2009 IFS 2009 P-1 2(d), 3a, 3b, 3c, 3d

2d) Find an orthogonal transformation of coordinates to reduce the quadratic form $q(x,y) = 2x^2 + 2xy + 2y^2$ to a canonical form. (10)

Matrix form of QF is

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

Characteristic Eqn: $|A - \lambda I| = 0$

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$\Rightarrow (\lambda - 2)^2 - 1 = 0$

$\Rightarrow \lambda^2 - 4\lambda + 3 = 0$

$(\lambda - 3)(\lambda - 1) = 0 \Rightarrow \boxed{\lambda = 1, 3}$

Eigen vector for $\lambda = 1$

$$(A-I)X = \begin{bmatrix} 2-1 & 1 \\ 1 & 2-1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} x + y = 0 \\ x = -y \end{matrix}$$

$V_1\ (x, y) = (1, -1)$

$$\boxed{X = NY} \rightarrow \begin{cases} x_1 = \frac{1}{\sqrt{2}} y_1 + \frac{1}{\sqrt{2}} y_2 \\ x_2 = \frac{-1}{2} y_1 + \frac{1}{2} y_2 \end{cases}$$

Eigenvector for $\lambda = 3$

$$(A - 3I)X = 0 \Rightarrow \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-x + y = 0 \Rightarrow x = y$$

$$v_2 = (x, y) = (1, 1)$$

vectors v_1 and v_2 are orthogonal.

$$v_1 v_2^T = 0$$

Modal Matrix: comprizing of eigenvectors.

$$M = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Normalized Model Matrix:

$$N = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

$$N^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}$$

Calculate

$$AN = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & 3/\sqrt{2} \end{bmatrix}$$

$$N^T(AN) = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$$

Canonical form: $y_1^2 + 3y_2^2 = [y_1, y_2] D [y_1, y_2]^T$

3(a)

The adiabatic law for the expansion of air is $PV^{1.4} = K$, where K is a constant. If at a given time the volume is observed to be 50 c.c. and the pressure is 30 kg per square cm, at what rate is the pressure changing if the volume is decreasing at the rate of 2 c.c. per second? (10m)

$$PV^{1.4} = K \Rightarrow K = 30(50)^{1.4}$$

$$P = KV^{-1.4}$$

$$\frac{dP}{dt} = K(-1.4)V^{-2.4} \cdot \frac{dV}{dt}$$

$$= -30(50)^{1.4} \frac{(1.4)}{(50)^{2.4}}(-2)$$

$$= \frac{3\cancel{0} \times 1.4 \times 2}{5\cancel{0}} = 1.68$$

$= 1.68$ kg per cm^2/second.

3(b) Determine the asymptotes of the curve
$x^3+x^2y-xy^2-y^3+2xy+2y^2-3x+y=0$. (10)

Asymptotes parallel to coordinate axes:
As the coefficients of highest degree terms of x and y i.e. x^3 and y^3 are constants, so the curve has no asymptotes parallel to x-axis or y-axis.

Oblique Asymptotes: Put $x=1$, $y=m$ in ~~the~~ third, second and first degree terms,

$$\phi_3(m) = 1+m-m^2-m^3$$

$$\phi_2(m) = 2m+2m^2$$

$$\phi_1(m) = -3+m$$

Slopes of the asymptotes are real roots of eqn $\phi_3(m)=0$.

$$1+m - m^2(1+m) = 0$$

$$(1+m)(1-m^2) = 0 \Rightarrow m = 1, -1, -1$$

For $m = 1$,

$$c = -\frac{\phi_2(m)}{\phi_3'(m)} = -\left(\frac{2m+2m^2}{1-2m-3m^2}\right) = \frac{-4}{-4}$$

$$= 1 \Rightarrow \boxed{y = x+1}$$

for, $m = -1$, c is given by

$$\frac{c^2}{2!}\phi_3''(m) + c\phi_2'(m) + \phi_1(m) = 0$$

$$\frac{c^2}{2}(-2-6m) + c(2+4m) + (-3+m) = 0$$

$$\frac{c^2}{2}(4) + c(-2) + (-4) = 0 \qquad (\because m = -1)$$

$$2c^2 - 2c - 4 = 0$$

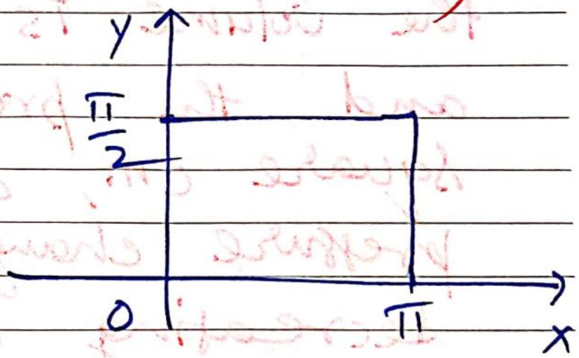
$$c^2 - c - 2 = 0$$

$$(c-2)(c+1) = 0 \Rightarrow c = 2, -1$$

$$\Rightarrow \boxed{y = -x+2}, \quad \boxed{y = -x-1}$$

3(c) Evaluate $\iint_D x \sin(x+y)\, dx\,dy$ where D is the region bounded by $0 \le x \le \pi$ and $0 \le y \le \frac{\pi}{2}$. (10)

$$\int_{x=0}^{\pi}\int_{y=0}^{\pi/2} x \sin(x+y)\, dx\,dy$$



$$= -\int_0^{\pi} x \left[\cos(x+y)\right]_{y=0}^{\pi/2} dx$$

$$= -\int_0^{\pi} x \left[\cos\left(\frac{\pi}{2}+x\right) - \cos(0+x)\right] dx$$

$$= -\int_0^{\pi} \left(-x\sin x - x\cos x\right) dx$$

$$= \int_0^{\pi} x(\sin x + \cos x)\, dx$$

$$= \left(x\left[-\cos x + \sin x\right]\right)_0^{\pi} - \int_0^{\pi} 1.(-\cos x + \sin x)\, dx$$

$$= \left[\pi(-(-1)) - 0\right] - \left(-\sin x - \cos x\right)_0^{\pi}$$

$$= \pi - 2.$$

3(d) Evaluate $\iiint (x+y+z+1)^4 \, dx\,dy\,dz$ over the region defined by $x \geq 0$, $y \geq 0$, $z \geq 0$ and $x+y+z \leq 1$. (10)

$$\int_{x=0}^{1} \int_{y=0}^{1-x} \int_{z=0}^{1-x-y} (x+y+z+1)^4 \, dz\,dy\,dx$$

$$= \int_{x=0}^{1} \int_{y=0}^{1-x} \frac{1}{5} \left[(x+y+z+1)^5\right]_{z=0}^{1-x-y} dy\,dx$$

$$= \frac{1}{5} \int_{x=0}^{1} \int_{y=0}^{1-x} 2^5 - (x+y+1)^5 \, dy\,dx$$

$$= \frac{1}{5} \int_0^1 \left[32y - \frac{1}{6}(x+y+1)^6\right]_{y=0}^{1-x} dx$$

$$= \frac{1}{5} \int_0^1 32(1-x) - \frac{1}{6}(2)^6 + \frac{1}{6}(x+1)^6 \, dx$$

$$= \frac{1}{5} \int_0^1 \frac{64}{3} - 32x + \frac{1}{6}(x+1)^6 \, dx$$

$$\frac{1}{5}\left[\frac{64}{3}x - 16x^2 + \frac{1}{42}(x+1)^7\right]_0^1 = \frac{117}{70}$$