2011

• lagrangian equation for simple pendulum

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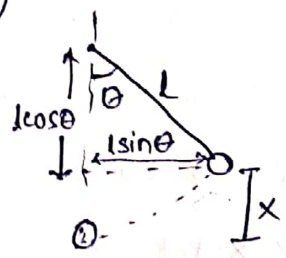

we have a pendulum of length l. At some time t, it is making an angle 'θ' with vertical and is at height 'x' from bottom.

so Potential energy $(V) = mgx = mg(l - l\cos\theta)$

$= mgl(1-\cos\theta)$

Kinetic energy $(T) = \frac{1}{2}mv^2$

$= \frac{1}{2}ml^2\omega^2$

$= \frac{1}{2}ml^2\dot{\theta}^2$

$L = T - V$

$= \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1-\cos\theta)$

from Hamilton principle of least action, we have

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

$$\frac{d}{dt}\left(ml^2\dot{\theta}\right) + mgl(\sin\theta) = 0$$

$$\frac{d\dot{\theta}}{dt} = -\frac{g}{l}\sin\theta$$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\sin\theta$$

for small value of θ $\sin\theta \approx \theta$

$$\boxed{\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta}$$

Equation of motion of Simple pendulum.

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(38) $\vec{q} = \dfrac{K^2(x\hat{j} - y\hat{i})}{x^2+y^2}$

$u = \dfrac{-K^2 y}{x^2+y^2}$ $\quad v = \dfrac{K^2 x}{x^2+y^2}$

$\dfrac{\partial u}{\partial x} = \dfrac{2K^2xy}{(x^2+y^2)^2}$ $\quad \dfrac{\partial v}{\partial y} = \dfrac{-2K^2xy}{(x^2+y^2)^2}$

$\dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = 0$

incompressible fluid satisfies continuity equation. so liquid motion possible.

Velocity potential

$u = -\dfrac{\partial\phi}{\partial x} = \dfrac{-K^2 y}{x^2+y^2}$

$\int d\phi = K^2 y \int \dfrac{dx}{x^2+y^2}$

$\phi = K^2 \left(\tan^{-1}\dfrac{x}{y}\right) + f(y)$

$v = -\dfrac{\partial\phi}{\partial y} = -\dfrac{K^2}{1+\frac{x^2}{y^2}}\left(\dfrac{-x}{y^2}\right) - f'(y)$

$\dfrac{K^2 x}{x^2+y^2} = \dfrac{K^2 x}{x^2+y^2} + f'(y)$

$f'(y) = 0$

$\therefore \boxed{\phi = K^2 \tan^{-1}\left(\dfrac{x}{y}\right) + C}$

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Again,

$$v = \frac{\partial \psi}{\partial x}$$

$$\psi = \int v \, dx$$

$$= \int \frac{k^2 x}{x^2+y^2} \, dx$$

$$= \frac{k^2}{2} \log(x^2+y^2) + g(y)$$

$$u = -\frac{\partial \psi}{\partial y} = -\frac{k^2}{2} \frac{2y}{x^2+y^2} + g'(y)$$

$$g'(y) = 0$$

$$g(y) = c_2$$

$$\boxed{\psi = \frac{k^2}{2} \log(x^2+y^2) + c_2}$$

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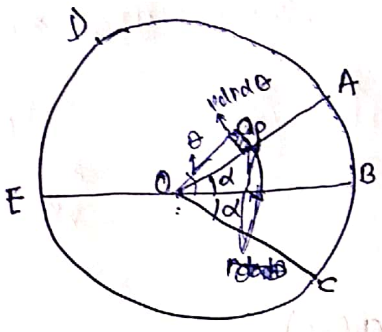

So we have a spherical sector OABCO of vertical angle 2α which is removed from sphere of radius a and centre O. This may be generated by OADEO of circle of radius a and centre at O about diameter EB.

consider an area $rdrd\theta$ at Point P.
By revolution of this area about point EB, we have a circular ring of radius $r\sin\theta$

$$dm = \rho \cdot (2\pi r\sin\theta)\, r dr d\theta$$

$$= 2\pi\rho r^2 \sin\theta\, dr d\theta$$

$$M = \int_{\theta=\alpha}^{\pi}\int_{r=0}^{a} 2\pi\rho r^2 \sin\theta\, dr d\theta = \frac{2\pi\rho a^3}{3}\int_{\alpha}^{\pi} \sin\theta\, d\theta$$

$$= \frac{2\pi\rho a^3}{3}(1+\cos\alpha)$$

$$\Rightarrow \rho = \frac{3M}{2\pi a^3(1+\cos\alpha)}$$

Again, MI of this elementary ring about EB, the line through centre and perpendicular to plane

$$= PN^2 dm$$

$$= r^2\sin^2\theta\, (2\pi\rho r^2)\sin\theta\, dr dr)$$

$$= 2\pi\rho r^4 \sin^3\theta\, dr d\theta$$

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MOI of remainder about EB (axis of symmetry)

$$= \int_{\alpha}^{\pi} \int_{0}^{a} 2\pi \rho r^4 \sin^3\theta \, dr \, d\theta$$

$$= \frac{2\pi\rho a^5}{5} \int_{\alpha}^{\pi} \sin^3\theta \, d\theta$$

$$= \frac{2\pi\rho a^5}{5} \int_{\alpha}^{\pi} \frac{1}{4} (3\sin\theta - \sin 3\theta) \, d\theta$$

$$= \frac{\pi\rho a^5}{10} \left[-3\cos\theta + \frac{\cos 3\theta}{3} \right]_{\alpha}^{\pi}$$

$$= \frac{\pi\rho a^5}{10} \left[3 - \frac{1}{3} + 3\cos\alpha - \frac{\cos 3\alpha}{3} \right]$$

$$= \frac{\pi\rho a^5}{30} \left[8 + 9\cos\alpha - [4\cos^3\alpha - 3\cos\alpha] \right]$$

$$= \frac{\pi a^5}{30} \times \frac{3M}{2\pi a^3 (1+\cos\alpha)} \times [8 + 12\cos\alpha - 4\cos^3\alpha]$$

$$= \frac{\pi a^2}{5} \frac{(1+\cos\alpha)^2 (2-\cos\alpha)}{(1+\cos\alpha)}$$

$$= \frac{\pi a^2}{5} (1+\cos\alpha)(2-\cos\alpha)$$

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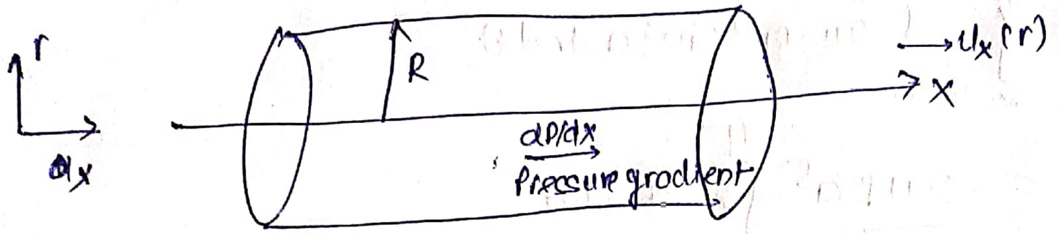
Poiseuille eqⁿ



We have a uniform circular cross-section of Radius R.

Pressure gradient $\frac{dP}{dx}$ is in axial direction x.

from continuity equation for steady flow

$$\frac{\partial u_x}{\partial x} = 0$$

Hence, axial velocity u_x is function of r alone.

from Navier stokes

$$\frac{\partial P}{\partial x} = \mu \left\{ \frac{\partial^2 u_x}{\partial r^2} + \frac{1}{r}\frac{\partial u_x}{\partial r} \right\}$$

$$\frac{\partial P}{\partial r} = 0$$

so Pressure ~~of~~ is function of x alone.

so $\frac{\partial P}{\partial x}$ can be replaced by $\frac{dP}{dx}$

$$\frac{d^2 u_x}{dr^2} + \frac{1}{r}\frac{du_x}{dr} = \frac{1}{\mu}\frac{dP}{dx}$$

$$\frac{du_x}{dr} = t$$

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$$\frac{d\tau}{dr} + \frac{\tau}{r} = \frac{1}{\mu}\frac{dP}{dx}$$

$$I.F = e^{\int \frac{1}{r} dr} = e^{\log r} = r$$

$$\tau \cdot r = \frac{1}{\mu}\frac{dP}{dx}\int r\, dr$$

$$\tau \cdot r = \frac{1}{\mu}\frac{dP}{dx}\frac{r^2}{2} + c_1$$

$$\frac{du_x}{dr} = \tau = \frac{1}{\mu}\left(\frac{dP}{dx}\right)\frac{r}{2} + \frac{c_1}{r}$$

$$u_x = \frac{1}{\mu}\left(\frac{dP}{dx}\right)\frac{r^2}{4} + c_1 \ln r + c_2$$

at $r=0$ $\ln r \to \infty$ $\quad u_x \to \infty$

~~so $\frac{du_x}{dr} \to \infty$~~

for finite velocity at $r=0$

$$\boxed{c_1 = 0}$$

$$u_x = \frac{1}{\mu}\left(\frac{dP}{dx}\right)\frac{r^2}{4} + c_2$$

at $r=a$ we have noslip condition so $u_x = 0$

$$0 = \frac{1}{\mu}\left(\frac{dP}{dx}\right)\frac{a^2}{4} + c_2$$

$$u_x = \frac{1}{4\mu}\left(-\frac{dP}{dx}\right)(a^2 - r^2)$$

as pressure is in decreasing direction with velocity

let say $P' = -dP/dx$

$$\boxed{u_x = \frac{1}{4\mu}(P')(a^2 - r^2)}$$

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