2018 8(a) IFoS

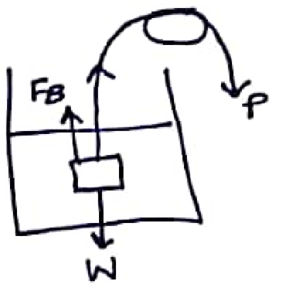
A body is immersed in a liquid. is balanced by a weight P to which it is attached by a thread passing over a fixed pulley and when half immersed, is balanced in the same manner by weight 2P. Prove that the density of body and liquid are in ratio 3 : 2.

Let ρ_s = density of body $\quad$ ρ_l = density of liquid

V = volume of body $\quad$ W = weight of body = $\rho_s V g$

F_B = Buoyant Force

Body immersed in liquid

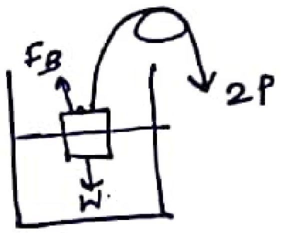


$$W = \rho_s V g$$
$$F_B = \rho_l V g$$

Balancing forces,

$$\rho_l V g + P = \rho_s V g \quad \text{— (1)}$$

Body half immersed



$$W = \rho_s V g$$
$$F_B = \rho_l \frac{V}{2} g$$

Balancing forces,

$$\rho_l \frac{V}{2} g + 2P = \rho_s V g \quad \text{— (2)}$$

(1) − (2),

$$P = \rho_l \frac{V}{2} g \quad \text{— (3)}$$

Putting (3) in (1),

$$3 \rho_l \frac{V}{2} g = \rho_s V g$$

$$\Rightarrow \boxed{\frac{\rho_s}{\rho_l} = \frac{3}{2}}$$

Proved

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8(b) Solve the DE

$$\frac{dy}{dx} - y = y^2(\sin x + \cos x)$$ (10)

$$\Rightarrow \frac{-1}{y^2}\frac{dy}{dx} + \frac{1}{y} = \sin x + \cos x$$

It is Bernoulli's equation

Let $\frac{1}{y} = z$, $\frac{-1}{y^2}\frac{dy}{dx} = \frac{dz}{dx}$

$$\therefore \frac{dz}{dx} + z = \sin x + \cos x$$

$$I.F. = e^{\int 1 \cdot dx} = e^x$$

Solution:

$$z \cdot e^x = \int e^x(\sin x + \cos x)\,dx$$

$$ze^x = \int e^x \sin x\,dx + \int e^x \cos x\,dx$$

$$\cancel{= e^x(-\cos x) - \int e^x}$$

$$= (\sin x)e^x - \cancel{\int(\cos x)e^x\,dx} + \cancel{\int e^x \cos x\,dx}$$

(Integrating by parts)

$$= e^x \sin x + C$$

$$z = \sin x + c e^{-x}$$

i.e. $\boxed{y(\sin x + c e^{-x}) = 1}$

$(A \times B) \times C = (A \cdot C) B - (B \cdot C) A$
$A \times (B \times C) = (A \cdot C) B - (A \cdot B) C$

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8(c) Prove that $a \times (b \times c) = (a \times b) \times c$, if and only if either $b = 0$ or c is collinear with a or b is perpendicular to both a and c. (10)

First, If $b = 0$, then $a \times (b \times c) = 0$
and $(a \times b) \times c = 0$, hence true.

If c is collinear with a i.e. $c = \lambda a$

$$a \times (b \times c) = a \times [b \times (\lambda a)]$$
$$= [a \cdot (\lambda a)] b - [a \cdot b] (\lambda a)$$
$$= \lambda [|a|^2 b - (a \cdot b) a]$$

$$(a \times b) \times c = (a \times b) \times (\lambda a)$$
$$= (a \cdot (\lambda a)) b - (b \cdot (\lambda a)) a$$
$$= \lambda [|a|^2 b - (a \cdot b) a]$$

$$\therefore \quad a \times (b \times c) = (a \times b) \times c.$$

If b is $\perp$ to a and c both
i.e. $b \cdot a = 0$, $b \cdot c = 0$

$$a \times (b \times c) = (a \cdot c) b - (a \cdot b) c$$
$$= (a \cdot c) b$$

$$(a \times b) \times c = (a \cdot c) b - (b \cdot c) a$$
$$= (a \cdot c) b$$

$$\therefore \quad (a \times b) \times c = a \times (b \times c).$$

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Conversely,
let $\quad a \times (b \times c) = (a \times b) \times c$

i.e. $\quad \cancel{(a \cdot c) b} - (a \cdot b) c = \cancel{(a \cdot c) b} - (b \cdot c) a$

$$(b \cdot c) a - (a \cdot b) c = 0$$

$$b \times (a \times c) = 0$$

This is possible, when either of the condition is met.

i) $b = 0$

ii) c is collinear with a, then $a \times c = 0$

iii) $b \cdot a = 0$ & $b \cdot c = 0$ i.e. b is perpendicular to both a and c.

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8(d) A particle is acted on by a force parallel to the axis of y whose acceleration is λy, initially projected with a velocity $a\sqrt{\lambda}$ parallel to x-axis at the point where $y = a$. Prove that it will describe a catenary. (10)

Given, $\dfrac{d^2y}{dt^2} = \lambda y$

$$\Rightarrow \quad 2\frac{dy}{dt}\cdot\frac{d^2y}{dt^2} = 2\lambda y\frac{dy}{dt}$$

[Multiplying by $2\frac{dy}{dt}$ and integrating]

$$\left(\frac{dy}{dt}\right)^2 = \lambda y^2 + C_1$$

When $t=0$, $\dfrac{dy}{dt} = 0$ and $y = a$

(Initial velocity is 0 in y-direction)

$$\therefore \quad C_1 = -\lambda a^2$$

$$\left(\frac{dy}{dt}\right)^2 = \lambda(y^2 - a^2)$$

$$\frac{dy}{dt} = \sqrt{\lambda}\sqrt{y^2 - a^2} \qquad \text{—— (1)}$$

Also, in x-direction, $\dfrac{d^2x}{dt^2} = 0$

(No acceleration in x-direction]

$\dfrac{dx}{dt} = C_2$; $t=0$, $\dfrac{dx}{dt} = a\sqrt{\lambda} \Rightarrow C_2 = a\sqrt{\lambda}$

$$\therefore \quad \frac{dx}{dt} = a\sqrt{\lambda} \qquad \text{—— (2)}$$

Dividing (1) by (2), $\dfrac{dy}{dx} = \dfrac{\sqrt{y^2 - a^2}}{a}$

i.e. $\dfrac{dy}{\sqrt{y^2-a^2}} = \dfrac{dx}{a} \Rightarrow \cosh^{-1}\dfrac{y}{a} = \dfrac{x}{a} + C_3$

Initially, $x=0$ and $y=a \Rightarrow C_3 = \cosh^{-1}(1) = 0$

$\therefore$ $\boxed{y = a\cosh(x/a)}$ ← Eqn of catenary.