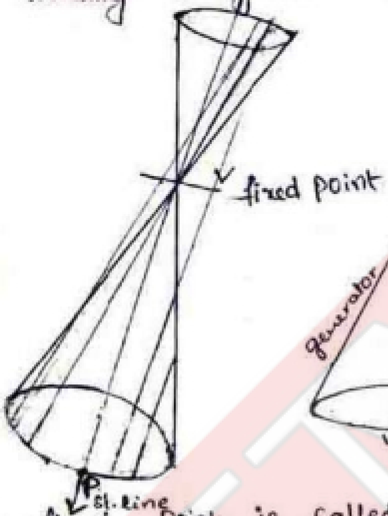
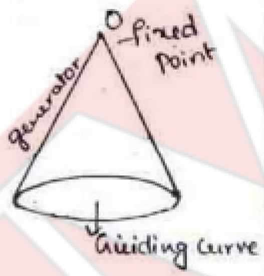
* Cones *

Set – V

Definition :-

A cone is a surface generated by a straight line which passes through a fixed point and satisfying one more condition i.e. intersecting a given curve (or) touching a given surface.





A fixed point is called the vertex and the given curve (or) surface is called the guiding curve [or guiding surface] of the cone.

The straight line is known as the generator of the cone.

→ A cone whose equation is of second degree is known as quadric cone (or) quadratic cone.

*** Equation of the cone with vertex at the origin :—**

To show that the equation of a cone whose vertex is the origin is homogeneous in x, y, z.

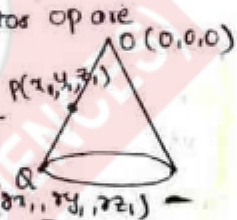
Sol^n :- Let the equation of the cone with vertex as the origin be

$$f(x, y, z) = 0 \quad \text{——} ①$$

Let $P(x_1, y_1, z_1)$ be any point on the cone.

$$\therefore f(x_1, y_1, z_1) = 0 \quad \text{——} ②$$

Equations of the generator OP are



$$\frac{x-0}{x_1-0} = \frac{y-0}{y_1-0} = \frac{z-0}{z_1-0}$$

$$\Rightarrow \frac{x}{x_1} = \frac{y}{y_1} = \frac{z}{z_1} \quad \text{——} ③$$

Any point on OP line is $Q(rx_1, ry_1, rz_1)$

Since the generator completely lies on the cone, then the point 'Q' lies on ~~completely~~ for all values 'r'

$$\therefore f(rx_1, ry_1, rz_1) = 0 \quad \text{for all values of 'r'} \quad \text{——} ④$$

from ② & ④ we have

$f(x_1, y_1, z_1)$ is homogeneous equation in x_1, y_1, z_1

$\therefore f(x, y, z) = 0$ is homogeneous in x, y, z.

Conversely, any homogeneous equation in x, y, z represents a cone whose vertex is the origin.

Sol^n :- Let the homogeneous equation be

$$f(x, y, z) = 0. \quad \text{——} ⑤$$

If $P(x_1, y_1, z_1)$ is any point on the above surface then $f(x_1, y_1, z_1) = 0$ —— (6)

Since the equation (5) is homogeneous, we have

$f(rx_1, ry_1, rz_1) = 0$ —— (7)

for all values of r.

But the point $Q(rx_1, ry_1, rz_1)$ is any point on the line OP

$\therefore$ Every point of the line OP lies on the surf (5)

$\therefore$ The surface is generated by the line through 'O'

$\therefore$ it represents a cone with vertex at the origin.

Note:— The second degree homogeneous

$ax^2 + by^2 + cz^2 + 2fyz + 2gxz + 2hxy = 0$

represents a cone with vertex at the origin

Note:— Method to make both equations homogeneous, when none of the two equations is a linear in x, y, z :—

(i) Make both equations homogeneous in x, y, z and t by introducing proper power of t, where t stands for 1.

(ii) Eliminate t from the two equations so obtained.

→ Find the equation to the cone with vertex at the origin and which pass through the curves given by the equations.

(i) $x^2+y^2+z^2-x-1=0$

$x^2+y^2+z^2-y-2=0$,

(ii) $x^2+y^2+z^2+x-2y+3z=4$,

$x^2+y^2+z^2+2x-3y+4z=5$.

Sol'n :- (i) The given equations

$x^2+y^2+z^2-x-1=0$ & $x^2+y^2+z^2+y-2=0$

$\Rightarrow x^2+y^2+z^2-xt-t^2=0$ —— (1) & $x^2+y^2+z^2+yt-2t^2=0$ —— (2)

where $t=1$

To eliminate t from (1) & (2)

Now (2) - (1) $\equiv -ty-tx+t^2=0$

$\Rightarrow t(t-x-y)=0$

$\Rightarrow t-x-y=0 \quad (\because t\neq 0)$

$\Rightarrow t=x+y$

$\therefore$ (1) $\equiv x^2+y^2+z^2-x(x+y)-(x+y)^2=0$

$\Rightarrow x^2+3xy-z^2=0$

which is the required equation of the cone.

→ Find the equations to the cone with vertex at the origin which pass through the curve

$ax^2+by^2+cz^2=1$, $lx+my+nz=p$.

Sol'n :- The given equations are

$ax^2+by^2+cz^2=1$ —— (1)

and $lx+my+nz=p$ —— (2)

(2) $\equiv \dfrac{lx+my+nz}{p}=1$

$① \equiv ax^2+by^2+cz^2 = (1)^2$

$① \equiv ax^2+by^2+cz^2 = \left(\frac{lx+my+nz}{p}\right)^2$

$\Rightarrow p^2(ax^2+by^2+cz^2) = (lx+my+nz)^2$

which is the required equation of the cone.

→ Find the equation to the cone at the origin which passes the curve $ax^2+by^2=2z$ —① , $lx+my+nz=p$ —②

Sol'n :- $② \equiv \frac{lx+my+nz}{p} = 1$

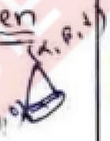
$① \equiv ax^2+by^2 = 2z\,(1)$

$\Rightarrow ax^2+by^2 = 2z\left[\frac{lx+my+nz}{p}\right]$

$\Rightarrow p(ax^2+by^2) = 2z(lx+my+nz)$

which is the required equation of the cone.

* Equation of a cone with a given vertex and a given base conic :-



To find the equation to the cone whose vertex is the point (α,β,γ) and base the conic

$f(x,y) = ax^2+by^2+2hxy+2fy+2gx+c=0,\ z=0.$

Sol'n :- The equations of the conic are $ax^2+by^2+cz^2+2hxy+2fy+2gx+c=0,\ z=0.$ —①

The equations of any line through (α,β,γ) are

$$\frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{z-\gamma}{n} \quad —②$$

This line meets the plane $z=0$.

$\therefore \frac{x-\alpha}{l} = \frac{y-\beta}{m} = \frac{0-\gamma}{n}$

$\Rightarrow \frac{x-\alpha}{l} = \frac{-\gamma}{n}\ /\ \frac{y-\beta}{m} = \frac{-\gamma}{n}$

$\Rightarrow x = \alpha - \frac{l\gamma}{n},\ y = \beta - \frac{m}{n}\gamma$

If this point $(x,y,0)$ lies on the given conic then

$$a\left[\alpha-\frac{l\gamma}{n}\right]^2 + b\left[\beta-\frac{m\gamma}{n}\right]^2 + 2h\left[\alpha-\frac{l\gamma}{n}\right]\left[\beta-\frac{m\gamma}{n}\right] + 2g\left[\alpha-\frac{l\gamma}{n}\right] + c = 0 \quad —③$$

This is the condition for line ② to intersect the conic ①.

Now eliminating l,m,n from ② & ③

Now putting the values of l,m,n from ② in ③ we have

$$a\left(\alpha-\frac{x-\alpha}{z-\gamma}\gamma\right)^2 + b\left(\beta-\frac{y-\beta}{z-\gamma}\gamma\right)^2 + 2h\left[\alpha-\frac{x-\alpha}{z-\gamma}\gamma\right]\left(\beta-\frac{y-\beta}{z-\gamma}\gamma\right) + 2f\left(\beta-\frac{y-\beta}{z-\gamma}\gamma\right) + 2g\left(\alpha-\frac{x-\alpha}{z-\gamma}\gamma\right) + c = 0$$

$$\Rightarrow a(\alpha z-\gamma x)^2 + b(\beta z-\gamma y)^2 + 2h(\alpha z-\gamma x)(\beta z-\gamma y) + 2f(\beta z-\gamma y)(z-\gamma) + 2g(\alpha z-\gamma x)(z-\gamma) + c(z-\gamma)^2 = 0 \quad —④$$

which is the required equation of the cone.

Note:— (i) The equation of the cone is satisfied by the coordinates of the vertex (α,β,γ) i.e putting α,β,γ for x,y,z in (4) we have

$$a(\alpha\gamma-\gamma\alpha)^2+b(\beta\gamma-\gamma\beta)+2h(\alpha\gamma-\gamma\alpha)(\beta\gamma-\gamma\beta)$$
$$+2f(\beta\gamma-\gamma\beta)(\gamma-\gamma)+2g(\alpha\gamma-\gamma\alpha)(\gamma-\gamma)+c(\gamma-\gamma)^2=0$$

$\Rightarrow 0=0$ which is true.

(ii) The equation of the cone (4) also satisfied by the equation of the base cone.

putting $z=0$ in (4) we have

$$a\gamma^2x^2+b\gamma^2y^2+2h\gamma^2xy+2f\gamma^2y+2g\gamma^2x+c\gamma^2=0$$

through out dividing with γ^2

$$ax^2+by^2+2hxy+2fy+2gx+c=0$$

→ find the equation of the cone whose vertex is (α,β,γ) and whose base is

(i) $ax^2+by^2=1,\ z=0$.

(ii) $y^2=4ax,\ z=0$.

1985 (iii) $x^2+y^2=4,\ z=2$, vertex $(1,1,1)$

Sol^n (i) The given base conic is

$$ax^2+by^2=1,\ z=0 \quad \text{—(1)}$$

Now equation of any line through $\omega(\alpha,\beta,\gamma)$

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \quad \text{—(2)}$$

it meets the plane $z=0$ where.

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{0-\gamma}{n}$$

$\Rightarrow x-\alpha=-\frac{l}{n}\gamma,\quad y-\beta=-\frac{m}{n}\gamma$

$\Rightarrow x=\alpha-\frac{l}{n}\gamma,\quad y=\beta-\frac{m}{n}\gamma$

This point lies on the conic (1)

$$a\left(\alpha-\frac{l}{n}\gamma\right)^2+b\left(\beta-\frac{m}{n}\gamma\right)^2=1 \quad \text{—(3)}$$

Now eliminating l,m,n from (2) & (3) we have

$$a\left[\alpha-\frac{x-\alpha}{z-\gamma}\gamma\right]^2+b\left[\beta-\frac{y-\beta}{z-\gamma}\gamma\right]^2=1$$

$$\Rightarrow a(\alpha z-\gamma x)^2+b(\beta z-\gamma y)^2=(z-\gamma)^2$$

∴ which is the required equation of the cone.

H.W → Obtain the locus of the lines which pass through a point (α,β,γ) and through points of the conic.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z=0$$

$$\left[\text{Ans: } \left(\frac{\alpha z-x\gamma}{a}\right)^2+\left(\frac{\beta z-y\gamma}{b}\right)^2=(z-\gamma)^2\right]$$

→ find the equation of the cone whose vertex is the point $(1,1,0)$ and whose guiding curve is $y=0,\ x^2+z^2=4$.

$$[\text{Ans: } x^2-3y^2+z^2-2xy+8y-4=0]$$

IFS-2009 → The section of a cone whose vertex is P and guiding curve the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ —(1), $z=0$ by the plane $\cancel{z}=0$ is a rectangular hyperbola show that the locus of P is $\frac{x^2}{a^2}+\frac{y^2+z^2}{b^2}=1$

Sol'n :- Let the vertex P be (α,β,γ) and given guiding curve the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z=0$ — ①

Now the equation of any line through $P(\alpha,\beta,\gamma)$ are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \quad — ②$$

it meets the plane $z=0$.

$$\therefore \frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{0-\gamma}{n}$$

$$\Rightarrow x-\alpha=-\frac{l}{n}\gamma,\quad y-\beta=-\frac{m}{n}\gamma$$

$$\Rightarrow x=\alpha-\frac{l\gamma}{n},\quad y=\beta-\frac{m}{n}\gamma,\quad z=0$$

This point lies on the ellipse ①

$$\therefore \frac{1}{a^2}\left[\alpha-\frac{l}{n}\gamma\right]^2+\frac{1}{b^2}\left[\beta-\frac{m}{n}\gamma\right]^2=1 \quad — ③$$

Now eliminating l,m,n from ② & ③ we have.

$$\frac{1}{a^2}\left(\alpha-\frac{x-\alpha}{z-\gamma}\gamma\right)^2+\frac{1}{b^2}\left(\beta-\frac{y-\beta}{z-\gamma}\gamma\right)^2=1$$

$$\Rightarrow \frac{1}{a^2}(\alpha z-\gamma x)^2+\frac{1}{b^2}(\beta z-\gamma y)^2=(z-\gamma)^2 \quad — ④$$

which is required equation of cone.

This meets the plane $x=0$

$$\therefore ④ \equiv \frac{1}{a^2}(\alpha z-0)^2+\frac{1}{b^2}(\beta z-\gamma y)^2=(z-\gamma)^2$$

$$\Rightarrow \frac{\alpha^2z^2}{a^2}+\frac{\beta^2z^2+\gamma^2y^2-2\beta\gamma yz}{b^2}=z^2+\gamma^2-2z\gamma$$

This will be a rectangular hyperbola in yz-plane.

if coefficient of y^2 + coefficient of $z^2=0$

$$\text{if } \frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{b^2}-1=0$$

$\therefore$ The locus of $P(\alpha,\beta,\gamma)$ is

$$\frac{x^2}{a^2}+\frac{y^2+z^2}{b^2}=1$$

→ Show that the equation of the cone whose vertex is the origin and whose base is the circle through the three points $(a,0,0)$, $(0,b,0)$, $(0,0,c)$ is $\sum a(b^2+c^2)yz=0$.

(or)

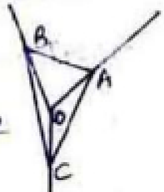
The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ meets the coordinate axes in A, B, C. Prove that the equation of the cone generated by lines drawn from O to meet the circle ABC is

$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0$$

Sol'n:- The equation of the plane is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \quad — ①$$

Since it meets the coordinate axes in A, B, C. The coordinates of A, B, C are $(a,0,0)$, $(0,b,0)$, $(0,0,c)$



Now the circle through A, B, C is the intersection of plane through A, B, C i.e i.e. plane ① and any sphere through the points A, B, C say the sphere OABC.

Now the sphere OABC through the points O(0,0,0), A(a,0,0), B(0,b,0) C(0,0,c) is $x^2+y^2+z^2-ax-by-cz=0$ ——②

∴ The guiding curve is the circle given by ① & ②

i.e. $x^2+y^2+z^2-ax-by-cz=0$;

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

$$② \equiv x^2+y^2+z^2-(ax+by+cz)(1)=0$$

$$\Rightarrow x^2+y^2+z^2-(ax+by+cz)\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)(1)=0.$$

$$\Rightarrow x^2+y^2+z^2-x^2-\frac{b}{a}xy-\frac{c}{a}zx-\frac{a}{b}xy-y^2-\frac{c}{b}yz-\frac{a}{c}zx-\frac{b}{c}yz-z^2=0$$

$$\Rightarrow -y+\left(\frac{b}{c}+\frac{c}{b}\right)-zx\left(\frac{c}{a}+\frac{a}{c}\right)-xy\left(\frac{a}{b}+\frac{b}{a}\right)=0$$

$$\Rightarrow yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0$$

$$\Rightarrow \Sigma a(b^2+c^2)yz=0 \quad ——③$$

which is required equation of the cone.

→ find the equation of the cone whose vertex is (1,2,3) and guiding curve the circle $x^2+y^2+z^2=4$, $x+y+z=1$.

Sol'n :- Any generator through (1,2,3) is

$$\frac{x-1}{l}=\frac{y-2}{m}=\frac{z-3}{n}=\frac{x+y+z-6}{l+m+n} \quad ——①$$

if it meets the plane $x+y+z=1$ then from ①, we have.

$$\frac{x-1}{l}=\frac{y-2}{m}=\frac{z-3}{n}=\frac{1-6}{l+m+n}$$

$$\Rightarrow \frac{x-1}{l}=\frac{y-2}{m}=\frac{z-3}{n}=\frac{-5}{l+m+n}$$

$$\Rightarrow x=1-\left[\frac{5l}{l+m+n}\right],\ y=2-\left[\frac{5m}{l+m+n}\right]$$ and $$z=3-\left[\frac{5n}{l+m+n}\right]$$

i.e. the generator ① meets the plane $x+y+z=1$ in the point

$$\left[\frac{m+n-4l}{l+m+n},\ \frac{2l-3m-2n}{l+m+n},\ \frac{3l+3m-2n}{l+m+n}\right]$$

If this point lies on $x^2+y^2+z^2=4$ we get

$$(m+n-4l)^2+(2l-3m-2n)^2+(3l+3m-2n)^2=4(l+m+n)^2 \quad ——②$$

Eliminating l,m,n between ① & ② we get.

$$[(y-2)+(z-3)-4(x-1)]^2+[2(x+1)-3(y-2)+2(z-3)]^2+[3(x+1)+3(y-2)-2(z-3)]^2$$

$$=4[(x-1)+(y-2)+(z-3)]^2$$

$$\Rightarrow (y+z-4x-1)^2+(2x-3y+2z-2)^2+(3x+3y-2z-3)^2=4(x+y+z-6)^2$$

$$\Rightarrow 5x^2+3y^2+z^2-6yz-4zx-2xy+6x+8y+10z-$$

which is the required equation.

→ show that the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ where $l^2+3m^2-3n^2=0$, is a generator of the cone $x^2+3y^2-3z^2=0$.

Sol^n :- The given line is

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \quad \text{--- (1)}$$

where $l^2+3m^2-3n^2=0$ — (2)

we can eliminate l,m,n from (1) & (2)

$l=x,\ m=y,\ n=z$

$$(2) \equiv x^2+3y^2-3z^2=0 \quad \text{--- (3)}$$

which is the required cone.

$\therefore$ (1) lies on the cone (3).

→ show that the lines through the point (α,β,γ) whose d.c's satisfies $al^2+bm^2+cn^2=0$ generate the cone.

$$a(x-\alpha)^2+b(y-\beta)^2+c(z-\gamma)^2=0$$

Sol^n :- Any line through the point (α,β,γ) is

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \quad \text{--- (1)}$$

where $al^2+bm^2+cn^2=0$ — (2)

Eliminate l,m,n from (1) & (2) we have

$$a(x-\alpha)^2+b(y-\beta)^2+c(z-\gamma)^2=0$$

which is the required cone.

Hence the result.

→ show that the equation of the cone whose vertex is at the origin and the d.c's of whose generator satisfy the relation $3l^2-4m^2+5n^2=0$ is $3x^2-4y^2+5z^2=0$.

* Enveloping Cone of a Sphere:

Definition:- the locus of the tangent from a given point to sphere is a cone called the enveloping cone or tangent cone from the point to the sphere.

(or)

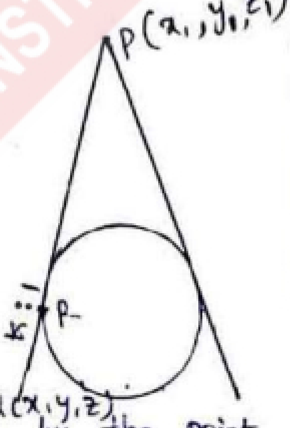
the cone formed by the tangent lines to a surface, drawn from a given point is called the enveloping cone of the surface with given point as its vertex.

→ To find Equation of the enveloping cone from the point (x_1, y_1, z_1) to the sphere $x^2+y^2+z^2=a^2$.

Sol'n:- The given equation of the sphere is $x^2+y^2+z^2=a^2$ —①

Let $P(x_1, y_1, z_1)$ be any given point.

Let $Q(x, y, z)$ be any point on a tangent from P to the given sphere.



Let PQ be divided by the point of contact R in the ratio K:1.

∴ The coordinates of R are

$$\left(\frac{kx+x_1}{k+1}, \frac{ky+y_1}{k+1}, \frac{kz+z_1}{k+1}\right)$$

Since this point R lies in the sphere ①

$$\therefore \left(\frac{kx+x_1}{k+1}\right)^2+\left(\frac{ky+y_1}{k+1}\right)^2+\left(\frac{kz+z_1}{k+1}\right)^2=a^2$$

$$\Rightarrow k^2x^2+x_1^2+2kxx_1+k^2y^2+y_1^2+2kyy_1+k^2z^2+z_1^2+2kzz_1=a^2(k^2+2k+1)$$

$$\Rightarrow k^2(x^2+y^2+z^2-a^2)+2k(xx_1+yy_1+zz_1-a^2)+(x_1^2+y_1^2+z_1^2-a^2)=0 \quad \text{—②}$$

which is quadratic in k.

Since the line PQ touches the sphere, the two values of k must be equal

∴ Discriminant of ② = 0

i.e. $b^2-4ac=0$

$$\therefore 4[xx_1+yy_1+zz_1-a^2]-4[x^2+y^2+z^2-a^2][x_1^2+y_1^2+z_1^2-a^2]=0$$

$$\Rightarrow [x^2+y^2+z^2-a^2][x_1^2+y_1^2+z_1^2-a^2]=[xx_1+yy_1+zz_1-a^2]^2$$

which is the required equation of the enveloping cone.

Note:- If $S=x^2+y^2+z^2-a^2$ so that $S=0$ is the equation of the sphere then

$S_1 = x_1^2 + y_1^2 + z_1^2 - a^2$ i.e. S_1 is the result of substituting the point (x_1, y_1, z_1) in S.

and $T = xx_1 + yy_1 + zz_1 - a^2$ the expression of the tangent plane at (x_1, y_1, z_1) to the sphere.

then the enveloping cone is $SS_1 = T^2$.

→ Find the enveloping cone of the sphere $x^2 + y^2 + z^2 - 2x + 4z = 1$ with vertex at $(1, 1, 1)$.

Sol'n :- The equation of the sphere is $x^2+y^2+z^2-2x+4z=1$ —①

and the given vertex is $P(1,1,1)$

Let $S = x^2+y^2+z^2-2x+4z-1$

and $x_1 = 1,\ y_1 = 1,\ z_1 = 1$

$\therefore S_1 = (1)^2 + (1)^2 + (1)^2 - 2(1) + 4(1) - 1$

$= 4$

and $T = xx_1 + yy_1 + zz_1 - (x+x_1) + 2(z+z_1) - 1$

$= x(1) + y(1) + z(1) - (x+1) + 2(z+1) - 1$

$= \cancel{x} + y + z - \cancel{x} - \cancel{1} + 2z + \cancel{2} - \cancel{1}$

$= y + 3z$

$\therefore$ Equation of the enveloping cone is $SS_1 = T^2$.

$\Rightarrow (x^2+y^2+z^2-2x+4z-1)(4) = (y+3z)^2$

$\Rightarrow 4x^2+4y^2+4z^2-8x+16z-4 = y^2+9z^2+6yz$

$\Rightarrow 4x^2+3y^2-5z^2-8x+16z-6yz-4 = 0$

→ Show that the plane $z=0$ cuts the enveloping cone of the sphere $x^2+y^2+z^2=11$ which has its vertex at $(2,4,1)$ in a rectangular hyperbola.

Sol'n :- The given equation of the sphere is $x^2+y^2+z^2=11$ —①

and given vertex $(2,4,1)$.

Let $S = x^2+y^2+z^2-11$ and $x_1 = 2$, $y_1 = 4$, $z_1 = 1$.

$\therefore S_1 = 4+16+1-11 = 10$

$T = xx_1 + yy_1 + zz_1 - 11$

$= 2x + 4y + z - 11$

$\therefore$ The equation of the enveloping cone is $SS_1 = T^2$.

$\Rightarrow (x^2+y^2+z^2-11)(10) = (2x+4y+z-11)^2$ —②

This meets the plane $z=0$.

$\therefore (x^2+y^2+0-11)(10) = (2x+4y+0-11)^2$

$\Rightarrow (x^2+y^2-11)10 - (2x+4y-11)^2 = 0$.

This represents a rectangular hyperbola in the xy-plane.

if coefficient of x^2 + coefficient of $y^2 = 0$

$\therefore (10-4) + (10-16) = 0$

$\Rightarrow 6-6 = 0$

$\Rightarrow 0 = 0$. which is true.

Hence the result.

* Quadratic cone through the axes :-

→ show that the general equation of a cone of second degree which pass through the axes is $fyz + gzx + hxy = 0$. where f, g, h are parameters.

sol'n :- The general equation of the cone with its vertex at the origin is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0 \quad \text{—(1)}$$

Since it passes through x-axis

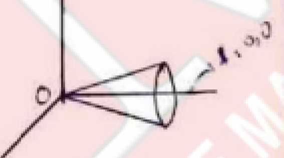
∴ the d.c's of x-axis are 1, 0, 0 must satisfy (1) $\frac{x}{1} = \frac{y}{0} = \frac{z}{0}$

$$\therefore a(1)^2 + b(0)^2 + c(0)^2 - 2f(0) + 2g(0) + 2h(0) = 0$$

$$\Rightarrow \boxed{a = 0}$$

Similarly the cone passes through the axes of y & z.

we have $b = 0, c = 0$.



$$\therefore (1) \equiv 0(x^2) + 0(y^2) + 0(z^2) + 2fyz + 2gzx + 2hxy = 0$$

$$\Rightarrow 2fyz + 2gzx + 2hxy = 0$$

$$\Rightarrow fyz + gzx + hxy = 0.$$

which is the required condition.

→ show that a cone can be found so as to contain any two given sets of three mutually perpendicular concurrent lines as generators.

(or)

show that a cone of second degree can be found to pass through any two sets of rectangular axes through the same origin.

sol'n :- Take the three lines of one set as coordinate axes (i.e. OX, OY, OZ).

Let the lines OX', OY', OZ' of the second set be $\frac{x}{l_1} = \frac{y}{m_1} = \frac{z}{n_1}$, $\frac{x}{l_2} = \frac{y}{m_2} = \frac{z}{n_2}$, $\frac{x}{l_3} = \frac{y}{m_3} = \frac{z}{n_3}$.

Now general equation of the cone through axes (i.e OX, OY, OZ) is $fyz + gzx + hxy = 0$ —(1)

If it passes through OX' & OY' then the d.c's l_1, m_1, n_1 & l_2, m_2, n_2 of OX', OY', OZ' satisfy (1)

$$\therefore fm_1n_1 + gn_1l_1 + hl_1m_1 = 0 \quad \text{—(2)}$$

$$fm_2n_2 + gn_2l_2 + hl_2m_2 = 0 \quad \text{—(3)}$$

Adding (2) & (3) we have

$$f(m_1n_1 + m_2n_2) + g(n_1l_1 + n_2l_2) + h(l_1m_1 + l_2m_2) = 0. \quad \text{—(4)}$$

But l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 are the d.c's of three mutually ⊥ lines.

$$\therefore m_1n_1 + m_2n_2 + m_3n_3 = 0 \Rightarrow m_1n_1 + m_2n_2 = -m_3n_3$$

$$n_1l_1 + n_2l_2 + n_3l_3 = 0 \Rightarrow n_1l_1 + n_2l_2 = -n_3l_3$$

$$\& \; l_1m_1 + l_2m_2 + l_3m_3 = 0 \Rightarrow l_1m_1 + l_2m_2 = -l_3m_3$$

Putting these values in (4) we have

$$-fm_3n_3 - gn_3l_3 - hl_3m_3 = 0$$

$\Rightarrow -fm_3n_3 + gn_3l_3 + hl_3m_3 = 0$

i.e. ① is satisfied by the d.c's l_3, m_3, n_3 of OZ'.

∴ The cone passes through the OZ'. i.e the cone passes through OX, OY, OZ and OX', OY', OZ'

i.e two sets of rectangular axes.

→ Find the equation of the cone which contains the three coordinate axes and the lines through the origin having direction cosines l_1, m_1, n_1 and l_2, m_2, n_2

Sol'n :- The equation of any cone through the three coordinate axes is $fyz + gzx + hxy = 0$ —①

Since it passes through lines with d.c's l_1, m_1, n_1 and l_2, m_2, n_2 and the d.c's of the generators satisfy the equation of the cone.

∴ $fm_1n_1 + gn_1l_1 + hl_1m_1 = 0$ —②

$fm_2n_2 + gn_2l_2 + hl_2m_2 = 0$ —③

∴ Eliminating f, g, h from ①, ②, ③

we have
$$\begin{vmatrix} yz & zx & xy \\ m_1n_1 & n_1l_1 & l_1m_1 \\ m_2n_2 & n_2l_2 & l_2m_2 \end{vmatrix} = 0$$

$\Rightarrow yz[n_1l_1l_2m_2 - m_2l_2l_1m_1]$

$-zx[m_1n_1l_2m_2 - m_2n_2l_1m_1]$

$+xy[m_1n_1n_2l_2 - m_2n_2n_1l_1] = 0$

$\Rightarrow l_1l_2yz[n_1m_2 - n_2m_1] + m_1m_2xz[n_2l_1 - n_1l_2]$

$+ n_1n_2xy[m_1l_2 - m_2l_1]$

which is required equation.

→ Find the equation to the cone which passes through the three coordinate axes as well as the two lines $\frac{x}{1} = \frac{y}{-2} = \frac{z}{3}$ —(I) ; $\frac{x}{3} = \frac{y}{-1} = \frac{z}{1}$ —(II)

Sol'n :- the equation of any cone through the three coordinate axes

$fyz + gzx + hxy = 0$ —①

Since it passes through the lines (I) & (II) and d.c's of the generators satisfy the equation of the cone.

$\therefore f(-2)(3) + g(3)(1) + h(1)(-2) = 0$

& $f(-1)(1) + g(1)(3) + h(3)(-1) = 0$

$\Rightarrow f(-6) + g(3) + h(-2) = 0$ —②

& $f(-1) + g(3) + h(-3) = 0$ —③

Eliminating f, g, h from ①, ② & ③ we get

$$\begin{vmatrix} yz & zx & xy \\ -6 & 3 & -2 \\ -1 & 3 & -3 \end{vmatrix} = 0.$$

$xy(-9+6) - zx(18-2) + xy(-18+3) = 0$

$\Rightarrow 3yz + 16zx + 15xy = 0$

→ Find the equation of the quadric cone which passes through the 3 coordinate axes and three mutually perpendicular lines.

$\frac{1}{2}x = y = -z$, $x = \frac{1}{3}y = \frac{1}{5}z$, $\frac{1}{8}x = \frac{-1}{11}y = \frac{1}{5}z$

Sol^n:- Now the equation of any cone through the coordinate axes is

$$fyz + gzx + hxy = 0 \quad \text{——(1)}$$

Since it passes through the line

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{-1}$$

$\therefore f(1)(-1) + g(-1)(2) + h(2)(1) = 0$

$\Rightarrow -f - 2g + 2h = 0$

$\Rightarrow f + 2g - 2h = 0$ ——(2)

Similarly (1) passes through $\frac{x}{1} = \frac{y}{3} = \frac{z}{5}$

$\therefore f(3)(5) + g(5)(1) + h(1)(3) = 0$

$\Rightarrow 15f + 5g + 3h = 0$ ——(3)

solving (2) & (3)

$$\frac{f}{6+10} = \frac{g}{-30-3} = \frac{h}{5-30}$$

$$\rightarrow \frac{f}{16} = \frac{g}{-33} = \frac{h}{-25}$$

$\therefore$ putting these values of f, g, h in (1) we get

$$16(yz) + (-33)zx + (-25)xy = 0$$

$$\Rightarrow 16yz - 33zx - 25xy = 0$$

which is the required equation of the cone and the generator line $\frac{x}{8} = \frac{y}{-11} = \frac{z}{5}$ also satisfy the equation of this cone.

Q8 → Planes through OX & OY include an angle α, show that their line of intersection lies on the cone $z^2(x^2+y^2+z^2) = x^2y^2\tan^2\alpha$.

Sol^n:- The equation of any plane through OX $(y=0, z=0)$ is $y + \lambda z = 0$ ——(1)

and the equation of any plane through OY $(x=0, z=0)$ is $x + \mu z = 0$ ——(2)

The angle between the two planes (1) & (2) is

$$\cos\alpha = \frac{0.1 + 1.0 + \mu\lambda}{\sqrt{1+\lambda^2}\sqrt{1+\mu^2}} = \frac{\mu\lambda}{\sqrt{1+\lambda^2}\sqrt{1+\mu^2}}$$

$$= \frac{\mu\lambda}{\sqrt{1+\mu^2+\lambda^2+\lambda^2\mu^2}}$$

$$\sec\alpha = \frac{1}{\cos\alpha} = \frac{\sqrt{1+\mu^2+\lambda^2+\lambda^2\mu^2}}{\mu\lambda}$$

$$\tan^2\alpha = \sec^2\alpha - 1$$

$$= \frac{1+\mu^2+\lambda^2+\lambda^2\mu^2}{\mu^2\lambda^2} - 1$$

$$= \frac{1+\lambda^2+\mu^2}{\mu^2\lambda^2} \quad \text{——(3)}$$

Eliminating λ, μ from (1), (2) & (3) we get

$$\tan^2\alpha = \frac{1 + \frac{y^2}{z^2} + \frac{x^2}{z^2}}{\left(\frac{y^2}{z^2}\right)\left(\frac{x^2}{z^2}\right)}$$

$$= \frac{z^2(x^2+y^2+z^2)}{x^2y^2}$$

$$\therefore z^2(x^2+y^2+z^2) = x^2y^2\tan^2\alpha.$$

which is the required eq^n of the cone.

* Condition for general second degree equation to represent a cone :-

To find the condition that the equation

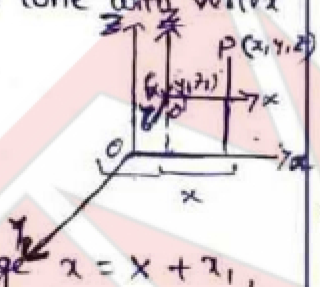
$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0.$$

may represent a cone

Sol^n :- The given equation is

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0 \quad \text{—①}$$

If it represents a cone with vertex at (x_1, y_1, z_1) say, then shifting the origin to the point (x_1, y_1, z_1)



so that we change $x = X + x_1$, $y = Y + y_1$ and $z = Z + z_1$.

$\therefore$ The transformed equation is

$$a(X+x_1)^2+b(Y+y_1)^2+c(Z+z_1)^2+2f(Y+y_1)(Z+z_1)+2g(Z+z_1)(X+x_1)$$
$$+2h(X+x_1)(Y+y_1)+2u(X+x_1)+2v(Y+y_1)+2w(Z+z_1)+d=0$$

$$\Rightarrow aX^2+bY^2+cZ^2+2fYZ+2gZX+2hXY+2X(ax_1+hy_1+gz_1+u)$$
$$+2Y(hx_1+by_1+fz_1+v)+2Z(gx_1+fy_1+cz_1+w)+$$
$$(ax_1^2+by_1^2+cz_1^2+2fy_1z_1+2gz_1x_1+2hx_1y_1+2ux_1+2vy_1+2wz_1+d)=0 \quad \text{—②}$$

Since ② represents a cone with vertex at the origin, so it must be homogeneous in X, Y, Z.

$\therefore$ Coefficient of $X=0$, coefficient of $Y=0$, coefficient of $Z=0$ and constant term $=0$.

i.e.
$$ax_1+hy_1+gz_1+u=0 \quad \text{—③}$$
$$hx_1+by_1+fz_1+v=0 \quad \text{—④}$$
$$gx_1+fy_1+cz_1+w=0 \quad \text{—⑤}$$

and
$$ax_1^2+by_1^2+cz_1^2+2fy_1z_1+2gz_1x_1+2hx_1y_1+2ux_1+2vy_1+2wz_1+d=0 \quad \text{—⑥}$$

Now ⑥ can be written as,

$$x_1(ax_1+hy_1+gz_1+u)+y_1(hx_1+by_1+fz_1+v)+z_1(gx_1+fy_1+cz_1+w)+ux_1+vy_1+wz_1+d=0.$$

$$\Rightarrow ux_1+vy_1+wz_1+d=0 \quad \text{—⑦}$$
(using ③, ④ & ⑤)

Eliminating x_1, y_1, z_1 from ③, ④, ⑤ & ⑦ we get

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & d \end{vmatrix} = 0$$

which is the required condition.

Note :- The vertex of the cone is obtained by solving any three of the four equations ③, ④, ⑤ and ⑦ for x_1, y_1, z_1.

→* Method for Numerical Questions:-

(i) Make the given equation homogeneous in x, y, z, t by introducing proper powers of t where $t=1$.

(ii) Let this be denoted by
$F(x, y, z, t) = 0$

(iii) Then the four equations ③, ④, ⑤ & ⑦ are obtained by equations $\frac{\partial F}{\partial x}=0$, $\frac{\partial F}{\partial y}=0$, $\frac{\partial F}{\partial z}=0$ and $\frac{\partial F}{\partial t}=0$ where ultimately $t=1$

(iv) Solve any three of the above four equations for x, y, z.

(v) Substitute these values of x, y, z in the fourth equation and if it is satisfied then the given equation represents a cone and values of x, y, z found in (iv) are the coordinates of the vertex.

→ Show that the equation
$4x^2-y^2+2z^2+2xy-3yz+12x-11y+6z+4=0$
represents a cone with vertex $(-1,-2,-3)$.

Sol'n :- Given equation is
$4x^2-y^2+2z^2+2xy-3yz+12x-11y+6z+4=0$
Making given equation homogeneous, we get.
$$f(x,y,z,t) = 4x^2-y^2+2z^2+2xy-3yz+12xt-11yt+6zt+4t^2=0$$

Now $\frac{\partial F}{\partial x}=0$

$\Rightarrow 8x+2y+12t=0$

$\Rightarrow \boxed{4x+y+6t=0}$

$\frac{\partial F}{\partial y}=0$

$\Rightarrow -2y+2x-3z-11t=0$

$\Rightarrow \boxed{2x-2y-3z-11t=0}$

$\frac{\partial F}{\partial z}=0 \Rightarrow \boxed{4z-3y+6t=0}$

and $\frac{\partial F}{\partial t}=0 \Rightarrow \boxed{12x-11y+6z+8t=0}$

putting $t=1$ in above equations, we get
$4x+y+6=0$ —— ②
$2x-2y-3z-11=0$ —— ③
$3y-4z-6=0$ —— ④
$12x-11y+6z+8=0$ —— ⑤

③×2 ≡ $4x-4y-6z-22=0$ —— ⑥

Now ⑥−② $5y+6z+28=0$

$\Rightarrow 10y+12z+56=0$ —— ⑦

④×3 $9y-12z-18=0$ —— ⑧

⑦+⑧ ≡ $19y+38=0$

$\Rightarrow \boxed{y=-2}$

④ ≡ $3(-2)-4z-6=0$

$\Rightarrow -4z=12$

$\Rightarrow \boxed{z=-3}$

② ≡ $4x-2+6=0$

$\Rightarrow 4x=-4$

$\Rightarrow \boxed{x=-1}$

$\therefore$ These values of x, y & z as $x=-1$, $y=-2$ & $z=-3$ satisfy ⑤

$\therefore$ The equation represents a cone and its vertex is $(-1,-2,-3)$.

→ Show that the equation $x^2-2y^2+3z^2-4xy+5yz-6zx+8x-19y-2z-20=0$ represents a cone with vertex $(1,-2,3)$.

→ Show that the equation $2y^2-8yz-4zx-8xy+6x-4y-2z+5=0$ represents a cone whose vertex is $(-7/6, 1/3, 5/6)$.

* Angle between two lines in which a plane through the vertex cuts a cone:-

→ Find the angle between the lines of intersection of the plane $x-3y+z=0$ and the cone $x^2-5y^2+z^2=0$

Sol'n:- the given plane is

$$x-3y+z=0 \quad \text{—①}$$

and given cone is $x^2-5y^2+z^2=0$ —②

Let the line of section be

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \quad \text{—③}$$

Since it lies on the plane ①

$\therefore$ It is ⊥lar to the normal to the plane.

$$\therefore al+bm+cn=0$$

$$\Rightarrow l-3m+n=0 \quad \text{—④}$$

Also the line ③ lies on the cone ②.

$\therefore$ Its d.c's satisfies the equation of the cone.

$$\therefore l^2-5m^2+n^2=0 \quad \text{—⑤}$$

$$④ \equiv \boxed{l=3m-n}$$

$$⑤ \Rightarrow (3m-n)^2-5m^2+n^2=0$$

$$\Rightarrow 9m^2+n^2-6mn-5m^2+n^2=0$$

$$\Rightarrow 4m^2+2n^2-6mn=0$$

$$\Rightarrow 4m^2+2n^2-4mn-2mn=0$$

$$\Rightarrow 4m(m-n)-2n(m-n)=0$$

$$\Rightarrow (4m-2n)(m-n)=0$$

$\Rightarrow m-n=0$ $\Rightarrow m=n$ | $4m-2n=0$, $4m=2n \Rightarrow m=\frac{1}{2}n$

$\Rightarrow m-n=0$, $\Rightarrow 0l+m-n=0$ | $0l+4m-2n=0$, also $l-3m+n=0$

from ① $l-3m+n=0$

Solving

$$\frac{l}{1-3}=\frac{m}{-1-0}=\frac{n}{0-1} \qquad \Big| \qquad \frac{l}{4-6}=\frac{m}{-2-0}=\frac{n}{0-4}$$

$$\frac{l}{-2}=\frac{m}{-1}=\frac{n}{-1} \qquad \Big| \qquad \Rightarrow \frac{l}{-2}=\frac{m}{-2}=\frac{n}{-4}$$

$$\Rightarrow \frac{l}{2}=\frac{m}{1}=\frac{n}{1} \qquad \Big| \qquad \Rightarrow \frac{l}{1}=\frac{m}{1}=\frac{n}{2}$$

Putting these values of l, m, n in ③ the required lines of section are $\frac{x}{2}=\frac{y}{1}=\frac{z}{1}$ & $\frac{x}{1}=\frac{y}{1}=\frac{z}{2}$

If θ is angle between two lines of section then $\cos\theta = \dfrac{2(1)+1(1)+2(1)}{\sqrt{4+1+1}\sqrt{1+1+4}}$

$$\cos\theta = \frac{|5|}{\sqrt{6}\sqrt{6}} = \frac{5}{6}$$

$$\therefore \theta = \cos^{-1}(5/6)$$

Formulae :-

Let the plane be $ux+vy+wz=0$ — ①

and the cone be

$$f(x,y,z) = ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \quad \text{— ②}$$

then the angle between the lines cutting by plane ① in the cone ② is given by

$$\tan\theta = \frac{2P\sqrt{u^2+v^2+w^2}}{(a+b+c)(u^2+v^2+w^2)-F(u,v,w)}$$

where $P^2 = \begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & 0 \end{vmatrix}$ and

$F(u,v,w) = au^2+bv^2+cw^2+2fvw+2gwu+2huv.$

Note :— ① The lines are $\perp$lar, if

$$F(u,v,w) = (a+b+c)(u^2+v^2+w^2)$$

② If the lines are coincident

$$\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & 0 \end{vmatrix} = 0$$

→ Find the angles between the lines of section of the planes & cones.

(i) $10x+7y-6z=0$ and $20x^2+7y^2-108z^2=0$

(ii) $4x-y-5z=0$ and $8yz+3zx-5xy=0$

(iii) $x+y+z=0$ and $6xy+3yz-2zx=0$

(iv) $x+y+z=0$ and $x^2yz+xy-3z^2=0$

Ans (i) $\cos^{-1}(16/21)$ (ii) $\pi/2$

(iii) $\pi/3$ (iv) $\pi/6$

→ Find the equations to the lines in which plane $2x+y-z=0$ cuts the cone $4x^2-y^2+3z^2=0$

Sol'n :- The given plane is

$$2x+y-z=0 \quad \text{— ①}$$

and cone is $4x^2-y^2+3z^2=0$ — ②

Let $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ be the equations of any one of the two lines in which the given plane meets the given cone.

$\therefore$ we have

$2l+m-n=0$ — ③, $4l^2-m^2+3n^2=0$ — ④

③ $\equiv n = 2l+m$

④ $\equiv 4l^2-m^2+3(2l+m)^2=0$

$\Rightarrow 4l^2-m^2+3(4l^2+4lm+m^2)=0$

$\Rightarrow 4l^2-m^2+12l^2+12lm+3m^2=0$

$\Rightarrow 16l^2+2m^2+12lm=0$

$\Rightarrow 8l^2+m^2+6lm=0$

$\Rightarrow 8\left(\frac{l}{m}\right)^2+6\left(\frac{l}{m}\right)+1=0$

$\Rightarrow l/m = \dfrac{-6\pm\sqrt{36-32}}{16} = -\frac{1}{4}$ (or) $-\frac{1}{2}$

$\therefore l/m = -1/4$ & $l/m = -1/2$

$\Rightarrow l/m+1/4=0$ & $l/m+1/2=0$

$\Rightarrow 4l+m=0$ & $2l+m=0$

from ③ we have

$2l+m-n=0$	$2l+m-n=0$
$\Rightarrow 4l+m+0n=0$	$2l+m+0n=0$
& $2l+m-n=0$	& $2l+m-n=0$

solving

$\frac{l}{-1}=\frac{m}{4}=\frac{n}{4-2}$	$\frac{l}{-1}=\frac{m}{2}=\frac{n}{2-2}$
$\frac{l}{-1}=\frac{m}{4}=\frac{n}{2}$	$\Rightarrow \frac{l}{-1}=\frac{m}{2}=\frac{n}{0}$

$\therefore$ The required lines are

$$\frac{x}{-1}=\frac{y}{4}=\frac{z}{2}\ ;\ \frac{x}{-1}=\frac{y}{2}=\frac{z}{0}$$

→ Find the equations of the lines of intersection of the following planes and cones.

(i) $x+3y-2z=0$ and $x^2+9y^2-4z^2=0$

(ii) $3x+4y+z=0$ and $15x^2-32y^2-7z^2=0$

2003 → (iii) $x+7y-5z=0$ and $3yz+14zx-30xy=0$

Ans:- (i) $x=2z,\ y=0$; $3y=2z,\ x=0$

(ii) $\frac{x}{-3}=\frac{y}{2}=\frac{z}{1}$; $\frac{x}{2}=\frac{y}{-1}=\frac{z}{-2}$

(iii) $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$; $\frac{x}{3}=\frac{y}{1}=\frac{z}{2}$

→ Show that the equation of the quadratic cone which contains the three coordinate axes and the lines in which the plane $x-5y-3z=0$, cuts the cone $7x^2+5y^2-3z^2=0$ is $yz+10zx+18xy=0$.

* Mutually Perpendicular generators of a Cone :-

→ The necessary & sufficient condition for cone $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ to have three mutually ⊥lar generators is that sum of coefficient of x^2, y^2, z^2 is zero. i.e. $a+b+c=0$.

→ If the general equation of the second degree

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$$

represents a cone, then the condition that it may have three mutually perpendicular generators is $a+b+c=0$.

This result follows on shifting the origin to vertex.

The coefficients of the second degree term remain unaffected.

→ Problems

(2006) Prove that the plane $ax+by+cz=0$ cuts the cone $yz+zx+xy=0$ in perpendicular lines if $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$.

Sol'n :- The equation of the plane is

$$ax+by+cz=0 \quad \text{—— ①}$$

and the cone is $yz+zx+xy=0$ —— ②

Comparing ② with

$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$

$\therefore a=0,\ b=0,\ c=0$

$\rightarrow a+b+c = 0+0+0$
$= 0$

∴ The cone ② has three mutually ⊥lar generators.

The plane ① will cut the cone ② in ⊥ lines if the normal to the plane ① through the vertex (0,0,0) [whose d.c's are proportional to a,b,c] lies on the cone ②

if $bc+ca+ab=0$ ($\because$ d.c's of the generator satisfy the equation of the cone).

if $\frac{1}{a}+\frac{1}{b}+\frac{1}{c}=0$

(on dividing throughout by abc)

which is the required condition.

→ Prove that the plane $lx+my+nz=0$ cuts the cone $(b-c)x^2+(c-a)y^2+(a-b)z^2+2fyz+2gzx+2hxy=0$ in perpendicular lines if $(b-c)l^2+(c-a)m^2+(a-b)n^2+2fmn+2gnl+2hlm=0$.

Sol'n: The given plane is $lx+my+nz=0$ —— ①

and cone is

$$(b-c)x^2+(c-a)y^2+(a-b)z^2+2fyz+2gzx+2hxy=0 \quad \text{—— ②}$$

Here the sum of the coefficients of $x^2, y^2, z^2 = (b-c)+(c-a)+(a-b)=0$.

$\therefore$ the cone (5) has three mutually $\perp$lar generators.

Now if the plane (1) cuts the cone (2) in perpendicular lines then normal to the plane (1) through vertex (0,0,0) i.e. $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ is the generator of the cone (2)

since the d.c's of the generator satisfy the cone equation.

i.e. l, m, n must satisfy (2)

$$\therefore (b-c)l^2+(c-a)m^2+(a-b)n^2 + 2fmn+2gnl+2hlm=0$$

which is the required condition.

→ If $x=\frac{1}{2}y=z$ represents one of a set of three mutually perpendiculars of the cone $11yz+6zx-14xy=0$, find the equations of other two.

(2008) → If $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ represents one of a set of three mutually perpendicular generators of the cone, $5yz-8zx-3xy=0$ find the equations of the other two.

sol'n:- The given cone is

$$5yz-8zx-3xy=0 \quad \text{—— (1)}$$

and one of its three $\perp$lar generators is

$$\frac{x}{1}=\frac{y}{2}=\frac{z}{3} \quad \text{—— (2)}$$

the other two $\perp$lar generators are the lines which plane through the vertex (0,0,0), and $\perp$ to line (2)

i.e. the plane $x+2y+3z=0$ —— (3)

Let a line of section of (1) & (3) be

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \quad \text{—— (4)}$$

Since (4) lies in the plane (3)

$\therefore$ It is $\perp$ to the normal to the plane.

$$\therefore l+2m+3n=0 \quad \text{—— (5)}$$

Also (4) lies on Cone (1)

$\therefore$ ie. the d.c's of (4) satisfies the equation of cone.

$$\therefore 5mn-8nl-3lm=0 \quad \text{—— (6)}$$

$$(5) \equiv l=-(2m+3n)$$

$$\therefore (6) \equiv 5mn+8n(2m+3n)+3m(2m+3n)=0$$

$$\Rightarrow 6m^2+30mn+24n^2=0$$

$$\Rightarrow m^2+5mn+4n^2=0$$

$$\Rightarrow (m+n)(m+4n)=0$$

$\Rightarrow m+n=0$	$m+4n=0$
$\Rightarrow 0l+m+n=0$	$\Rightarrow 0l+m+4n=0$
Also (5) $\equiv l+2m+3n=0$	Also $l+2m+3n=0$
solving:	Solving:
$\frac{l}{3-2}=\frac{m}{1-0}=\frac{n}{0-1}$	$\frac{l}{3-8}=\frac{m}{4-0}=\frac{n}{0-1}$
$\frac{l}{1}=\frac{m}{1}=\frac{n}{-1}$	$\frac{l}{-5}=\frac{m}{4}=\frac{n}{-1}$

$$\therefore (4) \equiv \frac{x}{1}=\frac{y}{1}=\frac{z}{-1} \;\&\; \frac{x}{-5}=\frac{y}{4}=\frac{z}{-1}$$

which are the other two generators.

→ Show that the cone whose vertex is origin and which passes through the curve of intersection of the surface $3x^2-y^2+z^2=3a^2$ and any plane at a distance 'a' from the origin, has three mutually perpendicular generators.

→ Show that the cone whose vertex at the origin and which passes through the curve of intersection of the sphere $x^2+y^2+z^2=3a^2$ and any plane at a distance 'a' from the origin has three mutually ⊥lar generators.

Sol^n:- Given sphere is

$$x^2+y^2+z^2=3a^2 \quad \text{——} (1).$$

Any plane at a distance 'a' from the origin is $lx+my+nz=a$ —(2) (normal form)

where l,m,n are d.c's of normal to the plane.

Making (1) homogeneous with the help of (2),

the equation of the cone whose vertex is the origin and base, the curve of intersection of (1)&(2) is

$$x^2+y^2+z^2=3(lx+my+nz)^2$$

$$\Rightarrow x^2(1-3l^2)+y^2(1-3m^2)+z^2(1-3n^2)$$
$$-6mnyz-6nlzx-6lmxy=0. \quad (3)$$

which is the required cone vertex at the origin

Now in (3), we have

Coefficient of x^2 + Coefficient of y^2 + Coefficient of $z^2 = (1-3l^2)+(1-3m^2)+(1-3n^2)$

$$= 3-3(l^2+m^2+n^2)$$
$$= 3-3(1) \quad (\because l^2+m^2+n^2=1)$$
$$= 0$$

∴ The cone (3) has three mutually ⊥lar generators.

→ Find the locus of the points from which three mutually perpendicular lines can be drawn to intersect the conic

$$z=0,\ ax^2+by^2=1.$$

Sol^n:- The given conic is

$$z=0,\quad ax^2+by^2=1 \quad \text{——} (1)$$

Let (α,β,γ) be the point from which three mutually ⊥ lines can be drawn to intersect the conic (1).

Any line through (α,β,γ) is

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \quad \text{——} (2)$$

Since it meets the plane $z=0$

$$\therefore \frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{0-\gamma}{n}$$

$$\Rightarrow x=\alpha-\frac{l}{n}\gamma,\quad y=\beta-\frac{m}{n}\gamma$$

∴ this point lies on (1) if

$$a\left(\alpha-\frac{l}{n}\gamma\right)^2+b\left(\beta-\frac{m}{n}\gamma\right)^2=1 \quad \text{--- (3)}$$

Now eliminate l, m, n from (2) & (3)

we have

$$a\left[\alpha-\frac{x-\alpha}{z-\gamma}\cdot\gamma\right]^2+b\left[\beta-\frac{y-\beta}{z-\gamma}\cdot\gamma\right]^2=1$$

$$\Rightarrow a[\alpha z-\gamma x]^2+b[\beta z-\gamma y]^2=[z-\gamma]^2$$

$$\Rightarrow a(\alpha z-\gamma x)^2+b(\beta z-\gamma y)^2-(z-\gamma)^2=0$$

This cone has three mutually $\perp$ar generators if

Coefficient of x^2 + coefficient of y^2 + coefficient of $z^2=0$

if $a\gamma^2+b\gamma^2+(a\alpha^2+b\beta^2-1)=0$

if $a\alpha^2+b\beta^2+(a+b)\gamma^2=1$

$\therefore$ Locus of the point (α, β, γ) is

$$ax^2+by^2+(a+b)z^2=1$$

Hence the result.

2007 Show that the plane $2x-y+2z=0$ cuts the cone $xy+yz+zx=0$ in perpendicular lines.

we find two lines when a plane cuts cone we get two lines these two lines $\perp$r.

* Tangent Plane :-

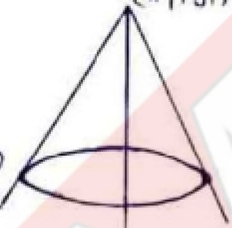
To find the equation of the tangent plane at the point (x_1,y_1,z_1) to the cone

$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$

Sol'n :- The given equation of the cone is $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ —(1)

Equations of any line through (x_1,y_1,z_1) are

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \quad \text{—(2)}$$



Any point on this line is

$(x_1+lr,\ y_1+mr,\ z_1+nr)$

If it lies on the Cone (1) then

$$a(x_1+lr)^2+b(y_1+mr)^2+c(z_1+nr)^2+2f(mr+y_1)(nr+z_1)+2g(nr+z_1)(lr+x_1)+2h(lr+x_1)(mr+y_1)=0$$

$$\Rightarrow r^2[al^2+bm^2+cn^2+2fmn+2gnl+2hlm]+2r[l(ax_1+hy_1+gz_1)+m[hx_1+by_1+fz_1)+n(gx_1+fy_1+cz_1)]+ax_1^2+by_1^2+cz_1^2+2fy_1z_1+2gz_1x_1+2hx_1y_1=0 \quad \text{—(3)}$$

which is a quadratic equation in r.

Since (x_1,y_1,z_1) lies on the Cone (1).

$$\therefore ax_1^2+by_1^2+cz_1^2+2fy_1z_1+2gz_1x_1+2hx_1y_1=0 \quad \text{—(4)}.$$

$$\therefore (3) \equiv r^2(al^2+bm^2+cn^2+2fmn+2gnl+2hlm)+2r[l(ax_1+hy_1+gz_1)+m(hx_1+by_1+fz_1)+n(gx_1+fy_1+cz_1)]+0=0 \quad \text{—(A)}$$

$$\Rightarrow r[r(\ )+2(\ )]=0$$

$\Rightarrow$ This equation has one root as zero

If the line (2) touches the Cone, then the two values of r in (A) must be equal.

But Since one root is zero.

$\therefore$ Other root is also zero.

i.e. the Coefficient of $r=0$.

$$\text{i.e. } l(ax_1+hy_1+gz_1)+m(hx_1+by_1+fz_1)+n(gx_1+fy_1+cz_1)=0 \quad \text{—(5)}$$

which is the condition for the line (2), to touch the cone (1) at (x_1,y_1,z_1)

To find the locus of tangent line (2), we have to eliminate l,m,n from (2) & (5)

$\therefore$ Putting the values of l,m,n from (2) in (5),

we have

$$(x-x_1)(ax_1+hy_1+gz_1)+(y-y_1)(hx_1+by_1+fz_1)+(z-z_1)(gx_1+fy_1+cz_1)=0$$

$$\Rightarrow x(ax_1+hy_1+gz_1)+y(hx_1+by_1+fz_1)+z(gx_1+fy_1+cz_1)=ax_1^2+by_1^2+cz_1^2+2fy_1z_1+2gz_1x_1+2hx_1y_1$$

$\Rightarrow x(ax_1+hy_1+gz_1)+$

$y(hx_1+by_1+fz_1)+z(gx_1+fy_1+cz_1)=0$ —— ⑥

which is the required ($\because$ from ④) equation of the tangent plane.

* Working Rule to Tangent plane at (x_1, y_1, z_1) :-

In the equation of given cone (or) any surface change x^2 to xx_1, y^2 to yy_1, z^2 to zz_1, yz to $\frac{1}{2}(yz_1+y_1z)$, zx to $\frac{1}{2}(zx_1+z_1x)$, xy to $\frac{1}{2}(xy_1+x_1y)$, x to $\frac{1}{2}(x+x_1)$, y to $\frac{1}{2}(y+y_1)$, z to $\frac{1}{2}(z+z_1)$.

The equation obtained by this method will be same as equation ⑥.

Note:-

→ The tangent plane at any point of a cone passes through its vertex.

→ The vertex of the cone ① is $(0,0,0)$ and it clearly lies on the ~~tangent~~ plane ⑥

→ the tangent plane at any point 'P' of a cone touches the cone (P intersection) along the generator through P.

Sol^n :- Let $P(x_1, y_1, z_1)$ be any point. The equation of the tangent plane at $P(x_1, y_1, z_1)$ is $x(ax_1+hy_1+gz_1)+y(hx_1+by_1+fz_1)$

$+z(gx_1+fy_1+cz_1)=0$ —— ①

The equations of OP the generator through P are

$$\frac{x-0}{x_1-0}=\frac{y-0}{y_1-0}=\frac{z-0}{z_1-0}$$

using $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$

$$\Rightarrow \frac{x}{x_1}=\frac{y}{y_1}=\frac{z}{z_1}=r \text{ (say)}$$

Any point on OP is $Q(rx_1, ry_1, rz_1)$

The equation of the tangent plane at Q is

$x(arx_1+hry_1+grz_1)+y(hrx_1+bry_1+frz_1)$

$+z(grx_1+fry_1+crz_1)=0.$

Dividing throughout by r.

$x(ax_1+hy_1+gz_1)+y(hx_1+by_1+fz_1)$

$+z(gx_1+fy_1+cz_1)=0.$

which is the same as the equation ① of the tangent plane at P.

$\therefore$ The tangent plane at P also touches the cone at any point of OP.

i.e. the generator through P.

$\therefore$ it touches the cone along OP.

This OP is called the generator of contact.

Note :- In the equation of the cone $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$

We generally use the following notations:

(1) $D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$

(2). A, B, C, F, G, H are the cofactors of a, b, c, f, g, h in D.

so that $A = bc - f^2$, $B = ca - g^2$,

$C = ab - h^2$, $F = gh - af$, $G = hf - bg$

$H = fg - ch$

(3). $BC - F^2 = D.a$

Similarly $CA - G^2 = Db$, $AB - H^2 = Dc$

$GH - AF = fD$, $HF - BG = gD$,

$FG - CH = hD$.

where $D = abc + 2fgh - af^2 - bg^2 - ch^2$

(4) $\begin{vmatrix} a & h & g & u \\ h & b & f & v \\ g & f & c & w \\ u & v & w & 0 \end{vmatrix}$

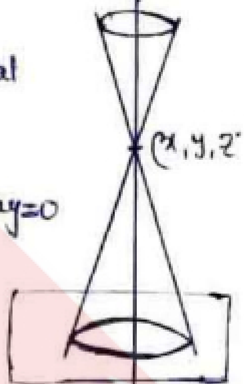
$= -(Au^2 + Bv^2 + Cw^2 + 2Fvw + 2Gwu + 2Huv)$

* Condition of tangency of a plane and cone:—

The condition that the plane $lx + my + nz = 0$ may touch the cone $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ is $Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0$.

* Reciprocal Cone :—

The locus of the normals to the tangent planes through vertex of the cone is another cone called the reciprocal cone.



* The equation of reciprocal cone of the cone

$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$

is $Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0$

where A, B, C, D, F, G, H are cofactors of a, b, c, f, g, h in

$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$

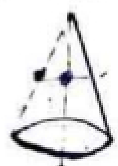
Problems :—

Show that the locus of the mid points of chords of the cone $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$ drawn parallel to the line $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ is the plane

$x(al + hm + gn) + y(hl + bm + fn) + z(gl + fm + cn) = 0$.

Sol'n :- Let $P(x_1, y_1, z_1)$ be the midpoint of one of the chords drawn ||el to the $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$

Then equation of this chord is



$\frac{x - x_1}{l} = \frac{y - y_1}{m} = \frac{z - z_1}{n}$ —— (1)

Any point on this line is

$(lr+x_1, mr+y_1, nr+z_1)$

If it lies on the cone

$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$

then $a(lr+x_1)^2+b(mr+y_1)^2+c(nr+z_1)^2$

$+2f(mr+y_1)(nr+z_1)+2g(nr+z_1)(lr+x_1)$

$+2h(lr+x_1)(mr+y_1)=0$

$\Rightarrow r^2(al^2+bm^2+cn^2+2fmn+2gnl+2hlm)$

$+2r[l(ax_1+hy_1+gz_1)+m(hx_1+by_1+fz_1)$

$+n(gx_1+fy_1+cz_1)]+$

$(ax_1^2+by_1^2+cz_1^2+2fy_1z_1+2gz_1x_1+2hx_1y_1)=0$

which is a quadratic in r.

Since $P(x_1, y_1, z_1)$ is the midpoint of the chord.

$\therefore$ The two values of r should be equal in magnitude but opposite in sign.

$\therefore$ Sum of roots $=0$ (or) the coefficient of $r=0$.

i.e. $l(ax_1+hy_1+gz_1)+m(hx_1+by_1+fz_1)$

$+n(gx_1+fy_1+cz_1)=0$.

$\Rightarrow x_1(al+hm+gn)+y_1(hl+bm+fn)$

$+z_1(gl+fm+cn)=0$

$\therefore$ The locus of $P(x_1, y_1, z_1)$ is

$x(al+hm+gn)+y(hl+bm+fn)+$

$z(gl+fm+cn)=0$ —— (2)

which is the required plane.

$\rightarrow$ Find the locus of the chords of the cone which are bisected at a fixed point.

Sol'n: Let $P(x_1, y_1, z_1)$ be the given fixed point and let any chord through P which is bisected at P be

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \quad \text{—— (1)}$$

Any point on this line is

$(lr+x_1, mr+y_1, nr+z_1)$

If it lies on the cone

$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$

then $a(lr+x_1)^2+b(mr+y_1)^2+c(nr+z_1)^2$

$+2f(mr+y_1)(nr+z_1)+2g(nr+z_1)(lr+x_1)$

$+2h(lr+x_1)(mr+y_1)=0$

$\Rightarrow r^2(al^2+bm^2+cn^2+2fmn+2gnl+2hlm)$

$+2r[l(ax_1+hy_1+gz_1)+m(hx_1+by_1+fz_1)$

$+n(gx_1+fy_1+cz_1)]$

$+ax_1^2+by_1^2+cz_1^2+2fy_1z_1+2gz_1x_1+2hx_1y_1=0$.

which is a quadratic in r.

Since $P(x_1, y_1, z_1)$ is the mid point of the chord (1)

$\therefore$ the two values of r should be equal in magnitude but opposite in sign.

$\therefore$ Coefficient of $r=0$.

$\therefore\ l(ax_1+hy_1+gz_1)+m(hx_1+by_1+fz_1)+n(gx_1+fy_1+cz_1)=0$ — (2)

Eliminating l,m,n from (1) & (2) the locus of the chords which are bisected at P_1 is

$(x-x_1)(ax_1+hy_1+gz_1)+(y-y_1)(hx_1+by_1+fz_1)+(z-z_1)(gx_1+fy_1+cz_1)=0$

which is the required equation.

→ Prove that the cones $ax^2+by^2+cz^2=0$ and $\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=0$ are reciprocal.

Sol^n:- The given first of the cone is $ax^2+by^2+cz^2=0$ — (1)

comparing with

$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$

we have $a=a,\ b=b,\ c=c$

$f=0,\ g=0,\ h=0.$

$\therefore\ A=bc-f^2=bc-0=bc$

Similarly $B=ca-g^2=ca-0=ca$

$C=ab-h^2=ab-0=ab$

$F=gh-af=0-0=0,$

$G=hf-bg=0,\quad H=fg-ch=0$

$\therefore$ The reciprocal cone of (1) is

$Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0$

$\Rightarrow bcx^2+cay^2+abz^2=0$

(on dividing through by abc)

$\Rightarrow \frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=0$

which is the second cone.

Note:- The condition for the cone

$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ — (1)

to have three mutually perpendicular tangent planes, if the reciprocal cone

$Ax^2+By^2+Cz^2+2Fyz+2Gzx+2Hxy=0$

has three mutually perpendicular generators for which $A+B+C=0$

i.e. $f^2+g^2+h^2=bc+ca+ab$

→ Prove that the perpendiculars drawn from the origin to the tangent plane to the cone $ax^2+by^2+cz^2=0$ lie on the cone $\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=0.$

Sol^n:- The given cone is

$ax^2+by^2+cz^2=0$ — (1)

we required to find the reciprocal cone of (1)

Comparing (1) with

$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$

we have. $a=a,\ b=b,\ c=c$

$f=0,\ g=0,\ h=0$

$\therefore\ A=bc-f^2=bc-0=bc$

$B=ca-g^2=ca-0=ca$

$C=ab-h^2=ab-0=ab$

$F = gh - af = 0 - 0$

$G = hf - bg = 0 - 0$

$H = fg - ch = 0 - 0$

$\therefore$ The reciprocal cone is

$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0$

$\Rightarrow \quad bcx^2 + cay^2 + abz^2 + 0 + 0 + 0 = 0$

$\Rightarrow \quad \frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = 0$

which is the required equation.

→ Show that the general equation of the cone which touches the three coordinate planes is

$$\sqrt{fx} \pm \sqrt{gy} \pm \sqrt{hz} = 0$$

Sol'n :- The general equation of a cone through the coordinate axes is

$$fyz + gzx + hxy = 0 \quad \text{—①}$$

Its reciprocal cone is

$$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0 \quad \text{—②}$$

where $A = bc - f^2 = 0 - f^2 = -f^2$

$B = ca - g^2 = -g^2$

$C = ab - h^2 = -h^2$

$F = gh - af = gh$

$G = hf - bg = hf$ & $H = fg - ch = fg$

$\therefore$ ② $\equiv -f^2x^2 - g^2y^2 - h^2z^2 + 2ghyz + 2hfzx + 2fgxy = 0$

$\Rightarrow f^2x^2 + g^2y^2 + h^2z^2 - 2ghyz - 2hfzx + 2fgxy = 4fgxy$

$\Rightarrow (fx + gy - hz)^2 = 4gfxy$

$\Rightarrow fx + gy - hz = \pm 2\sqrt{fgxy}$

$\Rightarrow fx \pm 2\sqrt{fgxy} + gy = hz$

$\Rightarrow (\sqrt{fx} \pm \sqrt{gy})^2 = hz$

$\Rightarrow \sqrt{fx} \pm \sqrt{gy} \pm \sqrt{hz} = 0$

→ Find the equation of the cone which touches three coordinate planes and the planes $x + 2y + 3z = 0$, $2x + 3y + 4z = 0$

Sol'n :- Required cone which touches the three coordinate planes and the planes $x + 2y + 3z = 0$, $2x + 3y + 4z = 0$ is reciprocal line of a cone which passes through normals through the origin ie which passes through the three coordinate axes and two normals

$$\frac{x}{1} = \frac{y}{2} = \frac{z}{3} \text{ —① } \& \frac{x}{2} = \frac{y}{3} = \frac{z}{4} \text{ —②}$$

Now any cone equation through the coordinate axis is

$$fyz + gzx + hxy = 0 \quad \text{—③}$$

If this cone passes through the lines ① & ②

$\therefore$ d.c's of these lines satisfy the equation of cone ③.

$\therefore\ 6f + 3g + 2h = 0$ and

$12f + 8g + 6h = 0$

$\therefore\ \frac{f}{2} = \frac{g}{-12} = \frac{h}{12} \Rightarrow \frac{f}{1} = \frac{g}{-6} = \frac{h}{6}$

$\therefore$ ③ $\equiv yz - 6zx + 6xy = 0$

$\Longrightarrow 2yz - 12zx + 12xy = 0$ —④

The required cone is the reciprocal cone of ④

Comparing ④ with

$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$

we have

$a = b = c = 0,\ f = 1,\ g = -6,\ h = 6$

$A = bc - f^2 = -1,\ B = ca - g^2 = -36$

$C = ab - h^2 = -36$

$F = gh - af = -36,\ G = hf - bg = 6$

$H = fg - ch = -6$

$\therefore$ The reciprocal cone is

$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0$

$-x^2 - 36y^2 - 36z^2 - 72yz + 12zx - 12xy = 0.$

$\Rightarrow x^2 + 36y^2 + 36z^2 + 72yz - 12zx + 12xy = 0$

which is the required equation of the cone which touches the three coordinate planes and the two given planes.

→ Prove that the cones

$ayz + bzx + cxy = 0$,

$(ax)^{1/2} + (by)^{1/2} + (cz)^{1/2} = 0$ are reciprocal.

Sol'n :- The given cones are

$ayz + bzx + cxy = 0$ —①

$(ax)^{1/2} + (by)^{1/2} + (cz)^{1/2} = 0$ —②

we required to find the reciprocal cone of ② is ①

② $\equiv \sqrt{ax} + \sqrt{by} + \sqrt{cz} = 0$

$\Longrightarrow \sqrt{ax} + \sqrt{by} = -\sqrt{cz}$

$\Rightarrow (\sqrt{ax} + \sqrt{by})^2 = cz$

$\Rightarrow ax + by + 2\sqrt{ax}\sqrt{by} = cz$

$\Rightarrow ax + by - cz = -2\sqrt{abxy}$

$\Rightarrow (ax + by - cz)^2 = 4abxy$

$\Rightarrow a^2x^2 + b^2y^2 + c^2z^2 + 2abxy - 2bcyz - 2aczx = 4abxy$

$\Rightarrow a^2x^2 + b^2y^2 + c^2z^2 - 2abxy - 2bcyz - 2aczx = 0$ —③

For the reciprocal cone

This is comparing with

$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$

$a = a^2,\ b = b^2,\ c = c^2,\ f = -bc,\ g = -ac,$
$h = -ab.$

$\therefore\ A = bc - f^2 = b^2c^2 - b^2c^2 = 0$

$B = ca - g^2 = c^2a^2 - a^2c^2 = 0$

$C = ab - h^2 = a^2b^2 - a^2b^2 = 0$

$F = gh - af = (-ac)(-ab) + a^2bc$

$= a^2bc + a^2bc = 2a^2bc$

$G = hf - bg = 2ab^2c,\quad H = 2abc^2$

the reciprocal cone is

$Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0$

$\Rightarrow 0 + 0 + 0 + 4a^2bc\,yz + 4ab^2c\,zx + 4abc^2xy = 0$

$\Rightarrow ayz + bzx + cxy = 0$

which is required equation.

→ Prove that the tangent planes to the cone $x^2-y^2+2z^2-3yz+4zx-5xy=0$ are perpendicular to the generator of the cone. $17x^2+8y^2+29z^2+2f\,yz+28yz-46zx-16xy=0$

Sol'n :- The given first cone is

$$x^2-y^2+2z^2-3yz+4zx-5xy=0 \quad \text{—— (1)}$$

we are required to find the reciprocal cone of (1)

∴ Comparing (1) with

$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$

we have $a=2,\ b=-2,\ c=4,$

$f=-3,\ g=4,\ h=-5$

continue this we get the solution.

(4)

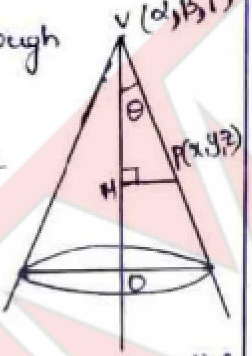
* The Right Circular Cone:–

Definition :– The surface generated by a straight line which passes through a fixed point and makes a constant angle with a fixed line through the fixed point is known as the right circular cone.

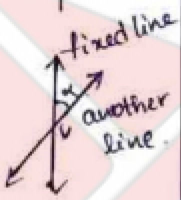
→ The fixed point is called the vertex.

→ The constant angle is called the semi-vertical angle.

→ The fixed line through the fixed point (i.e. vertex) is called the axis of the cone.



Note:– The section of a right circular cone by a plane perpendicular to its axis is a circle.

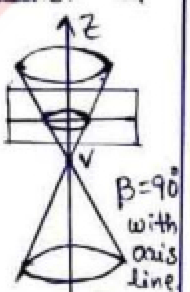


* Equations of a right circular Cone :–

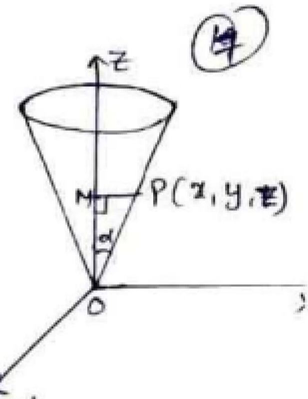
(a) Standard form

To show that the equation of the right circular cone whose vertex is the origin, axis OZ and semi-vertical angle α is

$$x^2+y^2 = z^2\tan^2\alpha.$$



Let $P(x,y,z)$ be any point on the line.



Draw $PM \perp OZ$

$\therefore \angle MOP = \alpha$

Now, in the right angled ΔOMP,

$$\frac{OM}{OP} = \cos\alpha \quad \text{——} \quad (1)$$

Now OM = Projection of OP on OZ whose d.c's are $0,0,1$

$$= 0(x-0)+0(y-0)+1(z-0)$$

$$= z \qquad [\text{Using } l(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1)]$$

Also $OP = \sqrt{x^2+y^2+z^2}$

$$(1) \equiv \frac{z}{\sqrt{x^2+y^2+z^2}} = \cos\alpha$$

$$\Rightarrow z^2 = (x^2+y^2+z^2)\cos^2\alpha$$

$$\Rightarrow z^2\sec^2\alpha = (x^2+y^2+z^2)$$

$$\Rightarrow z^2(1+\tan^2\alpha) = x^2+y^2+z^2$$

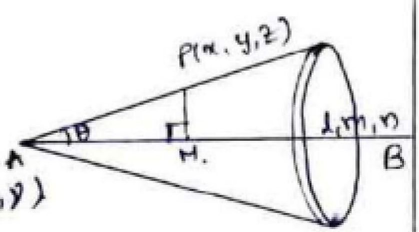
$$\Rightarrow x^2+y^2 = z^2\tan^2\alpha$$

which is a required equation

(b) General Form :–

To find the equation of a right circular cone whose vertex is (α,β,γ), semi vertical angle θ, and axis has d.c.'s l,m,n.

Sol^n:– Let $P(x,y,z)$ be any point on cone and AB, the axis of cone whose d.c's are l, m, n and passes through the vertex $A(\alpha, \beta, \gamma)$



Draw $PM \perp AB$

$\therefore$ $\angle PAM = \theta$, the semi verticle angle

$\therefore$ right angle $\triangle AMP$,

$$\frac{AM}{AP} = \cos\theta \quad \text{——— (1)}$$

AM = projection of AP on the AB line whose d.c's are l, m, n.

$$= l(x-\alpha) + m(y-\beta) + n(z-\gamma)$$

and $AP = \sqrt{(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2}$

$$(1) \equiv [l(x-\alpha) + m(y-\beta) + n(z-\gamma)]^2$$

$$= [(x-\alpha)^2 + (y-\beta)^2 + (z-\gamma)^2]\cos^2\theta \quad \text{——— (2)}$$

which is the required equation of the cone.

Note:- (i) Put $\alpha = \beta = \gamma = 0$ in (2) then the equation of the right circular cone whose vertex is origin and axis with the d.c's l, m, n and semi vertical angle θ is

$$(lx + my + nz)^2 = (x^2 + y^2 + z^2)\cos^2\theta$$

(ii) If OZ is the axis of cone and $(0,0,0)$ as the vertex and θ, the semi vertical angle, then putting $\alpha = \beta = \gamma = 0$, $l = 0, m = 0, n = 1$.

$$(2) \equiv z^2 = (x^2 + y^2 + z^2)\cos^2\theta$$

$$\Rightarrow z^2 \sec^2\theta = x^2 + y^2 + z^2$$

$$\Rightarrow z^2(1 + \tan^2\theta) = x^2 + y^2 + z^2$$

$$\Rightarrow z^2 \tan^2\theta = x^2 + y^2$$

(iii) The semi vertical angle of a right circular cone admitting sets of three mutually perpendicular generators is $\tan^{-1}\sqrt{2}$.

For this, the sum of the coefficients of x^2, y^2, z^2 in the equation of such a cone must be zero and this means that

$$1 + 1 - \tan^2\theta = 0 \Rightarrow \theta = \tan^{-1}\sqrt{2}.$$

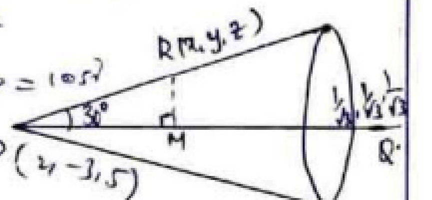
→ Find the equation to the right circular cone whose vertex is $P(2, -3, 5)$, axis PQ which makes equal angles with the axes and semi vertical angle is $30°$.

Sol^n:- Since the d.c's of the axis PQ which makes equal angles with the coordinate axes.

If a line PQ makes angles α, β, γ with axes, we take

$\alpha = \beta = \gamma$

$\Rightarrow \cos\alpha = \cos\beta = \cos\gamma$



$\Rightarrow l = m = n$

Since $l^2 + m^2 + n^2 = 1$

$$\Rightarrow 3l^2 = 1$$

$$\Rightarrow l = \pm\frac{1}{\sqrt{3}}$$

we take +ve sign.

$l = m = n = \frac{1}{\sqrt{3}}$.

Let $R(x,y,z)$ be any pt on the surface of the cone.

Draw $RM \perp PQ$

$\therefore \angle MPR = 30^\circ$

In the rt. angled $\triangle RMP$

$\cos 30^\circ = \frac{MP}{PR}$.

$\Rightarrow \frac{\sqrt{3}}{2} = \frac{MP}{PR}$ —(1)

Now MP = projection of PR on the axis PQ. —(2)

and d.r's of PR are $x-2,\ y+3,\ z-5$.

and d.c's of PQ are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

$\therefore (2) \equiv MP = \frac{1}{\sqrt{3}}(x-2) + \frac{1}{\sqrt{3}}(y+3) + \frac{1}{\sqrt{3}}(z-5)$

$= \frac{1}{\sqrt{3}}(x+y+z-4)$

and $PR = \sqrt{(x-2)^2+(y+3)^2+(z-5)^2}$

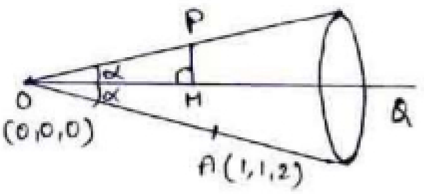
$\therefore (1) \equiv 5(x^2+y^2+z^2) - 8(yz+zx+xy) - 4x + 86y - 58z + 278 = 0.$

→ Find the equation of the circular cone which passes through the point $(1,1,2)$ and has its vertex at the origin and the axis the line $\frac{x}{2} = \frac{y}{-4} = \frac{z}{3}$.

Sol'n:- Let the d.c's of axis be l, m, n.

Given that the axis the line

$\frac{x}{2} = \frac{y}{-4} = \frac{z}{3}$



$\therefore$ The d.r's of $(0,0,0)$ the OQ are proportional to $(2,-4,3)$

$\therefore$ The actual d.c's of OQ are

$\frac{2}{\sqrt{29}}, \frac{-4}{\sqrt{29}}, \frac{3}{\sqrt{29}}$

Let α be the semi-vertical angle of the cone

Since $A(1,1,2)$ lies on the cone.

$\therefore$ The d.c's of OA are proportional to $1-0, 1-0, 2-0$.

$\therefore$ The actual d.c's are $\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}$

The semi vertical angle α of a right circular cone is the angle between the axis & the generator of the cone.

$\therefore \alpha$ is the angle between OQ & OA.

$\therefore \cos\alpha = l_1 l_2 + m_1 m_2 + n_1 n_2$

$= \left(\frac{2}{\sqrt{29}}\right)\left(\frac{1}{\sqrt{6}}\right) + \left(\frac{-4}{\sqrt{29}}\right)\left(\frac{1}{\sqrt{6}}\right) + \left(\frac{3}{\sqrt{29}}\right)\left(\frac{2}{\sqrt{6}}\right)$

$= \frac{1}{\sqrt{29}} \cdot \frac{1}{\sqrt{6}}\left[2-4+6\right]$

$$\boxed{\cos\alpha = \frac{4}{\sqrt{29}\cdot\sqrt{6}}}$$

Let $P(x,y,z)$ be any point on the cone.

Draw $PM \perp OQ$

$\therefore$ In the right angle ΔRMO

$\therefore \cos\alpha = \frac{MO}{PO}$

$\Rightarrow (MO)^2 = (PO)^2 \left(\frac{16}{29\times 6}\right)$ ——— ①

Now MO = Projection of PO on OQ

$= l(x_2-x_1) + m(y_2-y_1) + n(z_2-z_1)$

$= \frac{2}{\sqrt{29}}(x-0) + \left(\frac{-4}{\sqrt{29}}\right)(y-0) + \frac{3}{\sqrt{29}}(z-0)$

$= \frac{1}{\sqrt{29}}[2x-4y+3z]$

and $PO = \sqrt{x^2+y^2+z^2}$

$\therefore$ ① $\equiv \frac{1}{29}(2x-4y+3z)^2 = (x^2+y^2+z^2)\frac{16}{29\times 6}$

$\Rightarrow 3(4x^2+16y^2+9z^2-16xy-24yz+12xz)$

$= 8x^2+8y^2+8z^2$

$\Rightarrow 4x^2+40y^2+19z^2-48xy-72yz$

$+36xz = 0$

$\rightarrow$ Lines are drawn from the origin with the d.c's proportional $(1,2,2)$, $(2,3,6)$, $(3,4,12)$; find the direction cosines of the axis of right circular cone through them, and Prove that the semi vertical angle of the cone is $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

Sol'n :- Let l, m, n be the d.c's of the axis of the right circular cone.

Let O be the origin and P, Q, R be the given points.

Now the d.r's of OP, OQ, OR are $(1,2,2)$, $(2,3,6)$, $(3,4,12)$

The d.c's of OP, OQ and OR are $1/3, 2/3, 2/3$, $1/3$, $\frac{3}{7}$, $6/7$ and $\frac{3}{13}, \frac{4}{13}, \frac{12}{13}$.

Let α be the semi-vertical angle of the cone then

$\cos\alpha = \frac{1}{3}l + \frac{2}{3}m + \frac{2}{3}n = \frac{2}{7}l + \frac{3}{7}m + \frac{6}{7}n$

$= \frac{3}{13}l + \frac{4}{13}m + \frac{12}{13}n$. ——— Ⓐ

Now take first two members.

$\frac{1}{3}l + \frac{2}{3}m + \frac{2}{3}n = \frac{2}{7}l + \frac{3}{7}m + \frac{6}{7}n$

$\Rightarrow 7l + 14m + 14n = 6l + 9m + 18n$

$\Rightarrow l + 5m - 4n = 0$ ——— ①

From first & last we get

$2l + 7m - 5n = 0$ ——— ②

Solving, we get

$\frac{l}{-1} = \frac{m}{1} = \frac{n}{1} = \pm\frac{\sqrt{l^2+m^2+n^2}}{\sqrt{1+1+1}} = \pm\frac{1}{\sqrt{3}}$

$\therefore$ The d.c's of the axis are

$\frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

$\therefore$ Putting these values in Ⓐ we get

$\cos\alpha = \frac{1}{3}\left(-\frac{1}{\sqrt{3}}\right) + \frac{2}{3}\left(\frac{1}{\sqrt{3}}\right) + \frac{2}{3}\left(\frac{1}{\sqrt{3}}\right)$

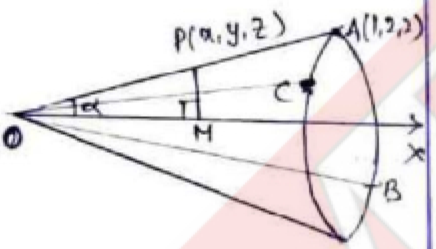
$= \frac{1}{3\sqrt{3}}(-1+2+2) = \frac{3}{3\sqrt{3}}$

$\cos\alpha = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

→ find the equation of the right circular cone generated by the straight lines drawn from the origin to cut the circle through the three points (1,2,2) (2,1,-2) and (2,-2,1)

Sol'n :- Let $A(1,2,2)$, $B(2,1,-2)$, $C(2,-2,1)$ be the given point.

Let l,m,n be the actual d.c's of the axis OX.



Then OA, OB, OC make the same angle α with the axis OX, where α is the semi-vertical angle.

The direction ratios of OA, OB, OC are (1,2,2), (2,1,-2), (2,-2,1)

∴ The d.c's of OA, OB, OC are

$\frac{1}{3}, \frac{2}{3}, \frac{2}{3}$; $\frac{2}{3}, \frac{1}{3}, \frac{-2}{3}$; $\frac{2}{3}, \frac{-2}{3}, \frac{1}{3}$

$\therefore \cos\alpha = \frac{1}{3}l + \frac{2}{3}m + \frac{2}{3}n = \frac{2}{3}l + \frac{1}{3}m - \frac{2}{3}n = \frac{2}{3}l + \left(\frac{-2}{3}\right)m + \frac{1}{3}n$ —— ①

from first two members we have

$\frac{l}{3} + \frac{2}{3}m + \frac{2}{3}n = \frac{2}{3}l + \frac{1}{3}m - \frac{2}{3}n$

$\Rightarrow l + 2m + 2n = 2l + m - 2n$

$\Rightarrow l - m - 4n = 0$ —— ②

from last two members we have

$3m - 3n = 0 \Rightarrow 0l + m - n = 0$ —— ③

Solving ② & ③ we have

$\frac{l}{5} = \frac{m}{1} = \frac{n}{1} = \pm\frac{\sqrt{l^2+m^2+n^2}}{\sqrt{25+1+1}} = \pm\frac{1}{\sqrt{27}}$

$\therefore l = \frac{5}{\sqrt{27}}$, $m = \frac{1}{\sqrt{27}}$, $n = \frac{1}{\sqrt{27}}$

$\therefore$ ① $\equiv \cos\alpha = \frac{1}{3}\left(\frac{5}{\sqrt{27}}\right) + \frac{2}{3}\left(\frac{1}{\sqrt{27}}\right) + \frac{2}{3}\left(\frac{1}{\sqrt{27}}\right)$

$= \frac{1}{\sqrt{27}}\left(\frac{5+2+2}{3}\right)$

$= \frac{1}{\sqrt{27}} \times 9 = \frac{9}{9\sqrt{3}} = \frac{1}{\sqrt{3}}$

$\therefore \boxed{\cos\alpha = \frac{1}{\sqrt{3}}}$

Let $P(x,y,z)$ be any point on the cone.

Draw $PM \perp OX$

$\therefore \angle MOP = \alpha$

In the right angle ΔOMP,

$\frac{OM}{OP} = \cos\alpha$

$(OM)^2 = (OP)^2 \frac{1}{3}$ —— ④

OM = projection of OP on O

$= l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$

continue this solution

→ If α is the semi-vertical angle of the right circular cone which passes through the lines OX, OY, $x = y = z$, show that $\cos\alpha = (9 - 4\sqrt{3})^{-1/2}$

Sol'n :- Let l,m,n be the d.c's of the axis of the cone. Since the axis makes the same angle α with each of the lines ox, oy and $x=y=z$.

Now the d.r.s of OX, OY, $x=y=z$ are $(1,0,0)$, $(0,1,0)$ and $(1,1,1)$

$\therefore$ The d.c.s of ox, oy and $x=y=z$ are $(1,0,0)$, $(0,1,0)$ and $\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$

$$\cos\alpha = l(1)+m(0)+n(0) = l(0)+m(1)+n(0)$$
$$= l\left(\tfrac{1}{\sqrt{3}}\right)+m\left(\tfrac{1}{\sqrt{3}}\right)+n\left(\tfrac{1}{\sqrt{3}}\right) \quad \text{--- (1)}$$

from first two no.s we have $l=m \Rightarrow l-m+0n=0$ --- (2)

from last two no.s we have $m=\frac{l}{\sqrt{3}}+\frac{m}{\sqrt{3}}+\frac{n}{\sqrt{3}}$

$$\Rightarrow \frac{l}{\sqrt{3}}+\left(\frac{1-\sqrt{3}}{\sqrt{3}}\right)m+\frac{n}{\sqrt{3}}=0$$

$$\Rightarrow l+(1-\sqrt{3})m+n=0 \quad \text{--- (3)}$$

solving (2) and (3)

$$\frac{l}{-1+0}=\frac{m}{0-1}=\frac{n}{1-\sqrt{3}+1} \Rightarrow \frac{l}{-1}=\frac{m}{-1}=\frac{n}{2-\sqrt{3}}$$

$$\therefore \frac{l}{-1}=\frac{m}{-1}=\frac{n}{2-\sqrt{3}}=\pm\frac{\sqrt{l^2+m^2+n^2}}{\sqrt{1+1+(2-\sqrt{3})^2}}$$

$$=\pm\frac{1}{\sqrt{2+4+3-4\sqrt{3}}}$$

$$=\pm\frac{1}{\sqrt{9-4\sqrt{3}}}$$

$$\frac{l}{-1}=\frac{-1}{\sqrt{9-4\sqrt{3}}} \Rightarrow l=\frac{+1}{\sqrt{9-4\sqrt{3}}}$$

$$m=\frac{+1}{\sqrt{9-4\sqrt{3}}}$$

$$n=\frac{-(2-\sqrt{3})}{\sqrt{9-4\sqrt{3}}}=\frac{\sqrt{3}-2}{\sqrt{9-4\sqrt{3}}}$$

from (1)

$$\cos\alpha=\frac{-1}{\sqrt{9-4\sqrt{3}}}$$

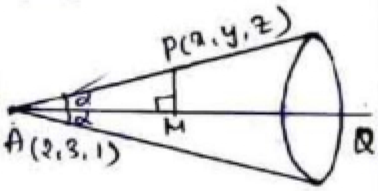
$$\alpha=\cos^{-1}\left(\frac{-1}{\sqrt{9-4\sqrt{3}}}\right)$$

$\rightarrow$ show that the equation of the right circular cone with vertex $(2,3,1)$, axis parallel to the line $-x=\frac{y}{2}=z$ and one of its generators having d.c's proportional to $(1,-1,1)$ is

$$x^2-8y^2+z^2+12xy-12yz+16zx-46x+36y+22z-19=0$$

Sol^n:- Let l,m,n be the d.c's of the axis of the right circular cone.

The given line $\frac{x}{-1}=\frac{y}{2}=\frac{z}{1}$ is $\parallel$el to the axis.



$\therefore$ the d.c's of the axis are proportional to $-1,2,1$.

$\therefore$ The actual d.c's are $-\frac{1}{\sqrt{6}},\frac{2}{\sqrt{6}},\frac{1}{\sqrt{6}}$

$$\therefore l=-\frac{1}{\sqrt{6}}, m=\frac{2}{\sqrt{6}}, n=\frac{1}{\sqrt{6}}.$$

Now the d.c's of its generator are proportional to $1,-1,1$.

The actual d.c's are

$\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

Now let α be the semi-vertical angle.

Then the semi vertical angle α of a right circular cone is the angle between the axis and the generator of the cone.

$\therefore \cos\alpha = l_1 l_2 + m_1 m_2 + n_1 n_2$

$= \left(\frac{-1}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{3}}\right) + \left(\frac{2}{\sqrt{6}}\right)\left(-\frac{1}{\sqrt{3}}\right) + \left(\frac{1}{\sqrt{6}}\right)\left(\frac{1}{\sqrt{3}}\right)$

$= \frac{1}{\sqrt{18}}[-1-2+1] = \frac{-2}{3\sqrt{2}}$

Let $P(x, y, z)$ be any point on the cone.

Draw $PM \perp AQ$

$\therefore \angle MAP = \alpha$

In right angle ΔAMP,

$\cos\alpha = \frac{AM}{AP}$

$\Rightarrow (AM)^2 = (AP)^2 \cos^2\alpha$ —— (1)

Now AM = Projection of AP on AQ

$= l(x_2 - x_1) + m(y_2 - y_1) + n(z_2 - z_1)$.

$= \left(\frac{-1}{\sqrt{6}}\right)(x-2) + \left(\frac{2}{\sqrt{6}}\right)(y-3) + \frac{1}{\sqrt{6}}(z-1)$

$= \frac{1}{\sqrt{6}}[-x + 2y + z + 2 - 6 - 1]$

$= \frac{1}{\sqrt{6}}[-x+2y+z-5]$

and $(AP)^2 = \sqrt{(x-2)^2 + (y-3)^2 + (z-1)^2}$

(1) $\equiv \frac{1}{\cancel{6}_2}[(-x+2y) + (z-2)]^2 = [(x-2)^2 + (y-3)^2 + (z-1)^2] \times \frac{\cancel{4}^2}{\cancel{9}\times\cancel{2}_3}$

$\Rightarrow 3[x^2 + 4y^2 - 4xy + z^2 + 4 - 4z + 2(-x+2y)(z-2)] = 4[x^2+y^2+z^2-4x-6y-2z+4+9+1]$

$\Rightarrow 3x^2 + 12y^2 - 12xy + 3z^2 + 12 - 12z + 6(-xz + 2x + 2yz - 4y) = 4x^2 + 4y^2 + 4z^2 - 16x - 24y - 8z + 56$

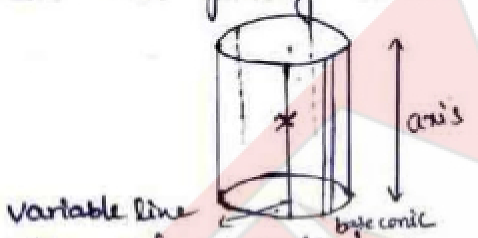
$x^2 - 8y^2 + z^2 + 12xy - 12yz + 6zx - 46x + 36y + 22z - 19 = 0$

* The Cylinder * Set – VI

Definition :–

The surface generated by a variable line which is always parallel to a fixed line and intersects a given curve (or touches a given surface) is called the cylinder.

→ The variable line is called the generator, the fixed line the axis and the given curve (or surface) the guiding curve.



*** Equation of a cylinder :–**

To find the equation of the cylinder whose generators are parallel to the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ and the base conic is

$$f(x,y) = ax^2+by^2+2hxy+2gx+2fy+c=0,\ z=0.$$

Sol^n :– The given line is

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \quad \text{—— (1)}$$

and the base conic is

$$f(x,y) = ax^2+by^2+2hxy+2gx+2fy+c=0,\ z=0 \quad \text{—— (2)}$$

Let (x_1, y_1, z_1) be any point on the generator of the cylinder and parallel to the line (1).

Then equations of generator line (i.e. a line through (x_1, y_1, z_1) and $\parallel$el to (1)) are

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} \quad \text{—— (3)}$$

It meets the plane $z=0$

$\therefore$ Putting $z=0$ in (3) we get

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{0-z_1}{n}$$

$$\therefore\ x = x_1 - \frac{l}{n}z_1,\quad y = y_1 - \frac{m}{n}z_1$$

$\therefore$ The point $\left(x_1 - \frac{l}{n}z_1,\ y_1 - \frac{m}{n}z_1\right)$

If this point lies on the conic then

$$a\left[x_1-\frac{l}{n}z_1\right]^2 + b\left[y_1-\frac{m}{n}z_1\right]^2 + 2h\left(x_1-\frac{l}{n}z_1\right)\left(y_1-\frac{m}{n}z_1\right) + 2g\left(x_1-\frac{l}{n}z_1\right) + 2f\left(y_1-\frac{m}{n}z_1\right) + c = 0$$

$\therefore$ The locus of (x_1, y_1, z_1) is

$$a\left(x-\frac{l}{n}z\right)^2 + b\left(y-\frac{m}{n}z\right)^2 + 2h\left(x-\frac{l}{n}z\right)\left(y-\frac{m}{n}z\right) + 2g\left(x-\frac{l}{n}z\right) + 2f\left(y-\frac{m}{n}z\right) + c = 0$$

$$\Rightarrow a(nx-lz)^2 + b(ny-mz)^2 + 2h(nx-lz)(ny-mz) + 2ng(nx-lz) + 2nf(ny-mz) + cn^2 = 0$$

which is the required equation of the cylinder.

Note :– If the generators are $\parallel$el to z-axis, then $l=0$, $m=0$ and $n=1$.

$\therefore$ The equation of the cylinder becomes $ax^2+by^2+2hxy+2gx+2fy+c=0$ which is free from z.

→ If we required to find the equation of the cylinder whose generators are ||lel to z-axis and intersect a given conic then eliminate z from the equations of the conic.

∴ If is given the equation of the cylinder.

→ If the generators are ||lel to x-axis then eliminate 'x' and if the generators are ||lel to y-axis then eliminate y from the equations of the conic to get equations of the cylinder.

Problems

→ Find the equation of a cylinder whose generating lines have the d.c's (l, m, n) and which passes through the circle $x^2+y^2=a^2, z=0$.

→ Find the equation to the cylinder whose generators are parallel to the line $\frac{x}{1}=\frac{y}{-2}=\frac{z}{3}$, and whose guiding curve is the ellipse. $x^2+2y^2=1$, $z=3$.

→ Find the equation of the cylinder whose generators intersect the curve $ax^2+by^2=2z$, $lx+my+nz=p$ and are parallel to z-axis.

Sol: The given base conic is

$$ax^2+by^2=2z, \quad lx+my+nz=p \quad (1)$$

Since the generators of the cylinder are ||lel to the z-axis.

∴ The required equation of the cylinder free from the z-coordinate.

Now eliminate z from the equations (1), to get the required cylinder.

From first equation of (1) we have.

$$z=\frac{ax^2+by^2}{2}$$

Putting in the second equation of (1),

$$lx+my+n\left(\frac{ax^2+by^2}{2}\right)=p$$

$$\Rightarrow 2lx+2my+n(ax^2+by^2)=2p$$

$$\Rightarrow n(ax^2+by^2)+2lx+2my-2p=0$$

which is the required cylinder.

→ Find the equation of the cylinder with generators parallel to x-axis and passing through the curve

$$ax^2+by^2+cz^2=1, \quad lx+my+nz=p.$$

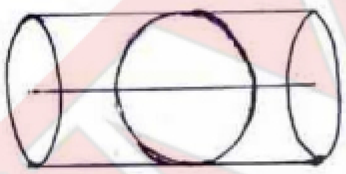
* Enveloping Cylinder of a Sphere:-

To find the equation to the cylinder whose generators touch the sphere $x^2+y^2+z^2=a^2$ and are parallel to the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ (or)

To find the locus of the tangent lines drawn to a sphere and parallel to a given line.

Sol'n :- The given sphere $x^2+y^2+z^2=a^2$ — ①

and the line



$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ — ②

Let (α,β,γ) be any point on the cylinder.

$\therefore$ Any line through (α,β,γ) $\parallel$el to ② is $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=r$ (say) — ③

Any point on this line is $(lr+\alpha,\ mr+\beta,\ nr+\gamma)$.

This point lies on the sphere ①, then

$$(\alpha+lr)^2+(\beta+mr)^2+(\gamma+nr)^2=a^2$$

$$\Rightarrow r^2(l^2+m^2+n^2)+2r(\alpha l+\beta m+\gamma n) + (\alpha^2+\beta^2+\gamma^2-a^2)=0 \quad ④$$

Clearly which is a quadratic in r.

Since the generator ③ is a tangent line of the given sphere.

$\therefore$ the two values of r given by ③ must be equal.

$\therefore$ The descriminant of ③ $=0$.

i.e. $b^2-4ac=0$.

$$[2(l\alpha+m\beta+n\gamma)]^2=4(l^2+m^2+n^2)(\alpha^2+\beta^2+\gamma^2-a^2)$$

The locus of (α,β,γ) is

$$(lx+my+nz)^2=(l^2+m^2+n^2)(x^2+y^2+z^2-a^2)$$

which is required equation of the cylinder and is known as the Enveloping cylinder of a sphere.

Problem

→ Find the enveloping cylinder of a sphere $x^2+y^2+z^2-2x+4y=1$ having its generators parallel to the line $x=y=z$.

(Ans: $x^2+y^2+z^2-xy-yz-zx-4x+5y-z-2=0$)

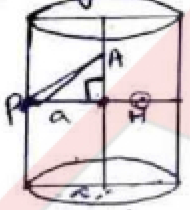
Sol'n:- Let (α,β,γ) be any point on the cylinder.

Continue in this way.

* Right Circular Cylinder :-

A surface generated by a line which intersects a fixed circle (is called guiding curve) and is ⊥lar to the plane of the circle is called right circular cylinder.

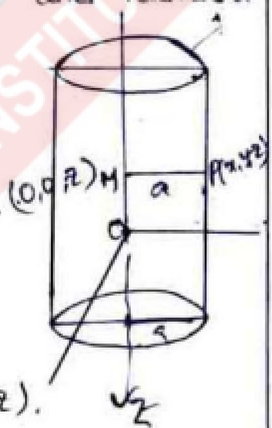
→ The normal to the plane of the circle through its centre is called the axis of the cylinder and the radius of the circle is the radius of the cylinder.



* Equation of Right Circular Cylinder :-

(a) Standard form :-

Show that the equation of the right circular cylinder whose axis is the z-axis and radius 'a' is $x^2+y^2=a^2$



Let $P(x,y,z)$ be any point on the cylinder.

Draw $PM \perp$ z-axis

$\therefore OM = z$ and the coordinates of $M(0,0,z)$.

$\therefore$ MP = radius of the cylinder (given).

But $MP=\sqrt{(x-0)^2+(y-0)^2+(z-z)^2}$

$=\sqrt{x^2+y^2}$

$\therefore \sqrt{x^2+y^2}=a$

$\Rightarrow x^2+y^2=a^2$,

which is required equation.

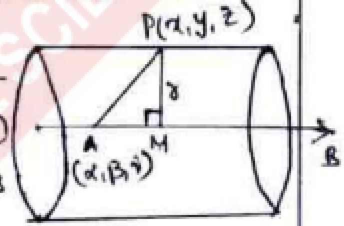
(b) General form :-

To find the equation to the right circular cylinder whose radius is 'r' and axis is the line

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}.$$

Sol'n :- Let AB be the axis of the cylinder whose equations are

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \quad \text{--- (1)}$$



where $A(\alpha,\beta,\gamma)$ is a point on it.

The d.r's of AB are l,m,n.

$\therefore$ The actual d.c's are

$$\frac{l}{\sqrt{\Sigma l^2}},\ \frac{m}{\sqrt{\Sigma l^2}},\ \frac{n}{\sqrt{\Sigma l^2}}$$

Let $P(x,y,z)$ be any pt on the cylinder.

Draw $PM \perp AB$ axis.

and join PA.

PM = radius of the cylinder = r

In the right angled ΔPAM,

$$AP^2 = AM^2 + PM^2 \quad \text{--- (2)}$$

$$(AP)^2 = (x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2$$

AM = Projection of AP on AB-axis.

$$= \frac{l}{\sqrt{\Sigma l^2}}(x-\alpha)+\frac{m}{\sqrt{\Sigma l^2}}(y-\beta)+\frac{n}{\sqrt{\Sigma l^2}}(z-\gamma)$$

$$= \frac{l(x-\alpha)+m(y-\beta)+n(z-\gamma)}{\sqrt{l^2+m^2+n^2}}$$

$$\text{①} \equiv (x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2 = \frac{l(x-\alpha)+m(y-\beta)+n(z-\gamma)}{l^2+m^2+n^2}+r^2$$

which is the required equation of the cylinder.

→ Find the equation of the right circular cylinder of radius 2 whose axis is the line

$$\frac{x-1}{2} = \frac{y-2}{2} = \frac{z-2}{2}$$

→ The axis of the a right circular cylinder of radius 2 is

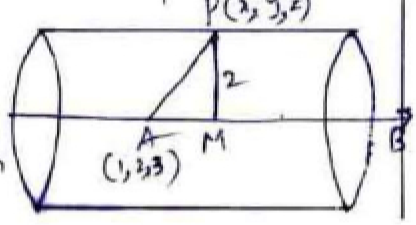
$$\frac{x-1}{2} = \frac{y}{3} = \frac{z-3}{1}$$

show that its equation is

$$10x^2+5y^2+13z^2-12xy-6yz-4zx-8x+30y-74z+59=0.$$

→ Find the equation of the right circular cylinder of radius 2 whose axis passes through (1,2,3) and has d.c's proportional to (2,-3,6).

Sol^n: Let AB be axis of the cylinder which passes through the point A(1,2,3) and has d.c's proportional 2,-3,6.



∴ Dividing each by

$$\sqrt{4+9+36} = \sqrt{49} = 7$$

∴ Actual d.c's are $\frac{1}{7}, \frac{2}{7}, \frac{3}{7}$.

Let P(x,y,z) be any point on the cylinder.

Draw $PM \perp AB$

∴ In right angled ΔAPM

$$AP^2 = AM^2 + PM^2$$

Continue this solution.

→ Find equation to the right circular cylinder whose guiding circle is

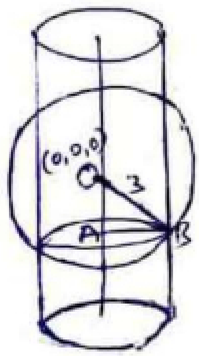
$$x^2+y^2+z^2=9, \quad x-y+z=3.$$

Note:- The axis of the cylinder is the line through centre of sphere and ⊥ to the plane of the circle and radius of the cylinder is equal to radius of circle.

Sol^n:- The sphere is $x^2+y^2+z^2=9$ —— ①

and plane is $x-y+z=3$ —— ②

The centre of the sphere is O(0,0,0) and its radius is OB=3.



OA = ⊥ distance of O(0,0,0) from the plane ②

$$= \frac{|0-0+0-3|}{\sqrt{1+1+1}} = \frac{|-3|}{\sqrt{3}} = \sqrt{3}$$

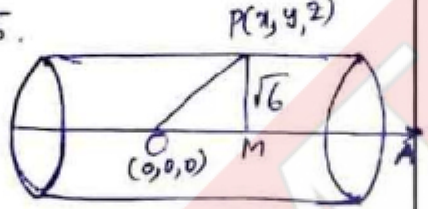
$\therefore$ AB = radius of the circle

$$= \sqrt{OB^2 - OA^2} = \sqrt{9-3} = \sqrt{6}$$

Again equation of the line through the centre O(0,0,0) of the sphere and $\perp^r$ to plane (2) are

$$\frac{x-0}{1} = \frac{y-0}{-1} = \frac{z-0}{1}$$

which is the axis of the cylinder and radius $\sqrt{6}$.



the d.c's of the axis are proportional to 1, -1, 1

$\therefore$ the actual d.c's are

$$\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$$

Let P(x,y,z) be any point on the cylinder.

Join OP and draw MP $\perp$ OA.

$$(OP)^2 = (OM)^2 + (MP)^2 \quad \text{—} (3)$$

Now $(OP)^2 = \sqrt{x^2+y^2+z^2}$

$(MP)^2 = 6$

& OM = Projection of OP on OA.

$$= \frac{1}{\sqrt{3}}(x) - \frac{1}{\sqrt{3}}(y) + \frac{1}{\sqrt{3}}(z)$$

$$= \frac{1}{\sqrt{3}}(x-y+z)$$

$$(3) \equiv x^2+y^2+z^2 = \frac{(x-y+z)^2}{3} + 6$$

$$\Rightarrow x^2+y^2+z^2+xy+yz-zx-9=0$$

which is the required equation of the cylinder.

→ Find the equation of the right circular cylinder whose guiding curve is the circle through the points (1,0,0), (0,1,0), (0,0,1).

Sol'n:- Let A(1,0,0), B(0,1,0) C(0,0,1) be the given points.

Then the circle through A, B, C is the intersection of the plane ABC and the sphere OABC.

Now the equation of the plane ABC is $\frac{x}{1} + \frac{y}{1} + \frac{z}{1} = 1$ [intercept form]

$$x+y+z = 1 \quad \text{—} (1)$$

and the sphere OABC is

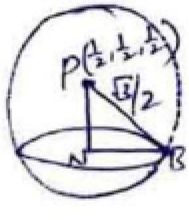
$$x^2+y^2+z^2-x-y-z = 0 \quad \text{—} (2)$$

(Using $x^2+y^2+z^2-ax-by-cz=0$)

the centre of the sphere is $P(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2})$

and radius $= \sqrt{\frac{1}{4}+\frac{1}{4}+\frac{1}{4}}$

$$= \frac{\sqrt{3}}{2}$$



from the right angle $\triangle$le. $\triangle$PMB

$$NB^2 = \sqrt{PB^2 - NP^2} \quad \text{—} (3)$$

NP = $\perp$ distance from 'P' to the plane.

$$= \frac{|\tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2}-1|}{\sqrt{1+1+1}} = \frac{1}{2\sqrt{3}}.$$

(2) $\equiv NB = \sqrt{\frac{3}{4} - \frac{1}{4\times 3}}$

$$= \sqrt{\frac{9-1}{4\times 3}} = \sqrt{\frac{8}{4\times 3}} = \sqrt{\frac{2}{3}}$$

which is the radius of the circle.

$\therefore$ This is also radius of the cylinder.

Now the equations of PN are

[i.e. through $P(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$ and $\perp$ lar to the plane (1)]

$$\frac{x-\frac{1}{2}}{1} = \frac{y-\frac{1}{2}}{1} = \frac{z-\frac{1}{2}}{1}$$

which is the axis of the cylinder.

Now the d.c's of the axis are proportional to 1, 1, 1.

The actual d.c's are $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

and the radius is $\sqrt{2/3}$.

continue in this way we get the solution.

2005 → find the right circular cylinder whose guiding curve is the circle through three points $(a,0,0)$, $(0,b,0)$ and $(0,0,c)$. find also the axis of the cylinder.