Basis and Dimension: Set - II

Basis: Let $V(F)$ be a vector space. and $S = \{\alpha_1, \alpha_2, \dots \alpha_n\} \subseteq V$

if (i) S is LI

(ii) $L(S) = V$ i.e, V spanned by S.

i.e, each vector in V is a l.c. of finite no. of elts of S.

then S is called basis of $V(F)$.

Ex:- $S = \{e_1, e_2, \dots e_n\} \subseteq V_n(F)$

where $e_1 = (1, 0, 0, \dots 0)$; $e_2 = (0, 1, 0, \dots 0) \dots e_n = (0, 0, \dots 0, 1)$

is basis of $V_n(F)$.

Soln (i) To prove S is LI.

$S = \{e_1, e_2 \dots e_n\} \subseteq V_n(F)$

where $e_1 = (1, 0, \dots 0)$, $e_2 = (0, 1, 0 \dots) \dots e_n = (0, 0, \dots 0, 1)$

Let $a_1, a_2, \dots a_n \in F$ then

$a_1e_1 + a_2e_2 + \dots + a_ne_n = 0$

$\Rightarrow a_1(1, 0, \dots 0) + a_2(0, 1, 0, \dots 0) + \dots + a_n(0, 0, \dots 0, 1) = (0, 0, \dots, 0)$

$\Rightarrow (a_1, a_2, \dots a_n) = (0, 0, 0, \dots 0)$

$\Rightarrow a_1 = a_2 = a_3 = \dots = a_n = 0$

$\therefore$ S is LI.

(ii) To prove that $L(S) = V_n(F)$.

we have always $L(S) \subseteq V_n(F)$ — (1)

Let $\alpha = (a_1, a_2, \dots a_n) = a_1(1, 0, 0 \dots 0) + a_2(0, 1, 0, \dots 0) + \dots + a_n(0, 0 \dots 0, 1) \in L$

$\therefore \alpha \in L(S)$

$\Rightarrow V_n(F) \subset L(S)$ — (2)

$\therefore$ from (1) & (2) $L(S) = V_n(F)$.

$\therefore$ S is a basis of $V_n(F)$

Note I. The set $\{(1, 0, 0, \dots 0), (0, 1, 0, \dots 0), \dots (0, 0, \dots 0, 1)\}$ is called the standard basis of $V_n(F)$.

[2]. $\{(1,0),(0,1)\}$ is a basis of $V_2(F)$

[3]. $\{(1,0,0),(0,1,0),(0,0,1)\}$ is a basis of $V_3(F)$.

Ex:- $S=\{1,i\}$ is a basis of $C(R)$.

Sol^n: S is LI

we have always $L(S)\subseteq C(R)$ — ①

Let $\alpha\in C(R)$ then $\alpha=a+bi$; $a,b\in R$

$=a\cdot1+b(i)\in L(S)$

$\therefore \alpha\in L(S)$

$\therefore C(R)\subseteq L(S)$ — ②

$\therefore$ from (1) & (2) $L(S)=C(R)$

$\therefore$ S is a basis of $C(R)$

Ex:- Let $F_3[x]=\{a_0+a_1x+a_2x^2 / a_0,a_1,a_2\in F\}$

then $\{1,x,x^2\}\subseteq F_3[x]$ is basis of $F_3[x]$ over F.

Sol^n Let $S=\{1,x,x^2\}\subseteq F[x]$

S is LI

we have always $L(S)\subseteq F[x]$ — ①

Let $\alpha=a_0+a_1x+a_2x^2\in F[x]$.

then $a_0+a_1x+a_2x^2=a_0(1)+a_1(x^2)+a_2(x^3)\in L(S)$

$\therefore \alpha\in L(S)$

$\therefore F[x]\subseteq L(S)$ — ②

$\therefore$ from (1) & (2) $L(S)=F[x]$

$\therefore$ S is a basis of $F[x]$.

→ S.T the set $\{(1,0,0),(0,1,0),(1,1,0),(1,2,3)\}\subseteq V_3(R)$ is not a basis of $V_3(R)$

Sol^n Let $S=\{(1,0,0),(0,1,0),(1,1,0),(1,2,3)\}\subseteq V_3(R)$

(i) To check whether the set S is LI or not:

Let $a,b,c,d\in R$ then

$a(1,0,0)+b(1,1,0)+c(1,2,3)+d(0,1,0)=(0,0,0)$

$\Rightarrow (a+b+c,\ b+2c+d,\ 3c)=(0,0,0)$

$\Rightarrow a+b+c=0$ — (1)

$b+2c+d=0$ — (2)

$3c=0$

$\Rightarrow \boxed{c=0}$

① ≡ $a+b=0 \Rightarrow \boxed{a=-b}$

② ≡ $b+d=0 \Rightarrow \boxed{b=-d}$

If $d=k \neq 0$ then $b=-k$ and $a=k$

$\therefore \exists$ non-zero values for a, b, d to satisfy the equations ①, ②

$\therefore$ The given set of vectors are LD.

$\therefore$ S is not a basis set of $V_3(\mathbb{R})$.

Note: Any subset of $V_n(F)$, (i.e, $S \subseteq V_n(F)$) having more than n elts will be LD and it cannot be a basis set of $V_n(F)$.

Defn: Finite Dimensional vector space (FDVS)

→ The vector space $V(F)$ is said to finite dimensional vector space or finitely generated if there exists a finite subset S of V s.t $V = L(S)$.

Note: If there exists no finite subset which spans V then V is called an infinite dimensional vector space.

Ex: Let $S = \{(1,0),(0,1)\} \subseteq V_2(F)$ then $V_2(F)$ is FDVS.

Soln Let $(a,b) \in V_2(F)$; $a, b \in F$

then $(a,b) = x(1,0) + y(0,1)$; $x, y \in F$

$\Rightarrow (a,b) = (x,0) + (0,y)$
$= (x,y)$

$\Rightarrow \boxed{x=a}$; $\boxed{y=b}$

$\therefore (a,b) = a(1,0) + b(0,1) \in L(S)$

$\therefore (a,b) \in L(S)$

$\therefore V_2(F) \subseteq L(S)$ — ①

W.K.T $L(S) \subseteq V_2(F)$ — ②

$\therefore$ from (1) & (2) we have $V_2(F) = L(S)$.

$\therefore V_2(F)$ is a FDVS.

→ Similarly $V_3(\mathbb{R}) = \{(a,b,c) / a,b,c \in \mathbb{R}\}$ is a FDVS.

Since $V_3(\mathbb{R}) = L(S)$

where $S = \{(1,0,0),(0,1,0),(0,0,1)\} \subseteq V_3(\mathbb{R})$.

Similarly $V_n(\mathbb{R}) = \{(a_1, a_2 \ldots a_n) / a_1, a_2 \ldots a_n \in \mathbb{R}\}$ is a FDVS.

Since $V_n(\mathbb{R}) = L(S)$

where $S = \{(1,0,0,\ldots 0), (0,1,0\ldots 0), \ldots (0,0,\ldots,0,1)\} \subseteq V_n(\mathbb{R})$

<u>Note</u>: A vector space may have more than one basis

<u>Ex</u> (1) $S = \{(1,0),(0,1)\}$ is a basis of $\mathbb{R}^2(\mathbb{R})$

(2) $T = \{(1,1),(1,0)\}$ is also a basis of $\mathbb{R}^2(\mathbb{R})$

<u>Sol</u>n Let $a, b \in \mathbb{R}$ then

$a(1,1) + b(1,0) = (0,0)$

$\Rightarrow (a+b, a) = (0,0)$

$\Rightarrow a+b = 0, \boxed{a=0}$

$\Rightarrow \boxed{b=0}$

$\therefore T$ is LI.

W.K.T $L(T) \subseteq \mathbb{R}^2(\mathbb{R})$ —①

Let $(a,b) \in \mathbb{R}^2(\mathbb{R})$ then

$(a,b) = b(1,1) + (a-b)(1,0)$

$\therefore \mathbb{R}^2(\mathbb{R}) \subseteq L(T)$ —②

$\therefore$ from (1) & (2) $\mathbb{R}^2(\mathbb{R}) = L(T)$.

$\therefore T$ is a basis of $\mathbb{R}^2(\mathbb{R})$.

→ Show that the set $S = \{1, x, x^2, \ldots x^n\}$ of $n+1$ polynomials is a basis for the vector space $F_n[x]$ of all polynomials of degree n over the field F.

<u>Soln</u> Given that $S = \{1, x, x^2, \ldots x^n\} \subset F_n[x]$.

(i) To prove S is LI

Let $a_0, a_1, a_2 \ldots a_n \in F$ then

$a_0(1) + a_1(x) + a_2x^2 + \ldots + a_n(x^n) = 0$ (zero polynomial)

$\Rightarrow a_0 + a_1x + a_2x^2 + \ldots + a_nx^n = 0 + 0x + 0x^2 \ldots + 0x^n$

$\Rightarrow a_0 = a_1 = a_2 = \ldots = a_n = 0$

$\therefore S$ is LI

(ii) To prove $L(S) = F_n[x]$

W.K.T $L(S) \subseteq F_n[x]$

Let $f(x)$ be any polynomial of degree n over F.

i.e, $f(x) \in F_n[x]$.

then $f(x) = b_0 + b_1x + b_2x^2 + \ldots + b_nx^n$; where $b_0, b_1, \ldots b_n \in F$

$\Rightarrow f(x) = b_0(1) + b_1(x) + b_2(x^2) + \cdots + b_n(x^n)$

$= $ L.C. of elts of S

$\in L(S)$

$\therefore f(x) \in L(S)$

$\therefore$ If $f(x) \in F_n[x]$ then $f(x) \in L(S)$

$\therefore F_n[x] \subseteq L(S)$ —(2)

from (1) & (2) $L(S) = F_n[x]$

$\therefore$ S is a basis of $F_n[x]$.

Note: The above basis 'S' is the standard basis of the vector space of all polynomials of degree n over F.

Infinite dimensional vector space :-

Defn: The vector space $V(F)$ is said to be infinite dimensional vector space or infinitely generated if there exists an infinite subset S of V s.t $L(S) = V$.

Ex: Show that the set $S = \{1, x, x^2, \ldots, x^n, \ldots\}$ is a basis of the vector space $F[x]$ of all polynomials over the field F.

Soln: Given that $S = \{1, x, x^2, \ldots x^n\} \subseteq F[x]$.

(i) To prove S is LI.

$S' = \{x^{m_1}, x^{m_2}, \ldots x^{m_n}\}$ be a finite subset of S having n vectors.

Here $m_1, m_2, \ldots m_n$ are non-negative integers.

Let $a_1, a_2, \ldots a_n \in F$ then $a_1 x^{m_1} + a_2 x^{m_2} + \cdots + a_n x^{m_n} = 0$ (zero poly)

$\Rightarrow a_1 x^{m_1} + a_2 x^{m_2} + \cdots + a_n x^{m_n} = 0x^{m_1} + 0x^{m_2} + \cdots + 0x^{m_n}$.

$\Rightarrow a_1 = a_2 = \cdots = a_n = 0$

$\therefore S'$ is LI

$\therefore$ Every finite subset of S is LI.

$\therefore S$ is LI.

(ii) To prove $L(S) = F[x]$.

w.k.T $L(S) \subseteq F[x]$ —(1)

Let $f(x) \in F[x]$

i.e., $f(x) = b_0 + b_1 x + b_2 x^2 + \cdots + b_m x^m$ be any polynomial of degree m in $F[x]$

$= b_0(1) + b_1(x) + b_2x^2 + \cdots + b_m x^m + 0x^{m+1} + 0x^{m+2} + \cdots$

$=$ L.C. of elts of S

$\in L(S)$

$\therefore f(x) \in L(S)$

$\therefore$ If $f(x) \in F[x]$ then $f(x) \in L(S)$

$\therefore F[x] \subseteq L(S)$. —— (2)

$\therefore$ from (1) & (2) we have $L(S) = F[x]$

$\therefore S$ is a basis of $F[x]$.

Note: 1. The vector space $F[x]$ is an infinite dimensional vectorspace. Because there exists no finite subsets of $F[x]$ which spans $F[x]$.

2. The vector space $F[x]$ has no finite basis.

Existence of basis of a finite dimensional vectorspace:

Theorem → Every finite dimensional vector space $V(F)$ has a basis (or).

If $S = \{\alpha_1, \alpha_2, \ldots \alpha_m\}$ spans $V(F)$. i.e, $L(S) = V$ then there exists a subset of S which forms a basis of V.

Proof: Let $V(F)$ be a finite dimensional vectorspace. then $\exists$ a finite subset S of V s.t $L(S) = V$.

i.e, Let $S = \{\alpha_1, \alpha_2, \ldots \alpha_m\} \subseteq V$ s.t $L(S) = V$.

If S is LI then 'S' itself is a basis of V.

If S is LD then there exists a vector $\alpha_i \in S$ is a linear combination of its preceding vectors $\alpha_1, \alpha_2, \ldots \alpha_{i-1}$

i.e, $\alpha_i = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_{i-1}\alpha_{i-1}$ —— (1)

where $a_1, a_2, \ldots a_{i-1} \in F$

Now if we omit this vector α_i from the set 'S'. then the remaining set 'S'' having $m-1$ vectors $\alpha_1, \alpha_2, \ldots \alpha_{i-1}, \alpha_{i+1}, \ldots \alpha_m$

i.e, $S' = \{\alpha_1, \alpha_2, \ldots \alpha_{i-1}, \alpha_{i+1}, \ldots \alpha_m\} \subseteq S$

Now we show that $L(S') = V$

Clearly $S' \subset S \Rightarrow L(S') \subset L(S)$

$\Rightarrow L(S') \subset V$ — (1) $(\because L(S) \subset V)$

Let $\alpha \in V$ then α is l.c. of elts of S.

$\therefore \alpha = b_1\alpha_1 + b_2\alpha_2 + \cdots + b_i\alpha_i + b_{i+1}\alpha_{i+1} + \cdots + b_m\alpha_m$

where $b_1, b_2, \ldots b_{i-1}, b_i, b_{i+1}, \ldots b_m \in F$

(I) $\Rightarrow \alpha = b_1\alpha_1 + b_2\alpha_2 + \cdots + b_{i-1}\alpha_{i-1} + b_i(a_1\alpha_1 + a_2\alpha_2 + \cdots + a_{i-1}\alpha_{i-1})$
$+ b_{i+1}\alpha_{i+1} + \cdots + b_m\alpha_m$

$= (b_1 + b_i a_1)\alpha_1 + (b_2 + b_i a_2)\alpha_2 + \cdots + (b_{i-1} + b_i a_{i-1})\alpha_{i-1}$
$+ b_{i+1}\alpha_{i+1} + \cdots + b_m\alpha_m$

$=$ l.c. of $\alpha_1, \alpha_2, \ldots \alpha_{i-1}, \alpha_{i+1}, \ldots \alpha_m$

$=$ l.c. of elts of the set S'

$\in L(S')$

$\therefore \alpha \in L(S')$

$\therefore V \subseteq L(S')$ — (2)

$\therefore$ from (1) & (2)

$V = L(S')$.

If S' is LI then S' is a basis of $V(F)$.

If S' is LD then proceeding ~~as~~ above we get new set S'' of $m-2$ vectors which generates V. i.e, $L(S'') = V$.

continuing in this way, after finite no. of steps, obtain a LI subset of S which generates V and therefore it is a basis of V.

At the most repeating the ~~proce~~ procedure we left with a subset having a single non-zero vector which generates V and we know that a set containing a single non-zero vector is LI.

$\therefore$ It forms a basis of V

Theorem: If $V(F)$ is a finite dimensional vector space, then any two bases of V have same number of elements.

Proof: Let $V(F)$ be a finite dimensional vector space. then it has a basis.

Let $S_1 = \{\alpha_1, \alpha_2, \ldots \alpha_m\}$ and $S_2 = \{\beta_1, \beta_2 \ldots \beta_n\}$ be two bases of V.

Now we shall prove that $m = n$.

If possible let $m \neq n$ then $m > n$ or $m < n$

Suppose $m > n$:

Since $\alpha_i \in V$ and S_2 is a basis of V, $\exists$ $a_{ji} \in F$ s.t

$$\alpha_i = a_{1i}\beta_1 + a_{2i}\beta_2 + \cdots + a_{ni}\beta_n \; ; \; i = 1, 2, \ldots m. \quad \text{—— (1)}$$

Now consider the relation

$$x_1\alpha_1 + x_2\alpha_2 + \cdots + x_m\alpha_m = 0, \; x_i \in F \quad \text{—— (2)}$$

from (1) & (2) we have

$$x_1(a_{11}\beta_1 + a_{21}\beta_2 + a_{31}\beta_3 + \cdots + a_{n1}\beta_n) + x_2(a_{12}\beta_1 + a_{22}\beta_2 + a_{32}\beta_3 + \cdots + a_{n2}\beta_n)$$

$$+ \cdots + x_m(a_{1m}\beta_1 + a_{2m}\beta_2 + \cdots + a_{nm}\beta_n) = 0$$

$$\Rightarrow (x_1a_{11} + x_2a_{12} + \cdots + x_m a_{1m})\beta_1 + (x_1a_{21} + x_2a_{22} + \cdots + x_m a_{2m})\beta_2$$

$$+ \cdots + (x_1a_{n1} + x_2a_{n2} + \cdots + x_m a_{nm})\beta_n = 0 \quad \text{—— (3)}$$

Since $\beta_1, \beta_2, \ldots \beta_n$ are LI.

from (3) we have

$$\left.\begin{array}{l} x_1a_{11} + x_2a_{12} + \cdots + x_m a_{1m} = 0 \\ x_1a_{21} + x_2a_{22} + \cdots + x_m a_{2m} = 0 \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ x_1a_{n1} + x_2a_{n2} + \cdots + x_m a_{nm} = 0 \end{array}\right\} \text{—— (4)}$$

$$\Rightarrow \left.\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m = 0 \\ \cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m = 0 \end{array}\right\} \text{—— (5)}$$

$\therefore$ This is a system of n homogeneous linear eqns in m unknown variables.

As $m > n$ i.e, $n < m$

i.e, no. of eqns are less than no. of unknowns.

$\therefore$ The above system (5) of eqns have a non-zero solution.

i.e, there exist $x_1, x_2, \ldots x_m$ in F not all zero to satisfy the eqn (2).

$\therefore \alpha_1, \alpha_2, \ldots \alpha_m$ are LD

which contradicts that S_1 is a basis of $V(F)$.

$\therefore$ Our assumption that $m>n$ wrong.

$\therefore m \not> n$.

Similarly $m \not< n$

$\therefore m = n$

i.e, Any two bases of a FDVS $V(F)$ have the same no. of elts.

Dimension of a vector space:

Defn: The no. of elts in any basis of a finite dimensional vector space $V(F)$ is called the dimension of the vector space $V(F)$. and is denoted by $\dim V$ or $\dim_F V$.

Note: [1]. If a vectorspace $V(F)$ has a finite basis having n vectors then $\dim V = n$

[2]. If $\dim V = n$ then V has a basis containing n vectors say $S = \{\alpha_1, \alpha_2 \dots \alpha_n\}$

it means the vectors $\alpha_1, \alpha_2, \dots \alpha_n$ are L.I. and each vector $\alpha \in V$ is expressible as

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n$$ where $a_1, a_2, \dots a_n \in F$.

Ex: 1) $\dim \mathbb{R}^2 = 2$.
Since $\{(1,0), (0,1)\}$ is a basis of $\mathbb{R}^2$

2) $\dim \mathbb{R}^3 = 3$
Since $\{(1,0,0), (0,1,0), (0,0,1)\}$ is a basis of $\mathbb{R}^3$.

3) $\dim \mathbb{R}^n = n$
Since $\{(1,0,0 \dots 0), (0,1,0 \dots 0), \dots (0,0,\dots 1)\}$ is a basis of $\mathbb{R}^n$.

4) $\dim_{\mathbb{R}} \mathbb{C} = 2$
Since $\{1, i\}$ is a basis of $\mathbb{C}$ over $\mathbb{R}$.

5) If F is any field then $\dim_F F = 1$
Since $\{1\}$, a set consisting of the unity elt of F is a basis of F over F.

Similarly $\dim_{\mathbb{R}} \mathbb{R} = 1$; $\dim_{\mathbb{C}} \mathbb{C} = 1$.

Note: Every non-zero elt of F will form a basis of F.

Theorem: (I)

→ A finite dimensional vector space V(F) has dimension n iff n is the maximum no. of linearly independent vectors in any subset of V.

proof: N.C: Let dim V = n and let $S = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$ be a basis of V. Then $\alpha_1, \alpha_2 \ldots \alpha_n$ are L.I

Let $T = \{\beta_1, \beta_2, \ldots \beta_m\}$ be any subset of V s.t $m > n$.

If we prove that T is LD set then n is maximum no. of LI vectors in any subset of V.

Since $\beta_i \in V$ and S is a basis of V,

$\exists\ a_{ji} \in F$ s.t

$$\beta_i = a_{1i}\alpha_1 + a_{2i}\alpha_2 + \cdots + a_{ni}\alpha_n \; ; \; i = 1, 2, \ldots m. \quad \text{——(1)}$$

Consider the relation

$$x_1\beta_1 + x_2\beta_2 + x_3\beta_3 + \cdots + x_m\beta_m = 0 \; ; \; x_i \in F \quad \text{——(2)}$$

from (1) & (2) we have

$$x_1(a_{11}\alpha_1 + a_{21}\alpha_2 + a_{31}\alpha_3 + \cdots + a_{n1}\alpha_n) + x_2(a_{21}\alpha_1 + a_{22}\alpha_2 + \cdots + a_{2n}\alpha_n)$$
$$+ \cdots\cdots + x_m(a_{1m}\alpha_1 + a_{2m}\alpha_2 + \cdots + a_{nm}\alpha_n) = 0$$

$$\Rightarrow (a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1m}x_m)\alpha_1 + (a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m)\alpha_2$$
$$+ \cdots\cdots + (a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \cdots + a_{nm}x_m)\alpha_n = 0$$

Since $\alpha_1, \alpha_2, \ldots \alpha_n$ are LI

$$\left.\begin{aligned} \therefore\ a_{11}x_1 + a_{12}x_2 + \cdots + a_{1m}x_m &= 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2m}x_m &= 0 \\ \cdots\cdots\cdots\cdots & \\ \cdots\cdots\cdots\cdots & \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nm}x_m &= 0 \end{aligned}\right\} \text{——(3)}$$

This is a system of n homogeneous linear equations in m unknown variables.

As $m > n$ i.e, $n < m$

i.e, the no. of equations are less than no. of unknowns.

∴ The above system (3) of equations have non-zero solution.

i.e., there exist non-zero values of $x_1, x_2, \dots x_m$ to satisfy the relation (2).

$\therefore \beta_1, \beta_2, \dots \beta_m$ are L.D. $(m>n)$

$\therefore$ n is the maximum no. of L.I vectors in any subset of V.

S.C:

Let n be the maximum of L.I vectors in any subset of V.

Let $S = \{\alpha_1, \alpha_2, \dots \alpha_n\}$ be a L.I subset of V.

Now we have to prove that S is a basis of V.

For this we are enough to prove that $V = L(S)$.

Since $S \subset V$

$\therefore L(S) \subset V$ —— (1)

Let $\alpha \in V$ and $S = \{\alpha_1, \alpha_2, \dots \alpha_n\}$ is a maximal L.I set.

$\therefore T = \{\alpha_1, \alpha_2, \dots \alpha_n, \alpha\}$ is L.D. —— (2)

$\Rightarrow \exists$ atleast one non-zero scalar $a_1, a_2, \dots a_n, a \in F$

s.t $a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n + a\alpha = 0$ —— (3)

If $a = 0$ then

$a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n = 0$

$\Rightarrow a_1 = a_2 = \dots = a_n = 0$ $(\because S$ is LI$)$

$\therefore a = a_1 = a_2 = \dots = a_n = 0$

which contradicts (2).

$\therefore a \neq 0$

$\therefore$ from (3) we have

$a\alpha = -a_1\alpha_1 - a_2\alpha_2 - \dots - a_n\alpha_n$

$\Rightarrow \alpha = \left(\frac{-a_1}{a}\right)\alpha_1 + \left(\frac{-a_2}{a}\right)\alpha_2 + \dots + \left(\frac{-a_n}{a}\right)\alpha_n$

$\Rightarrow \alpha$ is L.C. of elts of S.

$\Rightarrow \alpha \in L(S)$

$\therefore V \subset L(S)$ —— (4)

from (1) & (4) we have $V = L(S)$

$\therefore$ S is a basis containing n vectors.

$\therefore \dim V = n$.

→ **Theorem:**
If $\dim V = n$ then any $n+1$ vectors are LD.

proof: Theorem (I) first part.

→ **Extension theorem:**

Every finite linearly independent subset of a finite dimensional vector space V over F can be extended to form a basis of V

(or)

If V is a finite dimensional vectorspace over F and if $S_1 = \{\alpha_1, \alpha_2, \dots \alpha_r\}$ is any LI set of vectors in V, Prove that, unless 'S_1' is a basis, we can find the vectors $\alpha_{r+1}, \alpha_{r+2}, \dots \alpha_n$ in V s.t $\{\alpha_1, \alpha_2 \dots \alpha_r, \alpha_{r+1}, \alpha_{r+2}, \dots \alpha_n\}$ is a basis of $V(F)$.

proof: Let $\dim V = n$, then 'n' is the maximum no. of LI vectors in any subset of V.

Since $S_1 = \{\alpha_1 \alpha_2 \dots \alpha_r\}$ is any LI set of vectors in V.
if S_1 spans V i.e, $L(S_1) = V$. then it forms a basis of V (here $r = n$)

Let $S_2 = \{\alpha_1, \alpha_2, \alpha_3 \dots \alpha_r, \alpha_{r+1}, \dots \alpha_n\}$ be the maximal LI subset of V.

If we P.T $L(S_2) = V$ then S_2 is a basis of V.

Let $\alpha \in V$ then $T = \{\alpha_1, \alpha_2 - \alpha_r, \alpha_{r+1}, \alpha_{r+2} \dots \alpha_n, \alpha\}$. which contains $n+1$ (i.e $> n$)
it must be LD.

$\therefore \exists$ atleast one non-zero scalar $a_1, a_2, a_3 \dots a_n, a \in F$
s.t $a_1\alpha_1 + a_2\alpha_2 + \dots + a_r\alpha_r + \dots + a_n\alpha_n + a\alpha = 0$ — (1)

If possible let $a = 0$, then $a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n = 0$

$\Rightarrow a_1 = a_2 = \dots = a_n = 0$ ($\because S_2$ is LI)

$\Rightarrow a_1 = a_2 = a_3 = \dots = a_n = a = 0$

which is contradiction to T is LD.

$\therefore a \neq 0$

$\therefore$ (1) $\equiv$ $a\alpha = -(a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n)$

$\Rightarrow \alpha = \left(\frac{-a_1}{a}\right)\alpha_1 + \left(\frac{-a_2}{a}\right)\alpha_2 + \dots + \left(\frac{-a_n}{a}\right)\alpha_n$

$\in L(S_2)$.

$\therefore V \subset L(S_2)$ — (3)

w.k.T $L(S_2) \subseteq V$ — (4)

from (3) & (4)

$V = L(S_2)$

$\therefore S_2$ is a basis.

Theorem: If $\dim V = n$ and $\{\beta_1, \beta_2, \ldots \beta_m\}$ is LI subset of V then $m \leq n$.

(or)

If $\dim V = n$ then a LI subset S_1 of V cannot have more than 'n' elements.

proof: Let $\dim V = n$ then 'n' is the maximum no. of LI vectors in any subset V.

Let $S_1 = \{\beta_1, \beta_2, \ldots \beta_m\}$ be a LI subset of V.

If it contains more than 'n' elts then S_1 is LD.

$\therefore$ A LI subset S_1 of V cannot have more than n elts.

Theorem (II)

If $\dim V = n$ and $S = \{\alpha_1, \alpha_2 \ldots \alpha_n\}$ is a LI subset of V then S is a basis of V.

proof: Since S is LI subset of V, it can be extended to form a basis of V.

Since $\dim V = n$ & S contains n LI vectors.

$\therefore$ S itself forms a basis of V.

Theorem:

If $\dim V = n$ and $S = \{\alpha_1, \alpha_2, \ldots \alpha_n\}$ spans V then S is a basis of V.

proof Since $\dim V = n$

$\therefore$ Any basis of V has exactly n elts.

Since S spans V.

i.e., $L(S) = V$.

$\therefore$ there exists any subset of S which forms a basis of V. (By existence of a basis of a FDVS theorem)

Since no basis of V can have fewer than 'n' elts.

$\therefore$ S itself forms a basis of V.

Note: If a vectorspace V(F) is of dimension 'n' then any set of n linearly independent vectors in V forms a basis of V.

(This result in theorem (II))

→ **Theorem**: Let $S=\{\alpha_1, \alpha_2, \ldots \alpha_n\}$ be a basis of a finite dimensional vector space $V(F)$ of dimension 'n'. Then every elt α of V can be uniquely expressed as $\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots\cdots + a_n\alpha_n$ where $a_1, a_2, \ldots a_n \in F$.

Proof: Since $S=\{\alpha_1\, \alpha_2 \cdots \alpha_n\}$ is a basis of V.

$\therefore L(S) = V$.

$\therefore$ Any vector $\alpha \in V$ can be expressed as

$$\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots\cdots + a_n\alpha_n \quad \text{—— (1)}$$

<u>To show that (1) is unique representation:</u>

Let us suppose that —— (2)

$$\alpha = b_1\alpha_1 + b_2\alpha_2 + \cdots\cdots + b_n\alpha_n$$ where $b_1, b_2, \ldots b_n \in F$.

from (1) & (2)
we have $a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n = b_1\alpha_1 + b_2\alpha_2 + \cdots + b_n\alpha_n$

$\Rightarrow (a_1-b_1)\alpha_1 + (a_2-b_2)\alpha_2 + \cdots\cdots + (a_n-b_n)\alpha_n = 0$

$\Rightarrow a_1-b_1=0,\ a_2-b_2=0, \cdots\ a_n-b_n=0$ ($\because S$ is LI)

$\Rightarrow a_1=b_1;\ a_2=b_2;\ \cdots\cdots\ a_n=b_n$.

$\therefore$ (1) is a unique expression of V as a l.c. of $\alpha_1, \alpha_2, \ldots\ldots \alpha_n$.

<u>Row Reduced Echelon matrix</u>:-

An echelon matrix is called a row reduced echelon matrix or row canonical form iff

(i) the distinguished elts are equal to 1.

and (ii) these elements (distinguished) are the only non-zero elements in their respective coloumns.

<u>Note</u>: The first non-zero elts in the rows of an echelon matrix are called distinguished elts ~~of~~ A.

<u>Ex</u>:- $\begin{bmatrix} 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & 7 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

are all row reduced echelon matrices

Theorem: Prove that the non-zero rows of an echelon matrix are LI.

Proof: Let $R_1, R_2, \dots R_{n-1}, R_n$ be the non-zero rows of an echelon matrix A.

If possible let $R_n, R_{n-1} \dots R_2, R_1$ be the LD. then one of the rows say R_m is a l.c. of its preceding rows.

i.e, $R_m = a_{m+1} R_{m+1} + a_{m+2} R_{m+2} + \dots + a_n R_n$ — ①

Let k^{th} elt of R_m be its non-zero entry.

Since A is an echelon form,

$\therefore$ The k^{th} elt of each $R_{m+1}, R_{m+2}, \dots R_n$ is zero.

$\therefore$ ① $\equiv$ the k^{th} elt of R_m = k^{th} elt of $[a_{m+1} R_{m+1} + a_{m+2} R_{m+2} + \dots + a_n R_n]$

$= a_{m+1}(0) + a_{m+2}(0) + \dots + a_n(0)$

$= 0$

$\therefore k^{th}$ elt of $R_m = 0$

$\therefore$ which contradicts the assumption that k^{th} elt of R_m is non-zero.

$\therefore R_1, R_2, \dots R_{n-1}, R_n$ are LI.

Problems

① Give examples of two different bases of $V_3(\mathbb{R})$ or $\mathbb{R}^3$

Sol^n Let $V_3(\mathbb{R}) = \{(a,b,c) / a,b,c \in \mathbb{R}\}$

Let $S_1 = \{(1,0,0), (0,1,0), (0,0,1)\} \subseteq V_3(\mathbb{R})$

and $S_2 = \{(0,1,0), (0,0,1), (2,3,4)\} \subset V_3(\mathbb{R})$

Now we show that the sets S_1 & S_2 both form basis for $V_3(\mathbb{R})$.

(I) Let $S_1 = \{(1,0,0), (0,1,0), (0,0,1)\} \subseteq V_3(\mathbb{R})$

(i) To show S_1 is LI.

Let $a_1, a_2, a_3 \in F$. then

$a_1(1,0,0) + a_2(0,1,0) + a_3(0,0,1) = (0,0,0)$

$\Rightarrow (a_1,0,0)+(0,a_2,0)+(0,0,a_3)=(0,0,0)$

$\Rightarrow (a_1,a_2,a_3)=(0,0,0)$

$\Rightarrow a_1=a_2=a_3=0$

$\therefore S_1$ is LI

(ii) To show $L(S_1)=V_3(\mathbb{R})$

w.k.T $L(S_1)\subseteq V_3(\mathbb{R})$ —(1)

Let $\alpha \in V_3(\mathbb{R})$

i.e, $\alpha=(a,b,c)\in V_3(\mathbb{R})$

then $(a,b,c)=a(1,0,0)+b(0,1,0)+c(0,0,1)$
$\in L(S_1)$

$\therefore \alpha \in L(S_1)$

$\therefore V_3(\mathbb{R})\subseteq L(S_1)$ —(2)

$\therefore$ from (1) & (2) we have
$V_3(\mathbb{R})=L(S_1)$.

$\therefore S_1$ is a basis of $V_3(\mathbb{R})$

(ii). $S_2=\{(0,1,0),(0,0,1),(2,3,4)\}\subseteq V_3(\mathbb{R})$

Similar.

2006 → Let V be the vector space of all 2×2 matrices over the field F. prove that V has dimension 4 by exhibiting a basis for V which has 4 elements.

Soln: Let $V(F)=\left\{\begin{bmatrix} a & b\\ c & d\end{bmatrix} \Big/ a,b,c,d\in F\right\}$

Let $\alpha_1=\begin{bmatrix}1&0\\0&0\end{bmatrix}$, $\alpha_2=\begin{bmatrix}0&1\\0&0\end{bmatrix}$, $\alpha_3=\begin{bmatrix}0&0\\1&0\end{bmatrix}$, $\alpha_4=\begin{bmatrix}0&0\\0&1\end{bmatrix}$

be four elts of V.

Let $S=\{\alpha_1,\alpha_2,\alpha_3,\alpha_4\}\subseteq V$

(i) To show S is LI:

If $a_1,a_2,a_3,a_4\in F$ then $a_1\alpha_1+a_2\alpha_2+a_3\alpha_3+a_4\alpha_4=0$

$\Rightarrow a_1\begin{bmatrix}1&0\\0&0\end{bmatrix}+a_2\begin{bmatrix}0&1\\0&0\end{bmatrix}+a_3\begin{bmatrix}0&0\\1&0\end{bmatrix}+a_4\begin{bmatrix}0&0\\0&1\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$

$\Rightarrow \begin{bmatrix}a_1&0\\0&0\end{bmatrix}+\begin{bmatrix}0&a_2\\0&0\end{bmatrix}+\begin{bmatrix}0&0\\a_3&0\end{bmatrix}+\begin{bmatrix}0&0\\0&a_4\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$

$\Rightarrow \begin{bmatrix}a_1&a_2\\a_3&a_4\end{bmatrix}=\begin{bmatrix}0&0\\0&0\end{bmatrix}$

$\Rightarrow a_1=a_2=a_3=a_4=0$. $\therefore S$ is LI

(ii) To show $L(S) = V$.

w.k.t $L(S) \subseteq V$ — ①

Let $\alpha = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in V$ then

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= a\alpha_1 + b\alpha_2 + c\alpha_3 + d\alpha_4 \in L(S).$$

$\therefore \alpha \in L(S)$

$\therefore V \subseteq L(S)$ — (2)

$\therefore$ from ① & (2) $V = L(S)$

$\therefore S$ is a basis of V.

Since the no. of elts in the basis 'S' is 4.

$\therefore \dim V = 4$.

→ Let V be the vector space of 2×2 matrices over F. Find a basis $\{A_1, A_2, A_3, A_4\}$ for V s.t. $A_i^2 = A_i$ for each i.

Sol^n: $V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} / a, b, c, d \in F \right\}$

Let $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, $A_3 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$, $A_4 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

be any four ~~vector~~ elts of V s.t $A_i^2 = A_i$ for each i.

Let $S = \{A_1, A_2, A_3, A_4\} \subseteq V$.

(i) To show S LI.

(ii) To show $L(S) = V$.

→ S.T the real field $\mathbb{R}$ is a vector space of infinite dimension over the rational field $\mathbb{Q}$.

Sol^n. We prove that the set $\{1, \pi, \pi^2, \dots \pi^n\}$ is LI over $\mathbb{Q}$ for any +ve integer 'n'

Suppose $a_0(1) + a_1(\pi) + a_2\pi^2 + \dots\dots + a_n\pi^n = 0$; where $a_i \in \mathbb{Q}$ and all a_i's are not zero.

Then π is a root of the non-zero polynomial $a_0 + a_1x + a_2x^2 + \dots + a_nx^n$ over $\mathbb{Q}$.

This is impossible, since π is a transcendental number.

$\therefore \{1, \pi, \pi^2, \dots \pi^n\}$ is LI over $\mathbb{Q}$ for all +ve integer 'n'.

Hence $\mathbb{R}$ is ~~of~~ an infinite dimension over $\mathbb{Q}$.

→ Determine whether or not the vectors $(1,-3,2)$, $(2,4,1)$ and $(1,1,1)$ form a basis of $\mathbb{R}^3$.

Sol^n: W.K.T $\dim(\mathbb{R}^3) = 3$

if we show that the given three vectors are linearly independent they form a basis of $\mathbb{R}^3$.

Now form the matrix A, whose rows are given vectors ~~and~~

$$\therefore A = \begin{bmatrix} 1 & -3 & 2 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

$$|A| = \begin{vmatrix} 1 & -3 & 2 \\ 2 & 4 & 1 \\ 1 & 1 & 1 \end{vmatrix} = 1(3)+3(1)+2(-2)$$
$$= 3+3-4$$
$$= 2 \neq 0$$

$\therefore$ The given vectors are L.I.

$\therefore$ They form a basis of $\mathbb{R}^3$.

→ Let V be vector space of ordered pairs of complex numbers over the field $\mathbb{R}$. i.e, let V be the vector space $C(\mathbb{R})$

S.T the set $S = \{(1,0), (i,0), (0,1), (0,i)\}$ is a basis for V.

→ S.T the vectors $(1,0,-1)$, $(0,-3,2)$ and $(1,2,1)$ form a basis for the vector space $\mathbb{R}^3(\mathbb{R})$.

Sol^n W.K.T $\dim \mathbb{R}^3 = 3$

Now form the matrix A whose rows are given vectors.

$$\therefore A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -3 & 2 \\ 1 & 2 & 1 \end{bmatrix}$$

$$\text{Now } |A| = 1(-7) - 0(\ ) - 1(3)$$
$$= -7-3 = -10 \neq 0$$

$\therefore$ The given vectors are L.I.

$\therefore$ They form a basis for $V(F)$.

→ S.T the set $\{(1,i,0), (2i,1,1), (0,1+i,1-i)\}$ is a basis for $V_3(\mathbb{C})$.

Sol^n W.K.T $\dim V_3(\mathbb{C}) = 3$.

Now form the matrix whose rows are given vectors

$|\ \ | \neq 0$

→ S.T the set $S = \{(1,0,0), (1,1,0), (4,5,0)\}$ is not a basis of $\mathbb{R}^3$. (44)

Sol: W.K.T $\dim \mathbb{R}^3 = 3$

Form the matrix A whose rows are given set of vectors.

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 4 & 5 & 0 \end{bmatrix}$$

$\therefore |A| = 0$

$\therefore$ The given set S is LD.

$\therefore$ S cannot be a basis of $\mathbb{R}^3$.

1993 S.T the set $S = \{(1,0,0), (1,1,0), (1,1,1), (0,1,0)\}$ is a spanning set of $\mathbb{R}^3$ but not a basis of $\mathbb{R}^3$.

Sol: To show $L(S) = \mathbb{R}^3$.

W.K.T $L(S) \subseteq \mathbb{R}^3$ — (I)

Let $(a,b,c) \in \mathbb{R}^3$ then

$(a,b,c) = x_1(1,0,0) + x_2(1,1,0) + x_3(1,1,1) + x_4(0,1,0)$

$= (x_1 + x_2 + x_3,\ x_2 + x_3 + x_4,\ x_3)$

$\Rightarrow x_1 + x_2 + x_3 = a$ — (1)

$x_2 + x_3 + x_4 = b$ — (2)

$x_3 = c$ — (3)

(2) ≡ $x_2 + x_4 = b - c$ — (4)

(1) ≡ $x_1 + x_2 = a - c$ — (5)

Take $x_2 = b$; $x_4 = -c$ in (4)

(5) ≡ $x_1 = a - b - c$

$\therefore (a,b,c) = (a-b-c)(1,0,0) + b(1,1,0) + c(1,1,1) - c(0,1,0)$

$\in L(S)$

$\therefore \mathbb{R}^3 \subseteq L(S)$ — (II)

from (I) & (II): $\mathbb{R}^3 = L(S)$

Since $\dim \mathbb{R}^3 = 3$ and S contains $4 = (3+1)$ vectors.

$\therefore$ S is LD.

$\therefore$ S cannot be a basis of $\mathbb{R}^3$

→ Let $\{a, b, c\}$ be a basis for the vector space $\mathbb{R}^3$.

P.T the sets

$\{a+b, b+c, c+a\}$, $\{a, a+b, a+b+c\}$ are also bases of $\mathbb{R}^3$.

Sol^n Since $\{a, b, c\}$ is a basis of $\mathbb{R}^3$.

$\therefore \dim \mathbb{R}^3 = 3$

(i) Now let $x, y, z \in \mathbb{R}$ then

$$x(a+b) + y(b+c) + z(c+a) = 0$$

$$\therefore x = y = z = 0$$

$\therefore \{a+b, b+c, c+a\}$ is L.I

$\therefore$ It is a basis of $\mathbb{R}^3$.

(ii) Now let $x, y, z \in \mathbb{R}$ then

$$xa + y(a+b) + z(a+b+c) = 0$$

2008, 20M → Find the dimension of the subspace of $\mathbb{R}^4$ spanned by the set $\{(1,0,0,0), (0,1,0,0), (1,2,0,1), (0,0,0,1)\}$. Hence find a basis for the subspace.

Theorem:-
Let $V(F)$ be a vector space and a subset $S=\{\alpha_1,\alpha_2,\dots\alpha_n\}$ of $V(F)$ (i.e, $S\subseteq V$) be a linearly independent set. if $\alpha\in V(F)$ and $\alpha\notin L(S)$ then show that $S_1=\{\alpha,\alpha_1,\alpha_2\dots\alpha_n\}$ is a linearly independent set

(or)

If $S=\{\alpha_1,\alpha_2,\dots\alpha_n\}$ is a LI set of vectors in V and $\alpha\in V$ is such that $\alpha\notin L(S)$, then $\{\alpha,\alpha_1,\alpha_2,\dots\alpha_n\}$ is LI set.

proof: Let $a, a_1, a_2, \dots a_n\in F$

then $a\alpha+a_1\alpha_1+a_2\alpha_2+\dots\dots+a_n\alpha_n=0$ —— (1)

If $a\neq0$ then

$\alpha=\left(\frac{-a_1}{a}\right)\alpha_1+\left(\frac{-a_2}{a}\right)\alpha_2+\dots\dots+\left(\frac{-a_n}{a}\right)\alpha_n$

$\in L(S)$.

$\Rightarrow \alpha\in L(S)$ which is a contradiction to the hypothesis that $\alpha\notin L(S)$

$\therefore a=0$

(1) $\equiv a_1\alpha_1+a_2\alpha_2+\dots\dots+a_n\alpha_n=0$

$\Rightarrow a_1=a_2=\dots=a_n=0$ ($\because S$ is LI)

$\Rightarrow a=a_1=a_2=\dots\dots=a_n=0$

$\therefore S_1=\{\alpha,\alpha_1,\alpha_2,\dots\alpha_n\}$ is LI set.

Problem: $\Rightarrow$ Extend the set $\{(1,1,1),(1,0,0)\}$ to form a basis of $\mathbb{R}^3$.

Sol^n: Method(I)

Let $\mathbb{R}^3=\{(x,y,z)/x,y,z\in\mathbb{R}\}$

Let $S=\{(1,1,1),(1,0,0)\}\subseteq\mathbb{R}^3$.

Let $a,b\in\mathbb{R}$ then

$a(1,1,1)+b(1,0,0)=(0,0,0)$

$\Rightarrow a+b=0 \Rightarrow \boxed{b=0}$

$\boxed{a=0}$

$\therefore a=b=0$

$\therefore S$ is LI.

Now $L(S)=\{a(1,1,1)+b(1,0,0)/a,b\in\mathbb{R}\}$

$=\{(a+b,a,a)/a,b\in\mathbb{R}\}$

Let $\alpha=(0,0,1)\in V$ then $\alpha\notin L(S)$.

$\therefore$ The set $S'=\{(1,1,1),(1,0,0),(0,0,1)\}$ is LI.

$\therefore S'$ is a basis of $\mathbb{R}^3$.

Similarly $(0,1,0) \notin L(S)$.

$\therefore$ The set $\{(1,1,1), (1,0,0), (0,1,0)\}$ is LI set.

$\therefore$ It is also basis of $\mathbb{R}^3$.

Method II

$\mathbb{R}^3 = \{(x,y,z) / x,y,z \in \mathbb{R}\}$

$S = \{(1,1,1), (1,0,0)\}$ LI set.

Since $a(1,1,1) + b(1,0,0) = (0,0,0)$ where $a, b \in \mathbb{R}$

$\Rightarrow (a+b, a, a) = (0,0,0)$

$\Rightarrow a = b = 0$

$\therefore S$ is LI

W.K.T the vectors $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$ form a standard basis of $\mathbb{R}^3$.

$\therefore$ The vectors $\alpha = (1,1,1)$, $\beta = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$ span $\mathbb{R}^3$.

but any basis of $\mathbb{R}^3$ contains exactly 3 LI elts.

Let us check whether α, β, e_2 are LI or not:

Now form the matrix A whose rows the vectors α, β, e_2.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \Rightarrow |A| = 1(-1) - 1(0) = -1 \neq 0$$

$\therefore \alpha, \beta, e_2$ are LI vectors.

$\therefore$ The vectors form a basis.

Similarly the set $\{\alpha, \beta, e_3\}$ is also LI.

$\therefore$ It is a basis of $\mathbb{R}^3$.

Method III Let $S = \{(1,1,1), (1,0,0)\} \subseteq \mathbb{R}^3$

since $a(1,1,1) + b(1,0,0) = (0,0,0)$ where $a, b \in \mathbb{R}$

$\Rightarrow a = 0, b = 0$

$\therefore S$ is LI

W.K.T the vectors $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, & $e_3 = (0,0,1)$ form a standard basis of $\mathbb{R}^3$.

$\therefore$ The vectors $\alpha = (1,1,1)$, $\beta = (1,0,0)$, $e_2 = (0,1,0)$, $e_3 = (0,0,1)$ span $\mathbb{R}^3$.

but any basis of $\mathbb{R}^3$ contains exactly 3 L.I. elts.

Let us check whether α, β, e_2 are L.I or not :-

Now form the matrix A whose rows are the vectors α, β, e_2 reduce it to echelon matrix.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\underset{R_2 \to R_2 - R_1}{\sim} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 1 & 0 \end{bmatrix} \underset{R_3 \to R_3 + R_2}{\sim} \begin{bmatrix} 1 & 1 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix}$$

$\therefore$ The echelon matrix of A has no zero rows.

$\therefore$ The vectors α, β, e_2 are L.I.

$\therefore$ They form a basis of $\mathbb{R}^3$.

and also the vectors $(1,1,1)$, $(0,-1,-1)$ and $(0,0,-1)$ are L.I. ($\because$ The non-zero rows of matrix are L.I).

$\therefore$ These are also form a basis of $\mathbb{R}^3$.

Note: The extension of linearly independent vectors to a basis is not unique.

$\rightarrow$ Extend the set $\{(0,0,0,3), (1,1,0,0), (0,1,-1,0)\}$ to form a basis of $\mathbb{R}^4$.

$\rightarrow$ Extend the set $S = \{(1,1,0)\}$ to form two different bases of $\mathbb{R}^3$.

Sol^n Since $(1,1,0) \neq (0,0,0)$

$\therefore$ S is L.I set.

and $L(S) = \{a(1,1,0) / a \in \mathbb{R}\}$
$= \{(a,a,0) / a \in \mathbb{R}\}$

Since $(0,0,1) \notin L(S)$

$\therefore S_1 = \{(1,1,0), (0,0,1)\}$ is L.I.

Now $L(S_1) = \{a(1,1,0) + b(0,0,1) / a, b \in \mathbb{R}\}$
$= \{(a+b, a, b) / a, b \in \mathbb{R}\}$

Since $(0,1,1) \notin L(S)$

$S_2 = \{(1,1,0), (0,0,1), (0,1,1)\}$ is L.I

$\therefore S_2$ is a basis of $\mathbb{R}^3$.

Similarly $\{(1,1,0), (0,0,1), (0,1,0)\}$ is also basis of $\mathbb{R}^3$.

→ Extend the set $\{(3,-1,2)\}$ to two different bases for $\mathbb{R}^3$.

→ Can the set $\{(1,0,0,0), (0,1,0,0), (1,-1,0,0)\}$ be extended to form a basis of $\mathbb{R}^4$?

Sol^n The given set of vectors are not L.I vectors.

Since $1(1,0,0,0) + (-1)(0,1,0,0) + (-1)(1,-1,0,0) = (0,0,0,0)$.

$\therefore$ The given set of vectors cannot be extend to form a basis

→ Determine whether or not the following vectors form a basis.

(i) $(1,-1,0), (1,3,-1), (5,3,-2)$ of $\mathbb{R}^3(\mathbb{R})$

(ii) $(1,0,1), (1,1,0), (1,1,-1)$ of $\mathbb{R}^3(\mathbb{R})$

(iii) $(6,2,3,4), (0,5,-3,1), (0,0,7,-2), (0,0,0,4)$ of $\mathbb{R}^4(\mathbb{R})$

iv) $(1,-2,4,1), (2,-3,9,-1), (1,0,6,-5), (2,-5,7,5)$ of $\mathbb{R}^4(\mathbb{R})$

1998 → Given two linearly independent vectors $(1,0,1,0)$ and $(0,-1,1,0)$ of $\mathbb{R}^4$, find a basis of $\mathbb{R}^4$ which includes these two vectors.

→ Let V be the vectorspace of all 2×2 symmetric matrices over $\mathbb{R}$. Find a basis and the dimension of V.

$(\because A^T = A$ A is symmetric$)$

Sol^n $V = \left\{ \begin{bmatrix} a & b \\ b & c \end{bmatrix} / a, b, c \in \mathbb{R} \right\}$.

Let $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \subseteq V$

① To show S is L.I.

Let $x, y, z \in \mathbb{R}$ then

$x\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + y\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + z\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow x = y = z = 0$

(2) To show $L(S) = V$.

W.K.T $L(S) \subseteq V$ —— (i)

Let $\begin{bmatrix} a & b \\ b & c \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\in L(S)$.

$\therefore V \subseteq L(S)$ —— (ii)

$\therefore$ from (i) & (ii) $L(S) = V$.

$\therefore$ S is a basis of V.

$\therefore \dim V = 3$

→ Let V be the vectorspace of 3×3 symmetric matrices over F. then show that $\dim V = 6$ by exhibiting a basis of V.

Sol: Let $V = \left\{ \begin{bmatrix} a & h & g \\ h & b & e \\ g & e & c \end{bmatrix} \; a, b, c, h, g, e \in F \right\}$

Let $S = \left\{ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \right\} \subseteq V.$

Note: → Dimension of the vectorspace V of all 2×2 symmetric matrices is $3 = 2+1$

→ Dimension of the vectorspace V of all 3×3 symmetric matrices is $6 = 3+2+1$

→ Dimension of the vectorspace V of all $n\times n$ symmetric matrices is $n + (n-1) + (n-2) + \cdots\cdots + 3 + 2 + 1$.

$= \dfrac{n(n+1)}{2}$

→ V be the vector space of 2×2 antisymmetric matrices over F. Show that $\dim V = 1$ by exhibiting a basis of V.

Sol^n Let $V = \left\{ \begin{bmatrix} 0 & -a \\ a & 0 \end{bmatrix} / a \in F \right\}$

Let $S = \left\{ \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \right\} \subseteq V$.

→ Let V be the vector space of 3×3 antisymmetric matrices over F. Show that $\dim V = 3$ by exhibiting a basis of V.

Sol^n Let $V = \left\{ \begin{bmatrix} 0 & h & g \\ -h & 0 & e \\ -g & -e & 0 \end{bmatrix} / h, g, e \in F \right\}$

Let $S = \left\{ \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \right\} \subseteq V$.

Note: → The dimension of the vectorspace of 2×2 skew symmetric matrices over F is $1\,(=2-1)$

→ The dimension of the vector space of 3×3 skew symmetric matrices over F is $3\,(=(3-1)+(3-2))$

→ The dimension of the vector space of $n\times n$ skew symmetric matrices over F is $(n-1)+(n-2)+\cdots+2+1$
$$=\frac{n(n-1)}{2}$$

Note: Let V be the vector space of $m\times n$ matrices over a field F.
Let $E_{ij}\in V$ be the matrix with 1 as ij-entry and elsewhere zero then the set $\{E_{ij}\}$ is a basis of V and $\dim V = mn$.
(This basis is called the standard basis of V)

→ Let V(IR) be the real vectorspace of all 2×3 matrices with real entries. Find a basis for V(R). What is the dimension of V(R).?

Sol^n Let $S=\left\{\begin{bmatrix}1&0&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&1&0\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&1\\0&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\1&0&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&1&0\end{bmatrix},\begin{bmatrix}0&0&0\\0&0&1\end{bmatrix}\right\}\subseteq V(R)$

are L.I. and $L(S)=V$.

$\therefore$ S is a basis of V.

$\therefore \dim V = 6$

→ Let V be the set of all real valued functions $y=f(x)$ satisfying $\frac{d^2y}{dx^2}+4y=0$. Prove that V is a 2-dimensional real vector space.

Sol^n $\frac{d^2y}{dx^2}+4y=0 \Rightarrow (D^2+4)y=0$ —— ① where $D=\frac{d}{dx}$.

A.E of ① is $m^2+4=0$
$\Rightarrow m=\pm 2i$

$\therefore$ G.S. of ① is $y=c_1\cos 2x + c_2 \sin 2x$ —— ②
where c_1 and c_2 are any real constants.

$\therefore$ Since V is the set of all real valued functions
$y=f(x)$ satisfying $\frac{d^2y}{dx^2}+4y=0$

$\therefore V=\{y=c_1\cos 2x + c_2\sin 2x \,/\, c_1, c_2\in\mathbb{R}\}$ is a vectorspace.
(Here we must show V is a vectorspace).

Let $S=\{\cos 2x, \sin 2x\}\subseteq V$.

The wronksian of $y_1(x)=\cos 2x$, $y_2(x)=\sin 2x$

is $W(x)=\begin{vmatrix} y_1(x) & y_2(x)\\ y_1'(x) & y_2'(x)\end{vmatrix}$

$$= \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2 \neq 0.$$

$\therefore$ S is a LI subset of V.

By (2) $L(S) = V$.

$\therefore$ S is a basis of V over $\mathbb{R}$

$\therefore$ V is a two dimensional real vector space

→ Let V be the set of all real-valued functions $y = f(x)$ satisfying $\frac{d^3y}{dx^3} - 7\frac{dy}{dx} - 6y = 0$.
S.T $V(\mathbb{R})$ is a 3-dimensional real vector space. write down a basis of this vector space.

Sol: $m^3 - 7m - 6 = 0 \Rightarrow (m+1)(m^2 - m - 6) = 0$
$\Rightarrow (m+1)(m-3)(m+2) = 0$
$\Rightarrow m = -1, -2, 3.$

$$\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}$$

→ S.T the set of all real valued continuous functions $y = f(x)$ satisfying the differential equation $\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 6y = 0$ is a vectorspace over $\mathbb{R}$.
Give a basis for the vectorspace.

$m = -1, -2, -3.$

→ S.T the matrices $\begin{bmatrix} 1 & 5 \\ 5 & 2 \end{bmatrix}$, $\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$ and $\begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix}$ form a basis of $V(\mathbb{R})$.
where V is the vector space of all 2×2 symmetric matrices over reals.

→ S.T the dimension of the vector space $Q(\sqrt{2})$ over Q is 2.

Sol: Let $Q(\sqrt{2}) = \{a + b\sqrt{2} / a, b \in Q\}$.

Let $S = \{1, \sqrt{2}\} \subseteq Q\sqrt{2}$

(i) To show S is LI

Let $x, y \in Q$ then
$x(1) + y(\sqrt{2}) = 0 + 0\sqrt{2}$
$\Rightarrow x = y = 0$

(ii) $L(S) = Q(\sqrt{2})$.

W.k.T $L(S) \subseteq Q\sqrt{2}$ —(1)

and let $a + b\sqrt{2} \in Q\sqrt{2}$ then $a + b\sqrt{2} = a(1) + b\sqrt{2} \in L(S)$

$\therefore Q(\sqrt{2}) \subseteq L(S)$. —(2)

$\therefore$ from (1) & (2) $L(S) = Q(\sqrt{2})$.

$\therefore$ S is a basis and $\dim(Q\sqrt{2}) = 2$

→ S.T the dimension of vector space $Q(\sqrt{2},\sqrt{3})$ over Q is 4.

Sol^n Let $Q(\sqrt{2},\sqrt{3}) = \{a+b\sqrt{2}+c\sqrt{3}+d\sqrt{2}\sqrt{3} / a,b,c,d \in Q\}$.

Let $S = \{1, \sqrt{2}, \sqrt{3}, \sqrt{6}\}$

→ S.T $f_1(t) = 1$, $f_2(t) = t-2$, $f_3(t) = (t-2)^2$ form a basis of P_3, the space of polynomial with degree ≤ 2.

Express $3t^2-5t+4$ as a l.c. of f_1, f_2, f_3.

Sol^n Let $f(t) = 3t^2-5t+4 \in P_3$.

then $f(t) = x f_1(t) + y f_2(t) + z f_3(t)$ where $x, y, z \in F$.

$\Rightarrow 3t^2-5t+4 = x(1) + y(t-2) + z(t-2)^2$ —— ①

$= x+ty-2y+t^2z+4z-4tz$.

$\Rightarrow 3t^2-5t+4 = zt^2+(y-4z)t+(x-2y+4z)$

$\Rightarrow \boxed{z=3}$

$y-4z = -5$

$\Rightarrow \boxed{y=7}$

$x-2y+4z = 4$

$\Rightarrow \boxed{x=6}$

$\therefore$ ① $\equiv$ $3t^2-5t+4 = 6(1) + 7(t-2) + 3(t-2)^2$

$=$ l.c. of f_1, f_2, f_3

Dimension of a subspace:-

Theorem: If W is a subspace of a finite dimensional vectorspace $V(F)$ then W is finite dimensional and $\dim W \leq \dim V$.

Further $V=W \Leftrightarrow \dim V = \dim W$.

proof Given that W is a subspace of finite dimensional vector space $V(F)$.

Let $\dim V = n$

(i) To prove W is finite dimensional.

If possible suppose that W is not finite dimensional.

then W has infinite basis.

Take S_1 is an infinite basis of W.

$\therefore S_1$ is LI in W.

$\Rightarrow S_1$ is LI in V.

But S_1 is the infinite set.

$\therefore S_1$ is a LI subset of V having more than 'n' elts.

which is contradiction.

$\therefore$ our supposition is wrong.

$\therefore$ W is a finite dimensional.

Take dim W = m

Now we have to S.T $m \leq n$.

Let $S_1 = \{\alpha_1, \alpha_2, \ldots \alpha_m\}$ be a basis of W.

$\Rightarrow S_1$ is LI set in W.

$\Rightarrow S_1$ is LI set in V.

Any LI subset of vector space V(F) can be extended to form a basis of V.

$\therefore$ there exists a basis S of V s.t $S_1 \subseteq S$.

$\Rightarrow$ No. of elts in $S_1 \leq$ No. of elts in S.

$\Rightarrow m \leq n$

i.e, $\dim W \leq \dim V$.

(ii) If V = W then

W is a subspace of V and

V is " " W

$\therefore \dim W \leq \dim V$ & $\dim V \leq \dim W$.

$\Rightarrow \dim V = \dim W$.

Conversely suppose that $\dim V = \dim W$.

Let $\dim V = \dim W = n$ (say)

Let S be a basis of W.

then L(S) = W and S has 'n' LI vectors.

Also S is subset of V ($\because S \subset W \subset V$)

and S has n LI vectors (i.e, S is LI in V)

$\Rightarrow$ S is a basis of V.

$\Rightarrow L(S) = V$.

$\therefore \underline{V = W}$

Note 1. If $W = \{0\}$ then the dimension $W = 0$

2. If W is a proper subspace of a finite-dimensional vectorspace V, then W is finite dimensional and $\dim W < \dim V$

3. If V is finite dimensional and W is a subspace of V such that $\dim V = \dim W$. then $V = W$.

85, 86, 88 → If W_1, W_2 are two subspaces of a finite dimensional vectorspace $V(F)$, then $\dim(W_1+W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$.

proof: Given that W_1 & W_2 are two subspaces of $V(F)$.

$\therefore W_1+W_2$, $W_1 \cap W_2$ are also subspaces of $V(F)$.

Since W_1, W_2, W_1+W_2 & $W_1 \cap W_2$ are subspaces of finite dimensional vector space $V(F)$.

$\therefore W_1, W_2, W_1+W_2$ & $W_1 \cap W_2$ are all finite dimensional.

Let $\dim(W_1 \cap W_2) = K$ and

Let $S = \{\gamma_1, \gamma_2, \dots \gamma_K\} \subseteq W_1 \cap W_2$ be a basis of $W_1 \cap W_2$.

then $S \subseteq W_1$ and $S \subseteq W_2$

Since S is LI and $S \subseteq W_1$

$\therefore S$ can be extended to form a basis of W_1

Let $S_1 = \{\gamma_1, \gamma_2, \dots \gamma_K, \alpha_1, \alpha_2 \dots \alpha_m\}$ be a basis of W_1.

$\therefore \dim(W_1) = k+m$

Since S is LI and $S \subseteq W_2$.

$\therefore S$ can be extended to form a basis of W_2.

Let $S_2 = \{\gamma_1, \gamma_2 \dots \gamma_K, \beta_1, \beta_2, \dots \beta_t\}$ be a basis of W_2.

$\therefore \dim(W_2) = k+t$

$$\therefore \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2) = (k+m)+(k+t)-k$$
$$= k+m+t.$$

Now we have to show that $\dim(W_1+W_2) = k+m+t$.

forthis we have to show that the set

$S_3 = \{\gamma_1, \gamma_2, \dots \gamma_k, \alpha_1, \alpha_2, \dots \alpha_m, \beta_1, \beta_2, \dots \beta_t\}$ is a basis of W_1+W_2.

(i) To show S_3 is LI.

Let $c_1, c_2, \ldots, c_k, a_1, a_2, \ldots a_m, b_1, b_2 \ldots b_t \in F$

then

$$(c_1\gamma_1 + c_2\gamma_2 + \cdots + c_k\gamma_k) + a_1\alpha_1 + a_2\alpha_2 + \cdots + a_m\alpha_m + b_1\beta_1 + b_2\beta_2 + \cdots + b_t\beta_t = 0 \quad (1)$$

$$\Rightarrow b_1\beta_1 + b_2\beta_2 + \cdots + b_t\beta_t = -(c_1\gamma_1 + c_2\gamma_2 + \cdots + c_k\gamma_k + a_1\alpha_1 + a_2\alpha_2 + \cdots + a_m\alpha_m) \quad (2)$$

Now $-(c_1\gamma_1 + c_2\gamma_2 + \cdots + c_k\gamma_k + a_1\alpha_1 + a_2\alpha_2 + \cdots + a_m\alpha_m) \in W_1$ (3) ($\because$ it is l.c. of elts of S_1)

and $b_1\beta_1 + b_2\beta_2 + \cdots + b_t\beta_t \in W_2$ (4) ($\because$ it is a l.c. of elts of S_2)

(2) $\equiv$ $b_1\beta_1 + b_2\beta_2 + \cdots + b_t\beta_t \in W_1$ (5) (by (3))

$\therefore$ from (4) & (5) we have

$b_1\beta_1 + b_2\beta_2 + \cdots + b_t\beta_t \in W_1 \cap W_2$.

$\therefore$ It can be expressed as a l.c. of the basis of $W_1 \cap W_2$.

$\therefore$ we have

$b_1\beta_1 + b_2\beta_2 + \cdots + b_t\beta_t = d_1\gamma_1 + d_2\gamma_2 + \cdots + d_k\gamma_k$.

$\Rightarrow b_1\beta_1 + b_2\beta_2 + \cdots + b_t\beta_t - d_1\gamma_1 - d_2\gamma_2 - \cdots - d_k\gamma_k = 0$

Since S_2 is LI set

$\therefore b_1 = b_2 = \cdots = b_t = d_1 = d_2 = \cdots = d_k = 0$

$\therefore$ (1) $\equiv$ $c_1\gamma_1 + c_2\gamma_2 + \cdots + c_k\gamma_k + a_1\alpha_1 + a_2\alpha_2 + \cdots + a_m\alpha_m = 0$

Since S_1 is LI.

$\therefore c_1 = c_2 = c_3 = \cdots = c_k = a_1 = a_2 = \cdots = a_n = 0$

$\therefore c_1 = c_2 = c_3 = \cdots = c_k = a_1 = a_2 = \cdots = a_m = b_1 = b_2 = \cdots = b_t = 0$

$\therefore S_3$ is LI set.

(ii) To show $L(S_3) = W_1 + W_2$

W.K.T $L(S_3) \subseteq W_1 + W_2$ (A)

Let $\alpha \in W_1 + W_2$ then

$\alpha = \alpha_i + \alpha_j$ where $\alpha_i \in W_1$ and $\alpha_j \in W_2$.

Since α_i is a l.c. of the basis of W_1 and α_j is a l.c of the basis of W_2

$\therefore \alpha =$ l.c. of the basis of $W_1 +$ l.c. of the basis of W_2

$= $ l.c. of elts of S_3

$\in L(S_3)$

$\therefore$ If $\alpha \in W_1+W_2$ then $\alpha \in L(S_3)$

$\therefore W_1+W_2 \subseteq L(S_3)$

from (A) & (B) we have

$L(S_3) = W_1+W_2$

$\therefore S_3$ is a basis of W_1+W_2

$\therefore \dim(W_1+W_2) = k+m+t$

$\therefore \dim(W_1+W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$

Note: (1) If W_1 and W_2 are two subspaces of a FDVS $V(F)$ such that $W_1 \cap W_2 = \{0\}$ then $\dim(W_1+W_2) = \dim W_1 + \dim W_2$.

Defn Row-equivalence of two matrices:

A matrix A is said to be row-equivalent to a matrix B iff B can be obtained from A by a finite no. of elementary row operations.

Defn Coloumn-equivalence of two matrices:

A matrix A is said to be coloumn equivalent to a matrix B iff B can be obtained from A by a finite no. of elementary coloumn operations.

Note: Elementary row operations are:

(i) Interchange of the i^{th} & j^{th} rows: $R_i \leftrightarrow R_j$

(ii) Multiplying the i^{th} row by a non-zero scalar k: $R_i \to kR_i$

(iii) Adding to the i^{th} row k times the j^{th} row: $R_i \to R_i + kR_j$

Defn Row space of a matrix:

Let $A = [a_{ij}]$ be an $m \times n$ matrix over a field F

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

The rows of A are the m vectors
$R_1 = (a_{11}, a_{12}, \ldots a_{1n})$, $R_2 = (a_{21}, a_{22} \ldots a_{2n}) \ldots$
$\ldots \ldots R_m = (a_{m1}, a_{m2} \ldots a_{mn})$ as vectors in F^n or $V_n(F)$.
($\because$ each of these being an n-tuple over F).

The linear span of these vectors
i.e, $L(\{R_1, R_2, \ldots R_m\})$ is a subspace of F^n and is called the row space of A.

i.e, row sp(A) = Span $(R_1 R_2 \ldots R_m)$

Similarly, the space spanned by the coloumn vectors
i.e, $L(\{C_1, C_2, \ldots C_n\})$ is a subspace of F^m and is called the coloumn space of A.

where $C_1 = (a_{11}\ a_{21}, a_{31} \ldots a_{m1})$
$C_2 = (a_{12}\ a_{22}\ a_{32} \ldots a_{m2})$
$\vdots$
$C_n = (a_{1n}, a_{2n}, \ldots a_{mn})$

i.e, Col sp(A) = Span $(C_1, C_2, \ldots C_n)$

Note: [1]. Coloumn space of A is the same as ~~the~~ row space of A^T.
i.e, Col sp(A) = row sp(A^T).

Theorem: Row equivalent matrices have the same row space.

proof: Let A and B be two row equivalent matrices
Then by definition of row equivalence, each row of B is either a row of A or l.c. of rows of A.
$\therefore$ The row space of B is contained in the row space of A.
Sly applying the inverse elementary row operation B and obtain A.
$\therefore$ The row space of A is contained in the row space of B.
$\therefore$ The row space of A & B are same.

Note [1]. Coloumn equivalent matrices have the same coloumn space.

[2]. Let A and B be two row-reduced echelon matrices. Then A and B have the same row space iff they have the same non-zero rows.

→ Determine whether the following matrices have the same row space.

$$A = \begin{pmatrix} 1 & 1 & 5 \\ 2 & 3 & 13 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 & -2 \\ 3 & -2 & -3 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & -1 & -1 \\ 4 & -3 & -1 \\ 3 & -1 & 3 \end{pmatrix}$$

Sol^n The matrices have the same row space iff their row reduced echelon matrices have the same non-zero rows.

$$A = \begin{pmatrix} 1 & 1 & 5 \\ 2 & 3 & 13 \end{pmatrix} \underset{R_2 \to R_2 - 2R_1}{\sim} \begin{pmatrix} 1 & 1 & 5 \\ 0 & 1 & 3 \end{pmatrix} \underset{R_1 \to R_1 - R_2}{\sim} \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & -1 & -2 \\ 3 & -2 & -3 \end{pmatrix} \underset{R_2 \to R_2 - 3R_1}{\sim} \begin{pmatrix} 1 & -1 & -2 \\ 0 & 1 & 3 \end{pmatrix} \underset{R_1 \to R_1 + R_2}{\sim} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & -1 & -1 \\ 4 & -3 & -1 \\ 3 & -1 & 3 \end{pmatrix}$$

$R_2 \to R_2 - 4R_1$
$R_3 \to R_3 - 3R_1$

$$\sim \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 3 \\ 0 & 2 & 6 \end{pmatrix}$$

$R_3 \to R_3 - 2R_2$

$$\sim \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

$R_1 \to R_1 + R_2$

$$\sim \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}.$$

$\therefore$ A and C have the same row space.

and B has different row space.

Vector spaces - III (52)

→ Determine the following matrices have the same coloumn space.

$$A=\begin{pmatrix}1&3&5\\1&4&3\\1&1&9\end{pmatrix};\quad B=\begin{pmatrix}1&2&3\\-2&-3&-4\\7&12&17\end{pmatrix}$$

Sol^n: A and B have the same coloumn space iff A^T & B^T have same row space.

Now A^T & B^T reduce to row canonical form

$$A^T=\begin{pmatrix}1&1&1\\3&4&1\\5&3&9\end{pmatrix}$$

$R_2 \to R_2-3R_1$, $R_3\to R_3-5R_1$

$$\sim\begin{pmatrix}1&1&1\\0&1&-2\\0&-2&4\end{pmatrix}$$

$R_3\to R_3+2R_1$

$$\sim\begin{pmatrix}1&1&1\\0&1&-2\\0&0&0\end{pmatrix}$$

$R_1\to R_1-R_2$

$$\sim\begin{pmatrix}1&0&-1\\0&1&-2\\0&0&0\end{pmatrix}$$

$$B^T=\begin{pmatrix}1&-2&7\\2&-3&12\\3&-4&17\end{pmatrix}$$

$R_2\to R_2-2R_1$, $R_3\to R_3-3R_1$

$$\sim\begin{pmatrix}1&-2&7\\0&1&-2\\0&2&-4\end{pmatrix}$$

$R_1\to R_1+R_3$

$$\sim\begin{pmatrix}1&0&-1\\0&1&-2\\0&2&-4\end{pmatrix}$$

$R_3\to R_3-2R_1$

$$\sim\begin{pmatrix}1&0&-1\\0&1&-2\\0&0&0\end{pmatrix}$$

$\therefore$ A^T & B^T have the same row space.

$\therefore$ A and B have the same coloumn space.

Note: As the non-zero rows of an echelon matrix are LI and row equivalent matrices have same row space.

it follows that

Dimension of row space of A (i.e. dimension of subspace) = Maximum no. of LI rows of A.

= maximum no. of LI rows of echelon matrix of A

= no. of non-zero rows of echelon matrix of A

→ Reduce the matrix $A=\begin{bmatrix}0&1&-3&-1\\2&-1&4&0\\4&1&-1&-3\end{bmatrix}$ to row-reduced echelon form

Also find a basis for the row space and its dimension.

Sol^n $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 2 & -1 & 4 & 0 \\ 4 & 1 & -1 & -3 \end{bmatrix}$

$R_2 \rightarrow R_1$

$\sim \begin{bmatrix} 2 & -1 & 4 & 0 \\ 0 & 1 & -3 & -1 \\ 4 & 1 & -1 & -3 \end{bmatrix}$

$\xrightarrow{R_3 \to R_3 - 2R_1} \sim \begin{bmatrix} 2 & -1 & 4 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 3 & -9 & -3 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 3R_2} \sim \begin{bmatrix} 2 & -1 & 4 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_1 \to R_1 + R_2} \sim \begin{bmatrix} 2 & 0 & 1 & -1 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$R_1 \to \frac{1}{2}R_1$

$\sim \begin{bmatrix} 1 & 0 & 1/2 & -1/2 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

which is the row reduced echelon form.

$\therefore$ A basis for the rowspace is

$\left\{\left(1, 0, \frac{1}{2}, -\frac{1}{2}\right), (0, 1, -3, -1)\right\}$ and the dimension of rowspace is 2.

$\longrightarrow$ Let $U = \text{Span}(u_1, u_2, u_3)$ and $W = \text{Span}(v_1, v_2)$ be two subspaces of $\mathbb{R}^4$ where $u_1 = (1, 2, -1, 3)$, $u_2 = (2, 4, +1, -2)$, $u_3 = (3, 6, 3, -7)$

$v_1 = (1, 2, -4, 11)$, $v_2 = (2, 4, -5, 14)$; S.T. $U = W$.

Sol^n: form the matrix A whose rows are u_i's $(i = 1, 2, 3)$ and reduce it to row reduced echelon form.

$A = \begin{pmatrix} 1 & 2 & -1 & 3 \\ 2 & 4 & 1 & -2 \\ 3 & 6 & 3 & -7 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - 3R_1} \sim \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 3 & -8 \\ 0 & 0 & 6 & -16 \end{pmatrix} \xrightarrow{R_3 \to R_3 - 2R_2,\ R_2 \to \frac{R_2}{3}} \sim \begin{pmatrix} 1 & 2 & -1 & 3 \\ 0 & 0 & 1 & -8/3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim$

$\xrightarrow{R_1 \to R_1 + R_2} \sim \begin{pmatrix} 1 & 2 & 0 & 1/3 \\ 0 & 0 & 1 & -8/3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 0 & 1/3 \\ 0 & 0 & 1 & -8/3 \end{pmatrix}$

Now form the matrix whose rows are v_i's $(i = 1, 2)$ and reduce it to row reduced echelon form.

$B = \begin{pmatrix} 1 & 2 & -4 & 11 \\ 2 & 4 & -5 & 14 \end{pmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \sim \begin{pmatrix} 1 & 2 & -4 & 11 \\ 0 & 0 & 3 & -8 \end{pmatrix} \xrightarrow{R_2 \to \frac{1}{3}R_2} \sim \begin{pmatrix} 1 & 2 & -4 & 11 \\ 0 & 0 & 1 & -8/3 \end{pmatrix} \xrightarrow{R_1 \to R_1 + 4R_2} \sim \begin{pmatrix} 1 & 2 & 0 & 1/3 \\ 0 & 0 & 1 & -8/3 \end{pmatrix}.$

Since the non-zero rows of the row reduced matrices are same.

$\therefore$ The row spaces of A & B are equal.

$\therefore U = V.$

→ Find a basis and dimension of the subspace W of $\mathbb{R}^4$ generated by $(1,-4,-2,1)$, $(1,-3,-1,2)$, $(3,-8,-2,7)$. Also extend the basis of W to a basis of the whole space $\mathbb{R}^4$. (53)

Soln: Now form the matrix A whose rows are the given vectors and reduce it an echelon form.

$$A = \begin{bmatrix} 1 & -4 & -2 & 1 \\ 1 & -3 & -1 & 2 \\ 3 & -8 & -2 & 7 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1,\ R_3 \to R_3 - 3R_1} \begin{bmatrix} 1 & -4 & -2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 4 & 4 & 4 \end{bmatrix} \xrightarrow{R_3 \to R_3 - 4R_2} \begin{bmatrix} 1 & -4 & -2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

∴ The non-zero rows in the echelon matrix $(1,-4,-2,1)$ and $(0,1,1,1)$ form a basis of W and $\dim W = 2$

In particular, the original three given vectors are LD.

Since $\mathbb{R}^4$ is 4-dimensional vector space.

∴ we require for LI vectors which include the above two vectors.

∴ The vectors $(1,-4,-2,1)$, $(0,1,1,1)$, $(0,0,1,0)$, and $(0,0,0,1)$ are LI over $\mathbb{R}$. (Since they form an echelon matrix)

∴ These vectors form a basis of $\mathbb{R}^4$.

∴ It is an extension of the basis of W.

2004 → Let S be the space generated by vectors $\{(0,2,6),(3,1,6),(4,-4,-2)\}$. What is the dimension of the space S? Find basis for S.

1985 → Consider the basis $S = \{v_1, v_2, v_3\}$ of $\mathbb{R}^3$ where $v_1 = (1,1,1)$ $v_2 = (1,1,0)$, $v_3 = (1,0,0)$.

Express $(2,-3,5)$ in terms of the basis v_1, v_2, v_3.

→ Let W be the subspace of $\mathbb{R}^5$ spanned by $u_1 = (1,2,-1,3,4)$ $u_2 = (2,4,-2,6,8)$, $u_3 = (1,3,2,2,6)$, $u_4 = (1,4,5,1,8)$ and $u_5 = (2,7,3,3,9)$. Find a subset of the vectors which form a basis of W.

Soln: Method I

Let $\{u_1, u_2, u_3, u_4, u_5\}$ which spans W.

Since $u_2 = 2u_1$

∴ u_2 & u_1 are LD.

$\therefore$ Eliminate the vector u_2 from S.

$\therefore$ if $S_1 = \{u_1, u_3, u_4, u_5\}$ then subspace W of $\mathbb{R}^5$ spanned by S_1.

Now there exists no real number c s.t $u_3 = cu_1$

$\therefore u_3, u_1$ are L.I.

Sly $u_4 \neq cu_1$ & $u_5 \neq cu_1$.

Now let us check whether the vector u_4 is a l.c. of u_1, u_3, u_5 or not.

Let $u_4 = au_1 + bu_3 + cu_5$ where $a, b, c \in \mathbb{R}$

$u_4 = (1,4,5,1,8) = -1(1,2,-1,3,4) + 2(1,3,2,2,6) - 0(2,7,3,3,9)$

$\therefore u_4$ is l.c. of u_1, u_3 and u_5.

$\therefore S_1$ is LD.

$\therefore$ Eliminate the vector u_4 from S_1.

if $S_2 = \{u_1, u_3, u_5\}$ then subspace W of $\mathbb{R}^5$ spanned by S_2.

No vector of S_2 is a l.c. of others.

$\therefore S_2$ is LI subset of S.

$\therefore$ $\underline{S_2 \text{ is a basis of } W}$

<u>Method 2</u>: Form the matrix A whose rows are given vectors and reduce the matrix to an echelon form but with interchanging any zero rows.

$$A = \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 2 & 4 & -2 & 6 & 8 \\ 1 & 3 & 2 & 2 & 6 \\ 1 & 4 & 5 & 1 & 8 \\ 2 & 7 & 3 & 3 & 9 \end{bmatrix} \underset{\substack{R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - R_1 \\ R_4 \to R_4 - R_1 \\ R_5 \to R_5 - 2R_1}}{\sim} \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 & 2 \\ 0 & 2 & 6 & -2 & 4 \\ 0 & 3 & 5 & -3 & 1 \end{bmatrix}$$

$R_4 \to R_4 - 2R_3,\ R_5 \to R_5 - 3R_3$

$$\sim \begin{bmatrix} 1 & 2 & -1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 3 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -4 & 0 & -5 \end{bmatrix}$$

$\therefore$ The non-zero rows are the first, third and fifth rows.

$\therefore u_1, u_3, u_5$ form a basis of W

(54)

1986 Find a maximal LI subsystem in the system of vectors $v_1 = (2,-3,-4)$, $v_2 = (1,9,3)$, $v_3 = (-2,-4,1)$ and $v_4 = (3,7,-1)$

without changing any zero rows.

1988 Determine a basis of the subspace spanned by the vectors $v_1 = (1,2,3)$, $v_2 = (2,1,-1)$, $v_3 = (1,-1,-4)$, $v_4 = (4,2,-2)$.

→ Let V_1 and V_2 be the subspaces of R^4 generated by $\{(1,1,0,-1), (1,2,3,0), (2,3,3,-1)\}$ and $\{(1,2,2,-2), (2,3,2,-3), (1,3,4,-3)\}$ respectively. Find the dimension of (i) V_1 (ii) V_2 (iii) V_1+V_2 (iv) $V_1 \cap V_2$.

Sol^n Let $S_1 = \{(1,1,0,-1), (1,2,3,0), (2,3,3,-1)\}$ and
$S_2 = \{(1,2,2,-2), (2,3,2,-3), (1,3,4,-3)\}$.

(i) Form the matrix A whose rows are the vectors of S_1 and reduce it to an echelon matrix.

$$A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 3 & -1 \end{bmatrix} \xrightarrow{R_2 \to R_2 - R_1,\ R_3 \to R_3 - 2R_1} \sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & 1 \end{bmatrix} \xrightarrow{R_3 \to R_3 - R_2} \sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

∴ The echelon matrix of A has two non-zero rows.

∴ $\{(1,1,0,-1), (0,1,3,1)\}$ form a basis of V_1.

∴ $\dim V_1 = 2$

(ii) proceed as in (i) &

$\dim V_2 = 2$

(iii) Since V_1 and V_2 are two subspaces of $\mathbb{R}^4$.

∴ V_1+V_2 is also subspace of $\mathbb{R}^4$.

∴ V_1+V_2 is the space generated by all the six vectors. (i.e, S_1 & S_2)

Now form the matrix A whose rows are the given six vectors and reduce it to an echelon form.

$$A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 3 & -1 \\ 1 & 2 & 2 & -2 \\ 2 & 3 & 2 & -3 \\ 1 & 3 & 4 & -3 \end{bmatrix} \xrightarrow{\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 - 2R_1,\ R_4 \to R_4 - R_1 \\ R_5 \to R_5 - 2R_1,\ R_6 \to R_6 - R_1}} \sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 3 & 1 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & 2 & 4 & -2 \end{bmatrix}$$

$V_1+V_2 = L(V_1 \cup V_2)$

$R_3 \to R_3 - R_2$
$R_5 \to R_5 - R_4$
$R_6 \to R_6 - 2R_4$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \underset{R_3 \to R_3 - R_2}{\overset{R_3 \leftrightarrow R_4}{\sim}} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 1 & 3 & 1 \\ 0 & 0 & -1 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

$\therefore$ The echelon matrix of A has three non-zero rows.

$\therefore \dim(V_1 + V_2) = 3$

(iv) $\dim(V_1 \cap V_2) = \dim V_1 + \dim V_2 - \dim(V_1 + V_2)$
$= 2 + 2 - 3$
$= 1$

1996, In R^4, let W_1 be the space generated by $(1,1,0,-1)$ $(2,4,6,0)$ and $(-2,-3,-3,1)$ and let W_2 be the space generated by $(-1,-2,-2,2)$, $(4,6,4,-6)$ $(1,3,4,-3)$.
Find a basis for the space $W_1 + W_2$.

$\to$ Let $V = \mathbb{R}^4(\mathbb{R})$
$W = \{(a,b,c,d) \in \mathbb{R}^4 / a = b + c,\ c = b + d\}$.
Find a basis and the dimension of W.

Sol^n: Let $\alpha_1 = (1,1,0,-1)$ and $\alpha_2 = (0,1,-1,-2)$
then $\alpha_1, \alpha_2 \in W$ and are L.I.

Since $x\alpha_1 + y\alpha_2 = 0$ where $x, y \in \mathbb{R}$.

$\Rightarrow x(1,1,0,-1) + y(0,1,-1,-2) = 0$
$\Rightarrow (x, x+y, -y, -x-2y) = (0,0,0,0)$
$\Rightarrow x = 0 = y$.

To show W is spanned by α_1 & α_2.

Let $(a,b,c,d) \in W$ then $a = b + c$, & $c = b + d$ — (1)

Since $a(1,1,0,-1) + c(0,1,-1,-2)$
$= (a, a-c, c, -a+2c)$
$= (a,b,c,d)$ by (1)

$\therefore W$ is spanned by $\{\alpha_1, \alpha_2\}$.
$\therefore \{\alpha_1, \alpha_2\}$ is a basis of W and $\dim W = 2$

(OR) $W = \{(a,b,c,d) / \substack{a = b+c \\ c = b+d}\} \subseteq \mathbb{R}^4$

Let $\alpha = (a,b,c,d)$
$\therefore \alpha = (b+c, b, b+d, d)$ where $a = b+c$, $c = b+d$
$= (2b+d, b, b+d, d)$
$= b(2,1,1,0) + d(1,0,1,1)$

$W = L(\{\alpha_1, \alpha_2\})$
where $\alpha_1 = (2,1,1,0)$
$\alpha_2 = (1,0,1,1)$
are L.I.

$\therefore \{\alpha_1, \alpha_2\}$ is basis of W.

55

→ Let $A = \{(x, y, 0) / x, y \in \mathbb{R}\}$ and $B = \{(0, y, z) / y, z \in \mathbb{R}\}$ be two subspaces of $V = \mathbb{R}^3(\mathbb{R})$.
Find the dimension of $A + B$.

Sol^n Let $(x, y, 0) \in A$ then

$(x, y, 0) = x(1, 0, 0) + y(0, 1, 0)$

$\therefore A = L(\{e_1, e_2\})$ where $e_1 = (1, 0, 0)$, $e_2 = (0, 1, 0)$ are LI.

$\therefore$ The set $\{e_1, e_2\}$ is basis of A

$\therefore \dim A = 2$

Let $(0, y, z) \in B$ then

$(0, y, z) = y(0, 1, 0) + z(0, 0, 1)$

$\Rightarrow B = L(\{\alpha_1, \alpha_2\})$ where $\alpha_1 = (0, 1, 0)$, $\alpha_2 = (0, 0, 1)$ are LI.

$\therefore$ The set $\{\alpha_1, \alpha_2\}$ is a basis of B.

$\therefore \dim B = 2$

Now $A \cap B = \{(0, y, 0) / y \in \mathbb{R}\}$

Let $(0, y, 0) = y(0, 1, 0)$

$\therefore A \cap B = L(\{\beta\})$ where $\beta = (0, 1, 0)$ is LI.

$\therefore \{\beta\}$ is a basis of $A \cap B$.

$\therefore \dim(A \cap B) = 1$

Since $\dim(A + B) = \dim A + \dim B - \dim(A \cap B)$
$= 2 + 2 - 1$
$= 3.$

→ Find the two subspaces A and B of $V = \mathbb{R}^4(\mathbb{R})$ s.t. $\dim A = 2$, $\dim B = 3$ and $\dim(A \cap B) = 1$.

Sol^n Let $A = \{(x, y, 0, 0) / x, y \in \mathbb{R}\}$ and
$B = \{(0, y, z, t) / y, z, t \in \mathbb{R}\}$ be two subsets of $\mathbb{R}^4(\mathbb{R})$

It is easy to verify that A and B are subspaces of $V = \mathbb{R}^4(\mathbb{R})$.

Let $(x, y, 0, 0) \in A$ then

$(x, y, 0, 0) = x(1, 0, 0, 0) + y(0, 1, 0, 0)$.

$\Rightarrow A = L(\{e_1, e_2\})$.

where $e_1 = (1, 0, 0, 0)$ & $e_2 = (0, 1, 0, 0)$. are LI.

$\therefore$ The set $\{e_1, e_2\}$ is basis of A.

$\therefore \dim A = 2$

Let $(0, y, z, t) \in B$ then

$(0, y, z, t) = y(0, 1, 0, 0) + z(0, 0, 1, 0) + t(0, 0, 0, 1)$.

$\Rightarrow B = L(\{\alpha_1, \alpha_2, \alpha_3\})$

where $\alpha_1 = (0, 1, 0, 0)$

$\alpha_2 = (0, 0, 1, 0)$

$\alpha_3 = (0, 0, 0, 1)$ are LI.

$\therefore$ The set $\{\alpha_1, \alpha_2, \alpha_3\}$ is a basis of B.

$\therefore \dim B = 3$

Now $A \cap B = \{(0, y, 0, 0) / y \in \mathbb{R}\}$

Let $(0, y, 0, 0) \in A \cap B$ then

$(0, y, 0, 0) = y(0, 1, 0, 0)$

$A \cap B = L(\{\beta\})$

where $\beta = (0, 1, 0, 0)$ is LI.

$\therefore$ The set $\{\beta\}$ is a basis of $\mathbb{R}^4$.

$\therefore \dim(A \cap B) = 1$

$\therefore \dim(A+B) = \dim A + \dim B - \dim(A \cap B)$

$= 2+3-1$

$= 4$.

<u>Co ordinates</u> :

Let $B = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be a basis of $V(F)$.

Since $B = \{\alpha_i / i = 1, 2, \ldots n\}$ spans V, the vector $\alpha \in V$ is a l.c. of the α_i's.

i.e, $\alpha = a_1\alpha_1 + a_2\alpha_2 + \cdots + a_n\alpha_n$; $a_i \in F$.

Since the α_i's are LI.

The 'n' scalars $a_1, a_2, \ldots a_n$ are completely determined by the vector α and the basis set $B = \{\alpha_i / i = 1, 2, \ldots n\}$.

we call these scalars the coordinates of α in $\{\alpha_i\}$ and call the n-tuple $(a_1, a_2, \ldots a_n)$ the coordinate vector of α is relative to the basis $\{\alpha_i\}$ and is denoted by $[\alpha]_B$ or $[\alpha]$

i.e, $[\alpha] = (a_1, a_2, \ldots a_n)$.

problem → find the co-ordinate vector of $\alpha = (3,1,-4)$ in $\mathbb{R}^3$ of α relative to the basis $f_1 = (1,1,1)$, $f_2 = (0,1,1)$, $f_3 = (0,0,1)$.

Soln α is l.c. of f_1, f_2, f_3 using unknowns x, y and z

i.e, $\alpha = xf_1 + yf_2 + zf_3$

$\Rightarrow (3,1,-4) = x(1,1,1) + y(0,1,1) + z(0,0,1)$

$= (x,x,x) + (0,y,y) + (0,0,z)$

$= (x, x+y, x+y+z)$

$\Rightarrow \boxed{x=3}$,

$x+y=1 \Rightarrow \boxed{y=-2}$

$x+y+z = -4 \Rightarrow \boxed{z=-5}$

$\therefore [\alpha] = (3,-2,-5)$

H.W. → Find the co-ordinate vector of $\alpha = (3,1,-4)$ relative to the usual basis $e_1 = (1,0,0)$ $e_2 = (0,1,0)$ $e_3 = (0,0,1)$.

H.W. → Let V be the vectorspace of polynomials with degree ≤ 2

$V = \{at^2 + bt + c \mid a, b, c \in \mathbb{R}\}$

The polynomials $e_1 = 1$, $e_2 = t-1$, and $e_3 = (t-1)^2 = t^2 - 2t + 1$ form a basis for V. Let $\alpha = 2t^2 - 5t + 6$. find $[\alpha]$, the co-ordinate vector α relative to the basis $\{e_1, e_2, e_3\}$. (Ans: $[\alpha] = (3,-1,2)$)

→ find the coordinate vector $[u]$ relative to the basis $\{1, t, t^2, t^3\}$ of V. where $u = 2 - 3t + t^2 + 2t^3$.

Soln: u is a l.c. of $1, t, t^2, t^3$; using unknowns x, y, z, w.

i.e, $u = 1x + ty + t^2 z + t^3 w$.

$\Rightarrow 2-3t+t^2+2t^3 = x+yt+zt^2+wt^3$

$\Rightarrow x=2,\ y=-3,\ z=1,\ w=2$

$\therefore [u] = (2,-3,1,2)$

→ Let W be the space generated by the polynomials $v_1 = t^3-2t^2+4t+1$, $v_2 = 2t^3-3t^2+9t-1$, $v_3 = t^3+6t-5$ and $v_4 = 2t^3-5t^2+7t+5$. Find a basis and dimension of W.

Sol^n Since W is spanned by polynomials of degree 3.

$\therefore W$ is a subspace of the space $V_3(\mathbb{R})$.

(the space of all real polynomials of degree ≤ 3 and the zero polynomial).

w.r.t $\{1, t, t^2, t^3\}$ is a basis for $V_3(\mathbb{R})$.

$\therefore$ The co-ordinate vectors of v_1, v_2, v_3 & v_4 w.r.t the above basis are

$(1,4,-2,1), (-1,9,-3,2), (-5,6,0,1)$ and $(5,7,-5,2)$

Now form the matrix A whose rows are these co-ordinate vectors and reduce it to an echelon form

$$A = \begin{bmatrix} 1 & 4 & -2 & 1 \\ -1 & 9 & -3 & 2 \\ -5 & 6 & 0 & 1 \\ 5 & 7 & -5 & 2 \end{bmatrix} \underset{\substack{R_2 \to R_2+R_1 \\ R_3 \to R_3+5R_1 \\ R_4 \to R_4-5R_1}}{\sim} \begin{bmatrix} 1 & 4 & -2 & 1 \\ 0 & 13 & -5 & 3 \\ 0 & 26 & -10 & 6 \\ 0 & -13 & 5 & -3 \end{bmatrix}$$

$$\underset{\substack{R_3 \to R_3-2R_2 \\ R_4 \to R_4+R_2}}{\sim} \begin{bmatrix} 1 & 4 & -2 & 1 \\ 0 & 13 & -5 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

which is in the echelon form.

The non-zero rows of the echelon form of A form a basis of the subspace W.

i.e, The vectors $(1,4,-2,1)$ and $(0,13,-5,3)$ form a basis for W.

$\therefore$ A basis for W consists of polynomials t^3-2t^2+4t+1 and $3t^3-5t^2+13t$.

$\therefore \dim W = 2$

Quotient Space

Let W be any subspace of a vector space $V(F)$. 0050

Let $\alpha \in V$ then the set $W+\alpha = \{\gamma+\alpha / \gamma \in W\}$ is called a left coset of W in V generated by α.

Similarly the set $\alpha+W = \{\alpha+\gamma / \gamma \in W\}$ is called a right coset of W in V generated by α.

Note: → The right and left cosets $W+\alpha$ and $\alpha+W$ are both subsets of V.

→ Since $+^n$ is commutative in V

$\therefore W+\alpha = \alpha+W$.

i.e, the right coset is same as the left coset.

→ Simply $W+\alpha$ is called a coset of W in V generated by α.

→ We have $\alpha = 0 \in V$ and

$W+0 = W$

$\therefore W$ itself is a coset in V.

Theorem $\alpha \in W \Rightarrow W+\alpha = W$.

Proof: To prove $W+\alpha \subseteq W$.

Let $\gamma+\alpha \in W+\alpha$ then $\gamma \in W$.

Since W is a subspace of V.

$\therefore \gamma \in W, \alpha \in W \Rightarrow \gamma+\alpha \in W$.

$\therefore$ If $\gamma+\alpha \in W+\alpha$ then $\gamma+\alpha \in W$.

$\therefore W+\alpha \subseteq W$ —— (1)

To prove $W \subseteq W+\alpha$.

Let $\beta \in W$

Since W is a subspace of V.

$\therefore \alpha \in W \Rightarrow -\alpha \in W$

$\therefore \beta \in W, -\alpha \in W \Rightarrow \beta-\alpha \in W$

Now $\beta = \underbrace{(\beta-\alpha)+\alpha}_{\in W+\alpha}$ $\quad (\because \beta-\alpha \in W)$

$\therefore$ If $\beta \in W$ then $\beta \in W+\alpha$.

$\therefore W \subseteq W+\alpha$ —— (2)

$\therefore$ from (1) & (2) we have

$W = W+\alpha$.

$\therefore \alpha \in W \Rightarrow W+\alpha = W$.

Theorem: If $W+\alpha$ & $W+\beta$ are two cosets of W in V then

$W+\alpha = W+\beta \Leftrightarrow \alpha-\beta \in W$.

proof: Since $0 \in W \Rightarrow 0+\alpha \in W+\alpha$

$\Rightarrow \alpha \in W+\alpha$

$W+\alpha = \{o+\alpha / o \in W\}$

Since $W+\alpha = W+\beta$

$\Rightarrow \alpha \in W+\beta$. $(\because \alpha \in W+\alpha)$

$\Rightarrow \alpha-\beta \in W+(\beta-\beta)$

$\Rightarrow \alpha-\beta \in W$.

Conversely suppose that

$\alpha-\beta \in W \Rightarrow W+(\alpha-\beta) = W$

$\Rightarrow W+[(\alpha-\beta)+\beta] = W+\beta$

$\Rightarrow W+\alpha = W+\beta$.

$\therefore W+\alpha = W+\beta \Leftrightarrow \alpha-\beta \in W$.

Theorem: If W is any subspace of a vector space $V(F)$, then the set $\frac{V}{W}$ of all cosets $W+\alpha$ where $\alpha \in V$.

i.e., $\frac{V}{W} = \{W+\alpha / \alpha \in V\}$ is a vector space over F and scalar multiplication compositions defined as follows.

Let $W+\alpha \in \frac{V}{W}$, $W+\beta \in \frac{V}{W}$

$\Rightarrow (W+\alpha)+(W+\beta) = W+(\alpha+\beta)$ $\forall \alpha, \beta \in V$.

and $a(W+\alpha) = W+a\alpha$; $a \in F$; $\alpha \in V$.

proof:

we have $\alpha, \beta \in V \Rightarrow \alpha+\beta \in V$

$a \in F, \alpha \in V \Rightarrow a\alpha \in V$.

$\therefore W+(\alpha+\beta) \in \frac{V}{W}$ and $W+(a\alpha) \in \frac{V}{W}$

$\therefore \frac{V}{W}$ is closed w.r.t $+^n$ of cosets and scalar multiplication

Now we shall show that these two compositions are well defined

i.e, these compositions are independent of the particular representative chosen to denote a coset.

Let $W+\alpha = W+\alpha'$; $\alpha, \alpha' \in V$.

and $W+\beta = W+\beta'$; $\beta, \beta' \in V$.

we have $W+\alpha = W+\alpha' \Rightarrow \alpha-\alpha' \in W$

and $W+\beta = W+\beta' \Rightarrow \beta-\beta' \in W$

Since W is a subspace.

$\therefore \alpha-\alpha' \in W$, $\beta-\beta' \in W \Rightarrow (\alpha-\alpha')+(\beta-\beta') \in W$

$\Rightarrow (\alpha+\beta)-(\alpha'+\beta') \in W$

1-1: $f(a)=f(b) \Rightarrow a=b$

well defined: $a=b \Rightarrow f(a)=f(b)$ condition

$\Rightarrow W+(\alpha+\beta) = W+(\alpha'+\beta')$

$\Rightarrow (W+\alpha)+(W+\beta) = (W+\alpha')+(W+\beta')$

$\therefore$ Addition in $\frac{V}{W}$ is well defined.

Again $a\in F$, $\alpha-\alpha'\in W$

$\Rightarrow a(\alpha-\alpha')\in W$

$\Rightarrow a\alpha-a\alpha'\in W$

$\Rightarrow W+a\alpha = W+a\alpha'$

$\therefore$ Scalar multiplication in $\frac{V}{W}$ is well defined.

(I) (i) $W+\alpha, W+\beta, W+\gamma \in \frac{V}{W}$

$$\Rightarrow (W+\alpha)+[(W+\beta)+(W+\gamma)] = (W+\alpha)+[W+(\beta+\gamma)]$$
$$= W+[\alpha+(\beta+\gamma)]$$
$$= W+[(\alpha+\beta)+\gamma] \quad \text{(By Asso. prop. in V)}$$
$$= [W+(\alpha+\beta)]+(W+\gamma)$$
$$= [(W+\alpha)+(W+\beta)]+(W+\gamma)$$

$\therefore$ Asso. prop. is satisfied.

(ii) If $0\in V$ then $W+0 = W\in\frac{V}{W}$

If $W+\alpha\in\frac{V}{W}$ then $(W+0)+(W+\alpha) = W+(0+\alpha)$
$= W+\alpha$.

Similarly $(W+\alpha)+(W+0) = W+\alpha$.

$\therefore W+0 = W$ is an identity in $\frac{V}{W}$.

(iii) If $\alpha\in V$ then $-\alpha\in V$

If $W+\alpha\in\frac{V}{W}$ then $W+(-\alpha) = W-\alpha\in\frac{V}{W}$.

$$\therefore (W-\alpha)+(W+\alpha) = W+[(-\alpha)+\alpha]$$
$$= W+0$$
$$= W$$

Similarly $(W+\alpha)+(W-\alpha) = W+0 = W$

$\therefore W-\alpha$ is an inverse of $W+\alpha$ in V/W

(iv) $W+\alpha, W+\beta\in\frac{V}{W} \Rightarrow (W+\alpha)+(W+\beta) = W+(\alpha+\beta)$
$= W+(\beta+\alpha)$ (By comm. prop. in V)
$= (W+\beta)+(W+\alpha)$

$\therefore$ comm. prop. is satisfied.

(II) $a,b\in F$; $W+\alpha, W+\beta\in\frac{V}{W}$.

(i) $a[(W+\alpha)+(W+\beta)] = a[W+(\alpha+\beta)]$
$= W+a(\alpha+\beta)$
$= W+(a\alpha+a\beta)$
$= (W+a\alpha)+(W+a\beta) = a(W+\alpha)+a(W+\beta)$.

(ii) $(a+b)(W+\alpha) = W+(a+b)\alpha$
$= W+(a\alpha+b\alpha)$
$= (W+a\alpha)+(W+b\alpha)$
$= a(W+\alpha)+b(W+\alpha)$

(iii) $(ab)(W+\alpha) = W+(ab)\alpha$
$= W+a(b\alpha)$
$= a(W+b\alpha)$
$= a[b(W+\alpha)]$

(iv) $1(W+\alpha) = W+1\alpha$
$= W+\alpha$; $1\in F$.

$\therefore \frac{V}{W}$ is a vector space over F.

<u>Defn</u> If W is any subspace of a vectorspace V(F) and $\frac{V}{W} = \{W+\alpha / \alpha\in V\}$ is a vector space over F w.r.t coset $+^n$ and scalar $\times^n$. then the vectorspace $\frac{V}{W}(F)$ is called the quotient space of V relative to W.

Dimension of a Quotient Space:

<u>Theorem</u>: 1989 If W be a subspace of a finite dimensional vectorspace V(F) then $\dim\left(\frac{V}{W}\right) = \dim V - \dim W$.

<u>Proof</u>: Given that W is a subspace of a finite dimensional vectorspace V(F).

$\therefore$ W is also finite dimensional.

Let $\dim W = m$

Let $S = \{\alpha_1, \alpha_2, \ldots \alpha_m\}$ be a basis of W.

Since S is LI & $S\subseteq V$.

$\therefore$ S can be extended to form a basis of V.

Let $S' = \{\alpha_1, \alpha_2, \ldots \alpha_m, \beta_1, \beta_2, \ldots \beta_l\}$ be a basis of V.

$\therefore \dim(V) = m+l$

$\therefore \dim V - \dim W = m+l-m$
$= l$

Now we have to prove that $\dim\left(\frac{V}{W}\right) = l$.

for this we have to show that the set $S'' = \{W+\beta_1, W+\beta_2, \ldots W+\beta_l\}$ is a basis of V/W.

(i) <u>To show S'' is LI</u>:

Let $a_1, a_2, \ldots a_l \in F$

then $a_1(W+\beta_1)+a_2(W+\beta_2)+\cdots\cdots+a_l(W+\beta_l)=W$ $\left(\because W+0=W\in\frac{V}{W}\right)$

$\Rightarrow (W+a_1\beta_1)+(W+a_2\beta_2)+\cdots\cdots+(W+a_l\beta_l)=W$

$\Rightarrow W+(a_1\beta_1+a_2\beta_2+\cdots\cdots+a_l\beta_l)=W+0$ $(\because (W+\alpha)+(W+\beta)=W+(\alpha+\beta))$

$\Rightarrow (a_1\beta_1+a_2\beta_2+\cdots\cdots+a_l\beta_l)-0\in W$ $\left(\because W+\alpha=W+\beta \Leftrightarrow \alpha-\beta\in W\right)$

$\Rightarrow a_1\beta_1+a_2\beta_2+\cdots\cdots+a_l\beta_l\in W$

$\therefore$ It can be expressed as a l.c. of the basis of W.

$\Rightarrow a_1\beta_1+a_2\beta_2+\cdots\cdots+a_l\beta_l=b_1\alpha_1+b_2\alpha_2+\cdots\cdots+b_m\alpha_m$

$\Rightarrow a_1\beta_1+a_2\beta_2+\cdots\cdots+a_l\beta_l-b_1\alpha_1-b_2\alpha_2-\cdots\cdots-b_m\alpha_m=0$

$\therefore a_1=a_2=\cdots\cdots=a_l=b_1=b_2=\cdots\cdots=b_m=0$

$\Rightarrow a_1=a_2=a_3=\cdots\cdots=a_l=0$

$\therefore$ The set S'' is L.I.

(ii) To show $L(S'')=V/W$

w.k.T $L(S'')\subseteq V/W$ — (A)

Let $W+\alpha\in V/W$

Since S' is basis of V.

$\therefore$ the vector $\alpha\in V$ can be expressed as $c_1\alpha_1+c_2\alpha_2+\cdots+c_m\alpha_m+d_1\beta_1+d_2\beta_2+\cdots+d_l\beta_l$.

$\alpha=c_1\alpha_1+c_2\alpha_2+\cdots\cdots+c_m\alpha_m+d_1\beta_1+\cdots\cdots+d_l\beta_l$

$=\gamma+d_1\beta_1+d_2\beta_2+\cdots\cdots+d_l\beta_l$ where $\gamma=c_1\alpha_1+c_2\alpha_2+\cdots\cdots+c_m\alpha_m\in W$

$\Rightarrow W+\alpha=W+(\gamma+d_1\beta_1+d_2\beta_2+\cdots\cdots+d_l\beta_l)$

$=(W+\gamma)+d_1\beta_1+d_2\beta_2+\cdots\cdots+d_l\beta_l\}$

$=W+(d_1\beta_1+d_2\beta_2+\cdots\cdots+d_l\beta_l)$ $\left(\because \gamma\in W\Rightarrow W+\gamma=W\right)$

$=(W+d_1\beta_1)+(W+d_2\beta_2)+\cdots\cdots+(W+d_l\beta_l)$

$=d_1(W+\beta_1)+d_2(W+\beta_2)+\cdots\cdots+d_l(W+\beta_l)$

$\in L(S'')$.

$\therefore \frac{V}{W}=L(S'')$.

$\therefore S''$ is a basis of $\frac{V}{W}$.

$\therefore \dim\left(\frac{V}{W}\right)=l$.

$\rightarrow$ Determine $\dim\left(\frac{V}{W}\right)$; when $V=\mathbb{C}(\mathbb{R})$, $W=\mathbb{R}(\mathbb{R})$.

Sol^n W.K.T $\{1, i\}$ is a basis of $C(\mathbb{R})$
and $\{1\}$ is a basis of $\mathbb{R}(\mathbb{R})$

$\therefore \dim\left(\frac{V}{W}\right) = \dim V - \dim W$
$= 2-1$
$= 1$

→ Let P_n denote the vectorspace of all polynomials of degree $\leq n$ over $\mathbb{R}$.

Exhibit ~~a~~ basis of $\frac{P_4}{P_2}$, hence verify that
$$\dim\left(\frac{P_4}{P_2}\right) = \dim P_4 - \dim P_2.$$

Sol^n: We have
$P_2 = \{a_0 + a_1x + a_2x^2 / a_0, a_1, a_2 \in \mathbb{R}\}$ and
$P_4 = \{a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 / a_0, a_1, a_2, a_3 \in \mathbb{R}\}$

It is clear that P_2 is subspace of P_4. [$\frac{V}{W}$; W subspace of V]

W.K.T $\{1, x, x^2\}$ & $\{1, x, x^2, x^3, x^4\}$ are bases of P_2 and P_4 respectively.

$\therefore \dim(P_2) = 3$; $\dim(P_4) = 5$.

Now we define
$$\frac{P_4}{P_2} = \{P_2 + a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 / a_i \in \mathbb{R}\}$$
$$= \{(a_3x^3 + a_4x^4) + P_2 + (a_0 + a_1x + a_2x^2) / a_i \in \mathbb{R}\}$$
$$= \{a_3x^3 + a_4x^4 + P_2 / a_3, a_4 \in \mathbb{R}\}$$
($\because \alpha \in W \Rightarrow W + \alpha = W$
i.e., $\alpha = a_0 + a_1x + a_2x^2 \in P_2$
$\alpha \in P_2 \Rightarrow P_2 + \alpha = P_2$)

$\therefore \frac{P_4}{P_2}$ is a quotient space with the basis $\{P_2 + x^3, P_2 + x^4\}$

$$\therefore \dim\left(\frac{P_4}{P_2}\right) = 2.$$
$$= \dim(P_4) - \dim(P_2)$$