The Sphere

Sphere:

Defn: A sphere is the locus of a point which moves so that its distance from a fixed point always remains constant.

The fixed point is called the centre and the constant distance is called the radius of the sphere.

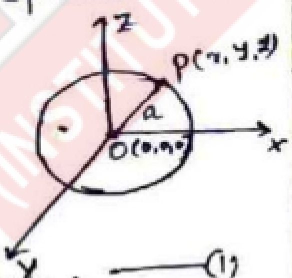
Equations of a sphere in different forms:

(a) **Standard form:**

To show that the eqn of the sphere whose centre is the origin and radius 'a' is $x^2+y^2+z^2=a^2$.

Proof: Let $P(x,y,z)$ be any point on the sphere.

Join OP.



Then

OP = radius of sphere $= a$ (given) ——(1)

By distance formula

$OP = \sqrt{x^2+y^2+z^2}$. ——(2)

From (1) & (2), we have

$\sqrt{x^2+y^2+z^2} = a$

$\Rightarrow x^2+y^2+z^2 = a^2$

which is the required eqn of the sphere.

(b) **Central form:**

To find the eqn of a sphere whose centre is (a,b,c) and radius is r.

proof: Let $C(a,b,c)$ be the centre of the sphere.

Let $P(x,y,z)$ be the point on the sphere.

Join CP. Then

CP = radius of sphere $= r$ (given)

$\Rightarrow CP^2 = r^2$

$\Rightarrow (x-a)^2+(y-b)^2+(z-c)^2 = r^2$

which is the required eqn.

(C) **General form:**

To prove that the equation $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ represents a sphere and find its centre and radius.

proof: The given eqn is

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad \text{—— (1)}$$

This can be written as

$$(x^2+2ux)+(y^2+2vy)+(z^2+2wz)+d=0$$

Adding $u^2+v^2+w^2$ to bothsides we get

$$(x^2+2ux+u^2)+(y^2+2vy+v^2)+(z^2+2wz+w^2)=-d+u^2+v^2+w^2$$

$$\Rightarrow (x+u)^2+(y+v)^2+(z+w)^2 = u^2+v^2+w^2-d.$$

$$\Rightarrow [x-(-u)]^2+[y-(-v)]^2+[z-(-w)]^2 = \left[\sqrt{u^2+v^2+w^2-d}\right]^2 \quad \text{—— (2)}$$

which is clearly of the central form of the sphere.

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2 \quad \text{—— (3)}$$

$\therefore$ (1) represents a sphere.

Now comparing (2) & (3).

we have

$a=-u$, $b=-v$, $c=-w$ and

$$r=\sqrt{u^2+v^2+w^2-d}$$

$\therefore$ Centre of the sphere (1) is $(-u,-v,-w)$ and the radius is $\sqrt{u^2+v^2+w^2-d}$.

Note: If $u^2+v^2+w^2-d<0$ then the radius of the sphere imaginary and the centre $(-u,-v,-w)$ is real. In this case the sphere is called pseudo-sphere (or) a virtual sphere.

Working rule for finding the centre and radius of the sphere:

(1) First of all make the coefficients of $x^2, y^2, z^2 = 1$ if they are not so.

(2) Centre is

$$\left[-\tfrac{1}{2}\text{coeff of } x,\ -\tfrac{1}{2}\text{coeff of } y,\ -\tfrac{1}{2}\text{coeff of } z\right]$$

and radius is

$$\sqrt{\left(\tfrac{1}{2}\text{coeff of } x\right)^2+\left(\tfrac{1}{2}\text{coeff of } y\right)^2+\left(\tfrac{1}{2}\text{coeff of } z\right)^2 - \text{Constant term.}}$$

(1) Conditions for a Sphere:

The given eqn represents a sphere if

(i) it is a second degree in x, y, z

(ii) coeff of x^2 = coeff of y^2 = coeff of z^2

and

(iii) it doesnot contain the terms involving the products xy, yz and zx.

(2) Since the general eqn of the sphere

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$

contains four unknown constants u, v, w, d.

So the sphere can be found to satisfy four conditions.

Four-point form:

To find the eqn of a sphere passing through the given points.

Sol^n: Let (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4) be the given four points.

Let the required eqn of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad \text{--- (1)}$$

Since it passes through (x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3) and (x_4, y_4, z_4).

$$\therefore (x_1^2+y_1^2+z_1^2)+2ux_1+2vy_1+2wz_1+d=0 \quad \text{--- (2)}$$

$$(x_2^2+y_2^2+z_2^2)+2ux_2+2vy_2+2wz_2+d=0 \quad \text{--- (3)}$$

$$(x_3^2+y_3^2+z_3^2)+2ux_3+2vy_3+2wz_3+d=0 \quad \text{--- (4)}$$

$$(x_4^2+y_4^2+z_4^2)+2ux_4+2vy_4+2wz_4+d=0 \quad \text{--- (5)}$$

Eliminating u, v, w, d from (1), (2), (3), (4) & (5) with the help of determinants, we have

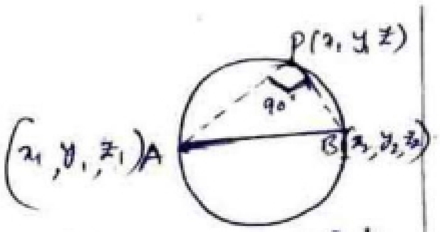
$$\begin{vmatrix} x^2+y^2+z^2 & x & y & z & 1 \\ x_1^2+y_1^2+z_1^2 & x_1 & y_1 & z_1 & 1 \\ x_2^2+y_2^2+z_2^2 & x_2 & y_2 & z_2 & 1 \\ x_3^2+y_3^2+z_3^2 & x_3 & y_3 & z_3 & 1 \\ x_4^2+y_4^2+z_4^2 & x_4 & y_4 & z_4 & 1 \end{vmatrix} = 0$$

which is the required equation.

Diameter form:-

To find the equation of the sphere on the join of (x_1, y_1, z_1) and (x_2, y_2, z_2) as diameter.

Sol^n: Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ be two given points.



Let $P(x, y, z)$ be any point on the sphere. Join AP and BP.

Since AB is diameter of the sphere then

$\angle APB$ = angle in the semi-circle $= 90°$

i.e, $AP \perp PB$ —— (1)

Now the d.r's of AP are $x-x_1,\ y-y_1,\ z-z_1$ and the d.r's of BP are $x-x_2,\ y-y_2,\ z-z_2$.

Since $AP \perp BP$.

$$\therefore (x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$$

which is the required eqn of the sphere

Problems:

→ Find the radius and centre of the sphere

$$x^2+y^2+z^2-2x+4y-6z=2.$$

Soln: This is comparing with the general eqn of the sphere

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0$$

we have

$2u=-2 \Rightarrow u=-1$ || $2v=4 \Rightarrow v=2$ || $2w=-6 \Rightarrow w=-3$

and $d=-2$

$\therefore$ centre is $(-u,-v,-w)$
$= (1,-2,3)$

radius $= \sqrt{u^2+v^2+w^2-d}$
$= \sqrt{1+4+9+2}$
$= \sqrt{16} = 4$

H.W → Find the centres and radius of the following spheres.

(i) $x^2+y^2+z^2-6x+8y-10z+1=0$

(ii) $x^2+y^2+z^2+2x-4y-6z+5=0$

(iii) $2x^2+2y^2+2z^2-2x+4y+2z+3=0$

→ Obtain the eqn of the sphere described on the join of the points $A(2,-3,4)$ $B(-5,6,-7)$ as diameter.

Soln: Now let the required eqn of sphere on the join of two points (x_1,y_1,z_1) & (x_2,y_2,z_2) as diameter be

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$$

$$(x-2)(x+5)+(y+3)(y-6)+(z-4)(z+7)=0$$

$$\Rightarrow x^2+y^2+z^2+3x-3y+3z-56=0$$

→ Find the eqn of the sphere through the points $(0,0,0)$, $(a,0,0)$, $(0,b,0)$, $(0,0,c)$.

Sol^n: Let the eqn of the sphere be

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$ —— ①

Since it passes through $(0,0,0)$

$\therefore d=0$

① ≡ $x^2+y^2+z^2+2ux+2vy+2wz=0$ —— ②

Since it passes through $(a,0,0)$, $(0,b,0)$ & $(0,0,c)$.

$\therefore$ we have

$a^2+2ua=0 \Rightarrow u=\frac{-a}{2}$

$b^2+2vb=0 \Rightarrow v=-\frac{b}{2}$

$c^2+2wc=0 \Rightarrow w=\frac{-c}{2}$

putting these values in ②

we get

$x^2+y^2+z^2-ax-by-cz=0$

which is the required equation.

→ Note: Eqn of the sphere OABC where $A(a,0,0)$, $B(0,b,0)$, $C(0,0,c)$ are three points on the axis is $x^2+y^2+z^2-ax-by-cz=0$.

→ Find the eqn of the sphere circumscribing the tetrahedron $x=0$, $y=0$, $z=0$, $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

Sol^n: Three planes out of the given four planes taken at a time determine one vertex of the tetrahedron.

Hence the vertices of the tetrahedron are

$(0,0,0)$, $(a,0,0)$, $(0,b,0)$, $(0,0,c)$.

Remaining solution similar to previous problem.

→ Find the eqn of the sphere passing through the three points $(3,0,2)$, $(-1,1,1)$, $(2,-5,4)$ and having its centre on the plane $2x+3y+4z=6$ is

$x^2+y^2+z^2+4y-6z=1$.

Sol^n: Let the eqn of the sphere be $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ —— ①

Since it passes through $(3,0,2)$, $(-1,1,1)$ and $(2,-5,4)$

$\therefore 9+4+6u+4w+d=0$

$\Rightarrow 6u+4w+13+d=0$ —— ②

$1+1+1-2u+2v+2w+d=0$

$\Rightarrow -2u+2v+2w+3+d=0$ —— (3)

$4+25+16+4u-10v+8w+d=0$

$\Rightarrow 4u-10v+8w+45+d=0$ —— (4)

Also centre $(-u,-v,-w)$ lies on the plane $2x+3y+4z-6=0$

$\therefore 2u+3v+4w+6=0$ —— (5)

Solving the above eqns (2), (3), (4) & (5) we get u, v, w, d values. putting these values in eqn (1).

which is the required sphere.

→ Find the eqn to a sphere passing through the points $(1,-3,4)$, $(1,-5,2)$, $(1,-3,0)$ and having centre on the plane $x+y+z=0$.

→ Obtain the sphere having its centre on the line $5y+2z=0=2x-3y$ and passing through the two points $(0,-2,-4)$, $(2,-1,-1)$.

Sol: Let the eqn of the sphere be

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$ —— (1)

Since its centre lies on the line $5y+2z=0=2x-3y$

$\therefore 5(-v)+2(-w)=0=2(-u)-3(-v)$

i.e, $5v+2w=0$ —— (2)

& $2u-3v=0$ —— (3)

Since the sphere passes through the points $(0,-2,-4)$ and $(2,-1,-1)$:

$\therefore 0+4+16+0-4v-8w+d=0$

$\Rightarrow -4v-8w+d+16=0$ —— (4)

& $4+1+1+4u-2v-2w+d=0$

$\Rightarrow 4u-2v-2w+d+6=0$ —— (5)

Solving the eqns (2), (3), (4) & (5) we get

$u=-3,\ v=-2,\ w=5$

$d=12$.

$\therefore$ (1) $\equiv$

$x^2+y^2+z^2-6x-4y+10z+12=0$

which is the required equation.

→ P.T the eqn $ax^2+ay^2+az^2+2ux+2vy+2wz+d=0$ represents a sphere. Find its radius and centre.

[Ans: $\frac{\sqrt{\Sigma u^2-ad}}{a}$, $\left(-\frac{u}{a}, -\frac{v}{a}, -\frac{w}{a}\right)$]

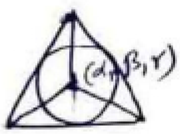
→ Find the eqn to the sphere through the points $(0,0,0)$, $(0,1,-1)$, $(-1,2,0)$, $(1,2,3)$.

→ Find the eqn of the sphere through the four points $(4,-1,2)$, $(0,-2,3)$, $(1,-5,-1)$ $(2,0,1)$.

→ Find the eqn of the sphere through the four points $(0,0,0)$, $(-a,b,c)$, $(a,-b,c)$, $(a,b,-c)$ and determine its radius.

→ Find the eqn of the sphere inscribed in the tetrahedron whose faces are $x=0$, $y=0$, $z=0$ and $x+2y+2z=1$.

Sol^n: The given faces are $x=0$, $y=0$, $z=0$ and $1-x-2y-2z=0$.



Let (α,β,γ) be the centre and 'r' the radius of the inscribed sphere. Then $\perp^r$ distances of the centre from all the four faces are equa and each equal to radius.

$$\therefore \frac{\alpha}{1}=\frac{\beta}{1}=\frac{\gamma}{1}=\frac{1-\alpha-2\beta-2\gamma}{\sqrt{1+4+4}}=r$$

$\therefore \alpha=\beta=\gamma=r$ and

$1-\alpha-2\beta-2\gamma=r$

Eliminating α,β,γ, we get

$1-r-2r-2r=r$

$\Rightarrow 8r=1$

$\Rightarrow r=\frac{1}{8}$.

$\therefore \alpha=\beta=\gamma=\frac{1}{8}$

$\therefore$ The centre is $(\alpha,\beta,\gamma)=\left(\frac{1}{8},\frac{1}{8},\frac{1}{8}\right)$ and the radius $(r)=\frac{1}{8}$

$\therefore$ The eqn of the sphere with centre $\left(\frac{1}{8},\frac{1}{8},\frac{1}{8}\right)$ and radius $\frac{1}{8}$ is

$$\left(x-\frac{1}{8}\right)^2+\left(y-\frac{1}{8}\right)^2+\left(z-\frac{1}{8}\right)^2=\left(\frac{1}{8}\right)^2$$

$$\Rightarrow x^2-\frac{1}{4}x+\frac{1}{64}+y^2-\frac{1}{4}y+\frac{1}{64}+z^2-\frac{1}{4}z+\frac{1}{64}=\frac{1}{64}.$$

$$\Rightarrow x^2+y^2+z^2-\frac{1}{4}(x+y+z)+\frac{1}{32}=0$$

$$\Rightarrow 32(x^2+y^2+z^2)-8(x+y+z)+1=0$$

which is the Required eqn.

→ A sphere is inscribed in the tetrahedron whose faces are $x=0, y=0, z=0$ $2x+6y+3z=14$.

Find its centre, radius and write down its equation.

2002 → Find the co-ordinates of the centre of the sphere inscribed in the tetrahedron formed by the planes whose equations are $x=0,\ y=0,\ z=0$ $x+y+z=a$.

→ Find the eqn of the sphere inscribed in the tetrahedron whose faces are

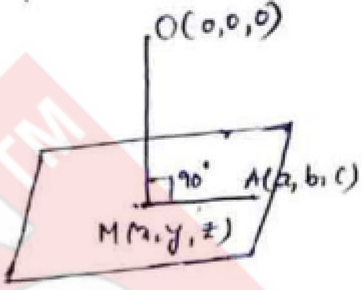
(i) $x=0,\ y=0,\ z=0,\ 2x-6y+3z+6=0$

2007 (ii) $x=0,\ y=0,\ z=0,\ 2x+3y+6z=6$

→ A plane passes through a fixed point (a,b,c), show that the locus of the foot of the perpendicular to it from the origin is ... sphere $x^2+y^2+z^2-ax-by-cz=0$.

Solⁿ: Let $A(a,b,c)$ be the fixed point on the variable plane α.

and let $M(x,y,z)$ be the foot of $\perp$ from the origin to the plane α.



$\therefore$ OM $\perp$ MA.

Now the d.r.'s of OM are x, y, z and the d.r.'s of MA are $x-a, y-b, z-c$.

Since OM $\perp$ MA

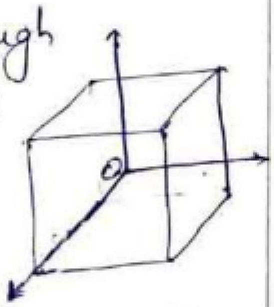
$\therefore x(x-a)+y(y-b)+z(z-c)=0$

$\Rightarrow x^2+y^2+z^2-ax-by-cz=0$

which is the required locus and it represents a sphere.

→ A point moves so that the sum of the squares of its distances from the six faces of a cube is constant. Show that its locus is a sphere.

Solⁿ: Take the centre of the cube as the origin and the planes through the centre parallel to its faces as co-ordinate planes.



Let each of the eadge of the cube be equal to '2a'.

Then the eqns of the three pairs of parallel faces of the cube are

$x=a,\ x=-a,\ y=a,\ y=-a$

and $z=a,\ z=-a$.

Now let $P(\alpha,\beta,\gamma)$ be any point in the locus.

Now the sum of squares of distances of P from the six faces is constant $=6k^2$ (say)

$$\left(\frac{\alpha-a}{1}\right)^2+\left(\frac{\alpha+a}{1}\right)^2+\left(\frac{\beta-a}{1}\right)^2+\left(\frac{\beta+a}{1}\right)^2+\left(\frac{\gamma-a}{1}\right)^2+\left(\frac{\gamma+a}{1}\right)^2=6k^2$$

$\Rightarrow 2(\alpha^2+\beta^2+\gamma^2)+6a^2=6k^2$

$\Rightarrow \alpha^2+\beta^2+\gamma^2=3(k^2-a^2)$

$\therefore$ Locus of $P(\alpha,\beta,\gamma)$ is

$x^2+y^2+z^2=3(k^2-a^2)$.

which clearly represents a sphere.

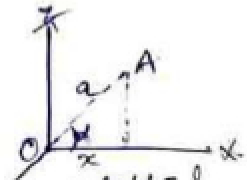
$\longrightarrow$ OA, OB, OC are three mutually perpendicular lines through the origin whose direction cosines are $l_1,m_1,n_1;\ l_2,m_2,n_2;\ l_3,m_3,n_3$. If $OA=a,\ OB=b,\ OC=c$. Show that the eqn of the sphere OABC is

$$x^2+y^2+z^2-x(al_1+bl_2+cl_3)-y(am_1+bm_2+cm_3)-z(an_1+bn_2+cn_3)=0$$

Soln: Since l_1,m_1,n_1 are the actual d.c's of OA and $OA=a$.

$\therefore$ The co-ordinates of A are $(l_1a,\ m_1a,\ n_1a)$. | using (lr, mr, nr)

Similarly, the co-ordinates of B & C are $(l_2a,\ m_2a,\ n_2a)$ and $(l_3a,\ m_3a,\ n_3a)$ respectively.



$\therefore \cos\alpha=l$

by from fig $\cos\alpha=\frac{x}{a}$

$\Rightarrow l=\frac{x}{a}$

$\Rightarrow x=la$

Also O is (0,0,0).

Now let the required eqn of the sphere be

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$ —— (1)

Since it passes through $O(0,0,0)$

$\therefore$ ① $\equiv$ $x^2+y^2+z^2+2ux+2vy+2wz=0$ — ②

$(\because d=0)$

Since it passes through

$A(l_1a, m_1a, n_1a)$

$\therefore l_1^2a^2+m_1^2a^2+n_1^2a^2+2ul_1a+2vm_1a+2wn_1a=0$

$\Rightarrow a+2ul_1+2vm_1+2wn_1=0$ — ③

$(\because l_1^2+m_1^2+n_1^2=0)$

Similarly for B and C

$b+2ul_2+2vm_2+2wn_2=0$ — ④

and $c+2ul_3+2vm_3+2wn_3=0$ — ⑤

Since the lines OA, OB, OC are mutually perpendicular.

$\therefore l_1, l_2, l_3;\ m_1, m_2, m_3;\ n_1, n_2, n_3$ are the d.c's of OX, OY, OZ referred to OA, OB, OC as axes.

So $l_1^2+l_2^2+l_3^2=1$, $m_1^2+m_2^2+m_3^2=1$

$n_1^2+n_2^2+n_3^2=1$ and

$l_1m_1+l_2m_2+l_3m_3=m_1n_1+m_2n_2+m_3n_3$

$=n_1l_1+n_2l_2+n_3l_3=0$

Now multiplying ③ by l_1; ④ by l_2; ⑤ by l_3 and adding, we get

$al_1+2ul_1^2+2vl_1m_1+2wn_1l_1$

$+bl_2+2ul_2^2+2vl_2m_2+2wn_2l_2$

$+cl_3+2ul_3^2+2vl_3m_3+2wn_3l_3=0$

$\Rightarrow al_1+bl_2+cl_3+2u(1)+2v(0)+2w(0)=0$

$\Rightarrow u=-\frac{1}{2}(al_1+bl_2+cl_3)$

Similarly $v=-\frac{1}{2}(am_1+bm_2+cm_3)$

$w=-\frac{1}{2}(an_1+bn_2+cn_3)$

Substituting in eqn ②, we get

$x^2+y^2+z^2-x(al_1+bl_2+cl_3)$

$-y(am_1+bm_2+cm_3)$

$-z(an_1+bn_2+cn_3)=0$

which is the required equation.

1996 → Find the eqn of the sphere which passes through the points $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ and has its radius as small as possible.

Sol^n: Let the eqn of the sphere be

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$ — ①

Since it passes through $(1,0,0)$

$\therefore 1+2u+d=0.$

Since (1) passes through $(0,1,0)$

$\therefore 1+2v+d=0$

Since (1) passes through $(0,0,1)$

$\therefore 1+2w+d=0$

from these, we have

$u=v=w=-\frac{(1+d)}{2}$

$\therefore (1) \equiv x^2+y^2+z^2-(1+d)x-(1+d)y-(1+d)z+d=0$ — (2)

Centre is $(-u,-v,-w)$

$=\left(\frac{1+d}{2},\frac{1+d}{2},\frac{1+d}{2}\right)$

If 'R' is the radius of the sphere, then

$R^2=u^2+v^2+w^2-d$

$=3\left(\frac{1+d}{2}\right)^2-d$

$=\frac{3}{4}(1+d^2+2d)-d$

$=\frac{1}{4}(3+3d^2+6d-4d)$

$=\frac{1}{4}(3+3d^2+2d)$

$=\frac{3}{4}\left(d^2+\frac{2}{3}d+1\right)$

$=\frac{3}{4}\left[\left(d+\frac{1}{3}\right)^2+\left(1-\frac{1}{9}\right)\right]$

$=\frac{3}{4}\left[\left(d+\frac{1}{3}\right)^2+\frac{8}{9}\right]$

If $\left(d+\frac{1}{3}\right)=0$ then R^2 is least (i.e, R is least)

$\Rightarrow d=-\frac{1}{3}$

$\therefore (2) \equiv x^2+y^2+z^2-\left(1-\frac{1}{3}\right)x-\left(1-\frac{1}{3}\right)y-\left(1-\frac{1}{3}\right)z-\frac{1}{3}=0$

$\Rightarrow x^2+y^2+z^2-\frac{2}{3}x-\frac{2}{3}y-\frac{2}{3}z-\frac{1}{3}=0$

$\Rightarrow 3(x^2+y^2+z^2)-2(x+y+z)-1=0$

which is the required eqn of the sphere.

1985 → A variable plane through a fixed point (a,b,c) cuts the co-ordinate axes in the points A, B, C. Show that the locus of the centres of the sphere OABC is $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2$.

Soln: Let the eqn of the plane be $\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma}=1$ — (1)

Since it passes through (a,b,c)

$\therefore \frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}=1$ — (2)

Since (1) cuts the axes in A, B, C.

$\therefore$ The co-ordinates of $A(\alpha,0,0)$, $B(0,\beta,0)$, $C(0,0,\gamma)$ and $(0,0,0)$

Let the eqn of the sphere OABC be

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$ — (3)

Since it passes through $O(0,0,0)$.

$\therefore x^2+y^2+z^2+2ux+2vy+2wz=0$ —— (4)

and since it passes through $A(\alpha,0,0)$.

$\therefore \alpha^2+2u\alpha=0$

$\Rightarrow 2u=-\alpha$

Similarly $2v=-\beta$, $2w=-\gamma$

putting these values in (4), we get,

$x^2+y^2+z^2-\alpha x-\beta y-\gamma z=0$ —— (5)

If (x_1,y_1,z_1) is the centre

then $x_1=+\frac{\alpha}{2}$, $y_1=\frac{\beta}{2}$, $z_1=\frac{\gamma}{2}$

$\Rightarrow \alpha=2x_1$, $\beta=2y_1$, $\gamma=2z_1$

$\therefore$ (2) $\equiv \frac{a}{2x_1}+\frac{b}{2y_1}+\frac{c}{2z_1}=1$

$\Rightarrow \frac{a}{x_1}+\frac{b}{y_1}+\frac{c}{z_1}=2$

$\therefore$ Locus of (x_1,y_1,z_1) is

$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=2$

→ A plane through a fixed point $(1,1,1)$ cuts the axes in A, B, C. Find the locus of the centre of the sphere OABC where O is origin.

→ A sphere of constant radius 2K passes through the origin and meets the axes in A, B, C. Find the locus of the centroid of the tetrahedron OABC.

Sol^n: Let co-ordinates of the points A, B, C be $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$ respectively.

The eqn of the sphere OABC is

$x^2+y^2+z^2-ax-by-cz=0$

Radius of this sphere

$=\sqrt{\left(\frac{a}{2}\right)^2+\left(\frac{b}{2}\right)^2+\left(\frac{c}{2}\right)^2}=2K$ (given)

squaring on both sides, we get

$a^2+b^2+c^2=16K^2$ —— (1)

Let (x_1,y_1,z_1) be the co-ordinates of the tetrahedron OABC, then

$x_1=\frac{0+a+0+0}{4} \Rightarrow x_1=\frac{a}{4}$

$\Rightarrow a=4x_1$

Similarly $b=4y_1$ & $c=4z_1$

putting these values of a, b, c in ①, we get

$16x_1^2 + 16y_1^2 + 16z_1^2 = 16k^2$

$\Rightarrow x_1^2 + y_1^2 + z_1^2 = k^2$.

$\therefore$ Locus of the centroid (x_1, y_1, z_1) is

$x^2 + y^2 + z^2 = k^2$

1988 → A sphere of constant radius k passes through the origin and meets the axes in A, B, C. prove that the centroid of the triangle ABC lies on the sphere

$9(x^2 + y^2 + z^2) = 4k^2$.

→ A variable sphere passes through the origin 'O' and meets the axes in A, B, C so that the volume of the tetrahedron OABC is constant. find the locus of the centre of the sphere.

2007 → A sphere of constant radius 'r' passes through the origin 'O' and cuts the axes in A, B, C. Find the locus of the foot of the perpendicular from 'O' to the plane ABC.

Sol^n: Let the co-ordinates of the points A, B, C be $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$ respectively.

Then the eq^n of the sphere OABC is

$x^2 + y^2 + z^2 - ax - by - cz = 0$

Its radius $= \sqrt{\left(\frac{a}{2}\right)^2 + \left(\frac{b}{2}\right)^2 + \left(\frac{c}{2}\right)^2}$

$= r$ (given)

$\Rightarrow a^2 + b^2 + c^2 = 4r^2$. —— ①

Now the eqn of the plane ABC is

$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ —— ②

D.r's of the $\perp^r$ to this plane ② are $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$.

$\therefore$ Eqns of the line through $O(0,0,0)$ and $\perp^r$ to the plane ② are

$\frac{x-0}{1/a} = \frac{y-0}{1/b} = \frac{z-0}{1/c}$

$\Rightarrow ax = by = cz$ ——— (3)

To find the locus of foot of $\perp^r$ from 'O' on the plane (2), i.e, the locus of the point of intersection of the plane (2) and line (3), we have to eliminate the unknown constants a, b, c from (1), (2) & (3).

Now from (3),

Let $ax = by = cz = \lambda$ (say)

$\Rightarrow a = \frac{\lambda}{x},\ b = \frac{\lambda}{y},\ c = \frac{\lambda}{z}$

putting these values in (1), we get

$$\lambda^2\left(\frac{1}{x^2} + \frac{1}{y^2} + \frac{1}{z^2}\right) = 4r^2$$

$$\Rightarrow \lambda^2(x^{-2} + y^{-2} + z^{-2}) = 4r^2. \quad \text{——— (4)}$$

and putting the values of a, b, c in (2), we get

$$\frac{1}{\lambda}(x^2+y^2+z^2) = 1.$$

$$\Rightarrow \frac{1}{\lambda^2}(x^2+y^2+z^2)^2 = 1 \quad \text{——— (5)}$$

Multiplying (4) & (5), we get

$$(x^2+y^2+z^2)^2(x^{-2}+y^{-2}+z^{-2}) = 4r^2.$$

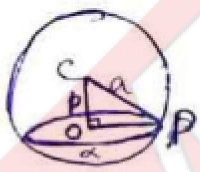
which is the required locus.

plane section of a sphere:

To prove that the section of a sphere by a plane is a circle.

proof: Let 'C' be the centre of the sphere, 'a' its radius and α be the plane.

Draw $CO \perp^r$ from 'C' on the plane α and let $CO = p$.



'O' is the fixed point and p is a fixed length.

Let 'P' be any point on the section of the sphere by the plane 'α'. Join CP & OP

$\therefore CO \perp OP$

$\therefore$ In the right angled ΔCOP we have

$OP^2 = CP^2 - CO^2$

$= a^2 - p^2$.

(or)

$OP = \sqrt{a^2-p^2}$ which is Constant.

and O is fixed point.

$\therefore$ P lies on a circle whose centre is 'O' and radius is $\sqrt{a^2-p^2}$

$\therefore$ The section of the sphere by a plane is a circle.

Equation of a circle:

Since the intersection of a sphere with a plane is a circle.

$\therefore$ in general a circle can be represented by the eqns of a sphere and a plane taken together.

i.e, the two eqns

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$

and $lx+my+nz=p$. taken together represent a circle.

Note:

$\boxed{I}$ (1) The centre of the circle is the foot of $\perp^r$ from the centre of the sphere on the plane and

(2) Radius of the circle $= \sqrt{a^2-p^2}$, where 'a' is the radius of the sphere and p the length of $\perp$ from

the centre of the sphere on the plane.

II. The section of a sphere by a plane passing through the centre of the sphere is called a great circle. Its centre and radius is the same as that of the sphere.

problems :-

→ find the centre and the radius of the circle.

$x^2+y^2+z^2-2y-4z=11$,
$x+2y+2z=15$

Sol: The given sphere is

$$x^2+y^2+z^2-2y-4z-11=0 \quad \text{—— (1)}$$

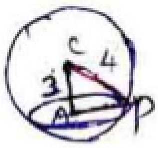
Its centre is

$(-u,-v,-w)=(0,1,2)$ and

radius $=\sqrt{1+4+11}=\sqrt{16}=4$

The given plane is

$$x+2y+2z=15 \quad \text{—— (2)}$$



Eqns (1) & (2) taken together represent a circle.

Now the centre of the circle is the foot of $\perp^r$ from the centre of the sphere (1) on the plane (2)

Now the d.r.'s of the normal to the plane (2) are 1, 2, 2.

∴ Eqns of the line CA through C and $\perp^r$ to plane (2) are

$$\frac{x-0}{1}=\frac{y-1}{2}=\frac{z-2}{2}=r \text{ (say)}.$$

Any point on the line is

$$(r,\ 2r+1,\ 2r+2) \quad \text{—— (3)}$$

Let it be A.

Since it lies on the plane (2).

$$\therefore (r)(1)+2(2r+1)+2(2r+2)=15$$

$$\Rightarrow 9r=9$$

$$\Rightarrow \boxed{r=1}$$

$\therefore$ (3) ≡ $A(1,\ 2(1)+1,\ 2(1)+2)$

$=(1,3,4)$

which is required centre of the circle.

Again $p=CA=$ distance from $C(0,1,2)$ to the plane $x+2y+2z=15$

$\therefore p = \left|\frac{0+2+4-15}{\sqrt{1+4+4}}\right|$

$= \frac{9}{3} = 3$

$\therefore$ The radius of the circle is

$AP = \sqrt{a^2 - p^2}$

$= \sqrt{(4)^2 - (3)^2}$

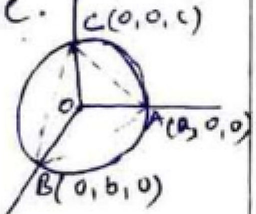
$= \sqrt{16-9}$

$= \sqrt{7}$

2001 → Find the eqns of the circle circumscribing the triangle formed by the three points $(a,0,0)$, $(0,b,0)$, $(0,0,c)$. Obtain also the co-ordinates of the centre of the circle.

Sol^n: Let the given points be $A(a,0,0)$, $B(0,b,0)$, $C(0,0,c)$

Then the circumcircle of $\triangle ABC$ is the intersection of the plane ABC and the sphere OABC.



Now the plane ABC is

$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ and the —— (1)

eqn of the sphere OABC is $x^2+y^2+z^2-ax-by-cz=0$ —— (2)

$\therefore$ The eqns of the circle of $\triangle ABC$ are

$x^2+y^2+z^2-ax-by-cz=0,$

$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

Remaining solution similar to previous problem.

→ Find the eqn of that plane which cuts the sphere $x^2+y^2+z^2=a^2$ in a circle whose centre is (α,β,γ).

Sol^n: Since 'O' is the centre of the sphere and $A(\alpha,\beta,\gamma)$ is the centre of the circle.

$\therefore OA \perp^r$ to the required plane of the circle

Now the d.r.'s of OA are $\alpha-0, \beta-0, \gamma-0$

$\Rightarrow \alpha, \beta, \gamma$.

$\therefore$ The coeff. of x, y, z in the eqn of the plane are α, β, γ. ($\because$ OA is $\perp^r$ to the plane of circle).

$\therefore$ Eqn of the plane of the circle is

$\alpha(x-\alpha)+\beta(x-\beta)+\gamma(x-\gamma)=0$

1989 Show that the centres of all sections of the sphere $x^2+y^2+z^2=a^2$ by planes through a point (α,β,γ) lie on the sphere $x(x-\alpha)+y(y-\beta)+z(z-\gamma)=0$.

Soln: Let (x_1,y_1,z_1) be the centre of one of the sections, then the eqn of the plane is

$x_1(x-x_1)+y_1(y-y_1)+z_1(z-z_1)=0$ ——(1)

Since it passes through the point (α,β,γ)

$\therefore x_1(\alpha-x_1)+y_1(\beta-y_1)+z_1(\gamma-z_1)=0$

$\Rightarrow x_1(x_1-\alpha)+y_1(y_1-\beta)+z_1(z_1-\gamma)=0$

$\therefore$ Locus of (x_1,y_1,z_1) is

$x(x-\alpha)+y(y-\beta)+z(z-\gamma)=0$

which is the required eqn of the sphere.

→ If 'r' be the radius of the circle

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$,

$lx+my+nz=0$.

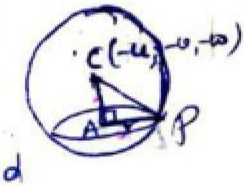
prove that

$(r^2+d)(l^2+m^2+n^2)=(mw-nv)^2+(nu-lw)^2+(lv-mu)^2$

Sol: The equations of the circle are

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$ ——(1)

and $lx+my+nz=0$ ——(2)



The centre of the sphere (1) is $C(-u,-v,-w)$ and its radius $CP=\sqrt{u^2+v^2+w^2-d}$

Also $CA=\perp^r$ distance of $C(-u,-v,-w)$ from the plane (2)

$$= \frac{|l(-u)+m(-v)+n(-w)|}{\sqrt{l^2+m^2+n^2}}$$

$$= \frac{lu+mv+nw}{\sqrt{l^2+m^2+n^2}}$$

In the right angled ΔCAP

$AP^2 = CP^2 - CA^2$

$$\Rightarrow r^2 = (u^2+v^2+w^2-d) - \frac{(lu+mv+nw)^2}{l^2+m^2+n^2}$$

$$\Rightarrow (r^2+d)(l^2+m^2+n^2) = (u^2+v^2+w^2)(l^2+m^2+n^2) - (lu+mv+nw)^2$$

By using the Lagrange identity

$$= (mw-nv)^2 + (nu-lw)^2 + (lv-mu)^2$$

Hence the result.

Lagrange's identity:

$$\Big[(l_1^2+m_1^2+n_1^2)(l_2^2+m_2^2+n_2^2) - (l_1l_2+m_1m_2+n_1n_2)^2 = (m_1n_2-n_1m_2)^2 + (n_1l_2-l_2n_1)^2 + (l_1m_2-m_1l_2)^2\Big]$$

Note: The four points are said to be concyclic if the circle through any three points passes through the fourth point.

→ Show that the following sets of points are concyclic.

(i) $(5,0,2)$, $(2,-6,0)$, $(7,-3,8)$, $(4,-9,6)$

(ii) $(-8,5,2)$, $(-5,2,2)$, $(-7,6,6)$, $(-4,3,6)$.

Sol: (i) Let the four given points be

$A(5,0,2)$, $B(2,-6,0)$, $(7,-3,8)$, $D(4,-9,6)$.

Let us find the eqns of the circle ABC:

To find the eqn of the plane ABC:

Any plane through A is

$l(x-5)+m(y)+n(z-2)=0$ —— (1)

Since it passes through B & C we get

$3l+6m+2n=0$ —— (2)

and $2l-3m+6n=0$ —— (3)

Solving (2) & (3), we get

$$\frac{l}{6} = \frac{m}{-2} = \frac{n}{-3}.$$

$\therefore$ (1) $\equiv$ $\boxed{6x-2y-3z-24=0}$ —— (4)

Now to find the eqn of the sphere OABC.

Let the eqn of the sphere through OABC be

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$ —— (5)

Since it passes through O(0,0,0)

$\therefore x^2+y^2+z^2+2ux+2vy+2wz=0$ —— (6)

and it passes through the points A, B, C.

$29+10u+4w=0$ —— (7)

$10+u-3v=0$ —— (8)

$61+7u-3v\cancel{+8w}=0$ —— (9)

Subtracting (8) from (9) we get

$51+6u+8w=0$ —— (10)

Multiplying (7) by 2 and subtract (10) from it we get

$7+14u=0$

$\Rightarrow u=-\frac{1}{2}$

(8) $\equiv 3v=u+10$

$=-\frac{1}{2}+10$

$\Rightarrow v=\frac{19}{6}$

(7) $\equiv w=-6.$

putting u, v, w in (6), we get

$x^2+y^2+z^2-x+\frac{19}{3}y-12z=0$ —— (11)

$\therefore$ The eqns of the circle through A, B, C are (i.e, intersection of sphere OABC and plane ABC)

$x^2+y^2+z^2-x+\frac{19}{3}y-12z=0$ —— (12)

and $6x-2y-3z-24=0$ —— (13)

The fourth point D(4, −9, 6) lies on circle ABC, if it lies both on the sphere (12) and plane (13).

Now D(4, −9, 6) lies on sphere (12) if

$16+81+36-4-57-72=0$

$\Rightarrow 0=0$ which is true.

Similarly D(4, −9, 6) lies on plane (13) if

$24+18-18-24=0$

$\Rightarrow 0=0$ which is true.

$\therefore$ D lies on the circle through A, B, C.

$\therefore$ The points A, B, C, D are concyclic.

Intersection of two spheres:

We now consider two spheres and assume that the given spheres have points in common, i.e, intersect.

Assuming that two given spheres intersect, we show that the locus of the points of intersection of two spheres is a circle.

The co-ordinates of points, if any, common to the two spheres

$$S_1 = x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0$$

$$S_2 = x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0.$$

satisfy both these eqns and, therefore also satisfy the eqn

$$S_1-S_2 = 2x(u_1-u_2)+2y(v_1-v_2)+2z(w_1-w_2)+d_1-d_2=0$$

which being a linear eqn in x, y, z represents a plane.

Now the points of intersection of the two spheres $S_1=0$, $S_2=0$ are the same as those of any one of these spheres and the plane $S_1-S_2=0$ and so it is a circle.

Note: The eqns of two spheres taken together also represents a circle.

→ Show that the sphere

$S_1 = x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0$ cuts $S_2 = x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0$ in a great cirle if

$$2(u_2^2+v_2^2+w_2^2)-d_2 = 2(u_1u_2+v_1v_2+w_1w_2)-d_1$$

$$\Rightarrow 2(u_1u_2+v_1v_2+w_1w_2) = 2r_2^2+d_1+d_2$$

where r_2 is the radius of the second sphere.

Sol: The plane of the circle, i.e, the plane in which their circle of intersection is $S_1 - S_2 = 0$

$\Rightarrow 2(u_1-u_2)x + 2(v_1-v_2)y + 2(w_1-w_2) + d_1 - d_2 = 0$ — (1)

The circle of intersection is the great circle of the sphere S_2 only when the above plane passes through the centre of the sphere S_2,

i.e, the plane passes through $(-u_2, -v_2, -w_2)$.

$\therefore 2(u_1-u_2)(-u_2) + 2(v_1-v_2)(-v_2) + 2(w_1-w_2)(-w_2) + d_1 - d_2 = 0$

$\Rightarrow 2(u_2^2+v_2^2+w_2^2) - d_2 = 2(u_1u_2+v_1v_2+w_1w_2) - d_1$

$\Rightarrow 2[(u_2^2+v_2^2+w_2^2) - d_2] + d_2 = 2(u_1u_2+v_1v_2+w_1w_2) - d_1$

$\Rightarrow 2r_2^2 + d_1 + d_2 = 2(u_1u_2+v_1v_2+w_1w_2)$

Spheres through a given circle:

Let the circle be given by the eqns

$S \equiv x^2+y^2+z^2+2ux+2vy+2wz+d = 0$ —— ①

$P \equiv lx+my+nz-p=0$ —— ②

then the eqn $S+\lambda P=0$

i.e, $x^2+y^2+z^2+2ux+2vy+2wz+d+\lambda(lx+my+nz)=0$ —— ③

The equation ③ clearly represents a sphere

[∵ (i) it is a second degree equation.

(ii) coefficients of x^2, y^2, z^2 are equal.

and (iii) it doesnot contain the product terms xy, yz, zx.]

Also the co-ordinates of the points which satisfy ① & ② both, also satisfy ③.

Hence (3) represents a sphere through the curve of intersection of ① & ②.

i.e, the given circle.

∴ The set of spheres through the circle $S=0$, $P=0$ is $\{S+\lambda P=0;\ \lambda$ is parameter$\}$.

Similarly if the circle be given by the intersection of two spheres.

$S \equiv x^2+y^2+z^2+2ux+2vy+2wz+d=0$

$S' \equiv x^2+y^2+z^2+2u'x+2v'y+2w'z+d'=0.$

then any sphere through this circle is $S+kS'=0$

∴ The set of spheres through the circle $S=0, S'=0$ is $\{S+kS'=0;\ k$ is the parameter$\}$.

→ The eqn of the plane of the circle through the two spheres $S=0, S'=0$ is

$S-S' = 2(u-u')x+2(v-v')y+2(w-w')z+d-d'=0$

From this we see that the eqn of any sphere through the circle $S=0, S'=0$ is of the form $S+k(S-S')=0$

where k is parameter.

This form is sometimes more convenient.

Note: The general eqn of the sphere through the circle
$x^2+y^2+2gx+2fy+c=0,\ z=0$ is
$x^2+y^2+z^2+2gx+2fy+2kz+c=0.$
where k is the parameter.

→ Find the equation of the sphere through the circle $x^2+y^2+z^2=9,\ 2x+3y+4z=5$ and the point $(1,2,3)$.

Sol: Given equations of the circle
$x^2+y^2+z^2=9$
$2x+3y+4z=5$ and the point $P(1,2,3)$.
Let the required eqn of sphere through a circle is
$$(x^2+y^2+z^2-9)+\lambda(2x+3y+4z-5)=0 \quad (3)$$
Since it passes through $P(1,2,3)$
$$(1+4+9-9)+\lambda(2+6+12-5)=0$$
$$5+\lambda(15)=0$$
$$\boxed{\lambda=-1/3}$$

$\therefore$ (3) ≡
$$x^2+y^2+z^2-9-\frac{1}{3}(2x+3y+4z-5)=0$$
$$\Rightarrow 3(x^2+y^2+z^2)-2x-3y-4z-22=0$$
which is the required eqn of the sphere.

(2000) → Find the eqn of the sphere through the circle $x^2+y^2+z^2=4,\ x+2y-z=2$ and the point $(1,-1,1)$.

→ Find the eqn to the sphere which passes through the point (α,β,γ) and the circle $x^2+y^2=a^2,\ z=0$

(Ans: $(x^2+y^2+z^2-a^2)\gamma+(a^2-\alpha^2-\beta^2-\gamma^2)z=0$)

Hint: $(x^2+y^2+z^2-a^2)+\lambda z=0$

→ Show that the eqn of the sphere having its centre on the plane $4x-5y-z=3$ and passing through the circle with equations
$x^2+y^2+z^2-2x-3y+4z+8=0,$
$x^2+y^2+z^2+4x+5y-6z+2=0$
is $x^2+y^2+z^2+7x+9y-11z-1=0$

Sol^n: The given circle is

$x^2+y^2+z^2-2x-3y+4z+8=0$ —— (1)

$x^2+y^2+z^2+4x+5y-6z+2=0$ —— (2)

(1)−(2) ≡ $3x+4y-5z-3=0$ —— (3)

Now circle represented by (1) & (2) is same as the circle given by (1)&(3).

Now any sphere through the circle given by (1)&(3) is

$x^2+y^2+z^2-2x-3y+4z+8+\lambda(3x+4y-5z-3)=0$ —— (4)

$\Rightarrow x^2+y^2+z^2+(3\lambda-2)x+(4\lambda-3)y+(-5\lambda+4)z+(8-3\lambda)=0$

its centre $(-u,-v,-w)$

$=\left(\frac{2-3\lambda}{2}, \frac{3-4\lambda}{2}, \frac{5\lambda-4}{2}\right)$

Since it lies in the plane $4x-5y-z=3$

we get $\boxed{\lambda=3}$

(4) ≡ $x^2+y^2+z^2+7x+9y-11z-1=0$.

→ Find the eqn of the sphere through the circle $x^2+y^2+z^2+2x+3y+6=0$, $x-2y+4z-9=0$.

and the centre of the sphere $x^2+y^2+z^2-2x+4y-6z+5=0$.

(Ans: $x^2+y^2+z^2+7y-8z+24=0$)

→ Show that the two circles

$x^2+y^2+z^2-y+2z=0$, $x-y+z-2=0$

$x^2+y^2+z^2+x-3y+z-5=0$,

$2x-y+4z-1=0$;

lie on the same sphere and find its equation.

Sol^n: The given circles are

$x^2+y^2+z^2-y+2z=0$, $x-y+z-2=0$ —— (1)

$x^2+y^2+z^2+x-3y+z-5=0$, $2x-y+4z-1=0$ —— (2)

Any sphere through (1) is

$x^2+y^2+z^2-y+2z+\lambda_1(x-y+z-2)=0$ —— (3)

and any sphere through (2) is

$x^2+y^2+z^2+x-3y+z-5+\lambda_2(2x-y+4z-1)=0$ —— (4)

The circles (1) & (2) will lie on same sphere

if the eqns (3) & (4) represent the same sphere for same values of λ_1, λ_2.

∴ Equating the coefficients of like terms in ③ & ④, we get

$\lambda_1 = 2\lambda_2 + 1$ —⑤, $-1-\lambda_1 = -\lambda_2 - 3$ —⑥

$2+\lambda_1 = 4\lambda_2 + 1$ —⑦, $-2\lambda_1 = -\lambda_2 - 5$ —⑧

Solving ⑤ & ⑥, we get

$\lambda_1 = 3, \quad \lambda_2 = 1.$

and these values clearly satisfy remaining two eqns ⑦ & ⑧

∴ Two circles ① & ② lie on the same sphere whose eqn is (putting $\lambda_1 = 3, \lambda_2 = 1$ in ③ & ④)

we get

$$x^2+y^2+z^2+3x-4y+5z-6=0$$

→ Show that the two circles

$$2(x^2+y^2+z^2)+8x-13y+17z-17=0,\quad 2x+y-3z+1=0;$$

$$x^2+y^2+z^2+3x-4y+3z=0,\quad x-y+2z-4=0;$$

lie on the same sphere and find its equation.

→ Prove that the circles

$$x^2+y^2+z^2-2x+3y+4z-5=0,\quad 5y+6z+1=0;$$

$$x^2+y^2+z^2-3x-4y+5z-6=0,\quad x+2y-7z=0$$

lie on the same sphere and find its eqn.

2010 model 12M → Prove that the plane $x+2y-z=4$ cuts the sphere $x^2+y^2+z^2-x+z-2=0$ in a circle of radius unity and find the eqn of sphere which has this circle for one of its great circle.

Solⁿ: The given sphere

$$x^2+y^2+z^2-x+z-2=0 \quad \text{—①}$$

and the plane

$$x+2y-z-4=0 \quad \text{—②}$$

Centre of the sphere ① is $C\left(\frac{1}{2},0,-\frac{1}{2}\right)$.

and its radius is $CP = \sqrt{\frac{1}{4}+0+\frac{1}{4}+2}$

$= \sqrt{5/2}$.

$CA = \perp^r$ distance from $C\left(\frac{1}{2},0,-\frac{1}{2}\right)$ to the plane ②

$$= \frac{\left|\frac{1}{2}+2(0)-\left(-\frac{1}{2}\right)-4\right|}{\sqrt{1+4+1}}$$

$$= \frac{3}{\sqrt{6}} = \sqrt{3/2}$$

Radius of circle

$AP = \sqrt{CP^2 - CA^2}$

$= \sqrt{\frac{5}{2} - \frac{3}{2}}$

$= \sqrt{1} = 1.$

$\therefore$ The plane (2) meets the sphere (1) in a circle of radius unity.

Now any sphere through the intersection of (1) & (2) is

$x^2 + y^2 + z^2 - x + z - 2 + K(x + 2y - z - 4) = 0$ —— (3)

If the circle of intersection of (1) & (2) is a great circle of sphere (3), then the centre $\left(\frac{1-K}{2}, -K, \frac{K-1}{2}\right)$ lies on the plane (2).

$\therefore \frac{1-K}{2} + 2(-K) - \left(\frac{K-1}{2}\right) - 4 = 0$

$\Rightarrow \boxed{K = -1}$

$\therefore$ (3) $\equiv x^2 + y^2 + z^2 - 2x - 2y + 2z + 2 = 0$

→ Obtain the eqn of the sphere having the circle $x^2 + y^2 + z^2 + 10y - 4z - 8 = 0,\ x + y + z = 3$ as the great circle.

→ Find the eqn to the sphere which passes through the circle $x^2 + y^2 = 4,\ z = 0$ and is cut by the plane $x + 2y + 2z = 0$ in a circle of radius 3.

Sol^n: The given circle is

$x^2 + y^2 - 4 = 0,\ z = 0.$

The eqns of this circle can be written as

$x^2 + y^2 + z^2 - 4 = 0,\ z = 0$

Any sphere through this circle is

$(x^2 + y^2 + z^2 - 4) + \lambda z = 0$ —— (1)

Its centre $= \left(0, 0, -\frac{\lambda}{2}\right)$

and radius $= \sqrt{\frac{\lambda^2}{4} + 4} = CP$

Now the sphere (1) cut by the plane

$x + 2y + 2z = 0$ —— (2)

in a circle of the radius 3.

Draw $CA \perp$ to the plane (2) from C.

$\therefore CA = \perp^r$ distance from $\left(0, 0, -\frac{\lambda}{2}\right)$ on the plane (2).

$= \frac{|0 + 0 - \lambda|}{\sqrt{1+4+4}} = \frac{\lambda}{3}$

Now from the right $\angle^d$ ΔCAP, $CA^2 + AP^2 = CP^2$

$\Rightarrow \frac{\lambda^2}{9} + 9 = \frac{\lambda^2}{4} + 4$

$\Rightarrow \lambda = \pm 6$

$\therefore$ (1) $\equiv x^2 + y^2 + z^2 \pm 6z - 4 = 0$

2008 → A sphere 'S' has points $(0,1,0)$, $(3,-5,2)$ at opposite ends of a diameter. Find the equation of the sphere S with the plane $5x-2y+4z+7=0$ as a great circle

→ A sphere S has points $(0,1,0)$, $(3,-5,2)$ at opposite ends of a diameter. Find the equation of the sphere having the intersection of the sphere S with the plane $5x-2y+4z+7=0$ as a great circle.

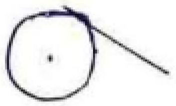
$x^2+y^2+2x+2y+2z+2=0$

2009 → Find the equation of the sphere having its centre on the plane $4x-5y-z=3$, and passing through the circle

$$x^2+y^2+z^2-12x-3y+4z+8=0$$
$$3x+4y-5z+3=0$$

Tangent plane (Line) property:

If a plane (line) touch as the sphere, then $\perp^r$ distance from the centre of the sphere on the plane (line) must be equal to its radius of the sphere.



Intersection of a sphere by a straightline:

To find the points where the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ meets the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$

Sol^n: The given line is

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=r \text{ (say)} \quad \text{——} (1)$$

and sphere

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad \text{——} (2)$$

Any point on the line (1) is

$$(lr+x_1, mr+y_1, nr+z_1) \quad \text{——} (3)$$

If it lies on the sphere (2)

$$\therefore (lr+x_1)^2+(mr+y_1)^2+(nr+z_1)^2+2u(lr+x_1)+2v(mr+y_1)+2w(nr+z_1)+d=0 \quad (4)$$

which is a quadratic in 'r'. hence it gives two values of 'r'

$\therefore$ These values putting in (3) we get two points of intersection.

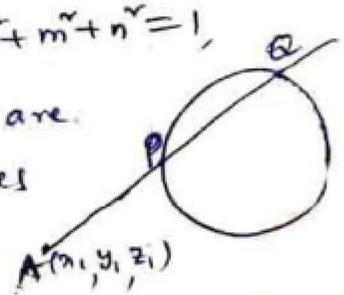
Note:

1. The eqn of the tangent plane at the point (x_1, y_1, z_1) to the sphere $x^2+y^2+z^2=a^2$ is $xx_1+yy_1+zz_1=a^2$.

2. The eqn of the tangent plane at the point (x_1, y_1, z_1) to the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ is $xx_1+yy_1+zz_1+u(x+x_1)+v(y+y_1)+w(z+z_1)+d=0$

Power of a point w.r.t a sphere:-

Let l, m, n be the actual d.c's of the line (1), so that $l^2+m^2+n^2=1$, and r_1, r_2 are the distances of the point $A(x_1, y_1, z_1)$ from the



points of intersections P and Q.

Now the eqn (4) reduces to

$$r^2 + 2r[l(u+x_1) + m(v+y_1) + n(w+z_1)] + x_1^2+y_1^2+z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d = 0$$

$[\because l^2+m^2+n^2=1]$

and $r_1 = AP$, $r_2 = AQ$ are its two roots.

$\therefore AP \cdot AQ = r_1 r_2$

$= x_1^2+y_1^2+z_1^2 + 2ux_1 + 2vy_1 + 2wz_1 + d.$

[$\because ax^2+bx+c=0$ product of two roots $= \frac{c}{a}$]

which is independent of the d.c.'s l, m, n and is thus constant.

i.e., if from a fixed point 'A', lines are drawn in any direction to intersect a given Sphere in P and Q, then AP. AQ is constant. This constant AP. AQ is called the power of the point A w.r.t the Sphere.

Note: The power of a point is obtained by substituting the co-ordinates of the point in the eqn of the sphere after making the R.H.S. zero.

→ Find the co-ordinates of the points where the line (i) $\frac{x+3}{4} = \frac{y+4}{3} = \frac{z-8}{5}$ intersects the sphere $x^2+y^2+z^2+2x-10y=23$.

(ii) $\frac{x+2}{4} = \frac{y+9}{3} = \frac{z-8}{-5}$ meets the sphere $x^2+y^2+z^2=49$.

Sol: (i) The given line is

$\frac{x+3}{4} = \frac{y+4}{3} = \frac{z-8}{-5} = r$ (say) —— (1)

and the sphere is

$x^2+y^2+z^2+2x-10y-23=0$ —— (2)

Any point on the line (1) is

$(4r-3,\ 3r-4,\ -5r+8)$ —— (3)

It lies on the sphere (2).

$\therefore$ (2) $\equiv$ $(4r-3)^2 + (3r-4)^2 + (-5r+8)^2 + 2(4r-3) - 10(3r-4) - 23 = 0$

$\Rightarrow 50r^2 - 150r + 100 = 0$

$\Rightarrow r^2 - 3r + 2 = 0$

$\Rightarrow (r-1)(r-2) = 0 \Rightarrow r = 1, 2$

putting $r=1, r=2$ in (3)

$\therefore$ (3) $\equiv$ The required points of intersection are $(1,-1,3)$ & $(5,2,-2)$.

→ Find the locus of the middle points of the system of parallel chords of the sphere

(i) $x^2+y^2+z^2+2ux+2vy+2wz+d=0$

(ii) $x^2+y^2+z^2=a^2$.

(or)

Show that the locus of the mid-points of a system of parallel chords of a sphere is a plane through its centre perpendicular to the given chords.

Sol^n: (i)

Let all chords of the system be parallel to the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ —— (1)

where l, m, n are actual d.c.'s.

Let (x_1, y_1, z_1) be the mid point of one of the chords.

Then its equations are

$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$

Any point on this line is $(lr+x_1, mr+y_1, nr+z_1)$

This lies on the sphere

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$

$\Rightarrow (lr+x_1)^2+(mr+y_1)^2+(nr+z_1)^2+2u(lr+x_1)+2v(mr+y_1)+2w(nr+z_1)+d=0$

$\Rightarrow r^2(l^2+m^2+n^2)+2r[l(u+x_1)+m(v+y_1)+n(w+z_1)]+x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d=0$ —— (2)

which is a quadratic in r.

Since (x_1, y_1, z_1) is the mid-point of the chord

∴ the roots of (2) must be equal and opposite.

i.e, the sum of roots is zero $\left(-\frac{b}{a}=0\right)$

i.e, the co-efficient of $r=0$

∴ $l(u+x_1)+m(v+y_1)+n(w+z_1)=0$

∴ Locus of the mid point (x_1, y_1, z_1) is

$l(u+x)+m(v+y)+n(w+z)=0$

$\Rightarrow l(x+u)+m(y+v)+n(z+w)=0$

which is clearly a plane through the centre $(-u, -v, -w)$ and $\perp^r$ to the line (1)

(ii) Ans: $lx+my+nz=0$

→ Show that the sum of the squares of the intercepts made by a given sphere on any three mutually $\perp^r$ straight lines through a fixed point is constant.

Sol^n: Let the given sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad \text{--- (1)}$$

and the three mutually $\perp^r$ lines through the fixed point $(0,0,0)$ (say), be

$$\frac{x}{l_1}=\frac{y}{m_1}=\frac{z}{n_1} \quad \text{--- (2)}; \qquad \frac{x}{l_2}=\frac{y}{m_2}=\frac{z}{n_2} \quad \text{--- (3)}$$

and $\frac{x}{l_3}=\frac{y}{m_3}=\frac{z}{n_3}$ --- (4)

where l_1, m_1, n_1 etc. are the actual d.c's, so that

$$\left.\begin{array}{l} l_1^2+m_1^2+n_1^2=1, \quad l_1^2+l_2^2+l_3^2=1 \\ l_1m_1+l_2m_2+l_3m_3=0, \\ l_1l_2+m_1m_2+n_1n_2=0 \ \ldots \text{etc.} \end{array}\right\} \text{--- (5)}$$

To find the intercept on line (2):

Any point on line (2) is (l_1r, m_1r, n_1r).

If it lies on the sphere (1), then

$$r^2(l_1^2+m_1^2+n_1^2)+2r(l_1u+m_1v+n_1w)+d=0$$

$$\Rightarrow r^2+2r(l_1u+m_1v+n_1w)+d=0 \quad (\because \text{from (5)})$$

It is a quadratic in r; let the two roots be r_1, r_2, which are the distances from O of the two points of intersection say A_1, A_2 of the line and the sphere.

$\therefore$ If L_1 is the length of intercept on the first line, then

$$L_1 = A_1A_2 = OA_2 - OA_1 = r_2 - r_1.$$

$$\therefore L_1^2 = (r_2-r_1)^2$$

$$= (r_1+r_2)^2 - 4r_1r_2$$

$$= 4(ul_1+vm_1+wn_1)^2 - 4d.$$

$$\left(\because r_1+r_2 = \frac{l_1u+m_1v+n_1w}{1}, \quad r_1r_2 = \frac{d}{1}\right)$$

$$= 4(u^2l_1^2+v^2m_1^2+w^2n_1^2+2uvl_1m_1+2vwm_1n_1+2wul_1n_1)-4d$$

Similarly,

$$L_2 = 4(u^2l_2^2+v^2m_2^2+w^2n_2^2+2uvl_2m_2+2vwm_2n_2+2wul_2n_2)-4d.$$

16(i)

$$L_3^2 = 4(u^2 l_3^2 + v^2 m_3^2 + w^2 n_3^2 + 2uv l_3 m_3 + 2vw m_3 n_3 + 2wu n_3 l_3) - 4d.$$

Adding, the sum of squares of the intercepts

$$= L_1^2 + L_2^2 + L_3^2$$

$$= 4\left[u^2(l_1^2+l_2^2+l_3^2) + v^2(m_1^2+m_2^2+m_3^2) + w^2(n_1^2+n_2^2+n_3^2) + 2uv(l_1m_1+l_2m_2+l_3m_3) + 2vw(m_1n_1+m_2n_2+m_3n_3) + 2wu(l_1n_1+l_2n_2+l_3n_3)\right] - d$$

$$= 4\left[u^2(1) + v^2(1) + w^2(1) + 2uv(0) + 2wv(0) + 2wu(0)\right] - 12d. \quad (\because \text{from } (5))$$

$$= 4(u^2+v^2+w^2) - 12d.$$

which is free from l_1, m_1, n_1 etc and is therefore constant for any set of lines.

Hence the result

(OR)

Let the equation of the given sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0$$ and

take the fixed point 'O' as the origin and any three mutually perpendicular lines through it as the co-ordinate axes.

The 'x-axis (i.e., $y=0=z$) meets the sphere in points given by

$$x^2+2ux+d=0;$$

so that if x_1, x_2 be its roots, the points of intersection are $(x_1,0,0)$, $(x_2,0,0)$.

Also we have

$x_1 + x_2 = -2u \quad ; \quad x_1 x_2 = d.$

$\therefore (\text{intercept on } x\text{-axis})^2 = (x_1 - x_2)^2$

$= (x_1 + x_2)^2 - 4x_1 x_2$

$= 4u^2 - 4d$

$= 4(u^2 - d).$

Similarly,

$(\text{intercept on } y\text{-axis})^2 = 4(v^2 - d)$

$(\text{intercept on } z\text{-axis})^2 = 4(w^2 - d).$

The sum of the squares of the intercepts.

$= 4(u^2 + v^2 + w^2 - 3d).$

$= 4(u^2 + v^2 + w^2) - 12d$

→ Show that the plane $lx+my+nz=p$ will touch the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$ if $(ul+mv+nw+p)^2 = (l^2+m^2+n^2)(u^2+v^2+w^2-d)$

IAS 2004 → Find the tangent planes to the sphere $x^2+y^2+z^2-4x+2y-6z+5=0$ which are parallel to the plane $2x+2y-z=0$

Sol^n: equation of sphere is
$$x^2+y^2+z^2-4x+2y-6z+5=0$$
its centre $(2,-1,3)$.
and radius $=\sqrt{4+1+9-5}$
$=3$

Any plane $\parallel$ to the plane $2x+2y-z=0$ is
$$2x+2y-z=k \text{ ——— (1)}$$

If it touches the sphere, then length of $\perp^r$ from the centre of sphere must be equal to the radius of the sphere.

$$\therefore \frac{|2(2)+2(-1)-(+3)-k|}{\sqrt{4+4+1}}=3$$
$$\Rightarrow |4-2-3-k| = 3\sqrt{9}$$
$$\Rightarrow -1-k=\pm 9$$
$$\Rightarrow k=-1\pm 9$$
$$\Rightarrow k=-10 \text{ or } 8$$

from (1), we have
$2x+2y-z=-10$ and
$2x+2y-z=8$.

$\therefore$ The required tangent planes are
$2x+2y-z+10=0$ and
$2x+2y-z-8=0$

2004 → Find the equations of tangent planes to the sphere $x^2+y^2+z^2-4x+2y-6z+5=0$, which are parallel to the plane $2x+y-z=4$.

→ Find the equation of the tangent plane to the sphere $3(x^2+y^2+z^2)-2x-3y-4z-22=0$ at the point $(1,2,3)$.

→ Find the value of 'a' for which the plane $x+y+z=a\sqrt{3}$ touches the sphere $x^2+y^2+z^2-2x-2y-2z-6=0$.

→ Find the equation of the sphere which touches the sphere $x^2+y^2+z^2-x+3y+2z-3=0$ at the point $(1,1,-1)$ and passes through the origin.

Sol^n: The given sphere is

$$x^2+y^2+z^2-x+3y+2z-3=0 \quad \text{--- (1)}$$

Equation of tangent plane at $(1,1,-1)$ to the sphere (1) is

$$x(1)+y(1)+z(-1)-\tfrac{1}{2}(x+1)+\tfrac{3}{2}(y+1)+(z-1)-3=0$$

$$\Rightarrow \tfrac{1}{2}x+\tfrac{5}{2}y-3=0$$

$$\Rightarrow x+5y-6=0 \quad \text{--- (2)}$$

The required sphere touching (1) at $(1,1,-1)$ is the sphere through the circle of intersection of (1) and the tangent plane at $(1,1,-1)$ to the sphere i.e, the plane (2).

Now any sphere through the circle of intersection of (1) & (2) is

$$x^2+y^2+z^2-x+3y+2z-3+k(x+5y-6)=0 \quad \text{--- (3)}$$

If it passes through the origin $(0,0,0)$,

then $-3-6k=0$

$$\Rightarrow \boxed{k=-\tfrac{1}{2}}$$

$\therefore$ (3) $\equiv$

$$x^2+y^2+z^2-x+3y+2z-3-\tfrac{1}{2}(x+5y-6)=0$$

$$2(x^2+y^2+z^2)-3x+y+4z=0$$

which is the required equation.

→ Show that the equation of the sphere which touches the sphere $4(x^2+y^2+z^2)+10x-25y-2z=0$ at the point $(1,2,-2)$ and passes through the point $(-1,0,0)$ is $x^2+y^2+z^2+2x-6y+1=0$

→ Find the equations of the sphere through the circle $x^2+y^2+z^2=1$, $2x+4y+5z=6$ and touching the plane $z=0$.

→ Find the eqns of the sphere which pass through the circle $x^2+y^2+z^2-2x+3y+4z-3=0$, $2x+y+z=4$ and touches the plane $3x+4y=14$.

→ Show that the plane $2x-2y+z+12=0$ touches the sphere $x^2+y^2+z^2-2x-4y+2z=3$ and find the point of contact.

Sol^n: The given plane is

$$2x-2y+z+12=0 \quad \text{--- (1)}$$

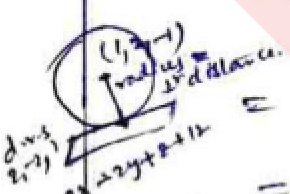
and the sphere is

$$x^2+y^2+z^2-2x-4y+2z-3=0$$

The centre of the sphere is $(1,2,-1)$ and its radius is

$$\sqrt{1+4+1+3}=3.$$

Also the $\perp^r$ distance of the centre $(1,2,-1)$ from the plane (1)



$$=\frac{2(1)-2(2)+(-1)+12}{\sqrt{4+4+1}}$$

$$=\frac{2-4-1+12}{3}$$

$$=\frac{9}{3}=3.$$

Since $\perp^r$ distance of the centre from the plane (1) = radius of the sphere

∴ The plane (1) touches the sphere (2).

The point of contact is the foot of perpendicular from the centre of the sphere on the plane.

Now the equations of the line through the centre $(1,2,-1)$ and $\perp^r$ to the plane (1) are

$$\frac{x-1}{2}=\frac{y-2}{-2}=\frac{z+1}{1}$$

Any point on this line is $(2r+1, -2r+2, r-1)$ — (3)

If it lies on the plane (1) then

$$2(2r+1)-2(-2r+2)+r-1+12=0$$

$$\Rightarrow 9r+9=0$$

$$\Rightarrow \boxed{r=-1}$$

∴ (3) ≡ $(2(-1)+1, -2(-1)+2, -1-1)$

$= (-1, 4, -2)$

which is the required point of contact.

(IAS 2014) → Find the co-ordinates of the points on the sphere $x^2+y^2+z^2-4x+2y=4$

the tangent planes at which are parallel to the plane
$2x-y+2z=1$
(Ans: $(4,-2,2)$, $(0,0,-2)$)

2008 $\Rightarrow$ Find the equation of the tangent line to the circle
$x^2+y^2+z^2+5x-7y+2z-8=0$,
$3x-2y+4z+3=0$
at the point $(-3,5,4)$.

Sol^n:
Note: The tangent line to a circle is the line of intersection of the tangent plane to the sphere at the given point and the plane of circle.

The given sphere is
$x^2+y^2+z^2+5x-7y+2z-8=0$ —— ①
and the plane of the circle is $3x-2y+4z+3=0$ —— ②

Now equation of the tangent plane at $P(-3,5,4)$ to the sphere ① is
$x(-3)+y(5)+z(4)+\frac{5}{2}(x-3)-\frac{7}{2}(y+5)+1(z+4)-8=0$
$\Rightarrow -x+3y+10z-58=0$
$\Rightarrow x-3y-10z+58=0$ —— ③

The eqns ② & ③ taken together represent the equation of the tangent line to the circle given by ① & ②.

To find the d.r.'s of the tangent line:

Omitting the constant terms in ② & ③, the equations are
$3x-2y+4z=0$
$x-3y-10z=0$

$\frac{x}{20+12}=\frac{y}{4+30}=\frac{z}{-9+2}$

$\Rightarrow \frac{x}{32}=\frac{y}{34}=\frac{z}{-7}$.

$\therefore$ The d.r.'s of the tangent line are $32, 34, -7$.

Also the tangent line passes through the given point $P(-3,5,4)$.

$\therefore$ The eqns of the tangent line to the circle at $P(-3,5,4)$ are $\frac{x+3}{32}=\frac{y-5}{34}=\frac{z-4}{-7}$.

$\rightarrow$ Find the eqn of tangent line to the circle $3x^2+3y^2+3z^2-2x-3y-4z-22=0$, $3x+4y+5z-26=0$ at the point $(1,2,3)$

→ Find the equations of the two tangent planes to the sphere $x^2+y^2+z^2=9$ which pass through the line $x+y=6$, $x-2z=3$.

Sol^n: The given line is

$x+y=6, \quad x-2z=3.$

Any plane through this line is

$$x+y-6+K(x-2z-3)=0 \quad \text{——(1)}$$

If it touches the sphere $x^2+y^2+z^2-9=0$, then $\perp^r$ distance from the centre $(0,0,0)$ on the plane (1) must be equal to the radius $(=3)$ of the sphere.

$$\therefore \frac{-6-3K}{\sqrt{(1+K)^2+1+4K^2}}=3$$

$\Rightarrow -2-K=\sqrt{5K^2+2K+2}$

$\Rightarrow (2-K)^2=5K^2+2K+2$

$\Rightarrow K^2+4K+4=5K^2+2K+2$

$\Rightarrow 4K^2-2K-2=0$

$\Rightarrow 2K^2-K-1=0$

$\Rightarrow 2K^2-2K+K-1=0$

$\Rightarrow 2K(K-1)+1(K-1)=0$

$(2K+1)(K-1)=0$

$\Rightarrow K=-\frac{1}{2}, 1.$

Putting these values in (1), the required planes are

$2x+y-2z=9$ and

$x+2y+2z=9$

→ Obtain the equations of the tangent planes to the sphere

(i) $x^2+y^2+z^2=9$ which can be drawn through the line $\frac{x-5}{2}=\frac{y-1}{-2}=\frac{z-1}{1}$

(ii) $x^2+y^2+z^2+6x-2z+1=0$ which pass through the line

$3(16-x)=3z=2y+30.$

(Hint: If the given line is symmetrical form then convert it into unsymmetrical form)

→ Find the equations of spheres that pass through the points $(4,1,0)$, $(2,-3,4)$, $(1,0,0)$ and touch the plane $2x+2y-z=11$.

Sol^n: Let the equation of the sphere be

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$ ——①

Since it passes through $(4,1,0)$

$\therefore 8u+2v+d+17=0$ ——②

Since ① passes through $(2,-3,4)$ & $(1,0,0)$

$\therefore$ we have

$4u-6v+8w+d+29=0$ ——③

$2u+d+1=0$ ——④

centre of the sphere ① is $(-u,-v,-w)$.

and radius $=\sqrt{u^2+v^2+w^2-d}$

Since the sphere touches the plane $2x+4y-z=11$

$\therefore$ length of the $\perp^r$ from the centre of the sphere to the plane $2x+4y-z-11=0$ must be equal to the radius of the sphere.

$$\therefore \frac{2(-u)+4(-v)-(-w)-11}{\sqrt{4+4+1}} = \sqrt{u^2+v^2+w^2-d}$$

$\Rightarrow (-2u-4v+w-11)^2 = 9(u^2+v^2+w^2-d)$

$\Rightarrow 5u^2+5v^2+8w^2-8uv+40vw+4uw-44v+22w-9d-121=0$ ——⑤

from ④ $\boxed{u=-\frac{1}{2}(d+1)}$ ——(i)

from ② $2v=-8u-d-17$ ——(ii)

$\therefore$ from (i) & (ii), we get

$\boxed{v=\frac{1}{2}(3d-13)}$

from ③, $w=\frac{5d-33}{4}$ ——(iii)

Substituting these values of u, v, w in ⑤, we get

$72d^2-747d+1935=0$

$8d^2-83d+215=0$

$\Rightarrow d=5, \frac{43}{8}$.

$\therefore$ Substituting $d=5$ in (i), (ii) & (iii),

we get

$u=-3,\ v=1,\ w=-2$.

$\therefore$ the required eq^n of the sphere is

$x^2+y^2+z^2-6x+2y-4z+5=0$

Also, sub. $d=\frac{43}{8}$ in (i), (ii), & (iii)

we get $u=\ ,\ v=\ ,\ w=$ (proceed like this.)

and the required eq^n of the sphere is

$16(x^2+y^2+z^2)-102x+50y-49z+86=0$

→ find the locus of the centre of the sphere of constant radius which passes through a given point and touches the given line.

Sol^n: Take x-axis to be given line and perpendicular from the given point on the x-axis as the z-axis, then the co-ordinates of the given point on the z-axis are of the form $(0,0,c)$.

Let the eqn of the sphere be

$$x^2+y^2+z^2+2ux+2vy+2wz+d=0 \quad \text{—(1)}$$

It passes through $(0,0,c)$.

$$\therefore c^2+2wc+d=0 \quad \text{—(2)}$$

Given that the radius of the sphere is constant say 'λ'.

$$\therefore u^2+v^2+w^2-d=\lambda^2 \quad \text{—(3)}$$

The sphere meets the x-axis. $(y=0, z=0)$.

where

$$x^2+2ux+d=0 \quad \text{—(4)}$$

Since the line i.e, x-axis touches the sphere, then the two values of x given by (4) must be equal

i.e, the discriminant of (4) is zero ($b^2-4ac=0$)

$$\therefore 4u^2-4d=0$$

$$\Rightarrow \boxed{u^2=d} \quad \text{—(5)}$$

Eliminating d from (2), (5) & (4), we get

$u^2+2wc+c^2=0$ [by adding (2) & (5)]

and $v^2+w^2=\lambda^2$ [subtracting (5) from (3)]

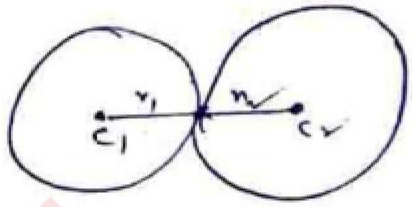
$\therefore$ The locus of the centre $(-u,-v,-w)$ of the sphere (1) is

$$x^2-2cz+c^2=0 \text{ and } y^2+z^2=\lambda^2.$$

which represents a curve of intersection of two surfaces.

Touching Spheres:

(i) Two spheres touch externally if the distance between their centres is equal to the sum of their radii.



(ii) Two spheres touch internally if the distance between their centres is equal to the difference of their radii.

Angle of intersection of two spheres:

The angle of intersection of two spheres is the angle between their tangent planes at a common point of intersection. Since the radii of the spheres to a common point are $\perp^r$ to the tangent planes at that point, so the angle between the radii of spheres at the common point is equal to the angle between the tangent planes.

i.e, the angle of intersection of the spheres.

To find the angle:

Let C_1, C_2 be the centres of the spheres of radii r_1 & r_2.

Let P be their common point of intersection.

Let $C_1C_2 = d$

The angle of intersection i.e, the angle between the tangent planes at P is the angle between the radii of the two spheres.

i.e, $\angle C_1PC_2 = \theta$, then

$$\cos\theta = \frac{(C_1P)^2 + (C_2P)^2 - (C_1C_2)^2}{2(C_1P)\cdot(C_2P)}.$$

[cosine formula]

$$= \frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}$$

$$\therefore \theta = \cos^{-1}\left(\frac{r_1^2 + r_2^2 - d^2}{2r_1r_2}\right)$$

Orthogonal Spheres:

Two spheres are said to be orthogonal if the angle of intersection of two spheres is a right angle.

i.e, if the two spheres cut orthogonally then the square of the distance between the centres of two spheres = sum of squares of radii.

i.e $\boxed{d^2 = r_1^2 + r_2^2}$

Condition of orthogonality of two spheres:

To find the condition that the spheres
$x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0$
$x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0$
to be orthogonal.

Sol^n: Two given spheres
$$x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0 \quad \text{---(1)}$$
$$x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0 \quad \text{---(2)}$$

If the spheres cut orthogonally then square of distance between their centres = sum of the squares of their radii. ---(3)

Now the centres of the spheres (1) & (2) are
$C_1(-u_1, -v_1, -w_1)$ and $C_2(-u_2, -v_2, -w_2)$. and their are
$\sqrt{u_1^2+v_1^2+w_1^2-d_1^2}$, $\sqrt{u_2^2+v_2^2+w_2^2-d}$

$\therefore$ (3) $\equiv (u_1-u_2)^2+(v_1-v_2)^2+(w_1-w_3)^2$
$=(u_1^2+v_1^2+w_1^2-d_1)+(u_2^2+v_2^2+w_2^2-d_2)$.

$\Rightarrow 2u_1u_2+2v_1v_2+2w_1w_2=d_1+d_2$.
which is the required condition.

$\rightarrow$ Two spheres of radii r_1 and r_2 cut orthogonally prove that the radius of the common circle is
$$\frac{r_1r_2}{\sqrt{r_1^2+r_2^2}}.$$

Sol^n: Let the common circle be $x^2+y^2=a^2;\ z=0$. ---(1)
where 'a' is radius of this circle.

Let the two given spheres through the circle (1) be
$$x^2+y^2+z^2-a^2+2\lambda_1z=0 \quad \text{---(2)}$$
and $$x^2+y^2+z^2-a^2+2\lambda_2z=0 \quad \text{---(3)}$$

Since r_1, r_2 are radii of the given spheres (2) & (3)
$$\therefore \left.\begin{aligned} r_1^2&=\lambda_1^2+a^2 \\ r_2^2&=\lambda_2^2+a^2\end{aligned}\right\} \quad \text{---(4)}$$

Since the spheres (2) & (3) cut orthogonally.

$\therefore 2\lambda_1\lambda_2 = -a^2-a^2$

$\Rightarrow \lambda_1\lambda_2 = -a^2$

$\Rightarrow \lambda_1^2\lambda_2^2 = a^4$

$\therefore$ (4) $\equiv (r_1^2-a^2)(r_2^2-a^2) = a^4$

$\Rightarrow r_1^2r_2^2 - a^2(r_1^2+r_2^2) + a^4 = a^4.$

$\Rightarrow a^2(r_1^2+r_2^2) = r_1^2r_2^2$

$\Rightarrow a^2 = \dfrac{r_1^2r_2^2}{r_1^2+r_2^2}$

$\Rightarrow \boxed{a = \dfrac{r_1r_2}{\sqrt{r_1^2+r_2^2}}}$

Hence, the result.

1995 → Two spheres of radii r_1 and r_2 cut orthogonally. Prove that the area of the common circle is $\dfrac{\pi r_1^2r_2^2}{r_1^2+r_2^2}$.

Sol^n: From the above problem the radius of the common circle is $a = \dfrac{r_1r_2}{\sqrt{r_1^2+r_2^2}}$

$\therefore$ The area of the common circle

$= \pi a^2 = \pi\dfrac{r_1^2r_2^2}{r_1^2+r_2^2}.$

→ Find the equation of the sphere that passes through the circle

$x^2+y^2+z^2-2x+3y-4z+6=0,$

$3x-4y+5z-15=0$ and cuts the sphere

$x^2+y^2+z^2+2x+4y-6z+11=0$ orthogonally.

Sol^n: Given equations of circle.

$x^2+y^2+z^2-2x+3y-4z+6=0,$
$3x-4y+5z-15=0$ —— (1)

and given sphere

$x^2+y^2+z^2+2x+4y-6z+11=0$ —— (2)

Any sphere through the circle (1) is

$x^2+y^2+z^2-2x+3y-4z+6+\lambda(3x-4y+5z-15)=0$ —— (3)

$\Rightarrow x^2+y^2+z^2+(-2+3\lambda)x+(3-4\lambda)y+(-4+5\lambda)z+6-15\lambda=0$

This will cut the sphere (2) orthogonally iff

$2\left(\dfrac{-2+3\lambda}{2}\right)(1)+2\left(\dfrac{3-4\lambda}{2}\right)(2)+2\left(\dfrac{-4+5\lambda}{2}\right)(-3) = (6-15\lambda)+11$

[$\because$ by using $2u_1u_2+2v_1v_2+2w_1w_2=d_1+d_2$]

$\Rightarrow \boxed{\lambda = -1/5}$

Putting this value of λ in ③, we get

$$5(x^2+y^2+z^2)-13x+19y-25z+45=0.$$

$\longrightarrow$ Find the equation of the sphere that passes through the two points $(0,3,0)$, $(-2,-1,-4)$ and cuts orthogonally the two spheres.

$$x^2+y^2+z^2+x-3z-2=0,$$
$$2(x^2+y^2+z^2)+x+3y+4=0.$$

2006 $\longrightarrow$ Find the equation of the sphere which touches the plane $3x+2y-z+2=0$ at the point $(1,-2,1)$ and also cuts orthogonally the sphere.

$$x^2+y^2+z^2-4x+6y+4=0$$

Sol'n :- Given plane $3x+2y-z+2=0$ —— ①

and the given sphere

$x^2+y^2+z^2-4x+6y+4=0$ —— ②

Since the required sphere touches the plane ① at $P(1,-2,1)$.

$\therefore$ Its centre lies on the normal to the plane at 'P'.

Now the equations of normal to the plane ① through $P(1,-2,1)$ are

$$\frac{x-1}{3} = \frac{y+2}{2} = \frac{z-1}{-1} = r \text{ (say)} \quad \text{—— ③}$$

Any point on this line $C(3r+1,\ 2r-2,\ -r+1)$.

Let this point be the centre of the required sphere.

Now the radius of the required sphere.

$$CP = \sqrt{(3r+1-1)^2+(2r-2+2)^2+(-r+1-1)^2}$$
$$= \sqrt{9r^2+4r^2+r^2}$$
$$= r\sqrt{14}$$

Since the required sphere cuts the sphere ② orthogonally.

$\therefore$ Square of distance between the centres = Sum of square of their radii. —— ④

Now the centre of the sphere ② $C'(2,-3,0)$

and radius $= \sqrt{4+9-4}$
$= 3$

$\therefore$ ④ $\equiv$

$$(3r+1-2)^2+(2r-2+3)^2+(-r+1-0)^2 = 9+14r^2$$

$$\Rightarrow r = -3/2$$

∴ The centre of the required sphere.

C $\left(-7/2, -5, 5/2\right)$.

and the radius $CP = \frac{-3}{2}\sqrt{14}$

$= \frac{3}{2}\sqrt{14}$ (numerically)

∴ The required sphere is

$$\left(x+7/2\right)^2+(y+5)^2+\left(z-5/2\right)^2=\left(\frac{3\sqrt{14}}{2}\right)^2$$

$$\Rightarrow x^2+y^2+z^2+7x+10y-5z+12=0.$$

2010 20M → Show that every sphere through the circle $x^2+y^2-2ax+r^2=0,\ z=0$ cuts orthogonally every sphere through the circle $x^2+z^2=r^2,\ y=0$.

Sol'n:- Any sphere through the first circle

$x^2+y^2-2ax+r^2=0,\ z=0$

i.e. the circle

$x^2+y^2+z^2-2ax+r^2=0,\ z=0$

is $x^2+y^2+z^2-2ax+r^2+\lambda_1 z=0$ ——(1)

Again any sphere through the second circle

$x^2+z^2=r^2,\ y=0.$

i.e. the circle $x^2+y^2+z^2=r^2$, $y=0$ is

$x^2+y^2+z^2-r^2+\lambda_2 y=0$ ——(2)

(1) & (2) will cut orthogonally if $2u_1u_2+2v_1v_2+2w_1w_2=d_1+d_2$

if $2(-a)(0)+2(0)\left(\frac{\lambda_2}{2}\right)+2\left(\frac{\lambda_1}{2}\right)(0)$

$= r^2-r^2$

if $0=0$

which is true.

Hence the result.

2007 → Show that the spheres $x^2+y^2+z^2-x+z-2=0$ and $3x^2+3y^2+3z^2-8x-10y+8z+14=0$ cut orthogonally. Find the centre and radius of their common circle.

Length of Tangent:

To find the length of the tangent from the point (x_1, y_1, z_1) to the sphere $x^2+y^2+z^2+2ux+2vy+2wz+d=0$

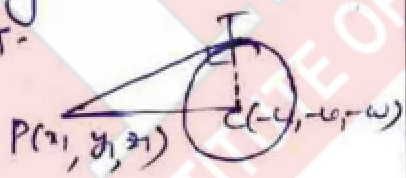
Let $P(x_1, y_1, z_1)$ be a point out side the sphere

$S = x^2+y^2+z^2+2ux+2vy+2wz+d=0$

Its centre is $C(-u,-v,-w)$.

and radius $= \sqrt{u^2+v^2+w^2-d}$

Now let the tangent from $P(x_1,y_1,z_1)$ to the sphere meet at T, then radius CT at T must be at right angles to the tangent PT.



$\therefore$ ΔPTC is right $\angle$d triangle.

$$\therefore PT^2 = PC^2 - CT^2$$

$$= (x_1+u)^2+(y_1+v)^2+(z_1+w)^2-(u^2+v^2+w^2-d)$$

$$= x_1^2+y_1^2+z_1^2+2ux_1+2vy_1+2wz_1+d.$$

$$= S_{11}$$

→ Find the length of the tangent drawn from the point $P(1,2,3)$ to the sphere $5(x^2+y^2+z^2)-x+10y+20z+8=0$

Sol^n: Let $P(1,2,3)$ be the given point.

Let the tangent from $P(1,2,3)$ to the sphere $x^2+y^2+z^2-\frac{1}{5}x+2y+4z+\frac{8}{5}=0$ meet at T.

$$\therefore (PT)^2 = x_1^2+y_1^2+z_1^2-2ux_1+2vy_1+2wz_1+d$$

$$= 1+4+9-2\left(\tfrac{1}{10}\right)(1)+2(1)(2)+2(2)(3)+\frac{8}{5}$$

$$(PT)^2 = \frac{157}{5}$$

$$\Rightarrow PT = \sqrt{\frac{157}{5}}$$

which is the required length of the tangent

Radical plane of two Spheres:

The locus of a point whose powers w.r.t two spheres are equal

i.e, the locus of a point

where the square of the lengths of the tangents to the two spheres are equal, is a plane called the radical plane of the two spheres.

Equation of Radical plane of two spheres:

To find the equation of the radical plane of the spheres $x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0$

$x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0$

Sol^n: The given spheres are

$S_1 \equiv x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0$ ——— (1)

$S_2 \equiv x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0$ ——— (2)

Let $P(x,y,z)$ be any point on the radical plane.

Then the power of P w.r.t sphere (1) = the power of P w.r.t sphere (2)

$\therefore x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1 = x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2$

$\Rightarrow 2(u_1-u_2)x+2(v_1-v_2)y+2(w_1-w_2)z+d_1-d_2=0$ ——— (3)

which is the required eqn.

Note: The radical plane of two spheres $S_1=0$, $S_2=0$ (in both of which the coefficients of x^2, y^2, z^2 are equal to unity) is $S_1-S_2=0$

→ The radical plane of two spheres is perpendicular to the line joining their centres.

Sol^n: Let the spheres be

$x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0$ ——— (1)

$x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0$ ——— (2)

The centres of (1) & (2) are $C_1(-u_1,-v_1,-w_1)$ & $C_2(-u_2,-v_2,-w_2)$.

$\therefore$ d.r.'s of line joining the centres C_1, C_2 are $u_1-u_2,\ v_1-v_2,\ w_1-w_2$.

Also the d.r.'s of the normal to the radical plane are proportional to $2(u_1-u_2),\ 2(v_1-v_2),\ 2(w_1-w_2)$.

or $u_1-u_2,\ v_1-v_2,\ w_1-w_2$.

($\because$ Radical plane of two spheres is $2(u_1-u_2)x+2(v_1-v_2)y+2(w_1-w_2)z+d_1-d_2=0$)

The normal to the radical plane is parallel to the line C_1C_2 (or) the line C_1C_2 is $\perp^r$ to the radical plane.

Note: If the spheres intersect then the plane of their common circle is their radical plane.

Radical line of three spheres:

The three radical planes of three spheres intersect in a line.

i.e, If $S_1=0$, $S_2=0$, $S_3=0$ be the three spheres then their radical planes $S_1-S_2=0$, $S_2-S_3=0$, $S_3-S_1=0$ clearly meet in the line $S_1=S_2=S_3$.
$\Leftrightarrow S_1-S_2=0, S_2-S_3=0$

This line is called the radical line or radical axis of three spheres.

Radical centre of four spheres:

The four radical lines of four spheres taken three at a time in a point which is called the radical centre of the four spheres.

Let the four spheres $S_1=0$, $S_2=0$, $S_3=0$, $S_4=0$.

Then the point common to the three planes $S_1=S_2=S_3=S_4$ is clearly common to the radical lines, taken three by three, of four spheres.

$S_1=S_2=S_3$
$S_1=S_3=S_4$
$S_2=S_3=S_4$
$S_1=S_2=S_4$

This point is the intersection of two lines $S_1-S_2=0$, $S_2-S_3=0$; $S_1-S_3=0$, $S_2-S_4=0$

This point is called radical centre.

Co-axial Spheres:

A system of spheres any two members of which have the same radical plane is called a co-axial system of spheres.

Equation of Co-axial system of spheres determined by two given spheres:

If $S_1=0$, $S_2=0$ be two spheres then $S_1+\lambda S_2=0$ represents a system of spheres, where λ is a parameter, such that any two members of the system have the same radical plane.

Let $S_1+\lambda_1 S_2=0$, $S_1+\lambda_2 S_2=0$ by any two members of the system $S_1+\lambda S_2=0$

Making the co-efficients x^2, y^2, z^2 unity in the two equations, we write them in the form

$$\frac{S_1+\lambda_1 S_2}{1+\lambda_1}=0, \quad \frac{S_1+\lambda_2 S_2}{1+\lambda_2}=0$$

The radical plane of these two spheres is

$$\frac{S_1+\lambda_1 S_2}{1+\lambda_1}-\frac{S_1+\lambda_2 S_2}{1+\lambda_2}=0$$

$$\Rightarrow (S_1+\lambda_1 S_2)(1+\lambda_2)-(S_1+\lambda_2 S_2)(1+\lambda_1)=0$$

$$\Rightarrow \lambda_2(S_1-S_2)-\lambda_1(S_1-S_2)=0$$

$$\Rightarrow (\lambda_2-\lambda_1)(S_1-S_2)=0$$

$$\Rightarrow S_1-S_2=0 \quad (\because \lambda_1\neq\lambda_2)$$

Since the radical plane is independent of λ_1, λ_2.

we see that every two members of the system have the same radical plane.

$\therefore S_1+\lambda S_2=0$ represents a system of co-axial spheres determined by two spheres $S_1=0$, $S_2=0$.

The co-axial system is also given by the eqn

$$S_1+\lambda(S_1-S_2)=0$$

Equation of co-axial system in the simplest form.

To prove that the equation of a co-axial system of spheres can be put in the form $x^2+y^2+z^2+2ux+d=0$ where 'u' is the parameter.

Soln: Let any two spheres of the system be

$$x^2+y^2+z^2+2u_1x+2v_1y+2w_1z+d_1=0 \quad \text{①}$$

$$x^2+y^2+z^2+2u_2x+2v_2y+2w_2z+d_2=0 \quad \text{②}$$

Now take the line joining the centres as the x-axis.

$\therefore$ y & z co-ordinates of their centres become zero.

i.e, $v_1=0,\ w_1=0,\ v_2=0,\ w_2=0$

and the equations of the spheres ① & ② become

$x^2+y^2+z^2+2u_1x+d_1=0$ — ③

$x^2+y^2+z^2+2u_2x+d_2=0$ — ④

Now the equation of their radical plane is

$2(u_1-u_2)x+d_1-d_2=0$

Let this be taken as the yz-plane i.e, $x=0$.

$\therefore d_1-d_2=0$

$\Rightarrow d_1=d_2=d$ (say)

$\therefore$ Eqns of spheres ③ & ④ become

$x^2+y^2+z^2+2u_1x+d=0$

$x^2+y^2+z^2+2u_2x+d=0$

$\therefore$ The equations of the co-axial system can be put in the form

$x^2+y^2+z^2+2ux+d=0$.

where 'u' is parameter.

Limiting points of a co-axial system:

The centres of two spheres of a co-axial system which have zero radius are called the limiting points of the system.

To find the limiting points of a system of co-axial spheres $x^2+y^2+z^2+2ux+d=0$

Solⁿ: The given system of co-axial spheres is $x^2+y^2+z^2+2ux+d=0$.

Its centre is $(-u,0,0)$ and radius $\sqrt{u^2-d}$

Since for limiting points, radius $=0$

$\therefore \sqrt{u^2-d}=0 \Rightarrow u^2-d=0$

$\Rightarrow u=\pm\sqrt{d}$

$\therefore$ The centre $(-u,0,0)$ becomes $(\sqrt{d},0,0)$ & $(-\sqrt{d},0,0)$, which are the reqd limiting points.

→ Find the limiting points of the co-axial system defined by the spheres

$x^2+y^2+z^2+3x-3y+6=0$ — ①

$x^2+y^2+z^2-6y-6z+6=0$ — ②

Solⁿ: The equation of any plane of circle two given spheres is ①–②

$\therefore\ 3x+3y+6z=0$
$\Rightarrow x+y+2z=0$

Now the eqn of co-axial system determined by the given spheres is

$x^2+y^2+z^2+3x-3y+\lambda(x+y+2z)=0$
$(\because S_1+\lambda(S_1-S_2)=0)$

$\Rightarrow x^2+y^2+z^2+(3+\lambda)x+(\lambda-3)y+2\lambda z+6=0$ — (3)

where 'λ' is parameter.

Its centre $=\left(-\frac{(3+\lambda)}{2}, -\frac{(\lambda-3)}{2}, -\lambda\right)$ — (4)

and radius
$=\sqrt{\left(\frac{3+\lambda}{2}\right)^2+\left(\frac{\lambda-3}{2}\right)^2+\lambda^2-6}$

For limiting point, equating this to zero, we get

$\left(\frac{3+\lambda}{2}\right)^2+\left(\frac{\lambda-3}{2}\right)^2+\lambda^2-6=0$

$\Rightarrow \lambda^2=1 \Rightarrow \boxed{\lambda=\pm 1}$.

$\therefore$ (4) $\equiv (-1,2,1)$ & $(-2,1,-1)$.
which are the required limiting points.

→ Show that the spheres which cut two given spheres along great circles all pass through two fixed points.

Sol^n: Let the two given spheres be

$x^2+y^2+z^2+2u_1x+d=0$ — (1)
$x^2+y^2+z^2+2u_2x+d=0$ — (2)

The eqn of another sphere be

$x^2+y^2+z^2+2ux+2vy+2wz+c=0$ — (3)

where u, v, w, c are different for different spheres.

If (3) cuts (1) in a great circle then the centre $(-u_1,0,0)$ of (1) must be lie on the radical plane. i.e. the plane of circle (1) & (3) is

$2(u-u_1)+2vy+2wz+c-d=0$

$\therefore 2(u-u_1)(-u_1)+2v(0)+2w(0)+c-d=0$

$\Rightarrow 2uu_1-2u_1^2-c+d=0$ — (4)

Similarly (3) cuts (2) in a great circle if

$2uu_2-2u_2^2-c+d=0$ — (5)

(5) − (4) $\equiv$
$2u(u_1-u_2)-2(u_1^2-u_2^2)=0$

$\Rightarrow u-(u_1+u_2)=0$

$\Rightarrow \boxed{u=u_1+u_2}$

(4) $\equiv \boxed{c=2u_1u_2+d}$

u & c are constants dependent on only u_1, u_2, d, the given quantities.

$\therefore$ The sphere ③ cuts x-axis where putting $y=0, z=0$ in ③.

$\therefore$ ③ $\equiv x^2+2ux+c=0$

The roots of this equation are constant; depending upon the constants u & c only.

$\therefore$ Every sphere ③ cuts the x-axis at the same two points.

Hence the result.

→ Find the limiting points of the co-axial system of spheres
$x^2+y^2+z^2-20x+30y-40z+29+\lambda(2x-3y+4z)=0$.

(Ans: $(2,-3,4), (-2,3,-4)$)

→ Prove that the every sphere that passes through the limiting points of a co-axial system cuts every sphere of the system orthogonally.

Sol^n: Let the system of co-axial sphere be

$$x^2+y^2+z^2+2\lambda x+d=0 \quad \text{—①}$$

The limiting points of system are $(\sqrt{d},0,0)$ & $(-\sqrt{d},0,0)$.

Let the eqn of the sphere through the limiting points be

$$x^2+y^2+z^2+2ux+2vy+2wz+c=0 \quad \text{—②}$$

Since it passes through the limiting points $(\sqrt{d},0,0)$ & $(-\sqrt{d},0,0)$

$\therefore d+2u\sqrt{d}+c=0$ &

$d-2u\sqrt{d}+c=0$

Solving these we get $u=0$ & $c=-d$.

$$② \equiv x^2+y^2+z^2+2vy+2wz-d=0 \quad \text{—③}$$

Since ③ & ① cut orthogonally if

$2u_1u_2+2v_1v_2+2w_1w_2=d_1+d_2$

$\Rightarrow 0(2\lambda)+2v(0)+2w(0)=d$

$0=0$ which is true.

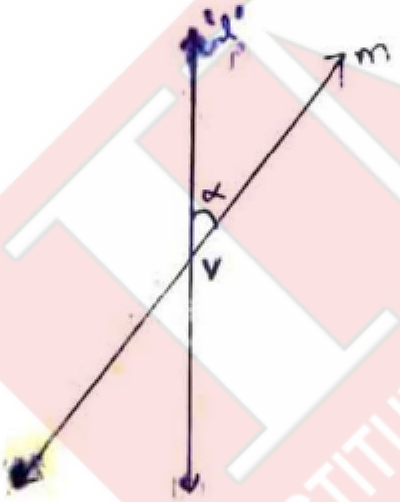
Hence the result.

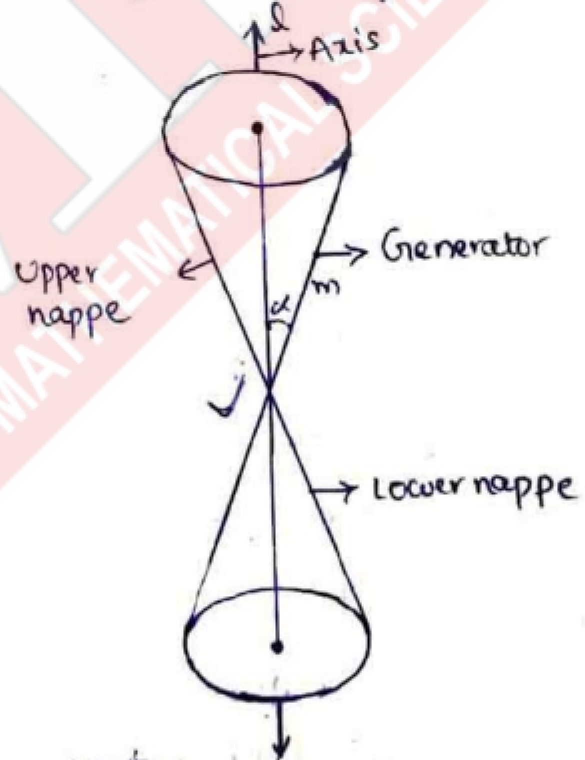
→ Show that the eqn $x^2+y^2+z^2+2\mu y+2\nu z-d=0$ where μ & ν are parameters represents a system of spheres passing through limiting points of the system $x^2+y^2+z^2+\lambda x+d=0$ and cutting every member of this system at right angles.

* Cone *

Def → Let 'l' be a fixed vertical line and 'm' be another line intersecting it at a fixed point V and inclined to it at an angle α.

Suppose we rotate the line 'm' around the line 'l' in such a way that the angle 'α' remains constant. Then the surface generated is a double-napped right circular hallow cone here in after referred as cone and extending indefinitely far in both directions.





The point V is called vertex.
The line 'l' is the axis of the cone. The rotating line 'm' is called a generator of the cone. The vertex separates the cone into two parts called nappes.