Revision is the most crucial part of mathematics preparation in Civil Services Examination. For that purpose, I had prepared short notes on each topic of the syllabus which I am sharing here. I would suggest aspirants to use this as just a reference for quick revision and not as your primary source. This optional requires good understanding of the topic which cannot be derived from short notes.

It is the process of preparing such notes that gives a candidate confidence about his or her preparation and brings conceptual clarity. Hence I would encourage aspirants to do this exercise themselves even if it is time consuming.

All the best!

Regards,
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AIR 8 - CSE 2015

ANALYTIC GEOMETRY

Co-ordinate Systems

(1) 3 axes divide space in 8 parts & each part is called octant.

(2) Basic geometry properties
Quadrilateral is said to be rectangle when opposite sides are equal & diagonals are equal.
Quadrilateral is said to be rhombus if 4 sides are equal but diagonals are unequal.
Diagonals bisect each other in parallelogram, rectangle, rhombus, square.

(3) If 3 points (a_i, b_i, c_i) are collinear then determinant

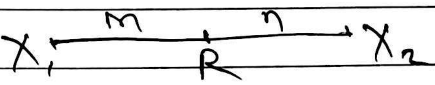
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

It is necessary condition but not sufficient.

(4) In questions of proving 4 points form rhombus or 3 pts. form triangle, find out lengths of sides & diagonals & prove.

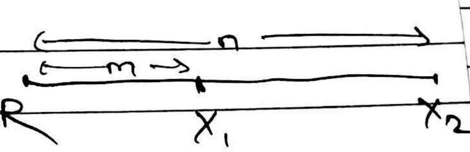
(5) Internal section m:n formula

$$R = \left\{ \frac{mx_2 + nx_1}{m+n}, \frac{my_2 + ny_1}{m+n}, \frac{mz_2 + nz_1}{m+n} \right\}$$



External section m:n formula

$$R = \left\{ \frac{nx_1 - mx_2}{n-m}, \frac{ny_1 - my_2}{n-m}, \frac{nz_1 - mz_2}{n-m} \right\}$$



note that $\frac{X_1R}{RX_2} = \frac{-m}{n}$ is same as $\frac{m}{-n}$. Whatever +, − combo you use in numerator, same should be used in denominator.

(6) Centroid of a tetrahedron ABCD divides any line joining A to centroid of opposite triangle BCD in ratio 3:1. Obv. centroid of ABCD is given by $\frac{x_1 + x_2 + x_3 + x_4}{4}$

Date
Page

(7) All edges equal means regular tetrahedron.

(8) XY plane divides line joining (x_1, y_1, z_1) & (x_2, y_2, z_2) in ratio $-z_1 : z_2$.

Note $-5:4$ (internally) ratio $=$ $5:4$ (externally) ratio

The plane $ax+by+cz+d=0$ divides segment joining (x_1, y_1, z_1) & (x_2, y_2, z_2) in ratio $-\frac{(ax_1+by_1+cz_1+d)}{(ax_2+by_2+cz_2+d)}$

(9) 3 vertices of ‖ given, 4th found by using prop that midpoints of diagonals are same.

(10) Locus
Set of all points in space satisfying a given condition.

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DIRECTION RATIOS / COSINES

① Direction cosines
If a line makes α, β, γ angles with X, Y, Z axes then $\cos\alpha, \cos\beta$ & $\cos\gamma$ are its direction cosines.
α, β, γ are called direction angles.

Direction Ratios
3 no. proportional to directional cosines.

If D.R. are a, b, c then d.c. are given by

$$\frac{\pm a}{\sqrt{a^2+b^2+c^2}}, \frac{\pm b}{\sqrt{a^2+b^2+c^2}}, \frac{\pm c}{\sqrt{a^2+b^2+c^2}}$$

Same +ve or −ve sign taken for all 3.

② Angle bet^n 2 lines
D.C. are (l_1, m_1, n_1) & (l_2, m_2, n_2) then

$$\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$$

$$\sin\theta = \pm\sqrt{\Sigma(l_1m_2 - l_2m_1)^2} \quad \text{i.e.} \quad \pm\sqrt{\Sigma \begin{vmatrix} l_1 & l_2 \\ m_1 & m_2 \end{vmatrix}^2}$$

③ Dealing with questions of relation bet^n direction cosines of 2 given lines

$ul + vm + wn = 0$, $al^2 + bm^2 + cn^2 = 0$

P.T. above ⊥ if $u^2(b+c) + v^2(a+c) + w^2(a+b) = 0$

→ eliminating n we get $al^2 + bm^2 + c\left[\frac{ul+mv}{w}\right]^2 = 0$

$$\therefore (aw^2 + cu^2)\left(\frac{l}{m}\right)^2 + 2cuv\left(\frac{l}{m}\right) + (bw^2 + cv^2) = 0$$

∴ Product of roots ⇒ $\frac{l_1}{m_1} \cdot \frac{l_2}{m_2} = \frac{bw^2 + cv^2}{aw^2 + cu^2}$

Similarly $\frac{l_1l_2}{bw^2+cv^2} = \frac{m_1m_2}{aw^2+cu^2} = \frac{n_1n_2}{av^2+bu^2}$ (for this form eq. in terms of $\left(\frac{l}{n}\right)$)

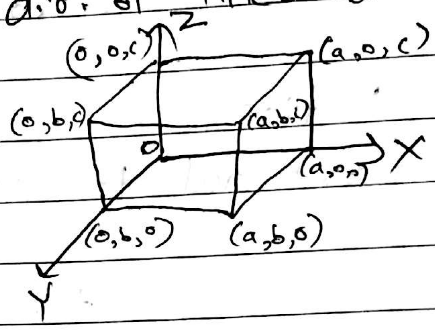
$\therefore \Sigma l_1l_2 = 0 \Rightarrow (\qquad\qquad) = 0$ Hence proved.

→ 2 lines are ∥ ⇒ d.c. are same ⇒ $b^2 = 4ac$ & so on.

Date
Page

(4) In questions where 3 concurrent lines are to be proven to be coplanar use $a \cdot [b \times c] = 0$

(5) In questions regarding d.r. of lines of a cube / ||$^{\text{l}}$ piped, form following image & use simple coordinates.



THE PLANE

(1) Surface s.t. line joining any 2 points lies wholly on the surface.

(2) Eq. of any plane passing through (x_1, y_1, z_1) is
: $A(x-x_1)+B(y-y_1)+C(z-z_1)=0$ (one point form)

Interecept form: $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

Normal form : ~~...~~ $lx+my+nz=p$
(l,m,n) are d.c.s of normal & p is distance from origin
p is always taken +ve.

Reduction from General to normal form
$$\frac{A}{\sqrt{\Sigma A^2}}x+\frac{B}{\sqrt{\Sigma A^2}}y+\frac{C}{\sqrt{\Sigma A^2}}z=\frac{D}{\sqrt{\Sigma A^2}} \quad (D \geq 0)$$

(3) Angle bet^n 2 planes $\Rightarrow$ $\frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\cdot\sqrt{a_2^2+b_2^2+c_2^2}} = \cos\theta$

$$\sin\theta = \frac{\sqrt{\Sigma(a_1b_2-a_2b_1)^2}}{\sqrt{\Sigma a_1^2}\sqrt{\Sigma a_2^2}}$$

(4) Eq. of plane through 3 points

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0$$

Similarly condⁿ for 4 points to be coplanar is above determinant for 4 points = 0.

Faster method for finding plane eq. would be cross product.

(5) Eq. of plane ⊥ to XY plane $AX+BY+D=0$

(6) Condition for pt. P to be circumcentre of A, B, C.
(a) ~~D~~ $PA = PB = PC$ & (b) *All 4 lie in same plane*

(7) Eq. of plane through line of intersection of 2 planes P_1 & P_2 → $P_1 + \lambda P_2 = 0$

(8) Plane eq. $ax+by+cz+d$ divides space into 3 part +ve, 0 or −ve.

(9) Length of ⊥ from a point to a plane

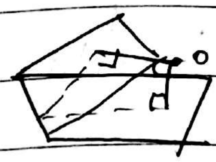
$$= \frac{|ax_1+by_1+cz_1+d|}{\sqrt{a^2+b^2+c^2}}$$

Distance betⁿ ॥ʟ planes = $\dfrac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}$

(10) Plane P_1 rotated angle α through a given line.
Plane P_1 rotated through intersection with $z=0$ angle α.
→ $lx+my=0$ rotated through
$lx+my+\lambda z=0$ $\therefore \cos\alpha = \dfrac{l^2+m^2}{\sqrt{(l^2+m^2)(l^2+m^2+\lambda^2)}}$ & so on.

(11) Whether origin lies in acute or obtuse angle betⁿ 2 planes.

1st ensure $a_1x+b_1y+c_1z+d_1=0$ & $a_2x+b_2y+c_2z+d_2=0$ has both d_1 & d_2 +ve.



if $a_1a_2+b_1b_2+c_1c_2 > 0 \Rightarrow$ angle of normals drawn at origin is acute

∴ Origin lies in Obtuse angle betⁿ planes.

∴ $a_1a_2+b_1b_2+c_1c_2 < 0 \Rightarrow$ origin lies in acute angle.

(13) Equation of bisector planes betⁿ 2 planes

$$\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = \pm \frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}$$

Let P_1, P_2 be 2 planes & B_1, B_2 its bisector planes.
Find $\cos\theta$ betⁿ P_1 & B_1. Then find $\tan\theta$.
If $\tan\theta < 1$; $\theta < 45$ ∴ B_1 bisects acute angle.
If $\tan\theta > 1$; $\theta > 45$ ∴ B_1 bisects obtuse angle.

If d_1 & $d_2 > 0 \Rightarrow \dfrac{P_1}{\sqrt{\Sigma a_1^2}} = \dfrac{P_2}{\sqrt{\Sigma a_2^2}}$ bisects angle containing origin.

(14) Orthogonal Projection of a point on a plane
= Foot of ⊥ from pt. to the plane
for finding foot of (x_1, y_1, z_1) on P; put (x_1+at, y_1+bt, z_1+ct) on plane $ax+by+cz+d=0$ & get the t.

(15) Area of triangle as sum of its projections on 3 planes

$$\Delta_x = \frac{1}{2}\begin{vmatrix} y_1 & z_1 & 1 \\ y_2 & z_2 & 1 \\ y_3 & z_3 & 1 \end{vmatrix} \quad \Delta_y = \frac{1}{2}\begin{vmatrix} x_1 & z_1 & 1 \\ x_2 & z_2 & 1 \\ x_3 & z_3 & 1 \end{vmatrix} \quad \Delta_z = \frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$

$$\Delta^2 = \Delta_x^2 + \Delta_y^2 + \Delta_z^2$$

(16) Vol^m of a tetrahedron ABCD $= \frac{1}{3} \times P \times \Delta$
where P is $\perp$ from A to BCD & Δ is area of ΔBCD.

$$\text{Vol}^m = \frac{1}{6} \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}$$

(17) Equation of a pair of planes:

$$(a_1x+b_1y+c_1z+d_1)(a_2x+b_2y+c_2z+d_2)=0$$

A homogeneous 2^{nd} degree equation represents a pair of plane if $\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$ $\quad ax^2+by^2+cz^2+2fyz+2gxz+2hxy=0$

How to remember? $\rightarrow$ $a \searrow b \searrow c$ The $h \leftarrow g$, $f \uparrow g$, $h \uparrow$, $g \leftarrow f \leftarrow$

Also remember that $2f$ is coeff. of yz, $2g$ of xz & so on.

(18) If $ax^2+by^2+cz^2+2fyz+2gxz+2hxy=0$ shows a pair of planes then angle bet^n them

$$\tan\theta = \frac{2\sqrt{f^2+g^2+h^2-ab-bc-ca}}{a+b+c}$$

THE STRAIGHT LINE

Date
Page

(1) Symmetrical form of eq.

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}.$$

Asymmetrical or general form

$$P_1=0=P_2$$

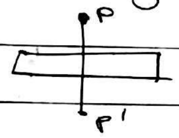
Parametric form

$$(x_1+lr,\ y_1+mr,\ z_1+nr)$$

Two point form

$$\frac{x-x_1}{x_1-x_2}=\frac{y-y_1}{y_1-y_2}=\frac{z-z_1}{z_1-z_2}$$

(2) Image of point in a plane



Find foot of $\perp$ & then go twice distance.

(3) To transform unsymmetrical to symmetric form, find cross product to get d.r.
& put $z=0$, solve x & y. That is one pt. on line.

(4) Angle betn a line & a plane = Complement of angle betn normal to plane & that line.

(5) Condition that a line is $\parallel$ to plane

$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ $\parallel$ to $ax+by+cz+d=0$ if

$$al+bm+cn=0 \ \& \ ax_1+by_1+cz_1+d\neq 0$$

if it lies in the plane then

$$al+bm+cn=0 \ \& \ ax_1+by_1+cz_1+d=0$$

(6) $\frac{x-x_1}{l_1}=\frac{y-y_1}{m_1}=\frac{z-z_1}{n_1}$ & $\frac{x-x_2}{l_2}=\frac{y-y_2}{m_2}=\frac{z-z_2}{n_2}$ intersect if

$$0=\begin{vmatrix} x_1-x_2 & y_1-y_2 & z_1-z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}$$

& eq. of intersecting plane is $\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$

(7) 2 lines $P_1 = 0 = P_2$ & $P_3 = 0 P_4$ intersect if

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = 0$$

(8) Changing from symmetrical to unsymmetric form

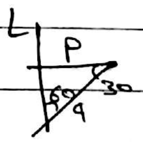
$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} \Rightarrow m(x-x_1) = l(y-y_1)$ &
$n(y-y_1) = m(z-z_1)$

(9) FINDING $\perp$ distance of pt. (α, β, γ) from line $\frac{(\ )}{l} = \frac{(\ )}{m} = \frac{(\ )}{n}$

→ Find foot by (x_1+lr, y_1+mr, z_1+nr) being $\perp$ point i.e.
$l(x_1+lr-\alpha) + m(y_1+mr-\beta) + n(z_1+nr-\gamma) = 0$
Thus get r & also you can find eq. of $\perp$ line.

(10) Find eq. of 2 lines through (1,1,1) that intersect line
$\frac{x-3}{2} = \frac{y-4}{1} = \frac{z-1}{1}$ in angle $60°$.

→ Use trigonometry



Find P & then q & then use parametric form of line L.

(11) Equation of any line intersecting 2 given lines
$u_1 = 0$ & $v_1 = 0$ — $u_2 = 0$ & $v_2 = 0$ is
$u_1 + k_1 v_1 = 0$ & $u_2 + K_2 v_2 = 0$ K, K_2 constants.

If given in symmetrical form; convert 2 lines to parametric form i.e. intersecting line connects 2 points
$(x_1 + l_1 r_1, y_1 + m_1 r_1, z_1 + n_1 r_1)$ & $(x_2 + l_2 r_2, y_2 + m_2 r_2, z_2 + n_2 r_2)$

12) SHORTEST DISTANCE BET^N 2 SKEW LINES

(a) Projection Method (both lines in symmetrical form)

$$\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1} \quad \& \quad \frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}.$$

Find d.c.s of $\perp$ by cross product.
Take its dot product with $((x_1-x_2), (y_1-y_2), (z_1-z_2))$
That gives the distance.

Now for points; put condition that a line bet^n parametric points of L_1, L_2 is $\parallel$ to $\perp$ d.c.

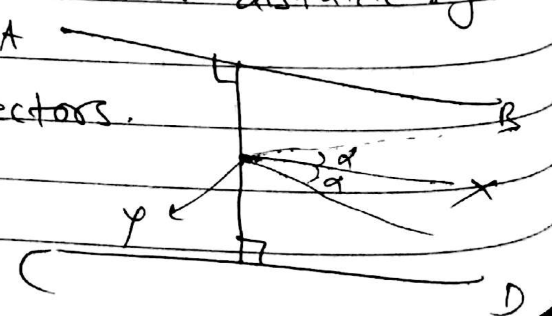
(b) Both are given in general form.
$P_1 = 0 = P_2$ & $P_3 = 0 = P_4$
find λ_1, λ_2 s.t. $P_1 + \lambda_1 P_2$ & $P_3 + \lambda_2 P_4$ are parallel.
Then find distance bet^n $\parallel$ planes.

Eq. of $\perp$ given by $P_1 + \lambda_3 P_2$ i.e. $\perp$ to $P_1 + \lambda_1 P_2$
& $P_3 + \lambda_4 P_4$ i.e $\perp$ to $P_3 + \lambda_2 P_4$.
Then $P_1 + \lambda_3 P_2 = 0 = P_3 + \lambda_4 P_4$ give eq. of line.

13) On questions involving d.r. of tetrahedron.
→ Put O at origin for OABC & assume d.c.s for OA, OB, OC & derive relations.

14) By proper choice of axes, eq of any 2 skew lines can be put as $y = x\tan\alpha$ & $z = c$
$y = -x\tan\alpha$ & $z = -c$

Take origin at midpoint of shortest distance segment.
Draw lines $\parallel$ to original lines.
Take X & Y axes as angle bisectors.



(15) Locus of a line intersecting given 3 lines

$u_1 = 0 = u_2$, $v_1 = 0 = v_2$, $w_1 = 0 = w_2$

→ That line should intersect $u_1 + \lambda_1 u_2 = 0$ & $v_1 + \lambda_2 v_2 = 0$

Put that in 3^{rd} line get condition of λ_1 & λ_2 then replace $\lambda_1 = \frac{-u_1}{u_2}$, $\lambda_2 = \frac{-v_1}{v_2}$

e.g. 3 lines $y = mx, z = c$ | $y = -mx, z = -c$ | $y = z$, $mx = -c$

→ $y - mx + k_1(z - c) = 0$ & $y + mx + k_2(z + c) = 0$

⇒ $(z + c) + k_1(z - c) = 0$ & $z - c + k_2(z + c) = 0$

∴ $k_1 = \frac{c - z}{z + c}$ & $k_2 = \frac{z + c}{c - z}$ ∴ $k_1 k_2 = 1$

$$\therefore \left(\frac{y - mx}{z - c}\right)\left(\frac{y + mx}{z + c}\right) = 1$$

Note it could be locus of line intersecting 2 lines & any other given curve. Method remains same.

(16) Intersection of 3 planes

→ Could be point, line or prism i.e. (no intersection)

Find symmetric eq. of intersection of 2 planes. Put parametric form point in 3^{rd} plane

if (i) One sol^n for t ⇒ intersect in a point

(ii) satisfied for all t (i.e. $5 = 5$ etc) ⇒ intersect in a line

(iii) Inconsistant eq. i.e. $0 = 2$ ⇒ no intersection. ∴ form a prism.

(17) On questions of prove P_1, P_2, P_3 will intersect in a line if ____ condition satisfied ; find $\frac{P_1 + \lambda P_3}{P_3 + \lambda P_2}$ & prove.

THE SPHERE

(1) Standard form $x^2+y^2+z^2=a^2$

Central form $(x-a)^2+(y-b)^2+(z-c)^2=r^2$

General form $x^2+y^2+z^2+2ux+2vy+2wz+d=0$

If $u^2+v^2+w^2-d<0 \Rightarrow$ pseudo sphere or virtual sphere

(2) A given eq. represents a sphere if

(a) It is a 2^{nd} degree equation

(b) Coeff. of $x^2=y^2=z^2$

(c) Doesn't contain term xy, yz or zx

(3) General eq. contains 4 unknowns (u,v,w,d) $\therefore$ 4 conditions needed to determine a sphere.

4 point form :- Eq. of sphere through P_1, P_2, P_3, P_4

$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2+y_2^2+z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2+y_3^2+z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2+y_4^2+z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

(4) Diameter form for (x_1,y_1,z_1) & (x_2,y_2,z_2) forming diameter

$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$

Eq. of OABC sphere $A(a,0,0), B(0,b,0), C(0,0,c)$ is

$x^2+y^2+z^2-ax-by-cz=0$

(5) Equation of a circle in 3-D

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$ & $lx+my+nz=p$

$\because$ section of sphere by plane is circle.

Centre is foot of $\perp$ drawn from centre of sphere.

Radius given by Pythagoras thm.

If plane passes through centre of sphere, it is called great circle. Radius & centre same as sphere.

(6) Eq. of circle via $(a,0,0)$, $(0,b,0)$ & $(0,0,c)$
→ $x^2+y^2+z^2-ax-by-cz=0$ &
$x/a+y/b+z/c=1$

(7) 4 points are concyclic if 4^{th} point lies on circle formed by 3 points. Find eq. of circle & show 4^{th} pt. satisfies it.
For eq. of circle, assume 4^{th} point is origin & find sphere & then eq. of plane.

(8) Equation of 2 spheres together represents a circle. & $S+kS'=0$ represents sphere passing through a given circle.

Also in circle is $S=0$ & plane $P=0$ then
$S+kP=0$ represents sphere passing through the circle

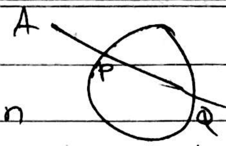
(9) For circle $x^2+y^2+2gx+2fy+c=0$ & $z=0$, the sphere family is given by $x^2+y^2+z^2+2gx+2fy+2kz+c=0$
(don't do S_1+kP here since S_1 is not sphere but cylinder)

(10) For a tangent plane/line, distance from centre is equal to radius.

eq. of tangent plane at (x_1, y_1, z_1) is
$xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d=0$

Power of a point w.r.t. sphere

(11) If from a fixed point A; lines are drawn to intersect sphere in P & Q then AP.AQ product is constant & equals power of A.

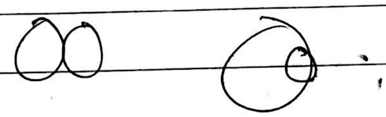


Power of A = $\left(\alpha^2+\beta^2+\gamma^2+2u\alpha+2v\beta+2w\gamma+d\right)$

(12) Tangent line to ~~sphere~~ circle is line of intersection of tangent plane at that point & the plane containing the circle.

In questions of tangent plane, generally best way is finding sol^n using $\perp$ from centre property.

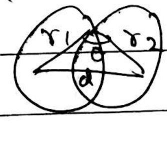
(13) Touching Spheres



Externally when $r_1+r_2=d$

Internally when $r_1-r_2=d$

(14) Angle between intersection of 2 spheres = Angle bet^n tangent planes at common point = Angle bet^n segment joining common point to each centre



$$\theta = \cos^{-1}\left(\frac{r_1^2+r_2^2-d^2}{2r_1r_2}\right)$$

Orthogonal spheres $\Rightarrow$ $r_1^2+r_2^2=d^2$

Condition for orthogonality $\Rightarrow$

$2u_1u_2+2v_1v_2+2w_1w_2=d_1+d_2$

(15) Length of tangent from a point $(x_1, y_1, z_1) =$
$\sqrt{x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d} = \sqrt{\text{Power}}$
First ensure coeff. of x^2 is 1. Can't have $5x^2+5y^2+\ldots$ & take sq. root

(16) Radicle plane of 2 spheres
Locus of points whose powers w.r.t. 2 spheres is equal. i.e. length of tangent equal.

Eq. of radicle plane of 2 spheres is $S_1 - S_2 = 0$.
& it is $\perp$ to line joining their centres.

If they intersect then plane of common circle → radicle.

(17) Radicle line of 3 spheres
Given by $S_1 - S_2 = 0$ & $S_2 - S_3 = 0$

Radicle centre of 4 spheres
Given by $S_1 - S_2 = 0$ $S_2 - S_3 = 0$
$S_1 - S_3 = 0$ $S_2 - S_4 = 0$

(18) Co-axial Spheres
System of spheres where any 2 spheres have same radicle plane.

If S_1 & S_2 are 2 spheres then $S_1 + \lambda S_2 = 0$ gives system of coaxial spheres.
Also given by $S_1 + \lambda(S_1 - S_2) = 0$

(19) Equation of coaxial ~~form of~~ spheres can be put in simple form $x^2+y^2+z^2+2ux+d=0$
Here u is the parameter.

(20) Limiting Point of co-axial spheres
= Centres of 2 spheres with 0 radius.

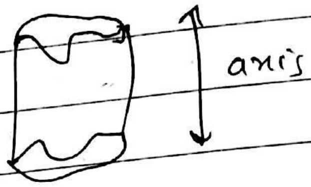
For $x^2+y^2+z^2+2ux+d=0$, limiting points are $(\sqrt{d},0,0)$ & $(-\sqrt{d},0,0)$.

(21) Any sphere passing through limiting point cut every other sphere of co-axial system orthogonally

THE CYLINDER

(1) Def^n

Surface generated by line which is always $\parallel$ to a given line (axis) & intersects guiding curve is called a cylinder



Guidying curve need not be a circle.

If generator is $\parallel$ to z-axis, then eq. of cylinder becomes $ax^2+by^2+2hxy+2gx+2fy+c=0$

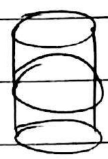
i.e. free from z.

(2) So to find eq. of cylinder with generators $\parallel$ to z-axis & intersect a given conic, then just eliminate z from eq. of the conic.

e.g. Guiding curve $ax^2+by^2=2z$ & $lx+my+nz=p$

$\therefore$ cylinder: $ax^2+by^2 = 2\left(\frac{p-lx-my}{n}\right)$

(3) Enveloping cylinder of a sphere & $\parallel$ to axis

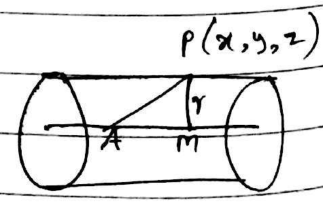


Let $(x'+lr)\,(y'+mr)\,(z'+nr)$ be a point on cylinder. Put this in sphere eq. & r should have ~~sam~~ repeated roots $\because$ tangent $\therefore$ $b^2=4ac$.

(4) Right Circular Cylinder

Guiding curve is circle & axis is $\perp$ to plane of circle.

Std. form: $x^2+y^2=a^2$



Gen. form: Find projection of AP on axis. The use pythagoras to find eq. of cylinder.

classmate

Date

(5) Find eq. of right circular cylinder where circle is $x^2+y^2+z^2=9$ & $x-y+z=3$.

→ Use that axis is line through centre of sphere & ⊥ to plane of circle. Radius of circle = rad. of cylinder

(careful! rad. of sphere is different!)

(6) Find eq. of cylinder whose guiding curve is circle passing through 3 points A, B & C.

→ find sphere imagining Origin as 4th point & plane. Proceed as point 5.

The Conicoid

① A 2^{nd} degree equation in x, y, z is called conicoid or quadric.

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$$

The qudric can be fixed by giving 9 conditions as shown by 9 independent coefficiente (when divided by d)

② $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ Ellipsoid

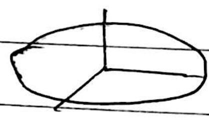
$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ Hyperboloid of one sheet

$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ Hyperboloid of two sheets

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2z}{c}$ Elliptic Paraboloid

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{2z}{c}$ Hyperbolic Paraboloid

③ Properties of Ellipsoid



Centre :- Origin. Since it bisects every chord passing through it.

Co-ordinate planes are the principle planes since surface is symmetric about them.

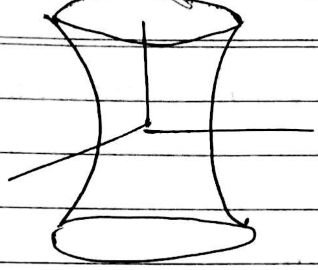
Intersection of principle planes is called principle axis. ∴ Co-ordinate axes are principle axes here.

Ellipsoid when cut by planes ‖ to coordinate planes gives ellipse. So it is essentially generated by ellipse of different sizes.

Page 143

(4) Hyperboloid of one sheet

$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$



looks like nuclear reactor.

Centre - origin

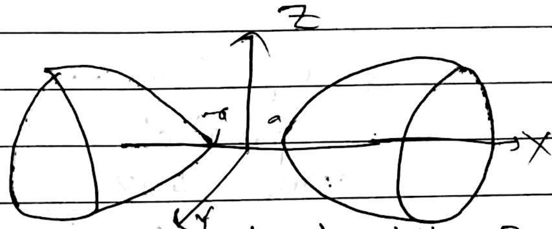
Principle planes - Coordinate planes

Principle Axes - Coordinate axes

Section by $z=k$ planes shows it is generated by ellipse of varing sizes.

(5) Hyperboloid of two sheets

$\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$



Same centre, principle planes / axes

looks like 2 Daggas kept on 2 sides

Section by $x=k$ shows surface formed by ellipses which turn imaginary for $|k|>a$

$\therefore$ No portion lies between $x=\pm a$.

(6) Central Conicoid $ax^2+by^2+cz^2=1$

A conicoid whose all chords through origin are bisected.

For all central conicoids, centre / principal planes - axes are as above i.e. coordinate planes - axes.

(7) Properties of Central Conicoid

(a) Every line cuts central conicoid in 2 points.
A chord of central conicoid through origin is called a diameter.

(b) Tangent plane at $(x_1,y_1,z_1) \to axx_1+byy_1+czz_1=1$

$lx+my+nz=p$ will be tangent if

$$\boxed{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p^2}$$

useful to remember in problems.

(c) Any tangent plane is of type

$$lx+my+nz = \sqrt{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}}$$

(d) Director Sphere.

Locus of point of intersection of 3 mutually $\perp$ tangent planes of conicoid is sphere called director sphere.

$$x^2+y^2+z^2 = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

(e) e.g. Find tangent plane of $4x^2-5y^2+7z^2+13=0$ parallel to plane $4x+20y-21z=0$.

Put $\dfrac{4xx_1-5yy_1+7zz_1+13}{4x+20y-21z+d}$ & get ratios.

e.g. Locus of foot of $\perp$ from origin to tangent planes is $a^2x^2+b^2y^2+c^2z^2 = (x^2+y^2+z^2)^2$.

→ comes easily if you know tangent plane given by $a^2l^2+b^2m^2+c^2n^2=p^2$ for std. ellipsoid.

In questions concerning foot of $\perp$ from origin, start by assumming (x_1,y_1,z_1) as foot & then put condition for tangent plane.

(d1) Director sphere of ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ becomes $\underline{x^2+y^2+z^2=a^2+b^2+c^2}$

Similarly Director Circle of ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ becomes $\underline{x^2+y^2=a^2+b^2}$

Director sphere is intersection of 3 $\perp$ tangent planes

Director circle is intersection of $\perp$ tangent lines.

Useful to simply remember the equation.

8) Normal

(a) Defined as $\perp$ to tangent plane through the point of contact.

i.e. $$\frac{x-x_1}{ax_1} = \frac{y-y_1}{by_1} = \frac{z-z_1}{cz_1} \quad (\text{for } ax^2+by^2+cz^2=1)$$

(b) From a fixed point (α, β, γ), we can draw 6 normals to an ellipsoids.

In all questions concerning normals drawn from a given point, use following tool

Let (x_1, y_1, z_1) be a foot of normal

$$\therefore \frac{\alpha-x_1}{ax_1} = \frac{\beta-y_1}{by_1} = \frac{\gamma-z_1}{cz_1} = \lambda$$

$$\Rightarrow (x_1, y_1, z_1) \rightarrow \left(\frac{\alpha a^2}{a^2+\lambda}, \frac{\beta b^2}{b^2+\lambda}, \frac{\gamma c^2}{c^2+\lambda}\right)$$

Based on varying λ; all questions can be solved.

(c) Six feet are point of intersection of ellipsoid with a cubic curve.

For proof, consider a general eq. of plane & put above λ formula in it. We get a cubic in λ.

(d) Six feet pass through a cone

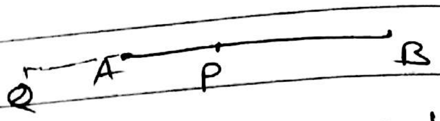
$$\frac{a^2\alpha(b^2-c^2)}{x} + \frac{b^2(c^2-a^2)\beta}{y} + \frac{c^2(a^2-b^2)\gamma}{z} = 0$$

Remember coeff. of $x^2 = y^2 = z^2 =$ constant $= 0$

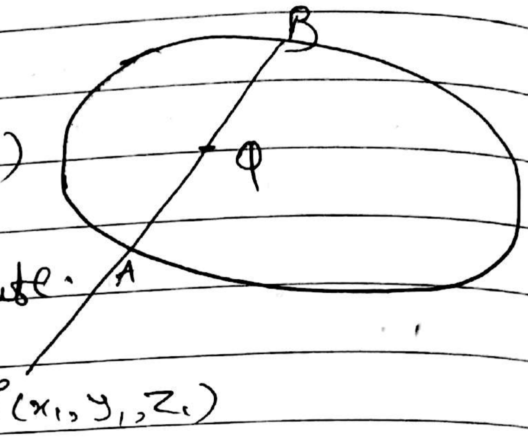
This could help in some questions of this cone.

⑨ POLAR PLANE OF A POINT

Harmonic division



P & Q cut AB in same ratio internally & externally. They are called harmonic conjugates of AB.



Let ~~chord~~ line through $P(x_1, y_1, z_1)$ ~~cut~~ ellipsoid in A & B then let Q be its harmonic conjugate.

Locus of such Q is called polar plane of P & is given by

$$axx_1 + byy_1 + czz_1 = 1$$

i.e. $T = 0$ (same as tangent)

Similarly we can find pole of a given polar plane $lx + my + nz = p$ of ~~ell~~ conicoid $ax^2 + by^2 + cz^2 = 1$

$$\left(\frac{l}{ap}, \frac{m}{bp}, \frac{n}{cp}\right)$$

⑩ 2 points such that polar planes pass through each other are called conjugate points. & planes are conjugate planes.

⑪ Polar / Conjugate lines

2 lines such that polar plane of any point on one line passes through other line are called polar lines.

For a line $\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n}$ & conicoid $ax^2 + by^2 + cz^2 = 0$

The polar line is given by

$$axx_1 + byy_1 + czz_1 = 1$$

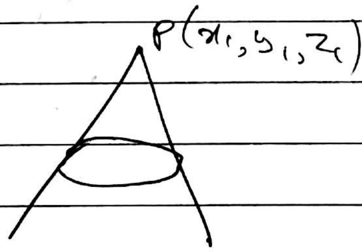
$$\& \; axl + bym + czn = 0$$

(remember as T ~~E~~ with (x_1, y_1, z_1) & T without constant for (l, m, n))

& If line PQ has points $P(x_1, y_1, z_1)$ & $Q(x_2, y_2, z_2)$ then polar of PQ is given by

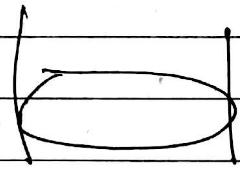
$$axx_1 + byy_1 + czz_1 = 1 \quad \& \quad axx_2 + byy_2 + czz_2 = 1$$

(12) Enveloping Cone



Eq. of cone from (x_1, y_1, z_1) enveloping conicoid $ax^2 + by^2 + cz^2 = 1$ given by $SS_1 = T^2$

(13) Enveloping Cylinder



To find enveloping cylinder of conicoid $ax^2 + by^2 + cz^2 = 1$ when given axis is $\parallel$ to $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$

We use (l, m, n) as points to get envelope in standard format let $S = ax^2 + by^2 + cz^2 - 1$

$S_1 = al^2 + bm^2 + cn^2$ (note we ~~don~~ ignore constant here)

$t = axl + bym + czn$ (〃)

$Ss_1 = t^2$ (remember we ignore constant only for S_1 & t & not for S i.e. when we put (l, m, n) ignore constant)

(14) SECTION WITH GIVEN CENTRE

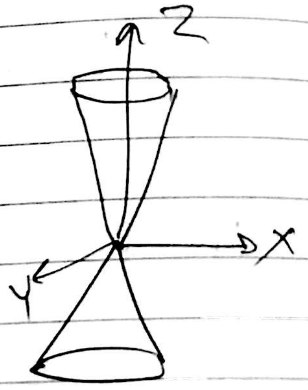
i.e. plane where any chord passing through (x_1, y_1, z_1) is bisected there.

$$T = S_1$$

i.e. $axx_1 + byy_1 + czz_1 = ax_1^2 + by_1^2 + cz_1^2$

CONE

(1) To trace cone $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$



Vertex is at origin.

It is generated by variable ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{k^2}{c^2}$$

(2) Std. equation of cone is $ax^2 + by^2 + cz^2 = 0$

It can be considered ~~centroid~~ central conicoid with vertex as centre.

Plane $lx + my + nz = 0$ touches cone if

$$\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} = 0$$

Again eq. of plane which cuts cone with conic whose centre is (x_1, y_1, z_1) given by $T = S_1$

(3) Normal plane through generator OP is a plane through OP & $\perp$ to tangent plane.

(4) e.g. P.T. locus of asymptotes drawn from origin to conicoid $ax^2 + by^2 + cz^2 = 1$ is the assymptotic cone $ax^2 + by^2 + cz^2 = 0$

$\rightarrow$ If $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ lies on conicoid then

$$al^2r^2 + bm^2r^2 + cn^2r^2 = 1$$
$$\therefore \quad al^2 + bm^2 + cn^2 = \frac{1}{r^2}$$

& asymptote $\Rightarrow$ it meets as $r \to \infty$ $\therefore$ $ax^2 + by^2 + cz^2 = 0$.

(5) P.T. Any plane whose normal lies on the cone $(b+c)x^2 + (c+a)y^2 + (a+b)z^2 = 0$ cuts the surface $ax^2 + by^2 + cz^2 = 1$ in rectangular hyperbola.

$\rightarrow$ Let plane $ux + vy + wz = 0$ cut conicoid in rectangular hyperbola.

Let asymptote of this hyperbola be

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \quad \therefore \quad ul + vm + wn = 0 \quad \text{— (1)}$$

now, (lr, mr, nr) meets conicoid in ∞.

$$\therefore \quad al^2 + bm^2 + cn^2 = \frac{1}{r^2} = 0 \quad (2)$$

Asymptotes of rect. hyperbola are $\perp$ & (1) & (2) $\Rightarrow$

$$u^2(b+c) + v^2(c+a) + w^2(a+b) = 0$$
$$\therefore \quad x^2(b+c) + y^2(c+a) + z^2(a+b) = 0$$

★ Finding eq. of director sphere of conicoid $ax^2 + by^2 + cz^2 = 1$

$\rightarrow$ Director sphere implies 3 concurrent $\perp$ tangent planes.

This doesn't mean we can draw a cone with 3 perpendicular tangents. The lines belonging to mutually $\perp$ tangent planes need not be multually $\perp$.

So, the derivation starts by assuming 3 $\perp$ tangent planes

$$l_1x + m_1y + n_1z = \left(\frac{l_1^2}{a} + \frac{m_1^2}{b} + \frac{n_1^2}{c}\right)^{1/2}$$
$$l_2x + m_2y + n_2z = \left(\frac{l_2^2}{a} + \frac{m_2^2}{b} + \frac{n_2^2}{c}\right)^{1/2}$$ & similarly with (l_3, m_3, n_3).

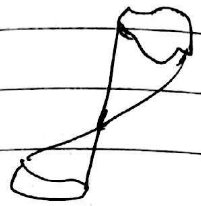
Then $\sum l_i^2 = 1$ & $\sum l_1 l_2 = 0$ & so on.

The point of director sphere satisfies all 3 equations.

& so on.

CONE in general

1. Cone is a surface generated by joining vertex to each point of the guiding curve with lines. Straight lines are called generators.



2. Eq. of cone with vertex at origin is homogeneous & converse is true.

3. Method to make non-homogeneous eq. homogeneous

(a) Introduce 4th variable t where $t=1$.

e.g. find cone with vertex as origin & guiding curve as $x^2+y^2+z^2-x-1=0$ & $x^2+y^2+z^2-y-2=0$

$\rightarrow$ $x^2+y^2+z^2-xt-t^2 = 0 = x^2+y^2+z^2-yt-2t^2$

$\therefore$ $t = x+y$

$\therefore$ $x^2+y^2+z^2-x(x+y)-(x+y)^2 = x^2+3xy-z^2=0$ is the required equation of cone.

(b) Some natural manipulations

guiding curve is $ax^2+by^2+cz^2=1$ & $lx+my+nz=p$

$\therefore$ Cone: $ax^2+by^2+cz^2 = \left(\dfrac{lx+my+nz}{p}\right)^2$

Guiding curve $ax^2+by^2=2z$ & $lx+my+nz=p$

$\therefore$ cone: $ax^2+by^2 = 2z\left(\dfrac{lx+my+nz}{p}\right)$

4. Finding eq. of cone with given vertex & guiding curve.

Vertex (α,β,γ) & conic: $ax^2+by^2+2hxy+2fy+2gx+c=0$, $z=0$

a line through vertex is $\dfrac{x-\alpha}{l} = \dfrac{y-\beta}{m} = \dfrac{z-\gamma}{n} = \dfrac{-\gamma}{n}$ ($\because z=0$)

$\therefore a\left[\alpha-\dfrac{l\gamma}{n}\right]^2 + b\left[\beta-\dfrac{m\gamma}{n}\right]^2 + 2h\left[\alpha-\dfrac{l\gamma}{n}\right]\left[\beta-\dfrac{m\gamma}{n}\right]$

$+\,2g\left[\alpha-\dfrac{l\gamma}{n}\right]+c=0$

now we replace

$\dfrac{l}{n} = \dfrac{x-\alpha}{z-\gamma}$, $\dfrac{m}{n} = \dfrac{y-\beta}{z-\gamma}$ & $\dfrac{l}{m} = \dfrac{x-\alpha}{y-\beta}$

(5) e.g. Find eq. of cone with vertex $(1,1,0)$ & guiding curve $x^2+z^2=4$, $y=0$

→ line $\frac{x-1}{l} = \frac{y-1}{m} = \frac{z}{n} = \frac{-1}{m}$ $\therefore x = 1-\frac{l}{m}$ $z = -\frac{n}{m}$

$\therefore \left(1-\frac{l}{m}\right)^2 + \frac{n^2}{m^2} = 4$ $\therefore m^2+l^2+n^2-2ml = 4m^2$

$\therefore -3m^2+l^2+n^2-2ml = 0$

$\therefore -3(y-1)^2+(x-1)^2+z^2-2(y-1)(x-1) = 0$

$\therefore x^2-3y^2+z^2-2xy+8y-4=0$

(6) Enveloping cone of a sphere $SS_1 = T^2$

(7) Cone of second degree passing through co-ordinate axes is $fyz+gzx+hxy=0$

A cone can be found to contain any 2 given set of 3 ⊥ axes as generators.

(8) Condition for general second degree equation to represent a cone

$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} = 0$$

Remember as $a \to h \to b$, $g \to f \to c$ & $u \to v \to w \to d$

& symmetric matrix.

An efficient way of checking if given eq. is cone

(a) First make it homogeneous by introducing 't' wherever necessary. Use $t=1$.

(b) Obtain 4 equations by making $\frac{dF}{dx}, \frac{dF}{dy}, \frac{dF}{dz}, \frac{dF}{dt} = 0$ each & put $t=1$

Solve any of 3 above for (x,y,z).

Put that in 4^{th} eqn. & if it is satisfied, it is a cone.

(9) Angle betⁿ 2 lines in which a plane through vertex cuts a cone:

e.g. find angle for plane: $x-3y+z=0$ & cone: $x^2-5y^2+z^2=0$

→ let line of section be $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$

We have $l-3m+n=0$ & $l^2-5m^2+n^2=0$

$\Rightarrow (4m-2n)(m-n)=0$ & get 2 lines.

(10) $ax^2+by^2+cz^2+2fyz+2gxz+2hxy+2ux+2vy+2wz+d=0$ represents a cone. (given)

Then it has 3 ⊥ generators if $a+b+c=0$

e.g. P.T. plane $ax+by+cz=0$ cuts the cone $yz+zx+xy=0$ in ⊥ lines if $1/a+1/b+1/c=0$

→ The cone has 3 ⊥ generators. ∴ ⊥ of plane will lie on cone if plane cuts in ⊥ lines.

∴ $ab+bc+ca=0$ (putting in plane eq.)

(11) Tangent Plane at any point passes through vertex & touches along whole generator.

Tangent obtained by replacing as follows

$x^2 \to xx_1$, $xy \to \frac{1}{2}(x_1y+xy_1)$, $x \to \frac{1}{2}(x+x_1)$ & so on.

(12) Condition of tangency of plane to a cone through origin

P: $lx+my+nz=0$ & cone $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$

condition is $Al^2+Bm^2+Cn^2+2Fmn+2Gln+2Hlm=0$

where A, B, C, F, G, H are cofactors in determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

(13) Reciprocal Cone

Locus of normals to tangent planes through vertex of cone is another cone called reciprocal cone.

let cone be $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$

Reciprocal is $Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0$

Again capital letters are cofactors.

(14) In locus problems, use following method

(a) Assume x', y', z' as point on the locus

(b) Present given locus condition in terms of (x', y', z')

(c) Replace it by (x, y, z)

(15) Condition for reciprocal cone through origin to have 3 $\perp$ generators is $A+B+C=0$ i.e.

$f^2+g^2+h^2 = ab+bc+ca$

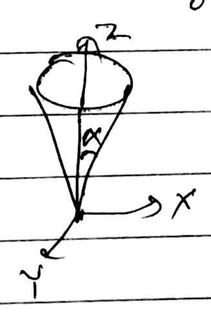
e.g. Show that gen. equation of cone which touches 3 co-ordinate planes is $\sqrt{fx} \pm \sqrt{gy} \pm \sqrt{hz} = 0$

→ Here we use if it touches coordinate planes; its reciprocal cone will have co-ordinate axes as generator

∴ Its reciprocal is $fyz+gxz+hxy=0$ & so on.

(16) Right Circular Cone

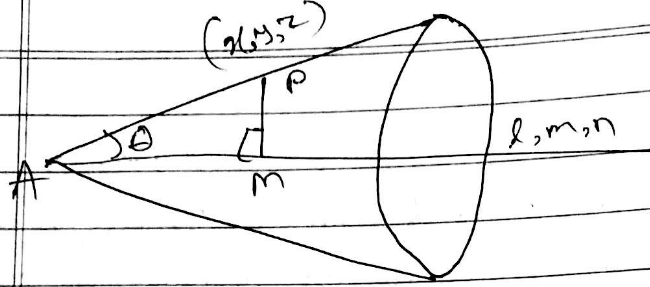
Surface generated when generators make constant angle with axis & all pass through vertex.



(a) Standard form: $x^2+y^2=z^2\tan^2\alpha$

(b) General form:- Vertex (α, β, γ), semi vertical angle θ & d.c. of axis (l, m, n)

$$\left[l(x-\alpha)+m(y-\beta)+n(z-\gamma)\right]^2 = \left((x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2\right)\cos^2\theta$$



use this trigonometry in all questions involving right circular cone.

$AM = AP\cos\theta$

Semi vertical angle of right circular cone with $3 \perp$ generators is $\tan^{-1}\sqrt{2}$.

(17) e.g. Lines are drawn from origin with d.c.s proportional to $(1,2,2)$, $(2,3,6)$ & $(3,4,12)$. Find d.c.s of axis of right circular cylinder through them & prove semi vertical angle is $\cos^{-1}(1/\sqrt{3})$.

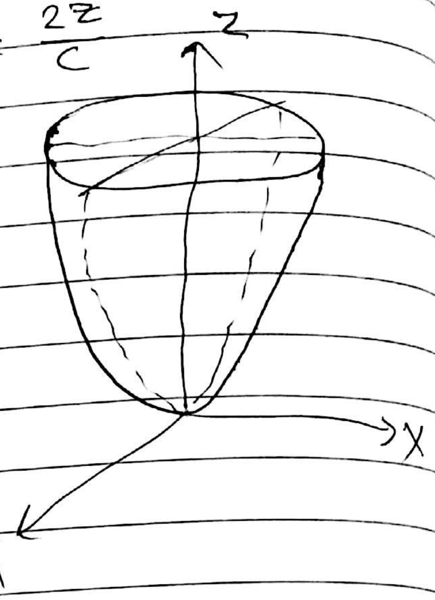
→ Use that if α is the angle;

$$\cos\alpha = \frac{1}{3}l + \frac{2}{3}m + \frac{2}{3}n = \frac{2}{7}l + \frac{3}{7}m + \frac{6}{7}n = \frac{3}{13}l + \frac{4}{13}m + \frac{12}{13}n$$

Solve 3 l,m,n eq. & get α also.

PARABOLOID

① Std. elliptic paraboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2z}{c}$



Simply generated by ellipses formed by $z=k$ planes.

② Std. Hyperbolic Paraboloid

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{2z}{c}$$

Sections by coordinate planes are downward & upward parabolas.

Generated by variable hyperbolas formed by $z=k$ planes. Bit difficult to draw figure.

③ General equation of paraboloid is

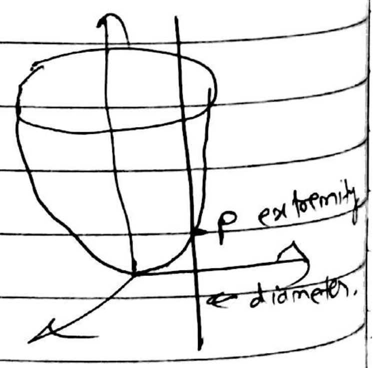
$$ax^2+by^2=2z$$

Elliptic if sign of a & b is same. Hyperbolic if sign of a & b is opposite.

Every line meets paraboloid in 2 points. ∴ Section by a plane is a conic.

④ A line ‖ to z-axis is called a diameter of paraboloid. The diameter cuts paraboloid in one point called its extremity & 2nd point at ∞.

∴ Diameter which is ⊥ to tangent plane at extremity is the axis & that extremity is the vertex.



(5) Tangent plane is $axx_1 + byy_1 = z + z_1$
This is also the polar plane.
& Enveloping cone is given by $SS_1 = T^2$
Plane section with given centre is $T = S_1$

(6) e.g. Find locus of points from which 3 ⊥ generators can be drawn to paraboloid $ax^2 + by^2 = 2z$
→ Find enveloping cone $SS_1 = T^2$. Put condition that $a+b+c=0$ for 3 ⊥ generators. Get locus equation.

(7) From a given point, 5 normals can be drawn to a paraboloid. In all questions concerning foot of normal, convert it to λ & solve. i.e.

$$x = \frac{a^2\alpha}{a^2+\lambda} \qquad y = \frac{b^2\beta}{b^2+\lambda} \qquad z = \gamma + \lambda$$

CONJUGATE DIAMETERS

① Let a system of ∥ chords of $ax^2+by^2+cz^2$ have d.c.s (l,m,n) & if (α,β,γ) be midpt. of any one of them, then locus of midpoints of ∥ chords is given by

$$al\alpha + bm y + cnz = 0$$

This plane passes through origin.

This plane is called diametral plane conjugate to d.c.s (l,m,n).

Also if given plane is $px+qy+rz=0$ then corresponding (l,m,n) satisfy

$$\frac{al}{p} = \frac{bm}{q} = \frac{cn}{r}$$

(∵ Every central plane is diametral plane for some direction)

② Following Conjugate Diameter discussion is confined to ellipsoids only.

* Diametral plane of OP :- Let P be any point on ellipsoid, then plane joining midpoints of chords ∥ to OP is called diametral plane of OP.

* Conjugate Semi-diameters
3 semi-diameters of ellipsoid are said to be conjugate if plane containing any 2 of them is diametral plane of 3^{rd} semi-diameter.

* Conjugate Planes
Any 3 planes are called conjugate planes if intersection of any 2 planes gives line whose diametral plane is the 3^{rd} plane.

(3) Useful relation betⁿ extremities of 3 conjugate-semi diameters if eq. of ellipsoid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ & extremities are (x_1,y_1,z_1), (x_2,y_2,z_2) & (x_3,y_3,z_3)

Here $\left(\frac{x_i}{a},\frac{y_i}{b},\frac{z_i}{c}\right)$ act like 3 ⊥ coordinate axes & have similar relations.

i.e. $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=1$

$\frac{x_1^2}{a^2}+\frac{x_2^2}{a^2}+\frac{x_3^2}{a^2}=1$ i.e. $x_1^2+x_2^2+x_3^2=a^2$

Similarly $\Sigma y_1^2=b^2$ $\Sigma z_1^2=c^2$

$\frac{x_1y_1}{ab}+\frac{x_2y_2}{ab}+\frac{x_3y_3}{ab}=0$ & $\Sigma x_1z_1=0$ & $\Sigma y_1z_1=0$

Remember we are NOT saying 3 semidiameters are ⊥ to each other; $\left(\frac{x_i}{a},\frac{y_i}{b},\frac{z_i}{c}\right)$ form ⊥ axes.

(4) Sum of squares of 3 conjugate semi diameters of ellipsoid is constant $= a^2+b^2+c^2$

Sum of squares of projection of 3 semi-diameters on any line or plane is constant.

(5) e.g. Find eq. of plane passing through extremities of conjugate diameters.

→ let eq. be $lx+my+nz=p$ ∴ $lx_1+my_1+nz_1=p$ & for other 2 pts.

Multiplication by x_1,x_2,x_3 & adding gives

$l=\frac{p}{a^2}(x_1+x_2+x_3)$ similarly get m & n.

Then eq. is

$\frac{x}{a^2}(x_1+x_2+x_3)+\frac{y}{b^2}(y_1+y_2+y_3)+\frac{z}{c^2}(z_1+z_2+z_3)=1$

REDUCTION OF GEN. EQ TO STD. FORM

(1) Some General Concepts First

General Eq.

$$F(x,y,z) = ax^2+by^2+cz^2+2fyz+2gxz+2hxy + 2ux+2vy+2wz+d = 0$$

The central planes are obtained by making

$$\frac{dF}{dx}=0, \quad \frac{dF}{dy}=0 \;\&\; \frac{dF}{dz}=0$$

The centre (α, β, γ) is given by intersection of above 3 planes. i.e.

$$ax+hy+gz+u=0$$
$$hx+by+fz+v=0$$
$$gx+fy+cz+w=0$$

Now, above 3 planes could have a point, line or plane in common. Hence we could have centre point, a line of centres or a plane of centres

(2) So when we transfer origin to a centre, we get equation closer to standard form i.e.

$$ax^2+by^2+cz^2+2fyz+2gxz+2hxy+d'=0$$

where $d' = u\alpha+v\beta+w\gamma+d$ $\quad (\alpha,\beta,\gamma)$ is centre.

(remember it is not $2u\alpha$ ~~but~~ $u\alpha$; but d remains same)

then w… want…

(3) Now we get even closer to standard eq. by transforming axes suitable.

We can get eq. as $\lambda_1 x^2+\lambda_2 y^2+\lambda_3 z^2+d'=0$

where λ are solⁿ of

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0$$

The new axes are formed by solving for (l_1, m_1, n_1) from

$(a-\lambda_1)l_1+hm_1+gn_1=0$, $hl_1+(b-\lambda_1)m_1+fn_1=0$ & $gl_1+fm_1+(c-\lambda_1)n_1=0$

Similarly putting λ_2 & λ_3 & solving gives other 2 axes.

This determinant is called discriminating cubic.

Reduction to canonical form in short

case 1, 2, 3

(1) First find $\lambda_1, \lambda_2, \lambda_3$ which are roots of $\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0$

If all are non-zero, find centre α, β, γ given by solving

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$$

Then use $d' = u\alpha + v\beta + w\gamma + d$ (note it is $u\alpha$ & not $2u\alpha$ like in eq.)

The eq. in canonical form is $\lambda_1 x^2 + \lambda_2 y^2 + \lambda_3 z^2 + d' = 0$

The new axes are given by solving for l, m, n

$$(a-\lambda_i)l + hm + gn = 0$$
$$hl + (b-\lambda)m + fn = 0$$
$$gl + fm + (c-\lambda)n = 0$$

(2) If one $\lambda_3 = 0$; find $D^* = \begin{vmatrix} u & v & w \\ h & b & f \\ g & f & c \end{vmatrix}$ In 2nd step (case 2) both are 0; In 3rd step only λ_3 is 0.

then we know for sure it has $\lambda_1 x^2 + \lambda_2 y^2$; we now only want to know whether z is present or not; for the D^*

If it is 0, then find any 1 solⁿ of $\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$ (It will have ∞ solⁿ here)

Again for this solⁿ (α, β, γ) find $d' = \alpha u + \beta v + \gamma w + d$

Then eq. is $\lambda_1 x^2 + \lambda_2 y^2 + d' = 0$

Axes found in same way.

(3) If determinant D^* is non zero; find axes l_3, m_3, n_3 corresponding to $\lambda_3 = 0$ then $K = ul_3 + vm_3 + wn_3$

& eq. is $\lambda_1 x^2 + \lambda_2 y^2 + 2kz = 0$

If in any case $\lambda_1 = \lambda_2$; it is conicoid of revolution.

Case 4 & Case 5

1) See if homogeneous part forms a perfect square. Then see if whole eq. can be shown as a quadratic in P where P is eq. of a plane.

$4x^2+9y^2+36z^2-36yz+24zx-12xy-10x+15y-30z+6=0$

$\therefore (2x-3y+6z)^2 = 5(2x-3y+6z)-6$

$\therefore (X-\frac{2}{7})(X-\frac{3}{7})=0$ $\quad \therefore$ Parallel planes $\left(X = \frac{2x-3y+6z}{7}\right)$

If this is not possible, it is a paraboloid.

Introduce λ so that you can convert it to $Y^2 = 4aX$.

For that make sure λ gives planes on LHS $\perp$ to plane on RHS.

$x^2+y^2+z^2-2yz+2zx-2xy+x-4y-z+1=0$

$(x-y+z)^2 = -x+4y+z-1$

$(x-y+z+\lambda)^2 = x(-1+2\lambda)+y(4-2\lambda)+z(1+2\lambda)-1+\lambda^2$

or $\perp$ we need

$(-1+2\lambda)-(4-2\lambda)+(1+2\lambda)=0 \Rightarrow \lambda = 1$

$\therefore 3\left[\frac{x-y+z+1}{\sqrt{3}}\right]^2 = \sqrt{6}\left[\frac{x+2y+z}{\sqrt{6}}\right]$ i.e. $3X^2 = \sqrt{6}\,Y$

2) In both cases, remember to mention planes in $lx+my+nz-p=0$ format finally by division by $\sqrt{a^2+b^2+c^2}$. This is to ensure you get X, Y as proper distances from $\perp$ planes.

Summary of generators

① hyperboloid of 1 sheet

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$$

Generator given by $\frac{x}{a}-\frac{z}{c}=\frac{1}{\lambda}\left(1-\frac{y}{b}\right)$ & $\frac{x}{a}+\frac{z}{c}=\lambda\left(1+\frac{y}{b}\right)$

and $\frac{x}{a}-\frac{z}{c}=\mu\left(1+\frac{y}{b}\right)$ & $\frac{x}{a}+\frac{z}{c}=\frac{1}{\mu}\left(1-\frac{y}{b}\right)$

So we have these 2 systems.

② Properties of these 2 systems
One generator of each system passes through every point of hyperboloid.

No 2 lines of same system intersect.

Any 2 generator of different system intersect.

③ Similarly
Generators of hyperbolic paraboloid

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z$$

$\rightarrow$ $\left(\frac{x}{a}+\frac{y}{b}\right)=\lambda z$, $\left(\frac{x}{a}-\frac{y}{b}\right)=\frac{2}{\lambda}$

& $\left(\frac{x}{a}+\frac{y}{b}\right)=\frac{2}{\mu}$, $\left(\frac{x}{a}-\frac{y}{b}\right)=\mu z$

Same properties.

4) Condition for a given line to be generator of given conicoid.

Let line be $\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n}$ & conicoid $ax^2+by^2+cz^2=1$

$\therefore$ A point on line is $(\alpha+lt, \beta+mt, \gamma+nt)$;

putting it on conicoid. This should be satisfied for all t.

$$a(\alpha+lt)^2 + b(\beta+mt)^2 + c(\gamma+nt)^2 = 1$$

$$\Rightarrow t^2(al^2+bm^2+cn^2) + 2t(al\alpha+bm\beta+cn\gamma) + (a\alpha^2+b\beta^2+c\gamma^2-1) = 0$$

for above to be true $\forall t$; coefficients $=0$

$$al^2+bm^2+cn^2=0$$

$$al\alpha+bm\beta+cn\gamma=0$$

$$a\alpha^2+b\beta^2+c\gamma^2=1$$

Quite intuitive.

Misc. Tips

(1) In questions involving coefficients like

$$\frac{yz}{q-r} + \frac{zx}{r-p} + \frac{xy}{p-q} = 0$$

→ It is easier to deal with $a = \frac{1}{q-r}$ $b = \frac{1}{r-p}$ $c = \frac{1}{p-q}$

$\therefore \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$ i.e. $ab + bc + ca = 0$ & so on.