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Q 5 (e). Prove that $\nabla^2 r^n = n(n+1) r^{n-2}$ & that $r^n \vec{r}$ is irrotational, where $r = |\vec{r}| = \sqrt{x^2+y^2+z^2}$

$\longrightarrow$ We have $\nabla^2 r^n = \nabla\cdot(\nabla r^n) = \text{div}(\text{grad } r^n)$

$= \text{div}(n r^{n-1} \text{grad } r)$

$= \text{div}\left(n r^{n-1} \frac{1}{r} \vec{r}\right) = \text{div}(n r^{n-2} \vec{r})$

$= (n r^{n-2}) \text{div } \vec{r} + \vec{r}\cdot(\text{grad } n r^{n-2})$

$= 3n r^{n-2} + \vec{r}\cdot[n(n-2) r^{n-3} \text{grad } r]$

$= 3n r^{n-2} + \vec{r}\cdot\left[n(n-2) r^{n-3} \frac{1}{r} \vec{r}\right]$

$= 3n r^{n-2} + \vec{r}\cdot[n(n-2) r^{n-4} \vec{r}]$

$= 3n r^{n-2} + n(n-2) r^{n-4} (\vec{r}\cdot\vec{r})$

$= 3n r^{n-2} + n(n-2) r^{n-4} r^2$

$= n r^{n-2} (3+n-2)$

$= n(n+1) r^{n-2}$

Now, $|\vec{r}| = \sqrt{x^2+y^2+z^2}$

$r^n \vec{r} = r^n \{x^2 i + y^2 j + z^2 k\}$

$r^n \vec{r}$ is irrotational if $\text{curl}(r^n \vec{r}) = 0$

$$\text{curl}(r^n \vec{r}) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$= i\left\{\frac{\partial}{\partial y}(z^2) - \frac{\partial}{\partial z}(y^2)\right\} + j\left\{\frac{\partial}{\partial z}(x^2) - \frac{\partial}{\partial x}(z^2)\right\} + \hat{k}\left\{\frac{\partial}{\partial x}(y^2) - \frac{\partial}{\partial y}(x^2)\right\}$

$= i(0) + j(0) + \hat{k}(0) = 0$

= ... vector.

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Using stokes' theorem, evaluate

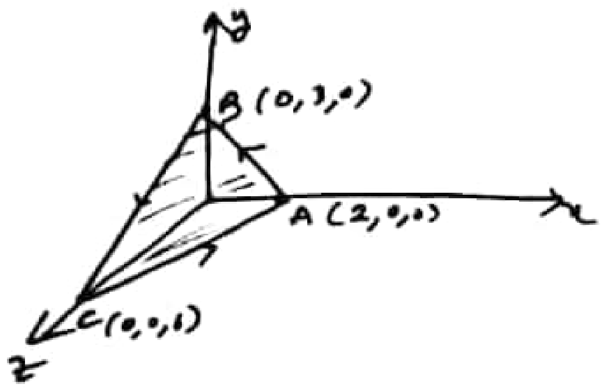
$$\oint_C [(x+y)\,dx + (2x-z)\,dy + (y+z)\,dz]$$

where C is boundary of triangle with vertices at $(2,0,0)$, $(0,3,0)$ and $(0,0,6)$

The given integral is of form $\oint_C \vec{F}\cdot d\vec{r}$

where $\vec{F} = (x+y)\hat{i} + (2x-z)\hat{j} + (y+z)\hat{k}$

$$\text{curl } \vec{F} = 2\hat{i} + \hat{k} \quad \text{— (1)}$$



Boundary (C) = ABCA

Area (S) = Triangle ABC

Using stokes' Theorem,

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\text{curl } \vec{F}\cdot\hat{n})\, dS$$

then $\hat{n}$ is normal vector to $\frac{x}{2}+\frac{y}{3}+\frac{z}{6}=1$

$$\hat{n} = \left(\frac{i}{2}+\frac{j}{3}+\frac{k}{6}\right)\Big/\frac{6}{\sqrt{14}} = \frac{3\hat{i}+2\hat{j}+k}{\sqrt{14}} \quad \text{— (2)}$$

[from (1) and (2)]

$$\text{curl } \vec{F}\cdot\hat{n} = \frac{7}{\sqrt{14}}$$

$$\oint_C \vec{F}\cdot d\vec{r} = \frac{7}{\sqrt{14}}\iint_S dS = \frac{7}{\sqrt{14}}\cdot(\text{Area of } \Delta ABC$$

Area of ΔABC

$\Delta^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$

$\Rightarrow \Delta^2 = \left(\frac{1}{2} \cdot 3 \cdot 6\right)^2 + \left(\frac{1}{2} \cdot 2 \cdot 6\right)^2 + \left(\frac{1}{2} \cdot 2 \cdot 3\right)^2$

$\Rightarrow \Delta^2 = 126$

$\Rightarrow \Delta = 3\sqrt{14}$ — (3)

Then,

$\oint_C \vec{F} \cdot d\vec{r} = \frac{7}{\sqrt{14}} \cdot 3\sqrt{14} = 21$

$$\boxed{\oint_C (x+y)\,dx + (2x-z)\,dy + (y+z)\,dz = 21}$$

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Q 8 c) Find the curvature & torsion of circular helix
$\vec{r} = a(\cos\theta, \sin\theta, \theta\cot\beta)$
β is constant angle at which it cuts its generators

$$\frac{d\vec{r}}{d\theta} = a\{-\sin\theta\, i + \cos\theta\, j + \cos\beta\, k\} \quad \text{--- (1)}$$

$$\frac{d^2 r}{d\theta^2} = a\{-\cos\theta\, i - \sin\theta\, j + 0\} \quad \text{--- (2)}$$

$$\frac{d^3 r}{d\theta^2} = a\{\sin\theta\, i - \cos\theta\, j + 0\} \quad \text{--- (3)}$$

Now, To find $\left|\frac{d\vec{r}}{d\theta} \times \frac{d^2\vec{r}}{d\theta^2}\right|$,

$$\cancel{\frac{d^2\vec{r}}{d\theta} \times d^2} \quad \frac{d\vec{r}}{d\theta} \times \frac{d^2\vec{r}}{d\theta^2} = \begin{vmatrix} i & j & k \\ -a\sin\theta & a\cos\theta & a\cos\beta \\ -a\cos\theta & -a\sin\theta & 0 \end{vmatrix}$$

$$= a^2\{(\sin\theta\cos\beta)\, i - (\cos\theta\cos\beta)\, j + k\}.$$

$$\therefore \left|\frac{d\vec{r}}{d\theta} \times \frac{d^2\vec{r}}{d\theta^2}\right| = \sqrt{a^4[(\sin\theta\cos\beta)^2 + (\cos\theta\cos\beta)^2 + 1^2]}$$

$$= a^2\sqrt{1+\cos^2\beta} \quad \text{--- (4)}$$

Also, $\left|\frac{d\vec{r}}{d\theta}\right| = a\sqrt{1+\cos^2\beta}$ --- (5)

Now, scalar triple prod+ $\left[\frac{d\vec{r}}{d\theta}\ \frac{d^2\vec{r}}{d\theta^2}\ \frac{d^3\vec{r}}{d\theta^3}\right]$

$$= \begin{vmatrix} -a\sin\theta & a\cos\theta & a\cos\beta \\ -a\cos\theta & -a\sin\theta & 0 \\ a\sin\theta & -a\cos\theta & 0 \end{vmatrix}$$

$$= a^3\cos\beta \quad \text{--- (6)}$$

Now, curvature $k = \dfrac{\left|\dfrac{d\vec{r}}{d\theta} \times \dfrac{d^2\vec{r}}{d\theta^2}\right|}{\left|\dfrac{d\vec{r}}{d\theta}\right|^3}$

From ④ & ⑤,

$$k = \frac{a^2\sqrt{1+\cos^2\beta}}{\left(a\sqrt{1+\cos^2\beta}\right)^3} = \boxed{\frac{1}{a(1+\cos^2\beta}}$$

Torsion, $T = \dfrac{\left[\dfrac{d\vec{r}}{dt}\ \dfrac{d^2\vec{r}}{dt^2}\ \dfrac{d^3\vec{r}}{dt^3}\right]}{\left|\dfrac{d\vec{r}}{dt} \times \dfrac{d^2\vec{r}}{dt^2}\right|^2}$

From ④ & 6

$$T = \frac{a^3\cos\beta}{a^4(1+\cos^2\beta)} = \boxed{\frac{\cos\beta}{a(1+\cos^2\beta)}}$$

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7d) Evaluate $\iint_S (\nabla\times\vec{F})\cdot\hat{n}\,dS$. where S is surface of cone, $z = 2-\sqrt{x^2+y^2}$, above xy-plane & $\vec{F} = (x-z)\hat{i} + (x^3+yz)\hat{j} - 3xy^2\hat{k}$

→ The xy-plane cuts the surface S of cone in the circle C, whose equations are

$x^2+y^2 = 4$ & $z=0$.

Let the parametric equations of curve C be

$x = \cancel{\sqrt{2}\cos t}$ $2\cos t$ $\quad y = \cancel{\sqrt{2}\sin t}$ $2\sin t$

By stoke's theorem

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n} \, ds = \oint_C \vec{F} \cdot d\vec{r}$$

$$= \int_C \left[(x-z)\hat{i} + (x^3+yz)\hat{j} - 3xy^2\hat{k}\right] \cdot (dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k})$$

$$= \int_C (x-z)\,dx + (x^3+yz)\,dy + (-3xy^2)\,dz$$

$$= \int_C x\,dx + x^3\,dy \quad \cdots \quad \because z = dz = 0$$

$$= \int_{t=0}^{2\pi} \left[x\frac{dx}{dt} + x^3\frac{dy}{dt}\right] dt$$

$$= \int_{t=0}^{2\pi} \left[2\cos t(-2\sin t) + 2^3\cos^3 t \cdot 2\cos t\right] dt$$

$$= \int_{t=0}^{2\pi} \left[-4\sin t\cos t + 16\cos^4 t\right] dt$$

$$= -2\int_{t=0}^{2\pi} \sin 2t \, dt + 16\int_{t=0}^{2\pi} \cos^4 t \, dt$$

$$= \left(\cos 2t\right]_0^{2\pi} + 16\left\{\frac{1}{32}\left[\sin 4x\right]_0^{2\pi} - \frac{1}{8}\left[x\right]_0^{2\pi} + \frac{1}{2}\left[x\right]_0^{2\pi} + \frac{1}{4}\left[\sin 2x\right]_0^{2\pi}\right\}$$

$$= 0 + 16\left\{0 - \frac{1}{8}(2\pi) + \frac{1}{2}(2\pi) + 0\right\}$$

$$= \boxed{12\pi}$$

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