Mains Test Series - 2021

Test - VII, Paper - I, full Syllabus

Answer Key

1(a). Find the dimension and a basis of the solution space W of the system $x+2y+2z-s+3t=0$, $x+2y+3z+s+t=0$, $3x+6y+8z+s+5t=0$.

Sol'n: we write the single matrix equation is

$$AX=0 \quad \text{--- (1)}$$

$$\begin{bmatrix} 1 & 2 & 2 & -1 & 3 \\ 1 & 2 & 3 & 1 & 1 \\ 3 & 6 & 8 & 1 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ s \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

where $A = \begin{bmatrix} 1 & 2 & 2 & -1 & 3 \\ 1 & 2 & 3 & 1 & 1 \\ 3 & 6 & 8 & 1 & 5 \end{bmatrix}$

$$\sim \begin{bmatrix} 1 & 2 & 2 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 2 & 4 & -4 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 3R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 2 & 2 & -1 & 3 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} R_3 \to R_3 - 2R_2$$

Clearly it is in echelon form.

Again we write a single matrix equation by using above, we get

$$x+2y+2z-s+3t=0$$

$$z+2s-2t=0$$

$\Rightarrow \boxed{z = 2t - 2s}$ and

$\boxed{x = -2y + 5s - 7t}$

Clearly y, s, t are free variables.

$\therefore$ dimension of $W = 3$

(i) Let $y = 1, s = 0, t = 0$ to obtain the solution

$v_1 = (-2, 1, 0, 0, 0)$

(ii) Let $y = 0, s = 1, t = 0$ to obtain the solution

$v_2 = (5, 0, -2, 1, 0)$

(iii) Let $y = 0, s = 0, t = 1$ to obtain the solution

$v_3 = (-7, 0, 2, 0, 1)$

The set $\{v_1, v_2, v_3\}$ is a basis of the solution space W.

1.(b) let $A = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ and C be a non-singular matrix of order 3×3. Find the eigen values of matrix B^3, where $B = C^{-1}AC$.

Sol: $A = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$

$$|A| = \begin{vmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{vmatrix} = 2((-1-3)) + 2(-1-1) + 2(3-1)$$

$$= 2 \times -4 + 2 \cancel{\times 2} \; \cancel{+ 2 \times 2}$$

$$|A| = -8 \neq 0$$

$\rightarrow$ A is linearly independent

$\rightarrow$ A is diagonalizable ——— (1)

let λ be an eigen value of A,

characteristic polynomial is

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 2-\lambda & -2 & 2 \\ 1 & 1-\lambda & 1 \\ 1 & 3 & -1-\lambda \end{vmatrix} = 0$$

$$0 = (2-\lambda)[(1-\lambda)^2 - 3] + 2[-1-\lambda-1] + 2[3-1+\lambda]$$

$$\Rightarrow (2-\lambda)(\lambda+2)(\lambda-2) = 0$$

$$\boxed{\lambda = 2, 2, -2}$$ ——— (2)

$\because$ A is diagonalizable (from eq^n (1)),

$\exists$ P invertible matrix. such that

$A = P^{-1}DP$ where $D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ ———

$$\Rightarrow A = P^{-1}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -2 \end{bmatrix} P$$

$A^2 = (P^{-1}DP)(P^{-1}DP)$

$A^2 = P^{-1}D(PP^{-1})DP$

$A^2 = P^{-1}DIDP$

$A^2 = P^{-1}D^2P$

Similarly, $A^3 = P^{-1}D^3P = P^{-1}\begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -8 \end{bmatrix} P$ — (3)

Now, since,

$B = C^{-1}AC$

$B = C^{-1}P^{-1}DPC$

$B^2 = (C^{-1}P^{-1}DPC)(C^{-1}P^{-1}DPC)$

$B^2 = C^{-1}P^{-1}DP(CC^{-1})P^{-1}DPC$

$B^2 = C^{-1}P^{-1}DPIP^{-1}DPC = C^{-1}P^{-1}DPP^{-1}DPC$

$B^2 = C^{-1}P^{-1}DIDPC = C^{-1}P^{-1}D^2PC$

Similarly,

$B^3 = C^{-1}P^{-1}D^3PC$

$\therefore B^3 = C^{-1}P^{-1}\begin{bmatrix} 8 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & -8 \end{bmatrix} PC$

where, PC is invertible [$\because$ P & C both are non-singular $\Rightarrow$ Invertible]

$\Rightarrow B^3$ is diagonalizable and diagonal elements are eigen values of B^3.

$\Rightarrow$ Eigen values of $B^3 = \lambda_1, \lambda_2, \lambda_3 = 8, 8, -8$

1.(c) Let $f : \mathbb{R} \to \mathbb{R}$ be such that

$$f(x) = \begin{cases} \dfrac{\sin(a+1)x + \sin x}{x} & \text{if } x<0 \\ c & \text{if } x=0 \\ \dfrac{(x+bx^2)^{1/2} - x^{1/2}}{bx^{3/2}} & \text{if } x>0 \end{cases}$$

Determine the values of a, b, c for which the function is continuous at $x=0$.

Sol'n: At $x=0$, $f(0)=c$

LHL: $\lim_{x\to 0-} f(x) = \lim_{x\to 0-} \dfrac{\sin(a+1)x + \sin x}{x}$

$$= \lim_{x\to 0-} \left[\frac{\sin(a+1)x}{x} + \frac{\sin x}{x}\right]$$

$$= \lim_{x\to 0-}\left[\frac{(a+1)\sin(a+1)x}{(a+1)x}\right] + \lim_{x\to 0-} \frac{\sin x}{x}$$

$$= (a+1)\lim_{x\to 0-} \frac{\sin(a+1)x}{(a+1)x} + \lim_{x\to 0-} \frac{\sin x}{x}$$

$$= (a+2)(1) + 1$$

$$= a+2 \qquad \left(\because \lim_{x\to 0} \frac{\sin x}{x} = 1\right)$$

RHL: $\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \dfrac{(x+bx^2)^{1/2} - x^{1/2}}{bx^{3/2}}$

$$= \lim_{x\to 0+} \frac{x^{1/2}(1+bx)^{1/2} - x^{1/2}}{bx^{3/2}}$$

$$= \lim_{x\to 0+} \frac{(1+bx)^{1/2}-1}{bx}$$

$$= \lim_{x\to 0+} \frac{\sqrt{1+bx}-1}{bx} \cdot \frac{\sqrt{1+bx}+1}{\sqrt{1+bx}+1}$$

$$= \lim_{x\to 0+} \frac{1+bx-1}{bx\left[\sqrt{1+bx}+1\right]}$$

$$= \lim_{x\to 0+} \frac{bx}{bx\left[\sqrt{1+bx}+1\right]}$$

$$= \lim_{x\to 0+} \frac{1}{\sqrt{1+bx}+1}$$

$$= \frac{1}{2}$$

This is the independent of b. so that b can have any non-zero real value

Since f is continuous at $x=0$

we have

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0+} f(x) = f(0)$$

$$\Rightarrow a+2 = \frac{1}{2} = c$$

$$\Rightarrow a+2 = \frac{1}{2} \ \& \ c = \frac{1}{2}$$

$$\Rightarrow a = -\frac{3}{2} \ \& \ c = \frac{1}{2}$$

$$\therefore a = -\frac{3}{2}, \ b \neq 0 \ \& \ c = \frac{1}{2}$$

(any real number)

1(d) Evaluate $\iint_D xy\,dA$, where D is the region bounded by the line $y=x-1$ and the parabola $y^2=2x+6$

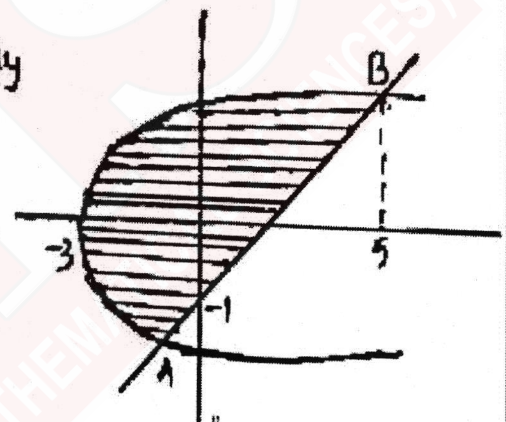
Sol^n: Solving the line and parabola we get

$A(-1,-2)$ and $B(5,4)$

$\therefore \iint_D xy\,dA$ where D is the region given by

$$D=\left\{(x,y)\in \mathbb{R}^2 : y\in[-2,4],\ \frac{y^2}{2}-3\le x\le y+1\right\}$$

$$\therefore \iint_D xy\,dA = \int_{-2}^{4}\int_{\frac{y^2}{2}-3}^{y+1} xy\,dx\,dy$$



$$= \int_{-2}^{4}\left[\frac{x^2}{2}\cdot y\right]_{\frac{y^2}{2}-3}^{y+1} dy$$

$$= \frac{1}{2}\int_{-2}^{4} y\left[(y+1)^2-\left(\frac{y^2}{2}-3\right)^2\right]dy$$

$$= \frac{1}{2}\int_{-2}^{4} y\left[\frac{-y^5}{4}+4y^3+2y^2-8y\right]dy$$

$$= \frac{1}{2}\left[\frac{-y^6}{24}+y^4+\frac{2y^3}{3}-4y^2\right]_{-2}^{4}$$

$$= 36$$

1.(e) Find the equation of a sphere touching the three coordinate planes. How many such spheres can be drawn?

Sol^n: Let the equation of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad \text{——} (1)$$

If the sphere touches the yz-plane i.e. $x=0$, then the length of the $\perp$lar from its centre $(-u,-v,-w)$ to the plane $x=0$ must be equal to its radius $=\sqrt{(u^2+v^2+w^2-d)}$

i.e. $\frac{-u}{1}=\sqrt{(u^2+v^2+w^2-d)}$

$$\Rightarrow u^2=u^2+v^2+w^2-d \Rightarrow v^2+w^2=d \quad \text{——} (2).$$

Similarly if the sphere (1) touches zx and xy-planes then we shall have

$$w^2+u^2=d \quad \text{——} (3) \quad \text{and} \quad u^2+v^2=d \quad \text{——} (4)$$

adding (2), (3) and (4) we get $2(u^2+v^2+w^2)=3d$

$$\Rightarrow u^2+v^2+w^2=\tfrac{3}{2}d$$

$$\Rightarrow u^2=\tfrac{1}{2}d \text{ from } (2).$$

Similarly from (3), (4) and (5) we get $v^2=\frac{1}{2}d=w^2$

$$\therefore u^2=\tfrac{1}{2}d=v^2=w^2=\lambda^2 \text{ (say)}$$

$$\Rightarrow u=\pm\lambda=v=w$$

Hence from (1) the required equation is

$$x^2+y^2+z^2\pm 2\lambda(x+y+z)+2\lambda^2=0$$

Since λ can taken an infinite number of values, so an infinite number of such spheres can be drawn but if the radius of the sphere is given then λ can be expressed interms of the given radius and then only eight such spheres can be possible as the sets of values of u, v & w can be taken in eight different ways.

2(a)(i) If $P=\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}$, then find P^{50}.

Sol^n: Given $P=\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}$

$$\Rightarrow P^2=\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}=\begin{bmatrix}1&2&3\\0&1&2\\0&0&1\end{bmatrix}$$

$$\Rightarrow P^3=\begin{bmatrix}1&2&3\\0&1&2\\0&0&1\end{bmatrix}\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}=\begin{bmatrix}1&3&6\\0&1&3\\0&0&1\end{bmatrix}$$

$$\Rightarrow P^4=\begin{bmatrix}1&3&6\\0&1&3\\0&0&1\end{bmatrix}\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}=\begin{bmatrix}1&4&1+2+3+4\\0&1&4\\0&0&1\end{bmatrix}$$

Similarly, $P^{50}=\begin{bmatrix}1&50&1+2+3+\cdots+50\\0&1&50\\0&0&1\end{bmatrix}$

Now, $1+2+3+\cdots+50=\dfrac{50(50+1)}{2}=1275$

$$\Rightarrow P^{50}=\begin{bmatrix}1&50&1275\\0&1&50\\0&0&1\end{bmatrix}$$

2(a) (ii) Find the dimension of the subspace
$W = \{(x, y, z, w) \in \mathbb{R}^4 \mid x+y+z+w=0,\ x+y+2z=0,\ x+3y=0\}$

Sol'n. $\because$

$$x+y+z+w=0$$
$$x+y+2z=0$$
$$x+3y=0$$

$\Rightarrow AX = 0$ —— (1)

where $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 1 & 3 & 0 & 0 \end{bmatrix}$; $X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$; $O = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

$$A \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 2 & -1 & -1 \end{bmatrix} \begin{matrix} R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1 \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & -1 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} R_2 \leftrightarrow R_3$$

we write a single matrix equation

$$x+y+z+w=0$$
$$2y-z-w=0$$
$$z-w=0$$

$\Rightarrow \boxed{z=w}$: $2y-w-w=0$
$\Rightarrow 2y=-2w$
$\Rightarrow \boxed{y=-w}$

$\therefore x-w+w+w=0 \Rightarrow \boxed{x=-w}$

$\therefore W = \{(-w, -w, w, w) \in \mathbb{R}^4 \mid w \in \mathbb{R}\}$

Clearly it contains only one free variable w.

$\therefore \boxed{\dim(W) = 1}$

2.(b) (i) Show that the height of an open cylinder of given surface and greatest volume is equal to the radius of its base.

Sol^n let r be the radius of the circular base; h, the height; s the surface and v the volume of the open cylinder so that

$$s = \pi r^2 + 2\pi r h \rightarrow (1)$$

$$v = \pi r^2 h \rightarrow (2)$$

Here, as given s is constant and v a variable. Also h, r are variables substituting the value of h as obtained from (i) in (ii) we get

$$v = \pi r^2\left(\frac{s-\pi r^2}{2\pi r}\right) = \frac{sr - \pi r^3}{2} \rightarrow (3)$$

which gives v in terms of one variable r.

As v must be necessarily non-negative, we have

$sr - \pi r^3 \geq 0 \Rightarrow \pi r^3 \leq sr \Rightarrow r \leq \sqrt{s/\pi}$

Also r is non-negative.

Thus 'r' varies in the interval $[0, \sqrt{s/\pi}]$.

Now $\dfrac{dv}{dr} = \dfrac{s - 3\pi r^2}{2}$.

So that $\frac{dv}{dr} = 0$ only when $r = \sqrt{s/3\pi}$, Negative value of r being inadmissible. Thus v has only one stationary value.

Now $V=0$ for the end points $r=0$ and $\sqrt{S/\pi}$ and positive for every other admissible value of r. Hence V is greatest for

$$r = \sqrt{S/3\pi}$$

Substituting this value of r in (i) we get

$$h = \frac{S-\pi r^2}{2\pi r} = \frac{S-\pi\left(S/3\pi\right)}{2\pi\sqrt{(S/3\pi)}}$$

$$= \frac{2S}{3}\cdot\frac{1}{2\pi}\sqrt{\left(\frac{3\pi}{S}\right)} = \sqrt{\left(\frac{S}{3\pi}\right)}$$

Hence $h=r$ for the cylinder of greatest volume and given surface.

2.(b)(ii) → If $z = (x+y) + (x+y)\,\phi(y/x)$, prove that

$$x\left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y \partial x}\right) = y\left(\frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x \partial y}\right)$$

Solution:

As z is a homogeneous function of degree 1, then

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z \quad \text{—(1)}$$

$$x^2\frac{\partial^2 z}{\partial x^2} + y^2\frac{\partial^2 z}{\partial y^2} - 2xy\frac{\partial^2 z}{\partial x \partial y} = 0 \quad \text{—(2)}$$

$$\frac{\partial z}{\partial x} = 1 + \phi(y/x) + (x+y)\,\phi'(y/x)(-y/x^2)$$

$$= 1 + \phi(y/x) - \frac{y(x+y)}{x^2}\phi'(y/x)$$

$$\frac{\partial z}{\partial y} = 1 + \phi(y/x) + \left(\frac{x+y}{x}\right)\phi'(y/x)$$

Substituting in (1) and (2), we have

$$x + y + \phi(y/x)(x+y) = z$$

$$\frac{\partial^2 z}{\partial x^2} = \phi'(y/x)(-y/x^2) + \frac{y(x+y)}{x^2}(y/x^2)\,\phi''(y/x)$$

$$+ \phi'(y/x)\left(y/x^2 - \frac{2y^2}{x^3}\right)$$

$$\frac{\partial^2 z}{\partial y^2} = \phi'(y/x)(1/x) + \frac{1}{x}\phi'(y/x) + \frac{(x+y)}{x^2}\phi'(y/x)$$

$$\frac{\partial^2 z}{\partial x \partial y} = -y/x^2\,\phi'(y/x) - y/x^2\,\phi'(y/x) - \frac{(x+y)}{x}\frac{y}{x^2}\cdot\phi''(y/x)$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{1}{x}\phi'(y/x) - \frac{(x+2y)}{x^2}\phi'(y/x) - y\frac{(x+y)}{x^3}\phi''(x)$$

$$x\left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y \partial x}\right) = \phi'(y/x)\left(\frac{2y^2}{x^2} + \frac{2y}{x}\right) + \phi''(y/x)\left(\frac{y^3}{x^3} + \frac{2y^2}{x^2} + y/x\right) \quad \text{——} (3)$$

$$y\left(\frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x \partial y}\right) = \phi'\left(\frac{y}{x}\right)\left(\frac{2y^2}{x^2} + \frac{2y}{x}\right) + \phi''(y/x)\left(\frac{y^3}{x^3} + \frac{2y^2}{x^2} + \frac{y}{x}\right) \quad \text{——} (4).$$

$\therefore$ from (3) & (4), we conclude that,

$$\boxed{x\left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y \partial x}\right) = y\left(\frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x \partial y}\right)}$$

Hence Proved.

2.(c) (i) A variable plane, which remains at a constant distance p from the origin, cuts the coordinate axes at A, B, and C. Show that the locus of the centroid of ΔABC is $x^{-2}+y^{-2}+z^{-2}=9p^{-2}$

Sol^n: Let the equation of the variable plane be

$$x/a+y/b+z/c=1 \quad \text{—— (1)}$$

The plane (1) meets the axes in A, B and C whose coordinates are $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$ respectively.

Also the distance of the plane (1) from $(0,0,0)$ is given as p, so we have,

$$p=\frac{1}{\sqrt{(1/a)^2+(1/b)^2+(1/c)^2}}$$

$$\Rightarrow \frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \quad \text{—— (2)}$$

The planes through A, B and C parallel to the coordinates planes are given by $x=a$, $y=b$ and $z=c$ respectively.

the required locus is obtained by eliminating a, b, c from the equations of these planes and the relation (2)

Substituting the values of a, b, c from (3) in (2), we have the required locus, as $x^{-2}+y^{-2}+z^{-2}=p^{-2}$.

2(c)(ii) Find the equation of the cylinder whose generators are parallel to the line $\frac{x}{1} = \frac{y}{(-2)} = \frac{z}{3}$ and passing through the curve $x^2+2y^2=1,\ z=0$.

Sol'n: Let $P(x_1, y_1, z_1)$ be any point on the cylinder, then the equations of the generator through P are $\frac{x-x_1}{1} = \frac{y-y_1}{-2} = \frac{z-z_1}{3}$ —— ①

This generator meets the plane $z=0$ in point given by

$$\frac{x-x_1}{1} = \frac{y-y_1}{-2} = \frac{0-z_1}{3}$$

i.e. in the point $\left[x_1-\frac{1}{3}z_1,\ y_1+\frac{2}{3}z_1, 0\right]$

∴ This generator ① intersects the given conic if

∴ The locus of $P(x_1, y_1, z_1)$ or the required equation of the cylinder is

$$\left(x-\tfrac{1}{3}z\right)^2+2\left[y+\tfrac{2}{3}z\right]^2=1$$

$$\Rightarrow \left(x^2+\tfrac{1}{9}z^2-\tfrac{2}{3}xz\right)+2\left(y^2+\tfrac{4}{9}z^2+\tfrac{4}{3}yz\right)=1$$

$$\Rightarrow 9x^2+z^2-6xz+18y^2+8z^2+24yz-9=0$$

$$\Rightarrow 3x^2+6y^2+3z^2-2xz+8yz-3=0.$$

3.(a) If H is a Hermitian matrix, show that $(I-iH)^{-1}(I+iH)^{-1} = (I+iH)^{-1}(I-iH) = U$, where U is a unitary matrix and that if λ is an eigen value of H, then $(1-i\lambda)/1+i\lambda$ is an eigen value of U.

Find U when $H = \begin{bmatrix} 1 & e^{i\alpha} \\ e^{-i\alpha} & -1 \end{bmatrix}$

Sol: Since H is a Hermitian matrix.

$\therefore H^\theta = H$ —①

we know that the characteristic roots of H are real.

$\therefore$ The roots of the equation $|H - I\lambda| = 0$ are real.

Neither i nor $-i$ is a root of the equation $|H-\lambda I| = 0$

$\Rightarrow |H - iI| \neq 0$ and $|H + iI| \neq 0$

$\Rightarrow (H - iI)$ is non-singular and $(H + iI)$ is non-singular.

$\Rightarrow (iH + I)$ & $(iH - I)$ are also non-singular.

$\Rightarrow (I + iH)$ & $(I - iH)$ are also non-singular. $\left(\because |A| \neq 0 \Rightarrow |-A| \neq 0\right)$

Given that $U = (I+iH)^{-1}(I-iH)$

consider $U^\theta = \left[(I+iH)^{-1}(I-iH)\right]^\theta$

$= (I-iH)^\theta \left[(I+iH)^{-1}\right]^\theta$

$= \left[I^\theta - (iH)^\theta\right]\left[(I^\theta + iH)^\theta\right]^{-1}$

$= (I - \bar{i}H^\theta)(I + \bar{i}H^\theta)^{-1}$

$= (I+iH)(I-iH)^{-1}$ $\left(\because H^\theta = H,\ \bar{i} = -i\right)$

Now $U^\theta U = (I+iH)(I-iH)^{-1}(I+iH)^{-1}(I-iH)$

$= (I+iH)(I+iH)^{-1}(I-iH)^{-1}(I-iH)$ $\left(\because (I+iH)^{-1}(I-iH)^{-1} = (I-iH)^{-1}(I+iH)^{-1}\right)$

$= I \cdot I$

$= I$

$\therefore U^\theta U = I$

$\therefore U$ is a unitary matrix.

To show that $U = (I-iH)(I+iH)^{-1}$

Since $(I-iH)(I+iH) = (I+iH)(I-iH)$

Premultiplying throughout by $(I+iH)^{-1}$ and post multiplying throughout by $(I+iH)^{-1}$, we get

$$(I+iH)^{-1}(I-iH)(I+iH)(I+iH)^{-1} = (I+iH)^{-1}(I+iH)(I-iH)(I+iH)^{-1}$$

$$\Rightarrow (I+iH)^{-1}(I-iH) = (I-iH)(I+iH)^{-1}$$

$$U = (I-iH)(I+iH)^{-1}.$$

$$\therefore U = (I-iH)(I+iH)^{-1} = (I+iH)^{-1}(I-iH).$$

Suppose λ is an eigen value of H and x is the corresponding eigen vector of H.

Then $Hx = \lambda x$

$\Rightarrow iHx = i\lambda x$

$\Rightarrow x + iHx = x + i\lambda x$

$\Rightarrow (I+iH)x = (1+i\lambda)x$ —— (1)

Similarly $(I-iH)x = (1-i\lambda)x$ —— (2)

Premultiplying (1) throughout by $(I+iH)^{-1}$ we get

$$x = (1+i\lambda)(I+iH)^{-1}x$$

$$\Rightarrow (1+i\lambda)^{-1}x = (I+iH)^{-1}x$$

$$\Rightarrow (I+iH)^{-1}x = (1+i\lambda)^{-1}x \text{ —— (3)}$$

Now pre-multiplying (2) by $(I+iH)^{-1}$ throughout, we get

$$(I+iH)^{-1}(I-iH)x = (1-i\lambda)(I+iH)^{-1}x$$

$$\Rightarrow [(I+iH)^{-1}(I-iH)]x = (1-i\lambda)(1+i\lambda)^{-1}x \text{ (by (3))}$$

$$\Rightarrow [(I+iH)^{-1}(I-iH)]x = [(1-i\lambda)(1+i\lambda)^{-1}]x$$

$\therefore x$ is a characteristic vector of A

$U = (I+iH)^{-1}(I-iH)$ and $(1-i\lambda)(1+i\lambda)^{-1}$ is the corresponding characteristic root of U.

Given that $H = \begin{pmatrix} 1 & e^{i\alpha} \\ e^{-i\alpha} & -1 \end{pmatrix}$

$\therefore I - iH = \begin{pmatrix} 1-i & -ie^{i\alpha} \\ -ie^{-i\alpha} & 1+i \end{pmatrix}$ and $I+iH = \begin{pmatrix} 1+i & ie^{i\alpha} \\ ie^{-i\alpha} & 1-i \end{pmatrix}$

$\Rightarrow (I+iH)^{-1} = \frac{1}{3}\begin{pmatrix} 1-i & -ie^{i\alpha} \\ -ie^{-i\alpha} & 1+i \end{pmatrix}$

Now

$U = (I+iH)^{-1}(I-iH)$

$= \frac{1}{3}\begin{pmatrix} 1-i & -ie^{i\alpha} \\ -ie^{-i\alpha} & 1+i \end{pmatrix}\begin{pmatrix} 1-i & -ie^{i\alpha} \\ -ie^{-i\alpha} & 1+i \end{pmatrix}$

$= \frac{1}{3}\begin{pmatrix} -1-2i & -2ie^{i\alpha} \\ -2ie^{-i\alpha} & -1+2i \end{pmatrix}$

3.(b)(i) Show that $\frac{v-u}{1+v^2} < \tan^{-1}v - \tan^{-1}u < \frac{v-u}{1+u^2}$, if $0<u<v$ and deduce that $\frac{\pi}{4}+\frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4}+\frac{1}{6}$.

Sol^n: Let $f(x) = \tan^{-1}x$

then $f'(x) = \frac{1}{1+x^2}$

Applying Lagrange's mean value theorem to f on $[u,v]$, we get

$$\frac{f(v)-f(u)}{v-u} = f'(c) \text{ where } u<c<v.$$

$$\Rightarrow \frac{\tan^{-1}v - \tan^{-1}u}{v-u} = \frac{1}{1+c^2} \quad \text{———} (1)$$

Now $c>u \Rightarrow \frac{1}{1+c^2} < \frac{1}{1+u^2}$

and $c<v \Rightarrow \frac{1}{1+c^2} > \frac{1}{1+v^2}$

$$\therefore \frac{1}{1+v^2} < \frac{1}{1+c^2} < \frac{1}{1+u^2}$$

$$\Rightarrow \frac{1}{1+v^2} < \frac{\tan^{-1}v - \tan^{-1}u}{v-u} < \frac{1}{1+u^2} \quad [\text{using}(1)]$$

$$\Rightarrow \frac{v-u}{1+v^2} < \tan^{-1}v - \tan^{-1}u < \frac{v-u}{1+u^2}$$

Taking $v = 4/3$ and $u=1$, we have

$$\frac{1/3}{1+\frac{16}{9}} < \tan^{-1}\frac{4}{3} - \tan^{-1}1 < \frac{1/3}{1+1}$$

$$\Rightarrow \frac{3}{25} < \tan^{-1}\frac{4}{3} - \frac{\pi}{4} < \frac{1}{6}$$

$$\Rightarrow \frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$$

3.(b)(ii) Examine the convergence of $\int_1^\infty \frac{dx}{x\sqrt{x^2+1}}$

Sol: let $f(x) = \frac{1}{x\sqrt{x^2+1}}$, (behaves like x^{-2} at ∞) and

$g(x) = 1/x^2$

So that $\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{x^2}{x\sqrt{x^2+1}} = 1$ (non-zero finite)

Hence, the two integrals $\int_1^\infty f\,dx$ and $\int_1^\infty g\,dx$ behaves alike

As $\int_1^\infty \frac{dx}{x^2}$ converges therefore $\int_1^\infty \frac{dx}{x\sqrt{x^2+1}}$ also converges.

3.(c) → Prove that, in general, three normals can be drawn from a given point to the paraboloid $x^2+y^2=2az$, but if the point lies on the surface

$$27a(x^2+y^2)+8(a-z)^3=0$$ then

two of the three normals coincide.

Solution:

The equations of the normal at (x_1, y_1, z_1) to the paraboloid $x^2+y^2=2az$ are

$$\frac{x-x_1}{x_1}=\frac{y-y_1}{y_1}=\frac{z-z_1}{-a}$$

This passes through a given point (α, β, γ) if

$$\frac{\alpha-x_1}{x_1}=\frac{\beta-y_1}{y_1}=\frac{\gamma-z_1}{z_1}=\lambda \text{ (say)}$$

These gives $\alpha - x_1 = \lambda x_1 \Rightarrow x_1 = \alpha/(1+\lambda)$
Similarly, $y_1 = \beta/(1+\lambda)$, $z_1 = \gamma + a\lambda$ } (1)

Also (x_1, y_1, z_1) lies on the given paraboloid, so

$$x_1^2+y_1^2=2az_1 \Rightarrow \left[\frac{\alpha}{1+\lambda}\right]^2+\left[\frac{\beta}{1+\lambda}\right]^2=2a(\gamma+a\lambda),$$

[from (1)]

$\Rightarrow \alpha^2+\beta^2 = 2a(\gamma+a\lambda)(1+\lambda)^2$ —— (2)

This being a cubic in λ gives three values of λ and so from (1) there are three points on the paraboloid normals at which pass through (α,β,γ).

The equation (2) can be rewritten as

$$f(\lambda) \equiv 2a(1+\lambda)^2(\gamma+a\lambda)-(\alpha^2+\beta^2)=0 \quad \text{—— (3)}$$

The condition that this equation has two equal roots is obtained by eliminating λ between $f(\lambda)=0$ and $f'(\lambda)=0$.

From (3), $f'(\lambda)=0$ means $2a(1+\lambda)^2(a)+4a(1+\lambda)(\gamma+a\lambda)=0$

$\Rightarrow a(1+\lambda)+2(\gamma+a\lambda)=0 \quad (\because 1+\lambda\neq 0)$

$\Rightarrow (a+2\gamma)+\lambda(3a)=0$

$\Rightarrow \lambda = -(a+2\gamma)/(3a)$

Substituting this value of λ in (3), we get,

$$2a\left[1-\frac{a+2\gamma}{3a}\right]^2\left[\gamma-\frac{a(a+2\gamma)}{3a}\right]=\alpha^2+\beta^2$$

$\Rightarrow 2a[2(a-\gamma)]^2[a(\gamma-a)]=27a^3(\alpha^2+\beta^2)$

$\Rightarrow 27a(\alpha^2+\beta^2)+8(a-\gamma)^3=0$

$\therefore$ Locus of the point (α,β,γ) is

$27a(x^2+y^2)+8(a-z)^3=0.$

Hence, proved.

4(a)(i) Let A be a 3×3 upper triangular matrix with real entries. If $a_{11}=1$, $a_{22}=2$ and $a_{33}=3$, determine α,β and γ such that $A^{-1}=\alpha A^2+\beta A+\gamma I$.

Sol'n: Let $A=\begin{bmatrix}1 & a & b\\ 0 & 2 & c\\ 0 & 0 & 3\end{bmatrix}_{3\times3}$ where $a,b,c\in\mathbb{R}$

then characteristic equation of A is

$|A-\lambda I|=0$ — ① where $I=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$

$$\Rightarrow \begin{vmatrix}1-\lambda & a & b\\ 0 & 2-\lambda & c\\ 0 & 0 & 3-\lambda\end{vmatrix}=0$$

$\Rightarrow (1-\lambda)(2-\lambda)(3-\lambda)=0$

$\Rightarrow \lambda^3-6\lambda^2+11\lambda-6=0$ — ②.

we know that by Caley's theorem, every square matrix satisfies its own characteristic equation.

$\therefore$ ② $\equiv A^3-6A^2+11A-6I=0$ — ③

$\Rightarrow A^2-6A+11I-6A^{-1}=0$ (by multiplying with A^{-1})

$\Rightarrow A^{-1}=\frac{1}{6}\left[A^2-6A+11I\right]$ — ④

Given that $A^{-1}=\alpha A^2+\beta A+\gamma I$

$\Rightarrow \frac{1}{6}\left[A^2-6A+11I\right]=\alpha A^2+\beta A+\gamma I$ (from ④)

$\Rightarrow A^2-6A+11I=(6\alpha)A^2+(6\beta)A+(6\gamma)I$

$\Rightarrow 6\alpha=1$, $6\beta=-6$, $6\gamma=11$

$\Rightarrow \alpha=1/6$, $\beta=-1$, $\gamma=11/6$.

4.(a)(ii) Show that the matrix $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ is diagonalizable. Also find the diagonal form and ~~diagonalizing~~ matrix P

Sol^n: The characteristic equation of A is $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix} -9-\lambda & 4 & 4 \\ -8 & 3-\lambda & 4 \\ -16 & 8 & 7-\lambda \end{vmatrix}=0$$

$C_1 \to C_1+C_2+C_3$

$$\Rightarrow \begin{vmatrix} -1-\lambda & 4 & 4 \\ -1-\lambda & 3-\lambda & 4 \\ -1-\lambda & 8 & 7-\lambda \end{vmatrix}=0$$

$$\Rightarrow (-1-\lambda)\begin{vmatrix} 1 & 4 & 4 \\ 1 & 3-\lambda & 4 \\ 1 & 8 & 7-\lambda \end{vmatrix}=0$$

$R_2 \to R_2-R_1$
$R_3 \to R_3-R_1$

$$\Rightarrow (1+\lambda)\begin{vmatrix} 1 & 4 & 4 \\ 0 & -1-\lambda & 0 \\ 0 & 4 & 3-\lambda \end{vmatrix}=0$$

$$\Rightarrow (1+\lambda)(1+\lambda)(3-\lambda)=0$$

$$\Rightarrow \lambda=-1,-1,3$$

The characteristic roots of A are $-1,-1,3$.

The characteristic roots of A corresponding to the eigen vectors X of A corresponding to the characteristic root -1 are given by

$[A-(-1)I]X=0$

$$\Rightarrow \begin{bmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2-R_1$
$R_3 \to R_3-2R_1$.

$$\sim \begin{bmatrix} -8 & 4 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The rank of the coefficient matrix = 1.

∴ The equations have 3−1 = 2 linearly independent solutions.

∴ we have $-8x_1+4x_2+4x_3=0$

$\Rightarrow -2x_1+x_2+x_3=0$

Let $x_2=k_1$ and $x_3=k_2$; k_1, k_2 are arbitrary constants.

$\therefore x_1=\frac{k_1+k_2}{2}$.

$$\therefore X=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}\frac{k_1+k_2}{2}\\k_1\\k_2\end{bmatrix}$$

$$=k_1\begin{bmatrix}1/2\\1\\0\end{bmatrix}+k_2\begin{bmatrix}1/2\\0\\1\end{bmatrix}$$

$$=k_1X_1+k_2X_2.$$

Here $X_1=\begin{bmatrix}1/2\\1\\0\end{bmatrix}$ & $X_2=\begin{bmatrix}1/2\\0\\1\end{bmatrix}$ are LI vectors of A corresponding to characteristic root −1.

∴ The geometric multiplicity of eigenvalue is equal to its algebraic multiplicity.

Now the eigen vectors X of A corresponding to the eigen value 3 are given by

$(A-3I)X=0$.

$$\Rightarrow \begin{bmatrix}-12&4&4\\-8&0&4\\-16&8&4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\sim\begin{bmatrix}4&-4&0\\-8&0&4\\-16&8&4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\quad R_1\to R_1-R_3$$

$$\sim\begin{bmatrix}4&-4&0\\0&-8&4\\0&-8&4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\quad \begin{matrix}R_2\to R_2+2R_1\\R_3\to R_3+4R_1\end{matrix}$$

$$\sim \begin{bmatrix} 4 & -4 & 0 \\ 4 & -8 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

The rank of the coefficient matrix = 2

$\therefore$ The equations have $3-2=1$ LI solution.

$\therefore$ we have $4x_1 - 4x_2 = 0 \Rightarrow x_1 = x_2$

$-8x_2 + 4x_3 = 0 \Rightarrow 2x_2 = x_3$

Taking $x_3 = 2$, then $x_2 = 1$ & $x_1 = 1$.

$\therefore$ $X_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ is an eigen vector of A corresponding to the eigen value 3.

$\therefore$ The geometric multiplicity of eigen value 3 is 1 and its algebraic multiplicity is also 1.

$\therefore$ A is similar to diagonal matrix.

$\therefore$ A is diagonalizable matrix.

Let $P = [X_1 \ X_2 \ X_3] = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

The columns of P are LI eigen vectors of A corresponding to the eigen values $-1, -1, 3$ respectively. The matrix P will transform A to diagonal form D is given by the relation

$P^{-1}AP = D$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

The transforming matrix $P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

and diagonal matrix

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

4.(b) Find the Maximum and minimum value of $x^2+y^2+z^2$ subjected to the conditions $\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}=1$ and $z=x+y$.

Sol^n. let $f=x^2+y^2+z^2$

$$\phi_1=\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}-1=0$$

and $\phi_2=x+y-z=0$.

Consider a function F of independent variables x,y,z where.

$$F=x^2+y^2+z^2+\lambda_1\left(\frac{x^2}{4}+\frac{y^2}{5}+\frac{z^2}{25}-1\right)+\lambda_2(x+y-z)$$

$$dF=\left(2x+\frac{x}{2}\lambda_1+\lambda_2\right)dx+\left(2y+\frac{2y}{5}\lambda_1+\lambda_2\right)dy$$
$$+\left(2z+\frac{2z}{25}\lambda_1-\lambda_2\right)dz$$

$[dF=F_x\,dx+F_y\,dy+F_z\,dz]$

As x,y,z are independent variables

$$\frac{\partial F}{\partial x}=0\Rightarrow 2x+\frac{x}{2}\lambda_1+\lambda_2=0$$

$$\frac{\partial F}{\partial y}=0\Rightarrow 2y+\frac{2y}{5}\lambda_1+\lambda_2=0$$

$$\frac{\partial F}{\partial z}=0\Rightarrow 2z+\frac{2y}{25}\lambda_1-\lambda_2=0.$$

$$\therefore x=\frac{-2\lambda_2}{\lambda_1+4},\quad y=\frac{-5\lambda_2}{2\lambda_1+10},\quad z=\frac{25\lambda_2}{2\lambda_1+50}\longrightarrow (1)$$

Substituting in $x+y=z$, we get.

$$\frac{-2\lambda_2}{\lambda_1+4}+\frac{(-5\lambda_2)}{5\lambda_1+10}-\frac{25\lambda_2}{2\lambda_1+50}=0.$$

$$\lambda_2\left[\frac{2}{\lambda_1+4}+\frac{5}{5\lambda_1+10}+\frac{25}{2\lambda_1+50}\right]=0.$$

if $\lambda_2\neq 0$, $\frac{2}{\lambda_1+4}+\frac{5}{5\lambda_1+10}+\frac{25}{2\lambda_1+50}=0\longrightarrow (2)$

if $\lambda_2=0$

then from (1)

$x=0, y=0, z=0$, but $(x,y,z)=(0,0,0)$ doesnot satisfy the other condition of the constraint.

$\therefore$ from (2) $\quad 17\lambda_1^2 + 245\lambda_1 + 750 = 0$

$$\Rightarrow \lambda_1 = -10,\ \lambda_1 = -75/17.$$

for $\lambda_1 = -10$, from (1)

$$x = \frac{\lambda_2}{3},\ y = \frac{\lambda_2}{2},\ z = \frac{5\lambda_2}{6} \rightarrow (3)$$

Now substituting (3) in $\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} = 1$

we get $\lambda_2^2\left[\frac{1}{36} + \frac{1}{20} + \frac{1}{36}\right] = 1$

$$\frac{19\lambda_2^2}{180} = 1 \Rightarrow \lambda_2^2 = \frac{180}{19} \text{ (or)}$$

$$\Rightarrow \lambda_2 = \pm 6\sqrt{\frac{5}{19}}.$$

putting $\lambda_2 = \pm 6\sqrt{\frac{5}{19}}$ in eqn (3)

from the corresponding stationary points are

$\left(2\sqrt{\frac{5}{19}}, 3\sqrt{\frac{5}{19}}, 5\sqrt{\frac{5}{19}}\right)$ and $\left(-2\sqrt{\frac{5}{19}}, -3\sqrt{\frac{5}{19}}, -5\sqrt{\frac{5}{19}}\right)$

The value of $x^2+y^2+z^2$ corresponding to these points is 10.

for $\lambda_1 = -75/17$.

from (1) $\quad x = \frac{34}{7}\lambda_2,\ y = \frac{-17}{4}\lambda_2,\ z = \frac{17}{28}\lambda_2 \rightarrow (2)$

which on substitution in $\frac{x^2}{4} + \frac{y^2}{5} + \frac{z^2}{25} = 1$ give

$$\lambda_2 = \pm\frac{140}{17\sqrt{646}}$$

Substituting $\lambda_2 = \pm\frac{140}{17\sqrt{646}}$ in (2)

Then the corresponding stationary points are

$\left(\frac{40}{\sqrt{646}}, \frac{-35}{\sqrt{646}}, \frac{5}{\sqrt{646}}\right)$ and $\left(\frac{-40}{\sqrt{646}}, \frac{35}{\sqrt{646}}, \frac{-5}{\sqrt{646}}\right)$

The value of $x^2+y^2+z^2$ corresponding to these points is $\frac{75}{17}$.

$\therefore$ The maximum value is 10 and the minimum value is $75/17$.

4.(c) Find the equations to the generating lines of the hyperboloid $\frac{x^2}{4}+\frac{y^2}{9}-\frac{z^2}{16}=1$ which pass through the points $(2,3,-4)$ and $(2,-1,\frac{4}{3})$

Sol^n: Any line through $(2,3,-4)$ and $(2,-1,4/3)$

is $\frac{x-2}{l}=\frac{y-3}{m}=\frac{z+4}{n}=r$ (say) —— ①

∴ Any point on this line is

$(lr+2,\ mr+3,\ nr-4)$ and it lies on the given hyperboloid if

$$\frac{(lr+2)^2}{4}+\frac{(mr+3)^2}{9}-\frac{(nr-4)^2}{16}=1$$

$$r^2\left(\frac{l^2}{4}+\frac{m^2}{9}-\frac{n^2}{16}\right)+2r\left(\frac{2l}{4}+\frac{3m}{9}+\frac{4n}{16}\right)=0 \quad \text{—— ②}$$

If the line ① is a generator of the given hyperboloid, then ① lies wholly on the hyperboloid and the conditions for which from ② are

$$\frac{l^2}{4}+\frac{m^2}{9}-\frac{n^2}{16}=0 \quad \text{and} \quad \frac{2l}{4}+\frac{3m}{9}+\frac{4n}{16}=0$$

$$\Rightarrow \frac{l^2}{4}+\frac{m^2}{9}-\frac{n^2}{16}=0 \quad \text{and} \quad \frac{l}{2}+\frac{m}{3}=-\frac{n}{4} \quad \text{—— ③}$$

Eliminating n, we get

$$\frac{l^2}{4}+\frac{m^2}{9}-\left(\frac{l}{2}+\frac{m}{3}\right)^2=0$$

$$\Rightarrow -\frac{1}{3}lm=0$$

$\Rightarrow$ either $l=0$ or $m=0$

When $l=0$, from ③, we get $\frac{m}{3}=-\frac{n}{4}$

When $m=0$, from ③, we get $\frac{l}{2}=-n/4$

$$\Rightarrow \frac{l}{1}=\frac{-n}{2}$$

Hence from (i), ~~eq~~uations of the required generator through $(2,3,-4)$

are $\dfrac{x-2}{0} = \dfrac{y-3}{3} = \dfrac{z+4}{-4}$

and $\dfrac{x-2}{1} = \dfrac{y-3}{0} = \dfrac{z+4}{-2}$

Similarly, we can find the generators through the point $(2,-1, 4/3)$ are

$\dfrac{x-2}{0} = \dfrac{y+1}{3} = \dfrac{z-(4/3)}{-4}$

and $\dfrac{x-2}{3} = \dfrac{y+1}{6} = \dfrac{z-(4/3)}{10}$

5.(a) → Solve $(3y^2 - 7x^2 + 7)x\,dx + (7y^2 - 3x^2 + 3)y\,dy = 0.$ —— (1)

Solution:

Let $X = x^2$, $Y = y^2$ so that

$dX = 2x\,dx$, $dY = 2y\,dy$

∴ From (1),

$$\frac{dY}{dX} = \frac{(7X - 3Y - 7)}{(7Y - 3X + 3)}$$

Now, putting $X = u + h$, $Y = v + K$

$dX = du$, $dY = dv$ so that

$$\frac{dv}{du} = \frac{7u - 3v + (7h - 3k - 7)}{7v - 3u + (7k - 3h + 3)} \quad \text{—— (2)}$$

choose h, k so that $(7h - 3k - 7) = 0$ and

$(7k - 3h + 3) = 0$ —— (3)

Solving (3), we get $h = 1$, $k = 0$.

∴ $X = u + 1$, $Y = v + 0$

$\Rightarrow u = X - 1$, $v = Y$

Eq^n (2) becomes

$$\frac{dv}{du} = \frac{7u - 3v}{7v - 3u} = \frac{7 - 3(v/u)}{7(v/u - 3)} \quad \text{—— (4)}$$

Taking $\frac{v}{u} = w \Rightarrow v = uw$

$$\frac{dv}{du} = w + u\frac{dw}{du} \quad \text{—(5)}$$

From (4) and (5)

$$w + u\frac{dw}{du} = \frac{7-3w}{7w-3}$$

$$\Rightarrow u\frac{dw}{du} = \frac{7-3w}{7w-3} - w$$

$$\Rightarrow u\frac{dw}{du} = \frac{7-7w^2}{7w-3}$$

$$\Rightarrow \frac{7w-3}{7(1-w^2)}\,dw = \frac{du}{u}$$

$$\Rightarrow \frac{du}{u} = \frac{1}{7}\left[-5.\frac{dw}{w+1} - 2\frac{dw}{w-1}\right]$$

$$\Rightarrow \log u + \log c' = \frac{1}{7}\left[-5\log(w+1) - 2\log(w-1)\right]$$

$$\Rightarrow 7\log uc' = -\left[\log(w+1)^5.(w-1)^2\right]$$

$$\Rightarrow \log(uc')^7 = -\left[\log(w+1)^5.(w-1)^2\right]$$

$$\Rightarrow \log u^7.c + \log\left[(w+1)^5.(w-1)^2\right] = 0$$

$$\Rightarrow c\,u^7.(w+1)^5.(w-1)^2 = 0$$

$$\Rightarrow c = \frac{1}{(w+1)^5.(w-1)^2.u^7}$$

$$\Rightarrow \quad c = \frac{1}{(v/u+1)^5 (v/u-1)^2 u^7}$$

$$\Rightarrow \quad c = \frac{1}{\left(\frac{v+u}{u}\right)^5 \left(\frac{v-u}{u}\right)^2 \cdot u^7}$$

$$\Rightarrow \quad c = \frac{1}{\frac{(v+u)^5}{\cancel{u^5}} \cdot \frac{(v-u)^2}{\cancel{u^2}} \cdot \cancel{u^7}}$$

$$\Rightarrow \quad c = \frac{1}{(v+u)^5 (v-u)^2}$$

$$\Rightarrow \quad c = \frac{1}{(Y-X+1)^2 (Y+X-1)^5}$$

$$\Rightarrow \quad \boxed{c = \frac{1}{(y^2-x^2+1)^2 (y^2+x^2-1)^5}}$$ which

is the required solution.

Hence, the result.

5(b) Solve $y_2 - 2y_1 + y = xe^x \log x$, $x>0$ by the method variation of parameters.

Sol^n: Given $(D^2-2D+1)y = xe^x \log x$, $x>0$, $D \equiv \frac{d}{dx}$ — (1)

Comparing (1) with $y_2 + Py_1 + Qy = R$, here $R = xe^x \log x$

Consider $(D^2-2D+1)y = 0 \Rightarrow (D-1)^2 y = 0$ — (2)

Auxiliary equation of (2) is $(D-1)^2 = 0$ so that $D = 1, 1$

$\therefore$ C.F of (1) $= (c_1 + c_2 x)e^x = c_1 e^x + c_2 x e^x$,

c_1 & c_2 being arbitrary constants. — (3)

let $u = e^x$, $v = xe^x$

Also here $R = xe^x \log x$ — (4)

Here
$$W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x} \neq 0. \quad \text{— (5)}$$

Then P.I of (1) $= uf(x) + vg(x)$ — (6)

where $f(x) = -\int \frac{vR}{W} dx = -\int \frac{xe^x . xe^x \log x}{e^{2x}} dx$

$$= -\int x^2 \log x \, dx, \text{ by (4) and (5)}$$

$$= -\left[\log x . \frac{x^3}{3} - \int \frac{1}{x} \frac{x^3}{3} dx\right] = -\left[\frac{1}{3}x^3 \log x - \frac{1}{9}x^3\right]$$

$g(x) = \int \frac{uR}{W} dx = \int \frac{e^x . xe^x \log x}{e^{2x}} dx = \int x \log x \, dx$, by (4) & (5)

$$= \frac{x^2}{2}\log x - \int \frac{x^2}{2} \cdot \frac{1}{x} dx = \frac{x^2}{2}\log x - \frac{x^2}{4}$$

$\therefore$ (6) $\Rightarrow$ P.I $= -e^x\left\{(x^3/3)\log x - \frac{x^3}{9}\right\} + xe^x\left\{\frac{x^2}{2}\log x - \frac{x^2}{4}\right\}$

P.I $= x^3 e^x \log x\left(\frac{1}{2} - \frac{1}{3}\right) - x^3 e^x\left(\frac{1}{4} - \frac{1}{9}\right) = \frac{1}{6}x^3 e^x \log x - \frac{5}{36}x^3 e^x$

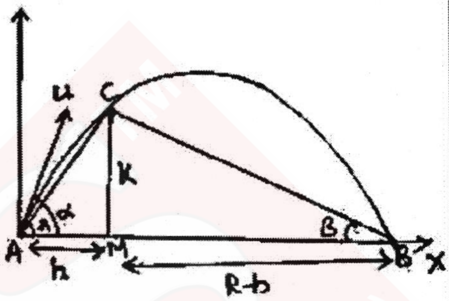
Hence the general solution of (1) is

$$y = c_1 e^x + c_2 xe^x + \frac{1}{6}x^3 e^x \log x - \frac{5}{36}x^3 e^x$$

5(c) A particle is thrown over a triangle from one end of a horizontal base and grazing over the vertex falls on the other end of the base. If A, B be the base angles of the triangle and α the angle of projection, Prove that

$$\tan\alpha = \tan A + \tan B.$$

Sol^n: Let A be the point of projection, u the velocity of projection and α the angle of projection.



The particle while grazing over the vertex C falls at the point B.

If AB = R, then $R = \dfrac{2u^2\sin\alpha\cos\alpha}{g}$ —(1)

Take the horizontal line AB as the x-axis and the vertical line AY as the y-axis. Let the coordinates of the vertex C be (h, k). Then the point (h, k) lies on the trajectory whose equation is

$$y = x\tan\alpha - \frac{1}{2}g\frac{x^2}{u^2\cos^2\alpha}$$

$$\therefore\ k = h\tan\alpha - \frac{1}{2}g\frac{h^2}{u^2\cos^2\alpha}$$

$$= h\tan\alpha\left[1 - \frac{gh}{2u^2\sin\alpha\cos\alpha}\right]$$

$$= h\tan\alpha\left[1-\frac{h}{R}\right] \quad \text{by (1)}$$

$$\therefore\ \frac{k}{h} = \tan\alpha\left(\frac{R-h}{R}\right) \quad \left[\because \text{from } \Delta CAM,\ \tan A = \frac{k}{h}\right]$$

$$\Rightarrow \tan A = \tan\alpha\left(\frac{R-h}{R}\right)$$

$$\therefore\ \tan\alpha = \tan A\left(\frac{R}{R-h}\right)$$

$$= \tan A\left(\frac{(R-h)+h}{R-h}\right)$$

$$= \tan A\left[1+\frac{h}{R-h}\right] = \tan A + \tan A\left(\frac{h}{R-h}\right)$$

$$= \tan A + \frac{k}{h}\cdot\frac{h}{R-h} \quad \left[\because \tan A = \frac{k}{h}\right]$$

$$= \tan A + k/(R-h)$$

But from the ΔCMB, $\tan B = k/(R-h)$

$$\therefore\ \tan\alpha = \tan A + \tan B$$

5(d) Find the directional derivative of the function $xy^2+yz^2+zx^2$ along the tangent to the curve $x=t,\ y=t^2,\ z=t^3$ at the point $(1,1,1)$.

Sol'n: Let $\phi(x,y,z) = xy^2+yz^2+zx^2$

Then $\text{grad}\,\phi = \frac{\partial\phi}{\partial x}\hat{i}+\frac{\partial\phi}{\partial y}\hat{j}+\frac{\partial\phi}{\partial z}\hat{k}$

$= (y^2+2zx)\hat{i}+(z^2+2xy)\hat{j}+(x^2+2yz)\hat{k}$

$= 3i+3j+3k$, at the point $(1,1,1)$

$=3(\hat{i}+\hat{j}+\hat{k})$

Also for the curve $x=t,\ y=t^2,\ z=t^3$, we have

$\frac{dx}{dt}=1,\quad \frac{dy}{dt}=2t,\quad \frac{dz}{dt}=3t^2.$

At the point $(1,1,1)$ on the curve $x=t,\ y=t^2,\ z=t^3$ we have $t=1$.

Now a vector along the tangent to the above curve at the point (x,y,z)

$= \left(\frac{dx}{dt}\right)\hat{i}+\left(\frac{dy}{dt}\right)\hat{j}+\left(\frac{dz}{dt}\right)\hat{k}$

$= i+2tj+3t^2k$

Putting $t=1$, a vector along the tangent to the curve at the point $(1,1,1) = \hat{i}+2\hat{j}+3\hat{k}$.

If $\hat{a}$ be the unit vector in the direction of this tangent, then $\hat{a} = \frac{i+2j+3k}{|i+2j+3k|} = \frac{\hat{i}+2\hat{j}+3\hat{k}}{\sqrt{14}}$

$\therefore$ The required directional derivative

$= \hat{a}\cdot\text{grad}\,\phi$ at $(1,1,1)$

$= \frac{i+2j+3k}{\sqrt{14}}\ 3(i+j+k)$

$= \frac{3}{\sqrt{14}}(1+2+3) = \frac{18}{\sqrt{14}}.$

5(e) (i). If A and B are irrotational, Prove that $A \times B$ is solenoidal

(ii). Prove that curl curl $(\phi \,\text{grad}\,\phi) = 0$.

Sol^n: (i) If A and B are irrotational, then curl $A = 0$ & curl $B = 0$

i.e. $\nabla \times A = 0$, $\nabla \times B = 0$

Now $\text{div}(A \times B) = B\cdot(\nabla \times A) - A\cdot(\nabla \times B)$

$= B\cdot(0) - A\cdot(0)$

$= 0$

$\therefore$ $A \times B$ is solenoidal.

(ii) we know that

$\text{Curl}(\phi A) = \text{grad}\,\phi \times A + \phi\, \text{curl}\, A$

putting grad ϕ in place of A, we get

$\text{Curl}(\phi\, \text{grad}\,\phi) = \text{grad}\,\phi \times \text{grad}\,\phi + \phi\, \text{curl}(\text{grad}\,\phi)$

$= 0 + \phi(0)$

$(\because \text{grad}\,\phi \times \text{grad}\,\phi = 0$ & $\text{curl}(\text{grad}\,\phi) = 0)$

$\therefore$ $\text{Curl}(\phi\, \text{grad}\,\phi) = 0$

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6(a)(i) Find the solution of the differential equation $y = 2xp - yp^2$ where $p = dy/dx$. Also find the singular solution.

Sol^n: Given $y = 2xp - yp^2$ —— ①

Solving ① for x, $x = y/2p + yp/2$ —— ②

Diff. ② w.r.t y and noting that $dy/dx = 1/p$

we get $\frac{1}{p} = \frac{1}{2p} - \frac{y}{2p^2}\frac{dp}{dy} + \frac{p}{2} + \frac{y}{2}\frac{dp}{dy}$

$$\Rightarrow \frac{y}{2}\frac{dp}{dy}\left(1 - \frac{1}{p^2}\right) + \frac{p}{2}\left(1 - \frac{1}{p^2}\right) = 0$$

$$\Rightarrow \frac{1}{2}\left(1 - \frac{1}{p^2}\right)\left(y\frac{dp}{dy} + p\right) = 0$$

omitting the first factor, for general solution we have

$$y\left(\frac{dp}{dy}\right) + p = 0$$

$$\Rightarrow \left(\frac{1}{p}\right)dp + \left(\frac{1}{y}\right)dy = 0$$

Integrating $\log p + \log y = \log c$

$$\Rightarrow py = c \Rightarrow p = c/y \quad —— ③$$

Eliminating p from ① & ③, the general solution is

$$y = (2xc)/y - (yxc^2)/y^2 \Rightarrow y^2 = 2cx - c^2 \quad —— ④$$

The p-disc. relation from ① i.e. $yp^2 - 2xp + y = 0$ is given by

$$4x^2 - 4y^2 = 0 \Rightarrow x^2 - y^2 = 0$$

$$\Rightarrow (x-y)(x+y) = 0$$

Hence $x - y = 0$ and $x + y = 0$ are singular solutions because these both discriminants and also satisfy ①.

6(a)(ii) Find the orthogonal trajectories of the family of curves $\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1$, where λ is a parameter

Sol: The given family of curves is

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1, \text{ where } \lambda \text{ as a parameter} \quad \text{--- (1)}$$

Differentiating (1) w.r.t. x, we get

$$\frac{2x}{a^2+\lambda}+\frac{2y}{b^2+\lambda}\frac{dy}{dx}=0$$

$$\Rightarrow x(b^2+\lambda)+y(a^2+\lambda)\frac{dy}{dx}=0$$

$$\Rightarrow \lambda\left(x+y\frac{dy}{dx}\right)=-\left(b^2x+a^2y\frac{dy}{dx}\right)$$

$$\Rightarrow \lambda=-\frac{\left(b^2x+a^2y\frac{dy}{dx}\right)}{x+y\frac{dy}{dx}}$$

$$\therefore a^2+\lambda=a^2-\frac{\left(b^2x+a^2y\frac{dy}{dx}\right)}{x+y\frac{dy}{dx}}$$

$$=\frac{(a^2-b^2)x}{x+y\frac{dy}{dx}}$$

and $$b^2+\lambda=b^2-\frac{\left(b^2x+a^2y\frac{dy}{dx}\right)}{x+y\frac{dy}{dx}}$$

$$=\frac{-(a^2-b^2)y\frac{dy}{dx}}{\left(x+y\frac{dy}{dx}\right)}$$

putting the above values of $a^2+\lambda$ and $b^2+\lambda$ in ①, we have

$$\frac{x^2\left\{x+y\frac{dy}{dx}\right\}}{(a^2-b^2)x} - \frac{y^2\left\{x+y\frac{dy}{dx}\right\}}{(a^2-b^2)\,y\frac{dy}{dx}} = 1$$

$$\Rightarrow \left\{x+y\frac{dy}{dx}\right\}\left\{x-y\frac{dx}{dy}\right\} = a^2-b^2 \quad \text{②}$$

which is the differential equation of the given family of curves ①

Replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in ②, the differential equation of the required orthogonal trajectories is

$$\left\{x+y\left(-\frac{dx}{dy}\right)\right\}\left[x-y\left\{-\frac{dy}{dx}\right\}\right) = a^2-b^2$$

$$\Rightarrow \left(x-y\frac{dx}{dy}\right)\left(x+y\frac{dy}{dx}\right) = a^2-b^2$$

Which is same as the differential equation ② of the given family of curves ①.

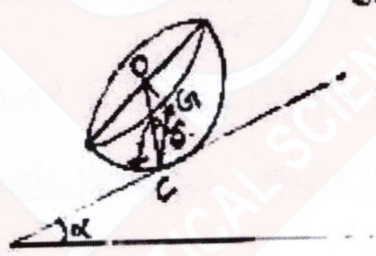
Hence the system of given curves ① is self orthogonal, i.e, each member of the given family of curves intersects its own members orthogonally.

6(b) → A uniform solid hemisphere rests on a rough plane inclined to the horizon at an angle ϕ with its curved surface touching the plane. Find the greatest admissible value of the inclination ϕ for equilibrium. If ϕ be less than this value is the equilibrium stable?

Sol^n: Let O be the centre of the base of the hemisphere and r be its radius. If C is the point of contact of the hemisphere and the inclined plane, then OC = r. Let G be the centre of gravity of the hemisphere.

Then OG = 3r/8. In the position of equilibrium the line CG must be vertical. ($\phi = \alpha$)

Since OC is ⊥lar to the inclined plane and CG is ⊥lar to the horizontal,



∴ $\angle OCG = \alpha$. Suppose in equilibrium the axis of the hemisphere makes an angle θ with the vertical. From △OGC, we have

$$\frac{OG}{\sin\alpha} = \frac{OC}{\sin\theta} \Rightarrow \frac{3r/8}{\sin\alpha} = \frac{r}{\sin\theta}$$

$$\therefore \sin\theta = \tfrac{8}{3}\sin\alpha \Rightarrow \theta = \sin^{-1}\left(\tfrac{8}{3}\sin\alpha\right)$$

Giving the position of equilibrium of the hemisphere

Since $\sin\theta < 1$, therefore $\tfrac{8}{3}\sin\alpha < 1$

$$\Rightarrow \sin\alpha < \tfrac{8}{3} \Rightarrow \alpha < \sin^{-1}\tfrac{3}{8}$$

Thus for the equilibrium to exist, we must have

$$\alpha < \sin^{-1}\tfrac{3}{8}.$$

Now let CG = h, Then

$$\frac{h}{\sin(\theta-\alpha)} = \frac{3r/8}{\sin\alpha} \quad \text{so that } h = \frac{3r\sin(\theta-\alpha)}{8\sin\alpha}$$

Here $\rho_1 = r$ & $\rho_2 = \infty$

The equilibrium will be stable if

$$h < \frac{\rho_1 \rho_2 \cos\alpha}{\rho_1 + \rho_2}$$

$$\Rightarrow \frac{1}{h} > \frac{\rho_1 + \rho_2}{\rho_1 \rho_2} \sec\alpha \Rightarrow \frac{1}{h} > \left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)\sec\alpha$$

$$\Rightarrow \frac{1}{h} > \frac{1}{r}\sec\alpha \qquad (\because \rho_1 = r,\ \rho_2 = \infty)$$

$$\Rightarrow h < r\cos\alpha$$

$$\Rightarrow \frac{3r\sin(\theta-\alpha)}{8\sin\alpha} < r\cos\alpha$$

$$\Rightarrow 3\sin(\theta-\alpha) < 8\sin\alpha . \cos\alpha$$

$$\Rightarrow 3\sin\theta\cos\alpha - 3\cos\theta\sin\alpha < 8\sin\alpha\cos\alpha$$

$$\Rightarrow 8\sin\alpha\cos\alpha - 3\sin\alpha\sqrt{\left(1-\frac{64}{9}\sin^2\alpha\right)} < 8$$

$$\left[\because \sin\theta = \tfrac{8}{3}\sin\alpha\right]$$

$$\Rightarrow -\sin\alpha\sqrt{(9-64\sin^2\alpha)} < 0 \quad \text{—— (2)}$$

$$\Rightarrow \sin\alpha\sqrt{(9-64\sin^2\alpha)} > 0$$

But from (1),

$\sin\alpha < 3/8$ i.e. $64\sin^2\alpha < 9$ i.e. $\sqrt{9-64\sin^2\alpha}$ is a +ve real number. Therefore the relation (2) is true Hence the equilibrium is stable.

6.(c)(i) → Show that $E = \mathbf{r}/r^2$ is irrotational. Find ϕ such that $E = -\nabla\phi$ and such that $\phi(a) = 0$ where $a > 0$.

Solution:

(a) $\vec{E}$ is irrotational if $\nabla \times \vec{E} = 0$

Now, $\nabla \times \vec{E} = \nabla \times \left(\frac{\vec{r}}{r^2}\right)$

$$= \nabla\left(\frac{1}{r^2}\right) \times \vec{r} + \frac{1}{r^2}(\nabla \times \vec{r})$$

$[\because \nabla \times (\phi A) = (\nabla\phi) \times A + \phi(\nabla \times A)]$

$$= -\frac{2}{r^4}(\vec{r} \times \vec{r}) + 0 \qquad [\because \nabla \times \vec{r} = 0]$$

$$= \frac{-2}{r^4}(0) \qquad [\because \vec{r} \times \vec{r} = 0]$$

$$= 0$$

i.e. $\nabla \times \vec{E} = 0$

$\Rightarrow$ **E is irrotational.** —— (1)

(b) Given $\vec{E} = -\nabla\phi = \frac{\vec{r}}{r^2}$

$$\Rightarrow \nabla\phi = -\vec{r}/r^2$$

$$\Rightarrow \hat{i}\frac{\partial\phi}{\partial x} + \hat{j}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z} = \frac{-\vec{r}}{r^2} \quad \text{—— (1)}$$

$$\frac{\partial\phi(r)}{\partial x} = \phi'(r)\frac{\partial r}{\partial x}$$

$$\Rightarrow \frac{\partial\phi(r)}{\partial x} = \phi'(r)\cdot\left(\frac{x}{r}\right) \qquad \left[\because r^2 = x^2+y^2+z^2,\ 2r\frac{\partial r}{\partial x} = 2x,\ \frac{\partial r}{\partial x} = \frac{x}{r}\right]$$

Similarly, $\frac{\partial\phi}{\partial y} = \phi'(r)\cdot\left(\frac{y}{r}\right)$ & $\frac{\partial\phi}{\partial z} = \phi'(r)\cdot\left(\frac{z}{r}\right)$

$\therefore$ from (1),
we have,

$$\phi'(r)\left(\frac{x}{r}\hat{i} + \frac{y}{r}\hat{j} + \frac{z}{r}\hat{k}\right) = -\frac{\vec{r}}{r^2}$$

$$\Rightarrow \quad \phi'(r)\cdot\frac{\vec{r}}{r} = -\frac{\vec{r}}{r^2}$$

$$\Rightarrow \quad \phi'(r) = -\frac{1}{r}$$

Integrating, we get

$$\phi(r) = -\log r + \log c$$

$$\Rightarrow \quad \phi(r) = \log\left(\frac{c}{r}\right) \quad \text{——} (2)$$

Given $\phi(a) = 0$

$\therefore$ from (2), $\phi(a) = -\log a + \log c$

$$\Rightarrow \quad 0 = -\log a + \log c$$

$$\Rightarrow \quad \log c = \log a$$

$$\Rightarrow \quad \boxed{c = a}$$

$\therefore$ from (2),

$$\boxed{\phi(r) = \log\frac{a}{r}}$$

Hence, the result.

6(c)(ii) ϕ_1 and ϕ_2 are two scalar functions such that $(\nabla^2+K^2)\phi_1=0$ and $(\nabla^2+K^2)\phi_2=0$ and $f=\nabla\times[r\phi_1+r\times\nabla\phi_2]$, show that $\text{div} f=0$, $(\nabla^2 \times K^2)f=0$.

Sol'n: $\text{div}\vec{f}=\nabla\cdot\vec{f}=\nabla\cdot[\nabla\times(\vec{r}\phi_1+\vec{r}\times\nabla\phi_2)]$

$$=\nabla\cdot\{[\nabla\times(\vec{r}\phi_1)]+\nabla\times(\vec{r}\times\nabla\phi_2)\}$$

$$=\nabla\cdot\{[\nabla\phi_1\times\vec{r}+\phi_1 \,\text{Curl}\,\vec{r}]\}+\nabla\cdot\{\nabla\times(\vec{r}\times\nabla\phi_2)\}$$

$$=\nabla\cdot\{\nabla\phi_1\times\vec{r}\}+\phi_1(0)\}+\nabla\cdot\{\nabla\times\vec{A}\}$$

where $\vec{A}=\vec{r}\times\nabla\phi_2$

$=0+0$ [$\because$ divergence of curl of a vector is zero]

$=0$

$$(\nabla^2+K^2)f=(\nabla^2+K^2)[\nabla\times(\vec{r}\phi_1+\vec{r}\times\nabla\phi_2)]$$

$$=(\nabla^2+K^2)\{[\vec{r}\times\nabla\phi_1+\phi_1(\nabla\times\vec{r})]+\nabla\times(\vec{r}\times\nabla\phi_2)\}$$

$$=(\nabla^2+K^2)(\vec{r}\times\nabla\phi_1)+(\nabla^2+K^2)[(\nabla^2\phi_2)\vec{r}-3\nabla\phi_2]$$

$$=(\nabla\times\vec{r})(\nabla^2+K^2)\phi_1+\nabla^2[(\nabla^2+K^2)\phi_2]\vec{r}$$
$$-3\nabla(\nabla^2+K^2)\phi_2$$

$=0+0-0=0.$ $\quad$ [$\because (\nabla^2+K^2)\phi_1=0$, $(\nabla^2+K^2)\phi_2=0$, $\nabla\times\vec{r}=0$).

(or)

$$\text{div} f=\nabla[\nabla\times\{(r\phi_1)+(\vec{r}\times\nabla\phi_2)\}]$$

$$=\nabla[\nabla\times r\phi_1]+\nabla\cdot[\nabla\times(r\times\nabla\phi_2)]$$

$$=[\nabla,\nabla,r\phi_1]+[\nabla,\nabla,r\times\nabla\phi_2]$$

$=0+0$

$=0.$

7(a) Solve $(x+2)y'' - (2x+5)y' + 2y = (x+1)e^x$.

Sol^n: Dividing by $(x+2)$, the given equation in standard form is

$$\frac{d^2y}{dx^2} - \frac{2x+5}{x+2}\frac{dy}{dx} + \frac{2}{x+2}y = \frac{x+1}{x+2}e^x \quad \text{—(1)}$$

Comparing (1) with $y'' + Py' + Qy = R$, we have

$$P = -(2x+5)/x+2, \quad Q = 2/(x+2), \quad R = [(x+1)/(x+2)]e^x \text{ — (2)}.$$

Here $2^2 + 2P + Q = 4 + 2\left[-\frac{2x+5}{x+2}\right] + \frac{2}{x+2} = \frac{4(x+2) - 2(2x+5)+2}{x+2} = 0$

showing that a part of C.F of (1) is

$$u = e^{2x} \quad \text{—— (3)}$$

Let the general solution be $y = uv$ —— (4)

Then v is given by $\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$

$$\frac{d^2v}{dx^2} + \left(-\frac{2x+5}{x+2} + \frac{2}{e^{2x}}\frac{de^{2x}}{dx}\right)\frac{dv}{dx} = \frac{x+1}{x+2}\frac{e^x}{e^{2x}}$$

$$\Rightarrow \frac{d^2v}{dx^2} + \left(4 - \frac{2x+5}{x+2}\right)\frac{dv}{dx} = \frac{x+1}{x+2}e^{-x}$$

$$\Rightarrow \frac{d^2v}{dx^2} + \frac{2x+3}{x+2}\frac{dv}{dx} = \frac{x+1}{x+2}e^{-x} \quad \text{— (5)}$$

Let $\frac{dv}{dx} = q$ so that $\frac{d^2v}{dx^2} = dq/dx$ — (6)

Then (5) becomes $\frac{dq}{dx} + \frac{2x+3}{x+2}q = \frac{x+1}{x+2}e^{-x}$ — (7)

Now, $E = \int \frac{2x+3}{x+2}dx = \int\left(2 - \frac{1}{x+2}\right)dx = 2x - \log(x+2)$

$\therefore$ I.F of (7) $= e^E = e^{2x-\log(x+2)} = e^{2x}e^{-\log(x+2)} = e^{2x}e^{\log(x+2)^{-1}}$
$= e^{2x}(x+2)^{-1}$

and solution is

$$\therefore q.e^{2x}(x+2)^{-1} = C_1 + \int \frac{x+1}{x+2}e^{-x}e^{2x}(x+2)^{-1}dx$$

$$= \int \frac{x+1}{(x+2)^2} e^x dx + c_1$$

$$= \int \frac{(x+2)-1}{(x+2)^2} e^x dx + c_1 = \int \frac{1}{x+2} e^x dx - \int \frac{1}{(x+2)^2} e^x dx + c_1$$

$$= \frac{1}{x+2} e^x - \int \left\{ -\frac{1}{(x+2)^2} \right\} e^x dx - \int \frac{1}{(x+2)^2} e^x dx + c_1$$

(Integrating by parts only the first integral)

$$\Rightarrow (qe^{2x})/(x+2) = (x+2)^{-1} e^x + c_1$$

$$\Rightarrow q = \frac{dv}{dx} = e^{-x} + c_1 e^{-2x}(x+2), \text{ by } (6)$$

$$\therefore dv = [e^{-x} + c_1 e^{-2x}(x+2)]dx.$$

Integrating $v = -e^{-x} + c_1 \int e^{-2x}(x+2)\,dx + c_2$

c_1, c_2 being arbitrary constants.

$$\Rightarrow v = -e^{-x} + c_1 \left[(x+2)(-\tfrac{1}{2})e^{-2x} - \int 1(-\tfrac{1}{2})e^{-2x} dx \right] + c_2$$

$$\Rightarrow v = -e^{-x} + c_1[(x+2)\times(-1/2)e^{-2x} - \tfrac{1}{4}e^{-2x}] + c_2$$

$$= -e^{-x} + (c_1/4)(2x+5) + c_2 \quad \text{---} (8).$$

from (3), (4) and (8), the required general solution is

$$y = uv = e^{2x}[-e^{-x} + (c_1/4)(2x+5) + c_2]$$

$$= c_1' e^{2x}(2x+5) + c_2 e^{2x} - e^x, \text{ where } c_1' = c_1/4.$$

7.(b) A uniform rod AB of length 2a movable about a hinge at A rests with other end against a smooth vertical wall. If α is the inclination of the rod to the vertical, Prove that the magnitude of reaction of the hinge is $\frac{1}{2}W\sqrt{4+\tan^2\alpha}$ where W is the weight of the rod.

Sol^n: Let a uniform rod AB of length 2a movable about the hinge at the end A rest with a smooth vertical wall CD.

Let W be the weight of the rod and G its middle point.

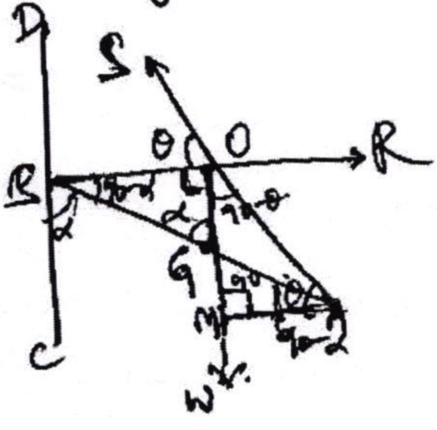
The rod is in equilibrium under the action of the following three forces only.

(i) R, the reaction of the wall at B acting at right angles to the wall.

(ii) S, the reaction of the hinge at A. and

(iii) W, the weight of the rod acting vertically downwards at its middle point G.

Since the force R and the line of action of W meet at O, therefore the reaction S of the hinge at A must also pass through O, as shown in the figure.



Let the rod AB and the reaction S make angles α and θ ~~respectively~~ with the vertical and horizontal respectively.

1.4

$\angle ABC = \alpha$, and $\angle OAM = \theta$. and $\angle ABO = 90 - \alpha$.

$\therefore \angle OGB = \alpha$ and $\angle AOM = 90 - \theta$.

In ΔOAB, by the trigonometrical theorem, we have

$(AG + BG)\cot OGB = AG \cot AOG - BG \cot BOG$.

$(a+a)\cot\alpha = a\cot(90-\theta) - a\cot 90$.

$2a\cot\alpha = a\tan\theta - 0$

$\tan\theta = 2\cot\alpha$. —— (1)

$\therefore$ the reaction S at the hinge makes an angle $\theta = \tan^{-1}(2\cot\alpha)$ with the horizontal.

Now by Lami's theorem at the point O, we have

$$\frac{S}{\sin 90} = \frac{W}{\sin(180-\theta)} = \frac{R}{\sin(90+\theta)}$$

$$\therefore S = \frac{W}{\sin\theta} = W\operatorname{cosec}\theta = W\sqrt{1+\cot^2\theta}$$

$$= W\sqrt{\left(1+\frac{1}{4}\tan^2\alpha\right)}$$

$$= \frac{W}{2}\sqrt{4+\tan^2\alpha}.$$

7(C) verify the divergence theorem for $A = 4xi - 2y^2 j + z^2 k$ taken over the region bounded by $x^2+y^2=4$, $z=0$ and $z=3$.

Sol^n: Let S denote the closed surface bounded by the cylinder $x^2+y^2=4$ and the planes $z=0, z=3$

Also let V be the volume bounded by the surface S. By Gauss divergence theorem,

we have $\iint_S F\cdot\hat{n}\, ds = \iiint_V \operatorname{div} F\, dV$

$$\text{volume integral} = \iiint_V \operatorname{div} F\, dv$$

$$= \iiint_V \nabla\cdot F\, dV$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(4x) + \frac{\partial}{\partial y}(-2y^2) + \frac{\partial}{\partial z}(z^2)\right] dV$$

$$= \iiint_V (4-4y+2z)\, dV$$

$$= 2\int_{z=0}^{3}\int_{x=-2}^{2}\int_{y=-\sqrt{4-x^2}}^{\sqrt{4-x^2}} (2-2y+z)\, dz\, dx\, dy$$

$$= 2\int_{z=0}^{3}\int_{x=-2}^{2}\left[2y - y^2 + zy\right]_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dz\, dx$$

$$= 2\int_{z=0}^{3}\int_{x=-2}^{2}\left[(2+z)2\sqrt{4-x^2} - 0\right] dz\, dx$$

$$= 4\int_{z=0}^{3}\int_{x=-2}^{2}\left[2\sqrt{4-x^2} + z\sqrt{4-x^2}\right] dz\, dx$$

$$= 4\int_{x=-2}^{2}\left[2z\sqrt{4-x^2} + \frac{z^2}{2}\sqrt{4-x^2}\right]_{z=0}^{3} dx$$

$$= 4\int_{x=-2}^{2} \left[6\sqrt{4-x^2} + \frac{9}{2}\sqrt{4-x^2}\right]dx$$

$$= 4\times\frac{21}{2}\int_{x=-2}^{2}\sqrt{4-x^2}\,dx$$

$$= \frac{84}{2}\times 2\int_{0}^{2}\sqrt{4-x^2}\,dx$$

$$= 84\left[\frac{x}{2}\sqrt{4-x^2} + 2\sin^{-1}\frac{x}{2}\right]_0^2$$

$$= 84\left[0 + 2\times\frac{\pi}{2}\right] = 84\pi$$

Now we shall evaluate the surface integral

$$\iint_S F\cdot\hat{n}\,ds$$

Here the surface S consists of three surfaces

i) the surface S_1 of the base i.e., the plane face $z=0$ of the cylinder.

ii) the surface S_2 of the top i.e, the plane $z=3$ of the cylinder and

iii) the surface S_3 of the convex position of the cylinder

For the surface S_1 i.e. $z=0$, $F = 4xi - 2y^2 j$, putting $z=0$ in F.

A unit vector $\hat{n}$ along the outward drawn normal to S_1 is obviously $-\hat{k}$.

$$\therefore \iint_{S_1} F\cdot n\,ds = \iint_{S_1}(4xi - 2y^2 j)\cdot(-k)\,ds$$

$$= \iint_{S_1} 0\,ds = 0.$$

For the surface S_2 i.e. $z=3$, $\vec{F} = 4x\hat{i} - 2y\hat{j} + 9\hat{k}$

Putting $z=3$ in F.

A unit vector $\hat{n}$ along the outward drawn normal to S_2 is obviously k.

$$\therefore \iint_{S_2} F.n\,ds = \iint_{S_2} (4xi - 2y^2 j + 9k).k\,ds$$

$$= \iint_{S_2} 9\,ds = 9\,2\pi(2)$$

$$= 36\pi \quad (\because \text{area of } S_2 = 2\pi r \text{ here } r=2)$$

For the convex portion S_3 i.e. $x^2+y^2=4$, a vector normal to S_3 is given by

$$\nabla(x^2+y^2) = 2xi + 2yj$$

$\therefore n =$ a unit vector along outward drawn normal at any point of S_3

$$= \frac{2xi+2yj}{\sqrt{4x^2+4y^2}} = \frac{xi+yj}{2} \quad (\because x^2+y^2=4 \text{ on } S_3)$$

$$\therefore \text{on } S_3 \quad F.\hat{n}\,ds = (4xi - 2y^2 j + z^2 k).\tfrac{1}{2}(xi+yj)$$

$$= 2x^2 - y^3$$

Also ds = elementry of area on the surface S_3

$= 2\,d\theta\,dz$, using cylindrical coordinates r, θ, z.

$$\therefore \iint_{S_3} F.\hat{n}\,ds = \iint_{S_3} (2x^2-y^3)\,2\,d\theta\,dz \quad \text{where } x = 2\cos\theta,\ y = 2\sin\theta$$

$$= \int_{z=0}^{3}\int_{\theta=0}^{2\pi} (8\cos^2\theta - 8\sin^3\theta)\,2\,d\theta\,dz$$

$$= 16\int_{\theta=0}^{2\pi} \left[\cos^2\theta - \sin^3\theta\right]\left[z\right]_{z=0}^{3} d\theta$$

$$= 48 \int_0^{2\pi} (\cos^2\theta - \sin^3\theta)\, d\theta$$

$$= 48 \left[4 \int_0^{\pi/2} \cos^2\theta\, d\theta - \int_0^{2\pi} \sin^3\theta\, d\theta \right]$$

$$= 48 \left[4 \cdot \tfrac{1}{2} \cdot \tfrac{\pi}{2} - 0 \right] \qquad \left(\because \sin^3(2\pi - \theta) = -\sin^3\theta \text{ Odd function } \int_0^{2\pi} \sin^3\theta\, d\theta = 0 \right)$$

$$= 48\pi$$

Hence the required surface integral

$$= \iint_S F\cdot\hat{n}\,ds = \iint_{S_1} F\cdot\hat{n}\,ds + \iint_{S_2} F\cdot\hat{n}\,ds + \iint_{S_3} F\cdot\hat{n}\,ds$$

$$= 0 + 36\pi + 48\pi$$

$$= 84\pi$$

$$\therefore \iint F\cdot\hat{n}\,ds = 84\pi$$

$$\therefore \iint_S F\cdot\hat{n}\,ds = \iiint_V \operatorname{div} F\, dv.$$

8(a) Solve $(D^4+2D^2+1)y=0$, where $y(0)=0$, $y'(0)=1$, $y''(0)=2$ and $y'''(0)=-3$.

Sol^n: Given that $(D^4+2D^2+1)y=0$ —— (1)

Equation (1) can be written as

$$y^{iv}+2y''+y=0$$

Taking Laplace transform of bothsides of above equation, we have

$$L\{y^{iv}\}+2L\{y''\}+L\{y\}=0$$

$$\Rightarrow p^4L\{y\}-p^3y(0)-p^2y'(0)-py''(0)-y'''(0)$$
$$+2[p^2L\{y\}-py(0)-y'(0)]+L\{y\}=0$$

$$\Rightarrow (p^4+2p^2+1)L\{y\}-p^2-2p+3+2(-1)=0$$

$$\Rightarrow (p^2+1)^2L\{y\}=p^2+2p-1$$

$$\Rightarrow L\{y\}=\frac{p^2+2p-1}{(p^2+1)^2}$$

$$=\frac{1}{p^2+1}+\frac{2p}{(p^2+1)^2}-\frac{2}{(p^2+1)^2}$$

$$y=L^{-1}\left\{\frac{1}{p^2+1}\right\}+L^{-1}\left\{\frac{2p}{(p^2+1)^2}\right\}-2L^{-1}\left\{\frac{1}{(p^2+1)^2}\right\}$$

$$=\sin t-L^{-1}\left\{\frac{d}{dp}\left(\frac{1}{p^2+1}\right)\right\}-2L^{-1}\left\{\frac{1}{(p^2+1)^2}\right\}$$

$$=\sin t-(-1)^1t\,L^{-1}\left(\frac{1}{p^2+1}\right)-2L^{-1}\left\{\frac{1}{(p^2+1)^2}\right\}$$

$$=\sin t+t\sin t-2L^{-1}\left\{\frac{1}{(p^2+1)^2}\right\} \quad \text{—— (2)}$$

Now let $L^{-1}\left\{\frac{1}{(p^2+1)^2}\right\}=L^{-1}\left\{\frac{1}{p}\,\frac{p}{(p^2+1)^2}\right\}$ —— (3)

Now, $L^{-1}\left\{\frac{p}{(p^2+1)^2}\right\}=L^{-1}\left\{-\frac{1}{2}\,\frac{d}{dp}\left\{\frac{1}{(p^2+1)}\right\}\right\}$

$$=-\frac{1}{2}L^{-1}\frac{d}{dp}\left(\frac{1}{p^2+1}\right)$$

$$= -\frac{1}{2}(t)(-1)^1 L^{-1}\left(\frac{1}{p^2+1}\right)$$

$$= \frac{1}{2} t \sin t = F(t) \text{ say}$$

$$\therefore (3) \equiv L^{-1}\left\{\frac{1}{p}\frac{p}{(p^2+1)^2}\right\} = \int_0^t F(x)\,dx \qquad \left[\because L^{-1}\left[\frac{f(p)}{p}\right] = \int_0^t F(x)\,dx \text{ if } L^{-1}\{f(p)\} = F(x)\right]$$

$$= \int_0^t \frac{1}{2} x \sin x\,dx$$

$$= \frac{1}{2}(\sin t - t\cos t)$$

$$\therefore (2) \equiv y = \sin t + t\cos t - 2\cdot\frac{1}{2}(\sin t - t\cos t)$$

$$= t(\sin t + \cos t)$$

8.(b) A Particle moves with a central acceleration which varies inversely as the cube of the distance. If it be projected from an apse at a distance a from the origin with a velocity which is $\sqrt{2}$ times the velocity for a circle of radius a, show that the equation to its path is $r\cos(\theta/\sqrt{2})=a$

Sol'n: Here the central acceleration varies inversely as the cube of the distance i.e., $P=\mu/r^3=\mu u^3$, where μ is a constant. If V is the velocity for a circle of radius a, then

$$\frac{V^2}{a}=[P]_{r=a}=\frac{\mu}{a^3}$$

$$V=\sqrt{(\mu/a^2)}$$

$\therefore$ the velocity of projection $v_1=\sqrt{2}V=\sqrt{(2\mu/a^2)}$

The differential equation of the path is

$$h^2\left[u+\frac{d^2u}{d\theta^2}\right]=\frac{P}{u^2}=\frac{\mu u^3}{u^2}=\mu u.$$

Multiplying bothsides by $2(du/d\theta)$ and integrating, we get

$$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu u^2+A \quad \text{—} \; (1)$$

where A is a constant.

But initially when $r=a$, i.e. $u=1/a$, $du/d\theta=0$ (at an apse), and $v=v_1=\sqrt{(2\mu/a^2)}$.

$\therefore$ from (1), we have

$$\frac{2\mu}{a^2}=h^2\left[\frac{1}{a^2}\right]=\frac{\mu}{a^2}+A$$

$\therefore$ $h^2=2\mu$ and $A=\mu/a^2$

Substituting the values of h^2 and A in (1), we have

$$2\mu\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu u^2+\frac{\mu}{a^2}$$

$$\Rightarrow 2\left(\frac{du}{d\theta}\right)^2=\frac{1}{a^2}+u^2-2u^2=\frac{1-a^2u^2}{a^2}$$

$$\Rightarrow \sqrt{2}\,a\frac{du}{d\theta}=\sqrt{(1-a^2u^2)} \Rightarrow d\theta/\sqrt{2}=\frac{a\,du}{\sqrt{(1-a^2u^2)}}$$

Integrating, $(\theta/\sqrt{2})+B=\sin^{-1}(au)$, where B is a constant.

But initially, when $u=1/a$, $\theta=0$, $\therefore$ $B=\sin^{-1}1=\frac{1}{2}\pi$

$\therefore$ $(\theta/\sqrt{2})+\frac{1}{2}\pi=\frac{1}{2}\sin^{-1}(au) \Rightarrow au=a/r=\sin\left\{\frac{1}{2}\pi+(\theta/\sqrt{2})\right\}$

$\Rightarrow a=r\cos(\theta/\sqrt{2})$, which is required equation of the path.

8.(C) If $A = 2yz\,i - (x+3y-2)\,j + (x^2+z)\,k$, evaluate $\iint_S (\nabla \times A).n\,ds$ over the surface of intersection of the cylinders $x^2+y^2=a^2$, $x^2+z^2=a^2$ which is included in the first octant.

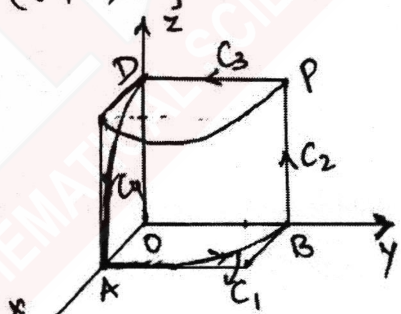
Sol'n: By Stoke's theorem

$$\iint_S (\nabla \times \vec{A}).\hat{n}\,ds = \int_C \vec{A}.d\vec{r} \quad \text{—— (1)}.$$

Here C is the Curve consisting of four arcs namely $C_1 : AB$, $C_2 : BP$, $C_3 : PD$, $C_4 : DA$. Thus we evaluate RHS of (1), along these four arcs one by one.

Along C_1: $z=0$, $x^2+y^2=a^2$ y varies from 0 to a.

$$\int_{C_1} \vec{A}.d\vec{r} = \int_{C_1} [2yz\,dx - (x+3y-2)\,dy + (x^2+z)\,dz]$$



$$= -\int_{C_1} (x+3y-2)\,dy$$

$$= -\int_0^a \left[\sqrt{a^2-y^2} + 3y - 2\right] dy$$

$$= -\left[\frac{y}{2}\sqrt{a^2-y^2} + \frac{a^2}{2}\sin^{-1}\frac{y}{a} + \frac{3}{2}y^2 - 2y\right]_0^a$$

$$\int_{C_1} \vec{A}.d\vec{r} = -\frac{\pi a^2}{2} - \frac{3a^2}{2} + 2a \quad \text{—— (2)}.$$

Along C_2: $x=0$, $y=a$; $dx=0$; $dz=0$ and y varies from 0 to a.

$$\therefore \int_{C_2} \vec{A}.d\vec{r} = \int_{C_2} z\,dz = \int_0^a z\,dz = \left[\frac{z^2}{2}\right]_0^a = \frac{a^2}{2} \quad \text{—— (3)}$$

Along C_3: $x=0$, $z=a$; $dx=0$; $dz=0$ and y varies from a to 0.

$$\therefore \int_{C_3} \vec{A}.d\vec{r} = \int_{C_3} (3y-2)\,dy = -\int_0^a (3y-2)\,dy$$

$$= \left[-\frac{3y^2}{2} + 2y\right]_a^0 = \frac{3a^2}{2} - 2a \quad \text{——} (4)$$

Along C_4: $y=0$, $x^2+z^2=a^2$, z varies from a to 0.

$$\therefore \int_{C_4} \vec{A}\cdot d\vec{r} = \int_a^0 (x^2+z)\,dz = \int_a^0 (a^2 - z^2 + z)\,dz$$

$$= \left(a^2 z - \frac{1}{3}z^3 + \frac{z^2}{2}\right)_a^0 = \frac{-2a^3}{3} - \frac{a^2}{2} \quad \text{——} (5)$$

Thus, the desired integral is sum of (2), (3), (4), (5)

i.e. $$\iint_S (\nabla\times\vec{A})\cdot\hat{n}\,ds = -\frac{\pi a^2}{4} - \frac{3a^2}{2} + 2a + \frac{a^2}{2} + \frac{3a^2}{2} - 2a - \frac{2a^3}{3} - \frac{a^2}{2}$$

$$= -\frac{\pi a^2}{4} - \frac{2a^3}{3}$$

$$= -\frac{a^2}{12}(3\pi + 8a)$$