CSE - 2018

Q1 (c) Determine if $\lim\limits_{x \to 1} (1-x) \tan \frac{\pi x}{2}$ exists or not. If the limit exists, then find its value.

Sol

$$\lim_{x \to 1} (1-x) \tan\left(\frac{\pi x}{2}\right)$$

$$\Rightarrow \lim_{z \to 1} \frac{(1-z)}{\cos\left(\frac{\pi x}{2}\right)} \sin\left(\frac{\pi x}{2}\right)$$

$$\cancel{L = \lim_{x \to 1}} \quad (a)$$

Assuming x be a real number

$\lim\limits_{x \to a} f(z)$ exist if and only if

$$\lim_{x \to a^+} f(z) = \lim_{z \to a^-} f(z)$$

i.e LHL = RHL

$$\text{LHL} = \lim_{x \to 1^-} (1-x) \tan\left(\frac{\pi x}{2}\right)$$

$$= \lim_{h \to 0} \left(1-(1-h)\right) \tan\left[\frac{\pi(1-h)}{2}\right]$$

$$\Rightarrow \lim_{h \to 0} h \tan\left[\frac{\pi}{2} - \frac{\pi h}{2}\right]$$

$$= \lim_{h \to 0} h \cot\left(\frac{\pi h}{2}\right) \quad \text{--- (1)}$$

$$\Rightarrow \lim_{h \to 0} \frac{h}{\tan\left(\frac{\pi h}{2}\right)} = \lim_{h \to 0} \frac{\pi/2\, h}{\frac{\pi}{2} \tan\frac{\pi h}{2}}$$

$$\Rightarrow \lim_{h \to 0} \frac{\pi/2\, h}{\tan(\pi/2\, h)} \cdot \frac{2}{\pi} \quad \left\{ \because \lim_{y \to 0} \frac{y}{\tan y} = 1 \right.$$

where $y = \frac{\pi}{2} h$

Similarly

$$RHL = \lim_{x \to 1^+} (1-x) \tan\left(\frac{\pi x}{2}\right)$$

$$\Rightarrow \lim_{h \to 0} \left(1 - (1+h)\right) \tan\left[\frac{\pi}{2}(1+h)\right]$$

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$$\Rightarrow \lim_{h \to 0} h \cot\left(\frac{\pi}{2} h\right) \quad \left\{ \because \tan(\pi/2 + \theta) = \cot\theta \right.$$

$$\Rightarrow \lim_{h \to 0} h \cot\left(\frac{\pi}{2} h\right) = \lim_{h \to 0} \frac{\pi/2\, h}{\tan(\pi/2\, h)} \cdot \frac{2}{\pi}$$

$$\Rightarrow \frac{2}{\pi}$$

$$\therefore LHL = R.H.L = \lim_{x \to a} f(x) = \frac{2}{\pi}$$

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Hence limit exist its value is $\boxed{\frac{2}{\pi}}$

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Q1(d) find the limit

$$\lim_{n\to\infty} \frac{1}{n^2} \sum_{r=0}^{n-1} \sqrt{n^2-r^2}$$

$$L = \lim_{n\to\infty} \frac{1}{n^2} \sum_{n=0}^{n-1} \sqrt{n^2-r^2}$$

$$\Rightarrow \quad l = \lim_{n\to\infty} \sum_{n=0}^{n-1} \frac{1}{n} \sqrt{1-\frac{r^2}{n^2}}$$

$$\left\{ \because \int_0^1 f(x)\,dx = \lim_{n\to 0} \left[\sum_{r=0}^{r=n-1} \frac{1}{n} f\left(\frac{r}{n}\right) \right] \right.$$

$$\Rightarrow \int_0^1 f(x)\,dx = \int_0^1 \sqrt{1-x^2}\,dx$$

$$l = \frac{1}{2}\left[x\sqrt{1-x^2} + \sin^{-1}x \right]_0^1$$

$$l = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}$$

$$\boxed{l = \frac{\pi}{4}}$$

Q4 (b) Evaluate the integral

$$\int_0^a \int_{x/a}^{x} \frac{x\,dy\,dx}{x^2+y^2}.$$

$$I = \int_{x=0}^{a} \int_{y=x/a}^{x} \frac{x\,dy\,dx}{x^2+y^2}$$

treating x as cont. bcoz integral limits are variable with respect to x, integrate first with respect to y

Thus $\int_{x=0}^{a} x \left[\frac{1}{x} \tan^{-1}\left(\frac{y}{x}\right) \right]_{y=x/a}^{x} dx$

$$= \int_{x=0}^{a} \left(\frac{\pi}{4} - \tan^{-1}\frac{1}{a} \right) dx$$

$$\boxed{I = \frac{\pi}{4}a - a\tan^{-1}\left(\frac{1}{a}\right)}$$

Q(c)

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Ques Let $f(x,y) = xy^2$ if, $y > 0$
$= -xy^2$; if $y \le 0$
determine which of $\frac{\partial f}{\partial x}(0,1)$ and $\frac{\partial f}{\partial y}(0,1)$ exists and which does not exist

$$f_x(0,1) = \frac{\partial f}{\partial x}(0,1) = \lim_{\delta x \to 0} \frac{f(0+\delta x, 1) - f(0,1)}{\delta x}$$

$$\Rightarrow \lim_{\delta x \to 0} \left[\frac{f(\delta x, 1) - f(0,1)}{\delta x} \right]$$

$$\Rightarrow \lim_{\delta x \to 0} \left[\frac{\delta x (1)^2 - 0}{\delta x} \right] = 1$$

$$\frac{\partial f}{\partial y}(0,1) = f_y(0,1) = \lim_{\delta y} \frac{f(0, 1+\delta y) - f(0,1)}{\delta y}$$

$$\lim_{\delta y \to 0} \frac{0(1+\delta y)^2 - 0}{\delta y} = 0$$

$f_x(0,1)$ and $f_y(0,1)$ both exist.

Qu find the maximum and the minimum values of $x^4 - 5x^2 + 4$ on the interval $[2, 3]$.

Sol^n

$f(x) = x^4 - 5x^2 + 4$

$f'(x) = 4x^3 - 10x$

Equating to zero $f'(x)$ we get

$4x^3 - 10x = x(4x^2 - 10) = 0$

$x = 0, \pm \sqrt{5}/2$

The f^n f has three critical points $\notin [2,3]$

Thus $f(x)$ is monotonic on $[2,3]$

Now to check, whether it is monotonic increasing or decreasing

$f'(2) = 4(2)^3 - 10(2) = 12 > 0$

$f'(3) = 4(3)^3 - 10(3) = 78 > 0$

Since $f'(2)$ and $f'(3) > 0$

$f'(3) > f(2)$

Hence ~~f'(x)~~ $f(x)$ is monotonically increasing on $[2,3]$

$f_{min} = (2)^4 - 5(2)^2 + 4 = 0$

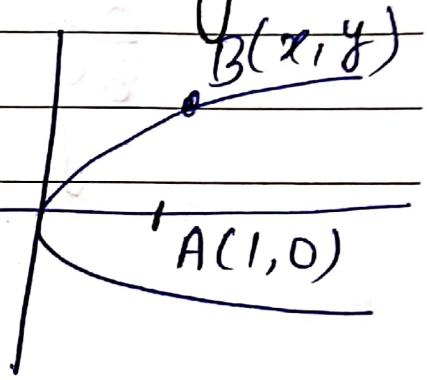
$f_{max} = f(x)\Big|_{at\ x=3} = (3)^4 - 5(3)^2 + 4$

$\Rightarrow 40$

$$\boxed{\text{So } f_{min} = 0 \text{ and } f_{max} = 40}$$

Ques 2(b) Find the shortest distance from the point (1,0) to the parabola $y^2 = 4x$

Soln Let $B(x,y)$ be a point on the parabola which is nearest to point $(1,0)$



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Now, distance b/w A and B is given by

$$AB = \sqrt{(x-1)^2 + y^2}$$

$$y^2 = 4ax = 4x$$

$$AB = \sqrt{(x-1)^2 + 4x}$$

$$AB^2 = x^2 + 1 + 2x$$

$$AB^2 = (x+1)^2$$

$$AB = \sqrt{\left(\frac{y^2}{4} - 1\right)^2 + y^2}$$

$$AB = \sqrt{\frac{y^4}{16} + \frac{y^2}{2} + 1}$$

$$Z = AB^2 = \frac{y^4}{16} + \frac{y^2}{2} + 1$$

$$\frac{\partial Z}{\partial y} = \frac{4y^3}{16} + \frac{2y}{2} = 0$$

$$y(y^2+4) = 0$$

$$\Rightarrow \quad y=0, \; y = \pm 2i$$

rejecting imaginary values

we get $y=0$, Thus $x=0$

Hence $(0,0)$ is closest to the point $(1,0)$

$$PO = \sqrt{(\cancel{x}0-1)^2 + 0} = 1$$