2016 - MECHANICS - CSE

5.c No forces $\Rightarrow V = c$ —(1)

No forces $\Rightarrow$ velocity is also constant

$$\Rightarrow T = \frac{1}{2} m v^2 = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2), \quad \text{—(2)}$$

where $\dot{x} = \frac{x}{t}, \dot{y} = \frac{y}{t}, \dot{z} = \frac{z}{t}$

Lagrangian $(L) = T - V$

$$\boxed{L = \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - c}$$

Hamilton's Principle Func. $(S) = \int_0^t L \, dt$

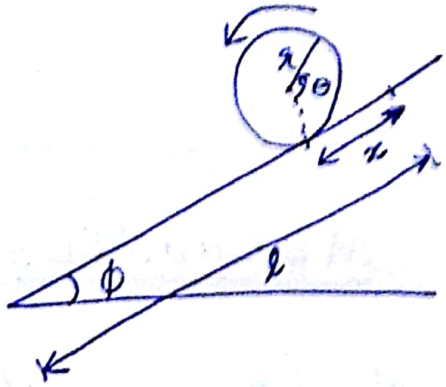
$$= L \int_0^t dt$$

$$= \left\{ \frac{1}{2} m (\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - c \right\} \times t$$

$$S = \frac{m}{2} \left(\frac{x^2}{t} + \frac{y^2}{t} + \frac{z^2}{t} \right) - ct$$

8.b Constraint $\rightarrow r\theta = x$

$$\Rightarrow r\dot{\theta} = \dot{x} \quad -(1)$$



$$T = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\{r^2\}\dot{\theta}^2$$

$$= \cancel{4}\, m\dot{x}^2 \qquad (\text{from } (1))$$

$$V = -mgx\sin\phi$$

Lagrangian $(L) = T - V$

$$\boxed{L = m\dot{x}^2 + mgx\sin\phi}$$

Lagrange's Eq. of Motion,

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$$

$$\Rightarrow 2m\ddot{x} - mg\sin\phi = 0$$

$$\Rightarrow \boxed{\ddot{x} = \frac{g}{2}\sin\phi}$$

Multiplying $2\left(\frac{dx}{dt}\right)$, and integrating,

$$\Rightarrow \left(\frac{dx}{dt}\right)^2 = \underline{gx\sin\phi + C = v^2}$$

Initial velocity $= 0$, at $x = 0$

$$\Rightarrow C = 0$$

$$\therefore \boxed{v^2 = gx\sin\phi}$$

At $\underline{x = l}$, $\underline{\underline{v = \sqrt{gl\sin\phi}}}$

 Raj Aryan

5.(b) Does a fluid with velocity $\vec{q} = \left[z - \frac{2x}{r},\ 2y - 3z - \frac{2y}{r},\ x - 3y - \frac{2z}{r}\right]$ posses vorticity, where $\vec{q}(u,v,w)$ is the velocity in the Cartesian frame, $\vec{r} = (x,y,z)$ and $r^2 = x^2+y^2+z^2$? What is the circulation in the circle $x^2+y^2=9$, $z=0$? (10)

Sol: Given $\vec{q} = \vec{q}(u,v,w) = \left[z - \frac{2x}{r},\ 2y - 3z - \frac{2y}{r},\ x - 3y - \frac{2z}{r}\right]$

i.e, $u = z - \frac{2x}{r}$, $v = 2y - 3z - \frac{2y}{r}$, $w = x - 3y - \frac{2z}{r}$ — ①

Let $\vec{\Omega} = \Omega_x \hat{i} + \Omega_y \hat{j} + \Omega_z \hat{k}$ be the vorticity vector

Thus, $\vec{\Omega} = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right)\hat{i} + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\hat{j} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{k}$ — ②

As $r^2 = x^2+y^2+z^2$

∴ from ①, we have

$u = z - \frac{2x}{\sqrt{x^2+y^2+z^2}}$, $v = 2y - 3z - \frac{2y}{\sqrt{x^2+y^2+z^2}}$, $w = x - 3y - \frac{2z}{\sqrt{x^2+y^2+z^2}}$

so that

$$\frac{\partial u}{\partial y} = -2x\left\{-\frac{1}{2}\frac{2y}{(x^2+y^2+z^2)^{3/2}}\right\} \quad ; \quad \frac{\partial u}{\partial z} = 1 - 2x\left\{-\frac{1}{2}\frac{2z}{(x^2+y^2+z^2)^{3/2}}\right\}$$

$$\frac{\partial v}{\partial x} = -2y\left\{-\frac{1}{2}\frac{2x}{(x^2+y^2+z^2)^{3/2}}\right\} \quad ; \quad \frac{\partial v}{\partial z} = -3 - 2y\left\{-\frac{1}{2}\frac{2z}{(x^2+y^2+z^2)^{3/2}}\right\}$$

$$\frac{\partial w}{\partial x} = 1 - 2z\left\{-\frac{1}{2}\frac{2x}{(x^2+y^2+z^2)^{3/2}}\right\} \quad ; \quad \frac{\partial w}{\partial y} = -3 - 2z\left\{-\frac{1}{2}\frac{2y}{(x^2+y^2+z^2)^{3/2}}\right\}$$

using the above values in ②, we get

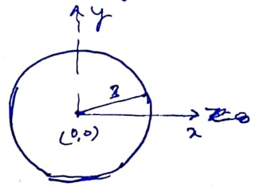
$$\Omega = 0\hat{i} + 0\hat{j} + 0\hat{k} = 0$$

The vorticity vector of the fluid motion is zero, so fluid doesn't posses vorticity

And the flow is irrotational.

IInd part Circulation in the circle $x^2+y^2=9,\ z=0$

Given $u = z - \frac{2x}{r}$, $v = 2y - 3z - \frac{2y}{r}$, $w = x - 3y - \frac{2z}{r}$ —(i)



Let $x = 3\cos\theta,\ y = 3\sin\theta,\ z = 0$ —(ii)

so that
$$\left.\begin{aligned} dx &= -3\sin\theta\, d\theta, \\ dy &= 3\cos\theta\, d\theta \\ dz &= 0 \end{aligned}\right\} \text{(iii)}$$

The circulation around the circle ~~is given~~ is can be given by

$$\Gamma = \int_c \vec{q}\cdot d\vec{r} = \int_0^{2\pi} \vec{q}\cdot d\vec{r}$$

$$= \int_0^{2\pi} u\,dx + v\,dy + w\,dz$$

$$= \int_0^{2\pi} (0 - 2\cos\theta)(-3\sin\theta\, d\theta) + (6\sin\theta - 2\sin\theta)(3\cos\theta\, d\theta) + 0$$

$$= \int_0^{2\pi} 6\sin\theta\cos\theta\, d\theta + 12\sin\theta\cos\theta\, d\theta$$

$$= 9\int_0^{2\pi} (2\sin\theta\cos\theta)\, d\theta$$

$$= 9\int_0^{2\pi} \sin 2\theta\, d\theta$$

Hence, circulation = 0

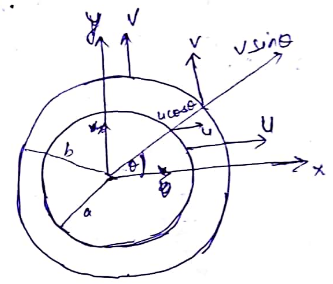
$$= 0$$

Thus curl q is zero everywhere inside the circle.

7. (b) The space between two concentric spherical shells of radii a, b $(a<b)$ is filled with a liquid of density ρ. If the shells are set in motion, the inner one with velocity U in the x-direction and outer one with velocity V in the y-direction, then show that the initial motion of the liquid is given by velocity potential

$$\phi = \frac{\left\{a^3 U\left(1+\frac{1}{2}b^3 r^{-3}\right)x - b^3 V\left(1+\frac{1}{2}a^3 r^{-3}\right)y\right\}}{b^3-a^3}$$

where $r^2 = x^2+y^2+z^2$, the co-ordinates being rectangular. Evaluate the velocity at any point of the liquid. (20)



Sol: Since the motion is irrotational, consequently the velocity potential ϕ exists such that $\nabla^2\phi = 0$ ... (1)

The Boundary conditions for ϕ are

(i) $\left(-\frac{\partial\phi}{\partial r}\right)_{r=a} = U\cos\theta$ ... (2)

(ii) $\left(-\frac{\partial\phi}{\partial r}\right)_{r=b} = V\sin\theta$ ... (3)

Above considerations suggest that ϕ must involve terms containing $\sin\theta$ and $\cos\theta$.

So, let us assume that

$$\phi = \left(Ar + \frac{B}{r^2}\right)\cos\theta + \left(Cr + \frac{D}{r^2}\right)\sin\theta \quad \ldots (4)$$

so that $\frac{\partial\phi}{\partial r} = \left(A - \frac{2B}{r^3}\right)\cos\theta + \left(C - \frac{2D}{r^3}\right)\sin\theta$ ... (5)

using (2), (3) and (5), we get

$$-\left(A - \frac{2B}{a^3}\right)\cos\theta - \left(C - \frac{2D}{a^3}\right)\sin\theta = U\cos\theta \quad \ldots (6)$$

$$\ldots (7)$$

and $-\left(A - \frac{2B}{b^3}\right)\cos\theta - \left(C - \frac{2D}{b^3}\right)\sin\theta = V\sin\theta$

Solving (6) and (7), we have

$A - \frac{2B}{a^3} = -U$ ... (a) ; $\left(C - \frac{2D}{a^3}\right) = 0$ ... (b)

and $A - \frac{2B}{b^3} = 0$ ... (c) ; $\left(C - \frac{2D}{b^3}\right) = -V$ ... (d) (a), (b), (c), (d), we get

$$= -\frac{ua^3}{a^3-b^3}, \quad B = -\frac{v\,a^3b^3}{2(a^3-b^3)}$$

$$C = \frac{-vb^3}{b^3-a^3}, \quad D = \frac{-v\,a^3b^3}{2(b^3-a^3)}$$

above values in eqn. (6), we get

$$\phi = \frac{-ua^3}{a^3-b^3}\left(r+\frac{b^3}{2r^2}\right)\cos\theta - \frac{vb^3}{b^3-a^3}\left(r+\frac{a^3}{2r^2}\right)\sin\theta \quad \text{--- (8)}$$

$$= \frac{1}{b^3-a^3}\, a^3u\left(1+\frac{b^3}{2r^3}\right)r\cos\theta - \frac{b^3v}{b^3-a^3}\left(1+\frac{a^3}{2r^3}\right)r\sin\theta$$

$$\phi = \frac{\{a^3u\left(1+\frac{1}{2}b^3r^{-3}\right)x - b^3v\left(1+\frac{1}{2}a^3r^{-3}\right)y}{b^3-a^3} \quad \text{--- (9)}$$ using $x = r\cos\theta$, $y = r\sin\theta$

Proved

calculation of velocity at any point

from (8),

$$\phi = \frac{ua^3}{b^3-a^3}\left(r+\frac{b^3}{2}r^{-2}\right)\cos\theta - \frac{vb^3}{b^3-a^3}\left(r+\frac{a^3}{2}r^{-2}\right)\sin\theta.$$

$$\therefore q_r = -\frac{\partial\phi}{\partial r} = -\left[\frac{ua^3}{b^3-a^3}\left(1-2\frac{b^3}{2}r^{-3}\right)\cos\theta - \frac{vb^3}{b^3-a^3}\left(1-2\frac{a^3r^{-3}}{2}\right)\sin\theta\right]$$

$$= \frac{vb^3\left(1-\frac{a^3}{r^3}\right)\sin\theta - ua^3\left(1-\frac{b^3}{r^3}\right)\cos\theta}{b^3-a^3}$$

Similarly,

$$q_\theta = -\frac{1}{r}\left(\frac{\partial\phi}{\partial\theta}\right) = \frac{-1}{r}\left[\frac{ua^3}{b^3-a^3}\left(r+\frac{b^3}{2}r^{-2}\right)(-\sin\theta) - \frac{vb^3}{b^3-a^3}\left(r+\frac{a^3}{2}r^{-2}\right)\cos\theta\right]$$

$$= \frac{vb^3}{b^3-a^3}\left(1+\frac{a^3}{2r^3}\right)\cos\theta + \frac{ua^3}{b^3-a^3}\left(1+\frac{b^3}{2r^3}\right)\sin\theta$$

$$= \frac{vb^3\left(1+\frac{a^3}{2r^3}\right)\cos\theta + ua^3\left(1+\frac{b^3}{2r^3}\right)\sin\theta}{(b^3-a^3)}$$

$\therefore q = \sqrt{q_r^2+q_\theta^2}$ is the required velocity.

As strength of source = m and there is spherical symmetry, the velocity due to source will be radial in direction (say q_r)

Then by conservation of mass

$$4\pi \rho m = \rho(4\pi r^2 q_r)$$

$$q_r = \frac{m}{r^2}$$

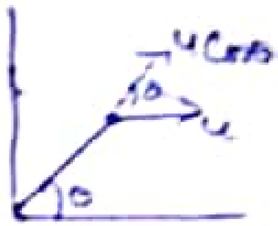
$$-\frac{\partial \phi_1}{\partial r} = \frac{m}{r^2}$$

$$\Rightarrow \boxed{\phi_1 = \frac{m}{r}}$$

Also, potential due to uniform flow of fluid

$$q'_r = u\cos\theta$$

$$\phi' = -ur\cos\theta$$



Hence, Total potential $= \phi_1 + \phi'$

$$\boxed{\phi = \frac{m}{r} - ur\cos\theta}$$

Also, $q_\theta = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = -u\sin\theta$

$$q_\varphi = 0$$

Streamlines are given by

$$\frac{dr}{-\partial\phi/\partial r} = \frac{r\,d\theta}{-\frac{1}{r}\frac{\partial\phi}{\partial\theta}} = \frac{r\sin\theta\,d\varphi}{-\frac{1}{r\sin\theta}\,\partial\phi/\partial\varphi}$$

$$\Rightarrow \frac{dr}{\frac{m}{r^2}+u\cos\theta} = \frac{r\,d\theta}{-u\sin\theta}, \quad \varphi = c$$

$$\Rightarrow -u\sin\theta\,dr = ur\cos\theta\,d\theta + \frac{m}{r}\,d\theta$$

$$\Rightarrow u(r\cos\theta\,d\theta + \sin\theta\,dr) + \frac{m}{r}\,d\theta = 0$$

$$\Rightarrow u\,d(r\sin\theta) + \frac{m}{r}\,d\theta = 0$$

$$\Rightarrow u\,r\sin\theta\,d(r\sin\theta) + m\sin\theta\,d\theta = 0$$

$$\Rightarrow u\,\frac{(r\sin\theta)^2}{2} + (-m\cos\theta) = \text{Const.}$$

$$\Rightarrow \boxed{u r^2\sin^2\theta - 2m\cos\theta = \text{Const.}}$$

Scanned with CamScanner