Scan Scanned

Keep it up!

Date : ......................

A CONSOLIDATED QUESTION PAPER-CUM-ANSWER BOOKLET

IMS™ (INSTITUTE OF MATHEMATICAL SCIENCES)

# MAINS TEST SERIES-2019

(JUNE-2019 *to* SEPT.-2019)

**Under the guidance *of* K. Venkanna**

# MATHEMATICS

206/250

**PAPER - I : LINEAR ALGEBRA, CALCULUS & 3D**

**TEST CODE: TEST-1: IAS(M)/09-JUNE.-2019**

**Time: 3 Hours** **Maximum Marks: 250**

## INSTRUCTIONS

1. This question paper-cum-answer booklet has **44** pages and has **34 PART/SUBPART** questions. Please ensure that the copy of the question paper-cum-answer booklet you have received contains all the questions.
2. Write your Name, Roll Number, Name of the Test Centre and Medium in the appropriate space provided on the right side.
3. A consolidated Question Paper-cum-Answer Booklet, having space below each part/sub part of a question shall be provided to them for writing the answers. Candidates shall be required to attempt answer to the part/sub-part of a question strictly within the pre-defined space. Any attempt outside the pre-defined space shall not be evaluated. "
4. Answer must be written in the medium specified in the admission Certificate issued to you, which must be stated clearly on the right side. No marks will be given for the answers written in a medium other than that specified in the Admission Certificate.
5. Candidates should attempt Question Nos. 1 and 5, which are compulsory, and any **THREE** of the remaining questions selecting at least **ONE** question from each Section.
6. The number of marks carried by each question is indicated at the end of the question. Assume suitable data if considered necessary and indicate the same clearly.
7. Symbols/notations carry their usual meanings, unless otherwise indicated.
8. All questions carry equal marks.
9. All answers must be written in blue/black ink only. Sketch pen, pencil or ink of any other colour should not be used.
10. All rough work should be done in the space provided and scored out finally.
11. The candidate should respect the instructions given by the invigilator.
12. The question paper-cum-answer booklet must be returned in its entirety to the invigilator before leaving the examination hall. Do not remove any page from this booklet.

**READ INSTRUCTIONS ON THE LEFT SIDE OF THIS PAGE CAREFULLY**

Name MAYUR KHANDELWAL

Roll No.

Test Centre

Medium English

**Do not write your Roll Number or Name anywhere else in this Question Paper-cum-Answer Booklet.**

I have read all the instructions and shall abide by them

Mayur

Signature of the Candidate

I have verified the information filled by the candidate above

Signature of the invigilator

**IMPORTANT NOTE:**

Whenever a question is being attempted, all its parts/ sub-parts must be attempted contiguously. This means that before moving on to the next question to be attempted, candidates must finish attempting all parts/ sub-parts of the previous question attempted. This is to be strictly followed. Pages left blank in the answer-book are to be clearly struck out in ink. Any answers that follow pages left blank may not be given credit.

**P.T.O.**

# DO NOT WRITE ON THIS SPACE

P.T.O.

# INDEX TABLE

| QUESTION | No. | PAGE NO. | MAX. MARKS | MARKS OBTAINED | |
|---|---|---|---|---|---|
| 1 | (a) | | | 08 | 40 |
| | (b) | | | 08 | |
| | (c) | | | 08 | |
| | (d) | | | 08 | |
| | (e) | | | 08 | |
| 2 | (a) | | | | |
| | (b) | | | | |
| | (c) | | | | |
| | (d) | | | | |
| 3 | (a) | | | 13 | 44 |
| | (b) | | | 08 | |
| | (c) | | | 06 | |
| | (d) | | | 17 | |
| 4 | (a) | | | | |
| | (b) | | | | |
| | (c) | | | | |
| | (d) | | | | |
| 5 | (a) | | | 08 | 34 |
| | (b) | | | 02 | |
| | (c) | | | 08 | |
| | (d) | | | 08 | |
| | (e) | | | 08 | |
| 6 | (a) | | | | |
| | (b) | | | | |
| | (c) | | | | |
| | (d) | | | | |
| 7 | (a) | | | 10 | 43 |
| | (b) | | | 09 | |
| | (c) | | | 13 | |
| | (d) | | | 11 | |
| 8 | (a) | | | 13 | 45 |
| | (b) | | | 13 | |
| | (c) | | | 19 | |
| | ~~(d)~~ | | | | |
| Total Marks | | | | | |

206/250

P.T.O.

# DO NOT WRITE ON THIS SPACE

**SECTION - A**

**1.** (a) Determine the angle between the vectors $\mathbf{u} = (1, 0, 0)$ and $v = (1, 0, 1)$ in $\mathbf{R}^3$. **(10)**

$\bar{u} = \hat{i} + 0\hat{j} + 0\hat{k} = \hat{i}$ , $\bar{v} = \hat{i} + 0\hat{j} + \hat{k} = \hat{i} + \hat{k}$

$$\cos\theta = \frac{\bar{u}\cdot\bar{v}}{|u||v|} = \frac{1}{1\cdot\sqrt{2}}$$

$$\Rightarrow \cos\theta = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \boxed{\theta = 45^\circ}$$

08

**1.** (b) The transformation $T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -y \\ -x \end{bmatrix}$ defines a reflection in the line $y = -x$. show that T is linear transformation. Determine the standard matrix of this transformation. Find the image of $\begin{bmatrix} 4 \\ 1 \end{bmatrix}$. **[10]**

$$T\left\{\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right\} = T\left\{\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \end{bmatrix}\right\} = \begin{bmatrix} -(y_1+y_2) \\ -(x_1+x_2) \end{bmatrix}$$

$$= \begin{bmatrix} -y_1 \\ -x_1 \end{bmatrix} + \begin{bmatrix} -y_2 \\ -x_2 \end{bmatrix}$$

$$= T\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + T\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$$

$\Rightarrow$ $\boxed{\text{T is linear transformation}}$

T-matrix $\longrightarrow$ $T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}0\\-1\end{bmatrix} = 0\begin{bmatrix}1\\0\end{bmatrix} + (-1)\begin{bmatrix}0\\1\end{bmatrix}$

$T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}-1\\0\end{bmatrix} = (-1)\begin{bmatrix}1\\0\end{bmatrix} + 0\begin{bmatrix}0\\1\end{bmatrix}$

T Matrix $= \begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix}$

Image of $\begin{bmatrix}4\\1\end{bmatrix}$

$T\begin{bmatrix}4\\1\end{bmatrix} = \begin{bmatrix}-1\\-4\end{bmatrix}$

08

**1.** (c) If $u = \sin^{-1}\frac{x^2+y^2}{x+y}$, show that :

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x\,\partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \tan^3 u.$$ **(10)**

Let $\sin u = \frac{x^2+y^2}{x+y}$ — degree 1.
$Z =$

$$x^2\frac{\partial^2 z}{\partial x^2} + y^2\frac{\partial^2 z}{\partial y^2} + 2xy\frac{\partial^2 z}{\partial x\partial y} = 1(1-1)z = 0 \quad -(1)$$

$$\frac{\partial z}{\partial x} = \cos u\frac{\partial u}{\partial x} \Rightarrow \frac{\partial^2 z}{\partial x^2} = \cos u\frac{\partial^2 u}{\partial x^2} - \sin u\left(\frac{\partial u}{\partial x}\right)^2 \quad -(2)$$

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = z$$

$$\frac{\partial^2 z}{\partial y^2} = \cos u \frac{\partial^2 u}{\partial y^2} - \sin u \left(\frac{\partial u}{\partial y}\right)^2 \quad \text{--- (3)}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \cos u \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial y}(-\sin u) \quad \text{--- (4)}$$

by (1), (2), (3), (4) —

$$\cos u \left(x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y}\right) - \sin u \left\{ x^2\left(\frac{\partial u}{\partial x}\right)^2 + 2xy\left(\frac{\partial u}{\partial y}\frac{\partial u}{\partial x}\right) + y^2\left(\frac{\partial u}{\partial y}\right)^2 \right\} = 0 \quad \text{--- (5)}$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \tan u \quad \text{--- (6)} \quad \Rightarrow \left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)^2 = \tan^2 u$$

$$\boxed{x^2 \frac{\partial^2 u}{\partial x^2} + y^2 \frac{\partial^2 u}{\partial y^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} = \frac{\tan^2 u \sin u}{\cos u} = \tan^3 u}$$

08

**1.** (d) Show that :

$$\int_0^{\pi/2} \frac{\sin^2 x\,dx}{\sin x + \cos x} = \frac{1}{\sqrt{2}} \log\left(1+\sqrt{2}\right)$$

**[10]**

$$\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\sin^2 x\,dx}{\sin x + \cos x} \quad \text{--- (1)} \quad = \int_0^{\pi/2} \frac{\sin^2\left(\frac{\pi}{2}-x\right)}{\sin\left(\frac{\pi}{2}-x\right) + \cos\left(\frac{\pi}{2}-x\right)}\,dx$$

$$\Rightarrow I = \int_0^{\pi/2} \frac{\cos^2 x\,dx}{\sin x + \cos x} \quad \text{--- (2)}$$

Adding (1) & (2) —

$$2I = \int_0^{\pi/2} \frac{\sin^2 x + \cos^2 x}{(\sin x + \cos x)} dx = \int_0^{\pi/2} \frac{dx}{\sin x + \cos x}$$

$$= \frac{1}{\sqrt{2}} \int_0^{\pi/2} \frac{dx}{\sin\left(x + \frac{\pi}{4}\right)} = \frac{1}{\sqrt{2}} \left[ \log \tan\left(\frac{x}{2} + \frac{\pi}{8}\right) \right]_0^{\pi/2}$$

$$= \frac{2}{\sqrt{2}} \log \cot \frac{\pi}{8} = \sqrt{2} \log (1 + \sqrt{2})$$

$$\boxed{I = \frac{1}{\sqrt{2}} \log (1 + \sqrt{2})}$$

08

1. (e) P is a point on the plane lx + my + nz = p. A point Q is taken on the line OP such that OP. OQ = $p^2$, prove that the locus of Q is
p(lx + my + nz) = $x^2 + y^2 + z^2$. **[10]**

Let O is Origin.

And line OP $\rightarrow$ $\frac{x}{l'} = \frac{y}{m'} = \frac{z}{n'} = r$

$l', m', n'$ are dcs.

Point on line OP = $(l'r, m'r, n'r)$

If P lies on $lx + my + nz = p$

$$l(l'r) + m(m'r) + n(n'r) = p$$

$$\Rightarrow r = \frac{p}{(ll' + mm' + nn')}$$

$OP = \dfrac{p}{(ll' + mm' + nn')}$ Let $Q = (h, k, t)$

$OP \cdot OQ = p^2$

$\dfrac{p\sqrt{h^2+k^2+t^2}}{(ll'+mm'+nn')} = p^2$

$\Rightarrow h^2+k^2+t^2 = p(ll'+mm'+nn')$

Substituting $l'$ by $x$, $m'$ by $y$, $n'$ by $z$
& $h$ by $x$, $k$ by $y$, $t$ by $z$

~~Share it!~~

08

$\Rightarrow \boxed{x^2+y^2+z^2 = p(lx+my+nz)}$

**2.** (a) (i) Find the kernel and range of the linear operator $T(x, y, z) = (x, y, 0)$ and describe transformation geometrically.

(ii) If $\alpha$, $\beta$ are any scalars, then prove that $A^2 - (\alpha + \beta)A + \alpha\beta I = (A - \alpha I)(A - \beta I)$, where A is any square matrix of order n and $I = I_n$.

**[23]**

**3.** (a) Consider the linear transformation $T(x, y) = (3x + 4y, 5x + 7y)$ of $\mathbf{R}^2 \to \mathbf{R}^2$.

(i) Prove that T is invertible and find the inverse of T.

(ii) Determine the preimage of the vector (1, 2) **[15]**

i) $T: \mathbb{R}^2 \to \mathbb{R}^2 \Rightarrow U = V$

thus T is invertible iff T is one-one.

$\Rightarrow$ T is one-one iff $N(T) = \{(0,0)\}$

Null space

$$\begin{aligned} 3x + 4y &= 0 \\ 5x + 7y &= 0 \end{aligned} \Rightarrow x = 0, y = 0$$

$\Rightarrow$ T is one-one & onto

$\Rightarrow$ T is invertible

Inverse of T

$$a = 3x + 4y$$
$$b = 5x + 7y$$

$$\Rightarrow y = 3b - 5a \text{ and } x = 7a - 4b$$

$$T^{-1}(a, b) = (7a - 4b, 3b - 5a)$$

$$\boxed{T^{-1}(x, y) = (7x - 4y, 3y - 5x)}$$

ii) Pre image of (1,2)

$y = 3b - 5a = 3(2) - 5(1) = 1$

$x = 7a - 4b = 7(1) - 4(2) = -1$

$\boxed{(-1,1)}$

-1.5

3. (b) Show that the following function is discontinuous at (0,0) :

$$f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x-y}, & x \neq y, \\ 0, & x = y \end{cases}.$$ **[10]**

Solution) let $f(x,y) = \begin{cases} \dfrac{x^3+y^3}{x-y} & x \neq y \\ 0 & x = y \end{cases}$

Let choose path $y = x - mx^3$ if $x \neq y$

$f(x,y) = \dfrac{x^3 + (x - mx^3)^3}{mx^3} = \dfrac{1 + (1 - mx^2)^3}{m}$

$$\underset{x\to 0}{Lt}\ \frac{1+(1-mx^2)^3}{m} = \frac{2}{m} \text{ can take}$$

any value.

$$\Rightarrow \boxed{\text{Given } f^n \text{ is discontinuous at } (0,0)}$$

08

3. (c) Evaluate $\lim_{x\to a}\left(2-\frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$ **[07]**

Let $y = \underset{x\to a}{Lt}\left(2-\frac{x}{a}\right)^{\tan\left(\frac{\pi x}{2a}\right)}$

$$y = \underset{x\to a}{Lt}\left(2-\frac{x}{a}\right)^{\tan\frac{\pi x}{2a}}$$

$1^\infty$ form

$$\log y = \underset{x\to a}{Lt}\ \tan\frac{\pi x}{2a}\log\left(2-\frac{x}{a}\right)$$

$$= \underset{x\to a}{Lt}\ \frac{\log\left(2-\frac{x}{a}\right)}{\cot\frac{\pi x}{2a}} \qquad \frac{0}{0} \text{ form}$$

By L Hospital rule. $\log y = \lim_{x \to a} \frac{\left(\frac{-1}{a}\right) / \left(2-\frac{x}{a}\right)}{-\left(\operatorname{cosec}^2 \frac{\pi x}{2a}\right)\frac{\pi}{2a}}$

$= \frac{2}{\pi}$

06

$\boxed{y = e^{2/\pi}}$

**3.** (d) (i) Find the equation of the sphere which touches the sphere $x^2 + y^2 + z^2 - x + 3y + 2z - 3 = 0$ at the point $(1, 1, -1)$ and passes through the origin.

(ii) Prove that the condition that the plane $ux + vy + wz = 0$ may cut the cone $ax^2 + by^2 + cz^2 = 0$ in perpendicular generators if
$(b + c) u^2 + (c + a) v^2 + (a + b) w^2 = 0$ **[18]**

i) tangent plane at $(1,1,-1)$ on sphere
$x^2+y^2+z^2-x+3y+2z-3=0$

$\Rightarrow x+y-z-\frac{(x+1)}{2}+\frac{3(y+1)}{2}+\frac{2(z-1)}{2}-3=0$

$\Rightarrow \frac{x}{2}+\frac{5y}{2}+\cancel{\frac{z}{2}}-3=0$

$\Rightarrow \boxed{x+5y-6=0}$

sphere through $x+5y-6=0$ & $x^2+y^2+z^2-x+3y+2z-3=0$

$\Rightarrow x^2+y^2+z^2-x+3y+2z-3+\lambda(x+5y-6)=0$

It passes through origin

08

$\Rightarrow -3+\lambda(-6)=0 \Rightarrow \lambda=-\frac{1}{2}$

$\Rightarrow \boxed{x^2+y^2+z^2-\frac{3x}{2}+\frac{y}{2}+2z=0}$

ii) generator $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ lies on cone & plane thus $\Rightarrow ul + vm + wn = 0$ & $al^2 + bm^2 + cn^2 = 0$

$$\Rightarrow al^2 + bm^2 + c\left(\frac{ul + vm}{w}\right)^2 = 0$$

$$\Rightarrow l^2(aw^2 + cu^2) + m^2(bw^2 + cv^2) + 2lm(cuv) = 0$$

$$\frac{l_1 l_2}{m_1 m_2} = \frac{bw^2 + cv^2}{aw^2 + cu^2}$$

By symmetry,

$$\frac{l_1 l_2}{bw^2 + cv^2} = \frac{m_1 m_2}{aw^2 + cu^2} = \frac{n_1 n_2}{bu^2 + av^2}$$

-09

$\perp$ if $l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

$$\Rightarrow \boxed{u^2(b+c) + v^2(a+c) + w^2(a+b) = 0}$$

**SECTION – B**

5. (a) Show that the function $h(x) = 4x^2 + 3x - 7$ lies in the space Span {f, g} generated by $f(x) = 2x^2 - 5$ and $g(x) = x + 1$. **[10]**

If $h(x)$ is spanned by $f(x)$ & $g(x)$

$\exists$ $a, b$ s.t.

$$h(x) = af(x) + bg(x)$$

$$\Rightarrow 4x^2 + 3x - 7 = a(2x^2 - 5) + b(x+1)$$

Comparing coefficients of $1, x, x^2$ –

$2a = 4$

$b - 5a = -7 \Rightarrow \boxed{a = 2, b = 3}$

$b = 3$

Thus $h(x) = 4x^2 + 3x - 7$ is spanned by $f(x) = 2x^2 - 5$ & $g(x) = x + 1$

$$\Rightarrow \boxed{h(x) = 2f(x) + 3g(x)}$$

08

**5.** (b) Show that the set {(1, 2, 0, 3), (4, 0, 5, 8), (8, 1, 5, 6)} is linearly independent in $\mathbf{R}^4$. The vectors form a basis for a three-dimensional subspace V of $\mathbf{R}^4$. Construct an orthonormal basis for V. **[10]**

??

Let form a matrix of elements.

$$\begin{bmatrix} 1 & 2 & 0 & 3 \\ 4 & 0 & 5 & 8 \\ 8 & 1 & 5 & 6 \end{bmatrix}$$

$R_2 \to R_2 - 4R_1$, $R_3 \to R_3 - 8R_1$.

$$\Rightarrow \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & -8 & 5 & -4 \\ 0 & -15 & 5 & -18 \end{bmatrix}$$

$R_3 \to R_3 - \frac{15R_2}{8} \Rightarrow \begin{bmatrix} 1 & 2 & 0 & 3 \\ 0 & -8 & 5 & -4 \\ 0 & 0 & -\frac{35}{8} & -\frac{21}{2} \end{bmatrix}$

No. of non-zero rows are 3 in echelon matrix = no. of elements

$\boxed{\text{Set } S \text{ is linearly independent}}$.

Basis of V → $\{(1\ 2\ 0\ 3), (0, -8, 5, -4), (0, 0, 5, 12)\}$

Orthonormal basis → $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -8 \\ 5 \\ -4 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 5 \\ 12 \end{bmatrix}$ Incorrect basis!

02

Use Gram Schmidth process!

5. (c) Applying Lagrange's mean value theorem to the function defined by f(x) = log (1 + x) for all x ≥ 0, show that $0 < [\log(1 + x)]^{-1} - x^{-1} < 1$ whenever x > 0.**[10]**

By Lagrange's MVT –

$$f'(c) = \frac{f(b) - f(a)}{b - a} \quad a < c < b$$

if $f(x)$ is continuous in $[a, b]$

$f(x)$ is differentiable in $(a, b)$

Clearly $\log(1+x)$ is continuous & differentiable in all $x \geq 0$

Let take $I = [0, x]$

$$f'(c) = \frac{1}{1+c}, = \frac{\log(1+x) - \log 1}{x}$$

$$\Rightarrow \quad \frac{\log(1+x)}{x} = \frac{1}{1+c}$$

$$\because \quad 0 < c < x \Rightarrow 1 < 1+c < 1+x \Rightarrow \frac{1}{1+x} < \frac{1}{1+c} < 1$$

$$\Rightarrow \quad \frac{1}{1+x} < \frac{\log(1+x)}{x} < 1$$

$$\Rightarrow \quad \frac{x}{1+x} < \log(1+x) < x$$

-08

$$\Rightarrow \frac{1}{x} < (\log(1+x))^{-1} < \frac{1+x}{x} \Rightarrow \boxed{0 < (\log(1+x))^{-1} - x^{-1} < 1}$$

**5.** (d) A sphere of constant radius 2k passes through origin and meets the axes in A, B and C. Find the locus of the centroid of the tetrahedron OABC. **[10]**

Let locus of tetrahedron centroid

$$= (a, b, c)$$

$$\Rightarrow \quad A = (4a, 0, 0)$$

$$B = (0, 4b, 0) \qquad O = (0,0,0)$$

$$C = (0, 0, 4c)$$

Sphere through O, A, B, C.

$$x^2 + y^2 + z^2 - 4ax - 4by - 4cz = 0$$

radius of sphere = $2k$

$\Rightarrow \sqrt{4a^2+4b^2+4c^2} = 2k$

$\Rightarrow a^2+b^2+c^2 = k^2$

08 $\Rightarrow$ $x^2+y^2+z^2 = k^2$

**5.** (e) Find the equations to the tangent planes to the hyperbolid $2x^2 - 6y^2 + 3z^2 = 5$ which pass through the line $x + 9y - 3z = 0 = 3x - 3y + 6z - 5$. **[10]**

Any plane passing through line

$x+9y-3z=0=3x-3y+6z-5$

is $x+9y-3z+\lambda(3x-3y+6z-5)=0$

$\Rightarrow x(1+3\lambda)+y(9-3\lambda)+z(-3+6\lambda)-5\lambda=0$

Comparing with $lx+my+nz=p$

$l=1+3\lambda$, $m=9-3\lambda$, $n=-3+6\lambda$, $p=5\lambda$

Let hyperboloid $\rightarrow \frac{2x^2}{5} - \frac{6y^2}{5} + \frac{3z^2}{5} = 1$

tangent plane condition for $ax^2+by^2+cz^2=1$ &

$lx+my+nz=p \Rightarrow \frac{l^2}{a}+\frac{m^2}{b}+\frac{n^2}{c}=p^2$

$\Rightarrow \frac{(1+3\lambda)^2}{2} - \frac{(3\lambda-9)^2}{6} + \frac{(6\lambda-3)^2}{3} = \frac{(5\lambda)^2}{5}$

$\Rightarrow 27\lambda^2+18\lambda+3-9\lambda^2-81+54\lambda+72\lambda^2+18-72\lambda = 30\lambda^2$

$\Rightarrow 60\lambda^2-60=0 \Rightarrow \lambda = \pm 1$

08

2 planes $\rightarrow \lambda = 1 \rightarrow 4x+6y+3z=5$

$\lambda=-1 \rightarrow 2x-12y+9z=5$

6. (a) (i) A square matrix A is said to be involutory if $A^2 = I$. Prove that the matrices $\begin{bmatrix} 1 & \alpha \\ 0 & -1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ \alpha & -1 \end{bmatrix}$ are involutory for all scalars $\alpha$.

Determine all $2 \times 2$ involutory matrices.

(ii) Determine the rank of the matrix

$$A = \begin{bmatrix} 1 & 2 & -1 & 0 \\ -1 & 3 & 0 & -4 \\ 2 & 1 & 3 & -2 \\ 1 & 1 & 1 & -1 \end{bmatrix}$$

[10+ =1 ]

**7.** (a) The function f defined by

$$f(x)=\begin{cases} x^2+3x+a, & \text{if } x \le 1 \\ bx+2, & \text{if } x>1 \end{cases}$$

is given to be derivable for every x. Find a and b. **[12]**

$f(x)$ is derivable $\Rightarrow$ $f(x)$ is continuous for every $x$.

$\Rightarrow$ $f(x)$ is continuous at $x=1$

$\Rightarrow$ $1+3+a = b+2$

$\Rightarrow$ $\boxed{b=a+2}$ — 1

If $f$ is derivable for all $x$ -

$$f'(x) = \begin{cases} 2x+3 & x \leq 1 \\ b & x > 1 \end{cases}$$

$$2(1)+3 = b \Rightarrow b = 5$$

$$\Rightarrow \quad a = b-2 = 3$$

$$\boxed{a = 3, b = 5}$$

10

**7.** (b) Determine the values of p and q for which

$$\lim_{x\to 0}\frac{x(1+p\cos x)-q\sin x}{x^3}$$

exists and equals 1. **(10)**

Since $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots$$

$$\lim_{x\to 0} \frac{x\left\{\left(1-\frac{x^2}{2}+\frac{x^4}{4!}\right)p\right\} - q\left(x - \frac{x^3}{3!} - \cdots\right)}{x^3}$$

$$= \lim_{x \to 0} \frac{x(1-a) - \frac{bx^3}{2} + \frac{ax^3}{6} + bx}{x^3}$$

$$= \lim_{x \to 0} \frac{x(1-a+b) + \frac{x^3}{6}(a-3b)}{x^3} = 1$$

$b - a + 1 = 0 \quad \Rightarrow \quad a = b+1$

$\frac{a-3b}{6} = 1 \qquad a = 3b+6$

$$\boxed{b = -5/2,\ a = -3/2}$$

09

**7.** (c) The temperature T at any point (x, y, z) in space is $T = 400\ xyz^2$. Find the highest temperature on the unit sphere $x^2 + y^2 + z^2 = 1$. **[15]**

$F(x, y, z) = 400xyz^2 + \lambda(x^2+y^2+z^2-1)$

$\frac{\partial F}{\partial x} = 0 \Rightarrow 400yz^2 + 2\lambda x = 0$ — (1)

$\frac{\partial F}{\partial y} = 0 \Rightarrow 400xz^2 + 2\lambda y = 0$ — (2)

$\frac{\partial F}{\partial y} = 0 \Rightarrow 400(2xyz) + 2\lambda z = 0$ — (3)

Solving (1), (2), (3) $\Rightarrow x = y,\ z = \sqrt{2}\, y$

$x^2+y^2+2z^2=1$

$\Rightarrow 2y^2+2y^2=1 \quad \Rightarrow \lambda=-100$

$\Rightarrow y=\frac{1}{2},\ x=\frac{1}{2},\ z=\frac{1}{\sqrt{2}}$

$d^2F = dx\left(2\lambda dx + 400 dy\, z^2 + 800 yz\, dz\right)$
$+ dy\left(2\lambda dy + 400 dx\, z^2 + 800 xz\, dz\right)$
$+ dz\left(2\lambda dz + 800 yz\, dx + 800 dy\, xz + 800 dz\, xy\right)$

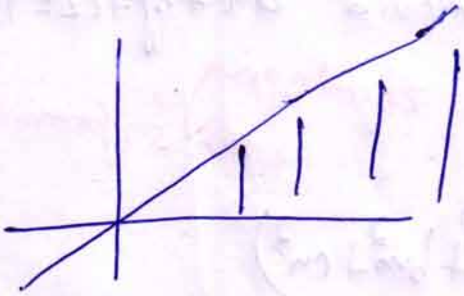
$= -200\left\{(dx)^2+(dy)^2+(dz)^2\right\}$
$+ 200\, dx\, dy + 400\, dx\, dz$
$+ 200\sqrt{2}\, dy\, dz + 200\, dx\, dy$
$+ 200\sqrt{2}\, dy\, dz$
$+ 200\, (dz)^2$
$+ 200\sqrt{2}\, dx\, dz < 0$

13

$\Rightarrow T_{max} = 400\, xyz^2$

$= 400\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} = \boxed{50 \text{ units}}$

7. (d) Evaluate the following integral $\int_0^{\infty}\int_0^{x} xe^{-x^2/y}\,dx\,dy$ by changing the order of integration. **[13]**



Changing order
$x \to y$ to $\infty$
$y - 0$ to $\infty$

$$I = \int_0^{\infty}\int_y^{\infty} x e^{-x^2/y}\,dx\,dy$$

$$= \int_0^{\infty}\left(\frac{-y}{2}e^{-x^2/y}\right)_y^{\infty} dy$$

$$= \int_0^{\infty}\frac{y}{2}e^{-y}\,dy$$

$$= \frac{1}{2}\left(-ye^{-y} - e^{-y}\right)_0^{\infty} = \boxed{\frac{1}{2}}$$

11

$\Rightarrow$ z-axis

**8.** (a) Show that the enveloping cylinder of the conicoid $ax^2 + by^2 + cz^2 = 1$ with generators perpendicular to x-axis meets the plane $z = 0$ in parabolas. **[15]**

Enveloping cylinder of conicoid $ax^2+by^2+cz^2=1$

is $\boxed{SS_1 = T^2}$ $\Leftarrow$ Enveloping cone's formula.

However the question is wrong! and so for scoring is awarded for gracious attempt!

$\Rightarrow (ax^2+by^2+cz^2-1)(al^2+bm^2+cn^2)$
$= (alx+bmy+cnz)^2$

generator $\perp$ to x-axis
$\hookrightarrow l=0$

$\Rightarrow (ax^2+by^2+cz^2-1)(bm^2+cn^2) = (bmy+cnz)^2$

at $x=0$ section -

$\Rightarrow (cz^2+by^2-1)(bm^2+cn^2) = b^2m^2y^2+c^2n^2z^2 + 2bmcnyz$

$\Rightarrow$ ~~$ax^2(bm^2+cn^2)+bcn^2y^2-(bm^2+cn^2)=0$~~

$\Rightarrow bcn^2y^2+bcm^2z^2-2bmcnyz-(bm^2+cn^2)=0$

parabola if $h^2=a'b'$

$h = bmcn$

$a' = bcn^2,\ b' = bcm^2$

$h^2 = a'b' \Rightarrow$ parabolas are section

3

**8.** (b) Find the surface generated by a line which intersects the lines y = a = z and x + 3z = a = y + z and is parallel tot he plane x + y = 0. **[15]**

line intersecting $y-a=0=z-a$

& $x+3z-a=0=y+z-a$

is $y-a+\lambda(z-a)=0=x+3z-a+\mu(y+z-a)=0$

$0x+y+\lambda z-a(\lambda+1)=0=x+\mu y+z(\mu+3)-a(\mu+1)=0$

dcs of line →

$$\frac{l}{\mu+3-\lambda\mu}=\frac{m}{\lambda}=\frac{n}{-1}$$

line is ∥ to plane $x+y=0$

thus –

$\mu+3-\lambda\mu+\lambda=0$

$\Rightarrow \lambda+\mu+3=\lambda\mu$

13

$$\Rightarrow \boxed{\frac{y-a}{z-a}+\frac{x+3z-a}{y+z-a}-3=\frac{(y-a)(x+3z-a)}{(z-a)(y+z-a)}}$$

$\Rightarrow \boxed{\cancel{(y-a)^2+zy-za}}$

can simplify it.

**8.** (c) Show that the surface represented by the equation

$x^2 + y^2 + z^2 - yz - xz - xy - 3x - 6y - 9z + 21 = 0$

is a paraboloid of revolution the coordinates of the focus being (1, 2, 3) and the equations to axis are $x = y - 1 = z - 2$. **[20]**

$$D = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = \begin{vmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{vmatrix}$$

$$= \frac{3}{4} + \frac{1}{2}\left(-\frac{1}{2} - \frac{1}{4}\right) - \frac{1}{2}\left(\frac{1}{4} + \frac{1}{2}\right)$$

$$= \frac{3}{4} - \frac{3}{4} = 0$$

$$|D - \lambda I| = \begin{vmatrix} 1-\lambda & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1-\lambda & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & -\frac{1}{2} \\ -\frac{1}{2} & \frac{3}{2}-\lambda & -\frac{1}{2} \\ -\frac{1}{2} & \lambda-\frac{3}{2} & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -1 & 1-\lambda \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} 1-\lambda & 0 & -\frac{1}{2} \\ -1 & 0 & \frac{1}{2}-\lambda \\ -\frac{1}{2} & -1 & 1-\lambda \end{vmatrix} = 0$$

19

$$(1-\lambda)\left(\frac{1}{2} - \lambda\right) = \frac{1}{2}$$

$$\Rightarrow \lambda = 0, \frac{3}{2}, \frac{3}{2}$$

find $l, m, n$ of axis $\rightarrow \lambda = 0$

$$\begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} l \\ m \\ n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow \quad 2l - m - n = 0, \quad 2m - l - n = 0, \quad 2n - l - m = 0$

$\Rightarrow \quad \boxed{l = m, \ m = n}$

$l, m, n = \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$

$$\cancel{\frac{\partial F/\partial x}{l} = \frac{\partial F/\partial y}{m} = \frac{\partial F/\partial z}{n} = 2k}$$

$\cancel{\Rightarrow 2x - z - y = 0, \ 2y - z - x = 6, \ 2z - y - x = 9}$

$\cancel{\Rightarrow \text{Let } z = 0 \Rightarrow x = y = 3}$

$\cancel{k = ux + vy + wz + d = 21 - \frac{9}{2} - \frac{18}{2} = \frac{15}{2}}$

paraboloid of revolution

$k = ul + vm + wn = -\frac{6\sqrt{3}}{2} = -3\sqrt{3}$

$\frac{3x^2}{2} + \frac{3y^2}{2} = 6\sqrt{3}z$

$\Rightarrow x^2 + y^2 = 4\sqrt{3}z$

$$\frac{\partial F/\partial x}{l} = \frac{\partial F/\partial y}{m} = \frac{\partial F/\partial z}{n} = 2k \Rightarrow x = 0, y = 1, z = 2$$

& $k(lx + my + nz) + ux + vy + wz + d = 0$

Eqn of axis $\rightarrow \frac{x}{1} = \frac{y-1}{1} = \frac{z-2}{1}$

focus lies on axis $\rightarrow (r, r+1, r+2) \Rightarrow OQ = a = (4\sqrt{3}/4) = \sqrt{3}$

$\Rightarrow \quad r^2 + r^2 + r^2 = 3 \Rightarrow r = 1 \Rightarrow \boxed{\text{focus} \rightarrow (1, 2, 3)}$

# ROUGH SPACE

$\int \operatorname{cosec} x \, dx$

$= \log \tan \frac{x}{2}$

$\frac{\frac{1}{2}\sec^2\frac{x}{2}}{\tan\frac{x}{2}}$

$\frac{\frac{1}{2}}{\sin\frac{x}{2}\cos\frac{x}{2}}$

$\frac{\pi}{4}$

$\frac{3\pi}{8}$ $\frac{\pi}{2}-\frac{\pi}{8}$ 5(b)

log

$\log \cot\frac{\pi}{8} - \log \tan\frac{\pi}{8}$

$\log \cot\frac{\pi}{8}$