F})\cdot \mathbf{n}\, dS$.

We have curl $\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k}$.

If $\mathbf{n}$ is a unit vector along outward drawn normal at any point (x, y, z) on the surface S *i.e.*, the surface $\phi(x, y, z) \equiv x^2 + y^2 + z^2 = a^2$, then

$$\mathbf{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}}{\sqrt{(4x^2 + 4y^2 + 4z^2)}} = \frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{a},$$

since on S, $x^2 + y^2 + z^2 = a^2$.

$$\therefore \iint_S (\text{curl } \mathbf{F})\cdot \mathbf{n}\, dS = \iint_S (\mathbf{i} + \mathbf{j} + \mathbf{k})\cdot\left(\frac{x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}}{a}\right) dS$$

$$= \frac{1}{a}\iint_S (x + y + z)\,dS.$$

To evaluate it we shall use polar spherical coordinates (r, θ, ϕ). We have $z = r\cos\theta$, $x = r\sin\theta\cos\phi$; $y = r\sin\theta\sin\phi$.

Here $r = a$. $\therefore$ $x = a\sin\theta\cos\phi$, $y = a\sin\theta\sin\phi$, $z = a\cos\theta$.

Also dS = an elementary area on the surface of the sphere at the point $(a, \theta, \phi) = a\,d\theta \,.\, a\sin\theta\, d\phi = a^2\sin\theta\, d\theta\, d\phi$.

$$\therefore \iint_S (\text{curl } \mathbf{F})\cdot \mathbf{n}\, dS$$

$$= \frac{1}{a}\int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} (a\sin\theta\cos\phi + a\sin\theta\sin\phi + a\cos\theta)\, a^2\sin\theta\, d\theta\, d\phi$$

$$= a^2 \int_{\theta=0}^{\pi/2}\int_{\phi=0}^{2\pi} (\sin^2\theta\cos\phi + \sin^2\theta\sin\phi + \cos\theta\sin\theta)\, d\theta\, d\phi$$

$$= a^2 \int_{\theta=0}^{\pi/2} \left[\sin^2\theta\sin\phi - \sin^2\theta\cos\phi + \phi\cos\theta\sin\theta\right]_{\phi=0}^{2\pi} d\theta$$

$$= a^2 \int_{\theta=0}^{\pi/2} 2\pi\cos\theta\sin\theta\, d\theta = 2\pi a^2 \cdot \tfrac{1}{2} = \pi a^2. \qquad \dots(3)$$

From (2) and (3), we see that

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F})\cdot \mathbf{n}\, dS.$$

This verifies Stoke's theorem.

Ex. 12. *Verify Stoke's theorem for the vector* $\mathbf{A} = 3y\,\mathbf{i} - xz\,\mathbf{j} + yz^2\,\mathbf{k}$, *where S is the surface of the paraboloid $2z = x^2 + y^2$ bounded by $z = 2$ and C is its boundary.* (Meerut 1973, 77)

Sol. The boundary C of the surface S is the circle in the plane $z = 2$ whose equations are $x^2 + y^2 = 4$, $z = 2$. The radius of this circle is 2 and centre $(0, 0, 2)$. Suppose $x = 2\cos t$, $y = 2\sin t$, $z = 2$, $0 \le t < 2\pi$ are parametric equations of C. By Stoke's theorem

$\oint_C \mathbf{A}\cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{A})\cdot \mathbf{n}\, dS$, where $\mathbf{n}$ is a unit vector along outward drawn normal to the surface S.

We have $\oint_C \mathbf{A}\cdot d\mathbf{r} = \oint_C (3y\,\mathbf{i} - xz\,\mathbf{j} + yz^2\,\mathbf{k})\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$

$$= \oint_C (3y\,dx - xz\,dy + yz^2\,dz)$$

$$= \oint_C (3y\,dx - 2x\,dy), \text{ since on } C,\ z = 2 \text{ and } dz = 0$$

$$= \int_{2\pi}^{0} \left(3y\frac{dx}{dt} - 2x\frac{dy}{dt}\right) dt$$

[Note that here the surface S lies below the curve C and so direction of C is positive if C is traversed in clockwise sense]

$$= -\int_0^{2\pi} [3\cdot 2\sin t\cdot(-2\sin t) - 2\cdot 2\cos t\cdot 2\cos t]\, dt$$

$$= -\int_0^{2\pi} [-12\sin^2 t - 8\cos^2 t]\, dt$$

$$= 4\left[12\int_0^{\pi/2}\sin^2 t\,dt + 8\cdot\int_0^{\pi/2}\cos^2 t\,dt\right]$$

$$= 4\left[12\cdot\frac{1}{2}\cdot\frac{\pi}{2}+8\cdot\frac{1}{2}\cdot\frac{\pi}{2}\right] = 4\cdot\frac{\pi}{4}\cdot 20 = 20\pi. \quad \ldots(1)$$

Let S_1 be the plane region bounded by the circle C. If S' is the surface consisting of the surfaces S and S_1, then S' is a closed surface. Let V be the volume bounded by S'.

By Gauss divergence theorem, we have

$$\iint_{S'} (\text{curl } \mathbf{A})\cdot\mathbf{n}\,dS = \iiint_V \text{div curl } \mathbf{A}\,dV$$

$$= 0, \text{ since div curl } \mathbf{A} = 0.$$

$$\therefore \iint_S (\text{curl } \mathbf{A})\cdot\mathbf{n}\,dS + \iint_{S_1} (\text{curl } \mathbf{A})\cdot\mathbf{n}\,dS = 0$$

$[\because S'$ consists of S and $S_1]$

or
$$\iint_S (\text{curl } \mathbf{A})\cdot\mathbf{n}\,dS = -\iint_{S_1} (\text{curl } \mathbf{A})\cdot\mathbf{n}\,dS$$

$$= -\iint_{S_1} (\text{curl } \mathbf{A})\cdot\mathbf{k}\,dS.$$

$[\because$ on S_1, $\mathbf{n} = \mathbf{k}]$

Now
$$\text{curl } \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y & -xz & yz^2 \end{vmatrix}$$

$$= \mathbf{i}\left[\frac{\partial}{\partial y}(yz^2) - \frac{\partial}{\partial z}(-xz)\right] - \mathbf{j}\left[\frac{\partial}{\partial x}(yz^2) - \frac{\partial}{\partial z}(3y)\right] + \mathbf{k}\left[\frac{\partial}{\partial x}(-xz) - \frac{\partial}{\partial y}(3y)\right]$$

$$= (z^2+x)\,\mathbf{i} - (z+3)\,\mathbf{k}.$$

$$\therefore \iint_S (\text{curl } \mathbf{A})\cdot\mathbf{n}\,dS = -\iint_{S_1} [(z^2+x)\,\mathbf{i} - (z+3)\,\mathbf{k}]\cdot\mathbf{k}\,dS$$

$$= \iint_{S_1} (z+3)\,dS = \iint_{S_1} 5\,dS, \text{ since on } S_1,\ z = 2$$

$= 5S_1$, where S_1 is the area of a circle of radius 2

$$= 5.\pi.2^2 = 20\pi. \quad \ldots(2)$$

From (1) and (2), we see that

$$\oint_C \mathbf{A}\cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{A})\cdot\mathbf{n}\,dS.$$

This verifies Stoke's theorem.

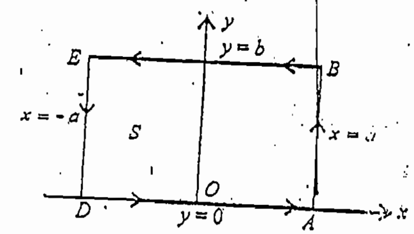
Ex. 13. *Verify Stoke's theorem for*

$$\mathbf{F} = (x^2+y^2)\,\mathbf{i} - 2xy\,\mathbf{j}$$

taken round the rectangle bounded by

$$x = \pm a,\ y = 0,\ y = b. \qquad \text{(Meerut 1967)}$$

Sol. We have



$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2+y^2 & -2xy & 0 \end{vmatrix}$$

$$= (-2y-2y)\,\mathbf{k} = -4y\mathbf{k}.$$

Also $\mathbf{n} = \mathbf{k}$.

$$\therefore \iint_S (\text{curl } \mathbf{F})\cdot\mathbf{n}\,dS = \int_{y=0}^{b}\int_{x=-a}^{a} (-4y\mathbf{k})\cdot\mathbf{k}\,dx\,dy$$

$$= -4\int_{y=0}^{b}\int_{x=-a}^{a} y\,dx\,dy = -4\int_{y=0}^{b}\left[xy\right]_{x=-a}^{a} dy$$

$$= -4\int_{y=0}^{b} 2ay\,dy = -4\left[ay^2\right]_0^b = -4ab^2.$$

Also $\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C [(x^2+y^2)\mathbf{i} - 2xy\,\mathbf{j}]\cdot(dx\,\mathbf{i} + dy\,\mathbf{j})$

$$= \oint_C [(x^2+y^2)\,dx - 2xy\,dy]$$

$$= \int_{DA} [(x^2+y^2)\,dx - 2xy\,dy] + \int_{AB} + \int_{BE} + \int_{ED}.$$

Along DA, $y = 0$ and $dy = 0$. Along AB, $x = a$ and $dx = 0$.
Along BE, $y = b$ and $dy = 0$. Along ED, $x = -a$ and $dx = 0$.

$$\therefore \oint_C \mathbf{F}\cdot d\mathbf{r} = \int_{x=-a}^{a} x^2\,dx + \int_{y=0}^{b} -2ay\,dy + \int_{x=a}^{-a} (x^2+b^2)\,dx + \int_{y=b}^{0} 2ay\,dy$$

$$= \int_{-a}^{a} x^2\,dx - \int_{-a}^{a} (x^2+b^2)\,dx - 4a\int_0^b y\,dy$$

$$= -\int_{-a}^{a} x^2\,dx - 4a\int_0^b y\,dy = -2ab^2 - 4a\left[\frac{y^2}{2}\right]_0^b = -4ab^2.$$

Thus $\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\text{curl }\mathbf{F})\cdot\mathbf{n}\,dS.$

Hence the theorem is verified.

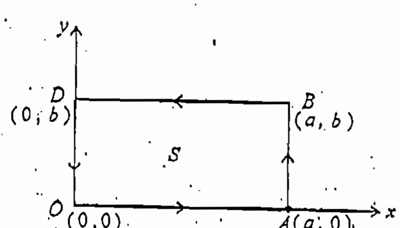
Ex. 14. *Verify Stoke's theorem for the function*

$$\mathbf{F} = x^2\mathbf{i} + xy\,\mathbf{j}$$

integrated along the rectangle, in the plane $z = 0$, whose sides are along the lines $x = 0$, $y = 0$, $x = a$ and $y = b$. (Meerut 1976)

Sol. We have

$$\text{curl }\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & xy & 0 \end{vmatrix}$$



$$= 0\mathbf{i} - 0\mathbf{j} + \left[\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(x^2)\right]\mathbf{k}$$

$$= y\,\mathbf{k}.$$

The closed curve C is the boundary of the rectangle $OABD$ traversed in anti-clockwise sense. The surface S bounded by C is the area of rectangle $OABD$.

Also $\mathbf{n}$ = unit normal vector to S = $\mathbf{k}$.

By Stoke's theorem, we have

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\text{curl }\mathbf{F})\cdot\mathbf{n}\,dS.$$

Let us verify it.

We have $\iint_S (\text{curl }\mathbf{F})\cdot\mathbf{n}\,dS = \int_{y=0}^{b}\int_{x=0}^{a} (y\,\mathbf{k})\cdot\mathbf{k}\,dx\,dy$

$$= \int_{y=0}^{b}\int_{x=0}^{a} y\,dx\,dy = \int_{x=0}^{a} \left[\frac{y^2}{2}\right]_{y=0}^{b} dx = \frac{b^2}{2}\int_0^a dx$$

$$= \frac{ab^2}{2} \qquad \ldots (1)$$

Also $\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C (x^2\mathbf{i} + xy\,\mathbf{j})\cdot(dx\,\mathbf{i} + dy\,\mathbf{j})$

$$= \oint_C (x^2\,dx + xy\,dy)$$

$$= \int_{OA} (x^2\,dx + xy\,dy) + \int_{AB} (x^2\,dx + xy\,dy) + \int_{BD} (x^2\,dx + xy\,dy) + \int_{DO} (x^2\,dx + xy\,dy).$$

Along OA, $y = 0$ and $dy = 0$, x varies from 0 to a;
along AB, $x = a$, $dx = 0$, y varies from 0 to b;
along BD, $y = b$, $dy = 0$, x varies from a to 0;
and along DO, $x = 0$, $dx = 0$, y varies from b to 0.

$$\therefore \oint_C \mathbf{F}\cdot d\mathbf{r} = \int_0^a x^2\,dx + \int_0^b ay\,dy + \int_a^0 x^2\,dx + \int_b^0 0\,dy$$

$$= \left[\frac{x^3}{3}\right]_0^a + a\left[\frac{y^2}{2}\right]_0^b + \left[\frac{x^3}{3}\right]_a^0 + 0$$

$$= \frac{a^3}{3} + \frac{ab^2}{2} - \frac{a^3}{3} = \frac{ab^2}{2} \qquad \ldots (2)$$

From (1) and (2), we see that

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F})\cdot \mathbf{n}\, dS.$$

This verifies Stoke's theorem.

Ex. 15. *Verify Stoke's theorem for the function*

$$\mathbf{F} = x^2\mathbf{i} + xy\,\mathbf{j},$$

integrated round the square, in the plane $z = 0$, whose sides are along the lines $x = 0$, $y = 0$, $x = a$, $y = a$. (Bombay 1970)

Sol. Proceed as in solved example 14.

Show that $\oint_C \mathbf{F}\cdot d\mathbf{r} = \frac{1}{2}a^3 = \iint_S (\text{curl } \mathbf{F})\cdot \mathbf{n}\, dS.$

Ex. 16. *Verify Stoke's theorem for a vector field defined by $\mathbf{F} = (x^2 - y^2)\,\mathbf{i} + 2xy\,\mathbf{j}$ in the rectangular region in the xy-plane bounded by the lines $x = 0$, $x = a$, $y = 0$ and $y = b$.* (Kanpur 1975)

Sol. Proceed as in solved example 14.

Ex. 17. *Verify Stoke's theorem for the function*

$$F(x, y, z) = xy\,\mathbf{i} + xy^2\mathbf{j}$$

integrated round the square with vertices $(1, 0, 0)$, $(1, 1, 0)$, $(0, 1, 0)$ and $(0, 0, 0)$, where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors along x-axis and y-axis respectively. (Meerut 1979)

Sol. Proceed as in solved example 14.

We observe that the z-coordinate of each vertex of the square is zero. Therefore the square lies in the xy-plane. Its vertices in the xy-plane are the points $(0, 0)$, $(1, 0)$, $(1, 1)$ and $(0, 1)$. Also here $\mathbf{n} = \mathbf{k}$.

Ex. 18. *Verify Stoke's theorem for $\mathbf{F} = -y^3\mathbf{i} + x^3\mathbf{j}$, where S is the circular disc $x^2 + y^2 \le 1$, $z = 0$.*

Sol. The boundary C of S is a circle in xy-plane of radius one and centre at origin.

Suppose $x = \cos t$, $y = \sin t$, $z = 0$, $0 \le t < 2\pi$ are parametric equations of C. Then

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C (-y^3\mathbf{i} + x^3\mathbf{j})\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \oint_C (-y^3\,dx + x^3\,dy) = \int_{t=0}^{2\pi} \left\{-y^3\frac{dx}{dt} + x^3\frac{dy}{dt}\right\} dt$$

$$= \int_0^{2\pi} [-\sin^3 t\,(-\sin t) + \cos^3 t\,(\cos t)]\, dt$$

$$= \int_0^{2\pi} (\cos^4 t + \sin^4 t)\, dt = 4\int_0^{\pi/2} (\cos^4 t + \sin^4 t)\, dt$$

$$= 4\left\{\frac{3.1}{4.2}\frac{\pi}{2} + \frac{3.1}{4.2}\frac{\pi}{2}\right\} = \frac{3\pi}{2}.$$

Also $\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -y^3 & x^3 & 0 \end{vmatrix} = (3x^2 + 3y^2)\,\mathbf{k}.$

Here $\mathbf{n} = \mathbf{k}$ because the surface S is the xy-plane.

$\therefore (\nabla \times \mathbf{F})\cdot\mathbf{n} = (3x^2 + 3y^2)\,\mathbf{k}\cdot\mathbf{k} = 3(x^2 + y^2).$

$$\therefore \iint_S (\nabla \times \mathbf{F})\cdot \mathbf{n}\, dS = 3\iint_S (x^2 + y^2)\, dS$$

$$= 3\int_{\theta=0}^{2\pi}\int_{r=0}^{1} r^2\, r\, d\theta\, dr, \text{ changing to polars}$$

$$= \frac{3}{4}\int_0^{2\pi} d\theta = \frac{3}{4}(2\pi) = \frac{3\pi}{2}.$$

Thus $\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F})\cdot \mathbf{n}\, dS = \frac{3\pi}{2}.$

Hence the theorem is verified.

Ex. 19. *Evaluate by Stoke's theorem*

$$\oint_C (e^x\,dx + 2y\,dy - dz)$$

where C is the curve $x^2 + y^2 = 4$, $z = 2$. (Meerut 1969; Agra 72)

Sol. $\oint_C (e^x\,dx + 2y\,dy - dz)$

$$= \oint_C (e^x\mathbf{i} + 2y\,\mathbf{j} - \mathbf{k})\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \oint_C \mathbf{F}\cdot d\mathbf{r}, \text{ where } \mathbf{F} = e^x\mathbf{i} + 2y\,\mathbf{j} - \mathbf{k}.$$

Now curl $\mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^x & 2y & -1 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = \mathbf{0}.$

$\therefore$ By Stoke's theorem

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F})\cdot \mathbf{n}\, dS$$

$= 0,$ since curl $\mathbf{F} = 0$.

Ex. 20. *Evaluate by Stoke's theorem*

$$\oint_C (yz\,dx + xz\,dy + xy\,dz)$$

where C is the curve $x^2 + y^2 = 1,\ z = y^2$. (Andhra 1989, Kanpur 80)

Sol. Here $\mathbf{F} = yz\,\mathbf{i} + xz\,\mathbf{j} + xy\,\mathbf{k}$.

$$\therefore \quad \text{Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}$$

$$= (x - x)\,\mathbf{i} - (y - y)\,\mathbf{j} + (z - z)\,\mathbf{k} = 0.$$

$\therefore$ By Stoke's theorem

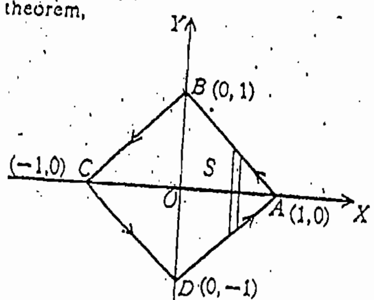
$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS$$

$= 0$, since curl $\mathbf{F} = 0$.

Ex. 21. *Evaluate* $\oint_C (xy\,dx + xy^2\,dy)$ *by Stoke's theorem where C is the positively oriented square with vertices* $(1, 0), (-1, 0), (0, 1)$ *and* $(0, -1)$.

Sol. We have $\oint_C (xy\,dx + xy^2\,dy) = \oint_C \mathbf{F} \cdot d\mathbf{r}$, where $\mathbf{F} = xy\,\mathbf{i} + xy^2\,\mathbf{j}$ and $\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j}$.

By Stoke's theorem,



$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS,$$

where S is any surface bounded by the square C and $\mathbf{n}$ is unit normal vector to the surface S.

Let us take the surface S as the area bounded by the square C. Since the square lies in the xy-plane, therefore $\mathbf{n} = \mathbf{k}$.

$$\text{Now curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & xy^2 & 0 \end{vmatrix} = (y^2 - x)\,\mathbf{k}.$$

$\therefore$ (Curl $\mathbf{F}$) $\cdot\, \mathbf{n} = (y^2 - x)\,\mathbf{k} \cdot \mathbf{k} = y^2 - x$.

$\therefore$ The given line integral

$$\oint_C (xy\,dx + xy^2\,dy) = \iint_S (y^2 - x)\, dS,$$ where S is the area of the square $ABCD$.

Equation of the st. line AB is $x + y = 1$ i.e., $y = 1 - x$ and the equation of the st. line BC is $-x + y = 1$ i.e., $y = x + 1$.

$$\therefore \iint_S (y^2 - x)\,dS = \int_{x=-1}^{0} \int_{y=-(x+1)}^{x+1} (y^2 - x)\,dx\,dy + \int_{x=0}^{1} \int_{y=-(1-x)}^{1-x} (y^2 - x)\,dx\,dy$$

$$= 2\int_{x=-1}^{0} \int_{y=0}^{x+1} (y^2 - x)\,dx\,dy + 2\int_{x=0}^{1} \int_{y=0}^{1-x} (y^2 - x)\,dx\,dy$$

[$\because$ the integrand $y^2 - x$ is an even function of y]

$$= 2\int_{x=-1}^{0} \left[\frac{y^3}{3} - xy\right]_{y=0}^{x+1} dx + 2\int_{x=0}^{1} \left[\frac{y^3}{3} - xy\right]_{y=0}^{1-x} dx$$

$$= 2\int_{-1}^{0} \left[\frac{1}{3}(x+1)^3 - x(x+1)\right] dx + 2\int_0^1 \left[\frac{(1-x)^3}{3} - x(1-x)\right] dx$$

$$= 2\left[\frac{1}{12}(x+1)^4 - \frac{x^3}{3} - \frac{x^2}{2}\right]_{-1}^{0} + 2\left[-\frac{1}{12}(1-x)^4 - \frac{x^2}{2} + \frac{x^3}{3}\right]_0^1$$

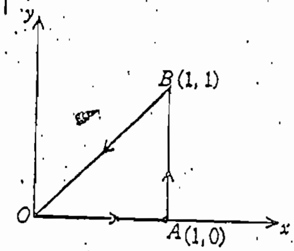
$$= 2\left[\frac{1}{12} - \frac{1}{3} + \frac{1}{2}\right] + 2\left[-\frac{1}{2} + \frac{1}{3} + \frac{1}{12}\right]$$

$$= \frac{2}{12} - \frac{2}{3} + 1 - 1 + \frac{2}{3} + \frac{2}{12} = \frac{4}{12} = \frac{1}{3}.$$

Ex. 22. (a) *Evaluate* $\oint_C \mathbf{F} \cdot d\mathbf{r}$ *by Stoke's theorem where* $\mathbf{F} = y^2\mathbf{i} + x^2\mathbf{j} - (x + z)\,\mathbf{k}$ *and C is the boundary of the triangle with vertices at* $(0, 0, 0), (1, 0, 0), (1, 1, 0)$.

Sol. We have

$$\text{Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 & x^2 & -(x+z) \end{vmatrix} = 0\mathbf{i} + \mathbf{j} + 2(x-y)\mathbf{k}.$$



Also we note that z co-ordinate of each vertex of the triangle is zero. Therefore the triangle lies in the xy-plane. So $\mathbf{n} = \mathbf{k}$.

$\therefore$ Curl $\mathbf{F} \cdot \mathbf{n} = [\mathbf{j} + 2(x-y)\mathbf{k}] \cdot \mathbf{k}$
$= 2(x-y).$

In the figure, we have only considered the x-y plane.

The equation of the line OB is $y = x$.

By Stoke's theorem

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS$$

$$= \int_{x=0}^{1} \int_{y=0}^{x} 2(x-y)\, dx\, dy = 2 \int_{x=0}^{1} \left[xy - \frac{y^2}{2} \right]_{y=0}^{x} dx$$

$$= 2 \int_0^1 \left[x^2 - \frac{x^2}{2} \right] dx = 2 \int_0^1 \frac{x^2}{2}\, dx = \int_0^1 x^2\, dx = \tfrac{1}{3}.$$

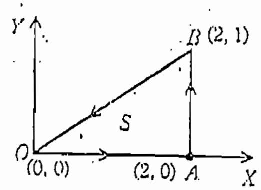
Ex. 22 (b). *If* $\mathbf{F} = (2x^2 + y^2)\mathbf{i} + (3y - 4x)\mathbf{j}$, *evaluate* $\oint_C \mathbf{F} \cdot d\mathbf{r}$ *where C is the boundary of the triangle with vertices at* (0, 0), (2, 0) *and* (2, 1).

(Andhra 1992)

Sol. We have

$$\text{Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x^2 + y^2 & 3y - 4x & 0 \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + (-4 - 2y)\mathbf{k}$$

$$= -2(2+y)\mathbf{k}$$



By Stoke's theorem,

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F}) \cdot \mathbf{n}\, dS,$$

where S is any surface bounded by the curve C and $\mathbf{n}$ is unit normal vector to the surface S.

Let us take the surface S as the area of the given triangle. Since the triangle lies in the xy-plane, therefore $\mathbf{n} = \mathbf{k}$.

$\therefore$ (curl $\mathbf{F}$) $\cdot \mathbf{n} = -2(2+y)\mathbf{k} \cdot \mathbf{k} = -2(2+y)$.

$$\therefore \oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S -2(2+y)\, dS,$$

where S is the area of the triangle OAB.

The equation of the st. line OB is $y = \frac{1}{2}x$.

$$\therefore \iint_S -2(2+y)\, dS$$

$$= \int_{x=0}^{2} \int_{y=0}^{x/2} -2(2+y)\, dx\, dy$$

[$\because$ for the region S, x varies from 0 to 2 and y varies from 0 to $x/2$]

$$= \int_{x=0}^{2} \left[-4y - y^2 \right]_{y=0}^{x/2} dx,$$

integrating with respect to y regarding x as constant

$$= \int_0^2 \left[-2x - \frac{x^2}{4} \right] dx = \left[-x^2 - \frac{x^3}{12} \right]_0^2$$

$$= -4 - \frac{8}{12} = -4 - \frac{2}{3} = -\frac{14}{3}.$$

$$\therefore \oint_C \mathbf{F} \cdot d\mathbf{r} = -\frac{14}{3}.$$

Ex. 23. *Evaluate by Stoke's theorem*

$$\oint_C (\sin z\, dx - \cos x\, dy + \sin y\, dz)$$

where C is the boundary of the rectangle

$$0 \le x \le \pi,\ 0 \le y \le 1,\ z = 3.$$

Sol. Here $\mathbf{F} = \sin z\, \mathbf{i} - \cos x\, \mathbf{j} + \sin y\, \mathbf{k}$.

$$\text{Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin z & -\cos x & \sin y \end{vmatrix} = \cos y\, \mathbf{i} + \cos z\, \mathbf{j} + \sin x\, \mathbf{k}.$$

Since the rectangle lies in the plane $z = 3$, therefore $\mathbf{n} = \mathbf{k}$.

$\therefore$ curl $\mathbf{F} \cdot \mathbf{n} = (\cos y\, \mathbf{i} + \cos z\, \mathbf{j} + \sin x\, \mathbf{k}) \cdot \mathbf{k} = \sin x.$

By Stoke's theorem

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F})\cdot \mathbf{n}\, dS$$

$$= \int_{y=0}^{1} \int_{x=0}^{\pi} \sin x\, dx\, dy = \int_{x=0}^{\pi} \sin x\, dx = 2.$$

Ex. 24. *By converting into a line integral evaluate*

$$\iint_S (\nabla \times \mathbf{A})\cdot \mathbf{n}\, dS, \text{ where } \mathbf{A} = (x - z)\,\mathbf{i} + (x^3 + yz)\,\mathbf{j} - 3xy^2\,\mathbf{k}$$

and S is the surface of the cone $z = 2 - \sqrt{(x^2 + y^2)}$ *above the xy-plane.* (Meerut 1974)

Sol. The xy-plane cuts the surface S of the cone $z = 2 - \sqrt{(x^2 + y^2)}$ in the circle C whose equations are $x^2 + y^2 = 4$, $z = 0$. Thus the boundary of the surface S is the circle C.

The surface S lies above the circle C. Let the parametric equations of the curve C be $x = 2\cos t$, $y = 2\sin t$, $z = 0$, $0 \le t < 2\pi$.

By Stoke's theorem, we have

$$\iint_S (\nabla \times \mathbf{A})\cdot \mathbf{n}\, dS = \oint_C \mathbf{A}\cdot d\mathbf{r}$$

$$= \int_C \left[(x - z)\,\mathbf{i} + (x^3 + yz)\,\mathbf{j} - 3xy^2\,\mathbf{k}\right]\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \int_C \left[(x - z)\,dx + (x^3 + yz)\,dy - 3xy^2 dz\right]$$

$$= \int_C (x\,dx + x^3 dy) \qquad [\because \text{ on } C, z = 0 \text{ and } dz = 0]$$

$$= \int_{t=0}^{2\pi} \left(x\frac{dx}{dt} + x^3 \frac{dy}{dt}\right) dt$$

$$= \int_0^{2\pi} [2\cos t\cdot(-2\sin t) + 8\cos^3 t\cdot 2\cos t]\, dt$$

$$= -2\int_0^{2\pi} 2\sin t\cos t\, dt + 16\int_0^{2\pi} \cos^4 t\, dt$$

$$= -2\int_0^{2\pi} \sin 2t\, dt + 16\cdot 4\int_0^{\pi/2} \cos^4 t\, dt$$

$$= -2\left[-\frac{\cos 2t}{2}\right]_0^{2\pi} + 64\cdot\frac{3\cdot 1}{4\cdot 2}\cdot\frac{\pi}{2}$$

$$= 0 + 12\pi = 12\pi.$$

Ex. 25. *By converting into a line integral evaluate*

$$\iint_S (\nabla \times \mathbf{F})\cdot \mathbf{n}\, dS$$

where $\mathbf{F} = (x^2 + y - 4)\,\mathbf{i} + 3xy\,\mathbf{j} + (2xy + z^2)\,\mathbf{k}$ *and S is the surface of the paraboloid* $z = 4 - (x^2 + y^2)$ *above the xy-plane.*

Sol. The xy-plane cuts the surface S of the paraboloid $z = 4 - (x^2 + y^2)$ in the circle C whose equations are $x^2 + y^2 = 4$, $z = 0$. Thus the boundary of the surface S is the circle C and the surface S lies above the circle C. Let the parametric equations of the curve C be $x = 2\cos t$, $y = 2\sin t$, $z = 0$, $0 \le t < 2\pi$.

By Stoke's theorem, we have

$$\iint_S (\nabla \times \mathbf{F})\cdot \mathbf{n}\, dS = \oint_C \mathbf{F}\cdot d\mathbf{r}$$

$$= \int_C \left[(x^2 + y - 4)\,\mathbf{i} + 3xy\,\mathbf{j} + (2xy + z^2)\,\mathbf{k}\right]\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \int_C \left[(x^2 + y - 4)\,dx + 3xy\,dy + (2xz + z^2)\,dz\right]$$

$$= \int_C \left[(x^2 + y - 4)\,dx + 3xy\,dy\right], \text{ since on } C, z = 0 \text{ and } dz = 0$$

$$= \int_{t=0}^{2\pi} \left[(x^2 + y - 4)\frac{dx}{dt} + 3xy\frac{dy}{dt}\right] dt$$

$$= \int_0^{2\pi} [(4\cos^2 t + 2\sin t - 4)(-2\sin t) + 3\cdot 2\cos t\cdot 2\sin t\cdot 2\cos t]\, dt$$

$$= -8\int_0^{2\pi} \cos^2 t\sin t\, dt - 4\int_0^{2\pi} \sin^2 t\, dt + 8\int_0^{2\pi} \sin t\, dt + 24\int_0^{2\pi} \cos^2 t\sin t\, dt$$

$$= 8\left[\frac{\cos^3 t}{3}\right]_0^{2\pi} - 4\cdot 2\cdot 2\cdot\int_0^{\pi/2} \sin^2 t\, dt + 8\left[-\cos t\right]_0^{2\pi} - 24\left[\frac{\cos^3 t}{3}\right]_0^{2\pi}$$

$$= 8\cdot 0 - 16\cdot\frac{1}{2}\cdot\frac{\pi}{2} + 8\cdot 0 - \frac{24}{3}\cdot 0 = -4\pi.$$

Ex. 26. *Evaluate* $\iint_S (\nabla \times \mathbf{F})\cdot \mathbf{n}\, dS$, *where* $\mathbf{F} = (y - z + 2)\,\mathbf{i} + (yz + 4)\,\mathbf{j} - xz\,\mathbf{k}$ *and S is the surface of the cube* $x = y = z = 0$, $x = y = z = 2$ *above the xy-plane.*

Sol. The xy-plane cuts the surface of the cube in a square. Thus the curve C bounding the surface S is the square, say $OABD$, in the xy-plane whose vertices in the xy-plane are the points

$O(0,0), A(2,0), B(2,2), D(0,2)$.

[Draw figure as in solved example 14].

By Stoke's theorem, we have

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \oint_C \mathbf{F} \cdot d\mathbf{r}$$

$$= \oint_C \left[(y - z + 2)\,\mathbf{i} + (yz + 4)\,\mathbf{j} - xz\,\mathbf{k}\right] \cdot (dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \int_C \left[(y - z + 2)\,dx + (yz + 4)\,dy - xz\,dz\right]$$

$$= \int_C \left[(y + 2)\,dx + 4\,dy\right] \qquad [\because \text{ on } C, z = 0 \text{ and } dz = 0]$$

$$= \int_{OA} \left[(y + 2)\,dx + 4\,dy\right] + \int_{AB} \left[(y + 2)\,dx + 4\,dy\right] + \int_{BD} \left[(y + 2)\,dx + 4\,dy\right] + \int_{DO} \left[(y + 2)\,dx + 4\,dy\right]$$

$$= \int_0^2 2\,dx + \int_0^2 4\,dy + \int_2^0 4\,dx + \int_2^0 4\,dy$$

[$\because$ on OA, $y = 0$, $dy = 0$ and x varies from 0 to 2;
on AB, $x = 2$, $dx = 0$ and y varies from 0 to 2;
on BD, $y = 2$, $dy = 0$, x varies from 2 to 0;
and on DO, $x = 0$, $dx = 0$, y varies from 2 to 0]

$$= 2\left[x\right]_0^2 + 4\left[y\right]_0^2 + 4\left[x\right]_2^0 + 4\left[y\right]_2^0$$

$$= 4 + 8 - 8 - 8 = -4.$$

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Ex. 27. *Apply Stoke's theorem to prove that*

$$\int_C (y\,dx + z\,dy + x\,dz) = -2\sqrt{2}\pi a^2$$

where C is the curve given by

$$x^2 + y^2 + z^2 - 2ax - 2ay = 0,\ x + y = 2a$$

and begins at the point $(2a, 0, 0)$ and goes at first below the z-plane.

(Meerut 1982; Agra 69)

Solution. The centre of the sphere $x^2 + y^2 + z^2 - 2ax - 2ay = 0$ is the point $(a, a, 0)$. Since the plane $x + y = 2a$ passes through the point $(a, a, 0)$, therefore the circle C is great circle of this sphere.

$\therefore$ Radius of the circle C

$= \text{radius of the sphere} = \sqrt{(a^2 + a^2)} = a\sqrt{2}$.

Now $\displaystyle\int_C (y\,dx + z\,dy + x\,dz) = \int_C (y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}) \cdot d\mathbf{r}$

$$= \iint_S \left[\operatorname{curl}(y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k})\right] \cdot \mathbf{n}\, dS,$$

where S is any surface of which circle C is boundary [Stoke's theorem].

$$\text{Now curl}\,(y\mathbf{i} + z\mathbf{j} + x\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix}$$

$$= -\mathbf{i} - \mathbf{j} - \mathbf{k} = -(\mathbf{i} + \mathbf{j} + \mathbf{k}).$$

Let us take S as the surface of the plane $x + y = 2a$ bounded by the circle C. Then a vector normal to S is grad $(x + y) = \mathbf{i} + \mathbf{j}$.

$\therefore$ $\mathbf{n}$ = unit normal to $S = \dfrac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$.

$$\therefore \int_C (y\,dx + z\,dy + x\,dz)$$

$$= \iint_S -(\mathbf{i} + \mathbf{j} + \mathbf{k}) \cdot \left(\frac{1}{\sqrt{2}}\mathbf{i} + \frac{1}{\sqrt{2}}\mathbf{j}\right) dS$$

$$= -\frac{2}{\sqrt{2}} \iint_S dS = -\frac{2}{\sqrt{2}}\ (\text{area of the circle of radius } a\sqrt{2})$$

$$= -\sqrt{2}\,(2\pi a^2).$$

Ex. 28. *Use Stoke's theorem to evaluate* $\displaystyle\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$, *where* $\mathbf{F} = y\,\mathbf{i} + (x - 2xz)\,\mathbf{j} - xy\,\mathbf{k}$ *and S is the surface of sphere* $x^2 + y^2 + z^2 = a^2$, *above the xy-plane.*

Sol. The boundary C of the surface S is the circle $x^2 + y^2 = a^2$, $z = 0$. Suppose $x = a\cos t$, $y = a\sin t$, $z = 0$, $0 \le t < 2\pi$ are parametric equations of C. By Stoke's theorem, we have

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS$$

$$= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \left[y\mathbf{i} + (x - 2xz)\,\mathbf{j} - xy\,\mathbf{k}\right] \cdot (\mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz)$$

$$= \int_C \left[y\,dx + (x - 2xz)\,dy - xy\,dz\right]$$

$$= \int_C (y\,dx + x\,dy) \qquad [\because \text{ on } C, z = 0 \text{ and } dz = 0]$$

$$= \int_0^{2\pi} \left(y\frac{dx}{dt} + x\frac{dy}{dt} \right) dt$$

$$= \int_0^{2\pi} [a \sin t\,(-a \sin t) + a \cos t\,(a \cos t)]\,dt$$

$$= a^2 \int_0^{2\pi} (\cos^2 t - \sin^2 t)\,dt = a^2 \int_0^{2\pi} \cos 2t\,dt = a^2 \left[\frac{\sin 2t}{2}\right]_0^{2\pi} = 0.$$

Ex. 29. *Evaluate the surface integral* $\iint_S$ *curl* $\mathbf{F}\cdot\mathbf{n}\,dS$ *by transforming it into a line integral, S being that part of the surface of the paraboloid* $z = 1 - x^2 - y^2$ *for which* $z \geq 0$, *and* $\mathbf{F} = y\mathbf{i} + z\mathbf{j} + x\mathbf{k}$.

(Bombay 1979)

Sol. The boundary C of the surface S is the circle $x^2 + y^2 = 1$, $z = 0$. Suppose $x = \cos t, y = \sin t, z = 0, 0 \leq t < 2\pi$ are parametric equations of C. By Stoke's theorem, we have

$$\iint_S \text{curl } \mathbf{F}\cdot\mathbf{n}\,dS = \int_C \mathbf{F}\cdot d\mathbf{r}$$

$$= \int_C (y\mathbf{i} + z\mathbf{j} + x\mathbf{k})\cdot(\mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz) = \int_C y\,dx + z\,dy + x\,dz$$

$$= \int_C y\,dx \qquad [\because \text{ on } C, z = 0 \text{ and } dz = 0]$$

$$= \int_0^{2\pi} y\frac{dx}{dt}\,dt = \int_0^{2\pi} \sin t\,(-\sin t)\,dt = -\int_0^{2\pi} \sin^2 t\,dt$$

$$= -4\int_0^{\pi/2} \sin^2 t\,dt = -4 \times \tfrac{1}{2} \times \frac{\pi}{2} = -\pi.$$

Ex. 30. *If* $\mathbf{F} = (y^2 + z^2 - x^2)\mathbf{i} + (z^2 + x^2 - y^2)\mathbf{j} + (x^2 + y^2 - z^2)\mathbf{k}$, *evaluate* $\iint$ *curl* $\mathbf{F}\cdot\mathbf{n}\,dS$ *taken over the portion of the surface* $x^2 + y^2 + z^2 - 2ax + az = 0$ *above the plane* $z = 0$, *and verify Stoke's theorem.*

Sol. The surface $x^2 + y^2 + z^2 - 2ax + az = 0$ meets the plane $z = 0$ in the circle C given by $x^2 + y^2 - 2ax = 0$, $z = 0$. The polar equation of the circle C lying in the xy-plane is $r = 2a\cos\theta$, $0 \leq \theta < \pi$. Also the equation $x^2 + y^2 - 2ax = 0$ can be written as

$(x - a)^2 + y^2 = a^2$. Therefore the parametric equations of the circle C can be taken as

$$x = a + a\cos t, y = a \sin t, z = 0, 0 \leq t < 2\pi.$$

Let S denote the portion of the surface $x^2 + y^2 + z^2 - 2ax + az = 0$ lying above the plane $z = 0$ and S_1 denote the plane region bounded by the circle C. By an application of divergence theorem, we have

$$\iint_S \text{curl } \mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_1} \text{curl } \mathbf{F}\cdot\mathbf{k}\,dS.$$

Now
$$\text{curl } \mathbf{F}\cdot\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 + z^2 - x^2 & z^2 + x^2 - y^2 & x^2 + y^2 - z^2 \end{vmatrix}\cdot\mathbf{k}$$

$$= \left[\frac{\partial}{\partial x}(z^2 + x^2 - y^2) - \frac{\partial}{\partial y}(y^2 + z^2 - x^2)\right]\mathbf{k}\cdot\mathbf{k}$$

$$= 2(x - y). \qquad [\because \mathbf{i}\cdot\mathbf{k} = \mathbf{j}\cdot\mathbf{k} = 0]$$

$$\therefore \iint_S \text{curl } \mathbf{F}\cdot\mathbf{n}\,dS = \iint_{S_1} \text{curl } \mathbf{F}\cdot\mathbf{k}\,dS = \iint_{S_1} 2(x - y)\,dS$$

$$= 2\int_{\theta=0}^{\pi} \int_{r=0}^{2a\cos\theta} (r\cos\theta - r\sin\theta)\,r\,d\theta\,dr, \text{ changing to polars.}$$

$$= 2\int_{\theta=0}^{\pi} (\cos\theta - \sin\theta)\left[\frac{r^3}{3}\right]_0^{2a\cos\theta} d\theta$$

$$= 2 \times \frac{8a^3}{3}\int_0^{\pi} (\cos\theta - \sin\theta)\cos^3\theta\,d\theta$$

$$= \frac{16a^3}{3}\int_0^{\pi} \cos^4\theta\,d\theta \qquad \left[\because \int_0^{\pi} \cos^3\theta\sin\theta\,d\theta = 0\right]$$

$$= 2 \times \frac{16a^3}{3}\int_0^{\pi/2} \cos^4\theta\,d\theta$$

$$= 2 \times \frac{16a^3}{3} \times \frac{3 \times 1}{4 \times 2} \times \frac{\pi}{2} = 2\pi a^3. \qquad \ldots(1)$$

Also
$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (y^2 + z^2 - x^2)\,dx + (z^2 + x^2 - y^2)\,dy + (x^2 + y^2 - z^2)\,dz$$

$$= \int_C (y^2 - x^2)\,dx + (x^2 - y^2)\,dy \qquad [\because \text{ on } C, z = 0 \text{ and } dz = 0]$$

$$= \int_0^{2\pi} (x^2 - y^2)\left(\frac{dy}{dt} - \frac{dx}{dt}\right) dt$$

$$= \int_0^{2\pi} [(a + a\cos t)^2 - a^2 \sin^2 t]\,(a\cos t + a\sin t)\,dt$$

$$= a^3 \int_0^{2\pi} (1 + \cos^2 t + 2\cos t - \sin^2 t)(\cos t + \sin t)\,dt$$

$$= a^3 \int_0^{2\pi} 2\cos^2 t\,dt; \text{ the other integrals vanish}$$

$$= 2a^3 \times 4 \int_0^{\pi/2} \cos^2 t\,dt = 8a^3 \times \tfrac{1}{2} \times \frac{\pi}{2} = 2\pi a^3 \qquad \ldots (2)$$

Comparing (1) and (2), we see that

$$\iint_S \text{curl}\,\mathbf{F} \cdot \mathbf{n}\,dS = \int_C \mathbf{F} \cdot d\mathbf{r}.$$

Hence Stoke's theorem is verified.

Ex. 31. *Show that*

$$\iint_S \phi\,\text{curl}\,\mathbf{F} \cdot d\mathbf{S} = \int_C \phi\mathbf{F} \cdot d\mathbf{r} - \iint_S (\text{grad}\,\phi \times \mathbf{F}) \cdot d\mathbf{S}.$$

Sol. Here C is the closed curve bounding the surface S. Applying Stoke's theorem to the vector $\phi\mathbf{F}$, we have

$$\oint_C (\phi\mathbf{F}) \cdot d\mathbf{r} = \iint_S \text{curl}\,(\phi\mathbf{F}) \cdot \mathbf{n}\,dS$$

$$= \iint_S [\nabla \times (\phi\mathbf{F})] \cdot d\mathbf{S}$$

$$= \iint_S [(\text{grad}\,\phi) \times \mathbf{F} + \phi\,\text{curl}\,\mathbf{F}] \cdot d\mathbf{S}$$

$$[\because \text{curl}\,(\phi\mathbf{A}) = (\text{grad}\,\phi) \times \mathbf{A} + \phi\,\text{curl}\,\mathbf{A}]$$

$$= \iint_S (\text{grad}\,\phi \times \mathbf{F}) \cdot d\mathbf{S} + \iint_S \phi\,\text{curl}\,\mathbf{F} \cdot d\mathbf{S}.$$

Hence by transposition, we have

$$\iint_S \phi\,\text{curl}\,\mathbf{F} \cdot d\mathbf{S} = \int_C \phi\,\mathbf{F} \cdot d\mathbf{r} - \iint_S (\text{grad}\,\phi \times \mathbf{F}) \cdot d\mathbf{S}.$$

Ex. 32. *If* $\mathbf{f} = \nabla\phi$ *and* $\mathbf{g} = \nabla\psi$ *are two vector point functions, such that*

$$\nabla^2\phi = 0,\ \nabla^2\psi = 0$$

show that

$$\iint_S (\mathbf{g} \cdot \nabla)\,\mathbf{f} \cdot d\mathbf{S} = \int_C (\mathbf{f} \times \mathbf{g}) \cdot d\mathbf{r} + \iint_S (\mathbf{f} \cdot \nabla)\mathbf{g} \cdot d\mathbf{S}.$$

Sol. Here C is the closed curve bounding the surface S. Applying Stoke's theorem to the vector $\mathbf{f} \times \mathbf{g}$, we have

$$\oint_C (\mathbf{f} \times \mathbf{g}) \cdot d\mathbf{r} = \iint_S \nabla \times (\mathbf{f} \times \mathbf{g}) \cdot \mathbf{n}\,dS$$

$$= \iint_S \text{curl}\,(\mathbf{f} \times \mathbf{g}) \cdot d\mathbf{S}$$

$$= \iint_S \left[(\mathbf{g} \cdot \nabla)\,\mathbf{f} - \mathbf{g}\,\text{div}\,\mathbf{f} - (\mathbf{f} \cdot \nabla)\,\mathbf{g} + \mathbf{f}\,\text{div}\,\mathbf{g}\right] \cdot d\mathbf{S}$$

$$= \iint_S \left[(\mathbf{g} \cdot \nabla)\,\mathbf{f} - \mathbf{g}\,\text{div}\,\nabla\phi - (\mathbf{f} \cdot \nabla)\,\mathbf{g} + \mathbf{f}\,\text{div}\,\nabla\psi\right] \cdot d\mathbf{S}$$

$$[\because \mathbf{f} = \nabla\phi \text{ and } \mathbf{g} = \nabla\psi]$$

$$= \iint_S [(\mathbf{g} \cdot \nabla)\,\mathbf{f} - 0\,\mathbf{g} - (\mathbf{f} \cdot \nabla)\,\mathbf{g} + 0\,\mathbf{f}] \cdot d\mathbf{S}$$

$[\because \text{div}\,\nabla\phi = \nabla \cdot \nabla\phi = \nabla^2\phi = 0$, given and similarly $\text{div}\,\nabla\psi = \nabla^2\psi = 0$, given$]$

$$= \iint_S (\mathbf{g} \cdot \nabla)\,\mathbf{f} \cdot d\mathbf{S} - \iint_S (\mathbf{f} \cdot \nabla)\,\mathbf{g} \cdot d\mathbf{S}.$$

Hence by transposition, we have

$$\iint_S (\mathbf{g} \cdot \nabla)\,\mathbf{f} \cdot d\mathbf{S} = \int_C (\mathbf{f} \times \mathbf{g}) \cdot d\mathbf{r} + \iint_S (\mathbf{f} \cdot \nabla)\,\mathbf{g} \cdot d\mathbf{S}.$$

Ex. 33. *Prove that a necessary and sufficient condition that* $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ *for every closed curve* C *lying in a simply connected region* R *is that* $\nabla \times \mathbf{F} = 0$ *identically.*

Sol. Sufficiency. Suppose R is simply connected and $\text{curl}\,\mathbf{F} = 0$ everywhere in R. Let C be any closed path in R. Since R is simply connected, therefore we can find a surface S in R having C as its boundary. Therefore by Stoke's theorem

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\text{curl}\,\mathbf{F}) \cdot \mathbf{n}\,dS = 0.$$

Necessity. Suppose $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C and assume that $\nabla \times \mathbf{F} \neq 0$ at some point A.

Then taking $\nabla \times \mathbf{F}$ as continuous, there must exist a region with A as an interior point, where $\nabla \times \mathbf{F} \neq 0$. Let S be a surface contained in this region whose normal $\mathbf{n}$ at each point is in the same direction as

$\nabla \times \mathbf{F}$, i.e. $\nabla \times \mathbf{F} = \lambda \mathbf{n}$ where λ is a positive constant. Let C be the boundary of S. Then by Stoke's theorem

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n}\, dS = \iint_S \lambda\, \mathbf{n} \cdot \mathbf{n}\, dS$$
$$= \lambda S > 0.$$

This contradicts the hypothesis that $\oint_C \mathbf{F} \cdot d\mathbf{r} = 0$ for every closed path C. Therefore we must have $\nabla \times \mathbf{F} = 0$ everywhere in R.

§ 10. Line integrals Independent of path.

Let $\mathbf{F}(x, y, z) = f(x, y, z)\,\mathbf{i} + g(x, y, z)\,\mathbf{j} + h(x, y, z)\,\mathbf{k}$ be a vector point function defined and continuous in a region R of space. Let P and Q be two points in R and let C be a path joining P to Q. Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int (f dx + g dy + h dz) \qquad \dots (1)$$

is called the line integral of $\mathbf{F}$ along C. In general the value of this line integral depends not only on the end points P and Q of the path C but also on C.

In other words, if we integrate from P to Q along different paths, we shall, in general, get different values of the integral. *The line integral (1) is said to be independent of path in R, if for every pair of end points P and Q in R the value of the integral is the same for all paths C in R starting from P and ending at Q.*

In this case the value of this line integral will depend on the choice of P and Q and not on the choice of the path joining P to Q.

Definition. *The experssion $f\,dx + g\,dy + h\,dz$ is said to be an exact differential if there exists a single valued scalar point function $\phi(x, y, z)$, having continuous first partial derivatives such that*

$$d\phi = f\,dx + g\,dy + h\,dz.$$

It can be easily seen that $f\,dx + g\,dy + h\,dz$ is an exact differential if and only if the vector function

$$\mathbf{F} = f\,\mathbf{i} + g\,\mathbf{j} + h\,\mathbf{k}$$

is the gradient of a single valued scalar function $\phi(x, y, z)$.

Because $\mathbf{F} = \text{grad}\,\phi$

if, and only if $f\,\mathbf{i} + g\,\mathbf{j} + h\,\mathbf{k} = \dfrac{\partial \phi}{\partial x}\mathbf{i} + \dfrac{\partial \phi}{\partial y}\mathbf{j} + \dfrac{\partial \phi}{\partial z}\mathbf{k}$

if, and only if $f = \dfrac{\partial \phi}{\partial x}$, $g = \dfrac{\partial \phi}{\partial y}$, $h = \dfrac{\partial \phi}{\partial z}$

if, and only if $f\,dx + g\,dy + h\,dz = \dfrac{\partial \phi}{\partial x}dx + \dfrac{\partial \phi}{\partial y}dy + \dfrac{\partial \phi}{\partial z}dz$

if, and only if $f\,dx + g\,dy + h\,dz = d\phi$.

Thus $\mathbf{F} = \text{grad}\,\phi$ if, and only if $f\,dx + g\,dy + h\,dz$ is an exact differential $d\phi$.

Theorem 1. *Let $f(x, y, z)$, $g(x, y, z)$ and $h(x, y, z)$ be continuous in a region R of space. Then the line integral*

$$\int (f\,dx + g\,dy + h\,dz)$$

is independent of path in R if and only if the differential form under the integral sign is exact in R. (Meerut 1968)

Or

Let $\mathbf{F}(x, y, z)$ be continuous in region R of space. Then the line integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

is independent of the path C in R joining P and Q if and only if $\mathbf{F} = \text{grad}\,\phi$ where $\phi(x, y, z)$ is a single-valued scalar function having continuous first partial derivatives in R. (Kerala 1975)

Proof. Suppose $\mathbf{F} = \text{grad}\,\phi$ in R. Let P and Q be any two points in R and let C be any path from P to Q in R.

Then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \nabla\phi \cdot d\mathbf{r}$$
$$= \int_C \left(\frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}\right) \cdot (dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$
$$= \int_C \left(\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz\right) = \int_C d\phi$$
$$= \int_P^Q d\phi = [\phi]_P^Q = \phi(Q) - \phi(P).$$

Thus the line integral depends only on points P and Q and not the path joining them. This is true of course only if $\phi(x, y, z)$ is single valued at all points P and Q.

Conversely suppose the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of the path C joining any two points P and Q in R. Let P be a fixed point (x_0, y_0, z_0) in R and let Q be any point (x, y, z) in R.

Let

$$\phi(x, y, z) = \int_{(x_0, y_0, z_0)}^{(x, y, z)} \mathbf{F} \cdot d\mathbf{r} = \int_{(x_0, y_0, z_0)}^{(x, y, z)} \left(\mathbf{F} \cdot \frac{d\mathbf{r}}{ds}\right) ds.$$

Differentiating both sides with respect to s, we get

$$\frac{d\phi}{ds} = \mathbf{F} \cdot \frac{d\mathbf{r}}{ds}.$$

But $$\frac{d\phi}{ds} = \frac{\partial \phi}{\partial x}\frac{dx}{ds} + \frac{\partial \phi}{\partial y}\frac{dy}{ds} + \frac{\partial \phi}{\partial z}\frac{dz}{ds}$$

$$= \left(\frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}\right) \cdot \left(\frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j} + \frac{dz}{ds}\mathbf{k}\right)$$

$$= \nabla \phi \cdot \frac{d\mathbf{r}}{ds}.$$

$$\therefore \quad \mathbf{F} \cdot \frac{d\mathbf{r}}{ds} = \nabla \phi \cdot \frac{d\mathbf{r}}{ds} \quad \text{or} \quad (\nabla \phi - \mathbf{F}) \cdot \frac{d\mathbf{r}}{ds} = 0.$$

Now this result is true irrespective of the path joining P to Q *i.e.* this result is true irrespective of the direction of $\frac{d\mathbf{r}}{ds}$ which is tangent vector to C. Therefore we must have

$$\nabla\phi - \mathbf{F} = 0$$

i.e., $$\nabla \phi = \mathbf{F}.$$

This completes the proof of the theorem.

Definition. *A vector field $\mathbf{F}(x, y, z)$ defined and continuous in a region R of space is said to be a conservative vector field if the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ is independent of the path C in R joining P and Q where P and Q are any two points in R.*

By theorem 1, vector field $\mathbf{F}(x, y, z)$ is conservative if and only if $\mathbf{F} = \nabla \phi$ where $\phi(x, y, z)$ is a single valued scalar function having continuous first partial derivatives in R. The function $\phi(x, y, z)$ is called the scalar potential *of the vector field* $\mathbf{F}$.

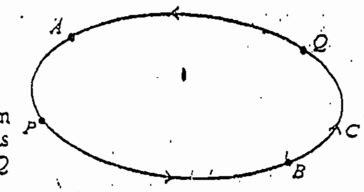
Theorem 2. *Let $\mathbf{F}(x, y, z)$ be a vector function defined and continuous in a region R of space. Then the line integral $\int_P^Q \mathbf{F} \cdot d\mathbf{r}$ is independent of the path joining any two points P and Q in R if and only if $\oint \mathbf{F} \cdot d\mathbf{r} = 0$ for every simple closed path in R.*

Proof. Let C be any simple closed path in R and let the line integral be independent of path in R. Take two points P and Q on C and subdivide C into two arcs PBQ and QAP. Then

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \int_{PBQAP} \mathbf{F} \cdot d\mathbf{r}$$

$$= \int_{PBQ} \mathbf{F} \cdot d\mathbf{r} + \int_{QAP} \mathbf{F} \cdot d\mathbf{r}$$

$$= \int_{PBQ} \mathbf{F} \cdot d\mathbf{r} - \int_{PAQ} \mathbf{F} \cdot d\mathbf{r}$$



$= 0$, since the integral from P to Q along a path through B is equal to the integral from P to Q along a path through A.

Conversely suppose that the integral under consideration is zero on every simple closed path in R. Let P and Q be any two points in R which join P to Q and do not cross. Then

$$\oint_{PBQAP} \mathbf{F} \cdot d\mathbf{r} = \int_{PBQ} \mathbf{F} \cdot d\mathbf{r} + \int_{QAP} \mathbf{F} \cdot d\mathbf{r} = \int_{PBQ} \mathbf{F} \cdot d\mathbf{r} - \int_{PAQ} \mathbf{F} \cdot d\mathbf{r}.$$

But as given, we have $\oint_{PBQAP} \mathbf{F} \cdot d\mathbf{r} = 0.$

$$\therefore \int_{PBQ} \mathbf{F} \cdot d\mathbf{r} - \int_{PAQ} \mathbf{F} \cdot d\mathbf{r} = 0$$

or $$\int_{PBQ} \mathbf{F} \cdot d\mathbf{r} = \int_{PAQ} \mathbf{F} \cdot d\mathbf{r}.$$

This completes the proof of the theorem.

Theorem 3. *Let $\mathbf{F}(x, y, z) = f\mathbf{i} + g\mathbf{j} + h\mathbf{k}$ be a continuous vector function having continuous first partial derivatives in a region R of space. If $\int f dx + g dy + h dz$ is independent of path in R and consequently $f dx + g dy + h dz$ is an exact differential in R then curl $\mathbf{F} = 0$ everywhere in R. Conversely, if R is simply connected and curl $\mathbf{F} = 0$ everywhere in R, then $f dx + g dy + h dz$ is an exact differential in R or $\int f dx + g dy + h dz$ is independent of path in R.* (Allahabad 1979)

Proof. Suppose $\int (f\,dx + g\,dy + h\,dz)$ is independent of path in R. Then $f dx + g dy + h dz$ is an exact differential in R. Therefore

$$\mathbf{F} = f\mathbf{i} + g\mathbf{j} + h\mathbf{k} = \text{grad}\,\phi.$$

$$\therefore \quad \text{curl}\,\mathbf{F} = \text{curl}\,(\text{grad}\,\phi) = 0.$$

Conversely suppose R is simply connected and curl $\mathbf{F} = 0$ everywhere in R. Let C be any simple closed path in R. Since R is simply connected, therefore we can find a surface S in R having C as its boundary. Therefore by Stoke's theorem

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F})\cdot \mathbf{n}\, dS = 0.$$

Thus $\oint_C \mathbf{F}\cdot d\mathbf{r}$ is zero for every simple closed path C in R.

Therefore $\int \mathbf{F}\cdot d\mathbf{r}$ is independent of path in R.

Therefore $\mathbf{F} = \nabla\phi$ and consequently $f dx + g dy + h dz$ is an exact differential $d\phi$.

Note. The assumption that R be simply connected is essential and cannot be omitted. It is obvious from the following example.

Example. Let $\mathbf{F} = -\dfrac{y}{x^2+y^2}\mathbf{i} + \dfrac{x}{x^2+y^2}\mathbf{j}$.

Here $\mathbf{F}$ is not defined at origin. In every region R of the xy-plane not containing the origin, we have

$$\text{Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -\dfrac{y}{x^2+y^2} & \dfrac{y}{x^2+y^2} & 0 \end{vmatrix}$$

$$= 0\mathbf{i} + 0\mathbf{j} + \left\{\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{y}{x^2+y^2}\right)\right\}\mathbf{k}$$

$$= \left\{\frac{x^2+y^2-2x^2}{(x^2+y^2)^2} + \frac{x^2+y^2-2y^2}{(x^2+y^2)^2}\right\}\mathbf{k} = 0\,\mathbf{k} = 0.$$

Suppose R is simply connected. For example let R be the region enclosed by a simple closed curve C not enclosing the origin. Then

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \oint_C \left(-\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy\right)$$

$$= \iint_R \left[\frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) - \frac{\partial}{\partial y}\left(-\frac{y}{x^2+y^2}\right)\right] dx\, dy,$$

$= 0.$ by Green's theorem in plane

Suppose R is not simply connected. Let R be the region of the xy-plane contained between concentric circles of radii $\frac{1}{2}$ and $\frac{3}{2}$ and having centre at origin. Obviously R is not simply connected. We have $z = 0$, everywhere in R. Let C be a closed curve in R. The parametric equations of C can be taken as $x = \cos t, y = \sin t, z = 0, 0 \le t < 2\pi$.

$$\text{We have } \oint_C \mathbf{F}\cdot d\mathbf{r} = \int\left(-\frac{y}{x^2+y^2}dx + \frac{y}{x^2+y^2}dy\right)$$

$$= \int_{t=0}^{2\pi}\left[-\frac{\sin t}{\cos^2 t+\sin^2 t}\frac{dx}{dt} + \frac{\cos t}{\cos^2 t+\sin^2 t}\frac{dy}{dt}\right]dt$$

$$= \int_0^{2\pi}(\sin^2 t + \cos^2 t)\, dt = 2\pi.$$

Thus we see that $\oint_C \mathbf{F}\cdot d\mathbf{r} \neq 0$.

Definition. Irrotational vector field: *A vector field* $\mathbf{F}$ *is said to be irrotational if curl* $\mathbf{F} = 0$.

We see that an irrotational field $\mathbf{F}$ is characterised by any one of the three conditions :

(i) $\mathbf{F} = \nabla\phi$,

(ii) $\nabla\times\mathbf{F} = 0$,

(iii) $\oint_C \mathbf{F}\cdot d\mathbf{r} = 0$ for every closed path.

Any one of these conditions implies the other two.

Solved Examples

Ex. 1. *Are the following forms exact ?*

(i) $x dx - y dy + z dz$. (ii) $e^y dx + e^x dy + e^z dz$.

(iii) $yz dx + xz dy + xy dz$. (iv) $y^2z^3 dx + 2xyz^3 dy + 3xy^2z^2 dz$.

Sol. (i) We have

$x dx - y dy + z dz = (x\mathbf{i} - y\mathbf{j} + z\mathbf{k})\cdot(dx\mathbf{i} + dy\mathbf{j} + dz\mathbf{k})$

$= \mathbf{F}\cdot d\mathbf{r}$, where

$\mathbf{F} = x\mathbf{i} - y\mathbf{j} + z\mathbf{k}$.

We have $\text{Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & -y & z \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = 0.$

$\therefore$ the given form is exact.

(ii) Here $\mathbf{F} = e^y\mathbf{i} + e^x\mathbf{j} + e^z\mathbf{k}$. We have

$$\text{Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ e^y & e^x & e^z \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + (e^x - e^y)\,\mathbf{k}.$$

Since curl $\mathbf{F} \neq 0$, therefore the given form is not exact.

(iii) Here $\mathbf{F} = yz\mathbf{i} + xz\mathbf{j} + xy\mathbf{k}$. We have

$$\text{Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix} = (x - x)\mathbf{i} - (y - y)\mathbf{j} + (z - z)\mathbf{k} = 0.$$

Since curl $\mathbf{F} = 0$, therefore the given form is exact.

(iv) Here $\mathbf{F} = y^2z^3\,\mathbf{i} + 2xyz^3\,\mathbf{j} + 3xy^2z^2\,\mathbf{k}$. We have

$$\text{Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2z^3 & 2xyz^3 & 3xy^2z^2 \end{vmatrix}$$

$$= (6xyz^2 - 6xyz^2)\,\mathbf{i} - (3y^2z^2 - 3y^2z^2)\,\mathbf{j} + (2yz^3 - 2yz^3)\,\mathbf{k}$$

$$= 0.$$

$\therefore$ the given form is exact.

Ex. 2. *In each of following cases show that the given differential form is exact and find a function ϕ such that the form equals $d\phi$:*

(i) $x dx - y dy - z dz$. (ii) $dx + z dy + y dz$.

(iii) $\cos x\,dx - 2yz\,dy - y^2\,dz$.

(iv) $(z^2 - 2xy)\,dx - x^2 dy + 2xz\,dz$.

Sol. (i) Here $\mathbf{F} = x\mathbf{i} - y\mathbf{j} - z\mathbf{k}$. We have

$$\text{Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & -y & -z \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = 0$$

$\therefore$ the given form is exact.

Let $\mathbf{F} = \nabla \phi$,

or $x\mathbf{i} - y\mathbf{j} - z\mathbf{k} = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}$. Then

$\frac{\partial \phi}{\partial x} = x$ whence $\phi = \frac{x^2}{2} + f_1(y, z)$ ... (1)

$\frac{\partial \phi}{\partial y} = -y$ whence $\phi = -\frac{y^2}{2} + f_2(x, z)$ ... (2)

$\frac{\partial \phi}{\partial z} = -z$ whence $\phi = -\frac{z^2}{2} + f_3(x, y)$. ... (3)

The constants of integration are functions of the variables not involved in the integration because the derivatives are partial. (1), (2), (3) each represents ϕ. These agree if we choose

$$f_1(y, z) = -\frac{y^2 + z^2}{2},\; f_2(x, z) = \frac{x^2 - z^2}{2},\; f_3(x, y) = \frac{x^2 - y^2}{2}.$$

$\therefore\ \phi = \frac{x^2 - y^2 - z^2}{2}$ to which may be added any constant.

Hence $\phi = \frac{x^2 - y^2 - z^2}{2} + C$, where C is a constant.

(ii) Here $\mathbf{F} = \mathbf{i} + z\mathbf{j} + y\mathbf{k}$. We have

$$\text{Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 1 & z & y \end{vmatrix} = 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = 0.$$

$\therefore$ the given form is exact.

Let $\mathbf{F} = \nabla \phi$

or $\mathbf{i} + z\,\mathbf{j} + y\,\mathbf{k} = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}$. Then

$\frac{\partial \phi}{\partial x} = 1$ whence $\phi = x + f_1(y, z)$ ... (1)

$\frac{\partial \phi}{\partial y} = z$ whence $\phi = zy + f_2(x, z)$ ... (2)

$\frac{\partial \phi}{\partial z} = y$ whence $\phi = yz + f_3(x, y)$ ... (3)

(1), (2), (3) each represents ϕ. These agree if we choose

$f_1(y, z) = zy,\; f_2(x, z) = x,\; f_3(x, y) = x$.

$\therefore \phi = x + yz$ to which may be added any constant.

$\therefore \phi = x + yz + C$.

(iii) Here $F = \cos x\,i - 2yz\,j - y^2 k$. We have

$$\text{Curl } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \cos x & -2yz & -y^2 \end{vmatrix}$$

$= (-2y + 2y)\,i + 0j + 0k = 0.$

$\therefore$ the given form is exact.

Let $F = \nabla \phi$,

or $\cos x\,i - 2yz\,j - y^2 k = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k$. Then

$\frac{\partial \phi}{\partial x} = \cos x$ whence $\phi = \sin x + f_1(y, z)$ ... (1)

$\frac{\partial \phi}{\partial y} = -2yz$ whence $\phi = -y^2 z + f_2(x, z)$ ... (2)

$\frac{\partial \phi}{\partial z} = -y^2$ whence $\phi = -y^2 z + f_3(x, y)$. ... (3)

(1), (2), (3) each represents ϕ. These agree if we choose $f_1(y, z) = -y^2 z$, $f_2(x, z) = \sin x$, $f_3(x, y) = \sin x$.

$\therefore \phi = \sin x - y^2 z$ to which may be added any constant.

$\therefore \phi = \sin x - y^2 z + C.$

(iv) Here $F = (z^2 - 2xy)\,i - x^2 j + 2xz\,k$. We have

$$\text{Curl } F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 - 2xy & -x^2 & 2xz \end{vmatrix} = 0i + 0j + 0k = 0.$$

$\therefore$ the given form is exact.

Let $F = \nabla \phi$

or $(z^2 - 2xy)\,i - x^2 j + 2xz\,k = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k$. Then

$\frac{\partial \phi}{\partial x} = z^2 - 2xy$ whence $\phi = z^2 x - x^2 y + f_1(y, z)$ ... (1)

$\frac{\partial \phi}{\partial y} = -x^2$ whence $\phi = -x^2 y + f_2(x, z)$ ... (2)

$\frac{\partial \phi}{\partial z} = 2xz$ whence $\phi = xz^2 + f_3(x, y)$. ... (3)

(1), (2), (3) each represents ϕ. These agree if we choose $f_1(y, z) = 0$, $f_2(x, z) = xz^2$, $f_3(x, y) = -x^2 y$.

$\therefore \phi = z^2 x - x^2 y$ to which may be added any constant.

$\therefore \phi = z^2 x - x^2 y + C.$

Ex. 3. *Show that*

$(y^2 z^3 \cos x - 4x^3 z)\,dx + 2z^3 y \sin x\,dy + (3y^2 z^2 \sin x - x^4)\,dz$

is an exact differential of some function ϕ *and find this function.*

Sol. Let $F = (y^2 z^3 \cos x - 4x^3 z)\,i + 2z^3 y \sin x\,j + (3y^2 z^2 \sin x - x^4)\,k$

We have curl F

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 z^3 \cos x - 4x^3 z & 2z^3 y \sin x & 3y^2 z^2 \sin x - x^4 \end{vmatrix}$$

$= i\left[\frac{\partial}{\partial y}(3y^2 z^2 \sin x - x^4) - \frac{\partial}{\partial z}(2z^3 y \sin x)\right]$

$\quad - j\left[\frac{\partial}{\partial x}(3y^2 z^2 \sin x - x^4) - \frac{\partial}{\partial z}(y^2 z^3 \cos x - 4x^3 z)\right]$

$\quad + k\left[\frac{\partial}{\partial x}(2z^3 y \sin x) - \frac{\partial}{\partial y}(y^2 z^3 \cos x - 4x^3 z)\right]$

$= (6yz^2 \sin x - 6z^2 y \sin x)\,i - \left[(3y^2 z^2 \cos x - 4x^3) - (3z^2 y^2 \cos x - 4x^3)\right] j + (2z^3 y \cos x - 2yz^3 \cos x)\,k$

$= 0i - 0j + 0k = 0.$

$\therefore$ there exists a scalar function $\phi(x, y, z)$ such that

$F = \nabla \phi.$

$\therefore \quad F \cdot dr = \nabla \phi \cdot dr$

or $\quad (y^2 z^3 \cos x - 4x^3 z)\,dx + 2z^3 y \sin x\,dy + (3y^2 z^2 \sin x - x^4)\,dz = d\phi.$

Hence $(y^2 z^3 \cos x - 4x^3 z)\,dx + 2z^3 y \sin x\,dy + (3y^2 z^2 \sin x - x^4)\,dz$ is an exact differential of some function ϕ.

Now $F = \nabla \phi \Rightarrow F = \frac{\partial \phi}{\partial x} i + \frac{\partial \phi}{\partial y} j + \frac{\partial \phi}{\partial z} k$

Equating the coefficients of i, j, k on both sides, we get

$\frac{\partial \phi}{\partial x} = y^2 z^3 \cos x - 4x^3 z$ whence $\phi = y^2 z^3 \sin x - x^4 z + f_1(y, z)$ ... (1)

$\frac{\partial \phi}{\partial y} = 2z^3 y \sin x$ whence $\phi = z^3 y^2 \sin x + f_2(z, x)$ ... (2)

$\frac{\partial \phi}{\partial z} = 3y^2 z^2 \sin x - x^4$ whence $\phi = y^2 z^3 \sin x - x^4 z + f_3(x, y)$ ... (3)

(1), (2), (3) each represents ϕ. These agree if we choose

$f_1(y,z) = 0,\ f_2(z,x) = -x^4z,\ f_3(x,y) = 0.$

$\therefore \phi = y^2z^3 \sin x - x^4z$ to which may be added any constant.

Hence $\phi = y^2z^3 \sin x - x^4z + C$.

Ex. 4. *Show that* $\mathbf{F} = (2xy + z^3)\,\mathbf{i} + x^2\mathbf{j} + 3xz^2\mathbf{k}$ *is a conservative force field. Find the scalar potential. Find also the work done in moving an object in this field from* $(1, -2, 1)$ *to* $(3, 1, 4)$.

Sol. The field F will be conservative if $\nabla \times \mathbf{F} = 0$.

We have

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3xz^2 \end{vmatrix} = 0.$$

Therefore F is a conservative force field.

Let $\mathbf{F} = \nabla\phi$

or $\quad (2xy + z^3)\,\mathbf{i} + x^2\mathbf{j} + 3xz^2\mathbf{k} = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}$. Then

$$\frac{\partial \phi}{\partial x} = 2xy + z^3 \text{ whence } \phi = x^2y + z^3x + f_1(y,z) \quad \ldots (1)$$

$$\frac{\partial \phi}{\partial y} = x^2 \text{ whence } \phi = x^2y + f_2(x,z) \quad \ldots (2)$$

$$\frac{\partial \phi}{\partial z} = 3xz^2 \text{ whence } \phi = xz^3 + f_3(x,y) \quad \ldots (3)$$

(1), (2), (3) each represents ϕ. These agree if we choose

$f_1(y,z) = 0,\ f_2(x,z) = z^3x,\ f_3(x,y) = x^2y.$

$\therefore \phi = x^2y + xz^3$ to which may be added any constant.

$\therefore \phi = x^2y + xz^3 + C.$

$$\text{Work done} = \int_{(1,-2,1)}^{(3,1,4)} \mathbf{F}\cdot d\mathbf{r}$$

$$= \int_{(1,-2,1)}^{(3,1,4)} d\phi = \Big[\phi\Big]_{(1,-2,1)}^{(3,1,4)}$$

$$= \Big[x^2y + xz^3\Big]_{(1,-2,1)}^{(3,1,4)} = 202.$$

Ex. 5. *Show that the vector field* F *given by*

$$\mathbf{F} = (y + \sin z)\,\mathbf{i} + x\,\mathbf{j} + x\cos z\,\mathbf{k}$$

is conservative. Find its scalar potential.

Sol. We have

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y + \sin z & x & x\cos z \end{vmatrix} = 0.$$

$\therefore$ the vector field F is conservative.

Let $\mathbf{F} = \nabla\phi$

or $\quad (y + \sin z)\,\mathbf{i} + x\,\mathbf{j} + x\cos z\,\mathbf{k} = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}$. Then

$$\frac{\partial \phi}{\partial x} = y + \sin z \text{ whence } \phi = xy + x\sin z + f_1(y,z) \quad \ldots (1)$$

$$\frac{\partial \phi}{\partial y} = x \text{ whence } \phi = xy + f_2(x,z) \quad \ldots (2)$$

$$\frac{\partial \phi}{\partial z} = x\cos z \text{ whence } \phi = x\sin z + f_3(x,y) \quad \ldots (3)$$

(1), (2), (3) each represents ϕ. These agree if we choose

$f_1(y,z) = 0,\ f_2(x,z) = x\sin z,\ f_3(x,y) = xy.$

$\therefore \phi = xy + x\sin z$ to which may be added any constant.

$\therefore \phi = xy + x\sin z + C.$

Ex. 6. *Show that the vector field*

$$\mathbf{F} = (2xy^2 + yz)\,\mathbf{i} + (2x^2y + xz + 2yz^2)\,\mathbf{j} + (2y^2z + xy)\,\mathbf{k}$$

is conservative.

Sol. We have

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^2 + yz & 2x^2y + xz + 2yz^2 & 2y^2z + xy \end{vmatrix}$$

$$= \mathbf{i}\left[\frac{\partial}{\partial y}(2y^2z + xy) - \frac{\partial}{\partial z}(2x^2y + xz + 2yz^2)\right]$$

$$- \mathbf{j}\left[\frac{\partial}{\partial x}(2y^2z + xy) - \frac{\partial}{\partial z}(2xy^2 + yz)\right]$$

$$+ \mathbf{k}\left[\frac{\partial}{\partial x}(2x^2y + xz + 2yz^2) - \frac{\partial}{\partial y}(2xy^2 + yz)\right]$$

$$= [(4yz + x) - (x + 4yz)]\,\mathbf{i} - (y - y)\,\mathbf{j} + [(4xy + z) - (4xy + z)]\,\mathbf{k}$$

$$= 0\mathbf{i} - 0\mathbf{j} + 0\mathbf{k} = 0.$$

$\therefore$ the vector field F is conservative.

Ex. 7. *Show that* $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ *is conservative and find* ϕ *such that* $\mathbf{F} = \nabla\phi$. (Kanpur 1980)

Sol. We have Curl F

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= \left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right)\mathbf{i} + \left(\frac{\partial x}{\partial z} - \frac{\partial z}{\partial x}\right)\mathbf{j} + \left(\frac{\partial y}{\partial x} - \frac{\partial x}{\partial y}\right)\mathbf{k}$$

$$= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = 0.$$

$\therefore$ the vector field F is conservative.

Let $\mathbf{F} = \nabla\phi$.

i.e., $x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}$. Then

$\frac{\partial \phi}{\partial x} = x$ whence $\phi = \frac{1}{2}x^2 + f_1(y, z)$ ... (1)

$\frac{\partial \phi}{\partial y} = y$ whence $\phi = \frac{1}{2}y^2 + f_2(z, x)$ ... (2)

$\frac{\partial \phi}{\partial z} = z$ whence $\phi = \frac{1}{2}z^2 + f_3(x, y)$ ... (3)

(1), (2), (3) each represents ϕ. These agree if we choose

$$f_1(y, z) = \frac{1}{2}y^2 + \frac{1}{2}z^2, f_2(z, x) = \frac{1}{2}z^2 + \frac{1}{2}x^2,$$

$$f_3(x, y) = \frac{1}{2}x^2 + \frac{1}{2}y^2.$$

$\therefore \phi = \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2$ to which may be added any constant.

Hence $\phi = \frac{1}{2}(x^2 + y^2 + z^2) + C$.

Ex. 8. *Show that*

$\mathbf{F} = (\sin y + z)\mathbf{i} + (x\cos y - z)\mathbf{j} + (x - y)\mathbf{k}$ *is a conservative vector field and find a function* ϕ *such that*

$\mathbf{F} = \nabla\phi$. (Bombay 1966)

Sol. We have curl F

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin y + z & x\cos y - z & x - y \end{vmatrix}$$

$$= \mathbf{i}\left[\frac{\partial}{\partial y}(x - y) - \frac{\partial}{\partial z}(x\cos y - z)\right] + \mathbf{j}\left[\frac{\partial}{\partial z}(\sin y + z) - \frac{\partial}{\partial x}(x - y)\right]$$

$$+ \mathbf{k}\left[\frac{\partial}{\partial x}(x\cos y - z) - \frac{\partial}{\partial y}(\sin y + z)\right]$$

$$= (-1 + 1)\mathbf{i} + (1 - 1)\mathbf{j} + (\cos y - \cos y)\mathbf{k}$$

$$= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = 0.$$

$\therefore$ the vector field F is conservative.

Let $\mathbf{F} = \nabla\phi$

i.e., $(\sin y + z)\mathbf{i} + (x\cos y - z)\mathbf{j} + (x - y)\mathbf{k} = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}$.

Then $\frac{\partial \phi}{\partial x} = \sin y + z$ whence $\phi = x\sin y + xz + f_1(y, z)$ ... (1)

$\frac{\partial \phi}{\partial y} = x\cos y - z$ whence $\phi = x\sin y - yz + f_2(z, x)$ ... (2)

$\frac{\partial \phi}{\partial z} = x - y$ whence $\phi = xz - yz + f_3(x, y)$ ... (3)

(1), (2), (3) each represents ϕ. These agree if we choose

$f_1(y, z) = -yz$, $f_2(z, x) = xz$, $f_3(x, y) = x\sin y$.

$\therefore \phi = x\sin y + xz - yz$.

$\therefore \phi = x\sin y + xz - yz$ to which may be added any constant.

Hence $\phi = x\sin y + xz - yz + C$.

Ex. 9. *Show that the vector field defined by*

$\mathbf{F} = (2xy - z^3)\mathbf{i} + (x^2 + z)\mathbf{j} + (y - 3xz^2)\mathbf{k}$ *is conservative, and find the scalar potential of* F. (Bombay 1970)

Sol. We have curl F

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy - z^3 & x^2 + z & y - 3xz^2 \end{vmatrix}$$

$$= \mathbf{i}\left[\frac{\partial}{\partial y}(y - 3xz^2) - \frac{\partial}{\partial z}(x^2 + z)\right] + \mathbf{j}\left[\frac{\partial}{\partial z}(2xy - z^3) - \frac{\partial}{\partial x}(y - 3xz^2)\right] + \mathbf{k}\left[\frac{\partial}{\partial x}(x^2 + z) - \frac{\partial}{\partial y}(2xy - z^3)\right]$$

$$= (1 - 1)\mathbf{i} + (-3z^2 + 3z^2)\mathbf{j} + (2x - 2x)\mathbf{k}$$

$$= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = 0.$$

$\therefore$ the vector field F is conservative.

Let $\mathbf{F} = \nabla\phi = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}$. Then

$\frac{\partial \phi}{\partial x} = 2xy - z^3$ whence $\phi = x^2y - z^3x + f_1(y, z)$ ... (1)

$$\frac{\partial\phi}{\partial y} = x^2 + z \text{ whence } \phi = x^2 y + zy + f_2(z,x) \quad \dots (2)$$

$$\frac{\partial\phi}{\partial z} = y - 3xz^2 \text{ whence } \phi = yz - xz^3 + f_3(x,y) \quad \dots (3)$$

(1), (2), (3) each represents ϕ. These agree if we choose

$f_1(y,z) = zy$, $f_2(z,x) = -z^3 x$, $f_3(x,y) = x^2 y$.

$\therefore \phi = x^2 y - z^3 x + zy$ to which may be added any constant.

Hence $\phi = x^2 y - z^3 x + zy + C$.

Ex. 10. *Evaluate*

$$\int_C 2xyz^2\,dx + (x^2z^2 + z\cos yz)\,dy + (2x^2yz + y\cos yz)\,dz$$

where C is any path from $(0, 0, 1)$ *to* $(1, \frac{1}{4}\pi, 2)$. (Meerut 1968)

Sol. We have $F = 2xyz^2\,\mathbf{i} + (x^2z^2 + z\cos yz)\,\mathbf{j} + (2x^2yz + y\cos yz)\,\mathbf{k}$.

$$\therefore \nabla \times F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xyz^2 & x^2z^2 + z\cos yz & 2x^2yz + y\cos yz \end{vmatrix}$$

$$= (2x^2z + \cos yz - yz\sin yz - 2x^2z - \cos yz + yz\sin yz)\,\mathbf{i} - (4xyz - 4xyz)\,\mathbf{j} + (2xz^2 - 2xz^2)\,\mathbf{k} = 0.$$

$\therefore$ the given line integral is independent of path in space.

Let $F = \nabla\phi$. Then

$$\frac{\partial\phi}{\partial x} = 2xyz^2 \text{ whence } \phi = x^2yz^2 + f_1(y,z) \quad \dots (1)$$

$$\frac{\partial\phi}{\partial y} = x^2z^2 + z\cos yz \text{ whence } \phi = x^2z^2y + \sin yz + f_2(x,z) \quad \dots (2)$$

$$\frac{\partial\phi}{\partial z} = 2x^2yz + y\cos yz \text{ whence } \phi = x^2yz^2 + \sin yz + f_3(x,y) \quad \dots (3)$$

(1), (2), (3) each represents ϕ. These agree if we choose

$f_1(y,z) = \sin yz$, $f_2(x,z) = 0$, $f_3(x,y) = 0$.

$\therefore \phi = x^2yz^2 + \sin yz$ to which may be added any constant.

The given line integral is therefore

$$\int_C d(x^2yz^2 + \sin yz) = \Big[x^2yz^2 + \sin yz\Big]_{(0,0,1)}^{(1,\pi/4,2)}$$

$$= \pi + \sin\tfrac{1}{2}\pi = \pi + 1.$$

Ex. 11. *Evaluate*

$$\int_C yz\,dx + (xz + 1)\,dy + xy\,dz,$$

where C is any path from $(1, 0, 0)$ *to* $(2, 1, 4)$. (Agra 1972 ; Meerut 64)

Sol. We have $F = yz\mathbf{i} + (xz + 1)\mathbf{j} + xy\mathbf{k}$.

$$\therefore \nabla \times F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz + 1 & xy \end{vmatrix}$$

$$= (x - x)\,\mathbf{i} - (y - y)\,\mathbf{j} + (z - z)\,\mathbf{k} = 0.$$

$\therefore$ the differential form $yz\,dx + (xz + 1)\,dy + xy\,dz$ is exact and the given line integral is independent of path.

Let $F = \nabla\phi$

or $yz\,\mathbf{i} + (xz + 1)\,\mathbf{j} + xy\,\mathbf{k} = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$. Then

$$\frac{\partial\phi}{\partial x} = yz \text{ whence } \phi = xyz + f_1(y,z) \quad \dots (1)$$

$$\frac{\partial\phi}{\partial y} = xz + 1 \text{ whence } \phi = xyz + y + f_2(x,z) \quad \dots (2)$$

$$\frac{\partial\phi}{\partial z} = xy \text{ whence } \phi = xyz + f_3(x,y) \quad \dots (3)$$

(1), (2), (3) each represents ϕ. These agree if we choose

$f_1(y,z) = y$, $f_2(x,z) = 0$, $f_3(x,y) = y$.

$\therefore \phi = xyz + y$ to which may be added any constant.

The given line integral is therefore

$$= \int_{(1,0,0)}^{(2,1,4)} d(xyz + y) = \Big[xyz + y\Big]_{(1,0,0)}^{(2,1,4)}$$

$$= [8 + 1 - 0 - 0] = 9.$$

Ex. 12. *Show that the form under the integral sign is exact and evaluate*

$$\int_{(0,2,1)}^{(2,0,1)} [ze^x\,dx + 3yz\,dy + (e^x + y^2)\,dz].$$

Sol. Here $F = ze^x\,\mathbf{i} + 2yz\,\mathbf{j} + (e^x + y^2)\,\mathbf{k}$.

We have
$$\text{curl } F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ze^x & 2yz & e^x + y^2 \end{vmatrix}$$

$$= (2y - 2y)\,\mathbf{i} - (e^x - e^x)\,\mathbf{j} + 0\mathbf{k} = 0.$$

$\therefore$ the form under the integral sign is exact and consequently the line integral is independent of path in space.

Let $F = \nabla\phi$

or $\quad ze^x \mathbf{i} + 2yz\,\mathbf{j} + (e^x + y^2)\,\mathbf{k} = \dfrac{\partial\phi}{\partial x}\mathbf{i} + \dfrac{\partial\phi}{\partial y}\mathbf{j} + \dfrac{\partial\phi}{\partial z}\mathbf{k}$. Then

$$\frac{\partial\phi}{\partial x} = ze^x \text{ whence } \phi = ze^x + f_1(y, z) \quad \ldots (1)$$

$$\frac{\partial\phi}{\partial y} = 2yz \text{ whence } \phi = y^2z + f_2(z, x) \quad \ldots (2)$$

$$\frac{\partial\phi}{\partial z} = e^x + y^2 \text{ whence } \phi = e^xz + y^2z + f_3(x, y) \quad \ldots (3)$$

(1), (2), (3) each represents ϕ. These agree if we choose

$f_1(y, z) = y^2z, f_2(x, z) = e^xz, f_3(x, y) = 0.$

$\therefore\ \phi = ze^x + y^2z$ to which may be added any constant. The given line integral is therefore

$$= \int_{(0,2,1)}^{(2,0,1)} d(ze^x + y^2z) = \Big[ze^x + y^2z\Big]_{(0,2,1)}^{(2,0,1)}$$

$$= [e^2 + 0 - 1 - 4] = e^2 - 5.$$

Ex. 13. *If* $F = \cos y\,\mathbf{i} - x\sin y\,\mathbf{j}$, *evaluate* $\int_C F\cdot dr$ *where C is the curve* $y = \sqrt{(1 - x^2)}$ *in the x-y plane from* (1, 0) *to* (0, 1).

Sol. We have $\int_C F\cdot dr = \int_C (\cos y\,dx - x\sin y\,dy)$

$$= \int_1^0 \cos\sqrt{(1 - x^2)}\,dx - \int_0^1 \sqrt{(1 - y^2)}\sin y\,dy.$$

It is difficult to evaluate the integrals directly. However we observe that

$$\text{Curl } F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \cos y & -x\sin y & 0 \end{vmatrix}$$

$$= 0\mathbf{i} + 0\mathbf{j} + (-\sin y + \sin y)\mathbf{k} = 0.$$

$\therefore$ the given line integral is independent of path.

Let $F = \nabla\phi$

or $\quad \cos y\,\mathbf{i} - x\sin y\,\mathbf{j} = \dfrac{\partial\phi}{\partial x}\mathbf{i} + \dfrac{\partial\phi}{\partial y}\mathbf{j} + \dfrac{\partial\phi}{\partial z}\mathbf{k}$. Then

$$\frac{\partial\phi}{\partial x} = \cos y \text{ whence } \phi = x\cos y + f_1(y, z) \quad \ldots (1)$$

$$\frac{\partial\phi}{\partial y} = -x\sin y, \text{ whence } \phi = x\cos y + f_2(x, z) \quad \ldots (2)$$

$$\frac{\partial\phi}{\partial z} = 0 \text{ whence } \phi = f_3(x, y). \quad \ldots (3)$$

From (1), (2), (3), we see that $\phi = x\cos y$.

The given line integral is equal to

$$\int_{(1,0)}^{(0,1)} d(x\cos y) = \Big[x\cos y\Big]_{(1,0)}^{(0,1)} = [0 - 1\cos 0] = -1.$$

Ex. 14. *Show that the vector field* F *given by*

$$F = (x^2 - yz)\,\mathbf{i} + (y^2 - zx)\,\mathbf{j} + (z^2 - xy)\,\mathbf{k}$$

is irrotational. Find a scalar ϕ *such that* $F = \nabla\phi$.

Sol. We have

$$\text{Curl } F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 - yz & y^2 - zx & z^2 - zy \end{vmatrix}$$

$$= (-x + x)\,\mathbf{i} - (-y + y)\,\mathbf{j} + (-z + z)\,\mathbf{k} = 0.$$

$\therefore$ The vector field F is irrotational.

Let $F = \nabla\phi$

or $\quad (x^2 - yz)\,\mathbf{i} + (y^2 - zx)\,\mathbf{j} + (z^2 - xy)\,\mathbf{k} = \dfrac{\partial\phi}{\partial x}\mathbf{i} + \dfrac{\partial\phi}{\partial y}\mathbf{j} + \dfrac{\partial\phi}{\partial z}\mathbf{k}$. Then

$$\frac{\partial\phi}{\partial x} = x^2 - yz \text{ whence } \phi = \frac{x^3}{3} - xyz + f_1(y, z) \quad \ldots (1)$$

$$\frac{\partial\phi}{\partial y} = y^2 - zx \text{ whence } \phi = \frac{y^3}{3} - xyz + f_2(x, z) \quad \ldots (2)$$

$$\frac{\partial\phi}{\partial z} = z^2 - xy \text{ whence } \phi = \frac{z^3}{3} - xyz + f_3(x, y). \quad \ldots (3)$$

(1), (2), (3) each represents ϕ. These agree if we choose

$$f_1(y, z) = \frac{y^3}{3} + \frac{z^3}{3},\ f_2(x, z) = \frac{x^3 + z^3}{3},\ f_3(x, y) = \frac{x^3 + y^3}{3}.$$

Therefore $\phi = \dfrac{x^3 + y^3 + z^3}{3} - xyz + C.$

Ex. 15. *Show that the following vector functions* F *are irrotational and find the corresponding scalar* ϕ *such that*

$$F = \nabla\phi.$$

(i) $F = (\sin y + z\cos x)\,\mathbf{i} + (x\cos y + \sin z)\,\mathbf{j} + (y\cos z + \sin x)\,\mathbf{k}$. (Calcutta 1975)

(ii) $F = (y\sin z - \sin x)\,\mathbf{i} + (x\sin z + 2yz)\,\mathbf{j} + (xy\cos z + y^2)\,\mathbf{k}$

(iii) $F = x^3\mathbf{i} + y^3\mathbf{j} + z^3\mathbf{k}$.

Sol. (i) We have curl **F**

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin y + z\cos x & x\cos y + \sin z & y\cos z + \sin x \end{vmatrix}$$

$$= \mathbf{i}\left[\frac{\partial}{\partial y}(y\cos z + \sin x) - \frac{\partial}{\partial z}(x\cos y + \sin z)\right] + \mathbf{j}\left[\frac{\partial}{\partial z}(\sin y + z\cos x) - \frac{\partial}{\partial x}(y\cos z + \sin x)\right] + \mathbf{k}\left[\frac{\partial}{\partial x}(x\cos y + \sin z) - \frac{\partial}{\partial y}(\sin y + z\cos x)\right]$$

$$= (\cos z - \cos z)\,\mathbf{i} + (\cos x - \cos x)\,\mathbf{j} + (\cos y - \cos y)\,\mathbf{k}$$

$$= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = 0.$$

∴ the vector **F** is irrotational.

Let $\mathbf{F} = \nabla\phi$

i.e., $(\sin y + z\cos x)\,\mathbf{i} + (x\cos y + \sin z)\,\mathbf{j} + (y\cos z + \sin x)\,\mathbf{k}$

$$= \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}.$$ Then

$$\frac{\partial\phi}{\partial x} = \sin y + z\cos x \text{ whence } \phi = x\sin y + z\sin x + f_1(y,z) \quad \ldots(1)$$

$$\frac{\partial\phi}{\partial y} = x\cos y + \sin z \text{ whence } \phi = x\sin y + y\sin z + f_2(z,x) \quad \ldots(2)$$

$$\frac{\partial\phi}{\partial z} = y\cos z + \sin x \text{ whence } \phi = y\sin z + z\sin x + f_3(x,y) \quad \ldots(3)$$

(1), (2), (3) each represents ϕ. These agree if we choose $f_1(y,z) = y\sin z$, $f_2(z,x) = z\sin x$, $f_3(x,y) = x\sin y$.

∴ $\phi = x\sin y + z\sin x + y\sin z$ to which may be added any constant.

Hence $\phi = x\sin y + z\sin x + y\sin z + C$.

(ii) Do yourself. Ans. $\phi = xy\sin z + \cos x + y^2 z + C$.

(iii) Do yourself. Ans. $\phi = \frac{1}{4}(x^4 + y^4 + z^4) + C$.

Ex. 16. *Find a, b, c if* $\mathbf{F} = (3x - 3y + az)\,\mathbf{i} + (bx + 2y - 4z)\,\mathbf{j} + (2x + cy + z)\,\mathbf{k}$ *is irrotational.* (Calicut 1974)

Sol. The vector **F** is irrotational if and only if curl **F** = 0. We have

$$\text{Curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x - 3y + az & bx + 2y - 4z & 2x + cy + z \end{vmatrix}$$

$$= \mathbf{i}\left[\frac{\partial}{\partial y}(2x + cy + z) - \frac{\partial}{\partial z}(bx + 2y - 4z)\right] + \mathbf{j}\left[\frac{\partial}{\partial z}(3x - 3y + az) - \frac{\partial}{\partial x}(2x + cy + z)\right] + \mathbf{k}\left[\frac{\partial}{\partial x}(bx + 2y - 4z) - \frac{\partial}{\partial y}(3x - 3y + az)\right]$$

$$= (c + 4)\,\mathbf{i} + (a - 2)\,\mathbf{j} + (b + 3)\,\mathbf{k}.$$

Now curl **F** = 0. If $(c + 4)\mathbf{i} + (a - 2)\mathbf{j} + (b + 3)\mathbf{k} = 0$

i.e., if $c + 4 = 0$, $a - 2 = 0$, $b + 3 = 0$

i.e., if $a = 2$, $b = -3$, $c = -4$.

Hence the given vector **F** is irrotational if $a = 2$, $b = -3$, $c = -4$.

Ex. 17. *Show that*

$$(2x\cos y + z\sin y)\,dx + (xz\cos y - x^2\sin y)\,dy + x\sin y\,dz = 0$$

is an exact differential equation and hence solve it.

Sol. The given differential equation is exact if there exists a scalar function $\phi(x,y,z)$ such that

$$(2x\cos y + z\sin y)\,dx + (xz\cos y - x^2\sin y)\,dy + x\sin y\,dz = d\phi.$$

Let $\mathbf{F} = (2x\cos y + z\sin y)\,\mathbf{i} + (xz\cos y - x^2\sin y)\,\mathbf{j} + x\sin y\,\mathbf{k}$.

We have curl **F**

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x\cos y + z\sin y & xz\cos y - x^2\sin y & x\sin y \end{vmatrix}$$

$$= \mathbf{i}\left[\frac{\partial}{\partial y}(x\sin y) - \frac{\partial}{\partial z}(xz\cos y - x^2\sin y)\right] + \mathbf{j}\left[\frac{\partial}{\partial z}(2x\cos y + z\sin y) - \frac{\partial}{\partial x}(x\sin y)\right] + \mathbf{k}\left[\frac{\partial}{\partial x}(xz\cos y - x^2\sin y) - \frac{\partial}{\partial y}(2x\cos y + z\sin y)\right]$$

$$= (x\cos y - x\cos y)\,\mathbf{i} + (\sin y - \sin y)\,\mathbf{j} + [(z\cos y - 2x\sin y) - (-2x\sin y + z\cos y)]\,\mathbf{k}$$

$$= 0\mathbf{i} + 0\mathbf{j} + 0\mathbf{k} = 0.$$

∴ the vector **F** is conservative.

Hence there exists a scalar function $\phi(x,y,z)$ such that

$$\mathbf{F} = \nabla\phi.$$

$$\therefore \mathbf{F}\cdot d\mathbf{r} = \nabla\phi\cdot d\mathbf{r}$$

or $(2x\cos y + z\sin y)\,dx + (xz\cos y - x^2\sin y)\,dy + x\sin y\,dz = d\phi$.

Hence the given differential equation is exact.

Now $\mathbf{F} = \nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$. Therefore

$\frac{\partial\phi}{\partial x} = 2x\cos y + z\sin y$ whence $\phi = x^2\cos y + xz\sin y + f_1(y, z)$ ... (1)

$\frac{\partial\phi}{\partial y} = xz\cos y - x^2\sin y$ whence $\phi = xz\sin y + x^2\cos y + f_2(z, x)$ ... (2)

$\frac{\partial\phi}{\partial z} = x\sin y$ whence $\phi = xz\sin y + f_3(x, y)$ ... (3)

(1), (2), (3) each represents ϕ. These agree if we choose

$f_1(y, z) = 0, f_2(z, x) = 0, f_3(x, y) = x^2\cos y$.

$\therefore \phi = x^2\cos y + xz\sin y$.

Now the given differential equation reduces to

$d\phi = 0$ whose solution is $\phi = C$

i.e., $x^2\cos y + xz\sin y = C$.

Ex. 18. *If* F *is irrotational in a simply connected region R, show that there exists a scalar field ϕ such that* $\mathbf{F} = grad\ \phi$. (Calicut 1975)

Sol. Since F is irrotational in a simply connected region R, therefore curl $\mathbf{F} = 0$ in R.

Let C be any simple closed path in R. Then by Stoke's theorem

$$\oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_S (\text{curl } \mathbf{F})\cdot\mathbf{n}\, dS, \text{ where } S \text{ is any surface in } R$$

whose boundary is the closed curve C.

$= 0$, since curl $\mathbf{F} = 0$.

$\therefore$ the line integral $\int_P^Q \mathbf{F}\cdot d\mathbf{r}$ is independent of the path joining any two points P and Q in R.

Let P be a fixed point (x_0, y_0, z_0) in R and let Q by any point (x, y, z) in R. Let

$$\phi(x, y, z) = \int_{(x_0, y_0, z_0)}^{(x,y,z)} \mathbf{F}\cdot d\mathbf{r} = \int_{(x_0, y_0, z_0)}^{(x,y,z)} \left(\mathbf{F}\cdot\frac{d\mathbf{r}}{ds}\right) ds.$$

Differentiating both sides with respect to s, we get

$$\frac{d\phi}{ds} = \mathbf{F}\cdot\frac{d\mathbf{r}}{ds}.$$

But $\frac{d\phi}{ds} = \frac{\partial\phi}{\partial x}\frac{dx}{ds} + \frac{\partial\phi}{\partial y}\frac{dy}{ds} + \frac{\partial\phi}{\partial z}\frac{dz}{ds}$

$$= \left(\frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}\right)\cdot\left(\frac{dx}{ds}\mathbf{i} + \frac{dy}{ds}\mathbf{j} + \frac{dz}{ds}\mathbf{k}\right) = \nabla\phi\cdot\frac{d\mathbf{r}}{ds}.$$

$\therefore \mathbf{F}\cdot\frac{d\mathbf{r}}{ds} = \nabla\phi\cdot\frac{d\mathbf{r}}{ds}$ or $(\nabla\phi - \mathbf{F})\cdot\frac{d\mathbf{r}}{ds} = 0$.

Now this result is true irrespective of the path joining P to Q i.e.,

this result is true irrespective of the direction of $\frac{d\mathbf{r}}{ds}$ which is tangent vector to C. Therefore we must have

$\nabla\phi - \mathbf{F} = 0$

i.e., $\nabla\phi = \mathbf{F}$ i.e., $\mathbf{F} = \text{grad } \phi$.

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Curvilinear Co-ordinate Systems

→ Here we are studying Non-Cartesian co-ordinate systems.

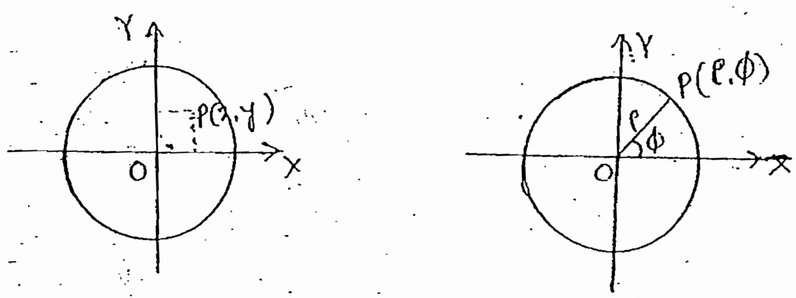
→ The non-Cartesian co-ordinate systems are special cases of the general orthogonal Curvilinear co-ordinates.

→ We have introduced Curvilinear co-ordinates by linking them with Cartesian co-ordinates.

→ The expressions for gradient, divergence, Curl and Laplacian operator are derived first in Curvilinear and then in polar co-ordinates.

These expressions will be useful in next section.

* Plane Polar Co-ordinate System:



(a) (b)

from the fig (a). It models a flat circular dinner plate. To specify any point P on its surface we have to draw two fixed coordinate axes at right angles to each other passing through O, the centre of the plate. This is the familiar Cartesian Co-ordinate system. we can locate the point of interest by giving its distance from the two axes.

The position of point P can also be uniquely defined by measuring its radial distance from the origin and the angle ϕ between the x-axis and the line joining the point to the origin as shown fig (b).

point O is called the pole.

The distance ρ is called the radius vector of the point P and ϕ is its polar angle. These two coordinates taken together are called plane coordinates of P.

Now we can specify the position of the point P as $P(\rho, \phi)$. we would now like to know how x and y are called related to ρ and ϕ.

To relate the x & y coordinates of P to its ρ & ϕ coordinates, we can easily write referring to fig (b).

$$x = \rho\cos\phi \quad (-\infty < x < \infty)$$

$$\text{and} \quad y = \rho\sin\phi, \quad (-\infty < y < \infty) \quad \text{——} \quad (1)$$

It is quite easy to change from Cartesian coordinates to plane polar coordinates. By squaring these relations and adding, we will get

$$\rho = \sqrt{x^2+y^2} \quad \text{——— (2)}$$

and on dividing one by the other, we can write

$$\phi = \tan^{-1}\left(\frac{y}{x}\right) \quad \text{——— (3)}$$

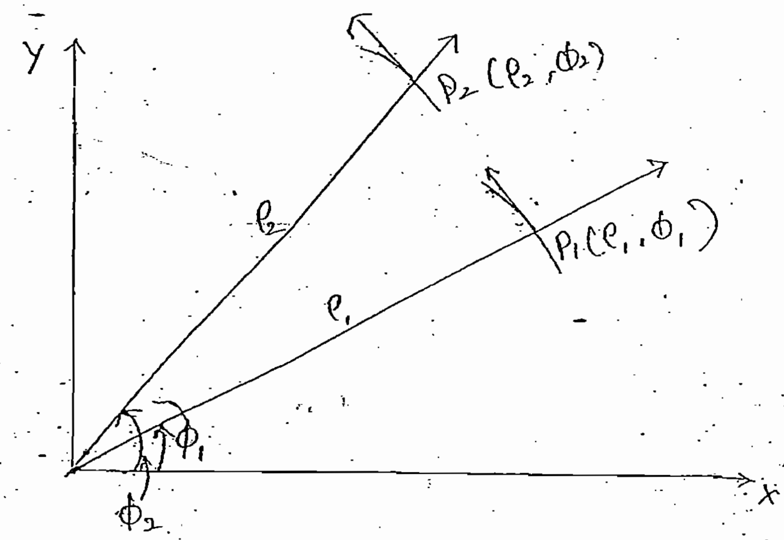
we take ρ to be +ve when it is measured from the origin along the line OP. similarly, ϕ is taken to be +ve in the anticlockwise direction from x-axis. the range of variation of these coordinates is given by

$$0 < \rho < \infty, \quad 0 \leq \phi \leq 2\pi$$

we know that cartesian coordinate system is orthogonal, i.e. the x and y-axes meet at right angles at any point in the plane and their directions are fixed.

we will note that each concurrent line cuts the concentric circles at one point only. when we draw a tangent at that point, we will note that the coordinate axes defining the directions of increasing ρ and increasing ϕ are at right angles. That is, the polar coordinates form an orthogonal coordinate system in the plane. But for two different points

$P_1(\rho_1, \phi_1)$ and $P_2(\rho_2, \phi_2)$, the directions of ρ and ϕ coordinate axes are not the same.



From this we may conclude that unlike in the Cartesian system, the directions of plane polar coordinate axes vary from point to point.

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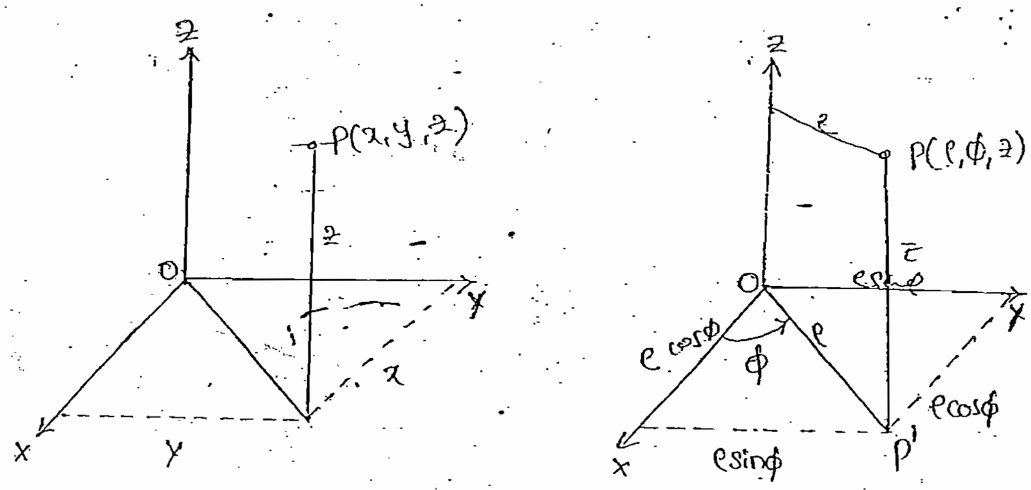
Cylindrical Co-ordinate System:

We know that in Cartesian co-ordinate system, the position of a point P in space is denoted by $P(x,y,z)$. We denote the cylindrical polar co-ordinates of this point by $P(\rho,\phi,z)$. By referring to below figure, we will note that ρ and z respectively denote the distance of P from the z-axis, and the xy-plane while ϕ is the angle which the line joining the origin with the projection of P on xy-plane makes with the x-axis. It is called the azimuthal angle.

If we recall the equation $x^2+y^2=\rho^2$ with ρ = constant defines a circle in a plane. This suggests that if you add a Cartesian z-axis to the plane polar co-ordinate system, so that z denotes the perpendicular distance from the xy-plane, we obtain a cylinder. That is, cylindrical polar co-ordinate system extends the plane polar co-ordinate system to three dimensions. So if a point has Cartesian co-ordinates (x,y,z) and cylindrical polar co-ordinates (ρ,ϕ,z).



Representation of a point in Cartesian and Cylindrical co-ordinates.

$$x = \rho\cos\phi \qquad (-\infty < x < \infty)$$

$$y = \rho\sin\phi \qquad (-\infty < y < \infty) \qquad \text{——— (1)}$$

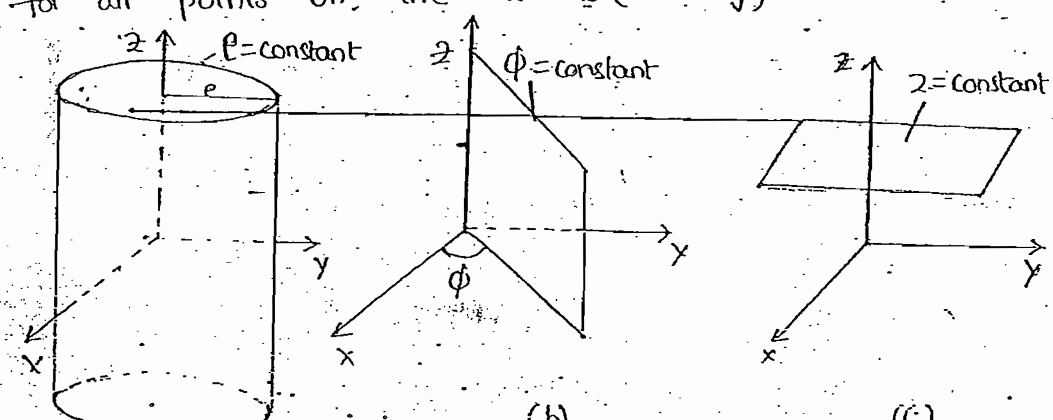
$$z = z \qquad (-\infty \leq z < \infty)$$

$$\rho = \sqrt{x^2 + y^2} \qquad (0 < \rho < \infty)$$

$$\phi = \tan^{-1}(y/x) \qquad (0 \leq \phi \leq 2\pi)$$

$$z = z \qquad (-\infty < z < \infty) \qquad \text{——— (2)}$$

In case of plane polar coordinates, ϕ is undefined at the origin. But in cylindrical co-ordinates ϕ is undefined for all points on the z-axis ($x = 0 = y$).



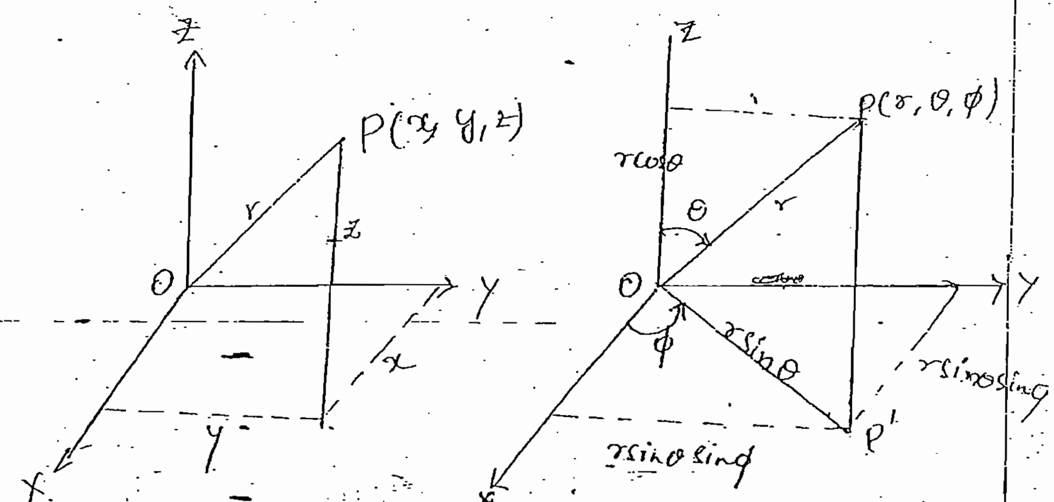
(b) (c)

Now refer to fig (a), which shows a cylinder of radius ρ and whose axis of symmetry is along the z-axis. We will note that for any point on the surface of cylinder, ρ is constant. That is, ρ = constant defines a circular cylindrical surface. This is also called the ϕ-z surface. For different values of ρ, we will obtain co-axial right circular cylinders. Their common axis of symmetry is z-axis.

- The ρz-surface defined by ϕ = constant is half-plane bounded on one edge by the z-axis (fig b)
- But the $\rho\phi$-surface given by z = Constant is a plane parallel to the xy plane (fig c) just as in the Cartesian Co-ordinate System.

Spherical polar co-ordinate System:

In spherical polar co-ordinate system, the position of a point is specified by the radial distance r, the polar angle θ and the azimuthal angle ϕ as shown in the figure. We can see that while θ is measured in the clockwise direction from the z-axes, ϕ is measured in the anticlockwise direction from the x-axis.



Representation of a point in cartesian and spherical polar co-ordinate system.

Referring to this figure we can see that the projection OP onto the xy-plane, $OP' = r\sin\theta$ while its projection on z-axis is $r\cos\theta$. The components of OP' along x and y-axes are $OP'\cos\phi$ and $OP'\sin\phi$. (i.e, $r\sin\theta\cos\phi$ and $r\sin\theta \cos\phi$)

Hence, for a point P having cartesian co-ordinates (x, y, z) and spherical polar co-ordinate (r, θ, ϕ),

we can write

$$\left.\begin{aligned} x &= OP' \cos\phi = r\sin\theta\cos\phi && (-\infty < x < \infty) \\ y &= OP' \sin\phi = r\sin\theta\sin\phi && (-\infty < y < \infty) \\ z &= OP\cos\theta = r\cos\theta && (-\infty < z < \infty) \end{aligned}\right\}$$

$$\left.\begin{aligned} r &= \sqrt{x^2+y^2+z^2} && (0 \le r < \infty) \\ \theta &= \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right) && (0 \le \theta \le \pi) \\ \text{and } \phi &= \tan^{-1}\left(\frac{y}{x}\right). && (0 \le \phi \le 2\pi) \end{aligned}\right\}$$

$\left[\because x^2+y^2 = r^2\sin^2\theta(\sin^2\phi + \cos^2\phi) = r^2\sin^2\theta\right.$

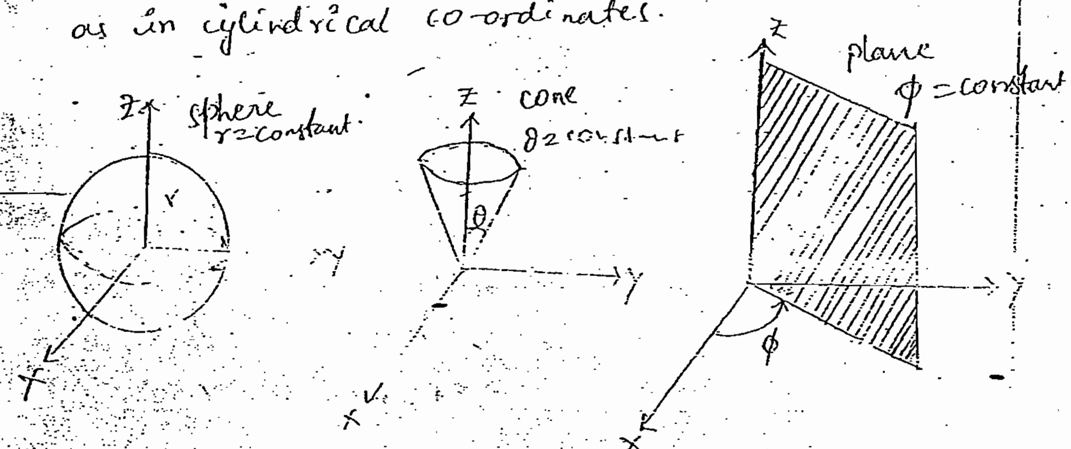
$\Rightarrow r\sin\theta = \sqrt{x^2+y^2}$

$\therefore \frac{r\sin\theta}{r\cos\theta} = \frac{\sqrt{x^2+y^2}}{z}$

$\Rightarrow \tan\theta = \frac{\sqrt{x^2+y^2}}{z}$

$\left.\Rightarrow \theta = \tan^{-1}\left(\frac{\sqrt{x^2+y^2}}{z}\right)\right]$

The surfaces defined by $r =$ constant, $\theta =$ constant and $\phi =$ constant are as shown below. It is instructive to note that in spherical co-ordinates, $\phi =$ constant is the half plane as in cylindrical co-ordinates.



P. 2

The co-ordinate surfaces are

$r =$ constant say, c_1 ; spheres having centre at the origin (or origin of constant $= 0$ i.e. $c_1 = 0$)

$\theta =$ constant say, c_2 ; cones having vertex at the origin (lines if $c_2 = 0$ or π, and the xy-plane if $c_2 = \pi/2$)

$\phi =$ constant say, c_3 ; planes through the z-axis.

[illegible]