IFoS - 2018

1 (c) (10 M) $\longrightarrow$ $u(x,y) = (x-1)^3 - 3xy^2 + 3y^2$

[Note: Exact same question is asked in CSE Mains 2018].

Let $f(z) = u+iv$ be a regular function where $z = x+iy$.

As $f(z)$ is a regular function, so it will satisfy C-R conditions. So,

$$\frac{\partial u}{\partial x} = 3(x-1)^2 - 3y^2 = \frac{\partial v}{\partial y}.$$

Integrating w.r.t y we get,

$$v(x,y) = 3(x-1)^2 y - y^3 + f_1(x)$$

Also $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow -6xy + 6y = -\left[6(x-1)y + f_1'(x)\right]$.

$$f_1'(x) = 0 \Rightarrow f_1(x) = K \text{ (constant)}$$

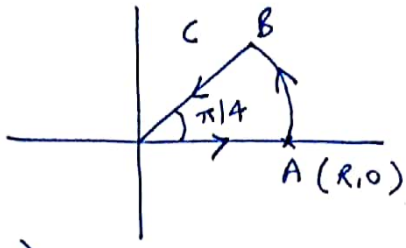
So we get, $\boxed{v = 3(x-1)^2 y - y^3 + K}$

2 (c) (10 M) $\longrightarrow$ P.T $\int_0^\infty \cos x^2 \, dx = \int_0^\infty \sin x^2 \, dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$.

[Note: In these type of questions selection of contour plays the most important role and it is advisable to remember the contours of $\int_0^\infty e^{-x^2} dx$, $\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x} dx$, among others].

Let C be the contour consisting of the line OA, $\underset{\downarrow \\ \text{circular}}{\text{arc AB}}$ & line BO in a positive orientation.

Let $f(z) = e^{iz^2}$ and we are going to integrate $f(z)$ along C



$\because$ $f(z)$ is analytic inside & on C

$\therefore$ $\displaystyle\int_C f(z)\,dz = 0$ (by Cauchy's theorem)

$$\int_C f(z)\,dz = \int_{OA} e^{iz^2}\,dz + \int_{AB} e^{iz^2}\,dz + \int_{BO} e^{iz^2}\,dz .$$

AB : Circular Arc : $z = Re^{i\theta}$ $\quad \theta \in (0, \frac{\pi}{4})$

BO : line segment : $z = te^{i\pi/4}$ $\quad t \in (0, R)$

Let us see the integral over AB $[z = Re^{i\theta}]$

$$\left| \int_0^{\pi/4} e^{iR^2 e^{2i\theta}} \cdot (iRe^{i\theta})\,d\theta \right| \leq \int_0^{\pi/4} \left| e^{iR^2 e^{2i\theta}} \right| \left| iRe^{i\theta} \right| d\theta$$

$$\leq R \int_0^{\pi/4} \left| e^{-R^2 \sin 2\theta} \right| \left| e^{iR^2 \cos 2\theta} \right| d\theta$$

$$\leq R \int_0^{\pi/4} e^{-R^2 \sin 2\theta}\,d\theta$$

$$\leq R \int_0^{\pi/4} e^{-R^2 \left(\frac{4\theta}{\pi}\right)}\,d\theta \qquad \left[\because \sin\theta \geq \frac{2\theta}{\pi} \Rightarrow \sin 2\theta \geq \frac{4\theta}{\pi} \right].$$

$$\leq \frac{\pi}{4R}\left(1 - e^{-R^2}\right)$$

<u>Now as $R \to \infty$</u> $\Rightarrow \left| \displaystyle\int_{AB} f(z)\,dz \right| = 0 \Rightarrow \displaystyle\int_{AB} f(z)\,dz = 0 .$

$$\text{Now, } \int_0^\infty (\cos x^2 + i \sin x^2)\,dx = \int_0^\infty e^{i\pi/4} \cdot e^{-t^2}\,dt$$

$$\boxed{-\int_0^R e^{it^2 \cdot e^{i\pi/2}} \cdot e^{i\pi/4}\,dt}$$

$$= e^{i\pi/4} \int_0^\infty e^{-t^2}\,dt$$

Now as we know $\int_0^\infty e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}$.

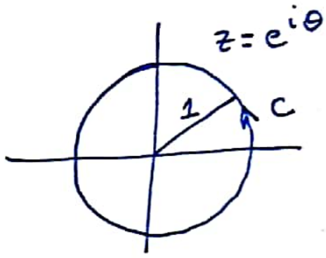
So. $\int_0^\infty (\cos x^2 + i\sin x^2)\,dx = \left(\frac{1+i}{\sqrt{2}}\right)\frac{\sqrt{\pi}}{2}$.

By comparing real & imaginary part we get,

$$\boxed{\int_0^\infty \cos x^2\,dx = \int_0^\infty \sin x^2\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}}.$$

$\xrightarrow[(10M)]{3(c)}$ Evaluate $\int_0^{2\pi} \cos^{2n}\theta\,d\theta$; n is a positive integer.

Let us take a unit circle as a contour



$z = e^{i\theta} = \cos\theta + i\sin\theta$

$\frac{1}{z} = e^{-i\theta} = \cos\theta - i\sin\theta$ $\Rightarrow \cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$

& also $dz = ie^{i\theta}d\theta \Rightarrow d\theta = \frac{dz}{iz}$.

$$\int_0^{2\pi} \cos^{2n}\theta\,d\theta = \oint_C \left[\frac{1}{2}\left(z+\frac{1}{z}\right)\right]^{2n}\frac{dz}{iz}.$$

Now we know that $\oint_C f(z)\,dz = 2\pi i \times (\text{Sum of Residue})$

We can clearly observe that in our question the only singularity is $z=0$. Residue at zero is the coefficient of $\frac{1}{z}$ in the series expansion of $f(z)$.

$$\text{Now } \oint \frac{1}{2^{2n}}\left(z+\frac{1}{z}\right)^{2n}\frac{1}{iz} = \frac{1}{2^{2n}}\sum_{k=0}^{2n} {}^{2n}C_k\, z^k \left(\frac{1}{z}\right)^{2n-k}\frac{1}{iz}.$$

$$= \frac{1}{2^{2n}i}\sum_{k=0}^{2n} z^{(2k-2n-1)}\,{}^{2n}C_k$$

We can clearly see that coefficient of $\left(\frac{1}{z}\right)$ is obtained by putting $k = n$ ~~in the earlier ex~~

So we get Residue at $(z=0) \equiv \frac{1}{2^{2n} i} \, {}^{2n}C_n$

$$\therefore \oint_C \left[\frac{1}{2}\left(z+\frac{1}{z}\right)\right]^{2n} \frac{dz}{iz} = 2\pi i \times \text{Res}\,(z=0)$$

$$= 2\pi \cancel{i} \times \frac{1}{2^{2n}\cancel{i}} \, {}^{2n}C_n = \frac{\pi}{2^{2n-1}} \, {}^{2n}C_n .$$

$$\boxed{\therefore \int_0^{2\pi} \cos^{2n}\theta \, d\theta = \frac{\pi}{2^{2n-1}} \, {}^{2n}C_n .}$$

Method 2 : It can easily be solved by using β & γ functions

$$\int_0^{2\pi} \cos^{2n}\theta \, d\theta = 4 \times \int_0^{\pi/2} \cos^{2n}\theta \, d\theta$$

Use the formula : $\displaystyle\int_0^{\pi/2} \sin^p x \cos^q x \, dx = \frac{1}{2} \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}$.

Using this we get $\displaystyle\int_0^{\pi/2} \cos^{2n}\theta \, d\theta = \frac{1}{2} \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{2n+1}{2}\right)}{\Gamma(n+1)}$

$$= \frac{1}{2} \frac{\sqrt{\pi}}{n!} \times \left(\frac{\Gamma n}{\Gamma n}\right) \Gamma\left(n+\frac{1}{2}\right)$$

Duplication formula

$$= \frac{1}{2} \frac{\sqrt{\pi}}{n!\,(n-1)!} \times \left[\Gamma(n)\,\Gamma\left(n+\frac{1}{2}\right)\right]$$

$$\frac{1}{2} \frac{\sqrt{\pi}}{n!\,(n-1)!} \left[\frac{\sqrt{\pi}}{2^{2n-1}} (2n-1)!\right] \times \frac{2n}{2n}$$

$$\Rightarrow \int_0^{\pi/2} \cos^{2n}\theta \, d\theta = \frac{\pi \, (2n!)}{2^{2n+1} \cdot (n!)^2}$$

$$\Rightarrow \boxed{\int_0^{2\pi} \cos^{2n}\theta \, d\theta = \frac{\pi}{2^{2n-1}} \, {}^{2n}C_n .}$$

Note: Do the proof of Duplication Formula

↳ Imp for exam .