REAL ANALYSIS TOPICS PYQs

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1. LIMIT, CONTINUITY & DIFFERENTIABILITY

1. 2c 2018

Show that if a function f defined on an open interval (a, b) of $I\!R$ is convex, then f is continuous. Show, by example, if the condition of open interval is dropped, then the convex function need not be continuous. 15

2. 4a 2018

Suppose $I\!R$ be the set of all real numbers and $f: I\!R \to I\!R$ is a function such that the following equations hold for all $x, y \in I\!R$:

(i) $f(x + y) = f(x) + f(y)$

(ii) $f(xy) = f(x)\,f(y)$

Show that $\forall x \in I\!R$ either $f(x) = 0$, or, $f(x) = x$. 20

3. 1b 2018 IFoS

(b) A function $f : [0, 1] \to [0, 1]$ is continuous on $[0, 1]$. Prove that there exists a point c in $[0, 1]$ such that $f(c) = c$. 10

4. 1c 2017

Find the supremum and the infimum of $\dfrac{x}{\sin x}$ on the interval $\left(0, \dfrac{\pi}{2}\right]$. *10*

5. 1b 2017 IFoS

1.(b) A function $f: I\!R \to I\!R$ is defined as below:

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational} \\ 1 - x & \text{if } x \text{ is irrational} \end{cases}$$

Prove that f is continuous at $x = \frac{1}{2}$ but discontinuous at all other points in $I\!R$. 10

6. 1b 2016

For the function $f : (0, \infty) \to \mathbb{R}$ given by

$$f(x) = x^2 \sin\frac{1}{x},\ 0 < x < \infty,$$

show that there is a differentiable function $g : \mathbb{R} \to \mathbb{R}$ that extends f. *10*

7. 3b 2016 IFoS

Examine the continuity of $f(x, y) = \begin{cases} \dfrac{\sin^{-1}(x+2y)}{\tan^{-1}(2x+4y)} & , (x, y) \neq (0, 0) \\ \frac{1}{2} & , (x, y) = (0, 0) \end{cases}$

at the point (0, 0). 8

8. 1a 2012 IFoS

1. Answer the following :

(a) Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 1 & , x \text{ is irrational} \\ -1 & , x \text{ is rational} \end{cases}$$

is discontinuous at every point in $\mathbb{R}$. 10

9. 2c 2010

(c) Define the function

$$f(x) = x^2 \sin\frac{1}{x}, \quad \text{if } x \neq 0$$
$$= 0, \quad \text{if } x = 0$$

Find $f'(x)$. Is $f'(x)$ continuous at $x = 0$? Justify your answer. 15

10. 1c 2010 IFoS

(c) If $f : \mathbb{R} \to \mathbb{R}$ is such that

$$f(x+y) = f(x) f(y)$$

for all x, y in $\mathbb{R}$ and $f(x) \neq 0$ for any x in $\mathbb{R}$, show that $f'(x) = f(x)$ for all x in $\mathbb{R}$ given that $f'(0) = f(0)$ and the function is differentiable for all x in $\mathbb{R}$. 10

11. 2d 2009

Show that if $f : [a, b] \to \mathbb{R}$ is a continuous function then $f([a, b]) = [c, d]$ for some real numbers c and d, $c \leq d$. 15

2. UNIFORM CONTINUITY

1. 2b 2020

Prove that the function $f(x) = \sin x^2$ is *not* uniformly continuous on the interval $[0, \infty[$. 15

2. 1b 2019 IFoS

Show that the function $f(x) = \sin\left(\frac{1}{x}\right)$ is continuous and bounded in $(0, 2\pi)$, but it is not uniformly continuous in $(0, 2\pi)$. 8

3. 4b 2016

Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $\lim\limits_{x \to +\infty} f(x)$ and $\lim\limits_{x \to -\infty} f(x)$ exist and are finite. Prove that f is uniformly continuous on $\mathbb{R}$. 15

4. 2b 2015 IFoS

(b) Let $X = (a, b]$. Construct a continuous function $f : X \to \mathbb{R}$ (set of real numbers) which is unbounded and not uniformly continuous on X. Would your function be uniformly continuous on $[a + \varepsilon, b]$, $a + \varepsilon < b$? Why ? 14

5. 2b IFoS 2014

(b) Show that the function $f(x) = \sin\frac{1}{x}$ is continuous but not uniformly continuous on $(0, \pi)$. 15

6. 3a (13m) 2013 IFoS

Show that the function $f(x) = x^2$ is uniformly continuous in $(0, 1)$ but not in $\mathbb{R}$.

7. 1b 2011

(b) Let $S = (0, 1]$ and f be defined by $f(x) = \frac{1}{x}$ where $0 < x \leq 1$ (in $\mathbb{R}$). Is f uniformly continuous on S ? Justify your answer. 12

3. SEQUENCES

1. 1c 2020

Prove that the sequence (a_n) satisfying the condition $|a_{n+1}-a_n| \leq \alpha |a_n - a_{n-1}|$, $0<\alpha<1$ for all natural numbers $n \geq 2$, is a Cauchy sequence. 10

2. 1a 2017

Let $x_1 = 2$ and $x_{n+1} = \sqrt{x_n + 20}$, $n = 1, 2, 3, \ldots$. Show that the sequence $x_1, x_2, x_3, \ldots$ is convergent. *10*

3. 1c 2016

Two sequences $\{x_n\}$ and $\{y_n\}$ are defined inductively by the following :

$$x_1 = \frac{1}{2},\ y_1 = 1 \text{ and } x_n = \sqrt{x_{n-1}\, y_{n-1}},\quad n = 2, 3, 4, \ldots$$

$$\frac{1}{y_n} = \frac{1}{2}\left(\frac{1}{x_n} + \frac{1}{y_{n-1}}\right),\quad n = 2, 3, 4, \ldots$$

Prove that

$$x_{n-1} < x_n < y_n < y_{n-1},\quad n = 2, 3, 4, \ldots$$

and deduce that both the sequences converge to the same limit l, where $\frac{1}{2} < l < 1$. *10*

4. 1c 2010

(c) Discuss the convergence of the sequence $\{x_n\}$

where $x_n = \dfrac{\sin\left(\frac{n\pi}{2}\right)}{8}$. 12

5. 1d 2010

(d) Define $\{x_n\}$ by $x_1 = 5$ and

$x_{n+1} = \sqrt{4 + x_n}$ for $n > 1$. 12

Show that the sequence converges to $\dfrac{(1+\sqrt{17})}{2}$.

4a. SERIES

1. 1d 2018

Find the range of $p(>0)$ for which the series :

$$\frac{1}{(1+a)^p} - \frac{1}{(2+a)^p} + \frac{1}{(3+a)^p} - \dots,\ a>0, \text{ is}$$

(i) absolutely convergent and (ii) conditionally convergent. 10

2. 1e 2012

Show that the series $\sum_{n=1}^{\infty} \left(\frac{\pi}{\pi+1}\right)^n n^6$ is convergent. 12

3. 2c 2009

Show that the series :

$$\left(\frac{1}{3}\right)^2 + \left(\frac{1\cdot 4}{3\cdot 6}\right)^2 + \dots\dots\dots + \left(\frac{1\cdot 4\cdot 7 \dots\dots (3n-2)}{3\cdot 6\cdot 9 \dots\dots 3n}\right)^2 + \dots\dots\dots$$

converges. 15

4b. ALTERNATING SERIES

1. 2a 2016

Show that the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n+1}$$

is conditionally convergent. (If you use any theorem(s) to show it, then you must give a proof of that theorem(s).) *15*

2. 1c 2015

Test the convergence and absolute convergence of the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^2+1}$. 10

4c. REARRANGEMENT OF TERMS

1. 4c 2017

Let $\sum_{n=1}^{\infty} x_n$ be a conditionally convergent series of real numbers. Show that there is a rearrangement $\sum_{n=1}^{\infty} x_{\pi(n)}$ of the series $\sum_{n=1}^{\infty} x_n$ that converges to 100. *20*

2. 1b 2015 IFoS

(b) Let $\sum_{n=1}^{\infty} a_n$ be an absolutely convergent series of real numbers.

Suppose $\sum_{n=1}^{\infty} a_{2n} = \frac{9}{8}$ and $\sum_{n=0}^{\infty} a_{2n+1} = \frac{-3}{8}$. What is $\sum_{n=1}^{\infty} a_n$?

Justify your answer. (Majority of marks is for the correct justification). *8*

5. RIEMANN'S INTEGRATION

1. 1c 2019 IFoS

Test the Riemann integrability of the function f defined by

$$f(x) = \begin{cases} 0 & \text{when } x \text{ is rational} \\ 1 & \text{when } x \text{ is irrational} \end{cases}$$

on the interval [0, 1]. 8

2. 1b 2018

Prove the inequality : $\frac{\pi^2}{9} < \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \frac{x}{\sin x} dx < \frac{2\pi^2}{9}$. 10

3. 2a 2017

Let

$$f(t) = \int_0^t [x]\, dx,$$

where [x] denotes the largest integer less than or equal to x.

(i) Determine all the real numbers t at which f is differentiable.

(ii) Determine all the real numbers t at which f is continuous but not differentiable. 15

4. 2b 2015

Is the function

$$f(x) = \begin{cases} \frac{1}{n}, & \frac{1}{n+1} < x \le \frac{1}{n} \\ 0, & x = 0 \end{cases}$$

Riemann integrable? If yes, obtain the value of $\int_0^1 f(x)\, dx$. 15

5. 2b 2014

Integrate $\int_0^1 f(x)\,dx$, where

$$f(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x} & , \quad x \in]0, 1] \\ 0 & , \quad x = 0 \end{cases}$$ 15

6. 4a 2014 IFoS P-1

Let f be a real valued function defined on [0, 1] as follows :

$$f(x) = \begin{cases} \frac{1}{a^{r-1}}, & \frac{1}{a^r} < x \leq \frac{1}{a^{r-1}}, \; r = 1, 2, 3 \ldots\ldots \\ 0 & x = 0 \end{cases}$$

where a is an integer greater than 2. Show that $\int_0^1 f(x)\,dx$ exists and is equal to $\frac{a}{a+1}$. 10

7. 1b 2014 IFoS

(b) Let f be defined on [0, 1] as

$$f(x) = \begin{cases} \sqrt{1-x^2}, & \text{if } x \text{ is rational} \\ 1-x & , \text{ if } x \text{ is irrational} \end{cases}$$

Find the upper and lower Riemann integrals of f over [0, 1]. 8

8. 4a 2014 IFoS

(a) Show that the function $f(x) = \sin x$ is Riemann integrable in any interval $[0, t]$ by taking the partition $P = \left\{0, \frac{t}{n}, \frac{2t}{n}, \frac{3t}{n}, \ldots, \frac{nt}{n}\right\}$ and $\int_0^t \sin x\,dx = 1 - \cos t$. 10

9. 1c 2013

Let $f(x) = \begin{cases} \dfrac{x^2}{2} + 4 & \text{if } x \geq 0 \\ \dfrac{-x^2}{2} + 2 & \text{if } x < 0 \end{cases}$

Is f Riemann integrable in the interval [–1, 2] ? Why ? Does there exist a function g such that $g'(x) = f(x)$? Justify your answer. *10*

10. 3d 2013

Let [x] denote the integer part of the real number x, i.e., if $n \leq x < n + 1$ where n is an integer, then $[x] = n$. Is the function $f(x) = [x]^2 + 3$ Riemann integrable in [–1, 2] ? If not, explain why. If it is integrable, compute $\int_{-1}^{2} ([x]^2 + 3)\, dx$. *10*

11. 3b 2012

(b) Let $f(x)$ be differentiable on [0, 1] such that $f(1) = f(0) = 0$ and $\int_0^1 f^2(x)\, dx = 1$. Prove that $\int_0^1 x f(x) f'(x)\, dx = -\frac{1}{2}$. 15

12. 4b 2012

(b) Give an example of a function $f(x)$, that is not Riemann integrable but $|f(x)|$ is Riemann integrable. Justify. 20

13. 1c 2011 IFoS

(c) Determine whether

$$f(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$$

is Riemann-integrable on $[0, 1]$ and justify your answer. 10

14. 2b 2011 IFoS

(b) Let the function f be defined by

$$f(x) = \frac{1}{2^t}, \quad \text{when } \frac{1}{2^{t+1}} < x \le \frac{1}{2^t}$$

$$(t = 0, 1, 2, 3, \ldots)$$

$$f(0) = 0$$

Is f integrable on $[0, 1]$? If f is integrable, then evaluate $\int_0^1 f\, dx$. 13

15. 1f 2010 P-1

(f) Show that the function

$$f(x) = [x^2] + |x - 1|$$

is Riemann integrable in the interval $[0, 2]$, where $[\alpha]$ denotes the greatest integer less than or equal to α. Can you give an example of a function that is not Riemann integrable on $[0, 2]$? Compute $\int_0^2 f(x)\, dx$, where $f(x)$ is as above. 12

6. UNIFORM CONVERGENCE

1. 2b 2020 IFoS

Show that the sequence of functions $\{f_n(x)\}$, where $f_n(x) = nx(1-x)^n$, does not converge uniformly on $[0, 1]$. 15

2. 3a 2019

Discuss the uniform convergence of

$$f_n(x) = \frac{nx}{1+n^2x^2}, \ \forall x \in \mathbb{R}\,(-\infty, \infty)$$

$$n = 1, 2, 3, \ldots$$ 15

3. 3b 2019 IFoS

(b) Show that the sequence $\{\tan^{-1} nx\}$, $x \geq 0$ is uniformly convergent on any interval $[a, b]$, $a > 0$ but is only pointwise convergent on $[0, b]$. *15*

4. 4b 2018 IFoS

Let $f_n(x) = \dfrac{x}{n+x^2}$, $x \in [0, 1]$. Show that the sequence $\{f_n\}$ is uniformly convergent on $[0, 1]$. 8

5. 1b 2016 IFoS

Examine the Uniform Convergence of

$$f_n(x) = \frac{\sin(nx+n)}{n}, \ \forall x \in \mathbb{R}, \ n = 1, 2, 3, \ldots$$ 8

6. 3a 2016 IFoS

If $f_n(x) = \frac{3}{x+n}$, $0 \le x \le 2$, state with reasons whether $\{f_n\}_n$ converges uniformly on [0, 2] or not. 10

7. 3b 2015

Test the series of functions $\sum_{n=1}^{\infty} \frac{nx}{(1+n^2x^2)}$ for uniform convergence. 15

8. 3b 2015 IFoS

(b) Let $f_n(x) = \frac{x}{1+nx^2}$ for all real x. Show that f_n converges uniformly to a function f. What is f ? Show that for $x \neq 0$, $f_n'(x) \to f'(x)$ but $f_n'(0)$ does not converge to $f'(0)$. Show that the maximum value $|f_n(x)|$ can take is $\frac{1}{2\sqrt{n}}$. 13

9. 2c 2013

Show that the series $\sum_{1}^{\infty} \frac{(-1)^{n-1}}{n+x^2}$, is uniformly convergent but not absolutely for all real values of x. 13

10. 1b 2012

(b) Let

$$f_n(x) = \begin{cases} 0, & \text{if } x < \frac{1}{n+1}, \\ \sin\frac{\pi}{x}, & \text{if } \frac{1}{n+1} \le x \le \frac{1}{n}, \\ 0, & \text{if } x > \frac{1}{n}. \end{cases}$$

Show that $f_n(x)$ converges to a continuous function but not uniformly. 12

11. 4b 2012 IFoS

(b) Examine the series

$$\sum_{n=1}^{\infty} u_n(x) = \sum_{n=1}^{\infty}\left[\frac{nx}{1+n^2x^2} - \frac{(n-1)x}{1+(n-1)^2x^2}\right]$$

for uniform convergence. Also, with the help of this example, show that the condition of uniform convergence of $\sum_{n=1}^{\infty} u_n(x)$ is sufficient but not necessary for the sum $S(x)$ of the series to be continuous. 13

12. 2b 2011

(b) Let $f_n(x) = nx(1-x)^n$, $x \in [0, 1]$

Examine the uniform convergence of $\{f_n(x)\}$ on $[0, 1]$. 15

13. 3b 2011

(b) Show that the series for which the sum of first n terms

$$f_n(x) = \frac{nx}{1+n^2x^2}, \quad 0 \le x \le 1$$

cannot be differentiated term-by-term at $x = 0$. What happens at $x \neq 0$? 15

14. 4b 2011

(b) Show that if $S(x) = \sum_{n=1}^{\infty} \frac{1}{n^3 + n^4 x^2}$, then its derivative

$$S'(x) = -2x \sum_{n=1}^{\infty} \frac{1}{n^2(1+nx^2)^2}, \text{ for all } x. \quad 20$$

15. 2d 2010

(d) Consider the series $\sum_{n=0}^{\infty} \frac{x^2}{(1+x^2)^n}$.

Find the values of x for which it is convergent and also the sum function.

Is the convergence uniform ? Justify your answer. 15

16. 3c 2010

(c) Let $f_n(x) = x^n$ on $-1 < x \leq 1$ for $n = 1, 2, \ldots$. Find the limit function. Is the convergence uniform ? Justify your answer. 15

17. 3c 2009

Show that :

$$\underset{x \to 1}{\text{Lt}} \sum_{n=1}^{\infty} \frac{n^2 x^2}{n^4 + x^4} = \sum_{n=1}^{\infty} \frac{n^2}{n^4 + 1}$$

Justify all steps of your answer by quoting the theorems you are using. 15

7. DIFFERENTIATION UNDER INTEGRAL SIGN

1. 1c 2019

Evaluate

$$\int_0^{\infty} \frac{\tan^{-1}(ax)}{x(1+x^2)}\,dx, \quad a>0,\ a\neq 1.$$ 10

8. THEORY OF REAL NUMBERS

1. 2d 2013

Show that every open subset of $\mathbb{R}$ is a countable union of disjoint open intervals. *14*

2. 3d 2009

Show that a bounded infinite subset of $\mathbb{R}$ must have a limit point. 15