Set – I

* Complex Analysis *

→ Introduction:–

In the field of real numbers, the equation $x^2+1=0$ has no solution. To permit the solution of this and similar equations (i.e $x^2-2x+3=0$ etc) the real number system was extended to the set of complex numbers. Euler introduced the symbol i with the property that $i^2=-1$. He also called i as the imaginary unit.

- A number of the form $a+ib$ where a, b are real numbers was called complex number.
- If we write $z=x+iy$ then z is called a complex variable.
- Also x is called real part of z and is denoted by $R(z)$ i.e $R(z)=x$ and y is called imaginary part of z and is denoted by $I(z)$ i.e $I(z)=y$
- Some times we express z as $z=(x,y)$
- If $x=0$ i.e $z=iy$ then z is called pure imaginary number
- The conjugate of $z=x+iy$ is $\bar{z}=x-iy$
- $Re(z) = x = \frac{z+\bar{z}}{2}$
- $I(z) = y = \frac{z-\bar{z}}{2i}$

* Fundamental operations with Complex Numbers

Addition : $(a+ib)+(c+id)=(a+c)+i(b+d)$

Subtraction: $(a+ib)-(c+id)=(a-c)+i(b-d)$

Multiplication: $(a+ib)(c+id)=(ac-bd)+i(bc+ad)$

Division : $$\frac{a+ib}{c+id} = \frac{(a+ib)(c-id)}{(c+id)(c-id)}$$

$$= \frac{(ac+bd)+i(bc-ad)}{c^2+d^2}$$

$$= \frac{ac+bd}{c^2+d^2} + i\left(\frac{bc-ad}{c^2+d^2}\right)$$

if $c^2+d^2 \neq 0$

* Absolute value

The absolute value (or) modulus of a complex number $z=a+ib$ is denoted by $|z|$ and is defined as

$$|z| = |a+ib| = \sqrt{a^2+b^2}$$

Evidently $|z|^2 = a^2+b^2$

$= (a+ib)(a-ib)$

$= z\bar{z}$

$\therefore |z|^2 = z\bar{z}$

Also $\overline{z_1 \cdot z_2} = \bar{z}_1 \, \bar{z}_2$

* Geometrical Representation of Complex Numbers:–

Consider the complex number $z=x+iy$

A complex number can be regarded as an ordered pair of reals. i.e $z=(x,y)$

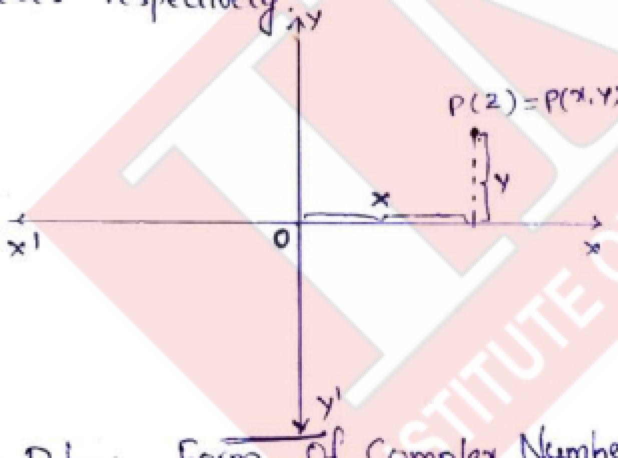
This form of z suggests that z can be represented by a point P whose coordinates are x & y relative to the rectangular axes X & Y

To each complex number there corresponds one and only one point in the xy-plane and conversely to each point in the plane there exists one and only one complex number. Due to this fact the complex number z is referred to the point z in this plane.

This plane is called complex plane or Gaussian plane or Argand plane.

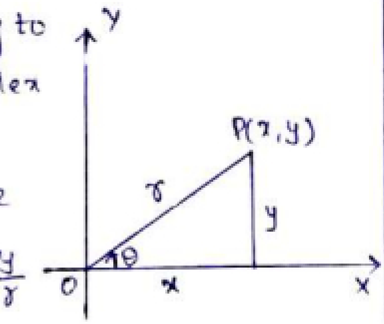
The representation of complex numbers is called Argand diagram.

The complex number $x+iy$ is called complex coordinate and x, y axes are called real and imaginary axes respectively.



Polar form of Complex Numbers:

Consider the point P in the complex plane corresponding to a non-zero complex number.



From the figure

$\cos\theta = \frac{x}{r}$ $\sin\theta = \frac{y}{r}$

$\Rightarrow x = r\cos\theta,\ y = r\sin\theta$

$\therefore r = \sqrt{x^2+y^2}$

$= |x+iy|$

$= |z|$

$\boxed{\therefore r = |z|}$

and $\tan\theta = \frac{y}{x} \Rightarrow \boxed{\theta = \tan^{-1}\left(\frac{y}{x}\right)}$

It follows that,

$z = x+iy = r(\cos\theta + i\sin\theta)$

$= re^{i\theta}$ —— (1)

It is called polar form of the complex number z.

r and θ are called polar coordinates of z.

→ r is called modulus (or) absolute value of z.

→ The angle θ which the line OP makes with the +ve x-axis, is called argument (or) amplitude of z and is denoted by $\theta = \arg(z)$ (or) $\theta = \text{amp}(z)$

→ The argument of z is not unique, since the equation (1) does not alter, if we replace θ by $2\pi+\theta$ so θ can have infinite number of values which differ from each other by 2π.

→ If a value of θ satisfies (1) and lies b/w $-\pi$ & π i.e. $-\pi < \theta \le \pi$ then that value of θ is called principal value of the argument.

Note:– It is evident from the definition of difference and modulus that $|z_1 - z_2|$

is the distance b/w two points z_1 & z_2

i.e $z_1 = x_1 + iy_1$ & $z_2 = x_2 + iy_2$

$\therefore |z_1 - z_2| = \sqrt{(x_1-x_2)^2 + (y_1-y_2)^2}$

It follows that for fixed complex number z_0 and a real number r the equation $|z - z_0| = r$ represents a circle with centre z_0 and radius r.

* **Point Set** – Any collection of points in the complex (two dimensional) plane is called a point set and each point is called a member (or) element of the point set.

– the set of complex numbers is denoted by 'C'.

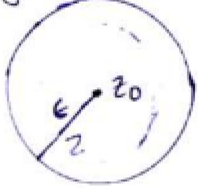
* **ϵ neighbourhood of a complex number z_0** :–

the set of all points $z \in C$ satisfying the condition $|z - z_0| < \epsilon$ is defined as ϵ-neighbourhood of the z_0.

– A deleted neighbourhood of z_0 is neighbourhood of z_0 in which the point z_0 is omitted

i.e. $0 < |z - z_0| < \epsilon$.

– In general ϵ-neighbourhood of z_0 is denoted by $N(z_0, \epsilon)$.

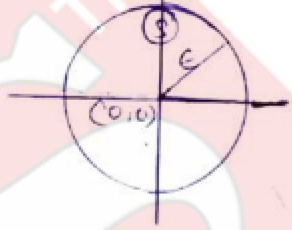


* **Bounded Set** :– A set 's' is said to be bounded if it is contained in some neighbourhood of the origin.

(or)

A set 's' is called bounded if we can find a constant ϵ such that $|z| < \epsilon \ \forall \ z \in S$.

– If a set is not bounded then it is said to be unbounded



* **Interior Point** :–

A point z_1 of a set 's' is said to be an interior point of the set s if there exist a neighbourhood of z_1 which is contained completely in the set S

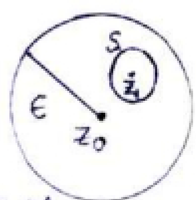
– If every neighbourhood of z_1 contains some points of S and some points that doesnot belong to 's' is called a boundary point

– A point z_0 which is neither interior nor boundary point is called exterior point.

Example :–

Let $A = \{z \in C / |z - z_0| < \epsilon\}$

$B = \{z \in C / |z - z_0| \le \epsilon\}$



In this example every point of A is an interior point but not B.

* **Open Set** – A set 's' is called an open set if every point in 's' is an

interior Point

Ex:- (i) the empty set

(ii) the set of all complex numbers.

(iii) $\{z : |z| > r\}$, $r \geq 0$

(iv) $\{z : r_1 < |z| < r_2\}$, $0 \leq r_1 < r_2$

* **Limit Point** :- A point z_0 is said to be a limit point of 's' if every deleted neighbourhood of z_0 contains a point of s.

— Limit point is also known as Cluster point (or) point of accumulation.

— The limit Point of the set may (or) may not belong to the set

Ex:- ① the limit points of open set $|z| < 1$ are $|z| \leq 1$.

i.e. all the points of the set and all the points on the boundary $|z| = 1$

② The set $\{1, \frac{1}{2}, \frac{1}{3}, \cdots \frac{1}{n}, \cdots\}$ has 0 as a limit point.

③ the set $\left\{\frac{3+2ni}{1+n} \,/\, n = 1, 2, 3 \cdots\right\}$

$$= \left\{\frac{3+2i}{2}, \frac{3+4i}{3}, \frac{3+6i}{4}, \cdots\right\}$$

has $2i$ as a limit point

* **Closed Set** : A set is said to be closed if it contains all its limit points

Ex:- ① the empty set

(2) the set of all complex numbers.

(3) $\{z : |z| > r\}$, $r \geq 0$

(4) $\{z : r_1 \leq |z| \leq r_2\}$, $0 \leq r_1 < r_2$

(5) the union of any two closed sets

* **Closure of a set** :- the union of a set and its limit points is called closure.

* **Domain (Region)** :-

— A set 's' of points in the complex plane is said to be connected set if any two of its points can be joined by a continuous curve, all of whose belong to 's'

— An open connected is called an open domain (or) open region

— If the boundary point of 's' are also added to an open domain. then it is called closed domain.

* **Complex Variable** :-

If a symbol z takes any one of the values of a set of complex numbers, then z is called a complex variable. (or)

Let D be an arbitrary non-empty point set of XOY-plane. If z is allowed to denote any point of D, then z is called a complex variable. and D is the domain of definition of z (or) simply domain.

* **Functions of a Complex Variable**

we say that 'w' is a function of the complex variable z with domain D and Range R, if D and R are two non-empty point sets of complex plane, if to each z in D there corresponds at least one w in R and to each w of R, there is at least one z of D to which w corresponds

Then we symbolically write

$w = f(z)$

— The variable z is sometimes called independent variable and 'w' is called dependent variable.

The value of a function at z=a is written as f(a).

Thus if $f(z) = z^2$, $f(2i) = (2i)^2 = -4$

— If we have only one value 'w' of R to each value of z in D, then we say that w is a single valued function of z (or) f(z) is single valued.

— If more than one value of w corresponds to each value of z, we say that w is a multi valued (or) multiple valued function.

Ex:- ① Let $w = z^2$. Then corresponding to each value of z we get only one value to w so w is a single valued function.

This is because

the function $w = z^2$ may be expressed as $w = f(z)$

$= f(x+iy)$

$= f(x, y)$

$= (x+iy)^2$

$= x^2 - y^2 + i(2xy)$

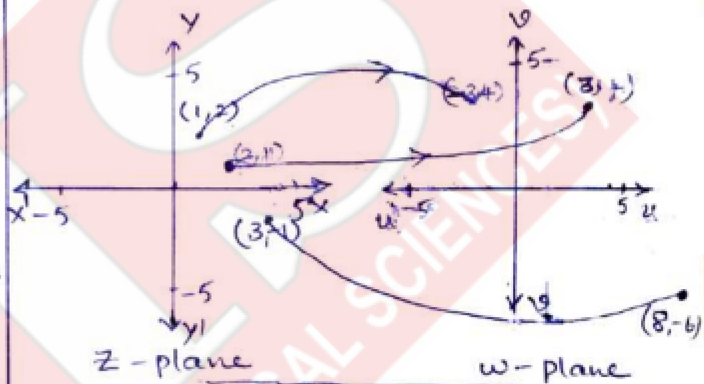
where $Re(w) = x^2 - y^2$

$= u(x,y)$ say

and $Im(w) = 2xy$

$= v(x,y)$ say

$\boxed{\because w = u+iv}$



Example ②. Let $w = z^{1/2}$

Here to each value of z we get two values to w. so we say multi-valued function

This is because:

$$w = z^{1/2} = (x+iy)^{1/2}$$

$$= \sqrt{r}\, e^{i\theta/2} \quad \text{where } x = r\cos\theta,\ y = r\sin\theta$$

Let $\theta = \theta_1$ then $w = \sqrt{r}\, e^{i\theta_1/2}$;

$\theta = \theta_1 + 2\pi$ then $w = \sqrt{r}\, e^{i(\theta_1+2\pi)/2}$

$$= \sqrt{r}\left[\cos(180 + \theta_1/2) + i\sin(180+\theta_1/2)\right]$$

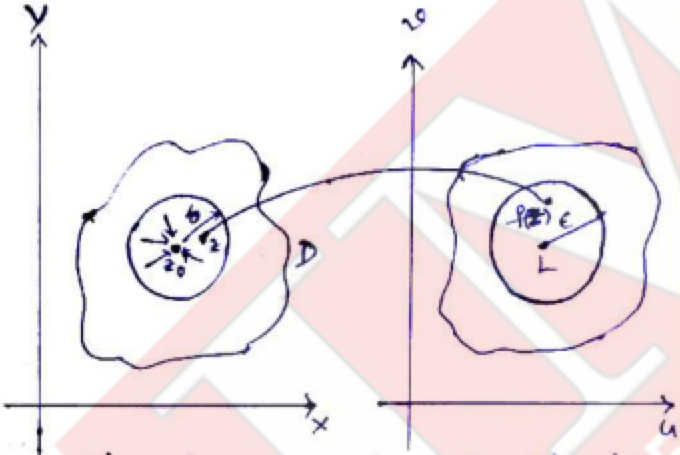
$$= \sqrt{r}\left[-\cos\theta_1/2 - i\sin\theta_1/2\right]$$

$$= -\sqrt{r}\, e^{i\theta_1/2}$$

we can verify that w gets the same values for θ_1 and $\theta_1+4\pi$.

* Limit of a Function:-

Let $f(z)$ be a function of a complex variable z. Then we say that $\lim_{z\to z_0} f(z)=L$, if for any given $\epsilon>0$ (however small), $\exists$ a $\delta>0$ (depending on ϵ) such that $|f(z)-L|<\epsilon$ whenever $0<|z-z_0|<\delta$



The above results can also be written as Let f be a function of two real variables x & y. we say that $\lim_{(x,y)\to(x_0,y_0)} f(x,y)=L$ if for each $\epsilon>0$, $\exists$ a $\delta>0$ such that $|f(x,y)-L|<\epsilon$ for every $0<\sqrt{(x-x_0)^2+(y-y_0)^2}<\delta$

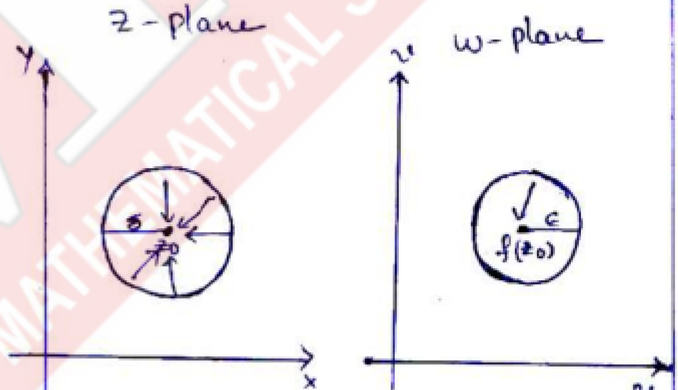
* Continuity of a Function:-

If $f(x_0,y_0)=L$, then we say that $f(x,y)$ is continuous at (x_0,y_0) (or) $f(z_0)=L$

i.e the value of the function at $z=z_0$ is equal to 'L', then we say that $f(z)$ is continuous at $z=z_0$;

(or)

$f(z)$ is continuous at $z=z_0$ if $\lim_{z\to z_0} f(z)=f(z_0)$ i.e if given $\epsilon>0$ (however small), $\exists$ a $\delta>0$ depending on ϵ such that $|f(z)-f(z_0)|<\epsilon$ whenever $|z-z_0|<\delta$



Note:- Here we are silent about how z approaches z_0, i.e. along which path it approaches z_0 is immaterial.

Note:- Let us consider $f(z)=z^2+3z+5$

Let $z=x+iy$ then $z^2=x^2-y^2+i2xy$

$\therefore f(z)=z^2+3z+5$

$=(x^2-y^2+3x+5)+i(2xy+3y)$

$=f_1(x,y)+if_2(x,y)$

In general we can write

$$f(z) = u(x,y) + iv(x,y)$$

(or) $w = u+iv$

where u & v are functions of real variables x & y

Theorem → The function $f(z)=u(x,y)+iv(x,y)$ is continuous at a point $z_0 = x_0+iy_0$ iff $u(x,y)$ & $v(x,y)$ are both continuous at the point (x_0,y_0)

Proof:- Suppose $f(z)$ is continuous at $z=z_0$ for every $\epsilon>0$, $\exists$ a $\delta>0$ such that $|f(z)-f(z_0)|<\epsilon$ whenever $|z-z_0|<\delta$.

Now we have

$|u(x,y)-u(x_0,y_0)| < |f(z)-f(z_0)| < \epsilon$ whenever $|z-z_0|<\delta$

$(\because z = x+iy \Rightarrow |x| \le |z|$ & $|y| \le |z|)$

$\therefore |u(x,y)-u(x_0,y_0)|<\epsilon$ whenever $|(x,y)-(x_0,y_0)|<\delta$

Similarly $|v(x,y)-v(x_0,y_0)|<\epsilon$ whenever $|(x,y)-(x_0,y_0)|<\delta$

$\therefore$ $u(x,y)$ & $v(x,y)$ are continuous at (x_0,y_0)

Conversely suppose that $u(x,y)$ & $v(x,y)$ are both continuous at (x_0,y_0)

$\therefore$ Given $\epsilon>0$ $\exists$ a $\delta>0$ such that

$|u(x,y)-u(x_0,y_0)|<\epsilon/2$ &
$|v(x,y)-v(x_0,y_0)|<\epsilon/2$

whenever $|(x,y)-(x_0,y_0)|<\delta$

Now we have

$$|f(z)-f(z_0)| = |u(x,y)+iv(x,y) - [u(x_0,y_0)+iv(x_0,y_0)]|$$

$$= |(u(x,y)-u(x_0,y_0)) + i(v(x,y) - v(x_0,y_0))|$$

$$\le |u(x,y)-u(x_0,y_0)| + |v(x,y)-v(x_0,y_0)|$$

$< \epsilon/2 + \epsilon/2$ whenever $|(x,y)-(x_0,y_0)|<\delta$

$= \epsilon$

$\therefore |f(z)-f(z_0)|<\epsilon$ whenever $|z-z_0|<\delta$

$\therefore$ $f(z)$ is continuous at $z=z_0$

Note:- In the case of a function of single real variable, there are only two directions to travel, a limit exists iff the RHL and LHL coincide.

— In the case of a function of two variables there are infinitely many directions are possible.

Problems:

→ Let $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & ; (x,y)\neq(0,0) \\ 0 & ; (x,y)=(0,0) \end{cases}$

whether this function is continuous (or) not

Sol^n:- Along either of the coordinate axes

Let us suppose that $(x,y)\to(0,0)$ along x-axis then $y=0$

$$\therefore f(x,y) = f(x,0) = \frac{x(0)}{x^2+0} = 0$$

$\therefore \underset{x\to 0}{Lt} f(x,0) = 0$

Similarly $\underset{y\to 0}{Lt} f(0,y) = 0$

$\therefore \underset{x\to 0}{Lt} f(x,0) = 0 = \underset{y\to 0}{Lt} f(0,y)$

$\therefore$ The function is tending to zero along coordinate axes.

Now if we approach $(0,0)$ along the straight line path $y=mx$ we get

$$\underset{x\to 0}{Lt} f(x,mx) = \underset{x\to 0}{Lt} \frac{mx^2}{x^2+m^2x^2} = \underset{x\to 0}{Lt} \frac{m}{1+m^2}$$
$$= \frac{m}{1+m^2}$$

Since the limit depends upon the value of m

$\therefore f(x,y)$ approaches different values along different straight lines.

$\therefore$ the limit at the origin does not exist

$\therefore$ the function is not continuous at $(0,0)$

$\rightarrow$ Let $f(x,y) = \begin{cases} \dfrac{x^2y^2}{(x+y^2)^3} ; & (x,y)\neq(0,0) \\ 0 ; & (x,y)=(0,0) \end{cases}$

Sol'n: $\underset{x\to 0}{Lt} f(x,0) = \underset{y\to 0}{Lt} f(0,y) = 0$

$\therefore$ the function is tending to zero along the coordinate axes.

Now if we approach $(0,0)$ along the straight line path $y=mx$, we get

$$\underset{x\to 0}{Lt} f(x,mx) = \underset{x\to 0}{Lt} \frac{m^2x^4}{(x+m^2x^2)^3}$$
$$= \underset{x\to 0}{Lt} \frac{m^2x}{(1+m^2x)^3}$$
$$= 0$$

$\therefore f(x,y)$ is tending to zero as $(x,y)\longrightarrow(0,0)$ along any straight line.

But along the parabola $x=y^2$

$$\underset{y\to 0}{Lt} f(y^2,y) = \underset{y\to 0}{Lt} \frac{y^4y^2}{(y^2+y^2)^3} = 1/8$$

$\therefore \underset{(x,y)\to(0,0)}{Lt} f(x,y)$ does not exist.

$\therefore f(x,y)$ is not continuous at $(0,0)$

$\rightarrow$ Let $f(x,y) = \begin{cases} \dfrac{x^3-2y^3}{x^2+y^2} ; & (x,y)\neq(0,0) \\ 0 ; & (x,y)=(0,0) \end{cases}$

show that $f(x,y)$ is continuous at $(0,0)$.

Method (1):

Now we have

$$|f(x,y)-f(0,0)| = \left|\frac{x^3-2y^3}{x^2+y^2} - (0,0)\right|$$
$$= \left|\frac{x^3-2y^3}{x^2+y^2}\right|$$

Now

$$|x^3-2y^3| \leq |x^3| + |2y^3|$$
$$= |x|^3 + 2|y|^3$$
$$= |x||x|^2 + 2|y||y|^2$$

$\leq |z||x|^2 + 2|z||y|^2$

$(\because z = x+iy$
$|x| \leq |z|$ &
$|y| \leq |z|)$

$= |z| (|x|^2 + 2|y|^2)$

$= |z| (x^2+2y^2)$

$< \sqrt{x^2+y^2}\, 2(x^2+y^2)$

$(\because x^2+2y^2 < 2(x^2+y^2))$

$\therefore \dfrac{|x^3-2y^3|}{|x^2+y^2|} < 2\sqrt{x^2+y^2}$

$\Rightarrow \left|\dfrac{x^3-2y^3}{x^2+y^2} - (0,0)\right| < 2|z-0|$
$< \epsilon$

whenever $|z-0| < \epsilon/_2 = \delta$ (choosing)

$\Rightarrow |f(x,y) - f(0,0)| < \epsilon$ whenever
$|z-0| < \delta$.

$\therefore \lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0)$

$\therefore f(x,y)$ is continuous at $(0,0)$

method (2):

Switching to polar coordinates,

$x = r\cos\theta$, $y = r\sin\theta$

we show that $|f(r,\theta)| < \epsilon$
whenever $|r| < \delta$

Now we have

$|f(x,y) - f(0,0)| = \left|\dfrac{x^3-2y^3}{x^2+y^2} - (0,0)\right|$

$\Rightarrow |f(r,\theta)| = \left|\dfrac{r^3\cos^3\theta - 2r^3\sin^3\theta}{r^2}\right|$

$= |r|\,|\cos^3\theta - 2\sin^3\theta|$

$\leq |r|\left[|\cos^3\theta| + 2|\sin^3\theta|\right]$

$\leq |r|\left[1+2(1)\right]$

$= 3|r| < \epsilon$

whenever $|r| < \epsilon/_3 = \delta$ (choosing)

$\therefore |f(r,\theta)| < \epsilon$ whenever $|r| < \delta$

H.W. If $f(0,0)=0$, which of the following functions are continuous at the origin?

(a) $f(x,y) = \dfrac{x^2y^2}{x^4+y^4}$

(b) $f(x,y) = \dfrac{x^2y^2}{(x^2+y^2)^2}$

(c) $f(x,y) = \dfrac{x^3y^2}{(x^2+y^2)^2}$ (continuous)

(d) $f(x,y) = \dfrac{x+ye^{-x^2}}{1+y^2}$ (continuous)

(e) $f(x,y) = \dfrac{(x+y^2)^2}{x^2+y^2}$

Sol'n: (c) $\lim\limits_{x\to 0} f(x,0) = 0 = \lim\limits_{y\to 0} f(0,y)$

$\therefore f(x,y)$ is tending to zero along the coordinate axes

choosing $y = mx$

$\lim\limits_{x\to 0} f(x,mx) = \lim\limits_{x\to 0} \dfrac{x^3m^2x^2}{(x^2+m^2x^2)^2}$

$= \lim\limits_{x\to 0} \dfrac{m^2x}{(1+m^2)}$

$= 0$

choosing $x = my$

$$\lim_{y\to 0} f(my, y) = \lim_{y\to 0} \frac{m^3 y^3 y^2}{(m^2y^2+y^2)^2}$$

$$= \lim_{y\to 0} \frac{m^3 y}{(m^2+1)^2}$$

$$= 0$$

$\therefore f(x,y)$ is tending to zero as $(x,y) \longrightarrow (0,0)$ along any straight line.

$\therefore f(x,y)$ is continuous at $(0,0)$

(or)

$$f(x,y) = \frac{x^3y^2}{(x^2+y^2)^2}; \ (x,y) \neq (0,0)$$

Now we have

$$|f(x,y) - f(0,0)| = |f(x,y) - (0,0)|$$

$$= \left|\frac{x^3y^2}{(x^2+y^2)^2}\right| \quad \text{--- (1)}$$

Now $|x^3y^2| = |x|^3|y|^2$

$= |x| (x^2 \cdot y^2)$

$\leq \sqrt{x^2+y^2}\,(x^2+y^2)(x^2+y^2)$

$(\because |x| \leq |z| \ \& \ |y| \leq |z|$

$\Rightarrow x^2 \leq |z|^2 \ \&$

$y^2 \leq |z|^2)$

$= (x^2+y^2)^2\sqrt{x^2+y^2}$

$$\Rightarrow \frac{|x^3y^2|}{|(x^2+y^2)^2|} \leq \sqrt{x^2+y^2}$$

$$\left|\frac{x^3y^2}{(x^2+y^2)^2}\right| \leq |z| < \epsilon$$

where $|z| < \frac{\epsilon}{1} = \delta$

$\Rightarrow |f(x,y) - f(0,0)| < \epsilon$ where $|z - 0| < \delta$

$\therefore f(x,y)$ is continuous at $(0,0)$

* Differentiability :-

If a function $f(z)$ is single valued in a domain D, then the derivative of $f(z)$ at $z = z_0$ is denoted by $f'(z_0)$ and is defined as $f'(z_0) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$

provided that the limit exists. In this case we say that $f(z)$ is differentiable at $z = z_0$.

(or)

A function 'f' is said to be differentiable at z, if

$\lim_{h \to 0} \frac{f(z+h)-f(z)}{h}$ exists and is denoted by $f'(z)$.

i.e. $f'(z) = \lim_{h \to 0} \frac{f(z+h)-f(z)}{h}$

Note :- 'h' approaches 'o' through points in the plane, not just along the real axis.

→ Every differentiable function $f(z)$ is continuous But the converse is not true.

Proof : Let $f(z)$ be differentiable at $z = z_0$

$$\therefore f'(z_0) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$$

Now we have

$$f(z)-f(z_0) = \frac{f(z)-f(z_0)}{z-z_0}(z-z_0)$$

$$\Rightarrow \lim_{z \to z_0}\left[f(z)-f(z_0)\right] = \lim_{z \to z_0}\left[\frac{f(z)-f(z_0)}{z-z_0} \times (z-z_0)\right]$$

$$= \lim_{z \to z_0}\left[\frac{f(z)-f(z_0)}{z-z_0}\right] \lim_{z \to z_0}(z-z_0)$$

$$= f'(z_0).(0)$$

$$= 0.$$

$$\Rightarrow \lim_{z \to z_0}\left(f(z)-f(z_0)\right) = 0$$

$$\Rightarrow \lim_{z \to z_0} f(z) = \lim_{z \to z_0} f(z_0)$$

$$= f(z_0)$$

$\therefore f(z)$ is continuous at $z = z_0$

Converse : For example

$$f(z) = |z|$$

$= \sqrt{x^2+y^2}$ is continuous at $(0,0)$

but not differentiable at $(0,0)$.

Since :

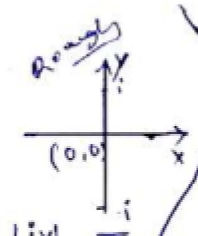
$$\lim_{z \to 0} f(z) = 0 = f(0)$$

$\therefore f(z)$ is continuous at $(0,0)$.

But $\lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0} = \lim_{z \to 0} \frac{|z|}{z}$

$$= \lim_{(x,y) \to (0,0)} \frac{\sqrt{x^2+y^2}}{x+iy}$$

$$= \lim_{(x,y) \to (0,0)} \frac{|x+iy|}{x+iy}$$



$\frac{|iy|}{iy} = \frac{\sqrt{y^2}}{iy} = \frac{1}{i} = -i$

$$= \begin{cases} 1 \text{ for +ve real values of } z \\ -1 \text{ for -ve real values of } z \\ -i \text{ for values of } z \text{ on the +ve imaginary axis} \\ i \text{ for values of } z \text{ on the -ve imaginary axis} \end{cases}$$

$\therefore f'(0)$ doesnot exist

(or)

Since $\underset{h\to 0}{Lt}\, f(z+h) = \underset{h\to 0}{Lt}\, |z+h|$

the value of the limit at $z=0$

is $\underset{h\to 0}{Lt}\, |0+h| = \underset{h\to 0}{Lt}\, |h| = 0$

Hence $|z|$ is continuous at the origin.

But $\underset{h\to 0}{Lt} \frac{f(z+h)-f(z)}{h} = \underset{h\to 0}{Lt} \frac{|z+h|-|z|}{h}$

$$= \underset{h\to 0}{Lt} \frac{|0+h|-0}{h}$$

$$= \underset{h\to 0}{Lt} \frac{|h|}{h} \quad \text{--- (1)}$$

Let $h = h_1 + ih_2$ and let $h \to 0$ along the real axis then $h_2 = 0$ & $h_1 \to 0$.

$\therefore \underset{h\to 0}{Lt} \frac{|h|}{h} = \underset{h_1\to 0}{Lt} \frac{|h_1|}{h_1} = \pm 1$.

If $h \to 0$ along the imaginary axis, $h_1 = 0$ and $h_2 \to 0$ then

$\underset{h\to 0}{Lt} \frac{|h|}{h} = \underset{h_2\to 0}{Lt} \frac{|ih_2|}{h_2} = \pm i$

$\therefore$ the function is behaving different along the different paths

$\Rightarrow$ the function is not differentiable at the origin

Note:- The rules of differentiation of real functions are also valid for complex function.

$\rightarrow$ Show that $f(z) = z^2$ is differentiable every where.

Sol'n: $f'(z) = \underset{h\to 0}{Lt} \frac{f(z+h)-f(z)}{h}$

$$= \underset{h\to 0}{Lt} \frac{(z+h)^2 - z^2}{h}$$

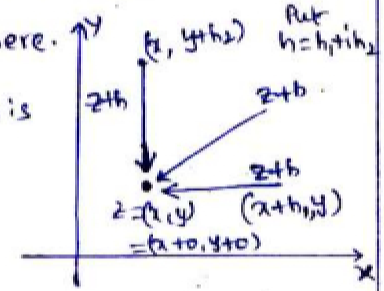
$$= \underset{h\to 0}{Lt} \frac{\cancel{z^2} + 2zh + h^2 - \cancel{z^2}}{h}$$

$$= \underset{h\to 0}{Lt} (2z+h)$$

$$= 2z$$

whatever be the path along which $h \to 0$ the limit exist and z^2 is defined everywhere.

$\therefore$ The function is differentiable Every where



$\rightarrow$ where is $|z|$ differentiable?

Sol'n: Let $f(z) = |z|$

Now $f'(z) = \underset{h\to 0}{Lt} \frac{f(z+h)-f(z)}{h}$

$$= \underset{h\to 0}{Lt} \frac{|z+h|-|z|}{h}$$

$$= \lim \frac{(|z+h|-|z|)(|z+h|+|z|)}{h(|z+h|+|z|)}$$

$$= \lim_{h\to 0} \frac{1}{|z+h|+|z|} \lim_{h\to 0} \frac{(z+h)(\overline{z+h}) - z\bar{z}}{h}$$

$$= \frac{1}{|z+0|+|z|} \lim_{h\to 0} \frac{(z+h)(\bar{z}+\bar{h}) - z\bar{z}}{h}$$

$$= \frac{1}{2|z|} \lim_{h\to 0} \frac{\cancel{z\bar{z}} + z\bar{h} + h\bar{z} + h\bar{h} - \cancel{z\bar{z}}}{h}$$

$$= \frac{1}{2|z|} \lim_{h\to 0} \left(\bar{z} + \bar{h} + z\frac{\bar{h}}{h}\right) \quad \text{--- (1)}$$

Let us approach z along a line parallel to x-axis

$\therefore$ h is a real number

$\therefore$ $\bar{h} = h$ Then from (1)

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

$$= \frac{1}{2|z|} \lim_{h\to 0} \bar{z} + h + \frac{z\cancel{h}}{\cancel{h}}$$

$$= \frac{1}{2|z|} \lim_{h\to 0} (\bar{z} + h + z)$$

$$= \frac{1}{2|z|} \lim_{h\to 0} (2x + h) \quad (\because z + \bar{z} = 2x)$$

$$= \frac{\cancel{2}x}{\cancel{2}|z|} = \frac{x}{|z|} \quad \text{--- (2)}$$

Again let us approach z along a line parallel to y-axis

$\therefore$ h is purely imaginary

$\therefore$ $h = ih_1$

Then from (1),

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

$$= \frac{1}{2|z|} \lim_{h_1\to 0} \left(\bar{z} - ih_1 + z\left(\frac{-ih_1}{ih_1}\right)\right)$$

$$= \frac{1}{2|z|} \lim_{h_1\to 0} (\bar{z} - ih_1 - z)$$

$$= \frac{1}{2|z|} \lim_{h_1\to 0} (-2iy - ih_1)$$

$$= \frac{1}{2|z|}(-2iy) = \frac{-iy}{|z|} \quad \text{--- (3)}$$

(2) & (3) are unequal if $x \neq 0$ and $y \neq 0$

$\therefore$ $|z|$ does not have a derivative if $z \neq 0$

Now if $z = 0$

$$\lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \lim_{h\to 0} \frac{|h|}{h}$$

$$\Rightarrow \lim_{h\to 0} \frac{f(z+h) - f(z)}{h} = \begin{cases} \pm 1 & \text{if approached parallel to } x\text{-axis} \\ \pm i & \text{if approached parallel to } y\text{-axis} \end{cases}$$

$\therefore$ the limit would not exist even now

$\therefore$ $f(z) = |z|$ is no where differentiable

* Cauchy – Riemann Equations:

Let $f(z) = u(x,y) + iv(x,y)$ and $f'(z)$ exists at $z = x+iy$. Then the first order partial derivatives of u and v exist at (x,y) and they must satisfy the Cauchy-Riemann equations $u_x = v_y$ and $u_y = -v_x$ there.

Also $f'(z) = u_x + iv_x$, where these partial derivatives are evaluated at (x,y).

Proof: It is given that $f'(z)$ exists at z

$$\therefore f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h} \text{ exists}$$

Since $f(z) = u(x,y) + iv(x,y)$

$$\Rightarrow f(x+iy) = u(x,y) + iv(x,y)$$

Now we can write

$$\frac{f(z+h)-f(z)}{h} = \frac{f((x+a)+i(y+b)) - f(x+iy)}{a+ib}$$

where $h = a+ib$

$$= \frac{u(x+a,y+b)+iv(x+a,y+b)-u(x,y)-iv(x,y)}{a+ib}$$

$$= \frac{u(x+a,y+b)-u(x,y)}{a+ib} + i\frac{v(x+a,y+b)-v(x,y)}{a+ib} \quad \text{—— (1)}$$

Let us suppose that $h\to 0$ along the real axis then $b=0$ and $a\to 0$ (along a line ll lel to x-axis)

$$\therefore \lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \lim_{a\to 0} \frac{u(x+a,y)-u(x,y)}{a} + i\lim_{a\to 0}\frac{v(x+a,y)-v(x,y)}{a}$$

Since the limit on the L.H.S exists the limits on the R.H.S must also exist.

In addition to this we can observe that the limits on R.H.S are nothing but partial derivatives of u & v with respect to x

$$\therefore f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= u_x + iv_x \quad \text{—— (2)}$$

Now let us suppose that $h\to 0$ along a line parallel to the imaginary axis through the point z

Then $a=0$ and $b\to 0$

$\therefore$ from (1),

$$f'(z) = \lim_{b\to 0} \frac{u(x,y+b)-u(x,y)}{ib} + \lim_{b\to 0} i\left(\frac{v(x,y+b)-v(x,y)}{ib}\right)$$

Since the limit on L.H.S. exists, the limits on R.H.S also exist. The limits on R.H.S are partial derivatives of u & v with respect to y.

$$\therefore f'(z) = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \quad \text{—— (3)}$$

Now comparing (2) & (3), we get

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Equating the real & imaginary parts, we get

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}; \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

i.e. $u_x = v_y$, $v_x = -u_y$ or $u_y = -v_x$

these equations are known as Cauchy - Riemann equations

Note: from ②, we have

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial}{\partial x}(u+iv) = \frac{\partial}{\partial x} f = \frac{\partial f}{\partial x}$$

Similarly from ③,

$$f'(z) = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = -i\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) = -i\frac{\partial}{\partial y}(u+iv) = -i\frac{\partial f}{\partial y}$$

From these we get

$$\boxed{f'(z) = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}}$$

this equation provides a method for calculating the derivative, if the derivative is known to exist.

Ex:- $f(z) = z^2$

$$= (x+iy)^2 = x^2-y^2+i(2xy)$$

is everywhere differentiable.

Sothat $f'(z) = \frac{\partial f}{\partial x}$

$$= 2x+i(2y)$$

$$= 2(x+iy) = 2z$$

→ Problems:

(1) Consider $f(z) = \bar{z}^2$

$$= (x-iy)^2$$

$$= x^2 - y^2 - 2ixy$$

$\therefore u(x,y) = x^2-y^2$, $v(x,y) = -2xy$

$$\therefore \frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial x} = -2y$$

$$\frac{\partial u}{\partial y} = -2y \qquad \frac{\partial v}{\partial y} = -2x$$

In general $\frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} \neq -\frac{\partial v}{\partial x}$

But at $(0,0)$ these conditions are satisfied

$\therefore$ The Cauchy Riemann (CR) conditions are not satisfied except at $(0,0)$

$\therefore$ the function is not differentiable except at $(0,0)$

(2) Consider $f(z) = z^2$

$$= (x+iy)^2$$

$$= x^2-y^2+2ixy$$

$\therefore u(x,y) = x^2-y^2$ & $v(x,y) = 2xy$

$$\frac{\partial u}{\partial x} = 2x \qquad \frac{\partial v}{\partial x} = 2y$$

$$\frac{\partial u}{\partial y} = -2y \qquad \frac{\partial v}{\partial y} = 2x$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \;\&\; \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}$$

$\therefore$ Cauchy - Riemann - conditions are satisfied every where

Now let us check whether the function is differentiable (or) not

For that consider

$$f'(z) = \lim_{h\to 0} \frac{f(z+h)-f(z)}{h}$$

$$= \lim_{h \to 0} \frac{(z+h)^2 - z^2}{h}$$

$$= \lim_{(a+ib)\to 0} \frac{(x+a)^2 - (y+b)^2 + 2i(x+a)(y+b) - x^2 + y^2 - 2ixy}{a+ib}$$

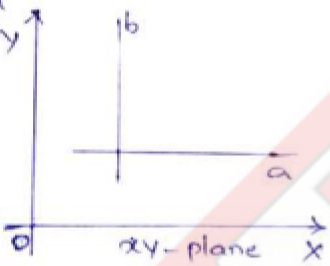
where $h = a+ib$

Let $h \to 0$ along the real axis of ab-plane, then $b = 0$ and $a \to 0$

$$f'(z) = \lim_{a\to 0} \frac{2ax + a^2 + 2i(ay)}{a}$$

$$= 2(x+iy)$$

$$= 2z$$



Let $h \to 0$ along the imaginary axis of the ab-plane

$\therefore a = 0$ and $b \to 0$

$$\therefore f'(z) = \lim_{b\to 0} \frac{-2by - b^2 + 2i(xb)}{ib}$$

$$= \lim_{b\to 0} \frac{-2y + 2ix}{i} = 2z$$

Since the limit is not depending on the value 'h' (or) a, b;

we say that the function is differentiable for all values of z

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<u>Note</u>:- The Cauchy – Riemann conditions are necessary condition only.

That is when a function $f(z)$ is differentiable then Cauchy-Riemann conditions are satisfied.

But they are not sufficient conditions

i.e. even though Cauchy-Riemann conditions are satisfied, the function may not be differentiable at that point.

This can be verified in the following example.

Let $f(z) = \begin{cases} \dfrac{xy^2}{x^2+y^2} & \text{when } z \neq 0 \\ 0 & \text{when } z = 0 \end{cases}$

Let us observe that the behaviour of the function at the origin.

Let us verify Cauchy-Riemann conditions at origin.

From the given function we get

$u = \dfrac{xy^2}{x^2+y^2}$ $\quad v = 0$ when $z \neq 0$

$u = 0$ $\quad v = 0$ when $z = 0$

$$\therefore \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$$

Now $\left.\dfrac{\partial u}{\partial x}\right|_{z=0} = \lim_{h\to 0} \dfrac{u(h,0) - u(0,0)}{h}$

$$= \lim_{h\to 0} \frac{0-0}{h} = 0$$

$\left.\dfrac{\partial u}{\partial y}\right|_{z=0} = \lim_{h\to 0} \dfrac{u(0,k) - u(0,0)}{h}$

$$= \lim_{k\to 0} \frac{0-0}{k} = 0$$

$\therefore \dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y} = 0$ & $\dfrac{\partial u}{\partial y} = \dfrac{-\partial v}{\partial x} = 0$

$\therefore$ Cauchy Riemann conditions are satisfied at the origin.

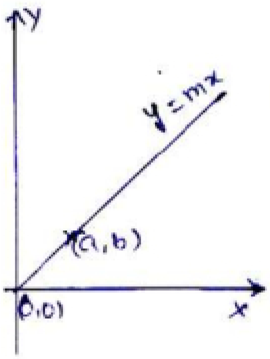
Now let us <u>check the differenbility at the origin.</u>

we say that the function is differentiable at the origin if

$\lim\limits_{h\to 0} \dfrac{f(c+h)-f(c)}{h}$ exists

Now let $h = a+ib$ and $h\to 0$ along a line $y = mx$ of xy-plane then

$$\lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0}\frac{f(h)}{h}$$



$$= \lim_{h\to 0}\frac{f(a+ib)}{a+ib}$$

i.e $(a+ib)\to 0$

$$= \lim_{(a+ib)\to 0}\frac{ab^2}{a^2+b^2}\cdot\frac{1}{a+ib}$$

$$= \lim_{a\to 0}\frac{a(ma)^2}{a^2+b^2}\cdot\frac{1}{a+ima}$$

$$= \lim_{a\to 0}\frac{m^2a^3}{a^2(1+m^2)}\cdot\frac{1}{a(1+im)}$$

$$= \lim_{a\to 0}\frac{m^2}{(1+im)(1+m^2)}$$

$$= \frac{m^2}{(1+im)(1+m^2)}$$

$\therefore$ the value of the limit dependens upon the value of m

$\therefore$ the derivative of $f(z)$ at $z=0$ doesnot exist.

$\therefore$ The function is not differentiable at the origin.

Note:- If the Cauchy-Riemann equations are not satisfied then the function is nowhere differentiable.

problems:

→ Determine where the following functions satisfy the Cauchy-Riemann equations and where the functions are differentiable

(a) $f(z) = x^2-y^2$ (b) $f(z) = 2xyi$

(c) $f(z) = z\,\mathrm{Re}(z)$ (d) $f(z) = z|z|$

(e) $f(x,y) = \begin{cases}\dfrac{xy}{x^2+y^2} & ; z\neq 0\\ 0 & ; z=0\end{cases}$

Sol^n: (a) $f(z) = x^2-y^2$

comparing with $f(z) = u(x,y)+iv(x,y)$

$u(x,y) = x^2-y^2$; $v(x,y) = 0$

$\dfrac{\partial u}{\partial x} = 2x$ $\quad \dfrac{\partial v}{\partial x} = 0$

$\dfrac{\partial u}{\partial y} = -2y$ $\quad \dfrac{\partial v}{\partial y} = 0$

$\therefore \dfrac{\partial u}{\partial x} \neq \dfrac{\partial v}{\partial y}$ & $\dfrac{\partial v}{\partial x} \neq -\dfrac{\partial u}{\partial y}$

But at $(0,0)$ these conditions are satisfied.

$\therefore$ Cauchy-Riemann equations are not satisfied except at $(0,0)$

Now let us check the differentibility at $(0,0)$.

We say that the function is differentiable at the origin if $\lim\limits_{h\to 0}\dfrac{f(0+h)-f(0)}{h}$ exists.

Let $h = a+ib$ and $h\to 0$ along a line $y = mx$ of xy-plane,

then $\lim\limits_{h\to 0}\dfrac{f(h)-f(0)}{h} = \lim\limits_{(a+ib)\to 0}\dfrac{f(a+ib)-0}{a+ib}$

$$= \lim_{(a+ib)\to 0}\frac{a^2-b^2}{a+ib}$$

$$= \lim_{a\to 0} \frac{a^2-(ma)^2}{a+i(ma)}$$

$$= \lim_{a\to 0} \frac{a^2(1-m^2)}{a(1+im)}$$

$$= 0$$

$\therefore$ At $(0,0)$ the given function $f(z) = x^2 - y^2$ is differentiable

(d) $f(z) = z|z|$

Sol'n: $f(z) = (x+iy)\sqrt{x^2+y^2}$

$$= x\sqrt{x^2+y^2} + iy\sqrt{x^2+y^2}$$

$u(x,y) = x\sqrt{x^2+y^2}$; $v(x,y) = y\sqrt{x^2+y^2}$

$$\frac{\partial u}{\partial x} = \frac{2x^2+y^2}{\sqrt{x^2+y^2}} \qquad \frac{\partial v}{\partial x} = \frac{xy}{\sqrt{x^2+y^2}}$$

$$\frac{\partial u}{\partial y} = \frac{xy}{\sqrt{x^2+y^2}} \qquad \frac{\partial v}{\partial y} = \frac{x^2+2y^2}{\sqrt{x^2+y^2}}$$

$\therefore \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial y}$ at $(x,y) \neq (0,0)$

Now let us check at the origin:

At $z=0$

$$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x}$$

$$= \lim_{x\to 0} \frac{x\sqrt{x^2}}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0,y)-u(0,0)}{y}$$

$$= \lim_{y\to 0} \frac{0-0}{y} = 0$$

and $$\frac{\partial v}{\partial x} = \lim_{x\to 0} \frac{v(x,0)-v(0,0)}{x}$$

$$= \lim_{x\to 0} \frac{0-0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{y\to 0} \frac{v(0,y)-v(0,0)}{y}$$

$$= 0$$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = \frac{-\partial v}{\partial x}$

$\therefore f(z)$ satisfies the Cauchy-Riemann equations at the origin.

Let us check the differentiability at $(0,0)$

Now $$f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z-0}$$

$$= \lim_{z\to 0} \frac{z|z|}{z} = \lim_{z\to 0} |z|$$

$$= \lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2}$$

Along any path $f'(0) = 0$

$\therefore f(z)$ is differentiable at $(0,0)$.

* Analytic Function :—

— Consider a single valued function $f(z)$ in a domain D

The function $f(z)$ is said to be analytic at a point $z = z_0$ if it is differentiable every where in the neighbourhood of z_0 (i.e. if there exists a neighbourhood $|z-z_0|<\delta$ at all points of which $f'(z)$ exists)

— Thus analyticity is a region based property

— A function $f(z)$ is analytic in a domain D if it is analytic at every point in the domain.

— A function $f(z)$ is analytic at every point in the complex plane is called an entire function.

— If $f'(z)$ exists at every point of a domain D except at a finite number of exceptional points then $f(z)$ is said to be analytic in D and is reffered to as analytic function in D. These exceptional points are called singular points (or) singularities of the function.

— If $f'(z)$ exists at every point of D, then we say that $f(z)$ is regular in D

— The terms regular and holomorphic are also sometimes used as synonyms for analytic.

* Singular Point :-

A point $z = z_0$ is said to be a singular point of a function $f(z)$ if $f'(z_0)$ does not exist.

Examples :

→ The function $f(z) = |z|^2$
$= x^2+y^2$ is differentiable only at origin but not differentiable at any other point so it is not analytic at any other point.

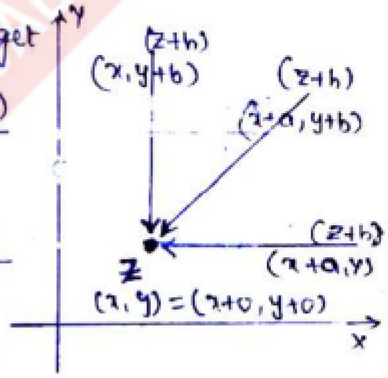
Sol'n: Consider $\dfrac{f(z+h)-f(z)}{h} = \dfrac{|z+h|^2-|z|^2}{h}$

Let $h = a+ib$ then $\dfrac{|(x+a)+i(y+b)|^2-|x+iy|^2}{a+ib}$

$\therefore \lim\limits_{h\to 0} \dfrac{f(z+h)-f(z)}{h} = \lim\limits_{h\to 0} \dfrac{(x+a)^2+(y+b)^2-(x^2+y^2)}{a+ib}$

$= \lim\limits_{(a+ib)\to 0} \dfrac{2ax+a^2+2by+b^2}{a+ib}$ —— ①

Let the point $z+h$ tends to z along a line parallel to real axis then as $h\to 0$, $b=0$ & $a\to 0$.



$\therefore$ from ①, we get

$\lim\limits_{h\to 0} \dfrac{f(z+h)-f(z)}{h}$

$= \lim\limits_{a\to 0} \dfrac{a(2x+a)}{a}$

$= 2x$ —— ②

Similarly let us suppose that $z+h$ tends to z along a line parallel to imaginary axis, then as $h\to 0$, $a=0$ & $b\to 0$.

$\therefore$ from ①, we get

$f'(z) = \lim\limits_{b\to 0} \dfrac{b(2y+b)}{ib} = -i2y$ —— ③

Comparing ② & ③ we say that the given function $|z|^2$ is not differentiable when $x\neq 0$, $y\neq 0$ i.e. other than origin.

At the origin, we have.

$$\lim_{h\to 0}\frac{f(z+h)-f(z)}{h}=\lim_{h\to 0}\frac{|0+h|^2-|0|^2}{h}$$

$$=\lim_{h\to 0}\frac{|h|^2}{h}$$

Now as $h\to 0$ along real axis, h is any real number say h_1

Then $\lim_{h_1\to 0}\frac{|h_1|^2}{h_1}=\lim_{h_1\to 0}\frac{h_1^2}{h_1}=0$

As $h\to 0$ along imaginary axis, h is an imaginary number say $h=ih_2$

then $\lim_{h\to 0}\frac{|h|^2}{h}=\lim_{h_2\to 0}\frac{|ih_2|^2}{h_2}$

$$=\lim_{h_2\to 0}\frac{h_2^2}{h_2}$$

$$=0$$

$\therefore$ At origin the given function $|z|^2$ is differentiable.

Ex(2): The function $f(z)=x^2y^2$ is differentiable at all points on each of the coordinate axis, but is still nowhere analytic.

Ex(3): All polynomials are entire functions and $f(z)=\frac{1}{1-z}$ is analytic anywhere except at $z=1$.

* Now we can state the necessary and sufficient condition for a function to be analytic in a domain D as below:

— Necessary and sufficient Conditions:

Let $f(z)=u(x,y)+iv(x,y)$ be defined in a domain D with $u(x,y)$ and $v(x,y)$ having continuous partial derivatives throughout D then the necessary and sufficient condition for a function $f(z)$ to be analytic in D is the satisfaction of Cauchy-Riemann Conditions

i.e $\frac{\partial u}{\partial x}=\frac{\partial v}{\partial y}$; $\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y}$

→ show that an analytic $g(z)$ is independent of $\bar{z}$.

Sol'n:– Let $z=x+iy$ then

$$x=\frac{z+\bar{z}}{2}\ \&\ y=\frac{z-\bar{z}}{2i}$$

$$\therefore \frac{\partial g}{\partial \bar{z}}=\frac{\partial}{\partial \bar{z}}(g(z))$$

$$=\frac{\partial}{\partial \bar{z}}\left(g(x+iy)\right)$$

$$=\frac{\partial g}{\partial x}\cdot\frac{\partial x}{\partial \bar{z}}+i\frac{\partial g}{\partial y}\frac{\partial y}{\partial \bar{z}}$$

$$=\frac{\partial g}{\partial x}\left(\tfrac{1}{2}\right)+\frac{\partial g}{\partial y}\left(-\tfrac{1}{2i}\right)$$

$$=\tfrac{1}{2}\left[\frac{\partial g}{\partial x}+i\frac{\partial g}{\partial y}\right]$$

Since $g(z)$ is analytic

$\therefore$ by Cauchy-Riemann Conditions, we have

$$\frac{\partial g}{\partial \bar{z}}=\tfrac{1}{2}\left(-i\frac{\partial g}{\partial y}+i\frac{\partial g}{\partial y}\right)$$

$$=0\qquad \left(\because f'(z)=\frac{\partial f}{\partial x}=-i\frac{\partial f}{\partial y}\right)$$

$\therefore$ then analytic function $g(z)$ is independent of $\bar{z}$

→ Prove [2007] that the function f defined by $f(z)=\begin{cases}\frac{z^5}{|z|^4}, & z\neq 0\\ 0, & z=0\end{cases}$

is not differentiable at $z=0$.

Sol^n: We say that $f(z)$ is not differentiable at the origin if

$\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$ does not exist.

Let $h=a+ib$

then $\lim_{h\to 0}\frac{f(h)-f(0)}{h}=\lim_{(a+ib)\to 0}\frac{f(a+ib)}{a+ib}$

$=\lim_{(a+ib)\to 0}\frac{(a+ib)^5}{|a+ib|^4}\times\frac{1}{(a+ib)}$

$=\lim_{(a+ib)\to 0}\frac{(a+ib)^4}{|a+ib|^4}$

$=\left[\lim_{(a+ib)\to 0}\frac{a+ib}{|a+ib|}\right]^4$ —— (1)

Let $h\to 0$ along the real axis of ab-plane then $b=0$ and $a\to 0$

$\therefore f'(0)=\left(\lim_{a\to 0}\frac{a}{|a|}\right)^4$

$=1$.

Let $h\to 0$ along the imaginary axis of ab-plane then $a=0$ and $b\to 0$.

$f'(0)=\left[\lim_{b\to 0}\frac{ib}{|ib|}\right]^4$

$=\left[\lim_{b\to 0}\frac{ib}{b}\right]^4=1$

Let $h\to 0$ along a line $y=mx$ of xy-plane then from (1)

$f'(0)=\left[\lim_{a\to 0}\frac{a+ima}{\sqrt{a^2+m^2a^2}}\right]^4$

$=\left[\lim_{a\to 0}\frac{1+im}{\sqrt{1+m^2}}\right]^4=\frac{1+im}{\sqrt{1+m^2}}$

$\therefore$ The value of the limit depends upon the value of m

$\therefore$ The derivative of $f(z)$ at $z=0$ does not exist

$\therefore$ The function is not differentiable at the origin

(or)

Now $f'(0)=\lim_{z\to 0}\frac{f(z)-f(0)}{z-0}$

$=\lim_{z\to 0}\frac{z^5}{|z|^4}\times\frac{1}{z}$

$=\lim_{z\to 0}\frac{z^4}{|z|^4}=\left[\lim_{z\to 0}\frac{z}{|z|}\right]^4$

$=\left[\lim_{(x+iy)\to 0}\frac{x+iy}{\sqrt{x^2+y^2}}\right]^4$ —— (1)

$z\to 0$ along a line $y=mx$ of xy-plane then from (1)

$f'(0)=\left[\lim_{x\to 0}\frac{x+imx}{\sqrt{x^2+m^2x^2}}\right]^4=\left[\lim_{x\to 0}\frac{1+im}{\sqrt{1+m^2}}\right]^4$

$=\left[\frac{1+im}{1+m^2}\right]^4$

$\therefore$ The value of the limit depends upon the value of m

$f'(z)$ does not exist at $z=0$

Problems:

→ $f(z)=e^z$ is entire function?

Sol^n: $f(z)=e^z$

$=e^{x+iy}$

$= e^x e^{iy}$

$= e^x(\cos y + i\sin y)$

$= e^x \cos y + ie^x \sin y$

Comparing with $f(z) = u(x,y) + iv(x,y)$

$u(x,y) = e^x \cos y$; $v(x,y) = e^x \sin y$

$\frac{\partial u}{\partial x} = e^x \cos y$; $\frac{\partial v}{\partial x} = e^x \sin y$

$\frac{\partial u}{\partial y} = -e^x \sin y$; $\frac{\partial v}{\partial y} = e^x \cos y$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\therefore$ Cauchy–Riemann Conditions are satisfied.

Since $e^x \cos y$ & $e^x \sin y$ are always continuous everywhere.

$\therefore$ The partial derivatives are continuous everywhere.

$\therefore f(z)$ is an analytic everywhere

→ place restriction on the Constants a, b & c so that the following functions are entire

(a) $f(z) = x + ay - i(bx + cy)$

(b) $f(z) = ax^2 - by^2 + icxy$

(c) $f(z) = e^x \cos ay + ie^x \sin(y+b) + c$

(d) $f(z) = a(x^2+y^2) + ibxy + c$

Sol^n :- (a) $f(z) = x + ay - i(bx + cy)$

Since $f(z)$ is entire function

$\therefore f(z)$ is analytic

$\therefore f(z)$ has continuous partial derivatives and satisfy Cauchy-Riemann equations.

Now $u(x,y) = x + ay$; $v(x,y) = -(bx + cy)$

$\frac{\partial u}{\partial x} = 1$; $\frac{\partial v}{\partial x} = -b$

$\frac{\partial u}{\partial y} = a$; $\frac{\partial v}{\partial y} = -c$

Since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow 1 = -c$

$\Rightarrow \boxed{c = -1}$

and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow a = b$

(c) $f(z) = e^x \cos ay + ie^x \sin(y+b) + c$

Since $f(z)$ is entire function

$\therefore f(z)$ is analytic.

$\therefore f(z)$ has continuous partial derivative & satisfies Cauchy–Riemann Conditions

Now $u(x,y) = e^x \cos ay + c$;

$v(x,y) = e^x \sin(y+b)$

$\frac{\partial u}{\partial x} = e^x \cos ay + 0$; $\frac{\partial v}{\partial x} = e^x \sin(y+b)$

$\frac{\partial u}{\partial y} = -ae^x \sin ay$; $\frac{\partial v}{\partial y} = e^x \cos(y+b)$

Since $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow e^x \cos ay = e^x \cos(y+b)$

$\Rightarrow \cos ay = \cos(y+b)$

$\Rightarrow a = 1, b = 2k\pi, k = 1, 2$

and c is any Complex number

* Problems Related to the test of Analyticity of a function:

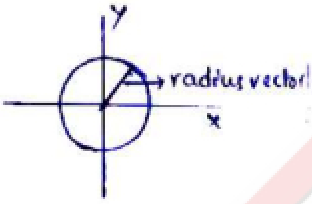
If $f(z) = \begin{cases} \dfrac{x^3y(y-ix)}{x^6+y^2} & ; z \neq 0 \\ 0 & ; z = 0 \end{cases}$

show that $\dfrac{f(z)-f(0)}{z} \to 0$ as $z \to 0$ along any radius vector but not $z \to 0$ in any manner.

i.e. f is not differentiable at $z=0$

Sol^n: $\dfrac{f(z)-f(0)}{z} = \dfrac{f(z)-0}{z}$

$= \dfrac{f(z)}{z}$



$= \dfrac{x^3y(y-ix)}{(x^6+y^2)z}$

$= \dfrac{-ix^3y(x+iy)}{(x^6+y^2)z}$

$= \dfrac{-ix^3y}{x^6+y^2}$

Along the path $y=mx$ (radius vector)

$\underset{z\to 0}{Lt} \dfrac{f(z)-f(0)}{z} = \underset{x\to 0}{Lt} \dfrac{-ix^3(mx)}{x^6+(mx)^2}$

$= \underset{x\to 0}{Lt} \dfrac{-ix^4 m}{x^2(x^4+m^2)}$

$= \underset{x\to 0}{Lt} \dfrac{-ix^2 m}{x^4+m^2}$

$= 0$

$\therefore \dfrac{f(z)-f(0)}{z} \longrightarrow 0$ as $z \longrightarrow 0$.

Now along the path $y = x^3$

$\underset{z\to 0}{Lt} \dfrac{f(z)-f(0)}{z} = \underset{x\to 0}{Lt} \dfrac{-ix^3y^3}{x^6+(x^3)^2}$

$= \underset{x\to 0}{Lt} \dfrac{-i}{2} = -\frac{i}{2} \neq 0$

$\therefore \underset{z\to 0}{Lt} \dfrac{f(z)-f(0)}{z} \neq 0$

along any path except radius vector.

→ Show that the function $f(z) = \sqrt{|xy|}$ is not analytic at $(0,0)$, although Cauchy-Riemann are satisfied at the point.

Sol^n: Given that $f(z) = \sqrt{|xy|}$

Here $u(x,y) = \sqrt{|xy|}$; $v(x,y) = 0$

At the point $(0,0)$

$\dfrac{\partial u}{\partial x} = \underset{x\to 0}{Lt} \dfrac{u(x,0)-u(0,0)}{x}$

$= \underset{x\to 0}{Lt} \dfrac{0-0}{x} = 0$

$\dfrac{\partial u}{\partial y} = \underset{y\to 0}{Lt} \dfrac{u(0,y)-u(0,0)}{y} = 0$

$\dfrac{\partial v}{\partial x} = \underset{x\to 0}{Lt} \dfrac{v(x,0)-v(0,0)}{x} = 0$

$\dfrac{\partial v}{\partial y} = 0$

$\therefore$ Cauchy-Riemann equations are satisfied at the point $(0,0)$.

Again $f'(0) = \underset{z\to 0}{Lt} \dfrac{f(z)-f(0)}{z-0}$

$= \underset{(x,y)\to(0,0)}{Lt} \dfrac{\sqrt{|xy|}-0}{(x+iy)}$

Let $(x,y) \longrightarrow (0,0)$ along $y = mx$

$\therefore f'(0) = \underset{x\to 0}{Lt} \dfrac{\sqrt{|mx^2|}}{x+imx}$

$$= \lim_{x\to 0} \left(\frac{\sqrt{|m|}}{1+im}\right)$$

$$= \frac{\sqrt{|m|}}{1+im}$$

which depends on m

$\therefore f'(0)$ does not exist

$\therefore f(z)$ is not analytic at $(0,0)$

1998, 2012 → Prove that the function $f(z) = u+iv$ where

$$f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}; & z \neq 0 \\ 0 & z = 0 \end{cases}$$

is continuous and Cauchy-Riemann equations are satisfied at the origin, yet $f'(z)$ does not exist at $z=0$

Sol^n:- $f(z) = u+iv$

$\Rightarrow u+iv = f(z)$

$$= \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2}; \; z \neq 0$$

$$= \frac{(x^3-y^3) + i(x^3+y^3)}{x^2+y^2}; \; z \neq 0$$

$\Rightarrow u = \dfrac{x^3-y^3}{x^2+y^2}$ $\quad v = \dfrac{x^3+y^3}{x^2+y^2}$

where $x \neq 0$, $y \neq 0$

(I) To prove that $f(z)$ is continuous everywhere

when $z \neq 0$ u & v both are rational functions of x & y with non-zero denominators.

$\therefore u$ & v are continuous at all those points for which $z \neq 0$

$\therefore f(z)$ is continuous at $z \neq 0$

At the origin

$u=0$, $v=0$ ($\because f(0)=0$)

$\therefore u$ & v are both continuous at the origin

$\therefore f(z)$ is continuous at $(0,0)$

$\therefore f(z)$ is continuous everywhere

(II) To show that Cauchy-Riemann Equations are satisfied at $z=0$;

Since $f(0)=0$

$\Rightarrow u(0,0) + iv(0,0) = 0$

$\Rightarrow u(0,0) = 0 = v(0,0)$

Now $\left(\dfrac{\partial u}{\partial x}\right)_{(0,0)} = \lim_{x\to 0} \dfrac{u(x,0)-u(0,0)}{x}$

$$= \lim_{x\to 0} \frac{\frac{x^3}{x^2} - 0}{x}$$

$$= \lim_{x\to 0} (1) = 1$$

$\left(\dfrac{\partial u}{\partial y}\right)_{(0,0)} = \lim_{y\to 0} \dfrac{u(0,y)-u(0,0)}{y}$

$$= \lim_{y\to 0} \frac{-y-0}{y} = -1$$

$\left(\dfrac{\partial v}{\partial x}\right)_{(0,0)} = \lim_{x\to 0} \dfrac{x-0}{x} = 1$

$\left(\dfrac{\partial v}{\partial y}\right)_{(0,0)} = \lim_{y\to 0} \dfrac{v(0,y)-v(0,0)}{y}$

$$= \lim_{y\to 0} \frac{y-0}{y} = 1$$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ at $z=0$

$\therefore$ Cauchy - Riemann equations are satisfied

(iii) To Prove that $f'(0)$ doesnot exist:

$$f'(0) = \lim_{z\to 0} \frac{f(z)-f(0)}{z-0}$$

$$= \lim_{(x,y)\to(0,0)} \frac{(x^3-y^3)+i(x^3+y^3)}{(x^2+y^2)(x+iy)}$$

Let $(x,y)\longrightarrow(0,0)$ along the coordinate axes:

$$f'(0) = \lim_{x\to 0} \frac{x^3(1+i)}{x^2(x)} \text{ (along x-axis)}$$

$$= \lim_{x\to 0} (1+i) = 1+i$$

and $f'(0) = \lim_{y\to 0} \frac{-y^3+i(y^3)}{y^2(iy)} = \frac{-1+i}{i}$

$$= 1+i$$

Along $y=x$, then

$$f'(0) = \lim_{x\to 0} \frac{2x^3 i}{2x^2(x+ix)} = \frac{i}{1+i}$$

$$= \tfrac{1}{2}(1+i)$$

Since the values of $f'(0)$ are not unique along different paths.

$\therefore f'(0)$ does not exist.

$\therefore f(z)$ is not analytic at $z=0$

1991 Showthat the function

$$f(z) = \begin{cases} e^{-z^{-4}} & z\neq 0 \\ 0 & z=0 \end{cases}$$

is not analytic at $z=0$ although Cauchy-Riemann equations are satisfied at the point How would you explain this?

Solⁿ - To show that Cauchy-Riemann equations are satisfied at $z=0$.

$$w = f(z) = u+iv$$

Since $f(z)=0$ for $z=0$

$$0 = f(0) = u(0,0)+iv(0,0)$$

$$\Rightarrow u(0,0)=0$$
$$v(0,0)=0$$

Since $f(z) = e^{-z^{-4}}$ for $z\neq 0$.

$$u+iv = e^{-\left[\frac{(x+iy)}{(x-iy)}(x-iy)\right]^{-4}}$$

$$= e^{-\left[\frac{x^2+y^2}{x-iy}\right]^{-4}}$$

$$= e^{-\frac{(x-iy)^4}{(x^2+y^2)^4}}$$

$$= e^{-\frac{1}{(x^2+y^2)^4}\left[x^2-y^2-2xyi\right]^2}$$

$$= e^{-\frac{1}{(x^2+y^2)^4}\left[x^4+y^4-4x^2y^2-2x^2y^2+4y^3xi-4x^3yi\right]}$$

$$= e^{-\frac{1}{(x^2+y^2)^4}\left[(x^4+y^4-6x^2y^2)-4ixy(x^2-y^2)\right]}$$

$$\therefore u(x,y) = e^{-\frac{(x^4+y^4-6x^2y^2)}{(x^2+y^2)^4}} \cos\left[\frac{4xy(x^2-y^2)}{(x^2+y^2)^4}\right]$$

$$v(x,y) = +e^{-\frac{(x^4+y^4-6x^2y^2)}{(x^2+y^2)^4}} \sin\left[\frac{4xy(x^2-y^2)}{(x^2+y^2)^4}\right]$$

At $z=0$

$$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0)-u(0,0)}{x}$$

$$= \lim_{x\to 0}\left(\frac{e^{-x^{-4}}}{x}\right) = \lim_{x\to 0}\left(\frac{e^{-1/x^4}}{x}\right)$$

$$= \lim_{x\to 0}\left(\frac{1}{x e^{1/x^4}}\right)$$

$$= \lim_{x\to 0}\left[\frac{1}{x}\left(\frac{1}{1+\frac{1}{x^4}+\left(\frac{1}{x^4}\right)^2+}\right)\right]$$

$$= \lim_{x\to 0}\left[\frac{1}{x+\frac{1}{x^3}+\frac{1}{x^7}+}\right]$$

$$= 0$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0}\frac{u(0,y)-u(0,0)}{y}$$

$$= \lim_{y\to 0}\frac{e^{-y^{-4}}}{y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0}\frac{v(x,0)-v(0,0)}{x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{y\to 0}\frac{v(0,y)-v(0,0)}{y} = 0$$

$\therefore$ Cauchy-Riemann equations are satisfied at $z=0$

(II) To show that $f(z)$ is not analytic at $z=0$.

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} e^{-z^{-4}}$$

Let $z\to 0$ along the path $z = re^{i\pi/4}$

$\therefore r\to 0$ as $z\to 0$

$$\therefore \lim_{z\to 0} f(z) = \lim_{r\to 0} e^{-\left[re^{i\pi/4}\right]^{-4}}$$

$$= \lim_{r\to 0} e^{-\left[r^{-4}\left(e^{i\pi/4}\right)^{-4}\right]}$$

$$= \lim_{r\to 0} e^{r^{-4}}$$

$$= \lim_{r\to 0} e^{1/r^4} = \infty$$

$\therefore \lim_{z\to 0} f(z)$ doesnot exist

$\therefore f(z)$ is not continuous at $z=0$

$\therefore f(z)$ is not differentiable at $z=0$

$\therefore f(z)$ is not analytic at $z=0$

(III) Explanation

The function $f(z)$ is analytic at $z=0$ if (i) Cauchy-Riemann equations satisfied at $z=0$

(ii) $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are all continuous at $z=0$

Here the first one is satisfied and the second one is not satisfied

$\therefore$ The function is not analytic at $z=0$

* Polar form of Cauchy-Riemann Equations :-

Let $f = u+iv$ be differentiable function with continuous partial derivatives (ie analytic) at a point $z = re^{i\theta}$, where u, v, r, θ are all real and $r \neq 0$.

Then the Cauchy - Riemann equations are

$$\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} ; \quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

Proof :- Since the partial derivatives are continuous, the chain rule may be applied.

Let us take into consideration of the relations $x = r\cos\theta \quad y = r\sin\theta$

then $\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial r}$

$$= \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\cdot\sin\theta \quad \text{——} (1)$$

Similarly $\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial \theta}$

$$= -\frac{\partial v}{\partial x}(r\sin\theta) + \frac{\partial v}{\partial y}(r\cos\theta) \quad \text{——} (2)$$

Since Cauchy - Riemann Conditions are Satisfied

i.e $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

from (2) we have

$$\frac{\partial v}{\partial \theta} = \frac{\partial u}{\partial y}(r\sin\theta) + \frac{\partial u}{\partial x}(r\cos\theta)$$

$$= r\left(\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right)$$

$$= r\frac{\partial u}{\partial r} \quad (\text{from } (1))$$

$$\therefore \frac{\partial v}{\partial \theta} = r\frac{\partial u}{\partial r}$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}} \quad \text{——} (\mathrm{I})$$

Similarly Consider

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}$$

$$= -r\sin\theta\frac{\partial u}{\partial x} + r\cos\theta\cdot\frac{\partial u}{\partial y} \quad \text{——} (3)$$

$$\frac{\partial v}{\partial r} = \frac{\partial v}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial r}$$

$$= \cos\theta\cdot\frac{\partial v}{\partial x} + \sin\theta\cdot\frac{\partial v}{\partial y}$$

$$= -\cos\theta\frac{\partial u}{\partial y} + \sin\theta\frac{\partial u}{\partial x} \quad (\because \text{by Cauchy-Riemann equations}) \quad \text{——} (4)$$

Comparing (3) & (4), we have

$$\boxed{-\frac{1}{r}\frac{\partial u}{\partial \theta} = \frac{\partial v}{\partial r}} \quad \text{——} (\mathrm{II})$$

$\therefore$ (I) and (II) are known as the polar form of Cauchy - Riemann Conditions.

→ why is not the polar form of Cauchy - Riemann Conditions valid at the origin.

Sol'n : At origin $r = 0$,

the Cauchy - Riemann Conditions of the polar form give

$$\frac{\partial u}{\partial \theta} = \frac{\partial v}{\partial \theta} = 0.$$

$\Rightarrow$ u, v are independent of θ which is not true

[generally u & v are depending on r & θ
$x = r\cos\theta, y = r\sin\theta$
if $x=0$
$x=0, y=0$
i.e. if $r=0$
$(x,y)=(0,0)$]

$\therefore$ Cauchy - Riemann Conditions are not valid at origin.

$\rightarrow$ If $f(z)$ is differentiable show that $|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$

$$= \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial v}{\partial y}\right)^2$$

$$= \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}$$

This last expression is the Jacobian of u & v with respect to x & y

Sol'n:- Let $f(z) = u(x,y) + iv(x,y)$ —— (1)

where $u(x,y)$ and $v(x,y)$ are functions of x & y.

Since $f(z)$ is differentiable

$\therefore$ the first order partial derivatives of u & v exist and they must satisfy the Cauchy-Riemann equations

i.e $u_x = v_y$ and $u_y = -v_x$ —— (2)

and $f'(z) = u_x + iv_x$

$$= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} \text{ (from (2))}$$

Since $z = x+iy$

$\Rightarrow |z| = \sqrt{x^2+y^2}$

$$\therefore |f'(z)| = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2}$$

$$= \sqrt{\left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2}$$

$$|f'(z)|^2 = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2$$

$$= \left(\frac{\partial v}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2$$

Now $|f'(z)|^2 = \frac{\partial u}{\partial x}\cdot\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x}\cdot\frac{\partial v}{\partial x}$

$$= \frac{\partial u}{\partial x}\cdot\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$$

(from (2))

$\rightarrow$ Let $f(z)$ and $g(z)$ be differentiable at z_0 with $f(z_0) = g(z_0) = 0$

If $g'(z_0) \neq 0$ then $\lim_{z\to z_0}\frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}$.

Sol'n:

$$\frac{f'(z_0)}{g'(z_0)} = \frac{\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}}{\lim_{z\to z_0}\frac{g(z)-g(z_0)}{z-z_0}}$$

$$= \lim_{z\to z_0}\left[\frac{f(z)-f(z_0)}{g(z)-g(z_0)}\right]$$

$$= \lim_{z\to z_0}\frac{f(z)}{g(z)} \quad (\because f(z_0) = g(z_0) = 0)$$

$\rightarrow$ Problems

(1) Let $f(z) = |z|^2$ & $g(z) = z$

then $\lim_{z\to 0}\frac{f(z)}{g(z)} = \frac{f'(0)}{g'(0)}$ [$\because f(0) = g(0) = 0$]

$$= \frac{\lim_{z\to 0}\frac{f(z)-f(0)}{z-0}}{\lim_{z\to 0}\frac{g(z)-g(0)}{z-0}}$$

$$= \frac{\lim_{z\to 0} \frac{|z|^2}{z}}{\lim_{z\to 0} \frac{z}{z}}$$

$$= \frac{\lim_{z\to 0} \frac{z\bar{z}}{z}}{\lim_{z\to 0} (1)}$$

$$= \lim_{z\to 0} \bar{z} = 0$$

→(2) for $f(z) = \sin z$

we have $\lim_{z\to 0} \frac{f(az)}{f(z)} = \lim_{z\to 0}\left(\frac{\sin az}{\sin z}\right)$

$$= \frac{a\cos(0)}{\cos(0)}$$

$$= a(1)$$

$$= a$$

where 'a' is any complex number.

→ Evaluate the following limits exist. (a) $\lim_{z\to 0} \frac{e^z-1}{3z}$ (b) $\lim_{z\to 0} \frac{z^2}{|z|}$

(c) $\lim_{z\to 0} \frac{2\sin z}{e^z-1}$ (d) $\lim_{z\to 0} z\sin 1/z$

Sol'n:- (b) $\lim_{z\to 0} \frac{z^2}{|z|}$

Let $f(z) = z^2$; $g(z) = |z|$

then $f(0) = g(0) = 0$.

$$\therefore \lim_{z\to 0} \frac{z^2}{|z|} = \frac{f'(0)}{g'(0)} = \frac{\lim_{z\to 0} \frac{f(z)-f(0)}{z-0}}{\lim_{z\to 0} \frac{g(z)-g(0)}{z-0}}$$

$$= \frac{\lim_{z\to 0} \frac{z^2}{z}}{\lim_{z\to 0} \frac{|z|}{z}}$$

$$= \frac{\lim_{z\to 0} z}{\lim_{z\to 0} \frac{|z|}{z}} \quad \text{——— (1)}$$

Now $\lim_{z\to 0} \frac{|z|}{z} = \begin{cases} 1 & \text{for +ve real values of } z \\ -1 & \text{for -ve real values of } z \\ -i & \text{for values of } z \text{ on the +ve imaginary axis} \\ i & \text{for values of } z \text{ on the -ve imaginary axis} \end{cases}$

∴ from (1)

$$\lim_{z\to 0} \frac{z^2}{|z|} = 0$$

(c) $\lim_{z\to 0} \frac{2\sin z}{e^z-1}$

Let $f(z) = 2\sin z$

$g(z) = e^z-1$ then $f(0)=g(0)=0$

Now $\lim_{z\to 0} \frac{2\sin z}{e^z-1} = \frac{f'(0)}{g'(0)}$

$$= \frac{2\cos(0)}{e^0} = \frac{2}{1}$$

$$= 2$$

(d) $\lim_{z\to 0} z\cdot\sin 1/z = \lim_{z\to 0} \frac{\sin 1/z}{1/z}$

$$= \lim_{z\to 0} \frac{\cos(1/z)(-1/z^2)}{(-1/z^2)}$$

$$= \lim_{z\to 0} \cos(1/z)$$

$$= \cos\infty$$

Limit does not exist

→ Let $f(z)$ be analytic with continuous partials in a domain D that excludes the origin then show that $f'(z) = e^{-i\theta}\frac{\partial f}{\partial r} = \frac{1}{iz}\frac{\partial f}{\partial \theta}$ at all points in D

Sol'n: Let $f(z)$ be analytic with continuous partials in a domain D.

Let $f(z) = u+iv$

where $u = u(x,y)$ &

$v = v(x,y)$

we know that

$$f'(z) = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y} \quad \text{--- (1)}$$

Putting $x = r\cos\theta$, $y = r\sin\theta$

$$\Rightarrow \boxed{r^2 = x^2+y^2}$$

Consider
$$\frac{\partial f}{\partial r} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial r}$$

$$= \frac{\partial f}{\partial x}\cos\theta + \frac{\partial f}{\partial y}\sin\theta$$

$$= -i\frac{\partial f}{\partial y}\cos\theta + \frac{\partial f}{\partial y}\sin\theta \text{ (from (1))}$$

$$= -i\frac{\partial f}{\partial y}(\cos\theta + i\sin\theta)$$

$$\therefore \frac{\partial f}{\partial r} = -i\frac{\partial f}{\partial y}e^{i\theta}$$

$$\Rightarrow -i\frac{\partial f}{\partial y} = e^{-i\theta}\frac{\partial f}{\partial r}$$

$$\Rightarrow \boxed{f'(z) = e^{-i\theta}\frac{\partial f}{\partial r}} \quad \text{from (1)}$$

Now consider

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$= \frac{\partial f}{\partial x}(-r\sin\theta) + \frac{\partial f}{\partial y}(r\cos\theta)$$

$$= -i\frac{\partial f}{\partial y}(-r\sin\theta) + \frac{\partial f}{\partial y}(r\cos\theta) \text{ (from (1))}$$

$$= \frac{\partial f}{\partial y}(r\cos\theta + ir\sin\theta)$$

$$\Rightarrow \frac{\partial f}{\partial \theta} = \frac{-1}{i}f'(z)(x+iy)$$

$$= if'(z).z \qquad (\because x = r\cos\theta,\ y = r\sin\theta)$$

$$\Rightarrow \boxed{f'(z) = \frac{1}{iz}\frac{\partial f}{\partial \theta}}$$

Note: If $f(z)$ is analytic at a point then $f(z)$ has derivatives of all orders at that point.

That means the real and imaginary parts have continuous partial derivatives of all orders at that point.

In particular the existence of $f'(z)$ tells us that $f'(z) = \frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y}$ is continuous (In view of 4th page theorem) the derivatives of its real and imaginary components are also continuous.

* Harmonic Functions:-

Definition:- A continuous real-valued function $U(x,y)$ defined in a domain D is said to be harmonic in D if it has continuous first and second order partials that satisfy the equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0$$

This equation is known as laplace's equation.

→ If a function $f(z)=u(x,y)+iv(x,y)$ is analytic in a domain D, then its real and imaginary parts u, v are harmonic in D.

Sol^n:- Given that the function $f(z)=u(x,y)+iv(x,y)$ is analytic in a domain

$$\therefore f'(z)=\frac{\partial f}{\partial x} \quad \text{——} (1)$$

(or)

$$f'(z)=-i\frac{\partial f}{\partial y} \quad \text{——} (2)$$

Since the analytic function has derivatives of all orders

$$\therefore (1) \equiv f''(z)=\frac{\partial}{\partial x}(f'(z))$$

$$=\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)$$

$$=\frac{\partial^2 f}{\partial x^2} \quad \text{——} (3)$$

$$(2) \equiv f''(z)=-i\frac{\partial}{\partial y}(f'(z))$$

$$=-i\frac{\partial}{\partial y}\left(-i\frac{\partial f}{\partial y}\right)=(-1)\frac{\partial^2 f}{\partial y^2} \quad \text{——} (4)$$

from (3) & (4) we have

$$\frac{\partial^2 f}{\partial x^2}=-\frac{\partial^2 f}{\partial y^2}$$

$$\Rightarrow \frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=0$$

$$\Rightarrow \nabla^2 f=0 \quad \text{——} (5)$$

which is valid for any analytic function $f(z)$. i.e. If $f(z)=u(x,y)+iv(x,y)$ is an analytic in a domain D then from (5)

$$\nabla^2 f=0$$

$$\Rightarrow \nabla^2(u+iv)=0$$

$$\Rightarrow \nabla^2 u+i\nabla^2 v=0$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0;\ \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0 \quad \text{——} (6)$$

$\therefore$ Both u and v are harmonic functions such functions u & v are called Conjugate harmonic functions (or) Conjugate functions simply i.e If $f(z)=u+iv$ is analytic then u, v both are harmonic functions since they satisfy Laplace's equation.

In such a case, u & v called Conjugate harmonic functions (or) v is called a harmonic conjugate of u

→ v is a harmonic conjugate of u iff u is a harmonic Conjugate of $-v$

Sol^n: Let $f(z)=u+iv$ be an analytic then v is harmonic conjugate of u

Since $f(z)$ is analytic

$\Rightarrow if(z)$ is also analytic

$$\therefore if(z)=i(u+iv)$$

$$=-v+iu$$

From this we say that u harmonic conjugate of $-v$

Let $if(z)$ be an analytic then u is haromic conjugate of $-v$

$\therefore if(z) = -v+iu$

$\therefore i[if(z)] = i[-v+iu]$

$\Rightarrow -f(z) = -iv-u$

$= -(u+iv)$

$\Rightarrow f(z) = u+iv$

$\therefore v$ is harmonic conjugate of u

Note:- Laplace's equation furnishes us with a necessary condition for a function to be the real (or imaginary) Part of analytic function

Ex: Verify whether $u(x,y) = x^2+y$ can be real part of an analytic function.

Sol'n:- If the given function $u(x,y)$ is to be real part of an analytic function

it has to satisfy the laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

But $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2+0 = 2$

$\neq 0$

$\therefore$ Hence it cannot be

* Calculation of Harmonic Conjugate :-

By using Cauchy-Riemann conditions, we can calculate the harmonic conjugate when real part of an analytic function is given

$\rightarrow$ show that $u = x^3-3y^2x$ is harmonic and determine its harmonic conjugate.

Sol'n: It is given that $u = x^3-3xy^2$ ——— ①

$\frac{\partial u}{\partial x} = 3x^2-3y^2 \Rightarrow \frac{\partial^2 u}{\partial x^2} = 6x$

$\frac{\partial u}{\partial y} = -6xy \Rightarrow \frac{\partial^2 u}{\partial y^2} = -6x$

$\therefore \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x-6x = 0$

$\therefore u$ is harmonic

Now let us try to find its harmonic conjugate.

Consider $\frac{\partial u}{\partial x} = 3x^2-3y^2 = \frac{\partial v}{\partial y}$.

$\Rightarrow \frac{\partial v}{\partial y} = 3x^2-3y^2$ (by Cauchy-Riemann equations)

Integrating with respect to y, we get,

$v = 3x^2y - y^3 + \phi(x)$ ——— ②

where $\phi(x)$ is a constant function of the integration.

Partially differentiating ② ~~partially~~ with respect to x the above relation. we get

$$\frac{\partial v}{\partial x} = 6xy + \phi(x) \quad \text{——} \ (3)$$

By Cauchy-Riemann equations,

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Longrightarrow -6xy = -(6xy + \phi'(x))$$

$$\Longrightarrow \phi'(x) = 0 \Rightarrow \phi(x) = c \ (\text{Constant})$$

$\therefore$ (2) $\equiv v = 3x^2y - y^3 + c$

$\therefore$ which is the required harmonic conjugate.

Hence the corresponding analytic function is $f(z) = (x^3 - 3xy^2) + i(3x^2y - y^3 + c)$

→ Show that the following functions are harmonic and determine their harmonic conjugates.

(a) $u = ax + by$; a & b are real constants

(b) $u = \frac{y}{x^2+y^2}$; $x^2+y^2 \neq 0$

(c) $u = x^3 - 3xy^2$

(d) $u = \text{Arg } z$; $-\pi < \text{Arg } z < \pi$

$\hookrightarrow [\text{Arg } z = \theta = \tan^{-1}(y/x)]$

(e) $u = e^{x^2-y^2}\cos(2xy)$ (f) $u = (x-1)^3 - 3xy^2 + 3y^2$

→ Show that the function $u(x,y) = x + e^{-x}\cos y$ is harmonic and find its harmonic conjugate, determine $f(z)$ interms of z

Sol'n:- It is given that

$$u(x,y) = x + e^{-x}\cos y \quad \text{——} \ (1)$$

$$\frac{\partial u}{\partial x} = 1 - e^{-x}\cos y \Rightarrow \frac{\partial^2 u}{\partial x^2} = e^{-x}\cos y$$

$$\frac{\partial u}{\partial y} = -e^{-x}\sin y \Rightarrow \frac{\partial^2 u}{\partial y^2} = -e^{-x}\cos y$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

$\therefore u(x,y)$ is harmonic function.

Now <u>let us try to find harmonic conjugate</u>

Consider $\frac{\partial u}{\partial x} = 1 - e^{-x}\cos y = \frac{\partial v}{\partial y}$

[by Cauchy-Riemann conditions]

$$\Longrightarrow \frac{\partial v}{\partial y} = 1 - e^{-x}\cos y$$

Integrating partially with respect to y we get

$$v = y - e^{-x}\sin y + \phi(x) \quad \text{——} \ (2)$$

where $\phi(x)$ is a constant function of the integration.

Differentiating (2) partially with respect to x, we get

$$\frac{\partial v}{\partial x} = e^{-x}\sin y + \phi'(x) \quad \text{——} \ (3)$$

by Cauchy-Riemann Conditions

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\Longrightarrow -e^{-x}\sin y = -(e^{-x}\sin y + \phi'(x))$$

$$\Longrightarrow \phi'(x) = 0 \Rightarrow \phi(x) = c \ (\text{constant})$$

$\therefore$ (2) $\equiv v = y - e^{-x}\sin y + c$

which is required harmonic conjugate of u.

Now the analytic function

$$f(z) = u + iv$$

$$= x + e^{-x}\cos y + i(y - e^{-x}\sin y + c)$$

$= (x+iy) + e^{-x}(\cos y - i\sin y) + ic$

$= z + e^{-x}e^{-iy} + ic$

$= z + e^{-(x+iy)} + ic$

$= z + e^{-z} + ic$

2005 → If $f(z) = u+iv$ is analytic function of the complex variable z and $u - v = e^x(\cos y - \sin y)$ determine $f(z)$ interms of z

Sol'n: It is given that $u-v = e^x(\cos y - \sin y)$ —— (I)

$\therefore \frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = e^x(\cos y - \sin y)$ —— (1)

and $\frac{\partial u}{\partial y} - \frac{\partial v}{\partial y} = -e^x(\sin y + \cos y)$ —— (2)

(2) $\equiv -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x} = -e^x(\sin y + \cos y)$

(by Cauchy-Riemann Conditions)

$\Rightarrow \frac{\partial v}{\partial x} + \frac{\partial u}{\partial x} = e^x(\sin y + \cos y)$ —— (3)

solving (1) & (3)

$\frac{\partial u}{\partial x} = e^x \cos y$ —— (4) & $\frac{\partial v}{\partial x} = e^x \sin y$ —— (5)

Integrating (5) with respect to x, we get

$v = e^x \sin y + \phi(y)$ —— (6)

Differentiating with respect to y, we get

$\frac{\partial v}{\partial y} = e^x \cos y + \phi'(y)$

by Cauchy-Riemann Conditions

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$

$\Rightarrow e^x \cos y = e^x \cos y + \phi'(y)$

$\Rightarrow \phi'(y) = 0 \Rightarrow \phi(y) = c$ (constant)

$\therefore$ (6) $\equiv \boxed{v = e^x \sin y + c}$ —— (7)

Now from (I)

$u = v + e^x(\cos y - \sin y)$

$\Rightarrow u = e^x \sin y + c + e^x(\cos y - \sin y)$ (from (7))

$\therefore \boxed{u = e^x(\cos y) + c}$ —— (8)

$\therefore f(z) = u + iv$

$= e^x \cos y + c + i(e^x \sin y + c)$

$= e^x[\cos y + i\sin y] + c + ic$

$= e^x e^{iy} + \lambda$ (where $\lambda = c+ic$)

$= e^{x+iy} + \lambda$

$\boxed{f(z) = e^z + \lambda}$

* Some Simple Methods to Construct an analytic function $f(z)$

(1) Milne - thomson's Method :-

Since $f(z) = u(x,y) + iv(x,y)$

and $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$

$\therefore f(z) = u\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right) + iv\left(\frac{z+\bar{z}}{2}, \frac{z-\bar{z}}{2i}\right)$ —— (1)

This relation can be regarded as a formal identity in two independent variables z & $\bar{z}$

Putting $\bar{z} = z$, we get

$x = z$, and $y = 0$

and $f(z) = u(z,0) + iv(z,0)$

we have $f(z) = u + iv$

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$$

$$= \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \text{ (By Cauchy-Riemann conditions)}$$

If we write

$$\frac{\partial u}{\partial x} = \phi_1(x,y) \ \& \ \frac{\partial u}{\partial y} = \phi_2(x,y)$$

$$\therefore f'(z) = \phi_1(x,y) - i\phi_2(x,y)$$

$$= \phi_1(z,0) - i\phi_2(z,0)$$

Integrating it, we get

$$f(z) = \int[\phi_1(z,0) - i\phi_2(z,0)]dz + C$$

where C is an arbitrary constant.

$\therefore f(z)$ is constructed when $u(x,y)$ is given.

Similarly,

if $v(x,y)$ is given, it can be shown that

$$f(z) = \int[\psi_1(z,0) + i\psi_2(z,0)]dz + C$$

where $\psi_1(x,y) = \frac{\partial v}{\partial y}$ &

$\psi_2(x,y) = \frac{\partial v}{\partial x}$

Problem

[1997] → Find the analytic function of which the real part is $e^x(x\cos y - y\sin y)$

Sol^n: Here $u(x,y) = e^x(x\cos y - y\sin y)$

$$\therefore \frac{\partial u}{\partial x} = e^x(x\cos y - y\sin y) + e^x\cos y$$

$$= \phi_1(x,y) \text{ (say)}$$

and $$\frac{\partial u}{\partial y} = e^x(-x\sin y - y\cos y - \sin y)$$

$$= \phi_2(x,y) \text{ (say)}$$

By Milne's method,

we have

$$f'(z) = \phi_1(z,0) - i\phi_2(z,0)$$

$$= e^z(z\cos(0) - 0) + e^z\cos(0) - i\,0]$$

$$= e^z z + e^z$$

$$= e^z(z+1)$$

integrating, we get

$$f(z) = \int e^z(z+1)\,dz + C$$

$$= e^z(z-1) + e^z + C$$

$$f(z) = e^z . z + C$$

Method 2

If the real part of an analytic function $f(z)$ is a given harmonic function $u(x,y)$ then $f(z) = 2u(z/2, z/2i) - u(0,0) + ci$ where c is real

Problem → Construct the analytic function $f(z) = u + iv$ where $u = \sin x \cosh y$

Sol^n: Here $u = \sin x \cosh y$

$$u(z/2, z/2i) = \sin(z/2)\cosh(z/2i)$$

$$= \sin(z/2)\left[\frac{e^{z/2i} + e^{-z/2i}}{2}\right]$$

$$= \sin(z/2)\left[\frac{e^{-zi/2} + e^{zi/2}}{2}\right]$$

$$= \sin(z/2)\left[\frac{e^{zi/2} + e^{-zi/2}}{2}\right]$$

$$= \sin(z/2)\cos(z/2) \quad \left[\because \cos x = \frac{e^{xi} + e^{-xi}}{2},\ \sin x = \frac{e^{xi} - e^{-xi}}{2}\right]$$

and $u(0,0) = 0$

$$\therefore f(z) = 2u(z/2, z/2i) - u(0,0) + ci$$

where c is real constant.

$$= 2\sin(z/2)\cos(z/2) - 0 + ci$$

$$= \sin z + ci$$

H.W. → $u(x,y) = x^3 - 3xy^2 + 3x + 1$

H.W. → $u(x,y) = y^3 - 3x^2y$

→ If $u - v = (x-y)(x^2+4xy+y^2)$ and $f(z) = u+iv$ is an analytic function of $z = x+iy$, find $f(z)$ in terms of z.

Sol'n: Since $f(z) = u+iv$

$\Rightarrow if(z) = iu - v$

Adding we get

$f(z) + if(z) = (u-v) + i(u+v)$

$\Rightarrow (1+i)f(z) = (u-v) + i(u+v)$

$\Rightarrow F(z) = U + iV$

where $F(z) = (1+i)f(z)$

$U = u-v$ and $V = u+v$

$$\frac{\partial U}{\partial x} = (x-y)\cdot(2x+4y) + (x^2+4xy+y^2) = \phi_1(x,y) \text{ say}$$

and $$\frac{\partial U}{\partial y} = (x-y)(4x+2y) - (x^2+4xy+y^2) = \phi_2(x,y) \text{ say}$$

By Milne's method

$$F'(z) = \phi_1(z,0) - i\phi_2(z,0)$$
$$= z(2z) + z^2 - i[z(4z) - z^2]$$
$$= 2z^2 + z^2 - i(4z^2 - z^2)$$
$$= 3z^2(1-i)$$

Integrating we get

$$F(z) = z^3(1-i) + c$$

$$\Rightarrow (1+i)f(z) = -i\left(1 - \frac{1}{i}\right)z^3 + c$$
$$= -i(1+i)z^3 + c$$

$$\Rightarrow f(z) = -iz^3 + \frac{c}{1+i}$$

$$\Rightarrow f(z) = -iz^3 + c_1, \text{ where } c_1 = \frac{c}{1+i}$$

→ If $u + v = \dfrac{2\sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x}$ and $f(z) = u+iv$ is an analytic function of $f(z)$ then find $f(z)$ in terms of z

Sol'n: Since $f(z) = u+iv$

$\Rightarrow if(z) = -v + iu$

Adding, we get

$f + if = (u-v) + i(u+v)$

$(1+i)f = (u-v) + i(u+v)$

$\Rightarrow F(z) = U + iV$

where $F(z) = (1+i)f$

$U = u-v$ & $V = u+v$

$F'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = \frac{\partial V}{\partial y} + i\frac{\partial V}{\partial x}$

Since $f(z)$ is analytic
$\Rightarrow F(z)$ is analytic
$\Rightarrow$ C-R equations are satisfied.
$\therefore \frac{\partial U}{\partial x} = \frac{\partial V}{\partial y}$ & $\frac{\partial U}{\partial y} = -\frac{\partial V}{\partial x}$

Since $V = u+v$

$$= \frac{2\sin 2x}{e^{2y} + e^{-2y} - 2\cos 2x}$$

$$\frac{\partial V}{\partial x} = \frac{(e^{2y}+e^{-2y}-2\cos 2x)(4\cos 2x) - (2\sin 2x)(4\sin 2x)}{(e^{2y}+e^{-2y}-2\cos 2x)^2} = \psi_2(x,y) \text{ (say)}$$

$$\frac{\partial V}{\partial y} = \frac{-(2e^{2y} - 2e^{-2y})2\sin 2x}{(e^{2y}+e^{-2y}-2\cos 2x)^2} = \psi_1(x,y) \text{ say}$$

By Milne's method

$F'(z) = \psi_1(z,0) + i\psi_2(z,0)$ where $\psi_1 = \frac{\partial V}{\partial y}$, $\psi_2 = \frac{\partial V}{\partial x}$

$$\therefore F'(z) = 0 + i\frac{(2-2\cos 2z)(4\cos 2z) - 8\sin^2 2z}{(2-2\cos 2z)^2}$$

$$= \frac{2i(\cos 2z - 1)}{(1-\cos 2z)^2} = \frac{-2i}{1-\cos 2z} = \frac{-2i}{2\sin^2 z}$$
$$= -i\operatorname{cosec}^2 z$$

$$\therefore F(z) = -i\int \operatorname{cosec}^2 z\, dz + c$$
$$= i\cot z + c$$

$$\therefore (1+i)f(z) = i\cot z + c$$

$$f(z) = \left(\frac{i}{1+i}\right)\cot z + \frac{c}{1+i}$$

$$f(z) = \frac{\cot z}{1-i} + c_1 \quad \text{where } c_1 = \frac{c}{1+i}$$

HW 2005 If $f(z) = u+iv$ and $u - v = e^x(\cos y - \sin y)$ find $f(z)$ in terms of z