IAS MATHEMATICS (OPT.)-2016

PAPER - 2 : SOLUTIONS

Q1(a) PYQ-2016

Let K be a field and K[X] be the ring of polynomials over K in a single variable X. For a polynomial $f \in K[X]$, let (f) denote the ideal in K[X] generated by f. Show that (f) is a maximal ideal in K[X] if and only if f is an irreducible polynomial over K.

Ans. Suppose first that (f) is a maximal ideal in K[X].

T.S: f is an irreducible polynomial over K.

Clearly, f is neither the zero polynomial nor a unit in K[X] as (f) is a maximal ideal in K[X]. ($\because$ $\{0\}$ nor K[X] is a maximal ideal in K[X]).

If $f(x) = g(x)\,h(x)$ is a factorization of f(x) over K,

then $\langle f(X)\rangle \subseteq \langle g(X)\rangle \subseteq K[X]$.

Thus, $\langle f(X)\rangle = \langle g(X)\rangle$ or $K[X] = \langle g(X)\rangle$.

In the first case, we must have,

$\deg f(X) = \deg g(X)$ and in the

second case, it follows that $\deg g(X) = 0$

$\Rightarrow \deg h(X) = \deg f(X)$.

Thus, $f(X)$ cannot be written as a product of two polynomials in $K[X]$ of lower degree.

$\therefore$ $f(x)$ is irreducible over K.

Conversely, suppose that $f(X)$ is irreducible over K.

T.S: $\langle f(X) \rangle$ is a maximal ideal in $K[X]$.

Now, suppose that $f(x)$ is irreducible over K.

Let I be any ideal of $K[X]$ such that

$\langle f(X) \rangle \subseteq I \subseteq K[X]$.

Because, $K[X]$ is a principal ideal domain, we know that $I = \langle g(X) \rangle$ for some $g(x)$ in $K[X]$.

$\Rightarrow f(X) \in \langle g(X) \rangle$

$\therefore$ ~~f(X)~~ $f(X) = g(x)\, h(X)$, where $h(x) \in K[X]$.

Since $f(x)$ is irreducible over K,

$\Rightarrow g(X)$ is constant or $h(X)$ is constant.

If $g(x)$ is constant, $\Rightarrow I = K[X]$

If $h(X)$ is constant, $\Rightarrow \langle f(x) \rangle = \langle g(x) \rangle = I$

$\therefore \langle f(X) \rangle$ is maximal in $K[X]$.

1(c) IAS-2016.

Two sequences $\{x_n\}$ and $\{y_n\}$ are defined inductively by the following:

$x_1 = \frac{1}{2}, y_1 = 1$ and $x_n = \sqrt{x_{n-1} y_{n-1}}$, $n = 2, 3, 4, \ldots$

$$\frac{1}{y_n} = \frac{1}{2}\left(\frac{1}{x_n} + \frac{1}{y_{n-1}}\right), n = 2, 3, 4 \ldots$$

prove that $x_{n-1} < x_n < y_n < y_{n-1}$, $n = 2, 3, \ldots$

and deduce that both the sequences converge to the same limit 'l', where $\frac{1}{2} < l < 1$.

Sol Given that $x_1 = 1/2, y_1 = 1$

$\therefore \frac{1}{2} = x_1 < y_1 = 1$ —— (1)

We have $x_2 = \sqrt{x_1 y_1} < \sqrt{y_1 y_1} = y_1$

$\therefore x_2 < y_1$ —— (2)

We have $\frac{1}{y_2} = \frac{1}{2}\left(\frac{1}{x_2} + \frac{1}{y_1}\right)$

$\Rightarrow \frac{1}{y_2} < \frac{1}{2}\left(\frac{1}{x_2} + \frac{1}{x_2}\right)$

$= \frac{1}{x_2}$ $\left(\because x_2 < y_1 \Rightarrow \frac{1}{x_2} > \frac{1}{y_1}\right)$

$\therefore \frac{1}{y_2} < \frac{1}{x_2} \Rightarrow y_2 > x_2$

$\therefore x_2 < y_2$ —— (3)

let us suppose that

$x_{n-1} < y_{n-1}$ — (4).

we have $x_n = \sqrt{x_{n-1} \cdot y_{n-1}}$

$< \sqrt{y_{n-1} \cdot y_{n-1}}$ (by (4))

$= y_{n-1} \ \forall \ n \geq 2.$

we have — (5)

$\frac{1}{y_n} = \frac{1}{2}\left(\frac{1}{x_n} + \frac{1}{y_{n-1}}\right)$

$< \frac{1}{2}\left(\frac{1}{x_n} + \frac{1}{x_n}\right)$

$= \frac{1}{x_n}$ $\left(\because x_n < y_{n-1} \Rightarrow \frac{1}{x_n} > \frac{1}{y_{n-1}} \Rightarrow \frac{1}{y_{n-1}} < \frac{1}{x_n}\right)$

$\therefore \frac{1}{y_n} < \frac{1}{x_n}$.

$\Rightarrow y_n > x_n$

$\Rightarrow x_n < y_n \ \forall \ n \geq 1$ — (6)

we have $x_n = \sqrt{x_{n-1} \cdot y_{n-1}}$

$> \sqrt{x_{n-1} \, x_{n-1}}$ (by (4))

$= x_{n-1}$

$\therefore x_n > x_{n-1} \ \forall \ n \geq 2$ — (7)

we have $\frac{1}{y_n} = \frac{1}{2}\left(\frac{1}{x_n} + \frac{1}{y_{n-1}}\right)$

$> \frac{1}{2}\left(\frac{1}{y_{n-1}} + \frac{1}{y_{n-1}}\right)$

$= \frac{1}{y_{n-1}}$ by (5)

$\therefore \frac{1}{y_n} > \frac{1}{y_{n-1}}$

$\Rightarrow y_n < y_{n-1} \quad \forall\, n \geq 2$ ——— (8)

from (6), (7) & (8),

we have

$x_{n-1} < x_n < y_n < y_{n-1} \quad \forall\, n \geq 2$

$\therefore \frac{1}{2} = x_1 < x_2 < x_3 < \cdots < y_3 < y_2 < y_1 = 1$

$\therefore (x_n)$ is an increasing and bounded above by 1.

$\therefore (x_n)$ converges.

and (y_n) is a decreasing and bounded below by $\frac{1}{2}$.

$\therefore (y_n)$ converges.

Let $\lim x_n = l$ then $\lim x_{n-1} = l$.

and

$\lim y_n = m$ then $\lim y_{n-1} = m$.

$\therefore l = \sqrt{lm}$ and $\frac{1}{m} = \frac{1}{2}\left(\frac{1}{l} + \frac{1}{m}\right)$

$\therefore l = m$

and $\frac{1}{2} < l < 1$.

1(d) 2016. Is $v(x,y) = x^3 - 3xy^2 + 2y$ a harmonic function? Prove your claim. If yes, find its conjugate harmonic function $u(x,y)$ and hence obtain the analytic function whose real and imaginary parts are u and v respectively.

Sol^n: given that $v(x,y) = x^3 - 3xy^2 + 2y$.

$$\therefore \frac{\partial v}{\partial x} = 3x^2 - 3y^2 \;;\; \frac{\partial^2 v}{\partial x^2} = 6x.$$

$$\frac{\partial v}{\partial y} = -6xy + 2 \;;\; \frac{\partial^2 v}{\partial y^2} = -6x$$

$$\therefore \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0.$$

$\therefore$ $v(x,y)$ is a harmonic function.

Let the harmonic conjugate be $u(x,y)$. Then

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \Rightarrow \frac{\partial u}{\partial y} = -3x^2 + 3y^2$$

$$\therefore u(x,y) = -3x^2y + y^3 + c_1(x).$$

Also

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} \Rightarrow \frac{\partial u}{\partial x} = -6xy + 2$$

$$\therefore u(x,y) = -3x^2y + 2x + c_2(y).$$

$$\therefore u(x,y) = -3x^2y + y^3 + 2x.$$

$\therefore$ The analytic function with real and imaginary parts as u & v respectively is,

$$w = f(u,v) = u + iv = (-3x^2y + y^3 + 2x) + i(x^3 - 3xy^2 + 2y)$$

1.(e)Find the maximum value of 5x + 2y with constraints $x + 2y \geq 1$, $2x + y \leq 1$, $x \geq 0$ and $y \geq 0$ by graphical method.

Solution

2016 1(e)

1) Use graphical method:-

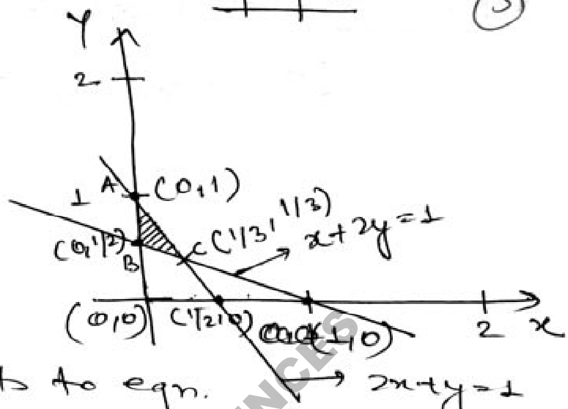
Max $= 5x + 2y$

S.C $x + 2y \geq 1$

$2x + y \leq 1$

$x \geq 0 \quad y \geq 0$

Graph (3)



By converting all constraints to eqn.

$x + 2y = 1$ —(i)

$2x + y = 1$ —(ii)

From eqn –(i) we get

Putting $x = 0$; $y = 1/2$

$y = 0$; $x = 1$

x	0	1
y	1/2	0

From eqn (ii) we get

Putting $x = 0$; $y = 1$

$y = 0$; $x = 1/2$

x	0	1/2
y	1	0

Shaded region ABC satisfies the condition

from (i) and (ii) we get $C = (1/3, 1/3)$

$x + 2y = 1 \times 2$

$2x + y = 1 \times 1$

$2x + 4y = 2$

$2x + y = 1$

$(-) \quad (-)$

$3y = 1$

$y = 1/3$

$x = 1/3$

$\therefore$ Max $= 5x + 2y$

at $A(0,1) = 5 \cdot 0 + 2 = 2$

$B(0, 1/2) = 5 \cdot 0 + 1 = 1$

$C(1/3, 1/3) = \frac{5}{3} + \frac{2}{3} = \frac{7}{3}$

$\therefore$ Max. value of Z is obtain at $C(1/3, 1/3) = 7/3$

Ans.

Q(c) IAS -2016

→ Show that the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$ is conditionally convergent. (If you use any theorem(s) to show it, then you must give a proof of that theorem(s).)

Sol Given that $\sum \frac{(-1)^{n+1}}{n+1} = \sum (-1)^{n+1} u_n$ say

Here $u_n = \frac{1}{n+1} \; \forall n \in \mathbb{N}$

$u_{n+1} = \frac{1}{n+2}$

$\because n+2 > n+1 \; \forall n \in \mathbb{N}$

$\Rightarrow \frac{1}{n+2} < \frac{1}{n+1} \; \forall n$

$\Rightarrow u_{n+1} < u_n \; \forall n \in \mathbb{N}$

$\Rightarrow u_n > u_{n+1} \; \forall n.$

$\therefore (u_n)$ is a decreasing.

and $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n+1} = 0.$

$\therefore$ By Leibnitz test, $\sum_{n=1}^{\infty} (-1)^{n+1} u_n$ converges.

Let us prove Leibnitz Test:

let $\sum_{n=1}^{\infty} (-1)^{n-1} u_n = u_1 - u_2 + u_3 - \cdots \cdots (-1)^{n-1} u_n + \cdots$

$u_n > 0 \; \forall n \in \mathbb{N}$

then (i) $u_{n+1} \leq u_n \; \forall n \in \mathbb{N}$

(ii) $\lim u_n = 0$

Let proof $S_{2n} = u_1 - u_2 + \cdots + u_{2n-1} - u_{2n}$

$S_{2n+1} = u_1 - u_2 + \cdots - u_{2n} + u_{2n+1}$.

we shall first show that the sequences (S_{2n}) and (S_{2n+1}) converge to the same limit say 's'.

we have $S_{2n+2} - S_{2n} = u_{2n+1} - u_{2n+2}$

$\geq 0 \; \forall n$ (by (i)).

$\therefore S_{2n+2} \geq S_{2n} \; \forall n$.

$\therefore (S_{2n})$ is monotonically increasing.

Again $S_{2n} = u_1 - u_2 + u_3 - \cdots + u_{2n-1} - u_{2n}$

$= u_1 - [(u_2 - u_3) + (u_4 - u_5) + \cdots + (u_{2n-2} - u_{2n-1}) + u_{2n}]$

$\leq u_1 \; \forall n$ (by (i))

$\therefore (S_{2n})$ is bounded above by u_1.

$\therefore (S_{2n})$ is a decreasing and bounded above by u_1

$\therefore (S_{2n})$ is convergent

let $S_{2n} = s$ and $\lim u_{2n+1} = 0$ (by (ii))

$\therefore \lim\limits_{n\to\infty}(S_{2n} + u_{2n+1}) = \lim\limits_{n\to\infty} S_{2n} + \lim\limits_{n\to\infty} u_{2n+1}$

$= S + 0$

$= S.$

$\therefore \lim\limits_{n\to\infty} S_{2n+1} = S$

$\therefore$ the sequences (S_{2n}) and (S_{2n+1}) converge to the same limit 'S'

Now we shall show that (S_n) cgs to 'S'

Let $S_n = u_1 - u_2 + \cdots + (-1)^{n-1} u_n$.

Let $\epsilon > 0$ be given however small and the sequences (S_{2n+1}) and (S_{2n}) both converge to 'S'

$\therefore$ there exist $p, q \in I^+$ such that

$|S_{2n} - S| < \epsilon \ \forall\ 2n > p$ and

$|S_{2n+1} - S| < \epsilon \ \forall\ 2n+1 > q$.

The above inequalities imply that $|S_n - S| < \epsilon \ \forall\ n > \max\{p, q\}$.

$\therefore (S_n)$ converges to 'S'

$\therefore \sum (-1)^{n-1} u_n$ also converges to 'S'.

Q.2 (b) 2016 → Let p be a prime number and Z_p denote the additive group of integers modulo p. Show that every non-zero elements of Z_p generates Z_p.

Solution → Let $Z_p = \{0, 1, 2, \ldots, p-1\}$

now $1(1) = 1$

$2(1) = 1+1 = 2$

$3(1) = 1+1+1 = 3$

$\vdots$

$(p-1)(1) = p-1$

$p(1) = p = 0$ etc.

$\therefore$ 1 is the generator of $(Z_p, +)$

i.e. $(Z_p, +) = \langle 1 \rangle$

$\therefore$ $\langle Z_p, + \rangle$ is a cyclic group generated by 1.

Similarly we can prove that $2, 3, \ldots, p-1$ are also generators.

2.(c) **Maximize** $z = 2x_1 + 3x_2 + 6x_3$

subject to $2x_1 + x_2 + x_3 \le 5$

$3x_2 + 2x_3 \le 6$

$x_1 \ge 0,\ x_2 \ge 0,\ x_3 \ge 0.$

Is the optimal solution unique ? Justify your answer.

Solution:

Max $z = 2x_1 + 3x_2 + 6x_3$

S.T $2x_1 + x_2 + x_3 \le 5$; $3x_2 + 2x_3 \le 6$; $x_i \ge 0$

Soln. Standard form of given problem.

Max $z = 2x_1 + 3x_2 + 6x_3 + 0s_1 + 0s_2$

$$\left.\begin{array}{l} \text{S.T} \quad 2x_1 + x_2 + x_3 + s_1 + 0s_2 = 5 \\ 0x_1 + 3x_2 + 2x_3 + 0s_1 + s_2 = 6 \end{array}\right] \text{-(i)}$$

$\therefore$ IBFS $= (x_1, x_2, x_3, s_1, s_2) = (0, 0, 0, 5, 6)$

Now,

	C_j	2	3	6	0	0			
CB	Basic	x_1	x_2	x_3	s_1	s_2	B	θ	
0	s_1	2	1	1	1	0	5	5	$\therefore$ 2 = key element
0	s_2	0	3	2	0	1	6	3 ←	$\therefore$ I.V = x_3 ; $\therefore$ O.V = s_2
Z_j =	$\Sigma C_B a_{ij}$	0	0	0	0	0			
C_j =	$C_j - Z_j$	2	3	↑6	0	0			
0	s_1	2	$-1/2$	0	1	$-1/2$	2	1 ←	2 = key element
6	x_3	0	$3/2$	1	0	$1/2$	3	—	$\therefore$ I.V = x_1 ; $\therefore$ O.V = s_1
Z_j	=	0	9	6	0	3			
C_j	=	2↑	-6	0	0	-3			
2	x_1	1	$-1/4$	0	$1/2$	$-1/4$	1		
6	x_3	0	$3/2$	1	0	$1/2$	3		
Z_j	=	2	$17/2$	6	1	$5/2$			
C_j		0	$-11/2$	0	-1	$-5/2$			

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From above table ; C_{ij}'s ≤ 0

And Yes, here optimal solution is unique.

As, all non-basic variable in the last table are less than zero.

$\therefore$ optimal Basic soln $= (x_1, x_2, x_3, s_1, s_2)$

$= (1, 0, 3, 0, 0)$

$\therefore$ Max $z = 2x_1 + 3x_2 + 6x_3$

$= 2\cdot(1) + 3(0) + 6(3)$

$= 2 + 18 = \underline{20}$.

3 (a). Let K be an extension of a field F. Prove that the elements of K, which are algebraic over F, form a subfield of K. Further, if $F \subset K \subset L$ are fields, L is algebraic over K and K is algebraic over F, then prove that L is algebraic over F.

IAS 2016

Ans. Let K be an extension of a field F

Let $a \in K$ be any arbitrary element.

Since $a \in K$ is algebraic over F, thus $\exists$ elements $\alpha_0, \alpha_1, \ldots \alpha_n \in F$, not all zero, such that

$$\alpha_0 a^n + \alpha_1 a^{n-1} + \ldots + \alpha_n = 0 \quad \text{—(1)}$$

TS: Element of K form a subfield of K.

As K is a field and let $a, b \in K$ be any two arbitrary elements, then we need to show that for a, b that are algebraic over F and forms a subfield of K i.e. $a-b$ and ab^{-1} are also the elements of K which are algebraic over F.

Clearly, $a-b$ and $ab^{-1} \in K$ ($\because$ K is a field)

Now, To show, $a-b$ and ab^{-1} are algebraic over F. Also, $\alpha_0 a^n + \alpha_1 a^{n-1} + \ldots \alpha_n = 0$

and, $\alpha_0 b^n + \alpha_1 b^{n-1} + \ldots + \alpha_n = 0$

Since, $a, b \in K$ are algebraic over F.

$\Rightarrow F(a,b)$ is a finite extension of F.

Hence, all elements of $F(a,b)$ are algebraic over F. [$\because$ A finite extension is an algebraic extension].

In particular, $a-b$ and ab^{-1}, $b \neq 0$ are all algebraic over F.

$\Rightarrow$ The elements of K form a subfield of K.

TS: If $F \subset K \subset L$ are fields, L is alg. over K and K is algebraic over F. Thus, we need to show L is algebraic over F.

As, Let $\alpha \in L$ and

let $m(X) = b_0 + b_1 X + \cdots + b_{n-1} X^{n-1} + X^n$ be the minimal polynomial of α over K.

$\Rightarrow b_i \in K$ and are therefore algebraic over F.

If $M = F(b_0, b_1, \ldots, b_{n-1})$

$\Rightarrow M$ is a finite extension of F.

($\because$ If E is generated over F by finitely many elements $\alpha_1, \ldots, \alpha_n$ algebraic over F (so that $E = F(\alpha_1, \ldots, \alpha_n)$), then E is a finite extension of F).

Since the coefficients of $m(x) \in M$

$\Rightarrow \alpha$ is algebraic over M.

$\Rightarrow M(\alpha)$ is a finite extension of M.

$\Rightarrow M(\alpha)$ is a finite extension of F

$\therefore \alpha$ is algebraic over F.

Since $\alpha \in L$ is arbitrary,

$\Rightarrow L$ is algebraic over F.

3(b) 2016. → Find the relative maximum and minimum values of the function $f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$.

Sol:- ~~$f(x) = 4x^3 -$~~

Given; $f(x,y) = x^4 + y^4 - 2x^2 + 4xy - 2y^2$

Now,

$$f_x = 4x^3 + (-2)2x + 4y = 4(x^3 - x + y).$$

$$f_y = 4y^3 + 4x - 4y \Rightarrow 4(y^3 - y + x)$$

$$f_x = f_y = 0$$

$$\Rightarrow (x,y) = (0,0), (\sqrt{2}, \sqrt{2}), (\sqrt{2}, -\sqrt{2})$$

$$f_{xx} = 12x^2 - 4$$

$$f_{yy} = 12y^2 - 4$$

$$f_{xy} = f_{yx} = 4$$

$$f_{xx} f_{yy} - f_{xy}^2 > 0$$

$$(12x^2 - 4)(12y^2 - 4) - 16 > 0$$

Also; $f_{xx} > 0$ & $f_{yy} > 0$ at $(x,y) = (\sqrt{2}, \sqrt{2})$ and $(x,y) = (\sqrt{2}, -\sqrt{2})$

Now,

Max → $f(\sqrt{2}, +\sqrt{2}) = 4 + 4 - 2\times2 + 4\times2 - 2\times2$

$$\boxed{f(\sqrt{2}, \sqrt{2}) = 8}$$

Min → $f(\sqrt{2}, -\sqrt{2}) = 4 + 4 - 2\times2 - 4\times2 - 2\times2$

$$\boxed{f(\sqrt{2}, -\sqrt{2}) = -8}$$

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4(a) 2016. → Show that every algebraically closed field is infinite.

Sol'n: Definition: A field F is said to be algebraically closed if each non-constant polynomial in $F[x]$ has a root in F.

→ Let F be a finite field and consider the polynomial

$$f(x) = 1 + \prod_{a \in F} (x-a)$$

The coefficients of $f(x)$ lie in the field F, and thus $f(x) \in F[x]$. Of course, $f(x)$ is a non-constant polynomial.

Note that for each $a \in F$, we have

$$f(a) = 1 \neq 0.$$

So the polynomial $f(x)$ has no root in F

Hence the finite field F is not algebraic closed.

It follows that every algebraically closed field must be infinite.

4(b) 2016 → Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\lim_{x \to +\infty}$ and $\lim_{x \to -\infty} f(x)$ exist and are finite. Prove that f is uniformly continuous on $\mathbb{R}$.

Sol'n: f is uniformly continuous on $\mathbb{R}$, if and only if for every $\epsilon > 0$ $\exists$ $\delta > 0$ such that $\forall$ $x, y \in \mathbb{R}$

if $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$

we also know that the limit exists.

$$\lim_{x \to \infty} f(x) = L$$

This means that for every $\epsilon > 0$ $\exists$ $N > 0$ (which depends upon ϵ) such that if $x > N$; then

$$|f(x) - L| < \epsilon$$

Finally, we know that if $f(x)$ is continuous on a finite closed interval, then it is uniformly continuous on that interval.

we need to show that

$\exists$ $\delta > 0$

such that $\forall$ $x, y \in \mathbb{R}$

if $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$

let $\epsilon > 0$, there is a positive number N, such that

$$|f(x) - L| < \epsilon/2 \quad \forall x > N \text{ or } x < -N$$

The function is continuous on the finite interval $[-N-1, N+1]$, hence f is also uniformly continuous on this compact interval.

Consequently, there exists a positive number $\delta > 1$, such that

$$|f(x) - f(y)| < \varepsilon \;\; \forall x, y \in [-N-1, N+1]$$

$$\text{and } |x-y| < \delta \text{ ——— } (1)$$

let, x and y be numbers, ~~then~~

$$|x-y| < \delta$$

$\Rightarrow |x-y| < 1$ and ~~thus~~ the

$x, y \in [-N-1, N+1]$ or $x, y > |N+1|$

In the latter case.

$$|f(x)-f(y)| = |f(x) - L + L - f(y)|$$

$$\leq |f(x)-L| + |f(y)-L|$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \text{ ——— } (2)$$

So, either (1) and (2) are always true

Thus, f is uniformly continuous on $\mathbb{R}$.

4(c) Prove that every power series represents an analytic function inside its circle of convergence.

Sol:- Let, the radius of convergence of the power series; $f(z) = \sum_{0}^{\infty} a_n z^n$ be R, so that we have

$$\frac{1}{R} = \overline{\lim} |a_n|^{1/n}$$

let $\phi(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$, The radius of convergence of this series is also 'R' for, we have.

$\lim |n a_n|^{1/n} = \overline{\lim}\, n^{1/n} |a_n|^{1/n} = \lim |a_n|^{1/n}$

$[\because \lim n^{1/n} = 1.]$

Suppose, z is any point within the circle of convergence, so that $|z| < R$. Then there exists a positive number r such that

$$|z| < r < R$$

For convenience, we write $|z| = \rho, |h| = \eta$

Then $\rho < r$. Also h may be so chosen that $\rho + \eta < r$.

Since, $\sum a_n z^n$ is convergent in $|z| < R$, $a_n r^n$ is bounded for $0 < r < R$, so that $|a_n r^n| < M$ where M is finite and positive.

we then have

$$\left| \frac{f(z+h) - f(z)}{h} - \phi(z) \right|$$

$$= \left| \sum_{n=0}^{\infty} \left[a_n \left\{ \frac{(z+h)^n - z^n}{h} - n z^{n-1} \right\} \right] \right|$$

$$= \left| \sum_{n=0}^{\infty} \left[a_n \left\{ \frac{n(n-1)}{1\cdot 2} z^{n-2} \cdot h + \cdots + h^{n-1} \right\} \right] \right|$$

$$\leq \sum_{n=0}^{\infty} |a_n| \left\{ \frac{n(n-1)}{1\cdot 2} |z|^{n-2} |h| + \cdots + |h|^{n-1} \right\}$$

$$\leq \sum_{n=0}^{\infty} \frac{M}{r^n} \left\{ \frac{n(n-1)}{1\cdot 2} \cdot |z|^{n-2} |h| + \cdots + |h|^{n-1} \right\}$$

$$\leq \sum_{n=0}^{\infty} \frac{M}{r^n} \left\{ \frac{n(n-1)}{1\cdot 2} \rho^{n-2} \cdot \eta + \cdots + \eta^{n-1} \right\}$$

$$= \sum_{n=0}^{\infty} \frac{M}{r^n} \cdot \frac{1}{\eta} \left\{ (\rho+\eta)^n - \rho^n - n\rho^{n-1}\eta \right\}$$

$$= \frac{M}{\eta} \sum_{n=0}^{\infty} \left\{ \left(\frac{\rho+\eta}{r}\right)^n - \left(\frac{\rho}{r}\right)^n - \frac{\eta n}{\rho}\left(\frac{\rho}{r}\right)^n \right\} \quad \text{——} (1)$$

Now; $\displaystyle\sum_{n=0}^{\infty} \left(\frac{\rho+\eta}{r}\right)^n = 1 + \frac{\rho+\eta}{r} + \left(\frac{\rho+\eta}{r}\right)^2 + \cdots$

$$\sum_{n=0}^{\infty} \left(\frac{\rho+\eta}{r}\right)^n = \frac{1}{1-\frac{\rho+\eta}{r}} = \frac{r}{r-\rho-\eta}$$

$$\sum_{n=0}^{\infty} \left(\frac{\rho}{r}\right)^n = 1 + \frac{\rho}{r} + \left(\frac{\rho}{r}\right)^2 + \cdots = \frac{1}{1-\frac{\rho}{r}} = \frac{r}{r-\rho}$$

To sum $\sum n \left(\frac{\rho}{r}\right)^n$, we put

$$S = \frac{\rho}{r} + 2\left(\frac{\rho}{r}\right)^2 + 3\left(\frac{\rho}{r}\right)^3 + 4\left(\frac{\rho}{r}\right)^4 + \cdots$$

$$S \cdot \frac{\rho}{r} = \left(\frac{\rho}{r}\right)^2 + 2\left(\frac{\rho}{r}\right)^3 + 3\left(\frac{\rho}{r}\right)^4 + \cdots$$

Subtracting we get,

$$S\left(1-\frac{\rho}{r}\right) = \frac{\rho}{r} + \left(\frac{\rho}{r}\right)^2 + \left(\frac{\rho}{r}\right)^3 + \cdots$$

$$= \frac{\rho/r}{1-\rho/r} = \frac{\rho}{r-\rho}$$

$$\boxed{\therefore\ S = \frac{\rho r}{(r-\rho)^2}}$$

Substituting these values in ①, we get

$$\left|\frac{f(z+h)-f(z)}{h} - \phi(z)\right| \le \frac{M}{\eta}\left\{\frac{r}{r-\rho-\eta} - \frac{r}{r-\rho} - \frac{\eta r}{(r-\rho)^2}\right\}$$

$$= \frac{M}{\eta}\left[\frac{r\cdot\eta^2}{(r-\rho-\eta)(r-\rho)^2}\right]$$

$$= \frac{Mr\eta}{(r-\rho-\eta)(r-\rho)^2}$$

which tends to zero as $\eta \to 0$

Hence; $\lim\limits_{h\to 0} \frac{f(z+h)-f(z)}{h} = \phi(z)$

It follows that $f(z)$ has the derivative $\phi(z)$. Thus $f(z)$, which is clearly one-valued, is also differentiable so that $f(z)$ is analytic within $|z|=R$.

Again since the radius of convergence of derived series $\phi(z) = \Sigma n a_n z^{n-1}$ is also R.

[$\because \lim n^{1/n}|a_n|^{1/n} = \lim |a_n|^{1/n}$.

since $[\lim n^{1/n} = 1]$; we see that $\phi(z)$ is also analytic in $|z| < R$. Successively differentiating and applying the theorems, we see that the sum function $f(z)$ of a power series possesses derivatives of all orders within ~~its~~ circle of convergence and that these derivatives are obtained by term by term differentiation of series.

2016

5(e) Ques) Find the general integral of the partial Differential Equation

$$(y+zx)p-(x+yz)q=x^2-y^2$$

Sol: The Lagrange's auxillary equations of the given equation are

$$\frac{dx}{y+zx}=\frac{dy}{-(x+yz)}=\frac{dz}{x^2-y^2}$$

$(y, x, 1)$

$$\frac{y\,dx+x\,dy+dz}{y^2+xyz-x^2-xyz+x^2-y^2}=\frac{y\,dx+x\,dy+dz}{0}$$

$$\Rightarrow d(xy)+dz=0$$

$$\Rightarrow xy+z=c_1.$$

For $(x, y, -z)$

$$\frac{x\,dx+y\,dy-z\,dz}{xy+x^2z-xy-y^2z-z(x^2-y^2)}=\frac{x\,dx+y\,dy-z\,dz}{0}$$

$$\Rightarrow x\,dx+y\,dy-z\,dz=0$$

$$\Rightarrow x^2+y^2-z^2=c_2. \text{ (By integration).}$$

$\therefore$ The general integral of the PDE $\phi(c_1, c_2)$

$$=\phi(xy+z, x^2+y^2-z^2).$$

Q. 6(c) 2016 IAS P-II

Let $f(x) = e^{2x}\cos 3x$, for $x \in [0,1]$. Estimate the value of $f(0.5)$ using Lagrange' interpolating polynomial of degree '3' over the nodes $x=0$, $x=0.3$, $x=0.6$ and $x=1$. Also, compute the error bound over the interval $[0,1]$ and the actual error $E(0.5)$?

Sol:- For nodes, $f(x)$ will be.

x	0	0.3	0.6	1
$f(x)$	1	1.1326	-0.7543	-7.315

at $x=0$; $f(0) = e^{2\times 0}\cdot\cos(3\cdot 0) = e^0\cdot\cos 0 = 1$.

at $x=0.3$; $f(0.3) = e^{0.6}\cdot\cos(0.9) = 1.326$

at $x=0.6$; $f(0.6) = e^{1.2}\cos(1.8) = -0.7543$

at $x=1$; $f(1) = e^2\cdot\cos 3 = -7.315$

Basic Lagrange's polynomial of degree n

$$L_k(x) = \frac{(x-x_0)(x-x_1)(x-x_2)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)}$$

& $P(x) = \sum_{r=0}^{3} f_r L_r(x)$.

Here;

$$L_0(x) = \frac{(x-0.3)(x-0.6)(x-1)}{(-0.3)(-0.6)(-1)} = \frac{(x-0.3)(x-0.6)(x-1)}{-0.18}$$

$$L_1(x) = \frac{(x-0)(x-0.6)(x-1)}{(0.3)(-0.3)(-0.7)} = \frac{x(x-0.6)(x-1)}{+(0.063)}$$

$$L_2(x) = \frac{(x-0)(x-0.3)(x-1)}{(0.6)(0.3)(-0.4)} = \frac{x(x-0.3)(x-1)}{-(0.072)}$$

$$L_3(x) = \frac{(x-0)(x-0.3)(x-0.6)}{1(0.7)(0.4)} = \frac{x(x-0.3)(x-0.6)}{0.28}$$

So; $f(x) = 1 L_0(x) + 1.1326\, L_1(x) - 0.7543\, L_2(x) - 7.315\, L_3(x)$

$$f(x) = \frac{-1}{0.18}(x-0.3)(x-0.6)(x-1) + \frac{1.1326}{0.063}(x)(x-0.6)(x-1)$$

$$+\frac{(-0.7543)}{-0.072}\left[x(x-0.3)(x-1)\right]+\frac{(-7.315)}{0.28}(x(x-0.3)(x-0.6))$$

~~Now~~

$$f(x) = -\frac{(x-0.3)(x-1)(x-0.6)}{0.18} + 17.98\,x(x-0.6)(x-1)$$
$$+10.4764(x)(x-0.3)(x-1)$$
$$-26.125(x)(x-0.3)(x-0.6)$$

Now, Substitute $x = 0.5$

So; $f(0.5) = -0.0555 + 0.4495 - 0.52382 + 0.26125$

$f(0.5) = \underline{\underline{0.131374}}$

Actual value from f_x; $f(0.5) = e^1(\cos 1.5)$

$= 0.192283$

So, using Lagrange's Interpolating polynomial. value of $f(x)$ at $x = 0.5$ is 0.131374.

using $f(x)$; $f(0.5) = 0.192283$.

So, actual Error of $E[0.5] = |f(0.5) - P(0.5)|$

$E[0.5] = |0.192283 - 0.131374| = 0.060909$

$$\boxed{E[0.5] = 0.060909}$$

To find Error bound over $[0,1]$

$$E_3(x,f) = \frac{f''''(\xi(x))}{4!}(x-0)(x-0.3)(x-0.6)(x-1)$$

We know that, $f(x)$ is ~~of~~ decreasing ~~from~~, 0 to 1,

So; $f(x) = e^{2x}.\cos 3x$.

$\underline{f''''(x) = -209\,e^{2x}\cos 3x - 96\,e^{2x}.\sin 3x}$

$f''''(x)$ is increasing from 0 to 1

$$\max_{x \in [0,1]} \left| f''''(x) \right| \leq 1428.75 \quad \text{at } x = 1$$

So, Also max of $x(x-0.3)(x-0.6)(x-1)$ is

$$g(x) = x^4 - 1.2x^3 + 0.45x^2 - 0.054x.$$

$$g'(x) = 4x^3 - 3.6x^2 + 0.9x - 0.054$$

$$x = 0.08787,\ 0.5121,\ 0.3$$

So; Error bound $= \left| \dfrac{f''''(\xi)}{4!} x(x-0.3)(x-0.6)(x-1) \right|$

$$= \left| \frac{1428.75}{4 \times 3 \times 2 \times 1} \times 0.00466 \right|$$

$$\boxed{\text{Error bound} = 0.277415625}$$

7(C) 2016 For an integral $\int_{-1}^{1} f(x)\,dx$, show that the two-point Gauss quadrature rule is given by $\int_{-1}^{1} f(x)\,dx = f\left(\frac{1}{\sqrt{3}}\right)+f\left(-\frac{1}{\sqrt{3}}\right)$. Using this rule, estimate $\int_{2}^{4} 2xe^{x}\,dx$.

Sol^n: Gaussian formula imposes a restriction on the limits of integration to be form -1 to 1.

In general, the limits of the integral $\int_a^b f(x)dx$ are changed to -1 to 1 by means of the transformation: $x = \frac{1}{2}(b-a)u + \frac{1}{2}(b+a)$

let us consider the Gauss's formula in the form:

$$\int_{-1}^{1} f(x)\,dx = \omega_1 f(x_1) + \omega_2 f(x_2) + \cdots + \omega_n f(x_n)$$
$$= \sum_{i=1}^{n} \omega_i f(x_i) \quad \text{—— (1)}$$

where ω_i and x_i are called the weights and abscissae respectively.

An advantage of this formula is that the abscissae and weights are symmetrical w.r.t the middle point of the interval.

In the equation (1), there are altogether $2n$ arbitrary constants (i.e. n weights & n abscissae) and therefore the weights and abscissae can be determined such that the formulae is exact when $f(x)$ is a polynomial of degree not exceeding $2n-1$.

Hence, we consider

$$f(x) = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots + c_{2n-1} x^{2n-1} \quad \text{---(2)}$$

$$\therefore \text{(1)} \equiv \int_{-1}^{1} f(x)\,dx = \int_{-1}^{1}\left[c_0 + c_1 x + c_2 x^2 + \cdots + c_{2n-1} x^{2n-1}\right] dx$$

$$= 2c_0 + \tfrac{2}{3} c_2 + \tfrac{2}{5} c_4 + \cdots \quad \text{---(3)}$$

by setting $x = x_i$ in (2), we obtain

$$f(x_i) = c_0 + c_1 x_i + c_2 x_i^2 + \cdots + c_{2n-1} x_i^{2n-1}$$

Substituting these values on the right hand side of (1), we obtain

$$\int_{-1}^{1} f(x)\,dx = \sum_{i=1}^{n} w_i f(x_i)$$

$$= w_1 f(x_1) + w_2 f(x_2) + \cdots + w_n f(x_n)$$

$$= w_1\left[c_0 + c_1 x_1 + c_2 x_1^2 + c_3 x_1^3 + \cdots + c_{2n-1} x_1^{2n-1}\right]$$
$$+ w_2\left[c_0 + c_1 x_2 + c_2 x_2^2 + c_3 x_2^3 + \cdots + c_{2n-1} x_2^{2n-1}\right]$$
$$+ \cdots$$
$$+ \cdots$$
$$+ w_n\left[c_0 + c_1 x_n + c_2 x_n^2 + c_3 x_n^3 + \cdots + c_{2n-1} x_n^{2n-1}\right]$$

$$= c_0\left[w_1 + w_2 + w_3 + \cdots + w_n\right] + c_1\left[w_1 x_1 + w_2 x_2 + \cdots w_n x_n\right]$$
$$+ c_2\left[w_1 x_1^2 + w_2 x_2^2 + w_3 x_3^2 + \cdots + w_n x_n^2\right] + \cdots$$
$$+ \cdots + c_{2n-1}\left[w_1 x_1^{2n-1} + w_2 x_2^{2n-1} + w_3 x_3^{2n-1} + \cdots w_n x_n^{2n-1}\right] \quad \text{---(4)}$$

But the equations (3) & (4) are identical for all values

of c_i, hence comparing coefficients of c_i,
we obtain $2n$ equations in $2n$ unknown ω_i & x_i ($i = 1, 2, \ldots n$)

$$\left.\begin{array}{l} \omega_1 + \omega_2 + \omega_3 + \cdots + \omega_n = 2 \\ \omega_1 x_1 + \omega_2 x_2 + \omega_3 x_3 + \cdots + \omega_n x_n = 0 \\ \omega_1 x_1^2 + \omega_2 x_2^2 + \omega_3 x_3^2 + \cdots + \omega_n x_n^2 = 2/3 \\ \quad \vdots \qquad \vdots \qquad \vdots \qquad \qquad \vdots \\ \omega_1 x_1^{2n-1} + \omega_2 x_2^{2n-1} + \omega_3 x_3^{2n-1} + \cdots + \omega_n x_n^{2n-1} = 0 \end{array}\right\} \text{——} (5)$$

* **one point formula:**

Gauss formula for $n=1$ is

$$\int_{-1}^{1} f(x)\,dx = \omega_1 f(x_1) \text{ —— } (6)$$

$\therefore$ The method has two knowns ω_1 & x_1

$\therefore$ from (5), $\omega_1 = 2$

&

$$\omega_1 x_1 = 0$$

$$\Rightarrow x_1 = 0 \quad (\because \omega_1 = 2)$$

$$\therefore (6) \equiv \boxed{\int_{-1}^{1} f(x)\,dx = f(0)} \text{ —— } (7)$$

* **Two-point formula**

Gauss formula for $n=2$ is

$$\int_{-1}^{1} f(x)\,dx = \sum_{i=1}^{2} \omega_i f(x_i)$$

$$= \omega_1 f(x_1) + \omega_2 f(x_2) \text{ —— } (8)$$

$\therefore$ The method has four unknowns ω_1, ω_2 & x_1, x_2

$\therefore$ from (5),

$$\left.\begin{array}{l} \omega_1 + \omega_2 = 2 \\ \omega_1 x_1 + \omega_2 x_2 = 0 \\ \omega_1 x_1^2 + \omega_2 x_2^2 = 2/3 \\ \omega_1 x_1^3 + \omega_2 x_2^3 = 0 \end{array}\right\} \text{——} (9)$$

solving these equations, we obtain

$w_1 = w_2 = 1$; $x_1 = -\frac{1}{\sqrt{3}}$, $x_2 = \frac{1}{\sqrt{3}}$

$$\therefore (8) \equiv \boxed{\int_{-1}^{1} f(x)dx = 1.f\left(-\tfrac{1}{\sqrt{3}}\right) + 1.f\left(\tfrac{1}{\sqrt{3}}\right)}$$

let us find $\int_2^4 2xe^x dx$!

let $f(x) = 2xe^x$; limits 2 to 4.

we have $x = \frac{1}{2}(b-a)u + \frac{1}{2}(b+a)$

$= \frac{1}{2}(2)u + \frac{1}{2}(6)$

$x = u + 3.$

if $x = 2$ then $u = -1$.

if $x = 4$ then $u = 1$.

and $f(u) = 2(u+3)e^{u+3}$.

and $dx = du$.

$$\therefore \int_{x=2}^{4} 2xe^x dx = \int_{u=-1}^{1} 2(u+3)e^{u+3} \cdot du.$$

$$= 2\left(\frac{-1}{\sqrt{3}}+3\right)e^{-1/\sqrt{3}}.e^3 + 2\left(\frac{1}{\sqrt{3}}+3\right)e^{\frac{1}{\sqrt{3}}}.e^3.$$

$$\left(\because f\left(\frac{-1}{\sqrt{3}}\right) = 2\left(\frac{-1}{\sqrt{3}}+3\right)e^{\frac{-1}{\sqrt{3}}+3}.\right.$$

$$\text{and } f\left(\tfrac{1}{\sqrt{3}}\right) = 2\left(\tfrac{1}{\sqrt{3}}+3\right)e^{\frac{1}{\sqrt{3}}+3}.$$

Q(a) → Find the temperature $u(x,t)$ in a bar of silver of length 10 cm and constant cross-section of area 1cm^2. Let density $\rho = 10.6\,g/cm^3$ thermal conductivity $K = 1.04$ cal/(cm sec°C) and specific heat $\sigma = 0.056$ cal/g °C. The bar is perfectly isolated laterally, with ends kept at 0°C and initial temperature $f(x) = \sin(0.1\pi x)$°C. Note that $u(x,t)$ follows the heat equation $u_t = c^2 u_{xx}$, where

$$c^2 = K/(\rho\sigma).$$

IAS 2016

Soln → These problem involves the solution of the differential equation with various Boundary and initial conditions. All problems start with the same separation of variables process which is described below:

$$u_t = c^2 u_{xx} \ ; \ \text{where} \ c^2 = \frac{K}{\rho\sigma} \ (\text{given})$$

Here, c^2 is the thermal diffusion co-efficient. The initial condition specifies the value of u at all values of x at $t=0$:

$$u(x,0) = f(x) = \sin(0.1\pi x) \quad (\text{given})$$

The boundary conditions at $x=0$ and $x=l$ (10 cm) can be function of time in general.

$$u(0,t) = u(l,t) = 0 \ - \ (\text{given})$$

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Now, we use the basic separation of variables approach:

$$u(x,t) = X(x)T(t)$$

$$u_t = X(x).T'(t) = c^2 u_{xx} = c^2 T(t)X''(x)$$

Dividing throughout by $c^2 X(x)T(t)$, we obtained the following result

$$\frac{1}{c^2}\cdot\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)}$$

The left hand side is a function of 't' only the right hand side is a function of 'x' only. The only way that this can be correct is both sides equal to a constant.

This also shows that the separation of variable solution works. In order to simplify, the solution, we choose the constant to be equal to $-\lambda^2$. This gives us two ordinary DE to solve

$$\frac{1}{c^2}\cdot\frac{T'(t)}{T(t)} = -\lambda^2 = \frac{X''(x)}{X(x)}$$

Then, $\dfrac{T'(t)}{T(t)} = -c^2\lambda^2 \Rightarrow \boxed{T(t) = Ae^{-c^2\lambda^2 t}}$

and $\dfrac{X''(x)}{X(x)} = -\lambda^2 \Rightarrow \boxed{X(x) = B\sin(\lambda x) + C\cos(\lambda x)}$

Thus, our general solution for $u(x,t) = X(x)T(t)$ becomes

$$u(x,t) = T(t)X(x) = Ae^{-c^2\lambda^2 t}[B\sin(\lambda x) + C\cos(\lambda x)]$$

$$u(x,t) = e^{-c^2\lambda^2 t}[c_1\sin(\lambda x) + c_2\cos(\lambda x)]$$

Substituting the boundary condition at $x=0$

$$u(0,t) = 0 = e^{-c^2\lambda^2 t}\left[c_1 \sin(0) + c_2 \cos(0)\right]$$

$$\boxed{u(0,t) = e^{-c^2\lambda^2 t} \cdot c_2}.$$

~~Equilibrium~~

Equation will be satisfied for all t if $c_2 = 0$, with $c_2 = 0$, we can apply solution to the boundary condition at $x = l$.

$$\boxed{u(l,t) = 0 = c_1 e^{-\lambda^2 c^2 t} \sin(\lambda l)}$$

This equation will be satisfied. If the sine term is zero (0). This will be true only if $\lambda l = n\pi$, where n is an integer.

Since, any integeral value of n gives a solution to the original differential equation, with the boundary conditions that $u=0$ at both boundaries, the most general solution is one that is a sum of all possible solutions, each multiplied by a different constant C_n. This general solution is written as follows—

$$u(x,t) = \sum_{n=1}^{\infty} C_n \cdot e^{-c^2\lambda^2 t} \sin(\lambda_n x)$$

$$\text{where ; } \lambda_n = \frac{n\pi}{l}$$

We still have to satisfy the initial condition that $u(x,0) = f(x)$. Substituting this condition for $t=0$ gives

$$f(x) = u(x,0) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{l}\right)$$

Since, the ODE for $X(x)$ forms a sturm-Liouville problem. The solution $C_n \sin\left(\frac{n\pi x}{l}\right)$ are

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eigen function solutions to the problem in interval $0 \le x \le l$. Thus, we can expand any initial condition function. $f(x)$ in terms of these eigen functions.

Multiplying both sides of equation by $\sin\left(\frac{m\pi x}{l}\right)$ and integrating both sides of the resulting equation from a lower limit of zero to an upper limit of 'l', we get the following result

$$\int_0^l f(x)\sin\left(\frac{m\pi x}{l}\right)dx = \int_0^l \sum_{n=1}^{\infty} C_n \sin\left(\frac{m\pi x}{l}\right)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \sum_{n=1}^{\infty}\int_0^l C_n \sin\left(\frac{m\pi x}{l}\right)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= C_m \int_0^l \sin^2\left(\frac{m\pi x}{l}\right)dx$$

[∵ Note: (1) We can reverse the order of summation and integration because these operation commute.

(2) The integrals in the summation all vanish unless $m=n$, leaving only this integral to evaluate.

$$C_m = \frac{\int_0^l f(x)\sin\left(\frac{m\pi x}{l}\right)dx}{\int_0^l \sin^2\left(\frac{m\pi x}{l}\right)dx}$$

$$C_m = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{m\pi x}{l}\right)dx$$

Given; $f(x) = \sin(0.1\pi x)$

Since; $l = 10 \Rightarrow 0.1\,cm^{-1} = 1/l$.

Thus, we can write the initial condition as

$$f(x) = \sin\left(\frac{\pi x}{l}\right)$$

Then

$$C_m = \frac{2}{l}\int_0^l \sin\left(\frac{\pi x}{l}\right)\cdot \sin\left(\frac{m\pi x}{l}\right)dx.$$

Then;

$$C_m = \begin{cases} 1 & ; m=1 \\ 0 & ; m\neq 1 \end{cases}$$

Substituting this result into the equation gives the following solution to the diffusion equation when $f(x) = \sin\left(\frac{\pi x}{l}\right)$ and the boundary temperature are zero.

$$\boxed{u(x,t) = e^{-\left(\frac{\pi}{l}\right)^2 c^2 t}\sin\left(\frac{\pi x}{l}\right)}$$

From the data given in the problem, thermal diffusivity 'C' can be computed as $\boxed{c^2 = \frac{K}{\rho\sigma}}$