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IAS MATHEMATICS (OPT.)-2012

PAPER - II : SOLUTIONS

2012.

1(a) P-II How many elements of order 2 are there in the group of order 16 generated by a and b such that the order of a is 8, the order of b is 2 and $bab^{-1} = a^{-1}$.

Ans. Let G be the group generated by a and b

$\therefore |a| = 8$ and $|b| = 2$

$\Rightarrow b^2 = e$

$\Rightarrow b = b^{-1}$

$bab^{-1} = a^{-1}$ — ①

$ba = a^{-1}b$

$\therefore a^m b^n \in G$

In the general form of elements of G as $b^n a^m$ can be written as:-

$ba = a^{-1}b$

$\therefore \underbrace{(b \ldots b)}_{n \text{ times}} \underbrace{(a \ldots a)}_{m \text{ times}} = \underbrace{(b \ldots b)}_{(n-1)} ba \underbrace{(a \ldots a)}_{(m-1)}$

$= (b \ldots b)(a^{-1}b)(a \ldots a)$

$= a^{-1} \underbrace{(b \ldots b)}_{n \text{ times}} \underbrace{(a \ldots a)}_{(m-1)}$

$= a^{-m} b^n$

$G = \{e, a, a^2, a^3, a^4, a^5, a^6, a^7, b, ab, a^2b, a^3b, a^4b, a^5b, a^6b, a^7b\}$

elements of order 2 are:-

i) b

ii) a^4

iii) $ab \cdot ab = abab^{-1} = aa^{-1} = e \quad \therefore |ab| = 2$

iv) $a^2b \cdot a^2b = a^2ba^2b^{-1} = a^2(a^2)^{-1} = e$

$\therefore |a^2b| = 2$

v) $|a^3b| = 2$

vi) $|a^4b| = 2$

vii) $|a^5b| = 2$

viii) $|a^6b| = 2$

ix) $|a^7b| = 2$

$(a^m b)(a^m b) = a^m(ba^m b)$

$= a^m(ba^m b^{-1})$

$= a^m(a^m)^{-1} = e$

$\therefore ab, a^2b, a^3b, a^4b, a^5b, a^6b, a^7b$ have order 2

$\therefore$ 9 elements are of order 2.

1(c) PAS 2012

Show that the function defined by

$$f(z) = \begin{cases} \dfrac{x^2y^5(x+iy)}{x^6+y^{10}}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

is not analytic at the origin though it satisfies Cauchy - Riemann equations at the origin.

Sol^n

Here $u+iv = \dfrac{x^2y^5(x+iy)}{x^4+y^{10}}$

$\therefore \quad u = \dfrac{x^3y^5}{x^4+y^{10}}, \quad v = \dfrac{x^2y^5}{x^4+y^{10}}$

At the origin,

$$u_x = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x \to 0} \frac{0-0}{x} = 0$$

$$u_y = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{y \to 0} \frac{0-0}{y} = 0$$

Similarly $v_x = 0$, $v_y = 0$

Hence Cauchy. Riemann equations are satisfied at the origin.

But $\displaystyle\lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{\substack{x \to 0 \\ y \to 0}} \left[\frac{x^2y^5(x-iy)}{x^4+y^{10}} - 0\right]\frac{1}{x+iy}$

Since $f(0) = 0$ and $z = x+iy$

$$= \lim_{\substack{x\to 0\\ y\to 0}} \frac{x^2 y^5}{x^4+y^{10}}$$

$$= \lim_{x\to 0} \frac{x^2 m^5 x^5}{x^4 + m^{10} x^{10}}$$

if $z\to 0$ along the radius vector $y = mx$

$$= \lim_{x\to 0} \frac{m^5 x^3}{1 - m^{10} x^6} = 0$$

and $= \lim_{x\to 0} \dfrac{x^2 x^2}{x^4 + x^4}$ if $z\to 0$ along the curve $y^5 = x^2$

$$= \frac{1}{2}$$

Showing that $f'(0)$ does not exist.

Hence $f(z)$ is not analytic at origin although Cauchy-Riemann equations are satisfied here.

2012

Q.1(d) For each hour per day that Ashok studies mathematics, it yields him 10 marks and for each hour that he studies physics, it yields him 5 marks. He can study at most 14 hours a day and he must get at least 40 marks in each. Determine graphically how many hours a day he should study mathematics and physics each, in order to maximize his marks?

Sol: Let 'x' represents no. of hours Ashok studies mathematics

and 'y' represents no. of hours Ashok studies physics

and 'z' represents the function.

According to given –

maximize: $z = 10x + 5y$

Subject to

$x + y \le 14$ (at most 14 hours a day)

$x \geq 4$ } at least 40 marks

$y \geq 8$ } in each subject.

and $x, y \geq 0$ ($\because$ marks can't be negative)

writing in equal form

Maximize $z = 10x + 5y$

Subject to $x + y = 14$

$x = 4$ [$\because$ 40 mark – 10 marks each hour]

$y = 8$

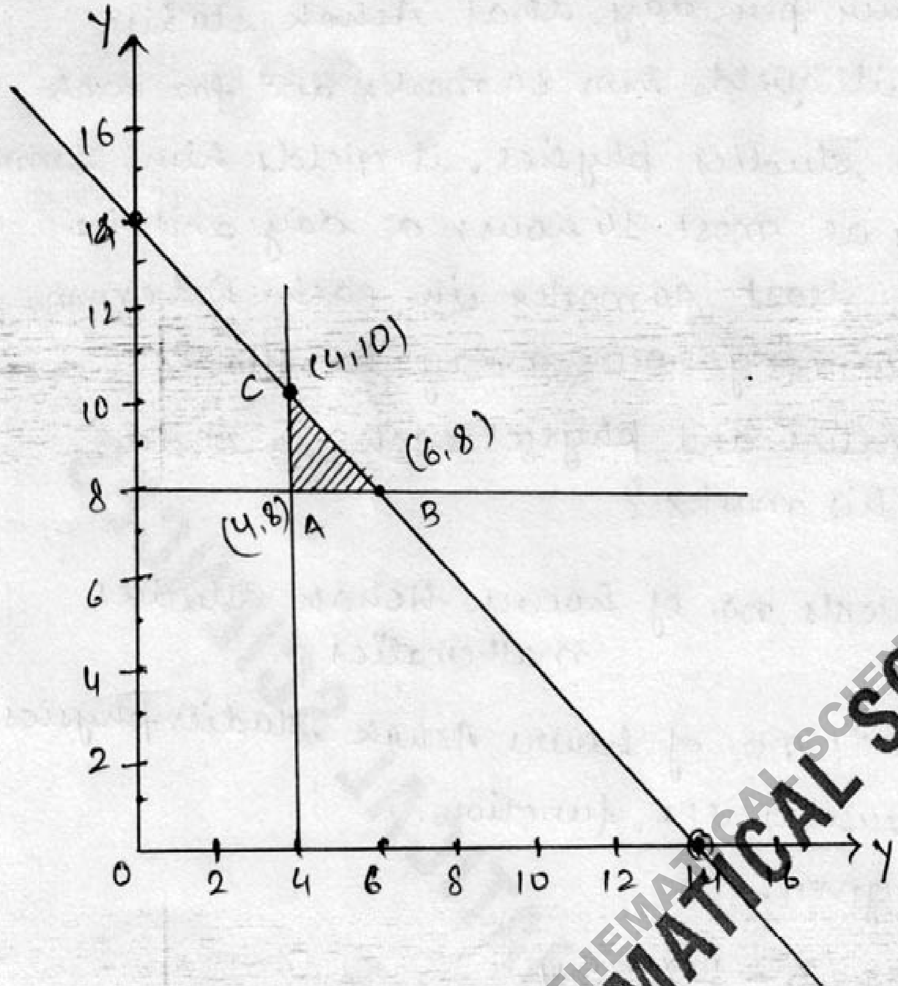
$x, y \geq 0$

Writing basic solution for $x + y = 14$

x	0	14
y	14	0

Considering only positive axes of x and y as

$x, y \geq 0$.



$A(4,8) \Rightarrow z = 10\times 4 + 5\times 8 = 80$

At $(6,8) \Rightarrow z = 10\times 6 + 5\times 8 = 100$ — Maximum

At $(4,10) \Rightarrow z = 10\times 4 + 5\times 10 = 90$

So, to get maximum marks, Ashok should study Mathematics for 6 hours and physics for 8 hours.

1(a) How many conjugacy classes does the permutation group S_5 of permutations of 5 numbers have? Write down one element in each class (preferably in terms of cycles).

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Ans.

$S_5 = \{I, (12), (13), (14) \text{ ——— } 120 \text{ elements}\}$

$[I] = \{\theta I \theta^{-1} \mid \theta \in S_5\} = \{I\}$

$[(12)] = \{\theta (12) \theta^{-1} \mid \theta \in S_5\}$

$= \{\theta(1)\ \theta(2) \mid \theta \in S_5\}$

= all 2-cycles of S_5

if $\theta = I$, $\theta(12)\theta^{-1} = (\theta(1)\ \theta(2))$

$= (1\ 2) \in [(1\ 2)]$

$[(1\ 2\ 3)] = \{(\theta(1)\ \theta(2)\ \theta(3)) \mid \theta \in S_5\}$

= all 3 cycles of S_5

if $\theta = (1\ 4\ 3)$

$\theta(1\ 2\ 3)\theta^{-1} = (\theta(1)\ \theta(2)\ \theta(3))$

$= (4\ 2\ 1) \in [(1\ 2\ 3)]$

$[(1\ 2\ 3\ 4)] = \{\theta(1)\ \theta(2)\ \theta(3)\ \theta(4)) \mid \theta \in S_5\}$

= all 4-cycles of S_5

if $\theta = (1\ 5\ 3\ 4\ 2)$

$\theta(1\ 2\ 3\ 4)\theta^{-1} = (\theta(1)\ \theta(2)\ \theta(3)\ \theta(4))$

$= (5\ 4\ 2) \in [(1\ 2\ 3\ 4)]$

$[(1\ 2\ 3\ 4\ 5)] = \{(\theta(1)\ \theta(2)\ \theta(3)\ (\theta(4)) \mid \theta \in S_5\}$

= all 5-cycles of S_5.

if $\theta = (1\ 2\ 3\ 5)$

$\theta(1\ 2\ 3\ 4\ 5)\theta^{-1} = \theta(1)\ \theta(2)\ \theta(3)\ \theta(4)\ \theta(5)$

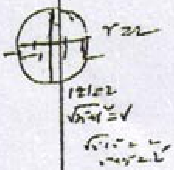
$= (2\ 3\ 5\ 4\ 1)$

$\in [(1\ 2\ 3\ 4\ 5)]$

IAS 2012 20 P-II

Use Cauchy integral formula to evaluate $\int_C \frac{e^{3z}}{(z+1)^4}\,dz$, where C is the circle $|z|=2$.

Sol: comparing the given-integral with $\int_C \frac{f(z)}{(z-z_0)^n}\,dz$,



we get

$f(z) = e^{3z}$, $z=-1$

· Since $f(z)$ is analytic in $|z|=2$

and $z_0=-1$ is a point inside $|z|=2$

∴ we apply Cauchy's Integral formula

$$\int_C \frac{dz}{(z-z_0)^4} = \frac{2\pi i}{n!} f^3(z_0) \quad \text{—} (1)$$

Now $f(z) = e^{3z}$ $\quad (\because z_0 = -1)$

$\Rightarrow f'(z) = 3e^{3z} \Rightarrow f'(-1) = 3e^{-3}$

$\Rightarrow f''(z) = 9e^{3z} \Rightarrow f''(-1) = 9e^{-3}$

$\Rightarrow f'''(z) = 27e^{3z} \Rightarrow f'''(-1) = 27e^{-3}$

∴ from (1), we have

$$\int_C \frac{dz}{(z+1)^4} = \frac{2\pi i}{3!} f'''(-1)$$

$$= \frac{2\pi i}{6}(27)e^{-3}$$

$$= 9\pi i e^{-3}$$

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③(A) Is the ideal generated by 2 and x in the polynomial ring $Z(x)$ of polynomials in a single variable x with coefficients in the ring of integers Z, a principal ideal? Justify your answer.

IAS —2011

Ans. Let $S = \{2, x\}$, $I = \langle S \rangle$

$I = \{p(x)2 + q(x)x \mid p(x), q(x) \in Z[x]\}$

Let $I = \langle a(x) \rangle$ [Assuming it is principal ideal]

$x \in \langle a(x) \rangle$, $x = a(x) \cdot b(x)$

$2 \in \langle a(x) \rangle$, $2 = a(x) \cdot c(x)$

$\deg(x) = \deg(a(x)) + \deg(b(x))$

$1 = \deg(a(x)) + \deg(b(x))$

(i) either $\deg(a(x)) = 0$ or $\deg(b(x)) = 0$

$\deg(2) = \deg(a(x)) + \deg(b(x))$

(ii) $\deg(a(x)) = 0$ and $\deg(c(x)) = 0$

$\therefore \deg(a(x)) = 0$

$\Rightarrow a(x) = \pm 1, \pm 2$

If $a(x) = \pm 1 \Rightarrow I = Z[x]$
$\neq \langle S \rangle$ (contradiction)

If $a(x) = \pm 2 \Rightarrow x = \pm 2[b(x)]$
$= \pm 2[b_0 + b_1(x) + b_2(x) \ldots]$
$\Rightarrow x = \pm 2b_1 x$
$\Rightarrow 1 = \pm 2b_1$ Not possible

Hence our assumption that $\langle S \rangle$ is a principal ideal is wrong.

(2012) Q.3(c) P-II

Expand $f(z) = \frac{1}{(z+1)(z+3)}$ in laurent's series valid for:

(a) $1 < |z| < 3$ (b) $|z| > 3$

(c) $0 < |z+1| < 2$ (d) $|z| < 1$.

Sol:-

$$f(z) = \frac{1}{(z+1)(z+3)}$$

$$f(z) = \frac{1}{2}\left[\frac{1}{z+1}\right] - \frac{1}{2}\left[\frac{1}{z+3}\right] \quad \text{——} (A)$$

$$f(z) = \frac{1}{2}\left[\underbrace{\frac{1}{z+1}}_{f(z_1)} - \underbrace{\frac{1}{z+3}}_{f(z_2)}\right] \quad \text{——} (1)$$

Here; $f_1(z) = \frac{1}{z+1}$ & $f_2(z) = \frac{1}{z+3}$

(a) $1 < |z| < 3$

let us expand the laurent's series valid for $1 < |z-0| < 3$.

Clearly, $f(z)$ is analytical $1 < |z| < 3$.

we have; $f_1(z) = \frac{1}{z+1} = \frac{1}{z(1+1/z)}$

$f_1(z) = \frac{1}{z}\left[1 + \frac{1}{z}\right]^{-1}$ This expansion is valid when $\frac{1}{|z|} < 1$

$\therefore f_1(z) = \frac{1}{z}\left[1 - \frac{1}{z} + \frac{1}{z^2} \cdots\cdots\right]$

$f_1(z) = \left[\frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} \cdots\cdots\right]$ is ~~valid for~~ $|z| > 1$ as $\left|\frac{1}{z}\right| < 1$ —— (2)

we have,

$$f_2(z) = \frac{1}{z+3} = \frac{1}{z(1+3/z)} = \frac{1}{z}\left[1+3/z\right]^{-1}$$

(or)

$$f_2(z) = \frac{1}{z+3} = \frac{1}{3(\frac{z}{3}+1)} = \frac{1}{3}\left[1+\frac{z}{3}\right]^{-1}$$

$$f_2(z) = \frac{1}{3}\left[1-\frac{z}{3}+\frac{z^2}{9}-\frac{z^3}{27}+\cdots\right] \quad \text{———} \cancel{(3)}$$

is valid for $\left|\frac{z}{3}\right|<1$ or $|z|<3$.

$$\therefore f_2(z) = \frac{1}{3}-\frac{z}{9}+\frac{z^2}{27}-\frac{z^3}{81}+\cdots \quad \text{———} (3)$$

$\therefore$ from (1)

$$f(z) = \frac{1}{2}\left[\underbrace{\left(\frac{1}{z}-\frac{1}{z^2}+\frac{1}{z^3}+\cdots\right]}_{\text{Principal part}}-\underbrace{\left(\frac{1}{3}-\frac{z}{9}+\frac{z^2}{27}+\cdots\right)}_{\text{Analytical Part}}\right]$$

is valid in $1<|z|<3$

$$\therefore f(z) = \frac{1}{2}\left[\frac{1}{z}-\frac{1}{3}-\frac{1}{z^2}+\frac{z}{9}+\frac{1}{z^3}-\frac{z^2}{27}+\cdots\right]$$

is valid for $\underline{1<|z|<3}$

(b) $|z|>3$

Now consider –

$$f_1(z) = \frac{1}{z+1} = \frac{1}{z}\left(1+\frac{1}{z}\right)^{-1}$$

$$f_1(z) = \frac{1}{z}\left[1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+\cdots\right]$$

$$f_1(z) = \left[\frac{1}{z}-\frac{1}{z^2}+\frac{1}{z^3}-\frac{1}{z^4}+\cdots\right]$$ is valid for

$\left|\frac{1}{z}\right|<1$; i.e $|z|>1$.

which is valid for $|z|>3$ [$\because |z|>3>1$ which is $\underline{\text{required range}}$.

$$f_2(z) = \frac{1}{z+3} = \frac{1}{z(1+3/z)} = \frac{1}{z}\left(1+3/z\right)^{-1}$$

$$f_2(z) = \frac{1}{z}\left[1 - \frac{3}{z} + \frac{9}{z^2} - \frac{27}{z^3} + \cdots\right]$$

$$f_2(z) = \frac{1}{z} - \frac{3}{z^2} + \frac{9}{z^3} - \frac{27}{z^4} + \cdots$$

This expansion if valid for $\left|\frac{3}{z}\right| < 1 \Rightarrow |z| > 3$

which is the required range.

$\therefore$ from ①

$$f(z) = \frac{1}{2}\left[\left(\frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \cdots\right]\right] - \frac{1}{2}\left[\frac{1}{z} - \frac{3}{z^2} + \frac{9}{z^3} - \cdots\right]$$

$$\boxed{f(z) = \frac{1}{z^2} - \frac{4}{z^3} + \frac{13}{z^4} - \cdots}$$

which is valid and required laurent's expansion valid for $|z| > 3$.

(e) (C) $0 < |z+1| < 2$

let $z+1 = u$ and $0 < |u| < 2$.

from ①

$$f(z) = \frac{1}{(z+1)(z+3)} = \frac{1}{u(u+2)} = \frac{1}{2u(1+u/2)}$$

$$f(z) = \frac{1}{2u}\left(1+u/2\right)^{-1} = \frac{1}{2u}\left[1 - \frac{u}{2} + \frac{u^2}{4} - \frac{u^3}{8} + \cdots\right]$$

$$f(z) = \frac{1}{2u} - \frac{1}{4} + \frac{u}{8} - \frac{u^2}{16} + \cdots$$

The expansion is valid only when

$$0 < |u/2| < 1 \Rightarrow 0 < |u| < 2$$

which is required range.

$\therefore$ The required laurent's series valid for $0 < |u| < 2$

i.e $0<|z+1|<2$ is

$$f(z)=\frac{1}{2(z+1)}-\frac{1}{4}+\frac{1}{8}(z+1)-\frac{(z+1)^2}{16}+\cdots$$

(d) $|z|<1$.

Now, consider $f_1(z)=\frac{1}{z+1}$

$$\therefore f_1(z)=\frac{1}{z+1}=(1+z)^{-1}=(1-z+z^2-z^3\cdots)$$

This expansion is valid only when $|z|<1$ which is the required range.

Now, $f_2(z)=\frac{1}{z+3}=\frac{1}{3(1+z/3)}=\frac{1}{3}(1+z/3)^{-1}$

$$f_2(z)=\frac{1}{3}\left[1-\frac{z}{3}+\frac{z^2}{9}-\frac{z^3}{27}+\cdots\right]$$

This expansion is only valid for:

$$\left|\frac{z}{3}\right|<1 \Rightarrow |z|<3$$

which is also valid for $|z|<1$.

$\therefore$ from (1)

$$f(z)=\frac{1}{2}\left[1-z+z^2-z^3+\cdots\right]-\frac{1}{6}\left[1-\frac{z}{3}+\frac{z^2}{9}-\cdots\right]$$

$$\boxed{f(z)=\frac{1}{3}-\frac{4}{9}z+\frac{13}{27}z^2+\cdots}$$

which is the required Laurent's expansion valid for $|z|<1$.

This is a Taylor's Series.

2012 4(a) P-II Describe the maximal ideals in the ring of Gaussian integers $\mathbb{Z}[i] = \{a+bi \mid a, b \in \mathbb{Z}\}$

Ans: As $\mathbb{Z}[i]$ is a principal ideal domain.

$\Rightarrow$ ~~Any~~ Any ideal in $\mathbb{Z}[i]$ is generated by 1 element.

If z is in $a+bi$,

and $z = vw$ for some v, w in $\mathbb{Z}[i]$

which are not units,

then $\langle z \rangle$ is contained in $\langle v \rangle$.

For this not to be the case, we need z to be prime, and in fact x is a prime.

Thus, the maximal ideals in $\mathbb{Z}[i]$ are precisely those generated by a prime in $\mathbb{Z}[i]$.

4(c) IAS 2012 P-II → By the method of Vogel, determine an initial basic feasible solution for the following transportation problem:

Products P_1, P_2, P_3 and P_4 have to be sent of destinations D_1, D_2 and D_3. The cost of sending product P_i to destinations D_j is C_{ij}, where the matrix.

$$[C_{ij}] = \begin{bmatrix} 10 & 0 & 15 & 5 \\ 7 & 3 & 6 & 15 \\ 0 & 11 & 9 & 13 \end{bmatrix}$$

The total requirements of destinations D_1, D_2 and D_3 are given by 45, 45, 95 respectively and the availability of the products P_1, P_2, P_3 and P_4 are respectively 25, 25, 55 and 70.

Sol^n → Calculating IBFS using vogel method:-

→ Calculating the rim value ie. the difference b/w the 2 least number of each row column.

→ Selecting the row or column with max. rim value.

→ Now select the minimum cost cell in that row or column and assign maximum value possible.

→ Now calculate the rim values again leaving the occupied cells and follow the same.

procedure as above untill all the cells are filled.

	P_1	P_2	P_3	P_4		
D_1	10 x	0	15	5	45	(5)
D_2	7 x	3	6	15	45	(3)
D_3	0 (25)	11	9	13	95	(9)
	25	35	55	70		
	(7)	(3)	(1)	(8)		

→ selecting 3 row; since max. value is a
Now selecting (3,1) cell with min. cost and allocating min (25, 95) = 25.

0 x	15 x	5 (45)	45	(5)
3	6	15	45	(3)
11	9	13	70	(2)
35	55	70		
(3)	(3)	(8)		

→ Selecting 3rd column and (1,3) cell allocating: min (45, 70) = 45

3	6	15	45	(3)
11	9	13	70	(2)
35	55	25		
(8)	(3)	(2)		

Selecting 1^{st} column and (1,1) cell

Allocating : min (35, 45) = 35

6 (10)	15 ×	10 (9)
9 (45)	13 25	70 (4)
55 (3) (9)	25 (2) (13)	

∴ final table is :—

10 ×	0 ×	15 ×	5 (45)
7 ×	3 (35)	6 (10)	15 ×
0 25	11 ×	9 (45)	13 (25)

The IBFS is :—

$= 5\times 45 + 3\times 35 + 6\times 10 + 0\times 25 + 45\times 9 + 13\times 25$

$= 1120.$

5(a). Solve the PDE $(D-2D')(D-D')^2 z = e^{x+y}$

SOLUTION

Auxiliary equation for the given PDE $(m-2)(m-1)^2 = 0$

$\Rightarrow$ m = 2, 1, 1.

$\therefore$ C.F. = $\phi_1(y+2x)+\phi_2(y+x)+x\phi_3(y+x)$

$$\text{Particular integral} = \frac{e^{x+y}}{(D-2D')(D-D')^2}$$

$$= \frac{1}{(D-D')^2}\frac{e^{x+y}}{(1-2)}$$

$$= \frac{-e^{x+y}}{(D-D')^2}$$

$$\text{P.I.} = \frac{-x^2}{2}e^{x+y}$$

$\therefore$ General solution y = C.F. + P.I.

$$y = \phi_1(y+2x)+\phi_2(y+x)+x\phi_3(y+x)-\frac{x^2}{2}e^{x+y}$$

Where ϕ_1, ϕ_2, ϕ_3 are arbitrary functions.

Q. 5(b) Use Newton-Raphson method to find the real root of the equation $3x = \cos x + 1$ correct to four decimal places.

Sol:-

$$f(x) = 3x - \cos x - 1$$

$$f'(x) = 3 + \sin x$$

Both $f(x)$ & $f'(x)$ are continuous

$f(1) = 1.4597$ which is closer to '0' other than nearby values.

Hence, we will take $x_0 = 1$

$x_0 = 1 \qquad f(x_0) = 1.45970$

$f'(x_0) = 3.84147 = 3.8415$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{1.45970}{3.8415} = 0.62002$$

Similarly $x_1 = 0.62002$

$f(x_1) = 0.04619$

$f'(x_1) = 3.58105$

$$x_2 = 0.62002 - \frac{0.4619}{3.58105} = 0.60712$$

$\therefore x_2 = 0.60712$

$f(x_2) = 0.00007$

$f'(x_2) = 3.57050$

$$x_3 = 0.60712 - \frac{0.00007}{3.57050} = 0.60710$$

$f(x_3) = 0.00000$

$f'(x_3) = 3.57049$

$$x_4 = 0.60710 - \frac{0.0000}{3.57049} = 0.60710$$

Hence; $\boxed{x_4 = 0.60710}$ is the required result.

6(a). Solve the PDE px + qy = 3z .....(1)

SOLUTION

Lagranges auxiliary equation of (1) is given by $\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{3z}$ .....(2)

$(2)\Rightarrow \frac{dx}{x}=\frac{dy}{y} \Rightarrow ln\, x = l\, ny + ln c_1$

$$x = c_1 y$$
$$x/y = c_1$$

$(2)\Rightarrow \frac{dy}{y}=\frac{dz}{3z} \Rightarrow lny = \frac{lnz}{3}+lnc_2$

$\frac{y^3}{z}=c_2$

$\therefore \quad \phi\left(x/y, y^3/z\right)=0$ ϕ being arbitrary functions.

6(b). A string of length 'l' is fixed at its ends. The string from the mid point is pulled up to a height k and then released from rest. Find the deflection y(x, t) of the vibrating string.

SOLUTION

Given string of length 'l' pulled upwards by k units at $n=\frac{l}{2}$.

$\therefore$ Initial Conditions:

$$u(x,0) = k\frac{x}{l/2} = \frac{2kx}{l} \quad \text{for } 0<x<l/2$$

$$= -k/_{l/2}(x-l) = \frac{2k(l-x)}{l} \quad \text{for } l/2<x<l$$

$$u_t(x,0) = 0$$

Boundary conditions

$$u(0,t) = 0$$
$$v(l,t) = 0$$

PDE of transverse vibrations of the given elastic string is given by

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}} \quad .....(1)$$

Let us suppose y(x, t) = X(x) T(t)

From boundary conditions we have

$$X(0)T(t) = 0$$
$$X(l)\ T(t) = 0$$

T(t) ≠ 0 for some t > 0.

$$\therefore \quad X(0) = X(l) = 0 \quad .....(2)$$

$\therefore$ From (1) $\quad X''T = 1/c^2\ XT''$

Let $\quad \frac{X''}{X} = \frac{T''}{c^2 T} = \mu$

$\therefore$ $X'' - \mu X = 0$ solve using the boundary conditions.

Case(i)

$\mu = 0 \Rightarrow X'' = 0$

$\therefore \quad X = X = Ax = B$

From(2)$\Rightarrow$ X(0) = 0; X(l) = 0 $\Rightarrow$ A = 0 and B = 0

$\therefore$ We reject $\mu = 0$.

Case(ii) $\quad \mu = +\lambda^2, \mu \neq 0$

$$X'' - \lambda^2 X = 0$$

Solving $\quad X = Ae^{\lambda x} + B\ e^{-\lambda x}$

from(2) $\Rightarrow$ X(0) = 0; X(l) = 0

A + B = D; $Ae^{\lambda l} + Be^{-\lambda l} = 0$

$\Rightarrow$ A = 0 and B = 0

$\therefore$ We reject $\mu = \lambda^2$

Case(iii) $\quad \mu = -\lambda^2 \Rightarrow X''+\lambda^2 X = 0$

solving $\quad X = A\cos\lambda x + B\sin\lambda x$

X(0) = 0; X(l) = 0 $\Rightarrow$ A = 0, $\sin\lambda l = 0$

$$\boxed{\lambda = \frac{n\pi}{l}}$$

$$\therefore \quad X_n(x) = B_n \sin\left(\frac{n\pi x}{l}\right)$$

we have $\quad T''+\lambda^2 c^2 T = 0$

solving $\quad T(t) = C\cos(\lambda ct) + D\sin(\lambda ct)$

$$T_n(t) = C\cos\left(\frac{n\pi ct}{l}\right) + D\sin\left(\frac{n\pi ct}{l}\right)$$

$\therefore \quad \mu(x, t) = \nabla(x)\tau(t)$

$$= \sum_{n=1}^{\infty}\left[E_n Cos\left(\frac{n\pi ct}{l}\right) + F_n \sin\left(\frac{n\pi ct}{l}\right)\right]\sin\left(\frac{n\pi x}{l}\right)$$

Applying initial conditions

$$u_t(x,0) = 0$$

$$\Rightarrow \left[-E_n \frac{n\pi c}{l}.\sin\frac{n\pi ct}{l} + F_n \frac{n\pi c}{l}\cos\left(\frac{n\pi ct}{l}\right)\right]\sin\frac{n\pi x}{l} = u_t(x,t)$$

$\therefore \quad u_t(x,0) \Rightarrow F_n = 0$

$$\therefore \quad u(x, t) = \sum_{n=1}^{\infty} E_n \cos\left(\frac{n\pi ct}{l}\right).\sin\left(\frac{n\pi x}{l}\right)$$

We have $\quad u(x, 0) = \dfrac{2kx}{l} \quad 0 < x < l/2$

$$= \frac{2k(l-x)}{l} \quad l/2 < x < l.$$

$$E_n = \frac{2}{l}\int_0^1 u(x,0)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$E_n = \frac{2}{l}\left[\int_0^{l/2}\frac{2kx}{l}.\sin\left(\frac{n\pi x}{l}\right)dx + \int_{l/2}^{l}\frac{2k(l-x)}{l}\sin\left(\frac{n\pi x}{l}\right)dx\right]$$

$$= \frac{4k}{l^2}\left[\frac{x\left(-\cos\left(\frac{n\pi x}{l}\right)\right)}{n\pi/l} - \frac{-\sin\left(\frac{n\pi x}{l}\right)}{(n\pi/l)^2}\right]_0^{l/2} + \frac{4k}{l^2}\left[\frac{(l-x)\cos\frac{n\pi x}{l}}{n\pi/l} - \frac{\sin\left(\frac{n\pi x}{l}\right)}{(n\pi/l)^2}\right]_{l/2}^{l}$$

$$= \frac{4k}{l^2}\left[\frac{l^2}{2n\pi} - \cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{l^2}{2n\pi}\cos\frac{n\pi}{2}(-1) - \frac{l^2}{n^2\pi^2}\sin n\pi + \frac{l^2}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right]$$

$$= \frac{8k}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

$$E_n = \begin{cases} \dfrac{8k}{\pi^2}\dfrac{(-1)^{n+1}}{(2m-1)^2}, & \text{if } n = 2m-1(\text{odd}) \ \& m = 1,2,3,... \\ 0 & , \text{ if } n = 2m \quad (\text{even}) \& m = 1,2,3,.. \end{cases}$$

Substituting the above value of E_n in ( ), the required displacement function is given by

$$\therefore \quad \boxed{u(x,t) = \frac{8k}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{(2n-1)^2}\cos\frac{(2n-1)\pi ct}{l}.\sin\frac{(2n-1)\pi x}{l}}$$

Required solution

Q. Solve the following system of eqⁿ by Gauss Sidel Mtd.
6(c) Method

$$3x + 20y - z = -18$$
$$20x + y - 2z = 17$$
$$2x - 3y + 20z = 25.$$

Sol:- We can rewrite the above three eqⁿ in the form

$$x = \frac{1}{20}(17 - y + 2z)$$

$$y = \frac{1}{20}(-18 - 3x + z)$$

$$z = \frac{1}{20}(25 + 3y - 2x)$$

Now, let us assume that initial values

$$(x^0, y^0, z^0) = (0, 0, 0)$$

By Gauss Siedel Iterative Method.

<u>First Iteration</u>

$$x^{k+1} = \frac{1}{20}\left[17 - y^k + 2z^k\right]$$

$$y^{k+1} = \frac{1}{20}\left[-18 - 3x^{k+1} + z^k\right]$$

$$z^{k+1} = \frac{1}{20}\left[25 - 2x^{k+1} + 3y^{k+1}\right]$$

Put $k = 0$

$$x' = \frac{1}{20}(17 - 0 + 0) = 0.8500$$

$$y' = \frac{1}{20}(-18 - 3 \times 0.8500 + 0) = -1.0275$$

$$z' = \frac{1}{20}\left[25 - 2 \times 0.8500 + 3(-1.0275)\right]$$

$$\boxed{z' = 1.0109}$$

Second Iteration: Put $K=1$

$$x^{(2)} = \frac{1}{20}\left[17 + 1.0275 + 2\times 1.0109\right] = 1.0025.$$

$$y^{(2)} = \frac{1}{20}\left[-18 - 3\times 1.0025 + 1.0109\right] = -0.9998$$

$$z^{(2)} = \frac{1}{20}\left[25 - 2\times 1.0025 + 3\times(-0.9998)\right] = 0.9998$$

Third iteration: $K=2$

$$x^{3} = \frac{1}{20}\left(17 + 0.9998 + 2\times 0.9998\right) = 1.0000$$

$$y^{3} = \frac{1}{20}\left[-18 - 3\times 1.0000 + 0.9998\right) = -1.0000$$

$$z^{3} = \frac{1}{20}\left[25 - 2\times 1 + 3(-1)\right] = 1.0000$$

Hence, the Solution of Iteration is $(1,-1,1)$.

Que: Find $\frac{dy}{dx}$ at $x=0.1$, from the following data

7(a)

x	0.1	0.2	0.3	0.4
y	0.9975	0.9900	0.9776	0.964

Sol: First we get to find the required polynomial in $f(x)$ – i.e. $y=f(x)$ of degree 2 or 3. According to the given set of data by Newton divided diff. formula.

$h=0.1$.

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$
0.1	0.9975			
		-0.075		
0.2	0.9900		-0.245	
		-0.124		0.17
0.3	0.9776		-0.24	$[x_0,x_1,x_2,x_3]$
		-0.172	(x_0,x_1,x_2)	
0.4	0.9604	$[x_0,x_1]$		

The formula Given is

$$y=f(x)=y_0+(x-x_0)[x_0,x_1]+(x-x_0)(x-x_1)\,[x_0,x_1,x_2]$$
$$+(x-x_0)(x-x_1)(x-x_2)\,[x_0,x_1,x_2,x_3]$$

$$y=0.9975+(x-0.1)(-0.075)+(x-0.1)(x-0.2)(-0.245)$$
$$+(x-0.1)(x-0.2)(x-0.3)(0.17)$$

$$y=0.9975+0.075-0.075x-0.245x^2$$
$$+0.0735x-0.00590+0.017x^3-0.0102x^2$$
$$+0.00187x-0.00012.$$

Now; $$\frac{dy}{dx}=-0.075-0.490x+0.735$$
$$+0.051x^2-0.0204x+0.00187$$

$$\left.\frac{dy}{dx}\right|_{x=0.1}=-0.075+(-0.0490)+0.735$$
$$+0.00051-0.00204+0.00187$$

$$\boxed{\left.\frac{dy}{dx}\right|_{x=0.1}=-0.05D16}$$

7(b) IAS-2012 P-II. → The edge $r=a$ of a circular plate is kept at temperature $f(\theta)$. The plate is insulated so that there is no loss of heat from either surface, Find the temperature distribution.

Sol^n: Here we have to take the solution in polar coordinates.

The solution is $u=(c_1\cos p\theta+c_2\sin p\theta)(c_3r^p+c_4r^{-p})$ —— (1)

Since the temperature remains finite at $r=0$

$\therefore c_4=0$

Also, if we increase θ by 2π, we arrive at the same point, so the solution (1) should be periodic with period 2π.

$\therefore p=n$, an integer.

Hence we may write the general solution as

$$u=\sum_{n=0}^{\infty}(c_1\cos n\theta+c_2\sin n\theta)c_3r^n$$

$$=\sum(A_n\cos n\theta+B_n\sin n\theta)r^n$$ where $c_1c_3=A_n$, $c_2c_3=B_n$ (say)

Applying to this, the condition $u=f(\theta)$ for $r=a$, we get

$$f(\theta)=\sum(A_n\cos n\theta+B_n\sin n\theta)a^n$$

$$\therefore a^nA_n=\frac{1}{\pi}\int_0^{2\pi}f(\theta)\cos n\theta\,d\theta$$

$$a^nB_n=\frac{1}{\pi}\int_0^{2\pi}f(\theta)\sin n\theta\,d\theta$$

Hence the result.

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8(b) Show that $\phi = x f(r)$ is a possible form for the velocity potential for an incompressible fluid motion. If the fluid velocity $\vec{q} \to 0$ as $r \to \infty$, find the surfaces of constant speed.

IAS-2012 P-II

Sol'n: $\phi = x f(r)$ ——— (1)

$q = -\nabla\phi = -\nabla[x f(r)] = -[f(r)\nabla x + x\nabla f(r)]$ — (2).

Now $r^2 = x^2+y^2+z^2 \Rightarrow 2r\left(\frac{\partial r}{\partial x}\right) = 2x \Rightarrow \frac{\partial r}{\partial x} = x/r$ — (3)

Similarly $\frac{\partial r}{\partial y} = y/r$ and $\frac{\partial r}{\partial z} = z/r$ — (4)

Also, $\nabla x = [i(\partial/\partial x) + j(\partial/\partial y) + k(\partial/\partial z)]x = i$

and $\nabla f(r) = [i(\partial/\partial x) + \hat{j}(\partial/\partial y) + k(\partial/\partial z)]f(r)$

$= i f'(r)(\partial r/\partial x) + j f'(r)(\partial r/\partial y) + k f'(r)\left(\frac{\partial r}{\partial z}\right)$

$= i f'(r)(x/r) + j f'(r)(y/r) + k f'(r)(z/r)$, by (3) & (4)

$= (1/r) f'(r)\,(ix+jy+kz) = \frac{1}{r} f'(r)\vec{r}$.

$\therefore$ (2) $\Rightarrow q = -f(r)i - (x/r)f'(r)\vec{r}$.

For a possible motion of an incompressible fluid, we have

$\nabla.q = 0$ (or) $\nabla.(-\nabla\phi) = 0 \Rightarrow \nabla^2\phi = 0$.

$\Rightarrow \left[\frac{\partial^2}{\partial x^2} + \partial^2/\partial y^2 + \partial^2/\partial z^2\right][x f(r)] = 0$; using (1)

Now, $\frac{\partial^2}{\partial x^2}[x f(r)] = \frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}\{x f(r)\}\right] = \frac{\partial}{\partial x}\left[f(r) + x\frac{\partial f(r)}{\partial x}\right]$

$= \frac{\partial f}{\partial x} + \frac{\partial f}{\partial x} + x\frac{\partial^2 f}{\partial x^2} = 2\frac{\partial f}{\partial x} + x\frac{\partial^2 f}{\partial x^2}$

Also $\frac{\partial^2}{\partial y^2}[x f(r)] = x\frac{\partial^2 f}{\partial y^2}$ and $\frac{\partial^2}{\partial z^2}[x f(r)] = x\frac{\partial^2 f}{\partial z^2}$

$\therefore$ (6) becomes $2\frac{\partial f}{\partial x} + x\left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) = 0$ — (7)

Now $\partial f/\partial x = df/dr \cdot \partial r/\partial x = f'(x/r)$ using (3) — (8)

and $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial}{\partial x}\left(f'\frac{x}{r}\right) = \frac{f'}{r} + x\frac{\partial}{\partial x}\left(\frac{f'}{r}\right)$

$= \frac{f'}{r} + x\frac{d}{dr}\left(\frac{f'}{r}\right)\frac{\partial r}{\partial x} = \frac{f'}{r} + x.\frac{rf''-f'}{r^2}.\frac{x}{r}$

$$\frac{\partial^2 f}{\partial x^2} = \frac{f'}{r} + \frac{x^2}{r^2} f'' - \frac{x^2}{r^3} f' \quad \text{——} \ (9)$$

Similarly $\frac{\partial^2 f}{\partial y^2} = \frac{f'}{r} + \frac{y^2}{r^2} f'' - \frac{y^2}{r^3} f'$ —— (10)

and $\frac{\partial^2 f}{\partial z^2} = \frac{f'}{r} + \frac{z^2}{r^2} f'' - \frac{z^2}{r^3} f'$ —— (11)

Adding (9), (10) and (11) we get

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{3f'}{r} + \frac{x^2+y^2+z^2}{r^2} f'' - \frac{x^2+y^2+z^2}{r^3} f'$$

$$= \frac{3f'}{r} + f'' - \frac{f'}{r}, \text{ as } x^2+y^2+z^2 = r^2$$

$$\therefore \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 2f'/r + f'' \quad \text{——} \ (12)$$

using (8) and (12), (7) reduces to

$$\frac{2f'(x)}{r} + x\left(\frac{2f'}{r} + f''\right) = 0 \Rightarrow f'' + \frac{4f'}{r} = 0$$

$$\Rightarrow f''/f' + 4/r = 0$$

Integrating $\log f' + 4\log r = \log c_1$ so that $f' = c_1 r^{-4}$ —— (13)

Integrating (13) $f = -(c_1/3) \times r^{-3} + c_2$, c_2 being an arbitrary constant. —— (14)

Substituting the values of f' and f from (13) & (14) in (5), we get

$$\vec{q} = -\{(c_1/3r^3) - c_2\}\vec{i} - (c_1 x/r^5)\vec{r} \quad \text{——} \ (15)$$

Given that $\vec{q} \to 0$ as $r \to \infty$ hence (15) shows that $c_2 = 0$

$$\therefore \text{from (15)}, \quad \vec{q} = \frac{c_1}{3r^3}\left(\vec{i} - \frac{3x\vec{r}}{r^2}\right) \quad \text{——} \ (16)$$

Now $q^2 = \vec{q}\cdot\vec{q} = \frac{c_1^2}{9r^6}\left(\vec{i} - \frac{3x\vec{r}}{r^2}\right)\cdot\left(\vec{i} - \frac{3x\vec{r}}{r^2}\right)$

$$= \frac{c_1^2}{9r^6}\left[\vec{i}\cdot\vec{i} - \frac{6x}{r^2}\vec{r}\cdot\vec{i} + \frac{9x^2}{r^4}\vec{r}\cdot\vec{r}\right]$$

$$= \frac{c_1^2}{9r^6}\left(1 - \frac{6x^2}{r^2} + \frac{9x^2 r^2}{r^4}\right) \text{ as } \vec{r}\cdot\vec{r} = r^2 \ \& \ \vec{r}\cdot\vec{i} = x$$

$$= \frac{c_1^2}{9r^6}\left(1 + \frac{3x^2}{r^2}\right)$$

$$= \frac{c_1^2}{9r^8}(r^2 + 3x^2)$$

Hence the required surfaces of constant speed are

$q^2 = \text{constant}$ (or) $\left(\frac{c_1^2}{9r^8}\right)(r^2+3x^2) = \text{constant}$

$$\Rightarrow (r^2+3x^2)r^{-8} = \text{Constant}$$