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IAS MATHEMATICS (OPT.)-2010

PAPER - II : SOLUTIONS

Que:- 1(a) Let $G = R - \{1\}$ be the set of all real numbers omitting -1. Define the binary relation $*$ on G by $a*b = a+b+ab$. Show $(G,*)$ is a group and it is abelian.

Soln:-

a) since 'G' is the set of all real numbers except '-1'. & '$*$' is an operation defined in $G = \mathbb{R} - \{-1\}$ such that,

$$a*b = a+b+ab. \quad \forall a,b \in G.$$

when,

$a,b \in G$,

$a*b = a+b+ab \in G$

$\therefore a*b \in G$.

$\therefore$ '$*$' is a binary operation on 'G'.

$\therefore a*b = a+b+ab$ —— (1) $\forall a,b \in G$.

NOTE:- if possible let, $a*b = 1$

$\Rightarrow a+b+ab = 1$

$\Rightarrow (a+1)+b(a+1) = 0$.

$\Rightarrow (a+1)(b+1) = 0$.

$\Rightarrow a+1 = 0$ or $b+1 = 0$

$\Rightarrow a = -1$ or $b = -1$

clearly which is contradiction to hypothesis, that $a \neq -1$, $b \neq -1$ in 'G'.

b)

(i) closure property :-

$\forall a,b \in G$.

$a*b = a+b+ab \in G$ by (1)

$\therefore$ 'G' is closed under '$*$'

ii) Associative property :-

$\forall\, a,b,c \in G \Rightarrow (a*b)*c = (a+b+ab)*c$

$= a+b+ab+c+(a+b+ab).c$

$= a+b+c+ab+bc+ca+abc.$

similarly,

$a*(b*c) = a+b+c+ab+bc+ca+abc$

$\therefore\ (a*b)*c = a*(b*c).$

$\therefore$ Associative law holds.

iii) Existence of left Identity :-

let, $a \in G$, $e \in G$, then $e*a = a$

Now, $e*a = a$

$\Rightarrow\ e+a+ea = a$

$\Rightarrow\ e(1+a) = 0$

$\Rightarrow\ e = 0 \in G.\ (\because a \neq -1).$

$\therefore\ e*a = 0*a$

$= 0+a+0(a)$

$= a.$

$\therefore \forall\, a \in G, \exists\, 0 \in G$, such that $0*a = a.$

$\therefore$ '0' is the left identity in 'G'.

iv) Existence of left inverse :-

let, $a \in G$, $b \in G$ then $b * a = e$.

Now, $b * a = e$

$\Rightarrow b + a + ba = 0$

$\Rightarrow b(1+a) = -a$

$\Rightarrow b = \frac{-a}{1+a} \in G. \ (\because a \neq -1)$

$$b * a = \frac{-a}{1+a} * a.$$

$$= \frac{-a}{1+a} + a + \left(\frac{-a}{1+a}\right) \cdot a$$

$$= \frac{-a}{1+a} + a - \frac{a^2}{1+a}$$

$$= \frac{-a + a(1+a) - a^2}{1+a}$$

$$= 0.$$

For each, $a \in G$, $\exists\ b = \frac{-a}{1+a} \in G$. , such that

$$\frac{-a}{1+a} * a = 0$$

$\therefore b = \frac{-a}{1+a}$ is left inverse of 'a' in 'G' w.r.t $*$.

$\therefore (G, *)$ is a group.

$\forall\ a, b \in G \Rightarrow a * b = a + b + ab$

$= b + a + ba$ (by comm. prop. of $\mathbb{R}$)

$= b * a$

$\therefore (G, *)$ is a commutative group.

1(f) IAS 2010 P-II → Construct the dual of the primal problem:

Maximize $z = 2x_1 + x_2 + x_3$; Subject to the constraints $x_1 + x_2 + x_3 \geq 6$, $3x_1 - 2x_2 + 3x_3 = 3$

$-4x_1 + 3x_2 - 6x_3 = 1$ and $x_1, x_2, x_3 \geq 0$.

Sol^n →

$$\text{Max } z = 2x_1 + x_2 + x_3 \quad \text{——} \quad (1)$$

Changing constraint 1 to ≤ types:—

$$-x_1 - x_2 - x_3 \leq -6$$

changing the constraint 2 from = to ≤ :—

$$3x_1 - 2x_2 + 3x_3 \leq 3 \ , \quad 3x_1 - 2x_2 + 3x_3 \geq 3$$

$$-3x_1 + 2x_2 - 3x_3 \leq -3$$

Changing the constraints 3 from = to ≤ :—

$$4x_1 + 3x_2 - 6x_3 \geq 1 \ ; \quad 4x_1 + 3x_2 - 6x_3 \leq 1$$

$$-4x_1 - 3x_2 + 6x_3 \leq -1$$

∴ The constraints are:—

$$-x_1 - x_2 - x_3 \leq -6$$

$$3x_1 - 2x_2 + 3x_3 \leq 3$$

$$-3x_1 + 3x_2 - 6x_3 \leq 1$$

$$-4x_1 - 3x_2 + 6x_3 \leq -1$$

Converting to duality:—

$$\text{Min } z_1 = -6y_1 + 3y_2 - 3y_3 + y_4 - y_5$$

S.C:-

$$-y_1 + 3y_2 - 3y_3 + 4y_4 - 4y_5 \geq 2$$

$$-y_1 - 2y_2 + 2y_3 + 3y_4 - 3y_5 \geq 1$$

$$-y_1 + 3y_2 - 3y_3 - 6y_4 + 6y_5 \geq 1$$

$$y_1, y_2, y_3, y_4, y_5 \geq 0$$

It can be written as :-

$$\text{Min } z_1 = -6y_1 + 3(y_2 - y_3) + 1(y_4 - y_5)$$

S.C. $-y_1 + 3(y_2 - y_3) + 4(y_4 - y_5) \geq 2$

$-y_1 - 2(y_2 - y_3) + 3(y_4 - y_5) \geq 1$

$-y_1 + 3(y_2 - y_3) - 6(y_4 - y_5) \geq 1$

let $y_2 - y_3 = u_1$, $y_4 - y_5 = u_2$

$\therefore$ Min $z = -6y_1 + 3u_1 + u_2$

S.C. $-y_1 + 3u_1 + 4u_2 \geq 2$

$-y_1 - 2u_1 + 3u_2 \geq 1$

$-y_1 + 3u_1 - 6u_2 \geq 1$

where $y_1 \geq 0$

u_1, u_2 are unrestricted.

1(c) Discuss the convergence of the sequence $\{x_n\}$ where $x_n = \frac{\sin\left(\frac{n\pi}{2}\right)}{8}$.

IAS 2010

→ Let $x_n = \frac{\sin\left(\frac{n\pi}{2}\right)}{8}$ ~~$\forall n$~~

Since $\underset{n\to\infty}{Lt}\, x_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{1}{8} & \text{if } n = 4m+1 \\ -\frac{1}{8} & \text{if } n = 4m+3 \end{cases}$

does not exist

$\therefore (x_n)$ does not converge.

1(e) 2010 P-II

Show that $u(x,y) = 2x - x^3 + 3xy^2$ is a harmonic function. Find a harmonic conjugate of $u(x,y)$. Hence find the analytic function f for which $u(x,y)$ is the real part.

Sol^n: $u(x,y) = 2x - x^3 + 3xy^2$

$$\frac{\partial u}{\partial x} = 2 - 3x^2 + 3y^2$$

$$\frac{\partial^2 u}{\partial x^2} = -6x$$

$$\frac{\partial u}{\partial y} = 6xy$$

$$\frac{\partial^2 u}{\partial y^2} = 6x$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -6x + 6x = 0.$$ Hence harmonic.

Now to find conjugate Harmonic.

$$\frac{\partial u}{\partial y} = 6xy \Rightarrow \frac{\partial v}{\partial x} = -6xy$$

$$v = -6x^2y + f(y)$$

Now $$\frac{\partial v}{\partial y} = -6x^2 + f'(y)$$

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x}$$

$$-6x^2 + f'(y) = 2 - 3x^2 + 3y^2$$

$$f'(y) = 3x^2 + 3y^2 + 2$$

$$f(y) = 3x^2y + y^3 + 2y + C$$

$$\therefore\ v = -6x^2y + 3x^2y + y^3 + 2y + C$$

$$v = y^3 - 3x^2y + 2y + C,$$ which is the harmonic conjugate of $u(x,y)$

$$F = u(x,y) + iv(x,y)$$

$$F = 2x - x^3 + 3xy^2 + i(y^3 - 3x^2y + 2y + C).$$

2010 2(a) Let $(\mathbb{R}^*, \cdot)$ be the multiplicative group of non-zero reals and $(GL(n,\mathbb{R}), \times)$ be the multiplicative group of $n \times n$ non singular real matrices. Show that the quotient group $\frac{GL(n,\mathbb{R})}{SL(n,\mathbb{R})}$ and $(\mathbb{R}^*, \circ)$ are isomorphic where, $SL(n,\mathbb{R}) = \{A \in GL(n,\mathbb{R}) \mid \det A = 1\}$

What is the centre of $GL(n,\mathbb{R})$?

Ans. Define $f : GL(n,\mathbb{R}) \to \mathbb{R}^*$

by $f(A) = \det A \quad \forall \; A \in GL(n,\mathbb{R})$

f is a well defined map as,

if $A = B \qquad$ for $A, B \in GL(n,\mathbb{R})$

$\Rightarrow \det A = \det B$

$\Rightarrow f(A) = f(B)$

f is a homomorphism as for $A, B \in GL(n,\mathbb{R})$

$$f(AB) = \det(AB) = \det(A) \cdot \det(B) = f(A) \cdot f(B)$$

Let $a \in \mathbb{R}^*$. Then, $A = I_n \in GL(n,\mathbb{R})$ and $f(A) = \det A = a$.

$\Rightarrow f$ is an epimorphism.

and so, by the first Isomorphism Theorem,

$$\frac{GL(n,\mathbb{R})}{\ker f} \cong \mathbb{R}^*$$

Now, $\ker f = \{A \in GL(n,\mathbb{R}) \mid \det A = 1\}$

$= SL(n,\mathbb{R})$

Hence, $$\frac{GL(n,\mathbb{R})}{SL(n,\mathbb{R})} \cong \mathbb{R}^*$$

Centre of $GL(n,\mathbb{R}) = \{A \in GL(n,\mathbb{R}) \mid AX = XA \;\forall\; X \in GL(n,\mathbb{R})\}$

= group of all nonzero scalar multiples of the identity matrix

$= kI_n$, for some constant k

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2(b) IAS -2010 → Let $C = \{ f : I = [0,1] \to \mathbb{R} \,/\, f \text{ is continuous} \}$.
Show that 'C' is a commutative ring with 1 under point-wise addition and multiplication.
Determine whether 'C' is an integral domain. Explain.

Sol. given that
$C = \{ f : I = [0,1] \to \mathbb{R} \,/\, f \text{ is continuous} \}$

Define (i) $(f+g)(x) = f(x) + g(x)$
(ii) $(fg)(x) = f(x) + g(x)$ $\forall x \in [0,1]$, $f, g \in C$.

(I) To show that $(C, +)$ is an abelian group

(i) closure property: since the sum of real valued continuous functions on $[0,1]$ is again a real valued continuous function on $[0,1]$.
i.e $\forall f, g \in C \Rightarrow f+g \in C$.

(ii) Asso. prop: since the sum of the functions is an associative.
i.e $\forall f, g, h \in C \Rightarrow f+(g+h) = (f+g)+h$

(iii) Existence of identity element:
$\forall f \in C \; \exists \, 0 \in C$ s.t
$f + 0 = f = 0 + f$
Here 0 is an identity element in C.

(iv) Existence of inverse element:
For each $f \in C \; \exists \, -f \in C$ s.t
$f + (-f) = 0 = (-f) + f$.
Here $-f$ is an inverse of f in C

(v) commutative prop:
$\forall f, g \in C \Rightarrow f+g = g+f$
$\therefore$ $(C, +)$ is an abelian group.

(II) To show that $(C, \times)$ is a semi-group:

(i) closure prop:-

since the product of two real valued continuous functions on $[0,1]$ is again real valued continuous function on $[0,1]$

i.e $\forall f, g \in C \Rightarrow fg \in C$

(ii) Asso. prop: since the multiplication of functions is asso.

i.e $\forall f, g, h \in C \Rightarrow$

$$(fg)h = f(gh).$$

$\therefore (C, \times)$ is a semi-group.

(III) Distributive laws:

$\forall f, g, h \in C$

(i) $f(g+h) = fg + fh$ (L.D.L)

(ii) $(g+h)f = gf + hf$ (R.D.L)

$\therefore (C, +, \times)$ is a ring.

(IV) comm. prop: $\forall f, g \in C \Rightarrow$

$fg = gf$ (by hyp.(ii))

(V) Existence of identity element:

$\forall f \in C \ \exists\ 1 \in C$ (i.e $g : [0,1] \to \mathbb{R}$ s.t $g(x) = 1 \ \forall x \in [0,1]$)

$f \cdot 1 = f = 1 f$

$\therefore (C, +, \times)$ is a commutative ring with unity.

clearly C is not an integral domain.

consider the functions f and g on $[0,1]$

by $f(x) = \begin{cases} \frac{1}{2} - x & ; 0 \le x \le \frac{1}{2} \\ 0 & ; \frac{1}{2} \le x \le 1 \end{cases}$

and $g(x) = \begin{cases} 0 ; & 0 \le x \le 1/2 \\ x - 1/2 ; & 1/2 \le x \le 1. \end{cases}$

then f and g are continuous functions

and $f \ne 0, g \ne 0$.

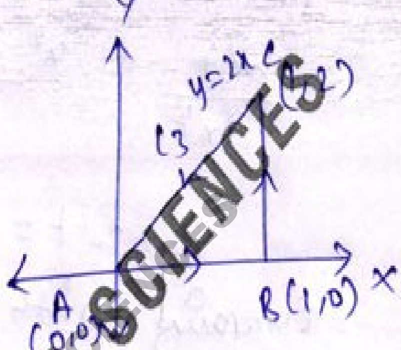
where as: $(gf)(x) = g(x) \cdot f(x)$

i.e $f \ne 0, g \ne 0 \Rightarrow fg = 0$. $\quad = 0 \;\; \forall x \in [0,1]$.

Q(a) 2010 (i) Evaluate the line integral $\int_C f(z)\,dz$ where $f(z)=z^2$, C is the boundary of the triangle with vertices $A(0,0)$ $B(1,0)$, $C(1,2)$ in that order.

Sol^n: To Evaluate $I=\int_C f(z)\,dz$

$$f(z)=z^2$$



$C=C_1+C_2+C_3$

$C_1: x: x\to 0$ to 1 : $dy=0$

$C_2: Y: Y\to 0$ to 2 : $dx=0$, $x=1$.

$C_3: z=(t+2ti)$: $t\to 1$ to 0.

$$I=\int_{C_1} x^2\,dx+\int_{C_2}(1+iy)^2(i\,dy)+\int_{C_3}(t+2ti)^2(dt+2i\,dt)$$

$$=\int_0^1 x^2\,dx+\int_0^2 (1+iy)^2(i\,dy)+\int_1^0 (1+2i)^2t^2(1+2i)\,dt.$$

$$=\frac{1}{3}+i\left[\frac{(1+iy)^3}{3i}\right]_0^2+(1+2i)^3\cdot\left[\frac{t^3}{3}\right]_1^0$$

$$=\frac{1}{3}+\frac{1}{3}\left[(1+2i)^3-1\right]+(1+2i)^3\left(-\tfrac{1}{3}\right)$$

$$=\underline{\underline{0}}.$$

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4(b) IAS-2010 P-II

Show that $e^{\frac{\lambda}{2}(z-\frac{1}{z})} = \sum_{n=-\infty}^{\infty} a_n z^n$

where $a_n = \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - \lambda\sin\theta)\,d\theta$.

$f(z) = \exp\left[\lambda/_2\left(z-\frac{1}{z}\right)\right]$

$f(z)$ is analytic at every point except at $z=0$ & $z=\infty$

i.e. it is analytic in the ring shaped region

$r \le |z| \le R$ where $r<R$.

Hence it can be expanded as a Laurent's series in the form

$$e^{\frac{1}{2}\lambda\left[z-\frac{1}{z}\right]} = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n}.$$

where $a_n = \frac{1}{2\pi i}\int_C e^{\frac{\lambda}{2}(z-\frac{1}{z})} \frac{dz}{z^{n+1}}$.

and $b_n = \frac{1}{2\pi i}\int_C e^{\frac{\lambda}{2}(z-\frac{1}{z})} \cdot \frac{dz}{z^{-n+1}}$.

where C is any circle with origin as centre.

Let us take C as $|z|=1$ so that $e^{i\theta} = z$

$dz = ie^{i\theta}d\theta$.

$$\therefore a_n = \frac{1}{2\pi i}\int_0^{2\pi} e^{\frac{\lambda}{2}(e^{i\theta}-e^{-i\theta})} \cdot \frac{ie^{i\theta}}{e^{(n+1)i\theta}}\,d\theta.$$

$$= \frac{1}{2\pi i}\int_0^{2\pi} ie^{\lambda i\sin\theta} \cdot e^{-ni\theta}\,d\theta.$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \left[\cos(n\theta-\lambda\sin\theta) - i\sin(n\theta-\lambda\sin\theta)\right] d\theta$$

$$a_n = \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - \lambda\sin\theta)\, d\theta.$$

Since $\int_0^{2\pi} \sin(n\theta - \lambda\sin\theta)\, d\theta$ vanishes by the property of definite integrals.

Now replace z by $-\frac{1}{z}$.

$$f(z) = e^{\frac{\lambda}{2}\left(\frac{-1}{z}+z\right)} = e^{\frac{\lambda}{2}\left(z-\frac{1}{z}\right)}$$

$$= \sum_{n=0}^{\infty} a_n\left(\frac{-1}{z}\right)^n + \sum_{n=1}^{\infty} b_n(-z)^n$$

$$= \sum_{n=0}^{\infty} a_n(-1)^n . z^{-n} + \sum_{n=1}^{\infty} b_n(-1)^n . z^n.$$

$\therefore\ b_n = (-1)^n a_n.$

$$\therefore\ f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} (-1)^n a_n z^{-n}.$$

$$= \sum_{n=-\infty}^{\infty} a_n z^n. \qquad n \in I.$$

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4(c) IAS 2010 P-II

Determine an optimal transportation programme so that the transportation cost of 340 tons of a certain type of material from three factories F_1, F_2, F_3 to five warehouses W_1, W_2, W_3, W_4, W_5 is minimized. The five warehouses must recieve 40 tons, 50 tons, 70 tons, 90 tons and 90 tons respectively. The availability of the material at F_1, F_2, F_2 is 100 tones, 120 tons, 120 tons respectively. The transportation costs per ton from factories to warehouses are given in the table below.

	W_1	W_2	W_3	W_4	W_5
F_1	4	1	2	6	9
F_2	6	4	3	5	7
F_3	5	2	6	4	5

using Vogel's approximation method to obtain the initial basic feasible solution.

Sol^n To find IBFS using VAM:—

→ firstly calculating all the rim values ie. difference b/w least two number in each row and column.

→ Select the maximum rim value and allocate the value possible, to the least cost cell in that row or column.

→ Now calculating the sum value again and continue the allocation till all the cells are filled.

	W_1	W_2	W_3	W_4	W_5		
f_1	4 (30)	1 ×	2 (70)	6 ×	9 ×	100	(1) (2) (2)
f_2	6 (10)	4 ×	3 ×	5 (20)	7 (90)	120	(1) (2) (1) (1) (1)
f_3	5 ×	2 (50)	6 ×	4 (70)	8 ×	120	(2) (1) (1) (1)
	40	50	70	90	90		
	(1)	(1)	(1)	(1)	(1)		
	(1)		(1)	(1)	(1)		
	(1)			(1)	(1)		
	(1)			(1)	(1)		
	(6)			(5)	(7)		

This is the required matrix for the IBFS.

Now to calculate the value of u_i and v_j

$$\Delta_{ij} = u_i + v_j - c_{ij}$$

$\Delta_{ij} = 0$ for basic cells.

$u_1 + v_1 = 4$ $\qquad$ $u_2 + v_1 = 6$

$u_1 + v_3 = 2$ $\qquad$ $u_2 + v_4 = 5$

$u_3 + v_2 = 2$ $\qquad$ $u_2 + v_5 = 7$

$u_3 + v_4 = 4$

Since 2nd row has maximum basic cells.

$\therefore$ Let $u_2 = 0$

$\Rightarrow v_1 = 6, \; v_4 = 5, \; v_5 = 7, \; v_3 = 4, \; v_2 = 3$

$u_3 = -1$, $u_1 = -2$

Now calculating the value for non-basic cells.

$$\Delta_{ij} = u_i + v_j - c_{ij}$$

$\Delta_{12} = -2+3-1 = 0$

$\Delta_{14} = -2+5-6 = -4$

$\Delta_{15} = -2+7-9 = -4$

$\Delta_{22} = 3+0-4 = -1$

$\Delta_{23} = 0+4-3 = 1$

$\Delta_{31} = -1+6-5 = 0$

$\Delta_{33} = -1+4-6 = -1$

$\Delta_{35} = -1+7-8 = -2$

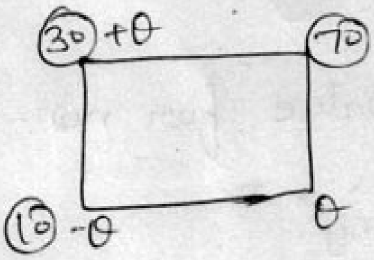
Selecting the Δ_{ij} with maximum value greater than 0.

$\therefore$ Selecting $\Delta_{23} = 1$

	W_1	W_2	W_3	W_4	W_5	a_i	u_i
F_1	4 (30)	1 (0)	2 (70)	6 (-4)	9 (-4)	100	-2
F_2	6 (10)	4 (-1)	3 (1)	5 (20)	7 (90)	120	0
F_3	5 (0)	2 (50)	6 (-3)	4 (70)	8 (-2)	120	-1
b_j	40	50	70	90	90		
v_j	6	3	4	5	7		

with cell (2,3) drawing a closed loop with the vertices as the basic cells.

Allocating value θ to (2,3) and subtracting and adding θ alternatively in closed loop.



Let $\theta = 10$

$\therefore$ The new table is given as:—

	W_1	W_2	W_3	W_4	W_5	
f_1	4 (40)	1 x	2 (60)	6 x	9 x	100
f_2	6 x	4 x	3 (10)	5 (20)	7 (90)	120
f_3	5 x	2 (50)	6 x	4 (70)	8 x	120.
	40	50	70	90	90	

→ Now calculating the value of u_i and v_j again:-

$\Delta_{ij} = 0$ (for basic cells).

$\therefore \quad u_1 + v_1 = 4 \qquad u_2 + v_3 = 3$

$u_1 + v_3 = 2 \qquad u_2 + v_4 = 5$

$u_3 + v_2 = 2 \qquad u_2 + v_5 = 7$

Since row 2 has maximum basic cells.

$\therefore$ let $u_2 = 0$

$v_3 = 3, \quad v_4 = 5, \quad v_5 = 7, \quad v_1 = 5, \quad v_2 = 3$

$u_1 = -1, \quad u_3 = -1$

→ Now calculating Δ_{ij} for non basic cells drawing the new table:-

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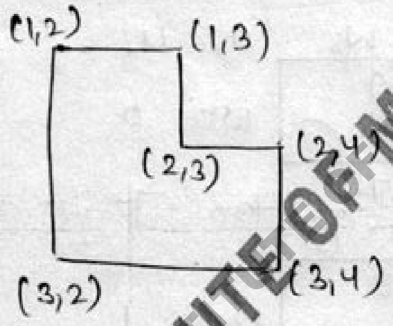
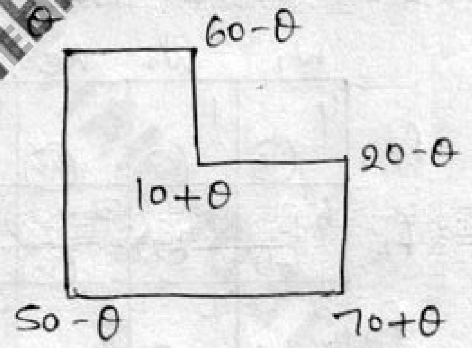
	W_1	W_2	W_3	W_4	W_5	a_i	u_i
f_1	4, 40	1, 1	2, 60	6, −2	9, −3	100	−1
f_2	6, −1	4, −1	3, 10	5, 20	7, 90	120	0
f_3	5, −1	2, 50	6, −4	4, 70	8, −2	120	−1
b_j	40	50	70	90	90		
v_j	5	3	3	5	7		

Selecting $\Delta_{12} = 1$ since it has maximum value greater than 0.

Making a closed loop starting and ending of cell (1,2) and all the vertices are basic cells.





Let $\theta = 20$

$\therefore$ New table is :—

4, 40	1, 20	2, 40	6, x	9, x	100
6, x	4, x	3, 30	5, x	7, 90	120
5, x	2, 30	6, x	4, 90	8, x	120
40	50	70	90	90	

→ Calculating the value of u_i and v_j again $\Delta_{ij} = 0$ for basic cells.

$u_1 + v_1 = 4$ $\quad$ $u_2 + v_3 = 3$

$u_1 + v_2 = 1$ $\quad$ $u_2 + v_5 = 7$

$u_1 + v_3 = 2$ $\quad$ $u_3 + v_2 = 2$

$u_3 + v_4 = 4$

Let $u_1 = 0$

$v_1 = 4$, $v_2 = 1$, $v_3 = 2$, $v_4 = 3$, $v_5 = 6$

$u_2 = 1$, $u_3 = 1$

Now calculating the Δ_{ij} for non-basic cells $\Delta_{ij} = u_i + v_j - c_{ij}$

The new table is:—

	W_1	W_2	W_3	W_4	W_5	a_i	u_i
F_1	4 (40)	1 (20)	2 (40)	6 (-3)	9 (-3)	100	0
F_2	6 (-1)	4 (-2)	3 (30)	5 (-1)	7 (90)	120	1
F_3	5 (0)	2 (30)	6 (-3)	4 (90)	8 (-1)	120	1
b_j	40	50	70	90	90		
v_j	4	1	2	3	6.		

Since all $\Delta_{ij} \le 0$

∴ Hence optimality obtained

The optimal solution is:—

$= 4 \times 40 + 1 \times 20 + 2 \times 40 + 3 \times 30 + 7 \times 90 + 2 \times 30 + 4 \times 90$

$= 1400.$

5(a). Solve the PDE $(D^2-D')(D-2D')Z=e^{2x+y}+xy$

SOLUTION

Given $(D^2-D')(D-2D')Z=e^{2x+y}+xy$...(1)

complementary function corresponding to $(D-2D')$ is given by $\phi(y+2x)$

For the non-linear part $(D^2-D')z = 0$.

Let the trail solution be $Z=Ae^{hx+ky}$ ...(2)

Put in $(D^2-D')Z = 0$

$$Ae^{hx+ky}(h^2)-Ae^{hx+ky}(k) = 0$$

$$(h^2 - k)\,Ae^{hx+ky} = 0$$

$$\therefore \quad h^2 = k$$

$$\therefore \quad \text{C.F.} = \sum Ae^{hx+h^2y}+\phi(y+2x)$$

$$\text{P.I.} = \frac{e^{2x+y}+xy}{(D^2-D')(D-2D')}$$

$$= \frac{e^{2x+y}}{(2^2-1)(D-2D')}+\frac{xy}{(-2D')(1-D/2D')(-D')(1-D^2/D')}$$

$$= \frac{e^{2x+y}}{3(D-2D')}+\frac{(1-D/2D')^{-1}\left(1-\frac{D^2}{D'}\right)^{-1}xy}{2D'^2}$$

$$= \frac{+xe^{2x+y}}{3}+\frac{\left(1-\frac{D}{2D'}\right)^{-1}\left(1+\frac{D^2}{D}+\frac{D^4}{D'^2}\right)xy}{2D'^2}$$

$$= \frac{x\,e^{2x+y}}{3}+\frac{(1+D/2D')(xy)}{2D'}$$

$$= \frac{x}{3}e^{2x+y}+\frac{1}{2D'^2}\left(xy+\frac{y^2}{4}\right)$$

$$\text{P.I.} = \frac{x}{3}e^{2x+y}+\frac{xy^3}{12}+\frac{y^4}{96}$$

$$\therefore \quad Z=\phi(2x+y)+\sum Ae^{hx+h^2y}+\frac{x}{3}e^{2x+y}+\frac{xy^3}{12}+\frac{y^4}{96}$$

5(b). Find the surface satisfying the PDE $(D^2-2DD'+D'^2)z=0$ and the condition that $bz=y^2$ when $x=0$ and $az=x^2$ when $y=0$.

SOLUTION

Given $\quad \left(D^2-2DD'+D'^2\right)Z=0$

$$(D-D')^2 z=0.$$

Since it is homogenous equations.

General solutions

$$\boxed{Z=\phi_1(y+x)+x\phi_2(y+x)}$$

when x = 0; bz = y^2

$$z = \phi_1(y)+0.\phi_2(y)$$

$$\frac{y^2}{b} = \phi_1(y)$$

$$\therefore \quad \phi_1(x+y) = \frac{(x+y)^2}{b}$$

when y = 0 ; az = x^2

$$\frac{x^2}{a} = \frac{(x+0)^2}{b}+x\phi_2(x)$$

$$\phi_2(x) = \frac{x}{a}-\frac{x}{b}$$

$$\phi_2(x+y) = (x+y)\left(\frac{1}{a}-\frac{1}{b}\right)$$

$$\therefore \quad Z = \boxed{\frac{(x+y)^2}{b}+x(x+y)\left(\frac{1}{a}-\frac{1}{b}\right)}$$

Q. 5(c) Find the positive root of the equation $10xe^{-x^2}-1=0$. Correct upto 6 decimal places by using Newton-Raphson method, upto three iteration?

Sol:- Given $f(x) = 10xe^{-x^2}-1$

$f(2) = -0.633687$

This value is closer to '0'. So we would approach this value.

$f'(x) = 10e^{-x^2}(1-2x^2)$

and $f(x)$ & $f'(x)$ both are continuous function

$x_0 = 2$; $f(x_0) = -0.633687$

$f'(x_0) = -1.282095$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 2 - \frac{0.633687}{1.282095} = 1.505741$$

$f(x_1) = 0.559888$; $f'(x_1) = -3.661615$

$$x_2 = 1.505741 + \frac{0.559888}{3.661615} = 1.658648$$

$f(x_2) = 0.059159$

$f'(x_2) = -2.874975.$

$$x_3 = 1.658648 + \frac{0.059159}{2.874975} = 1.679225.$$

$$\boxed{\therefore x_3 = 1.679225.}$$

which is the required root of the eq^n $10x \cdot e^{-x^2}-1=0$ is $x_3 = 1.679225$

5(e) IAS-2010 P-II → A uniform lamina is bounded by parabolic arc, of latus rectum $4a$, and a double ordinate at a distance b from the vertex. If $b = \frac{1}{3}a(7+4\sqrt{7})$, show that two of the principal axes at the end of a latus rectum are the tangent and normal there.

Sol^n: Let the equation of the parabola be $y^2 = 4ax$ — ①

∴ Coordinates of the end L of L.R. LL' are $(a, 2a)$.

Differentiating ① we get $\frac{dy}{dx} = \frac{2a}{y}$

$$\therefore \text{ At } L(a,2a),\ \frac{dy}{dx} = \frac{2a}{2a} = 1$$

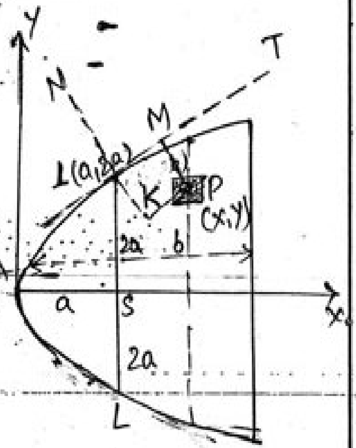
∴ Equation of the tangent LT at L is

$$y - 2a = 1\cdot(x-a) \Rightarrow y - x - a = 0 \quad \text{—②}$$

and the equation of the normal LN at L is

$$y - 2a = -\frac{1}{1}(x-a) \quad \text{—③}$$

$$\Rightarrow y + x - 3a = 0$$



Consider an element $\delta x \delta y$ at the point $P(x,y)$ of the lamina, then PM = length of ⊥lar from P on tangent LT given by ②

$$= \frac{y-x-a}{\sqrt{1+1}} = \frac{y-x-a}{\sqrt{2}}$$

and PK = length of ⊥lar from P on the normal LN given by ③

$$= \frac{y+x-3a}{\sqrt{2}}$$

P.I. of the element about LT and LN

$$PM.PK.\delta m = \left(\frac{y-x-a}{\sqrt{2}}\right)\left(\frac{y+x-3a}{\sqrt{2}}\right)\rho\,\delta x\,\delta y.$$

If the tangent and normal at L are the principal axes, then the P.I of the lamina about these will be zero.

i.e. P.I of the lamina about LT and LN.

$$= \int_{x=0}^{b} \int_{y=-2\sqrt{ax}}^{2\sqrt{ax}} \left(\frac{y-x-a}{\sqrt{2}}\right)\left(\frac{y+x-3a}{\sqrt{2}}\right) \cdot \rho \, dx\, dy = 0$$

$$\Rightarrow \frac{\rho}{2} \int_0^b \int_{-2\sqrt{ax}}^{2\sqrt{ax}} \{y^2 - 4ay + (3a^2 + 2ax - x^2)\} dx\, dy = 0$$

$$\Rightarrow \int_0^b \left\{\frac{1}{3}y^3 - 2ay^2 + (3a^2 + 2ax - x^2)y\right\}_{-2\sqrt{ax}}^{2\sqrt{ax}} dx = 0$$

$$\Rightarrow 2\int_0^b \left\{\frac{8}{3} ax\sqrt{ax} + 2(3a^2 + 2ax - x^2)\sqrt{ax}\right\} dx = 0$$

$$\Rightarrow \int_0^b \left(\frac{8}{3} a^{3/2} x^{3/2} + 6a^{5/2} x^{1/2} + 4a^{3/2} x^{3/2} - 2a^{1/2} x^{5/2}\right) dx = 0$$

$$\Rightarrow \left[\frac{16}{15} a^{3/2} b^{5/2} + 4a^{5/2} b^{3/2} + \frac{8}{5} a^{3/2} b^{5/2} - \frac{4}{7} a^{1/2} b^{7/2}\right] = 0$$

$$\Rightarrow \frac{16}{15} ab + 4a^2 + \frac{8}{5} ab - \frac{4}{7} b^2 = 0$$

$$\Rightarrow b^2 - \frac{14}{3} ab - 7a^2 = 0$$

$$\Rightarrow b = \frac{\frac{14}{3} a \pm \sqrt{\left\{\frac{196}{9} a^2 + 28a^2\right\}}}{2} = \frac{1}{2}\left(\frac{14}{3} \pm \frac{8}{3}\sqrt{7}\right) a$$

$$\Rightarrow b = \frac{a}{3}(7 + 4\sqrt{7})$$ (leaving −ve sign, as b can not be −ve)

Hence if $b = \frac{a}{3}(7 + 4\sqrt{7})$

then the principal axes at L are the tangents & normal there.

5(b) IAS 2010 P-II In an incompressible fluid, the vorticity at every point is constant in magnitude and direction. Show that the components of velocity u, v, w are solutions of Laplace's equation.

Soln Let $\vec{W} = \xi\hat{i} + \eta\hat{j} + \zeta\hat{k}$, $\vec{q} = u\hat{i} + v\hat{j} + w\hat{k}$

Vorticity is constant in magnitude and direction.

$\Rightarrow$ ξ, η, ζ are constant.

$$\Rightarrow \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = \xi = \text{const}, \quad \frac{1}{2}\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) = \eta = \text{Const}$$

$$\frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \zeta = \text{Const.}$$

$$\therefore \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = \text{Const} \quad \text{---} (1)$$

$$\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} = \text{Const.} \quad \text{---} (2)$$

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \text{Const.} \quad \text{---} (3)$$

Differentiation of (2) and (3) w.r.to z and y gives

$$\frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 w}{\partial x \partial z}, \quad \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y}$$

Equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

Observe that

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

$$= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y \partial x} + \frac{\partial^2 w}{\partial x \partial z}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

$$= \frac{\partial}{\partial x}(0) = 0$$

$\therefore \nabla^2 u = 0$ Similarly we can prove –

$$\nabla^2 v = 0, \ \nabla^2 w = 0$$

It means that components of velocity are solutions of Laplace's equation.

6(a). Solve the following PDE pz+qy = x, $x_0(s)=s$ $y_0(s)=1$ $z_0(s)=2s$ by the method of characterstics.

SOLUTION

$$f(x,y,z,p,q) = pz+qy-x \quad \ldots(1)$$

given $x_0(s)=s$ $y_0(s)=1$ $z_0(s)=2s$ ...(2)

Solving for p_0,q_0.

$$Z_0'(s) = p_0X_0'(s)+q_0y_0'(s)$$

$$2 = p_0 + 0$$

from(1) $\Rightarrow p_0(2s)+q_0(1)-s = 0$

$$q_0 = -3s$$

$\therefore$ $(x_0,y_0,z_0,p_0,q_0) = (s,1,2s,2,-3s)$ ...(3)

Charastic equations can be written as

$$X'(t) = f_p = z \quad \ldots(4)$$

$$y'(t) = fq = y \quad \ldots(5)$$

$$z'(t) = pf_p+qf_q=pz+qy=x \quad \ldots(6)$$

$$p'(t) = -f_x-pf_z=-(-1)-p(p)=1-p^2 \quad \ldots(7)$$

$$q'(t) = -f_y-qf_z = -q-pq \quad \ldots(8)$$

from(5) $y'(t) = y$

$$\frac{dy}{y} = dt \Rightarrow y = c_1e^t$$

$\Rightarrow$ $\boxed{y=e^t}$ $(\because y_0=1)$

(4)+(5)+(6) $\Rightarrow \dfrac{dx}{dt}+\dfrac{dy}{dt}+\dfrac{dz}{dt} = x+y+z$

$$\therefore \quad \frac{d(x+y+z)}{(x+y+z)} = dt$$

$$\therefore \quad (x+y+z) = c_1e^t$$

$$x+z = c_1e^t - e^t \quad \ldots(9)$$

$$(4)-(6) \Rightarrow \frac{dx}{dt}-\frac{dz}{dt} = z-x$$

$$(x-z) = c_3e^{-t} \quad \ldots(10)$$

Put initial condition in (9)

$$s+2s = c_1-1$$

$$c_1 = 3s+1$$

$$\therefore \quad (x+z) = 3se^t$$

Putting initial condition is (10)

$$(x-z) = +c_3e^{-t}$$

$$s-2s = c_3$$

$$x-z = -se^{-t}$$

$$\therefore \quad x = \frac{s(3e^t-e^{-t})}{2}; \quad ..(11)$$

$$y = e^t; \qquad ..(12)$$

$$z = \frac{s(3e^t + e^{-t})}{2} \qquad ..(13)$$

$$\frac{(11)}{(13)} \equiv \frac{x}{z} = \frac{3e^t - e^{-t}}{3e^t + e^{-t}}$$

$$\frac{x}{z} = \frac{3y - 1/y}{3y + 1/y} \qquad \boxed{\because e^t = y}$$

$\therefore$ $\boxed{z = x\left(\frac{3y^2+1}{3y^2-1}\right)}$ required solutions.

6(b). Reduce the following 2nd order PDE into canonical form and find its general solutions $xu_{xx} + 2x^2u_{xy} - u_x = 0.$

SOLUTION

Given $\qquad xr + 2x^2s - p = 0 \qquad \ldots(1)$

comparing with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$

$R = x,\ S = 2x^2,\ T = 0$

$S^2 - 4RT = 4x^4 - 0 = 4x^4 > 0$. Hyperbolic λ-quadratic is given by $R\lambda^2+S\lambda+T=0$.

$$x\lambda^2 + 2x^2\lambda = 0$$

$$\therefore \qquad \lambda = 0 \quad \lambda = -2x$$

Hence characteristic equation

$$\frac{dy}{dx} + \lambda_1 = 0\ ; \qquad \frac{dy}{dx} + \lambda_2 = 0$$

$$\frac{dy}{dx} = 0 \qquad \frac{dy}{dx} - 2x = 0$$

$$y = c_1, \qquad y - x^2 = c_2$$

$\therefore$ Let $\qquad u = y \qquad \ldots(2)$

$$v = y - x^2 \qquad \ldots(3)$$

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}$$

$$= \frac{\partial z}{\partial u}(0) + \frac{\partial z}{\partial v}(-2x) \qquad \ldots(4)$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}.\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}$$

$$= \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} \qquad \ldots(5)$$

$$r = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(-2x\frac{\partial z}{\partial v}\right)$$

$$= -2\frac{\partial z}{\partial v} - 2x\left[\frac{\partial^2 z}{\partial u \partial v}.\frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial v^2}\frac{\partial v}{\partial x}\right]$$

$$= -2\frac{\partial z}{\partial v} + 4x^2\frac{\partial^2 z}{\partial v^2} \qquad \ldots(6)$$

$$s = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial^2 z}{\partial u^2}.\frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial u \partial v}\frac{\partial v}{\partial x} + \frac{\partial^2 z}{\partial u \partial v}\frac{\partial u}{\partial x} + \frac{\partial^2 z}{\partial v^2}\frac{\partial v}{\partial x}$$

$$= -2x\left(\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right) \quad \ldots(7)$$

Using (4)(5)(6)(7) given equation (1) transform to

$$x\left[-2\frac{\partial z}{\partial v} + 4x^2\frac{\partial^2 z}{\partial v^2}\right] + 2x^2\left[-2x\left(\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right)\right] - \left[-2x\frac{\partial z}{\partial v}\right] = 0$$

$\boxed{\frac{\partial^2 z}{\partial u \partial v} = 0}$ Required canonical form.

$$\therefore \quad \frac{\partial z}{\partial u} = \phi_1(u)$$

$$z = \int \phi_1(u) + \phi_2(v)$$

$\boxed{z = \int \phi_1(y)dy + \phi_2(y - x^2)}$ general solution

6(c). Solve the following heat equations

$u_t - u_{xx} = 0$, $0 < x < 2$, $t > 0$, $u(0, t) = u(2, t) = 0$, $t > 0$. $u(x, 0) = x(2-x)$, $0 \le x \le 2$.

SOLUTION

Heat flow equation $u_t - u_{xx} = 0$.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad \ldots(1)$$

Boundary conditions $u(0,t) = u(2,t) = 0$..(2) t>0.

Initial conditions $u(x,0) = x(2-x)$ ...(3)

Let the trail solution be

$$u(x,t) = X(x)T(t)$$

From(2) $X(0)T(t) = X(2)T(t) = 0$

For some $t > 0$, there exist t such that $T(t) \neq 0$.

$\therefore$ $X(0) = X(2) = 0.$ ...(4)

From(1) $XT' = X''T$

$$\frac{T'}{T} = \frac{X''}{X} = \mu \quad \text{(say)}$$

Solving $X'' - \mu x = 0;$

$X(0) = X(2) = 0$

Case(i)

$\mu = 0$ $X'' = 0 \Rightarrow X = Ax + B$.

Putting(4) $A = 0$, $B = 0$

$\therefore$ we reject $\mu = 0$.

Case(ii) $\mu = \lambda^2$ $X'' - \lambda^2 X = 0 \Rightarrow X = Ae^{\lambda x} + Be^{-\lambda x}$

Putting (4) $\Rightarrow A = 0$; $B = 0$

$\therefore$ we reject $\mu = \lambda^2$

Case(iii) $\mu = -\lambda^2$, $\lambda \neq 0$

$X'' + \lambda^2 X = 0$

$$\Rightarrow \quad X = A\cos\lambda x + B\sin\lambda x$$

$$X(0) = A + B(0) = 0$$

$$X(2) = B\sin(2\lambda) = 0$$

$$\therefore \quad 2\lambda = n\pi$$

$$\lambda = n\pi/2$$

$$\therefore \quad X_n = B_n \sin\left(\frac{n\pi x}{2}\right)$$

Corresponding

$$\frac{T'}{T} = \mu$$

$$\frac{T'}{T} = -\lambda^2$$

$$\therefore \quad T = c\,e^{-\lambda^2 t} \quad T_n = c_n e^{-n^2\pi^2 t/4}$$

$$\therefore \quad u_n(x,t) = X_n(x)T_n(t)$$

$$u(x,t) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{2}\right) e^{-n^2\pi^2 t/4}$$

$$u(x,0) = \sum_{n=1}^{\infty} D_n \sin\left(\frac{n\pi x}{2}\right) = x(2-x)$$

$$D_n = \frac{2}{2}\int u(x)\sin\left(\frac{n\pi x}{2}\right)dx$$

$$D_n = \frac{2}{2}\int_0^2 x(2-x)\sin\left(\frac{n\pi x}{2}\right)dx$$

$$D_n = \frac{16}{(n\pi)^3}\left[1-(-1)^n\right]$$

$$\boxed{u(x,t) = \frac{16}{\pi^3}\sum_{n=1}^{\infty}\left(\frac{1-(-1)^n}{n^3}\right)\sin\left(\frac{n\pi x}{2}\right).e^{-n^2\pi^2 t/4}}$$

Q. Given the system of equations

Q.7(a)

$$2x + 3y = 1$$
$$2x + 4y + 3z = 2$$
$$2y + 6z + Aw = 4$$
$$4z + Bw = C$$

State the ~~stability~~ (Solvability) condition and uniqueness condition for the system. Give the solution when it exists.

Sol- We will solve it by Reducing it to Echelon form by using Gauss Elimination method.

The given system of eqⁿ $Eq^n = A \times B$

Then, Augumented matrix $= [A|B]$

from the system of equation

$$A = \left[\begin{array}{cccc|c} 2 & 3 & 0 & 0 & 1 \\ 2 & 4 & 3 & 0 & 2 \\ 0 & 2 & 6 & A & 4 \\ 0 & 0 & 4 & B & C \end{array}\right] \quad R_2 \to R_2 - R_1$$

$$A \sim \left[\begin{array}{cccc|c} 2 & 3 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 2 & 6 & A & 4 \\ 0 & 0 & 4 & B & C \end{array}\right] \quad R_3 \to R_3 - 2R_2$$

$$A \sim \left[\begin{array}{cccc|c} 2 & 3 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 4 & A & 2 \\ 0 & 0 & 4 & B & C \end{array}\right] \quad R_4 \to R_4 - R_3$$

$$A \sim \left[\begin{array}{cccc|c} 2 & 3 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 4 & A & 2 \\ 0 & 0 & 0 & B-A & C-2 \end{array}\right]$$

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Now, for unique solution the basis of Both Matrix, must be equal and equal to the rank of Matrix 1 & 4.

(i) Hence, $B - A \neq 0$; $C - 2 \neq 0$

$B \neq A$; $C \neq 2$ (for unique Solution)

(ii) For inifinitely many solution.

$B - A = 0$, $\Rightarrow B = A$, $C = 2$

(iii) for No solution

$B - A = 0$ $\quad C - 2 \neq 0$

$B = A$ $\quad C \neq 2$

Now, to find any Solution

let $\boxed{B = 3,\ A = 1,\ C = 3}$

$$\left|\begin{array}{cccc|c} 2 & 3 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 4 & 1 & 2 \\ 0 & 0 & 0 & 2 & 3 \end{array}\right|$$

$2w = 3$; $w = 1/\cancel{4}3$

$4z + w = 2 \Rightarrow z = 1/8$

$y + z = 1 \Rightarrow y = 7/8$

$2x + 3y + w = 1 \Rightarrow x = -\frac{25}{16}$

Be any Solution.

Q. 7(b) Find the value of integral $\int_1^5 \log_{10} x\,dx$ by using Simpson's $\frac{1}{3}$ rule correct upto 4 decimal place. Take 8 subintervals in your computation?

Sol: Given $\int_1^5 \log_{10} x\,dx. \Rightarrow f(x) = \log_{10} x$

$a = 1$, $b = 5$ $\Rightarrow h = \frac{b-a}{8} = \frac{5-1}{8}$

$h = 0.5$.

Hence, the frequency chart can be shown as

x	1	1.5	2	2.5	3	3.5	4	4.5	5
y	0	0.1761	0.3010	0.3979	0.4771	0.5441	0.6021	0.6532	0.6990

Now, By applying the Simpson's $\frac{1}{3}$ rule.

Now, Applying Simpson's formula.

$$\therefore \int_2^{10} \log_{10} x\,dx = \frac{h}{3}\left[(y_0 + y) + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)\right]$$

$$\therefore \int_2^{10} \log_{10} x\,dx = \frac{h}{3}\left[7.0848 + 2.764 + 0.6990\right]$$

$$\int_2^{10} \log_{10} x\,dx = \frac{h}{3}\left[10.5478\right]$$

$$\int_2^{10} \log_{10} x\,dx = \frac{0.5}{3} \times 10.5478$$

$$\boxed{\int_2^{10} \log_{10} x\,dx = 1.757966\ldots}$$

8(a) A sphere of radius a and mass M rolls down a rough plane inclined at angle α to the horizontal.

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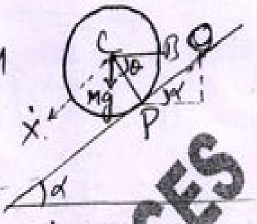
If x be the distance of the point of contact of the sphere from a fixed point on the plane, find the acceleration by using Hamilton's equations.

Sol^n: Let a sphere of radius a and mass M roll down a rough plane inclined at an angle α starting initially from a fixed point O of the plane. In time t, let the sphere roll down a distance x and during this time let it turn through an angle θ. Since there is no slipping.



$$\therefore x = OA = \text{arc } AB = a\theta$$

so that $\dot{x} = a\dot{\theta}$.

If T and V are kinetic and potential energies of the sphere, then $T = \frac{1}{2}Mk^2\dot{\theta}^2 + \frac{1}{2}M\dot{x}^2 = \frac{1}{2}M\frac{2}{5}a^2\dot{\theta}^2 + \frac{1}{2}M(a\dot{\theta})^2$

$$\Rightarrow T = \frac{7}{10}M\dot{x}^2$$

and $V = -Mg\,OL = -Mg\,x\sin\alpha$. (Since the sphere move down the plane)

$$\therefore L = T - V = \frac{7}{10}M\dot{x}^2 + Mgx\sin\alpha$$

Here x is the only generalised co-ordinate

$$p_x = \frac{\partial L}{\partial \dot{x}} = \frac{7}{5}M\dot{x} \quad \text{—— (1)}$$

Since L doesnot contain t explicitly.

$$H = T + V = \frac{7}{10}M\dot{x}^2 - Mgx\sin\alpha = \frac{7}{10}M\left(\frac{5}{7M}p_x\right)^2 - Mgx\sin\alpha \quad (\text{from (1)})$$

$$= \frac{5}{14M}p_x^2 - Mgx\sin\alpha.$$

Hence the two Hamilton's equations are

$$\dot{p}_x = -\frac{\partial H}{\partial x} = Mg\sin\alpha \quad \text{—— (2)}, \qquad \dot{x} = \frac{\partial H}{\partial p_x} = \frac{5}{7M}p_x \quad \text{—— (3)}$$

Differentiating (3) and using (2), we get

$$\ddot{x} = \frac{5}{7M}\dot{p}_x = \frac{5}{7M}Mg\sin\alpha \Rightarrow \ddot{x} = \frac{5}{7}g\sin\alpha.$$

which gives the required acceleration

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$$= \frac{8}{9}(1) + \frac{5}{9}\left(\frac{5}{8} + \frac{5}{8}\right)$$

$$= \frac{8}{9} + \frac{50}{72} = 1.5833$$

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When a pair of equal and opposite rectilinear vortices are situated in a long circular cylinder at equal distance from its axis, show that path of each vortex is given by the equation,

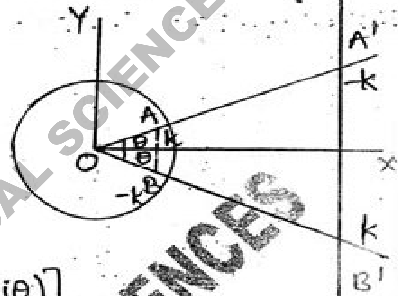
$$(r^2\sin^2\theta - b^2)(r^2-a^2)^2 = 4a^2b^2r^2\sin^2\theta.$$

Sol^n: Let x-axis be the axis of the cylinder. Consider the vortices $+k$ at $A(r,\theta)$ & $-k$ at $B(-r,\theta)$ inside the cylinder s.t distances of A and B from the axis are equal. Evidently, AB is ⊥lar to x-axis. The image of vortex $+k$ at A w.r.to the cylinder is a vortex $-k$ at A', the inverse point of A. Similarly the image of vortex $-k$ at B is a vortex $+k$ at B'.

$$OB\cdot OB' = a^2 = OA\cdot OA',$$

where a is the radius of the cylinder. Then

$$OB' = \frac{a^2}{r} = OA' \text{ as } OB = OA = r$$



The complex potential due to this system at $P(z)$ is.

$$W = \frac{ik}{2\pi}\left[\log(z-re^{i\theta}) - \log\left(z-\frac{a^2}{r}e^{i\theta}\right) - \log(z-re^{-i\theta}) + \frac{ik}{2\pi}\log\left(z-\frac{a^2}{r}e^{-i\theta}\right)\right]$$

The motion of the vortex at A is due to other vortices. If W' be the complex potential for the motion of A, then

$$w' = W - \frac{ik}{2\pi}\log(z-re^{i\theta}) \text{ at } z = re^{i\theta}$$

$$= \frac{ik}{2\pi}\left[-\log\left(z-\frac{a^2e^{i\theta}}{r}\right) - \log(z-re^{-i\theta}) - \log\left(z-\frac{a^2}{r}e^{-i\theta}\right)\right]$$

at $z = re^{i\theta}$

$$w' = \frac{-ik}{2\pi}\left[\log\left(re^{i\theta}-\frac{a^2}{r}e^{i\theta}\right) + \log(re^{i\theta}-re^{-i\theta}) - \log\left(re^{i\theta}-\frac{a^2}{r}e^{-i\theta}\right)\right]$$

$$= \frac{-ik}{2\pi}\log(r^2-a^2)e^{i\theta} - \log r + \log(2ir\sin\theta)$$

$$- \left[\log\{(r^2-a^2)\cos\theta + i\sin\theta(r^2+a^2)\} + \log r\right]$$

$$\psi = \frac{-k}{2\pi}\left[\log|(r^2-a^2)e^{i\theta}| + \log|2ir\sin\theta| - \log|\{(r^2-a^2)\cos\theta + i\sin\theta(r^2+a^2)\}|\right]$$

$$= \frac{-k}{2\pi}\left[\log(r^2-a^2) + \log 2r\sin\theta\right] - \frac{1}{2}\log\left\{(r^2-a^2)^2\cos^2\theta + \sin^2\theta(r^2+a^2)^2\right\}$$

streamlines are given by $\psi = \text{const.}$ i.e.

$$\log\left\{\frac{(r^2-a^2)^2(2r\sin\theta)^2}{(r^2-a^2)^2\cos^2\theta + (r^2+a^2)^2\sin^2\theta}\right\} = \text{const} = \log 4b^2$$

$$\Rightarrow (r^2-a^2)^2 r^2\sin^2\theta = b^2\left[r^4 + a^4 - 2r^2a^2\cos 2\theta\right]$$

$$\Rightarrow (r^2-a^2)^2\left[r^2\sin^2\theta - b^2\right] = 4r^2a^2b^2\sin^2\theta$$ This completes the proof.