2017

Ques1 Prove that
$\nabla^2 r^n = n(n+1) r^{n-2}$ & that
$r^n \vec{r}$ is irrotational, where
$r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$

We have,
$$\nabla^2 r^n = \sum \frac{\partial^2}{\partial x^2}(r^n)$$

$$\Rightarrow \sum \frac{\partial}{\partial x}\left(n r^{n-1} \cdot \frac{\partial r}{\partial x}\right)$$

$\because \quad r^2 = x^2+y^2+z^2$
$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\Rightarrow \sum \frac{\partial}{\partial x}\left(n r^{n-1} \frac{x}{r}\right)$$

$$\Rightarrow \sum \frac{\partial}{\partial x}\left(n r^{n-2} x\right)$$

$$\Rightarrow \sum n \cdot r^{n-3}(n-2) \cdot \frac{x^2}{r} + n r^{n-2} \cdot 1$$

$$\Rightarrow n(n-2) r^{n-4}(x^2+y^2+z^2) + 3 \cdot n \cdot r^{n-2}$$

$$(n)(n-2) r^{n-4} r^2 + 3 n r^{n-2}$$

$$n(n-2) r^{n-2} + 3 n r^{n-2}$$

$- n r^{n-2}(n+1)$

$n(n+1) r^{n-2}$

(ii) for $r^n \vec{r}$ to be irrotational,

$\nabla \times r^n \vec{r} = 0$

$\Rightarrow \sum \hat{i} \times \frac{\partial}{\partial x}(r^n \vec{r})$

$\Rightarrow \sum \hat{i} \times \left(n(r^{n-1}) \cdot \frac{x}{r} \vec{r} + \hat{i} \times (r^n \hat{i})\right.$

$\Rightarrow \sum \hat{i} \times n r^{n-2} \cdot x \vec{r} + 0$

$\Rightarrow n r^{n-2} \sum \hat{i} \times x \vec{r}$

$\Rightarrow n r^{n-2} \left[\hat{i} \times x(x\hat{i}+y\hat{j}+z\hat{k}) + \hat{j} \times y(x\hat{i}+y\hat{j}+z\hat{k})\right.$
$\left. + \hat{k} \times z(x\hat{i}+y\hat{j}+z\hat{k})\right]$

$\Rightarrow n r^{n-2}\left[xy\hat{k} - xz\hat{j} - xy\hat{k} + yz\hat{i} + zx\hat{j} - zy\hat{i}\right]$

$\Rightarrow n r^{n-2}[0]$

$\Rightarrow 0$

Thus its irrotational.

Ques 2 Using Stokes theorem, evaluate

$$\left[\oint_C (x+y)\,dx + (2x-z)\,dy + (y+z)\,dz\right]$$

where C is the boundary of the triangle with vertices at (2, 0, 0), (0, 3, 0), (0, 0, 6).

Ans

We know as per Stoke's theorem,

$$\oint_C F\cdot dr = \iint_S (\nabla\times F)\cdot\hat{n}\, dS$$

$$F = (x+y)\hat{i} + (2x-z)\hat{j} + (y+z)k$$

$$\nabla\times F = \begin{vmatrix} \hat{i} & \hat{j} & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x+y & 2x-z & y+z \end{vmatrix}$$

$$= \hat{i}(1+1) - \hat{j}(0-0) + k(2-1)$$

$$= 2\hat{i} + k$$

Eq$^n$ of the plane passing through the given 3 points,

$$S: \frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1$$

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$$\hat{n} = \frac{\nabla S}{|\nabla S|} = \frac{\hat{i}/2 + \hat{j}/3 + \hat{k}/6}{\sqrt{\frac{1}{4} + \frac{1}{9} + \frac{1}{36}}}$$

$$= \frac{(\hat{i}/2 + \hat{j}/3 + \hat{k}/6)}{\sqrt{14}} \cdot 6$$

$$(\nabla \times f) \cdot \hat{n} = \cancel{\frac{6}{\sqrt{14}}} \frac{6}{\sqrt{14}} \left(1 + \frac{1}{6}\right)$$

$$= \frac{7}{\sqrt{14}}$$

Taking proj$^{on}$ of the surface S on the xy plane as R,

$$ds = \frac{dx\,dy}{|\hat{n} \cdot k|}$$

$$= dx\,dy \cdot \sqrt{14}$$

$$\therefore \iint_S (\nabla \times f) \cdot \hat{n}\, ds = \iint_R (\nabla \times f) \cdot \hat{n}\; dx\,dy\,\sqrt{14}$$

$$= \iint_R \frac{7}{\sqrt{14}} \sqrt{14}\; dx\,dy$$

$$= 7 \iint_R dx\,dy$$

$$= 7 \int_{x=0}^{2} \int_{y=0}^{\frac{6-3x}{2}} dx\,dy$$

$$= 7 \int_{x=0}^{2} \frac{6-3x}{2}$$

$$= 7 \left[ 3x - \frac{3x^2}{4} \right]_0^2$$

$$= 7 \left[ 6 - \frac{12}{4} \right]$$

$$= 7(3)$$

$$= 21$$

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Ques 3

Evaluate

$$\iint_S (\nabla \times f) \cdot \hat{n}\, ds$$ where S is

the surface of the cone $z = 2 - \sqrt{x^2+y^2}$ above xy plane & $\vec{f} = (x-z)\hat{i} + (x^3+yz)\hat{j} - 3xy^2\hat{k}$

Ans: We know as per Stokes Theorem,

$$\oint f \cdot dr = \iint_S (\nabla \times f) \cdot \hat{n}\, ds$$

$$dr = dx\,i + dy\,j + dz\,k$$

$$f \cdot dr = (x-z)\,dx + (x^3+yz)\,dy + (-3xy^2)\,dz$$

C is the boundary of the surface $z = 2-\sqrt{x^2+y^2}$, i.e.,

$$C: \quad x^2+y^2 = 4, \text{ a circle.}$$

Now,

$$\int_C f \cdot dr = \int_C (x-z)\,dx + (x^3+yz)\,dy + (-3xy^2)\,dz$$

On C, $z = 0 \Rightarrow dz = 0$

Taking $x = 2\cos\theta$
$y = 2\sin\theta$

$$\int_C f \cdot dr = \int_{\theta=0}^{2\pi} 2(\cos\theta)(-2\sin\theta) + (8\cos^3\theta)(2\cos\theta)$$

$$= \int_{\theta=0}^{2\pi} \left(-4\cos\theta \cdot \sin\theta + 16\cos^4\theta\right) d\theta$$

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$$= \int_{\theta=0}^{2\pi} -2\sin(2\theta)\,d\theta + \int_{\theta=0}^{2\pi} 16 \cdot \cos^4\theta \, d\theta$$

$$= \frac{-2}{(-2)}\Big[\cos(2\theta)\Big]_{\theta=0}^{2\pi} + 16 \cdot 2 \cdot 2 \int_0^{\pi/2} \cos^4\theta \, d\theta$$

$$= (1-1) + 16 \cdot 2 \cdot 2 \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}$$

$$= 12\pi \;//$$

Ques 4

Find the curvature & torsion of the circular helix
$\vec{r} = a(\cos\theta, \sin\theta, \cot\beta)$
$\beta$ is the constant angle at which it cuts its generators.

Ans

We have,

$$\vec{r} = a\cos\theta\, i + a\sin\theta\, j + a\cot\beta$$

We know that,

$$\text{Curvature} = K = \frac{\left|\frac{d\vec{r}}{d\theta} \times \frac{d^2\vec{r}}{d\theta^2}\right|}{\left|\frac{d\vec{r}}{d\theta}\right|^3}$$

$$\frac{d\vec{r}}{d\theta} = -a\sin\theta\, i + a\cos\theta\, j$$

$$\frac{d^2\vec{r}}{d\theta^2} = -a\cos\theta\, i - a\sin\theta\, j$$

$$\frac{d\vec{r}}{d\theta} \times \frac{d^2\vec{r}}{d\theta^2} = \begin{vmatrix} i & j & k \\ -a\sin\theta & +a\cos\theta & 0 \\ -a\cos\theta & -a\sin\theta & 0 \end{vmatrix}$$

$$= i(0) - j(0) + k(a^2)$$
$$= a^2 k$$

$$\left|\frac{d\vec{r}}{d\theta} \times \frac{d^2\vec{r}}{d\theta^2}\right| = a^2$$

$$\left|\frac{dr}{d\theta}\right|^3 = (a^2)^{3/2}$$
$$= a^3$$

$$\therefore K = \frac{a^2}{a^3} = \frac{1}{a}$$

Now, Torsion $= \tau = \dfrac{\left[\dfrac{d\vec{r}}{d\theta}\ \dfrac{d^2\vec{r}}{d\theta^2}\ \dfrac{d^3\vec{r}}{d\theta^3}\right]}{\left|\dfrac{d\vec{r}}{dt} \times \dfrac{d^2\vec{r}}{dt^2}\right|}$

$$\left[\frac{d\vec{r}}{d\theta}\ \frac{d^2\vec{r}}{d\theta^2}\ \frac{d^3\vec{r}}{d\theta^3}\right] = \left(\frac{d\vec{r}}{d\theta} \times \frac{d^2\vec{r}}{d\theta^2}\right)\cdot\left(\frac{d^3\vec{r}}{d\theta^3}\right)$$
$$= (a^2 \hat{k})\cdot(a\sin\theta\,\hat{i} - a\cos\theta\,\hat{j})$$
$$= 0.$$

Hence $\tau = 0.$