LINEAR ALGEBRA

~~CSE~~

: IFoS-2013 :

① Find the dimension and a basis of the solution space W of the system

$$x+2y+2z-s+3t=0$$
$$x+2y+3z+s+t=0$$
$$3x+6y+8z+s+5t=0$$

→ The given system can be written as $AX=0$ where $A=\begin{bmatrix}1&2&2&-1&3\\1&2&3&1&1\\3&6&8&1&5\end{bmatrix}$ and $X=\begin{bmatrix}x\\y\\z\\s\\t\end{bmatrix}$. Converting the system into echelon form:

$$\begin{bmatrix}1&2&2&-1&3\\1&2&3&1&1\\3&6&8&1&5\end{bmatrix}\begin{bmatrix}x\\y\\z\\s\\t\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$R_2\to R_2-R_1,\ R_3\to R_3-3R_1$

$$\begin{bmatrix}1&2&2&-1&3\\0&0&1&2&-2\\0&0&2&4&-4\end{bmatrix}\begin{bmatrix}x\\y\\z\\s\\t\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$R_3\to R_3-2R_2$

$$\Rightarrow \begin{bmatrix}1&2&2&-1&3\\0&0&1&2&-2\\0&0&0&0&0\end{bmatrix}\begin{bmatrix}x\\y\\z\\s\\t\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Clearly, the system is in echelon form. Now,

$z+2s-2t=0$ & $x+2y+2z-s+3t=0$

$z=-2s+2t$

$$x=-2y-2z+s-3t$$
$$=-2y-2(-2s+2t)+s-3t$$
$$=-2y+5s-7t$$

$$\therefore \begin{bmatrix}x\\y\\z\\s\\t\end{bmatrix}=\begin{bmatrix}-2y+5s-7t\\y\\-2s+2t\\s\\t\end{bmatrix}=y\begin{bmatrix}-2\\1\\0\\0\\0\end{bmatrix}+s\begin{bmatrix}5\\0\\-2\\1\\0\end{bmatrix}+t\begin{bmatrix}-7\\0\\2\\0\\1\end{bmatrix}$$

∴ Basis of solution space is $\{(-2\ 1\ 0\ 0\ 0)^T, (5\ 0\ -2\ 1\ 0)^T, (-7\ 0\ 2\ 0\ 1)^T\}$

Dim. of solution space = 3.

② Find the characteristic eq^n of the matrix $A=\begin{bmatrix}2&1&1\\0&1&0\\1&1&2\end{bmatrix}$ and hence find the matrix represented by

$A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I$.

→ Char. eq^n of A is given by $|A-\lambda I|=0 \Rightarrow \begin{vmatrix}2-\lambda&1&1\\0&1-\lambda&0\\1&1&2-\lambda\end{vmatrix}=0$

$\Rightarrow (2-\lambda)\left[(1-\lambda)(2-\lambda)\right] + 1(0-0) - 1(\lambda-1) = 0$

$\Rightarrow \lambda^3 - 5\lambda^2 + 7\lambda - 3 = 0$ — (1) which is the required char. equation.

By Cayley Hamilton's Theorem, A satisfies its char. equation (1).

Therefore $A^3 - 5A^2 + 7A - 3I = 0 \Rightarrow A^3 = 5A^2 - 7A + 3I$

Premultiplying with A,

$A^4 = 5A^3 - 7A^2 + 3A$

Similarly :

$A^4 = 5A^3 - 7A^2 + 3A \Rightarrow A^4 - 5A^3 = -7A^2 + 3A$ — (1)

$A^5 = 5A^4 - 7A^3 + 3A^2$

$A^6 = 5A^5 - 7A^4 + 3A^3$

$A^7 = 5A^6 - 7A^5 + 3A^4$

$A^8 = 5A^7 - 7A^6 + 3A^5$

$\Rightarrow A^8 - 5A^7 + 7A^6 - 3A^5 = 0$ — (2)

(1) + (2)

$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I$

$= 0 + (-7)A^2 + 3A + 8A^2 - 2A + I$

$= A^2 + A + I$

$$A^2 + A + I = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 8 & 5 & 5 \\ 0 & 3 & 0 \\ 5 & 5 & 8 \end{bmatrix}$$

(3) Let $\mathbb{F}$ be a subfield of complex numbers and T is a function from $\mathbb{F}^3 \to \mathbb{F}^3$ defined as $T(x_1, x_2, x_3) = (x_1 + x_2 + 3x_3,\ 2x_1 - x_2,\ -3x_1 + x_2 - x_3)$. What are the conditions on (a,b,c) such that (a,b,c) be in the nullspace of T? Also, find the nullity of T.

$\rightarrow N_A(T) = \{(x_1, x_2, x_3) \in F \mid T(x_1, x_2, x_3) = (0,0,0)\}$.

Let $(a,b,c) \in N_A(T)$. Then, $T(a,b,c) = (0,0,0)$.

i.e. $(a+b+3c,\ 2a-b,\ -3a+b-c) = (0,0,0)$

$\Rightarrow a+b+3c = 0, \quad 2a - b = 0, \quad -3a + b - c = 0.$

$2a = b \longrightarrow -3a + 2a - c = 0$

$\Rightarrow c = -a.$

$a + b + 3c = 0 \Rightarrow a + 2a - 3a = 0$ hence it satisfies the values found.

$\therefore$ The required conditions are $b = 2a,\ c = -a$.

i.e. $N_A(T) = \{(a, 2a, -a) \mid a \in \mathbb{F}\}$.

Clearly, the basis of $N_A(T) = \{(1, 2, -1)\}$.

$\therefore$ Nullity $(T) = 1$.

(4) Let V be the vector space of 2x2 matrices over $\mathbb{R}$ and let $M = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$. Let $F: V \to V$ be a linear map defined by $F(A) = MA$. Find a basis and dimension of :

(i) The kernel of W of F.

(ii) The image of U of F.

(i) Kernel $W = \{ A \in V \mid F(A) = 0 \}$.

Let $A \in V$, such that $F(A) = 0 \Rightarrow A \in W$. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

Then $F(A) = 0 \Rightarrow MA = 0 \Rightarrow \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} a-c & b-d \\ -2a+2c & -2b+2d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

On comparing: $a = c$, $b = d$.

$\therefore A = \begin{bmatrix} a & b \\ a & b \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$.

$\therefore$ Basis of Kernel $W = \left\{ \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \right\}$.

Hence $\dim W = 2$.

(ii) Image $U = \{ A \in V \mid F(X) = A,\ X \in V \}$.

Basis of 2×2 matrices over $\mathbb{R}$ can be taken as

$S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

$F\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = M\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix}$

$F\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = M\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$.

$F\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = M\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 2 & 0 \end{bmatrix}$

$F\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = M\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}$.

$\therefore U =$ Span of $\left\{ \begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix} \right\}$.

But U is L.D. set as $\begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} = -\begin{bmatrix} -1 & 0 \\ 2 & 0 \end{bmatrix}$ &

$\begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix} = -\begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix}$.

The only L.I. matrices set in $U = \left\{ \begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -2 \end{bmatrix} \right\}$ or $\left\{ \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 2 & 0 \end{bmatrix} \right\}$.

$\therefore$ Basis of $U = \left\{ \begin{bmatrix} 0 & -1 \\ 0 & 2 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 2 & 0 \end{bmatrix} \right\}$.

Hence $\dim U = 2$

(ii) Inconsistency:

For inconsistency, $Rank(A) \neq Rank(A|B)$ This can only be possible when $Rank(A|B) > Rank(A)$. For this to happen,

$a^2-4a+4=0 \Rightarrow a=2$.

Then, $Rank(A) = 2 < 3 = Rank(A|B)$.

Hence, for $a=2$, we have no solution.

2(c) (b) Let $H = \begin{bmatrix} 1 & i & 2+i \\ -i & 2 & 1-i \\ 2-i & 1+i & 2 \end{bmatrix}$ be a Hermitian matrix. Find a non-singular matrix P such that P^THP is a diagonal matrix.

$R_2 \to R_2 + iR_1$, $R_3 \to R_3 - (2-i)R_1$

$\rightarrow$ $[H|I] = \left[\begin{array}{ccc|ccc} 1 & i & 2+i & 1 & 0 & 0 \\ -i & 2 & 1-i & 0 & 1 & 0 \\ 2-i & 1+i & 2 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & i & 2+i & 1 & 0 & 0 \\ 0 & 1 & i & i & 1 & 0 \\ 0 & -i & -3 & -(2-i) & 0 & 1 \end{array}\right]$

$C_2 \to C_2 - iC_1$
$C_3 \to C_3 - (2+i)C_1$ $\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -i & -(2+i) \\ 0 & 1 & i & i & 2 & 1-2i \\ 0 & -i & -3 & -(2-i) & 2i+1 & 6 \end{array}\right]$

$R_3 \to R_3 + iR_2$ $\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -i & -(2+i) \\ 0 & 1 & i & i & 2 & 1-2i \\ 0 & 0 & -4 & -3+i & 4i+1 & 8+i \end{array}\right]$

$R_2 \to 4R_2 + iR_3$ $\sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -i & -(2+i) \\ 0 & 4 & 0 & i-1 & 4+i & 3 \\ 0 & 0 & -4 & -3+i & 4i+1 & 8+i \end{array}\right]$

$\therefore P = \begin{bmatrix} 1 & -i & -(2+i) \\ i-1 & 4+i & 3 \\ -3+i & 4i+1 & 8+i \end{bmatrix}$ & $P^THP = D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & -4 \end{bmatrix}$

Signature of H: No. of +ve values in diagonal form $-$ No. of $-$ve values in diagonal form

$\Rightarrow 2-1 = 1$