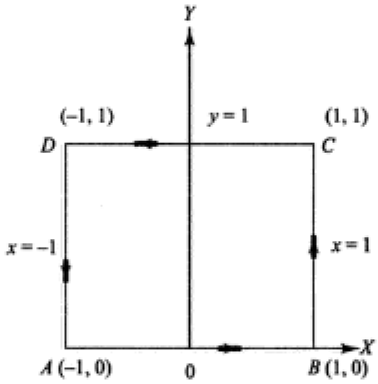
4.24 Verify Cauchy's theorem for the integral of z^3 taken over the boundary of the rectangle with vertices $-1, 1, 1+i, -1+i$.

Solution Let $f(z) = z^3$, since $f(z)$ is analytic within and on the boundary of the rectangle (say, C) and also $f'(z)$ is continuous at each point within and on C. Hence, applying Cauchy's theorem, we get

$$\oint_C z^3 dz = 0$$



Along AB, $y = 0 \Rightarrow dy = 0$ $(-1 \le x \le 1)$
Along BC, $x = 1 \Rightarrow dx = 0$ $(0 \le y \le 1)$
Along CD, $y = 1 \Rightarrow dy = 0$ $(-1 \le x \le 1)$
Along DA, $x = -1 \Rightarrow dx = 0$ $(0 \le y \le 1)$

Fig. 4.21

Now, consider, $\oint_C z^3 dz$

$$= \oint_C (x+iy)^3 (dx + idy)$$

$$= \oint_C [(x+iy)^3 dx + i\,(x+iy)^3 dy]$$

$$= \int_{AB} (x+iy)^3 dx + i\,(x+iy)^3 dy$$

$$+ \int_{BC} [(x+iy)^3 dx + i\,(x+iy)^3 dy]$$

$$+ \int_{CD} [(x+iy)^3 dx + i\,(x+iy)^3 dy]$$

$$+ \int_{DA} [(x+iy)^3 dx + i\,(x+iy)^3 dy]$$

$$= \int_{-1}^{1} x^3 dx + \int_0^1 i(1+iy)^3 dy + \int_1^{-1} (x+i)^3 dx + \int_1^0 i(-1+iy)^3 dy$$

$$= 0 + i\int_0^1 (1+iy)^3 dy + \int_1^{-1} (x+i)^3 dx + i\int_1^0 (iy-1)^3 dy$$

$$= i\left[\frac{(1+iy)^4}{4 \times i}\right]_0^1 + \left[\frac{(x+i)^4}{4}\right]_1^{-1} + i\left[\frac{(iy-1)^4}{4 \times i}\right]_1^0$$

$$= \frac{1}{4}\left[(1+iy)^4\right]_0^1 + \frac{1}{4}\left[(x+i)^4\right]_1^{-1} + \frac{1}{4}\left[(iy-1)^4\right]_1^0$$

$$= \frac{1}{4}[\{(1+i)^4 - 1\} + \{(i-1)^4 - (i+1)^4\} + \{1 - (i-1)^4\}] = 0$$

This verifies the Cauchy's theorem.

Example 16. *Show that the function* $f(z) = u + iv$, *where*

$$f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, & z \neq 0 \\ = 0, & z = 0 \end{cases}$$

satisfies the Cauchy-Riemann equations at $z = 0$. *Is the function analytic at* $z = 0$? *Justify your answer.* *(MDU Dec 2009)*

Solution. $$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2+y^2} = u + iv$$

$$u = \frac{x^3 - y^3}{x^2+y^2}, \qquad v = \frac{x^3+y^3}{x^2+y^2}$$

[By differentiation the value of $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ at (0, 0) we get $\frac{0}{0}$, so we apply first principle method]

At the origin

$$\frac{\partial u}{\partial x} = \lim_{h\to 0} \frac{u(0+h,\ 0) - u(0,\ 0)}{h} = \lim_{h\to 0} \frac{\frac{h^3}{h^2}}{h} = 1 \qquad \text{(Along } x\text{- axis)}$$

$$\frac{\partial u}{\partial y} = \lim_{k\to 0} \frac{u(0,\ 0+k) - u(0,\ 0)}{k} = \lim_{k\to 0} \frac{\frac{-k^3}{k^2}}{k} = -1 \qquad \text{(Along } y\text{- axis)}$$

$$\frac{\partial v}{\partial x} = \lim_{h\to 0} \frac{v(0+h,\ 0) - v(0,\ 0)}{h} = \lim_{h\to 0} \frac{\frac{h^3}{h^2}}{h} = 1 \qquad \text{(Along } x\text{- axis)}$$

$$\frac{\partial v}{\partial y} = \lim_{k\to 0} \frac{v(0,\ 0+k) - v(0,\ 0)}{k} = \lim_{k\to 0} \frac{\frac{k^3}{k^2}}{k} = 1 \qquad \text{(Along } y\text{-axis)}$$

Thus we see that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \text{and} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence, Cauchy-Riemann equations are satisfied at $z = 0$.

Again $$f'(0) = \lim_{z\to 0} \frac{f(0+z)-f(0)}{z} = \lim_{z\to 0} \left[\frac{\dfrac{x^3-y^3+i\,(x^3+y^3)}{x^2+y^2} - (0)}{x+i\,y} \right]$$

$$= \lim_{z\to 0} \left[\frac{x^3-y^3+i(x^3+y^3)}{x^2+y^2} \cdot \frac{1}{x+iy} \right]$$

Now let $z \to 0$ along $y = x$, then

$$f'(0) = \lim_{x\to 0} \frac{x^3-x^3+i(x^3+x^3)}{x^2+x^2}\left(\frac{1}{x+ix}\right)$$

$$= \frac{2i}{2(1+i)} = \frac{i}{1+i} = \frac{i(1-i)}{(1+i)(1-i)} = \frac{i+1}{1+1} = \frac{1}{2}(1+i) \qquad \text{... (1)}$$

Again let $z \to 0$ along $y = 0$, then

$$f'(0) = \lim_{x\to 0} \frac{x^3+ix^3}{x^2} \cdot \frac{1}{x} = (1+i) \qquad [\text{Increment} = z] \qquad \text{... (2)}$$

From (1) and (2), we see that $f'(0)$ is not unique. Hence the function $f(z)$ is not analytic at $z = 0$. **Ans.**

Example 17. *Show that the function*

$$f(z) = e^{-z^{-4}}, \qquad (z \neq 0) \quad \text{and}$$
$$f(0) = 0$$

is not analytic at $z = 0$,
although, Cauchy-Riemann equations are satisfied at the point. How would you explain this.

Solution. $$f(z) = u+iv = e^{-z^{-4}} = e^{-(x+iy)^{-4}} = e^{-\frac{1}{(x+iy)^4}}$$

$$\Rightarrow \qquad u+iv = e^{-\frac{(x-iy)^4}{(x^2+y^2)^4}} = e^{-\frac{1}{(x^2+y^2)^4}[(x^4+y^4-6x^2y^2)-i4xy(x^2-y^2)]}$$

$$\Rightarrow \qquad u+iv = e^{-\frac{x^4+y^4-6x^2y^2}{(x^2+y^2)^4}} . e^{\frac{-i4xy(x^2-y^2)}{(x^2+y^2)^4}}$$

$$\Rightarrow \qquad u+iv = e^{-\frac{x^4+y^4-6x^2y^2}{(x^2+y^2)^4}} \left[\cos\frac{4xy(x^2-y^2)}{(x^2+y^2)^4} - i\sin\frac{4xy(x^2-y^2)}{(x^2+y^2)^4}\right]$$

Equating real and imaginary parts, we get

$$u = e^{-\frac{x^4+y^4-6x^2y^2}{(x^2+y^2)^4}\cos\frac{4xy(x^2-y^2)}{(x^2+y^2)^4}}, \quad v = e^{-\frac{x^4+y^4-6x^2y^2}{(x^2+y^2)^4}\sin\frac{4xy(x^2-y^2)}{(x^2+y^2)^4}}$$

At $z = 0$

$$\frac{\partial u}{\partial x} = \lim_{h\to 0}\frac{u(0+h,\ 0)-u(0,\ 0)}{h} = \lim_{h\to 0}\frac{e^{-h^{-4}}}{h} = \lim_{h\to 0}\frac{1}{he^{\frac{1}{h^4}}}$$

$$= \lim_{h\to 0}\left[\frac{1}{h\left[1+\frac{1}{h^4}+\frac{1}{2!h^8}+\frac{1}{3!h^{12}}+\ldots..\right]}\right], \quad \left(e^x = 1+x+\frac{x^2}{2!}+\ldots..\right)$$

$$= \lim_{h\to 0}\left[\frac{1}{\left[h+\frac{1}{h^3}+\frac{1}{2\,h^7}+\frac{1}{6h^{11}}\ldots..\right]}\right] = \frac{1}{0+\infty} = \frac{1}{\infty} = \mathbf{0}$$

$$\frac{\partial u}{\partial y} = \lim_{k\to 0}\frac{u(0,\ 0+k)-u(0,\ 0)}{k} = \lim_{k\to 0}\frac{e^{-k^{-4}}}{k} = \lim_{k\to 0}\frac{1}{k\ e^{\frac{1}{k^4}}} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{h\to 0}\frac{v(0+h,\ 0)-v(0,\ 0)}{h} = \lim_{h\to 0}\frac{e^{-h^{-4}}}{h} = \lim_{h\to 0}\frac{1}{h.e^{\frac{1}{h^4}}} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{k\to 0}\frac{v(0,\ 0+k)-v(0,\ 0)}{k} = \lim_{k\to 0}\frac{e^{-k^{-4}}}{k} = \lim_{k\to 0}\frac{1}{k.e^{\frac{1}{k^4}}} = 0$$

Hence $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ ($C-R$ equations are satisfied at $z = 0$)

But $f'(0) = \lim_{z\to 0}\frac{f(z)-f(0)}{z} = \lim_{z\to 0}\frac{e^{-z^{-4}}}{z}$

Along $z = re^{i\frac{\pi}{4}}$ $f'(0) = \lim_{r\to 0}\frac{e^{-r^{-4}}\cdot e^{-\left(e^{i\frac{\pi}{4}}\right)^{-4}}}{re^{i\frac{\pi}{4}}} = \lim_{r\to 0}\frac{e^{-r^{-4}}\cdot e^{-\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)^{-4}}}{re^{i\frac{\pi}{4}}}$

$$= \lim_{r\to 0}\frac{e^{-r^{-4}}e^{-\cos\pi}}{re^{i\frac{\pi}{4}}} = \lim_{r\to 0}\frac{e^{-r^{-4}}\cdot e}{re^{i\frac{\pi}{4}}} = \infty$$

Showing that $f'(z)$ does not exist at $z = 0$. Hence $f(z)$ is not analytic at $z = 0$. **Proved.**

Example 18. *Examine the nature of the function*

$$f(z) = \frac{x^2 y^5 (x+iy)}{x^4 + y^{10}}; z \neq 0$$

$$f(0) = 0$$

in the region including the origin.

Solution. Here $\quad f(z) = u + iv = \dfrac{x^2 y^5 (x+iy)}{x^4 + y^{10}}; z \neq 0$

Equating real and imaginary parts, we get

$$u = \frac{x^3 y^5}{x^4 + y^{10}}, \quad v = \frac{x^2 y^6}{x^4 + y^{10}}$$

$$\frac{\partial u}{\partial x} = \lim_{h \to 0} \frac{u(0+h, 0) - u(0,0)}{h} = \lim_{h \to 0} \frac{\frac{0}{h^4}}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{k \to 0} \frac{u(0, 0+k) - u(0,0)}{k} = \lim_{k \to 0} \frac{\frac{0}{k^{10}}}{k} = \lim_{k \to 0} \frac{0}{k} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{h \to 0} \frac{v(0+h, 0) - v(0,0)}{h} = \lim_{h \to 0} \frac{\frac{0}{h^4}}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{k \to 0} \frac{v(0, 0+k) - v(0,0)}{k} = \lim_{k \to 0} \frac{\frac{0}{k^{10}}}{k} = \lim_{k \to 0} \frac{0}{k} = 0$$

From the above results, it is clear that

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence, C-R equations are satisfied at the origin.

But $\quad f'(0) = \lim_{z \to 0} \dfrac{f(0+z) - f(0)}{z} = \lim_{\substack{x \to 0 \\ y \to 0}} \left[\dfrac{x^2 y^5 (x+iy)}{x^4 + y^{10}} - 0 \right] \cdot \dfrac{1}{x+iy}$ (Increment $= z$)

$$= \lim_{\substack{x \to 0 \\ y \to 0}} \frac{x^2 y^5}{x^4 + y^{10}}$$

Let $z \to 0$ along the radius vector $y = mx$, then

$$f'(0) = \lim_{x \to 0} \frac{m^5 x^7}{x^4 + m^{10} x^{10}} = \lim_{x \to 0} \frac{m^5 x^3}{1 + m^{10} x^6} = \frac{0}{1} = 0 \quad \text{... (1)}$$

Again let $z \to 0$ along the curve $y^5 = x^2$

$$f'(0) = \lim_{x \to 0} \frac{x^4}{x^4 + x^4} = \frac{1}{2} \quad \text{... (2)}$$

(1) and (2) shows that $f'(0)$ does not exist. Hence, $f(z)$ is not analytic at origin although Cauchy-Riemann equations are satisfied there. **Ans.**

Example 21. *If $u(x, y)$ and $v(x, y)$ are harmonic functions in a region R, prove that the function*

$$\left[\left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+i\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)\right]$$

is an analytic function of $z = x + iy$. (*R.G.P.V., Bhopal, III Semester, Dec. 2004*)

Solution. Since $u(x, y)$ and $v(x, y)$ are harmonic functions in a region R, therefore

$$\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0 \quad \ldots (1) \qquad \text{and} \qquad \frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0 \qquad \ldots (2)$$

Let $$F(z) = R+iS = \left(\frac{\partial u}{\partial y}-\frac{\partial v}{\partial x}\right)+i\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)$$

Equating real and imaginary parts, we get

$$R = \frac{\partial u}{\partial y}-\frac{\partial v}{\partial x},$$

$$\frac{\partial R}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}-\frac{\partial^2 v}{\partial x^2} \quad \ldots (3) \qquad \frac{\partial R}{\partial y} = \frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 v}{\partial x \partial y} \qquad \ldots (4)$$

$$S = \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}$$

$$\frac{\partial S}{\partial x} = \frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 v}{\partial x \partial y} \quad \ldots (5) \qquad \frac{\partial S}{\partial y} = \frac{\partial^2 u}{\partial x \partial y}+\frac{\partial^2 v}{\partial y^2} \qquad \ldots (6)$$

Putting the value of $\frac{\partial^2 u}{\partial x^2}$ from (1) in (5), we get

$$\frac{\partial S}{\partial x} = -\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial x \partial y} \qquad \ldots (7)$$

Putting the value of $\frac{\partial^2 v}{\partial y^2}$ from (2) in (6), we get

$$\frac{\partial S}{\partial y} = \frac{\partial^2 u}{\partial x \partial y}-\frac{\partial^2 v}{\partial x^2} \qquad \ldots (8)$$

From (3) and (8), $$\frac{\partial R}{\partial x} = \frac{\partial S}{\partial y}$$

From (4) and (7), $$\frac{\partial R}{\partial y} = -\frac{\partial S}{\partial x}$$

Therefore, C-R equations are satisfied and hence the given function is analytic. **Proved.**

Example 26. *If $u - v = (x - y)(x^2 + 4xy + y^2)$ and $f(z) = u + iv$ is an analytic function of $z = x + iy$, find $f(z)$ in terms of z.*

Solution. $u + iv = f(z) \quad \Rightarrow \quad iu - v = if(z)$

Adding these, $(u - v) + i(u + v) = (1 + i) f(z)$

Let $U + iV = (1 + i) f(z)$ where $U = u - v$ and $V = u + v$

$$F(z) = (1 + i) f(z)$$

$$U = u - v = (x - y)(x^2 + 4xy + y^2)$$

$$= x^3 + 3x^2y - 3xy^2 - y^3$$

$$\frac{\partial U}{\partial x} = 3x^2 + 6xy - 3y^2$$

$$\frac{\partial U}{\partial y} = 3x^2 - 6xy - 3y^2$$

We know that $dV = \frac{\partial V}{\partial x} \cdot dx + \frac{\partial V}{\partial y} dy = -\frac{\partial U}{\partial y} \cdot dx + \frac{\partial U}{\partial x} \cdot dy$ [C-R equations]

On putting the values of $\frac{\partial U}{\partial x}$ and $\frac{\partial U}{\partial y}$, we get

$$= (-3x^2 + 6xy + 3y^2)\, dx + (3x^2 + 6xy - 3y^2) \,.\, dy$$

Integrating, we get

$$V = \int (-3x^2 + 6xy + 3y^2) dx + \int (-3y^2) dy$$

(y as constant) (Ignoring terms of x)

$$= -x^3 + 3x^2y + 3xy^2 - y^3 + c$$

$$F(z) = U + iV$$

$$= (x^3 + 3x^2y - 3xy^2 - y^3) + i(-x^3 + 3x^2y + 3xy^2 - y^3) + ic$$

$$= (1 - i)x^3 + (1 + i)3x^2y - (1 - i)3xy^2 - (1 + i)y^3 + ic$$

$$= (1 - i)x^3 + i(1 - i)3x^2y - (1 - i)3xy^2 - i(1 - i)y^3 + ic$$

$$= (1 - i)[x^3 + 3ix^2y - 3xy^2 - iy^3] + ic$$

$$= (1 - i)(x + iy)^3 + iC = (1 - i)z^3 + ic$$

$$(1 + i) f(z) = (1 - i) z^3 + ic, \qquad [F(z) = (1 + i) f(z)]$$

$$f(z) = \frac{1-i}{1+i} z^3 + \frac{ic}{1+i} = -\frac{i(1+i)}{(1+i)} z^3 + \frac{i(1-i)}{(1+i)(1-i)} c = -iz^3 + \frac{1+i}{2} c \qquad \textbf{Ans.}$$

Example 27. *If $f(z) = u + iv$ is an analytic function of $z = x + iy$ and $u - v = e^{-x}[(x - y) \sin y - (x + y) \cos y]$ find $f(z)$.* (*U.P. III Semester, 2009-2010*)

Solution. We know that,

$$f(z) = u + iv \qquad \ldots (1)$$

$$i f(z) = iu - v \qquad \ldots (2)$$

$$F(z) = U + iV$$

$$U = u - v = e^{-x}[(x - y) \sin y - (x + y) \cos y]$$

$$\frac{\partial U}{\partial x} = -e^{-x}[(x - y) \sin y - (x + y) \cos y] + e^{-x}[\sin y - \cos y]$$

$$\frac{\partial U}{\partial y} = e^{-x}[(x - y) \cos y - \sin y - (x + y)(-\sin y) - \cos y]$$

We know that,

$$dV = \frac{\partial V}{\partial x}dx + \frac{\partial V}{\partial y}dy = -\frac{\partial U}{\partial y}dx + \frac{\partial U}{\partial x}dy \qquad \text{[C – R equations]}$$

$$= -e^{-x}\,[(x-y)\cos y - \sin y + (x+y)\sin y - \cos y]\,dx$$
$$- e^{-x}\,[(x-y)\sin y - (x+y)\cos y - \sin y + \cos y]\,dy$$
$$= -e^{-x}\,x\,\{(\cos y + \sin y)\,dx - e^{-x}\,(-y\cos y - \sin y + y\sin y - \cos y)\,dx$$
$$- e^{-x}\,[(x-y)\sin y - (x+y)\cos y - \sin y + \cos y]\,dy$$

$$V = (\cos y + \sin y)\,(x\,e^{-x} + e^{-x}) + e^{-x}\,(-y\cos y - \sin y + y\sin y - \cos y) + C$$

$$F(z) = U + iV$$

$$F(z) = e^{-x}\,[(x-y)\sin y - (x+y)\cos y] + i\,e^{-x}\,[x\cos y + \cos y + x\sin y + \sin y$$
$$- y\cos y - \sin y + y\sin y - \cos y] + iC$$
$$= e^{-x}\,[\{x\sin y - y\sin y - x\cos y - y\cos y\} + i\{x\cos y + x\sin y - y\cos y + y\sin y\}] + iC$$
$$= e^{-x}\,[(x+iy)\sin y - (x+iy)\cos y + (-y+ix)\sin y + (-y+ix)\cos y] + iC$$
$$= e^{-x}\,[(x+iy)\sin y - (x+iy)\cos y + i\,(x+iy)\sin y + i\,(x+iy)\cos y] + iC$$
$$= e^{-x}\,(x+iy)\,[\sin y - \cos y + i\sin y + i\cos y] + iC$$
$$= e^{-x}\,(x+iy)\,[(1+i)\sin y + i\,(1+i)\cos y] + iC$$

$$(1+i)\,f(z) = e^{-x}\,(x+iy)\,(1+i)\,(\sin y + i\cos y) + iC$$

$$f(z) = e^{-x}\,(x+iy)\,(\sin y + i\cos y) + \frac{iC}{1+i}$$

$$= i\,z\,e^{-x}\,(\cos y - i\sin y) + \frac{iC}{1+i}$$

$$= i\,z\,e^{-x}\,e^{-iy} = i\,z\,e^{-(x+iy)} = i\,z\,e^{-z} + \frac{iC}{1+i}$$ **Ans.**

Let $\phi_1(x, y) = -e^{-x}\,[(x-y)\sin y - (x+y)\cos y] + e^{-x}\,[\sin y - \cos y]$

$\phi_1(z, 0) = -e^{-z}\,[z\sin 0 - z\cos 0] + e^{-z}\,[\sin 0 + \cos 0]$

$= -e^{-z}\,[z-1]$

Let $\phi_2(x, y) = e^{-x}\,[(x-y)\cos y - \sin y + (x+y)\sin y - \cos y]$

$\phi_2(z, 0) = e^{-z}\,[(z)\cos 0 - \sin 0 + z\sin 0 - \cos 0]$

$= e^{-z}\,[z-1]$

$$F(z) = U + iV$$

$$F'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y} = f_1(z, 0) - i\,f_2(z, 0)$$

$$= e^{-z}\,(z-1) - i\,e^{-z}\,(z-1) = (1-i)\,e^{-z}\,(z-1) = (1-i)\,e^{-z}\,(z-1)$$

$$F(z) = (1-i)\left[z\,\frac{e^{-z}}{-1} - \int\frac{e^{-z}}{-1}\,dz\right] + C = (1-i)\,[-z\,e^{-z} - e^{-z}] + C$$

$$(1+i)\,f(z) = (-1+i)\,(z+1)\,e^{-z} + C$$

$$f(z) = \frac{(-1+i)}{1+i}(z+1)\,e^{-z} + C = \frac{(-1+i)\,(1-i)}{(1+i)\,(1-i)}(z+1)\,e^{-z} + C$$

$$= i\,(z+1)\,e^{-z} + C$$ **Ans.**

Example 28. *Let $f(z) = u(r, \theta) + iv(r, \theta)$ be an analytic function and $u = -r^3 \sin 3\theta$, then construct the corresponding analytic function $f(z)$ in terms of z.*

Solution. $u = -r^3 \sin 3\theta$

$$\frac{\partial u}{\partial r} = -3r^2 \sin 3\theta, \qquad \frac{\partial u}{\partial \theta} = -3r^3 \cos 3\theta$$

We know that $$dv = \frac{\partial v}{\partial r} dr + \frac{\partial v}{\partial \theta} d\theta$$

$$= \left(-\frac{1}{r}\frac{\partial u}{\partial \theta}\right) dr + \left(r \frac{\partial u}{\partial r}\right) d\theta \qquad \left(\begin{array}{l} C - R \text{ equations} \\ \dfrac{\partial u}{\partial r} = \dfrac{1}{r}\dfrac{\partial v}{\partial \theta} \\ \dfrac{\partial u}{\partial \theta} = -r\dfrac{\partial v}{\partial r} \end{array}\right)$$

$$= -\frac{1}{r}(-3r^3 \cos 3\theta)\, dr + r(-3r^2 \sin 3\theta)\, d\theta$$

$$= 3r^2 \cos 3\theta \cdot dr - 3r^3 \sin 3\theta\, d\theta$$

$$v = \int (3r^2 \cos 3\theta) dr - c = r^3 \cos 3\theta + c$$

$$f(z) = u + iv = -r^3 \sin 3\theta + ir^3 \cos 3\theta + ic = ir^3(\cos 3\theta + i \sin 3\theta) + ic$$

$$= ir^3 e^{i3\theta} + ic = i(re^{i\theta})^3 + ic = iz^3 + ic$$ **Ans.**

Example 35. *If $u - v = (x - y)(x^2 + 4xy + y^2)$ and $f(z) = u + iv$ is an analytic function of $z = x + iy$, find $f(z)$ in terms of z by Milne Thomson method.*

Solution. We know that

$$f(z) = u + iv \quad \ldots (1)$$

$$i f(z) = i u - v \quad \ldots (2)$$

Adding (1) and (2), we get

$$(1 + i) f(z) = (u - v) + i(u + v)$$

$$F(z) = U + iV$$

$$U = u - v = (x - y)(x^2 + 4xy + y^2)$$

$$\frac{\partial U}{\partial x} = (x^2 + 4xy + y^2) + (x - y)(2x + 4y)$$

$$= x^2 + 4xy + y^2 + 2x^2 + 4xy - 2xy - 4y^2 = 3x^2 + 6xy - 3y^2$$

$$\phi_1(x, y) = 3x^2 + 6xy - 3y^2$$

$$\phi_1(z, 0) = 3z^2$$

$$\frac{\partial U}{\partial y} = -(x^2 + 4xy + y^2) + (x - y)(4x + 2y)$$

$$= -x^2 - 4xy - y^2 + 4x^2 + 2xy - 4xy - 2y^2 = 3x^2 - 6xy - 3y^2$$

$$\phi_2(x, y) = 3x^2 - 6xy - 3y^2$$

$$\phi_2(z, 0) = 3z^2$$

$$F(z) = \mathrm{U} + i\mathrm{V}$$

$$F'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y} = \phi_1(z, 0) - i\,\phi_2(z, 0) = 3z^2 - i\,3z^2$$

$$= 3(1 - i) z^2$$

$$F(z) = (1 - i) z^3 + C$$

$$(1 + i) f(z) = (1 - i) z^3 + C$$

$$f(z) = \frac{1-i}{1+i} z^3 + \frac{C}{1+i} = \frac{(1-i)(1-i)}{(1+i)(1-i)} z^3 + C_1$$

$$= \frac{1 - 2i + (-i)^2}{1+1} z^3 + C_1 = \frac{1 - 2i - 1}{2} z^3 + C_1 = -i z^3 + C_1 \qquad \textbf{Ans.}$$

Example 36. *If $f(z) = u + iv$ is an analytic function of z and $u - v = \dfrac{\cos x + \sin x - e^{-y}}{2\cos x - 2\cosh y}$, prove that*

$$f(z) = \frac{1}{2}\left[1 - \cot\frac{z}{2}\right] \text{ when } f\left(\frac{\pi}{2}\right) = 0. \quad \text{(R.G.P.V. Bhopal, III Semester, Dec. 2007)}$$

Solution. We know that $\quad f(z) = u + iv$

$\therefore \quad i\,f(z) = iu - v$ [Multiplying by i]

On adding, we get $\quad (1 + i)\,f(z) = (u - v) + i(u + v)$

Let $\quad F(z) = U + iV$

We have, $\quad U = u - v = \dfrac{\cos x + \sin x - e^{-y}}{2\cos x - 2\cosh y}$

$$\Rightarrow \quad U = \frac{\cos x + \sin x - \cosh y + \sinh y}{2\cos x - 2\cosh y} \quad [\because e^{-y} = \cosh y - \sinh y]$$

$$= \frac{\cos x - \cosh y}{2(\cos x - \cosh y)} + \frac{\sin x + \sinh y}{2(\cos x - \cosh y)} = \frac{1}{2} + \frac{\sin x + \sinh y}{2(\cos x - \cosh y)} \quad \ldots(1)$$

Differentiating (1) w.r.t. x partially, we get

$$\frac{\partial U}{\partial x} = \frac{1}{2}\left[\frac{(\cos x - \cosh y)\cos x - (\sin x + \sinh y)(-\sin x)}{(\cos x - \cosh y)^2}\right]$$

$$= \frac{1}{2}\left[\frac{(\cos^2 x + \sin^2 x - \cosh y\cos x + \sinh y\sin x)}{(\cos x - \cosh y)^2}\right]$$

Replacing x by z and y by 0 in (2), we get

$$\phi_1(z, 0) = \frac{1}{2}\left[\frac{1 - \cos z}{(\cos z - 1)^2}\right] = \frac{-(\cos z - 1)}{2(\cos z - 1)^2} = \frac{-1}{2(\cos z - 1)} = \frac{1}{2(1 - \cos z)}$$

Differentiating (1) partially w.r.t. y, we get

$$\frac{\partial U}{\partial y} = \frac{1}{2}\left[\frac{(\cos x - \cosh y)\,.\cosh y - (\sin x + \sinh y)(-\sinh y)}{(\cos x - \cosh y)^2}\right]$$

$$= \frac{1}{2}\left[\frac{(\cos x\cosh y) + \sin x\sinh y - (\cosh^2 y - \sinh^2 y)}{(\cos x - \cosh y)^2}\right]$$

$$\phi_2(x, y) = \frac{1}{2}\left[\frac{\cos x\cosh y + \sin x\sinh y - 1}{(\cos x - \cosh y)^2}\right] \quad \ldots(3)$$

Replacing x by z and y by 0 in (3), we have

$$\phi_2(z, 0) = \frac{1}{2}\left[\frac{\cos z - 1}{(\cos z - 1)^2}\right] = \frac{1}{2}\cdot\frac{1}{\cos z - 1} = \frac{1}{2}\cdot\left(\frac{-1}{1 - \cos z}\right)$$

$$F'(z) = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y} \qquad \text{[C–R equations]}$$

$$= \phi_1(z, 0) - i\,\phi_2(z, 0)$$

By Milne Thomson Method,

$$F(z) = \int\left[\phi_1(z, 0) - i\phi_2(z, 0)\right] dz$$

$$= \int\left[\frac{1}{2}\cdot\frac{1}{(1-\cos z)} + \frac{i}{2}\cdot\frac{1}{1-\cos z}\right]dz$$

$$= \frac{1+i}{2}\int\frac{1}{2\sin^2 z/2}dz = \frac{1+i}{4}\int \operatorname{cosec}^2(z/2)\,dz$$

$$= \left(\frac{1+i}{4}\right)\cdot\frac{(-\cot z/2)}{\left(\frac{1}{2}\right)} + C = -\left(\frac{1+i}{2}\right)\cot\frac{z}{2} + C$$

$$\Rightarrow \qquad (1+i)\,f(z) = -\left(\frac{1+i}{2}\right)\cot\frac{z}{2} + C \qquad \Rightarrow \quad f(z) = -\frac{1}{2}\cot\frac{z}{2} + \frac{C}{1+i} \qquad ...(4)$$

On putting $z = \frac{\neq}{2}$ in (4), we get

$$f\left(\frac{\pi}{2}\right) = -\frac{1}{2}\cot\frac{\pi}{4} + \frac{C}{1+i}$$

$$0 = -\frac{1}{2} + \frac{C}{1+i} \qquad \Rightarrow \qquad \frac{C}{1+i} = \frac{1}{2} \qquad \left[f\left(\frac{\pi}{2}\right) = 0, \text{ given}\right]$$

On putting the value of $\frac{C}{1+i}$ in (4), we get

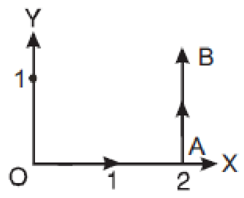
$$f(z) = -\frac{1}{2}\cot\frac{z}{2} + \frac{1}{2}$$

Hence, $$f(z) = \frac{1}{2}\left(1 - \cot\frac{z}{2}\right)$$ **Proved.**

Example 44. *Evaluate* $\int_0^{2+i} (\bar{z})^2 dz$ *along the real axis from z* = *0 to z* = 2 *and then along a line parallel to y-axis from z* = *2 to z* = *2* + *i.*

(R.G.P.V., Bhopal, III Semester, June 2005)

Solution. $\int_0^{2+i} (\bar{z})^2 dz = \int_0^{2+i} (x - iy)^2 (dx + i dy)$



$$= \int_{OA} (x^2)\,dx + \int_{AB} (2 - i\,y)^2\, i dy$$

[Along OA, $y = 0$, $dy = 0$, x varies 0 to 2.
Along AB, $x = 2$, $dx = 0$ and y varies 0 to 1]

$$= \int_0^2 x^2 dx + \int_0^1 (2 - i\,y)^2\, i\,dy$$

$$= \left[\frac{x^3}{3}\right]_0^2 + i\int_0^1 (4 - 4i\,y - y^2)\,d\,y = \frac{8}{3} + i\left[4y - 2i\,y^2 - \frac{y^3}{3}\right]_0^1$$

$$= \frac{8}{3} + i\left[4 - 2i - \frac{1}{3}\right] = \frac{8}{3} + \frac{i}{3}(11 - 6i) = \frac{1}{3}(8 + 11i + 6) = \frac{1}{3}(14 + 11i)$$

Which is the required value of the given integral. **Ans.**