9 The Riemann Integral

(Integration of Bounded Functions on R)

At elementary stage, the subject of integration is generally introduced as the inverse of differentiation, so that a function F is called an integral of a given function f, if $F'(x) = f(x)$, for all values of x belonging to the domain of the function f. Historically speaking the subject arose in connection with the evaluation of areas of plane regions and thus amounted to finding out the limit of a sum when the number of terms tended to infinity, each term tending to zero. Realisation that the subject could be looked upon as inverse of differentiation came afterwards. The reference to integration from summation point of view was always associated with the geometrical concepts.

To formulate an independent theory of integration, the German mathematician, Riemann, gave a purely arithmetic treatment to the subject and thus developed the subject entirely free from the intuitive dependence on geometrical concepts. Many refinements and generalisations of the subject followed, the most noteworthy being *Lebesgue theory* of integration.

The present chapter is, thus, based on a definition of the Riemann integral which depends very explicitly on the order structure of the real line. Accordingly, we begin by discussing the integration of real bounded functions on intervals. Integration over sets other than intervals is beyond the scope of the present discussion.

The function will always be assumed to be bounded unless otherwise stated.

1. DEFINITIONS AND EXISTENCE OF THE INTEGRAL

Let $[a, b]$ be a given closed interval.

Partition. By a partition of $[a, b]$, we mean a finite set P of points $x_0, x_1, x_2, \ldots, x_n$, where

$$a = x_0 \leq x_1 \leq \ldots \leq x_{n-1} \leq x_n = b$$

The partition P consists of $n + 1$ points. Clearly any number of partitions of $[a, b]$ can be considered. $[x_0, x_1]$, $[x_1, x_2]$, ..., $[x_{i-1}, x_i]$, ..., $[x_{n-1}, x_n]$ are the sub-intervals of $[a, b]$. We shall use the same symbol Δx_i to denote the ith sub-interval $[x_{i-1}, x_i]$ as its sub-length $x_i - x_{i-1}$. Thus,

$$\Delta x_i = x_i - x_{i-1} \ (i = 1, 2, \ldots, n)$$

Let f be a *bounded real-valued* function on $[a, b]$. Evidently f is bounded on each sub-interval corresponding to each partition P. Let M_i, m_i be the bounds (supremum and infimum) of f in Δx_i.

From the two sums,

$$U(P,f)=\sum_{i=1}^{n} M_i\Delta x_i = M_1\Delta x_1 + M_2\Delta x_2 + \ldots + M_n\Delta x_n$$

$$L(P,f)=\sum_{t=1}^{n} M_i\Delta x_i = m_1\Delta x_1 + b = m_2\Delta x_2 + \ldots + m_n\Delta x_n$$

respectively called the *Upper* and the *Lower* (Darboux) *sums* of f corresponding to the partition P.

If M, m are the bounds of f in $[a, b]$, we have

$$m \le m_i \le M_i \le M$$

$$\Rightarrow \qquad m\,\Delta x_i \le m_i\,\Delta x_i \le M_i\,\Delta x_i \le M\,\Delta x_i$$

Putting $i = 1, 2, \ldots, n$ and adding all the inequalities, we get

$$m(b-a) \le L(P,f) \le U(P,f) \le M(b-a),\ b \ge a \qquad \ldots(1)$$

Now each partition gives rise to a pair of sums, the upper and the lower sums. By considering all partitions of $[a, b]$, we get a set U of upper sums and a set L of lower sums. The inequality (1) shows that both these sets are bounded and so each set has the supremum and the infimum. The *infimum* of the set of upper sums is called the *Upper integral* and the supremum of the set of lower sums is called the *Lower integral* over $[a, b]$. Thus

$$\overline{\int_a^b} f\,dx = \inf U \quad \text{or} \quad \inf\{U(P,f): P \text{ is a partition of } [a,b]\}$$

$$\overline{\int_a^b} f\,dx = \sup L \quad \text{or} \quad \sup\{L(P,f): P \text{ is a partition of } [a,b]\}$$

These two integrals may or may not be equal.

Definition 1 (*Darboux's condition of integrability*). When the two integrals are equal, *i.e.*,

$$\overline{\int_a^b} f dx = \underline{\int_a^b} f dx = \int_a^b f dx$$

we say that f is *Riemann Integrable* (or simply *integrable*) over $[a, b]$ and the common value of these integrals is called the *Riemann Integral* (or simply the *integral*) of f over $[a, b]$.

The fact that f is integrable over $[a, b]$, we express by writing

$$f \in R[a,b] \text{ or } R \text{ simply.}$$

Evidently from equation (1),

$$m(b-a) \le \int_a^b f\,dx \le M(b-a),\ b \ge a \qquad \ldots(2)$$

Thus, the upper and lower integrals are defined for *every bounded function* but they may not necessarily be equal for every bounded function. There exist functions for which these integrals are not equal, such functions are not integrable. Thus the question of their equality, and hence the question of the integrability of a function, is a more delicate one and will be our main concern in the next few pages.

Note: For the sake of convenience, whenever the scope for confusion is not there, we shall omit the limits of integration and write simply,

$$\overline{\int} f\,dx, \quad \underline{\int} f\,dx, \quad \int f\,dx$$

Remarks:

1. The statement that $\int_a^b f\,dx$ exists, implies that the function f is *bounded* and *integrable* over $[a, b]$.
2. We have introduced the concept of integrability of a function subject to two very important limitations, *viz.* (*i*) the function is bounded, (*ii*) the interval is finite.
3. From equations (1) and (2), it follows that when $b > a$,

$$m(b-a) \le L(P,f) \le \int_a^b f\,dx \le U(P,f) \le M(b-a)$$

4. Since the upper integral is the greatest lower bound of the set of upper sums, therefore corresponding to any $\varepsilon_1 > 0, \exists$ an upper sum (or $\exists$ a partition P_1) such that

$$U(P_1 f) < \overline{\int_a^b} f\,dx + \varepsilon_1$$

Similarly,

$$L(P_2\, f) > \underline{\int_a^b} f\,dx - \varepsilon_2$$

5. $U(P,f) - L(P,f) = \sum\limits_i M_i\,\Delta x_i - \sum\limits_i m_i\,\Delta x_i = \sum\limits_i (M_i - m_i)\,\Delta x_i$

$(M_i - m_i)$ being the oscillation of f in the sub-interval Δx_i, $U(n, f) - L(P,f)$ is called the *oscillatory sum* denoted by $\omega(P,f)$ and is non-negative.

Example 1. Show that a constant function k is integrable and

$$\int_a^b k\,dx = k(b-a)$$

▪ For any partition P of the interval $[a, b]$, we have

$$\begin{aligned} L(P,f) &= k\,\Delta x_1 + k\,\Delta x_2 + \ldots + k\,\Delta x_n \\ &= k(\Delta x_1 + \Delta x_2 + \ldots + \Delta x_n) \\ &= k(b-a) \end{aligned}$$

$$\Rightarrow \quad \underline{\int_a^b} k\,dx = \sup\, L(P,f) = k(b-a)$$

$$\begin{aligned} \overline{\int_a^b} k\,dx &= \inf\, U(P,f) \\ &= \inf\,(k\,\Delta x_1 + k\,\Delta x_2 + \ldots + k\,\Delta x_n) \\ &= k(b-a) \end{aligned}$$

Thus,

$$\underline{\int_a^b} k\, dx = \overline{\int_a^b} k\, dx = k(b-a)$$

which implies that the function k is integrable and

$$\int_a^b k\, dx = k(b-a)$$

Example 2. Show that the function f defined by

$$f(x) = \begin{cases} 0, & \text{when } x \text{ is rational,} \\ 1, & \text{when } x \text{ is irrational} \end{cases}$$

is not integrable on any interval.

■ Let us consider a partition P of an interval $[a, b]$.

$$U(P, f) = \sum_{i=1}^{n} M_1 \Delta x_1$$

$$= 1\,\Delta x_1 + 1\,\Delta x_2 + \ldots + 1\,\Delta x_n = b - a$$

$$\overline{\int_a^b} f\, dx = \inf U(P, f) = b - a$$

$$\underline{\int_a^b} f\, dx = \sup L(P, f)$$

$$= \sup \{0\Delta x_1 + 0\Delta x_2 + \ldots + 0\Delta x_n\} = 0$$

Thus,

$$\overline{\int} f\, dx \neq \underline{\int} f\, dx$$

Hence, the function f is not integrable.

Example 3. Show that x^2 is integrable on any interval $[0, k]$.

■ Let us consider the partition P of $[0, k]$ obtained by dividing the interval into n equal parts. Thus $[0, k/n, 2k/n, \ldots, nk/n]$ is the partition P, $[(i-1)(k/n)]^2$ and $[i(k/n)]^2$ are the lower and upper bounds of the function in Δx_i, and the length of each such interval is k/n.

$$\therefore \quad U(P, x^2) = \frac{k^3}{n^3}(1^2 + 2^2 + 3^2 + \ldots + n^2)$$

$$= \frac{k^3}{n^3} \cdot \frac{n}{6}(n+1)(2n+1)$$

$$= \frac{k^3}{6}\left(1 + \frac{1}{n}\right)\left(2 + \frac{1}{n}\right)$$

and

$$L(P, x^2) = \frac{k^3}{n^3}\{0 + 1^2 + 2^2 + \dots + (n-1)^2\}$$
$$= \frac{k^3}{6}\left(1 - \frac{1}{n}\right)\left(2 - \frac{1}{n}\right)$$

$\therefore$
$$\inf U(P, x^2) = \frac{k^3}{3} = \sup L(P, x^2)$$

Hence, the function x^2 is integrable and

$$\int_0^k x^2\, dx = k^3/3$$

Ex. 1. Show that $(3x + 1)$ is integrable on $[1, 2]$ and

$$\int_1^2 (3x+1)\,dx = \frac{11}{2}$$

Ex. 2. A function f is bounded on $[a, b]$. Show that

(*i*) when k is a positive constant,

$$\overline{\int_a^b} kf\ dx = k\overline{\int_a^b} f\ dx;\ \underline{\int_a^b} kf\ dx = k\underline{\int_a^b} f\ dx$$

and (*ii*) when k is a negative constant,

$$\overline{\int_a^b} kf\ dx = k\underline{\int_a^b} f\ dx;\ \underline{\int_a^b} kf\ dx = k\overline{\int_a^b} f\ dx$$

Deduce that if f is bounded and integrable over $[a, b]$, then so is kf, where k is any constant and then

$$\int_a^b kf\,dx = k\int_a^b f\,dx$$

[*Hint*: If M_i, m_i M_i, m_i are bounds of f in Δx_1, then kM_i, km_i, are bound of kf in Δx_i, if k is positive. But if k is negative, the bounds of kf are km_i, kM_i]

1.1 A Definition

The meaning of $\int_a^b f\,dx$ when $b \le a$. If f is bounded and integrable on $[b, a]$, for $a > b$, we define

$$\int_a^b f\,dx = -\int_b^a f\,dx, \text{ when } a > b$$

Also

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$$\int_a^b f\,dx = 0 \quad \text{when } a = b$$

1.2 Inequalities for Integrals

In an earlier section we have proved that for a bounded integrable function f,

$$m(b-a) \le \int_a^b f\ dx \le M(b-a), \text{ when } b \ge a \qquad \text{...(3)}$$

If $b < a$, so that $a > b$, then as proved above

$$m(a-b) \le \int_b^a f\ dx \le M(a-b), \text{ when } a > b$$

$$\Rightarrow \qquad -m(a-b) \ge -\int_b^a f\ dx \ge -M(a-b)$$

$$\Rightarrow \qquad m(b-a) \ge \int_a^b f\ dx \ge M(b-a), \text{ when } b < a \qquad \text{...(4)}$$

We shall now make some interesting deductions from these two inequalities.

Deduction 1. If f is bounded and integrable on $[a, b]$, then there exists a number λ lying between the bounds of f such that

$$\int_a^b f\,dx = \lambda(b-a)$$

Deduction 2. If f is continuous and integrable on $[a, b]$, then there exists a number c between a and b such that

$$\int_a^b f\,dx = (b-a)\,f(c)$$

Deduction 3. If f is bounded and integrable on $[a, b]$ and k is a number such that $|f(x)| \le k$ for all $x \in [a, b]$, then

$$\left|\int_a^b f\,dx\right| \le k\,|b-a|$$

Let M, m be the bounds of $f(x)$. Now

$$|f(x)| \le k, \ \forall\, x \in [a, b]$$

$$\therefore \qquad -k \le f(x) \le k \ \Rightarrow\ -k \le m \le f(x) \le M \le k$$

which for $b \ge a$, implies that

$$-k(b-a) \le m(b-a) \le \int_a^b f\,dx \le M(b-a) \le k(b-a)$$

$\Rightarrow$
$$\left|\int_a^b f\,dx\right| \le k(b-a)$$

If $b < a$, so that $a > b$, we have

$$\left|\int_b^a f\,dx\right| \le k\,|a-b| \quad \Rightarrow \quad \left|\int_a^b f\,dx\right| \le k\,|b-a|$$

The result is trivial for $a = b$.

Deduction 4. If f is bounded and integrable on $[a, b]$, and $f(x) \ge 0$ for all $x \in [a, b]$, then

$$\int_a^b f\,dx \begin{cases} \ge 0, & \text{if } b \ge a \\ \le 0, & \text{if } b \le a \end{cases}$$

Since $f(x) \ge 0$, for all $x \in [a, b]$, therefore, the lower bound $m \ge 0$.

The result follows from the inequalities (3) and (4) above.

Deduction 5. If f, g are bounded and integrable on $[a, b]$, such that $f \ge g$, then

$$\int_a^b f\,dx \ge \int_a^b g\,dx \quad \text{when } b \ge a$$

and

$$\int_a^b f\,dx \le \int_a^b g\,dx \quad \text{when } b \le a$$

Now

$$f \ge g \Rightarrow f - g \ge 0, \quad \forall\, x \in [a, b]$$

Hence using Deduction 4, we have

$$\int_a^b (f-g)\,dx \ge 0 \qquad \text{if} \quad b \ge a$$

or

$$\int_a^b f\,dx \ge \int_a^b g\,dx \quad \text{if} \quad b \ge a$$

Similarly

$$\int_a^b f\,dx \le \int_a^b g\,dx \quad \text{if} \quad b \ge a$$

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Note: We have assumed the result $\int (f-g)\,dx = \int f\,dx - \int g\,dx$. It will however be proved in § 5, Theorem 6.

2. REFINEMENT OF PARTITIONS

Definition. For any partition P, the length of the largest sub-interval is called the *norm* or *mesh* of the partition and is denoted as $\mu(P)$ (or simply μ).

$$\therefore \qquad \mu(P) = \max \Delta x_i \ (1 \le i \le n)$$

A partition P^* is said to be a *refinement* of P if $P^* \supseteq P$, *i.e.*, every point of P is a point of P^*.

We also say that P^* *refines* P or that P^* is *finer than* P.

If P_1 and P_2 are two partitions, then we say that P^* is their *common refinement* if $P^* = P_1 \cup P_2$.

Theorem 1. *If P^* is a refinement of a partition P, then for a bounded function f,*

(*i*) $L(P^*, f) \ge L(P, f)$, *and*

(*ii*) $U(P^*, f) \le U(P, f)$.

To prove (*i*), suppose first that P^* contains just one point more than P.

Let this extra point be ξ, and suppose that this point is in Δx_i, that is, $x_{i-1} < \xi < x_i$.

As the function is bounded over the entire interval $[a, b]$, it is bounded in every sub-interval Δx_i $(i = 1, 2, \ldots, n)$. Let w_1, w_2, m_i be the infimum (g.l.b.) of f in the intervals $[x_{i-1}, \xi]$, $[\xi, x_i]$, $[x_{i-1}, x_i]$, respectively.

Clearly $m_i \le w_1$, $m_i \le w_2$.

$$\begin{aligned}\therefore \quad L(P^*, f) - L(P, f) &= w_1(\xi - x_{i-1}) + w_2(x_i - \xi) - m_i(x_i - x_{i-1}) \\ &= (w_1 - m_i)(\xi - x_{i-1}) + (w_2 - m_i)(x_i - \xi) \ge 0\end{aligned}$$

($\because$ each bracket is positive)

If P^* contains p points more than P, we repeat the above reasoning p times and arrive at

$$L(P^*, f) \ge L(P, f)$$

Similarly, we can prove that

$$U(P^*, f) \le U(P, f)$$

Corollary. *If a refinement P^* of P contains p points more than P, and $|f(x)| \le k$, for all $x \in [a, b]$, then*

$$L(P, f) \le L(P^*, f) \le L(P, f) + 2pk\mu$$
$$U(P, f) \ge U(P^*, f) \ge U(P, f) - 2pk\mu$$

Proceeding as in the above theorem, when P^* contains one point more than P, we get

$$L(P^*, f) - L(P, f) = (w_1 - m_i)(\xi - x_{i-1}) + (w_2 - m_i)(x_i - \xi)$$

Since $|f(x)| \le k$, for all $x \in [a, b]$, therefore

$$-k \le m_i \le w_1 \le k$$
$$\Rightarrow \qquad 0 \le w_1 - m_i \le 2k$$

Similarly,

$$0 \le w_2 - m_i \le 2k$$

$$\therefore \quad L(P^*, f) - L(P, f) \le 2k(\xi - x_{i-1}) + 2k(x_i - \xi)$$
$$= 2k\ \Delta x_i$$
$$\le 2k\mu, \text{ where } \mu \text{ is norm of } P$$

Now supposing that each additional point is introduced one by one, by repeating the above reasoning p times, we get

$$L(P^*, f) \le L(P, f) + 2pk\mu$$

Also
$$L(P, f) \le L(P^*, f)$$

$$\therefore \quad L(P, f) \le L(P^*, f) \le L(P, f) + 2pk\mu$$

Similarly, the other result may be proved.

Ex. If P^* is a refinement of P, then

$$U(P^*, f) - L(P^*, f) \le U(P, f) - L(P, f)$$

Theorem 2. *For any two partitions P_1, P_2,*

$$L(P_1, f) \le U(P_2, f)$$

i.e., no upper sum can ever be less than any lower sum.

Let P^* be a common refinement of P_1, P_2, so that

$$P^* = P_1 \cup P_2$$

Using Theorem 1, we get

$$L(P_1, f) \le L(P^*, f) \le U(P^*, f) \le U(P_2, f) \qquad \text{...(1)}$$

$$\therefore \quad L(P_1, f) \le U(P_2, f)$$

Corollary. *For any bounded function f,*

$$\underline{\int} f\,dx \le \overline{\int} f\,dx$$

By keeping P_2 fixed and taking the l.u.b. over all partitions P_1, (1) of Theorem 2 gives

$$\underline{\int} f\,dx \le U(P_2, f) \qquad \text{...(2)}$$

Taking the g.l.b. over all P_2 in equation (2), we get

$$\underline{\int} f\,dx \le \overline{\int} f\,dx$$

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3. DARBOUX'S THEOREM

If f is a bounded function on $[a, b]$, *then to every* $\varepsilon > 0$, *there corresponds* $\delta > 0$ *such that*

$$(i)\quad U(P,f) < \overline{\int_a^b} f\,dx + \varepsilon,$$

$$(ii)\quad L(P,f) > \underline{\int_a^b} f\,dx - \varepsilon$$

for every partition P of $[a, b]$ *with norm* $\mu(P) < \delta$.

Let us first attend to (i).

As f is bounded, therefore $\exists\, k > 0$, such that

$$|f(x)| \le k, \ \forall\, x \in [a, b]$$

Again, since the upper integral is the infimum (g.l.b.) of the set of upper sums, therefore to every $\varepsilon > 0$ there exists a partition $P_1 = \{x_0, x_1, x_2, \ldots, x_p\}$ of $[a, b]$ such that

$$U(P_1, f) < \overline{\int_a^b} f dx + \frac{1}{2}\varepsilon \qquad \ldots(1)$$

The partition P_1 has $p - 1$ points besides $x_0\ (= a)$ and $x_p\ (= b)$.

Let δ be a positive number such that $2k(p-1)\delta = \frac{1}{2}\varepsilon$ ...(2)

Let P be any partition with norm $\mu(P) < \delta$.

Let, further, P^* be a refinement of P and P_1, so that $P^* = P \cup P_1$.

As P^* is a refinement of P having at the most $p - 1$ more points than P, therefore (using Corollary, Theorem 1), we get

$$\begin{aligned} U(P, f) - 2k(p-1)\delta &\le U(P^*, f) \\ &\le U(P_1, f) \\ &< \overline{\int_a^b} f dx + \frac{1}{2}\varepsilon \qquad \text{[using equation (1)]} \end{aligned}$$

$$\Rightarrow \quad U(P,f) < \overline{\int_a^b} f\,dx + \frac{1}{2}\varepsilon - \frac{1}{2}\varepsilon = \overline{\int_a^b} f\,dx + \varepsilon \qquad \text{[using equation (2)]}$$

for any partition P with norm $\mu(P) < \delta$.

Similarly, we can prove the other part.

Note: The definition of infimum also leads to the fact that

$$U(P, f) < \overline{\int_a^b} f\,dx + \varepsilon$$

but this implies that for every $\varepsilon > 0$ there exists at least one partition P with this property. The importance of Darboux's Theorem lies in the fact that it asserts the existence of an infinite number of partitions which have this property, with the only restriction that their norm $\mu(P) < \delta$, depends on the choice of ε.

4. CONDITIONS OF INTEGRABILITY

We have stated earlier that a bounded function is said to be integrable when the upper and the lower integrals are equal. We now formalise and give the necessary and sufficient conditions for the integrability of a function in two forms.

Theorem 3. *First form. A necessary and sufficient condition for the integrability of a bounded function f is that to every $\varepsilon > 0$, there corresponds $\delta > 0$ such that for every partition P of $[a, b]$ with norm $\mu(P) < \delta$,*

$$U(P, f) - L(P, f) < \varepsilon$$

The condition is necessary. The bounded function f is integrable,

$$\therefore \quad \underline{\int_a^b} f\,dx = \overline{\int_a^b} f\,dx = \int_a^b f\,dx$$

Let ε be any positive number. By Darboux's Theorem there exists $\delta > 0$ such that for every partition P with norm $\mu(P) < \delta$,

$$U(P,f) < \overline{\int_a^b} f\,dx + \frac{1}{2}\varepsilon = \int_a^b f\,dx + \frac{1}{2}\varepsilon \qquad ...(1)$$

$$L(P,f) > \underline{\int_a^b} f\,dx - \frac{1}{2}\varepsilon = \int_a^b f\,dx - \frac{1}{2}\varepsilon \qquad ...(2)$$

or

$$-L(P,f) < -\int_a^b f\,dx + \frac{1}{2}\varepsilon \qquad ...(3)$$

From equations (1) and (3), we get on adding

$$U(P, f) - L(P, f) < \varepsilon$$

for every partition P with norm $\mu(P) < \delta$.

The condition is sufficient. Let ε be any positive number. For any partition P with norm $\mu(P) < \delta$ (depending on ε), we are given that

$$U(P, f) - L(P, f) < \varepsilon$$

Also for any partition P, we know that

$$L(P,f) \le \underline{\int_a^b} f\,dx \le \overline{\int_a^b} f\,dx \le U(P, f)$$

$$\Rightarrow \quad \overline{\int_a^b} f\,dx - \underline{\int_a^b} f\,dx \le U(P,f) - L(P,f) < \varepsilon$$

Since ε is an arbitrary positive number, therefore we see that a non-negative number is less than every positive number. Hence it must be equal to zero,

$$\Rightarrow \quad \overline{\int_a^b} f\,dx - \underline{\int_a^b} f\,dx = 0$$

$$\Rightarrow \quad \overline{\int_a^b} f\,dx = \underline{\int_a^b} f\,dx$$

so that f is integrable.

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Note: The theorem can also be stated as follows:
A necessary and sufficient condition for the integrability of a bounded function f is that

$$\lim \{U(P,f)-L(P,f)\}=0$$

when the norm $\mu(P)$ of the partition P tends to 0.

Theorem 4. ***Second form.*** *A bounded function f is integrable on $[a, b]$ iff for every $\varepsilon>0$ there exists a partition P of $[a, b]$, such that*

$$U(P,f)-L(P,f)<\varepsilon$$

The condition is necessary. Suppose the function f is integrable, so that

$$\underline{\int_a^b} f\,dx=\overline{\int_a^b} f\,dx=\int_a^b f\,dx$$

Let ε be a positive number.

Since the upper and the lower integrals are the infimum and the supremum, respectively, of the upper and the lower sums, therefore $\exists$ partitions P_1 and P_2 such that

$$U(P_1,f)<\overline{\int_a^b} f\,dx+\frac{1}{2}\varepsilon=\int_a^b f\,dx+\frac{1}{2}\varepsilon$$

$$L(P_2,f)>\underline{\int_a^b} f\,dx-\frac{1}{2}\varepsilon=\int_a^b f\,dx-\frac{1}{2}\varepsilon$$

Let P be the common refinement of P_1 and P_2, *i.e.*, $P=P_1 \cup P_2$,

$$\therefore \quad U(P,f)\le U(P_1,f)<\int_a^b f\,dx+\frac{1}{2}\varepsilon<L(P_2,f)+\varepsilon$$

$$\le L(P,f)+\varepsilon$$

Thus $\exists$ a partition P such that

$$U(P,f)-L(P,f)<\varepsilon$$

The condition is sufficient. Let ε be any positive number.

Let P be a partition for which

$$U(P,f)-L(P,f)<\varepsilon$$

For any partition P, we know that

$$L(P,f)\le \underline{\int_a^b} f\,dx\le \overline{\int_a^b} f\,dx\le U(P,f)$$

$$\therefore \quad \overline{\int_a^b} f\,dx-\underline{\int_a^b} f\,dx\le U(P,f)-L(P,f)<\varepsilon$$

The non-negative number, being less than every positive number ε, must be zero.

$$\therefore \quad \overline{\int_a^b} f\,dx=-\underline{\int_a^b} f\,dx$$

so that f is integrable.

Note: Comparison of the two forms indicates that from the necessary point of view, the first form is stronger than the second but from the sufficiency view point the second form is stronger than the first.

4.1 **Deduction 6.** *A function f is integrable over $[a, b]$ iff there is a number I lying between $L(P,f)$ and $U(P,f)$ such that for any $\varepsilon > 0$, $\exists$ a partition P of $[a, b]$ such that*

$$|U(P,f) - I| < \varepsilon, \text{ and } |I - L(P,f)| < \varepsilon.$$

Necessary: As $f \in R[a, b]$ therefore for $\varepsilon > 0$, $\exists$ a partition P of $[a, b]$ such that

$$|U(P,f) - L(P,f)| < \varepsilon$$

If I is a number between $L(P,f)$ and $U(P,f)$, then

$$|U(P,f) - I| < |U(P,f) - L(P,f)| < \varepsilon$$

and

$$|I - L(P,f)| < |U(P,f) - L(P,f)| < \varepsilon$$

Hence, the result.

Sufficient: For $\varepsilon > 0$, $\exists$ a partition P such that

$$|U(P,f) - I| < \frac{1}{2}\varepsilon \text{ and } |I - L(P,f)| < \frac{1}{2}\varepsilon$$

$$\therefore \quad |U(P,f) - L(P,f)| \le |U(P,f) - I| + |I - L(P,f)|$$

$$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$$

$$\Rightarrow \quad f \in R[a, b]$$

Deduction 7. A function f is integrable over $[a, b]$ iff there is a number I such that for any $\varepsilon > 0, \exists\, \delta$ such that for all partitions P with mesh $\mu(P) < \delta$,

$$|U(P,f) - I| < \varepsilon, \text{ and } |I - L(P,f)| < \varepsilon$$

The proof is similar to that of deduction 6 and is left to the reader.

Note: We know that $U(P^*,f) < U(P,f)$ when $P^* \supset P$. Therefore, the upper sum becomes smaller and smaller as the partition gets finer and finer. Thus the upper integral $\overline{\int_a^b} f\,dx$, which is the g.l.b. of the set U of upper sums, can be looked upon as $\lim\limits_{\mu(P)\to 0} U(P,f)$ or that for $\varepsilon > 0, \exists\, \delta > 0$ such that for all partitions P with mesh $\mu(P) < \delta, |U(P,f) - \overline{\int} f\,dx| < \varepsilon$. Thus Deduction 7 merely states that the upper and the lower sums converge to the same quantity I and moreover $I = \int_a^b f\,dx$.

CHAPTER 9

5. INTEGRABILITY OF THE SUM AND DIFFERENCE OF INTEGRABLE FUNCTIONS

Theorem 5. *If f_1 and f_2 are two bounded and integrable functions on $[a, b]$, then $f = f_1 + f_2$ is also integrable on $[a, b]$, and*

$$\int_a^b f_1\,dx + \int_a^b f_2\,dx = \int_a^b f\,dx$$

Clearly f is bounded on $[a, b]$.

Let $P = \{a = x_0, x_1, \ldots, x_n = b\}$ be any partition of $[a, b]$ and M_i', m_i'; M_i'', m_i'', M_i, m_i be the bounds of f_1, f_2 and f respectively in Δx_i.

[$M_i' + M_i''$ and $m_i' + m_i''$ are rough upper and rough lower bounds while M_i and m_i, the supremum and the infimum of f in Δx_i.]

$$m_i' + m_i'' \le m_i \le M_i \le M_i' + M_i'' \qquad ...(1)$$

Multiplying by Δx_i and adding all these inequalities for $i = 1,2,3,\ldots n$, we get

$$L(P,f_1) + L(P,f_2) \le L(P,f) \le U(P,f) \le U(P,f_1) + U(P,f_2) \qquad ...(2)$$

Let ε be any positive number.

Since f_1, f_2 are integrable therefore, we can choose $\delta > 0$ such that, for any partition P with norm $\mu(P) < \delta$, we have

$$\left.\begin{aligned} U(P,f_1) - L(P,f_1) < \frac{1}{2}\varepsilon \\ U(P,f_2) - L(P,f_2) < \frac{1}{2}\varepsilon \end{aligned}\right\} \qquad ...(3)$$

Thus for any partition P with norm $\mu(P) < \delta$ we have, from (2) and (3),

$$U(P,f) - L(P,f) \le U(P,f_1) + U(P,f_2) - L(P,f_1) - L(P,f_2)$$
$$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$$

Thus, the function f is integrable.

Now we proceed to prove the second part.

Since f_1, f_2 are integrable and ε is any positive number, therefore, by Darboux's Theorem, $\exists\, \delta > 0$ such that for all partitions P whose norm $\mu(P) < \delta$, we have

$$U(P,f_1) < \int_a^b f_1\,dx + \frac{1}{2}\varepsilon, \text{ and } U(P,f_2) < \int_a^b f_2 dx + \frac{1}{2}\varepsilon \qquad ...(4)$$

Also,

$$\int_a^b f\,dx \le U(P,f) \le U(P,f_1) + U(P,f_2) \qquad \text{[using (2)]}$$

$$< \int_a^b f_1\,dx + \int_a^b f_2\,dx + \varepsilon \qquad \text{[using (4)]}$$

Since ε is arbitrary, we conclude that

$$\int_a^b f\,dx \le \int_a^b f_1\,dx + \int_a^b f_2\,dx \qquad ...(5)$$

Proceeding with $(-f_1),(-f_2)$ in place of f_1, f_2 respectively, we get

$$\int_a^b (-f)\,dx \le \int_a^b (-f_1)\,dx + \int_a^b (-f_2)\,dx$$

or

$$\int_a^b f\,dx \ge \int_a^b f_1\,dx + \int_a^b f_2\,dx \qquad ...(6)$$

From equations (5) and (6),

$$\int_a^b f\,dx = \int_a^b f_1\,dx + \int_a^b f_2\,dx$$

Note: When f_1 is integrable, for $\varepsilon > 0, \exists\, \delta_1 > 0$ such that for $\mu(P) < \delta_1, U(P, f) - L(P, f_1) < \frac{1}{2}\varepsilon$. Similarly $\exists\, \delta_2 > 0$ for the functions f_2. However, $\delta = \min(\delta_1, \delta_2)$ works for both the functions. It is this δ which was used for (3) above, δ for (4) was also selected by a similar reasoning.

Ex. Prove the above theorem by using Theorem 4 for condition of integrability.

Theorem 6. *If f_1, f_2 are two bounded and integrable functions on $[a, b]$, then $f = f_1 - f_2$ is also integrable on $[a, b]$ and*

$$\int_a^b f\,dx = \int_a^b f_1\,dx - \int_a^b f_2\,dx$$

Let $f = f_1 + (-f_2)$, so that f is bounded on $[a, b]$.

Let $P = \{a = x_0, x_1, x_2, ..., x_n = b\}$ be any partition of $[a, b]$ and M_i', m_i'; M_i'', m_i''; M_i, m_i be the bounds of f_1, f_2 and f respectively in Δx_i. Clearly the bounds of $(-f_2)$ are $-m_i''$ and $-M_i''$.

$$\therefore \qquad m_i' - M_i'' \le m_i \le M_i \le M_i' - m_i'' \qquad ...(1)$$

Multiplying by Δx_i and adding all these inequalities for $i = 1, 2, ..., n$, we get

$$L(P, f_1) - U(P, f_2) \le L(P, f) \le U(P, f) \le U(P, f_1) - L(P, f_2) \qquad ...(2)$$

Let $\varepsilon > 0$ be a given number.

Since f_1, f_2 are integrable, therefore $\exists\, \delta > 0$ such that for any partition P whose norm $\mu(P) < \delta$, we have

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$$\left.\begin{aligned} U(P,f_1)-L(P,f_1)<\frac{1}{2}\varepsilon \\ U(P,f_2)-L(P,f_2)<\frac{1}{2}\varepsilon \end{aligned}\right\} \quad \ldots(3)$$

Thus for any partition P with norm $\mu(P) < \delta,$ we have from equations (2) and (3),

$$U(P,f) - L(P,f) \le U(P,f_1) - L(P,f_2) - L(P,f_1) + U(P,f_2)$$

$$<\frac{1}{2}\varepsilon+\frac{1}{2}\varepsilon=\varepsilon$$

Thus, the function f is integrable.

Let us now prove the second part.

Since f_1, f_2 are integrable and ε is any positive number, therefore, by Darboux's Theorem $\exists\, \delta>0$ such that for all partitions P with norm $\mu(P) < \delta,$ we have

$$U(P,f_1) < \int_a^b f_1 dx + \frac{1}{2}\varepsilon, \text{ and } L(P,f_2) > \int_a^b f_2\, dx - \frac{1}{2}\varepsilon \quad \ldots(4)$$

$$\therefore \quad \int_a^b f\, dx \le U(P,f)$$

$$\le U(P,f_1) - L(P,f_2) \quad \text{[using (8)]}$$

$$< \int_a^b f_1\, dx - \int_a^b f_2\, dx + \varepsilon$$

Since ε is arbitrary, we conclude that

$$\int_a^b f\, dx \le \int_a^b f_1\, dx - \int_a^b f_2\, dx$$

Proceeding with $(-f_1)$ and $(-f_2)$ in place of f_1 and f_2 respectively, we get

$$\int_a^b f\, dx \ge \int_a^b f_1\, dx - \int_a^b f_2\, dx$$

$$\therefore \quad \int_a^b f\, dx = \int_a^b f_1\, dx - \int_a^b f_2\, dx$$

Ex. Prove the above theorem by using Theorem 4.

Theorem 7. (*i*) *If a bounded function f is integrable on* $[a, b]$, *then it is also integrable on* $[a, c]$ *and* $[c, b]$, *where c is a point of* $[a, b]$.

(*ii*) *Conversely, if f is bounded and integrable on* $[a, c]$, $[c, b]$, *then it is also integrable on* $[a, b]$.

(*iii*) *Also in either case,*

$$\int_a^b f\, dx = \int_a^c f\, dx + \int_c^b f\, dx, \quad a \le c \le b$$

(*i*) Since $f \in R[a, b]$ therefore for $\varepsilon > 0, \exists$ a partition P such that

$$U(P,f) - L(P,f) < \varepsilon$$

Let P^* be a refinement of P such that

$$P^* = P \cup \{c\}$$

$$\therefore \quad L(P,f) \le L(P^*,f) \le U(P^*,f) \le U(P,f) \quad \text{...(1)}$$

$$\Rightarrow \quad U(P^*,f) - L(P^*,f) \le U(P,f) - L(P,f) < \varepsilon \quad \text{...(2)}$$

Let P_1, P_2 denote the sets of points of P^* between $[a, c]$, $[c, b]$ respectively. Clearly P_1, P_2 are partitions of $[a, c]$, $[c, b]$ respectively and $P^* = P_1 \cup P_2$.

Also

$$U(P^*,f) = U(P_1,f) + U(P_2,f) \quad \text{...(3)}$$

and

$$L(P^*,f) = L(P_1,f) + L(P_2,f) \quad \text{...(4)}$$

$$\therefore \quad \{U(P_1,f) - L(P_1,f)\} + \{U(P_2,f) - L(P_2,f)\} = U(P^*,f) - L(P^*,f) < \varepsilon$$

Since each bracket on the left is non-negative, it follows that partitions P_1, P_2 exist such that

$$U(P_1,f) - L(P_1,f) < \varepsilon/2$$

$$U(P_2,f) - L(P_2,f) < \varepsilon/2$$

$\Rightarrow$ Integrable f is on $[a, c]$ and $[c, b]$

(*ii*) Let $f \in R$ over $[a, c]$ and $[c, b]$.

Therefore for $\varepsilon > 0$, we can find partitions P_1, P_2 of $[a, c]$, $[c, b]$ respectively such that

$$U(P_1,f) - L(P_1,f) < \frac{1}{2}\varepsilon \quad \text{...(5)}$$

$$U(P_2,f) - L(P_2,f) < \frac{1}{2}\varepsilon \quad \text{...(6)}$$

Let $P^* = P_1 \cup P_2$.

Clearly P^* is a partition of $[a, b]$.

$$\therefore \quad U(P^*,f) - L(P^*,f) = U(P_1,f) + U(P_2,f) - L(P_1,f) - L(P_2,f) < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$$

Thus for $\varepsilon > 0, \exists$ a partition P^* of $[a, b]$ such that

$$U(P^*,f) - L(P^*,f) < \varepsilon$$

$$\Rightarrow \quad f \in R \text{ over } [a, b]$$

(*iii*) We know that for any functions S and T, if $W = S + T$, then $\inf W > \inf S + \inf T$

Now for any partitions P_1, P_2 of $[a, c]$, $[c, b]$ respectively, if $P^* = P_1 \cup P_2$, then

$$U(P^*,f) = U(P_1,f) + U(P_2,f)$$

Hence, on taking the infimum for all partitions, we get

$$\underline{\int_a^b} f\,dx \ge \underline{\int_a^c} f\,dx + \underline{\int_a^b} f\,dx$$

But since f is integrable on $[a, c]$, $[c, b]$, $[a, b]$,

$$\therefore \qquad \int_a^b f\,dx \ge \int_a^c f\,dx + \int_c^b f\,dx \qquad ...(7)$$

Proceeding with $(-f)$ in place of f, we get

$$\int_a^b f\,dx \le \int_a^c f\,dx + \int_c^b f\,dx \qquad ...(8)$$

Equations (7) and (8), imply that

$$\int_a^b f\,dx = \int_a^c f\,dx + \int_c^b f\,dx$$

Ex. Do the above theorem by using Theorem 3 for conditions of integrability.

5.1* Integrability of the Product, Quotient and the Modulus of Integrable Functions

Before taking up the main theorems, let us prove a simple but useful lemma.

Lemma. *The oscillation of a bounded function f on an interval $[a, b]$ is the supremum of the set $\{|f(x_1) - f(x_2)| : x_1, x_2 \in [a, b]\}$ of numbers.*

Let M, m be the bounds of f, on $[a, b]$. Now

$$m \le f(x_1), f(x_2) \le M, \ \forall\, x_1, x_2 \in [a, b]$$

$$\Rightarrow \qquad |f(x_1) - f(x_2)| \le M - m, \ \forall\, x_1, x_2 \in [a, b] \qquad ...(1)$$

$\Rightarrow$ $\qquad M - m$ is an upper bound of the set in question.

Let $\varepsilon > 0$ be any given number.

Since M is supremum of f, therefore $\exists\, x' \in [a, b]$ such that

$$f(x') > M - \frac{1}{2}\varepsilon \qquad ...(2)$$

Similarly, $\exists\, x'' \in [a, b]$ such that

$$f(x'') < m + \frac{1}{2}\varepsilon \qquad ...(3)$$

Equations (2) and (3) imply that $\exists\, x', x'' \in [a, b]$ such that

$$f(x') - f(x'') > M - m - \varepsilon \qquad ...(4)$$

$$\Rightarrow \qquad |f(x') - f(x'')| > M - m - \varepsilon$$

(1) and (4) imply that $M-m$ is an upper bound and no number less than $M-m$ can be an upper bound of the set in question.

$$\therefore \qquad M-m=\sup\{|f(x_1)-f(x_2)|: x_1, x_2 \in [a, b]\}$$

Theorem 8*. *If f_1 and f_2 are two bounded and integrable functions on $[a, b]$, then their product $f_1 f_2$ is also bounded and integrable on $[a, b]$.*

Since f_1, f_2 are bounded therefore, $\exists\, k>0$, such that for all $x \in [a, b]$,

$$|f_1(x)| \le k, \; |f_2(x)| \le k$$

$$\Rightarrow \qquad |(f_1 f_2)(x)| = |f_1(x) f_2(x)| \le k^2, \;\; \forall\, x \in [a, b]$$

$$\Rightarrow \qquad f_1 \cdot f_2 \text{ is bounded}$$

Let $P=\{a=x_0, x_1, x_2, \ldots, x_n=b\}$ be any partition of $[a, b]$.

Let M_i', m_i'; M_i'', m_i''; M_i, m_i be the bounds of f_1, f_2 and $f_1 f_2$ respectively in Δx_i.

We have for all $x_1, x_2 \in \Delta x_i$,

$$\begin{aligned}(f_1 f_2)(x_2)-(f_1 f_2)(x_1) &= f_1(x_2) f_2(x_2) - f_1(x_1) f_2(x_1)\\ &= f_2(x_2)[f_1(x_2)-f_1(x_1)] + f_1(x_1)[f_2(x_2)-f_2(x_1)]\end{aligned}$$

$$\begin{aligned}\Rightarrow \quad |(f_1 f_2)(x_2)-(f_1 f_2)(x_1)| &\le |f_2(x_2)|\cdot|f_1(x_2)-f_1(x_1)| + |f_1(x_1)|\cdot|f_2(x_2)-f_2(x_1)|\\ &\le k(M_i'-m_i') + k(M_i''-m_i'')\end{aligned}$$

$$\Rightarrow \qquad M_i - m_i \le k(M_i'-m_i') + k(M_i''-m_i'') \qquad \ldots(1)$$

Now let $\varepsilon>0$ be a given number.

Since f_1, f_2 are integrable, therefore $\exists\, \varepsilon>0$ such that for any partition P with norm $\mu(P)<\delta$,

$$U(P, f_1)-L(P, f_1)<\varepsilon/2k, \quad U(P, f_2)-L(P, f_2)<\varepsilon/2k$$

Hence, from (1), multiplying by Δx_i and adding all such inequalities, we have for any partition P with norm $\mu(P)<\delta$,

$$\begin{aligned}U(P, f_1 f_2)-L(P, f_1 f_2) &\le k[U(P, f_1)-L(P, f_1)] + k[U(P, f_2)-L(P, f_2)]\\ &< k(\varepsilon/2k)+k(\varepsilon/2k)=\varepsilon\end{aligned}$$

implying that $f_1 f_2$ is integrable on $[a, b]$.

Theorem 9. *If f_1, f_2 are two bounded and integrable functions on $[a, b]$ and there exists a number $\lambda>0$ such that $|f_2(x)| \ge \lambda$, for all x in $[a, b]$, then f_1/f_2 is bounded and integrable on $[a, b]$.*

Since f_1, f_2 are bounded, therefore, $\exists$ positive number k such that

$$|f_1(x)| \le k, \; \lambda \le |f_2(x)| \le k, \;\; \forall\, x \in [a, b]$$

$$\Rightarrow \qquad |(f_1/f_2)(x)| = |f_1(x)/f_2(x)| \le k/\lambda, \;\; \forall\, x \in [a, b]$$

$$\Rightarrow \qquad f_1/f_2 \text{ is bounded}$$

Let $P=\{a=x_0, x_1, \ldots, x_n=b\}$ be a partition of $[a, b]$ and let M_i', m_i'; M_i'', m_i''; m_i, M_i be the bounds of $f_1, f_2, f_1/f_2$, respectively in Δx_i. We have for all $x_1, x_2 \in \Delta x_i$,

$$|(f_1/f_2)(x_2)-(f_1/f_2)(x_1)| = \left|\frac{f_1(x_2)}{f_2(x_2)} - \frac{f_1(x_1)}{f_2(x_1)}\right|$$

$$= \left| \frac{f_2(x_1)[f_1(x_2) - f_1(x_1)] - f_1(x_1)[f_2(x_2) - f_2(x_1)]}{f_2(x_2)\, f_2(x_1)} \right|$$

$$\leq (k/\lambda^2)|f_1(x_2) - f_1(x_1)| + (k/\lambda^2)|f_2(x_2) - f_2(x_1)|$$

$$\leq (k/\lambda^2)(M_i' - m_i') + (k/\lambda^2)(M_i'' - m_i'')$$

$$\Rightarrow \quad M_i - m_i \leq (k/\lambda^2)(M_i' - m_i') + (k/\lambda^2)(M_i'' - m_i'') \qquad ...(1)$$

Now, let $\varepsilon > 0$ be a given number.

Since f_1, f_2 are integrable, therefore $\exists\ \delta > 0$ such that for any partition P with norm $\mu(P) < \delta$,

$$U(P, f_1) - L(P, f_1) < \varepsilon \lambda^2 / 2k$$

$$U(P, f_2) - L(P, f_2) < \varepsilon \lambda^2 / 2k$$

Hence from equation (1), for any partition P with $\mu(P) < \delta$, we have

$$U(P, f_1/f_2) - L(P, f_1/f_2) \leq (k/\lambda^2)\,[U(P, f_1) - L(P, f_1)] + (k/\lambda^2)\,[U(P, f_2) - L(P, f_2)]$$

$$< (k/\lambda^2)(\varepsilon\lambda^2/2k) + (k/\lambda^2)(\varepsilon\lambda^2/2k) = \varepsilon$$

implying that f_1/f_2 is integrable on $[a, b]$.

Theorem 10. *If f is bounded and integrable on $[a, b]$, then $|f|$ is also bounded and integrable on $[a, b]$. Moreover*

$$\left| \int_a^b f\, dx \right| \leq \int_a^b |f|\, dx$$

Since f is bounded, therefore $\exists\ k > 0$, such that

$$|f(x)| \leq k, \quad \forall\ x \in [a, b]$$

$\Rightarrow$ the function $|f|$ is bounded.

Again, since f is integrable, $\exists$ a partition $P = \{a = x_0, x_1, \ldots, x_n = b\}$ of $[a, b]$ such that

$$U(P, f) - L(P, f) < \varepsilon$$

Let $M_i, m_i;\ M_i', m_i'$ be the bounds of f and $|f|$ in Δx_i.

We have for all $x_1, x_2 \in \Delta x_i$

$$||f|(x_2) - |f|(x_1)| = ||f(x_2)| - |f(x_1)|| \leq |f(x_2) - f(x_1)| \leq M_i - m_i$$

$$\Rightarrow \quad M_i' - m_i' \leq M_i - m_i$$

This implies that for any partition P,

$$U(P, |f|) - L(P, |f|) \leq U(P, f) - L(P, f) < \varepsilon$$

Hence, $|f|$ is integrable on $[a, b]$.

Since $f(x), -f(x) \leq |f(x)| = |f|(x)$, for all x in $[a, b]$, therefore by Deduction 5, we have

$$\int_a^b f\, dx \leq \int_a^b |f|\, dx$$

and
$$-\int_a^b f\,dx = \int_a^b (-f)\,dx \le \int_a^b |f|\,dx$$

$\Rightarrow$
$$\left|\int_a^b f\,dx\right| \le \int_a^b |f|\,dx$$

Note: *The converse of the above theorem is not true.* Consider, for example, the function,

$$f(x) = \begin{cases} 1, & \text{when } x \text{ is rational} \\ -1, & \text{when } x \text{ is irrational} \end{cases}$$

Here

$$\overline{\int_a^b} f\,dx = b - a, \quad \underline{\int_a^b} f\,dx = -(b - a)$$

$\Rightarrow$ f is not integrable

But $|f(x)| = 1$, for all x, so that $\int_a^b |f|\,dx$ exists and is equal to $(b - a)$.

Thus, $|f|$ is integrable while f is not.

Theorem 11. *If f is integrable on $[a, b]$, then f^2 is also integrable on $[a, b]$.*

Since f is bounded on $[a, b]$, therefore $|f|$ is also bounded on $[a, b]$.

Thus $\exists\ M > 0$, such that $|f(x)| \le M$, for all x in $[a, b]$.

Again, since f is integrable, therefore $|f|$ is also integrable on $[a, b]$, and therefore for $\varepsilon > 0, \exists$ a partition P of $[a, b]$ such that

$$U(P, |f|) - L(P, |f|) < \frac{\varepsilon}{2M}$$

Again, since $|f^2(x)| = |f(x)|^2 \le M^2$, therefore f^2 is bounded.

If M_i, m_i be the bounds of $|f|$ and M_i', m_i' those of f^2 in Δx_i, then $M_i' = M_i^2, m_i' = m_i^2$.

Also

$$\begin{aligned} U(P, f^2) - L(P, f^2) &= \sum_{i=1}^{n} (M_i' - M_i')\Delta x_i \\ &= \sum_{i=1}^{n} (M_i^2 - M_i^2)\Delta x_i \\ &= \sum_{i=1}^{n} (M_i - m_i)(M_1 + m_i)\Delta x_i \\ &\le 2M \left\{\sum_{i=1}^{n} (M_i - m_i)\Delta x_i\right\} \\ &= 2M\,\{U(P, |f|) - L(P, |f|)\} \\ &< 2M\,\frac{\varepsilon}{2M} = \varepsilon \end{aligned}$$

$\Rightarrow$
$$f^2 \in R\,[a, b]$$

Corollary. If f_1 and f_2 are both integrable on $[a, b]$, then $f_1 f_2$ is also integrable on $[a, b]$.

Since f_1, f_2 are integrable on $[a, b]$, therefore, f_1^2, f_2^2 and $(f_1 + f_2)^2$ are all integrable on $[a, b]$.

Also

$$f_1 f_2 = \frac{1}{2}\left\{(f_1 + f_2)^2 - f_1^2 - f_2^2\right\}$$

therefore, $f_1 f_2 \in R\ [a, b]$

6. THE INTEGRAL AS A LIMIT OF SUMS (*Riemann sums*)

Earlier, we arrived at the integral of a function via the upper and the lower sums. The numbers M_i, m_i which appear in these sums are not necessarily the values of the function f (they are values of f if f is continuous). We shall now show that $\int f\,dx$ can also be considered as the limit of a sequence of sums in which M_i and m_i are replaced by the values of f.

Corresponding to a partition P of $[a, b]$, let us choose points $t_1, t_2, \ldots, t_n$ such that $x_{i-1} \le t_i \le x_i$ $(i = 1, 2, \ldots, n)$ and let us consider the sum

$$S(P, F) = \sum_{i=1}^{n} f(t_i)\Delta x_i$$

The sum $S(P, f)$ is called *a Riemann sum of f over* $[a, b]$ *relative to* P.

It may be noted that t_i are arbitrary and that t_i can be any point whatsoever of Δx_i.

We say that $S(P, f)$ converges to A as $\mu(P) \to 0$, *i.e.*,

$$\lim_{\mu(P)\to 0} S(P, f) = A$$

if, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|S(P, f) - A| < \varepsilon$$

for every partition $P = \{x_0, x_1, \ldots, x_n\}$ of $[a, b]$ with norm $\mu(P) < \delta$ and for every choice of points t_i in $[x_{i-1}, x_i]$.

Definition 2. (*Second definition of integrability*). *A function f is said to be integrable on* $[a, b]$ *if* $\lim S(P, f)$ *exists as* $\mu(P) \to 0$, *and then*

$$\lim_{\mu(P)\to 0} S(P, f) = \int_a^b f\,dx.$$

Note: Since $\mu(P) \to 0$ when $n \to \infty$, therefore $\lim\limits_{\mu(P)\to 0}$ can be replaced by $\lim\limits_{n\to\infty}$ in the above definition.

We have, thus given two definitions of integrability. The equivalence of the two will now be established.

Def. 1 ⇒ Def. 2. Let a bounded function f be integrable according to the former definition, so that

$$\int_a^b f\,dx = \overline{\int_a^b} dx = \underline{\int_a^b} f\,dx$$

Let ε be any positive number.

By Darboux's Theorem, there exists $\delta > 0$ such that for every partition P with norm $\mu(P) < \delta$,

$$U(P,f) < \overline{\int_a^b} f\,dx + \varepsilon = \int_a^b f\,dx + \varepsilon \quad \text{...(1)}$$

and

$$L(P,f) > \underline{\int_a^b} f\,dx - \varepsilon = \int_a^b f\,dx - \varepsilon \quad \text{...(2)}$$

If t_i is any point of Δx_i, we have

$$L(P,f) \le \sum_{i=1}^{n} f(t_i)\,\Delta x_i \le U(P,f) \quad \text{...(3)}$$

From equations(1), (2) and (3), we deduce that for any $\varepsilon > 0$, $\exists\, \delta > 0$ such that for every partition P with norm $\mu(P) < \delta$,

$$\int_a^b f\,dx - \varepsilon < \sum_{i=1}^{n} f(t_i)\,\Delta x_i < \int_a^b f\,dx + \varepsilon$$

$$\Rightarrow \quad \left| \sum_{i=1}^{n} f(t_i)\,\Delta x_i - \int_a^b f\,dx \right| < \varepsilon$$

Thus, the function is integrable according to Def. 2 also.

Remark: Thus $f \in R \Rightarrow \lim S(P,f)$ exists.

Def. 2 ⇒ Def. 1. Let a function f be integrable according to the second definition, *i.e.*,

$$\lim_{\mu(P)\to 0} S(P,f) \text{ exists}$$

In other words, to every number $\varepsilon > 0$, there exists $\delta > 0$ such that for every partition $P = \{x_0, x_1, \ldots, x_n\}$ with norm $\mu(P) < \delta$ and for every choice of points t_i in Δx_i, $\exists$ a number A, such that

$$\left| \sum_{i=1}^{n} f(t_i)\,\Delta x_i - A \right| < \varepsilon$$

It will first be shown that f is bounded.

Let, if possible, f be not bounded.

Taking $\varepsilon = 1$, there exists a partition P such that for every choice of t_i in Δx_i

$$\left| \sum f(t_i)\Delta x_i - A \right| < 1$$

$$\Rightarrow \quad \left| \sum f(t_i)\Delta x_i \right| < |A| + 1$$

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As f is not bounded in $[a, b]$ it must also be so in at least one subinterval, say Δx_m.

Let us take $t_i = x_i$ when $i \neq m$ so that every t_i except t_m is fixed and accordingly every term of the sum $\Sigma f(t_i)\, \Delta x_i$ except the term $f(t_m)\, \Delta x_m$ is also fixed. Since f is not bounded in Δx_m, we can choose a point t_m in Δx_m, such that

$$|\Sigma f(t_i)\Delta x_i| > |A| + 1$$

and thus we arrive at a contradiction.

Hence, the function f is bounded on $[a, b]$.

Now, let ε be any positive number. Thus there exists $\delta > 0$ such that for all partitions P with $\mu(P) < \delta$, we have

$$A - \frac{1}{2}\varepsilon < S(P, f) < A + \frac{1}{2}\varepsilon \qquad ...(4)$$

We choose one such P. If we let the points t_i range over the intervals Δx_i and take the l.u.b. and the g.l.b. of the numbers $S(P, f)$ obtained in this way, (4) yields

$$A - \frac{1}{2}\varepsilon < L(P, f) \leq U(P, f) < A + \frac{1}{2}\varepsilon \qquad ...(5)$$

$$\Rightarrow \qquad U(P, f) - L(P, f) < \varepsilon$$

Also

$$L(P, f) \leq \underline{\int_a^b} f\, dx \leq \overline{\int_a^b} f\, dx \leq U(P, f)$$

$$\therefore \qquad \overline{\int_a^b} f\, dx - \underline{\int_a^b} f\, dx \leq U(P, f) - L(P, f) < \varepsilon$$

so that a non-negative number is less than every positive number.

$$\Rightarrow \qquad \overline{\int_a^b} f\, dx - \underline{\int_a^b} f\, dx = 0$$

or

$$\overline{\int_a^b} f\, dx = \underline{\int_a^b} f\, dx$$

so that, the function is integrable.

6.1 Example 4. Show that $\int_1^2 f\, dx = \frac{11}{2}$, where $f(x) = 3x + 1$.

■ Let $P = \{1 = x_0, x_1, x_2, \ldots, x_n = 2\}$ be a partition which divides $[1, 2]$ into n equal sub-intervals, each of length $\frac{2-1}{n} = \frac{1}{n}$, so that

$$\mu(P) = \frac{1}{n} \to 0 \text{ as } n \to \infty$$

$$x_i = 1 + \frac{i}{n}, i = 1, 2, \ldots, n$$

$$\Delta x_i = \frac{1}{n}, i = 1, 2, \ldots, n$$

$$\sum_{i=1}^{n} \Delta x_i = n.\frac{1}{n} = 1$$

Let $t_i = x_i$, when $i = 1, 2, \ldots, n$.

$$\therefore \quad S(P,f) = \sum_{i=1}^{n} f(t_i)\, \Delta x_i = \sum_{i=1}^{n} f(x_i)\, \Delta x_i = \sum_{i=1}^{n} (3x_i + 1)\, \Delta x_i$$

$$= \sum_{i=1}^{n} \left\{3\left(1 + \frac{i}{n}\right) + 1\right\} \Delta x_i = 4 \sum_{i=1}^{n} \Delta x_i + \frac{3}{n^2} \sum_{i=1}^{n} i$$

$$= 4 + \frac{3}{n^2}\frac{n(n+1)}{2} = \frac{11}{2} + \frac{3}{2n}$$

Proceeding to limits when $\mu(P) \to 0$,

$$\lim_{\mu(P)\to 0} S(P,f) = \frac{11}{2}$$

Since, the limit exists, the function is integrable and

$$\int_1^2 f\, dx = \lim S(P,f) = \frac{11}{2}$$

Example 5. Compute $\int_{-1}^{1} f\, dx$, where $f(x) = |x|$.

▪ The function f is bounded and continuous on $[-1, 1]$, and

$$f(x) = \begin{cases} -x, & x \le 0 \\ x, & x \ge 0 \end{cases}$$

Let the partition $P = \{-1 = x_0, x_1, \ldots, x_n = 0 = y_0, y_1, \ldots, y_n = 1\}$ divides $[-1, 1]$ into $2n$ equal sub-intervals each of length $1/n$, so that

$$\mu(P) = 1/n \to 0 \text{ as } n \to \infty;$$

$$x_i = -1 + i/n, \quad i = 1, 2, \ldots, n$$

$$y_i = i/n, \quad i = 1, 2, \ldots, n$$

$$\Delta x_i = 1/n = \Delta y_i$$

$$\sum_{i=1}^{n} \Delta x_i = 1 = \sum_{i=1}^{n} \Delta y_i$$

$$\left.\begin{array}{llll} \text{Let} & t_i \in \Delta x_i & \text{and} & t_i' \in \Delta y_i \\ \text{and let} & t_i = x_i & \text{and} & t_i' = y_i \end{array}\right\}, i = 1, 2, \ldots, n$$

$$S(P,f) = \sum_{i=1}^{n} f(t_i)\,\Delta x_i + \sum_{i=1}^{n} f(t_i')\,\Delta y_i$$

$$= \sum_i (-x_i)\,\Delta x_i + \sum_i y_i\,\Delta y_i$$

$$= \sum_i \left(1 - \frac{i}{n}\right)\Delta x_i + \sum_i \frac{i}{n}\Delta y_i$$

$$= \sum_{i=1}^{n} \Delta x_i - \frac{1}{n^2}\sum_{i=1}^{n} i + \frac{i}{n^2}\sum_{i=1}^{n} i$$

$$\therefore \quad \lim_{\mu(P)\to 0} S(P,f) = 1$$

and since the limit exists, the function is integrable and

$$\int_{-1}^{-b} |x|\,dx = \lim S(P,f) = 1.$$

6.2 Some Applications

I. *If f_1 and f_2 are functions and where $f = f_1 \pm f_2$ are bounded and integrable on $[a, b]$, then*

$$\int_a^b f\,dx = \int_a^b f_1\,dx \pm \int_a^b f_2\,dx$$

Let ε be any positive number and $f = f_1 + f_2$.

Since f_1, f_2 are integrable, therefore $\exists\,\delta > 0$ such that for every partition $P = \{x_0, x_1, x_2, \ldots, x_n\}$ with norm $\mu(P) < \delta$ and for every choice of points t_i in Δx_i.

$$\left.\begin{aligned} \left|\sum_i f_1(t_i)\Delta x_i - \int_a^b f_1\,dx\right| < \frac{1}{2}\varepsilon \\ \left|\sum_i f_2(t_i)\Delta x_i - \int_a^b f_2\,dx\right| < \frac{1}{2}\varepsilon \end{aligned}\right\}$$

$$\Rightarrow \quad \left|\sum \{(f_1 + f_2)(t_i)\}\,\Delta x_i - \left\{\int_a^b f_1\,dx + \int_a^b f_2\,dx\right\}\right| < \varepsilon$$

$$\therefore \quad \int_a^b f\,dx = \int_a^b (f_1 + f_2)\,dx = \int_a^b f_1\,dx + \int_a^b f_2\,dx$$

The case of $f = f_1 - f_2$ may be discussed similarly.

Corollary. *If* $f_1 \in R$ and $f_2 \in R$ *over* $[a, b]$, *and* c_1, c_2 *any two constants, then* $c_1 f_1 + c_2 f_2 \in R$ $[a, b]$ *and*

$$\int_a^b (c_1 f_1 + c_2 f_2)\, dx = c_1 \int_a^b f_1\, dx + c_2 \int_a^b f_2\, dx$$

Let $f = c_1 f_1 + c_2 f_2$.

For any partition P, we can write

$$S(P, f) = \sum_i f(t_i)\, \Delta x_i = c_1 \sum_i f_1(t_i)\, \Delta x_i + c_2 \sum_i f_2(t_i)\, \Delta x_i$$

$$= c_1 S(P, f_1) + c_2 S(P, f_2)$$

Since f_1, f_2 are integrable, for $\varepsilon > 0$, we can choose $\delta > 0$ such that for all partition P with $\mu(P) < \delta$, we have

$$\left| S(P, f_1) - \int_a^b f_1\, dx \right| < \varepsilon$$

and

$$\left| S(P, f_2) - \int_a^b f_2\, dx \right| < \varepsilon$$

$$\therefore \qquad \left| S(P, f) - c_1 \int_a^b f_1\, dx - c_2 \int_a^b f_2\, dx \right| < c_1\, \varepsilon + c_2\, \varepsilon$$

$\Rightarrow$ $\lim_{\mu(P) \to 0} S(P, f)$ exists and equals $c_1 \int_a^b f_1\, dx + c_2 \int_a^b f_2\, dx$

Hence, $(c_1 f_1 + c_2 f_2) \in R$ and

$$\int_a^b (c_1 f_1 + c_2 f_2) dx = c_1 \int_a^b f_1\, dx + c_2 \int_a^b f_2\, dx$$

II. *If a function f is bounded and integrable on each of the intervals* $[a, c]$, $[c, b]$, $[a, b]$ *where c is a point of* $[a, b]$, *then*

$$\int_a^b f\, dx = \int_a^c f\, dx + \int_c^b f\, dx$$

Let $\varepsilon > 0$ be given.

As f is integrable on each of the intervals $[a, c]$, $[c, b]$ and $[a, b]$, there exists $\delta > 0$ such that for every partition $P = \{a = x_0, x_1, \ldots, x_n = b\}$ containing the point c, with norm $\mu(P) < \delta$ and for every choice of points t_i in Δx_i,

$$\left| \sum_{[a,c]} f(t_i) \Delta x_i - \int_a^c f\, dx \right| < \frac{1}{3} \varepsilon$$

$$\left|\sum_{[c,b]} f(t_i)\Delta x_i - \int_c^b f\,dx\right| < \frac{1}{3}\varepsilon$$

$$\left|\sum_{[a,b]} f(t_i)\Delta x_i - \int_a^b f\,dx\right| < \frac{1}{3}\varepsilon$$

But

$$\sum_{[a,c]} f(t_i)\,\Delta x_i + \sum_{[c,b]} f(t_i)\,\Delta x_i = \sum_{[a,b]} f(t_i)\,\Delta x_i$$

Therefore, we deduce that

$$\left|\int_a^b f\,dx - \int_a^c f\,dx - \int_c^b f\,dx\right| < \varepsilon$$

$$\Rightarrow \qquad \int_a^b f\,dx = \int_a^c f\,dx + \int_c^b f\,dx$$

III. *A function f is integrable over $[a, b]$ iff for $\varepsilon > 0, \exists\, \delta > 0$ such that if P, P' are any two partitions of $[a, b]$ with mesh less than δ, then*

$$|S(P,f) - S(P',f)| < \varepsilon$$

[This is the analog for Riemann sums of the Cauchy property of sequences.]

First, let $f \in R\,[a, b]$, and $\int_a^b f\;dx = I$

$\therefore$ For $\varepsilon > 0, \exists\, \delta > 0$ such that for all partitions P, P' of $[a, b]$ with mesh less than δ and all positions of t_i in Δx_i,

$$|S(P,f) - I| < \frac{1}{2}\varepsilon \qquad \text{...(1)}$$

$$|S(P',f) - I| < \frac{1}{2}\varepsilon \qquad \text{...(2)}$$

$$\Rightarrow |S(P,f) - S(P',f)| \le |S(P,f) - I| + |S(P',f) - I| < \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon$$

Conversely, for $\varepsilon > 0\; \exists\, \delta_1 > 0$ such that for any partitions P, P' with less than δ_1 we have

$$|S(P,f) - S(P',f)| < \frac{1}{2}\varepsilon \qquad \text{...(3)}$$

We know that for a given partition P' and every choice of t_i in Δx_i, $S(P',f)$ is bounded by $L(P',f)$ and $U(P',f)$, which for all partitions of $[a, b]$ are, in turn, bounded by $m\,(b - a)$ and $M\,(b - a)$, where m, M are bounds of f.

Thus, the sequence $\{S(P',f)\}$ of Riemann sums is bounded. As every bounded sequence has a limit point, let the sequence have a limit point I, mesh so that

$$\lim_{\mu(P')\to 0} S(P',f) = I$$

Hence, for $\varepsilon > 0, \exists\, \delta_2 > 0$ such that for partition P' with mesh

$$\mu(P') < \delta_2$$

$$\left|S(P',f) - I\right| < \frac{1}{2}\varepsilon \qquad ...(4)$$

Let $\delta = \min(\delta_1, \delta_2)$.

$$\therefore \quad |S(P,f) - I| \le |S(P,f) - S(P',f)| + |S(P',f) - I|$$

$$< \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon, \text{ for } \mu(P) < \delta$$

$$\Rightarrow \quad \lim_{\mu(P)\to 0} S(P,f) = I$$

Thus, $\lim_{\mu(P)\to 0} S(P,f)$ exists and hence, the function f is integrable.

7. SOME INTEGRABLE FUNCTIONS

Theorem 12. *Every continuous function is integrable.*

We shall prove that a function f which is continuous on $[a, b]$ is also integrable on $[a, b]$.

Let $\varepsilon > 0$ be given.

Let us choose a positive number η, such that

$$\eta(b - a) < \varepsilon$$

Since f is continuous on the closed interval $[a, b]$, therefore, it is bounded and is *uniformly continuous* on $[a, b]$, which implies that there exists $\delta > 0$, such that

$$|f(t_1) - f(t_2)| < \eta, \text{ if } |t_1 - t_2| < \delta, \text{ and } t_1, t_2 \in [a, b] \qquad ...(1)$$

We, now, choose a partition P with norm $\mu(P) < \delta$.

Then by equation (1), we have

$$M_i - m_i \le \eta \quad (i = 1, 2, \ldots, n)$$

$$\therefore \quad U(P,f) - L(P,f) = \sum_{i=1}^{n} (M_i - m_i)\,\Delta x_i$$

$$\le \eta \sum_{i=1}^{n} \Delta x_i$$

$$= \eta(b - a) < \varepsilon \qquad ...(2)$$

Thus, f is integrable [§ 4 Theorem 3].

Corollary. *If a function f is continuous, then to every $\varepsilon > 0$ there corresponds $\delta > 0$ such that*

$$\left|\sum_{i=1}^{n} f(t_i)\,\Delta x_i - \int_a^b f\,dx\right| < \varepsilon$$

for every partition $P = \{x_0, x_1, x_2, \ldots, x_n\}$ of $[a, b]$ with $\mu(P) < \delta$, *and for every choice of points* t_i in $[x_{i-1}, x_i]$.

Follows from (2) above, since the two numbers $\sum f(t_i)\,\Delta x_i$ and $\int f\,dx$ lie between $U(P,f)$ and $L(P,f)$.

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Remark: For continuous functions, $\lim\limits_{\mu(P)\to 0} \Sigma f(t_i)\,\Delta x_i$ exists and equals

$$\int_a^b f\,dx.$$

Note: Continuity is a sufficient condition for integrability. It is not a necessary condition. Functions exist which are integrable but not continuous. See examples 6, 7 and 8. Many more may be constructed.

Theorem 13. *If a function f is monotonic on* $[a, b]$, *then it is integrable on* $[a, b]$.

We shall prove the theorem when f is monotonic increasing (the proof for the other case is analogous). Clearly f is bounded.

Let $\varepsilon > 0$ be given.

Let us choose a number $\eta < \dfrac{\varepsilon}{f(b) - f(a) + 1}$.

We, now, choose a partition $P = \{x_0, x_1, \dots, x_n\}$ of $[a, b]$ with norm $\mu(P) < \eta$. Since f is monotonic increasing, therefore

$$M_i = f(x_i),\ m_i = f(x_{i-1}), \quad (i = 1, 2, \dots, n)$$

$$\begin{aligned} U(P,f) - L(P,f) &= \sum_{i=1}^{n} (M_i - m_i)\,\Delta x_i \\ &= \sum_{i=1}^{n} \{f(x_i) - f(x_{i-1})\}\,\Delta x_i \\ &< \eta \sum_{i=1}^{n} \{f(x_i) - f(x_{i-1})\} \\ &< \frac{\varepsilon}{f(b) - f(a) + 1} \cdot \{f(b) - f(a)\} \\ &< \varepsilon. \end{aligned}$$

Hence, f is integrable.

Note: $f(b) - f(a) + 1$ have been taken to cover the case when $f(a) = f(b)$.

Theorem 14. *A bounded function f, having a finite number of points of discontinuity on* $[a, b]$, *is integrable on* $[a, b]$.

Let M, m be the bounds of f.

Let $\varepsilon > 0$ be a given number.

Let there be p points of discontinuity of f on $[a, b]$.

We now consider a partition P of $[a, b]$ such that all the points of discontinuity get enclosed in p non-overlapping sub-intervals, the sum of whose lengths $< \frac{1}{2}\varepsilon/(M - m)$. The oscillation of f in each of these sub-intervals being $\le (M - m)$, their total contribution to the difference $\{U(P,f) - L(P,f)\}$ is less than

$$\frac{\varepsilon}{2(M-m)}(M-m)=\frac{1}{2}\varepsilon.$$

The function f is continuous in the remaining portion of $[a, b]$, *i.e.*, in the $(p + 1)$ sub-intervals of $[a, b]$ excluding the sub-intervals considered above.

As in Theorem 12, the contribution to the difference $\{U(P, f) - L(P, f)\}$ from each of these $(p+1)$ sub-intervals can be made $< \dfrac{\varepsilon}{2(p+1)}$, so that the total contribution to $\{U(P, f) - L(P, f)\}$ by these $(p + 1)$ sub-intervals is less than $\dfrac{\varepsilon}{2(p+1)}(p+1)=\dfrac{1}{2}\varepsilon$

Thus, for the partition P of $[a, b]$,

$$U(P,f) - L(P,f) < \varepsilon.$$

Hence, the function f is integrable.

Theorem 15. *A bounded function f is integrable on $[a, b]$, if the set of its points of discontinuity has only a finite number of limit points.*

Let the set of points of discontinuity of f have a finite number p of limit points. Let M, m be the bounds of f.

The limit points may be enclosed in p non-overlapping subintervals of $[a, b]$, the sum of whose lengths $\leq \frac{1}{2}\varepsilon/(M-m)$. So that their total contribution to $\{U(P, f) - L(P, f)\}$ is $< \frac{1}{2}\varepsilon$

Only a finite number of points of discontinuity of f can be outside these subintervals, *i.e.*, the function f has a finite number of points of discontinuity on $[a, b]$ excluding the p subintervals enclosing the limit points. Therefore, as in theorem 14, the total contribution to $\{U(P,f) - L(P,f)\}$ from these portions of $[a, b]$ can be made $< \frac{1}{2}\varepsilon$.

Thus for such a partition P of $[a, b]$,

$$U(P,f) - L(P,f) < \varepsilon.$$

Hence, the function f is integrable on $[a, b]$.

Example 6. A function f is defined on $[-1, 1]$ as follows:

$$f(x) = \begin{cases} k, \text{ positive constant when } x \neq 0 \\ 0, \text{ when } x = 0. \end{cases}$$

Show that f is integrable on $[-1, 1]$ and that the value of the integral is $2k$.

- The function has only one point of discontinuity, 0, and is therefore integrable (Theorem 14). Proceeding as in § 1.1 Example 1, it may be shown that the value of the integral is $2k$.

Example 7. A function f is defined on $[0, 1]$ as follows:

$$f(x) = \begin{cases} 0, \text{ when } x \text{ is irrational or zero, and} \\ 1/q, \text{ when } x \text{ is any non-zero rational number } p/q \text{ with least positive integers } p \text{ and } q. \end{cases}$$

Show that f is integrable on $[0, 1]$ and the value of the integral is zero.

- The function f is bounded with bounds 0 and 1.

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Let ε be any positive number.

There exists the largest integer $q \in \mathbf{N}$ such that $1/q > \frac{1}{2}\varepsilon$ or $q < 2/\varepsilon$, so that there are only a finite number of points p/q for which $I/q > \frac{1}{2}\varepsilon$. Let us call such points as exceptional points.

Thus at those rational points which are exceptional points, f has a value $1/q > \frac{1}{2}\varepsilon$, while at the other rational points, the value of f is $1/q > \frac{1}{2}\varepsilon$. The function is zero at the irrational points.

Also, every interval contains rational as well as irrational points.

Thus the oscillation of f in any interval which includes no exceptional points is less than $\frac{1}{2}\varepsilon$ and that in an interval which includes the exceptional points it is at the most equal to 1.

Let us consider a partition P of [0, 1] so as to enclose the exceptional points (finite in number) in subintervals, the sum of whose lengths is less than $\frac{1}{2}\varepsilon$. Thus the contribution to $[U(P,f) - L(P,f)]$ made by these is less than $\frac{1}{2}\varepsilon$.

The contribution to $[U(P,f) - L(P,f)]$ by the remaining portion of [0, 1] is evidently less than $\frac{1}{2}\varepsilon$.

Hence, $U(P,f) - L(P,f) < \varepsilon$, so that the function f is integrable.

Note: The above function has a discontinuity at each *rational*, while continuity at each *irrational*.

Example 8. Show that the function f defined as follows:

$$f(x) = \frac{1}{2^n}, \text{ when } \frac{1}{2^{n+1}} < x \leq \frac{1}{2^n}, \ (n = 0, 1, 2, \ldots),$$

$$f(0) = 0,$$

is integrable on [0, 1], although it has an infinite number of points of discontinuity.

■ Now

$$\begin{aligned} f(x) &= 1, \text{ when } \frac{1}{2} < x \leq \frac{1}{1} \\ &= \frac{1}{2}, \text{ when } \frac{1}{2^2} < x \leq \frac{1}{2} \\ &= \frac{1}{2^2}, \text{ when } \frac{1}{2^3} < x \leq \frac{1}{2^2} \\ &\ \vdots \\ &= \frac{1}{2^{n-1}}, \text{ when } \frac{1}{2^n} < x \leq \frac{1}{2^{n-1}} \\ &\ \vdots \\ &= 0, \text{ when } x = 0. \end{aligned}$$

Thus, we notice that f is bounded and monotonic increasing on [0, 1].
Hence, f is integrable [Theorem13].

Aliter. f is continuous on $[0, 1]$, except at the set of points

$$0, \frac{1}{2}, \left(\frac{1}{2}\right)^2, \left(\frac{1}{2}\right)^3, \ldots$$

which have only one limit point, 0, and hence f is integrable.

8. INTEGRATION AND DIFFERENTIATION (*The Primitive*)

We shall first show that integration and differentiation are in a certain sense, inverse operations, then define the primitive of a function and go on to prove a theorem which is usually called the fundamental theorem of calculus.

Theorem 16. *If a function f is bounded and integrable on* $[a, b]$, *then the function F defined as*

$$F(x) = \int_a^x f(t)\,dt,\ a \le x \le b$$

is continuous on $[a, b]$, *and furthermore, if f is continuous at a point c of* $[a, b]$, *then F is derivable at c and*

$$F'(c) = f(c).$$

Since f is bounded therefore $\exists$ a number $K > 0$, such that

$$|f(x)| \le K,\ \forall\, x \in [a, b].$$

If x_1, x_2 are two points of $[a, b]$ such that $a \le x_1 < x_2 \le b$, then

$$|F(x_2) - F(x_1)| = \left|\int_{x_1}^{x_2} f(t)\,dt\right|$$

$$\le K(x_2 - x_1) \qquad (\S\ 1.3 \text{ Deduction } 3)$$

Thus for a given $\varepsilon > 0$, we see that

$$|F(x_2) - F(x_1)| < \varepsilon,\ \text{ if } |x_2 - x_1| < \varepsilon/K.$$

Hence, the function F is continuous (in fact uniformly) on $[a, b]$.

Let f be continuous at a point c of $[a, b]$, so that for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(c)| < \varepsilon,\ \text{ for } |x - c| < \delta$$

Let $c - \delta < s \le c \le t < c + \delta$

$$\therefore \qquad \left|\frac{F(t) - F(s)}{t - s} - f(c)\right| = \left|\frac{1}{t - s}\int_s^t \{f(x) - f(c)\}\,dx\right|$$

$$\le \frac{1}{t - s}\int_s^t |f(x) - f(c)|\,dx < \varepsilon.$$

$$\Rightarrow \qquad F'(c) = f(c),$$

i.e., continuity of f at any point of $[a, b]$ implies derivability of **F** at that point.

Note: As c is any point of $[a, b]$, we have for all $x \in [a, b]$,

$$F'(t) = f(t) \qquad \Rightarrow F' = f$$

i.e., continuity of f on $[a, b]$ implies derivability of F on $[a, b]$.

This theorem is sometimes referred to as the *First Fundamental Theorem of Integral Calculus*.

Definition. A derivable function F, if it exists such that its derivative F' is equal to a given function f, is called a *primitive* of f.

The above theorem shows that a sufficient condition for a function to admit of a primitive is that it is continuous. Thus every continuous function f possesses a primitive F, where

$$F(x) = \int_a^x f(t)\, dt$$

Remark: We shall now show, with the help of an example, that continuity of a function is not a necessary condition for the existence of a primitive, in other words, "*functions possessing primitives are not necessarily continuous*". Consider the function f on $[0, 1]$, where

$$f(x) = \begin{cases} 2x, \sin(1/x) - \cos(1/x), & \text{if } x \neq 0 \\ 0, & \text{if } x = 0. \end{cases}$$

It has a primitive F, where

$$F(x) = \begin{cases} x^2 \sin 1/x, & \text{if } x \neq 0. \\ 0, & \text{if } x = 0. \end{cases}$$

Clearly $F'(x) = f(x)$ but $f(x)$ is not continuous at $x = 0$, *i.e.*, $f(x)$ is not continuous on $[0, 1]$ [Example 4, Ch. 6]. In fact, all this amounts to saying that the derivative of a function is not necessarily continuous.

9. THE FUNDAMENTAL THEOREM OF CALCULUS

Theorem 17. *A function f is bounded and integrable on $[a, b]$, and there exists a function F such that $F' = f$ on $[a, b]$, then*

$$\int_a^b f\, dx = F(b) - F(a).$$

Since the function $F' = f$ is bounded and integrable, therefore for every given $\varepsilon > 0, \exists\, \delta > 0$ such that for every partition $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$, with norm $\mu(P) < \delta$,

$$\left.\begin{aligned} &\left| \sum_{i=1}^{n} f(t_i)\, \Delta x_i - \int_a^b f\, dx \right| < \varepsilon \\ &\text{or} \\ &\lim_{(P)\to 0} \sum_{i=1}^{n} f(t_i)\, \Delta x_i = \int_a^b f\, dx, \end{aligned}\right\} \qquad \ldots(1)$$

for every choice of points t_i in Δx_i.

Since we have freedom in the selection of points t_i in Δx_i, we choose them in a particular way as follows:

By Lagrange's Mean Value Theorem, we have

$$F(x_i) - F(x_{i-1}) = F'(t_i)\,\Delta x_i \quad (i = 1, 2, \ldots, n)$$
$$= f(t_i)\,\Delta x_i$$

$$\Rightarrow \quad \sum_{i=1}^{n} f(t_i)\,\Delta x_i = \sum_{i=1}^{n} \{F(x_i) - F(x_{i-1})\}$$
$$= F(b) - F(a).$$

Hence, from (1)
$$\int_a^b f\,dx = F(b) - F(a)$$

It is sometimes referred to as *The Second Fundamental Theorem of Integral Calculus.*

9.1 Solved Examples

Example 9. Show that $\int_0^1 f\,dx = \frac{2}{3}$, where f is the integrable function in Example 8.

▪ The function f is integrable

$$\therefore \quad \int_{1/2^n}^{1} f\,dx = \int_{1/2}^{1} f\,dx + \int_{1/2^2}^{1/2} f\,dx + \int_{1/2^3}^{1/2^2} f\,dx + \ldots + \int_{1/2^n}^{1/2^{n-1}} f\,dx$$
$$= \int_{1/2}^{1} dx + \frac{1}{2}\int_{1/2^2}^{1/2} dx + \frac{1}{2^2}\int_{1/2^3}^{1/2^2} dx + \ldots + \frac{1}{2^{n-1}}\int_{1/2^n}^{1/2^{n-1}} dx$$
$$= \frac{1}{2} + \frac{1}{2}\left(\frac{1}{2} - \frac{1}{2^2}\right) + \frac{1}{2^2}\left(\frac{1}{2^2} - \frac{1}{2^3}\right) + \ldots + \frac{1}{2^{n-1}}\left(\frac{1}{2^{n-1}} - \frac{1}{2^n}\right)$$
$$= \frac{1}{2}\left\{1 + \frac{1}{2^2} + \left(\frac{1}{2^2}\right)^2 + \left(\frac{1}{2^2}\right)^3 + \ldots + \left(\frac{1}{2^2}\right)^{n-1}\right\}$$
$$= \frac{1}{2}\cdot\frac{1-\frac{1}{4}}{1-\frac{1}{4}} = \frac{2}{3}\left(1 - \frac{1}{4^n}\right)$$

Proceeding to limits when $n \to \infty$, we get
$$\int_0^1 f\,dx = \frac{2}{3}$$

Example 10. Show that the function $[x]$, where $[x]$ denotes the greatest integer not greater than x, is integrable in $[0, 3]$, and
$$\int_0^3 [x]\,dx = 3$$

■ The function is bounded and has only three points of finite discontinuity at 1, 2, 3.

Let ε be a given positive number.

Consider a partition P where,

$$P = \{0 = x_0, x_1, \ldots, x_l, y_0, y_1, \ldots, y_m, z_0, z_1, \ldots, z_n = 3\}$$

where $y_0 = 1,\ z_0 = 2$.

Now

$$U(P,f) = \Sigma\, 0\cdot\Delta x_i + 1\cdot(y_0 - x_l) + \Sigma\, 1\cdot\Delta y_i + 2(z_0 - y_m) + \Sigma\, 2\cdot\Delta z_i + 3(z_n - z_{n-1})$$

$$= 1 + 2 + \{(y_0 - x_l) + (z_0 - y_m) + (z_n - z_{n-1})\}$$

Let us select P such that

$$(y_0 - x_i) + (z_0 - y_m) + (z_n - z_{n-1}) < \varepsilon$$

$\therefore$ $$U(P,f) < 3 + \varepsilon$$

Again,

$$L(P,f) = \Sigma 0\cdot\Delta x_i + \Sigma 1\cdot\Delta y_i + (z_0 - y_m) + \Sigma 2\cdot\Delta z_i = 3$$

$\therefore$ $$U(P,f) - L(P,f) < \varepsilon$$

so that the function is integrable, and therefore

$$\overline{\int_0^3} [x]\, dx = \underline{\int_0^3} [x]\, dx = 3$$

Aliter. Since the function is bounded and has only three points of discontinuity therefore it is integrable and

$$\int_0^3 [x]\, dx = \int_0^1 [x]\, dx + \int_1^2 [x]\, dx + \int_2^3 [x]\, dx$$

$$= \int_0^1 0\, dx + \int_1^2 dx + \int_2^3 2\, dx = 3$$

1, 2 and 3 being respectively the points of discontinuity of the three integrals on the right.

Example 11. Let f is a non-negative continuous function on $[a, b]$ and

$$\int_a^b f\, dx = 0. \text{ Prove that } f(x) = 0, \text{ for all } x \in [a, b].$$

■ Suppose that, for some $c \in\,]a, b[,\ f(c) > 0$.

Then, for $\varepsilon = \frac{1}{2} f(c) > 0$, continuity of f at c implies that, there exists a $\delta > 0$ such that

$$f(x) > \frac{1}{2} f(c),\quad \forall\, x \in\,]c - \delta, c + \delta[$$

Now,

$$\int_a^b f(x)\,dx = \int_a^{c-\delta} f(x)\,dx + \int_{c-\delta}^{c+\delta} f(x)\,dx + \int_{c+\delta}^{b} f(x)\,dx$$

$$\geq \int_{c-\delta}^{c+\delta} f(x)\,dx \qquad (\because f(x) \geq 0,\ \forall\, x \in [a, b])$$

$$> \frac{1}{2} f(c) \int_{c-\delta}^{c+\delta} dx = \delta\, f(x) > 0.$$

which is a contradiction. Thus $f(x) = 0,\ \forall\, x \in \,]a, b[$.

Similarly, $f(a) \ngtr 0$, and $f(b) \ngtr 0$. Hence, the result follows.

Example 12. Show that $\int_0^t \sin x\, dx = 1 - \cos t$.

▮ The function $\sin x$ is bounded and continuous in any interval $[0, t]$ and is therefore integrable. [To be more specific, take $t \leq \pi/2$.]

Consider a partition $P = \left\{0, \frac{t}{n}, \frac{2t}{n}, \ldots, \frac{nt}{n}\right\}$ of $[0, t]$.

Now

$$U(P, \sin x) = \frac{t}{n}\left\{\sin\frac{t}{n} + \sin\frac{2t}{n} + \sin\frac{3t}{n} + \ldots + \sin\frac{nt}{n}\right\}$$

$$= \frac{t}{n}\left(\cos\frac{t}{2n} - \cos\,(n+1)\,\frac{t}{n}\right)\Big/\left(2\sin\frac{t}{2n}\right)$$

$$= \left\{\cos\frac{t}{2n} - \cos\,(n+1)\,\frac{t}{n}\right\}\Big/\left(\frac{\sin t/2n}{t/2n}\right)$$

and

$$L(P, \sin x) = \frac{t}{n}\left\{\sin\frac{t}{n} + \sin\frac{2t}{n} + \ldots + \sin\overline{n-1}\,\frac{t}{n}\right\}$$

$$= \left\{\cos\frac{t}{2n} - \cos\frac{nt}{n}\right\}\Big/\left(\frac{\sin t/2n}{t/2n}\right).$$

In the limit,

$$U(P, \sin x) = L(P, \sin x) = 1 - \cos t$$

$\therefore$

$$\int_0^t \sin x\, dx = 1 - \cos t$$

CHAPTER 9

Example 13. Evaluate $\int_0^1 \left(2x \sin \frac{1}{x} - \cos \frac{1}{x}\right) dx$

■ The function

$$f(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x}, & x \in]0, 1[\\ 0, & x = 0 \end{cases}$$

is not continuous on [0, 1] (it is discontinuous at $x = 0$), but it is bounded and continuous on] 0, 1] and thus Riemann-integrable on [0, 1].

The function

$$g(x) = \begin{cases} x^2 \sin \frac{1}{x}, & x \in]0, 1] \\ 0, & x = 0 \end{cases}$$

is differentiable on [0, 1] and satisfies

$$g'(x) = f(x), \ \forall\, x \in [0, 1]$$

$$\therefore \quad \int_0^1 \left(2x \sin \frac{1}{x} - \cos \frac{1}{x}\right) dx = g(1) - g(0) = \sin 1$$

Note: If f is not integrable but $f(x) = g'(x), \ \forall\, x \in [a, b]$, then

$$\int_a^b f\, dx \neq g(b) - g(a).$$

This can be seen by the following example, which has a primitive without being Riemann-integrable.

Example 14. Prove with the help of an example that the equation (1)

$$\int_a^b f'(x)\, dx = f(b) - f(a), \text{ is not always valid.}$$

■ Let f be defined on [0, 1] as follows:

$$f(x) = x^2 \cos(\pi/x^2), \ \text{if } 0 < x \leq 1, \ f(0) = 0.$$

Then f is differentiable on [0, 1] and

$$f'(x) = 2x \cos(\pi/x^2) + (2\pi/x) \sin(\pi/x^2), \ \text{if } 0 < x \leq 1,$$

$$f'(0) = 0$$

f' is not bounded on [0, 1] and therefore, it is not Riemann-integrable, *i.e.*, $\int_0^1 f'(x)\, dx$ does not exist.

Therefore the equation (1) fails to hold.

Note: In fact, there are functions f with bounded derivatives f' that are not Riemann-integrable, but these are much more difficult to construct.

Ex. Find the error in the following:

(*i*) If $f(x) = \dfrac{-1}{x-1}$, $f'(x) = \dfrac{1}{(x-1)^2}$, hence

$$\int_0^2 \frac{1}{(x-1)^2}\, dx = f(2) - f(0) = \frac{-1}{2-1} - \left(\frac{-1}{0-1}\right) = -2$$

(*ii*) If $f(x) = 2\sqrt{x}$, then $f'(x) = \dfrac{1}{\sqrt{x}}$; hence

$$\int_0^1 \frac{1}{\sqrt{x}}\, dx = 2\sqrt{1} - 0 = 2.$$

Note: In both the parts $f'(x)$ is not bounded.

10. MEAN VALUE THEOREMS OF INTEGRAL CALCULUS

We have two mean value theorems for derivatives. In the same way we have two mean value theorems for integrals as well.

10.1 First Mean Value Theorem

Theorem 18. *If a function f is continuous on* $[a, b]$, *then there exists a number* ξ *in* $[a, b]$ *such that*

$$\int_a^b f\, dx = f(\xi)(b-a)$$

f is continuous, therefore $f \in R[a, b]$.

Let m, M be the infimum and supremum of f in $[a, b]$. Then as in § 1.1, we have

$$m(b-a) \le \int_a^b f\, dx \le M(b-a)$$

Hence, there is a number $\mu \in [m, M]$, such that

$$\int_a^b f\, dx = \mu\,(b-a)$$

Since f is continuous on $[a, b]$, it attains every value between its bounds m, M. Therefore, there exists a number $\xi \in [a, b]$ such that $f(\xi) = \mu$.

$\therefore$
$$\int_a^b f\, dx = f(\xi)(b-a).$$

CHAPTER 9

Remark: The preceding theorem says that the condition of continuity is necessary for the function to assume its mean value in the given interval, for example the function $f(x)$ is defined on [2, 5] as follows:

$$f(x) = \begin{cases} 1 & \text{if } 2 \leq x < 3 \\ 3 & \text{if } 3 \leq x \leq 5 \end{cases}$$

Now

$$\int_2^5 f\,dx + \int_2^3 1\,dx + \int_3^5 3\,dx = 7$$

and so the mean value of the function is

$$\frac{1}{5-2}\int_2^5 f\,dx = \frac{7}{3},$$

which the function fails to assume on the interval.

10.2 The Generalised First Mean Value Theorem

Theorem 19. *If f and g are integrable on $[a, b]$ and g keeps the same sign over $[a, b]$, then there exists a number μ lying between the bounds of f such that*

$$\int_a^b fg\,dx = \mu \int_a^b g\,dx$$

Let g be positive over $[a, b]$.

If m, M are the bounds of f, we have for all $x \in |a, b|$,

$$m \leq f(x) \leq M$$

$$\Rightarrow \quad mg(x) \leq f(x)\,g(x) \leq Mg(x)$$

$$\therefore \quad m\int_a^b g(x)\,dx \leq \int_a^b f(x)\,g(x)\,dx \leq M\int_a^b g(x)\,dx, \text{ when } b \geq a \qquad ...(1)$$

Let μ be a number lying between m and M.

$$\therefore \quad \int_a^b fg\,dx = \int_a^b g\,dx \qquad ...(2)$$

Corollary. If in addition to the conditions of the above theorem, f is continuous on $[a, b]$, then $\exists$ a number $\xi \in [a, b]$, such that

$$\int_a^b fg\,dx = f(\xi)\int_a^b g\,dx \qquad ...(3)$$

Notes:

1. If $g(x) \leq 0$, the sign of the inequality changes in (1) but (2) and (3) remain unchanged.
2. If $b \leq a$, the sign of inequality changes in (1) but (2) and (3) remain unaffected.

EXERCISE

1. Show that $\int_0^1 x^4\, dx = \frac{1}{5}$.

2. If f is continuous and non-negative on $[a, b]$. Then show that

$$\int_a^b f\, dx \geq 0.$$

3. Determine whether f is Riemann-integrable on $[0, 1]$ and justify your answer.

(i) $f(x) = \dfrac{1}{x+1}$

(ii) $f(x) = \left| x - \dfrac{1}{4} \right|$

(iii) $f(x) = x \cos \dfrac{1}{x}, \quad x \neq 0$
$\quad = 0, \quad x = 0$

(iv) $f(x) = \dfrac{1}{x-1}, \quad x \neq 1$
$\quad = 0, \quad x = 1$

(v) $f(x) = \sin \dfrac{1}{x}, \quad$ if x is irrational
$\quad = 0, \quad$ otherwise

[*Hint* (v): Let P be any partition of $[0, 1]$, then $L(P, f) = 0$. Suppose $\frac{2}{\pi} \in [x_{i-1}, x_i]$. If $\frac{2}{\pi} \leq x \leq 1$, then $\sin \frac{1}{x} \geq \sin 1$, and hence

$$U(P, f) \geq (\sin 1)\,(1 - x_{i-1}) \geq (\sin 1)\,(1 - 2/\pi)$$

It follows that

$$\underline{\int} f\, dx = 0 \text{ and } \overline{\int} f\, dx \geq (\sin 1)\left(1 - \frac{2}{\pi}\right) > 0.$$

Thus, f is not Riemann-integrable.

4. Prove that the function f defined on $[0, 1]$ as

$$f(x) = 2n, \text{ if } x = \frac{1}{n} \text{ where } n = 1, 2, \ldots$$
$$= 0, \text{ otherwise}$$

is not Riemann-integrable on $[0, 1]$.

(*Hint:* $\lim_{x \to 0} f(x) = \infty$, f is not bounded above and so not Riemann integrable on $[0, 1]$)

5. Prove that the function f defined as

$$f(x) = \begin{cases} x, & \text{when } x \text{ is rational} \\ -x, & \text{when } x \text{ is not rational} \end{cases}$$

is not integrable over $[a, b]$; but $|f|$ is integrable.

6. Integrate on $[0, 2]$ the function $f(x) = x[x]$, where $[x]$ denotes the greatest integer not greater than x.

7. f and g are two bounded functions on $[a, b]$ such that $f(x) = g(x)$ except for a finite number of points x in $[a, b]$. If g is integrable on $[a, b]$, then prove that f is so and in this case

$$\int_a^b f\,dx = \int_a^b g\,dx$$

[*Hint*: $F(x) = f(x) - g(x) = 0$ except at a finite number of points of $[a, b]$, so that $F(x)$ has a finite number of points of discontinuity on $[a, b]$].

8. If $f(y, x) = 1 + 2x$, for y rational and $f(y, x) = 0$, for y irrational, find $F(y)$, where

$$F(y) = \int_0^1 f(y, x)\,dx$$

Is F integrable on $[0, 1]$?

9. Evaluate

$$\overline{\int_0^2} f(x)\,dx$$

$$\text{where } f(x) = \begin{cases} 0, & \text{when } x = n/(n+1),\ (n+1)/n,\ (n = 1, 2, 3 \ldots) \\ 1, & \text{elsewhere.} \end{cases}$$

Is f integrable on $[0, 2]$? Examine for continuity the function f so defined at the point $x = 1$.

10. f is bounded and integrable on $[a, b]$, show that

$$\int_a^b [f(x)]^2\,dx = 0$$

if and only if $f(c) = 0$, at every point, c, of continuity of f.

11. If f is integrable over $[a, b]$, under what conditions is $1/f$ integrable over $[a, b]$? State and prove a theorem about the integrability of $1/f$ over $[a, b]$.

12. A function f is defined on $[0, 1]$ as follows:

$$f(x) = \frac{1}{a^{r-1}} \text{ and } f(0) = 0, \text{ where } \frac{1}{a^r} < x \le \frac{1}{a^{r-1}}, \text{ for } r = 1, 2, 3, \ldots,$$

where a is an integer greater than 2.

Show that $\displaystyle\int_0^1 f\,dx$ exists and is equal to $\dfrac{a}{a+1}$.

13. A function f is defined on $[0, 1]$, for positive integral value of r such that

$$f(x) = (-1)^{r-1}; \text{ where } 1/(r+1) < x < 1/r,\ (r = 1, 2, 3, \ldots)$$
$$f(0) = 0.$$

Show that $\displaystyle\int_0^1 f\,dx = \log 4 - 1$.

14. A function f is defined on [0, 1] by $f(x) = 2rx$, where $1/(r+1) < x < 1/r$, $(r = 1, 2, 3, \ldots)$, then show that $f \in R$ [0, 1] and its integral is $\pi^2/6$.

11. INTEGRATION BY PARTS

Theorem 20. *If f and g are integrable on* $[a, b]$, *and*

$$F(x) = A + \int_a^x f(x)\,dx,\ G(x) = B + \int_a^x g(x)\,dx$$

where A and B are constants, then

$$\int_a^b F(x)\,g(x)\,dx = \left[F(x)\,G(x)\right]_a^b - \int_a^b G(x)\,f(x)\,dx$$

[Here $\left[F(x)\,G(x)\right]_a^b$ denotes the difference $F(b)\,G(b) - F(a)\,G(a)$.]

Let $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ be a partition of $[a, b]$.

We have

$$\begin{aligned}\left[F(x)\,G(x)\right]_a^b &= \sum_{i=1}^{n} [F(x_i)\,G(x_i) - F(x_{i-1})\,G(x_{i-1})]\\ &= \Sigma\, G(x_i)[F(x_i) - F(x_{i-1})] + \Sigma\, F(x_{i-1})[G(x_i) - G(x_{i-1})]\\ &= \Sigma\, G(x_i) \int_{x_{i-1}}^{x_i} f(x)\,dx + \Sigma\, F(x_{i-1}) \int_{x_{i-1}}^{x_i} g(x)\,dx \qquad \ldots(1)\end{aligned}$$

Let $\Delta f_i = f(x_i) - f(x_{i-1})$, and $\Delta g_i = g(x_i) - g(x_{i-1})$ be the oscillation of f and g in Δx_i.

Now for all $x \in \Delta x_i$, we have

$$\begin{cases} \left|f(x) - f(x_i)\right| \le \left|f(x_i) - f(x_{i-1})\right| = \Delta f_i \\ \left|g(x) - g(x_{i-1})\right| \le \Delta g_i \end{cases}$$

$$\Rightarrow \quad \begin{cases} f(x_i) - \Delta f_i \le f(x) \le f(x_i) + \Delta f_i \\ g(x_{i-1}) - \Delta g_i \le g(x) \le g(x_{i-1}) + \Delta g_i \end{cases}$$

$$\Rightarrow \quad \begin{cases} [f(x_i) - \Delta f_i]\,\Delta x_i \le \displaystyle\int_{x_{i-1}}^{x_i} f(x)\,dx \le [f(x_i) + \Delta f_i]\,\Delta x_i \\ [g(x_{i-1}) - \Delta g_i]\,\Delta x_i \le \displaystyle\int_{x_{i-1}}^{x_i} g(x)\,dx \le [g(x_{i-1}) + \Delta g_i]\,\Delta x_i \end{cases}$$

$$\Rightarrow \begin{cases} \int_{x_i-1}^{x_i} f(x)dx = [f(x_i)+\theta_i \Delta f_i]\Delta x \\ \int_{x_i-1}^{x_i} g(x)dx = [g(x_i)+\theta_i \Delta g_i]\Delta x_i \end{cases} \quad ...(2)$$

where $-1 \le \theta_i, \theta_i' \le 1$. ...(3)

Hence, from equations (1), (2) and (3), we get

$$[F(x)G(x)]_a^b = \Sigma\, G(x_i)\, f(x_i)\, \Delta x_i + \Sigma\, F(x_{i-1})g(x_{i-1})\, \Delta x_i + \sigma \quad ...(4)$$

where

$$\sigma = \Sigma\,[G(x_i)\, \Delta f_i\, \theta_i + F(x_{i-1})\Delta g_i\, \theta_i']\, \Delta x_i$$

Since F and G, being continuous, are bounded, therefore a number k exists such that

$$|F(x)| \le k, |G(x)| \le k, \quad \forall\, x \in [a, b]$$

$$\therefore \quad |\sigma| \le k(\Sigma\, \Delta f_i + \Sigma\, \Delta g_i)\, \Delta x_i$$

In the limit when $\mu(P) \to 0$, $\sigma \to 0$, and (4) gives

$$[F(x)G(x)]_a^b = \int_a^b G(x) f(x)\, dx + \int_a^b F(x) g(x)\, dx$$

or

$$\int_a^b F(x) g(x)\, dx = [F(x)G(x)]_a^b - \int_a^b G(x) f(x)\, dx$$

Hence, the proof.

Corollary. If a function g is bounded and integrable on $[a, b]$ and if a function f is derivable and its derivative f' is bounded and integrable on $[a, b]$, then

$$\int_a^b f(x) g(x)\, dx = \left[f(x) \int_a^x g(x)\, dx \right]_a^b - \int_a^b \left\{ f'(x) \int_a^x g(x)\, dx \right\} dx$$

$$= f(b) \int_a^b g(x) dx - \int_a^b \left\{ f'(x) \int_a^x g(x)\, dx \right\} dx$$

If, however, both the derivatives f' and g' are assumed to be bounded and integrable, a much shorter and simpler proof exists, which follows.

Theorem 21. ***A particular case.*** *If f and g are both differentiable on $[a, b]$ and if f' and g' are both integrable on $[a, b]$ then*

$$\int_a^b f(x) g'(x)\, dx = [f(x)\cdot g(x)]_a^b - \int_a^b g(x) f'(x)\, dx$$

Since f and g are differentiable and hence continuous on $[a, b]$, therefore $f, g \in R[a, b]$. Again, since f, g, f' and g' are all integrable over $[a, b]$, therefore $fg', gf' \in R[a, b]$.

Let $F(x) = f(x)\, g(x)$, for all $x \in [a, b]$

$$\therefore \quad F'(x) = f(x)g'(x) + g(x)\, f'(x)$$

$$\therefore \quad \int_a^b F'(x)\, dx = \int_a^b \{f(x)g'(x) + g(x)\, f'(x)\}\, dx$$

$$= \int_a^b f(x)g'(x)\, dx + \int_a^b g(x)\, f'(x)\, dx \qquad \ldots(1)$$

Also by Fundamental Theorem (17),

$$\int_a^b F'(x)\, dx = F(b) - F(a) = [f(x)g(x)]_a^b \qquad \ldots(2)$$

From equations (1) and (2), we get

$$\int_a^b f(x)g'(x)\, dx = [f(x)g(x)]_a^b - \int_a^b g(x)f'(x)dx.$$

12. CHANGE OF VARIABLE IN AN INTEGRAL

Theorem 22. *If (i) $f \in R[a, b]$, (ii) ϕ is a derivable, strictly monotonic function on $[\alpha, \beta]$, where $a = \phi(\alpha), b = \phi(\beta)$, and (iii) $g' \in R\ [\alpha, \beta]$, then*

$$\int_a^b f(x)\, dx = \int_\alpha^\beta f(\phi(y))\phi'(y)\, dy$$

$$\left[\text{Change of variable in } \int_a^b f(x)\, dx \text{ by putting } x = \phi(y).\right]$$

Let ϕ be strictly monotonic increasing on $[\alpha, \beta]$.

Since ϕ is strictly monotonic, it is invertible, *i.e.*,

$$x = \phi(y) \qquad \Rightarrow y = \phi^{-1}(x), \ \forall\ x \in [a, b]$$

so that

$$\alpha = \phi^{-1}(a) \quad \text{and} \quad \beta = \phi^{-1}(b)$$

Let $P = \{a = x_0, x_1, x_2, \ldots, x_n = b\}$ be any partition of $[a, b]$ and

$Q = \{\alpha = y_0, y_1, y_2, \ldots, y_n = \beta\}$, $y_i = \phi^{-1}(x_i)$ the corresponding partition of $[\alpha, \beta]$.

By Lagrange's Mean Value Theorem,

$$\Delta x_i = \phi(y_i) - \phi(y_{i-1}) = \phi'(\eta_i)\,\Delta y_i, \quad \eta_i \in \Delta y_i \qquad \text{...(1)}$$

Let

$$\xi_i = \phi(\eta_i), \text{ where } \xi_i \in \Delta x_i \qquad \text{...(2)}$$

Now,

$$S(P, f) = \sum_{i=1}^{n} f(\xi_i)\,\Delta x_i = \sum_{i=1}^{n} f(\phi(\eta_i))\phi'(\eta_i)\,\Delta y_i \qquad \text{...(3)}$$

Uniform continuity of ϕ implies that $\mu(Q) \to 0$ as $\mu(P) \to 0$

Also $f \in R$ implies $\lim\limits_{\mu(P)\to 0} S(P, F)$ exists, and therefore the limit of the R.H.S. of (3) as $\mu(Q) \to 0$

also exists and equals the integral $\int_{\alpha}^{\beta} f(\phi(y))\phi'(y)\,dy$.

Hence, letting $\mu(P) \to 0$ in (3), we get

$$\int_{a}^{b} f(x)\,dx = \int_{\alpha}^{\beta} f(\phi(y))\phi'(y)\,dy.$$

Remarks:

1. If $\phi'(y) \neq 0$ for any $y \in [\alpha, \beta]$, then ϕ is strictly monotonic in $[\alpha, \beta]$. Hence, the condition of strict monotonicity of ϕ in the statement of the theorem, may be replaced by

$$\phi'(y) \neq 0, \quad \forall\, y \in [\alpha, \beta]$$

2. The theorem holds even if $\phi'(y) = 0$ for a finite number of values of y. For, in that case, the interval $[\alpha, \beta]$ can be divided into a finite number of sub-intervals, in each of which ϕ is strictly monotonic. Repetition of the argument for each of the sub-intervals in turn and the addition of the results, gives the required result.

13. SECOND MEAN VALUE THEOREM

The theorem is due to the great German mathematician, Karl Weierstrass, and the proof depends upon the *Abel's Lemma* and the *Bonnett's Theorem*. The theorem in fact is a generalisation of the Bonnett's Theorem, which, in reality, is the second mean value theorem under slightly more restricted conditions.

Abel's Lemma. *If $\{b_n\}$ is a positive monotone decreasing sequence and k, K denote respectively the least and the greatest values of the sums $\sum_{r=m}^{p} u_r$, for $p = m, m+1, \ldots, n$, then*

$$b_m k \leq \sum_{r=m}^{n} b_r u_r \leq b_m K$$

Let $S_p = \sum_{r=m}^{p} u_r$, then

$$\sum_{r=m}^{n} b_r u_r = b_m u_m + b_{m+1} u_{m+1} + \ldots + b_n u_n$$

$$= b_m S_m + b_{m+1}(S_{m+1} - S_m) + \ldots + b_n (S_n - S_{n-1})$$
$$= (b_m - b_{m+1})\, S_m + (b_{m+1} - b_{m+2})\, S_{m+1} + \ldots,$$
$$+ (b_{n-1} - b_n)\, S_{n-1} + b_n S_n$$

All brackets on the right are non-negative.

$\therefore$
$$k(b_m - b_{m+1} + b_{m+1} - b_{m+2} + \ldots + b_{n-1} - b_n + b_n)$$
$$\le \sum_{r=m}^{n} b_r u_r \le K(b_m - b_{m+1} + \ldots - b_n + b_n)$$

$\Rightarrow$
$$b_m k \le \sum_{r=m}^{n} b_r u_r \le b_m K$$

In particular ($m = 1$), the lemma may be stated in the form:

If $b_1, b_2, \ldots, b_n$ is a positive monotone decreasing set and k, K denote respectively the least and the greatest values of the partial sums, $\sum_{r=1}^{p} u_r, 1 \le p \le n$ of the numbers, $u_1, u_2, \ldots, u_n$, then

$$b_1 k \le \sum_{r=1}^{n} b_r u_r \le b_1 K.$$

Theorem 23. ***Second mean value theorem.*** *If $\int_a^b f\, dx$ and $\int_a^b g\, dx$ both exist and f is monotone on [a, b], then there exists $\xi \in [a, b]$ such that*

$$\int_a^b fg\, dx = f(a) \int_a^{\xi} g\, dx + f(b) \int_{\xi}^{b} g\, dx$$

We first prove the *Bonnett's form* of the theorem, where in addition to the hypothesis of the theorem, the monotone function is *positive* and *monotone decreasing* on $[a, b]$.

If $\int_a^b \phi\, dx$ and $\int_a^b g\, dx$ both exist and ϕ is positive and monotone decreasing on [a, b], then there exists a point $\xi \in [a, b]$ such that

$$\int_a^b \phi g\, dx = \phi(a) \int_a^{\xi} g\, dx$$

Let $P = \{a = x_0, x_1, \ldots, x_n = b\}$ be a partition of $[a, b]$, and M_i, m_i the bounds of g in Δx_i.
Let $t_1 = a$ and t_i $(i \ne 1)$ be any point of Δx_i.
In the sub-interval Δx_i, we know

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$$m_i \, \Delta x_i \le \int_{x_{i-1}}^{x_i} g \, dx \le M_i \, \Delta x_i$$

and

$$m_i \, \Delta x_i \le g(t_i) \, \Delta x_i \le M_i \, \Delta x_i$$

Letting $i = 1, 2, 3, \ldots, p, \, (p \le n)$ and adding vertically, we get

$$\sum_{i=1}^{p} m_i \, \Delta x_i \le \int_a^{xp} g \, dx \le \sum_{i=1}^{p} M_i \, \Delta x_i$$

and

$$\sum_{i=1}^{p} m_i \, \Delta x_i \le \sum_{i=1}^{p} g(t_i) \, \Delta x_i \le \sum_{i=1}^{p} M_i \, \Delta x_i$$

which gives

$$\left| \int_a^{xp} g \, dx - \sum_{i=1}^{p} g(t_i) \, \Delta x_i \right| \le \sum_{i=1}^{p} (M_i - m_i) \, \Delta x_i \le \omega(P, g)$$

$$\Rightarrow \qquad \int_a^{xp} g \, dx - \omega(P,g) \le \sum_{i=1}^{p} g(t_i) \, \Delta x_i \le \int_a^{xp} g \, dx + \omega(P,g)$$

where $\omega(P,g)$ denotes the oscillatory sum, $U(P,g) - L(P,g)$.

Now, $\int_a^t g \, dx$, being a continuous function (§ 8 Th. 16), is bounded. Let A, B be its bounds, so that we have

$$B - \omega(P,g) \le \sum_{i=1}^{p} g(t_i) \, \Delta x_i \le A + \omega(P,g) \qquad \ldots(1)$$

Using the Abel's lemma, where

$$b_i = \phi(t_i), \qquad u_i = g(t_i) \, \Delta x_i$$
$$k = B - \omega(P,g), \qquad K = A + \omega(P,g)$$

we get

$$\phi(a)[B - \omega(P,g)] \le \sum_{i=1}^{n} \phi(t_i) g(t_i) \, \Delta x_i \le \phi(a)[A + \omega(P,g)]$$

Taking the limit when $\mu(P) \to 0$, we get

$$B\phi(a) \le \int_a^b \phi g \, dx \le A\phi(a)$$

$$\Rightarrow \qquad \int_a^b \phi g \, dx = \mu \phi(a) \qquad \ldots(2)$$

where μ is some number between B and A.

The function $\int_a^b g(x)\,dx$ being continuous, must assume, for some $\xi \in [a, b]$, the value μ which lies between its bounds. Thus, we get

$$\int_a^b \phi g\,dx = \phi(a)\int_a^\xi g\,dx \quad \text{...(3)}$$

Let us now prove the theorem proper, due to Weierstrass.

Let, first, f be monotone *decreasing*, so that the function ϕ where $\phi = f - f(b)$ is positive and monotone decreasing. Therefore, by what has been proved above, there exists a number ξ between a and b such that

$$\int_a^b g[f - f(b)]\,dx = [f(a) - f(b)]\int_a^\xi g\,dx$$

$$\int_a^b fg\,dx = f(a)\int_a^\xi g\,dx + f(b)\left[\int_a^b g\,dx - \int_a^\xi g\,dx\right]$$

$$= (fa)\int_a^\xi g\,dx + f(b)\int_\xi^b g\,dx \quad \text{...(4)}$$

Let now f be monotone *increasing*, so that $(-f)$ is monotone decreasing and therefore from (4),

$$\int_a^b (-f)g\,dx = -f(a)\int_a^\xi g\,dx - f(b)\int_\xi^b g\,dx$$

$$\Rightarrow \quad \int_a^b fg\,dx = f(a)\int_a^\xi g\,dx + f(b)\int_\xi^b g\,dx$$

Hence, the theorem.

Note: It may be easily verified that the theorem holds for $a > b$ also.

13.1 Second Mean Value Theorem (*A particular case*)

Theorem 24. *If f is monotonic and f, f' and g are all continuous in $[a, b]$, then there exists $\xi \in [a, b]$ such that*

$$\int_a^b f(x)g(x)\,dx = f(a)\int_a^\xi g(x)\,dx + f(b)\int_\xi^b g(x)\,dx$$

Let $G(x) = \int_a^x g(t)\,dt$

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Clearly $G(a) = 0$, and under the given conditions, $G(x)$ is differentiable and $G'(x) = g(x)$.

$$\therefore \qquad \int_a^b f(x)g(x)\,dx = \int_a^b f(x)G'(x)\,dx$$

$$= \left[f(x)G(x)\right]_a^b - \int_a^b G(x)f'(x)\,dx$$

on integrating by parts.

Since G, being continuous, is integrable and f is monotone and continuous on $[a, b]$, therefore on using generalised First Mean Value Theorem, $\exists\ \xi \in [a, b]$ such that

$$\int_a^b f(x)g(x)\,dx = f(b)G(b) - G(\xi)\int_a^b f'(x)\,dx$$

$$= f(b)G(b) - G(\xi)\{f(b) - f(a)\}$$

$$= f(b)\{G(b) - G(\xi)\} + f(a)\,G(\xi)$$

$$= f(b)\int_a^b g(x)\,dx + f(a)\int_a^\xi g(x)\,dx$$

Example 15. If a function f is continuous on $[0, 1]$, show that

$$\lim_{x\to\infty}\int_0^1 \frac{nf(x)}{1+n^2x^2}\,dx = \frac{\pi}{2}f(0)$$

■ Let us put

$$\int_0^1 \frac{nf(x)}{1+n^2x^2}\,dx = \int_0^{1/\sqrt{n}} \frac{nf(x)}{1+n^2x^2}\,dx + \int_{1/\sqrt{n}}^1 \frac{nf(x)}{1+n^2x^2}\,dx$$

By generalised first mean value theorem,

$$\int_0^{1/\sqrt{n}} \frac{nf(x)}{1+n^2x^2}\,dx = f(\xi)\int_0^{1/\sqrt{n}} \frac{n}{1+n^2x^2}\,dx, \text{ where } 0 \le \xi \le \frac{1}{\sqrt{n}}$$

$$= f(\xi)\tan^{-1}\sqrt{n} \to \frac{\pi}{2}f(0) \quad \text{as } n \to \infty$$

Again, since f is continuous on $[0, 1]$, it is bounded and therefore there exists K such that

$$|f(x)| \le K, \quad \forall\ x \in [0, 1]$$

$$\left|\int_{1/\sqrt{n}}^1 \frac{nf(x)}{1+n^2x^2}\,dx\right| = K\left|\int_{1/\sqrt{n}}^1 \frac{n}{1+n^2x^2}\,dx\right|$$

$$= K \mid \tan^{-1} n - \tan^{-1} \sqrt{n}$$
$$\to 0 \text{ as } n \to \infty$$

Hence, the result.

Example 16. ***Riemann-Lebesgue Lemma.*** If a function f is bounded and integrable on $[a, b]$, show that

$$\lim_{n\to\infty} \int_a^b f(x) \cos nx \, dx = 0 \text{ and } \lim_{n\to\infty} \int_a^b f(x) \sin nx \, dx = 0.$$

▪ Let $I_n = \int_a^b f(x) \cos nx \, dx.$

Let, further, $\varepsilon > 0$ be an arbitrary number.

Since f is bounded and integrable on $[a, b]$, there exists a partition $P = \{a = x_0, x_1, x_2, \ldots, x_p = b\}$ such that the oscillatory sum

$$U(P,f) - L(P,f) = \sum_{i=1}^{p} (M_i - m_i) \Delta x_i < \varepsilon/2$$

where M_i, m_i are bounds of f in Δx_i.

Now

$$I_n = \sum_{i=1}^{p} \int_{x_{i-1}}^{x_i} f(x) \cos nx \, dx$$

$$= \sum_{i=1}^{p} f(x_{i-1}) \int_{x_{i-1}}^{x_i} \cos nx \, dx + \sum_{i=1}^{p} \int_{x_{i-1}}^{x_i} \{f(x) - f(x_{i-1})\} \cos nx \, dx$$

$\therefore$
$$|I_n| \le \sum_{i=1}^{p} |f(x_{i-1})| \left| \int_{x_{i-1}}^{x_i} \cos nx \, dx \right|$$
$$+ \sum_{i=1}^{p} \int_{x_{i-1}}^{x_i} \left| \{f(x) - f(x_{i-1})\} \cos nx \right| dx$$

But for all $x \in \Delta x_i$, we have

$$\left| f(x) - f(x_{i-1}) \right| \le M_i - m_i \; (i = 1, 2, \ldots, p)$$

so that
$$\left| \{f(x) - f(x_{i-1})\} \cos nx \right| \le M_i - m_i$$

and
$$\sum_{i=1}^{p} \int_{x_{i-1}}^{x_i} \left| \{f(x) - f(x_{i-1})\} \cos nx \right| dx \le \sum_{i=1}^{p} (M_i - m_i)(x_i - x_{i-1})$$

$$< \frac{1}{2}\varepsilon$$

Also
$$\left|\int_{x_{i-1}}^{x_i} \cos nx\, dx\right| = \left|\frac{1}{n}\{\sin nx_i - \sin nx_{i-1}\}\right|$$

$$\leq \frac{1}{n}\left\{\left|\sin nx_i\right| + \left|\sin nx_{i-1}\right|\right\} \leq \frac{2}{n}$$

$\therefore$
$$\left|I_n\right| \leq \frac{2}{n}\sum_{i=1}^{P}\left|f(x_{i-1})\right| + \frac{1}{2}\varepsilon$$

For a fixed $P, f(x_{i-1})$ is a fixed quantity. Also there exists a positive number m such that for all $n \geq m$,

$$\frac{2}{n}\sum_{i=1}^{P}\left|f(x_{i-1})\right| < \frac{1}{2}\varepsilon$$

$\therefore$
$$\left|I_n\right| < \varepsilon, \quad \forall\, n \geq m$$

$\Rightarrow$
$$\lim_{n\to\infty}\int_a^b f(x)\cos nx\, dx = 0$$

It may similarly be shown that

$$\lim_{n\to\infty}\int_a^b f(x)\sin nx\, dx = 0.$$

Notes:

1. For another proof see Fourier series.

2. In particular, if $f(x)$ is bounded and integrable in $\left[0, \frac{1}{2}\pi\right]$, then

$$\lim_{n\to\infty}\int_0^{\pi/2} f(x)\sin nx\, dx = 0.$$

If we assign the value 0 at the origin to $\left(\frac{1}{x} - \frac{1}{\sin x}\right)$, it becomes continuous, bounded and integrable in $\left[0, \frac{1}{2}\pi\right]$, so that

$$\lim_{n\to\infty}\int_0^{\pi/2} f(x)\left(\frac{1}{x} - \frac{1}{\sin x}\right)\sin nx\, dx = 0$$

$\Rightarrow$
$$\lim_{n\to\infty}\int_0^{\pi/2} f(x)\frac{\sin nx}{x}\, dx = \lim_{n\to\infty}\int_0^{\pi/2} f(x)\frac{\sin nx}{\sin x}\, dx.$$

Example 17. Show that lim I_n, where

$$I_n = \int_0^{\delta} \frac{\sin nx}{x} dx, n \in N \text{ exists and equals } \pi/2$$

- Since $\lim_{x\to 0} \frac{\sin nx}{x} = n$, the integrand becomes continuous for every value of x, if we assign to it the value n at $x = 0$.

I. Let us first prove that the integral exists (is convergent).

Let $I_n = \int_0^{\delta} \frac{\sin nx}{x} dx$

Putting $nx = t$, we get

$$I_n = \int_0^{n\delta} \frac{\sin t}{t} dt$$

$$\therefore \quad |I_{n+p} - I_n| = \left| \int_{n\delta}^{(n+p)\delta} \frac{\sin t}{t} dt \right|, p \geq 1$$

$$\leq \int_{n\delta}^{(n+p)\delta} \frac{|\sin t|}{t} dt$$

Since $|\sin t|$ keeps the same sign (positive) and $1/t$ is positive and monotonic decreasing in $[n\delta, (n+p)\delta]$, using Bonnett's form of the second mean value theorem, we get

$$|I_{n+p} - I_n| \leq \frac{1}{n\delta} \int_{n\delta}^{(n+p)\delta} |\sin t| dt$$

$$\leq \frac{2}{n\delta} \leq \varepsilon, \forall n > \frac{2}{\varepsilon\delta}$$

Hence by Cauchy's principle of convergence, $\{I_n\}$ converges.

II. It will now be shown that

$$\lim_{n\to\infty} I_n = \lim_{n\to\infty} \int_0^{\pi/2} \frac{\sin nx}{\sin x} dx$$

Let us write

$$\int_0^{\pi/2} \frac{\sin nx}{x} dx = \int_0^{\delta} \frac{\sin nx}{x} dx + \int_{\delta}^{\pi/2} \frac{\sin nx}{x} dx$$

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Using the preceding example,

$$\int_{\delta}^{\pi/2} \frac{\sin nx}{x}\,dx \to 0 \text{ as } n \to \infty$$

$$\Rightarrow \qquad \lim_{n\to\infty} I_n = \lim_{n\to\infty} \int_0^{d} \frac{\sin nx}{x}\,dx = \lim_{n\to\infty} \int_0^{\pi/2} \frac{\sin nx}{x}\,dx$$

The function f, where

$$f(x) = \begin{cases} \dfrac{1}{x} - \dfrac{1}{\sin x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

is continuous and therefore bounded and integrable in $[0, \pi/2]$, so that using the preceding example, we get

$$\lim_{n\to\infty} \int_0^{\pi/2} \left(\frac{1}{x} - \frac{1}{\sin x}\right) \sin nx\,dx = 0$$

$$\Rightarrow \qquad \lim_{n\to\infty} I_n = \lim_{n\to\infty} \int_0^{\pi/2} \frac{\sin nx}{x}\,dx = \lim_{n\to\infty} \int_0^{\pi/2} \frac{\sin nx}{\sin x}\,dx.$$

III. Let us now proceed to evaluate $\lim I_n$ and this we do by letting $n \to \infty$ through odd integral values.

We know that

$$\frac{\sin(2n+1)}{\sin x} = 2\left\{\frac{1}{2} + \cos 2x + \cos 4x + \ldots + \cos 2nx\right\}$$

Integrating, we get

$$\int_0^{\pi/2} \frac{\sin(2n+1)}{\sin x}\,dx = \frac{\pi}{2},\ \forall n \in \mathbf{N}$$

$$\lim I_n = \lim \int_0^{\pi/2} \frac{\sin x}{x}\,dx = \lim \int_0^{\pi/2} \frac{\sin nx}{x}\,dx$$

$$= \lim \int_0^{\pi/2} \frac{\sin(2x+1)}{\sin x}\,dx = \frac{\pi}{2}$$

Note: It will be defined later, in improper integrals, that

$$\int_0^{\infty} f\,dx = \lim_{x\to\infty} \int_0^x f\,dx$$

$$\int_0^{x} \frac{\sin x}{x}\,dx = \frac{x}{2}$$

EXERCISE

1. If f is continuous and positive on $[a, b]$, then show that $\int_a^b f\,dx$ is also positive.
2. Can the number ξ of the first mean value theorem always belong to $]a, b[$? Find the function f on some closed interval which satisfies the conditions of this theorem but for which ξ must be an end point of the interval.
3. How far can the Lagrange's mean value theorem for derivatives be used to make the statement : If f is the derivative of some function on $[a, b]$, then there exists a number $\xi \in [a,b]$ such that

$$\int_a^b f\,dx = f(\xi)\,(b-a)$$

[*Hint:* In Lagrange's mean value theorem, $\xi \in]a, b[$.]

4. If f is continuous on $[a, b]$ and $F(x) = \int_a^x f(t)\,dt$, for all $x \in [a, b]$, apply (if possible) the Lagrange's mean value theorem for derivatives to F over $[a, b]$. State the resulting theorem for F. Also state the first mean value theorem for integrals to $\int_a^b f\,dx$, and compare the two statements.

Also try to relate Cauchy mean value theorem with the generalised first mean value theorem for integrals.

5. If f and g are integrable and

$$\int_a^b f\,dx = \int_a^b g\,dx$$

then show that

$$f(\xi) = g(\xi), \text{ for some } \xi \in [a, b].$$

6. In the second mean value theorem show that f must be monotonic, by proving that the theorem does not hold in $\left[-\frac{1}{2}\pi, \frac{1}{2}\pi\right]$, if $f(x) = \cos x$, $g(x) = x^2$.

$\left[\text{Note that } \int_{-\pi/2}^{\pi/2} x^2 \cos x\,dx > 0.\right]$

7. Show that the second mean value theorem does not hold good in $[-1, 1]$, for $f(x) = g(x) = x^2$. Also test the validity of the first (generalised) mean value theorem.

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8. Verify the two mean value theorems in $[-1, 1]$ for the functions $f(x) = e^x$, $g(x) = x$.

9. If $f \in R[a,b]$ and $F(x) = \int_a^x f(t)\,dt$, for all $x \in [a, b]$, then show that F is of bounded variation on $[a, b]$.

10. If $f, g \in R[a,b]$, then prove the following:

(i) $\left|\int_a^b fg\right| \le \left(\int_a^b f^2\right)^{1/2} \left(\int_a^b g^2\right)^{1/2}$, and

(ii) $\left(\int_a^b (f+g)^2\right)^{1/2} \le \left(\int_a^b f^2\right)^{1/2} + \left(\int_a^b g^2\right)^{1/2}$.