Main Test Series -2018
Test-08 - Paper-II
Answer Key

1(a) Show that the set $G = \{f_1, f_2, f_3, f_4, f_5, f_6\}$ of six transformations on the set of complex numbers defined by $f_1(z) = z$, $f_2(z) = 1-z$, $f_3(z) = \frac{z}{z-1}$, $f_4(z) = \frac{1}{z}$, $f_5(z) = \frac{1}{1-z}$ and $f_6(z) = \frac{z-1}{z}$ is a non-abelian group of order 6. with respect to composition of mappings.

Solⁿ: Let $G = \{f_1, f_2, f_3, f_4, f_5, f_6\}$

Suppose we denote multiplicatively the composition known as the composite or product of two functions.

If $f: A \to B$ and $g: B \to C$ then by definition $(gf): A \to C$ such that $(gf)(x) = g[f(x)]$ $\forall x \in A$.

The function gf is called the composite of the functions g and f.

we prepare the composition table as follows.

Since the function f_1 is the identity function,

therefore

$f_1f_1 = f_1$, $f_1f_2 = f_2 = f_2f_1$, $f_1f_3 = f_3 = f_3f_1$, $f_1f_6 = f_6 = f_6f_1$.

$f_4f_1 = f_4 = f_1f_4$, $f_1f_5 = f_5 = f_5f_1$,

Now, $f_2f_2(z) = f_2(1-z) = 1-(1-z) = z = f_1(z)$.

$\therefore f_2f_2 = f_1$

$(f_2f_3)(z) = f_2\left(f_3(z)\right) = f_2\left(\frac{z}{z-1}\right) = 1-\frac{z}{z-1} = \frac{-1}{z-1} = \frac{1}{1-z}$
$= f_5(z)$

$\therefore f_2f_3 = f_5$

$(f_2f_4)(z) = f_2\left(f_4(z)\right) = f_2\left(\frac{1}{z}\right) = 1-\frac{1}{z} = \frac{z-1}{z} = f_6$

$\therefore f_2f_4 = f_6$

$(f_2f_5)(z) = f_2\left(f_5(z)\right) = f_2\left(\frac{1}{1-z}\right) = 1-\frac{1}{1-z} = \frac{-z}{1-z} = \frac{z}{z-1}$
$= f_3(z)$

$\therefore f_2f_5 = f_3$

$(f_2f_6)(z) = f_2\left(f_6(z)\right) = f_2\left(\frac{z-1}{z}\right) = 1-\frac{z-1}{z} = \frac{1}{z} = f_4$

$\therefore f_2f_6 = f_4$

Similarly calculating the other products we get the composition table as given below

composition of two function	f_1	f_2	f_3	f_4	f_5	f_6
f_1	f_1	f_2	f_2	f_4	f_5	f_6
f_2	f_2	f_1	f_5	f_6	f_3	f_4
f_3	f_3	f_6	f_1	f_5	f_4	f_2
f_4	f_4	f_5	f_6	f_1	f_2	f_3
f_5	f_5	f_4	f_2	f_3	f_6	f_1
f_6	f_6	f_3	f_4	f_2	f_1	f_5

we make the following observations:

(i) All the entries in the composition table are elements of the set G, therefore G is closed w.r.t to the given composition.

(ii) we know that the composite of functions is an associative composition.

i.e, If $f: A \to B$, $g: B \to C$, $h: C \to D$

then $h(gf) = (hg)f$.

(iii) The identity function f_1 is the identity element.

(iv) Each function possesses inverse.

Thus $f_1^{-1} = f_1$, $f_2^{-1} = f_2$, $f_3^{-1} = f_3$, $f_4^{-1} = f_4$,

$f_5^{-1} = f_6$, $f_6^{-1} = f_5$.

(v) The composition is not commutative

Since $f_2 f_3 = f_5$ and $f_3 f_2 = f_6$.

Thus $f_2 f_3 \neq f_3 f_2$.

The set G contains 6 elements.

Hence G is a finite non-abelian group of order 6 with respect to the composite composition.

Note: Here we see in the composition table that the entries in the second row donot coincide with the corresponding entries in the second column. Thus $f_2 f_3 \neq f_3 f_2$.

Therefore the composition is not commutative.

1(b) → Prove that a group of order 30 can have atmost 7 Subgroups of order 5.

Sol^n: Let $(G, *)$ be a group of order 30, then

$$o(G) = 30.$$

To Prove that G has atmost 7 Subgroups of order '5'.
If possible suppose that G has 8 subgroups of order '5' say

$H_1, H_2, H_3 - - - H_8$ such that $o(H_1) = o(H_2) = \cdots = o(H_8) = 5$

and $H_1, H_2, \cdots H_8$ are different. —— (1)

let $H_1 \cap H_2 \cap H_3 \cap \cdots \cap H_8 = H$

then $H \le G$ and $H \le H_1, H \le H_2, \cdots H \le H_8$.

let $H \le H_1$

then by Lagrange's theorem.

$$\frac{o(H_1)}{o(H_2)} = \frac{5}{o(H)}$$

$$\Rightarrow o(H) = 1 \text{ or } 5$$

if $o(H) = 1$ then $H = \{e\}$ ——

if $o(H) = 5$ then $\boxed{H = H_1}$ —— (2)

$= o(H_1)$

Case(a): If $H = H_1$ then $H_1 \cap H_2 = H \Rightarrow H_2 \subseteq H_1$

but $o(H_2) = o(H_1)$

$\Rightarrow H_2 = H_1$

which is contradiction to the fact H_1 & H_2 are different.

Case(b): If $H = \{e\}$

then $H_1 \cap H_2 \cap H_3 \cap \cdots \cap H_8 = \{e\}$

$\therefore H_1 \cup H_2 \cup H_3 \cup \cdots H_8$ has $1+(4\times 8)$
$= 33$ elements.

which is a contradiction.

$\therefore$ our assumption that G has 8 subgroups of order 5 is wrong.

$\therefore$ G has almost 7 subgroups of order 5.

I(C) Discuss the convergence of the series.

$$1+\frac{3}{7}x+\frac{3.6}{7.10}x^2+\frac{3.6.9}{7.10.13}x^3+\cdots, x>0.$$

Sol'n: Leaving the first term, we have

$$u_n = \frac{3.6.9\ldots(3n)}{7.10.13\ldots(3n+4)}\cdot x^n$$

$$\Rightarrow u_{n+1} = \frac{3.6.9\ldots(3n)(3n+3)}{7.10.13\ldots(3n+4)(3n+7)}\cdot x^{n+1}$$

$$\therefore u_n/u_{n+1} = \frac{3n+7}{3n+3}\cdot\frac{1}{x}$$

$$= \frac{1+7/3n}{1+1/n}\cdot\frac{1}{x}$$

$$\therefore \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \frac{1}{x}$$

$\therefore$ By Ratio test,

$\sum u_n$ Converges if $\frac{1}{x} > 1$ i.e. $x < 1$

and $\sum u_n$ diverges if $\frac{1}{x} < 1$ i.e. $x > 1$

If $x = 1$ then the ratio test fails.

when $x = 1$, $\frac{u_n}{u_{n+1}} = \frac{3n+7}{3n+3}$

Now we apply Gauss Test,

$$\therefore \frac{u_n}{u_{n+1}} = \frac{3n+7}{3n+3} = \frac{3n\left(1+\frac{7}{3n}\right)}{3n\left(1+\frac{1}{n}\right)}$$

$$= \left(1+\frac{7}{3n}\right)\left(1+\frac{1}{n}\right)^{-1}$$

$$= \left(1+\frac{7}{3n}\right)\left(1-\frac{1}{n}+\frac{1}{n^2}-\frac{1}{n^3}+\cdots\right)$$

$$= \left(1-\frac{1}{n}+\frac{1}{n^2}\cdots\right)+\left(\frac{7}{3n}-\frac{7}{3n^2}+\cdots\right)$$

$$= 1+\frac{4}{3n}-\frac{4}{3n^2}+\cdots$$

$$= 1+\left(\frac{4}{3}\right)\left(\frac{1}{n}\right)+o\left(\frac{1}{n^2}\right)$$

Comparing it with

$$\frac{u_n}{u_{n+1}} = 1+\frac{\lambda}{n}+o\left(\frac{1}{n^2}\right)$$

where $\lambda = 4/3 > 1$

$\therefore$ By Gauss Test,

$\sum u_n$ is convergent.

$\therefore$ The given series converges if $x \leq 1$ and diverges if $x > 1$.

1(d) Show that the function $u = \sin x \cosh y + 2\cos x \sinh y + x^2 - y^2 + 4xy$ satisfies Laplace's equation and determine the corresponding analytic function $f(z) = u + iv$.

Sol^n: Here $u = \sin x \cosh y + 2\cos x \sinh y + x^2 - y^2 + 4xy$

$$\therefore \frac{\partial u}{\partial x} = \cos x \cosh y - 2\sin x \sinh y + 2x + 4y$$
$$= \phi_1(x, y) \quad (\text{say})$$

$$\frac{\partial u}{\partial y} = \sin x \sinh y + 2\cos y \cosh y - 2y + 4x$$
$$= \phi_2(x, y) \quad (\text{say})$$

$$\frac{\partial^2 u}{\partial x^2} = -\sin x \cosh y - 2\cos x \sinh y + 2 \quad \text{——} \ (1)$$

$$\frac{\partial^2 u}{\partial y^2} = \sin x \cosh y + 2\cos x \sinh y - 2 \quad \text{——} \ (2)$$

Adding (1) and (2) we see that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

i.e. u satisfies Laplace's equation. Hence u is a harmonic function.

By Milne's method we have

$$f'(z) = \phi_1(z, 0) - i\phi_2(z, 0)$$
$$= (\cos z + 2z) - i(2\cos z + 4z)$$

Integrating, we get

$$f(z) = \int (1-2i)(\cos z + 2z)\,dz + c$$
$$\Rightarrow f(z) = (1-2i)(\sin z + z^2) + c$$

1(e) If $x_1=2,\ x_2=3,\ x_3=1$ is a feasible solution of the LPP

Maximize $Z = x_1+2x_2+4x_3$

subject to $2x_1+x_2+4x_3=11$

$3x_1+x_2+5x_3=14$

$x_1, x_2, x_3 \geq 0.$

find a basic feasible solution of the problem.

Sol^n: The given system of equations may be put in matrix notations as $\begin{pmatrix} 2 & 1 & 4 \\ 3 & 1 & 5 \end{pmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{pmatrix} 11 \\ 14 \end{pmatrix}$

$\Rightarrow AX=B$

where $\begin{pmatrix} 2 & 1 & 4 \\ 3 & 1 & 5 \end{pmatrix}$; $B=\begin{pmatrix} 11 \\ 14 \end{pmatrix}$; $X=\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

Let the columns of A be denoted by

$A_1=\begin{pmatrix} 2 \\ 3 \end{pmatrix},\ A_2=\begin{pmatrix} 1 \\ 1 \end{pmatrix},\ A_3=\begin{pmatrix} 4 \\ 5 \end{pmatrix}$

Here $\rho(A)=2$.

$\therefore$ A basic solution to the given system of equations exist with not more than two variables different from zero.

Also, the column vectors A_1, A_2, A_3 are linear dependent (we can easily verify)

$A_1\lambda_1+A_2\lambda_2+A_3\lambda_3=0.$

$\Rightarrow 2\lambda_1+\lambda_2+4\lambda_3=0$

$3\lambda_1+\lambda_2+5\lambda_3=0$

Clearly this is a system of two equations in three unknowns $\lambda_1, \lambda_2, \lambda_3$.

Let us choose one of the λ_s arbitrarily say $\lambda_1 = 1$.

$\therefore \lambda_2 + 4\lambda_3 = -2$

$\lambda_2 + 5\lambda_3 = -3$

Solving, we get $\lambda_2 = 2,\ \lambda_3 = -1$

To reduce the no. of +ve variables, the variable to be driven to zero is found by choosing r for which

$$\frac{x_r}{\lambda_r} = \text{Min}\left\{ \frac{x_i}{\lambda_i} \Big/ \lambda_i > 0 \right\}$$

$$= \text{Min}\left\{ \frac{x_1}{\lambda_1}, \frac{x_2}{\lambda_2}, \frac{x_3}{\lambda_3} \right\} = \text{Min}\left\{ \frac{2}{1}, \frac{3}{2}, \frac{1}{-1} \right\} = \frac{3}{2}$$

Thus, we can remove vector A_2 for which $\frac{x_2}{\lambda_2} = \frac{3}{2}$ and obtain new solution with not more than two non-negative (non-zero) variables

The variables of new are given by

$\hat{x}_1 = x_1 - \frac{3}{2}(1) = 2 - \frac{3}{2} = \frac{1}{2}$

$\hat{x}_2 = x_2 - \frac{3}{2}(2) = 3 - 3 = 0$

$\hat{x}_3 = x_3 - \frac{3}{2}(-1) = 1 + \frac{3}{2} = \frac{5}{2}$.

$\therefore$ The basic feasible solution is

$x_1 = \frac{1}{2},\ x_3 = \frac{5}{2}$ with $x_2 = 0$ (non-basic)

2(b)(i) Examine the convergence of the integral

$$\int_1^2 \frac{dx}{(1+x)\sqrt{2-x}}$$

Sol^n: Here $f(x) = \dfrac{1}{(1+x)\sqrt{2-x}}$

2 is the only point of infinite discontinuity of f on $[1,2]$.

Take $g(x) = \dfrac{1}{\sqrt{2-x}}$, then

$$\lim_{x\to 2-} \frac{f(x)}{g(x)} = \lim_{x\to 2-} \frac{1}{1+x} = \frac{1}{3}$$

which is non-zero and finite.

$\therefore$ By Comparison test,

$\int_1^2 f(x)\,dx$ and $\int_1^2 g(x)\,dx$ converge (or) diverge together.

But $\int_1^2 g(x)\,dx = \int_1^2 \dfrac{dx}{\sqrt{2-x}}$ converges ($\because n = \frac{1}{2} < 1$)

$\therefore \int_1^2 f(x)\,dx = \int_1^2 \dfrac{dx}{(1+x)\sqrt{2-x}}$ is convergent.

2(b)(iii) Prove that $\prod_{n=1}^{\infty}\left(1-\frac{1}{n^{4/3}}\right)e^{\frac{1}{n^{4/3}}}$ is absolutely convergent

Soln: Here $1+a_n = \left(1-\frac{1}{n^{2/3}}\right)e^{\frac{1}{n^{2/3}}}$

$$= \left(1-\frac{1}{n^{2/3}}\right)\left(1+\frac{1}{n^{2/3}}+\frac{1}{2!\,n^{4/3}}+\frac{1}{3!\,n^{6/3}}+\cdots\right)$$

$$= 1-\frac{1}{n^{4/3}}+\frac{1}{2n^{4/3}}+\frac{1}{6n^{6/3}}-\frac{1}{2n^{6/3}}+\cdots$$

$$\Rightarrow a_n = -\frac{1}{2n^{4/3}}-\frac{1}{3n^{4/3}}-\cdots$$

$$= \frac{1}{n^{4/3}}\left(-\frac{1}{2}-\frac{1}{3n^{2/3}}+\cdots\right)$$

$$\Rightarrow |a_n| = \frac{1}{n^{4/3}}\left|-\frac{1}{2}-\frac{1}{3n^{2/3}}\cdots\right|$$

Comparing $\sum|a_n|$ with $\sum\frac{1}{n^{4/3}}$

we have $\lim_{n\to\infty}\frac{|a_n|}{\frac{1}{n^{4/3}}} = \frac{1}{2}$, a finite quantity.

But $\sum\frac{1}{n^{4/3}}$ is convergent.

$\therefore \sum_n |a_n|$ is convergent

$\Rightarrow \sum a_n$ is absolutely convergent

$\Rightarrow \prod_{n=1}^{\infty}(1+a_n) = \prod_{n=1}^{\infty}\left(1-\frac{1}{n^{2/3}}\right)e^{\frac{1}{n^{2/3}}}$ is

absolutely convergent.

2(c) Use the method of Contour integration to Prove that

$$\int_0^\infty \frac{\cos mx}{x^4+x^2+1}\,dx = \frac{\pi}{6} e^{-m\sqrt{3}/2}\left[\sqrt{3}\cos\frac{m}{2} + 3\sin\frac{m}{2}\right] \text{ (or)}$$

$$= \frac{\pi}{\sqrt{3}} e^{-m\sqrt{3}/2} \sin\left(\frac{m}{2}+\frac{1}{6}\pi\right).$$

Sol^n. Consider the integral

$$\int_C f(z) \text{ , where } f(z) = \frac{e^{imz}}{z^4+z^2+1}$$

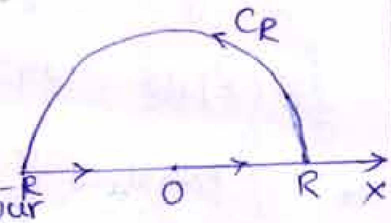
taken round a closed Contour C Consisting of the upper half of a large circle $|z|=R$ and the real axis from $-R$ to R.

poles of $f(z)$ are given by $z^4+z^2+1=0$

i.e. by $(z^2-1)(z^4+z^2+1)=0$

$$z^6-1=0 \Rightarrow z=(1)^{1/6}$$

$$\Rightarrow z=\{e^{2\pi n i}\}^{1/6} = e^{n\pi i/3}$$



The only poles which are within the Contour are $z=e^{i\pi/3}$ and $z=e^{2\pi/3}$

The other values of z either correspond to the factor (z^2-1), (or) lie outside the Semi-circle.

Let one of them $e^{\pi i/3}=\alpha$ (say), then another $e^{2i\pi/3}=\alpha^2$.

Residue of $f(z)$ at $z=\alpha$ is $\left[\dfrac{e^{imz}}{\frac{d}{dz}\left[z^4+z^2+1\right]}\right]_{z=\alpha}$

$$= \frac{e^{im\alpha}}{4\alpha^3+2\alpha} = -\frac{3+i\sqrt{3}}{12} e^{im/2} \cdot e^{-m\sqrt{3}/2}$$

Similarly residue at $z=\alpha^2$ is

$$= \frac{3-i\sqrt{3}}{12} \cdot e^{-im/\sqrt{2}} \cdot e^{-m\sqrt{3}/2}$$

$\therefore$ Sum of residues

$$= -\frac{e^{-m\sqrt{3}/2}}{12}\left[3\left(e^{im/2}-e^{-im/2}\right)+i\sqrt{3}\left(e^{im/2}+e^{-im/2}\right)\right]$$

$$= -\frac{e^{-m\sqrt{3}/2}}{12}\left[6i\sin m/2 + 2i\sqrt{3}\cos m/2\right]$$

$$= -\frac{ie^{-m\sqrt{3}/2}}{\sqrt{3}}\left[\frac{\sqrt{3}}{2}\sin m/2 + \frac{1}{2}\cos m/2\right]$$

$$= -\frac{ie^{-m\sqrt{3}/2}}{\sqrt{3}}\sin\left(\frac{m}{2}+\frac{1}{6}\pi\right)$$

Hence by Cauchy's residue theorem

$$\int_C f(z)\,dz = 2\pi i \times \text{Sum of residues within the Contour C.}$$

i.e. $\int_{-R}^{R} f(x)\,dx + \int_{C_R} f(z)\,dz = 2\pi i \times \text{Sum of residues}$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{e^{imx}}{x^4+x^2+1}\,dx + \int_{C_R}\frac{e^{imz}}{z^4+z^2+1}\,dz$$

$$= 2\pi i \cdot \frac{-e^{-m\sqrt{3}/2}}{\sqrt{3}}\sin\left(\frac{m}{2}+\frac{1}{6}\pi\right) \quad \text{—— (1)}$$

Now, $\left|\int_{C_R}\frac{e^{imz}}{z^4+z^2+1}\,dz\right|$

$$\le \int_{C_R}\frac{|e^{imz}|}{|z^4+z^2+1|}|dz| \le \int_{C_R}\frac{|e^{imz}|\,|dz|}{z^4+z^2}$$

$$\le \int_{C_R}\frac{|e^{imz}|\,|dz|}{|z|^4-|z|^2}$$

$$= \int_0^{\pi}\frac{e^{-mR\sin\theta}\,R\,d\theta}{R^4-R^2}$$

$$\le 2\int_0^{\frac{1}{2}\pi}\frac{e^{-2mR\theta/\pi}}{R^3-R}\,d\theta$$

$$= \frac{\pi(1-e^{-mR})}{mR(R^3-R)} \text{ which } \to 0 \text{ as } R \to \infty$$

Hence making $R \to \infty$, relation (1) becomes

$$\int_{-\infty}^{\infty} \frac{e^{imx}}{x^4+x^2+1}\,dx = \frac{2\pi e^{-m\sqrt{3}/2}}{\sqrt{3}} \sin\left(\frac{m}{2}+\frac{1}{6}\pi\right)$$

Equating real parts we have

$$\int_{-\infty}^{\infty} \frac{\cos mx}{x^4+x^2+1}\,dx = \frac{2\pi}{\sqrt{3}} e^{-m\sqrt{3}/2} \sin\left(\frac{m}{2}+\frac{1}{6}\pi\right)$$

(or) $$\int_{0}^{\infty} \frac{\cos mx}{x^4+x^2+1}\,dx = \frac{\pi}{\sqrt{3}} e^{-m\sqrt{3}/2} \sin\left(\frac{m}{2}+\frac{1}{6}\pi\right)$$

3(b) Show that the function $f(x)=x^2$ is uniformly continuous in $(0,1)$ but not in $\mathbb{R}$.

Sol^n: ~~Let $\epsilon>0$ be given.~~

We shall show that the function $f(x)=x^2 \; \forall x\in\mathbb{R}$ is uniformly continuous on $(0,1)$.

Let $\epsilon>0$ be given

Let $x_1, x_2 \in (0,1)$. We have

$$\begin{aligned} |f(x_1)-f(x_2)| &= |x_1^2-x_2^2| \\ &= |(x_1-x_2)(x_1+x_2)| \\ &= |x_1-x_2|\,|x_1+x_2| \\ &\le |x_1-x_2|\cdot(|x_1|+|x_2|) \quad (\because |x_1+x_2|\le|x_1|+|x_2|) \\ &\le 2\cdot|x_1-x_2| \end{aligned}$$

$$\left(\begin{aligned} \because\ & x_1,x_2\in(0,1) \\ & \Rightarrow |x_1|<1 \text{ and } |x_2|<2 \\ & \Rightarrow |x_1|+|x_2|<2 \end{aligned}\right)$$

Then $|f(x_1)-f(x_2)|<\epsilon$ whenever $|x_1-x_2|<\epsilon/2$.

Thus given $\epsilon>0$, there exists $\delta=\frac{\epsilon}{2}>0$ such that

$|f(x_1)-f(x_2)|<\epsilon$ whenever $|x_1-x_2|<\delta$ $\forall x_1,x_2\in(0,1)$.

Hence $f(x)=x^2$ is uniformly continuous in $(0,1)$.

Now we shall show that f is not uniformly continuous on $\mathbb{R}$.

Let $\epsilon>0$ be given.

we shall show that for each $\delta>0$, $\exists\, x_1, x_2 \in \mathbb{R}$ such that $|x_1-x_2|<\delta \Rightarrow |f(x_1)-f(x_2)| \geq \epsilon$.

The sequence $\langle a_n \rangle$ defined by

$a_n = \sqrt{n+2\epsilon} - \sqrt{n}$ converges to 0,

Since $a_n = \dfrac{2\epsilon}{\sqrt{n+2\epsilon}+\sqrt{n}} \longrightarrow 0$ as $n\to\infty$.

$\therefore$ Given $\delta>0$, $\exists$ a +ve integer m such that

$|a_n - 0| < \delta \quad \forall n \geq m$.

$\Rightarrow \sqrt{n+2\epsilon}-\sqrt{n} < \delta \quad \forall n \geq m$.

Let us take $x_1 = \sqrt{m+2\epsilon}$ and $x_2 = \sqrt{m}$, then

$|x_1-x_2|<\delta$ whereas $|f(x_1)-f(x_2)| = |m+2\epsilon-m|$
$= 2\epsilon > \epsilon$.

Hence f is not uniformly continuous on $\mathbb{R}$.

3(c) A Company is spending Rs 1000 on transportation of its units from plants to four distribution centres. The supply and demand of units, with unit cost of transportation are given below.

plants	D_1	D_2	D_3	D_4	availabilities
P_1	19	30	50	12	7
P_2	70	30	40	60	10
P_3	40	10	60	20	18
	5	8	7	15	

(Distribution centers)

What can be the maximum saving by optimal scheduling

3(c) Sol^n Using Vogel's Approximation method the initial basic feasible solution is given by

(5) 19	30	50	(2) 12
70	30	(7) 40	(3) 60
40	(8) 10	60	(10) 20

The initial BFS has $m+n-1 = 3+4-1 = 6$ allocations (basic variables) in independent positions.

Now finding the values of u_i and v_j:

As the maximum no. of basic cells exists in the ~~first row and~~ 3rd column.

putting $v_4 = 0$

and the values of u_i's and v_j's and also the net evaluations $\Delta_{ij} = u_i + v_j - c_{ij}$ for all unoccupied cells are exhibited as shown below.

				u_i
(5) 19	30 -ve	50 -ve	(2) 12	12
70 -ve	30 +ve	(7) 40	(3) 60	60
40 -ve	(8) 10	60 +ve	(10) 20	20
7	-10	-20	0	

Since the cell (2,1) has a positive value so, the current solution is not optimal.

$\therefore$ The cell (2,1) enters the basis.

we allocate unknown quantity θ, to this cell (2,1) and identify a loop involving basic cells around this entering cell.

Add and subtract θ, alternately to and from the transition cells of the loop subject to the rim requirements as shown below.

(5) 19	30	50	(2) 12
70	θ 30	(7) 40	$(3-\theta)$ 60
40	$(8-\theta)$ 10	60	$(10+\theta)$ 20

Now assign maximum value θ so that one basic variable becomes zero and other basic variables ≥ 0.

Taking $\theta = 3$.

$\therefore x_{24} = 0$

$\therefore$ The cell (2,4) leaves the basis.

$\therefore$ The new basic feasible solution is as shown below.

(5) 19	30	50	(2) 12
70	(3) 30	(7) 40	60
40	(5) 10	60	(13) 20

Now compute the net evaluations as shown below

				u_i
(5) 19	(-ve) 30	(-ve) 50	(2) 12	0
(-ve) 70	(3) 30	(7) 40	(-ve) 60	28
(-ve) 40	(5) 10	-ve 60	(13) 20	8
$v_j \to$ 19	2	12	12	

Since all the net evaluations are ≤ 0.

$\therefore$ The current basic feasible solution is optimal.

$\therefore$ The optimal transportation cost $= 19\times 5 + 12\times 2 + 30\times 3 + 7\times 40 + 5\times 10 + 13\times 20 = 799$

$\therefore$ Maximum Saving $= 1000 - 799$
$= 201$

4(a)(ii) Discuss the irreducibility of $f(x) = x^4+1$, over rationals.

Sol^n: Given that $f(x) = x^4+1$

replacing x by $x-1$,

$\therefore$ we have $f(x-1) = (x-1)^4+1$

$$= x^4-4x^3+6x^2-4x+1+1$$

$$= 2-4x+6x^2-4x^3+x^4$$

we write $a_0=2$, $a_1=-4$, $a_2=6$, $a_3=-4$, $a_4=1$

Then $P=2$ divides a_0, a_1, a_2, a_3; but $P \nmid a_4$ and $P^2 \nmid a_0$

By Eisenstein criterion of irreducibility, $f(x-1)$ is irreducible over $\mathbb{Q}$. Hence $f(x) = x^4+1$ is irreducible over $\mathbb{Q}$.

for otherwise

$f(x) = g(x)h(x)$; $g(x)h(x) \in \mathbb{Q}[x]$

and $\deg g(x) > 0$ and $h(x) > 0$

$\Rightarrow f(x-1) = g(x-1)h(x-1)$

where $g(x-1), h(x-1) \in \mathbb{Q}[x]$ are both of positive degree and so $f(x-1)$ is reducible over $\mathbb{Q}$, a contradiction.

4(b) Find the analytic function of the following function is real part:

$$e^{-x}\{(x^2-y^2)\cos y + 2xy\sin y\}.$$

Solⁿ: Here $u(x,y) = e^{-x}\{(x^2-y^2)\cos y + 2xy\sin y\}$

$$\therefore \frac{\partial u}{\partial x} = e^{-x}\{2x\cos y + 2y\sin y\} - e^{-x}\{(x^2-y^2)\cos y + 2xy\sin y\}$$

$$= \phi_1(x,y) \text{ (say)}$$

$$\& \ \frac{\partial u}{\partial y} = e^{-x}\{-(x^2-y^2)\sin y - 2y\cos y + 2x\sin y + 2xy\cos y\}$$

$$= \phi_2(x,y) \text{ (say)}$$

By Milne's method we have

$$f'(z) = \phi_1(z,0) - i\phi_2(z,0)$$

$$= e^{-z}\{2z-(0)\} - e^{-x}\{z^2\} - i\{0\}e^{-z}$$

$$= e^{-z}\{2z - z^2\}$$

$$f'(z) = e^{-z}\{2z-z^2\}$$

Integrating, we get

$$f(z) = \int e^{-z}(2z-z^2)\,dz$$

$$= e^{-z}z^2 + ic.$$

4(c) Use Cauchy's theorem and/or Cauchy integral formula to evaluate the following integrals (i) $\displaystyle\int_{|z|=4} \frac{z^4}{(z-1)^3}\,dz$ (ii) $\displaystyle\int_{|z-1-i|=5/4} \frac{z^{1/2}}{z-1}\,dz$.

4(c)(i) Given that $\int\limits_{|z|=4} \frac{z^4}{(z-i)^3} dz$.

Here $f(z) = z^4$ is analytic in $|z| = 4$.

Comparing the given integral with $\int_C \frac{f(z)}{(z-z_0)^n} dz$

since $f(z) = z^4$ is analytic in $|z|=4$.

$\therefore$ we can apply the Cauchy's integral formule

$$\int\limits_{|z|=4} \frac{f(z)}{(z-z_0)^3} dz = \frac{2\pi i f''(z_0)}{2!}$$

$$f(z) = z^4 \Rightarrow f'(z) = 4z^3$$

$$f''(z) = 12z^2$$

$$\int \frac{z^4}{(z-i)^3} dz = \frac{2\pi i}{2!} f''(z_0)$$

$$= \frac{2\pi i}{2!} (12)(i)^2 \quad (\because z_0 = i)$$

$$= -12\pi i.$$

(ii) Given $\int\limits_{|z-1-i|=5/4} \frac{z^{1/2}}{z-1} dz$ ...

we have $|z-1-i| = \frac{5}{4}$ is a circle with centre at $z = 1+i$ and radius $5/4$.

i.e., at the centre $(1,1)$.

Also $z-1 = 0$

$\Rightarrow z = 1$

$\Rightarrow z = (1,0)$ $\therefore z = 1$ lies inside C.

Let $f(z) = z^{1/2}$ which is clearly analytic at every point with in and on C.

$$\therefore \int \frac{f(z)}{(z-1)} dz = \int \frac{f(z)}{z-z_0} dz, \text{ where } z_0 = 1.$$

$$= 2\pi i\, f(z_0) \quad \text{(By Cauchy's integral formula)}$$

$$= 2\pi i (1)^{1/2}$$

$$= 2\pi i.$$

4(d) Construct the dual of the LPP.

Maximize $z = 4x_1 + 9x_2 + 2x_3$

Subject to

$$2x_1 + 3x_2 + 2x_3 \le 7$$

$$3x_1 - 2x_2 + 4x_3 = 5$$

$$x_1, x_2, x_3 \geq 0$$

Sol'n: The equality constraint can be written as

$$3x_1 - 2x_2 + 4x_3 \le 5$$

$$3x_1 - 2x_2 + 4x_3 \geq 5$$

Since the problem is of maximization type all constraints should be of $(\leq)$ type.

now multiply $3x_1-2x_2+4x_3 \geq 5$ by -1, we get

$$-3x_1+2x_2-4x_3 \leq -5$$

we may Rewrite the primal

$$\text{Max } z = 4x_1+9x_2+2x_3$$

subject to

$$2x_1+3x_2+2x_3 \leq 7$$
$$3x_1-2x_2+4x_3 \leq 5$$
$$-3x_1+2x_2-4x_3 \leq -5$$

Let y_1, y_2 & y_3 be the dual variables associated with the above 3 constraints.

Then the dual problem is given by

$$\text{Minimize } w = 7y_1+5y_2-5y_3$$

subject to

$$2y_1+3y_2-3y_3 \geq 4$$
$$3y_1-2y_2+2y_3 \geq 9$$
$$2y_1+4y_2-4y_3 \geq 2$$
$$y_1, y_2, y_3 \geq 0.$$

This can be written as

$$\text{Min } w = 7y_1+5(y_2-y_3)$$

subject to

$$2y_1+3(y_2-y_3) \geq 4$$
$$3y_1-2(y_2-y_3) \geq 9$$
$$2y_1+4(y_2-y_3) \geq 2$$

Here the new variable $y_2-y_3 = y_1'$ becomes unrestricted in sign being the difference of two non-negative variable.

$\therefore$ The above dual problem takes the form

$$\text{Min } w = 7y_1+y_1'$$

sub to $2y_1+3y_1' \geq 4$, $3y_1-2y_1' \geq 9$, $2y_1+4y_1' \geq 2$, $y_1 \geq 0$, y_1' is unrestricted

5(a). Find the complete Integral of $(x+y)(p+q)^2+(x-y)(p-q)^2=1$

Sol'n: To reduce the given equation to standard form

$$f(p,q)=0$$

put $x+y=X^2$ and $x-y=Y^2$

$$\therefore p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial X}\frac{\partial X}{\partial x}+\frac{\partial z}{\partial Y}\frac{\partial Y}{\partial x}$$

$$=\frac{1}{2X}\frac{\partial z}{\partial X}+\frac{1}{2Y}\frac{\partial z}{\partial Y}$$

and $q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial X}\frac{\partial X}{\partial y}+\frac{\partial z}{\partial Y}\frac{\partial Y}{\partial y}$

$$=\frac{1}{2X}\frac{\partial z}{\partial X}-\frac{1}{2Y}\frac{\partial z}{\partial Y}$$

So that $p+q=\frac{1}{X}\frac{\partial z}{\partial X}$ and $p-q=\frac{1}{Y}\frac{\partial z}{\partial Y}$

Putting in the given equation it reduces to

$$\left(\frac{\partial z}{\partial X}\right)^2+\left(\frac{\partial z}{\partial Y}\right)^2=1$$

$\Rightarrow P^2+Q^2=1$ where $P=\frac{\partial z}{\partial X}$, $Q=\frac{\partial z}{\partial Y}$ —(1)

Equation (1) is of the form $f(P,Q)=0$

$\therefore$ Its solution is given by

$$z=aX+bY+c \quad —(2)$$

$\therefore P=\frac{\partial z}{\partial X}=a$ & $Q=\frac{\partial z}{\partial Y}=b$

putting (1), we get

$$a^2+b^2=1\Rightarrow \boxed{b=\sqrt{1-a^2}}$$

putting in (2), the Complete Integral is

$$z=aX+(\sqrt{1-a^2})Y+c$$

$$\Rightarrow z=a\sqrt{(x+y)}+(\sqrt{1-a^2})\sqrt{(x-y)}+c$$

which Contains two arbitrary Constants a and c and is the complete integral of the given equation.

5(b) solve the partial differential equation.

$$\frac{\partial^3 z}{\partial x^3} - 2\frac{\partial^3 z}{\partial x^2 \partial y} - \frac{\partial^3 z}{\partial x \partial y^2} + 2\frac{\partial^3 z}{\partial y^3} = e^{x+y}$$

Sol'n: Here $D^3 - 2D^2D' - DD'^2 + 2D'^3$

$= D^2(D - D'^2) - DD'(D - D') - 2D'^2(D - D')$

$= (D - D')(D^2 - DD' - 2D'^2) = (D - D')(D - 2D')(D + D')$

so given equation becomes $(D - D')(D - 2D')(D + D')z = e^{x+y}$

Its $C.F = \phi_1(y + x) + \phi_2(y + 2x) + \phi_3(y - x)$

$$P.I = \frac{1}{(D - D')(D - 2D')(D + D')} e^{x+y}$$

$$= \frac{1}{D - D'} \frac{1}{(1-2)(1+1)} \iint e^v \, dv \, dv \quad \text{where } v = x + y$$

$$= \frac{1}{D - D'}\left(-\frac{1}{2}\right)e^v = -\frac{1}{2}\frac{1}{D - D'} e^{x+y}$$

$$= -\frac{1}{2}\frac{x}{1!}e^{x+y}$$

$\therefore$ solution is $z = \phi_1(y + x) + \phi_2(y + 2x) + \phi_3(y - x) - \frac{1}{2}xe^{x+y}$.

5(c) → The bacteria concentration in a reservoir varies as $C = 4e^{-2t} + e^{-0.1t}$. Using Newton Raphson method, calculate the time required for the bacteria concentration to be 0.5.

Sol'n. Given that $C = 4e^{-2t} + e^{-0.1t}$.

Now the bacteria concentration to be 0.5 $C = 0.5$

i.e, $0.5 = 4e^{-2t} + e^{-0.1t}$

i.e. $f(t) = 4e^{-2t} + e^{-0.1t} - 0.5 = 0$

Now we have to find the time required for the bacteria concentration to be 0.5.

∴ we want to find the roots of the equation $f(t) = 4e^{-2t} + e^{-0.1t} - 0.5 = 0$.

By Newton Raphson method,

$$t_{n+1} = t_n - \frac{f(t_n)}{f'(t_n)}$$

$$f(t) = 4e^{-2t} + e^{-0.1t} - 0.5$$

$$f'(t) = -8e^{-2t} - 0.1e^{-0.1t}$$

Let the initial time be $t_0 = 0$.

Then $t_1 = t_0 - \frac{f(t_0)}{f'(t_0)}$

$= 0 - \frac{4.5}{-8.1}$

$= 0.5555$

$t_2 = t_1 - \frac{f(t_1)}{f'(t_1)} = 0.5555 - \frac{1.7628}{-2.72843}$

$= 1.20168$

Continue in this way, we get $t = 6.932$.

5(d) Prove that following Boolean expression
$$(A+B)(\bar{A}\bar{C}+C)(\overline{\bar{B}+AC}) = \bar{A}B.$$

Sol'n: $(A+B)(\bar{A}\bar{C}+C)(\overline{\bar{B}+AC})$

$$= (A+B)(\bar{A}\bar{C}+C)(\bar{\bar{B}}\cdot\overline{AC})$$

$$= [A\bar{A}\bar{C} + AC + \bar{A}\bar{C}B + BC]$$
$$[B(\bar{A}+\bar{C})]$$

$$= (AC+\bar{A}\bar{C}B+BC)\cdot(B\bar{A}+B\bar{C}) \quad (\because A\bar{A}=0)$$

$$= AC\cdot B\bar{A} + AC\cdot B\bar{C} + \bar{A}\bar{C}B\cdot B\bar{A} + \bar{A}\bar{C}B\cdot B\bar{C} + BC\cdot B\bar{A} + BC\cdot B\bar{C}$$

$$= 0+0+\bar{A}B\bar{C} + \bar{A}\bar{C}B + BC\bar{A} + 0$$

$$= \bar{A}B(\bar{C}+\bar{C}+C)$$

$$= \bar{A}B$$

5(e) Find the stream function ψ for the given velocity potential $\phi = cx$, where c is constant.

Sol'n: The velocity potential $\phi = cx$ represents fluid flow because it satisfies Laplace equation $\nabla^2\phi = 0$

Since $-\frac{\partial\phi}{\partial x} = -c = u$ and $u = -\frac{\partial\psi}{\partial y}$

$$\therefore \frac{\partial\psi}{\partial y} = c \Rightarrow \psi = cy + f(x) \quad \text{--- (1)}$$

Differentiating with regard to x, we have
$$\frac{\partial\psi}{\partial x} = f'(x)$$

But $\frac{\partial\varphi}{\partial x} = v = -\frac{\partial\varphi}{\partial y} \Rightarrow \frac{\partial\psi}{\partial x} = 0$, as $\frac{\partial\varphi}{\partial y} = 0$

$$\Rightarrow f'(x) = 0 \Rightarrow f(x) = \text{Const.} \quad \text{--- (2)}$$

The stream function ψ is given as
$\psi = \text{Const} + cy$, which represents Parallel flow in which streamlines are parallel to x-axis.

6(a)(ii). Form a partial differential equation by eliminating the function 'f' from: $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$.

Sol'n: Given that $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$ —— ①

Differentiating ① partially w.r.t x & y, we get

$$\frac{\partial z}{\partial x} = 2f'\left[\frac{1}{x} + \log y\right]\left(-\frac{1}{x^2}\right)$$

$$\Rightarrow -x^2\frac{\partial z}{\partial x} = 2f'\left[\frac{1}{x} + \log y\right] \text{ —— ②}$$

and $\frac{\partial z}{\partial y} = 2y + 2f'\left[\frac{1}{x} + \log y\right]\left(\frac{1}{y}\right)$

$$\Rightarrow y\frac{\partial z}{\partial y} - 2y^2 = 2f'\left[\frac{1}{x} + \log y\right] \text{ —— ③}$$

from ② & ③ we get

$$-x^2\frac{\partial z}{\partial x} = y\frac{\partial z}{\partial y} - 2y^2$$

$$\Rightarrow x^2\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2y^2$$

$$\Rightarrow x^2p + yq = 2y^2$$

which is the required partial differential equation.

6(b) A rocket is launched from the ground. Its acceleration is registered during the first 80 seconds and is given in the table below. Using Simpson's $\frac{1}{3}$rd rule, find the velocity of the rocket at $t = 80$ seconds.

t(sec):	0	10	20	30	40	50	60	70	80
f(cm/sec²):	30	31.63	33.34	35.47	37.75	40.33	43.25	46.69	50.67

Sol^n: Since acceleration is defined as the rate of change of velocity.

we have $\frac{dv}{dt} = a$ (or) $v = \int_0^{80} a\,dt$

Using Simpson's $\frac{1}{3}$ rule, we have

$$v = \frac{h}{3}\left[(y_0 + y_8) + 4(y_1 + y_3 + y_5 + y_7) + 2(y_2 + y_4 + y_6)\right]$$

$$= \frac{10}{3}\left[(30 + 50.67) + 4(31.63 + 35.47 + 40.33 + 46.69) + 2(33.34 + 37.75 + 43.25)\right]$$

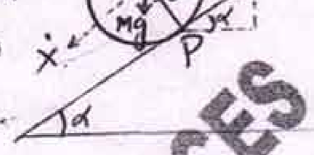
$= 3086.1$ m/s

Therefore the required velocity is given by

$v = 3.0861$ km/sec.

6(C) A sphere of radius a and mass M rolls down a rough plane inclined at angle α to the horizontal.
If x be the distance of the point of contact of the sphere from a fixed point on the plane, find the acceleration by using Hamilton's equations.

Sol^n: Let a sphere of radius a and mass M roll down a rough plane inclined at an angle α starting initially from a fixed point O of the plane. In time t, let the sphere roll down a distance x and during the time let it turn through an angle θ. Since there is no slipping.



$$\therefore x = OA = \text{arc } AB = a\theta.$$

$$\text{so that } \dot{x} = a\dot{\theta}.$$

If T and V are kinetic and potential energies of the sphere, then $T = \frac{1}{2}Mk^2\dot{\theta}^2 + \frac{1}{2}M\dot{x}^2 = \frac{1}{2}M\frac{2}{5}a^2\dot{\theta}^2 + \frac{1}{2}M(a\dot{\theta})^2$

$$\Rightarrow T = \frac{7}{10}M\dot{x}^2$$

and $V = -Mg\,OL = -Mg\,x\sin\alpha$. (Since the sphere move down the plane)

$$\therefore L = T - V = \frac{7}{10}M\dot{x}^2 + Mgx\sin\alpha$$

Here x is the only generalised co-ordinate

$$\therefore p_x = \frac{\partial L}{\partial \dot{x}} = \frac{7}{5}M\dot{x}. \quad \text{——— (1)}$$

Since L doesnot contain t explicitly.

$$\therefore H = T + V = \frac{7}{10}M\dot{x}^2 - Mgx\sin\alpha = \frac{7}{10}M\left(\frac{5}{7M}p_x\right) - Mgx\sin\alpha \quad (\text{from (1)})$$

$$= \frac{5}{14M}p_x^2 - Mgx\sin\alpha.$$

Hence the two Hamilton's equations are

$$\dot{p}_x = -\frac{\partial H}{\partial x} = Mg\sin\alpha \text{ —— (2)}, \quad \dot{x} = \frac{\partial H}{\partial p_x} = \frac{5}{7M}p_x \text{ —— (3)}$$

Differentiating (3) and using (2), we get

$$\ddot{x} = \frac{5}{7M}\dot{p}_x = \frac{5}{7M}Mg\sin\alpha \Rightarrow \ddot{x} = \frac{5}{7}g\sin\alpha.$$

which gives the required acceleration

7(a) Reduce the second-order partial differential equation $x^2\frac{\partial^2 u}{\partial x^2} - 2xy\frac{\partial^2 u}{\partial x\partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

Sol^n: Let $x = e^X$, $y = e^Y$

So that $X = \log x$, $Y = \log y$ —— ①

Also let $D = \frac{\partial}{\partial x}$, $D' = \frac{\partial}{\partial y}$ and $D_1 = \frac{\partial}{\partial X}$, $D_1' = \frac{\partial}{\partial Y}$.

Then the given equation becomes

$$[D_1(D_1-1) - 2D_1D_1' + D_1'(D_1'-1) + D_1 + D_1']u = 0$$

$$(D_1^2 - 2D_1D_1' + D_1'^2)u = 0$$

$$(D_1 - D_1')^2 u = 0$$

Hence the required general solution is

$$C.F = \phi_1(Y+X) + X\phi_2(Y+X)$$

$$= \phi_1(\log y + \log x) + \log x\, \phi_2(\log y + \log x)$$

$$= \phi_1(\log xy) + \log x \cdot \phi_2(\log xy)$$

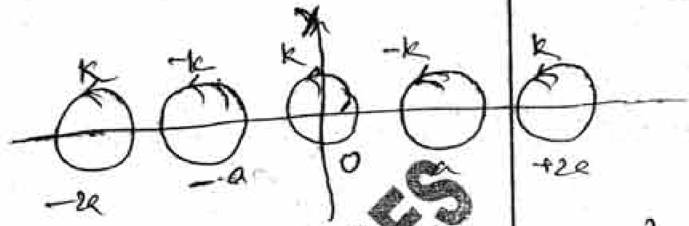
$$= F_1(xy) + \log x\, F_2(xy)$$

7(C) → An infinite row of equidistant rectilinear vortices is at a distance a apart. The vortices are of the same numerical strength k but they are alternately of opposite signs. Find the complex function that determines the velocity potential and the stream function. Show that the vortices remain at rest and draw streamlines. Show also that if α be the radius of a vortex, the amount of flow between any vortex and the next is $\frac{k}{\pi} \log \cot \frac{\pi\alpha}{2a}$.

Sol'n

7(c) Sol^n: Let the vortices each of strength k be placed at $(0,0)$, $(\pm 2a, 0)$, $(\pm 4a, 0)$ ... and vortices each of strength $-k$ be placed at $(\pm a, 0)$, $(\pm 3a, 0)$, $(\pm 5a, 0)$, ...

The complex potential at any point $P(z)$ is given by



$$W = \frac{ik}{2\pi}\log z + \frac{ik}{2\pi}\left[\log(z-2a) + \log(z+2a) + \log(z-4a) + \log(z+4a) + \cdots\right]$$
$$-\frac{ik}{2\pi}\left[\log(z-a) + \log(z+a) + \log(z-3a) + \log(z+3a) + \cdots\right]$$
$$= \frac{ik}{2\pi}\log\left[\frac{z(z^2-2^2a^2)(z^2-4^2a^2)\cdots}{(z^2-a^2)(z^2-3^2a^2)(z^2-5^2a^2)\cdots}\right]$$
$$= \frac{ik}{2\pi}\log\left[\frac{\frac{z}{2a}\left\{1-\left(\frac{z}{2a}\right)^2\right\}\left\{1-\left(\frac{z}{4a}\right)^2\right\}\cdots}{\left\{1-\left(\frac{z}{a}\right)^2\right\}\left\{1-\left(\frac{z}{3a}\right)^2\right\}\cdots}\right] + \text{constant}$$
$$= \frac{ik}{2\pi}\log\frac{\sin(\pi z/2a)}{\cos(\pi z/2a)} = \frac{ik}{2\pi}\log\tan\left(\frac{\pi z}{2a}\right)$$

$$W = \frac{ik}{2\pi}\log\tan\left(\frac{\pi z}{2a}\right) \quad \text{——— (1)}$$
$$\phi + i\psi = \frac{ik}{2\pi}\log\tan\left(\frac{\pi z}{2a}\right) \quad \text{——— (2)}$$
$$\phi - i\psi = -\frac{ik}{2\pi}\log\tan\left(\frac{\pi \bar{z}}{2a}\right) \quad \text{——— (3)}$$

(2)−(3) gives $2i\psi = \frac{ik}{2\pi}\log\left(\tan\frac{\pi z}{2a}\right)\left(\tan\frac{\pi\bar{z}}{2a}\right)$

$$\psi = \frac{k}{4\pi}\log\left[\sin\left(\frac{\pi z}{2a}\right)\sin\left(\frac{\pi\bar{z}}{2a}\right)\Big/\cos\left(\frac{\pi z}{2a}\right)\cos\left(\frac{\pi\bar{z}}{2a}\right)\right]$$
$$= \frac{k}{4\pi}\log\left[\frac{\cosh\frac{\pi y}{a} - \cos\frac{\pi x}{a}}{\cosh\frac{\pi y}{a} + \cos\frac{\pi x}{a}}\right] \quad \text{——— (4)}$$

Stream lines are given by ψ = constant

i.e. $\cosh\frac{\pi y}{a} = b\cos\frac{\pi x}{a}$

(2)+(3) gives $2\phi = \frac{ik}{2\pi}\log\frac{\tan(\pi z/2a)}{\tan(\pi\bar{z}/2a)}$

$$\phi = \frac{ik}{4\pi}\log\left[\frac{\sin(\pi z/2a)\cos(\pi\bar{z}/2a)}{\sin(\pi\bar{z}/2a)\cos(\pi\bar{z}/2a)}\right]$$

$$= \frac{ik}{4\pi}\log\frac{\sin(\pi x/2a)+i\sinh(\pi y/a)}{\sin(\pi x/2a)-i\sinh(\pi y/a)}$$

$$= \frac{-k}{4\pi}\left[\tan^{-1}\frac{\sinh(\pi y/a)}{\sin(\pi x/a)}+\tan^{-1}\frac{\sinh(\pi y/a)}{\sin(\pi x/a)}\right]$$

$$\phi = -\frac{k}{2\pi}\tan^{-1}\frac{\sinh(\pi y/a)}{\sin(\pi x/a)} \quad \text{—— (5)}$$

Required velocity potential and stream function are given by (4) and (5).

2nd part: consider the motion of the vortex $+k$ at the origin. Then $W_1 = W - \frac{ik}{2\pi}\log z$

Let u_0, v_0 be the velocity components of the vortex at $(0,0)$.

$$-u_0 - iv_0 = -\left(\frac{dW_1}{dz}\right)_{z=0} = \frac{ik}{2\pi}\left[\frac{\sec \pi z/2a}{\tan \pi z/2a}\cdot\frac{\pi}{2a}-\frac{1}{z}\right]_{z=0} = 0$$

This shows that the vortex at $(0,0)$ is at rest.
Similarly we can prove that every vortex is at rest.

3rd part: To determine the flow ψ at any point of x-axis, i.e., at $(x,0)$ is

$$\psi = \frac{k}{4\pi}\log\left[\frac{1-\cos(\pi x/a)}{1+\cos(\pi x/a)}\right] = \frac{k}{4\pi}\cdot 2\log\tan\left(\frac{\pi x}{2a}\right) \quad \text{(by (4))}$$

$$\psi(x,0) = \frac{k}{4\pi}\log\tan\left(\frac{\pi x}{2a}\right)$$

flow between two consecutive vortices

$$= 2\text{ flow across } (a-\alpha,0) \text{ to } (\alpha,0)$$

$$= \psi(a-\alpha,0) - \psi(\alpha,0)$$

$$= \frac{k}{2\pi}\log\left\{\tan\frac{\pi}{2a}(a-\alpha)\Big/\tan\left(\frac{\pi\alpha}{2a}\right)\right\}$$

$$= \frac{k}{2\pi}\log\left(\cot\frac{\pi\alpha}{2a}\right)^2 = \frac{k}{\pi}\log\cot\left(\frac{\pi\alpha}{2a}\right).$$

8(a) A string of length l is fixed at its ends. The string from the mid-point is pulled upto a height k and then released from rest. Find the deflection $y(x,t)$ of the vibrating string.

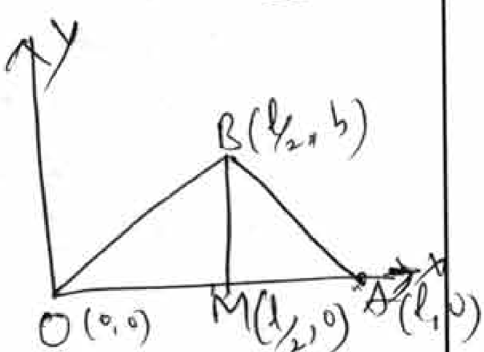
Sol: The displacement function $y(x,t)$ is the solution of the wave equation.

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \quad \text{—— (1)} \quad \text{(or)} \quad \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

Subject to the boundary conditions:

$$y(0,t) = y(l,t) = 0 \quad \forall t \geq 0. \quad \text{—— (2)}$$

Initial position of the string at $t=0$ is made up of two straight line segments OB and BA as shown in the figure and the string is released from rest.



The equation of OB is given by

$$y - 0 = \frac{h-0}{\frac{l}{2}-0}(x-0)$$

$$\Rightarrow y = \frac{2hx}{l} \quad \text{for } 0 \leq x \leq l/2$$

The equation of BA is given by

$$y - 0 = \frac{h-0}{\frac{l}{2}-l}(x-l)$$

$$\Rightarrow y = \frac{2h(l-x)}{l} \quad \text{for } \frac{l}{2} \leq x \leq l$$

Hence, the initial displacement is given by

$$= y(x,0) = f(x) = \begin{cases} 2hx/l, & 0 \le x \le l/2 \\ \frac{2h(l-x)}{l}, & \frac{l}{2} \le x \le l \end{cases} \quad \text{--- (3)}$$

and the initial velocity $= \left(\frac{\partial y}{\partial t}\right)_{t=0} = 0$ --- (4)

Suppose that (1) has the solution of the form

$y(x,t) = X(x)T(t)$ --- (5)

Substituting this value of y in (1), we have

$XT'' = c^2 X''T \Rightarrow \frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = \mu$ say

$\Rightarrow \underline{X'' - \mu X = 0}$ (6) & $T'' - \mu c^2 T = 0$ --- (7)

Using (2), (5) gives

$X(0)T(t) = 0$ and $X(l)T(t) = 0$ --- (8)

$\Rightarrow \boxed{X(0) = 0 \;\&\; X(l) = 0}$ (9) $(\because T(t) = 0$ leads $y \equiv 0 \;\forall t)$

we now solve (6) under B.C. (9) for $T(t) \ne 0$.

Three cases arise.

Case (1): Let $\mu = 0$. Then the solution of (6) is

$X(x) = Ax + B$. --- (10)

Using B.C. (9), (10) gives $X(0) = A(0) + B \Rightarrow B = 0$

and $X(l) = Al + B \Rightarrow 0 = Al + 0 \Rightarrow A = 0$

$\Rightarrow X(x) = 0$

This leads to $y \equiv 0$, which doesnot satisfy I.C. (3) & (4)

so we reject $\mu = 0$

Case (2): Let $\mu = \lambda^2$, $\lambda \ne 0$. Then the solution of (6) is $X(x) = Ae^{\lambda x} + Be^{-\lambda x}$ --- (11)

using B.C. (9), (11) gives $X(x) = 0$

This leads to $y \equiv 0$ which does not satisfy (3) & (4).

Case(3): Let $\mu = -\lambda^2$.

Then the solution of (6) is

$X(x) = A\cos\lambda x + B\sin\lambda x$ — (13)

Using B.C (9), (12) gives $X(0) = 0 = A + B(0)$
$\Rightarrow A = 0$

and $X(l) = 0 = 0 + B\sin\lambda l \Rightarrow B\sin\lambda l = 0$
$\Rightarrow \sin\lambda l = 0$

Now $\sin\lambda l = 0$
$\Rightarrow l\lambda = n\pi$
$\Rightarrow \lambda = \frac{n\pi}{l}$, $n = 1, 2, 3, \ldots$

from (12), we have
$X(x) = B\sin\frac{n\pi x}{l}$, $n = 1, 2, \ldots$

Hence non-zero solutions $X_n(x)$ of (6) are given by

$X_n(x) = B_n \sin\left(\frac{n\pi x}{l}\right)$ — (13)

from (7), $T'' - \mu c^2 T = 0$
$\Rightarrow T'' + \lambda^2 c^2 T = 0$
$\Rightarrow T'' + \frac{n^2\pi^2}{l^2} c^2 T = 0$

whose general solution is

$T_n(t) = C_n \cos\left(\frac{n\pi c t}{l}\right) + D_n \sin\left(\frac{n\pi c t}{l}\right)$

$\therefore y_n(t) = X_n(x) T_n(t)$
$= B_n \sin\left(\frac{n\pi x}{l}\right)\left[C_n \cos\left(\frac{n\pi c t}{l}\right) + D_n \sin\left(\frac{n\pi c t}{l}\right)\right]$
$= \left[E_n \cos\left(\frac{n\pi c t}{l}\right) + F_n \sin\left(\frac{n\pi c t}{l}\right)\right]\sin\left(\frac{n\pi x}{l}\right)$ — (14)

are solutions of (1) satisfying (2)

Here $E_n = B_n C_n$ and $F_n = B_n D_n$ are new arbitrary constants.

Inorder to obtain a solution also satisfying (3) & (4), we consider more general solution.

$$y(x,t) = \sum_{n=1}^{\infty} y_n(x,t)$$

$$y(x,t) = \sum_{n=1}^{\infty}\left(E_n \cos\frac{n\pi ct}{l} + F_n \sin\frac{n\pi ct}{l}\right)\sin\frac{n\pi x}{l} \quad \text{--- (15)}$$

Differentiating (15) partially w.r.t to t, we get

$$\frac{\partial y}{\partial t} = \sum_{n=1}^{\infty}\left\{-E_n \frac{n\pi c}{l}\sin\frac{n\pi ct}{l} + \frac{n\pi c}{l}F_n \cos\frac{n\pi ct}{l}\right\}\sin\frac{n\pi x}{l}. \quad \text{--- (16)}$$

putting $t=0$ in (16)

$$0 = \sum_{n=1}^{\infty} \frac{n\pi c}{l} F_n \sin\frac{n\pi x}{l}$$

$$\text{where } F_n = \frac{2}{n\pi c}\int_0^l (0)\sin\frac{n\pi x}{l}dx = 0$$

$$y(x,t) = \sum_{n=1}^{\infty} E_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l}.$$

$$\text{where } E_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}dx. \quad \text{--- (17)}$$

$$\therefore E_n = \frac{2}{l}\left[\int_0^{l/2} f(x)\sin\frac{n\pi x}{l}dx + \int_{l/2}^{l} f(x)\sin\frac{n\pi x}{l}dx.\right]$$

$$= \frac{2}{l}\left[\int_0^{l/2}\frac{2hx}{l}\sin\frac{n\pi x}{l}dx + \int_{l/2}^{l}\frac{2h(l-x)}{l}\sin\frac{n\pi x}{l}dx\right]$$

$$= \frac{4h}{l^2}\int_0^{l/2} x\sin\frac{n\pi x}{l}dx + \frac{4h}{l^2}\int_{l/2}^{l}(l-x)\sin\frac{n\pi x}{l}dx.$$

$$= \frac{4h}{l^2}\left[(x)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - (1)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right]_0^{l/2}$$

$$+ \frac{4h}{l^2}\left[(l-x)\left(-\frac{l}{n\pi}\cos\frac{n\pi x}{l}\right) - (-1)\left(-\frac{l^2}{n^2\pi^2}\sin\frac{n\pi x}{l}\right)\right]_{l/2}^{l}$$

$$= \frac{4h}{l^2}\left(-\frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right) + \frac{4h}{l^2}\left(\frac{l^2}{2n\pi}\cos\frac{n\pi}{2} + \frac{l^2}{n^2\pi^2}\sin\frac{n\pi}{2}\right)$$

$$= \frac{8h}{n^2\pi^2}$$

$$= \begin{cases} \dfrac{8(-1)^{m+1} h}{(2m-1)^2\pi^2} & \text{if } n = 2m-1 \text{ (odd). and } m = 1, 2, 3, \ldots \\ 0 & \text{if } n = 2m \text{ (even) and } m = 1, 2, 3 \ldots \end{cases}$$

Substituting the above value of E_n in (17), the required displacement function is given by

$$y(x,t) = \frac{8h}{\pi^2} \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{(2m-1)^2} \sin\frac{(2m-1)\pi x}{l} \cos\frac{(2m-1)\pi ct}{l}.$$

8(c) Convert $(0.231)_5$, $(104.231)_5$ and $(247)_7$ to base 10.

Sol'n: (a) $(0.231)_5 = 2\times 5^{(-1)} + 3\times 5^{-2} + 1\times 5^{-3}$

$$= \frac{2}{5} + \frac{3}{25} + \frac{1}{125}$$

$$= 0.4 + 0.12 + 0.008$$

$$= (0.528)_{10}$$

(b) $(104.231)_5 = 1\times 5^2 + 0\times 5^1 + 4\times 5^0 + 2\times 5^{-1} + 3\times 5^{-2} + 1\times 5^{-3}$

$$= 25 + 0 + 4 + 0.4 + 0.12 + 0.008$$

$$= (29.528)_{10}$$

(c) $(247)_7$

This question is wrong. Since under base 7 the digit must be lies between 0 to 6.

8(d) A ring slides on a smooth circular hoop of equal mass and of radius a which can turn a vertical plane about a fixed point O in its circumference. If θ and ϕ be the inclination to the vertical of the radius through O and of the radius through the ring, Prove that the principal coordinates are $2\theta+\phi$ and $(\phi-\theta)$, and the periods of small oscillations are $2\pi\sqrt{a/2g}$ and $2\pi\sqrt{2a/g}$.

Sol'n: Let M be the mass of each of the ring and the circular hoop of radius a and centre C, which can turn about the point O of its circumference. At time t, let us radius OC of the hoop make an angle θ with the vertical. At this time t, let the ring be at P, such that CP make an angle ϕ with the vertical. Initially the ring was at the end A of diameter ~~OA which was vertical~~.

Referred to O as origin, the horizontal and vertical lines through O as axes, the coordinates (x_c, y_c) of the centre C and (x_p, y_p) of the point P are given by

$x_c = a\sin\theta,\ y_c = a\cos\theta$

$x_p = a(\sin\theta + \sin\phi),\ y_p = a(\cos\theta+\cos\phi)$

$\therefore\ v_c^2 = \dot{x}_c^2 + \dot{y}_c^2 = (a\cos\theta\dot{\theta})^2 + (-a\sin\theta\dot{\theta})^2 = a^2\dot{\theta}^2$

& $v_p^2 = \dot{x}_p^2 + \dot{y}_p^2 = a^2(\cos\theta\dot{\theta} + \cos\phi\dot{\phi})^2 + a^2(-\sin\theta\dot{\theta} - \sin\phi\dot{\phi})^2$

$= a^2\{\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\cos(\theta-\phi)\} = a^2(\dot{\theta}^2+\dot{\phi}^2+2\dot{\theta}\dot{\phi})$

($\because \theta, \phi$ are small)

If T be the kinetic energy and W the work function of the system, then we have

T = K.E of the loop + K.E of the ring

$= \left[\frac{1}{2}Mk^2\dot{\theta}^2 + \frac{1}{2}M.v_c^2\right] + \left[\frac{1}{2}Mv_p^2\right]$

$= \frac{1}{2}M(a^2\dot{\theta}^2 + a^2\dot{\theta}^2) + \frac{1}{2}Ma^2(\dot{\theta}^2+\dot{\phi}^2+2\dot{\theta}\dot{\phi})$

$= \frac{1}{2}Ma^2(3\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi})$

and $W = Mgy_c + Mgy_p + C = Mga(2\cos\theta + \cos\phi) + C$

$\therefore$ Lagrange's θ- equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\theta}\right) - \left(\frac{\partial T}{\partial \theta}\right) = \frac{\partial W}{\partial \theta}$

i.e. $\frac{d}{dt}\left[\frac{1}{2}Ma^2(6\dot\theta + 2\dot\phi)\right] = -2Mga\sin\theta = -2Mga\theta$ ($\because \theta$ is small)

$\Rightarrow 3\ddot\theta + \ddot\phi = -2c\theta$, (where $c = g/a$) —— (1)

And Lagrange's ϕ-equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot\phi}\right) - \frac{\partial T}{\partial \phi} = \frac{\partial W}{\partial \phi}$

$\Rightarrow \frac{d}{dt}\left[\frac{1}{2}Ma^2(2\dot\phi + 2\dot\theta)\right] = -Mga\sin\phi = -Mga\phi$, $\because \phi$ is small

$\Rightarrow \ddot\theta + \ddot\phi = -c\phi$ (where $c = g/a$) —— (2)

Multiplying (2) by λ and adding to (1), we get

$$(3+\lambda)\ddot\theta + (1+\lambda)\ddot\phi = -c(2\theta + \lambda\phi) \quad \text{—— (3)}$$

Now choose λ such that

$$\frac{3+\lambda}{1+\lambda} = \frac{2}{\lambda} \Rightarrow \lambda^2 + \lambda - 2 = 0$$
$$\Rightarrow (\lambda-1)(\lambda+2) = 0 \Rightarrow \lambda = 1, -2$$

when $\lambda = 1$, (3) reduce to

$$D^2(2\theta + \phi) = -\tfrac{1}{2}c(2\theta + \phi) \quad \text{—— (4)}$$

And when $\lambda = -2$, (3) reduce to

$$D^2(\phi - \theta) = -2c(\phi - \theta) \quad \text{—— (5)}$$

Putting $2\theta + \phi = X$ and $\phi - \theta = Y$ in (4) & (5), we have

$D^2X = -\frac{g}{2a}X$ and $D^2Y = -\frac{2g}{a}Y$, $\because c = g/a$

which represents two independent S.H.M

Thus the principal coordinates are X & Y,

$\Rightarrow 2\theta + \phi$ and $\phi - \theta$.

$2\pi\Big/\sqrt{\frac{g}{2a}}$ and $2\pi\Big/\sqrt{\frac{2g}{a}}$ i.e.

$2\pi\sqrt{2a/g}$ and $2\pi\sqrt{a/2g}$