2014

Page:

① a] let G be the set of all real 2×2 matrices $\begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$, where $xz \neq 0$ Show that G is a Group under matrix multiplication. let N denote the subset $\left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} : a \in R \right\}$. is N a normal subgroup of G? justify your answer. (10)

ans:- Things to know:-

Group:- A non ϕ set G, together with a binary operation $*$ is said to form a Group, if it satisfies the following postulates

(i) Associativity: $a*(b*c) = (a*b)*c$.
for all $a, b, c \in G$

ii) Existence of identity:
$\exists$ an element $e \in G \ni$
$a*e = e*a = a$ for all $a \in G$

iii) Existence of inverse:-
for every $a \in G$ $\exists$ $a' \in G \ni$
$a*a' = a'*a = e$

Scanned with CamScanner

Normal Subgroup:-

A subgroup H of a Group G is called a normal subgroup of G if $Ha = aH$ for all $a \in G$.

or

$\forall\ a \in G \qquad a^{-1} H a = H$

soln:- 1] let $a = \begin{bmatrix} x_1 & y_1 \\ 0 & z_1 \end{bmatrix}$, $b = \begin{bmatrix} x_2 & y_2 \\ 0 & z_2 \end{bmatrix}$

$x_1 z_1 \neq 0 \qquad x_2 z_2 \neq 0$ --- ①

$$a \cdot b = \begin{bmatrix} x_1 x_2 & x_1 y_2 + y_1 z_2 \\ 0 & z_1 z_2 \end{bmatrix}$$

$x_1 x_2 \cdot z_1 z_2 \neq 0$ --- by ①

$\therefore\ a, b \in G \Rightarrow a \cdot b \in G$

$\therefore$ $(\cdot)$ is a binary operation on G or closour property holds

2] $a = \begin{bmatrix} x_1 & y_1 \\ 0 & z_1 \end{bmatrix}$ $b = \begin{bmatrix} x_2 & y_2 \\ 0 & z_2 \end{bmatrix}$ $c = \begin{bmatrix} x_3 & y_3 \\ 0 & z_3 \end{bmatrix}$

$x_1 z_1 \neq 0 \qquad x_2 z_2 \neq 0 \qquad x_3 z_3 \neq 0$

consider, $a \cdot (b \cdot c)$

$$= \begin{bmatrix} x_1 & y_1 \\ 0 & z_1 \end{bmatrix} \begin{bmatrix} x_2 x_3 & x_2 y_3 + y_2 z_3 \\ 0 & z_2 z_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 x_2 x_3 & x_1 x_2 y_3 + x_1 y_2 z_3 + y_1 z_2 z_3 \\ 0 & z_1 z_2 z_3 \end{bmatrix}$$

$(a \cdot b) \cdot c$

$$= \begin{bmatrix} x_1 x_2 & x_1 y_2 + y_1 z_2 \\ 0 & z_1 z_2 \end{bmatrix} \begin{bmatrix} x_3 & y_3 \\ 0 & z_3 \end{bmatrix}$$

$$= \begin{bmatrix} x_1 x_2 x_3 & x_1 x_2 y_3 + x_1 y_2 z_3 + y_1 z_2 z_3 \\ 0 & z_1 z_2 z_3 \end{bmatrix}$$

$\therefore \; a \cdot (b \cdot c) = (a \cdot b) \cdot c$

$\therefore$ $(\cdot)$ is associative.

3) let $e = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ be identity.

$$\begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$$

$$\begin{bmatrix} ax + 0 & bx + dy \\ 0 & zd \end{bmatrix} = \begin{bmatrix} x & y \\ 0 & z \end{bmatrix}$$

$zd = z$

$d = 1$

$ax = x \Rightarrow a = 1$

$bx + y = y$

$bx = 0$

$\Rightarrow b = 0$ -- $\{ \because xz \neq 0 \Rightarrow x \neq 0, z \neq 0$

$\therefore \; e = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

is identity element.

let a^{-1} be inverse of a

$$\therefore \begin{bmatrix} x & y \\ 0 & z \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ 0 & z_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \begin{bmatrix} x x_1 & x y_1 + y z_1 \\ 0 & z z_1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$x x_1 = 1$$

$$\Rightarrow x_1 = 1/x$$

$$z z_1 = 1 \Rightarrow z_1 = 1/z$$

$$x y_1 + y z_1 = 0$$

$$x y_1 = - y z_1$$

$$y_1 = -\frac{y}{x} z_1$$

$$y_1 = -\frac{y}{x} \cdot \frac{1}{z}$$

$$\because xz \neq 0$$

$\therefore y_1$ is defined

$$\therefore \begin{bmatrix} 1/x & -\frac{y}{xz} \\ 0 & \frac{1}{z} \end{bmatrix}$$ is inverse.

To show $A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ is normal

subgroup of g.

let $x = \begin{bmatrix} x_1 & y_1 \\ 0 & z_1 \end{bmatrix}$ $x^{-1} = \begin{bmatrix} \frac{1}{x_1} & \frac{-y_1}{x_1 z_1} \\ 0 & \frac{1}{z_1} \end{bmatrix}$

$\therefore$ consider,

$x^{-1} A x$

$$= \begin{bmatrix} \frac{1}{x_1} & \frac{-y_1}{x_1 z_1} \\ 0 & \frac{1}{z_1} \end{bmatrix} \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ 0 & z_1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{x_1} & \frac{-y_1}{x_1 z_1} \\ 0 & \frac{1}{z_1} \end{bmatrix} \begin{bmatrix} x_1 & y_1 + a z_1 \\ 0 & z_1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & \frac{-y_1^2}{x_1 z_1} + \frac{a(-y_1)}{x_1} \\ 0 & 1 \end{bmatrix}$$

whe $\frac{-y_1^2}{x_1 z_1} + \frac{a(-y_1)}{x_1} \in R$

$\therefore$ $x^{-1} A x \in A$

$\therefore$ A is normal subgroup of G.

2] a] Show that $\mathbb{Z}_7$ is a field. then find $([5]+[6])^{-1}$ and $(-[4])^{-1}$

Soln: $\mathbb{Z}_7 = \{[0], [1] [2] [3] [4] [5] [6]\}$

we have to show that $[\mathbb{Z}_7, +_7, \cdot_7)$ is field.

composition table w.r.t $+_7$

$+_7$	[0]	[1]	[2]	[3]	[4]	[5]	[6]
[0]	[0]	[1]	[2]	[3]	[4]	[5]	[6]
[1]	[1]	[2]	[3]	[4]	[5]	[6]	[0]
[2]	[2]	[3]	[4]	[5]	[6]	[0]	[1]
[3]	[3]	[4]	[5]	[6]	[0]	[1]	[2]
[4]	[4]	[5]	[6]	[0]	[1]	[2]	[3]
[5]	[5]	[6]	[0]	[1]	[2]	[3]	[4]
[6]	[6]	[0]	[1]	[2]	[3]	[4]	[5]

As we can see from composition table,
[0] is additive identity of $\mathbb{Z}_7$
[0] is in each row $\Rightarrow \forall a \in \mathbb{Z}_7$ additive inverse $-a$ exist.

$$([3]+[2])+[5] = [5]+[5]$$
$$= [3]$$
$$[3]+([2]+[5]) = [3]+[0]$$
$$= [3]$$

here associtivity.

$\times_7$	[1]	[2]	[3]	[4]	[5]	[6]
[1]	[1]	[2]	[3]	[4]	[5]	[6]
[2]	[2]	[4]	[6]	[1]	[3]	[5]
[3]	[3]	[6]	[2]	[5]	[1]	[4]
[4]	[4]	[1]	[5]	[2]	[6]	[3]
[5]	[5]	[3]	[1]	[6]	[4]	[2]
[6]	[6]	[5]	[4]	[3]	[2]	[1]

as we can see [1] is multiplicative identity

[1] is in each row $\Rightarrow$
$\forall\ [a] \in \mathbb{Z}_7 \setminus [0]$
$\exists\ [a]^{-1} \ni$
$[a][a]^{-1} = [1]$

$([a] \times [b]) \times [c]$
$= [(a \times b)] \times [c]$
$= [a \times b \times c]$
$= [a] \times [b \times c]$
$= [a] \times ([b] \times [c])$
Associative property satisfied.

Transpose of this composition table is same.
$\Rightarrow (\mathbb{Z}_7, +_7, \cdot_7)$ is commutative

$$([5]+[6])^{-1}$$

$$= [4]^{-1}$$

$$= [2]$$

$$(-[4])^{-1} = ([3])^{-1}$$

$$= [5].$$

(4) Prove that the set $Q(\sqrt{5}) = \{a + b\sqrt{5}; a, b \in Q\}$ is commutative ring with identity. (15)

Sol:- Ring:-

A non-ϕ set R, together with two binary compositions + and $\cdot$ is said to form a Ring If the following axioms are satisfied.

(1) $a + (b + c) = (a + b) + c \quad \forall a, b, c \in R$

2) $a + b = b + a$ for $a, b \in R$

3) $\exists$ some element 0 in R

$\ni a + 0 = 0 + a = a$ for all $a \in R$

4] For each $a \in R$ $\exists$ an element $(-a) \in R \ni a + (-a) = (-a) + a = 0$

5] $a \cdot (b \cdot c) = (a \cdot b) \cdot c$ for all $a, b, c \in R$

6] $a \cdot (b + c) = a \cdot b + a \cdot c$

7) $a \cdot b = b \cdot a$ -- commutative.

$x = a + b\sqrt{5}$, $a, b \in Q$

$y = c + d\sqrt{5}$ $c, d \in Q$

$x + y = (a + c) + (b + d)\sqrt{5}$

$a + c \in Q$ $b + d \in Q$

$\therefore x + y \in Q\sqrt{5}$

$\therefore$ closure property holds.

Scanned with CamScanner

$\Rightarrow (a+b\sqrt{5}) + (p+c\sqrt{5} + d+e\sqrt{5})$

$= (a+b\sqrt{5}) (p+d + (c+e)\sqrt{5})$

$= (a+p+d) + (b+c+e)\sqrt{5}$

$= (a+p+(b+c)\sqrt{5}) + (d+e\sqrt{5})$

$= ((a+b\sqrt{5}) + p+c\sqrt{5}) + (d+e\sqrt{5})$

$\therefore$ associativity holds.

$\Rightarrow$ let $c+d\sqrt{5} \in Q$

$c=d=0$

$\therefore$ 0 is additive identity.

$\Rightarrow$ let $x = a+b\sqrt{5} \in Q\sqrt{5}$ additive

$c+d\sqrt{5}$ is inverse of x

$\therefore a+b\sqrt{5} + c+d\sqrt{5} = 0$

$\therefore (a+c) + (b+d)\sqrt{5} = 0+0\sqrt{5}$

$\therefore a+c=0 \quad b+d=0$

$\therefore c=-a, \; d=-b$

$\therefore -a-b\sqrt{5}$ is additive inverse

$\Rightarrow x = a+b\sqrt{5}, \quad y = c+d\sqrt{5}$

$x \cdot y = (a+b\sqrt{5})(c+d\sqrt{5})$

$= ac + ad\sqrt{5} + bc\sqrt{5} + 5bd$

$= (ac+5bd) + (ad+bc)\sqrt{5}$

$\in Q(\sqrt{5})$

hence closure hold.

$\Rightarrow$ $x = a+b\sqrt{5}$ $\quad y = c+d\sqrt{5}$ $\quad z = e+f\sqrt{5}$

$x.(y.z)$

$= (a+b\sqrt{5})((c+d\sqrt{5}).(e+f\sqrt{5}))$

$= (a+b\sqrt{5}).(ce+5fd+\sqrt{5}(cf+ed))$

$= ace+5afd+5bcf+5bed$

$\quad + \sqrt{5}(acf+aed+bce+5bfd)$

$(x.y).z$

$= ((a+b\sqrt{5}).(c+d\sqrt{5}))(e+f\sqrt{5})$

$= (ac+5bd+\sqrt{5}(ad+bc))$

$\quad + (e+f\sqrt{5})$

$= (ace+5bde+5adf+5bcf$

$\quad + \sqrt{5}(acf+5bdf+ade+bce)$

associativity. holds.

$\Rightarrow$ $(a+b\sqrt{5})(c+d\sqrt{5}+e+f\sqrt{5})$

$= (a+b\sqrt{5})((c+e)+(d+f)\sqrt{5})$

$= (a+b\sqrt{5})(c+e)+(a+b\sqrt{5})(d+f)\sqrt{5}$

$= ac+ae+bc\sqrt{5}+be\sqrt{5}$

$+ (ad+df)\sqrt{5}+(bd+bf)5$

$= ac+5bd+\sqrt{5}(ad+bc)$

$\quad + ae+5bf+\sqrt{5}(af+be)$

$= (a+b\sqrt{5})(c+d\sqrt{5})+(a+b\sqrt{5})(e+f\sqrt{5})$

Scanned with CamScanner

$\Rightarrow$ $(a+b\sqrt{5})(c+d\sqrt{5})$

$= ac+bd5 + \sqrt{5}(ad+bc)$

$= ca+5db + \sqrt{5}(da+cb)$

$= (c+d\sqrt{5})(a+b\sqrt{5})$

here commutative.

let $x = a+b\sqrt{5}$ is arbitrary element of $Q(\sqrt{5})\setminus\{0\}$

$y = c+d\sqrt{5}$ is multiplicative identity.

$x \cdot y = x$

$(a+b\sqrt{5})(c+d\sqrt{5}) = a+b\sqrt{5}$

$(ac+5bd)+(ad+bc)\sqrt{5} = a+b\sqrt{5}$

$\Rightarrow ac+5bd = a$

$ad+bc = b$

$a(c-1)+5bd = 0$ $\qquad$ $b(c-1)+ad=0$

$c-1 = -\dfrac{5bd}{a} = -\dfrac{ad}{b}$

$\Rightarrow 5b^2d = a^2d$

$d(5b^2-a^2)=0$

$5b^2 \neq a^2$

$\therefore d=0$

$a(c-1)+5b0=0$

$a(c-1)=0$

$a\neq 0 \therefore c=1$

$\therefore$ identity $=1$ exist.

3) Show that the set $\{a+b\omega : \omega^3=1\}$ where $a, b \in R$, is a field w.r.t usual addition & multiplicn. — (13)

Soln:- Let $R = \{a+b\omega : \omega^3 = 1\}$

To show $(R, +)$ is Group

$\forall\ a_1+b_1\omega,\ a_2+b_2\omega$

$\Rightarrow$ $(a_1+b_1\omega)+(a_2+b_2\omega)$

$= (a_1+a_2)+(b_1+b_2)\omega$

$\because a_1+a_2 \in R,\ b_1+b_2 \in R$

hence closure property holds.

$\Rightarrow$ $(a_1+b_1\omega)+(a_2+b_2\omega)$

$= (a_1+a_2)+(b_1+b_2)\omega$

$= (a_2+a_1)+(b_2+b_1)\omega$

$= (a_2+b_2\omega)+(a_1+b_1\omega)$

hence commutative w.r.t +

$\Rightarrow$ $[(a_1+b_1\omega)+(a_2+b_2\omega)]+(a_3+b_3\omega)$

$= (a_1+a_2)+(b_1+b_2)\omega + a_3+b_3\omega$

$= (a_1+a_2+a_3)+(b_1+b_2+b_3)\omega$

$= (a_1+b_1\omega)+(a_2+b_2\omega+a_3+b_3\omega)$

associative holds.

$\Rightarrow$ $0+0\omega = 0 \in R$

$\therefore 0$ is identity element

$\Rightarrow$ $\forall\ a+b\omega\ \exists\ c+d\omega \ni$

$a+b\omega + c+d\omega = 0$

$(a+c) + (b+d)\omega = 0 + 0\omega$

$a = -c \quad b = -d$

$\therefore\ -a - b\omega$ is additive inverse.

$\therefore$ $(R,+)$ is commutative group.

$(R^*, \cdot)$

$\Rightarrow$ ~~let~~ $(a+b\omega)\cdot(c+d\omega)$

$= ac + ad\omega + bc\omega + bd\omega^2$

$= ac + ad\omega + bc\omega + bd(-1-\omega) \quad \{1+\omega+\omega^2 = 0\}$

$= (ac - bd) + (ad + bc - bd)\omega$

$\therefore$ closour property holds.

$\Rightarrow$ Associative property is satisfy over complex no.

$\Rightarrow$ $\cdot\ (a+b\omega)(c+d\omega) = (a+b\omega)$

~~at~~ $\cdot$ let $c=1,\ d=0$

$(a+b\omega)(1) = a+b\omega$

$\therefore\ 1 \in R^*$

identity holds

iv) $(a+b\omega)(c+d\omega)=1$

$$c+d\omega = \frac{1}{a+b\omega} \times \frac{a+b\omega^2}{a+b\omega^2}$$

$$= \frac{a+b\omega^2}{a^2+ab\omega^2+ab\omega+b^2}$$

$$= \frac{a+b\omega^2}{a^2+ab(\omega+\omega^2)+b^2}$$

$$= \frac{a+b\omega^2}{a^2-ab+b^2}$$

$$= \frac{a\cancel{+}-b-b\omega}{a^2-ab+b^2}$$

$$= \frac{a-b}{a^2-ab+b^2} + \frac{(-b\omega)}{a^2+b^2-ab}$$

$$= p+q\omega$$

$p, q \in R$

here inverse exist.

$\Rightarrow$ $(a+b\omega)(c+d\omega)$

$$= ac+ad\omega+bc\omega+bd\omega^2$$

$$= ac+bc\omega+ad\omega+bd\omega^2$$

$$= c(a+b\omega)+d(a\omega+b\omega^2)$$

$$= (c+d\omega)(a+b\omega)$$

$\therefore$ commutativity holds.

$(R^*, \cdot)$ is abelian group.

Scanned with CamScanner

$\Rightarrow$ Distributive property.

$(a+b\omega) \times [(c+d\omega) + (e+f\omega)]$

$= (a+b\omega)\,[(c+e) + (d+f)\omega]$

$= a(c+e) + (ad+af)\omega + bc\omega + be\omega + bd\omega^2 + bf\omega^2$

$= (a+b\omega)(c+d\omega) + (a+b\omega)\cdot(e+f\omega)$

hence $(R, +, \cdot)$ is field.