5 Functions of a Single Variable (I)

LIMIT AND CONTINUITY

In the chapter on sequences, we considered functions whose domain was the set **N** of natural numbers. We shall now consider real valued functions with domain as any interval, open or closed.

1. LIMITS

Let f be a function defined for all points in a neighbourhood N of a point c except possibly at the point c itself.

Definition 1. The function f is said to tend to a limit l as x tends to (or approaches) c if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$0 < |x - c| < \delta \Rightarrow |f(x) - l| < \varepsilon$$

or

$$|f(x) - l| < \varepsilon, \text{ when } 0 < |x - c| < \delta$$

or

$$f(x) \in]l - \varepsilon, l + \varepsilon[, \forall x \in]c - \delta, c + \delta[\text{ except possibly } c$$

In symbols, we then write

$$\lim_{x \to c} f(x) = l$$

Definition 2. The function f is said to tend to $+\infty$ as x tends to c (or in symbols, $\lim_{x \to c} f(x) = +\infty$) if for each $G > 0$ (however large) there exists a $\delta > 0$ such that

$$f(x) > G, \text{ when } |x - c| < \delta$$

The function f is said to tend to $-\infty$ as x tends to c (or in symbols, $\lim_{x \to c} f(x) = -\infty$), if for each $G > 0$ (however large) there exists a $\delta > 0$ such that

$$f(x) < -G, \text{ when } |x - c| < \delta$$

Definition 3. The function f is said to tend to a limit l as x tends to ∞ (or in symbols, $\lim_{x \to \infty} f(x) = l$) if for each $\varepsilon > 0$, there exists a $k > 0$, such that

$$|f(x) - l| < \varepsilon, \text{ when } x > k$$

The function f is said to tend to $+\infty$ as x tends to ∞ (or in symbols, $\lim_{x\to\infty} f(x)=\infty$) if for each $G>0$ (however large) there exists a $k>0$, such that

$$f(x)>G, \text{ when } x>k$$

1.1 Left Hand and Right Hand Limits

While defining the limit of a function f as x tends to c, we consider values of $f(x)$ when x is very close to c. The values of x may be greater or less than c. If we restrict x to values less than c, then we say that x tends to c from below or from the left and write it symbolically as $x \to c-0$ or simply $x \to c-$. The limit of f with this restriction on x, is called the *left hand limit*. Similarly, if x takes only values greater than c, then x is said to tend to c from above or from the right, and is denoted symbolically as $x \to c+0$ or $x \to c+$. The limit of f is then called the *right hand limit*.

Definition 5. A function f is said to tend to a limit l as x tends to c from the left if for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x)-l|<\varepsilon, \text{ when } c-\delta<x<c$$

In symbols, we then write

$$\lim_{x\to c-0} f(x)=l \text{ or } f(c-0)=l \text{ or } f(c-)=l$$

Definition 6. A function f is said to tend to a limit l as x tends to c from the right if for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x)-l|<\varepsilon, \text{ when } c<x<c+\delta$$

In symbols, we then write

$$\lim_{x\to c+0} f(x)=l \text{ or } f(c+0)=l \text{ or } f(c+)=l$$

It may be noted that $\lim_{x\to c} f(x)$ exists if and only if both the limits, the left hand and the right hand, exist and are equal.

One-sided infinite limit may also be defined in the same way as above.

Example 1. Find the right hand and the left hand limits of a function defined as follows:

$$f(x)=\begin{cases} \dfrac{|x-4|}{x-4}, & x\neq 4, \\ 0, & x=4. \end{cases}$$

■ Now, when $x>4$, $|x-4|=x-4$.

$$\therefore \quad \lim_{x\to 4+0} f(x)=\lim_{x\to 4+0}\frac{|x-4|}{x-4}=\lim_{x\to 4+0}\frac{x-4}{x-4}=\lim_{x\to 4+0} 1=1$$

Again when $x<4$, $|x-4|=-(x-4)$.

$$\therefore \quad \lim_{x\to 4-0} f(x)=\lim_{x\to 4-0}\frac{-(x-4)}{x-4}=\lim_{x\to 4-0}(-1)=-1$$

so that

$$\lim_{x\to 4+0} f(x) \neq \lim_{x\to 4-0} f(x)$$

Hence $\lim_{x\to 4} f(x)$ does not exist.

Example 2. Evaluate $\lim_{x\to 0+} \dfrac{1}{1+e^{-1/x}}$.

[As $x \to 0+$, we feel that $1/x$ increases indefinitely, $e^{1/x}$ increases indefinitely, $e^{-1/x}$ tends to 0, $1+e^{-1/x}$ tends to 1; thus the required limit may be 1.]

We have to show that for a given $\varepsilon > 0$, $\exists$ a $\delta > 0$ such that

$$\left|\frac{1}{1+e^{-1/x}} - 1\right| < \varepsilon, \text{ when } 0 < x < \delta$$

Now

$$\left|\frac{1}{1+e^{-1/x}} - 1\right| = \left|\frac{-e^{-1/x}}{1+e^{-1/x}}\right| = \frac{1}{e^{1/x}+1} < \varepsilon,$$

when $e^{1/x} + 1 > \dfrac{1}{\varepsilon}$ or $\dfrac{1}{x} > \log\left(\dfrac{1}{\varepsilon} - 1\right)$

$$\Rightarrow \qquad 0 < x < \frac{1}{\log(1/\varepsilon - 1)}, \text{ for } 0 < \varepsilon < 1$$

Thus choosing $\delta = \dfrac{1}{\log(1/\varepsilon - 1)}$, we see that if $0 < \varepsilon < 1$,

$$\left|\frac{1}{1+e^{-1/x}} - 1\right| < \varepsilon, \text{ when } 0 < x < \delta$$

Again when $\varepsilon \geq 1$,

$$\left|\frac{1}{1+e^{-1/x}} - 1\right| < \varepsilon \Rightarrow e^{1/x} > \frac{1}{\varepsilon} - 1$$

which is true for all values of x, so that any $\delta > 0$ would work.

Thus for any $\varepsilon > 0$ we are able to find a $\delta > 0$ such that

$$\left|\frac{1}{1+e^{-1/x}} - 1\right| < \varepsilon, \text{ when } 0 < x < \delta$$

$$\therefore \qquad \lim_{x\to 0+} \frac{1}{1+e^{-1/x}} = 0.$$

Example 3. Prove that $\lim_{x\to 0} x\sin\frac{1}{x} = 0$.

- Now,

$$\left|x\sin\frac{1}{x}\right| = |x|\cdot\left|\sin\frac{1}{x}\right| \le |x|$$

Thus choosing a $\delta = \varepsilon$, we see that

$$\left|x\sin\frac{1}{x}\right| < \varepsilon, \text{ when } 0<|x|<\delta$$

$$\Rightarrow \qquad \lim_{x\to 0} x\sin\frac{1}{x} = 0$$

Example 4. If $\lim_{x\to a} f(x)$ exists, prove that it must be unique.

- Let, if possible, $f(x)$ tend to limits l_1 and l_2.

Hence for any $\varepsilon > 0$, it is possible to choose a $\delta > 0$ such that

$$|f(x)-l_1| < \frac{1}{2}\varepsilon, \text{ when } 0<|x-a|<\delta$$

$$|f(x)-l_2| < \frac{1}{2}\varepsilon, \text{ when } 0<|x-a|<\delta$$

Now

$$|l_1 - l_2| = |l_1 - f(x) + f(x) - l_2|$$
$$\le |l_1 - f(x)| + |f(x) - l_2| < \varepsilon,$$

when $0<|x-a|<\delta$

i.e., $|l_1 - l_2|$ is less than any positive number ε (however small) and so must be equal to zero. Thus $l_1 = l_2$.

Example 5. Show that $\lim_{x\to 3} \frac{1}{(x-3)^4} = \infty$.

- Let G be any positive number, however large.

Now $\left|\frac{1}{(x-3)^4}\right| > G$

or $$\frac{1}{(x-3)^4} > G, \text{ when } (x-3)^4 < \frac{1}{G} \text{ or when } 0 < |x-3| < \frac{1}{G^{1/4}}$$

Choosing $\delta = \frac{1}{G^{1/4}}$, we get the required result.

Example 6. Prove that $\lim_{x \to 0} \log|x| = -\infty$.

Given $G > 0$, choose $\delta = e^{-G}$. Now if $0 < |x - 0| < \delta$ we have $|x| < e^{-G}$, and so $\log|x| < -G$, consequently $\lim_{x \to 0} \log|x| = -\infty$.

Example 7. Show that $\lim_{x \to 1} 2^{1/(x-1)}$ does not exist.

We first consider the left hand limit. Let $\varepsilon > 0$ be given. Choose a positive integer m such that, $1/2^m < \varepsilon$.

Take $\delta = \frac{1}{m}$ and let x satisfy $1 - \delta < x < 1$. Now $-\delta < (x-1) < 0$, and so $\frac{1}{x-1} < -\frac{1}{\delta} < 0$.

Thus $\left|2^{1/(x-1)} - 0\right| = 2^{1/(x-1)} < 2^{-1/\delta} < 2^{-m} < \varepsilon$

and hence $\lim_{x \to 1-} 2^{1/(x-1)} = 0$.

Next, consider x to be on the right of 1.

Let $\delta > 0$ be arbitrary and choose a positive integer m_0 such that $\frac{1}{m_0} < \delta$. Then if $n \geq m_0$, $1 + \frac{1}{n} \in]1, 1+\delta[$ and $2^{\frac{1}{1+\frac{1}{n}-1}} = 2^n$, which is unbounded. Therefore $\lim_{x \to 1+} 2^{1/(x-1)}$ does not exist.

1.2 Theorems on Limits

Let f and g be two real functions with domain D. We define four new functions $f \pm g$, fg, f/g on domain D by setting

$$(f+g)\,x = f(x) + g(x)$$
$$(f-g)\,x = f(x) - g(x)$$
$$(f \cdot g)\,x = f(x) \cdot g(x)$$
$$(f/g)\,x = f(x)/g(x), \text{ if } g(x) \neq 0 \text{ for any } x \in D$$

We shall now prove a theorem concerning the limits of these functions.

Theorem 1. *If f and g are two functions defined on some neighbourhood of c such that*

$$\lim_{x\to c} f(x) = l, \lim_{x\to c} g(x) = m$$

then

(i) $\lim\limits_{x\to c}(f+g)x = \lim\limits_{x\to c} f(x) + \lim\limits_{x\to c} g(x) = l + m.$

(ii) $\lim\limits_{x\to c}(f-g)x = \lim\limits_{x\to c} f(x) - \lim\limits_{x\to c} g(x) = l - m.$

(iii) $\lim\limits_{x\to c}(f\cdot g)x = \lim\limits_{x\to c} f(x) \cdot \lim\limits_{x\to c} g(x) = lm.$

(iv) $\lim\limits_{x\to c}(f/g)x = \lim\limits_{x\to c} f(x)/\lim\limits_{x\to c} g(x) = l/m$, *if* $m \neq 0$.

(i) Since $\lim\limits_{x\to c} f(x) = l$, $\lim\limits_{x\to c} g(x) = m$, therefore for any $\varepsilon > 0$, $\exists$ positive numbers δ_1, δ_2 such that

$$|f(x) - l| < \frac{1}{2}\varepsilon, \text{ when } 0 < |x - c| < \delta_1$$

$$|g(x) - m| < \frac{1}{2}\varepsilon, \text{ when } 0 < |x - c| < \delta_2.$$

If $\delta = \min(\delta_1, \delta_2)$, then for $0 < |x - c| < \delta$,

$$\left|f(x) - l < \frac{1}{2}\right|\varepsilon, \ |g(x) - m| < \frac{1}{2}\varepsilon$$

and

$$\begin{aligned}|(f+g)x - (l+m)| &= |f(x) - l + g(x) - m| \\ &\le |f(x) - l| + |g(x) - m| \\ &< \varepsilon\end{aligned}$$

$\Rightarrow$ $|(f+g)x - (l+m)| < \varepsilon$, when $0 < |x - c| < \delta$

$\Rightarrow$ $\lim\limits_{x\to c}(f+g)x = l + m$

(ii) Proof is similar to part (*i*).

(iii) Let $\varepsilon > 0$ be given.

Now

$$\begin{aligned}|(f\cdot g)x - lm| &= |f(x)(g(x) - m) + m(f(x) - l)| \\ &\le |f(x)| \cdot |g(x) - m| + |m| \cdot |f(x) - l| \qquad \ldots(1)\end{aligned}$$

Since $\lim\limits_{x\to c} f(x) = l$, therefore for $\varepsilon = 1$, $\exists$ a $\delta_1 > 0$, such that

$$|f(x) - l| < 1, \text{ when } 0 < |x - c| < \delta_1.$$

Now

$$|f(x)| = |f(x) - l + l| \le |f(x) - l| + |l|$$

$$< 1 + |l|, \text{ when } 0 < |x - c| < \delta_1 \qquad \ldots(2)$$

Again, since $\lim_{x \to c} g(x) = m$, therefore $\exists$ a $\delta_2 > 0$ such that

$$|g(x) - m| < \frac{\frac{1}{2}\varepsilon}{1 + |l|}, \text{ when } 0 < |x - c| < \delta_2. \qquad \ldots(3)$$

Also, as $\lim_{x \to c} f(x) = l$, therefore $\exists$ a $\delta_3 > 0$ such that

$$|f(x) - l| < \frac{\frac{1}{2}\varepsilon}{1 + |m|}, \text{ when } 0 < |x - c| < \delta_3. \qquad \ldots(4)$$

Let $\delta = \min(\delta_1, \delta_2, \delta_3)$.

Then, from (1), (2), (3) and (4), we have

$$|(f \cdot g)x - lm| < \frac{(1 + |l|)\frac{1}{2}\varepsilon}{1 + |l|} + \frac{|m|\varepsilon/2}{1 + |m|} < \varepsilon,$$

when $0 < |x - c| < \delta$.

Hence

$$\lim_{x \to c} (fg)(x) = lm$$

(*iv*) *Lemma.* Show that if $\lim_{x \to c} g(x) = m > 0$, then $\exists$ a $\delta_1 > 0$ such that

$$|g(x)| > \frac{1}{2}|m|, \text{ when } 0 < |x - c| < \delta_1$$

Since $\lim_{x \to c} g(x) = m$, therefore for $\varepsilon = \frac{1}{2}|m| > 0$, $\exists$ a $\delta_1 > 0$ such that

$$|g(x) - m| < \frac{1}{2}|m|, \text{ when } 0 < |x - c| < \delta_1$$

Also

$$|m| = |m - g(x) + g(x)| \le |m - g(x)| + |g(x)|$$

$$< \frac{1}{2}|m| + |g(x)|, \text{ when } 0 < |x - c| < \delta_1$$

i.e.,

$$|g(x)| > \frac{1}{2}|m|, \text{ when } 0 < |x - c| < \delta_1$$

$\Rightarrow$ $\exists$ a deleted neighbourhood of c on which $g(x)$ does not vanish.

CHAPTER 5

Let us now attend to the main theorem.

$$\left|(f/g)x - l/m\right| = \frac{mf(x) - lg(x)}{mg(x)}$$

$$\leq \frac{|f(x) - l|}{|g(x)|} + \frac{|l| \cdot |g(x) - m|}{|m| \cdot |g(x)|}$$

$$< \frac{2|f(x) - l|}{|m|} + \frac{2|l| \cdot |g(x) - m|}{|m|^2} \qquad \ldots(1)$$

where $0 < |x - c| < \delta_1$

Let $\varepsilon > 0$ be given.

Since $\lim_{x \to c} f(x) = l$, $\lim_{x \to c} g(x) = m$, therefore $\exists$ positive numbers δ_2 and δ_3 such that

$$|f(x) - l| < \frac{1}{4}\varepsilon|m|, \text{ when } 0 < |x - c| < \delta_2 \qquad \ldots(2)$$

and

$$|g(x) - m| < \frac{1}{4}\varepsilon\frac{|m|^2}{|l| + 1}, \text{ when } 0 < |x - c| < \delta_3 \qquad \ldots(3)$$

Let $\delta = \min(\delta_1, \delta_2, \delta_3)$.

Thus from (1), (2) and (3), we have

$$|(f/g)x - l/m| < \varepsilon, \text{ when } 0 < |x - c| < \delta$$

Hence $\lim_{x \to c} (f/g)x = l/m$, provided $m \neq 0$.

Example 8. Evaluate:

(*i*) $\lim_{x \to -1} \frac{(x + 2)(3x - 1)}{x^2 + 3x - 2}$

(*ii*) $\lim_{x \to 0} \frac{\sqrt{4 + x} - 2}{x}$

(*iii*) $\lim_{x \to 0+} \frac{\sin x}{\sqrt{x}}$.

■ (*i*) $\lim_{x \to -1} \frac{(x + 2)(3x - 1)}{x^2 + 3x - 2} = \frac{\lim_{x \to -1}(x + 2) \cdot \lim_{x \to -1}(3x - 1)}{\lim_{x \to -1}\left(x^2 + 3x - 2\right)}$

$$= \frac{1 \cdot (-4)}{-4} = 1.$$

(ii) $$\lim_{x \to 0} \frac{\sqrt{4+x}-2}{x} = \lim_{x \to 0} \frac{\sqrt{4+x}-2}{x} \cdot \frac{\sqrt{4+x}+2}{\sqrt{4+x}+2}$$

$$= \lim_{x \to 0} \frac{1}{\sqrt{4+x}+2} = \frac{1}{4}.$$

(iii) $$\lim_{x \to 0+} \frac{\sin x}{\sqrt{x}} = \left(\lim_{x \to 0+} \frac{\sin x}{x} \right) \cdot \left(\lim_{x \to 0+} \sqrt{x} \right) = 1 \cdot 0 = 0.$$

Example 9. Evaluate $\lim\limits_{x \to 1} \dfrac{x^2-1}{x-1}$.

■ Let us evaluate the left hand and the right hand limits.

When $x \to 1-0$, put $x = 1-h, h > 0$.

Hence $h \to 0+$ as $x \to 1-0$, so that

$$\lim_{x \to 1-0} \frac{x^2-1}{x-1} = \lim_{h \to 0+} \frac{(1-h)^2-1}{-h} = \lim_{h \to 0+} \frac{-h(2-h)}{-h}$$

$$= \lim_{h \to 0+} (2-h) = 2$$

Again when $x \to 1+0$, put $x = 1+h, h > 0$.

$$\therefore \quad \lim_{x \to 1+0} \frac{x^2-1}{x-1} = \lim_{h \to 0+} \frac{(1+h)^2-1}{h} = \lim_{h \to 0+} (2+h) = 2.$$

So that both, the left hand and the right hand, limits exist and are equal. Hence limit of the given function exists and equals 2.

Note: Since $x \neq 1$, division by $(x-1)$ is permissible.

$$\therefore \quad \lim_{x \to 1} \frac{x^2-1}{x-1} = \lim_{x \to 1} (x+1) = 2.$$

Example 10. Evaluate $\lim\limits_{x \to 0} \dfrac{e^{1/x}}{e^{1/x}+1}$.

■ Now when $x \to 0+$, $1/x \to \infty$, $e^{-1/x} \to 0$ and when $x \to 0-$, $1/x \to -\infty$, $e^{1/x} \to 0$.

$$\therefore \quad \lim_{x \to 0+} \frac{e^{1/x}}{e^{1/x}+1} = \lim_{x \to 0+} \frac{1}{1+e^{-1/x}} = 1$$

and

$$\lim_{x\to 0+} \frac{e^{1/x}}{e^{1/x}+1} = \frac{0}{1} = 0$$

so that the left hand limit is not equal to the right hand limit.

Hence $\lim\limits_{x\to 0} \dfrac{e^{1/x}}{e^{1/x}+1}$ does not exist.

Example 11. Find $\lim\limits_{x\to 0} e^x \operatorname{sgn}(x+[x])$, where the signum function is defined as

$$\operatorname{sgn}(x) = \begin{cases} 1, \text{ if } x > 0 \\ 1, \text{ if } x = 0 \\ -1, \text{ if } x < 0 \end{cases}$$

and $[x]$ means the greatest integer $\leq x$.

■ $\text{L.H.L.} = \lim\limits_{h\to 0+} e^{0-h} \operatorname{sgn}[0-h+(0-h)] = \lim\limits_{h\to 0+} \left(-e^{-h}\right) = -1$

$\text{R.H.L.} = \lim\limits_{h\to 0+} e^{0+h} \operatorname{sgn}[0+h+(0+h)] = \lim\limits_{h\to 0+} e^{h} = 1$

$\therefore$ $\lim\limits_{x\to 0} e^x \operatorname{sgn}(x+[x])$ does not exist.

EXERCISE

Evaluate:

1. $\lim\limits_{x\to 2} \dfrac{x^2-4}{x-2}$

2. $\lim\limits_{x\to 0-} \dfrac{|x|}{x}$ and $\lim\limits_{x\to 0+} \dfrac{|x|}{x}$

3. $\lim\limits_{x\to 0} \dfrac{3x+|x|}{7x-5|x|}$

4. $\lim\limits_{x\to 1} \dfrac{1}{(x-1)}\left(\dfrac{1}{x+3} - \dfrac{2}{3x+5}\right)$

5. $\lim\limits_{x\to 0} \dfrac{1-2\cos x+\cos 2x}{x^2}$

6. Show that $\lim\limits_{x\to 0} \dfrac{xe^{1/x}}{1+e^{1/x}} = 0$.

7. Show that $\lim\limits_{x \to 0} \dfrac{e^{1/x} - 1}{e^{1/x} + 1}$ does not exist.

8. Show that $\lim\limits_{x \to 0} \dfrac{e^{1/x} - e^{-1/x}}{e^{1/x} + e^{-1/x}}$ does not exist.

9. If $\lim\limits_{x \to c} f(x) = l$, then show that $\lim\limits_{x \to c} |f(x)| = |l|$.

[*Hint.* $\big||f(x)| - |l|\big| \le |f(x) - l|, \forall x$

10. If $\lim\limits_{x \to c} g(x) = m$ and $m \neq 0$, then show that $\lim\limits_{x \to c} \dfrac{1}{g(x)} = \dfrac{1}{m}$.

ANSWERS

1. 4;
2. −1, 1;
3. Does not exist;
4. 1/32;
5. −1.

1.3 Limit of a Function (*Sequential approach*)

We have defined the limit of a function f, defined in a neighbourhood (taken as an interval) of a point c, to be l as x approaches c, if for any $\varepsilon > 0 \; \exists \; \delta > 0$ such that

$$|f(x) - l| < \varepsilon, \text{ when } 0 < |x - c| < \delta \qquad \ldots(1)$$

We now give an alternative definition in terms of limits of sequences.

Definition. A number l is called the limit of a function f as x tends to c if the limit of the sequence $\{f(x_n)\}$ exists and is equal to l for any sequence $\{x_n\}$, $x_n \neq c$ for any n, convergent to c. Thus

$$\lim_{\substack{x_n \to c \\ x_n \neq c}} f(x) = l \qquad \ldots(2)$$

Here, as in all similar cases, it is tactly implied that x_n *tending to c runs through a set of values for which f is defined.*

We now show that the two definitions are equivalent.

Let f have a limit l as $x \to c$ in the sense of the first definition, so that for any $\varepsilon > 0$, there exists $\delta > 0$, such that

$$|f(x) - l| < \varepsilon, \text{ for } 0 < |x - c| < \delta$$

Again, let there be a sequence $\{x_n\}$, $x_n \neq c$ for any n, converging to c, so that there exists a natural number m such that

$$|x_n - c| < \delta, \text{ for } n \geq m$$

Hence,

$$|f(x_n) - l| < \varepsilon, \text{ for } n \geq m,$$

i.e., the sequence $\{f(x_n)\}$ converges to l. Also, since this property holds for every sequence $\{x_n\}$, $x_n \neq c$, tending to c, we have proved that the limit exists in the sense of the second definition too.

Conversely, let a function f have a limit l and x tends to c in the sense of the second definition and suppose that it has no limit in the sense of the first definition. Then there exists at least one value of ε, say ϵ_0 for which there is no δ of the first definition, this means that for any δ there is a value $x = x^{(\delta)}$ belonging to the set satisfying $0 < |x - c| < \delta$ such that $\left|f\left(x^{(\delta)}\right) - l\right| \geq \varepsilon_0$.

Let δ take up successively the values $1, \frac{1}{2}, \frac{1}{3}, \ldots$ For each of them there is value x_k such that

$$|x_k - c| < \frac{1}{k}\,(x_k \neq c)$$

and

$$|f(x_k) - l| \geq \epsilon_0\,(k = 1, 2, 3, \ldots)$$

These relations show that $x_k\,(x_k \neq c)$ tends to c while $f(x_k)$ does not tend to l, which is a contradiction. Thus our supposition that the limit does not exist in the sense of the first definition is disproved.

We have thus proved the equivalence of the two definitions.

We now prove a very useful theorem due to Cauchy.

Cauchy's Criterion for Finite Limits

Theorem 2. *A function f tends to a finite limit as x tends to c if and only if for every $\varepsilon > 0$, there exists a neighbourhood N of c such that*

$$|f(x') - f(x'')| < \varepsilon, \text{ for all } x', x'' \in N;\ x', x'' \neq c$$

Necessary. Let $\lim\limits_{x \to c} f(x) = l$, a finite number.

Therefore, for any $\varepsilon > 0$, there exists a deleted neighbourhood $N(c)$ such that $|f(x) - l| < \frac{1}{2}\varepsilon$, for $x \in N(c)$. Let $x', x'' \in N(c)$ so that

$$|f(x') - f(x'')| \leq |f(x') - l| + |l - f(x'')| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$$

Sufficient. Let for any $\varepsilon > 0$, there exist a deleted neighbourhood $N(c)$ of c such that $|f(x') - f(x'')| < \varepsilon$, for all $x', x'' \in N(c)$.

Let $\{x_n\}$, $x_n \neq c$ for any n, be an arbitrary sequence tending to c such that there exists a natural number m_0 such that $x_n, x_m \in N(c)$ for $n, m \geq m_0$.

Then

$$|f(x_n) - f(x_m)| < \varepsilon, \text{ for } n, m \geq m_0$$

Consequently, by Cauchy's general principle, sequence $\{f(x_n)\}$ tends to a limit.

We have thus proved for any sequence of numbers (x_n), $x_n \neq c$, converging to c, $\lim f(x_n)$ exists. Now, we prove that all these limits, $\lim f(x_n)$, corresponding to all possible different sequences tending to c, are equal to each other.

Let, if possible, $\{x_n\}$ and $\{x_n'\}$, $x_n \neq c$, $x_n' \neq c$ be two sequences tending to c such that sequences $\{f(x_n)\}$ and $\{f(x_n')\}$ tend to l and l' respectively. Let us construct the sequence $\{x_1, x_1', x_2, x_2', ...\}$ which converges to c. Therefore, by what has been proved above, the sequence $\{f(x_1), f(x_1'), f(x_2), ...\}$ converges, which is only possible if $l = l'$.

Hence the theorem.

We may similarly prove that:

A function f tends to a finite limit as x tends to ∞ *if and only if for every* $\epsilon > 0$ *there exists* $G > 0$ *such that*

$$|f(x') - f(x'')| < \epsilon, \text{ for all } x', x'' > G$$

Example 12. Show that $\lim\limits_{x \to 0} \frac{1}{x} \sin\frac{1}{x}$ does not exist.

- Let $f(x) = \frac{1}{x} \sin\frac{1}{x}$. The function f is defined for every real number $x \neq 0$. Now for each natural number n, let $x_n = \frac{2}{\pi(4n+1)}$, and so

$$f(x_n) = (4n+1)\pi/2 \sin(2n\pi + \pi/2) = (4n+1)\pi/2 \to \infty \text{ as } n \to \infty$$

$$\therefore \quad \lim_{n \to \infty} f(x_n) = \infty, \text{ when } \{x_n\} = \left\{\frac{2}{(4n+1)\pi}\right\} \text{ converges to zero.}$$

Again, by taking $x_n = 1/n\pi$, we see $f(x_n) = n\pi \cdot 0 = 0$ for every natural number n, and so $\lim\limits_{x \to 0} f(x) \neq \infty$.

Therefore, $\lim\limits_{x \to 0} f(x)$ does not exist.

Example 13. Find $\lim\limits_{x \to -\infty} \left(x^2 \, sgn(\cos x)\right)$.

- Let $x = -2n\pi$, so when $x \to -\infty$, $n \to \infty$.

Now

$$x^2 \, sgn(\cos x) = (-2n\pi)^2 \, sgn \cos(-2n\pi) = 4n^2\pi^2$$

$$\therefore \quad \lim_{x \to -\infty} x^2 \, \text{sgn}(\cos x) = \infty$$

Again, let $x = -(2n+1)\pi$

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$\therefore \quad x^2 \, sgn\,(\cos x) = \left(-(2n+1)\pi\right)^2 sgn\,\cos\left(-(2n+1)\pi\right) = -(2n+1)^2\pi^2$

and so

$$\lim_{x \to -\infty} x^2 \, sgn\,(\cos x) = -\infty$$

Hence

$$\lim_{x \to -\infty} x^2 \, sgn\,(\cos x) \text{ does not exist.}$$

2. CONTINUOUS FUNCTIONS

Let f be a function defined on an interval $[a, b]$. We shall now consider the behaviour of f at points of $[a, b]$.

2.1 Continuity at a Point

Definition 1 (Continuity at an internal point). A function f is said to be continuous at a point c, $a < c < b$, if

$$\lim_{x \to c} f(x) = f(c)$$

In other words, the function is continuous at c, if for each $\varepsilon > 0$, $\exists\, \delta > 0$ such that

$$|f(x) - f(c)| < \varepsilon, \text{ when } |x - c| < \delta$$

Definition 2. A function f is said to be *continuous from the left* at c if

$$\lim_{x \to c-0} f(x) = f(c)$$

Also f is *continuous from the right at c* if

$$\lim_{x \to c+0} f(x) = f(c)$$

Clearly a function is continuous at c if and only if it is continuous from the left as well as from the right.

Definition 3 (Continuity at an end point). A function f defined on a closed interval $[a, b]$ is said to be continuous at the end point a if it is continuous from the right at a, *i.e.*

$$\lim_{x \to a+0} f(x) = f(a)$$

Also the function is continuous at the end point b of $[a, b]$ if

$$\lim_{x \to b-0} f(x) = f(b)$$

Thus a function f is continuous at a point c if

(*i*) *$\lim_{x \to c} f(x)$ exists, and*

(*ii*) *limit equals the value of the function at $x = c$.*

2.2 Continuity in an Interval

A function f is said to be continuous in an interval $[a, b]$ if it is continuous at every point of the interval.

2.3 Discontinuous Functions

A function is said to be *discontinuous* at a point c of its domain if it is not continuous there at c. The point c is then called a *point of discontinuity* of the function.

Types of discontinuities

(*i*) A function f is said to have *a removable discontinuity* at $x = c$ if $\lim\limits_{x \to c} f(x)$ exists but is not equal to the value $f(c)$ (which may or may not exist) of the function. Such a discontinuity can be removed by assigning a suitable value to the function at $x = c$.

(*ii*) f is said to have a *discontinuity of the first kind* at $x = c$ if $\lim\limits_{x \to c-0} f(x)$ and $\lim\limits_{x \to c+0} f(x)$ both exist but are not equal.

(*iii*) f is said to have a *discontinuity of the first kind from the left* at $x = c$ if $\lim\limits_{x \to c-0} f(x)$ exists but is not equal to $f(c)$.

Discontinuity of the first kind from the right is similarly defined.

(*iv*) f is said to have a *discontinuity of the second kind* at $x = c$ if neither $\lim\limits_{x \to c-0} f(x)$ nor $\lim\limits_{x \to c+0} f(x)$ exists.

(*v*) f is said to have a *discontinuity of the second kind from the left* at $x = c$ if $\lim\limits_{x \to c-0} f(x)$ does not exist.

Discontinuity of the second kind from the right may be defined similarly.

2.4 Theorems on Continuity

Theorem 3. *If f, g be two functions continuous at a point c then the functions $f + g$, $f - g$, fg are also continuous at c and if $g(c) \neq 0$, then f/g is also continuous at c.*

The proof is left as an exercise.

Theorem 4. *A function f defined on an interval I is continuous at a point $c \in I$ iff for every sequence $\{c_n\}$ in I converging to c, we have*

$$\lim_{n \to \infty} f(c_n) = f(c)$$

First let us suppose that the function f is continuous at a point $c \in I$, and $\{c_n\}$ is a sequence in I such that $\lim\limits_{n \to \infty} c_n = c$.

Since f is continuous at c, therefore for any given $\in > 0, \exists$ a $\delta > 0$ such that

$$|f(x)-f(c)|<\varepsilon, \text{ when } 0<|x-c|<\delta \qquad \ldots(1)$$

Again, since $\lim_{n\to\infty} c_n = c$, therefore $\exists$ a positive integer m, such that

$$|c_n - c| < \delta, \forall\, n \geq m \qquad \ldots(2)$$

From (1), putting $x = c_n$, we have

$$|f(c_n)-f(c)|<\varepsilon, \text{ when } |c_n - c|<\delta$$

$$\Rightarrow \quad |f(c_n)-f(c)|<\varepsilon, \quad \forall\, n \geq m \qquad \text{[using 2]}$$

$\Rightarrow$ The sequence $\{f(c_n)\}$ converges to $f(c)$

or
$$\lim_{n\to\infty} f(c_n) = f(c)$$

Let us now suppose that f is *not continuous* at c, we shall now show that though $\exists$ a sequence $\{c_n\}$ in I converging to c, yet the sequence $\{f(c_n)\}$ does not converge to $f(c)$.

Since f is not continuous at c, therefore, there exists an $\varepsilon > 0$ such that for every $\delta > 0, \exists$ an $x \in I$ such that

$$|f(x)-f(c)| \geq \varepsilon, \text{ when } |x-c|<\delta$$

$\therefore$ By taking $\delta = 1/n$, we find that for each positive integer n, there is a $c_n \in I$ such that

$$|f(c_n)-f(c)| \geq \varepsilon, \text{ when } |c_n - c| < \frac{1}{n}$$

Thus the sequence $\{f(c_n)\}$ does not converge to $f(c)$, while the sequence $\{c_n\}$ converges to c.

Notes:

1. If $\lim c_n = c \Rightarrow \lim f(c_n) \neq f(c)$, then f is not continuous at c.
2. Compare with § 1.3.

Example 14. Examine the following function for continuity at the origin.

$$f(x)=\begin{cases}\dfrac{xe^{1/x}}{1+e^{1/x}} & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases}$$

■ Now

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0-} \frac{xe^{1/x}}{1+e^{1/x}} = 0$$

and

$$\lim_{x\to 0+} f(x) = \lim_{x\to 0+} \frac{x}{e^{-1/x}+1} = 0$$

Thus,

$$\lim_{x\to 0-} f(x) = \lim_{x\to 0+} f(x) = \lim_{x\to 0} f(x) = 0$$

Also

$$\lim_{x\to 0} f(x) = f(0)$$

Thus, the function is continuous at the origin.

Example 15. Show that the function defined as:

$$f(x) = \begin{cases} \dfrac{\sin 2x}{x} & \text{when } x \neq 0 \\ & \text{when } x = 0 \end{cases}$$

has removable discontinuity at the origin.

▪ Now

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \frac{\sin 2x}{2x} \cdot 2 = 2$$

so that

$$\lim_{x\to 0} f(x) \neq f(0)$$

Hence the limit exists, but is not equal to the value of the function at the origin. Thus the function has a removable discontinuity at the origin.

Note: The discontinuity can be removed by redefining the function at the origin such as $f(0) = 2$.

Example 16. Show that the function defined by

$$f(x) = \begin{cases} x \sin 1/x, & \text{when } x \neq 0 \\ 0, & \text{when } x = 0 \end{cases}$$

is continuous at $x = 0$.

▪ Now

$$\lim_{x\to 0} f(x) = \lim_{x\to 0} \left(x \sin \frac{1}{x} \right) = 0$$

so that

$$\lim_{x\to 0} f(x) = f(0)$$

Hence, f is continuous at $x = 0$.

Example 17. A function f is defined on **R** by

$$f(x) = \begin{cases} -x^2 & \text{if } x \leq 0 \\ 5x - 4 & \text{if } 0 < x \leq 1 \\ 4x^2 - 3x & \text{if } 1 < x < 2 \\ 3x + 4 & \text{if } x \geq 2 \end{cases}$$

Examine f for continuity at $x = 0, 1, 2$. Also discuss the kind of discontinuity, if any.

■ Now

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} \left(-x^2\right) = 0$$

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} (5x - 4) = -4$$

so that

$$\lim_{x \to 0-} f(x) = f(0) \neq \lim_{x \to 0+} f(x)$$

Thus the function has a discontinuity of the first kind from the right at the origin.

Again

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1-} (5x - 4) = 1$$

$$\lim_{x \to 1+} f(x) = \lim_{x \to 1+} \left(4x^2 - 3x\right) = 1$$

so that

$$\lim_{x \to 1-} f(x) = \lim_{x \to 1+} f(x) = 1 = f(1)$$

$$\Rightarrow \qquad \lim_{x \to 1} f(x) = f(1)$$

Thus the function is continuous at $x = 1$.

Again

$$\lim_{x \to 2-} f(x) = \lim_{x \to 2-} \left(4x^2 - 3x\right) = 10$$

$$\lim_{x \to 2+} f(x) = \lim_{x \to 2+} (3x + 4) = 10$$

Also

$$f(2) = 10 \Rightarrow \lim_{x \to 2} f(x) = f(2)$$

Thus, the function is continuous at $x = 2$.

Example 18. Is the function f, where $f(x) = \dfrac{x - |x|}{x}$ continuous?

■ For $x < 0$, $f(x) = \dfrac{x + x}{2} = 2$, continuous.

For $x > 0$, $f(x) = \dfrac{x - x}{x} = 0$, continuous.

The function is not defined at $x = 0$.

Thus $f(x)$ is continuous for all x except at zero.

Example 19. Discuss the kind of discontinuity, if any of the function is defined as follows:

$$f(x) = \begin{cases} \dfrac{x - |x|}{x} & \text{when } x \neq 0 \\ 2 & \text{when } x = 0 \end{cases}$$

- The function is continuous at all points except possibly the origin.
  Let us test at $x = 0$.
  Now,

$$\lim_{x \to 0-} f(x) = \lim_{x \to 0-} \frac{x + x}{x} = 2$$

$$\lim_{x \to 0+} f(x) = \lim_{x \to 0+} \frac{x - x}{x} = 0$$

and

$$f(0) = 2$$

Thus the function has discontinuity of the first kind from the right at $x = 0$.

Example 20. If $[x]$ denotes the largest integer $\leq x$, then discuss the continuity at $x = 3$ for the function

$$f(x) = x - [x], \forall\, x \geq 0$$

- Now

$$\lim_{x \to 3-} f(x) = \lim_{x \to 3-} \{x - [x]\} = 3 - 2 = 1$$

$$\lim_{x \to 3+} f(x) = \lim_{x \to 3+} \{x - [x]\} = 3 - 3 = 0$$

and

$$f(3) = 0$$

Thus the function has a discontinuity of the first kind from the left at $x = 3$.

Note: The function is continuous at all points except the integral values 1, 2, 3, ...

Example 21. Prove that the *Dirichlet's* function f defined on **R** by

$$f(x) = \begin{cases} 1, & \text{when } x \text{ is irrational} \\ -1, & \text{when } x \text{ is rational} \end{cases}$$

is discontinuous at every point.

- First, let a be any *rational* number so that $f(a) = -1$.
  Since in any interval there lie an infinite number of rational and irrational numbers, therefore, for each positive integer n, we can choose an irrational number a_n such that $|a_n - a| < \frac{1}{n}$.

  Thus the sequence $\{a_n\}$ converges to a.
  But $f(a_n) = 1$ for all n, and $f(a) = -1$, so that

$$\lim_{n \to \infty} f(a_n) \neq f(a)$$

Thus by Theorem 4, § 2.4, the function is discontinuous at any rational number a.

Hence, the function is discontinuous at all rational points.

Next, let b be any *irrational* number. For each positive integer n we can choose a rational number b_n such that $|b_n - b| < \frac{1}{n}$. Thus the sequence $\{b_n\}$ converges to b.

But $f(b_n) = -1$ for all n and $f(b) = 1$.

$$\therefore \qquad \lim_{n \to \infty} f(b_n) \neq f(b)$$

Hence, the function is discontinuous at all irrational points.

Example 22. Show that the function $f(x)$ defined on **R** by

$$f(x) = \begin{cases} x, & \text{when } x \text{ is irrational} \\ -x, & \text{when } x \text{ is rational} \end{cases}$$

is continuous only at $x = 0$.

- First, let $a \neq 0$ be any *rational* number, so that $f(a) = -a$. Since in every interval there lie an infinite number of rational and irrational numbers, therefore, for each positive integer n, we can choose an irrational number a_n such that

$$|a_n - a| < \frac{1}{n}$$

Thus the sequence $\{a_n\}$ converges to a.

But

$$\lim_{n \to \infty} f(a_n) = \lim_{n \to \infty} a_n = a$$

Thus

$$\lim_{n \to \infty} f(a_n) \neq f(a), \ a \neq 0$$

so that by Theorem 4, § 2.4, the function is discontinuous at any rational number, other than zero.

In a similar way the function may be shown to be discontinuous at every *irrational* point.

It may be seen from above, that the function is continuous at $x = 0$ (*i.e.*, $a = 0$). However, it can be shown to be continuous at $x = 0$ as follows:

Let $\varepsilon > 0$ be given and let $\delta = \varepsilon$ (or any $\delta < \varepsilon$), then

$|x| < \delta \Rightarrow |f(x) - f(0)| = |-x| = |x| < \varepsilon$, when x is rational and

$|x| < \delta \Rightarrow |f(x) - f(0)| = |x| < \varepsilon$, when x is irrational

Thus

$$|x| < \delta \quad \Rightarrow \quad |f(x) - f(0)| < \varepsilon$$

or

$$\lim_{x \to 0} f(x) = f(0)$$

Hence, the function is continuous at $x = 0$.

EXERCISE

Investigate the continuity at the indicated point:

1. $$f(x)=\begin{cases}\dfrac{\sin(x-c)}{x-c}, & \text{if } x\neq c\\ 0, & \text{if } x=c\end{cases}\quad \text{at } x=c$$

2. $$f(x)=\begin{cases}\dfrac{\tan^{-1}x}{x}, & \text{if } x\neq 0\\ 1, & \text{if } x=0\end{cases}\quad \text{at } x=0$$

3. $$f(x)=\begin{cases}\dfrac{x^3-8}{x^2-4}, & \text{if } x\neq 2\\ 3, & \text{if } x=2\end{cases}\quad \text{at } x=2$$

4. $f(x)=x-|x|$, at $x=0$

5. $$f(x)=\begin{cases}x^m\sin\dfrac{1}{x}, & \text{if } x\neq 0,\ m>0\\ 0, & \text{if } x=0\end{cases}\quad \text{at } x=0$$

6. $$f(x)=\begin{cases}(1+x)^{1/x}, & \text{if } x\neq 0\\ 1, & \text{if } x=0\end{cases}\quad \text{at } x=0$$

7. $$f(x)=\begin{cases}\dfrac{e^{1/x^2}}{1-e^{1/x^2}}, & \text{when } x\neq 0\\ 1, & \text{when } x=0\end{cases}\quad \text{at } x=0$$

8. $$f(x)=\begin{cases}\dfrac{e^{1/x}-e^{-1/x}}{e^{1/x}+e^{-1/x}}, & \text{when } x\neq 0\\ 1, & \text{when } x=0\end{cases}\quad \text{at } x=0$$

9. Examine the continuity at $x=1$.

$$f(x)=\begin{cases}2x, & \text{when } 0\le x<1\\ 3, & \text{when } x=1\\ 4x, & \text{when } 1<x\le 2\end{cases}$$

10. Obtain the points of discontinuity of the function f, defined on $[0, 1]$ as follows:

$$f(0)=0,\ f(x)=\frac{1}{2}-x,\ \text{if } 0<x<\frac{1}{2},$$

$$f\left(\frac{1}{2}\right)=\frac{1}{2},\ f(x)=\frac{3}{2}-x,\ \text{if } \frac{1}{2}<x<1$$

$$f(1)=1$$

Also examine the kind of discontinuities.

11. Show that the function f defined on **R** by

$$f(x)=\begin{cases}1, & \text{if } x \text{ is rational}\\ 0, & \text{if } x \text{ is irrational}\end{cases}$$

is discontinuous at every point.

12. Show that the function f defined by

$$f(x)=\begin{cases}x, & \text{if } x \text{ is rational}\\ 0, & \text{if } x \text{ is irrational}\end{cases}$$

is continuous only at $x = 0$.

13. Show that the function f defined by

$$f(x)=\begin{cases}[x+1]\sin\dfrac{1}{x}, & x\in\,]-1,0[\,\cup\,]0,1[\\ 0, & \text{otherwise}\end{cases}$$

has discontinuity of the second kind at $x = 0$ and discontinuity of the first kind at $x = 1$.

14. Show that the function f defined by

$$f(x)=\begin{cases}(1+x)\operatorname{sgn} x+\operatorname{sgn}|x|-1, & \text{if } x \text{ is rational}\\ \operatorname{sgn} x, & \text{if } x \text{ is irrational}\end{cases}$$

has discontinuity of the second kind at $x \neq 0$ and discontinuity of the first kind at $x = 0$.

ANSWERS

1. Removable discontinuity
2. Continuous
3. Continuous
4. Continuous
5. Continuous
6. Removable discontinuity
7. Removable discontinuity
8. Discontinuity of the first kind from the left
9. Discontinuity of the first kind
10. Discontinuity of the first kind from the right at 0, discontinuity of the first kind at $x = \frac{1}{2}$, discontinuity of the first kind from the left at $x = 1$.

3. FUNCTIONS CONTINUOUS ON CLOSED INTERVALS

We shall now study some properties of functions which are continuous on closed intervals. In fact, we shall show that a function which is continuous on a closed interval, is bounded, attains its bounds and assumes every value between the bounds.

Theorem 5. *If a function is continuous in a closed interval, then it is bounded therein.*

Let f be a function defined and continuous in a closed interval I.

We shall show that if the function f is not bounded, then it fails to be continuous at some point of the closed interval I.

Let, if possible, f is *not bounded above*, so that for each positive integer $n \;\exists$ a point $x_n \in I$ such that $f(x_n) > n$.

Now $\{x_n\}$, being a sequence in the closed interval I, is bounded and has at least one limit point, say, ξ.

A closed interval is a closed set and so $\xi \in I$.

Further, since ξ is a limit point of the sequence $\{x_n\}$, therefore, there exists a subsequence (x_{n_k}) of $\{x_n\}$ such that $x_{n_k} \to \xi$ as $k \to \infty$.

Also since $f(x_{n_k}) > n_k$, for all k, therefore the sequence $\{f(x_{n_k})\}$ diverges to ∞.

Thus $\exists$ a point ξ of I such that a sequence (x_{n_k}) in I converges to ξ, but

$$\lim_{k \to \infty} f\left(x_{n_k}\right) \neq f(\xi)$$

Thus f is not continuous at ξ, which is a contradiction and hence the function is bounded above.

By considering a function 'f', it can be shown in a similar way that the function f is also bounded below.

Hence, the function is bounded.

Theorem 6. *If a function f is continuous on a closed interval $[a, b]$, then it attains its bounds at least once in $[a, b]$.*

If f is a constant function, then evidently it attains its bounds at every point of the interval.

Let f be a function which is not a constant.

Since f is continuous on the closed interval $[a, b]$, therefore, it is bounded. Let m and M be the infimum and supremum of f. It is to be shown that $\exists$ point α, β of $[a, b]$ such that

$$f(\alpha) = m, \; f(\beta) = M$$

Let us consider the case of the *supremum*.

Suppose f does not attain the supremum M so that the function does not take the value M for any point $x \in [a, b]$, *i.e.*,

$$f(x) \neq M, \text{ for any } x \in [a, b]$$

Now consider the function

$$g(x) = \frac{1}{M - f(x)}, \quad \forall\, x \in [a, b]$$

which is positive for all values of x in $[a, b]$.

Evidently the function g is continuous and so bounded in $[a, b]$. Let k (>0) be its supremum.

$$\therefore \quad \frac{1}{M - f(x)} \leq k, \quad \forall\, x \in [a, b]$$

$$\Rightarrow \quad f(x) \leq M - \frac{1}{k}, \quad \forall\, x \in [a, b]$$

which contradicts the hypothesis that M is the supremum of f in $[a, b]$. Hence our supposition that f does not attain the value M leads to a contradiction and therefore f attains its supremum for at least one value in $[a, b]$.

It may similarly be shown that the function also attains its infimum m.

Hence, the function attains its bounds at least once in $[a, b]$.

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Note: It may be observed from the two preceding theorems, that the function f, continuous on the closed interval $[a, b]$, has the least and the greatest values m and M, *i.e.*, the range set of f is bounded with m and M as its smallest and greatest elements. Thus the range set of f is a subset of $[m, M]$. We shall, in fact, show later that the range set of f is $[m, M]$ itself and that f takes up every value between m and M.

ILLUSTRATIONS

1. The function $f(x) = \dfrac{1}{1+|x|}$, for real x, is continuous and bounded and attains its supremum for $x = 0$ but does not attain the infimum.

2. The function $f(x) = -\dfrac{1}{1+|x|}$, (for all $x \in \mathbf{R}$, is continuous and bounded, attains its infimum but not the supremum.

3. The function $f(x) = x$, for all $x \in]0, 1[$ is continuous and bounded but attains neither the infimum nor the supremum.

Theorem 7. *If a function f is continuous at an interior point c of an interval $[a, b]$ and $f(c) \neq 0$, then $\exists$ a $\delta > 0$ such that $f(x)$ has the same sign as $f(c)$, for every $x \in]c - \delta, c + \delta[$.*

Since the function f is continuous at an interior point c of $[a, b]$, therefore for any $\varepsilon > 0, \exists$ a $\delta > 0$ such that

$$|f(x) - f(c)| < \varepsilon, \ \forall\, x \in]c - \delta, c + \delta[$$

or

$$f(c) - \varepsilon < f(x) < f(c) + \varepsilon, \ \forall\, x \in]c - \delta, c + \delta[$$

When $f(c) > 0$, taking ε to be less than $f(c)$ we find that

$$f(x) > 0, \ \forall\, x \in]c - \delta, c + \delta[$$

When $f(c) < 0$, taking ε to be less than $-f(c)$ we find that

$$f(x) < 0, \ \forall\, x \in]c - \delta, c + \delta[$$

Hence the theorem.

Corollary. If f is continuous at the end point b of $[a, b]$ and $f(b) \neq 0$, then there exists an interval $]b - \delta, b[$ such that $f(x)$ has the sign of $f(b)$ for all x in $]b - \delta, b]$.

A similar result holds for continuity at a.

Note: When c is an interior point of the interval, the theorem may be restated as:

If a function f is continuous at an interior point c of an interval $[a, b]$ and $f(c) \neq 0$, then $\exists$ a neighbourhood N of c wherein $f(x)$ has the same sign as $f(c)$, for all $x \in N$.

Theorem 8. *If a function f is continuous on a closed interval $[a, b]$ and $f(a)$ and $f(b)$ are of opposite signs $(f(a)\cdot f(b) < 0)$, then there exists at least one point $\alpha \in]a, b[$ such that $f(\alpha) = 0$.*

Let us suppose that $f(a) > 0$ and $f(b) < 0$.

Let S consists of those points of $[a, b]$ for which $f(x)$ is positive, *i.e.*,

$$S = \{x : a \le x \le b \wedge f(x) > 0\}$$

Now

$$f(a) > 0 \quad \Rightarrow \quad a \in S \Rightarrow S \ne \phi$$

Also S is bounded above by b.

Hence by the order completeness property, S has the supremum, say α, where $a \le \alpha \le b$.

We shall now show that

(*i*) $\alpha \ne a$, $\alpha \ne b$, and

(*ii*) $f(\alpha) = 0$.

(*i*) First we show that $\alpha \ne a$

Since $f(a) > 0$, therefore $\exists$ a $\delta > 0$ such that

$$f(x) > 0, \quad \forall\, x \in [a, a+\delta[$$

$\Rightarrow$ $[a, a+\delta[\subseteq S$

$\Rightarrow$ the supremum α of S is greater than or equal to $a+\delta$

$\Rightarrow$ $\alpha \ne a$

Now we shall show that $\alpha \ne b$.

Since $f(b) < 0$, therefore $\exists$ a $\delta > 0$ such that

$$f(x) < 0, \quad \forall\, x \in]b-\delta, b]$$

$\Rightarrow$ $b - \delta$ is an upper bound of S

$\Rightarrow$ $\alpha \le b - \delta \Rightarrow \alpha \ne b$

(*ii*) We shall now show that $f(\alpha) \not> 0$ and $f(\alpha) \not< 0$.

If $f(\alpha) > 0$, then $\exists$ a $\delta > 0$ such that

$$f(x) > 0, \quad \forall\, x \in]\alpha-\delta, \alpha+\delta[$$

$\Rightarrow$ $]\alpha-\delta, \alpha+\delta[\subseteq S$

Let us choose a positive $\delta_2 < \delta$ such that $\alpha + \delta_2 \in]\alpha-\delta, \alpha+\delta[$

A member $\alpha + \delta_2$ of S is greater than the supremum α of S, which is a contradiction.

$\therefore$ $$f(\alpha) \not> 0$$

Let now $f(\alpha) < 0$, so that $\exists$ a $\delta_1 > 0$ such that ...(1)

$$f(x) < 0, \quad \forall\, x \in]\alpha-\delta_1, \alpha+\delta_1[$$

Again, since α is the supremum of S, therefore, there exists a member β of S, where $\alpha - \delta_1 < \beta \le \alpha$ such that

CHAPTER 5

$$f(\beta) > 0$$

But from (1), $f(\beta) < 0$, which is a contradiction.

$$f(\alpha) \nless 0$$

$\therefore$

Thus, it follows that $f(\alpha) = 0$.

Theorem 9. *Intermediate value theorem.* *If a function f is continuous on $[a, b]$ and $f(a) \neq f(b)$, then it assumes every value between f (a) and f (b).*

Let A be any number between $f(a)$ and $f(b)$. We shall show that there exists a number $c \in]a, b[$ such that $f(c) = A$.

Consider a function ϕ defined on $[a, b]$ such that

$$\phi(x) = f(x) - A$$

Clearly $\phi(x)$ is continuous on $[a, b]$.

Also

$$\phi(a) = f(a) - A, \text{ and } \phi(b) = f(b) - A$$

so that $\phi(a)$ and $\phi(b)$ are of opposite signs.

Thus the function ϕ is continuous on $[a, b]$ and $\phi(a)$ and $\phi(b)$ are of opposite signs; therefore, by the previous theorem, $\exists\, c \in]a, b[$ such that

$$\phi(c) = 0$$

$$\Rightarrow \qquad f(c) - A = 0 \Rightarrow f(c) = A$$

Corollary. A function f, which is continuous on a closed interval $[a, b]$, assumes every value between its bounds.

Since the function f is continuous on the closed interval $[a, b]$, therefore, it is bounded and attains its bounds on $[a, b]$, *i.e.*, $\exists$ two numbers α, β in $[a, b]$ such that

$$f(\alpha) = M \text{ and } f(\beta) = m,$$

where M and m are, respectively, the supremum and the infimum of f.

Since f is continuous on $[a, b]$, therefore, it is continuous on $[\beta, \alpha]$ or $[\alpha, \beta]$ as the case may be, and consequently assumes every value between $f(\alpha)$ and $f(\beta)$.

Thus the function assumes every value between its bounds.

We may sum up in other words:

The range of a continuous function whose domain is a closed interval is as well a closed interval.

Or, in still better words:

The image of a closed interval under a continuous function (mapping) is a closed interval.

Example 23. Show that the function defined on $[0, 1]$ as

$$f(x) = 2x + 1, \forall\, x \in]0, 1]$$
$$f(0) = 0$$

does not satisfy the conclusion of the intermediate value theorem.

- The function f is bounded, but is not continuous on $[0, 1]$, since f fails to be right-continuous at $x = 0$.

$f(1) = 3 = M$ and $f(0) = 0 = m$, but there is no $c \in]0, 1[$ with $f(c) = 1$. It is easy to see that none of the intermediate values $x \in]0, 3[$ are assumed by f.

Theorem 10. *Fixed point theorem.* *If f is continuous on $[a, b]$ and $f(x) \in [a, b]$, for every $x \in [a, b]$ then f has a fixed point, i.e., there exists a point, $c \in [a, b]$ such that $f(c) = c$.*

Suppose f is continuous on $[a, b]$ and $f(x) \in [a, b]$ for every $x \in [a, b]$. If $f(a) = a$ or $f(b) = b$, then the theorem is proved, hence we assume that $f(a) > a$ and $f(b) < b$.

Let $g(x) = f(x) - x, \ \forall\, x \in [a, b]$

Now $g(a) > 0$, $g(b) < 0$ and g is continuous on $[a, b]$. '0' is an intermediate value of g on $[a, b]$. Hence by intermediate value theorem, there exists a point $c \in]a, b[$ such that $g(c) = 0$. Then $f(c) = c$.

Definition. A function f defined on $[a, b]$ is said to satisfy the *intermediate-value property* on $[a, b]$ if for every $x_1, x_2 \in [a, b]$ with $x_1 < x_2$ and for every A between $f(x_1)$ and $f(x_2)$ there is a $c \in]x_1, x_2[$ with $f(c) = A$.

A function which satisfies the intermediate-value property on $[a, b]$ need not be continuous on $[a, b]$. For example, the function $f(x) = \sin 1/x$ with $f(0) = 0$ defined on $[-2/\pi, 2/\pi]$ satisfies the intermediate value property but is not continuous at $x = 0$.

Ex. 1 If f satisfies the intermediate-value property on $[a, b]$, then prove that f has no discontinuities (of the first kind and removable on $[a, b]$).

Ex. 2 Prove that if f is one-to-one and satisfies the intermediate-value property on $[a, b]$, then f is continuous on $[a, b]$.

(*Hint*: Monotone functions have no discontinuities of the second kind).

Ex. 3 If f is continuous on $[a, b]$ and $f(a) = f(b)$, then show that there exist $x, y \in]a, b[$ such that $f(x) = f(y)$.

4. UNIFORM CONTINUITY

Let f be a function defined on an interval I. Then by definition, the function is continuous at any point $c \in I$ if for any $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$|f(x) - f(c)| < \varepsilon, \text{ when } |x - c| < \delta.$$

For continuity at any other point $d \in I$, for the same ε, a $\delta_1 > 0$ would exist (not necessarily equal to δ). There is in fact a δ corresponding to each point of I. The number δ in general depends on the selection of ε and the point c. However, if a δ could be found which depends only on ε and not on the selection of the point c, such a δ would work for the whole interval I on which f is continuous. In such a case, f is said to be *uniformly continuous* on I. Thus, the notion of uniform continuity is *global* in character in as much as we talk of uniform continuity only on an interval.

The notion of continuity is, however, *local* in character in as much as we can talk of continuity at a point.

It may seem to a beginner that the infimum of the set consisting of δ's corresponding to different points of I would work for the whole of I. But the infimum may be zero. In general, therefore, a δ

which may work for the entire interval may not exist, so that every continuous function may not be uniformly continuous.

Definition. A function f defined on an interval I is said to be ***uniformly continuous*** on I if to each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$|f(x_2) - f(x_1)| < \varepsilon, \text{ for arbitrary points } x_1, x_2 \text{ of } I \text{ for which } |x_1 - x_2| < \delta$$

4.1 We shall now prove two theorems on uniform continuity.

Theorem 11. *A function which is uniformly continuous on an interval is continuous on that interval.*

Let a function f be uniformly continuous on an interval I, so that for a given $\varepsilon > 0$, there corresponds a $\delta > 0$ such that

$$|f(x_1) - f(x_2)| < \varepsilon, \text{ where } x_1, x_2 \text{ are any two points of } I \text{ for which}$$

$$|x_1 - x_2| < \delta$$

Let $x \in I$, then on taking $x_1 = x$, we find that for $\varepsilon > 0$, $\exists\ \delta > 0$ such that

$$|f(x) - f(x_2)| < \varepsilon, \text{ when } |x - x_2| < \delta.$$

Hence the function is continuous at every point $x_2 \in I$, *i.e.*, the function f is continuous on I.

Theorem 12. *A function which is continuous on a closed interval is also uniformly continuous on that interval.*

Let a function f be continuous on a closed interval I. Let, if possible, f be not uniformly continuous on I. Then there exists an $\varepsilon > 0$ such that for any $\delta > 0$, there are numbers $x, y \in I$ for which

$$|f(x) - f(y)| \nless \varepsilon, \text{ when } |x - y| < \delta$$

In particular for each positive integer n, we can find real numbers x_n, y_n in I such that

$$|f(x_n) - f(y_n)| \nless \varepsilon, \text{ when } |x_n - y_n| < 1/n \quad \text{...(1)}$$

Now $\{x_n\}$ and $\{y_n\}$ being sequences in the closed interval I, they are bounded and so each has at least one limit point, say ξ and η respectively.

As a closed interval is a closed set,

$$\therefore \quad \xi \in I, \eta \in I$$

Further since ξ is a limit point of $\{x_n\}$, there exists a convergent subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \to \xi$ as $k \to \infty$.

Similarly, there exists a convergent subsequence $\{y_{n_k}\}$ of $\{y_n\}$ such that $y_{n_k} \to \eta$ as $k \to \infty$.

Again from (1), we find that

$$\left|f(x_{n_k}) - f(y_{n_k})\right| \nless \varepsilon, \text{ when } \left|x_{n_k} - y_{n_k}\right| < 1/n_k \le 1/k \quad \text{...(2)}$$

The second inequality shows that

$$\lim_{k \to \infty} x_{n_k} = \lim_{k \to \infty} y_{n_k}$$

$$\Rightarrow \quad \xi = \eta$$

From the first inequality we find that in case the sequences $\{f(x_{n_k})\}$ and $\{f(y_{n_k})\}$ converge, the limits to which they converge are different.

We thus have two sequences $\{x_{n_k}\}$ and $\{y_{n_k}\}$ both of which converge to ξ but $\{f(x_{n_k})\}$ and $\{f(y_{n_k})\}$ do not converge to the same limit.

So by Theorem 4 § 2.4, f is not continuous at ξ, for, otherwise, the two sequences $\{f(x_{n_k})\}$ and $\{f(y_{n_k})\}$ would converge to the same point $f(\xi)$.

Thus we arrive at a contradiction and so the hypothesis that f is not uniformly continuous on I is false.

Hence, f is uniformly continuous on I.

Example 24. Show that the function $f(x) = 1/x$ is not uniformly continuous on $]0, 1]$.

■ Clearly the function is continuous on $]0, 1]$.

It will be uniformly continuous on the given interval if for a given $\varepsilon > 0, \exists$ a $\delta > 0$, independent of the choice of points x and c in $]0, 1]$, such that

$$\left|\frac{1}{x} - \frac{1}{c}\right| < \varepsilon, \text{ when } |x - c| < \delta$$

or

$$\left|\frac{c - x}{cx}\right| < \varepsilon, \text{ when } c - \delta < x < c + \delta \qquad ...(1)$$

If we take $c = \delta$, then the interval $]c - \delta, c + \delta[$ becomes $]0, 2\delta[$. Also condition (1) must hold for any x in this interval.

But

$$\frac{\delta - x}{\delta x} \to \infty \text{ as } x \to 0,$$

i.e., if we choose x sufficiently close to zero, then condition (1) is violated.

Hence, $1/x$ is not uniformly continuous on $]0, 1]$.

Aliter.

If $\varepsilon = \dfrac{1}{2}$ and δ is any positive number, then for $n > \dfrac{1}{\delta}$,

$$\left|\frac{1}{n} - \frac{1}{n+1}\right| = \frac{1}{n(n+1)} < \frac{1}{n} < \delta$$

Therefore taking $x_1 = \dfrac{1}{n}$ and $x_2 = \dfrac{1}{n+1}$, as any two points of the interval $]0, 1]$, we have

$$|f(x_1) - f(x_2)| = |n - (n+1)| = 1 > \varepsilon, \text{ whenever } |x_1 - x_2| < \delta.$$

Hence, f is not uniformly continuous on $]0, 1]$.

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Note: The function $f(x) = 1/x$ is uniformly continuous on $[a, \infty[$ where $a > 0$.

Example 25. Prove that $f(x) = \sin x^2$ is not uniformly continuous on $[0, \infty[$.

- Let $\varepsilon = \frac{1}{2}$ and δ be any positive number such that for $n > \pi/\delta^2$

$$\left|\sqrt{\frac{n\pi}{2}} - \sqrt{\frac{(n+1)\pi}{2}}\right| < \delta$$

Therefore, taking $x_1 = \sqrt{\frac{n\pi}{2}}$ and $x_2 = \sqrt{\frac{(n+1)\pi}{2}}$, as any two points of the interval $[0, \infty[$

$$|f(x_1) - f(x_2)| = \left|\sin\frac{n\pi}{2} - \sin\frac{(n+1)\pi}{2}\right| = 1 > \varepsilon,$$

$$|x_1 - x_2| < \delta$$

Hence $f(x) = \sin x^2$ is not uniformly continuous on $[0, \infty[$.

Example 26. Prove that

$$f(x) = \sin\frac{1}{x}, x \neq 0$$
$$= 0, \quad x = 0$$

is not uniformly continuous on $[0, \infty[$.

- Let $\varepsilon = \frac{1}{2}$ and $\delta > 0$ be such that $\frac{1}{n(2n\pi + \pi)} < \delta$, $\forall\, n \geq m$. Taking $x = \frac{2}{2m\pi}$ and $y = \frac{2}{(2m+1)\pi}$ be any two points of $[0, \infty[$, then

$$|x - y| = \left|\frac{2}{2m\pi} - \frac{2}{(2m+1)\pi}\right| = \frac{1}{m(2m+1)\pi} < \delta$$

does not imply

$$|f(x) - f(y)| = \left|\sin m\pi - \sin\left(m\pi + \frac{\pi}{2}\right)\right| = 1 < \varepsilon$$

Hence, $\sin\frac{1}{x}$ is not uniformly continuous on $]0, \infty[$.

Example 27. Show that the function $f(x) = x^2$ is uniformly continuous on $[-1, 1]$.

- Let x_1, x_2 be any two points of $[-1, 1]$, then

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$$|f(x_1) - f(x_2)| = |x_1^2 - x_2^2| = |x_1 - x_2| \cdot |x_1 + x_2|$$

$$< \varepsilon, \text{ when } |x_1 - x_2| < \frac{1}{2}\varepsilon = \delta$$

(where δ is independent of the choice of x_1, x_2).

Thus, for any $\varepsilon > 0, \exists$ a $\delta = \frac{1}{2}\varepsilon$ such that for any choice of x_1, x_2 in [–1, 1], we have

$$|f(x_1) - f(x_2)| < \varepsilon, \text{ when } |x_1 - x_2| < \frac{1}{2}\varepsilon = \delta$$

Thus, the function f is uniformly continuous on [–1, 1].

Example 28. Prove that $\sin x$ is uniformly continuous on $[0, \infty[$.

■ Let $\varepsilon > 0$ be given. Let $x, y \in [0, \infty[$, then

$$|\sin x - \sin y| = \left|2\sin\left(\frac{x-y}{2}\right)\cos\left(\frac{x+y}{2}\right)\right| \le 2\frac{|x-y|}{2}$$

$$= |x - y| < \varepsilon, \text{ whenever } |x - y| < \varepsilon$$

Therefore, taking $\delta = \varepsilon$

$$|\sin x - \sin y| < \varepsilon, \text{ whenever } |x - y| < \delta$$

Hence, $\sin x$ is uniformly continuous on $[0, \infty[$.

EXERCISE

1. Show that the following functions are uniformly continuous in the given interval:
   (i) $f(x) = x^2$ in [1, 2]
   (ii) $f(x) = x^3$ in [0, 1]
   (iii) $f(x) = \sqrt{x}$ in $[0, 2]$
   (iv) $f(x) = x/(1 + x^2)$ on **R**.
2. Show that the function $f(x) = x^2$ is not uniformly continuous on $[0, \infty[$.
3. Show that the function $f(x) = 1/x^2$ is uniformly continuous on $[a, \infty[$, where $a > 0$; but not uniformly continuous on $]0, \infty[$.
4. Determine which of the following functions are uniformly continuous on the indicated intervals:
   (i) $f(x) = x^3$ on $[0, \infty[$
   (ii) $f(x) = \tan^{-1} x$ on **R**
   (iii) $f(x) = \dfrac{\sin x}{x}$ on $]0, \infty[$
   (iv) $f(x) = \begin{cases} \sin \pi x & \text{for } x \in]0, 1] \\ x^2 - 1 & \text{for } x \in]1, 2[\end{cases}$

5. If f and g are uniformly continuous on the same interval, prove that $f+g$ and $f-g$ are also uniformly continuous on the interval.
6. Prove that if f and g are each uniformly continuous on the bounded open interval $]a, b[$, then the product fg is uniformly continuous on $]a, b[$.
7. Prove that if f and g are each uniformly continuous on the interval I and if in addition each function is bounded on I, then the product fg is uniformly continuous on I. Is boundedness of each function on I necessary for the uniform continuity of the product?

   [*Hint*: Boundedness of each function on I is not necessary; consider $f(x) = g(x) = \sqrt{x}$ on $[0, \infty[$].
8. Show by an example that a continuous bounded function on the bounded open interval $]a, b[$ need not be uniformly continuous on $]a, b[$.
9. Prove or give a counter example:

   If $f(x)$ is continuous and bounded on **R**, then f is uniformly continuous on **R**.

   [*Hint*: False, $f(x) = \sin x^2,\ x \in \mathbf{R}$]
10. If f is continuous on bounded open interval $]a, b[$, then prove that f is uniformly continuous on $]a, b[$ iff $f(a+)$ and $f(b-)$ both exist.
11. If f is continuous on $[a, \infty[$ (or $]-\infty, b]$) and $\lim_{x\to\infty} f(x)$ $\left(\text{or } \lim_{x\to-\infty} f(x)\right)$ exists, then prove that f is uniformly continuous on $[a, \infty[$ (or $]-\infty, b]$). Is the converse true?
12. Prove that, if f is continuous on **R**, then f is uniformly continuous on every bounded interval of **R**. Is f then uniformly continuous on **R**?
13. If f is continuous on **R** and $\lim_{x\to\infty} f(x)$ and $\lim_{x\to-\infty} f(x)$ both exist, then prove that f is uniformly continuous on **R**.

ANSWERS

4. (*i*) No (*ii*) Yes (*iii*) Yes (*iv*) Yes.