IFoS-17

Q.5a Form partial differential equation by eliminating arbitary functions ϕ & ψ from relation $z=\phi(x^2-y)+\psi(x^2+y)$.

$\rightarrow$ Given,

$$z=\phi(x^2-y)+\psi(x^2+y) \quad \text{——— (1)}$$

Now, diff (1) partially w.r.t. x & y, we get,

$$\frac{\partial z}{\partial x}=\phi'(x^2-y)\times 2x+\psi'(x^2+y)\times 2x$$

$$\therefore \frac{\partial z}{\partial x}=2x\left(\phi'(x^2-y)+\psi'(x^2+y)\right) \quad \text{——— (2)}$$

$$\& \ \frac{\partial z}{\partial y}=-\phi'(x^2-y)+\psi'(x^2+y) \quad \text{——— (3)}$$

diff. (2) & (3) partially w.r.t. x & y respectively.

$$\therefore \frac{\partial^2 z}{\partial x^2}=2\left(\phi'(x^2-y)+\psi'(x^2+y)\right)+2x\left(\phi''(x^2-y)\times 2x+\psi''(x^2+y)\times 2x\right)$$

$$\frac{\partial^2 z}{\partial x^2}=2\left[\phi'(x^2-y)+\psi'(x^2+y)\right]+4x^2\left[\phi''(x^2-y)+\psi'(x^2+y)\right] \quad \text{——— (4)}$$

$$\frac{\partial^2 z}{\partial y^2}=\phi''(x^2-y)+\psi''(x^2+y) \quad \text{——— (5)}$$

$\therefore$ from (2), (4) & (5); we get.

$$\frac{\partial^2 z}{\partial x^2}=2\times\frac{1}{2x}\times\frac{\partial z}{\partial x}+4x^2\times\frac{\partial^2 z}{\partial y^2}$$

$$\therefore \boxed{x\frac{\partial^2 z}{\partial x^2}=4x^3\frac{\partial^2 z}{\partial y^2}+\frac{\partial z}{\partial x}}$$

which is required pde

Q.6-a) Solve the partial differential ~~function~~ equation:-

$$(x-y)\frac{\partial z}{\partial x} + (x+y)\frac{\partial z}{\partial y} = 2xz$$

$\longrightarrow$ Let, $\frac{\partial z}{\partial x} = p$, $\frac{\partial z}{\partial y} = q$, $\cancel{2xz=R}$.

& $x-y=P$, $x+y=Q$, $2xz=R$.

$\therefore (x-y)\frac{\partial z}{\partial x} + (x+y)\frac{\partial z}{\partial y} = 2xz$ — (1)

$\therefore$ (1) $\equiv$ $(x-y)p + (x+y)q = 2xz$. — (2)

compare (2) with $Pp + Qq = R$

Now, longrange's Auxillary equations of (2) are,

$$\frac{dx}{x-y} = \frac{dy}{x+y} = \frac{dz}{2xz} \qquad \text{—— (3)}$$

Taking first two fractions of (2), we get,

$$\frac{dx}{x-y} = \frac{dy}{x+y}$$

$\therefore$ $(x+y)\,dx - (x-y)dy = 0$.

$\therefore (xdx + ydy) + (ydx - xdy) = 0$

dividing both sides by x^2+y^2, we get,

$$\frac{xdx+ydy}{x^2+y^2} + \frac{ydx - xdy}{x^2+y^2} = 0.$$

$\therefore \frac{1}{2} d\left(\log(x^2+y^2)\right) + d\left(\tan^{-1}\left(\frac{y}{x}\right)\right) = 0.$

$$\therefore \boxed{\frac{1}{2}\log(x^2+y^2) + \tan^{-1}\frac{y}{x} = c_1} \qquad \text{—— (4)}$$

using multipliers $1, 1, -\frac{1}{z}$,

each fraction of (3) $= \dfrac{dx + dy - \frac{1}{z}dz}{(x-y) + (x+y) - \frac{1}{z} \times 2xz}$

$= \dfrac{dx + dy - \frac{1}{z}dz}{0}$

$\therefore \quad dx + dy - \frac{1}{z}dz = 0$

$\therefore \boxed{x + y - \log z = c_2}$ —— (5)

from (4) & (5), required g.s. of (1) is:

$$\boxed{f\left(\frac{1}{2}\log(x^2+y^2) + \tan^{-1}\frac{y}{x},\ x+y-\log z\right) = 0}$$

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Q.6 b) Find the surface which is orthogonal to the family of surfaces $z(x+y) = c(3z+1)$ & which passes through circle $x^2+y^2=1$, $z=1$

$\Rightarrow$ Let,

$$f = \frac{z(x+y)}{3z+1} \quad \& \quad P = \frac{z}{3z+1}, \quad Q = \frac{z}{3z+1}, \quad R = \frac{x+y}{(3z+1)^2}$$

Surfaces are generated by integral curves of

$$\frac{dx}{z(3z+1)} = \frac{dy}{z(3z+1)} = \frac{dz}{x+y} \quad \text{———} \quad (1)$$

taking first two fractions of (1); we get ~~$4(x+y) = x$~~.

or $$\frac{dx}{\cancel{z}(3\cancel{z}+1)} = \frac{dy}{\cancel{z}(3\cancel{z}+1)}$$

i.e., $dx = dy$

Integrating, we get,

$$\boxed{x - y = c_1} \quad \text{———} \quad (2)$$

Now, $$\frac{x\,dx + y\,dy}{z(x+y)(3z+1)} = \frac{dz}{x+y};$$

$\Rightarrow$ $x\,dx + y\,dy = z(3z+1)\,dz$.

Integrating we get.

$$x^2+y^2 = 2z^3 + z^2 + c_2$$

$$\therefore \boxed{x^2+y^2-2z^3-z^2 = c_2} \quad \text{———} \quad (3)$$

Thus any surface which is orthogonal to given surface is given by $f(c_1, c_2) = f(x-y,\ x^2+y^2-2z^3-z^2)$

for any f (of one variable) $\Rightarrow$ $x^2+y^2-2z^3-z^2 = f(x-y)$ is solution.

For perticular surface passing through circle, $x^2+y^2=1, z=1$

take, $f=-2$,

$\therefore$ Required surface is

$$\boxed{x^2+y^2 = 2z^3+z^2-2}$$

Q.6 c> Find complete integral of $xp - yq = xq f(z - px - qy)$ where

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}$$

$\rightarrow$ Let, $F(x,y,z,p,q) = xp - yq - xqf(z - px - qy) = 0$ ——— (1)

Chorpit's auxilory eq^ns are.

$$\frac{dp}{\partial F/\partial x + p(\partial F/\partial z)} = \frac{dq}{\partial F/\partial y + q(\partial F/\partial z)} = \frac{dz}{-p(\partial F/\partial p) - q(\partial F/\partial q)} = \frac{dx}{-(\partial F/\partial p)} = \frac{\partial y}{-(\partial F/\partial q)} \text{ ——— (2)}$$

$$\Rightarrow \frac{dp}{p - qf + xqpf' - pqxf'} = \frac{dp}{-q + xq^2 f' - xq^2 f'} \text{ ——— (3)} \quad (\text{from (1) \& (2)})$$

Each fraction of (3) $= \dfrac{x\,dp + y\,dq}{xp - yq - qxf} = \dfrac{x\,dp + y\,dq}{0}$ (by 2)

$\Rightarrow x\,dp + y\,dq = 0$

$\therefore \underline{x\,dp + y\,dp} + \underline{p\,dx + q\,dy} = p\,dx + q\,dy$ (adding $p\,dx + q\,dy$ on both sides)

$\therefore dz - d(xp) + d(yp) = 0$ or $dz = p\,dx + q\,dy$.

Integrating, we get

$z - xp - yp = a$ ——— (4) (a = constant).

$\therefore \underline{xp + yq = a - z}$ ——— (5)

from (4), (1) becomes,

$xp - yq = xq f(a)$ ——— (6)

substract, (6) from (5); $2yq = z - q - xq f(a)$

$\Rightarrow q = (z - a)/(2y + xf(a))$ ——— (7)

Using (7), (5) $\equiv$ $p = \dfrac{(z-a)(y + xf(a))}{x(2y + xf(a))}$ ——— (8)

using (7) & (8),

$dz = p\,dx + q\,dy$ becomes

$$dz = (z-a)\left[\frac{[y + x f(a)]\,dx}{x(2y + x f(a))} + \frac{dy}{2y + x f(a)}\right]$$

~~Integrating, we get~~

$\cancel{2\log(z-a)}$

$$\therefore \frac{2\,dz}{z-a} = \frac{2y\,dx + 2x f(a)\,dx + 2x\,dy}{x(2y + x f(a))}$$

Integrating we get,

$$2\log(z-a) = \log\left(2xy + x^2 f(a)\right) + \log b$$

i.e., $\boxed{(z-a)^2 = b\left(2xy + x^2 f(a)\right)}$.

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Q.6d> A tightly stretched string with fixed end points $x=0$ & $x=l$ is initially in a position given by $y = y_0 \sin^3\left(\frac{\pi x}{l}\right)$. It is released from rest from this position, find the displacement $y(x,t)$.

$\Rightarrow$ Given,

$$y = y_0 \sin^3\left(\frac{\pi x}{l}\right)$$

~~We ha~~

Boundary condition:- $y(0,t) = y(\pi,t) = 0 \quad \forall\, t \geq 0$

Initial condition:-

$$\text{Initial velocity} = \left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 \quad \text{for } 0 \leq x \leq \pi \qquad \text{---(1)}$$

$$\text{Initial displacement} = y(x,0) = y_0 \sin^3\left(\frac{\pi x}{l}\right) \qquad \text{---(2)}$$

We know that, solution of one dimensional wave satisfying given Boundary ~~condi~~ & initial condition can be given as,

$$y(x,t) = \sum_{n=1}^{\infty}\left(E_n \cos\frac{n\pi ct}{l} + F_n \sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l} \qquad \text{---(3)}$$

where $E_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\,dx.$

$$F_n = \frac{2}{n\pi c}\int_0^l g(x)\sin\frac{n\pi x}{l}\,dx.$$

For initial velocity, differentiate (2) partially w.r.t. 't', we get,

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty}\left\{-\frac{n\pi c}{l}E_n \sin\frac{n\pi c t}{l} + \frac{n\pi c}{l}F_n \cos\frac{n\pi c}{l}t\right\}\sin\frac{n\pi}{l}x. \qquad \text{---(4)}$$

put $t=0$ in ③ & ④ & by using initial conditions,

we get,

$$③ \Rightarrow y(x,0) = y_0 \sin^3\left(\frac{\pi x}{l}\right) = \sum_{n=1}^{\infty} E_n \sin \frac{n\pi}{l} x. \quad \text{—— (5)}$$

$$④ \Rightarrow \left(\frac{\partial y}{\partial t}\right)_{t=0} = 0 = \sum_{n=1}^{\infty} \frac{n\pi c}{l} F_n \sin \frac{n\pi x}{l} \quad \text{—— (6)}$$

where $F_n = \frac{2}{n\pi c} \int_0^l (0) \sin \frac{n\pi x}{l} \, dx. = 0.$

$$\therefore y(x,0) = y_0 \sin^3\left(\frac{\pi x}{l}\right) = \sum_{n=1}^{\infty} E_n \sin \frac{n\pi}{l} x$$

$$\left[\sin 3\theta = 3\sin\theta - 4\sin^3\theta. \quad \therefore \sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4} \right]$$

$$\therefore \frac{y_0 \times 3\sin\left(\frac{\pi x}{l}\right) - \sin\left(\frac{3\pi x}{l}\right)}{4} = E_1 \sin \frac{\pi x}{l} + E_2 \sin \frac{2\pi x}{l} + E_3 \sin \frac{3\pi x}{l} + \cdots$$

comparing coefficients of similar terms, we have,

$$E_1 = \frac{3y_0}{4}, \quad E_2 = 0, \quad E_3 = -\frac{y_0}{4}, \quad E_4 = E_5 = E_6 = \cdots E_n = 0.$$

Putting above values in eqn ③,

required displacement is given by

$$\boxed{y(x,t) = \frac{3y_0}{4} \sin\left(\frac{\pi x}{l}\right) \cos\left(\frac{\pi c t}{l}\right) - \frac{y_0}{4} \sin \frac{3\pi}{l} x \cos\left(\frac{3\pi c}{l}\right) t}$$

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Q.8d Solve Laplace's eqn $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$ subject to conditions

$u(0,y)=u(l,y)=u(x,0)=0$ & $u(x,a)=\sin\left(\frac{n\pi x}{l}\right)$,

⟹ Given,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \quad \text{——— (1)}$$

Three possible solns of (1) are.

$$u=(c_1e^{px}+c_2e^{-px})(c_3\cos py+c_4\sin py) \quad \text{——— (2)}$$

$$u=(c_5\cos px+c_6\sin px)(c_7e^{py}+c_8e^{-py}) \quad \text{——— (3)}$$

$$u=(c_9x+c_{10})(c_{11}y+c_{12}) \quad \text{——— (4)}$$

We solving (1) to satisfying boundary conditions,

$$u(0,y)=0, \quad u(l,y)=0 \quad \text{——— (5)}$$

$$u(x,0)=0, \quad u(x,a)=\sin(n\pi x/l) \quad \text{——— (6)}$$

using (5) & (2),

we get

$$c_1+c_2=0 \quad \& \quad c_1e^{pl}+c_2e^{-pl}=0$$

solving these eqns, we get $c_1=c_2=0$ which is trivial soln

similarly by using (5), & (4) we get trivial soln.

Hense, possible soln. for problem is solution (3),

using (5) & (3), we have,

$$c_5(c_7e^{py}+c_8e^{-py})=0 \quad \text{i.e., } c_5=0$$

$$\therefore (3) \equiv u=c_6\sin px(c_7e^{py}+c_8e^{-py}) \quad \text{——— (7)}$$

$\therefore$ either $c_6 = 0$ or $\sin(pl) = 0$

If we take $c_6 = 0$ we get a trivial sol^n.

$\therefore \sin(pl) = 0$ ~~is~~
where $pl = n\pi$ or $p = n\pi/l$ where $n = 0, 1, 2, 3, \ldots$

from ⑦ $\equiv$ $u = c_6 \sin n\pi x/l . (c_1 e^{n\pi y/l} + c_8 e^{-n\pi y/l})$

using ⑥ $\equiv$ we get, $0 = c_6 \sin(n\pi x/l)(c_7 + c_8)$

i.e., $c_8 = -c_7$.

$\therefore$ Sol^n ~~for~~ problem is

$$u(x,y) = b_n \sin \frac{n\pi x}{l}(e^{n\pi y/l} - e^{-n\pi y/l})$$

where $b_n = c_6 c_7$

from ⑥;

$$u(x,0) = \sin\left(\frac{n\pi x}{l}\right) = b_n \sin \frac{n\pi x}{l}(e^{n\pi a/l} - e^{-n\pi a/l})$$

$\therefore$ we get, $\quad b_n = \dfrac{1}{e^{n\pi a/l} - e^{-n\pi a/l})}$

Hense, required eq^n is

$$u(x,y) = \frac{e^{n\pi y/l} - e^{-n\pi y/l}}{e^{n\pi a/l} - e^{-n\pi a/l}} \sin \frac{n\pi x}{l} = \frac{\sinh(n\pi y/l)}{\sinh(n\pi a/l)} \sin \frac{n\pi x}{l}.$$

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