CSE 2010

1. Find D.D. of $f(x,y) = x^2y^3 + xy$ at $(2,1)$ in $\hat{a}$ which makes angle $\frac{\pi}{3}$ with x-axis.

direction ratios of desired unit vector are $\hat{a} = \cos\frac{\pi}{3}\hat{i} + \sin\frac{\pi}{3}\hat{j} = \frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}$

$DD = \nabla f \cdot \hat{a}$

$\Rightarrow \nabla f|_{(2,1)} = (2xy^3 + y)\hat{i} + (3x^2y^2 + x)\hat{j}$

$= 5\hat{i} + 14\hat{j}$

$DD = (5\hat{i} + 14\hat{j}) \cdot \left(\frac{1}{2}\hat{i} + \frac{\sqrt{3}}{2}\hat{j}\right)$

$= \frac{5}{2} + 7\sqrt{3}$ Ans

2. Show $\vec{V} = xyz(yz\hat{i} + xz\hat{j} + xy\hat{k})$ is conservative.

i.e. TST $\nabla \times \vec{v} = 0$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2z^2 & x^2yz^2 & x^2y^2z \end{vmatrix}$$

$= \hat{i}(2x^2yz - 2x^2yz) + \hat{j}(2xy^2z - 2xy^2z) + \hat{k}(2xyz^2 - 2xyz^2) = 0\hat{i} + 0\hat{j} + 0\hat{k}$

so $\vec{V}$ is conservative.

3. Prove $\text{div}(f\vec{V}) = f(\text{div}\,\vec{V}) + (\text{grad}\,f)\vec{V}$; f is scalar

TST $\nabla\cdot(f\vec{V}) = f(\nabla\cdot\vec{V}) + (\nabla f)\cdot\vec{V}$

$$\nabla\cdot(f\vec{V}) = \sum \hat{i}\cdot\frac{\partial(f\vec{V})}{\partial x}$$

$$= \sum \hat{i}\cdot\left(\frac{\partial f}{\partial x}\vec{V}\right) + \sum \hat{i}\cdot\left(f\frac{\partial \vec{V}}{\partial x}\right)$$

using $\boxed{\frac{\partial}{\partial x}(\phi\vec{A}) = \frac{\partial\phi}{\partial x}\vec{A} + \phi\frac{\partial\vec{A}}{\partial x}}$

$$= \sum \frac{\partial f}{\partial x}\hat{i}\cdot\vec{V} + f\sum \hat{i}\cdot\frac{\partial\vec{V}}{\partial x}$$

$$= \nabla f\cdot\vec{V} + f\,\nabla\cdot\vec{V}$$

$$= \nabla f\cdot\vec{V} + f\,\text{div}\,\vec{V}$$

Hence proved.

Q-4 Use divergence thm $\iint_S \vec{V}\cdot\hat{n}\,dA$

$\vec{V} = x^2z\hat{i} + y\hat{j} - xz^2\hat{k}$; S is boundary of region bounded by paraboloid $z = x^2+y^2$ and the plane $z = 4y$

$$\iint \vec{V}\cdot\hat{n}\,dA = \iiint_V (\nabla\cdot\vec{V})\,dV$$

V = volume bounded by region S

ROUGH

intersection of plane $z = 4y$ with paraboloid $z = x^2 + y^2$ is

$$x^2 + y^2 = 4y \Rightarrow x^2 + y^2 - 4y + 4 = 4$$

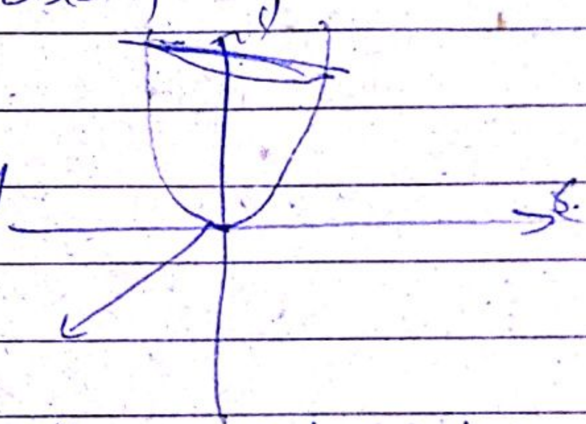
$$x^2 + (y-2)^2 = (2)^2$$

which is circle with centre $(0,2)$ and radius 2

$$\nabla \cdot \vec{V} = 2x2 + 1 - 2x2 = 1$$

so $I = \iiint 1 \, dV = \iiint dx\,dy\,dz$

$$= \iint \left(\int_{z=x^2+y^2}^{4y} 1 \, dz \right) dx\,dy$$



$$= \iint (4y - x^2 - y^2) \, dx\,dy$$

Put $x = r\cos\theta$,
$y = 2 + r\sin\theta$
$0 \le \theta \le 2\pi$ → converting into polar co-ordinates

Then

$$I = \int_{r=0}^{2} \int_{\theta=0}^{2\pi} \left(4(2 + r\sin\theta) - r^2\cos^2\theta - (2 + r\sin\theta)^2\right) r \, dr \, d\theta$$

$$= \iint (8 + 4r\sin\theta - r^2\cos^2\theta - 4 - r^2\sin^2\theta - 4r\sin\theta) r \, dr\,d\theta$$

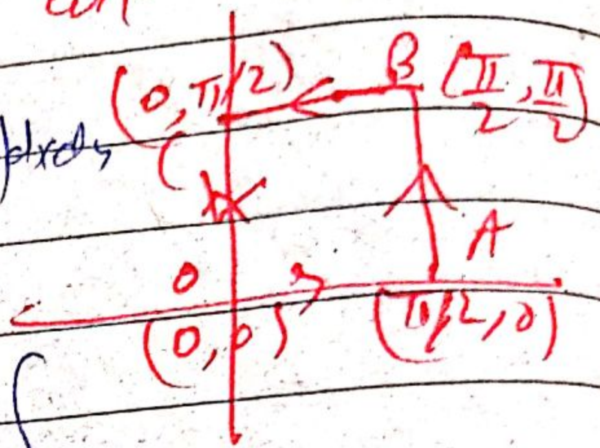
$$= \int_{\theta=0}^{2\pi} \int_0^2 (4r - r^3) \, dr\,d\theta = \left[2r^2 - \frac{r^4}{4}\right]_0^2 \times 2\pi$$

$$= (8 - 4) 2\pi = \boxed{8\pi} \text{ Ans.}$$

5. Verify Green's Th^m

$$\oint e^{-x} \sin y \, dx + e^{-x} \cos y \, dy$$

C: boundary of square with vertices



$$\oint_C P\,dx + Q\,dy = \iint_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy$$

$$\oint_C = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$

OA : $y = 0, \; 0 \le x \le \pi \quad dy = 0$

AB : $x = \frac{\pi}{2} \quad 0 \le y \le \frac{\pi}{2}, \; dx = 0$

BC : $y = \frac{\pi}{2}; \; \frac{\pi}{2} \le x \le 0, \; dy = 0$

CO : $x = 0; \; \frac{\pi}{2} \le y \le 0; \; dx = 0.$

$$I = \int_{OA} \left[e^{-x}(0) + 0\right] + \int_{AB} e^{-\frac{\pi}{2}} \cos y \, dy$$

$$+ \int_{BC} e^{-x} \sin \frac{\pi}{2} \, dx + \int_{CO} \cos y \, dy$$

$$= e^{-\frac{\pi}{2}} \sin y \Big|_0^{\pi/2} + \frac{e^{-x}}{-1} \Big|_{\pi/2}^{0} + \sin y \Big|_{\pi/2}^{0}$$

$$= \boxed{e^{-\frac{\pi}{2}} - 1} + \cancel{e^{-\frac{\pi}{2}}} - e^{0} = 2\left(e^{-\frac{\pi}{2}} - 1\right)$$

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$$\frac{\partial Q}{\partial x} = -e^{-x}\cos y; \quad \frac{\partial P}{\partial y} = e^{-x}\cos y$$

$$\iint (-e^{-x}\cos y - e^{-x}\cos y)\,dx\,dy$$

$$= -\iint 2e^{-x}\cos y\,dx\,dy$$

$$= -2\int_{x=0}^{\pi/2} e^{-x}\,dx \times \int_0^{\pi/2} \cos y\,dy$$

$$= -2 \times \frac{e^{-x}}{-1}\Big|_0^{\pi/2} \times \sin y\Big|_0^{\pi/2}$$

$$= -2 \times \left[-e^{-\pi/2} + 1\right] \times 1$$

$$= 2(e^{-\pi/2} - 1) \quad \text{--- } (2)$$

Hence verified