Chapter 4

2017

4.1 Section-A

Question-1(a) **Let** A **be a square matrix of order 3 such that each of its diagonal elements is '**a**' and each of its off-diagonal elements is 1. If** $\mathrm{B} = \mathrm{bA}$ **is orthogonal, determine the values of** a **and** b**.**

[8 Marks]

Solution: Given,

$$A = \begin{bmatrix} a & 1 & 1 \\ 1 & a & 1 \\ 1 & 1 & a \end{bmatrix} \quad \therefore B = bA = \begin{bmatrix} ba & b & b \\ b & ba & b \\ b & b & ba \end{bmatrix}$$

B is orthogonal i.e., $BB^T = I$

$$\therefore \begin{bmatrix} ba & b & b \\ b & ba & b \\ b & b & ba \end{bmatrix} \begin{bmatrix} ba & b & b \\ b & ba & b \\ b & b & ba \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore \quad b^2a^2 + 2b^2 = 1$$

and

$$2b^2a + b^2 = 0$$

$$b^2\left(a^2 + 2\right) = 1$$

and

$$b^2\left(2a + 1\right) = 0$$

$$\therefore b^2 = 0$$

or

$$\left(2a^2 + 1\right) = 0$$

But $b = 0$ is not possible, as first equation will not be satisfied

$$\therefore \quad a = \frac{-1}{2} \quad \Rightarrow \quad b^2\left(\frac{1}{4} + 2\right) = 1 \Rightarrow b = \pm\frac{2}{3}$$

.

Question-1(b) Let V be the vector space of all 2×2 matrices over the field R. Show that W is not a subspace of V, where (i) W contains all 2×2 matrices with zero determinant. (ii) W consists of all 2×2 matrices A such that $A^2 = A$.

[8 Marks]

Solution: (i)

$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in R \right\} = M_2(R)$$

It is a vector space over field R. W= Set of all 2×2 matrices with determinant zero. Let

$$w_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

and

$$w_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

det

$$(w_1) = 0 = \det(\omega_2) \quad \therefore \omega_1, w_2 \in W$$

But

$$w_1 + w_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\det(\omega_1 + \omega_2) = 1 \neq 0$$

$$\therefore w_1 + w_2 \notin W$$

(ii) W consists of all 2×2 matrices A such that $A^2 = A$. Let,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad , \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$A^2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, B^2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\therefore \quad A, B \in W$$

But

$$A + B = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$$

$$(A+B)^2 = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix} \neq (A+B)$$

$$\therefore (A+B) \notin W$$

Hence, W is not a subspace of V

Question-1(c) Using the Mean Value Theorem, show that (i) $f(x)$ is constant in $[a, b]$, if $f'(x) = 0$ in $[a, b]$
(ii) $f(x)$ is a decreasing function in (a, b), if $f'(x)$ exists and is < 0 everywhere in (a, b)

[8 Marks]

Solution: (i) Let x_1 and x_2 be two distinct points in interval $[a, b]$, and let

$$x_1 < x_2$$

.

$$\therefore [x_1, x_2] \subseteq [a, b]$$

Then f is continuous on $[x_1, x_2]$ and f is differential on $[x_1, x_2]$ Using LMVT, there exist some,

$$c \in [x_1, x_2]$$

such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$$

ie.

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0 \quad [\because f'(x) = 0 \quad \forall x \in [a, b]]$$

$\Rightarrow$

$$f(x_2) - f(x_1) = 0$$

ie

$$f(x_1) = f(x_2)$$

Hence, $f(x)$ is constant function. as x_1 and x_2 were arbitrary in $[a, b]$.
(ii) Let x_1 and x_2 be any two distinct points in $[a, b]$ and $x_1 < x_2$

$$\therefore [x_1, x_2] \subseteq [a, b]$$

f is differentiable on (a, b), hence it is differentiable on

$$(x_1, x_2) \subseteq (a, b)$$

and continuous also. Using LMVT, there exist some $c \in (x_1, x_2)$ such that

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$$

$$f(x_2) - f(x_1) = f'(c) \cdot (x_2 - x_1)$$

Now, since

$$x_2 > x_1 \quad \therefore \quad (x_2 - x_1) > 0$$

and $f'(x) < 0$ on (a, b)

$$\therefore f'(c) < 0$$

as

$$c \in (x, x_2) \subseteq (a, b)$$

$$\therefore f(x_2) - f(x_1) < 0$$

$$x_1 < x_2 \Rightarrow f(x_1) > f(x_2)$$

$\therefore f(x)$ is decreasing function on (a, b)

Question-1(d) **Jacobian** $\mathrm{J} = \frac{\partial(\mathrm{u}, \mathrm{v})}{\partial(\mathrm{x}, \mathrm{y})}$**, and hence show that** $\mathrm{u}, \mathrm{v}.$ **are independent unless**

$$\frac{a}{A} = \frac{b}{B} = \frac{h}{H}$$

[8 Marks]

Solution:

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$= \begin{vmatrix} 2ax + 2hy & 2hx + 2by \\ 2Ax + 2Hy & 2Hx + 2By \end{vmatrix}$$

$$= 4(ax + 2hy)(Hx + By) - 4(Ax + Hy)(hx + by)$$

$$= 4\left[\left(aHx^2 + xy(hH + aB) + hBy^2\right) - \left(Ahx^2 + xy(Hh + Ab) + bHy^2\right)\right]$$

$$= -4\left[(aH - Ah)x^2 + (aB - Ab)xy + (Bh - bH)y^2\right]$$

u and v are independent, if $J = 0$ ie

$$aH - Ah = 0; \quad aB - Ab = 0, \quad Bh - bH = 0$$

$$\therefore \quad \frac{a}{A} = \frac{h}{H}; \quad \frac{a}{A} = \frac{b}{B}, \quad \frac{h}{H} = \frac{b}{B}$$

ie

$$\frac{a}{A} = \frac{b}{B} = \frac{h}{H}$$

Question-1(e) **Find the equations of the planes parallel to the plane** $3x - 2y + 6z + 8 = 0$ **and at a distance 2 from it.**

[8 Marks]

Solution: Equation of any plane parallel to given plane is

$$3x - 2y + 6z + k = 0$$

Distance between two planes

$$\frac{|k - 8|}{\sqrt{9 + 4 + 36}} = 2$$

ie

$$|k - 8| = 14$$

$$\therefore \quad k - 8 = 14 \quad or \quad k - 8 = -14$$

$$k = 22 \quad or \quad k = -6$$

Hence, required equations of planes are

$$3x - 2y + 6z + 22 = 0$$

or

$$3x - 2y + 6z - 6 = 0$$

Question-2(a) State the Cayley-Hamilton theorem. Verify this theorem for the matrix

$$\mathbf{A} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$ **. Hence find** A^{-1}

[10 Marks]

Solution: Every square matrix satisfies its characteristic equation, given by, $|A - \lambda I| = 0$ from the given matrix

$$\begin{vmatrix} 1-\lambda & 0 & 2 \\ 0 & -1-\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\left(\lambda^2 + \lambda - 1\right) = 0$$

$\Rightarrow$

$$\lambda^3 - 2\lambda + 1 = 0 - (1)$$

Now,

$$A^2 = \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & -1 \\ 0 & -1 & +1 \end{bmatrix}$$

$$A^3 = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 4 \\ 0 & -3 & 2 \\ 0 & 2 & -1 \end{bmatrix}$$

$$A^3 - 2A + I = \begin{bmatrix} 1 & 0 & 4 \\ 0 & -3 & 2 \\ 0 & 2 & -1 \end{bmatrix} - 2 \begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= 0$$

Hence, A satisfies its characteristic equation, given by (1).

Now, To find A^{-1},

$$A^3 - 2A + I = 0$$
$$A^{-1}A^3 - 2A^{-1}A + A^{-1} = 0$$
$$A^{-1} = -A^2 + 2I$$

$$A^{-1} = -\begin{bmatrix} 1 & 2 & 2 \\ 0 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} 0 & -2 & -2 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Question-2(b) Show that

$$\int_0^{\pi/2} \sin^p\theta\cos^q\theta d\theta = \frac{1}{2}\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}, p, q > -1$$

Hence, evaluate the following integrals:

(i) $\int_0^{\pi/2}\sin^4 x\cos^5 x dx$

(ii) $\int_0^1 x^3\left(1-x^2\right)^{5/2}dx$

(iii) $\int_0^1 x^4(1-x)^3 dx$

[10 Marks]

Solution: We define,

$$T(m) = \int_0^\infty x^{m-1}\cdot e^{-x}dx, m > 0$$

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx, \quad m,n > 0$$

We put,

$$\sin^2\theta = x \quad \therefore \quad 2\sin\theta\cos\theta d\theta = dx$$

$$\int_0^{\pi/2}\sin^2\theta\cos^2\theta d\theta = \int_0^{\pi/2}\left(\sin^2\theta\right)^{\frac{p-1}{2}}\cdot\left(cos^2\theta\right)^{\frac{q-1}{2}}\cdot\sin\theta\cos\theta d\theta$$

$$= \int_0^1 x^{\frac{p-1}{2}}(1-x)^{\frac{q-1}{2}}\cdot\frac{dx}{2}$$

$$= \frac{1}{2}\beta\left(\frac{p+1}{2},\frac{q+1}{2}\right)$$

$$= \frac{1}{2}\cdot\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)}$$

$$\left[\because \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\right]$$

(i)

$$\int_0^{\pi/2} \sin^4 x \cos^5 x dx = \frac{1}{2} \cdot \frac{\Gamma\left(\frac{4+1}{2}\right)\Gamma\left(\frac{5+1}{2}\right)}{\Gamma\left(\frac{4+5+2}{2}\right)}$$

$$= \frac{1}{2} \cdot \frac{\Gamma\left(\frac{5}{2}\right) \cdot \Gamma(3)}{\Gamma(\frac{1}{2})}$$

$$\left(\because \Gamma(m) = (m-1)!, \quad \Gamma(\frac{1}{2}) = \sqrt{\pi}, \quad \Gamma(m) = (m-1)\Gamma(m-1)\right)$$

$$= \frac{\frac{1}{2} \times \frac{3}{2} \times \frac{1}{2} \times \Gamma\left(\frac{1}{2}\right) \cdot 2!}{\frac{9}{2} \times \frac{7}{2} \times \frac{5}{2} \times \frac{3}{2} \times \frac{1}{2} \times \Gamma(\frac{1}{2})}$$

$$= \frac{8}{315}$$

(ii) Put,

$$x^2 = y \quad \Rightarrow \quad 2xdx = dy$$

or

$$dx = \frac{1}{2x} dy$$

$$\int_0^1 x^3 \left(1 - x^2\right)^{5/2} dx = \int_0^1 y(1-y)^{5/2} \frac{dy}{2}$$

$$= \frac{1}{2}\beta\left(1+1, \frac{5}{2}+1\right) = \frac{1}{2}\beta\left(2, \frac{7}{2}\right)$$

$$= \frac{1}{2} \frac{\Gamma(2) \cdot \Gamma(\frac{7}{2})}{\Gamma\left(2 + \frac{7}{2}\right)} = \frac{1}{2} \cdot \frac{\Gamma(2)\Gamma(\frac{7}{2})}{\Gamma(\frac{11}{2})}$$

$$= \frac{1}{2} \cdot \frac{(1)!\Gamma(\frac{7}{2})}{\frac{9}{2} \cdot \frac{7}{2} \cdot \Gamma(\frac{7}{2})} = \frac{2}{63}$$

(iii)

$$\int_0^1 x^4 (1-x)^3 dx = \beta(4+1, 3+1) = \beta(5,4)$$

$$= \frac{\Gamma(5)\Gamma(4)}{\Gamma(5+4)} = \frac{\Gamma(5) \cdot 3!}{8 \times 7 \times 6 \times 5 \cdot \Gamma(5)}$$

$$= \frac{1}{280}$$

Question-2(c) Find the maxima and minima for the function

$$f(x, y) = x^3 + y^3 - 3x - 12y + 20$$

Also find the saddle points (if any) for the function.

[10 Marks]

Solution:

$$f_x = 3x^2 - 3, \quad f_y = 3y^2 - 12$$

for critical points, $f_x = 0$ and $f_y = 0$

$$\therefore x = \pm 1, \quad y = \pm 2$$

The function has four stationary points

$$(1, 2), (-1, 2), (1, -2), (-1, -2)$$

Now,

$$f_{xx} = 6x \quad , \quad f_{yy} = 6y \quad , f_{xy} = 0$$

At

$$(1, 2), \quad f_{xx} = +6 > 0, \quad f_{yy} = 12 > 0, f_{xy} = 0$$
$$f_{xx} \cdot f_{yy} - f_{xy} = 6 \cdot 12 - 0 = 72 > 0$$
$$\therefore \quad (1, 2)\text{is point of minima}$$

. At $(-1, 2)$

$$f_{xx} = -6 < 0, \quad f_{yy} = 12 > 0, \quad f_{xy} = 0$$
$$f_{xx} \cdot f_{yy} - f_{xy}^2 = -6 \times 12 = -72 < 0$$

function is neither maximum, nor minimum at $(-1, 2)$ At (1,-2)

$$f_{xx} = 6 > 0, \quad fy_y = -12 < 0, \quad f_{xy} = 0$$
$$F_{xx} \cdot f_{yy} - f_{xy}^2 = -72 < 0$$

function is neither maximum, nor minimum at $(1, -2)$ $At(-1, -2)$

$$f_{xx} = -6 < 0, \quad f_{yy} = -12 < 0, \quad f_{xy} = 0$$
$$f_{xx} \cdot f_{yy} - f_{xy}^2 = 72 > 0$$

(-1,-2) is print of maxima. Point of Maxima is (-1,-2) Point of Minima is (1,2) Stationary points like (-1,2) and (1,-2) which are not extreme points are saddle points. Result Used:

$$f(a, b) \text{ is an extreme value of } f(x, y),$$

if

$$f_x(a, b) = 0 = f_y(a, b),$$

and

$$f_{xx} \cdot f_{yy} - f_{xy}^2 > 0 \text{ at } (a, b)$$

and this extreme value is maximum if

$$f_{xx}(a, b) < 0$$

or minimum if

$$f_{xx}(a, b) > 0$$

Question-2(d) Show that the angles between the planes given by the equation $2x^2 - y^2 + 3z^2 - xy + 7zx + 2yz = 0$ is $\tan^{-1} \frac{\sqrt{50}}{4}$.

[10 Marks]

Solution: Let the equations of two planes be

$$a_1x + b_1y + z = 0$$

and

$$a_2x + b_2y + 3z = 0$$

($\because$ planes passes through origin).
Now the combined equation is

$$(a_1x + b_1y + z)(a_2x + b_2y + 3z) = 0$$

$$a_1a_2x^2 + b_1b_2y^2 + 3z^2 + xy\left(a_1b_2 + a_2b_1\right) + xz\left(3a_1 + a_2\right) + yz\left(b_2 + 3b_1\right) = 0$$

Comparing the coefficients in the given eqn

$$a_1a_2 = 2, \quad b_1b_2 = -1, a_1b_2 + a_2b_1 = -1$$

$$3a_1 + a_2 = 7, \quad 3b_1 + b_2 = 2$$

$\Rightarrow$

$$a_1 = 2, \quad a_2 = 1 \quad ; \quad b_1 = 1, \quad b_2 = -1$$

Equations of planes are

$$2x + y + z = 0; \quad x - y + 3z = 0$$

Angle between the planes

$$\cos\theta = \frac{2 \cdot 1 + 1(-1) + 1(3)}{\sqrt{4+1+1}\sqrt{1+1+9}} = \frac{4}{\sqrt{6} \cdot \sqrt{11}} = \frac{4}{\sqrt{66}}$$

$$\therefore \tan\theta = \frac{\sqrt{50}}{4} \quad \therefore \quad \theta = \tan^{-1} \frac{\sqrt{50}}{4}$$

Alternate solution: Let θ be the angle between pair of planes given by the general homogeneous equation of second degree

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

$$\Rightarrow \tan\theta = \tfrac{2\sqrt{(f^2+g^2+h^2-ab-bc-ca)}}{a+b+c}$$

Here, equation of pair of planes is,

$$2x^2 - y^2 + 3z^2 - xy + 7zx + 2yz = 0$$

$$a = 2, b = -1, c = 3, f = 1, g = \frac{7}{2}, h = \frac{-1}{2}$$

$$\therefore \tan\theta = \frac{2\sqrt{\left(1 + \frac{49}{4} + \frac{1}{4} + 2 + 3 - 6\right)}}{2 - 1 + 3}$$

$$= \frac{2}{4}\sqrt{\left(\frac{50}{4}\right)} = \frac{\sqrt{50}}{4}$$

$$\therefore \quad \theta = \tan^{-1} \frac{\sqrt{50}}{4}$$

Question-3(a) Reduce the following matrix to a row-reduced echelon form and hence find its rank:

$$\mathbf{A} = \begin{bmatrix} -1 & 2 & -1 & 0 \\ 2 & 4 & 4 & 2 \\ 0 & 0 & 1 & 5 \\ 1 & 6 & 3 & 2 \end{bmatrix}$$

[10 Marks]

Solution:

$$A = \begin{bmatrix} -1 & 2 & -1 & 0 \\ 2 & 4 & 4 & 2 \\ 0 & 0 & 1 & 5 \\ 1 & 6 & 3 & 2 \end{bmatrix}$$

$$R_2 \to R_2 + 2R_1 \quad ; \quad R_4 \to R_4 + R_1$$

$$A \sim \begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & 8 & 2 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 8 & 2 & 2 \end{bmatrix}$$

$$R_4 \to R_4 - R_2$$

$$\sim \begin{bmatrix} -1 & 2 & -1 & 0 \\ 0 & 8 & 2 & 2 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Above form is row echelon form of A. we have 3 non-zero rows

$$\therefore \quad Rank(A) = 3$$

.

Question-3(b) Given that the set $\{u, v, w\}$ is linearly independent, examine the sets
$(i)\{\mathbf{u}+\mathbf{v}, \mathbf{v}+\mathbf{w}, \mathbf{w}+\mathbf{u}\}$
$(ii)\{\mathbf{u}+\mathbf{v}, \mathbf{u}-\mathbf{v}, \mathbf{u}-2\mathbf{v}+2\mathbf{w}m\}$ **for linear independence.**

[10 Marks]

Solution: Set of vectors $\{v_1, v_2, \ldots v_n\}$ is L.I. if

$$c_1v_1 + c_2v_2 + \ldots + c_nv_n = 0$$

implies that

$$c_1 = c_2 .... = c_n = 0$$

(i) Let $a, b, c \in R$.

Consider,

$$a(u+v)+b(v+w)+c(\omega+u)=0$$

i.e.

$$(a+c)u+(a+b)v+(b+c)w=0$$

Since $\{u, v, \omega\}$ are L.I.

$$\therefore a+c=0, \quad a+b=0 \quad , \quad b+c=0$$

Adding these three,

$$2(a+b+c)=0$$

. Hence,

$$a=0, b=0, c=0$$

$\therefore$ Given set is L.I.

(ii) Again let $a, b, c \in R$ and consider

$$a(u+v)+b(u-v)+c(u-2v+2w)=0$$

$$(a+b+c)u+(a-b-2c)v+2cw=0$$

$$\Rightarrow a+b+c=0 \qquad [\because \{u, v, \omega\} \text{ are L.I}]$$

$$a-b-2c=0$$

$$\therefore 2c=0 \qquad \therefore \quad c=0 \Rightarrow a=0; b=0$$

Hence, given set is $L.I.$

Question-3(c) Evaluate the integral $\int_0^\infty \int_0^\infty \mathrm{e}^{-\left(x^2+y^2\right)} dxdy$, by changing to polar coordinates. Hence show that $\int_0^\infty \mathrm{e}^{-\mathrm{x}^2} \mathrm{dx} = \dfrac{\sqrt{\pi}}{2}$.

[10 Marks]

Solution: The region of integration is first quadrant of xy-plane.
Hence $'r'$ varies from 0 to ∞ and θ varies from 0 to $\dfrac{\pi}{2}$.

$$\begin{aligned} I &= \int_0^\infty \int_0^\infty e^{-\left(x^2+y^2\right)} dxdy \\ &= \int_0^{\pi/2} \int_0^\infty e^{-r^2} \cdot r dr d\theta = -\frac{1}{2} \int_0^{\pi/2} \left[e^{-r^2}\right]_0^\infty d\theta \\ &= \frac{-1}{2} \int_0^{\pi/2} (0-1) d\theta = \frac{1}{2} \times \frac{\pi}{2} = \frac{\pi}{4} \end{aligned}$$

Also,

$$\begin{aligned} I &= \int_0^\infty e^{-x^2} dx \times \int_0^\infty e^{-y^2} dy \\ &= \left(\int_0^\infty e^{-x^2} dx\right)^2 = \frac{\pi}{4} \\ \therefore \int_0^\infty e^{-x^2} dx &= \frac{\sqrt{\pi}}{2} \end{aligned}$$

Question-3(d) Find the angle between the lines whose direction cosines are given by the relations $l+\ \mathrm{m}+\mathrm{n}=0$ and $2\mathrm{lm}+2\ln-\mathrm{mn}=0$.

[10 Marks]

Solution: Eliminating n between the given relations

$$\begin{aligned} n &= -(l+m) \\ 2lm-2l(l+m)+m(l+m) &= 0 \\ 2lm-2l^2-2lm+ml+m^2 &= 0 \\ \text{or} 2\ell^2-lm-m^2 &= 0 \\ 2l^2-2lm+1m-m^2 &= 0 \\ 2l(l-m)+m(l-m) &= 0 \\ (l-m)(2l+m) &= 0 \end{aligned}$$

If $2l+m=0$, then from $l+m+n=0$, $\quad n=l$

$$\therefore \quad \frac{l}{1}=\frac{m}{-2}=\frac{n}{1} \qquad \ldots(1)$$

If $l-m=0$, then $l+m+n=0$, $n=-2m$

$$\frac{l}{1}=\frac{m}{1}=\frac{n}{-2} \qquad \ldots(2)$$

Hence, angle between the lines with direction ratios given by (1) and (2) is

$$\cos\theta=\frac{1\cdot 1+(-2)\cdot 1+1\cdot(-2)}{\sqrt{1+4+1}\cdot\sqrt{1+1+4}}=\frac{-3}{6}=\frac{-1}{2}$$

$$\cos\theta=\frac{-1}{2} \quad \therefore \quad \theta=120^\circ$$

Question-4(a) Find the eigenvalues and the corresponding eigenvectors for the matrix $A=\begin{bmatrix}0 & -2\\ 1 & 3\end{bmatrix}$. Examine whether the matrix A is diagonalizable. Obtain a matrix D (if it is diagonalizable) such that $\mathrm{D}=\mathrm{P}^{-1}\,\mathrm{A}P$.

[10 Marks]

Solution: Characteristic eqn,

$$c\begin{vmatrix}-\lambda & -2\\ 1 & 3-\lambda\end{vmatrix}=0 \quad \Rightarrow \quad \begin{array}{c}\lambda^2-3\lambda+2=0\\ (\lambda-1)(\lambda-2)=0\end{array}$$

Eigen values, $\lambda=1, \quad \lambda=2$

For $\lambda = 1$, Let eigenvector be, $v \begin{bmatrix} x \\ y \end{bmatrix}$

$$\therefore Av = \lambda v$$

i.e, $(A - \lambda)v = 0$

$$\therefore \begin{bmatrix} -1 & -2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

ie.

$$x + 2y = 0 \quad \therefore \text{ Eigen-vector is } \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

For

$$\lambda = 2, \quad \begin{bmatrix} -2 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

ie

$$x + y = 0 \quad \therefore \text{Eigen vector is } \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Here, algebraic multiplicity of each eigenvalue is equal to geometric multiplicity. $\therefore A$ is diagonizable.

$$P = \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix}, P^{-1} = \begin{bmatrix} -1 & -1 \\ -1 & -2 \end{bmatrix}$$
$$\therefore D = P^{-1}AP = \begin{bmatrix} -1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 0 & -2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

Question-4(b) A function $f(x, y)$ is defined as follows:

$$f(x,y) = \begin{cases} \dfrac{x^2y^2}{x^2+y^2}, & \textbf{if } (x,y) \neq (0,0) \\ 0, & \textbf{if } (x,y) = (0,0) \end{cases}$$

Show that $\mathrm{f_{xy}}(0,0) = \mathrm{f_{yx}}(0,0)$

[10 Marks]

Solution:

$$f_{xy}(0,0) = \lim_{h \to 0} \frac{f_y(0+h,0) - f_y(0,0)}{h} - (1)$$
$$f_y(0,0) = \lim_{k \to 0} \frac{f(0,k) - f(0,0)}{k} = \lim_{k \to 0} \frac{0}{k} = 0$$
$$fy(h,0) = \lim_{k \to 0} \frac{f(h,k) - f(h,0)}{k} = \lim_{k \to 0} \frac{1}{k} \cdot \frac{h^2k^2}{(h^2+k^2)}$$
$$= 0$$

Hence, from (1), $\quad f_{xy}(0,0) = \lim_{h \to 0} \dfrac{0-0}{h} = 0$ Now,

$$f_{yx}(0,0) = \lim_{k \to 0} \frac{f_x(0,k) - f_x(0,0)}{k} - (2)$$

$$f_x(0,k) = \lim_{h\to 0} \frac{f(h,k) - f(0,k)}{h} = \lim_{h\to 0} \frac{1}{h} \cdot \frac{h^2k^2}{h^2+k^2}$$
$$= 0$$

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{0-0}{h} = 0$$

Hence, from(2), $f_{yx} = \lim_{k\to 0} \frac{0-0}{k} = 0$

$$\therefore \quad f_{xy}(0,0) = f_{yx}(0,0)$$

Question-4(c) Find the equation of the right circular cone with vertex at the origin and whose axis makes equal angles with the coordinate axes and the generator is the line passing through the origin with direction ratios $(1,-2,2)$.

[10 Marks]

Solution: The vertex of the cone is O(0,0,0) and since its axis makes equal angles with the coordinate axes, so the equations of its axis can be taken as

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{1} \quad (\because l = m = n)$$

Also, d.r.'s of its generator are (1,-2,2) If θ is the semi-vertical angle of the cone, then

$$\cos\theta = \frac{1\cdot 1 + 1\cdot(-2) + 1\cdot 2}{\sqrt{1+1+1}\cdot\sqrt{1+4+4}} = \frac{1}{3\sqrt{3}} - (1)$$

If $P(x,y,z)$ is any general point on the cone, then OP is a generator and d.r.'s of OP are $(x-0, y-0, z-0)$ ie, (x,y,z) Also OP makes an angle θ with the axis

$$\therefore \cos\theta = \frac{x\cdot 1 + y\cdot 1 + z\cdot 1}{\sqrt{1+1+1}\cdot\sqrt{x^2+y^2+2^2}} - (2)$$

From (1) and (2) ,

$$l\frac{x+y+z}{\sqrt{3}\cdot\sqrt{x^2+y^2+z^2}} = \frac{1}{3\sqrt{3}} \text{ or } 9(x+y+z)^2 = x^2+y^2+z^2$$
$$\text{or } 4\left(x^2+y^2+z^2\right) + 9(xy+yz+zx) = 0$$

Question-4(d) Find the shortest distance and the equation of the line of the shortest distance between the lines

$$\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1}$$

and

$$\frac{x+3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}$$

[10 Marks]

Solution: Any point on first line $P(3r+3,-r+8,r+3)$ Any point on second line $Q(-3t-3,2t-7,4t+6)$ D.r.'s of PQ are

$$(3r+3t+6,r-2t+15,r-4t-3)\quad -(1)$$

If PQ is the shortest distance (SD) between the given lines, then PQ is perpendicular both lines.

$$\therefore 3(3r+3t+6)-1(-r-2t+15)+1(r-4t-3)=0$$

and

$$-3(3r+3t+6)+2(-r-2t+5)+4(r-4t-3)=0$$

or

$$11r+7t=0\text{and}7r+29t=0$$

Solving, we get $r=0,t=0$

$$\therefore Points, P(3,+8,3)\text{and}Q(-3,-7,6)$$

D.r's of PQ are $(6,15,-3)$ or $(2,5,-1)$

$$\begin{aligned}SD=PQ&=\sqrt{(3+3)^2+(8+7)^2+(3-6)^2}\\&=\sqrt{36+225+9}=3\sqrt{30}\end{aligned}$$

Also, PQ is line through $P(3,8,3)$ and with d.r.'s $(2,5,-1)$, so it equation is

$$\frac{x-3}{2}=\frac{y-8}{5}=\frac{z-3}{-1}$$

4.2 Section-B

Question-5(a) Solve

$$\left(2D^3-7D^2+7D-2\right)y=e^{-8x}.where, D=\frac{d}{dx}$$

[8 Marks]

Solution: Let

$$2D^3 - 7D^2 + 7D - 2 = f(D) \quad -(1)$$
$$\therefore \qquad f(D)y = e^{-8x} - (2)$$

Auxiliary equation of (2) is, $f(D) = 0$ or

$$2D^3 - 7D^2 + 7D - 2 = 0$$

roots are , $D = 1, 2, \frac{1}{2}$.
Complementary function (CF) of (2) is

$$y = c_1e^x + c_2e^{2x} + c_3e^{x/2} - (3)$$

For finding particular integral $(PI)-$

$$f(D)y = Q(x)$$

where,

$$Q(x) = e^{-8x}$$

$$\begin{aligned} PI &= \frac{1}{f(D)}Q(x) \\ &= \frac{1}{(D-1)(D-2)\left(D-\frac{1}{2}\right)2} \cdot \left(e^{-8x}\right) \\ &= \frac{1}{(-8-1)(-8-2)(-16-1)} \cdot e^{-8x} \\ &= \frac{1}{-1530}e^{-8x} - (4) \end{aligned}$$

General solution, $y = y_c + y_p$

$$y = c_1e^x + c_2e^{2x} + c_3e^{x/2} - \frac{e^{-8x}}{1530}$$

Question-5(b) Solve the differential equation

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^4$$

[8 Marks]

Solution:

$$x^2\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - 4y = x^4$$

Using the substitution,

$$x = e^z, \text{i.e}, z = \log x$$

The equation becomes

$$(D_1(D_1-1)-2D_1-4)y=e^{4z},\quad D_1=\frac{d}{dz}$$
$$\therefore\left(D_1^2-3D_1-4\right)y=e^{4z}-(2)$$

Auxiliary equation of (2) is

$$D_1^2-3D_1-4=0\therefore D_1=4,-1$$
$$\therefore C\cdot F=y_c=c_1e^{-z}+c_2e^{4z}-(3)$$

$$\begin{aligned}PI&=\frac{1}{D_1^2-3D_1-4}\left(e^{4z}\right)\\&=\frac{1}{(D_1-4)(D_1+1)}\left(e^{4z}\right)\\&=\frac{1}{(D_1-4)}\cdot\frac{e^{4_2}}{(4+1)}\\&=\frac{z}{5}\cdot e^{4z}-(4)\end{aligned}$$

$$\begin{aligned}\therefore y&=y_c+y_p\\&=c_1e^{-z}+c_2e^{4z}+\frac{ze^{4z}}{5}\\&=c_1e^{-\log x}+c_2e^{4\log x}+\log x\cdot\frac{e^{4\log x}}{5}\\&=\frac{c_1}{x}+c_2x^4+\frac{1}{5}x^4\cdot\log x\end{aligned}$$

Question-5(c) A particle is undergoing simple harmonic motion of period T about a centre O and it passes through the position $P(OP=b)$ with velocity v in the direction OP. Prove that the time that elapses before it returns to P is $\frac{\mathrm{T}}{\pi}\tan^{-1}\left(\frac{\mathrm{vT}}{2\pi\mathrm{b}}\right)$.

[8 Marks]

Solution:

$$x=a\sin\omega t, v=a\omega\cos\omega t,\quad\omega=\frac{2\pi}{T}$$

At

$$\begin{aligned}x&=b\\b&=a\sin\omega t_b\\\Rightarrow t_b&=\frac{1}{\omega}\sin^{-1}\frac{b}{a}\\&=\frac{T}{2\pi}\sin^{-1}\frac{b}{a}\\t_a&=\frac{T}{4}=\frac{2\pi}{\omega}\cdot\frac{1}{4}\end{aligned}$$

$$
\begin{aligned}
v &= a\omega \cos \omega t_b \\
\therefore \quad v &= a\omega \left(1 - \sin^2 \omega t_b\right)^{1/2} \\
&= a\omega \left(1 - \frac{b^2}{a^2}\right)^{1/2} \\
&= \omega\sqrt{a^2 - b^2} \\
\Rightarrow a^2 &= \frac{v^2}{\omega^2} + b^2 \\
&= \left(\frac{vT}{2\pi}\right)^2 + b^2
\end{aligned}
$$

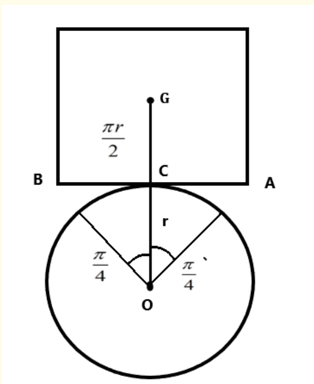
Time required,

$$
\begin{aligned}
t = 2\left(t_a - t_b\right) &= 2\left(\frac{T}{4} - \frac{T}{2\pi}\sin^{-1}\frac{b}{a}\right) \\
&= \frac{T}{\pi}\left(\frac{\pi}{2} - \sin^{-1}\frac{b}{a}\right) = \frac{T}{\pi}\cos^{-1}\frac{b}{a} \\
&= \frac{T}{\pi}\cdot\tan^{-1}\frac{\sqrt{a^2 - b^2}}{b} \\
&= \frac{T}{\pi}\tan^{1}\left(\frac{Tv}{2\pi b}\right) \\
&\left[\because a^2 - b^2 = \left(\frac{vT}{2\pi}\right)^2\right]
\end{aligned}
$$

Question-5(d) A heavy uniform cube balances on the highest point of a sphere whose radius is r. If the sphere is rough enough to prevent sliding and if the side of the cube be $\frac{\pi r}{2}$, then prove that the total angle through which the cube can swing without falling is $90°$.

[8 Marks]

Solution: If G is the centre of gravity of the cube, then for equilibrium the line OCG must be vertical. First we show that the equilibrium of the cube is stable.



Here, P_1 = the radius of curvature of the upper body at the point of contact = ∞ and, P_2 = the radius of curvature of the lower body at the point of contact = r h = height of the centre of gravity, G of the upper body above the point C = half of the edge of cube

$$= \frac{\pi r}{4}$$

The equilibrium will be stable if

$$\frac{1}{h} > \frac{1}{P_1} + \frac{1}{P_2} \quad \text{i.e,}\ \frac{1}{\pi r/4} > \frac{1}{\infty} + \frac{1}{r}$$

ie.

$$\frac{4}{\pi r} > \frac{1}{r} \quad i.e, \quad 4 > \pi$$

which is true.

Hence, the equilibrium is stable. So, if the cube is slightly displaced, it will tend to come back to its original position of equilibrium.
During a swing to the right, the cube will not fall down till the right hand corner A of the lowest edge comes in contact with the sphere.
If θ is the angle through which the cube twins when the right hand corner A of the lowest edge comes in contact within sphere,
$\therefore \quad r\theta =$ half the edge of the cube $= \dfrac{\pi r}{4}$
$\therefore \quad \theta = \pi/4$

Similarly, the cube can turn through an angle of $\pi/4$ to the left side on the sphere.
Hence the total angle through which the cube can swing (or rock) without falling is $2\dfrac{\pi}{4}$ ie, $\dfrac{\pi}{2}$.

Question-5(e) Prove that

$$\nabla^2 r^n = n(n+1)r^{n-2}$$

and that $\mathrm{r}^{\mathrm{n}}\overrightarrow{\mathrm{r}}$ **-is irrotational, where** $\mathrm{r} = |\overrightarrow{\mathrm{r}}| = \sqrt{\mathrm{x}^2 + \mathrm{y}^2 + \mathrm{z}^2}$.

[8 Marks]

Solution:

$$\begin{aligned}
\nabla^2 r^n &= \nabla \cdot (\nabla r^n) = \operatorname{div}(grad r^n)\\
&= \operatorname{div}\left(n r^{n-1} grad r\right)\\
&= \operatorname{div}\left(n r^{n-1}\frac{\vec{r}}{r}\right) = \operatorname{div}\left(n r^{n-2}\vec{r}\right)\\
&= \left(n r^{n-2}\right)\operatorname{div}\vec{r} + \vec{r}\cdot\left(g\,\mathrm{rad}\, n r^{n-2}\right)\\
&= 3n r^{n-2} + \vec{r}\cdot\left[n(n-2)r^{n-3} grad r\right]\\
&= 3n r^{n-2} + \vec{r}\cdot\left[n(n-2)r^{n-3}\cdot\frac{\vec{r}}{r}\right]\\
&= 3n r^{n-2} + \vec{r}\cdot\left(n(n-2)r^{n-4}\vec{r}\right]\\
&= n r^{n-2}(3+n-2)\\
&= n(n+1)r^{n-2}
\end{aligned}$$

Now,

$$|\vec{r}| = \sqrt{x^2+y^2+z^2}$$

$$r^n\vec{r} = r^n\{xi+yj+zk\}$$

$$\operatorname{curl}(\vec{r}) = \begin{vmatrix} i & j & k\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ x & y & z\end{vmatrix}$$

$$= i(0)+j(0)+k(0) = 0$$

$$[\because \operatorname{curl}(\phi A) = (grad\phi)\times A + \phi\operatorname{curl}A)]$$

Hence, $r^n\vec{r}$ is irrotational.

Question-6(a) Solve the differential equation

$$\left(\frac{dy}{dx}\right)^2 + 2\cdot\frac{dy}{dx}\cdot y\cot x = y^2$$

[15 Marks]

Solution:

$$\left(\frac{dy}{dx}\right)^2 + 2\cdot\frac{dy}{dx}\cdot y\cot x = y^2 - (1)$$

Put

$$\frac{dy}{dx} = p$$

$$p^2 + 2py\cot x = y^2$$

$\Rightarrow$

$$p^2 + 2py\cot x + y^2\cot^2 x = y^2 + y^2\cot^2 x$$

$\Rightarrow$

$$(p+y\cot x)^2 = y^2\operatorname{cosec}^2 x$$

$$\therefore\quad p + y\cot x = \pm y\operatorname{cosec} x$$

i.e.,

$$\left.\begin{aligned}\frac{dy}{dx} + y(\cot x - \operatorname{cosec} x) = 0\\ \frac{dy}{dx} + y(\cot x + \operatorname{cosec} x) = 0\end{aligned}\right\} \begin{gathered}\text{components}\\ \text{of eqn.}\end{gathered}$$

$$\frac{dy}{y} + (\cot x - \operatorname{cosec} x)dx = 0$$

Integrating

$$\log y + \log\sin x - \log\left(\tan\frac{x}{2}\right) = \log C$$
$$\therefore \log y = \log C + \log\tan\frac{x}{2} - \log\sin x$$
$$y = \frac{c \times \tan x/2}{\sin x} = \frac{c \cdot \sin\frac{x}{2}}{\cos\frac{x}{2} \cdot 2\sin\frac{x}{2}\cos\frac{x}{2}}$$
$$y = \frac{c}{2\cos^2\frac{x}{2}} = \frac{c}{1+\cos x}$$

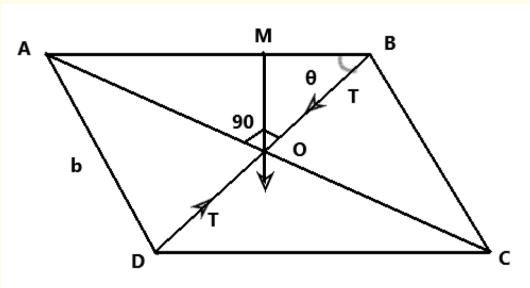
ie $y(1+\cos x) = C$ is one solution. Similarly, solving the second equation, we get, $y(1-cosx) = C$

$$\therefore \text{ General solution of (1)} is, \quad [y(1+\cos x) - c][y(1-cosx) - C] = 0$$

Question-6(b) A string of length a, forms the shorter diagonal of a rhombus formed of four uniform rods, each of length b and weight W, which are hinged together. If one of the rods is supported in a horizontal position, then prove that the tension of the string is $\frac{2\mathrm{W}\left(2\mathrm{b}^2-\mathrm{a}^2\right)}{\mathrm{b}\sqrt{4\mathrm{b}^2-\mathrm{a}^2}}$.

[10 Marks]

Solution: Let T be the tension in the string BD. The total weight out of the rods AB, BC, CD and DA can be taken as acting of the point of intersection O of the diagonals AC and BD. We have, $\angle AOB = 90°$



Let $\angle ABO = \theta$, Draw $OM \perp toAB$ Give the system a small symmetrical displacement in which θ change from to $\theta + \delta\theta$. The line AB remains fixed. The points O, C and D change. The angle AOB will remain 90°. $BD = 2BO = 2ABcos\theta = 2bcos\theta$ ($\because$ length $BD = a$ at equilibrinm. It changes during displacement, and depends on angle θ) The depth of O below the fixed line

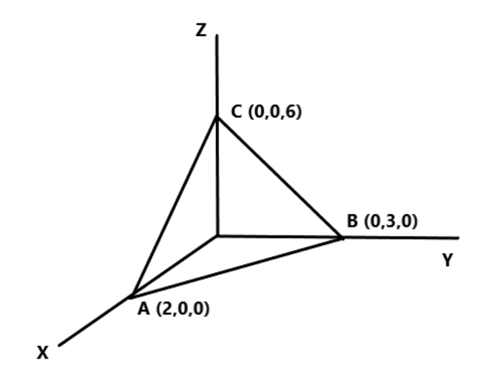
$$AB = MO = (BO)\sin\theta = (ABcos\theta)\sin\theta$$

ie
$$MO = b\sin\theta\cos\theta$$
By the principle of virtual work,
$$-T\delta(2bcos\theta) + 4W\delta(b\sin\theta\cos\theta) = 0$$
or
$$2bT\sin\theta\delta\theta + 4bW\left(cos^2\theta - \sin^2\theta\right)\delta\theta = 0$$
or
$$2b\left[T\sin\theta - 2W\left(\sin^2\theta - cos^2\theta\right)\delta\theta = 0\right.$$
or
$$T\sin\theta - 2W\left(\sin^2\theta - \cos^2\theta\right) = 0 \quad (\because \delta\theta \neq 0)$$
or
$$T = \frac{2W\left(\sin^2\theta - cos^2\theta\right)}{\sin\theta} = \frac{2W\left(1 - 2\cos^2\theta\right)}{\sqrt{1-\cos^2\theta}}$$
In the position of equilibrium,
$$l \therefore BD = a \text{ or } BO = \frac{a}{2}$$
$$\therefore \quad \cos\theta = \frac{BO}{AB} = \frac{a}{2b}$$
$$\therefore T = \frac{2W\left(1 - 2\cdot\frac{a^2}{4b^2}\right)}{\sqrt{1-\frac{a^2}{4b^2}}}$$
$$\therefore T = \frac{2W\left(2b^2 - a^2\right)}{b\sqrt{4b^2 - a^2}}$$

Question-6(c) Using Stokes' theorem, evaluate
$$\int_{\mathrm{C}}[(x+y)dx + (2x-z)dy + (y+z)dz]$$
where C **is the boundary of the triangle with vertices at** $(2,0,0)$**,** $(0,3,0)$ **and** $(0,0,6)$.

[10 Marks]

Solution: The given integral is of the form $\oint_c F\cdot dr$, where
$$F = (x+y)i + (2x-z)j + (y+z)k$$
$$dx = idx + jdy + kdz$$
C: Boundary of Triangle ABC S: Area of Triangle ABC



$$\text{curl}(F) = \begin{vmatrix} i & j & k \\ \frac{2}{2x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix} = 2i + k - (1)$$

Using Stokes' theorem,

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S (\text{curl}\,\vec{F} \cdot \hat{n}) dS$$

Here

$$\hat{n}, \text{is unit normal vector to} \frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1$$

$$\hat{n} = \frac{6}{\sqrt{14}} \left(\frac{i}{2} + \frac{j}{3} + \frac{k}{6} \right) = \frac{1}{\sqrt{14}} (3i + 2j + k) - (2)$$

$$\text{curl}(\vec{F}) \cdot \hat{n} = \frac{1}{\sqrt{14}} (6+1) = \frac{7}{\sqrt{14}} \quad [\text{from}(1)\&(2)]$$

$$\therefore \oint_C \vec{F} \cdot d\vec{r} = \frac{7}{\sqrt{14}} \iint_S dS = \frac{7}{\sqrt{14}} (\text{ Area of } \triangle ABC)$$

$$= \frac{7}{\sqrt{14}} \times 3\sqrt{14} = 21$$

$$[Area(\triangle ABC) \Rightarrow \Delta^2 = \Delta_x^2 + \Delta_y^2 + \Delta_z^2$$

$$\Delta^2 = \left(\frac{1}{2} \times 3 \cdot 6 \right)^2 + \left(\frac{1}{2} \cdot 2 \cdot 6 \right)^2 + \left(\frac{1}{2} \cdot 2 \cdot 3 \right)^2 = 126]$$

Question-7(a) Solve the differential equation

$$\mathrm{e}^{3\mathrm{x}} \left(\frac{\mathrm{dy}}{\mathrm{dx}} - 1 \right) + \left(\frac{\mathrm{dy}}{\mathrm{dx}} \right)^3 \mathrm{e}^{2\mathrm{y}} = 0$$

[10 Marks]

Solution: Let $e^x = X, \quad e^y = Y$

$$\therefore e^x dx = dx, \quad e^y dy = dy$$

$$\Rightarrow \quad \frac{dY}{dX} = \frac{e^y}{e^x} \frac{dy}{dx}$$

$$\Rightarrow P = \frac{Y}{X}p$$

$$\Rightarrow p = \frac{X}{Y}P$$

The given ODE becomes

$$e^{3x}\left(\frac{dy}{dx} - 1\right) + \left(\frac{dy}{dx}\right)^3 e^{2y} = 0$$

$$X^3\left(\frac{X}{Y}P - 1\right) + \left(\frac{X}{Y}P\right)^3 Y^2 = 0$$

$$XP - Y + P^3 = 0$$

$$\Rightarrow Y = XP + P^3$$

, which is in Clairaut's form $y = xp + f(p)$
Hence, the general solution is

$$Y = Xc + c^3$$

$$\Rightarrow e^y = ce^x + c^3$$

Question-7(b) A planet is describing an ellipse about the Sun as a focus. Show that its velocity away from the Sun is the greatest when the radius vector to the planet is at a right angle to the major axis of path and that-the velocity then is $\frac{2\pi\ \mathrm{ae}}{\mathrm{T}\sqrt{1-\mathrm{e}^2}}$, where 2a is the major axis, e is the eccentricity and T is the periodic time.

[10 Marks]

Solution: The polar equation of the elliptic orbit B

$$\frac{l}{a} = 1 + e\cos\theta \quad d \quad lu = 1 + e\cos\theta \quad -(1)$$

We know,

$$h^2 = r^2\dot{\theta} \quad d \quad \dot{\theta} = hu^2 \quad \left(u = \frac{1}{r}\right) - (2)$$

Also,

$$\frac{dr}{dt} = \frac{dr}{d\theta}\frac{d\theta}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\left(hu^2\right) = -h\frac{du}{d\theta}$$

for maximum value of $\frac{dr}{dt}$, we have

$$\frac{d}{d\theta}\left(\frac{dr}{dt}\right) = 0 \quad d \quad \frac{d}{d\theta}\left(-h\frac{\partial u}{d\theta}\right) = 0 \quad d \quad \frac{d^2u}{d\theta^2} = 0$$

$$(\because \text{h is constant})$$

From (1) ,

$$\frac{du}{d\theta} = \frac{-e}{l}\sin\theta \quad \& \quad \frac{d^2u}{d\theta^2} = -\frac{e}{l}\cos\theta$$

$$\therefore \frac{d^2u}{d\theta^2} = 0 \Rightarrow \frac{-e}{l}\cos\theta = 0 \Rightarrow \cos\theta = 0, \text{i.e}, \theta = \frac{\pi}{2}$$

This proves the first part. For maximum value of $\frac{dr}{dt}$,

$$\frac{du}{d\theta} = -\frac{e}{l}\sin\frac{\pi}{2} = -\frac{e}{l} \quad -(3)$$

From

$$(2)\&(3), \left(\frac{dr}{dt}\right)_{\max} = \frac{he}{l} = \sqrt{\mu l}\cdot\frac{e}{l} = \int\frac{\mu}{l}e - (4)$$

As,

$$l = \frac{b^2}{a} = a\left(1-e^2\right)$$

and

$$T = \frac{2\pi a^{3/2}}{\sqrt{\mu}}$$

$$\therefore \sqrt{l} = \sqrt{a\left(1-e^2\right)}$$

and

$$\sqrt{\mu} = \frac{2\pi a^{3/2}}{T}$$

Substituting in (4)

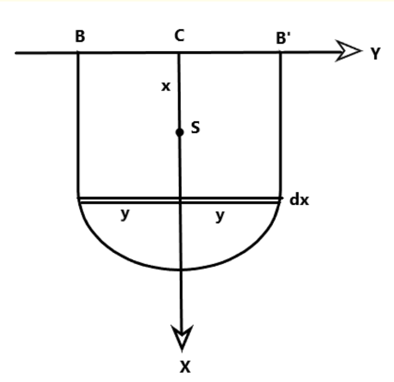
$$\left(\frac{dr}{dt}\right)_{\max} = \frac{2\pi a^{3/2}e}{T\sqrt{a\left(1-e^2\right)}} = \frac{2\pi ae}{T\sqrt{1-e^2}}$$

[l = semi-latus rectum]

Question-7(c) A semi-ellipse bounded by its minor axis is just immersed in a liquid, the density of which varies as the depth. If the minor axis lies on the surfare, then find the eccentricity in order that the focus may be the centre of pressure.

[10 Marks]

Solution: BAB' is the semi-ellipse immersed in a liquid with minor axis BB' in the surface. Consider the elementary strip of width dx at a distance x from c.



$$\therefore P = \rho g x = kx \cdot gx = kgx^2$$

$$ds = 2ydx$$

But,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

ie.

$$y = \frac{b}{a}\sqrt{a^2 - x^2}$$

$$\therefore ds = \frac{2b}{a}\sqrt{a^2 - x^2}dx$$

$$\therefore \bar{x} = \frac{\int_0^a xpds}{\int_0^a pds} = \frac{\int_0^a x \cdot \text{kg}\, x^2 \cdot \frac{2b}{a}\sqrt{a^2 - x^2}dx}{\int_0^a \text{kg}\, x^2 \frac{2b}{a}\sqrt{a^2 - x^2}dx}$$

$$= \frac{\int_0^a x^3\sqrt{a^2 - x^2}dx}{\int_0^a x^2\sqrt{a^2 - x^2}dx}$$

Put,

$$x = a\sin\theta, \quad dx = acos\theta d\theta$$

$$\therefore \quad \bar{x} = \frac{\int_0^{\pi/2} a^3 \sin^3\theta \cdot a^2\cos^2\theta d\theta}{\int_0^{\pi/2} a^2\sin^2\theta \cdot a^2\cos^2\theta d\theta}$$

$$= \left(\frac{2 \cdot 1}{5 \cdot 3}a\right) / \left(\frac{1 \cdot 1}{4 \cdot 2} \cdot \frac{\pi}{2}\right)$$

$$= \frac{32}{15\pi}a$$

$$CS = \text{ Distance of focus from } C$$

$$= ae = \frac{32a}{15\pi}$$

$$\therefore e = \frac{32}{15\pi}$$

Question-7(d) Evaluate

$$\iint_{\mathrm{S}} (\nabla \times \overrightarrow{\mathrm{f}}) \cdot \hat{\mathrm{n}}\mathrm{dS}$$

$$y = 3 = \frac{-3}{2}(3 \cdot 0)$$

where S is the surface of the cone, $z = 2 - \sqrt{x^2 + y^2}$ above xy-plane and

$$\vec{f} = (x - z)\hat{i} + \left(x^3 + yz\right)\hat{j} - 3xy^2\hat{k}$$

[10 Marks]

Solution: The xy-plane cuts the surface S of cone in the circle, C whose equation is $x^2+y^2=4; z=0$. parametric eqn: $x=2\cos t$, $y=2\sin t$ By Stokes' theorem,

$$\iint_S (\nabla\times\vec{f})\cdot\hat{n}dS=\oint_C \vec{F}\cdot d\vec{r}$$

$$=\int_c (x-2)dx+\left(x^3+yz\right)dy+\left(-3xy^2\right)dz$$

$$=\int_c xdx+x^3dy \quad (\because z=dz=0)$$

$$=\int_{t=0}^{2\pi}\left[x\frac{dx}{dt}+x^3\frac{dy}{dt}\right]dt$$

$$=\int_{t=0}^{2\pi}\left(2\cos t(-2\sin t)+8\cos^3 t\cdot 2\cos t\right]dt$$

$$=\int_{t=0}^{2\pi}\left(-2(\sin 2t)+16\cos^4 t\right)dt$$

$$=-2\int_0^{2\pi}\sin 2tdt+16\int_0^{2\pi}\left(\frac{1+cos2t}{2}\right)^2dt$$

$$=[\cos 2t]_0^{2\pi}+16\left\{\frac{1}{32}\left[\sin 4x\right]_0^{2\pi}-\frac{1}{8}[n]_0^{2\pi}+\frac{1}{2}[x]_0^{2\pi}+\frac{1}{4}[\sin 2x]_0^{2\pi}\right\}$$

$$=0+16\left\{0-\frac{2\pi}{8}+\frac{2\pi}{2}+0\right\}$$

$$=12\pi$$

Question-8(a) **Solve** $\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{dx}^2}+4\mathrm{y}=\tan 2\mathrm{x}$ **by using the method of variation of parameter.**

[10 Marks]

Solution: Comparing with,

$$y_2+Py_1+Qy=R$$

$$P=0,\quad Q=4, R=\tan 2x$$

Auxiliary

$$\text{eqn}\left(D^2+4\right)=0 \quad \therefore \quad D=\pm 2i$$

$$y_c=c_1\cos 2x+c_2\sin 2x \quad -(1)$$

Using method of variation of parameters, let the complete solution be given by

$$y=A\cos 2x+B\sin 2x$$

, where A and B are functions of x.
Then,

$$u(x)=\cos 2x, v(x)=\sin 2x,\quad R(x)=\tan 2x$$

.

$$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

$$\begin{aligned} A &= \int \frac{-vR}{w} dx = \int \frac{-\sin 2x \cdot \tan 2x}{2} dx \\ &= \int \frac{-\sin^2 2x}{2\cos 2x} dx = \frac{-1}{2} \int \frac{1-\cos^2 2x}{\cos 2x} dx \\ &= \frac{-1}{2} \int (\sec 2x - \cos 2x) dx \\ &= -\frac{1}{4} (\log|\sec 2x + \tan 2x| - \sin 2x) \end{aligned}$$

$$\begin{aligned} \text{and, } B &= \int \frac{uR}{w} dx = \int \frac{\cos 2x \cdot \tan 2x}{2} dx \\ &= \frac{1}{2} \int \sin 2x dx = -\frac{1}{4} \cos 2x \end{aligned}$$

$$\therefore y = \frac{-1}{4} (\log \mid \sec 2x + \tan 2x \mid - \sin 2x) \cos 2x - \frac{1}{4} \cos 2x \cdot \sin 2x - (2),$$

where y is the general solution of the given DE.

Question-8(b) A particle moves in a straight line, its acceleration directed towards a fixed point O in the line and is always equal to $\mu \left(\frac{a^5}{x^2}\right)^{\frac{1}{3}}$ when it is at a distance x from O. If it starts from rest at a distance a from O, then prove that it will arrive at O with a velocity $a\sqrt{6\mu}$ -after time $\frac{8}{15}\sqrt{\frac{6}{\mu}}$.

[10 Marks]

Solution: Acceleration,

$$\frac{d^2x}{dt^2} = -\mu \cdot \frac{a^{5/3}}{x^{2/3}}$$

$$2\left(\frac{d^2x}{dt^2}\right) \cdot \frac{dx}{dt} = -2\mu \frac{a^{5/3}}{x^{2/3}} \cdot \frac{dx}{dt}$$

Integrating both sides w.r.t $\left(\frac{dx}{dt}\right)$ from rest to final point (0).

$$\left(\frac{dx}{dt}\right)^2 \Bigg|_0^{v_0} = -\ 2\mu a^{5/3} \frac{x^{1/3}}{1/3}\Bigg|_a^0$$

$$v_0^2 = 6\mu a^{5/3} \cdot a^{1/3} = 6\mu a^2$$

$$v_0 = a\sqrt{6\mu}$$

$$\left(\frac{dx}{dt}\right)^2 \Bigg|_0^{dx/dt} = -\ 6\mu a^{5/3} \cdot x^{1/3}\big|_a^x$$

$$\left(\frac{dx}{dt}\right)^2 = 6\mu a^{5/3}\left(a^{1/3} - x^{1/3}\right)$$

$$\frac{dx}{\sqrt{a^{1/3} - x^{1/3}}} = -\int_0^{t_0} \sqrt{6\mu a^{5/3}}dt$$

Put,

$$x^{1/3} = a^{1/3}\sin^2\theta \Rightarrow x = a\sin^6\theta \Rightarrow dx = 6a\sin^5\theta\cos\theta$$

$$x = 0 \to \theta = \pi/2$$

$$x = 0 \to \theta = 0$$

$$\int_{\pi/2}^{0} \frac{6a\sin^5\theta\cos\theta}{a^{1/6}\left(1-\sin^2\theta\right)^{1/2}}d\theta = -\int_0^{t_0}\sqrt{6\mu a^{5/3}}dt$$

$$\sqrt{6\mu a^{5/3}}t_0 = \int_0^{\pi/2} 6a^{5/6}\sin^5\theta d\theta$$

$$\sqrt{6\mu}a^{5/6}t_0 = 6a^{5/6}\cdot\frac{4\cdot 2}{1\cdot 3\cdot 5} = \frac{16}{5}a^{5/6}$$

$$\therefore t_0 = \frac{16}{5}\cdot\frac{1}{\sqrt{6\mu}}$$

$$t_0 = \frac{8}{15}\cdot\sqrt{\frac{6}{\mu}}$$

Question-8(c) Find the curvature and torsion of the circular helix

$$\overrightarrow{\mathrm{r}} = \mathrm{a}(\cos\theta, \sin\theta, \theta\cot\beta),$$

β is the constant angle at which it cuts its generators.

[10 Marks]

Solution: Curvature,

$$\kappa = \frac{\left|\frac{d\vec{r}}{d\theta}\times\frac{d^2\vec{r}}{d\theta^2}\right|}{\left|\frac{d\vec{r}}{d\theta}\right|^3} - (1)$$

Torsion,

$$\tau = \frac{\left[\frac{d\vec{r}}{d\theta}\,\frac{d^2\vec{r}}{d\theta^2}\,\frac{d^3\vec{r}}{d\theta^3}\right]}{\left|\frac{d\vec{r}}{d\theta}\times\frac{d^2\vec{r}}{d\theta^2}\right|^2} - (2)$$

$$\vec{r} = a(\cos\theta i + \sin\theta j + \theta\cot\beta k)$$

$$\frac{d\vec{r}}{d\theta} = a(-\sin\theta i + \cos\theta j + \cot\beta k)$$

$$\frac{d^2\vec{r}}{d\theta^2} = a(-\cos\theta i - \sin\theta j)$$

$$\frac{d^3\vec{r}}{d\theta^3} = a(\sin\theta i - cos\theta j)$$

$$\frac{d\vec{r}}{d\theta} \times \frac{d^2\vec{r}}{d\theta^2} = \begin{vmatrix} i & j & k \\ -a\sin\theta & a\cos\theta & acot\beta \\ -a\cos\theta & -a\sin\theta & 0 \end{vmatrix}$$

$$= i\left(a^2\sin\theta\cot\beta\right) - j\left(a^2\cos\theta\cot\beta\right)$$

$$+k\left(a^2\sin^2\theta + a^2\cos^2\theta\right)$$

$$= a^2[(\sin\theta\cot\beta)i - (\cos\theta\cot\beta)j + k]$$

$$\left|\frac{d\vec{r}}{d\theta} \times \frac{d^2\vec{r}}{d\theta^2}\right| = a^2\sqrt{(\sin\theta\cot\beta)^2 + (\cos\theta cot\beta)^2 + 1^2}$$

$$= a^2\sqrt{1+\cot^2\beta} = a^2\operatorname{cosec}\beta$$

$$\left|\frac{\partial\vec{r}}{d\theta}\right| = a\sqrt{\sin^2\theta + \cos^2\theta + \cot^2\beta} = a\operatorname{cosec}\beta$$

$$\therefore \quad \kappa = \frac{a^2\cdot\operatorname{cosec}\beta}{(a\operatorname{cosec}\beta)^3} = \frac{1}{a}\sin^2\beta$$

For torsion, scalar triple product is

$$\left[\frac{d\vec{r}}{d\theta}\frac{d^2\vec{r}}{d\theta^2}\frac{d^3\vec{r}}{d\theta^3}\right] = \begin{vmatrix} -a\sin\theta & acos\theta & a\cot\beta \\ -a\cos\theta & -a\sin\theta & 0 \\ a\sin\theta & -a\cos\theta & 0 \end{vmatrix}$$

$$= a\cot\beta\left(a^2\cos^2\theta + a^2\sin^2\theta\right)$$

$$= a^3\cot\beta$$

$$\tau = \frac{a^3\cot\beta}{\left(a^2\operatorname{cosec}\beta\right)^2}$$

$$= \frac{1}{a}\cdot\frac{\cos\beta}{\sin\beta}\times\sin^2\beta$$

$$= \frac{1}{a}\sin\beta\cos\beta$$

Question-8(d) If the tangent to a curve makes a constant angle α, with a fixed line, then prove that $\kappa\cos\alpha \pm \tau\sin\alpha = 0$ Conversely, if $\frac{\mathrm{K}}{\mathrm{t}}$ is constant, then show that the tangent makes a constant angle with a fixed direction.

[10 Marks]

Solution: Let e, be the unit vector parallel to the given fixed line so that as given

$$t\cdot e = \cos\alpha \qquad \ldots(1)$$

Differentiating. we get

$$\frac{dt}{ds} \cdot e = 0 \qquad \kappa n \cdot e = 0 \quad (\text{ frenet's first })$$
$$\therefore \quad n \cdot e = 0 \qquad \ldots (2)$$

Hence, n is $\perp$ to e. Thus, the vectors b, t, e are coplanar.

$$\therefore \quad b \cdot e = \pm \sin \alpha \qquad \ldots (3)$$

Differentiating (2) and applying Frenet-Serret formula,

$$\frac{dn}{ds} \cdot e = 0$$

ie

$$-(\kappa t + \tau b) \cdot e = 0$$
$$\therefore \quad \kappa \cos \alpha \pm \tau \sin \alpha = 0$$

from (1)&(3) Conversely: het

$$\frac{\kappa}{\tau} = \frac{1}{a},$$

a is some scalar constant. or

$$\frac{1}{\kappa} = \frac{a}{\tau}$$

ie,

$$\sigma = ap$$

As

$$\frac{dt}{ds} = \frac{1}{p} n and \frac{db}{ds} = \frac{1}{\sigma} n$$
$$\therefore p \frac{dt}{ds} = n = \sigma \frac{db}{ds}$$

or

$$\frac{dt}{ds} = \frac{\sigma}{p} \cdot \frac{db}{ds} = a \frac{db}{ds}$$

Integrating, we get

$$t = ab + c \ ,$$

where c is a constant vector.
Multiplying scalarly with t, we get

$$t \cdot t = ab \cdot t + c \cdot t$$

$$1 = 0 + ct, i.e \quad t \cdot c = 1$$

Hence, the tangent makes a constant angle with a fixed direction.