Example 1. *Evaluate*

$$\int_C \mathbf{F} \cdot d\mathbf{r},$$

where

$$\mathbf{F} = c\,[-3a \sin^2\theta \cos\theta\, \mathbf{i} + a\,(2 \sin\theta - 3 \sin^3\theta)\, \mathbf{j} + b \sin 2\theta\, \mathbf{k}]$$

and the curve C is given by

$$\mathbf{r} = a \cos\theta\, \mathbf{i} + a \sin\theta\, \mathbf{j} + b\theta\, \mathbf{k};$$

θ varying from π/4 to π/2.

Solution. We have

$$\frac{d\mathbf{r}}{d\theta} = -a \sin\theta \mathbf{i} + a \cos\theta \mathbf{j} + b\mathbf{k}$$

$$\therefore \quad \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{\pi/4}^{\pi/2} \mathbf{F} \cdot \frac{d\mathbf{r}}{d\theta} d\theta \qquad \text{[Refer (3), § 11.1.1]}$$

$$= c\left(a^2 + b^2\right) \int_{\pi/4}^{\pi/2} \sin 2\theta \, d\theta = \frac{1}{2} c\left(a^2 + b^2\right).$$

Example 5. *If* $\mathbf{F} = (2y+3)\mathbf{i} + xz\,\mathbf{j} + (yz-x)\,\mathbf{k}$, *evaluate* $\int_C \mathbf{F}\cdot d\mathbf{r}$ *along the following paths C :*

(*a*) $x = 2t^2,\ y = t,\ z = t^3$ *from* $t = 0$ *to* $t = 1$.

(*b*) *The straight lines from (0, 0, 0) to (0, 0, 1) then to (0, 1, 1) and then to (2, 1, 1).*

(*c*) *The straight line joining (0, 0, 0) to (2, 1, 1).*

Solution. (*a*) On putting the values of x, y, z in terms of t.

$$\mathbf{F} = (2t+3)\mathbf{i} + 2t^5\mathbf{j} + \left(t^4 - 2t^2\right)\mathbf{k}$$

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = 2t^2\mathbf{i} + t\mathbf{j} + t^3\mathbf{k}$$

$$\therefore \quad d\mathbf{r} = \left(4t\mathbf{i} + \mathbf{j} + 3t^2\mathbf{k}\right) dt.$$

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \left[4t\,(2t+3) + 1\cdot 2t^5 + 3t^2\left(t^4 - 2t^2\right)\right] dt$$

$$= \int_0^1 \left(8t^2 + 12t + 2t^5 + 3t^6 - 6t^4\right) dt$$

$$= \frac{288}{35}.$$

(*b*) $$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (F_1 dx + F_2 dy + F_3 dz)$$

$$= \int \left[(2y+3)\,dx + xz\,dy + (yz-x)\,dz\right] \qquad \text{...(1)}$$

Along C_1 the line joining (0, 0, 0) to (0, 0, 1), $x = 0$, $y = 0$.

$\therefore$ $dx = 0$, $dy = 0$ and z varies from 0 to 1.

$$\therefore \quad \int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 (yz - x)\,dz = 0, \ \because\ x = 0,\ y = 0.$$

Along C_2 the line joining (0, 0, 1) to (0, 1, 1), $x = 0$, $z = 1$.

$\therefore$ $dx = 0$, $dz = 0$ and y varies from 0 to 1.

$$\therefore \quad \int_2 \mathbf{F}\cdot d\mathbf{r} = \int_0^1 xz\,dy = 0, \ \because\ x = 0.$$

Along C_3, the line joining (0, 1, 1) to (2, 1, 1), $y = 1$, $z = 1$.

$\therefore$ $dy = 0$, $dz = 0$ and x varies from 0 to 2.

$$\therefore \quad \int_{C_3} \mathbf{F}\cdot d\mathbf{r} = \int_0^2 (2y+3)\,dx = \int_0^2 5dx$$

$$= 10.$$

$$\therefore \quad \int_C \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot d\mathbf{r} + \int_{C_2} \mathbf{F}\cdot d\mathbf{r} + \int_{C_3} \mathbf{F}\cdot d\mathbf{r}$$

$$= 0 + 0 + 10 = 10.$$

(c) The parametric equations of the line joining (0, 0, 0) to (2, 1, 1) is $\frac{x}{2} = \frac{y}{1} = \frac{z}{1} = t.$

or $\quad x = 2t, \; y = t, \; z = t$ and t varies from 0 to 1.

$$\therefore \quad \int_C \mathbf{F}\cdot d\mathbf{r} = \int_C (2y+3)\,dx + xz\,dy + (yz - x)\,dz$$

$$= \int_0^1 \left[(2t+3)\,2 + 2t^2 + \left(t^2 - 2t\right)\right] dt$$

$$= \int_0^1 \left(3t^2 + 2t + 6\right) dt$$

$$= \left[t^3 + t^2 + 6t\right]_0^1 = 8.$$

Example 7. *Evaluate the surface integral*

$$\int_S (yz\mathbf{i} + zx\mathbf{j} + xy\mathbf{k}) \cdot d\mathbf{a}, \qquad \text{(Avadh 2004)}$$

where S is the surface of the sphere $x^2 + y^2 + z^2 = 1$ *in the first octant.*

Solution. The sphere is parametrically given by

$$x = \sin\theta\cos\phi, \quad y = \sin\theta\sin\phi, \quad z = \cos\theta$$

and the part in the first octant corresponds to

$$0 \le \theta \le \frac{1}{2}\pi, \quad 0 \le \phi \le \frac{1}{2}\pi.$$

Thus, the required surface integral

$$= \iint_S (yz\,dy\,dz + zx\,dz\,dx + xy\,dx\,dy)$$

$$= \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \left[yz \frac{\partial(y,z)}{\partial(\theta,\phi)} + zx \frac{\partial(z,x)}{\partial(\theta,\phi)} + xy \frac{\partial(x,y)}{\partial(\theta,\phi)} + \right] d\theta\, d\phi$$

Now

$$\frac{\partial(y,z)}{\partial(\theta,\phi)} = \begin{vmatrix} \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \\ \frac{\partial z}{\partial\theta} & \frac{\partial z}{\partial\phi} \end{vmatrix} = \begin{vmatrix} \cos\theta\sin\phi & \sin\theta\cos\phi \\ -\sin\theta & 0 \end{vmatrix}$$

$$= \sin\theta\cos^2\phi.$$

$$\frac{\partial(z,x)}{\partial(\theta,\phi)} = \begin{vmatrix} \frac{\partial z}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \end{vmatrix} = \begin{vmatrix} -\sin\theta & 0 \\ \cos\theta\cos\phi & -\sin\theta\sin\phi \end{vmatrix}$$

$$= \sin^2\theta\sin\phi.$$

$$\frac{\partial(x,y)}{\partial(\theta,\phi)} = \begin{vmatrix} \frac{\partial x}{\partial\theta} & \frac{\partial x}{\partial\phi} \\ \frac{\partial y}{\partial\theta} & \frac{\partial y}{\partial\phi} \end{vmatrix} = \begin{vmatrix} \cos\theta\cos\phi & -\sin\theta\sin\phi \\ \cos\theta\sin\phi & -\sin\theta\cos\phi \end{vmatrix}$$

$$= \sin\theta\cos\theta.$$

$\therefore$ integral

$$= \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} [\sin\theta\cos\theta\sin\phi\sin^2\theta\cos\phi +$$

$$\sin\theta\cos\theta\cos\phi\sin^2\theta\sin\phi + \sin^2\theta\sin\phi\cos\phi\sin\theta\cos\theta]\, d\theta\, d\phi$$

$$= \int_0^{\frac{1}{2}\pi} \int_0^{\frac{1}{2}\pi} \left[2\sin^3\theta\cos\theta\sin\phi\cos\phi + \sin^3\theta\cos\theta\sin\phi\cos\phi\right] d\theta\, d\phi$$

$$= \frac{7}{20}.$$

Example 9. *Evaluate* $\int_V (2x+y)\,dV$, *where V is the closed region bounded by the cylinder* $z = 4 - x^2$ *and the plane* $x = 0$, $y = 0$, $y = 2$ *and* $z = 0$.

Solution. The given cylinder $z = 4 - x^2$ meets the x-axis ($y = 0, z = 0$) at $x^2 = 4$ or $x = 2$, *i.e.*, at the point (2, 0, 0). It meets z-axis at $z = 4$, *i.e.*, at (0, 0, 4). Therefore, the limits of integration are as under.

$$z = 4 \text{ to } z = 4 - x^2, \quad y = 0 \text{ to } y = 2 \text{ and } x = 0 \text{ to } x = 2.$$

Also $dV = dx\,dy\,dz$.

$$\therefore \quad \int_V (2x+y)\,dV = \iiint (2x+y)\,dx\,dy\,dz$$

$$= \iint (2x+y)\,[z]_0^{4-x^2}\,dx\,dy$$

$$= \int_{x=0}^{2} \int_{y=0}^{2} (2x+y)\left(4-x^2\right) dx\,dy$$

$$= \int_0^2 \left[2x\left(4-x^2\right)y + \left(4-x^2\right)\frac{y^2}{2}\right]_{y=0}^{2} dx$$

$$= \int_0^2 \left[4x\left(4-x^2\right)+2\left(4-x^2\right)\right] dx$$

$$= 2\int_0^2 \left(4+8x-x^2-2x^3\right) dx$$

$$= \frac{80}{3}.$$

Example 2. *If* $\overrightarrow{OA} = a\mathbf{i}$, $\overrightarrow{OB} = a\mathbf{j}$, $\overrightarrow{OC} = a\mathbf{k}$, *form three coterminous edges of a cube and S denotes the surface of the cube, evaluate*

$$\int_S \left\{\left(x^3 - yz\right)\mathbf{i} - 2x^2y\mathbf{j} + 2\mathbf{k}\right\} \cdot \mathbf{n}\, dS,$$

by expressing it as a volume integral. Also verify the result by direc evaluation of the surface integral. (Garhwal 2004

Solution. Let

$$\mathbf{F} = \left(x^3 - yz\right)\mathbf{i} - 2x^2y\mathbf{j} + 2\mathbf{k}$$

$$\therefore \quad \text{div}\,\mathbf{F} = \frac{\partial\left(x^3 - yz\right)}{\partial x} + \frac{\partial\left(-2x^2y\right)}{\partial y} + \frac{\partial(2)}{\partial z}$$

$$= 3x^2 - 2x^2 = x^2.$$

$$\therefore \quad \int_S \mathbf{F}\cdot\mathbf{n}\, dS = \int_V \text{div}\,\mathbf{F}\, dv = \iiint_V x^2 dx\, dy\, dz$$

$$= \left[\int_0^a dz\right]\left[\int_0^a dy\right]\left[\int_0^a x^2 dx\right]$$

$$= a \cdot a \cdot \frac{1}{2}a^3 = \frac{1}{2}a^5.$$

We shall now evaluate the surface integral directly. The surface S consists of *six* faces.

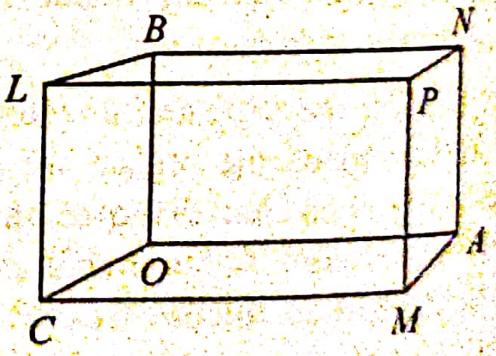

Fig. 11.9.

Over the surface $AOBN$

$$\int_S \mathbf{F}\cdot\mathbf{n}\, dS$$

$$= \int_S \left\{\left(x^3 - yz\right)\mathbf{i} - 2x^2y\mathbf{j} + 2\mathbf{k}\right\} \cdot (-\mathbf{k})\, dS$$

$$= \int_0^a \int_0^a -2dx\, dy = -2a^2.$$

Over the surface $PLCM$,

$$\int_S \mathbf{F}\cdot\mathbf{n}\, dS = \int_S \left\{\left(x^3 - yz\right)\mathbf{i} - 2x^2y\mathbf{j} + 2\mathbf{k}\right\} \cdot \mathbf{k}\, dS$$

$$= \int_0^a \int_0^a 2dx\, dy = 2a^2.$$

Similarly over the faces *NPMA, BLCO, AOCM, NBLP,* the corresponding surface integrals are respectively

$$\int_S \mathbf{F}\cdot\mathbf{i}\, dS = \iint_S \left(x^3 - yz\right) dy\, dz = \int_0^a \int_0^a \left(a^3 - yz\right) dy\, dz$$

$$= a^5 - \frac{1}{4}a^4,$$

$$\int_S \mathbf{F}\cdot(-\mathbf{i})\, dS = -\iint_S \left(x^3 - yz\right) dy\, dz = \int_0^a \int_0^a yz\, dy\, dz = \frac{1}{4}a^4$$

$$\int_S \mathbf{F}\cdot(-\mathbf{j})\, dS = \int_0^a \int_0^a 2x^2 y\, dx\, dy = 0$$

$$\int_S \mathbf{F}\cdot\mathbf{j}\, dS = -\int_0^a \int_0^a 2x^2 y\, dx\, dy = -2a \int_0^a \int_0^a x^2 dx\, dz$$

$$= \frac{-2}{3}a^5.$$

Adding we see that over the whole surface

$$\int_S \mathbf{F}\cdot\mathbf{n}\, dS = -2a^2 + 2a^2 + a^5 - \frac{1}{4}a^4 + \frac{1}{4}a^4 + 0 - \frac{2}{3}a^5 = \frac{1}{3}a^5.$$

Hence the verification.

Example 7. *Find the directional derivative of* $\phi = x^2yz + 4xz^2$ *at* $(1, -2, -1)$ *in the direction* $2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$. *In which direction the directional derivative will be maximum and what is its magnitude. Also find a unit normal to the surface* $x^2yz + 4xz^2 = 6$ *at the point* $(1, -2, -1)$, *find the equation of tangent plane and normal at the point* $(1, -2, -1)$.

Solution. We have

$$\frac{\partial \phi}{\partial x} = 2xyz + 4z^2, \quad \frac{\partial \phi}{\partial y} = x^2 z, \quad \frac{\partial \phi}{\partial z} = x^2 y + 8xz.$$

$$\text{grad } \phi = \Sigma \mathbf{i} \frac{\partial \phi}{\partial x} = (2xyz + 4z^2)\,\mathbf{i} + x^2 z\,\mathbf{j} + (x^2 y + 8xz)\,\mathbf{k}.$$

or, $$\text{grad } \phi = 8\mathbf{i} - \mathbf{j} - 10\mathbf{k} \text{ at the point } (1, -2, -1).$$

Now, if $\hat{\mathbf{a}}$ be a unit vector then directional derivative of ϕ along the direction of $\hat{\mathbf{a}}$ is $\hat{\mathbf{a}} \cdot \text{grad } \phi$.

A unit vector in the direction $(2\mathbf{i} - \mathbf{j} - 2\mathbf{k})$ is

$$\frac{2\mathbf{i}-\mathbf{j}-2\mathbf{k}}{\sqrt{(4+1+4)}} = \frac{1}{3}(2\mathbf{i}-\mathbf{j}-2\mathbf{k})$$

$\therefore$ Directional derivative

$$= \frac{1}{3}(2\mathbf{i}-\mathbf{j}-2\mathbf{k})\cdot(8\mathbf{i}-\mathbf{j}-10\mathbf{k})$$

$$= \frac{1}{3}(16+1+20) = \frac{37}{3}.$$

We know that the directional derivative is maximum in the direction of normal which is the direction of grad ϕ. Hence, directional derivative is maximum along grad $\phi = 8\mathbf{i} - \mathbf{j} - 10\mathbf{k}$. Maximum value of this directional derivative is $|\text{grad } \phi|$

$$= \sqrt{(64+1+100)} = \sqrt{(165)}.$$

A unit normal to the surface $= \dfrac{\text{grad } \phi}{|\text{grad } \phi|}$

$$= \frac{8\mathbf{i}-\mathbf{j}-10\mathbf{k}}{\sqrt{(165)}}.$$

Tangent plane to the surface at (1, – 2, – 1).

Let P be any point on the tangent plane at A $(1, -2, -1)$ and the position vectors of P and A be $\mathbf{r}$ and $\mathbf{r}_0$.

$\therefore$ $\quad \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathbf{r}_0 = \mathbf{i} - 2\mathbf{j} - \mathbf{k}$

$$\overrightarrow{AP} = \mathbf{r} - \mathbf{r}_0 = (x-1)\,\mathbf{i} + (y+2)\,\mathbf{j} + (z+1)\,\mathbf{k}.$$

grad ϕ is normal to the surface and as such it is perpendicular to AP.

$\therefore$ $\quad \text{grad } \phi \,.\, \overrightarrow{AP} = 0.$

$\Rightarrow$ $\quad (8\mathbf{i} - \mathbf{j} - 10\mathbf{k}) \,.\, [(x-1)\,\mathbf{i} + (y+2)\,\mathbf{j} + (z+1)\,\mathbf{k}] = 0$

$\Rightarrow$ $\quad 8\,(x-1) - (y+2) - 10\,(z+1) = 0$

$\Rightarrow$ $\quad 8x - y - 10z = 20.$

Normal to the surface.

If p be any point on the normal line at A, then $(\mathbf{r} - \mathbf{r}_0)$ lies along the normal, *i.e.*, along grad ϕ, hence

$$(\mathbf{r} - \mathbf{r}_0) \times \text{grad } \phi = 0$$

$$\Rightarrow \quad \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x-1 & y+2 & z+1 \\ 8 & -1 & -10 \end{vmatrix} = 0$$

$$\Rightarrow \mathbf{i}\{(z+1)-10(y+2)\}+\mathbf{j}\{10(x-1)+8(z+1)\}$$
$$-\mathbf{k}\{(x-1)+8(y+2)\}=0$$

Equating to zero the coefficients of **i**, **j** and **k**, we get

$$\frac{z+1}{10}=\frac{y+2}{1};\ \frac{z+1}{10}=\frac{x-1}{-8};\ \frac{x-1}{-8}=\frac{y+2}{1}$$

$$\therefore \quad \frac{x-1}{-8}=\frac{y+2}{1}=\frac{z+1}{10}.$$

Note. It could be written directly as the equation of a line through (1, – 2, – 1) and with direction ratios as 8, – 1, – 10.

Example 3. *Prove that* $\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$. (Ajmer 97)

Solution. Let $\phi = f(r)$; $\therefore \quad \nabla^2 f(r) = \nabla^2 \phi = \Sigma \frac{\partial^2 \phi}{\partial x^2}$.

$$\frac{\partial \phi}{\partial x} = f'(r) \frac{\partial r}{\partial x} = \frac{x}{r} f'(r)$$

$$\therefore \quad \frac{\partial^2 \phi}{\partial x^2} = \frac{r\left\{1 \cdot f'(r) + x f''(r) \cdot \frac{x}{r}\right\} - x f'(r) \cdot \frac{x}{r}}{r^2}$$

$$= \frac{1}{r} f'(r) + \frac{x^2}{r^2} f''(r) - \frac{x^2}{r^3} f'(r)$$

$$\therefore \quad \Sigma \frac{\partial^2 \phi}{\partial x^2} = \frac{3}{r} f'(r) + \frac{\Sigma x^2}{r^2} f''(r) - \frac{\Sigma x^2}{r^3} f'(r)$$

$$= \frac{3}{r} f'(r) + f''(r) - \frac{1}{r} f'(r)$$

$$= f''(r) + \frac{2}{r} f'(r).$$

Example 9. *Show that*

$$\textit{curl}\,\frac{\mathbf{a}\times\mathbf{r}}{r^3}=-\frac{\mathbf{a}}{r^3}+\frac{3\mathbf{r}}{r^3}(\mathbf{a}\,.\,\mathbf{r}),$$

where **a** *is a constant vector.*

(Kolkata 99, Gorakhpur 98, 2000, Lucknow 97)

Solution. Now,

$$\text{curl}\,\frac{\mathbf{a}\times\mathbf{r}}{r^3}=\Sigma\mathbf{i}\times\frac{\partial}{\partial x}\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right)$$

Also

$$\frac{\partial}{\partial x}\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right)=-\frac{3}{r^4}\frac{\partial r}{\partial x}(\mathbf{a}\times\mathbf{r})+\frac{1}{r^3}\left(\mathbf{a}\times\frac{\partial\mathbf{r}}{\partial x}\right)$$

$$=-\frac{3x}{r^5}(\mathbf{a}\times\mathbf{r})+\frac{1}{r^3}\mathbf{a}\times\mathbf{i}.$$

$$\therefore\quad \mathbf{i}\times\frac{\partial}{\partial x}\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right)=-\frac{3x}{r^5}\left[(\mathbf{i}\,.\,\mathbf{r})\,\mathbf{a}-(\mathbf{i}\,.\,\mathbf{a})\,\mathbf{r}\right]+\frac{1}{r^3}\left[(\mathbf{i}\,.\,\mathbf{i})\mathbf{a}-(\mathbf{i}\,.\,\mathbf{a})\,\mathbf{i}\right]$$

$$=-\frac{3x^2}{r^5}\mathbf{a}+\frac{3\mathbf{r}}{r^5}(\mathbf{i}x\cdot\mathbf{a})+\frac{1}{r^3}\mathbf{a}-\frac{1}{r^3}(\mathbf{i}\cdot\mathbf{a})\,\mathbf{i}.$$

$$\therefore \quad \Sigma \mathbf{i} \times \frac{\partial}{\partial x}\left(\frac{\mathbf{a} \times \mathbf{r}}{r^3}\right) = -\frac{3\mathbf{a}}{r^5} r^2 + \frac{3\mathbf{r}}{r^5} \mathbf{r} \cdot \mathbf{a} + \frac{3}{r^3} \mathbf{a} - \frac{1}{r^3} \mathbf{a}$$

$$= -\frac{\mathbf{a}}{r^3} + \frac{3\mathbf{r}}{r^5} \mathbf{a} \,.\, \mathbf{r}.$$

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Example 10. *The equation of motion of a particle P of mass m is given by* $m(d^2\mathbf{r}/dt^2) = f(r)\,\hat{\mathbf{r}}$, *where* $\mathbf{r}$ *is the position vector of P measured from an origin O,* $\hat{\mathbf{r}}$ *is a unit vector in the direction of* $\mathbf{r}$ *and* $f(r)$ *is a function of the distance of P from O, show that* $\mathbf{r}\times\left(\dfrac{d\mathbf{r}}{dt}\right) = \mathbf{c}$, *where* $\mathbf{c}$ *is a constant vector.*

Solution. We have

$$m\left(\frac{d^2\mathbf{r}}{dt^2}\right) = f(r)\,\hat{\mathbf{r}}$$

$$\Rightarrow \qquad m\mathbf{r}\times\left(\frac{d^2\mathbf{r}}{dt^2}\right) = f(r)\,\mathbf{r}\times\hat{\mathbf{r}}$$

$$\Rightarrow \qquad m\mathbf{r}\times\left(\frac{d^2\mathbf{r}}{dt^2}\right) = 0 \Rightarrow \mathbf{r}\times\left(\frac{d^2\mathbf{r}}{dt^2}\right) = 0 \qquad \text{...(1)}$$

$$\left[\because \mathbf{r}\times\hat{\mathbf{r}} = \mathbf{r}\times\frac{\mathbf{r}}{r} = 0\right]$$

But, $$\frac{d}{dt}\left(\mathbf{r}\frac{d\mathbf{r}}{dt}\right) = \frac{d\mathbf{r}}{dt}\times\frac{d\mathbf{r}}{dt} + \mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2} = \mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}$$

Hence, (1) becomes

$$\frac{d}{dt}\left(\mathbf{r}\times\frac{d\mathbf{r}}{dt}\right) = 0$$

Integrating, $$\mathbf{r}\times\frac{d\mathbf{r}}{dt} = \mathbf{c}.$$

Example 1. *A particle moves along the curve $x = 2t^2$, $y = t^2 - 4t$, $z = 3t - 5$, where t is the time. Find the component of its velocity and acceleration at time $t = 1$ in the direction* $\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$.

Solution. $\mathbf{r} = 2t^2\mathbf{i} + (t^2 - 4t)\,\mathbf{j} + (3t - 5)\,\mathbf{k}$

$$\therefore \quad \text{Velocity} = \frac{d\mathbf{r}}{dt} = 4t\,\mathbf{i} + (2t - 4)\,\mathbf{j} + 3\mathbf{k}$$

$$= 4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k} \text{ at } t = 1.$$

we know that $\dfrac{\mathbf{A}\cdot\mathbf{B}}{|\mathbf{A}|}$ is the projection of **B** in the direction of **A.**

Hence, component of velocity vector in the direction of $\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$

$$= \frac{(\mathbf{i} - 3\mathbf{j} + 2\mathbf{k})}{\sqrt{(1+9+4)}}\cdot(4\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) = \frac{16}{\sqrt{14}}.$$

Again acceleration $\dfrac{d^2\mathbf{r}}{dt^2} = 4\mathbf{i} + 2\mathbf{j}$

Hence, component of acceleration in the given direction is

$$\frac{\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}}{\sqrt{14}}\cdot(4\mathbf{i} + 2\mathbf{j}) = -\frac{2}{\sqrt{14}}$$

vectors are linearly dependent.

Example 3. *If* **a, b, c** *be three non-zero, non-coplanar vectors, find a relation between the vectors* **a** + *3***b** + *4***c**, **a** – 2**b** + *3***c**, **a** + *5***b** – 2**c**, 6**a** + *14***b** + 4**c**.

Solution. Let

$$\mathbf{a} + 3\mathbf{b} + 4\mathbf{c} = x_1 (\mathbf{a} - 2\mathbf{b} + 3\mathbf{c}) + x_2 (\mathbf{a} + 5\mathbf{b} - 2\mathbf{c}) + x_3 (6\mathbf{a} + 14\mathbf{b} + 4\mathbf{c})$$

$$\Rightarrow \quad \mathbf{a} + 3\mathbf{b} + 4\mathbf{c} = (x_1 + x_2 + 6x_3)\, \mathbf{a} + (-2x_1 + 5x_2 + 14x_3)\, \mathbf{b} + (3x_1 - 2x_2 + 4x_3)\, \mathbf{c}$$

Equating the coefficients of **a, b, c** on both sides, we get

$$x_1 + x_2 + 6x_3 = 1, \quad -2x_1 + 5x_2 + 14x_3 = 3, \quad 3x_1 - 2x_2 + 4x_3 = 4$$

$$\Rightarrow \qquad x_1 = -2, \quad x_2 = -3, \; x_3 = 1.$$

$$\therefore \quad \mathbf{a} + 3\mathbf{b} + 4\mathbf{c} = -2 (\mathbf{a} - 2\mathbf{b} + 3\mathbf{c}) - 3 (\mathbf{a} + 5\mathbf{b} - 2\mathbf{c}) + (6\mathbf{a} + 14\mathbf{b} + 4\mathbf{c}).$$

Example 1. *Given two vectors*

$$\mathbf{a} = \mathbf{i} + \mathbf{j} - \mathbf{k}; \quad \mathbf{b} = \mathbf{i} - \mathbf{j} + \mathbf{k},$$

find a unit vector **c**, *perpendicular to the vector* **a** *and coplanar with* **a** *and* **b**. *Find also a vector* **d** *perpendicular to both* **a** *and* **c**.

Solution. Any vector coplanar with **a** and **b** is

$$\lambda(\mathbf{i} + \mathbf{j} - \mathbf{k}) + \mu(\mathbf{i} - \mathbf{j} + \mathbf{k}) = (\lambda + \mu)\,\mathbf{i} + (\lambda - \mu)\,\mathbf{j} + (-\lambda + \mu)\,\mathbf{k}$$

This will be perpendicular to **a** if

$$(\lambda + \mu) \,.\, 1 + (\lambda - \mu) \,.\, 1 + (-\lambda + \mu)(-1) = 0$$

$$\Rightarrow \lambda + \mu + \lambda - \mu + \lambda - \mu = 0 \quad \Rightarrow 3\lambda - \mu = 0 \quad \Rightarrow \mu = 3\lambda.$$

Thus, the unit vector perpendicular to **a** is

$$\lambda(4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) \text{ where } \lambda\sqrt{24} = 1 \Rightarrow \lambda = \frac{1}{\sqrt{24}}$$

Thus

$$\mathbf{c} = \frac{1}{\sqrt{24}}(4\mathbf{i} - 2\mathbf{j} + 2\mathbf{k}) = \frac{1}{\sqrt{6}}(2\mathbf{i} - \mathbf{j} + \mathbf{k}).$$

If $p\mathbf{i} + q\mathbf{j} + r\mathbf{k}$ is the required unit vector **d** then, because of its perpendicularity to both **a** and **c**, we have,

$$p + q - r = 0; \quad 2p - q + r = 0$$

which give $p = 0, \quad q = r.$ [By taking $\mathbf{a} \,.\, \mathbf{d} = 0$ and $\mathbf{c} \,.\, \mathbf{d} = 0$]

Thus, the required vector **d** is given by $(1/\sqrt{2})(\mathbf{j} + \mathbf{k})$.

Example 2. *Given two vectors*

$$\mathbf{a} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}; \quad \mathbf{b} = -\mathbf{i} + 2\mathbf{j} - \mathbf{k}.$$

Find the projection of **a** *on* **b** *and that of* **b** *on* **a**.

Solution. Let θ denote the angle between the vectors **a** and **b**, so that $|\mathbf{b}|\cos\theta$ is the projection of **b** on **a** and $|\mathbf{a}|\cos\theta$ is the projection of **a** on **b**.

We have

$$|\mathbf{b}|\cos\theta = \frac{\mathbf{a}\,.\,\mathbf{b}}{|\mathbf{a}|} = \frac{-9}{\sqrt{14}},$$

and

$$|\mathbf{a}|\cos\theta = \frac{\mathbf{a}\,.\,\mathbf{b}}{|\mathbf{b}|} = \frac{-9}{\sqrt{6}}$$

Hence the results.

Example 6. *If* **a**, **b** *are vectors and* a, b *their lengths, show that*

$$\left(\frac{\mathbf{a}}{a^2}-\frac{\mathbf{b}}{b^2}\right)^2=\left(\frac{\mathbf{a}-\mathbf{b}}{ab}\right)^2$$

Solution. $$\left(\frac{\mathbf{a}}{a^2}-\frac{\mathbf{b}}{b^2}\right)^2=\left(\frac{\mathbf{a}}{a^2}-\frac{\mathbf{b}}{b^2}\right)\cdot\left(\frac{\mathbf{a}}{a^2}-\frac{\mathbf{b}}{b^2}\right)$$

$$=\frac{\mathbf{a}\,.\,\mathbf{a}}{a^4}-\frac{2\mathbf{a}\,.\,\mathbf{b}}{a^2b^2}+\frac{\mathbf{b}\,.\,\mathbf{b}}{b^4}=\frac{a^2}{a^4}-\frac{2\mathbf{a}\,.\,\mathbf{b}}{a^2b^2}+\frac{b^2}{b^4}$$

$$=\frac{1}{a^2}-\frac{2\mathbf{a}\,.\,\mathbf{b}}{a^2b^2}+\frac{1}{b^2}=\frac{b^2-2\mathbf{a}\,.\,\mathbf{b}+a^2}{a^2b^2}$$

$$=\frac{(\mathbf{a}-\mathbf{b})\,.\,(\mathbf{a}-\mathbf{b})}{a^2b^2}=\frac{(\mathbf{a}-\mathbf{b})^2}{a^2b^2}.$$

Example 1. *Find the angle between the lines AB, AC where A, B, C are the three points with rectangular cartesian coordinates*

$$(1, 2, -1), \quad (2, 0, 3), \quad (3, -1, 2)$$

respectively.

Solution. In terms of usual notation, the position vectors of the given points are

$$\vec{OA} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}, \quad \vec{OB} = 2\mathbf{i} + 3\mathbf{k}, \quad \vec{OC} = 3\mathbf{i} - \mathbf{j} + 2\mathbf{k},$$

O being the origin. Thus, we have

$$\vec{AB} = \vec{OB} - \vec{OA} = \mathbf{i} - 2\mathbf{j} + 4\mathbf{k}, \quad \vec{AC} = \vec{OC} - \vec{OA} = 2\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}.$$

$$\therefore \quad \vec{AB} \,.\, \vec{AC} = (\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}) \,.\, (2\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}) = 20.$$

Also
$$\vec{AB}^2 = (\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}) \,.\, (\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}) = 21.$$

$$\vec{AC}^2 = (2\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}) \,.\, (2\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}) = 22.$$

If θ denote the angle between AB and AC, we have

$$\cos\theta = \frac{\vec{AB} \,.\, \vec{AC}}{|\vec{AB}|\,|\vec{AC}|} = \frac{20}{\sqrt{21}\sqrt{22}}$$