Mathematics Mains Test -2

Units: Modern Algebra + Real Analysis + Complex + LPP

Time : 3 hours	Maximum marks: 250

Instructions

1. Question paper contains **8** questions out of this candidate need to answer **5** questions in the following pattern
2. Candidate should attempt question No's **1** and **5** compulsorily and any **three** of the remaining questions selecting **atleast one** question from each section.
3. The number of marks carried by each question is indicated at end of each question.
4. Assume suitable data if considered necessary and indicate the same clearly.
5. Symbols/Notations carry their usual meanings, unless otherwise indicated.

Section- A

1.

a) Find all solutions of $x^2 - x + 2 = 0$ over $z_3[i]$. (10 marks)

b) Show that the mapping $f: G \to G'$ such that $f(x + iy) = x$ where G is a group of complex numbers under addition, G' is a group of real numbers under addition, is a homomorphism onto and find $kerf$. (10 marks)

c) i) Show that set of fourth roots of unity, is a cyclic group w.r.t. multiplication.
ii) Prove that the group $(\{1,2,3,4\}, \times_5)$ is cyclic and write its generators. (10 marks)

d) Prove that $\lim_{n\to\infty} \frac{1}{n}[1 + 2^{1/2} + 3^{1/3} + \cdots + n^{1/n}]=1$. (10 marks)

e) Test for what values of α the series $\sum \frac{1}{n^{\alpha + \beta/n}}$ converges and diverges. (10 marks)

2.

a) Examine the convergence of $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n}(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n})$ (10 marks)

b) Show that the series $\sum_{n=1}^{\infty}(-1)^{n+1}(\sqrt{n+1} - \sqrt{n})$ is conditionally convergent. (15 marks)

c) Examine the continuity of f defined by $f(x) = \lim_{n\to\infty} \frac{x^n}{1+x^n e^x}$ $\forall x \geq 0$. (15 marks)

d) Test the convergence of i) $\sum_{n=1}^{\infty} tan\frac{1}{n}$ ii) $\sum_{n=1}^{\infty} \frac{1}{n} sin\frac{1}{n}$ (10 marks)

3.

a) If $a_1 > 0, b_1 > 0$ and $a_n = \sqrt{a_{n-1}b_{n-1}}$ and $b_n = \frac{2a_{n-1}b_{n-1}}{a_{n-1}+b_{n-1}}$, prove that
   I. $\{a_n\}$ and $\{b_n\}$ are monotonic, the one increasing and the other decreasing.
   II. $\{a_n\}$ and $\{b_n\}$ both converge to the common limit. (15 marks)

b) Show that the sequence of functions $< f_n >$ where $f_n(x) = nx(1-x)^n$ is not uniformly convergent on [0 1]. (15 marks)

c) Show that the series $\frac{x}{1+x^2} + \left(\frac{2^2 x}{1+2^3x^2} - \frac{x}{1+x^2}\right) + \left(\frac{3^2 x}{1+3^3x^2} - \frac{2^2 x}{1+2^3x^2}\right) + \cdots$ does not converge uniformly on [0 1]. (15 marks)

d) Test the convergence of $1 - \frac{1}{3.2^2} + \frac{1}{5.3^2} - \frac{1}{7.4^2} + \cdots$. (5 marks).

4.

a) Show that $H = \{1, -1\}$ is a normal subgroup of the group $G = \{1, -1, i, -i\}$ under multiplication. Also write the composition table for the quotient group $G/_H$ (10 marks)

b) Show that $(Z_6 = \{0,1,2,3,4,5\}, +_6)$ is a group. Hence prove that $S = \{0,2,4\}, T = \{0,3\}$ are subgroups of Z_6 . Is $S \cup T$ a subgroup of Z_6? and also prove that intersection of two subgroups of a group is a subgroup of a group. (20 marks)

c) In a group G for $a, b \in G, O(a) = 5, b \neq e \; and \; aba^{-1} = b^2$. Find $O(b)$. And also show that all groups of order 5 and less are commutative. (20 marks)

Section- B

5.

a) A company produces two types of hats. Each hat of the first type requires twice as much labor time as the second type. If all hats are of second type only, the company can produce a total of 500 hats a day. The market limits daily sales of the first and second type to 150 and 250 hats. Assuming that the profits per hats are Rs 8 for type A and Rs 5 for type B, formulate the problem as a linear programming model in order to determine the number of hats to be produced of each type so as to maximize the profit. (10 marks)

b) Solve the following LPP graphically

Maximize $Z = 5x_1 + 3x_2$

Subject to

$$x_1 + x_2 \leq 2$$
$$5x_1 + 2x_2 \leq 10$$
$$3x_1 + 8x_2 \leq 12$$
$$x_1, x_2 \geq 0$$

(10marks)

c) Show that the function $f(z) = sinx \; coshy + icosx \; sinhy \;\; is$ continuous as well as analytic everywhere. (10 marks)

d) If $u = e^x(xcosy - ysiny)$, find the analytic function $u + iv$. (10 marks)

e) Prove that the function $2x - x^3 + 3xy^2$ is harmonic and find the harmonic conjugate. (10 marks)

6.

a) If $f(z) = u + iv$ is an analytic function of $z = x + iy$ *and* $u - v = \frac{e^y - cosx + sinx}{coshy - cosx}$, find $f(z)$ subject to the condition $f\left(\frac{\pi}{2}\right) = \frac{3-i}{2}$.

(15 marks)

b) If $f(z) = \frac{1}{(Z-1)(Z-2)}$ then find the Laurent's series expansion of $f(z)$ which are valid in the regions

i) $0 < |z-1| < 1$

ii) $0 < |z-2| < 1$

iii) $0 < |z| < 2$

iv) $|z| < 1$

v) $|z| > 2$ (25 marks)

c) If $f(z) = \frac{Z+1}{Z^2+9}$ then find the singularities and residues of the given function.

(10 marks)

7.

a) Find the optimal solution of the following transportation problem.

	A	B	C	D	E	Supply
1	10	2	3	15	9	35
2	5	10	15	2	4	40
3	15	5	14	7	15	20
4	20	15	13	25	8	30
Demand	20	20	40	10	35	

(25 marks)

b) Solve the following assignment problem

	I	II	III	IV	V
A	2	9	2	7	1
B	6	8	7	6	1
C	4	6	5	3	1
D	4	2	7	3	1
E	5	3	9	5	1

(20 marks)

c) Obtain dual of following LPP

Maximize $Z = 6x_1 + 4x_2 + 6x_3 + x_4$

Subject to $4x_1 + 5x_2 + 4x_3 + 8x_4 = 21$

$3x_1 + 7x_2 + 8x_3 + 2x_4 \leq 48$

$x_1, x_2, x_3, x_4 \geq 0$ (5 marks)

8.

a) Solve the following LPP using Simplex method.

Maximize $Z = 12x_1 + 16x_2$

Subject to $10x_1 + 20x_2 \leq 120$

$8x_1 + 8x_1 \leq 80,\ x_1, x_2 \geq 0.$ (20 marks)

b) If C denotes positively oriented boundary of surface whose sides lies along the lines $X = \pm 2, Y = \pm 2$, then evaluate

I. $\int_C \frac{e^{-Z}}{Z - \frac{\pi i}{2}}\, dz$

II. $\int_C \frac{CosZ}{Z(Z^2+8)}\, dz$

III. $\int_C \frac{CoshZ}{Z^4}\, dz$ (15 marks)

c) Show that $\int_0^{2\pi} \frac{d\theta}{5+3sin\theta} = \frac{\pi}{2}$ (15 marks)