5(a) Construct a PDE of all surfaces of revolution having the z-axis as the axis of rotation. (8) (IAS 1997)

Sol: From the coordinate geometry of three dimensions, equation of any surface of revolution having z-axis as the axis of rotation may be taken as

$$z = \phi[(x^2+y^2)^{1/2}] \qquad \text{— ①}$$

where ϕ is an arbitrary function.

Differentiating (1) partially w.r.t x and y,

$$p = \frac{\partial z}{\partial x} = \phi'[(x^2+y^2)^{1/2}]\,\frac{1}{2}(x^2+y^2)^{-1/2}.2x$$

$$q = \frac{\partial z}{\partial y} = \phi'[(x^2+y^2)^{1/2}]\,\frac{1}{2}(x^2+y^2)^{-1/2}.2y$$

Dividing these two,

$$\frac{p}{q} = \frac{x}{y} \quad \text{or} \quad \boxed{py = qx}$$

which is the required PDE.

5(b) Using Newton-Raphson method, find the value of $(37)^{1/3}$, correct to four decimal places. (8)

Sol: Let $x = (37)^{1/3}$

We consider the function,

$$f(x) \equiv x^3 - 37 = 0$$

By Newton-Raphson Method,

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^3 - 37}{3x_n^2} = \frac{2x_n^3 + 37}{3x_n^2}$$

We note that,

$f(3) = -10$ & $f(4) = 27$

Let $x_0 = 3$, then

i) for $n=0$,

$$x_1 = \frac{2x_0^3 + 37}{3x_0^2} = 3.37037$$ [use Calci]

ii) for $n = 1$,

$$x_2 = \frac{2x_1^3 + 37}{3x_1^2} = 3.33265$$

iii) for $n = 2$, $x_3 = 3.33222$

iv) for $n = 3$, $x_4 = 3.33222$

} Same upto 5 places

∴ Value of $(37)^{1/3}$, correct to four decimal places is

$$(37)^{1/3} = 3.3322$$

5(c) (i) Convert $(14231)_8$ into an equivalent binary number and then find the equivalent decimal number.

1	4	2	3	1
001	100	010	011	001

$\therefore (14231)_8 = (1100010011001)_2$

12 11 10 9 8 7 6 5 4 3 2 1 0

And, for decimal,

$1 \times 2^{12} + 1 \times 2^{11} + 1 \times 2^{7} + 1 \times 2^{4} + 1 \times 2^{3} + 1 \times 2^{0}$

$= 4096 + 2048 + 128 + 16 + 8 + 1$

$= (6297)_{10}$ [verify with Calci]

(ii) Convert $(43503)_{10}$ into an equivalent binary number and then find the equivalent hexadecimal number.

$\therefore (43503)_{10}$

$= (1010100111101111)_2$

2	43503	
2	21751	1
2	10875	1
2	5437	1
2	2718	1
2	1359	0
2	679	1
2	339	1
2	169	1
2	84	1
2	42	0
2	21	0
2	10	1
2	5	0
2	2	1
	1	0

Hexadecimal System

$\underbrace{1010}_{A}\ \underbrace{1001}_{9}\ \underbrace{1110}_{E}\ \underbrace{1111}_{F}$

$\therefore (43503)_{10} = (A9EF)_{16}$

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5(d) Find the condition on a, b, c (real numbers) such that the dynamical system with equations

$$\dot{p} = aq - q^2, \quad \dot{q} = bp + cq \text{ is Hamiltonian.}$$

Compute also the Hamiltoniain of the system. (8)

Sol: Given,

$$\left.\begin{aligned} \dot{p} &= aq - q^2 \\ \dot{q} &= bp + cq \end{aligned}\right\} \quad \text{---(1)}$$

As,

$$\left.\begin{aligned} \dot{p} &= \frac{dP}{dt} = -\frac{\partial H}{\partial q} \\ \dot{q} &= \frac{dq}{dt} = \frac{\partial H}{\partial p} \end{aligned}\right\} \quad \text{---(2)}$$

$$\therefore \quad -\frac{\partial H}{\partial q} = aq - q^2$$

Integrating,

$$H = \frac{q^3}{3} - \frac{aq^2}{2} + f(P)$$

Differentiating partially w.r.t. p

$$\frac{\partial H}{\partial p} = f'(p)$$

$$\Rightarrow \quad f'(p) = bp + cq \quad [\text{using (1) \& (2)}]$$

As, $f'(p)$ is function of p only

$$\therefore \quad c = 0$$

i.e. $f'(p) = bp \Rightarrow f(p) = \frac{bp^2}{2}$

$$\therefore \quad \boxed{H = \frac{q^3}{3} - \frac{aq^2}{2} + \frac{bp^2}{2}} \quad \& \quad c = 0.$$

5(e) Find the general solution of the PDE
$p\tan x + q\tan y = \tan z$

(8).

Sol: Lagrange's auxiliary equations are

$$\frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z} \quad \text{—— (1)}$$

Taking first two parts of (1)

$$\cot x\,dx - \cot y\,dy = 0$$

Integrating, $\log\sin x - \log\sin y = \log c_1$

or $$\frac{\sin x}{\sin y} = c_1 \quad \text{—— (2)}$$

Taking last two parts of (1)

$$\cot y\,dy - \cot z\,dz = 0$$

Integrating, $\log\sin y - \log\sin z = \log c_2$

or $$\frac{\sin y}{\sin z} = c_2 \quad \text{—— (3)}$$

From (2) and (3), the required general solution is

$$\frac{\sin x}{\sin y} = \phi\left(\frac{\sin y}{\sin z}\right),$$

ϕ being an arbitrary function.