Previous Years' Paper (Solved)

IFS Main Examination, 2019

MATHEMATICS

PAPER-I

INSTRUCTIONS: *There are* **eight** *questions in all, out of which* **five** *are to be attempted. Question Nos. 1 and 5 are compulsory. Out of the remaining six questions,* **three** *are to be attempted selecting at.least* **one** *question from each of the two Sections A and B. Answers must be written in* **English** *only.*

SECTION–A

1. (*a*) Let $T : R^3 \to R^3$ be a linear operator on R^3 defined by $T(x, y, z) = (2y + z, x - 4y, 3x)$. Find the matrix of T in the basis $\{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$. **(8)**

(*b*) The eigenvalues of a real symmetric matrix A are -1, 1 and -2. The corresponding eigenvectors are $\frac{1}{\sqrt{2}}(-1\ 1\ 0)^T$, $(0\ 0\ 1)^T$ and $\frac{1}{\sqrt{2}}(-1\ -1\ 0)^T$ respectively. Find the matrix A^4. **(8)**

(*c*) Find the volume lying inside the cylinder $x^2 + y^2 - 2x = 0$ and outside the paraboloid $x^2 + y^2 = 2z$, while bounded by *xy*-plane. **(8)**

(*d*) Justify by using Rolle's theorem or mean value theorem that there is no number *k* for which the equation $x^3 - 3x + k = 0$ has two distinct solutions in the interval $[-1, 1]$. **(8)**

(*e*) If the coordinates of the points A and B are respectively $(b \cos \alpha, b \sin \alpha)$ and $(a \cos \beta, a \sin \beta)$ and if the line joining A and B is produced to the point $M(x, y)$ so that $AM : MB = b : a$, then show that

$$x\cos\frac{\alpha+\beta}{2} + y\sin\frac{\alpha+\beta}{2} = 0.$$ **(8)**

2. (*a*) Determine the extreme values of the function $f(x, y) = 3x^2 - 6x + 2y^2 - 4y$ in the region $\{(x, y) \in \mathbb{R}^2 : 3x^2 + 2y^2 \le 20\}$. **(10)**

(*b*) Consider the singular matrix

$$A = \begin{bmatrix} -1 & 3 & -1 & 1 \\ -3 & 5 & 1 & -1 \\ 10 & -10 & -10 & 14 \\ 4 & -4 & -4 & 8 \end{bmatrix}$$

Given that one eigenvalue of A is 4 and one eigenvector that does not correspond to this eigenvalue 4 is $(1\ 1\ 0\ 0)^T$. Find all the eigenvalues of A other than 4 and hence also find the real numbers *p*, *q*, *r* that satisfy the matrix equation $A^4 + pA^3 + qA^2 + rA = 0$. **(15)**

(*c*) A line makes angles $\alpha, \beta, \gamma, \delta$ with the four diagonals of a cube. Show that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}.$$ **(15)**

3. (*a*) Show that the equation $3^x + 4^x = 5^x$ has exactly one root. **(10)**

(*b*) The dimensions of a rectangular box are linear functions of time—$l(t)$, $w(t)$ and $h(t)$. If the length and width are increasing at the rate 2 cm/sec and the height is decreasing at the rate 3 cm/sec, find the rates at which the volume V and the surface area S are changing with respect to time. If $l(0) = 10$, $w(0) = 8$ and $h(0) = 20$, is V increasing or decreasing, when $t = 5$ sec? What about S, when $t = 5$ sec? **(15)**

(c) Show that the shortest distance between the straight lines

$$\frac{x-3}{3}=\frac{y-8}{-1}=\frac{z-3}{1} \text{ and}$$

$$\frac{x-3}{-3}=\frac{y+7}{2}=\frac{z-6}{4}$$

is $3\sqrt{30}$. Find also the equation of the line of shortest distance. **(15)**

4. (a) Using elementary row operations, reduce the matrix

$$A=\begin{bmatrix} 2 & 1 & 3 & 0 \\ 3 & 0 & 2 & 5 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 3 \end{bmatrix}$$

to reduced echelon form and find the inverse of A and hence solve the system of linear equations $AX = b$, where $X = (x, y, z, u)^T$ and $b = (2, 1, 0, 4)^T$. **(15)**

(b) Find the centroid of the solid generated by revolving the upper half of the cardioid $r = a(1 + \cos\theta)$ bounded by the line $\theta = 0$ about the initial line. Take the density of the solid as uniform. **(10)**

(c) A variable plane is parallel to the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=0$ and meets the axes at the points A, B and C. Prove that the circle ABC lies on the cone

$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0. \quad \textbf{(15)}$$

SECTION–B

5. (a) Solve the differential equation

$$(D^2 + 1)y = x^2 \sin 2x; \; D \equiv \frac{d}{dx}. \quad \textbf{(8)}$$

(b) Solve the differential equation

$(px - y)(py + x) = h^2 p$, where $p = y'$. **(8)**

(c) A 2 metres rod has a weight of 2 N and has its centre of gravity at 120 cm from one end. At 20 cm, 100 cm and 160 cm from the same end are hung loads of 3 N, 7 N and 10 N respectively. Find the point at which the rod must be supported if it is to remain horizontal. **(8)**

(d) Let $\bar{r} = \bar{r}(s)$ represent a space curve. Find $\frac{d^3\bar{r}}{ds^3}$ in terms of $\bar{T}$, $\bar{N}$ and $\bar{B}$, where $\bar{T}$, $\bar{N}$ and $\bar{B}$ represents tangent, principal normal and binormal respectively. Compute $\frac{d\bar{r}}{ds}\cdot\left(\frac{d^2\bar{r}}{ds^2}\times\frac{d^3\bar{r}}{ds^3}\right)$ in terms of radius of curvature and the torsion. **(8)**

(e) Evaluate $\int_{(0,0)}^{(2,1)} (10x^4 - 2xy^3)dx - 3x^2y^2dy$ along the path $x^4 - 6xy^3 = 4y^2$. **(8)**

6. (a) Solve by the method of variation of parameters the differential equation

$$x''(t) - \frac{2x(t)}{t^2} = t, \text{ where } 0 < t < \infty. \quad \textbf{(15)}$$

(b) Find the law of force for the orbit $r^2 = a^2 \cos 2\theta$ (the pole being the centre of the force). **(15)**

(c) Verify Stokes' theorem for $\bar{V} = (2x - y)\hat{i} - yz^2\hat{j} - y^2z\hat{k}$, where S is the upper half surface of the sphere $x^2 + y^2 + z^2 = 1$ and C is its boundary. **(10)**

7. (a) Find the general solution of the differential equation $\ddot{x} + 4x = \sin^2 2t$

Hence find the particular solution satisfying the conditions

$$x\left(\frac{\pi}{8}\right)=0 \text{ and } \dot{x}\left(\frac{\pi}{8}\right)=0. \quad \textbf{(15)}$$

(*b*) A vessel is in the shape of a hollow hemisphere surmounted by a cone held with the axis vertical and vertex uppermost. If it is filled with a liquid so as to submerge half the axis of the cone in the liquid and height of the cone be double the radius (r) of its base, find the resultant downward thrust of the liquid on the vessel in terms of the radius of the hemisphere and density (ρ) of the liquid. **(15)**

(*c*) Derive the Frenet-Serret formulae. Verify the same for the space curve $x = 3 \cos t$, $y = 3 \sin t$, $z = 4t$. **(10)**

8. (*a*) Find the general solution of the differential equation

$(x - 2)y'' - (4x - 7)y' + (4x - 6)y = 0.$ **(10)**

(*b*) A shot projected with a velocity u can just reach a certain point on the horizontal plane through the point of projection. So in order to hit a mark h metres above the ground at the same point, if the shot is projected at the same elevation, find increase in the velocity of projection. **(15)**

(*c*) Derive $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$

in spherical coordinates and compute

$$\nabla^2\left(\frac{x}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\right)$$

in spherical coordinates. **(15)**

PAPER-II

INSTRUCTIONS: *There are* ***eight*** *questions in all, out of which* ***five*** *are to be attempted. Question Nos.* ***1*** *and* ***5*** *are compulsory. Out of the remaining* ***six*** *questions,* ***three*** *are to be attempted selecting at least* ***one*** *question from each of the two Sections* ***A*** *and* ***B****. Answers must be written in* ***English*** *only.*

SECTION–A

1. (*a*) Let R be an integral domain. Then prove that ch R (characteristic of R) is 0 or a prime. **(8)**

(*b*) Show that the function $f(x) = \sin\left(\dfrac{1}{x}\right)$ is continuous and bounded in $(0, 2\pi)$, but it is not uniformly continuous in $(0, 2\pi)$. **(8)**

(*c*) Test the Riemann integrability of the function f defined by:

$$f(x) = \begin{cases} 0 & \text{when } x \text{ is rational} \\ 1 & \text{when } x \text{ is irrational} \end{cases}$$

on the interval $[0, 1]$. **(8)**

(*d*) Using Cauchy's Integral formula, evaluate the integral $\oint_c \dfrac{dz}{(z^2+4)^2}$

where $c : |z - i| = 2$. **(8)**

(*e*) A firm manufactures two products A and B on which the profits earned per unit are ₹ 3 and ₹ 4 respectively. Each product is processed on two machines M1 and M2. Product A requires one minute of processing time on M1 and two minutes on M2, while B requires one minute on M1 and one minute on M2. Machine M1 is available for not more than 7 hours 30 minutes, while machine M2 is available for 10 hours during any working day. Find the number of units of products A and B to be manufactures to get maximum profit, using graphical method. **(8)**

2. (*a*) Let I and J be ideals in a ring R. Then prove that the quotient ring (I + J)/J is isomorphic to the quotient ring I/(I ∩ J). **(10)**

Explanatory Answers

Paper-I

1.(*a*) $T : R^3 \to R^3$ defined by

$$T(x, y, z) = (2y + z, x - 4y, 3x)$$

Basis point

$$(u, v, w) = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$$

$$T(u) = (2 \times 1 + 1, 1 - 4 \times 1, 3 \times 1) = (3, -3, 3)$$

$$T(v) = (2 \times 1 + 0, 1 - 4 \times 1, 3 \times 1) = (2, -3, 3)$$

$$T(w) = (2 \times 0 + 0, 1 - 4 \times 0, 3 \times 1) = (0, 1, 3)$$

$$T(u, v, w) = \begin{bmatrix} u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \\ u_3 & v_3 & w_3 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 0 \\ -3 & -3 & 1 \\ 3 & 3 & 3 \end{bmatrix}.$$

1.(*b*) It is proved that $A = P.D.P^{-1}$, where D is the diagonal matrix formed by eigenvalues of A and P is the matrix whose columns are the eigenvectors of A.

$$\text{This is } D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$\text{and} \quad P = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \end{bmatrix}$$

Now $|P| = -2$

$$\text{and} \quad P^{-1} = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \end{bmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

Thus, $A = PDP^{-1}$

$$= \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \frac{1}{\sqrt{2}}\begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & 0 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -1 & 0 & -1 \\ 1 & 0 & -1 \\ 0 & \sqrt{2} & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 2 & 2 & 0 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -3 & -1 & 0 \\ -1 & -3 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

$$\text{Now,} \quad A^2 = \frac{1}{4}\begin{bmatrix} -3 & -1 & 0 \\ -1 & -3 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}\begin{bmatrix} -3 & -1 & 0 \\ -1 & -3 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 9+1 & 3+1 & 0 \\ 3+1 & 1+9 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

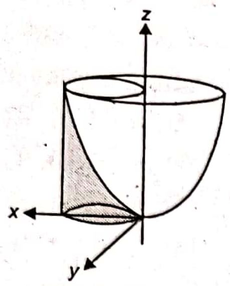
$$= \frac{1}{4}\begin{bmatrix} 10 & 4 & 0 \\ 4 & 10 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$A^4 = \frac{1}{16}\begin{bmatrix} 10 & 4 & 0 \\ 4 & 10 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 10 & 4 & 0 \\ 4 & 10 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$= \frac{1}{16}\begin{bmatrix} 100+16 & 40+40 & 0 \\ 40+40 & 16+100 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$A^4 = \frac{1}{16}\begin{bmatrix} 116 & 80 & 0 \\ 80 & 116 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

1.(c) The cylinder $x^2 + y^2 = 2x$ lies over the circular disk D which can be described as



$$\left\{(r,\theta)\left|\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}, 0 \le r \le 2r\cos\theta\right.\right\}$$

in polar co-ordinates.

The region is that if we write (x, y, z) $= (r\cos\theta, r\sin\theta, z)$ for any point in the cylinder, then

$$r^2 = x^2 + y^2 \le 2x = 2r\cos\theta$$

i.e., $\quad r \le 2\cos\theta$

As $\quad 2\cos\theta \ge r \ge 0,$

it follows that

$$\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$$

The height of the solid is the z-value of the paraboloid from the xy-plane.

Hence, the volume V of the solid is

$$2x\iint_D (x^2 + y^2).dA = 2x\int_{-\pi/2}^{\pi/2}\int_0^{2\cos\theta} r^2.r.dr.d\theta$$

$$= 2\times\frac{3\pi}{2} = 3\pi.$$

1.(d) Given equation, $x^3 - 3x + k = 0$

From mean-value theorem, we know that if $f(x)$ has a derivative on the interval (a, b) and also continuous in the interval $[a, b]$.

Then, at some value $c \in (a, b)$

$$f'(c) = \frac{f(b) - f(a)}{(b-a)}$$

Now, $\quad f(x) = x^3 - 3x + k = 0$

$$f'(x) = 3x^2 - 3$$

$$f'(c) = 3c^2 - 3$$

$$f(b) = f(1) = (1)^3 - 3(1) + k$$

$$= k - 2$$

$$f(a) = f(-1) = (-1)^3 - 3(-1) + k$$

$$= -1 + 3 + k = k + 2$$

from, $\quad f'(c) = \dfrac{f(a) - f(b)}{b-a}$

$$3c^2 - 3 = \frac{k+2-k+2}{1-(-1)}$$

$$3c^2 - 3 = 2$$

$$c = \pm\sqrt{\frac{5}{3}}$$

Hence, there is no number k for which the given equation has two distinct solution in the interval $[-1, 1]$.

1.(e) As M divides AB externally in the ratio $b : a$

$$x = \frac{b(a\cos\beta) - a(b\cos\alpha)}{b-a}$$

$$y = \frac{b(a\sin\beta) - a(b\sin\alpha)}{b-a}$$

$$\Rightarrow \frac{x}{y} = \frac{\cos\beta - \cos\alpha}{\sin\beta - \sin\alpha} = \frac{2\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}}{2\cos\frac{\alpha+\beta}{2}\sin\frac{\beta-\alpha}{2}}$$

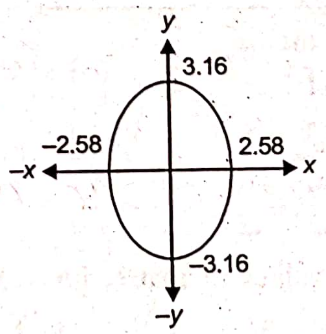
$$\Rightarrow x\cos\frac{\alpha+\beta}{2} + y\sin\frac{\alpha+\beta}{2} = 0.$$

2.(a) Given function,

$$f(x, y) = 3x^2 - 6x + 2y^2 - 4y$$

Given domain:

$$\{(x, y) \in R^2 : 3x^2 + 2y^2 \le 20\}$$



So, $\quad -2.58 \le x \le 2.58$

and $\quad -3.16 \le y \le 3.16$

This is equation of ellipse.

Now, critical point in the interior of the domain $\quad f_x = 6x - 6$

$$f_y = 4y - 4$$

Setting these partial derivatives equal to zero, we find that $f_x = 6x - 6 = 0 \Rightarrow x = 1$

$f_y = 4y - 4 = 0 \Rightarrow y = 1$

Critical point $(x, y) = (1, 1)$

The boundary of the domain contains all points (x, y) with $3x^2 + 2y^2 = 20$

We can substitute this equation in the formula for f to obtain simple function:

$$f(x, y) = (3x^2 + 2y^2) - (6x + 4y)$$
$$= 20 - (6x + 4y)$$

Notice that (x, y) can not take an arbitrary values since (x, y) is a point on the ellipse

Here, $-2.58 \le x \le 2.58$

and, $-3.16 \le y \le 3.16$

The values of f at these points are shown in the following table:

Point	Value of f	
(1, 1)	$20 - 6 \times 1 - 4 \times 1$ $= 10$	
(–2.58, 0)	$20 - 6 \times (-2.58) - 0$ $= 20 + 15.48 = 35.48$	Maximum
(2.58, 0)	$20 - 6 \times (2.58) - 0$ $= 20 - 15.48 = 4.52$	
(0, –3.16)	$20 - 0 - 4(-3.16)$ $= 20 + 12.64 = 32.64$	
(0, 3.16)	$20 - 0 - 4(3.16)$ $= 20 - 12.64 = 7.36$	Minimum

2.(b) Given matrix

$$A = \begin{bmatrix} -1 & 3 & -1 & 1 \\ -3 & 5 & 1 & -1 \\ 10 & -10 & -10 & 14 \\ 4 & -4 & -4 & 8 \end{bmatrix}$$

Eigen values of matrix $|A - \lambda 1| = 0$

$$\begin{bmatrix} -1-\lambda & 3 & -1 & 1 \\ -3 & 5-\lambda & 1 & -1 \\ 10 & -10 & -10-\lambda & 14 \\ 4 & -4 & -4 & 8-\lambda \end{bmatrix} = 0$$

$$(-1-\lambda)\begin{vmatrix} 5-\lambda & 1 & -1 \\ -10 & -10-\lambda & 14 \\ -4 & -4 & 8-\lambda \end{vmatrix} + \ldots$$

on expending the remaining term = 0

On solving that we get,

$\lambda_1 = 4, \lambda_2 = 4, \lambda_3 = 2$ and $\lambda_4 = 0$

and corresponding eigenvectors are

$$V_1 = (1, 0, 4, 1)$$
$$V_2 = (0, 0, 1, 1)$$
$$V_3 = (1, 1, 0, 0)$$

and $V_4 = (2, 1, 1, 0)$

Thus, given eigen vector $(1, 1, 0, 0)^T$ are corresponded for eigenvalue $\lambda = 2$

Now, Given matrix eqation:

$$A^4 + pA^3 + qA^2 + rA = 0$$

Sum of roots, $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = -p$

$$-4 + 4 + 2 + 0 = -p$$

$\therefore$ $p = -2$

Also, $\lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_4 + \lambda_4\lambda_1 = q$

$$-4 \times 4 + 4 \times 2 + 2 \times 0 + 0 \times (-4) = q$$

$\therefore$ $q = -8$

and,

$$\lambda_1\lambda_2\lambda_3 + \lambda_2\lambda_3\lambda_4 + \lambda_3\lambda_4\lambda_1 + \lambda_4\lambda_1\lambda_2 = -r$$
$$-4 \times 4 \times 2 + 0 + 0 + 0 = -r$$

$\therefore$ $r = 32$

Hence, $p = -2$, $q = -8$, $r = 32$.

2.(c) Let PQ be any line with direction cosines l, m, n. Let the angle between PQ and OG is α, angle between PQ and AD is β, angle between PQ and BE is γ and the angle between PQ and CF is δ.

Diagonals	Direction ratios	Direction cosines
OG	$a - 0, a - 0, a - 0$	$\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$
AD	$0 - a, a - 0, a - 0$	$\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$
BE	$0 - a, 0 - a, a - 0$	$\frac{-1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$
CF	$a - 0, 0 - a, a - 0$	$\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

Using $\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$, we have,
Now

$$\cos\alpha = l\left(\frac{1}{\sqrt{3}}\right) + m\left(\frac{1}{\sqrt{3}}\right) + n\left(\frac{1}{\sqrt{3}}\right) \quad ...(i)$$

$$\cos\beta = l\left(\frac{-1}{\sqrt{3}}\right) + m\left(\frac{1}{\sqrt{3}}\right) + n\left(\frac{1}{\sqrt{3}}\right) \quad ...(ii)$$

$$\cos\gamma = l\left(\frac{-1}{\sqrt{3}}\right) + m\left(\frac{-1}{\sqrt{3}}\right) + n\left(\frac{1}{\sqrt{3}}\right) \quad ...(iii)$$

$$\cos\delta = l\left(\frac{1}{\sqrt{3}}\right) + m\left(\frac{-1}{\sqrt{3}}\right) + n\left(\frac{1}{\sqrt{3}}\right) \quad ...(iv)$$

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$

$$= \frac{4}{3}(l^2 + m^2 + n^2) = \frac{4}{3}(1) = \frac{4}{3}.$$

3.(a) The given equation is

$$3^x + 4^x = 5^x \quad ...(1)$$

Dividing both the sides by 5^x, we get

$$\left(\frac{3}{5}\right)^x + \left(\frac{4}{5}\right)^x = 1 \quad ...(2)$$

let $\sin\theta = \frac{3}{5}$ then $\cos\theta = \frac{4}{5}$

hence the equation (2) is reduced to

$(\sin\theta)^x + (\cos\theta)^x = 1$

which is true for $x = 2$.

i.e. $x = 2$ is the only root of equation (1) infact, this is a well-known theorem known as Fermas theorem.

It states that

$$a^n + b^n \neq c^n \quad \text{for} \quad n > 2$$

where $a, b, c \in z$ and $n \in N$.

3.(b) Length of the rectangle box $= l(t)$
Width of the rectangle box $= w(t)$
Height of the rectangle box $= h(t)$
Since, length and width are increasing, we have

$$\frac{dl}{dt} = \pm 2 \text{ cm/sec} \Rightarrow \frac{dw}{dt} = +2 \text{ cm/sec}$$

And the height is decreasing, we have

$$\frac{dh}{dt} = -3 \text{ cm/sec}$$

Now, volume of the rectangular box

$$V = l \times w \times h$$

$$\Rightarrow \quad \frac{dV}{dt} = \frac{dl}{dt} \times wh + lh\frac{dw}{dt} + lw\frac{dh}{dt}$$

From question, we have,

$$l(0) = 10, \; w(0) = 8 \text{ and } h(0) = 20$$

$$\therefore \quad \frac{dV}{dt} = 8 \times 20 \times 2 + 10 \times 20 \times 2 + 10 \times 8 \times (-3)$$

$$\frac{dV}{dt} = 480 \text{ cm}^3/\text{sec}$$

Surface Area of the rectangular box

$$S = 2(lw + wh + lh)$$

$$\Rightarrow \quad \frac{dS}{dt} = 2\left(w.\frac{dl}{dt} + l.\frac{dw}{dt} + w.\frac{dh}{dt} + h.\frac{dw}{dt} + h.\frac{dl}{dt} + l.\frac{dh}{dt}\right)$$

$$\Rightarrow \quad \frac{dS}{dt} = 2[8(2) + 10(2) + 8(-3) + 20(2) + 20(2) + 10(-3)]$$

$$= 2[16 + 20 - 24 + 40 + 40 - 30]$$

$$= 2[62]$$

$$\therefore \quad \frac{dS}{dt} = 124 \text{ cm}^2/\text{sec}$$

Hence, volume and surface Area both are increasing with time.

Now, $V(0) = l(0).\, w(0).h(0)$

$= 10 \times 8 \times 20 = 1600 \text{ cm}^3$

Now, $V(5) = 1600 + 480 \times 5$

$= 1600 + 2400 = 4000 \text{ cm}^3$.

Again, $S(0) = 2(l(0).w(0) + w(0).h(0) + h(0).l(0))$

$= 2(10 \times 8 + 8 \times 20 + 20 \times 10)$

$S(0) = 2(80 + 160 + 200) = 880 \text{ cm}^2$

$S(5) = 880 + 124 \times 5$

$= 880 + 620 = 1500 \text{ cm}^2$.

3.(c) Given lines are

$$(x - 3)/3 = (y - 8)/-1 = (z - 3)/1 = r_1 \text{ (say)} \quad ...(1)$$

and $(x + 3)/-3 + (y + 7)/2 = (z - 6)/4 = r_2$ (say) ...(2)

Any point on line (1) is $P(3r_1 + 3, 8 - r_1, r_1 + 3)$ and on line (2) is $Q(-3 - 3r_2, 2r_2 - 7, 4r_2 + 6)$.

If PQ is the line of the shortest distance, then direction ratios of PQ

$= (3r_1 + 3) - (-3 - 3r_2), (8 - r_1) - (2r_2 - 7), (r_1 + 3) - (4r_2 + 6)$.

That is, $3r_1 + 3r_2 + 6, -r_1 - 2r_2 + 15, r_1 - 4r_2 - 3$

As PQ is perpendicular to lines (1) and (2), therefore,

$3(3r_1 + 3r_2 + 6) - 1(-r_1 - 2r_2 + 15) + 1(r_1 - 4r_2 + 3) = 0$

$\Rightarrow \quad 11r_1 + 7r_2 = 0 \quad ...(3)$

and $-3(3r_1 + 3r_2 + 6) + 2(-r_1 - 2r_2 + 15) + 4(r_1 - 4r_2 + 3) = 0$

$\Rightarrow \quad 7r_1 + 11r_2 = 0 \quad ...(4)$

On solving equations (3) and (4), we get

$$r_1 = r_2 = 0$$

So, points P(3, 8, 3) and Q(–3, –7, 6).

Therefore, length of the shortest distance

$$PQ = \sqrt{\{(-3-3)^2 + (-7-8)^2 + (6-3)^2\}} = 3\sqrt{30}$$

Direction ratios of shortest distance line are 2, 5 and –1.

Therefore equation of the shortest distance line is $(x - 3)/2 = (y - 8)/5 = (z - 3)/-1$.

4.(a) Row Operation 1: $\begin{bmatrix} 2 & 1 & 3 & 0 \\ 3 & 0 & 2 & 5 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 3 \end{bmatrix}$

multiply the 1st row by **1/2**

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 3 & 0 & 2 & 5 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 3 \end{bmatrix}$$

Row Operation 2: $\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 3 & 0 & 2 & 5 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 3 \end{bmatrix}$

add **–3** times the 1st row to the 2nd row

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & \frac{-3}{2} & \frac{-5}{2} & 5 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 3 \end{bmatrix}$$

Row Operation 3: $\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & \frac{-3}{2} & \frac{-5}{2} & 5 \\ 1 & 1 & 1 & 1 \\ 2 & 1 & 1 & 3 \end{bmatrix}$

add **–1** times the 1st row to the 3rd row

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & \frac{-3}{2} & \frac{-5}{2} & 5 \\ 0 & \frac{1}{2} & \frac{-1}{2} & 1 \\ 2 & 1 & 1 & 3 \end{bmatrix}$$

Row Operation 4: $\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & \frac{-3}{2} & \frac{-5}{2} & 5 \\ 0 & \frac{1}{2} & \frac{-1}{2} & 1 \\ 2 & 1 & 1 & 3 \end{bmatrix}$

add **–2** times the 1st row to the 4th row

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & \frac{-3}{2} & \frac{-5}{2} & 5 \\ 0 & \frac{1}{2} & \frac{-1}{2} & 1 \\ 0 & 0 & -2 & 3 \end{bmatrix}$$

Row Operation 5: $\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & \frac{-3}{2} & \frac{-5}{2} & 5 \\ 0 & \frac{1}{2} & \frac{-1}{2} & 1 \\ 0 & 0 & -2 & 3 \end{bmatrix}$

multiply the 2nd row by **−2/3**

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 1 & \frac{5}{3} & \frac{-10}{3} \\ 0 & \frac{1}{2} & \frac{-1}{2} & 1 \\ 0 & 0 & -2 & 3 \end{bmatrix}$$

Row Operation 6: $\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 1 & \frac{5}{3} & \frac{-10}{3} \\ 0 & \frac{1}{2} & \frac{-1}{2} & 1 \\ 0 & 0 & -2 & 3 \end{bmatrix}$

add **−1/2** times the 2nd row to the 3rd row

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 1 & \frac{5}{3} & \frac{-10}{3} \\ 0 & 0 & \frac{-4}{3} & \frac{8}{3} \\ 0 & 0 & -2 & 3 \end{bmatrix}$$

Row Operation 7: $\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 1 & \frac{5}{3} & \frac{-10}{3} \\ 0 & 0 & \frac{-4}{3} & \frac{8}{3} \\ 0 & 0 & -2 & 3 \end{bmatrix}$

multiply the 3rd row by **−3/4**

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 1 & \frac{5}{3} & \frac{-10}{3} \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -2 & 3 \end{bmatrix}$$

Row Operation 8: $\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 1 & \frac{5}{3} & \frac{-10}{3} \\ 0 & 0 & 1 & -2 \\ 0 & 0 & -2 & 3 \end{bmatrix}$

add **2** times the 3rd row to the 4th row

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 1 & \frac{5}{3} & \frac{-10}{3} \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Row Operation 9: $\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 1 & \frac{5}{3} & \frac{-10}{3} \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$

multiply the 4th row by **−1**

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 1 & \frac{5}{3} & \frac{-10}{3} \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Row Operation 10: $\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 1 & \frac{5}{3} & \frac{-10}{3} \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

add **2** times the 4th row to the 3rd row

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 1 & \frac{5}{3} & \frac{-10}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Row Operation 11: $\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 1 & \frac{5}{3} & \frac{-10}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

add 10/3 times the 4th row to the 2nd row

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 1 & \frac{5}{3} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Row Operation 12: $\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 1 & \frac{5}{3} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

add **–5/3** times the 3rd row to the 2nd row

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Row Operation 13: $\begin{bmatrix} 1 & \frac{1}{2} & \frac{3}{2} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

add **–3/2** times the 3rd row to the 1st row

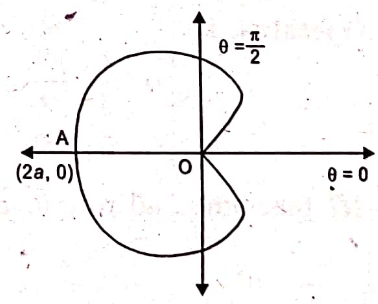
$$\begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Row Operation 14: $\begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

add **–1/2** times the 2nd row to the 1st row

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

4.(*b*) The volume of the solid is generated by revolving the upper half of the cardioid about the initial line $\theta = 0$. For the region above the initial line, θ varies from 0 to π.



$$V = \int_0^{\pi} \frac{2}{3}\pi r^3 \sin\theta\, d\theta$$

$$= \frac{2\pi}{3}\int_0^{\pi} a^3 (1+\cos\theta)^3 \sin\theta\, d\theta$$

Putting,

$(1 + \cos\theta) = t,$

$\sin\theta\, d\theta = dt$

when, $\theta = 0,\ t = 0$

$\theta = \pi,\ t = 2$

$$V = \frac{2\pi}{3} a^3 \int_0^2 t^3\, dt$$

$$= \frac{2\pi}{3} a^3 \left| \frac{t^4}{4} \right| = \frac{8}{3}\pi a^3.$$

4.(c) Let the variable plane, which is parallel to the given plane, be

$x/a + y/b + z/c = k$ ...(i)

This meets the axes in A$(ak, 0, 0)$, B$(0, bk, 0)$ and C$(0, 0, ck)$.

Hence, equation of the sphere OABC is

$x^2 + y^2 + z^2 - akx - bky - ckz = 0$

or $x^2 + y^2 + z^2 - k(ax + by + cz) = 0$ ...(ii)

The circle ABC lies on both, the plane (i) and the sphere (ii). Hence, (i) and (ii) together represent the circle ABC and the locus of the circle ABC will be obtained by eliminating k from (i) and (ii). Thus, the locus of circle ABC is

$(x^2 + y^2 + z^2) - (x/a + y/b + z/c)(ax + by + cz) = 0$

or $yz(b/c + c/b) + zx(c/a + a/c) + xy(a/b + b/a) = 0$.

5.(a) Given differential equation:

$$(D^2 + 1)y = x^2 . \sin(2x)$$

Let $y_h = e^{mx}$ be a trial solution of the corresponding homogeneous equation $(D^2 + 1)y = 0$, for some real or complex values of m.

That is $(D^2 + 1)e^{mx} = 0$

$\Rightarrow$ $(m^2 + 1)e^{mx} = 0$

$\Rightarrow$ $(m^2 + 1) = 0$

{Since $e^{mx} \neq 0$ for any m}

$\Rightarrow$ $m = \pm i$

So, the solution of the homogeneous equation is e^{ix} and e^{-ix}.

Thus, y_h is some linear combination of e^{ix} and e^{-ix}:

$$y_h = C_1 e^{ix} + C_2 e^{-ix}$$

$C_1, C_2 \in R$ being arbitrary

Now, Let y_p be the particular solution of the given differential equation.

Then, $(D^2 + 1)y_p = x^2 . \sin(2x)$

$$\Rightarrow \quad y_p = \frac{1}{(D^2+1)} x^2 . \sin(2x)$$

$$= \text{Imaginary part of } \frac{1}{(D^2+1)} x^2 . e^{2ix}$$

$$= \text{Imaginary part of } e^{2ix} . \frac{1}{(D+2i)^2+1} . x^2$$

$$= \text{Imaginary part of } e^{2ix} . \frac{1}{(D^2+4iD-3)} . x^2$$

= Imaginary part of

$$e^{2ix} . \frac{1}{-3\left(1 - \frac{D^2+4iD}{3}\right)} . x^2$$

= Imaginary part of

$$e^{2ix} . -\frac{1}{3}\left[1 - \frac{D^2+4iD}{3}\right]^{-1} . x^2$$

= Imaginary part of e^{2ix},

$$-\frac{1}{3}\left[1 + \frac{D^2+4iD}{3} + \frac{(D^2+4iD)^2}{9}\right] x^2$$

= Imaginary part of e^{2ix},

$$-\frac{1}{3}\left[x^2 + \frac{2}{3} + \frac{8x}{3} i - \frac{16(2)}{9}\right]$$

= Imaginary part of e^{2ix},

$$-\frac{1}{3}\left[x^2 + \frac{8x}{3} i - \frac{26}{9}\right]$$

= Imaginary part of $(\cos 2x + i \sin 2x)$,

$$\frac{-1}{3}\left[x^2 + \frac{8x}{3} i - \frac{26}{9}\right]$$

$$= -\frac{1}{3}\left[\frac{8x}{3} \cos 2x + \left(x^2 - \frac{26}{9}\right) \sin 2x\right]$$

$\therefore$ The complete solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$\therefore \quad y = C_1 e^{ix} + C_2 e^{-ix}$$

$$-\frac{1}{3}\left[\frac{8x}{3} \cos 2x + \left(x^2 - \frac{26}{9}\right) \sin 2x\right].$$

5.(b) Hint is $x^2 = X$ and $y^2 = Y$

$\Rightarrow$ $2x\, dx = dX$, $2y dy = dY$

$$\frac{dY}{dX} = \frac{2ydy}{2xdx} = \frac{y\, dy}{x\, dx}$$

Let $\frac{dY}{dX} = P$ and $\frac{dy}{dx} = p$

$$\therefore \qquad P = \frac{y}{x} p$$

$$\Rightarrow \qquad py = xP$$

$$\Rightarrow \qquad p = \frac{Px}{y} \qquad ...(2)$$

(1) $\Rightarrow p^2xy + p(x^2 - y^2 - h^2) - xy = 0$ ...(3)

(2) and (3)

$$\Rightarrow \left(\frac{P^2x^2}{y^2}\right)xy + \left(\frac{Px}{y}\right)(x^2 - y^2 - h^2) - xy = 0$$

Dividing by (x/y), we get:

$P^2x^2 + P(x^2 - y^2 - h^2) - y^2 = 0$

$\Rightarrow P^2X + P(X - Y - h^2) - Y = 0$

$\Rightarrow P^2X + PX - PY - h^2P - Y = 0$

$$\Rightarrow \qquad Y(1 + P) = X(P^2 + P) - h^2P$$

$$\Rightarrow \qquad Y = \frac{P(P+1)}{1+P}X - \frac{h^2P}{1+P}$$

$$\Rightarrow \qquad Y = PX - \frac{h^2P}{P+1}$$

which is in Clairaut's form.

Its solution is given by

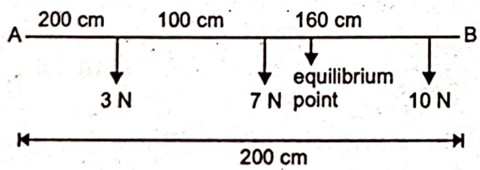
$$Y = cX - \frac{h^2c}{c+1}$$

where c is an arbitrary constant.

$\therefore$ The general solution of (1) is

$$y^2 = cx^2 - \frac{h^2c}{c+1}.$$

5.(c)



Let AB is a rod of length 200 cm.
and equilibrium point is at distance x cm from end A.

From equation of momentum

$$(160 - x) \times 10 = (x - 100)7 + (x - 20)3$$

$$1600 - 10x = 7x - 700 + 3x - 60$$

$$10x + 7x + 3x = 1600 + 700 + 60$$

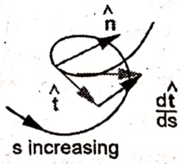
$$20x = 2360$$

$$x = \frac{2360}{20} \Rightarrow x = 118 \text{ cm}$$

Hence, rod must be supported at point 118 cm from end A, so that to remain horizontal.

5.(d) 1. Unit tangent $\hat{t}$: Obvious choice is

$$\hat{t} = dr(s)/ds$$



2. Principal Normal $\hat{n}$

Proved earlier that $|a(t)|$ = const then $a \cdot da/dt = 0$.

So $\qquad \hat{t} = \hat{t}(s)$

$$|\hat{t}| = \text{const}$$

$$\Rightarrow \qquad \hat{t}\,.\,d\hat{t}/ds = 0$$

Hence, the principal normal $\hat{n}$ is defined from

$$\kappa\hat{n} = d\hat{t}/ds$$

where $\kappa \geq 0$ is the curve's curvature.

3. The Binormal $\hat{b}$: The third number of a local r-h set is the binormal,

$$\hat{b} = \hat{t} \times \hat{n}$$

Tangent $\hat{t}$, Normal $\hat{n} : d\hat{t}/ds = \kappa\hat{n}$,

Binormal $\qquad \hat{b} = \hat{t} \times \hat{n}$

- Since $\hat{b}.\hat{t} = 0$, if we differentiate wrt s..

$$\frac{d\hat{b}}{ds}.\hat{t} + \hat{b}.\frac{d\hat{t}}{ds} = \frac{d\hat{b}}{ds}.\hat{t} + \hat{b}.\kappa\hat{n} = 0$$

from which

$$\frac{d\hat{b}}{ds}.\hat{t} = 0$$

- This means that $d\hat{b}/ds$ is along the direction of $\hat{n}$:

$$\frac{d\hat{b}}{ds} = -\tau(s)\hat{n}(s)$$

where τ is the space curve's torsion.

Tangent $\hat{t}$, Normal $\hat{n}$, Binormal $\hat{b} = \hat{t} \times \hat{n}$

$$d\hat{t}/ds = \kappa\hat{n}$$

$$d\hat{b}/ds = -\tau(s)\hat{n}(s)$$

- Differentiating $\hat{n}.\hat{t} = 0$

$$(d\hat{n}/ds).\hat{t} + \hat{n}.(d\hat{t}/ds) = 0$$

$$(d\hat{n}/ds).\hat{t} + \hat{n}.\kappa\hat{n} = 0$$

$$(d\hat{n}/ds).\hat{b} = -\kappa$$

- Now do the same to $\hat{n}.\hat{b}=0$:

$$(d\hat{n}/ds).\hat{b}+\hat{n}.(d\hat{b}/ds)=0$$

$$(d\hat{n}/ds).\hat{b}+\hat{n}.(-\tau)\hat{n}=0$$

$$(d\hat{n}/ds).\hat{b}=+\tau$$

- Hence, $\dfrac{d\hat{n}}{ds}=-\kappa(s)\hat{t}(s)+\tau(s)\hat{b}(s)$.

5.(e) First, note that the derivative of the coefficient of "dx", $10x^4-2xy^3$, with respect to y, is $-6xy^2$ and that the derivative of the coefficient of "dy", $-3x^2y^2$ is with respect to x also $-6xy^2$. That tells us that the integral is independent of the path. One method of doing this would be to choose some simple path, say the straight line between (0, 0) and (2, 1) or perhaps the "broken line" from (0, 0) to (2, 0) and then from (2, 0) to (2, 1).

But a more fundamental method is to use the "fundamental theorem of calculus"–find a function, $f(x, y)$, such that

$df = fxdx + fydy = (10x^4 - 2xy^3)dy - 3x^2y^3dy$.

That means we must have $fx = 10x^4 - 2xy^3$ and, since that partial derivative is just the derivative with respect to x, treating y as a constant, we take the antiderivative treating y as a constant: $f(x, y) = 2x^5 - x^2y^3 + u(y)$. Notice the "$u(y)$". That is the "constant of integration" but, since we are treating y as a constant, it may, in fact be a function of y. Differentiating that function with respect to y, $fy = -3x^2y^2 + u'(y)$ and that must be equal to $-3x^2y^2$.

So $\quad -3x^2y^2 + u' = -3x^2y^2$

and u really is a constant.

We have, $\quad f(x, y) = 2x^5 - x^2y^3 + C$.

Evaluate that at (2, 1) and (0, 0) and subtract.

7.(a) The A.E. is

$$m^2+4=0$$

i.e., $\quad m^2=-4$

$$m=\pm 2i$$

$$\text{C.F.} = C_1\cos 2x + C_2 \sin 2x$$

$$\text{P.I.} = \frac{1}{D^2+4}\sin^2(2t)$$

Where, $\sin^2 2t = \dfrac{1-\cos 4t}{2}$

$$=\frac{1}{D^2+4}\left[\frac{1-\cos 4t}{2}\right]$$

$$=\frac{1}{2}\frac{1}{D^2+4}(1-\cos 4t)$$

$$=\frac{1}{2}\left[\frac{e^{0x}}{D^2+4}-\frac{\cos 4t}{D^2+4}\right]$$

$$=\frac{1}{2}[\text{P.I.}_1-\text{P.I.}_2]$$

$$\text{P.I.}_1 = \frac{e^{0x}}{D^2+4}=\frac{1}{4} \qquad (D\to 0)$$

$$\text{P.I.}_2 = \frac{1}{D^2+4}\cos 4t \qquad (D^2\to -2^2)$$

$$=\frac{1}{-2^2+4}\cos 4t \qquad (Dr=0)$$

Differentiate the Dr and multiplying by 't'.

$$=\frac{1}{2D}x\cos 2x\times\frac{D}{D}$$

$$=\frac{-t\sin 4t.4}{2D^2} \qquad (D^2\to -2^2)$$

$$=\frac{-4t\sin 4t}{2(-2^2)}=\frac{4t\sin 4t}{8}=\frac{t\sin 4t}{2}$$

$$\text{P.I.} = \frac{1}{2}\left[\frac{1}{4}-\frac{t\sin 4t}{4}\right]=\frac{1}{8}-\frac{t\sin 4t}{4}$$

∴ The general solution is

$$y = \text{C.F.} + \text{P.I.}$$

$$y = C_1\cos 2x + C_2\sin 2x + \frac{1}{8}-\frac{t}{4}\sin 4t.$$

7.(b) Let r be the radius of the base of the hemisphere or cone then the height of the base = $2r$ (given)

i.e., $\quad$ VK = $2r$ or VL = LK = r

Also from similar triangles VLF and VKB, we have

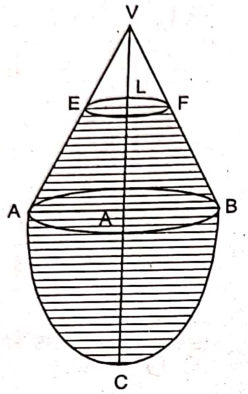
$$\frac{LF}{KB}=\frac{VL}{VK}=\frac{r}{2r}=\frac{1}{2}$$

or $\quad \text{LF} = \frac{1}{2}\text{KB} = \frac{1}{3}r$

$\therefore$ The volume of the frustum

$$\text{ABFE} = \frac{\pi}{3}h\left(r_1^2 + r_2^2 + r_1r_2\right)$$

$$= \frac{1}{3}\pi r\left[r^2 + \left(\frac{1}{2}r\right)^2 + r\left(\frac{1}{2}r\right)\right] = \frac{7}{12}\pi r^3$$



And the volume of the hemisphere $= \frac{2}{3}\pi r^3$.

$\therefore$ Weight of the liquid contained in the vessel = (volume of the frustum + volume of the hemisphere) $g\rho$

$$= \left(\frac{7}{12}\pi r^3 + \frac{2}{3}\pi r^3\right)g\rho = \frac{5}{4}\pi r^3 g\rho,$$

where ρ is the density of the liquid.

Now the resultant vertical thrust on the vessel, which is partly pressed upwards and partly downwards = weight of the liquid contained

$$= \frac{5}{4}\pi r^3 g\rho.$$

7.(c) In differential geometry, the **Frenet-Serret formulas** describe the kinematic properties of a particle moving along a continuous, differentiable curve in three-dimensional Euclidean space $\mathbb{R}^3$, or the geometric properties of the curve itself irrespective of any motion. More specifically, the formulas describe the derivatives of the so-called **tangent, normal, and binormal** unit vectors in terms of each other.

The tangent, normal and binormal unit vectors, often called **T**, **N** and **B** or collectively the **Frenet-Serret frame** or **TNB** frame, together form an orthonormal basis spanning $\mathbb{R}^3$ and are defined as follows:

- T is the unit vector tangent to the curve, pointing in the direction of motion.
- N is the Normal unit vector, the derivative of T with respect to the arc length parameter of the curve, divided by its length.
- B is the binormal unit vector, the cross product of T and N.

The Frenet-Serret formulas are:

$$\frac{dT}{ds} = \kappa N; \quad \frac{dN}{ds} = -\kappa T + \tau B,$$

$$\frac{dB}{ds} = -\tau N,$$

where d/ds is the derivative with respect to arclength, κ is the curvature, and τ is the torsion of the curve. The two scalars κ and τ effectively define the curvature and torsion of a space curve. The associated collection, **T, N, B,** κ, and τ, is called the **Frenet-Serret apparatus**. Intuitively, curvature measures the failure of a curve to be a straight line, while torsion measures the failure of a curve to be planar.

The Frenet-Serret formulas are also known as Frenet-Serret theorem, and can be stated more concisely using matrix notation.

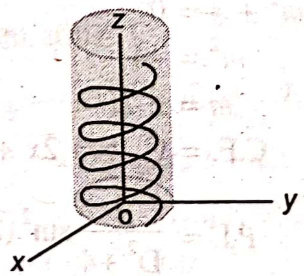
$$\begin{bmatrix} T' \\ N' \\ B' \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix}\begin{bmatrix} T \\ N \\ B \end{bmatrix}$$

This matrix is skew-symmetric.

The space curve is a circular helix (see adjacent figure). Since $t = z/4$, the curve has equations $x = 3\cos(z/4)$, $y = 3\sin(z/4)$ and therefore lies on the cylinder $x^2 + y^2 = 9$.

(*a*) The position vector for any point on the curve is

$$r = 3\cos t\, i + 3\sin t\, j + 4t\, k$$



Then, $\frac{dr}{dt} = -3 \sin t\, i + 3 \cos i\, j + 4\, k$

$$\frac{ds}{dt} = \left|\frac{dr}{dt}\right| = \sqrt{\frac{dr}{dt}\cdot\frac{dr}{dt}}$$

$$= \sqrt{(-3\sin t)^2 + (3\cos t)^2 + 4^2} = 5$$

Thus, $T = \frac{dr}{ds} = \frac{dr/dt}{ds/dt}$

$$= -\frac{3}{5}\sin t\, i + \frac{3}{5}\cos t\, j + \frac{4}{5}k$$

(*b*) $\frac{dT}{dt} = \frac{d}{dt}\left(-\frac{3}{5}\sin t\, i + \frac{3}{5}\cos t\, j + \frac{4}{5}k\right).$

$$= -\frac{3}{5}\cos t\, i - \frac{3}{5}\sin t\, j$$

$$\frac{dT}{ds} = \frac{dT/dt}{ds/dt}$$

$$= -\frac{3}{25}\cos t\, i - \frac{3}{25}\sin t\, j$$

Since $\frac{dT}{ds} = \kappa N$

$$\left|\frac{dT}{ds}\right| = |\kappa|\,|N| = \kappa \text{ as } \kappa \geq 0$$

Then,

$$\kappa = \left|\frac{dT}{ds}\right| = \sqrt{\left(-\frac{3}{25}\cos t\right)^2 + \left(-\frac{3}{25}\sin t\right)^2}$$

$= \frac{3}{25}$ and $\rho = \frac{1}{\kappa}$ From $\frac{dT}{ds} = \kappa N$,

We obtain, $N = \frac{1}{\kappa}\frac{dT}{ds} = -\cos t\, i - \sin t\, j.$

(*c*) $B = T \times N$

$$= \begin{vmatrix} -\frac{3}{5}\sin t & \frac{3}{5}\cos t & \frac{4}{5} \\ -\cos t & -\sin t & 0 \end{vmatrix}$$

$$= \frac{4}{5}\sin t\, i - \frac{4}{5}\cos t\, j + \frac{3}{5}k$$

$$\frac{dB}{dt} = \frac{4}{5}\cos t\, i + \frac{4}{5}\sin t\, j$$

$$\frac{dB}{ds} = \frac{dB/dt}{ds/dt} = \frac{4}{25}\cos t\, i + \frac{4}{25}\sin t\, j$$

$$-\tau N = -\tau(-\cos t\, i - \sin t\, j)$$

$$= \frac{4}{25}\cos t\, i + \frac{4}{25}\sin t\, j$$

or $\tau = \frac{4}{25}$ and $\sigma = \frac{1}{\tau} = \frac{25}{4}.$

Paper-II

1.(*a*) Suppose $chR = n \neq 0$ and suppose n is not a prime. Then $n = m_1 \cdot m_2$ where m_1 and m_2 are proper divisors of n. For any $a \in R$, $a \neq 0$, we have,

$$0 = na^2 = (m_1 m_2)a^2$$
$$= (m_1 a)(m_2 a)$$

Since R is an integral domain,

$$m_1 a = 0 \text{ or } m_2 a = 0$$

Suppose $m_1 a = 0$.

Then we show that $m_1 x = 0$ for any $x \in R$.

Now $m_1(xa) = (m_1 x)a = x(m_1 a) = x_0 = 0$

Since, $a \neq 0$, and R is an integral domain $m_1 x = 0$.

Thus $m_1 x = 0$ for all x, $m_1 < n$.

This contradicts the assumption that $chR = n$.

Hence n is a prime.

Corollary. The *ch* of a field is either 0 or a prime.

1.(*b*) It suffices to show that for some $\varepsilon > 0$ there is no $\delta > 0$ such that $|x - y| < \delta$ implies $|\sin(1/x) - \sin(1/y)| < \varepsilon$.

We shall do this for $\varepsilon = 1$. Given $\delta > 0$, we choose $x, y \in (0, \delta)$ so that $\sin(1/x) = 1$ and $\sin(1/y) = -1$. (For example, we can let x (resp. y) be the reciprocal of $(n + 1/2)\pi$ (resp. of $(n + 3/2)\pi$) for some integer $n > 1/\delta$).

Then $|x - y| < \delta$ but $|\sin(1/x) - \sin(1/y)| = 2 > \varepsilon$.