Find the Solution of

$z = \frac{1}{2}(p^2+q^2) + (p-x)(q-y)$ which pass through x-axis

This is Cauchy Strip problem.

This is present in Dr. Raisinghania book & SuccessClap Question Bank

This Solution is SIMPLER than the Textbook Soln

Step-by-step Procedure

→ pass thru x-axis ⇒

$x_0 = \lambda, \quad y_0 = 0, \; z_0 = 0$

→ Find p_0, q_0
- → use Main Eqn
- → Use $dz = p\,dx + q\,dy$

→ Main eqn puts ⇒

$$0 = \frac{1}{2}(p_0^2 + q_0^2) + (p_0 - x_0)(q - y_0)$$

$$= \frac{1}{2}(p_0^2 + q_0^2) + (p_0 - \lambda)(q_0)$$

→ $dz_0 = p_0\,dx_0 + q_0\,dy_0$

$dz_0 = 0$

$dy_0 = 0$

$dx_0 = d\lambda$

$0 = p_0\,d\lambda + q_0(0)$

$\Rightarrow p_0\,d\lambda = 0$

$\Rightarrow p_0 = 0$

Put in previous eqn

$$0 = \frac{1}{2}(0^2 + q_0^2) + (0 - \lambda) q_0$$

$$q_0^2 = 2\lambda q_0 \quad \Rightarrow q_0 = 2\lambda$$

we got Initial values

$x_0 = \lambda,\ y_0 = 0,\ z = 0,\ p_0 = 0,\ q_0 = 2\lambda$

→ Strip eqn: $f = \frac{p^2+q^2}{2} + pq - py - qx + xy - z = 0$

$$\frac{dx}{dt} = \frac{\partial f}{\partial p} = p + q - y \quad -(1)$$

$$\frac{dy}{dt} = \frac{\partial f}{\partial q} = q + p - x \quad -(2)$$

$$\frac{dz}{dt} = p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q} \quad -(3)$$

$$= p(p+q-y) + q(q+p-x)$$

$$\frac{dp}{dt} = -\frac{\partial f}{\partial x} - p\frac{\partial f}{\partial z} = p + q - y \quad -(4)$$

$$\frac{dq}{dt} = -\frac{\partial f}{\partial y} - q\frac{\partial f}{\partial z} = p + q - x \quad -(5)$$

→ (1) & (4) $\Rightarrow \frac{dx}{dt} = \frac{dp}{dt} = [\ ]$

↓ Integrate

$x = p + C_1$

$\rightarrow$ ② & ⑤ $\Rightarrow \frac{dy}{dt} = \frac{dq}{dt} = [\,]$

$\downarrow$ Integrate

$$y = q + C_2$$

$\rightarrow$ Find C_1, C_2 with intial value

$x_0 = p_0 + C_1 \Rightarrow \lambda = 0 + C_1 \Rightarrow C_1 = \lambda$

$y_0 = q_0 + C_2 \Rightarrow 0 = 2\lambda + C_2 \Rightarrow C_2 = -2\lambda$

$$\boxed{\begin{aligned} x &= p + \lambda \\ y &= q - 2\lambda \end{aligned}}$$

$\rightarrow$ Observe :

$$\frac{dp}{dt} + \frac{dq}{dt} - \frac{dy}{dt} = (p+q-y) + (p+q-x) - (q+p-x)$$

$$\frac{d(p+q-y)}{dt} = p+q-y$$

$\downarrow$ Integrate

$$p+q-y = Ae^{t}$$

Similarly

$$\frac{dp}{dt} + \frac{dq}{dt} - \frac{dx}{dt} = (p+q-y) + (p+q-x) - (p+q-y)$$

$$= p+q-x$$

$$\frac{d}{dt}(p+q-x) = p+q-x$$

$$(p+q-x) = Be^{t}$$

$$p+q-y = Ae^{t} \Rightarrow p_0+q_0-y_0 = Ae^{(0)}$$

$$0+2\lambda-0 = A \Rightarrow A = 2\lambda$$

$$p+q-x = Be^{t} \Rightarrow p_0+q_0-x_0 = Be^{0}$$

$$0+2\lambda-\lambda = B \Rightarrow B = \lambda$$

$$\rightarrow p+q-y = 2\lambda e^{t}$$

we have
$$y = q-2\lambda$$
$$x = p+\lambda$$

$$p+q-q+2\lambda = 2\lambda e^{t}$$
$$p = 2\lambda(e^{t}-1)$$

$$\rightarrow p+q-x = \lambda e^{t}$$
$$p+q-p-\lambda = \lambda e^{t}$$
$$q = \lambda(e^{t}-1)$$

$$\rightarrow \frac{dz}{dt} = p(p+q-y)+q(p+q-x)$$
$$= p \cdot 2\lambda e^{t} + q \cdot \lambda e^{t}$$
$$= 4\lambda^2 e^{t}(e^{t}-1) + \lambda^2 e^{t}(e^{t}+1)$$
$$= 5\lambda^2 e^{2t} - 3\lambda^2 e^{t}$$

$\downarrow$ Integrate

$$z = \frac{5\lambda^2}{2} e^{2t} - 3\lambda^2 e^{t} + c$$

$$t=0 \Rightarrow z = z_0 = 0 \qquad 0 = \frac{5\lambda^2}{2} e^{0} - 3\lambda^2 e^{0} + c$$

$$c = \frac{\lambda^2}{2}$$

$$x = p + \lambda = \lambda(2e^t - 1)$$

$$y = q - 2\lambda = \lambda(e^t - 1)$$

$$z = \lambda^2\left(\frac{5}{2}e^{2t} - 3e^t + \frac{1}{2}\right)$$

$$\frac{x}{y} = \frac{2e^t - 1}{e^t - 1} \Rightarrow e^t = \frac{y - x}{2y - x}$$

we have ~~xy=~~ $x - 2y = \lambda(2e^t - 1) - 2\lambda(e^t - 1)$

$$= -\lambda + 2\lambda$$

$$= \lambda$$

$$\boxed{x - 2y = \lambda}$$

$\downarrow$ Put in z

$$z = (x-2y)^2\left[\frac{5}{2}\left(\frac{y-x}{2y-x}\right)^2 - 3\left(\frac{y-x}{2y-x}\right) + \frac{1}{2}\right]$$