Chapter 10

2011

10.1 Section-A

Question-1(a) Let V be the vector space of 2×2 matrices over the field of real numbers $\mathbb{R}$. Let $\mathrm{W} = \{\mathrm{A} \in \mathrm{V} | \ Trace(\mathrm{A}) = 0\}$. Show that W is a subspace of V. Find a basis of W and dimension of W.

[10 Marks]

Solution: Given, v is a vector space of 2×2 matrices over R i.e.

$$V = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \cdot \in R \right\}$$

and W is a subset of V such that Trace $(A) = 0$ when $A \in W$.

Clearly $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \in W$ i.e., W is not empty. Now, let $A_1, A_2 \in W$ then,

$$\text{Trace}\,(A_1) = 0 \text{ and Trace}\,(A_2) = 0$$

then,

$$\begin{aligned} \text{tr}\,(xA_1 + yA_2) &= x\,\text{Tr}\,(A_1) + y\,\text{Tr}\,(A_2) \\ &= x \cdot 0 + y \cdot 0 \\ &= 0 \end{aligned}$$

$$\text{i.e.,} \quad Trace(xA_1 + yA_2) = 0$$

$$\Rightarrow \quad xA_1 + yA_2 \in W$$

If $\begin{bmatrix} x & y \\ z & w \end{bmatrix} \in W$ then $x + w = 0$ i.e., it can have at maximum three free variables.
Hence, dimension of $W = 4 - 1 = 3$ and the basis of W are

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{and} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

i.e.,

$$W = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}.$$

Question 1(b) Find the linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^3$ which has its range the subspace spanned by $(1, 0, -1)$, $(1, 2, 2)$.

[10 Marks]

Solution: Let T be the required linear transformation such that the range of it is spanned by $(1, 0, -1), (1, 2, 2)$ As $\{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ are the standard basis of R_3. Hence, we can assume

$$\begin{aligned} T(1, 0, 0) &= (1, 0, -1) \\ T(0, 1, 0) &= (1, 2, 2) \\ \text{and } T(0, 0, 1) &= (0, 0, 0) \end{aligned}$$

$$\text{Also, } (x, y, z) = x(1, 0, 0) + y(0, 1, 0) + z(0, 0, 1)$$

$$\begin{aligned} \Rightarrow \quad T(x, y, z) &= xT(1, 0, 0) + yT(0, 1, 0) + zT(0, 0, 1) \\ &= x(1, 0, -1) + y(1, 2, 2) + z(0, 0, 0) \\ &= (x + y, 2y, -x + 2y) \\ \text{i.e.,} \quad T(x, y, z) &= (x + y, 2y, -x, +2y) \end{aligned}$$

is the required transformation.

Question-1(c) Show that the function defined by

$$f(x, y) = \begin{cases} \dfrac{x^3 + y^3}{x - y}, & x \neq y \\ 0, & x = y \end{cases}$$

is discontinuous at the origin but possesses partial derivatives f_x and f_y there at.

[10 Marks]

Solution: The given function is

$$f(x, y) = \begin{cases} \dfrac{x^3 + y^3}{x - y}, & x \neq y \\ 0, & x = y \end{cases}$$

The above function is continuous at origin if it is equal to $\lim_{(x,y)\to(0,0)} f(x, y) = f(0, 0) = 0$, irrespective of the path taken by the function to approach origin.

$$\text{Now} \lim_{(x,y)\to(0,0)} f(x, y) = \lim_{(x,y)\to(0,0)} \frac{x^3 + y^3}{x - y}$$

Let $y = x - mx^3$ be the path through which this (x, y) approaches to origin, then. Clearly

$y \to 0$ when $x \to 0$. Then,

$$\begin{aligned}\lim_{(x,y)\to(0,0)} f(x,y) &= \lim_{x\to 0} \frac{x^3 + (x - mx^3)^3}{x - x + mx^3} \\ &= \lim_{x\to 0} \frac{x^3 + x^3 - 3x^2 \cdot n^2x^3 + 3x \cdot m^2x^6 - m^3x^9}{mx^3} \\ &= \lim_{x\to 0} \frac{2x^3 - 3mx^5 + 3m^2x^7 - m^3x^9}{mx^3} \\ &= \frac{2}{m}\end{aligned}$$

$\implies$ It approaches to different values depending on the value of m.
$\implies$ The function is discontinuous at the origin.

Again,

$$\begin{aligned}f(x,y) &= \frac{x^3 + y^3}{x - y} \\ f_x(0,0) &= \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} \\ &= \lim_{h\to 0} \frac{h^3/h - 0}{h} = 0 \\ \text{and } f_y(0,0) &= \lim_{k\to 0} \frac{f(0,0+k) - f(0,0)}{k} \\ &= \lim_{k\to 0} \frac{\frac{k^3}{-k} - 0}{k} = 0\end{aligned}$$

$\implies$ $f_x(0,0)$ and $f_y(0,0)$ exist at origin.

Question-1(d) Let the function f **be defined by**

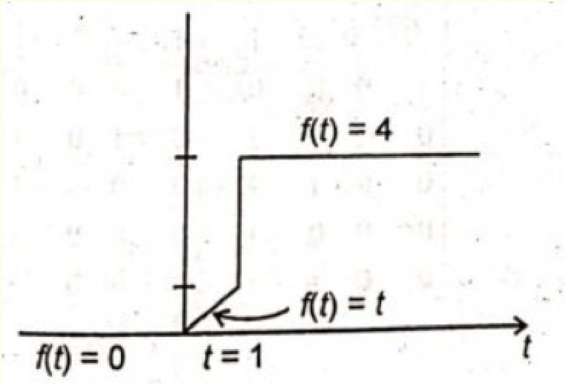
$$f(t) = \begin{cases} 0, & \text{for } t < 0 \\ t, & \text{for } \quad 0 \le t \le 1 \\ 4, & \text{for } t > 1 \end{cases}$$

(i) Determine the function $\mathrm{F(x)} = \int_0^{\mathrm{x}} \mathrm{f(t)dt}$
(ii) Where is F **non-differentiable? Justify your answer.**

[10 Marks]

Solution: The given function f is defined as

$$f(t) = \begin{cases} 0, & \text{for } t < 0 \\ t, & \text{for } \quad 0 \le t \le 1 \\ 4, & \text{for } t > 1 \end{cases}$$



Now we have to calculate

$$F(x) = \int_0^x f(t)dt$$

Case I: When $0 < x \leq 1$
then

$$\begin{aligned} F(x) &= \int_0^x f(t)dt \\ &= \int_0^x tdt \\ &= \frac{x^2}{2} \end{aligned}$$

Case II: $x > 1$
then

$$\begin{aligned} F(x) &= \int_0^{x'} f(t)dt \\ &= \int_0^1 f(t)dt + \int_1^x f(t)dt \\ &= \int tdt + \int 4dt \\ &= \frac{1}{2} + 4(x-1) \\ &= 4x - \frac{7}{2} \\ &= 4x - \frac{7}{2} \end{aligned}$$

i.e., $\quad F(x) = 4x - \frac{7}{2} \quad x > 1$

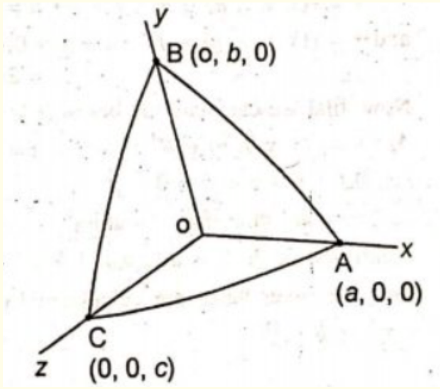
$$F(x) = \begin{cases} \frac{x^2}{2}, & \text{for } 0 < x \leq 1 \\ 4x - \frac{7}{2}, & \text{for } x > 1 \end{cases}$$

Clearly the function $F(x)$ is not differentiable at $x = 1$.

Question-1(e) A variable plane is at a constant distance p from the origin and meets the axes at A, B, C. Prove that the locus of the centroid of the tetrahedron $OABC$ is $\dfrac{1}{x^2}+\dfrac{1}{y^2}+\dfrac{1}{z^2}=\dfrac{16}{p^2}$.

[10 Marks]

Solution: Let the required plane cut the axes at A, B, C such that $A=(a,0,0), B=(0,b,0)$ and $C=(0,0,c)$



Then the equation of this plane is given by

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

Now from question, the length of perpendicular to this plane from origin is p. Then,

$$\frac{|0+0+0-1|}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}=p$$

or,

$$\frac{1}{p^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$$

Again, let (α,β,γ) be the centroid of the tetrahedron. then, $\alpha=\frac{0+a+0+0}{4}$

$$\beta=\frac{0+0+b+0}{4}$$
$$\gamma=\frac{0+0+0+c}{4}$$

or, $a=4\alpha,\quad b=4\beta,\quad c=4\gamma$

putting a,b,c is equation (2), we get

$$\frac{1}{p^2}=\frac{1}{16\alpha^2}+\frac{1}{16\beta^2}+\frac{1}{16\gamma^2}$$

or,

$$\frac{1}{\alpha^2}+\frac{1}{\beta^2}+\frac{1}{\gamma^2}=\frac{16}{p^2}$$

Hence, locus of (α,β,γ) is given by $\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{16}{p^2}$
Hence, proved.

Question-2(a) Let

$$V = \{(x, y, z, u) \in R^4 : y + z + u = 0\}$$

$$W = \{(x, y, z, u) \in R^4 : x + y = 0, z = 2u\}$$

be two subspaces of R^4. Find bases for $V, W, V + W$ and $V \cap W$.

[10 Marks]

Solution:

$$\begin{aligned}
V &= \{(x, y, z, u) \in R^4 : \quad y + z + u = 0\} \\
&= \{(x, -z - u, z, u) \in R^4\} \\
&= \{x(1, 0, 0, 0) + z(0, -1, 1, 0) + u(0, -1, 0, 1)\} \\
&= \text{Span}\{(1, 0, 0, 0), (0, -1, 1, 0), (0, -1, 0, 1)\} \\
W &= \{(x, y, z, u) \in R^4 : x + y = 0, z = 2u\} \\
&= \{(-y, y, 2u, u) \in R^4\} \\
&= \{y(-1, 1, 0, 0) + u(0, 0, 2, 1)\} \\
&= \text{Span}\{(-1, 1, 0, 0), (0, 0, 2, 1)\}
\end{aligned}$$

Now the bases of $V + W$ are given by the number of independent rows in the matrix formed by the bases of V and W in the form of row vector

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

i.e., No. of independent rows in above matrix $= 4$

i.e., Dimension of $V + W = 4$. i.e. The bases of $V + W$ are given by

$$(1, 0, 0, 0), (0, -1, 0, 1), (0, 0, 1, -1)(0, 0, 0, 1)$$

Now we calculate the bases of $V \cap W$. Clearly it should satisfy

$$y + z + u = 0$$

$$x + y = 0, z = 2u$$

i.e., there is only one free variable in this subspace. Choose $u = 1$ is the free variable then,

$$z = 2, y = -3, x = 3$$

i.e., $\{(3, -3, 2, 1)\}$ is the basis of $V \cap W$ and the dimension of this subspace is 1.

Question-2(b) Find the characteristic polynomial of the matrix

$$A = \begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$$

and hence compute A^{10}.

[10 Marks]

Solution: The given matrix

$$A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{bmatrix}$$

Then, the characteristic equation of this polynomial is given by

$$|A - \lambda l| = 0$$

$$\begin{vmatrix} 3-\lambda & 1 & 1 \\ 2 & 4-\lambda & 2 \\ -1 & -1 & 1-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)\{(4-\lambda)(1-\lambda)+2\} - 1\}2(1-\lambda)+2\} + 1\{-2+4-\lambda\} = 0$$

$$16 - 20\lambda + 8\lambda^2 - \lambda^3 = 0$$

$$\lambda^3 - 8\lambda^2 + 20\lambda - 16 = 0$$

Hence, by Cayley-Hamilton Theorem it should be satisfy the by matrix A. i.e.,

$$A^3 - 8A^2 + 20A - 16I = 0$$

or, the characteristic polynomial is given by

$$A^3 - 8A^2 + 20A - 16I = 0$$

from the given expression, it is difficult to calculate A^{10}.

Question-2(c) Let $A = \begin{pmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{pmatrix}$**. Find an invertible matrix** P **such that** $P^{-1}AP$ **is a diagonal matrix.**

[10 Marks]

Solution: The given matrix A is $\begin{bmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{bmatrix}$ The characteristic equation of this matrix is given by

$$|A - \lambda| = 0$$

$$\begin{vmatrix} |-\lambda & -3 & 3 \\ 0 & -5-\lambda & 6 \\ 0 & -3 & 4-\lambda \end{vmatrix} = 0$$

$$(1-\lambda) \mid (\lambda+5)(\lambda-4) + 18\} = 0$$

$$(1-\lambda) \mid \lambda^2 + \lambda - 20 + 18\} = 0$$

$$\Rightarrow (1-\lambda)\,(\lambda^2 + \lambda - 2) = 0$$

$$(1-\lambda)\,(\lambda^2 + 2\lambda - \lambda - 2) = 0 \Rightarrow \quad \lambda = 1, 1, -2$$

i.e., $\lambda = 1, 1, -2$ are the eigen values of the matrix A.

Now, for $\lambda = 1$, the eigen vector is given by $[A - I][X] = 0$ where $[X] = [x, yz]^T$

$$\begin{bmatrix} 0 & -3 & 3 \\ 0 & -6 & 6 \\ 0 & -3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & -3 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-3y + 3z = 0$$

$$-y + z = 0$$

. Clearly, this will possess two eigenvectors as there are two free variables satisfying the above condition.
Hence, the eigen vectors corresponding to $\lambda = 1$ is given by, $\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$ and $\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T$

For $\lambda = -2$, the eigenvector is given by

$$[A + 2I][X] = 0$$

$$\begin{bmatrix} 3 & -3 & 3 \\ 0 & -3 & 6 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \quad x - y + z = 0$$

$$-y + 2z = 0$$

i.e. It'll possesses only one free variable.

Choose $z = 1$ as the free variable then. $y = 2$ and $x = 1$ i.e., $[121]^T$ is the required eigen vector.

Hence, the invertible matrix (P) is given by

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}^T$$

It will reduce the matrix A to a diagonal matrix by operation $P^{-1}AP = D$ where

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Verification As

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$\Rightarrow \mid P \mid = -1$ Now

$$P^{-1} = \frac{\text{Adj}\, P}{|P|} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & 1 & -1 \end{bmatrix}$$

$$AP = \begin{bmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -4 \\ 0 & 1 & -2 \end{bmatrix}$$

$$P^{-1}AP = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & -4 \\ 0 & 1 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

Question-2(d) Find an orthogonal transformation to reduce the quadratic form $5x^2 + 2y^2 + 4xy$ to a canonical form.

[10 Marks]

Solution: The given quadratic form is

$$5x^2 + 2y^2 + 4xy$$

its associated matrix A can be written as

$$\begin{bmatrix} 5 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} A \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Apply congruent operation $R_2 \to R_2 - \frac{2}{5}R_1$ and $C_2 \to C_2 - \frac{2}{5}C_1$ we get:

$$\begin{bmatrix} 5 & 0 \\ 0 & \frac{6}{5} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -\frac{2}{5} & 1 \end{bmatrix} A \begin{bmatrix} 1 & -\frac{2}{5} \\ 0 & 1 \end{bmatrix}$$

Now, apply $R_1 \to R_1 \cdot \frac{1}{\sqrt{5}}$ and $C_i = \frac{1}{\sqrt{5}} \cdot C_1$ we get

$$\begin{bmatrix} 1 & 0 \\ 0 & \frac{6}{5} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} & 0 \\ -\frac{2}{5} & 1 \end{bmatrix}, A \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ 0 & 1 \end{bmatrix}$$

Apply $R_2 \to \sqrt{\frac{5}{6}}R_2$ and $C_2 \to \sqrt{\frac{5}{6}}C_2$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} & 0 \\ -\sqrt{\frac{2}{15}} & \sqrt{\frac{5}{6}} \end{bmatrix} A \begin{bmatrix} \frac{1}{\sqrt{5}} & -\sqrt{\frac{2}{15}} \\ 0 & \sqrt{\frac{5}{6}} \end{bmatrix}$$

Hence the orthogonal transformation is

$$\begin{bmatrix} \frac{1}{\sqrt{5}} & -\sqrt{\frac{2}{15}} \\ 0 & \sqrt{\frac{5}{6}} \end{bmatrix}$$

Question-3(a) Show that the equation $3^x + 4^x = 5^x$ has exactly one root.

[8 Marks]

Solution: The given equation is

$$3^x + 4^x = 5^x$$

Dividing both the sides by 5^x, we get

$$\left(\frac{3}{5}\right)^x + \left(-\frac{4}{5}\right)^x = 1$$

let $\sin\theta = \frac{3}{5}$ then $\cos\theta = \frac{4}{5}$ hence the equation (2) is reduced to

$$(\sin\theta)^x + (\cos\theta)^x = 1$$

which is true for $x = 2$ i.e. $x = 2$ is the only root of equation (1). This result is also known as known as Fermat theorem. It states that $a^n + b^n \neq c^n$ for $n > 2$ where $a, b, c \in z$ and $n \in N$.

Question-3(b) Test for convergence the integral $\int_0^\infty \sqrt{\mathrm{xe^{-x}}}\mathrm{dx}$.

[8 Marks]

Solution: The given integral is

$$\int_0^\infty \sqrt{xe^{-x}}dx = \int_0^\infty \sqrt{x}e^{-x/2}dx$$

Let $y = \frac{x}{2}$ then

$$\begin{aligned} x &= 2y \\ dx &= 2dy \\ &= \int_0^\infty \sqrt{2}ye^{-y}2dy \\ &= 2\sqrt{2}\int_0^\infty \sqrt{y}e^{-y}dy \end{aligned}$$

Let

$$f(y) = \sqrt{y}e^{-y} = \frac{e^{-y}}{y^{-1/2}}$$

Clearly the function has an infinite discontinuity at $y = 0$ and $y = \infty$.
Hence, we have to examine the convergence at both $y = 0$ and $y = \infty$.
Consider, $y = \infty$,

$$\int_0^\infty \sqrt{y}e^{-y}dy = \int_0^1 \sqrt{y}e^{-y}dy + \int_1^\infty \sqrt{y}e^{-y}dy$$

We test the two integrals on the right for convergence at 0 and ∞ respectively.
Convergence at $y = 0$:
Let $g(y) = \sqrt{y}$ such that

$$\lim_{y\to 0} \frac{f(y)}{g(y)} = e^{-y} \to 1 \quad \text{as } x \to 0$$

However

$$\int_0^1 g(y)dy = \left.\frac{y^{3/2}}{32}\right|_0^1 \qquad \ldots (1)$$

converges

$$\Rightarrow \int_0^1 \sqrt{y}e^{-y}dy$$

.
Convergence at ∞:
Let $g(y) = \frac{1}{y^2}$ then

$$\begin{aligned} \lim_{y\to\infty} \frac{f(y)}{g(y)} &= \lim_{y\to\infty} \frac{\sqrt{y}e^{-y}}{17y^2} \\ &= \lim_{y\to\infty} \frac{y^{5/2}}{e^y} \to 0 \text{ as } y \to \infty \end{aligned}$$

As $\int_1^\infty g(y)dy$ converges if $g(y) = \frac{1}{y^2}$ i.e., $\int_1^\infty \sqrt{y}e^{-y}dy$ converges $\ldots (2)$

From (1) and (2),
$\int_0^\infty \sqrt{y}e^{-y}dy$ converges.

The given integral

$$
\begin{aligned}
2\sqrt{2}\int_0^\infty \sqrt{y}e^{-y}dy &= 2\sqrt{2}\int_0^\infty y^{\frac{3}{2}-1}e^{-y}dy \\
&= 2\sqrt{2}\left(\frac{3}{2}\right) \\
&= 2\sqrt{2}\cdot\frac{1}{2}\cdot\sqrt{\pi} \\
&= \sqrt{2\pi}
\end{aligned}
$$

Question-3(c) Show that the area of the surface of the sphere $x^2+y^2+z^2=a^2$ cut off by $x^2+y^2=ax$ is $2(\pi-2)$a^2.

[12 Marks]

Solution: The given sphere is $x^2+y^2+z^2=a^2$

$$
\begin{aligned}
\therefore \quad \frac{\partial z}{\partial x} &= -\frac{x}{z} \\
\frac{\partial z}{\partial y} &= -\frac{y}{z} \\
\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2} &= \frac{1}{z}\sqrt{x^2+y^2+z^2} \\
&= \frac{a}{\sqrt{a^2-x^2-y^2}}
\end{aligned}
$$

Now the surface area is

$$
\begin{aligned}
\iint ds &= \iint \sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2}\,dxdy \\
&= 4\iint \frac{a}{\sqrt{a^2-x^2-y^2}}dxdy
\end{aligned}
$$

over half the circle $x^2+y^2=ax$

let $x=r\cos\theta$, $y=r\sin\theta$ then

$$x^2+y^2=ax$$

becomes $r=a\cos\theta$ and $dxdy=rd\theta dr$

$$\begin{aligned}
\therefore \quad S &= 4a \int_0^{\pi/2} \int_0^{a\cos\theta} \frac{r d\theta dr}{a^2 - r^2} \\
&= 4a \int_0^{\pi/2} \left[-\sqrt{a^2 - r^2}\right]^{n\cos\theta} d\theta \\
&= 4a \int_0^{\pi/2} (1 - \sin\theta) d\theta \\
&= 4a^2 [\theta + \cos\theta]_0^{\pi/2} \\
&= 4a^2 \left[\left(\frac{\pi}{2} + 0\right) - (0 + 1)\right] \\
&= 2a^2(\pi - 2) \text{ units}
\end{aligned}$$

i.e.,

$$S = 2(\pi - 2)a^2 \text{ units}$$

Proved.

Question-3(d) Show that the function defined by

$$f(x, y, z) = 3\log\left(x^2 + y^2 + z^2\right) - 2x^2 - 2y^3 - 2z^3, (x, y, z) \neq (0, 0, 0)$$

has only one extreme value, $\log\left(\dfrac{3}{e^2}\right)$

[12 Marks]

Solution: The given function is

$$f(x, y, z) = 3\log\left(x^2 + y^2 + z^2\right) - 2x^3 - 2y^3 - 2z^3 \quad (x, y, z) \neq (0, 0, 0)$$

for extremum value

$$f_x = f_y = f_z = 0$$

Now,

$$\begin{aligned}
f_x &= 3 \cdot \frac{2x}{x^2 + y^2 + z^2} - 6x^2 \\
&= \frac{6x}{x^2 + y^2 + z^2} - 6x^2 \\
&= 0
\end{aligned}$$

$$\Rightarrow \frac{6x\left[1 - x\left(x^2 + y^2 + z^2\right)\right]}{\left(x^2 + y^2 + z^2\right)} = 0 \text{ as, } (x, y, z) \neq (0, 0, 0)$$

$$\begin{aligned}
\Rightarrow \quad 1 - x\left(x^2 + y^2 + z^2\right) &= 0 \\
x\left(x^2 + y^2 + z^2\right) &= 1
\end{aligned}$$

Similarly,

$$f_y = 0$$

$$\Rightarrow \quad y\left(x^2 + y^2 + z^2\right) = 1$$

and

$$f_z = 0$$

$$\Rightarrow \quad z\left(x^2+y^2+z^2\right)=1$$

From equations (1),(2) and (3), we get $x=y=z$, i.e.,

$$x\left(x^2+x^2+x^2\right)=1$$

$$3x^3=1$$

$$\Rightarrow x=\left(\frac{1}{3}\right)^{\frac{1}{3}}=\frac{1}{3^{1/3}}$$

$$x=y=z=\frac{1}{3^{1/3}}$$

Hence, the value of $f(x,y,z)$ at the point $\left(\frac{1}{3^{1/3}}\cdot\frac{1}{3^{1/3}},\frac{1}{3^{1/3}}\right)$ is given by

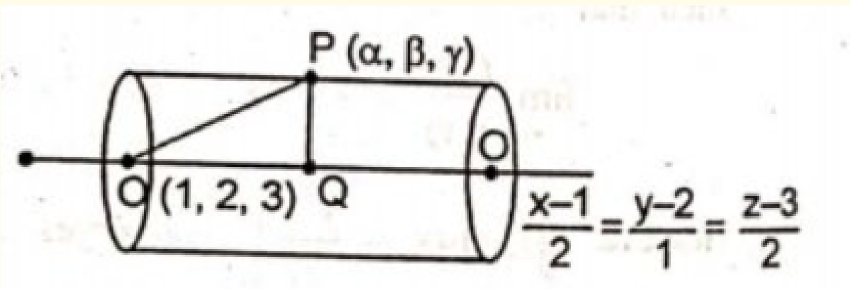
$$f\left(\frac{1}{3^{1/3}},\frac{1}{3^{1/3}},\frac{1}{3^{1/3}}\right)=3\log\left(\frac{1}{3^{2/3}}+\frac{1}{3^{2/3}}+\frac{1}{3^{2/3}}\right)-2\left(\frac{1}{3}+\frac{1}{3}+\frac{1}{3}\right)$$
$$=3\log\left(\frac{3}{3^{2/3}}\right)-2$$
$$=3\log 3^{1/3}-2=\frac{3}{3}\log 3-2$$
$$=\log 3-2$$
$$=\log\left(\frac{3}{e^2}\right)$$

i.e., the only extreme value of $f(x,y,z)$ is $\log\left(\frac{3}{e^2}\right)$

Question-4(a) Find the equation of the right circular cylinder of radius 2 **whose axis is the line** $\dfrac{x-1}{2}=\dfrac{y-2}{1}=\dfrac{z-3}{2}$.

[10 Marks]

Solution: Let OO' be the axis of the right circular cylinder which has radius 2 .



From question, The equation of the line OO'. is

$$\frac{x-1}{2}=\frac{y-2}{1}=\frac{z-3}{2}$$

clearly it passes through (1,2,3) let

$$O\equiv(1,2,3)$$

Now, let $P(\alpha, \beta, \gamma)$ be a point which lies on the cylinder, then from the figure, the projection of OP on the line OO' is given by

$$(\alpha - 1)\frac{2}{3} + (\beta - 2) \cdot \frac{1}{3} + (\gamma - 3) \cdot \frac{2}{3}$$

Now in right angled triangle OPQ we have,

$$OP^2 = PQ^2 + OQ^2$$

$$\Rightarrow (\alpha-1)^2+(\beta-2)^2+(\gamma-3)^2 = 2^2+\tfrac{1}{9}\left[2(\alpha-1)+(\beta-2)+(\gamma-3)\right]^2$$

$$\Rightarrow 9\left[(\alpha-1)^2+(\beta-2)^2+(\gamma-3)^2\right] = 36+\left[2\alpha+\beta+2\gamma-10\right]^2$$

$$\Rightarrow 9\left[\alpha^2+\beta^2+\gamma^2+14-2\alpha-4\beta-6\gamma\right]36+$$

$$\left[4\alpha^2+\beta^2+4\gamma^2+100+4\alpha\beta+4\beta\gamma+8\alpha\gamma-40\alpha-20\beta-40\gamma\right]$$

$$\Rightarrow 5\alpha^2+8\beta^2+5\gamma^2-4\alpha\beta-8\alpha\gamma+2\alpha-16\beta+4\gamma-10=0$$

Hence, equation of right circular cylinder is given by the locus (α, β, γ) i.e.,

$$5x^2+8y^2+5z^2-4xy-4yz-8xz+22x-16y+4z-10=0$$

Question-4(b) Find the tangent planes to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ which are parallel to the plane $lx+my+nz=0$.

[10 Marks]

Solution: The equation of ellipsoid is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad \ldots(1)$$

Let equation of tangent plane which is parallel to given plane is

$$lx+my+nz=p \qquad \ldots(2)$$

Let it touches ellipsoid at point (x, y, z). We know that equation of tangent plane at point (x, y, z) is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=1 \qquad \ldots(3)$$

If (2) and (3) are identical then

$$\frac{x_1}{a^2l}=\frac{y_1}{b^2m}=\frac{z_1}{c^2n}=\frac{1}{p}$$

$$\Rightarrow \quad x_1=\frac{a^2l}{p}, y_1=\frac{b^2m}{p}, z=\frac{c^2n}{p}$$

Point (x, y, z) lies on ellipsoid (1),

$$\therefore \quad \frac{1}{a^2}\left(\frac{a^2 l}{p}\right)^2 + \frac{1}{b^2}\left(\frac{b^2 m}{p}\right)^2 + \frac{1}{c^2}\left(\frac{c^2 n}{p}\right)^2 = 1$$
$$\Rightarrow a^2 l^2 + b^2 m^2 + c^2 n^2 = p^2$$

Using (2), equation of plane is

$$lx + my + nz = \pm\sqrt{a^2 l^2 + b^2 m^2 + c^2 n^2}$$

Question-4(c) Prove that the semi-latus rectum of any conic is a harmonic mean between the segments of any focal chord.

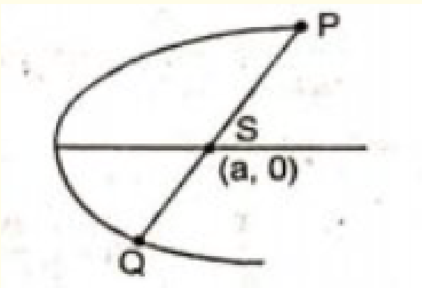
[8 Marks]

Solution: Consider a parabola conic whose equation is

$$y^2 = 4ax$$

then the length of semi-latus rectum $= 2a$

Let $S = (a, 0)$ be the focus of this parabola and PSQ be any focal chord of this parabola,



let $\quad P \equiv (at^2, 2at)$ then

$$Q \equiv \left(\frac{a}{t^2}, \frac{-2a}{t}\right)$$
$$SP^2 = a^2\left(1 - t^2\right)^2 + 4a^2 t^2$$
$$= a^2\left(1 + t^2\right)^2$$
$$\Rightarrow SP = a\left(1 + t^2\right)$$
$$\text{Similarly, } SQ = a\left(1 + \frac{1}{t^2}\right)$$

Now, the harmonic mean of SP and SQ is given by

$$\begin{aligned}\frac{2\cdot SP\cdot SQ}{SP+SQ} &= \frac{2a^2\left(1+t^2\right)\left(1+\frac{1}{t^2}\right)}{a\left[1+t^2+1+\frac{1}{t^2}\right]} \\ &= \frac{2a\left[1+t^2+\frac{1}{t^2}+1\right]}{\left[2+t^2+\frac{1}{t^2}\right]} \\ &= \frac{2a\left[2+t^2+\frac{1}{t^2}\right]}{\left[2+t^2+\frac{1}{t^2}\right]} = 2a\end{aligned}$$

which is equal to semi-latus rectum.

Question-4(d) Tangent planes at two points P and Q of a paraboloid meet in the line RS. Show that the plane through RS and middle point of PQ is parallel to the axis of the paraboloid.

[12 Marks]

Solution: Let standard equation of paraboloid be

$$2cz = ax^2 + by^2$$

and the given points be $P\left(x_1, y_1, z_1\right)$ & $Q\left(x_2, y_2, z_2\right)$.
Tangent planes at P and Q are given by:

$$\begin{aligned} c\left(z+z_1\right) &= ax_1x + by_1y \qquad \ldots(1) \\ c\left(z+z_2\right) &= ax_2x + by_2y \qquad \ldots(2)\end{aligned}$$

Hence, equation of plane passing through line of intersection of (1) and (2) is given by:

$$\left(ax_1\right)x + (by)y - cz - cz + \lambda\left[\left(ax_2\right)x + \left(by_2\right)y - \left(z - cz_2\right) = 0 \qquad \ldots(3)$$

Middle point of $p\varphi, m\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}, \frac{2_1+2_2}{2}\right)$ lies on the above plane.

Hence, we obtain the value of λ as

$$\begin{aligned}\lambda &= \frac{-\left[-ax_1\left(\frac{x_1+x_2}{2}\right) + by_1\left(\frac{y_1+y_2}{2}\right) - c\left(\frac{z+z_2}{2}\right) - \left(z_1\right)\right]}{\left[ax_2\left(\frac{x_1+x_2}{2}\right) + by_2\left(\frac{y_1+y_2}{2}\right) - c\left(\frac{2+z_2}{2}\right) - \left(z_2\right)\right.} \\ &= \frac{-\left[ax_1x_2 + by_1y_2 - c\left(z_1+z_2\right)\right]}{\left[ax_2x_1 + by_2y_1 - c\left(2_1+z_2\right)\right]} \\ &= -1\end{aligned}$$

P and Q lies on paraboloid, therefore

$$\begin{aligned} ax_1^2 + by_1^2 - 2cz &= 0 \\ ax_2^2 + by_2^2 - 2cz_2 &= 0\end{aligned}$$

Hence equation of plane (from(3))

$$a\left(x, +x_2\right)x + b(y, -y,)y = c\left(z+z_2\right)$$

D.R of Normal of this plane are

$$\left\langle a\left(x_1+x_2\right), b\left(y_1+y_2\right), 0\right\rangle$$

Axis of the paraboloid is $z-axis$ D.R. of z-axis are $(0,0,1)$

$$\therefore a\left(x_1-x_2\right)\times 0+b\left(y_1-y_2\right)\times 0+0x_1=0$$

Therefore, the above plane is parallel to axis of paraboloid.

10.2 Section-B

Question-5(a) Find the family of curves whose tangents form an angle $\pi/4$ with hyperbolas $xy=c$.

[10 Marks]

Solution: The given curve is

$$xy=C$$
$$\Rightarrow y=\frac{C}{x}$$
$$\frac{dy}{dx}=\frac{-C}{x^2}=m_2(\text{say})$$

From the question,

$$\frac{\pi}{4}=\tan^{-1}\left(\frac{m_1-m_2}{1+m_1m_2}\right)$$
$$\tan\frac{\pi}{4}=\frac{\frac{dy}{dx}+\frac{C}{x^2}}{1-\frac{dy}{dx}\cdot\frac{C}{x^2}}$$
$$\Rightarrow 1-\frac{dy}{dx}\cdot\frac{C}{x^2}=\frac{dy}{dx}+\frac{C}{x^2}$$
$$\Rightarrow \frac{dy}{dx}\left[1+\frac{C}{x^2}\right]=1-\frac{C}{x^2}$$
$$\Rightarrow \frac{dy}{dx}=\frac{x^2-C}{x^2+C}$$
$$=\frac{x^2+C-2C}{x^2+C}$$
$$=1-\frac{2C}{x^2+C}$$
$$\therefore dy=\left(1-\frac{2C}{x^2+C}\right)dx$$

Integrating both the sides, we get

$$y=x-\frac{2C}{\sqrt{C}}\tan^{-1}\frac{x}{\sqrt{C}}+C^1$$

$(C^1=$ integration constants$)$

$$y=x-2\sqrt{C}\tan^{-1}\frac{x}{\sqrt{C}}+C^1$$

is the required family of curves.

Question-5(b) Solve: $\dfrac{d^2y}{dx^2} - 2\tan x\dfrac{dy}{dx} + 5y = \sec x \cdot e^x$

[10 Marks]

Solution: Here $P = -2\tan x, Q = -\left(a^2+1\right)$ and $R = e^x \sec x$.
We choose $u = e^{-\frac{1}{2}\int P dx} = e^{\int \tan x dx} = e^{\log \sec x} = \sec x$.
Putting $y = uv$ in the given equation, it reduces to its normal form

$$\frac{d^2\nu}{dx^2} + Xv = Y \qquad \ldots(1)$$

$$\begin{aligned} \text{where } X &= Q - \frac{1}{2}\frac{dP}{dx} - \frac{1}{4}P^2 = -\left(a^2+1\right) - \frac{1}{2}\cdot\left(-2\sec^2 x\right) - \frac{1}{4}\cdot 4\tan^2 x \\ &= -a^2 - 1 + \sec^2 x - \tan^2 x = -a^2 - 1 + 1 = -a^2 \\ \text{and } Y &= Re^{\frac{1}{2}\int P dx} = e^x \sec x e^{-\int \tan x dx} = \left(e^x \sec x\right)\left(1/\sec x\right) \\ &= e^x \end{aligned}$$

Hence, the normal form (1) of the given differential equation is

$$\frac{d^2v}{dx^2} - a^2 v = e^x, \text{ or } \left(D^2 - a^2\right)v = e^x \qquad \ldots(2)$$

Now (2) is a linear differential equation with constant coefficients.
A.E. is $m^2 - a^2 = 0$, or $m^2 = a^2$ giving $m = \pm a$. $\therefore$ C.F. of the solution of (2) $= c_1 e^{ax} + c_2 e^{-ax}$.

$$P.I. = \frac{1}{D^2 - a^2}e^x = \frac{1}{1^2 - a^2}e^x$$

.
$\therefore$ the solution of (2) is $v = c_1 e^{ax} + c_2 e^{-ax} + \frac{e^x}{1-a^2}$.
Hence the general solution of the given differential equation is

$$y = uv = \left(c_1 e^{ax} + c_2 e^{-ax}\right)\sec x + \frac{e^x \sec x}{1 - a^2}$$

Question-5(c) The apses of a satellite of the Earth are at distances r_1 and r_2 from the centre of the Earth. Find the velocities at the apses in terms of r_1 and r_2.

[10 Marks]

Solution: Let the satellite of the Earth moves under the inverse square law $= \frac{\mu}{r^2}$
Clearly the satellite will move in elliptical orbit and the velocity at a distance r is given by

$$v^2 = \mu\left[\frac{2}{r} - \frac{1}{a}\right]$$

where $2a$ = major axis of elliptical orbit Now at apse

$$r_1 = a + ae$$

and

$$r_2 = a - ae \quad \Rightarrow 2a = r_1 + r_2$$

Now from question at $r = r_1, v = v_1$

$$\begin{aligned}
v_1^2 &= \mu\left[\frac{2}{r_1} - \frac{1}{a}\right] \cdots \\
&= \mu\left[\frac{2}{r_1} - \frac{2}{r_1 + r_2}\right] \\
v_1^2 &= 2\mu\left[\frac{1}{r_1} - \frac{1}{r_1 + r_2}\right] \\
&= 2\mu\left[\frac{r_1 + r_2 - r_1}{r_1(r_1 + r_2)}\right] \\
v_1^2 &= \frac{2\mu r_2}{r_1 + r_2}
\end{aligned}$$

$$\Rightarrow \quad v_1 = \sqrt{\frac{2\mu r_2}{r_1(r_1 + r_2)}}$$

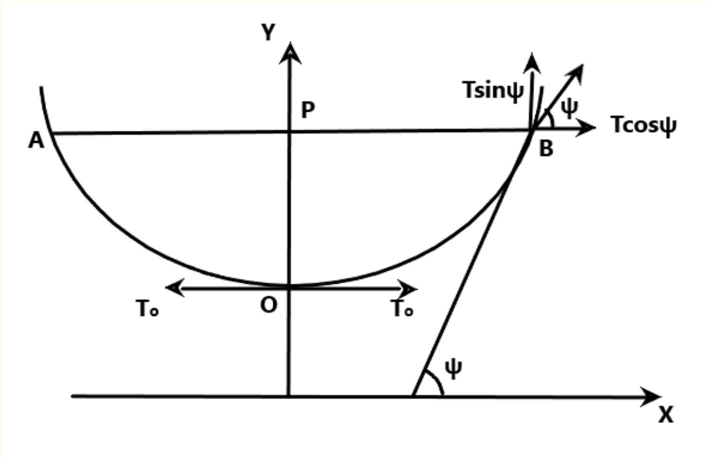
Similarly,

$$v_2 = \sqrt{\frac{2\mu r_1}{r_2(r_1 + r_2)}}$$

Question-5(d) A cable of length 160 meters and weighing 2 kg per meter is suspended from two points in the same horizontal plane. The tension at the points of support is 200 kg. Show that the span of the cable is $120\cosh^{-1}\left(\frac{5}{3}\right)$ and also find the sag.

[10 Marks]

Solution: Weight of the cable $(= 160 \times 2 = 320)$ will act at middle point of AB, i.e., at point O.



In equilibrium,

$$W = 2T\sin\psi$$
$$\implies 3202 \times 200\sin\psi$$
$$\implies \sin\psi = \frac{4}{5}$$

We know that equation of common catenary is given by:

$$y = c\cosh\left(\frac{x}{c}\right) ...(i)$$

and $T = wy$, where w = weight per unit length $= 2kg/m$
At point A or B,

$$T = 200kg$$
$$\implies y = T/w = 100m$$

Also,

$$T = T_0\cos\psi = wc$$
$$\therefore 22 \times \frac{3}{5} = 2c \;\; (\because \sin\psi = \frac{4}{5})$$
$$\implies c = 60m$$

Now, for span, we put y=100 m in equation of catenary (i),

$$\implies 100 = 60\cosh\left(\frac{x}{60}\right) \Rightarrow x = 60\cos h^{-1}\left(\frac{5}{3}\right)$$

$$Span = 2x = 120\cos h^{-1}(5/3)$$
$$Sag = y - c = 100 - 60 = 40 \text{ m}$$

.

Question-5(e) Evaluate the line integral $\oint_C (\sin x dx + y^2 dy - dz)$, where C is the circle $x^2 + x^2 = 16, z = 3$, by using Stokes' theorem.

[10 Marks]

Solution: The given line integral is

$$\int_C \left(\sin x dx + y^2 dy - dz\right)$$

where C is the circle $x^2 + y^2 = 16, z = 3$

$$\int_C \left(\sin x\hat{i} + y^2\hat{j} - \hat{k}\right) \cdot (dxi + dy\hat{j} + dz\hat{k}) = \int_C \bar{F} \cdot dr$$

where $\bar{F} = \sin x\hat{i} + y^2\hat{j} - \hat{k}$ Now from the Stokes' theorem

$$\int_C \bar{F} \cdot dr = \oiint_S (\bar{\nabla} \times \bar{F}) \cdot d\bar{s}$$

where S is the surface enclosed by the curves. Now,

$$(\bar{\nabla} \times \bar{F}) \cdot \bar{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin x & y^2 & -1 \end{vmatrix} \cdot K = 0$$

$$\Rightarrow \quad \oint \bar{F} \cdot d\bar{r} = 0$$

Question-6(a) Solve: $p^2 + 2py \cot x = y^2$ **where** $p = \frac{dy}{dx}$

[10 Marks]

Solution:

$$p^2 + 2py \cot x = y^2$$
$$p^2 + 2py \cot x - y^2 = 0$$

Solving the above equation, for the quadratic in p, we get

$$p = \frac{-2y \cot x \pm \sqrt{4y^2 \cot^2 x + 4y^2}}{2}$$
$$p = -y \cot x \pm y \operatorname{cosec} x$$

Case I: When $p = -y \cot x + y \operatorname{cosec} x$
then

$$\frac{dy}{dx} = y(-\cot x - \operatorname{cosec} x)$$
$$\frac{dy}{y} = (-\cot x + \operatorname{cosec} x)dx$$

$$\frac{dy}{y} = \left(-\frac{\cos x}{\sin x} + \frac{1}{\sin x}\right) dx$$
$$= \left(\frac{2\sin^2 x/2}{2\sin\frac{x}{2}\cos\frac{x}{2}}\right) dx$$
$$r\frac{dy}{y} = \tan\frac{x}{2} dx$$

integrating both the sides, we get

$$\log y = 2\log\sec\frac{x}{2} + \log C_1$$
$$< \log C_1 \text{ (integration constant)}$$
$$y = C_1 \sec^2\frac{x}{2}$$
$$y - C_1\sec^2\frac{x}{2} = 0$$

is one solution.
Case II: when $p = -y\cot x - y\,\mathrm{cosec}\, x$
then,

$$\frac{dy}{dx} = -y(\cot x - \mathrm{cosec}\, x)dx$$
$$\frac{dy}{y} = (-\cot x - \mathrm{cosec}\, x)dx$$
$$\frac{dy}{y} = -\left(\frac{1+\cos x}{\sin x}\right) dx = \frac{-\cos x/2}{\sin x/2}dx$$

Integrating both the sides, we get

$$\log y = 2\log\mathrm{cosec}\frac{x}{2} + \log C_2$$
$$< \log C_2 = \text{ integration constant}$$
$$y - C_2\,\mathrm{cosec}^2\frac{x}{2} = 0$$

is another solution.
Hence, the required solution is given by,

$$\left(y - C_1\sec^2\frac{x}{2}\right)\left(y - C_2\,\mathrm{cosec}^2\frac{x}{2}\right) = 0$$

where C_1 and C_2 are arbitrary constant.

Question-6(b) Solve:

$$(x^4D^4 + 6x^3D^3 + 9x^2D^2 + 3xD + 1)y = (1 + \log x)^2$$

where $D \equiv \dfrac{d}{dx}$

[15 Marks]

Solution: The given differential equation is

$$\left[x^4D^4 + 6x^3D^3 + 9x^2D^2 + 3xD + 1 \mid y = \langle 1 + \log x\rangle^2\right.$$

It is solved by putting $x = e^z$ then reducing the above equation in the form of y and

$$x = e^z$$

since,

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{dz}\cdot\frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}\\
x\frac{dy}{dx} &= \frac{dy}{dz}\\
x\frac{dy}{dx} &= D_1 y \quad (\text{where } D_1 = \frac{d}{dz}\ )\\
\frac{d^2y}{dx^2} &= \frac{d}{dx}\left(\frac{dy}{dx}\right)\\
&= \frac{d}{dz}\left(\frac{dy}{dx}\right)\cdot\frac{dz}{dx}\\
&= \frac{d}{dz}\left(\frac{1}{x}\cdot\frac{dy}{dz}\right)\cdot\frac{1}{x}\\
&= \frac{1}{x}\left[\frac{1}{x}\cdot\frac{d^2y}{dz^2} - \frac{1}{x}\frac{dy}{dz}\right]\\
x^2\frac{d^2y}{dx^2} &= \frac{d^2y}{dz^2} - \frac{dy}{dz}\\
x^2\frac{d^2y}{dx^2} &= D_1\left(D_1 - 1\right)y\\
\text{Similarly, } x^3\frac{d^3y}{dx^3} &= D_1\left(D_1 - 1\right)\left(D_1 - 2\right)y\\
\text{and } x^4\frac{d^4y}{dx^4} &= D_1\left(D_1 - 1\right)\left(D_1 - 2\right)\left(D_1 - 3\right)y
\end{aligned}$$

Putting these values in equation (1), we get

$$[D_1\left(D_1 - 1\right)\left(D_1 - 2\right)\left(D_1 - 3\right) + 6D_1\left(D_1 - 1\right)\left(D_1 - 2\right) +$$

$$9D_1\left(D_1 - 1\right) + 3D_1 + 1]\, y = \left(1 + e^2\right)^2$$

$$\Rightarrow \left[D_1\left(D_1^3 - 6D_1^2 + 11D_1 - 6\right) + 6D_1\left(D_1^2 - 3D_1 + 2\right) +\right.$$

$$\left.9D_1\left(D_1 - 1\right) + 3D_1 + 1\right] y = 1 + 2e^2 + e^2$$

$$\left(D_1^4 + 2D_1^2 + 1\right)y = 1 + 2e^z + e^{2z}$$

The auxiliary equation is given by

$$m^4 + 2m^2 + 1 = 0$$

$$\Rightarrow \left(m^2 + 1\right)^2 = 0$$

$$\Rightarrow \quad m = \pm i, \pm i$$

Hence, the complementary function is given by

$$y = \left(C_1 + C_2 z\right)\cos z + \left(C_3 + C_4 z\right)\sin z$$

where C_1, C_2, C_3 and C_4 are arbitrary constant Now, the Particular Integral is given by

$$\begin{aligned} y &= \frac{1}{(1+2D_1^2+D_1^4)}\left(1+2e^z+e^{2z}\right) \\ &= \frac{1}{(1+D_1^2)^2} + 2\cdot\frac{1}{(1+D_1^2)^2}e^z + \frac{e^{2z}}{(1+D_1^2)^2} \\ &= 1 + \frac{2\cdot e^z}{(1+1)^2} + \frac{e^{2z}}{(1+4)^2} = 1 + \frac{e^z}{2} + \frac{e^{2z}}{25} \end{aligned}$$

i.e., the general solution is given by

$$\begin{aligned} y &= \text{C.F.} + \text{P.I.} \\ &= (C_1 + C_2 z)\cos z + (C_3 + C_4 z)\sin z + 1 + \frac{e^z}{2} + \frac{e^{2z}}{25} \end{aligned}$$

Putting the value of z, we get

$$y = (C_1 + C_2\log x)\cos(\log x) + (C_3 + C_4\log x)\sin(\log x) + 1 + \frac{x}{2} + \frac{x^2}{25}$$

is the required solution.

Question-6(c) Solve:

$$\left(\mathrm{D}^4 + \mathrm{D}^2 + 1\right)\mathrm{y} = \mathrm{ax}^2 + \mathrm{be}^{-\mathrm{x}}\sin 2\mathrm{x}$$

, where $D = \dfrac{d}{dx}$

[15 Marks]

Solution:

$$\left(D^4 + D^2 + 1\right)y = ax^2 + be^{-x}\sin 2x$$

The auxiliary equation is given by

$$m^4 + m^2 + 1 = 0$$

$$m^4 + 2m^2 + 1 - m^2 = 0$$

$$\left(m^2+1\right)^2 - m^2 = 0$$

$$\Rightarrow \left(m^2+m+1\right)\left(m^2-m+1\right) = 0$$

$$m = \frac{-1\pm\sqrt{1-4}}{2}, \frac{1+\sqrt{1-4}}{2}$$

$$m = \frac{-1}{2} \pm \frac{\sqrt{3}i}{2}$$

and

$$m = \frac{1}{2} \pm \frac{\sqrt{3}i}{2}$$

are the roots of auxiliary equation.// Hence, the complementary function is given by

$$y = e^{-x/2}\left[C_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + C_2 \sin\left(\frac{\sqrt{3}}{2}x\right)\right] +$$
$$e^{x/2}\left[C_3 \cos\left(\frac{\sqrt{3}x}{2}\right) + C_4 \sin\left(\frac{\sqrt{3}x}{2}\right)\right]$$

Now, the Particular Integral is given by

$$y = \frac{1}{(1 + D^2 + D^4)}\left\{ax^2 + be^{-x}\sin 2x\right\}$$

Consider

$$\begin{aligned}
\frac{a}{(1 + D^2 + D^4)}x^2 &= a\left\langle 1 + D^2 + D^4\right\rangle^{-1} x^2 \\
&= a\left\langle 1 - \left(D^2 + D^4\right) + \left(D^2 + D^4\right)^2 + \dots\right\rangle x^2 \\
&= a\left[\left(x^2\right) - D^2(x)^2\right] = a\left[x^2 - 2\right]
\end{aligned}$$

Now Consider,

$$\begin{aligned}
\frac{1}{(1 + D^2 + D^4)}e^{-x}\sin 2x &= e^{-x}\frac{1}{[1 + (D-1)^2 + (D-1)^4)}\sin 2x \\
&= e^{-x}\,\frac{1}{[1 + D^2 - 2D + 1 + D^4 - 4D^3 + 6D^2} = 4D + 1\Big]^{\sin 2x} \\
&= e^{-x}\frac{1}{(D^4 - 4D^3 + 7D^2 - 6D + 3)}\sin 2x \\
&= e^{-x}\frac{1}{\left(D^2\right)^2 - 4D\left(D^2\right) + 7\cdot D^2 - 6D + 3}\sin 2x \\
&= e^{-x}\frac{1}{\left(-2^2\right)^2 - 4D\left(-2^2\right) + 7\left(-2^2\right) - 6D + 3}\sin 2x \\
&= e^{-x}\frac{1}{16 + 16D - 28 - 6D + 3}\sin 2x \\
&= e^{-x}\frac{1}{10D - 9}\sin 2x \\
&= e^{-x}\frac{(10D + 9)}{(100D^2 - 81)}\sin 2x \\
&= e^{-x}\frac{(10D + 9)\sin 2x}{-400 - 81} \\
&= \frac{-e^{-x}}{481}(20\cos 2x + 9\sin 2x) \\
\therefore y &= \frac{1}{(1 + D^2 + D^4)}\left(ax^2 + be^{-x}\sin 2x\right) \\
&= a\left(x^2 - 2\right) - \frac{be^{-x}}{481}(20\cos 2x + 9\sin 2x)
\end{aligned}$$

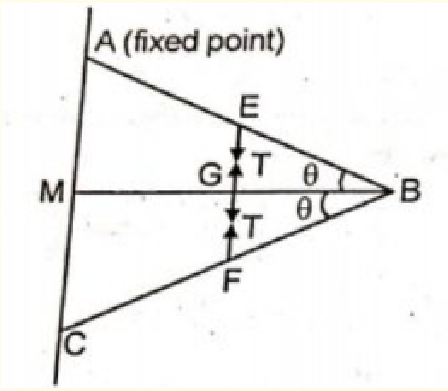
Hence, the general solution is given by,

$$y = e^{\frac{-x}{2}}\left[C_1 \cos\left(\frac{\sqrt{3}}{2}x\right) + C_2 \sin\left(\frac{\sqrt{3}}{2}x\right)\right] + e^{\frac{x}{2}}\left[C_3 \cos\left(\frac{\sqrt{3}}{2}x\right) + C_4 \sin\left(\frac{\sqrt{3}}{2}x\right)\right]$$
$$+a\left(x^2 - 2\right) - \frac{be^{-x}}{481}(20\cos 2x + 9\sin 2x)$$

Question-7(a) One end of a uniform rod AB, of length 2a and weight W, is attached by a frictionless joint to a smooth wall and the other end B is smoothly hinged to an equal rod BC. The middle points of the rods are connected by an elastic cord of natural length a and modulus of elasticity 4W. Prove that the system can rest in equilibrium in a vertical plane with C in contact with the wall below A, and the angle between the rod is $2\sin^{-1}\left(\frac{3}{4}\right)$.

[13 Marks]

Solution: AB and BC are two rods each of length $2a$ and weight W smoothly jointed together at B. The end A of the rod AB is attached to a smooth vertical wall and the end C of the rod BC is in contact with the wall. The middle points E and F of rods AB and BC are connected by an clastic string of natural length a.



Let T be the tension in the string EF. The total weight 2 W of the two rods can be taken acting at the middle point of EF. The line BG is horizontal and meets AC at its middle point M. Let $\angle\text{ABM} = \theta = \angle\text{CBM}$

Give the system a small symmetrical displacement about BM in which θ changes to $\theta+\delta\theta$. The point A remains fixed, the point G is slightly displaced, the length EF changes, the lengths of the rods AB and BC do not change. We have $\text{EF} = 2\text{EG} = 2\ \text{EB}\sin\theta = 2a\sin\theta$ Also the depth of G below the fixed point

$$\text{A} = \text{AM} = \text{AB}\sin\theta = 2a\sin\theta$$

The equation of virtual work is

$$-T\delta(2a\sin\theta) + 2W\delta(2a\sin\theta) = 0$$

$$(-2aT\cos\theta + 4aW\cos\theta)\delta\theta = 0$$

$$2a\cos\theta(-T + 2W)\delta\theta = 0$$

$$-T + 2W = 0$$

$$[\because \delta\theta \neq 0 \text{ and } \cos\theta \neq 0]$$

$$T = 2W$$

Also, by Hooke's law the tension T in the elastic string EF is given by

$$T = \lambda \cdot \frac{2a\sin\theta - a}{a} \left(\begin{array}{c} \text{where } \lambda \text{ is the modulus} \\ \text{elasticity of the sthing} \end{array} \right)$$
$$T = 4W(2\sin\theta - 1)$$

Equating the two values of T, we

$$2W = 4W(2\sin\theta - 1)$$
$$1 = 2(2\sin\theta - 1)$$
$$1 = 4\sin\theta - 2$$
$$3 = 4\sin\theta$$
$$\sin\theta = \frac{3}{4}$$
$$\theta = \sin^{-1}\left(\frac{3}{4}\right)$$

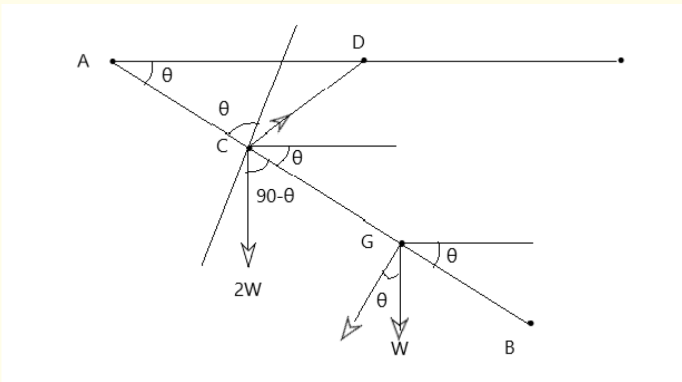
$\therefore$ In equilibrium, the whole angle between AB and BC

$$\implies 2\theta = 2\sin^{-1}\left(\frac{3}{4}\right)$$

Question-7(b) AB is a uniform rod, of length 8 a, which can turn freely about the end A, which is fixed. C is a smooth ring, whose weight is twice that of the rod, which can slide on the rod, and is attached by a string CD to a point D in the same horizontal plane as the point A. If AD and CD are each of length a, fix the position of the ring and the tension of the string when the system is in equilibrium. Show also that the action on the rod at the fixed end A is a horizontal force equal to $\sqrt{3}$W, where W is the weight of the rod.

[14 Marks]

Solution: Given $AD = CD = a$



Force at ring C along the rod AB

$$2W\cos(90 - \theta) = T\cos\theta$$
$$2W\sin\theta = T\cos\theta$$

Moments about A

$$
\begin{aligned}
2W \times ACCA\theta + W \times AGGS\theta &= T(\sin\theta AC) \\
2W\cos\theta \cdot AC - T\sin\theta(AC) &= -W\cos\theta(AG) \\
(T\sin\theta - 2W\cos\theta)AC &= W\cos\theta \cdot 4a \\
2\phi\left(\frac{\sin^2\theta}{\cos\theta} - \cos\theta\right)AC &= \mu\cos\theta \cdot 4a \\
\frac{\sin^2\theta - \cos^2\theta}{\cos\theta} \cdot 2\pi\cos\theta &= 2a\cos\theta \\
-\cos 2\theta &= \cos\theta \\
\Rightarrow \quad 2\theta &= 1\pi - \theta \\
\Rightarrow \quad \theta &= \frac{\pi}{3}
\end{aligned}
$$

$$T = 2W\tan\frac{\pi}{3} \Rightarrow T = 2\sqrt{3}W$$

Horizontal component at A,

$$
\begin{aligned}
&= T\cos\theta = T\cos\frac{\pi}{3} \\
2\sqrt{3} &= \frac{1}{3}
\end{aligned}
$$

Vertical Component

$$
\begin{aligned}
&= 3W - T\sin\theta \\
&= 3W - T\sin\frac{\pi}{3} \\
&= 3W - 2\sqrt{3}W\frac{2\sqrt{3}}{2} \\
&= 0
\end{aligned}
$$

So, only action at A is the horizontal force.

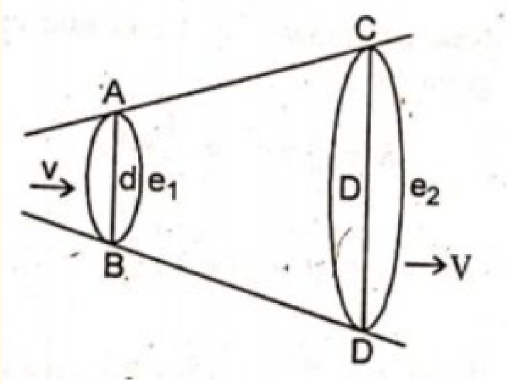
Question-7(c) A stream is rushing from a boiler through a conical pipe, the diameter of the ends of which are D and d; If V and v be the corresponding velocities of the stream and if the motion is supposed to be that of the divergence from the vertex of cone, prove that

$$\frac{v}{\mathrm{V}} = \frac{\mathrm{D}^2}{\mathrm{d}^2}\mathrm{e}^{\left(v^2 - \mathrm{V}^2\right)/2K}$$

where K is the pressure divided by the density and supposed constant.

[13 Marks]

Solution: Let e_1 and e_2 be the densities of steam at the ends of the conical pipe AB and CD. By the principle of conservation of mass, the mass of the steam that enters and leaves at the ends AB and CD are the same. Thus we have



$$\pi\left(\frac{1}{2}d\right)^2 ve_1 = \pi\left(\frac{1}{2}D\right)^2 Ve_2^-$$

$$\frac{v}{V} = \frac{D^2e_2}{d^2e_1}$$

let p be the pressure, e the density and u the velocity at distance r from AB, then the equation of motion is given by

$$u\frac{\partial u}{\partial r} = -\frac{1}{e}\frac{\partial p}{\partial r}$$

$$p = Ke$$

$$u\frac{\partial u}{\partial r} = -\frac{K}{e}\frac{\partial e}{\partial r}$$

By integrating, we have

$$\frac{1}{2}u^2 = -K\log e + K\log E$$

where E is an arbitrary constant

$$\log\frac{e}{E} = -\frac{u^2}{2K}$$

$$e = E\exp\left(-\frac{u^2}{2K}\right)$$

Again

$$e = e_1 \text{ when } u = v$$

$$c_1 = E\exp\left(-\frac{v^2}{2K}\right)$$

and

$$e = e_2 \text{ when } u = v$$

$$e_2 = E\exp\left(\frac{-V^2}{2K}\right)$$

$$\frac{e_1}{e_2} = \frac{\exp\left(-v^2/2K\right)}{\exp\left(-V^2/2K\right)}$$

from (1) and (2), we have

$$\frac{v}{V} = \frac{D^2}{d^2}\exp\cdot\left(\frac{v^2-V^2}{2K}\right)$$

Proved.

Question-8(a) Find the curvature, torsion and the relation between the arc length S and parameter u for the curve: $\vec{\mathbf{r}} = \vec{\mathbf{r}}(\mathrm{u}) = 2\log_{\mathbf{e}} \mathbf{u}\hat{\mathbf{i}} + 4\mathbf{u}\hat{\mathbf{j}} + (2\mathbf{u}^2 + 1)\,\hat{\mathbf{k}}$

[10 Marks]

Solution: The parametric equation of the given curve is

$$\vec{r} = \vec{r}(u) = 2\log_e u\hat{i} + 4u\hat{j} + \left(2u^2 + 1\right)\hat{k}$$

where u is the parameter. Now,

$$\dot{\vec{r}} = \frac{d\bar{r}}{du} = \frac{2.1}{u}\hat{i} + 4\dot{j} + (4ut)\hat{k}$$

and

$$\ddot{\vec{r}} = \frac{d^2r}{du^2} = \frac{-2}{u^2}\hat{i} + 0 + 4\hat{k}$$

and

$$\dddot{\vec{r}} = \frac{d^3r}{du^2} = \frac{4}{u^3}\hat{i}$$

Now, we know that the curvature (k) of the curve is given by

$$K = \frac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{\vec{r}}|^3}$$

$$Now, \dot{\vec{r}} \times \ddot{\vec{r}} = \begin{bmatrix} i & j & k \\ \frac{2}{u} & 4 & 4u \\ \frac{-2}{u^2} & 0 & 4 \end{bmatrix}$$

$$= 16\hat{i} - \frac{16}{u}\hat{j} + \frac{8}{u^2}\hat{k}$$

$$\Rightarrow |\dot{\vec{r}} \times \ddot{\vec{r}}| = \sqrt{16^2 + \left(\frac{16}{u}\right)^2 + \left(\frac{8}{u^2}\right)^2}$$

$$= \frac{8\left(1 + 2u^2\right)}{u^2}$$

$$\text{and } |\dot{\vec{r}}| = |\frac{2\hat{i} + 4\hat{j} + 4u\hat{k}\ |}{u}$$

$$= 2\sqrt{\left(\frac{1}{u}\right)^2 + 2^2 + (2u)^2}$$

$$= \frac{2\left(1 + 2u^2\right)}{u}$$

$$\therefore K = \frac{|\dot{\vec{r}} \times \ddot{\vec{r}}|}{|\dot{\vec{r}}|^3}$$

$$= \frac{8\left(1 + 2u^2\right)}{u^2} \cdot \frac{u^3}{8\left(1 + 2u^2\right)^3}$$

$$\therefore \quad = \frac{u}{\left(1 + 2u^2\right)^2}$$

$$\therefore \quad = \frac{u}{\left(1 + 2u^2\right)^2}$$

and the torsion (T) is given by

$$T = \frac{[\dot{\vec{r}} \quad \ddot{\vec{r}} \quad \dddot{\vec{r}}]}{|\dot{\vec{r}} \times \ddot{\vec{r}}|^2}$$

$$\text{Now, } [\dot{\vec{r}} \quad \ddot{\vec{r}} \quad \dddot{\vec{r}}] = \begin{bmatrix} \frac{2}{u} & 4 & 4u \\ \frac{-2}{u^2} & 0 & 4 \\ \frac{4}{u^3} & 0 & 0 \end{bmatrix} = \frac{64}{u^3}$$

$$\therefore \quad = \frac{64}{u^3} \cdot \frac{u^4}{64\left(1+2u^2\right)^2}$$

$$= \frac{u}{\left(1+2u^2\right)^2}$$

$$\text{Torsion } (T) = \frac{u}{\left(1+2u^2\right)^2}$$

Now, the arc length S is given by the

$$\int ds = \int \frac{d\vec{r}}{du} \mid \cdot du$$

$$\because \quad \frac{d\vec{r}}{du} = \frac{2}{u}\hat{i} + 4\hat{j} + 4u\hat{k}$$

$$\Rightarrow \left|\frac{d\vec{r}}{du}\right| = \sqrt{\frac{4}{u^2} + 16 + 16u^2}$$

$$= \frac{2\left(1+2u^2\right)}{u}$$

$$\therefore \quad \int ds = \int 2\left(\frac{1+2u^2}{u}\right) du$$

$$= 2\int \frac{1}{u} du + 4\int u du$$

$$\therefore \quad S = 2\log u + 2u^2$$

$$\Rightarrow \quad S = 2\left(u^2 + \log u\right)$$

is the required relation between S and parameter u.

Question-8(b) Prove the vector identity: $curl(\vec{f} \times \vec{g}) = \vec{f} \operatorname{div} \vec{g} - \vec{g} \operatorname{div} \vec{f} + (\vec{g} \cdot \nabla)\vec{f} - (\vec{f} \cdot \nabla)\vec{g}$ **and verify it for the vectors** $\vec{f} = x\hat{i} + z\hat{j} + y\hat{k}$ **and** $\vec{g} = y\hat{i} + z\hat{k}$.

[10 Marks]

Solution: The given vector identity is

$$\nabla \times (\bar{f} \times \vec{g}) = \bar{f}(\bar{\nabla} \cdot \bar{g}) - \bar{g}(\bar{\nabla} \cdot \bar{f}) + \quad (\bar{g} \cdot \bar{\nabla})\bar{f} - (\bar{f} \cdot \bar{\nabla})\bar{g}$$

$$\begin{aligned}
\text{L.H.S.} &= \bar{\nabla} \times (\bar{f} \times \bar{g}) \\
&= \Sigma i \times \frac{\partial}{\partial x}(\bar{f} \times \bar{g}) \\
&= \sum \bar{i} \times \left[\left(\frac{\partial \bar{f}}{\partial x} \times \bar{g}\right) + \left(\bar{f} \times \frac{\partial \bar{g}}{\partial x}\right)\right] \\
&= \Sigma \bar{i} \times \left(\frac{\partial \bar{f}}{\partial x} \times \bar{g}\right) + \Sigma \bar{i} \times \left(\bar{f} \times \frac{\partial \bar{g}}{\partial x}\right)
\end{aligned}$$

Now, consider

$$\bar{i} \times \left(\frac{\partial \bar{f}}{\partial x} \times \bar{g}\right) = (\bar{g} \cdot \bar{i})\frac{\partial \bar{f}}{\partial x} - \left(\bar{i} \cdot \frac{\partial \bar{f}}{\partial x}\right)\bar{g} \quad \left(\begin{array}{c}\text{using relation } \bar{A} \times (\bar{B} \times \bar{C}) \\ = (\bar{A} \cdot \bar{C})\bar{B} - (\bar{A} \cdot \bar{B})\bar{C}\end{array}\right)$$

$$\begin{aligned}
\therefore \quad \Sigma \bar{i} \times \left(\frac{\partial \bar{f}}{\partial x} \times \bar{g}\right) &= \Sigma(\bar{g} \cdot \bar{i})\frac{\partial \bar{f}}{\partial x} - \Sigma\left(\bar{i} \cdot \frac{\partial \bar{f}}{\partial x}\right)\bar{g} \\
&= \bar{g} \cdot \left(\Sigma i \frac{\partial}{\partial x}\right)\bar{f} - \Sigma\left(i \frac{\partial}{\partial x} \cdot \bar{f}\right)\bar{g} \\
&= (\bar{g} \cdot \bar{\nabla})\bar{f} - (\bar{\nabla} \cdot \bar{f})\bar{g} \\
&= (\bar{g} \cdot \bar{\nabla})\bar{f} - \bar{g}(\bar{\nabla} \cdot \bar{f})
\end{aligned}$$

Similarly for

$$\begin{aligned}
\Sigma i \times \left(\bar{f} \times \frac{\partial \bar{g}}{\partial x}\right) &= \Sigma\left(\bar{i} \cdot \frac{\partial \bar{g}}{\partial x}\right) \cdot \bar{f} - \Sigma(\bar{i} \cdot \bar{f})\frac{\partial \bar{g}}{\partial x} \\
&= \bar{f}(\bar{\nabla} \cdot \bar{g}) - (\bar{f} \cdot \bar{\nabla})\bar{g}
\end{aligned}$$

hence

$$\bar{\nabla}(\bar{f} \times \bar{g}) = f(\bar{\nabla} \cdot \bar{g}) - g(\bar{\nabla} \cdot \bar{f}) + (\bar{g} \cdot \bar{\nabla})\bar{f} - (\bar{f} \cdot \bar{\nabla})\bar{g}$$

which proves the identity.

Now, if $\vec{f} = x\hat{i} + z\hat{j} + y\hat{k}$ and $\quad \vec{g} = y\hat{i} + z\hat{k}$ then

$$\begin{aligned}
\bar{f} \times \bar{g} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & z & y \\ y & 0 & z \end{vmatrix} \\
&= z^2\hat{i} + \left(y^2 - xz\right)\hat{j} - yz\hat{k}
\end{aligned}$$

Now

$$\begin{aligned}
\bar{\nabla} \times (\bar{f} \times \bar{g}) &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z^2 & y^2 - xz & -yz \end{vmatrix} \\
&= \hat{i}\left\{\frac{-\partial}{\partial y}(yz) - \frac{\partial}{\partial z}\left(y^2 - xz\right)\right\} - \hat{j}\left\{\frac{\partial}{\partial x}(-yz) - \frac{\partial}{\partial z}\left(z^2\right)\right\} \\
&\quad + \hat{k}\left\{\frac{\partial}{\partial x}\left(y^2 - xz\right) - \frac{\partial z^2}{\partial y}\right\} \\
&= \hat{i}(-z + x) + \hat{j}(2z) + \hat{k}(-z) \\
&= (x - z)\hat{i} + 2z\hat{j} - z\hat{k} \\
\bar{g} &= (y\hat{i} + z\hat{k})
\end{aligned}$$

$$\Rightarrow \quad \bar\nabla\cdot\bar g = 1 \text{ and } \quad \bar f = (x\hat i + z\hat j + y\hat k) \Rightarrow \quad \bar\nabla\cdot\bar f = 1$$

Also

$$\begin{aligned}(\bar g\cdot\bar\nabla)\bar f &= \left(\frac{y\partial}{\partial x} + \frac{z\partial}{\partial z}\right)(x\hat i + z\hat j + y\hat k)\\ &= (y\hat i + \vec j)\\ (\bar f\cdot\bar\nabla)\bar g &= \left(x\frac{\partial}{\partial x} + z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z}\right)(y\hat i + z\hat k)\\ &= (z\hat i + y\hat k)\end{aligned}$$

Hence,

$$\begin{aligned}\bar f(\bar\nabla\cdot\bar g) - \bar g(\bar\nabla\cdot\bar f) + (\bar g\cdot\bar\nabla)\bar f - (\bar f\cdot\bar\nabla)\bar g &= \vec f - \bar g + (y\hat i + z\hat j) - (z\hat i + y\hat k)\\ &= (x\hat i + z\hat j + y\hat k) - (y\hat i + z\hat k) + (y\hat i + z\hat j) - (z\hat i + y\hat k)\\ &= (x - z)\hat i + 2z\hat j - z\hat k\end{aligned}$$

The given identity is verified for the vector $\bar f = x\hat i + z\hat j + y\hat k$ and $\vec g = y\hat i + z\hat k$

Question-8(c) Verify Green's theorem in the plane for

$$\oint_C \left[\left(3x^2 - 8y^2\right)dx + (4y - 6xy)dy\right]$$

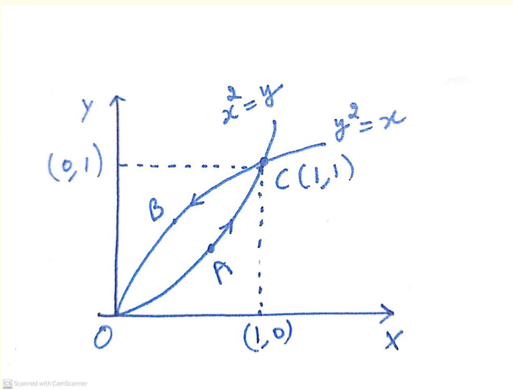
where C is the boundary of the region enclosed by the curves $y = \sqrt{x}$ and $y = x^2$.

[10 Marks]

Solution: The Green's theorem in a plane is given by

$$\oint_C Mdx + Ndy = \oiint_S \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)dxdy$$

where S is the area enclosed by the boundary of the curve C shown as shaded portion of the figure:



Now from question, the given curve is $y\ \sqrt{x}$ and $y = x^2$ and we have to verify Green's theorem for

$$\oint_C \left[\left(3x^2 - 8y^2\right) dx + (4y - 6xy)dy\right] = \int_{OAC} \left[3x^2 - 8y^2\right) dx + (4y - 6xy)dy +$$

$$\oint_{CBO} \left(3x^2 - 8y^2\right) dx + (4y - 6xy)dy$$

Consider $\int_{OAC} (3x^2 - 8y^2)\, dx + (4y - 6xy)dy$ along OAC path $y = \sqrt{x}$ or $x = y^2$

$$\therefore \quad dx = 2ydy$$

$\therefore$ the integral can be changed to

$$\begin{aligned}\int_{y=0}^{1} \left(3y^4 - 8y^2\right) 2ydy + \left(4y - 6y^3\right) dy &= \int_{y=0}^{1} \left(6y^5 + 16y^2 + 4y - 6y^3\right) dy \\ &= \int_{y=0}^{1} \left(6y^5 - 22y^2 + 4y\right) dy \\ &= \left[y^6 - \frac{22}{4}y^4 + \frac{4}{2}y^2\right]_0^1 \\ &= \left(1 - \frac{22}{4} + 2\right) \\ &= 3 - \frac{11}{2} = \frac{-5}{2}\end{aligned}$$

Consider $\int_{CBO} (3x^2 - 8y^2)\, dx + (4y - 6xy)dy$ along this path $y = x^2$

$$\begin{aligned}y = \int_{x=1}^{0} \left(3x^2 - 8x^4\right) dx + \left(4x^2 - 6x^3\right) 2xdx &= \int_{x=1}^{0} \left(3x^2 - 8x^4 + 8x^3 - 12x^4\right) dx \\ &= \int_{1}^{0} \left(3x^2 + 8x^3 - 20x^4\right) dx \\ &= 3\frac{x^3}{3} + 8\frac{x^4}{4} - 20\frac{x^5}{5}\bigg|_1^0 = 1\end{aligned}$$

$$\therefore \int_C \left(3x^2 - 8y^2\right) dx + (4y - 6xy)dy = \frac{-5}{2} + 1 = \frac{-3}{2}$$

Now consider the integral

$$\begin{aligned}\iint_S \left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) dxdy &= \iint (-16y - (-6y))dxdy \\ &= -10 \iint ydxdy \\ &= -\,10 \int_{x=0}^{1} \int_{y=x^2}^{\sqrt{x}} ydxdy \\ &= -10 \int_0^1 \frac{y^2}{2} \int_{x^2}^{\sqrt{x}} dx \\ &= -\,5 \int_0^1 \left(x - 4^4\right) dx \\ &= -\frac{3}{2}\end{aligned}$$

Hence the Green's theorem is verified.

Question-8(d) The position vector $\overrightarrow{r}$ of a particle of mass 2 units at any time t, referred to fixed origin and axes, is

$$\vec{r} = \left(t^2 - 2t\right)\hat{i} + \left(\frac{1}{2}t^2 + 1\right)\hat{j} + \frac{1}{2}t^2\hat{k}$$

At time $t = 1$, find its kinetic energy, angular momentum, time rate of change of angular momentum and the moment of the resultant force, acting at the particle, about the origin.

[10 Marks]

Solution: The position vector of the particle of mass 2 unit at time t is given by

$$\vec{r} = \left(t^2 - 2t\right)\hat{i} + \left(\frac{1}{2}t^2 + 1\right)\hat{j} + \frac{1}{2}t^2\hat{k}$$

Now we know that the kinetic energy is given by

$$K = \frac{1}{2}mv^2 = \frac{1}{2}m(\vec{v}\cdot\vec{v}) \because \quad \vec{r} = \left(t^2 - 2t\right)\hat{i} + \left(\frac{1}{2}t^2 + 1\right)\hat{j} + \frac{t^2}{2}\vec{k}$$

Hence

$$\begin{aligned}\vec{v} &= \frac{dr}{dt} = (2t-2)\hat{i} + t\hat{j} + t\hat{k}\\ \therefore \quad \vec{v}\cdot\bar{v} &= 4(t-1)^2 + t^2 + t^2\\ &= 2\left[(t-1)^2 + t^2\right]\\ \therefore \quad K &= \frac{1}{2}\cdot 2\cdot 2\left[2(t-1)^2 + t^2\right]\end{aligned}$$

$\therefore$ At $t = 1,$ the K.E. is given by

$$K = 2\left[2(1-1)^2 + 1^2\right] = 2 \text{ Units.}$$

$$\begin{aligned}\vec{v}|_{t=1} &= \hat{j} + \hat{k}\\ \vec{r}|_{t=1} &= -\hat{i} + \frac{3}{2}\hat{j} + \frac{1}{2}\hat{k}\\ \therefore \quad \vec{L}\Big|_{t=1} &= \vec{r}\times m\vec{v} = \begin{bmatrix}\hat{i} & \hat{j} & \hat{k}\\ -1 & \frac{3}{2} & \frac{1}{2}\\ 0 & 2 & 2\end{bmatrix}\\ \Rightarrow \quad \vec{L} &= 2(\hat{i} + \hat{j} + \hat{k})\end{aligned}$$

Since,

$$\vec{L} = \vec{r}\times m\vec{v}$$

differentiating both sides with respect to t we get

$$\frac{d\bar{L}}{dt} = \frac{dr}{dt} \times m\vec{v} + \vec{r} \times m\frac{d\bar{v}}{dt} = \vec{r} \times m\frac{d\bar{v}}{dt}$$

$$\because \quad \vec{v} = (2t-2)\hat{i} + \hat{t}\hat{j} + t\hat{k}$$

$$\Rightarrow \quad \frac{d\bar{v}}{dt} = 2\hat{i} + \hat{j} + \hat{k}$$

$$\therefore \left.\frac{d\bar{L}}{dt}\right|_{t=1} = 2\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & \frac{3}{2} & \frac{1}{2} \\ 2 & 1 & 1 \end{bmatrix} = 2\hat{i} + 4\hat{j} - 8\hat{k}$$

Finally, the moment of the resultant force is given by,

$$\vec{\tau} = \vec{r} \times \vec{F} = \vec{r} \times m\frac{d\bar{v}}{dt}$$

$$\Rightarrow \quad \vec{\tau}|_{t=1} = 2\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & \frac{3}{2} & \frac{1}{2} \\ 2 & 1 & 1 \end{bmatrix}$$

$$= 2\hat{i} + 4\hat{j} - 8\hat{k}$$

Thus, at $t = 1$, Kinetic energy = 2 units.

Angular momentum $(\vec{L}) = 2(\hat{i} + \hat{j} - \hat{k})$ units

Time Rate of change of angular momentum = $(2\hat{i} + 4\hat{j} - 8\hat{k})$ units and moment of the resultant force = $(2\hat{i} + 4\hat{j} - 8\hat{k})$ units.