Set - II

Gradient, Divergence and Curl

§ 1. Partial Derivatives of Vectors.

Suppose **r** is a vector depending on more than one scalar variable. Let $\mathbf{r}=\mathbf{f}(x, y, z)$ *i.e.* let $\mathbf{r}$ be a function of three scalar variables x, y and z. The partial derivative of **r** with respect to x is defined as

$$\frac{\partial \mathbf{r}}{\partial x}=\lim_{\delta x \to 0} \frac{\mathbf{f}(x+\delta x, y, z)-\mathbf{f}(x, y, z)}{\delta x}$$

if this limit exists. Thus $\partial \mathbf{r}/\partial x$ is nothing but the ordinary derivative of **r** with respect to x provided the other variables y and z are regarded as constants. Similarly we may define the partial derivatives $\dfrac{\partial \mathbf{r}}{\partial y}$ and $\dfrac{\partial \mathbf{r}}{\partial z}$.

Higher partial derivatives can also be defined as in Scalar Calculus. Thus, for example,

$$\frac{\partial^2 \mathbf{r}}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial \mathbf{r}}{\partial x}\right), \quad \frac{\partial^2 \mathbf{r}}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial \mathbf{r}}{\partial y}\right), \quad \frac{\partial^2 \mathbf{r}}{\partial z^2}=\frac{\partial}{\partial z}\left(\frac{\partial \mathbf{r}}{\partial z}\right),$$

$$\frac{\partial^2 \mathbf{r}}{\partial x\, \partial y}=\frac{\partial}{\partial x}\left(\frac{\partial \mathbf{r}}{\partial y}\right), \quad \frac{\partial^2 \mathbf{r}}{\partial y\, \partial x}=\frac{\partial}{\partial y}\left(\frac{\partial \mathbf{r}}{\partial x}\right).$$

If **r** has continuous partial derivatives of the second order at least, then, $\dfrac{\partial^2 \mathbf{r}}{\partial x\, \partial y}=\dfrac{\partial^2 \mathbf{r}}{\partial y\, \partial x}$ *i.e.* the order of differentiation is immaterial. If $\mathbf{r}=\mathbf{f}(x, y, z)$ the total differential $d\mathbf{r}$ of **r** is given by

$$d\mathbf{r}=\frac{\partial \mathbf{r}}{\partial x}\, dx+\frac{\partial \mathbf{r}}{\partial y}\, dy+\frac{\partial \mathbf{r}}{\partial z}\, dz.$$

§ 2. The Vector Differential Operator Del. (∇).

The vector differential operator ∇ (read as *del* or *nabla*) is defined as

$$\nabla \equiv \frac{\partial}{\partial x}\, \mathbf{i}+\frac{\partial}{\partial y}\, \mathbf{j}+\frac{\partial}{\partial z}\, \mathbf{k} \equiv \mathbf{i}\, \frac{\partial}{\partial x}+\mathbf{j}\, \frac{\partial}{\partial y}+\mathbf{k}\, \frac{\partial}{\partial z}$$

and operates distributively.

The vector operator ∇ can generally be treated to behave as an ordinary vector. It possesses properties like ordinary vectors. The symbols $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ can be treated as its components along **i**, **j**, **k**.

§ 3. Gradient of a scalar Field. Definition.

Let $f(x, y, z)$ be defined and differentiable at each point (x, y, z) in a certain region of space (i.e., defines a differentiable scalar field). Then the gradient of f, written as ∇f or grad f, is defined as

$$\nabla f=\left(\frac{\partial}{\partial x}\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\frac{\partial}{\partial z}\mathbf{k}\right)f=\frac{\partial f}{\partial x}\mathbf{i}+\frac{\partial f}{\partial y}\mathbf{j}+\frac{\partial f}{\partial z}\mathbf{k}.$$

[Kerala 1975; Allahabad 79]

It should be noted that ∇f is a vector whose three successive components are $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$, and $\frac{\partial f}{\partial z}$. Thus the gradient of a scalar field defines a vector field. *If f a scalar point function, then ∇f is a vector point function.*

§ 4. Formulas involving gradient.

Theorem 1. Gradient of the sum of two scalar point functions. *If f and g are two scalar point functions, then*

$$grad\ (f+g)=grad\ f+grad\ g$$

or

$$\nabla (f+g)=\nabla f+\nabla g.$$

Proof. We have $\nabla (f+g)=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)(f+g)$

$$=\mathbf{i}\frac{\partial}{\partial x}(f+g)+\mathbf{j}\frac{\partial}{\partial y}(f+g)+\mathbf{k}\frac{\partial}{\partial z}(f+g)$$

$$=\mathbf{i}\frac{\partial f}{\partial x}+\mathbf{i}\frac{\partial g}{\partial x}+\mathbf{j}\frac{\partial f}{\partial y}+\mathbf{j}\frac{\partial g}{\partial y}+\mathbf{k}\frac{\partial f}{\partial z}+\mathbf{k}\frac{\partial g}{\partial z}$$

$$=\left(\mathbf{i}\frac{\partial f}{\partial x}+\mathbf{j}\frac{\partial f}{\partial y}+\mathbf{k}\frac{\partial f}{\partial z}\right)+\left(\mathbf{i}\frac{\partial g}{\partial x}+\mathbf{j}\frac{\partial g}{\partial y}+\mathbf{k}\frac{\partial g}{\partial z}\right)$$

$$=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)f+\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)g$$

$$=\nabla f+\nabla g=\text{grad } f+\text{grad } g.$$

Similarly, we can prove that $\nabla (f-g)=\nabla f-\nabla g.$

Theorem 2. Gradient of a constant. *The necessary and sufficient condition for a scalar point function to be constant is that*

$$\nabla f=\mathbf{0}.$$

Proof. If $f(x, y, z)$ is constant, then

$$\frac{\partial f}{\partial x}=0, \frac{\partial f}{\partial y}=0, \frac{\partial f}{\partial z}=0.$$

Therefore $\text{grad } f=\mathbf{i}\frac{\partial f}{\partial x}+\mathbf{j}\frac{\partial f}{\partial y}+\mathbf{k}\frac{\partial f}{\partial z}=0\mathbf{i}+0\mathbf{j}+0\mathbf{k}=\mathbf{0}$.

Hence the condition is necessary.

Conversely, let $\text{grad } f=\mathbf{0}$. Then $\mathbf{i}\frac{\partial f}{\partial x}+\mathbf{j}\frac{\partial f}{\partial y}+\mathbf{k}\frac{\partial f}{\partial z}=\mathbf{0}$.

Therefore $\frac{\partial f}{\partial x}=0, \frac{\partial f}{\partial y}=0, \frac{\partial f}{\partial z}=0.$

$\therefore$ f must be independent of x, y and z.

$\therefore$ f must be a constant. Hence the condition is sufficient.

Theorem 3. Gradient of the product of two scalar point functions. *If f and g are scalar point functions, then*

$$\text{grad}\,(fg)=f\,\text{grad}\,g+g\,\text{grad}\,f$$

or

$$\nabla\,(fg)=f\,\nabla g+g\,\nabla f.$$

[Meerut 1972; Bombay 69]

Proof. We have $\nabla\,(fg)=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)(fg)$

$$=\mathbf{i}\frac{\partial}{\partial x}(fg)+\mathbf{j}\frac{\partial}{\partial y}(fg)+\mathbf{k}\frac{\partial}{\partial z}(fg)$$

$$=\mathbf{i}\left(f\frac{\partial g}{\partial x}+g\frac{\partial f}{\partial x}\right)+\mathbf{j}\left(f\frac{\partial g}{\partial y}+g\frac{\partial f}{\partial y}\right)+\mathbf{k}\left(f\frac{\partial g}{\partial z}+g\frac{\partial f}{\partial z}\right)$$

$$=f\left(\mathbf{i}\frac{\partial g}{\partial x}+\mathbf{j}\frac{\partial g}{\partial y}+\mathbf{k}\frac{\partial g}{\partial z}\right)+g\left(\mathbf{i}\frac{\partial f}{\partial x}+\mathbf{j}\frac{\partial f}{\partial y}+\mathbf{k}\frac{\partial f}{\partial z}\right)$$

$$=f\,\nabla g+g\,\nabla f=f\,\text{grad}\,g+g\,\text{grad}\,f.$$

In particular if c is a constant, then

$$\nabla\,(cf)=c\,\nabla f+f\,\nabla c=c\,\nabla f+\mathbf{0}=c\,\nabla f.$$

Theorem 4. Gradient of the Quotient of two scalar functions. *If f and g are two scalar point functions, then*

$$\nabla\left(\frac{f}{g}\right)=\frac{g\,\nabla f-f\,\nabla g}{g^2}.$$

[Jiwaji 1982]

Proof. We have $\nabla\left(\frac{f}{g}\right)=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\left(\frac{f}{g}\right)$

$$=\mathbf{i}\frac{\partial}{\partial x}\left(\frac{f}{g}\right)+\mathbf{j}\frac{\partial}{\partial y}\left(\frac{f}{g}\right)+\mathbf{k}\frac{\partial}{\partial z}\left(\frac{f}{g}\right).$$

But $\frac{\partial}{\partial x}\left(\frac{f}{g}\right)=\frac{g\frac{\partial f}{\partial x}-f\frac{\partial g}{\partial x}}{g^2}$, $\frac{\partial}{\partial y}\left(\frac{f}{g}\right)=\frac{g\frac{\partial f}{\partial y}-f\frac{\partial g}{\partial y}}{g^2}$, and $\frac{\partial}{\partial z}\left(\frac{f}{g}\right)=\frac{g\frac{\partial f}{\partial z}-f\frac{\partial g}{\partial z}}{g^2}$.

$$\therefore\quad \nabla\left(\frac{f}{g}\right)=\frac{1}{g^2}\left\{\mathbf{i}\left(g\frac{\partial f}{\partial x}-f\frac{\partial g}{\partial x}\right)+\mathbf{j}\left(g\frac{\partial f}{\partial y}-f\frac{\partial g}{\partial y}\right)+\mathbf{k}\left(g\frac{\partial f}{\partial y}-f\frac{\partial g}{\partial z}\right)\right\}$$

$$=\frac{1}{g^2}\left\{g\left(\mathbf{i}\frac{\partial f}{\partial x}+\mathbf{j}\frac{\partial f}{\partial y}+\mathbf{k}\frac{\partial f}{\partial z}\right)-f\left(\mathbf{i}\frac{\partial g}{\partial x}+\mathbf{j}\frac{\partial g}{\partial y}+\mathbf{k}\frac{\partial g}{\partial z}\right)\right\}$$

$$=\frac{1}{g^2}\left\{g\nabla f-f\nabla g\right\}.$$

Solved Examples

Ex. 1. *If* $\mathbf{F}=e^{xy}\,\mathbf{i}+(x-2y)\,\mathbf{j}+x\sin y\,\mathbf{k}$, *calculate*

(i) $\frac{\partial \mathbf{F}}{\partial x}$, (ii) $\frac{\partial \mathbf{F}}{\partial y}$, (iii) $\frac{\partial^2 \mathbf{F}}{\partial x^2}$, (iv) $\frac{\partial^2 \mathbf{F}}{\partial x\,\partial y}$, (v) $\frac{\partial^2 \mathbf{F}}{\partial y^2}$.

Sol. (i) $$\frac{\partial \mathbf{F}}{\partial x}=\left[\frac{\partial}{\partial x}(e^{xy})\right]\mathbf{i}+\left[\frac{\partial}{\partial x}(x-2y)\right]\mathbf{j}+\left[\frac{\partial}{\partial x}(x\sin y)\right]\mathbf{k}$$

$$=(ye^{xy})\,\mathbf{i}+(1)\,\mathbf{j}+(\sin y)\,\mathbf{k}=ye^{xy}\,\mathbf{i}+\mathbf{j}+\sin y\,\mathbf{k}.$$

(ii) $$\frac{\partial \mathbf{F}}{\partial y}=\left[\frac{\partial}{\partial y}(e^{xy})\right]\mathbf{i}+\left[\frac{\partial}{\partial y}(x-2y)\right]\mathbf{j}+\left[\frac{\partial}{\partial y}(x\sin y)\right]\mathbf{k}$$
$$=xe^{xy}\,\mathbf{i}-2\mathbf{j}+x\cos y\,\mathbf{k}.$$

(iii) $$\frac{\partial^2 \mathbf{F}}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial \mathbf{F}}{\partial x}\right)=\frac{\partial}{\partial x}[ye^{xy}\,\mathbf{i}+\mathbf{j}+\sin y\,\mathbf{k}]$$
$$=\left[\frac{\partial}{\partial x}(ye^{xy})\right]\mathbf{i}+\frac{\partial}{\partial x}\mathbf{j}+\left[\frac{\partial}{\partial x}(\sin y)\right]\mathbf{k}$$
$$=y^2e^{xy}\,\mathbf{i}+0+0\,\mathbf{k}=y^2e^{xy}\,\mathbf{i}.$$

(iv) $$\frac{\partial^2 \mathbf{F}}{\partial x\,\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial \mathbf{F}}{\partial x}\right)=\frac{\partial}{\partial y}[ye^{xy}\,\mathbf{i}+\mathbf{j}+\sin y\,\mathbf{k}]$$
$$=\left[\frac{\partial}{\partial y}(ye^{xy})\right]\mathbf{i}+\frac{\partial}{\partial y}\mathbf{j}+\left(\frac{\partial}{\partial y}\sin y\right)\mathbf{k}$$
$$=(e^{xy}+xye^{xy})\,\mathbf{i}+0+\cos y\,\mathbf{k}$$
$$=e^{xy}(xy+1)\,\mathbf{i}+\cos y\,\mathbf{k}.$$

(v) $$\frac{\partial^2 \mathbf{F}}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial \mathbf{F}}{\partial y}\right)=\frac{\partial}{\partial y}[xe^{xy}\,\mathbf{i}-2\mathbf{j}+x\cos y\,\mathbf{k}]$$
$$=\left[\frac{\partial}{\partial y}(xe^{xy})\right]\mathbf{i}+\frac{\partial}{\partial y}(-2\mathbf{j})+\left[\frac{\partial}{\partial y}(x\cos y)\right]\mathbf{k}$$
$$=x^2e^{xy}\,\mathbf{i}+0-x\sin y\,\mathbf{k}=x^2e^{xy}\,\mathbf{i}-x\sin y\,\mathbf{k}.$$

Ex. 2. *If* $\phi(x, y, z)=xy^2z$ *and* $\mathbf{f}=xz\,\mathbf{i}-xy\,\mathbf{j}+yz^2\,\mathbf{k}$, *show that* $\dfrac{\partial^3}{\partial x^2\,\partial z}(\phi\,\mathbf{f})$ *at* $(2, -1, 1)$ *is* $4\,\mathbf{i}+2\,\mathbf{j}$. **[Garhwal 1985]**

Sol. We have $\phi\,\mathbf{f}=xy^2z(xz\,\mathbf{i}-xy\,\mathbf{j}+yz^2\,\mathbf{k})$

$$=x^2y^3z^2\,\mathbf{i}-x^2y^3z\,\mathbf{j}+xy^3z^3\,\mathbf{k}.$$

$$\therefore\quad \frac{\partial}{\partial x}(\phi\mathbf{f})=2xy^2z^2\,\mathbf{i}-2xy^3z\,\mathbf{j}+y^3z^3\,\mathbf{k},$$

$$\frac{\partial^2}{\partial x^2}(\phi\mathbf{f})=\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(\phi\mathbf{f})\right]=\frac{\partial}{\partial x}[2xy^2z^2\,\mathbf{i}-2xy^3z\,\mathbf{j}+y^3z^3\,\mathbf{k}]$$

$$=2y^2z^2\,\mathbf{i}-2y^3z\,\mathbf{j}+0\mathbf{k}=2y^2z^2\,\mathbf{i}-2y^3z\,\mathbf{j}$$

and $$\frac{\partial^3}{\partial x^2\,\partial z}(\phi\mathbf{f})=\frac{\partial}{\partial z}\left[\frac{\partial^2}{\partial x^2}(\phi\mathbf{f})\right]=\frac{\partial}{\partial z}(2y^2z^2\,\mathbf{i}-2y^3z\,\mathbf{j})$$

$$=4y^2z\,\mathbf{i}-2y^3\,\mathbf{j}.$$

$\therefore$ at the point $(2, -1, 1)$, $\dfrac{\partial^3}{\partial x^2\,\partial z}(\phi\mathbf{f})=4.(-1)^2\cdot 1\,\mathbf{i}$

$$-2\cdot(-1)^3\,\mathbf{j}=4\,\mathbf{i}+2\,\mathbf{j}.$$

Ex. 3. *If* $\mathbf{f}=(2x^2y-x^4)\,\mathbf{i}+(e^{xy}-y\sin x)\,\mathbf{j}+x^2\cos y\,\mathbf{k}$, *verify that*

$$\frac{\partial^2\mathbf{f}}{\partial y\,\partial x}=\frac{\partial^2\mathbf{f}}{\partial x\,\partial y}.$$ **[Agra 1981]**

Sol. We have $$\frac{\partial\mathbf{f}}{\partial x}=\left[\frac{\partial}{\partial x}(2x^2y-x^4)\right]\mathbf{i}+\left[\frac{\partial}{\partial x}(e^{xy}-y\sin x)\right]\mathbf{j}+\left[\frac{\partial}{\partial x}(x^2\cos y)\right]\mathbf{k}$$

$$=(4xy-4x^3)\,\mathbf{i}+(ye^{xy}-y\cos x)\,\mathbf{j}+2x\cos y\,\mathbf{k}.$$

$$\therefore\quad \frac{\partial^2\mathbf{f}}{\partial x\,\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial\mathbf{f}}{\partial x}\right)=\frac{\partial}{\partial y}[(4xy-4x^3)\,\mathbf{i}+(ye^{xy}-y\cos x)\,\mathbf{j}+2x\cos y\,\mathbf{k}]$$

$$=\left[\frac{\partial}{\partial y}(4xy-4x^3)\right]\mathbf{i}+\left[\frac{\partial}{\partial y}(ye^{xy}-y\cos x)\right]\mathbf{j}+\left[\frac{\partial}{\partial y}(2x\cos y)\right]\mathbf{k}$$

$$=(4x)\,\mathbf{i}+(e^{xy}+xy\,e^{xy}-\cos x)\,\mathbf{j}+(-2x\sin y)\,\mathbf{k}$$

$$=4x\,\mathbf{i}+(e^{xy}+xy\,e^{xy}-\cos x)\,\mathbf{j}-2x\sin y\,\mathbf{k}. \quad \text{...(1)}$$

Again $$\frac{\partial\mathbf{f}}{\partial y}=\left[\frac{\partial}{\partial y}(2x^2y-x^4)\right]\mathbf{j}+\left[\frac{\partial}{\partial y}(e^{xy}-y\sin x)\right]\mathbf{j}+\left[\frac{\partial}{\partial y}(x^2\cos y)\right]\mathbf{k}$$

$$=2x^2\,\mathbf{i}+(xe^{xy}-\sin x)\,\mathbf{j}-x^2\sin y\,\mathbf{k}.$$

$$\therefore \quad \frac{\partial^2 \mathbf{f}}{\partial y\, \partial x} = \frac{\partial}{\partial x}\left(\frac{\partial \mathbf{f}}{\partial y}\right) = \left[\frac{\partial}{\partial x}(2x^2)\right]\mathbf{i} + \left[\frac{\partial}{\partial x}(xe^{xy} - \sin x)\right]\mathbf{j} - \left[\frac{\partial}{\partial x}(x^2 \sin y)\right]\mathbf{k}$$

$$= 4x\,\mathbf{i} + (e^{xy} + xye^{xy} - \cos x)\,\mathbf{j} - 2x \sin y\,\mathbf{k}. \qquad \ldots(2)$$

From (1) and (2), we have $\dfrac{\partial^2 \mathbf{f}}{\partial x\, \partial y} = \dfrac{\partial^2 \mathbf{f}}{\partial y\, \partial x}$.

Ex. 4. *If* $\mathbf{u} = xyz\,\mathbf{i} + xz^2\,\mathbf{j} - y^3\,\mathbf{k}$ *and* $\mathbf{v} = x^3\,\mathbf{i} - xyz\,\mathbf{j} + x^2 z\,\mathbf{k}$, *calculate* $\dfrac{\partial^2 \mathbf{u}}{\partial y^2} \times \dfrac{\partial^2 \mathbf{v}}{\partial x^2}$ *at the point* (1, 1, 0)

Sol. We have $\dfrac{\partial \mathbf{u}}{\partial y} = \left[\dfrac{\partial}{\partial y}(xyz)\right]\mathbf{i} + \left[\dfrac{\partial}{\partial y}(xz^2)\right]\mathbf{j} - \left[\dfrac{\partial}{\partial y}(y^3)\right]\mathbf{k}$

$$= xz\,\mathbf{i} + 0\,\mathbf{j} - 3y^2\,\mathbf{k} = xz\,\mathbf{i} - 3y^2\,\mathbf{k}$$

and $\dfrac{\partial^2 \mathbf{u}}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial \mathbf{u}}{\partial y}\right) = 0\,\mathbf{i} - 6y\,\mathbf{k} = -6y\,\mathbf{k}$.

Again $\dfrac{\partial \mathbf{v}}{\partial x} = \left[\dfrac{\partial}{\partial x}(x^3)\right]\mathbf{i} - \left[\dfrac{\partial}{\partial x}(xyz)\right]\mathbf{j} + \left[\dfrac{\partial}{\partial x}(x^2 z)\right]\mathbf{k}$

$$= 3x^2\,\mathbf{i} - yz\,\mathbf{j} + 2xz\,\mathbf{k}$$

and $\dfrac{\partial^2 \mathbf{v}}{\partial x^2} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial \mathbf{v}}{\partial x}\right) = 6x\,\mathbf{i} - 0\,\mathbf{j} + 2z\,\mathbf{k} = 6x\,\mathbf{i} + 2z\mathbf{k}$.

$$\therefore \quad \frac{\partial^2 \mathbf{u}}{\partial y^2} \times \frac{\partial^2 \mathbf{v}}{\partial x^2} = (-6y\mathbf{k}) \times (6x\mathbf{i} + 2z\mathbf{k})$$

$$= -36xy\,\mathbf{k} \times \mathbf{i} - 12yz\,\mathbf{k} \times \mathbf{k}$$

$$= -36xy\mathbf{j} \qquad [\because \ \mathbf{k} \times \mathbf{i} = \mathbf{j} \text{ and } \mathbf{k} \times \mathbf{k} = \mathbf{0}]$$

$\therefore$ at the point (1, 1, 0), we have $\dfrac{\partial^2 \mathbf{u}}{\partial y^2} \times \dfrac{\partial^2 \mathbf{v}}{\partial x^2} = -36\mathbf{j}$.

IAS-2012

Ex. 5. *If* $\mathbf{A} = x^2 yz\,\mathbf{i} - 2xz^3\,\mathbf{j} + xz^2\,\mathbf{k}$, $\mathbf{B} = 2z\,\mathbf{i} + y\,\mathbf{j} - x^2\,\mathbf{k}$, *find the value of* $\dfrac{\partial^2}{\partial x \partial y}(\mathbf{A} \times \mathbf{B})$ *at* (1, 0, −2). **[Kanpur 1985, 81]**

Sol. We have $\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x^2 yz & -2xz^3 & xz^2 \\ 2z & y & -x^2 \end{vmatrix}$

$$= (2x^3 z^3 - xyz^2)\,\mathbf{i} + (2xz^3 + x^4 yz)\,\mathbf{j} + (x^2 y^2 z + 4xz^4)\,\mathbf{k}.$$

$$\therefore \quad \frac{\partial}{\partial y}(\mathbf{A} \times \mathbf{B}) = -xz^2\,\mathbf{i} + x^4 z\,\mathbf{j} + 2x^2 yz\,\mathbf{k}.$$

Again $\dfrac{\partial^2}{\partial x \partial y}(\mathbf{A} \times \mathbf{B}) = \dfrac{\partial}{\partial x}\left\{\dfrac{\partial}{\partial y}(\mathbf{A} \times \mathbf{B})\right\}$

$$=-z^2\,\mathbf{i}+4x^3z\,\mathbf{j}+4xyz\,\mathbf{k}. \qquad \ldots(1)$$

Putting $x=1, y=0$ and $z=-2$ in (1), we get the required derivative at the point $(1, 0, -2)=-4\mathbf{i}-8\mathbf{j}$.

Ex. 6. *If $f(x, y, z)=3x^2y-y^3z^2$, find grad f at the point $(1, -2, -1)$.* **[Agra 1978; Rohilkhand 83]**

Sol. We have

$$\text{grad } f=\nabla f=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)(3x^2y-y^3z^2)$$

$$=\mathbf{i}\frac{\partial}{\partial x}(3x^2y-y^3z^2)+\mathbf{j}\frac{\partial}{\partial y}(3x^2y-y^3z^2)+\mathbf{k}\frac{\partial}{\partial z}(3x^2y-y^3z^2)$$

$$=\mathbf{i}(6xy)+\mathbf{j}(3x^2-3y^2z^2)+\mathbf{k}(-2y^3z)$$

$$=6xy\,\mathbf{i}+(3x^2-3y^2z^2)\,\mathbf{j}-2y^3z\mathbf{k}.$$

Putting $x=1, y=-2, z=-1$, we get

$$\nabla f=6(1)(-2)\,\mathbf{i}+\{3(1)^2-3(-2)^2(-1)^2\}\,\mathbf{j}-2(-2)^3(-1)\,\mathbf{k}$$

$$=-12\mathbf{i}-9\mathbf{j}-16\mathbf{k}$$

Ex. 7. *If $r=|\mathbf{r}|$ where $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, prove that*

(i) $\nabla f(r)=f'(r)\nabla r$, (ii) $\nabla r=\frac{1}{r}\mathbf{r}$, **[Rohilkhand 1984]**

(iii) $\nabla f(r)\times\mathbf{r}=0$, (iv) $\nabla\left(\frac{1}{r}\right)=-\frac{\mathbf{r}}{r^3}$, 2007 **[Meerut 1991]**

(v) $\nabla \log|\mathbf{r}|=\frac{\mathbf{r}}{r^2}$, **[Meerut 1991; Kanpur 88]**

(vi) $\nabla r^n=nr^{n-2}\mathbf{r}$. **[Kanpur 1986; Agra 86; Rohilkhand 90; Garhwal 84]**

Sol. If $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, then $r=|\mathbf{r}|=\sqrt{(x^2+y^2+z^2)}$.

$$\therefore \quad r^2=x^2+y^2+z^2.$$

(i) $$\nabla f(r)=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)f(r)$$

$$=\mathbf{i}\frac{\partial}{\partial x}f(r)+\mathbf{j}\frac{\partial}{\partial y}f(r)+\mathbf{k}\frac{\partial}{\partial z}f(r)$$

$$=\mathbf{i}f'(r)\frac{\partial r}{\partial x}+\mathbf{j}f'(r)\frac{\partial r}{\partial y}+\mathbf{k}f'(r)\frac{\partial r}{\partial z}$$

$$=f'(r)\left(\mathbf{i}\frac{\partial r}{\partial x}+\mathbf{j}\frac{\partial r}{\partial y}+\mathbf{k}\frac{\partial r}{\partial z}\right)=f'(r)\nabla r.$$

(ii) We have $\nabla r=\mathbf{i}\frac{\partial r}{\partial x}+\mathbf{j}\frac{\partial r}{\partial y}+\mathbf{k}\frac{\partial r}{\partial z}$.

Now $r^2=x^2+y^2+z^2$; $\therefore \quad 2r\frac{\partial r}{\partial x}=2x$ *i.e.*, $\frac{\partial r}{\partial x}=\frac{x}{r}$.

Similarly $\frac{\partial r}{\partial y}=\frac{y}{r}$ and $\frac{\partial r}{\partial z}=\frac{z}{r}$.

$\therefore \quad \nabla r=\frac{x}{r}\,\mathbf{i}+\frac{y}{r}\,\mathbf{j}+\frac{z}{r}\,\mathbf{k}=\frac{1}{r}\,(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})=\frac{1}{r}\,\mathbf{r}=\hat{\mathbf{r}}.$

(iii) We have as in part (i), $\nabla f(r)=f'(r)\,\nabla r$.

But as in part (ii) $\nabla r=\frac{1}{r}\,\mathbf{r}$.

$$\therefore \quad \nabla f(r)=f'(r)\,\frac{1}{r}\,\mathbf{r}.$$

$\therefore \nabla f(r)\times\mathbf{r}=\left\{f'(r)\,\frac{1}{r}\,\mathbf{r}\right\}\times\mathbf{r}=\left\{\frac{1}{r}\,f'(r)\right\}(\mathbf{r}\times\mathbf{r})$

$=\mathbf{0}$, since $\mathbf{r}\times\mathbf{r}=\mathbf{0}$.

(iv) We have $\nabla\left(\frac{1}{r}\right)=\mathbf{i}\,\frac{\partial}{\partial x}\left(\frac{1}{r}\right)+\mathbf{j}\,\frac{\partial}{\partial y}\left(\frac{1}{r}\right)+\mathbf{k}\,\frac{\partial}{\partial z}\left(\frac{1}{r}\right)$

$=\mathbf{i}\,\left(-\frac{1}{r^2}\frac{\partial r}{\partial x}\right)+\mathbf{j}\,\left(-\frac{1}{r^2}\frac{\partial r}{\partial y}\right)+\mathbf{k}\left(-\frac{1}{r^2}\frac{\partial r}{\partial z}\right)$

$=-\frac{1}{r^2}\left(\frac{\partial r}{\partial x}\,\mathbf{i}+\frac{\partial r}{\partial y}\,\mathbf{j}+\frac{\partial r}{\partial z}\,\mathbf{k}\right)$

$=-\frac{1}{r^2}\left(\frac{x}{r}\,\mathbf{i}+\frac{y}{r}\,\mathbf{j}+\frac{z}{r}\,\mathbf{k}\right)$ [see part (ii)]

$=-\frac{1}{r^3}\,(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})=-\frac{1}{r^3}\,\mathbf{r}.$

(v) We have $\nabla\log|\mathbf{r}|=\nabla\log r$

$=\mathbf{i}\,\frac{\partial}{\partial x}\log r+\mathbf{j}\frac{\partial}{\partial y}\log r+\mathbf{k}\,\frac{\partial}{\partial z}\log r$

$=\frac{1}{r}\,\frac{\partial r}{\partial x}\,\mathbf{i}+\frac{1}{r}\frac{\partial r}{\partial y}\,\mathbf{j}+\frac{1}{r}\frac{\partial r}{\partial z}\mathbf{k}=\frac{1}{r}\left(\frac{x}{r}\,\mathbf{i}+\frac{y}{r}\,\mathbf{j}+\frac{z}{r}\,\mathbf{k}\right)$

$=\frac{1}{r^2}\,(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})=\frac{1}{r^2}\,\mathbf{r}.$

(vi) We have $\nabla r^n=\mathbf{i}\,\frac{\partial}{\partial x}\,r^n+\mathbf{j}\,\frac{\partial}{\partial y}\,r^n+\mathbf{k}\,\frac{\partial}{\partial z}\,r^n$

$=\mathbf{i}\,nr^{n-1}\,\frac{\partial r}{\partial x}+\mathbf{j}\,nr^{n-1}\frac{\partial r}{\partial y}+\mathbf{k}\,nr^{n-1}\,\frac{\partial r}{\partial z}=nr^{n-1}\left(\mathbf{i}\,\frac{\partial r}{\partial x}+\mathbf{j}\,\frac{\partial r}{\partial y}+\mathbf{k}\,\frac{\partial r}{\partial z}\right)$

$=nr^{n-1}\,\nabla r$

$=nr^{n-1}\,\frac{1}{r}\,\mathbf{r}$ $\quad\left[\because\quad \nabla r=\frac{\mathbf{r}}{r}\text{ as in part (ii)}\right]$

$=nr^{n-2}\,\mathbf{r}.$

Ex. 8. *Prove that* $f(u)\,\nabla u=\nabla\int f(u)\,du$.

Sol. We have $\nabla \int f(u)\, du$

$$=\Sigma\, \mathbf{i}\, \frac{\partial}{\partial x}\left\{\int f(u)\, du\right\}$$ [by def. of gradient]

$$=\Sigma\, \mathbf{i}\left\{\frac{d}{du}\int f(u)\, du\right\}\frac{\partial u}{\partial x}=\Sigma \mathbf{i}\, f(u)\frac{\partial u}{\partial x}=f(u)\,\Sigma\, \mathbf{i}\,\frac{\partial u}{\partial x}=f(u)\,\nabla u.$$

Ex. 9. *Show that*

(*i*) *grad* $(\mathbf{r}\cdot\mathbf{a})=\mathbf{a}$, (*ii*) *grad* $[\mathbf{r},\ \mathbf{a},\ \mathbf{b}]=\mathbf{a}\times\mathbf{b}$,

where **a** *and* **b** *are constant vectors.* **[Rohilkhand 1981; Kanpur 87]**

Sol. (i) Let $\mathbf{a}=a_1\,\mathbf{i}+a_2\,\mathbf{j}+a_3\,\mathbf{k}$. Then a_1, a_2, a_3 are constants. Also $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$.

$\therefore$ $\mathbf{r}\cdot\mathbf{a}=a_1x+a_2y+a_3z.$

$\therefore$ grad $(\mathbf{r}\cdot\mathbf{a})=\nabla\,(\mathbf{r}\cdot\mathbf{a})=\nabla\,(a_1x+a_2y+a_3z)$

$$=\mathbf{i}\,\frac{\partial}{\partial x}(a_1x+a_2y+a_3z)+\mathbf{j}\,\frac{\partial}{\partial y}(a_1x+a_2y+a_3z)+\mathbf{k}\,\frac{\partial}{\partial z}(a_1x+a_2y+a_3z)$$

$$=a_1\,\mathbf{i}+a_2\,\mathbf{j}+a_3\,\mathbf{k}=\mathbf{a}.$$

(ii) grad $[\mathbf{r},\ \mathbf{a},\ \mathbf{b}]=$ grad $\{\mathbf{r}\cdot(\mathbf{a}\times\mathbf{b})\}$, where $\mathbf{a}\times\mathbf{b}$ is a constant vector

$=\mathbf{a}\times\mathbf{b}$ as in part (i).

Ex. 10. *If* $\phi\,(x, y, z)=x^2y+y^2x+z^2$, *find* $\nabla\phi$ *at the point* $(1, 1, 1)$. **[Agra 1979]**

Sol. We have $\nabla\phi=\dfrac{\partial\phi}{\partial x}\,\mathbf{i}+\dfrac{\partial\phi}{\partial y}\,\mathbf{j}+\dfrac{\partial\phi}{\partial z}\,\mathbf{k}$

$$=\left[\frac{\partial}{\partial x}(x^2y+y^2x+z^2)\right]\mathbf{i}+\left[\frac{\partial}{\partial y}(x^2y+y^2x+z^2)\right]\mathbf{j}+\left[\frac{\partial}{\partial z}(x^2y+y^2x+z^2)\right]\mathbf{k}$$

$$=(2xy+y^2)\,\mathbf{i}+(x^2+2xy)\,\mathbf{j}+2z\,\mathbf{k}.$$

Putting $x=1$, $y=1$, $z=1$, we get

$\nabla\phi$ at the point $(1, 1, 1)=3\mathbf{i}+3\mathbf{j}+2\mathbf{k}.$

Ex. 11. *Find grad f, where f is given by*

$f=x^3-y^3+xz^2$, *at the point* $(1, -1, 2)$. **[Agra 1977]**

Sol. We have grad $f=\nabla f=\dfrac{\partial f}{\partial x}\,\mathbf{i}+\dfrac{\partial f}{\partial y}\,\mathbf{j}+\dfrac{\partial f}{\partial z}\,\mathbf{k}$

$$=\left[\frac{\partial}{\partial x}(x^3-y^3+xz^2)\right]\mathbf{i}+\left[\frac{\partial}{\partial y}(x^3-y^3+xz^2)\right]\mathbf{j}+\left[\frac{\partial}{\partial z}(x^3-y^3+xz^2)\right]\mathbf{k}$$

$$=(3x^2+z^2)\,\mathbf{i}+(-3y^2)\,\mathbf{j}+2xz\,\mathbf{k}.$$

Putting $x=1$, $y=-1$, $z=2$, we get

∇f at the point $(1, -1, 2) = 7\,\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$.

S 2011

Ex. 12 *If* $u = x + y + z$, $v = x^2 + y^2 + z^2$, $w = yz + zx + xy$, *prove that*

$(grad\ u) \cdot [(grad\ v) \times (grad\ w)] = 0.$ [Kohlapur 1978]

Sol. We have grad $u = \frac{\partial u}{\partial x}\mathbf{i} + \frac{\partial u}{\partial y}\mathbf{j} + \frac{\partial u}{\partial z}\mathbf{k}$

$= 1\,\mathbf{i} + 1\,\mathbf{j} + 1\,\mathbf{k} = \mathbf{i} + \mathbf{j} + \mathbf{k},$

grad $v = \frac{\partial v}{\partial x}\mathbf{i} + \frac{\partial v}{\partial y}\mathbf{j} + \frac{\partial v}{\partial z}\mathbf{k} = 2x\,\mathbf{i} + 2y\,\mathbf{j} + 2z\,\mathbf{k}$

and grad $w = \frac{\partial w}{\partial x}\mathbf{i} + \frac{\partial w}{\partial y}\mathbf{j} + \frac{\partial w}{\partial z}\mathbf{k}$

$= (y+z)\,\mathbf{i} + (z+x)\,\mathbf{j} + (x+y)\,\mathbf{k}.$

$\therefore$ grad $u \cdot [(\text{grad } v) \times (\text{grad } w)]$ = scalar triple product of the vectors grad u, grad v and grad w

$$= \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & z+x & x+y \end{vmatrix} = 2 \begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ y+z & z+x & x+y \end{vmatrix}$$

$$= 2 \begin{vmatrix} 1 & 1 & 1 \\ x+y+z & x+y+z & x+y+z \\ y+z & z+x & x+y \end{vmatrix}, \text{ by } R_2 + R_3$$

$$= 2(x+y+z) \begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ y+z & z+x & x+y \end{vmatrix}$$

$= 2(x+y+z).0,$

the first two rows of the determinant being identical

$= 0.$

Ex. 13. *If* $\mathbf{F} = \left(y\frac{\partial f}{\partial z} - z\frac{\partial f}{\partial y}\right)\mathbf{i} + \left(z\frac{\partial f}{\partial x} - x\frac{\partial f}{\partial z}\right)\mathbf{j} + \left(x\frac{\partial f}{\partial y} - y\frac{\partial f}{\partial x}\right)\mathbf{k},$

prove that

(*i*) $\mathbf{F} = \mathbf{r} \times \nabla f$, (*ii*) $\mathbf{F} \cdot \mathbf{r} = 0$, (*iii*) $\mathbf{F} \cdot \nabla f = 0$.

Sol. We have $\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$ and $\mathbf{r} = x\mathbf{i} + y\,\mathbf{j} + z\mathbf{k}$

(i) $$\mathbf{r}\times\nabla f=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ x & y & z\\ \partial f/\partial x & \partial f/\partial y & \partial f/\partial z\end{vmatrix}$$

$$=\left(y\frac{\partial f}{\partial z}-z\frac{\partial f}{\partial y}\right)\mathbf{i}+\left(z\frac{\partial f}{\partial x}-x\frac{\partial f}{\partial z}\right)\mathbf{j}+\left(x\frac{\partial f}{\partial y}-y\frac{\partial f}{\partial x}\right)\mathbf{k}=\mathbf{F}$$

(ii) $\mathbf{F}\cdot\mathbf{r}=(\mathbf{r}\times\nabla f)\cdot\mathbf{r}$ [$\because$ $\mathbf{F}=\mathbf{r}\times\nabla f$]

$=0$, because the value of a scalar triple product having two vectors equal is zero.

(iii) $\mathbf{F}\cdot\nabla f=(\mathbf{r}\times\nabla f)\cdot\nabla f=[\mathbf{r},\nabla f,\nabla f]$

$=0$, because the value of a scalar triple product having two vectors equal is zero.

Ex. 14. *Prove that* $\mathbf{A}\cdot\left(\nabla\frac{1}{r}\right)=-\frac{\mathbf{A}\cdot\mathbf{r}}{r^3}$.

Sol. First prove that $\nabla\frac{1}{r}=-\frac{\mathbf{r}}{r^3}$.

[For its complete solution see Ex. 7, part (iv)]

$$\therefore\quad \mathbf{A}\cdot\nabla\left(\frac{1}{r}\right)=\mathbf{A}\cdot\left(-\frac{\mathbf{r}}{r^3}\right)=-\frac{\mathbf{A}\cdot\mathbf{r}}{r^3}.$$

Ex. 15. *Prove that* $\nabla r^{-3}=-3r^{-5}\,\mathbf{r}$. **[Agra 1974]**

Sol. We have

$$\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}\ \text{ and }\ r=|\mathbf{r}|=\sqrt{(x^2+y^2+z^2)}$$

so that $$r^2=x^2+y^2+z^2.$$

Now $$\nabla r^{-3}=\left(\frac{\partial}{\partial x}r^{-3}\right)\mathbf{i}+\left(\frac{\partial}{\partial y}r^{-3}\right)\mathbf{j}+\left(\frac{\partial}{\partial z}r^{-3}\right)\mathbf{k}$$

$$=-3r^{-4}\frac{\partial r}{\partial x}\mathbf{i}-3r^{-4}\frac{\partial r}{\partial y}-3r^{-4}\frac{\partial r}{\partial z}\mathbf{k}$$

$$=-3r^{-4}\left(\frac{\partial r}{\partial x}\mathbf{i}+\frac{\partial r}{\partial y}\mathbf{j}+\frac{\partial r}{\partial z}\mathbf{k}\right).\quad\ldots(1)$$

Differentiating both sides of $r^2=x^2+y^2+z^2$ partially w.r.t. x, we have

$$2r\frac{\partial r}{\partial x}=2x\quad\text{or}\quad\frac{\partial r}{\partial x}=\frac{x}{r}.$$

Similarly $\frac{\partial r}{\partial y}=\frac{y}{r}$ and $\frac{\partial r}{\partial z}=\frac{z}{r}$.

$\therefore$ from (1), we have

$$\nabla r^{-3} = -3r^{-4}\left(\frac{x}{r}\,\mathbf{i} + \frac{y}{r}\,\mathbf{j} + \frac{z}{r}\,\mathbf{k}\right)$$

$$= -3r^{-5}(x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = -3r^{-5}\,\mathbf{r}.$$

Ex. 16. *Prove that* $\nabla\phi \cdot d\mathbf{r} = d\phi$.

Sol. We have $\nabla\phi = \dfrac{\partial\phi}{\partial x}\,\mathbf{i} + \dfrac{\partial\phi}{\partial y}\,\mathbf{j} + \dfrac{\partial\phi}{\partial z}\,\mathbf{k}$. ...(1)

Also $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \Rightarrow d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k}$. ...(2)

From (1) and (2), $\nabla\phi \cdot d\mathbf{r}$

$$= \left(\frac{\partial\phi}{\partial x}\,\mathbf{i} + \frac{\partial\phi}{\partial y}\,\mathbf{j} + \frac{\partial\phi}{\partial z}\,\mathbf{k}.\right)\cdot(dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k})$$

$$= \frac{\partial\phi}{\partial x}\,dx + \frac{\partial\phi}{\partial y}\,dy + \frac{\partial\phi}{\partial z}\,dz = d\phi.$$

Ex. 17. *Show that*

$$\frac{d\phi}{ds} = \nabla\phi \cdot \frac{d\mathbf{r}}{ds},\ \textit{where}\ \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

and ϕ *is a function of* x, y *and* z.

Sol. We have $\nabla\phi = \dfrac{\partial\phi}{\partial x}\,\mathbf{i} + \dfrac{\partial\phi}{\partial y}\,\mathbf{j} + \dfrac{\partial\phi}{\partial z}\,\mathbf{k}$ ...(1)

Also $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \Rightarrow \dfrac{d\mathbf{r}}{ds} = \dfrac{dx}{ds}\,\mathbf{i} + \dfrac{dy}{ds}\,\mathbf{j} + \dfrac{dz}{ds}\,\mathbf{k}$. ...(2)

From (1) and (2), we have

$$\nabla\phi \cdot \frac{d\mathbf{r}}{ds} = \left(\frac{\partial\phi}{\partial x}\,\mathbf{i} + \frac{\partial\phi}{\partial y}\,\mathbf{j} + \frac{\partial\phi}{\partial z}\,\mathbf{k}\right)\cdot\left(\frac{dx}{ds}\,\mathbf{i} + \frac{dy}{ds}\,\mathbf{j} + \frac{dz}{ds}\,\mathbf{k}\right)$$

$$= \frac{\partial\phi}{\partial x}\frac{dx}{ds} + \frac{\partial\phi}{\partial y}\frac{dy}{ds} + \frac{\partial\phi}{\partial z}\frac{dz}{ds} = \frac{d\phi}{ds}.$$

Ex. 18. ρ *and* p *are two scalar point functions such that* ρ *is a function of* p; *show that*

$$\nabla\rho = \frac{d\rho}{dp}\,\nabla p.$$

Sol. We have $\nabla\rho = \dfrac{\partial\rho}{\partial x}\,\mathbf{i} + \dfrac{\partial\rho}{\partial y}\,\mathbf{j} + \dfrac{\partial\rho}{\partial z}\,\mathbf{k}$. ...(1)

Since ρ is a function of p, therefore

$$\frac{\partial\rho}{\partial x} = \frac{d\rho}{dp}\frac{\partial p}{\partial x},\ \frac{\partial\rho}{\partial y} = \frac{d\rho}{dp}\frac{\partial p}{\partial y},\ \frac{\partial\rho}{\partial z} = \frac{d\rho}{dp}\frac{\partial p}{\partial z}.$$

$\therefore$ from (1), we have

$$\nabla\rho = \frac{d\rho}{dp}\frac{\partial p}{\partial x}\,\mathbf{i} + \frac{d\rho}{dp}\frac{\partial p}{\partial y}\,\mathbf{j} + \frac{d\rho}{dp}\frac{\partial p}{\partial z}\,\mathbf{k}$$

$$=\frac{df}{dp}\left(\frac{\partial p}{\partial x}\mathbf{i}+\frac{\partial p}{\partial y}\mathbf{j}+\frac{\partial p}{\partial z}\mathbf{k}\right)=\frac{df}{dp}\nabla p.$$

Ex. 19. *If* $\phi=(3r^2-4r^{1/2}+6r^{-1/3})$, *show that*

$$\nabla\phi=2\,(3-r^{-3/2}-r^{-7/3})\,\mathbf{r}.$$

Sol. We have $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ and $r=|\mathbf{r}|=\sqrt{(x^2+y^2+z^2)}$ so that $r^2=x^2+y^2+z^2$.

Now ϕ is a function of r.

$$\therefore\quad \nabla\phi=\frac{d\phi}{dr}\nabla r \qquad \text{[See Ex. 7 part (i)]}$$

$$=[6r-4.\tfrac{1}{2}r^{-1/2}+6.(-\tfrac{1}{3})\,r^{-4/3}]\,\nabla r$$

$$=(6r-2r^{-1/2}-2r^{-4/3})\,\frac{1}{r}\,\mathbf{r}\left[\because\quad \nabla r=\frac{1}{r}\mathbf{r}\right]$$

$$=(6-2r^{-3/2}-2r^{-7/3})\,\mathbf{r}=2\,(3-r^{-3/2}-r^{-7/3})\,\mathbf{r}.$$

Ex. 20. (*i*) *Interpret the symbol* $\mathbf{a}\cdot\nabla$.

(*ii*) *Show that* $(\mathbf{a}\cdot\nabla)\,\phi=\mathbf{a}\cdot\nabla\,\phi$.

(*iii*) *Show that* $(\mathbf{a}\cdot\nabla)\,\mathbf{r}=\mathbf{a}$.

Sol. (i) Let $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$. Then

$$\mathbf{a}\cdot\nabla=(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\cdot\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)$$

$$=a_1\frac{\partial}{\partial x}+a_2\frac{\partial}{\partial y}+a_3\frac{\partial}{\partial z}.$$

Thus the symbol $\mathbf{a}\cdot\nabla$ stands for the operator

$$a_1\frac{\partial}{\partial x}+a_2\frac{\partial}{\partial y}+a_3\frac{\partial}{\partial z}.$$

(ii) $$(\mathbf{a}\cdot\nabla)\,\phi=\left(a_1\frac{\partial}{\partial x}+a_2\frac{\partial}{\partial y}+a_3\frac{\partial}{\partial z}\right)\phi.$$

Also $$\mathbf{a}\cdot\nabla\phi=(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})\cdot\left(\frac{\partial\phi}{\partial x}\mathbf{i}+\frac{\partial\phi}{\partial y}\mathbf{j}+\frac{\partial\phi}{\partial z}\mathbf{k}\right)$$

$$=a_1\frac{\partial\phi}{\partial x}+a_2\frac{\partial\phi}{\partial y}+a_3\frac{\partial\phi}{\partial z}.$$

Hence $(\mathbf{a}\cdot\nabla)\,\phi=\mathbf{a}\cdot\nabla\phi$.

(iii) $$(\mathbf{a}\cdot\nabla)\,\mathbf{r}=\left(a_1\frac{\partial}{\partial x}+a_2\frac{\partial}{\partial y}+a_3\frac{\partial}{\partial z}\right)\mathbf{r}$$

$$=a_1\frac{\partial\mathbf{r}}{\partial x}+a_2\frac{\partial\mathbf{r}}{\partial y}+a_3\frac{\partial\mathbf{r}}{\partial z}.$$

But $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$. $\therefore\ \dfrac{\partial\mathbf{r}}{\partial x}=\mathbf{i},\ \dfrac{\partial\mathbf{r}}{\partial y}=\mathbf{j},\ \dfrac{\partial\mathbf{r}}{\partial z}=\mathbf{k}.$

$$\therefore\quad (\mathbf{a}\cdot\nabla)\,\mathbf{r}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}=\mathbf{a}.$$

§ 8. Divergence of a vector point function.

Definition. *Let* **V** *be any given differentiable vector point function. Then the divergence of* **V**, *written as,*

$$\nabla \cdot \mathbf{V} \quad \textit{or} \quad \textit{div } \mathbf{V},$$

is defined as div $\mathbf{V} = \nabla \cdot \mathbf{V} = \left(\mathbf{i} \dfrac{\partial}{\partial x} + \mathbf{j} \dfrac{\partial}{\partial y} + \mathbf{k} \dfrac{\partial}{\partial z} \right) \cdot \mathbf{V}$

$$= \mathbf{i} \cdot \frac{\partial \mathbf{V}}{\partial x} + \mathbf{j} \cdot \frac{\partial \mathbf{V}}{\partial y} + \mathbf{k} \cdot \frac{\partial \mathbf{V}}{\partial z} = \Sigma \mathbf{i} \cdot \frac{\partial \mathbf{V}}{\partial x}.$$

[Sagar 1983; Kerala 74; Bombay 70]

It should be noted that div **V** is a scalar quantity. Thus *the divergence of a vector point function is a scalar point function.*

Theorem. *If* $\mathbf{V} = V_1 \mathbf{i} + V_2 \mathbf{j} + V_3 \mathbf{k}$ *is a differentiable vector point function, then* $\quad \textit{div } \mathbf{V} = \dfrac{\partial V_1}{\partial x} + \dfrac{\partial V_2}{\partial y} + \dfrac{\partial V_3}{\partial z}.$

Proof. We have by definition

$$\text{div } \mathbf{V} = \nabla \cdot \mathbf{V} = \mathbf{i} \cdot \frac{\partial \mathbf{V}}{\partial x} + \mathbf{j} \cdot \frac{\partial \mathbf{V}}{\partial y} + \mathbf{k} \cdot \frac{\partial \mathbf{V}}{\partial z}.$$

Now $\mathbf{V} = V_1 \mathbf{i} + V_2 \mathbf{j} + V_3 \mathbf{k}$; $\quad \therefore \quad \dfrac{\partial \mathbf{V}}{\partial x} = \dfrac{\partial V_1}{\partial x} \mathbf{i} + \dfrac{\partial V_2}{\partial x} \mathbf{j} + \dfrac{\partial V_3}{\partial x} \mathbf{k}.$

$$\therefore \quad \mathbf{i} \cdot \frac{\partial \mathbf{V}}{\partial x} = \mathbf{i} \cdot \left(\frac{\partial V_1}{\partial x} \mathbf{i} + \frac{\partial V_2}{\partial x} \mathbf{j} + \frac{\partial V_3}{\partial x} \mathbf{k} \right) = \frac{\partial V_1}{\partial x}.$$

Similarly $\quad \mathbf{j} \cdot \dfrac{\partial \mathbf{V}}{\partial y} = \dfrac{\partial V_2}{\partial y}$ and $\mathbf{k} \cdot \dfrac{\partial \mathbf{V}}{\partial z} = \dfrac{\partial V_3}{\partial z}.$

Hence $\quad \text{div } \mathbf{V} = \dfrac{\partial V_1}{\partial x} + \dfrac{\partial V_2}{\partial y} + \dfrac{\partial V_3}{\partial z}.$

Solenoidal Vector. Definition. *A vector* **V** *is said to be solenoidal if div* $\mathbf{V} = 0$. **[Meerut 1991 S; Calcutta 75]**

§ 9. Curl of a vector point function. Definition.

Let **f** *be any given differentiable vector point function. Then the curl or rotation of* **f**, *written as* $\nabla \times \mathbf{f}$, *curl* **f** *or rot* **f** *is defined as*

$$curl\ \mathbf{f} = \nabla \times \mathbf{f} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \times \mathbf{f}$$

$$= \mathbf{i} \times \frac{\partial \mathbf{f}}{\partial x} + \mathbf{j} \times \frac{\partial \mathbf{f}}{\partial y} + \mathbf{k} \times \frac{\partial \mathbf{f}}{\partial z} = \Sigma \mathbf{i} \times \frac{\partial \mathbf{f}}{\partial x}.$$

[Sagar 1983; Bombay 86; Punjab 88]

It should be noted that curl **f** is a vector quantity. *Thus the curl of a vector point function is a vector point function.*

Theorem. *If* $\mathbf{f} = f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}$ *is a differentiable vector point function, then*

$$curl\ \mathbf{f} = \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\mathbf{k}.$$

Proof. We have by definition

$$\text{curl } \mathbf{f} = \nabla \times \mathbf{f} = \mathbf{i} \times \frac{\partial \mathbf{f}}{\partial x} + \mathbf{j} \times \frac{\partial \mathbf{f}}{\partial y} + \mathbf{k} \times \frac{\partial \mathbf{f}}{\partial z}$$

$$= \mathbf{i} \times \frac{\partial}{\partial x}(f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}) + \mathbf{j} \times \frac{\partial}{\partial y}(f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k}) + \mathbf{k} \times \frac{\partial}{\partial z}(f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k})$$

$$= \mathbf{i} \times \left(\frac{\partial f_1}{\partial x}\mathbf{i} + \frac{\partial f_2}{\partial x}\mathbf{j} + \frac{\partial f_3}{\partial x}\mathbf{k}\right) + \mathbf{j} \times \left(\frac{\partial f_1}{\partial y}\mathbf{i} + \frac{\partial f_2}{\partial y}\mathbf{j} + \frac{\partial f_3}{\partial y}\mathbf{k}\right) + \mathbf{k} \times \left(\frac{\partial f_1}{\partial z}\mathbf{i} + \frac{\partial f_2}{\partial z}\mathbf{j} + \frac{\partial f_3}{\partial z}\mathbf{k}\right)$$

$$= \left(\frac{\partial f_2}{\partial x}\mathbf{k} - \frac{\partial f_3}{\partial x}\mathbf{j}\right) + \left(-\frac{\partial f_1}{\partial y}\mathbf{k} + \frac{\partial f_3}{\partial y}\mathbf{i}\right) + \left(\frac{\partial f_1}{\partial z}\mathbf{j} - \frac{\partial f_2}{\partial z}\mathbf{i}\right)$$

$$= \left(\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial f_1}{\partial z} - \frac{\partial f_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial f_2}{\partial x} - \frac{\partial f_1}{\partial y}\right)\mathbf{k}.$$

Note. It should be noted that the expression for curl **f** can be written immediately if we treat the operator ∇ as a vector quantity. Thus

$$\text{Curl } \mathbf{f} = \nabla \times \mathbf{f} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \times (f_1\mathbf{i} + f_2\mathbf{j} + f_3\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_2 & f_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ f_1 & f_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ f_1 & f_2 \end{vmatrix}\mathbf{k}$$

$$=\left(\frac{\partial f_3}{\partial y}-\frac{\partial f_2}{\partial z}\right)\mathbf{i}+\left(\frac{\partial f_1}{\partial z}-\frac{\partial f_3}{\partial x}\right)\mathbf{j}+\left(\frac{\partial f_2}{\partial x}-\frac{\partial f_1}{\partial y}\right)\mathbf{k}.$$

But we must take care that in the expansion of the determinant the operators $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ must *precede* the functions f_1, f_2, f_3.

Irrotational vector. Definition. *A vector* $\mathbf{f}$ *is said to be irrotational if* $\nabla\times\mathbf{f}=\mathbf{0}$. **[Meerut 1991 S]**

§ 10. The Laplacian operator ∇^2.

The Laplacian operator ∇^2 *is defined as*

$$\nabla^2\equiv\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}.$$

If f is a scalar point function, then

$$\nabla^2 f=\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}+\frac{\partial^2 f}{\partial z^2}.$$

It should be noted that $\nabla^2 f$ is also a scalar quantity.

If $\mathbf{f}$ is a vector point function, then

$$\nabla^2\mathbf{f}=\frac{\partial^2\mathbf{f}}{\partial x^2}+\frac{\partial^2\mathbf{f}}{\partial y^2}+\frac{\partial^2\mathbf{f}}{\partial z^2}.$$

It should be noted that $\nabla^2\mathbf{f}$ is also a vector quantity.

Laplace's equation. *The equation* $\nabla^2 f=0$ *is called Laplace's equation.* A function which satisfies Laplace's equation is called a **harmonic function**.

Solved Examples

Ex. 1. *Prove that div* $\mathbf{r}=3$.

[Agra 1978; Rohilkhand 81; Kanpur 75; Gorakhpur 88]

Sol. We have $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$.

By definition, div $\mathbf{r}=\nabla\cdot\mathbf{r}=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\cdot\mathbf{r}$

$$=\mathbf{i}\cdot\frac{\partial\mathbf{r}}{\partial x}+\mathbf{j}\cdot\frac{\partial\mathbf{r}}{\partial y}+\mathbf{k}\cdot\frac{\partial\mathbf{r}}{\partial z}$$

$$=\mathbf{i}\cdot\mathbf{i}+\mathbf{j}\cdot\mathbf{j}+\mathbf{k}\cdot\mathbf{k}\quad\left[\because\ \frac{\partial\mathbf{r}}{\partial x}=\mathbf{i},\ \frac{\partial\mathbf{r}}{\partial y}=\mathbf{j},\ \frac{\partial\mathbf{r}}{\partial z}=\mathbf{k}\right]$$

$$=1+1+1=3.$$

Ex. 2. *Prove that curl* $\mathbf{r}=\mathbf{0}$.

[Agra 1968; Kanpur 75, 79; Rohilkhand 76; Gorakhpur 88]

Sol. We have by definition

$$\text{Curl } \mathbf{r} = \nabla \times \mathbf{r} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \times \mathbf{r}$$

$$= \mathbf{i} \times \frac{\partial \mathbf{r}}{\partial x} + \mathbf{j} \times \frac{\partial \mathbf{r}}{\partial y} + \mathbf{k} \times \frac{\partial \mathbf{r}}{\partial z}.$$

Now $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. $\therefore \quad \dfrac{\partial \mathbf{r}}{\partial x} = \mathbf{i}, \dfrac{\partial \mathbf{r}}{\partial y} = \mathbf{j}, \dfrac{\partial \mathbf{r}}{\partial z} = \mathbf{k}$.

$\therefore \quad \text{Curl } \mathbf{r} = \mathbf{i} \times \mathbf{i} + \mathbf{j} \times \mathbf{j} + \mathbf{k} \times \mathbf{k} = 0 + 0 + 0 = 0.$

Ex. 3. *If* $\mathbf{f} = x^2 y\,\mathbf{i} - 2xz\,\mathbf{j} + 2yz\,\mathbf{k}$, *find*

(*i*) *div* **f**, (*ii*) *curl* **f**, (*iii*) *curl curl* **f**. **[Agra 1986]**

Sol. (i) We have

$$\text{div } \mathbf{f} = \nabla \cdot \mathbf{f} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot (x^2 y\,\mathbf{i} - 2xz\,\mathbf{j} + 2yz\,\mathbf{k})$$

$$= \frac{\partial}{\partial x}(x^2 y) + \frac{\partial}{\partial y}(-2xz) + \frac{\partial}{\partial z}(2yz) = 2xy + 0 + 2y = 2y(x+1).$$

(ii) We have $\text{curl } \mathbf{f} = \nabla \times \mathbf{f} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2 y & -2xz & 2yz \end{vmatrix}$

$$= \left[\frac{\partial}{\partial y}(2yz) - \frac{\partial}{\partial z}(-2xz)\right]\mathbf{i} - \left[\frac{\partial}{\partial x}(2yz) - \frac{\partial}{\partial z}(x^2 y)\right]\mathbf{j}$$

$$+ \left[\frac{\partial}{\partial x}(-2xz) - \frac{\partial}{\partial y}(x^2 y)\right]\mathbf{k}$$

$$= (2z + 2x)\,\mathbf{i} - 0\,\mathbf{j} + (-2z - x^2)\,\mathbf{k} = (2x + 2z)\,\mathbf{i} - (x^2 + 2z)\,\mathbf{k}.$$

(iii) We have $\text{curl curl } \mathbf{f} = \nabla \times (\nabla \times \mathbf{f})$

$= \nabla \times [(2x + 2z)\,\mathbf{i} - (x^2 + 2z)\,\mathbf{k}]$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2x + 2z & 0 & -x^2 - 2z \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(-x^2 - 2z)\right]\mathbf{i} - \left[\frac{\partial}{\partial x}(-x^2 - 2z) - \frac{\partial}{\partial z}(2x + 2z)\right]\mathbf{j}$$

$$+ \left[0 - \frac{\partial}{\partial y}(2x + 2z)\right]\mathbf{k}$$

$$= 0\,\mathbf{i} - (-2x - 2)\,\mathbf{j} + (0 - 0)\,\mathbf{k} = (2x + 2)\,\mathbf{j}.$$

Ex. 4. *Find the divergence and curl of the vector*
$\mathbf{f}=(x^2-y^2)\,\mathbf{i}+2xy\,\mathbf{j}+(y^2-xy)\,\mathbf{k}$. **[Agra 1982]**

Sol. We have div $\mathbf{f}=\nabla\cdot\mathbf{f}$

$$=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\cdot[(x^2-y^2)\,\mathbf{i}+2xy\,\mathbf{j}+(y^2-xy)\,\mathbf{k}]$$

$$=\frac{\partial}{\partial x}(x^2-y^2)+\frac{\partial}{\partial y}(2xy)+\frac{\partial}{\partial z}(y^2-xy)$$

$$=2x+2x+0=4x.$$

Also curl $\mathbf{f}=\nabla\times\mathbf{f}$

$$=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\times[(x^2-y^2)\,\mathbf{i}+2xy\,\mathbf{j}+(y^2-xy)\,\mathbf{k}]$$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^2-y^2 & 2xy & y^2-xy\end{vmatrix}$$

$$=\left[\frac{\partial}{\partial y}(y^2-xy)-\frac{\partial}{\partial z}(2xy)\right]\mathbf{i}+\left[\frac{\partial}{\partial z}(x^2-y^2)-\frac{\partial}{\partial x}(y^2-xy)\right]\mathbf{j}$$
$$+\left[\frac{\partial}{\partial x}(2xy)-\frac{\partial}{\partial y}(x^2-y^2)\right]\mathbf{k}$$

$$=[(2y-x)-0]\,\mathbf{i}+(0+y)\,\mathbf{j}+(2y+2y)\,\mathbf{k}$$

$$=(2y-x)\,\mathbf{i}+y\mathbf{j}+4y\,\mathbf{k}.$$

Ex. 5. *Find div* $\mathbf{F}$ *and curl* $\mathbf{F}$ *where*
$\mathbf{F}=\textit{grad}\,(x^3+y^3+z^3-3xyz)$.

Sol. We have $\mathbf{F}=\text{grad}\,(x^3+y^3+z^3-3xyz)$

$$=\mathbf{i}\frac{\partial}{\partial x}(x^3+y^3+z^3-3xyz)+\mathbf{j}\frac{\partial}{\partial y}(x^3+y^3+z^3-3xyz)$$
$$+\mathbf{k}\frac{\partial}{\partial z}(x^3+y^3+z^3-3xyz)$$

$$=(3x^2-3yz)\,\mathbf{i}+(3y^2-3xz)\,\mathbf{j}+(3z^2-3xy)\,\mathbf{k}.$$

Now div $\mathbf{F}=\nabla\cdot\mathbf{F}$

$$=\frac{\partial}{\partial x}(3x^2-3yz)+\frac{\partial}{\partial y}(3y^2-3xz)+\frac{\partial}{\partial z}(3z^2-3xy)$$

$$=6x+6y+6z=6\,(x+y+z).$$

Also curl $\mathbf{F}=\nabla\times\mathbf{F}$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy\end{vmatrix}$$

$$=\left[\frac{\partial}{\partial y}(3z^2-3xy)-\frac{\partial}{\partial z}(3y^2-3xz)\right]\mathbf{i}$$
$$+\left[\frac{\partial}{\partial z}(3x^2-3yz)-\frac{\partial}{\partial x}(3z^2-3xy)\right]\mathbf{j}$$
$$+\left[\frac{\partial}{\partial x}(3y^2-3xz)-\frac{\partial}{\partial y}(3x^2-3yz)\right]\mathbf{k}$$
$$=(-3x+3x)\,\mathbf{i}+(-3y+3y)\,\mathbf{j}+(-3z+3z)\,\mathbf{k}$$
$$=0\mathbf{i}+0\mathbf{j}+0\mathbf{k}=\mathbf{0}.$$

Ex. 6. *Given $\phi=2x^3y^2z^4$, find div (grad ϕ).*

Sol. We have grad $\phi=$ grad $(2x^3y^2z^4)$

$$=\mathbf{i}\frac{\partial}{\partial x}(2x^3y^2z^4)+\mathbf{j}\frac{\partial}{\partial y}(2x^3y^2z^4)+\mathbf{k}\frac{\partial}{\partial z}(2x^3y^2z^4)$$
$$=6x^2y^2z^4\,\mathbf{i}+4x^3yz^4\,\mathbf{j}+8x^3y^2z^3\,\mathbf{k}.$$

Now div (grad ϕ) $=\nabla\cdot$(grad ϕ) $=\nabla\cdot(6x^2y^2z^4\,\mathbf{i}+4x^3yz^4\,\mathbf{j}+8x^3y^2z^3\,\mathbf{k})$

$$=\frac{\partial}{\partial x}(6x^2y^2z^4)+\frac{\partial}{\partial y}(4x^3yz^4)+\frac{\partial}{\partial z}(8x^3y^2z^3)$$
$$=12xy^2z^4+4x^3z^4+24x^3y^2z^2.$$

Ex. 7. *If $\mathbf{f}=xy^2\,\mathbf{i}+2x^2yz\,\mathbf{j}-3yz^2\,\mathbf{k}$, find div $\mathbf{f}$ and curl $\mathbf{f}$. What are their values at the point $(1,-1,1)$?*

[Rohilkhand 1982]

Sol. We have div $\mathbf{f}=\nabla\cdot\mathbf{f}$

$$=\frac{\partial}{\partial x}(xy^2)+\frac{\partial}{\partial y}(2x^2yz)+\frac{\partial}{\partial z}(-3yz^2)$$
$$=y^2+2x^2z-6yz.$$

$\therefore$ div $\mathbf{f}$ at $(1,-1,1)=(-1)^2+2.1^2.1-6.(-1).1=1+2+6=9.$

Also curl $\mathbf{f}=\nabla\times\mathbf{f}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ xy^2 & 2x^2yz & -3yz^2\end{vmatrix}$

$$=\left[\frac{\partial}{\partial y}(-3yz^2)-\frac{\partial}{\partial z}(2x^2yz)\right]\mathbf{i}-\left[\frac{\partial}{\partial x}(-3yz^2)-\frac{\partial}{\partial z}(xy^2)\right]\mathbf{j}$$
$$+\left[\frac{\partial}{\partial x}(2x^2yz)-\frac{\partial}{\partial y}(xy^2)\right]\mathbf{k}$$
$$=(-3z^2-2x^2y)\,\mathbf{i}-(0-0)\,\mathbf{j}+(4xyz-2xy)\,\mathbf{k}$$
$$=-(3z^2+2x^2y)\,\mathbf{i}+(4xyz-2xy)\,\mathbf{k}.$$

$\therefore$ curl $\mathbf{f}$ at $(1,-1,1)=-[3\cdot1^2+2.1^2.(-1)]\,\mathbf{i}$
$+[4.1.(-1).1-2.1.(-1)]\,\mathbf{k}$

$$=-\mathbf{i}-2\mathbf{k}.$$

Ex. 8. *If* $\mathbf{F}=x^2z\,\mathbf{i}-2y^3z^2\,\mathbf{j}+xy^2z\,\mathbf{k}$, *find div* $\mathbf{F}$, *curl* $\mathbf{F}$ *at* $(1, -1, 1)$. [Garhwal 1979; Madras 78]

Sol. We have div $\mathbf{F}=\nabla\cdot\mathbf{F}$

$$=\frac{\partial}{\partial x}(x^2z)+\frac{\partial}{\partial y}(-2y^3z^2)+\frac{\partial}{\partial z}(xy^2z)$$

$$=2xz-6y^2z^2+xy^2.$$

$\therefore$ div $\mathbf{F}$ at $(1, -1, 1)=2.1.1-6.(-1)^2.1^2+1.(-1)^2$

$$=2-6+1=-3.$$

Also curl $\mathbf{F}=\nabla\times\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^2z & -2y^3z^2 & xy^2z\end{vmatrix}$

$$=\left[\frac{\partial}{\partial y}(xy^2z)-\frac{\partial}{\partial z}(-2y^3z^2)\right]\mathbf{i}+\left[\frac{\partial}{\partial z}(x^2z)-\frac{\partial}{\partial x}(xy^2z)\right]\mathbf{j}$$

$$+\left[\frac{\partial}{\partial x}(-2y^3z^2)-\frac{\partial}{\partial y}(x^2z)\right]\mathbf{k}$$

$$=(2xyz+4y^3z)\,\mathbf{i}+(x^2-y^2z)\,\mathbf{j}+(0-0)\,\mathbf{k}$$

$$=2\,(xyz+2y^3z)\,\mathbf{i}+(x^2-y^2z)\,\mathbf{j}.$$

$\therefore$ curl $\mathbf{F}$ at $(1, -1, 1)=2\,[1.(-1).1+2.(-1)^3.1]\,\mathbf{i}$

$$+[1^2-(-1)^2.1]\,\mathbf{j}$$

$$=2(-1-2)\,\mathbf{i}+(1-1)\,\mathbf{j}=-6\mathbf{i}+0\mathbf{j}$$

$$=-6\mathbf{i}.$$

Ex. 9. *If* $\mathbf{F}=(y^2+z^2-x^2)\,\mathbf{i}+(z^2+x^2-y^2)\,\mathbf{j}+(x^2+y^2-z^2)\,\mathbf{k}$, *find div* $\mathbf{f}$ *and curl* $\mathbf{f}$.

Sol. We have div $\mathbf{f}=\nabla\cdot\mathbf{f}$

$$=\frac{\partial}{\partial x}(y^2+z^2-x^2)+\frac{\partial}{\partial y}(z^2+x^2-y^2)+\frac{\partial}{\partial z}(x^2+y^2-z^2)$$

$$=-2x-2y-2z=-2\,(x+y+z).$$

Also curl $\mathbf{f}=\nabla\times\mathbf{f}$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y^2+z^2-x^2 & z^2+x^2-y^2 & x^2+y^2-z^2\end{vmatrix}$$

$$=\left[\frac{\partial}{\partial y}(x^2+y^2-z^2)-\frac{\partial}{\partial z}(z^2+x^2-y^2)\right]\mathbf{i}$$

$$+\left[\frac{\partial}{\partial z}(y^2+z^2-x^2)-\frac{\partial}{\partial x}(x^2+y^2-z^2)\right]\mathbf{j}$$

$$+\left[\frac{\partial}{\partial x}(z^2+x^2-y^2)-\frac{\partial}{\partial y}(y^2+z^2-x^2)\right]\mathbf{k}$$

$$=(2y-2z)\,\mathbf{i}+(2z-2x)\,\mathbf{j}+(2x-2y)\,\mathbf{k}$$

$$=2(y-z)\,\mathbf{i}+2(z-x)\,\mathbf{j}+2(x-y)\,\mathbf{k}.$$

Ex. 10. *If* $\mathbf{f}=(x+y+1)\,\mathbf{i}+\mathbf{j}+(-x-y)\,\mathbf{k}$, *prove that* $\mathbf{f}\cdot$*curl* $\mathbf{f}=0$. **[Kanpur 1988; Agra 86]**

Sol. We have curl $\mathbf{f}=\nabla\times\mathbf{f}$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x+y+1 & 1 & -x-y \end{vmatrix}$$

$$=\left[\frac{\partial}{\partial y}(-x-y)-\frac{\partial}{\partial z}(1)\right]\mathbf{i}+\left[\frac{\partial}{\partial z}(x+y+1)-\frac{\partial}{\partial x}(-x-y)\right]\mathbf{j}$$

$$+\left[\frac{\partial}{\partial x}(1)-\frac{\partial}{\partial y}(x+y+1)\right]\mathbf{k}$$

$$=(-1-0)\,\mathbf{i}+(0+1)\,\mathbf{j}+(0-1)\,\mathbf{k}=-\mathbf{i}+\mathbf{j}-\mathbf{k}.$$

$\therefore\quad \mathbf{f}\cdot\text{curl }\mathbf{f}=[(x+y+1)\,\mathbf{i}+\mathbf{j}+(-x-y)\,\mathbf{k}]\cdot(-\mathbf{i}+\mathbf{j}-\mathbf{k})$

$$=(x+y+1)\cdot(-1)+1\cdot1+(-x-y)\cdot(-1)$$

$$=-x-y-1+1+x+y=0.$$

Ex. 11. *If* $u=3x^2y$ *and* $v=xz^2-2y$, *then find* grad $[(\text{grad } u)\cdot(\text{grad } v)]$.

Sol. We have grad $u=\mathbf{i}\dfrac{\partial}{\partial x}(3x^2y)+\mathbf{j}\dfrac{\partial}{\partial y}(3x^2y)+\mathbf{k}\dfrac{\partial}{\partial z}(3x^2y)$

$$=6xy\,\mathbf{i}+3x^2\,\mathbf{j}+0\,\mathbf{k}=6xy\,\mathbf{i}+3x^2\,\mathbf{j}.$$

Also grad $v=\mathbf{i}\dfrac{\partial}{\partial x}(xz^2-2y)+\mathbf{j}\dfrac{\partial}{\partial y}(xz^2-2y)+\mathbf{k}\dfrac{\partial}{\partial z}(xz^2-2y)$

$$=z^2\,\mathbf{i}-2\mathbf{j}+2xz\,\mathbf{k}.$$

$\therefore\quad (\text{grad } u)\cdot(\text{grad } v)=(6xy\,\mathbf{i}+3x^2\,\mathbf{j})\cdot(z^2\,\mathbf{i}-2\mathbf{j}+2xz\,\mathbf{k})$

$$=6xyz^2-6x^2.$$

Hence grad $[(\text{grad } u)\cdot(\text{grad } v)]=\text{grad }(6xyz^2-6x^2)$

$$=\mathbf{i}\frac{\partial}{\partial x}(6xyz^2-6x^2)+\frac{\partial}{\partial y}(6xyz^2-6x^2)+\mathbf{k}\frac{\partial}{\partial z}(6xyz^2-6x^2)$$

$$=(6yz^2-12x)\,\mathbf{i}+(6xz^2)\,\mathbf{j}+(12xyz)\,\mathbf{k}.$$

Ex. 12. *If* $u=x^2-y^2+4z$, *show that* $\nabla^2 u=0$.

Sol. We have $\nabla^2 u=\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}+\dfrac{\partial^2}{\partial z^2}\right)u$

$$=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}$$

Now $\dfrac{\partial u}{\partial x}=\dfrac{\partial}{\partial x}(x^2-y^2+4z)=2x.$

$\therefore \quad \dfrac{\partial^2 u}{\partial x^2}=\dfrac{\partial}{\partial x}\left(\dfrac{\partial u}{\partial x}\right)=\dfrac{\partial}{\partial x}(2x)=2.$

Again $\dfrac{\partial u}{\partial y}=\dfrac{\partial}{\partial y}(x^2-y^2+4z)=-2y.$

$\therefore \quad \dfrac{\partial^2 u}{\partial y^2}=\dfrac{\partial}{\partial y}\left(\dfrac{\partial u}{\partial y}\right)=\dfrac{\partial}{\partial y}(-2y)=-2.$

Finally $\dfrac{\partial u}{\partial z}=\dfrac{\partial}{\partial z}(x^2-y^2+4z)=4.$

$\therefore \quad \dfrac{\partial^2 u}{\partial z^2}=\dfrac{\partial}{\partial z}\left(\dfrac{\partial u}{\partial z}\right)=\dfrac{\partial}{\partial z}(4)=0.$

$\therefore \quad \dfrac{\partial^2 u}{\partial x^2}+\dfrac{\partial^2 u}{\partial y^2}+\dfrac{\partial^2 u}{\partial z^2}=2-2+0=0.$

Hence $\nabla^2 u=0.$

Ex. 13. *If* $\mathbf{f}=f_1\,\mathbf{i}+f_2\,\mathbf{j}+f_3\,\mathbf{k}$, *show that*

$\nabla\cdot\mathbf{f}=\nabla f_1\cdot\mathbf{i}+\nabla f_2\cdot\mathbf{j}+\nabla f_3\cdot\mathbf{k}.$

Sol. We have $\nabla f_1=\dfrac{\partial f_1}{\partial x}\,\mathbf{i}+\dfrac{\partial f_1}{\partial y}\,\mathbf{j}+\dfrac{\partial f_1}{\partial z}\,\mathbf{k}.$

$\therefore \quad \nabla f_1\cdot\mathbf{i}=\left(\dfrac{\partial f_1}{\partial x}\,\mathbf{i}+\dfrac{\partial f_1}{\partial y}\,\mathbf{j}+\dfrac{\partial f_1}{\partial z}\,\mathbf{k}\right)\cdot\mathbf{i}=\dfrac{\partial f_1}{\partial x}.$

Similarly $\nabla f_2\cdot\mathbf{j}=\dfrac{\partial f_2}{\partial y}$ and $\nabla f_3\cdot\mathbf{k}=\dfrac{\partial f_3}{\partial z}.$

$\therefore \quad \nabla f_1\cdot\mathbf{i}+\nabla f_2\cdot\mathbf{j}+\nabla f_3\cdot\mathbf{k}$

$$=\frac{\partial f_1}{\partial x}+\frac{\partial f_2}{\partial y}+\frac{\partial f_3}{\partial z}=\text{div}\,\mathbf{f}=\nabla\cdot\mathbf{f}.$$

Ex. 14. *Prove that* $\nabla\cdot(r^3\mathbf{r})=6r^3.$

Sol. We have $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}.$

$\therefore \quad r^3\mathbf{r}=r^3(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})=r^3x\mathbf{i}+r^3y\mathbf{j}+r^3z\mathbf{k}.$

$\therefore \quad \nabla\cdot(r^3\mathbf{r})=\text{div}\,(r^3\mathbf{r})=\dfrac{\partial}{\partial x}(r^3x)+\dfrac{\partial}{\partial y}(r^3y)+\dfrac{\partial}{\partial z}(r^3z)$

$$=r^3+3r^2x\frac{\partial r}{\partial x}+r^3+3r^2y\frac{\partial r}{\partial y}+r^3+3r^2z\frac{\partial r}{\partial z}$$

$$=3r^3+3r^2\left(x\frac{\partial r}{\partial x}+y\frac{\partial r}{\partial y}+z\frac{\partial r}{\partial z}\right) \qquad \ldots(1)$$

Now $r^2=x^2+y^2+z^2.$

$\therefore \quad 2r\dfrac{\partial r}{\partial x}=2x$ or $\dfrac{\partial r}{\partial x}=\dfrac{x}{r}.$

Similarly $\frac{\partial r}{\partial y}=\frac{y}{r}$ and $\frac{\partial r}{\partial z}=\frac{z}{r}$.

$\therefore$ from (1), $\nabla\cdot(r^3\mathbf{r})=3r^3+3r^2\left(x.\frac{x}{r}+y.\frac{y}{r}+z.\frac{z}{r}\right)$

$$=3r^3+3r^2\left(\frac{x^2+y^2+z^2}{r}\right)=3r^3+3r^2.\frac{r^2}{r}$$

$$=3r^3+3r^3=6r^3.$$

Ex. 15. *Find the constants a, b, c so that the vector* $\mathbf{F}=(x+2y+az)\,\mathbf{i}+(bx-3y-z)\,\mathbf{j}+(4x+cy+2z)\,\mathbf{k}$ *is irrotational.*

Sol. The vector **F** is irrotational if curl $\mathbf{F}=\mathbf{0}$.

We have curl $\mathbf{F}=\nabla\times\mathbf{F}$

$$=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+2y+az & bx-3y-z & 4x+cy+2z \end{vmatrix}$$

$$=\left[\frac{\partial}{\partial y}(4x+cy+z)-\frac{\partial}{\partial z}(bx-3y-z)\right]\mathbf{i}$$

$$+\left[\frac{\partial}{\partial z}(x+2y+az)-\frac{\partial}{\partial x}(4x+cy+2z)\right]\mathbf{j}$$

$$+\left[\frac{\partial}{\partial x}(bx-3y-z)-\frac{\partial}{\partial y}(x+2y+az)\right]\mathbf{k}$$

$$=(c+1)\,\mathbf{i}+(a-4)\,\mathbf{j}+(b-2)\,\mathbf{k}.$$

$\therefore$ curl $\mathbf{F}=\mathbf{0}\Rightarrow(c+1)\,\mathbf{i}+(a-4)\,\mathbf{j}+(b-2)\,\mathbf{k}=\mathbf{0}$

$$\Rightarrow c+1=0,\ a-4=0,\ b-2=0$$

$$\Rightarrow c=-1,\ a=4,\ b=2.$$

Hence the vector **F** is irrotational if $a=4$, $b=2$, $c=-1$.

Ex. 16. *Determine the constant a so that the vector* $\mathbf{V}=(x+3y)\,\mathbf{i}+(y-2z)\,\mathbf{j}+(x+az)\,\mathbf{k}$ *is solenoidal.* **[Kanpur 1978]**

Sol. A vector **V** is said to be solenoidal if div $\mathbf{V}=0$.

We have div $\mathbf{V}=\nabla\cdot\mathbf{V}=\frac{\partial}{\partial x}(x+3y)+\frac{\partial}{\partial y}(y-2z)+\frac{\partial}{\partial z}(x+az)$

$$=1+1+a=2+a.$$

Now div $\mathbf{V}=0$ if $2+a=0$ *i.e.* if $a=-2$.

Ex. 17. *Show that the vector* $\mathbf{V}=(\sin y+z)\,\mathbf{i}+(x\cos y-z)\,\mathbf{j}+(x-y)\,\mathbf{k}$ *is irrotational.*

Sol. A vector **V** is said to be irrotational if curl $\mathbf{V}=\mathbf{0}$.

We have curl $\mathbf{V}=\nabla\times\mathbf{V}$

$$=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin y+z & x\cos y-z & x-y \end{vmatrix}$$

$$=\left[\frac{\partial}{\partial y}(x-y)-\frac{\partial}{\partial z}(x\cos y-z)\right]\mathbf{i}-\left[\frac{\partial}{\partial x}(x-y)-\frac{\partial}{\partial z}(\sin y+z)\right]\mathbf{j}$$

$$+\left[\frac{\partial}{\partial x}(x\cos y-z)-\frac{\partial}{\partial y}(\sin y+z)\right]\mathbf{k}$$

$$=(-1+1)\,\mathbf{i}-(1-1)\,\mathbf{j}+(\cos y-\cos y)\,\mathbf{k}=\mathbf{0}.$$

$\therefore$ $\mathbf{V}$ is irrotational.

Ex. 18. *If* $\mathbf{V}$ *is a constant vector, show that*

(*i*) *div* $\mathbf{V}=0$, (*ii*) *curl* $\mathbf{V}=\mathbf{0}$.

Sol. (i) We have $\text{div}\ \mathbf{V}=\mathbf{i}\cdot\frac{\partial \mathbf{V}}{\partial x}+\mathbf{j}\cdot\frac{\partial \mathbf{V}}{\partial y}+\mathbf{k}\cdot\frac{\partial \mathbf{V}}{\partial z}$

$$=\mathbf{i}\cdot\mathbf{0}+\mathbf{j}\cdot\mathbf{0}+\mathbf{k}\cdot\mathbf{0}=0.$$

(ii) We have $\text{curl}\ \mathbf{V}=\mathbf{i}\times\frac{\partial \mathbf{V}}{\partial x}+\mathbf{j}\times\frac{\partial \mathbf{V}}{\partial y}+\mathbf{k}\times\frac{\partial \mathbf{V}}{\partial z}$

$$=\mathbf{i}\times\mathbf{0}+\mathbf{j}\times\mathbf{0}+\mathbf{k}\times\mathbf{0}=\mathbf{0}.$$

Ex. 19. *If* $\mathbf{a}$ *is a constant vector, find*

(*i*) *div* $(\mathbf{r}\times\mathbf{a})$, **[Rohilkhand 1981, 84]**

2007 → (*ii*) *curl* $(\mathbf{r}\times\mathbf{a})$. **[Rohilkhand 1984; Indore 83]**

Sol. We have $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$.

Let $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$. Then the scalars a_1, a_2, a_3 are all constants.

We have $\mathbf{r}\times\mathbf{a}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x & y & z \\ a_1 & a_2 & a_3 \end{vmatrix}$

$$=(a_3y-a_2z)\,\mathbf{i}+(a_1z-a_3x)\,\mathbf{j}+(a_2x-a_1y)\,\mathbf{k}.$$

(i) $\text{div}\,(\mathbf{r}\times\mathbf{a})=\frac{\partial}{\partial x}(a_3y-a_2z)+\frac{\partial}{\partial y}(a_1z-a_3x)+\frac{\partial}{\partial z}(a_2x-a_1y)$

$$=0+0+0=0.$$

(ii) $\text{curl}\,(\mathbf{r}\times\mathbf{a})=\nabla\times(\mathbf{r}\times\mathbf{a})$

$$=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_3y-a_2z & a_1z-a_3x & a_2x-a_1y \end{vmatrix}$$

$$=\left[\frac{\partial}{\partial y}(a_2x-a_1y)-\frac{\partial}{\partial z}(a_1z-a_3x)\right]\mathbf{i}-\left[\frac{\partial}{\partial x}(a_2x-a_1y)\right.$$
$$\left.-\frac{\partial}{\partial z}(a_3y-a_2z)\right]\mathbf{j}+\left[\frac{\partial}{\partial x}(a_1z-a_3x)-\frac{\partial}{\partial y}(a_3y-a_2z)\right]\mathbf{k}$$

$$=-2a_1\mathbf{i}-2a_2\mathbf{j}-2a_3\mathbf{k}=-2(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})=-2\mathbf{a}.$$

Ex. 20. *If* $\mathbf{V}=e^{xyz}(\mathbf{i}+\mathbf{j}+\mathbf{k})$, *find curl* $\mathbf{V}$.

[Meerut 1991 P; Kanpur 87; Agra 83]

Sol. We have curl $\mathbf{V}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ e^{xyz} & e^{xyz} & e^{xyz} \end{vmatrix}$

$$=\left[\frac{\partial}{\partial y}(e^{xyz})-\frac{\partial}{\partial z}(e^{xyz})\right]\mathbf{i}+\left[\frac{\partial}{\partial z}(e^{xyz})-\frac{\partial}{\partial x}(e^{xyz})\right]\mathbf{j}$$
$$+\left[\frac{\partial}{\partial x}(e^{xyz})-\frac{\partial}{\partial y}(e^{xyz})\right]\mathbf{k}$$

$$=e^{xyz}(xz-xy)\,\mathbf{i}+e^{xyz}(xy-yz)\,\mathbf{j}+e^{xyz}(yz-xz)\,\mathbf{k}.$$

Ex. 21. *Evaluate div* $\mathbf{f}$ *where*

$$\mathbf{f}=2x^2z\mathbf{i}-xy^2z\mathbf{j}+3y^2x\,\mathbf{k}.$$

[Kanpur 1988]

Sol. We have

$$\text{div }\mathbf{f}=\nabla\cdot\mathbf{f}=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\cdot(2x^2z\mathbf{i}-xy^2z\mathbf{j}+3y^2x\,\mathbf{k})$$
$$=\frac{\partial}{\partial x}(2x^2z)+\frac{\partial}{\partial y}(-xy^2z)+\frac{\partial}{\partial z}(3y^2x)$$
$$=4xz-2xyz+0=2xz(2-y).$$

Ex. 22. *Show that* $\nabla^2(x/r^3)=0$.

[Meerut 1991 P; Rohilkhand 92]

Sol. $\nabla^2\left(\dfrac{x}{r^3}\right)=\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}+\dfrac{\partial^2}{\partial z^2}\right)\left(\dfrac{x}{r^3}\right)$.

Now $\dfrac{\partial^2}{\partial x^2}\left(\dfrac{x}{r^3}\right)=\dfrac{\partial}{\partial x}\left\{\dfrac{\partial}{\partial x}\left(\dfrac{x}{r^3}\right)\right\}=\dfrac{\partial}{\partial x}\left\{\dfrac{1}{r^3}-\dfrac{3x}{r^4}\dfrac{\partial r}{\partial x}\right\}$

$$=\frac{\partial}{\partial x}\left\{\frac{1}{r^3}-\frac{3x}{r^4}\frac{x}{r}\right\}\left[\because\quad r^2=x^2+y^2+z^2\text{ gives }\frac{\partial r}{\partial x}=\frac{x}{r}\right]$$

$$=\frac{\partial}{\partial x}\left\{\frac{1}{r^3}-\frac{3x^2}{r^5}\right\}=-\frac{3}{r^4}\frac{\partial r}{\partial x}-\frac{6x}{r^5}+\frac{15x^2}{r^6}\frac{\partial r}{\partial x}$$

$$=-\frac{3}{r^4}\frac{x}{r}-\frac{6x}{r^5}+\frac{15x^2}{r^6}\frac{x}{r}=-\frac{9x}{r^5}+\frac{15x^3}{r^7}.$$

Again $\dfrac{\partial^2}{\partial y^2}\left(\dfrac{x}{r^3}\right)=\dfrac{\partial}{\partial y}\left\{\dfrac{\partial}{\partial y}\left(\dfrac{x}{r^3}\right)\right\}=\dfrac{\partial}{\partial y}\left\{-\dfrac{3x}{r^4}\dfrac{\partial r}{\partial y}\right\}$

$$=\frac{\partial}{\partial y}\left\{-\frac{3x}{r^4}\frac{y}{r}\right\} \qquad \left[\because \quad \frac{\partial r}{\partial y}=\frac{y}{r}\right]$$

$$=\frac{\partial}{\partial y}\left(-\frac{3xy}{r^5}\right)=-\frac{3x}{r^5}+\frac{15xy}{r^6}\frac{\partial r}{\partial y}=-\frac{3x}{r^5}+\frac{15xy^2}{r^7}.$$

Similarly $\dfrac{\partial^2}{\partial z^2}\left(\dfrac{x}{r^3}\right)=-\dfrac{3x}{r^5}+\dfrac{15xz^2}{r^7}.$

Therefore adding we get

$$\nabla^2\left(\frac{x}{r^3}\right)=\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\left(\frac{x}{r^3}\right)$$

$$=-\frac{9x}{r^5}+\frac{15x^3}{r^7}-\frac{3x}{r^5}+\frac{15xy^2}{r^7}-\frac{3x}{r^5}+\frac{15xz^2}{r^7}$$

$$=-\frac{15x}{r^5}+\frac{15x}{r^7}(x^2+y^2+z^2)=-\frac{15x}{r^5}+\frac{15x}{r^7}r^2=0.$$

§ 11. Important Vector Identities.

1. *Prove that div* $(\mathbf{A}+\mathbf{B})=$ *div* $\mathbf{A}+$ *div* $\mathbf{B}$

or $\qquad \nabla\cdot(\mathbf{A}+\mathbf{B})=\nabla\cdot\mathbf{A}+\nabla\cdot\mathbf{B}.$ **[Meerut 1992]**

Proof. We have

$$\text{div}(\mathbf{A}+\mathbf{B})=\nabla\cdot(\mathbf{A}+\mathbf{B})=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\cdot(\mathbf{A}+\mathbf{B})$$

$$=\mathbf{i}\cdot\frac{\partial}{\partial x}(\mathbf{A}+\mathbf{B})+\mathbf{j}\cdot\frac{\partial}{\partial y}(\mathbf{A}+\mathbf{B})+\mathbf{k}\cdot\frac{\partial}{\partial z}(\mathbf{A}+\mathbf{B})$$

$$=\mathbf{i}\cdot\left(\frac{\partial\mathbf{A}}{\partial x}+\frac{\partial\mathbf{B}}{\partial x}\right)+\mathbf{j}\cdot\left(\frac{\partial\mathbf{A}}{\partial y}+\frac{\partial\mathbf{B}}{\partial y}\right)+\mathbf{k}\cdot\left(\frac{\partial\mathbf{A}}{\partial z}+\frac{\partial\mathbf{B}}{\partial z}\right)$$

$$=\left(\mathbf{i}\cdot\frac{\partial\mathbf{A}}{\partial x}+\mathbf{j}\cdot\frac{\partial\mathbf{A}}{\partial y}+\mathbf{k}\cdot\frac{\partial\mathbf{A}}{\partial z}\right)+\left(\mathbf{i}\cdot\frac{\partial\mathbf{B}}{\partial x}+\mathbf{j}\cdot\frac{\partial\mathbf{B}}{\partial y}+\mathbf{k}\cdot\frac{\partial\mathbf{B}}{\partial z}\right)$$

$$=\nabla\cdot\mathbf{A}+\nabla\cdot\mathbf{B}=\text{div}\,\mathbf{A}+\text{div}\,\mathbf{B}.$$

2. *Prove that curl* $(\mathbf{A}+\mathbf{B})=$ *curl* $\mathbf{A}+$ *curl* $\mathbf{B}$

or $\qquad \nabla\times(\mathbf{A}+\mathbf{B})=\nabla\times\mathbf{A}+\nabla\times\mathbf{B}.$

Proof. We have curl $(\mathbf{A}+\mathbf{B})=\nabla\times(\mathbf{A}+\mathbf{B})$

$$=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\times(\mathbf{A}+\mathbf{B})=\Sigma\,\mathbf{i}\times\frac{\partial}{\partial x}(\mathbf{A}+\mathbf{B})=\Sigma\,\mathbf{i}\times\left(\frac{\partial\mathbf{A}}{\partial x}+\frac{\partial\mathbf{B}}{\partial x}\right)$$

$$=\Sigma \mathbf{i}\times\frac{\partial \mathbf{A}}{\partial x}+\Sigma \mathbf{i}\times\frac{\partial \mathbf{B}}{\partial x}=\text{curl } \mathbf{A}+\text{curl } \mathbf{B}.$$

3. *If* **A** *is a differentiable vector function and* ϕ *is a differentiable scalar function, then*

$$\textit{div}\,(\phi\mathbf{A})=(\textit{grad}\;\phi)\cdot\mathbf{A}+\phi\;\textit{div}\;\mathbf{A}$$

or

$$\nabla\cdot(\phi\mathbf{A})=(\nabla\phi)\cdot\mathbf{A}+\phi\,(\nabla\cdot\mathbf{A}).$$

[Meerut B. Sc. Physics 1983; Gorakhpur 85; Garhwal 84; Rohilkhand 82; Agra 81; Bombay 86]

Proof. We have

$$\text{div}\,(\phi\mathbf{A})=\nabla\cdot(\phi\mathbf{A})=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\cdot(\phi\mathbf{A})$$

$$=\mathbf{i}\cdot\frac{\partial}{\partial x}(\phi\mathbf{A})+\mathbf{j}\cdot\frac{\partial}{\partial y}(\phi\mathbf{A})+\mathbf{k}\cdot\frac{\partial}{\partial z}(\phi\mathbf{A})$$

$$=\Sigma\left\{\mathbf{i}\cdot\left(\frac{\partial}{\partial x}(\phi\mathbf{A})\right)\right\}=\Sigma\left\{\mathbf{i}\cdot\left(\frac{\partial\phi}{\partial x}\mathbf{A}+\phi\frac{\partial\mathbf{A}}{\partial x}\right)\right\}$$

$$=\Sigma\left\{\mathbf{i}\cdot\left(\frac{\partial\phi}{\partial x}\mathbf{A}\right)\right\}+\Sigma\left\{\mathbf{i}\cdot\left(\phi\frac{\partial\mathbf{A}}{\partial x}\right)\right\}$$

$$=\Sigma\left\{\left(\frac{\partial\phi}{\partial x}\mathbf{i}\right)\cdot\mathbf{A}\right\}+\Sigma\left\{\phi\left(\mathbf{i}\cdot\frac{\partial\mathbf{A}}{\partial x}\right)\right\}$$

[Note $\mathbf{a}\cdot(m\mathbf{b})=(m\mathbf{a})\cdot\mathbf{b}=m\,(\mathbf{a}\cdot\mathbf{b})$]

$$=\left\{\Sigma\frac{\partial\phi}{\partial x}\mathbf{i}\right\}\cdot\mathbf{A}+\phi\Sigma\left(\mathbf{i}\cdot\frac{\partial\mathbf{A}}{\partial x}\right)=(\nabla\phi)\cdot\mathbf{A}+\phi\,(\nabla\cdot\mathbf{A}).$$

4. *Prove that curl* $(\phi\mathbf{A})=(\textit{grad}\;\phi)\times\mathbf{A}+\phi$ *curl* **A**

or

$$\nabla\times(\phi\mathbf{A})=(\nabla\phi)\times\mathbf{A}+\phi\,(\nabla\times\mathbf{A}).$$

[Agra 1968; Meerut 67, 68, 72; Bombay 68; Kanpur 76; Punjab 63]

Proof. We have

$$\text{curl}\;(\phi\mathbf{A})=\nabla\times(\phi\mathbf{A})=\left(\mathbf{i}\frac{\partial}{\partial x}+\mathbf{j}\frac{\partial}{\partial y}+\mathbf{k}\frac{\partial}{\partial z}\right)\times(\phi\mathbf{A})$$

$$=\Sigma\left\{\mathbf{i}\times\frac{\partial}{\partial x}(\phi\mathbf{A})\right\}=\Sigma\left\{\mathbf{i}\times\left(\frac{\partial\phi}{\partial x}\mathbf{A}+\phi\frac{\partial\mathbf{A}}{\partial x}\right)\right\}$$

$$=\Sigma\left\{\mathbf{i}\times\left(\frac{\partial\phi}{\partial x}\mathbf{A}\right)\right\}+\Sigma\left\{\mathbf{i}\times\left(\phi\frac{\partial\mathbf{A}}{\partial x}\right)\right\}$$

$$=\Sigma\left\{\left(\frac{\partial\phi}{\partial x}\mathbf{i}\right)\times\mathbf{A}\right\}+\Sigma\left\{\phi\left(\mathbf{i}\times\frac{\partial\mathbf{A}}{\partial x}\right)\right\}$$

[Note that $\mathbf{a}\times(m\mathbf{b})=(m\mathbf{a})\times\mathbf{b}=m\,(\mathbf{a}\times\mathbf{b})$]

$$=\left\{\Sigma\left(\frac{\partial\phi}{\partial x}\mathbf{i}\right)\right\}\times\mathbf{A}+\phi\Sigma\left(\mathbf{i}\times\frac{\partial\mathbf{A}}{\partial x}\right)=(\nabla\phi)\times\mathbf{A}+\phi\,(\nabla\times\mathbf{A}).$$

IFS 2003 or

5. *Prove that div* $(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot curl\ \mathbf{A}-\mathbf{A}\cdot curl\ \mathbf{B}$

$$\nabla\cdot(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot(\nabla\times\mathbf{A})-\mathbf{A}\cdot(\nabla\times\mathbf{B}).$$

[Agra 1984, 85; Kanpur 87; Calicut 74; Allahabad 79; Gorakhpur 88]

Proof. We have

$$\text{div}\,(\mathbf{A}\times\mathbf{B})=\Sigma\left\{\mathbf{i}\cdot\frac{\partial}{\partial x}(\mathbf{A}\times\mathbf{B})\right\}=\Sigma\left\{\mathbf{i}\cdot\left(\frac{\partial\mathbf{A}}{\partial x}\times\mathbf{B}+\mathbf{A}\times\frac{\partial\mathbf{B}}{\partial x}\right)\right\}$$

$$=\Sigma\left\{\mathbf{i}\cdot\left(\frac{\partial\mathbf{A}}{\partial x}\times\mathbf{B}\right)\right\}+\Sigma\left\{\mathbf{i}\cdot\left(\mathbf{A}\times\frac{\partial\mathbf{B}}{\partial x}\right)\right\}$$

$$=\Sigma\left\{\left(\mathbf{i}\times\frac{\partial\mathbf{A}}{\partial x}\right)\cdot\mathbf{B}\right\}-\Sigma\left\{\mathbf{i}\cdot\left(\frac{\partial\mathbf{B}}{\partial x}\times\mathbf{A}\right)\right\}$$

[Note $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$ and $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})=-\mathbf{a}\cdot(\mathbf{c}\times\mathbf{b})$]

$$=\left\{\Sigma\left(\mathbf{i}\times\frac{\partial\mathbf{A}}{\partial x}\right)\right\}\cdot\mathbf{B}-\Sigma\left\{\left(\mathbf{i}\times\frac{\partial\mathbf{B}}{\partial x}\right)\cdot\mathbf{A}\right\}=(\text{curl}\,\mathbf{A})\cdot\mathbf{B}-\left\{\Sigma\left(\mathbf{i}\times\frac{\partial\mathbf{B}}{\partial x}\right)\right\}\cdot\mathbf{A}$$

$$=(\text{curl}\,\mathbf{A})\cdot\mathbf{B}-(\text{curl}\,\mathbf{B})\cdot\mathbf{A}=\mathbf{B}\cdot\text{curl}\,\mathbf{A}-\mathbf{A}\cdot\text{curl}\,\mathbf{B}.$$

IFS -2002 2000

6. *Prove that*

$$curl\,(\mathbf{A}\times\mathbf{B})=(\mathbf{B}\cdot\nabla)\,\mathbf{A}-\mathbf{B}\ div\ \mathbf{A}-(\mathbf{A}\cdot\nabla)\,\mathbf{B}+\mathbf{A}\ div\ \mathbf{B}.$$

[Meerut 1991; Agra 74; Allahabad 77; Ravishankar 82]

Proof. We have curl $(\mathbf{A}\times\mathbf{B})=\nabla\times(\mathbf{A}\times\mathbf{B})$

$$=\Sigma\left\{\mathbf{i}\times\frac{\partial}{\partial x}(\mathbf{A}\times\mathbf{B})\right\}=\Sigma\left\{\mathbf{i}\times\left(\mathbf{A}\times\frac{\partial\mathbf{B}}{\partial x}+\frac{\partial\mathbf{A}}{\partial x}\times\mathbf{B}\right)\right\}$$

$$=\Sigma\left\{\mathbf{i}\times\left(\mathbf{A}\times\frac{\partial\mathbf{B}}{\partial x}\right)\right\}+\Sigma\left\{\mathbf{i}\times\left(\frac{\partial\mathbf{A}}{\partial x}\times\mathbf{B}\right)\right\}$$

$$=\Sigma\left\{\left(\mathbf{i}\cdot\frac{\partial\mathbf{B}}{\partial x}\right)\mathbf{A}-(\mathbf{i}\cdot\mathbf{A})\frac{\partial\mathbf{B}}{\partial x}\right\}+\Sigma\left\{(\mathbf{i}\cdot\mathbf{B})\frac{\partial\mathbf{A}}{\partial x}-\left(\mathbf{i}\cdot\frac{\partial\mathbf{A}}{\partial x}\right)\mathbf{B}\right\}$$

$$=\Sigma\left\{\left(\mathbf{i}\cdot\frac{\partial\mathbf{B}}{\partial x}\right)\mathbf{A}\right\}-\Sigma\left\{(\mathbf{A}\cdot\mathbf{i})\frac{\partial\mathbf{B}}{\partial x}\right\}+\Sigma\left\{(\mathbf{B}\cdot\mathbf{i})\frac{\partial\mathbf{A}}{\partial x}\right\}-\Sigma\left\{\left(\mathbf{i}\cdot\frac{\partial\mathbf{A}}{\partial x}\right)\mathbf{B}\right\}$$

$$=\left\{\Sigma\left(\mathbf{i}\cdot\frac{\partial\mathbf{B}}{\partial x}\right)\right\}\mathbf{A}-\left\{\mathbf{A}\cdot\Sigma\mathbf{i}\frac{\partial}{\partial x}\right\}\mathbf{B}+\left\{\mathbf{B}\cdot\Sigma\mathbf{i}\frac{\partial}{\partial x}\right\}\mathbf{A}-\left\{\Sigma\left(\mathbf{i}\cdot\frac{\partial\mathbf{A}}{\partial x}\right)\right\}\mathbf{B}$$

$$=(\text{div}\,\mathbf{B})\,\mathbf{A}-(\mathbf{A}\cdot\nabla)\,\mathbf{B}+(\mathbf{B}\cdot\nabla)\,\mathbf{A}-(\text{div}\,\mathbf{A})\,\mathbf{B}.$$

2004

7. *Prove that*

$$grad\,(\mathbf{A}\cdot\mathbf{B})=(\mathbf{B}\cdot\nabla)\,\mathbf{A}+(\mathbf{A}\cdot\nabla)\,\mathbf{B}+\mathbf{B}\times curl\ \mathbf{A}+\mathbf{A}\times curl\ \mathbf{B}.$$

[Allahabad 1980, 82; Rohilkhand 78; Jiwaji 81]

Proof. We have

$$\text{grad}\,(\mathbf{A}\cdot\mathbf{B})=\nabla(\mathbf{A}\cdot\mathbf{B})=\Sigma\mathbf{i}\frac{\partial}{\partial x}(\mathbf{A}\cdot\mathbf{B})=\Sigma\mathbf{i}\left(\mathbf{A}\cdot\frac{\partial\mathbf{B}}{\partial x}+\frac{\partial\mathbf{A}}{\partial x}\cdot\mathbf{B}\right)$$

$$=\Sigma\left\{\left(\mathbf{A}\cdot\frac{\partial\mathbf{B}}{\partial x}\right)\mathbf{i}\right\}+\Sigma\left\{\left(\mathbf{B}\cdot\frac{\partial\mathbf{A}}{\partial x}\right)\mathbf{i}\right\}. \quad \ldots(1)$$

Now we know that $\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\, \mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\, \mathbf{c}$.

$$\therefore \quad (\mathbf{a} \cdot \mathbf{b})\, \mathbf{c} = (\mathbf{a} \cdot \mathbf{c})\, \mathbf{b} - \mathbf{a} \times (\mathbf{b} \times \mathbf{c}).$$

$$\therefore \quad \left(\mathbf{A} \cdot \frac{\partial \mathbf{B}}{\partial x}\right) \mathbf{i} = (\mathbf{A} \cdot \mathbf{i}) \frac{\partial \mathbf{B}}{\partial x} - \mathbf{A} \times \left(\frac{\partial \mathbf{B}}{\partial x} \times \mathbf{i}\right)$$

$$= (\mathbf{A} \cdot \mathbf{i}) \frac{\partial \mathbf{B}}{\partial x} + \mathbf{A} \times \left(\mathbf{i} \times \frac{\partial \mathbf{B}}{\partial x}\right).$$

$$\text{Thus } \Sigma \left\{\left(\mathbf{A} \cdot \frac{\partial \mathbf{B}}{\partial x}\right) \mathbf{i}\right\} = \Sigma \left\{(\mathbf{A} \cdot \mathbf{i}) \frac{\partial \mathbf{B}}{\partial x}\right\} + \Sigma \left\{\mathbf{A} \times \left(\mathbf{i} \times \frac{\partial \mathbf{B}}{\partial x}\right)\right\}$$

$$= \left\{\mathbf{A} \cdot \Sigma \mathbf{i} \frac{\partial}{\partial x}\right\} \mathbf{B} + \mathbf{A} \times \Sigma \left(\mathbf{i} \times \frac{\partial \mathbf{B}}{\partial x}\right)$$

$$= (\mathbf{A} \cdot \nabla)\, \mathbf{B} + \mathbf{A} \times (\nabla \times \mathbf{B}). \qquad \ldots(2)$$

$$\text{Similarly } \Sigma \left\{\left(\mathbf{B} \cdot \frac{\partial \mathbf{A}}{\partial x}\right) \mathbf{i}\right\} = (\mathbf{B} \cdot \nabla)\, \mathbf{A} + \mathbf{B} \times (\nabla \times \mathbf{A}). \qquad \ldots(3)$$

Putting the values from (2) and (3) in (1), we get

$$\text{grad}\,(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\, \mathbf{B} + \mathbf{A} \times (\nabla \times \mathbf{B}) + (\mathbf{B} \cdot \nabla)\, \mathbf{A} + \mathbf{B} \times (\nabla \times \mathbf{A}).$$

Note. If we put **A** in place of **B**, then

$$\text{grad}\,(\mathbf{A} \cdot \mathbf{A}) = 2\,(\mathbf{A} \cdot \nabla)\, \mathbf{A} + 2\mathbf{A} \times (\nabla \times \mathbf{A})$$

or $$\tfrac{1}{2}\, \text{grad}\, \mathbf{A}^2 = (\mathbf{A} \cdot \nabla)\, \mathbf{A} + \mathbf{A} \times \text{curl}\, \mathbf{A}.$$

2003

8. *Prove that div grad* $\phi = \nabla^2 \phi$

i.e. $\nabla \cdot (\nabla \phi) = \nabla^2 \phi$. **[Rohilkhand 1981; Garhwal 85]**

Proof. We have

$$\nabla \cdot (\nabla \phi) = \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{j} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial \phi}{\partial x} \mathbf{i} + \frac{\partial \phi}{\partial y} \mathbf{j} + \frac{\partial \phi}{\partial z} \mathbf{k}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial z}\right)$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \phi = \nabla^2 \phi.$$

9. *Prove that curl of the gradient of* ϕ *is zero*

i.e. $\nabla \times (\nabla \phi) = 0$, *i.e. curl grad* $\phi = 0$.

[Meerut 1991S; Rohilkhand 81; Agra 74; Garhwal 82; Kerala 74; Jiwaji 83]

Proof. We have $\text{grad}\, \phi = \dfrac{\partial \phi}{\partial x} \mathbf{i} + \dfrac{\partial \phi}{\partial y} \mathbf{j} + \dfrac{\partial \phi}{\partial z} \mathbf{k}$.

$$\therefore \quad \text{curl grad}\, \phi = \nabla \times \text{grad}\, \phi$$

$$= \left(\mathbf{i} \frac{\partial}{\partial x} + \mathbf{i} \frac{\partial}{\partial y} + \mathbf{k} \frac{\partial}{\partial z}\right) \times \left(\frac{\partial \phi}{\partial x} \mathbf{i} + \frac{\partial \phi}{\partial y} \mathbf{j} + \frac{\partial \phi}{\partial z} \mathbf{k}\right)$$

$$=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial \phi}{\partial x} & \frac{\partial \phi}{\partial y} & \frac{\partial \phi}{\partial z} \end{vmatrix}$$

$$=\left(\frac{\partial^2\phi}{\partial y\,\partial z}-\frac{\partial^2\phi}{\partial z\,\partial y}\right)\mathbf{i}+\left(\frac{\partial^2\phi}{\partial z\,\partial x}-\frac{\partial^2\phi}{\partial x\,\partial z}\right)\mathbf{j}+\left(\frac{\partial^2\phi}{\partial x\,\partial y}-\frac{\partial^2\phi}{\partial y\,\partial x}\right)\mathbf{k}$$

$$=0\mathbf{i}+0\mathbf{j}+0\mathbf{k}=\mathbf{0},$$

provided we suppose that ϕ has continuous second partial derivatives so that the order of differentiation is immaterial.

10 *Prove that div curl* $\mathbf{A}=0$, *i.e.*, $\nabla\cdot(\nabla\times\mathbf{A})=0$.

[Meerut 1992; Kanpur 89; Agra 82; Rohilkhand 90]

Proof. Let $\mathbf{A}=A_1\mathbf{i}+A_2\mathbf{j}+A_3\mathbf{k}$.

Then curl $\mathbf{A}=\nabla\times\mathbf{A}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$

$$=\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)\mathbf{i}+\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)\mathbf{j}+\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)\mathbf{k}.$$

Now div curl $\mathbf{A}=\nabla\cdot(\nabla\times\mathbf{A})$

$$=\frac{\partial}{\partial x}\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)$$

$$=\frac{\partial^2 A_3}{\partial x\,\partial y}-\frac{\partial^2 A_2}{\partial x\,\partial z}+\frac{\partial^2 A_1}{\partial y\,\partial z}-\frac{\partial^2 A_3}{\partial y\,\partial x}+\frac{\partial^2 A_2}{\partial z\,\partial x}-\frac{\partial^2 A_1}{\partial z\,\partial y}$$

$=0$, assuming that $\mathbf{A}$ has continuous second partial derivatives.

IFS 2001 2002

11. *Prove that*

$\nabla\times(\nabla\times\mathbf{A})=\nabla(\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}$. **[Meerut B.Sc. Physics 1983; Kanpur 86; Allahabad 81; Rohilkhand 90]**

Proof. Let $\mathbf{A}=A_1\mathbf{i}+A_2\mathbf{j}+A_3\mathbf{k}$.

Then $\nabla\times\mathbf{A}=\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$

$$=\left(\frac{\partial A_3}{\partial y}-\frac{\partial A_2}{\partial z}\right)\mathbf{i}+\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)\mathbf{j}+\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)\mathbf{k}.$$

$$\therefore\ \nabla\times(\nabla\times\mathbf{A})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ \dfrac{\partial A_3}{\partial y}-\dfrac{\partial A_2}{\partial z} & \dfrac{\partial A_1}{\partial z}-\dfrac{\partial A_3}{\partial x} & \dfrac{\partial A_2}{\partial x}-\dfrac{\partial A_1}{\partial y}\end{vmatrix}$$

$$=\Sigma\left[\left\{\frac{\partial}{\partial y}\left(\frac{\partial A_2}{\partial x}-\frac{\partial A_1}{\partial y}\right)-\frac{\partial}{\partial z}\left(\frac{\partial A_1}{\partial z}-\frac{\partial A_3}{\partial x}\right)\right\}\mathbf{i}\right]$$

$$=\Sigma\left[\left\{\left(\frac{\partial^2 A_2}{\partial y\ \partial x}+\frac{\partial^2 A_3}{\partial z\ \partial x}\right)-\left(\frac{\partial^2 A_1}{\partial y^2}+\frac{\partial^2 A_1}{\partial z^2}\right)\right\}\mathbf{i}\right]$$

$$=\Sigma\left[\left\{\frac{\partial}{\partial x}\left(\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}\right)-\left(\frac{\partial^2 A_1}{\partial y^2}+\frac{\partial^2 A_1}{\partial z^2}\right)\right\}\mathbf{i}\right]$$

$$=\Sigma\left[\left\{\frac{\partial}{\partial x}\left(\frac{\partial A_1}{\partial x}+\frac{\partial A_2}{\partial y}+\frac{\partial A_3}{\partial z}\right)-\left(\frac{\partial^2 A_1}{\partial x^2}+\frac{\partial^2 A_1}{\partial y^2}+\frac{\partial^2 A_1}{\partial z^2}\right)\right\}\mathbf{i}\right]$$

$$=\Sigma\left[\left\{\frac{\partial}{\partial x}(\nabla\cdot\mathbf{A})-(\nabla^2 A_1)\right\}\mathbf{i}\right]$$

$$=\Sigma\left[\left\{\frac{\partial}{\partial x}(\nabla\cdot\mathbf{A})\right\}\mathbf{i}\right]-\nabla^2\Sigma A_1\mathbf{i}=\nabla\ (\nabla\cdot\mathbf{A})-\nabla^2\mathbf{A}.$$

Solved Examples

Ex. 1. *Prove that grad $f(u)=f'(u)$ grad u.*

Sol. We have

$$\text{grad } f(u)=\mathbf{i}\,\frac{\partial}{\partial x}f(u)+\mathbf{j}\,\frac{\partial}{\partial y}f(u)+\mathbf{k}\,\frac{\partial}{\partial z}f(u)$$

$$=\mathbf{i}\,f'(u)\frac{\partial u}{\partial x}+\mathbf{j}\,f'(u)\,\frac{\partial u}{\partial y}+\mathbf{k}\,f'(u)\,\frac{\partial u}{\partial z}$$

$$=f'(u)\left[\mathbf{i}\,\frac{\partial u}{\partial x}+\mathbf{j}\,\frac{\partial u}{\partial y}+\mathbf{k}\,\frac{\partial u}{\partial z}\right]=f'(u)\text{ grad } u.$$

Ex. 2. *Taking* $\mathbf{F}=x^2y\ \mathbf{i}+xz\ \mathbf{j}+2yz\ \mathbf{k}$, *verify that* *div curl* $\mathbf{F}=0$. **[Agra 1986; Rohilkhand 85]**

Sol. We have Curl $\mathbf{F}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ x^2y & xz & 2yz\end{vmatrix}$

$$=\left[\frac{\partial}{\partial y}(2yz)-\frac{\partial}{\partial z}(xz)\right]\mathbf{i}-\left[\frac{\partial}{\partial x}(2yz)-\frac{\partial}{\partial z}(x^2y)\right]\mathbf{j}$$
$$+\left[\frac{\partial}{\partial x}(xz)-\frac{\partial}{\partial y}(x^2y)\right]\mathbf{k}$$

$=(2z-x)\,\mathbf{i}-0\,\mathbf{j}+(z-x^2)\,\mathbf{k}=(2z-x)\,\mathbf{i}+(z-x^2)\,\mathbf{k}.$

Now div curl $\mathbf{F}=$ div $[(2z-x)\,\mathbf{i}+(z-x^2)\,\mathbf{k}]$

$$=\frac{\partial}{\partial x}(2z-x)+\frac{\partial}{\partial z}(z-x^2)=-1+1=0.$$

Ex. 2. *Verify that curl grad $f=\mathbf{0}$, where*

$f=x^2y+2xy+z^2.$ [Agra 1973]

Sol. We have grad $f=(\partial f/\partial x)\,\mathbf{i}+(\partial f/\partial y)\,\mathbf{j}+(\partial f/\partial z)\,\mathbf{k}$

$=(2xy+2y)\,\mathbf{i}+(x^2+2x)\,\mathbf{j}+2z\mathbf{k}.$

$\therefore$ curl grad $f=\nabla\times[(2xy+2y)\,\mathbf{i}+(x^2+2x)\,\mathbf{j}+2z\mathbf{k}]$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ 2xy+2y & x^2+2x & 2z\end{vmatrix}$$

$$=\left[\frac{\partial}{\partial y}(2z)-\frac{\partial}{\partial z}(x^2+2x)\right]\mathbf{i}+\left[\frac{\partial}{\partial z}(2xy+2y)-\frac{\partial}{\partial x}(2z)\right]\mathbf{j}$$
$$+\left[\frac{\partial}{\partial x}(x^2+2x)-\frac{\partial}{\partial y}(2xy+2y)\right]\mathbf{k}$$

$=(0-0)\,\mathbf{i}+(0-0)\,\mathbf{j}+(2x+2-2x-2)\,\mathbf{k}$

$=0\mathbf{i}+0\mathbf{j}+0\mathbf{k}=\mathbf{0}.$

Ex. 4. *Prove that*

curl $(\psi\nabla\phi)=\nabla\psi\times\nabla\phi=-$*curl* $(\phi\nabla\psi).$ [Bombay 1986]

Sol. We know that curl $(\phi\mathbf{A})=(\nabla\phi)\times\mathbf{A}+\phi$ curl $\mathbf{A}$.

In the above formula replacing ϕ by ψ and $\mathbf{A}$ by $\nabla\phi$, we have

curl $(\psi\nabla\phi)=(\nabla\psi)\times\nabla\phi+\psi$ curl $\nabla\phi$

$=\nabla\psi\times\nabla\phi+\mathbf{0}$

[$\because$ curl $\nabla\phi=$ curl grad $\phi=\mathbf{0}$]

$=\nabla\psi\times\nabla\phi.$ ...(1)

Similarly curl $(\phi\nabla\psi)=(\nabla\phi)\times\nabla\psi+\phi$ curl $\nabla\psi$

$=\nabla\phi\times\nabla\psi+\mathbf{0}$

$=\nabla\phi\times\nabla\psi$

$=-\nabla\psi\times\nabla\phi.$ ...(2)

From (1) and (2), we have

curl $(\psi\nabla\phi)=\nabla\psi\times\nabla\phi=-$curl $(\phi\nabla\psi).$

Ex. 5. *Show that curl* $(\mathbf{a}\cdot\mathbf{r})\,\mathbf{a}=\mathbf{0}$, *where* $\mathbf{a}$ *is a constant vector.*

Sol. We know that curl $(\phi\mathbf{A})=(\nabla\phi)\times\mathbf{A}+\phi$ curl $\mathbf{A}$.

Replacing ϕ by $\mathbf{a}\cdot\mathbf{r}$ and $\mathbf{A}$ by $\mathbf{a}$ in the above formula, we have

$$\text{curl }(\mathbf{a}\cdot\mathbf{r})\,\mathbf{a}=[\nabla(\mathbf{a}\cdot\mathbf{r})]\times\mathbf{a}+(\mathbf{a}\cdot\mathbf{r})\text{ curl }\mathbf{a}. \quad \ldots(1)$$

But if $\mathbf{a}$ is a constant vector, then curl $\mathbf{a}=\mathbf{0}$ and $\nabla(\mathbf{a}\cdot\mathbf{r})=\mathbf{a}$.

$\therefore$ from (1), we have

$$\text{curl }(\mathbf{a}\cdot\mathbf{r})\,\mathbf{a}=\mathbf{a}\times\mathbf{a}+\mathbf{0}=\mathbf{0}+\mathbf{0}=\mathbf{0}.$$

Ex. 6. *Find* $\nabla\phi$ *and* $|\nabla\phi|$ *when*

$$\phi=(x^2+y^2+z^2)\,e^{-(x^2+y^2+z^2)^{1/2}}.$$

Sol. Let $r^2=x^2+y^2+z^2$. Then we can write $\phi=r^2e^{-r}$.

Now $\nabla\phi=\dfrac{\partial\phi}{\partial x}\mathbf{i}+\dfrac{\partial\phi}{\partial y}\mathbf{j}+\dfrac{\partial\phi}{\partial z}\mathbf{k}.$

We have $\dfrac{\partial\phi}{\partial x}=\dfrac{\partial\phi}{\partial r}\dfrac{\partial r}{\partial x}=[2re^{-r}-r^2e^{-r}]\dfrac{\partial r}{\partial x}.$

But $r^2=x^2+y^2+z^2.$

Therefore $2r\dfrac{\partial r}{\partial x}=2x$ or $\dfrac{\partial r}{\partial x}=\dfrac{x}{r}.$

So $\dfrac{\partial\phi}{\partial x}=re^{-r}(2-r)\dfrac{x}{r}=(2-r)\,e^{-r}x.$

Similarly $\dfrac{\partial\phi}{\partial y}=(2-r)\,e^{-r}y$ and $\dfrac{\partial\phi}{\partial z}=(2-r)\,e^{-r}z.$

Therefore $\nabla\phi=(2-r)\,e^{-r}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})=(2-r)\,e^{-r}\mathbf{r}.$

Also $|\nabla\phi|=|(2-r)\,e^{-r}\mathbf{r}|=(2-r)\,e^{-r}|\mathbf{r}|=(2-r)\,e^{-r}r.$

Ex. 7. *Prove that div* $(r^n\mathbf{r})=(n+3)\,r^n$.

[Gorakhpur 1985; Rohilkhand 78; Kanpur 87]

Sol. We have

$$\text{div }(\phi\mathbf{A})=\phi\,(\text{div }\mathbf{A})+\mathbf{A}\cdot\text{grad }\phi.$$

Putting $\mathbf{A}=\mathbf{r}$ and $\phi=r^n$ in this identity, we get

$$\begin{aligned}\text{div }(r^n\mathbf{r})&=r^n\text{ div }\mathbf{r}+\mathbf{r}\cdot\text{grad }r^n\\&=3r^n+\mathbf{r}\cdot(nr^{n-1}\text{ grad }r)\qquad[\because\ \text{div }\mathbf{r}=3\text{ and grad }f(u)=f'(u)\text{ grad }u]\\&=3r^n+\mathbf{r}\cdot\left[nr^{n-1}\frac{1}{r}\mathbf{r}\right]\qquad\left[\because\ \text{grad }r=\hat{\mathbf{r}}=\frac{1}{r}\mathbf{r}\right]\\&=3r^n+nr^{n-2}(\mathbf{r}\cdot\mathbf{r})=3r^n+nr^{n-2}r^2=(n+3)\,r^n.\end{aligned}$$

Ex. 8. *Prove that* $\nabla^2(r^n\mathbf{r})=n\,(n+3)\,r^{n-2}\,\mathbf{r}$. **[Kanpur 1988]**

Sol. We have $\nabla^2(r^n\mathbf{r})=\nabla[\nabla\cdot(r^n\mathbf{r})]=\text{grad }[\text{div }(r^n\mathbf{r})]$

$=\text{grad }[(\text{grad }r^n)\cdot\mathbf{r}+r^n\text{ div }\mathbf{r}]$

$$=\text{grad }[(nr^{n-2}\,\mathbf{r})\cdot\mathbf{r}+3r^n]=\text{grad }[nr^{n-2}\,\mathbf{r}^2+3r^n]$$
$$=\text{grad }[nr^{n-2}\,r^2+3r^n]=\text{grad }[(n+3)\,r^n]$$
$$=(n+3)\text{ grad }r^n=(n+3)\,nr^{n-2}\,\mathbf{r}=n\,(n+3)\,r^{n-2}\,\mathbf{r}.$$

Ex. 9. *Prove that div* $\left(\frac{\mathbf{r}}{r^3}\right)=0.$ **[Banaras 1978]**

Sol. We have $\text{div}\left(\frac{1}{r^3}\,\mathbf{r}\right)=\text{div }(r^{-3}\,\mathbf{r})$

$$=r^{-3}\text{ div }\mathbf{r}+\mathbf{r}\cdot\text{grad }r^{-3}=3r^{-3}+\mathbf{r}\cdot(-3r^{-4}\text{ grad }r)$$
$$=3r^{-3}+\mathbf{r}\cdot\left(-3r^{-4}\frac{1}{r}\,\mathbf{r}\right)$$
$$=3r^{-3}-3r^{-5}\,(\mathbf{r}\cdot\mathbf{r})=3r^{-3}-3r^{-5}\,r^2=3r^{-3}-3r^{-3}=0.$$

$\therefore$ the vector $r^{-3}\,\mathbf{r}$ is solenoidal.

Ex. 10. *Prove that div* $\hat{\mathbf{r}}=2/r.$ **[Kanpur 1979]**

Sol. $\text{div }(\hat{\mathbf{r}})=\text{div}\left(\frac{1}{r}\,\mathbf{r}\right).$ Now proceed as in Ex. 7.

Alternative Method.

$$\text{div }\hat{\mathbf{r}}=\text{div}\left(\frac{1}{r}\,\mathbf{r}\right)=\text{div}\left[\frac{1}{r}\,(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\right]$$
$$=\text{div}\left(\frac{x}{r}\,\mathbf{i}+\frac{y}{r}\,\mathbf{j}+\frac{z}{r}\,\mathbf{k}\right)=\frac{\partial}{\partial x}\left(\frac{x}{r}\right)+\frac{\partial}{\partial y}\left(\frac{y}{r}\right)+\frac{\partial}{\partial z}\left(\frac{z}{r}\right)$$
$$=\left(\frac{1}{r}-\frac{x}{r^2}\,\frac{\partial r}{\partial x}\right)+\left(\frac{1}{r}-\frac{y}{r^2}\,\frac{\partial r}{\partial y}\right)+\left(\frac{1}{r}-\frac{z}{r^2}\,\frac{\partial r}{\partial z}\right)$$

Now $r^2=x^2+y^2+z^2$. $\therefore$ $2r\,\frac{\partial r}{\partial x}=2x$ *i.e.* $\frac{\partial r}{\partial x}=\frac{x}{r}\cdot$

Similarly $\frac{\partial r}{\partial y}=\frac{y}{r}$ and $\frac{\partial r}{\partial z}=\frac{z}{r}\cdot$

$$\therefore\quad \text{div }\hat{\mathbf{r}}=\frac{3}{r}-\left(\frac{x}{r^2}\,\frac{x}{r}+\frac{y}{r^2}\,\frac{y}{r}+\frac{z}{r^2}\,\frac{z}{r}\right)$$
$$=\frac{3}{r}-\frac{x^2+y^2+z^2}{r^3}=\frac{3}{r}-\frac{r^2}{r^3}=\frac{3}{r}-\frac{1}{r}=\frac{2}{r}\cdot$$

Ex. 11. *Prove that vector* $f(r)\,\mathbf{r}$ *is irrotational.*

[Agra 1974; Kanpur 75]

Sol. The vector $f(r)\,\mathbf{r}$ will be irrotational if

$$\text{curl }[f(r)\,\mathbf{r}]=\mathbf{0}.$$

We know that $\text{Curl }(\phi\mathbf{A})=(\text{grad }\phi)\times\mathbf{A}+\phi\text{ curl }\mathbf{A}.$

Putting $\phi=f(r)$ and $\mathbf{A}=\mathbf{r}$ in this identity, we get

$$\text{Curl }[f(r)\,\mathbf{r}]=[\text{grad }f(r)]\times\mathbf{r}+f(r)\text{ curl }\mathbf{r}$$
$$=[f'(r)\text{ grad }r]\times\mathbf{r}+f(r)\,\mathbf{0}\qquad[\because\ \text{curl }\mathbf{r}=\mathbf{0}]$$

$$=\left[f'(r)\frac{1}{r}\mathbf{r}\right]\times\mathbf{r}=f'(r)\frac{1}{r}(\mathbf{r}\times\mathbf{r})=\mathbf{0},\text{ since }\mathbf{r}\times\mathbf{r}=\mathbf{0}.$$

$\therefore$ The vector $f(r)\,\mathbf{r}$ is irrotational.

IFS 2003 2008

Ex. 12. *Prove that* $\nabla^2 f(r)=f''(r)+\frac{2}{r}f'(r)$. **[Agra 1977]**

Sol. We know that if ϕ is a scalar function then

$$\nabla^2\phi=\nabla\cdot(\nabla\phi).$$

$$\therefore\quad \nabla^2 f(r)=\nabla\cdot\{\nabla f(r)\}=\text{div }\{\text{grad } f(r)\}$$

$$=\text{div }\{f'(r)\text{ grad } r\}=\text{div}\left\{\frac{1}{r}f'(r)\,\mathbf{r}\right\}$$

$$=\frac{1}{r}f'(r)\text{ div }\mathbf{r}+\mathbf{r}\cdot\text{grad}\left\{\frac{1}{r}f'(r)\right\}$$

$$=\frac{3}{r}f'(r)+\mathbf{r}\cdot\left[\frac{d}{dr}\left\{\frac{1}{r}f'(r)\right\}\text{grad } r\right]$$

$$=\frac{3}{r}f'(r)+\mathbf{r}\cdot\left[\left\{-\frac{1}{r^2}f'(r)+\frac{1}{r}f''(r)\right\}\frac{1}{r}\mathbf{r}\right]$$

$$=\frac{3}{r}f'(r)+\left[\frac{1}{r}\left\{-\frac{1}{r^2}f'(r)+\frac{1}{r}f''(r)\right\}\right](\mathbf{r}\cdot\mathbf{r})$$

$$=\frac{3}{r}f'(r)+\left[\frac{1}{r}\left\{-\frac{1}{r^2}f'(r)+\frac{1}{r}f''(r)\right\}\right]r^2$$

$$=\frac{3}{r}f'(r)-\frac{1}{r}f(r)+f''(r)=f''(r)+\frac{2}{r}f'(r).$$

Ex. 13. *If* $\nabla^2 f(r)=0$, *show that*

$$f(r)=\frac{c_1}{r}+c_2,$$

where $r^2=x^2+y^2+z^2$ *and* c_1, c_2 *are arbitrary constants.*

[Bombay 1989]

Sol. As shown in the preceding example, if

$r^2=x^2+y^2+z^2$, then $\nabla^2 f(r)=f''(r)+\frac{2}{r}f'(r)$.

$\therefore$ if $\nabla^2 f(r)=0$, then

$$f''(r)+\frac{2}{r}f'(r)=0\quad\text{or}\quad\frac{f''(r)}{f'(r)}=-\frac{2}{r}.$$

Integrating with respect to r, we get

$$\log f'(r)=-2\log r+\log c,\text{ where } c \text{ is a constant}$$

$$=\log\frac{c}{r^2}.$$

$$\therefore\quad f'(r)=\frac{c}{r^2}.$$

Again integrating,

$$f(r)=-\frac{c}{r}+c_2 \text{ where } c_2 \text{ is a constant}$$

$$=\frac{c_1}{r}+c_2, \text{ replacing } -c \text{ by } c_1.$$

Ex. 14. *Prove that* $\nabla^2\left(\frac{1}{r}\right)=0$ *or* $div\left(grad\,\frac{1}{r}\right)=0.$

[Meerut 1991S; Agra 84; Rohilkhand 81; Kanpur 79]

Sol. We have

$$\nabla^2\left(\frac{1}{r}\right)=\nabla\cdot\left(\nabla\frac{1}{r}\right)=\text{div}\left(\text{grad }\frac{1}{r}\right)$$

$$=\text{div}\left(-\frac{1}{r^2}\text{ grad } r\right)=\text{div}\left(-\frac{1}{r^2}\frac{1}{r}\mathbf{r}\right)=\text{div}\left(-\frac{1}{r^3}\mathbf{r}\right)$$

$$=\left(-\frac{1}{r^3}\right)\text{div }\mathbf{r}+\mathbf{r}\cdot\text{grad}\left(-\frac{1}{r^3}\right)=-\frac{3}{r^3}+\mathbf{r}\cdot\left[\frac{d}{dr}\left(-\frac{1}{r^3}\right)\text{ grad } r\right]$$

$$=-\frac{3}{r^3}+\mathbf{r}\cdot\left(\frac{3}{r^4}\frac{1}{r}\mathbf{r}\right)=-\frac{3}{r^3}+\frac{3}{r^5}(\mathbf{r}\cdot\mathbf{r})=-\frac{3}{r^3}+\frac{3}{r^5}r^2=0.$$

$\therefore$ $1/r$ is a solution of Laplace's equation.

Ex. 15. *Prove that div grad* $r^n=n(n+1)r^{n-2}$,

i.e. $\nabla^2 r^n=n(n+1)r^{n-2}$.

[Kanpur 1978, 80; Rohilkhand 81; Agra 84; Jiwaji 83]

Sol. We have $\nabla^2 r^n=\nabla\cdot(\nabla r^n)=\text{div}(\text{grad } r^n)$

$$=\text{div}(nr^{n-1}\text{ grad } r)=\text{div}\left(nr^{n-1}\frac{1}{r}\mathbf{r}\right)=\text{div}(nr^{n-2}\mathbf{r})$$

$$=(nr^{n-2})\text{ div }\mathbf{r}+\mathbf{r}\cdot(\text{grad } nr^{n-2})$$

$$=3nr^{n-2}+\mathbf{r}\cdot[n(n-2)r^{n-2}\text{ grad } r]$$

$$=3nr^{n-2}+\mathbf{r}\cdot\left[n(n-2)r^{n-3}\frac{1}{r}\mathbf{r}\right]$$

$$=3nr^{n-2}+\mathbf{r}\cdot[n(n-2)r^{n-4}\mathbf{r}]=3nr^{n-2}+n(n-2)r^{n-4}(\mathbf{r}\cdot\mathbf{r})$$

$$=3nr^{n-2}+n(n-2)r^{n-4}r^2=nr^{n-2}(3+n-2)=n(n+1)r^{n-2}.$$

Note. If $n=-1$, then $\nabla^2(r^{-1})=(-1)(-1+1)r^{-3}=0.$

Ex. 16. *Prove that curl grad* $r^n=\mathbf{0}$.

[Rohilkhand 1992; Garhwal 81]

Sol. Let $r^n=\phi$. Now proceed as in identity 9 of § 11.

Ex. 17. *If* $\mathbf{r}$ *is the position vector of the point* (x, y, z) *show that curl* $(r^n\mathbf{r})=\mathbf{0}$, *where* r *is the module of* $\mathbf{r}$. [Kanpur 1978]

Sol. We know that curl $(\phi\mathbf{A})=(\nabla\phi)\times\mathbf{A}+\phi$ curl $\mathbf{A}$. Putting $\phi=r^n$ and $\mathbf{A}=\mathbf{r}$ in this identity, we get

$$\text{curl }(r^n\mathbf{r})=(\nabla r^n)\times\mathbf{r}+r^n\text{ curl }\mathbf{r}$$

$$=(nr^{n-1}\nabla r)\times\mathbf{r}+r^n\,\mathbf{0}$$

$[\because \nabla f(r)=f'(r)\nabla r$ and curl $\mathbf{r}=$ curl $(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})=\mathbf{0}]$

$$=\left(nr^{n-1}\frac{1}{r}\mathbf{r}\right)\times\mathbf{r} \qquad \left[\because \nabla r=\frac{1}{r}\mathbf{r}\right]$$

$$=nr^{n-2}(\mathbf{r}\times\mathbf{r})=nr^{n-2}\,\mathbf{0} \qquad [\because \mathbf{r}\times\mathbf{r}=\mathbf{0}]$$

$$=\mathbf{0}.$$

2006 **Ex. 18.** *Prove that $r^n\,\mathbf{r}$ is an irrotational vector for any value of n but is solenoidal only if $n+3=0$.* **[Agra 1976; Rohilkhand 78]**

Sol. Let $\mathbf{F}=r^n\,\mathbf{r}$.

The vector $\mathbf{F}$ is irrotational if curl $\mathbf{F}=\mathbf{0}$. Proceeding as in Ex. 17 show that curl $(r^n\,\mathbf{r})=\mathbf{0}$ for any value of n.

$\therefore$ $r^n\,\mathbf{r}$ is an irrotational vector for any value of n.

The vector $\mathbf{F}$ is solenoidal if div $\mathbf{F}=0$. Proceeding as in Ex. 7, show that div $(r^n\,\mathbf{r})=(n+3)\,r^n$.

$\therefore$ the vector $r^n\,\mathbf{r}$ is solenoidal only if $(n+3)\,r^n=0$ *i.e.*, only if $n+3=0$ *i.e.*, only if $n=-3$.

2007 **Ex. 19.** *If $\mathbf{u}=(1/r)\,\mathbf{r}$, show that $\nabla\times\mathbf{u}=\mathbf{0}$.* **[Kanpur 1979]**

Sol. We have $\nabla\times\mathbf{u}=$ curl $\mathbf{u}=$ curl $[(1/r)\,\mathbf{r}]$.

We know that curl $(\phi\mathbf{A})=(\nabla\phi)\times\mathbf{A}+\phi$ curl $\mathbf{A}$.

Replacing ϕ by $1/r$ and $\mathbf{A}$ by $\mathbf{r}$ in this identity, we have

curl $[(1/r)\,\mathbf{r}]=[\nabla(1/r)]\times\mathbf{r}+(1/r)$ curl $\mathbf{r}$

$$=\left[\left(-\frac{1}{r^2}\right)\nabla r\right]\times\mathbf{r}+(1/r)\,\mathbf{0}$$

$[\because$ curl $\mathbf{r}=\mathbf{0}$ and $\nabla f(r)=f'(r)\nabla r]$

$$=\left[-\frac{1}{r^2}\frac{1}{r}\mathbf{r}\right]\times\mathbf{r} \qquad \left[\because \nabla r=\frac{1}{r}\mathbf{r}\right]$$

$$=-\frac{1}{r^3}(\mathbf{r}\times\mathbf{r})=-\frac{1}{r^3}\,\mathbf{0}=\mathbf{0}.$$

Hence $\nabla\times\mathbf{u}=\mathbf{0}$ if $\mathbf{u}=(1/r)\,\mathbf{r}$.

Ex. 20. *If $\mathbf{u}=(1/r)\,\mathbf{r}$ find grad (div $\mathbf{u}$).* **[Kanpur 1976]**

Sol. Proceeding as in Ex. 10, first show that div $\mathbf{u}=$ div $[(1/r)\,\mathbf{r}]=2/r$.

$\therefore$ grad (div $\mathbf{u}$) $=$ grad $(2/r)=(-2/r^2)$ grad r

$[\because$ grad $f(r)=f'(r)$ grad $r]$

$$=-\frac{2}{r^2}\left(\frac{1}{r}\mathbf{r}\right)=-\frac{2}{r^3}\mathbf{r}.$$

Ex. 21. *If $\nabla^2 f(r)=0$ show that $f(r)=c_1\log r+c_2$ where $r^2=x^2+y^2$ and c_1, c_2 are arbitrary constants.* **[Poona 1970]**

Sol. We have $\nabla^2 f(r)=\left(\dfrac{\partial^2}{\partial x^2}+\dfrac{\partial^2}{\partial y^2}\right)f(r)=\dfrac{\partial^2}{\partial x^2}f(r)+\dfrac{\partial^2}{\partial y^2}f(r)$

Now $\frac{\partial}{\partial x} f(r)=f'(r) \cdot \frac{\partial r}{\partial x}$.

But from $r^2=x^2+y^2$, we have $2r \frac{\partial r}{\partial x}=2x$ or $\frac{\partial r}{\partial x}=\frac{x}{r}$.

$\therefore \quad \frac{\partial}{\partial x} f(r)=f'(r) \cdot \frac{x}{r}=\frac{x}{r} f'(r)$.

$$\therefore \quad \frac{\partial^2}{\partial x^2} f(r)=\frac{\partial}{\partial x}\left[\frac{x}{r} f'(r)\right]$$

$$=\frac{1}{r} f'(r)-\frac{x}{r^2} \frac{\partial r}{\partial x} f'(r)+\frac{x}{r} f''(r) \cdot \frac{\partial r}{\partial x}$$

$$=\frac{1}{r} f'(r)-\frac{x}{r^2} \cdot \frac{x}{r} f'(r)+\frac{x}{r} \cdot \frac{x}{r} f''(r)$$

$$=\frac{1}{r} f'(r)-\frac{x^2}{r^3} f'(r)+\frac{x^2}{r^2} f''(r).$$

Similarly, by symmetry,

$$\frac{\partial^2}{\partial y^2} f(r)=\frac{1}{r} f'(r)-\frac{y^2}{r^3} f'(r)+\frac{y^2}{r^2} f''(r).$$

$$\therefore \quad \nabla^2 f(r)=\frac{2}{r} f'(r)-\frac{x^2+y^2}{r^3} f'(r)+\frac{x^2+y^2}{r^2} f''(r)$$

$$=\frac{2}{r} f'(r)-\frac{r^2}{r^3} f'(r)+\frac{r^2}{r^2} f''(r)$$

$$=\frac{1}{r} f'(r)+f''(r).$$

$\therefore \quad$ if $\nabla^2 f(r)=0$, then

$$f''(r)+\frac{1}{r} f'(r)=0 \text{ or } \frac{f''(r)}{f'(r)}=-\frac{1}{r}.$$

Integrating with respect to r, we get

$\log f'(r)=-\log r+\log c_1$, where c_1 is a constant

$=\log (c_1/r)$.

$\therefore f'(r)=c_1/r$.

Again integrating,

$f(r)=c_1 \log r+c_2$, where c_2 is a constant.

Hence $f(r)=c_1 \log r+c_2$, where c_1, c_2 are arbitrary constants.

2003

Ex. 22. *Prove that* $\frac{1}{2} \nabla \mathbf{a}^2=(\mathbf{a} \cdot \nabla) \mathbf{a}+\mathbf{a} \times$ *curl* $\mathbf{a}$.

[Kanpur 1986]

Sol. Proceed as in identity 7 by taking $\mathbf{A}=\mathbf{a}$ and $\mathbf{B}=\mathbf{a}$

We have $\frac{1}{2} \nabla \mathbf{a}^2=\frac{1}{2} \nabla(\mathbf{a} \cdot \mathbf{a})$

$$=\frac{1}{2} \Sigma \mathbf{i} \frac{\partial}{\partial x}(\mathbf{a} \cdot \mathbf{a})=\frac{1}{2} \Sigma \mathbf{i}\left(\mathbf{a} \cdot \frac{\partial \mathbf{a}}{\partial x}+\frac{\partial \mathbf{a}}{\partial x} \cdot \mathbf{a}\right)$$

$$=\frac{1}{2}\Sigma 2\mathbf{i}\left(\mathbf{a}\cdot\frac{\partial \mathbf{a}}{\partial x}\right)=\Sigma\left(\mathbf{a}\cdot\frac{\partial \mathbf{a}}{\partial x}\right)\mathbf{i}.$$

We know that $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=(\mathbf{A}\cdot\mathbf{C})\,\mathbf{B}-(\mathbf{A}\cdot\mathbf{B})\,\mathbf{C}$.

$\therefore\quad (\mathbf{A}\cdot\mathbf{B})\,\mathbf{C}=(\mathbf{A}\cdot\mathbf{C})\,\mathbf{B}-\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$.

$$\therefore\quad \left(\mathbf{a}\cdot\frac{\partial \mathbf{a}}{\partial x}\right)\mathbf{i}=(\mathbf{a}\cdot\mathbf{i})\frac{\partial \mathbf{a}}{\partial x}-\mathbf{a}\times\left(\frac{\partial \mathbf{a}}{\partial x}\times\mathbf{i}\right)=(\mathbf{a}\cdot\mathbf{i})\,\frac{\partial \mathbf{a}}{\partial x}+\mathbf{a}\times\left(\mathbf{i}\times\frac{\partial \mathbf{a}}{\partial x}\right).$$

$$\text{Thus }\Sigma\left\{\left(\mathbf{a}\cdot\frac{\partial \mathbf{a}}{\partial x}\right)\mathbf{i}\right\}=\Sigma\left\{(\mathbf{a}\cdot\mathbf{i})\,\frac{\partial \mathbf{a}}{\partial x}\right\}+\Sigma\left\{\mathbf{a}\times\left(\mathbf{i}\times\frac{\partial \mathbf{a}}{\partial x}\right)\right\}$$

$$=\left\{\mathbf{a}\cdot\Sigma\mathbf{i}\,\frac{\partial}{\partial x}\right\}\mathbf{a}+\mathbf{a}\times\Sigma\left(\mathbf{i}\times\frac{\partial \mathbf{a}}{\partial x}\right)=(\mathbf{a}\cdot\nabla)\mathbf{a}+\mathbf{a}\times(\nabla\times\mathbf{a})$$

$$=(\mathbf{a}\cdot\nabla)\,\mathbf{a}+\mathbf{a}\times\text{curl }\mathbf{a}.$$

Hence $\frac{1}{2}\nabla\, a^2=(\mathbf{a}\cdot\nabla)\,\mathbf{a}+\mathbf{a}\times\text{ curl }\mathbf{a}$.

Ex. ~~23~~. *Show that curl* $\mathbf{a}\,\phi(r)=\frac{1}{r}\phi'(r)\,\mathbf{r}\times\mathbf{a}$, *where* $\mathbf{a}$ *is a constant vector.* **[Kanpur 1982]**

Sol. We know that curl $(\phi\mathbf{A})=(\nabla\phi)\times\mathbf{A}+\phi$ curl $\mathbf{A}$. Replacing ϕ by $\phi(r)$ and $\mathbf{A}$ by $\mathbf{a}$ in this identity, we have

curl $[\mathbf{a}\,\phi(r)]=[\nabla\,\phi(r)]\times\mathbf{a}+\phi(r)$ curl $\mathbf{a}$

$=[\phi'(r)\,\nabla r]\times\mathbf{a}+\phi(r)\,\mathbf{0}$

[$\because$ $\mathbf{a}$ is a constant vector $\Rightarrow$ curl $\mathbf{a}=\mathbf{0}$]

$=\left[\phi'(r)\frac{1}{r}\mathbf{r}\right]\times\mathbf{a}$ $\qquad\left[\because\ \nabla r=\frac{1}{r}\mathbf{r}\right]$

$=\frac{1}{r}\,\phi'(r)\,\mathbf{r}\times\mathbf{a}.$

Ex. 24. *Prove that* $\nabla^2(\phi\psi)=\phi\nabla^2\psi+2\nabla\phi\cdot\nabla\psi+\psi\nabla^2\phi$.

[Meerut 1972; Bombay 86]

Sol. We have $\nabla^2(\phi\psi)=\nabla\cdot[\nabla(\phi\psi)]$

$=\nabla\cdot[\phi\,(\nabla\psi)+\psi\,(\nabla\phi)]=\nabla\cdot[\phi\,(\nabla\psi)]+\nabla\cdot[\psi(\nabla\phi)]$

$=\phi\nabla\cdot(\nabla\psi)+(\nabla\psi)\cdot(\nabla\phi)+\psi\nabla\cdot(\nabla\phi)+(\nabla\psi)\cdot(\nabla\phi)$

$=\phi\nabla^2\psi+2\nabla\phi\cdot\nabla\psi+\psi\nabla^2\phi.$

Ex. 25. *Prove that div* $(\nabla\phi\times\nabla\psi)=0$.

Sol. We know that

div $(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot$curl $\mathbf{A}-\mathbf{A}\cdot$Curl $\mathbf{B}$.

$\therefore$ div $(\nabla\phi\times\nabla\psi)=(\nabla\psi)\cdot($Curl $\nabla\phi)-(\nabla\phi)\cdot($Curl $\nabla\psi)$

$=(\nabla\psi)\cdot\mathbf{0}-(\nabla\phi)\cdot\mathbf{0}$ $\qquad$ [$\because$ curl grad $\phi=\mathbf{0}$]

$=0.$

2004 **Ex. 26.** *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are irrotational, prove that* $\mathbf{A}\times\mathbf{B}$ *is solenoidal.* **[Bombay 1988; Kanpur 77, 79]**

Sol. If **A** and **B** are irrotational, then

$$\text{curl } \mathbf{A}=\mathbf{0}, \text{ curl } \mathbf{B}=\mathbf{0}.$$

Now $\text{div } (\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot\text{curl } \mathbf{A}-\mathbf{A}\cdot\text{curl } \mathbf{B}=\mathbf{B}\cdot\mathbf{0}-\mathbf{A}\cdot\mathbf{0}=0.$

Since div $(\mathbf{A}\times\mathbf{B})$ is zero, therefore $\mathbf{A}\times\mathbf{B}$ is solenoidal.

Ex. 27. *Prove that curl* $(\phi \text{ grad } \phi)=\mathbf{0}$.

Sol. We know that

$$\text{curl } (\phi\mathbf{A})=\text{grad } \phi\times\mathbf{A}+\phi \text{ curl } \mathbf{A}.$$

Putting grad ϕ in place of **A**, we get

$$\text{curl } (\phi \text{ grad } \phi)=\text{grad } \phi\times\text{grad } \phi+\phi \text{ curl grad } \phi$$
$$=\mathbf{0}+\phi\mathbf{0}.$$

Here grad $\phi\times$ grad $\phi=\mathbf{0}$, since it is the cross product of two equal vectors. Also curl grad $\phi=\mathbf{0}$.

$$\therefore \quad \text{curl } (\phi \text{ grad } \phi)=\mathbf{0}+\mathbf{0}=\mathbf{0}.$$

Ex. 28. *If f and g are two scalar point functions, prove that* $div\ (f\nabla g)=f\ \nabla^2 g+\nabla f\cdot\nabla g.$ **[Meerut 1972]**

Sol. We have $\nabla g=\dfrac{\partial g}{\partial x}\ \mathbf{i}+\dfrac{\partial g}{\partial y}\ \mathbf{j}+\dfrac{\partial g}{\partial z}\ \mathbf{k}.$

Therefore $f\nabla g=f\dfrac{\partial g}{\partial x}\ \mathbf{i}+f\dfrac{\partial g}{\partial y}\ \mathbf{j}+f\dfrac{\partial g}{\partial z}\ \mathbf{k}.$

So $\text{div } (f\nabla g)=\dfrac{\partial}{\partial x}\left(f\dfrac{\partial g}{\partial x}\right)+\dfrac{\partial}{\partial y}\left(f\dfrac{\partial g}{\partial y}\right)+\dfrac{\partial}{\partial z}\left(f\dfrac{\partial g}{\partial z}\right)$

$$=f\left(\frac{\partial^2 g}{\partial x^2}+\frac{\partial^2 g}{\partial y^2}+\frac{\partial^2 g}{\partial z^2}\right)+\left(\frac{\partial f}{\partial x}\ \frac{\partial g}{\partial x}+\frac{\partial f}{\partial y}\ \frac{\partial g}{\partial y}+\frac{\partial f}{\partial z}\ \frac{\partial g}{\partial z}\right)$$

$$=f\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)$$
$$+\left(\frac{\partial f}{\partial x}\ \mathbf{i}+\frac{\partial f}{\partial y}\ \mathbf{j}+\frac{\partial f}{\partial z}\ \mathbf{k}\right)\cdot\left(\frac{\partial g}{\partial x}\ \mathbf{i}+\frac{\partial g}{\partial y}\ \mathbf{j}+\frac{\partial g}{\partial z}\ \mathbf{k}\right)$$

$$=f\nabla^2 g+\nabla f\cdot\nabla g.$$

IAS 2011

Ex. 29. *A vector function* **f** *is the product of a scalar function and the gradient of a scalar function. Show that*

$$\mathbf{f}\cdot curl\ \mathbf{f}=0.$$ **[Kerala 1975]**

Sol. Let $\mathbf{f}=\psi \text{ grad } \phi$, where ψ and ϕ are scalar functions. We have curl $\mathbf{f}=$ curl $(\psi \text{ grad } \phi)$.

We know that curl $(\phi\mathbf{A})=(\text{grad } \phi)\times\mathbf{A}+\phi$ curl **A**.

$$\therefore \quad \text{curl } (\psi \text{ grad } \phi)=(\text{grad } \psi)\times(\text{grad } \phi)+\psi \text{ (curl grad } \phi)$$
$$=(\text{grad } \psi)\times(\text{grad } \phi) \quad [\because \text{ curl grad } \phi=\mathbf{0}]$$

Now $\mathbf{f}\cdot\text{curl } \mathbf{f}=(\psi \text{ grad } \phi)\cdot\{(\text{grad } \psi)\times(\text{grad } \phi)\}$

$$=[\psi \text{ grad } \phi, \text{ grad } \psi, \text{ grad } \phi]=\psi\ [\text{grad } \phi, \text{ grad } \psi, \text{ grad } \phi]$$

$=0$, since the value of a scalar triple product is zero if two vectors are equal.

Ex. 30. *Given that* $\rho\,\mathbf{F}=\nabla p$, *where* $\rho, p, \mathbf{F}$ *are point functions, prove that* $\mathbf{F}\cdot curl\ \mathbf{F}=0$. **[Kerala 1975]**

Sol. We have $\mathbf{F}=(1/\rho)\,\nabla p$, where $1/\rho$ and p are scalar functions. Now proceed as in Ex. 29.

Ex. 31. *Prove that* $\nabla\cdot\left\{r\nabla\left(\frac{1}{r^3}\right)\right\}=\frac{3}{r^4}$

or, $\qquad div\,[r\ grad\ r^{-3}]=3r^{-4}$. **[Meerut 1991P]**

Sol. We have $\nabla\left(\frac{1}{r^3}\right)=\text{grad } r^{-3}$

$$=\frac{\partial}{\partial x}(r^{-3})\,\mathbf{i}+\frac{\partial}{\partial y}(r^{-3})\,\mathbf{j}+\frac{\partial}{\partial z}(r^{-3})\,\mathbf{k}.$$

Now $\frac{\partial}{\partial x}(r^{-3})=-3r^{-4}\frac{\partial r}{\partial x}$. But $r^2=x^2+y^2+z^2$.

Therefore $2r\frac{\partial r}{\partial x}=2x$ or $\frac{\partial r}{\partial x}=\frac{x}{r}$.

So $\frac{\partial}{\partial x}(r^{-3})=-3r^{-4}\frac{x}{r}=-3r^{-5}x$.

Similarly $\frac{\partial}{\partial y}(r^{-3})=-3r^{-5}y$ and $\frac{\partial}{\partial z}(r^{-3})=-3r^{-5}z$.

Therefore $\nabla\left(\frac{1}{r^3}\right)=-3r^{-5}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$.

$$\therefore\quad r\nabla\left(\frac{1}{r^3}\right)=-3r^{-4}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k}).$$

$$\therefore\quad \nabla\cdot\left(r\nabla\frac{1}{r^3}\right)=\frac{\partial}{\partial x}(-3r^{-4}x)+\frac{\partial}{\partial y}(-3r^{-4}y)+\frac{\partial}{\partial z}(-3r^{-4}z).$$

Now $\frac{\partial}{\partial x}(-3r^{-4}x)=12\,r^{-5}\frac{\partial r}{\partial x}x-3r^{-4}$

$$=12\,r^{-5}\frac{x}{r}x-3r^{-4}=12r^{-6}x^2-3r^{-4}.$$

Similarly $\frac{\partial}{\partial y}(-3r^{-4}y)=12r^{-6}y^2-3r^{-4}$

and $\qquad \frac{\partial}{\partial z}(-3r^{-4}z)=12r^{-6}z^2-3r^{-4}$.

Hence $\nabla\cdot\left(r\nabla\frac{1}{r^3}\right)=12r^{-6}(x^2+y^2+z^2)-9r^{-4}$

$$=12r^{-6}r^2-9r^{-4}=12r^{-4}-9r^{-4}=3r^{-4}$$

Ex. 32. *Prove that* $\mathbf{a}\cdot\left(\nabla\frac{1}{r}\right)=-\frac{\mathbf{a}\cdot\mathbf{r}}{r^3}$.

Sol. We have

$$\text{grad } \frac{1}{r}=-\frac{1}{r^2} \text{ grad } r=-\frac{1}{r^2}\frac{1}{r}\mathbf{r}=-\frac{1}{r^3}\mathbf{r}.$$

$$\therefore \quad \mathbf{a}\cdot\left(\nabla \frac{1}{r}\right)=\mathbf{a}\cdot\left(-\frac{1}{r^3}\mathbf{r}\right)=-\frac{\mathbf{a}\cdot\mathbf{r}}{r^3}.$$

Ex. 33. *Prove that*

$$\mathbf{b}\cdot\nabla\left(\mathbf{a}\cdot\nabla \frac{1}{r}\right)=\frac{3(\mathbf{a}\cdot\mathbf{r})(\mathbf{b}\cdot\mathbf{r})}{r^5}-\frac{\mathbf{a}\cdot\mathbf{b}}{r^3}$$

where **a** *and* **b** *are constant vectors.*

Sol. As shown in the last example, we have

$$\mathbf{a}\cdot\nabla\frac{1}{r}=-\frac{\mathbf{a}\cdot\mathbf{r}}{r^3}.$$

$$\therefore \quad \mathbf{b}\cdot\nabla\left(\mathbf{a}\cdot\nabla\frac{1}{r}\right)=\mathbf{b}\cdot\nabla\left(-\frac{\mathbf{a}\cdot\mathbf{r}}{r^3}\right)=\mathbf{b}\cdot\Sigma\mathbf{i}\frac{\partial}{\partial x}\left(-\frac{\mathbf{a}\cdot\mathbf{r}}{r^3}\right)$$

$$=\mathbf{b}\cdot\Sigma\mathbf{i}\left\{-\frac{1}{r^3}\frac{\partial}{\partial x}(\mathbf{a}\cdot\mathbf{r})+(\mathbf{a}\cdot\mathbf{r})\frac{\partial}{\partial x}\left(-\frac{1}{r^3}\right)\right\}$$

$$=\mathbf{b}\cdot\Sigma\mathbf{i}\left\{-\frac{1}{r^3}\left(\mathbf{a}\cdot\frac{\partial\mathbf{r}}{\partial x}\right)+3(\mathbf{a}\cdot\mathbf{r})r^{-4}\frac{\partial r}{\partial x}\right\}$$

[$\because$ **a** is a constant vector]

$$=\mathbf{b}\cdot\Sigma\mathbf{i}\left\{-\frac{\mathbf{a}\cdot\mathbf{i}}{r^3}+\frac{3x}{r^5}(\mathbf{a}\cdot\mathbf{r})\right\} \quad \left[\because \frac{\partial\mathbf{r}}{\partial x}=\mathbf{i} \text{ and } \frac{\partial r}{\partial x}=\frac{x}{r}\right]$$

$$=\mathbf{b}\cdot\Sigma\left\{-\frac{1}{r^3}(\mathbf{a}\cdot\mathbf{i})\,\mathbf{i}+\frac{3}{r^5}(\mathbf{a}\cdot\mathbf{r})\,x\mathbf{i}\right\}$$

$$=\mathbf{b}\cdot\left\{-\frac{1}{r^3}\mathbf{a}+\frac{3}{r^5}(\mathbf{a}\cdot\mathbf{r})\,\mathbf{r}\right\}$$

[$\because$ $\Sigma(\mathbf{a}\cdot\mathbf{i})\,\mathbf{i}=\mathbf{a}$, and $\Sigma x\mathbf{i}=\mathbf{r}$]

$$=-\frac{\mathbf{a}\cdot\mathbf{b}}{r^3}+\frac{3(\mathbf{a}\cdot\mathbf{r})(\mathbf{b}\cdot\mathbf{r})}{r^5}.$$

Ex. 34. *Prove that div* $(\mathbf{A}\times\mathbf{r})=\mathbf{r}\cdot$*curl* $\mathbf{A}$. **[Rohilkhand 1979]**

Sol. We know that

$$\text{div }(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot\text{curl }\mathbf{A}-\mathbf{A}\cdot\text{curl }\mathbf{B}.$$

$$\therefore \quad \text{div }(\mathbf{A}\times\mathbf{r})=\mathbf{r}\cdot\text{curl }\mathbf{A}-\mathbf{A}\cdot\text{curl }\mathbf{r}$$

$$=\mathbf{r}\cdot\text{curl }\mathbf{A}-\mathbf{A}\cdot\mathbf{0} \qquad [\because \text{ curl }\mathbf{r}=\mathbf{0}]$$

$$=\mathbf{r}\cdot\text{curl }\mathbf{A}.$$

Ex. 35. *If* **a** *is a constant vector, prove that*

div $\{r^n(\mathbf{a}\times\mathbf{r})\}=0.$ **[Allahabad 1980; Rohilkhand 77]**

Sol. We have

$$\text{div }(\phi\mathbf{A})=\phi\text{ div }\mathbf{A}+\mathbf{A}\cdot\text{grad }\phi.$$

$$\therefore \quad \text{div }\{r^n(\mathbf{a}\times\mathbf{r})\}=r^n\text{ div }(\mathbf{a}\times\mathbf{r})+(\mathbf{a}\times\mathbf{r})\cdot\text{grad } r^n$$

$$=r^n\text{ div }(\mathbf{a}\times\mathbf{r})+(\mathbf{a}\times\mathbf{r})\cdot(nr^{n-1}\text{ grad } r)$$

$=r^n\,(\mathbf{r}\cdot\text{curl }\mathbf{a}-\mathbf{a}\cdot\text{curl }\mathbf{r})+(\mathbf{a}\times\mathbf{r})\cdot\left(nr^{n-1}\dfrac{1}{r}\,\mathbf{r}\right)$

$=r^n\,(\mathbf{r}\cdot 0-\mathbf{a}\cdot 0)+nr^{n-2}\,(\mathbf{a}\times\mathbf{r})\cdot\mathbf{r}$

[$\because$ curl of constant vector is zero and curl $\mathbf{r}=0$]

$=nr^{n-2}\,[\mathbf{a},\mathbf{r},\mathbf{r}]$

$=0$, since a scalar triple product having two equal vectors is zero.

Ex. 36. *Prove that*

$$\nabla\cdot(U\nabla V-V\nabla U)=U\nabla^2V-V\nabla U.$$

[Meerut 1969; Bombay 89; Agra 70]

Sol. We have $\nabla\cdot(U\nabla V-V\nabla U)$

$=\nabla\cdot(U\nabla V)-\nabla\cdot(V\nabla U).$

Now $\nabla\cdot(U\nabla V)=U\{\nabla\cdot(\nabla V)\}+(\nabla U)\cdot(\nabla V)$

$=U\nabla^2V+(\nabla U)\cdot(\nabla V).$

Interchanging U and V, we get

$$\nabla\cdot(V\nabla U)=V\nabla^2U+(\nabla V)\cdot(\nabla U).$$

$\therefore\quad \nabla\cdot(U\nabla V-V\nabla U)$

$=[U\nabla^2V+(\nabla U)\cdot(\nabla V)]-[V\nabla^2U+(\nabla V)\cdot(\nabla U)]$

$=U\nabla^2V-V\nabla^2U.$

Ex. 37. *If* **a** *and* **b** *are constant vectors, prove that*

(i) *div* $[(\mathbf{r}\times\mathbf{a})\times\mathbf{b}]=-2\mathbf{b}\cdot\mathbf{a}.$ **[Rohilkhand 1979]**

(ii) *curl* $[(\mathbf{r}\times\mathbf{a})\times\mathbf{b}]=\mathbf{b}\times\mathbf{a}.$ **[Rohilkhand 1979; Gorakhpur 87]**

Sol. (i) We have $(\mathbf{r}\times\mathbf{a})\times\mathbf{b}=(\mathbf{b}\cdot\mathbf{r})\,\mathbf{a}-(\mathbf{b}\cdot\mathbf{a})\,\mathbf{r}.$

$\therefore\quad$ div $[(\mathbf{r}\times\mathbf{a})\times\mathbf{b}]=$ div $[(\mathbf{b}\cdot\mathbf{r})\,\mathbf{a}-(\mathbf{b}\cdot\mathbf{a})\,\mathbf{r}]$

$=$ div $[(\mathbf{b}\cdot\mathbf{r})\,\mathbf{a}]-$ div $[(\mathbf{b}\cdot\mathbf{a})\,\mathbf{r}]$ ...(1)

But div $(\phi\mathbf{A})=\phi$ div $\mathbf{A}+\mathbf{A}\cdot$grad ϕ.

Taking $\phi=\mathbf{b}\cdot\mathbf{r}$ and $\mathbf{A}=\mathbf{a}$, we get

div $[(\mathbf{b}\cdot\mathbf{r})\,\mathbf{a}]=(\mathbf{b}\cdot\mathbf{r})$ div $\mathbf{a}+\mathbf{a}\cdot$grad $(\mathbf{b}\cdot\mathbf{r})$.

Since $\mathbf{a}$ is a constant vector, therefore div $\mathbf{a}=0$.

Also let $\quad\mathbf{b}=b_1\,\mathbf{i}+b_2\,\mathbf{j}+b_3\,\mathbf{k}.$

Then $\mathbf{b}\cdot\mathbf{r}=(b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k})\cdot(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$

$=b_1x+b_2y+b_3z$ where b_1, b_2, b_3 are constants.

$\therefore\quad$ grad $(\mathbf{b}\cdot\mathbf{r})=b_1\mathbf{i}+b_2\mathbf{j}+b_3\mathbf{k}=\mathbf{b}.$

$\therefore\quad$ div $[(\mathbf{b}\cdot\mathbf{r})\,\mathbf{a}]=\mathbf{a}\cdot\mathbf{b}.$ ...(2)

Again div $[(\mathbf{b}\cdot\mathbf{a})]\,\mathbf{r}=(\mathbf{b}\cdot\mathbf{a})$ div $\mathbf{r}+\mathbf{r}\cdot$grad $(\mathbf{b}\cdot\mathbf{a})$.

But div $\mathbf{r}=3$. Also grad $(\mathbf{b}\cdot\mathbf{a})=0$ because $\mathbf{b}\cdot\mathbf{a}$ is constant.

$\therefore\quad$ div $[(\mathbf{b}\cdot\mathbf{a})\,\mathbf{r}]=3\,(\mathbf{b}\cdot\mathbf{a}).$ ...(3)

Substituting the values from (2) and (3) in (1), we get

$$\text{div }[(\mathbf{r}\times\mathbf{a})\times\mathbf{b}]=(\mathbf{a}\cdot\mathbf{b})-3\,(\mathbf{b}\cdot\mathbf{a})=-2\mathbf{b}\cdot\mathbf{a}.$$

(ii) $\text{Curl}\,[(\mathbf{r}\times\mathbf{a})\times\mathbf{b}]=\text{curl}\,[(\mathbf{b}\cdot\mathbf{r})\,\mathbf{a}-(\mathbf{b}\cdot\mathbf{a})\,\mathbf{r}]$

$=\text{curl}\,[(\mathbf{b}\cdot\mathbf{r})\,\mathbf{a}]-\text{curl}\,[(\mathbf{b}\cdot\mathbf{a})\,\mathbf{r}].$

But $\quad \text{curl}\,(\phi\mathbf{A})=\text{grad}\,\phi\times\mathbf{A}+\phi\,\text{curl}\,\mathbf{A}.$

$\therefore\quad \text{curl}\,[(\mathbf{b}\cdot\mathbf{r})\,\mathbf{a}]=[\text{grad}\,(\mathbf{b}\cdot\mathbf{r})]\times\mathbf{a}+(\mathbf{b}\cdot\mathbf{r})\,\text{curl}\,\mathbf{a}$

$=\mathbf{b}\times\mathbf{a}\ [\because\ \text{curl}\,\mathbf{a}=\mathbf{0}\text{ and grad}\,(\mathbf{b}\cdot\mathbf{r})=\mathbf{b}]$

Also $\quad \text{curl}\,[(\mathbf{b}\cdot\mathbf{a})\,\mathbf{r}]=[\text{grad}\,(\mathbf{b}\cdot\mathbf{a})]\times\mathbf{r}+(\mathbf{b}\cdot\mathbf{a})\,\text{curl}\,\mathbf{r}$

$=\mathbf{0}\ [\because\ \text{grad}\,(\mathbf{b}\cdot\mathbf{a})=\mathbf{0}\text{ and curl}\,\mathbf{r}=\mathbf{0}]$

$\therefore\quad \text{curl}\,[(\mathbf{r}\times\mathbf{a})\times\mathbf{b}]=\mathbf{b}\times\mathbf{a}-\mathbf{0}=\mathbf{b}\times\mathbf{a}.$

2001
IFS-2002

Ex. 38. *If* **a** *is a constant vector, prove that*

$$\textit{curl}\,\frac{\mathbf{a}\times\mathbf{r}}{r^3}=-\frac{\mathbf{a}}{r^3}+\frac{3\mathbf{r}}{r^5}(\mathbf{a}\cdot\mathbf{r}).$$

[Rajasthan 1981]

Sol. We have

$$\text{curl}\,\frac{\mathbf{a}\times\mathbf{r}}{r^3}=\nabla\times\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right)=\Sigma\left\{\mathbf{i}\times\frac{\partial}{\partial x}\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right)\right\}.$$

Now $$\frac{\partial}{\partial x}\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right)=-\frac{3}{r^4}\frac{\partial r}{\partial x}(\mathbf{a}\times\mathbf{r})+\frac{1}{r^3}\left(\mathbf{a}\times\frac{\partial\mathbf{r}}{\partial x}\right)+\frac{1}{r^3}\left(\frac{\partial\mathbf{a}}{\partial x}\times\mathbf{r}\right)\quad\ldots(1)$$

Now $\dfrac{\partial\mathbf{a}}{\partial x}=\mathbf{0}$ because **a** is a constant vector.

Also $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}.\qquad\therefore\quad \dfrac{\partial\mathbf{r}}{\partial x}=\mathbf{i}.$

Further $\dfrac{\partial r}{\partial x}=\dfrac{x}{r}.$

$\therefore$ (1) becomes

$$\frac{\partial}{\partial x}\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right)=-\frac{3}{r^4}\frac{x}{r}(\mathbf{a}\times\mathbf{r})+\frac{1}{r^3}(\mathbf{a}\times\mathbf{i})$$

$$=-\frac{3x}{r^5}(\mathbf{a}\times\mathbf{r})+\frac{1}{r^3}(\mathbf{a}\times\mathbf{i}).$$

$$\therefore\quad \mathbf{i}\times\frac{\partial}{\partial x}\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right)=-\frac{3x}{r^5}\,\mathbf{i}\times(\mathbf{a}\times\mathbf{r})+\frac{1}{r^3}\,\mathbf{i}\times(\mathbf{a}\times\mathbf{i})$$

$$=-\frac{3x}{r^5}\Big[(\mathbf{i}\cdot\mathbf{r})\,\mathbf{a}-(\mathbf{i}\cdot\mathbf{a})\,\mathbf{r}\Big]+\frac{1}{r^3}[(\mathbf{i}\cdot\mathbf{i})\,\mathbf{a}-(\mathbf{i}\cdot\mathbf{a})\,\mathbf{i}]$$

$$=-\frac{3x}{r^5}\,x\mathbf{a}+\frac{3x}{r^5}\,a_1\mathbf{r}+\frac{1}{r^3}\,\mathbf{a}-\frac{1}{r^3}\,a_1\mathbf{i}$$

$[\because\ \mathbf{i}\cdot\mathbf{r}=x\text{ and }\mathbf{i}\cdot\mathbf{a}=a_1\text{ if }\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}]$

$$=-\frac{3x^2}{r^5}\,\mathbf{a}+\frac{3}{r^5}\,a_1x\,\mathbf{r}+\frac{1}{r^3}\,\mathbf{a}-\frac{1}{r^3}\,a_1\mathbf{i}.$$

$$\therefore\quad \Sigma\left\{\mathbf{i}\times\frac{\partial}{\partial x}\left(\frac{\mathbf{a}\times\mathbf{r}}{r^3}\right)\right\}$$

$$=\left\{-\frac{3}{r^5}\Sigma x^2\right\}\mathbf{a}+\left\{\frac{3}{r^5}\Sigma a_1 x\right\}\mathbf{r}+\frac{3}{r^3}\mathbf{a}-\frac{1}{r^3}\Sigma a_1\mathbf{i}$$

$$=-\frac{3}{r^5}r^2\mathbf{a}+\frac{3}{r^5}(\mathbf{r}\cdot\mathbf{a})\mathbf{r}+\frac{3}{r^3}\mathbf{a}-\frac{1}{r^3}\mathbf{a}$$

$[\because \quad \Sigma x^2=r^2,\ \Sigma a_1 x=\mathbf{r}\cdot\mathbf{a},\ \Sigma a_1\mathbf{i}=\mathbf{a}]$

$$=-\frac{\mathbf{a}}{r^3}+\frac{3}{r^5}(\mathbf{a}\cdot\mathbf{r})\mathbf{r}.$$

Ex. 39. *Prove that* $\operatorname{div}\left\{\frac{f(r)\,\mathbf{r}}{r}\right\}=\frac{1}{r^2}\frac{d}{dr}(r^2 f)$. [Agra 1981]

Sol. We have

$$\operatorname{div}\left\{\frac{f(r)\,\mathbf{r}}{r}\right\}=\operatorname{div}\left\{\frac{f(r)}{r}(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\right\}$$

$$=\frac{\partial}{\partial x}\left\{\frac{f(r)}{r}x\right\}+\frac{\partial}{\partial y}\left\{\frac{f(r)}{r}y\right\}+\frac{\partial}{\partial z}\left\{\frac{f(r)}{r}z\right\} \quad \ldots(1)$$

Now $\frac{\partial}{\partial x}\left\{\frac{f(r)}{r}x\right\}=\frac{f(r)}{r}+x\frac{d}{dr}\left\{\frac{f(r)}{r}\right\}\frac{\partial r}{\partial x}$

$$=\frac{f(r)}{r}+x\left\{\frac{f'(r)}{r}-\frac{1}{r^2}f(r)\right\}\frac{x}{r}=\frac{f(r)}{r}+\frac{x^2}{r^2}f'(r)-\frac{x^2}{r^3}f(r).$$

Similarly $\frac{\partial}{\partial y}\left\{\frac{f(r)}{r}y\right\}=\frac{f(r)}{r}+\frac{y^2}{r^2}f'(r)-\frac{y^2}{r^3}f(r)$

and $\frac{\partial}{\partial z}\left\{\frac{f(r)}{r}z\right\}=\frac{f(r)}{r}+\frac{z^2}{r^2}f'(r)-\frac{z^2}{r^3}f(r).$

Putting these values in (1), we get

$$\operatorname{div}\left\{\frac{f(r)\,\mathbf{r}}{r}\right\}=\frac{3}{r}f(r)+\frac{r^2}{r^2}f'(r)-\frac{r^2}{r^3}f(r)$$

$$=\frac{2}{r}f(r)+f'(r)=\frac{1}{r^2}\left[2rf(r)+r^2f'(r)\right]=\frac{1}{r^2}\frac{d}{dr}\left[r^2 f(r)\right].$$

Ex. 40. *Evaluate* $\operatorname{div}\{\mathbf{a}\times(\mathbf{r}\times\mathbf{a})$, *where* $\mathbf{a}$ *is a constant vector.* [Kanpur 1976]

Sol. We have $\operatorname{div}\{\mathbf{a}\times(\mathbf{r}\times\mathbf{a})\}$

$=\operatorname{div}(\mathbf{a}\times\mathbf{b})$, where $\mathbf{b}=\mathbf{r}\times\mathbf{a}$

$=\mathbf{b}\cdot(\nabla\times\mathbf{a})-\mathbf{a}\cdot(\nabla\times\mathbf{b})$ [See identity 5 of § 11]

$=-\mathbf{a}\cdot(\nabla\times\mathbf{b})$ [$\because$ $\mathbf{a}$ is a constant vector $\Rightarrow \nabla\times\mathbf{a}=\mathbf{0}$]

$=-\mathbf{a}\cdot[\nabla\times(\mathbf{r}\times\mathbf{a})]$. ...(1)

Now $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$.

Let $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$. Then the scalars a_1, a_2, a_3 are all constants.

Proceeding as in Ex. 19 part (ii) after § 10, we have

$\operatorname{curl}(\mathbf{r}\times\mathbf{a})=\nabla\times(\mathbf{r}\times\mathbf{a})=-2\mathbf{a}$, [Do it here]

Hence from (1), we have

$$\text{div}\,\{\mathbf{a}\times(\mathbf{r}\times\mathbf{a})\}=-\mathbf{a}\cdot(-2\mathbf{a})=2\mathbf{a}\cdot\mathbf{a}=2a^2.$$

Ex. 41. *If* $\mathbf{a}$ *and* $\mathbf{b}$ *are constant vectors, then show that*

$$\nabla\cdot(\mathbf{a}\cdot\mathbf{b}\mathbf{r})=3\mathbf{a}\cdot\mathbf{b}.$$

Sol. We know that $\nabla\cdot(\phi\mathbf{A})=(\nabla\phi)\cdot\mathbf{A}+\phi(\nabla\cdot\mathbf{A})$

[See identity 3 of § 11]

Replacing ϕ by $\mathbf{a}\cdot\mathbf{b}$ and $\mathbf{A}$ by $\mathbf{r}$ in the above identity, we get

$$\nabla\cdot(\mathbf{a}\cdot\mathbf{b}\mathbf{r})=[\nabla(\mathbf{a}\cdot\mathbf{b})]\cdot\mathbf{r}+(\mathbf{a}\cdot\mathbf{b})\,(\nabla\cdot\mathbf{r}). \qquad \text{...(1)}$$

Since $\mathbf{a}$ and $\mathbf{b}$ are constant vectors, therefore $\mathbf{a}\cdot\mathbf{b}$ is a constant scalar.

$$\therefore\quad \nabla\,(\mathbf{a}\cdot\mathbf{b})=\mathbf{0}.$$

Also $\nabla\cdot\mathbf{r}=\nabla\cdot(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$

$$=\frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(z)=1+1+1=3.$$

Substituting these values in (1), we get

$$\nabla\cdot(\mathbf{a}\cdot\mathbf{b}\mathbf{r})=\mathbf{0}\cdot\mathbf{r}+3(\mathbf{a}\cdot\mathbf{b})=3\mathbf{a}\cdot\mathbf{b}.$$

Ex. 42. *Prove that* $\nabla^2\left[\nabla\cdot\left(\frac{\mathbf{r}}{r^2}\right)\right]=2r^{-4}$.

Sol. We know that $\nabla\cdot(\phi\mathbf{A})=\phi(\nabla\cdot\mathbf{A})+\mathbf{A}\cdot(\nabla\phi)$. ...(1)

Putting $\mathbf{A}=\mathbf{r}$ and $\phi=1/r^2$ in this identity, we get

$$\nabla\cdot\left(\frac{\mathbf{r}}{r^2}\right)=\frac{1}{r^2}(\nabla\cdot\mathbf{r})+\mathbf{r}\cdot\left(\nabla\frac{1}{r^2}\right)$$

$$=\frac{3}{r^2}+\mathbf{r}\cdot\left[-\frac{2}{r^3}\nabla r\right]$$

$$\left[\because\quad \nabla\cdot\mathbf{r}=3 \text{ and } \nabla f(r)=f'(r)\,\nabla r\right]$$

$$=\frac{3}{r^2}+\mathbf{r}\cdot\left(-\frac{2}{r^3}\frac{1}{r}\mathbf{r}\right) \qquad \left[\because\quad \nabla r=\frac{1}{r}\mathbf{r}\right]$$

$$=\frac{3}{r^2}-\frac{2}{r^4}(\mathbf{r}\cdot\mathbf{r})=\frac{3}{r^2}-\frac{2}{r^4}\,r^2=\frac{1}{r^2}.$$

$$\therefore\quad \nabla^2\left[\nabla\cdot\left(\frac{\mathbf{r}}{r^2}\right)\right]=\nabla^2\left(\frac{1}{r^2}\right)=\nabla\cdot\left(\nabla\frac{1}{r^2}\right)$$

$$=\nabla\cdot\left(-\frac{2}{r^3}\nabla r\right)=\nabla\cdot\left(-\frac{2}{r^3}\frac{1}{r}\mathbf{r}\right)$$

$$=\nabla\cdot\left(-\frac{2}{r^4}\mathbf{r}\right)$$

$$=\left(-\frac{2}{r^4}\right)(\nabla\cdot\mathbf{r})+\mathbf{r}\cdot\left[\nabla\left(\frac{-2}{r^4}\right)\right], \text{ using the identity (1)}$$

$$=-\frac{2}{r^4}\cdot 3+\mathbf{r}\cdot\left[\frac{8}{r^5}\nabla r\right]$$

$$=-\frac{6}{r^4}+\mathbf{r}\cdot\left(\frac{8}{r^5}\frac{1}{r}\mathbf{r}\right)=-\frac{6}{r^4}+\frac{8}{r^6}\mathbf{r}\cdot\mathbf{r}$$

$$=-\frac{6}{r^4}+\frac{8}{r^6}r^2=-\frac{6}{r^4}+\frac{8}{r^4}=\frac{2}{r^4}=2r^{-4}.$$

Ex: 43. *Prove that curl* $[\mathbf{r}\times(\mathbf{a}\times\mathbf{r})]=3\mathbf{r}\times\mathbf{a}$, *where* **a** *is a constant vector.* **[Gorakhpur 1983]**

Sol. Curl $[\mathbf{r}\times(\mathbf{a}\times\mathbf{r})]$

$=\nabla\times[(\mathbf{r}\cdot\mathbf{r})\,\mathbf{a}-(\mathbf{r}\cdot\mathbf{a})\,\mathbf{r}]$ $\quad[\because\ \mathbf{a}\times(\mathbf{b}\times\mathbf{c})=(\mathbf{a}\cdot\mathbf{c})\,\mathbf{b}-(\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}]$

$=\nabla\times[r^2\mathbf{a}-(\mathbf{r}\cdot\mathbf{a})\,\mathbf{r}]$ $\quad[\because\ \mathbf{r}\cdot\mathbf{r}=\mathbf{r}^2=r^2]$

$=\nabla\times(r^2\mathbf{a})-\nabla\times[(\mathbf{r}\cdot\mathbf{a})\,\mathbf{r}]$

$[\because\ \nabla\times(\mathbf{A}+\mathbf{B})=\nabla\times\mathbf{A}+\nabla\times\mathbf{B}]$

$=(\nabla r^2)\times\mathbf{a}+r^2(\nabla\times\mathbf{a})-[\nabla(\mathbf{r}\cdot\mathbf{a})]\times\mathbf{r}-(\mathbf{r}\cdot\mathbf{a})(\nabla\times\mathbf{r})$

$[\because\ \nabla\times(\phi\mathbf{A})=(\nabla\phi)\times\mathbf{A}+\phi(\nabla\times\mathbf{A})]$

$=(2r\,\nabla r)\times\mathbf{a}+r^2\mathbf{0}-[\nabla(\mathbf{r}\cdot\mathbf{a})]\times\mathbf{r}-(\mathbf{r}\cdot\mathbf{a})\,\mathbf{0}$

$[\because\ \nabla f(r)=f'(r)\,\nabla r;\ \nabla\times\mathbf{a}=\mathbf{0}$, **a** being a constant vector; and $\nabla\times\mathbf{r}=\mathbf{0}]$

$=\left(2r\,\frac{1}{r}\mathbf{r}\right)\times\mathbf{a}-[\nabla(\mathbf{r}\cdot\mathbf{a})]\times\mathbf{r}$

$=2\mathbf{r}\times\mathbf{a}-\mathbf{a}\times\mathbf{r}$ $\quad[\because\ \nabla(\mathbf{r}\cdot\mathbf{a})=\mathbf{a}$, if **a** is a constant vector. See Ex. 9 after § 4. Do it here]

$=2\mathbf{r}\times\mathbf{a}+\mathbf{r}\times\mathbf{a}=3\mathbf{r}\times\mathbf{a}.$

Ex. 44. *Prove that* $\nabla\times(\mathbf{F}\times\mathbf{r})=2\mathbf{F}-(\nabla\cdot\mathbf{F})\,\mathbf{r}+(\mathbf{r}\cdot\nabla)\,\mathbf{F}.$ **[Allahabad 1980]**

Sol. We know that

$\nabla\times(\mathbf{A}\times\mathbf{B})=\mathbf{A}\,(\nabla\cdot\mathbf{B})-\mathbf{B}\,(\nabla\cdot\mathbf{A})+(\mathbf{B}\cdot\nabla)\,\mathbf{A}-(\mathbf{A}\cdot\nabla)\,\mathbf{B}.$

[See identity 6 after § 11]

Putting $\mathbf{A}=\mathbf{F}$ and $\mathbf{B}=\mathbf{r}$ in this identity, we get

$\nabla\times(\mathbf{F}\times\mathbf{r})=\mathbf{F}\,(\nabla\cdot\mathbf{r})-\mathbf{r}\,(\nabla\cdot\mathbf{F})+(\mathbf{r}\cdot\nabla)\,\mathbf{F}-(\mathbf{F}\cdot\nabla)\,\mathbf{r}.$ ...(1)

Now $\nabla\cdot\mathbf{r}=\nabla\cdot(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$

$$=\frac{\partial}{\partial x}(x)+\frac{\partial}{\partial y}(y)+\frac{\partial}{\partial z}(z)=1+1+1=3. \quad ...(2)$$

If $\mathbf{F}=F_1\mathbf{i}+F_2\mathbf{j}+F_3\mathbf{k}$, then

$$\mathbf{F}\cdot\nabla=(F_1\mathbf{i}+F_2\mathbf{j}+F_3\mathbf{k})\cdot\left(\mathbf{i}\,\frac{\partial}{\partial x}+\mathbf{j}\,\frac{\partial}{\partial y}+\mathbf{k}\,\frac{\partial}{\partial z}\right)$$

$$=F_1\,\frac{\partial}{\partial x}+F_2\,\frac{\partial}{\partial y}+F_3\,\frac{\partial}{\partial z}.$$

$$\therefore\quad (\mathbf{F}\cdot\nabla)\,\mathbf{r}=\left(F_1\frac{\partial}{\partial x}+F_2\,\frac{\partial}{\partial y}+F_3\,\frac{\partial}{\partial z}\right)(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})$$

$$=F_1\,\frac{\partial}{\partial x}\,(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})+\ldots+$$

$$=F_1\mathbf{i}+F_2\mathbf{j}+F_3\mathbf{k}=\mathbf{F}. \qquad \ldots(3)$$

$\therefore$ from (1), (2) and (3), we get

$$\nabla\times(\mathbf{F}\times\mathbf{r})=3\mathbf{F}-(\nabla\cdot\mathbf{F})\,\mathbf{r}+(\mathbf{r}\cdot\nabla)\,\mathbf{F}-\mathbf{F}$$

$$=2\mathbf{F}-(\nabla\cdot\mathbf{F})\,\mathbf{r}+(\mathbf{r}\cdot\nabla)\,\mathbf{F}.$$

Ex. 43. *If* **a** *and* **b** *are constant vectors, prove that*

grad $[(\mathbf{r}\times\mathbf{a})\cdot(\mathbf{r}\times\mathbf{b})]=(\mathbf{b}\times\mathbf{r})\times\mathbf{a}+(\mathbf{a}\times\mathbf{r})\times\mathbf{b}.$

[Kanpur 1977]

Sol. We have $(\mathbf{r}\times\mathbf{a})\cdot(\mathbf{r}\times\mathbf{b})$

$$=\begin{vmatrix}\mathbf{r}\cdot\mathbf{r} & \mathbf{r}\cdot\mathbf{b}\\ \mathbf{a}\cdot\mathbf{r} & \mathbf{a}\cdot\mathbf{b}\end{vmatrix}, \text{ by Lagrange's identity}$$

$$=(\mathbf{a}\cdot\mathbf{b})\,(\mathbf{r}\cdot\mathbf{r})-(\mathbf{r}\cdot\mathbf{b})\,(\mathbf{a}\cdot\mathbf{r}).$$

$\therefore$ grad $[(\mathbf{r}\times\mathbf{a})\cdot(\mathbf{r}\times\mathbf{b})]$

$$=\text{grad}\,[(\mathbf{a}\cdot\mathbf{b})\,(\mathbf{r}\cdot\mathbf{r})-(\mathbf{r}\cdot\mathbf{b})\,(\mathbf{a}\cdot\mathbf{r})]$$

$$=\text{grad}\,[(\mathbf{a}\cdot\mathbf{b})\,(\mathbf{r}\cdot\mathbf{r})]-\text{grad}\,[(\mathbf{r}\cdot\mathbf{b})\,(\mathbf{a}\cdot\mathbf{r})]$$

$$=(\mathbf{a}\cdot\mathbf{b})\,\text{grad}\,(\mathbf{r}\cdot\mathbf{r})+(\mathbf{r}\cdot\mathbf{r})\,\text{grad}\,(\mathbf{a}\cdot\mathbf{b})$$

$$-(\mathbf{r}\cdot\mathbf{b})\,\text{grad}\,(\mathbf{a}\cdot\mathbf{r})-(\mathbf{a}\cdot\mathbf{r})\,\text{grad}\,(\mathbf{r}\cdot\mathbf{b}) \qquad \ldots(1)$$

$[\because\ \text{grad}\,(\phi\psi)=\phi\ \text{grad}\ \psi+\psi \text{ and } \phi]$

Now grad $(\mathbf{r}\cdot\mathbf{r})=\text{grad}\ \mathbf{r}^2=\text{grad}\,(x^2+y^2+z^2)$

$$=2x\mathbf{i}+2y\mathbf{j}+2z\mathbf{k}=(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})=2\mathbf{r}.$$

Also if **a** and **b** are constant vectors, then $\mathbf{a}\cdot\mathbf{b}$ is a constant scalar and so grad $(\mathbf{a}\cdot\mathbf{b})=\mathbf{0}$.

Further if **a** is a constant vector, then grad $(\mathbf{r}\cdot\mathbf{a})=\mathbf{a}$. Similarly **b** is a constant vector implies grad $(\mathbf{r}\cdot\mathbf{b})=\mathbf{b}$.

Putting the above values in (1), we have

$$\text{grad}\,[(\mathbf{r}\times\mathbf{a})\cdot(\mathbf{r}\times\mathbf{b})]=(\mathbf{a}\cdot\mathbf{b})\,(2\mathbf{r})+(\mathbf{r}\cdot\mathbf{r})\,\mathbf{0}-(\mathbf{r}\cdot\mathbf{b})\,\mathbf{a}-(\mathbf{a}\cdot\mathbf{r})\,\mathbf{b}$$

$$=[(\mathbf{a}\cdot\mathbf{b})\,\mathbf{r}-(\mathbf{r}\cdot\mathbf{a})\,\mathbf{b}]+[(\mathbf{a}\cdot\mathbf{b})\,\mathbf{r}-(\mathbf{r}\cdot\mathbf{b})\,\mathbf{a}]$$

$$=(\mathbf{b}\times\mathbf{r})\times\mathbf{a}+(\mathbf{a}\times\mathbf{r})\times\mathbf{b}.$$

Ex. 46. *Prove that curl* $[r^n(\mathbf{a}\times\mathbf{r})]=(n+2)\,r^n\mathbf{a}-nr^{n-2}\,(\mathbf{r}\cdot\mathbf{a})\,\mathbf{r}$, *where* **a** *is a constant vector.* **[Rohilkhand 1977]**

Sol. We know that curl $(\phi\mathbf{A})=(\text{grad }\phi)\times\mathbf{A}+\phi$ curl $\mathbf{A}$.

Putting $\phi=r^n$ and $\mathbf{A}=\mathbf{a}\times\mathbf{r}$ in this identity, we have

$$\text{curl }[r^n(\mathbf{a}\times\mathbf{r})]=(\nabla r^n)\times(\mathbf{a}\times\mathbf{r})+r^n \text{ curl }(\mathbf{a}\times\mathbf{r}). \quad \ldots(1)$$

Now $\nabla r^n=nr^{n-1}\nabla r=nr^{n-1}(1/r)\,\mathbf{r}=nr^{n-2}\mathbf{r}$.

$$\because \quad (\nabla r^n)\times(\mathbf{a}\times\mathbf{r})=(nr^{n-2}\mathbf{r})\times(\mathbf{a}\times\mathbf{r})$$
$$=nr^{n-2}\,\mathbf{r}\times(\mathbf{a}\times\mathbf{r})=nr^{n-2}[(\mathbf{r}\cdot\mathbf{r})\,\mathbf{a}-(\mathbf{r}\cdot\mathbf{a})\,\mathbf{r}]$$
$$=nr^{n-2}[r^2\mathbf{a}-(\mathbf{r}\cdot\mathbf{a})\,\mathbf{r}]$$
$$=nr^n\mathbf{a}-nr^{n-2}(\mathbf{r}\cdot\mathbf{a})\,\mathbf{r}. \quad \ldots(2)$$

Also $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$.

Let $\mathbf{a}=a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k}$, where the scalars a_1, a_2, a_3 are all constants.

Then
$$\mathbf{a}\times\mathbf{r}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_1 & a_2 & a_3\\ x & y & z\end{vmatrix}$$
$$=(a_2z-a_3y)\,\mathbf{i}+(a_3x-a_1z)\,\mathbf{j}+(a_1y-a_2x)\,\mathbf{k}.$$

$$\therefore \quad \text{curl }(\mathbf{a}\times\mathbf{r})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ a_2z-a_3y & a_3x-a_1z & a_1y-a_2x\end{vmatrix}$$

$$=\left[\frac{\partial}{\partial y}(a_1y-a_2x)-\frac{\partial}{\partial z}(a_3x-a_1z)\right]\mathbf{i}+\left[\frac{\partial}{\partial z}(a_2z-a_3y)\right.$$
$$\left.-\frac{\partial}{\partial x}(a_1y-a_2x)\right]\mathbf{j}+\left[\frac{\partial}{\partial x}(a_3x-a_1z)-\frac{\partial}{\partial y}(a_2z-a_3y)\right]\mathbf{k}$$
$$=(a_1+a_1)\,\mathbf{i}+(a_2+a_2)\,\mathbf{j}+(a_3+a_3)\,\mathbf{k}=2\,(a_1\mathbf{i}+a_2\mathbf{j}+a_3\mathbf{k})=2\mathbf{a} \quad \ldots(3)$$

Substituting from (2) and (3) in (1), we get

$$\text{curl }[r^n(\mathbf{a}\times\mathbf{r})]=nr^n\,\mathbf{a}-nr^{n-2}(\mathbf{r}\cdot\mathbf{a})\,\mathbf{r}+r^n\,(2\mathbf{a})$$
$$=(n+2)\,r^n\,\mathbf{a}-nr^{n-2}(\mathbf{r}\cdot\mathbf{a})\,\mathbf{r}.$$

Ex. 47. *Prove that* $\mathbf{a}\cdot\{\nabla(\mathbf{v}\cdot\mathbf{a})-\nabla\times(\mathbf{v}\times\mathbf{a})\}=\textit{div}\;\mathbf{v}$, *where* $\mathbf{a}$ *is a constant unit vector.*

Sol. We know that

$$\nabla(\mathbf{A}\cdot\mathbf{B})=(\mathbf{A}\cdot\nabla)\,\mathbf{B}+(\mathbf{B}\cdot\nabla)\,\mathbf{A}+\mathbf{A}\times(\nabla\times\mathbf{B})+\mathbf{B}\times(\nabla\times\mathbf{A}).$$

$$\therefore \quad \nabla(\mathbf{v}\cdot\mathbf{a})=(\mathbf{v}\cdot\nabla)\,\mathbf{a}+(\mathbf{a}\cdot\nabla)\,\mathbf{v}+\mathbf{v}\times(\nabla\times\mathbf{a})+\mathbf{a}\times(\nabla\times\mathbf{v}) \quad \ldots(1)$$

Let $\mathbf{v}=v_1\,\mathbf{i}+v_2\,\mathbf{j}+v_3\,\mathbf{k}$.

Then $\mathbf{v}\cdot\nabla=(v_1\,\mathbf{i}+v_2\,\mathbf{j}+v_3\,\mathbf{k})\cdot\left(\mathbf{i}\,\frac{\partial}{\partial x}+\mathbf{j}\,\frac{\partial}{\partial y}+\mathbf{k}\,\frac{\partial}{\partial z}\right)$

$$=v_1\,\frac{\partial}{\partial x}+v_2\,\frac{\partial}{\partial y}+v_3\,\frac{\partial}{\partial z}.$$

$\because\quad (\mathbf{v}\cdot\nabla)\,\mathbf{a}=\left(v_1\,\frac{\partial}{\partial x}+v_2\,\frac{\partial}{\partial y}+v_3\,\frac{\partial}{\partial z}\right)\mathbf{a}$

$$=v_1\,\frac{\partial\mathbf{a}}{\partial x}+v_2\,\frac{\partial\mathbf{a}}{\partial y}+v_3\,\frac{\partial\mathbf{a}}{\partial z}$$

$=\mathbf{0}$, because $\mathbf{a}$ is a constant vector.

Also $\nabla\times\mathbf{a}=\mathbf{0}$, $\mathbf{a}$ being a constant vector.

$\therefore$ from (1), we have

$$\nabla\,(\mathbf{v}\cdot\mathbf{a})=(\mathbf{a}\cdot\nabla)\,\mathbf{v}+\mathbf{a}\times(\nabla\times\mathbf{v}) \quad \ldots(2)$$

Also we know that

$$\nabla\times(\mathbf{A}\times\mathbf{B})=(\nabla\cdot\mathbf{B})\,\mathbf{A}-(\nabla\cdot\mathbf{A})\,\mathbf{B}+(\mathbf{B}\cdot\nabla)\,\mathbf{A}-(\mathbf{A}\cdot\nabla)\,\mathbf{B}.$$

$\therefore\quad \nabla\times(\mathbf{v}\times\mathbf{a})=(\nabla\cdot\mathbf{a})\,\mathbf{v}-(\nabla\cdot\mathbf{v})\,\mathbf{a}+(\mathbf{a}\cdot\nabla)\,\mathbf{v}-(\mathbf{v}\cdot\nabla)\,\mathbf{a}$

$$=-(\nabla\cdot\mathbf{v})\,\mathbf{a}+(\mathbf{a}\cdot\nabla)\,\mathbf{v} \quad \ldots(3)$$

$[\because\quad \nabla\cdot\mathbf{a}=0$ and $(\mathbf{v}\cdot\nabla)\,\mathbf{a}=\mathbf{0}]$

Subtracting (3) from (2), we get

$$\nabla\,(\mathbf{v}\cdot\mathbf{a})-\nabla\times(\mathbf{v}\times\mathbf{a})=\mathbf{a}\times(\nabla\times\mathbf{v})+(\nabla\cdot\mathbf{v})\,\mathbf{a} \quad \ldots(4)$$

Multiplying both sides of (4) scalarly by $\mathbf{a}$, we get

$\mathbf{a}\cdot[\nabla\,(\mathbf{v}\cdot\mathbf{a})-\nabla\times(\mathbf{v}\times\mathbf{a})]=\mathbf{a}\cdot[\mathbf{a}\times(\nabla\times\mathbf{v})]+\mathbf{a}\cdot[(\nabla\cdot\mathbf{v})\,\mathbf{a}]$

$=[\mathbf{a},\,\mathbf{a},\,\nabla\times\mathbf{v}]+(\nabla\cdot\mathbf{v})\,(\mathbf{a}\cdot\mathbf{a})$

$=0+(\nabla\cdot\mathbf{v})\,\mathbf{a}^2$, since the scalar triple product

$[\mathbf{a},\,\mathbf{a},\,\nabla\times\mathbf{v}]=0$

$=\nabla\cdot\mathbf{v}\qquad[\because\quad \mathbf{a}^2=|\,\mathbf{a}\,|^2=1$, $\mathbf{a}$ being a unit vector]

$=\text{div}\,\mathbf{v}$.

Ex. 48. *If* $\mathbf{a}$ *is a constant vector, then prove that*

(i) $\nabla\,(\mathbf{a}\cdot\mathbf{u})=(\mathbf{a}\cdot\nabla)\,\mathbf{u}+\mathbf{a}\times \textit{curl}\,\mathbf{u}$,

(ii) $\nabla\cdot(\mathbf{a}\times\mathbf{u})=-\mathbf{a}\cdot\textit{curl}\,\mathbf{u}$,

(iii) $\nabla\times(\mathbf{a}\times\mathbf{u})=\mathbf{a}\;\textit{div}\,\mathbf{u}-(\mathbf{a}\cdot\nabla)\,\mathbf{u}$.

Sol. (i) Proceed exactly as in Ex. 47.

We have $\nabla\,(\mathbf{a}\cdot\mathbf{u})=(\mathbf{a}\cdot\nabla)\,\mathbf{u}+(\mathbf{u}\cdot\nabla)\,\mathbf{a}+\mathbf{a}\times(\nabla\times\mathbf{u})$

$+\mathbf{u}\times(\nabla\times\mathbf{a})$.

Since **a** is a constant vector, therefore

$$(\mathbf{u}\cdot\nabla)\,\mathbf{a}=\mathbf{0} \text{ and } \nabla\times\mathbf{a}=\mathbf{0}.$$

$\therefore\quad \nabla(\mathbf{a}\cdot\mathbf{u})=(\mathbf{a}\cdot\nabla)\,\mathbf{u}+\mathbf{a}\times\text{curl }\mathbf{u}.$

(ii) We know that $\nabla\cdot(\mathbf{A}\times\mathbf{B})=\mathbf{B}\cdot(\nabla\times\mathbf{A})-\mathbf{A}\cdot(\nabla\times\mathbf{B})$.

$\therefore\quad \nabla\cdot(\mathbf{a}\times\mathbf{u})=\mathbf{u}\cdot(\nabla\times\mathbf{a})-\mathbf{a}\cdot(\nabla\times\mathbf{u})$

$=\mathbf{u}\cdot\mathbf{0}-\mathbf{a}\cdot(\nabla\times\mathbf{u})$, since $\nabla\times\mathbf{a}=\mathbf{0}$,

a being a constant vector

$=0-\mathbf{a}\cdot\text{curl }\mathbf{u}=-\mathbf{a}\cdot\text{curl }\mathbf{u}.$

(iii) Proceed exactly as in Ex. 47, using the identity for $\nabla\times(\mathbf{A}\times\mathbf{B})$.