Chapter 11

2010

11.1 Section-A

Question-1(a) Show that the set

$$\mathrm{P[t]} = \left\{\mathrm{at}^2 + \mathrm{bt} + \mathrm{c/a, b, c} \in \mathbb{R}\right\}$$

forms a vector space over the field $\mathbb{R}$. Find a basis for this vector space. What is the dimension of this vector space?

[8 Marks]

Solution: From question $\mathrm{P}(t) = \{at^2 + bt + c\}$
Let, $f(t)$ and $g(t) \in p(t)$ then, $f(t) = a_1t^2 + b_1t + c_1$ and $g(t) = a_2t^2 + b_2t + c_2$ then,

$$f(t) + g(t) = (a_1 + a_2)\,t^2 + (b_1 + b_2)\,t + (c_1 + c_2)$$

$$\Rightarrow f(t) + g(t) \in p(t)$$

$\because a_1, a_2, b_1, b_2, c_1, c_2 \in \mathbb{R}$
Also, $f(t) = g(t)$ iff $a_1 = a_2, b_1 = b_2, c_1 = c_2$
and, $kf(t) = (ka_1)\,t^2 + (kb_1)\,t + kc_1 = i \in p(t)$

$$\begin{aligned} f(t) + g(t) &= (a_1 + a_2)\,t^2 + (b_1 + b_2)\,t + (c_1 + c_2) \\ &= (a_2 + a_1)\,t^2 + (b_2 + b_1)\,t + (c_2 + c_1) \\ &= g(t) + f(t) \end{aligned}$$

$\Rightarrow$ Set is commutative.
Also, if $b(t) = a_3t^2 + b_3t + c_3$ then $f(t) + (g(t) + h(t)\} = \{f(t) + g(t)\} + h(t)$ Existence of identity $0 = 0.t^2 + 0.t + 0$ i.e., $0 \in p(t) \Rightarrow 0 + f(t) = f(t)$

Existence of additive inverse of each member as $f(t) \in p(t)$ then $-f(t) \in p(t)$ and $-f(t) + f(t) = 0$
$\therefore -f(t)$ is the additive inverse of $f(t)$ i.e. $P(t)$ is an abelian group w.r.t. addition of polynomial of less than or equal to degree. Hence: $p(t)$ is vector space.

Question-1(b) Determine whether the quadratic form is positive definite.

$$\mathrm{q = x^2 + y^2 + 2xz + 4yz + 3z^2}$$

[8 Marks]

Solution: The associated symmetric matrix of the given quadratic form can be written as:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} \quad \text{i.e.} q = \begin{bmatrix} x & y & z \end{bmatrix} \mathrm{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

to ascertain the positive definite, we have to apply the congruent operation in the above matrix.i.e.

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 2 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathrm{A} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Apply congrucnt operation $R_3 \to R_3 - R_1$ & $C_3 \to C_3 - C_1$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \mathrm{A} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Apply congruent operation $R_3 \to R_3 - 2R_2$& $C_3 \to C_3 - 2C_2$, we get

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & -2 & 1 \end{bmatrix} \mathrm{A} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

As all the roots of scalar matrix in the left hand side are not positive. Hence, the given quadratic form is not positive.

Question-1(c) Prove that between any two real roots of $\mathrm{e^x \sin x} = 1$, there is at least one real root of $e^x \cos x + 1 = 0$.

[8 Marks]

Solution: The given function is $f(x) = e^x \sin x - 1 = 0$ or, $\sin x - e^{-x} = 0$
Now, $f(x) = \sin x - e^{-x} = 0$
If x_1 and x_2 are two roots of $f(x) = 0$ then by Rolle's theorem $\exists$ at least one real root of $f'(x) = 0$ lies between x_1 and x_2.

$$\because f'(x) = \cos x + e^{-x} = 0$$

i.e. $e^x \cos x + 1 = 0$ has a root lies between two real roots of $e^x \sin x = 1$

Question-1(d) Let f be a function defined on $\mathbb{R}$ such that

$$f(x+y) = f(x) + f(y), \quad x, y \in \mathbb{R}$$

If f is differentiable at one point of $\mathbb{R}$, then prove that f is differentiable on $\mathbb{R}$

[8 Marks]

Solution: Given,

$$f(x+y) = f(x) + f(y) \quad \ldots \quad (1)$$

Let f be differentiable at a and c be any general point.

Then,

$$\begin{aligned} \underset{h\to 0}{Lt} \frac{f(a+h)-f(a)}{h} &= \underset{h\to 0}{Lt} \frac{f(a)+f(h)-f(a)}{h} \text{ (from (1))} \\ &= \underset{h\to 0}{Lt} \frac{f(h)}{h} \text{ (exists, } \because f \text{ is diff at } a) \end{aligned}$$

Hence,

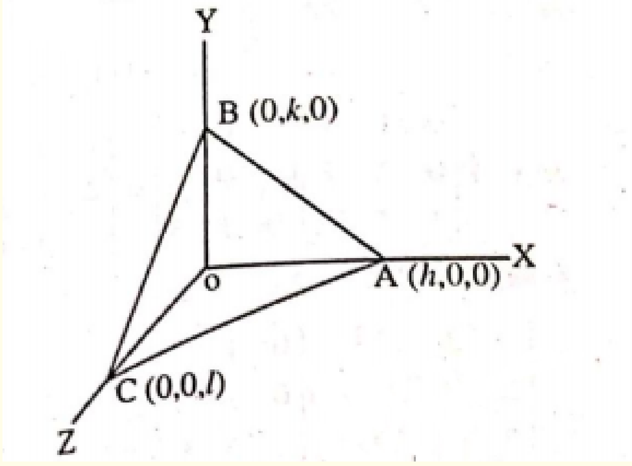
$$\begin{aligned} \underset{h\to 0}{Lt} \frac{f(c+h)-f(c)}{h} &= \underset{h\to 0}{Lt} \frac{f(c)+f(h)-f(c)}{h} \\ &= \underset{h\to 0}{Lt} \frac{f(h)}{h} \text{ exists} \quad \ldots \quad (2) \end{aligned}$$

As c was arbitrary point on $\mathbb{R}$, hence f is differentiable on $\mathbb{R}$.

Question-1(e) If a plane cuts the axes in $\mathrm{A, B, C}$ and $\mathrm{(a, b, c)}$ are the coordinates of the centroid of the triangle ABC, then show that the equation of the plane is $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 3$.

[8 Marks]

Solution: Let the co-ordinate of A $\equiv (h, 0, 0)$, B $= (0, k, 0)$ and C $\equiv (0, 0, l)$ then, equation of plane ABC is $\frac{x}{h} + \frac{y}{k} + \frac{z}{l} = 1$.



Now, (a, b, c) is the centroid of ΔABC then

$$a = \frac{h+0+0}{3}, b = \frac{0+k+0}{3}, c = \frac{0+0+l}{3}$$

α, $\quad h = 3a, k = 3b, l = 3c$
i.e. equation of the plane ABC can be rewritten as

$$\frac{x}{3a} + \frac{y}{3b} + \frac{z}{3c} = 1$$

or

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 3$$

Question-1(f) Find the equations of the spheres passing through the circle

$$x^2 + y^2 + z^2 - 6x - 2z + 5 = 0, y = 0$$

and touching the plane $3y + 4z + 5 = 0$.

[8 Marks]

Solution: The equation of the given circle is

$$\left.\begin{array}{r} x^2 + y^2 + z^2 - 6x - 2z + 5 = 0 \\ y = 0 \end{array}\right\} \cdots (1)$$

Equation of any sphere passing through the circle (I) is given by

$$x^2 + y^2 + z^2 - 6x - 2z + 5 + \lambda y = 0 \quad \cdots (2)$$

Centre of sphere (2) is $\left(3, -\frac{\lambda}{2}, 1\right)$ and radius of this sphere is $\sqrt{\frac{\lambda^2}{4} + 5}$.
Now, if the plane $3y + 4z + 5 = 0 \ldots (3)$ is a tangent plane to (2), then,

$$\frac{\left|3\left(\frac{-\lambda}{2}\right)+4+5\right|}{5} = \sqrt{\frac{\lambda^2+20}{4}}$$

$$\Rightarrow \left|\frac{9-\frac{3\lambda}{2}}{5}\right| = \sqrt{\frac{\lambda^2+20}{4}}$$

$$\Rightarrow \frac{3(6-\lambda)}{10} = \sqrt{\frac{\lambda^2+20}{4}}$$

$$\Rightarrow \frac{9(6-\lambda)^2}{100} = \frac{\lambda^2+20}{4}$$

$$\Rightarrow 9\left(\lambda^2 - 12\lambda + 36\right) = 25\left(\lambda^2 + 20\right)$$

$$\Rightarrow 25\lambda^2 + 500 = 9\lambda^2 - 108\lambda + 324$$

$$\Rightarrow 16\lambda^2 + 108\lambda + 176 = 0$$

$$\Rightarrow 4\lambda^2 + 27\lambda + 44 = 0$$

$$\Rightarrow 4\lambda^2 + 11\lambda + 16\lambda + 44 = 0$$

$$\Rightarrow \quad \lambda(4\lambda + 11) + 4(4\lambda + 11) = 0$$

$$\Rightarrow (\lambda + 4)(4\lambda + 11) = 0$$

$$\Rightarrow \lambda = -4 \quad \text{or,} \quad \lambda = -\frac{11}{4}$$

Hence, the equation of sphere is given by $x^2 + y^2 + z^2 - 6x - 2z + 5 - 4y = 0$ and $4\left(x^2 + y^2 + z^2 - 6x - 2z + 5\right) - 11y = 0$.

Question-2(a) Show that the following vectors form a basis for R^3

$$\alpha_1 = (1, 0, -1), \quad \alpha_2 = (1, 2, 1), \quad \alpha_3 = (0, -3, 2)$$

Find the components of $(1, 0, 0)$ w.r.t. the basis $\{\alpha_1, \alpha_2, \alpha_3\}$.

[10 Marks]

Solution: To show that $\alpha_1, \alpha_2, \alpha_3$, form a basis of $\mathbb{R}^3$. It is sufficient to show that they are linearly independent. i.e. $\exists\, ax, y, z \in \mathbb{R}$ such that

$$x\alpha_1 + y\alpha_2 + z\alpha_3 = (0, 0, 0)$$

then $x = y = z = 0$

$$x(1, 0, -1) + y(1, 2, 1) + z(0, -3, 2) = (0, 0, 0)$$

$$(x + y, 2y - 3z - x + y + 2z) = (0, 0, 0)$$

Comparing the co-efficients, we get,

$$\begin{aligned} x + y &= 0z \cdots (1) \\ 2y - 3z &= 0 \cdots (2) \\ -x + y + 2z &= 0 \cdots (3) \end{aligned}$$

(1) and (3) $\Rightarrow 2y + 2z = 0 \cdot (4)$
(2) and (4) $\Rightarrow 5z = 0$ or $z = 0$
$\Rightarrow y = 0$ i.e. $\quad x = y = z = 0$

Hence, $\{\alpha_1, \alpha_2, \alpha_3\}$ are linearly independent. Also dimension = 3, hence, they form a basis of $\mathbb{R}^3$.

Now, let $(1,0,0) = a\alpha_1 + b\alpha_2 + c\alpha_3$ then,

$$(1,0,0) = a(1,0,-1) + b(1,2,1) + c(0,-3,2)$$

$$\Rightarrow (1,0,0) = (a+b, 2b-3c, -a+b+2c)$$

$$\Rightarrow a+b=1, \quad 2b-3c=0 \quad -a+b+2c=0$$

$$\therefore \quad a+b=1 \Rightarrow a=(1-b)$$

$$\Rightarrow 2b = 3c \Rightarrow c = \tfrac{2}{3}b$$

$$-a+b+2c=0$$

$$\Rightarrow b-1+b+\tfrac{4}{3}b = 0$$

$$2b + \tfrac{4}{3}b = 1$$

$$\Rightarrow \tfrac{10b}{3} = 1$$

$$b = \tfrac{3}{10}$$

$$\therefore a = 1 - \tfrac{3}{10} = \tfrac{7}{10}$$

$$\therefore \mathrm{C} = \tfrac{2}{3} \cdot \tfrac{3}{10} = \tfrac{1}{5}$$

$$\therefore \quad (1,0,0) = \tfrac{7}{10}\alpha_1 + \tfrac{3}{10}\alpha_2 + \tfrac{1}{5}\alpha_3$$

Question-2(b) Find the characteristic polynomial of $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}$. Verify Cayley-Hamilton theorem for this matrix and hence find its inverse.

[10 Marks]

Solution: Let the given matrix be A= $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix}$

then, the characteristic equation of A is given by,

$$|A - \lambda.I| = 0$$

$$\Rightarrow \begin{bmatrix} -\lambda & 0 & 1 \\ 1 & -\lambda & 2 \\ 0 & 1 & 3-\lambda \end{bmatrix} = 0$$

$$\Rightarrow -\lambda \cdot \{\lambda(\lambda-3)-2\}+1(1)=0$$

$$\Rightarrow -\lambda\,(\lambda^2-3\lambda-2\}+1=0$$

$$\Rightarrow -\lambda^3+3\lambda^2+2\lambda+1=0$$

$$\Rightarrow \lambda^3-3\lambda^2-2\lambda-1=0$$

Now, by Cayley-Hamilton theorem, it should also satisfy the matrix A i.e.

$$A^3-3A^2-2A-I=0\cdots(1)$$

To prove the identity (1), we will calculate A^3 and A^2.

$$A^2=\begin{bmatrix}0&0&1\\1&0&2\\0&1&3\end{bmatrix}\begin{bmatrix}0&0&1\\1&0&2\\0&1&3\end{bmatrix}$$
$$A^2=\begin{bmatrix}0&1&3\\0&2&7\\1&3&11\end{bmatrix}$$

&

$$\mathrm{A}^3=\begin{bmatrix}0&1&3\\0&2&7\\1&3&11\end{bmatrix}\begin{bmatrix}0&0&1\\1&0&2\\0&1&3\end{bmatrix}$$
$$\mathrm{A}^3=\begin{bmatrix}1&3&11\\2&7&25\\3&11&39\end{bmatrix}$$

Now,

$$\mathrm{A}^3-3\ \mathrm{A}^2-2\ \mathrm{A}-\mathrm{I}=\begin{bmatrix}1&3&11\\2&7&25\\3&11&39\end{bmatrix}-3\begin{bmatrix}0&1&3\\0&2&7\\1&3&11\end{bmatrix}-2\begin{bmatrix}0&0&1\\1&0&2\\0&1&3\end{bmatrix}-\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$
$$=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}$$

Hence, Cayley-Hamilton theorem is verified.
Now,

$$A^3-3A^2-2A-I=0$$

$$\Rightarrow I=A^3-3A^2-2A$$

multiply both the sides by $\cdot\ \mathrm{A}^{-1}$, we get

$$A^{-1}=\begin{bmatrix}0&1&3\\0&2&7\\1&3&11\end{bmatrix}-3\begin{bmatrix}0&0&1\\1&0&2\\0&1&3\end{bmatrix}-\begin{bmatrix}2&0&0\\0&2&0\\0&0&2\end{bmatrix}$$
$$\Rightarrow A^{-1}=\begin{bmatrix}-2&1&0\\1&0&0\end{bmatrix}$$

Question-2(c) Let $A = \begin{pmatrix} 5 & -6 & -6 \\ -1 & .4 & 2 \\ 3 & -6 & -4 \end{pmatrix}$. **Find an invertible matrix P such that P^{-1} AP is a diagonal matrix.**

[10 Marks]

Solution: The such invertible matrix can be formed with the help of eigenvectors of matrix A. The characteristic equation of matrix is given by

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} -\lambda & -6 & -6 \\ -1 & 4-\lambda & 2 \\ 3 & -6 & -4-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (5-\lambda)((\lambda-4)(\lambda+4)+12\} +6\{\lambda+4-6\}-6\{6-3(4-\lambda)\} = 0$$

$$\Rightarrow 4 - 8\lambda + 5\lambda^2 - \lambda^3 = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda)^2 = 0$$

Hence, eigenvalues of matrix A is given by

$$\lambda = 1, 2, 2$$

Now, corresponding to $\lambda = 2$, the eigenvector is obtained through

$$[\mathrm{A}-2\mathrm{I}]\mathrm{X} = \begin{bmatrix} 5-2 & -6 & -6 \\ -1 & 2 & 2 \\ 3 & -6 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \propto \begin{bmatrix} 3 & -6 & -6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 - 2x_2 - 2x_3 = 0$

This imples that there are two free variables. Putting $x_2 = 0, x_3 = 1$, we get the eigenvector $[2, 0, 1]$ and by putting $x_2 = 1, x_3 = 0$, we get the eigenvector $[2, 1, 0]$.

Hence, the two eigenvectors corresponding to $i = 2$ are $[2, 0, 1]$ and $[2, 1, 0]$.

Now, the eigenvector corresponding to $\lambda = 1$ is given by

$$\begin{bmatrix} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \propto \begin{bmatrix} 1 & -3/2 & -3/2 \\ 1 & 3 & 2 \\ 3 & -6 & -5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Apply $R_1 \to \frac{1}{4}R_1$, $R_2 \to R_2 - R_1 \& R_3 \to R_3 - 3R_1$, we get:

$$\begin{bmatrix} 1 & -3/2 & -3/2 \\ 0 & 3/2 & 1/2 \\ 0 & -3/2 & -1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow 2x_1 - 3x_2 - 3x_3 = 0$$

$$\Rightarrow 3x_2 + x_3 = 0$$

There is only one free variable say $x_2 = 1$ then $\quad x_3 = -3$ & $\quad 2x_1 - 3 + 9 = 0 \quad x_1 = -3$

$\therefore \quad (-3, 1, -3)$

Hence, the invertible matrix P can be written as

$$P = \begin{bmatrix} 2 & 2 & -3 \\ 0 & 1 & 1 \\ 1 & 0 & -3 \end{bmatrix} \text{ and } |\mathrm{P}| = -6 + 2 + 3 = -1$$

$$\therefore \quad P^{-1} = -\begin{bmatrix} -3 & 1 & -1 \\ 6 & -3 & 2 \\ 5 & -2 & 2 \end{bmatrix}^{\top} = \begin{bmatrix} 3 & -6 & -5 \\ -1 & 3 & 2 \\ 1 & -2 & -2 \end{bmatrix}$$

Hence,

$$\mathrm{P}^{-1}\mathrm{AP} = \begin{bmatrix} 3 & -6 & -5 \\ -1 & 3 & 2 \\ 1 & -2 & -2 \end{bmatrix} \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix} \begin{bmatrix} 2 & 2 & -3 \\ 0 & 1 & 1 \\ 1 & 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

which is a diagonal matrix.

Question-2(d) Find the rank of the matrix

$$\begin{pmatrix} 1 & 2 & 1 & 1 & 2 \\ 2 & 4 & 3 & 4 & 7 \\ -1 & -2 & 2 & 5 & 3 \\ 3 & 6 & 2 & 1 & 3 \\ 4 & 8 & 6 & 8 & 9 \end{pmatrix}$$

[10 Marks]

Solution: The rank of any matrix is equal to number of non-zero rows in the echelon form of the given matrix. Now, Let

$$A = \begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 2 & 4 & 3 & 4 & 7 \\ -1 & -2 & 2 & 5 & 3 \\ 3 & 6 & 2 & 1 & 3 \\ 4 & 8 & 6 & 8 & 9 \end{bmatrix}$$

Apply $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 + R_1$, $\mathrm{R}_4 \to \mathrm{R}_4 - 3\mathrm{R}_1$ and $\mathrm{R}_5 \to \mathrm{R}_5 - 4\mathrm{R}_1$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 3 & 6 & 5 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & 2 & 4 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & -4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -5 \end{bmatrix}$$

Apply $\mathrm{R}_3 \to -\frac{1}{4}\mathrm{R}_3, \mathrm{R}_5 \to \mathrm{R}_5 - 5\mathrm{R}_3$, $\mathrm{R}_3 \to \mathrm{R}_3 - 3\mathrm{R}_2, \mathrm{R}_4 \to \mathrm{R}_4 + \mathrm{R}_2$ and $R_5 \to R_5 - 2R_2$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 1 & 1 & 2 \\ \hline 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

No. of non zero rows in echelon form = 3 i.e. Rank of the given matrix = 3.

Question-3(a) Discuss the convergence of the integral

$$\int_0^\infty \frac{dx}{1 + x^4 \sin^2 x}$$

[10 Marks]

Solution: Consider the integral $\mathrm{I} = \int_0^{\pi x} \frac{dx}{1+x^4 \sin^2 x} \propto$, $\quad I = \sum_{n=1}^{n} \int_{(r-1)x}^{n} \frac{dx}{1+x^4 \sin^2 x}$ Now for $\int_{(r-1)x}^{r} \frac{dx}{1+x^4 \sin^2 x}$.

Let $x = (r-1)\pi + y$ then $\quad dx = dy$

$\therefore$ Above integral reduces to

$$\int_0^\pi \frac{dy}{1 + [(r-1)\pi + y]^4 \sin^2[(r-1)\pi + y]} = \int_n^\pi \frac{dy}{1 + \{(r-1)\pi + y\}^4 \sin^2 y}$$

$$< \int_0^\pi \frac{dy}{1 + \{(r-1)\pi\}^4 \sin^2 y}$$

$$2\int_0^{\pi/2} \frac{\operatorname{cosec}^2 y dy}{\operatorname{cosec}^2 + (r-1)^4\pi^4} = 2\int_0^{\frac{\pi}{2}} \frac{\cos ec^2 y dy}{1 + (r-1)^4\pi^4 + \cos^2 y} \,|$$

$$= 2.\frac{1}{\sqrt{1 + (r-1)^4\pi^4}} \cot^{-1} \frac{\cot y}{\sqrt{1 + (r-1)^4\pi^4}}$$

$$= \frac{2}{\sqrt{1 + (r-1)^4\pi^4} \cdot \frac{\pi}{2}}$$

ie.

$$\int_{(r-1)x}^{n} \frac{dx}{1+x^4\sin^2\alpha} < \frac{\pi}{\sqrt{1+(r-1)^4\pi^4}}$$
$$= \frac{\pi}{(r-1)^2\pi^2} - \frac{1}{r^2\pi^2}$$

i.e.

$$\sum_{r=1}^{n} \int \frac{dx}{1+x^4\sin^2\alpha} < \sum_{r=1}^{n} \frac{1}{\pi^2 r^2}$$
$$\therefore \underset{n\to\infty}{Lt} \int_0^{n\pi} \frac{dx}{1+x^4\sin^2\alpha} < \sum \frac{1}{r^2}$$

which is convergent. Hence,

$$\int_0^{\infty} \frac{dx}{1+x^4\sin^2 x}$$

is convergent.

Question-3(b) Find the extreme value of xyz if $x+y+z=a$.

[10 Marks]

Solution: Define a Lagrangian function $\mathrm{F}(x,y,z,\lambda) = xyz + \lambda(x+y+z-a)$
Then for extremum value

$$d\,\mathrm{F} = 0$$
$$\Rightarrow yzdx + xzdy + xydz + \lambda(dx+dy+dz) = 0$$
$$\Rightarrow (yz+\lambda)dx + (xz+\lambda)dy + (xy+\lambda)dz = 0$$

Equating the co-efficients, we get

$$yz+\lambda = 0; \quad xz+\lambda = 0; \quad xy+\lambda = 0$$
$$yz+\lambda - xz - \lambda = 0$$
$$\Rightarrow z(x-y) = 0$$
$$\Rightarrow z = 0 \text{ or } x = y$$

However, $z = 0 \Rightarrow \lambda = 0$ which further led to

$$x = y = 0$$

Hence, $x - y$ is the acceptable solution.
Similarly from $xz+\lambda = 0$ and $xy+\lambda = 0$ we get

$$y = z \text{ i.e. } x = y = z$$

is the condition for extremum of Lagrangian function.

Also,

$$x + y + z = 0 \Rightarrow 3x = a$$

or

$$x = y = z = \frac{a}{3}$$

Hence, the extremum value of

$$xyz = \frac{a^3}{27}$$

Question-3(c) Let

$$f(x,y) = \begin{cases} \dfrac{xy\left(x^2 - y^2\right)}{x^2 + y^2}, & \text{if } (x,y) \neq (0,0) \\ 0, & \text{if } (x,y) = (0,0) \end{cases}$$

Show that: $(i) f_{xy}(0,0) \neq f_{yx}(0,0)$
(ii)f **is differentiable at** $(0,0)$.

[10 Marks]

Solution:

$$f_{xy}(0,0) = \underset{k \to 0}{Lt} \frac{f_x(0,k) - f_x(0,0)}{k}$$

$$f_{yx}(0,0) = \underset{k \to 0}{Lt} \frac{f_x(h,0) - f_x(0,0)}{h}$$

Now,

$$\begin{aligned} f_x(0,k) &= \underset{h \to 0}{Lt} \frac{f(h,k) - f(h,0)}{h} \\ &= \underset{h \to 0}{Lt} \frac{\frac{hk\left(h^2 - k^2\right)}{h^2 + k^2} - 0}{h} \\ &= \underset{h \to 0}{Lt} \frac{k\left(h^2 - k^2\right)}{h^2 + k^2} \\ &= -k \end{aligned}$$

$$\begin{aligned} f_x(0,0) &= \underset{h \to 0}{Lt} \frac{f(h,0) - f(0,0)}{h} \\ &= \underset{h \to 0}{Lt} \frac{0}{h} \\ &= 0 \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad f_{xy}(0,0) &= \underset{h \to 0}{Lt} \frac{f_x(0,k) - f_x(0,0)}{k} \\ &= \underset{k \to 0}{Lt} - \frac{k - 0}{k} \\ &= -1 \end{aligned}$$

Also,

$$f_y(h,0) = \underset{k\to 0}{Lt}\frac{f(h,k)-f(h,0)}{k}$$
$$= \underset{k\to 0}{Lt}\frac{h\frac{k\left(h^2-k^2\right)}{h^2+k^2}-0}{k}$$
$$= h$$

$$f_y(0,0) = \underset{k\to 0}{Lt}\frac{f_y(h,0)-f_y(0,0)}{k}$$
$$= \underset{k\to 0}{Lt}\frac{0-0}{k}$$
$$= 0$$

$$\therefore \quad f_{yx}(0,0) = \underset{h\to 0}{Lt}\frac{f_y(h,0)-f_y(0,0)}{h}$$
$$= \underset{h\to 0}{Lt}\frac{h-0}{h}$$
$$= 1$$

i.e.$f_{yx}(0,0) = 1$ also $f_{xy}(0,0) = -1$

Hence,$f_{yx}(0,0) \neq f_{xy}(0,0)$.
Further, $f_x(0,0) = 0 = f_y(0,0)$

Also, when $x^2+y^2 \neq 0$, then

$$|f_x| = \frac{|x^4y+4x^2y^3-y^5|}{\left(x^2+y^2\right)^2}$$
$$\leq \frac{6\left(x^2+y^2\right)^{5/2}}{\left(x^2+y^2\right)^2}$$
$$= 6\left(x^2+y^2\right)^{1/2}$$

Evidently,

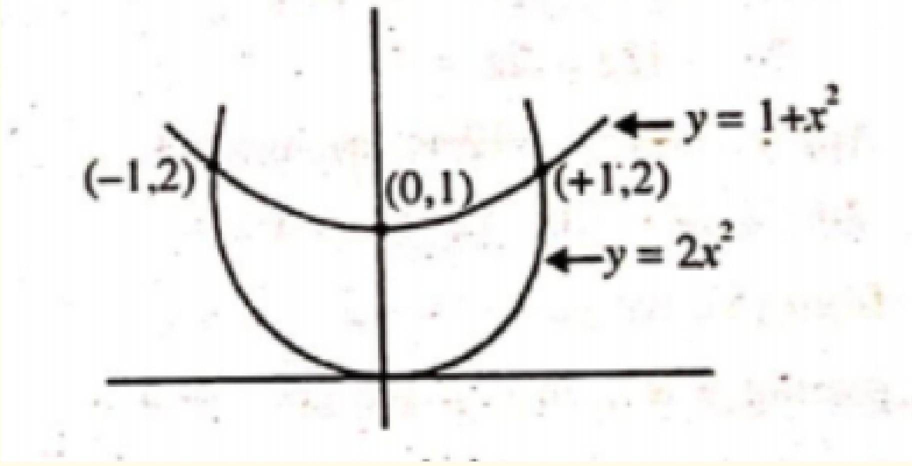
$$\lim_{(x,y)\to(0,0)} f_x(x,y) = f_x(0,0)$$

Thus, f_x is continuous at (0,0) and $f_y(0,0)$ exists $\Rightarrow f$ is differentiable at $(0,0)$.

Question-3(d) Evaluate $\iint_D (x+2y)dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1+x^2$.

[10 Marks]

 CHAPTER 11. 2010



Solution:
We have to calculate

$$\iint (x+2y)dA = \cdot \int_{y=0-1}^{1} \int_{y=0}^{1} (x+2y)dxdy$$
$$= \int_{y=0}^{1} \frac{x^2}{2} + 2xy|' \, dy$$
$$= 4\int_{y=0}^{1} ydy = 4 \times \frac{1}{2}$$
$$= 2 \text{ units}$$

Question-4(a) Prove that the second degree equation represents a cone

$$x^2 - 2y^2 + 3z^2 + 5yz - 6zx - 4xy + 8x - 19y - 2z - 20 = 0$$

whose vertex is $(1, -2, 3)$.

[10 Marks]

Solution: The given equation is

$$f(x, y, z) = x^2 - 2y^2 + 3z^2 + 5yz - 6zx - 4xy + 8x - 19y - 2z - 20 = 0$$

Making homogeneous with the help of new variable t, to calculate the vertex of cone. i.e.

$$\mathrm{F}(x, y, z, t) = x^2 - 2y^2 + 3z^2 + 5yz - 6z - 4xy + 8xt - 19xt - 2z - 20t^2 = 0$$

Now, differentiating partially with respect to x, y, z, t and then putting $t = 1$, we get,

$$F_x = 2x - 6z - 4y + 8 = 0$$

$$\Rightarrow x - 2y - 3z + 4 = 0$$

$$F_y = -4y + 5z - 4x - 19 = 0$$

$$\Rightarrow 4x + 4y - 5z + 19 = 0$$

$$F_z = 6z - 6x + 5y - 2 = 0$$

$$\Rightarrow 6x - 5y - 6z + 2 = 0$$

$$F_t = 8x - 19y - 2z - 40 = 0$$

$$8x - 19y - 2z - 40 = 0$$

Now, if $f(x, y, z) = 0$ represent a cone the value of $x, y; z$ obtained from solving (1),(2) and (3) should satisfy (4) and that value represent the vertex of the cone.

Apply $(2) - 4 \times (1)$, we get

$$12y + 7z + 3 = 0$$

Apply $(3) - 6 \times (1)$ we get

$$7y + 12z - 2z = 0$$

Apply $7 \times (5) - 12 \times (6)$, we get,

$$-95z + 285 = 0 \Rightarrow z = 3$$

from (5), we get,

$$y = -2$$

putting y & z in (1), we get $x = 1$ i.e.

$$(x, y, z) = (1, -2, 3)$$

Now putting this in (4) we get,

$$8 + 38 - 6 - 40 = 0$$

Hence, the given second degree equation represent a cone with vertex $(1, -2, 3)$

Question-4(b) If the feet of three normals drawn from a point P to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ lie in the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, prove that the feet of the other three normals lie in the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} + 1 = 0$.

[10 Marks]

Solution: Let the co-ordinates of the given point be (x_1, y_1, z_1). Now the co-ordinates (α, β, γ) of the feet of six normals from (x_1, y_1, z_1) to given ellipsoid are given by:

$$\alpha = \frac{a^2 x_1}{a^2 + \lambda}, \beta = \frac{b^2 y_1}{b^2 + \lambda}, \gamma = \frac{c^2 z_1}{c^2 + \lambda}$$

where λ is a parameter.
Now, (α, β, γ) lies on ellipsoid.

$$\Rightarrow \frac{a^2 x_1^2}{(a^2 + \lambda)^2} + \frac{b^2 y_1^2}{(b^2 + \lambda)^2} + \frac{c^2 z_1^2}{(c^2 + \lambda)^2} = 1 \; ldots(1)$$

which gives six values of $\lambda^{\cdot}$

Now, if three of six lie on plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ then

$$\frac{ax_1}{a^2+\lambda}+\frac{by_1}{b^2+\lambda}+\frac{cz_1}{c^2+\lambda}-1=0\ldots(2)$$

(satisfied by three value of λ).

Let the other three feet lie on

$$\frac{x}{a'}+\frac{y}{b'}+\frac{z}{c'}-p'=0$$

then

$$\frac{a^2x_1}{a'\left(a'+\lambda\right)}+\frac{b^2y_1}{b'\left(b^2+\lambda\right)}+\frac{c^2z_1}{c'\left(c^2+\lambda\right)}-p'=0\ldots(3)$$

(2) and (3) in combined form represent a conic passing through the feet of six normals, which is represented by equation (1) also.
Comparing coefficients, we get

$$\frac{a^3}{a'\left(a^2+\lambda\right)^2}=\frac{a^2}{\left(a^2+\lambda\right)^2}$$

$$\Rightarrow\frac{1}{a'}=\frac{1}{a}$$

Similarly $b'=b$, $c'=c$ and $\quad p'=-1$
$\Rightarrow$ The equation of other plane is given by:

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}+1=0$$

Question-4(c) If $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ represents one of the three mutually perpendicular generators of the cone $5\mathrm{yz}-8\mathrm{zx}-3\mathrm{xy}=0$, find the equations of the other two.

[10 Marks]

Solution: Let

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{h}$$

represent one of other two generator as this is perpendicular to given generator

$$\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$$

. Hence,

$$l+2m+3n=0$$

Also

$$5mn-8\ln-3lm=0$$

$$\Rightarrow 5mn-l(3m+8n)=0$$

$$\Rightarrow 5mn + (2m+3n)(3m+8n) = 0 < \text{using } (1) >$$

$$\Rightarrow 6m^2 + 30mn + 24n^2 = 0$$

$$\Rightarrow m^2 + 5mn + 4n^2 = 0$$

$$\Rightarrow m^2 + mn + 4mn + 4n^2 = 0$$

$$\Rightarrow m(m+n) + 4n(m+n) = 0$$

$$\Rightarrow (m+n)(m+4n) = 0$$

$$m + n = 0 \Rightarrow \frac{m}{1} = \frac{n}{-1}$$

$$1 + 2 - 3 = 0 \Rightarrow l = 1$$

then, i.e. $\frac{x}{1} = \frac{y}{1} = \frac{z}{-1}$ represent one generator if $m + 4n = 0$, then, $\frac{m}{-4} = \frac{n}{1}$ then, $l - 8 + 3 = 0 \Rightarrow l = 5 \Rightarrow \frac{x}{5} = \frac{y}{-4} = \frac{z}{1}$ represent other generator.

Hence, the equation of two other generators are

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{-1}$$

&

$$\frac{x}{5} = \frac{y}{-4} = \frac{z}{1}$$

Question-4(d) Prove that the locus of the point of intersection of three tangent planes to the ellipsoid $\frac{\mathrm{x}^2}{\mathrm{a}^2} + \frac{\mathrm{y}^2}{\mathrm{b}^2} + \frac{\mathrm{z}^2}{\mathrm{c}^2} = 1$, which are parallel to the conjugate diametral planes of the ellipsoid $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = 1$ is $\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = \frac{a^2}{\alpha^2} + \frac{b^2}{\beta^2} + \frac{c^2}{\gamma^2}$.

[10 Marks]

Solution: Let (x_1, y_1, z_1) (x_2, y_2, z_2) & (x_3, y_3, z_3) be the end point of conjugate diametrical planes of ellipsoid $\frac{x^2}{\alpha^1} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = 1$ then equation of plane parallel to these conjugate diameterical planes are given by,

$$\frac{xx_1}{\alpha^2} + \frac{yy_1}{\beta^2} + \frac{zz_1}{\gamma^2} = d_1; \quad \frac{xx_2}{\alpha^2} + \frac{yy_2}{\beta^2} + \frac{zz_2}{\gamma^2} = d_2; \quad \text{and} \frac{xx_3}{\alpha^2} + \frac{yy_3}{\beta^2} + \frac{zz_3}{\gamma^2} = d_3$$

Now, three planes are tangent planes to

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

then, by the properties of tangent planes.

$$\frac{a^2x_1^2}{\alpha^4}+\frac{b^2y_1^2}{\beta^4}+\frac{c^2z_1^2}{\gamma^4}=d_1^2\frac{a^2x_2^2}{\alpha^4}+\frac{b^2y_2^2}{\beta^4}+\frac{c^2z_2^2}{\gamma^4}$$
$$=d_2^2\frac{a^2x_3^2}{\alpha^4}+\frac{b^2y_3^2}{\beta^4}+\frac{c^2z_3^2}{\gamma^4}$$
$$=d_3^2$$

adding above three equation we get,

$$\frac{a^2}{\alpha^4}\left(x_1^2+x_2^2+x_3^2\right)+\frac{b^2}{\beta^4}\left(y_1^2+y_2^2+y_3^2\right)+\frac{c^2}{\gamma^4}\left(z_1^2+z_2^2+z_3^2\right)=d_1^2+d_2^2+d_3^2$$

$$\Rightarrow \tfrac{a^2}{\alpha^4}\alpha^2+\tfrac{b^2}{\beta^4}\beta^2+\tfrac{c^2}{\gamma^4}\gamma^2=d_1^2+d_2^2+d_3^2$$

(By properties of conjugate diametrical planes)

$$\Rightarrow \tfrac{a^2}{\alpha^2}+\tfrac{b^2}{\beta^2}+\tfrac{c^2}{\gamma^2}=d_1^2+d_2^2+d_3^2$$

Also,

$$\left(\frac{xx_1}{\alpha^2}+\frac{yy_1}{\beta^2}+\frac{zz_1}{\gamma^2}\right)^2+\left(\frac{xx_2}{\alpha^2}+\frac{yy_2}{\beta^2}+\frac{zz_2}{\gamma^2}\right)^2+\left(\frac{xx_1}{\alpha^2}+\frac{yy_1}{\beta^2}+\frac{zz_3}{\gamma^2}\right)^2=d_1^2+d_2^2+d_3^2$$

$$\Rightarrow \tfrac{x^2}{\alpha^4}\Sigma x_1^2+\tfrac{y^2}{\beta^4}\Sigma y_1^2+\tfrac{z^2}{\gamma^4}\Sigma z_1^2=d_1^2+d_2^2+d_3^2$$

(other term of equation vanishes)

$$\Rightarrow \tfrac{x^2}{\alpha^4}\alpha^2+\tfrac{y^2}{\beta^4}\beta^2+\tfrac{z^2}{\gamma^4}\gamma^2=\tfrac{a^2}{\alpha^2}+\tfrac{b^2}{\beta^2}+\tfrac{c^2}{\gamma^2}$$

$$\Rightarrow \tfrac{x^2}{\alpha^2}+\tfrac{y^2}{\beta^2}+\tfrac{z^2}{\gamma^2}=\tfrac{a^2}{\alpha^2}+\tfrac{b^2}{\beta^2}+\tfrac{c^2}{\gamma^2}$$

which is the locus of the point of intersection of tangent planes of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}+1$$

11.2 Section-B

Question-5(a) Show that $\cos(\mathrm{x}+\mathrm{y})$ **is an integrating factor of**

$$ydx+[y+\tan(x+y)]dy=0$$

Hence solve it.

[8 Marks]

Solution: The given differential equation is

$$ydx+[y+\tan(x+y)]dy=0 \quad \cdots(1)$$

Now, if $\cos(x+y)$ is an I.F. of the above equation, then it should reduce it into exact form.

$$y\cos(x+y)dx + \left[\begin{array}{l} y\cos(x+y) \\ +\sin(x+y)]dy = 0 \end{array}\right.$$

Now, if it is exact then $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ where, $M = y\cos(x+y)$

$$\begin{aligned} \text{N} &= y\cos(x+y) + \sin(x+y) \\ \frac{\partial M}{\partial y} &= \cos(x+y) - y\sin(x+y) \\ \frac{\partial N}{\partial x} &= -y\sin(x+y) + \cos(x+y) \end{aligned}$$

i.e. (1) becomes exact after multiplication by $\cos(x+y)$

Hence, solution of the equation is given by

$$\int y\cos(x+y)dx + \int \{y\cos(x+y) + \sin(x+y)\}dy$$

$$y\sin(x+y) + 0 = c$$

as there is no term independent of x is contained in second integral.

Question-5(b) Solve:

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \sin x$$

[8 Marks]

Solution: For complementary function, the auxiliary equation is given by

$$m^2 - 2m + 1 = 0$$

$$\Rightarrow \quad m = 1, 1$$

Hence, complementary function

$$y = (c_1 + c_2 x)\, e^x$$

where, c_1, c_2 are arbitrary constants.
Now, the particular integral is given by,

$$\begin{aligned} y &= \frac{1}{(D-1)^2} xe^x \sin x \\ &= e^x \cdot \frac{1}{(D+1-1)^2} x\sin x \\ &= e^x \frac{1}{D^2} x\sin x = e^x \frac{1}{D} \int x\sin x dx \\ &= e^x \frac{1}{D}[-x\cos x + \sin x] \\ &= e^x \left[\int (\sin x - x\cos x)dx\right] \\ &= e^x[-\cos x - \{x\sin x + \cos x\}] \\ &= -xe^x \sin x - 2\cos x \end{aligned}$$

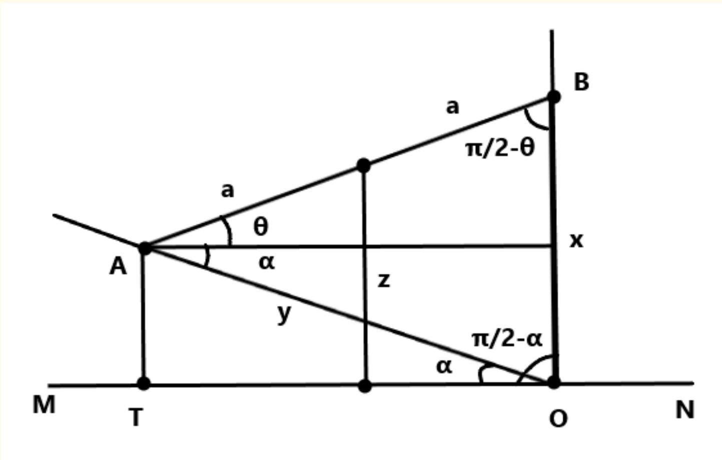
Hence, General solution is given

$$y = (c_1 + c_2 x)\, e^x - x e^x \sin x - 2\cos$$

Question-5(c) A uniform rod AB rests with one end on a smooth vertical wall and the other on a smooth inclined plane, making an angle α with the horizon. Find the positions of equilibrium and discuss stability.

[8 Marks]

Solution: Let rod AB is resting with one end on inclined plane AO and other end on smooth wall BO.
Let $AO = y, \quad BO = x, \quad AB = 2a.$



In triangle ABO,

$$\frac{2a}{\sin\left(\frac{\pi}{2} - \alpha\right)} = \frac{x}{\sin(\theta + \alpha)} = \frac{y}{\sin\left(\frac{\pi}{2} - \theta\right)}$$

$$\frac{2a}{\cos\alpha} = \frac{x}{\sin(\theta + \alpha)} = \frac{y}{\cos\theta}$$

$$\therefore \quad x = \frac{2a\sin(\theta + \alpha)}{\cos\alpha}; y = \frac{2a\,\mathrm{cs}\,\theta}{\cos\alpha}$$

$z =$ height of centre of gravity
of rod AB° from fined plane mN

$$z = \frac{1}{2}[AT + BO] = \frac{1}{2}[y\sin\alpha + x]$$

$$= \frac{1}{2}\left[\frac{2a\,\mathrm{cs}\,\theta \cdot \sin\alpha}{\mathrm{cs}\,\alpha} + \frac{2a\sin(\theta + \alpha)}{\mathrm{cs}\,\alpha}\right)$$

$$z = \frac{a}{\cos\alpha}[-\cos\theta - \sin\alpha + \sin(-\theta + \alpha)]$$

$$= \frac{a}{\cos\alpha}[\sin\theta - \cos\alpha + 2 - \cos\theta - \sin\alpha]$$

For stability,

$$\frac{dz}{d\theta} = -0$$

$$\frac{a}{\cos\alpha[\cos\theta - \cos\alpha - 2\sin\theta - \sin\alpha]} = 0$$

i.e.

$$\cos\theta - \cos\alpha = 2\sin\theta sin\alpha$$

$$=| \tan\theta = \frac{1}{2}\cot\alpha \quad \ldots \quad (1)$$

$$\frac{dz}{d\theta} = \frac{a}{\cos\alpha}[\cos\theta \cdot \cos\alpha - 2\sin\theta\sin\alpha]$$

$$\begin{aligned}\frac{d^2z}{d\theta^2} &= \frac{a}{\cos\alpha}[-\sin\theta\cos\alpha - 2\cos\theta\sin\alpha] \\ &= -\frac{a}{\cos\alpha}(\sin\theta\cos\alpha + 2\cos\theta\sin\alpha) \\ &= a \text{ negative quantity because } \theta \text{ and } \alpha \text{ are acute angles.}\end{aligned}$$

Thus, in the position of equilibrium, given by condition (1),

$$\frac{d^2z}{d\theta^2}$$

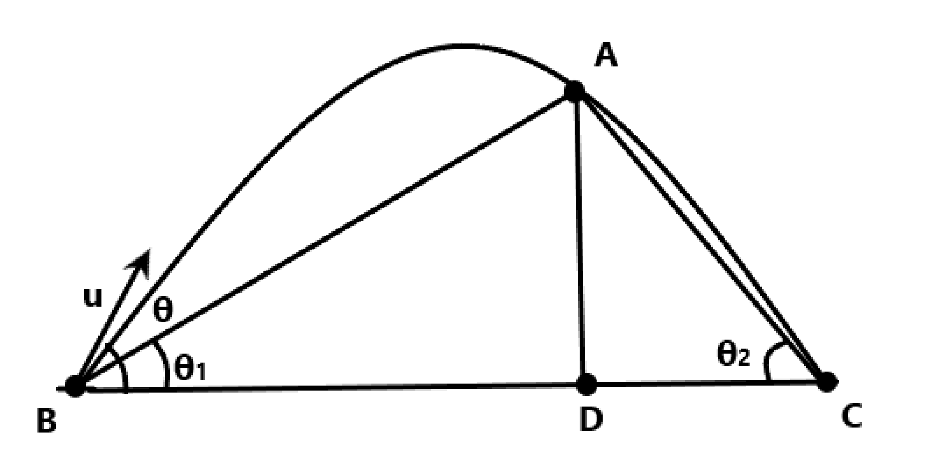
is negative which means z is maximum.
Hence, the equilibrium is unstable.

Question-5(d) A particle is thrown over a triangle from one end of a horizontal base and grazing the vertex falls on the other end of the base. If θ_1 and θ_2 be the base angles and θ be the angle of projection, prove that,

$$\tan\theta = \tan\theta_1 + \tan\theta_2$$

[8 Marks]

Solution: Given:
1)$\angle ABC = \theta_1$
2) $\angle ACB = \theta_2$
3) Angle of projection $= \theta_3$



Let the initial velocity be $'u'$ and $AD = h$

$$\Rightarrow \tan\theta_1 = \frac{AD}{BD} \Rightarrow BD = h\cot\theta_1 \cdots (1)$$

Again,

$$\tan\theta_2 = \frac{AD}{CD}$$
$$CD = h\cot\theta_2 \cdots (2)$$
$$BC = BD + CD \cdots (3)$$

Putting (1) and (2) in (3) , we get,

$$BC = h \cdot [cot\theta_1 + [\cot\theta_2] \cdots (4)$$

Thus the range of the projectile is given in equation (4) , that is BC

Now, Range, $R = \frac{u^2 \sin 2\theta}{g}$ – (5) where g= gravitational acceleration Using (4) and (5),

$$h\left[(\cot\theta_1 + \cot\theta_2)\right] = \frac{u^2}{g}\sin 2\theta$$
$$\Rightarrow \frac{u^2}{g} = h \cdot \frac{[(\cot\theta + cot\theta_2)]}{\sin 2\theta} \cdots (6)$$

At any instant $'t'$, equation of projectile is given as:

$$y = -u\sin\theta t - \frac{1}{2}yt^2 \quad \text{and } x = \text{u}\cos\theta$$

$$\Rightarrow \quad y = x\tan\theta - \frac{1}{2}g\frac{x^2}{u^2\cos^2\theta}...(7)$$

Using (6) in (y) we get:

$$y = x\tan\theta - \frac{sin2\theta.x^2}{2h\left[\cot\theta_1 + \cot\theta_2\right]\cdot\cos^2\theta}....(8)$$

At the point A, $\quad x = h\cdot\cot\theta_1 \quad$ and $\quad y = h$
Hence, putting these values in (8) we get,

$$h = h\cot\theta_1\tan\theta - \frac{2\sin\theta\cos\theta}{2h\cos^2\theta}\cdot\frac{h^2\cot^2\theta_1}{[\cot\theta_1 + \cot\theta_2]}$$

$$1 = \cot\theta_1 \tan\theta - \frac{\tan\theta \cot^2\theta_1}{[60 + \theta_1 + (0 + \theta)}$$

$$1 = \tan\theta \mid \frac{\cot\theta_1 \cot\theta_2}{\cot\theta + \cot\theta}\Big]$$

$$\Rightarrow \quad \tan\theta = \frac{[\cot\theta_1 + \cot\theta_2]}{\cot\theta \cot\theta_2}$$

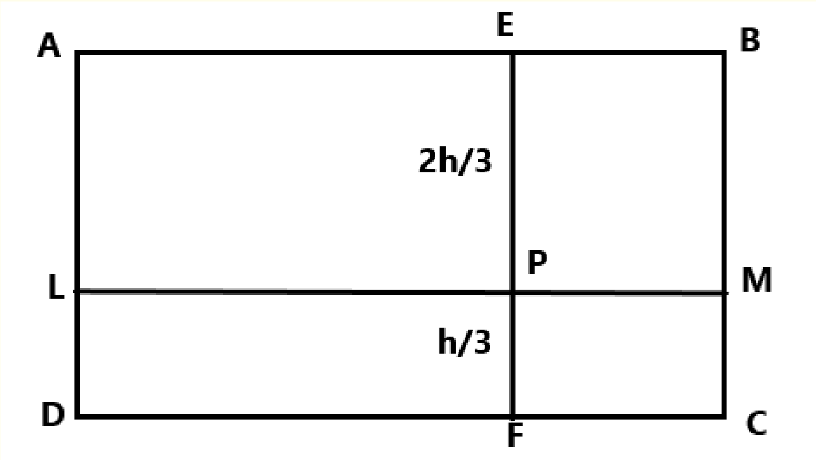
$$\therefore \tan\theta = \tan\theta_1 + \tan\theta_2$$

Hence proved.

Question-5(e) Prove that the horizontal line through the centre of pressure of a rectangle immersed in a liquid with one side in the surface, divides the rectangle in two parts, the fluid pressure on which, are in the ratio, $4 : 5$.

[8 Marks]

Solution: Let LM be the horizontal line through P, the centre of pressure of rectangle $ABCD$ is immersed in liquid with the side AB in the surface.



Let

$$AB = a$$

and

$$AD = h \Rightarrow EP = 2/3h$$

$$\begin{aligned}
P &= \text{Pressure on area } ABCD\\
&= w\cdot(\text{Area } ABCD)\cdot(\text{depth of its } C.G. \text{ below the free surface}))\\
&= w\cdot(ah)\left(\frac{h}{2}\right)\\
&= \frac{1}{2}wah^2\\
P_1 &= \text{presure on area } ALMB\\
&= w-(\text{Area } ALMB)\cdot(\text{depth of its} C.G.\text{below the free surface})\\
&= -w\cdot\left(a\cdot\frac{2}{3}h\right)\mid\left(\frac{1}{2}\cdot\frac{2}{3}h\right)\\
&= \frac{2}{9}wah^2\\
P_2 &= \text{ Pressure on area } LDCM\\
&= P-P_1\\
&= w\,\mathrm{ah}^2\left(\frac{1}{2}-\frac{2}{9}\right)\\
&= \frac{5}{18}wah^2
\end{aligned}$$

$$\therefore\ \frac{P_1}{P_2}=\frac{2}{9}\times\frac{18}{5}\cdot\frac{wah^2}{wah^2}=\frac{4}{5}$$

Question-5(f) Find the directional derivate of $\overrightarrow{\mathrm{V}}^2$, where, $\overrightarrow{\mathrm{V}}=\mathrm{xy}^2\overrightarrow{\mathrm{i}}+\mathrm{zy}^2\overrightarrow{\mathrm{j}}+\mathrm{xz}^2\overrightarrow{\mathrm{k}}$ at the point $(2,0,3)$ in the direction of the outward normal to the surface $x^2+y^2+z^2=14$ at the point $(3,2,1)$.

[8 Marks]

Solution: The unit normal vector at point (3,2,1) of the surface

$$x^2+y^2+z^2=14$$

is given by

$$\frac{3\hat{i}+2\hat{j}+\hat{k}}{\sqrt{14}}=\hat{n}(\text{say})$$

Now,

$$\overline{\mathrm{V}}=xy^2\hat{i}+zy^2\hat{j}+xz^2\hat{k}$$

then,

$$\overline{\mathrm{v}}^2=\left(x^2y^4+z^2y^4+x^2z^4\right)$$

then,

$$\nabla\overline{\mathrm{V}}^2=\left(2xy^4+2xz^4\right)\hat{i}+\left(4x^2y^3+4y^3z^2\right)\hat{j}+\left(2y^4z+4x^2z^3\right)\hat{k}$$

Hence, required directional derivative at point (2,0,3) is given by:

$$\left[\left(2xy^4+2xz^4\right)\hat{i}+\left(4x^2y^3+4y^3z^2\right)\hat{j}+\left(2y^4z+4x^2z^3\right)\hat{k}\right]\cdot\left[\frac{3\hat{i}+2\hat{j}+\hat{k}}{\sqrt{14}}\right]$$

$$= \frac{81 \times 4 \times 3 + 16 \times 27 \times 4}{\sqrt{14}}$$
$$= \frac{2700}{\sqrt{14}}$$

Question-6(a) Solve the following differential equation

$$\frac{dy}{dx} = \sin^2(x - y + 6)$$

[8 Marks]

Solution: Let $z = x - y + 6$ then,

$$\frac{dz}{dx} = 1 - \frac{dy}{dx}$$

or,

$$\frac{dy}{dx} = 1 - \frac{dz}{dx}$$
$$1 - \frac{dz}{dx} = \sin^2 z$$

or,

$$\frac{dz}{dx} = \cos^2 z$$

or,

$$\sec^2 z dz = dx$$

After integrating, we get:

$$\tan z = x + c$$

or,

$$\tan(x - y + 6) = x + c$$

where $c =$ arbitrary constant.

Question-6(b) Find the general solution of

$$\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} + \left(x^2 + 1\right) y = 0$$

[12 Marks]

Solution: The above equation is solved by reducing it to normal form. i.e. (removal of 1st derivative). Let, $y = uv$ be the solution of above equation then. The above equation can be reduced to

$$\frac{d^2v}{dx^2}+\left(P+\frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx}+\left(\frac{d^2u}{dx^2}+2x\frac{du}{dx}+u\right)v=0\ldots(1)$$

Now, to remove 1st derivative, we should equate

$$P+\frac{2}{u}\frac{du}{dx}=0$$

or

$$\frac{du}{u}+xdx=0$$

then, (1) is reduced to

$$\frac{d^2v}{dx^2}+\mathrm{I}v=0$$

where,

$$I=Q-\frac{1}{4}P^2-\frac{1}{2}\frac{dp}{dx}$$
$$Q=\left(x^2+1\right),\quad P=2x$$
$$1=\left(x^2+1\right)-x^2-1=0$$
$$\therefore\frac{d^2v}{dx^2}=0\Rightarrow v=(c_1+c_2x)$$

where, c_1 and c_2 are arbitrary constant Hence,

$$y=(c_1+c_2x)\,e^{-x^{1/2}}$$

$$y=c_1e^{-x^2/2}+c_2xe^{-x^2/2}$$

is the general solution of the given equation.

Question-6(c) Solve

$$\left(\frac{d}{dx}-1\right)^2\left(\frac{d^2}{dx^2}+1\right)^2y=x+e^x$$

[10 Marks]

Solution: The complementary function is given by

$$y=(c_1+c_zx)\,e^x+(c_3+c_4x)\sin x+(c_5+c_6x)\cos x$$

The particular integral is given by:

$$\begin{aligned}
y &= \frac{1}{(D-1)^2\left(D^2+1\right)^2}\left(x+e^x\right) \\
&= \frac{1}{(1-D)^2\left(1+D^2\right)^2}x + \frac{1}{(D-1)^2\left(D^2+1\right)^2}e^x \\
&= \left[1+2D+3D^2+\cdots\right]\left(1-2D^2+3D^4-\cdots\cdots\right)x + \frac{x^2e^x}{2.4} \\
&= (x+2) + \frac{x^2e^x}{8}
\end{aligned}$$

Hence, the general solution is given by

$$y = \left(c_1+c_2x\right)e^x + \left(c_3+c_4x\right)\sin x + \left(c_5+c_6x\right)\cos x + (x+2) + \frac{x^2e^x}{8}$$

Question-6(d) Solve by the method of variation of parameters the following equation

$$\left(x^2-1\right)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = \left(x^2-1\right)^2$$

[10 Marks]

Solution: The above equation can be written as

$$y_2 - \frac{2x}{x^2-1}y_1 + \frac{2}{x^2-1}y = \left(x^2-1\right) \quad \ldots(1)$$

Clearly, x and x^2+1 is solution of reduced differential equation (i.e. making right hand side to zero).
Let, $y = \mathrm{A}x + \mathrm{B}\left(x^2+1\right)$ be the solution of (1) where A and B are function of x. put a condition $A_1x + B_1\left(x^2+1\right) = 0$.
Now,

$$\begin{gathered}
y = \mathrm{Ax} + \mathrm{B}\left(x^2+1\right), \\
y_1 = \mathrm{A} + 2\,\mathrm{B}x, \\
y_2 = \mathrm{A}_1 + 2\,\mathrm{B}_1x + 2\,\mathrm{B}
\end{gathered}$$

Putting y, y_1 & y_2 in equation (1), we get

$$A_1 + 2B_1x + 2B - \frac{2x}{x^2-1}(A+2Bx) + \frac{2}{x^2-1}\left[\mathrm{Ax} + \mathrm{B}\left(x^2+1\right)\right] = \left(x^2-1\right)$$

or,

$$\mathrm{A}_1 + 2\,\mathrm{B}_1x = x^2 - 1$$

also,

$$\mathrm{A}_1x + \mathrm{B}_1\left(x^2+1\right) = 0$$

or,

$$\mathrm{B}_1\left(2x^2 - x^2 - 1\right) = x\left(x^2-1\right)$$

or,

$$B_1 = x \quad \Rightarrow \quad B = \frac{x^2}{2} + c_1$$

also,

$$A_1 + 2x^2 = x^2 - 1$$
$$\Rightarrow A_1 = -\left(x^2 + 1\right)$$
$$\therefore A = -\frac{x^3}{3} - x + c_2$$

$$\begin{aligned}
\therefore y &= \text{Ax} + \text{B}\left(x^2 + 1\right) \\
&= \left(c_2 - x - \frac{x^3}{3}\right)\dot{x} + \left(\frac{x^2}{2} + c_1\right)\left(x^2 + 1\right) \\
&= c_1\left(x^2 + 1\right) + c_2 x - x^2 - \frac{x^4}{3} + \frac{x^4}{2} + \frac{x^2}{2} \\
&= c_1\left(x^2 + 1\right) + c_2 x - \frac{x^2}{2} + \frac{x^4}{6}
\end{aligned}$$

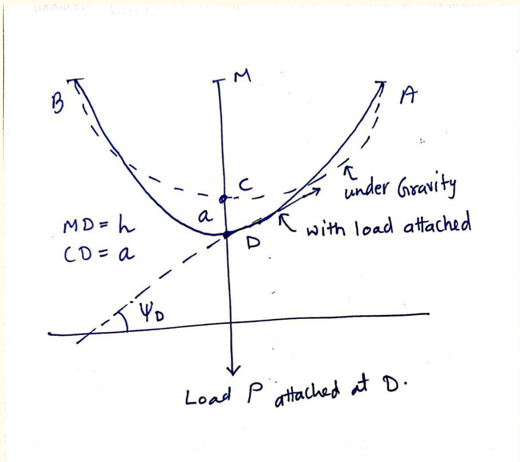
i.e. the general solution is

$$y = c_1\left(x^2 + 1\right) + c_2 x - \frac{x^2}{2} + \frac{x^4}{6}$$

Question-7(a) A uniform chain of length $2l$ and weight W, is suspended from two points A and B in the same horizontal line. A load P is now hung from the middle point D of the chain and the depth of this point below AB is found to be h. Show that each terminal tension is,

$$\frac{1}{2}\left[P \cdot \frac{l}{h} + W \cdot \frac{h^2 + i^2}{2hl}\right]$$

[14 Marks]



Solution:

Initially AB hangs under gravity. But when load P is attached to middle point D such that $AD = BD = l$, then let T_D be the tension at D along tangent at D to AD and BD.

Let C be the lowest point of catenary such that $CD = a$.

Sag of catenary $= h$.

Let ψ_D be the angle that T_D at D makes with horizontal.

$$\Rightarrow 2T_D \sin\psi_D = P$$

Also,

$$T_D \sin\psi_D = wS \quad (\because T_x = wC; \quad T_y = ws)$$

Now, since $w = \frac{W}{2l}$ and $s = CD = a$, therefore,

$$T_D \sin\psi_D = \frac{W}{2l}a$$
$$\frac{P}{2} = \frac{W}{2l}a$$
$$\implies a = \frac{P}{W}l$$

Let y_A be the height and s_A be the arc length at A and similarly let y_D be the height and s_D be the arc length at D. Then,

$s_A = l + a$ and $s_D = a$;
$y_D = h = y_A \Rightarrow y_D = y_A - h$

Also, $c^2 + s^2 = y^2$ (given)

$$\Rightarrow c^2 + s_A^2 = y_A^2; c^2 + s_D^2 = y_D^2$$
$$\Rightarrow y_A^2 - y_D^2 = s_A^2 - s_D^2 = (l+a)^2 - a^2$$
$$\Rightarrow y_A^2 - (y_A - h)^2 = (l+a)^2 - a^2$$
$$\Rightarrow y_A = \frac{l^2 + h^2 + 2al}{2h}$$

Also, terminal tension at A or B is given by:

$$\begin{aligned} T &= wy_A \\ &= \frac{W}{2l} \times \frac{l^2 + h^2 + 2al}{2h} \\ &= \frac{W}{4lh}\left[l^2 + h^2 + 2 \times \frac{P}{W}l^2\right] \\ &= \frac{1}{2}\left[P\frac{l}{h} + W\frac{l^2 + h^2}{2lh}\right] \end{aligned}$$

Question-7(b) A particle moves with a central acceleration $\dfrac{\mu}{(\text{ distance })^2}$, it is projected with velocity V at a distance R. Show that its path is a rectangular hyperbola if the angle of projection is,

$$\sin^{-1}\left[\frac{\mu}{\mathrm{VR}\left(\mathrm{V}^2-\dfrac{2\mu}{\mathrm{R}}\right)^{1/2}}\right]$$

[13 Marks]

Solution: If the particle describes a hyperbola under the central acceleration

$$\frac{\mu}{(\text{ distance })^2},$$

then the velocities V of the particle at distance r from centre of force is given by,

$$\mathrm{V}^2=\mu\left(\frac{2}{r}+\frac{1}{a}\right)$$

where $2a$ = transverse axis.
As particle is projected with velocity V at distance R, then from (1), we have,

$$\mathrm{V}^2=\mu\left(\frac{2}{\mathrm{R}}+\frac{1}{a}\right) \quad \text{or}$$

$$\frac{\mu}{a}=\mathrm{V}^2-\frac{2\mu}{\mathrm{R}}$$

If α is required angle of projection to describe a rectangular hyperbola, then at the point of projection from the relation $h=vp$ we have

$$h=\mathrm{Vp}=\mathrm{VR}\sin\alpha \quad [\because p=r\sin\phi \ \&\ \text{initially } r=R,\phi=\alpha]$$

Also,

$$\begin{aligned} h&=\sqrt{\mu l}\\ &=\sqrt{\mu\cdot b^2/a}\\ &=\sqrt{\mu a} \quad [b=a \text{ for rectangular hyperbola}] \end{aligned}$$

from (3) and (4) we have,

$$\begin{aligned} \mathrm{VR}\sin\alpha&=\sqrt{\mu a}\\ \Rightarrow \sin\alpha&=\frac{\sqrt{\mu a}}{\mathrm{VR}}\\ &=\frac{\mu\sqrt{a}}{\mathrm{VR}\sqrt{\mu}}\\ &=\frac{\mu}{\mathrm{VR}\sqrt{\mu a}} \end{aligned}$$

from (2)

$$\Rightarrow \quad \sin\alpha=\frac{\mu}{\mathrm{VR}\sqrt{\mathrm{V}^2-\frac{2\mu}{\mathrm{R}}}}$$

$$\Rightarrow \quad \alpha = \sin^{-1}\left\{\frac{\mu}{\mathrm{VR}\sqrt{\mathrm{V}^2 - \frac{2\mu}{\mathrm{R}}}}\right\}$$

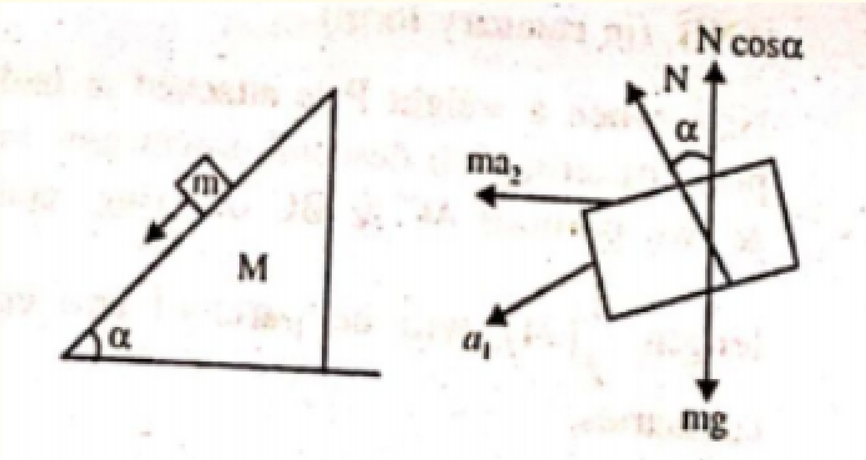
which is required angle of projection.

Question-7(c) A smooth wedge of mass M is placed on a smooth horizontal plane and a particle of mass m slides down its slant face which is inclined at an angle α to the horizontal plane, Prove that the acceleration of the wedge is,

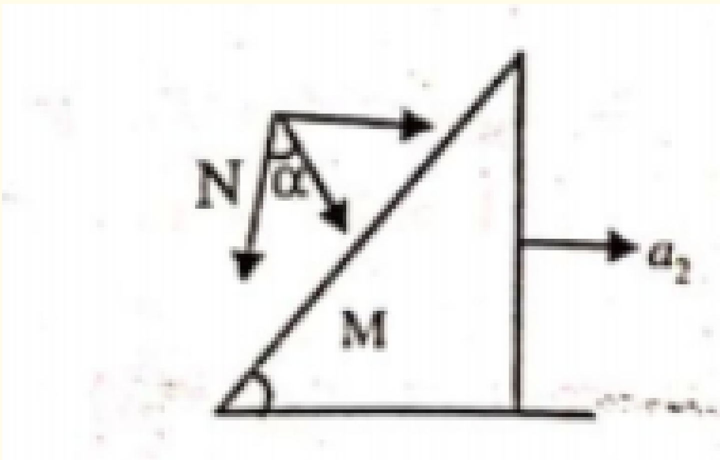
$$\frac{mg\sin\alpha\cos\alpha}{M + m\sin^2\alpha}$$

[13 Marks]

Solution: Let a_1 and a_2 be the acceleration of m and M respectively.



Then from free body diagram.



$$\begin{aligned} mg - N\cos\alpha &= ma_1\sin\alpha ma_2 + \mathrm{N}\sin\alpha \\ &= ma_1\cos\alpha \end{aligned}$$

Also, $\mathrm{N}\sin\alpha = \mathrm{M}a_2$ (1) $\times \cos\alpha -$ (2) $\times \sin\alpha$ we get,

$$mg\cos\alpha - N - ma_2\sin\alpha = 0$$

putting N from (3), we get,

$$\Rightarrow mg\cos\alpha - \frac{Ma_2}{\sin\alpha} - ma_2\sin\alpha = 0$$
$$\Rightarrow a_2\left(\mathrm{M} + m\sin^2\alpha\right) = mg\sin\alpha\cos\alpha$$
$$\therefore a_2 = \frac{mg\sin\alpha\cos\alpha}{M + m\sin^2\alpha}$$

Question-8(a) (i) Show that $\overrightarrow{\mathrm{F}} = (2\mathrm{x}y + \mathrm{z}^3)\overrightarrow{\mathrm{i}} + \mathrm{x}^2\overrightarrow{\mathrm{j}} + 3\mathrm{z}^2\mathrm{x}\overrightarrow{\mathrm{k}}$ is a conservative field. Find its scalar potential and also the work done in moving a particle from $(1, -2, 1)$ to $(3, 1, 4)$.

[5 Marks]

Solution: Field F will be conservative then $\bar{\nabla} \times \overline{\mathrm{F}} = 0$ i.e.

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} = 0$$

Now

$$\bar{\nabla} \times \overline{\mathrm{F}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy + z^3 & x^2 & 3z^2x \end{vmatrix}$$
$$= \hat{i}.0 - \hat{j}\cdot\left(3z^2 - 3z^2\right) + \hat{k}(2x - 2x)$$
$$= 0$$
$$\text{i.e.}\bar{\nabla} \times \overline{\mathrm{F}} = 0$$

$\Rightarrow \overrightarrow{\mathrm{F}}$ is conservative field.

Hence, $\overrightarrow{\mathrm{F}}$ can be written as $\overrightarrow{\mathrm{F}} = \nabla\mathrm{U}$ where U is scalar function.

Now,

$$\frac{\partial U}{\partial x} = 2xy + z^3$$
$$\Rightarrow U = x^2y + xz^3 + f_1(y, z)\frac{\partial \mathrm{U}}{\partial y}$$
$$= x^2$$
$$\Rightarrow \mathrm{U} = x^2y + f_2(x, z)\frac{\partial U}{\partial z}$$
$$= 3z^2x,$$
$$\Rightarrow \mathrm{U} = xz^3 + f_3(x, y)$$

above three expression which represent same potential function, we get,

$$U = x^2y + xz^3$$

. Now. work done in moving a particle from $(1. - 2, 1)$ to $(3, 1, 4)$

$$\begin{aligned}\Rightarrow U(3,1,4) - \mathrm{U}(1,-2,1) &= 3.^21 + 3.4^3 - (1(-2)+1)\\ &= 202 \text{ units.}\end{aligned}$$

Question-8(a) (ii) Show that, $\nabla^2 f(r) = \left(\frac{2}{r}\right) f'(r) + f''(r)$, where

$$r = \sqrt{x^2+y^2+z^2}$$

[5 Marks]

Solution:

$$\begin{aligned}\nabla^2 f(r) &= \bar{\nabla}\cdot(\nabla f(r))\\ &= \bar{\nabla}\cdot\left(f'(r)\frac{\vec{r}}{r}\right)\\ &= \bar{\nabla}\cdot\left(\frac{f'(r)}{r}\vec{r}\right)\\ &= \left(\bar{\nabla}\frac{f'(r)}{r}\right)\cdot\bar{r} + \frac{f'(r)}{r}(\bar{\nabla}\cdot\bar{r})\left[\frac{f''(r)\bar{r}}{r} + f'(r)\left(-\frac{1}{r^2}\right)\frac{\vec{r}}{r}\right]\cdot\vec{r} + 3\frac{f'(r)}{r}\\ &= \frac{f''(r)}{r^2}(\vec{r}\cdot\vec{r}) - \frac{f'(r)}{r} + \frac{3f'(r)}{r}\\ &= f''(r) + \frac{2f'(r)}{r}\\ \text{i.e.}\nabla^2 f(r) &= f''(r) + \frac{2f'(r)}{r}\end{aligned}$$

Question-8(b) Use divergence theorem to evaluate,

$$\iint_{\mathrm{s}} \left(\mathrm{x}^3\mathrm{dydz} + \mathrm{x}^2\mathrm{ydzdx} + \mathrm{x}^2\mathrm{zdydx}\right)$$

where S is the sphere, $x^2+y^2+z^2=1$.

[10 Marks]

Solution: By divergence theorem, we have

$$\iint_s F_1 dydz + F_2 dzdx + F_3 dxdy = \iiint \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dxdydz$$

$$\Rightarrow \int \left(x^3 dydz + x^2 ydxdz + x^2 zdxdy\right) = \iiint \left\{\frac{\partial x^3}{\partial x} + \frac{\partial \left(x^2 y\right)}{\partial y} + \frac{\partial \left(x^2 z\right)}{\partial z}\right\} dxdydz$$

$$= \iiint_{x^+ + y^2 + z^2 = 1} 5x^2 dxdydz$$

Converting the above integral into polar form, we get,

$$\int_{r=0} \int_{-0}^{\pi} \int_{-0}^{2\pi} \left(5r^2 \cos^2\theta \cos^2\phi\right)\left(r^2 \sin\theta dr d\theta d\phi\right) = \int_{r=0}^{1} 5r^4 dr \int \cos^2\theta \sin\theta d\theta \int_0^{2\pi} \cos^2\phi d\phi$$

$$= 5 \cdot \frac{1}{5} \cdot \frac{2}{3} \cdot \frac{\pi}{2}$$

$$= \frac{\pi}{3}$$

Question-8(c) If $\vec{A} = 2y\vec{i} - z\vec{j} - x^2\vec{k}$ and S is the surface of the parabolic cylinder $y^2 = 8x$ in the first octant bounded by the planes $y = 4$, $z = 6$, evaluate the surface integral,

$$\iint_S \overrightarrow{A} \cdot \hat{n} \overrightarrow{dS}$$

[10 Marks]

Solution: A vector normal to the parabolic cylinder is given by.

$$\nabla\left(8x - y^2\right) = 8\bar{i} - 2y\hat{y}$$

$$\Rightarrow \hat{n} = \frac{8\hat{i} - 2y\hat{j}}{\sqrt{64 + 4y^2}}$$

$$= \frac{4\bar{i} - \bar{y}\bar{j}}{\sqrt{16 + y^2}}$$

$$\Rightarrow \iint_S \bar{A} \cdot \hat{n} d\bar{S} = \iint \left(2y\bar{i} - z\bar{j} + x^2\hat{k}\right) \cdot \frac{(4\bar{i} - \bar{y})}{\sqrt{16 + y^2}} \cdot \frac{dydz}{|\hat{i}\hat{n}|}$$

$$= \iint \left(2y\bar{i} - z\hat{j} + x^2\hat{k}\right) \frac{(4\bar{i} - \hat{y})}{\sqrt{16 + y^2}} \cdot \frac{dydz}{\frac{4}{\sqrt{16 + y^2}}}$$

$$= \frac{1}{4} \int (8 + z) ydydz = \frac{1}{4} \int_{z=0}^{6} (8 + z) \frac{16}{2} dz$$

$$= \frac{1}{4} \int_0^6 (64 + 8z) dz = \frac{1}{4} \left[64z + 4z^2\right]_{2=0}^{6}$$

$$= \frac{4}{4} \left[16z + z^2\right]_0^6 = 96 + 36$$

$$= 132 \text{ Units.}$$

Question-8(d) Use Green's theorem in a plane to evaluate the integral, $\int_{\mathrm{C}} \left[\left(2\mathrm{x}^2 - \mathrm{y}^2\right) \mathrm{dx} + \left(\mathrm{x}^2 + \mathrm{y}^2\right) \mathrm{dy} \right]$, where C is the boundary of the surface in the xy-plane enclosed by $y = 0$ and the semi-circle, $y = \sqrt{1 - x^2}$.

[10 Marks]

Solution: The Green's theorem in a plane is defined as

$$\int \mathrm{M}dx + \mathrm{N}dy = \iint \left(\frac{\partial \mathrm{N}}{\partial x} - \frac{\partial \mathrm{M}}{\partial \mathrm{y}} \right) dxdy$$

$$\begin{aligned}
\int \left(2x^2 - y^2\right) dx + \left(x^2 + y^2\right) dy &= \iint (2x + 2y) dydx \\
2 \int_{x=1}^{1} \int_{0}^{\sqrt{1-x^2}} (x + y) dydx &= 2 \int_{-1}^{1} \left[\left(x\sqrt{1 - x^2} + \frac{1 - x^2}{2} \right) \right] dx \\
&= \frac{2 \times 2}{2} \int_0^1 \frac{1 - x^2}{0} dx \quad [\text{ other integral vanishes }] \\
&= 2 \left(x - \frac{x^3}{3} \right)_0^1 \\
&= \frac{4}{3}
\end{aligned}$$