IAS MATHEMATICS (OPT.)-2008

PAPER - II : SOLUTIONS

12M
1.

1.(a) Let R_0 be the set of all real numbers except zero. Define a binary operation $*$ on R_0 as:

$a*b = |a|b$ where $|a|$ denotes absolute value of a.

Does $(R_0, *)$ form a group ? Examine.

Sol^n:

Let R_0 be the set of all real numbers except zero. i.e. $R_0 = \mathbb{R} - \{0\}$ and $*$ is an operation defined in R_0 such that

$$a*b = |a|\,b \quad \forall a, b \in R_0 \quad \text{--- (1)}$$

where $|a|$ denotes absolute value of a.

(i) Closure prop:

$\forall a, b \in R_0$

$a*b = |a|\,b \in R_0$

$\therefore R_0$ is closed under $*$.

(ii) Associative prop:

$$\forall a, b, c \in R_0 \Rightarrow (a*b)*c = (|a|b)*c$$
$$= \big|\,|a|b\,\big|\,c$$
$$= \big|\,|a|\,\big|\,|b|\,c.$$
$$= |a|\,|b|\,c.$$

Now
$$a*(b*c) = |a|\,(b*c)$$
$$= |a|\,(|b|c)$$
$$= |a|\,|b|\,c.$$

$$\therefore (a*b)*c = a*(b*c).$$

$\therefore$ Associative law holds.

(iii) Existence of right identity:

Let $a \in R_0$, $b \in R_0$ such that $a*b = a$

$$\Rightarrow |a|\, b = a \quad (\text{by } ①)$$

$$\Rightarrow b = \frac{a}{|a|}$$

$$\Rightarrow b = \begin{cases} -1 & \text{if } a<0 \\ 1 & \text{if } a>0 \end{cases}$$

$\therefore \forall\, a \in R_0$, $\exists$ an element $b \in R_0$ where $b = \begin{cases} -1 & \text{if } a<0 \\ 1 & \text{if } a>0 \end{cases}$

such that $a * b = a$.

but the identity element b is not unique

According to the definition of a group, the identity element is unique.

$\therefore$ Identity property does not satisfy under $*$.

$\therefore (R_0, *)$ is not a group.

HEAD OFFICE:25/8, Old Rajinder Nagar Market, Delhi-60. (M) 9999197625, 011-45629987. BRANCH OFFICE(DELHI): 105-106, Top Floor, Mukherjee Tower, Mukherjee Nagar, Delhi-9. BRANCH OFFICE(HYDERBAD): H.No. 1-10-237, 2nd Floor, Room No. 202 R.K'S-Kancham's Blue Sapphire Ashok Nagar Hyd-20. (M) 09652351152, 09652661152

12M → 2. Suppose that there is a positive even integer n such that $a^n = a$ for all the elements 'a' of some ring R. Show that

1.(b)

$a+a=0$ for all $a \in R$ and $a+b=0 \Rightarrow a=b$ for all $a,b \in R$.

Sol^n: Suppose that there is a positive even integer such that $a^n = a \;\forall\, a \in R$ (ring)

Let $a^2 = a \;\forall\, a \in R$ ——①.

$\forall\, a \in R,\ a+a \in R$ (by closure property of R)

we have $(a+a)^2 = a+a$ (by ①)

$\Rightarrow (a+a)(a+a) = a+a$

$\Rightarrow a(a+a) + a(a+a) = a+a$

$\Rightarrow (a^2+a^2) + (a^2+a^2) = a+a$

$\Rightarrow (a+a) + (a+a) = a+a$ (by ①)

$\Rightarrow (a+a) + (a+a) = (a+a) + 0$ (by the +ve identity property of the ring R)

$\Rightarrow \boxed{a+a=0}$. (by the +ve left cancellation of the ring) ——②.

Now for all $a \in R,\ b \in R$,

we have $a+b=0$

$\Rightarrow a+b = a+a$ (by using ②)

$\Rightarrow b = a$ (by LCL w.r.t +n)

$\Rightarrow \boxed{a=b}$

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For $x > 0$, show $\frac{x}{1+x} < \log(1+x) < x$

Sol'n : Let $f(t) = \log(1+t) \ \forall\ t \in [0, x]$

where $x > 0$

It is continuous & differentiable in $[0, x]$

and $f'(t) = \frac{1}{1+t} \ \forall\ t \in (0, x)$

By Lagranges Mean value theorem

$\exists\ c \in (0, x)$ such that

$$f'(c) = \frac{f(x) - f(0)}{x - 0}$$

$$\Rightarrow \frac{1}{1+c} = \frac{\log(1+x) - \log 1}{x}$$

$$\Rightarrow \frac{1}{1+c} = \frac{\log(1+x) - 0}{x}$$

$$\Rightarrow \frac{1}{1+c} = \frac{\log(1+x)}{x} \quad \text{——} \ (1)$$

Since $c \in (0, x)$

$\Rightarrow \quad 0 < c < x$

$\Rightarrow \quad 1 < 1+c < 1+x$

$\Rightarrow \quad 1 > \frac{\log(1+x)}{x} > \frac{1}{1+x}$ (by (1))

$\Rightarrow \quad x > \log(1+x) > \frac{x}{1+x}$ $(\because x > 0)$

$\Rightarrow \quad \frac{x}{1+x} < \log(1+x) < x$

HEAD OFFICE:25/8, Old Rajinder Nagar Market, Delhi-60. (M) 9999197625, 011-45629987. BRANCH OFFICE(DELHI): 105-106, Top Floor, Mukherjee Tower, Mukherjee Nagar, Delhi-9. BRANCH OFFICE(HYDERBAD): H.No. 1-10-237, 2nd Floor, Room No. 202 R.K'S-Kancham's Blue Sapphire Ashok Nagar Hyd-20. (M) 09652351152, 09652661152

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1.(d) If f is a continuous function of x satisfying the functional equation $f(x+y) = f(x) + f(y)$ Show that $f(x) = ax$, where 'a' is a constant (or) $f(x) = x f(1)$ $\forall x \in \mathbb{R}$

Sol^n: Given that 'f' is continuous

and $f(x+y) = f(x) + f(y)$ —— ①

Taking $x = 0 = y$ in ①

from ① $f(0+0) = f(0) + f(0)$

$\Rightarrow f(0) = f(0) + f(0)$

$\Rightarrow f(0) + 0 = f(0) + f(0)$

$\Rightarrow f(0) = 0$

Taking $y = -x$

from ① $f(x + (-x)) = f(x) + f(-x)$

$\Rightarrow f(0) = f(x) + f(-x)$

$\Rightarrow 0 = f(x) + f(-x)$

$\Rightarrow f(-x) = -f(x)$

If x be a +ve integer,

We have
$f(x) = f(1+1+1+ \cdots +1)$

$= f(1) + f(1) + f(1) + \cdots + f(1)$

$= x f(1)$
$= ax$, say

where $f(1) = a$

Now, let x be a $-ve$ integer.

We write $x = -y$ so that y is $+ve$ integer.

We have $f(x) = f(-y)$

$$= -f(y) \quad [\because f(-y) = -f(y)]$$

$$= -ay$$
$$= a(-y)$$
$$= ax$$

Again let $x = \frac{p}{q}$ be a rational number;

q being $+ve$

We have $f(p) = f(p/q \cdot q)$

$$= f\left(\frac{p}{q} + \frac{p}{q} + \dots\dots\dots q \text{ times}\right)$$

$$= f(p/q) + f(p/q) + \dots\dots q \text{ times}$$

$$= q f(p/q)$$

$$\therefore f(p) = q f(p/q)$$

$$\Rightarrow ap = q f(p/q) \quad (\because f(p) = ap)$$

$$\Rightarrow f(p/q) = a \cdot p/q$$

$$\Rightarrow f(x) = ax \quad (\because p/q = x)$$

Now, suppose that x is any real number.

Let $\{x_n\}$ be a sequence of rational numbers

such that $\lim x_n = x$

We have, x_n, being rational

$$f(x_n) = a x_n \quad \text{——} \quad (2)$$

Let $n \to \infty$

As f is a continuous function

We obtain from (2)

$$f(x) = ax \quad \forall x$$

Hence the result

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5. Find the residue of $F(z)=\dfrac{\cot z\coth z}{z^3}$ at $z=0$.

Solution We have, as in Method 2 of Problem 7.4(b),

$$F(z)\frac{\cos z\cosh z}{z^3\sin z\sinh z}=\frac{\left(1-\frac{z^2}{2!}+\frac{z^4}{4!}-\cdots\right)\left(1+\frac{z^2}{2!}+\frac{z^4}{4!}+\cdots\right)}{z^3\left(z-\frac{z^3}{3!}+\frac{z^5}{5!}-\cdots\right)\left(z+\frac{z^3}{3!}+\frac{z^5}{5!}+\cdots\right)}$$

$$=\frac{\left(1-\frac{z^4}{6}+\cdots\right)}{z^5\left(1-\frac{z^4}{90}+\cdots\right)}=\frac{1}{z^5}\left(1-\frac{7z^4}{45}+\cdots\right)$$

and so the residue (coefficient of $1/z$) is $-7/45$.

Another method. The result can also be obtained by finding

$$\lim_{z\to 0}\frac{1}{4!}\frac{d^4}{dz^4}\left\{z^5\frac{\cos z\cosh z}{z^3\sin z\sinh z}\right\}$$

but this method is much more laborious than that given above.

HEAD OFFICE:25/8, Old Rajinder Nagar Market, Delhi-60. (M) 9999197625, 011-45629987. BRANCH OFFICE(DELHI): 105-106, Top Floor, Mukherjee Tower, Mukherjee Nagar, Delhi-9. BRANCH OFFICE(HYDERBAD): H.No. 1-10-237, 2nd Floor, Room No. 202 R.K'S-Kancham's Blue Sapphire Ashok Nagar Hyd-20. (M) 09652351152, 09652661152

1(f) → IAS 2008 P-II

find the dual of the following linear programming problem:

$$\text{Max}\cdot Z = 2x_1 - x_2 + x_3$$

such that

$$x_1 + x_2 - 3x_3 \le 8$$
$$4x_1 - x_2 + x_3 = 2$$
$$2x_1 + 3x_2 - x_3 \ge 5$$
$$x_1, x_2, x_3 \ge 0.$$

Sol^n →

$$\text{Max } z = 2x_1 - x_2 + x_3$$

Converting all constraints to ($\le$) sign.

$$x_1 + x_2 - 3x_3 \le 8 \quad \text{—— (1)}$$

Constraint 2 can be written as:-

$$4x_1 - x_2 + x_3 \le 2 \quad \text{—— (2)}$$
$$4x_1 - x_2 + x_3 \ge 2$$
$$\Rightarrow -4x_1 + x_2 - x_3 \le -2 \quad \text{—— (3)}$$

Constraints 3, multiplying by -1 on both sides

$$-2x_1 - 3x_2 + x_3 \le -5 \quad \text{—— (4)}$$

∴ Constraints are

$$x_1 + x_2 - 3x_3 \le 8$$
$$4x_1 - x_2 + x_3 \le 2$$
$$-4x_1 + x_2 - x_3 \le -2$$
$$-2x_1 - 3x_2 + x_3 \le -5$$

Converting it into duality:-

→ Converting maximum objective function to min

→ Converting al ($\le$) constraints to ($\ge$)

→ Constracting the new constraints and fn by interchanging variables.

$$\text{Min } z^* = 8u + 2v - 2w - 5z$$

S.C. :

$$u + 4v - 4w - 2z \geq 2$$
$$u - v + w - 3z \geq -1$$
$$-3u + v - w + z \geq 1$$
$$u, v, w, z \geq 0$$

$$\text{Min } z^* = 8u + 2(v - w) - 5z$$

S.C.

$$u + 4(v - w) - 2z \geq 2$$
$$u - 1(v - w) - 3z \geq -1$$
$$-3u + 1(v - w) + z \geq 1$$

Let $v - w = y$

$$\therefore \text{Min } z^* = 8u + 2y - 5z$$

$$S.C. = u + 4y - 2z \geq 2$$
$$u - y - 3z \geq -1$$
$$-3u + y + z \geq 1 \quad ; u, z \geq 0,$$

y is unrestricted

This is the required dual form.

15(1) (2) (7)

2.(a) 3. Let G and $\bar{G}$ be two groups and let $\phi: G \to \bar{G}$ be a homomorphism. for any element $a \in G$

(i) prove that $o(\phi(a)) / o(a)$

(ii) $\ker\phi$ is a normal subgroup of G.

Sol Let $(G, *)$ and $(\bar{G}, *)$ be two given groups

let $\phi: G \to \bar{G}$ be the given homomorphism.

(i) for any element $a \in G$,

we have to prove that $o(\phi(a)) / o(a)$

i.e $\frac{o(a)}{o(\phi(a))}$

Let $a \in G$ and $o(a) = m$ then $a^m = e$ where m is the least +ve integer.

$\therefore \phi(a^m) = \phi(e)$

$\Rightarrow \phi[a.a.\cdots a.a \text{ (m times)}] = e'$ (identity in $\bar{G}$)

$\Rightarrow \phi(a).\phi(a) \cdots \phi(a) \text{ (m times)} = e'$

($\because \phi$ is homo)

$\Rightarrow (\phi(a))^m = e' \Rightarrow o(\phi(a)) \le m$.

$\therefore$ If n is the order of $\phi(a)$ in $\bar{G}$ then n must be a divisor of m. i.e n/m

i.e $o(\phi(a))/o(a)$.

($\because$ If the element a of a group G is of order 'n' then $a^m = e \Leftrightarrow n$ is a divisor of m).

(ii) To prove that $\text{Ker}\,\phi$ is a normal subgroup of G. i.e $(\text{Ker}\,\phi) \Delta G$.

Let $K = \text{Ker}\,\phi$ then $K = \{x \in G \mid \phi(x) = e' \text{ (identity } \bar{G})\}$

Since $e \in G$,

$\phi(e) = e' \Rightarrow e \in K$

$\therefore K$ is non-empty subset of G.

Let $a, b \in K$ then $\phi(a) = e'$, $\phi(b) = e'$.

Now we have

$$\begin{aligned}\phi(ab^{-1}) &= \phi(a)\,\phi(b^{-1}) \\ &= \phi(a)\,[\phi(b)]^{-1} \\ &= e'(e')^{-1} \\ &= e'\,e' \\ &= e'.\end{aligned}$$

$\therefore \phi(ab^{-1}) = e'$.

$\therefore ab^{-1} \in K$.

$\therefore K$ is a subgroup of G.

Let $x \in G$,

$\phi(xax^{-1}) = \phi(x)\,\phi(a)\,[\phi(x)]^{-1}$ ($\because \phi$ is hom)

$= \phi(x)\,e'\,[\phi(x)]^{-1}$

$= \phi(x)\,[\phi(x)]^{-1}$ (by identity prop of $\bar{G}$)

$= e'$ (by inverse prop of $\bar{G}$).

$\therefore \phi(xax^{-1}) = e'$

$\Rightarrow xax^{-1} \in K$

$\therefore K$ is a normal subgroup of G.

HEAD OFFICE:25/8, Old Rajinder Nagar Market, Delhi-60. (M) 9999197625, 011-45629987. BRANCH OFFICE(DELHI): 105-106, Top Floor, Mukherjee Tower, Mukherjee Nagar, Delhi-9. BRANCH OFFICE(HYDERBAD): H.No. 1-10-237, 2nd Floor, Room No. 202 R.K'S-Kancham's Blue Sapphire Ashok Nagar Hyd-20. (M) 09652351152, 09652661152

ISM 2.(b) → 4. Let R be a ring with unity. If the product of any two non-zero elements is non-zero, prove that $ab=1 \Rightarrow ba=1$

Whether Z_6 has the above property or not explain. Is Z_6 an integral domain.

Sol:- Given that $(R,+,\times)$ is a ring with unity element.

i.e $\forall\ a \in R, \exists\ 1 \in R$ s.t $a.1 = 1.a = a$. —— ①

let $a \neq 0, b \neq 0$ be two elements in R

$\Rightarrow ab \neq 0$ in R.

i.e R has no zero divisors.

Let us suppose that $ab = 1$

$\Rightarrow aba = 1.a$

$\Rightarrow aba = a.1$ (by using ①)

$\Rightarrow aba - a.1 = 0$

$\Rightarrow a(ba-1) = 0$

$\Rightarrow$ either $a = 0$ or $ba-1 = 0$

but $a \neq 0$

$\therefore ba - 1 = 0$

$\Rightarrow ba = 1$.

$\therefore \boxed{ab = 1 \Rightarrow ba = 1}$

Now

$(\mathbb{Z}_6 = \{0,1,2,3,4,5\}, +_6, \times_6)$ be the given ring.

form the composition tables for $\mathbb{Z}_6$ w.r.t $+_6$ and $\times_6$

Table(i):

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

Table(ii):

$\times_6$	0	1	2	3	4	5
0	0	0	0	0	0	0
1	0	1	2	3	4	5
2	0	2	4	0	2	4
3	0	3	0	3	0	3
4	0	4	2	0	4	2
5	0	5	4	3	2	1

Clearly the ring $\mathbb{Z}_6$ has unity 1 but the product of the two non-zero elements not necessarily non-zero.

for example: $2, 3 \in \mathbb{Z}_6 \Rightarrow 2 \times_6 3 = 0$.

$\therefore$ $\mathbb{Z}_6$ does not contain the above property and $\mathbb{Z}_6$ is not an integral domain because $\mathbb{Z}_6$ has the zero divisors.

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2.(c) Discuss the Convergence of the series

$$\frac{x}{2}+\frac{1.3}{2.4}x^2+\frac{1.3.5}{2.4.6}x^3+\cdots,\ (x>0)$$

Sol^n:

Let $u_n = \frac{1.3.5\ldots(2n-1)}{2.4.6\ldots(2n)}x^n$

$$u_{n+1} = \frac{1.3.5\ldots(2n-1)(2n+1)}{2.4.6\ldots(2n)(2n+2)}x^{n+1}$$

Now $\frac{u_n}{u_{n+1}} = \frac{2n+2}{2n+1}\cdot\frac{1}{x} = \frac{1+\frac{1}{n}}{1+\frac{1}{2n}}\cdot\frac{1}{x}$

$\therefore \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \frac{1}{x}$

$\therefore$ By Ratio test Σu_n Converges if $\frac{1}{x}>1$

i.e., $x<1$ & diverges if $\frac{1}{x}<1$ i.e. $x>1$

If $x=1$ then the ratio fails.

but when $x=1$, $\frac{u_n}{u_{n+1}} = \frac{2n+2}{2n+1}$

Clearly which is not involving in e.

So we apply the Raabe's test.

$$n\left(\frac{u_n}{u_{n+1}}-1\right) = \frac{n}{2n+1} = \frac{1}{2+\frac{1}{n}}$$

Now $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = \frac{1}{2}<1$

$\therefore$ By Raabe's Test, Σu_n is divergent.

Hence Σu_n is convergent if $x<1$

and divergent if $x\geq 1$

3(b) Que:- Prove that, Every Integral domain can be embedded in a field.

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Sol^n:- let 'D' be an integral Domain with at least two elements.

consider, $S = \{(a,b) \mid a, b \in D, b \neq 0\}$.

Then $S \neq \phi$ and $S \subset D \times D$.

For all $(a,b), (c,d) \in S$ define a relation '$\sim$' on 'S' as $(a,b) \sim (c,d)$, if and only if $ad = bc$.

We now prove that '$\sim$' is an equivalence relation on 'S'.

1) For each $(a,b) \in S$, we have $ab = ba$. which implies that $(a,b) \sim (a,b)$.

2) $(a,b), (c,d) \in S$ and $(a,b) \sim (c,d)$

$$\Rightarrow ad = bc \Rightarrow cb = da \Rightarrow (c,d) \sim (a,b).$$

3) $(a,b), (c,d), (e,f) \in S$.

$$(a,b) \sim (c,d), (c,d) \sim (e,f) \Rightarrow ad = bc, cf = de$$

$$\Rightarrow (ad)f = (bc)f, \ cf = de$$

$$\Rightarrow (af)d = b(de) \qquad (\because d \neq 0)$$

$$\Rightarrow af = be.$$

$$\Rightarrow (a,b) \sim (e,f)$$

$\therefore$ '$\sim$' is an equivalance relation on 'S'. The equivalance relation '$\sim$' partitions the set 'S' into equivalence classes which are either identical or disjoint.

HEAD OFFICE:25/8, Old Rajinder Nagar Market, Delhi-60. (M) 9999197625, 011-45629987. BRANCH OFFICE(DELHI): 105-106, Top Floor, Mukherjee Tower, Mukherjee Nagar, Delhi-9. BRANCH OFFICE(HYDERBAD): H.No. 1-10-237, 2nd Floor, Room No. 202 R.K'S-Kancham's Blue Sapphire Ashok Nagar Hyd-20. (M) 09652351152, 09652661152

For $(a,b) \in S$, let a/b denote the equivalence class of (a,b). Then $a/b = \{(x,y) \sim (a,b)\}$.

If a/b, c/d are the equivalence classes of $(a,b), (c,d) \in S$ then either $a/b = c/d$ or $a/b \cdot c/d = \phi$. It is evident that $a/b = c/d$ if and only if $ad = bc$.

let, F denotes the set of all the equivalence classes or the set of quotients.

Then $F = \left\{ \frac{a}{b} \,\middle|\, (a,b) \in S \right\}$.

since 'D' has at least two elements, say, 0, $a \in D$. we have quotients $\frac{0}{a}, \frac{a}{a} \in F$ and $\frac{0}{a} \neq \frac{a}{a}$.

$\therefore$ the set 'F' has at least two elements.

For $\frac{a}{b}, \frac{c}{d} \in F$ define addition (+) and multiplication

$$\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd} \quad \text{and} \quad \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

since 'D' is without zero divisor,

$b \neq 0, d \neq 0 \in D \Rightarrow bd \neq 0$.

So, $\frac{ad+bc}{bd}, \frac{ac}{bd} \in F$.

Now, we prove that the addition & multiplication defined above are well defined.

let, $\frac{a}{b} = \frac{a'}{b'}$ and $\frac{c}{d} = \frac{c'}{d'}$

then $ab' = a'b$ and $cd' = c'd$

Now (I) $\Rightarrow ab'dd' = a'bdd'$ & $bb'cd' = bb'c'd$.

$\Rightarrow ab'dd' + bb'cd' = a'bdd' + bb'c'd$

$\Rightarrow (ad+bc)b'd' = (a'd'+b'c')bd$

$$\Rightarrow \frac{ad+bc}{bd} = \frac{a'd'+b'c'}{b'd'}$$

Also (I) $\Rightarrow ab'cd' = a'bc'd$

$\Rightarrow (ac)(b'd') = (a'c')(bd)$

$$\Rightarrow \frac{ac}{bd} = \frac{a'c'}{b'd'}.$$

$\therefore$ addition and multiplication of quotients are well defined binary operation on 'F'.

We now prove that $(F,+,\cdot)$ is a field.

1) For $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in F$: $\left(\frac{a}{b}+\frac{c}{d}\right)+\frac{e}{f} = \frac{ad+bc}{bd}+\frac{e}{f}$.

$$= \frac{(ad+bc)f+(bd)e}{(bd)f}.$$

$$= \frac{a(df)+(cf+de)b}{b(df)}$$

$$= \frac{a}{b}+\frac{cf+de}{df}$$

$$= \frac{a}{b}+\left(\frac{c}{d}+\frac{e}{f}\right)$$

$\therefore$ addition is associative.

2) For $\frac{a}{b}, \frac{c}{d} \in F$; $\frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$.

$$= \frac{bc+ad}{db}$$

$$= \frac{c}{d} + \frac{a}{b}.$$

$\therefore$ addition is commutative.

3) For $u \neq 0 \in D$, we have $\frac{0}{u} \in F$ such that,

$$\frac{0}{u} + \frac{a}{b} = \frac{0b+ua}{ub} = \frac{ua}{ub} = \frac{a}{b} \quad \forall \frac{a}{b} \in F.$$

$\therefore \frac{0}{u} \in F$ is zero element.

4) let $\frac{a}{b} \in F$. Then $\frac{-a}{b} \in F$ such that

$$\frac{a}{b} + \frac{-a}{b} = \frac{ab+(-a)b}{b^2} = \frac{0}{b^2} = \frac{0}{u}$$

$(\because 0u = 0b^2)$

$\therefore$ every element in 'F' has additive inverse.

5) For $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in F$; $\left(\frac{a}{b} \cdot \frac{c}{d}\right) \cdot \frac{e}{f} = \frac{ac}{bd} \cdot \frac{e}{f}$.

$$= \frac{(ac)e}{(bd)f}$$

$$= \frac{a(ce)}{b(df)}$$

$$= \frac{a}{b} \frac{ce}{df} = \frac{a}{b} \cdot \left(\frac{c}{d} \cdot \frac{e}{f}\right)$$

$\therefore$ multiplication is associative.

6) For $\frac{a}{b}, \frac{c}{d} \in F$; $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd} = \frac{ca}{db} = \frac{c}{d} \cdot \frac{a}{b}$

$\therefore$ multiplication is commutative.

7) For $u \neq 0 \in D$ we have $\frac{u}{u} \in F$, such that

$$\frac{a}{b} \cdot \frac{u}{u} = \frac{au}{bu} = \frac{a}{b} \quad \forall \frac{a}{b} \in F.$$

$\therefore \frac{u}{u} \in F$ is the unity element.

8) let $\frac{a}{b} \in F$ and $\frac{a}{b} \neq \frac{0}{u}$.

Then $au \neq 0$ which implies that $a \neq 0$, as $u \neq 0$.

$\therefore b \neq 0$ & $a \neq 0 \implies \frac{b}{a} \in F$

$\therefore$ For $\frac{a}{b} \left(\neq \frac{0}{u}\right) \in F$ there exists $\frac{b}{a} \in F$, s.t

$$\frac{a}{b} \cdot \frac{b}{a} = \frac{ab}{ba} = \frac{u}{u} \qquad (\because (ab)u = (ba)u).$$

$\therefore$ every non-zero element in 'F' has multiplicative inverse.

9) For $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in F$; $\frac{a}{b} \cdot \left(\frac{c}{d} + \frac{e}{f}\right) = \frac{a}{b} \cdot \frac{cf+de}{df}$.

$$= \frac{a(cf+de)}{b(df)}$$

$$= \frac{(acf + ade)(bdf)}{(bdf)(bdf)}$$

$$= \frac{acf.bdf + ade.bdf}{bdf.bdf}$$

$$= \frac{acf}{bdf} + \frac{ade}{bdf}$$

$$= \frac{ac}{bd} + \frac{ae}{bdf} = \frac{a}{b}\cdot\frac{c}{d} + \frac{a}{b}\cdot\frac{e}{f}$$

Similarly, we can prove that,

$$\left(\frac{c}{d}+\frac{e}{f}\right)\cdot\frac{a}{b} = \frac{c}{d}\cdot\frac{a}{b} + \frac{e}{f}\cdot\frac{a}{b}$$

$\therefore$ multiplication is distributive over addition.

In view of (1), (2), (3), (4), (5), (6), (7), (8) & (9).

$(F,+,\cdot)$ is a field.

Now, we have to prove that 'D' is embedded in the field 'F', that is we have to show there exists isomorphism of 'D' into 'F'.

Define mapping $\phi: D \to F$ by

$$\phi(a) = \frac{ax}{x} \;\forall\; a \in D \text{ and } x(\neq 0) \in D.$$

$a, b \in D$ and $\phi(a) = \phi(b) \Rightarrow \dfrac{ax}{x} = \dfrac{bx}{x}$

$$\Rightarrow (ax)x = (bx)x.$$

$$\Rightarrow (a-b)x^2 = 0$$

$$\Rightarrow a-b = 0 \quad \because x^2 \neq 0.$$

$$\Rightarrow a = b$$

$\therefore \phi$ is one-one.

For $a, b \in D$; $\phi(a+b) = \frac{(a+b)x}{x} = \frac{(a+b)xx}{xx}$

$$= \frac{axx + bxx}{xx}$$

$$= \frac{ax}{x} + \frac{bx}{x} = \phi(a) + \phi(b)$$

$$\phi(ab) = \frac{(ab)x}{x} = \frac{(ab)xx}{xx} = \frac{ax}{x} \cdot \frac{bx}{x} = \phi(a) \cdot \phi(b)$$

$\therefore$ 'ϕ' is a homomorphism.

Hence, 'ϕ' is an isomorphism of 'D' into 'F'.

$\therefore$ the integral domain 'D' is embedded in the field 'F'.

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3.(c) Let f be a continuous function on $[0,1]$. Using first mean value theorem on Integration, Prove that

$$\lim_{n\to\infty}\int_0^1 \frac{nf(x)}{1+n^2x^2}dx = \frac{\pi}{2}f(0).$$

Sol^n: Let f be a continuous function on $[0,1]$.

Now

$$\int_0^1 \frac{nf(x)}{1+n^2x^2}dx = \int_0^{1/\sqrt{n}} \frac{nf(x)}{1+n^2x^2}dx + \int_{1/\sqrt{n}}^1 \frac{nf(x)}{1+n^2x^2}dx \quad \text{——} (1)$$

By the first mean value theorem.

$$\int_0^{1/\sqrt{n}} \frac{nf(x)}{1+n^2x^2}dx = f(\xi)\int_0^{1/\sqrt{n}} \frac{n}{1+n^2x^2}dx, \text{ where } 0\le \xi \le \frac{1}{\sqrt{n}}$$

$$= f(\xi)\int_0^{1/\sqrt{n}} \frac{n}{n^2\left(\frac{1}{n^2}+x^2\right)}dx$$

$$= f(\xi)\frac{1}{n}\left[\frac{\tan^{-1}\left(\frac{x}{1/n}\right)}{1/n}\right]_0^{1/\sqrt{n}}$$

$$= f(\xi)\left[\tan^{-1}\sqrt{n}-0\right]$$

$$= f(\xi)\tan^{-1}\sqrt{n} \to \frac{\pi}{2}f(0) \text{ as } n\to\infty \quad \text{——} (2).$$

$\left[\because \text{as } n\to\infty,\ \tan^{-1}\sqrt{n}\to \pi/2\right.$

$\& \ 0\le\xi\le\frac{1}{\sqrt{n}}\to 0 \text{ as } n\to\infty$

i.e, $0\le \xi\le 0$

$\left.\therefore \xi = 0\right]$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

<u>First mean value theorem:</u>

If $\int_a^b f(x)dx$ & $\int_a^b \phi(x)dx$ both exist and $\phi(x)$ keeps the same sign, +ve or −ve, throughout the interval of integration, then $\exists$ a number μ, lying b/w the bounds of f such that

$$\int_a^b f(x)\phi(x)dx = \mu\int_a^b \phi(x)dx$$

Here $\phi(x) = \frac{n}{1+n^2x^2}$

Again, since f is continuous on $[0,1]$, it is bounded and therefore there exists $K>0$ such that $|f(x)|<K \ \forall x\in[0,1]$

$$\therefore \left| \int_{1/\sqrt{n}}^{1} \frac{n f(x)}{1+n^2x^2} dx \right| \leq K \left| \int_{1/\sqrt{n}}^{1} \frac{n}{1+n^2x^2} dx \right|$$

$$= K \left| \frac{n}{n^2} \int_{1/\sqrt{n}}^{1} \frac{1}{\frac{1}{n^2}+x^2} dx \right|$$

$$= K \left| \frac{1}{n} \left[\frac{\tan^{-1} nx}{1/n} \right]_{1/\sqrt{n}}^{1} \right|$$

$$= K \left| \left[\tan^{-1} nx \right]_{1/\sqrt{n}}^{1} \right|$$

$$= K \left| \tan^{-1}(n) - \tan^{-1}\sqrt{n} \right|$$

$$\longrightarrow 0 \text{ as } n \to \infty \quad \text{—— (3)}$$

$\therefore$ from (1), (2) & (3)

we have

$$\int_0^1 \frac{n f(x)}{1+n^2x^2} dx = \frac{\pi}{2} f(0) + 0$$

$$= \frac{\pi}{2} f(0) \quad \text{as } n \to \infty$$

$$\lim_{n\to\infty} \int_0^1 \frac{n f(x)}{1+n^2x^2} = \frac{\pi}{2} f(0)$$

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3.(d)(ii) Prove that $\frac{\tan x}{x} > \frac{x}{\sin x}$, $x \in (0, \pi/2)$

Sol'n: $\frac{\tan x}{x} - \frac{x}{\sin x} = \frac{\tan x \sin x - x^2}{x \sin x}$

Since $x \sin x > 0 \ \forall x \in (0, \pi/2)$

$\therefore$ We are enough to show that

$\tan x . \sin x - x^2 > 0 \ \forall \ x \in (0, \pi/2)$

Let $f(x) = \tan x . \sin x - x^2 \ \forall \ x \in (0, \pi/2)$

$\Rightarrow f'(x) = \sec^2 x \sin x + \tan x \cos x - 2x$

$= \sin x (\sec^2 x + 1) - 2x$

We cannot decide about the sign of $f'(x)$ (because of the presence of $2x$ term)

Let $g(x) = f'(x) \ \forall \ x \in (0, \pi/2)$

$\Rightarrow g'(x) = \cos x (\sec^2 x + 1) + \sin x (2 \sec^2 x \tan x) - 2$

$= \sec x + \cos x - 2 + 2 \sin^2 x \sec^3 x$

$= (\sqrt{\sec x} - \sqrt{\cos x})^2 + 2 \sin^2 x \sec^3 x$

Since $g'(x) > 0 \ \forall \ x \in (0, \pi/2)$

$\Rightarrow g(x)$ is an increasing function $(0, \pi/2)$

$\Rightarrow g(0) < g(x)$ in $0 < x < \pi/2$

Since $g(0) = 0$

$\therefore g(x) > 0$

$\Rightarrow f'(x) > 0$ whenever $0 < x < \pi/2$

$\therefore f$ is an increasing function in

$0 < x < \pi/2$

$\Rightarrow \quad f(0) < f(x)$

$\Rightarrow \quad 0 < f(x)$

$\Rightarrow \quad \tan x \sin x - x^2 > 0$ in $(0, \pi/2)$

$$\Rightarrow \quad \frac{\tan x \sin x - x^2}{x \sin x} > 0$$

$$\Rightarrow \quad \frac{\tan x}{x} - \frac{x}{\sin x} > 0$$

$$\Rightarrow \frac{\tan x}{x} > \frac{x}{\sin x} \text{ whenever } 0 < x < \pi/2$$

IAS-2008 4(b) P-II

Let $f(z)$ be an entire function satisfying $|f(z)| \le k|z|^2$ for some constant k and all z. Show that $f(z) = az^2$ for some constant 'a'.

Soln: Given that f is an entire function.

$\therefore f$ has a Taylor series expansion around $z=0$.

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

By cauchy estimates. $|f^n(z)| \le \frac{n!M}{R^n}$

M is bound of $f(z)$.

for $n=3$, $|f^{(3)}(z)| \le \frac{3!R^2}{R^3}$

$|f(z)| \le k|z|^2$.

$\le kR^2 < \frac{3}{R}$

Since f is entire, we can let $R \to \infty$

$\Rightarrow f^{(3)}(z) = 0 \ \forall z$

Hence the claim.

$f(z) = f(0) + f'(0) + f''(0)z^2$

Using the inequality, we have

$|f(z)| \le k|z|^2$.

Clearly $f(0) = 0$ & $f'(0) = 0$.

Using the inequality once again

$|f(0)| \le 0$ and $f'(z) f(0) + \frac{z^2}{2} f''(0)$

$\therefore f(z) = k_1 z^2$

5(a). Find the general solution of the paritial differential equation $(2xy-1)p+(z-2x^2)q=2(x-yz)$ and also find the particular solution which passes through the lines x=1, y=0.

SOLUTION

Lagranges auxiliary equation can be written as

$$\frac{dx}{2xy-1}=\frac{dy}{z-2x^2}=\frac{dz}{2(x-yz)} \qquad \ldots(1)$$

using multipliers $(z,1,x)$

$$(1)\equiv \frac{zdx+dy+xdz}{2xyz-z+z-2x^2+2x^2-2xyz}=\frac{\mathrm{xdx}+\mathrm{ydy}+\frac{1}{2}\mathrm{dz}}{0}$$

$\therefore$ zdx + dy + xdz = 0 $\Rightarrow$ $\boxed{\mathrm{xz}+\mathrm{y}=\mathrm{c}_1}$

using multipliers $\left(\mathrm{x},\mathrm{y},\frac{1}{2}\right)$

$$(1)\equiv \frac{\mathrm{xdx}+\mathrm{ydy}+\frac{1}{2}\mathrm{dx}}{2\mathrm{x}^2\mathrm{y}-\mathrm{x}+\mathrm{yz}-2\mathrm{x}^2\mathrm{y}+\mathrm{x}-\mathrm{yz}}=\frac{xdx+ydy+\frac{1}{2}dz}{0}$$

$$\therefore \quad xdx+ydy+\frac{1}{2}dz=0$$

$$\boxed{x^2+y^2+z=c_2}$$

Required solution is

$\therefore$ $\phi(xz+y,x^2+y^2+z)=0$

where ϕ is arbitrary function.

5(b). Find general solution of the Partial Differential Equation $(D^2+DD'-6D'^2)z=y\cos x$ where $D=\frac{d}{dx}, D'=\frac{d}{dy}$.

SOLUTION

Homogenous equations of (1) is given by $(D^2+DD'-6D'^2)z=0$

Auxiliary function is given by

$m^2+m-6=0 \Rightarrow m=-3, 2$

$\therefore$ C.F. $=\phi_1(y-3x)+\phi_2(y+2x)$

ϕ_1, ϕ_2 being arbitrary functions.

Particular integral

$$= \frac{y\cos x}{(D+3D')(D-2D')} \quad \textit{Let } y+2x=a$$

$$= \frac{1}{(D+3D')}\int (a-2x)\cos x\,dx$$

$$= \frac{1}{D+3D'}\left[(a-2x)\sin x-(-2)\int \sin x\,dx\right]$$

$$= \frac{1}{D+3D'}\left[y\sin x-2\cos x\right] \qquad \text{put } a=y+2x$$

$$= \int\left[(b+3x)\sin x-2\cos x\right]dx \qquad \boxed{\begin{array}{l} y-3x=b \\ y=3x+b \end{array}}$$

$$= (b+3x)(-\cos x)+\int 3\cos x\,dx-2\int \cos x\,dx$$

$$y_p = -y\cos x+\sin x$$

$$\boxed{\therefore G.S.=\phi_1(y-3x)+\phi_2(y+2x)-y\cos x+\sin x}$$

5(c) Determine an approximate root of the equation $\cos x - xe^x = 0$ using Regula-Falsi method, correct to 4 decimal place (13)

P-II-2008

Sol^n: $f(x) = \cos x - xe^x$.

So that $f(0) = 1$ and $f(1) = \cos 1 - e$
$= -2.17798$

$\therefore f(0)\, f(1) < 0$

Hence the root lies between 0 and 1.

Take $x_0 = 0$ and $x_1 = 1$.

$\therefore f(x_0) = 1$ and $f(x_1) = -2.17798$

By the method of false position, we get

$$x_2 = \frac{x_0 f(x_1) - x_1 f(x_0)}{f(x_1) - f(x_0)} \quad \text{—(1)}$$

$$= \frac{0(-2.17798) - 1(1)}{-2.17798 - 1}$$

$$= 0.31467$$

$\therefore$ The first approximation to the root is

$x_2 = 0.31467$

Now $f(x_2) = 0.51987 > 0$.

$\therefore f(x_2) \cdot f(x_1) < 0$.

$\therefore$ The root lies b/w 0.31467 and 1

Take $x_0 = 0.31467$ and $x_1 = 1$

$\therefore f(x_0) = 0.51987$ and $f(x_1) = -2.17798$

From (1),

$$x_3 = \frac{(0.3146)(-2.17798) - 1(0.51987)}{-2.17798 - 0.51987}$$

$$x_3 = 0.44673$$

$\therefore$ The 2nd approximation to the root is

$x_3 = 0.44673$

HEAD OFFICE:25/8, Old Rajinder Nagar Market, Delhi-60. (M) 9999197625, 011-45629987. BRANCH OFFICE(DELHI): 105-106, Top Floor, Mukherjee Tower, Mukherjee Nagar, Delhi-9. BRANCH OFFICE(HYDERBAD): H.No. 1-10-237, 2nd Floor, Room No. 202 R.K'S-Kancham's Blue Sapphire Ashok Nagar Hyd-20. (M) 09652351152, 09652661152

Now repeating this process, the successive approximations are

$x_4 = 0.49402$, $x_5 = 0.50995$,

$x_6 = 0.51520$, $x_7 = 0.51692$, $x_8 = 0.51748$.

$x_9 = 0.51767$, $x_{10} = 0.51775$, etc.

$\therefore$ The approximate root is 0.5177 correct to 4 decimal places

5(f) $\longrightarrow$ IAS-2008 P-II

If the velocity potential of a fluid is $\phi = (z/r^3)\tan^{-1}(y/x)$ where $r^2 = x^2+y^2+z^2$, then show that the streamlines lies on the surface $x^2+y^2+z^2 = c(x^2+y^2)^{2/3}$, c being an arbitrary constant.

Sol^n $\longrightarrow$

The velocity potential ϕ is given by

$$\phi(x,y,z) = (x^2+y^2+z^2)^{-3/2}\, z\tan^{-1}(y/x)$$

$$= r^{-3} z\tan^{-1}(y/x) \text{ —— } (1)$$

where $$r^2 = x^2+y^2+z^2 \text{ —— } (2)$$

so that $$\frac{\partial r}{\partial x} = \frac{x}{r},\ \frac{\partial r}{\partial y} = \frac{y}{r},\ \frac{\partial z}{\partial r} = \frac{z}{r} \text{ —— } (3)$$

$$u = -\frac{\partial\phi}{\partial x} = 3zxr^{-5}\tan^{-1}\frac{y}{x} + \frac{zyr^{-3}}{x^2+y^2}$$

$$v = -\frac{\partial\phi}{\partial y} = 3zyr^{-5}\tan^{-1}\frac{y}{x} - \frac{zxr^{-3}}{x^2+y^2}$$

$$w = -\frac{\partial\phi}{\partial z} = 3z^2r^{-5}\tan^{-1}\frac{y}{x} - r^{-3}\tan^{-1}\frac{y}{x}$$

The equation of line flow are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

i.e. $$\frac{dx}{3zxr^{-5}\tan^{-1}\frac{y}{x} + \frac{zyr^{-3}}{x^2+y^2}} = \frac{dy}{3zyr^{-5}\tan^{-1}\frac{y}{x} - \frac{zxr^{-3}}{x^2+y^2}}$$

$$= \frac{dz}{(3z^2r^{-5} - r^{-3})\tan^{-1}\frac{y}{x}} \text{ —— } (4)$$

each member of equation (4) is

$$= \frac{x\,dx + y\,dy + z\,dz}{(3x^2+3y^2+3z^2)r^{-2}-1} = \frac{x\,dx + y\,dy}{(3x^2+3y^2)/r^2}$$

or, $$\frac{x\,dx + y\,dy + z\,dz}{2} = \frac{r^2(x\,dx + y\,dy)}{3(x^2+y^2)}$$

or, $$\frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2+y^2+z^2} = \frac{2}{3}\,\frac{2x\,dx + 2y\,dy}{x^2+y^2} \quad \text{--- (5)}$$

Integrating (5)

$$\log(x^2+y^2+z^2) = (2/3)\log(x^2+y^2) + \log c$$

or, $x^2+y^2+z^2 = c\,(x^2+y^2)^{2/3}$, c being an arbitrary constant — (6)

(6) gives the required series of the surface.

6(a) find the steady state temperature distribution in a thin rectangular plate bounded by the lines $x=0, x=a, y=0, y=b$. The edges $x=0, x=a, y=0$ are kept at temperature zero while the edge $y=b$ is kept at $100^\circ C$.

IAS-2008 P-II

Sol: The steady state temperature $u(x,y)$ is the solution of Laplace's equation

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \quad \text{——(1)}$$

subject to boundary conditions

$$u(0,y)=u(a,y)=0 \quad \text{——(2)}$$

$$u(x,0)=0 \quad \text{——(3)}$$

$$u(x,b)=100 \quad \text{——(4)}$$

suppose (1) has a solution of the form

$$u(x,y)=X(x)\,Y(y) \quad \text{——(5)}$$

from (1), $X''Y+XY''=0$

$$\Rightarrow \frac{X''}{X}=-\frac{Y''}{Y}=\mu \text{ (say)}$$

$$\Rightarrow X''-\mu X=0 \quad \text{——(6)} \quad \text{and} \quad Y''+\mu Y=0 \quad \text{——(7)}$$

using B.C. (2), (5) gives $X(0)Y(y)=0$ & $X(a)Y(y)=0$

$$\Rightarrow X(0)=0 \ \& \ X(a)=0 \quad \text{——(8)}$$

since $Y(y)\neq 0$ since otherwise $u\equiv 0$ which doesnot satisfy (4)

we now solve (6) under B.C (8):

Three cases arise.

case(i): Let $\mu=0$. Then (6) $\equiv X''=0 \Rightarrow X=Ax+B$

Using B.C. (8) $A = B = 0$

$\Rightarrow X(x) = 0$ which leads to $u = 0$.

so reject $\mu = 0$

Case(2): Let $\mu = \lambda^2$ where $\lambda \neq 0$

$\therefore$ from (6): $X'' - \lambda X = 0$

$\Rightarrow X(x) = Ae^{\lambda x} + Be^{-\lambda x}$

Using B.C. (8): we get $A = B = 0$

$\Rightarrow X(x) = 0$

This leads to $u = 0$

so reject $\mu = \lambda^2$

Case(3): Let $\mu = -\lambda^2$, where $\lambda \neq 0$.

Then solution of (6) is given by

$X(x) = A\cos\lambda x + B\sin\lambda x$

Using B.C (8): $A = 0$ and $A\cos\lambda a + B\sin\lambda a = 0$

$\Rightarrow B\sin\lambda a = 0$

$\Rightarrow \sin\lambda a = 0$ $(\because B \neq 0)$ otherwise we will get $X(x) = 0$ and hence $u = 0$)

$\Rightarrow \lambda a = n\pi$

$\Rightarrow \lambda = \frac{n\pi}{a}$, $n = 1, 2, \ldots$ — (9)

Hence non zero solution $X_n(x)$ of (6) are given by

$X_n(x) = B_n \sin\left(\frac{n\pi x}{a}\right)$ — (10)

and $\mu = -\lambda^2 = -\frac{n^2\pi^2}{a^2}$.

$\therefore$ from (7): $Y'' - \frac{n^2\pi^2}{a^2} Y = 0$

whose general solution is — (11)

$Y_n(y) = C_n e^{\frac{n\pi y}{a}} + D_n e^{-\frac{n\pi y}{a}}$

Using (3), (5) gives $0 = X(x)Y(0)$ so that $Y(0) = 0$ $(\because X(x) \neq 0$, for otherwise we will get $u = 0)$

But $Y(0) = 0 \Rightarrow Y_n(y) = 0$ —

Using the initial values

$p_0 = c_3 e^{t_0}$ and $q_0 = c_4 e^{t_0}$

$\Rightarrow -\frac{\lambda}{2} = c_3$ and $\frac{\lambda}{\sqrt{2}} = c_4$

$\therefore \boxed{p = -\frac{\lambda}{2} e^t}$ and $\boxed{q = \frac{\lambda}{\sqrt{2}} e^t}$

from (8) & (9)

$x = 2p + 2\lambda = 2\left(-\frac{\lambda}{2}\right) + 2\lambda = -\lambda e^t + 2\lambda$

i.e. $\boxed{x = -\lambda e^t + 2\lambda}$

and $y = -2\left(\frac{\lambda}{\sqrt{2}}\right)e^t + \sqrt{2}\lambda = -\sqrt{2}\lambda e^t + \sqrt{2}\lambda$

$\Rightarrow \boxed{y = \lambda\sqrt{2}(1 - e^t)}$

from (3), $z'(t) = 2z$

$\Rightarrow \frac{dz}{z} = 2dt$

$\Rightarrow \log z = 2t + \log c_5$

$\Rightarrow$ using initial values, we get

$\log z_0 = 2t_0 + \log c_5 \Rightarrow c_5 = z_0$

$\therefore \log z = 2t + \log z_0$

$\Rightarrow z = z_0 e^{2t}$

$\Rightarrow \boxed{z = -\frac{\lambda^2}{4} e^{2t}}$

$\therefore$ The required characteristics of (1) are given by

$\boxed{x = \lambda(2 - e^t)}$ (i) ; $\boxed{y = \lambda\sqrt{2}(1 - e^t)}$ (ii) ; $\boxed{z = -\frac{\lambda^2}{4} e^{2t}}$ (iii)

Now eliminating e^t and λ from (i), (ii) & (iii)

from (i) ; $\lambda = \frac{x}{2 - e^t}$

from (ii) $y = \frac{x}{(2-e^t)}\sqrt{2}(1-e^t) = \frac{\sqrt{2}x(1-e^t)}{2-e^t}$

$\Rightarrow y(2-e^t) = \sqrt{2}x(1-e^t)$

$\Rightarrow 2y - ye^t = \sqrt{2}x - \sqrt{2}xe^t$

$\Rightarrow (\sqrt{2}x - y)e^t = \sqrt{2}x - 2y$

$\Rightarrow e^t = \frac{\sqrt{2}x - 2y}{\sqrt{2}x - y}$

from (i) $\lambda = \frac{x}{2 - \left(\frac{\sqrt{2}x - 2y}{\sqrt{2}x - y}\right)} = \frac{(\sqrt{2}x - y)x}{2\sqrt{2}x - 2y - \sqrt{2}x + 2y}$

$\Rightarrow \lambda = \frac{(\sqrt{2}x - y)}{\sqrt{2}}$

$\Rightarrow \boxed{\lambda = \frac{\sqrt{2}x - y}{\sqrt{2}}}$

from (iii):

$z = -\frac{\lambda^2 e^{2t}}{4}$

$= -\frac{1}{4}\left(\frac{\sqrt{2}x - y}{\sqrt{2}}\right)^2 \left(\frac{\sqrt{2}x - 2y}{\sqrt{2}x - y}\right)^2$

$= -\frac{(\sqrt{2}x - 2y)^2}{8}$

$= -\frac{[2x^2 + 4y^2 - 4\sqrt{2}xy]}{8}$

$= -\frac{[x^2 - 2\sqrt{2}xy + 2y^2]}{4}$

$\Rightarrow \boxed{4z + (x - \sqrt{2}y)^2 = 0}$ which is the required integral surface of (1) putting $y=0$ in (11), and using $Y_n(y)=0$,

we have $0=C_n+D_n \Rightarrow D_n=-C_n$.

Then (16) reduces to

$$Y_n(y)=C_n\left(e^{n\pi y/a}-e^{-\frac{n\pi y}{a}}\right)=2\sinh\left(\frac{n\pi y}{a}\right)$$

$\therefore u_n(x,y)=X_n(x)\,Y_n(y)$

$$u_n(x,y)=E_n\sin\left(\frac{n\pi x}{a}\right)\sinh\frac{n\pi y}{a}$$

are solutions of (1) satisfying (2) and (3).

Here $E_n=2B_nC_n$.

In order to satisfy (4), we now consider more general solution given by

$$u(x,y)=\sum u_n(x,y)=\sum_{n=1}^{\infty}E_n\sin\frac{n\pi x}{a}\sinh\frac{n\pi y}{a} \quad \text{--- (12)}$$

putting $y=b$ in (12) and using (4),

we get

$$f(x)=100=\sum_{n=1}^{\infty}\left(E_n\sinh\frac{n\pi b}{a}\right)\sin\frac{n\pi x}{a}$$

where $E_n\sinh\left(\frac{n\pi b}{a}\right)=\frac{2}{a}\int f(x)\sin\frac{n\pi x}{a}\,dx$

$$\Rightarrow E_n=\frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)}\int(100)\sin\frac{n\pi x}{a}\,dx$$

$$=\frac{200}{a\sinh\left(\frac{n\pi b}{a}\right)}\left[\frac{-\cos\frac{n\pi x}{a}}{n\pi/a}\right]_0^a$$

$$\Rightarrow E_n=\frac{200}{n\pi}\left[1-(-1)^n\right]\operatorname{cosech}\left(\frac{n\pi b}{a}\right)=\begin{cases}0, & \text{if } n=2m,\ m=1,2,3,\dots\\ \frac{(400)\operatorname{cosec}\{(2m-1)\pi b/a\}}{(2m-1)\pi}, & \text{if } n=2m-1,\ m=1,2,3,\dots\end{cases}$$

$\therefore$ (12) reduces to

$$u(x,y)=\sum_{m=1}^{\infty}E_{2m-1}\sin\frac{(2m-1)\pi x}{a}\sinh\frac{(2m-1)\pi y}{a}$$

$$(\text{or})\quad u(x,y)=\frac{400}{\pi}\sum_{m=1}^{\infty}\sin\frac{(2m-1)\pi x}{a}\sinh\frac{(2m-1)\pi y}{a}\operatorname{cosech}\frac{(2m-1)\pi b}{a}$$

6(b). Find complete and singular integrals of $2xz - px^2 - 2qxy + pq = 0$ using charpit's method.

SOLUTION

Refer 2007 IAS

Singular solution found by eliminating a, b in $\phi(x, y, z, a, b) = 0, \frac{\partial \phi}{\partial a} = 0, \frac{\partial \phi}{\partial a} = 0, \frac{\partial \phi}{\partial b} = 0$.

$f \Rightarrow z = ax^2 + by - ab$. ...(1)

By reducing to clarauit form

$$\frac{\partial f}{\partial a} = x^2 - b = 0 \quad ...(2)$$

$$\frac{\partial f}{\partial b} = y - a = 0 \quad ...(3)$$

form(2) and (3) is solving $\frac{\partial f}{\partial a} = 0 \frac{\partial f}{\partial b} = 0$.

$x^2 = b \quad y = a$

$\therefore$ putting $a = y$, $b = x^2$ in (1)

$$z = yx^2 + x^2y - x^2y$$

$z = yx^2$ Required singular solutions.

6(c). Reduce $\frac{\partial^2 z}{\partial x^2} = x^2 \frac{\partial^2 z}{\partial y^2}$ to canonical form.

SOLUTION

Given $r - x^2 t = 0$ …(1)

comparing with $Rr + Ss + Tt + F(p,q,x,y,z) = 0$

$R = 1 \quad S = 0 \quad T = -x^2$

$S^2 - 4RT = 0 - 4(1)(-x^2) = 4x^2 > 0$

$\therefore$ Hyperbolic λ-quadratic is given by

$\lambda^2 - x^2 = 0$ …(2)

$$\lambda = \pm x$$

Characteristic equation is given by

$$\frac{dy}{dx} + x = 0; \qquad \frac{\partial y}{\partial x} - x = 0$$

$$y + \frac{x^2}{2} = c_1; \qquad y - \frac{x^2}{2} = c_2$$

Let $u = y + \frac{x^2}{2} \qquad v = y - \frac{x^2}{2}$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial Z}{\partial V}\frac{\partial v}{\partial x}$$

$$= x\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

$$= \left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(x\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\right)$$

$$= \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) + x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

$$= \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) + x\left(\frac{\partial^2 z}{\partial u^2}\frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial u \partial v}\frac{\partial v}{\partial x}\right) - x\left(\frac{\partial^2 z}{\partial u \partial v}\frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial v^2}\frac{\partial v}{\partial x}\right)$$

$$= \left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) + x^2\left(\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right) \quad \text{…(3)}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial^2 z}{\partial u^2}\frac{\partial u}{\partial y} + \frac{\partial^2 z}{\partial u \partial v}\frac{\partial v}{\partial y} + \frac{\partial^2 z}{\partial u \partial v}\frac{\partial u}{\partial y} + \frac{\partial^2 z}{\partial v^2}\frac{\partial v}{\partial y}$$

$$= \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \qquad \ldots(4)$$

Putting (3) and (4) in given equation

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} + x^2\left(\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right) - x^2\left(\frac{\partial^2 z}{\partial v^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right) = 0$$

$$\boxed{\frac{\partial^2 z}{\partial u \partial v} = \frac{1}{4x^2}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)}$$

Required Canonical forms.

MATHEMATICS by K. Venkanna NA.P-2-2008

Que) 7(a)(i) The following values of the function $f(x) = \sin x + \cos x$ are given:

x	$10°$	$20°$	$30°$
$f(x)$	1.1585	1.2817	1.3360

Construct the quadratic interpolating polynomial that fits the data. Hence, calculate $f(\pi/12)$ & compare with exact value.

Sol:-

$$h = \frac{20-10}{1} = 10°$$

$$y_0 = f(x_0) = 1.1585$$

The forward difference table is given by:-

x	y	Δy	$\Delta^2 y$
$10°$	1.1585		
		0.1232	
$20°$	1.2817		-0.0689
		0.0543	
$30°$	1.3360		

Here, $\Delta y_0 = 0.1232$

$\Delta^2 y_0 = -0.0689$

$$u = \frac{x - x_0}{h} = \frac{x-10}{10}$$

The formula is

$$y = f(x_0) = y_0 + u\Delta y_0 + u(u-1)\Delta^2 y_0$$

$$y = 1.1585 + \frac{(x-10)}{10}(0.1232) + \left(\frac{x-10}{10}\right)\left(\frac{x-20}{20}\right)(-0.0689).$$

HEAD OFFICE:25/8, Old Rajinder Nagar Market, Delhi-60. (M) 9999197625, 011-45629987. BRANCH OFFICE(DELHI): 105-106, Top Floor, Mukherjee Tower, Mukherjee Nagar, Delhi-9. BRANCH OFFICE(HYDERBAD): H.No. 1-10-237, 2nd Floor, Room No. 202 R.K'S-Kancham's Blue Sapphire Ashok Nagar Hyd-20. (M) 09652351152, 09652661152

$$y = 1.1585 + 0.01232x - 0.1232 - 0.0003x^2 + 0.009x - 0.06$$

$$y = f(x) = -0.0003x^2 + 0.0213x + 0.9753$$

$\Rightarrow$ put $x = \frac{\pi}{12}$

$$y = f(\pi/12) = -0.0003\left(\frac{\pi^2}{144}\right) + 0.0213 \times \frac{\pi}{12} + 0.9753$$

$$\boxed{y = f\left(\frac{\pi}{12}\right) = 1.2273}$$

The exact value at $\frac{\pi}{12}$ = i.e $15°$

$$f(\pi/12) = \sin 15° + \cos 15°$$

$$f(\pi/12) = \underline{\underline{1.2247}}.$$

Hence, the result.

Q.w) 7(a)(ii) Apply Gauss-Sidel method to calculate (x, y, z) from the system

$$-x-y+6z=42$$
$$6x-y-z=11.33$$
$$-x+6y-z=32$$

with the initial value

$(x,y,z)=(4.67, 7.62, 9.05)$

upto two iteration.

Sol- Given equations are.

$$-x-y+6z=42$$
$$6x-y-z=11.33$$
$$-x+6y-z=32$$

$x_0=4.67$, $y_0=7.62$, $z_0=9.05$

Then; the above equations can be rewritten as

$$x=\frac{11.33+y+z}{6}$$

$$y=\frac{32+x+z}{6}$$

$$z=\frac{42+x+y}{6}$$

The general formula is

$$x^{k+1}=\frac{11.33+y^k+z^k}{6}$$

$$y^{k+1}=\frac{32+x^{k+1}+z^k}{6}$$

$$z^{k+1}=\frac{42+x^{k+1}+y^{k+1}}{6} \qquad k=0,1,2,3\ldots$$

Now; first iteration ; with initial values

$x_0 = 4.67 \quad z_0 = 9.05$

$y_0 = 7.62$

$$x^1 = \frac{11.33 + 7.62 + 9.05}{6} = 4.6666$$

$$y^1 = \frac{32 + 4.6666 + 9.05}{6} = 7.6194$$

$$z^1 = \frac{42 + 4.6666 + 7.6194}{6} = 9.04768$$

Now for 2nd Iteration: $x_0 = 4.6666$

$y_0 = 7.6194$

$z_0 = 9.04768$

$$x_2 = \frac{11.33 + 7.6194 + 9.04768}{6} = 4.66618$$

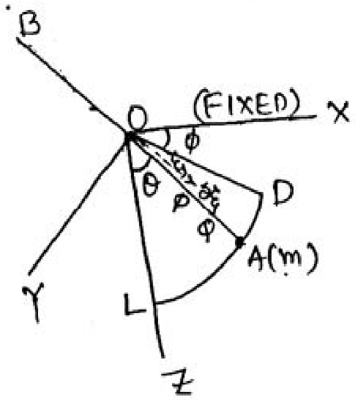
$$y_2 = \frac{32 + 4.66618 + 9.04768}{6} = 7.61897$$

$$z_2 = \frac{42 + 4.66618 + 7.61897}{6} = 9.0475$$

Hence, After two iteration, the solution is given by $(4.66618, 7.61897, 9.0475)$.

IAS 2008 8(a)
→ A uniform rod, of mass 3m and length 2l, has its middle point fixed and a mass m attached at one extremity. The rod when in a horizontal position is set rotating about a vertical axis through its centre with an angular velocity equal to $\sqrt{(2ng)/l}$; show that the heavy end of the rod will fall till the inclination of the rod to the vertical is $\cos^{-1}\{\sqrt{n^2+1}-n\}$, and will then rise again.

Sol'n: Let AB be the rod of mass 3m & length 2l. The middle point O of the rod is fixed and a mass m attached at the extremity A. Initially let the rod rest along OX in the plane of the paper. Let a line OY $\perp$lar to OX in the plane of the paper and a line OZ $\perp$lar to the plane of the paper be taken as axes of Y and Z respectively. At time t, let the rod turn through an angle ϕ to OX i.e. the plane OAL containing the rod and Z axis make an angle ϕ with X-Z plane. And let θ be the inclination of the rod with OZ at this time t. If P is a point of the rod at a distance OP $= \xi$, from O then coordinates of P are given by



$$x_p = \xi \sin\theta\cos\phi,\ y_p = \xi\sin\theta\sin\phi,\ z_p = \xi\cos\theta.$$

$\therefore$ If v_p and v_A are the velocities of the point P and A respectively, then

$$v_p^2 = \dot{x}_p^2 + \dot{y}_p^2 + \dot{z}_p^2$$

$$= (\xi\cos\theta\cos\phi\dot{\theta} - \xi\sin\theta\sin\phi\dot{\phi})^2 + (\xi\cos\theta\sin\phi\dot{\theta} + \xi\sin\theta\cos\phi\dot{\phi})^2 + (-\xi\sin\theta\dot{\theta})^2$$

$= \xi^2 (\dot\theta^2 + \dot\phi^2 \sin^2\theta)$

$\therefore$ At A, $\xi = OA = l$, $\therefore v_A^2 = l^2 (\dot\theta^2 + \dot\phi^2 \sin^2\theta)$.

Let $PQ = \delta\xi$ be an element of the rod at P, then mass of this element,

$$\delta m = \frac{3m}{2l} \cdot \delta\xi$$

$\therefore$ K.E of the element $PQ = \frac{1}{2}\delta m \cdot v_P^2 = \frac{1}{2}\xi^2 (\dot\theta^2 + \dot\phi^2 \sin^2\theta) \cdot \frac{3m}{2l}\delta\xi$

$\therefore$ K.E of the rod $AB = \frac{3m}{4l} \int_{-l}^{l} \xi^2 (\dot\theta^2 + \dot\phi^2 \sin^2\theta)\, d\xi$

$$= \frac{1}{2} m (\dot\theta^2 + \dot\phi^2 \sin^2\theta) l^2.$$

and K.E of mass m at $A = \frac{1}{2} m v_A^2 = \frac{1}{2} m l^2 (\dot\theta^2 + \dot\phi^2 \sin^2\theta)$

$\therefore$ The total K.E of the system

$T =$ K.E of the rod + K.E of the particle

$= ml^2 (\dot\theta^2 + \dot\phi^2 \sin^2\theta)$

The work function $W = -mg \cdot z_A = mgl\cos\theta$

$\therefore$ Lagrange's θ-equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\theta}\right) - \frac{\partial T}{\partial \theta} = \frac{\partial W}{\partial \theta}$

i.e $\frac{d}{dt}(2ml^2\dot\theta) - 2ml^2\dot\phi^2 \sin\theta\cos\theta = -mgl\sin\theta$

$\Rightarrow 2l\ddot\theta - 2l\dot\phi^2 \sin\theta\cos\theta = -g\sin\theta \quad \text{—— (1)}$

And Lagrange's ϕ-equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\phi}\right) - \frac{\partial T}{\partial \phi} = \frac{\partial W}{\partial \phi}$

$\Rightarrow \frac{d}{dt}(2ml^2\dot\phi \sin^2\theta) = 0 \Rightarrow \frac{d}{dt}(\dot\phi\sin^2\theta) = 0 \quad \text{—— (2)}$

Integrating (2) we get $\dot\phi \sin^2\theta = C$ (Const)

($\because$ Rod was horizontal)

But initially when $\theta = \pi/2$

$\dot\phi = \sqrt{(2ng/l)} \quad \text{—— (3)}$

$\therefore C = \sqrt{(2ng/l)}$

$\therefore \dot\phi \sin^2\theta = \sqrt{(2ng/l)}$

Substituting the value of $\dot{\phi}$ from (3) in (1) we get

$$2l\ddot{\theta} - 2l\cdot\frac{2ng}{l\sin^4\theta}\sin\theta\cos\theta = -g\sin\theta$$

$$\Rightarrow 2l\ddot{\theta} - 4ng\cot\theta\,\mathrm{cosec}^2\theta = -g\sin\theta \quad \text{—— (4)}$$

Multiplying both sides by $\dot{\theta}$ & integrating, we get

$$l\dot{\theta}^2 + 2ng\cot^2\theta = g\cos\theta + D$$

But initially when $\theta = \pi/2$, $\dot{\theta} = 0$. $\therefore D = 0$

$$\therefore l\dot{\theta}^2 + 2ng\cot^2\theta = g\cos\theta \quad \text{—— (5)}$$

The rod will fall till $\dot{\theta} = 0$ $\Rightarrow 2n\cos^2\theta - \cos\theta\sin^2\theta = 0$

$\Rightarrow 2ng\cot^2\theta = g\cos\theta$

$\Rightarrow \cos\theta(2n\cos\theta - \sin^2\theta) = 0.$

$\therefore$ either $\cos\theta = 0$ i.e. $\theta = \pi/2$

$\Rightarrow 2n\cos\theta - \sin^2\theta = 0 \Rightarrow 2n\cos\theta - (1-\cos^2\theta) = 0$

$\Rightarrow \cos^2\theta + 2n\cos\theta - 1 = 0$

$$\therefore \cos\theta = \frac{-2n \pm \sqrt{(4n^2+4)}}{2} \Rightarrow \cos\theta = -n + \sqrt{(n^2+1)}, \text{ Leaving } -\text{ve sign.}$$

$\because$ $-$ve value of $\cos\theta$ is inadmissible as θ cannot be obtuse.

$$\therefore \theta = \cos^{-1}\left[\sqrt{(n^2+1)} - n\right].$$

from (4), we have $$2l\ddot{\theta} = \frac{g(4n\cos\theta - \sin^4\theta)}{\sin^3\theta} \quad \text{—— (6)}$$

when $\cos\theta = -n + \sqrt{(n^2+1)}$, $\cos^2\theta = 2n^2+1-2n\sqrt{(n^2+1)}$

$$\therefore 4n\cos\theta - \sin^4\theta = 4n\cos\theta - (1-\cos^2\theta)^2$$

$$= 4n\left[-n+\sqrt{(n^2+1)}\right] - \left[-2n^2 + 2n\sqrt{(n^2+1)}\right]^2$$

$$= -8n^2 - 8n^4 + 4n\sqrt{(n^2+1)} + 8n^3\sqrt{(n^2+1)}$$

$$= 4n\sqrt{(n^2+1)}\left[-n+\sqrt{(n^2+1)}\right]^2, \text{ which is +ve}$$

$\because \theta$ is acute angle $\therefore \sin^3\theta$ is also +ve

$\therefore$ when $\theta = \cos^{-1}\left[\sqrt{(n^2+1)} - n\right]$, from (6), we see that $\ddot{\theta}$ is +ve. Hence from this position the rod will rise again.