Forest 2016

PDE

(5a) Obtain the PDF governing the eq^n $\phi(u,v)=0$, $u = xyz$, $v = x+y+z$.

$\cancel{\frac{\partial u}{\partial x} = yz}$; $\cancel{\frac{\partial u}{\partial y} = xz}$; $\cancel{\frac{\partial u}{\partial z} = xy}$

$\cancel{\frac{\partial v}{\partial x} = 1}$; $\cancel{\frac{\partial v}{\partial y} = 1}$; $\cancel{\frac{\partial v}{\partial z} = 1}$

$\phi(u) = v$
or $\phi(v) = u$

The above eq^n can be written as.

$$xyz = \phi(x+y+z) \quad \text{—— } (1)$$

Diff. (1) wrt. x,

$$yz = \phi'(x+y+z)\left(1+\frac{\partial z}{\partial x}\right)$$

$$yz = \phi'(x+y+z)(1+p)$$

$$yz/(1+p) = \phi'(x+y+z) \quad \text{—— } (2)$$

Diff. (1) wrt y.

$$xz = \phi'(x+y+z)\left(1+\frac{\partial z}{\partial y}\right)$$

$$xz = \phi'(x+y+z)(1+q)$$

$$xz/(1+q) = \phi'(x+y+z) \quad \text{—— } (3)$$

Equating (2) & (3),

$$\frac{yz}{1+p} = \frac{xz}{1+q}$$

$$y\cancel{z}(1+q) = x\cancel{z}(1+p) \Rightarrow y + yq = x + xp$$

$$\Rightarrow xp - yq = y - x$$

$$\cancel{yz + yzq = xz + xzp}$$

$$\boxed{Pp + Qq = R}$$

$xzp - yzp$

which is the required PDE.

5(b) Find the general solⁿ of PDE $xy^2\frac{\partial z}{\partial x} + y^3\frac{\partial z}{\partial y} = (3xy^2 - 4x^3)$

ie. $xy^2 p + y^3 q = (3xy^2 - 4x^3)$ —— (1)

Clearly which is in the form of $Pp + Qq = R$.

here $P = xy^2$; $Q = y^3$; $R = 3xy^2 - 4x^3$

Now the Lagrange's Auxiliary eqⁿs of (1) are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

ie. $$\frac{dx}{xy^2} = \frac{dy}{y^3} = \frac{dz}{3xy^2 - 4x^3} \quad \text{—— (2)}$$

Taking first two fraction of (2), we get

$$\frac{dx}{xy^2} = \frac{dy}{y^3}$$

ie $\frac{dx}{x} = \frac{dy}{y}$ on integrating, we get

$\log x = \log y + \log c_1$

ie $x = yc_1$

$\frac{x}{y} = c_1$ —— (A)

Taking last two fractions of (2), we get

$$\frac{dy}{y^3} = \frac{dz}{3xy^2 - 4x^3}$$

Let $x = c_1 y$

$$\frac{dy}{y^3} = \frac{dz}{3c_1y^3 - 4c_1^3y^3} \Rightarrow \frac{dy}{y^3} = \frac{dz}{y^3(3c_1 - 4c_1^3)}$$

$y = \log(3c_1 - 4c_1^3) + \log c_2$ (B)

$e^y = (3c_1 - 4c_1^3)c_2 \Rightarrow c_2 = \frac{e^y}{3c_1 - 4c_1^3}$

Ⓐ & Ⓑ are required general solⁿ of

$f(u,v)=0$

ie $f\left(\frac{x}{y}, \frac{e^y}{3c_1-4c_1^3}\right)=0$.

Q6(a) Find the general solution of PDE $xy^2p+y^3q=(3xy^2-4x^3)$

$\left(p=\frac{\partial z}{\partial y}, q=\frac{\partial z}{\partial y}\right)$

6(b) Find the particular integral of $\frac{\partial^2 z}{\partial x^2}-2\frac{\partial^2 z}{\partial x\partial y}+\frac{\partial^2 z}{\partial y^2}=2x\cos y$

or $r-2s+t=2x\cos y$

Solⁿ Given eqⁿ in symbolic form is .

$(D^2-2DD'+D'^2)z=2x\cos y$

$PI=\frac{1}{D^2-2DD'+D'^2}\cdot 2x\cos y$

AE :- $m^2-2m+1=0$

$m^2-m-m+1=0$

$m(m-1)-1(m-1)=0$

$(m-1)^2=0$

$m=1,1$

$$\boxed{\begin{array}{ll} D=mD' & m=1,1 \\ y=c-mx & \\ \;\;=c-x & \end{array}}$$

ie.

$PI=\frac{1}{(D-D')(D-D')}\cdot 2x\cos y=\frac{1}{(D-D')}\left[\frac{1}{(D-D')}2x\cos y\right]$

$=\frac{1}{(D-D')}\int 2x\cos(c-x)\,dx$

$=\frac{1}{(D-D')}\left[-2x\sin(c-x)-\int(-2\sin(c-x))\,dx\right]$

$=\frac{1}{(D-D')}\left[-2x\sin(c-x)+\cancel{2}\cos(c-x)\right]$

$=\frac{1}{(D-D')}(-2x\sin y+\cos y)$

$$= \int \left[2x \sin(c-x) + \cos(c-x) \right] dx$$

$$= (-2)\left(-x\cos(c-x) - \int \cos(c-x)\,dx\right) + \left(-\sin(c-x)\right)$$

$$= -2x\cos(c-x) + \sin(\cancel{c-x}) - \sin(\cancel{c-x})$$

$$= -2x\cos((c-x))$$

$$= -2x\cos y$$

6 (c) A uniform rod of length L whose surface is thermally insulated is initially at temperature $\theta = \theta_0$. At time $t=0$, one end is suddenly cooled at $\theta = 0$ and subsequently maintained at this temperature; the other end remains thermally insulated. Find the temperature distribution $\theta(x,t)$.

CSE 2016

5(a) Find the General equation of surfaces orthogonal to the family of spheres given by $x^2+y^2+z^2 = cz$.

Let $f = \dfrac{x^2+y^2+z^2}{z}$

The Surfaces orthogonal to the above system are the surfaces generated by the integral curves of the equations

$$\frac{dx}{\partial f/\partial x} = \frac{dy}{\partial f/\partial y} = \frac{dz}{\partial f/\partial z}$$

$$\Rightarrow \quad \frac{dx}{} = \frac{dy}{} = \frac{dz}{}$$

which have solutions

Thus any surface which is orthogonal to the given surfaces has equation of the form

5(e) Find the general integral of the PDE

$$(y+zx)p - (x+yz)q = x^2 - y^2 \quad \text{— (1)}$$

Eq (1) is clearly of the form $Pp + Qq = R$

Here $P = y+zx$, $Q = -(x+yz)$, $R = x^2 - y^2$

Now the Lagrange's auxiliary eqns of (1) are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

ie. $$\frac{dx}{y+zx} = \frac{dy}{-(x+yz)} = \frac{dz}{x^2-y^2} \quad \text{— (2)}$$

~~using the multipliers $x, y, -z$~~

~~each fraction of (2) $= \frac{x\,dx + y\,dy - z\,dz}{xy + zx^2 - xy + y^2z - zx^2 + zy^2}$~~

using the multipliers, $x, y, -z$

each fraction of (2) $= \dfrac{x\,dx + y\,dy - z\,dz}{\cancel{xy} + \cancel{zx^2} - \cancel{xy} - \cancel{zy^2} - \cancel{x^2z} + \cancel{y^2z}} = 0$

$\Rightarrow x\,dx + y\,dy - z\,dz = 0$

On Integrating, we get

$$\boxed{x^2 + y^2 - z^2 = 2C_1}$$

(A)

$x, 1, +1, 0$

$= \dfrac{-dx + dy}{\cancel{-x+yz-x^2+y^2}}$

$-dy$

$+x \pm yz \quad -y-zx-x+y z$

$+z(x/y)$

$y, x, 1$

$y\,dx + x\,dy + dz$

$0, -1, 1, R$

using the multipliers $y, x, 1$, we get

$$\frac{y\,d\cancel{y}x + x\,dy + dz}{\cancel{y^2} + \cancel{xyz} - \cancel{x^2} - \cancel{xyz} + \cancel{x^2} - \cancel{y^2}} = 0$$

$\Rightarrow \quad y\,dx + x\,dy + dz = 0$

on integrating, we get

$$\boxed{xy + z = C_2} \quad \text{(B)}$$

$\therefore$ from (A) and (B), the required general solution of (1) is $f(u, v) = 0$

$u(x,y,z) = c_1$
$v(x,y,z) = c_2$
be 2 independent soln of (5).

ie. $f(x^2 + y^2 - z^2,\ xy + z) = 0$

6(a) Determine the characteristics of the equation $z = p^2 - q^2$ and find the integral surface which passes through the parabola $4z + x^2 = 0$, $y = 0$.

Let $f(x, y, z, p, q) = z - p^2 + q^2 = 0$ —(1)

we are to find integral surface of (1) which passes through the parabola $4z + x^2 = 0$ & $y = 0$ whose parametric eqn's are $x = 2\lambda$, $y = 0$, $z = -\lambda^2$

~~Let~~ ie. $x_0 = f_1(\lambda) = 2\lambda$; $y_0 = f_2(\lambda) = 0$ and $z_0 = f_3(\lambda) = -\lambda^2$ be initial values.

Now we find the initial values p_0 and q_0 by the relations $f_3'(\lambda) = p_0 f_1'(\lambda) + q_0 f_2'(\lambda)$ and

$f(f_1(\lambda), f_2(\lambda), f_3(\lambda), p_0, q_0) = 0$

ie. $f(2\lambda, 0, -\lambda^2, p_0, q_0) = 0$

$-2\lambda = 2P_0 + 0$ & $-\lambda^2 - (-\lambda)^2 + q_0^2 = 0$

$\Rightarrow P_0 = -\lambda$ $\quad -\lambda^2 - \lambda^2 + q_0^2 = 0$

$q_0^2 = 2\lambda^2 \Rightarrow q_0 = \pm\sqrt{2}\lambda$

Now characteristic eqⁿˢ of (1) are

$x'(t) = f_p = -2p$ — (2)

$y'(t) = f_q = 2q$ — (3)

$z'(t) = pf_p + qf_q = -2(p^2 - q^2) = -2z$ — (4)

$p'(t) = -(f_x + pf_z) = -p$ — (5)

$q'(t) = -(f_y + qf_z) = -q$ — (6)

Now $x'(t) = -2p$ (using 2 & 5)

$= 2p'(t)$

$dx = 2\,dp$

$x = 2p + C_1$ — (7)

$\boxed{\Rightarrow x = 2C_3e^{-t} + C_1}$

(using (3) & (6))

$y'(t) = 2q = -2q'(t)$

$dy = -2\,dq$

$y = -2q + C_2$ — (8)

$\boxed{\Rightarrow y = -2C_4e^{-t} + C_2}$

Now using the intial values in (7) & (8), we get

$x_0 = 2P_0 + C_1$ $\quad y_0 = -2q_0 + C_2$

$2\lambda = (-2\lambda) + C_1$ $\quad 0 = -2(\pm\sqrt{2}\lambda) + C_2$

$\Rightarrow C_1 = 4\lambda$ $\quad C_2 = \pm 2\sqrt{2}\lambda$

using ⑤,

$p'(t) = -p$

$\frac{dp}{dt} = -p$

$\frac{dp}{p} = -dt$

$\log p = -t + \log c_3$

$p = c_3 e^{-t}$ —— ⑨

using ⑥

$q'(t) = -q$

$\Rightarrow q = c_4 e^{-t}$ —— ⑩

using ④

$z'(t) = -2z$

$\frac{dz}{dt} = -2z$

$\frac{dz}{z} = -2\,dt$

$\log z = -2t + \log c_5$

$z = c_5 e^{-2t}$ —— ⑪

Again using initial values in ⑨, ⑩ & ⑪, we get (at t=0)

$c_3 = -\lambda \quad , \quad c_4 = \pm\sqrt{2}\,\cancel{\lambda} \quad , c_5 = -\lambda^2$

∴ characteristic curves passing through the intial curve are given by

$x(\lambda, t) = -2\lambda e^{-t} + 4\lambda$
$= 2\lambda(2 - e^{-t})$

$y(\lambda, t) = -2(\pm\sqrt{2}\lambda)e^{-t} + \pm 2\sqrt{2}\lambda$
$= \pm 2\sqrt{2}\lambda(1 - e^{-t})$

$z(\lambda, t) = -\lambda^2 e^{-2t}$

$\Rightarrow \boxed{4z + (x \pm \sqrt{2}y)^2 = 0}$ is the required surface.

7 (a) Solve the Pde $\frac{\partial^3 z}{\partial x^3} - \frac{2\partial^3 z}{\partial x^2 \partial y} - \frac{\partial^3 z}{\partial x \partial y^2} + \frac{2\partial^3 z}{\partial y^3} = e^{x+y}$

In