ANALYTIC GEOMETRY

: IFoS - 2015 :

① (e) The tangent at $(a\cos\theta, b\sin\theta)$ on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meets the auxiliary circle in two points. The chord joining them subtends a right angle at the centre. Find the eccentricity of the ellipse.

→ The equation of tangent to the ellipse at $(a\cos\theta, b\sin\theta)$ is given by

$$\frac{x \cdot a\cos\theta}{a^2} + \frac{y\, b\sin\theta}{b^2} = 1 \quad \text{—— ①}$$

$$\Rightarrow \frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1 \quad \text{—— ②}$$

The joint equation of the lines joining the points of intersection of ② and the auxiliary circle $x^2+y^2=a^2$ to the origin which is the centre of the circle is

$$x^2+y^2 = a^2\left[\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta\right]^2.$$

Since thes lines are at right angles, sum of coeff of x^2 & y^2 is zero.

$$\therefore\ 1-\cos^2\theta + 1 - \frac{a^2}{b^2}\sin^2\theta = 0 \quad \Rightarrow \quad \sin^2\theta\left(1-\frac{a^2}{b^2}\right)+1 = 0$$

$$\Rightarrow \sin^2\theta\,(b^2-a^2) + b^2 = 0 \Rightarrow \sin^2\theta\,(a^2(1-e^2)-a^2) + a^2(1-e^2) = 0$$

$[b^2 = a^2(1-e^2)]$

$$\Rightarrow \sin^2\theta\,(-a^2e^2) + a^2 - a^2e^2 = 0$$

$$\Rightarrow (1+\sin^2\theta)a^2e^2 = a^2 \Rightarrow e^2 = (1+\sin^2\theta)^{-1}$$

$$\Rightarrow \boxed{e = (1+\sin^2\theta)^{-1/2}}$$

③ (d) Find the equation of the plane containing the straight line $y+z=1, x=0$ and parallel to the straight line $x-z=1,\ y=0$

→ Eqn. of any plane containing the line $y+z=1, x=0$ is

$$\lambda x + y + z = 1 \quad \text{—— ①}$$

This plane ① is parallel to the line $x-z=1, y=0 \Rightarrow \frac{x}{1} = \frac{y}{0} = \frac{z+1}{1}$ whose drs are $1, 0, 1$. Then this line is parallel to normal to the plane ① whose drs are $\lambda, 1, 1$

Therefore, condition of perpendicularity is $\lambda \cdot 1 + 1 \cdot 0 + 1 \cdot 1 = 0$

$\lambda = -1$

① $\equiv -x+y+z=1 \Rightarrow \boxed{x-y-z+1=0}$ which is the required plane.

4(b) Find the locus of a variable straight line that always intersect $x=1, y=0$; $y=1, z=0$; $z=1, x=0$

→ Plane passing throug':

(i) $x=1, y=0$ is $x + \lambda_1 y - 1 = 0$

(ii) $y=1, z=0$ is $y + \lambda_2 z - 1 = 0$

(iii) $z=1, x=0$ is $z + \lambda_3 x - 1 = 0$

These planes will intersect a line if ~~they are coplanar the condition for which is~~

$$\begin{vmatrix} 1 & \lambda_1 & 0 & -1 \\ 0 & 1 & \lambda_2 & -1 \\ \lambda_3 & 0 & 1 & -1 \end{vmatrix} = 0$$

$$\sim \begin{vmatrix} \cancel{1} & \lambda_1 - 1 & \cancel{-\lambda_2} & 0 \\ \cancel{\lambda_3} & 1 & \lambda_2 - 1 & \cancel{0} \\ \lambda_3 & 0 & 1 & \cancel{-1} \end{vmatrix} = \cancel{0} \Rightarrow \cancel{\lambda_3} \quad \lambda_1 \cancel{\lambda_2} \quad + \lambda_2 \cancel{y} \quad \cancel{\lambda_2 - 1}$$

i.e. $\begin{vmatrix} 1 & \lambda_1 & 0 \\ 0 & 1 & \lambda_2 \\ \lambda_3 & 0 & 1 \end{vmatrix} = 0 \Rightarrow 1 + \lambda_1 \lambda_2 \lambda_3 = 0.$

$\Rightarrow 1 + \frac{(1-x)}{y} \frac{(1-y)}{z} \frac{(1-z)}{x} = 0$

$\Rightarrow \boxed{xyz + (1-x)(1-y)(1-z) = 0}$ which is the reqd locus.

4(c) Find the locus of poles of chords which are normal to the parabolas $y^2 = 4ax$

→ The equation of any normal to the parabola $y^2 = 4ax$ is

$y = mx - 2am - am^3$ —— ①

Let (x_1, y_1) be the pole of ① wrt the parabola.

Then ① is the polar of (x_1, y_1) wrt the parabola i.e. $yy_1 = 2a(x + x_1)$ —— (2)

comparing (2) & (3), we have

$$\frac{2a}{m} = \frac{y_1}{1} = \frac{2ax_1}{-2am - am^3} \Rightarrow x_1 = -2a - am^2 \text{ \& } y_1 = \frac{2a}{m}.$$

Eliminating m between these two eqns, $x_1 = -2a - a \cdot \frac{4a^2}{y_1^2}$

$\Rightarrow \quad y_1^2(x_1 + 2a) + 4a^3 = 0.$

$\therefore$ Required locus of pole (x_1, y_1) is $y^2(x + 2a) = 4a^3$