Set – VII

The conicoid

A surface whose equation is of the second degree in x, y, z is called the conicoid i.e, the general equation of second degree in x, y, z

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy+2ux+2vy+2wz+d=0$$

represents a locus called a conicoid or Quadric.

– The above equation contains ten unknown constants which can be reduced to nine effective constants by dividing the equation throughout by 'a'.

Thus a conicoid can be determined with the help of nine conditions which give rise to nine independent relations between the constants.

– By suitable transformation of axes, the above general equation can be reduced to one of the following standard forms.

(1) $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$: Ellipsoid

(2) $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$; Hyperboloid of one sheet.

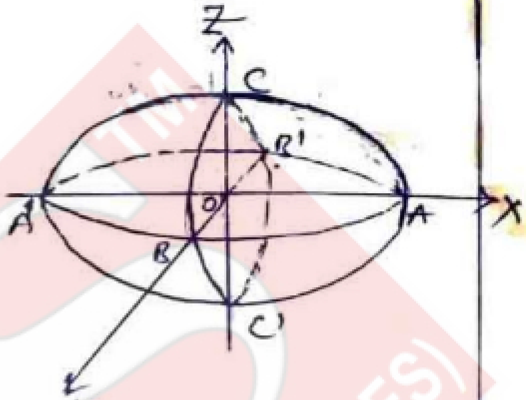
(3) $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$; Hyperboloid of two sheets

(4) $\frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{2z}{c}$; Elliptic paraboloid.

(5) $\frac{x^2}{a^2}-\frac{y^2}{b^2}=\frac{2z}{c}$; Hyperbolic paraboloid.

* Shapes of surfaces.

(1) Ellipsoid : $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.



(i) centre: If (α,β,γ) is a point on the ellipsoid, then $(-\alpha,-\beta,-\gamma)$ is also a point on it.

The middle point of the join of these points is $(0,0,0)$, the origin.

Thus (α,β,γ), $(-\alpha,-\beta,-\gamma)$ are the points on a straightline through the origin and are equidistance from the origin.

Hence origin bisects every chord which passes through it and is therefore the centre of the surface.

(ii) Symmetry: Since there are only even powers of x, the surface is symmetrical about yz-plane.

Similarly, the surface is symmetrical about xz and xy-planes.

If the point (α,β,γ) satisfies the eqn ①, then $(\alpha,\beta,-\gamma)$ also satisfies it. The line joining (α,β,γ) $(\alpha,\beta,-\gamma)$ is bisected at right angle by the xy-plane. It follows that the xy-plane bisects all chords perpendicular to it.

Similarly other co-ordinate planes also bisect chords $\perp^r$ to them.

These three planes are called principal planes. (2)

The three lines of intersection of three principal planes taken in pairs are called principal axes.

In the present case co-ordinate axes are the principal axes.

(iii) Intersection with axes:

The surface meets x-axis $(y=0, z=0)$

$\therefore$ we have $\frac{x^2}{a^2}=1 \Rightarrow x=\pm a$

i.e., the surface meets the x-axis in the points $A(a,0,0)$ and $A'(-a,0,0)$.

Similarly it meets y-axis $(x=0, z=0)$ at $B(0,b,0)$ and $B'(0,-b,0)$

and z-axis $(x=0, y=0)$ at $C(0,0,c)$ and $(0,0,-c)$

(iv) Sections by co-ordinate planes: The surface meets the yz-plane i.e., $x=0$:

we have $\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$.

which is an ellipse in ~~that~~ plane. (i.e yz-plane)

Similarly, it meets the zx-plane $(y=0)$ in the ellipse $\frac{x^2}{a^2}+\frac{z^2}{c^2}=1$ in that plane

and it meets the xy-plane $(z=0)$ in the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in that plane.

(v) Generated by a variable curve:

The surface meets the plane $z=k$ in a curve

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1-\frac{k^2}{c^2} \;;\; z=k,\ -c\le k\le c \quad \text{--- (1)}$$

The surface is generated by the variable ellipse (1) in which k takes different values and whose plane is ∥ to the xy-plane ($z=0$) and centre $(0,0,k)$ moves on the z-axis.

The ellipse (1) is real only if $1-\frac{k^2}{c^2}>0$

i.e, $k^2<c^2$

i.e, $|k|<c$.

i.e, k lies between $-c$ and c.

Similarly x and y cannot be numerically greater than 'a' & b respectively.

So that we have for every point (x,y,z) on the surface

$-a\le x\le a$, $-b\le y\le b$, $-c\le z\le c$.

Hence, the surface lies between the planes $x=a$, $x=-a$; $y=b$, $y=-b$;

$z=c$, $z=-c$.

and therefore is a closed surface.

Hyperboloid of one sheet: $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$.

(i) The origin bisects all chords which pass through it and is therefore the centre of the surface.

(ii) It is symmetrical about each of the co-ordinate planes for only even powers x, y, z occur in its eqn; co-ordinate planes are the principal planes and co-ordinate axes are the principal axes of the surface.

(iii) It meets the x-axis at $A(a,0,0)$, $A'(-a,0,0)$; the y-axis at $B(0,b,0)$, $B'(0,-b,0)$; and the z-axis in imaginary points. ($\because$ putting $x=0, y=0$; we get $-\frac{z^2}{c^2}=1$, $z^2=-c^2$)

(iv) Its section by the yz-plane $(x=0)$ is the hyperbola $\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ (i.e, DE, D'E')

- Its section by the zx-plane $(y=0)$ is the hyperbola $\frac{x^2}{a^2}-\frac{z^2}{c^2}=1$ (i.e, FG, F'G')

- Its section by the xy-plane $(z=0)$ is the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

(v) The sections by the planes $z=k$ which are parallel to the xy-plane are the similar ellipses

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1+\frac{k^2}{c^2},\ z=k \quad \text{—— (1)}$$

whose centre lie on z-axis and which increase in size as k increases.

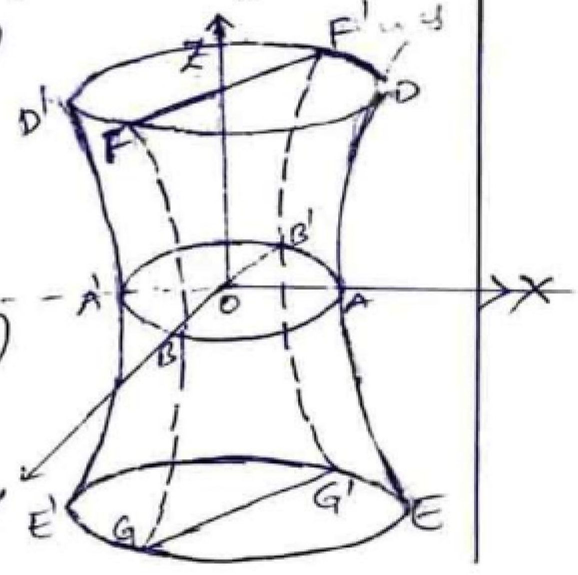
There is no limit to the increase of k.

The surface may, therefore be generated by the variable ellipse (1) where k varies from $-\infty$ to $+\infty$.

The shape of the surface as shown in the figure.

(which is like Juggler's dabru) i.e. in Hindi — Hurka

It is known as hyperboloid of one sheet.



The hyperboloid of two sheets: $\frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$.

(i) origin is the centre; co-ordinate planes are the principal planes and co-ordinate axes are the principal axes of the surface.

(ii) It is symmetrical about each of the co-ordinate planes for only even powers of x, y, z occur in its equation.

(iii) It meets the x-axis at $A(a,0,0)$, $A'(-a,0,0)$ and the y and z-axes in imaginary points.

(iv) Its section by the xy-plane ($z=0$) is the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ (i.e. ACB, $A'C'B'$)

- Its section by the zx-plane ($y=0$) is the hyperbola $\frac{x^2}{a^2}-\frac{z^2}{c^2}=1$. (i.e, DAE, $D'A'E'$)

- Its section by the yz-plane ($x=0$), is the imaginary ellipse $\frac{y^2}{b^2}+\frac{z^2}{c^2}=-1$

(v) The surface cuts the plane $x=k$ in an ellipse.

$$\frac{y^2}{b^2}+\frac{z^2}{c^2}=\frac{k^2}{a^2}-1,\quad x=k.$$

which increases in size as k^2 increases, but is real when $\frac{k^2}{a^2}-1>0$.

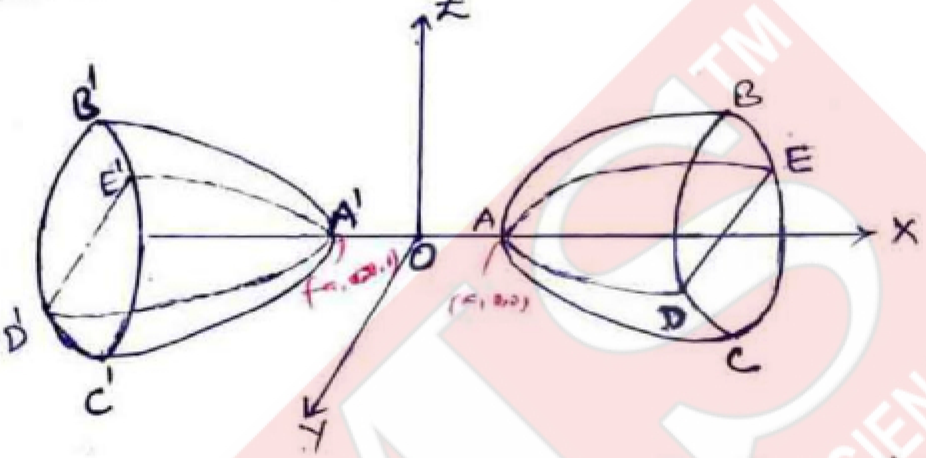
i.e, $k^2>a^2$.

i.e, $|k|>a$

i.e, when k does not lie between $-a$ and a.

Thus ~~no~~ portion of the surface lies between the ~~planes~~ $x = \pm a$.

The surface thus consists of two detached portions as shown in the figure.
It is known as hyperboloid of two sheets.



Its shape is like that of two tablas placed as shown in the figure.

* central conicoid :—

A conicoid whose all chords through the origin are bisected at the origin is called a central conicoid.

The equation $ax^2+by^2+cz^2=1$ — (1)

in general, represents a central conicoid.

All the above three equations

$$\left[\text{viz } \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,\ \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1,\ \frac{x^2}{a^2}-\frac{y^2}{b^2}-\frac{z^2}{c^2}=1\right]$$

are covered by this equation.

(i) when a, b, c are all +ve, (1) represents an ellipsoid.

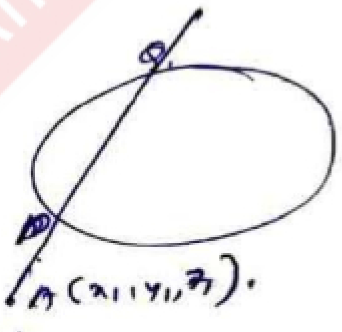
(ii) when two are +ve and one -ve, it represents a hyperboloid of one sheet, and (iii) when two are −ve and +ve it represents a hyperboloid of two sheets.

The above equation for all values of a, b, c (−ve or +ve) represents a surface whose centre is origin and co-ordinate planes, the three principal planes.

The equation $ax^2+by^2+cz^2=1$ is called the standard form of central conicoid.

* Intersection of a line and a conicoid:—

To find the points of intersection of the line $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ with the conicoid $ax^2+by^2+cz^2=1$.



Sol The given line is

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \quad \text{--- (1)}$$

and the conicoid is

$$ax^2+by^2+cz^2=1 \quad \text{--- (2)}.$$

Any point on the line (1) is

$$(lr+x_1,\ mr+y_1,\ nr+z_1) \quad \text{--- (3)}$$

If it lies on the conicoid (2), then

$$a(lr+x_1)^2+b(mr+y_1)^2+c(nr+z_1)^2=1.$$

$$\Rightarrow r^2(al^2+bm^2+cn^2)+2r(alx_1+bmy_1+cnz_1)+(ax_1^2+by_1^2+cz_1^2-1)=0. \quad \text{--- (4)}$$

which is a quadratic in r, giving two values of r.

putting these values of r in (3), (5)
we get the two points of intersection
P and Q.

Hence, every line meets a central
conicoid in two points.

The two values r_1 and r_2 of r obtained from the equation (4) are the measures of the distances of the points of intersection P and Q from the point (x_1, y_1, z_1) if (l, m, n) are the direction cosines of the line.

NOTE: The equation (4) will frequently be used in what follows:

Def → A chord of a central conicoid which passes through the centre is called a diameter.

→ prove that the sum of the squares of the reciprocals of any three mutually perpendicular diameters of an ellipsoid is constant.

Sol Let the ellipsoid be $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ —(1)

Let l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 be the actual d.c's of three mutually $\perp$ diameters say PCP', QCQ', RCR' and let $2r_1, 2r_2, 2r_3$ be the lengths of the diameters.

Since the diameters of the ellipsoid are bisected at the centre
$C(0,0,0)$, $CP = CP' = r_1$, $CQ = CQ' = r_2$
$CR = CR' = r_3$.

Now as P is at a distance r_1 from $C(0,0,0)$ and d.c's of CP are l_1, m_1, n_1

$\therefore$ The co-ordinates of P are $(l_1 r_1, m_1 r_1, n_1 r_1)$

Since P lies on ellipsoid (1)

$$\therefore \frac{l_1^2 r_1^2}{a^2} + \frac{m_1^2 r_1^2}{b^2} + \frac{n_1^2 r_1^2}{c^2} = 1.$$

$$\Rightarrow \frac{1}{r_1^2} = \frac{l_1^2}{a^2} + \frac{m_1^2}{b^2} + \frac{n_1^2}{c^2}$$

now we have

$$\therefore \frac{1}{(PP')^2} = \frac{1}{(2r_1)^2} = \frac{1}{4r_1^2}$$

$$= \frac{1}{4}\left(\frac{l_1^2}{a^2} + \frac{m_1^2}{b^2} + \frac{n_1^2}{c^2}\right) \quad (2)$$

Similarly $\frac{1}{(QQ')^2} = \frac{1}{4}\left(\frac{l_2^2}{a^2} + \frac{m_2^2}{b^2} + \frac{n_2^2}{c^2}\right) \quad (3)$

and $\frac{1}{RR'^2} = \frac{1}{4}\left(\frac{l_3^2}{a^2} + \frac{m_3^2}{b^2} + \frac{n_3^2}{c^2}\right) \quad (4)$

Adding (2), (3) and (4), we have

$$\frac{1}{PP'^2} + \frac{1}{QQ'^2} + \frac{1}{RR'^2} = \frac{1}{4}\left[\frac{1}{a^2}(l_1^2 + l_2^2 + l_3^2) + \frac{1}{b^2}(m_1^2 + m_2^2 + m_3^2) + \frac{1}{c^2}(n_1^2 + n_2^2 + n_3^2)\right]$$

$$= \frac{1}{4}\left[\frac{1}{a^2}(1) + \frac{1}{b^2}(1) + \frac{1}{c^2}(1)\right]$$

$$= \frac{1}{4}\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)$$

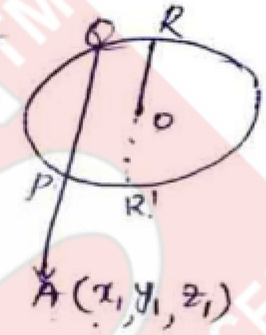
$=$ constant. ($\because l_1, m_1, n_1, \ldots$ etc are the d.c's of three mutually $\perp$ lines).

6

→ A line through a given point A meets the central conicoid in P, Q. If R'OR is the diameter parallel to APQ, prove that $AP \cdot AQ : OR^2$ is constant.

Sol^n: Let $A(x_1, y_1, z_1)$ be the given point and let the conicoid be $ax^2 + by^2 + cz^2 = 1$. ——— (1)

Let l, m, n be the actual d.c.'s of the line through A which meets the conicoid in P and Q.



∴ Equations of the line APQ passing through $A(x_1, y_1, z_1)$ and with d.c.'s l, m, n are

$$\frac{x-x_1}{l} = \frac{y-y_1}{m} = \frac{z-z_1}{n} \quad \text{——— (2)}$$

Any point on this line is $(lr+x_1,\ mr+y_1,\ nr+z_1)$

If it lies on the conicoid (1), then

$$a(lr+x_1)^2 + b(mr+y_1)^2 + c(nr+z_1)^2 = 1$$

$$\Rightarrow r^2(al^2+bm^2+cn^2) + 2r(alx_1 + bmy_1 + cnz_1) + (ax_1^2 + by_1^2 + cz_1^2 - 1) = 0 \quad \text{——— (3)}$$

which is a quadratic in 'r'.

Since l, m, n are the actual d.c.'s of the line (2).

∴ The two values of r in (2) are the lengths AP and AQ.

$$\therefore AP \cdot AQ = \text{product of roots}$$

$$= \frac{ax_1^2 + by_1^2 + cz_1^2 - 1}{al^2+bm^2+cn^2} \quad \text{——— (4)}$$

(r_1, r_2 are roots of $ax^2+bx+c=0$, $r_1 r_2 = \frac{c}{a}$)

Now the equations of the diameter OR through O(0,0,0) and || to line ② are

$$\frac{x-0}{l}=\frac{y-0}{m}=\frac{z-0}{n}.$$

If $OR = r'$, then the co-ordinates of R are (lr', mr', nr').

Since R lies on the conicoid ①, then

$$al^2r'^2+bm^2r'^2+cn^2r'^2=1$$

$$\Rightarrow r'^2(al^2+bm^2+cn^2)=1$$

$$\Rightarrow r'^2=OR^2=\frac{1}{al^2+bm^2+cn^2} \quad \text{—⑤}$$

Dividing ④ & ⑤, we get

$$\frac{AP\cdot AQ}{OR^2}=\frac{ax_1^2+by_1^2+cz_1^2-1}{al^2+bm^2+cn^2}\times\frac{al^2+bm^2+cn^2}{1}$$

$$=ax_1^2+by_1^2+cz_1^2-1$$

$$=\text{constant}.$$

Hence the result.

→ A is given point and POP' any diameter of a central conicoid. If OQ and OQ' are the diameters parallel to AP and AP', prove that $\frac{AP^2}{OQ^2}+\frac{AP'^2}{OQ'^2}$ is constant.

Sol: Let the central conicoid be $ax^2+by^2+cz^2=1$ —①

Let A be the point (α,β,γ) and $P(x_1,y_1,z_1)$ $P'(-x_1,-y_1,-z_1)$ extremities of diameter POP'.

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The d.c's of AP are proportional to

$x_1-\alpha,\ y_1-\beta,\ z_1-\gamma$ | using $x_2-x_1,\ y_2-y_1,\ z_2-z_1$.

Dividing each by $\sqrt{(x_1-\alpha)^2+(y_1-\beta)^2+(z_1-\gamma)^2} = AP$,

$\therefore$ the actual d.c's of AP are

$$l = \frac{x_1-\alpha}{AP}\ ;\ m = \frac{y_1-\beta}{AP}\ \&\ n = \frac{z_1-\gamma}{AP}. \quad \text{—(2)}$$

$\therefore$ d.c's of OQ (|| to AP) are also l, m, n.

Then if $OQ = r$, the co-ordinates of Q are (lr, mr, nr).

Since Q lies on the conicoid (1).

$$\therefore (1) \equiv a(lr)^2+b(mr)^2+c(nr)^2 = 1$$

$$\Rightarrow r^2(al^2+bm^2+cn^2) = 1$$

$$\Rightarrow (OQ)^2\left[a\left(\frac{x_1-\alpha}{AP}\right)^2+b\left(\frac{y_1-\beta}{AP}\right)^2+c\left(\frac{z_1-\gamma}{AP}\right)^2\right] = 1$$

($\because$ from (2) & $OQ=r$)

$$\Rightarrow \frac{OQ^2}{AP^2}\left[a(x_1-\alpha)^2+b(y_1-\beta)^2+c(z_1-\gamma)^2\right] = 1.$$

$$\Rightarrow \frac{AP^2}{OQ^2} = a(x_1-\alpha)^2+b(y_1-\beta)^2+c(z_1-\gamma)^2 \quad \text{—(3)}.$$

Now changing (x_1, y_1, z_1) to $(-x_1, -y_1, -z_1)$,

i.e., P changes to P' and Q changes Q'.

$$\therefore \frac{AP'^2}{OQ'^2} = a(-x_1-\alpha)^2+b(-y_1-\beta)^2+c(-z_1-\gamma)^2$$

$$= a(x_1+\alpha)^2+b(y_1+\beta)^2+c(z_1+\gamma)^2 \quad \text{—(4)}$$

Adding (3) and (4), we have

$$\frac{Ap^2}{OQ^2}+\frac{Ap'^2}{OQ'^2} = a\left[(x_1+\alpha)^2+(x_1-\alpha)^2\right]+b\left[(y_1+\beta)^2+(y_1-\beta)^2\right] + c\left[(z_1+\gamma)^2+(z_1-\gamma)^2\right].$$

$$= 2a(x_1^2+\alpha^2)+2b(y_1^2+\beta^2)+2c(z_1^2+\gamma^2)$$

$$= 2(a\alpha^2+b\beta^2+c\gamma^2)+2(ax_1^2+by_1^2+cz_1^2).$$

$$= 2(a\alpha^2+b\beta^2+c\gamma^2)+2(1)$$

$$= 2(a\alpha^2+b\beta^2+c\gamma^2+1) \quad \left(\because p(x_1,y_1,z_1) \text{ lie on } (1) \; \therefore ax_1^2+by_1^2+cz_1^2=1\right)$$

$$= \text{constant}.$$

Hence the Result.

Tangent plane:

To find the equation of tangent plane of the central conicoid at the point (x_1,y_1,z_1) of the $ax^2+by^2+cz^2=1$.

Sol^n: The given conicoid is $ax^2+by^2+cz^2=1$ —— (1)

Equation of a line through (x_1,y_1,z_1) is

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \quad \text{—— (2)}$$

Any point on (2) is $(lr+x_1,\ mr+y_1,\ nr+z_1)$

If it lies on (1), then

$$a(lr+x_1)^2+b(mr+y_1)^2+c(nr+z_1)^2=1.$$

$$\Rightarrow r^2(al^2+bm^2+cn^2)+2r(alx_1+bmy_1+cnz_1)+(ax_1^2+by_1^2+cz_1^2-1)=0 \quad \text{—— (3)}$$

But (x_1,y_1,z_1) lies on (1)

$$\therefore (1) \equiv ax_1^2+by_1^2+cz_1^2=1. \quad \text{—— (4)}$$

$\therefore$ (3) becomes

$r^2(al^2+bm^2+cn^2)+2r(alx_1+bmy_1+cnz_1)=0$ (8)

$r[r(al^2+bm^2+cn^2)+2(alx_1+bmy_1+cnz_1)]=0$ — (5)

$\Rightarrow r=0.$

Since the line (2) touches the conicoid (1)

$\therefore$ it cuts (1) at two coincident points,

which is so if the two values of 'r'

in (5) are equal.

But since one root of (5) is zero,

$\therefore$ the other must also be zero.

$\therefore$ coefficient of $r=0$

i.e, $alx_1+bmy_1+cnz_1=0$ — (6)

Eliminating l, m, n from (2) & (6),

the locus of line (2) is

$ax_1(x-x_1)+by_1(y-y_1)+cz_1(z-z_1)=0$

$ax_1(x-x_1)+by_1(y-y_1)+cz_1(z-z_1) = ax_1^2+by_1^2+cz_1^2$

$\Rightarrow axx_1+byy_1+czz_1 = ax_1^2+by_1^2+cz_1^2$

$\Rightarrow axx_1+byy_1+czz_1=1$ ($\because$ by (4))

which is the required equation of

tangent plane at (x_1, y_1, z_1)

* Condition of Tangency:-

To find the condition that the plane $lx+my+nz=p$ should touch the conicoid $ax^2+by^2+cz^2=1$.

Sol^n: The given plane is $lx+my+nz=p$ — (1)

and the conicoid is $ax^2+by^2+cz^2=1$ — (2)

Let the plane (1) touch the conicoid (2).

at the point (x_1, y_1, z_1).

Then ① should be identical with the tangent plane at (x_1, y_1, z_1) to ②.

Now the equation of tangent plane at (x_1, y_1, z_1) to ② is

$$axx_1 + byy_1 + czz_1 = 1. \quad \text{--- ③}$$

comparing ① and ③, we have

$$\frac{ax_1}{l} = \frac{by_1}{m} = \frac{cz_1}{n} = \frac{1}{p}.$$

$$\therefore x_1 = \frac{l}{ap}, \; y_1 = \frac{m}{bp}, \; z_1 = \frac{n}{cp} \quad \text{--- ④}$$

But since (x_1, y_1, z_1) being the point of contact lies on the conicoid ②

$$\therefore a\left(\frac{l}{ap}\right)^2 + b\left(\frac{m}{bp}\right)^2 + c\left(\frac{n}{cp}\right)^2 = 1$$

$$\Rightarrow \boxed{\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} = p^2}$$

which is the required condition of tangency

Note: From ④, the point of contact is

$$\left(\frac{l}{ap}, \frac{m}{bp}, \frac{n}{cp}\right).$$

→ Find the equations of two tangent planes of the conicoid $ax^2 + by^2 + cz^2 = 1$ which are parallel to the plane $lx + my + nz = 0$.

Sol: The given conicoid is $ax^2 + by^2 + cz^2 = 1$ --- ①

and any plane $\parallel$ to $lx + my + nz = 0$ is

$$lx + my + nz = p \quad \text{--- ②}$$

If ② touches ①, then

$$\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p^2$$

$$p=\pm\sqrt{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}}.$$

$\therefore$ (2) $\equiv$; the required tangent planes are

$$lx+my+nz=\pm\sqrt{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}} \quad \text{--- (3)}$$

Note: The equation (3) represents tangent planes for all values of l, m, n.

Thus any tangent plane to conicoid (1) is

$$lx+my+nz=\sqrt{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}}.$$

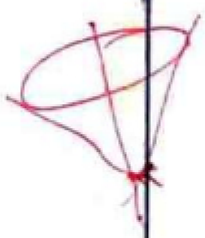
Director sphere:

To find the locus of the point of intersection of three mutually perpendicular tangent planes to the central conicoid $ax^2+by^2+cz^2=1$.

sol^n: The given conicoid is

$$ax^2+by^2+cz^2=1 \quad \text{--- (1)}$$

Let $l_1x+m_1y+n_1z=\sqrt{\frac{l_1^2}{a}+\frac{m_1^2}{b}+\frac{n_1^2}{c}}$ --- (2)



$$l_2x+m_2y+n_2z=\sqrt{\frac{l_2^2}{a}+\frac{m_2^2}{b}+\frac{n_2^2}{c}} \quad \text{--- (3)}$$

and $l_3x+m_3y+n_3z=\sqrt{\frac{l_3^2}{a}+\frac{m_3^2}{b}+\frac{n_3^2}{c}}$ --- (4)

be three mutually $\perp^r$ tangent planes

so that $l_1l_2+m_1m_2+n_1n_2=0$, etc and
$l_1m_1+l_2m_2+l_3m_3=0$ etc. and
$l_1^2+m_1^2+n_1^2=1$ etc. and $l_1^2+l_2^2+l_3^2=1$. etc. } --- (5)

The co-ordinates of the point of intersection satisfy the three equations ②, ③, ④ and its locus is therefore obtained by eliminating l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 from equations squaring and adding ②, ③, ④.

we have

$$(l_1x + m_1y + n_1z)^2 + (l_2x + m_2y + n_2z)^2 + (l_3x + m_3y + n_3z)^2$$
$$= \left(\frac{l_1^2}{a} + \frac{m_1^2}{b} + \frac{n_1^2}{c}\right) + \left(\frac{l_2^2}{a} + \frac{m_2^2}{b} + \frac{n_2^2}{c}\right) + \left(\frac{l_3^2}{a} + \frac{m_3^2}{b} + \frac{n_3^2}{c}\right)$$

$$\Rightarrow x^2(l_1^2+l_2^2+l_3^2) + y^2(m_1^2+m_2^2+m_3^2) + z^2(n_1^2+n_2^2+n_3^2)$$
$$+2xy(l_1m_1+l_2m_2+l_3m_3) + 2yz(m_1n_1+m_2n_2+m_3n_3)$$
$$+2zx(n_1l_1+n_2l_2+n_3l_3) = \frac{1}{a}(l_1^2+l_2^2+l_3^2) + \frac{1}{b}(m_1^2+m_2^2+m_3^2) + \frac{1}{c}(n_1^2+n_2^2+n_3^2)$$

$$\Rightarrow x^2(1) + y^2(1) + z^2(1) + 2xy(0) + 2yz(0) + 2zx(0)$$
$$= \frac{1}{a}(1) + \frac{1}{b}(1) + \frac{1}{c}(1) \quad (\because \text{from } ⑤)$$

$$\Rightarrow x^2+y^2+z^2 = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

which is the required locus and is a sphere concentric with the conicoid and is known as the <u>director sphere</u>.

→ Show that the length of the $\perp^r$ from the origin on the tangent plane at the point (x', y', z') of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ is given by $\frac{1}{p^2} = \frac{x'^2}{a^4}+\frac{y'^2}{b^4}+\frac{z'^2}{c^4}$.

Sol^n: The given ellipsoid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ —— ①

The equation of the tangent plane at (x', y', z') to ① is $\frac{xx'}{a^2}+\frac{yy'}{b^2}+\frac{zz'}{c^2}-1=0$ —— ②

Let $$l_1x+m_1y+n_1z=\sqrt{\frac{l_1^2}{a}+\frac{m_1^2}{b}+\frac{n_1^2}{c}} \quad ...(2)$$

$$l_2x+m_2y+n_2z=\sqrt{\frac{l_2^2}{a}+\frac{m_2^2}{b}+\frac{n_2^2}{c}} \quad ...(3)$$

and $$l_3x+m_3y+n_3z=\sqrt{\frac{l_3^2}{a}+\frac{m_3^2}{b}+\frac{n_3^2}{c}} \quad ...(4)$$

be three mutually $\perp$ tangent planes so that

$$\left.\begin{array}{l} l_1l_2+m_1m_2+n_1n_2=0 \text{ etc. and } l_1m_1+l_2m_2+l_3m_3=0 \text{ etc.} \\ \text{and } l_1^2+m_1^2+n_1^2=1 \text{ etc. and } l_1^2+l_2^2+l_3^2=1 \text{ etc.} \end{array}\right\} \quad ...(5)$$

The co-ordinates of the point of intersection satisfy the three equations (2), (3), (4) and its locus is therefore obtained by eliminating l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3 from equations. Squaring and adding (2), (3), (4), we have

$$(l_1x+m_1y+n_1z)^2+(l_2x+m_2y+n_2z)^2+(l_3x+m_3y+n_3z)^2$$
$$=\left(\frac{l_1^2}{a}+\frac{m_1^2}{b}+\frac{n_1^2}{c}\right)+\left(\frac{l_2^2}{a}+\frac{m_2^2}{b}+\frac{n_2^2}{c}\right)+\left(\frac{l_3^2}{a}+\frac{m_3^2}{b}+\frac{n_3^2}{c}\right)$$

or $$x^2(l_1^2+l_2^2+l_3^2)+y^2(m_1^2+m_2^2+m_3^2)+z^2(n_1^2+n_2^2+n_3^2) +2xy(l_1m_1+l_2m_2+l_3m_3)+2yz(m_1n_1+m_2n_2+m_3n_3) +2zx(n_1l_1+n_2l_2+n_3l_3)$$
$$=\frac{1}{a}(l_1^2+l_2^2+l_3^2)+\frac{1}{b}(m_1^2+m_2^2+m_3^2)+\frac{1}{c}(n_1^2+n_2^2+n_3^2)$$

or $$x^2(1)+y^2(1)+z^2(1)+2xy(0)+2yz(0)+2zx(0)$$
$$=\frac{1}{a}(1)+\frac{1}{b}(1)+\frac{1}{c}(1) \quad \textit{Using (5)}$$

or $$x^2+y^2+z^2=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$$

which is the required locus and is a sphere concentric with the conicoid and is known as the *director sphere*.

Example 1: *Show that the length of the perpendicular from the origin on the tangent plane at the point (x', y', z') of the ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \text{ is given by}$$

$$\frac{1}{p^2}=\frac{x'^2}{a^4}+\frac{y'^2}{b^4}+\frac{z'^2}{c^4}$$

Sol. The given ellipsoid is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad ...(1)$$

The equation of the tangent plane at (x', y', z') to (1) is

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}+\frac{zz'}{c^2}=1$$

or $$\frac{xx'}{a^2}+\frac{yy'}{b^2}+\frac{zz'}{c^2}-1=0 \quad ...(2)$$

If p is the $\perp$ distance from the origin $(0, 0, 0)$ on (2), we have

$$p = \frac{0+0+0-1}{\sqrt{\frac{x'^2}{a^4}+\frac{y'^2}{b^4}+\frac{z'^2}{c^4}}}$$

or
$$-\frac{1}{p} = \sqrt{\frac{x'^2}{a^4}+\frac{y'^2}{b^4}+\frac{z'^2}{c^4}}$$

Squaring, $\frac{1}{p^2} = \frac{x'^2}{a^4}+\frac{y'^2}{b^4}+\frac{z'^2}{c^4}$

which proves the required result.

Example 2. *If P, Q are any two point on the ellipsoid, the plane through the centre and the line of intersection of the tangent planes at P, Q bisects PQ.*

Sol. Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be any two points on the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad \ldots(1)$$

$\therefore$
$$\left.\begin{aligned}\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}&=1\\ \text{and}\quad \frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}+\frac{z_2^2}{c^2}&=1\end{aligned}\right] \quad \ldots(2)$$

[$\because$ P, Q lie on (1)

Now equations of the tangents planes at P and Q to (1) are

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=1 \quad \text{or} \quad \frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}-1=0 \quad \ldots(3)$$

and
$$\frac{xx_2}{a^2}+\frac{yy_2}{b^3}+\frac{zz_2}{c^2}=1 \quad \text{or} \quad \frac{xx_2}{a^2}+\frac{yy_2}{b^2}+\frac{zz_2}{c^2}-1=0 \quad \ldots(4)$$

Now any plane through the line of intersection of (3) and (4) is

$$\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}-1\right)+k\left(\frac{xx_2}{a^2}+\frac{yy_2}{b^2}+\frac{zz_2}{c^2}-1\right)=0 \ldots(5)$$

If it passes through the centre $(0, 0, 0)$ of the ellipsoid, then

$$0-1+k(0-1)=0 \quad \text{or} \quad k=-1.$$

$\therefore$ From (5),
$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}-1 -\left(\frac{xx_2}{a^2}+\frac{yy_2}{b^2}+\frac{zz_2}{c^2}-1\right)=0$$

or
$$\frac{x(x_1-x_2)}{a^2}+\frac{y(y_1-y_2)}{b^2}+\frac{z(z_1-z_2)}{c^2}=0 \quad \ldots(6)$$

Now mid-point of PQ is

$$M\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}+\frac{z_1+z_2}{2}\right).$$

It lies on (6) if

$$\frac{(x_1+x_2)(x_1-x_2)}{2a^2}+\frac{(y_1+y_2)(y_1-y_2)}{2b^2}+\frac{(z_1+z_2)(z_1-z_2)}{2c^2}=0$$

or if $$\frac{x_1^2-x_2^2}{2a^2}+\frac{y_1^2-y_2^2}{2b^2}+\frac{z_1^2-z_2^2}{2c^2}=0$$

or if $$\frac{1}{2}\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}\right)-\frac{1}{2}\left(\frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}+\frac{z_2^2}{c^2}\right)=0$$

or if $\frac{1}{2}(1)-\frac{1}{2}(1)=0$ | *Using* (2)

or if $\frac{1}{2}-\frac{1}{2}=0$ which is true. Hence the result.

Example 3. (*a*) *A tangent plane to the ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

meets the co-ordinate axes in A, B and C. Find the locus of the centroid of the (i) triangle ABC, (ii) tetrahedron OABC.

(Agra 1985, 87 ; Kanpur 1983)

(*b*) *If P be the point of contact of a tangent plane to the ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

which meets the axes in A, B, C and PD, PE, PF are perpendiculars drawn from P to the axes, prove that

$$OD\,.\,OA=a^2,\ OE\,.\,OB=b^2,\ OF\,.\,OC=c^2.$$

Sol. (*a*) Let $P(x_1, y_1, z_1)$ be any point on the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad ...(1)$$

$$\therefore \quad \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=1 \quad ...(2)$$

Equation of tangent plane at $P(x_1, y_1, z_1)$ to (1) is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=1 \quad ...(3)$$

This meets X-axis $(y=0, z=0)$

where $$\frac{xx_1}{a^2}=1 \quad \therefore \quad x=\frac{a^2}{x_1}.$$

Thus (3) meets X-axis in the point $A\left(\frac{a^2}{x_1}, 0, 0\right)$ Similarly it meets Y-axis in $B\left(0, \frac{b^2}{y_1}, 0\right)$, and Z-axis in $C\left(0, 0, \frac{c^2}{z_1}\right)$

(*i*) Then if $G(\alpha, \beta, \gamma)$ be the centroid of $\triangle ABC$.

$$\alpha=\frac{\frac{a^2}{x_1}+0+0}{3}=\frac{a^2}{3x_1}, \text{ similarly } \beta=\frac{b^2}{3y_1}, \gamma=\frac{c^2}{3z_1}$$

which give $$x_1=\frac{a^2}{3\alpha},\ y_1=\frac{b^2}{3\beta},\ z_1=\frac{c^2}{3\gamma}.$$

Putting these values of (x_1, y_1, z_1) in (2), we get

$$\frac{1}{a^2} \cdot \frac{a^4}{9\alpha^2} + \frac{1}{b^2} \cdot \frac{b^4}{9\beta^2} + \frac{1}{c^2} \cdot \frac{c^4}{9\gamma^2} = 1$$

or
$$\frac{a^2}{\alpha^2} + \frac{b^2}{\beta^2} + \frac{c^2}{\gamma^2} = 9.$$

$\therefore$ Locus of G (α, β, γ) is [changing (α, β, γ) to (x, y, z)]

$$\frac{a^2}{x^2} + \frac{b^2}{y^2} + \frac{c^2}{z^2} = 9.$$

(*ii*) **Please try yourself.**

$$\left[\text{Ans. } \frac{a^2}{x^2} + \frac{b^2}{y^2} + \frac{c^2}{z^2} = 16 \right]$$

(*b*) Let $P(x_1, y_1, z_1)$ be the point of contact.

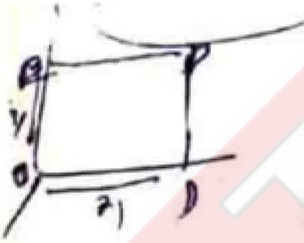
Then the equation of the tangent plane at $P(x_1, y_1, z_1)$ to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} + \frac{zz_1}{c^2} = 1 \quad \text{...(1)}$$

If PD, PE, PF are $\perp$s drawn from P on the axes, then $OD = x_1$, $OE = y_1$, $OF = z_1$. (*Def. of co-ordinates*)

Now the plane (1) meets X-axis $(y=0, z=0)$ in the point A where

$$\frac{xx_1}{a^2} = 1 \quad \text{or} \quad xx_1 = a^2$$



or
$$OA \,.\, OD = a^2 \quad | \because x = OA, \; x_1 = OD$$

Similarly (1) meets Y axis $(z=0, x=0)$ in the point B where

$$\frac{yy_1}{b^2} = 1 \quad \text{or} \quad yy_1 = b^2$$

or
$$OB \,.\, OE = b^2.$$

Similarly we can prove that $OC \,.\, OF = c^2$.

Hence the result.

Example 4. *The tangent plane to the surface $x^2 + 12y^2 + 4z^2 = 8$ at the point $(1, \frac{1}{2}, 1)$ meets the co-ordinate axes at A, B, C. Find the centroid of $\triangle ABC$.* (Agra, 1986)

Sol. The tangent plane to the given surface at $(1, \frac{1}{2}, 1)$ is

$$x(1) + 12y(\tfrac{1}{2}) + 4z(1) = 8$$

or
$$x + 6y + 4z = 8$$

which meets the co-ordinate axes at A, B and C.

$\Rightarrow$ $A(8, 0, 0)$, $B(0, 4/3, 0)$, $C(0, 0, 2)$

$\therefore$ Centroid of $\triangle ABC$ is,

$$\frac{8+0+0}{3}, \quad \frac{0+\frac{4}{3}+0}{3} + \frac{0+0+2}{3}$$

i.e.,
$$\left(\frac{8}{3}, \frac{4}{9}, \frac{2}{3}\right).$$

Example 5. *A tangent plane to the conicoid $ax^2+by^2+cz^2=1$ meets the co-ordinate axes in P, Q and R. Find the locus of the centroid of the $\triangle PQR$.*

Sol. Any tangent plane to the given conicoid is

$$lx+my+nz=\sqrt{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}} \quad \text{...(i)}$$

This plane meets the x-axis at P, so the co-ordinates of P, are

$$\left[\left(\frac{1}{l}\right)\sqrt{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}},\ 0,\ 0\right]$$

putting $y=0=z$ in (i).

Similarly Q and R are

$$\left[0,\ \frac{1}{m}\sqrt{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}},\ 0\right]$$

and $$\left[0,\ 0,\ \frac{1}{n}\sqrt{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}}\right].$$

$\therefore$ If (x_1, y_1, z_1) be the centroid of $\triangle PQR$, then

$$x_1=\frac{1}{3}\left[\frac{1}{l}\sqrt{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}}+0+0\right]$$

$$=\frac{1}{3l}\sqrt{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}}$$

$$y_1=\frac{1}{3m}\sqrt{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}},$$

and $$z_1=\frac{1}{3n}\sqrt{\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}}$$

$$(3lx_1)^2=\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}$$

or $$\frac{9l^2}{a}=\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}\right)\cdot\frac{1}{ax_1^2}$$

Similarly, $$\frac{9m^2}{b}=\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}\right)\frac{1}{by_1^2}$$

and $$\frac{9n^2}{c}=\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}\right)\frac{1}{cz_1^2}.$$

Adding,

$$9\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}\right)=\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}\right)\left(\frac{1}{ax_1^2}+\frac{1}{by_1^2}+\frac{1}{cz_1^2}\right)$$

or $$\frac{1}{ax_1^2}+\frac{1}{by_1^2}+\frac{1}{cz_1^2}=9$$

$\therefore$ The required locus is

$$\frac{1}{ax^2}+\frac{1}{by^2}+\frac{1}{cz^2}=9.$$

Example 6. *Show that the plane*

(*a*) $x+2y+3z=2$ *touches the conicoid* $x^2-2y^2+3z^2=2$.

(*b*) $3x+12y-6z-17=0$ *touches the conicoid* (Bundelkhand 1985)

$$3x^2-6y^2+9z^2+17=0.$$

Find also the point of contact in each case.

Sol. (*a*) Let the plane $x+2y+3z=2$ ...(1)

touch the conicoid $x^2-2y^2+3z^2=2$ ...(2)

at the point (x_1, y_1, z_1).

The equation of tangent plane at (x_1, y_1, z_1) to (2) is

$$xx_1-2yy_1+3zz_1=2 \quad ...(3)$$

Since (3) and (1) are identical, $\therefore$ comparing (3) and (1), we get

$$\frac{x_1}{1}=\frac{-2y_1}{2}=\frac{3z_1}{3}=\frac{2}{2}$$

$\therefore \quad x_1=1, \quad y_1=-1, \quad z_1=1$ ...(4)

Now the plane (1) will touch (2) if the point of contact (x_1, y_1, z_1) *i.e.*, $(1, -1, 1)$ lies on the conicoid (2) *i.e.*, if

$$(1)^2-2(-1)^2+3(1)^2=2 \quad \text{or} \quad 1-2+3=2 \text{ or } 2=2$$

which is true.

Hence the plane (1) touches the conicoid (2) and the point of contact is (x_1, y_1, z_1) *i.e.*, $(1, -1, 1)$. | *from* (1)

(*b*) **Please try yourself.** [**Ans.** $(-1, 2, \frac{3}{2})$]

Example 7. *Find the equations to the tangent planes to the surfaces*

(*a*) $4x^2-5y^2+7z^2+13=0$, *parallel to the plane*

$$4x+20y-21z=0.$$

(*b*) $x^2-2y^2+3z^2=2$, *parallel to the plane*

$$x-2y+3z=0.$$ (M.D.U. 1987, 85)

Sol. (*a*) Any plane ∥ to $4x+20y-21z=0$ is

$$4x+20y-21z=K \quad ...(1)$$

| *Type* $lx+my+nz=p$

The given conicoid is

$$4x^2-5y^2+7z^2+13=0$$

or
$$4x^2-5y^2+7z^2=-13$$

or
$$-\frac{4}{13}x^2+\frac{5}{13}y^2-\frac{7}{13}z^2=1 \quad ...(2)$$

[*Form* $ax^2+by^2+cz^2=1$]

(1) will touch the conicoid (1) if

$$\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p^2$$

or if $\dfrac{(4)^2}{\left(-\frac{4}{13}\right)}+\dfrac{(20)^2}{\left(\frac{5}{13}\right)}+\dfrac{(-21)^2}{\left(-\frac{7}{13}\right)}=K^2$

or if $$-52+1040-819=K^2$$

or if $$K^2=1040-871=169 \qquad \therefore \quad K=\pm 13$$

Putting these values of K in (1), the required tangent planes are $$4x+20y-21z=\pm 13.$$

(*b*) Please try yourself. [Ans. $x-2y+3z=\pm 2$]

Example 8. *Find the co-ordinates of the point of contact of the plane $4x-6y+3z=5$ and the conicoid $2x^2-6y^2+3z^2=5$.*

(Bundelkhand 1984)

Sol. Let the plane

$$4x-6y+3z=5 \qquad \ldots(i)$$

touch the conicoid $$2x^2-6y^2+3z^2=5 \qquad \ldots(ii)$$

at (x_1, y_1, z_1).

The tangent plane to (*ii*) at (x_1, y_1, z_1) is

$$2xx_1-6yy_1+3zz_1=5 \qquad \ldots(iii)$$

As (*i*) and (*iii*) represent the same plane, so comparing them, we have

$$\frac{2x_1}{4}=\frac{-6y_1}{-6}=\frac{3z_1}{3}=\frac{5}{5}$$

which gives $x_1=2, y_1=1, z_1=1$

$\therefore$ Required point is (2, 1, 1).

Example 9. (*a*) *Find the equations to the two tangent planes which contain the line given by*

$$7x-10y-30=0,\ 5y-3z=0$$

and touch the conicoid $7x^2+5y^2+3z^2=60.$ [V. Imp.]

(Agra 1986, 84 ; M.D.U. 1984)

(*b*) *Find the equations of the tangent planes to*

$$2x^2-6y^2+3z^2=5$$

which pass through the line $x+9y-3z=0=3x-3y+6z-5.$

[Imp.] (M.D.U. 1986 ; K.U. 1983)

(*c*) *Find the equations of the tangent planes to*

$$7x^2-3y^2-z^2+21=0$$

which pass through the line

$$7x-6y+9=0,\ z=3.$$

Sol. (*a*) The given line is

$$7x+10y-30=0,\ 5y-3z=0$$

Any plane through this line is

$$7x+10y-30+k(5y-3z)=0$$

or $$7x+5(k+2)y-3kz=30 \qquad \ldots(1)$$

| *Form* $lx+my+nz=p$

The given ellipsoid is

$$7x^2+5y^2+3z^2=60$$

or $$\frac{7}{60}x^2+\frac{5}{60}y^2+\frac{3}{60}z^2=1 \qquad \ldots(2)$$

| Form $ax^2+by^2+cz^2=1$

The plane (1) touches the ellipsoid (2) if

$$\frac{(7)^2}{\left(\frac{7}{60}\right)}+\frac{25(k+2)^2}{\left(\frac{5}{60}\right)}+\frac{(-3k)^2}{\left(\frac{3}{60}\right)}=(30)^2$$

| Using $\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p^2$

or $$\frac{49\times 60}{7}+\frac{25(k+2)^2\times 60}{5}+\frac{9k^2\times 60}{3}=900$$

or $$420+300(k^2+4k+4)+180k^2=900$$

or $$480k^2+1200k+720=0$$

or $$2k^2+5k+3=0 \quad \text{or} \quad (2k+3)(k+1)=0$$

$\therefore$ $$k=-\frac{3}{2} \text{ or } -1$$

Putting these values of k in (1), the required tangent planes are

$$7x+5\left(-\frac{3}{2}+2\right)y+\frac{9}{2}z=30 \text{ and } 7x+5(-1+2)y+3z=30$$

or $$7x+\frac{5}{2}y+\frac{9}{2}z=30 \text{ and } 7x+5y+3z=30$$

or $$14x+5y+9z=60 \text{ and } 7x+5y+3z=30.$$

(b) The given line is

$$x+9y-3z=0=3x-3y+6z-5$$

Any plane through this line is

$$x+9y-3z+k(3x-3y+6z-5)=0$$

or $$x(1+3k)+3(3-k)y-3(1-2k)z=5k \qquad \ldots(1)$$

| Type $lx+my+nz=p$

and the given conicoid is

$$2x^2-6y^2+3z^2=5$$

or $$\frac{2}{5}x^2-\frac{6}{5}y^2+\frac{3}{5}z^2=1 \qquad \ldots(2)$$

| Form $ax^2+by^2+cz^2=1$

The plane (1) touches (2) if

$$\frac{(1+3k)^2}{\left(\frac{2}{5}\right)}+\frac{9(3-k)^2}{\left(-\frac{6}{5}\right)}+\frac{9(1-2k)^2}{\left(\frac{3}{5}\right)}=(5k)^2$$

| Using $\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p^2$

or $$\frac{5}{2}(1+3k)^2-\frac{45}{6}(3-k)^2+\frac{45}{3}(1-2k)^2=25k^2$$

or $$(1+3k)^2-3(3-k)^2+6(1-2k)^2=10k^2$$

| On dividing throughout by $\frac{5}{2}$

or $$1+9k^2+6k-3(9+k^2-6k)+6(1+4k^2-4k)-10k^2=0$$

or $$1+9k^2+6k-27-3k^2+18k+6+24k^2-24k-10k^2=0$$

or $20k^2-20=0$ or $k^2=1$ $\therefore k=\pm 1.$

Putting these values of k in (1), the required tangent planes are

$$(1+3)x+3(3-1)y-3(1-2)z=5$$

and $$(1-3)x+3(3+1)y-3(1+2)z=-5$$

or $4x+6y+3z-5=0$ and $-2x+12y-9z+5=0$

or $4x+6y+3z-5=0$ and $2x-12y+9z-5=0.$

(c) Please try yourself.

[Ans. $7x-6y-4z+21=0$; $14x-12y-z+21=0$]

Example 10. *Find the equations of the tangent plane to the surface $3x^2-6y^2+9z^2+17=0$ parallel to the plane $x+4y-2z=0$.*

(Kanpur 1987)

Sol. Any plane parallel to the given plane is

$$x+4y-2z=p \qquad \ldots(i)$$

If this plane touches the ellipsoid

$$3x^2-6y^2+9z^2+17=0$$

or $$3x^2-6y^2+9z^2=-17$$

or $$\left(-\frac{3}{17}\right)x^2-\left(\frac{6}{17}\right)y^2-\left(\frac{9}{17}\right)z^2=1$$

then the condition of tangency is

$$\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p^2$$

or $$\left(\frac{1^2}{(-3/17)}\right)+\left(\frac{4^2}{(6/17)}\right)+\left(\frac{2^2}{(-9/17)}\right)=p^2$$

or $$p^2=\left(-\frac{17}{3}\right)+\left(\frac{136}{3}\right)-\left(\frac{68}{9}\right)$$

or $$9p^2=-51+408-68=289$$

or $$p=\pm\sqrt{\frac{289}{9}}=\pm\left(\frac{17}{3}\right)$$

$\therefore$ From (1) the required tangent planes are

$$x+4y-2z=\pm\left(\frac{17}{3}\right)$$

or $$3x+12y-6z=\pm 17.$$

Example 11. *Find the equations to the tangent planes to $7x^2-3y^2-z^2+21=0$ which pass through the line $7x+6y+9=0$, $z=3$.* (M.D.U. 1983; K.U. 1985, 81)

Sol. Please try yourself [Ans. $7x-6y-4z+21=0$, $14x-12y-z+21=0$]

Example 12. *Find the condition that the line $\frac{x-2}{l}=\frac{y-1}{m}=\frac{z-3}{n}$ may touch the ellipsoid $3x^2+8y^2+z^2=c^2$.* (Agra 1983)

Sol. Given line is

$$\frac{x-2}{l}=\frac{y-1}{m}=\frac{z-3}{n}$$

see page 8

Given ellipsoid is

$$3x^2+8y^2+z^2=c^2$$

$$\Rightarrow \frac{3}{c^2}x^2+\frac{8}{c^2}y^2+\frac{1}{c^2}z^2=1.$$

The condition for becoming a tangent line is

$$a\alpha l+b\beta m+c\gamma n=0$$

i.e. $$\frac{3}{c^2}.(2).\,l+\frac{8}{c^2}.(1)\,m+\frac{1}{c^2}.3\,.\,n=0.$$

$$\Rightarrow \quad 6l+8m+3n=0.$$

Example 13. *If P is the point of contact of a tangent plane ABC to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ and PD, PE, PF are perpendiculars from P on the axes, prove that $OD.OA=a^2$, $OE.OB=b^2$, $OF.OC=c^2$; A, B, C being the points where the tangent plane at P meets the co-ordinate axes.*

Sol. Let $P\to(\alpha,\beta,\gamma)$ so that the equation of tangent plane ABC is

$$\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}+\frac{\gamma z}{c^2}=1$$

It meets the co-ordinate axes in A, B, C.

$$\therefore \quad OA=\frac{a^2}{\alpha},\quad OB=\frac{b^2}{\beta} \text{ and } OC=\frac{c^2}{\gamma}$$

Also PD, PE, PF are perpendiculars from P on the axes.

$$\therefore \quad OD=\alpha,\ OE=\beta \text{ and } OF=\gamma$$

Hence $OD\,.\,OA=\alpha\,.\,\frac{a^2}{\alpha}=a^2$

$$OE\,.\,OB=\beta\,.\,\frac{b^2}{\beta}=b^2$$

and $$OF\,.\,OC=\gamma\,.\,\frac{c^2}{\gamma}=c^2.$$

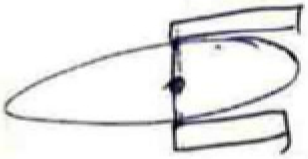
Example 14. *Show that the tangent planes at the extremities of any diameter of an ellipsoid are parallel.*

Sol. Let the equation of the ellipsoid be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad \ldots(i)$$

As its centre is (0, 0, 0) so any diameter of this ellipsoid is a line through (0, 0, 0) and its equation is given by

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \qquad \ldots(ii)$$



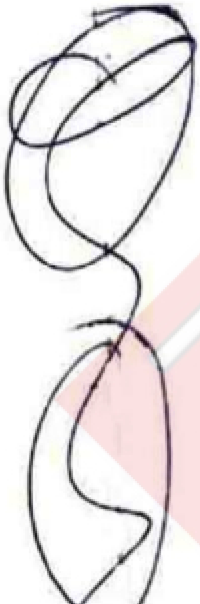
Any point on this diameter is (lr, mr, nr).

If this point lies on (i), then

$$\frac{l^2r^2}{a^2}+\frac{m^2r^2}{b^2}+\frac{n^2r^2}{c^2}=1$$

or $$r^2\left[\frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{c^2}\right]=1$$

or $$r=\pm\frac{1}{\sqrt{\left(\frac{l}{a}\right)^2+\left(\frac{m}{b}\right)^2+\left(\frac{n}{c}\right)^2}}=\pm k, \qquad \ldots(iii)$$



$\therefore$ The extremities of the diameter (ii) are

(lk, mk, nk) and $(-lk, -mk, -nk)$

where k is given by (iii).

Now the equation of the tangent plane to the ellipsoid (i) at (lk, mk, nk) is given by

$$\frac{xlk}{a^2}+\frac{ymk}{b^2}+\frac{znk}{c^2}=1$$

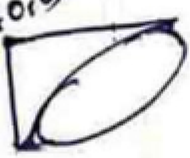
or $$\frac{lx}{a^2}+\frac{my}{b^2}+\frac{nz}{c^2}=\frac{1}{k} \qquad \ldots(iv)$$

Similarly the equation of the tangent plane to (i) at the other extremity $(-lk, -mk, -nk)$ of (ii) is

$$\frac{lx}{a^2}+\frac{my}{b^2}+\frac{nz}{c^2}=-\frac{1}{k} \qquad \ldots(v)$$

Since the equations (iv) and (v) differ in the constant terms only, so these represent parallel planes (each being a linear equation in x, y, z).

(k, 0, 0)



Example 15. *Through a fixed point $(k, 0, 0)$ pairs of perpendicular lines are drawn to the conicoid $ax^2+by^2+cz^2=1$. Show that the plane through any pair touches the cone*

$$\frac{(x-k)^2}{(b+c)(ak^2-1)}+\frac{y^2}{c(ak^2-1)-a}+\frac{z^2}{b(ak^2-1)-a}=0.$$

Sol. Any line through the point $(k, 0, 0)$ is

$$\frac{x-k}{l}=\frac{y-0}{m}=\frac{z-0}{n} \qquad \ldots(i)$$

Any point on (i) is $(k+lr, mr, nr)$, which is at a distance r from $(k, 0, 0)$.

$\therefore$ The distance of the points where the line (i) meets the given conicoid

$$ax^2+by^2+cz^2=1 \quad \ldots(ii)$$

are given by

$$a(k+lr)^2+b(mr)^2+c(nr)^2=1$$

or $$r^2(al^2+bm^2+cn^2)+2ak\,lr+(ak^2-1)=0 \quad \ldots(iii)$$

If the line (i) touches (ii) at $(k, 0, 0)$, then the two values of r given by (iii) must be coincident and the condition for the same is "$B^2=4AC$".

i.e. $$(2akl)^2=4(al^2+bm^2+cn^2)(ak^2-1)$$

or $$(al^2+bm^2+cn^2)(ak^2-1)=a^2k^2/2 \quad \ldots(iv)$$

Now let the two perpendicular tangent lines through $(k, 0, 0)$ be

$$\frac{x-k}{l_1}=\frac{y}{m_1}=\frac{z}{n_1},$$

and $$\frac{x-k}{l_2}=\frac{y}{m_2}=\frac{z}{n_2}, \quad \ldots(v)$$

Then from (iv), we get

$$(al_1^2+bm_1^2+cn_1^2)(ak^2-1)=a^2k^2l_1^2$$

and $$(al_2^2+bm_2^2+cn_2^2)(ak^2-1)=a^2k^2l_2^2$$

Adding these, we get

$$[a(l_1^2+l_2^2)+b(m_1^2+m_2^2)+c(n_1^2+n_2^2)](ak^2-1)=a^2k^2(l_1^2+l_2^2) \quad \ldots(vi)$$

If the line $\dfrac{(x-k)}{l_3}=\dfrac{(y-0)}{m_3}=\dfrac{(z-0)}{n_3}$ be the normal to the plane containing the tangent lines given by (v), then we obtain a set of three mutually perpendicular lines for whcih we have the relations $l_1^2+l_2^2+l_3^2=1=l_3^2+m_3^2+n_3^2$ etc.

i.e., $$l_1^2+l_2^2=m_3^2+n_3^2.$$

Similarly $m_1^2+m_2^2=n_3^2+l_3^2$, $n_1^2+n_2^2=l_3^2+m_3^2$

Substituting these values in (vi), we get

$$[a(m_3^2+n_3^2)+b(n_3^2+l_3^2)+c(l_3^2+m_3^2)](ak^2-1)=a^2k^2(m_3^2+n_3^2)$$

or $$l_3^2[(b+c)(ak^2-1)]+m_3^2[(a+c)(ak^2-1)-a^2k^2]+n_3^2[(a+b)(ak^2-1)-a^2k^2]=0$$

or $$l_3^2(b+c)(ak^2-1)+m_3^2[c(ak^2-1)-a]+n_3^2[b(ak^2-1)-a]=0 \quad \ldots(vii)$$

Eliminating l_3, m_3, n_3, between (vii) and $\dfrac{x-k}{l_3}=\dfrac{y}{m_3}=\dfrac{z}{n_3}$ we find that the normal to the plane containing the lines given by (v) generates the cone

$$(x-k)^2(b+c)(ak^2-1)+y^2[c(ak^2-1)-a]+z^2[b(ak^2-1)-a]=0$$

and the plane itself touches the reciprocal cone

$$\frac{(x-k)^2}{(b+c)(ak^2-1)}+\frac{y^2}{c(ak^2-1)-a}+\frac{z^2}{b(ak^2-1)-a}=0.$$

Example 16. *Prove that the equation to the two tangent planes to the conicoid*

$$ax^2+by^2+cz^2=1$$

which pass through the line

$$u\equiv lx+my+nz-p=0, \qquad u'\equiv l'x+m'y+n'z-p'=0 \text{ is}$$

$$u^2\left(\frac{l'^2}{a}+\frac{m'^2}{b}+\frac{n'^2}{c}-p'^2\right)-2uu'\left(\frac{ll'}{a}+\frac{mm'}{b}+\frac{nn'}{c}-pp'\right)$$
$$+u'^2\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}-p^2\right)=0$$

[V. Imp.]

Sol. Any plane through the line

$$u=0,\ u'=0 \text{ is } u+ku'=0 \qquad \ldots(1)$$

i.e., $$(lx+my+nz-p)+k(l'x+m'y+n'z-p')=0$$

or $$(l+kl')x+(m+km')y+(n+kn')z=p+kp' \qquad \ldots(2)$$

| *Form* $lx+my+nz=p$

This will be tangent plane to the conicoid

$$ax^2+by^2+cz^2=1$$

if $$\frac{(l+kl')^2}{a}+\frac{(m+km')^2}{b}+\frac{(n+kn')^2}{c}=(p+kp')^2$$

| *Using* $\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p^2$

or if $$k^2\left(\frac{l'^2}{a}+\frac{m'^2}{b}+\frac{n'^2}{c}-p'^2\right)+2k\left(\frac{ll'}{a}+\frac{mm'}{b}+\frac{nn'}{c}-pp'\right)$$
$$+\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}-p^2\right)=0$$

Putting $k=\frac{-u}{u'}$, from (1), the required equation is

$$\frac{u^2}{u'^2}\left(\frac{l'^2}{a}+\frac{m'^2}{b}+\frac{n'^2}{c}-p'^2\right)-2\frac{u}{u'}\left(\frac{ll'}{a}+\frac{mm'}{b}+\frac{nn'}{c}-pp'\right)$$
$$+\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}-p^2\right)=0$$

or $$u^2\left(\frac{l'^2}{a}+\frac{m'^2}{b}+\frac{n'^2}{c}-p'^2\right)-2uu'\left(\frac{ll'}{a}+\frac{mm'}{b}+\frac{nn'}{c}-pp'\right)$$
$$+u'^2\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}-p^2\right)=0$$

Hence the result.

Example 17. (a) *Show that the plane*

$$lx+my+nz=p$$

will touch the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1, \text{ if } a^2l^2+b^2m^2+c^2n^2=p^2.$$

[V. Imp.] (*K.U. 1987*)

(b) *Tangent planes are drawn to the ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

through the point (α, β, γ). *Prove that the perpendiculars to them through the origin generate the cone*

$$(\alpha x+\beta y+\gamma z)^2=a^2x^2+b^2y^2+c^2z^2$$

[Imp.] (*Allahabad 1984, 83, 80*)

(c) *Tangent planes are drawn to the conicoid*

$$ax^2+by^2+cz^2=1$$

through the point (α, β, γ). *Prove that the perpendiculars to them from the origin generate the cone*

$$(\alpha x+\beta y+\gamma z)^2=\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}.$$

[Imp.] (*K.U. 1986 ; Kanpur 1981*)

Sol. (a) The given plane is

$$lx+my+nz=p \quad \ldots(1)$$

and the ellipsoid is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \quad \ldots(2)$$

Let the plane (1) touch the ellpsoid (2) at (x_1, y_1, z_1). Then the plane (1) will be identical with the tangent plane at (x_1, y_1, z_1) to the surface (2) *i.e.*,

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}=1 \quad \ldots(3)$$

Comparing (1) and (3), we get

$$\frac{\frac{x_1}{a^2}}{l}=\frac{\frac{y_1}{b^2}}{m}=\frac{\frac{z_1}{c^2}}{n}=\frac{1}{p}$$

or

$$\frac{x_1}{a^2l}=\frac{y_1}{b^2m}=\frac{z_1}{c^2n}=\frac{1}{p}$$

$$\therefore \quad x_1=\frac{a^2l}{p},\ y_1=\frac{b^2m}{p},\ z_1=\frac{c^2n}{p} \quad \ldots(4)$$

Since (x_1, y_1, z_1) being the point of contact lies on the ellipsoid (2).

$$\therefore \quad \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=1$$

or $$\frac{1}{a^2}\left(\frac{a^2 l}{p}\right)^2+\frac{1}{b^2}\left(\frac{b^2 m}{p}\right)^2+\frac{1}{c^2}\left(\frac{c^2 n}{p}\right)^2=1 \qquad \Big|\ \textit{Using (4)}$$

or $$a^2l^2+b^2m^2+c^2n^2=p^2 \qquad \Big|\ \text{C.T.M.}$$

which is the required condition.

(*b*) Any plane through (α, β, γ) is

$$l(x-\alpha)+m(y-\beta)+n(z-\gamma)=0$$

or $$lx+my+nz=l\alpha+m\beta+n\gamma \qquad ...(1)$$

$\Big|$ *Form* $lx+my+nz=p$

This will touch the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

if $$a^2l^2+b^2m^2+c^2n^2=(l\alpha+m\beta+n\gamma)^2 \qquad ...(2)$$

$\Big|$ *Using* $a^2l^2+b^2m^2+c^2n^2=p^2$ [Part (*a*)]

Now equations of the normal to (1) through (0, 0, 0) are

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \qquad ...(3)$$

$\therefore$ Locus of the line (3) is [*eliminating* l, m, n *from* (2) *and* (3)]

$$a^2x^2+b^2y^2+c^2z^2=(\alpha x+\beta y+\gamma z)^2$$

which being a second degree homogeneous equation in x, y, z represents a cone.

(*c*) Any plane through (α, β, γ) is

$$l(x-\alpha)+m(y-\beta)+n(z-\gamma)=0$$

$$lx+my+nz=l\alpha+m\beta+n\gamma \qquad ...(1)$$

$\Big|$ *Form* $lx+my+nz=p$

If it is the tangent plane to the conicoid

$$ax^2+by^2+cz^2=1,$$

then $$\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=(l\alpha+m\beta+n\gamma)^2 \qquad ...(2)$$

$\Big|$ *Using* $\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p^2$

Now d.c.'s of the normal to plane (1) are proportional to l, m, n

$\Big|$ *Co-effis. of* x, y, z

$\therefore$ Equations of $\perp$ to (1) through (0, 0, 0) *i.e.*, the normal to (1) through the origin are

$$\frac{x-0}{l}=\frac{y-0}{m}=\frac{z-0}{n} \text{ or } \frac{x}{l}=\frac{y}{m}=\frac{z}{n} \qquad ...(3)$$

To find the locus of line (3), *we have to eliminate* l, m, n, *from* (3) *and* (2). Putting the values of l, m, n from (3) in (2), we get

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}=(\alpha x+\beta y+\gamma z)^2$$

which is the required equation of cone.

Example 18. *Obtain the tangent planes for the ellipsoid*

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c^2}=1$$

which are parallel to $lx+my+nz=0.$

If 2r is the distance between the planes, show that a line through the origin and perpendicular to the planes lies on the cone

$$x^2(a^2-r^2)+y^2(b^2-r^2)+z^2(c^2-r^2)=0.$$

[V. Imp.] (K.U. 1983 ; Rohilkhand 1982)

Sol. Any plane ∥ to $lx+my+nz=0$ is

$$lx+my+nz=p \qquad \ldots(1)$$

This will touch the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

if $\quad a^2l^2+b^2m^2+c^2n^2=p^2$

or if $\quad p=\pm\sqrt{a^2l^2+b^2m^2+c^2n^2}$

Putting these values of p in (1), the required tangent planes are

$$lx+my+nz=\sqrt{a^2l^2+b^2m^2+c^2n^2} \qquad \ldots(2)$$

and $$lx+my+nz=-\sqrt{a^2l^2+b^2m^2+c^2n^2} \qquad \ldots(3)$$

Now one point on the plane (2) is (putting $x=0,\ y=0$)

$$P=\left[0,\ 0,\ \frac{\sqrt{a^2l^2+b^2m^2+c^2n^2}}{n}\right]$$

Now distance between two ∥ planes is the ⊥ distance of a point on one plane from the other.

Since $2r$ is given to be the distance between two ∥ planes (2) and (3).

$\therefore\quad 2r=\perp$ distance of P from the plane (3)

$$=\frac{0+0+n.\dfrac{\sqrt{a^2l^2+b^2m^2+c^2n^2}}{n}+\sqrt{a^2l^2+b^2m^2+c^2n^2}}{\sqrt{l^2+m^2+n^2}}$$

$$=\frac{2\sqrt{a^2l^2+b^2m^2+c^2n^2}}{\sqrt{l^2+m^2+n^2}}$$

or $\quad r\sqrt{l^2+m^2+n^2}=\sqrt{a^2l^2+b^2m^2+c^2n^2}$

Squaring $r^2(l^2+m^2+n^2)=a^2l^2+b^2m^2+c^2n^2$

or $$l^2(a^2-r^2)+m^2(b^2-r^2)+n^2(c^2-r^2)=0 \qquad \ldots(4)$$

Now equations of the line through (0, 0, 0) and ⊥ to the tangent planes (2) or (3) are

$$\frac{x-0}{l}=\frac{y-0}{m}=\frac{z-0}{n} \text{ or } \frac{x}{l}=\frac{y}{m}=\frac{z}{n} \qquad \ldots(5)$$

Eliminating l, m, n from (5) and (4) [by putting the values of l, m, n from (5) in (4)], the required locus is

$$x^2(a^2-r^2)+y^2(b^2-r^2)+z^2(c^2-r^2)=0$$

which is a cone, being a homogenous equation in x, y, z.

Example 19. *If the line of intersection of two perpendicular tangent planes to the ellipsoid whose equation referred to rectangular axes is*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

passes through the fixed point $(0, 0, k)$, *show that it lies on the cone*

$$x^2(b^2+c^2-k^2)+y^2(c^2+a^2-k^2)+(z-k)^2(a^2+b^2)=0.$$

[Imp.]

Sol. The equation of the ellipsoid is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad \ldots(1)$$

Any line through $(0, 0, k)$ is

$$\frac{x-0}{l}=\frac{y-0}{m}=\frac{z-k}{n} \text{ or } \frac{x}{l}=\frac{y}{m}=\frac{z-k}{n} \qquad \ldots(2)$$

and any plane through this line (2) is

$$Lx+My+N(z-k)=0 \qquad \ldots(3)$$

where $$Ll+Mm+Nn=0 \qquad \ldots(4)$$

If the plane (3) *i.e.*, $Lx+My+Nz=Nk$ touches the ellipsoid (1), then

$$L^2a^2+M^2b^2+N^2c^2=N^2k^2 \qquad | \text{ Using } l^2a^2+m^2b^2+n^2c^2=p^2$$

or $$L^2a^2+M^2b^2+N^2(c^2-k^2)=0 \qquad \ldots(5)$$

If the tangent planes are $\perp$, their normals are also $\perp$.

Now the lines whose d.c.'s L, M, N, are given by the equations (4) and (5), are normals to the planes (3).

From (4), $L=-\dfrac{Mm+Nn}{l}$. Putting this in (5), we get

$$\frac{(Mm+Nn)^2}{l^2}a^2+M^2b^2+N^2(c^2-k^2)=0$$

or $$a^2(Mm+Nn)^2+b^2l^2M^2+l^2(c^2-k^2)N^2=0$$

or $$M^2(a^2m^2+b^2l^2)+2MNmna^2+N^2(c^2l^2+a^2n^2-l^2k^2)=0$$

Dividing throughout by N^2, we get

$$\frac{M^2}{N^2}(a^2m^2+b^2l^2)+2mna^2\frac{M}{N}+(c^2l^2+a^2n^2-l^2k^2)=0 \qquad \ldots(6)$$

which is a quadratic in $\dfrac{M}{N}$. If L_1, M_1, N_1 and L_2, M_2, N_2 are the d.c.'s of the two lines, then, $\dfrac{M_1}{N_1}, \dfrac{M_2}{N_2}$ are the roots of (6), so that

$$\frac{M_1M_2}{N_1N_2}=\frac{c^2l^2+n^2a^2-l^2k^2}{a^2m^2+b^2l^2}=\frac{(c^2-k^2)l^2+n^2a^2}{m^2a^2+l^2b^2}$$

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$$\therefore \quad \frac{M_1M_2}{(c^2-k^2)l^2+n^2a^2}=\frac{N_1N_2}{m^2a^2+l^2b^2} \quad \ldots(7)$$

Similarly, eliminating M between (4) and (5), we have

$$\frac{N_1N_2}{l^2b^2+m^2a^2}=\frac{L_1L_2}{b^2n^2+(c^2-k^2)m^2} \quad \ldots(8)$$

$$\therefore \quad \frac{L_1L_2}{(c^2-k^2)m^2+b^2n^2}=\frac{M_1M_2}{(c^2-k^2)l^2+n^2a^2}=\frac{N_1N_2}{a^2m^2+b^2l^2}$$

| From (7) and (8)

Since the two normals with d.c.'s L_1, M_1, N_1 and L_2, M_2, N_2 are $\perp$ $\therefore$

$$L_1L_2+M_1M_2+N_1N_2=0$$

or $$(c^2-k^2)m^2+b^2n^2+(c^2-k^2)l^2+n^2a^2+a^2m^2+b^2l^2=0$$

or $$l^2(b^2+c^2-k^2)+m^2(c^2+a^2-k^2)+n^2(a^2+b^2)=0 \quad \ldots(9)$$

Eliminating l, m, n from (2) and (9), the line (2) generates the cone

$$x^2(b^2+c^2-k^2)+y^2(c^2+a^2-k^2)+(z-k)^2(a^2+b^2)=0.$$

Hence the result.

Example 20. *Find the locus of the feet of perpendiculars from the origin to the tangent planes to the surface $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ which cuts off from the axes, intercepts the sum of whose reciprocals is equal to constant $\frac{1}{k}$.* (K.U. 1983)

Sol. The given ellipsoid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ...(1)

Let $P(x_1, y_1, z_1)$ be the foot of $\perp$ from $O(0, 0, 0)$ to any tangent plane to (1)

d.r.'s of OP are x_1-0, y_1-0, z_1-0 | $x_2-x_1, y_2-y_1, z_2-z_1$

or x_1, y_1, z_1

$\because$ OP is $\perp$ to the tangent plane,

$\therefore$ d.r.'s of OP are co-effs. of x, y, z in the equation of tangent plane.

$\therefore$ Equation of tangent plane (1) is

$$xx_1+yy_1+zz_1=p \quad \ldots(2)$$

$\because$ Plane (2) touches (1)

$$\therefore \quad a^2x_1^2+b^2y_1^2+c^2z_1^2=p^2 \quad \ldots(3)$$

| *Using $a^2l^2+b^2m^2+c^2n^2=p^2$*

Plane (2) meets X-axis ($y=0, z=0$) where

$$xx_1=p \quad \text{or} \quad x=\frac{p}{x_1}$$

Thus the plane cuts off intercept from the X-axis which is $\frac{p}{x_1}$. Similarly it cuts off intercepts $\frac{p}{y_1}$ and $\frac{p}{z_1}$ from Y and Z-axis.

The sum of reciprocals of intercepts $=\frac{1}{k}$ (given)

or $$\frac{x_1}{p}+\frac{y_1}{p}+\frac{z_1}{p}=\frac{1}{k}$$

or $$\frac{x_1+y_1+z_1}{p}=\frac{1}{k}$$

$\therefore$ $$p=k(x_1+y_1+z_1)$$

Eliminating p [By putting this value of p in (3)], we get

$$a^2x_1^2+b^2y_1^2+c^2z_1^2=k^2(x_1+y_1+z_1)^2$$

$\therefore$ Locus of foot of $\perp$ (x_1, y_1, z_1) is

$$a^2x^2+b^2y^2+c^2z^2=k^2(x+y+z)^2.$$

Example 21. *Prove that the locus of the foot of the perpendicular drawn from the centre of the ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

to any of its tangent planes is $a^2x^2+b^2y^2+c^2z^2=(x^2+y^2+z^2)^2$. [Imp.]

(*Kanpur 1981 ; K.U. 1986, 82 ; M.D.U. 1983 ; Allahabad 1986*)

Sol. The given conicoid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ...(1)

Let $P(x_1, y_1, z_1)$ be the foot of $\perp$ from centre $O(0, 0, 0)$ of (1) to any tangent plane to (1)

d.r.'s of OP are x_1, y_1, z_1

$\therefore$ Equation of tangent plane is

$xx_1+yy_1+zz_1=p$ ...(2) [$\because$ Tangent plane is $\perp$ to OP and hence co-effs. of x, y, z in tangent plane are d.r.'s of OP]

$\because$ Plane (2) touches ellipsoid (1)

$\therefore$ $a^2x_1^2+b^2y_1^2+c^2z_1^2=p^2$ ...(3) [*Using* $a^2l^2+b^2m^2+c^2n^2=p^2$]

Also $P(x_1, y_1, z_1)$ lies on (2) [$\because$ P is the foot of $\perp$ from (0, 0, 0) to the tangent plane (2)]

$\therefore$ $$x_1^2+y_1^2+z_1^2=p^2 \quad ...(4)$$

Eliminating p from (3) and (4) (by equating its values), we get

$$a^2x_1^2+b^2y_1^2+c^2z_1^2=(x_1^2+y_1^2+z_1^2)^2$$

$\therefore$ Locus of (x_1, y_1, z_1), the foot of $\perp$ is

$$a^2x^2+b^2y^2+c^2z^2=(x^2+y^2+z^2)^2.$$

Example 22. *If P is the point on the ellipsoid* $x^2+2y^2+\frac{1}{3}z^2=1$, *such that the perpendicular from the origin on the tangent plane at P is of unit length, show that P lies on one or other of the planes* $3y=\pm z$.

Sol. Let P be the point (x_1, y_1, z_1)

Since it lies on the ellipsoid

$$x^2+2y^2+\tfrac{1}{3}z^2=1 \quad \ldots(1)$$

$$\therefore \quad x_1^2+2y_1^2+\tfrac{1}{3}z_1^2=1 \quad \ldots(2)$$

Now equation of tangent plane at P (x_1, y_1, z_1) to (1) is

$$xx_1+2yy_1+\tfrac{1}{3}zz_1=1$$

or
$$3xx_1+6yy_1+zz_1-3=0 \quad \ldots(3)$$

The $\perp$ distance from (0, 0, 0) to (3) $=1$ (given)

$$\therefore \quad \pm\frac{0+0+0-3}{\sqrt{9x_1^2+36y_1^2+z_1^2}}=1$$

or
$$9x_1^2+36y_1^2+z_1^2=9$$

or
$$x_1^2+4y_1^2+\frac{1}{9}z_1^2=1 \quad \ldots(4)$$

Subtracting (2) from (4), we have

$$2y_1^2+\left(\frac{1}{9}-\frac{1}{3}\right)z_1^2=0$$

or
$$2y_1^2-\frac{2}{9}z_1^2=0$$

or
$$9y_1^2-z_1^2=0$$

$\therefore$ Locus of $P(x_1, y_1, z_1)$ is $9y^2-z^2=0$

or
$$(3y-z)(3y+z)=0$$

i.e.,
$$3y-z=0 \text{ or } 3y+z=0$$

Thus P lies either on $3y-z=0$ or $3y+z=0$

i.e., P lies on one of the planes $3y=\pm z$.

Hence the result.

Example 23. (*a*) *Prove that the locus of the point of intersection of three mutually perpendicular tangent planes to the ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

is
$$x^2+y^2+z^2=a^2+b^2+c^2.$$

[V. Imp.] (*Rajputana 1983*)

(*b*) *Prove that the locus of points from which three mutually perpendicular tangent planes can be drawn to touch the ellipse*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, z=0$$

is the sphere $x^2+y^2+z^2=a^2+b^2$. [Imp.]

Sol. (*a*) Please try yourself as in Art. 9.

(*b*) Let
$$lx+my+nz=p \quad \ldots(1)$$

be a tangent plane to the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, z=0 \quad \ldots(2)$$

The plane (1) meets the plane $z=0$ in the line $lx+my=p$.

The plane (1) touches the ellipse (2) if in the plane $z=0$ *i.e.*, the XY-plane, the line $lx+my=p$ touches ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

i.e., if $\qquad p=\sqrt{a^2l^2+b^2m^2}$ $\qquad$ [From co-ordinate Geometry]

Putting the value of p in (1), any tangent plane to the ellipse (2) is

$$lx+my+nz=\sqrt{a^2l^2+b^2m^2}.$$

Now let the three mutually $\perp$ tangent planes to the ellipse (1), be

$$l_1x+m_1y+n_1z=\sqrt{a^2l_1^2+b^2m_1^2}$$
$$l_2x+m_2y+n_2z=\sqrt{a^2l_2^2+b^2m_2^2}$$
$$l_3x+m_3y+n_3z=\sqrt{a^2l_3^2+b^2m_3^2}$$

Squaring and adding these equations

$$(l_1x+m_1y+n_1z)^2+(l_2x+m_2y+n_2z)^2+(l_3x+m_3y+n_3z)^2$$
$$=(a^2l_1^2+b^2m_1^2)+(a^2l_2^2+b^2m_2^2)+(a^2l_3^2+b^2m_3^2)$$

or $$x^2\Sigma l_1^2+y^2\Sigma m_1^2+z^2\Sigma n_1^2+2xy\Sigma l_1m_1+2yz\Sigma m_1n_1+2zx\Sigma n_1l_1=a^2\Sigma l_1^2+b^2\Sigma m_1^2$$

or $$x^2(1)+y^2(1)+z^2(1)+2xy(0)+2yz(0)+2zx(0)=a^2(1)+b^2(1)$$

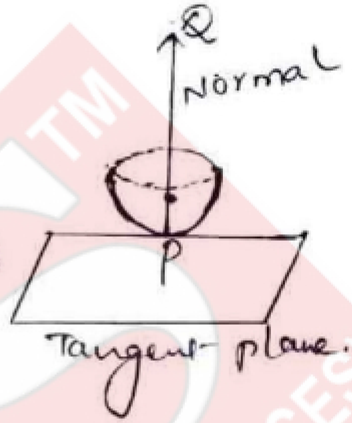
or $$x^2+y^2+z^2=a^2+b^2$$ $\qquad$ $\because$ l_1, m_1, n_1 etc., are the d.c.'s of 3 mutually $\perp$ lines

which is the required sphere.

Set-VII

Normal:

The normal at any point P of a surface (quadric) is a line through the point of contact P and perpendicular to the tangent plane at P.



Equations of the normal:

To find the equations of the normal at the point (x_1, y_1, z_1) of the conicoid $ax^2+by^2+cz^2=1$.

Sol^n: The given conicoid is $ax^2+by^2+cz^2=1$ ——— (1)

Equation of tangent plane at (x_1,y_1,z_1) to (1) is $axx_1+byy_1+czz_1=1$ ——— (2)

The d.c.'s of the normal to this plane are proportional to ax_1, by_1, cz_1

$\therefore$ Equations of the normal at $P(x_1,y_1,z_1)$ to (1) [i.e, a line through (x_1,y_1,z_1) and $\perp^r$ to the tangent plane (2)] are

$$\frac{x-x_1}{ax_1}=\frac{y-y_1}{by_1}=\frac{z-z_1}{cz_1} \quad \text{——— (3)}$$

Actual d.c.'s form:

If p is the length of the perpendicular distance from the centre (0,0,0) to the tangent plane (2)

then $p=\dfrac{1}{\sqrt{a^2x_1^2+b^2y_1^2+c^2z_1^2}}$

The d.c.'s of the normal at (x_1, y_1, z_1) to ① are proportional to ax_1, by_1, cz_1.

Dividing each by $\sqrt{a^2x_1^2+b^2y_1^2+c^2z_1^2}$

$\therefore$ The actual direction cosines are

$$\frac{ax_1}{\sqrt{a^2x_1^2+b^2y_1^2+c^2z_1^2}}, \frac{by_1}{\sqrt{a^2x_1^2+b^2y_1^2+c^2z_1^2}}, \frac{cz_1}{\sqrt{a^2x_1^2+b^2y_1^2+c^2z_1^2}}$$

$$\Rightarrow ax_1p, \ by_1p, \ cz_1p \quad \left(\because p = \frac{1}{\sqrt{a^2x_1^2+b^2y_1^2+c^2z_1^2}}\right)$$

$\therefore$ The equations of normal at (x_1, y_1, z_1) in actual direction cosines form are

$$\frac{x-x_1}{ax_1p} = \frac{y-y_1}{by_1p} = \frac{z-z_1}{cz_1p}.$$

→ S.T. The equations of the normal at the point (x_1, y_1, z_1) of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

$$\frac{x-x_1}{x_1/a^2} = \frac{y-y_1}{y_1/b^2} = \frac{z-z_1}{z_1/c^2}.$$

Note: The equations of the normal at (x_1, y_1, z_1) in the actual d.c.'s form are

$$\frac{x-x_1}{\frac{px_1}{a^2}} = \frac{y-y_1}{\frac{py_1}{b^2}} = \frac{z-z_1}{\frac{pz_1}{c^2}}.$$

where p = length of $\perp^r$ distance from the centre $(0,0,0)$ to the tangent plane of the ellipsoid.

(18)

→ The normal at a point P of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ meets the principal planes in G_1, G_2, G_3

(i) Show that $PG_1 : PG_2 : PG_3 = a^2 : b^2 : c^2$

(ii) If $PG_1^2 + PG_2^2 + PG_3^2 = k^2$, find the locus of P.

Solⁿ: The given ellipsoid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ —— (1)

Let $P(x_1, y_1, z_1)$ be any point on the surface.

then the equations of the normal at $P(x_1, y_1, z_1)$ to (1) in actual d.r.s form are

$$\frac{x-x_1}{\frac{p x_1}{a^2}} = \frac{y-y_1}{\frac{p y_1}{b^2}} = \frac{z-z_1}{\frac{p z_1}{c^2}} = r \text{ (say)}$$ —— (2)

where 'r' denotes the distance of any point on the normal from $P(x_1, y_1, z_1)$.

Any point on this normal is

$$\left(x_1 + \frac{r p x_1}{a^2},\ y_1 + \frac{r p y_1}{b^2},\ z + \frac{r p z_1}{c^2}\right)$$

If it lies on the YZ plane i.e, $x=0$,

then $x_1 + \frac{r p x_1}{a^2} = 0 \Rightarrow 1 + \frac{pr}{a^2} = 0$

$$\therefore r = \frac{-a^2}{p}.$$

$$\text{i.e } \boxed{PG_1 = \frac{-a^2}{p}}.$$

Similarly $\boxed{PG_2 = \frac{-b^2}{p}}$ & $\boxed{PG_3 = \frac{-c^2}{p}}$

$\therefore$ (i) $PG_1 : PG_2 : PG_3 = \frac{-a^2}{p} : \frac{-b^2}{p} : \frac{-c^2}{p}$

$$= a^2 : b^2 : c^2.$$

(ii) We are given that, $PG_1^2 + PG_2^2 + PG_3^2 = k^2$

$$\Rightarrow \frac{a^4}{p^2} + \frac{b^4}{p^2} + \frac{c^4}{p^2} = k^2$$

$$\Rightarrow \boxed{\frac{1}{p^2} = \frac{k^2}{a^4+b^4+c^4}}$$ —— (3)

But $p = \perp$. distance from $(0,0,0)$ on the tangent plane

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} + \frac{zz_1}{c^2} = 1 \text{ at } (x_1, y_1, z_1) \text{ to } (1).$$

$$= \frac{|-1|}{\sqrt{\frac{x_1^2}{a^4} + \frac{y_1^2}{b^4} + \frac{z_1^2}{c^4}}}$$

$$\therefore \frac{1}{p^2} = \frac{x_1^2}{a^4} + \frac{y_1^2}{b^4} + \frac{z_1^2}{c^4} = \frac{k^2}{a^4+b^4+c^4} \quad \text{(from (3))}$$

$\therefore$ Locus of $P(x_1, y_1, z_1)$ is [changing (x_1, y_1, z_1) to (x, y, z)]

$$\frac{x^2}{a^4} + \frac{y^2}{b^4} + \frac{z^2}{c^4} = \frac{k^2}{a^4+b^4+c^4} \quad \text{—— (4)}$$

Also p lies on (1),

Thus p lies on the curve of intersection of two ellipsoides (1) and (4).

→ Find the length of the normal chord through $P(x_1, y_1, z_1)$ of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, and prove that if it is equal to $4PG_3$, where G_3 is the point in which the normal chord meets the plane XOY, then p lies on the cone

$$\frac{x^2}{a^6}(2c^2 - a^2) + \frac{y^2}{b^6}(2c^2 - b^2) + \frac{z^2}{c^4} = 0$$

Sol^n: The given ellipsoid is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ —— (1)

Equations of the normal at $P(x_1, y_1, z_1)$ to (1) in the actual d.c's form are

$$\frac{x - x_1}{\frac{px_1}{a^2}} = \frac{y - y_1}{\frac{py_1}{b^2}} = \frac{z - z_1}{\frac{pz_1}{c^2}} = r \text{ (say)}$$

where $p = \frac{1}{\sqrt{\frac{x_1^2}{a^4} + \frac{y_1^2}{b^4} + \frac{z_1^2}{c^4}}}$ —— (2)

(19)

Any point on the normal at a distance 'r' from P is

$$A\left(x_1+\frac{px_1}{a^2}r,\ y_1+\frac{py_1}{b^2}r,\ z_1+\frac{pz_1}{c^2}r\right)$$

If 'r' is the length of the normal chord, then this point must lie on the ellipsoid (1).

$$\therefore \frac{1}{a^2}\left(x_1+\frac{px_1}{a^2}r\right)^2+\frac{1}{b^2}\left(y_1+\frac{py_1}{b^2}r\right)^2+\frac{1}{c^2}\left(z_1+\frac{pz_1}{c^2}r\right)^2=1.$$

$$\Rightarrow r^2p^2\left(\frac{x_1^2}{a^6}+\frac{y_1^2}{b^6}+\frac{z_1^2}{c^6}\right)+2pr\left(\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}+\frac{z_1^2}{c^4}\right)+\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1=0 \quad \text{--- (3)}$$

But since $P(x_1,y_1,z_1)$ lies on (1)

$$\therefore \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=1$$

$\therefore$ (3) becomes

$$r^2p^2\left(\frac{x_1^2}{a^6}+\frac{y_1^2}{b^6}+\frac{z_1^2}{c^6}\right)+2pr\left(\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}+\frac{z_1^2}{c^4}\right)=0$$

$$\Rightarrow rp\left(\frac{x_1^2}{a^6}+\frac{y_1^2}{b^6}+\frac{z_1^2}{c^6}\right)+2\left(\frac{1}{p^2}\right)=0\ (\because \text{from (2)})$$

$$\Rightarrow r=\frac{-2}{p^3\left(\frac{x_1^2}{a^6}+\frac{y_1^2}{b^6}+\frac{z_1^2}{c^6}\right)}$$

which is the length of the required normal chord.

Again the normal meets the xy-plane i.e, $z=0$ in G_3

where $z_1+\frac{pz_1}{c^2}r=0$

$$\Rightarrow r=-\frac{c^2}{p}.$$

$$\therefore PG_3=-\frac{c^2}{p}$$

Now if length of normal $= 4PG_3$, then

$$-\frac{2}{p^3\left[\frac{x_1^2}{a^6}+\frac{y_1^2}{b^6}+\frac{z_1^2}{c^6}\right]} = -4\cdot\frac{c^2}{p}.$$

$$\Rightarrow \frac{1}{p^2} = 2c^2\left(\frac{x_1^2}{a^6}+\frac{y_1^2}{b^6}+\frac{z_1^2}{c^6}\right)$$

$$\Rightarrow \frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}+\frac{z_1^2}{c^4} = 2c^2\left(\frac{x_1^2}{a^6}+\frac{y_1^2}{b^6}+\frac{z_1^2}{c^6}\right)$$

$(\because$ from ②$)$

$$\Rightarrow \frac{x_1^2}{a^6}(2c^2-a^2)+\frac{y_1^2}{b^6}(2c^2-b^2)+\frac{z_1^2}{c^4}=0$$

$\therefore$ Locus of $P(x_1, y_1, z_1)$ is

$$\frac{x^2}{a^6}(2c^2-a^2)+\frac{y^2}{b^6}(2c^2-b^2)+\frac{z^2}{c^4}=0$$

which is the required surface

Example 3. *The normal at a variable point P of the ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

meet the plane XOY in A and AQ is drawn parallel to OZ and equal to AP. Prove that the locus of Q is given by

$$\frac{x^2}{a^2-c^2}+\frac{y^2}{b^2-c^2}+\frac{z^2}{c^2}=1.$$

Find the locus of R if OR is drawn from the centre equal and parallel to AP.

Sol. The given ellipsoid is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad \text{...(1)}$$

Let $P(x_1, y_1, z_1)$ be the variable point on (1).

Then equations of the normal at $P(x_1, y_1, z_1)$ to (1) in the actual d.c.'s form are

$$\frac{x-x_1}{\frac{px_1}{a^2}}=\frac{y-y_1}{\frac{py_1}{b^2}}=\frac{z-z_1}{\frac{pz_1}{c^2}}=r \text{ (say)}$$

where $$p=\frac{1}{\sqrt{\left(\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}+\frac{z_1^2}{c^4}\right)}} \qquad \text{...(2)}$$

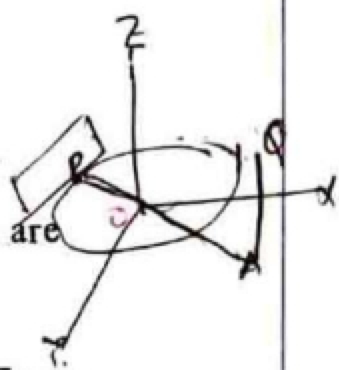
Any point on the normal at a distance r from P is

$$A\left(x_1+\frac{px_1}{a^2}r,\ y_1+\frac{py_1}{b^2}r,\ z_1+\frac{pz_1}{c^2}r\right) \qquad \text{...(3)}$$

The normal meets the XOY plane, *i.e.*, $z=0$, in A where

$$z_1+\frac{pz_1}{c^2}r=0 \quad \text{or} \quad r=\frac{-c^2}{p}$$

$$\therefore \qquad AP=r=\frac{-c^2}{p}.$$



Putting this value of r in (3), the co-ordinates of A are

$$\left(x_1-\frac{c^2x_1}{a^2},\ y_1-\frac{c^2y_1}{b^2},\ 0\right)$$

$\therefore$ Equations of line AQ through A and ∥ to OZ are

$$\frac{x-\left(x_1-\frac{c^2x_1}{a^2}\right)}{0}=\frac{y-\left(y_1-\frac{c^2y_1}{b^2}\right)}{0}=\frac{z-0}{1}=r \text{(say)}$$

where $r=\frac{-c^2}{p}$

If each member $=AQ=AP=\dfrac{-c^2}{p}$, then the co-ordinates of Q are given by

$$x=x_1-\frac{c^2x_1}{a^2},\quad y=y_1-\frac{c^2y_1}{b^2},\quad z=\frac{-c^2}{p}$$

or $$x=\frac{(a^2-c^2)}{a^2}x_1,\ y=\frac{b^2-c^2}{b^2}y_1,\quad z=\frac{-c^2}{p}\qquad z^2=\frac{c^4}{p^2}\ \ldots(4)$$

The locus of Q is obtained by eliminating (x_1, y_1, z_1) from the equations (4). Now

$$\boxed{\frac{z^2}{c^2}=\frac{c^2}{p^2}=\cancel{c^2\left(\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}+\frac{z_1^2}{c^4}\right)}}\qquad \Big|\ \textit{Using } (2)$$

$$\left(\frac{z}{c^2}\right)^2=\frac{1}{p^2}$$

$$\frac{z^2}{c^4}=\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}+\frac{z_1^2}{c^4}$$

$$z^2=c^2\left(\frac{x_1^2}{a^4}+\frac{y_1^2}{b^4}+\frac{z_1^2}{c^4}\right)$$

$$=\frac{c^2x_1^2}{a^4}+\frac{c^2y_1^2}{b^4}+\frac{z_1^2}{c^2}$$

$$=\frac{c^2x_1^2}{a^4}+\frac{c^2y_1^2}{b^4}+\left(1-\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}\right)\quad \Big|\ \because \text{P lies on (1)}\ \therefore \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=1$$

$$=\frac{(c^2-a^2)x_1^2}{a^4}+\frac{c^2-b^2}{b^4}y_1^2+1$$

$$=\frac{c^2-a^2}{a^4}\left(\frac{a^2x}{a^2-c^2}\right)^2+\frac{c^2-b^2}{b^4}\left(\frac{b^2y}{b^2-c^2}\right)+1\qquad \Big|\ \textit{From } (4)$$

$$=-\frac{x^2}{a^2-c^2}-\frac{y^2}{b^2-c^2}+1$$

or $$\frac{x^2}{a^2-c^2}+\frac{y^2}{b^2-c^2}+\frac{z^2}{c^2}=1$$

which is the required locus of Q.

Second part. Equations of OR, a line through O(0, 0, 0) and parallel to normal at P, are

$$\frac{x-0}{\frac{px_1}{a^2}}=\frac{y-0}{\frac{py_1}{b^2}}=\frac{z-0}{\frac{pz_1}{c^2}}=AP=-\frac{c^2}{p}\quad \text{for R}$$

Then if R be (x, y, z)

$$\therefore\quad x=-x_1\frac{c^2}{a^2},\quad y=-y_1\frac{c^2}{b^2},\ z=-z_1$$

so that $$x_1=\frac{-a^2x}{c^2},\qquad y_1=\frac{-b^2y}{c^2},\ z_1=-z.$$

But (x_1, y_1, z_1) lies on (1) $\therefore\ \dfrac{x_1^2}{a^2}+\dfrac{y_1^2}{b^2}+\dfrac{z_1^2}{c^2}=1$

or $$\frac{1}{a^2}\cdot\frac{a^4x^2}{c^4}+\frac{1}{b^2}\cdot\frac{b^4y^2}{c^4}+\frac{z^2}{c^2}=1$$

or $$a^2x^2+b^2y^2+c^2z^2=c^4$$

which is the required locus of R.

Example 4. *The normals to an ellipsoid at the points P, P′ meet a principal plane in G, G′; show that the plane which bisects PP′ at right angles, bisects GG′.*

Sol. Let the ellipsoid be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1 \qquad \ldots(1)$$

and let the principal plane be $\quad x=0 \qquad \ldots(2)$

Let the points P, P′ be (x_1, y_1, z_1) and (x_2, y_2, z_2) respectively. Then since P, P′ lie on the ellipsoid (1),

$$\therefore \quad \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=1$$

$$\frac{x_2^2}{a^2}+\frac{y_2^2}{b^2}+\frac{z_2^2}{c^2}=1$$

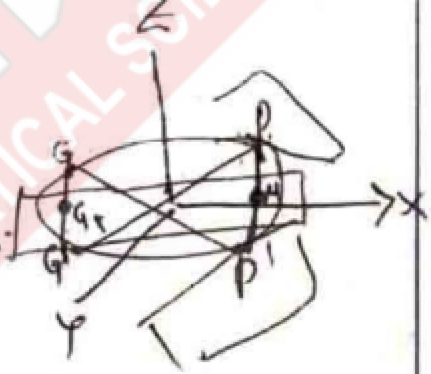
Subtracting, $\quad \frac{1}{a^2}(x_1^2-x_2^2)+\frac{1}{b^2}(y_1^2-y_2^2)+\frac{1}{c^2}(z_1^2-z_2^2)=0 \qquad \ldots(3)$

The normal at $P(x_1, y_1, z_1)$ to (1) is

$$\frac{x-x_1}{\frac{x_1}{a^2}}=\frac{y-y_1}{\frac{y_1}{b^2}}=\frac{z-z_1}{\frac{z_1}{c^2}}$$

This meets the plane $x=0$, where

$$\frac{0-x_1}{\frac{x_1}{a^2}}=\frac{y-y_1}{\frac{y_1}{b^2}}=\frac{z-z_1}{\frac{z_1}{c^2}}$$



or

$$-a^2=\frac{y-y_1}{\frac{y_1}{b^2}}=\frac{z-z_2}{\frac{z_1}{c^2}}$$

$$\therefore \quad y=y_1-y_1\frac{a^2}{b^2}, \qquad z=z_1-z_1\cdot\frac{a^2}{c^2}.$$

Thus the point G is $\left(0,\ y_1-\frac{a^2}{b^2}y_1,\ z_1-\frac{a^2}{c^2}z_1\right)$

Similarly G′ is $\left(0,\ y_2-\frac{a^2}{b^2}y_2,\ z_2-\frac{a^2}{c^2}z_2\right)$

The mid-point of GG′ is

$$G_1\left[0,\ \frac{y_1+y_2}{2}-\frac{a^2}{b^2}\left(\frac{y_1+y_2}{2}\right),\ \frac{z_1+z_2}{2}-\frac{a^2}{c^2}\left(\frac{z_1+z_2}{2}\right)\right]$$

i.e., $\quad G_1\left[0,\ \frac{y_1+y_2}{2}\left(1-\frac{a^2}{b^2}\right),\ \frac{z_1+z_2}{2}\left(1-\frac{a^2}{c^2}\right)\right]$

Now mid-point of PP′ is $M\left[\frac{x_1+x_2}{2},\ \frac{y_1+y_2}{2},\ \frac{z_1+z_2}{2}\right]$

and the d.c.'s of PP′ are proportional to $x_1-x_2,\ y_1-y_2,\ z_1-z_2$

$\therefore$ Equation of the plane through M, the mid-point of PP′ and $\perp$ to PP′ is

$$(x_1-x_2)\left(x-\frac{x_1+x_2}{2}\right)+(y_1-y_2)\left(y-\frac{y_1+y_2}{2}\right)+(z_1-z_2)\left(z-\frac{z_1+z_2}{2}\right)=0.$$

This passes through G_1 if

$$(x_1-x_2)\left(0-\frac{x_1+x_2}{2}\right)+(y_1-y_2)\left[\frac{y_1+y_2}{2}\left(1-\frac{a^2}{b^2}\right)-\frac{y_1+y_2}{2}\right]$$
$$+(z_1-z_2)\left[\frac{z_1+z_2}{2}\left(1-\frac{a^2}{c^2}\right)-\frac{z_1+z_2}{2}\right]=0$$

or if $\quad -\frac{1}{2}(x_1^2-x_2^2)-\frac{a^2}{2b^2}(y_1^2-y_2^2)-\frac{a^2}{2c^2}(z_1^2-z_2^2)=0$

or if $\frac{1}{a^2}(x_1^2-x_2^2)+\frac{1}{b^2}(y_1^2-y_2^2)+\frac{1}{c^2}(z_1^2-z_2^2)=0$

which is true by (3). Hence the result.

Example 5. *The normals at P and P′, points of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ meet the plane $z=0$ in G_3 and G_3' and make angles θ and θ' with PP′. Show that*

$$PG_3 \,.\, \cos\theta+P'G_3'\cos\theta'=0.$$

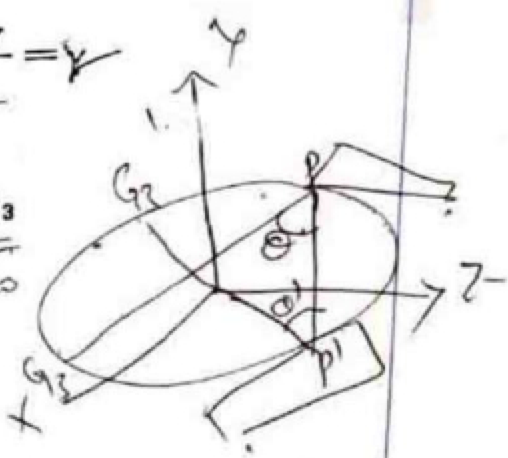
Sol. Let $P\to(\alpha,\beta,\gamma)$ and
$P'\to(\alpha',\beta',\gamma')$

Equations of normal at P are

$$\frac{x-\alpha}{\frac{p\alpha}{a^2}}=\frac{y-\beta}{\frac{p\beta}{b^2}}=\frac{z-\gamma}{\frac{p\gamma}{c^2}}=\gamma\text{ (say)}$$

It meets the plane $z=0$ where

$$\frac{x-\alpha}{\frac{p\alpha}{a^2}}=\frac{y-\beta}{\frac{p\beta}{b^2}}=\frac{0-\gamma}{\frac{p\gamma}{c^2}}=\gamma$$



$$\Rightarrow \qquad \gamma=-\frac{c^2}{p}=PG_3$$

$z=0$

Similarly

$$P'G_3'=-\frac{c^2}{p'}$$

D.C.'s of normal at P are

$$\frac{p\alpha}{a^2},\ \frac{p\beta}{b^2},\ \frac{p\gamma}{c^2}$$

D.C.'s of normal at P′ are

$$\frac{p'\alpha'}{a^2},\ \frac{p'\beta'}{b^2},\ \frac{p'\gamma'}{c^2}$$

D.R.'s of PP' are $\alpha'-\alpha, \beta'-\beta, \gamma'-\gamma$

$\therefore$ D.C.'s of PP' are

$$\frac{\alpha'-\alpha}{PP'}, \quad \frac{\beta'-\beta}{PP'}, \quad \frac{\gamma'-\gamma}{PP'}$$

Since θ is the angle between the normal at P and the line PP', we have

$$\cos\theta = \frac{p\alpha}{a^2} \cdot \frac{\alpha'-\alpha}{PP'} + \frac{p\beta}{b^2} \cdot \frac{\beta'-\beta}{PP'} + \frac{p\gamma}{c^2} \cdot \frac{\gamma'-\gamma}{PP'}$$

$$\therefore \quad PG_3 \cos\theta = -\frac{c^2}{p} \; \frac{p}{PP'}\left[\frac{\alpha(\alpha'-\alpha)}{a^2} + \frac{\beta(\beta'-\beta)}{b^2} + \frac{\gamma(\gamma'-\gamma)}{c^2}\right]$$

$$= -\frac{c^2}{PP'}\left[\frac{\alpha\alpha'}{a^2} + \frac{\beta\beta'}{b^2} + \frac{\gamma\gamma'}{c^2} - \left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2}\right)\right]$$

$$= -\frac{c^2}{PP'}\left[\frac{\alpha\alpha'}{a^2} + \frac{\beta\beta'}{b^2} + \frac{\gamma\gamma'}{c^2} - 1\right]$$

$\because$ $P(\alpha, \beta, \gamma)$ lies on the given ellipsoid

$\therefore$ $\dfrac{\alpha^2}{a^2} + \dfrac{\beta^2}{b^2} + \dfrac{\gamma^2}{c^2} = 1$

Similarly $P'G_3' \cos\theta'$

$$= -\frac{c^2}{PP'}\left[1 - \left(\frac{\alpha\alpha'}{a^2} + \frac{\beta\beta'}{b^2} + \frac{\gamma\gamma'}{c^2}\right)\right]$$

$$= \frac{c^2}{PP'}\left[\left(\frac{\alpha\alpha'}{a^2} + \frac{\beta\beta'}{b^2} + \frac{\gamma\gamma'}{c^2}\right) - 1\right]$$

$$= -PG_3 \cos\theta$$

$\Rightarrow$ $P'G_3' \cos\theta' + PG_3 \cos\theta = 0.$

Example 6. *Prove that two normals to the ellipsoid*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

lie in the plane

$$lx + my + nz = 0$$

and the line joining their feet has direction cosines proportional to

$$a^2(b^2-c^2)mn, \; b^2(c^2-a^2)nl, \; c^2(a^2-b^2)lm.$$

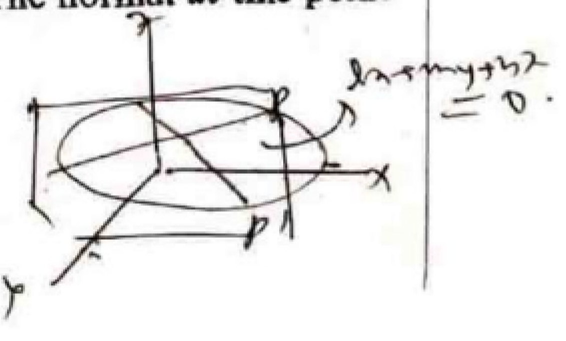
Also obtain the co-ordinates of these points.

Sol. The given ellipsoid is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \quad \ldots(1)$$

Let P (x_1, y_1, z_1) be any point on (1). The normal at this point P is

$$\frac{x-x_1}{\frac{x_1}{a^2}} = \frac{y-y_1}{\frac{y_1}{b^2}} = \frac{z-z_1}{\frac{z_1}{c^2}}$$



It lies on the plane $lx+my+nz=0$

if $$lx_1+my_1+nz_1=0 \quad ...(2)$$

and $$l\left(\frac{x_1}{a^2}\right)+m\left(\frac{y_1}{b^2}\right)+n\left(\frac{z_1}{c^2}\right)=0 \quad ...(3)$$

Solving (2) and (3) by cross-multiplication,

$$\frac{x_1}{\frac{mn}{c^2}-\frac{mn}{b^2}}=\frac{y_1}{\frac{nl}{a^2}-\frac{nl}{c^2}}=\frac{z_1}{\frac{lm}{b^2}-\frac{lm}{a^2}}$$

or $$\frac{x_1}{mna^2(c^2-b^2)}=\frac{y_1}{nlb^2(a^2-c^2)}=\frac{z_1}{lmc^2(b^2-a^2)}$$

or $$\frac{\frac{x_1}{a}}{mna(b^2-c^2)}=\frac{\frac{y_1}{b}}{nlb(c^2-a^2)}=\frac{\frac{z_1}{c}}{lmc(a^2-b^2)}$$

$$=\frac{\sqrt{\Sigma\frac{x_1^2}{a^2}}}{\sqrt{\Sigma m^2n^2a^2(b^2-c^2)^2}}=\frac{\pm 1}{\sqrt{\Sigma a^2m^2n^2(b^2-c^2)^2}}$$

$=\pm\frac{1}{d}$ (say) $\quad$ | $\because$ $P(x_1, y_1, z_1)$ lies on (1), $\therefore$ $\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=1$

where $$d=\sqrt{\Sigma a^2m^2n^2(b^2-c^2)^2}$$

$\therefore$ The required two points are

$$\left[\frac{a^2mn(b^2-c^2)}{d}, \frac{b^2nl(c^2-a^2)}{d}, \frac{c^2lm(a^2-b^2)}{d}\right]$$

and $$\left[-\frac{a^2mn(b^2-c^2)}{d}, -\frac{b^2nl(c^2-a^2)}{d}, -\frac{c^2lm(a^2-b^2)}{d}\right].$$

| *On taking —ve sign*

The d.c.'s of the line joining these two points are proportional

to $$-\frac{a^2mn(b^2-c^2)}{d}-\frac{a^2mn(b^2-c^2)}{d}, \ldots\ldots, \ldots\ldots$$

| *Using* $x_2-x_1, y_2-y_1, z_2-z_1$

i.e., $$a^2mn(b^2-c^2), b^2nl(c^2-a^2), c^2lm(a^2-b^2)$$

Hence the result.

Number of normals from a given point.

(23)

(I) To prove that there are six points on an ellipsoid the normals at which pass through a given point (α, β, γ).

Sol^n: Let the ellipsoid be $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. ——— (1)

Equations of the normal at (x_1, y_1, z_1) are

$$\frac{x-x_1}{x_1/a^2}=\frac{y-y_1}{y_1/b^2}=\frac{z-z_1}{z_1/c^2}=\lambda$$

If it passes through (α, β, γ) then

$$\frac{\alpha-x_1}{x_1/a^2}=\frac{\beta-y_1}{y_1/b^2}=\frac{\gamma-z_1}{z_1/c^2}=\lambda \text{ (say)}$$

From first and last members, we have

$$\alpha-x_1=\frac{\lambda x_1}{a^2}$$

$$\Rightarrow \alpha = x_1\left(1+\frac{\lambda}{a^2}\right)$$

$$= x_1\left(\frac{a^2+\lambda}{a^2}\right)$$

$$\left.\begin{aligned}&\Rightarrow x_1=\frac{\alpha a^2}{a^2+\lambda}\\ &\text{Similarly, } y_1=\frac{\beta b^2}{b^2+\lambda}\ ;\ z_1=\frac{\gamma c^2}{c^2+\lambda}\end{aligned}\right\} \text{——— (2)}$$

Since (x_1, y_1, z_1) lies on (1), we have

$$\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}=1$$

$$\Rightarrow \frac{1}{a^2}\left[\frac{a^2\alpha}{a^2+\lambda}\right]^2+\frac{1}{b^2}\left[\frac{b^2\beta}{b^2+\lambda}\right]^2+\frac{1}{c^2}\left[\frac{c^2\gamma}{c^2+\lambda}\right]^2=1. \quad (\because \text{from (2)})$$

$$\Rightarrow \frac{a^2\alpha^2}{(a^2+\lambda)^2}+\frac{b^2\beta^2}{(b^2+\lambda)^2}+\frac{c^2\gamma^2}{(c^2+\lambda)^2}=1.$$

$$\Rightarrow a^2\alpha^2(a^2+\lambda)^2(c^2+\lambda)^2+b^2\beta^2(a^2+\lambda)^2(c^2+\lambda)^2+ c^2\gamma^2(a^2+\lambda)^2(b^2+\lambda)^2=(a^2+\lambda)^2(b^2+\lambda)^2(c^2+\lambda)^2$$

which, being an equation of the sixth degree, gives six values of λ, to each of which there corresponds a point ($\cancel{\alpha,\beta,\gamma}$ x_1, y_1, z_1), as obtained from (2).

∴ There are six points on a central quadric (i.e., ellipsoid) the normals at which pass through a given point,

i.e., through a given point, six normals, in general, can be drawn to a central quadric.

Note: Foot of normal:

From (2) $\left(\frac{a^2\alpha}{a^2+\lambda}, \frac{b^2\beta}{b^2+\lambda}, \frac{c^2\gamma}{c^2+\lambda}\right)$ are the co-ordinates of the foot of normal.

Cubic curve through the feet of six normals from a point:

→ To show that the feet of the normals from (α,β,γ) to the ellipsoid are the six points of intersection of the ellipsoid and a certain cubic curve.

Sol^n: Let the ellipsoid be $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ——— (1)

If the normal at (x_1, y_1, z_1) to the ellipsoid passes through the given point (α, β, γ),

then $x_1 = \dfrac{a^2\alpha}{a^2+\lambda}$, $y_1 = \dfrac{b^2\beta}{b^2+\lambda}$, $z_1 = \dfrac{c^2\gamma}{c^2+\lambda}$. (from the above ①, equation ②)

The feet of the normals (x_1, y_1, z_1) lie on the curve (changing (x_1, y_1, z_1) to (x, y, z)).

$$x = \frac{a^2\alpha}{a^2+\lambda}, \quad y = \frac{b^2\beta}{b^2+\lambda}, \quad z = \frac{c^2\gamma}{c^2+\lambda}$$

where λ is a parameter. —— ②

To prove ~~that~~ ~~the~~ curve ② is a cubic curve.

To test the degree of curve we see its intersection with any arbitrary plane.

The curve ② meets an arbitrary plane

$$ux + vy + cz + d = 0 \quad \text{—— ③}$$

$$\Rightarrow u\frac{a^2\alpha}{a^2+\lambda} + v\frac{b^2\beta}{b^2+\lambda} + w\frac{c^2\gamma}{c^2+\lambda} + d = 0$$

$$\Rightarrow ua^2\alpha(b^2+\lambda)(c^2+\lambda) + vb^2\beta(a^2+\lambda)(c^2+\lambda) + wc^2\gamma(a^2+\lambda)(b^2+\lambda) + d(a^2+\lambda)(b^2+\lambda)(c^2+\lambda) = 0$$

which is a cubic in λ, giving three values of λ.

Thus the curve ② is cubic curve.

Since feet of the normals also lie on the ellipsoid ①, we can conclude that feet of the six normals from a given point are the six points of intersection of the ellipsoid and a cubic curve.

→ Show that in general six normals can be drawn from a given point (f, g, h) to the conicoid $ax^2+by^2+cz^2=1$. prove also that the six feet of the normals from (f, g, h) to the conicoid are the intersections of the conicoid with a cubic curve.

* Quadric cone through six concurrent normals:

To show that the six normals from (α, β, γ) to the ellipsoid lie on a cone of second degree.

Sol^n: Let the ellipsoid be $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$. —— (1)

Now since the normal at (x_1, y_1, z_1) passes through (α, β, γ) we have

$$x_1=\frac{a^2\alpha}{a^2+\lambda}, \quad y_1=\frac{b^2\beta}{b^2+\lambda}, \quad z_1=\frac{c^2\gamma}{c^2+\lambda} \quad \text{—— (2)}$$

(from (I); equation (2))

Let the equations of the normal from (α, β, γ) to the ellipsoid be —— (3)

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$$

But w.k.T the equations of the normal at (x_1, y_1, z_1) in the actual d.c.'s form are

$$\frac{x-x_1}{\frac{px_1}{a^2}}=\frac{y-y_1}{\frac{py_1}{b^2}}=\frac{z-z_1}{\frac{pz_1}{c^2}}$$

(from pg-1 (back) Note condition)

where $p=$ length of $\perp^r$ distance from the centre $(0,0,0)$ to the tangent plane of (1)

Then $l = \frac{p x_1}{a^2}$

$= \frac{p}{a^2} \frac{a^2 \alpha}{a^2+\lambda}$ (from ②)

$= \frac{p\alpha}{a^2+\lambda}$

$\Rightarrow a^2+\lambda = \frac{p\alpha}{l}$ ——④

Similarly $b^2+\lambda = \frac{p\beta}{m}$ ——⑤ ; $c^2+\lambda = \frac{p\gamma}{n}$ ——⑥

Multiplying ④, ⑤ & ⑥ by b^2-c^2, c^2-a^2, a^2-b^2

and adding,

we get

$$(b^2-c^2)(a^2+\lambda)+(c^2-a^2)(b^2+\lambda)+(a^2-b^2)(c^2+\lambda)$$
$$= (b^2-c^2)\frac{p\alpha}{l} + (c^2-a^2)\frac{p\beta}{m} + (a^2-b^2)\frac{p\gamma}{n}$$

$$\Rightarrow 0+\lambda(0) = \frac{p\alpha}{l}(b^2-c^2)+(c^2-a^2)\frac{p\beta}{m}+(a^2-b^2)\frac{p\gamma}{n}$$

$$\Rightarrow \frac{\alpha}{l}(b^2-c^2)+\frac{\beta}{m}(c^2-a^2)+\frac{\gamma}{n}(a^2-b^2)=0 \quad \text{——⑦}$$

Eliminating l, m, n from ③ and ⑦, the locus of the normals ③ is

$$\frac{\alpha(b^2-c^2)}{x-\alpha}+\frac{\beta(c^2-a^2)}{y-\beta}+\frac{\gamma(a^2-b^2)}{z-\gamma}=0$$

$$\Rightarrow \alpha(b^2-c^2)(y-\beta)(z-\gamma)+\beta(c^2-a^2)(x-\alpha)(z-\gamma)$$
$$+\gamma(a^2-b^2)(x-\alpha)(y-\beta)=0$$

which is a cone of second degree.

Hence the result.

(1983) prove that the feet of the six normals from (α, β, γ) to the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ lie on the curve of intersection of the ellipsoid and the cone $\frac{a^2(b^2-c^2)\alpha}{x}+\frac{b^2(c^2-a^2)\beta}{y}+\frac{c^2(a^2-b^2)\gamma}{z}=0$

Sol^n: The ellipsoid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Equations ~~of~~ at (x_1, y_1, z_1) are

$$\frac{x-x_1}{x_1/a^2}=\frac{y-y_1}{y_1/b^2}=\frac{z-z_1}{z_1/c^2}.$$

If it passes through (α, β, γ) then

$$\frac{\alpha-x_1}{x_1/a^2}=\frac{\beta-y_1}{y_1/b^2}=\frac{\gamma-z_1}{z_1/c^2}=\lambda \text{ (say)}$$

Then six feet of the normals from (α, β, γ) are given by

$$x_1=\frac{a^2\alpha}{a^2+\lambda}$$

$$y_1=\frac{b^2\beta}{b^2+\lambda} \text{ and } z_1=\frac{c^2\gamma}{c^2+\lambda}$$

These give

$$a^2+\lambda=\frac{a^2\alpha}{x_1}, \quad b^2+\lambda=\frac{b^2\beta}{y_1}$$

$$\text{and } c^2+\lambda=\frac{c^2\gamma}{z_1}.$$

Multiplying these equations by b^2-c^2, c^2-a^2, a^2-b^2 and adding, we get | **Note this step**

$$0+\lambda(0)=\frac{a^2\alpha(b^2-c^2)}{x_1}+\frac{b^2\beta(c^2-a^2)}{y_1}+\frac{c^2\gamma(a^2-b^2)}{z_1}$$

$\therefore$ (x_1, y_1, z_1) *i.e.*, the feet of the normals, lie on the cone

$$\frac{a^2\alpha(b^2-c^2)}{x}+\frac{b^2(c^2-a^2)\beta}{y}+\frac{c^2(a^2-b^2)\gamma}{z}=0$$

Also the feet (x_1, y_1, z_1) of the normals lie on the ellipsoid (1). Thus the feet of the six normals lie on the curve of intersection of the ellipsoid and the above cone.

Note. *In the equation of the cone through the feet of six normals from a point to an ellipsoid.*

Co-eff. of $x^2=0$, co-eff. of $y^2=0$, co-eff. of $z^2=0$, constant term$=0$. **[Remember]**

Example 2. *If P, Q, R ; P', Q', R' are the feet of six normals from a point to the ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

and the plane PQR is given by

$$lx+my+nz=p\ ;$$

then the plane $P'Q'R'$ is given by

$$\frac{x}{a^2l}+\frac{y}{b^2m}+\frac{z}{c^2n}+\frac{1}{p}=0. \qquad (K.U.\ 1984)$$

[Imp.]

Sol. The equation of the ellipsoid is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

or
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0 \qquad \text{...(1)}$$

and that the plane PQR is $lx+my+nz-p=0$ ...(2)

Let the required equation of plane $P'Q'R'$ be

$$l'x+m'y+n'z-p'=0 \qquad \text{...(3)}$$

The joint equation of the planes PQR and $P'Q'R'$ is

$$(lx+my+nz-p)(l'x+m'y+n'z-p')=0 \qquad \text{...(4)}$$

The eqution of conicoid through the points of intersection of the ellipsoid (1) and pair of planes (4) is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)+k(lx+my+nz-p)(l'x+m'y+n'z-p')=0 \qquad \text{...(5)}$$

If it is the same as the equation of the cone through the feet P, Q, R ; P′, Q′, R′ of the six normals from the given point to the ellipsoid, then

$$Co\text{-}eff.\ of\ x^2=0 \quad i.e., \ \frac{1}{a^2}+kll'=0 \quad \text{or} \quad l'=-\frac{1}{kla^2}$$

$$Co\text{-}eff.\ of\ y^2=0 \quad i.e., \ \frac{1}{b^2}+kmm'=0 \quad \therefore \quad m'=-\frac{1}{kmb^2}$$

$$Co\text{-}eff.\ of\ z^2=0 \quad i.e., \ \frac{1}{c^2}+knn'=0 \quad \therefore \quad n'=-\frac{1}{knc^2}$$

$$Constant\ term=0 \quad i.e., \ -1+kpp'=0 \quad \therefore \quad p'=\frac{1}{kp}.$$

Putting these values of l', m', n', p' in (3), the required plane P′Q′R′ is

$$-\frac{x}{kla^2}-\frac{y}{kmb^2}-\frac{z}{knc^2}-\frac{1}{kp}=0$$

or
$$\frac{x}{la^2}+\frac{y}{mb^2}+\frac{z}{nc^2}+\frac{1}{p}=0.$$

Hence the result.

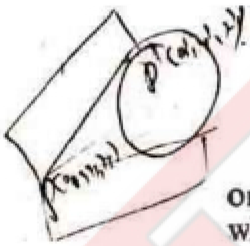
Article 14. Plane of Contact.

To find the equation of plane of contact of the point (x_1, y_1, z_1) with respect to conicoid $ax^2+by^2+cz^2=1$.

Let (x', y', z') be the point of contact any tangent plane to the conicoid $ax^2+by^2+cz^2=1$ ...(1)

$\therefore$ Tangent plane at (x', y', z') to (1) is

$$axx'+byy'+czz'=1$$



If it passes through the given point (x_1, y_1, z_1), then

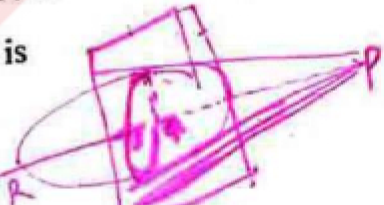
$$ax_1x'+by_1y'+cz_1z'=1$$

$\therefore$ Locus of the points of contact (x', y', z') is

$$ax_1x+by_1y+cz_1z=1$$

or
$$\mathbf{axx_1+byy_1+czz_1=1}$$

which is the required plane of contact.



Article 15. Polar plane of a point.

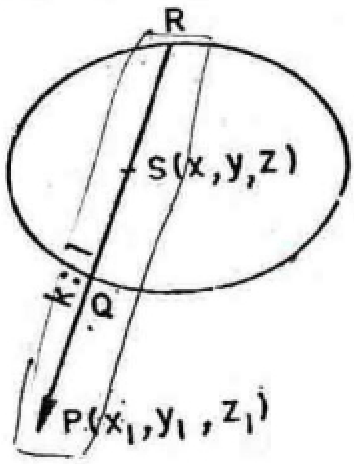
To find the equation of the polar plane of the point (x_1, y_1, z_1) w.r.t. the central conicoid $ax^2+by^2+cz^2=1$.

(M.D.U. 1985 ; K U. 1986, 85 ; Manipur 1983)

The given conicoid is

$$ax^2+by^2+cz^2=1 \qquad ...(1)$$

Let $P(x_1, y_1, z_1)$ be the given point and let PQR be any line through P which meets (1) in Q and R.



Let $S(x, y, z)$ be the harmonic conjugate of P w.r.t. Q and R.

Let Q divide PS in the ratio $k:1$.

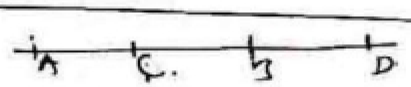
Then co-ordinates of Q are

$$\left(\frac{kx+x_1}{k+1}, \frac{ky+y_1}{k+1}, \frac{kz+z_1}{k+1}\right).$$

Also the points C & D are called harmonic conjugates w.r.t A and B. Thus C is harmonic conjugate of D and vice versa.

AC:BC
AD:BD

Harmonic Division:



If a line AB is divided internally & externally in the same same ratio at C & D then C and D are said to divide AB harmonically.

Since Q lies on the conicoid (1),

$$\therefore \quad a\left(\frac{kx+x_1}{k+1}\right)^2+b\left(\frac{ky+y_1}{k+1}\right)^2+c\left(\frac{kz+z_1}{k+1}\right)^2=1$$

or $$a(kx+x_1)+b(ky+y_1)^2+c(kz+z_1)^2-(k+1)^2=0$$

or $$k^2(ax^2+by^2+cz^2-1)+2k(axx_1+byy_1+czz_1-1)$$
$$+(ax_1^2+by_1^2+cz_1^2-1)=0 \quad \ldots(2)$$

which is a quadratic equation in k.

Since PS is divided harmonically, *i.e.*, internally and externally in the same ratio at Q and R, $\therefore$ the quadratic (2) has equal and opposite roots.

$\therefore$ Sum of roots$=0$ *i.e.*, coeff. of $k=0$

or $$axx_1+byy_1+czz_1-1=0$$

or $$\mathbf{axx_1+byy_1+czz_1=1} \quad | \text{ C.T.M.}$$

which is the equation of required polar plane of P.

Cor. *If P lies on the conicoid, the polar plane at P becomes the tangent plane at P.*

Article 16. Pole of a given plane.

To find the pole of the plane $lx+my+nz=p$, w.r.t. the conicoid $ax^2+by^2+cz^2=1$.

Let (x_1, y_1, z_1) be the required pole.

Then the polar plane of (x_1, y_1, z_1) w.r.t. conicoid

$$ax^2+by^2+cz^2=1$$

i.e., $$axx_1+byy_1+czz_1=1 \quad \ldots(1)$$

must be identical with the given plane

$$lx+my+nz=p \quad \ldots(2)$$

Comparing (1) and (2), we have

$$\frac{ax_1}{l}=\frac{by_1}{m}=\frac{cz_1}{n}=\frac{1}{p}$$

$$\therefore \quad x_1=\frac{l}{ap},\ y_1=\frac{m}{bp},\ z_1=\frac{n}{cp}.$$

Thus the pole is $\left(\frac{\mathbf{l}}{\mathbf{ap}}, \frac{\mathbf{m}}{\mathbf{bp}}, \frac{\mathbf{n}}{\mathbf{cp}}\right)$.

Example. *Prove that the locus of the poles of the tangent planes to $a^2x^2+b^2y^2-c^2z^2=1$ with respect to $\alpha^2x^2+\beta^2y^2+\gamma^2z^2=1$ is the hyperboloid of one sheet. Find its equation.*

Sol. Let $lx+my+nz=p$ ...(1) be a tangent plane

to $a^2x^2+b^2y^2-c^2z^2=1$ ...(2)

$$\therefore \quad \frac{l^2}{a^2}+\frac{m^2}{b^2}+\frac{n^2}{-c^2}=p^2 \ldots(3) \quad \Big| \text{ Using } \frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p^2$$

condition of tangency

Let (x_1, y_1, z_1) be the pole of plane (1) w.r.t.

$$\alpha^2x^2+\beta^2y^2+\gamma^2z^2=1 \quad \ldots(4)$$

$\therefore$ Equation of polar plane of (x_1, y_1, z_1) w.r.t. (4) is

$$\alpha^2xx_1+\beta^2yy_1+\gamma^2zz_1=1 \quad \ldots(5)$$

Comparing (1) and (5),

$$\frac{\alpha^2x_1}{l}=\frac{\beta^2y_1}{m}=\frac{\gamma^2z_1}{n}=\frac{1}{p}$$

From first and fourth members

$l=\alpha^2px_1$, similarly $m=\beta^2py_1$ and $n=\gamma^2pz_1$

Putting these values of l, m, n in (3) [To eliminate l, m, n], we have

$$\frac{\alpha^4p^2x_1^2}{a^2}+\frac{\beta^4p^2y_1^2}{b^2}+\frac{\gamma^4p^2z_1^2}{-c^2}=p^2$$

Cancelling p^2, $\dfrac{\alpha^4x_1^2}{a^2}+\dfrac{\beta^4y_1^2}{b^2}-\dfrac{\gamma^4z_2^2}{c^2}=1$

$\therefore$ Locus of (x_1, y_1, z_1) [pole of (1) *w.r.t.* (4)] is

$$\frac{\alpha^4}{a^2}x^2+\frac{\beta^4}{b^2}y^2-\frac{\gamma^4}{c^2}z^2=1$$

which is a hyperboloid of one sheet. [$\because$ co-effs. of x^2 and y^2 are positive, but co-eff. of z^2 is negative]

Article 17. Conjugate points and conjugate planes.

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points and let the conicoid be

$$ax^2+by^2+cz^2=1. \quad \ldots(1)$$

Then polar plane of (x_1, y_1, z_1) *w.r.t.* (1) is

$$axx_1+byy_1+czz_1=1$$

If it passes through $Q(x_2, y_2, z_2)$, then

$$ax_1x_2+by_1y_2+cz_1z_2=1$$

The symmetry of this result shows that the polar plane of Q also passes through P.

The two points such that the polar plane of each passes through the other are called the **conjugate points.**

Similarly it can be easily shown that if the pole of a plane S_1 lies on another plane S_2, then pole of S_2 must lie on S_1. *Two such planes (as S_2 and S_1 here) are called conjugate planes.*

Article 18. Polar lines

Two lines such that the polar plane of any point on one line passes through the other line are called **conjugate lines** *or* **polar lines.**

Polar of a line.

To find the equations of the polar of the line

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \text{ w.r.t. the conicoid } ax^2+by^2+cz^2=1.$$

(*K.U. 1986*)

The given conicoid is $ax^2+by^2+cz^2=1$ ...(1)

and the given line is $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ ...(2)

Any point on line (2) is $(lr+x_1, mr+y_1, nr+z_1)$.

It's polar plane w.r.t. conicoid (1) is

$$ax(lr+x_1)+by(mr+y_1)+cz(nr+z_1)=1$$

or $$axx_1+byy_1+czz_1-1+r(alx+bmy+cnz)=0.$$

This passes through the line

$$\left.\begin{aligned} axx_1+byy_1+czz_1-1=0 \\ alx+bmy+cnz=0 \end{aligned}\right\}$$

for all values of r.

Hence the equations of the polar line of (2) are

$$\mathbf{axx_1+byy_1+czz_1=1,\ alx+bmy+cnz=0.}$$

Method to write down the polar of

$$\frac{\mathbf{x-x_1}}{\mathbf{l}}=\frac{\mathbf{y-y_1}}{\mathbf{m}}=\frac{\mathbf{z-z_1}}{\mathbf{n}}$$

w.r.t. a central conicoid (equation in the standard form).

1. *Write down the polar plane of* (x_1, y_1, z_1) *w.r.t. conicoid thus getting* $axx_1+byy_1+czz_1=1.$

2. *Write down the polar plane of* (l, m, n) *and omit the constant term thus getting* $alx+bmy+cnz=0.$

3. *The above two equations are the required equations of the polar.*

Example 1. *Show that the equations of the polar of the line*

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$$

w.r.t. quadric $x^2-2y^2+3z^2=4$ *are* $\frac{x+6}{3}=\frac{y-2}{3}=z-2.$ (Kanpur 58)

Sol. The given line is $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ ...(1)

and the conicoid is $x^2-2y^2+3z^2=4.$ ...(2)

Any point on (1) is $(2r+1, 3r+2, 4r+3)$.

Polar plane of this point w.r.t. (2) is

$$x(2r+1)-2y(3r+2)+3z(4r+3)=4$$

or $$x+2rx-6yr-4y+12rz+9z-4=0$$

or $$(x-4y+9z-4)+2r(x-3y+6z)=0$$

which passes through the line

$$\left.\begin{aligned} x-4y+9z-4=0 \\ x-3y+6z=0 \end{aligned}\right]\begin{aligned} &...(3) \text{ for all values of } r \\ &...(4) \end{aligned}$$

$\therefore$ Equations (3) and (4) are the equations of the polar line of (1) w.r.t. conicoid (2).

To reduce the line given by (3) and (4) in symmetrical form.

To find d.r.'s of this line [omitting constant terms in (3) and (4)], we get

$$x-4y+9z=0$$
$$x-3y+6z=0$$

$$\therefore \quad \frac{x}{-24+27}=\frac{y}{9-6}=\frac{z}{-3+4} \text{ or } \frac{x}{3}=\frac{y}{3}=\frac{z}{1}.$$

Thus the d.r.'s of polar line are 3, 3, 1.

For any point put $z=2$ in (3) and (4).

(*As suggested by the question*)

$$\therefore \quad x-4y+14=0 \quad \text{and} \quad x-3y+12=0$$

Solving, we have $x=-6$, $y=2$. Also $z=2$.

Hence one point on the polar is $(-6, 2, 2)$.

Thus the equations of the polar line in the symmetrical form are

$$\frac{x+6}{3}=\frac{y-2}{3}=\frac{z-2}{1}.$$

Hence the result.

Example 2. *Find the condition that the lines*

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \text{ and } \frac{x-\alpha'}{l'}=\frac{y-\beta'}{m'}=\frac{z-\gamma'}{n'}$$

should be polar with respect to the conicoid $ax^2+by^2+cz^2=1$.

Sol. The polar of the line $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}$ w.r.t. the conicoid $ax^2+by^2+cz^2=1$ is given by

$$a\alpha x+b\beta y+c\gamma z-1=0, \; alx+bmy+cnz=0 \qquad ...(1)$$

[*See Article 18 above*]

But
$$\frac{x-\alpha'}{l'}=\frac{y-\beta'}{m'}=\frac{z-\gamma'}{n'} \qquad ...(2)$$

is given to be polar. Hence (2) should be identical with (1), *i.e.*, line (2) should lie on both the planes given by (1).

For this the point $(\alpha', \beta', \gamma')$ should lie on both the planes and the line (2) should be $\perp$ to the normal of each of the planes in (1).

$\therefore$ The required conditions are

$$a\alpha\alpha'+b\beta\beta'+c\gamma\gamma'=1$$
$$a\alpha l'+b\beta m'+c\gamma n'=0$$
$$a\alpha' l+b\beta' m+c\gamma' n=0$$
$$all'+bmm'+cnn'=0.$$

Example 3. *Find the locus of straight lines drawn through a fixed point (α, β, γ) at right angles to their polars with respect to*

$$ax^2+by^2+cz^2=1.$$

(M.D.U. 1984 ; Kanpur 1982, 88 ; Lucknow 1980)

Sol. Any line through (α, β, γ) is $\dfrac{x-\alpha}{l}=\dfrac{y-\beta}{m}=\dfrac{z-\gamma}{n}$ ...(1)

The polar of (1) w.r.t. the given conicoid is

$$\left.\begin{array}{l} a\alpha x+b\beta y+c\gamma z=1 \\ alx+bmy+cnz=0 \end{array}\right\} \quad ...(2)$$

Omitting the constant terms in (2), the d.c.'s of line (2) are, given by

$$a\alpha x+b\beta y+c\gamma z=0$$
$$alx+bmy+cnz=0$$

$$\therefore \quad \frac{x}{bc(n\beta-m\gamma)}=\frac{y}{ca(l\gamma-n\alpha)}=\frac{z}{ab(m\alpha-l\beta)}$$

Thus the d.c.'s of line (2) are proportional to

$$bc(n\beta-m\gamma),\ ca(l\gamma-n\alpha),\ ab(m\alpha-l\beta).$$

The lines (1) and (2) are $\perp$ | *given*

$$\therefore \quad lbc(n\beta-m\gamma)+mca(l\gamma-n\alpha)+nab(m\alpha-l\beta)=0$$

or $$\alpha mna(b-c)+\beta nlb(c-a)+\gamma lmc(a-b)=0$$

or $$\Sigma \frac{\alpha}{l}\left(\frac{1}{c}-\frac{1}{b}\right)=0 \quad ...(3)$$

on dividing by *lmn abc*.

To find the locus of (1), eliminating l, m, n from (1) and (3), we have

$$\Sigma \left(\frac{\alpha}{x-\alpha}\right)\left(\frac{1}{c}-\frac{1}{b}\right)=0 \quad \text{or} \quad \Sigma \left(\frac{\alpha}{x-\alpha}\right)\left(\frac{1}{b}-\frac{1}{c}\right)=0.$$

Example 4. *If P, Q are the points,* (x_1, y_1, z_1) (x_2, y_2, z_2), *the polar of PQ w.r.t.* $ax^2+by^2+cz^2=1$ *is given by*

$$axx_1+byy_1+czz_1=1,\ axx_2+byy_2+czz_2=1.$$

Sol. Equations of line PQ are $\dfrac{x-x_1}{x_2-x_1}=\dfrac{y-y_1}{y_2-y_1}=\dfrac{z-z_1}{z_2-z_1}$

The polar of this line w.r.t. the given conicoid is

$$axx_1+byy_1+czz_1=1 \quad ...(1)$$

and $$ax(x_2-x_1)+by(y_2-y_1)+cz(z_2-z_1)=0 \quad ...(2)$$

Adding (1) and (2), we have $axx_2+byy_2+czz_2=0$ ...(3)

Hence (1) and (3) are the required equations of of the polar.

Observations. *It is clear from the equations (1) and (3) that the polar of PQ is the line of intersection of polar planes of P and Q.*

Example 5. *Find the polar plane of the point* $(2, -3, 4)$ *with respect to the conicoid* $x^2+2y^2+z_2=4$. (Bundelkhand 1984)

Sol. Required polar plane is

$$x(2)+2y(-3)+z(4)=4$$

or $$x-3y+2z=2.$$

Example 6. *Find the locus of straight line through a fixed point (α, β, γ) whose polar lines with respect to the quadratics $ax^2+by^2+cz^2=1$ and $a'x^2+b'y^2+c'z^2=1$ are coplanar.*

Sol. Any line through (α, β, γ) is

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=\gamma \text{ (say)} \qquad \ldots(i)$$

The equations of the polar line of (1) w.r.t.

$ax^2+by^2+cz^2=1$ are

$$a\alpha x+b\beta y+c\gamma z=1 \qquad \ldots(ii)$$

$$alx+bmy+cnz=0 \qquad \ldots(iii)$$

And the equations of polar line of (*i*) w.r.t.

$a'x^2+b'y^2+c'z^2=1$ are

$$a'\alpha x+b'\beta y+c'\gamma z=1 \qquad \ldots(iv)$$

$$a'lx+b'my+c'nz=0 \qquad \ldots(v)$$

From (*ii*) and (*iv*), we have

$$(a-a')\,\alpha x+(b-b')\,\beta y+(c-c')\,\gamma z=0 \qquad \ldots(vi)$$

From (*iii*) and (*v*), solving simultaneously, we have

$$\frac{lx}{(bc'-b'c)}=\frac{my}{(ca'-c'a)}=\frac{nz}{(ab'-a'b)} \qquad \ldots(vii)$$

Eliminating x, y, z between (*vi*) and (*vii*), we get

$$\frac{(a-a')\alpha(bc'-b'c)}{l}+\frac{(b-b')\beta(ca'-c'a)}{m}+\frac{(c-c')\gamma(ab'-a'b)}{n}=0 \qquad \ldots(viii)$$

Eliminating l, m, n between (*i*) and (*viii*), we get the locus of the line as

$$\sum\frac{(a-a')\alpha(bc'-b'c)}{(x-\alpha)}=0.$$

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Example 7. *Prove that the locus of the poles of the tangent planes of $ax^2+by^2+cz^2=1$ with respect to $a'x^2+b'y^2+c'z^2=1$ is the conicoid* $\dfrac{(a'x)^2}{a}+\dfrac{(b'y)^2}{b}+\dfrac{(c'z)^2}{c}=1$. (Allahabad 1982 ; Kanpur 1986)

Sol. Let $\qquad lx+my+nz=p \qquad \ldots(i)$

be the tangent plane to the conicoid

$$ax^2+by^2+cz^2=1 \qquad \ldots(ii)$$

Then $$\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p^2 \qquad \ldots(iii)$$

Let (α, β, γ) be the pole of the plane (*i*) w.r. to

$a'x^2+b'y^2+c'z^2=1$. Then we have $a'\alpha x+b'\beta y+c'\gamma z=1$ ---(iv)

Comparing (*i*) and (*iv*), we get

$$\frac{a'\alpha}{l}=\frac{b'\beta}{m}=\frac{c'\gamma}{n}=\frac{1}{p} \qquad \ldots(v)$$

Eliminating l, m, n between (*iii*) and (*v*), we get

$$\frac{(a'\alpha p)^2}{a}+\frac{(b'\beta p)^2}{b}+\frac{(c'\gamma p)^2}{\cancel{p^2}\, c}=p^2$$

or $$\frac{(a'\alpha)^2}{a}+\frac{(b'\beta)^2}{b}+\frac{(c'\gamma)^2}{c}=1$$

$\therefore$ The required locus of (α, β, γ) is

$$\frac{(a'x)^2}{a}+\frac{(b'y)^2}{b}+\frac{(c'z)^2}{c}=1.$$

Example 8. *Show that the locus of the pole of the plane $lx+my+nz=p$ with respect to the system of conicoids $\Sigma[x^2/(a^2+k)=1$ is a straight line perpendicular to the given plane, where k is a parameter.*

Sol. Let (α, β, γ) be the pole of the plane

$$lx+my+nz=p \qquad \ldots(i)$$

with respect to the conicoid

$$\frac{x^2}{(a^2+k^2)}+\frac{y^2}{(b^2+k)}+\frac{z^2}{(c^2+k)}=1 \qquad \ldots(ii)$$

The polar plane of (α, β, γ) w.r.t. this conicoid is

$$\frac{\alpha x}{(a^2+k)}+\frac{\beta y}{(b^2+k)}+\frac{\gamma z}{(c^2+k)}=1 \qquad \ldots(iii)$$

Since (i) and (iii) represents the same plane, therefore comparing them, we get

$$\frac{\alpha/(a^2+k)}{l}=\frac{\beta/(b^2+k)}{m}=\frac{\gamma/(c^2+k)}{n}=\frac{1}{p}$$

where $\alpha=(a^2+k)\,\dfrac{l}{p},\ \beta=(b^2+k)\,\dfrac{m}{p},\ \gamma=(c^2+k)\,\dfrac{n}{p}$

or $$\frac{\alpha-(a^2l/p)}{l}=\frac{k}{p}=\frac{\beta-(b^2m/p)}{m}=\frac{\gamma-(c^2n/p)}{n}$$

$\therefore$ The locus of (α, β, γ) is

$$\frac{x-(a^2l/p)}{l}=\frac{y-(b^2m/p)}{m}=\frac{z-(c^2n/p)}{n}$$

which is a straight line and its direction cosines being l, m, n is perpendicular to the plane (i).

Article 19. Enveloping Cone

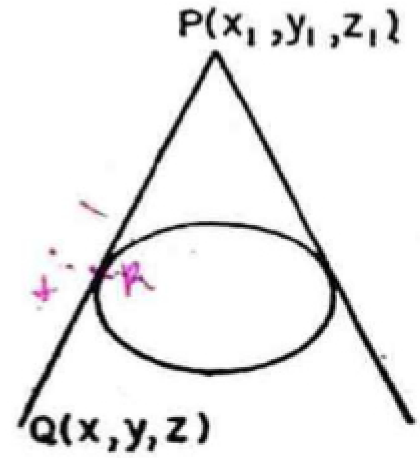
To find the equation of enveloping cone from the point (x_1, y_1, z_1) to the central conicoid $ax^2+by^2+cz^2=1$. *(M.D.U. 1984)*

The given conicoid is

$$ax^2+by^2+cz^2=1 \qquad \ldots(1)$$

Let P be the point (x_1, y_1, z_1).

Let Q (x, y, z) be *any* point on a tangent from P to the conicoid.



The point which divides PQ in the ratio $k:1$ is

$$\left(\frac{kx+x_1}{k+1}, \frac{ky+y_1}{k+1}, \frac{kz+z_1}{k+1}\right).$$

If it lies on (1), then

$$a\left(\frac{kx+x_1}{k+1}\right)^2+b\left(\frac{ky+y_1}{k+1}\right)^2+c\left(\frac{kz+z_1}{k+1}\right)^2=1$$

which simplifies to

$$k^2(ax^2+by^2+cz^2-1)+2k(axx_1+byy_1+czz_1-1)+(ax_1^2+by_1^2+cz_1^2-1)=0 \quad \ldots(2)$$

which is a quadratic in k.

Since the line PQ *touches* the conicoid (1), $\therefore$ (2) must have equal roots.

$$\therefore \quad 4(axx_1+byy_1+czz_1-1)^2 = 4(ax^2+by^2+cz^2-1)(ax_1^2+by_1^2+cz_1^2-1) \quad | \textit{ Using } b^2=4ac$$

or
$$(\mathbf{axx_1+byy_1+czz_1-1})^2 = (\mathbf{ax^2+by^2+cz^2-1})(\mathbf{ax_1^2+by_1^2+cz_1^2-1})$$

which is the required equation.

Remember. If S=0 is the given surface, then with usual notations the enveloping cone is given by $SS_1=T^2$.

Example 1. *Find the locus of points from which three mutually perpendicular tangents can be drawn to the surface $ax^2+by^2+cz^2=1$.* **[Imp.]**

Sol. Let $P(x_1, y_1, z_1)$ be the point.

Then the three mutually $\perp$ tangents drawn from P will be three mutually $\perp$ generators of the enveloping cone with P as vertex. The equation of the enveloping cone is $SS_1=T^2$

or $\quad (ax^2+by^2+cz^2-1)(ax_1^2+by_1^2+cz_1^2-1)=(axx_1+byy_1+czz_1-1)^2$

Since this cone has three mutually $\perp$ generators,

$\therefore$ Co-eff. of x^2+coeff. of y^2+coeff. of $z^2=0$

i.e., $\quad a(by_1^2+cz_1^2-1)+b(ax_1^2+cz_1^2-1)+c(ax_1^2+by_1^2-1)=0$

or $\quad a(b+c)x_1^2+b(c+a)y_1^2+c(a+b)z_1^2=a+b+c$

$\therefore$ Locus of $P(x_1, y_1, z_1)$ is [*changing* (x_1, y_1, z_1) to (x, y, z)]

$$a(b+c)x^2+b(c+a)y^2+c(a+b)z^2=a+b+c.$$

Example 2. *The section of the enveloping cone of the ellipsoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ whose vertex is P by the plane $z=0$ is (i) a parabola, (ii) a rectangular hyperbola. Find the locus of P.* **[Imp.]** (*M.D.U. 1985*)

Sol. Let $P(x_1, y_1, z_1)$ be the vertex of enveloping cone of the ellipsoid

$$S \equiv \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0 \quad \ldots(1)$$

The enveloping cone of (1) is $SS_1=T^2$

i.e.,
$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}-1\right)^2$$

This meets the plane $z=0$, where

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2$$

or $\frac{x^2}{a^2}\left(\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)+\frac{y^2}{b^2}\left(\frac{x_1^2}{a^2}+\frac{z_1^2}{c^2}-1\right)-\frac{2x_1y_1}{a^2b^2}xy+\ldots=0$ ...(2)

(1) and (2) represent a parabola in the XY plane if

$$\frac{1}{a^2}\left(\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right).\ \frac{1}{b^2}\left(\frac{x_1^2}{a^2}+\frac{z_1^2}{c^2}-1\right)=\frac{x_1^2y_1^2}{a^4b^4}$$

| *Using* $ab=h^2$ if the equation is $ax^2+2hxy+by^2+\ldots=0$

or $\frac{1}{a^2b^2}\left(\frac{x_1^2y_1^2}{a^2b^2}+\frac{y_1^2z_1^2}{b^2c^2}-\frac{y_1^2}{b^2}+\frac{z_1^2x_1^2}{a^2c^2}+\frac{z_1^4}{c^4}-\frac{z_1^2}{c^2}-\frac{x_1^2}{a^2}-\frac{z_1^2}{c^2}+1\right)=\frac{x_1^2y_1^2}{a^4b^4}$

or $\left(\frac{y_1^2z_1^2}{b^2c^2}+\frac{z_1^2x_1^2}{c^2a^2}+\frac{z_1^4}{c^4}-\frac{z_1^2}{c^2}\right)-\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)=0$

or $\frac{z_1^2}{c^2}\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)-\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)=0$

or $\left(\frac{z_1^2}{c^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)=0$

$\therefore$ Locus of $P(x_1, y_1, z_1)$ is $\left(\frac{z^2}{c^2}-1\right)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)=0$

$\therefore$ Either $\frac{z^2}{c^2}-1=0$ or $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0.$

Rejecting the second equation [$\because$ it is the given ellipsoid and P does not lie on it], the locus is $\frac{z^2}{c^2}-1=0$ or $z=\pm c$.

(*Kanpur 1988*)

(*ii*) The equation (2) represents a rectangular hyperbola in the XY plane if co-eff. of x^2+co-eff. $y^2=0$

i.e., if $\frac{1}{a^2}\left(\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)+\frac{1}{b^2}\left(\frac{x_1^2}{a^2}+\frac{z_1^2}{c^2}-1\right)=0$

$\therefore$ Locus of $P(x_1, y_1, z_1)$ is

$$\frac{1}{a^2}\left(\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)+\frac{1}{b^2}\left(\frac{x^2}{a^2}+\frac{z^2}{c^2}-1\right)=0$$

or $$\frac{x^2}{a^2b^2}+\frac{y^2}{a^2b^2}+\frac{z^2}{c^2}\left(\frac{1}{a^2}+\frac{1}{b^2}\right)-\left(\frac{1}{a^2}+\frac{1}{b^2}\right)=0$$

or $$\frac{x^2+y^2}{a^2b^2}+\frac{z^2(a^2+b^2)}{a^2b^2c^2}=\frac{a^2+b^2}{a^2b^2}$$

or $$\frac{x^2+y^2}{a^2+b^2}+\frac{z^2}{c^2}=1.$$

Example 3. *Find the locus of luminous point if the ellipsoid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ *casts a circular shadow on the plane* $z=0$.

(Kanpur 1988)

Sol. Let $P(x_1, y_1, z_1)$ be the luminous point.

The enveloping cone of the given ellipsoid with vertex at P is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)\left(\frac{x_1^2}{a^4}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)$$
$$=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}+\frac{zz_1}{c^2}-1\right)^2 \qquad \Big|\ \textit{Using } SS_1=T^2$$

This meets the plane $z=0$, where

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)=\left(\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1\right)^2$$

This will be a circle if the *co-eff. of* $xy=0$

and *co-eff. of* x^2 = *co-eff. of* y^2

i.e., if
$$\frac{x_1y_1}{a^2b^2}=0 \qquad \ldots(1)$$

and
$$\frac{1}{a^2}\left(\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)=\frac{1}{b^2}\left(\frac{x_1^2}{a^2}+\frac{z_1^2}{c^2}-1\right) \qquad \ldots(2)$$

From (1) either $x_1=0$ or $y_1=0$.

Case I. If $x_1=0$ from (2), we have

$$\frac{1}{a^2}\left(\frac{y_1^2}{b^2}+\frac{z_1^2}{c^2}-1\right)=\frac{1}{b^2}\left(\frac{z_1^2}{c^2}-1\right)$$

$$\therefore \quad \frac{y_1^2}{a^2b^2}+\frac{z_1^2(b^2-a^2)}{a^2b^2c^2}=\frac{b^2-a^2}{a^2b^2}$$

or
$$\frac{y_1^2}{b^2-a^2}+\frac{z_1^2}{c^2}=1.$$

Thus the locus of $P(x_1, y_1, z_1)$ is $x=0,\ \frac{y^2}{b^2-a^2}+\frac{z^2}{c^2}=1$ which is an ellipse in the YZ plane.

Case II. If $y_1=0$, (2) gives

$$\frac{1}{a^2}\left(\frac{z_1^2}{c^2}-1\right)=\frac{1}{b^2}\left(\frac{x_1^2}{a^2}+\frac{z_1^2}{c^2}-1\right)$$

$\therefore$ Locus of $P(x_1, y_1, z_1)$ is

$$y=0,\ \frac{1}{a^2}\left(\frac{z^2}{c^2}-1\right)=\frac{1}{b^2}\left(\frac{x^2}{a^2}+\frac{z^2}{c^2}-1\right)$$

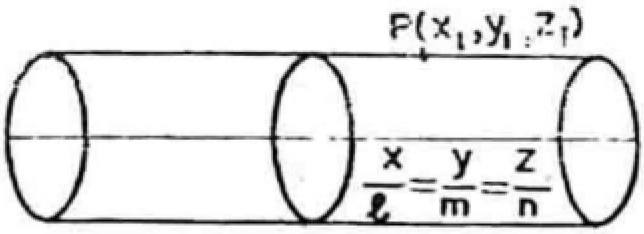
or
$$y=0,\ \frac{x^2}{a^2-b^2}+\frac{z^2}{c^2}=1.$$

Article 20. Enveloping Cylinder

To find the equation of the enveloping cylinder of the central conicoid $ax^2+by^2+cz^2=1$ whose generators are parallel to the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$. *(Garhwal 1986)*

The given conicoid is $ax^2+by^2+cz^2=1$ ...(1)

and the given line is $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ ...(2)



Let $P(x_1, y_1, z_1)$ be any point on a tangent ∥ to the line (2).
| **Note this step**

The equations of the tangent line through (x_1, y_1, z_1) and ∥ to (2) are

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=r \text{ (say)}.$$

Any point on this line is $(lr+x_1, mr+y_1, nr+z_1)$.

If it lies on (1), then $a(lr+x_1)^2+b(mr+y_1)^2+c(nr+z_1)^2=1$

or $r^2(al^2+bm^2+cn^2)+2r(alx_1+bmy_1+cnz_1)+(ax_1^2+by_1^2+cz_1^2-1)=0$ ...(3)

Since the line (2) touches the conicoid (1), ∴ (3) has equal roots.

$\therefore \quad 4(alx_1+bmy_1+cnz_1)^2 - 4(al^2+bm^2+cn^2)(ax_1^2+by_1^2+cz_1^2-1)=0$ | *Using* $b^2-4ac=0$

or $\quad (alx_1+bmy_1+cnz_1)^2=(al^2+bm^2+cn^2)[ax_1^2+by_1^2+cy_1^2-1]$

$\therefore$ Locus of (x_1, y_1, z_1) is

$$(alx+bmy+cnz)^2=(al^2+bm^2+cn^2)(ax^2+by^2+cz^2-1)$$

which is the required equation of enveloping cylinder.

Method to write down the enveloping cylinder.

If $S\equiv ax^2+by^2+cz^2-1$, so that $S=0$ is the equation of central conicoid then $s_1=al^2+bm^2+cn^2$, *i.e.*, s_1 is obtained by putting (l, m, n) in S and neglecting the constant term.

$t=alx+bmy+cnz$, where t is the expression for the tangent plane at (l, m, n) after omitting the constant term, then the enveloping cylinder is $\quad Ss_1=t^2$. **[Remember]**

IFS-2003

Example 1. *Prove that the enveloping cylinder of the ellipsoid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ *whose generators are parallel to the lines* $\frac{x}{0}=\frac{y}{\pm\sqrt{a^2-b^2}}=\frac{z}{c}$, *meet the plane* $z=0$ *in circles.*

Sol. The given ellipsoid is $S\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0$ ...(1)

and the given lines are $\frac{x}{0}=\frac{y}{\pm\sqrt{a^2-b^2}}=\frac{z}{c}$ ...(2)

The equation of enveloping cylinder is $Ss_1=t^2$

i.e., $$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)\left[\frac{1}{a^2}(0)^2+\frac{1}{b^2}(\pm\sqrt{a^2-b^2})^2+\frac{1}{c^2}(c)^2\right]=\left[\frac{1}{a^2}(0)x+\frac{1}{b^2}(\pm\sqrt{(a^2-b^2)})y+\frac{1}{c^2}\cdot cz\right]^2$$

or $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1\right)\left(\frac{a^2-b^2}{b^2}+1\right)=\left(\pm\frac{\sqrt{a^2-b^2}}{b^2}y+\frac{z}{c}\right)^2$

This meets the plane $z=0$ where

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{a^2-b^2}{b^2}+1\right)=\left(\frac{\pm\sqrt{a^2-b^2}}{b^2}.\,y+0\right)^2$$

or $$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right).\frac{a^2}{b^2}=\frac{(a^2-b^2)\,y^2}{b^4}$$

or $$x^2+\frac{a^2}{b^2}y^2-a^2=\frac{(a^2-b^2)\,y^2}{b^2}=\frac{a^2}{b^2}y^2-y^2$$

or $x^2+y^2=a^2$, which is a circle.

2008

Example 2. *Show that the enveloping cylinder of the ellipsoid $ax^2+by^2+cz^2=1$ with generators parallel to Z-axis meet the plane $z=0$ in ellipse.* or in parabolas.

Sol. Please try yourself as above.

[**Hint.** *Remember that an equation in x, y represents a parabola in XY plane if its second degree terms form a perfect square.*]

Article 21. Section with a given centre. **[V. Imp.]**

To find the locus of chords of the conicoid

$$\mathbf{ax^2+by^2+cz^2=1}$$

which are bisected at the given point (x_1, y_1, z_1).

The given conicoid is $\quad ax^2+by^2+cz^2=1 \quad$ ...(1)

Any chord through (x_1, y_1, z_1) is $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$ ...(2)

Any point on this chord is $(lr+x_1, mr+y_1, nr+z_1)$

If it lies on (1), then

$$a(lr+x_1)^2+b(mr+y_1)^2+c(nr+z_1)^2=1$$

or $r^2(al^2+bm^2+cn^2)+2r(alx_1+bmy_1+cnz_1)+(ax_1^2+by_1^2+cz_1^2-1)=0$ ...(3)

which is a quadratic in r.

If l, m, n are the actual d.c.'s of line (2), then here r is the distance of any point common to the conicoid (1) and the chord (2) from the given point (x_1, y_1, z_1).

If (x_1, y_1, z_1) is the middle point of chord (2), the points of intersection of (1) and (2) should be equidistant and on either side of (x_1, y_1, z_1), *i.e.*, the two values of r should be equal and opposite or the sum of roots in (3) is zero.

$\therefore$ Co-eff. of $r=0$ giving $alx_1+bmy_1+cnz_1=0$ ...(4)

Eliminating l, m, n from (2) and (4), the locus of chords (2) is

$$ax_1(x-x_1)+by_1(y-y_1)+cz_1(z-z_1)=0$$

or $$\mathbf{axx_1+byy_1+czz_1=ax_1^2+by_1^2+cz_1^2}$$

or $\quad axx_1+byy_1+czz_1-1=ax_1^2+by_1^2+cz_1^2-1 \quad ...(5)$

which is of the form $\quad T=S_1$. [Remember]

Note. The plane (5) meets the given conicoid in a conic whose centre is (x_1, y_1, z_1).

Article 22. To find the locus of middle points of a system of chords of the conicoid $ax^2+by^2+cz^2=1$ which are parallel to the line $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$. *(M.D.U. 1983)*

Any chord through (x_1, y_1, z_1) drawn $\parallel$ to $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ is

$$\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n} \quad ...(1)$$

Any point on this chord is $(lr+x_1, mr+y_1, nr+z_1)$

This lies on the given conicoid $ax^2+by^2+cz^2=1$ if

$$a(lr+x_1)^2+b(mr+y_1)^2+c(nr+z_1)^2=1$$

or $r^2(al^2+bm^2+cn^2)+2r(alx_1+bmy_1+cnz_1)+(ax_1^2+by_1^2+cz_1^2-1)=0 \quad ...(2)$

If (x_1, y_1, z_1) is the mid point of (1), then the two values of r in (2) must be equal in magnitude but opposite in sign, *i.e.*, its sum of two roots is zero or the co-eff. of $r=0$

$\therefore \quad alx_1+bmy_1+cnz_1=0$

$\because$ Locus of (x_1, y_1, z_1) the mid-point is

$$alx+bmy+cnz=0$$

which is a plane through the centre of the conicoid.

Example 1. *Find the equation to the plane which cuts the surface (a) $2x^2+3y^2+5z^2=4$ in a conic whose centre is at the point (1, 2, 3).*

(b) $x^2+4y^2-5z^2=1$ in a conic whose centre is at the point (2, 3, 4).

Sol. (*a*) The given conicoid is $S\equiv 2x^2+3y^2+5z^2-4=0$

[*Make R.H.S.=0*]

Here $\quad S_1=2(1)^2+3(2)^2+5(3)^2-4 \quad$ | *Putting (1, 2, 3) in S*

$\quad =2+12+45-4=55$

and $\quad T=2x(1)+3y(2)+5z(3)-4$

$\quad =2x+6y+15z-4.$

The required plane is $T=S_1$, *i.e.*,

$$2x+6y+15z-4=55 \quad \text{or} \quad 2x+6y+15z=59.$$

(*b*) **Please try yourself** as above. [**Ans.** $x+6y-10z+20=0$]

Example 2. *Show that centre of the conic given by*

$$ax^2+by^2+cz^2=1, \; lx+my+nz=p$$

is the point $\quad \left(\frac{lp}{ap_0^2}, \frac{mp}{bp_0^2}, \frac{np}{cp_0^2}\right)$

where $\quad l^2+m^2+n^2=1 \quad$ and $\quad p_0=\sqrt{\sum \frac{l^2}{a}}$.

Sol. Let (x_1, y_1, z_1) be the centre of the section of the conicoid $ax^2+by^2+cz^2=1$ by the plane $lx+my+nz=p$.

Then the plane with (x_1, y_1, z_1) as the centre of section is $(T=S_1)$

i.e., $$axx_1+byy_1+czz_1-1=ax_1^2+by_1^2+cz_1^2-1$$

or $$axx_1+byy_1+czz_1=ax_1^2+by_1^2+cz_1^2 \quad \ldots(1)$$

The plane should be identical with $lx+my+nz=p \quad \ldots(2)$

Comparing the co-efficients in (1) and (2), we have

$$\frac{ax_1}{l}=\frac{by_1}{m}=\frac{cz_1}{n}=\frac{ax_1^2+by_1^2+cz_1^2}{p}=k \text{ (say)},$$

From these we have, $x_1=\frac{lk}{a}, y_1=\frac{mk}{b}, z_1=\frac{nk}{c} \quad \ldots(3)$

and $$ax_1^2+by_1^2+cz_1^2=pk \quad \ldots(4)$$

Putting the values of x_1, y_1, z_1 from (3) in (4), we have

$$a\left(\frac{l^2k^2}{a^2}\right)+b\left(\frac{m^2k^2}{b^2}\right)+c\left(\frac{n^2k^2}{c^2}\right)=pk$$

or $$\left(\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}\right)k=p \text{ or } p_0^2k=p \quad \therefore \quad k=\frac{p}{p_0^2}.$$

Putting this value of k in (3), the centre of section (x_1, y_1, z_1)

is $$\left(\frac{lp}{ap_0^2}, \frac{mp}{bp_0^2}, \frac{np}{cp_0^2}\right).$$ Hence the result.

Example 3. *Find the locus of centres of all plane sections of a conicoid*

(*a*) *which pass through a fixed point.*

(*b*) *which are at a constant distance from the centre.*

(*c*) *which are parallel to a given line.*

(*d*) *which pass through a given line.*

Sol. Let (x_1, y_1, z_1) be the centre of plane section of the conicoid
$$ax^2+by^2+cz^2=1. \quad \ldots(1)$$

Then equation of the plane with (x_1, y_1, z_1) as centre is

$$axx_1+byy_1+czz_1-1=ax_1^2+by_1^2+cz_1^2-1 \quad | \text{ Using } T=S_1$$

or $$axx_1+byy_1+czz_1=ax_1^2+by_1^2+cz_1^2 \quad \ldots(2)$$

(*a*) The plane (2) passes through a fixed point say (α, β, γ), then

$$a\alpha x_1+b\beta y_1+c\gamma z_1=ax_1^2+by_1^2+cz_1^2$$

$\therefore$ Locus of (x_1, y_1, z_1) is $ax^2+by^2+cz^2=a\alpha x+b\beta y+c\gamma z$ which is a conicoid.

(*b*) The plane (2) is at a constant distance k (say) from the centre (0, 0, 0).

$$\therefore \quad \frac{ax_1^2+by_1^2+cz_1^2}{\sqrt{a^2x_1^2+b^2y_1^2+c^2z_1^2}}=k$$

or $$(ax_1^2+by_1^2+cz_1^2)^2=k^2(a^2x_1^2+b^2y_1^2+c^2z_1^2)$$

$\therefore$ Locus of (x_1, y_1, z_1) is $(ax^2+by^2+cz^2)^2=k^2(a^2x^2+b^2y^2+c^2z^2)$.

(c) Let the given line be $\dfrac{x}{l}=\dfrac{y}{m}=\dfrac{z}{n}$. ...(3)

The plane (2) will be $\parallel$ to line (3) if its normal is $\perp$ to (3), *i.e.*,

if $$l(ax_1)+m(by_1)+n(cz_1)=0$$

or $$alx_1+bmy_1+cnz_1=0.$$

$\therefore$ Locus of (x_1, y_1, z_1) is $alx+bmy+cnz=0$ which is a plane.

(d) Let the given line be $\dfrac{x-\alpha}{l}=\dfrac{y-\beta}{m}=\dfrac{z-\gamma}{n}$. ...(4)

The line (4) will lie on plane (2) if (*i*) the line (4) is $\perp$ to normal to the plane (2) *i.e.*,

$$lax_1+mby_1+ncz_1=0 \quad \text{...(5)}$$

and (*ii*) one point (α, β, γ) on (4) lies on the plane (2) *i.e.*,

$$a\alpha x_1+b\beta y_1+c\gamma z_1=ax_1^2+by_1^2+cz_1^2 \quad \text{...(6)}$$

From (5) and (6), the locus of (x_1, y_1, z_1) is

$$alx+bmy+cnz=0,\ ax^2+by^2+cz^2=a\alpha x+b\beta y+c\gamma z$$

which being the intersection of a plane and a conicoid represents a conic.

Example 4. *Find the centre of the conic given by the equations* $2x-2y-5z+5=0,\ 3x^2+2y^2-15z^2=4$.

Sol. The conicoid is $\quad S\equiv 3x^2+2y^2-15z^2-4=0$ ...(1)

Let (x_1, y_1, z_1) be the centre of the given conic. Then equation of the plane which cuts (1) in a conic with centre (x_1, y_1, z_1) is $T=S_1$

i.e., $$3xx_1+2yy_1-15zz_1-4=3x_1^2+2y_1^2-15z_1^2-4$$

or $$3xx_1+2yy_1-15zz_1-(3x_1^2+2y_1^2-15z_1^2)=0 \quad \text{...(2)}$$

Now this is the same as the given plane

$$2x-2y-5z+5=0 \quad \text{...(3)}$$

Comparing (2) and (3), we get

$$\frac{3x_1}{2}=\frac{2y_1}{-2}=\frac{-15z_1}{-5}=\frac{-(3x_1^2+2y_1^2-15z_1^2)}{5}=k \text{ (say)}$$

$$\therefore \quad x_1=\frac{2}{3}k,\ y_1=-k,\ z_1=\frac{k}{3} \quad (4)$$

and $$3x_1^2+2y_1^2-15z_1^2=-5k.$$

Putting the values of x_1, y_1, z_1 in the last equation, we get

$$3\left(\frac{4}{9}k^2\right)+12k^2-15\left(\frac{k^2}{9}\right)=-5k$$

or $$\frac{4}{3}k^2+2k^2-\frac{5}{3}k^2=-5k$$

or $$4k^2+6k^2-5k^2=-15k \quad \text{or} \quad 5k^2=-15k \quad \therefore\ k=-3.$$

$\therefore$ From (4), the centre is (x_1, y_1, z_1) *i.e.*, $(-2, 3, -1)$.

Example 5. *Prove that the centres of sections of*

$$ax^2+by^2+cz^2=1$$

by the planes which are at a constant distance p from the origin lie on the surface

$$(ax^2+by^2+cz^2)^2=p^2(a^2x^2+b^2y^2+c^2z^2).$$

Sol. If (α, β, γ) be the centre of the section of the given ellipsoid then equation of this section of the sphere is "$T=S_1$"

i.e. $$(a\alpha x+b\beta y+c\gamma z-1)=(a\alpha^2+b\beta^2+c\gamma^2-1)$$

or $$-a\alpha x-b\beta y-c\gamma z+(a\alpha^2+b\beta^2+c\gamma^2)=0 \quad \text{...(i)}$$

The distance of this plane (*i*) from the origin (0, 0, 0) is given as p.

$$\therefore \quad p=\frac{a\alpha^2+b\beta^2+c\gamma^2}{\sqrt{[(a\alpha)^2+(b\beta)^2+(c\gamma)^2]}}$$

or $$p^2(a^2\alpha^2+b^2\beta^2+c^2\gamma^2)=(a\alpha^2+b\beta^2+c\gamma^2)^2$$

$\therefore$ The locus of the centre (α, β, γ) is

$$p^2(a^2x^2+b^2y^2+c^2z^2)=(ax^2+by^2+cz^2)^2.$$

Example 6. *Prove that the centre of the section of the ellipsoid* $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ *by the plane ABC whose equation is* $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ *is the centroid of the triangle ABC.*

Sol. The equation of the ellipsoid is

$$S\equiv\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0 \quad \text{...(i)}$$

and the equation of the plane ABC is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \quad \text{...(ii)}$$

Let (α, β, γ) be the centre of the section (*i*) by the plane (*ii*) then the equation of this section is "$T=S_1$"

i.e. $$\frac{x\alpha}{a^2}+\frac{y\beta}{b^2}+\frac{\gamma z}{c^2}-1=\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}=1$$

or $$\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}+\frac{\gamma z}{c^2}=\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2} \quad \text{...(iii)}$$

The equation (*ii*) and (*iii*) represent the same plane, so comparing them, we get

$$\frac{\left(\frac{\alpha}{a^2}\right)}{\left(\frac{1}{a}\right)}=\frac{\left(\frac{\beta}{b^2}\right)}{\left(\frac{1}{b}\right)}=\frac{\left(\frac{\gamma}{c^2}\right)}{\left(\frac{1}{c}\right)}=\frac{\left(\frac{\alpha^2}{a^2}\right)+\left(\frac{\beta^2}{b^2}\right)+\left(\frac{\gamma^2}{c^2}\right)}{1}=k \quad \text{(say)}$$

$\therefore \quad \alpha=ak, \beta=bk, \gamma=ck$ and

$$\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}=k$$

$$\therefore \quad \left(\frac{a^2k^2}{a^2}\right)+\left(\frac{b^2k^2}{b^2}\right)+\left(\frac{c^2k^2}{c^2}\right)=k$$

or $\quad 3k^2=k \quad$ or $\quad k=\frac{1}{3}$.

$\therefore \quad \alpha=ak=\frac{1}{3}a, \quad \beta=bk=\frac{1}{3}b, \quad \gamma=ck=\frac{1}{3}c.$

$\therefore$ The centre of the section of (i) by the plane (ii) is (α, β, γ) or $(\frac{1}{3}a, \frac{1}{3}b, \frac{1}{3}c)$

Also the co-ordinates of the vertices of $\triangle ABC$ are

$$A(a, 0, 0), B(0, b, 0), C(0, 0, c)$$

$\therefore$ The co-ordinates of the centroid of $\triangle ABC$ are $(\frac{1}{3}a, \frac{1}{3}b, \frac{1}{3}c)$.

Hence the centre of the section of (i) by (ii) is the centre of $\triangle ABC$.

Example 7. *Find the locus of the mid-points of the chords of the conicoid $ax^2+by^2+cz^2=1$ which passes through (α, β, γ).*

(Allahabad 1981 ; Kanpur 1979 ; Lucknow 1982)

Sol. Let (x_1, y_1, z_1) be the mid-point of the chord of the given conicoid. Then the locus of the chords of the given conicoid with (x_1, y_1, z_1) as mid-point is "$T=S_1$"

where $\quad T=axx_1+byy_1+czz_1-1 \quad$ and

$$S_1=ax_1^2+by_1^2+cz_1^2-1$$

i.e. $\quad axx_1+byy_1+czz_1-1=ax_1^2+by_1^2+cz_1^2-1$

or $\quad axx_1+byy_1+czz_1=ax_1^2+by_1^2+cz_1^2$

If it posses through (α, β, γ), we have

$$a\alpha x_1+b\beta y_1+c\gamma z_1=ax_1^2+by_1^2+cz_1^2$$

$\therefore$ The required locus of the mid-point (x_1, y_1, z_1) of the chords of the given conicoid is

$$ax^2+by^2+cz^2=a\alpha x+b\beta y+c\gamma z$$

or $\quad ax(x-\alpha)+by(y-\beta)+cz(z-\gamma)=0.$

Example 8. *Show that the line joining a point P to the centre of a conicoid $ax^2+by^2+cz^2=1$ passes through the centre of the section of the conicoid by the polar plane of P.*

Sol. Let (x', y', z') be the co-ordinates of the point P. Then the polar plane of $P(x', y', z')$ with respect to the given conicoid is

$$axx'+byy'+czz'=1 \qquad \ldots(i)$$

Let (α, β, γ) be the centre of the section of the given conicoid by the plane (i), then equation of this plane section can also be written as

"$T=S_1$" or

$$a\alpha x+b\beta y+c\gamma z-1=a\alpha^2+b\beta^2+c\gamma^2-1$$

or $\quad a\alpha x+b\beta y+c\gamma z=a\alpha^2+b\beta^2+c\gamma^2 \qquad \ldots(ii)$

Since the equations (i) and (ii) represent the same plane, so comparing them, we get

$$\frac{\alpha}{x'}=\frac{\beta}{y'}=\frac{\gamma}{z'} \qquad \ldots(iii)$$

Also the equations of the line joining the point $P(x', y', z')$ to the centre $(0, 0, 0)$ of the given conicoid is

$$\frac{x}{x'}=\frac{y}{y'}=\frac{z}{z'}$$

If this line passes through the centre (α, β, γ) of the section of given conicoid be the plane (i), then

$$\frac{\alpha}{x'}=\frac{\beta}{y'}=\frac{\gamma}{z'}$$

which is true by virtue of (iii).

Hence the line joining $P(x', y', z')$ to the centre of the given conicoid passes through the centre (α, β, γ) of the section of the conicoid by the polar plane (i) of P.

Example 9. *Prove that the section of the ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

whose centre is at the point $\left(\frac{1}{3a}, \frac{1}{3b}, \frac{1}{3c}\right)$ *passes through the extremities of the axes.* (Rohilkhand 1985)

Sol. The ellipsoid is

$$S \equiv \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}-1=0 \qquad \ldots(i)$$

The equation of the section of this ellipsoid with

$\left(\frac{1}{3a}, \frac{1}{3b}, \frac{1}{3c}\right)$ as its centre is "$T=S_1$"

i.e.,
$$\frac{x\cdot\frac{1}{3a}}{a^2}+\frac{y\cdot\frac{1}{3b}}{b^2}+\frac{z\cdot\frac{1}{3c}}{c^2}-1$$

$$=\frac{\left(\frac{1}{3a^2}\right)^2}{a^2}+\frac{\left(\frac{1}{3b}\right)^2}{b^2}+\frac{\left(\frac{1}{3c}\right)^2}{c^2}-1$$

or
$$\frac{x}{3a}+\frac{y}{3b}+\frac{z}{c}=\frac{1}{9}+\frac{1}{9}+\frac{1}{9}=\frac{1}{3}$$

or
$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$$

which is the plane evidently passing through $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$, the three extremities of the axes of the ellipsoid given by (i).

Example 10. *Find the locus of centres of sections of* $ax^2+by^2+cz^2=1$ *which touch* $\alpha x^2+\beta y^2+\gamma z^2=1$. (Rohilkhand 1983)

Sol. Let (x_1, y_1, z_1) be the centre of the section of conicoid

$$ax^2+by^2+cz^2=1 \qquad \ldots(i)$$

The equation of the section is $T=S_1$

or $\quad axx_1+byy_1+czz_1-1=ax_1^2+by_1^2+cz_1^2-1$

or $\quad ax_1x+by_1y+cz_1z=(ax_1^2+by_1^2+cz_1^2) \quad \ldots(i)$

If the plane (i) touches the conicoid $\alpha x^2+\beta y^2+\gamma z^2=1$, then we must have

$$\text{“}\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p^2\text{”}$$

$$\therefore \quad \frac{(ax_1)^2}{\alpha}+\frac{(by_1)^2}{\beta}+\frac{(cz_1)^2}{\gamma}=(ax_1^2+by_1^2+cz_1^2)^2$$

$\therefore$ The required locus of (x_1, y_1, z_1) is

$$\frac{a^2x^2}{\alpha}+\frac{b^2y^2}{\beta}+\frac{c^2z^2}{\gamma}=(ax^2+by^2+cz^2)^2.$$

Example 11. *Prove that the middle point of the chords of $ax^2+by^2+cz^2=1$ which are parallel to $x=0$ and touch $x^2+y^2+z^2=\gamma^2$ lies on the surface*

$$by^2(bx^2+by^2+cz^2-bx^2)+cz^2(-cx^2+by^2+cz^2-c\gamma^2)=0.$$

(Kanpur 1982 ; Rohilkhand 1983)

Sol. The equation of any line having (α, β, γ) as mid-point and parallel to the plane $x=0$ is

$$\frac{x-\alpha}{0}=\frac{y-\beta}{m}=\frac{z-\gamma}{n}=\lambda \text{ (say)} \quad \ldots(i)$$

where m and n are variables.

Any point on this line is $(\alpha, \beta+m\gamma, \gamma+n\lambda)$. If this point lies on the conicoid $ax^2+by^2+cz^2=1$, then we have

$$a\alpha^2+b(\beta+m\lambda)^2+c(\gamma+n\lambda)^2=1$$

or $\quad \lambda(bm^2+cn^2)+2\lambda(b\beta m+c\gamma n)+(a\alpha^2+b\beta^2+c\gamma^2-1)=0 \quad \ldots(ii)$

$\therefore$ (α, β, γ) is the mid-point of the chord (i) of the given conicoid, so that sum of the roots of equation (ii), which is a quadratic in λ, must be zero

i.e., $\quad bm\beta+cn\gamma=0 \quad \ldots(iii)$

Also the line (i) touches the sphere

$$x^2+y^2+z^2=r^2$$

$\therefore$ The length of perpendicular from the centre (0, 0, 0) of the sphere to (i) must be equal to the radius r of the sphere

i.e., $\quad \left[\begin{vmatrix}-\alpha & -\beta \\ 0 & m\end{vmatrix}^2+\begin{vmatrix}-\beta & -\gamma \\ m & n\end{vmatrix}^2+\begin{vmatrix}-\gamma & -\alpha \\ n & 0\end{vmatrix}^2\right]\div(m^2+n^2)=r^2$

or $\quad m^2\alpha^2+(n\beta-m\gamma)^2+\alpha^2n^2=r^2(m^2+n^2)$

or $\quad (r^2-\alpha^2)(m^2+n^2)=(n\beta-m\gamma)^2$

or $\quad (r^2-\alpha^2)\left[\left(\frac{m}{n}\right)^2+1\right]=\left[\beta-\left(\frac{m}{n}\right)\gamma\right]^2 \quad \ldots(iv)$

Also from (iii), we have $\dfrac{m}{n}=\dfrac{(-c\gamma)}{(b\beta)}$.

Substituting the value in (*iv*), we get

$$(r^2-\alpha^2)\left[\left(\frac{c^2\gamma^2}{b^2\beta^2}\right)+1\right]=\left[\beta+\left(\frac{c\gamma}{b\beta}\right)\right]^2$$

or $$(r^2-\alpha^2)[c^2\gamma^2+b^2\beta^2]=[b\beta^2+\gamma^2c]^2$$

$\therefore$ The required locus of (α, β, γ) is

$$(r^2-x^2)(c^2z^2+b^2y^2)=(by^2+cz^2)^2$$

or $c^2r^2z^2+b^2r^2y^2-c^2x^2z^2-b^2y^2x^2=b^2y^4+c^2z^4-2bcy^2z^2$

or $by^2(bx^2+by^2+cz^2-bx^2)+cz^2(cz^2+by^2-cx^2-cr^2)=0.$

CONE

Article 23. To trace the cone $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$.

The given surface is $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$ ...(1)

(*i*) **Symmetry.** Since the equation (1) contains even powers of x, y, z, so the surface is symmetrical about the YZ, ZX, and XY planes.

(*ii*) **Axes intersection.** The cone meets X-axis $(y=0,\ z=0)$ where $\frac{x^2}{a^2}=0$ or $x=0, 0$, *i.e.*, in two coincident points.

Thus cone meets X-axis at the origin. Similarly, it meets Y and Z-axis also at the origin.

(*iii*) **Sections by co-ordinate planes.** The cone (1) meets the YZ plane $(x=0)$, where $\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$ or $y=\pm\frac{b}{c}z$ which are two straight lines in that plane [on opposite sides of Z-axis and making equal angles with it].

Similarly, the cone (1) meets ZX plane $(y=0)$ in two lines $x=\pm\frac{a}{c}z$ which are equally inclined to Z-axis and on opposite sides of it.

Again it meets the XY plane $(z=0)$, where $\frac{x^2}{a^2}+\frac{y^2}{b^2}=0$ which is a point ellipse in *that plane*.

(*iv*) **Generated by a variable curve.** The surface (1) meets the plane $z=k$ [where putting $z=k$ in (1)].

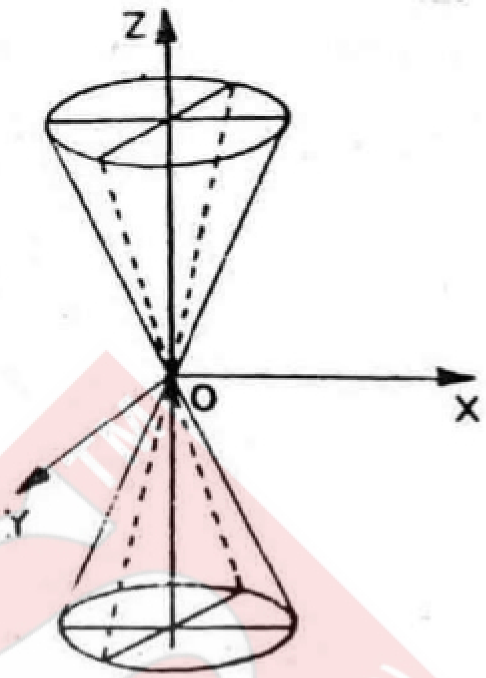
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{k^2}{c^2}=0 \quad \text{or} \quad \frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{k^2}{c^2}$$

Thus the cone (i) is generated by the variable ellipse

$$z=k, \frac{x^2}{a^2}+\frac{y^2}{b^2}=\frac{k^2}{c^2} \ldots(2) \ (k \text{ varies})$$

whose pla ie is ∥ to XY plane and centre $(0, 0\ k)$ moves on Z-axis. The ellipse (2) is real for all values of k +ve or −ve and the semi-axes $\frac{ak}{c}$, $\frac{bk}{c}$ increase as k increases numerically and $\to \infty$ as $k \to \infty$.

∴ The cone extends to infinity both above and below the XY-plane.



Hence the shape is as shown in the adjoining figure.

Note 1. The standard equation of the cone is of the form

$$ax^2+by^2+cz^2=0.$$

Note 2. A cone can be regarded as a central conicoid whose centre is the vertex.

Article 24. Some important results about the cone

$$ax^2+by^2+cz^2=0.$$

(*i*) The **tangent plane** at (x_1, y_1, z_1) and the **plane of contact** of (x_1, y_1, z_1) and **polar plane** of (x_1, y_1, z_1) w.r.t. given cone is

$$axx_1+byy_1+czz_1=0.$$

(*ii*) The plane $lx+my+nz=0$ **touches** the cone if

$$\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=0.$$

(*iii*) The equation of plane which cuts the cone in a conic with centre (x_1, y_1, z_1) is given by $T=S_1$.

☞ *The student is advised to prove these results* as in Arts. 7, 8, 14, 15, 21.

Example 1. *Find the equation of the normal plane of the cone*

$$ax^2+by^2+cz^2=0,$$

through the generator

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}.$$

Sol. [The **normal plane** *through the generator OP of a cone (vertex O) is the plane through OP and ⊥ to the tangent plane at any point of OP.*] **[Remember]**

The given cone is $\quad ax^2+by^2+cz^2=0 \quad \ldots(1)$

and the generator is $\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \quad \ldots(2)$

Any plane through the line (2) is

$$Ax+By+Cz=0 \quad \text{...(3)}$$

where $$Al+Bm+Cn=0 \quad \text{...(4)}$$

Any point on (2) is Q (lr, mr, nr). The tangent plane at Q to (1) is

$$ax(lr)+by(mr)+cz(nr)=0 \text{ or } alx+bmy+cnz=0 \quad \text{...(5)}$$

If (3) is the normal plane through (2), then (3) is $\perp$ to (5).

$$\therefore \quad Aal+Bbm+Ccn=0 \quad \text{...(6)}$$

Solving (4) and (6) by cross-multiplication, we have

$$\frac{A}{mn(c-b)}=\frac{B}{nl(a-c)}=\frac{C}{lm(b-a)}$$

Putting these values of A, B, C in (3) and taking out $-$ve sign common, we have

$$mn(b-c)x+nl(c-a)y+lm(a-b)z=0.$$

Dividing throughout by lmn, we get

$$\frac{(b-c)x}{l}+\frac{(c-a)y}{m}+\frac{(a-b)z}{n}=0,$$

which is the required normal plane.

Example 2. *Lines are drawn through the origin perpendicular to normal planes of the cone*

$$ax^2+by^2+cz^2=0.$$

Show that they generate the cone

$$\frac{a(b-c)^2}{x^2}+\frac{b(c-a)^2}{y^2}+\frac{c(a-b)^2}{z^2}=0.$$

Sol. Let the line OP given by

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$$

be any generator of the cone

$$ax^2+by^2+cz^2=0 \quad \text{...(1)}$$

Since the d.c.'s of the generator satisfy the equation of the cone,

$$\therefore \quad al^2+bm^2+cn^2=0 \quad \text{...(2)}$$

Also equation of the normal plane through OP to (1) is

$$\left(\frac{b-c}{l}\right)x+\left(\frac{c-a}{m}\right)y+\left(\frac{a-b}{n}\right)z=0 \quad \text{...(3)}$$

[*See Example* 1, above]

Equations of the line through (0, 0, 0) $\perp$ to (3) are

$$\frac{x}{\left(\frac{b-c}{l}\right)}=\frac{y}{\left(\frac{c-a}{m}\right)}=\frac{z}{\left(\frac{a-b}{n}\right)}$$

or $$\frac{l}{\frac{(b-c)}{x}}=\frac{m}{\frac{(c-a)}{y}}=\frac{n}{\frac{(a-b)}{z}} \quad \text{...(4)}$$

To find the locus of line (4), we have to eliminate l, m, n from (4) and (2). Putting the values of l, m, n from (4) in (2), we get

$$a\left(\frac{b-c}{x}\right)^2+b\left(\frac{c-a}{y}\right)^2+c\left(\frac{a-b}{z}\right)^2=0$$

or
$$\frac{a(b-c)^2}{x^2}+\frac{b(c-a)^2}{y^2}+\frac{c(a-b)^2}{z^2}=0$$

which is the required cone.

Example 3. *Prove that if a plane cuts the cone*

$$ax^2+by^2+cz^2=0$$

in perpendicular generators, it touches the cone

$$\frac{x^2}{b+c}+\frac{y^2}{c+a}+\frac{z^2}{a+b}=0.$$

Sol. Let the plane be $ux+vy+wz=0$ ...(1)

and the cone is $ax^2+by^2+cz^2=0$ ...(2)

Let a line of section of (1) and (2) be

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n}.$$

Since it lies on (1) and (2) both.

$\therefore$ $ul+vm+wn=0$ and $al^2+bm^2+cn^2=0$ ...(3)

The two lines given by (3) are $\perp$ if

$$u^2(b+c)+v^2(c+a)+w^2(a+b)=0 \qquad ...(4)$$

Now the plane (1) will touch the cone

$$\frac{x^2}{b+c}+\frac{y^2}{c+a}+\frac{z^2}{a+b}=0$$

if
$$\frac{u^2}{\left(\frac{1}{b+c}\right)}+\frac{v^2}{\left(\frac{1}{c+a}\right)}+\frac{w^2}{\left(\frac{1}{a+b}\right)}=0$$

$\Big|$ Using $\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=0$

or if $u^2(b+c)+v^2(c+a)+w^2(a+b)=0$ which is true by (4).

Hence the result.

Remember. *Two lines given by*

$$ul+vm+wn=0 \text{ and } al^2+bm^2+cn^2=0$$

are $\perp$ *if* $u^2(b+c)+v^2(c+a)+w^2(a+b)=0$.

Euample 4. *Show that the perpendicular tangent planes to*

$$ax^2+by^2+cz^2=0$$

intersect in generators of the cone

$$a(b+c)x^2+b(c+a)y^2+c(a+b)z^2=0 \qquad \text{[Imp.]}$$

Sol. The given cone is $ax^2+by^2+cz^2=0$ ...(1)

Any tangent plane to (1) is $lx+my+nz=0$ ...(2)

where $\frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=0$ ...(3)

Let the line of intersection of two tangent planes, (through origin) be

$$\frac{x}{L}=\frac{y}{M}=\frac{z}{N} \quad ...(4)$$

Since it lies on (2), $\therefore$ $Ll+Mm+Nn=0$ ...(5)

The two lines given by (5) and (3) are $\perp$,

$$\therefore \quad L^2\left(\frac{1}{b}+\frac{1}{c}\right)+M^2\left(\frac{1}{c}+\frac{1}{a}\right)+N^2\left(\frac{1}{a}+\frac{1}{b}\right)=0$$

Eliminating L, M, N from this and (4), the required locus is

$$x^2\left(\frac{1}{b}+\frac{1}{c}\right)+y^2\left(\frac{1}{c}+\frac{1}{a}\right)+z^2\left(\frac{1}{a}+\frac{1}{b}\right)=0$$

or $$a(b+c)x^2+b(c+a)y^2+c(a+b)z^2=0.$$

Example 5. (*a*) *The locus of the asymptotes drawn from the origin to the conicoid*

$$ax^2+by^2+cz^2=1$$

is the asymptotic cone

$$ax^2+by^2+cz^2=0.$$

(*b*) *Prove that the hyperboloids*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1 \text{ and } -\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

have the same asymptotic cone.

Sol. [Def. *An asymptote meets the given surface at two points an infinity*]. **[Remember]**

(*a*) The conicoid is $ax^2+by^2+cz^2=1$ ...(1)

Let the asymptote through (0, 0, 0) be

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \quad ...(2)$$

Any point on (2) is (lr, mr, nr).

If it lies on (1), then

$$al^2r^2+bm^2r^2+cn^2r^2=1$$

or $$al^2+bm^2+cn^2=\frac{1}{r^2}$$

Since the asymptote (2) meets (1) an infinity $\therefore$ $r=\infty$.

$$\therefore \quad al^2+bm^2+cn^2=\frac{1}{\infty}=0 \quad ...(3)$$

Eliminating l, m, n from (2) and (3), the locus of (2) is

$$ax^2+by^2+cz^2=0$$

which is a cone.

(*b*) Please try yourself as in part (*a*).

Example 6. *Any plane whose normal lies on the cone*

$$(b+c)x^2+(c+a)y^2+(a+b)z^2=0$$

cuts the surface

$$ax^2+by^2+cz^2=1$$

in a rectangular hyperbola. **[Imp.]**

Sol. Let the plane be $ux+vy+wz=0$ ...(1)

This cuts the surface $ax^2+by^2+cz^2=1$

in rectangular hyperbola.

Let the asymptote of this hyperbola be

$$\frac{x}{l}=\frac{y}{m}=\frac{z}{n} \quad ...(2)$$

The asymptote (2) lies on plane (1)

$$\therefore \quad ul+vm+wn=0 \quad ...(3)$$

Also any point on (2) is (lr, mr, nr). This point will lie on the surface $ax^2+by^2+cz^2=1$, if

$$r^2(al^2+bm^2+cn^2)=1 \quad \text{or} \quad al^2+bm^2+cn^2=\frac{1}{r^2}.$$

But $r\to\infty$, as the asymptote cuts the surface at ∞

$\therefore$ We have $al^2+bm^2+cn^2=0$ ...(4)

The asymptote of a rectangular hyperbola are $\perp$. Thus the two lines given by (3) and (4) are $\perp$.

$\therefore \quad u^2(b+c)+v^2(c+a)+w^2(a+b)=0$ | *Refer Ex.* 15 (*i*), *page* 35

This shows that the normal $\frac{x}{u}=\frac{y}{v}=\frac{z}{w}$

to plane (1) lies on the cone

$$(b+c)x^2+(c+a)y^2+(a+b)z^2=0.$$