$$\therefore \quad c = \frac{a}{\log \cot (\theta/2)}$$

Substituting this value of c in (3), we get

$$\frac{l}{2a} \sin \theta \log \cot \frac{\theta}{2} = \cosh \mu (\pi - \theta) + \cos \theta.$$

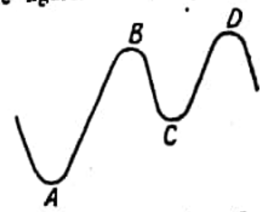
The part of the string between the pegs is

$$2s = 2c \tan \left(\tfrac{1}{2}\pi - \theta\right) = 2c \cot \theta$$

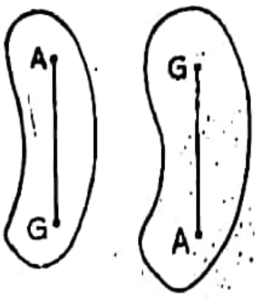
$$= \frac{2a \cot \theta}{\log \cot (\theta/2)}.$$

Stable and Unstable Equilibrium

§ 1. Introduction. Consider the motion of a body on a smooth curve in a vertical plane as shown in the figure. Obviously the body can rest at points A, B, C and D which are points of maxima or minima of the curve. If the body be slightly displaced from its position of rest at A or C (*i.e.*, the points of minima), it will tend to return to its original position of rest, while if displaced from its position of rest at B or D (*i.e.*, the points of maxima), it will tend to move still further away from its original position of rest. In the first case the equilibrium of the body is said to be *stable* and in the second case it is said to be *unstable*.



Take one more illustration. Consider the equilibrium of a rigid body fixed at one point say A. For the equilibrium of the body the centre of gravity G of the body must lie on the vertical line through the point of support A. There arise three cases.



Case 1. Suppose that the centre of gravity G lies below the point of support A. In this case if the body be slightly displaced from its position of equilibrium its centre of gravity will be raised. If the body be then let free, the force of gravity will bring the body back to its original position of equilibrium. In this case the body is said to be in *stable equilibrium*.

Case 2. Next suppose that the centre of gravity G lies above the point of support A. In this case if the body be slightly displaced from its position of equilibrium, its centre of gravity will be lowered. If the body be then let free, the force of gravity will still further move away the body from its original position of equilibrium. In this case the body is said to be *in unstable equilibrium*.

Case 3. If the centre of gravity G is at the point of support A, the body will still be in equilibrium when displaced. In this case we say that the body is in a state of *neutral equilibrium*.

Remark. It can be seen that among all the possible positions of the body, in the case 1 the height of the centre of gravity of the body above some fixed plane is minimum and in the case 2 it is maximum.

§ 2. Definitions. [Meerut 77, 90, 90 P]

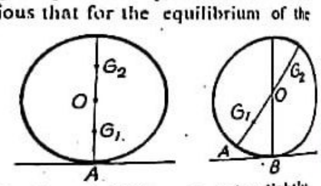
(i) Stable equilibrium. *A body is said to be in stable equilibrium if when slightly displaced from its position of equilibrium, the forces acting on the body tend to make it return towards its position of equilibrium.*

(ii) Unstable equilibrium. *The equilibrium of a body is said to be unstable if when slightly displaced from its position of equilibrium, the forces acting on the body tend to move the body further away from its position of equilibrium.*

(iii) Neutral Equilibrium. *A body is said to be in neutral equilibrium if the forces acting on it are such that they keep the body in equilibrium in any slightly displaced position.*

Further examples of stable, unstable and neutral equilibrium.

Consider the case of a heavy sphere resting on a horizontal plane. Suppose the centre of gravity of the sphere is not at its geometric centre O. It is obvious that for the equilibrium of the sphere its point of contact A with the plane, its geometric centre O and its centre of gravity must be in the same vertical line. One position of equilibrium is in which the centre of gravity G_1 is



below the geometric centre O. In this case if the sphere be slightly displaced it would tend to come back to its original position of equilibrium. This is the position of **stable** equilibrium. The other position of equilibrium is in which the centre of gravity G_2 is above the geometric centre O. In this case if the sphere be slightly displaced it would not come back to its original position of equilibrium but would go further away from that position. This is the position of **unstable equilibrium.**

If, however, the centre of gravity of the sphere is at its geometric centre O, the sphere will still be in equilibrium when displaced. In this case the equilibrium is **neutral.**

If a right circular cone rests on a horizontal plane with its base in contact with the plane and its axis vertical, its equilibrium is stable. But if it rests with its vertex in contact with the plane and its axis vertical, its equilibrium is unstable. Again if it rests along a generator, it is in neutral equilibrium.

The equilibrium of a pendulum is stable when it is displaced from its vertical position of equilibrium, for it returns towards the vertical position again. Any top heavy thing or a stick placed vertically on a finger is an example of unstable equilibrium.

§ 3. The Work Function. Suppose a material system is acted upon by a system of forces X, Y, Z parallel to the axes of coordinates. If during a small displacement whose projections on the coordinate axes are dx, dy, dz the work done by these forces is dW, then

$$dW = X\,dx + Y\,dy + Z\,dz.$$

The forces X, Y, Z generally depend upon the position of the particle. If we confine ourselves to the class of forces which are single-valued and are functions of x, y, z (and not of time t), then integrating the above equation from some standard position (x_0, y_0, z_0) to any position (x, y, z), we have

$$W = \int_{(x_0, y_0, z_0)}^{(x, y, z)} (X\,dx + Y\,dy + Z\,dz).$$

Such a function W is called the *work function*. It is work done by the forces in displacing the body from standard position to any position.

If W_A and W_B are the values of the work function at two positions A and B, then $W_B - W_A$ gives the work done by the forces in displacing the body from A to B.

If $X\,dx + Y\,dy + Z\,dz$ is an exact differential, the forces are called *conservative forces*.

§ 4. Work function test for the nature of stability of equilibrium.

Let A be the position of equilibrium of a rigid body under the action of a given system of forces and let W be the work function of the system in this position A. Suppose the body undergoes a small displacement and takes a position B near to

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the position of equilibrium A, then the value of the work function in the position B will be $W+dW$. Therefore the work done by the forces in displacing the body from the equilibrium position A to the nearby position B is dW. Since the body is in equilibrium in the position A, therefore by the principle of virtual work, we have $dW=0$. *Hence the work function W is stationary (maximum or minimum) in the position of equilibrium.*

First suppose that W is maximum at the equilibrium position A. Imagine that the body is slightly displaced to a position B and let W' be the work function there. Since W is maximum at A, therefore, $W' < W$, so that $W'-W$ is negative. It means that in displacing the body from A to B the work done by the forces is negative *i.e.*, the work is done against the forces and hence the forces will have a tendency to bring the body back to the original position of equilibrium A. Hence the equilibrium at A is stable.

Next suppose that W is minimum at the equilibrium position A. If W' is the value of the work function in a slightly displaced position B of the body, then in this case $W' > W$, so that $W'-W$ is positive. It means that in displacing the body from A to B the work done by the forces is positive *i.e.*, the work has been done by the forces and so the forces will have a tendency to move the body further away from the position of equilibrium. Hence in this case the equilibrium at A is unstable.

Thus in the positions of equilibrium of the body the work function W is either maximum or minimum. If it is maximum, the equilibrium is stable and if it is minimum, the equilibrium is unstable.

§ 5. Potential energy test for the nature of stability of equilibrium. [Lucknow 79; Meerut 83, 83P, 87P, 87S, 88, 88P, 89P]

Potential energy of a body. *The potential energy of a body acted upon by a conservative system of forces, is defined as its capacity to do work by virtue of the position it has acquired. It is measured by the amount of work it can do in passing from the present position to some standard position.* If W be the work function of the body in any position referred to some standard position, and V be the potential energy of the body in that position referred to the same standard position, then $V=-W$. If V_A and V_B are the values of the potential energy at the two positions A and C, then V_A-V_B is the work done by the forces in displacing the body from A to B.

Let A be the position of equilibrium of a rigid body under the action of a given system of forces and V be the potential energy of the body in this position A. Suppose the body undergoes a small displacement and takes a position B near to the position of equilibrium A, then the potential energy of the body in the position B will be $V+dV$. Therefore the work done by the forces in displacing the body from the equilibrium position A to the nearby position B is $V-(V+dV)$ *i.e.*, $-dV$. Since the body is in equilibrium in the position A, therefore by the principle of virtual work, we have $-dV=0 \Rightarrow dV=0$. *Hence the potential energy V is stationary (maximum or minimum) in the position of equilibrium.*

First suppose that V is minimum at the equilibrium position A. Imagine that the body is slightly displaced to a position B and let V' be the potential energy there. Since V is minimum at A, therefore $V' > V$, so that $V-V'$ is negative. It means that in displacing the body from A to B the work done by the forces acting on the body is negative *i.e.*, the work is done against the forces and so the forces will have a tendency to bring the body back to the original position of equilibrium A. Hence the equilibrium at A is stable.

Thus we see that *in the position of stable equilibrium, the potential energy of the body is minimum.* **[Meerut 87, 87P, 88, 88P]**

Next suppose that V is maximum at the equilibrium position A. If V' is the value of the potential energy in a slightly displaced position B of the body, then in this case $V' < V$, so that $V-V'$ is positive. It means that in displacing the body from A to B the work done by the forces is positive *i.e.*, the work is done by the forces and so the forces will tend to move the body further away from the position of equilibrium. Hence the equilibrium at A is unstable.

Thus in the positions of equilibrium of the body the potential energy V is either maximum or minimum. If it is minimum, the equilibrium is stable, and if it is maximum, the equilibrium is unstable.

For example, whenever gravitational energy is the only form of potential energy involved, the height of the centre of gravity of the body above a fixed horizontal plane must be a minimum for stable equilibrium and maximum for unstable equilibrium.

§ 6. z-test for the nature of stability.

Suppose a body is in equilibrium under its weight only *i.e.*, the force of gravity is the only external force acting on the body. Let z be the height of the centre of gravity of the body above a fixed horizontal plane. Express z as a function of some variable θ, i.e., let $z=f(\theta)$. By the principle of virtual work, for the equilibrium of the body, we must have

$$-W\,\delta z=0, \text{ where } W \text{ is the weight of the body}$$

$$\Rightarrow \delta z=0 \Rightarrow \frac{dz}{d\theta}\,\delta\theta=0 \Rightarrow \frac{dz}{d\theta}=0.$$

Thus the equilibrium positions of the body are given by the equation $dz/d\theta=0$. So in the position of equilibrium the height of the centre of gravity of the body above a fixed level must be either maximum or minimum.

Suppose the equation $dz/d\theta=0$ on solving gives $\theta=\alpha, \beta, \gamma$ etc. as the positions of equilibrium.

To test the nature of equilibrium at the position $\theta=\alpha$. We find $d^2z/d\theta^2$ for $\theta=\alpha$. If it is positive, then z is minimum for $\theta=\alpha$. So if we give a slight displacement to the body, the height of its centre of gravity will be raised and then on being set free the body will tend to come back to its original position of equilibrium. Therefore in this case the equilibrium is stable.

Again if $d^2z/d\theta^2$ for $\theta=\alpha$ is negative, then z is maximum for $\theta=\alpha$. So if we give a slight displacement to the body, the height of its centre of gravity will be lowered and then on being set free the force of gravity will still displace the body further away from its original position of equilibrium. Therefore in this case the equilibrium is unstable.

Thus the equilibrium positions of the body are given by the equation $dz/d\theta=0$. If for a root $\theta=\alpha$ of this equation, $d^2z/d\theta^2$ is positive, then z is minimum and the equilibrium is stable. But if for $\theta=\alpha$, $d^2z/d\theta^2$ is negative, then z is maximum and the equilibrium is unstable.

If however $d^2z/d\theta^2=0$ for $\theta=\alpha$, then we consider $d^3z/d\theta^3$ and $d^4z/d\theta^4$. Then for the position of equilibrium $\theta=\alpha$, we must have $d^3z/d\theta^3=0$, and the equilibrium is stable or unstable according as for this position $d^4z/d\theta^4$ is positive or negative.

Similar tests apply for the other positions of equilibrium $\theta=\beta, \gamma$ etc.

Remark. If $z=f(\theta)$ represents the depth of the centre of gravity of the body below some fixed horizontal plane, then the conditions for the stability and unstablility of the equilibrium are reversed. In this case for equilibrium position we must have $dz/d\theta=0$. If for a root $\theta=\alpha$ of this equation $d^2z/d\theta^2$ is positive, then z is

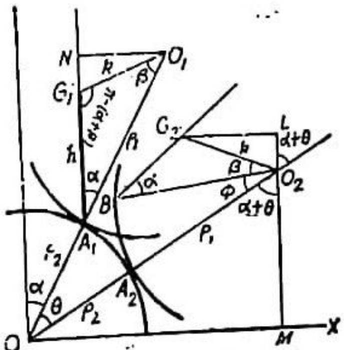
minimum and the equilibrium is unstable. But if for $\theta=\alpha$, $d^2z/d\theta^2$ is negative, then z is maximum and the equilibrium is stable. If however $d^2z/d\theta^2=0$ for $\theta=\alpha$, then we consider higher differential coefficients of z and conclude similarly.

§ 7. Stability of a body resting on a fixed rough surface.

Theorem. *A body rests in equilibrium upon another fixed body. The portions of the two bodies in contact have radii of curvatures ρ_1 and ρ_2 respectively. The centre of gravity of the first body is at a height h above the point of contact and the common normal makes an angle α with the vertical; it is required to prove that the equilibrium is stable or unstable according as $h <$ or $> \dfrac{\rho_1\rho_2}{\rho_1+\rho_2}\cos\alpha$.*

[Meerut 81]

Let O and O_1 be the centres of curvature of the lower and upper bodies in the position of rest and A_1 be their point of contact. In this position of equilibrium the common normal OA_1O_1 makes an angle α with the vertical OY. If G_1 is the centre of gravity of the upper body, then for equilibrium the line A_1G_1 must be vertical. It is given that $A_1G_1=h$.



Let $O_1G_1=k$

and $\angle OO_1G_1=\beta$.

Suppose the upper body is slightly displaced by pure rolling over the lower body which is fixed. Let A_2 be the new point of contact. O_2 is the new position of O_1 and the point A_1 of the upper body rolls up to the position B so that O_2B is the new position of the original normal O_1A_1. Also G_2 is the new position of G_1 so that $O_2G_2=O_1G_1=k$.

Suppose the common normal at A_2 makes angles θ and ϕ with the original normals OA_1 and O_2B.

We have $O_1A_1=\rho_1$ and $OA_1=\rho_2$. Also $O_2A_2=\rho_1$ and $OA_2=\rho_2$. Since the upper body rolls on the lower body without slipping therefore arc A_1A_2=arc A_2B i.e., $\rho_2\theta=\rho_1\phi$.

$$\therefore \frac{d\phi}{d\theta}=\frac{\rho_2}{\rho_1}. \qquad \ldots(1)$$

Let z be the height of G_2 above the fixed horizontal line OX.

Then $z = LM = LO_2 + O_2M$

$= O_2G_2 \cos \angle G_2O_2L + OO_2 \cos(\alpha+\theta)$

$= k \cos[\pi - (\alpha+\theta+\phi+\beta)] + (\rho_1+\rho_2)\cos(\alpha+\theta)$

$[\because O_2G_2 = O_1G_1 = k]$

$= (\rho_1+\rho_2)\cos(\alpha+\theta) - k\cos(\alpha+\theta+\phi+\beta).$

$$\therefore \frac{dz}{d\theta} = -(\rho_1+\rho_2)\sin(\alpha+\theta) + k\sin(\alpha+\theta+\phi+\beta)\left(1+\frac{d\phi}{d\theta}\right)$$

[$\because$ α, β are constants and θ, ϕ are the only variables]

$$= -(\rho_1+\rho_2)\sin(\alpha+\theta) + k\sin(\alpha+\theta+\phi+\beta)\left(1+\frac{\rho_2}{\rho_1}\right) \quad \text{[from (1)]}$$

$$= \frac{\rho_1+\rho_2}{\rho_1}\left[-\rho_1\sin(\alpha+\theta) + k\sin(\alpha+\theta+\phi+\beta)\right]$$

and
$$\frac{d^2z}{d\theta^2} = \frac{\rho_1+\rho_2}{\rho_1}\left[-\rho_1\cos(\alpha+\theta) + k\cos(\alpha+\theta+\phi+\beta)\left(1+\frac{d\phi}{d\theta}\right)\right]$$

$$= \frac{\rho_1+\rho_2}{\rho_1}\left[-\rho_1\cos(\alpha+\theta) + k\cos(\alpha+\theta+\phi+\beta)\left(1+\frac{\rho_2}{\rho_1}\right)\right]$$

$$= \frac{\rho_1+\rho_2}{\rho_1^2}\left[-\rho_1^2\cos(\alpha+\theta) + k(\rho_1+\rho_2)\cos(\alpha+\theta+\phi+\beta)\right].$$

In the position of equilibrium $\theta=0$ and $\phi=0$.

Thus the equilibrium is stable or unstable according as $d^2z/d\theta^2$ is positive or negative for $\theta=\phi=0$,

i.e., according as $k(\rho_1+\rho_2)\cos(\alpha+\beta) >$ or $< \rho_1^2 \cos\alpha$.

But from the $\Delta A_1G_1O_1$, we have

$h = A_1G_1 = A_1N - G_1N = A_1O_1\cos\alpha - O_1G_1\cos\angle O_1G_1N$

$= \rho_1\cos\alpha - k\cos(\alpha+\beta).$

$\therefore \quad k\cos(\alpha+\beta) = \rho_1\cos\alpha - h.$

Hence the equilibrium is stable or unstable according as

$(\rho_1+\rho_2)(\rho_1\cos\alpha - h) >$ or $< \rho_1^2\cos^2\alpha$

i.e., $(\rho_1+\rho_2)\rho_1\cos\alpha - (\rho_1+\rho_2)h >$ or $< \rho_1^2\cos^2\alpha$

i.e., $(\rho_1+\rho_2)h <$ or $> (\rho_1+\rho_2)\rho_1\cos\alpha - \rho_1^2\cos^2\alpha$

i.e., $(\rho_1+\rho_2)h <$ or $> \rho_1\rho_2\cos\alpha$

i.e., $h <$ or $> \dfrac{\rho_1\rho_2}{\rho_1+\rho_2}\cos\alpha.$

Cor. If $\alpha=0$, the above conditions give that the equilibrium is stable or unstable according as

$h <$ or $> \dfrac{\rho_1\rho_2}{\rho_1+\rho_2}$ *i.e.*, $\dfrac{1}{h} >$ or $< \dfrac{\rho_1+\rho_2}{\rho_1\rho_2}$

or $\dfrac{1}{h} >$ or $< \dfrac{1}{\rho_1} + \dfrac{1}{\rho_2}.$

Thus suppose that a body rests in equilibrium upon another body which is fixed and the portions of the two bodies in contact have radii of curvatures ρ_1 and ρ_2 respectively. The C.G. of the first body is at a height h above the point of contact and the common normal coincides with the vertical. Then the equilibrium is stable or unstable according as

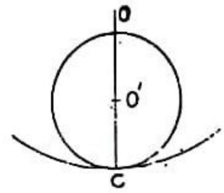
$$\frac{1}{h} > \text{ or } < \frac{1}{\rho_1} + \frac{1}{\rho_2}.$$

If the portions of the bodies in contact are spheres of radii r_1 and r_2, then in the above condition we put $\rho_1=r_1$ and $\rho_2=r_2$. Thus the equilibrium is stable or unstable according as

$$\frac{1}{h} > \text{ or } < \frac{1}{r_1} + \frac{1}{r_2}.$$

If the surface of the upper body at the point of contact is plane, then $\rho_1=\infty$ and if the surface of the lower body at the point of contact is plane, then $\rho_2=\infty$.

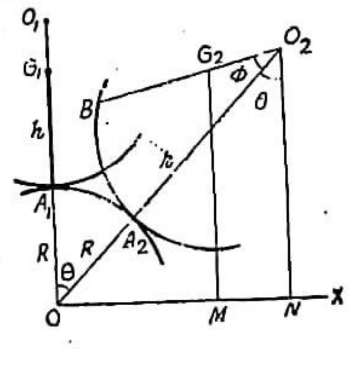
If the surface of the lower body at the point of contact instead of being convex is concave, then ρ_2 is to be taken with negative sign.



On account of its importance we shall now give an independent proof in case the surfaces in contact are spherical.

§ 8. *A body rests in equilibrium upon another fixed body, the portions of the two bodies in contact being spheres of radii r and R respectively and the straight line joining the centres of the spheres being vertical; if the first body be slightly displaced, to find whether the equilibrium is stable or unstable, the bodies being rough enough to prevent any sliding.* **[Lucknow 76]**

Let O be the centre of the spherical surface of the lower body which is fixed and O_1 that of the upper body which rests on the lower body, A_1 being their point of contact and the line OO_1 being vertical. If G_1 is the centre of gravity of the upper body, then for the equilibrium of the upper body, the line A_1G_1 must be vertical; let A_1G_1 be h. The figure is a section of the bodies by a vertical plane through G_1.



Suppose the upper body is slightly displaced by pure rolling over the lower body. Let A_2 be the new point of contact, O_2 is the new position of O_1 and the point A_1 of the upper body rolls up to the position B so that O_2B is the new position of O_1A_1. Also G_2 is the new position of G_1 so that $BG_2=A_1G_1=h$.

Let $\therefore A_1OA_2=\theta$ and $\therefore BO_2A_2=\phi$;

so that $\therefore G_2O_2N=\theta+\phi$.

We have $O_1A_1=r$ and $OA_1=R$. Also $O_2A_2=O_2B=r$ and $OA_2=R$. Since the upper body rolls on the lower body without slipping, therefore

arc $A_1A_2=$ arc A_2B *i.e.*, $R\theta=r\phi$ *i.e.*, $\phi=(R/r)\,\theta$.

Now in order to find the nature of equilibrium, we should find the height z of the centre of gravity G_2 in the new position above the fixed horizontal line OX. We have

$$z=G_2M=O_2N=O_2G_2\cos(\theta+\phi)$$
$$=OO_2\cos\theta-(O_2B-BG_2)\cos(\theta+\phi)$$
$$=(R+r)\cos\theta-(r-h)\cos(\theta+\phi)$$
$$=(R+r)\cos\theta-(r-h)\cos\{\theta+(R/r)\,\theta\} \qquad [\because\ \phi=(R/r)\,\theta]$$
$$=(R+r)\cos\theta-(r-h)\cos\left\{\frac{\theta\,(r+R)}{r}\right\}.$$

For equilibrium, we have $dz/d\theta=0$

i.e., $$-(R+r)\sin\theta+(r-h)\sin\left\{\frac{\theta\,(r+R)}{r}\right\}\cdot\frac{r+R}{r}=0.$$

This is satisfied by $\theta=0$.

Now $$\frac{d^2z}{d\theta^2}=-(R+r)\cos\theta+(r-h)\cos\left\{\frac{\theta\,(r+R)}{r}\right\}\cdot\left(\frac{r+R}{r}\right)^2.$$

$$\therefore\quad\left(\frac{d^2z}{d\theta^2}\right)_{\theta=0}=-(R+r)+(r-h)\left(\frac{r+R}{r}\right)^2$$
$$=\left(\frac{r+R}{r}\right)^2\left\{(r-h)-\frac{r^2}{R+r}\right\}=\left(\frac{r+R}{r}\right)^2\left\{r-\frac{r^2}{R+r}-h\right\}$$
$$=\left(\frac{r+R}{r}\right)^2\left\{\frac{rR}{R+r}-h\right\}.$$

This will be positive if

$$\frac{rR}{R+r}>h \text{ i.e., } \frac{1}{h}>\frac{R+r}{rR} \text{ i.e., } \frac{1}{h}>\frac{1}{r}+\frac{1}{R}$$

and negative, if $\frac{rR}{R+r}<h$ *i.e.*, $\frac{1}{h}<\frac{1}{r}+\frac{1}{R}$.

Hence the equilibrium is stable or unstable according as

$$\frac{1}{h}>\frac{1}{r}+\frac{1}{R} \quad\text{or}\quad \frac{1}{h}<\frac{1}{r}+\frac{1}{R}.$$

Here R is the radius of the lower body and r that of the upper body and h is the height of the C.G. of the upper body above the point of contact.

Now it remains to discuss the case when

$1/h=1/r+1/R$ *i.e.*, $h=rR/(R+r)$.

In this case $d^2z/d\theta^2=0$. Hence we find $d^3z/d\theta^3$ and $d^4z/d\theta^4$

We have $$\frac{d^3z}{d\theta^3}=(R+r)\sin\theta-(r-h)\sin\left\{\frac{\theta\,(r+R)}{r}\right\}\cdot\left(\frac{r+R}{r}\right)^3,$$

and $$\frac{d^4z}{d\theta^4}=(R+r)\cos\theta-(r-h)\cos\left\{\frac{\theta\,(r+R)}{r}\right\}\cdot\left(\frac{r+R}{r}\right)^4.$$

Obviously $\left(\frac{d^3z}{d\theta^3}\right)_{\theta=0}=0.$

Also $$\left(\frac{d^4z}{d\theta^4}\right)_{\theta=0}=(R+r)-(r-h)\left(\frac{r+R}{r}\right)^4$$
$$=(R+r)\left\{1-\frac{r-h}{r}\left(\frac{r+R}{r}\right)^3\right\}$$
$$=(R+r)\left\{1-\frac{r-h}{r}\cdot\frac{R+r}{r}\cdot\left(\frac{R+r}{r}\right)^2\right\}$$
$$=(R+r)\left\{1-\left(r-\frac{rR}{R+r}\right)\cdot\frac{R+r}{r^2}\cdot\left(\frac{R+r}{r}\right)^2\right\} \qquad \left[\because\ h=\frac{rR}{r+R}\right]$$
$$=(R+r)\left\{1-\frac{r^2}{R+r}\cdot\frac{R+r}{r^2}\cdot\left(\frac{R+r}{r}\right)^2\right\}$$
$$=(R+r)\left\{1-\left(\frac{R+r}{r}\right)^2\right\}$$
$$=(R+r)\left\{1-\left(1+\frac{R}{r}\right)^2\right\},$$ which is negative.

This shows that z is maximum and so in this case the equilibrium is unstable.

Hence if $\frac{1}{h}>\frac{1}{r}+\frac{1}{R}$, then equilibrium is stable

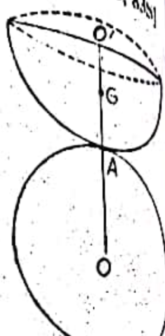
and if $\frac{1}{h}\leqslant\frac{1}{r}+\frac{1}{R}$, the equilibrium is unstable.

Remark. If the upper body has a plane face in contact with the lower body of radius R, then obviously $r=\infty$. And if the lower body be plane, then $R=\infty$.

Illustrative Examples

Ex. 1. *A hemisphere rests in equilibrium on a sphere of equal radius; show that the equilibrium is unstable when the curved, and* *stable when the flat surface of the hemisphere rests on the sphere.* **[Meerut 79, 82, 83S]**

Sol. (i) When the curved surface of the hemisphere rests on the sphere. A hemisphere of centre O' rests on a sphere of centre O with its curved surface in contact with the sphere. The point of contact is A and $OA=O'A=a$ (say). Also the line OAO' is vertical.



If G is the centre of gravity of the hemisphere, then G lies on $O'A$ and $O'G=\frac{3}{8}a$.

Here ρ_1 = the radius of curvature of the upper body at the point of contact = the radius of the hemisphere $= a$,

and ρ_2 = the radius of curvature of the lower body at the point of contact $= a$.

Also h = the height of the centre of gravity of the upper body above the point of contact A

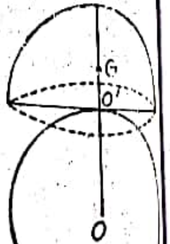
$$=AG=O'A-O'G=a-\tfrac{3}{8}a=\tfrac{5}{8}a.$$

We have $$\frac{1}{h}=\frac{1}{5a/8}=\frac{8}{5a},$$

and $$\frac{1}{\rho_1}+\frac{1}{\rho_2}=\frac{1}{a}+\frac{1}{a}=\frac{2}{a}=\frac{10}{5a}.$$

Thus $\dfrac{1}{h}<\dfrac{1}{\rho_1}+\dfrac{1}{\rho_2}$. Hence the equilibrium is unstable in this case.

(ii) When the flat surface of the hemisphere rests on the sphere. In this case a hemisphere of centre O' rests on a sphere of centre O and equal radius a with its flat surface (*i.e.* the plane base) in contact with the sphere. The point of contact is O' and G is the C.G. of the hemisphere.



Here ρ_1 = the radius of curvature of the upper body at the point of contact $=\infty$,

[Note that the base of the hemisphere touches the sphere along a straight line]

and ρ_2 = the radius of curvature of the lower body at the point of contact = the radius of the sphere $= a$.

Also h = the height of the C.G. of the hemisphere above the point of contact $O' = O'G = \frac{3}{8}a$.

We have $$\frac{1}{h}=\frac{1}{3a/8}=\frac{8}{3a},$$

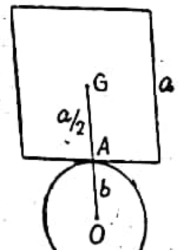
and $$\frac{1}{\rho_1}+\frac{1}{\rho_2}=\frac{1}{\infty}+\frac{1}{a}=0+\frac{1}{a}=\frac{1}{a}=\frac{3}{3a}.$$

Obviously $\dfrac{1}{h}>\dfrac{1}{\rho_1}+\dfrac{1}{\rho_2}$. Hence in this case the equilibrium is stable.

Remark. Remember that for a straight line the radius of curvature at any point is infinity, and for a circle the radius of curvature at any point is equal to the radius of the circle.

Ex. 2. *A uniform cubical box of edge a is placed on the top of a fixed sphere, the centre of the face of the cube being in contact with the highest point of the sphere. What is the least radius of the sphere for which the equilibrium will be stable?* **[Meerut 72, 84S]**

Sol. A uniform cubical box of edge a is placed on the top of a fixed sphere of centre O. The point of contact is A. If G is the C.G. of the box, then for equilibrium the line OAG must be vertical. Let the radius of the sphere be b.



The figure shows the vertical section of the bodies through the point of contact A.

Here ρ_1 = the radius of curvature of the upper body at the point of contact $=\infty$,

and ρ_2 = the radius of curvature of the lower body at the point of contact $= b$.

Also h = the height of the C.G. of the box above the point of contact A = half the edge of the box $=\frac{1}{2}a$.

The equilibrium will be stable, if

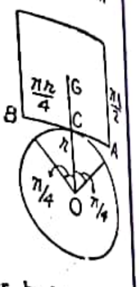
$$\frac{1}{h}>\frac{1}{\rho_1}+\frac{1}{\rho_2}\ \text{i.e.,}\ \frac{1}{\frac{1}{2}a}>\frac{1}{\infty}+\frac{1}{b}\ \text{i.e.,}\ \frac{2}{a}>\frac{1}{b}\ \text{i.e.,}\ b>\tfrac{1}{2}a.$$

Hence the least value of b for the equilibrium to be stable is $\frac{1}{2}a$.

Ex. 3. *A heavy uniform cube balances on the highest point of a sphere whose radius is r. If the sphere is rough enough to prevent sliding and if the side of the cube be $\pi r/2$, show that the cube can rock through a right angle without falling.* **[Meerut 82(S)]**

h above pt of contact (NOT plane of contact)

Sol. A heavy uniform cube balances on the highest point C of a sphere whose centre is O and radius r. The length of a side of the cube is $\pi r/2$. If G is the C.G. of the cube, then for equilibrium the line OCG must be vertical. In the figure we have shown a cross section of the bodies by a vertical plane through the point of contact C.



First we shall show that the equilibrium of the cube is stable.

Here ρ_1 = the radius of curvature of the upper body at the point of contact $C=\infty$,

and ρ_2 = the radius of curvature of the lower body at the point of contact $= r$.

Also h = the height of the centre of gravity G of the upper body above the point of contact C = half the edge of the cube $=\pi r/4$.

The equilibrium will be stable if

$$\frac{1}{h} > \frac{1}{\rho_1}+\frac{1}{\rho_2} \text{ i.e., } \frac{1}{\pi r/4} > \frac{1}{\infty}+\frac{1}{r}$$

i.e.,
$$\frac{4}{\pi r} > \frac{1}{r} \text{ i.e., } \frac{4}{\pi} > 1 \text{ i.e., } 4 > \pi$$

which is so because the value of π lies between 3 and 4.

Hence the equilibrium is stable. So if the cube is slightly displaced, it will tend to come back to its original position of equilibrium. During a swing to the right, the cube will not fall down till the right hand corner A of the lowest edge comes in contact with the sphere.

If θ is the angle through which the cube turns when the right hand corner A of the lowest edge comes in contact with the sphere, we have

$$r\theta = \text{half the edge of the cube} = \pi r/4,$$

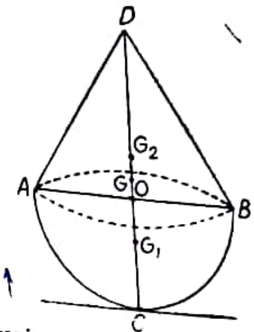
so that $\theta = \pi/4$.

Similarly the cube can turn through an angle $\pi/4$ to the left side on the sphere. Hence the total angle through which the cube can swing (or rock) without falling is $2 \cdot \frac{1}{4}\pi$ i.e., $\frac{1}{2}\pi$.

Ex. 4. *A body, consisting of a cone and a hemisphere on the same base, rests on a rough horizontal table the hemisphere being in contact with the table; show that the greatest height of the cone so that the equilibrium may be stable, is $\sqrt{3}$ times the radius of the hemisphere.* **[Meerut 81]**

∴P

Sol. AB is the common base of the hemisphere and the cone and COD is their common axis which must be vertical for equilibrium. The hemisphere touches the table at C.



Let H be the height OD of the cone and r be the radius OA or OC of the hemisphere. Let G_1 and G_2 be the centres of gravity of the hemisphere and the cone respectively. Then

$OG_1 = 3r/8$ and $OG_2 = H/4$.

h↑

If h be the height of the centre of gravity of the combined body composed of the hemisphere and the cone $\boxed{\text{above the point of contact } C}$, then using the formula $x=\dfrac{w_1x_1+w_2x_2}{w_1+w_2}$, we have

$$h=\frac{\frac{1}{3}\pi r^2 H . CG_2+\frac{2}{3}\pi r^3 . CG_1}{\frac{1}{3}\pi r^2 H+\frac{2}{3}\pi r^3}=\frac{\frac{1}{3}\pi r^2 H\,(r+\frac{1}{4}H)+\frac{2}{3}\pi r^3 . \frac{5}{8}r}{\frac{1}{3}\pi r^2 H+\frac{2}{3}\pi r^3}$$

$$=\frac{H\,(r+\frac{1}{4}H)+\frac{5}{4}r^2}{H+2r}.$$

Here ρ_1 = the radius of curvature at the point of contact C of the upper body which is spherical $= r$,

and ρ_2 = the radius of curvature of the lower body at the point of contact $=\infty$.

∴ the equilibrium will be stable if

$$\frac{1}{h} > \frac{1}{\rho_1}+\frac{1}{\rho_2} \text{ i.e., } \frac{1}{h} > \frac{1}{r}+\frac{1}{\infty} \text{ i.e., } \frac{1}{h} > \frac{1}{r}$$

i.e., $h < r$

i.e., $\dfrac{H\,(r+\frac{1}{4}H)+\frac{5}{4}r^2}{H+2r} < r$ i.e., $Hr+\frac{1}{4}H^2+\frac{5}{4}r^2 < Hr+2r^2$.

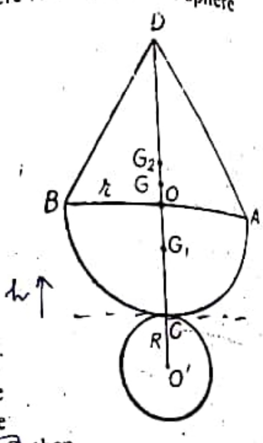
i.e., $\frac{1}{4}H^2 < \frac{3}{4}r^2$ i.e., $H^2 < 3r^2$ i.e., $H < r\sqrt{3}$.

Hence the greatest height of the cone consistent with the stable equilibrium of the body is $\sqrt{3}$ times the radius of the hemisphere.

Ex. 5. *A solid homogeneous hemisphere of radius r has a solid right circular cone of the same substance constructed on the base; the hemisphere rests on the convex side of the fixed sphere of radius R. Show that the length of the axis of the cone consistent with stability for a small rolling displacement is*

$$\frac{r}{R+r}\,[\sqrt{\{(3R+r)(R-r)\}}-2r].$$

Sol. Let O be the centre of the common base AB of the hemisphere and the cone. The hemisphere rests on a fixed sphere of radius R and centre O', their point of contact being C. For equilibrium the line $O'COD$ must be vertical. Let H be the length of the axis OD of the cone. It is given that $OB=OC=r=$ the radius of the hemisphere.



If G_1 and G_2 are the centres of gravity of the hemisphere and the cone respectively, then

$OG_1=3r/8$ and $OG_2=H/4$.

Let G be the centre of gravity of the combined body composed of the hemisphere and the cone. If h be the height of G above the point of contact C then

$$h=\frac{\frac{2}{3}\pi r^3.\frac{5}{8}r+\frac{1}{3}\pi r^2H.(r+\frac{1}{4}H)}{\frac{2}{3}\pi r^3+\frac{1}{3}\pi r^2 H}=\frac{H(r+\frac{1}{4}H)+\frac{5}{4}r^2}{H+2r}.$$

Here $\rho_1=$ the radius of curvature at the point of contact C of the upper body $=r$.

and $\rho_2=$ the radius of curvature at C of the lower body $=R$.

The equilibrium will be stable if

$$\frac{1}{h}>\frac{1}{\rho_1}+\frac{1}{\rho_2} \text{ i.e., } \frac{1}{h}>\frac{1}{r}+\frac{1}{R}$$

i.e., $$\frac{H+2r}{H(r+\frac{1}{4}H)+\frac{5}{4}r^2}>\frac{R+r}{rR}$$

i.e., $(R+r)\{Hr+\frac{1}{4}H^2+\frac{5}{4}r^2\}-rR(H+2r)<0$

i.e., $\frac{1}{4}H^2(R+r)+H\{(R+r)r-rR\}+\frac{5}{4}r^2(R+r)-2r^2R<0$

i.e., $H^2(R+r)+4r^2H+5r^3-3r^2R<0$

i.e., $H^2(R+r)+4r^2H-r^2(3R-5r)<0$

i.e., $$H^2+\frac{4r^2}{R+r}H-\frac{r^2(3R-5r)}{R+r}<0$$

i.e., $$\left(H+\frac{2r^2}{R+r}\right)^2-\frac{4r^4}{(R+r)^2}-\frac{r^2(3R-5r)}{R+r}<0$$

i.e., $$\left(H+\frac{2r^2}{R+r}\right)^2-\frac{4r^4+r^2(3R-5r)(R+r)}{(R+r)^2}<0$$

i.e., $$\left(H+\frac{2r^2}{R+r}\right)^2-\frac{r^2[4r^2+3R^2-2rR-5r^2]}{(R+r)^2}<0$$

$$\left(H+\frac{2r^2}{R+r}\right)^2-\frac{r^2(3R^2-2rR-r^2)}{(R+r)^2}<0$$

i.e., $$\left(H+\frac{2r^2}{R+r}\right)^2<\frac{r^2(3R+r)(R-r)}{(R+r)^2}$$

i.e., $$H+\frac{2r^2}{R+r}<\frac{r}{R+r}\sqrt{\{(3R+r)(R-r)\}}$$

i.e., $$H<\frac{r}{R+r}\sqrt{\{(3R+r)(R-r)\}}-\frac{2r^2}{R+r}$$

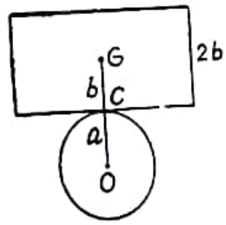
i.e., $$H<\frac{r}{R+r}[\sqrt{\{(3R+r)(R-r)\}}-2r].$$

Therefore the greatest value of H consistent with the stability of equilibrium is

$$\frac{r}{R+r}[\sqrt{\{(3R+r)(R-r)\}}-2r].$$

Ex. 6. *A uniform beam, of thickness 2b, rests symmetrically on a perfectly rough horizontal cylinder of radius a; show that the equilibrium of the beam will be stable or unstable according as b is less or greater than a.*

Sol. C is the point of contact of the beam and the cylinder and G is the centre of gravity of the beam. The figure shows the cross section of the bodies by a vertical plane through C. For equilibrium the line OCG is vertical.



Here $\rho_1=$ radius of curvature of the upper body at the point of contact $C=\infty$,

$\rho_2=$ radius of curvature of the lower body at $C=a$.

Also $h=$ the height of C.G. of the beam above the point of contact $C=\frac{1}{2}$ (thickness of the beam) $=\frac{1}{2}.2b=b$.

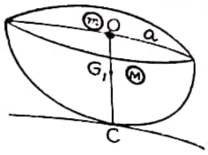
The equilibrium is stable or unstable according as

$$\frac{1}{h}>\text{ or }<\frac{1}{\rho_1}+\frac{1}{\rho_2} \quad \text{i.e., } \frac{1}{h}>\text{ or }<\frac{1}{\infty}+\frac{1}{a}$$

i.e., $$\frac{1}{h}>\text{ or }<\frac{1}{a}$$

i.e., $h<$ or $>a$ i.e., $b<$ or $>a$.

Ex. 7. (a). *A uniform solid hemisphere rests in equilibrium upon a rough horizontal plane with its curved surface in contact with the plane and a particle of mass m is fixed at the centre of the plane face. Show that for any value of m, the equilibrium is stable.*



Sol. C is the point of contact of the hemisphere and the plane and O is the centre of the base of the hemisphere. Let M be the mass of the hemisphere and a be its radius. A particle of mass m is placed at O. The mass M of the hemisphere acts at G_1 where $OG_1 = 3a/8$.

If h be the height of the centre of gravity of the combined body consisting of the hemisphere and the mass m. above the point of contact C, then

$$h = \frac{M \cdot \frac{5}{8}a + m \cdot a}{M + m}$$

Here ρ_1 = the radius of curvature of the upper body at the point of contact $C = a$,

and ρ_2 = the radius of curvature of the lower body at the point of contact $C = \infty$.

The equilibrium will be stable if

$$\frac{1}{h} > \frac{1}{\rho_1} + \frac{1}{\rho_2} \text{ i.e., } \frac{1}{h} > \frac{1}{a} + \frac{1}{\infty} \text{ i.e., } \frac{1}{h} > \frac{1}{a} \text{ i.e., } h < a$$

i.e., $$\frac{\frac{5}{8}aM + am}{M + m} < a \quad \text{i.e., } \frac{5}{8}aM + am < aM + am$$

i.e., $\frac{5}{8}aM < aM$

i.e., $\frac{5}{8}a < a$, which is so whatever may be the value of m.

Hence for any value of m. the equilibrium is stable.

Ex. 7 (b). *A uniform hemisphere rests in equilibrium with its base upwards on the top of a sphere of double its radius. Show that the greatest weight which can be placed at the centre of the plane face without rendering the equilibrium unstable is one-eighth of the weight of the hemisphere.*

Sol. Draw figure yourself. Here a hemisphere rests on the top of a sphere. The base of the hemisphere is upwards. Let $2r$ be the radius of the sphere and r that of the hemisphere.

If W be the weight of the hemisphere and w be the weight placed at the centre of the base of the hemisphere, then

$$h = \frac{W \cdot \frac{5}{8}r + w \cdot r}{W + w}.$$

Here $\rho_1 = r$ and $\rho_2 = 2r$. The equilibrium will be stable if

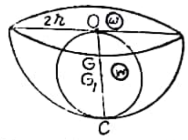
$$\frac{1}{h} > \frac{1}{r} + \frac{1}{2r} \text{ i.e., } \frac{1}{h} > \frac{3}{2r} \text{ i.e., } \frac{W + w}{\frac{5}{8}rW + wr} > \frac{3}{2r}$$

i.e., $$2W + 2w > \frac{15}{8}W + 3w \text{ i.e., } \frac{1}{8}W > w$$

i.e., $$w < \frac{1}{8}W,$$

which proves the required result.

Ex. 8 (a). *A solid sphere rests inside a fixed rough hemispherical bowl of twice its radius. Show that, however large a weight is attached to the highest point of the sphere, the equilibrium is stable.*



Sol. Let r be the radius of the solid sphere which rests inside a fixed rough hemispherical bowl of radius $2r$. Their point of contact is C and O is the highest point of the sphere so that $OC = 2r$. Let W and w be weights of the sphere and the weight attached to the highest point of the sphere. The weight W of the sphere acts at the middle point G_1 of its diameter OC.

If h is the height of the centre of gravity of the combined body consisting of the sphere and the weight w attached to O, then

$$h = \frac{W \cdot r + w \cdot 2r}{W + w}.$$

Here ρ_1 = the radius of curvature of the upper body at the point of contact C = the radius of the sphere $= r$,

and ρ_2 = the radius of curvature of the lower body at the point of contact $C = -2r$, the negative sign is taken because the surface of the lower fixed body i.e., the bowl at C is concave.

The equilibrium will be stable if

$$\frac{1}{h} > \frac{1}{\rho_1} + \frac{1}{\rho_2} \text{ i.e., } \frac{1}{h} > \frac{1}{r} - \frac{1}{2r} \text{ i.e., } \frac{1}{h} > \frac{1}{2r} \text{ i.e., } h < 2r$$

i.e., $$\frac{Wr + 2wr}{W + w} < 2r$$

i.e., $$Wr + 2wr < 2Wr + 2wr \text{ i.e., } Wr < 2Wr.$$

which is so whatever be the value of w.

Hence, however large a weight is attached to the highest point of the sphere, the equilibrium is stable.

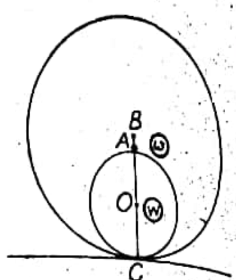
Ex. 8 (b). *A solid sphere rests inside a fixed rough hemispherical bowl of thrice its radius. Find the conditions and nature of equilibrium if a large weight is attached to the highest point of the sphere.* [Meerut 84, 85S]

Sol. Proceed exactly as in part (a). Equilibrium will be stable if weight of the sphere > weight attached.

Ex. 9. *A sphere of weight W and radius a lies within a fixed spherical shell of radius b, and a particle of weight w is fixed to the*

...d of the vertical diameter. Prove that the equilibrium is stable if $\frac{W}{w} > \frac{b-2a}{a}$.

Sol C is the point of contact of the sphere and the spherical shell, O is the centre of the sphere, CA is the vertical diameter of the sphere and B is the centre of the spherical shell. We have $OC=a$ and $BC=b$.



The weight W of the sphere acts at O and a particle of weight w is attached to A. If h be the height of the centre of gravity of the combined body consisting of the sphere and the weight w attached at A, then

$$h=\frac{W.a+w.2a}{W+w}=\frac{W+2w}{W+w}a.$$

Here $\rho_1=a$ and $\rho_2=-b$.

The equilibrium will be stable if

$$\frac{1}{h}>\frac{1}{\rho_1}+\frac{1}{\rho_2} \text{ i.e., } \frac{1}{h}>\frac{1}{a}-\frac{1}{b} \text{ i.e., } \frac{W+w}{a(W+2w)}>\frac{b-a}{ab}$$

i.e., $(W+w)ab > a(b-a)(W+2w)$

i.e., $(W+w)b > (b-a)(W+2w)$

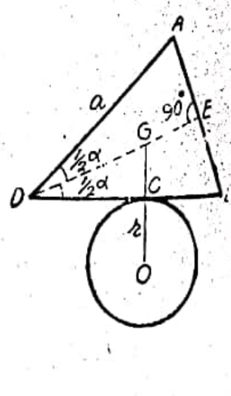
i.e., $W\{b-(b-a)\} > w\{2(b-a)-b\}$ i.e., $Wa > w(b-2a)$

i.e., $\frac{W}{w} > \frac{b-2a}{a}$.

Ex. 10. *A lamina in the form of an isosceles triangle, whose vertical angle is α, is placed on a sphere, of radius r, so that its plane is vertical and one of its equal sides is in contact with the sphere; show that, if the triangle be slightly displaced in its own plane, the equilibrium is stable if $\sin\alpha < 3r/a$, where a is one of the equal sides of the triangle.*

Sol. DAB is an isosceles triangular lamina in which

$DA=DB=a$ and $\angle ADB=\alpha$.



The centre of gravity G of the lamina lies on its median DE which is perpendicular to AB and also bisects the angle ADB. We have

$DG=\frac{2}{3}DE=\frac{2}{3}a\cos\frac{1}{2}\alpha$.

The lamina rests on a fixed sphere whose centre is O and radius r. Their point of contact is C. For equilibrium the line OCG must be vertical.

If h be the height of the C.G. of the lamina abo[ve the point] of contact C, then

$$h=GC=DG\sin\tfrac{1}{2}\alpha=\tfrac{2}{3}a\cos\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\alpha=\tfrac{1}{3}a[\sin\alpha]$$

Here $\rho_1=$ the radius of curvature of the upper [body at the] point of contact $C=\infty$.

and $\rho_2=$ the radius of curvature of the lower fixed body at the point $C=r$.

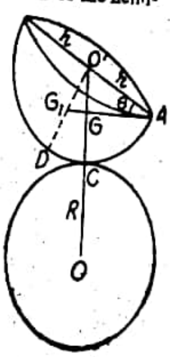
The equilibrium will be stable if

$$\frac{1}{h}>\frac{1}{\rho_1}+\frac{1}{\rho_2} \text{ i.e., } \frac{1}{h}>\frac{1}{\infty}+\frac{1}{r} \text{ i.e., } \frac{1}{h}>\frac{1}{r}.$$

i.e., $h<r$ i.e., $\frac{1}{3}a\sin\alpha<r$ i.e., $\sin\alpha<3r/a$.

Ex. 11. *A heavy hemispherical shell of radius r has a particle attached to a point on the rim, and rests with the curved surface in contact with a rough sphere of radius R at the highest point. Prove that if $R/r > \sqrt{5}-1$, the equilibrium is stable, whatever be the weight of the particle.* **[Meerut 90P]**

Sol. Let O' be the centre of the base of the hemispherical shell of radius r. Let a weight be attached to the rim of the hemispherical shell at A. The centre of gravity G_1 of the hemispherical shell is on its symmetrical radius $O'D$ and $O'G_1=\frac{1}{2}O'D=\frac{1}{2}r$.



Let G be the centre of gravity of the combined body consisting of the hemispherical shell and the weight at A. Then G lies on the line AG_1.

The hemispherical shell rests with its curved surface in contact with a rough sphere of radius R and centre O at the highest point C. For equilibrium the line $OCGO'$ must be vertical but AG_1 need not be horizontal.

Let $CG=h$. Also here $\rho_1=r$ and $\rho_2=R$.

The equilibrium will be stable if

$$\frac{1}{h}>\frac{1}{\rho_1}+\frac{1}{\rho_2} \text{ i.e., } \frac{1}{h}>\frac{1}{r}+\frac{1}{R} \text{ i.e., } \frac{1}{h}>\frac{R+r}{rR}$$

i.e., $$h<\frac{rR}{R+r}. \quad ...(1)$$

The value of h depends on the weight of the particle attached at A. So the equilibrium will be stable, whatever be the weight of the particle attached at A, if the relation (1) holds even for the maximum value of h.

Now h will be maximum if $O'G$ is minimum *i.e.*, if $O'G$ is perpendicular to AG_1 or if $\triangle AO'G$ is right angled.

Let $\angle O'AG=\theta$. Then from right angled $\triangle AO'G_1$,

$$\tan\theta=\frac{O'G_1}{O'A}=\frac{\frac{1}{2}r}{r}=\frac{1}{2}. \quad \therefore \quad \sin\theta=\frac{1}{\sqrt{5}}.$$

$\therefore$ the minimum value of $O'G$

$$=O'A\sin\theta=r\,(1/\sqrt{5})=r/\sqrt{5}.$$

$\therefore$ the maximum value of $h=r-$the minimum value of $O'G$

$$=r-\frac{r}{\sqrt{5}}=\frac{r\,(\sqrt{5}-1)}{\sqrt{5}}.$$

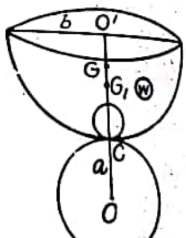
Hence the equilibrium will be stable, whatever be the weight of the particle at A, if

$$\frac{r\,(\sqrt{5}-1)}{\sqrt{5}}<\frac{rR}{R+r} \quad \text{i.e., if} \quad \frac{\sqrt{5}-1}{\sqrt{5}}<\frac{R}{R+r}$$

i.e., if $\quad (\sqrt{5}-1)\,R+(\sqrt{5}-1)\,r<R\sqrt{5}$

i.e., if $\quad (\sqrt{5}-1)\,r<R \quad$ i.e., if $\quad R/r>\sqrt{5}-1.$

Ex. 12. *A thin hemispherical bowl, of radius b and weight W rests in equilibrium on the highest point of a fixed sphere, of radius a, which is rough enough to prevent any sliding. Inside the bowl is placed a small smooth sphere of weight w; show that the equilibrium is not stable unless $w<W\dfrac{a-b}{2b}$.*



Sol. O is the centre, a the radius and C the highest point of the fixed sphere. A hemispherical bowl of radius b and weight W rests on the highest point C of this sphere and inside the bowl is placed a small smooth sphere of weight w. The weight W of the bowl acts at G_1 where $O'G_1=\frac{1}{2}O'C$.

First we want to find out the height of the C.G. of the combined body consisting of the hemispherical bowl of weight W and sphere of weight w above the point of contact C. If the upper bowl be slightly displaced, the small smooth sphere placed inside it moves in such a way that the line of action of its weight w always passes through O', the centre of the base of the bowl. Hence so far as the question of the stability of the bowl is concerned the weight w of the small sphere may be taken to act at the centre O' of the bowl. If h be the height of the centre of gravity G of the combined body (*i.e.*, hemispherical shell of weight W and sphere of weight w) above the point of contact C, then

w acts at O'

$$h=\frac{W\cdot\frac{1}{2}b+w\cdot b}{W+w}=\frac{(W+2w)\,b}{2\,(W+w)}.$$

Here $\rho_1=b$ and $\rho_2=a$. Hence the equilibrium will be stable if

$$\frac{1}{h}>\frac{1}{\rho_1}+\frac{1}{\rho_2} \text{ i.e., } \frac{1}{h}>\frac{1}{b}+\frac{1}{a} \text{ i.e., } \frac{1}{h}>\frac{a+b}{ab} \text{ i.e., } h<\frac{ab}{a+b}$$

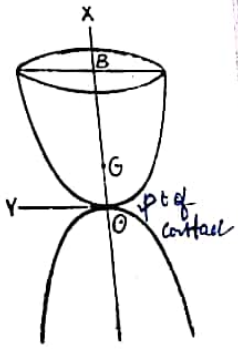
i.e., $\quad \dfrac{(W+2w)\,b}{2\,(W+w)}<\dfrac{ab}{a+b}$ i.e., $\dfrac{W+2w}{2\,(W+w)}<\dfrac{a}{a+b}$

i.e., $\quad (a+b)\,(W+2w)<2a\,(W+w)$

i.e., $\quad w\,(2a+2b-2a)<W\,(2a-a-b)$

i.e., $\quad 2wb<W\,(a-b)$ i.e., $w<\dfrac{W\,(a-b)}{2b}.$

Ex. 13. *A solid frustum of a paraboloid of revolution of height h and latus rectum $4a$, rests with its vertex on the vertex of a paraboloid of revolution, whose latus rectum is $4b$; show that the equilibrium is stable if $h<\dfrac{3ab}{a+b}$.*



Sol. The point of contact of the two bodies is O and $OB=h$.

Let the equation of the generating parabola of the upper paraboloid be

$$y^2=4ax.$$

The parabola $y^2=4ax$ passes through the origin and the y-axis is tangent at the origin. If ρ be the radius of curvature of this parabola at the origin, then by Newton's formula for the radius of curvature at the origin, we have

$$\rho=\lim_{x\to 0}\frac{y^2}{2x}=\lim_{x\to 0}\frac{4ax}{2x}=\lim_{x\to 0}2a=2a.$$

$\therefore$ the radius of curvature of the parabola $y^2=4ax$ at the vertex (*i.e.*, at the origin) is $\boxed{2a}$

So here, $\rho_1=$ the radius of curvature of the lower body at the point of contact $=2a$,

and $\quad \rho_2=$ the radius of curvature of the upper body at the point of contact $=2b$.

If H be the height of the centre of gravity G of the upper body above the point of contact O, then

$$H=OG=\bar{x}=\frac{\int x\,dm}{\int dm}=\frac{\int_0^h x\,\pi y^2\,dx}{\int_0^h xy^2\,dx}$$

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$$=\frac{\int_0^h x.4ax\,dx}{\int_0^h 4ax\,dx}=\frac{\int_0^h x^2\,dx}{\int_0^h x\,dx}=\frac{\left[\frac{x^3}{3}\right]_0^h}{\left[\frac{x^2}{2}\right]_0^h}=\frac{\frac{h^3}{3}}{\frac{h^2}{2}}=\tfrac{2}{3}h.$$

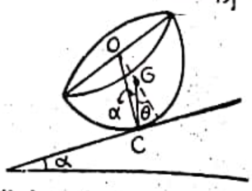
Now the equilibrium will be stable if

$$\frac{1}{H}>\frac{1}{\rho_1}+\frac{1}{\rho_2} \text{ i.e., } \frac{3}{2h}>\frac{1}{2a}+\frac{1}{2b}$$

i.e., $$\frac{3}{h}>\frac{a+b}{ab} \text{ i.e., } \frac{h}{3}<\frac{ab}{a+b} \text{ i.e., } h<\frac{3ab}{a+b}.$$

Ex. 14. *A solid hemisphere rests on a plane inclined to the horizon at an angle $\alpha < \sin^{-1} \frac{3}{8}$, and the plane is rough enough to prevent any sliding. Find the position of equilibrium and show that it is stable.* **[Meerut 90; Lucknow 79]**

Sol. Let O be the centre of the base of the hemisphere and r be its radius. If C is the point of contact of the hemisphere and the inclined plane, then $OC=r$. Let G be the centre of gravity of the hemisphere.



Then $OG=3r/8$. In the position of equilibrium the line CG must be vertical.

Since OC is perpendicular to the inclined plane and CG is perpendicular to the horizontal, therefore $\angle OCG=\alpha$. Suppose in equilibrium the axis of the hemisphere makes an angle θ with the vertical. From $\triangle OGC$, we have

$$\frac{OG}{\sin\alpha}=\frac{OC}{\sin\theta} \text{ i.e., } \frac{3r/8}{\sin\alpha}=\frac{r}{\sin\theta}.$$

$\therefore$ $\sin\theta=\frac{8}{3}\sin\alpha$, or $\theta=\sin^{-1}(\frac{8}{3}\sin\alpha)$,

giving the position of equilibrium of the hemisphere.

Since $\sin\theta<1$, therefore $\frac{8}{3}\sin\alpha<1$

i.e., $$\sin\alpha<\tfrac{3}{8} \text{ i.e., } \alpha<\sin^{-1}\tfrac{3}{8}.$$

Thus for the equilibrium to exist, we must have

$$\alpha<\sin^{-1}\tfrac{3}{8}.$$

Now let $CG=h$. Then

$$\frac{h}{\sin(\theta-\alpha)}=\frac{3r/8}{\sin\alpha}, \text{ so that } h=\frac{3r\sin(\theta-\alpha)}{8\sin\alpha}.$$

Here $\rho_1=r$ and $\rho_2=\infty$.

The equilibrium will be stable if

$$\boxed{h<\frac{\rho_1\rho_2\cos\alpha}{\rho_1+\rho_2}}$$ [See § 7]

$$\frac{1}{h}>\frac{\rho_1+\rho_2}{\rho_1\rho_2}\sec\alpha \text{ i.e., } \boxed{\frac{1}{h}>\left(\frac{1}{\rho_1}+\frac{1}{\rho_2}\right)\sec\alpha}$$

i.e., $$\frac{1}{h}>\frac{1}{r}\sec\alpha \qquad [\because \rho_1=r,\ \rho_2=\infty]$$

i.e., $$h<r\cos\alpha$$

i.e., $$\frac{3r\sin(\theta-\alpha)}{8\sin\alpha}<r\cos\alpha \qquad \text{[substituting for } h]$$

i.e., $$3\sin(\theta-\alpha)<8\sin\alpha\cos\alpha$$

or, $$3\sin\theta\cos\alpha-3\cos\theta\sin\alpha<8\sin\alpha\cos\alpha$$

or, $$8\sin\alpha\cos\alpha-3\sin\alpha\sqrt{(1-\tfrac{64}{9}\sin^2\alpha)}<8\sin\alpha\cos\alpha$$

$$[\because \sin\theta=\tfrac{8}{3}\sin\alpha]$$

or, $$-\sin\alpha\sqrt{(9-64\sin^2\alpha)}<0$$

or, $$\sin\alpha\sqrt{(9-64\sin^2\alpha)}>0. \qquad ...(2)$$

But from (1),

$$\sin\alpha<\tfrac{3}{8} \text{ i.e., } 64\sin^2\alpha<9 \text{ i.e., } \sqrt{(9-64\sin^2\alpha)}$$

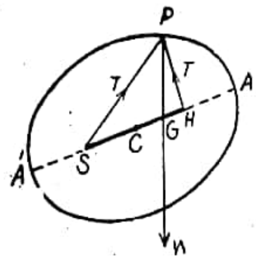
is a positive real number. Therefore the relation (2) is true. Hence the equilibrium is stable.

Ex. 15. *A rod SH, of length 2c and whose centre of gravity G is at a distance d from its centre, has a string, of length 2c sec α, tied to its two ends and the string is then slung over a small smooth peg P; find the position of equilibrium and show that the position which is not vertical is stable.*

Sol. We have

$SP+PH=$ the length of the string $=2c\sec\alpha$,

as is given. The middle point of the rod SH is C and its centre of gravity is G such that $CG=d$.



Since in an ellipse the sum of the focal distances of any point on it is constant and is equal to the length $2a$ of its major axis, therefore the peg P must lie on an ellipse whose foci are S and H and for which the length of the major axis $2a=2c\sec\alpha$, so that

$$a=c\sec\alpha.$$

Now $SH=2c$ (given) and so $CH=c$. But $CH=ae$, where e is the eccentricity of this ellipse.

$$\therefore \quad ae=c.$$

If b be the length of the semi minor axis of this ellipse, then

$$b^2=a^2(1-e^2)=a^2-a^2e^2=c^2\sec^2\alpha-c^2=c^2\tan^2\alpha.$$

Hence the equation of this ellipse with C as origin and CH as x-axis is

$$\frac{x^2}{c^2\sec^2\alpha}+\frac{y^2}{c^2\tan^2\alpha}=1,$$

or $$x^2\sin^2\alpha+y^2=c^2\tan^2\alpha.$$

Shifting the origin to the point G $(d, 0)$, it becomes

$$(x+d)^2\sin^2\alpha+y^2=c^2\tan^2\alpha.$$

Changing to polar coordinates, it becomes

$$(r\cos\theta+d)^2\sin^2\alpha+r^2\sin^2\theta=c^2\tan^2\alpha \qquad \ldots(1)$$

where G is the pole and GH is the initial line so that for the point P, $GP=r$ and $\angle PGH=\theta$.

If we find the value of θ for which r is maximum or minimum and regard the corresponding point P of the ellipse for the position of the peg and make PA vertical, we shall find the inclined position of equilibrium.

From (1),

$$r^2\cos^2\theta\sin^2\alpha+2rd\cos\theta\sin^2\alpha+d^2\sin^2\alpha+r^2-r^2\cos^2\theta=c^2\tan^2\alpha$$

or $$r^2\cos^2\theta\cos^2\alpha-2rd\cos\theta\sin^2\alpha+(c^2\tan^2\alpha-r^2-d^2\sin^2\alpha)=0.$$

This is a quadratic in $\cos\theta$. Therefore

$$\cos\theta=\frac{2rd\sin^2\alpha\pm\sqrt{[4r^2d^2\sin^4\alpha-4r^2\cos^2\alpha\,(c^2\tan^2\alpha-r^2-d^2\sin^2\alpha)]}}{2r^2\cos^2\alpha}$$

$$=\frac{d\sin^2\alpha\pm\sqrt{[d^2\sin^4\alpha-c^2\sin^2\alpha+r^2\cos^2\alpha+d^2\sin^2\alpha\cos^2\alpha]}}{r\cos^2\alpha}$$

$$=\frac{d\sin^2\alpha\pm\sqrt{[r^2\cos^2\alpha-(c^2-d^2)\sin^2\alpha]}}{r\cos^2\alpha}$$

For real values of $\cos\theta$, we must have

$$r^2\cos^2\alpha>(c^2-d^2)\sin^2\alpha \text{ i.e., } r^2>(c^2-d^2)\tan^2\alpha.$$

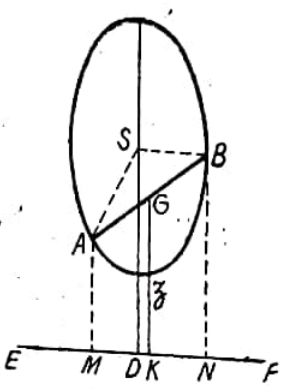
Therefore the least value of r is $\sqrt{(c^2-d^2)}\tan\alpha$ and in that case

$$\cos\theta=\frac{d\sin^2\alpha}{r\cos^2\alpha}=\frac{d\sin^2\alpha}{\sqrt{(c^2-d^2)}\tan\alpha\cos^2\alpha}=\frac{d\tan\alpha}{\sqrt{(c^2-d^2)}}.$$

This gives the position of equilibrium in which the rod is not vertical. Since in this case r, the depth of the C.G. of the rod below the peg, is minimum, therefore the equilibrium is unstable.

The other two positions of equilibrium are when P is at A or A' i.e., when the rod is vertical.

Ex. 16. *A smooth ellipse is fixed with its axis vertical and in it is placed a beam with its ends resting on the arc of the ellipse; if the length of the beam be not less than the latus rectum of the ellipse, show that when it is in stable equilibrium, it will pass through the focus.*



Sol. Let S be a focus and EF be the corresponding directrix of the ellipse. Referred to S as pole and the perpendicular SD from the focus to the directrix as the initial line, the polar equation of the ellipse is

$$l/r=1+e\cos\theta. \qquad \ldots(1)$$

Let AB be the beam and G its middle point i.e., its centre of gravity. Let z be the height of G above the fixed line EF. Then

$$z=GK=\tfrac{1}{2}(AM+BN).$$

But by the definition of the ellipse,

$$\frac{AS}{AM}=e \text{ and } \frac{BS}{BN}=e, \text{ so that } AM=\frac{1}{e}AS \text{ and } BN=\frac{1}{e}BS.$$

$$\therefore\quad z=\tfrac{1}{2}\left[\frac{AS}{e}+\frac{BS}{e}\right]=\frac{1}{2e}(AS+BS). \qquad \ldots(2)$$

Now z will be minimum if $AS+BS$ is minimum i.e., if A, S and B lie on the same straight line i.e., if the beam AB passes through the focus S. But z is minimum implies that the equilibrium of the beam is stable. Hence the equilibrium of the beam is stable when it passes through the focus S.

In this case when the beam passes through the focus S, we have

$$AB=AS+BS$$

$$=\frac{l}{1+e\cos\theta}+\frac{l}{1+e\cos(\pi+\theta)}, \qquad \text{by (1)}$$

[Note that if the vectorial angle of B is θ then that of A is $\pi+\theta$]

$$=\frac{l}{1+e\cos\theta}+\frac{l}{1-e\cos\theta}=\frac{2l}{1-e^2\cos^2\theta}.$$

$\therefore$ the length of the beam AB will be least when $1-e^2\cos^2\theta$ is greatest i.e., when $\cos\theta=0$ or $\theta=\frac{1}{2}\pi$.

Then $AB=2l=$ length of the latus rectum of the ellipse.

Therefore the least length of the beam is equal to the length of the latus rectum of the ellipse.

Problems based upon z-test

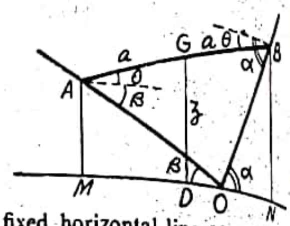
Ex. 17. *A uniform beam of length $2a$ rests with its ends on two smooth planes which intersect in a horizontal line. If the inclinations of the planes to the horizontal are α and β $(\alpha>\beta)$, show that the* *inclination θ of the beam to the horizontal in one of the equilibrium positions is given by*

$$\tan\theta=\tfrac{1}{2}(\cot\beta-\cot\alpha)$$

and show that the beam is unstable in this position. **[Kanpur 80]**

Sol. Let AB be a uniform beam of length $2a$ resting with its ends A and B on two smooth inclined planes OA and OB. Suppose the beam makes an angle θ with the horizontal. We have

$\angle AOM=\beta$ and $\angle BON=\alpha$.



The centre of gravity of the beam AB is its middle point G. Let z be the height of G above the fixed horizontal line MN. We shall express z as a function of θ.

We have, $z=GD=\tfrac{1}{2}(AM+BN)$

$$=\tfrac{1}{2}(OA\sin\beta+OB\sin\alpha). \quad \ldots(1)$$

Now in the triangle OAB, $\angle OAB=\beta+\theta$, $\angle OBA=\alpha-\theta$ and $\angle AOB=\pi-(\alpha+\beta)$. Applying the sine theorem for the ΔOAB, we have

$$\frac{OA}{\sin(\alpha-\theta)}=\frac{OB}{\sin(\beta+\theta)}=\frac{AB}{\sin[\pi-(\alpha+\beta)]}=\frac{2a}{\sin(\alpha+\beta)}.$$

$$\therefore\quad OA=\frac{2a\sin(\alpha-\theta)}{\sin(\alpha+\beta)},\ OB=\frac{2a\sin(\beta+\theta)}{\sin(\alpha+\beta)}.$$

Substituting for OA and OB in (1), we have

$$z=\tfrac{1}{2}\left[\frac{2a\sin(\alpha-\theta)}{\sin(\alpha+\beta)}\sin\beta+\frac{2a\sin(\beta+\theta)}{\sin(\alpha+\beta)}\sin\alpha\right]$$

$$=\frac{a}{\sin(\alpha+\beta)}\left[\sin(\alpha-\theta)\sin\beta+\sin(\beta+\theta)\sin\alpha\right]$$

$$=\frac{a}{\sin(\alpha+\beta)}\left[(\sin\alpha\cos\theta-\cos\alpha\sin\theta)\sin\beta+(\sin\beta\cos\theta+\cos\beta\sin\theta)\sin\alpha\right]$$

$$=\frac{a}{\sin(\alpha+\beta)}[\sin\theta(\sin\alpha\cos\beta-\cos\alpha\sin\beta)+2\cos\theta\sin\alpha\sin\beta].$$

$$\therefore\quad \frac{dz}{d\theta}=\frac{a}{\sin(\alpha+\beta)}[\cos\theta(\sin\alpha\cos\beta-\cos\alpha\sin\beta)-2\sin\theta\sin\alpha\sin\beta] \quad \ldots(2)$$

For equilibrium of the beam, we have $\dfrac{dz}{d\theta}=0$

i.e., $\cos\theta(\sin\alpha\cos\beta-\cos\alpha\sin\beta)-2\sin\theta\sin\alpha\sin\beta=0$

or $2\sin\theta\sin\alpha\sin\beta=\cos\theta(\sin\alpha\cos\beta-\cos\alpha\sin\beta)$

i.e., $$\frac{\sin\theta}{\cos\theta}=\tfrac{1}{2}\left(\frac{\sin\alpha\cos\beta-\cos\alpha\sin\beta}{\sin\alpha\sin\beta}\right)$$

or $$\tan\theta=\tfrac{1}{2}(\cot\beta-\cot\alpha). \quad \ldots(3)$$

This gives the required position of equilibrium of the beam.

Differentiating (2), we have

$$\frac{d^2z}{d\theta^2}=\frac{a}{\sin(\alpha+\beta)}[-\sin\theta(\sin\alpha\cos\beta-\cos\alpha\sin\beta)-2\cos\theta\sin\alpha\sin\beta]$$

$$=\frac{-2a\sin\alpha\sin\beta\cos\theta}{\sin(\alpha+\beta)}[\tfrac{1}{2}\tan\theta(\cot\beta-\cot\alpha)+1]$$

$$=\frac{-2a\sin\alpha\sin\beta\cos\theta}{\sin(\alpha+\beta)}[\tan^2\theta+1] \quad \text{[by (3)]}$$

$=$ a negative quantity because θ, α and β are all acute angles and $\alpha+\beta<\pi$.

Thus in the position of equilibrium $d^2z/d\theta^2$ is negative *i.e.*, z is maximum. Hence the equilibrium is unstable.

Ex. 18. *A uniform heavy beam rests between two smooth planes, each inclined at an angle $\frac{1}{4}\pi$ to the horizontal, so that the beam is in a vertical plane perpendicular to the line of action of the planes. Show that the equilibrium is unstable when the beam is horizontal.*

Sol. Draw figure as in Ex. 17, taking $\alpha=\beta=\frac{1}{4}\pi$. If the beam makes an angle θ with the horizontal and z be the height of the C.G. of the beam above the fixed horizontal line MN, then proceeding as in Ex. 17, we have

$$z=\frac{a}{\sin\frac{1}{2}\pi}[\sin(\tfrac{1}{4}\pi-\theta)\sin\tfrac{1}{4}\pi+\sin(\tfrac{1}{4}\pi+\theta)\sin\tfrac{1}{4}\pi]$$

$$=a\left[\left(\frac{1}{\sqrt{2}}\cos\theta-\frac{1}{\sqrt{2}}\sin\theta\right)\frac{1}{\sqrt{2}}+\left(\frac{1}{\sqrt{2}}\cos\theta+\frac{1}{\sqrt{2}}\sin\theta\right)\frac{1}{\sqrt{2}}\right]$$

$=a\cos\theta$.

$\therefore\quad dz/d\theta=-a\sin\theta$.

For equilibrium of the beam, we have $dz/d\theta=0$ *i.e.*, $\sin\theta=0$ *i.e.*, $\theta=0$.

$\therefore$ the beam rests in a horizontal position.

Now $d^2z/d\theta^2=-a\cos\theta$.

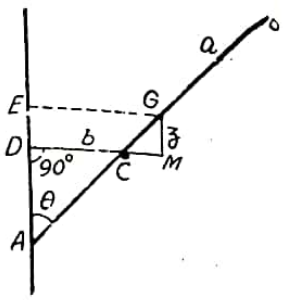
When $\theta=0$, $d^2z/d\theta^2=-a\cos 0=-a$, which is negative.

Thus in the position of equilibrium $d^2z/d\theta^2$ is negative *i.e.*, z is maximum. Hence the equilibrium is unstable.

Ex. 19. *A heavy uniform rod rests with one end against a smooth vertical wall and with a point in its length resting on a smooth peg; find the position of equilibrium and show that it is unstable.*

[Lucknow 81; Meerut 80, 82, 84P, 85, 85P, 86S, 87S, 92]

Sol. Let AB be a uniform rod of length $2a$. The end A of the rod rests against a smooth vertical wall and the rod rests on a smooth peg C whose distance from the wall is say b i.e., $CD=b$.



Suppose the rod makes an angle θ with the wall. The centre of gravity of the rod is at its middle point G. Let z be the height of G above the fixed peg C i.e., $GM=z$. We shall express z in terms of θ. We have

$$z=GM=ED=AE-AD$$
$$=AG\cos\theta-CD\cot\theta$$
$$=a\cos\theta-b\cot\theta.$$

$\therefore \quad dz/d\theta=-a\sin\theta+b\operatorname{cosec}^2\theta$

and $\quad d^2z/d\theta^2=-a\cos\theta-2b\operatorname{cosec}^2\theta\cot\theta.$

For equilibrium of the rod, we have $dz/d\theta=0$

i.e., $\quad -a\sin\theta+b\operatorname{cosec}^2\theta=0$

or $\quad a\sin\theta=b\operatorname{cosec}^2\theta, \quad$ or $\quad \sin^3\theta=b/a$

or $\quad \sin\theta=(b/a)^{1/3}, \quad$ or $\quad \theta=\sin^{-1}(b/a)^{1/3}.$

This gives the position of equilibrium of the rod.

Again $d^2z/d\theta^2=-(a\cos\theta+2b\operatorname{cosec}^2\theta\cot\theta)$

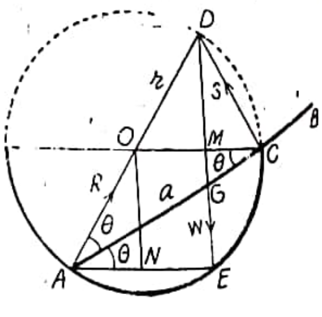
$=-$negative for all acute values of θ.

Thus $d^2z/d\theta^2$ is negative in the position of equilibrium and so z is maximum. Hence the equilibrium is unstable.

Ex. 20. *A heavy uniform rod, length 2a, rests partly within and partly without a fixed smooth hemispherical bowl of radius r, the rim of the bowl is horizontal, and one point of the rod is in contact with the rim; if θ be the inclination of the rod to the horizon, show that* $2r\cos 2\theta=a\cos\theta$.

Show also that the equilibrium of the rod is stable.

Sol. Let AB be the rod of length $2a$ with its centre of gravity at G. A point C of its length is in contact with the rim of the bowl of radius r and centre O.



The rod is in equilibrium under the action of three forces. The reaction R of the bowl at A is along the normal AO and the reaction S of the rim at C is perpendicular to the rod. Let these reactions meet in a point

D. Since the line AOD passes through the centre O of the bowl and $\angle ACD$ is a right angle, therefore AOD is a diameter of the sphere of which the bowl is a part.

The third force on the rod is its weight W acting vertically downwards through its middle point G. Since the three forces must be concurrent, therefore the line DG is vertical.

Suppose the line DG meets the surface of the bowl at the point E. Join AE; then AE is horizontal because $\angle AED=90°$, being the angle in a semi-circle.

We have $\angle BAE=\theta=\angle ACO \qquad [\because AE$ is parallel to $OC]$

$\qquad =\angle OAC. \qquad [\because OA=OC]$

$\therefore \quad \angle DAE=2\theta.$

Suppose z is the depth of the centre of gravity G of the rod below the fixed horizontal line OC. Then

$$z=MG=ME-GE=ON-GE$$
$$=OA\sin 2\theta-AG\sin\theta=r\sin 2\theta-a\sin\theta.$$

$\therefore \quad dz/d\theta=2r\cos 2\theta-a\cos\theta.$

For the equilibrium of the rod, we must have $dz/d\theta=0$

i.e., $2r\cos 2\theta-a\cos\theta=0$ i.e., $2r\cos 2\theta=a\cos\theta$.

This gives the position of equilibrium of the rod.

Again $\quad d^2z/d\theta^2=-4r\sin 2\theta+a\sin\theta$

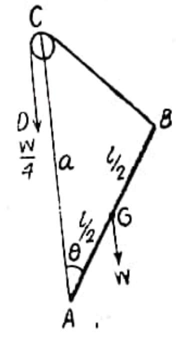
$=-2(2r\sin 2\theta)+a\sin\theta$

$=-2.DE+GE$, which is negative because $DE>GE$.

Thus the depth z of the C.G. of the rod below a fixed horizontal line is maximum. Hence the equilibrium is stable.

Ex. 21. *One end A of a uniform rod AE of weight W and length l is smoothly hinged at a fixed point, while B is tied to a light string which passes over a small smooth pulley at a distance a vertically above A and carries a weight* $W/4$. *If* $l<a<2l$, *show that the system is in stable equilibrium when AB is vertically upwards, and that there is also a configuration of equilibrium in which the rod is at a certain angle to the vertical.*

Sol. Let AB be the rod of length l hinged at the fixed point A. The weight W of the rod acts through its middle point G. Let h be the length of the string BCD which is attached to E and passes over a smooth pulley at C, AC being vertical and equal to a. The string carries a weight $W/4$ at its other end D. Let $\angle BAC=\theta$.



From $\triangle BAC$,

$BC=\sqrt{(AB^2+AC^2-2AB.AC.\cos\theta)}$

$=\sqrt{(l^2+a^2-2la\cos\theta)}.$

$\therefore$ the length of the portion CD of the string hanging vertically

$=b-BC=b-\sqrt{(l^2+a^2-2la\cos\theta)}$.

The weight W acts at the point G whose height above the fixed point A is $AG\cos\theta$ *i.e.*, $\frac{1}{2}l\cos\theta$. The weight $W/4$ acts at D whose height above A is $a-b+\sqrt{(l^2+a^2-2la\cos\theta)}$.

Hence if z be the height, above the fixed point A, of the centre of gravity of the system consisting of the weight W and $W/4$, then

$$(W+\tfrac{1}{4}W)z=W.\tfrac{1}{2}l\cos\theta+\tfrac{1}{4}W\{a-b+\sqrt{(l^2+a^2-2la\cos\theta)}\}$$

i.e., $5z=2l\cos\theta+a-b+\sqrt{(l^2+a^2-2la\cos\theta)}$.

$$\therefore\quad 5\frac{dz}{d\theta}=-2l\sin\theta+\frac{al\sin\theta}{\sqrt{(l^2+a^2-2la\cos\theta)}}$$

and
$$5\frac{d^2z}{d\theta^2}=-2l\cos\theta+\frac{al\cos\theta}{\sqrt{(l^2+a^2-2la\cos\theta)}}-\frac{a^2l^2\sin^2\theta}{(l^2+a^2-2la\cos\theta)^{3/2}}$$

For the equilibrium of the system we must have $dz/d\theta=0$. Obviously $dz/d\theta$ vanishes when $\sin\theta=0$ *i.e.*, $\theta=0$ *i.e.*, the rod AB is vertically upwards. Thus the system is in equilibrium when the rod AB is vertically upwards.

For $\theta=0$, we have
$$5\frac{d^2z}{d\theta^2}=-2l+\frac{al}{\sqrt{(l^2+a^2-2la)}}$$
$$=-2l+\frac{al}{a-l},\text{ if } a>l$$
$$=-l\frac{(a-2l)}{a-l},$$

which is positive if $l<a<2l$.

Thus if $l<a<2l$, then for $\theta=0$, $d^2z/d\theta^2$ is positive *i.e.*, z is minimum. Hence this is a stable position of equilibrium.

Again $dz/d\theta$ also vanishes when

$$-2+\frac{a}{\sqrt{(l^2+a^2-2l\,a\cos\theta)}}=0\quad\text{or}\quad 4=\frac{a^2}{(l^2+a^2-2l\,a\cos\theta)}$$

or $\quad 4l^2+4a^2-8\,la\cos\theta=a^2$

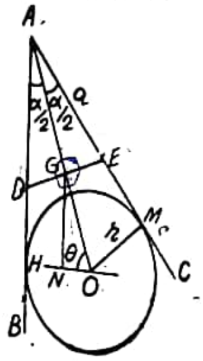
or $\quad \cos\theta=\dfrac{3a^2+4l^2}{8la}$, which gives a real value of θ when $l<a<2l$.

So there is also a configuration of equilibrium in which the rod is inclined to the vertical.

Ex. 22. *Two equal uniform rods are firmly jointed at one end so that the angle between them is α, and they rest in a vertical plane on a smooth sphere of radius r. Show that they are in a stable or unstable equilibrium according as the length of the rod is $>$ or $<$ $4r$ cosec α.* **[Lucknow 80]**

Sol. Let AB and AC be two rods jointed at A and placed in a vertical plane on a smooth sphere of centre O and radius r. We have $\angle BAC=\alpha$. Since the rods are tangential to the sphere, therefore $\angle BAO=\angle CAO=\frac{1}{2}\alpha$.

Suppose $AB=AC=2a$.



If D and E are the middle points of the rods AB and AC, then the combined C.G. of the rods is at the middle point G of ED which must be on AO. Suppose the rod AC touches the sphere at M. We have,

$OM=r$, $AE=a$, $\angle AMO=90°$, $\angle AGE=90°$.

Suppose AO makes an angle θ with the horizontal line OH through the fixed point O. Let z be the height of the C.G. of the system above the horizontal through O. Then

$$z=GN=OG\sin\theta=(AO-AG)\sin\theta$$
$$=(r\operatorname{cosec}\tfrac{1}{2}\alpha-a\cos\tfrac{1}{2}\alpha)\sin\theta.$$
$$\therefore\ dz/d\theta=(r\operatorname{cosec}\tfrac{1}{2}\alpha-a\cos\tfrac{1}{2}\alpha)\cos\theta.$$

For the equilibrium of the rods, we must have $dz/d\theta=0$ *i.e.*, $(r\operatorname{cosec}\frac{1}{2}\alpha-a\cos\frac{1}{2}\alpha)\cos\theta=0$ *i.e.*, $\boxed{\cos\theta=0}$ *i.e.*, $\theta=\frac{1}{2}\pi$.

Thus in the position of equilibrium of rods, the line AO must be vertical.

Also
$$d^2z/d\theta^2=-(r\operatorname{cosec}\tfrac{1}{2}\alpha-a\cos\tfrac{1}{2}\alpha)\sin\theta$$
$$=-r\operatorname{cosec}\tfrac{1}{2}\alpha+a\cos\tfrac{1}{2}\alpha,\text{ for }\boxed{\theta=\tfrac{1}{2}\pi.}$$

The equilibrium will be stable or unstable according as the height z of the C.G. of the system is minimum or maximum in the position of equilibrium,

i.e., according as $\boxed{d^2z/d\theta^2}$ is positive or negative at $\theta=\frac{1}{2}\pi$

i.e., according as $a\cos\frac{1}{2}\alpha>$ or $<r\operatorname{cosec}\frac{1}{2}\alpha$

i.e., according as $2a>$ or $<\dfrac{2r}{\cos\frac{1}{2}\alpha\sin\frac{1}{2}\alpha}$

i.e., according as $2a>$ or $<\dfrac{4r}{\sin\alpha}$

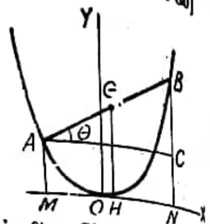
i.e., according as $2a>$ or $<4r\operatorname{cosec}\alpha$.

Ex. 23. *A uniform rod, of length $2l$, is attached by smooth rings at both ends of a parabolic wire, fixed with its axis vertical and vertex downwards, and of latus rectum $4a$. Show that the angle θ which the rod makes with the horizontal in a slanting position of* *equilibrium is given by* $\cos^2\theta=2a/l$, *and that, if these positions exist they are stable.*

Show also that the positions in which the rod is horizontal are stable or unstable according as the rod is below or above the focus. **[Meerut 80]**

Sol. Let AB be the rod of length $2l$. Take OX and OY as coordinate axes, so that the equation of the parabola be written as

$$x^2=4ay.$$



Let the coordinates of the point A be $(2at, at^2)$ and let the rod AB make an angle θ with the horizontal AC. Then the coordinates of B are $(2at+2l\cos\theta,\ at^2+2l\sin\theta)$. Since B lies on the parabola $x^2=4ay$, therefore

$$(2at+2l\cos\theta)^2=4a\,(at^2+2l\sin\theta)$$

or $\quad 8atl\cos\theta+4l^2\cos^2\theta=8al\sin\theta$

or $\quad (2al\cos\theta)\,t=2al\sin\theta-l^2\cos^2\theta$

or $\quad t=\tan\theta-(l/2a)\cos\theta. \qquad \ldots(1)$

The centre of gravity of the rod AB is at its middle point G. If z be the height of G above the fixed horizontal line OX, then

$$z=GH=\tfrac{1}{2}\,(AM+BN)$$
$$=\tfrac{1}{2}\,[at^2+(at^2+2l\sin\theta)]=at^2+l\sin\theta$$
$$=a\,[\tan\theta-(l/2a)\cos\theta]^2+l\sin\theta \qquad \text{[from (1)]}$$
$$=(l^2/4a)\cos^2\theta+a\tan^2\theta=(1/4a)\,[l^2\cos^2\theta+4a^2\tan^2\theta].$$

$\therefore \quad dz/d\theta=(1/4a)\,[-2l^2\cos\theta\sin\theta+8a^2\tan\theta\sec^2\theta]$

$$=(1/2a)\sin\theta\,[-l^2\cos\theta+4a^2\sec^3\theta].$$

For the equilibrium of the rod, we must have $dz/d\theta=0$

i.e. $\quad (1/2a)\sin\theta\,(-l^2\cos\theta+4a^2\sec^3\theta)=0.$

$\therefore$ either $\sin\theta=0$ *i.e.*, $\theta=0$, which gives the horizontal position of rest of the rod

or $\quad -l^2\cos\theta+4a^2\sec^3\theta=0$ *i.e.* $l^2\cos\theta=4a^2/\cos^3\theta$

i.e., $\cos^4\theta=4a^2/l^2$ *i.e.*, $\cos^2\theta=2a/l$, which gives the inclined position of rest of the rod.

Now, $\quad d^2z/d\theta^2=(1/2a)\cos\theta\,[-l^2\cos\theta+4a^2\sec^3\theta]$

$$+(1/2a)\sin\theta\,[l^2\sin\theta+12a^2\sec^3\theta\tan\theta]. \qquad \ldots(2)$$

When $\cos^2\theta=2a/l$ *i.e.*, when $-l^2\cos\theta+4a^2\sec^3\theta=0$, we have

$$d^2z/d\theta^2=(1/2a)\sin\theta\,[l^2\sin\theta+12a^2\sec^3\theta\tan\theta]$$
$$=(1/2a)\sin^2\theta\,[l^2+12a^2\sec^4\theta], \text{ which is } >0.$$

Hence in the inclined position of rest of the rod, z is minimum and so the equilibrium is stable.

Again when the rod is horizontal *i.e.*, $\theta=0$, we have, from (2)

$$\frac{d^2z}{d\theta^2}=\frac{8a^2-2l^2}{4a}=\frac{4a^2-l^2}{2a}.$$

The equilibrium in this case is stable or unstable according as $d^2z/d\theta^2$ is positive or negative

i.e., $\quad$ according as $4a^2-l^2>$ or <0

i.e., $\quad$ according as $2a>$ or $<l$

i.e., $\quad$ according as $2l<$ or $>4a$

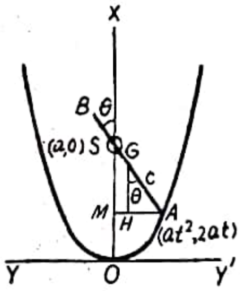
i.e., $\quad$ according as the rod is below or above the focus.

Ex. 24. *A uniform smooth rod passes through a ring at the focus of a fixed parabola whose axis is vertical and vertex below the focus, and rests with one end on the parabola. Prove that the rod will be in equilibrium if it makes with the vertical an angle* θ *given by the equation*

$$\cos^4\tfrac{1}{2}\theta=a/2c$$

where $4a$ *is the latus rectum and* $2c$ *the length of the rod. Investigate also the stability of equilibrium in this position.* **[Lucknow 81]**

Sol. Let the equation of the parabola be $\quad y^2=4ax.$



Let AB be the rod of length $2c$ with its end A on the parabola and passing through a ring at the focus S. Let the coordinates of A be $(at^2, 2at)$; the coordinates of the focus S are $(a, 0)$. If the rod AB makes an angle θ with the vertical OX, then

$\tan\theta=$ the gradient of the line AB

$$=\frac{2at-0}{at^2-a}=\frac{2t}{t^2-1}=\frac{-2t}{1-t^2}.$$

$$\therefore \quad \frac{2\tan\frac{1}{2}\theta}{1-\tan^2\frac{1}{2}\theta}=\frac{2\,(-t)}{1-(-t)^2}, \quad \text{or} \quad \tan\tfrac{1}{2}\theta=-t.$$

Let z be the height of the centre of gravity G of the rod AB above the fixed horizontal line YOY'. Then

$$z=OM+HG=OM+AG\cos\theta$$
$$=at^2+c\cos\theta$$
$$[\because \ OM=x\text{-coordinate of } A \text{ and } AG=\tfrac{1}{2}AB]$$
$$=a\tan^2\tfrac{1}{2}\theta+c\cos\theta.$$

$\therefore \quad dz/d\theta=2a\,(\tan\tfrac{1}{2}\theta\sec^2\tfrac{1}{2}\theta)\cdot\tfrac{1}{2}-c\sin\theta$

$$=a \tan \tfrac{1}{2}\theta \sec^2 \tfrac{1}{2}\theta - c.2 \sin \tfrac{1}{2}\theta \cos \tfrac{1}{2}\theta$$
$$=\sin \tfrac{1}{2}\theta\, [a \sec^3 \tfrac{1}{2}\theta - 2c \cos \tfrac{1}{2}\theta].$$

For the equilibrium of the rod, we must have $dz/d\theta=0$

i.e., $\sin \tfrac{1}{2}\theta\, (a \sec^3 \tfrac{1}{2}\theta - 2c \cos \tfrac{1}{2}\theta)=0.$

$\therefore$ either $\sin \tfrac{1}{2}\theta=0$ *i.e.,* $\theta=0$,

which gives the vertical position of equilibrium,

or $a \sec^3 \tfrac{1}{2}\theta - 2c \cos \tfrac{1}{2}\theta=0$ *i.a.,* $a \sec^3 \tfrac{1}{2}\theta = 2c \cos \tfrac{1}{2}\theta$

i.e., $\cos^4 \tfrac{1}{2}\theta = a/2c$, which gives the inclined position of rest of the rod.

Now

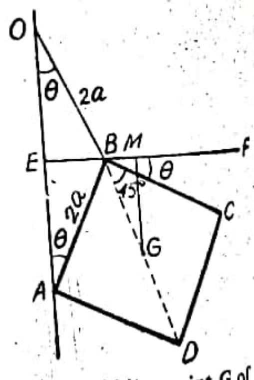
$$\frac{d^2z}{d\theta^2}=\tfrac{1}{4}\cos \tfrac{1}{2}\theta\, [a \sec^3 \tfrac{1}{2}\theta - 2c \cos \tfrac{1}{2}\theta]$$
$$+\sin \tfrac{1}{2}\theta \left[\frac{3a}{2} \sec^3 \tfrac{1}{2}\theta \tan \tfrac{1}{2}\theta + c \sin \tfrac{1}{2}\theta\right]$$
$$=\tfrac{1}{4}\cos \tfrac{1}{2}\theta\, [a \sec^3 \tfrac{1}{2}\theta - 2c \cos \tfrac{1}{2}\theta] + \sin^2 \tfrac{1}{2}\theta\, [\tfrac{3}{2}a \sec^4 \tfrac{1}{2}\theta + c],$$

which is >0 when $\cos^4 \tfrac{1}{2}\theta = a/2c$

i.e., when $a \sec^3 \tfrac{1}{2}\theta - 2c \cos \tfrac{1}{2}\theta = 0$.

Thus in the inclined position of equilibrium of the rod, $d^2z/d\theta^2$ is positive *i.e.,* z is minimum. Hence the equilibrium is stable in the inclined position of rest of the rod.

Ex. 25. *A square lamina rests with its plane perpendicular to a smooth wall one corner being attached to a point in the wall by a fine string of length equal to the side of the square. Find the position of equilibrium and show that it is stable.*



Sol. $ABCD$ is a square lamina of side $2a$. It is suspended from the point O in the wall by a fine string OB of length $2a$. The corner A of the lamina touches the wall and the plane of the lamina is perpendicular to the wall.

Let $\angle BAO=\theta$.

Then $\angle AOB=\angle BAO=\theta$. [$\because$ $AB=OB$]

Since BC is perpendicular to AB and the horizontal line EF is perpendicular to AO, therefore $\angle FBC=\theta$.

The centre of gravity of the lamina is the middle point G of the diagonal BD. We have

$$BG=\tfrac{1}{2}BD=\tfrac{1}{2}.2a\sqrt{2}=a\sqrt{2},$$

$\angle CBD=45°$ and $\angle FBG=45°+\theta$.

If z be the depth of G below the fixed point O, then

$$z=OE+MG=2a\cos\theta+BG\sin(45°+\theta)$$
$$=2a\cos\theta+a\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos\theta+\frac{1}{\sqrt{2}}\sin\theta\right)$$
$$=3a\cos\theta+a\sin\theta.$$

$\therefore$ $dz/d\theta=-3a\sin\theta+a\cos\theta.$

For equilibrium, $dz/d\theta=0$

i.e., $-3a\sin\theta+a\cos\theta=0$ *i.e.,* $\tan\theta=\tfrac{1}{3}$.

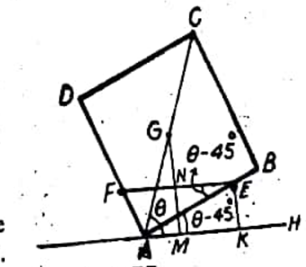
This gives the position of equilibrium *i.e.,* in equilibrium the side AB of the lamina makes an angle $\tan^{-1}\tfrac{1}{3}$ with the wall.

Now $d^2z/d\theta^2=-3a\cos\theta-a\sin\theta$

$$=-a\left(3\times\frac{3}{\sqrt{10}}+\frac{1}{\sqrt{10}}\right), \text{ when } \tan\theta=\tfrac{1}{3}$$
$$=\text{a negative number.}$$

Thus in the position of equilibrium the depth z of the C.G. of the lamina below the fixed point O is maximum. Hence the equilibrium is stable.

Ex. 26. *A square lamina rests in a vertical plane on two smooth pegs which are in the same horizontal line. Show that there is only one position of equilibrium unless the distance between the pegs is greater than one-quarter of the diagonal of the square, but that if this condition is satisfied, there may be three positions of equilibrium and that the symmetrical position will be stable, but the other two positions of equilibrium will be unstable.*



Sol. $ABCD$ is a square lamina resting on the pegs E and F which are in the same horizontal line. Let $EF=c$ and $AC=2d$. Suppose the diagonal AC makes an angle θ with the horizontal AH. Then

$\angle EAK=\theta-\angle CAB=\theta-45°$.

The C.G. of the lamina is the middle point G of the diagonal AC.

Let z be the height of G above the fixed line EF.

Then $z=GN=GM-NM=GM-EK$

$$=AG\sin\theta-AE\sin(\theta-45°)$$
$$=d\sin\theta-EF\cos(\theta-45°)\sin(\theta-45°)$$
$$=d\sin\theta-\tfrac{1}{2}c\sin 2(\theta-45°)$$
$$=d\sin\theta-\tfrac{1}{2}c\sin(2\theta-90°)$$

$=d \sin\theta+\frac{1}{2}c \sin(90°-2\theta)=d\sin\theta+\frac{1}{2}c\cos 2\theta.$

$\therefore \quad dz/d\theta=d\cos\theta-c\sin 2\theta.$

For equilibrium,

$$dz/d\theta=0 \text{ i.e., } d\cos\theta-c\sin 2\theta=0$$

i.e., $\quad d\cos\theta-2c\sin\theta\cos\theta=0$ i.e., $\cos\theta\,(d-2c\sin\theta)=0.$

$\therefore \quad \cos\theta=0$ i.e., $\theta=\frac{1}{2}\pi,$

or $\quad d-2c\sin\theta=0$ i.e., $\sin\theta=d/2c$ i.e., $\theta=\sin^{-1}(d/2c).$

In the position of equilibrium given by $\theta=\frac{1}{2}\pi$, the diagonal AC is vertical and the square rests symmetrically on the pegs.

In the position of equilibrium given by $\theta=\sin^{-1}(d/2c)$, if $d/2c<1$, the diagonal AC is not vertical but is inclined at some angle to the vertical. So it gives inclined position of equilibrium.

But we know that $\quad \sin\theta=\sin(\pi-\theta).$

Hence we shall have two inclined positions of equilibrium given by

$$\theta=\sin^{-1}(d/2c) \text{ and } \theta=\pi-\sin^{-1}(d/2c).$$

The inclined position of equilibrium is possible only when

$d/2c<1 \quad [\because \sin\theta<1$ for inclined position]

i.e., when $d<2c$ i.e., when $c>\frac{1}{2}d$ i.e., when $c>\frac{1}{4}.(2d)$

i.e., when the distance between the pegs $>\frac{1}{4}$ (length of the diagonal).

Thus there is only one position of equilibrium (*i.e.*, the symmetrical position) unless the distance between the pegs is greater than one-quarter of the diagonal of the square. Also if $2c>d$, there are three positions of equilibrium.

To determine the nature of equilibrium when $2c>d$.

We have,

$$d^2z/d\theta^2=-d\sin\theta-2c\cos 2\theta$$

$=-d\sin\theta-2c\,(1-2\sin^2\theta)=-d\sin\theta-2c+4c\sin^2\theta.$

For the symmetrical position of equilibrium $\theta=\frac{1}{2}\pi,$

$$d^2z/d\theta^2=-d-2c+4c=2c-d>0, \quad \text{because } 2c>d.$$

$\therefore \quad d^2z/d\theta^2$ is positive when $\theta=\frac{1}{2}\pi$ and so z is minimum for $\theta=\frac{1}{2}\pi$. Hence the symmetrical position of equilibrium given by $\theta=\frac{1}{2}\pi$ is stable.

For the inclined position of equilibrium given by $\sin\theta=d/2c$, we have

$$\frac{d^2z}{d\theta^2}=-d.\frac{d}{2c}-2c+4c.\frac{d^2}{4c^2}=-\frac{d^2}{2c}+\frac{d^2}{c}-2c$$

$$=\frac{d^2-4c^2}{2c}<0, \text{ because } 2c>d.$$

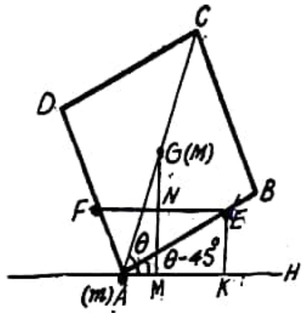
$\therefore \quad d^2z/d\theta^2$ is negative when $\sin\theta=d/2c$ and so z is maximum for the inclined positions of equilibrium. Hence the inclined positions of equilibrium are unstable.

Remark. When $2c<d$, there is only one position of equilibrium *i.e.*, the symmetrical position of equilibrium. For this position of equilibrium, $d^2z/d\theta^2=2c-d$ which is <0, because $2c<d$. Hence z is maximum and the equilibrium is unstable.

Ex. 27. *A uniform square board of mass M is supported in a vertical plane on two smooth pegs on the same horizontal level. The distance between the pegs is a and the diagonal of the square is D, where $D>4a$. If one diagonal is vertical and a mass m is attached to its lower end, prove that the equilibrium is stable, if*

$$4am>M.(D-4a).$$

Sol. $ABCD$ is a square board resting on the pegs E and F which are in the same horizontal line.



We have

$EF=a$ and $AC=D.$

The mass M of the lamina acts at the middle point G of AC and there is a mass m attached at A. Suppose the diagonal AC makes an angle θ with the horizontal AH. Then

$$\angle EAK=\theta-45°=\angle FEA.$$

The height of G (*i.e.*, the point where M acts) above EF

$=GN=GM-NM=GM-EK=AG\sin\theta-AE\sin(\theta-45°)$

$=\frac{1}{2}D\sin\theta-EF\cos(\theta-45°)\sin(\theta-45°)$

$=\frac{1}{2}D\sin\theta-\frac{1}{2}a\sin 2\,(\theta-45°)=\frac{1}{2}D\sin\theta-\frac{1}{2}a\sin(2\theta-90°)$

$=\frac{1}{2}D\sin\theta+\frac{1}{2}a\sin(90°-2\theta)=\frac{1}{2}D\sin\theta+\frac{1}{2}a\cos 2\theta.$

Also the depth of A (*i.e.*, the point where m acts) below EF

$=EK=AE\sin(\theta-45°)=EF\cos(\theta-45°)\sin(\theta-45°)$

$=\frac{1}{2}a\sin(2\theta-90°)=-\frac{1}{2}a\cos 2\theta.$

Let z be the height of C.G. of the system consisting of the masses M and m above the fixed line EF. Then

$$z=\frac{M\,(\frac{1}{2}D\sin\theta+\frac{1}{2}a\cos 2\theta)+m\,[-(-\frac{1}{2}a\cos 2\theta)]}{M+m}$$

$$=\frac{\frac{1}{2}MD\sin\theta+(M+m).\frac{1}{2}a\cos 2\theta}{M+m}.$$

$$\therefore \quad \frac{dz}{d\theta}=\frac{1}{M+m}[\tfrac{1}{2}MD\cos\theta-a(M+m)\sin 2\theta].$$

For equilibrium, $dz/d\theta=0$,

$$\tfrac{1}{2}MD\cos\theta-2a(M+m)\sin\theta\cos\theta=0$$

i.e., $\cos\theta\,[\tfrac{1}{2}MD-2a(M+m)\sin\theta]=0.$

$\therefore$ either $\cos\theta=0$ i.e., $\theta=\tfrac{1}{2}\pi$.

or $\tfrac{1}{2}MD-2a(M+m)\sin\theta=0$

i.e. $\sin\theta=MD/\{4a(M+m)\}$.

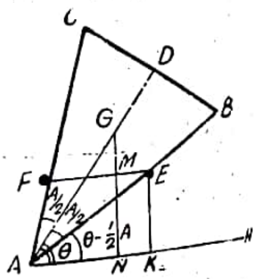
Now $\theta=\tfrac{1}{2}\pi$ means the diagonal AC is vertical.

We have $$\frac{d^2z}{d\theta^2}=\frac{1}{M+m}[-\tfrac{1}{2}MD\sin\theta-2a(M+m)\cos 2\theta]$$

$$=\frac{1}{M+m}[-\tfrac{1}{2}MD+2a(M+m)],\text{ for }\theta=\tfrac{1}{2}\pi.$$

The equilibrium is stable at $\theta=\frac{1}{2}\pi$ if z is minimum at $\theta=\frac{1}{2}\pi$ i.e., if $d^2z/d\theta^2$ is positive at $\theta=\frac{1}{2}\pi$ i.e., if $-\frac{1}{2}MD-2a(M+m)>0$ or, $4am > MD-4aM$ or, $4am > M(D-4a)$.

Ex. 28 (a). *A uniform isosceles triangular lamina ABC rests in equilibrium with its equal sides AB and AC in contact with two smooth pegs in the same horizontal line at a distance c apart. If the perpendicular AD upon BC is h, show that there are three positions of equilibrium, of which the one with AD vertical is stable and the other two are unstable, if $h < 3c$ cosec A; whilst if $h \geqslant 3c$ cosec A, there is only one position of equilibrium, which is unstable.*



Sol. ABC is an isosceles triangular lamina resting on two smooth pegs E and F which are in the same horizontal line and $EF=c$. The perpendicular AD from A upon BC is of length h. We have

$$\angle BAD=\angle CAD=\tfrac{1}{2}A.$$

The weight of the lamina acts at its centre of gravity G, where

$$AD=\tfrac{2}{3}AD=\tfrac{2}{3}h.$$

Suppose AD makes an angle θ with the horizontal AH, so that

$$\angle BAH=\theta-\tfrac{1}{2}A.$$

Let z be the height of G above the fixed horizontal line EF. Then

$$z=GM=GN-MN=GN-EK=AG\sin\theta-AE\sin(\theta-\tfrac{1}{2}A)$$

$$=\tfrac{2}{3}h\sin\theta-AE\sin(\theta-\tfrac{1}{2}A). \qquad ...(1)$$

Since EF is parallel to AK, therefore

$$\angle FEA=\angle EAK=\theta-\tfrac{1}{2}A.$$

Now in the $\triangle AEF$, we have

$$\angle EFA=\pi-\{A+(\theta-\tfrac{1}{2}A)\}=\pi-(\theta+\tfrac{1}{2}A).$$

Applying the sine theorem of trigonometry for the $\triangle AEF$, we have

$$\frac{AE}{\sin\angle EFA}=\frac{EF}{\sin\angle FAE}$$

i.e., $$\frac{AE}{\sin\{\pi-(\theta+\frac{1}{2}A)\}}=\frac{c}{\sin A}.$$

$$\therefore \quad AE=\frac{c}{\sin A}\sin(\theta+\tfrac{1}{2}A).$$

Substituting this value of AE in (1), we have

$$z=\tfrac{2}{3}h\sin\theta-\frac{c}{\sin A}\sin(\theta+\tfrac{1}{2}A)\sin(\theta-\tfrac{1}{2}A)$$

$$=\tfrac{2}{3}h\sin\theta-\frac{c}{2\sin A}[\cos A-\cos 2\theta]$$

$$=\tfrac{2}{3}h\sin\theta-\frac{c}{2}\cot A+\frac{c}{2\sin A}\cos 2\theta.$$

$$\therefore \quad \frac{dz}{d\theta}=\tfrac{2}{3}h\cos\theta-\frac{c}{\sin A}\sin 2\theta. \qquad ...(2)$$

For equilibrium, $\boxed{dz/d\theta=0}$

i.e., $$\tfrac{2}{3}h\cos\theta-\frac{2c}{\sin A}\sin\theta\cos\theta=0$$

i.e., $$2\cos\theta\left[\tfrac{1}{3}h-\frac{c\sin\theta}{\sin A}\right]=0.$$

$\therefore$ either $\cos\theta=0$ i.e. $\theta=\frac{1}{2}\pi$

or $$\tfrac{1}{3}h-\frac{c\sin\theta}{\sin A}=0 \text{ i.e. } \sin\theta=\frac{h\sin A}{3c}=\frac{h}{3c\operatorname{cosec}A}.$$

Now $\theta=\frac{1}{2}\pi$ gives the position of equilibrium in which AD is vertical and the triangle rests symmetrically on the pegs. The values of θ given by $\sin\theta=h/(3c\operatorname{cosec}A)$ are real and not equal to $\frac{1}{2}\pi$ if $h<3c\operatorname{cosec}A$. Since $\sin(\pi-\theta)=\sin\theta$, therefore if $h<3c\operatorname{cosec}A$, the equation $\sin\theta=h/(3c\operatorname{cosec}A)$ gives two inclined positions of equilibrium, one θ and the other $\pi-\theta$. Thus if $h<3c\operatorname{cosec}A$, there are three positions of equilibrium, one symmetrical and the other two inclined.

If $h \geqslant 3c$ cosec A, then the equation $\sin \theta = h/(3c \operatorname{cosec} A)$ either gives no real value of θ or the value of θ given by it is also equal to $\frac{1}{2}\pi$. Thus in this case the symmetrical position of equilibrium, $\theta = \frac{1}{2}\pi$, is the only position of equilibrium.

Nature of equilibrium.

From (2), $$\frac{d^2z}{d\theta^2} = -\tfrac{2}{3}h \sin \theta - \frac{2c}{\sin A} \cos 2\theta.$$

For $\theta = \frac{1}{2}\pi$, $$\frac{d^2z}{d\theta^2} = -\tfrac{2}{3}h + \frac{2c}{\sin A} = \tfrac{2}{3}(-h + 3c \operatorname{cosec} A), \quad \ldots(3)$$

which is positive or negative according as
$$h < \text{ or } > 3c \operatorname{cosec} A.$$

Thus for $\theta = \frac{1}{2}\pi$, z is minimum or maximum according as
$$h < \text{ or } > 3c \operatorname{cosec} A.$$

Hence for $\theta = \frac{1}{2}\pi$, the equilibrium is stable or unstable according as
$$h < \text{ or } > 3c \operatorname{cosec} A.$$

For $\theta = \frac{1}{2}\pi$, $d^2z/d\theta^2 = 0$ when $h = 3c$ cosec A. In this case we can see that $d^3z/d\theta^3 = 0$ and $d^4z/d\theta^4 = -6c$ cosec A, which is negative. So in this case z is maximum and the equilibrium is unstable. Thus the symmetrical position of equilibrium is stable or unstable according as
$$h < \text{ or } \geqslant 3c \operatorname{cosec} A.$$

Now we consider the inclined positions of equilibrium. From (3), we can write
$$\frac{d^2z}{d\theta^2} = -\tfrac{2}{3}h \sin \theta - \frac{2c}{\sin A}(1 - 2\sin^2 \theta). \quad \ldots(4)$$

For the inclined positions of equilibrium, $\sin \theta = (h \sin A)/3c$. Putting $\sin \theta = (h \sin A)/3c$ in (4), we get
$$\frac{d^2z}{d\theta^2} = -\frac{2h}{3} \cdot \frac{h \sin A}{3c} - \frac{2c}{\sin A} + \frac{4c}{\sin A} \cdot \frac{h^2 \sin^2 A}{9c^2}$$
$$= \frac{2h^2}{9c} \sin A - \frac{2c}{\sin A} = \frac{2}{9c} \sin A\,(h^2 - 9c^2 \operatorname{cosec}^2 A),$$

which is negative since for inclined positions of equilibrium
$$h < 3c \operatorname{cosec} A.$$

Thus for the inclined positions of equilibrium, z is maximum and so they are positions of unstable equilibrium.

Remark. For inclined positions of equilibrium to exist, we must have $h < 3c$ cosec A. For these positions of equilibrium, θ is given by $\sin \theta = (h \sin A)/3c$.

Now $\frac{1}{2}A < \theta \Rightarrow \sin \frac{1}{2}A < \sin \theta \Rightarrow \sin \frac{1}{2}A < (h \sin A)/3c$

$$\Rightarrow \sin \tfrac{1}{2}A < \frac{2h \sin \frac{1}{2}A \cos \frac{1}{2}A}{3c} \Rightarrow h > \tfrac{3}{2}c \sec \tfrac{1}{2}A.$$

Thus for inclined positions of equilibrium, we must have
$$\tfrac{3}{2}c \sec \tfrac{1}{2}A < h < 3c \operatorname{cosec} A.$$

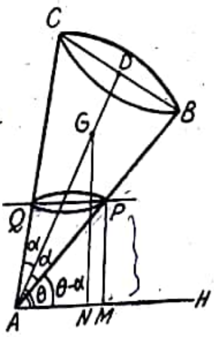
Ex. 28. (b) *An isosceles triangular lamina of an angle 2α and height h rests between two smooth pegs at the same level, distant $2c$ apart; prove that if*
$$3c \sec \alpha < h < 6c \operatorname{cosec} 2\alpha,$$
the oblique positions of equilibrium exist, which are unstable. Discuss the stability of the vertical position.

Sol. Proceed as in Ex. 28 (a). The complete question has been solved there.

Ex. 29 (a). *A smooth solid right circular cone, of height h and vertical angle 2α, is at rest with its axis vertical in a horizontal circular hole of radius a. Show that if $16a > 3h \sin 2\alpha$, the equilibrium is stable, and there are two other positions of unstable equilibrium; and that if $16a < 3h \sin 2\alpha$, the equilibrium is unstable, and the position in which the axis is vertical is the only position of equilibrium.*

Sol. ABC is a solid right circular cone whose height AD is h and vertical angle BAC is 2α. It rests in a horizontal circular hole PQ of radius a, so that $PQ = 2a$. We have
$$\angle BAD = \angle CAD = \alpha.$$

The weight of the cone acts at its centre of gravity G, where
$$AG = \tfrac{3}{4}AD = \tfrac{3}{4}h.$$



Suppose AD makes an angle θ with the horizontal AH, so that
$$\angle BAH = \theta - \alpha.$$

Let z be the height of G above the fixed horizontal line PQ. Then

$z = GN - PM$ via sine rule

$= AG \sin \theta - AP \sin (\theta - \alpha)$

$= \frac{3}{4}h \sin \theta - AP \sin (\theta - \alpha). \quad \ldots(1)$

Since PQ is parallel to AM, therefore
$$\angle QPA = \angle PAM = \theta - \alpha.$$

Now in the $\triangle APQ$, we have
$$\angle PQA = \pi - \{2\alpha + (\theta - \alpha)\} = \pi - (\theta + \alpha).$$

Applying the sine theorem of trigonometry for the ΔAPQ, we have

$$\frac{AP}{\sin\{\pi-(\theta+\alpha)\}}=\frac{PQ}{\sin 2\alpha}.$$

$\therefore \quad AP=\dfrac{2a}{\sin 2\alpha}\sin(\theta+\alpha)$, because $PQ=2a$.

Putting the value of AP in (1), we have

$$z=\tfrac{3}{4}h\sin\theta-\frac{2a\sin(\theta+\alpha)}{\sin 2\alpha}\sin(\theta-\alpha)$$

$$=\tfrac{3}{4}h\sin\theta-\frac{a}{\sin 2\alpha}[\cos 2\alpha-\cos 2\theta]$$

$$=\tfrac{3}{4}h\sin\theta-a\cot 2\alpha+\frac{a}{\sin 2\alpha}\cos 2\theta.$$

$$\therefore \quad \frac{dz}{d\theta}=\tfrac{3}{4}h\cos\theta-\frac{2a}{\sin 2\alpha}\sin 2\theta. \qquad \ldots(2)$$

For equilibrium, $dz/d\theta=0$

i.e., $\quad \tfrac{3}{4}h\cos\theta-\dfrac{4a}{\sin 2\alpha}\sin\theta\cos\theta=0$

i.e., $\quad \cos\theta\left[\tfrac{3}{4}h-\dfrac{4a\sin\theta}{\sin 2\alpha}\right]=0.$

$\therefore$ either $\cos\theta=0$ *i.e.*, $\theta=\tfrac{1}{2}\pi$,

or $\quad \tfrac{3}{4}h-\dfrac{4a\sin\theta}{\sin 2\alpha}=0$ *i.e.*, $\sin\theta=\dfrac{3h\sin 2\alpha}{16a}$.

Now $\theta=\frac{1}{2}\pi$ gives the position of equilibrium in which the axis AD of the cone is vertical. The values of θ given by

$$\sin\theta=(3h\sin 2\alpha)/16a$$

are real and not equal to $\frac{1}{2}\pi$ if $\sin\theta<1$ *i.e.*, if $16a>3h\sin 2\alpha$. Since $\sin(\pi-\theta)=\sin\theta$, therefore if $16a>3h\sin 2\alpha$, the equation

$$\sin\theta=(3h\sin 2\alpha)/16a$$

gives two oblique positions of equilibrium one θ and the other $\pi-\theta$. Thus if $16a>3h\sin 2\alpha$, there are three positions of equilibrium, one in which the axis AD is vertical and the other two inclined.

If $16a<3h\sin 2\alpha$, the equation

$$\sin\theta=(3h\sin 2\alpha)/16a$$

gives no real value of θ. Thus in this case the only position of equilibrium is that in which the axis of the cone is vertical.

Nature of equilibrium

From (2), $\quad \dfrac{d^2z}{d\theta^2}=-\tfrac{3}{4}h\sin\theta-\dfrac{4a}{\sin 2\alpha}\cos 2\theta. \qquad \ldots(3)$

For $\quad \theta=\frac{1}{2}\pi$,

$$\frac{d^2z}{d\theta^2}=-\tfrac{3}{4}h+\frac{4a}{\sin 2\alpha}=\frac{1}{4\sin 2\alpha}[-3h\sin 2\alpha+16a],$$

which is positive or negative according as

$$16a>\text{ or }<3h\sin 2\alpha.$$

Thus for $\theta=\frac{1}{2}\pi$, z is minimum or maximum according as

$$16a>\text{ or }<3h\sin 2\alpha.$$

Hence the vertical position of equilibrium is stable or unstable according as $16a>$ or $<3h\sin 2\alpha$.

Now we consider the inclined positions of equilibrium given by

$$\sin\theta=(3h\sin 2\alpha)/16a.$$

These exist only if $16a>3h\sin 2\alpha$. From (3), we can write

$$\frac{d^2z}{d\theta^2}=-\tfrac{3}{4}h\sin\theta-\frac{4a}{\sin 2\alpha}(1-2\sin^2\theta).$$

Putting $\sin\theta=(3h\sin 2\alpha)/16a$ in it, we get

$$\frac{d^2z}{d\theta^2}=-\tfrac{3}{4}h.\frac{3h\sin 2\alpha}{16a}-\frac{4a}{\sin 2\alpha}+\frac{8a}{\sin 2\alpha}.\frac{9h^2\sin^2 2\alpha}{256a^2}$$

$$=\frac{9h^2}{64a}\sin 2\alpha-\frac{4a}{\sin 2\alpha}=\frac{9h^2\sin^2 2\alpha-256a^2}{64a\sin 2\alpha}=\frac{(3h\sin 2\alpha)^2-(16a)^2}{64a\sin 2\alpha}$$

which is negative since for inclined positions of equilibrium

$$16a>3h\sin 2\alpha.$$

Thus for the inclined positions of equilibrium, z is maximum and so they are positions of unstable equilibrium.

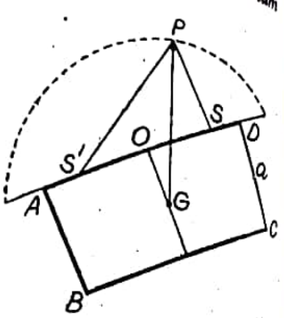
Ex. **29. (b)** *A smooth cone is placed with vertex downwards in a circular horizontal hole. Prove that the position of equilibrium with the axis vertical is unstable or stable according as it is, or, is not, the only possible position of equilibrium.*

Sol. Proceed as in Ex. 29 (a). Also take help from Ex. 28.

Ex. 30. (a) *A rectangular picture hangs in a vertical position by means of a string, of length l, which after passing over a smooth nail has its ends attached to two points symmetrically situated in the upper edge of the picture at a distance c apart. If the height of the picture is a, show that there is no position of equilibrium in which a side of the picture is inclined to the horizon if $la>c\sqrt{(c^2+a^2)}$, whilst if $\quad la<c\sqrt{(c^2+a^2)}$. there are two such positions which are both stable.*

Show also that in the latter case the position in which the side is vertical is stable for some and unstable for other displacements.

Sol. $ABCD$ is a rectangular picture which hangs by means of a string of length l passing over the peg P, the ends of the string being attached to two points S and S' symmetrically situated in the upper edge AD of the picture such that $SS'=c$. If O is the middle point of AD, then O is also the middle point of SS' because S and S' are symmetrically situated in AD. Therefore



$$OS=OS'=\tfrac{1}{2}c.$$

If G be the centre of gravity of the picture, then $OG=\frac{1}{2}a$, as height CD of the picture is given to be a.

We have $\quad SP+S'P=l. \quad$ ...(1)

From the relation (1), it is obvious that P lies on an ellipse whose foci are S and S' and the length say 2α, of whose major axis is l, so that $\alpha=\frac{1}{2}l$.

We have $OS=\alpha e$, where e is the eccentricity of the ellipse.

$$\therefore \quad \alpha e=\tfrac{1}{2}c.$$

If β be the semi major axis of the ellipse, then

$\beta^2=\alpha^2-\alpha^2e^2=\frac{1}{4}l^2-\frac{1}{4}c^2=\frac{1}{4}(l^2-c^2)$, so that $\beta=\frac{1}{2}\sqrt{(l^2-c^2)}$.

The centre of the ellipse is the middle point O of SS'. Take O as origin, OS as x-axis and a line perpendicular to OS through O as y-axis. Then the coordinates of G are $(0, -\frac{1}{2}a)$. Let the co-ordinates of P be $(\alpha \cos \theta, \beta \sin \theta)$.

Since the line PG is vertical, therefore if z be the depth of G below the fixed point P, then $z=PG$.

Now z is maximum or minimum according as z^2 or PG^2 is maximum or minimum.

Let, $\quad u=PG^2=(\alpha \cos \theta-0)^2+(\beta \sin \theta+\frac{1}{2}a)^2$

$$=\alpha^2 \cos^2 \theta+\beta^2 \sin^2 \theta+a\beta \sin \theta+\tfrac{1}{4}a^2.$$

$$\therefore \quad \frac{du}{d\theta}=2(\beta^2-\alpha^2) \sin \theta \cos \theta+a\beta \cos \theta.$$

For equilibrium,

$$\therefore \quad dz/d\theta=0 \text{ i.e., } du/d\theta=0,$$

i.e., $\quad \cos \theta\, [2(\beta^2-\alpha^2) \sin \theta+a\beta]=0.$

$\therefore$ either $\cos \theta=0$ i.e., $\theta=\frac{1}{2}\pi$,

or $\quad \sin \theta=\dfrac{a\beta}{2(\alpha^2-\beta^2)}=\dfrac{a.\frac{1}{2}\sqrt{(l^2-c^2)}}{2\,[\frac{1}{4}l^2-\frac{1}{4}(l^2-c^2)]}=\dfrac{a\sqrt{(l^2-c^2)}}{c^2}, \quad$ ...(2)

after substituting the values of α and β.

Here, $\theta=\frac{1}{2}\pi$ gives the position of equilibrium, symmetrical about the peg P, in which the sides AB and CD of the picture hang vertically.

There is no inclined position of equilibrium if the value of $\sin \theta$ given by (2) is > 1,

i.e., if $\quad a\sqrt{(l^2-c^2)} > c^2$, i.e., if $a^2l^2-a^2c^2 > c^4$

i.e., if $\quad a^2l^2 > c^2(c^2+a^2)$ i.e., if $al > c\sqrt{(a^2+c^2)}$.

Thus if $al > c\sqrt{(a^2+c^2)}$, there is no position of equilibrium in which a side of the picture is inclined to the horizon. In this case the symmetrical position $\theta=\frac{1}{2}\pi$ is the only position of equilibrium.

But if the value of $\sin \theta$ given by (2) is < 1,

i.e., $\quad a\sqrt{(l^2-c^2)} < c^2$, or $al < c\sqrt{(a^2+c^2)}$,

then (2) gives real values of θ. Since $\sin \theta=\sin(\pi-\theta)$, therefore when $al < c\sqrt{(a^2+c^2)}$, we have two inclined positions of equilibrium given by (2). In these positions the side CD may be inclined towards either side of the vertical. In this case there are in all three positions of equilibrium, one symmetrical, given by $\theta=\frac{1}{2}\pi$, and the other two, which are inclined, given by (2).

Nature of the positions of equilibrium.

We have,

$$d^2u/d\theta^2=2(\beta^2-\alpha^2)(\cos^2 \theta-\sin^2 \theta)-a\beta \sin \theta$$
$$=2(\beta^2-\alpha^2)(1-2 \sin^2 \theta)-a\beta \sin \theta. \quad \text{...(3)}$$

For the symmetrical position of equilibrium given by $\theta=\frac{1}{2}\pi$,

$$\frac{d^2u}{d\theta^2}=-2(\beta^2-\alpha^2)-a\beta$$
$$=-2\,[\tfrac{1}{4}(l^2-c^2)-\tfrac{1}{4}l^2]-a.\tfrac{1}{2}\sqrt{(l^2-c^2)}$$
$$=\tfrac{1}{2}c^2-\tfrac{1}{2}a\sqrt{(l^2-c^2)}=\tfrac{1}{2}[c^2-a\sqrt{(l^2-c^2)}],$$

which is positive or negative according as $a\sqrt{(l^2-c^2)} <$ or $> c^2$

i.e., according as $\quad al <$ or $> c\sqrt{(a^2+c^2)}$.

Thus if $al < c\sqrt{(c^2+a^2)}$, then u and so also z is minimum. Since z is the depth of G below the fixed point P, therefore the equilibrium is unstable in this case. Again if $al > c\sqrt{(c^2+a^2)}$, then u and so also z is maximum, and the equilibrium is stable. Hence the symmetrical equilibrium position $\theta=\frac{1}{2}\pi$ is unstable if

$$al < c\sqrt{(c^2+a^2)}$$

and stable if $al > c\sqrt{(c^2+a^2)}$.

Now consider the inclined positions of equilibrium given by

$$\sin\theta=\{a\sqrt{(l^2-c^2)}\}/c^2,$$

which give real values of θ only if

$$a\sqrt{(l^2-c^2)} < c^2, \text{ or } al < c\sqrt{(c^2+a^2)}.$$

In this case putting $\sin\theta=\{a\sqrt{(l^2-c^2)}\}/c^2$ in (3), we get

$$\frac{d^2u}{d\theta^2}=2\left[\tfrac{1}{4}(l^2-c^2)-\tfrac{1}{4}l^2\right]\left[1-2.\frac{a^2(l^2-c^2)}{c^4}\right]$$
$$-a.\tfrac{1}{2}\sqrt{(l^2-c^2)}.\frac{a}{c^2}\sqrt{(l^2-c^2)}$$
$$=-\frac{c^2}{2}+\frac{a^2(l^2-c^2)}{c^2}-\frac{a^2(l^2-c^2)}{2c^2}$$
$$=-\frac{c^2}{2}+\frac{a^2(l^2-c^2)}{2c^2}=\frac{1}{2c^2}[a^2(l^2-c^2)-c^4],$$

which is negative because $a\sqrt{(l^2-c^2)} < c^2$.

Thus in this case u and so also z is maximum and the equilibrium is stable. Hence if $al < c\sqrt{(c^2+a^2)}$, there are two inclined positions of equilibrium and they are both stable.

Ex. **30 (b)**. *A rectangular picture-frame hangs from a small perfectly smooth pulley by a string of length* 2a *attached symmetrically to two points on the upper edge at a distance* 2c *apart. Prove that if the depth of the picture is less than*

$$2c^2/\sqrt{(a^2-c^2)},$$

there are three positions of equilibrium of which the symmetrical one is unstable. If the depth exceeds the above value the symmetrical position of equilibrium is the only one and is stable.

Sol. Proceed as in Ex. 30 (a).

Centre of Gravity

§ **1. Centre of Gravity.** On account of the attraction of the earth (*known as gravity*) every particle on the surface of the earth is attracted towards the centre of the earth by a force proportional to its mass, called the **weight of the particle.**

A rigid body is considered as a collection of particles, rigidly connected with one another, on which are acting the weights of the particles. Such weights are considered to be parallel forces. The resultant of these forces is called the weight of the body and it always passes through a fixed point. This point is called the *centre of gravity* of the body.

Definition. *The* **centre of gravity** *of a body is the point, fixed relative to the body, through which the line of action of the weight of the body, always passes, whatever be the position of the body, provided that its size and shape remain unaltered.*

Centre of gravity is usually written in brief as C.G.

Note 1. The centre of gravity of a body does not necessarily lie in the body itself.

Note 2. The centre of mass (C.M.) of a body practically coincides with its centre of gravity (C.G.). Sometimes the words 'centroid' or 'centre of inertia' are used in place of the centre of gravity.

§ **2. Determination of the C.G. by integration.** If a number of particles of masses $m_1, m_2, m_3, \ldots$ be placed at the points (x_1, y_1), (x_2, y_2), $(x_3, y_3), \ldots$ referred to two rectangular axes, then the coordinates $(\bar{x}, \bar{y})$ of the centre of gravity (C.G.) of the body consisting of those particles are given by

$$\bar{\mathbf{x}}=\frac{\Sigma \mathbf{m}_1\mathbf{x}_1}{\Sigma \mathbf{m}_1} \text{ and } \bar{\mathbf{y}}=\frac{\Sigma \mathbf{m}_1\mathbf{y}_1}{\Sigma \mathbf{m}_1}.$$

These results are a simple consequence of a theorem on the moments of a system of parallel forces and their resultant.

In the case of continuous distribution of matter, the summations can be replaced by definite integrals. Then the C.G. of the body, is given by