Mains Test Series – 2020

COMMON TEST – [TEST-13 for Batch-I] & [TEST-5 for Batch-II]

full Syllabus (Paper-I)

1(a) Let W_1 and W_2 be subspaces of a vector space V such that the set-theoretic union of W_1 and W_2 is also a subspace. Prove that one of the spaces W_1 is contained in the other.

Sol'n: Let $W_1 \cup W_2$ be a subspace of V(F).

then Prove that $W_1 \subset W_2$ (or) $W_2 \subset W_1$

If possible suppose that $W_1 \not\subset W_2$ and $W_2 \not\subset W_1$

If $W_1 \not\subset W_2$

let $\alpha \in W_1$ then $\alpha \notin W_2$

if $W_2 \subset W_1$

let $\beta \in W_2$ then $\beta \notin W_1$

Now $\alpha \in W_1$, $\beta \in W_2$

$\Rightarrow \alpha, \beta \in W_1 \cup W_2$

$\Rightarrow \alpha+\beta \in W_1 \cup W_2$ ($\because W_1 \cup W_2$ is a subspace)

$\Rightarrow \alpha+\beta \in W_1$

Now $\alpha+\beta \in W_1$, $\alpha \in W_1$

$\Rightarrow (\alpha+\beta) - \alpha \in W_1$ ($\because W_1$ is a subspace)

$\Rightarrow \beta \in W_1$

which is contradiction to $\beta \notin W_1$

and $\alpha+\beta \in W_2$, $\beta \in W_2$

$\Rightarrow (\alpha+\beta) - \beta \in W_2$ ($\because W_2$ is a subspace)

$\Rightarrow \alpha \in W_2$

which contradiction to $\alpha \notin W_2$.

$\therefore$ our assumption that $W_1 \not\subset W_2$ and $W_2 \not\subset W_1$ is wrong.

$\therefore W_1 \subset W_2$ and $W_2 \subset W_1$

1(b). Let $T: M_{22} \to M_{22}$ be defined by $T(A) = A - A^T$. Give M_{22} the standard basis

$$S = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\} = \{e_1, e_2, e_3, e_4\}$$

and find the matrix for T with respect to S.

Sol'n: First we calculate the images of the basis vectors.

$$T\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0e_1 + 0e_2 + 0e_3 + 0e_4$$

$$T\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = 0e_1 + e_2 - e_3 + 0e_4$$

$$T\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 0e_1 - e_2 + e_3 + 0e_4$$

$$T\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0e_1 + 0e_2 + 0e_3 + 0e_4$$

$$\therefore \quad M = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

1(c) If $u = At^{-1/2} e^{-x^2/4a^2 t}$, Prove that $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$.

Sol^n: we have $u = At^{-1/2} e^{-x^2/4a^2 t}$

$$\Rightarrow \frac{\partial u}{\partial x} = At^{-1/2} \cdot e^{-x^2/4a^2 t} \left(\frac{-2x}{4a^2 t}\right)$$

$$= \frac{-x}{2a^2 t} u$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = -\frac{1}{2a^2 t}\left[u + x\frac{\partial u}{\partial x}\right]$$

$$= -\frac{1}{2a^2 t}\left[u + x\left(\frac{-ux}{2a^2 t}\right)\right]$$

$$= \frac{u}{4a^4 t^2}\left(-2a^2 t + x^2\right)$$

Again,

$$\frac{\partial u}{\partial t} = At^{-1/2} \cdot e^{-x^2/4a^2 t}\left(\frac{x^2}{4a^2 t^2}\right) - A\frac{1}{2}t^{-3/2} e^{-x^2/4a^2 t}$$

$$= At^{-1/2} \cdot e^{-x^2/4a^2 t}\left[\frac{x^2}{4a^2 t^2} - \frac{1}{2t}\right]$$

$$= \frac{u}{4a^2 t}\left(x^2 - 2a^2 t\right)$$

Clearly $\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2}$.

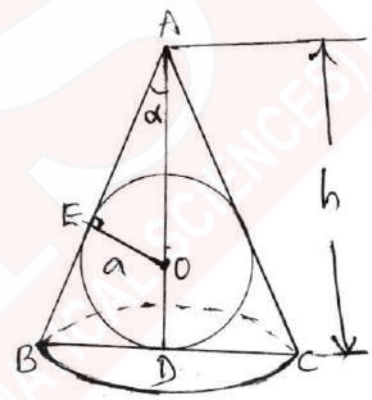
1(d) Find the altitude and the semi-vertical angle of a cone of least volume which can be circumscribed to a sphere of radius a.

Sol'n: Let h be the height and α the semi-vertical angle of the cone so that its radius $BD = h\tan\alpha$.

$\therefore$ The volume V of the cone is given by

$$V = \frac{1}{3}\pi(h\tan\alpha)^2 h = \frac{1}{3}\pi h^3\tan^2\alpha$$

Now we must express $\tan\alpha$ interms of h.



In the right angle ΔAEO

$$EA = \sqrt{(OA^2 - a^2)}$$
$$= \sqrt{[(h-a)^2 - a^2]}$$
$$= \sqrt{(h^2 - 2ha)}$$

$$\therefore \tan\alpha = \frac{EO}{EA} = \frac{a}{\sqrt{(h^2-2ha)}}$$

Thus $V = \frac{1}{3}\pi h^3 \cdot \frac{a^2}{h^2-2ha} = \frac{1}{3}\pi a^2 \cdot \frac{h^2}{h-2a}$

$$\therefore \frac{dV}{dh} = \frac{1}{3}\pi a^2 \frac{(h-2a)2h - h^2 \cdot 1}{(h-2a)^2} = \frac{1}{3}\pi a^2 \frac{h(h-4a)}{(h-2a)^2}.$$

Thus $\frac{dV}{dh} = 0$ for $h = 4a$, the other value $(h=0)$ being not possible.

Also $\frac{dV}{dh}$ is $-$ve when h is slightly $< 4a$, and it is +ve when h is slightly $> 4a$. Hence V is minimum (i.e least) when $h = 4a$ and $\alpha = \sin^{-1}\left(\frac{a}{OA}\right) = \sin^{-1}\left(\frac{a}{3a}\right) = \sin^{-1}\left(\frac{1}{3}\right)$.

1(e) Prove that the circles $x^2+y^2+z^2-2x+3y+4z-5=0$, $5y+6z+1=0$ and $x^2+y^2+z^2-3x-4y+5z-6=0$, $x+2y-7z=0$ lies on the same sphere and find its equation. Also find the value of a for which $x+y+z=a\sqrt{3}$ touches the sphere.

Sol^n: The equation of any sphere through the first circle is

$$(x^2+y^2+z^2-2x+3y+4z-5)+\lambda(5y+6z+1)=0 \quad \text{---} (1)$$

Similarly the equation of any sphere through the second circle is

$$(x^2+y^2+z^2-3x-4y+5z-6)+\mu(x+2y-7z)=0 \quad \text{---} (2)$$

If the given circles lies on the same sphere then (1) & (2) should represent the same sphere, so comparing the coefficients of x, y, z and constant terms in (1) & (2) we get

$$-2=-3+\mu \quad \text{---} (3); \qquad 3+5\lambda=-4+2\mu \quad \text{---} (4)$$

$$4+6\lambda=5-7\mu \quad \text{---} (5); \qquad -5+\lambda=-6 \quad \text{---} (6)$$

from (3) & (6) we get $\mu=1$, $\lambda=-1$

These values of λ & μ satisfy (4) and (5), hence the given circles lie on the same sphere. putting $\lambda=-1$ in (1) we get the required equation of the sphere as

$$x^2+y^2+z^2-2x-2y-2z-6=0 \quad \text{---} (7)$$

Of the centre in $(1,1,1)$ & radius $=\sqrt{(1^2+1^2+1^2+6)}=3$.

If the plane $x+y+z=a\sqrt{3}$ ——— (8).

touches the sphere (7), then the length of the $\perp$lar from the centre $(1,1,1)$ of the sphere to the

Plane (8) must be equal to the radius of the sphere.

i.e. $\dfrac{1+1+1-a\sqrt{3}}{\sqrt{1^2+1^2+1^2}} = 3$

$\Rightarrow 3 - a\sqrt{3} = \pm 3\sqrt{3}$

$\Rightarrow a\sqrt{3} = 3 \pm 3\sqrt{3}$

$\Rightarrow a = \sqrt{3} + 3.$

2(a) (i) Let V be the space of 2×2 matrices over F. Find basis $\{A_1, A_2, A_3, A_4\}$ for V such that $A_i^2 = A_i$ for each i.

(ii) Let V be a vector space over a subfield F of the complex numbers. Suppose α, β and γ are linearly independent vectors in V. Prove that $(\alpha+\beta), (\beta+\gamma)$, and $(\gamma+\alpha)$ are linearly independent.

Sol'n: (i) $V = \left\{\begin{bmatrix} a & b \\ c & d \end{bmatrix} / a, b, c, d \in F\right\}$

Let $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, $A_4 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

be any four elements of **V** such that $A_i^2 = A_i$ for each i.

Let $S = \{A_1, A_2, A_3, A_4\} \subseteq V$

(i) To show S LI.

(ii) To show $L(S) = V$.

(ii) Sol'n: Let $a, b, c \in F$ then

$a(\alpha+\beta) + b(\beta+\gamma) + c(\gamma+\alpha) = 0$

$\Rightarrow (a+c)\alpha + (a+b)\beta + (b+c)\gamma = 0$

Since α, β, γ are LI.

$\therefore a + c = 0$ — (1)

$a + b = 0$ — (2)

$b + c = 0$ — (3)

(1) − (3) $\equiv a - b = 0$ — (4)

(2) + (4) $\equiv 2a = 0 \Rightarrow \boxed{a=0}$

(4) $\equiv \boxed{b=0}$ and (3) $\equiv \boxed{c=0}$

$\therefore \alpha+\beta, \beta+\gamma, \gamma+\alpha$ are LI.

2(b) → Show that the function $f: R^2 \to R$ defined by setting

$$f(x,y) = \begin{cases} x\sin\frac{1}{x} + y\sin\frac{1}{y}, & \text{when } xy \neq 0 \\ x\sin\frac{1}{x}, & \text{when } x \neq 0, y = 0 \\ y\sin\frac{1}{y}, & \text{when } x = 0, y \neq 0 \\ 0, & \text{when } x = y = 0 \end{cases}$$

continuous but not differentiable at $(0,0)$.

Sol^n: we have

$$|f(x,y) - f(0,0)| = \left| x\sin\frac{1}{x} + y\sin\frac{1}{y} - 0 \right|$$
$$\leq |x|\,\left|\sin\frac{1}{x}\right| + |y|\,\left|\sin\frac{1}{y}\right|$$
$$\leq |x| + |y| \qquad (\because |\sin\theta| \leq 1)$$

Let $\epsilon > 0$ be given, choose $\delta = \epsilon/2$

Then $|f(x,y) - f(0,0)| < \epsilon$ if $|x| < \delta, |y| < \delta$.

Hence the given function is continuous at $(0,0)$.

Now $f_x(0,0) = \lim\limits_{h \to 0} \dfrac{f(0+h,0) - f(0,0)}{h}$

$$= \lim_{h \to 0} \frac{h\sin\frac{1}{h} - 0}{h}$$

$$= \lim_{h \to 0} \sin\frac{1}{h}$$

which does not exist.

Similarly $f_y(0,0)$ doesnot exist.

$\therefore$ f is not differentiable. [$\because$ If either of f_x, f_y does not exist at (a,b) then f is not differentiable at (a,b)].

2(C) (i) The plane $x-2y+3z=0$ is rotated through a right angle about its line of intersection with the plane $2x+3y-4z-5=0$, find the equation of the plane in its new position.

Sol'n: The equation of any plane through the line of intersection of the plane $x-2y+3z=0$ and $2x+3y-4z-5=0$ is

$$(x-2y+3z)+\lambda(2x+3y-4z-5)=0$$

$$\Rightarrow (1+2\lambda)x+(3\lambda-2)y+(3-4\lambda)z-5\lambda=0 \quad \text{—} \; (1)$$

It is given that the angle between the plane $x-2y+3z=0$ and (1) is a right angle, so the angle between their normals is a right angle. Also the d.r's of their normals are $1,-2,3$ and $1+2\lambda,\ 3\lambda-2,\ 3-4\lambda$.

$$\therefore 1(1+2\lambda)-2(3\lambda-2)+3(3-4\lambda)=0$$

$$\Rightarrow 1+2\lambda-6\lambda+4+9-12\lambda=0$$

$$\Rightarrow 16\lambda=14 \Rightarrow \lambda=7/8.$$

$\therefore$ from (1) the required equation is

$$[1+2(7/8)]x+[3(7/8)-2]y+[3-4(7/8)]z-5(7/8)=0$$

$$\Rightarrow 22x+5y-4z-35=0.$$

2(c)(ii) Find the S.D between lines.

$$\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1} \text{ and } \frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}$$

Find also its equations and the points in which it meets the given lines.

sol^n: The given lines are $\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}$ —— ①

& $\frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}$ —— ②

Any point on the line ① is $(3-3\gamma, 8-\gamma, 3+\gamma)$, say point P.

Any point on the line ② is $(-3-3\gamma', -7+2\gamma', 6+4\gamma')$ say point Q.

Then the direction ratio's of the line PQ are

$(3+3\gamma)-(-3-3\gamma'),\ (8-\gamma)-(-7+2\gamma'),\ (3+\gamma)-(6+4\gamma')$

$$3\gamma+3\gamma'+6,\ -\gamma-2\gamma'+15,\ \gamma-4\gamma'-3 \quad \text{—— ③}$$

Now if PQ is the S.D b/w the given lines then PQ is ⊥lar to both ① and ②, the conditions for the same are

$$3(3\gamma+3\gamma'+6)-1(-\gamma-2\gamma'+15)+1(\gamma-4\gamma'-3)=0$$

and $-3(3\gamma+3\gamma'+6)+2(-\gamma-2\gamma'+15)+4(\gamma-4\gamma'-3)=0$

$\Rightarrow 11\gamma+7\gamma'=0$ and $7\gamma+29\gamma'=0$

solving these we find $\gamma=0$ and $\gamma'=0$

Substituting these values of γ and γ', we find that the coordinates of P & Q are $(3,8,3)$ & $(-3,-7,6)$ respectively.

And d.r's of the line PQ from ③ are $6,15,-3$ (or) $2,5,-1$

Now the required S.D $= PQ = \sqrt{\{3-(-3)\}^2+\{8-(-7)\}^2+(3-6)^2}$

$$=\sqrt{36+225+9}=3\sqrt{30}.$$

Also PQ is a line through P(3,8,3) and of direction ratios 2,5,−1 so its equations are $\frac{x-3}{2}=\frac{y-8}{5}=\frac{z-3}{-1}$.

3(a) → We consider the 5×5 matrix $A = \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 2 & 4 & 1 & 10 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

and the following problems concerning A

(a) Find an investible matrix P such that PA is a row-reduced echelon matrix R.

(b) Find a basis for the row space W of A.

(c) Say which vectors $\{b_1, b_2, b_3, b_4, b_5\}$ are in W.

(d) Find the coordinate matrix of each vector $(b_1, b_2, b_3, b_4, b_5)$ in W in the ordered basis chosen in (b).

(e) Write each vector $(b_1, b_2, b_3, b_4, b_5)$ in W as a linear combination of the rows of A.

(f) Give an explicit description of the vectorspace V of all 5×1 column matrices X such that $AX = 0$.

Sol^n: To solve these problems we form the augmented matrix A' of the system $AX = Y$ and apply an appropriate sequence of row operations to A'.

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 0 & y_1 \\ 1 & 2 & -1 & -1 & 0 & y_2 \\ 0 & 0 & 1 & 4 & 0 & y_3 \\ 2 & 4 & 1 & 10 & 1 & y_4 \\ 0 & 0 & 0 & 0 & 1 & y_5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 3 & 0 & y_1 \\ 0 & 0 & -1 & -4 & 0 & -y_1+y_2 \\ 0 & 0 & 1 & 4 & 0 & y_3 \\ 0 & 0 & 1 & 4 & 1 & -2y_1+y_4 \\ 0 & 0 & 0 & 0 & 1 & y_5 \end{bmatrix} \begin{array}{l} R_2 \to R_2 - R_1 \\ R_4 \to R_4 - 2R_1 \end{array}$$

$R_2 \to -R_2$
$R_3 \to R_2 + R_3,\ R_4 \to R_4 + R_2$ $\qquad$ $R_3 \leftrightarrow R_5$

$$\sim \begin{bmatrix} 1 & 2 & 0 & 3 & 0 & y_1 \\ 0 & 0 & 1 & 4 & 0 & y_1-y_2 \\ 0 & 0 & 0 & 0 & 0 & -y_1+y_2+y_3 \\ 0 & 0 & 0 & 0 & 1 & -3y_1+y_2+y_4 \\ 0 & 0 & 0 & 0 & 1 & y_5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 3 & 0 & y_1 \\ 0 & 0 & 1 & 4 & 0 & y_1-y_2 \\ 0 & 0 & 0 & 0 & 1 & y_5 \\ 0 & 0 & 0 & 0 & 0 & -y_1+y_2+y_3 \\ 0 & 0 & 0 & 0 & 0 & -3y_1+y_2+y_4-y_5 \end{bmatrix}$$

(i) If

$$PY = \begin{bmatrix} y_1 \\ y_1 - y_2 \\ y_5 \\ -y_1 + y_2 + y_3 \\ -3y_1 + y_2 + y_4 - y_5 \end{bmatrix}$$

for all Y, then

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & 1 & 0 & 0 \\ -3 & 1 & 0 & 1 & -1 \end{bmatrix}$$

hence PA is the row reduced echelon matrix.

$$R = \begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

It should be stressed that the matrix P is not unique. There are, in fact many invertible matrices P such that $PA = R$.

(ii) As a basis for W we may take the non-zero rows

$$\rho_1 = (1\ 2\ 0\ 3\ 0)$$

$$\rho_2 = (0\ 0\ 1\ 4\ 0)$$

$$\rho_3 = (0\ 0\ 0\ 0\ 1) \quad \text{of } R.$$

(iii) The row-space W consists of all vectors of the form

$$\beta = c_1\rho_1 + c_2\rho_2 + c_3\rho_3$$

$$= (c_1, 2c_1, c_2, 3c_1 + 4c_2, c_3)$$

where c_1, c_2, c_3 are arbitrary scalars. Thus $(b_1, b_2, b_3, b_4, b_5)$ is in W if and only if

$$(b_1, b_2, b_3, b_4, b_5) = b_1\rho_1 + b_3\rho_2 + b_5\rho_3$$

which is true if and only if

$$b_2 = 2b_1$$
$$b_4 = 3b_1 + 4b_3$$

these equations are instances of the general system, and using them we may tell at a glance whether a given vector lies in W. Thus $(-5, -10, 1, -11, 20)$ is a linear combination of the rows of A, but $(1, 2, 3, 4, 5)$ is not.

(iv) The coordinate matrix of the vector $(b_1, 2b_1, b_3, 3b_1+4b_3, b_5)$ in the basis $\{\rho_1, \rho_2, \rho_3\}$ is evidently.

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}.$$

(v) There are many ways to write the vectors in W as linear combinations of the rows of A.

$$\beta = (b_1, 2b_1, b_3, 3b_1+4b_3, b_5)$$
$$= [b_1, b_3, b_5, 0, 0]\cdot R$$
$$= [b_1, b_3, b_5, 0, 0]\cdot PA$$
$$= [b_1, b_3, b_5, 0, 0]\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ -1 & 1 & 1 & 0 & 0 \\ -3 & 1 & 0 & 1 & -1 \end{bmatrix}\cdot A$$
$$= [b_1+b_3, -b_3, 0, 0, b_5]\cdot A$$

In particular, with $\beta = (-5, -10, 1, -11, 20)$ we have

$$\beta = (-4, -1, 0, 0, 20)\begin{bmatrix} 1 & 2 & 0 & 3 & 0 \\ 1 & 2 & -1 & -1 & 0 \\ 0 & 0 & 1 & 4 & 0 \\ 2 & 4 & 1 & 10 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

(vi) The equations in the system $RX = 0$ are

$$x_1 + 2x_2 + 3x_4 = 0$$
$$x_3 + 4x_4 = 0$$
$$x_5 = 0$$

Thus V consists of all columns of the form

$$X = \begin{bmatrix} -2x_2 - 3x_4 \\ x_2 \\ -4x_4 \\ x_4 \\ 0 \end{bmatrix}$$

where x_2 and x_4 are arbitrary.

(vii) The columns

$$\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{bmatrix} -3 \\ 0 \\ -4 \\ 1 \\ 0 \end{bmatrix}$$

form a basis of V.

(viii) The equation $Ax = y$ has solutions x if and only if

$$-y_1 + y_2 + y_3 = 0$$
$$-3y_1 + y_2 + y_4 - y_5 = 0$$

3(b)(i) Test for convergence of the integral

$$\int_0^\infty \frac{x\tan^{-1}x}{(1+x^4)^{1/3}}\,dx.$$

Sol'n: Let $f(x) = \dfrac{x\tan^{-1}x}{(1+x^4)^{1/3}}$

$$= \frac{\tan^{-1}x}{x^{1/3}(1-x^{-4})^{1/3}} \quad (\sim x^{-1/3} \text{ at } \infty)$$

and $g(x) = \dfrac{1}{x^{1/3}}$

So that $\dfrac{f(x)}{g(x)} = \dfrac{\tan^{-1}x}{(1+x^{-4})^{1/3}} \to \dfrac{\pi}{2}$ as $x \to \infty$

Hence $\int_1^\infty f\,dx$ and $\int_1^\infty g\,dx$ behave alike.

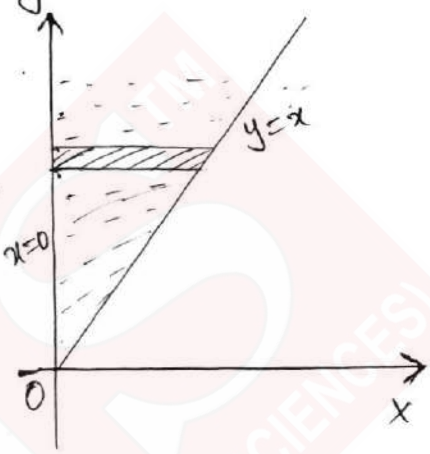
Since $\int_1^\infty \dfrac{dx}{x^{1/3}}$ diverges, therefore $\int_1^\infty \dfrac{x\tan^{-1}x}{(1+x^4)^{1/3}}dx$ also diverges.

3(b)(ii) Let $E = \{(x, y) \in \mathbb{R}^2 \,/\, 0 < x < y\}$. Then evaluate $\iint\limits_E y e^{-(x+y)}\,dx\,dy$

Sol^n: Let $I = \iint\limits_E y e^{-(x+y)}\,dx\,dy$

where $E = \{(x, y) \in \mathbb{R}^2 \,/\, 0 < x < y\}$

$$= \int_{y=0}^{\infty} \int_{x=0}^{y} y e^{-(x+y)}\,dx\,dy$$



$$= \int_{y=0}^{\infty} y \left(\frac{e^{-x}}{-1}\right)_0^y e^{-y}\,dy$$

$$= \int_{y=0}^{\infty} y e^{-y} \left(e^{-x}\right)_y^0\,dy$$

$$= \int_0^{\infty} y e^{-y}\left(1 - e^{-y}\right)dy$$

$$= \int_0^{\infty} \left(y e^{-y} - y e^{-2y}\right)dy$$

$$= \int_0^{\infty} e^{-y} y^{2-1}\,dy - \int_0^{\infty} e^{-2y} y^{2-1}\,dy$$

$$\left(\because \int_0^{\infty} e^{-ax} x^{n-1}\,dx = \frac{\Gamma n}{a^n}\right)$$

$$= \frac{\Gamma 2}{1^2} - \frac{\Gamma 2}{2^2}$$

$$= 1 - \tfrac{1}{4} \qquad \left(\because \Gamma(n+1) = n!\right)$$

$$= \tfrac{3}{4}$$

3(C) If the feet of the three normals from P to the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ lie on the plane $x/a+y/b+z/c=1$ prove that the feet of the other three lie on the plane $x/a+y/b+z/c+1=0$ and P lies on the line

$$a(b^2-c^2)x=b(c^2-a^2)y=c(a^2-b^2)z.$$

Sol: Let (x_1,y_1,z_1) be the given point, then the feet of the six normals from it to the given ellipsoid are given by

$$\alpha=\frac{a^2x_1}{a^2+\lambda},\quad \beta=\frac{b^2y_1}{b^2+\lambda},\quad \gamma=\frac{c^2z_1}{c^2+\lambda} \quad \text{--- (1)}$$

and the six values of λ are given by the equation

$$\frac{a^2x_1^2}{(a^2+\lambda)^2}+\frac{b^2y_1^2}{(b^2+\lambda)^2}+\frac{c^2z_1^2}{(c^2+\lambda)^2}-1=0 \quad \text{--- (2)}$$

Now the equation of the plane PQR is given as $lx+my+nz=p$ and then three of the six feet of the normals lie on it, so we have

$$\frac{la^2x_1}{a^2+\lambda}+\frac{mb^2y_1}{b^2+\lambda}+\frac{nc^2z_1}{c^2+\lambda}-p=0 \quad \text{--- (3)}$$

substituting the values from (1) in the equation of the plane PQR.

Let the equation of the plane $P'Q'R'$ be

$$l'x + m'y + n'z = p' \quad \text{—— (4)}$$

Since the remaining three feet of the normal lie on it so we have

$$\frac{l'a^2x_1}{a^2+\lambda} + \frac{m'b^2y_1}{b^2+\lambda} + \frac{n'c^2z_1}{c^2+\lambda} - p' = 0 \quad \text{—— (5)}$$

Hence we find that equation (2) is the product of (3) and (5), so comparing them we have

$$\frac{ll'a^4x_1^2}{(a^2+\lambda)^2} = \frac{a^2x_1^2}{(a^2+\lambda)^2}$$

or $ll'a^2 = 1$

or $l' = \frac{1}{a^2 l}$.

similarly

$$m' = \frac{1}{b^2m}, \quad n' = \frac{1}{c^2n} \text{ and } p' = -\frac{1}{p}$$

Here $l = 1/a$, $m = 1/b$, $n = 1/c$, $p = 1$

$\therefore l' = 1/a^2 l = \frac{1}{a}$, $m' = 1/b$, $n' = 1/c$,

$p' = -1/p = -1$.

The six feet of the normals, therefore lie on

$$\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}-1\right)\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}+1\right)=0$$

or, $$\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)^2-1=0$$

or $$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)+2\left(\frac{yz}{bc}+\frac{zx}{ac}+\frac{xy}{ab}\right)=1$$

or $$\frac{yz}{bc}+\frac{zx}{ac}+\frac{xy}{ab}=0$$ $\because$ the feet also lie on $\Sigma(x^2/a^2)=1$

Hence the feet of the normal lie on

$$\frac{yz}{bc}+\frac{zx}{ca}+\frac{xy}{ab}=0 \quad \text{——— (6)}$$

we know that if the normals be drawn from the point $P(x_1, y_1, z_1)$ then the feet of the normals lie on the cone

$$\frac{a^2x_1(b^2-c^2)}{x}+\frac{b^2y_1(c^2-a^2)}{y}+\frac{c^2z_1(a^2-b^2)}{z}=0$$

or $$a^2x_1(b^2-c^2)yz+b^2y_1(c^2-a^2)zx+c^2z_1(a^2-b^2)xy=0 \quad \text{——— (7)}$$

comparing (6) and (7), we get

$$\frac{a^2(b^2-c^2)x_1}{1/(bc)}=\frac{b^2(c^2-a^2)y_1}{1/(ca)}=\frac{c^2(a^2-b^2)z_1}{1/(ab)}$$

or $\quad a(b^2-c^2)x_1 = b(c^2-a^2)y_1 = c(a^2-b^2)z_1$

$\therefore$ The required locus of $P(x_1, y_1, z_1)$ is the line

$$a(b^2-c^2)x = b(c^2-a^2)y = c(a^2-b^2)z$$

Hence proved.

4(a)iii) Let $T: P_1 \to P_2$ be defined by $T(a+bx) = ax + (b/2)x^2$. Give P_1 and P_2 the standard bases $S = \{1, x\}$ and $T = \{1, x, x^2\}$ respectively. Find the matrix of T with respect to these bases. Do the same for $L: P_2 \to P_1$ defined by $L(a+bx+cx^2) = b + 2cx$.

Sol'n: Now $T(1) = x$, so

$$(T(1))_T = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

Likewise, $T(x) = \frac{1}{2}x^2$, so

$$(T(x))_T = \begin{pmatrix} 0 \\ 0 \\ \frac{1}{2} \end{pmatrix}$$

Therefore the matrix M_T representing T is

$$M_T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

For the transformation L,

$$L(1) = 0, \quad L(x) = 1, \quad L(x^2) = 2x$$

Thus $(L(1))_S = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $(L(x))_S = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $(L(x^2))_S = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$

and the matrix M_L representing L is

$$M_L = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

Note that

$$M_L M_T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2$$

so that we could call M_L a left inverse of M_T. However,

$$M_T M_L = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \neq I_3$$

and M_T is not a left inverse of M_L. Note that T is just antidifferentiation with arbitrary constant set to zero. when we antidifferentiate and then differentiate, we get the original function back. This is reflected by $M_L M_T = I$.

4(b) The ellipsoid with equation $x^2+2y^2+z^2=4$ is heated so that its temperature at (x,y,z) is given by $T(x,y,z)=70+10(x+z)$. Find the hottest and coldest points on the ellipsoid.

Sol^n: Given that the ellipsoid $x^2+2y^2+z^2=4$ —— ①

is heated so that its temperature at a point (x,y,z) is given by

$$T(x,y,z)=70+10(x+z) \quad \text{—— ②}$$

we have to find the hottest and coldest points on the ellipsoid.

Now, by using Lagrange's multiplier method.

Consider the function

$$F=70+10(x+z)+\lambda(x^2+y^2+z^2-4)$$

$$dF=[10+2x\lambda]dx+4y\lambda dy+(10+2z\lambda)dz$$

For stationary values $F_x=F_y=F_z=0$.

$$\Rightarrow 10+2x\lambda=0 \quad \text{—— ③}$$

$$4y\lambda=0$$

$$10+2z\lambda=0 \quad \text{—— ④}$$

Now $10+2x\lambda=0 \Rightarrow \lambda\neq 0$

and so $0=4y\lambda \Rightarrow y=0$.

from ③ & ④, we have

$$-2x\lambda=-2z\lambda$$

$$\Rightarrow x=z$$

Substituting $x=z$ & $y=0$ in equation ①, we have

$$x^2 + 0 + x^2 = 4$$
$$2x^2 = 4$$
$$x = \pm\sqrt{2}$$

$\therefore$ the points are $(\sqrt{2}, 0, \sqrt{2})$ and $(-\sqrt{2}, 0, -\sqrt{2})$

At $(\sqrt{2}, 0, \sqrt{2})$;

$$T(x, y, z) = 70 + 10(2\sqrt{2})$$
$$= 70 + 20\sqrt{2}$$
$$= 70 + 28.28 = 98.28 \approx 98$$

At $(-\sqrt{2}, 0, -\sqrt{2}) = 70 - 20\sqrt{2}$
$$= 70 - 28.28$$
$$= 41.72$$
$$\approx 42$$

Thus the hottest point on the ellipsoid is $(\sqrt{2}, 0, \sqrt{2})$ and the coldest point on the ellipsoid is $(-\sqrt{2}, 0, -\sqrt{2})$.

4(c) Find the locus of the points from which three mutually perpendicular tangents can be drawn to the paraboloid

$$(x^2/a^2)-(y^2/b^2)=2z.$$

Sol: Here we are to apply the condition that the enveloping cone, of the given paraboloid, with vertex at (α,β,γ) may have three mutually perpendicular generators.

Now the equation of the enveloping cone of the given paraboloid with vertex at the point (α,β,γ) is

$$SS_1=T^2 \quad \text{——— (1)}$$

Here

$$S=(x^2/a^2)-(y^2/b^2)-2z,$$

$$S_1=(\alpha^2/a^2)-(\beta^2/b^2)-2\gamma,$$

and

$$T=(\alpha x/a^2)-(\beta y/b^2)-(z+\gamma)$$

$\therefore$ From (1), the equation of the enveloping cone of the given paraboloid with vertex at (α,β,γ) is

$$\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}-2z\right)\left(\frac{\alpha^2}{a^2}-\frac{\beta^2}{b^2}-2\gamma\right)$$

$$=\left(\frac{\alpha x}{a^2}-\frac{\beta y}{b^2}-z-\gamma\right)^2$$

Also we know that if this cone has three mutually perpendicular generators, then sum of coefficients of x^2, y^2 and z^2 in it must be zero.

$$\left[\frac{1}{a^2}\left(\frac{\alpha^2}{a^2}-\frac{\beta^2}{b^2}-2\gamma\right)-\frac{\alpha^2}{a^4}\right]+\left[-\frac{1}{b^2}\left(\frac{\alpha^2}{a^2}-\frac{\beta^2}{b^2}-2\gamma\right)-\frac{\beta^2}{b^4}\right]+[-1]=0$$

or $$-\frac{1}{a^2}\left(\frac{\beta^2}{b^2}+2\gamma\right)-\frac{1}{b^2}\left(\frac{\alpha^2}{a^2}-2\gamma\right)-1=0$$

or $$\alpha^2+\beta^2-2\gamma(a^2-b^2)+a^2b^2=0$$

$\therefore$ Required locus of the point (α, β, γ) is

$$x^2+y^2-2(a^2-b^2)z+a^2b^2=0$$

5(a) (i) Solve $dy/dx = (x+y-2)/(y-x-4)$

(ii) Solve $(2xy^4e^y + 2xy^3 + y)dx + (x^2y^4e^y - x^2y^2 - 3x)dy = 0$.

Sol^n: Let $x = X+h$ and $y = Y+k$

so that $\frac{dy}{dx} = \frac{dY}{dX}$ —(1).

Then given equation gives $\frac{dY}{dX} = \frac{X+Y+(h+k-2)}{Y-X+(k-h-4)}$ —(2)

Choose h, k such that $h+k-2$ and $k-h-4=0$ —(3)

Solving (3), $h=-1, k=3$. Then (1) gives $X = x+1$ & $Y = y-3$ —(4)

Using (3), (2) becomes $\frac{dY}{dX} = \frac{X+Y}{Y-X} = \frac{1+(Y/X)}{(Y/X)-1}$ —(5)

Let $Y/X = v$ i.e. $Y = vX$ so that

$$\frac{dY}{dX} = v + X\left(\frac{dv}{dX}\right) \text{ —(6)}$$

from (5) and (6), $v + X\frac{dv}{dX} = \frac{1+v}{v-1}$

$$\Rightarrow X\frac{dv}{dX} = \frac{1+2v-v^2}{v-1}$$

$$\Rightarrow \frac{(v-1)dv}{1+2v-v^2} = \frac{dX}{X}$$

$$\Rightarrow \frac{(2-2v)dv}{1+2v-v^2} = -2\frac{dX}{X}$$

Integrating $\log(1+2v-v^2) + 2\log X = \log C$

$$\Rightarrow X^2(1+2v-v^2) = C$$

$$\Rightarrow X^2\{1+2(Y/X) - (Y/X)^2\} = C$$

$$\Rightarrow X^2 + 2XY - Y^2 = C$$

$$\Rightarrow (x+1)^2 + 2(x+1)(y-3) - (y-3)^2 = C, \text{ using (3)}.$$

5(a)(ii) Sol^n: Given eq^n $(2xy^4e^y + 2xy^3 + y)\,dx + (x^2y^4e^y - x^2y^2 - 3x)\,dy = 0$ —— ①.

Comparing ① with $M\,dx + N\,dy = 0$, we get

$$M = 2xy^4e^y + 2xy^3 + y \quad \& \quad N = x^2y^4e^y - x^2y^2 - 3x \text{ —— ②}$$

Here $\frac{\partial M}{\partial y} = 8xy^3e^y + 2xy^4e^y + 6xy^2 + 1$ & $\frac{\partial N}{\partial x} = 2xy^4e^y - 2xy^2 - 3$.

$\therefore \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -4(2xy^3e^y + 2xy^2 + 1) = -\frac{4}{y}(2xy^4e^y + 2xy^3 + y) = \frac{-4M}{y}$

$\Rightarrow \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = -\frac{4}{y}$, which is a function of y alone.

$$\Rightarrow \text{I.F of ①} = e^{\int(-4/y)\,dy}$$

$$= e^{-4\log y} = 1/y^4$$

Multiplying ① by $1/y^4$, we have

$$\left\{2xe^y + 2x/y + 1/y^3\right\}dx + \left\{x^2e^y - x^2/y^2 - 3(x/y^4)\right\}dy = 0$$

whose solution as usual is

$$\int \left\{2xe^y + (2x/y) + (1/y^3)\right\}dx = c$$

(Treating y as constant)

$$\Rightarrow x^2e^y + (x^2/y) + (x/y^3) = c.$$

5(b) (i) Find $L\{F(t)\}$, where $F(t) = \begin{cases} \cos(t - \frac{2}{3}\pi), & t > \frac{2\pi}{3} \\ 0, & t < \frac{2\pi}{3} \end{cases}$

(ii) Find $L^{-1}\left\{\frac{(p+1)e^{-\pi p}}{p^2+p+1}\right\}$.

Sol^n: (i) Let $\phi(t) = \cos t$

$$\therefore F(t) = \begin{cases} \phi(t - 2\pi/3), & t > 2\pi/3 \\ 0, & t < \frac{2\pi}{3} \end{cases}$$

we have $L\{\phi(t)\} = L\{\cos t\} = \frac{p}{p^2+1} = f(p)$.

$\therefore$ from second shifting theorem

$$L\{F(t)\} = e^{(-2\pi/3)p} \cdot f(p) = e^{-2\pi p/3} \cdot \frac{p}{p^2+1}$$

(ii) we have

$$L^{-1}\left\{\frac{p+1}{p^2+p+1}\right\} = L^{-1}\left\{\frac{(p+\frac{1}{2})+\frac{1}{2}}{(p+\frac{1}{2})^2+\frac{3}{4}}\right\} = e^{-t/2}\, L^{-1}\left\{\frac{p+\frac{1}{2}}{p^2+3/4}\right\}$$

$$= e^{-t/2}\, L^{-1}\left\{\frac{p}{p^2+(\sqrt{3}/2)^2}\right\} + \frac{1}{2}e^{-t/2}\, L^{-1}\left\{\frac{1}{p^2+(\sqrt{3}/2)^2}\right\}$$

$$= e^{-t/2}\cos\left(\sqrt{3}t/2\right) + \frac{1}{2}e^{-t/2}\left(2/\sqrt{3}\right)\sin\left(\sqrt{3}t/2\right)$$

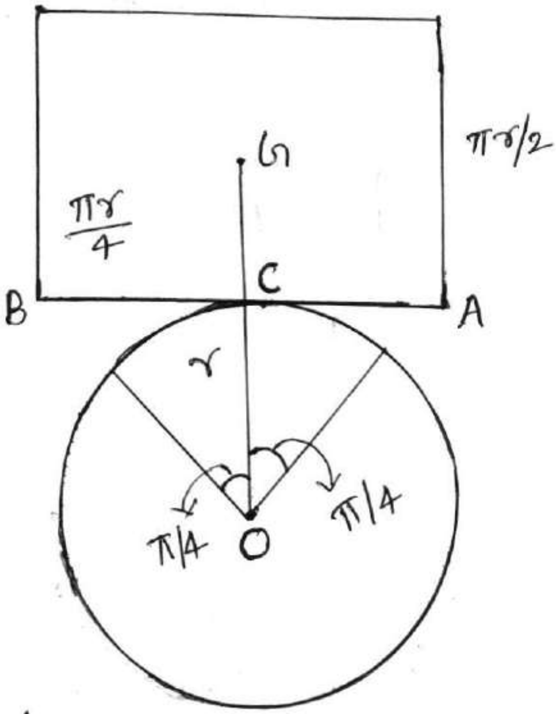
$$= \frac{e^{-t/2}}{\sqrt{3}}\left[\sqrt{3}\cos\left(\sqrt{3}t/2\right) + \sin\left(\sqrt{3}t/2\right)\right]$$

$$\therefore L^{-1}\left\{\frac{(p+1)e^{-\pi p}}{p^2+p+1}\right\} = \begin{cases} \frac{e^{-(t-\pi)/2}}{\sqrt{3}}\left[\sqrt{3}\cos\frac{\sqrt{3}}{2}(t-\pi) + \sin\frac{\sqrt{3}}{2}(t-\pi)\right], & t > \pi \\ 0, & t < \pi \end{cases}$$

$$= \frac{e^{-(t-\pi)/2}}{\sqrt{3}}\left[\sqrt{3}\cos\frac{\sqrt{3}}{2}(t-\pi) + \sin\frac{\sqrt{3}}{2}(t-\pi)\right] H(t-\pi).$$

5(c) A heavy uniform cube balances on the heighest point of a sphere whose radius is r. If the sphere is rough enough to prevent sliding and if the side of the cube be $\pi r/2$, show that the cube can rock through a right angle without falling.

Sol: A heavy uniform cube balances on the heighest point C of a sphere whose centre is O and radius r. The length of a side of the cube is $\pi r/2$. If G is the C.G. of the cube, then for equilibrium the line OCG must be vertical.



In the figure we have shown a cross section of the bodies by a vertical plane through the point of contact C.

First we shall show that the equilibrium of the cube is stable.

Here ρ_1 = the radius of curvature of the upper body at the point of contact $C = \infty$, and ρ_2 = the radius of curvature of the lower body at the point of contact $C = r$.

Also h = the height of the centre of gravity G of the upper body above the point of contact C = half the edge of the cube $= \pi r/4$.

The equilibrium will be stable if

$$\frac{1}{h} > \frac{1}{\rho_1} + \frac{1}{\rho_2} \quad \text{i.e;} \quad \frac{1}{\pi r/4} > \frac{1}{\infty} + \frac{1}{r}$$

$$\text{i.e,} \quad \frac{4}{\pi r} > \frac{1}{r} \quad \text{i.e,} \quad \frac{4}{\pi} > 1 \quad \text{i.e,} \quad 4 > \pi$$

which is so because the value of π lies between 3 and 4.

Hence the equilibrium is stable. so if the cube is slightly displaced, it will tends to come back to its original position of equilibrium. During a swing to the right, the cube will not fall down till the right hand corner A of the lowest edge comes in contact with the sphere.

If θ is the angle through which the cube turns when the right hand corner A of the lowest edge comes in contact with the sphere,

We have

$r\theta$ = half the edge of the cube $= \pi r/4$,
so that $\theta = \pi/4$.

Similarly the cube can turn through an angle $\pi/4$ to the left side on the sphere. Hence the total angle through which the cube can swing (or rock) without falling is

$$2 \cdot \frac{1}{4}\pi \quad \text{i.e.,} \quad \frac{1}{2}\pi$$

5(d) A point moves in a straight line so that its distance s from a fixed point at any time t is proportional to t^n. If v be the velocity and f the acceleration at any time t, show that
$v^2 = nfs/(n-1)$.

Sol: Here, distance $s \propto t^n$.

$$\therefore \text{ let } s = Kt^n, \quad \text{——— (1)}$$

where K is a constant of proportionality.

Differentiating (1), w.r.t. 't', we have

the velocity $v = ds/dt = Knt^{n-1}$ ——— (2)

Again differentiating (2),

the acceleration $f = dv/dt$

$$= Kn(n-1)t^{n-2} \quad \text{——— (3)}$$

$$\therefore v^2 = (Knt^{n-1})^2 = K^2n^2t^{2n-2}$$

$$= \frac{n \cdot \{Kn(n-1)t^{n-2}\} \cdot Kt^n}{(n-1)}$$

$$= \frac{nfs}{(n-1)},$$ Substituting from (1) and (3).

5(e) (i) Prove that $F = (y^2 \cos x + z^3)i + (2y\sin x - 4)j + (3xz^2+2)k$ is a conservative force field.

(ii) Find the scalar potential for F.

(iii) Find the workdone in moving an object in this field from $(0,1,-1)$ to $(\pi/2, -1, 2)$.

Sol'n: The field f will be conservative if $\nabla \times F = 0$

we have

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2\cos x + z^3 & 2y\sin x - 4 & 3xz^2+2 \end{vmatrix}$$

$$= i(0-0) + j(\cancel{3z^2 - 3z^2}) + k(2y\cos x - 2y\cos x) = 0.$$

$\therefore$ F is a conservative force field.

Let $\vec{F} = \nabla\phi$.

$$(y^2\cos x + z^3)\hat{i} + (2y\sin x - 4)\hat{j} + (3xz^2+2)\hat{k} = \frac{\partial\phi}{\partial x}i + \frac{\partial\phi}{\partial y}j + \frac{\partial\phi}{\partial z}k$$

Then

$$\frac{\partial\phi}{\partial x} = y^2\cos x + z^3 \Rightarrow \phi = y^2\sin x + z^3x + f_1(y,z) \quad (1)$$

$$\frac{\partial\phi}{\partial y} = 2y\sin x - 4 \Rightarrow \phi = y^2\sin x - 4y + f_2(x,z) \quad (2)$$

$$\frac{\partial\phi}{\partial z} = 3xz^2 + 2 \Rightarrow \phi = xz^3 + 2z + f_3(x,y) \quad (3)$$

These agree if (1), (2), (3) each represents ϕ. we choose

$f_1(y,z) = -4y + 2z$, $f_2(x,z) = xz^3$, $f_3(x,y) = y^2\sin x$

$\therefore \phi = y^2\sin x + xz^3 - 4y + 2z + C$ where C is a constant

$$\text{work done} = \int_{(0,1,-1)}^{(\pi/2,-1,2)} f\cdot dr = \int_{(0,1,-1)}^{(\pi/2,-1,2)} d\phi = [\phi]_{(0,1,-1)}^{(\pi/2,-1,2)}$$

$$= \left(y^2\sin x + xz^3 - 4y + 2z\right)_{0,1,-1}^{\pi/2,-1,2} = (1 + 4\pi + 8) - (-6)$$

$$= 4\pi + 15.$$

6(a) (i) Find the orthogonal trajectories of the family of curves $r = a(1+\cos\theta)$, where a is the parameter.

(ii) Solve: $p^3 - 4xyp + 8y^2 = 0$

(iii) Find the values of λ for which all solutions of $x^2\left(\frac{d^2y}{dx^2}\right) - 3x\left(\frac{dy}{dx}\right) - \lambda y = 0$ tend to zero $x \to \infty$.

Sol^n: The given family of curves is $r = a(1+\cos\theta)$ —— (1)

Taking logarithm of bothsides.

$$\log r = \log a + \log(1+\cos\theta) \quad \text{—— (2)}$$

Differentiating (2) w.r.t θ $\frac{1}{r}\frac{dr}{d\theta} = (-\sin\theta)(1+\cos\theta)$ —— (3)

which is the differential equation (1), Replacing $\frac{dr}{d\theta}$ by $-r^2\left(\frac{d\theta}{dr}\right)$ in (3), the differential equation of the required orthogonal trajectories is

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \frac{-\sin\theta}{1+\cos\theta} = -\frac{2\sin\theta/2\cos\theta/2}{1+2\cos^2\theta/2-1} = -\tan\theta/2$$

$$\Rightarrow r\left(\frac{d\theta}{dr}\right) = \tan\theta/2$$

$$\Rightarrow \left(\frac{1}{r}\right)dr = \cot(\theta/2)\,d\theta$$

Integrating, $\log r = 2\log\sin\theta/2 + \log c$

$$\Rightarrow r = c\sin^2\theta/2$$

$$\Rightarrow r = c\{(1-\cos\theta)/2\}$$

$$\Rightarrow r = b(1-\cos\theta), \text{ taking } b = c/2.$$

which is the equation of the required trajectories with b parameter.

6(a)(ii) Sol'n: Given $p^3 - 4xyp + 8y^2 = 0$, where $p = dy/dx$ —— ①

Solving ① for x, $x = \left(\frac{1}{4y}\right)p^2 + \left(\frac{1}{p}\right)(2y)$ —— ②

Differentiating ② w.r.t y and writing $1/p$ for $\frac{dx}{dy}$, we get

$$\frac{1}{p} = -\frac{p^2}{4y^2} + \frac{2p}{4y}\frac{dp}{dy} + \frac{2}{p} - \frac{2y}{p^2}\frac{dp}{dy}$$

$$\Rightarrow \frac{p^2}{4y} - \frac{1}{p} - \frac{dp}{dy}\left(\frac{p}{2y} - \frac{2y}{p^2}\right) = 0$$

$$\Rightarrow \left(\frac{p^2}{4y} - \frac{1}{p}\right) - \frac{2y}{p}\frac{dp}{dy}\left(\frac{p^2}{4y} - \frac{1}{p}\right) = 0$$

$$\Rightarrow \left(\frac{p^2}{4y} - \frac{1}{p}\right)\left(1 - \frac{2y}{p}\frac{dp}{dy}\right) = 0 \text{ —— ③}$$

Neglecting the first factor which does not involve dp/dy, ③ reduces to

$$1 - \frac{2y}{p}\left(\frac{dp}{dy}\right) = 0$$

$$\left(\frac{2}{p}\right)dp = \frac{1}{y}dy$$

Integrating $2\log p = \log y + \log c'$

$$\Rightarrow p^2 = c'y \text{ —— ④}$$

we now proceed to eliminate p b/w ① & ④.

Rewriting ①, $p(p^2 - 4xy) = -8y^2$

Squaring, $p^2(p^2 - 4xy)^2 = 64y^4$

$$\Rightarrow c'y(c'y - 4xy)^2 = 64y^4, \text{ using ④ —— ⑤}$$

Let $c' = 4c$, where c in new arbitrary constant.

Then ⑤ gives

$$4cy(4cy - 4xy)^2 = 64y^4$$

$$c(c-x)^2 = y \text{ —— ⑥}.$$

6(a)(ii) Sol'n: Given $(x^2D^2 - 3xD + \lambda)y = 0$, where $D \equiv d/dx$ — ①

Let $x = e^z$ so that $z = \log x$

Also let $D_1 \equiv d/dz$ — ②

Then $xD = D_1$ and $x^2D^2 = D_1(D_1 - 1)$ & so ① reduces to

$\{D_1(D_1-1) + 3D_1 - 1\}y = 0$

$(D_1^2 + 2D_1 - \lambda)y = 0$ — ③

Its auxiliary equations is $D_1^2 + 2D_1 - \lambda = 0$,

giving $D_1 = \{-2 \pm (4+4\lambda)^{1/2}\}/2$

$= -1 \pm (1+\lambda)^{1/2}$ where $\lambda \geq -1$.

Hence the required general solution is given by

$y = C_1 e^{-[1-(1+\lambda)^{1/2}]z} + C_2 e^{-\{1+(1+\lambda)^{1/2}\}z}$

$\equiv C_1 x^{-[1-(1+\lambda)^{1/2}]} + C_2 x^{-[1+(1+\lambda)^{1/2}]}$

using ② — ⑤

Since all solutions ④ must tend to zero as $x \to \infty$, λ must be chosen to satisfy the following condition

$1 - (1+\lambda)^{1/2} > 0$

$\Rightarrow (1+\lambda)^{1/2} < 1$

so that $\lambda < 0$ — ⑥

④ & ⑥ $\Rightarrow -1 \leq \lambda < 0$, which are the required values of λ.

6(b) A uniform chain of length l hangs between two points A and B which are at a horizontal distance a from one another, with B at a vertical distance b above A. Prove that the parameter of the catenary is given by

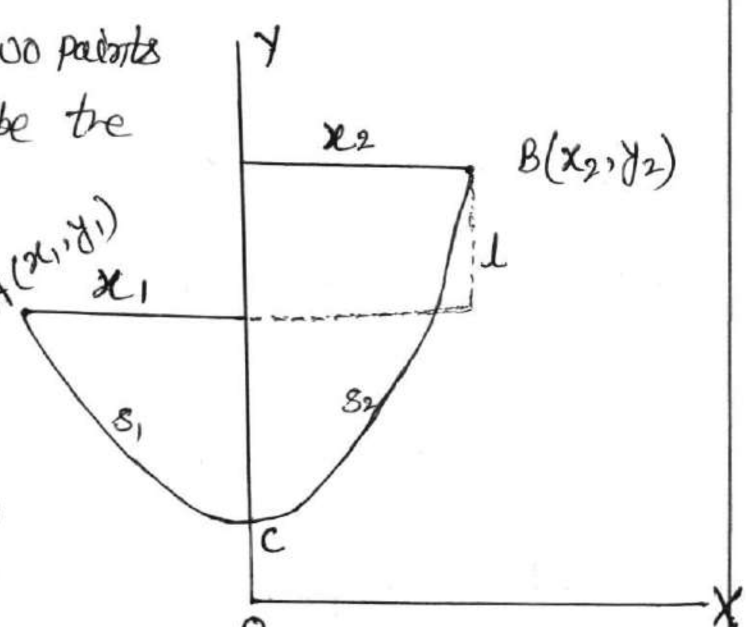
$$2c\ \sinh(a/2c) = \sqrt{(l^2-b^2)}.$$

Prove also that, if the tensions at A and B are T_1 and T_2 respectively,

$$T_1 + T_2 = W\sqrt{\left(1+\frac{4c^2}{l^2-b^2}\right)} \text{ and } T_2 - T_1 = Wb/l,$$

where W is the weight of the chain.

Sol: A uniform chain of length l and weight W hangs between two points A and B. Let C be the vertex, OX the directrix, OY the axis and c the parameter of the catenary in which the



chain hangs. Let (x_1, y_1) and (x_2, y_2) be the coordinates of the points A and B respectively and let arc $CA = s_1$ and arc $CB = s_2$.

We have $s_1 + s_2 = l$.

Since the horizontal distance between A and B is a, therefore

$$x_1 + x_2 = a.$$

Again since the vertical distance of B above A is b, therefore

$$y_2 - y_1 = b.$$

Let w be the weight per unit length of the chain. Then

$$W = lw, \text{ or } w = W/l.$$

By the formula $s = c \sinh(x/c)$, we have

$$s_1 = c \sinh(x_1/c) \text{ and}$$
$$s_2 = c \sinh(x_2/c).$$

$$\therefore l = s_1 + s_2 = c[\sinh(x_1/c) + \sinh(x_2/c)] \quad \text{—— (1)}$$

Again by the formula $y = c \cosh(x/c)$, we have

$$y_1 = c \cosh(x_1/c) \text{ and}$$
$$y_2 = c \cosh(x_2/c)$$

$$\therefore b = y_2 - y_1 = c[\cosh(x_2/c) - \cosh(x_1/c)] \quad \text{—— (2)}$$

Squaring and subtracting (1) and (2), we have

$$l^2 - b^2 = c^2\Big[-\{\cosh^2(x_1/c) - \sinh^2(x_1/c)\} - \{\cosh^2(x_2/c) - \sinh^2(x_2/c)\} + 2\{\cosh(x_1/c)\cosh(x_2/c) + \sinh(x_1/c)\sinh(x_2/c)\}\Big]$$

$$= c^2\left[-1-1+2\cosh(x_1/c+x_2/c)\right]$$

$$= c^2\left[-2+2\cosh\{(x_1+x_2)/c\}\right]$$

$$= 2c^2\left\{\cosh\frac{a}{c}-1\right\}$$

$$= c^2\left\{1+2\sinh^2\frac{a}{2c}-1\right\}$$

$$= 4c^2\sinh^2\frac{a}{2c} \quad \text{——— (3)}$$

$\therefore$ c is given by $2c\sinh(a/2c)=\sqrt{(l^2-b^2)}$.

[Remember that

$$\cosh(\alpha+\beta)=\cosh\alpha\cdot\cosh\beta+\sinh\alpha\cdot\sinh\beta$$

and $\cosh 2\alpha = 1+2\sinh^2\alpha$]

Now let T_1 and T_2 be the tensions at the points A and B respectively. Then by the formula $T=wy$, we have

$$T_1 = wy_1, \quad T_2 = wy_2.$$

$$\therefore T_2-T_1 = w(y_2-y_1) = wb = (W/l)b = Wb/l.$$

Also, $T_1+T_2 = w(y_1+y_2) = \frac{W}{l}(y_1+y_2) = W\frac{y_1+y_2}{s_1+s_2}$

$$= W\frac{c\cosh(x_1/c)+c\cosh(x_2/c)}{c\sinh(x_1/c)+c\sinh(x_2/c)}$$

$$= W \frac{\cosh(x_1/c) + \cosh(x_2/c)}{\sinh(x_1/c) + \sinh(x_2/c)}$$

$$= W \frac{2\cosh\frac{1}{2}(x_1/c + x_2/c)\,\cosh\frac{1}{2}(x_1/c - x_2/c)}{2\sinh\frac{1}{2}(x_1/c + x_2/c)\,\sinh\frac{1}{2}(x_1/c - x_2/c)}$$

$$= W \coth\left(\frac{x_1 + x_2}{2c}\right)$$

$$= W \coth\frac{a}{2c}$$

$$= W\sqrt{\left[1 + \operatorname{cosech}^2\frac{a}{2c}\right]} \quad \left[\because \coth^2\alpha = 1 + \operatorname{cosech}^2\alpha\right]$$

$$= W\sqrt{\left[1 + \frac{4c^2}{l^2 - b^2}\right]}$$

Substituting for $\operatorname{cosech}^2(a/2c)$ from (3).

6(C) (I) Given the space curve $x=t$, $y=t^2$, $z=\frac{2}{3}t^3$, find (i) the curvature K, (ii) the torsion τ.

(II) Evaluate by Green's theorem in plane:

$\int_C (e^{-x}\sin y\,dx + e^{-x}\cos y\,dy)$, where C is the rectangle with vertices $(0,0)$, $(\pi,0)$, $(\pi,\frac{1}{2}\pi)$, $(0,\frac{1}{2}\pi)$.

Sol'n: (I) Given that $\vec{r} = ti + t^2 j + \frac{2}{3}t^3 k$

$$\Rightarrow \frac{d\vec{r}}{dt} = i + 2tj + 2t^2 k$$

$$\frac{d^2\vec{r}}{dt^2} = 0i + 2j + 4tk$$

and $$\frac{d^3\vec{r}}{dt^3} = 0i + 0j + 4k$$

Now $$\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2} = \begin{vmatrix} i & j & k \\ 1 & 2t & 2t^2 \\ 0 & 2 & 4t \end{vmatrix}$$

$$= i(8t^2 - 4t^2) + j(0 - 4t) + k(2 - 0)$$

$$= 4t^2 i - 4tj + 2k$$

$$\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right| = \sqrt{16t^4 + 16t^2 + 4}$$

$$= 2\sqrt{4t^4 + 4t^2 + 1}$$

$$= 2\sqrt{(2t^2+1)^2}$$

$$= 2(2t^2+1)$$

$$\left[\frac{d\vec{r}}{dt}\ \frac{d^2\vec{r}}{dt^2}\ \frac{d^3\vec{r}}{dt^3}\right] = \frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2} \cdot \frac{d^3\vec{r}}{dt^3}$$

$$= (4t^2 i - 4tj + 2k) \cdot 4k = 8$$

$$\therefore \text{Curvature}(K) = \frac{\left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right|}{\left|\frac{d\vec{r}}{dt}\right|^3}$$

$$= \frac{2(2t^2+1)}{\left[\sqrt{1+4t+4t^4}\right]^3}$$

$$= \frac{2(2t^2+1)}{(2t^2+1)^3} = \frac{2}{(2t^2+1)^2}$$

$$\text{Torsion } (\tau) = \left[\frac{d\vec{r}}{dt} \; \frac{d^2\vec{r}}{dt} \; \frac{d^3\vec{r}}{dt^3}\right] \Big/ \left|\frac{d\vec{r}}{dt} \times \frac{d^2\vec{r}}{dt^2}\right|^2$$

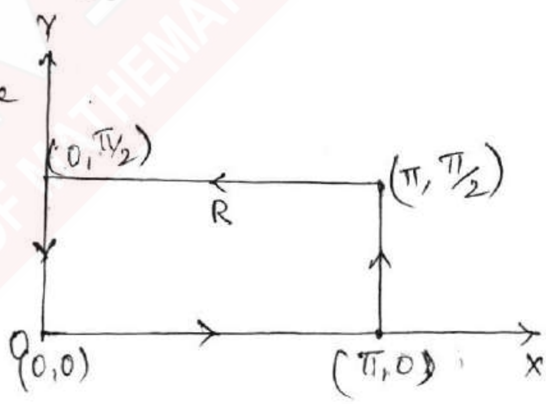
$$= \frac{8}{4(2t^2+1)^2} = \frac{2}{(2t^2+1)^2}$$

$$\therefore K = \tau = \frac{2}{(2t^2+1)^2}$$

$\therefore$ For the space curve $x=t$, $y=t^2$, $z=\frac{2t^3}{3}$ the curvature (K) and Torsion(τ) are same at every point.

6(C)II, Sol'n:

By Green's theorem in plane we have



$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \oint_C (M\,dx + N\,dy)$$

Here $M = e^{-x}\sin y$, $N = e^{-x}\cos y$

$$\therefore \frac{\partial M}{\partial x} = -e^{-x}\cos y, \quad \frac{\partial M}{\partial y} = e^{-x}\cos y$$

Hence the given line integral

$$= \iint_R (-e^{-x}\cos y - e^{-x}\cos y)\,dx\,dy$$

where R is the region enclosed by the rectangle C.

$$= \int_{x=0}^{\pi} \int_{y=0}^{\pi/2} -2e^{-x} \cos y \, dx \, dy$$

$$= \int_{x=0}^{\pi} -2e^{-x} \left[\sin y\right]_{y=0}^{\pi/2} dx$$

$$= \int_{0}^{\pi} -2e^{-x} dx = 2\left[e^{-x}\right]_0^{\pi} = 2\left(e^{-\pi} - 1\right)$$

7(a) Apply the method of variation of parameters to solve the equation $(x+2)y_2-(2x+5)y_1+2y=(x+1)e^x$.

Sol^n: putting the given equation in standard form

$y_2+Py_1+Qy=R$, we get

$$y_2-\frac{2x+5}{x+2}y_1+\frac{2}{x+2}y=\frac{x+1}{x+2}e^x \quad \text{—}(1)$$

Consider $y_2-\dfrac{2x+5}{x+2}y_1+\dfrac{2}{x+2}y=0$ —(2)

Comparing (2) with $y_2+Py_1+Qy=R$, we have

$P=-(2x+5)/x+2$, $Q=2/(x+2)$ and $R=0$

Here $2^2+2P+Q=4-\dfrac{2(2x+5)}{x+2}+\dfrac{2}{x+2}=0$

Hence $u=e^{2x}$ is an integral of (2). Let the complete solution of (2) be $y=uv$. Then (2) reduces to

$$\frac{d^2v}{dx^2}+\left[P+\frac{2}{u}\frac{du}{dx}\right]\frac{dv}{dx}=\frac{R}{u}$$

$$\Rightarrow \frac{d^2v}{dx^2}+\left[-\frac{2x+5}{x+2}+\frac{1}{e^{2x}}\times 2e^{2x}\right]\frac{dv}{dx}=0$$

$$\Rightarrow \frac{d^2v}{dx^2}+\frac{2x+3}{2x+2}\frac{dv}{dx}=0 \quad \text{—}(3)$$

Putting $\dfrac{dv}{dx}=q$ so that $\dfrac{d^2v}{dx^2}=\dfrac{dq}{dx}$, (3) becomes

$$\frac{dq}{dx}+\left[2-\frac{1}{x+2}\right]q=0\Rightarrow\frac{dq}{q}+\left[2-\frac{1}{x+2}\right]dx=0$$

Integrating, $\log q-\log a'-\log(x+2)=a'$ being an arbitrary constant.

$$\Rightarrow q=a'(x+2)e^{-2x}\Rightarrow\frac{dv}{dx}=a'(x+2)e^{-2x}$$

Integrating by chain rule of integration by parts, we have

$$v = a'\left[(x+2)\left(-\tfrac{1}{2}e^{-2x}\right) - (1)\left(\tfrac{1}{4}e^{-2x}\right)\right] + b$$

$$v = -(a'/4)\,e^{-2x}(2x+4+1) + b = a(2x+5) + b, \text{ where } a = -(a'/4)$$

Hence solution of ② is

$$y = uv = e^{2x}\{a(2x+5)e^{-2x} + b\} = a(2x+5) + be^{2x}$$

Thus, $a(2x+5) + be^{2x}$ is C.F of ①, a and b being arbitrary constants.

Let $y = A(2x+5) + Be^{2x}$ ——— ④

be the complete solution of ①. Then A & B are functions of x which are so chosen that ① will be satisfied. Differentiating ④, we get

$$y_1 = A_1(2x+5) + 2A + B_1e^{2x} + 2Be^{2x} \quad \text{——— ⑤}$$

Choose A & B such that $A_1(2x+5) + B_1e^{2x} = 0$ ——— ⑥

Then ⑤ reduces to $y_1 = 2A + 2Be^{2x}$ ——— ⑦

Differentiating ⑦, $y_2 = 2A_1 + 2B_1e^{2x} + 4Be^{2x}$ ——— ⑧

Using ④, ⑦ & ⑧, ① reduces to

$$2A_1 + 2B_1e^{2x} = [(x+1)/(x+2)]e^{x} \quad \text{——— ⑨}$$

Multiplying ⑥ by ② & substracting it from ⑨, we get

$$A_1(-4x-8) = \frac{x+1}{x+2}e^{x} \Rightarrow A_1 = \frac{dA}{dx} = -\frac{(x+1)}{4(x+2)^2}e^{x} \quad \text{——— ⑩}$$

Integrating, $A = -\frac{1}{4}\int \frac{x+1}{(x+2)^2}e^{x}\,dx + c_1$, c_1 being an arbitrary constant.

$$A = c_1 - \tfrac{1}{4}\int \frac{(x+2)-1}{(x+2)^2}e^{x}\,dx = c_1 - \tfrac{1}{4}\int e^{x}\{(x+2)^{-1} - (x+2)^{-2}\}\,dx$$

$$A = c_1 - \tfrac{1}{4}e^{x}(x+2)^{-1} \text{ as } \int e^{x}[f(x) + f'(x)]\,dx = e^{x}f(x) \quad \text{——— ⑪}$$

from (6) & (10), $\therefore B_1 = \frac{dB}{dx}$

$$= \frac{(2x+5)(x+1)e^{-x}}{4(x+2)^2} = \frac{(2x^2+7x+5)e^{-x}}{4(x+2)^2}$$

$$= \frac{2(x+2)^2-(x+3)}{4(x+2)^2}e^{-x}$$

$$\frac{dB}{dx} = \frac{1}{2}e^{-x} - \frac{(x+2)+1}{4(x+2)^2}e^{-x} = \frac{1}{2}e^{-x} + \frac{1}{4}e^{-x}\left[-(x+2)^{-1}-(x+2)^{-2}\right]$$

Integrating $B = C_2 - \frac{1}{2}e^{-x} + \frac{1}{4}e^{-x}(x+2)^{-1}$, C_2 being arbitrary Constant — (12)

using formula $\int e^{ax}[af(x)+f'(x)]dx = e^{ax}f(x)$ for $a=-1$]

Using (10) & (11) in (3), the required solution is

$$y = \left[C_1 - \frac{1}{4}e^{x}(x+2)^{-1}\right](2x+5) + \left[C_2 - \frac{1}{2}e^{-x} + \frac{1}{4}e^{-x}(x+2)^{-1}\right]e^{2x}$$

$$y = C_1(2x+5) + C_2e^{2x} + \frac{1}{4}e^{x}\left[\frac{1}{x+2} - 2 - \frac{2x+5}{x+2}\right]$$

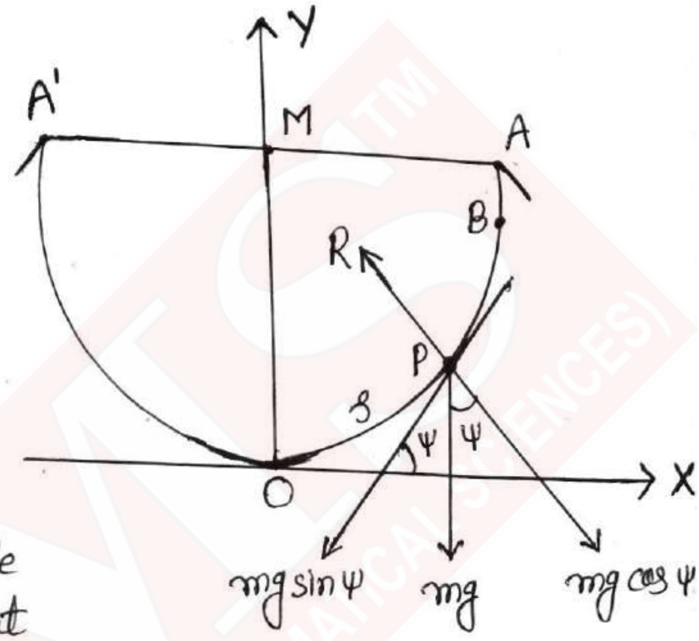
$$y = C_1(2x+5) + C_2e^{2x} - e^{x}$$

7(b). A particle is projected with velocity V from the cusp of a smooth inverted cycloid down the arc, show that the time of reaching the vertex is

$$\sqrt{(a/g)}\tan^{-1}\left[\left(\sqrt{4ag}\right)/V\right]$$

Sol:

Let a particle be projected with velocity V from the cusp A of a smooth inverted cycloid down the arc. If P is the position of the particle at time t such that



the tangent at P is inclined at an angle ψ to the horizontal and arc $OP = s$, then the equations of motion of the particle are

$$m\frac{d^2s}{dt^2} = -mg\sin\psi \quad \text{——— (1)}$$

and $$m\frac{v^2}{\rho} = R - mg\cos\psi \quad \text{——— (2)}$$

for the cycloid, $s = 4a\sin\psi$ ——— (3)

from (1) and (3), we have

$$\frac{d^2s}{dt^2} = -\frac{g}{4a}s.$$

Multiplying both sides by $2(ds/dt)$ and Integrating,

We have

$$v^2 = \left(\frac{ds}{dt}\right)^2 = -\frac{g}{4a}s^2 + A$$

But initially at the cusp A,

$$s = 4a \text{ and } (ds/dt)^2 = V^2$$

$$\therefore V^2 = -(g/4a)\cdot 16a^2 + A \quad \text{or} \quad A = V^2 + 4ag.$$

$$\therefore v^2 = \left(\frac{ds}{dt}\right)^2 = V^2 + 4ag - \frac{g}{4a}s^2$$

$$= \left(\frac{g}{4a}\right)\left[\frac{4a}{g}(V^2+4ag) - s^2\right]$$

or $$\frac{ds}{dt} = -\frac{1}{2}\sqrt{(g/a)}\sqrt{\left[\frac{4a}{g}(V^2+4ag) - s^2\right]}$$

(-ive sign is taken because the particle is moving in the direction of s decreasing.)

or, $$dt = -2\sqrt{(a/g)}\cdot\frac{ds}{\sqrt{[(4a/g)(V^2+4ag)-s^2]}}$$

Integrating, the time t_1 from the cusp A to the vertex O is given by

$$t_1 = -2\sqrt{(a/g)}\int_{s=4a}^{0}\frac{ds}{\sqrt{[(4a/g)(V^2+4ag)-s^2]}}$$

$$= 2\sqrt{(a/g)}\int_0^{4a}\frac{ds}{\sqrt{[(4a/g)(V^2+4ag)-s^2]}}$$

$$= 2\sqrt{(a/g)}\left[\sin^{-1}\frac{s}{2\sqrt{(a/g)}\sqrt{(V^2+4ag)}}\right]_0^{4a}$$

$$= 2\sqrt{(a/g)} \cdot \sin^{-1}\left\{ \frac{2\sqrt{(ag)}}{\sqrt{(v^2+4ag)}} \right\}$$

$$= 2\sqrt{(a/g)} \cdot \theta, \quad \text{------------} \quad (4)$$

where $\theta = \sin^{-1}\left\{ \frac{2\sqrt{(ag)}}{\sqrt{(v^2+4ag)}} \right\}$

we have $\sin\theta = \frac{2\sqrt{(ag)}}{\sqrt{(v^2+4ag)}}$.

$$\therefore \cos\theta = \sqrt{(1-\sin^2\theta)}$$

$$= \sqrt{\left[1 - \frac{4ag}{v^2+4ag}\right]}$$

$$= \frac{v}{\sqrt{(v^2+4ag)}}.$$

$$\therefore \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{2\sqrt{(ag)}}{v} = \frac{\sqrt{(4ag)}}{v}$$

$$\theta = \tan^{-1}\left[\sqrt{(4ag)}/v\right]$$

$\therefore$ from (4), the time of reaching the vertex is

$$= 2\sqrt{(a/g)} \cdot \tan^{-1}\left[\sqrt{(4ag)}/v\right].$$

7(C) (I) Find the angle of intersection at $(4,-3,2)$ of spheres $x^2+y^2+z^2=29$ and $x^2+y^2+z^2+4x-6y-8z-47=0$

(II) (i) Prove that $r^n r$ is an irrotational vector for any value of n but is solenoidal only if $n+3=0$

(ii) If $u=\left(\frac{1}{r}\right)r$, show that $\nabla\times u=0$

(iii) If $u=\left(\frac{1}{r}\right)r$ find grad(div u).

Sol'n: (I) Let $f_1=x^2+y^2+z^2-29$ and $f_2=x^2+y^2+z^2+4x-6y-8z-47$

Then $\text{grad} f_1 = 2xi+2yj+2zk$

and $\text{grad} f_2 = (2x+4)i+(2y-6)j+(2z-8)k$

Let $n_1=\text{grad} f_1$ at the point $(4,-3,2)$

and $n_2=\text{grad} f_2$ at the point $(4,-3,2)$.

Then $n_1 = 8i-6j+4k = 2(4i-3j+2k)$

and $n_2 = 12i-12j-4k = 4(3i-3j-k)$.

The vectors n_1 and n_2 are along normals to the two spheres at the point $(4,-3,2)$ and the angle θ b/w these two vectors is the angle of intersection of the two spheres at the point $(4,-3,2)$.

we have $\cos\theta = \dfrac{n_1 \cdot n_2}{|n_1||n_2|}$

$$= \frac{8(12+9-2)}{2\sqrt{(16+9+4)}\cdot 4\sqrt{(9+9+1)}}$$

$$= \frac{19}{\sqrt{29}\cdot\sqrt{19}}$$

$$\theta = \cos^{-1}\sqrt{\frac{19}{29}}.$$

7(c)(ii) (i) Let $F = r^n r$

The vector F is irrotational if curl F = 0.

we know that $\text{Curl}(\phi A) = (\nabla\phi)\times A + \phi \,\text{Curl}\, A$.

Putting $\phi = r^n$ and $A = r$ in this identity, we get

$$\text{curl}(r^n r) = \nabla r^n \times r + r^n \,\text{curl}\, r$$

$$= (n r^{n-1} \nabla r)\times r + r^n 0$$

$[\because \nabla f(r) = f'(r)\nabla r$ and $\text{curl}\, r = \text{Curl}(xi + yj + zk) = 0]$

$$= \left(n r^{n-1} \frac{1}{r} r\right)\times r \qquad \left[\because \nabla r = \frac{1}{r} r\right]$$

$$= n r^{n-2}(r \times r)$$

$$= n r^{n-2}\, 0 \qquad [\because r\times r = 0]$$

$$= 0$$

$\therefore$ $r^n r$ is an irrotational vector for any value of n.

$\therefore$ The vector F is solenoidal if div F = 0, show that $\text{div}(r^n r) = (n+3)r^n$.

we have $\text{div}(\phi A) = \phi\,(\text{div}\, A) + A\cdot \text{grad}\,\phi$

Putting $A = r$ and $\phi = r^n$ in this identity, we get

$$\text{div}(r^n r) = r^n \,\text{div}\, r + r\cdot \text{grad}\, r^n$$

$$= 3r^n + r\cdot(n r^{n-1}\,\text{grad}\, r)$$

$[\because \text{div}\, r = 3$ & $\text{grad}\, f(u) = f'(u)\,\text{grad}\, u]$

$$= 3r^n + r\cdot\left[n r^{n-1} \frac{1}{r} r\right] \qquad \left[\because \text{grad}\, r = \hat{r} = \frac{1}{r} r\right]$$

$$= 3r^n + n r^{n-2}(r\cdot r)$$

$$= 3r^n + n r^{n-2} r^2 = (n+3) r^n$$

$\therefore$ the vector $r^n r$ is solenoidal only if $(n+3)r^n = 0$

i.e. only if $n+3 = 0$ i.e. only if $n = -3$.

(ii) we have $\nabla\times\vec{u} = \nabla\times\left(\frac{1}{r}\vec{r}\right)$

we know that $\text{Curl}\,\phi A = (\nabla\phi)\times A + \phi\,\text{Curl}\, A$

Replacing ϕ by $\frac{1}{r}$ and A by $\vec{r}$ in this identity

we have

$$\text{Curl}\left(\frac{1}{r}\vec{r}\right) = \left[\nabla\left(\frac{1}{r}\right)\right]\times\vec{r} + \frac{1}{r}\,\text{Curl}\,\vec{r}$$

$$= \left[-\frac{1}{r^2}\nabla r\right]\times\vec{r} + \frac{1}{r}(0)$$

$$= \left(-\frac{1}{r^2}\frac{1}{r}\vec{r}\right)\times\vec{r} \qquad \left(\because \nabla r = \frac{1}{r}\vec{r}\right)$$

$$= -\frac{1}{r^3}(\vec{r}\times\vec{r}) = -\frac{1}{r^3}(0) = 0$$

Hence $\nabla\times u = 0$.

(iii) To find grad $(\text{div}\,\vec{u})$:

$$\text{div}(\vec{u}) = \text{div}\left(\frac{1}{r}\vec{r}\right) = \text{div}\,\frac{1}{r}(x\hat{i}+y\hat{j}+z\hat{k})$$

$$= \text{div}\left(\frac{x}{r}\hat{i}+\frac{y}{r}\hat{j}+\frac{z}{r}\hat{k}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{x}{r}\right)+\frac{\partial}{\partial y}\left(\frac{y}{r}\right)+\frac{\partial}{\partial z}\left(\frac{z}{r}\right)$$

$$= \left(\frac{1}{r}-\frac{x}{r^2}\frac{\partial r}{\partial x}\right)+\left(\frac{1}{r}-\frac{y}{r^2}\frac{\partial r}{\partial y}\right)+\left(\frac{1}{r}-\frac{z}{r^2}\frac{\partial r}{\partial z}\right)$$

Now $r^2 = x^2+y^2+z^2$

$$\therefore 2r\frac{\partial r}{\partial x} = 2x \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}$$

Similarly $\frac{\partial y}{\partial r} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$

$$\therefore \text{div}\,\vec{u} = \frac{3}{r} - \left(\frac{x}{r^2}\cdot\frac{x}{r}\right) + \left(\frac{y}{r^2}\cdot\frac{y}{r}\right) + \left(\frac{z}{r^2}\cdot\frac{z}{r}\right)$$

$$= \frac{3}{r} - \frac{x^2+y^2+z^2}{r^3} = \frac{3}{r} - \frac{r^2}{r^3} = \frac{3}{r} - \frac{1}{r} = \frac{2}{r}$$

$$\therefore \text{grad}(\text{div}\,u) = \text{grad}\left(\frac{2}{r}\right) = \left(-\frac{2}{r^2}\right)\text{grad}\,r$$

$$= \frac{-2}{r^2}\left(\frac{1}{r}\vec{r}\right)$$

$$= \frac{-2}{r^3}\vec{r}.$$

8(a) By using Laplace transform method solve the initial value problem $(D^3-2D^2+5D)y=0$ if $y(0)=0$, $y'(0)=1$

Sol^n: Taking the Laplace transform of bothsides of the given equation, we have

$$L\{y'''\}-2L\{y''\}+5L\{y'\}=0$$

$$p^3L\{y\}-p^2y(0)-py'(0)-y''(0)-2[p^2L\{y\}-py(0)-y'(0)]+5[pL\{y\}-y(0)]=0$$

$(p^3-2p^2+5p)L\{y\}-p-A-2[-1]+5.0=0$ where $y''(0)=A$

$$L\{y\}=\frac{A-2+p}{p^3-2p^2+5p}$$

$$=\frac{A-2}{p(p^2-2p+5)}+\frac{1}{p^2-2p+5}$$

$$=\frac{A-2}{5p}-\frac{A-2}{5}\cdot\frac{p-2}{p^2-2p+5}+\frac{1}{p^2-2p+5}$$

$$=\frac{A-2}{5p}-\frac{A-2}{5}\cdot\frac{(p-1)-1}{(p-1)^2+4}+\frac{1}{(p-1)^2+4}$$

$$=\frac{A-2}{5p}-\frac{A-2}{5}\cdot\frac{(p-1)}{(p-1)^2+4}+\frac{A+3}{10}\cdot\frac{2}{(p-1)^2+4}$$

$$\therefore y=\frac{A-2}{5}\cdot L^{-1}\left\{\frac{1}{p}\right\}-\frac{A-2}{5}L^{-1}\left\{\frac{p-1}{(p-1)^2+4}\right\}$$

$$=\frac{A-2}{5}-\frac{A-2}{5}e^t.\cos 2t+\frac{A+3}{10}e^t\sin 2t$$

Since $y\left(\frac{\pi}{8}\right)=1$

$$\therefore 1=\frac{A-2}{5}-\frac{A-2}{5}\cdot e^{\pi/8}\frac{1}{\sqrt2}+\frac{A+3}{10}e^{\pi/8}\frac{1}{\sqrt2}$$

$$\Rightarrow \frac{7-A}{5}=\frac{e^{\pi/8}}{10\sqrt2}(-2A+4+A+3)$$

$$\Rightarrow \left(\frac{7-A}{5}\right)\cdot\left[1-\frac{e^{\pi/8}}{2\sqrt2}\right]=0\Rightarrow A=7$$

Hence the required solution is

$$y=1+e^t(\sin 2t-\cos 2t)$$

8(b). A particle describes the curve $r^n = a^n \cos n\theta$ under a force to the pole. Find the law of force.

Sol: The equation of the curve is $r^n = a^n \cos n\theta$.
putting $r = 1/u$, we have

$$\frac{1}{u^n} = a^n \cos n\theta \text{ (or) } a^n u^n = \sec n\theta \quad \text{——— (1)}$$

Taking logarithm of both sides of (1), we have

$$n \log a + n \log u = \log \sec n\theta.$$

Differentiating w.r.t. 'θ', we have

$$\frac{n}{u}\frac{du}{d\theta} = \frac{1}{\sec n\theta} n \sec n\theta \tan n\theta$$

or $$\frac{du}{d\theta} = u \tan n\theta$$

Differentiating again w.r.t. 'θ', we have

$$\frac{d^2u}{d\theta^2} = \frac{du}{d\theta}\tan n\theta + u(\sec^2 n\theta)\cdot n$$

$$= u \tan n\theta \cdot \tan n\theta + un \sec^2 n\theta$$

$$\left[\because du/d\theta = u \tan n\theta\right]$$

$$= u \tan^2 n\theta + un \sec^2 n\theta \quad \text{——— (2)}$$

The differential equation of the central orbit is

$$\frac{P}{h^2u^2} = u + \frac{d^2u}{d\theta^2}.$$

$$\therefore P = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right)$$

$= h^2u^2(u + u\tan^2 n\theta + un\sec^2 n\theta)$

[putting the value of $\frac{d^2u}{d\theta^2}$ from ②]

$= h^2u^3(\sec^2 n\theta + n\sec^2 n\theta)$

$= h^2u^3(1+n)\sec^2 n\theta$

[substituting for $\sec n\theta$ from ①]

$= h^2(1+n)u^3\cdot(a^n u^n)^2$

$= h^2a^{2n}(1+n)u^{2n+3}$

$= \dfrac{h^2a^{2n}(1+n)}{r^{2n+3}}$

$\therefore P \propto 1/r^{2n+3}$ i.e., the force varies inversely as the $(2n+3)$th power of the distance from the pole.

8(c) If $F=(y^2+z^2-x^2)i+(z^2+x^2-y^2)j+(x^2+y^2-z^2)k$, evaluate $\iint \text{curl } F\cdot\hat{n}\,ds$ taken over the portion of the surface $x^2+y^2+z^2-2ax+az=0$ above the plane $z=0$, and verify Stoke's theorem.

Sol'n: The surface $x^2+y^2+z^2-2ax+az=0$ meets the plane $z=0$ in the circle C given by $x^2+y^2-2ax=0,\ z=0$.

The polar equation of the circle C lying in the xy-plane is $r=2a\cos\theta$, $0\le\theta<\pi$.

Also the equation $x^2+y^2-2ax=0$ can be written as $(x-a)^2+y^2=a^2$. Therefore the parametric equations of the circle C can be taken as

$x=a+a\cos t$, $y=a\sin t$, $z=0$, $0\le t\le 2\pi$.

Let S denote the portion of the surface $x^2+y^2+z^2-2ax+az=0$ lying above the plane $z=0$ and S_1 denote the plane region bounded by the circle C. By an application of divergence theorem, we have

$$\iint_S \text{curl } F\cdot\hat{n}\,ds=\iint_{S_1}\text{curl } F\cdot k\,ds$$

Now
$$\text{curl } F\cdot k=\begin{vmatrix} i & j & k\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ y^2+z^2-x^2 & z^2+x^2-y^2 & x^2+y^2-z^2\end{vmatrix}\cdot\hat{k}$$

$$=\left[\frac{\partial}{\partial x}(z^2+x^2-y^2)-\frac{\partial}{\partial y}(y^2+z^2-x^2)\right]\hat{k}\cdot\hat{k}$$

$$=2(x-y)$$

$$\therefore \iint_S \text{curl } F\cdot\hat{n}\,ds=\iint_{S_1}\text{curl } F\cdot\hat{k}\,ds=\iint_{S_1}2(x-y)\,ds$$

$$=\int_{\theta=0}^{2\pi}\int_{r=0}^{2a\cos\theta}(r\cos\theta-r\sin\theta)\,r\,d\theta\,dr$$

$$= 2\int_{\theta=0}^{\pi}(\cos\theta - \sin\theta)\left[\frac{r^3}{3}\right]_0^{2a\cos\theta} d\theta$$

$$= \frac{16a^3}{3}\int_0^{\pi}\cos^4\theta\, d\theta \cdot \quad \left(\because \int_0^{\pi}\cos^3\theta\sin\theta\, d\theta = 0\right)$$

$$= 2\times\frac{16a^3}{3}\int_0^{\pi/2}\cos^4\theta\, d\theta = 2\times\frac{16a^3}{3}\times\frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2}$$

$$= 2\pi a^3 \quad \text{——— (1)}$$

Also $\int_C F\cdot dr = \int_C (y^2+z^2-x^2)\,dx + (z^2+x^2-y^2)\,dy + (x^2+y^2-z^2)\,dz$

$$= \int_C (y^2-x^2)\,dx + (x^2-y^2)\,dy \quad (\because \text{on } C,\ z=0\ \&\ dz=0)$$

$$= \int_0^{2\pi}(x^2-y^2)(dy-dx) = \int_0^{2\pi}(x^2-y^2)\left(\frac{dy}{dt}-\frac{dx}{dt}\right)dt$$

$$= \int_0^{2\pi}\left[(a+\cos t)^2 - a^2\sin^2 t\right](a\cos t + a\sin t)\,dt$$

$$= a^3\int_0^{2\pi}(1+\cos^2 t + 2\cos t - \sin^2 t)(\cos t + \sin t)\,dt$$

$$= a^3\int_0^{2\pi} 2\cos^2 t\, dt, \text{ other integrals vanish}$$

$$= 2a^3\int_0^{2\pi}\cos^2 t\, dt$$

$$= 2a^3\times 4\int_0^{\pi/2}\cos^2 t$$

$$= 8a^3\times\frac{1}{2}\times\frac{\pi}{2}$$

$$= 2a^3\pi \quad \text{——— (2)}.$$

$\therefore$ from (1) & (2) we have

$$\iint \text{Curl } F\cdot\hat{n}\, dS = \int_C F\cdot d\hat{r}$$

Hence Stoke's theorem is verified.