CSE-2014 → Paper II

5) (b) Apply Newton Raphson-Method to determine a root of the equation $\cos x - xe^x = 0$ correct upto four decimal places.

⇒ Here, $f(x) = \cos x - xe^x$, ⇒ $f'(x) = -\sin x - e^x - xe^x$

Now, $f(0) = 1 > 0$ and $f(1) = -2.17798 < 0$

∴ one real root of $f(x) = 0$ lie between 0 and 1

taking $x_0 = 0$, the successive approximations of the root are computed in the following table:

n	x_n	$f(x_n)$	$f'(x_n)$	$h_n = -\frac{f(x_n)}{f'(x_n)}$	$x_{n+1} = x_n + h_n$
0	0	1	−1	1	1
1	1	−2.17798	−6.27803	−0.34692	0.65308
2	0.65308	−0.46064	−3.78395	−0.12174	0.58134
3	0.58134	−0.20396	−3.37727	−0.06039	0.52095
4	0.52095	−0.00974	−3.05842	−0.00318	0.51777
5	0.51777	−0.00004	−3.04219	−0.00001	0.51776

∴ The approximate root is, 0.5178, Correct upto four decimal places.

(c) Use five subintervals to integrates $\int_0^1 \frac{dx}{1+x^2}$ using Trapezoidal rule.

⇒ Here, $f(x) = \frac{1}{1+x^2}$, $a = 0$, $b = 1$, $n = 5$ ⇒ $h = \frac{1-0}{5}$

$= 0.2$

x_i ($i=0$ to 5)	y_i ($i=0$ to 5)	y_i $i=0,5$	y_i $i=1,2,3,4$
0.0	1.00000	1.00000	—
0.2	0.96154	—	0.96154
0.4	0.86207	—	0.86207
0.6	0.73529	—	0.73529
0.8	0.60976	—	0.60976
1.0	0.50000	0.50000	—
		$\Sigma y_i = 1.50000 (= Y_1)$	$\Sigma y_i = 3.16866 (= Y_2)$

Now, by the Trapezoidal Rule,

$$\int_0^1 \frac{dx}{1+x^2} = \frac{h}{2}\left[(y_0+y_5)+2(y_1+y_2+y_3+y_4)\right]$$

$$= \frac{h}{2}\left[Y_1 + 2Y_2\right]$$

$$= \frac{0.2}{2}\left[1.50000 + 2\times 3.16866\right]$$

$$= 0.783732$$

$= 0.7837$ (correct upto four decimal place)

6)(b) Solve the system of equations,

$$2x_1 - x_2 = 7$$
$$-x_1 + 2x_2 - x_3 = 1$$
$$-x_2 + 2x_3 = 1$$

using Gauss-Seidel iteration method (perform 3-iteration).

$\Rightarrow$ Given system of equations is diagonally dominant,

Now, we ~~write~~ re-arrange the system as,

$$x_1^{(k+1)} = \frac{1}{2}\left[7 + x_2^{(k)}\right]$$
$$x_2^{(k+1)} = \frac{1}{2}\left[1 + x_1^{(k+1)} + x_3^{(k)}\right]$$
$$x_3^{(k+1)} = \frac{1}{2}\left[1 + x_2^{(k+1)}\right]$$

we take the initial guess values as: $x_1^{(0)} = 0, x_2^{(0)} = 0, x_3^{(0)} = 0$

$\therefore$
$$\left.\begin{aligned} x_1^{(1)} &= \tfrac{1}{2}[7+0] = 3.5 \\ x_2^{(1)} &= \tfrac{1}{2}[1+3.5+0] = 2.25 \\ x_3^{(1)} &= \tfrac{1}{2}[1+2.25] = 1.625 \end{aligned}\right\} \text{ when } k=0$$

$$\left.\begin{aligned} x_1^{(2)} &= \tfrac{1}{2}[7+2.25] = 4.625 \\ x_2^{(2)} &= \tfrac{1}{2}[1+4.625+1.625] = 3.625 \\ x_3^{(2)} &= \tfrac{1}{2}[1+3.625] = 2.3125 \end{aligned}\right\} \text{ when } k=1$$

$$x_1^{(3)} = \frac{1}{2}[7+3.625] = 5.3125$$
$$x_2^{(3)} = \frac{1}{2}[1+5.3125+2.3125] = 4.3125$$ when $k=2$
$$x_3^{(3)} = \frac{1}{2}[1+4.3125] = 2.65625$$

$$x_1^{(4)} = \frac{1}{2}[7+4.3125] = 5.6563$$
$$x_2^{(4)} = \frac{1}{2}[1+5.6563+2.65625] = 4.656275$$ when $k=3$
$$x_3^{(4)} = \frac{1}{2}[1+4.656275] = 2.8281375$$

$$x_1^{(5)} = \frac{1}{2}[7+4.656275] = 5.8281375$$
$$x_2^{(5)} = \frac{1}{2}[1+5.8281375+2.8281375] = 4.8281375$$ when $k=4$
$$x_3^{(5)} = \frac{1}{2}[1+4.8281375] = 2.91406875$$

$$x_1^{(6)} = \frac{1}{2}[7+4.8281375] = 5.91406875$$
$$x_2^{(6)} = \frac{1}{2}[1+5.91406875+2.91406875] = 4.91406875$$ when $k=5$
$$x_3^{(6)} = \frac{1}{2}[1+4.91406875] = 2.957034375$$

$\therefore$ the approximate solution of the given system of equation is $x_1=6$, $x_2=5$, $x_3=3$

6)(c) Use Runge-kutta formula of fourth order to find the value of y at $x=0.8$, where $\frac{dy}{dx}=\sqrt{x+y}$, $y(0.4)=0.41$, take the step length $h=0.2$

$\Rightarrow$ Here $f(x,y)=\sqrt{x+y}$

For $y(0.6)$: $x_0=0.4$, $y_0=0.41$, $h=0.2$

$$\therefore K_1 = hf(x_0,y_0) = 0.2\times f(0.4,0.41) = 0.18$$
$$K_2 = hf(x_0+\tfrac{h}{2}, y_0+\tfrac{K_1}{2}) = 0.2\times f(0.5,0.50) = 0.20$$
$$K_3 = hf(x_0+\tfrac{h}{2}, y_0+\tfrac{K_2}{2}) = 0.2\times f(0.5,0.51) = 0.200998$$
$$K_4 = hf(x_0+h, y_0+K_3) = 0.2\times f(0.6,0.610998) = 0.220091$$

$$\therefore K = \frac{1}{6}(K_1+2K_2+2K_3+K_4)$$
$$= \frac{1}{6}\times[1.202087]$$
$$= 0.20034783$$

$\therefore y_1 = y(0.6) = y_0 + K$

$= 0.41 + 0.20034783$

$= 0.61034783$

For $y(0.8)$: $x_1 = 0.6$, $y_1 = 0.61034783$, $h = 0.2$

$K_1 = hf(x_1, y_1) = 0.2 f(0.6, 0.61034783) = 0.2200316$

$K_2 = hf(x_1 + \frac{h}{2}, y_1 + \frac{K_1}{2}) = 0.2(0.7, 0.7203636) = 0.238358$

$K_3 = hf(x_1 + \frac{h}{2}, y_1 + \frac{K_2}{2}) = 0.2f(0.7, 0.729527) = 0.239126$

$K_4 = hf(x_1 + h, y_1 + K_3) = 0.2f(0.8, 0.849474) = 0.256864$

$\therefore y_2 = y(0.8) = y_1 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$

$= 0.61034783 + \frac{1}{6} \times 1.4318636$

$= 0.61034783 + 0.23864393$

$= 0.848992$

$\therefore$ The value of y at $x = 0.8$ is 0.849, correct upto three decimal places.