Mains Test Series – 2020

Test – 2 (Paper – II)

Answer Key

Modern Algebra, Real Analysis, Complex Analysis & LPP

1(a) which of the following multiplication tables defined on the set $G=\{a,b,c,d\}$ form a group? support your answer in each case.

(i)

$\circ$	a	b	c	d
a	a	c	d	a
b	b	b	c	d
c	c	d	a	b
d	d	a	b	c

(ii)

$\circ$	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

(iii)

$\circ$	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

$\circ$	a	b	c	d
a	a	b	c	d
b	b	a	c	d
c	c	d	a	d
d	d	d	b	c

Sol^n: let $G=\{a,b,c,d\}$

Corresponding to table (i):

G is not a group w.r.t '$\circ$' (multiplication) because an associating property does not hold.

i.e $a,b,c \in G \Rightarrow$

L.H.S $(a\circ d)\circ c = c\circ c = a$

R.H.S $a \circ (b \circ c) = a \circ c = d$

$\therefore$ LHS $\neq$ RHS

Corresponding to table (ii):

Clearly G is a group w.r.t 'o' (multiplication) and inverse of each element is it self and $o(G) = 4$

$\therefore$ G is known klein – 4 group.

Corresponding to table (iii):

Clearly G is a group w.r.t 'o' (multiplication)

Corresponding to table (iv):

Clearly G is not a group w.r.t 'o' (multiplication) because an every row and every Column does not Contain an identity element 'a'.

$\therefore$ Identity prop. does not satisfy.

1(b) Show that the ring Z_p of integers modulo p is a field if and only if p is prime.

Sol'n: Let Z_p be a field. Let if possible, p be not prime

Then $p = ab$, where $1 < a, b < p$; $a, b \in Z$

$\Rightarrow ab \equiv 0 \pmod{p}$

$\Rightarrow ab = 0$ in Z_p, where $a \neq 0, b \neq 0 \in Z_p$

$\Rightarrow Z_p$ has zero divisors

$\Rightarrow Z_p$ is not an integral domain.

This is a contradiction, since Z_p is a field implies Z_p is an integral domain.

We know Z_p is a finite commutative ring.

Now we show that Z_p is an integral domain.

Let $a, b \in Z_p$ be such that $ab = 0$ in Z_p. Then p divides ab

$\Rightarrow p \mid a$ (or) $p \mid b$, since p is prime

$\Rightarrow a = 0$ or $b = 0$ in Z_p

So $ab = 0$ in $Z_p \Rightarrow a = 0$ (or) $b = 0$ in Z_p $(a, b \in Z_p)$

Hence Z_p is a finite integral domain and so Z_p is a field.

1(C) Show that the function $f(x) = 1/x$, $x > 0$ is continuous in $(0,1)$ but not uniformly continuous.

Sol'n: we have $\lim_{x\to c} f(x) = \lim_{x\to c} \frac{1}{x} = \frac{1}{c} = f(c)$.

so $f(x)$ is continuous at each point c in $]0,1[$ and hence $f(x)$ is continuous in $]0,1[$.

Now we shall show that $f(x)$ is not uniformly continuous in $]0,1[$.

For any $\delta > 0$, we can find a positive integer m such that $\frac{1}{m} < \delta$.

Let $x_1 = \frac{1}{m}$ and $x_2 = \frac{1}{2m}$. Then $0 < x_1 < 1$ and $0 < x_2 < 1$ so that $x_1, x_2 \in]0,1[$.

we have $|x_1 - x_2| = \left|\frac{1}{m} - \frac{1}{2m}\right| = \left|\frac{1}{2m}\right| = \frac{1}{2m} < \frac{1}{m} < \delta$.

and $|f(x_1) - f(x_2)| = \left|\frac{1}{x_1} - \frac{1}{x_2}\right| = |m - 2m| = |-m| = m > \frac{1}{2}$.

[$\because m$ is a +ve integer]

Thus if we take $\varepsilon = \frac{1}{2} > 0$, then whatever $\delta > 0$ we try there exist $x_1, x_2 \in]0,1[$ such that

$$|x_1 - x_2| < \delta \text{ but } |f(x_1) - f(x_2)| > \varepsilon = \frac{1}{2}.$$

In this way for $\varepsilon = \frac{1}{2} > 0$, there exists no $\delta > 0$ such that $|f(x_1) - f(x_2)| < \varepsilon$ whenever

$$|x_1 - x_2| < \delta, \; x_1, x_2 \in]0,1[.$$

Hence $f(x) = \frac{1}{x}$ is not uniformly continuous in $]0,1[$.

1(d) Prove that the function f defined by

$$f(z) = \begin{cases} \frac{z^5}{|z|^4}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$ is not differentiable at $z=0$.

Sol'n: The given function $f(z) = \begin{cases} \frac{z^5}{|z|^4}, & z \neq 0 \\ 0, & z = 0 \end{cases}$

$$= \begin{cases} \frac{(x^5 - 10x^3y^2 + 5xy^4) + i(5x^4y - 10x^2y^3 + y^5)}{(x^2+y^2)^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

At the origin

$$\frac{\partial u}{\partial x} = \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x\to 0} \frac{x^5/x^4}{x} = 1.$$

$$\frac{\partial u}{\partial y} = \lim_{y\to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x\to 0} \frac{v(x,0) - v(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{y\to 0} \frac{v(0,y) - v(0,0)}{y} = \lim_{y\to 0} \frac{y^5/y^4}{y} = 1$$

$\therefore$ we see that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Hence Cauchy-Riemann conditions are satisfied at $z=0$.

Taking $h = re^{i\theta} \neq 0$ with $r \neq 0$, it follows that for $h \neq 0$.

$$f'(0) = \frac{f(h) - f(0)}{h} = \frac{h^4}{|h|^4} = e^{i4\theta}$$

and therefore, as $h \to 0$ along different paths, the difference quotient doesnot yield a unique value. Thus f is not differentiable at the origin.

1(e) → A manufacturer has three machines I, II and III installed in his factory. Machines I and II are capable of being operated for at the most 12hrs, where as machine III must be operated at least for 5 hrs a day. He produces only two items M and N each requiring the use of all the three machines. The number of hrs required for producing 1 unit of each of the items M and N on the three machines are given in the following table:

Item	Number of hrs required on machines.		
	I	II	III
M	1	2	1
N	2	1	1.25

He makes a profit of Rs. 600 and Rs. 400 on item M and N respectively. How many of each item should he produce so as to maximize his profit assuming that he can sell all the items that he produces? What will be the maximum profit?

Sol'n: Let x_1 and x_2 be the number of items M and N respectively.

For machine I: Machine I takes 1 hour for the item M and 2 hrs for the item N. Then, we have

$$x_1 + 2x_2 \leq 12$$

For machine II: Machine II takes 2 hour for the item M and 1 hour for the item N. Then, we have

$$2x_1 + x_2 \leq 12$$

For machine III: Machine III takes 2 hour for the item M and 1 hrs for the item N. Then, we have

$$x_1 + 1.25x_2 \geq 5$$

Objective function: Here the objective function is to maximize the total profit on items M and N:

Total profit = Rs. $(600x_1 + 400x_2)$

Therefore, the given problem can be written as

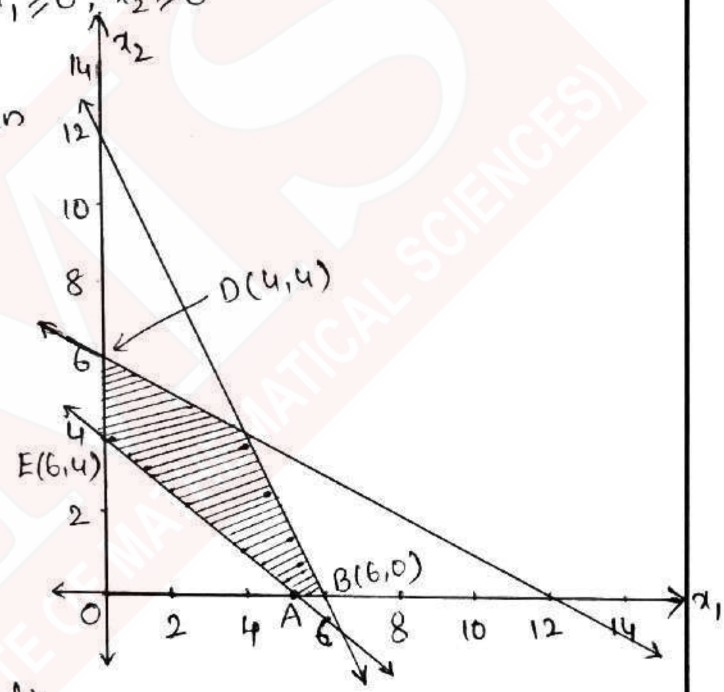
$$\text{Maximize } Z = 600x_1 + 400x_2$$

$$\text{subject to } x_1 + 2x_2 \le 12$$

$$2x_1 + x_2 \le 12$$

$$x_1 + 1.25x_2 \ge 5$$

$$\text{and } x_1 \ge 0,\ x_2 \ge 0$$



Step-1: Converting the given constraints into equations as:

$x_1 + 2x_2 = 12 \Rightarrow (12,0), (0,6)$

$2x_1 + x_2 = 12 \Rightarrow (6,0), (0,12)$

$x_1 + 1.25x_2 = 5 \Rightarrow (5,0), (0,4)$

Step-2: Draw these lines on x_1x_2 - plane. After making draw these lines, the feasible region (shaded) is shown fig. by ABCDEA.

The corners of the shaded regions are given by the points A (5,0), B (6,0) and C (4,4), D (0,6) and E (0,4).

Step-3: Let us evaluate Z at these corners.

at A (5,0), $Z = 600\times5 + 400\times0 = 3000$

at B (6,0), $Z = 3600$; at C (4,4), $Z = 4000$

at D (0,4), $Z = 2400$; at E (0,4), $Z = 1600$

Clearly, Z is maximum at $x_1 = 4$, $x_2 = 4$ and its maximum value is 4000.
Hence, the manufacturer has to produce 4 units of each item to obtain maximum profit of Rs. 4000.

2(a)(i) In S_3 give an example of two elements x, y such that $(x.y)^2 \neq x^2.y^2$.

Sol'n: Let $S_3 = \{e, \varphi, \varphi^2, \phi, \phi.\varphi, \varphi.\phi\}$ be a permutation group w.r.t permutation x^n defined as: $\varphi^3 = e$, $\phi^2 = e$ and

$$\phi.\varphi^i.\phi = \varphi^{-i};\ i \in N$$

let $x = \varphi$ and $y = \phi$

we have
$$(xy)^2 = (\varphi.\phi)^2$$
$$= (\varphi.\phi)(\varphi.\phi)$$
$$= \varphi.(\phi.\varphi.\phi) \quad \text{(by associ. in } S_3)$$
$$= \varphi.\varphi^{-1}$$
$$= e$$

where as
$$x^2y^2 = \varphi^2.\phi^2$$
$$= \varphi^2.e$$
$$= \varphi^2$$

$$\therefore (xy)^2 \underline{=} x^2y^2$$

2(a)(ii) Construct a multiplication table for $Z_2[i]$, the ring of Gaussian Integers modulo 2. Is this ring a field? Is it an integral domain?

Sol'n: let $Z_2[i] = \{a+ib \,/\, a, b \in (Z_2 = \{0,1\} +_2, \times_2)\}$

be the ring of Gaussian integers. w.r.t

$+^n$ & $\times^n$ under modulo 2.

then $Z_2[i] = \{0, 1, i, 1+i\}$

let us construct multiplication table with non-zero elements:

$\times$	1	i	$1+i$
1	1	i	$1+i$
i	i	1	$1+i$
$1+i$	$1+i$	$1+i$	0

$i^2 = -1$
= inverse of 1
= 1

$(1+i)(1+i) = 2+2i$

clearly $Z_2[i]$ is not a field. because from the above table, inverse of $1+i$ does not exist under $\times^n$ and also $Z_2[i]$ is not an integral domain because it has zero-divisors.

i.e $(1+i)\cdot(1+i) = 0$ for which $1+i \neq 0$.

2(b) Find three elements σ in S_9 with the property that $\sigma^3 = (1\,5\,7)(2\,8\,3)(4\,6\,9)$.

Sol'n: observe that if we start with a 9-cycle

$\sigma_{(say)} = (a_1\, a_2\, a_3\, a_4\, a_5\, a_6\, a_7\, a_8\, a_9)$ and cube it

we get $\sigma^3 = (a_1\, a_4\, a_7)(a_2\, a_5\, a_8)(a_3\, a_6\, a_9)$

$\therefore$ we can take

$$\sigma = (1\ 2\ 3\ 4\ 5\ 6\ 7\ 8\ 9)$$

But $(1\,5\,7)(2\,8\,3)(4\,6\,9) = (1\,5\,7)(4\,6\,9)(2\,8\,3)$

$= (2\,8\,3)(1\,5\,7)(4\,6\,9)$

So these given by

$(1\ 2\ 4\ 5\ 8\ 6\ 7\ 3\ 9)$, $(1\ 4\ 2\ 5\ 6\ 8\ 7\ 9\ 3)$

and $(2\ 1\ 4\ 8\ 5\ 6\ 3\ 7\ 9)$

2(c) Test the convergence and absolute convergence of the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+1}$.

Sol^n: we check the convergence of the given series by applying the Leibnitz Test:

$\sum (-1)^{n+1} . u_n$; $u_n > 0 \ \forall n$; is an alternating series, which converges if (i) $u_n > u_{n+1} \ \forall n$

(ii) $\lim_{n\to\infty} u_n = 0$

Given series is an alternating series

$\sum_{n=1}^{\infty} (-1)^{n+1} . \frac{n}{n^2+1}$; Here $u_n = \frac{n}{n^2+1}$

we consider; $f(x) = \frac{x}{x^2+1}$

then $f'(x) = \frac{1-x^2}{1+x^2} < 0 \ \forall \ x > 1$

Thus, f is a decreasing function and

hence $u_n \geq u_{n+1} \rightarrow$ [1st condition of Leibnitz test is satisfied]

Now $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{n}{n^2+1} = 0$

Hence 2nd condition of Leibnitz test satisfied.

Since both conditions of Leibnitz's test are satisfied.

$\therefore$ Given series is convergent.

we have $\sum \left| (-1)^{n+1} \frac{n}{n^2+1} \right| = \sum \frac{n}{n^2+1}$

$$= \sum v_n \text{ (say)}$$

Here $v_n = \dfrac{n}{n^2+1} \quad \forall\, n \in N$

$$= \frac{1}{n\left(1+\frac{1}{n^2}\right)} \quad \forall n \text{ ——— } (A)$$

Let $v_n' = \frac{1}{n} \quad \forall n$

we have $\lim\limits_{n\to\infty} \dfrac{v_n}{v_n'} = 1 \neq 0$

$\therefore$ By comparison test

$\sum v_n$ & $\sum v_n'$ are converge (or) diverge together.

$\because \sum v_n' = \sum \frac{1}{n}$ is divergent by P-test where $P=1$

$\therefore \sum v_n$ is also divergent

$\therefore \sum (-1)^{n+1} \dfrac{n}{n^2+1}$ is convergent but not absolutely convergent.

2(d) Evaluate $\int_0^{2\pi} \frac{d\theta}{(a+b\cos^2\theta)^2}$, where $a>b>0$.

Sol'n: Let $I = \int_0^{2\pi} \frac{d\theta}{(a+b\cos^2\theta)^2}$ ——— ①

Take C as unit circle $|z|=1$ or $z=e^{i\theta}$, $dz = ie^{i\theta}d\theta$

$$\Rightarrow \frac{dz}{iz} = d\theta, \quad 2\cos\theta = e^{i\theta}+e^{-i\theta} = z+z^{-1} = \frac{z^2+1}{z}$$

$$I = \int_0^{2\pi} \frac{4\,d\theta}{[2a+b(1+\cos 2\theta)]^2}$$

put $2\theta=\phi$, $2d\theta = d\phi$

$$\therefore \quad I = \int_0^{4\pi} \frac{2\,d\phi}{[2a+b(1+\cos\phi)]^2}$$

$$= 2\int_0^{2\pi} \frac{2\,d\phi}{[2a+b(1+\cos\phi)]^2}$$

$$I = \int_C \frac{4\,d\theta}{[2a+b(1+\cos\theta)]^2}$$

$$= 4\int_C \frac{1}{\left[2a+b\left\{1+\left(\frac{z^2+1}{2z}\right)\right\}\right]^2}\cdot\left(\frac{dz}{iz}\right)$$

$$= \frac{16}{i}\int_C \frac{z\,dz}{[2(2a+b)z+(z^2+1)b]^2}$$

$$\Rightarrow I = \frac{16}{ib^2}\int_C \frac{z\,dz}{[z^2+1+2cz]^2}, \quad c = \frac{2a+b}{b}$$

$$\Rightarrow I = \frac{16}{ib^2}\int_C f(z)\,dz \quad \text{——— ②}$$

where $f(z) = \frac{z}{(z^2+2cz+1)^2}$ ——— ③

For poles: $(z^2+2(z+1)^2=0$ (or) $z^2+2cz+1=0$

$$\Rightarrow z = \frac{-2c \pm 2\sqrt{c^2-1}}{2} = -c \pm \sqrt{(c^2-1)}$$

Take $\alpha = -c + \sqrt{(c^2-1)}$, $\beta = -(c+\sqrt{(c^2-1)})$

Then $\alpha\beta = 1$, $|\alpha|<1$, $|\beta|<1$

$z=\alpha$ is only pole of order 2 inside c.

$$f(z) = \frac{z}{[(z-\alpha)(z-\beta)]^2} = \frac{\phi(z)}{(z-\alpha)^2}$$

where $\phi(z) = \dfrac{z}{(z-\beta)^2}$, $\quad \text{Res}(z=\alpha) = \dfrac{\phi'(\alpha)}{\underline{|1}}$ —— (2)

$$\phi'(z) = \frac{(z-\beta)^2 - 2(z-\beta)}{(z-\beta)^4} = \frac{(z-\beta)-2z}{(z-\beta)^3} = -\frac{(\beta+z)}{(z-\beta)^3}$$

$$\Rightarrow \phi'(\alpha) = \frac{-(\alpha+\beta)}{(\alpha-\beta)^3} = -\frac{(-2c)}{(2\sqrt{c^2-1})^3} = \frac{2c}{8(c^2-1)^{3/2}} \cdots$$

$$\Rightarrow \text{Res}(z=\alpha) = \frac{\phi'(\alpha)}{\underline{|1}}$$

$$\int_C f(z)\,dz = 2\pi i \,\text{Res}(z=\alpha) = \frac{2\pi i}{4(c^2-1)^{3/2}}$$

Using (2), $I = \dfrac{16}{ib^2} \cdot \dfrac{2\pi i c}{4(c^2-1)^{3/2}} = \dfrac{8\pi c}{b^2(c^2-1)^{3/2}}$ —— (4)

Now $c^2-1 = \left(\dfrac{2a+b}{b}\right)^2 - 1 = \dfrac{(2a+b)^2-b^2}{b^2} = \dfrac{4a(a+b)}{b^2}$

Putting in (4),

$$I = \frac{8\pi}{b^2}\cdot\left(\frac{2a+b}{b}\right)\left\{\frac{b^2}{4a(a+b)}\right\}^{3/2}$$

$$I = \frac{\pi(2a+b)}{[a(a+b)]^{3/2}}$$

3(a) Let $z = \cos\theta + i\sin\theta$ be in T where $\theta \in \mathbb{Q}$. Prove that the order of z is infinite.

Sol'n: For $u \in \mathbb{C}$, $e^u = 1 \Leftrightarrow u = 2k\pi i$ for some $k \in \mathbb{Z}$.

Given $z = e^{i\theta}$, with $\theta \in \mathbb{Q}$.

Suppose $|z| = n$.

Then $e^{in\theta} = 1$.

This means that $in\theta = 2k\pi i$ for some $k \in \mathbb{Z} - \{0\}$.

Hence $\pi = \frac{n\theta}{2k} \in \mathbb{Q}$, which is a contradiction.

3(b) Examine the convergence of $\int_0^2 \frac{\log x}{\sqrt{2-x}}\,dx$.

Sol^n: Here $f(x) = \frac{\log x}{\sqrt{2-x}}$

clearly both 0 and 2 are points of infinite discontinuity of f on $[0,2]$. we may write

$$\int_0^2 f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx \quad \text{——— ①}$$

To test Convergence of $\int_0^1 f(x)\,dx$ at $x=0$

Since $f(x)$ is negative on $(0,1]$, we consider $-f(x)$

Take $g(x) = \frac{1}{x^n}$

$$\lim_{x\to 0+} \frac{-f(x)}{g(x)} = \lim_{x\to 0+} \frac{-x^n \log x}{\sqrt{2-x}} = 0 \text{ if } n>0$$

$$\left[\because \lim_{x\to 0+} x^n \log x = 0 \text{ if } n>0\right]$$

$\therefore$ Taking n between 0 and 1, the integral $\int_0^1 g(x)\,dx$ is convergent.

$\therefore$ By comparison test, $\int_0^1 -f(x)\,dx$ is also convergent.

To test the convergence of $\int_1^2 f(x)\,dx$ at $x=2$.

Take $g(x) = \frac{1}{\sqrt{2-x}}$

$$\lim_{x\to 2-} \frac{f(x)}{g(x)} = \lim_{x\to 2-} \log x = \log 2$$ which is non-zero and finite.

$\therefore$ By comparison test, $\int_1^2 f(x)\,dx$ and $\int_1^2 g(x)\,dx$ converge or diverge together.

But $\int_1^2 g(x)\,dx = \int_1^2 \frac{dx}{\sqrt{2-x}}$ $\quad\left|\right.$ form $\int_a^b \frac{dx}{(b-x)^n}$

is convergent. $\left(\because n = \frac{1}{2} < 1\right)$

$\therefore \int_1^2 f(x)\,dx$ is also convergent.

Hence, from (1), $\int_0^2 f(x)\,dx$ is convergent.

3(C) $f(x)$ is defined as follows:

$$f(x) = \begin{cases} \frac{1}{2}(b^2-a^2) & \text{for } 0<x<a \\ \frac{1}{2}b^2 - \frac{x^2}{6} - \frac{a^3}{3x} & \text{for } a<x<b \\ \frac{1}{3}\frac{b^3-a^3}{x} & \text{for } x>b \end{cases}$$

Prove that $f(x)$ and $f'(x)$ are continuous but $f''(x)$ is discontinuous.

Sol'n: Clearly $f(x)$ is continuous for $0<x<a$, $a<x<b$ and $x>b$.

The problem points are $x=a,b$

At $x=a$: $\lim_{x\to a-} f(x) = \lim_{x\to a} \frac{1}{2}(b^2-a^2) = \frac{1}{2}(b^2-a^2)$

$\lim_{x\to a+} f(x) = \lim_{x\to a} \frac{1}{2}b^2 - \frac{x^2}{6} - \frac{a^3}{3x} = \frac{1}{2}b^2 - \frac{a^2}{6} - \frac{a^3}{3a} = \frac{1}{2}(b^2-a^2)$

Since $\lim_{x\to a-} f(x) = \lim_{x\to a+} f(x) = f(a) = \frac{1}{2}(b^2-a^2)$

$\therefore f(x)$ is continuous at $x=a$

At $x=b$:

$\lim_{x\to b-} f(x) = \lim_{x\to b} \frac{1}{2}(b^2) - \frac{x^2}{6} - \frac{a^3}{3x} = \frac{b^2}{2} - \frac{b^2}{6} - \frac{a^3}{3b} = \frac{1}{3b}(b^3-a^3)$

$\lim_{x\to b+} f(x) = \lim_{x\to b} \frac{1}{3}\left(\frac{b^3-a^3}{x}\right) = \frac{1}{3b}(b^3-a^3)$

$\therefore \lim_{x\to b-} f(x) = \lim_{x\to b+} f(x) = f(b) = \frac{1}{3b}(b^3-a^3)$

$\therefore f(x)$ is continuous at $x=b$.

Thus $f(x)$ is continuous everywhere.

Clearly $f(x)$ is differentiable for all $x>0$ except possibly at $x=a$, $x=b$. which we examine next:

LHD at $x=a$ is $\frac{d}{dx}\left(\frac{1}{2}(b^2-a^2)\right)=0$

R.H.D at $x=a$ is $\frac{d}{dx}\left(\frac{b^2}{2}-\frac{x^2}{6}-\frac{a^3}{3x}\right)_{x=a} = \frac{-a}{3}+\frac{a^3}{3a^2}=0$

$\therefore$ LHD = RHD

$\therefore f'(x)$ exists at $x=a$ and $f'(a)=0$.

LHD at $x=b$ is $\frac{d}{dx}\left(\frac{1}{2}b^2-\frac{x^2}{6}-\frac{a^3}{3x}\right)_{x=b} = \frac{-b}{3}+\frac{a^3}{3b^2}=\frac{a^3-b^3}{3b^2}$

RHD at $x=b$ is $\frac{d}{dx}\left(\frac{1}{3}\frac{b^3-a^3}{x}\right)_{x=1} = -\frac{1}{3}\left(\frac{b^3-a^3}{x^2}\right)_{x=b} = \frac{a^3-b^3}{b^2}$

Thus $f(x)$ is differentiable at $x=b$ also.

Moreover it can easily be seen that

$$f'(x)=\begin{cases} 0, & 0<x\le a \\ -\frac{x}{3}+\frac{a^3}{3x^2}, & 0<x\le b \\ \frac{1}{3}\left(\frac{a^3-b^3}{x^2}\right), & x>b \end{cases}$$

It is obvious that $f'(x)$ is continuous for $0<x<a$, $a<x<b$ and $x>b$.

<u>Now we have to check at $x=a$ and $x=b$:</u>

$\underset{x\to a-}{Lt} f'(x)=0$ & $\underset{x\to a+}{Lt} f'(x) = -a/3+a^3/3a^2 = 0 = f'(a)$.

$\therefore f'(x)$ is continuous at $x=a$

$\underset{x\to b-}{Lt} f'(x) = -b/3+a^3/3b^2 = \frac{a^3-b^3}{3b^2}$ and $\underset{x\to b+}{Lt} f'(x) = \frac{a^3-b^3}{3b^2}$

$\therefore f'(x)$ is continuous at $x=b$. So it is <u>continuous everywhere.</u>

$$f''(x)=\begin{cases} 0 & 0<x<a \\ -\frac{1}{3}-\frac{2a^3}{3x^3} & a<x<b \\ -\frac{2}{3}\left(\frac{a^3-b^3}{x^3}\right) & x>b \end{cases}$$

$f''(x)$ does not exist at $x=a$, because LHD of $f'(x)$ at $x=a$ is 0. while RHD of $f'(x)$ at $x=a$ is $-\frac{1}{3}-\frac{2a^3}{3a^3}$.

$f''(x)$ does not exist at $x=b$, because LHD of $f'(x)$ at $x=b$ is $-\frac{1}{3}-\frac{2a^3}{3b^3}$

while RHD of $f'(x)$ at $x=b$ is $-\frac{2}{3}\frac{a^3-b^3}{b^3} = -\frac{2a^3}{3b^3}+\frac{2}{3}$

Hence $f''(x)$ is not continuous at $x=a$ and $x=b$.

3(d) Determine an optimal transportation programme so that the transportation cost of 340 tons of a certain type of material from three factories F_1, F_2, F_3 to five warehouses W_1, W_2, W_3, W_4, W_5 is minimized. The five warehouses must receive 40 tons, 50 tons, 70 tons, 90 tons and 90 tons respectively. The availability of the material at F_1, F_2, F_3 is 100 tons, 120 tons, 120 tons respectively. The transportation costs per ton from factories to warehouses are given in the table below:

	W_1	W_2	W_3	W_4	W_5
F_1	4	1	2	6	9
F_2	6	4	3	5	7
F_3	5	2	6	4	8

Use Vogel's approximation method to obtain the initial basic feasible solution.

Sol^n: In the given transportation problem

Total requirement = 340

Total availability = 340.

$\therefore$ The given problem is balanced.

Using the Vogel's Approximation method, the initial basic feasible solution is as shown below.

4 (30)	1	2 (70)	6	9	100
6 (10)	4	3	5 (20)	7 (90)	120
5	2 (50)	6	4 (70)	8	120
40	50	70	90	90	

The no. of allocations $= m+n-1$

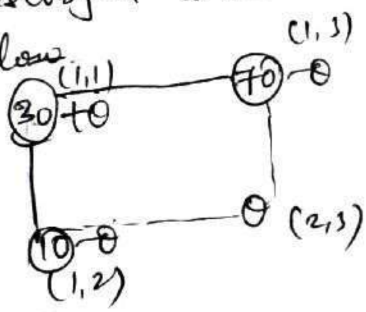
$= 5+3-1 = 7$ (basic variables)

Now finding the values of u_i and v_j's

As the maximum no. of basic cells must exist in the 2nd row.
putting $u_2 = 0$ and the values of u_i's and v_j's and also the net evaluations $\Delta_{ij} = u_i + v_j - c_{ij}$ for all unoccupied cells are exhibited below.

					u_i
4 (30)	1 (0)	2 (70)	6 -ve	9 -ve	-2
6 (10)	4 -ve	3 (1)	5 (20)	7 (90)	0
5 (0)	2 (50)	6 -ve	4 (70)	8 (-ve)	-1
6	3	4	5	7	v_j

Since the net evaluation in the cell (2,3) is +ve. Therefore the current basic feasible solution is not optimal.
∴ The cell (2,3) enters the basis.
We allocate the unknown quantity θ, to this cell (2,3) and identify a loop involving basic cells around this entering cell. Add and subtract θ, alternately to and from the transition cells of the loop subject to the rim requirements as shown below.



Let $\theta = 10$.
$x_{23} = 0$ (non-basic)
∴ The cell (2,3) leaves the basis.

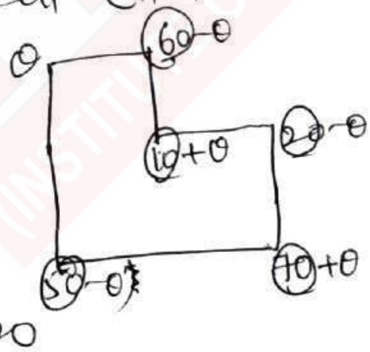
The new basic feasible solution is shown below.

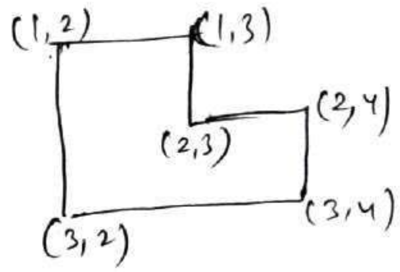
4 (40)	1	2 (60)	6	9
6	4	3 (10)	5 (20)	7 (90)
5	2 (50)	6	4 (70)	8

As the no. of allocations $= m+n-1 = 3+5-1 = 7$.
we compute the net evaluations which are shown below.

					u_i
4 (40)	1 (1)	2 (60)	6 (−)	9 (−)	−1
6 (−)	4 (−)	3 (10)	5 (20)	7 (90)	0
5 (−)	2 (50)	6 (−)	4 (70)	8 (−)	−1
v_j: 5	3	3	5	7	

Since the net evaluation in the cell (1,2) is +ve ∴ the current basic feasible solution is not optimal.
Making a closed loop starting and ending of the cell (1, 2).





Let $\theta = 20$
$x_{24} = 0$.
the cell (2,4) leaves the basis. The new basic feasible solution is shown below.

4 (40)	1 (20)	2 (40)	6	9
6	4	3 (30)	5	7 (90)
5	2 (30)	6	4 (90)	8

The no. of allocations $= m+n-1 = 7$.
Now compute the net evaluations which are shown below.

					u_i
4 (40)	1 (20)	2 (40)	6	9	0
6	4	3 (30)	5	7 (90)	1
5	2 (30)	6	4 (90)	8	1
v_j: 4	1	2	3	6	

Since all $\Delta_{ij} \le 0$.

$\therefore$ The current basic feasible solution is optimal.

$\therefore$ The optimal transportation cost

$= 4\times40 + 1\times20 + 2\times40 + 3\times30 +$
$7\times90 + 2\times30 + 4\times90$

$= 1400/-$

4(a) Let $R = \left\{ \begin{bmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{bmatrix} \in M_2(\mathbb{C}) \,\middle|\, \bar{\alpha}, \bar{\beta} \text{ denote the conjugates of } \alpha, \beta \right\}$

Define addition + and multiplication • in R by usual matrix addition and matrix multiplication. Show that R is a division ring but not a field.

Sol'n: Let $A = \begin{bmatrix} a+ib & c+id \\ -(c-id) & a-ib \end{bmatrix}$ and $B = \begin{bmatrix} r+it & u+iv \\ -(u-iv) & r-it \end{bmatrix} \in R$

Adding A & B

$$A+B = \begin{bmatrix} (a+r)+i(b+t) & (c+u)+i(d+v) \\ -(c+u)-i(d+v) & (a+r)-i(b+t) \end{bmatrix} \in R$$

and $AB = \begin{bmatrix} a+ib & c+id \\ -(c-id) & a-ib \end{bmatrix} \begin{bmatrix} r+it & u+iv \\ -(u-iv) & r-it \end{bmatrix} \in R$ [property of matrices and complex numbers]

Here $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in R$ is zero element for any $A \in R$,

we have $A \in R$ such that $A + (-A) = O$.

Further, distributive properties hold.

Hence R is a ring with identity $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in R$

Now $\begin{bmatrix} i & 1 \\ -1 & -i \end{bmatrix}, \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \in R$

and $\begin{bmatrix} i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} = \begin{bmatrix} 2i & 0 \\ 0 & -2i \end{bmatrix} \in R$

$$\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} \begin{bmatrix} i & 1 \\ -1 & -i \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

$\therefore$ R is a non-commutative ring.

Let $\begin{bmatrix} a+ib & c+id \\ -(c-id) & a-ib \end{bmatrix}$ be a non-zero element of R.

Then either $a+ib \neq 0$ or $c+id \neq 0$

i.e. $a^2+b^2 \neq 0$ or $c^2+d^2 \neq 0$

Hence $a^2+b^2+c^2+d^2 \neq 0$

Let $k = a^2+b^2+c^2+d^2$

Note that $\frac{1}{k}\begin{bmatrix} a+ib & c-id \\ -(c-id) & a-ib \end{bmatrix}$ be a non-zero element of R.

Then either $a+ib \neq 0$ or $c+id \neq 0$.

i.e, $a^2+b^2 \neq 0$ or $c^2+d^2 \neq 0$

Hence $a^2+b^2+c^2+d^2 \neq 0$

Let $k = a^2+b^2+c^2+d^2$

Note that $\frac{1}{k}\begin{bmatrix} a-ib & c-id \\ -(c+id) & a+ib \end{bmatrix} \in R$

is the inverse of $\begin{bmatrix} a+ib & c+id \\ -(c-id) & a-ib \end{bmatrix} \in R$

Hence each non-zero element of R has an inverse in R. Hence R is a division ring. But R is non-commutative, Clearly R is not a field.

4(b) (i) Define $\{x_n\}$ by $x_1 = 5$ and $x_{n+1} = \sqrt{4+x_n}$ for $n>1$. Show that the sequence converges to $\frac{1+\sqrt{17}}{2}$.

(ii) Test the Riemann integrability of the function f defined by

$$f(x) = \begin{cases} 0 & \text{when } x \text{ is rational} \\ 1 & \text{when } x \text{ is irrational} \end{cases}$$

on the interval $[0,1]$.

Sol'n: Given $x_{n+1} = \sqrt{4+x_n} \quad \forall\, n>1$ —— (1)

and $x_1 = 5$

when $n=1$

$x_2 = \sqrt{4+x_1} = \sqrt{4+5}$

$x_2 = \sqrt{9} = 3 < x_1$

$\therefore x_2 < x_1$

Let $x_{k+1} = x_k$

$\Rightarrow 4 + x_{k+1} < 4 + x_k$

$\sqrt{4+x_{k+1}} < 4+x_k$

$x_{k+2} < x_{k+1} \quad \forall\, k>2$

$\therefore x_{k+1} < x_k \quad \forall\, k>1$

$\therefore x_{n+1} < x_n \quad \forall\, n>1.$

$\therefore \{x_n\}$ is monotonically decreasing

Now $x_1 = 5 < 12$

$x_2 = \sqrt{x_1+4} = 3 < 12$

Let $x_K < 12$

$4 + x_K < 16$

$\sqrt{4+x_K} < \sqrt{16}$

$x_{K+1} < 4 < 12$

i.e. $x_{K+1} < 12$

$\Rightarrow x_{n+1} < 12 \quad \forall\, n > 1$ [by mathematical Induction]

$\Rightarrow x_n < 12 \quad \forall\, n \geq 1$

$\therefore \{x_n\}$ is bounded above by 12.

Hence, $\{x_n\}$ is convergent.

Let, $\lim\limits_{n\to\infty} x_n = l$

$\Rightarrow \lim\limits_{n\to\infty} x_{n+1} = l$

$\because x_{n+1} = \sqrt{4+x_n}$

$\lim\limits_{n\to\infty} x_{n+1} = \lim\limits_{n\to\infty} \sqrt{4+x_n}$

$l = \sqrt{4+l}$

Squaring on both sides

$l^2 = 4 + l$

$\Rightarrow l^2 - l - 4 = 0$

$\therefore l = \dfrac{+1 \pm \sqrt{1+16}}{2} = \dfrac{1 \pm \sqrt{17}}{2}$

$l = \dfrac{1+\sqrt{17}}{2}$; but $l \neq \dfrac{1-\sqrt{17}}{2}$ because $x_1 = 5 > 0$

$\therefore \boxed{l = \dfrac{1+\sqrt{17}}{2}}$ Hence proved.

4(b)(ii) Given function 'f' is defined by

$$f(x) = \begin{cases} 0 \ ; & \text{when } x \text{ is rational} \\ 1 \ ; & \text{when } x \text{ is irrational} \end{cases}$$

on the interval $[0,1]$

Clearly, $f(x)$ is bounded on $[0,1]$, because

$0 \leq f(x) \leq 1 \ \forall x \in [0,1]$.

Let $P = \{0 = x_0, x_1, x_2, x_3 \dots x_n = 1\}$ be a partition of $[0,1]$.

Let $I_r = [x_{r-1}, x_r]$; $r = 1, 2, 3, \dots n$. be r^{th} subinterval of $[a,b]$.

$\therefore M_r = 1$; $m_r = 0$

$$U(P,f) = \sum_{r=1}^{n} M_r \delta_r = \sum_{r=1}^{n} 1 \cdot \delta_r = 1 - 0 = 1$$

and $$L(P,f) = \sum_{r=1}^{n} m_r \delta_r = \sum_{r=1}^{n} 0 \cdot \delta_r = 0.$$

Now, $$\underline{\int_0^1} f(x)\,dx = \text{Lub}\{L(P,f)\}_{P \in P[0,1]}$$

and $$\overline{\int_0^1} f(x)\,dx = \text{glb}\{U(P,f)\}_{P \in P[0,1]}$$
$$= 1 - 0 = 1$$

$$\therefore \underline{\int_0^1} f(x)\,dx \neq \overline{\int_0^1} f(x)\,dx$$

$\therefore f(x)$ is not Riemann integrable on $[0,1]$

$\therefore$ Every bounded function need not be a Riemann integrable.

4(d). How many basic solutions are there in the following linearly independent set of equations? Find all of them.

$$2x_1 - x_2 + 3x_3 + x_4 = 6$$
$$4x_1 - 2x_2 - x_3 + 2x_4 = 10$$

Sol^n: Here $2x_1 - x_2 + 3x_3 + x_4 = 6$
$4x_1 - 2x_2 - x_3 + 2x_4 = 10$

The given system of equations can be written in the matrix form as $Ax = b$

where $A = \begin{bmatrix} 2 & -1 & 3 & 1 \\ 4 & -2 & -1 & 2 \end{bmatrix}$, $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$, $b = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$

Since the rank of A is 2 (i.e. $m=2$), the maximum number of linearly independent columns of A is 2. Thus we consider any of the 2×2 submatrices as basic matrix B.

$$\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}, \begin{bmatrix} -1 & 3 \\ -2 & -1 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ -2 & 2 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$$

Let $B = \begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}$

A basic solution to the system is obtained by taking $x_3 = x_4 = 0$ and solving the system

$$\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$$

$$\Rightarrow 2x_1 - x_2 = 6$$
$$4x_1 - 2x_2 = 10$$

As, $\begin{vmatrix} 2 & -1 \\ 4 & -2 \end{vmatrix} = -4 + 4 = 0$

$\Rightarrow$ B is not LI.

If, $B = \begin{bmatrix} -1 & 3 \\ -2 & -1 \end{bmatrix}$, then $x_1 = x_4 = 0$

$$\begin{bmatrix} -1 & 3 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$$

$$-x_2 + 3x_3 = 6 \Rightarrow -2x_2 + 6x_3 = 12$$
$$-2x_2 - x_3 = 10$$

by solving $x_3 = 2/7$, $x_2 = -36/7$

$\therefore (0, -36/7, 2/7, 0)^T$ is one of the basic solution.

If $B = \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}$, then $x_1 = x_2 = 0$

$$\therefore \begin{bmatrix} 3 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$$

$$3x_3 + x_4 = 6 \Rightarrow 6x_3 + 2x_4 = 12$$
$$-x_3 + 2x_4 = 10$$

by solving $x_3 = 2/7$, $x_4 = 36/7$

$\therefore (0, 0, 2/7, 36/7)^T$ is one of the basic solution.

If $B = \begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}$, $x_2 = x_4 = 0$

$$\begin{bmatrix} 2 & 3 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 10 \end{bmatrix}$$

$$\Rightarrow \begin{array}{l} 2x_1 + 3x_3 = 6 \\ 4x_1 - x_3 = 10 \Rightarrow 12x_1 - 3x_3 = 30 \end{array}$$

by solving $x_1 = 18/7$, $x_3 = 2/7$

$\therefore (18/7, 0, 2/7, 0)^T$ is one of the basic solution.

If $B = \begin{bmatrix} -1 & 1 \\ 2 & 2 \end{bmatrix}$

$$\begin{vmatrix} -1 & 1 \\ 2 & 2 \end{vmatrix} = -2 + 2 = 0$$

$\Rightarrow$ B is not LI.

If $B = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$ then $|B| = \begin{vmatrix} 2 & 1 \\ 4 & 2 \end{vmatrix} = 4 - 4 = 0$

$\therefore$ B is not LI.

Thus, $(0, 0, 2/7, 36/7)^T$, $(18/7, 0, 2/7, 0)^T$ and $(0, -36/7, 2/7, 0)^T$ are the basic solutions.

5(a) Prove that all cyclic groups of infinite order are isomorphic to Z.

Sol^n: Let G be a cyclic group with infinite order and suppose that a is a generator of G.

Define a map $\phi: Z \to G$ by $\phi: n \to a^n$. Then

$$\phi(m+n) = a^{m+n} = a^m a^n = \phi(m)\phi(n).$$

To show that ϕ is injective, suppose that m and n are two elements in Z, where $m \neq n$. We can assume that $m > n$. We must show that $a^m \neq a^n$. Let us suppose the contrary; that is, $a^m = a^n$. In this case $a^{m-n} = e$, where $m-n > 0$, which contradicts the fact that a has infinite order. Our map is onto since any element in G can be written as a^n for some integer n and $\phi(n) = a^n$.

5(b) Investigate what derangement of the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}$ ------ will reduce its sum to zero.

Sol'n: The given series is

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\cdots\cdots=\sum(-1)^{n-1}\cdot\frac{1}{n}$$

It is conditionally convergent with sum $\log 2$.

Let it be deranged by taking alternately α positive and β negative terms so that $k=\frac{\alpha}{\beta}$.

Sum of the deranged series $=\log 2+\frac{1}{2}\log k$

But it is given to be zero.

$$\therefore \log 2+\frac{1}{2}\log k=0$$

$$\Rightarrow \log k=-2\log 2$$

$$\Rightarrow \log k=\log 2^{-2}$$

$$\Rightarrow \log k=\log \tfrac{1}{4}$$

$$\Rightarrow k=\frac{\alpha}{\beta}=\frac{1}{4}$$

Hence to get the sum zero, one +ve term should be followed by four negative terms.

The deranged series is

$$1-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}-\frac{1}{8}+\frac{1}{3}-\frac{1}{10}-\frac{1}{12}-\frac{1}{14}-\frac{1}{16}+\frac{1}{5}-\cdots\cdots$$

5(C) → Determine $f(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$ is Riemann-integrable on [0,1] and justify your answer.

Sol^n:

The function

$$f(x) = \begin{cases} 2x\sin\frac{1}{x} - \cos\frac{1}{x} & ; x \in (0,1] \\ 0 & ; x = 0. \end{cases}$$

is not continuous on [0,1].
i.e, it is discontinuous at $x=0$.
i.e, $x=0$ is the only point of discontinuity.
but it is bounded and continuous on [0,1].
and thus Riemann-integrable on [0,1].

The function

$$g(x) = \begin{cases} x^2\sin\frac{1}{x}, & x \in (0,1] \\ 0 & x = 0. \end{cases}$$

is differentiable on [0,1] and satisfies $g'(x) = f(x)$, $\forall x \in [0,1]$.

$$\therefore \int_0^1 \left(2x\sin\frac{1}{x} - \cos\frac{1}{x}\right)dx = g(1) - g(0)$$

$$= \sin 1.$$

5(d) If $f(z) = u+iv$ is analytic function and $u-v = e^x(\cos y - \sin y)$, find $f(z)$ interms of z.

Sol'n: $f(z) = u+iv$

$\Rightarrow if(z) = iu - v$

Adding we have

$(u-v) + i(u+v) = (1+i)f(z) = F(z)$ say

put $u-v = U$ and $u+v = V$

then $F(z) = U+iV$

Now $U = e^x(\cos y - \sin y)$

$\therefore \frac{\partial U}{\partial x} = e^x(\cos y - \sin y) = \phi_1(x,y)$

and $\frac{\partial U}{\partial y} = e^x(-\sin y - \cos y) = \phi_2(x,y)$

By milnes method, we have

$$F'(z) = \phi_1(z,0) - i\phi_2(z,0)$$

$$= (e^z + ie^z) = (1+i)e^z$$

Integrating, we get

$$F(z) = \int (1+i)e^z \, dz + C$$

i.e, $(1+i)f(z) = (1+i)e^z + C$

$$f(z) = e^z + \frac{C}{1+i}$$

$$= e^z + C_1 \quad \text{where } C_1 = \frac{C}{1+i}$$

$$\boxed{\therefore f(z) = e^z + C_1}$$

5(e) $x_1=4,\ x_2=1,\ x_3=3$ is a feasible solution of the system of equations.

$2x_1-3x_2+x_3=8$
$x_1+2x_2+3x_3=15$

Reduce the feasible solution to two different basic feasible solutions.

Soln: The given system of equations can be put in matrix notation as

$$AX=B$$

where $A=\begin{bmatrix}2 & -3 & 1\\ 1 & 2 & 3\end{bmatrix}$ $B=\begin{pmatrix}8\\15\end{pmatrix}$ $X=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$

Here $\rho(A)=2$.

$\therefore$ A basic solution to the given system of eqns exist with not more than two variables different from zero.

Let the columns of A be denoted by

$A_1=\begin{pmatrix}2\\1\end{pmatrix}$ $A_2=\begin{pmatrix}-3\\2\end{pmatrix}$ $A_3=\begin{pmatrix}1\\3\end{pmatrix}$

Since the vectors A_1, A_2, A_3 are linear dependent.

$\therefore \exists$ scalars $\lambda_1, \lambda_2, \lambda_3$ not all zero such that

$A_1\lambda_1+A_2\lambda_2+A_3\lambda_3=0$

$$\Rightarrow \left.\begin{array}{l}2\lambda_1-3\lambda_2+\lambda_3=0\\ \lambda_1+2\lambda_2+3\lambda_3=0\end{array}\right\} \quad (1)$$

This is a system of two equations in three variables.

Let us choose one of the λ's arbitrarily

say $\lambda_1=2$

$\therefore 3\lambda_2-\lambda_3=4$

$-2\lambda_2-3\lambda_3=2$

Solving, we get $\lambda_2=\frac{10}{11},\ \lambda_3=\frac{-14}{11}$

To reduce the no. of +ve variables, the variable to be driven to zero is found by choosing r for which

$\frac{x_r}{\lambda_r} = \underset{i}{\text{Min}} \left\{ \frac{x_i}{\lambda_i} \,\middle|\, \lambda_i > 0 \right\}$

$= \text{Min} \left\{ \frac{x_1}{\lambda_1}, \frac{x_2}{\lambda_2} \right\} = \text{Min} \left\{ \frac{4}{2}, \frac{1}{10/11} \right\} = \left\{ 2, \frac{11}{10} \right\}$ $\left(\because \lambda_3 = \frac{-14}{11}\right)$

$= \frac{11}{10}$.

Therefore, the variable x_2 should be reduced to zero and the vector A_2 can be removed for which $\frac{x_2}{\lambda_2} = \frac{11}{10}$. and obtain new solution with not more than two non-negative (non-zero) variables.

The values of new variables are given by

$\hat{x}_1 = x_1 - \frac{x_2}{\lambda_2}\lambda_1 = 4 - \frac{11}{10}(2) = \frac{9}{5}$

$\hat{x}_3 = x_3 - \frac{x_2}{\lambda_2}\lambda_3 = 3 - \frac{11}{10}\left(\frac{-14}{11}\right) = \frac{22}{5}$.

Hence the basic feasible solution to given system of equations is given by

$x_1 = \frac{9}{5}, \quad x_2 = 0, \quad x_3 = \frac{22}{5}$.

For another BFS,

choose $\lambda_3 = 1$.

$\therefore$ (1) $\Rightarrow$ $2\lambda_1 - 3\lambda_2 = -1$

$\lambda_1 + 2\lambda_2 = -3$

solving, we get $\lambda_1 = -\frac{11}{7}, \ \lambda_2 = \frac{-5}{7}$.

Now $\frac{x_r}{\lambda_r} = \text{min}\left\{ \frac{x_3}{\lambda_3} \right\} = 3$ $\quad (\because \lambda_1, \lambda_2 \text{ are } -\text{ve})$

$\therefore$ The variable $x_3 = 0$ and the vector A_3 can be removed for $\frac{x_3}{\lambda_3} = 3$ and obtain new solution.

The values of new variables are given by

$\hat{x}_1 = 4 - 3\left(\frac{-11}{7}\right) = \frac{61}{7}$ and $\hat{x}_2 = 1 - 3\left(\frac{-5}{7}\right) = \frac{22}{7}$.

$\therefore$ The basic feasible solution is given by

$x_1 = \frac{61}{7}, \quad x_2 = \frac{22}{7}, \quad x_3 = 0$

6(a) (i) Let $\beta \in S_7$ and suppose $\beta^4 = (2\ 1\ 4\ 3\ 5\ 6\ 7)$. Find β. What are the possibilities for β if $\beta \in S_9$?

(ii) Let $\beta = (1\ 2\ 3)(1\ 4\ 5)$. Write β^{99} in disjoint cycle form.

Sol^n: (i) Since $\beta^{28} = (\beta^4)^7 = I$ (identity permutation)

we know that $o(\beta) / 28$

so that $o(\beta) \neq 1, 2$ (or) 4

If $o(\beta) = 14$ then β is written in disjoint cycle form would need at least one 7-cycle and one 2-cycle. But that requires atleast 9 symbols and we have only 7.

Likewise $o(\beta) = 28$ requires atleast one 7-cycle and one 4-cycle.

$$\text{So } o(\beta) = 7$$

Thus $\beta = \beta^8 = (\beta^4)^2 = (2\ 4\ 5\ 7\ 1\ 3\ 6)$

In S_9, $\beta = (2\ 4\ 5\ 7\ 1\ 3\ 6)(8)(9)$

(or)

$$\beta = (2\ 4\ 5\ 7\ 1\ 3\ 6)(8\ 9)$$

(ii)

$$\beta = (1\ 2\ 3)\ (1\ 4\ 5)$$

$$\beta = (1\ 4\ 5\ 2\ 3)$$

$$\beta^{99} = \beta^4 = \beta^{-1}$$

because order of β is 5.

$$\therefore (\beta^{99}) = (\beta)^{5\times 19} (\beta)^4 :$$

$$= (\beta)^4$$

$$= \beta^{-1}$$

$$\therefore \beta^{-1} = (1\ 3\ 2\ 5\ 4)$$

$$\therefore \beta^{99} = (1\ 3\ 2\ 5\ 4)$$

6(c) Is the ideal $M = \{\bar{0}, \bar{3}, \bar{6}, \bar{9}\}$ a maximal ideal of $Z/(12)$, the ring of integers modulo 12? Justify your answer.

Sol^n: Let $U = \langle 12 \rangle = 12Z = \{\ldots -36, -24, -12, 0, 12, 24, \ldots\}$

The only ideals of Z which contain U are Z, and $12Z$ and

$$2Z = \{\ldots -4, -2, 0, 2, 4, \ldots\}$$

$$3Z = \{\ldots -6, -3, 0, 3, 6 \ldots\}$$

$$4Z = \{\ldots -8, -4, 0, 4, 8 \ldots\}$$

$$6Z = \{\ldots -12, -6, 0, 6, 12 \ldots\}$$

Hence all the ideals of Z/U are $Z/U = \{\bar{0}, \bar{1} \ldots \overline{11}\}$

$U/U = \bar{0} = U$, $\quad 2Z/U = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}\}$,

$$3Z/U = \{\bar{0}, \bar{3}, \bar{6}, \bar{9}\}$$

$$4Z/U = \{\bar{0}, \bar{4}, \bar{8}\}$$

$$6Z/U = \{\bar{0}, \bar{6}\} \quad (\text{Here } \bar{n} = n + U)$$

Here $M = \{\bar{0}, \bar{3}, \bar{6}, \bar{9}\}$ is a maximal ideal of

$Z/(12) = \{\bar{0}, \bar{1}, \bar{2}, \ldots \overline{11}\}$

Since there does not exist any proper ideal between $M = \{\bar{0}, \bar{3}, \bar{6}, \bar{9}\}$ and $Z/(12) = \{\bar{0}, \bar{1} \ldots \overline{11}\}$

$\Rightarrow$ {Notice that $\{\bar{0}, \bar{2}, \bar{4}, \bar{6}, \bar{8}, \overline{10}\}$ is also a maximal ideal of $Z/(12)$.

However $\{\bar{0}\}$, $\{\bar{0}, \bar{4}, \bar{8}\}$, $\{\bar{0}, \bar{6}\}$ are not maximal ideals of $Z/(12)$, since for example,
$\{\bar{0}, \bar{6}\} \subset \{\bar{0}, \bar{3}, \bar{6}, \bar{9}\} \subset Z/(12)$}.

6(d) Every Euclidean domain is a principal ideal domain. Is a converse true? Justify your answer.

sol'n: Let R be an Euclidean ring.

Let U be an ideal of R.

To prove that U is a principal ideal.

Let $U=\{0\}$, where $0\in R$

Then $U=\{0\}$ is the ideal generated by $0\in R$

$\therefore$ U is a principal ideal of R.

Let $U\neq\{0\}$ then U contains non-zero elts.

then $\exists\ x\in U(\subseteq R)$ and $x\neq 0$ so that the set $\{d(x)/x\neq 0\}$ is a non-empty set of non-negative integers.

$\therefore$ By well ordering principle there exists $b\neq 0$ in U such that $d(b)\leq d(x)$ where $x\neq 0\in U$.

Now we prove that $U=\langle b\rangle$

Let $a\in U$

By division algorithm, $\exists\ q, r\in R$ such that $a=bq+r$ where $r=0$ (or) $d(r)<d(b)$

Since $b\in U$, $q\in R \Rightarrow bq\in U$ ($\because$ U is an ideal)

Since $a\in U$, $bq\in U \Rightarrow a-bq=r\in U$

if $r\neq 0$ then $d(r)<d(b)$ which contradiction to the fact $d(b)\leq d(x)\ \forall\ x\neq 0\in U$.

$\therefore r=0$ and hence $a=bq$.

$\therefore$ $U=\{bq/q\in R\}=\langle b\rangle$ is the Principal ideal generated by 'b' $(\neq 0)$ in U.

Hence every ideal U of R is a principal ideal.

$\therefore$ R is a principal ideal ring.

Note: 1. If U is an ideal of Euclidean ring R then U is a principal ideal of R so that

$$U = \langle b \rangle$$
$$= \{bq \mid q \in R\}$$

2. The converse of the above theorem need not be true.

Ex: $R = \left\{ a + b\left(\frac{1+\sqrt{19}i}{2}\right) \middle/ a, b \in Z \right\}$, the ring of complex numbers is a principal ideal ring but not Euclidean.

7(a) Prove that $\frac{x}{1+x} < \log(1+x) < x$ for all $x>0$. Deduce that $\log\frac{2n+1}{n+1} < \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n} < \log 2$, n being a positive integer.

Sol'n: Let $f(t) = \log(1+t)\ \forall\ t \in [0,x]$ where $x>0$.

and $f'(t) = \frac{1}{1+t}\ \forall\ t \in (0,x)$

By Lagrange's mean value theorem

$\exists\ c \in (0,x)$ such that

$$f'(c) = \frac{f(x)-f(0)}{x-0}$$

$$\Rightarrow \frac{1}{1+c} = \frac{\log(1+x) - \log 1}{x}$$

$$\Rightarrow \frac{1}{1+c} = \frac{\log(1+x)}{x} \quad \text{--- (1)}$$

Since $c \in (0,x)$

$\Rightarrow 0 < c < x$

$\Rightarrow 1 < 1+c < 1+x$

$\Rightarrow 1 > \frac{1}{1+c} > \frac{1}{1+x}$

$\Rightarrow 1 > \frac{\log(1+x)}{x} > \frac{1}{1+x}$ (by (1))

$\Rightarrow x > \log(1+x) > \frac{x}{1+x}$

i.e. $\frac{x}{1+x} < \log(1+x) < x$

Now, we have $\log(1+x) < x$

Let $x = \frac{1}{n+1}$

then $\log\left(1+\frac{1}{n+1}\right) < \frac{1}{n+1}$ i.e. $\log\left(\frac{n+2}{n+1}\right) < \frac{1}{n+1}$

$\log\left(1+\frac{1}{n+2}\right) < \frac{1}{n+2}$ i.e, $\log\left(\frac{n+3}{n+2}\right) < \frac{1}{n+2}$

Similarly $\log\left(\frac{n+4}{n+3}\right) < \frac{1}{n+3}$

$\log\left(\frac{n+5}{n+4}\right) < \frac{1}{n+4}$

$\vdots$

$\log\left(1+\frac{1}{n+n}\right) < \frac{1}{n+n}$ i.e, $\log\left(\frac{2n+1}{2n}\right) < \frac{1}{n+n}$

$\therefore$ Adding all the above, we get

$$\log\left(\frac{n+2}{n+1}\right)+\log\left(\frac{n+3}{n+2}\right)+\log\left(\frac{n+4}{n+3}\right)+\cdots+\log\left(\frac{2n+1}{2n}\right) < \frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{n+n}$$

$$\Rightarrow \log\left(\frac{n+2}{n+1}\cdot\frac{n+3}{n+2}\cdot\frac{n+4}{n+3}\cdots\frac{2n}{2n-1}\cdot\frac{2n+1}{2n}\right) < \frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n}$$

$$\Rightarrow \log\left(\frac{2n+1}{n+1}\right) < \frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{2n} \quad \text{—— (2)}$$

Also, we have $\frac{x}{1+x} < \log(1+x)$

Let $x=\frac{1}{n}$, then $\frac{1/n}{1+1/n} < \log\left(1+\frac{1}{n}\right) \Rightarrow \frac{1}{n+1} < \log\left(\frac{n+1}{n}\right)$

$x=\frac{1}{n+1}$ then $\frac{1/(n+1)}{1+\frac{1}{n+1}} < \log\left(1+\frac{1}{n+1}\right) \Rightarrow \frac{1}{n+2} < \log\left(\frac{n+2}{n+1}\right)$

$\vdots$

$x=\frac{1}{2n-1}$ then $\frac{\frac{1}{2n-1}}{1+\frac{1}{2n-1}} < \log\left(1+\frac{1}{2n-1}\right) \Rightarrow \frac{1}{2n} < \log\frac{2n}{2n-1}$

Adding, we get

$$\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{n+n} < \log\left(\frac{n+1}{n}\right)+\log\left(\frac{n+2}{n+1}\right)+\cdots\log\left(\frac{2n}{2n-1}\right)$$

$$= \log\frac{n+1}{n}\cdot\frac{n+2}{n+1}\cdot\frac{n+3}{n+2}\cdots\frac{2n}{2n-1}$$

$$= \log\left(\frac{2n}{n}\right) = \log 2$$

$$\therefore \frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n} < \log 2 \quad \text{—— (3)}$$

$\therefore$ from (2) & (3) $\log\left(\frac{2n+1}{n+1}\right) < \frac{1}{n+1}+\frac{1}{n+2}+\cdots+\frac{1}{2n} < \log 2$

7(b) Show that if $f_n(x) = \frac{n^2x}{1+n^4x^2}$, then $\langle f_n \rangle$ converges non-uniformly on $[0,1]$.

Sol'n: Here $f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{n^2x}{1+n^4x^2}$

$$= \lim_{n\to\infty} \frac{x/n^2}{\frac{1}{n^4}+x^2} = 0 \quad \forall\, x \in [0,1]$$

$$|f_n(x) - f(x)| = \left|\frac{n^2x}{1+n^4x^2} - 0\right| = \left|\frac{n^2x}{1+n^4x^2}\right|$$

Let $y = \frac{n^2x}{1+n^4x^2}$

then $$\frac{dy}{dx} = \frac{(1+n^4x^2).n^2 - n^2x.2n^4x}{(1+n^4x^2)^2} = \frac{n^2(1-n^4x^2)}{(1+n^4x^2)^2}$$

For maximum (or) minimum $\frac{dy}{dx} = 0$

$$\Rightarrow 1 - n^4x^2 = 0 \Rightarrow x = \frac{1}{n^2}$$

Also $$\frac{d^2y}{dx^2} = n^2.\frac{(1+n^4x^2)^2(-2n^4x) - (1-n^4x^2).2(1+n^4x^2).2n^4x}{(1+n^4x^2)^4}$$

$$= \frac{n^2[-2n^4x(1+n^4x^2) - 4n^4x(1-n^4x^2)]}{(1+n^4x^2)^3}$$

$$= \frac{-2n^6x[(1+n^4x^2) + 2(1-n^4x^2)]}{(1+n^4x^2)^3}$$

$$\left.\frac{d^2y}{dx^2}\right]_{x=1/n^2} = \frac{-2n^6.\frac{1}{n^2}\left(1+n^4.\frac{1}{n^4}\right)}{\left(1+n^4.\frac{1}{n^4}\right)^3}$$

$$= \frac{-2n^4}{8} = -\frac{n^4}{8} < 0$$

$\Rightarrow y$ is maximum when $x = \frac{1}{n^2}$ and maximum value of

$$y = \frac{n^2 \cdot \frac{1}{n^2}}{1 + n^4 \cdot \frac{1}{n^4}} = \frac{1}{2}$$

Also $x = \frac{1}{n^2} \to 0$ as $n \to \infty$

$$\therefore M_n = \max_{x \in [0,1]} |f_n(x) - f(x)|$$

$$= \max_{x \in [0,1]} \left| \frac{n^2 x}{1 + n^4 x^2} \right|$$

$$= \frac{1}{2}$$

which doesnot tend to 0 as $n \to \infty$.

Hence $\langle f_n \rangle$ converges non-uniformly on $[0,1]$.

7(c) obtain $\frac{\partial^2 f(0,0)}{\partial x \partial y}$ and $\frac{\partial^2 f(0,0)}{\partial y \partial x}$ for the function

$$f(x,y) = \begin{cases} \frac{xy(3x^2-2y^2)}{x^2+y^2} & ,(x,y) \neq (0,0) \\ 0 & ,(x,y)=(0,0) \end{cases}$$

Also, discuss the continuity of $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ at (0,0).

Sol^n: Given $f(x,y) = \begin{cases} \frac{xy(3x^2-2y^2)}{x^2+y^2} & ,(x,y) \neq (0,0) \\ 0 & ,(x,y)=(0,0) \end{cases}$

Now; $f_{xy}(0,0) = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h}$

$$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k}$$

$$= \lim_{k\to 0} \frac{0-0}{k} = \frac{0}{k} = 0$$

$$f_y(h,0) = \lim_{k\to 0} \frac{f(h,k) - f(h,0)}{k}$$

$$= \lim_{k\to 0} \frac{\frac{hk(3h^2-2k^2)}{h^2+k^2} - 0}{k}$$

$$= \lim_{k\to 0} \frac{h\cancel{k}(3h^2-2k^2)}{\cancel{k}(h^2+k^2)}$$

$$= \frac{h(3h^2)}{h^2} = 3h$$

$$f_{xy}(0,0) = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h} = \frac{3h-0}{h}$$

$$\boxed{f_{xy}(0,0) = 3}$$

Again $f_{yx}(0,0) = \lim_{k\to 0} \dfrac{f_x(0,k) - f_x(0,0)}{k}$

But $f_x(0,0) = \lim_{h\to 0} \dfrac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0} \dfrac{0-0}{h} = 0$

$$f_x(0,k) = \lim_{h\to 0} \frac{f(h,k)-f(0,k)}{h}$$

$$= \lim_{h\to 0} \frac{hk(3h^2-2k^2)}{h(h^2+k^2)}$$

$$= \frac{-k \cdot 2k^2}{k^2} = -2k$$

$\therefore f_{yx}(0,0) = \lim_{k\to 0} \dfrac{f_x(0,k) - f_x(0,0)}{k}$

$$= \frac{-2k}{k} = -2$$

$\therefore f_{xy}(0,0) \neq f_{yx}(0,0)$

$\therefore f(x,y)$ is not a continuous function.

7(d) Show that $\iint_D \frac{(x-y)}{(x+y)^3}\, dx\,dy$ does not exist, where

$D = \{(x,y) \in R^2 / 0 \le x \le 1, 0 < y < 1\}$.

Sol^n: The integrand $\frac{x-y}{(x+y)^3}$ is bounded over the square region D: $[0,1;0,1]$ except at the origin which is a point of infinite discontinuity.

Note: On rectangle $R = [a,b; c,d]$ the double integral $\iint_R f\,dx\,dy$ exists, and also $\int_c^d \{\int_a^b f\,dx\}\,dy$, $\int_a^b \{\int_c^d f\,dy\}\,dx$ exist, then all these three integrals are equal.

Note that the existence of repeated integral doesnot ensure the existence of the other repeated integral or the double integral.

Now

$$\int_0^1 \left\{\int_0^1 \frac{x-y}{(x+y)^3}\, dx\right\} dy = \lim_{\epsilon \to 0+} \int_\epsilon^1 \left[\int_0^1 \frac{x-y}{(x+y)^3}\, dx\right] dy$$

$$= \lim_{\epsilon \to 0+} \int_\epsilon^1 \left\{\int_0^1 \left[\frac{1}{(x+y)^2} - \frac{2y}{(x+y)^3}\right] dx\right\} dy$$

$$= \lim_{\epsilon \to 0+} \int_\epsilon^1 \left[-\frac{1}{x+y} + \frac{y}{(x+y)^2}\right]_0^1 dy$$

$$= \lim_{\epsilon \to 0+} \int_0^1 \left[-\frac{1}{1+y} + \frac{1}{y} + \frac{y}{(1+y)^2} - \frac{y}{y}\right] dy$$

$$= \lim_{\epsilon \to 0^+} \int_{\epsilon}^{1} -\frac{1}{(1+y)^2}\,dy$$

$$= \lim_{\epsilon \to 0} \left[\frac{1}{1+y}\right]_{\epsilon}^{1}$$

$$= \lim_{\epsilon \to 0^+} \left(\frac{1}{2} - \frac{1}{1+\epsilon}\right)$$

$$= -\frac{1}{2}.$$

Similarly,

$$\int_0^1 \left\{\int_0^1 \frac{x-y}{(x+y)^3}\,dy\right\} dx = \lim_{\epsilon \to 0^+} \int_{\epsilon}^{1} \left\{\int_0^1 \frac{x-y}{(x+y)^3}\,dy\right\} dx$$

$$= \frac{1}{2}.$$

Thus, the two repeated integrals exist and are unequal.

$\therefore$ double integral on $[0,1;0,1]$ cannot exist.

8(a) Using Cauchy's theorem / Cauchy's integral formula, Calculate the following integrals:

(i) $\int_C \frac{\cosh(\pi z)\,dz}{z(z^2+1)}$, where C is circle $|z|=2$.

(ii) $\int_C \frac{e^{az}\,dz}{(z-\pi i)}$, where C is the ellipse $|z-2|+|z+2|=6$.

(iii) $\int_C \frac{(\sin z)^2\,dz}{\left(z-\frac{\pi}{3}\right)^3}$, where C is circle $|z|=1$.

Sol^n: Let $I = \int_C \frac{\cosh(\pi z)}{z(z^2+1)}\,dz$ and C is $|z|=2$

Take $f(z) = \cosh(\pi z) = \cos(i\pi z)$

Then $I = \int_C \frac{f(z)}{z(z^2+1)}\,dz$.

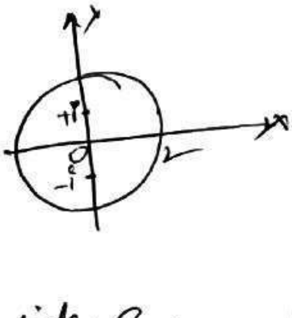
$$= \int_C \left(\frac{A}{z} + \frac{B}{z-i} + \frac{C}{z+i}\right) f(z)\,dz \quad \text{--- (1)}$$

Here $\frac{1}{z(z^2+1)} = \frac{1}{z(z-i)(z+i)} = \frac{A}{z} + \frac{B}{z-i} + \frac{C}{z+i}$

Now $A = \frac{1}{(z-i)(z+i)} = 1$ at $z=0$

$B = \frac{1}{z(z+i)} = -\frac{1}{2}$ at $z=i$

$C = \frac{1}{z(z-i)} = -\frac{1}{2}$ at $z=-i$



Here $z=0, i, -i$ are points inside C. and therefore $f(z)$ is analytic at these points

$\therefore$ By using Cauchy's integral formula

$$\int_C \frac{f(z)}{z-a} = 2\pi i\, f(a)$$

from (i),

we have $I = 2\pi i\left[A f(0) + B f(i) + C f(-i)\right]$

$= 2\pi i\left[1\cdot\cos(0) - \frac{1}{2}\cos(i^2\pi) + \frac{1}{2}\cos(-i^2\pi)\right]$

$= 2\pi i\left[1 + \frac{1}{2} + \frac{1}{2}\right]$

$= 4\pi i.$

(ii) Let $I = \int_C \frac{e^{az}}{z-\pi i}\,dz$; C is ellipse

C is ellipse $|z-2| + |z+2| = 6$

$\left((x-2)^2+y^2\right)^{1/2} = 6 - \left[(x+2)^2+y^2\right]^{1/2}$

Squaring, we get

$x^2+y^2+4-4x = 36+x^2+y^2+4x+4-12\left[(x+2)^2+y^2\right]^{1/2}$

$\Rightarrow 12(x^2+y^2+4x+4)^{1/2} = 36+8x$

$\Rightarrow 3(x^2+y^2+4x+4)^{1/2} = 9+2x$

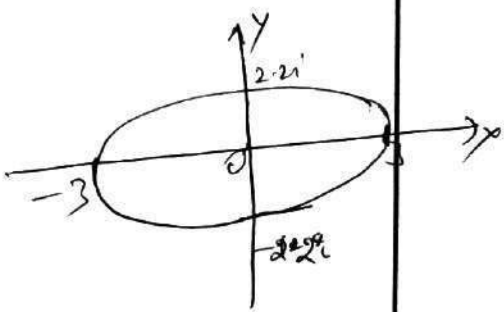
Again squaring

$9(x^2+y^2+4x+4) = (9+2x)^2$

$\Rightarrow 9(x^2+y^2+4x+4) = 81+4x^2+36x$

$\Rightarrow 5x^2+9y^2 = 45$

$\Rightarrow \frac{x^2}{9} + \frac{y^2}{5} = 1$



comparing $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

we get $a=3,\ b=\sqrt{5}\approx 2.2$ (appr.)

Clearly $z=\pi i = 3.14i$ lies outside C.

Hence $\frac{e^{az}}{z-\pi i}$ is analytic within and upon C

$\therefore I = 0$, by Cauchy's theorem.

(iii) Let $I = \int_C \frac{(\sin z)^2}{\left(z-\frac{\pi}{6}\right)^3} dz$. where C is circle $|z|=1$

~~By~~ n^{th} derivative formula of Cauchy's Integral formula,

$$f^n(a) = \frac{n!}{2\pi i} \int_C \frac{f(z)}{(z-a)^{n+1}} dz \quad \text{———} \ (1)$$

Here $a = \frac{\pi}{6} = \frac{3.14}{6} = 0.52 < 1$

$\therefore$ a lies inside $|z|=1$ which is contour C.

and $f(z) = (\sin z)^2$ is analytic within and on C.

$\therefore$ By (1),

$$f^2(a) = \frac{2!}{2\pi i} \int_C \frac{(\sin z)^2}{\left(z-\frac{\pi}{6}\right)^3} dz$$

i.e, $\int_C \frac{(\sin z)^2}{\left(z-\frac{\pi}{6}\right)^3} dz = \pi i f^2\left(\frac{\pi}{6}\right)$ ——— (2)

$f(z) = (\sin z)^2 \Rightarrow f'(z) = 2\sin z \cos z = \sin 2z$

$f''(z) = 2\cos 2z$

$\Rightarrow f''\left(\frac{\pi}{6}\right) = 2\cdot\cos 2\left(\frac{\pi}{6}\right) = 2\cos\frac{\pi}{3} = 2\cdot\frac{1}{2} = 1$

$\therefore$ from (2) $\int_C \frac{(\sin z)^2}{\left(z-\frac{\pi}{6}\right)^3} dz = \pi i$

8(b) (i) Show that the function e^{-1/z^2} has no singularities.
(ii) Find residue of $f(z) = e^z \operatorname{cosec}^2 z$ at all poles in the finite plane.

Sol'n: (i) Let $f(z) = e^{-1/z^2} = \frac{1}{e^{1/z^2}}$

Poles of $f(z)$ are given by $e^{1/z^2} = 0$,

which is not possible for any value of z, real or complex.

Zeros of $f(z)$ are given by

$$e^{-1/z^2} = 0 = e^{-\infty} \quad \text{so that} \quad \frac{-1}{z^2} = \infty$$

This $\Rightarrow$ $z = 0$. (repeated twice)

Hence $z=0$ is a zero of order two so that there is no limit of the zero. Consequently there is no singularity. Finally, $f(z)$ is free from any singularity.

(ii) It is given that $f(z) = e^z \operatorname{cosec}^2 z$

$$= \frac{e^z}{\sin^2 z} = \frac{e^z}{(\sin z)^2}$$

$\therefore$ the function has double pole at $z = 0, \pm\pi, \pm 2\pi, \ldots$

i.e. $z = m\pi$, where $m = 0, \pm 1, \pm 2, \ldots$

Using the general formula for calculating the residues we get

$$a_{-1} = \frac{1}{(1)!} \lim_{z \to m\pi} \frac{d}{dz}\left[(z-m\pi)^2 \frac{e^z}{(\sin z)^2}\right]$$

$$= \lim_{z \to m\pi} \frac{(\sin z)^2 [2(z-m\pi)e^z + (z-m\pi)^2 e^z] - e^z (z-m\pi)^2 2\sin z \cos z}{(\sin z)^4}$$

$$= \lim_{z \to m\pi} \frac{e^z (z-m\pi)^2 [\sin z - 2\cos z] + 2e^z (z-m\pi)\sin z}{(\sin z)^3}$$

Let $z - m\pi = t \Rightarrow z = m\pi + t$

when $z \to m\pi$, $t \to 0$ then this limit

can be written as

$$= \lim_{t \to 0} e^{m\pi+t} \left\{ \frac{t^2 [\sin(m\pi+t) - 2\cos(m\pi+t)] + 2t \sin(m\pi+t)}{[\sin(m\pi+t)^3]} \right\}$$

either m is even or odd, the limit becomes

$$\lim_{t \to 0} e^{m\pi+t} \left\{ \frac{t^2(\sin t - 2\cos t) + 2t \sin t}{(\sin t)^3} \right\}$$

$$= \lim_{t \to 0} e^{m\pi+t} \left\{ \lim_{t \to 0} \frac{t^2(\sin t - 2\cos t) + 2t\sin t}{(\sin t)^3} \right\}$$

$$= e^{m\pi} \left\{ \lim_{t \to 0} \frac{t^2(\sin t - 2\cos t) + 2t\sin t}{\sin^3 t} \right\}$$

This limit can be easily evaluated by applying L'Hospital's rule many times after making the following adjustment.

$$= e^{m\pi} \lim_{t \to 0} \left\{ \frac{t^2(\sin t - 2\cos t) + 2t\sin t}{t^3} \cdot \frac{t^3}{\sin^3 t} \right\}$$

$$= e^{m\pi} \lim_{t \to 0} \left[\frac{t^2 \sin t - 2\cos t + 2t\sin t}{t^3} \right] \lim_{t \to 0} \frac{t^3}{\sin^3 t}$$

$$= e^{m\pi} \lim_{t \to 0} \left[\frac{t^2 \sin t - 2\cos t + 2t \sin t}{t^3} \right]$$

$= e^{m\pi}$ (by using L'Hospital's Rule many times)

The residues are $e^{m\pi}$, $m = 0, \pm 1, \pm 2, \ldots$

8(c) A company has a team of four salesman and there are four districts where the company wants to start its business. After taking into account the capabilities of salesmen and the nature of districts, the company estimates that the profit per day in rupees for each salesman in each district is as follows.

Districts

Salesmen	D_1	D_2	D_3	D_4
A	16	10	14	11
B	14	11	15	15
C	15	15	13	12
D	13	12	14	15

Solⁿ: It is an assignment problem of maximization, so we first convert it into minimization problem by subtracting all the elements of the given matrix from the maximum element 16, we get

0	6	2	5
2	5	1	1
1	1	3	4
3	4	2	1

Subtract the smallest element of each row from the elements of that row, the reduced matrix is given by

0	6	2	5
1	4	0	0
0	0	2	3
2	3	1	0

Now subtracting the smallest element of each column from the elements of that column, we get

0	6	2	5
1	4	0	0
0	0	2	3
2	3	1	0

cover all the zero by minimum no. of horizontal and vertical lines. we required exactly four lines to cover all the zeros

0	6	2	5
1	4	0	0
0	0	2	3
2	3	1	0

As $r = 4 = n$, optimal assignment can be made at this stage.

0	6	2	5
1	4	0	$\cancel{0}$
$\cancel{0}$	0	2	3
2	3	1	0

Hence the optimal assignment of the given problem is

$A \to D_1$, $B \to C_3$, $C \to D_2$, $D \to D_4$.

and the maximum profit per day of given problem is

$16 + 15 + 15 + 15 = 61$

8(d) Solve the following problem by Big-M-method:
Max $z = x_1 + 2x_2 + 3x_3 - x_4$ Subject to: $x_1 + 2x_2 + 3x_3 = 15$, $2x_1 + x_2 + 5x_3 = 20$, $x_1 + 2x_2 + x_3 + x_4 = 10$ and $x_1, x_2, x_3, x_4 \geq 0$.

Sol'n: The objective function of the given LPP is of maximization type.

Now we write the given LPP in standard form.

$$\text{Max } z = x_1 + 2x_2 + 3x_3 - x_4 - MA_1 - MA_2$$

Subject to
$$x_1 + 2x_2 + 3x_3 + A_1 = 15$$
$$2x_1 + x_2 + 5x_3 + A_2 = 20$$
$$x_1 + 2x_2 + x_3 + x_4 = 10$$
$$x_1, x_2, x_3, x_4, A_1, A_2 \geq 0$$

where A_1, A_2 are artificial variables. Now the initial basic feasible solution is

$$x_1 = x_2 = x_3 = 0 \text{ (Non-basic)}$$
$$A_1 = 15,\ A_2 = 20,\ x_4 = 10$$

Note that the decision variable $x_4 \geq 0$ will provide a column $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ of initial basic matrix and that will reduce addition of no. of artificial variables.

For which $z = -35M - 10$

Now we put the above information in the simplex tableau.

	C_j	1	2	3	-1	-M	-M		
C_B	Basis	x_1	x_2	x_3	x_4	A_1	A_2	b	θ
-M	A_1	1	2	3	0	1	0	15	5
-M	A_2	2	1	(5)	0	0	1	20	4 →
-1	x_4	1	2	1	1	0	0	10	10
$z_j = \sum C_B a_{ij}$		-3M	-3M	8M	-1	-M	-M	-35M	-10
$C_j = c_j - z_j$		1+3M	2+3M	4+8M ↑	0	0	0		

From the above table, x_3 is the entering variable, A_2 is the outgoing variable and omit the column, for this variable in the next table. (5) is the key element make it unity and all other elements in its column equal to zero.

Then the revised simplex table is:

	C_j	1	2	3	-1	-M		
C_B	Basis	x_1	x_2	x_3	x_4	A_1	b.	
-M	A_1	$-1/5$	$(7/5)$	0	0	1	3	$\frac{15}{7} \rightarrow$
3	x_3	$2/5$	$1/5$	1	0	0	4	20
-1	x_4	$3/5$	$9/5$	0	1	0	6	$\frac{30}{9}$
$Z_j = \Sigma a_{ij} C_B$		$+\frac{M}{5}+\frac{3}{5}$	$-\frac{7M}{5}-\frac{6}{5}$	3	-1	-M	-3M+6	
$C_j = c_j - Z_j$		$\frac{-M}{5}+\frac{2}{5}$	$\frac{7M}{5}+\frac{16}{5}$ ↑	0	0	0		

x_2 is the entering variable, A_1 is the outgoing variable and omit the column for this variable in the next table, $7/5$ is the key element.

	C_j	1	2	3	-1		
C_B	Basis	x_1	x_2	x_3	x_4	b	θ
2	x_2	$-1/7$	1	0	0	$15/7$	—
3	x_3	$3/7$	0	1	0	$25/7$	$25/3$
-1	x_4	$(6/7)$	0	0	1	$15/7$	$15/7 \longrightarrow$
$Z_j = \Sigma a_{ij} C_B$		$1/7$	2	3	-1	$90/7$	
$C_j = c_j - Z_j$		$6/7$ ↑	0	0	0		

Introduce x_1, drop x_4, $6/7$ is the key element.

	C_j	1	2	3	-1	
C_B	Basis	x_1	x_2	x_3	x_4	b
2	x_2	0	1	0	$1/6$	$5/2$
3	x_3	0	0	1	$-1/2$	$5/2$
1	x_1	1	0	0	$7/6$	$5/2$
$Z_j = \Sigma a_{ij} C_B$		1	2	3	0	15
$C_j = c_j - Z_j$		0	0	0	-1	

Since all $C_j \le 0$, an optimal solution has been reached.

∴ The optimum basic feasible solution is

$$x_1 = 5/2,\ x_2 = 5/2,\ x_3 = 5/2,\ x_4 = 0,\ Z_{max} = 15$$