Date : ....

A CONSOLIDATED QUESTION PAPER-CUM-ANSWER BOOKLET

IMS™ (INSTITUTE OF MATHEMATICAL SCIENCES)

196/250

MAINS TEST SERIES-2019

(JUNE-2019 *to* SEPT.-2019)

Under the guidance *of* K. Venkanna

MATHEMATICS

PAPER - II : MODERN ALGEBRA, REAL ANALYSIS, COMPLEX & LPP

TEST CODE: TEST-2: IAS(M)/16-JUNE.-2019

Time: 3 Hours

Maximum Marks: 250

INSTRUCTIONS

1. This question paper-cum-answer booklet has 50 pages and has 33 PART/SUBPART questions. Please ensure that the copy of the question paper-cum-answer booklet you have received contains all the questions.
2. Write your Name, Roll Number, Name of the Test Centre and Medium in the appropriate space provided on the right side.
3. A consolidated Question Paper-cum-Answer Booklet, having space below each part/sub part of a question shall be provided to them for writing the answers. Candidates shall be required to attempt answer to the part/sub-part of a question strictly within the pre-defined space. Any attempt outside the pre-defined space shall not be evaluated. "
4. Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
5. Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any **THREE** of the remaining questions selecting at least **ONE** question from each Section.
6. The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
7. Symbols/notations carry their usual meanings, unless otherwise indicated.
8. All questions carry equal marks.
9. All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
10. All rough work should be done in the space provided and scored out finally.
11. The candidate should respect the instructions given by the invigilator.
12. The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any

READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY

Name Mayur Khandelwal

Roll No.

Test Centre

Medium English

Do not write your Roll Number or Name anywhere else in this Question Paper-cum-Answer Booklet.

I have read all the instructions and shall abide by them

Mayur

Signature of the Candidate

I have verified the information filled by the candidate above

Signature of the invigilator

IMPORTANT NOTE:
Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed. Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

P.T.O.

DO NOT WRITE ON THIS SPACE

INDEX TABLE

QUESTION	No.	PAGE NO.	MAX. MARKS	MARKS OBTAINED	
1	(a)			08	
	(b)			08	40
	(c)			08	
	(d)			08	
	(e)			08	
2	(a)			10	
	(b)			06	42
	(c)			13	
	(d)			13	
3	(a)				
	(b)				
	(c)				
	(d)				
4	(a)				
	(b)				
	(c)				
	(d)				
5	(a)			08	
	(b)			08	
	(c)			07	38
	(d)			07	
	(e)			08	
6	(a)				
	(b)				
	(c)				
	(d)				
7	(a)			06	
	(b)			08	34
	(c)			13	
	(d)			07	
8	(a)			08	
	(b)			06	42
	(c)			11	
	(d)			17	
			Total Marks		

196/250

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SECTION – A

1. (a) Are the following groups cyclic groups ? Give reasons.
   (i) The Klein 4-group.
   (ii) The dihedral group D_4.
   (iii) The group of all roots (real or complex) of the equation $x^n - 1 = 0$.
   (iv) The group Q^* of non-zero rationals, for multiplication. **[10]**

i) Klein 4-group $= \{e, i, j, k\}$ where $i^2 = j^2 = k^2 = e$

$G_1 = \langle e, 1, j, k\}$ is $\boxed{\text{not cyclic}}$ since - no element is of order 4.

ii) Dihedral group $D_4 = \{e, a, b, ab\}$ where $a^2 = b^2 = e$ & $ab = ba$ $\Rightarrow (ab)^2 = e$

$\boxed{\text{not cyclic}}$ since no element of order 4.

iii) $G = \{x \in C / x^n - 1 = 0\} = \{e^{\frac{i2k\pi}{n}} / k = 0, 1, 2 \ldots n-1\}$

$\boxed{G \text{ is cyclic}}$ since $G = \langle e^{\frac{i2\pi}{n}} \rangle$

iv) $Q^* = \{x \in Q / x \neq 0\}$

$(Q^*, \cdot)$ is $\boxed{\text{not cyclic}}$ group.

since no element $a \in Q$ s.t. $Q^* = \langle a \rangle$

08

1. (b) Let R be a commutative ring with unit element whose only ideals are (0) and R itself. Then R is a field. **[10]**

Let R be a commutative ring with unity.
then $aR = \{ar / r \in R\}$ is ideal of R
where $a \in R$
since – $a(r_1 - r_2) = ar_1 - ar_2$ and $r_1, r_2 \in R$
thus $ar_1 - ar_2 = a(r_1 - r_2) \in aR$ — (1)
and $r'(ar) = a(r'r) \in aR$ — (2)
$arr' = a(rr') \in aR$ — (3)

08

$\Rightarrow$ aR is proper ideal since $a \in aR \Rightarrow aR \neq \phi$
Since there are no proper ideals
$aR = R$ $(\because aR \neq 0)$
$\exists$ $b \in R$ s.t. $ab = 1$
Since R is Commutative $ab = 1 = ba$
thus $\forall$ $a \in R$ which are non-zero
$\exists$ $b = a^{-1}$ s.t. $ab = 1 = ba$
further R is Commutative & with unity.
$\Downarrow$
R is field

1. (c) For $u_1 > 0$, the sequence u_n defined by

$u_{n+1} = 1 + \frac{1}{u_n} \forall n$, converges to $\left(\frac{\sqrt{5}+1}{2}\right)$ **(10)**

Let $u_1 > 0$, $u_2 = 1 + \frac{1}{u_1}$, $u_3 = 1 + \frac{1}{u_2}$ $\Rightarrow u_n > 0$ $\forall n \in N$

Now $u_{n+1} - u_n = 1 + \frac{1}{u_n} - u_n = -\frac{(u_n^2 - u_n - 1)}{u_n}$

$u_{n+1} < u_n$ for $u_n > \frac{\sqrt{5}+1}{2}$

$u_{n+1} > u_n$ for $u_n < \frac{\sqrt{5}+1}{2}$

If $u_1 > \frac{\sqrt{5}+1}{2}$ $\Rightarrow$ Monotonically decreasing $u_{n+1} < u_n$

& $u_1, u_2, u_n > 1$

bonded

If $u_1 < \frac{\sqrt{5}+1}{2}$ $\Rightarrow$ Monotonically increasing & bounded

In both cases u_n Cvgs.

& $\lim_{n\to\infty} u_n = u_{n+1}$

$\Rightarrow 1 + \frac{1}{u_n} = u_n$

$\Rightarrow u_n^2 - u_n - 1 = 0$

$\Rightarrow \boxed{\lim_{n\to\infty} u_n = \frac{1+\sqrt{5}}{2}}$

08

1. (d) Prove that the function $u = e^{-x}(x \cos y + y \sin y)$. is harmonic and find the corresponding analytic function. **(10)**

u is harmonic if $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

Let $\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left((x\cos y + y\sin y)(-e^{-x}) + e^{-x}(\cos y)\right)$

$= (x\cos y + y\sin y)e^{-x} - e^{-x}\cos y - e^{-x}\cos y$ — (1)

$\frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(e^{-x}(-x\sin y + y\cos y + \sin y)\right)$

$= e^{-x}(-x\cos y + 2\cos y - y\sin y)$ — (2)

Clearly $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow$ u is harmonic

By Cauchy's equation— $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = -(x\cos y + y\sin y)e^{-x} + e^{-x}\cos y$

$v = -(x\sin y - y\cos y + \sin y)e^{-x} + e^{-x}\sin y + f(x)$

$= (y\cos y - x\sin y)e^{-x} + F(x)$

$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \Rightarrow v = (y\cos y + \sin y)e^{-x} + \sin y(xe^{-x} - e^{-x}) + F(y)$

$= (y\cos y - x\sin y)e^{-x} + F(y)$

Making $F(x) = F(y) = C \Rightarrow$ $\boxed{v = (y\cos y - x\sin y)e^{-x} + C}$

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1. (e) Write the dual of the problem.

Min $Z = 2x_2 + 5x_3$

Subject to $x_1 + x_2 \geq 2$

$2x_1 + x_2 + 6x_3 \leq 6$

$x_1 - x_2 + 3x_3 = 4$

$x_1, x_2, x_3 \geq 0.$

[10]

Problem can be rewritten as –

$$\text{Min. } Z = 2x_2 + 5x_3 + 0x_1$$

$$x_1 + x_2 \geq 2$$

$$-2x_1 - x_2 - 6x_3 \geq -6$$

$$x_1 - x_2 + 3x_3 \geq 4$$

$$-x_1 + x_2 - 3x_3 \geq -4 \qquad x_1, x_2, x_3 \geq 0$$

Dual of problem becomes –

$$\text{Max } Z = \cancel{-6u_1 + 4u_2 - 4u_3} \; 2u_1 - 6u_2 + 4u_3 - 4u_4$$

$$\text{s.t. } u_1 - 2u_2 + u_3 - u_4 \leq 0$$

$$u_1 - u_2 - u_3 + u_4 \leq 2$$

$$u_1, u_2, u_3, u_4 \geq 0 \qquad -6u_2 + 3u_3 - 3u_4 \leq 5$$

Dual can be rewritten as (by $u_3 - u_4 = u_3''$)

$$\text{Max } Z = 2u_1 - 6u_2 + 4u_3''$$

$$\text{s.t. } u_1 - 2u_2 + u_3'' \leq 0$$

$$u_1 - u_2 - u_3'' \leq 2$$

$$-6u_2 + 3u_3'' \leq 5$$

$$u_1, u_2 \geq 0$$

u_3'' is unrestricted.

08

2. (a) (i) Show that every subgroup of an abelian group is normal.
(ii) Is the converse of Problem (i) true ? If yes, prove it, if no, give an example of a non-abelian group all of whose subgroups are normal. **[12]**

i) Let G be abelian group.

Let $H \leq G \Rightarrow H$ is any subgroup of G.

$\Rightarrow$ Let $a = g^{-1}hg$ for $\forall g \in G$ and $h \in H$

$\Rightarrow a = h(g^{-1}g)$ (Since G is abelian and $h \in H \leq G$)

$\Rightarrow a = h \in H$

Thus $\forall g \in G$ & $h \in H$ $g^{-1}hg \in H$

$\Downarrow$

Every H is normal subgroup.

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ii) Converse of Problem is not true.

Since let take quaternion group.

$G = \{1, -1, i, -i, j, -j, k, -k\}$

G is non-abelian group.

Its subgroups are $\{1\}$, G, H_1, H_2, H_3, H_4

$\{1\}$, G are improper normal subgroups

$H_1 = \{1, -1\} \Rightarrow g^{-1}(1)g = 1 \in H_1$

$g^{-1}(-1)g = -1 \in H_1$

$H_2 = \{1, -1, -i, i\}$ $g^{-1}(i)g = g^{-1}(-i)g \in H_2$

e.g. $(k\,i\,(-k)) = -i \in H_2$

Similarly H_3, H_4 are normal subgroups.

$\Rightarrow$ Thus $\exists$ a non-abelian group whose all subgroups are normal.

2. (b) Suppose that N and M are two normal subgroups of G and that $N \cap M = (e)$. show that for any $n \in N$, $m \in M$, $nm = mn$. **[08]**

Let N, M are normal subgroup of G. &

$N \cap M = \{e\}$

Consider a element $a = n^{-1}m^{-1}nm \in G$

$a = n^{-1}m^{-1}nm = (n^{-1}m^{-1}n)m = m_1 m \in M$

(since M is normal subgroup)

thus for $m^{-1} \in M$ & $n \in N \subseteq G$

$n^{-1}m^{-1}n \in M$

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Similarity –

$$a = n^{-1}m^{-1}nm = n^{-1}(m^{-1}nm)$$
$$= n^{-1}n'$$
$$\in N$$

$\Rightarrow a \in M \ \& \ a \in N \Rightarrow a \in M \cap N$

$\Rightarrow a = e \quad (\because M \cap N = \{e\})$

$\Rightarrow n^{-1}m^{-1}nm = e$

$\Rightarrow m^{-1}nm = n \Rightarrow \boxed{nm = mn}$

2. (c) Test the convergence of

(i) $\int_0^1 \frac{dx}{\sqrt{1-x^3}}$

(ii) Prove that the integral $\int_0^\infty x^{m-1}e^{-x}dx$ is convergent if and only if $m > 0$. **[15]**

i) $\int_0^1 \frac{dx}{\sqrt{1-x^3}} = \int_0^1 \frac{dx}{\sqrt{(1-x)(1+x+x^2)}}$

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$x = 1$ is only point of discontinuity.

Let $f(x) = \frac{1}{\sqrt{(1-x)(1+x+x^2)}}$

$g(x) = \frac{1}{\sqrt{1-x}}$

$$\underset{x \to 1}{Lt} \frac{f(x)}{g(x)} = \frac{1}{\sqrt{1+x+x^2}} = \frac{1}{\sqrt{3}} \neq 0$$

Since $\int_0^1 g(x)\,dx$ is Cvgt. $\Rightarrow \int_0^1 \frac{dx}{\sqrt{1-x^3}}$ is Cvgt

ii) Let $\int_0^\infty x^{m-1} e^{-x} dx$ and $f(x) = x^{m-1} e^{-x}$

For $m \geq 1$ there is no point of discontinuity.

$\int_0^\infty x^{m-1} e^{-x} dx$ is Convergent.

Since $\int_0^1 f(x)\,dx$ is cvgt. &

$\int_1^\infty f(x)\,dx$ is cvgt

because let $g(x) = \frac{1}{x^2}$ $\underset{x \to \infty}{Lt} \frac{x^{m+1}}{e^x} = 0$

$\rightarrow$ Cvgt.

For $m < 1$ — 0 is point of Infinite discontinuity.

$\int_0^\infty = \int_0^1 + \int_1^\infty$ | $\int_0^1 \rightarrow$ let $g(x) = x^{m-1}$ $\underset{x \to 0}{Lt} \frac{f(x)}{g(x)} = 1$

x^{m-1} is Cvgt for $m > 0$

$\Rightarrow \int_0^1 f(x)\,dx$ Cvgt for $x > 0$

$\int_1^\infty f(x)\,dx$ is Cvgt. ~~[illegible]~~ $\Rightarrow \int f(x)\,dx$ is cvgt. for $m > 0$

2. (d) Apply the method of contour integration to prove that

$\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\, d\theta = \frac{\pi}{6}$. **[15]**

Let $z = e^{i\theta} \Rightarrow z^2 = e^{i2\theta}$ — (1)

$$\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\, d\theta = \text{Real} \int_0^{2\pi} \frac{e^{i2\theta}\, d\theta}{5+4\cos\theta} \quad \text{— (2)}$$

By (1) $\rightarrow$ $\frac{dz}{iz} = d\theta$ — (3)

Putting (1) & (3) in (2) —

$$\int_C \frac{z^2\, dz}{\left(5+\left(z+\frac{1}{z}\right)2\right) iz} = \int_C \frac{z^2\, dz}{(2z^2+5z+2)\, i}$$

C: —  Unit circle

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$$\oint f(z)\,dz = \int \frac{z^2\, dz}{2i\left(z^2+\frac{5z}{2}+1\right)} = \int \frac{z^2\, dz}{2i\,(z+2)\left(z+\frac{1}{2}\right)}$$

poles are — $z = -2, -\frac{1}{2}$

$-\frac{1}{2}$ lies in C.

Residue $\int \frac{f(z)}{dz}$ at $z = -\frac{1}{2}$ $\left(\frac{z^2}{2i(z+2)} \right)_{z=-\frac{1}{2}}$

$$= \frac{(1/4)}{(2i)(3/2)} = \frac{1}{12i}$$

$$\int_C f(z)\,dz = 2\pi i \times \frac{1}{12i} = \frac{\pi}{6}$$

$$\Rightarrow \boxed{\int_0^{2\pi} \frac{\cos 2\theta}{5+4\cos\theta}\, d\theta = Re\left(\frac{\pi}{6}\right) = \frac{\pi}{6}}$$

3. (a) Prove, by an example, that we can find three groups $E \subset F \subset G$, where E is normal in F, F is normal in G, but E is not normal in G. **[10]**

3. (b) Given an example of an integral domain which has an infinite number of elements, yet is of finite characteristic. **[10]**

SECTION – B

5. (a) Give an example of a group which is not a cyclic group, but every proper subgroup of which is cyclic. **[10]**

Let take Quaternion group – $i^2 = j^2 = k^2 = -1$

08 $G = \{1, -1, i, -i, j, -j, k, -k\}$ $ij = -ji = k$

$ki = -ik = j$

G is not cyclic group. $jk = -kj = i$

proper subgroups of G are –

$H_1 = \{1, -1, i, -i\}$ $H_2 = \{1, -1, j, -j\}$

$H_3 = \{1, -1, k, -k\}$ $H_4 = \{1, -1\}$

All proper subgroups of G are cyclic since.

$H_1 = \langle i \rangle$, $H_2 = \langle j \rangle$

$H_3 = \langle k \rangle$, $H_4 = \langle -1 \rangle$

5. (b) Consider the following rings R and R' with four elements

$R = \{a, b, c, d\}$ with + and • defined by

+	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

•	a	b	c	d
a	a	a	a	a
b	a	b	a	b
c	a	a	c	c
d	a	b	c	d

and $R' = \{x, y, z, t\}$ with + and • defined by

+	x	y	z	t
x	x	y	z	t
y	y	z	t	x
z	z	t	x	y
t	t	x	y	z

•	x	y	z	t
x	x	x	x	x
y	x	y	z	t
z	x	z	x	z
t	x	t	z	y

Investigate whether R and R' are isomorphic. [10]

In $(R,+) \to (R^+,+)$ $f(a) = x$, $f(d) = y$

$\Downarrow$

$f(e) = e'$

In $(R,\times) \to (R^+,\times) \to f(d) = y$ $(\because f(e) = e')$

Let $f(b) = z$, $f(c) = t$

not homomorphism

since $f(bc) = f(a) = x$

$f(b)f(c) = zt = z$ $x \neq z$

Now Let $f(b) = t$, $f(c) = z$

again not homomorphism.

$f(bc) = f(a) = x$

-08

$f(b)f(c) = tz = z$ $x \neq z$

not homomorphism

$\Downarrow$

R & R are not isomorphic

5. (c) The sequence $nx/(1 + n^2 x^2)$ is not uniformly convergent over **R** but it is uniformly convergent on $\{x : |x| > k > 0\}$. **[10]**

Let $\langle f_n(x) \rangle = \left\langle \frac{nx}{1+n^2x^2} \right\rangle$

Here $f(x) = \lim_{n\to\infty} \frac{nx}{1+n^2x^2} = 0$

$$|f_n(x) - f(x)| = \left| \frac{nx}{1+n^2x^2} \right|$$

by M_n test – $\frac{d}{dx}\left(\frac{nx}{1+n^2x^2}\right) = 0 \Rightarrow \boxed{x = \frac{1}{n}}$

$$\frac{d^2}{dx^2}\left(\frac{nx}{1+n^2x^2}\right) < 0 \text{ at } x = \frac{1}{n}$$

$\Rightarrow$ thus $M_n = \left| \frac{n \times \frac{1}{n}}{1 + n^2/n^2} \right| = \frac{1}{2}$

$M_n = \frac{1}{2} \not\to 0$ as $\frac{1}{n} \to 0$

thus $\boxed{\text{not uniformly convergent}}$ since R contains 0

for $|x| > k > 0$ $\left| \frac{nx}{1+n^2x^2} \right| \leq \frac{kn}{1+n^2k^2} = M_n$

$M_n = \frac{kn}{1+k^2n^2} \to 0$ as $\frac{1}{n} \to 0$ thus uniformly convergent

07

5. (d) If $f(z)=\frac{x^3y(y-ix)}{x^6+y^2}, z\neq 0$ and $f(0) = 0$, show that $\frac{f(z)-f(0)}{z}\to 0$ as $z\to 0$ along any radius vector but not as $z\to 0$ in any manner. **[10]**

Let $f(z)=f(x+iy)=\begin{cases}\dfrac{x^3y(y-ix)}{x^6+y^2} & z\neq 0\\ 0 & z=0\end{cases}$

~~$f'(0)=$~~

Now -

$$g(z)=\frac{f(z)-f(0)}{z}=\frac{x^3y(y-ix)}{(x^6+y^2)(x+iy)}\times\frac{i}{i}$$

$$g(z)=\frac{x^3y}{i(x^6+y^2)}$$

Along radius vector $y=mx$

$$g(z)=\frac{mx^4}{i(x^6+m^2x^2)}=\frac{mx^2}{i(x^4+m^2)}$$

$$\lim_{z\to 0} g(z)=\lim_{x\to 0}\frac{mx^2}{i(x^4+m^2)}=\boxed{0}$$

Along any other manner it does not $\to 0$ since let

$y=mx^3$ $\quad g(z)=\dfrac{mx^6}{i(x^6+m^2x^6)}=\dfrac{m}{i(1+m^2)}\neq 0$

Hence proved.

07

5. (e) A firm plans to purchase at least 200 quintals of scrap containing high quality metal X and low quality metal. Y. It decides that the scrap to be purchased must contain at least 100 quintals of X-metal and not more than 35 quintals of Y-metal. The firm can purchase the scrap from two suppliers (A and B) in unlimited quantities. The percentage of X and Y metals in terms of weight in the scraps supplied by A and B is given below :

Metals	supplier A	Supplier B
X	25%	75%
Y	10%	20%

The price of A's scrap is Rs. 200 per quintal and that of B's is Rs. 400 per quintal. Formulate this problem as LP model and solve it graphically to determine the quantities that the firm should buy from the two suppliers so as to minimize total purchase cost. [10]

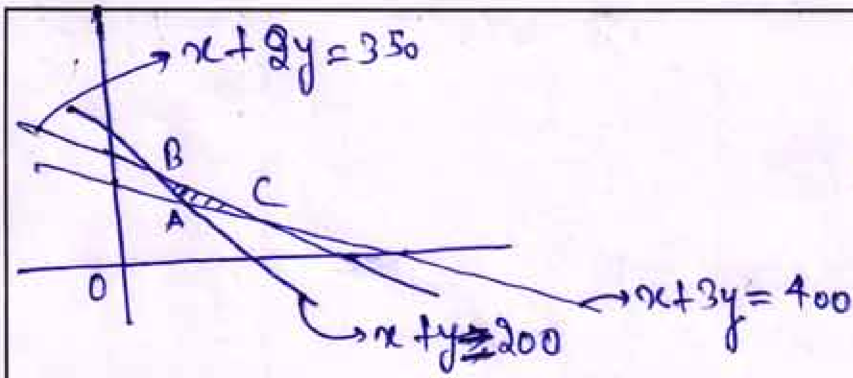
Let scrap bought from A is x
& from B is y.

$$x+y \geq 200$$
$$0.25x+0.75y \geq 100$$
$$0.10x+0.20y \leq 35$$
$x, y \geq 0$

$$\boxed{\begin{aligned} &\text{Min cost} = 200x+400y \\ &\text{s.t.} \quad x+y \geq 200 \\ &\qquad x+3y \geq 400 \\ &\qquad x+2y \leq 350 \\ &\qquad\qquad x, y \geq 0 \end{aligned}}$$

8'

Solution →



$A \to (100, 100)$
$B \to (50, 150)$
$C \to (250, 50)$

$Z_A = 200\times 100 + 400\times 100 = 60000$

$Z_B = 200\times 50 + 150\times 400 = 70000$

$Z_C = 200\times 250 + 50\times 400 = 70000$

Min cost when $x=100, y=100$
Total Cost = 60000 Rs.

6. (a) (i) Let G be the group of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where a, b, c, d are integers modulo p, p a prime number, such that $ad - bc \neq 0$. G forms a group relative to matrix multiplication. What is $o(G)$?

(ii) Let H be the subgroup of the G of part (a) defined by

$$H = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \,\middle|\, ad - bc = 1 \right\}.$$

What is $o(H)$? **[15]**

7. (a) Prove that the following sets are bounded

$\{n^{1/n} : n \in N\}, \left\{\left(1+\frac{1}{n}\right)^n : n \in N\right\}$, $\{a^{1/n} : a > 0 \text{ and } n \in N\}$.

Give supremum and infimum of each of these sets. [08]

A) $S = \{n^{1/n} : n \in N\}$

$= \{1, 2^{1/2}, 3^{1/3} \_\_\_\}$

$\underset{n\to\infty}{Lt}\ n^{1/n} = 1$

06

$\text{Max } n^{1/n} \Rightarrow \frac{d}{dn}(n^{1/n}) = 0 \Rightarrow 1 - \log n = 0$

$\Rightarrow n = e$

$e^{1/e}$ is supremum and $\in S$

infimum is $1 \in S$

B) $S = \left\{ \left(1+\frac{1}{n}\right)^n, n \in N \right\}$

$\left(1+\frac{1}{n}\right)^n$ is increasing f^n with $\lim_{n\to\infty} = e$

Supremum $S = e \notin S$ Infimum $S = 2 \in S$

C) $S = \{a^{1/n} : a>0 \ \& \ n \in N\}$

$= \{a, a^{1/2}, a^{1/3} - - \}$

If $a=1$ 1 is infimum as well as sup.

If $0<a<1$ 1 is sup $\notin S$ and a is infimum $\in S$

if $a>1$ 1 is inf. $\notin S$ and a is sup. $\in S$

7. (b) Show that $\prod_1^{\infty}\left(1-\frac{1}{4n^2}\right)$ converges and its limit lies between $\frac{1}{2}$ and 1. **[10]**

$\prod_1^{\infty}\left(1-\frac{1}{4n^2}\right) = \prod_1^{\infty}(1-b_n) \quad 0<b_n<1$

$\prod(1-b_n)$ cvgs if $\sum b_n$ cvgs

$\sum \frac{1}{4n^2}$ cvgs since $p>1$ by p-test

$\Downarrow$

$\prod\left(1-\frac{1}{4n^2}\right)$ Converges

08

$$\prod\left(1-\frac{1}{4n^2}\right)=\left(1-\frac{1}{4}\right)\left(1-\frac{1}{16}\right)$$

$$< 1 \times 1 \times 1$$

$$\boxed{< 1}$$

$$\prod\left(1-\frac{1}{4n^2}\right)=\frac{3}{4}\times\frac{15}{16}\times\frac{35}{36}-$$

$$=\cancel{\frac{1}{2}\times\frac{3}{2}\times\frac{3}{4}\times\frac{5}{4}\times}$$

$$\Rightarrow \boxed{\frac{1}{2}<\prod\left(1-\frac{1}{4n^2}\right)<1} = \frac{1}{2}\times\left(\frac{3}{2}\times\frac{3}{4}\right)\times\left(\frac{5}{4}\times\frac{5}{6}\right)\times\left(\frac{7}{6}\times\frac{7}{8}\right)$$

$$> \frac{1}{2}\times 1\times 1\times 1$$

$$> \frac{1}{2}$$

7. (c) For x > – 1, test for convergence

$$1+\frac{2^2}{3.4}x+\frac{2^2.4^2}{3.4.5.6}x^2+\frac{2^2.4^2.6^2}{3.4.5.6.7.8}x^3+....$$

[15]

$$\sum f_n(x)=\sum\frac{2^2.4^2\ldots(2n)^2}{3.4.\ldots(2n+1)(2n+2)}x^n$$

$$\frac{u_n}{u_{n+1}}=\frac{1}{x}\times\frac{2^2 4^2\ldots(2n)^2}{2^2 4^2\ldots(2n)^2(2n+1)^2}\times\frac{3.4.\ldots(2n+5)(2n+7)}{3.4.\ldots(2n+1)(2n+3)}$$

$$=\frac{1}{x}\frac{(2n+5)(2n+7)}{(2n+1)(2n+1)}$$

$$\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\frac{1}{x}$$

$\sum u_n$ Cvg if $\frac{1}{x} > 1 \Rightarrow x < 1$

dvgt if $\frac{1}{x} < 1 \Rightarrow x > 1$

at $x = 1$ ratio test fails.

at $x=1 \rightarrow \frac{u_n}{u_{n+1}} = \frac{(2n+5)(2n+7)}{(2n+1)^2}$

$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = n\left(\frac{20n+34}{4n^2+4n+1}\right) = 5 > 1$

↳ Rabees test

$\Rightarrow$ at $x=1$ $\sum u_n$ Cvgt.

thus $-1 < x \leq 1$ — Cvgt.

$x > 1 \longrightarrow$ dvgt.

7. (d) Let a function $f : \mathbf{R} \to \mathbf{R}$ satisfy the equation $f(x + y) = f(x) + f(y)$, $\forall$ x, y $\in \mathbf{R}$. Show that

(i) If f is continuous at the point x = a, then it is continuous for all x $\in \mathbf{R}$.

(ii) If f is continuous then $f(x) = kx$, for some constant k.

[15]

i) Let $f(x+y) = f(x) + f(y)$

Let f is continuous at $x = a$.

$\Rightarrow \quad |f(a+y) - f(a)| < \epsilon$ for $|x-a| < \delta$

$\Rightarrow \quad |f(y)| < \epsilon$ for $\forall x \in R$

thus at any arbitrary point x'

$|f(x'+y) - f(x')| = |f(y)| < \epsilon$

for $|x-x'| < \delta$

$\Rightarrow$ $f(x)$ is Continuous for all $x \in R$

-07

ii) $f(1+1) = f(1) + f(1)$

$f(2) = 2f(1)$

$f(3) = 3f(1)$

$\vdots$

$\boxed{f(x) = f(1)x = kx \quad \text{where } k = f(1)}$

If possible try it again otherwise consult key

~~Or~~

~~Assuming f is differentiable~~

Or

$f'(x+y) = f'(x)$ (partially differentiating w.r.t x)

$\Rightarrow$ At $y=-x$ $\Rightarrow$ $f'(x) = f'(0)$

$\Rightarrow$ $f(x) = f'(0)x$

$= kx$

where $k = f'(0)$

8. (a) Using Cauchy's/Cauchy's integral formula.

(i) Evaluate $\oint_c \frac{\sin 3z}{z+\pi/2} dz$ if c is the circle $|z| = 5$.

(ii) Evaluate $\oint_c \frac{e^{3z}}{z-\pi i} dz$ if C is : (A) the circle $|z-1| = 4$, (B) the ellipse $|z-2| + |z+2| = 6$. **[10]**

i) $\oint_C \frac{\sin 3z}{z+\frac{\pi}{2}} dz$ $C: |z| = 5$

pole $z = -\frac{\pi}{2}$ lies within $|z| = 5$

thus Cauchy formulare applies

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$\frac{n!}{2\pi i} \oint \frac{f(z)}{(z-z_0)^{n+1}} dz = \frac{d^n}{dz^n} f(z)$

$$\Rightarrow \frac{0!}{2\pi i} \int_C \frac{f(z)}{\left(z+\frac{\pi}{2}\right)} = \left(f(z)\right)_{z=-\frac{\pi}{2}}$$

Here $f(z) = \sin 3z$

$$\Rightarrow \int_{|z|=5} \frac{\sin 3z}{z+\pi/2} = \sin\left(\frac{-3\pi}{2}\right) \times 2\pi i$$

$$= 2\pi i$$

ii) $z = \pi i$ is pole of $\frac{e^{3z}}{z-\pi i}$

A) $z = \pi i$ lies within $|z-1| = 4$

$$\Rightarrow \int_{|u|=4} \frac{e^{3u} \cdot e^{3}}{u+1-\pi i} du \qquad \text{Let } z-1 = u$$

$$= e^{3} e^{3(\pi i-1)} \times 2\pi i$$

$$= -2\pi i$$

B) $z = \pi i$ lies outside $|z-2| + |z+2| = 6$

Since $2\sqrt{\pi^2+4} > 6$

thus $\oint_C \frac{e^{3z}}{z-\pi i} = 0$

$C: |z-2| + |z+2| = 6$

8. (b) Find the Laurent expansion of $\frac{z}{(z+1)(z+2)}$ about the singularity $z = -2$. Specify the region of convergence. **[07]**

let $z+2 = u$

$$\frac{z}{(z+1)(z+2)} = \frac{(u-2)}{u(u+1)}$$

$$= -\frac{2}{u} + \frac{3}{u+1}$$

$$= -\frac{2}{u} + 3(1+u)^{-1}$$

for $|u|<1$ $= -\frac{2}{u} + 3(1-u+u^2-u^3+ \text{---})$

$$= -\frac{2}{u} + 3 - 3u + 3u^2 - 3u^3 + \text{---}$$

for $|u|>1$ $= -\frac{2}{u} + \frac{3}{u}\left(1 - \frac{1}{u} + \frac{1}{u^2} \text{--}\right)$

06 $= \frac{1}{u} - \frac{3}{u^2} + \frac{3}{u^3} - \frac{3}{u^4} + \text{---}$

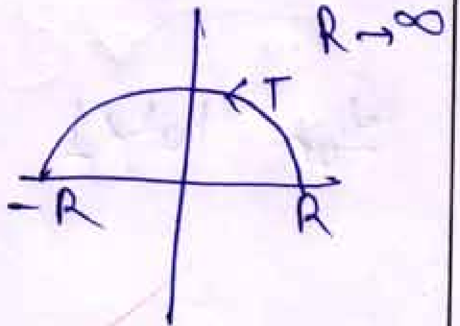
$|u|<1 \Rightarrow 0<|z+2|<1$ is region of Convergence

8. (c) By integrating $e^{iz} / (z - ai)$, $(a > 0)$ round a suitable contour, prove that $\int_{-\infty}^{\infty} \frac{a\cos x + x\sin x}{x^2 + a^2} dx = 2\pi e^{-a}$. **[13]**

$$\int_{-\infty}^{\infty} \frac{a\cos x + x\sin x}{x^2+a^2} dx = \text{Imginary part of } \int_{-\infty}^{\infty} \frac{e^{ix}(x+ai)}{x^2+a^2} dx$$

$$= \text{Imaginary part of } \int_{-\infty}^{\infty} \frac{e^{ix}}{x-ia} dx$$

Let $\int f(z) dz = \int_C \frac{e^{iz}}{z-ia}$

C: 

poles of $f(z) = z = ai$

Since $a \neq 0$ $z = ai$ lies within C

Residue of $\int f(z) dz$ at $z = ai$

$$= \left(e^{iz}\right)_{z=ai}$$

$$= e^{-a}$$

$$\int_C f(z) dz = 2\pi i e^{-a}$$

& $\int_T f(z) dz = 0$ by Jordan's lemma

$$\Rightarrow \int_{-\infty}^{\infty} \frac{e^{-ix}\,dx}{x-ia} = 2\pi i e^{-a}$$

$$\Rightarrow \operatorname{Img} \int_{-\infty}^{\infty} \frac{e^{-ix}\,dx}{x-ia} = 2\pi e^{-a}$$

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} \frac{a\cos x + x\sin x}{x^2+a^2}\,dx = 2\pi e^{-a}}$$

8. (d) Solve the following transportation problem

		To		Available
From	2	7	4	5
	3	3	1	8
	5	4	7	7
	1	6	2	14
Required	7	9	18	34

[20]

Balanced problem since Required = Available

Let find Initial basic feasible solution using Vogel's approx method.

2	7	4	5	(2)
3	3	1 (8)	~~8~~	(2)
5	4	7	7	(1)
1	6	2	14	(1)
7	9	~~18~~ 10		
(1)	(1)	(1)		

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2	7	4 ×	5	(2)
5	4	7 ×	7	(1)
1	6	2 (10)	~~14~~ 4	(1)
7	9	~~10~~		
(1)	(2)	(2)		

(5)	2	7	5
(1)	5	4	7
(5)	1 (4)	6 ×	~~4~~
	3 ~~7~~	9	
	(1)	(2)	

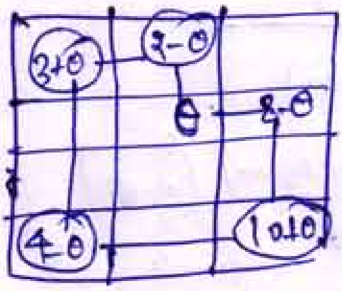
2 (3)	7 (2)	5	(5)
5 ×	4 (7)	7	(1)
3	9		
(3)	(3)		

I.B.F.S =

				u ↓
	2 (3)	7 (2)	4 (-1)	0
	3 (-3)	3 (2)	1 (8)	-2
	5 (-)	4 (7)	7 (-)	-3
	1 (4)	6 0	2 (10)	-1
v →	2	7	3	

basic variables are $\boxed{3+4-1 = 6}$ variables.

$u_i + v_j - c_{ij} > 0$ for $(2,2)$



$\theta_{min} = 2$

$\Longrightarrow$

			u_i
2 (5)	7 (-1)	4 (-1)	1
3 (-1)	3 (2)	1 (6)	-1
5 (-1)	4 (7)	7 (-1)	0
1 (2)	6 (-1)	2 (12)	0
1	4	2	

all $u_i + v_j - c_{ij} < 0 \Rightarrow$ Optimality is achieved

2 (5)	7	4
3	3 (2)	1 (6)
5	4 (7)	7
1 (2)	6	2 (12)

Min. cost $= 2\times5 + 3\times2 + 6\times1 + 4\times7 + 2\times1 + 12\times2$

$= 10 + 6 + 6 + 28 + 2 + 24$

$= \boxed{76}$

ROUGH SPACE

$\frac{(2n-1)(2n-1)}{(2n-1)\cdot 2n}$

$\frac{e^{in}(n+ai)}{(n-ai)}$

$(\cos n + i \sin n)(n+ai)$

3 5 7 9

$r \log n$

$\frac{A}{u} + \frac{B}{u-1}$

$A + B = 1$

$A = -2$

$B = 3$

$\frac{3}{u-1} - \frac{2}{u}$

$\frac{1}{2} \times \frac{3}{2} \times \frac{3}{4} \times \frac{5}{4} \times \frac{5}{8}$

$\frac{3}{4} \times \frac{15}{16}$

$\frac{5}{4} \times \frac{7}{8}$

$\log\left(1 - \frac{1}{3n}\right)\left(1 + \frac{1}{3n}\right)$

$1 - \left\{ \frac{1}{\cdot} \right.$

$\left(1 - \frac{1}{3}\right)\left(1 + \frac{1}{3}\right)$

$\frac{1}{3} + \frac{3}{2} +$