CSE - Real Analysis - 2010

Q.) Discuss the convergence of $\{x_n\}$

$$x_n = \frac{\sin \frac{nx}{2}}{8}$$

Soln: Let us calculate the limit point of x_n

$$\lim_{n\to\infty} x_n = \begin{cases} \frac{1}{8} \; ; & n = 1, 5, \ldots, 4n+1 \\ 0 \; ; & n = 0, 2, \ldots, 2n \\ -\frac{1}{8} \; ; & n = 3, 7, \ldots 2n+3 \end{cases}$$

Since $\{x_n\}$ doesn't converge to a unique limit point $\Rightarrow \{x_n\}$ is NOT convergent

Q) Show that $\{x_n\}$ where

$$x_1 = 5$$
$$x_{n+1} = \sqrt{4 + x_n} \; ; \; n > 1$$

converges to $\frac{1+\sqrt{17}}{2}$

Soln:

$$x_1 = 5$$
$$x_2 = \sqrt{4+5} = 3$$

So $x_2 < x_1$

Assume $x_n > x_{n+1}$

$\Rightarrow \quad 4 + x_n > 4 + x_{n+1}$

$\Rightarrow \quad \sqrt{4+x_n} > \sqrt{4 + x_{n+1}}$

$\Rightarrow \quad x_{n+1} > x_{n+2}$

So, $x_{n+1} > x_{n+2}$ (by induction)

So, $\{x_n\}$ is a monotonically decreasing sequence — (i)

Also, $x_1 > 0$, $x_2 > 0$

$\Rightarrow$ Assume $x_n > 0$

$$\Rightarrow 4 + x_n > 0$$
$$\Rightarrow \sqrt{4 + x_n} > 0$$
$$\Rightarrow x_{n+1} > 0$$

So, $\{x_n\}$ is bounded below. — (ii)

From (i) & (ii), $\{x_n\}$ is a convergent sequence.

Let $\lim\limits_{n \to \infty} x_n = l$

Then $\lim\limits_{n \to \infty} x_{n+1} = \lim\limits_{n \to \infty} \sqrt{4 + x_n}$

$$\Rightarrow l = \sqrt{4 + l}$$
$$\Rightarrow l^2 = 4 + l$$
$$\Rightarrow l^2 - l + 4 = 0$$
$$\Rightarrow l = \frac{+1 \pm \sqrt{1 + 16}}{2}$$
$$\Rightarrow l = \frac{1 + \sqrt{17}}{2}, \quad \frac{1 - \sqrt{17}}{2}$$

But since $x_n > 0$

$$\Rightarrow l = \frac{1 + \sqrt{17}}{2}$$

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Q) $f(x) = \begin{cases} x^2 \sin \frac{1}{x} & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$

Soln : $f'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h}$

$= \lim_{h \to 0} \frac{h^2 \sin 1/h}{h}$

$= \lim_{h \to 0} h \sin 1/h = 0$

and $f'(x) = x^2 \cos\left(\frac{1}{x}\right)\left(\frac{-1}{x^2}\right) + 2x \sin \frac{1}{x} \; ; x \neq 0$

$= 2x \sin \frac{1}{x} - \cos \frac{1}{x} \quad ; \quad x \neq 0$

So,

$f'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x} & ; \quad x \neq 0 \\ 0 & ; \quad x = 0 \end{cases}$

$\because \lim_{x \to 0} \left(2x \sin \frac{1}{x} - \cos \frac{1}{x} \right)$

does not exist $\Rightarrow f'(x)$ is NOT continuous at $x = 0$

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Q) Consider $\sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}$

Find values of x for which it is convergent and also the sum function. Is the convergence uniform?

Soln: Let $a_n = \frac{x^2}{(1+x^2)^n}$

Then $a_{n+1} = \frac{x^2}{(1+x^2)^{n+1}}$

$$\lim_{n\to\infty} \frac{a_n}{a_{n+1}} = \frac{x^2}{(1+x^2)^n} \times \frac{(1+x^2)^{n+1}}{x^2}$$

$$= (1+x^2)$$

For convergence : $1+x^2 > 1 \Rightarrow x^2 > 0$
which is true for all values of x
$\Rightarrow$ It converges for $x \in \mathbb{R}$.

Sum Function $S_n(x) =$

$$x^2 + \frac{x^2}{(1+x^2)} + \frac{x^2}{(1+x^2)^2} + \dots + \frac{x^2}{(1+x^2)^n}$$

$$= x^2 \left[1 + \frac{1}{(1+x^2)} + \frac{1}{(1+x^2)^2} + \dots + \frac{1}{(1+x^2)^n} \right]$$

$$= x^2 \times \frac{1\left[1 - \frac{1}{(1+x^2)^n}\right]}{\frac{1+x^2-1}{(1+x^2)}}$$

$$= x^2 \times \frac{(1+x^2)^n - 1}{(1+x^2)^n} \times \frac{(1+x^2)}{x^2}$$

$$= \frac{(1+x^2)^n - 1}{(1+x^2)^{n-1}}$$

Now,

$$S_n(x) = \frac{(1+x^2)^n - 1}{(1+x^2)^{n-1}}$$

$$\lim_{n\to\infty} S_n(x) = 0 \;;\; x = 0$$

$$\lim_{n\to\infty} \frac{(1+x^2)^n}{(1+x^2)^{n-1}} \;;\; x \neq 0$$

$$= \lim_{n\to\infty} (1+x^2) \;;\; x \neq 0$$

$$= 1 + x^2 \;;\; x \neq 0$$

So, $S(x) = \lim_{n\to\infty} S_n(x) = \begin{cases} 0 \;;\; x = 0 \\ 1 + x^2 \;;\; x \neq 0 \end{cases}$

which is discontinuous

$\Rightarrow$ Convergence is NOT uniform.

Q) $f_n(x) = x^n \;;\; -1 \leq x \leq 1$. Find the limit function. Is the convergence uniform?

Soln: $f(x) = \lim_{n\to\infty} f_n(x) = \begin{cases} 0 \;;\; x \in (-1, 1) \\ 1 \;;\; x = 1 \end{cases}$

For the convergence to be uniform, the limit function $f(x)$ must be continuous, which is not the case here

$\Rightarrow$ Convergence is NOT uniform.

CSE - Real Analysis - 2011

Q) $S = (0,1)$; $f(x) = 1/x$; $0 < x \le 1$. Is f uniformly continuous on S.

Soln: Consider $a_n = \frac{1}{n}$ on S

and $b_n = \frac{1}{n+1}$ on S

Then $|a_n - b_n| = \left|\frac{1}{n} - \frac{1}{n+1}\right| \to 0$

for $n > M$ (Archmedis Theorem)

But

$|f(a_n) - f(b_n)| = 1 \not\to 0$ on S

So, f is NOT uniformly continuous on S.

Q) $f_n(x) = nx(1-x)^n$, $x \in [0,1]$; Examine the uniform convergence on $[0,1]$

Soln: $f_n(0) = f_n(1) = 0$

$\lim_{n \to \infty} f_n(x)$ on $[0,1)$

let $a_n = nx(1-x)^n$

$a_{n+1} = (n+1)x(1-x)^{n+1}$

$\frac{a_{n+1}}{a_n} = \frac{(n+1)}{n}(1-x) = (1-x) < 1$

So, $\lim_{n \to \infty} f_n(x) = 0$ on $(0,1)$

So, the pointwise limit
$f(x) = 0$ on $[0,1]$

Then applying M_n Test:

$M_n = \sup |f_n(x) - f(x)|$ on $[0,1]$

$= \sup |nx(1-x)^n|$ on $[0,1]$

Let $\phi(x) = nx(1-x)^n$
$\phi'(x) = n(1-x)^n + nx(1-x)^{n-1}$
$= n(1-x)^{n-1}[1-x+x]$
$= n(1-x)^{n-1}$
Then $\phi'(x) = 0$ for $x = 1$
and $\phi'(x) > 0$ on $[0,1]$
So, $\sup nx(1-x)^n = 0$

So, $M_n = 0$
and $\lim_{n \to \infty} M_n = 0$

$\Rightarrow f_n(x)$ is Uniformly Convergent

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Q) Shortest distance from origin to hyperbola
$x^2 + 8xy + 7y^2 = 225$

Soln: Let $u = x^2 + y^2$
and $F = x^2 + y^2 + \lambda(x^2 + 8xy + 7y^2 - 225)$
Applying Lagrange's Method of Multipliers:

$$\frac{\partial F}{\partial x} = 0 = \frac{\partial F}{\partial y}$$

So, $2x + \lambda(2x + 8y) = 0$

$\Rightarrow x + \lambda(x + 4y) = 0 \quad - (i)$

$2y + \lambda(8x + 14y) = 0$

$\Rightarrow y + \lambda(4x + 7y) = 0 \quad - (ii)$

$\cancel{(i) \times x + (ii) \times y}$

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Also,

$\cancel{x^2 + 7y^2 + 8xy = 225}$

$\Rightarrow$ Now, (I) $\times x$ + (II) $\times y$

$\Rightarrow u + \lambda(x^2 + 8xy + 7y^2) = 0$

$\Rightarrow u + \lambda(225) = 0$

$\Rightarrow \lambda = \frac{-u}{225}$ — (III)

from (I), (II) and (III):

$$\frac{-x}{x+4y} = \frac{-y}{4x+7y} = -\frac{u}{225}$$

$$\Rightarrow \frac{x+4y}{x} = \frac{225}{u}$$

$$\Rightarrow \frac{225}{u} - 1 = \frac{4y}{x} \quad \text{— (A)}$$

and $\frac{4x+7y}{y} = \frac{225}{u}$

$$\Rightarrow \frac{225}{u} - 7 = \frac{4x}{y} \quad \text{— (B)}$$

Now, (A) $\times$ (B):

$$\frac{(225-u)}{u}\frac{(225-7u)}{u} = 16$$

$\Rightarrow 9u^2 + 1800u - (225)^2 = 0$

$\Rightarrow u = 25, -225$

$\because u = x^2 + y^2$

$\Rightarrow u = 25$

$\Rightarrow$ Shortest distance $= 5$.

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Q) Show that the series for which sum of n terms $f_n(x) = \frac{nx}{1+n^2x^2}$; $0 \le x \le 1$ can't be differentiated term by term at $x = 0$. What happens at $x \neq 0$?

Soln: $\lim_{n \to \infty} f_n(x) = f(x) = 0$

Let $M_n = \sup |f_n(x) - f(x)|$

$$= \sup \left| \frac{nx}{1+n^2x^2} \right|$$

Assume $\phi(x) = \frac{nx}{1+n^2x^2}$

$$\phi'(x) = \frac{n(1+n^2x^2) - nx \times 2n^2x}{(1+n^2x^2)^2}$$

$$= \frac{n + n^3x^2 - 2n^3x^2}{(1+n^2x^2)^2}$$

$$= \frac{n(1-n^2x^2)}{(1+n^2x^2)^2} = 0 \text{ for } x = \frac{1}{n}$$

in $[0,1]$

and $\phi'(x) > 0$ for $x < 1/n$

and $\phi'(x) < 0$ for $x > 1/n$

So, $M_n = \frac{n \cdot 1/n}{2} = \frac{1}{2}$

and $\lim_{n \to \infty} M_n = \frac{1}{2} \neq 0$

So, $f_n(x)$ is NOT uniformly continuous on $[0,1]$ $\Rightarrow$ It can't be differentiated term by term.

on $x \in (0,1]$

$\because f'(x) > 0$ for $x < 1/n$

and it reduces to less than 0 after $x > 1/n$

$\Rightarrow \sup f_n(x) = \dfrac{n\epsilon}{1+n^2\epsilon^2}$

where $\epsilon \to 0$ on $(0,1]$

$\Rightarrow M_n = \sup f_n(x) = 0$

So, $f_n(x)$ is uniformly convergent on an interval NOT containing zero.

$\Rightarrow$ It can be differentiated term by term.

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Q) Show that if :

Q) $S(x) = \sum_{n=1}^{\infty} \frac{1}{n^3 + n^4 x^2}$, then

$S'(x) = -2x \sum_{n=1}^{\infty} \frac{1}{(1+nx^2)^2}$ for all x

Soln: $S'(x) = \sum_{n=1}^{\infty} \frac{-2x}{(1+nx^2)^2}$

Then $\left| \frac{-2x}{(1+nx^2)^2} \right| = \left| \frac{2x}{(1+nx^2)^2} \right|$

and

$$\left| \frac{2x}{(1+nx^2)^2} \right| \leq \left| \frac{2x}{n^2 x^4} \right| \sim \frac{1}{n^2}$$

and $\sum \frac{1}{n^2}$ is convergent (P test)

So, $S'(x)$ is uniformly convergent (by Weirestrass M. Test)

So, $S(x)$ can be differentiated term by term

$$\Rightarrow S'(x) = \frac{d}{dx} \sum_{n=1}^{\infty} \frac{1}{n^3+n^4x^2}$$

$$= \sum_{n=1}^{\infty} \frac{d}{dx} \left(\frac{1}{n^3+n^4x^2} \right)$$

$$= S'(x)$$