11 Years
Solved Papers
2009-2019

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Civil Services Main Examination

TOPICWISE PREVIOUS YEARS' SOLVED PAPERS

Mathematics
Paper-II

11 Years

Previous Years Solved Papers

Civil Services Main Examination

(2009-2019)

Mathematics

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Topicwise Presentation

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10.8.2020

Previous Years Solved Papers of

Civil Services Main Examination

Mathematics : Paper-II

CONTENTS

3 Complex Analysis

1. Analytic Functions

1.1 Let $f(z) = \dfrac{a_0 + a_1 z + ... + a_{n-1}z^{n-1}}{b_0 + b_1 z + ... + b_n z^n}$, $b_n \neq 0$. Assume that the zeroes of the denominator are simple.

Show that the sum of the residues of $f(z)$ at its poles is equal to $\dfrac{a_n - 1}{b_n}$.

(2009 : 12 Marks)

Solution:

Proof : As the zeroes of denominator are simple, so $b_0 + b_1 z + ... + b_n z^n$ can be written as

$$b_n(z - z_0)(z - z_1)(z - z_2) ..... (z - z_{n-1})$$

or
$$f(z) = \frac{a_0 + a_1 z + ...a_{n-1}z^{n-1}}{b_n(z - z_0)(z - z_1)(z - z_2).....(z - z_n - 1)} \qquad ...(1)$$

As maximum degree of 'z' in numerator is $n - 1$ and degree of denominator is n (i.e., degree of denominator is greater than degree of numerator).

So, $f(z)$ can be written as

$$f(z) = \frac{A_0}{z - z_0} + \frac{A_1}{z - z_1} + \frac{A_2}{z - z_2} + ... \frac{A_{n-1}}{z - z_{n-1}} \qquad ...(2)$$

$$= \frac{A_0(z - z_1)(z - 2).....(z - z_{n-1}) + A_1(z - z_0)(z - z_2) - (z - z_{n-1}) + ...A_n(z - z_0)(z - z_{n-2})}{(z - z_0)(z - z_1).....(z - z_{n-1})} \qquad ...(3)$$

Now, we know that residues of $f(z)$ are $A_0, A_1, A_2, ..., A_{n-1}$.

According to problem, we need to calculate $A_0 + A_1 + A_2 ..... A_{n-1}$.

From (3), the coefficient of z^{n-1} in numerator will be calculated as :

$$A_0 + A_1 + A_2 ..... A_{n-1}$$

because on observing

$$\begin{aligned}
& A_0(z - z_1)(z - z_2) - (z - z_{n-1}) \\
+ \; & A_1(z - z_0)(z - z_2) - (z - z_{n-1}) \\
+ \; & A_2(z - z_0)(z - z_1)(z - z_3) - (z - z_{n-1}) \\
+ \; & \vdots \\
+ \; & A_{n-1}(z - z_0)(z - z_1).....(z - z_{n-2})
\end{aligned}$$

In each of terms respectively coefficient of z^{n-1} will be $A_0, A_1, A_2 ..... A_{n-1}$. So, the overall coefficient of z^{n-1} in numerator will be $A_0 + A_1 + A_2 ..... A_{n-1}$.

Now, comparing this coefficient of 2^{n-1} with (1) i.e.,

$$\frac{\dfrac{a_0}{b_n} + \dfrac{a_1}{b_n}z + \dfrac{a_2}{b_n}z^2 ..... \dfrac{a_{n-1}}{b_n}z^{n-1}}{(z - z_0)(z - z_1).....(z - z_{n-1})} \qquad ...(4)$$

In this fraction, coefficient of z^{n-1} will be $\frac{a_{n-1}}{b_n}$.
(as denominator $(z-z_0)(z-z_1)$ ..... $(z-z_{n-1})$ is common (3) and (4))

$$\therefore \quad A_0 + A_1 + A_2 \ldots. A_{n-1} = \frac{a_{n-1}}{b_n}$$

or $$\text{Sum of residues} = \frac{a_{n-1}}{b_n}. \text{ Thus Proved.}$$

1.2 **Show that $u(x, y) = 2x - x^3 + 3xy^2$ is a harmonic function. Find a harmonic conjugate of $u(x, y)$. Hence, find the analytic function f for which $u(x, y)$ is the real part.**

(2010 : 12 Marks)

Solution:

Given
$$u(x, y) = 2x - x^3 + 3xy^2$$
$$\frac{\partial u}{\partial x} = 2 - 3x^2 + 3y^2, \quad \frac{\partial^2 u}{\partial x^2} = -6x$$
$$\frac{\partial u}{\partial y} = 6xy, \quad \frac{\partial^2 y}{\partial y^2} = 6x$$
$$\therefore \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = -6x + 6x = 0$$

$\therefore$ $u(x, y)$ is a harmonic function.
Let v be its harmonic conjugate.

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2 - 3x^2 + 3y^2$$
$$\Rightarrow \quad v = 2y - 3x^2y + y^3 + f(x) \qquad \ldots(1)$$

Also,
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = 6xy$$
$$\Rightarrow \quad \frac{\partial v}{\partial x} = -6xy$$
$$\Rightarrow \quad v = -3x^2y + g(y) \quad \ldots(2)$$ ($f(x)$ and $g(y)$ are some arbitrary functions)

Comparing (1) and (2), we get
$$v = 2y + y^3 - 3x^2y + c, \text{ where } c \text{ is a constant.}$$

Now, let
$$f = u(x, y) + iv(x, y)$$
$$\Rightarrow \quad f = (2x - x^3 + 3xy^2) + i(2y + y^3 - 3x^2y + c)$$
$$f(z) = \int (\phi_1(z,0) - i\phi_2(z,0))dz$$
$$= \int (2 - 3z^2)dz = 2z - z^3 + A, \text{ where } A \text{ is a constant}$$
$$\Rightarrow \quad f(z) = 2z - z^3 + A$$

1.3 (i) **Evaluate the line integral $\int_c f(z)dz$ where $f(z) = z^2$, c is the boundary of the triangle with vertices $A(0, 0)$, $B(1, 0)$, $C(1, 2)$ in that order.**

(ii) **Find the image of the finite vertical strip $R: x = 5$ to $x = 9$, $-\pi \le y \le \pi$ of z-plane under exponential function.**

(2010 : 15 Marks)

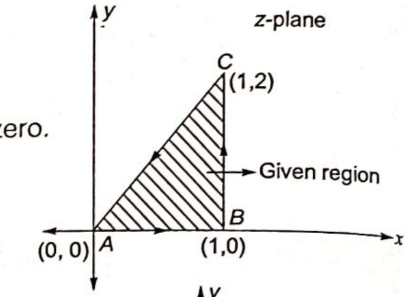
Solution:

(i) Given : $f(z) = z^2$

Now, $f(z) = z^2$ is analytic in whole given region.

Line integral of cled boundary curve of analytic function is always zero.

$\therefore \int_c f(z)dz$ in the given region is 0.



(ii) Given : x varies from 5 to 9 and y varies from $-\pi$ to π.

Now, $$z = x + iy$$

Image under exponential function will be

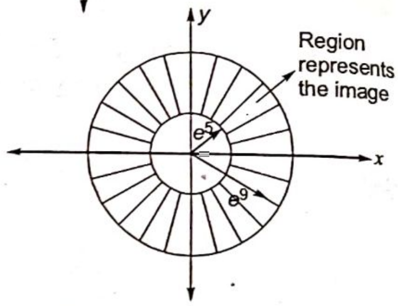
$$e^z = e^{x+iy} = e^x . e^{iy}$$

As x varies from 5 to 9, $\therefore$ e^x varies from e^5 to e^9.

Also, e^{i4} denotes angle variation.

So, image is given by region in figure.

So, the region is bounded by two circles of radii e^5 and e^9.



1.4 If $f(z) = u + iv$ is an analytic function of $z = x + iy$ and $u - v = \dfrac{e^y - \cos x + \sin x}{\cosh y - \cos x}$, find $f(z)$ subject to the condition, $f\left(\dfrac{\pi}{2}\right) = \dfrac{3-i}{2}$.

(2011 : 12 Marks)

Solution:

Let $$f(z) = u + iv$$

$\Rightarrow$ $$if(z) = iu - v$$

$\Rightarrow$ $$(1+i)f(z) = u - v + i(u+v) = U + iV$$

$\therefore$ $$U = u - v = \frac{e^y - \cos x + \sin x}{\cosh y - \cos x}$$

$$= \frac{\cos hy + \sin hy - \cos x + \sin x}{\cos hy - \cos x}$$

$$= 1 + \frac{\sin hy + \sin x}{\cos hy - \cos x}$$

Let $$\frac{\partial U}{\partial x} = \phi_1(x, y) \text{ and } \frac{\partial U}{\partial y} = \phi_2(x, y)$$

$\therefore$ $$\phi_1(x, y) = \frac{\partial U}{\partial x} = \frac{\cos x(\cos hy - \cos x) - \sin x(\sin hy + \sin x)}{(\cos hy - \cos x)^2}$$

$\therefore$ $$\phi_1(z, 0) = \frac{\cos z(1 - \cos z) - \sin^2 z}{(1-\cos z)^2} = \frac{\cos z - 1}{(1 - \cos z)^2}$$

$$= \frac{-1}{1 - \cos z} = \frac{-1}{2}\operatorname{cosec}^2\frac{z}{2}$$

$$\phi_2(x, y) = \frac{\partial U}{\partial y} = \frac{\cos hy(\cos hy - \cos x) - \sin hy(\sin hy + \sin x)}{(\cos hy - \cos x)^2}$$

$\therefore$ $$\phi_2(z, 0) = \frac{1 - \cos z}{(1 - \cos z)^2} = \frac{1}{1 - \cos z}$$

$$= \frac{1}{2}\operatorname{cosec}^2\frac{z}{2}$$

∴ By Milne's Method

$$(1+i)f(z) = \int[\phi_1(z,0) - i\phi_2(z,0)]dz + C$$

$$= \int\left(-\frac{1}{2}\operatorname{cosec}^2\frac{z}{2} - i\frac{1}{2}\operatorname{cosec}^2\frac{z}{2}\right)dz + C$$

$$= -\frac{1}{2}(1+i)\int\operatorname{cosec}^2\frac{z}{2}dz + C$$

$$= (1+i)\cot\frac{z}{2} + C$$

∴ $$f(z) = \cot\frac{z}{2} + \frac{C}{1+i}$$

$$= \cot\frac{z}{2} + d,\ d = \frac{C}{1+i}$$

At $z = \frac{\pi}{2}$, $$f(z) = \frac{3-i}{2}$$

∴ $$d = f\left(\frac{\pi}{2}\right) - \cot\frac{\pi}{4} = \frac{1-i}{2}$$

∴ $$f(z) = \cot\frac{z}{2} + \frac{1-i}{2}$$

1.5 Show that the function defined by

$$f(z) = \begin{cases} \dfrac{x^3y^5(x+iy)}{x^6+y^{10}}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

is not analytic at the origin though it satisfies Cauchy-Riemann equations at the origin.

(2012 : 12 Marks)

Solution:

It is given that

$$f(z) = \begin{cases} \dfrac{x^3y^5(x+iy)}{x^6+y^{10}}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

We have, $$\frac{f(z)-f(0)}{z} = \frac{\dfrac{x^3y^5(x+iy)}{x^6+y^{10}} - 0}{x+iy}$$

$$= \frac{x^3y^5}{x^6+y^{10}}$$

If $z = 0$ along $y = x$, then

$$\lim_{z\to 0}\frac{f(z)-f(0)}{z} = \lim_{x\to 0}\frac{x^8}{x^6+x^{10}} = \lim_{x\to 0}\frac{x^2}{1+x^4} = 0 \quad \text{...(i)}$$

If $z = 0$ along $y^{10} = x^6$, then

$$\lim_{z\to 0}\frac{f(z)-f(0)}{z} = \lim_{x\to 0}\frac{x^3\cdot x^3}{x^6+x^6} = \lim_{x\to 0}\frac{x^6}{2x^6} = \frac{1}{2} \quad ...(ii)$$

Hence, $f(z)$ is not analytic as the limits in (i) and (ii) are not unique.

Again
$$f(z) = \frac{x^3y^5(x+iy)}{x^6+y^{10}}$$
$$= \frac{x^4y^5}{x^6+y^{10}} + i\frac{x^3y^6}{x^6+y^{10}} = u + iv$$

$\therefore$
$$\frac{\partial u}{\partial x} = \lim_{x\to 0}\frac{u(x,0)-u(0,0)}{x}$$
$$= \lim_{x\to 0}\frac{0-0}{x} = 0$$
$$\frac{\partial u}{\partial y} = \lim_{y\to 0}\frac{u(0,y)-u(0,0)}{y} = \lim_{y\to 0}\frac{0-0}{y} = 0$$
$$\frac{\partial v}{\partial x} = \lim_{x\to 0}\frac{v(x,0)-v(0,0)}{x} = \lim_{x\to 0}\frac{0-0}{x} = 0$$
$$\frac{\partial v}{\partial y} = \lim_{y\to 0}\frac{v(0,y)-v(0,0)}{y} = \lim_{y\to 0}\frac{0-0}{y} = 0$$

Since,
$$\frac{\partial y}{\partial x} = \frac{\partial v}{\partial y}$$

and
$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\therefore$ Cauchy-Riemann equations are satisfied at the origin.

1.6 Use Cauchy integral formula to evauate $\int_c \frac{e^{3z}}{(z+1)^4}dz$, where c is the circle $|z| = 2$.

(2012 : 15 Marks)

Solution:

From Cauchy Integral Formula, we have

$$f^{(n)}(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)dz}{(z-a)^{n+1}}$$

Put $a = -1$, $n = 3$

$$f^{(3)}(-1) = \frac{3!}{2\pi i}\int_C \frac{f(z)dz}{(z+1)^4} \quad ...(i)$$

Take $f(z) = e^{3z}$, then
$$f^{(n)}(z) = 3^n e^{3z}$$

$\therefore$
$$f^{(3)}(-1) = 3^3e^{-3} = \frac{27}{e^3}$$

$\therefore$ from (i),
$$\frac{27}{e^3} = \frac{3!}{2\pi i}\int_C \frac{e^{3z}dz}{(z+1)^4}$$

$\Rightarrow$
$$\int_C \frac{e^{3z}dz}{(z+1)^4} = \frac{9\pi i}{e^3}$$

1.7 Prove that if $be^{a+1} < 1$ where a and b are positive and real, then the function $z^n e^{-a} - be^z$ has n zeros in the unit circle.

(2013 : 10 Marks)

Solution:

Approach : We use Rouchi's theorem which says that if $f(z)$ and $g(z)$ are analytic functions in a simply connected domain. And $|f(z) + g(z)| < |f(z)|$ on C a rectifiable curve in the domain then $f(z)$ and $g(z)$ has same number of zeros in the domain.

Let
$$g(z) = z^n e^{-a} - be^z$$
and
$$f(z) = -z^n e^{-a}$$
$\therefore$
$$f(z) + g(z) = -be^z$$
$$|f(z) + g(z)| = |-be^z| = be \text{ on } |z| = 1$$
and
$$|f(z)| = |-z^n e^{-a}| = e^{-a}|z|^n = e^{-a} \text{ on } |z| = 1$$
and since
$$be^{a+1} < 1 \Rightarrow be < e^{-a} \text{ on } |z| = 1$$
$\therefore$
$$|f(z) + g(z)| < |f(z)|$$

Also $f(z)$ and $g(z)$ are analytic on and inside the unit circle. Hence conditions of Rouchi's theorem are satisfied and so $f(z)$ and $g(z)$ has same number of zeros inside unit circle, i.e.,
$$g(z) = z^n e^{-a} - be^z$$
has n zeros inside unit circle.

1.8 Prove that the function $f(z) = u + iv$, where
$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2}, \ z \neq 0; \ f(0) = 0$$
satisfies Cauchy-Riemann equations at the origin, but the derivative of f at $z = 0$ does not exist.

(2014 : 10 Marks)

Solution:

Here,
$$u = \frac{x^3 y^3}{x^2 + y^2}$$
$$v = \frac{x^3 + y^3}{x^2 + y^2} \qquad \text{where } z \neq 0$$

Here we see the both u and v are rational and finite for all values of $z \neq 0$. So u and v are continuous at all those points for which $z \neq 0$.

Hence $f(z)$ is continuous.

Where $z \neq 0$

At the origin $u = 0$, $v = 0$, [since $f(0) = 0$)

Hence u and v are both continuous at the origin consequently $f(z)$ is continuous at the origin.

$\therefore$ $f(z)$ is continuous every-where at the origin.

$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x,0) - u(0,0)}{x} = \lim_{x \to 0}\left(\frac{x}{x}\right) = 1$$
$$\frac{\partial u}{\partial x} = \lim_{y \to 0} \frac{u(0,y) - u(0,0)}{y} = \lim_{x \to \infty}\left(\frac{-y}{y}\right) = -1$$
$$\frac{\partial u}{\partial x} = \lim_{x \to 0} \frac{u(x,0) - v(0,0)}{x} = \lim_{x \to 0}\left(\frac{x}{x}\right) = 1$$
$$\frac{\partial u}{\partial y} = \lim_{y \to 0} \frac{v(0,y) - v(0,0)}{y} = \lim_{y \to 0}\left(\frac{y}{y}\right) = 1$$

$\therefore$ We see that $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Hence Cauchy-Riemann conditions are satisfied at $z = 0$.

Again,
$$f'(0) = \lim_{x \to 0} \frac{f(z) - f(0)}{z}$$
$$= \lim_{x \to 0} \left[\frac{x^3 - y^3 + i(x^3 + y^3)}{x^2 + y^2} \cdot \frac{1}{x + iy} \right]$$

Now, let $z \to 0$ along $y = x$, then

$$f'(0) = \lim_{x \to 0} \frac{x^3 - x^3 + i(x^3 + x^3)}{x^2 + x^2} \cdot \frac{1}{x + ix}$$
$$= \lim_{x \to 0} \frac{2i}{2(1+i)} = \frac{1}{2}(1-i)$$

Again let $z \to 0$ along $y = 0$, then

$$f'(0) = \lim_{x \to 0} \frac{x^3(1+i)}{x^3} = 1 + i$$

$\therefore$ we see that $f'(0)$ is not unique.

i.e. the values of $f'(0)$ are not the same as $z \to 0$ along different curves.

$\therefore$ $f'(z) =$ does not exist at the origin.

1.9 Show that the function $v(x, y) = \ln(x^2 + y^2) + x + y$ is harmonic. Find its conjugate harmonic function $u(x, y)$. Also, find the corresponding analytic function $f(z) = u + iv$, in terms of z.

(2015 : 10 Marks)

Solution:

Given,
$$v(x, y) = \ln(x^2 + y^2) + x + y$$
$$\frac{\partial v}{\partial x} = \frac{2x}{x^2 + y^2} + 1, \quad \frac{\partial^2 v}{\partial x^2} = \frac{2(y^2 - x^2)}{(x^2 + y^2)^2}$$
$$\frac{\partial v}{\partial y} = \frac{2y}{x^2 + y^2} + 1, \quad \frac{\partial^2 v}{\partial y^2} = \frac{2(x^2 - y^2)}{(x^2 + y^2)^2}$$

Now,
$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = \frac{2(y^2 - x^2)}{(x^2 + y^2)^2} + \frac{2(x^2 - y^2)}{(x^2 + y^2)^2} = 0$$

$\therefore$ v is a harmonic function.

Given, u is harmonic conjugate of v.

$\therefore$ By Cauchy-Reimann equations,

$$u_x = v_y \text{ and } u_y = -v_x$$
$$\frac{\partial u}{\partial x} = \frac{2y}{x^2 + y^2} + 1$$

$\Rightarrow$
$$u = 2\tan^{-1}\left(\frac{x}{y}\right) + x + f_1(y) \quad \text{...(i)}$$

Similarly,
$$\frac{\partial x}{\partial y} = \frac{-2x}{x^2 + y^2} - 1$$

$\Rightarrow$
$$u = -2\tan^{-1}\left(\frac{y}{x}\right) - y + f_2(x) \quad \text{...(ii)}$$

From (i) and (ii)

$$u = 2\tan^{-1}\left(\frac{x}{y}\right) + x - y + c \text{, where } c \text{ is a constant.}$$

$$v_x = \frac{2x}{x^2+y^2} + 1 = \psi_2$$

$$v_y = \frac{2y}{x^2+y^2} + 1 = \psi_1$$

$\therefore$ By Milne Thomson's method

$$f(z) = \int (\psi_1(z,0) + i(\psi_2(z,0))dz$$

$$= \int \left[1 + i\left(\frac{2z}{z^2}+1\right)\right]dz$$

$$f(z) = z + i(2\log z + z) + k, \text{ where } k \text{ is a constant.}$$

1.10 Is $v(x, y) = x^3 - 3xy^2 + 2y$ a harmonic function? Prove your claim. If yes, find its conjugate harmonic function $u(x, y)$ and hence obtain the analytic function whose real and imaginary parts are u and v respectively.

(2016 : 10 Marks)

Solution:

Let f be the function

$$f = u + iv \quad \text{(given)}$$

Given :

$$v = x^3 - 3xy^2 + 2y$$

$\therefore$

$$v_x = 3x^2 - 3y^2,\ v_{xx} = 6x \quad \text{...(i)}$$

$$v_y = -6xy + 2,\ v_{yy} = -6x \quad \text{...(ii)}$$

Now,

$$v_{xx} + v_{yy} = +6x - 6x = 0$$

$\Rightarrow$ v is a harmonic function.

u is harmonic conjugate of v.

By Cauchy-Riemann equations,

$$u_x = v_y \Rightarrow u_x = -6xy + 2 \quad \text{(from (ii))}$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} = -6xy + 2$$

$$\Rightarrow \quad \int dy = \int (-6xy+2)dx$$

$$\Rightarrow \quad u = -3x^2y + 2x + g(y) \quad \text{...(iii)}$$

where g is an arbitrary function of .....

Also,

$$u_y = -v_x \Rightarrow u_y = -(3x^2 - 3y^2) \quad \text{(from (i))}$$

$$\Rightarrow \quad \frac{\partial u}{\partial y} = -3x^2 + 3y^2$$

$$\Rightarrow \quad \int du = \int (-3x^2+3y^2)dy$$

$$\Rightarrow \quad u = -3x^2y + y^3 + n(x), \text{ where } n(x) \text{ is an arbitrary fucntion of } x \text{....(iv)}$$

From (iii) & (iv)

$$u = -3x^2y + 2x + y^3 + c, \text{ where } c \text{ is a constant}$$

So,

$$f = (-3x^2y + 2x + y^3 + c) + i(x^3 - 3xy^2 + 2y)$$

1.11 Prove that every power series represents an analytic function inside its circle of convergence.

(2016 : 20 Marks)

Solution:

Let the series be $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and R be its radius of convergence.

Then, $$\frac{1}{R} = \overline{\lim}|a_n|^{1/n}$$

Let $$\phi(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$$

We know that radius of convergence of $\phi(z)$ and $f(z)$ would be the same. Let z be any arbitrary point inside circle of convergence. Then, we have $|z| < R$. So, there exists a positive number r such that $|z| < r < R$. Let's say $|z| = \alpha$, then it can be said that $\alpha < r$. We can choose a positive number $|n| = \beta$ such that

$$\alpha + \beta < r$$

Since, the power series $f(z)$ is convergent in the area $|z| < R$, therefore $a_n r^n$ is bounded for $0 < r < R$, so there exists a positive number M such that $|a_n r^n| < M$.

Now, $$\left|\frac{f(z+n)-f(z)}{n} - \phi(z)\right| = \left|\sum_{n=0}^{\infty} a_n \left\{\frac{(z+n)^n - z^n}{n} - nz^{n-1}\right\}\right|$$

$$= \left|\sum_{n=0}^{\infty}\left[a_n\left\{\frac{n(n-1)}{2}z^{n-2}h + \dots\dots + h^{n-1}\right\}\right]\right|$$

$$\le \sum_{n=0}^{\infty}|a_n|\left\{\frac{n(n-1)}{2}|z|^{n-2}|h| + \dots\dots + |h|^{n-1}\right\}$$

$$\le \sum_{n=0}^{\infty}\frac{M}{r^n}\left\{\frac{n(n-1)}{2}\alpha^{n-2}\beta + \dots\dots + \beta^{n-1}\right\}$$

$$\le \sum_{n=0}^{\infty}\frac{M}{r^n}\cdot\frac{1}{\beta}\{(\alpha+\beta)^n - \alpha^n - n\alpha^{n-1}\beta\}$$

$$= \frac{M}{\beta}\sum_{n=0}^{\infty}\left\{\left(\frac{\alpha+\beta}{r}\right)^n - \left(\frac{\alpha}{r}\right)^n - \frac{n\beta}{\alpha}\left(\frac{\alpha}{r}\right)^n\right\} \quad \dots(i)$$

Now, $$\sum_{n=0}^{\infty}\left(\frac{\alpha+\beta}{r}\right)^n = 1 + \frac{\alpha+\beta}{r} + \left(\frac{\alpha+\beta}{r}\right)^2 + \dots\dots$$

$$= \left(1 - \frac{\alpha+\beta}{r}\right)^{-1} = \frac{r}{r-\alpha-\beta}$$

and $$\sum_{n=0}^{\infty}\left(\frac{\alpha}{r}\right)^n = 1 + \left(\frac{\alpha}{r}\right) + \left(\frac{\alpha}{r}\right)^2 + \dots\dots = \left(1 - \frac{\alpha}{r}\right)^{-1} = \frac{r}{r-\alpha}$$

Again, putting $S = \sum_{n=0}^{\infty} n\left(\frac{a}{r}\right)^n$, we have

$$S = \frac{\alpha}{r} + 2\left(\frac{\alpha}{r}\right)^2 + 3\left(\frac{\alpha}{r}\right)^3 + \dots\dots \quad \dots(ii)$$

$$\Rightarrow \quad S\left(\frac{\alpha}{r}\right) = \left(\frac{\alpha}{r}\right)^2 + 2\left(\frac{\alpha}{r}\right)^3 + \dots\dots \quad \dots(iii)$$

Subtracting (ii), (iii), we get

$$S\left(1-\frac{\alpha}{r}\right) = \frac{\alpha}{r}+\left(\frac{\alpha}{r}\right)^2+\left(\frac{\alpha}{r}\right)^3+\ldots..$$

$$\Rightarrow \quad \frac{S(r-\alpha)}{r} = \frac{\alpha}{r}\left(1-\frac{\alpha}{r}\right)^{-1} = \frac{\alpha}{r-\alpha}$$

$$\Rightarrow \quad S = \frac{r\alpha}{(r-\alpha)^2}$$

Substituting these values in (i), we get

$$\Rightarrow \quad \left|\frac{f(z+n)-f(z)}{n}-\phi(z)\right| \leq \frac{M}{\beta}\left[\frac{r}{r-\alpha-\beta}-\frac{r}{r-\alpha}-\frac{r\beta}{(r-\alpha)^2}\right]$$

$$= \frac{M}{\beta}\left[\frac{r(r-\alpha)^2-r(r-\alpha)(r-\alpha-\beta)-r\beta(r-\alpha-\beta)}{(r-\alpha)^2(r-\alpha-\beta)}\right]$$

$$= \frac{M}{\beta}\cdot\frac{r\beta^2}{(r-\alpha-\beta)(r-\alpha)^2}$$

$$= \frac{Mr\beta}{(r-\alpha-\beta)(r-\alpha)^2} \to 0 \text{ as } \beta \to 0$$

$$\therefore \quad \lim_{n\to 0}\frac{f(z+n)-f(z)}{n} = \phi(z), \text{ which implies } f'(z) = \phi(z).$$

Thus, $f(z)$ is a single valued differentiable function. Consequently, it is analytic within the circle $|z| = R$. So, it is analytic inside its circle of convergence.

1.12 Let $f = u + iv$ be an analytic function on the unit disc $D = \{\mathbb{Z} \in \mathbb{C} : |z| < 1\}$. Show that $\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0 = \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}$ at all points of D.

(2017 : 15 Marks)

Solution:

Every analytic function satisfies Cauchy-Seimann equations :

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \ldots(i)$$

Also, because u and v are the real and imaginary parts of an analytic function, so derivatives of u and v, of all orders, exist and are continuous functions of x and y.

So that we have
$$\frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x} \quad \ldots(ii)$$

Differentiating (i)

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y};\ \frac{\partial^2 u}{\partial y^2} = -\frac{\partial^2 v}{\partial y \partial x}$$

Adding these two,

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2} = 0 \quad \text{(by virtue of (ii))}$$

Similarly,
$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2} = 0$$

Hence, both u and v satisfy Laplace equation.

1.13 Prove that the function : $u(x, y) = (x-1)^3 - 3xy^2 + 3y^2$ is harmonic and find its harmonic conjugate and corresponding analytic function $f(z)$ in terms of z.

(2018 : 10 Marks)

Solution:

Given :

$$u(x, y) = (x-1)^3 - 3xy^2 + 3y^2$$

$$\frac{\partial y}{\partial x} = 3(x-1)^2 - 3y^2, \frac{\partial u}{\partial y} = -6xy + 6y$$

$$\frac{\partial^2 u}{\partial x^2} = 6(x-1), \frac{\partial^2 u}{\partial y^2} = -6x + 6$$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6(x-1) - 6x + 6 = 0$$

So, $u(x, y)$ is harmonic.

Let $v(x, y)$ be its harmonic conjugate.

∴ By Cauchy-Euler equations,

$$u_x = v_y \text{ and } v_y = -v_x$$

i.e.,
$$3(x-1)^2 - 3y^2 = \frac{\partial v}{\partial y} \Rightarrow v = 3(x-1)^2 y - y^3 + f(x) \quad \text{...(i)}$$

$$\Rightarrow \quad v = 3x^2 y - 6xy + f(y) \quad \text{...(ii)}$$

From (i) and (ii), we get

$$v = 3(x-1)^2 y - y^3 + c \quad \text{(where } c \text{ is a constant)}$$

So, harmonic conjugate,
$$v = 3(x-1)2y - y^3 + c$$

By Milne-Thomson's method,

$$f(z) = \int (u_x(z,0) - u_y(z,0))dz$$

$$= \int 3(z-1)^2 dz$$

$$= (z-1)^3 + c$$

1.14 Suppose $f(z)$ is analytic function on a domain D in ¢ and satisfies the equation $Im\, f(z) = (\text{Re } f(z))^2$, $z \in D$. Show that $f(z)$ is constant in D.

(2019 : 10 Marks)

Solution:

Given f(z) is an analytic function as domain D in ¢.

Also
$$Im\, f(z) = (\text{Re } f(z))^2; z \in D \quad \text{...(1)}$$

If
$$f(z) = 4 + iv$$

$$Im\, f(z) = v \text{ and Re } f(z) = 4 \quad \text{...(2)}$$

∵ $f(z)$ is analytic function.

$$\therefore \quad \left.\begin{aligned} \frac{\partial y}{\partial x} = \frac{\partial v}{\partial y} \\ \frac{\partial y}{\partial y} = \frac{\partial v}{\partial x} \end{aligned}\right\} \quad \text{...(3)}$$

From (1) and (2),
$$v = u^2$$

$$\therefore \quad \left.\begin{aligned} \frac{\partial v}{\partial x} = 24\frac{\partial y}{\partial x} \\ \frac{\partial v}{\partial y} = 24\frac{\partial y}{\partial y} \end{aligned}\right\} \quad \text{...(4)}$$

From (3) and (4)

$$\frac{\partial v}{\partial x} = 0 \text{ and } \frac{\partial v}{\partial y} = 0$$

$\therefore$

$$\frac{\partial f}{\partial z} = \frac{\partial y}{\partial x} + \frac{\partial v}{\partial x} \Rightarrow \frac{\partial f}{\partial z} = 0$$

By integrating $\Rightarrow$

$$f(z) = C \text{ (Constant)}$$

Hence, the result.

2. Taylor's Series

2.1 If the function $f(z)$ is analytic and one valued in $|z - a| < R$, prove that for $0 < r < R$, $f'(a) = \frac{1}{\pi r}\int_0^{2\pi} P(\theta)e^{-i\theta}d\theta$, where $P(\theta)$ is the real part of $f(a + re^{i\theta})$.

(2011 : 15 Marks)

Solution:

The function $f(z)$ is given to be analytic in $|z - a| < R$ and $r < R$, therefore $f(z)$ is also analytic inside the circle C defined by $|z - a| = r$.

$$\therefore \quad f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)}{(z-a)^2}dz \qquad \text{...(i)}$$

Expanding $f(z)$ in a Taylor's series about $z = a$, we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

$$\Rightarrow \quad f(z) = f(a+re^{i\theta}) = \sum_{n=0}^{\infty} a_n r^n e^{in\theta} \qquad (\because z - a = re^{i\theta})$$

$$\Rightarrow \quad \overline{f(z)} = \sum_{n=0}^{\infty} \bar{a}_n r^n e^{-in\theta}$$

$$\therefore \quad \frac{1}{2\pi i}\int_C \frac{\overline{f(z)}}{(z-a)^2}dz = \frac{1}{2\pi i}\int_0^{2\pi} \frac{\Sigma \bar{a}_n r^n e^{-in\theta}}{r^2 e^{i2\theta}} \cdot rie^{i\theta}d\theta$$

$$= \frac{1}{2\pi}\sum \bar{a}_n r^{n-1}\int_0^{2\pi} e^{-i(n+1)\theta}d\theta = 0 \qquad \text{...(ii)}$$

From (i) and (ii), we have

$$f'(a) = \frac{1}{2\pi i}\int_C \frac{f(z)f + \overline{f(z)}}{(z-a)^2}dz$$

$$= \frac{1}{2\pi i}\int_C \frac{2 \text{ real part of } f(z)}{(z-a)^2}dz$$

$$= \frac{1}{\pi i}\int_0^{2\pi} \frac{\text{real part of } f(a+re^{i\theta})ire^{i\theta}d\theta}{r^2 e^{i2\theta}} \qquad [\because z = a + re^{i\theta}]$$

$$= \frac{1}{\pi r}\int_0^{2\pi} P(\theta)e^{-i\theta}d\theta \text{, where } P(\theta) \text{ is the real part of } f(a + re^{i\theta}).$$

2.2 **Describe the maximal ideals in the ring of Gaussian integers $Z[i] = \{a + bi | a, b \in Z\}$.**

(2012 : 20 Marks)

Solution:

Let us define

$$\delta(a+ib) = |a+ib|^2 = a^2 + b^2 \text{ for all } a + ib \in z[i]$$

$$\delta(u) = \delta(a+ib) = a^2 + b^2 > 0 \text{ for all } u \neq 0 \text{ in } z[i]$$

Again for any $u, v \in z[i]$

$$\delta(uv) = |u|^2|v|^2 \geq |u|^2 = \delta(u)$$

Let $u, v \in z[i]$, $v \neq 0$

Then,

$$u = a + ib,\ v = c + id,\ a, b, c, b \in z \text{ and } (c, d) \neq (0, 0)$$

$\therefore$

$$\frac{u}{v} = \frac{a+ib}{c+id} = \frac{(a+ib)}{(c+id)} \times \frac{c+id}{c+id}$$

$$= \frac{(a+ib)(c+id)}{c^2+d^2} = \alpha + i\beta \text{ (say)}$$

where α and β are rational numbers. Then we can find integers m and n such that $|m - \alpha| \leq \frac{1}{2}$ and $|n - \beta| \leq \frac{1}{2}$.

$\therefore$

$$u = (\alpha + i\beta)v = (m + in)v + [(\alpha - m) + i(\beta - n)]v \quad \text{...(i)}$$

Now

$$[(\alpha - m) + i(\beta - n)]v = [(\alpha + i\beta) - (m + in)]v$$

$$= (\alpha + i\beta)v - (m + in)v$$

$$= u - (m + in)v \in z[i]$$

as $z[i]$ is a ring and $u, v, m + in \in z[i]$

Let

$$r = [(\alpha - m) + i(\beta - n)]v$$

Then

$$\delta(r) = |r|^2 = [(\alpha - m)^2 + (\beta - n)^2]\,|v|^2 \quad \text{...(ii)}$$

$$\leq \left(\frac{1}{4} + \frac{1}{4}\right)|v|^2$$

Taking $q = m + in$, we have,

$$u = vq + r \quad \text{(using (i) and (ii))}$$

where either $r = 0$ or $\delta(r) < \delta(v)$

Hence, $z[i]$ is a Euclidean domain.

As every Euclidean domain is a principal ideal domain (PID).

$\therefore$ $z[i]$ is PID.

But in a PID, every non-zero not unit element a is prime element iff $<a>$ is a maximal ideal.

$\therefore$ Maximal ideals in $z[i]$ are the ideals generated by prime elements.

2.3 **For a function $f: \mathbb{C} \to \mathbb{C}$ and $n \geq 1$, let fn denote the n^{th} derivative of f and $f(0) = f$. Let f be an entire function such that for some $n \geq 1$, $f^n\left(\frac{1}{k}\right) = 0$ for all $K = 1, 2, 3, \ldots$. Show that f is a polynomial.**

(2017 : 15 Marks)

Solution:

Since $f(z)$ is entire, $\therefore$ $f(z)$ is analytic. Hence, $f(z)$ can be expressed as Taylor's series around $z = 0$ as

$$f(z) = \sum_{m=1}^{\infty} a_m \frac{1}{z^m} + \sum_{m=0}^{\infty} a_m z^m$$

$$f^n(z) = \sum_{m=1}^{\infty}(-1)^n(m)(m+1).....(m+(n-1))a_{-m}\frac{1}{z^{m+n}} +$$

$$\sum_{m=n}^{\infty}(m)(m-1).....(m-(n-1))a_m z^{m-n}$$

Now,

$$f^n\left(\frac{1}{K}\right) = \sum_{m=1}^{\infty}(-1)^m(m)(m+1).....(m+(n-1))a_{-m}\cdot K^{m+n} +$$

$$\sum_{m=n}^{\infty}m(m-1).....(m-n+1)a_m\left(\frac{1}{K}\right)^{m-n}$$

As $f^n\left(\frac{1}{K}\right) \to 0$ for all K = 1, 2, 3, ...

Let us take $K \to \infty$

We get $a_{-n} \to 0$ and $a_n = 0$

Similarly, if we take $n + 1$, $a_{n+1} = 0$

$\therefore$ $$a_n = a_{n+1} = a_{n+2} = ... = 0$$

$\therefore$ $$f(z) = a_0 + a_1 z + a_2 z^2 + ... + a_{n-1}z^{n-1}$$

which is a polynomial function.

2.4 Show that an isolated singular point z_0 of a function $f(z)$ is a pole of order m if and only if $f(z)$ can be written in the form $f(z) = \dfrac{\phi(z)}{(z-z_0)^m}$ where $\phi(z)$ is analytic and non-zero at z_0.

Moreover
$$\operatorname*{Res}_{z=z_0} f(z) = \frac{\phi^{(m-1)}(z_0)}{(m-1)!} \text{if } m \ge 1$$

(2019 : 15 Marks)

Solution:

Since $f(z)$ has a pole of order, m, then by definition, for $0 < |z - z_0| < R$.

$$f(z) = \sum_{n=0}^{\infty}a_n(z-z_0)^n + \frac{b_2}{z-z_0} + \frac{b_2}{(z-z_0)^2} + \frac{b_m}{(z-z_0)^m}, b_m + 0$$

$$\Rightarrow f(z) = \frac{1}{(z-z_0)^m}\left[\sum_{n=0}^{\infty}a_n(z-z_0)^{m+n} + b_1(z-z_0)^{m-1} + b_2(z-z_0)^{m-2} + b_2(z-z_0)^{m-2} + b_m\right]$$

$$\Rightarrow \qquad f(z) = \frac{\phi(z)}{(z-z_0)^m}$$

Clearly, $\phi(z_0) = b_m \neq 0$ and is analytic at z_0, as it has Taylor series expansion about z_0.

Conversely, suppose $f(z)$ can be written in the form :

$$f(z) = \frac{\phi(z)}{(z-z_0)^m}, \text{ then}$$

$$\phi(z) = \phi(z_0) + \phi(z_0)(z-z_0) + \frac{\phi^m(z_0)}{2!}(z-z_0)^2 + ... +$$

$$\frac{\phi^{(m-1)}(z_0)}{(m-1)!}(z-z_0)^{m-1} + ...$$

Since, $$\phi(z_0) \neq 0$$

$f(z)$ has a pole of order m.

With residue, $$b_1 = \frac{\phi^{(m-1)}(z_0)}{(m-1)!}$$

In case of simple pole, i.e., $m-1$,

$$\text{Res } z = z_0, f(z_0) = \phi(z_0).$$ Hence, proved.

3. Singularity

3.1 Determine all entire functions f(z) such taht 0 is a removal singularity of $f\left(\frac{1}{z}\right)$.

(2017 : 10 Marks)

Solution:

For any complex valued function, $f(z)$

$$f(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^{-n}$$

Since, $f(z)$ is entire, so all b_n are zero.

$$\therefore \quad f(z) = \sum_{n=0}^{\infty} a_n z^n$$

$$f\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} a_n \left(\frac{1}{z}\right)^n$$

$$= a_0 + \frac{a_1}{z} + \frac{a_2}{z_2} + \ldots\ldots$$

$f\left(\frac{1}{z}\right)$ is a removable singularity.

4. Laurant's Series

4.1 Find the Laurent series of the function $f(z) = \exp\left[\frac{\lambda}{2}\left(z - \frac{1}{z}\right)\right]$ as $\sum_{n=-\infty}^{\infty} c_n z^n$ for $0 < |z| < \infty$

where $$c_n = \frac{1}{\pi}\int_0^{\pi} \cos(n\phi - \lambda \sin\phi)d\phi \; ; \; n = 0, \pm 1, \pm 2, \ldots\ldots$$

with λ a given complex number and taking the unit circle c given by $z = e^{i\phi}(-\pi \le \phi \le \pi)$ as contour in this region.

(2010 : 15 Marks)

Solution:

O is the only point of singularity here. So, we can expand $\exp\left(\frac{\lambda}{2}\left(z - \frac{1}{z}\right)\right)$ as Laurent series valid for every value of $z \neq 0$, i.e., for $|z| > 0$.

Thus, we have

$$|z| \neq 0$$

$$\exp\left(\frac{\lambda}{2}\left(z - \frac{1}{z}\right)\right) = \sum_{n=-\infty}^{\infty} c_n z^n$$

when

$$c_n = \frac{1}{2\pi i}\int_c \frac{f(z)}{z^{n+1}}dz$$

where c is any circle with its centre at the origin. Taking c as the circle with radius unity, we have

$$c_n = \frac{1}{2\pi i}\int_c \frac{e^{\lambda/2\left(z-\frac{1}{z}\right)}}{z^{n+1}}dz$$

Putting $z = e^{i\phi}$, we obtain

$$c_n = \frac{1}{2\pi i}\int_0^{2\pi} \frac{e^{\frac{\lambda}{2}\left(e^{i\phi}-e^{-i\phi}\right)}}{e^{i(n+1)\phi}}\cdot ie^{i\phi}d\phi$$

$\Rightarrow$
$$c_n = \frac{1}{2\pi}\int_0^{2\pi} e^{i\lambda\sin\phi}\cdot e^{-in\phi}d\phi$$

$\Rightarrow$
$$c_n = \frac{1}{2\pi}\int_0^{2\pi}\cos(\lambda\sin\phi - n\phi)d\phi + \frac{i}{2\pi}\int_0^{2\pi}\sin(\lambda\sin\phi - n\phi)d\phi \qquad ...(1)$$

Taking $\phi = 2\pi - \theta$, we have

$$\int_0^{2\pi}\sin(\lambda\sin\phi - n\phi)d\phi - \left(-\int_{2\pi}^{0}\sin[-\lambda\sin\theta - 2n\pi + n\theta]d\theta\right)$$

$$= \int_0^{2\pi}\sin(\lambda\sin\phi - n\phi)d\phi$$

So that
$$\int_0^{2\pi}\sin(\lambda\sin\phi - n\phi)d\phi = 0 \qquad ...(2)$$

$\therefore$
$$c_n = \frac{1}{2\pi}\int_0^{2\pi}\cos(\lambda\sin\phi - n\phi)d\phi + 0 \quad \text{(Putting value from (2) in (1))}$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\cos(n\phi - \lambda\sin\phi)d\phi$$

$$= \frac{2}{2\pi}\int_0^{\pi}\cos(n\phi - \lambda\sin\phi)d\phi$$

$\Rightarrow$
$$c_n = \frac{1}{\pi}\int_0^{\pi}\cos(n\phi - \lambda\sin\phi)d\phi \qquad ...(3)$$

So, given expression can be expressed as $\sum_{-\infty}^{\infty} c_n z^n$ where c_n is given by above equation (3).

4.2 Find the Laurent series for the function

$$f(z) = \frac{1}{1-z^2} \text{ with centre } z = 1.$$

(2011 : 15 Marks)

Solution:

We have
$$f(z) = \frac{1}{1-z^2} = \frac{1}{(1-z)(1+z)}$$

$$= \frac{1}{z}\left(\frac{1}{1-z}+\frac{1}{1+z}\right) \quad ...(i)$$

First we will find Laurent expansion for $\phi(z) = \dfrac{1}{1+z}$ about $z = 1$.

We write, $$\phi(z) = \sum_{n=0}^{\infty} a_n(z-1)^n \quad ...(ii)$$

where $$a_n = \frac{\phi^n(1)}{n!}$$

But $$\phi^n(z) = (-1)(-2) ..... (-n)(1+2)^{-(n+1)}$$
$$= (-1)^n n!(1+z)^{-(n+1)}$$

$\therefore$ $$\frac{\phi^n(1)}{n!} = \frac{(-1)^n}{2^{n+1}}$$

$\therefore$ $$a_n = \frac{\phi^n(1)}{n!} = \frac{(-1)^n}{2^{n+1}}$$

From (ii), $$\phi(z) = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}(z-1)^n$$
$$= \sum_{n=0}^{\infty}\frac{(1-z)^n}{2^{n+1}}$$

$\therefore$ From (i), $$f(z) = \frac{1}{2}\left(\frac{1}{1-z}+\sum_{n=0}^{\infty}\frac{(1-z)^n}{2^{n+1}}\right)$$

This is the required expansion.

4.3 Expand the function $f(z) = \dfrac{1}{(z+1)(z+3)}$ in Laurent series valid for :

(i) $1 < |z| < 3$; (ii) $|z| > 3$; (iii) $0 < |z+1| < 2$; (iv) $|z| < 1$

(2012 : 15 Marks)

Solution:

Given : $$f(z) = \frac{1}{(z+1)(z+3)}$$
$$= \frac{1}{2(z+1)} - \frac{1}{2(z+3)} \quad \text{(using partial fractions)}$$

(i) $$1 < |z| < 3$$

$\Rightarrow$ $$\frac{1}{|z|} < 1 \text{ and } \frac{|z|}{3} < 1$$

$\therefore$ $$f(z) = \frac{1}{2(z+1)} - \frac{1}{2(z+3)}$$
$$= \frac{1}{2z\left(1+\frac{1}{z}\right)} - \frac{1}{6\left(1+\frac{z}{3}\right)}$$
$$= \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1} - \frac{1}{6}\left(1+\frac{z}{3}\right)^{-1}$$

$$= \frac{1}{2z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}.....\right)-\frac{1}{6}\left(1-\frac{z}{3}+\frac{z^2}{9}-\frac{z^3}{27}.....\right)$$

$$= .....+\frac{1}{2z^3}-\frac{1}{2z^2}+\frac{1}{2z}+\frac{1}{6}-\frac{Z}{18}+\frac{z^2}{54}-\frac{z^3}{162}.....$$

(ii) $|z| > 3$

$\Rightarrow$ $\frac{3}{|z|} < 1$

$\therefore$
$$f(z) = \frac{1}{2z}\left(1+\frac{1}{z}\right)^{-1}-\frac{1}{2z}\left(1+\frac{3}{z}\right)^{-1}$$

$$= \frac{1}{2z}\left(1-\frac{1}{z}+\frac{1}{z^2}-\frac{1}{z^3}+.....\right)-\frac{1}{2z}\left(1-\frac{3}{z}+\frac{9}{z^2}-\frac{27}{z^3}.....\right)$$

$$= \frac{1}{z^2}-\frac{4}{z^3}+\frac{13}{z^4}-\frac{40}{25}+.....$$

(iii) $0 < |z+1| < 2$

Put $z + 1 = 4$

Then $0 < |z+1| < 2 \;Þ\; 0 < |4| < 2$

$\therefore$
$$f(z) = \frac{1}{(z+1)(z+3)} = \frac{1}{4(4+2)}$$

$$= \frac{1}{24}\left(1+\frac{4}{2}\right)^{-1}$$

$$= \frac{1}{24}\left(1-\frac{4}{2}+\frac{4^2}{4}-\frac{4^3}{8}.....\right)$$

$$= \frac{1}{24}-\frac{1}{4}+\frac{4}{8}-\frac{4^2}{16}+.....$$

$$= \frac{1}{2(z+1)}-\frac{1}{4}+\frac{z+1}{8}-\frac{(z+1)^2}{16}+.....$$

(iv) $|z| < 1$

Then
$$f(z) = \frac{1}{2(z+1)}-\frac{1}{6\left(1+\frac{z}{3}\right)}$$

$$= \frac{1}{2}(1-z+z^2-z^3.....)-\frac{1}{6}\left(1-\frac{z}{3}+\frac{z^2}{9}-\frac{z^3}{27}.....\right)$$

$$= \frac{1}{3}-\frac{4}{9}z+\frac{13}{17}z^2.....$$

4.4 Expand in Laurent series the function $f(z) = \dfrac{1}{z^2(z-1)}$ about $z = 0$ and $z = 1$.

(2014 : 10 Marks)

Solution:

Given that
$$f(z) = \frac{1}{z^2(z-1)}$$

About $z = 0$, the Laurent Series is given by

$$\frac{1}{z^2(z-1)} = \frac{1}{-z^2(1-z)} = -\frac{1}{z^2}(1-z)^{-1}$$

$$= -\frac{1}{z^2}(1+z+z^2+z+.....)$$

$$= -\left(\frac{1}{z^2}+\frac{1}{z}+1+z+z^2+.....\right)$$

Let $z-1 = u \Rightarrow z = u+1$ and

$$\frac{1}{z^2(z-1)} = \frac{1}{(u+1)^2u} = \frac{1}{u}(1+u)^{-2}$$

$$= \frac{1}{u}\left[1-(2)(u)+\frac{(-2)(-3)}{2!}(u)^2+\frac{(-2)(-3)(-4)}{3!}(u)^3+.....\right]$$

$$= \frac{1}{u}[1-2u+3u^2-4u^3+.....]$$

4.5 Find all possible Taylor's and Laurent's series expansions of the function $f(z) = \dfrac{2z-3}{z^2-3z+2}$ about the point $z = 0$.

(2015 : 20 Marks)

Solution:

Given :
$$f(z) = \frac{2z-3}{(z^2-3z+2)} = \frac{2z-3}{(z-2)(z-1)}$$

Taylor Series :

$$f(z) = \frac{1}{z-1}+\frac{1}{z-2}$$

$$= \frac{1}{0-1}+(-z)\left(\frac{-1}{z-1}\right)^z_{z=0} + \frac{z^2}{2!}\times\frac{2}{(z-1)^3}\bigg|_{z=0} + \frac{z^3}{3!}\times\frac{-6}{(z-1)^4}\bigg|_{z=1} +$$

$$..... + \frac{1}{0-2} + z\times\frac{-1}{(z-2)^2}\bigg|_{z=0} + \frac{2z^2}{2!z}\times\frac{1}{(z-2)^3}\bigg|_{z=0}$$

$$+ \frac{z^3}{3!}\times\frac{-6}{(z-2)^4}\bigg|_{z=0}$$

$$= (-1-z-z^2-z^3-.....)+\left(\frac{-1}{2}-\frac{z}{4}-\frac{z^2}{8}-\frac{z^3}{16}-.....\right)$$

Laurent Series :

$$f(z) = \frac{1}{z-1}+\frac{1}{z-2}$$

$$= \frac{-1}{1-z}-\frac{1}{z-2}$$

$$= -1(1-z)^{-1}-\frac{1}{2}\left(1-\frac{2}{z}\right)^{-1}$$

$$= -1(1+z+z^2+z^3+.....)-\frac{1}{2}\left(1+\frac{z}{2}+\left(\frac{z}{2}\right)^2+\left(\frac{z}{3}\right)^3+.....\right)$$

If $|z| < 1$ and $\left|\frac{z}{2}\right| < 1 \Rightarrow |z| < 2$

$$\therefore \quad f(z) = -(1+z+z^2+.....)-\frac{1}{2}\left(1+\frac{z}{2}+\frac{z^2}{4}+.....\right) \text{ around point } O.$$

4.6 Find the Laurent's series which represent function $\frac{1}{(1+z^2)(z+2)}$ which (i) $|z| < 1$ (ii) $1 < |z| < 2$ (iii) $|z| > 2$

(2018 : 15 Marks)

Solution:

Let
$$f(z) = \frac{1}{(1+z^2)(z+2)} = \frac{1}{5}\left(\frac{1}{z+2}-\frac{z-2}{z^2+1}\right)$$

(i) $|z| < 1$

$$f(z) = \frac{1}{5}\left[\frac{1}{2}\left(1+\frac{z}{2}\right)^{-1}-(z-2)(1+z^2)^{-1}\right]$$

as $\left|\frac{z}{2}\right| < 1 \Rightarrow |z| < 2$ and $|z|^2 < 1 \Rightarrow |1z| < 1$

Both conditions are satisfied in this region.

So,
$$f(z) = \frac{1}{10}\left[1-\frac{z}{2}+\frac{z^2}{2^2}-.....\right]-\frac{z-2}{5}[1-z^2+z^4-z^6+.....]$$

(ii) $1 < |z| < 2$, we have

$$f(z) = \frac{1}{5}\left[\frac{1}{2}\left(1+\frac{z}{2}\right)^{-1}-\frac{z-2}{z^2}\left(1+\frac{1}{z^2}\right)^{-1}\right]$$

Here, $\left|\frac{z}{2}\right| < 1 \Rightarrow |z| < 2$ and $\left|\frac{1}{z^2}\right| < 1$ or $|z| > 1$

Both conditions are satisfied in this region.

$$\therefore \quad f(z) = \frac{1}{10}\left(1-\frac{z}{2}+\frac{z^2}{2^2}+\frac{z^3}{2^3}+.....\right)-\frac{z-2}{5z^2}\left(1-\frac{1}{z^2}+\frac{1}{z^4}+.....\right)$$

(iii) $|z| > 2$

$$f(z) = \frac{1}{5}\left[\frac{1}{2}\left(1+\frac{2}{z}\right)^{-1}-\frac{1}{z^2}(z-2)\left(1+\frac{1}{z^2}\right)^{-1}\right]$$

$$\Rightarrow \quad f(z) = \frac{1}{5z}\left(1-\frac{2}{z}+\frac{2^2}{z^2}-.....\right)-\frac{z-2}{5z^2}\left(1-\frac{1}{z^2}+\frac{1}{z^4}-.....\right) \quad (\text{for } |z| > 2)$$

4.7 Obtain the first three terms of the Laurent series expansion of the function $f(z) = \frac{1}{(e^z-1)}$ about the point $z = 0$ valid in the region $0 < |z| < 2\pi$.

(2019 : 10 Marks)

Solution:

Given, $$f(z) = \frac{1}{(e^z - 1)}$$

about the point $z = 0$ valid in region $0 < |z| < 2\pi >$, we have that

$$\frac{1}{(e^z-1)} = \frac{1}{z\left\{1+\left(\frac{z}{2!}\right)+\left(\frac{z^2}{3!}\right)+.....\right\}}$$, from where we get

$$p(z) = \left(\frac{z}{2!}+\frac{z^2}{3!}+\frac{z^3}{4!}+.....\right)$$

If $0 < |z| < 1$, so $$p(z) = \left(\frac{z}{2!}+\frac{z^2}{3!}+\frac{z^3}{4!}+.....\right)\angle 1$$

$\because$ at $z = 1 \Rightarrow$ $$p(z) = \left[\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+.....\right]\angle 1$$

Then, $$\frac{1}{e^z-1} = \frac{1}{z}-\frac{p(z)}{z}+\frac{p(z)^2}{z}+\frac{p(z)^3}{z}+.....$$

$$\frac{1}{e^z-1} = \frac{1}{z}-\frac{1}{2}-\frac{z}{3!}+\frac{z}{(2!)^2}-\frac{z^2}{4!}+\frac{2z^2}{(2!3!)}-\frac{7^2}{(2!)^3}$$

$$\frac{1}{e^z-1} = \frac{1}{z}-\frac{1}{2}+\frac{z}{12}+z^2\left[\frac{1}{6}-\frac{1}{24}-\frac{1}{8}\right]+0(z)^3$$

$$\frac{1}{e^z-1} = \frac{1}{z}-\frac{1}{2}+\frac{z}{12}+z^2(0)+0(z^3)$$

$$\frac{1}{e^z-1} = \frac{1}{z}-\frac{1}{2}+\frac{z}{12}+0(z^3) \qquad ...(1)$$

Also, first z terms asked coefficient $\left(\frac{1}{z}\right)$ is Res$[f(z)] \Rightarrow \lim_{z\to 0}\frac{z}{e^z-1} = 1$ So, verifies.

If $1 < |z| < 2\pi$

$\because$ asked about $z = 0$, so (1) holds as Laurent series is unique.

Hence, $$\frac{1}{e^z-1} = \frac{1}{z}-\frac{1}{2}+\frac{z}{12}+0(z^3)$$

5. Cauchy's Residue Theorem

5.1 Using Cauchy's residue theorem, evaluate the integral

$$I = \int_0^{\pi} \sin^4\theta \, d\theta$$

(2013 : 15 Marks)

Solution:

Using trignometry

$$\sin^4\theta = \left[\frac{1}{2}(1-\cos 2\theta)\right]^2 = \frac{1}{4}(1+\cos^2 2\theta - 2\cos 2\theta)$$

$$= \frac{1}{8}[2+(1+\cos 4\theta)-2\cos 2\theta]$$

$$= \frac{1}{8}[\cos 4\theta - 2\cos 2\theta + 3]$$

$$I = \int_0^{\pi} \sin^4\theta\, d\theta = \frac{1}{2}\int_0^{2\pi} \sin^4\theta\, d\theta$$

$$= \frac{1}{2}\int_0^{2\pi} \frac{1}{8}[\cos 4\theta - 2\cos 2\theta + 3]d\theta$$

$$= \frac{1}{16}\int_0^{2\pi} [\cos 4\theta - 2\cos 2\theta + 3]d\theta$$

$$= \text{Real part of } \frac{1}{16}\int_0^{2\pi} (e^{i4\theta} - 2e^{i2\theta} + 3)d\theta$$

Let $z = e^{i\theta}$, $\quad dz = ie^{i\theta}d\theta \;\Rightarrow\; d\theta = \frac{dz}{iz}$

$$= \text{R.P. of } \frac{1}{16}\oint_C \frac{(z^4 - 2z^2 + 3)dz}{iz}$$

where C is the unit circle.

$$\oint_C \frac{z^4 - 2z^2 + 3}{z}dz = 2\pi i \text{ Sum of Residue of } f(z)$$

where $\quad f(z) = \frac{z^4 - 2z^2 + 3}{z}$

Now $f(z)$ has one pole at $z = 0$ inside unit circle.

$$\text{Residue at } z = 0 = \lim_{z\to 0} zf(z)$$

$$= 3$$

$$\therefore \quad \oint f(z) = 6\pi i$$

$$\therefore \quad \frac{1}{16i}\oint_C f(z) = \frac{3}{8}\pi$$

$$\therefore \quad \int_0^{\pi} \sin^4\theta\, d\theta = \text{R.p } \frac{3}{8}\pi = \frac{3\pi}{8}$$

5.2 Evaluate the integral $\int_0^{\pi} \frac{d\theta}{\left(1+\frac{1}{2}\cos\theta\right)^2}$ using residues.

(2014 : 20 Marks)

Solution:

Let $\quad I = \int_0^{\pi} \frac{d\theta}{\left(1+\frac{1}{2}\cos\theta\right)^2}$

$$= \frac{1}{2}\int_0^{2\pi} \frac{4d\theta}{(2+\cos\theta)^2}$$

$$= \int_0^{2\pi} \frac{2d\theta}{(2+\cos\theta)^2}$$

Let the contour 'C' be the unit circle $|z| = 1$ with centre at the origin.

Let $z = e^{i\theta}$ then $\cos\theta = \frac{1}{2}\left(z+\frac{1}{z}\right)$

$\Rightarrow$ $dz = ie^{i\theta}d\theta$

$\Rightarrow$ $d\theta = \frac{dz}{iz}$

$\therefore$ $$I = \int_0^{2\pi} \frac{2d\theta}{(2+\cos\theta)^2} = \int_0^{2\pi} \frac{2dz}{iz\left(2+\frac{z^2+1}{2z}\right)^2}$$

$$= \frac{1}{i}\int_0^{2\pi} \frac{8z}{(z^2+4z+1)^2}dz$$

$$= \frac{8}{i}\int_C \frac{z}{(z^2+4z+1)^2}dz$$

$$= \frac{8}{i}\int_C f(z)dz$$

where $$f(z) = \frac{z}{(z^2+4z+1)^2} \quad \text{...(i)}$$

Now the poles of $f(z)$ are given by

$$(z^2 + 4z + 1)^2 = 0 \Rightarrow z = \frac{-4\pm\sqrt{16-4}}{2} = \frac{-4\pm\sqrt{12}}{2} = -2\pm\sqrt{3} \text{ (twice)}$$

$\therefore$ $f(z)$ has poles of order at $z = -2\pm\sqrt{3}$ (twice)

Let $\alpha = -2+\sqrt{3}$, $\beta = -2-\sqrt{3}$

Clearly $|\beta| > 1$

Since $|\alpha\beta| = 1 \Rightarrow |\alpha| < 1$

Hence, the θdy pole inside C is $z = \alpha$ of order 2. $(\because |\beta| > 1)$

$\therefore$ $$\int_C f(z)dz = \int \frac{z\,dz}{(z^2+4z+1)^2}$$

$$= 2\pi i \text{ (residue at } z = \alpha)$$

Now the residue at $z = \alpha$ is

$$\lim_{z\to\alpha} \frac{d}{dz}(z-\alpha)^2 \frac{z}{(z^2+4z+1)^2} = \lim_{z\to\alpha} \frac{d}{dz}\frac{z}{(z-\beta)^2}$$

$$= \lim_{z\to\alpha} \frac{-(\beta+z)}{(z-\beta)^3}$$

$$= \frac{-(\alpha+\beta)}{(\alpha-\beta)^3}$$

$$= \frac{-(-4)}{(2\sqrt{3})^3}$$

$$= \frac{(4)}{8(3\sqrt{3})} = \frac{1}{6\sqrt{3}}$$

$$\therefore \qquad \int_C f(z)dz = 2\pi i\left(\frac{1}{6\sqrt{3}}\right) = \frac{2\pi i}{6\sqrt{3}} = \frac{\sqrt{3}i}{3\sqrt{3}}$$

$\therefore$ from (i)
$$\int_0^{2\pi} \frac{2d\theta}{(2+\cos\theta)^2} = \frac{8}{i}\left(\frac{\pi i}{3\sqrt{3}}\right) = \frac{8\pi}{3\sqrt{3}}$$

$$\therefore \qquad I = \int_0^{\pi} \frac{d\theta}{\left(1+\dfrac{\cos\theta}{2}\right)^2} = \frac{8\pi}{3\sqrt{3}}$$

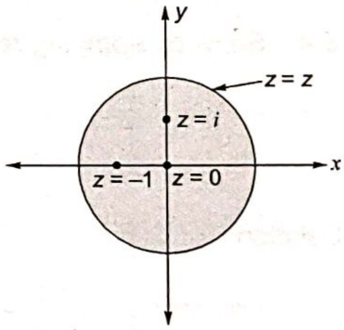
5.3 State Cauchy's Residue theorem. Using it, evaluate the integral $\int_C \frac{e^2+1}{z(z+1)(z-i)^2}dz$. $\{C : |z| = 2\}$

(2015 : 15 Marks)

Solution:

Cauchy's Residue theorem states that in a given region, integral of a function along the closed curve is equal to $2\pi i$ times sum of its residues.

Given,
$$I = \int_C \frac{e^2+1}{z(z+1)(z-i)^2}dz, \ C: |z| = 2$$
$$= \int_C f(z)dz$$



In the given region, $|z| = 2$, $f(z)$ is analytic everywhere except at $z = 0, -1, i$.

$z = 0$ is a pole of order 1.

$z = -1$ is a pole of order 1.

$z = i$ is a pole of order 2.

Now, **Residue at $z = 0$:**

$$\lim_{z\to 0} \frac{z(e^2+1)}{z(z+1)(z-i)^2} = \frac{1+1}{1(-i)^2} = \frac{2}{(-1)}$$
$$= -2$$

Residue at z = –1 :

$$\lim_{z\to -1} \frac{(z+1)(e^2+1)}{z(z+1)(z-1)^2} = \frac{1+e^{-1}}{(-1)(-1-i)^2} = \frac{-(1+e^{-1})}{1-1+2i}$$
$$= \frac{(1+e^{-1})}{2}i$$

Residue at $z = i$:

$$\lim_{z\to i} \frac{d}{dz}(z-1)^2 \frac{e^2+1}{z(z+1)(z-1)^2} = \lim_{z\to i} \frac{d}{dz} \cdot \frac{e^2+1}{z(z+1)}$$
$$= \lim_{z\to i} \left\{ \frac{-(e^z+1)}{z^2(z+1)} - \frac{(e^z+1)}{z(z+1)^2} + \frac{e^z}{z(z+1)} \right\}$$

$$= \frac{+(1+e^{i})}{(+1)(1+i)} - \frac{(1+e^{i})}{i(1+i)^2} + \frac{e^{i}}{i(i+1)}$$

$$= \frac{1+e^{i}}{1+i} - \frac{1+e^{i}}{2} + \frac{e^{i}}{i(i+1)}$$

$$= \frac{1-i}{2} + \frac{1+e^{i}}{2} = \frac{2-i+e^{i}}{2}$$

$$\int_C f(z)dz = 2\pi i \text{ (sum of residues)}$$

$$= 2\pi i\left(-2 + \frac{(1+e^{-1})i}{2} + \frac{2-i+e^{i}}{2}\right)$$

$$= 2\pi i\left(-2 + \frac{i+ie^{-1}+2-i+e^{i}}{2}\right)$$

$$\Rightarrow \quad \int_C f(z)dz = \pi i(e^{i} + ie^{-1} + 2 - 4)$$

$$= \pi i\left(e^{i} + \frac{i}{e} - 2\right)$$

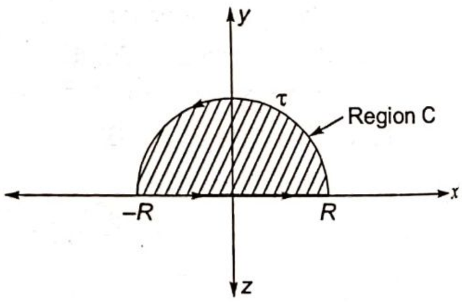
$$\therefore \quad \int_C f(z)dz = \pi i\left(e^{i} + \frac{i}{e} - 2\right)$$

5.4 Show by applying residue theorem that

$$\int_0^\infty \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3}, \quad a > 0$$

(2018 : 15 Marks)

Solution:



Consider $\int_{-R}^{R} \frac{dz}{(z^2+a^2)^2} + \int_{z} \frac{dz}{(z^2+a^2)^2}$

Now, $$\int_{-c}^{a} \frac{dz}{(z^2+a^2)^2} = \int_{c}^{a} \frac{dz}{(z+ai)^2(z-ai)^2}$$

In the above region, $z = ai$ is a pole of order 2.

$\therefore$ Residue at $z = ai$: $\lim_{z\to ai} \frac{d}{dz} \frac{(z-ai)^2}{(z+ai)^2(z-ai)^2}$

$$= \lim_{z\to ai} \frac{-2}{(z+ai)^3} = \frac{2}{8\times a^3 \times i} = \frac{-i}{4a^3}$$

By Cauchy's residue theorem,

$$\int_C \frac{dz}{(z^2+a^2)^2} = 2\pi i \text{ (sum of residues)}$$

$$= 2\pi i \times \frac{-i}{4a^3} = \frac{\pi}{2a^3} \qquad \text{...(i)}$$

Now, $$\int_L \frac{dz}{(z^2+a^2)^2} = \int_{-R}^{R} \frac{dz}{(z^2+a^2)^2} + \int_{\tau} \frac{dz}{(z^2+a^2)^2}$$

as $R \to \infty$, $$\int_{\tau} \frac{dz}{(z^2+a^2)^2} = 0$$

$\therefore$ $$\int_{C} \frac{dz}{(z^2+a^2)^2} = \int_{-\infty}^{\infty} \frac{dz}{(z^2+a^2)^2} = 2\int_{0}^{\infty} \frac{dz}{(z^2+a^2)^2}$$

$\therefore$ From (i)

$$2\int_{0}^{\infty} \frac{dz}{(z^2+a^2)^2} = \frac{\pi}{2a^3}$$

$\Rightarrow$ $$\int_{0}^{\infty} \frac{dz}{(z^2+a^2)^2} = \frac{\pi}{4a^3}$$

or $$\int_{0}^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{\pi}{4a^3},\ a > 0$$

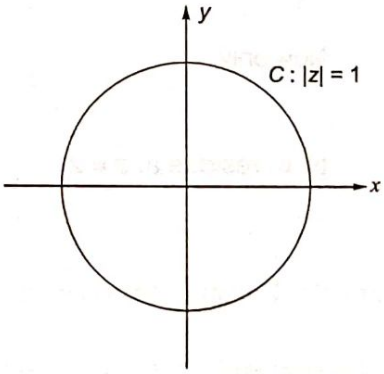
6. Contour Integration

6.1 If α, β, γ are real numbers such that $\alpha^2 > \beta^2 + \gamma^2$, show that :

$$\int_{0}^{2\pi} \frac{d\theta}{\alpha + \beta\cos\theta + \gamma\sin\theta} = \frac{2\pi}{\sqrt{\alpha^2 - \beta^2 - \gamma^2}}$$

(2009 : 30 Marks)

Solution:



Proof : Let $$z = e^{i\theta}$$

then, $$dz = ie^{i\theta}d\theta = izd\theta$$

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - z^{-1}}{2i}$$

$$\sin\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + z^{-1}}{2}$$

Put values of $d\theta$, $\cos\theta$ and $\sin\theta$ in the L.H.S. and the curve becomes a circle : $|z| = 1$

i.e., $$\int_{0}^{2\pi} \frac{d\theta}{\alpha + \beta\cos\theta + \gamma\sin\theta} = \oint_{C} \frac{dz}{iz\left\{\alpha + \beta\left(\frac{z+z^{-1}}{2}\right) + \gamma\left(\frac{z-z^{-1}}{2i}\right)\right\}}$$

$$= \oint_{C} \frac{dz}{\frac{iz}{2i}\{2i\alpha + \beta i(z+z^{-1}) + \gamma(z-z^{-1})\}}$$

$$= \oint_{C} \frac{2zdz}{z\{2i\alpha z + \beta i(z^2+1) + 8(z^2-1)\}}$$

$$= \oint_{C} \frac{2dz}{z^2(\gamma+\beta i) + 2i\alpha z + \beta i - \gamma}$$

$$= \oint_{C} \frac{2dz}{(\gamma+\beta i)\left\{z^2 + \frac{2i\alpha z}{\gamma+\beta i} + \frac{\beta i - \gamma}{\gamma + \beta i}\right\}} = \oint_{C} f(z)dz$$

where C is the circle of unit radius with centre at the origin.

Now, poles of $f(z)$ are given by

$$z^2 + \frac{2i\alpha z}{\gamma+\beta_i} + \frac{\beta_i - \gamma}{\gamma+\beta_i} = 0$$

$$z = \frac{\dfrac{-2i\alpha}{\gamma+\beta_i} \pm \sqrt{\left(\dfrac{2i\alpha}{\gamma+\beta_i}\right)^2 - \dfrac{4(\beta_i-\gamma)}{(\gamma+\beta_i)}}}{2}$$

$$z = \frac{-2i\alpha}{2(\gamma+\beta_i)} \pm \frac{2}{2}\sqrt{\frac{i^2\alpha^2 - (\beta_i-\gamma)(\gamma+\beta_i)}{(\gamma+\beta_i)^2}}$$

$$z = \frac{-i\alpha \pm \sqrt{-\alpha^2 - \{(\beta_i)^2 - \gamma^2\}}}{(\gamma+\beta_i)}$$

$$z = \frac{-i\alpha \pm \sqrt{-\alpha^2+\gamma^2+\beta^2}}{\gamma+\beta_i}$$

$$z = \frac{-i\alpha \pm \sqrt{-1}\sqrt{\alpha^2-\beta^2-\gamma^2}}{\gamma+\beta_i}$$

$$z = \frac{-i\alpha \pm i\sqrt{\alpha^2-\beta^2-\gamma^2}}{\gamma+\beta_i} \qquad \text{(as given that } \alpha^2 > \beta^2+\gamma^2\text{)}$$

Now only
$$z = \frac{-i\alpha + i\sqrt{\alpha^2-\beta^2-\gamma^2}}{\gamma+\beta_i} \text{ lies inside the circle : } |z| = 1$$
$$= z_0 \text{ (say)}$$

Now, residue at $z = z_0$

$$= \lim_{z\to z_0} \frac{(z-z_0)2}{(\gamma+\beta_i)\left(z^2 + \dfrac{2i\alpha z}{\gamma+\beta_i} + \dfrac{\beta_i-\gamma}{\gamma+\beta_i}\right)}$$

$$= \lim_{z\to z_0} \frac{2(z-z_0)}{(\gamma+\beta_i)(z-z_0)(z-z_1)} \quad \left(\text{where } z_1 = \frac{-i\alpha - i\sqrt{\alpha^2-\beta^2-\gamma^2}}{\gamma+\beta_i}\right)$$

$$= \frac{2}{(\gamma+\beta_i)(z_0-z_1)}$$

where,
$$z_0 - z_1 = \left(\frac{-i\alpha + i\sqrt{\alpha^2-\beta^2-\gamma^2}}{\gamma+\beta_i}\right) - \left(\frac{-i\alpha - i\sqrt{\alpha^2-\beta^2-\gamma^2}}{\gamma+\beta_i}\right)$$

$$= \frac{2i\sqrt{\alpha^2-\beta^2-\gamma^2}}{\gamma+\beta_i}$$

$$= \frac{1}{i\sqrt{\alpha^2-\beta^2-\gamma^2}}$$

Now,
$$\int_0^{2\pi} \frac{d\theta}{\alpha+\beta\cos\theta+\gamma\sin\theta} = \oint_C f(z)\,dz$$

$$= 2\pi i(\text{Residue at } z = z0)$$

$$= 2\pi i \times \frac{1}{i\sqrt{\alpha^2 - \beta^2 - \gamma^2}} \qquad \text{(By Cauchy's Residue Theorem)}$$

$$= \frac{2\pi}{\sqrt{\alpha^2 - \beta^2 - \gamma^2}}. \text{ Proved.}$$

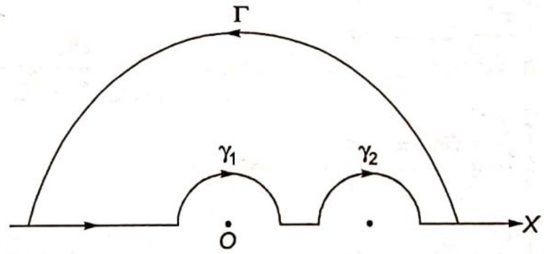
6.2 Evaluate by Contour integration $\int_0^1 \frac{dx}{(x^2 - x^3)^{1/3}}$.

(2011 : 15 Marks)

Solution:

$$\int_0^1 \frac{dx}{(x^2 - x^3)^{1/3}} = \int_\infty^1 \frac{-\frac{dx}{x^2}}{\left(\frac{1}{x^2} - \frac{1}{x^3}\right)^{1/3}}$$

$$= \int_1^\infty \frac{dx}{x(x-1)^{1/3}} = \int_0^\infty \frac{x^{-1/3}dx}{x+1} = \int_0^\infty \frac{x^{\frac{2}{3}-1}dx}{x+1}$$



Let
$$\int_C f(z)dz = \int_C \frac{z^{a-1}}{1-z}dz$$

where C is the contour consisting of a large semi-circle Γ, $|z| = R$ in the upper half plane indented at $z = 0$, $z = 1$.

V_1 and V_2 are the semi-circles in the upper half plane with radii P_1 and P_2 and centres $z = 0$, $z = 1$ respectively.

By Cauchy's residue theorem, we have

$$\int_C f(z)dz = \int_\Gamma f(z)dz + \int_{-R}^{-\rho_1} f(x)dz + \int_{\gamma_1} f(z)dz + \int_{P_1}^{1-\rho_2} f(x)dx + \int_{\gamma_2} f(z)dz + \int_{1+P_2}^{R} f(x)dx = 0 \qquad \text{...(i)}$$

($\because f(z)$ has no pole inside C)

Now,
$$\left|\int_\Gamma f(z)dz\right| \le \int_0^\pi \left|\frac{R^{a-1}e^{i(a-1)\theta}}{1-Re^{i\theta}} \cdot iRe^{i\theta}\right| d\theta \le \int_0^\pi \frac{R^a}{R-1}d\theta$$

$$= \frac{R^a\pi}{R-1} \to 0 \text{ as } R \to \infty \text{ as } a < 1.$$

Since,
$$\lim_{z\to 0} zf(z) = \lim_{z\to 0} \frac{z^a}{1-z} = 0,\ a > 0$$

$\therefore$
$$\lim_{P_1\to 0} \int_{\gamma_1} f(z)dz = -i(\pi - 0)0 = 0$$

$$\lim_{z\to 1}(z-1)f(z) = \lim_{z\to 1}(z-1)\frac{z^{a-1}}{1-z} = -1$$

$$\therefore \quad \lim_{\rho_2\to 0}\int_{\gamma_2} f(z)dz = ip$$

Hence, as $\rho_1 \to 0, \rho_2 \to 0, R \to \infty$, we have from (i)

$$\int_{-\infty}^{0} f(x)dx + \int_{0}^{1} f(x)dx + i\pi + \int_{1}^{\infty} f(x)dx = 0$$

$$\Rightarrow \quad \int_{-\infty}^{0} f(x)dx + \int_{0}^{\infty} f(x)dx = -i\pi$$

$$\Rightarrow \quad \int_{0}^{\infty} f(-x)dx + \int_{0}^{\infty} f(x)dx = -i\pi$$

(Putting $-x$ for x in the first integral)

$$\Rightarrow \quad \int_{0}^{\infty} \frac{(-x)^{a-1}}{1+x}dx + \int_{0}^{\infty}\frac{x^{a-1}}{1-x}dx = -i\pi$$

$$\Rightarrow \quad \int_{0}^{\infty} \frac{(-1)^{a-1}x^{a-1}}{1+x}dx + \int_{0}^{\infty}\frac{x^{a-1}}{1-x}dx = -i\pi$$

$$\Rightarrow \quad \int_{0}^{\infty} \frac{(e^{i\pi})^{a-1}x^{a-1}}{1+x}dx + \int_{0}^{\infty}\frac{x^{a-1}}{1-x}dx = -i\pi$$

$$\Rightarrow \quad \int_{0}^{\infty} \frac{e^{-i\pi}e^{ia\pi}x^{a-1}}{1+x}dx + \int_{0}^{\infty}\frac{x^{a-1}}{1-x}dx = -i\pi$$

$$\Rightarrow \quad -\int_{0}^{\infty} \frac{e^{ia\pi}x^{a-1}}{1+x}dx + \int_{0}^{\infty}\frac{x^{a-1}}{1-x}dx = -i\pi$$

Equating imaginary parts, we have

$$-\int \frac{\sin a\pi x^{a-1}}{1+x}dx = -\pi$$

$$\Rightarrow \quad \int \frac{x^{a-1}}{1+x}dx = \frac{\pi}{\sin a\pi}$$

Putting $a = \frac{2}{3}$, we have

$$\int \frac{x^{-1/3}}{1+x}dx = \frac{\pi}{\sin\frac{2\pi}{3}} = \frac{\pi}{\sin\frac{2\pi}{3}} = \frac{\pi}{\sqrt{3}/2} = \frac{2\pi}{\sqrt{3}}$$

6.3 Evaluate by contour integration :

$$I = \int_{0}^{2\pi} \frac{d\theta}{1-2a\cos\theta + a^2}, a^2 < 1$$

(2012 : 15 Marks)

Solution:

Given :

$$I = \int_{0}^{2\pi} \frac{d\theta}{1-2a\cos\theta + a^2}, a^2 < 1$$

Put $z = e^{i\theta} \Rightarrow$

$$dz = ie^{i\theta}\, d\theta = iz\, d\theta$$

Then

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{1}{2}\left(z+\frac{1}{z}\right)$$

Let C be a unit circle.

Then,

$$I = \int_C \frac{1}{1+a^2-\left(z+\frac{1}{z}\right)a}\cdot\frac{dz}{iz}$$

$$= -\frac{1}{ia}\int_C \frac{dz}{(z-a)\left(z-\frac{1}{a}\right)} = \int_C f(z)dz$$

Now, $z = a$ and $z = \frac{1}{a}$ are simple poles of $f(z)$.

Since, $0 < a < 1$, therefore the simple pole $z = a$ lies inside C.
Residue at $z = a$ is

$$\lim_{z\to a}(z-a)f(z) = \lim_{z\to a}\frac{-(z-a)}{ia(z-a)\left(z-\frac{1}{a}\right)} = \frac{1}{i(1-a^2)}$$

Hence by Cauchy's residue theorem, we have

$$\int_0^{2\pi}\frac{d\theta}{1+a^2-2a\cos\theta} = 2\pi i\ \text{(sum of the residues inside } C)$$

$$= 2\pi i\cdot\frac{1}{i(1-a^2)} = \frac{2\pi}{1-a^2}$$

6.4 Let $r : [0, 1] \to C$ be the curve

$$r(t) = e^{2\pi it},\ 0 \le t \le 1$$

Find giving justifications, the value of the contour integral $\int_r \frac{dz}{4z^2-1}$.

(2016 : 15 Marks)

Solution:

Given : $r : [0, 1] \to C$

and $$r(t) = e^{2\pi it}\ (0 \le t \le 1)$$

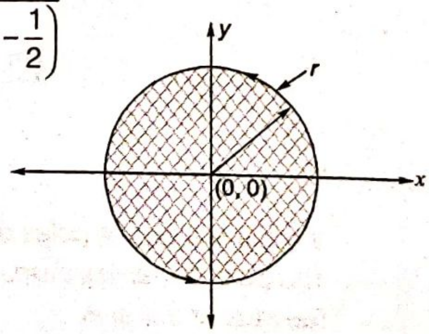
r represents the curve as shown in figure on complex plane.

Now, let

$$I = \int_r \frac{dz}{4z^2-1}$$

$\Rightarrow$

$$I = \int_r \frac{dz}{(2z+1)(2z-1)} = \int_r f(x)dx = \int_r \frac{dz}{4\left(z+\frac{1}{2}\right)\left(z-\frac{1}{2}\right)}$$



$$f(z) = \frac{1}{4x^2-1}$$

Now, in the region bounded by r, $z = -\frac{1}{2}$, $z = \frac{1}{2}$ are poles of order 1.

$\therefore$ Residue at $z = \frac{1}{2}$, R:

$$R_1 = \lim_{z \to \frac{1}{2}} \frac{\left(z - \frac{1}{2}\right) \times 1}{4\left(z + \frac{1}{2}\right)\left(z - \frac{1}{2}\right)} = \frac{1}{4\left(\frac{1}{2} + \frac{1}{2}\right)} = \frac{1}{4}$$

Residue at $z = -\frac{1}{2}$: R_2 :

$$R_2 = \lim_{z \to -\frac{1}{2}} \frac{\left(z + \frac{1}{2}\right) \times 1}{4\left(z - \frac{1}{2}\right)\left(z + \frac{1}{2}\right)} = -\frac{1}{4}$$

By Cauchy's residue theorem,

$$\int_r \frac{dz}{4z^2 - 1} = 2\pi i \text{ (sum of residues)}$$

$$= 2\pi i(R_1 + R_2) = 2\pi i\left(\frac{1}{4} - \frac{1}{4}\right)$$

$$\Rightarrow \quad \int_r \frac{dz}{4z^2 - 1} = 0$$

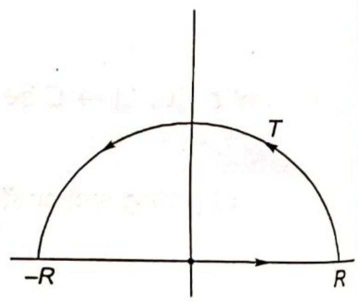
6.5 **Using contour integral method, prove that** $\int_0^\infty \frac{x \sin mx}{a^2 + x^2} dx = \frac{\pi}{2} e^{-ma}$.

(2017 : 15 Marks)

Solution:

Let
$$I = \int_C \frac{e^{imz}}{a^2 + z^2} dz = \int_C f(z) dx$$

where C is the contour consisting of a large semi-circle, T of radius R containing all the poles of the integrand in the upper half plane and the part of real axis from $-R$ to R.



By Cauchy-Residue Theorem,

$$\int_C f(z) dz = \int_{-R}^{R} \frac{e^{imx}}{a^2 + x^2} dx + \int_T \frac{e^{imz}}{a^2 + z^2} dz$$

$$= 2\pi i \text{ (sum of residues)}$$

Since,
$$\lim_{z \to \infty} \frac{1}{(z^2 + a^2)} = 0$$

Therefore,
$$\lim_{Z \to \infty} \int_\Gamma f(z) dz = 0, \text{ by Jordon Lemma}$$

$$\therefore \quad \lim_{R \to \infty} \int_{-R}^{R} \frac{e^{-imx}}{a^2 + x^2} dx = 2\pi i \text{ (sum of residues)}$$

or
$$\int_{-\infty}^{\infty} \frac{e^{imx}}{a^2 + x^2} dx = 2\pi i \text{ (sum of residues)}$$

$z = \pm ai$ are simple poles of $f(z)$.

The pole, $z = ai$ lies inside, C.

Residue at $z = ai$ is

$$\lim_{z \to ai}(z-ai)f(z) = \lim_{z \to ai}\frac{(z-ai)e^{imz}}{a^2+z^2}$$

$$= \lim_{z \to ai}\frac{e^{imz}}{z+ai} = \frac{e^{-ma}}{2ia}$$

From (i)

$$\int_{-\infty}^{\infty}\frac{e^{imx}}{x^2+a^2}dx = 2\pi i\left(\frac{e^{-ma}}{2ia}\right) = \frac{\pi}{a}e^{-ma}$$

Equating real parts on both sides,

$$\int_{-\infty}^{\infty}\frac{\cos mx}{x^2+a^2}dx = \frac{\pi}{a}e^{-ma}$$

or

$$\int_{0}^{\infty}\frac{\cos mx}{x^2+a^2}dx = \frac{\pi}{2a}e^{-ma}$$

Differentiating both sides w.r.t. m

$$\int_{0}^{\infty}\frac{x\sin mx}{x^2+a^2}dx = \frac{\pi}{2}e^{-ma}$$

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