2019 IFS P-1

SEC-B.

DATE 20/12/2020.

5(a) Solve: $(D^2+1)y = x^2 \sin 2x$

Auxiliary Eqn: $D^2+1 = 0 \Rightarrow D = \pm i$

C.F. $y = c_1 \cos x + c_2 \sin x$

$$P.I. = \frac{1}{D^2+1} x^2 \sin 2x$$

$$= \text{Img part of } \frac{1}{D^2+1} x^2 e^{i2x}$$

$$= \text{Im}\left[e^{i2x} \frac{1}{(D+2i)^2+1} x^2\right]$$

$$\left(\because \frac{1}{f(D)} V e^{ax} = e^{ax} \frac{1}{f(D+a)} V\right)$$

$$= \text{Im}\left(e^{i2x} \frac{1}{D^2+4iD-4+1} x^2\right)$$

$$= \text{Im}\left[\frac{-e^{i2x}}{3} \cdot \left(1-\left(\frac{D^2+4Di}{3}\right)\right)^{-1} x^2\right]$$

$$= \text{Im}\left[\frac{-e^{i2x}}{3}\left(1+\frac{D^2+4Di}{3}+\frac{16D^2i^2}{9}+\dots\right)x^2\right]$$

~~= Im~~ {using Binomial Expansion & neglecting higher powers of D.}

$$= \text{Im}\left(\frac{-e^{i2x}}{3}\left(x^2+\frac{(-26)}{9}+\frac{8xi}{3}\right)\right]$$

$$= \text{Im} -\frac{1}{3}(\cos 2x + i \sin 2x)\left(x^2-\frac{26}{9}+i\frac{8x}{3}\right)$$

$$= -\frac{1}{3}\left[(\sin 2x)\left(x^2-\frac{26}{9}\right)+(\cos 2x)\frac{8x}{3}\right]$$

$\therefore$ Complete Solution: $\boxed{y = C.F. + P.I.}$

5(b) Solve the D.E.

$$(px-y)(py+x) = h^2 p \quad ; \quad p = y'.$$

$$p^2xy + px^2 - py^2 - xy = h^2 p$$

Put,

$$x^2 = u \quad , \quad y^2 = v$$

$$P = \frac{dv}{du} = \frac{y}{x} p \qquad \left(\begin{array}{l} 2x\,dx = du \\ 2y\,dy = dv \end{array} \right)$$

i.e. $p = \frac{x}{y} P = \sqrt{\frac{u}{v}}\, P$

$\therefore$ The given D.E. transforms to

$$\frac{u}{v} P^2 \sqrt{u}\sqrt{v} + \sqrt{\frac{u}{v}}\, Pu - \sqrt{\frac{u}{v}}\, P.v - \sqrt{u}\sqrt{v}$$
$$= h^2 \sqrt{\frac{u}{v}}\, P$$

$$uP^2\sqrt{u} + \sqrt{u}.uP - \sqrt{u}\,vP - \sqrt{u}\,v = h^2\sqrt{u}\,P$$

i.e. $uP^2 + uP - vP - v = h^2 P$

$$u(P^2+P) - v(P+1) = h^2 P$$

$$uP - v = \frac{h^2 P}{P+1}$$

$$\left[v = uP - \frac{h^2 P}{P+1} \right]$$

This is in clairaut's form : $y = px + f(p)$

So, replacing P with c, we have general Solution :

$$v = cu - \frac{ch^2}{c+1}$$

i.e. $\boxed{y^2 = cx^2 - \frac{ch^2}{c+1}}$

IFoS 2019 (S&D)

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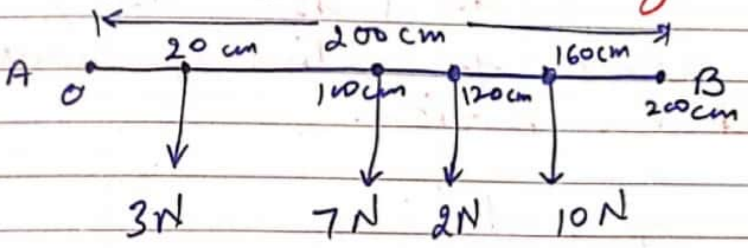
5(c) A 2 m rod has a weight of 2N and has its centre of gravity at 120 cm from one end. At 20 cm, 100 cm and 160 cm from the same end are hung loads of 3N, 7N and 10N respectively. Find the point at which the rod must be supported if it is to remain horizontal.



Varignon's Theorem: Moment of a force about a point is equal to the sum of the moments of the forces' components about the point.

Let us take moments about point A.

Resultant of all forces $= 3+7+2+10$

$= 22\,N$

$\therefore \ (22 \times r) = 3\times20 + 7\times100 + 2\times120 + 10\times160$

$= 2600$

$\therefore \ r = \frac{2600}{22} = \frac{1300}{11} = 118.18\ cm.$

Hence rod must be supported at a point 118.18 cm from end A.

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5(d) Let $\vec{r} = \vec{r}(s)$ represent a space curve. Find $\frac{d^3\vec{r}}{ds^3}$ in terms of $\vec{T}$, $\vec{N}$ and $\vec{B}$, where $\vec{T}, \vec{N}$ and $\vec{B}$ represent tangent, principal normal and binormal respectively. Compute,

$$\frac{d\vec{r}}{ds} \cdot \left(\frac{d^2\vec{r}}{ds^2} \times \frac{d^3\vec{r}}{ds^3} \right)$$

in terms of radius of curvature and the torsion.

$$\vec{T} = \frac{d\vec{r}}{ds}$$

$$\kappa\vec{N} = \frac{d\vec{T}}{ds} = \frac{d}{ds}\left(\frac{d\vec{r}}{ds}\right) = \frac{d^2\vec{r}}{ds^2}$$

ie. $$\frac{d^2\vec{r}}{ds^2} = \kappa\vec{N}$$

$$\Rightarrow \frac{d^3\vec{r}}{ds^3} = \kappa\frac{d\vec{N}}{ds} + \frac{d\kappa}{ds}\vec{N} \quad (*)$$

$$\frac{d^3\vec{r}}{ds^3} = \kappa(\vec{B}\tau - \kappa\vec{T}) + \frac{d}{ds}\left(\left\|\frac{d\vec{T}}{ds}\right\|\right)\cdot\vec{N} \quad (*)$$

(Serret Frenet $\Rightarrow \frac{d\vec{N}}{ds} = \tau\vec{B} - \kappa\vec{T}$)

=

$$\frac{d^2\vec{r}}{ds^2} \times \frac{d^3\vec{r}}{ds^3} = \kappa\vec{N} \times \left[\kappa(\vec{B}\tau - \kappa\vec{T}) + \frac{d\kappa}{ds}\vec{N}\right]$$

using (*)

$$= \kappa^2\tau(\vec{N}\times\vec{B}) - \kappa^3(\vec{N}\times\vec{T})$$

$$= \kappa^2\tau\vec{T} - \kappa^3\vec{B}$$

$$\therefore \frac{d\vec{r}}{ds} \cdot \left(\frac{d^2\vec{r}}{ds^2} \times \frac{d^3\vec{r}}{ds^3} \right)$$

$$= \vec{T}\cdot(\kappa^2\tau\vec{T} - \kappa^3\vec{B})$$

$$= \kappa^2\tau \qquad (\because \vec{T}\cdot\vec{B} = 0)$$

5(e) Evaluate $\int_{(0,0)}^{(2,1)} (10x^4 - 2xy^3)dx - 3x^2y^2dy$

along the path $x^4 - 6xy^3 = 4y^2$.

The integral is of the form

$$\int_C M dx + N dy$$

where $M = 10x^4 - 2xy^3$

$N = -3x^2y^2$

$$\frac{\partial M}{\partial y} = -6xy^2 , \quad \frac{\partial N}{\partial x} = -6xy^2$$

Method-1

As $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ Hence the given integral is path-independent.

It means we can use any path,

Let the path consists of straight line L_1: from $(0,0)$ to $(2,0)$ and then L_2: from $(2,0)$ to $(2,1)$.

Along L_1: $y = 0 \Rightarrow dy = 0$

Along L_2: $x = 2 \Rightarrow dx = 0$

$\therefore$ Value of integral $\int_{x=0}^{2} 10x^4 dx + \int_{y=0}^{1} -3(2)^2 y^2 dy$

$$= 2x^5 \Big|_0^2 - 4y^3 \Big|_0^1 = 64 - 4 = 60.$$

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Method-2 :

As $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore (10x^4 - 2xy^3)dx - (3x^2y^2)dy$ is an exact differential of $(2x^5 - x^2y^3)$.

$$\therefore \int_{(0,0)}^{(2,1)} (10x^4 - 2xy^3)dx - 3x^2y^2dy$$

$$= \int_{(0,0)}^{(2,1)} d(2x^5 - x^2y^3),$$

$$= (2x^5 - x^2y^3) \Big|_{(0,0)}^{(2,1)}$$

$$= 64 - 4 = 60.$$

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