IMS™

(INSTITUTE OF MATHEMATICAL SCIENCES)

UPSC-CSE 2021

Mains

MATHEMATICS

Optional Paper-I

Solutions

SECTION-A

1.(a) If $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ then show that $A^2 = A^{-1}$ (without finding A^{-1})

Soln

We have

$$A^2 = AA = \begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}$$

we have $A^3 = A^2A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 2 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 & 1 \\ 2 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$\therefore \boxed{A^3 = I}$ — (1)

Since $|A| = 1 \neq 0$ $\therefore A^{-1}$ exists.

$\therefore (1) \equiv A^3A^{-1} = IA^{-1}$

$\Rightarrow \boxed{A^2 = A^{-1}}$

Hence proved

1(b) Find the matrix associated with linear operator on $V_3(\mathbb{R})$ defined by $T(a,b,c) = (a+b, a-b, 2c)$ with respect to the ordered basis
$B = \{(0,1,1),(1,0,1),(1,1,0)\}$.

Sol Let $T: V_3(\mathbb{R}) \to V_3(\mathbb{R})$ be a linear operator defined by $T(a,b,c) = (a+b, a-b, 2c)$

Let us find matrix of the linear operator with respect to ordered basis
$B = \{(0,1,1),(1,0,1),(1,1,0)\}$.

We have

$$\left.\begin{aligned} T(0,1,1) &= (1,-1,2) \\ T(1,0,1) &= (1,1,2) \\ T(1,1,0) &= (2,0,0) \end{aligned}\right\} \quad \text{--- (1)}$$

Let $(a,b,c) \in V_3(\mathbb{R})$ then

$(a,b,c) = x(0,1,1) + y(1,0,1) + z(1,1,0)$ --- (2)

where $x,y,z \in \mathbb{R}$.

$\Rightarrow$ $y+z = a$ --- (i)

$x+z = b$, --- (ii)

$x+y = c$ --- (iii)

(iii) $-$ (ii) $\equiv$ $\boxed{y-z = c-b}$ --- (iv)

(i) + (iv) $\equiv$ $2y = a+c-b$

$\Rightarrow \boxed{y = \dfrac{a-b+c}{2}}$

(iv) $\equiv \dfrac{a-b+c}{2} - c + b = z$

$\Rightarrow z = \dfrac{a-b+c-2c+2b}{2}$

$\Rightarrow \boxed{z = \dfrac{a+b-c}{2}}$

(ii) $x = b - \left(\dfrac{a+b-c}{2}\right)$

$\Rightarrow x = \dfrac{2b-a-b+c}{2}$

$\Rightarrow \boxed{x = \dfrac{-a+b+c}{2}}$

$\therefore$ (2) $\equiv$

$(a,b,c) = \left(\dfrac{-a+b+c}{2}\right)(0,1,1) + \left(\dfrac{a-b+c}{2}\right)(1,0,1)$
$+ \left(\dfrac{a+b-c}{2}\right)(1,1,0)$. —— (3).

$\therefore$ ~~from~~ (1),

$T(0,1,1) = (1,-1,2) = 0(0,1,1) + 2(1,0,1) + (-1)(1,1,0)$

$T(1,0,1) = (1,1,2) = 1(0,1,1) + 1(1,0,1) + 0(1,1,0)$

$T(1,1,0) = (2,0,0) = -1(0,1,1) + 1(1,0,1) + 1(1,1,0)$.

$\therefore [T:B] = \begin{bmatrix} 0 & 1 & -1 \\ 2 & 1 & 1 \\ -1 & 0 & 1 \end{bmatrix}$

which is the required Matrix.

1(c) Given $\Delta(x) = \begin{vmatrix} f(x+\alpha) & f(x+2\alpha) & f(x+3\alpha) \\ f(\alpha) & f(2\alpha) & f(3\alpha) \\ f'(\alpha) & f'(2\alpha) & f'(3\alpha) \end{vmatrix}$

where f is a real valued differentiable function and α is a constant. Find $\lim_{x\to 0} \frac{\Delta(x)}{x}$.

Sol'n: we have

$$\Delta(x) = \begin{vmatrix} f(x+\alpha) & f(x+2\alpha) & f(x+3\alpha) \\ f(\alpha) & f(2\alpha) & f(3\alpha) \\ f'(\alpha) & f'(2\alpha) & f'(3\alpha) \end{vmatrix}.$$

where $f(\alpha)$, (2α), $f(3\alpha)$ are constants.

$$\Rightarrow \Delta(x) = f(x+\alpha)\begin{vmatrix} f(2\alpha) & f(3\alpha) \\ f'(2\alpha) & f'(3\alpha) \end{vmatrix} - f(x+2\alpha)\begin{vmatrix} f(\alpha) & f(3\alpha) \\ f'(\alpha) & f'(3\alpha) \end{vmatrix} + f(x+3\alpha)\begin{vmatrix} f(\alpha) & f(2\alpha) \\ f'(\alpha) & f'(2\alpha) \end{vmatrix}$$

$$\therefore \Delta(x) = Af(x+\alpha) - Bf(x+2\alpha) + Cf(x+3\alpha) \text{ say} \quad \text{——— (1)}$$

where A, B, C are constants.

$$\therefore \Delta'(x) = Af'(x+\alpha) - Bf'(x+2\alpha) + Cf'(x+3\alpha)$$

$$= \begin{vmatrix} f(2\alpha) & f(3\alpha) \\ f'(2\alpha) & f'(3\alpha) \end{vmatrix} f'(x+\alpha) - \begin{vmatrix} f(\alpha) & f(3\alpha) \\ f'(\alpha) & f'(3\alpha) \end{vmatrix} f'(x+2\alpha) + \begin{vmatrix} f(\alpha) & f(2\alpha) \\ f'(\alpha) & f'(2\alpha) \end{vmatrix} f'(x+3\alpha)$$

$$\Rightarrow \Delta'(x) = \begin{vmatrix} f'(x+\alpha) & f'(x+2\alpha) & f''(x+3\alpha) \\ f(\alpha) & f(2\alpha) & f(3\alpha) \\ f'(\alpha) & f'(2\alpha) & f'(3\alpha) \end{vmatrix} \quad \text{——— (2)}$$

we have

$$\underset{x\to 0}{Lt} \frac{\Delta(x)}{x} = \underset{x\to 0}{Lt} \frac{\Delta'(x)}{1} \left(\frac{0}{0}\right)$$

$$= 0$$

1(d). Show that between any two roots of $e^x \cos x = 1$, there exists at least one root of $e^x \sin x - 1 = 0$

Solution:-

Let a and b are the two roots of the equation $e^x . \cos x = 1$.

then, $e^a . \cos a = 1$

and $e^b . \cos b = 1$

Let f be the function defined as

$f(x) = e^{-x} - \cos x$

we observe that:

(i) $f(x)$ is continuous in $[a, b]$ as e^{-x} and $\cos x$ both are continuous function.

(ii) $f'(x) = -e^{-x} + \sin x$, therefore function is differentiable in $]a, b[$

(iii) $f(a) = e^{-a} - \cos a = e^{-a}.[1 - e^a \cos a] = 0$

and $f(b) = e^{-b} - \cos b = e^{-b}[1 - e^b \cos b] = 0$

$\Rightarrow f(a) = f(b) = 0$

Thus, $f(x)$ satisfies all the conditions of Rolle's theorem in $[a, b]$.

Hence, there exist at least one value of x in $]a,b[$ such that $f'(c)=0$.

$f'(x) = -e^{-x} + \sin x$

$f'(c) = -e^{-c} + \sin c = 0$

$-e^{-c} + \sin c = 0$

$-1 + e^{c}.\sin c = 0$

or $e^{c}.\sin c - 1 = 0$

Thus, c is the root of the equation $e^{x}.\sin x - 1 = 0$

Hence, between any two root of the equation $e^{x}.\cos x = 1$, there exists at least one root of $e^{x}.\sin x - 1 = 0$, 1(e) Find the equation of the cylinder whose generators are parallel to the line $\frac{x}{1} = \frac{y}{(-2)} = \frac{z}{3}$ and passing through the curve $x^2+2y^2=1,\ z=0$.

Sol^n: Let $P(x_1, y_1, z_1)$ be any point on the cylinder, then the equations of the generator through P

are $\frac{x-x_1}{1} = \frac{y-y_1}{-2} = \frac{z-z_1}{3}$ —— ①

This generator meets the plane $z=0$ in point given by

$$\frac{x-x_1}{1} = \frac{y-y_1}{-2} = \frac{0-z_1}{3}$$

i.e. in the point $\left[x_1 - \frac{1}{3}z_1,\ y_1 + \frac{2}{3}z_1, 0\right]$

∴ This generator ① intersects the given conic if

∴ The locus of $P(x_1, y_1, z_1)$ or the required equation of the cylinder is

$$\left(x - \tfrac{1}{3}z\right)^2 + 2\left[y + \tfrac{2}{3}z\right]^2 = 1$$

$$\Rightarrow \left(x^2 + \tfrac{1}{9}z^2 - \tfrac{2}{3}xz\right) + 2\left(y^2 + \tfrac{4}{9}z^2 + \tfrac{4}{3}yz\right) = 1$$

$$\Rightarrow 9x^2 + z^2 - 6xz + 18y^2 + 8z^2 + 24yz - 9 = 0$$

$$\Rightarrow 3x^2 + 6y^2 + 3z^2 - 2xz + 8yz - 3 = 0.$$

2.(a) Show that the planes which cut $ax^2+by^2+cz^2=0$ in perpendicular generators, touch the cone
$\Sigma\left[a^2/(b+c)\right]=0$

Sol^n: Let the plane be $ux+vy+wz=0$.

Let $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ be one of the lines in which the plane $ux+vy+wz=0$ meets the cone $ax^2+by^2+cz^2=0$, then we have

$ul+vm+wn=0$ —(1) and $al^2+bm^2+cn^2=0$ —(2)

Eliminating n between (1) & (2), we get

$al^2+bm^2+c\left[-(ul+vm)/w\right]^2=0$

$\Rightarrow (aw^2+cu^2)l^2+2cuvlm+(bw^2+cv^2)m^2=0$

$\Rightarrow (aw^2+cu^2)\frac{l^2}{m^2}+2cuv\frac{l}{m}+(bw^2+cv^2)=0$

If its roots are $\frac{l_1}{m_1}$ and $\frac{l_2}{m_2}$, then we have $\frac{l_1}{m_1}\cdot\frac{l_2}{m_2}$ = product of the roots

$=\frac{bw^2+cv^2}{aw^2+cu^2}$

$\Rightarrow \frac{l_1l_2}{bw^2+cv^2}=\frac{m_1m_2}{aw^2+cu^2}=\frac{n_1n_2}{av^2+bu^2}$ (by symmetry)

If the lines are perpendicular then $l_1l_2+m_1m_2+n_1n_2=0$

$$\Rightarrow (bw^2 + cv^2) + (cu^2 + aw^2) + (av^2 + bu^2) = 0.$$

$$\Rightarrow (b+c)u^2 + (c+a)v^2 + (a+b)w^2 = 0 \quad \text{---} (3)$$

The direction cosines of the normal to the plane $ux + vy + wz = 0$ are u, v, w and if this plane touches the given cone then,

$$Au^2 + Bv^2 + Cw^2 + 2Fvw + 2Gwu + 2Huv = 0 \quad \text{---} (4)$$

Here $A = bc - f^2 = \dfrac{1}{(c+a)(a+b)} - 0 = \dfrac{1}{(a+b)(c+a)}$

Similarly, $B = ca - g^2 = \dfrac{1}{(a+b)(c+b)} =$

and $C = \dfrac{1}{(b+c)(c+a)}$, $F = gh - af = 0$

$G = 0, H = 0$

(Here from the given cone $= \dfrac{1}{b+c}$, $b = \dfrac{1}{c+a}$, $c = \dfrac{1}{a+b}$, $f = g = h = 0$)

$\therefore$ from (4), the required condition is

$$\frac{1}{(a+b)(c+a)}u^2 + \frac{1}{(a+b)(c+b)}v^2 + \frac{1}{(b+c)(c+a)}w^2 = 0$$

$$\Rightarrow \frac{(b+c)u^2 + (c+a)v^2 + (a+b)w^2}{(a+b)(a+c)(b+c)} = 0$$

i.e., $(b+c)u^2 + (c+a)v^2 + (a+b)w^2 = 0$

which is same as (3).

which is the required condition.

2(b) Given that $f(x,y) = |x^2 - y^2|$.

Find $f_{xy}(0,0)$ and $f_{yx}(0,0)$.

~~Hence show that~~

$f_{xy}(0,0) = f_{yx}(0,0)$.

we have

$$f_x(0,0) = \underset{h\to 0}{Lt} \frac{f(h,0) - f(0,0)}{h}$$

$$= \underset{h\to 0}{Lt} \frac{|h^2|}{h} = \underset{h\to 0}{Lt} \frac{|h|^2}{h} = 0.$$

$$f_y(0,0) = \underset{k\to 0}{Lt} \frac{f(0,k) - f(0,0)}{k}$$

$$= \underset{k\to 0}{Lt} \frac{|-k^2|}{k} = \underset{k\to 0}{Lt} \frac{|k|^2}{k} = 0.$$

we have

$$f_{xy}(0,0) = \underset{h\to 0}{Lt} \frac{f_y(h,0) - f_y(0,0)}{h} \quad \text{--- (1)}$$

we have $f_y(h,0) = \underset{k\to 0}{Lt} \dfrac{f(h,k) - f(h,0)}{k}$

$$= \underset{k\to 0}{Lt} \frac{|h^2 - k^2| - |h^2|}{k}$$

$$= \underset{k\to 0}{Lt} \frac{|h^2 - k^2| - |h|^2}{k}.$$

$$= \underset{k\to 0}{Lt} \frac{|h^2-k^2| - h^2}{k}$$

$$= \underset{k\to 0}{Lt} \frac{(h^2-k^2) - h^2}{k} \quad \text{if } h^2 > k^2$$

$$= \underset{k\to 0}{Lt} \frac{-k^2}{k} = 0.$$

$$\therefore \text{(1)} \equiv f_{xy}(0,0) = \underset{h\to 0}{Lt} \frac{0-0}{h} = 0.$$

we have

$$f_{yx}(0,0) = \underset{k\to 0}{Lt} \frac{f_x(0,k) - f_x(0,0)}{k} \quad \text{--- (2)}$$

we have

$$f_x(0,k) = \underset{h\to 0}{Lt} \frac{f(h,k) - f(0,k)}{h}$$

$$= \underset{h\to 0}{Lt} \frac{|h^2-k^2| - |-k^2|}{h}$$

$$= \underset{h\to 0}{Lt} \frac{|h^2-k^2| - |k|^2}{h}$$

$$= \underset{h\to 0}{Lt} \frac{|h^2-k^2| - k^2}{h}$$

$$= \underset{h\to 0}{Lt} \frac{-(h^2-k^2) - k^2}{h} \quad \text{if } h^2 < k^2$$

$$= \underset{h\to 0}{Lt} \frac{-h^2}{h} = 0.$$

$$\therefore \text{(2)} \equiv f_{yx}(0,0) = \underset{k\to 0}{Lt} \frac{0-0}{k} = 0.$$

$$\therefore \boxed{f_{yx}(0,0) = 0}. \quad \therefore f_{xy}(0,0) = f_{yx}(0,0)$$

2(c) Show that $S = \{(x, 2y, 3x) / x, y \text{ are real numbers}\}$ is a subspace of $\mathbb{R}^3(\mathbb{R})$.
Find two bases of 'S'.
Also find the dimension of 'S'.

Sol: Let $\mathbb{R}^3(\mathbb{R}) = \{(x, y, z) / x, y, z \in \mathbb{R}\}$ be a given vector space.

let $S = \{(x, 2y, 3x) / x, y \in \mathbb{R}\} \subseteq \mathbb{R}^3$.

since $(0,0,0) \in \mathbb{R}^3 \Rightarrow (0,0,0) \in S$.

$\therefore S \neq \phi$.

let $\alpha = (x_1, 2y_1, 3x_1)$, $\beta = (x_2, 2y_2, 3x_2) \in S$

Let $a, b \in \mathbb{R}$ then we have

$$a\alpha + b\beta = (ax_1, 2ay_1, 3ax_1) + (bx_2, 2by_2, 3bx_2)$$
$$= (ax_1 + bx_2, 2(ay_1 + by_2), 3(ax_1 + bx_2))$$
$$\in S. \quad (\because ax_1 + bx_2, ay_1 + by_2 \in \mathbb{R}).$$

$\therefore S$ is a subspace of $\mathbb{R}^3(\mathbb{R})$.

let $\alpha = (x, 2y, 3x) \in S$; $x, y \in \mathbb{R}$

then $\alpha = x(1,0,3) + y(0,2,0)$
$\in L(S_1)$; where $S_1 = \{(1,0,3), (0,2,0)\} \subseteq S$.

$\therefore \alpha \in S \Rightarrow \alpha \in L(S_1)$

$\therefore S \subseteq L(S_1)$.

Clearly $L(S_1) \subseteq S$.

$\therefore \boxed{L(S_1) = S}$

Since no vector is a scalar multiplication of the other in S_1

$\therefore S_1$ is L.I.

$\therefore \boxed{S_1 \text{ is a basis of } S.}$

and the number of elements in basis S_1 is 2.

$\therefore \boxed{\dim(S) = 2}$

The other basis of S is

$\{(2,0,6), (0,4,0)\}$ etc.

3(a)(i) If $u = x^2+y^2$, $v = x^2-y^2$ where
$x = r\cos\theta$, $y = r\sin\theta$
~~then~~ find $\frac{\partial(u,v)}{\partial(r,\theta)}$.

Sol We have

$$\frac{\partial(u,v)}{\partial(r,\theta)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(r,\theta)}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \times \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

$$= \begin{vmatrix} 2x & 2y \\ 2x & -2y \end{vmatrix} \times \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$$

$$= (-4xy - 4xy)(\cos^2\theta + \sin^2\theta)r$$

$$= -8xyr = -8(r\cos\theta)(r\sin\theta)r$$

$$= -8r^3\sin\theta\cos\theta.$$

$$= -4r^3(\sin 2\theta)$$

3(a)(ii) If $\int_0^x f(t)\,dt = x + \int_x^1 t\,f(t)\,dt$, then find the value of $f(1)$.

Sol: Given $\int_0^x f(t)\,dt = x + \int_x^1 t\,f(t)\,dt$.

$$\frac{d}{dx}\left(\int_0^x f(t)\,dt\right) = \frac{d}{dx}\left[x + \int_x^1 t\,f(t)\,dt\right]$$

By using the Leibnitz's rule,

$$0 + f(x) \cancel{-0} = 1 + 0 + 0 - x f(x)$$

$$\Rightarrow f(x) = 1 - x f(x)$$

$$\Rightarrow f(x) + x f(x) = 1$$

$$\Rightarrow (1+x) f(x) = 1$$

$$\Rightarrow f(x) = \frac{1}{1+x}$$

$$\therefore f(1) = \frac{1}{1+1} = \frac{1}{2}.$$

3(a)(iii) Express $\int_a^b (x-a)^m (b-x)^n \, dx$ interms of Beta function.

Soln: put $x = a + (b-a)z$

$\Rightarrow dx = (b-a) z$

when $x = a, \; z = 0$

when $x = b, \; z = 1$

$\therefore \int_a^b (x-a)^m (b-x)^n \, dx$

$= \int_0^1 [(b-a)z]^m [b-a-(b-a)z]^n (b-a) dz$

$= \int_0^1 (b-a)^m z^m (b-a)^n (1-z)^n (b-a) \, dz$

$= (b-a)^{m+n+1} \int_0^1 z^m (1-z)^n \, dz$

$= (b-a)^{m+n+1} B(m+1, n+1)$

$\therefore \int_a^b (x-a)^m (b-x)^n \, dx = (b-a)^{m+n+1} B(m+1, n+1)$

if $m > -1, n > -1$.

(3)(b) A sphere of constant radius r passes through the origin O and cuts the axes in A, B, C. Prove that the locus of the foot of perpendicular from O to the plane ABC is given by $(x^2+y^2+z^2)(x^{-2}+y^{-2}+z^{-2})=4r^2$

Sol^n Let A, B and C be $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$

Then the equation of sphere through O, A, B, C is $x^2+y^2+z^2-ax-by-cz=0$

centre $\left(\frac{a}{2},\frac{b}{2},\frac{c}{2}\right)$

radius $= r = \sqrt{\frac{a^2}{4}+\frac{b^2}{4}+\frac{c^2}{4}} \Rightarrow 4r^2 = a^2+b^2+c^2$ —(1)

Equation of plane ABC - $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ —(2)

The equation of the line through O perpendicular to the plane (2)

$$\frac{x}{(1/a)} = \frac{y}{(1/b)} = \frac{z}{(1/c)} \quad —(3)$$

Any point on it $\left(\frac{k}{a},\frac{k}{b},\frac{k}{c}\right)$

if it is foot of perpendicular let (x_1,y_1,z_1) then

$x_1 = \frac{k}{a}, \ y_1 = k/b, \ z_1 = k/c$

$\Rightarrow a = k/x_1, \ b = k/y_1, \ c = k/z_1$

From eqn (1) $\rightarrow$

$$4r^2 = k^2\left(\frac{1}{x_1^2} + \frac{1}{y_1^2} + \frac{1}{z_1^2}\right) \quad \text{--- (4)}$$

From eqn (3) $\rightarrow$

$$\frac{x}{(1/a)} = \frac{y}{(1/b)} = \frac{z}{(1/c)} = k$$

$$\Rightarrow \cancel{x^2} \cancel{+y^2}. \ \frac{x^2}{(x/a)} = \frac{y^2}{(y/b)} = \frac{z^2}{(z/c)} = k$$

$$\Rightarrow \frac{x^2+y^2+z^2}{\frac{x}{a}+\frac{y}{b}+\frac{z}{c}} = \frac{k}{1}$$

$$\Rightarrow \frac{x^2+y^2+z^2}{1} = k$$ (From equation of plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$)

Putting value of k in eqn (4) $\rightarrow$

$$4r^2 = (x^2+y^2+z^2)^2(x_1^{-2}+y_1^{-2}+z_1^{-2})$$

Locus of (x_1, y_1, z_1)

$$\Rightarrow \boxed{4r^2 = (x^2+y^2+z^2)^2(x^{-2}+y^{-2}+z^{-2})}$$

3(c)(i). Prove that the eigen vectors, corresponding to two distinct eigen values of a real symmetric matrix, are orthogonal.

Sol'n: Let X_1, X_2 be two eigen vectors corresponding to two distinct eigen values λ_1, λ_2 of real symmetric matrix A.

Then $AX_1 = \lambda_1 X_1$ — ① & $AX_2 = \lambda_2 X_2$ — ②.

Here λ_1, λ_2 are real and X_1, X_2 are real vectors

Now $\lambda_1 X_2^T X_1 = X_2^T (\lambda_1 X_1)$

$= X_2^T (AX_1)$ (by ①)

$= (X_2^T A) X_1$

$= (X_2^T A^T) X_1$ ($\because A^T = A$)

$= (AX_2)^T X_1$

$= (\lambda_2 X_2)^T X_1$ (by ②)

$= \lambda_2 X_2^T X_1$

$(\lambda_1 - \lambda_2) X_2^T X_1 = 0$

$\Rightarrow X_2^T X_1 = 0$ ($\because \lambda_1$ & λ_2 are distinct $\Rightarrow \lambda_1 - \lambda_2 \neq 0$)

$\therefore X_1$ & X_2 are orthogonal.

3(c)(ii) For two square matrices A and B of order '2', show that $\text{trace}(AB) = \text{trace}(BA)$.
Hence show that $AB - BA \neq I_2$ where I_2 is an identity matrix of order '2'.

Sol Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times2}$; $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}_{2\times2}$

be two matrices of order 2.

Then we have

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}$$

$$\therefore \boxed{\text{trace}(AB) = a_{11}b_{11} + a_{12}b_{21} + a_{21}b_{12} + a_{22}b_{22}} \quad (1)$$

we have

$$BA = \begin{bmatrix} b_{11}a_{11} + b_{12}a_{21} & b_{11}a_{12} + b_{12}a_{22} \\ b_{21}a_{11} + b_{22}a_{21} & b_{21}a_{12} + b_{22}a_{22} \end{bmatrix}$$

$$\therefore \text{trace}(BA) = b_{11}a_{11} + b_{12}a_{21} + b_{21}a_{12} + b_{22}a_{22}$$

clearly $\text{trace}(AB) = \text{trace}(BA)$.

Also $AB - BA \neq I_2$.

4(c)(i) Reduce the following matrix to a row-reduced echelon form and hence also find its rank:

$$A = \begin{bmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 2 & 2 & 0 \\ 2 & 6 & 2 & 6 & 2 \\ 3 & 9 & 1 & 10 & 6 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 2 & 2 & 0 \\ 0 & 0 & -2 & -2 & 0 \\ 0 & 0 & -5 & -2 & 3 \end{bmatrix} \quad \begin{matrix} R_3 \to R_3 - 2R_1 \\ R_4 \to R_4 - 3R_1 \end{matrix}$$

$$R_2 \to \tfrac{1}{2} R_2$$

$$\sim \begin{bmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & -2 & 0 \\ 0 & 0 & -5 & -2 & 3. \end{bmatrix}$$

$$R_3 \to R_3 + 2R_2 \qquad R_4 \to R_4 + 5R_2$$

$$\sim \begin{bmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 3 \end{bmatrix} \quad R_3 \leftrightarrow R_4$$

$$\sim \begin{bmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 2 & 4 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad R_3 \to \tfrac{1}{3} R_3$$

clearly which is echelon form

The number of non-zero rows = 3.

$R_1 \to R_1 - 2R_1$

Also

$$A \sim \begin{bmatrix} 1 & 3 & 0 & 2 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$R_1 \to R_1 - 2R_3$
$R_2 \to R_2 - R_3$

$$\sim \begin{bmatrix} 1 & 3 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

clearly which is the row-reduced echelon form.

$$\therefore \boxed{\rho(A) = 3.}$$

4a(ii) → Find the eigen values and the corresponding eigen vectors of the Matrix $A=\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}$, over the complex field.

Sol we have

$$|A-\lambda I|=0 \;;\; I=\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$$\Rightarrow \begin{vmatrix}0-\lambda & -i\\ i & 0-\lambda\end{vmatrix}=0$$

$$\Rightarrow \lambda^2+i^2=0$$

$$\Rightarrow \lambda^2-1=0$$

$$\Rightarrow \boxed{\lambda=\pm 1}$$

if $\lambda=1$: we have $(A-\lambda I)X=0$

$$\Rightarrow (A-I)X=0$$

$$=\begin{bmatrix}-1 & -i\\ i & -1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$$\Rightarrow -x-yi=0 \text{ ——(i)}$$

$$ix-y=0 \text{ ——(ii)}$$

$$\text{(ii)}\times i = -x-iy=0$$

which is same as (i)

$$\therefore \boxed{y=ix}$$

$$\therefore X=\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}x\\ix\end{bmatrix}=x\begin{bmatrix}1\\i\end{bmatrix}=xX_1.$$

$\therefore$ $X_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$ is an eigen vector over 'C'.

<u>if $\lambda = -1$:</u>

$(A+I)X = 0 \Rightarrow \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\Rightarrow x - iy = 0$ —— (i)

$ix + y = 0$ —— (ii)

$(i) \times i \equiv ix + y = 0$ which is same as (ii)

$\therefore$ $\boxed{x = iy}$

$\therefore X = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} iy \\ y \end{bmatrix} = y \begin{bmatrix} i \\ 1 \end{bmatrix} = yX_2$

Here $X_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$ is an eigen vector over C (field).

$X_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$ and $X_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$ are eigen vectors over the field 'C'.

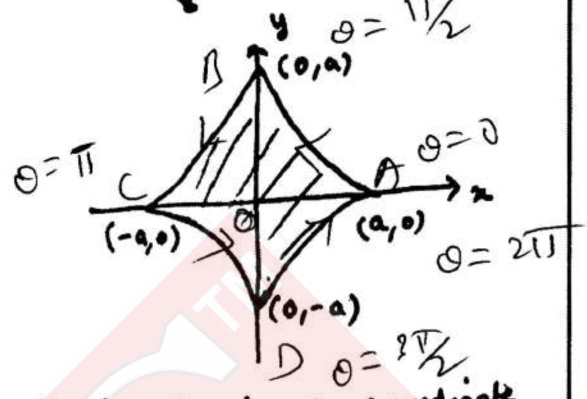
4.(b) → Show that the entire area of the Astroid :
$$x^{2/3} + y^{2/3} = a^{2/3} \text{ is } \frac{3}{8}\pi a^2.$$

Soln: We can express the astroid with parametric equation

$$y = r^3 \sin^3\theta$$
$$x = r^3 \cos^3\theta$$

where $0 \le r \le a^{1/3}$

and $0 \le \theta \le 2\pi$



∴ Area of astroid is given

$$A = \iint dx\,dy$$

While transforming the coordinate we multiply by the Jacobian of transformation to get double integral in transformed coordinates.

$$\therefore \iint dx\,dy = \iint J\left(\frac{x,y}{r,\theta}\right) dr\,d\theta \quad -①$$

$$J\left(\frac{x,y}{r,\theta}\right) = \frac{\partial x}{\partial r}\cdot\frac{\partial y}{\partial \theta} - \frac{\partial x}{\partial \theta}\cdot\frac{\partial y}{\partial r} \quad -②$$

$$\frac{\partial x}{\partial r} = 3r^2\cos^3\theta \qquad \frac{\partial y}{\partial \theta} = 3\sin^2\theta\cdot\cos\theta\cdot r^3$$

$$\frac{\partial x}{\partial \theta} = -3\cos^2\theta\cdot\sin\theta\cdot r^3 \qquad \frac{\partial y}{\partial r} = 3r^2\sin^3\theta$$

$$\therefore J\left(\frac{x,y}{r,\theta}\right) = 3r^2\cos^3\theta\,(3r^3\sin^2\theta\cdot\cos\theta) - (-3r^3\cos^2\theta\sin\theta)(3r^2\sin^3\theta)$$
$$= 9r^5\cos^4\theta\sin^2\theta + 9r^5\sin^4\theta\cos^2\theta$$
$$= 9r^5\cos^2\theta\sin^2\theta\,(\cos^2\theta + \sin^2\theta)$$
$$= 9r^5\cos^2\theta\sin^2\theta = \frac{9}{4}r^5\sin^2 2\theta$$
$$= \frac{9}{8}r^5(1-\cos 4\theta)$$

$$\therefore \iint dx\,dy = \int_{r=0}^{a^{1/3}}\int_{\theta=0}^{2\pi} \frac{9}{8}r^5(1-\cos 4\theta)\,dr\,d\theta$$

$$= \int_{\theta=0}^{2\pi}(1-\cos 4\theta)d\theta \int_{r=0}^{a^{1/3}} \frac{9}{8}r^5\,dr$$

$$= \left[\theta + \frac{\sin 4\theta}{4}\right]_{\theta=0}^{2\pi} \times \left[\frac{9}{8}\cdot\frac{r^6}{6}\right]_0^{a^{1/3}} =$$

$$= (2\pi - 0) + (0-0) \times \left[\frac{3}{16}(a^{1/3})^6 - \frac{3}{16}(0)^6\right]$$

$$= 2\pi \times \frac{3}{16} a^2$$

$$= \frac{3\pi a^2}{8}$$

$$\therefore A = \iint dx\,dy = \frac{3\pi a^2}{8}$$

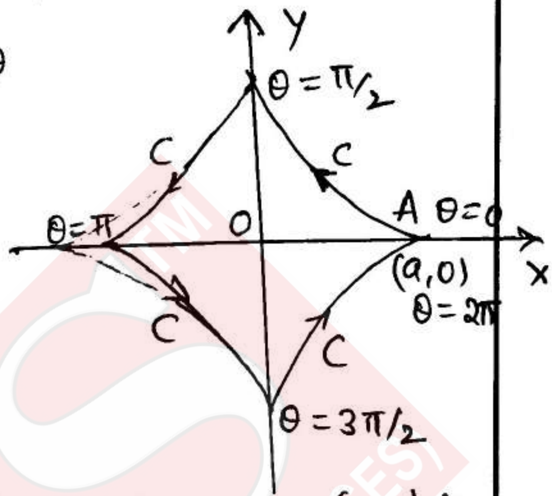
$\Rightarrow$ Area enclosed by Astroid $= \underline{\underline{\dfrac{3\pi a^2}{8}}}$

(Or)

4(b) Sol'n (or).

Sol'n: The parametric equations of the given curve $x^{2/3}+y^{2/3}=a^{2/3}$ can be taken as

$x=a\cos^3\theta$, $y=a\sin^3\theta$



Here C is the simple closed curve traversed in +ve direction by the whole arcs of the given hypocycloid.

At the point A, $\theta=0$ and when after one complete round in anti clockwise sense along the curve C we come back to A, then at A, $\theta=2\pi$.

The area bounded by the given hypocycloid.

$$=\frac{1}{2}\oint_C (x\,dy-y\,dx), \text{ by Green's theorem}$$

$$=\frac{1}{2}\int_{\theta=0}^{2\pi}\left(x\frac{dy}{d\theta}-y\frac{dx}{d\theta}\right)d\theta, \text{ where } x=a\cos^3\theta, y=a\sin^3\theta$$

$$=\frac{1}{2}\int_0^{2\pi}\left[a\cos^3\theta\cdot 3a\sin^2\theta\cos\theta-a\sin^3\theta\left(-3a\cos^2\theta\sin\theta\right)\right]d\theta$$

$$=\frac{3a^2}{2}\int_0^{2\pi}(\cos^4\theta\sin^2\theta+\sin^4\theta\cos^2\theta)d\theta$$

$$=2\,\frac{3a^2}{2}\int_0^{\pi}(\cos^4\theta\sin^2\theta+\sin^4\theta\cos^2\theta)d\theta$$

$$=4\cdot\frac{3a^2}{2}\int_0^{\pi/2}(\cos^4\theta\sin^2\theta+\sin^4\theta\cos^2\theta)d\theta$$

$$=6a^2\left[\frac{3.1.1}{6.4.2}\cdot\frac{\pi}{2}+\frac{3.1.1}{6.4.2}\cdot\frac{\pi}{2}\right]=6a^2\,\frac{\pi}{16}=\frac{3\pi a^2}{8}$$

4(c) Find the equation of the plane containing the lines:

$$\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7},$$

$$\frac{x-2}{1} = \frac{y-4}{3} = \frac{z-6}{5}.$$

Also find the point of intersection of the given lines.

Sol Given that

$$\frac{x+1}{3} = \frac{y+3}{5} = \frac{z+5}{7} (= r_1) \text{ say} \quad \text{---} (1)$$

$$\frac{x-2}{1} = \frac{y-4}{3} = \frac{z-6}{5} (= r_2) \text{ say} \quad \text{---} (2)$$

The equation of plane containing the line (1) and (2) given by

$$\begin{vmatrix} x+1 & y+3 & z+5 \\ 3 & 5 & 7 \\ 1 & 3 & 5 \end{vmatrix} = 0$$

$$\Rightarrow (x+1)(4) - (y+3)(8) + (z+5)(4) = 0$$

$$\Rightarrow (x+1) - 2y - 6 + z + 5 = 0$$

$$\Rightarrow \boxed{x - 2y + z = 0} \quad \text{---} (3)$$

Any general point on line (1) is

$(3r_1-1, 5r_1-3, 7r_1-5)$ —— (4)

Any general point on line (2) is

$(r_2+2, 3r_2+4, 5r_2+6)$ —— (5).

If a line (1) and (2) are intersecting then from (4) and (5), we have

$3r_1-1 = r_2+2 \Rightarrow 3r_1-r_2-3=0$ —— (6)

$5r_1-3 = 3r_2+4 \Rightarrow 5r_1-3r_2-7=0$ —— (7)

$7r_1-5 = 5r_2+6 \Rightarrow 7r_1-5r_2-11=0$ —— (8)

from (6) and (7), we have

$$\frac{r_1}{7-9} = \frac{r_2}{-15+21} = \frac{1}{-9+5}$$

$$\Rightarrow \frac{r_1}{-2} = \frac{r_2}{6} = \frac{1}{-4}$$

$$\Rightarrow \boxed{r_1 = \tfrac{1}{2}} \quad \boxed{r_2 = \tfrac{-3}{2}}$$

$\therefore$ from (4), $\left(\tfrac{3}{2}-1, \tfrac{5}{2}-3, \tfrac{7}{2}-5\right)$

$= \left(\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{3}{2}\right)$

from (5), $\left(-\tfrac{3}{2}+2, -\tfrac{9}{2}+4, -\tfrac{15}{2}+6\right)$

$= \left(\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{3}{2}\right)$, is the required point of intersection of two lines and satisfies the given plane.

SECTION-B

5.(a) Solve $(D^2+2)y = x^2e^{3x} + e^x\cos 2x$

Sol'n: Here auxiliary equation is $D^2+2=0$
$\Rightarrow D = \pm i\sqrt{2}$

$\therefore$ C.F $= c_1\cos(x\sqrt{2}) + c_2\sin(x\sqrt{2})$

c_1, c_2 being arbitrary constants

P.I Corresponding to x^2e^{3x}

$$= \frac{1}{D^2+2}x^2e^{3x} = e^{3x}\frac{1}{(D+3)^2+2}x^2$$

$$= e^{3x}\frac{1}{D^2+6D+11}x^2$$

$$= e^{3x}\frac{1}{11\left(1+\frac{6D}{11}+\frac{D^2}{11}\right)}x^2$$

$$= \frac{e^{3x}}{11}\left\{1+\left(\frac{6D+D^2}{11}\right)\right\}^{-1}x^2$$

$$= \frac{e^{3x}}{11}\left\{1-\frac{6D+D^2}{11}+\left(\frac{6D+D^2}{11}\right)^2-\cdots\right\}x^2$$

$$= \frac{e^{3x}}{11}\left(1-\frac{6D}{11}-\frac{D^2}{11}+\frac{36D^2}{121}+\cdots\right)x^2$$

$$= \frac{e^{3x}}{11}\left(1-\frac{6D}{11}+\frac{25D^2}{121}+\cdots\right)x^2$$

$$= \frac{e^{3x}}{11}\left(x^2-\frac{12x}{11}+\frac{50}{121}\right)$$

$$= e^{3x}(121x^2-132x+50)/(11)^3$$

P.I Corresponding to $e^x\cos 2x$

$$= \frac{1}{D^2+2}e^x\cos 2x$$

$$= e^x\frac{1}{(D+1)^2+2}\cos 2x$$

$$= e^x \frac{1}{D^2+2D+3} \cos 2x$$

$$= e^x \frac{1}{-2^2+2D+3} \cos 2x$$

$$= e^x \frac{1}{2D-1} \cos 2x$$

$$= e^x \frac{2D+1}{4D^2-1} \cos 2x$$

$$= e^x \frac{2D+1}{4(-2^2)-1} \cos 2x$$

$$= -\frac{e^x}{17}(2D+1) \cos 2x$$

$$= -\frac{e^x}{17}(-4\sin 2x + \cos 2x)$$

$\therefore$ The required solution is

$$y = c_1 \cos(x\sqrt{2}) + c_2 \sin(x\sqrt{2}) + \frac{1}{(11)^3} \times e^{3x}(121x^2 - 132x + 50)$$
$$- \frac{1}{17} e^x (\cos 2x - 4\sin 2x).$$

5(b) Solve the initial value problem

$\frac{d^2y}{dx^2}+4y=e^{-2x}\sin 2x$; $y(0)=y'(0)=0$

using Laplace transform method.

Sol^n: Given that $\frac{d^2y}{dx^2}+4y=e^{-2x}\sin 2x$; $y(0)=0$, $y'(0)=0$

i.e, $y''+4y=e^{-2x}\sin 2x$ —— (1)

Taking Laplace transform of both sides of (1), we have

$$L(y'')+4L(y)=L(e^{-2x}\sin 2x)$$

$$s^2L\{y\}-s\,y(0)-y'(0)+4L\{y\}=f(s+2)$$

$$s^2L(y)+4L(y)=\frac{2}{(s+2)^2+2^2}$$

$$(s^2+4)L(y)=\frac{2}{s^2+4s+8}$$

$L(\sin bt)=\frac{b}{s^2+b^2}=f(s)$
by first shifting theorem,
$\therefore L(e^{-at}\sin bt)=\frac{b}{(s+a)^2+b^2}$

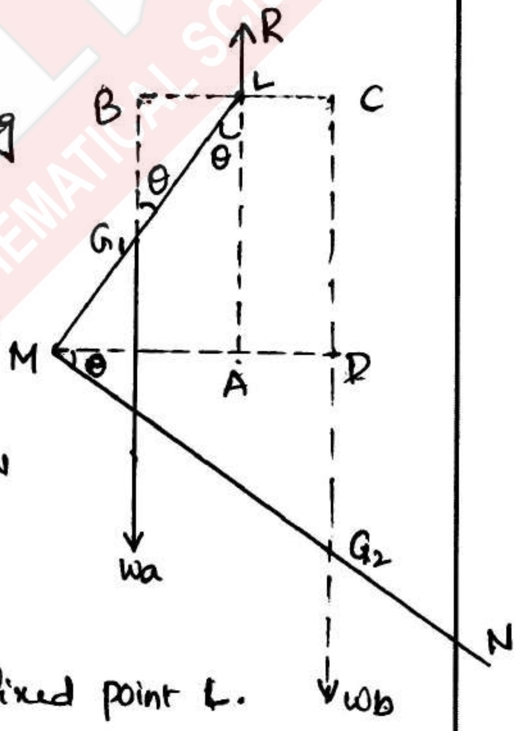
$$\Rightarrow L(y)=\frac{2}{(s^2+4)(s^2+4s+8)}$$

$$y=L^{-1}\left[\frac{2}{(s^2+4)(s^2+4s+8)}\right]$$

(please try yourself)

5(c) Two rods LM and MN are joined rigidly at the point M such that $(LM)^2+(MN)^2=(LN)^2$ and they are hanged freely in equilibrium from a fixed point L. Let w be the weight per unit length of both the rods which are uniform. Determine the angle, which the rod LM makes with the vertical direction, in terms of lengths of the rods.

Sol'n: Let LM and MN be two uniform rods of length a and b, rigidly joined at M so that $\angle LMN=90°$. Let the rods hang freely in equilibrium from the fixed point L.



The rods will be in equilibrium under the action of the following three forces only.

(i) wa, the weight of the rod LM acting vertically downwards at its middle point G_1.

(ii) wb, the weight of the rod MN acting vertically downwards at its middle point G_2, and

(iii) R, the reaction at the fixed point L.

Since the two forces wa and wb are parallel, therefore the third force R will also be parallel to them i.e., the reaction R at L will act in the verical direction.

Given that $\angle MLA = \theta$; $\therefore$ $\angle LG_1B = \theta$.

Also $\angle DMN = \angle LMN - \angle LMD = 90^\circ - (90^\circ - \theta) = \theta$.

To avoid the reaction R, taking moments of all the forces about the point L, we have

$$wa \cdot LB = wb \cdot LC$$

$$\Rightarrow a \cdot LB = b \cdot AD \qquad [\because LC = AD]$$

$$\Rightarrow a \cdot LB = b(MD - MA)$$

$$\Rightarrow a \cdot LG_1 \sin\theta = b(MG_2 \cos\theta - LM \sin\theta)$$

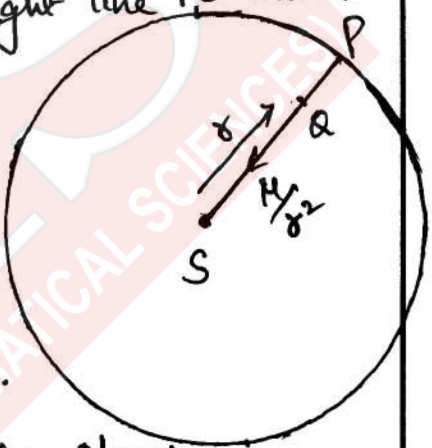
$$\Rightarrow a \cdot \tfrac{1}{2}a \sin\theta = b\left(\tfrac{1}{2}b\cos\theta - a\sin\theta\right)$$

$$\Rightarrow (a^2 + 2ab)\sin\theta = b^2\cos\theta$$

$$\Rightarrow \tan\theta = \frac{b^2}{a^2 + 2ab}$$

5(d) If a planet, which revolves around the Sun in a circular orbit, is suddenly stopped in its orbit, then find the time in which it would fall into the Sun. Also, find the ratio of its falling time to the period of revolution of the planet.

Sol^n: Let a planet describing a circular path of radius a and centre S (the Sun) be stopped at the point P of its path. Then it will begin to move towards S along the straight line PS under the acceleration $\mu/(\text{distance})^2$.



If Q is the position of the planet at time t such that $SQ = r$, then the acceleration at Q is μ/r^2 directed towards S.

$\therefore$ the equation of motion of the planet at Q is

$$v\frac{dv}{dr} = -\frac{\mu}{r^2}$$ (-ive sign is taken as the acceleration at Q is in the direction of r decreasing)

$$\Rightarrow v\frac{dv}{dr} = -\frac{\mu}{r^2}\,dr.$$

Integrating $\frac{v^2}{2} = \frac{\mu}{r} + A$, where A is a constant.

But at P, $r = SP = a$ and $v = 0$ [Note that the planet begins to move along PS with zero velocity at P]

$\therefore 0 = \frac{\mu}{a} + A \Rightarrow A = \frac{-\mu}{a}$

$\therefore \frac{v^2}{2} = \frac{\mu}{r} - \frac{\mu}{a} = \frac{\mu(a-r)}{ar}$

$$\Rightarrow v = \frac{dr}{dt} = -\sqrt{(2\mu/a)} \cdot \sqrt{\left(\frac{a-r}{r}\right)}$$

(−ve sign is taken because r decreases as t increases)

$$\Rightarrow dt = -\sqrt{\left(\frac{a}{2\mu}\right)} \cdot \sqrt{\left(\frac{r}{a-r}\right)}\, dr. \quad \text{——— (1)}$$

If t_1 is the time taken by the planet from P to S, then integrating (1), we have

$$\int_0^{t_1} dt = -\sqrt{\left(\frac{a}{2\mu}\right)} \int_{r=a}^{0} \sqrt{\left(\frac{r}{a-r}\right)}\, dr$$

$$\Rightarrow t_1 = \sqrt{\left(\frac{a}{2\mu}\right)} \int_0^{\pi/2} \sqrt{\left(\frac{a\cos^2\theta}{a-a\cos^2\theta}\right)} \cdot 2a\cos\theta\sin\theta\, d\theta,$$

putting $r = a\cos^2\theta$, so that $dr = -2a\cos\theta\sin\theta\, d\theta$

$$= a\sqrt{\left(\frac{a}{2\mu}\right)} \int_0^{\pi/2} 2\cos^2\theta\, d\theta = a\sqrt{\left(\frac{a}{2\mu}\right)} \int_0^{\pi/2} (1+\cos 2\theta)\, d\theta$$

$$= a\sqrt{\left(\frac{a}{2\mu}\right)} \left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \frac{\pi a^{3/2}}{2\sqrt{(2\mu)}}.$$

But the time period T of the planet's revolution is given by $T = \dfrac{2\pi a^{3/2}}{\sqrt{\mu}}$

$$\therefore \frac{t_1}{T} = \frac{1}{4\sqrt{2}} = \frac{\sqrt{2}}{8} \Rightarrow t_1 = \left(\sqrt{2}/8\right)T$$

i.e, the time taken by the planet from P to S is $\sqrt{2}/8$ times the period of the planet's revolution.

5(e) Show that $\nabla^2\left[\nabla\cdot\left(\frac{\vec{r}}{r}\right)\right] = \frac{2}{r^4}$, where $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$.

Sol^n: we know that $\nabla\cdot(\phi A) = \phi(\nabla\cdot A) + A\cdot(\nabla\phi)$ — (1)

putting $A = r$ and $\phi = \frac{1}{r^2}$ in this identity,

we get $\nabla\cdot\left(\frac{\vec{r}}{r^2}\right) = \frac{1}{r^2}(\nabla\cdot r) + r\cdot\left(\nabla\frac{1}{r^2}\right)$

$= \frac{3}{r^2} + \vec{r}\cdot\left[-\frac{2}{r^3}\nabla r\right]$

$\left[\because \nabla\cdot r = 3 \text{ and } \nabla f(r) = f'(r)\nabla r\right]$

$\left[\because \nabla r = \frac{1}{r}\vec{r}\right]$

$= \frac{3}{r^2} + \vec{r}\cdot\left(-\frac{2}{r^3}\frac{1}{r}\vec{r}\right)$

$= \frac{3}{r^2} - \frac{2}{r^4}(\vec{r}\cdot\vec{r}) = \frac{3}{r^2} - \frac{2}{r^4}r^2 = \frac{1}{r^2}$

$\therefore \nabla^2\left[\nabla\cdot\left(\frac{r}{r^2}\right)\right] = \nabla^2\left(\frac{1}{r^2}\right) = \nabla\cdot\left(\nabla\frac{1}{r^2}\right)$

$= \nabla\cdot\left(\frac{-2}{r^3}\nabla r\right)$

$= \nabla\cdot\left(-\frac{2}{r^3}\frac{1}{r}\vec{r}\right)$

$= \nabla\cdot\left(-\frac{2}{r^4}\vec{r}\right)$

$= \left(-\frac{2}{r^4}\right)(\nabla\cdot\vec{r}) + \vec{r}\cdot\left[\nabla\left(\frac{-2}{r^4}\right)\right]$, using the identity (1)

$= -\frac{2}{r^4}\cdot 3 + \vec{r}\left[\frac{8}{r^5}\nabla r\right]$

$= -\frac{6}{r^4} + \vec{r}\left(\frac{8}{r^5}\frac{1}{r}\vec{r}\right)$

$= -\frac{6}{r^4} + \frac{8}{r^6}\vec{r}\cdot\vec{r} = -\frac{6}{r^4} + \frac{8}{r^6}r^2$

$= -\frac{6}{r^4} + \frac{8}{r^4}$

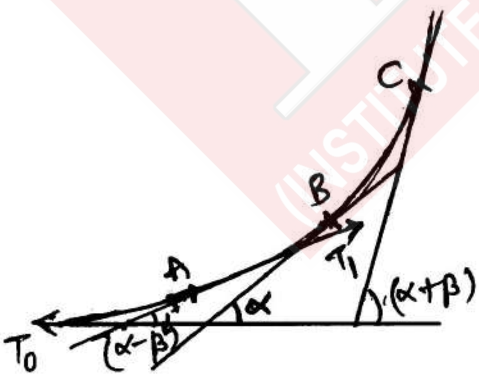
$= 2/r^4 = 2r^{-4}$.

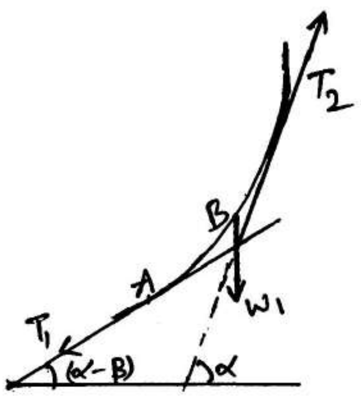
6(a) A heavy string, which is not of uniform density, is hung up from two points. Let T_1, T_2, T_3 be the tensions at the intermediate points A, B, C of the catenary respectively where its inclinations to the horizontal are in arithmetic progression with common difference β. Let w_1 and w_2 be the weights of the parts AB and BC of the string respectively. Prove that

(i) Harmonic mean of T_1, T_2 and $T_3 = \dfrac{3T_2}{1+2\cos\beta}$

(ii) $\dfrac{T_1}{T_3} = \dfrac{w_1}{w_2}$

Sol'n: Let O be the lowest point of the Catenary and T_1, T_2, T_3 be the tensions at A, B and C. respectively. If T_0 be the tension at O, then considering the equilibrium of the portion OA of the chain,
$$T_0 = T_1\cos(\alpha-\beta) \quad \text{———} (1).$$





Similarly considering the equilibrium of portions OB and OC, we get $T_0 = T_1\cos(\alpha-\beta) = T_2\cos\alpha = T_3\cos(\alpha+\beta)$ ——— (2).

Again considering the equilibrium of AB in vertical direction gives

$$T_3 \sin(\alpha+\beta) - T_2 \sin\alpha = w_2 \quad \text{——— (4).}$$

(ii) multiplying (3) by T_3 and (4) by T_1 and subtracting, we get

$$w_1 T_3 - w_2 T_1 = T_2 T_3 \sin\alpha - T_1 T_3 \sin(\alpha-\beta) - T_1 T_3 \sin(\alpha+\beta) + T_1 T_2 \sin\alpha$$

$$= T_1 T_3 \left\{ \frac{\sin\alpha \cos(\alpha-\beta)}{\cos\alpha} - \sin(\alpha-\beta) - \sin(\alpha+\beta) + \frac{\sin\alpha \cos(\alpha+\beta)}{\cos\alpha} \right\} \text{ from (2)}$$

$$= T_1 T_3 \left[\frac{\sin\alpha}{\cos\alpha} \cos(\alpha-\beta) + \cos(\alpha+\beta) - \sin(\alpha-\beta) - \sin(\alpha+\beta) \right]$$

$$= T_1 T_3 \left[2\sin\alpha\cos\beta - 2\sin\alpha\cos\beta \right]$$

$$= 0$$

$$\therefore w_1 T_3 = w_2 T_1$$

$$\Rightarrow \frac{T_1}{T_3} = \frac{w_1}{w_2}$$

This concludes the problem.

(i) Try yourself.

6(b) Solve the equation:

$$\frac{d^2y}{dx^2}+(\tan x-3\cos x)\frac{dy}{dx}+2y\cos^2x=\cos^4x.$$

completely by demonstrating all the steps involved.

Sol^n: Given that

$$\frac{d^2y}{dx^2}+(\tan x-3\cos x)\frac{dy}{dx}+2y\cos^2x=\cos^4x \quad (1)$$

Clearly it is in the form of

$$\frac{d^2y}{dx^2}+P(x)\frac{dy}{dx}+Q(x)y=R(x)$$

where $P(x)=\tan x-3\cos x$;

$Q(x)=2\cos^2x$ and $R(x)=\cos^4x$.

Let us solve (1) by changing the independent variable x to the new independent variable z where z is function of x ie $z=f(x)$ (say)

We have $\boxed{\frac{dy}{dx}=\frac{dy}{dz}\cdot\frac{dz}{dx}}$

We have $\frac{d^2y}{dx^2}=\frac{d}{dx}\left[\frac{dy}{dx}\right]=\frac{d}{dx}\left[\frac{dy}{dz}\cdot\frac{dz}{dx}\right]$.

$$\Rightarrow \frac{d^2y}{dx^2}=\frac{dy}{dz}\frac{d^2z}{dx^2}+\frac{dz}{dx}\frac{d}{dx}\left(\frac{dy}{dz}\right).$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{dy}{dz}\cdot\frac{d^2z}{dx^2} + \frac{dz}{dx}\cdot\frac{d}{dz}\left(\frac{dy}{dz}\right)\cdot\frac{dz}{dx}$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{dy}{dz}\cdot\frac{d^2z}{dx^2} + \left(\frac{dz}{dx}\right)^2\cdot\frac{d^2y}{dz^2}.$$

$\therefore$ (1) $\equiv$

$$\frac{dy}{dz}\frac{d^2z}{dx^2} + \left(\frac{dz}{dx}\right)^2\cdot\frac{d^2y}{dz^2} + (\tan x - 3\cos x)\frac{dy}{dz}\cdot\frac{dz}{dx} + 2y\cos^2 x = \cos^4 x.$$

$$\Rightarrow \frac{dy}{dz}\frac{d^2z}{dx^2} + \left(\frac{dz}{dx}\right)^2\frac{d^2y}{dz^2} + P(x)\frac{dy}{dz}\cdot\frac{dz}{dx} + Q(x)y = R(x) \text{ say}$$

$$\Rightarrow \left(\frac{dz}{dx}\right)^2\frac{d^2y}{dz^2} + \left(\frac{d^2z}{dx^2} + P(x)\frac{dz}{dx}\right)\frac{dy}{dz} + Q(x)y = R(x)$$

$$\Rightarrow \frac{d^2y}{dz^2} + \left[\frac{\frac{d^2z}{dx^2} + P(x)\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}\right]\frac{dy}{dz} + \frac{Q(x)}{\left(\frac{dz}{dx}\right)^2}y = \frac{R(x)}{\left(\frac{dz}{dx}\right)^2}. \qquad (2)$$

let us choose z

s.t. $\dfrac{Q}{\left(\frac{dz}{dx}\right)^2} = \text{constant} = 2$. say

$$\Rightarrow \frac{2\cos^2 x}{\left(\frac{dz}{dx}\right)^2} = 2$$

$$\Rightarrow \boxed{\frac{dz}{dx} = \cos x} \Rightarrow \boxed{z = \sin x} \quad (3)$$

we have $\frac{dz}{dx} = \cos x \Rightarrow \frac{d^2z}{dx^2} = -\sin x$

$$\therefore \frac{d^2z}{dx^2} + P(x)\frac{dz}{dx} = -\sin x + (\tan x - 3\cos x)\cos x$$

$$= -\sin x + \sin x - 3\cos^2 x$$

$$= -3\cos^2 x$$

$\therefore$ (2) $\equiv$

$$\frac{d^2y}{dz^2} + \left[\frac{-3\cos^2 x}{\cos^2 x}\right]\cdot\frac{dy}{dz} + \frac{2\cos^2 x}{\cos^2 x}y = \frac{\cos^4 x}{\cos^2 x}$$

$$\Rightarrow \frac{d^2y}{dz^2} + [-3]\frac{dy}{dz} + 2y = \cos^2 x.$$

$$\Rightarrow \frac{d^2y}{dz^2} - 3\frac{dy}{dz} + 2y = 1 - \sin^2 x$$

$$\Rightarrow \boxed{\frac{d^2y}{dz^2} - 3\frac{dy}{dz} + 2y = 1 - z^2} \quad (4)$$

To find y_c:

A.E is $m^2 - 3m + 2 = 0$

$$\Rightarrow (m-1)(m-2) = 0$$

$$\Rightarrow \boxed{m = 1, 2}$$

$$\therefore \boxed{y_c(z) = Ae^z + Be^{2z}}$$

To find y_p:

$$y_p = \frac{1}{D^2-3D+2}(1-z^2)$$

$$= \frac{1}{2}\left[1+\left(\frac{D^2-3D}{2}\right)\right]^{-1}(1-z^2)$$

$$= \frac{1}{2}\left(1-\left(\frac{D^2-3D}{2}\right)+\left(\frac{D^2-3D}{2}\right)^2 - \cdots\right](1-z^2)$$

$$= \frac{1}{2}\left[1+\left(\frac{D^2-3D}{2}\right)+\left(\frac{D^4+9D^2-6D^3}{4}\right)-\cdots\right](1-z^2)$$

$$= \frac{1}{2}\left[(1-z^2)+\frac{(-2-3(-2z))}{2}+\frac{9(-2)}{4}\right]$$

$$= \frac{1}{2}\left[1-z^2+\frac{(-2+6z)}{2}-\frac{9}{2}\right]$$

$$= \frac{1}{2}\left[\frac{\cancel{2}-2z^2-\cancel{2}+6z-9}{2}\right]$$

$$= \frac{1}{4}\left[-2z^2+6z-9\right]$$

$\therefore$ General solution of (2) is given by $y(z) = y_c(z) + y_p(z)$

$$\Rightarrow y(z) = Ae^{z}+Be^{2z}+\frac{1}{4}\left[-2z^2+6z-9\right]$$

$$\Rightarrow y(x) = Ae^{\sin x}+Be^{2\sin x}+\frac{1}{4}\left[-2\sin^2 x+6\sin x-9\right].$$

which is the required general solution of (1).

6(C) Evaluate $\int_C \vec{F}\cdot d\vec{r}$, where C is an arbitrary closed curve in the xy-plane and $\vec{F} = \dfrac{-y\hat{i}+x\hat{j}}{x^2+y^2}$.

Sol'n: In the xy-plane $r = x\hat{i}+y\hat{j}$ so that

$$dr = dx\,\hat{i} + dy\,\hat{j}.$$

$$\therefore \int_C F\cdot dr = \int_C \left(\frac{-y\hat{i}+x\hat{j}}{x^2+y^2}\right)\cdot(dx\,\hat{i}+dy\,\hat{j})$$

$$= \int_C \frac{-y\,dx + x\,dy}{x^2+y^2}$$

we change to polar coordinates by putting

$x = r\cos\theta$, $y = r\sin\theta$.

$\therefore dx = -r\sin\theta\,d\theta + \cos\theta\,dr$

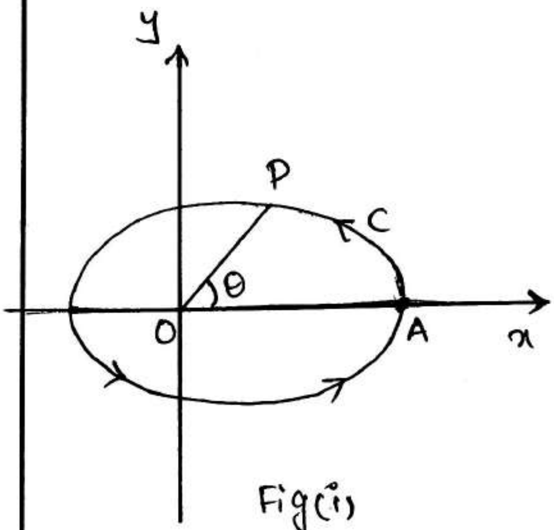
and $dy = r\cos\theta\,d\theta + \sin\theta\,dr$

$$\therefore \frac{-y\,dx + x\,dy}{x^2+y^2} = \frac{1}{r^2}\Big[-r\sin\theta(-r\sin\theta\,d\theta + \cos\theta\,dr) + r\cos\theta(r\cos\theta\,d\theta + \sin\theta\,dr)\Big]$$

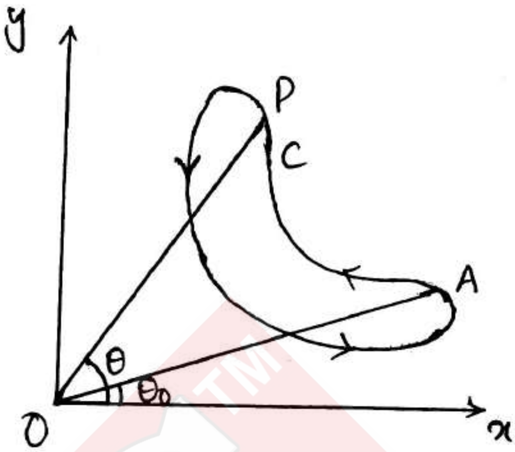
$$= \frac{r^2(\cos^2\theta+\sin^2\theta)}{r^2}\,d\theta$$

$$= d\theta$$

$$\therefore \int_C F\cdot dr = \int_C d\theta \quad \text{——— (1).}$$



Fig(i)



Fig(ii)

<u>Case I</u>. If the origin O lies inside the closed curve C as in fig (i), then for the curve C at the point A, we have $\theta = 0$ and when after a complete round we come back to A, then at A, $\theta = 2\pi$, so from (1)

$$\int_C F \cdot dr = \int_{\theta=0}^{2\pi} d\theta = 2\pi.$$

<u>Case II</u>: If the origin O lies outside the closed curve C as in fig (ii), then for the curve C at the point A, we have $\theta = \theta_0$ and when after a complete round along C we come back to A, then also at A, $\theta = \theta_0$, so from (1).

$$\int_C F \cdot dr = \int_{\theta=\theta_0}^{\theta_0} d\theta = [\theta]_{\theta_0}^{\theta_0} = 0.$$

7(a) Verify Gauss divergence theorem for $\vec{F} = 2x^2y\hat{i} - y^2\hat{j} + 4xz^2\hat{k}$ taken over the region in the first octant bounded by $y^2+z^2=9$ and $x=2$.

Sol^n: Let V be the volume enclosed by the closed surface S. Then by Gauss divergence theorem, we have

$$\iint_S F\cdot n\,ds = \iiint_V \operatorname{div} F\,dV$$

$$= \iiint_V \nabla\cdot(2x^2y\hat{i} - y^2\hat{j} + 4xz^2)\,dV$$

$$= \iiint_V \left[\frac{\partial}{\partial x}(2x^2y) + \frac{\partial}{\partial y}(-y^2) + \frac{\partial}{\partial z}(4xz^2)\right]dV$$

$$= \iiint_V (4xy - 2y + 8xz)\,dV,$$ where V is the volume in the first octant bounded by the cylinder $y^2+z^2=9$ and the planes $x=0$, $x=2$.

$$= 2\int_{x=0}^{2}\int_{z=0}^{3}\int_{y=0}^{\sqrt{9-z^2}} (2xy - y + 4xz)\,dx\,dz\,dy$$

$$= 2\int_{x=0}^{2}\int_{z=0}^{3}\left[xy^2 - \tfrac{1}{2}y^2 + 4xzy\right]_{y=0}^{\sqrt{(9-z^2)}} dx\,dz$$

$$= 2\int_{x=0}^{2}\int_{z=0}^{3}\left[x(9-z^2) - \tfrac{1}{2}(9-z^2) + 4xz\sqrt{(9-z^2)}\right]dx\,dz$$

$$= 2\int_{z=0}^{3} \left[\frac{x^2}{2}(9-z^2) - \frac{x}{2}(9-z^2) + 2x^2 z\sqrt{(9-z^2)}\right]_{x=0}^{2} dz$$

$$= 2\int_{0}^{3}\left[2(9-z^2) - (9-z^2) + 8z\sqrt{(9-z^2)}\right]dz$$

$$= 2\int_{0}^{3}\left[9 - z^2 - 4(-2z)(9-z^2)^{1/2}\right]dz$$

$$= 2\left[9z - \frac{z^3}{3} - 4\cdot\frac{2}{3}(9-z^2)^{3/2}\right]_0^3$$

$$= 2\left[27 - 9 + \frac{8}{3}\cdot 27\right]$$

$$= 2(18 + 72)$$

$$= 180$$

7(b) → Find all possible solutions of the differential equation:

$$y^2 \log y = xy \frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2.$$

Sol^n: solving for x,

$$x = \frac{y\log y}{p} - \frac{p}{y} \quad \text{——} \quad (1)$$

Differentiating (1) w.r.t y and remembering that

$$\frac{dx}{dy} = 1/p,$$

we get $\frac{1}{p} = \left(\log y + y\cdot\frac{1}{y}\right)\frac{1}{p} - y\log y \cdot \frac{1}{p^3}\frac{dp}{dy} - \left[-\frac{1}{y^2}\cdot p + \frac{1}{y}\frac{dp}{dy}\right]$

$$\Rightarrow \frac{1}{p} = \frac{\log y}{p} + \frac{1}{p} + \frac{p}{y^2} - \frac{dp}{dy}\left(\frac{y\log y}{p^2} + \frac{1}{y}\right)$$

$$\Rightarrow 0 = \frac{p}{y}\left(\frac{y\log y}{p^2} + \frac{1}{y}\right) - \frac{dp}{dy}\left(\frac{y\log y}{p^2} + \frac{1}{y}\right)$$

$$\Rightarrow \left(\frac{y\log y}{p^2} + \frac{1}{y}\right)\left(\frac{p}{y} - \frac{dp}{dy}\right) = 0$$

Omiting the first factor, we have

$$\frac{p}{y} - \frac{dp}{dy} = 0$$

$$\Rightarrow \frac{dp}{p} = \frac{dy}{y}$$

Integrating $\log p = \log y + \log c$

$$\Rightarrow p = yc \quad \text{——} \quad (2).$$

Putting the value of p given by (2) in (1),

we get $x = (\log y)/c - c$

$$\Rightarrow \log y = cx + c^2 \quad \text{——} \quad (3)$$

which is the required general solution

$(1) \equiv p^2 + xyp - y^2 \log y = 0$ ——— (A).

$(3) \equiv c^2 + cx - \log y = 0$ ——— (B)

To find singular solution(s):

p-disc. is given by

$$(xy)^2 - 4(1)\left[-y^2 \log y\right] = 0$$

$$\Rightarrow x^2y^2 + 4y^2 \log y = 0$$

$$\Rightarrow y \cdot y\left[x^2 + 4\log y\right] = 0$$ ——— (B')

c-discr. is given by

$$x^2 - 4(1)\left[-\log y\right] = 0$$

$$\Rightarrow x^2 + 4\log y = 0.$$ ——— (B''),

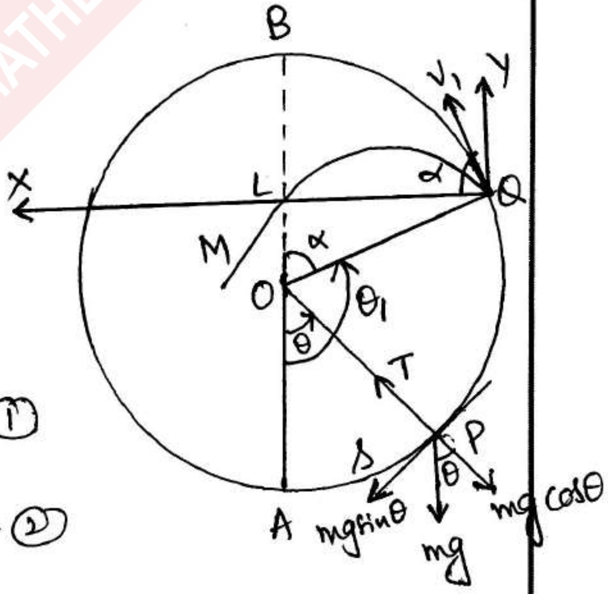
from both the discriminants,

$x^2 + 4\log y = 0$ is a common factor and satisfying given ODE.

$\therefore x^2 + 4\log y = 0$ is a singular. solution

7(C). A heavy particle hangs by an inextensible string of length a from a fixed point and is then projected horizontally with a velocity $\sqrt{2gh}$. If $\frac{5a}{2} > h > a$, then prove that the circular motion ceases when the particle has reached the height $\frac{1}{3}(a+2h)$ from the point of projection. Also, prove that the greatest height ever reached by the particle above the point of projection is $(4a-h)(a+2h)^2/27a^2$.

Sol'n: Let a particle of mass m be attached to one end of a string of length a whose other end is fixed at O. The particle is projected horizontally with a velocity $u = \sqrt{(2gh)}$ from A. If P is the position of the particle at time t such that $\angle AOP = \theta$ and arc $AP = s$, then the equations of motion of the particle are



$$m\frac{d^2s}{dt^2} = -mg\sin\theta \quad \text{--- (1)}$$

and $$m\frac{v^2}{a} = T - mg\cos\theta \quad \text{--- (2)}$$

Also $$s = a\theta \quad \text{--- (3)}$$

from (1) and (3), we have $a\dfrac{d^2\theta}{dt^2} = -g\sin\theta$.

Multiplying bothsides by $2a\dfrac{d\theta}{dt}$ and integrating, we have

$$v^2 = \left(a\frac{d\theta}{dt}\right)^2 = 2ag\cos\theta + A.$$

But at the point A, $\theta = 0$ and $v = u = \sqrt{2gh}$

$$\therefore A = 2gh - 2ag$$

$$\therefore v^2 = 2ag\cos\theta + 2gh - 2ag \quad \text{—— (4)}$$

from (2) and (4), we have

$$T = \frac{m}{a}(v^2 + ag\cos\theta) = \frac{m}{a}(3ag\cos\theta + 2gh - 2ag)$$

If the particle leaves the circular path at Q where $\theta = \theta_1$, then $T = 0$ when $\theta = \theta_1$.

$$\therefore 0 = \frac{m}{a}(3ag\cos\theta_1 + 2gh - 2ag) \Rightarrow \cos\theta_1 = -\frac{2h-2a}{3a}.$$

Since $\frac{5}{2}a > h > a$ i.e. $5a > 2h > 2a$, therefore $\cos\theta_1$ is $-$ve and its absolute value is < 1. So θ_1 is real and $\frac{1}{2}\pi < \theta_1 < \pi$.

Thus the particle leaves the circular path at Q before arriving at the highest point.

Height of the point Q above A

$$= AL = AO + OL = a + a\cos(\pi - \theta_1)$$

$$= a - a\cos\theta_1$$

$$= a + a\cdot\frac{2h-2a}{3a} = \frac{1}{3}(a+2h).$$

i.e, the particle leaves the circular path when it has reached a height $\frac{1}{3}(a+2h)$ above the point of projection.

If v_1 is the velocity of the particle at the point Q, then from (4), we have

$$v_1^2 = 2ag\cos\theta_1 + 2gh - 2ag$$

$$= -2ag\frac{(2h-2a)}{3a} + 2g(h-a)$$

$$= 2g(h-a)\left(1-\tfrac{2}{3}\right) = \tfrac{2}{3}g(h-a)$$

If $\angle LOQ = \alpha$, then $\alpha = \pi - \theta_1$.

$$\therefore \cos\alpha = \cos(\pi-\theta_1) = -\cos\theta_1 = \frac{2(h-a)}{3a}$$

Thus the particle leaves the circular path at the point Q with velocity $v_1 = \sqrt{\{\tfrac{2}{3}g(h-a)\}}$ at an angle $\alpha = \cos^{-1}\{2(h-a)/3a\}$ to the horizontal and will subsequently describe a parabolic path.

Maximum height of the particle above the point Q

$$= H = \frac{v_1^2\sin^2\alpha}{2g} = \frac{v_1^2}{2g}(1-\cos^2\alpha) = \tfrac{1}{3}(h-a)\left[1-\frac{4}{9a^2}(h-a)^2\right]$$

$$= \frac{1}{27a^2}(h-a)\left[9a^2-4(h^2-2ah+a^2)\right]$$

$$= \frac{(h-a)}{27a^2}\left[5a^2+8ah-4h^2\right] = \frac{1}{27a^2}(h-a)(a+2h)(5a-2h).$$

$\therefore$ Greatest height ever reached by the particle above the point of projection A.

$$= AL + H = \tfrac{1}{2}(a+2h) + \frac{1}{27a^2}(h-a)(a+2h)(5a-2h)$$

$$= \frac{1}{27a^2}(a+2h)\left[9a^2+(h-a)(5a-2h)\right]$$

$$= \frac{1}{27a^2}(a+2h)\left[4a^2+7ah-2h^2\right]$$

$$= \frac{1}{27a^2}(a+2h)(a+2h)(4a-h)$$

$$= \frac{1}{27a^2}(4a-h)(a+2h)^2.$$

8.(a)(i) Find the orthogonal trajectories of the family of curves $x^2/(a^2+\lambda)+y^2/(b^2+\lambda)=1$, where λ is a parameter.

Solⁿ: Given $x^2/(a^2+\lambda)+y^2/(b^2+\lambda)=1$ ——— (1)

Differentiating (1), $\dfrac{2x}{a^2+\lambda}+\dfrac{2y}{b^2+\lambda}\dfrac{dy}{dx}=0$

or $\dfrac{x}{a^2+\lambda}+\dfrac{y}{b^2+\lambda}\dfrac{dy}{dx}=0$

or $x(b^2+\lambda)+y(a^2+\lambda)\dfrac{dy}{dx}=0$

or $\lambda\left(x+y\dfrac{dy}{dx}\right)=-\left(b^2x+a^2y\dfrac{dy}{dx}\right)$

$\therefore\ \lambda=-\{b^2x+a^2y(dy/dx)\}/\{x+y(dy/dx)\}$

$\therefore\ a^2+\lambda=a^2-\dfrac{b^2x+a^2y(dy/dx)}{x+y(dy/dx)}=\dfrac{(a^2-b^2)x}{x+y(dy/dx)}$

and $b^2+\lambda=b^2-\dfrac{b^2x+a^2y(dy/dx)}{x+y(dy/dx)}=\dfrac{-(a^2-b^2)y(dy/dx)}{x+y(dy/dx)}$.

Putting the above values of $(a^2+\lambda)$ and $(b^2+\lambda)$ in (1), we have

$$\frac{x^2\{x+y(dy/dx)\}}{(a^2-b^2)x}-\frac{y^2\{x+y(dy/dx)\}}{(a^2-b^2)y(dy/dx)}=1$$

or $\{x+y(dy/dx)\}\{x-y(dx/dy)\}=a^2-b^2$, ——— (2)

which is the differential equation of the family (1).
Replacing dy/dx by $(-dx/dy)$ in (2), the differential

equation of the required orthogonal trajectories is

$$\{x+y(-dx/dy)\}\{x-y(-dy/dx)\} = a^2-b^2$$

or $\{x+y(dy/dx)\}\{x-y(dx/dy)\} = a^2-b^2$, —— ③

which is the same as the differential equation ② of the given family of curves ①. Hence, the system of the given curves ① is self orthogonal, i.e., each member of the given family of curves intersects its own members orthogonally.

8.(a)(ii) Solve by the method of variation of parameters
$x^2y'' - 2x(1+x)y' + 2(x+1)y = x^3$.

Sol^n: Re-writing the given equation in standard form, we get
$$\frac{d^2y}{dx^2} - \frac{2(1+x)}{x}\frac{dy}{dx} + \frac{2(x+1)}{x^2}y = x \quad \text{—— (1)}$$

C.F of (1), i.e. solution of
$$\frac{d^2y}{dx^2} - \frac{2(1+x)}{x}\frac{dy}{dx} + \frac{2(x+1)}{x^2}y = 0 \quad \text{—— (2)}$$

Comparing (2) with $y'' + Py' + Qy = R$,
$$P = -\frac{2(1+x)}{x}, \quad Q = \frac{2(x+1)}{x^2} \text{ and } R = 0 \quad \text{—— (3)}$$

Here $P + xQ = 0$, showing that $u = x$ —— (4) is a part of C.F of (2)

Let the complete solution of (1) be $y = uv$ —— (5)

Then v is given by $\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$

$$\Rightarrow \frac{d^2v}{dx^2} + \left[-\frac{2(1+x)}{x} + \frac{2}{x}\frac{dx}{dx}\right]\frac{dv}{dx} = 0$$

$$\Rightarrow \frac{d^2v}{dx^2} - 2\frac{dv}{dx} = 0 \text{ ; using (3) and (4)}$$

$$\Rightarrow (D^2 - 2D)v = 0 \quad \text{where } D \equiv d/dx \quad \text{—— (6)}$$

Auxiliary equation of (6) is $D^2 - 2D = 0 \Rightarrow D = 0, 2$.

$\therefore$ Solution of (6) is $y = c_1e^{0x} + c_2e^{2x} = c_1 + c_2e^{2x}$,
c_1 & c_2 being arbitrary constants —— (7)

from (4), (5) and (7), the complete solution of (1),
i.e. C.F of (1) is given by
$$y = x(c_1 + c_2e^{2x}) \Rightarrow y = c_1x + c_2xe^{2x} \quad \text{—— (8)}$$

Let $y = Ax + Bxe^{2x}$ ——— (9)

be the complete solution of (1). Then A and B are function of x which are so choosen that (1) will be satisfied. Differentiating (9) w.r.t 'x' we get

$$y' = A + A_1 x + B(e^{2x} + 2xe^{2x}) + B_1 x e^{2x} \quad \text{——— (10)}$$

where $A_1 = dA/dx$ and $B_1 = dB/dx$. choose A & B such that

$$A_1 x + B_1 x e^{2x} = 0 \quad \text{——— (11)}$$

Then (10) reduces to $y' = A + Be^{2x}(1+2x)$ ——— (12)

Differentiating (12), $y'' = A_1 + B_1 e^{2x}(1+2x) + B\{2e^{2x}(1+2x) + 2e^{2x}\}$ ——— (13)

Substituting the values of y, y' and y'' given by (9), (12) and (13) is (1), we have

$$x^2\{A_1 + B_1 e^{2x}(1+2x) + 4Be^{2x}(1+x)\} - 2x(1+x)\{A + Be^{2x}(1+2x)\} + 2(x+1)(Ax + Bxe^{2x}) = x^3$$

$$\Rightarrow A_1 x^2 + x^2 B_1 e^{2x}(1+2x) = x^3 \Rightarrow A_1 + B_1(1+2x)e^{2x} = x \quad \text{——— (14)}$$

solving (11) & (14) for A_1 and B_1, we have

$$A_1 = dA/dx = -1/2 \quad \text{and} \quad B_1 = \frac{dB}{dx} = \frac{1}{2}e^{-2x}$$

Integrating these $A = (-x/2) + c_1$ & $B = -\frac{1}{4}e^{-2x} + c_2$

Substituting the above values of A and B is (9).

the required solution is

$$y = \{(-x/2) + c_1\}x + \{(-1/4)e^{-2x} + c_2\}xe^{2x}$$

$$y = c_1 x + c_2 x e^{2x} - (x^2/2) - x/4$$

8(b) Describe the motion and path of a particle of mass m which is projected in a vertical plane through a point of projection with velocity u in a direction making an angle θ with the horizontal direction. further, if particles are projected from that point in the same vertical plane with velocity $4\sqrt{g}$, then determine the locus of vertices of their paths.

(Try your self.)

8(c) Using Stoke's theorem, evaluate $\iint_S (\nabla\times\vec{F})\cdot\hat{n}\,ds$ where $\vec{F} = (x^2+y-4)\hat{i} + 3xy\hat{j} + (2xy+z^2)\hat{k}$ and S is the surface of the paraboloid $z = 4-(x^2+y^2)$ above the xy-plane. Here, $\hat{n}$ is the unit outward normal vector on S.

Sol^n: The xy-plane cuts the surface S of the paraboloid $z=4-(x^2+y^2)$ in the circle C whose equations are $x^2+y^2=4$, $z=0$. Thus the boundary of the surface S is the circle C and the surface S lies above the circle C. Let the parametric equations of the curve C be

$x=2\cos t$, $y=2\sin t$, $z=0$, $0\le t<2\pi$.

By Stoke's theorem, we have

$$\iint_S (\nabla\times F)\cdot n\,ds = \oint_C F\cdot dr$$

$$= \int_C [(x^2+y-4)\hat{i} + 3xy\hat{j} + (2xy+z^2)\hat{k}]\cdot(dx\hat{i}+dy\hat{j}+dz\hat{k})$$

$$= \int_C [(x^2+y-4)dx + 3xy\,dy + (2xz+z^2)dz]$$

$$= \int_C [(x^2+y-4)dx + 3xy\,dy], \text{ since on } C,\ z=0 \text{ and } dz=0$$

$$= \int_{t=0}^{2\pi} \left[(x^2+y-4)\frac{dx}{dt} + 3xy\frac{dy}{dt}\right]dt$$

$$= \int_0^{2\pi} [(4\cos^2 t + 2\sin t - 4)(-2\sin t) + 3.2\cos t.2\sin t.2\cos t]\,dt$$

$$= -8\int_0^{2\pi} \cos^2 t \sin t\, dt - 4\int_0^{2\pi} \sin^2 t\, dt + 8\int_0^{2\pi} \sin t\, dt$$
$$+ 24\int_0^{2\pi} \cos^2 t \sin t\, dt$$

$$= 8\left[\frac{\cos^3 t}{3}\right]_0^{2\pi} - 4.2.2\int_0^{\pi/2} \sin^2 t\, dt + 8\left[-\cos t\right]_0^{2\pi}$$
$$-24\left[\frac{\cos^3 t}{3}\right]_0^{2\pi}$$

$$= 8.0 - 16.\frac{1}{2}.\frac{\pi}{2} + 8.0 - \frac{24}{3}.0$$

$$= -4\pi$$