G-20 (P-2 IFS 2020)

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6(a) Find the general solution and singular solution of the PDE

$$6yz - 6pxy - 3qy^2 + pq = 0 \quad — (*)$$

(10+5 = 15)

Sol: Using Charpit Method

$$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

Here, $f = 6yz - 6pxy - 3qy^2 + pq$

$$\frac{dx}{q - 6xy} = \frac{dy}{p - 3y^2} = \frac{dz}{p(q-6xy) + q(p-3y^2)} = \frac{dp}{0} = \frac{dq}{6px - 6z}$$

So, we get, $p = k$, where k is an arbitrary constant.

Using $p = k$ in the given D.E., we get

$$q = \frac{6y(z-kx)}{3y^2 - k} \quad (\text{using } p = k \text{ in } *)$$

$$\therefore \; dz = p\,dx + q\,dy = k\,dx + \frac{6y(z-kx)}{3y^2 - k}\,dy$$

$$\frac{dz - k\,dx}{z - kx} = \frac{6y\,dy}{3y^2 - k}$$

Integrating,

$$z - kx = k_1(3y^2 - k)$$

where k_1 is an arbitrary constant.

This is the general solution of the PDE.

Now, we find the envelope, if it exists.

Let $F(k, k_1) \equiv z - kx - k_1(3y^2 - k) = 0$ — (*)

Then, $F_k = -x + k_1 = 0$ and

$F_{k_1} = k - 3y^2 = 0$

$\Rightarrow$ $x = k_1$ and $3y^2 = k$

$\therefore$ (*) gives us

$$z - 3y^2x - x(3y^2 - 3y^2) = 0$$

i.e. $\boxed{z = 3xy^2}$

is a singular solution of the given PDE.

6(b) Find the Lagrange interpolating polynomial that fits the following data values.

x :	-1	2	3	5
$f(x)$:	-1	10	25	60

Also, interpolate at $x = 2.5$, correct to three decimal places. (15)

Sol: Lagrange's interpolating polynomial is

$$f(x) = \frac{(x-2)(x-3)(x-5)}{(-1-2)(-1-3)(-1-5)} f(-1) + \frac{(x+1)(x-3)(x-5)}{(2+1)(2-3)(2-5)} f(2)$$

$$+ \frac{(x+1)(x-2)(x-5)}{(3+1)(3-2)(3-5)} f(3) + \frac{(x+1)(x-2)(x-3)}{(5+1)(5-2)(5-3)} f(5)$$

$$= \frac{1}{72}(x^3-10x^2+31x-30) + \frac{10}{9}(x^3-7x^2+7x+15)$$

$$+ \frac{-25}{8}(x^3-6x^2+3x+10) + \frac{5}{3}(x^3-4x^2+x+6)$$

$$f(x) = -\frac{x^3}{3} + \frac{25}{6}x^2 + \frac{x}{2} - 5 \quad \text{[use Calci]}$$

Now,

$$f(2.5) = 17.08334 \quad \text{[use Calci]}$$

$$f(2.5) = 17.083$$

correct upto three decimal places.

6(c) In a fluid flow, the velocity vector is given by

$$\vec{V} = 2x\hat{i} + 3y\hat{j} - 5z\hat{k}$$

Determine the eqn of the streamline through a point $A = (4, 8, 1)$. (10)

Sol: Eqn of streamline is given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

$$\frac{dx}{2x} = \frac{dy}{3y} = \frac{dz}{-5z}$$

Taking the first two & integrating

$$3 \log x = 2 \log y + \log C_1$$

$$\Rightarrow \boxed{x^3 = C_1 y^2} \quad \text{— (1)}$$

Solving the last two equations,

$$-5 \log y = 3 \log z \cancel{-} \log C_2$$

$$\boxed{y^5 z^3 = C_2} \quad \text{— (2)}$$

streamlines are given by the curves of intersection of (2) and (3)

i.e. $\phi(C_1, C_2) = 0$

where ϕ is an arbitrary function.

$$\phi\left(x^3/y^2, \; y^5 z^3\right) = 0$$

Putting $(x, y, z) = (4, 8, 1)$, we get eqn of stream line, through this point i.e. intersection of curves ~~[illegible]~~ $x^3 = y^2$ and $y^5 z^3 = 32768$.