PDE-I

PARTIAL DIFFERENTIAL EQUATIONS

Partial diff. eqn: An eqn involving the derivatives of a dependent variable w.r.t more than one independent variable, is called a PDE.

Ex: (1) $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = kz^2$

(2) $\frac{\partial^2 z}{\partial x^2} = k\left(\frac{\partial^3 z}{\partial x^3}\right)^2$

(3) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

Order of PDE: The order of the highest order derivative involving in a differential eqn. is called the order of the PDE.

The examples (1), (2) and (3) orders are one, three & two respectively.

Degree of PDE: The degree (i.e, power) of the highest order derivative involving in the diff. eqn is called the degree of PDE.

The above examples (1), (2) & (3) degrees are one, two and one.

Linear partial diff. eqn: A partial diff eqn is said to be linear if (i) the dependent variable say z and all its partial derivatives occur in first degree only and (ii) no product of dependent variable (or) partial derivatives occur.

Ex: (1) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x \partial y} = 0$

(2) $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$

} are linear

(3) $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = kz^2$

(4) $\frac{\partial^2 z}{\partial x^2} = k\left(\frac{\partial^3 z}{\partial x^3}\right)^2$

} are not linear.

→ An eqn which is not linear is called non-linear PDE.

→ In the case of two independent variables x and y will usually be taken as the independent and z as the dependent variable.

→ The partial diff. coefficients $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ are denoted by p & q.

i.e, $p = \frac{\partial z}{\partial x}$ & $q = \frac{\partial z}{\partial y}$.

→ The second order partial derivatives $\frac{\partial^2 z}{\partial x^2}, \frac{\partial^2 z}{\partial x \partial y}, \frac{\partial^2 z}{\partial y^2}$ are denoted by r, s, t.

i.e, $r = \frac{\partial^2 z}{\partial x^2}$, $s = \frac{\partial^2 z}{\partial x \partial y}$, and $t = \frac{\partial^2 z}{\partial y^2}$

Note: 1. In the case of n independent variables, we take them to be $x_1, x_2, \ldots x_n$ and z as the dependent variable. In this case we use the following notations.

$p_1 = \frac{\partial z}{\partial x_1}$, $p_2 = \frac{\partial z}{\partial x_2}$, $p_3 = \frac{\partial z}{\partial x_3}$ ...... $p_n = \frac{\partial z}{\partial x_n}$.

2. Some times the partial derivatives are also denoted by suffixes.

$u_x = \frac{\partial u}{\partial x}$, $u_y = \frac{\partial u}{\partial y}$, $u_{xx} = \frac{\partial^2 u}{\partial x^2}$, $u_{xy} = \frac{\partial^2 u}{\partial x \partial y}$ and so on.

Formation (Derivation) of PDE:

Partial diff. equs can derived in two ways.

(I) By the elimination of arbitrary constants from a relation b/w x, y and z.

and (II) By the elimination of arbitrary functions of three variables.

I. By the elimination of arbitrary constants:

Let z be a function of x and y such that

$f(x, y, z, a, b) = 0$ —— (1) where a & b are arbitrary constants

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Differentiating (1) partially w.r.t x & y we get,

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial x} = 0 \quad \text{and} \quad \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\cdot\frac{\partial z}{\partial y} = 0$$

i.e, $\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} p = 0$ ——(2) and $\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} q = 0$ ——(3)

Now eliminating 'a' and 'b' from (2) & (3) we obtain an eqn of the form

$$f(x, y, z, p, q) = 0 \text{ ——(4)}$$

which is the required PDE of first order.

Note: If the number of arbitrary constants to be eliminated is equal to the number of independent variables then the derived partial differential eqn is of the first order.

But if the number of arbitrary constants to be eliminated is greater than number of independent variables then the derived partial diff. eqns will be of the second order (or) higher orders.

$\boxed{\text{II}}$. By the elimination of arbitrary functions.

Suppose we have a relation between x, y and z of the type $f(u, v) = 0$ ——(1)

where u and v are known as functions of x, y & z and f is arbitrary function of u & v.

Now we treat z dependent variable and x & y are independent variables.

Differentiating (1) w.r.t x we get,

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial x} + \frac{\partial u}{\partial z}\cdot\frac{\partial z}{\partial x}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial x} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial x} + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right) = 0$$

$$\Rightarrow \frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x} + 0 + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right) + \frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x} + 0 + \frac{\partial v}{\partial z}\cdot\frac{\partial z}{\partial x}\right) = 0$$

$$\Rightarrow \frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right)=0 \qquad \left(\because p=\frac{\partial z}{\partial x}\right)$$

$$\Rightarrow \frac{\partial f}{\partial u}\Big/\frac{\partial f}{\partial v} = -\frac{\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right)}{\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p\right)} \quad \text{——— (2)}$$

Similarly differentiating (1) w.r.t y we get

$$\frac{\partial f}{\partial u}\Big/\frac{\partial f}{\partial v} = -\frac{\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right)}{\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right)} \quad \text{——— (3)} \quad \left(\because q=\frac{\partial z}{\partial y}\right)$$

Now eliminating f from (2) & (3) we get

$$\frac{\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}\cdot p}{\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\cdot p} = \frac{\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q}{\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q}$$

$$\Rightarrow \left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right)\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right)=\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right)\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p\right)$$

$$\Rightarrow \left(\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z}-\frac{\partial v}{\partial y}\frac{\partial u}{\partial z}\right)p+\left(\frac{\partial v}{\partial x}\frac{\partial u}{\partial z}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\right)q = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial x}$$

$$\Rightarrow P\cdot p+Q\cdot q=R \quad \text{——— (4)}$$

where
$$P=\frac{\partial u}{\partial y}\frac{\partial v}{\partial z}-\frac{\partial v}{\partial y}\frac{\partial u}{\partial z}=\frac{\partial(u,v)}{\partial(y,z)}$$
$$Q=\frac{\partial v}{\partial x}\frac{\partial u}{\partial z}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}=\frac{\partial(u,v)}{\partial(z,x)}$$
$$R=\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}=\frac{\partial(u,v)}{\partial(x,y)}.$$

the eqn (4) is a PDE of the first order.

<u>Note</u>: $\boxed{1}$. If the given relation between x, y, z contains two arbitrary functions then the derived partial diff. eqn will contain partial derivatives of an order higher than two except in particular cases.

$\boxed{2}$. The PDE (4) derived in $\boxed{\text{II}}$ is a linear i.e, powers of p & q are both unity while the PDE (4) derived in $\boxed{\text{I}}$ need not be linear.

Type-I

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(1) → Form a PDE by elimination of arbitrary constants a & b from the eqn $z = ax+by+a^2+b^2$

Soln: Given eqn is $z = ax+by+a^2+b^2$ —— (1)

Diff. (1) partially w.r.t x & y, we get

$$\frac{\partial z}{\partial x} = a \text{ —— (2)}$$

$$\frac{\partial z}{\partial y} = b \text{ —— (3)}$$

Now eliminating a, b from (1), (2) & (3) we get

$$z = \frac{\partial z}{\partial x}x + \frac{\partial z}{\partial y}y + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2$$

which is the required PDE.

(2) → Eliminate a and b from $z = axe^y + \frac{1}{2}a^2e^{2y} + b$.

Soln: Given eqn is $z = axe^y + \frac{1}{2}a^2e^{2y} + b$. —— (1)

Diff (1) partially w.r.t x & y, we get

$$\frac{\partial z}{\partial x} = ae^y \text{ —— (2)}$$

$$\frac{\partial z}{\partial y} = axe^y + a^2e^{2y}$$

$$= x(ae^y) + (ae^y)^2 \text{ —— (3)}$$

Now sub (2) in eqn (3)

$$(3) \equiv \frac{\partial z}{\partial y} = x\left(\frac{\partial z}{\partial x}\right) + \left(\frac{\partial z}{\partial x}\right)^2$$

which is the required PDE

→ Form a PDE by eliminating arbitrary constants from the following relations.

(3) $z = ax+(1-a)y+b$; a, b

(4) $az+b = a^2x+y$; a, b

(5) $z = (x-a)^2+(y-b)^2$; a, b

(6) $z = a(x+y)+b$; a, b.

(7) $z = ax+by+ab$; a, b

(8) $z = ax+a^2y^2+b$; a, b

(9) $z = (x+a)(y+b)$; a, b

→ (10) $z = Ae^{pt}\sin px$; p, t A

→ Form a PDE by eliminating a, b, c from $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Sol: Given $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ —— (1)

differentiating (1) w.r.t x & y, we get

$$\frac{2x}{a^2}+\frac{2z}{c^2}\cdot\frac{\partial z}{\partial x}=0 \Rightarrow \frac{x}{a^2}+\frac{z}{c^2}\frac{\partial z}{\partial x}=0 \quad \text{—— (2)}$$

and $$\frac{2y}{b^2}+\frac{2z}{c^2}\cdot\frac{\partial z}{\partial y}=0 \Rightarrow \frac{y}{b^2}+\frac{z}{c^2}\frac{\partial z}{\partial y}=0 \quad \text{—— (3)}$$

Differentiating (2) w.r.t x and (3) w.r.t y, we find

$$\frac{1}{a^2}+\frac{1}{c^2}\left(\frac{\partial z}{\partial x}\right)^2+\frac{z}{c^2}\frac{\partial^2 z}{\partial x^2}=0 \Rightarrow c^2+a^2\left(\frac{\partial z}{\partial x}\right)^2+a^2 z\frac{\partial^2 z}{\partial x^2}=0 \quad \text{—— (4)}$$

and $$\frac{1}{b^2}+\frac{1}{c^2}\left(\frac{\partial z}{\partial y}\right)^2+\frac{z}{c^2}\frac{\partial^2 z}{\partial y^2}=0 \Rightarrow c^2+b^2\left(\frac{\partial z}{\partial y}\right)^2+b^2 z\frac{\partial^2 z}{\partial y^2}=0 \quad \text{—— (5)}$$

from (2) $$c^2=-\frac{a^2 z}{x}\frac{\partial z}{\partial x} \quad \text{—— (6)}$$

Sub (6) in (4), we get

$$(4) \equiv -\frac{a^2 z}{x}\frac{\partial z}{\partial x}+a^2\left(\frac{\partial z}{\partial x}\right)^2+a^2 z\frac{\partial^2 z}{\partial x^2}=0$$

$$\Rightarrow a^2\left[-\frac{z}{x}\frac{\partial z}{\partial x}+\left(\frac{\partial z}{\partial x}\right)^2+z\frac{\partial^2 z}{\partial x^2}\right]=0$$

(or) $$-z\frac{\partial z}{\partial x}+x\left(\frac{\partial z}{\partial x}\right)^2+zx\frac{\partial^2 z}{\partial x^2}=0$$

$$\Rightarrow xz\frac{\partial^2 z}{\partial x^2}+x\left(\frac{\partial z}{\partial x}\right)^2-z\frac{\partial z}{\partial x}=0 \quad \text{—— (7)}$$

Similarly, from (3) & (5),

$$zy\frac{\partial^2 z}{\partial y^2}+y\left(\frac{\partial z}{\partial y}\right)^2-z\frac{\partial z}{\partial y}=0 \quad \text{—— (8)}$$

Eqns (7) & (8) are two possible forms of the required equations of order 2.

IAS-96 Q1) Find the differential eqn of all spheres of radius λ having centre in the xy-plane. (I.A.S-96) 4

Sol: The eqn of any sphere of radius λ, having centre $(h,k,0)$ in the xy-plane is given by

$(x-h)^2+(y-k)^2+(z-0)^2=\lambda^2$ where h and k are arbitrary constants.

$\Rightarrow (x-h)^2+(y-k)^2+z^2=\lambda^2$. — ①

Differentiating eqn ① partially w.r.t x & y

$$(x-h)+z\frac{\partial z}{\partial x}=0 \Rightarrow (x-h)=-zp \quad \text{— ②} \quad \left(\because \frac{\partial z}{\partial x}=p\right)$$

$$(y-k)+z\frac{\partial z}{\partial y}=0 \Rightarrow (y-k)=-zq \quad \text{— ③} \quad \left(\because \frac{\partial z}{\partial y}=q\right)$$

Sub ② & ③ in eq ①

$$z^2p^2+z^2q^2+z^2=\lambda^2$$

$$z^2(p^2+q^2+1)=\lambda^2.$$

which is the required partial differential equation

IAS-97 ② Form the differential eqn by eliminating a and b from $z=(x^2+a)(y^2+b)$

Sol: Given $z=(x^2+a)(y^2+b)$. — ①

Differentiating ① partially w.r.t x & y, we get

$$\frac{\partial z}{\partial x}=2x(y^2+b)$$

$$p=2x(y^2+b)$$

$$\Rightarrow y^2+b=p/2x \quad \text{— ②}$$

$$\frac{\partial z}{\partial y}=2y(x^2+a)$$

$$q=2y(x^2+a) \quad \text{— ③}$$

$$\Rightarrow x^2+a=q/2y$$

Sub ② and ③ in eq ①

$$z=\frac{p}{2x}\frac{q}{2y}$$

$$\Rightarrow z=\frac{pq}{4xy} \Rightarrow 4xyz=pq.$$

which is the required partial differential equation

(3) IAS-98 Find the differential equation of the set of all right circular cones whose axes coincide with z-axis. (6)

Sol: The general equation of the set of all right circular cones whose axes coincide with z-axis, having semivertical angle α and vertex at $(0,0,c)$ is given by

$$x^2+y^2 = (z-c)^2 \tan^2\alpha. \quad \text{—} (1)$$

where α and c are arbitrary constants.

Differentiating eq (1) partially w.r.t x & y,

$$2x = 2(z-c)\tan^2\alpha \cdot \frac{\partial z}{\partial x}.$$

$$\Rightarrow x = p(z-c)\tan^2\alpha. \quad \text{—} (2)$$

$$2y = 2(z-c)\frac{\partial z}{\partial y}\tan^2\alpha$$

$$\Rightarrow y = (z-c)q\tan^2\alpha. \quad \text{—} (3)$$

from (3) $z-c = \dfrac{y}{q\tan^2\alpha}$ — (4)

Sub (4) in eq (2)

$$x = \frac{py}{q\tan^2\alpha}\tan^2\alpha.$$

$$x = py/q$$

$\Rightarrow$ $qx = py$. Which is the required equation.

(4) IAS-98. Eliminate a, b and c from $z = a(x+y) + b(x-y) + abt + c$.

Given $z = a(x+y) + b(x-y) + abt + c$. — (1)

Differentiating eq (1) partially w.r.t x, y & t

$$\frac{\partial z}{\partial x} = a+b \quad \text{—} (2)$$

$$\frac{\partial z}{\partial y} = a - b \quad \text{——— (3)}$$

$$\frac{\partial z}{\partial t} = ab \quad \text{——— (4)}$$

W.K.T $\quad 4ab = (a+b)^2 - (a-b)^2$

from (2), (3) & (4)

$$4 \cdot \frac{\partial z}{\partial t} = \left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2$$

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Show that the differential equation of all cones which have their vertex at the origin is $px+qy=z$. Verify that $yz+zx+xy=0$ is a surface satisfying the above equation.

Soln: The eqn of any cone with vertex at origin is

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0 \quad \text{——— (1)}$$

where a, b, c, f, g, h are parameters.

Differentiating eq (1) partially w.r.t x & y.

$$2ax+2cz\frac{\partial z}{\partial x}+2fy\frac{\partial z}{\partial x}+2g\left(x\frac{\partial z}{\partial x}+z\right)+2hy=0$$

$$ax+gz+hy+p(cz+fy+gx)=0 \quad \text{——— (2)}$$

$$2by+2cz\frac{\partial z}{\partial y}+2f\left(y\frac{\partial z}{\partial y}+z\right)+2bx+2gx\frac{\partial z}{\partial y}=0$$

$$by+fz+hx+q(cz+fy+gx)=0 \quad \text{——— (3)}$$

Multiplying (2) by x and (3) by y and adding, we have

$$ax^2+gzx+hxy+\underline{p(czx+fyx+gx^2)}+by^2+fzy+hxy+q\underline{(czy+fy^2+gxy)}=0$$

$$\Rightarrow (ax^2+by^2+gzx+fyz+2hxy)+px(cz+fy+gx)+qy(cz+fy+gx)=0$$

$$\Rightarrow (ax^2+by^2+gzx+fyz+2hxy)+(px+qy)(cz+fy+gx)=0 \quad \text{——— (4)}$$

from eqn (1) $\quad ax^2+by^2+2hxy+gzx+fyz=-cz^2-fyz-gzx \quad \text{(5)}$

(8)

sub eqn (5) in eqn (4)

$-(cz^2+fyz+gxz)+(cz+fy+gx)(px+qy)=0$

$\Rightarrow -z(cz+fy+gx)+(cz+fy+gx)(px+qy)=0$

$\Rightarrow (cz+fy+gx)(px+qy-z)=0$

$\Rightarrow px+qy-z=0$ —— (A)

which is the required differential equation.

Given surface is $yz+zx+xy=0$ —— (6)

Differentiating eq (6) partially w.r.t x and y

$y\frac{\partial z}{\partial x}+\frac{\partial z}{\partial x}x+z+y=0$ and $y\frac{\partial z}{\partial y}+z+x\frac{\partial z}{\partial y}+x=0$

$\Rightarrow yp+px+z+y=0$ $\Rightarrow yq+z+xq+x=0$

$\Rightarrow p(x+y)+z+y=0$ —— (7) $\Rightarrow (x+y)q+(x+z)=0$

$\Rightarrow p=-\frac{(z+y)}{x+y}$ $\Rightarrow q=-\frac{(x+z)}{x+y}$ —— (8)

sub (7) and (8) in eqn (A)

$$px+qy-z=\frac{-(z+y)x}{x+y}+\frac{-(x+z)y}{x+y}-z$$

$$=\frac{-xz-xy-xy-zy-xz-yz}{x+y}$$

$$=\frac{-2xz-2xy-2yz}{x+y}$$

$$=\frac{-2(xz+yz+xy)}{x+y}$$

$$=0$$

$\boxed{\because xy+yz+zx=0 \text{ by eqn (6)}}$

$\therefore$ eqn (6) is a surface satisfying eqn (A)

Type-II

(1) form a PDE by eliminating the arbitrary function ϕ from $z = e^{ny}\phi(x-y)$. —— ①

Soln: Differentiating ① partially w.r.t x & y, we get

$$\frac{\partial z}{\partial x} = e^{ny}\phi'(x-y)$$

$$\Rightarrow p = e^{ny}\phi'(x-y) \quad \text{—— (2)} \quad \left(\because \frac{\partial z}{\partial x} = p\right)$$

and $q = ne^{ny}\phi(x-y) + e^{ny}\phi'(x-y)(-1)$

$$q = ne^{ny}\phi(x-y) - e^{ny}\phi'(x-y) \quad \text{—— (3)}$$

sub ① & ② in eqn ③

$$q = nz - p$$

$$\Rightarrow \boxed{p + q = nz}$$

which is the required PDE of order one.

(2) form a PDE by eliminating the arbitrary functions f and F from $z = f(x+ay) + F(x-ay)$

Soln Given $z = f(x+ay) + F(x-ay)$ —— ①

Diff ① partially w.r.t x & y, we get

$$\frac{\partial z}{\partial x} = f'(x+ay) + F'(x-ay) \quad \text{—— (2)}$$

and $\frac{\partial z}{\partial y} = af'(x+ay) - aF'(x-ay)$ —— (3)

Diff. (2) & (3) partially w.r.t x & y respectively, we get

$$\frac{\partial^2 z}{\partial x^2} = f''(x+ay) + F''(x-ay) \quad \text{—— (4)}$$

$$\frac{\partial^2 z}{\partial y^2} = a^2 f''(x+ay) + a^2 F''(x-ay)$$

$$\Rightarrow \frac{\partial^2 z}{\partial y^2} = a^2[f''(x+ay) + F''(x-ay)] \quad \text{—— (5)}$$

Now sub (4) in (5)

we get $\boxed{\frac{\partial^2 z}{\partial y^2} = a^2\frac{\partial^2 z}{\partial x^2}}$ which is the required PDE.

2007 (3) $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$; f is an arbitrary function.

Sol^n: Given $z = y^2 + 2f\left(\frac{1}{x}+\log y\right)$ —— ①

Diff ① partially w.r.t x & y, we get

$p = 2f'\left(\frac{1}{x}+\log y\right)\cdot\left(-\frac{1}{x^2}\right)$ —— ②

and $q = 2y + 2f'\left(\frac{1}{x}+\log y\right)\cdot\frac{1}{y}$ —— ③

② $\Rightarrow$ $2f'\left(\frac{1}{x}+\log y\right) = -px^2$ —— ④

Sub ④ in ③, we get

$q = 2y - px^2\cdot\frac{1}{y}$

$\Rightarrow qy = 2y^2 - px^2$

$\Rightarrow \boxed{px^2 + qy = 2y^2}$

which is the required PDE of order one.

(4) $z = x^n f(y/x)$: f is an arbitrary function.

Sol^n Given $z = x^n f(y/x)$ —— ①

Diff ① partially w.r.t x & y, we get

$\frac{\partial z}{\partial x} = nx^{n-1} f(y/x) + x^n f'(y/x)\cdot(-y/x^2)$ —— ②

and $\frac{\partial z}{\partial y} = x^n f'(y/x)\cdot\left(\frac{1}{x}\right)$

$\Rightarrow x\frac{\partial z}{\partial y} = x^n f'(y/x)$ —— ③

② $\Rightarrow$ $\frac{\partial z}{\partial x} = n\cdot\frac{x^n}{x} f(y/x) + x^n f'(y/x)\cdot\left(-\frac{y}{x^2}\right)$ —— ④

Sub. ① & ③ in ④, we get

$\frac{\partial z}{\partial x} = \frac{n}{x}z + x\left(\frac{\partial z}{\partial y}\right)\left(-\frac{y}{x^2}\right)$

$\Rightarrow \frac{\partial z}{\partial x} = \frac{nz}{x} - \frac{y}{x}\left(\frac{\partial z}{\partial x}\right)$

$\Rightarrow \boxed{x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz}$ which is the required PDE

→ form partial diff. equs by eliminating the arbitrary functions from the following equations.

(5) $z = f(x+iy) + F(x-iy)$.

(6) $z = e^{ax+by} f(ax-by)$

(7) $lx+my+nz = \phi(x^2+y^2+z^2)$

(8) $z = f(x^2-y^2)$

(9) $z = f(x^2+y^2)$

(10) $z = f(y/x)$

(11) → $\phi(x+y+z, x^2+y^2-z^2) = 0$.

Sol: Given $\phi(x+y+z, x^2+y^2-z^2) = 0$

Let $u = x+y+z$, $v = x^2+y^2-z^2$

Then the given equ is $\phi(u,v) = 0$ —— (1)

Diff (1) w.r.t x partially, we get

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0.$$

$$\frac{\partial\phi}{\partial u}(1+p) + \frac{\partial\phi}{\partial v}(2x-2zp) = 0 \quad \text{—— (2)}$$

$$\left(\because \frac{\partial u}{\partial x} = 1, \frac{\partial u}{\partial z} = 1; \frac{\partial v}{\partial x} = 2x, \frac{\partial v}{\partial z} = -2z\right)$$

Diff (1) w.r.t y partially, we get

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) + \frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) = 0$$

$$\frac{\partial\phi}{\partial u}(1+q) + \frac{\partial\phi}{\partial v}(2y-2qz) = 0 \quad \text{—— (3)}$$

from (2) $$\frac{\partial\phi}{\partial u}\Big/\frac{\partial\phi}{\partial v} = \frac{-2(x-zp)}{1+p} \quad \text{—— (4)}$$

and from (3) $$\frac{\partial\phi}{\partial u}\Big/\frac{\partial\phi}{\partial v} = \frac{-2(y-qz)}{1+q} \quad \text{—— (5)}$$

From (4) & (5)

$$\frac{-\cancel{2}(x-zp)}{(1+p)} = \frac{-\cancel{2}(y-qz)}{(1+q)}$$

$\Rightarrow (1+q)(x-zp) = (1+p)(y-qz)$

$\Rightarrow x+qx-zp-\cancel{zpq} = y+py-qz-\cancel{pqz}$

$\Rightarrow \boxed{p(y+z)-(x+z)q = x-y}$

Which is the required PDE of order one

(12) → $z = f(x^2-y) + g(x^2+y)$

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Sol: Given $z = f(x^2-y) + g(x^2+y)$ —— ①

Diff ① partially w.r.t x & y, we get

$$\frac{\partial z}{\partial x} = f'(x^2-y)\,2x + g'(x^2+y)\,2x.$$

$$= 2x\left[f'(x^2-y) + g'(x^2+y)\right] \text{ —— ②}$$

and $\frac{\partial z}{\partial y} = f'(x^2-y)(-1) + g'(x^2+y)$ —— ③

Diff ② & ③ partially w.r.t x & y respectively

$$\frac{\partial^2 z}{\partial x^2} = 2x\left[f''(x^2-y)(2x) + g''(x^2+y)\cdot 2x\right] + 2\left[f'(x^2-y) + g'(x^2+y)\right]$$

$$= 4x^2\left[f''(x^2-y) + g''(x^2+y)\right] + 2\left[f'(x^2-y) + g'(x^2+y)\right] \text{ —— ④}$$

$$\frac{\partial^2 z}{\partial y^2} = f''(x^2-y) + g''(x^2+y) \text{ —— ⑤}$$

from eqn ② $f'(x^2-y) + g'(x^2+y) = \frac{1}{2x}\left(\frac{\partial z}{\partial x}\right)$ —— ⑥

Sub ⑤ & ⑥ in ④

$$\frac{\partial^2 z}{\partial x^2} = 4x^2\left(\frac{\partial^2 z}{\partial y^2}\right) + 2\cdot\left(\frac{1}{2x}\right)\left(\frac{\partial z}{\partial x}\right)$$

$$\Rightarrow \boxed{x\frac{\partial^2 z}{\partial x^2} = 4x^3\frac{\partial^2 z}{\partial y^2} + \frac{\partial z}{\partial x}}$$

which is the required PDE.

Equations solvable by direct integrations

We now consider the PDE's which can be solved by direct integration. In place of the usual constants of integration, we must use arbitrary functions of the variable held fixed.

(1) Solve $\dfrac{\partial^3 z}{\partial x^2 \partial y} + 18xy^2 + \sin(2x-y) = 0$

Sol:

Integrating twice w.r.t x and keeping y fixed, we get

$$\frac{\partial^2 z}{\partial x \partial y} + 9x^2y^2 - \frac{1}{2}\cos(2x-y) = f(y)$$

$$\Rightarrow \frac{\partial z}{\partial y} + 3x^3y^2 - \frac{1}{4}\sin(2x-y) = xf(y) + g(y)$$

Now integrating w.r.t y & keeping x fixed, we get

$$z + x^3y^3 - \frac{1}{4}\cos(2x-y) = x\int f(y)\,dy + \int g(y)\,dy + w(x).$$

Taking $\int f(y)\,dy = u(y)$

$\int g(y)\,dy = v(y)$

$$z + x^3y^3 - \frac{1}{4}\cos(2x-y) = x\,u(y) + v(y) + w(x).$$

Where u, v, w are arbitrary functions.

(2) Solve $\dfrac{\partial^2 z}{\partial x^2} + z = 0$; given that when $x = 0$, $z = e^y$ and $\dfrac{\partial z}{\partial x} = 1$

$\boxed{\text{Ans: } z = \sin x + e^y \cos x}$

→ Solve the following eqns:

(3) $\dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{x}{y} + a$

(4) $\dfrac{\partial^2 z}{\partial x^2} = xy$

(5) $\dfrac{\partial^2 u}{\partial x \partial t} = e^{-t}\cos x$

(6) $\dfrac{\partial^2 z}{\partial x^2} = a^2 z$; given that when $x = 0$, $\dfrac{\partial z}{\partial x} = a\sin x$ and $\dfrac{\partial z}{\partial y} = 0$

PDE of order one:-

Classification of first order partial diff. eqns are: (1) linear (2) semi-linear (3) quasilinear and (4) non-linear eqns.

(1) **Linear eqn:** A first order eqn $f(x, y, z, p, q) = 0$ is known as linear if it is linear in p, q and z.

i.e, if the given eqn is of the form

$$P(x,y)p + Q(x,y)q = R(x,y)z + S(x,y)$$

Ex: (1) $yx^2p + xy^2q = xyz + x^2y^3$

(2) $p + q = z + xy$.

(2) **Semi-linear:** A first order partial diff. eqn $f(x,y,z,p,q)=0$ is known as semi-linear eqn if it is linear in p and q and the coefficients of p & q are functions x & y only.

i.e, if the given eqn is of the form

$$P(x,y)p + Q(x,y)q = R(x,y,z)$$

Ex: (1) $xyp + x^2yq = x^2y^2z^2$

(2) $yp + xq = \frac{x^2z^2}{y^2}$.

(3) **Quasi-linear eqn:** A first order PDE $f(x,y,z,p,q)=0$ is known as quasi-linear eqn, if it is linear p & q.

i.e, if the given eqn of the form

$$P(x,y,z)p + Q(x,y,z)q = R(x,y,z)$$

Ex: (1) $x^2zp + y^2zq = xy$

(2) $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$.

(4) **Non-linear:** A first order PDE $f(x,y,z,p,q)=0$ which doesnot come under above three types, is known as a

non-linear eqn.

Ex: (1) $p^2+q^2=1$

(2) $pq=z$

(3) $x^2p^2+y^2q^2=z^2$.

Defn: A linear PDE of the first order is known as Lagrange's linear eqn, is of the form $Pp+Qq=R$ — ①
where P, Q, R are functions of x, y, z.
This eqn is called a quasi-linear equation.
This eqn ① is obtained by eliminating an arbitrary function f from $f(u,v)=0$ — ②
where u, v are functions of x, y, z.

Theorem: The general solution of the linear PDE
1989 $Pp+Qq=R$ — ① is $f(u,v)=0$ — ② where f is arbitrary function
and $u(x,y,z)=c_1$ and $v(x,y,z)=c_2$ — ③ form a solution of the equations $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$ — ④
where P, Q, R are functions of x, y, z

Proof: Now diff. ② partially w.r.t x & y, we get

$$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p\right)=0$$

and $$\frac{\partial f}{\partial u}\left(\frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q\right)+\frac{\partial f}{\partial v}\left(\frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q\right)=0$$

Now eliminating $\frac{\partial f}{\partial u}$, $\frac{\partial f}{\partial v}$, we get

$$\begin{vmatrix} \frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}p & \frac{\partial v}{\partial x}+\frac{\partial v}{\partial z}p \\ \frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}q & \frac{\partial v}{\partial y}+\frac{\partial v}{\partial z}q \end{vmatrix}=0$$

$$\Rightarrow \left(\frac{\partial u}{\partial y}\cdot\frac{\partial v}{\partial z}-\frac{\partial u}{\partial z}\frac{\partial v}{\partial y}\right)p+\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial x}-\frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\right)q =\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)$$

$\Rightarrow Pp + Qq = R$

where $P = \left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}\right)$

$Q = \frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}$

$R = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}$.

which is of the same form of eqn ①

$\therefore$ ② is g.s. of ①.

Now consider $u(x,y,z) = c_1$ & $v(x,y,z) = c_2$

where c_1 & c_2 are arbitrary constants.

By differentiating, we get

$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0$

$\Rightarrow \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0$ —— (i)

and $dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz = 0$

$\Rightarrow \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy + \frac{\partial v}{\partial z}dz = 0$ —— (ii)

By cross multiplication we get

$$\frac{dx}{\frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}} = \frac{dy}{\frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}} = \frac{dz}{\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}}.$$

$$\Rightarrow \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

which is same as the eqn ④

$\therefore u(x,y,z) = c_1$ & $v(x,y,z) = c_2$ are solutions of ④.

Note: Equations ④ are called Lagrange's auxiliary eqns (or) subsidiary eqns for ①.

→ **Working Rule for solving of Lagrange's eqn $Pp + Qq = R$:**

Step 1: Write the given eqn in standard form $Pp + Qq = R$ ——①

Step 2: Write the Lagrange's auxiliary eqns for ①.

namely $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ —— ②

Step 3: Solve these simultaneous eqns ② by using the well known methods.

Let $u(x,y,z)=c_1$ & $v(x,y,z)=c_2$ be two independent solutions of ②.

Step 4: Write the g.s. of ① as $f(u,v)=0$ (or) $u=\phi(v)$ (or) $v=\phi(u)$

Methods to solve the simultaneous eqns $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$:

Given eqns are $\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}$ ——①

where P,Q,R are functions of x,y,z

It can be solved in three method

Consider the three sets of eqns.

$$\frac{dx}{P}=\frac{dy}{Q} \;;\; \frac{dx}{P}=\frac{dz}{R},\; \frac{dy}{Q}=\frac{dz}{R} \quad \text{——②}$$

Method 1: If any two eqns of ② are integrable, by the method of variables seperable, we find their general solutions and that pair of solutions form the complete solution of the system ①.

Method 2: If one eqn of ② only integrable, by the method of variables seperable, we can find its g.s and this solution may be used to find the solution of another set of eqn ②. The pair of these solutions give the g.s. of the given equation ①.

Method 3: If no eqn of ② is integrable then we write

$$\frac{dx}{P}=\frac{dy}{Q}=\frac{dz}{R}=\frac{l_1dx+m_1dy+n_1dz}{l_1P+m_1Q+n_1R}=\frac{l_2dx+m_2dy+n_2dz}{l_2P+m_2Q+n_2R}$$

where l_1,m_1,n_1 ; l_2,m_2,n_2 are real numbers or functions of x,y,z

Case(i) If we choose l_1, m_1, n_1 and l_2, m_2, n_2 such that $l_1P + m_1Q + n_1R = 0$ and $l_2P + m_2Q + n_2R = 0$ then $l_1dx + m_1dy + n_1dz = 0$ and $l_2dx + m_2dy + n_2dz = 0$ which on integration gives two eqns.

$\therefore$ These eqns together give the complete solution.

Case(ii): If we choose l_1, m_1, n_1 & l_2, m_2, n_2 such that $l_1P + m_1Q + n_1R \neq 0$; $\dfrac{l_1dx + m_1dy + n_1dz}{l_1P + m_1Q + n_1R} = d\phi$

and $l_2P + m_2Q + n_2R \neq 0$; $\dfrac{l_2dx + m_2dy + n_2dz}{l_2P + m_2Q + n_2R} = d\psi$

then $\phi(x, y, z) = c_1$, $\psi(x, y, z) = c_2$ will become the g.s. of system ①.

Note: l_1, m_1, n_1 and l_2, m_2, n_2 are called multipliers.

Problems based on Method [I]

(1) Solve $x^2p + y^2q = z^2$.

Soln: Given $x^2p + y^2q = z^2$ —— ①

Clearly which is in the form of $Pp + Qq = R$

Here $P = x^2$; $Q = y^2$; $R = z^2$

Now the Lagrange's auxiliary eqns of ① are

$$\frac{dx}{x^2} = \frac{dy}{y^2} = \frac{dz}{z^2} \quad \text{—— ②}$$

Now taking the first two fractions of ②, we get

$$\frac{dx}{x^2} = \frac{dy}{y^2} \Rightarrow \boxed{-\frac{1}{x} + \frac{1}{y} = c_1} \quad \text{—— (3)}$$

Now taking the first and the last fractions of ②,

we get $\frac{dx}{x^2} = \frac{dz}{z^2} \Rightarrow \boxed{-\frac{1}{x} + \frac{1}{z} = c_2}$ —— (4)

$\therefore$ From (3) & (4) the required g.s. of ① is

$$f\left(-\frac{1}{x} + \frac{1}{y}, -\frac{1}{x} + \frac{1}{z}\right) = 0$$

where f is an arbitrary function.

(2) Solve $\left(\frac{y^2z}{x}\right)p + xzq = y^2$

Soln: Given that $\left(\frac{y^2z}{x}\right)p + xzq = y^2$ —— ①

Clearly which is in the form of $Pp + Qq = R$

Here $P = \frac{y^2z}{x}$; $Q = xz$ and $R = y^2$.

Now the Lagrange's auxiliary eqns of ① are

$$\frac{dx}{y^2z/x} = \frac{dy}{xz} = \frac{dz}{y^2} \quad \text{—— ②}$$

Taking the first two fractions of ②, we get

$$\frac{x\,dx}{y^2z} = \frac{dy}{xz} \Rightarrow \frac{x\,dx}{y^2} = \frac{dy}{x}$$

$$\Rightarrow x^2dx = y^2dy$$

$$\Rightarrow x^3 - y^3 = c_1 \quad \text{—— (3)}$$

Taking the first and last fractions of ②, we get

$$\frac{x\,dx}{y^2z} = \frac{dz}{y^2} \Rightarrow x\,dx = z\,dz$$

$$\Rightarrow x^2 - z^2 = 2c_2 \quad \text{—— (4)}$$

$\therefore$ The g.s. of ① is $f(x^3-y^3,\ x^2-z^2)=0$

where f is arbitrary function.

(3) Solve $a(p+q)=z$

(4) Solve $\frac{\partial z}{\partial x}+\frac{\partial z}{\partial y}=\sin x$

(5) Solve $zp=-x$.

(6) Solve $p\tan x+q\tan y=\tan z$.

(7) Solve $y^2p-xyq=x(z-2y)$

Sol^n: Given that $y^2p-xyq=x(z-2y)$ ——①

which is in the form of $Pp+Qq=R$

$P=y^2$; $Q=-xy$, $R=x(z-2y)$.

Now the Lagrange's auxiliary eqns of ① are

$$\frac{dx}{y^2}=\frac{dy}{-xy}=\frac{dz}{x(z-2y)} \quad \text{——②}$$

Taking the first two fractions of ②

$$\frac{dx}{y^2}=\frac{dy}{-xy} \Rightarrow -x\,dx=y\,dy$$

$$\Rightarrow \boxed{x^2+y^2=c_1} \quad \text{——③}$$

Taking the last two fractions of ②, we get

$$\frac{dy}{-xy}=\frac{dz}{x(z-2y)}$$

$$\Rightarrow \frac{dz}{dy}=\frac{2y-z}{y}$$

$$\Rightarrow \frac{dz}{dy}+\left(\frac{1}{y}\right)z=2 \quad \text{——④}$$

$$I.F.=e^{\int\frac{1}{y}dy}=e^{\log y}=y$$

G.S. of ④ is

$$z\cdot y=\int 2y\,dy+c_2$$

$$zy=y^2+c_2$$

$$\rightarrow \boxed{zy-y^2=c_2} \quad \text{——④}$$

$\therefore$ The required g.s. of ① is $f(x^2+y^2,\ zy-y^2)=0$

where f is an arbitrary function.

Problems based on Method (2):

(1) Solve $p+3q=5z+\tan(y-3x)$

Sol: Given that $p+3q=5z+\tan(y-3x)$ —— (1)

Comparing (1) with $Pp+Qq=R$

$P=1,\ Q=3,\ R=5z+\tan(y-3x)$

Now the Lagrange's A.Eqns of (1) are

$$\frac{dx}{1}=\frac{dy}{3}=\frac{dz}{5z+\tan(y-3x)} \quad \text{—— (2)}$$

Now taking first two fractions of (2), we get

$$\frac{dx}{1}=\frac{dy}{3} \Rightarrow \frac{dy}{3}=dx \Rightarrow dy=3dx$$
$$\Rightarrow \boxed{y-3x=c_1} \quad \text{—— (3)}$$

Now taking last two fractions of (2), we get

$$\frac{dy}{3}=\frac{dz}{5z+\tan(y-3x)}$$

$$\Rightarrow \frac{dy}{3}=\frac{dz}{5z+\tan c_1} \quad (\text{from (3)})$$

$$\Rightarrow \frac{1}{3}y=\frac{1}{5}\log(5z+\tan c_1)+c_2$$

$$\Rightarrow \frac{1}{3}y-\frac{1}{5}\log[5z+\tan c_1]=c_2$$

$$\Rightarrow \frac{1}{3}y-\frac{1}{5}\log[5z+\tan(y-3x)]=c_2 \quad \text{—— (4)}$$

$\therefore$ G.S. of (1) is

$$f\left(y-3x,\ \frac{1}{3}y-\frac{1}{5}\log(5z+\tan(y-3x))\right)=0$$

where f is an arbitrary function.

(2) Solve $z(z^2+xy)(px-qy)=x^4$.

Sol: Given that $z(z^2+xy)(px-qy)=x^4$

$$\Rightarrow z(z^2+xy)px-z(z^2+xy)qy=x^4$$

$$\Rightarrow xz(z^2+xy)p+[-yz(z^2+xy)]q=x^4 \quad \text{—— (1)}$$

Comparing (1) with $Pp+Qq=R$

$P = xz(z^2+xy)$; $Q = -yz(z^2+xy)$.

Now Lagrange's A.E's of (1) are

$$\frac{dx}{xz(z^2+xy)} = \frac{dy}{-yz(z^2+xy)} = \frac{dz}{x^4} \quad \text{——— (2)}$$

Taking first two fractions of (2), we get

$$\boxed{xy = c_1} \quad \text{——— (3)}$$

Taking first and last fractions of (2), we get

$$\frac{dx}{zx(z^2+xy)} = \frac{dz}{x^4}$$

$$\Rightarrow \frac{dx}{xz(z^2+c_1)} = \frac{dz}{x^4} \quad \text{(from (3))}$$

$$\Rightarrow \frac{dx}{z(z^2+c_1)} = \frac{dz}{x^3} \Rightarrow x^3\,dx = (z^3+c_1 z)\,dz$$

$$\Rightarrow \frac{x^4}{4} = \frac{z^4}{4} + \frac{c_1 z^2}{2} + c_2$$

$$\Rightarrow \boxed{x^4 - z^4 - 2z^2(xy) = 4c_2} \quad \text{——— (4)}$$

$\therefore$ G.S of (1) is

$$f(xy,\ x^4 - z^4 - 2z^2xy) = 0$$

where f is an arbitrary function.

(3) Solve $xzp + yzq = xy$

(4) Solve $p - 2q = 3x^2\sin(y+2x)$

Problems based on Method (3): Case(i)

(1) Solve $(mz-ny)p + (nx-lz)q = ly-mx$

Soln Given that $(mz-ny)p + (nx-lz)q = ly-mx$ ——— (1)

Comparing (1) with $Pp + Qq = R$

$P = mz-ny$; $Q = nx-lz$; $R = ly-mx$

Now the Lagrange's A.E. of (1) are

$$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx} \quad \text{——— (2)}$$

Now using the multipliers x, y & z.

each fraction of ② $= \dfrac{x\,dx + y\,dy + z\,dz}{0}$

$\Rightarrow x\,dx + y\,dy + z\,dz = 0$

Integrating, we get

$\boxed{x^2 + y^2 + z^2 = 2c_1}$ ——— ③

Again using the multipliers l, m, n

each fraction of ② $= \dfrac{l\,dx + m\,dy + n\,dz}{0}$

$\Rightarrow l\,dx + m\,dy + n\,dz = 0$

$\Rightarrow \boxed{lx + my + nz = c_2}$ ——— ④

$\therefore$ from ③ & ④, the required g.s. of ① is

$f(u, v) = 0$

(2) $\rightarrow$ $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$.

Multipliers are x, y, z & $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$.

(3) $\rightarrow$ $x(y - z)p + y(z - x)q = z(x - y)$

Multipliers are $1, 1, 1$ & $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$.

(4) $\rightarrow$ $x(y^2 - z^2)p - y(z^2 + x^2)q = z(x^2 + y^2)$

multipliers are x, y, z & $\frac{1}{x}, -\frac{1}{y}, -\frac{1}{z}$

(5) $\rightarrow$ $(y^2 + z^2)p - xyq = -zx$.

Multipliers: x, y, z & (no need to take multipliers by method(i))

2004 (6) $\rightarrow$ $x(y^2 + z)p - y(x^2 + z)q = z(x^2 - y^2)$

(7) $\rightarrow$ Solve $(x - y)p + (x + y)q = 2xz$. ——— ①

Sol^n: $\dfrac{dx}{x - y} = \dfrac{dy}{x + y} = \dfrac{dz}{2xz}$ ——— ②

Taking first two fractions of ②, we get

$\dfrac{dx}{x - y} = \dfrac{dy}{x + y} \Rightarrow (x + y)\,dx + (y - x)\,dy = 0$

$\Rightarrow (x\,dx + y\,dy) + (y\,dx - x\,dy) = 0$

$\Rightarrow \dfrac{x\,dx + y\,dy}{x^2 + y^2} + \dfrac{y\,dx - x\,dy}{x^2 + y^2} = 0$

$\frac{1}{2} d \log(x^2+y^2) + d\left(\tan^{-1}\left(\frac{y}{x}\right)\right) = 0$

$\boxed{\frac{1}{2} \log(x^2+y^2) + \tan^{-1}\left(\frac{y}{x}\right) = c_1}$ —— (3)

using the multipliers $1, 1, -\frac{1}{z}$

each fraction of (1) $= \dfrac{dx+dy-\frac{1}{z}dz}{(x-y)+(x+y)-\frac{1}{z}(2xz)}$

$= \dfrac{dx+dy-\frac{1}{z}dz}{2x-2x}$

$= \dfrac{dx+dy-\frac{1}{z}dz}{0}$

$\Rightarrow \boxed{x+y-\log z = c_2}$ —— (4)

$\therefore$ from (3) & (4) the required g.s. of (1) is

$f(u,v) = 0.$

Case (ii)

1997 → solve $(y+z)p + (z+x)q = x+y$ —— (1)

Sol^n $\dfrac{dx}{y+z} = \dfrac{dy}{z+x} = \dfrac{dz}{x+y}$ —— (2)

Using the multipliers $1, -1, 0$

each fraction of (1) $= \dfrac{dx-dy}{y-x}.$

$= \dfrac{d(x-y)}{-(x-y)}$ —— (3)

Again using the multipliers $0, 1, -1.$

each fraction of (1) $= \dfrac{dy-dz}{z-y}$

$= \dfrac{dy-dz}{-(y-z)}$ —— (4)

Finally using multipliers $1, 1, 1.$

each fraction of (1) $= \dfrac{dx+dy+dz}{2(x+y+z)}$

$= \dfrac{d(x+y+z)}{2(x+y+z)}$ —— (5)

$\therefore$ from (3), (4) & (5), we have

$\dfrac{d(x-y)}{x-y} = \dfrac{d(y-z)}{-(y-z)} = \dfrac{d(x+y+z)}{2(x+y+z)}$ —— (6)

Taking first two fractions of (6)

$$\frac{d(x-y)}{x-y} = \frac{d(y-z)}{(y-z)}$$

$$\Rightarrow \log(x-y) = \log(y-z) + \log c_1$$

$$\Rightarrow \boxed{\frac{x-y}{y-z} = c_1} \quad \text{--- (7)}$$

Now Taking the last two fractions of (6), we get

$$\log(x+y+z) + \log(y-z)^2 = \log c_2$$

$$\Rightarrow \boxed{(x+y+z)(y-z)^2 = c_2} \quad \text{--- (8)}$$

$\therefore$ from (7) & (8),
the required g.s. of (1) is

$$f\left(\frac{x-y}{y-z},\ (y-z)^2(x+y+z)\right) = 0$$

1996 → Solve $y^2(x-y)p + x^2(y-x)q = z(x^2+y^2)$ --- (1)

Sol^n: $\dfrac{dx}{y^2(x-y)} = \dfrac{dy}{x^2(y-x)} = \dfrac{dz}{z(x^2+y^2)}$ --- (2)

from (2) $\dfrac{dx}{y^2(x-y)} = \dfrac{dy}{x^2(y-x)}$

$$\Rightarrow \frac{dx}{y^2} = \frac{dy}{-x^2}$$

$$\Rightarrow x^2dx + y^2dy = 0$$

$$\Rightarrow \boxed{x^3 + y^3 = 3c_1} \quad \text{--- (3)}$$

Choosing the multipliers 1, −1, 0; we get

each fraction of (2) $= \dfrac{dx-dy}{(x-y)(x^2+y^2)}$ --- (4)

Now equating third fraction of (2) & the fraction (4)

we get, $\dfrac{dz}{z(x^2+y^2)} = \dfrac{d(x-y)}{(x-y)(x^2+y^2)}$

$$\log z = \log(x-y) + \log c_2$$

$$\Rightarrow \boxed{\frac{z}{x-y} = c_2} \quad \text{--- (5)}.$$

$\therefore$ From (4) & (5) the required g.s. of (1) is $f(u,v) = 0$

→ Solve $(x^2-y^2-z^2)p + 2xy\,q = 2xz$. multipliers; x, y, z

→ $(1+y)p + (1+x)q = z$; multipliers; $1, 1, 0$

→ $xzp + yzq = xy$; multipliers, $\frac{1}{x}, \frac{1}{y}, 0$

→ Solve $(x^2-yz)p + (y^2-zx)q = z^2-xy$

Soln: Given that $(x^2-yz)p+(y^2-zx)q = z^2-xy$ —— ①

Comparing with $Pp+Qq=R$.

$P = x^2-yz$; $Q = y^2-zx$; $R = z^2-xy$

Now the Lagrange's A.E's are

$$\frac{dx}{x^2-yz} = \frac{dy}{y^2-zx} = \frac{dz}{z^2-xy} \quad \text{—— ②}$$

Now using the multipliers $1,-1,0$ and $0,1,-1$

we get, each fraction of ② =

$$\frac{dx-dy}{x^2-y^2-yz+zx} = \frac{dy-dz}{y^2-z^2-zx+xy}$$

$$\Rightarrow \frac{dx-dy}{x^2-y^2+z(x-y)} = \frac{dy-dz}{y^2-z^2+x(y-z)}$$

$$\Rightarrow \frac{dx-dy}{(x-y)(x+y+z)} = \frac{dy-dz}{(y-z)(x+y+z)}$$

$$\Rightarrow \frac{dx-dy}{x-y} = \frac{dy-dz}{y-z}$$ on integration

$$\Rightarrow \boxed{\frac{x-y}{y-z} = c_1} \quad \text{—— ③}$$

Using the multipliers $1,1,1$; we get

each fraction of ② $= \dfrac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx}$ —— ④

Again using the multipliers x, y, z, we get

each ~~from~~ fraction of ② $= \dfrac{xdx+ydy+zdz}{x^3+y^3+z^3-3xyz}$

$$= \frac{xdx+ydy+zdz}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)} \quad \text{—— ⑤}$$

from (4) & (5) we have

$$\frac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx} = \frac{x\,dx+y\,dy+z\,dz}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}$$

$\Rightarrow (x+y+z)(dx+dy+dz) = x\,dx+y\,dy+z\,dz$

$\Rightarrow \frac{(x+y+z)^2}{2} = \frac{x^2}{2}+\frac{y^2}{2}+\frac{z^2}{2}+c_2$

$\Rightarrow (x+y+z)^2-(x^2+y^2+z^2) = 2c_2$

$\Rightarrow \boxed{xy+yz+zx = c_2}$ ——(6)

$\therefore$ from (3) & (6), the required g.s of (1) is $f\left(\frac{x-y}{y-z}, xy+yz+zx\right) = 0$ where f is an arbitrary function

2 → Solve $\cos(x+y)\,p + \sin(x+y)\,q = z$ ——(1)

Soln: $\frac{dx}{\cos(x+y)} = \frac{dy}{\sin(x+y)} = \frac{dz}{z}$ ——(2)

Now using the multipliers 1, 1, 0 and 1, −1, 0.

each fraction of (2) $= \frac{dx+dy}{\cos(x+y)+\sin(x+y)} = \frac{dx-dy}{\cos(x+y)-\sin(x+y)}$ ——(3)

from (2) & (3) we have

$$\frac{dz}{z} = \frac{dx+dy}{\cos(x+y)+\sin(x+y)} = \frac{dx-dy}{\cos(x+y)-\sin(x+y)} \quad \text{——(4)}$$

Now taking last two fractions of (4)

$$\frac{dx+dy}{\cos(x+y)+\sin(x+y)} = \frac{dx-dy}{\cos(x+y)-\sin(x+y)}$$

$$\frac{\cos(x+y)-\sin(x+y)}{\cos(x+y)+\sin(x+y)}\,d(x+y) = d(x-y)$$

$\Rightarrow \log[\cos(x+y)+\sin(x+y)] = (x-y)+\log c_1$

$\Rightarrow \boxed{[\cos(x+y)+\sin(x+y)]\,e^{y-x} = c_1}$ ——(5)

Now taking first two fractions of (4), we get

$$\frac{dz}{z} = \frac{dx+dy}{\cos(x+y)+\sin(x+y)}$$

$$\Rightarrow \frac{dz}{z} = \frac{\frac{1}{\sqrt{2}}(dx+dy)}{\sin\left(x+y+\frac{\pi}{4}\right)}$$

$$\Rightarrow \frac{dz}{z} = \frac{1}{\sqrt{2}}\operatorname{cosec}\left(x+y+\frac{\pi}{4}\right)d(x+y)$$

$$\Rightarrow \sqrt{2}\log z = \log\left|\tan\left(\frac{x+y+\frac{\pi}{4}}{2}\right)\right| + \log c_2$$

$$\Rightarrow \log z^{\sqrt{2}} = \log \tan\left(\frac{x}{2}+\frac{y}{2}+\frac{\pi}{8}\right) + \log c_2$$

$$\Rightarrow z^{\sqrt{2}}\cot\left(\frac{x}{2}+\frac{y}{2}+\frac{\pi}{8}\right) = c_2 \quad \text{--- (6)}$$

$\therefore$ From (5) & (6) the required g.s. of (1) is

$$f\left\{\left[\cos(x+y)+\sin(x+y)\right]e^{y-x},\ z^{\sqrt{2}}\cot\left(\frac{x+y}{2}+\frac{\pi}{8}\right)\right\} = 0$$

where f is an arbitrary function.

Q3 → Solve $(x^3+3xy^2)p + (y^3+3x^2y)q = 2z(x^2+y^2)$

Multipliers $1, +1, 0$ & $1, -1, 0$

$\frac{1}{x}, \frac{1}{y}, 0$

Q → $p+q = x+y+z$.

Multipliers : $1, 1, 1$

Q → $(2x^2+y^2+z^2-2yz-zx-xy)p + (x^2+2y^2+z^2-yz-2zx-xy)q$

$= x^2+y^2+2z^2-yz-zx-2xy$

Multipliers : $1, -1, 0$; $0, 1, -1$; $-1, 0, 1$

* The linear eqn containing more than two independent variables:

The generalisation of Lagrange's method is as follows:

Let the linear eqn with 'n' independent variables $x_1, x_2, \ldots, x_n$ be

$$P_1 p_1 + P_2 p_2 + \ldots P_n p_n = R \longrightarrow (1).$$

where $P_1, P_2, \ldots, P_n$ and R are fns of $x_1, x_2, \ldots, x_n$ and Z

Here p_i denotes $\frac{\partial z}{\partial x_i}$; $i = 1, 2, \ldots, n$.

Then g.s of (1) is given by

$$f(u_1, u_2, \ldots, u_n) = 0$$

where $u_i(x_1, x_2, \ldots, x_n, z) = c_i$; $i = 1, 2, \ldots, n$.

are independent solns of the auxiliary eqns

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \ldots = \frac{dx_n}{P_n} = \frac{dz}{R}.$$

solve

→ $x_2 x_3 p_1 + x_3 x_1 p_2 + x_1 x_2 p_3 + x_1 x_2 x_3 = 0$.

Given eqn is $x_2 x_3 p_1 + x_3 x_1 p_2 + x_1 x_2 p_3 = -x_1 x_2 x_3 \longrightarrow (1)$

Comparing (1) with

$$P_1 p_1 + P_2 p_2 + P_3 p_3 = R.$$

$\therefore P_1 = x_2 x_3,\ P_2 = x_3 x_1,\ P_3 = x_1 x_2$ & $R = -x_1 x_2 x_3$

Now the Lagrange's auxiliary eqns of (1) are

$$\frac{dx_1}{x_2 x_3} = \frac{dx_2}{x_3 x_1} = \frac{dx_3}{x_1 x_2} = \frac{dz}{-x_1 x_2 x_3} \longrightarrow (2)$$

Taking first two fractions of (2),

$$\frac{dx_1}{x_2 x_3} = \frac{dx_2}{x_3 x_1} \Rightarrow \frac{dx_1}{x_2} = \frac{dx_2}{x_1} \Rightarrow x_1 dx_1 = x_2 dx_2 \Rightarrow \boxed{x_1^2 - x_2^2 = c_1} \longrightarrow (3)$$

Taking ~~the~~ 2nd & third fractions of (2) (24)

$$\frac{dx_2}{x_3 x_1} = \frac{dx_3}{x_1 x_2} \Rightarrow x_2\,dx_2 = x_3\,dx_3$$

$$\Rightarrow \boxed{x_2^2 - x_3^2 = c_2} \quad \text{---}(4)$$

Taking the first and ~~forth~~ fourth fractions of (2)

we get $\boxed{x_1^2 + 2z = c_3}$ ---(5)

$\therefore$ from (3), (4) & (5), the g.s of (1) is

$f(x_1^2 - x_2^2,\ x_2^2 - x_3^2,\ x_1^2 + 2z) = 0.$

where f is arbitrary fn.

$\longrightarrow$ $x_2 x_3 z p_1 + x_3 x_1 z p_2 + x_1 x_2 z p_3 = x_1 x_2 x_3.$

94. $\longrightarrow$ $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z\left(\frac{\partial u}{\partial z}\right) = xyz.$ ---(1)

A.E's $\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z} = \frac{du}{xyz}.$

95. $\longrightarrow$ $(y+z+w)\frac{\partial w}{\partial x} + (z+x+w)\frac{\partial w}{\partial y} + (x+y+w)\frac{\partial w}{\partial z} = x+y+z$

A.E $\frac{dx}{y+z+w} = \frac{dy}{z+x+w} = \frac{dz}{x+y+w} = \frac{dw}{x+y+z}$ ---(2)

each fraction of (2) =

$$\frac{dx-dy}{y-x} = \frac{dy-dz}{z-y} = \frac{dz-dw}{w-z} = \frac{dx+dy+dz+dw}{3(x+y+z+w)}.$$

$\longrightarrow$ $(x_3 - x_2)p_1 + x_2 p_2 - x_3 p_3 + x_2^2 - (x_2 x_1 + x_2 x_3) = 0$

$\longrightarrow$

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2000 P.T if $x_1^3+x_2^3+x_3^3=1$ when $z=0$, the soln of the eqn $(s-x_1)p_1+(s-x_2)p_2+(s-x_3)p_3 = s-z$

can be given in the form

$$s^3\{(x_1-z)^3+(x_2-z)^3+(x_3-z)^3\}^4=(x_1+x_2+x_3-3z)^3$$

where $s=x_1+x_2+x_3+z$ and $p_i=\frac{\partial z}{\partial x_i}$, $i=1,2,3$.

soln Given that

$$(s-x_1)p_1+(s-x_2)p_2+(s-x_3)p_3=s-z \quad \text{——(1)}$$

where $s=x_1+x_2+x_3+z$. ——(2)

∴ the Lagrange's A.E's of (1) are

$$\frac{dx_1}{s-x_1}=\frac{dx_2}{s-x_2}=\frac{dx_3}{s-x_3}=\frac{dz}{s-z}$$

$$\Rightarrow \frac{dx_1}{x_2+x_3+z}=\frac{dx_2}{x_3+x_1+z}=\frac{dx_3}{x_1+x_2+z}=\frac{dz}{x_1+x_2+x_3} \quad \text{(by using (2))} \quad \text{——(3)}$$

each fraction of (3) is equal to

$$=\frac{dx_1+dx_2+dx_3-3dz}{2(x_1+x_2+x_3)+3z-3(x_1+x_2+x_3)}$$

$$=\frac{dx_1+dx_2+dx_3-3dz}{-(x_1+x_2+x_3)+3z}$$

$$=\frac{dx_1+dx_2+dx_3-3dz}{-(x_1+x_2+x_3-3z)}=\frac{d(x_1+x_2+x_3-3z)}{-(x_1+x_2+x_3-3z)} \quad \text{——(4)}$$

Again, each fraction of (3) $=\frac{dx_1+dx_2+dx_3+dz}{3(x_1+x_2+x_3+z)}$

$$=\frac{d(x_1+x_2+x_3+z)}{3(x_1+x_2+x_3+z)} \quad \text{——(5)}$$

from (4) and (5)

$$\frac{d(x_1+x_2+x_3-3z)}{-(x_1+x_2+x_3-3z)}=\frac{d(x_1+x_2+x_3+z)}{3(x_1+x_2+x_3+z)} \Rightarrow \frac{d(x_1+x_2+x_3+z)}{x_1+x_2+x_3+z}+\frac{3d(x_1+x_2+x_3-3z)}{x_1+x_2+x_3-3z}=0$$

integrating

$$\log(x_1+x_2+x_3+z)+3\log(x_1+x_2+x_3-3z)=\log a.$$

$\Rightarrow (x_1+x_2+x_3+z)(x_1+x_2+x_3-3z)^3 = a$. —— (6) $u(ii)$

where a is an arbitrary constant.

Given That $x_1^3+x_2^3+x_3^3=1$ when $z=0$

$\therefore$ Eqn (6) gives $(x_1+x_2+x_3)^3(x_1+x_2+x_3)=a$

$\Rightarrow a = (x_1+x_2+x_3)^4$. —— (7)

from (6) & (7)

$$\boxed{(x_1+x_2+x_3+z)(x_1+x_2+x_3-3z)^3 = (x_1+x_2+x_3)^4} \quad \text{—— (8)}$$

Now each fraction of (3) $= \dfrac{dx_1-dz}{-(x_1-z)}$

$$= \frac{3(x_1-z)^2\, d(x_1-z)}{-3(x_1-z)^3}$$

$$= \frac{d(x_1-z)^3}{-3(x_1-z)^3} \quad \text{—— (9)}$$

By symmetry, each fraction of (3) is also

$$= \frac{d(x_2-z)^3}{-3(x_2-z)^3} = \frac{d(x_3-z)^3}{-3(x_3-z)^3} \quad \text{—— (10)}$$

using (9) and (10)

each fraction of (3) is

$$= \frac{d(x_1-z)^3}{-3(x_1-z)^3} = \frac{d(x_2-z)^3}{-3(x_2-z)^3} = \frac{d(x_3-z)^3}{-3(x_3-z)^3}$$

$$= \frac{d\left[(x_1-z)^3+(x_2-z)^3+(x_3-z)^3\right]}{-3\left[(x_1-z)^3+(x_2-z)^3+(x_3-z)^3\right]} \quad \text{—— (11)}$$

from (4) and (11), we have

$$\frac{3\,d(x_1+x_2+x_3-3z)}{(x_1+x_2+x_3-3z)} = \frac{d\left[(x_1-z)^3+(x_2-z)^3+(x_3-z)^3\right]}{(x_1-z)^3+(x_2-z)^3+(x_3-z)^3}$$

Integrating

$$3\log(x_1+x_2+x_3-3z)+\log b = \log\left[(x_1-z)^3+(x_2-z)^3+(x_3-z)^3\right]$$

$$(x_1-z)^3+(x_2-z)^3+(x_3-z)^3 = b\left(x_1+x_2+x_3-3z\right)^3 \quad \text{—(12)}$$

where b is an arbitrary constant.

Given that $x_1^3+x_2^3+x_3^3=1$ when $z=0$.

from eqn (12)

$$x_1^3+x_2^3+x_3^3 = b(x_1+x_2+x_3)^3.$$

$$\Rightarrow \quad 1 = b(x_1+x_2+x_3)^3$$

$$\Rightarrow \boxed{b = \frac{1}{(x_1+x_2+x_3)^3}}.$$

$\therefore$ (12) $\equiv$

$$(x_1-z)^3+(x_2-z)^3+(x_3-z)^3 = \frac{(x_1+x_2+x_3-3z)^3}{(x_1+x_2+x_3)^3}$$

Raising both sides to power 4, we get

$$\Rightarrow \left[(x_1-z)^3+(x_2-z)^3+(x_3-z)^3\right]^4 = \frac{(x_1+x_2+x_3-3z)^{12}}{(x_1+x_2+x_3)^{12}} \quad \text{—(13)}$$

Raising both sides of eqn (8) to power 3, we have

$$(x_1+x_2+x_3+z)^3(x_1+x_2+x_3-3z)^9 = (x_1+x_2+x_3)^{12} \quad \text{—(14)}$$

multiplying the corresponding sides of (13) and (14), we have.

$$\left[(x_1-z)^3+(x_2-z)^3+(x_3-z)^3\right]^4(x_1+x_2+x_3+z)^3 = (x_1+x_2+x_3-3z)^3$$

$$s^3\left[(x_1-z)^3+(x_2-z)^3+(x_3-z)^3\right]^4 = (x_1+x_2+x_3-3z)^3$$

Since $x_1+x_2+x_3+z=s$.

Integral surfaces passing through a given curve:-

To find the integral surface of the general solution of the linear partial differential eqn $Pp+Qq=R$ which passes through a given curve.

Let $Pp+Qq=R$ — ① be the given eqn.

Let its auxiliary eqns give the following two independent solutions.

$u(x,y,z)=c_1$ & $v(x,y,z)=c_2$ — ②

then g.s. of ① is $f(u,v)=0$ where f is arbitrary function arising from a relation $f(c_1,c_2)=0$ — ③ between the constants c_1 & c_2.

we have to consider the problem of determing the function f in special cases.

Method(1): If we want to find integral surface passing through the given curve whose eqn in parametric form is given by $x=x(t)$, $y=y(t)$, $z=z(t)$. where t is parameter.

then ② may be expressed as

$u(x(t),y(t),z(t))=c_1$ and $v(x(t),y(t),z(t))=c_2$ — ④

Now eliminating the parameter 't' from ④, we get a relation involving c_1 & c_2.

Finally we replace c_1 & c_2 with the help of ② and obtain the required integral surface.

Method(2): we want to find the integral surface passing through the given curve which is determined by the following eqns $\phi(x,y,z)=0$ & $\psi(x,y,z)=0$ ⑤

Now we eliminate x,y,z from the four eqns ② & ⑤ and obtain a relation between c_1 & c_2.

finally, replace c_1 by $u(x, y, z)$ & c_2 by $v(x, y, z)$ in that relation and obtain the required integral surface.

Problems (Based on second Method)

→ Find the integral of the PDE $(x-y)p+(y-x-z)q=z$ through the circle $z=1$, $x^2+y^2=1$

Sol^n: Given that $(x-y)p+(y-x-z)q=z$ —— ①

Lagrange's A.Es are

$$\frac{dx}{x-y}=\frac{dy}{y-x-z}=\frac{dz}{z} \quad \text{——} ②$$

Using the multipliers 1, 1, 1

each fraction of ② $= \frac{dx+dy+dz}{0}$

$$\Rightarrow dx+dy+dz=0$$

$$\Rightarrow \boxed{x+y+z=c_1} \quad \text{——} ③$$

Taking last two eqns of ②

$$\frac{dy}{y+y-c_1}=\frac{dz}{z}$$ $\left[\because \text{from } ③ \; x+y+z=c_1 \Rightarrow y-c_1=-x-z\right]$

$$\frac{dy}{2y-c_1}=\frac{dz}{z}$$

$$\frac{1}{2}\log(2y-c_1)=\log z+\log c$$

$$\log(2y-c_1)=\log z^2c^2$$

$$2y-c_1=z^2c_2 \quad \text{where } c^2=c_2$$

$$\Rightarrow \frac{2y-(x+y+z)}{z^2}=c_2 \quad (\text{from } ③)$$

$$\Rightarrow \frac{y-x-z}{z^2}=c_2 \quad \text{——} ④$$

The curve is given by $z=1$, $x^2+y^2=1$

Taking $z=1$ in ③ & ④, we get

$x+y=c_1-1$ —— ⑤ & $y-x=c_2+1$ —— ⑥

But $2(x^2+y^2) = (x+y)^2+(y-x)^2$ — (7)

Now using (5) & (6) in (7), we get

$2(1) = (c_1-1)^2+(c_2+1)^2$

$\Rightarrow 2 = c_1^2+c_2^2-2c_1+2c_2+2$

$\Rightarrow c_1^2+c_2^2-2c_1+2c_2 = 0$ — 8

putting the values c_1 & c_2 in (8), we get

$$(x+y+z)^2+\frac{(y-x-z)^2}{z^4}-2(x+y+z)+2\frac{(y-x-z)}{z^2}=0$$

$$\Rightarrow z^4(x+y+z)^2+(y-x-z)^2-2z^4(x+y+z)+2z^2(y-x-z)=0$$

→ Find the eqn of the integral surface of the diff. eqn $(x^2-yz)p+(y^2-zx)q = z^2-xy$ — (1)
which passes through the line $x=1, y=0$

Soln:

$\frac{x-y}{y-z} = c_1$ — (2)

$xy+yz+zx = c_2$ — (3)

The given curve is $x=1, y=0$ — (4)

using (4) in (2) & (3) we get

$\boxed{-\frac{1}{z} = c_1}$ $\boxed{z = c_2}$ — (5)

from (5) $\left(-\frac{1}{z}\right)(z) = c_1c_2$

$\Rightarrow \boxed{c_1c_2 = -1}$ — (6)

using (2) & (3) in (6), we get

$$\left(\frac{x-y}{y-z}\right)(xy+yz+zx) = -1$$

97 → Find the eqn of surface satisfying $4yzp+q+2y=0$ and passing through $y^2+z^2=1$; $x+z=2$

Method I

98 → Find the integral surface of the linear PDE $x(y^2+z)p-y(x^2+z)q = (x^2-y^2)z$ which contains the straight line $x+y=0, z=1$ — (1)

Solⁿ: $xyz = c_1$ —② ; $x^2+y^2-2z = c_2$ —③

Method-I

The given curve $x+y=0$ & $z=1$ —④

Taking t as a parameter

put $x=t$ in ④, we get

$y=-t$ and $z=1$

$\therefore x=t\ ;\ y=-t\ ;\ z=1.$ —⑤

Using ⑤ in ② & ③, we get

$t(-t)(1)=c_1$ & $t^2+t^2-2=c_2$

$\Rightarrow -t^2=c_1$ & $2t^2-2=c_2$

$\Rightarrow t^2=-c_1$

$\Rightarrow 2(-c_1)-2=c_2$

$\Rightarrow 2c_1+c_2+2=0$ —⑥

Using ② and ③ in ⑥, we get

$2(xyz)+x^2+y^2-2z+2=0$

$\Rightarrow x^2+y^2+2xyz-2z+2=0$

(or)

2nd Method

Now eliminating x, y, z from ②, ③ & ④

we get $xy=c_1$ & $x^2+y^2-2=c_2$

$\Rightarrow (x+y)^2-2xy-2=c_2$

$\Rightarrow 0-2(c_1)-2=c_2$

$\Rightarrow 2c_1+c_2+2=0$

$\therefore$ From ② & ③, we get

$$\boxed{x^2+y^2-2z+2xyz+2=0}$$

2008 12M → Find the general solution of PDE

$(2xy-1)p+(z-2x^2)q = 2(x-yz)$

and also find the particular solution which passes through the lines $x=1,\ y=0$.

multipliers $(z, 1, x)$ $(x, y, 1/2)$

2005 → 2009 Find the particular integral of $x(y-z)p + y(z-x)q = z(x-y)$ which represents a surface passing through $x=y=z$.

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Sol Given eqn is

$$x(y-z)p + y(z-x)q = z(x-y) \quad \text{—— (1)}$$

Lagrange's A.E's are

$$\frac{dx}{x(y-z)} = \frac{dy}{y(z-x)} = \frac{dz}{z(x-y)} \quad \text{—— (2)}$$

Taking $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ as multipliers, we get each fraction of (2) $= \dfrac{\frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz}{0}$

$$\therefore \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0$$

$$\Rightarrow \log(xyz) = \log c_1$$

$$\Rightarrow \boxed{xyz = c_1} \quad \text{—— (2)}$$

Again taking the multipliers as 1, 1, 1, we get each fraction of (2) $= \dfrac{dx + dy + dz}{0}$

$$\therefore dx + dy + dz = 0$$

$$\Rightarrow \boxed{x+y+z = c_2} \quad \text{—— (3)}$$

but curve is $x = y = z$ —— (4)

Now we eliminate x, y, z from (2), (3) & (4) we get.

$$x^3 = c_1 \quad \& \quad 3x = c_2$$

$$\Rightarrow \left(\frac{c_2}{3}\right)^3 = c_1$$

$$\Rightarrow c_2^3 - 27c_1 = 0;$$

$$\Rightarrow (x+y+z)^3 - 27(xyz) = 0$$

which is reqd surface of (1)