~~1(b)~~ 3(a)
~~3(c)~~, 7(d),

Date : 01.09.2019

A CONSOLIDATED QUESTION PAPER-CUM-ANSWER BOOKLET

IMS™
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MAINS TEST SERIES-2019

(JUNE-2019 *to* SEPT.-2019)

Under the guidance *of* K. Venkanna

MATHEMATICS

194/250

PAPER - II : FULL SYLLABUS

TEST CODE: TEST-16: IAS(M)/01-SEPT.-2019

Time : 3 Hours | **Maximum Marks : 250**

INSTRUCTIONS

1. This question paper-cum-answer booklet has 48 pages and has 34 PART/SUBPART questions. Please ensure that the copy of the question paper-cum-answer booklet you have received contains all the questions.
2. Write your Name, Roll Number, Name of the Test Centre and Medium in the appropriate space provided on the right side.
3. A consolidated Question Paper-cum-Answer Booklet, having space below each part/sub part of a question shall be provided to them for writing the answers. Candidates shall be required to attempt answer to the part/sub-part of a question strictly within the pre-defined space. Any attempt outside the pre-defined space shall not be evaluated."
4. Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
5. Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any THREE of the remaining questions selecting at least ONE question from each Section.
6. The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
7. Symbols/notations carry their usual meanings, unless otherwise indicated.
8. All questions carry equal marks.
9. All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
10. All rough work should be done in the space provided and scored out finally.
11. The candidate should respect the instructions given by the invigilator.
12. The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name: UTKARSH KUMAR

Roll No.: 0858342

Test Centre:

Medium: ENGLISH

Do not write your Roll Number or Name anywhere else in this Question Paper-cum-Answer Booklet.

I have read all the instructions and shall abide by them

Utkarsh

Signature of the Candidate

I have verified the information filled by the candidate above

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IMPORTANT NOTE:
Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed. Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

P.T.O.

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Total Marks					

194/250

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SECTION – A

1. (a) Prove that a non-empty subset H of an group G is normal subgroup of $G \Leftrightarrow$ *for* all $x, y \in H$, $g \in G$, $(gx)(gy)^{-1} \in H$. **[10]**

① Let H be a normal subgroup.

$\forall\ x, y \in H,$

$xy^{-1} \in H \quad (y \in H \Rightarrow y^{-1} \in H \Rightarrow xy^{-1} \in H)$

$\Rightarrow \forall\ g \in G,\ g(xy^{-1})g^{-1} \in H \quad (H \triangleleft G)$

$\Rightarrow\ gx(gy)^{-1} \in H.$

② Let $\forall\ x, y \in H,\ g \in G,\ gx(gy)^{-1} \in H.$

Consider any $h \in H.$

Note: $x = e, y = e \Rightarrow e \in H.$

Now, take $x = h, y = e.$

$\Rightarrow \forall\ g \in G,\ ghg^{-1} \in H$

$\Rightarrow H \triangleleft G.$

07

So, a non empty subset H is normal subgroup of G iff $\forall\ x, y \in H, g \in G,$ $gx(gy)^{-1} \in H.$

1. (b) Let G be the group of non- zero complex numbers under multiplication, and let N be the set of complex numbers of absolute value 1. Show that G/N is isomorphic to the group of all positive real numbers under multiplication. **[10]**

To show: $\frac{G}{N} \simeq \mathbb{R}^+$

We define homomorphism $\phi : G \to \mathbb{R}^+$

as $\phi(a+ib) = \sqrt{a^2+b^2}$.

① $a+ib = c+id$

$\Rightarrow \sqrt{a^2+b^2} = \sqrt{c^2+d^2}$

$\Rightarrow \phi(a+ib) = \phi(c+id)$

$\boxed{\text{well defined}}$ ~~and one-one~~

② $\forall\ a \in \mathbb{R}^+$, $\exists\ a+0\cdot i \in G$

such that $\phi(a+0\cdot i) = a$ $\boxed{\text{onto}}$

③ $\phi((a+ib)(c+id)) = \phi(ac-bd+i(ad+bc))$

$= \sqrt{(ac-bd)^2+(ad+bc)^2}$

$= \sqrt{a^2c^2+b^2d^2+a^2d^2+b^2c^2} = \sqrt{(a^2+b^2)}\sqrt{(c^2+d^2)}$

$= \phi(a+ib)\cdot\phi(c+id)$. homomorphism

08

Show one one

④ $\ker\phi = \{x \mid \phi(x) = 1\} = N$.

$\therefore \phi$ is a homomorphism isomorphism

By first principle of homomorphism,

$\frac{G}{N} \simeq \mathbb{R}^+$.

1\. (c) Define a compact set. prove that the range of a continuous function defined on a compact set is compact. **[10]**

1. (d) Use Cauchy's theorem and/or Cauchy integral formula to evaluate the following integrals.

(i) $\int_{|z|=1} \frac{z+3}{z^4+az^3}\,dz; (|a|>|)$ (ii) $\int_{|z|=4} \frac{z^4}{(z-i)^3}\,dz$ **[10]**

(i) $I = \int_{|z|=1} \frac{z+3}{z^3(z+a)}\,dz.$

I has a pole of 3rd degree at 0.

$\therefore I = \frac{2\pi i}{2!} f^{(2)}(0)$, where $f(z) = \frac{z+3}{z+a}$.

$\Rightarrow f'(z) = \frac{a-3}{(z+a)^2} \Rightarrow f^{(2)}(z) = \frac{-2(a-3)}{(z+a)^3}.$

$I = \frac{2\pi i}{2!} \cdot \frac{-2(a-3)}{a^3} = -\frac{2\pi i(a-3)}{a^3}.$

(ii) $I = \int_{|z|=4} \frac{z^4}{(z-i)^3}\,dz.$

I has a pole of 3rd degree at $z=i$.

$\therefore I = \frac{2\pi i}{2!} f^{(2)}(i)$, where $f(z) = z^4$

$f'(z) = 4z^3, \quad f^{(2)}(z) = 12z^2.$

$\Rightarrow f^{(2)}(i) = 12\cdot(i)^2 = -12.$

08

$\therefore I = \frac{2\pi i}{2!}\cdot(-12) = \underline{\underline{-12\pi i}}$

1. (e) Find all the basic feasible solutions of the following problem:

$2x_1 + 3x_2 + x_3 + x_4 = 8$

$x_1 - 2x_2 + 6x_3 - 7x_4 = -3$

and choose the one which

Maximize $z = 2x_1 + 3x_2 + 4x_3 + 7x_4$ **[10]**

There are 2 equations in 4 independent variables

$\Rightarrow$ We make two of them vanish and find other two.

① $x_1 = x_2 = 0$

$\Rightarrow x_3 + x_4 = 8,\ 6x_3 - 7x_4 = -3$

$\Rightarrow x_3 = \frac{53}{13},\ x_4 = \frac{51}{13}.\quad z = 43.769$

② $x_1 = x_3 = 0 \Rightarrow 2x_2 + x_4 = 8,\ -2x_2 - 7x_4 = -3$

$\Rightarrow x_2 = \frac{53}{19},\ x_4 = \frac{-7}{19},\ z = 5.789$

③ $x_1 = x_4 = 0 \Rightarrow 3x_2 + x_3 = 8,\ -2x_2 + 6x_3 = -3$

$\Rightarrow x_2 = \frac{51}{20},\ x_3 = \frac{7}{20},\ z = 9.05$

④ $x_2 = x_3 = 0 \Rightarrow 2x_1 + x_4 = 8,\ x_1 - 7x_4 = -3$

$\Rightarrow x_1 = \frac{53}{15},\ x_4 = \frac{14}{15},\ z = 13.60$

⑤ $x_2 = x_4 = 0 \Rightarrow 2x_1 + x_3 = 8,\ x_1 + 6x_3 = -3$

$\Rightarrow x_1 = \frac{51}{11},\ x_3 = \frac{-14}{11},\ z = -0.4545$

⑥ $x_3 = x_4 = 0 \Rightarrow 2x_1 + 3x_2 = 8,\ x_1 - 2x_2 = -3$

$\Rightarrow x_1 = 1,\ x_2 = 2,\ z = 8.$

08

$\therefore$ These are the 6 basic feasible solutions, of these $(0, 0, \frac{53}{13}, \frac{51}{13})$ maximises z.

2. (a) (i) Show by counter example that the distributive laws in the definition of a ring is not redundant.

(ii) In the ring of integers modulo 10 (*i.e* z_{10} , $\oplus_{10}$, $\odot_{10}$), find the subfields. **[16]**

2. (b) (i) If $u = x^2 \tan^{-1}\left(\frac{y}{x}\right) - y^2 \tan^{-1}\left(\frac{x}{y}\right)$

Show that $x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} = 2u$.

(ii) Let f be defined on [0, 1] as

$$f(x) = \begin{cases} \sqrt{1-x^2} & \text{, if } x \text{ is rational} \\ 1-x & \text{, if } x \text{ is irrational} \end{cases}$$

Find the upper and lower Riemann integrals of f over [0, 1]. **[16]**

3. (a) Let A be an ideal of a commutative ring R and B = {x ∈ R : x^n ∈ A for some positive integer n}. Is B an ideal of R ? Justify your answer. **[15]**

Yes, B is an ideal of R.

① We show that B is a subring.

$\forall\, a, b \in B, \ \exists\, n_0, m_0 : \ a^{n_0} \in A, \ b^{m_0} \in A.$

$\Rightarrow \forall\, n \geq n_0, \ m \geq m_0 : \ a^n, b^m \in A$ — ①

$(a^{n_0} \in A \Rightarrow a^{n_0} \cdot a^{n-n_0} \in A$ because it is an ideal$)$

$\Rightarrow (a-b)^{n_0+m_0} = \sum_{i=0}^{n_0+m_0} \binom{n_0+m_0}{n_0} a^i \, b^{n_0+m_0-i}$ — ②

For $i = 0, 1, \ldots n_0$, $b^{n_0+m_0-i} \in A$, (using ①)

$i = n_0, n_0+1, \ldots, n_0+m_0$, $a^i \in A$ (using ①)

$\Rightarrow$ each term in ② is in A

$\Rightarrow (a-b)^{n_0+m_0} \in A \Rightarrow a-b \in B.$

Also, $(ab)^{n_0} = a^{n_0} b^{n_0}$ ($\because$ commutative ring)

$a^{n_0} \in A \Rightarrow a^{n_0} b^{n_0} \in A$ (A is ideal)

$\Rightarrow (ab)^{n_0} \in A \Rightarrow ab \in B.$

$\therefore$ B is a subring.

(2) Now, $\forall\ a \in B,\ b \in R,$

$\exists\ n_0 \in \mathbb{N} :\ a^{n_0} \in A.$

$(ab)^{n_0} = a^{n_0} \cdot b^{n_0} \in A$ ($a^{n_0} \in A$, A is ideal)

$\Rightarrow ab \in B.$

$\therefore$ B is an ideal of R.

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3. (b) (A) Show that the series $\sum(-1)^n\left[\sqrt{n^2+1}-n\right]$ is conditionally convergent.

(B) (i) Check whether or not the following function is Riemann integrable on [0, 1]:

$$f(x)=\begin{cases}\dfrac{\sin x}{x} & \text{if } x\neq 0\\ 1 & \text{if } x=0\end{cases}$$

(ii) Let $f:[-1,1]\to[0,1]$ bedefined by $f(x)=|x|$. Check whether it is Riemann integrable. **[18]**

(A) $u_n = (-1)^n\left(\sqrt{n^2+1}-n\right) = \dfrac{(-1)^n}{\sqrt{n^2+1}+n}$

As $n\to\infty$, $\dfrac{1}{\sqrt{n^2+1}+n}\to 0$.

Also, $\dfrac{1}{\sqrt{n^2+1}+n} > \dfrac{1}{\sqrt{(n+1)^2+1}+n+1}$

$\therefore \left\langle \dfrac{1}{\sqrt{n^2+1}+n}\right\rangle$ is a decreasing sequence with terms tending to 0 as $n\to\infty$.

$\Rightarrow \sum u_n$ is convergent (by Leibnitz's test). — (1)

We now consider $\sum v_n$, $v_n = |u_n| = \dfrac{1}{\sqrt{n^2+1}+n}$

$\lim\limits_{n\to\infty}\dfrac{v_n}{1/n} = \lim\limits_{n\to\infty}\dfrac{n}{\sqrt{n^2+1}+n} = \dfrac{1}{2}$

Limit is non zero finite $\Rightarrow \sum v_n$ behaves same as $\sum\frac{1}{n}$.

But $\sum\frac{1}{n}$ diverges $\therefore \sum|u_n|$ diverges. (2)

By (1) and (2), the series is conditionally convergent.

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(B) The function $\frac{\sin x}{x}$ is continuous at all points in (0, 1].

Also, $\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin x}{x} = 1 = f(0)$.

$\Rightarrow$ f is continuous at 0.

$\Rightarrow$ f is continuous at all points in domain

$\Rightarrow$ f is Riemann integrable over $[0,1]$

(ii) $f(x) = |x| = \begin{cases} x, & x \in (0,1] \\ 0, & x = 0 \\ -x, & x \in [-1,0) \end{cases}$

Clearly $f(x)$ is continuous in $(0,1]$ and $[-1,0)$.

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} x = 0 = f(0)$

$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} (-x) = 0 = f(0)$

$\therefore$ $f(x)$ is continuous at 0.

$\Rightarrow$ $f(x)$ is continuous at all points in its domain

$\Rightarrow$ $f(x)$ is Raimann integrable over $[-1,1]$

In fact $\int_{-1}^{1} f(x)dx = \int_{-1}^{0} (-x)dx + \int_{0}^{1} f(x)dx$

$= 1 + 1 = \underline{2}$.

3. (c) Obtain an optimal basic feasible solution to the following transportation problem.

	D_1	D_2	D_3	D_4	a_i
O_1	19	30	50	10	7
O_2	70	30	40	60	9
O_3	40	8	70	20	18
b_j	5	8	7	14	

[17]

We use Vogel's approximation method to find ~~the~~ a basic feasible solution.

19 (5)	30	50	10 (2)	~~7~~ 2 (9)(40)
70	30	40 (7)	60 (2)	9 (10)(20)
40	8 (8)	70	20 (10)	~~18~~ 10 (12)(20)(50)
~~5~~ (21)	8 (22)	7 (10)(30)	~~14~~ ~~4~~ 2 (10)(50)	

We check optimality by computing shadow costs.

19 (5)	30 [-32]	50 [-10]	10 (2)	$u_1 = 0$
70 [-1]	30 [18]	40 (7)	60 (2)	$u_2 = 50$
40 [-11]	8 (8)	70 [-70]	20 (10)	$u_3 = 10$

$v_1 = 19 \quad v_2 = -2 \quad v_3 = -10 \quad v_4 = 10$

One shadow cost is positive $\Rightarrow$ optimality not achieved.

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30 (θ)   60 ($2-\theta$)

8 ($8-\theta$)   20 ($10+\theta$)

$\xrightarrow{\theta = 2}$

30 (2)   60

8 (6)   20 (12)

We compute shadow costs again.

19 (5)	30 [-32]	50 [-42]	10 (2)	$u_1 = 0$
70 [-19]	30 (2)	40 (7)	60 [-18]	$u_2 = 32$
40 [-11]	8 (6)	70 [-52]	20 (12)	$u_3 = 10$

$v_1 = 19 \quad v_2 = -2 \quad v_3 = 8 \quad v_4 = 10$

All shadow costs are negative.

$\Rightarrow$ Optimality has been achieved.

Optimal allocation:

O_1 : 5 to D_1, 2 to D_4

O_2 : 2 to D_2, 7 to D_3

O_3 : 6 to D_2, 12 to D_4.

Optimal cost $= 5 \times 19 + 2 \times 10 + 2 \times 30$
$+ 7 \times 40 + 6 \times 8 + 12 \times 20$

$= \underline{743}$

9½

SECTION – B

5. (a) Solve $(x^2 - yz)\,p + (y^2 - zx)\,q = z^2 - xy$. **[10]**

Compare to $Pp + Qq = R$,

$P = x^2 - yz$, $Q = y^2 - zx$, $R = z^2 - xy$

Lagrange subsidiary equations:

$$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy}$$

① $$\frac{y\,dx + z\,dy}{x(xy - z^2)} = \frac{dz}{z^2 - xy}$$

$$\Rightarrow\ y\,dx + z\,dy + x\,dz = 0$$

② $$\frac{z\,dx + x\,dy}{y(xy - z^2)} = \frac{dz}{z^2 - xy}$$

$$\Rightarrow\ z\,dx + x\,dy + y\,dz = 0$$

Adding ①, ②: $(y\,dx + x\,dy) + (z\,dy + y\,dz) + (z\,dx + x\,dz) = 0$

$$\Rightarrow\ xy + yz + zx = c_1.$$

eqⁿ ① × x − eqⁿ ② × y:

$$(x^2 - y^2)\,dz + z(x\,dy - y\,dx) = 0$$

$$\Rightarrow\ \frac{dz}{z} + \frac{x\,dy - y\,dx}{x^2} \cdot \frac{1}{1 - \frac{y^2}{x^2}} = 0$$

$$\Rightarrow\ \log z + \frac{1}{2}\log\left(\frac{1 + y/x}{1 - y/x}\right) = \log c_2$$

$$\Rightarrow\ z \cdot \sqrt{\frac{x+y}{x-y}} = c_2.$$

$$\therefore\ \phi\left(xy + yz + zx,\ z\sqrt{\frac{x+y}{x-y}}\right) = 0.$$

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5. (b) Solve $\left(D^2-6DD'+9D'^2\right)z=\tan(y+3x)$ **[10]**

We consider the homogeneous equation

$$(D^2 - 6DD' + 9D'^2)z = 0$$

ie $$(D - 3D')^2 z = 0$$

with solutions $z_{cf} = \phi_1(y+3x) + x\phi_2(y+3x)$

Particular solution.

$$z_{pi} = \frac{1}{(D-3D')^2} \cdot \tan(y+3x).$$

$$= x \cdot \frac{1}{2(D-3D')} \cdot \tan(y+3x)$$ [$\because D-3D'$ vanishes on putting $D\to 3$, $D'\to 1$]

$$= x^2 \cdot \frac{1}{2} \tan(y+3x)$$

$\therefore$ Complete solution is given by

$$z = z_{cf} + z_{pi}$$

$$= \phi_1(y+3x) + x\phi_2(y+3x) + \frac{x^2}{2}\tan(y+3x)$$

08

5. (c) The current i in an electric circuit is given by i = $10e^{-t}\sin 2\pi t$ where t is in seconds. Using Newton's method, find the value of t correct to 3 decimal places for i = 2 amp. **[10]**

$i = 2 \Rightarrow f(t) = 10e^{-t}\sin 2\pi t - 2$.

By Newton's method,

$$t_{k+1} = t_k - \frac{f(t_k)}{f'(t_k)}$$

$$= t_k - \frac{(10e^{-t}\sin 2\pi t - 2)}{10e^{-t}(2\pi\cos 2\pi t - \sin 2\pi t)}$$

$k=0$: $t_0 = 0 \Rightarrow t_1 = 0 - \frac{(10\times1\times0-2)}{10\times1(2\pi\times1-0)}$

$= 0.03183$

$k=1$: $t_1 = 0.03183 \Rightarrow t_2 = t_1 - \frac{(10e^{-t_1}\sin 2\pi t_1 - 2)}{10e^{-t_1}(2\pi\cos 2\pi t_1 - \sin 2\pi t_1)}$

$= 0.03314$

08

$t_2 = 0.3314 \Rightarrow t_3 = 0.03314$

$\therefore$ t = 0.033 sec is the time at which i = 2 amp, correct to 3 decimal places.

5. (d) Draw a switching circuit that realizes the following switching function. If possible, draw a simpler switching circuit. **[10]**

x	y	z	f(x,y,z)
1	1	1	0
1	1	0	1
1	0	1	1
1	0	0	1
0	1	1	0
0	1	0	1
0	0	1	0
0	0	0	1

$$f(x,y,z) = xy\bar{z} + x\bar{y}z + x\bar{y}\bar{z} + \bar{x}y\bar{z} + \bar{x}\bar{y}\bar{z}$$

$$= xy\bar{z} + x\bar{y}(z+\bar{z}) + \bar{x}\bar{z}(y+\bar{y})$$

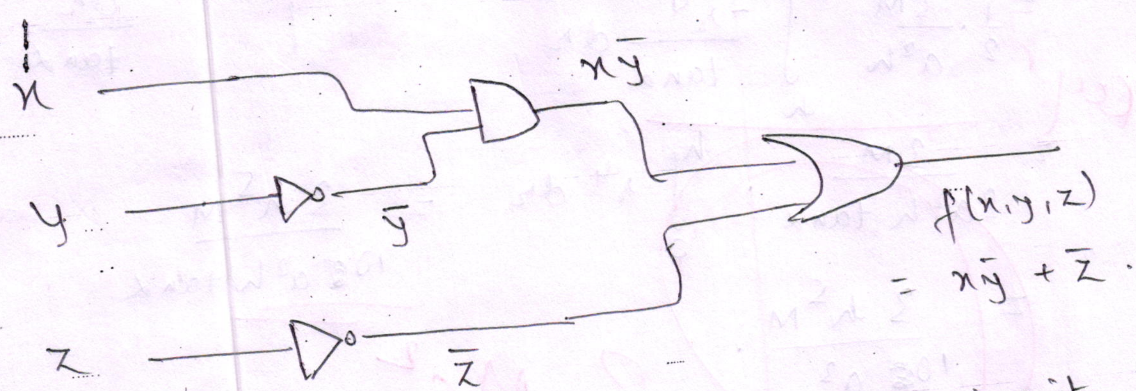
$$= xy\bar{z} + x\bar{y} + \bar{x}\bar{z}$$

$$= x(y\bar{z} + \bar{y}) + \bar{x}\cdot\bar{z}$$

$$= x(\bar{z}+\bar{y}) + \bar{x}\bar{z} = x\bar{y} + \bar{z}(x+\bar{x})$$

$$= x\bar{y} + \bar{z}.$$

The corresponding switching circuit is

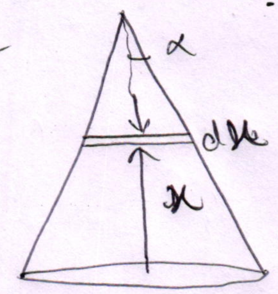


This is the simplest switching circuit in terms of number of gates.

08

5. (e) Find the M.I. of a right solid cone of mass M, height h and radius of whose base is a, about its axis. **[10]**

Let the right solid cylinder be as shown in figure.



$\tan\alpha = \dfrac{a}{h}$.

We consider a disc at height x from base.

Mass of disk $= M.\dfrac{\pi r^2\,dx}{\frac{1}{3}\pi a^2 h}$, radius, $r = (h-x)\tan\alpha$

$\therefore$ Moment of inertia, $dI = \dfrac{M}{2}.\dfrac{\pi r^4\,dx}{\frac{1}{3}\pi a^2 h}$

$\Rightarrow$ Moment of inertia

$$= \frac{1}{2}\int_h^0 M\frac{\pi r^4\,dx}{\frac{1}{3}\pi a^2 h}, \quad x = h - \frac{r}{\tan\alpha} \Rightarrow dx = \frac{-dr}{\tan\alpha}$$

$$= \frac{1}{2}.\frac{3M}{a^2h}\int_h^0 \frac{-r^4}{\tan\alpha}\,dr$$

$$= \frac{3M}{2a^2h\tan\alpha}\int_0^h r^4\,dr = \frac{3h^5M}{10a^2h\tan\alpha}$$

$$= \frac{3h^5M}{10a^3}.$$

06

Refer key

$\frac{3}{10}Ma^2$

6. (a) Form a partial differential equation by eliminating the arbitrary function ϕ from $\phi(x^2+y^2+z^2, z^2-2xy)=0$. **[07]**

$$\phi(x^2+y^2+z^2,\ z^2-2xy) = 0$$

① $\frac{\partial}{\partial x}\left(\phi(x^2+y^2+z^2,\ z^2-2xy)\right) = 0$

$\Rightarrow \frac{\partial \phi}{\partial u}(u,v)\left[\frac{\partial u}{\partial x}\right] + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial x} = 0, \quad v = z^2 - 2xy, \quad u = x^2+y^2+z^2$

~~$\Rightarrow 2x + 2zp + 2zp - 2y = 0$~~

$\Rightarrow \phi_u \cdot (2x + 2zp) + \phi_v(2zp - 2y) = 0.$

② $\frac{\partial}{\partial y}(\phi(u,v)) = 0$

$\Rightarrow \phi_u \cdot \frac{\partial u}{\partial y} + \phi_v \cdot \frac{\partial v}{\partial y} = 0$

$\Rightarrow \phi_u \cdot (2y + 2zq) + \phi_v(2zq - 2x) = 0.$

ϕ is arbitrary $\Rightarrow$ ①, ② must satisfy $\forall\ \phi_u, \phi_v$.

$\therefore \frac{2x + 2zp}{2y + 2zq} = \frac{2zp - 2y}{2zq - 2x}$

$\Rightarrow \frac{x + pz}{y + qz} = \frac{y - pz}{x - qz}$

$\Rightarrow x^2 - y^2 + (p-q)(x+y)z = 0$

$\Rightarrow x - y + (p-q)z = 0$

ie $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right) \cdot \frac{1}{x-y} = -z.$

06

6. (b) Find a surface satisfying the equation $D^2z = 6x + 2$ and touching $z = x^3 + y^3$ along its section by the plane $x + y + 1 = 0$. **[12]**

$$D^2 z = 6x + 2.$$

Homogeneous equation $D^2 z = 0$ has solution

$$z_{cf} = x\phi_1(y) + \phi_2(y).$$

Particular solution is

$$z_{pi} = \frac{1}{D^2}(6x+2) = \frac{1}{D}\cdot(3x^2+2x)$$

$$= x^3 + x^2.$$

$\therefore$ $z = x\phi_1(y) + \phi_2(y) + x^3 + x^2$

is the complete solution to given equation.

This curve touches $z = x^3 + y^3$ along section by $x + y + 1 = 0$.

$\Rightarrow$ gradients must be same at these points.

$\Rightarrow$ $(\phi_1(y) + 3x^2 + 2x,\ x\phi_1'(y) + \phi_2'(y),\ -1)$

$= (3x^2,\ 3y^2,\ -1)$ along section by $x+y+1=0$

$\Rightarrow$ $\phi_1(y) + 3x^2 + 2x = 3x^2$

$\Rightarrow$ $\phi_1(-1-x) = -2x$

$\Rightarrow$ $\phi_1(x) = -2(-1-x) = 2+2x.$

Also, $x\phi_1'(y) + \phi_2'(y) = 3y^2$

$\Rightarrow$ $2x + \phi_2'(y) = 3y^2$

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$\Rightarrow \phi_2'(y) = 3y^2 - 2x = 3y^2 + 2y + 2$

$\Rightarrow \phi_2(y) = y^3 + y^2 + 2y + c.$

$\therefore z = x(2+2y) + y^3 + y^2 + 2y + c + x^3 + x^2.$

Also, the curve's section by plane contains $(0, -1, -1)$.

Our surface must pass through this.

$\Rightarrow -1 = -1 + 1 - 2 + c \Rightarrow c = 1.$

$\therefore$ Required surface is

$$z = x^3 + y^3 + x^2 + y^2 + 2x + 2y + 2xy + 1.$$
$$= x^3 + x^2 + 2x + 1 + y^3 + y^2 + 2y + 2xy.$$

6. (c) Reduce $x^2\left(\partial^2 z/\partial x^2\right) - y^2\left(\partial^2 z/\partial y^2\right) = 0$ to canonical form and hence solve it. **[13]**

$$x^2 \frac{\partial^2 z}{\partial x^2} - y^2 \frac{\partial^2 z}{\partial y^2} = 0.$$

Auxiliary equation $x^2 m^2 - y^2 = 0$ has solution $m = \pm \frac{y}{x}$.

$$\frac{dy}{dx} \pm \frac{y}{x} = 0 \Rightarrow xy = c_1, \frac{x}{y} = c_2.$$

$\therefore$ Choose $u = xy$, $v = \frac{x}{y}$.

$$\frac{\partial z}{\partial x} = z_u \cdot y + z_v \cdot \frac{1}{y}$$

$$\frac{\partial^2 z}{\partial x^2} = z_{uu} y^2 + 2 z_{uv} + z_{vv} \cdot \frac{1}{y^2}.$$

$$\frac{\partial z}{\partial y} = z_u \cdot x + z_v \cdot \left(\frac{-x}{y^2}\right)$$

$$\Rightarrow \frac{\partial^2 z}{\partial y^2} = z_{uu}\cdot x^2 + 2z_{uv}\left(-\frac{x^2}{y^2}\right) + z_{vv}\cdot\frac{x^2}{y^4} + \frac{2x z_v}{y^3}$$

Putting in given equation,

$$x^2\left[y^2 z_{uu} + 2z_{uv} + \frac{1}{y^2}z_{vv}\right] - y^2\left[x^2 z_{uu} - \frac{2x^2}{y^2}z_{uv} + \frac{x^2}{y^4}z_{vv} + \frac{2x}{y^3}z_v\right] = 0$$

$$\Rightarrow 4x^2 z_{uv} - \frac{2x}{y}z_v = 0$$

$$\Rightarrow \boxed{2u\, z_{uv} = z_v}$$

Let $T = z_v$. $\frac{\partial T/\partial u}{T} = \frac{1}{2u} \Rightarrow T = \phi(v)\cdot\sqrt{u}$

$T = z_v = \sqrt{u}\cdot\phi(v) \Rightarrow \boxed{z = \sqrt{u}\cdot\psi_1(v) + \psi_2(u)}$

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6. (d) Obtain temperature distribution y(x, t) in a uniform bar of unit length whose one end is kept at 10°C and the other end is insulated. Further it is given that y(x, 0) = 1 – x, 0 < x < 1. **[18]**

We proceed by method of separation of variables. $y(x, t) = X(x)\,T(t) + f(x)$

where $f(x)$ is steady state distribution

Differential equation is $c^2 y_{xx} = y_t$

For steady state, $y_t = 0$

$\Rightarrow y_{xx} = 0 \Rightarrow f(x) = ax + b$.

$x = 1$ is insulated $\Rightarrow \frac{\partial f}{\partial x} = 0 \Rightarrow a = 0$

$f(x) = 10$ at $x = 0$

$\therefore f(x) = 10, \quad 0 \le x \le 1$

$\therefore y(x,t) = X(x)\, T(t) + 10.$

$\Rightarrow \quad c^2 X'' T = X T'$

$\Rightarrow \quad \frac{X''}{X} = \frac{1}{c^2}\frac{T'}{T} = \lambda$ (constant)

Case 1: $\lambda > 0 \Rightarrow X'' - \lambda X = 0$

$X = c_1 e^{\sqrt{\lambda}x} + c_2 e^{-\sqrt{\lambda}x}.$

$X(0) = 0 \Rightarrow c_1 + c_2 = 0$

$X'(1) = 0 \Rightarrow c_1 e^{\sqrt{\lambda}} - c_2 e^{-\sqrt{\lambda}} = 0$ $\Big]$ $c_1 = c_2 = 0$. Trivial solution rejected

Case 2: $\lambda = 0 \Rightarrow X'' = 0$

$X = a + bx$

$X'(1) = 0 \Rightarrow b = 0, \quad X(0) = 0 \Rightarrow a = 0.$ Trivial solution rejected

Case 3: $\lambda < 0 \Rightarrow X'' - \lambda X = 0$

$X = c_1 \cos\sqrt{-\lambda}\,x + c_2 \sin\sqrt{-\lambda}\,x.$

$X(0) = 0 \Rightarrow c_1 = 0$

$X'(1) = 0 \Rightarrow \sqrt{-\lambda} = (2n+1)\frac{\pi}{2}.$

Also, $T' - \lambda c^2 T = 0 \Rightarrow T = d \cdot e^{+\lambda c^2 t}$

$\therefore y_n(x,t) = c_n e^{\lambda_n c^2 t} \cdot \sin\sqrt{-\lambda_n}\,x + 10$

$\Rightarrow y(x,t) = \sum_{n=0}^{\infty} c_n e^{-(2n+1)^2\frac{\pi^2}{4}\cdot c^2 t} \sin(2n+1)\frac{\pi}{2}x + 10$

At $t=0$, $y(x,0) = \sum_{n=0}^{\infty} c_n \sin(2n+1)\frac{\pi}{2} + 10$

$= 1 - x.$

By Fourier's sine theory,

$c_n = \frac{2}{1}\int_0^1 (9-x)\sin(2n+1)\frac{\pi}{2}x$

$$= 2\left[9\left(\cos\frac{(2n+1)\pi x}{2}\right)\frac{2}{\pi(2n+1)} + x\cdot\left(\cos\frac{(2n+1)\pi x}{2}\right)\cdot\frac{2}{\pi(2n+1)} - \left(\sin\frac{(2n+1)\pi}{2}x\right)\cdot\frac{2^2}{(\pi(2n+1))^2}\right]_0^1$$

$$= 2\cdot\left[\frac{-\cancel{2}18}{\pi(2n+1)} - \frac{2^2}{(\pi(2n+1))^2}(-1)^n\right] = \frac{-4}{\pi(2n+1)}\left[9 + \frac{2(-1)^n}{\pi(2n+1)}\right]$$

$$\therefore y(x,t) = \left[\sum_{n=0}^{\infty}\frac{-4}{\pi(2n+1)}\left[9 + \frac{2\cdot(-1)^n}{\pi(2n+1)}\right]\cdot e^{-\frac{(2n+1)^2\pi^2}{4}\cdot c^2 t}\cdot\sin\frac{(2n+1)\pi x}{2}\right] + 10$$

7. (a) Convert the following binary numbers to the base indicated:
   (i) $(10111011001.101110)_2$ to octal
   (ii) $(10111011001.10111000)_2$ to hexadecimal
   (iii) $(0.101)_2$ to decimal

[10]

(i) $(10\ 111\ 011\ 001\ .\ 101\ 110)_2$

$= (2731.56)_8$

(ii) $(101\ 1101\ 1001\ .\ 1011\ 1000)_2$

$= (5D9.B8)_{16}$

(iii) $(0.101)_2 = (1\times 2^{-1} + 0\times 2^{-2} + 1\times 2^{-3})_{10}$

$= (0.5 + 0.125)_{10} = (0.625)_{10}$

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7. (b) A missile is launched from a ground station. The acceleration during its first 80 seconds of flight, as recorded, is given in the following table:

t(s)	0	10	20	30	40	50	60	70	80
$a(m/s^2)$	30	31.63	33.34	35.47	37.75	40.33	43.25	46.69	50.67

compute the velocity of the missile when t = 80s, using Simpson's 1/3 rule. **[10]**

$$\because a = \frac{dv}{dt} \Rightarrow v = \int_0^{80} a\,dt$$

This is an equi-spaced data.

$\Rightarrow$ By Simpson's 1/3 rule,

$$v = \frac{10}{3}\sum_{n=0}^{3}\left[a(20n) + 4a(20n+10) + a(20(n+1))\right]$$

08

$$= \frac{10}{3}\left[a(0) + 4\cdot a(10) + 2\cdot a(20) + 4\cdot a(30) + 2\cdot a(40) + 4\cdot a(50) + 2\cdot a(60) + 4a(70) + a(80)\right]$$

$$= \frac{10}{3}[30 + 4\times 31.63 + 2\times 33.34 + 4\times 35.47 + 2\times 37.75 + 4\times 40.33 + 2\times 43.25 + 4\times 46.69 + 50.67]$$

~~= 308.61 metre/sec.~~

$= 3086.1$ metre/sec.

7. (c) Using fourth order Runge-Kutta method find the solution of the initial value problem
$y' = 1/(x + y)$, $y(0) = 1$
in the range $0.5 \le x \le 2.0$, by taking $h = 0.5$. **[15]**

$$f(x,y) = y' = \frac{1}{x+y}.$$

$y(0) = 1$, $h = 0.5$

$$\Rightarrow y(0.5) = y(0) + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4],$$

where $k_1 = hf(0, 1) = 0.5$

$k_2 = hf(0 + \frac{h}{2}, 1 + \frac{k_1}{2}) = 0.33333$

$k_3 = hf(0 + \frac{h}{2}, 1 + \frac{k_2}{2}) = 0.35294$

$k_4 = hf(0 + h, 1 + k_3) = 0.26984$

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$\Rightarrow y(0.5) = 1.35706, \quad h = 0.5$

$\Rightarrow y(1) = y(0.5) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

where $k_1 = hf(0.5, 1.35706) = 0.26924$

$k_2 = hf(0.5 + \frac{h}{2}, 1.35706 + \frac{k_1}{2}) = 0.22305$

$k_3 = hf(0.5 + \frac{h}{2}, 1.35706 + \frac{k_2}{2}) = 0.22537$

$k_4 = hf(0.5 + h, 1.35706 + k_3) = 0.19362$

$\Rightarrow y(1) = 1.58368$

$\Rightarrow y(1.5) = y(1) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

where $k_1 = hf(1, 1.58368) = 0.19352$

$k_2 = 0.17062$

$k_3 = 0.17129$

$k_4 = 0.15361$

$\Rightarrow y(1.5) = 1.75551$

$\Rightarrow y(2) = y(1.5) + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$

where $k_1 = 0.15359$

$k_2 = 0.13958$

$k_3 = 0.13985$

$k_4 = 0.12836$

$\Rightarrow y(2) = 1.89565$

x	0.5	1.0	1.5	2.0
y	1.35706	1.58368	1.75551	1.89565

7\. (d) Draw a flow chart to declare the results for the following examination system: 60 candidates take the examination. Each candidate writes one major and two minor papers.

A candidate is declared to have passed in the examination if he/ she gets a minimum of 40 in all the three papers separately and an average of 50 in all the three papers put together.

Remaining candidates fail in the examination with an exemption in major if they obtain 60 and above and exemption in each minor if they obtain 50 and more in that minor.

[15]

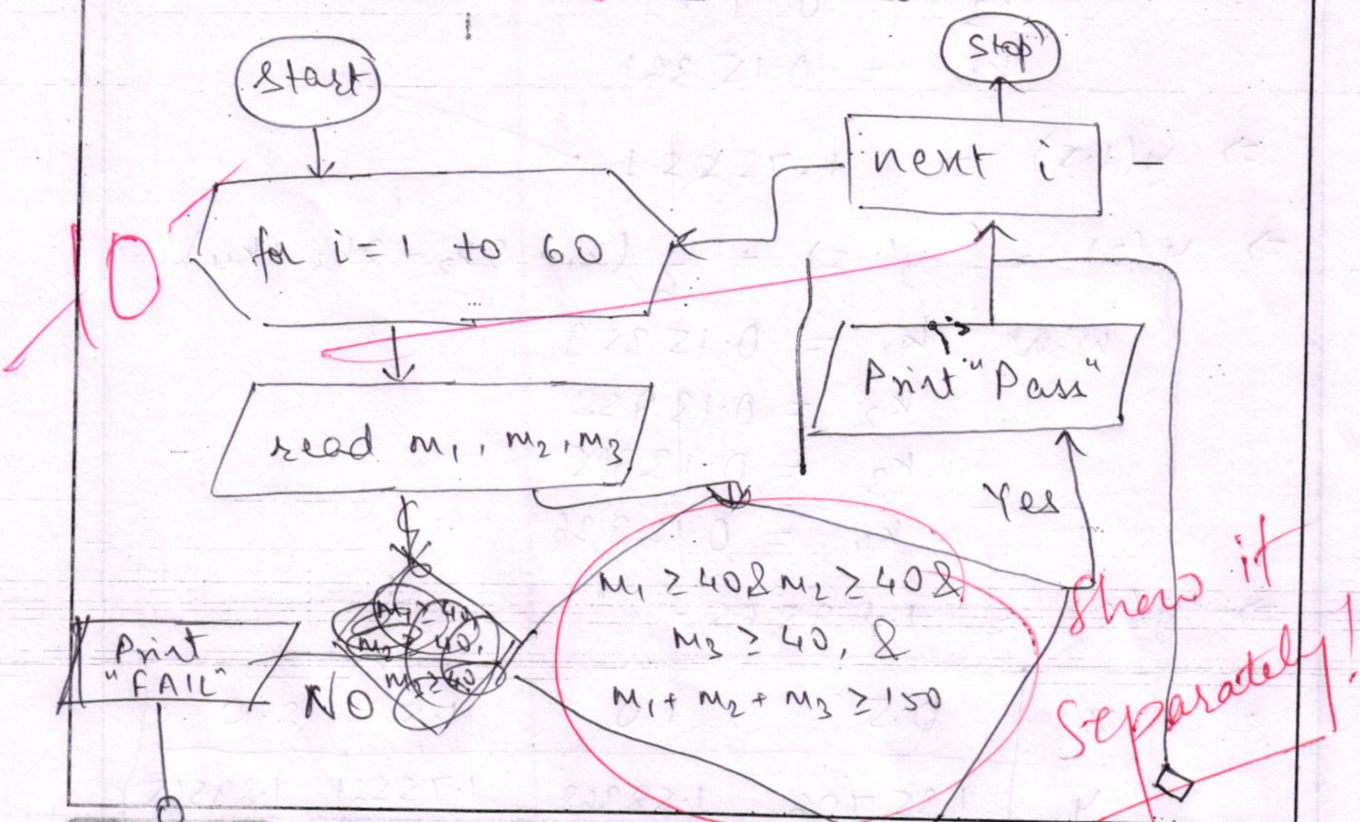
If m_1, m_2, m_3 are marks of a candidate, following cases happen:

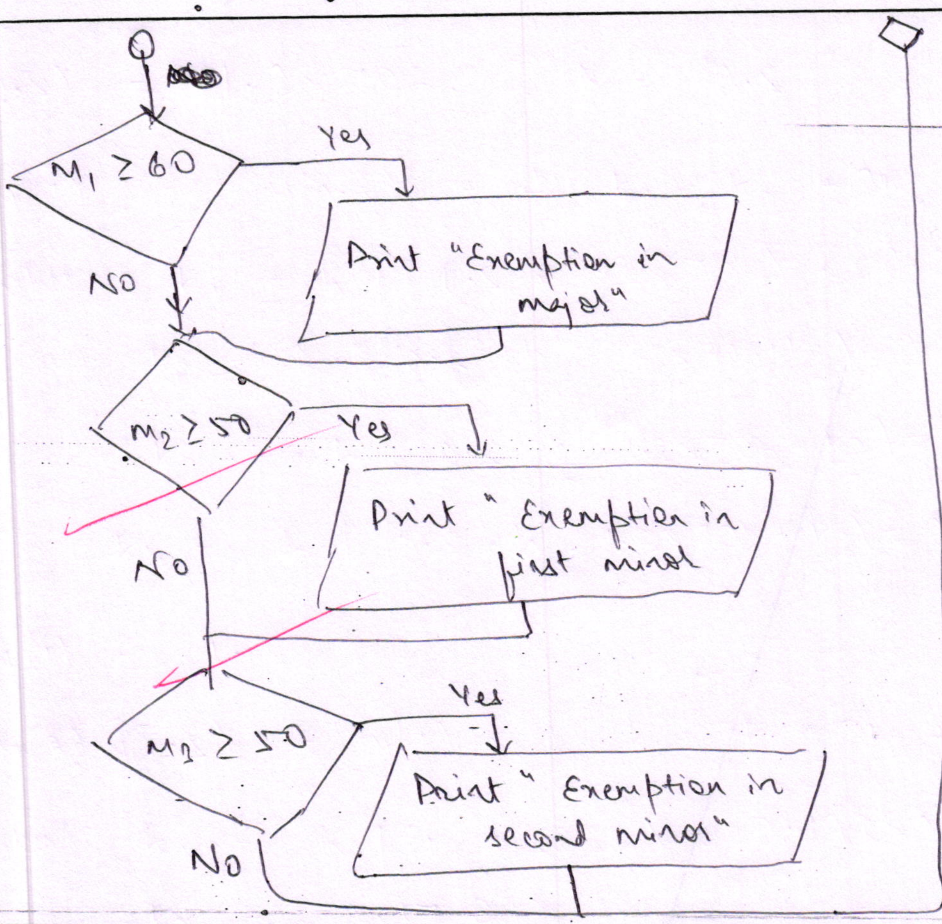
① $m_1, m_2, m_3 \geq 40$, $m_1 + m_2 + m_3 \geq 150$
$\Rightarrow$ Pass

② else, Fail.

Also, $m_1 \geq 60 \Rightarrow$ Exemption in major
$m_2 \geq 50 \Rightarrow$ Exemption in first minor
$m_3 \geq 50 \Rightarrow$ Exemption in second minor

We repeat this procedure 60 times to cover all candidates as shown below:





8. (a) Use Hamilton's equations to find the equation of motion of the (i) simple pendulum & (ii) compound pendulum. **[18]**