2

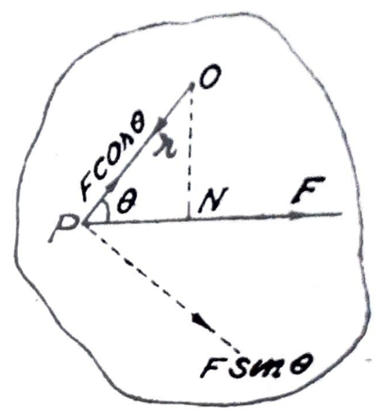
Equilibrium of a Rigid Body

(Coplanar Forces)

§ 1. Moment of a force about a point :

The **moment** *of a force applied on a rigid body about a point is defined as the amount of the tendency of the force to rotate the body about that point.*

Let **F** be the force acting at the point P of a rigid body. Let **r** be the position vector of the point P referred to a point O of the body. If θ is the angle between **F** and OP, then the components of **F** along and perpendicular to PO are $F \cos \theta$ and $F \sin \theta$ respectively.



(Fig. 2·1)

The tendency of the component $F \cos \theta$ is to move P along PO. But if the point O is fixed then due to the rigidity of the body the distance PO does not change, and hence the effect of $F \cos \theta$ is nulified. The tendency of the component $F \sin \theta$ is to turn the body perpendicular to OP. Thus when O is fixed, then the net effect of the force **F** acting at P is to turn P in a direction perpendicular to OP. The amount of the tendency of **F** to turn the body about O is $OP.F \sin \theta$.

$\therefore$ Moment of F about $O = OP.F \sin \theta$

$= F.OP \sin \theta = F.ON,$

where ON is the perpendicular from O on **F**.

Thus the moment of a force **F** *about the point O is the product of the magnitude of the force* **F** *and the perpendicular distance of O from the line of action of the force* **F**.

Now the moment of the force F about O

$= OP.F \sin \theta = |\mathbf{r}| \, |\mathbf{F}| \sin \theta$

and its direction is perpendicular to the plane of the vectors **r** and **F**, in the sense of a right hand screw rotated from **r** to **F**.

Hence the vector moment **M** *of the force* **F** *about O is defined by the vector* $\mathbf{M}=\mathbf{r}\times\mathbf{F}$.

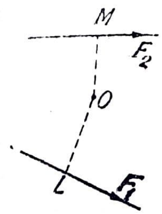
It can be easily seen that the vector moment of a force **F** about any point O is equal to $\mathbf{r}\times\mathbf{F}$ where **r** is the position vector with respect to O of any point P on the line of action of the force.

It is to be noted that the moment of the force about a point vanishes if the point lies on the line of action of the force.

Sign of the moment.

Let two forces $\mathbf{F}_1$ and $\mathbf{F}_2$ act on a lamina in its plane as shown in the figure.

If O is a point in the plane of the lamina about which it can turn, then the tendency of the forces $\mathbf{F}_1$ and $\mathbf{F}_2$ is to rotate the lamina in opposite directions.



(Fig. 2·2)

The moment of the force $\mathbf{F}_1$ about O has the tendency to rotate the lamina in the counter-clockwise direction and is taken as **positive** while the moment of the force $\mathbf{F}_2$ about O having the tendency to rotate the body in clockwise direction is taken as **negative.**

§ 2. General Theorems of Moments :

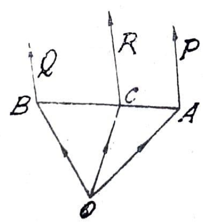
Theorem 1. *The sum of the moments of two like parallel forces acting on a rigid body about any arbitrary point is equal to the moment of their resultant about the same point.*

Proof. Let two like and parallel forces **P** and **Q** act at the points A and B of a body. If **R** is the resultant of the force, then

$$\mathbf{R}=\mathbf{P}+\mathbf{Q} \qquad \text{...(1)}$$

acting at C, such that

$$P.AC=Q.BC. \qquad \text{...(2)}$$



(Fig. 2·3)

The sum of the moments of the forces **P** and **Q** about any arbitrary point O

$$=\vec{OA}\times\mathbf{P}+\vec{OB}\times\mathbf{Q}$$

$$=(\vec{OC}+\vec{CA})\times\mathbf{P}+(\vec{OC}+\vec{CB})\times\mathbf{Q}$$

$$=\vec{OC}\times(\mathbf{P}+\mathbf{Q})+\vec{CA}\times\mathbf{P}+\vec{CB}\times\mathbf{Q}$$

$$=\vec{OC}\times\mathbf{R}-\vec{AC}\times\mathbf{P}+\vec{CB}\times\mathbf{Q}. \qquad \text{...(3)}$$

Now $\vec{AC}\times\mathbf{P}$ and $\vec{CB}\times\mathbf{Q}$ are both perpendicular to the plane containing **P**, **Q**, $\vec{AC}$, $\vec{CB}$ and are in the same directions.

Also $|\vec{AC}\times\mathbf{P}|=AC.P.\sin\theta$, $|\vec{CB}\times\mathbf{Q}|=CB.Q.\sin\theta$.

$\therefore \quad |\vec{AC}\times\mathbf{P}|=|\vec{CB}\times\mathbf{Q}| \qquad [\because \ AC.P=CB.Q, \text{ from (2)}]$

$\therefore \quad \vec{AC}\times\mathbf{P}=\vec{CB}\times\mathbf{Q}.$

Hence from (3), we have

the sum of the moments of the forces **P** and **Q** about O

$$=\vec{OC}\times\mathbf{R}$$

= moment of the resultant **R** about O.

Hence the theorem.

Theorem II. *If a number of coplanar forces acting at a point of a rigid body have a resultant, then the vector sum of the moments of all the forces about any arbitrary point is equal to the moment of the resultant about the point.*

Proof. Let the coplanar forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ acting at a point P of a rigid body have the resultant **R**.

$$\therefore \quad \mathbf{R}=\mathbf{F}_1+\mathbf{F}_2+\ldots+\mathbf{F}_n. \qquad \text{...(1)}$$

Let O be an arbitrary point and **r** be the position vector of the point P with respect to the point O.

Moment of the force **R** about O

$$=\mathbf{r}\times\mathbf{R}$$

$$=\mathbf{r}\times(\mathbf{F}_1+\mathbf{F}_2+\ldots+\mathbf{F}_n)$$

$$=\mathbf{r}\times\mathbf{F}_1+\mathbf{r}\times\mathbf{F}_2+\ldots+\mathbf{r}\times\mathbf{F}_n$$

= vector sum of the moments of the forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ about O.

Hence the theorem.

§ 3. Couple. (*Definition*). [Meerut 76]

A system of two equal and unlike parallel forces, whose lines of action are not the same, is called a **couple** *or a* **torque.**

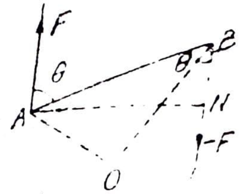
Since the couple consists of two equal and parallel forces in opposite directions, therefore the algebraic sum of the resolved parts of these forces in any direction is zero. Thus a couple has no tendency of motion of translation of the body in any direction.

Hence a couple cannot be replaced by a single force.

The couple only tends to rotate a body about a line perpendicular to the plane of the couple. A line perpendicular to the plane of a couple is called the *axis* of the couple.

Moment of a Couple.

Let two equal unlike and parallel forces **F** and $-$**F** act on a body at the points A and B respectively and let O be any point. Then **M**, the sum of the moments of the forces forming the couple about O, is given by



(Fig. 2·4)

$$\mathbf{M}=\vec{OA}\times\mathbf{F}+\vec{OB}\times(-\mathbf{F})$$

$$=(\vec{OB}+\vec{BA})\times\mathbf{F}-\vec{OB}\times\mathbf{F}$$

$$=\vec{BA}\times\mathbf{F},$$

which is independent of the point O. *Hence the moment of the couple is constant.*

We have,

$$|\mathbf{M}|=|\vec{BA}\times\mathbf{F}|=BA.F\sin\theta,$$

where θ is the angle between **F** and $\vec{BA}$

or $\qquad |\mathbf{M}|=F.(BA\sin\theta)=F.AN,$

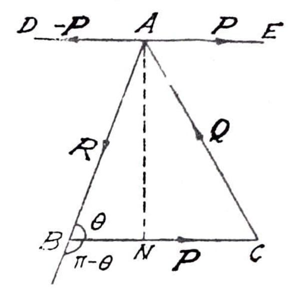
where AN is the perpendicular distance between the forces forming the couple. The perpendicular distance between the lines of action of the forces forming a couple is called the *arm* of the couple.

Thus *the moment of a couple is a vector whose magnitude is equal to the product of the magnitude of a force forming the couple and the perpendicular distance between the two forces forming the couple.* Direction of the moment vector of the couple is perpendicular to the plane containing the forces and is taken positive if the tendency of the couple is to rotate the body in counterclockwise direction.

§ **4.** *If three forces acting in one plane upon a rigid body at different points of it be represented in magnitude, direction and line of action by the sides of a triangle, taken in order, then they are equivalent to a couple whose moment is equal to twice the area of the triangle.*

Proof. Let the three forces **P, Q, R** acting in one plane upon a rigid body at different points of it be represented by the sides BC, CA and AB respectively of a triangle ABC.

At A, apply two equal and opposite forces **P** and $-$**P** parallel to BC along AE and AD respectively.



(Fig. 2·5)

The three forces **P, Q, R** acting at A along AE, CA and AB respectively are represented in magnitude, direction and line of action by the sides of $\triangle ABC$ taken in order. Therefore by the triangle law of forces, these forces are in equilibrium. Thus only two equal unlike and parallel forces **P** and $-$**P** along BC and AD are left.

Hence the system of given forces is equivalent to a couple consisting of two equal unlike and parallel forces **P** and $-$**P** along BC and AD respectively. The moment **M** of the couple is given by

$$\mathbf{M}=\vec{AB}\times\mathbf{P}.$$

We have $|\mathbf{M}|=AB.BC\sin(\pi-\theta)=BC.AB\sin\theta$

$=BC.AN$, where AN is the perpendicular from A on BC

$=2\left(\frac{1}{2}BC.AN\right)$

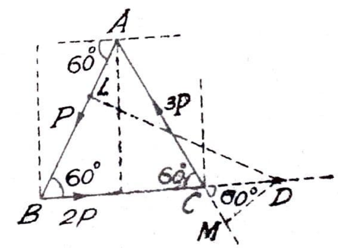
$=2$ (area of the triangle ABC).

Hence the result.

Illustrative Examples

Ex. 1. *Three forces P, $2P$, $3P$ act along the sides AB, BC, CA of an equilateral triangle ABC; find the magnitude and direction of their resultant, and find also the point in which its line of action meet the side BC.*

Sol. Let the forces P, $2P$, $3P$ act along the sides AB, BC, CA respectively, of an equilateral triangle ABC.



(Fig. 2·6)

The algebraic sums of the resolved parts of the forces along and perpendicular to BC are given by

$$X=2P-P\cos 60°-3P\cos 60°=0$$

and $$Y=-P\sin 60°+3P\sin 60°=P\sqrt{3}.$$

Hence the resultant $R=\sqrt{(X^2+Y^2)}=P\sqrt{3}.$

The angle that the resultant R makes with BC

$$=\tan^{-1}\left(\frac{Y}{X}\right)=\tan^{-1}\left(\frac{P\sqrt{3}}{0}\right)=\tan^{-1}\infty=\frac{\pi}{2}$$

$\therefore$ the resultant is perpendicular to the side BC.

Let the line of action of the resultant meet the side BC at the point D. Since the sum of moments of the forces about any point is equal to the moment of the resultant about that point, therefore the sum of moments of forces about the point D must be equal to zero.

i.e., $$P.DL-3P.DM=0,$$

or $$P.BD\sin 60°-3P.CD\sin 60°=0$$

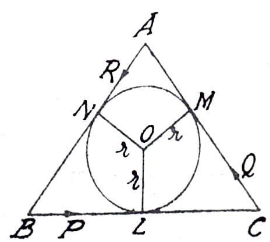
or $$BD-3(BD-BC)=0$$

or $$BD=\frac{3}{2}BC.$$

Ex. 2. *Three forces P, Q, R act along the sides BC, CA, AB of a triangle ABC, taken in order, if their resultant passes through the incentre of $\triangle ABC$, then prove that*

$$P+Q+R=0.$$ **[Raj. T.D.C. 79(S)]**

Sol. Let the forces P, Q, R act along the sides BC, CA, AB respectively of a triangle ABC, taken in order. Let O be the incentre of the $\triangle ABC$ and r be the radius of the incribed circle of the $\triangle ABC$. If the resultant of the forces P, Q and R passes through the incentre O, then the algebraic sum of the moments of the forces about O must be equal to zero. So, we have



(Fig. 2·7)

$$r.P+r.Q+r.R=0$$

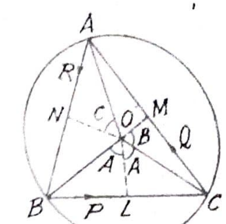
or $$r(P+Q+R)=0$$

or $$P+Q+R=0. \quad [\because r\neq 0]$$

Ex. 3. *The resultant of the forces P, Q, R acting along the sides BC, CA, AB respectively of a triangle ABC passes through its circumcentre. Show that*

$$P\cos A+Q\cos B+R\cos C=0.$$

Sol. Let the forces P, Q, R act along the sides BC, CA, AB respectively of a triangle ABC, taken in order. Let OL, OM, ON be the perpendicular bisectors of the sides BC, CA, AB respectively of the $\triangle ABC$. Then O is the circum-centre of the $\triangle ABC$, and we have $OA=OB=OC$. If the resultant of the forces, P, Q, R passes through the circum-centre O, then the algebraic sum of the moments of these forces about O must be equal to zero



(Fig. 2·8)

i.e., $$P.OL+Q.OM+R.ON=0. \quad ...(1)$$

Now $\angle BOC=2\angle BAC=2A$; $\therefore$ $\angle BOL=\angle COL=A$.

Similarly $\angle COM=\angle AOM=B$, and $\angle AON=\angle BON=C$.

$\therefore$ $OL=OB\cos A$, $OM=OC\cos B$, $ON=OA\cos C$.

Substituting in (1), we have

$$OB\cos A.P+OC\cos B.Q+OA\cos C.R=0$$

or $$P\cos A+Q\cos B+R\cos C=0 \quad [\because OA=OB=OC]$$

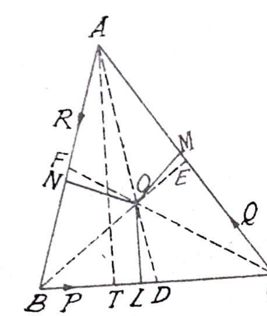
Ex. 4. *Three forces P, Q, R act along the sides BC, CA, AB of a triangle ABC taken in order. If their resultant passes through the centroid of the triangle ABC, prove that*

$$\frac{P}{\sin A}+\frac{Q}{\sin B}+\frac{R}{\sin C}=0.$$ **[Raj. T.D.C. 79(S)]**

Sol. Let the forces P, Q, R act along the sides BC, CA, AB of a triangle ABC, taken in order. If the resultant of these forces passes through the centroid O of the $\triangle ABC$, then the algebraic sum of the moments of the forces about O must be equal to zero. So, we have

$$P.OL+Q.OM+R.ON=0, \quad ...(1)$$

where OL, OM, ON are perpendiculars from O on the sides BC, CA, AB of the triangle ABC.



(Fig. 2·9)

Draw AT perpendicular to BC. From similar triangles DOL and DAT, we have

$$\frac{OL}{AT}=\frac{DO}{DA}=\frac{1}{3}. \quad \therefore \quad OL=\frac{AT}{3}.$$

Now if S is the area of the triangle ABC, then

$$S=\tfrac{1}{2}BC.AT=\tfrac{1}{2}a.AT \text{ or } AT=2S/a.$$

$\therefore\ OL=\frac{1}{3}AT=\frac{2S}{3a}$. Similarly $OM=\frac{2S}{3b}$ and $ON=\frac{2S}{3b}$

Substituting in (1), we have

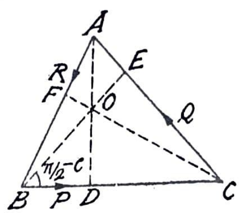
$$P.\frac{2S}{3a}+Q\frac{2S}{3b}+R.\frac{2S}{3c}=0$$

or $$\frac{P}{a}+\frac{Q}{b}+\frac{R}{c}=0$$

or $$\frac{P}{\sin A}+\frac{Q}{\sin B}+\frac{R}{\sin C}=0 \quad \left[\because \frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c}\right].$$

Ex. 5. *Three forces P, Q, R act along the sides BC, CA, AB of a triangle ABC taken in order; if their resultant passes through the orthocentre of the triangle, prove that*
$P \sec A+Q \sec B+R \sec C=0$. **[P.C.S. 75; Raj. T.D.C. 79(S)]**

Sol. Let the forces P, Q, R act along the sides BC, CA, AB of a triangle ABC, taken in order. If the resultant of these forces passes through the orthocentre O, then the algebraic sum of the moments of the forces about O must be equal to zero. So, we have



(Fig. 2·10)

$$P\,OD+Q.OE+R\,OF=0, \qquad ...(1)$$

where D, E, F are the feet of the perpendiculars drawn from the vertices A, B, C to the opposite sides.

In the $\triangle CBE$, $\angle CBE=\frac{1}{2}\pi-C$.

$\therefore$ from $\triangle OBD$, $\tan(\frac{1}{2}\pi-C)=\frac{OD}{BD}$, or $OD=BD\cot C$.

But from $\triangle ABD$, $BD=AB\cos B=c\cos B$.

$\therefore\ OD=c\cos B\cot C=\frac{c}{\sin c}\cos B\cos C=k\cos B\cos C$,

where $$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k \text{ (say)}.$$

Similarly $OE=k\cos C\cos A$ and $OF=k\cos A\cos B$.
Substituting in (1), we have
$P.k\cos B\cos C+Q.k\cos C\cos A+R.k\cos A\cos B=0$.
Dividing by $k\cos A\cos B\cos C$, we have
$$P\sec A+Q\sec B+R\sec C=0.$$

Ex. 6. *Three forces P, Q, R act along the sides BC, CA, AB of a triangle ABC, taken in order; their resultant passes through the centre of the inscribed and circumscribed circles, prove that*

$P:Q:R=(\cos B-\cos C):(\cos C-\cos A):(\cos A-\cos B)$.

Sol. Let the forces P, Q, R act along the sides BC, CA, AB of a triangle ABC, taken in order.

If the resultant passes through the centre of the inscribed circle (*i.e.*, circum-centre), then from Ex. 2, we have

$$P+Q+R=0, \qquad ...(1)$$

and if the resultant passes through the centre of the circumscribed circle (*i.e.*, circum-centre), then from Ex. 3, we have

$$P\cos A+Q\cos B+R\cos C=0. \qquad ...(2)$$

From (1) and (2), we have

$$\frac{P}{\cos C-\cos B}=\frac{Q}{\cos A-\cos C}=\frac{C}{\cos B-\cos A}$$

or $P:Q:R=(\cos B-\cos C):(\cos C-\cos A):(\cos A-\cos B)$

Ex. 7. *Forces P, Q, R act along the sides BC, CA, AB of a triangle ABC, taken in order and their resultant passes through the incentre and the centre of gravity of the triangle, prove that*
$P:Q:R=a(b-c):b(c-a):c(a-b)$.

[I.A.S. 75; I.E.S. 74]

Sol. Let the forces P, Q, R act along the sides BC, CA, AB of a triangle ABC, taken in order.

If the resultant passes through the incentre of the triangle then from Ex. 2, we have

$$P+Q+R=0, \qquad ...(1)$$

and if the resultant passes through the centre of gravity (centroid) of the triangle, then from Ex. 4, we have

$$\frac{P}{\sin A}+\frac{Q}{\sin B}+\frac{R}{\sin C}=0. \qquad ...(2)$$

From (1) and (2), have

$$\frac{P}{1.\frac{1}{\sin C}-1.\frac{1}{\sin B}}=\frac{Q}{1.\frac{1}{\sin A}-1.\frac{1}{\sin C}}=\frac{R}{1.\frac{1}{\sin B}-1.\frac{1}{\sin A}}$$

or $$P:Q:R=\frac{\sin B-\sin C}{\sin B\sin C}:\frac{\sin C-\sin A}{\sin A\sin C}:\frac{\sin A-\sin B}{\sin A\sin B}$$

$=\sin A(\sin B-\sin C):\sin B(\sin C-\sin A):$
$\sin C(\sin A-\sin B)$.

[Multiplying by $\sin A\sin B\sin C$].

Since $$\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=\frac{1}{K}\text{ (say)},$$

$\therefore\ \sin A=ak,\ \sin B=bk,\ \sin C=ck.$

$\therefore \quad P:Q:R=ak.(bk-ck):bk\,(ck-ak):ck\,(ak-bk)$
$=a\,(b-c):b\,(c-a):c\,(a-b)$. [Dividing by k^2]

Ex. 8. *Three forces act along the sides of a triangle ABC, taken in order and their resultant passes through the orthocentre and the centre of gravity of the triangle; show that the forces are in the ratio of*

$\sin 2A \sin (B-C) : \sin 2B \sin (C-A) : \sin 2C \sin (A-B)$.

Sol. Let the forces, P, Q, R act along the sides BC, CA, AB of a triangle ABC, taken in order.

If the resultant passes through the orthocentre of the triangle ABC, then from Ex. 5, we have

$$\frac{P}{\cos A}+\frac{Q}{\cos B}+\frac{R}{\cos C}=0, \qquad \ldots(1)$$

and if the resultant passes through the centre of gravity of the triangle ABC, then from Ex. 4, we have

$$\frac{P}{\sin A}+\frac{Q}{\sin B}+\frac{R}{\sin C}=0. \qquad \ldots(2)$$

From (1) and (2), we have

$$\frac{P}{\dfrac{1}{\cos B \sin C}-\dfrac{1}{\sin B \cos C}}=\frac{Q}{\dfrac{1}{\cos C \sin A}-\dfrac{1}{\sin C \cos A}}$$

$$=\frac{R}{\dfrac{1}{\cos A \sin B}-\dfrac{1}{\sin A \cos B}}$$

or $$P:Q:R=\frac{\sin B \cos C-\sin C \cos B}{\sin B \cos B \sin C \cos C}:\frac{\sin C \cos A-\cos C \sin A}{\sin A \cos A \sin C \cos C}:\frac{\sin A \cos B-\cos A \sin B}{\sin A \cos A \sin B \cos B}$$

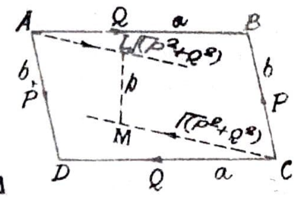
$$=\frac{\sin (B-C)}{4 \sin B \cos B \sin C \cos C}:\frac{\sin (C-A)}{4 \sin A \cos A \sin C \cos C}:\frac{\sin (A-B)}{4 \sin A \cos A \sin B \cos B}$$

$$=\frac{\sin (B-C)}{\sin 2B \sin 2C}:\frac{\sin (C-A)}{\sin 2A \sin 2C}:\frac{\sin (A-B)}{\sin 2A \sin 2B}$$

$=\sin 2A \sin (B-C) : \sin 2B \sin (C-A) : \sin 2C \sin (A-B)$.

Ex. 9. *ABCD is a rectangle such that $AB=CD=a$ and $BC=DA=b$. Forces P act along AD and CB. Forces Q act along AB and CD. Prove that the perpendicular distance between the resultant of the forces P, Q at A and the resultant of the forces P Q at C is* $\dfrac{Pa-Qb}{\sqrt{(P^2+Q^2)}}$.

Sol. The forces Q at A along AB and Q at C along CD from a couple of moment $-Q.AD=-Q.b$ (negative sign is taken, because the tendency of the couple is to rotate the rectangle in clockwise direction).



(Fig. 2.11)

The forces P at A along AD and P at C along CB form a couple of moment $P.AB=P.a$. The above two couples are equivalent to a single couple of moment $(Pa-Qb)$. Thus the whole system of forces is equivalent to a couple of moment $(Pa-Qb)$.

Also the resultant of the forces P, Q at A is $\sqrt{(P^2+Q^2)}$ inclined at an angle $\tan^{-1}(P/Q)$ to AB and the resultant of the forces P, Q at C is $\sqrt{(P^2+Q^2)}$ inclined of an angle $\tan^{-1}(P/Q)$ to CD. Thus the whole system of forces is equivalent to two equal and parallel forces each equal to $\sqrt{(P^2+Q^2)}$ at A and C in opposite directions, and so they form a couple whose moment is

$$\sqrt{(P^2+Q^2)}\,LM=\sqrt{(P^2+Q^2)}\,p,$$

where p is the perpendicular distance between these forces.

Hence $$\sqrt{(P^2+Q^2)}\,p=Pa-Qb.$$

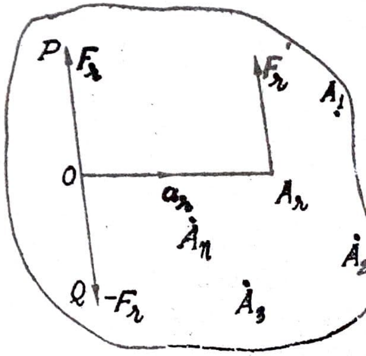
$$\therefore \quad p=\frac{Pa-Qb}{\sqrt{(P^2+Q^2)}}.$$

§ 2·5. **Theorem.**

Any system of coplanar forces acting on a rigid body is equivalent to a single forces acting at an arbitrarily chosen point together with a single couple.

[Meerut 71, 77, 78, 79 (S), 82, 84 (R), 85, 85 (P), 86, 87, 88 (P) Gorakhpur 75; Jiwaji 73; Raj. T. D. C. 78. 80]

Proof. Let a system of coplanar forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ act at the points $A_1, A_2, \ldots, A_n$ respectively of a rigid body. Let O be an arbitrary point and $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ the position vectors of the points $A_1, A_2, \ldots, A_n$ respectively with respect to the origin O.



(Fig. 2·12)

Consider the force $\mathbf{F}_r$ acting at the point A_r whose position vector referred to the origin O

$\mathbf{a}_r$. Apply two forces $\mathbf{F}_r$ and $-\mathbf{F}_r$ at O parallel to $\mathbf{F}_r$ along OP and OQ respectively as shown in the figure. Since two equal forces are applied at a point in opposite directions, therefore they do not affect the equilibrium of the body. Now the forces $\mathbf{F}_r$ at A_r and $-\mathbf{F}_r$ at O along OQ form a couple of moment

$$\vec{OA_r} \times \mathbf{F}_r = \mathbf{a}_r \times \mathbf{F}_r$$

and besides this couple a single force $\mathbf{F}_r$ parallel to OP is left at O.

Thus the single force $\mathbf{F}_r$ acting at A_r (whose position vector is $\mathbf{a}_r$) is equivalent to a force $\mathbf{F}_r$ acting at O and a couple of moment $\mathbf{a}_r \times \mathbf{F}_r$. Similarly all the n forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ acting at the point $A_1, A_2, \ldots, A_n$ whose position vectors referred to O are

$$\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$$

respectively are equivalent to single forces $\mathbf{F}_1, \mathbf{F}_2, \ldots \mathbf{F}_n$ acting at O and couples of moments

$$\mathbf{a}_1 \times \mathbf{F}_1, \mathbf{a}_2 \times \mathbf{F}_2, \ldots, \mathbf{a}_n \times \mathbf{F}_n.$$

Hence the given system of forces will be equivalent to forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ acting at O, together with couples of moments

$$\mathbf{a}_1 \times \mathbf{F}_1, \mathbf{a}_2 \times \mathbf{F}_2, \ldots, \mathbf{a}_n \times \mathbf{F}_n.$$

If $\mathbf{R}$ is the resultant of the concurrent forces $\mathbf{F}_1, \mathbf{F}_2, \mathbf{F}_n$ acting at O, then we have

$$\mathbf{R} = \mathbf{F}_1 + \mathbf{F}_2 + \ldots + \mathbf{F}_n = \sum_{r=1}^{n} \mathbf{F}_r$$

and if $\mathbf{M}$ is the moment of the resultant couple of the above couples, then

$$\mathbf{M} = \mathbf{a}_1 \times \mathbf{F}_1 + \mathbf{a}_2 \times \mathbf{F}_2 + \ldots + \mathbf{a}_n \times \mathbf{F}_n$$

$$= \sum_{r=1}^{n} \mathbf{a}_r \times \mathbf{F}_r$$

Hence the system of coplanar forces acting on a body is equivalent to a single force $\mathbf{R} = \sum_{r=1}^{n} \mathbf{F}_r$ acting at an arbitrarily chosen point O, together with a couple of moment

$$\mathbf{M} = \sum_{r=1}^{n} \mathbf{a}_r \times \mathbf{F}_r.$$

§ 2·6. Necessary and sufficient Conditions for equilibrium of a rigid body.

The necessary and sufficient conditions for the equilibrium of a rigid body under the action of a system of coplanar forces acting at different points of it are that the sums of the resolved parts of the forces in any two mutually perpendicular directions vanish separately and the sum of the moments of the forces about any point in the plane of the forces vanishes.. **[Meerut 73, 73 (S), 75 (S), 76 (P), 88; Raj. T.D.C. 80 (S); Gorakhpur 80; Gurunanak 73]**

Proof. Necessary Conditions. Let a rigid body under the action of a system of coplanar forces $\mathbf{F}_1, \mathbf{F}_2, \ldots, \mathbf{F}_n$ acting at different points $A_1, A_2, \ldots, A_n$ of the body whose position vectors referred to any origin O are $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ be in equilibrium.

We have proved in § 2·5 that the above system of forces is equivalent to a single force $\mathbf{R} = \sum_{r=1}^{n} \mathbf{F}_r$ acting at O and a single couple of moment $\mathbf{M} = \sum_{r=1}^{n} \mathbf{a}_r \times \mathbf{F}_r$.

The single force $\mathbf{R}$ and the couple $\mathbf{M}$, have respectively the tendency of motion of translation and the motion of rotation of the body. In case the forces acting on the body keep it in equilibrium, the forces have no tendency of motion of translation or of motion of rotation of the body. Hence the necessary conditions for the equilibrium of the body are that $\mathbf{R} = 0$ and $\mathbf{M} = 0$.

Let (x_r, y_r) be the coordinates of the point A_r referred to some rectangular axes through the point O, then $\mathbf{a}_r = x_r \mathbf{i} + y_r \mathbf{j}$.

Also if X_r, Y_r are the resolved parts parallel to the axes of the force $\mathbf{F}_r$ acting at the point A_r then $\mathbf{F}_r = X_r \mathbf{i} + Y_r \mathbf{i}$.

$$\therefore \quad \mathbf{R} = \sum_{r=1}^{n} \mathbf{F}_r = \sum_{r=1}^{n} (X_r \mathbf{i} + Y_r \mathbf{j}) = \left(\sum_{r=1}^{n} X_r \right) \mathbf{i} + \left(\sum_{r=1}^{n} Y_r \right) \mathbf{j}$$

or $\quad \mathbf{R} = X\mathbf{i} + Y\mathbf{j}$, where $X = \sum_{r=1}^{n} X_r$, $Y = \sum_{r=1}^{n} Y_r$

and $\quad \mathbf{M} = \sum_{r=1}^{n} (\mathbf{a}_r \times \mathbf{F}_r) = \sum_{r=1}^{n} \{(x_r \mathbf{i} + y_r \mathbf{j}) \times (X_r \mathbf{i} + Y_r \mathbf{j})\}$

$= \sum_{r=1}^{n} (x_r Y_r - y_r X_r)\, \mathbf{i}\times\mathbf{j}$. $[\because\ \mathbf{j}\times\mathbf{i}=-\mathbf{i}\times\mathbf{j},\ \mathbf{i}\times\mathbf{i}=0=\mathbf{j}\times\mathbf{j}]$

Now if $\mathbf{R}=0$, then $X=\sum_{r=1}^{n} X_r=0$ and $Y=\sum_{r=1}^{n} Y_r=0$

and if $\mathbf{M}=0$, then $M=\sum_{r=1}^{n}(x_r Y_r - y_r X_r)=0$.

Obviously M is the algebraic sum of the scalar moments of the forces about the point O.

Hence the necessary conditions of equilibrium of the body are that, the sum of the resolved parts of the forces in any two mutually perpendicular directions vanish separately and the sum of the moments of the forces about any point in the plane of the forces vanishes.

Sufficient Conditions. If the sum of the resolved parts of the forces in any two mutually perpendicular directions vanish separately and the sum of the moments of the forces about any point in the plane of the forces vanishes *i.e.*, $X=0$, $Y=0$ and $M=0$,

then $\mathbf{R}=X\mathbf{i}+Y\mathbf{j}=0$

and $\mathbf{M}=0$.

Thus there is neither the motion of translation nor the motion of rotation of the body, therefore the body is in equilibrium.

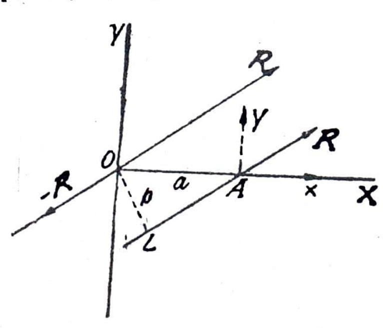
Hence the necessary and sufficient conditions for the equilibrium of a body are that $X=0$, $Y=0$ and $M=0$.

§ 2·7. Equation of the resultant.

To find the equation of the line of action of the resultant of the system of coplanar forces acting at different points of a body.

[Meerut 80 (S); Gorakhpur 79, 82; Raj. T.D.C. 79 (S), 80]

Let the system of coplanar forces acting at different points of a rigid body be reduced to a single force **R** acting at O (origin); together with a couple of moment **M**.



(Fig, 2·13)

The couple of moment **M** can be replaced by two equal unlike parallel forces one $-\mathbf{R}$ at O and the other **R** parallel to it at a distance p

from O such that

$$R.p=M \text{ or } p=M/R, \text{ provided } R\neq 0.$$

The forces **R** and $-\mathbf{R}$ at O balance. Hence the given system reduces to a single resultant force **R** acting along a line at a distance M/R from the origin O, and parallel to the force **R** acting at O.

Let the line of action of this single resultant force **R** meet the x-axis OX at the point A such that $OA=a$. Then

$$\vec{OA}=a\mathbf{i}.$$

If X and Y are the resolved parts of **R** parallel to the axes, then

$$\mathbf{R}=X\mathbf{i}+Y\mathbf{j},$$

and the moment M of the forces about O is given by

$$M=X.0+Y\,a=aY.$$

$$\therefore\quad a=M/Y.$$

$$\therefore\quad \vec{OA}=a\mathbf{i}=(M/Y)\mathbf{i}=\mathbf{a}, \text{ (say)}.$$

$\therefore$ Vector equation of the line of action of the single resultant force **R**, passing through A (whose position vector is **a**) is given by

$$\mathbf{r}=\mathbf{a}+t\mathbf{R}$$

or

$$\mathbf{r}=(M/Y)\,\mathbf{i}+t\mathbf{R}, \quad ...(1)$$

where

$$\mathbf{r}=x\mathbf{i}+y\mathbf{j}.$$

Cartesian form. From (1), we have

$$x\mathbf{i}+y\mathbf{j}=\left(\frac{M}{Y}\right)\mathbf{i}+t\,(X\mathbf{i}+Y\mathbf{j})$$

Equating the coefficients of **i** and **j** on both sides, we have

$$x=\frac{M}{Y}+tX \quad \text{and} \quad y=tY.$$

Eliminating t from these equations, we have

$$x=\frac{M}{Y}+\frac{y}{Y}X,$$

or

$$xY-yX=M,$$

which is the equation of the line of action of the resultant in cartesian form.

Remark. In the above equation of the single resultant force, we have

X=the algebraic sum of the resolved parts of the force along the x-axis OX,

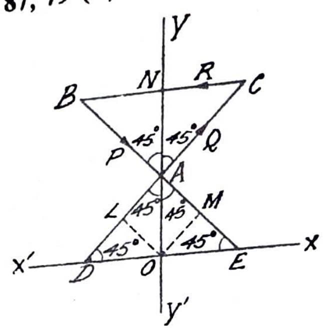
Y=the algebraic sum of the resolved parts of the force along the y-axis OY,

and M=the algebraic sum of the moments of the forces about the origin O,

Illustrative Examples

Ex. 10. *Three forces P, Q, R act along the sides of a triangle formed by the line $x+y=1$, $y-x=1$ and $y=2$. Find the equation of the line of action of the resultant.* **[Raj. T.D.C. 87, 79 (S) ; Gorakhpur 81]**

Sol. Let the three forces P, Q, R act along the lines $x+y=1$, $y-x=1$ and $y=2$ *i.e.*, the lines BA, AC and CB respectively.



(Fig. 2·14)

By the coordinate geometry,

$$OA=1,\ ON=2,$$
$$\angle OAD=\angle OAE=\angle ODA$$
$$=\angle OEA=45°.$$

Let X, Y be the algebraic sums of the resolved parts of the forces along OX and OY and M the algebraic sum of the moments of the forces about the origin O. Then the equation of the line of action of the resultant is given by

$$xY-yX=M. \qquad \text{...(1)}$$

Now $\quad X=P\cos 45°+Q\cos 45°-R=\dfrac{1}{\sqrt{2}}(P+Q-\sqrt{2}R),$

$$Y=-P\sin 45°+Q\sin 45°=\frac{1}{\sqrt{2}}(-P+Q),$$

and $\quad M=-P.OM-Q.OL+R.ON$

$$=-P.OA\sin 45°-Q.OA\sin 45°+R.2$$

$$=-\frac{1}{\sqrt{2}}(P+Q)+2R.$$

Substituting in (1) the equation of the line of action of the resultant is given by

$$x.\frac{1}{\sqrt{2}}(-P+Q)-y.\frac{1}{\sqrt{2}}(L+Q-\sqrt{2}R)=-\frac{1}{\sqrt{2}}(P+Q)+2R$$

or $\quad x(-P+Q)-y(P+Q-\sqrt{2}R)=-(P+Q)+2\sqrt{2}R$

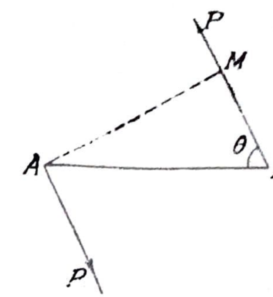
or $\quad (P-Q)x+(P+Q-\sqrt{2}R)y=P+Q-2\sqrt{2}R.$

Ex. 11. *Two equal unlike parallel forces acting at fixed points A and B form a couple of moment G. If their lines of action are turned through one right angle they form a couple of moment H. Show that when they both act at right angles to AB, they form a couple of moment $\sqrt{(G^2+H^2)}$.*

Sol. Let two equal unlike parallel forces P, P act at fixed point A and B. Then the moment G of this couple is given by

$$G=P.AM=P.AB\sin\theta, \qquad \text{...(1)}$$

where θ is the angle between the force P and the line AB.



(Fig. 2·15)

If the lines of action of the forces P, P are turned through one right angles, then the moment H of the new couple is obtained by replacing θ by $90°+\theta$ in (1) and is thus given by

$$H=P.AB\sin(90°+\theta)=P.AB\cos\theta. \qquad \text{...(2)}$$

From (1) and (2), we have

$$\sqrt{(G^2+H^2)}=P.AB \qquad \text{...(3)}$$

Also if the forces act at right angles to AB, then the moment of the couple

$$=P.AB=\sqrt{(G^2+H^2)}, \qquad \text{[from (3)]}$$

which is the required result.

Ex. 12. *If a system of forces in one plane reduces to a couple whose moment is G and when each force is turned round its point of application through a right angle it reduces to a couple of moment H. Prove that when each force is turned through an angle α, the system is equivalent to a couple whose moment is $G\cos\alpha+H\sin\alpha$.* **[I.F.S. 77]**

For what value of α will the moment of the new couple be equal to the moment of the old couple ?

Sol. Let a system of forces $F_1, F_2, \ldots, F_n$ act on a body at the points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ respectively. Let the forces be inclined at angles $\theta_1, \theta_2, \ldots, \theta_n$ to the x-axis. If X_r, Y_r are the resolved parts of the force F_r along the axes, then

$$X_r=F_r\cos\theta_r \text{ and } Y_r=F_r\sin\theta_r.$$

If X and Y are the sums of the resolved part of the forces along the axes and G the sum of the moments of the forces about the origin O, then

$$X=\sum_{r=1}^{n} X_r=\sum_{r=1}^{n} F_r\cos\theta_r,\quad Y=\sum_{r=1}^{n} Y_r=\sum_{r=1}^{n} F_r\sin\theta_r$$

and $\quad G=\sum_{r=1}^{n}(x_rY_r-y_rX_r)=\sum_{r=1}^{n} F_r(x_r\sin\theta_r-y_r\cos\theta_r). \qquad \text{...(1)}$

Since the system of forces reduces to a couple only.

$$\therefore \quad X=\sum_{r=1}^{n} F_r \cos\theta_r=0 \text{ and } Y=\sum_{r=1}^{n} F_r \sin\theta_r=0. \quad \ldots(2)$$

When each force is turned round its point of application through a right angle, then the sum of the resolved parts of the forces along x-axis

$$=\sum_{r=1}^{n} F_r \cos(90^\circ+\theta_r)=-\sum_{r=1}^{n} F_r \sin\theta_r=0$$

and the sum of the resolved parts of the forces along y axis

$$=\sum_{r=1}^{n} F_r \sin(90^\circ+\theta_r)=\sum_{r=1}^{n} F_r \cos\theta_r=0.$$

$\therefore$ in this case the single force is zero. Thus in this case the system of forces reduces to a single couple only. This moment H of the couple in this case, is given [from (1)] by

$$H=\sum_{r=1}^{n} F_r \{x_r \sin(90^\circ+\theta_r)-y_r \cos(90^\circ+\theta_r)\}$$

or

$$H=\sum_{r=1}^{n} F_r (x_r \cos\theta_r+y_r \sin\theta_r). \quad \ldots(3)$$

Again when each force is turned through an angle α, then sum of the resolved parts of the forces along the x-axis

$$=\sum_{r=1}^{n} F_r \cos(\theta_r+\alpha)$$

$$=\cos\alpha \sum_{r=1}^{n} F_r \cos\theta_r-\sin\alpha \sum_{r=1}^{n} F_r \sin\theta_r=0 \quad \text{[from (1)]}$$

and the sum of the resolved parts of the forces along y axis

$$=\sum_{r=1}^{n} F_r \sin(\theta_r+\alpha)$$

$$=\cos\alpha \sum_{r=1}^{n} F_r \sin\theta_r+\sin\alpha \sum_{r=1}^{n} F_r \cos\theta_r=0. \quad \text{[from (2)]}$$

Thus the single force is zero and hence the system of forces reduces to a single couple only. The moment of the couple, say M, in this case is given [from (1)] by,

$$M=\sum_{r=1}^{n} F_r \{x_r \sin(\theta_r+\alpha)-y_r \cos(\theta_r+\alpha)\}$$

$$=\sum_{r=1}^{n} F_r \{x_r (\sin\theta_r \cos\alpha+\cos\theta_r \sin\alpha)$$

$$-y_r (\cos\theta_r \cos\alpha-\sin\theta_r \sin\alpha)\}$$

$$=\cos\alpha . \sum_{r=1}^{n} F_r (\sin\theta_r-y_r \cos\theta_r)$$

$$+\sin\alpha \sum_{r=1}^{n} F_r (x_r \cos\theta_r+y_r \sin\theta_r)$$

or $M=G\cos\alpha+H\sin\alpha$. ...(4)

2nd part. If the moment of the new couple M is equal to the moment of the old couple G, then

$$G\cos\alpha+H\sin\alpha=G$$

or $$H\sin\alpha=G(1-\cos\alpha)$$

or $$\frac{H}{G}=\frac{1-\cos\alpha}{\sin\alpha}=\frac{2\sin^2\frac{1}{2}\alpha}{2\sin\frac{1}{2}\alpha\cos\frac{1}{2}\alpha}=\tan\tfrac{1}{2}\alpha.$$

$$\therefore \quad \alpha=2\tan^{-1}(H/G),$$

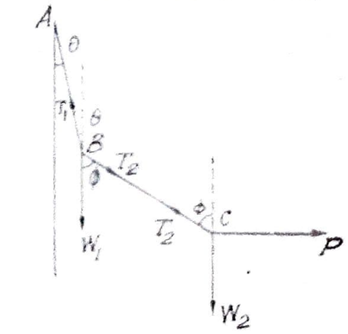
which is the required angle.

Ex. 13. *Weights W_1, W_2, are fastened to a light inextensible string ABC at the points B, C, the end A being fixed. Prove that, if a horizontal force P is applied at C and in equilibrium AB and BC are inclined at angles θ, ϕ to the vertical, then*

$$P=(W_1+W_2)\tan\theta=W_2\tan\phi.$$

[Meerut 81 (S), 82 (S), 84, 85]

Sol. Let W_1, W_2, be the weights fastened at the points B and C of a light inextensible string ABC, fixed at the point A. When a horizontal force P is applied at the end C, then AB and BC are inclined at angles θ and ϕ to the vertical. Let T_1 and T_2 be the tensions in the strings AB and AC respectively. The point B is in equilibrium under the action of the forces T_1, T_2 and W_1 and the point C is in equilibrium under the action of the forces T_2, P and W_2 as shown in the figure. For the equilibrium of B and C, resolving the forces acting at B and C horizontally and vertically, we have



(Fig. 2.16)

$$T_1 \sin\theta = T_2 \sin\phi \qquad \ldots(1)$$

for B,
$$T_1 \cos\theta = W_1 + T_2 \cos\phi \qquad \ldots(2)$$

and
$$T_2 \sin\phi = P \qquad \ldots(3)$$

for C,
$$T_2 \cos\phi = W_2. \qquad \ldots(4)$$

and

Dividing (3) by (4), we have $\tan\phi = P/W_2$

or
$$P = W_2 \tan\phi \qquad \ldots(5)$$

From (1) and (3), we have

$$T_1 \sin\theta = P \quad \text{or} \quad T_1 = \frac{P}{\sin\theta}$$

and from (4),
$$T_2 = \frac{W_2}{\cos\phi}.$$

Substituting for T_1 and T_2 in (2), we have

$$\frac{P}{\sin\theta} \cos\theta = W_1 + \frac{W_2}{\cos\phi} \cos\phi$$

or
$$P = (W_1 + W_2) \tan\theta. \qquad \ldots(6)$$

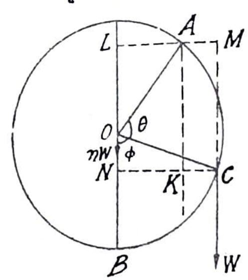
From (5) and (6), we have

$$P = (W_1 + W_2) \tan\theta = W_2 \tan\phi.$$

Ex. 14. *A uniform circular disc of weight nW has a heavy particle of weight W attached to a point C, on its rim. If the disc is suspended from a point A on its rim, B, is the lowest point; and if suspended from B, A is the lowest point. Show that the angle subtended by AB at the centre of the disc is $2\sec^{-1}\{2(n+1)\}$*

[Meerut 82(S)]

Sol. Let O be the centre of the circular disc of weight nW suspended from the point A. When a weight W is attached to a point C of the disc, B is the lowest point.



(Fig. 2·17 (i))

The disc is in equilibrium under the action of the following forces :

(i) nW, the weight of the disc acting vertically down-wards at its centre O,

(ii) W, the weight attached to the point C of the disc, acting vertically downwards,

and (iii) the reaction at the point of suspension A.

Let $\angle AOC = \theta$ and $\angle COB = \phi$.

$$\therefore \quad \angle AOL = \pi - (\theta + \phi).$$

To avoid the reaction at A, taking moment of the forces about the point A, we have

$$nW.AL = W.AM$$

or
$$n.AL = CK = CN - NK = CN - AL$$

or
$$(n+1)\, AL = CN$$

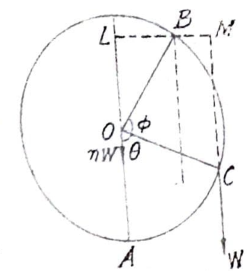
or
$$(n+1)\, a \sin\{\pi - (\theta+\phi)\} = a \sin\phi,$$

where $OA = OC = a =$ the radius of the circle

or
$$(n+1)\sin(\theta+\phi) = \sin\phi. \qquad \ldots(1)$$

Now when the disc is suspended from B, then A is the lowest point. [Fig. (ii)]

In this case also the disc is in equilibrium under the action of three forces, weights nW and W at O and C respectively and the reaction at the point B.



(Fig. 2·17 (ii))

Taking moments about the point B and proceeding as in the preceding case, we get

$$(n+1)\sin(\phi+\theta) = \sin\theta \qquad \ldots(2)$$

From (1) and (2), we have

$$\sin\theta = \sin\phi.$$

$$\therefore \quad \theta = \phi.$$

Substituting in (1), we have

$$(n+1)\sin 2\theta = \sin\theta$$

or
$$2(n+1)\sin\theta\cos\theta - \sin\theta = 0$$

or
$$\sin\theta\{2(n+1)\cos\theta - 1\} = 0$$

or
$$2(n+1)\cos\theta - 1 = 0$$

[$\because$ if $\sin\theta = 0$, then $\theta = 0$, which is impossible]

or
$$\sec\theta = 2(n+1)$$

or
$$\theta = \sec^{-1}\{2(n+1)\}.$$

$\therefore$ the required angle $AOB = \theta + \phi = 2\theta = 2\sec^{-1}\{2(n+1)\}$.

Ex. 15. *A rod is movable in a vertical plane about a smooth hinge at one end, and at the other end is fastened a weight $W/2$, the weight of the rod being W. This end is fastened by a string of length l to a point at a height c vertically over the hinge. Show that the tension of the string is lW/c.*

Sol. Let a rod AB of length $2a$ (say) be movable in a vertical plane about a smooth hinge at the end A. A weight $W/2$ is attached at the other end B of the rod and this end is fastened by a string BC of length l to a point C at a height $AC = c$ vertically over the hinge at A. The rod is in equilibrium under the action of the following forces :

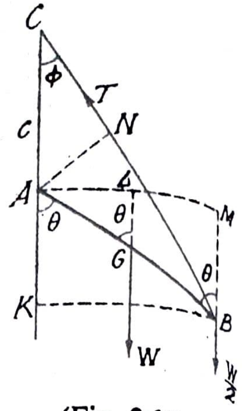
(i) W, weight of the rod at its mid-point G, acting vertically down-wards,

(ii) $W/2$, weight attached at the end B, acting vertically down-wards,

(iii) T, tension in the string along BC,

and (iv) the reaction at the hinge at A.

Let θ and ϕ be the angles of inclination of the rod and the string respectively to the vertical.



(Fig. 2·18)

To avoid reaction at A, taking moments about the point A, we have

$$T.AN = W\ AL + \tfrac{1}{2}W\ AM$$

or $$T.AC \sin\phi = W.AG \sin\theta + \tfrac{1}{2}W.AB \sin\theta$$

or $$T.c \sin\phi = W.a \sin\theta + \tfrac{1}{2}W.2a \sin\theta \quad [\because\ AB = 2a]$$

or $$T = W\frac{2a \sin\theta}{c \sin\phi}. \quad \ldots(1)$$

Now from the $\triangle CBK$, $BK = BC \sin\phi = l \sin\phi$

and from the $\triangle ABK$, $BK = AB \sin\theta = 2a \sin\theta$.

$$\therefore\ l \sin\phi = 2a \sin\theta. \quad \ldots(2)$$

$\therefore$ from (1) and (2), we get

$$T = Wl/c.$$

Ex. 16. *A beam of weight W, is divided by its centre of gravity G into two portions AG and GB, whose lengths are a and b respectively. The beam rests in a vertical plane on a smooth floor AC and against a smooth wall CB. A string is attached to a hook at C and to the beam at a point D. If T be the tension of the string, and α and β be the inclinations of the beam and string respectively to the horizon, show that*

$$T = W\frac{a \cos\alpha}{(a+b)\sin(\alpha-\beta)}.$$

Sol. Let G be the centre of gravity of a beam AB resting on a smooth floor AC and against a smooth vertical wall CB. We have $AG = a$ and $BG = b$. CD is a string attached to a hook at C and to a point D of the beam.

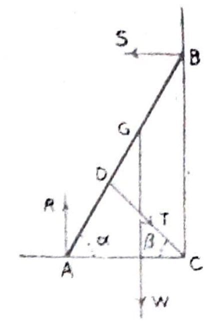
The beam AB is in equilibrium under the action of the following forces :

(i) the reaction R of the floor at A acting vertically upwards.

(ii) the reaction S of the wall at B acting perpendicular to the wall.

(iii) the weight W of the beam acting vertically down-wards through its centre of gravity G,

and (iv) the tension T in the string acting along DC.



(Fig. 2·19)

Resolving the forces horizontally, we have

$$S = T\cos\beta. \quad \ldots(1)$$

Also taking moments about A, we have

$$S.AB \sin\alpha = W.AG \cos\alpha + T.AC \sin\beta. \quad \ldots(2)$$

[Note that the length of the perpendicular drawn from A to DC is $AC \sin\beta$].

Substituting the value of S from (1) in (2), we have

$$T\cos\beta\,(a+b)\sin\alpha = W.a\cos\alpha + T(a+b)\cos\alpha\sin\beta.$$

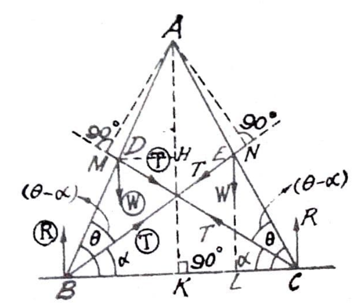
$$[\because\ AC = AB\cos\alpha = (a+b)\cos\alpha]$$

$$\therefore\ T(a+b)(\sin\alpha\cos\beta - \cos\alpha\sin\beta) = W.a\cos\alpha,$$

or $$T = W\frac{a\cos\alpha}{(a+b)\sin(\alpha-\beta)}.$$

Ex. 17. *Two equal beams AB and AC connected by a hinge at A, are placed in a vertical plane with their extremities B and C resting on a smooth horizontal plane; they are kept from falling by strings connecting B and C with the middle points of the beams. Show that the ratio of the tension of each string to the weight of the beam is $\frac{1}{8}\sqrt{(1+9\cot^2\theta)}$, θ being the inclination of each beam to the horizontal.*

Sol. Let the two beams AB and AC hinged at A be placed in a vertical plane with their extremities B and C resting on a smooth horizontal plane. The points B and C are connected to the middle points E and D of the opposite beams by strings BE and CD. Let T be the tension in each of these strings. Let W be the weight of each beam. The weights of the



(Fig. 2·20)

beams AB and AC act vertically down-wards through their middle points D and E respectively. By symmetry the reactions of the floor at B and C will be equal, say each equal to R. These reactions act at right angles to the floor.

Considering the combined equilibrium of the beams AB and AC and resolving the forces vertically, we have

$$2R=2W \quad \text{or} \quad R=W. \qquad \ldots(1)$$

[Note that while considering the combined equilibrium of the beams AB and AC the mutual action and reaction at the hinge A and the tensions in the strings BE and CD will not be considered because all these forces being mutually equal and opposite will balance each other].

Given that $\quad ABC=\angle ACB=\theta.$

Let $\quad \angle EBC=\angle DCB=\alpha.$

Then $\quad \angle ECD=\angle DBE=\theta-\alpha.$

Let $\quad AB=AC=2a.$

From $\triangle ELC$, we have

$$EL=EC \sin\theta=a\sin\theta.$$

Also $BL=\frac{3}{4}BC=\frac{3}{4}\, 4a\cos\theta=3a\cos\theta.$

[Note that $BC=2.2a\cos\theta=4a\cos\theta$]

$\therefore$ from $\triangle BLE$, we have

$$\cot\alpha=\frac{BL}{EL}=\frac{3a\cos\theta}{a\sin\theta}=3\cot\theta. \qquad \ldots(2)$$

Now we shall consider the equilibrium of the beam AB alone. The forces acting on this beam have been enclosed within circles in the figure and are as given below :

(i) the weight W of the beam acting vertically downwards through its middle point D,

(ii) the reaction R of the floor at B,

(iii) the tension T in the string BE acting at B along BE,

(iv) the tension T in the string DC acting at D along DC,

and (v) the reaction of the hinge at A, not shown in the figure.

To avoid the reaction of the hinge at A, taking moments about A, we have

$$T.AN+T.AM+W.DH-R.BK=0$$

or $\quad T.AB\sin(\theta-\alpha)+T.AC\sin(\theta-\alpha)+W.AD\cos\theta-R.AB\cos\theta=0$

or $\quad T.2a\sin(\theta-\alpha)+T.2a\sin(\theta-\alpha)+W.a\cos\theta-R.2a\cos\theta=0$

or $\quad 4T\sin(\theta-\alpha)+W\cos\theta-2W\cos\theta=0$

[$\because$ $R=W$, from (1)]

or $\quad T=\dfrac{W\cos\theta}{4\sin(\theta-\alpha)}=\dfrac{W\cos\theta}{4(\sin\theta\cos\alpha-\cos\theta\sin\alpha)}$

$=\dfrac{W\cot\theta.\operatorname{cosec}\alpha}{4(\cot\alpha-\cot\theta)}$,

dividing the Nr. and Dr. by $\sin\alpha\sin\theta$

$=\dfrac{W\sqrt{(1+\cot^2\alpha)}\cot\theta}{4(\cot\alpha-\cot\theta)}$.

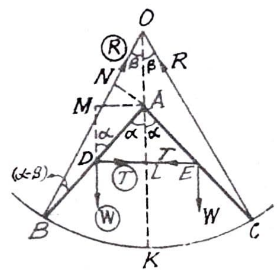
Substituting $\cot\alpha=3\cot\theta$, from (2), we have

$$T=\frac{W\sqrt{(1+9\cot^2\theta)}.\cot\theta}{4(3\cot\theta-\cot\theta)}=\frac{W}{8}\sqrt{(1+9\cot^2\theta)}.$$

$\therefore \quad T/W=\frac{1}{8}\sqrt{(1+9\cot^2\theta)}.$

Ex. 18. *Two equal uniform rods AB, AC each of weight W are freely joined at A and rest with the extremities B and C on the inside of a smooth circular hoop, whose radius is greater than the length of either rod, the whole being in a vertical plane and the middle points of the rods being joined by a light string, show that if the string is stretched, its tension is W* ($\tan\alpha-2\tan\beta$), *where* 2α *is the angle between the rods, and* β *the angle either rod subtends at the centre.* **[Meerut 70(S)]**

Sol. Let the two rods AB and AC each of weight W, freely jointed at A, be placed with the extremities B and C on the inside of a smooth circular hoop of centre O. The middle points D and E of the rods are connected by a string. Let T be the tension in the string. Also let R and R be the equal reactions at B and C passing through the centre O of the hoop.



(Fig. 2·21)

In the equilibrium position, the forces acting on the system are as follows :

(i) The weights W and W of the rods acting vertically downwards at the middle points D and E,

(ii) the tension T in the string at both ends D and E in inward directions.

(iii) the reactions R and R at B and C, passing through O, the centre of the hoop, and (iv) the mutual action and reaction of the hinge at A not shown in the figure.

Given that $\angle BAK = \angle CAK = \alpha$

and $\angle BOK = \angle COK = \beta$. $\therefore \angle ABO = \alpha - \beta$,

Considering the combined equilibrium of the rods AB and AC and resolving the forces vertically, we have

$$R \cos \beta + R \cos \beta = W + W,$$

or $$R = \frac{W}{\cos \beta}. \qquad \ldots(1)$$

Now we shall consider the equilibrium of the rod AB alone. Except for the reaction of the hinge at A, the forces acting on the rod AB have been shown in the figure by enclosing them within circles. Taking moments about A, we have

$$T.AL + W.AM = R.AN$$

or $$T.AD \cos \alpha + W.AD \sin \alpha = \frac{W}{\cos \beta}.AB \sin (\alpha - \beta)$$

$$\left[\because R = \frac{W}{\cos \beta}, \text{ from } (1)\right]$$

or $$T.\frac{AB}{2} \cos \alpha + W.\frac{AB}{2} \sin \alpha = \frac{W}{\cos \beta}.AB \sin (\alpha - B)$$

$$\left[\because AD = \frac{AB}{2}\right]$$

or $$T \cos \alpha = 2W \frac{\sin (\alpha - \beta)}{\cos \beta} - W \sin \alpha$$

or $$T = W.\frac{2 \sin (\alpha - \beta) - \sin \alpha \cos \beta}{\cos \alpha \cos \beta}$$

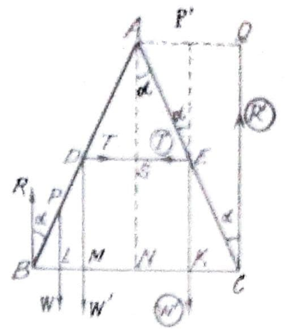
$$= W\frac{2 (\sin \alpha \cos \beta - \cos \alpha \sin \beta) - \sin \alpha \cos \beta}{\cos \alpha \cos \beta}$$

$$= W.\frac{\sin \alpha \cos \beta - 2 \cos \alpha \sin \beta}{\cos \alpha \cos \beta}$$

$$= W (\tan \alpha - 2 \tan \beta).$$

Ex. 19. *A step ladder in the form of the letter A with each of its legs inclined at an angle α to the vertical is placed on a horizontal floor and is held up by a cord connecting the middle points of its legs, there being no friction anywhere. Show that when a weight W is placed on one of the steps at a height from the floor equal to $1/n$ of the ladder, the increase in the tension of the cord is*

$(1/n)\ W \tan \alpha$.



(Fig. 2·22)

Sol. Let AB and AC be the two legs each of weight W' of the step ladder placed on a horizontal plane. The middle points D and E are connected by a cord DE. Let T be the tension in the cord acting at both ends D and E in inward directions, when a weight W is placed at the point P of the leg AB, such that

$$PL = \frac{1}{n} AN.$$

In the equilibrium position, the forces acting on the system are as follows :

(i) weights of the two legs W' and W' acting vertically downwards through their middle points D and E.

(ii) the tension T in the cord DE acting at both ends D and E in inward directions.

(iii) the reactions R and R' of the floor at the points B and C respectively acting at right angles to the floor,

(iv) the weight W placed at the point P on the leg Ab.

and (v) the mutual action and reaction at the hinge A not shown in the figure.

From the similar triangles PBL and ABN, we have

$$\frac{BL}{BN} = \frac{PL}{AN} = \frac{1}{n} \qquad \left[\because PL = \frac{1}{n} AN\right]$$

or $$BL = \frac{1}{n} BN = \frac{1}{2n} BC \qquad [\because BC = 2BN]$$

Since D and E are the middle points of AB and AC which are equal, therefore

$BM = MN = NK = KC$. Thus, we have

$$BM = \frac{1}{4} BC \quad \text{and} \quad BK = \frac{3}{4} BC.$$

Taking moments of all the forces acting on the whole ladder about the point B, we have

$$R'\ BC = W.BL + W'\ BM + W'.BK$$

(the moments of T and T at D and E are equal but with opposite signs and so they cancel each other, also the moments of the action and reaction of the hinge at A cancel each other), or $\quad R'.BC = W\frac{1}{2n}BC + W'.\frac{1}{4}BC + W'.\frac{3}{4}BC$

or $\quad R' = \frac{1}{2n}W + W'$. ...(1)

Now taking moments of the forces acting on the leg AC only (which are enclosed within circles in the figure) about the point A, we have

$$T.AS + W'.AP = R'.AQ$$

or $\quad T.AE\cos\alpha + W'.AE\sin\alpha = \left(\frac{W}{2n} + W'\right) AC\sin\alpha,$ [from (1)]

or $\quad T.\frac{AC}{2}\cos\alpha + W'.\frac{AC}{2}\sin\alpha = \left(\frac{W}{2n} + W'\right) AC\sin\alpha$ [$\because$ $AE = \frac{1}{2}AC$] ...(2)

or $\quad T = (W/n + W')\tan\alpha.$

If T' is the tension in the chord DE, when there is no weight W placed at the point P, then putting $T = T'$ and $W = 0$ in (2), we get

$$T' = W'\tan\alpha.$$

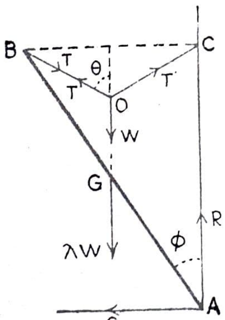
$\therefore$ The required increase in the tension of the cord

$$= T - T' = (W/n + W')\tan\alpha - W'\tan\alpha = (1/n)\,W\tan\alpha.$$

Ex. 20. *AB is a straight rod, of length 2a and weight λW, resting with the lower end A on the ground at the foot of a vertical wall AC, B and C at the same vertical height 2b above A. A heavy ring, of weight W, is free to move along a string of length 2l, which joins B and C. If the system be in equilibrium with the ring at the middle point of the string, show that* $l^2 = a^2 - b^2\,\frac{\lambda(\lambda+2)}{(\lambda+1)^2}$.

Sol. Let AB be a rod of length $2a$ and weight λW, resting with the lower end A on the ground at the foot of the vertical wall AC. Let the ring of weight W be in equilibrium at the middle point O of the string BOC of length $2l$ joining B and C.

If G is the middle point of the rod, then the vertical line through G parallel to the wall bisects BC and by symmetry also passes through O.



We have $OB = OC = l$ and $AG = BG = a$. (Fig. 2·23)

Let T be the tension in the string and R, S the vertical and horizontal components of the reaction at A. Also let

$$\angle BOC = 2\theta \quad \text{and} \quad \angle BAC = \phi.$$

Considering the equilibrium of the ring at O and resolving the forces vertically, we have

$$2T\cos\theta = W. \quad ...(1)$$

Now we shall consider the equilibrium of the rod AB. The forces acting on the rod AB are :

(i) its weight λW at G,

(ii) the tension T at B along BO, and

(iii) the reaction at A.

To avoid the reaction at A, we shall take moments of the forces acting on the rod above the point A.

We have

$$\angle ABO = \angle ABC - \angle OBC = (\tfrac{1}{2}\pi - \phi) - (\tfrac{1}{2}\pi - \theta) = \theta - \phi.$$

The length of the perpendicular drawn from A to BO produced

$$= AB\sin(\theta - \phi) = 2a\sin(\theta - \phi).$$

Also the length of the perpendicular drawn from A to the vertical line through $G = AG\sin\phi = a\sin\phi$.

Therefore taking moments of the forces acting on the rod AB about the point A, we have

$$T.2a\sin(\theta - \phi) = \lambda W\,a\sin\phi$$

or $\quad 2T\sin(\theta - \phi) = \lambda.2T\cos\theta\sin\phi$ [$\because$ from (1), $W = 2T\cos\theta$]

or $\quad \sin(\theta - \phi) = \lambda\cos\theta\sin\phi$ [$\because$ $T \neq 0$]

or $\quad \sin\theta\cos\phi - \cos\theta\sin\phi = \lambda\cos\theta\sin\phi$

or $\quad \sin\theta\cos\phi = (\lambda + 1)\cos\theta\sin\phi.$ ...(2)

Now from $\triangle OBC$, we have $BC = 2l\sin\theta$ and from $\triangle ABC$, we have $BC = 2a\sin\phi$.

$$\therefore \quad l\sin\theta = a\sin\phi.$$

Also $\quad 2b - AC = AB\cos\phi = 2a\cos\phi.$

so that $\quad a\cos\phi = b.$

Pntting $\sin\theta = (a/l)\sin\phi$ in (2), we have

$$(a/l)\sin\phi\cos\phi = (\lambda + 1)\cos\theta\sin\phi$$

or $\quad a\cos\phi = l(\lambda + 1)\cos\theta$ [$\because$ $\sin\phi \neq 0$]

or $\quad b = l(\lambda + 1)\cos\theta.$ [$\because$ $a\cos\phi = b$]

$\therefore \quad b^2 = (\lambda+1)^2 l^2\cos^2\theta = (\lambda+1)^2 l^2(1 - \sin^2\theta)$

$\quad = (\lambda+1)^2(l^2 - l^2\sin^2\theta)$

$\quad = (\lambda+1)^2(l^2 - a^2\sin^2\phi)$ [$\because$ $l\sin\theta = a\sin\phi$]

$\quad = (\lambda+1)^2[l^2 - a^2(1 - \cos^2\phi)]$

$$=(\lambda+1)^2\,[l^2-a^2+a^2\cos^2\phi]$$
$$=(\lambda+1)^2\,[l^2-a^2+b^2]$$

$$\therefore \quad \frac{b^2}{(\lambda+1)^2}=l^2-a^2+b^2$$

or $$l^2=a^2-b^2+\frac{b^2}{(\lambda+1)^2}=a^2-b^2\left[1-\frac{1}{(\lambda+1)^2}\right]$$

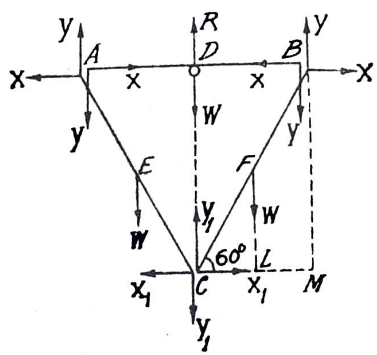
$$=a^2-b^2\,\frac{\lambda^2+2\lambda}{(\lambda+1)^2}=a^2-b^2\,\frac{\lambda\,(\lambda+2)}{(\lambda+1)^2}.$$

Ex. 21. *Three equal uniform rods each of weight W; are smoothly joined so as to form an equilateral triangle. If the system be supported at the middle point of one of the rods, show that the action at the lowest angle is* $\sqrt{3}\,W/6$ *and that at each of the others is*

$$W\sqrt{\left(\frac{13}{12}\right)}.$$

Also find the reaction at the support. **[Meerut 67, 77]**

Sol. Let three equal uniform rods AB, BC, CA each of weight W and length $2a$ be joined smoothly at A, B, C so as to form an equilateral triangle ABC. The weights of the rods will act vertically downwards at their middle points D, E and F. The system is supported at the middle point D of the rod AB. Let R be the reaction of the support at D acting perpendicular to AB. Since the ends A and B are in symmetrical positions, therefore the reactions on these ends will be equal. Let X and Y be the horizontal and vertical components of the reactions at A and B. Also let X_1 and Y_1 be the horizontal and vertical components of the reaction at C.



(Fig. 2·24)

All the forces acting on the system are shown in the figure.

For the equilibrium of the rod AB, resolving vertically the forces acting on it, we have

$$R=Y+Y+W,$$

or $$R=2Y+W. \quad \ldots(1)$$

For the equilibrium of the rod AC, resolving horizontally and vertically the forces acting on it, we have

$$X+X_1=0, \quad \ldots(2)$$

and $$Y_1+W=Y. \quad \ldots(3)$$

For the equilibrium of the rod BC, resolving vertically the forces acting on it, we have

$$Y+Y_1=W. \quad \ldots(4)$$

Taking moments about C, of the forces acting on the rod BC only, we have

$$W.CL+X.CD=Y.CM$$

or $$W\,CF\cos 60^\circ+X.BC\sin 60^\circ=Y.BC\cos 60^\circ$$

or $$W\frac{a}{2}+X.2a.\frac{\sqrt{3}}{2}=Y.2a.\tfrac{1}{2}$$

or $$W+2\sqrt{3}X=2Y. \quad \ldots(5)$$

Solving (3) and (4), we get $Y=W$ and $Y_1=0$,

$\therefore$ from (5), $2\sqrt{3}X=2W-W=W$ or $X=\dfrac{W\sqrt{3}}{6}$.

Then from (2), $X_1=-X=-\dfrac{W\sqrt{3}}{6}$.

$\therefore$ Reaction at A or B

$$=\sqrt{(X^2+Y^2)}=\left(\frac{1}{12}W^2+W^2\right)=W\sqrt{\left(\frac{13}{12}\right)}$$

and reaction at $C=\sqrt{(X_1^2+Y_1^2)}=\dfrac{\sqrt{3}}{6}W$.

Substituting $Y=W$ in (1), the reaction at the support at D
$$=R=2W+W=3W.$$

§ 2·8. Equilibrium of a rigid body under the action of three forces only.

Theorem. *If three forces acting upon a rigid body keep it in equilibrium, they must either meet in a point or be parallel.*

[Meerut 83; Raj. TDC 79, 80(S); Gorakhpur 70]

Proof. Let the three forces **P**, **Q** and **R** acting at the points A, B and C respectively of a rigid body, keep it in equilibrium.

Since the forces are in equilibrium, therefore

$$\mathbf{P}+\mathbf{Q}+\mathbf{R}=0. \quad \ldots(1)$$

Also the sum of the moments of the forces about any point must be zero.

$\therefore$ taking moments of the forces about the point A, we have

$$\vec{AB}\times\mathbf{Q}+\vec{AC}\times\mathbf{R}=0,$$

$$\vec{AB}\times Q=-\vec{AC}\times R$$

or $$\vec{AB}\times Q=\vec{CA}\times R=n \text{ (say)}.$$

or Thus n is a vector perpendicular to $\vec{AB}$, Q, $\vec{CA}$ and R, *i.e.*, the vector n is perpendicular to the plane ABC and the vectors Q and R passing through the points B and C respectively. Hence the forces Q and R must act in the plane ABC.

Since the forces Q and R are coplanar, hence their lines of action must either intersect or be parallel.

Case I. If the forces Q and R intersect, then for equilibrium, their resultant must be equal and opposite to the third force P. Thus the third force P must also act in the plane ABC and pass through the point of intersection of the forces Q and R. Hence the three forces are coplanar and concurrent.

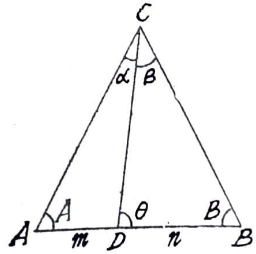
Case II. If the forces Q and R are parallel then their resultant is also parallel to them and for equilibrium it must be equal and opposite to the third force P. Hence the three forces are coplanar and parallel. Hence the theorem.

§ 2·9. Two Important Trigonometrical Theorems.

If a line CD be drawn through the vertex C of a triangle ABC meeting the opposite side AB in D and dividing it into two parts m and n and the angle C into two parts α *and* β, *and if* $\angle CDB=\theta$, *then*

(*i*) $(m+n)\cot\theta$
$=m\cot\alpha-n\cot\beta$

and (*ii*) $(m+n)\cot\theta$
$=n\cot A-m\cot B.$



(Fig. 2·25)

Illustrative Examples

Ex. 22. *A heavy rod AB rests with its ends on two smooth inclined planes which face each other and are inclined at angles* α *and* β *to the horizontal. The centre of gravity of the rod divides it into two parts a and b. Find the inclination of the rod to the horizontal and the reactions of the planes.*

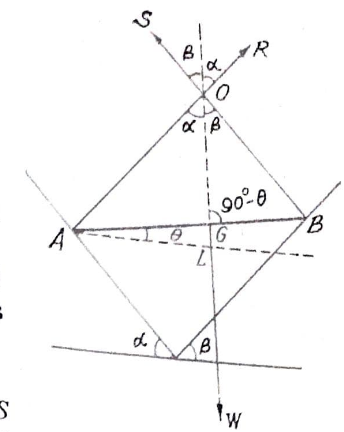
Sol. Let AB be a rod resting with its ends A and B on two smooth inclined planes inclined at angles α and β to the horizontal.

The rod is in equilibrium under the action of the following three forces only :

(i) R, the reaction of the inclined plane at A acting at right angles to this plane,

(ii) S, the reaction of the inclined plane at B acting at right angles to this plane,

and (iii) W, the weight of the rod acting vertically down-wards through its centre of gravity G, where $AG=a$ and $BG=b$.



(Fig. 2·26)

Since the two forces R and S meet at O, therefore the line of action of the third force W must also pass through O as shown in the figure.

Thus the line OG is vertical.

Now the angle between two lines is equal to the angle between their perpendicular line. Since AO is perpendicular to the inclined plane and OG is perpendicular to the vertical, therefore

$$\angle AOG=\alpha.$$

Similarly $$\angle BOG=\beta.$$

If the rod makes an angle θ with the horizontal, then

$$\angle OGB=\angle AGL=90°-\theta,$$

where AL is the horizontal line through A.

Now applying the well known theorem of trigonometry for the $\triangle OAB$, we have

$$(AG+BG)\cot\angle OGB=AG\cot\angle AOG-BG\cot\angle BOG$$

or $$(a+b)\cot(90°-\theta)=a\cot\alpha-b\cot\beta$$

or $$(a+b)\tan\theta=a\cot\alpha-b\cot\beta.$$

$$\therefore\quad \theta=\tan^{-1}\left[\frac{a\cot\alpha-b\cot\beta}{a+b}\right].$$

Again, by Lami's theorem at the point O, we have

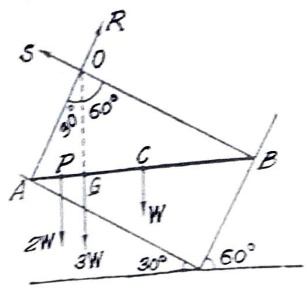
$$\frac{R}{\sin(180°-\beta)}=\frac{W}{\sin(\alpha+\beta)}=\frac{S}{\sin(180°-\alpha)}$$

or $$\frac{R}{\sin\beta}=\frac{W}{\sin(\alpha+\beta)}=\frac{S}{\sin\alpha}.$$

$$\therefore \quad R=\frac{W \sin \beta}{\sin (\alpha+\beta)} \quad \text{and} \quad S=\frac{W \sin \alpha}{\sin (\alpha+\beta)}.$$

Ex. 23. *A uniform beam rests with its ends on two smooth inclined plane which make angles of 30° and 60° with the horizontal respectively. A weight equal to twice that of the beam can slide along its length. Find the position of the sliding weight when the beam rests in a horizontal position.* **[Meerut 76(P)]**

Sol. Let the beam AB rest in a horizontal position with its ends A and B on two smooth inclined planes inclined at angles 30° and 60° with the horizontal respectively. In the equilibrium position let P be the position of the sliding weight $2W$, whereas W is the weight of the beam acting at its middle point C. Let G be the point where the resultant $3W$ of the weights W of the beam at C and $2W$ at P, acts.



(Fig. 2·27)

The reactions R and S of the inclined planes act perpendicular to the planes at A and B and meet at O.

For equilibrium the vertical line of action of the resultant weight $3W$ at G will also pass through O as shown in the figure and so the line OG will be vertical.

Clearly $\angle AOG=30°$, $\angle BOG=60°$ and OG is perpendicular to AB which is horizontal.

Now in $\quad \Delta AOG,\ AG=OG \tan 30°=OG/\sqrt{3}$

and in $\quad \Delta BOG,\ BG=OG \tan 60°=\sqrt{3}OG.$

$$\therefore \quad \frac{BG}{AG}=\frac{3}{1} \quad \text{or} \quad \frac{BG}{AG}+1=3+1 \quad \text{or} \quad \frac{BG+AG}{AG}=4$$

or $$\frac{AB}{AG}=4 \quad \text{or} \quad AG=\tfrac{1}{4}AB. \qquad \ldots(1)$$

But the resultant of the weight W at C and $2W$ at P act at G. So taking moments about A, we have

$$AG=\frac{W.AC+2W.AP}{W+2W}$$

or $$AG=\tfrac{1}{3}\left(\tfrac{1}{2}AB+2AP\right). \qquad \ldots(2)$$

From (1) and (2), we have

$$\tfrac{1}{3}\left(\tfrac{1}{2}AB+2AP\right)=\tfrac{1}{4}AB$$

or $$\tfrac{1}{2}AB+2AP=\tfrac{3}{4}AB$$

or $$2AP=\left(\tfrac{3}{4}-\tfrac{1}{2}\right) AB=\tfrac{1}{4}AB$$

or $$AP=\tfrac{1}{8}AB.$$

$$\therefore \quad BP=AB-AP=AB-\tfrac{1}{8}AB=\tfrac{7}{8}AB.$$

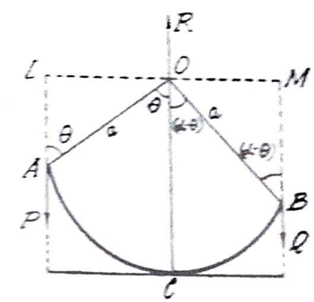
$$\therefore \quad \frac{AP}{BP}=\frac{1}{7}.$$

Hence the beam will be in equilibrium in the horizontal position on the inclined planes when the sliding weight is at the point which divides the beam in the ratio 1 : 7.

Ex. 24. *A rigid wire, without weight, in the form of the arc of a circle subtending an angle α at its centre and having two weights P and Q at its extremities, rests with its concavity downwards, upon a smooth horizontal plane. Show that, if θ be the inclination to the vertical of the radius to the end at which P is suspended, then*

$$\tan \theta=\frac{Q \sin \alpha}{P+Q \cos \alpha}.$$

Sol. Let ACB be a rigid wire without weight, in the form of the arc of a circle of radius a subtending an angle α at the centre *i.e.*, $\angle AOB=\alpha$. Let C be the point of contact of the wire with the smooth horizontal plane and P and Q be the two weights at the ends A and B respectively.



(Fig. 2·28)

The wire is in equilibrium under the action of following three forces :

(i) P, the weight at A, acting vertically downwards,

(ii) Q, the weight at B, acting vertically downwards,

and (iii) R, the reaction at the point of contact C, acting at right angles to the horizontal plane.

The line of action of the reaction R will pass through the centre O of the circle.

Given that $\angle AOC=\theta$; $\therefore \ \angle BOC=\alpha-\theta$.

To avoid the reaction R, taking moments of all the forces about O, we have

$$P.OL=Q.OM$$

or $$P\, a \sin \theta=Q.a \sin (\alpha-\theta),$$

$$P \sin \theta = Q (\sin \alpha \cos \theta - \cos \alpha \sin \theta),$$

or $$(P + Q \cos \alpha) \sin \theta = Q \sin \alpha \cos \theta.$$

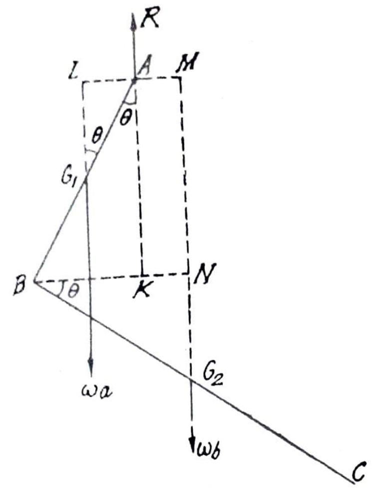
or $$\therefore \tan \theta = \frac{Q \sin \alpha}{P + Q \cos \alpha}.$$

Ex. 25. *Two uniform rods AB, BC rigidly joined at B so that angle ABC is a right angle, hang freely in equilibrium from a fixed point A. The lengths of the rods are a and b, and their weights are wa and wb. Prove that if AB makes an angle θ with the vertical*

$$\tan \theta = b^2/(a^2 + 2ab).$$

[Raj. T. D. C. 80; Jiwaji 72; Meerut 86]

Sol. Let AB and BC be two uniform rods of length a and b, rigidly joined at B so that $\angle ABC = 90°$. Let the rods hang freely in equilibrium from the fixed point A.



(Fig. 2·29)

The rods will be in equilibrium under the action of the following three forces only.

(i) wa, the weight of the rod AB acting vertically downwards at its middle point G_1,

(ii) wb, the weight of the rod BC acting vertically downwards at its middle point G_2,

and (iii) R, the reaction at the fixed point A.

Since the two forces wa and wb are parallel, therefore the third force R will also be parallel to them *i.e.*, the reaction R at A will act in the vertical direction.

Given that $\angle BAK = \theta$; $\therefore$ $\angle AG_1L = \theta$.

Also $\angle NBC = \angle ABC - \angle ABN = 90° - (90° - \theta) = \theta$.

To avoid the reaction R, taking moments of all the forces about the point A, we have

$$wa . AL = wb . AM$$

or $$a . AL = b . KN$$

or $$a . AL = b\,(BN - BK) \qquad [\because AM = KN]$$

or $$a . AG_1 \sin \theta = b\,(BG_2 \cos \theta - AB \sin \theta)$$

or $$a . \tfrac{1}{2}a \sin \theta = b\,(\tfrac{1}{2}b \cos \theta - a \sin \theta)$$

or $$(a^2 + 2ab) \sin \theta = b^2 \cos \theta$$

or $$\tan \theta = b^2/(a^2 + 2ab)$$

Ex. 26. *A uniform bar AB, of weight $2W$ and length a, is free to turn about a smooth hinge at its upper end A, and a horizontal force is applied to the end B so that bar is in equilibrium with B at a distance l from the vertical through A. Prove that the reaction at the hinge is equal to*

$$W\,[(4a^2 - 3l^2)/(a^2 - l^2)]^{1/2}.$$

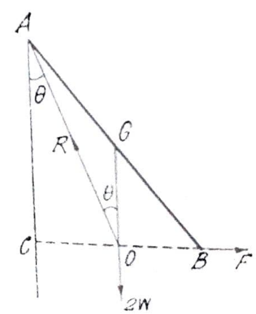
[Meerut 76 (P), 87]

Sol. The uniform bar AB of weight $2W$ and length a, free to turn about a smooth hinge at its upper end A, is in equilibrium under the action of the following forces :

(i) W, the weight of the bar AB acting vertically downwards at its middle point,

(ii) F, the horizontal force at the end B,

and (iii) R, the reaction at the hinge A.



(Fig. 2·30)

Since the bar is in equilibrium under the action of three forces only and two forces F and $2W$ meet at O, therefore the third force R must also pass through O, as shown in the figure.

Given that $AB = a$, $BC = l$.

Since G is the middle point of AB and GO is parallel to AC, therefore O is also the middle point of BC.

Let $\angle OAC = \theta$; then $\angle AOG = \theta$.

In the triangle OAC,

$$\tan \theta = \frac{OC}{AC} = \frac{\tfrac{1}{2}BC}{\sqrt{(AB^2 - BC^2)}} = \frac{l}{2\sqrt{(a^2 - l^2)}}.$$

By Lami's theorem at the point O, we have

$$\frac{R}{\sin 90^\circ}=\frac{2W}{\sin (90^\circ+\theta)}=\frac{F}{\sin (180^\circ-\theta)}$$

$\therefore \quad R=\dfrac{2W}{\cos\theta}=2W\sec\theta=2W\sqrt{(1+\tan^2\theta)}$

or $\quad R=2W\sqrt{\left[1+\dfrac{l^2}{4(a^2-l^2)}\right]} \quad \left[\because \tan\theta=\dfrac{l}{2\sqrt{(a^2-l^2)}}\right]$

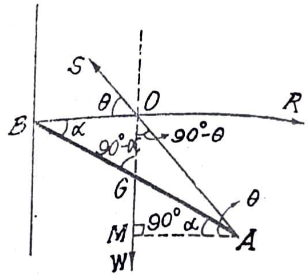
or $\quad R=W\,[(4a^2-3l^2)/(a^2-l^2)]^{1/2}$.

Ex. 27. *A uniform rod AB movable about a hinge at A rests with one end in contact with a smooth wall. If α be the inclination of the rod to the horizontal, show that the reaction at the hinge is*

$$\tfrac{1}{2}W\sqrt{(3+\operatorname{cosec}^2\alpha)}$$

and that it makes an angle $\tan^{-1}(2\tan\alpha)$ with the horizontal.

Sol. Let a uniform rod AB movable about the hinge at the end A rest with the end B in contact with a smooth vertical wall. Let W be the weight of the rod and G its middle point.



(Fig. 2·31)

The rod is in equilibrium under the action of the following three forces only :

(i) R, the reaction of the wall at B acting at right angles to the wall,

(ii) S, the reaction of the hinge at A,

and (iii) W, the weight of the rod acting vertically downwards at its middle point G

Since the force R and the line of action of W meet at O, therefore the reaction S of the hinge at A must also pass through O, as shown in the figure.

Let the rod AB and the reaction S make angles α and θ respectively with the horizontal, *i.e.*,

$$\angle ABO=\alpha \text{ and } \angle OAM=\theta.$$

$$\therefore \quad \angle OGB=90^\circ-\alpha \text{ and } \angle AOM=90^\circ-\theta.$$

In $\triangle OAB$, by the trigonometrical theorem, we have

$$(AG+BG)\cot OGB=AG\cot AOG-BG\cot BOG$$

or $\quad (a+a)\cot(90^\circ-\alpha)=a\cot(90^\circ-\theta)-a\cot 90^\circ,$

(where $AG=BG=a$, say)

or $\quad 2a\tan\alpha=a\tan\theta,$

or $\quad \tan\theta=2\tan\alpha \quad \ldots(1)$

$\therefore$ the reaction S at the hinge makes an angle

$$\theta=\tan^{-1}(2\tan\alpha)$$

with the horizontal.

Now by Lami's theorem at the point O, we have

$$\frac{S}{\sin 90^\circ}=\frac{W}{\sin(180^\circ-\theta)}=\frac{R}{\sin(90^\circ+\theta)}.$$

$$\therefore \quad S=\frac{W}{\sin\theta}=W\operatorname{cosec}\theta=W\sqrt{(1+\cot^2\theta)}$$

$$=W(1+\tfrac{1}{4}\cot^2\alpha)$$

$[\because \cot\theta=\tfrac{1}{2}\cot\alpha, \text{ from } (1)]$

$$=\tfrac{1}{2}W\sqrt{(4+\cot^2\alpha)}=\tfrac{1}{2}W\sqrt{(3+\operatorname{cosec}^2\alpha)}$$

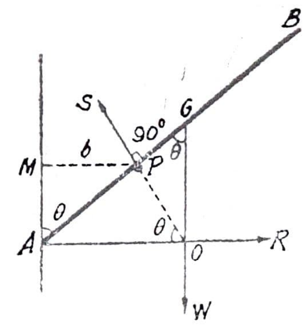
$[\because 1+\cot^2\alpha=\operatorname{cosec}^2\alpha]$.

Ex. 28. *A uniform beam, of length 2a, rests in equilibrium against a smooth vertical wall and upon a peg as a distance b from the wall, show that the inclination of the beam to the vertical is*

$$\sin^{-1}(b/a)^{1/3}.$$

[Gorakhpur 80, 82; Meerut 75 (S), 89 (P)]

Sol. Let AB be a beam of length $2a$ resting in equilibrium with one end A on a smooth vertical wall and a point P upon a peg at a distance b from the wall *i.e.*, $\quad PM=b,\ AB=2a.$



(Fig. 2·32)

The beam is in equilibrium under the action of the following three forces ;

(i) W, the weight of the beam acting vertically downwards at its middle point G.

(ii) R, the reaction at A of the wall, acting at right angles to the wall,

and (iii) S, the reaction at P of the peg, acting at right angles to the beam AB.

Since the two forces R and S meet at O, therefore the third force W must also pass through O, as shown in the figure.

We have, $\angle PAM=\theta$, $\therefore$ $\angle AGO=\theta$ and $\angle AOP=\theta$.

In $\triangle\ APM$, $\sin\theta=\dfrac{PM}{AP}$, ...(1)

in $\triangle\ APO$, $\sin\theta=\dfrac{AP}{AO}$, ...(2)

and in $\triangle\ OAG$, $\sin\theta=\dfrac{AO}{AG}$. ...(3)

Multiplying (1), (2) and (3), we have

$$\sin^3\theta=\frac{PM}{AP}\cdot\frac{AP}{AO}\cdot\frac{AO}{AG}=\frac{PM}{AG}=\frac{b}{a},$$

or $\sin\theta=(b/a)^{1/3}$, or $\theta=\sin^{-1}(b/a)^{1/3}$.

Ex. 29. *A uniform rod, of length a, hangs against a smooth vertical wall being supported by means of a string, of length l, tied to one end of the rod, the other end of the string attached to a point in the wall; show that the rod can rest inclined to the wall at an angle θ given by*

$$\cos^2\theta=\frac{l^2-a^2}{3a^2}.$$

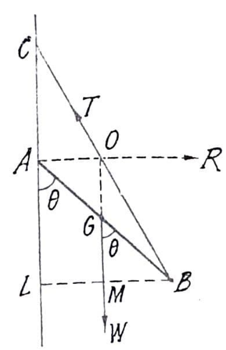
What are the limits of the ratio of a : l that the equilibrium may by possible ?

Sol. Let AB be a rod of weight W and length a whose one end A is resting on a smooth wall and the other end B is attached to point C on the wall by a string BC. Thus the rod is in equilibrium under the action of the following three forces :

(i) W, the weight of the rod acting vertically downwards at its middle point G,

(ii) R, the reaction of the wall acting at the end A, at right angles to the wall,

and (iii) T, the tension in the string along BC.



(Fig. 2·33)

Since the two forces W and R meet at O, therefore the third force T must also pass through O, as shown in the figure.

In $\triangle ABC$, G is the middle point of AB and GO is parallel to AC, hence O is also the middle point of BC.

$\therefore$ $OB=OC=l/2$, $AG=BG=a/2$.

Given that $\angle BAL=\theta$; $\therefore$ $\angle BGM=\theta$.

In $\triangle ABL$, $AL=AB\cos\theta=a\cos\theta$

and in $\triangle BGM$, $BM=BG\sin\theta=\frac{1}{2}a\sin\theta$.

In right angled $\triangle OMB$, we have

$$OB^2=OM^2+BM^2$$

or $$OB^2=AL^2+BM^2$$

or $$(\tfrac{1}{2}l)^2=(a\cos\theta)^2+(\tfrac{1}{2}a\sin\theta)^2$$

or $$l^2=4a^2\cos^2\theta+a^2(1-\cos^2\theta)-a^2+3a^2\cos^2\theta$$

or $$3a^2\cos^2\theta=l^2-a^2.\quad\therefore\ \cos^2\theta=\frac{l^2-a^2}{3a^2}.$$

Second part. The equilibrium is possible, if θ is real,

i.e., if $$0<\cos^2\theta<1,$$

or $$0<\frac{l^2-a^2}{3a^2}<1,$$

or $$0<l^2-a^2<3a^2,$$

or $$a^2<l^2<4a^2,$$

or $$\frac{a^2}{l^2}<1<4\frac{a^2}{l^2}.$$

or $$\frac{a}{l}<1 \text{ and } \frac{a}{l}>\frac{1}{2}.$$

$\therefore$ for equilibrium, $\frac{1}{2}<a/l<1$.

Ex. 30. *One end of a uniform rod of weight W is hinged and the other is tied by a string to a point in the same horizontal line as the hinge. If the rod and the string both be inclined at an angle θ to the horizon, in the position of equilibrium, prove that the reaction of the hinge is* $(W/4)\sqrt{(9+\cot^2\theta)}$.

[Meerut 71, 78, 79 (S), 82, 84 (R), 85, 89P; Raj. T.D.C. 80 (S)]

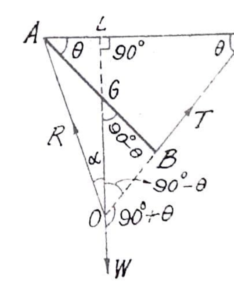
Also find the tension of the string.

Sol. Let AB be a uniform rod of weight W, hinged at the end A and let BC be a string such that A and C are in the same horizontal line. The rod is in equilibrium under the action of following three forces :

(i) W, the weight of the rod acting vertically downwards at its middle point G,

(ii) T, the tension in the string along BC,

and (iii) R, the reaction at the hinge A.



(Fig. 2·34)

Since the two forces W and T meet at O, therefore the reaction R must also pass through O as shown in the figure,

Given that $\angle BAC=\theta=\angle BCA$.

$\therefore \quad \angle BGO=\angle AGL=90^\circ-\theta$.

In the right angled triangle OLC,

$\angle COL=90^\circ-\theta$.

Let $\angle AOL=\alpha$.

In the $\triangle OAB$, by the trigonometrical theorem, we have

$$(AG+GB)\cot OGB=AG\cot AOG-BG\cot BOG,$$

or $$(a+a)\cot(90^\circ-\theta)=a\cot\alpha-a\cot(90^\circ-\theta),\quad \text{where } AG=BG=a$$

or $$2\tan\theta=\cot\alpha-\tan\theta$$

or $$3\tan\theta=\cot\alpha. \qquad ...(1)$$

$$\therefore \quad \tan\alpha=\tfrac{1}{3}\cot\theta.$$

By Lami's theorem at the point O, we have

$$\frac{R}{\sin(90^\circ+\theta)}=\frac{W}{\sin(90^\circ-\theta+\alpha)}=\frac{T}{\sin(180^\circ-\alpha)}$$

or $$\frac{R}{\cos\theta}=\frac{W}{\cos(\theta-\alpha)}=\frac{T}{\sin\alpha}. \qquad ...(2)$$

$$\therefore \quad R=\frac{W\cos\theta}{\cos(\theta-\alpha)}=\frac{W\cos\theta}{\cos\theta\cos\alpha+\sin\theta\sin\alpha}$$

$$=\frac{W\sec\alpha}{1+\tan\theta\tan\alpha}$$ [Dividing the Nr. and Dr. by $\cos\theta\cos\alpha$]

$$=\frac{W\sqrt{(1+\tan^2\alpha)}}{1+\tan\theta.\tan\alpha}$$

$$=\frac{W\sqrt{(1+\frac{1}{9}\cot^2\theta)}}{1+\tan\theta.\frac{1}{3}\cot\theta} \qquad \left[\because \tan\alpha=\frac{1}{3}\cot\theta\right]$$

$$=\tfrac{1}{4}W\sqrt{(9+\cot^2\theta)}.$$

Also from (2), we have

$$T=\frac{W\sin\alpha}{\cos(\theta-\alpha)}=\frac{W\sin\alpha}{\cos\theta\cos\alpha+\sin\theta\sin\alpha}$$

$$=\frac{W\operatorname{cosec}\theta}{\cot\theta\cot\alpha+1}$$ [Dividing the Nr. and Dr. by $\sin\theta\sin\alpha$]

$$=\frac{W\operatorname{cosec}\theta}{\cot\theta.3\tan\theta+1} \qquad \left[\because \tan\alpha=\frac{1}{3}\cot\theta\right]$$

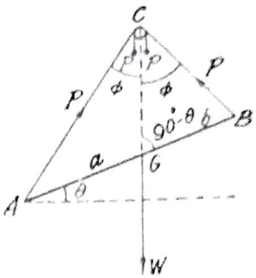
$$=\tfrac{1}{4}W\operatorname{cosec}\theta.$$

Ex. 31. *Equal weights P and P are attached to two strings ACP and BCP passing over a smooth peg C. AB is a heavy beam of weight W, whose centre of gravity is a feet from A and b feet from B; show that AB is inclined to the horizon at an angle*

$$\tan^{-1}\left[\frac{a-b}{a+b}\tan\left(\sin^{-1}\frac{W}{2P}\right)\right].$$

[Meerut 89 (S)]

Sol. Let AB be a beam of weight W, P and P two equal weights attached to the strings ACP and BCP passing over a smooth peg C. Now the tension of a string remains unaltered while passing over a smooth peg. Therefore the tension in each of the strings AC and BC will be equal to P.



(Fig 2·35)

The beam AB is in equilibrium under the action of the following three forces :

(i) P, the tension in the string AC,

(ii) P, the tension in the string BC,

and (iii) W, the weight of the beam acting vertically downwards at its centre of gravity G, where $AG=a$ and $BG=b$.

Since the two forces P and P meet at C, hence the line of action of the third force W must also pass through C. The resultant of two equal forces P along AC and P along BC will bisect the angle ACP between them and for equilibrium this resultant must be equal and opposite to the third force W.

$$\therefore \quad \angle ACG=\angle BCG=\phi \text{ (say)}.$$

Resolving vertically the forces acting on the beam, we have

$$W=P\cos\phi+P\cos\phi \quad \text{or} \quad W=2P\cos\phi$$

or $$\phi=\cos^{-1}\left(\frac{W}{2P}\right). \qquad ...(1)$$

In the $\triangle CAB$, by the trigonometrical theorem, we have

$$(AG+BG)\cot CGB=AG\cot ACG-BG\cot BCG,$$

or $$(a+b)\cot(90^\circ-\theta)=a\cot\phi-b\cot\phi,$$

or $$(a+b)\tan\theta=(a-b)\cot\phi$$

or $$\tan\theta=\frac{a-b}{a+b}\tan\left(\frac{\pi}{2}-\phi\right)$$

$$=\frac{a-b}{a+b}\tan\left(\frac{\pi}{2}-\cos^{-1}\frac{W}{2P}\right), \text{ from (1)}$$

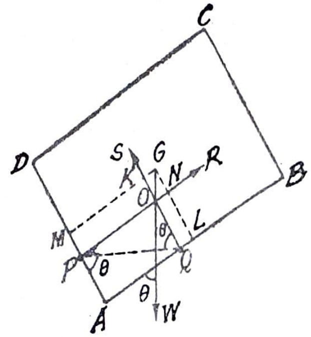
$$=\frac{a-b}{a+b}\tan\left(\sin^{-1}\frac{W}{2P}\right), \left[\because \sin^{-1}\alpha=\frac{\pi}{2}-\cos^{-1}\alpha\right]$$

or $$\theta=\tan^{-1}\left[\frac{a-b}{a+b}\tan\left(\sin^{-1}\frac{W}{2P}\right)\right].$$

Ex. 32. *A uniform rectangular board rests vertically in equilibrium with its sides a and b on two smooth pegs in the same horizontal line at a distance c apart. Prove that the side of length a makes with the vertical an angle θ given by*

$$2c\cos 2\theta=b\cos\theta-a\sin\theta.$$ **[Meerut 79, 83]**

Sol. Let $ABCD$ be a rectangular board of weight W and sides $AB=a$ and $AD=b$ resting in a vertical plane on two smooth pegs P and Q in the same horizontal line at a distance c apart *i.e.*, $PQ=c$.



(Fig. (2·36)

The board is in equilibrium under the action of the following three forces only :

(i) R, the reaction of the peg P, acting at right angles to AD,

(ii) S, the reaction of the peg Q, acting at right angles to AB,

and (iii) W, the weight of the board acting vertically downwards at its centre of gravity G.

Since the two forces R and S meet at O, therefore the line of action of third force W must also pass through O, as shown in the figure.

Now let GL and GM be the perpendiculars from G to the sides AB and AD be the rectangle. Let GL meet the line of action of R at N and GM meet the line of action of S at K. Clearly GN is perpendicular to R and GK is perpendicular to S,

and $$AL=\tfrac{1}{2}AB,\ AM=\tfrac{1}{2}AD.$$

If the side AB makes an angle θ with the vertical, then the other side AD will make an angle θ with the horizontal.

$$\therefore\ \angle APQ=\theta \text{ and } \angle PQO=\angle APQ=\theta.$$

Resolving the forces acting on the board horizontally, we have
$$R\sin\theta=S\cos\theta. \qquad \ldots(1)$$

Taking moments about G of all the forces acting on the board, we have
$$R.GN=S.GK$$

or $$R(GL-NL)=S(GM-MK)$$
or $$R(AM-OQ)=S(AL-OP)$$
or $$R(\tfrac{1}{2}AD-PQ\cos\theta)=S(\tfrac{1}{2}AB-PQ\sin\theta). \qquad \ldots(2)$$

Dividing (2) by (1), we have
$$\frac{AD-2PQ\cos\theta}{\sin\theta}=\frac{AB-2PQ\sin\theta}{\cos\theta}$$

or $$(AB-2PQ\sin\theta)\sin\theta=(AD-2PQ\cos\theta)\cos\theta$$
or $$2PQ(\cos^2\theta-\sin^2\theta)=AD\cos\theta-AB\sin\theta \qquad \ldots(3)$$
or $$2c\cos 2\theta=b\cos\theta-a\sin\theta.$$
$$[\because\ PQ=c,\ AD=b \text{ and } AB=a.]$$

Ex. 33. (a) *A square of side 2a is placed with its plane vertical between two smooth pegs which are in the same horizontal line at a distance c apart; show that it will be in equilibrium when the inclination of one of its edges to the horizon is either*

$$\pi/4 \quad or \quad \tfrac{1}{2}\sin^{-1}[(a^2-c^2)/c^2]$$

[Gorakhpur 81; Meerut 80, 83 (P), 88 (P); P.C.S. 81]

Sol. This is the same question as in the preceding example 32 except that here $ABCD$ is a square and $AB=AD=2a$.

Proceeding as in the preceding example, from the equation (3) we have $2PQ(\cos^2\theta-\sin^2\theta)=AD\cos\theta-AB\sin\theta$.

Putting $AB=AD=2a$ and $PQ=c$, we have
$$2c(\cos^2\theta-\sin^2\theta)=2a(\cos\theta-\sin\theta)$$
or $$c(\cos\theta-\sin\theta)(\cos\theta+\sin\theta)-a(\cos\theta-\sin\theta)=0$$
or $$(\cos\theta-\sin\theta)\{c(\cos\theta+\sin\theta)-a\}=0.$$
$$\therefore \text{ either } \cos\theta-\sin\theta=0 \text{ i.e., } \tan\theta=1 \text{ i.e., } \theta=\pi/4$$
or $$c(\cos\theta+\sin\theta)-a=0$$
i.e., $$\cos\theta+\sin\theta=a/c.$$

Squaring both sides, we get
$$\cos^2\theta+\sin^2\theta+2\sin\theta\cos\theta=a^2/c^2$$
or $$1+\sin 2\theta=a^2/c^2$$
or $$\sin 2\theta=(a^2/c^2)-1=(a^2-c^2)/c^2.$$
$$\therefore\ \theta=\tfrac{1}{2}\sin^{-1}[(a^2-c^2)/c^2].$$

Hence in equilibrium, the inclination of the edge of the square to the horizontal is either $\pi/4$

or $$\tfrac{1}{2}\sin^{-1}[(a^2-c^2)/c^2].$$

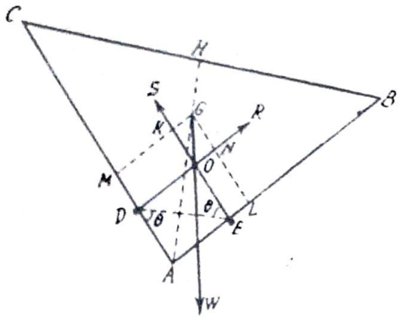
Ex. 33. (b). *A square lamina of side 2a rests in a vertical plane on two smooth pegs in a horizontal line. Show that if the sum of the distances of the pegs from the lowest corner is equal to a, there is equilibrinm.* **[Raj. T.D.C. 79; Gorakhpur 79, 82]**

Sol. Proceed as in Ex. 33 (a). For the equilibrium of the square, we have

$$a=c(\cos\theta+\sin\theta)=c\cos\theta+c\sin\theta$$
$$=PQ\cos\theta+PQ\sin\theta=AP+AQ$$
$=$sum of the distances of the pegs from the corner A.

Ex. 34. *A uniform triangular lamina ABC has a right angle at A and rests in a vertical plane with its sides in contact with two smooth pegs D, E in the same horizontal plane, the vertex A being downwards. Prove that the inclination θ of AC to the horizontal is given by* $AC\cos\theta-AB\sin\theta=3\,DE\cos 2\theta$. **[Meerut 70]**

Sol. Let ABC be a uniform triangular lamina, right angle



(Fig. 2·37)

at A, and of weight W, resting with its sides AC and AB in contact with two smooth pegs D and E in the same horizontal plane.

The lamina is in equilibrium under the action of the following three forces :

(i) R, the reaction of the peg D, at right angles to AC i.e., parallel to AB,

(ii) S, the reaction of the peg E, at right angles to AB i.e., parallel to AC

and (iii) W, the weight of the lamina acting vertically downwards at its centre of gravity G which divides the median AH in the ratio $2:1$.

Since the two forces R and S meet at O, therefore the third force W must also pass through O, as shown in the figure.

Let GL be the perpendicular from G on AB, meeting the line of action of R at N, and GM be the perpendicular from G on AC

meeting the line of action of S at K. Clearly GL and GM are parallel to the sides AC and AB respectively.

We have, $GL=$the perpendicular distance of the centroid of the $\triangle ABC$ from the line AB,

$=\frac{1}{3}\times$(sum of the perpendicular distances of the vertices of the $\triangle ABC$ from the line AB)

$=\frac{1}{3}.(0+0+AC)=\frac{1}{3}AC.$

Similarly $GM=\frac{1}{3}AB$.

Given that side AC makes an angle θ with the horizontal i.e.,

$\angle ADE=\theta$. $\quad\therefore\quad \angle DEO=\angle ADE=\theta$

Resolving the forces acting on the lamina horizontally, we get

$$R\sin\theta=S\cos\theta. \quad \text{...(1)}$$

Taking moments about G of all the forces acting on the lamina, we have $\quad R.GN=S\,GK$

or $\quad R.(GL-NL)=S\,(GM-MK)$

or $\quad R\,(\frac{1}{3}AC-OE)=S\,(\frac{1}{3}AB-OD)\quad [\because\ GL=\frac{1}{3}AC,\ GM=\frac{1}{3}AB]$

or $\quad R\,(\frac{1}{3}AC-DE\cos\theta)=S\,(\frac{1}{3}AB-DE\sin\theta) \quad \text{...(2)}$

$[\because\ OE=DE\cos\theta$ and $OD=DE\sin\theta]$

Dividing (2) by (1), we have

$$\frac{AC-3DE\cos\theta}{\sin\theta}=\frac{AB-3DE\sin\theta}{\cos\theta}$$

or $\quad (AC-3DE\cos\theta)\cos\theta=(AB-3DE\sin\theta)\sin\theta$

or $\quad AC\cos\theta-AB\sin\theta=3DE(\cos^2\theta-\sin^2\theta)$

or $\quad AC\cos\theta-AB\sin\theta=3DE\cos 2\theta$

Ex. 35. *An isosceles triangular lamina with its plane vertical rests with its vertex downwards, between two smooth pegs in the same horizontal line. Show that there will be equilibrium if the base make an angle $\sin^{-1}(\cos^2\alpha)$ with the vertical, 2α being the vertical angle of the lamina and the length of the base being three times the distance between the pegs.* **[Meerut 84(P)]**

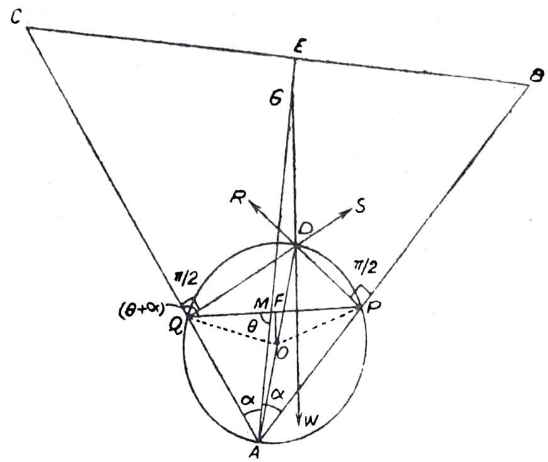
Sol. Let ABC be the isosceles triangular lamina of vertical angle 2α resting with its plane vertical, between two smooth pegs P and Q in the same horizontal line. Given that the base, $BC=3PQ$.

Let the base BC of the lamina make an angle θ with the vertical,

$\therefore$ the median AE will make an angle θ with the horizontal

i.e. $\quad \angle AMQ=\theta$.

The lamina is in equilibrium under three forces only and since two forces R and S, reactions at P and Q, acting perpendi-



(Fig. 2·38)

cular to the sides of the triangle, meet at D, hence vertical line of action of the third force W, weight of the lamina, will also pass through D.

Now $\because$ $\angle APD=\pi/2=\angle AQD$,

$\therefore$ the circle with AD as diameter will pass through P and Q.

If O is the centre of the circle, then we have

$$\angle POQ=2\angle PAQ=4\alpha.$$

$\therefore$ $\angle POF=\angle QOF=2\alpha$, where OF is perpendicular on PQ.

Now $PQ=2PF=2.OP \sin POF=2R \sin 2\alpha$

where $R=OP=OQ=$Radius of the circle

or $$PQ=AD \sin 2\alpha \quad \ldots(1)$$

$\because$ $\angle MQC$ is the exterior angle for triangle MAQ

$\therefore$ $\angle MQC=\angle QAM+\angle AMQ=\alpha+\theta$

Now $\angle PAD=\angle PQD$ (both being in the same segment).

$$=\angle MQC-\angle DQC=\alpha+\theta-\pi/2.$$

$\therefore$ $\angle GAD=\angle GAP-\angle PAD=\alpha-(\alpha+\theta-\pi/2)=\pi/2-\theta.$

But $\angle AGD=\pi/2-\theta$

$\therefore$ $\angle AGD=\angle GAD$. Thus $AD=GD$.

Also $ADG=\pi-\angle AGD-\angle GAD=\pi-2(\pi/2-\theta)=2\theta.$

In ΔGAD, we have

$$\frac{AG}{\sin ADG}=\frac{AD}{\sin AGD}$$

or $$\frac{\frac{2}{3} AE}{\sin 2\theta}=\frac{AD}{\sin(\pi/2-\theta)}$$

or $$\frac{\frac{2}{3} CE \cot\alpha}{\sin 2\theta}=\frac{PQ/\sin 2\alpha}{\cos\theta} \qquad \because \text{ from (1), } AD=\frac{PQ}{\sin 2\alpha}$$

or $$2CE \cot\alpha \cos\theta \sin 2\alpha=3PQ \sin 2\theta$$

or $$BC \cos\theta.2 \cos^2\alpha=BC.2 \sin\theta \cos\theta$$

or $$\sin\theta=\cos^2\alpha \qquad \because 2CE=BC=3PQ$$

$\therefore$ $$\theta=\sin^{-1}(\cos^2\alpha).$$

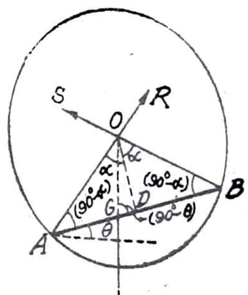
Ex. 36. *A beam whose centre of gravity divides it into two portions a and b, is placed inside a smooth sphere, show that if θ be its inclination to the horizontal in the position of equilibrium and 2α be the angle subtended by the beam at the centre of the sphere, then*

$$\tan\theta=\left(\frac{b-a}{b+a}\right)\tan\alpha. \qquad \text{[Meerut 69, 79, 82(P)]}$$

Sol. Let a beam AB of weight W be in equilibrium inside a smooth sphere of centre O and radius r(say).

If G is the centre of gravity of the beam then $AG=a$ and $BG=b$.

The beam AB is in equilibrium under the action of the following three forces only :



(Fig. 2·39)

(i) R, the reaction at A along the normal to the sphere at A and so passing through the centre of the sphere,

(ii) S, the reaction at B along the normal BO to the sphere at B,

and (iii) W, the weight of the beam acting vertically downwards through its centre of gravity G.

Since the two forces R and S meet at O, hence the line of action of the third force W must also pass through O, as shown in the figure.

Given that $\angle AOB=2\alpha$. If OD is perpendicular from O to AB, then $\angle AOD=\angle BOD=\alpha$.

$\therefore$ $$\angle OAB=\angle OBA=90°-\alpha.$$

The beam makes an angle θ to the horizontal.

$\therefore$ $$\angle OGB=90°-\theta.$$

In the ΔOAB, by the trigonometrical theorem, we have

$$(AG+GB)\cot OGB = BG \cot OAB - AG \cot OBA$$

$$(a+b)\cot(90°-\theta) = b\cot(90°-\alpha) - a\cot(90°-\alpha)$$

or

$$\tan\theta = \left(\frac{b-a}{b+a}\right)\tan\alpha.$$

or

Ex. 37. *A rod rests wholly within a smooth hemispherical bowl, of radius r, its centre of gravity dividing the rod into two portions of lengths a and b. Show that, if θ be the inclination of the rod to the horizon in the position of equilibrium, then*

$$\sin\theta = \frac{(b-a)}{2\sqrt{(r^2-ab)}}.$$

Sol. Refer figure of Ex. 36 on page 71.

Proceed as in the last example 36 to get

$$\tan\theta = \left(\frac{b-a}{b+a}\right)\tan\alpha. \qquad \text{...(1)}$$

Since D is perpendicular from O to AB, therefore D is the middle point of AB.

$$\therefore \quad AD = BD = \frac{AB}{2} = \frac{AG+BG}{2} = \frac{a+b}{2}.$$

$$\therefore \quad \text{from } \Delta OAD,\ \sin\alpha = \frac{AD}{OA} = \frac{a+b}{2r} \qquad [\because \ OA=r]$$

$$\therefore \quad \tan\alpha = \frac{1}{\cot\alpha} = \frac{1}{\sqrt{(\operatorname{cosec}^2\alpha - 1)}} = \frac{1}{\sqrt{[\{4r^2/(a+b)^2\}-1]}}$$

$$= \frac{(a+b)}{\sqrt{\{4r^2-(a+b)^2\}}}.$$

Substituting in (1), we have

$$\tan\theta = \left(\frac{b-a}{b+a}\right) \cdot \frac{(a+b)}{\sqrt{\{4r^2-(a+b)^2\}}} = \frac{(b-a)}{\sqrt{\{4r^2-(a+b)^2\}}}$$

$$\therefore \quad \sin\theta = \frac{1}{\operatorname{cosec}\theta} = \frac{1}{\sqrt{(1+\cot^2\theta)}}$$

$$= \frac{1}{\sqrt{\left[1+\left\{\frac{4r^2-(a+b)^2}{(b-a)^2}\right\}\right]}} = \frac{(b-a)}{\sqrt{[(b-a)^2+4r^2-(a+b)^2]}}$$

$$= \frac{(b-a)}{\sqrt{(4r^2-4ab)}} = \frac{(b-a)}{2\sqrt{(r^2-ab)}}.$$

Alternative method. We have

$$GD = AD - AG = \tfrac{1}{2}(a+b) - a = \tfrac{1}{2}(b-a).$$

Also $\quad OD = \sqrt{(OB^2 - BD^2)} = \sqrt{\{r^2 - \tfrac{1}{4}(a+b)^2\}}.$

Now $\quad OG^2 = OD^2 + GD^2 = r^2 - \tfrac{1}{4}(a+b)^2 + \tfrac{1}{4}(b-a)^2 = r^2 - ab;$

$$\therefore \quad OG = \sqrt{(r^2-ab)}.$$

From ΔOGD, we have

$$\cos(90°-\theta) = \frac{GD}{OG} = \frac{\tfrac{1}{2}(b-a)}{\sqrt{(r^2-ab)}}.$$

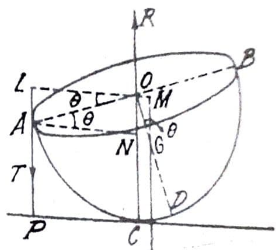
$$\therefore \quad \sin\theta = \frac{b-a}{2\sqrt{(r^2-ab)}}.$$

Ex. 38. *A hemisphere of radius a and weight W is placed with its curved surface on a smooth table and a string of length $l\,(<a)$ is attached to a point on its rim and to a point on the table. Find the position of equilibrium and prove that the tension of the string is*

$$\frac{3W}{8} \cdot \frac{(a-l)}{\sqrt{(2al-l^2)}}$$

[Meerut 89]

Sol. Let O be the centre of the base of a hemi-sphere of radius a and weight W placed on a smooth table. If OD is the axis of the hemi-sphere, then its centre of gravity G is on OD such that $OG = \tfrac{3}{8}a$.



(Fig. 2.40)

Let AP be a string of length $l\,(<a)$ attached to a point A on the rim of the hemisphere and to a point P of the horizontal table.

The hemisphere is in equilibrium under the action of the following three forces only:

(i) W, the weight of the hemisphere acting vertically downwards through its centre of gravity G,

(ii) R, the reaction of the table at the point of contact C, acting at right angles to the table and also passing through the centre O,

and (iii) T, the tension in the string AP, acting along AP.

Since the two forces W and R are vertical, therefore the third force T will also be vertical as shown in the figure.

Hence the hemisphere is in equilibrium with the string AP vertical.

Now $AP = l$, $OC = a$ $\therefore$ $ON = OC - NC = OC - AP = a - l$.

Let OL and OM be the perpendiculars from O to the lines of action of T and W.

If the base of the hemisphere is inclined at an angle θ to the horizontal, then

$$\angle AOL=\theta; \quad \therefore \quad \angle AON=90^\circ-\theta.$$

$$\therefore \quad \angle OGM=\angle GON=90^\circ-\angle AON=90^\circ-(90^\circ-\theta)=\theta.$$

To avoid the reaction R, taking moments of all forces about O, we have

$$T.OL=W.OM$$

or $$T.OA\cos\theta=W.OG\sin\theta$$

or $$T=\tfrac{3}{8}W\tan\theta. \quad \ldots(1)$$

$$[\because \quad OG=\tfrac{3}{8}a \text{ and } OA=a]$$

Now from $\triangle OAN$, $\sin\theta=\dfrac{ON}{OA}=\dfrac{a-l}{a}$.

$$\therefore \quad \tan\theta=\frac{1}{\cot\theta}=\frac{1}{\sqrt{(\operatorname{cosec}^2\theta-1)}}=\frac{1}{\sqrt{[\{a^2/(a-l)^2\}-1]}}$$

$$=\frac{(a-l)}{\sqrt{(2al-l^2)}}.$$

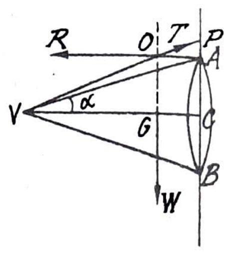
Substituting in (1), we have

$$T=\frac{3W}{8}\cdot\frac{(a-l)}{\sqrt{(2al-l^2)}}.$$

Ex. 39. *A solid cone of height h and semi-vertical angle α, is placed with its base against a smooth vertical wall and is supported by a string attached to its vertex and to a point in the wall; show that the greatest possible length of the string is $h\sqrt{(1+\frac{16}{9}\tan^2\alpha)}$.*

[Raj. T.D.C. 79, 81]

Sol. Let VAB be a solid cone of height $VC=h$ and semi-vertical angle α, placed with its base against a smooth vertical wall and supported by a string VP attached to the vertex V and to a point P in the wall. the length of the string will be greatest when the cone is just at the point of turning about the highest point A of its base. In this case only the point A of the base of the cone is in contact with the wall and so the reaction of the wall is at the point A only.



(Fig. 2·41)

The cone is in equilibrium under the action of the following three force only :

(i) R, the reaction at A acting at right angles to the wall,

(ii) W, the weight of the cone acting vertically downwards through its centre of gravity G, where $CG=\frac{1}{4}VC=\frac{1}{4}h$ and so $VG=\frac{3}{4}h$

and (iii) T, the tension in the string VP acting along VP.

Since the two forces R and W meet at O, therefore the third force T must also pass through O as shown in the figure *i.e.*, the string VP passes through O as shown in the figure.

From the similar triangle VOG and VPC, we have

$$\frac{VO}{VP}=\frac{VG}{VC}=\frac{\frac{3}{4}h}{h}=\frac{3}{4}.$$

$$\therefore \quad VP=\tfrac{4}{3}VO. \quad \ldots(1)$$

Now from $\triangle VOG$,

$$VO^2=VG^2+OG^2=(\tfrac{3}{4}h)^2+(h\tan\alpha)^2$$

$$[\because \quad OG=AC=VC\tan\alpha=h\tan\alpha]$$

or $$VO^2=\frac{9}{16}h^2\left(1+\frac{16}{9}\tan^2\alpha\right)$$

or $$VO=\frac{3}{4}h\sqrt{\left(1+\frac{16}{9}\tan^2\alpha\right)}.$$

$\therefore$ from (1), the greatest possible length of the string *i.e.*,

$$VP=\frac{4}{3}VO=h\sqrt{\left(1+\frac{16}{9}\tan^2\alpha\right)}.$$

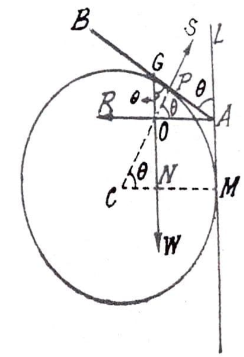
Ex. 40. *A uniform rod rests on a fixed smooth sphere with its lower end pressing against a smooth vertical wall which touches the sphere. If θ is the angle which the rod makes with the vertical, when in equilibrium, prove that $a=2l\sin\frac{1}{2}\theta\cos^3\frac{1}{2}\theta$, where l is the length of the rod and a the radins of the sphere.*

Sol. Let a uniform rod AB rest with a point P on a fixed smooth sphere of centre C and radius a.

The lower end A of the rod presses the wall at the point A. The rod is in equilibrium under the action of the following three forces only :

(i) R, the reaction of the wall at A acting at right angles to the wall,

(ii) S, the reaction of the sphere at P acting along the normal CP which is at right angles to the rod,



(Fig. 2·42)

and (iii) W, the weight of the rod acting vertically downwards through its middle point G.

Since the two forces R and S meet at O, therefore the third force W must also pass through O as shown in the figure.

Let the rod make an angle θ with tne vertical

i.e., $\angle LAB=\theta$, $\therefore$ $\angle AGO=\angle LAB=\theta$

If CM is perpendicular from C to the wall, then M is the point of contact of the sphere and the wall.

$\therefore$ $CM=a$ (radius).

We have

$$\angle OCM=\angle POA=90^\circ-\angle PAO=90^\circ-(90^\circ-\theta)=0.$$

In $\triangle AOG$, $OA=AG\sin\theta=\frac{1}{2}l\sin\theta$, ...(1)

because $AG=\frac{1}{2}AB=\frac{1}{2}l$.

In $\triangle AOP$, $OP=OA\cos\theta=\frac{1}{2}l\sin\theta\cos\theta$.

In $\triangle OCN$, $\cos\theta=\dfrac{CN}{CO}=\dfrac{CM-NM}{CP-OP}=\dfrac{a-OA}{a-\frac{1}{2}l\sin\theta\cos\theta}$.

[$\because$ $NM=OA$]

$\therefore$ $a-\frac{1}{2}l\sin\theta=\cos\theta.(a-\frac{1}{2}l\sin\theta\cos\theta)$

[$\because$ $OA=\frac{1}{2}l\sin\theta$, from (1)]

or $a(1-\cos\theta)=\frac{1}{2}l\sin\theta\,(1-\cos^2\theta)$

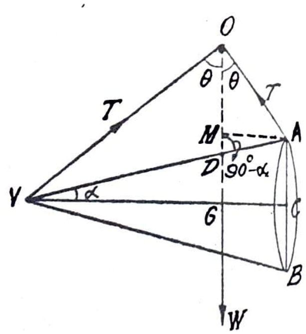
or $a=\frac{1}{2}l\sin\theta.(1+\cos\theta)$

or $a=\frac{1}{2}l.2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta.2\cos^2\frac{1}{2}\theta$

or $a=2l\sin\frac{1}{2}\theta\cos^3\frac{1}{2}\theta$.

Ex. 41. *The altitude of a cone is h and the radius of its base is r; a string is fastened to the vertex and to a point on the circumference of the circular base, and is then put over a smooth peg; show that, if the cone rests with its axis horizontal, the length of the string must be $\sqrt{(h^2+4r^2)}$.*

Sol. Let VAB be a cone of height h and let r be the radius of the base of this cone. Let a string VOA be fastened to the vertex V and a point A on the circumference of the circular base of the cone and then put over a smooth peg O. Let the cone be in equilibrium with its axis VC horizontal. Since the peg O is smooth, hence the tensions in the two parts VO and OA of the string will be equal.



(Fig. 2·43)

The cone is in equilibrium under the action of the following three forces only :

(i) T, the tension in the string VO, acting along VO,

(ii) T, the tension in the string AO, acting along AO,

and (iii) W, the weight of the cone, acting vertically downwards at its centre of gravity G where $VG:GC=3:1$.

Since the two forces T and T meet at O, hence the line of action of the third force W must also pass through O, as shown in the figure.

The resultant of two equal forces T and T at O will bisect the angle between them and must be equal and opposite in direction to the third force W. Hence OG is the bisector of $\angle VOA$. Suppose OG meets VA at D and AM is perpendicular from A to OG.

Since in the $\triangle VAC$, DG is parallel to AC, therefore

$$\frac{VD}{DA}=\frac{VG}{GC}=\frac{3}{1}$$

Thus $VD:DA=3:1$.

Let α be the semi-vertical angle of the cone i.e., $\angle AVC=\alpha$.

Then $\angle ODA=\angle VDG=90^\circ-\alpha$.

$\therefore$ in the $\triangle OVA$, by the trigometrical theorem, we have

$$(3+1)\cot(90^\circ-\alpha)=3\cot\theta-\cot\theta$$

or $4\tan\alpha=2\cot\theta$

or $\cot\theta=2\tan\alpha$. ...(1)

Now from the right angled $\triangle VGO$, we have

$$VO=VG\operatorname{cosec}\theta$$

and from the right angled triangle AMO, we have

$$AO=MA\operatorname{cosec}\theta.$$

If l be the length of the string VOA, then

$l=VO+AO=VG\operatorname{cosec}\theta+MA\operatorname{cosec}\theta=(VG+MA)\operatorname{cosec}\theta$

$=(VG+GC)\operatorname{cosec}\theta$ [$\because$ $MA=GC$]

$=VC\operatorname{cosec}\theta=h\operatorname{cosec}\theta=h\sqrt{(1+\cot^2\theta)}$

$=h\sqrt{(1+4\tan^2\alpha)}$ [$\because$ from (1), $\cot\theta=2\tan\alpha$]

$=h\sqrt{[1+4(r^2/h^2)]}$ [$\because$ $\tan\alpha=r/h$]

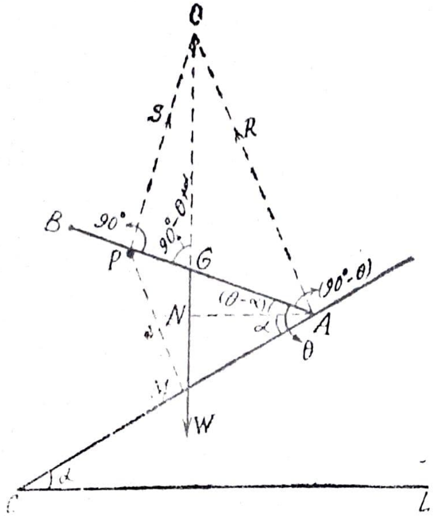
$=\sqrt{(h^2+4r^2)}$.

Ex. 42. *A uniform rod, of length $2a$, has one end resting on a smooth plane of inclination α to the horizon, and is supported by a horizontal rail which is parallel to the plane and at a distance c from it. Show that the inclination θ of the rod to the inclined plane is given by the equation*

$$c\sin\alpha=a\sin^2\theta.\cos(\theta-\alpha).$$

[P.C.S. 81]

Sol. Let AB be a uniform rod of length $2a$, and weight W, resting with one end A on a smooth plane inclined at an angle α to the horizon and supported by a horizontal rail at P, which is parallel to the inclined plane at a distance c from it, *i.e.*, $PM=c$.



(Fig. 2·44)

The rod is in equilibrium under the action of the following three force only :

(i) R, the reaction at A acting at right angles to the inclined plane,

(ii) S, the reaction of the rail at P, acting at right angles to the rod AB,

and (iii) W, the weight of the rod acting vertically downwards at its centre of gravity G.

Since the lines of action of two forces R and S meet at O, therefore the line of action of the third force W must also pass through O, as shown in the figure.

Let AN be the perpendicular from A to the vertical line of action of the weight W.

We have $\angle NAC=\angle ACL=\alpha$.

Given that $\angle BAC=\theta$; $\therefore$ $\angle GAN=\angle GAC-\angle NAC=\theta-\alpha$.

Now $\angle OGP=\angle AGN=90°-\angle GAN=90°-(\theta-\alpha)$,

$\angle OAP=90°-\angle BAC=90°-\theta$, and $\angle OPA=90°$.

Also from the $\triangle PAM$. $\sin\theta=\dfrac{PM}{AP}=\dfrac{c}{AP}$, $\quad [\because PM=c]$

so that $\quad AP=\dfrac{c}{\sin\theta}$.

Now in the $\triangle OAP$, by the trigonometrical theorem, we have

$$(AG+PG)\cot OGP=PG\cot OAP-AG\cot OPA$$

$$AP\cot\{90°-(\theta-\alpha)\}=(AP-AG)\cot(90°-\theta)-AG\cot 90°$$

or $\quad AP\tan(\theta-\alpha)=(AP-AG)\tan\theta$

or $\quad AP[\tan\theta-\tan(\theta-\alpha)]=AG\tan\theta$

or $\quad \dfrac{c}{\sin\theta}\cdot\left[\dfrac{\sin\theta}{\cos\theta}-\dfrac{\sin(\theta-\alpha)}{\cos(\theta-\alpha)}\right]=a.\dfrac{\sin\theta}{\cos\theta}$

$$\left[\because\ AP=\frac{c}{\sin\theta}\ \text{and}\ AG=\frac{AB}{2}=a\right]$$

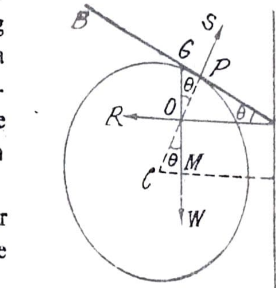
or $\quad c[\sin\theta\cos(\theta-\alpha)-\cos\theta\sin(\theta-\alpha)]=a\sin^2\theta\cos(\theta-\alpha)$

or $\quad c\sin\{\theta-(\theta-\alpha)\}=a\sin^2\theta.\cos(\theta-\alpha)$

or $\quad c\sin\alpha=a\sin^2\theta.\cos(\theta-\alpha)$.

Ex. 43. *A uniform rod of length $2l$ has a ring at one end which slides along a smooth vertical wire, the rod rests touching a smooth cylinder of radius a whose axis is horizontal and at a distance c from the wire. If θ be the angle the rod makes with the horizontal, show that* $\quad l\cos^3\theta=c\pm a\sin\theta$. **[Meerut 81, 84, 85(P)]**

Sol. Let a uniform rod AB of length $2l$ and weight W have a ring at the end A which slides along a smooth vertical wire. In an equilibrium position let the rod touch the sphere at the point P and make an angle θ with the horizontal.



(Fig 2·45 (i))

The rod is in equilibrium under the action of the following three forces :

(i) R, the reaction of the wire on the ring at A acting at right angles to the wire,

(ii) S, the reaction of the sphere at the point P acting at right angles to the rod along the normal CP of the sphere,

and (iii) W, the weight of the rod acting vertically downwards through its middle point G.

Since the two forces R and S meet at O, therefore the third force W must also pass through O as shown in the figure (i).

Given, $\angle BAO=\theta$; $\therefore$ $\angle POG=\theta=\angle COM$, where CL is the line perpendicular to the axis of the cylinder and the wall.

Also $CL=c$, $CP=a$ (radius of the cylinder), and $AG=BG=l$.

In $\quad \triangle OCM$, $\sin\theta=\dfrac{CM}{OC}$. $\quad$ ...(1)

Now in $\triangle AGO$, $OA=AG\cos\theta=l\cos\theta$.

$\therefore\quad CM=CL-ML=c-OA=c-l\cos\theta$;

also $\qquad OC=CP-OP=a-OA \sin\theta$

$\qquad [\because \quad OP=OA \sin\theta$, from $\triangle OAP]$

$\qquad =a-l\cos\theta\sin\theta.$

Substituting in (1), we have

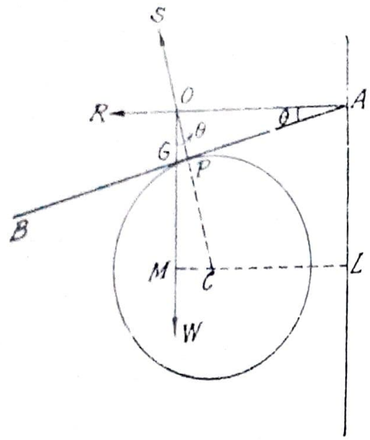
$$\sin\theta=\frac{c-l\cos\theta}{a-l\cos\theta\sin\theta}$$

or $\qquad (a-l\cos\theta\sin\theta)\sin\theta=c-l\cos\theta$

or $\qquad a\sin\theta-l\cos\theta\sin^2\theta=c-l\cos\theta$

or $\qquad l\cos^3\theta=c-a\sin\theta. \qquad \ldots(2)$

In case the ring at A is above the point of contact P, then the forces acting on the rod in equilibrium are shown in the figure (ii).



(Fig. 2·45 (ii))

Now in $\triangle COM$, $\sin\theta=\dfrac{CM}{OC}$.

In $\qquad \triangle AGO,\ AO=AG\cos\theta=l\cos\theta,$

and in $\qquad \triangle AOP,\ OP=AO\sin\theta=l\cos\theta\sin\theta.$

$\therefore \qquad CM=ML-CL=AO-CL=l\cos\theta-c$

and $\qquad OC=OP+PC=l\cos\theta\sin\theta+a.$

Substituting in (3), we have

$$\sin\theta=\frac{l\cos\theta-c}{l\cos\theta\sin\theta+a}$$

or $\qquad \sin\theta.(l\cos\theta\sin\theta+a)=l\cos\theta-c$

or $\qquad l\cos\theta.\sin^2\theta+a\sin\theta=l\cos\theta-c$

or $\qquad l\cos\theta(1-\cos^2\theta)+a\sin\theta=l\cos\theta-c$

or $\qquad l\cos^3\theta=c+a\sin\theta. \qquad \ldots(4)$

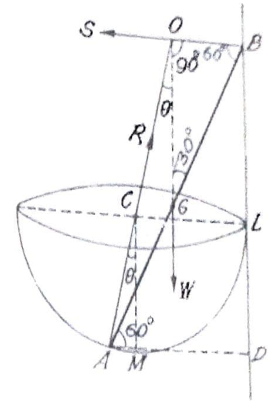
From (2) and (4), we have

$$l\cos^3\theta=c\pm a\sin\theta$$

Ex. 44. *A smooth hemispherical bowl of diameter a is placed so that its edge touches a smooth vertical wall. A heavy uniform rod is in equilibrium inclined at an angle $60°$ to the horizontal with one end resting on the surface of the bowl and the other end against the wall. Show that the length of the rod is $a[1+1/\sqrt{(13)}]$.*

[Raj T.D.C. 81]

Sol. Let a hemispherical bowl of centre C and diameter a be placed so that its edge touches a smooth vertical wall. Let AB be a heavy uniform rod of weight W and length $2a$, placed with one end A resting on the surface of the bowl and the other end B against the wall. In equilibrium the rod make an angle $60°$ to the horizontal.



(Fig. 2·46)

The rod is in equilibrium under the action of the following three forces :

(i) R, the reaction of the bowl at A along the normal at A *i.e.*, the reaction R passes through the centre C of the base of the bowl,

(ii) S, the reaction of the wall at B, acting at right angles to the wall,

and (iii) W, the weight of the rod acting vertically downwards at its middle point G.

The two forces R and S meet at O, hence the line of action of the third force W must also pass through O as shown in the figure. Let AD be the horizontal line through A and CM be the perpendicular from C to AD. Let $\angle ACM=\angle AOG=\theta$.

Given that $\angle GAD=60°$, we have

$$\angle ABO=60° \text{ and } \angle BGO=30°.$$

In the triangle OAG, by the trigonometerical theorem, we have

$$(AG+BG)\cot OGB=AG\cot AOG-BG\cot BOG$$

or $\qquad (\frac{1}{2}a+\frac{1}{2}a)\cot 30°=\frac{1}{2}a\cot\theta-\frac{1}{2}a\cot 90°$

or $\qquad 2a\sqrt{3}=a\cot\theta \quad \text{or} \quad \cot\theta=2\sqrt{3}. \qquad \ldots(1)$

Now from $\triangle ACM$,

$$AM=AC\sin\theta=\frac{a}{2\operatorname{cosec}\theta}=\frac{a}{2\sqrt{(1+\cot^2\theta)}}=\frac{a}{2\sqrt{(13)}}.$$

$\therefore \quad AD = AM + MD = AM + CL \qquad [\because \; MD = CL]$

$$= \frac{a}{2\sqrt{(13)}} + \frac{a}{2} = \frac{a}{2}\left[1 + \frac{1}{\sqrt{(13)}}\right].$$

Now to find AB, we have from $\triangle ABD$.

$$\frac{AD}{AB} = \cos 60^\circ = \tfrac{1}{2}.$$

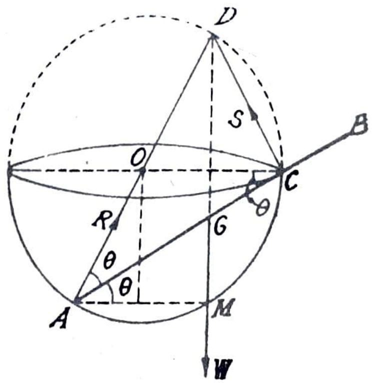
$$\therefore \quad AB = 2AD = 2 \cdot \frac{a}{2}\left[1 + \frac{1}{\sqrt{(13)}}\right] = a\left[1 + \frac{1}{\sqrt{(13)}}\right].$$

Ex 15. *A heavy uniform rod, of length 2a, rests partially inside and parally outside a fixed hemispherical bowl of radius r, the rim of the bowl is horizontal, and one point of the rod is in contact with the rim. If θ be the inclination of the rod to the horizontal, show that* $a \cos \theta = 2r \cos 2\theta$.

Also show that the greatest value of θ is $\sin^{-1} (\frac{1}{3}\sqrt{3})$.

[Gorakhpur 70; Meerut 81(S), 84(P)]

Sol. Let the rod AB, of length $2a$ and weight W, rest partially inside and partially outside a fixed hemispherical bowl of centre O and radius r. Let the rod rest with the end A in contact of the spherical surface and the point C in contact with the rim of the bowl.



(Fig. 2·47)

The rod is in equilibrium under the action of the following three forces only :

(i) R, the reaction of the spherical surface at A along the normal to the sphere at A and so passing through the centre O of the base of the bowl,

(ii) S, the reaction at C, acting at right angles to the rod,

and (iii) W, the weight of the rod acting vertically downwards at the middle point G of the rod.

Since the forces R and S meet at D, hence the line of action of the third force W must also pass through D as shown in the figure. Thus the line DG is vertical.

$\because \; \angle ACD = 90^\circ$ and the line AD passes through the centre of the sphere of which the bowl is a part, therefore the point D must be on this sphere *i.e.* AD is the diameter of the sphere of which the bowl is a part. Suppose DG when produced meets the

surface of the bowl at M. Join AM. Then $\angle AMD = 90^\circ$ because the diameter AD of the sphere subtends a right angle at the point M on the surface of the sphere. Hence the line AM is horizontal.

If the rod makes an angle θ to the horizontal, then $\angle CAM = \theta$.

$\because \; OC$ and AM are parallel and AC meets them,

$\therefore \; \angle OCA = \angle CAM = \theta$.

But $OC = OA$ (being radii of the bowl).

$\therefore \; \angle CAO = \angle OCA = \theta$.

$\therefore \; \angle DAM = 2\theta$.

Now from $\triangle DAM$, $AM = AD \cos 2\theta = 2r \cos 2\theta$ ...(1)

and from $\triangle GAM$, $AM = AG \cos \theta = a \cos \theta$. ...(2)

From (1) and (2), we have

$$a \cos \theta = 2r \cos 2\theta,$$

as the condition for the equilibrium of the rod. ...(3)

Second part. From (3), we have

$$a \cos \theta = 2r\,(2 \cos^2 \theta - 1)$$

or $\quad 4r \cos^2 \theta - a \cos \theta - 2r = 0.$

$$\therefore \quad \cos \theta = \frac{a \pm \sqrt{(a^2 + 32r^2)}}{8r}.$$

Neglecting the −ive sign because $\cos \theta$ cannot be −ive, θ being an acute angle, we have

$$\cos \theta = \frac{a + \sqrt{(a^2 + 32r^2)}}{8r}. \qquad ...(4)$$

Now θ is greatest when $\cos \theta$ is least. From (4) it is clear that $\cos \theta$ is least when a is least as r remains constant. Thus when the length of the rod is least, $\cos \theta$ is least. Now the least value of the length of the rod AB is AC with the other end B of the rod just in contact with the rim.

Thus $\cos \theta$ is least when $AB = AC$

or when $2a = AD \cos \theta$ or $2a = 2r \cos \theta$ or $\cos \theta = a/r$ ...(5)

$\therefore$ from (4) and (5), we have

$$\frac{a + \sqrt{(a^2 + 32r^2)}}{8r} = \frac{a}{r}$$

or $\quad a + \sqrt{(a^2 + 32r^2)} = 8a$, or $\sqrt{(a^2 + 32r^2)} = 7a$.

Squaring both sides, we have

$$a^2 + 32r^2 = 49a^2 \quad \text{or} \quad 48a^2 = 32r^2$$

or $\quad \dfrac{a^2}{r^2} = \dfrac{32}{48} = \dfrac{2}{3}$ or $\dfrac{a}{r} = \sqrt{(2/3)}$.

$\therefore$ from (5), $\cos \theta = \sqrt{(2/3)}$.

$$\therefore \quad \sin \theta = \sqrt{(1 - \cos^2 \theta)} = \sqrt{(1 - 4/3)} = 1/\sqrt{3} = \frac{\sqrt{3}}{3}$$

or $\theta = \sin^{-1} (\sqrt{3}/3)$.

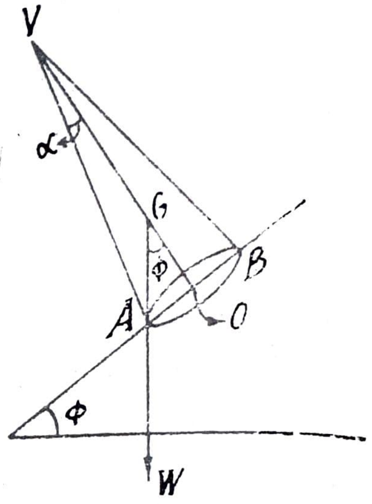
Hence the greatest value of θ is $\sin^{-1} (\sqrt{3}/3)$.

Ex. 46. *A heavy right cone of semi-vertical angle α rests in limiting equilibrium with its plane base upon an inclined plane of inclination θ to the horizon. Show that the cone will topple over or not, according as $\tan \theta >$ or $< 4 \tan \alpha$. Examine the case when $\tan \theta = 4 \tan \alpha$.* **[Meerut 72, 78]**

Sol. If the cone is on the point of slipping *i.e.* in limiting equilibrium on an inclined plane of inclination θ to the horizontal, then

$$\tan \theta = \mu = \tan \lambda \qquad \ldots(1)$$

where μ is the coefficient of friction and λ is the angle of friction.



(Fig. 2·48)

Let ϕ be the inclination of the inclined plane to the horizon when the cone is on the point of toppling over. In this case the vertical line through the centre of gravity G of the cone must just fall within the base of the cone *i.e.* must just pass through the point A.

Let O be the centre of the base of the cone, r the radius of the base and h the height of the cone. When GA is the vertical line through the centre of gravity G of the cone VAB of semi-vertical angle α, we have

$$\angle AGO = \phi, \text{ and } \tan \phi = \frac{AO}{OG} = \frac{r}{h/4} = 4 \frac{r}{h} = 4 \tan \alpha.$$

Now the cone will topple over or not, according as

$$\lambda > \text{ or } < \phi$$

or $$\tan \lambda > \text{ or } < \tan \phi.$$

or $$\tan \theta > \text{ or } < 4 \tan \alpha.$$

[$\because$ $\tan \lambda = \mu = \tan \theta$, from (1)]

When $\tan \theta = 4 \tan \alpha$, we have

$$\mu = 4 \tan \alpha$$

or $$\tan \lambda = \tan \phi, \text{ or } \lambda = \phi.$$

In this case the cone is on the point of both slipping and toppling over.