Sol. (*a*) Given $p^3 - p\,(y + 3) + x = 0$, where $p = dy/dx$. ... (1)

Solving for x, $x = p\,(y + 3) - p^3$. ... (2)

Differentiating (2) w.r.t. 'y' and writing $1/p$ for dx/dy, we get

$$\frac{1}{p} = p + (y+3)\frac{dp}{dy} - 3\,p^2\frac{dp}{dy} \quad \text{or} \quad \frac{1}{p} - p = \frac{dp}{dy}(y + 3 - 3p^2)$$

or $$\frac{1-p^2}{p}\frac{dy}{dp} = y + 3 - 3p^2 \quad \text{or} \quad \frac{dy}{dp} = \frac{p}{1-p^2}[y + 3(1-p^2)]$$

or $(dy/dp) - \{p/(1 - p^2)\}\,y = 3p$, which is linear equation. ... (3)

Its I.F. $= e^{-\int[p/(1-p^2)]dp} = e^{(1/2)\times\log(1-p^2)} = (1-p^2)^{1/2}$.

$\therefore$ Solution of (3) is $y\,(1 - p^2)^{1/2} = c + \int 3p\,(1-p^2)^{1/2}dp$. ... (4)

Putting $1 - p^2 = v$ so that $-2p\,dp = dv$ or $pdp = -(1/2)\,dv$, (4) gives

$$y\,(1 - p^2)^{1/2} = c - (3/2)\times\int v^{1/2}dv = c - v^{3/2} = c - (1-p^2)^{3/2} \qquad \text{... (5)}$$

$y = c\,(1 - p^2)^{-1/2} - (1 - p^2)$, where $|\,p\,| < 1$, c being an arbitrary constant

Putting this value of y in (2), we get

$x = p\,[c\,(1 - p^2)^{-1/2} - 1 + p^2 + 3] - p^3$ or $x = cp\,(1 - p^2)^{-1/2} + 2p$. ... (6)

(5) and (6) together form the solution of (1) in parametric form, p being treated as parameter.

Example 5. *Solve the equation,*

$(2x^2 + 3y^2 - 7)\,x\,dx - (3x^2 + 2y^2 - 8)\,y\,dy = 0$ (*U.P. II Semester, Summer 2005*)

Solution. We have

$$(2x^2 + 3y^2 - 7)\,x\,dx - (3x^2 + 2y^2 - 8)\,y\,dy = 0$$

Re-arranging (1), we get $\dfrac{x}{y}\dfrac{dx}{dy} = \dfrac{3x^2+2y^2-8}{2x^2+3y^2-7}$

Applying componendo and dividendo rule, we get

$$\frac{x\,dx + y\,dy}{x\,dx - y\,dy} = \frac{5x^2+5y^2-15}{x^2-y^2-1} \quad \Rightarrow \quad \frac{x\,dx+y\,dy}{x^2+y^2-3} = 5\left(\frac{x\,dx - y\,dy}{x^2-y^2-1}\right)$$

Multiplying by 2 both the sides, we get

$$\Rightarrow \quad \left(\frac{2x\,dx+2y\,dy}{x^2+y^2-3}\right) = 5\left(\frac{2x\,dx-2y\,dy}{x^2-y^2-1}\right)$$

Integrating both sides, we get

$$\log(x^2 + y^2 - 3) = 5\log(x^2 - y^2 - 1) + \log C$$

$$\Rightarrow \quad x^2 + y^2 - 3 = C\,(x^2 - y^2 - 1)^5$$ **Ans.**

where C is arbitrary constant of integration.

Example 7. *Solve* $(px^2 + y^2)(px + y) = (p + 1)^2$ *by using the substitutions* $X = x + y$ *and* $Y = xy$.

Sol. Given equation is $(px^2 + y^2)(px + y) = (p + 1)^2$ ...(1)

We have $X = x + y$ and $Y = xy$

$\therefore$ $\frac{dX}{dx} = 1 + p$ and $\frac{dY}{dx} = y + xp$

$\therefore$ $\frac{dY}{dX} = \frac{y + xp}{1 + p}$

Let $P = \frac{dY}{dX}$. $\therefore$ $P = \frac{xp + y}{1 + p}$.

(1) $\Rightarrow$ $(px^2 + y^2 + xyp + xy - xyp - xy)(px + y) = (p + 1)^2$ [Note this step]

$\Rightarrow$ $[(px + y)(x + y) - (p + 1)xy](px + y) = (p + 1)^2$

$\Rightarrow$ $(px + y)^2(x + y) - (p + 1)xy(px + y) = (p + 1)^2$

$$\Rightarrow \quad \frac{(px + y)^2 (x + y)}{(p + 1)^2} - \frac{xy(px + y)}{p + 1} = 1$$

$$\Rightarrow \quad (x + y)\left(\frac{px + y}{p + 1}\right)^2 - xy\left(\frac{px + y}{p + 1}\right) = 1$$

$\Rightarrow$ $XP^2 - YP = 1$ $\Rightarrow$ $YP = XP^2 - 1$ $\Rightarrow$ $Y = PX - \frac{1}{P}$.

This is a Clairaut's equation.

$\therefore$ Its solution is $Y = cX - \frac{1}{c}$. (By replacing P by c)

$\Rightarrow$ $\mathbf{xy = c(x + y) - \frac{1}{c}}$

This is the general solution of the given equation.

23. Show that $L\left(\frac{\cos\sqrt{t}}{\sqrt{t}}\right) = \sqrt{\left(\frac{\pi}{p}\right)}e^{-1/4p}$.

using the laplace transform

$$\int_0^\infty e^{-st} f(t)dt = \int_0^\infty e^{-st} \frac{e^{i\sqrt{t}} + e^{-i\sqrt{t}}}{2\sqrt{t}} dt$$

Here I used the exponential form of cosine.

Changing variables to $v = \sqrt{t}$, we transform the integral to

$$\int_0^\infty e^{-sv^2} \left(e^{iv} + e^{-iv}\right) dv$$

Now we can rearrange the integral and using completeing the square to yield

$$e^{-\frac{1}{4s}} \int_0^\infty e^{-s\left(v-\frac{i}{2s}\right)^2} + e^{-s\left(v+\frac{i}{2s}\right)^2} dv$$

Now, this next step is more of a trick i.e. not the most robust step you will ever see! but changing the variables once again $x = \sqrt{s}\left(v - \frac{i}{2s}\right)$ and $y = \sqrt{s}\left(v + \frac{i}{2s}\right)$

$$\frac{1}{\sqrt{s}} e^{-\frac{1}{4s}} \left[\int_0^\infty e^{-x^2} dx + \int_0^\infty e^{-y^2} dy\right]$$

Both upper limits hold here as well. So now you can see that we have the standard normal integrals which is equal to $\sqrt{\pi}$, but in this case we only integrated the positive domain, which leads both these integrals to equate to $\frac{\sqrt{\pi}}{2}$.

Subbing in the results we finally arrive at the above solution from maple as

$$\sqrt{\frac{\pi}{s}} e^{-\frac{1}{4s}}.$$

Ex. 35. *Find the curve for which sum of the reciprocals of the radius vector and the polar subtangent is constant.* **[I.A.S. 1996]**

Sol. Reciprocal of polar substangent = $\left(r^2 \frac{d\theta}{dr}\right)^{-1} = \frac{1}{r^2}\frac{dr}{d\theta}$.

Given that $\frac{1}{r} + \frac{1}{r^2}\frac{dr}{d\theta} = k$, where k is constant or $\frac{1}{r^2}\frac{dr}{d\theta} = k - \frac{1}{r} = \frac{kr-1}{r}$

or $d\theta = \frac{dr}{r(kr-1)}$ or $d\theta = \left(\frac{k}{kr-1} - \frac{1}{r}\right) dr$.

Integrating, $\theta + c = \log\{(kr-1)/r\}$ or $kr - 1 = re^{\theta+c}$ or $kr - 1 = c'r\,e^{\theta}$, where c' $(= e^c)$ is an arbitrary constant.

Ex. 22. (b). *Find the curves for which the portion of y-axis cut off between the origin and the tangent varies as the cube of the abscissa of the point of contact.* **(I.C.S. 1992)**

Sol. The intercept made by the tangent on y-axis is

$$y - x\,(dy/dx).$$

$\therefore$ the differential equation of the required family of curves is

$$y - x\frac{dy}{dx} = kx^3, \text{ where } k \text{ is the given constant of proportionality}$$

or $$x\frac{dy}{dx} - y = -kx^3$$

or $$\frac{dy}{dx} - \frac{1}{x}y = -kx^2, \text{ which is linear.}$$

$$\text{I.F.} = e^{\int(-1/x)\,dx} = e^{-\log x} = e^{\log(1/x)} = 1/x.$$

$\therefore$ the solution is $y.\,(1/x) = \int(-kx^2)\cdot(1/x)\,dx + c,$

where c is an arbitrary constant

or $$y/x = -\tfrac{1}{2}kx^2 + c.$$

Hence the required family of curves is

$y = -\frac{1}{2}kx^3 + cx$, where c is the parameter.

Ex. 22. (c). *Determine a family of curves for which the ratio of the y-intercept of the tangent to the radius vector is a constant.*

(I.C.S. 1995)

Sol. The intercept made by the tangent on y-axis at any point (x, y) on a curve is $y - x\,(dy/dx)$ and the length of the radius vector drawn to the point (x, y) is $\sqrt{(x^2 + y^2)}$.

$\therefore$ the differential equation of the required family of curves is

$$y - x\frac{dy}{dx} = k\sqrt{(x^2+y^2)}$$, where k is the given constant of proportionality

or $$\frac{dy}{dx} = \frac{y - k\sqrt{(x^2+y^2)}}{x}, \qquad \ldots(1)$$

which is homogeneous.

Putting $y = vx$ and $\frac{dy}{dx} = v + x\frac{dv}{dx}$ in (1), we get

$$v + x\frac{dv}{dx} = \frac{vx - k\sqrt{(x^2+v^2x^2)}}{x} = v - k\sqrt{(1+v^2)}$$

or $$x\frac{dv}{dx} = -k\sqrt{(1+v^2)} \text{ or } \frac{dv}{\sqrt{(1+v^2)}} = -\frac{k}{x}dx.$$

Integrating, we get

$$\log[v + \sqrt{(v^2+1)}] = -k\log x + \log c,$$

where c is an arbitrary constant

or $$\log[v + \sqrt{(v^2+1)}] = \log(c/x^k)$$

or $$v + \sqrt{(v^2+1)} = c/x^k$$

or $$(y/x) + \sqrt{\{(y^2/x^2)+1\}} = c/x^k$$

or $y + \sqrt{(y^2+x^2)} = c/x^{k-1}$, which is the required family of curves having c as the parameter.

Ex. 22. (d). *Determine the curve for which the radius of curvature is proportional to the slope of the tangent.* **(I.C.S. 1993)**

Sol. According to question radius of curvature $\rho = k\tan\psi$, where k is given constant of proportionality.

$$\therefore \quad \frac{ds}{d\psi} = k\tan\psi$$

or $$ds = k\tan\psi\, d\psi.$$

Integrating both sides, we get

$$s = k\log\sec\psi + c,$$ where c is constant of integration.

Let us take $s = 0$ when $\psi = 0$.

Then $0 = 0 + c$ or $c = 0$.

Hence the intrinsic equation of the required curve is

$$s = k\log\sec\psi.$$

Ex. 31. *Assume that a spherical rain drop evaporates at a rate proportional to its surface area. If its radius originally is 3 mm, and one hour later has been reduced to 2 mm, find an expression for the radius of the rain drop at any time.* **[I.A.S. 1997]**

Sol. Let r mm be the radius of the rain drop at time t hours from start. If V and S be volume and surface area of the rain drop, then we have

$V = (4/3)\,\pi r^3$ cubic mm and $S = 4\pi r^2$ sq. mm.... (1)

Given $dV/dt = -kS$, where k (> 0) is the constant of proportionality.

Using (1), this $\Rightarrow$ $4\pi r^2\,(dr/dt) = -k\,(4\pi r^2)$ or $dr = -k\,dt$

Integrating $r = -kt + c$, where c is an arbitrary constant. ... (2)

Now, initially when $t = 0$, $r = 3$ mm. Then (2) $\Rightarrow$ $c = 3$

Hence (2) reduces to $r = 3 - kt$ ... (3)

Again, given that $r = 2$ mm when $t = 1$ hour. Hence (3) reduces to $2 = 3 - k$ so that $k = 1$.

With $k = 1$, (3) reduces to $r = 3 - t$, which is the required expression for radius r at any time t.

Ex. 19. *Find the family of curves whose tangent form an angle $\pi/4$ with the hyperbolas $xy = c$.* **[I.A.S. 1994, 2006]**

Sol. Here the required angle is given by

$$\tan^{-1}\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right) = \frac{\pi}{4} \quad \text{or} \quad \tan\left(\frac{\pi}{4}\right) = \frac{m_1 - m_2}{1 + m_1 m_2} \quad \text{... (1)}$$

where $m_1 = dy/dx$ for the required family at (x, y)

and m_2= value of the dy/dx for the second curve $(xy = c) = -c/x^2$, as $y = c/x \Rightarrow dy/dx = -(c/x^2)$

Putting values of m_1 and m_2 in (1), we get

$$1 = \frac{\dfrac{dy}{dx} + \dfrac{c}{x^2}}{1 - \dfrac{c}{x^2} \times \dfrac{dy}{dx}} \quad \text{or} \quad 1 - \left(\frac{c}{x^2} \times \frac{dy}{dx}\right) = \frac{dy}{dx} + \frac{c}{x^2} \quad \text{or} \quad \left(1 + \frac{c}{x^2}\right)\frac{dy}{dx} = 1 - \frac{c}{x^2}$$

$$\text{or} \quad dy = \frac{x^2 - c}{x^2 + c}\,dx \quad \text{or} \quad dy = \left[\frac{x^2 + c - 2c}{x^2 + c}\right]dx = \left[1 - \frac{2c}{x^2 + c}\right]dx.$$

Integrating, $y = x - 2c\,(1/\sqrt{c})\tan^{-1}(x/\sqrt{c}) + c'$

or $y = x - 2\sqrt{c}\,\tan^{-1}(x/\sqrt{c}) + c'$, where c' is an arbitrary constant.

Ex. 35. *Find the curve for which sum of the reciprocals of the radius vector and the polar subtangent is constant.* **[I.A.S. 1996]**

Sol. Reciprocal of polar substangent $= \left(r^2 \dfrac{d\theta}{dr}\right)^{-1} = \dfrac{1}{r^2}\dfrac{dr}{d\theta}$.

Given that $\dfrac{1}{r} + \dfrac{1}{r^2}\dfrac{dr}{d\theta} = k$, where k is constant or $\dfrac{1}{r^2}\dfrac{dr}{d\theta} = k - \dfrac{1}{r} = \dfrac{kr - 1}{r}$

$$\text{or} \quad d\theta = \frac{dr}{r\,(kr - 1)} \quad \text{or} \quad d\theta = \left(\frac{k}{kr - 1} - \frac{1}{r}\right)dr.$$

Integrating, $\theta + c = \log\{(kr - 1)/r\}$ or $kr - 1 = re^{\theta + c}$ or $kr - 1 = c' r\, e^{\theta}$, where c' $(= e^c)$ is an arbitrary constant.

Ex. 5(*a*). *Solve* $dy/dx = (x + y - 1)^2/4(x - 2)^2$. **[Srivenkateshwar 2003]**

Sol. Given $dy/dx = (x + y - 1)^2/4(x - 2)^2$. ... (1)

Put $x = X + h$ and $y = Y + k$ so that $dx = dX$ and $dy = dY$.... (2)

Then, from (1), $$\frac{dY}{dX} = \frac{(X+h+Y+k-1)^2}{4(X+h-2)^2} = \frac{(X+Y+h-k-1)^2}{4(X+h-2)^2} \quad ... (3)$$

Choose h and k such that $h - k - 1 = 0$ and $h - 2 = 0$ so that $h = 2$ and $k = -1$.

Then, (2) $\Rightarrow$ $X = x - h = x - 2$ and $Y = y - k = y + 1$. ... (4)

Also, from (3), $dY/dX = (X + Y)^2/4X^2$ which is homogeneous ... (5)

Putting $Y = vX$ so that $dY/dX = v + X\,(dv/dX)$, (5) becomes

$$v + X\frac{dv}{dX} = \frac{(X+vX)^2}{4X^2} \quad \text{or} \quad X\frac{dv}{dX} = \frac{(v+1)^2}{4} - v = \frac{(1-v)^2}{4}.$$

Separating the variables, $4\,(1 - v)^{-2}\,dv = (1/X)\,dX$.

Integrating, $4\,(1 - v)^{-1} = \log X + c$ or $4\,(1 - Y/X)^{-1} = \log X + c$.

or $$\frac{4X}{X-Y} = \log X + c \quad \text{or} \quad \frac{4\,(x-2)}{x-y-3} = \log\,(x-2) + c, \text{ by (4)}$$

Ex. 7. *Prove that* $1/(x + y + 1)^4$ *is an integrating factor of* $(2xy - y^2 - y)\,dx + (2xy - x^2 - x)\,dy = 0$, *and find the solution of this equation.*

Sol. Multiplying the given equation by $1/(x + y + 1)^4$, we have

$$(2xy - y^2 - y)\,(x + y + 1)^{-4}\,dx + (2xy - x^2 - x)\,(x + y + 1)^{-4}\,dy = 0. \quad ... (1)$$

Comparing (1) with $M\,dx + N\,dy = 0$, we have

$M = (2xy - y^2 - y)\,(x + y + 1)^{-4}$ and $N = (2xy - x^2 - x)\,(x + y + 1)^{-4}$

Now, $$\partial M/\partial y = (2x - 2y - 1)\,(x + y + 1)^{-4} - 4\,(2xy - y^2 - y)\,(x + y + 1)^{-5}$$
$$= (x + y + 1)^{-5}\,\{(2x - 2y - 1)\,(x + y + 1) - 4\,(2xy - y^2 - y)\}$$

$$\Rightarrow \quad \partial M/\partial y = (x + y + 1)^{-5}\,(2x^2 + 2y^2 - 8xy + x + y + 1) \quad ... (2)$$

and $$\partial N/\partial x = (2y - 2x - 1)\,(x + y + 1)^{-4} - 4\,(2xy - x^2 - x)\,(x + y + 1)^{-5}$$
$$= (x + y + 1)^{-5}\,\{(2y - 2x - 1)\,(x + y + 1) - 4\,(2xy - x^2 - x)\}$$

$$\Rightarrow \quad \partial N/\partial x = (x + y + 1)^{-5}\,(2x^2 + 2y^2 - 8xy + x + y - 1). \quad ... (3)$$

From (2) and (3), $\partial M/\partial y = \partial N/\partial x$ and so (1) is exact. Solution of (1) is

$$\underset{[\text{Treating } y \text{ as constant}]}{\int M\,dx} + \int (\text{terms free from } x \text{ in } N)\,dy = c$$

or $$\int \frac{2xy - y^2 - y}{(x+y+1)^4}\,dx = c \quad \text{or} \quad \int \frac{2y\,(x+y+1-y-1) - (y^2+y)}{(x+y+1)^4}\,dx = c$$

or $$\int (x+y+1)^{-4}\{2y\,(x+y+1) - 2y\,(y+1) - (y^2+y)\}\,dx = c$$

or $$2y\int (x+y+1)^{-3}\,dx - [2y\,(y+1) + y\,(y+1)]\int (x+y+1)^{-4}\,dx = c \quad ... (3)$$

Intergrating (3) w.r.t. x while treating y as constant, we get.

or $$\frac{y\,(y+1)}{(x+y+1)^3} - \frac{y}{(x+y+1)^2} = c \quad \text{or} \quad \frac{y\,(y+1) - y\,(x+y+1)}{(x+y+1)^3} = c$$

or $y^2 + y - xy - y^2 - y = c\,(x + y + 1)^3$ or $c\,(x + y + 1)^3 + xy = 0$.

Ex. 8(a). *Solve* $\dfrac{x\,dx + y\,dy}{x\,dy - y\,dx} = \sqrt{\left(\dfrac{a^2 - x^2 - y^2}{x^2 + y^2}\right)}$ **[Delhi Maths (H) 2009; I.A.S. 1999; Kumaun 1998; Garhwal 2010]**

Sol. We transform the given equation to polars, by taking $x = r\cos\theta$, $y = r\sin\theta$,

so that $x^2 + y^2 = r^2$ ... (1)

and $y/x = \tan\theta$. ... (2)

From (1), $2x\,dx + 2y\,dy = 2r\,dr$ or $x\,dx + y\,dy = r\,dr$ ... (3)

(2) $\Rightarrow$ $(x\,dy - y\,dx)\,x^2 = \sec^2\theta\,d\theta$ $\Rightarrow$ $x\,dy - y\,dx = r^2\,d\theta$. ... (4)

Using (1), (3) and (4) the given equation reduces to

$(r\,dr)/(r^2\,d\theta) = \{(a^2 - r^2)/r^2\}^{1/2}$ or $d\theta = \{1/(a^2 - r^2)^{1/2}\}\,dr$.

Integrating, $\theta + c = \sin^{-1}(r/a)$ or $\tan^{-1}(y/x) + c = \sin^{-1}\{(x^2 + y^2)^{1/2}/a\}$.

Ex. 9(a). *Show that the equation* $(4x + 3y + 1)\,dx + (3x + 2y + 1)\,dy = 0$ *represents a family of hyperbolas having as asymptotes the lines* $x + y = 0$ *and* $2x + y + 1 = 0$. **[I.A.S. 1998]**

Sol. Given $(4x + 3y + 1)\,dx + (3x + 2y + 1)\,dy = 0$. ... (1)

Comparing (1) with $M\,dx + N\,dy = 0$, $M = 4x + 3y + 1$ and $N = 3x + 2y + 1$.

Here $\partial M/\partial y = 3 = \partial N/\partial x$ and so (1) is exact and as usual, its solution is given by

$$\int (4x + 3y + 1)\,dx + \int (2y + 1)\,dy = c$$

[Treating y as constant]

$2x^2 + 3xy + x + y^2 + y + c = 0$, c being arbitrary is constant ... (2)

Comparing (2) with standard form of conic section

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we have, $a = 2$, $h = 3/2$, $b = 1$.

Here $h^2 - ab = (9/4) - 2 =$ positive quantity $\Rightarrow$ (2) is hyperbola.

Since the equation of the hyperbola and asymptotes differ by a constant, so the combined equations of two asymptotes of the hyperbola (2) may be taken as

$2x^2 + 3xy + y^2 + x + y + k = 0$, where k is some constant. ... (3)

Comparing (3) with standard equation of pair of lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we have $a = 2$, $h = 3/2$, $b = 1$, $g = 1/2$, $f = 1/2$, $c = k$.

Condition for (3) to represent two lines is $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

or $2k + 2 \cdot (1/2) \cdot (1/2) \cdot (3/2) - 2 \cdot (1/4) - 1 \cdot (1/4) - k \cdot (9/4) = 0$ $\Rightarrow$ $k = 0$.

Hence the required equation of the two asymptotes of (2) is

$2x^2 + 3xy + y^2 + x + y = 0$ or $(x + y)(2x + y + 1) = 0$,

showing that $x + y = 0$ and $2x + y + 1 = 0$ are the required asymptotes.

Ex. 5. *Primitive* $(2xy^4 e^y + 2xy^3 + y)\, dx + (x^2y^4 e^y - x^2y^2 - 3x)\, dy = 0$, is

(*a*) $x^2e^y + (x^2/y) + (x/y^3) = c$ (*b*) $x^2e^v - (x^2/y) + (x/y^3) = c$

(*c*) $x^2e^v + (x^2/y) - (x/y^3) = c$ (*d*) $x^2e^v - (x^2/y) - (x/y^3) = c$. **[I.A.S. (Prel.) 1994]**

Sol. Ans. (*a*). Dividing throughout by y^4, we have

$$\{2xe^y + (2x/y) + (1/y^3)\}\, dx + \{x^2e^y - (x^2/y^2) - (3x/y^4)\}\, dy = 0.$$

$$(2xe^y\, dx + x^2e^y\, dy) + \left(\frac{2x\, dx}{y} - \frac{x^2\, dy}{y^2}\right) + \left(\frac{dx}{y^3} - \frac{3x\, dy}{y^4}\right) = 0$$

or $$d\,(x^2e^x) + d\,(x^2/y) + d\,(x/y^3) = 0.$$

Integrating, $x^2e^x + (x^2/y) + (x/y^3) = c$, c being an arbitrary constants.

..2

Ex. 1. *Solve the ordinary differential equation $(\cos 3x) \times (dy/dx) - 3y \sin 3x = (1/2) \times \sin 6x + \sin^2 3x,\ 0 < x < \pi/2$* **[I.A.S. 2007]**

Sol. Re-writing the given equation, we have

$$(dy/dx) - (3 \tan 3x)y = \sec 3x\ \{(1/2) \times \sin 6x + \sin^2 3x\} \qquad \text{... (1)}$$

which is linear whose I.F. $= e^{\int(-3\tan 3x)dx} = e^{\log\cos 3x} = \cos 3x$ and hence its solution is

$$y \cos 3x = \int \cos 3x \sec 3x\, \{(1/2) \times \sin 6x + \sin^2 3x\}\, dx + c$$

or $$y \cos 3x = \int \{(1/2) \times \sin 6x + (1/2) \times (1 - \cos 6x)\}\, dx + c$$

or $$y \cos 3x = -(1/12) \times \cos 6x + x/2 - (1/12) \times \sin 3x + c$$

or $$y \cos 3x = (1/12) \times (6x - \cos 6x - \sin 6x) + c,\ c \text{ being an arbitrary constant}$$

Ex. 2. *Find the solution of the equation $(1/y)dy + xy^2dx = -4x\, dx$* **[I.A.S. 2007]**

Sol. Re-writing the given equation, we have

$$\frac{dy}{y} + x(4 + y^2)\, dx = 0 \qquad \text{or} \qquad \frac{dy}{y(y^2+4)} + xdx = 0$$

or $$\frac{1}{4}\left(\frac{1}{y} - \frac{y}{y^2+4}\right)dy + x\, dx = 0 \qquad \text{or} \qquad \left(\frac{2y}{y^2+4} - \frac{2}{y}\right)dy = 8x\, dx$$

Integrating, $\log(y^2 + 4) - 2\log y - \log c = 4x^2$, c being an arbitrary constant

or $\log\{(y^2 + 4)/cy^2\} = 4x^2$ or $y^2 + 4 = cy^2e^{4x^2}$

Ex. 3 (a). *A particle falls from rest in a medium whose resistance varies as the velocity. Find the relation between velocity (v) and the distance (x).* **[M.S. Univ. T.N. 2007]**

(b) *A particle falls from rest in a medium whose resistance varies as the velocity of the particle. Find the distance fallen by the particle and its velocity at time t.* **[I.A.S. 2007]**

Sol. Let a particle of mass m fall from rest under gravity from a fixed point O. Let P be the position of the particle at any time t such that $OP = x$. Let v be its velocity at P. Let kv be the force of resistance per unit mass so that mkv is resistance of the medium on the particle acting in vertical upward direction.Then, the equation of the particle at any time t is given by

$$m\ddot{x} = mg - mkv \qquad \text{or} \qquad \ddot{x} = g(1 - kv/g) \qquad \dots(1)$$

Let V be the terminal velocity of the particle so that $v = V$ when $\ddot{x} = 0$. Then, (1) yields $0 = g(1 - kV/g)$ giving $k = g/V$. Hence, (1) reduces to

$$\ddot{x} = g(1 - v/V) \qquad \text{or} \qquad \ddot{x} = (g/V) \times (V - v) \qquad \dots(2)$$

Part (a). Since $\ddot{x} = v(dv/dx)$, $\quad (2) \Rightarrow \qquad v(dv/dx) = (g/V) \times (V - v)$

$$\text{or} \qquad \frac{v}{V - v}\, du = \frac{g}{V}\, dx \qquad \text{or} \qquad \left(\frac{V}{V - v} - 1\right) dv = \frac{g}{V}\, dx$$

Integrating, $\quad -V \log(V - v) - v = gx/V + A$, A being an arbitrary constant $\qquad \dots(3)$

Initially at O, when $x = 0$, $v = 0$. Hence, (3) givens $\quad A = -V \log V$.

Thus, (3) becomes $\qquad -V \log(V - v) - v = gx/V - V \log V$

$$\text{or} \qquad \frac{gx}{V} = V \log \frac{V}{V - v} - v \qquad \text{or} \qquad x = \frac{V^2}{g} \log \frac{V}{V - v} - \frac{Vv}{g}$$

Part (b). To find the velocity of the particle at any time t

Since $\ddot{x} = dv/dt$, (2) reduces to $\qquad dv/dt = (g/V) \times (V - v)$

or $\qquad \{V/(V - v)\}dv = g\, dt$

Integrating, $\quad -V \log(V - v) = gt + B$, B being an arbitrary constant $\qquad \dots(4)$

Initially at O, when $t = 0$, $v = 0$. Hence, (4) givens $\quad B = -V \log V$

Hence (4) reduces to $\qquad -V \log(V - v) = gt - V \log V$

$\log\{(V - v)/V\} = -gt/V \qquad$ so that $\qquad v = V(1 - e^{-gt/V}) \qquad \dots(5)$

To find the distance fallen by the particle at any time *t*

Since $v = dx/dt$, (5) reduces to $\qquad dx/dt = V(1 - e^{-gt/V})$

so that $\qquad dx = V(1 - e^{-gt/V})dt$

Integrating, $\quad x = Vt + (V^2/g) \times e^{-gt/V} + C$, C being an arbitrary constant $\qquad \dots(6)$

Initially at O, when $t = 0$, $x = 0$. Hence (6) yields $\qquad C = -(V^2/g)$

Hence (6) reduces to $\qquad x = Vt - (V^2/g) \times (1 - e^{-gt/V})$.

Ex. 4. *Solve the following differential equations:*

(*i*) $\dfrac{dy}{dx} = \dfrac{\sin x + x\cos x}{y\,(2\log y + 1)}$ (*ii*) $\dfrac{dy}{dx} = \dfrac{x\,(2\log x + 1)}{\sin y + y\cos y}$. **[I.A.S. (Prel.) 2009]**

Sol. (*i*) Re-writing the given equation, $(\sin x + x\cos x)\,dx = (2y\log y + y)\,dy$.

Integrating, $$-\cos x + \int x\cos x\,dx = 2\int y\log y\,dy + (y^2/2) + c. \qquad \text{... (1)}$$

Now, $$\int x\cos x\,dx = x\sin x - \int \sin x\,dx, \text{ integrating by parts}$$

or $$\int x\cos x\,dx = x\sin x + \cos x. \qquad \text{... (2)}$$

Also, $$\int y\log y\,dy = (\log y)\times(y^2/2) - \int \{(1/y)\times(y^2/2)\}\,dy, \text{ integrating by parts}$$

or $$\int y\log y\,dy = (y^2/2)\times\log y - y^2/4 \qquad \text{... (3)}$$

Using (2) and (3), (1) reduces to

$$-\cos x + x\sin x + \cos x = 2\ \{(y^2/2)\times\log y - y^2/4\} + y^2/2 + c.$$

or $x\sin x = y^2\log y + c$, c being an arbitrary constant.

(*ii*) Proceed exactly as in part (*i*). **Ans.** $x^2\log x = y\sin y + c$.

Ex. 8. *Solve* $\sqrt{(1 + x^2 + y^2 + x^2y^2)} + xy\,(dy/dx) = 0.$

Sol. Re-writing the given differential equation, we have

$$\sqrt{[(1+x^2)(1+y^2)]} + xy\,(dy/dx) = 0 \quad \text{or} \quad \sqrt{(1+x^2)}\sqrt{(1+y^2)} + xy\,(dy/dx) = 0$$

or $$\frac{\sqrt{(1+x^2)}\,dx}{x} + \frac{y\,dy}{\sqrt{(1+y^2)}} = 0 \quad \text{or} \quad \frac{(1+x^2)\,dx}{x\sqrt{(1+x^2)}} + \frac{y\,dy}{\sqrt{(1+y^2)}} = 0.$$

Integrating, $$\int \frac{dx}{x(1+x^2)^{1/2}} + \int \frac{x\,dx}{(1+x^2)^{1/2}} + \int \frac{y\,dy}{(1+y^2)^{1/2}} = \text{C}. \qquad \text{... (1)}$$

Now, $$\int \frac{dx}{x(1+x^2)^{1/2}} = \int \frac{(-1/t^2)\,dt}{(1/t)\sqrt{1+(1/t)^2}}, \text{ putting } x = \frac{1}{t}$$

$$= -\int \frac{dt}{\sqrt{(t^2+1)}} = -\log\{t + \sqrt{t^2+1}\}$$

$$= -\log\left\{\frac{1}{x} + \sqrt{\left(\frac{1}{x^2}+1\right)}\right\} = -\log\left\{\frac{1+\sqrt{(1+x^2)}}{x}\right\}$$

$$= \log x - \log\{1 + (1+x^2)^{1/2}\} \qquad \text{... (2)}$$

Again, $$\int \frac{x\,dx}{(1+x^2)^{1/2}} = \int \frac{t\,dt}{2\sqrt{t}}, \text{ putting } 1 + x^2 = t$$

$$= \frac{1}{2}\int t^{-1/2}\,dt = t^{1/2} = (1+x^2)^{1/2}. \qquad \text{... (3)}$$

Similarly, $$\int \frac{y\,dy}{(1+y^2)^{1/2}} = (1+y^2)^{1/2}. \qquad \text{... (4)}$$

Using (2), (3) and (4), (1) gives the required solution as

$$\log x - \log\{1 + (1+x^2)^{1/2}\} + (1+x^2)^{1/2} + (1+y^2)^{1/2} = \text{C}.$$

Ex. 2. *Solve* $(x+y)^2 (dy/dx) = a^2$. **[Meerut 1997; Indore 1998; I.A.S. (Prel.) 1994; Delhi Maths (G) 1997; Ravishankar 1992]**

Sol. Let $x + y = v$. ... (1)

Differentiating, $1 + (dy/dx) = dv/dx$ or $dy/dx = dv/dx - 1$... (2)

Using (1) and (2), the given equation becomes

$$v^2 \left(\frac{dv}{dx} - 1\right) = a^2 \quad \text{or} \quad v^2 \frac{dv}{dx} = a^2 + v^2$$

or $$dx = \frac{v^2}{v^2 + a^2} dv \quad \text{or} \quad dx = \left[1 - \frac{a^2}{a^2 + v^2}\right] dv.$$

Integrating, $x + c = v - a^2 \times (1/a) \times \tan^{-1} (v/a)$, where c is arbitrary constant

or $$x + c = x + y - a \tan^{-1} \left(\frac{x+y}{a}\right) \quad \text{or} \quad y - a \tan^{-1} \left(\frac{x+y}{a}\right) = c.$$

Ex. 4. *Solve:* $x \cos (y/x) (y\,dx + x\,dy) = y \sin (y/x) (x\,dy - y\,dx)$ ... (1)

or $$\left(x \cos \frac{y}{x} + y \sin \frac{y}{x}\right) y - \left(y \sin \frac{y}{x} - x \cos \frac{y}{x}\right) x \frac{dy}{dx} = 0. \quad \text{... (2)}$$

[Mysore 2004; Kanpur 1996; Lucknow 1997]

Sol. Rewriting (1), we get (2). So (1) and (2) are the same equations.

From (2), $$\frac{dy}{dx} = \frac{\{x \cos (y/x) + y \sin (y/x)\}\, y}{\{y \sin (y/x) - x \cos (y/x)\}\, x}$$

or $$\frac{dy}{dx} = \frac{[\cos (y/x) + (y/x) \sin (y/x)]\,(y/x)}{[(y/x) \sin (y/x) - \cos (y/x)]} \quad \text{... (3)}$$

Take $y/x = v$, *i.e.*, $y = vx$, so that $dy/dx = v + x\,(dv/dx)$.... (4)

Using (4), (3) becomes $$v + x \frac{dv}{dx} = \frac{v (\cos v + v \sin v)}{v \sin v - \cos v}$$

or $$x \frac{dv}{dx} = \frac{v (\cos v + v \sin v)}{v \sin v - \cos v} - v = \frac{2v \cos v}{v \sin v - \cos v} \text{ or } 2 \frac{dx}{x} = \frac{v \sin v - \cos v}{v \cos v} dv = \left[\frac{\sin v}{\cos v} - \frac{1}{v}\right] dv.$$

Integrating, $2 \log x = -\log \cos v - \log v + \log c$, c being an arbitrary constant.

or $\log x^2 = \log (c/v \cos v)$ or $x^2 v \cos v = c$ or $xy \cos (y/x) = c$. $[\because\ v = y/x]$

Ex. 6. *Solve* $(2x^2 + 3y^2 - 7)\, x\, dx - (3x^2 + 2y^2 - 8)\, y\, dy = 0$ **[I.A.S. 1995]**

Sol. Given $(2x^2 + 3y^2 - 7)\, x\, dx - (3x^2 + 2y^2 - 8)\, y\, dy = 0$. ... (1)

Let $x^2 = u$ and $y^2 = v$ so that $2x\, dx = du$ and $2y\, dy = dv$.... (2)

From (1) and (2), $(2u + 3v - 7)\, du - (3u + 2v - 8)\, dv = 0$

or $dv/du = (2u + 3v - 7)/(3u + 2v - 8)$. ... (3)

Taking $u = U + h$, $v = V + k$ so that $dv/du = dV/dU$, ... (4)

the given equation becomes $\dfrac{dV}{dU} = \dfrac{2U + 3V + (2h + 2k - 7)}{3U + 2V + (3h + 2k - 8)}$. ... (5)

Choose h, k so that $2h + 3k - 7 = 0$ and $3h + 2k - 8 = 0$.... (6)

Solving (3), we get $h = 2$, $k = 1$ so that from (4), we have

$U = u - 2$ and $V = v - 1$.

or $U = x^2 - 2$ and $V = y^2 - 1$, by (2)... (7)

Then (5) becomes $\dfrac{dV}{dU} = \dfrac{2U + 3V}{3U + 2V} = \dfrac{2 + 3(V/U)}{3 + 2(V/U)}$. ... (8)

Take $V/U = w$, *i.e.*, $V = wU$ so that $dV/dU = w + U\,(dw/dU)$... (9)

From (8) and (9), $w + U\dfrac{dw}{dU} = \dfrac{2 + 3w}{3 + 2w}$ or $U\dfrac{dw}{dU} = \dfrac{2(1 - w^2)}{3 + 2w}$

or $$\frac{2dU}{U} = \frac{3 + 2w}{1 - w^2}\, dw = \left[\frac{3}{1 - w^2} - \frac{-2w}{1 - w^2}\right] dw.$$

Integrating, $2 \log U = \dfrac{3}{2} \log \dfrac{1 + w}{1 - w} - \log (1 - w^2) + \dfrac{1}{2} \log c$, c being an arbitrary constant

or $$4 \log U = 3 \log \left(\frac{1 + w}{1 - w}\right) - 2 \log (1 - w^2) + \log c$$

or $\log \dfrac{U^4}{c} = \log \left(\dfrac{1 + w}{1 - w}\right)^3 - \log (1 - w^2)^2$ or $\log \dfrac{U^4}{c} = \log \left[\left(\dfrac{1 + w}{1 - w}\right)^3 \cdot \dfrac{1}{(1 - w^2)^2}\right]$

or $\dfrac{U^4}{c} = \dfrac{(1 + w)^3}{(1 - w)^5 (1 + w)^2}$ or $(1 - w)^5\, U^4 = c\,(1 + w)$

or $\left(1 - \dfrac{V}{U}\right)^5 U^4 = c\left(1 + \dfrac{V}{U}\right)$ or $(U - V)^5 = C\,(U + V)$

or $(x^2 - y^2 - 1)^5 = c\,(x^2 + y^2 - 3)$, by (7).

Ex. 2. *Test whether the equation $(x + y)^2\, dx - (y^2 - 2xy - x^2)\, dy = 0$ is exact and hence solve it.* **[I.A.S. 1995]**

Sol. The given equation can be re-written as $(x^2 + 2xy + y^2)\, dx + (x^2 + 2xy - y^2)\, dy = 0$ ..(1)

Comparing (1) with $M\,dx + N\,dy = 0$, *here* $M = x^2 + 2xy + y^2$, $N = x^2 + 2xy - y^2$.

$\therefore$ $\partial M/\partial y = 2x + 2y$ and $\partial N/\partial x = 2x + 2y$ so that $\partial M/\partial y = \partial N/\partial x$.

Hence (1) is exact and hence its solution is

$$\underset{\text{[Treating } y \text{ as constant]}}{\int M\, dx} + \int (\text{terms in } N \text{ not containing } x)\, dy = c'$$

or
$$\underset{\text{[Treating } y \text{ as constant]}}{\int (x^2 + 2xy + y^2)\, dx} + \int (-y^2)\, dy = c'$$

or $\quad x^3/3 + 2y \times (x^2/2) + y^2x - y^3/3 = c/3$, taking $c' = c/3$

or $\quad x^3 + y^3 + 3xy\,(x + y) = c$, c being an arbitrary constant.

Ex. 4. *Solve $(1 + e^{x/y})\, dx + e^{x/y}\, \{1 - (x/y)\}\, dy = 0$.* **[I.A.S. Prel. 2007;Osmania 2005]**

Sol. Comparing the given equation with $M\,dx + N\,dy = 0$, $M = 1 + e^{x/y}$, $N = e^{x/y}\,\{1 - (x/y)\}$.

$\therefore$ $\quad \partial M/\partial y = e^{x/y}\,(-x/y^2)$, $\quad \partial N/\partial x = e^{x/y}\,(-1/y) + (1 - x/y)\, e^{x/y}\,(1/y) = (-x/y^2)\, e^{x/y}$

Thus, $\partial M/\partial y = \partial N/\partial x$ and so the given equation is exact.

Its solution is
$$\underset{\text{[Treating } y \text{ as constant]}}{\int M\, dx} + \int (\text{terms in } N \text{ not containing } x)\, dy = c$$

or
$$\underset{\text{[Treating } y \text{ as constant]}}{\int (1 + e^{x/y})\, dx} = c \qquad \text{or} \qquad x + ye^{x/y} = c.$$

Ex. 7. *Solve $\{y\,(1 + 1/x) + \cos y\}\, dx + (x + \log x - x \sin y)\, dy = 0$*

[Delhi Maths. (G) 1993; I.A.S. 1993; Osmania 2005]

Sol. Comparing the given equation with $M\,dx + N\,dy = 0$, here

$M = y\,(1 + 1/x) + \cos y$ and $N = x + \log x - x \sin y$

$\therefore$ $\quad \partial M/\partial y = 1 + (1/x) - \sin y = \partial N/\partial x$

Hence, the given equation is exact and so its solution is

$$\underset{\text{[Treating } y \text{ as constant]}}{\int M\, dx} + \int (\text{terms in } N \text{ not containing } x)\, dy = c$$

or
$$\underset{\text{[Treating } y \text{ as constant]}}{\int (y + y/x + \cos y)\, dx} + 0 = c \qquad \text{or} \qquad yx + y \log x + x \cos y = c$$

Ex. 10(*a*). *Solve $y \sin 2x\, dx - (1 + y^2 + \cos^2 x)\, dy = 0$.* **[I.A.S. 1996; Lucknow 1994]**

Sol. Re-writing the given equation, $\quad y \sin 2x\, dx - \{1 + y^2 + \tfrac{1}{2}(1 + \cos 2x)\}\, dy = 0$. ... (1)

Comparing (1) with $M\,dx + N\,dy = 0$, $M = y \sin 2x$, $N = -(3/2) - y^2 - (1/2) \cos 2x$,

$\therefore$ $\partial M/\partial y = 2 \cos 2x = \partial N/\partial x$. Hence (1) is exact and its solution is

$$\underset{\text{[Treating } y \text{ as constant]}}{\int M\, dx} + \int (\text{terms in } N \text{ not containing } x)\, dy = c'$$

or
$$\underset{\text{[Treating } y \text{ as constant]}}{\int y \sin 2x\, dx} + \int \{(-3/2) - y^2\}\, dy = c'$$

or $\quad y \times (-1/2) \times \cos 2x - (3/2) \times y - y^3/3 = -c/6$, taking $c' = -c/6$

$\therefore$ Required solution is $\quad 3y \cos 2x + 9y + 2y^3 = c$, c being an arbitrary constant.

Ex. 12. *Show $(4x + 3y + 1)\,dx + (3x + 2y + 1)\,dy = 0$ is a family of hyperbolas with a common axis and tangent at the vertex.* **[I.A.S. 2000]**

Sol. Given $\qquad (4x + 3y + 1)\,dx + (3x + 2y + 1)\,dy = 0 \qquad$ ... (1)

Comparing (1) with $M\,dx + N\,dy = 0$ here, $\quad M = 4x + 3y + 1, \quad N = 3x + 2y + 1.$

Here $\partial M/\partial y = 3 = \partial N/\partial x$ and so (1) is exact. Its solution is

$$\underset{\text{[Treating } y \text{ as constant]}}{\int (4x + 3y + 1)\,dx} + \underset{\text{[Integrating terms free from } x\text{]}}{\int (3x + 2y + 1)\,dy} = 0$$

or $\qquad 2x^2 + 3xy + x + y^2 + y + k = 0$, where k is an arbitrary constant. $\qquad$ ... (2)

Comparing (2) with standard form of conic section $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, here

$a = 2, \quad b = 1, \quad h = 3/2, \quad g = 1/2, \quad f = 1/2, \quad c = k \qquad$ ... (3)

Then $\qquad h^2 - ab = (9/4) - 2 =$ positive quantity,

showing that (2) represents *a* family of hyperbolas, k being the parameter, with common axis and tangent at vertex.

Ex. 13. *Find the values of constant λ such that $(2xe^y + 3y^2)\,(dy/dx) + (3x^2 + \lambda e^y) = 0$ is exact. Further, for this value of λ, solve the equation.* **[I.A.S. 2002]**

Sol. Re-writing the given equation, $\qquad (3x^2 + \lambda e^y)\,dx + (2xe^y + 3y^2)\,dy = 0 \quad$ .. (1)

Comparing (1) with $M\,dx + N\,dy = 0$, here $\quad M = 3x^2 + \lambda e^y \quad$ and $\quad N = 2xe^y + 3y^2$.

Now, for (1) to be exact we must have

$\partial M/\partial y = \partial N/\partial x \qquad$ so that $\qquad \lambda e^y = 2e^y \qquad$ giving $\qquad \lambda = 2.$

$\therefore$ (1) becomes $\qquad (3x^2 + 2e^y)\,dx + (2xe^y + 3y^2)\,dy = 0 \qquad$ ... (3)

Equation (3) in exact and hence its solution is its solution is

$$\underset{\text{[Treating } y \text{ as constant]}}{\int M\,dx} + \int (\text{terms in N not containing } x)\,dy = c$$

or $\qquad \int (3x^2 + 2e^x)\,dx + \int (3y^2)\,dy = c \qquad$ or $\qquad x^3 + 2e^x + y^3 = c$

15. $\quad (1 + xy)\,y\,dx + x\,(1 - xy)\,dy = 0.$

[Calcutta 1995; I.A.S. 1994; Meerut 1993; Kanpur 1994; Ravishankar 1996; G.N.D.U. Amritsar 2010]

[Hint. Re-writing, the given equation is $\qquad y\,dx + x\,dy + xy\,(y\,dx - x\,dy) = 0$

or $\qquad d\,(xy) + x^2y^2\left(\dfrac{dx}{x} - \dfrac{dy}{y}\right) = 0 \qquad$ or $\qquad d\,(xy) + x^2y^2\,d\left(\log\dfrac{x}{y}\right) = 0$

or $\qquad \dfrac{1}{x^2y^2}\,d\,(xy) + d\left(\log\dfrac{x}{y}\right) = 0 \qquad$ or $\qquad d\left(\log\dfrac{x}{y} - \dfrac{1}{xy}\right) = 0,$

Integrating, $\qquad \log\,(x/y) - 1/(xy) = c$, where c is an arbitrary constant.

Rule II. *If the given equation $M\,dx + N\,dy = 0$ is homogeneous and $(Mx + Ny) \neq 0$, then $1/(Mx + Ny)$ is an integrating factor.*

Proof. Re-writing $M\,dx + N\,dy$, we have

$$M\,dx + N\,dy = \frac{1}{2}\left\{(Mx+Ny)\left(\frac{dx}{x}+\frac{dy}{y}\right)+(Mx-Ny)\left(\frac{dx}{x}-\frac{dy}{y}\right)\right\}$$

$$\Rightarrow \qquad \frac{M\,dx+N\,dy}{Mx+Ny} = \frac{1}{2}\left\{\left(\frac{dx}{x}+\frac{dy}{y}\right)+\frac{Mx-Ny}{Mx+Ny}\left(\frac{dx}{x}-\frac{dy}{y}\right)\right\} \qquad \ldots (1)$$

Since $M\,dx + N\,dy = 0$ is a homogeneous equation, M and N must be of the same degree in variables x and y and hence we may write

$$\frac{Mx-Ny}{Mx+Ny} = \text{some function of } \frac{x}{y} = f\left(\frac{x}{y}\right), \text{ say} \qquad \ldots (2)$$

Using (2), (1) reduces to

$$\frac{M\,dx+N\,dy}{Mx+Ny} = \frac{1}{2}\left\{\left(\frac{dx}{x}+\frac{dy}{y}\right)+f\left(\frac{x}{y}\right)\left(\frac{dx}{x}-\frac{dy}{y}\right)\right\}$$

$$=\frac{1}{2}\left\{d(\log xy)+f(e^{\log(x/y)})d\left(\log\frac{x}{y}\right)\right\}=\frac{1}{2}\left\{d(\log xy)+g\left(\log\frac{x}{y}\right)d\left(\log\frac{x}{y}\right)\right\}$$

[on assuming $f(e^{\log(x/y)}) = g\{\log(x/y)\}$]

$$= d\left[(1/2)\times\log xy+(1/2)\times\int g\{\log(x/y)\}\,d\{\log(x/y)\}\right]$$

showing that $1/(Mx+Ny)$ is an I.F. for the given equation $M\,dx + N\,dy = 0$.

Rule III. *If the equation $M\,dx + N\,dy = 0$ is of the form $f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0$, then $1/(Mx - Ny)$ is an integrating factor of $M\,dx + N\,dy = 0$ provided $(Mx - Ny) \neq 0$.* **[I.A.S. 1991]**

Proof. Suppose that $\qquad M\,dx + N\,dy = 0 \qquad \ldots (1)$

is of the form $\qquad f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0. \qquad \ldots (2)$

Comparing (1) and (2), we have

$$\frac{M}{y\,f_1(xy))} = \frac{N}{x\,f_2(xy)} = \mu \text{ (say)}$$

$$\Rightarrow \qquad M = \mu\,y\,f_1(xy) \qquad \text{and} \qquad N = \mu\,x\,f_2(xy). \qquad \ldots (3)$$

Re-writing $M\,dx + N\,dy$, we have

$$M\,dx + N\,dy = \frac{1}{2}\left\{(Mx+Ny)\left(\frac{dx}{x}+\frac{dy}{y}\right)+(Mx-Ny)\left(\frac{dx}{x}-\frac{dy}{y}\right)\right\}$$

$$\Rightarrow \frac{M\,dx+N\,dy}{Mx-Ny} = \frac{1}{2}\left\{\frac{Mx+Ny}{Mx-Ny}\left(\frac{dx}{x}+\frac{dy}{y}\right)+\left(\frac{dx}{x}-\frac{dy}{y}\right)\right\}$$

$$=\frac{1}{2}\left\{\frac{f_1(xy)+f_2(xy)}{f_1(xy)-f_2(xy)}\,d(\log xy)+d\left(\log\frac{x}{y}\right)\right\}, \text{ using (3)}$$

$$= \frac{1}{2}\left\{f(xy)\,d(\log xy)+d\left(\log\frac{x}{y}\right)\right\}, \quad \text{where } \frac{f_1(xy)+f_2(xy)}{f_1(xy)-f_2(xy)} = f(xy)$$

$$= \frac{1}{2}\left\{f(e^{\log xy})\,d(\log xy)+d\left(\log\frac{x}{y}\right)\right\}=\frac{1}{2}\left\{g(\log xy)\,d(\log xy)+d\log\left(\frac{x}{y}\right)\right\}$$

[on assuming that $f(e^{\log xy}) = g(\log xy)$]

$$= d\left\{(1/2)\times\log(x/y)+(1/2)\times\int g(\log xy)\,d(\log xy)\right\},$$

showing that $Mx - Ny$ is an I.F. of $M\,dx + N\,dy = 0$.

Ex. 2. *Solve* $y\,(1 + xy)\,dx + x\,(1 - xy)\,dy = 0.$ **[I.A.S. (Prel.) 2006; Meerut 1993; G.N.D.U. Amritsar 2010]**

Sol. Given $$(1 + xy)\,y\,dx + (1 - xy)\,x\,dy = 0. \qquad \dots (1)$$

Comparing (1) with $M\,dx + N\,dy = 0$, $M = (1 + xy)\,y$ and $N = (1 - xy)\,x$, showing that (1) is of the form $f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0$.

Again, $$Mx - Ny = xy\,(1 + xy) - xy\,(1 - xy) = 2x^2y^2 \neq 0,$$
showing that I.F. of (1) $= 1/(Mx - Ny) = 1/(2x^2y^2)$.

On multiplying (1) by $1/(2x^2y^2)$, we have $$\frac{1}{2}\left(\frac{1}{x^2y} + \frac{1}{x}\right)dx + \frac{1}{2}\left(\frac{1}{xy^2} - \frac{1}{y}\right)dy = 0,$$

which must be exact and so by the usual rule, solution of (2) is
$$\int\left(\frac{1}{2x^2y} + \frac{1}{2x}\right)dx + \int\left(-\frac{1}{2y}\right)dy = \frac{1}{2}\log c \qquad \text{or} \qquad \frac{1}{-2xy} + \frac{1}{2}\log x - \frac{1}{2}\log y = \frac{1}{2}\log c$$
[Treating y as constant]

or $\log(x/y) - \log c = 1/(xy)$ or $\log(x/cy) = 1/(xy)$ or $x = cy\,e^{1/(xy)}$.

Rule IV. *If* $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x}\right)$ *is a function x alone say* $f(x)$, *then* $e^{\int f(x)dx}$ *is an integrating factor of* $M\,dx + N\,dy = 0$. **[I.A.S. 1977, 94]**

Proof. Given equation is $$M\,dx + N\,dy = 0 \qquad \dots (1)$$
and $$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x) \qquad \text{so that} \qquad N f(x) = \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \qquad \dots (2)$$

Multiplying both sides of (1) by $e^{\int f(x)dx}$, we have $$M_1\,dx + N_1\,dy = 0, \qquad \dots (3)$$
where $$M_1 = M e^{\int f(x)dx} \qquad \text{and} \qquad N_1 = N e^{\int f(x)dx} \qquad \dots (4)$$

From (4), $$\frac{\partial M_1}{\partial y} = \frac{\partial M}{\partial y} e^{\int f(x)dx} \qquad \dots (5)$$

and $$\frac{\partial N_1}{\partial x} = \frac{\partial N}{\partial x} e^{\int f(x)dx} + N e^{\int f(x)dx} f(x) = e^{\int f(x)dx}\left\{\frac{\partial N}{\partial x} + N\,f(x)\right\}$$
$$= e^{\int f(x)dx}\,(\partial N/\partial x + \partial M/\partial y - \partial N/\partial x), \text{ by (2)}$$

so that $$\frac{\partial N_1}{\partial x} = e^{\int f(x)dx}\frac{\partial M}{\partial y}. \qquad \dots (7)$$

$\therefore$ From (6) and (7), $$\partial M_1/\partial dy = \partial N_1/\partial x,$$

showing the $M_1 dx + N_1 dy = 0$ must be exact and hence $e^{\int f(x)dx}$ is its I.F.

Ex. 1. *Solve* $(2xy^4e^y + 2xy^3 + y)\, dx + (x^2y^4\, e^y - x^2y^2 - 3x)\, dy = 0.$ ... (1)

Sol. Comparing (1) with $M\,dx + N\,dy = 0$, we get

$M = 2xy^4e^y + 2xy^3 + y$ and $N = x^2y^4e^y - x^2y^2 - 3x$. ... (2)

Here $\partial M/\partial y = 8xy^3e^y + 2xy^4e^y + 6xy^2 + 1$ and $\partial N/\partial x = 2xy^4e^y - 2xy^2 - 3$.

$$\therefore \quad \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -4(2xy^3e^y + 2xy^2 + 1) = -\frac{4}{y}(2xy^4e^y + 2xy^3 + y) = -\frac{4M}{y}$$

$$\Rightarrow \quad \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = -\frac{4}{y}, \text{ which is a function of } y \text{ alone.}$$

$$\Rightarrow \quad \text{I.F. of (1)} = e^{\int(-4/y)\,dy} = e^{-4\log y} = (1/y^4).$$

Multiplying (1) by $1/y^4$, we have

$\{2xe^y + (2x/y) + (1/y^3)\}\, dx + \{x^2e^y - (x^2/y^2) - 3\,(x/y^4)\}\, dy = 0$ whose solution as usual is

$\int \{2xe^y + (2x/y) + (1/y^3)\}\, dx = c$ or $x^2e^y + (x^2/y) + (x/y^3) = c$.

[Treating y as constant]

Ex. 2. (a) *Solve* $(1 - x^2)\,(dy/dx) + 2xy = x\sqrt{(1 - x^2)}$. **[Kerala 2001]**

(b) solve $(1 - x^2)(dy/dx) + 2xy = x\sqrt{1 - x^2}$, $y(0) = 1$ **[Delhi Maths (Prog) 2007]**

Sol. The given equation is $\dfrac{dy}{dx} + \dfrac{2x}{1 - x^2}\,y = \dfrac{x}{(1 - x^2)^{1/2}}$. ... (1)

Comparing (1) with $dy/dx + Py = Q$, here $P = 2x/(1 - x^2)$

Here $\int P\,dx = \int \dfrac{2x}{1 - x^2}\,dx = -\log(1 - x^2)$ hence I.F. of (1) $= e^{\int P\,dx} = \dfrac{1}{1 - x^2}$

So the required solution is

$$\frac{y}{1 - x^2} = \int \frac{x}{\sqrt{(1 - x^2)}} \times \frac{1}{1 - x^2}\, dx = -\frac{1}{2}\int t^{-3/2}\, dt + c, \text{ putting } 1 - x^2 = t \text{ and } -2x\,dx = dt$$

or $\dfrac{y}{1 - x^2} = t^{-1/2} + c = c + \dfrac{1}{\sqrt{t}}$ or $\dfrac{y}{1 - x^2} = \dfrac{1}{(1 - x^2)^{1/2}} + c$, as $t = 1 - x^2$ ... (2)

(b) First do upto equation (2) as in Ex. 2(a). Putting $x = 0$ and $y = 1$ in (2), we have $1 = 1 + c$ so that $c = 0$. Hence (2) becomes

$y/(1 - x^2) = 1/(1 - x^2)^{1/2}$ or $y = (1 - x^2)^{1/2}$

Ex. 6. (*a*) *Solve* $(1 + y^2)\, dx = (\tan^{-1} y - x)\, dy$. **[Delhi Maths 2007]**

[Agra 2005; Delhi Maths(G) 2004; Lucknow 1996; Calicut 2004; Utkal 2003]

Sol. Re-writing the given equaion, $$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1} y}{1+y^2}. \qquad \text{... (1)}$$

which is of the form dx/dy + $P_1 x = Q_1$. Comparing it with (1) here $P_1 = 1/(1 + x^2)$

$$\therefore \int P_1\, dy = \int \frac{1}{1+x^2}\, dy = \tan^{-1} y \quad \text{and hence I.F. of (1)} = e^{\int P_1 dy} = e^{\tan^{-1} y}.$$

Hence the required solution is $$xe^{\tan^{-1} y} = \int e^{\tan^{-1} y} \cdot \frac{\tan^{-1} y}{1+y^2}\, dy = c.$$

or $$xe^{\tan^{-1} y} = \int e^t \cdot t\, dt + c, \text{ putting } \tan^{-1} y = t \text{ and } (dy)/(1+y) = dt$$

or $$xe^{\tan^{-1} y} = te^t - e^t + c \qquad \text{or} \qquad xe^{\tan^{-1} y} = e^{\tan^{-1} y} (\tan^{-1} y - 1) + c$$

or $$x = \tan^{-1} y - 1 + ce^{\tan^{-1} y},$$ c being an arbitrary constant.

Ex. 6. (*b*) Solve $(1 + y^2) + (x - e^{-\tan^{-1} y})(dy/dx) = 0$. **[I.A.S. 2006]**

Sol. Re-writing the given equation, we have

$$\frac{dx}{dy} + \frac{x - e^{-\tan^{-1} y}}{1+y^2} = 0 \qquad \text{or} \qquad \frac{dx}{dy} + \frac{1}{1+y^2}\, x = \frac{e^{-\tan^{-1} y}}{1+y^2} \qquad \text{... (1)}$$

Its I.F. $= e^{\int \{1/(1+y^2)\} dy} = e^{\tan^{-1} y}$ and so its solution is

$$xe^{\tan^{-1} y} = \int \left(e^{\tan^{-1} y} \times \frac{e^{-\tan^{-1} y}}{1+y^2} \right) dy + c \qquad \text{or} \qquad xe^{\tan^{-1} y} = \tan^{-1} y + c \qquad \text{... (2)}$$

Ex. 7. *Solve* $\dfrac{dy}{dx} + \dfrac{y}{(1-x^2)^{3/2}} = \dfrac{x + (1-x^2)^{1/2}}{(1-x^2)^2}$. **[I.A.S. (Prel.) 2005]**

Sol. Comparing the given equation with $(dy/dx) + Py = Q$, here

$$P = \frac{1}{(1-x^2)^{3/2}} \qquad \text{and} \qquad Q = \frac{x + (1-x^2)^{1/2}}{(1-x^2)^2} \qquad \text{... (1)}$$

Hence, $$\int P\, dx = \int \frac{1}{(1-x^2)^{3/2}}\, dx = \int \frac{\cos\theta\, d\theta}{\cos^3\theta}, \text{ putting } x = \sin\theta$$

$$= \int \sec^2\theta\, d\theta = \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{x}{(1-x^2)^{1/2}}.$$

Hence, $$\text{I.F. of (1)} = e^{\int P\, dx} = e^{x/(1-x^2)^{1/2}} \qquad \text{... (2)}$$

Solution of the given differential equation is $$y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\, dx + c. \qquad \text{... (3)}$$

Now, $$\int Q\,(\text{I.F.})\, dx = \int \frac{x + (1-x^2)^{1/2}}{(1-x^2)^2}\, e^{x/(1-x^2)^{1/2}}\, dx \qquad \text{... (4)}$$

Put $x/(1-x^2)^{1/2} = t.$ ... (5)

From (5), $$\frac{(1-x^2)^{1/2}\cdot 1 - x\,(1/2)\,(1-x^2)^{-1/2}\cdot(-2x)}{1-x^2}\,dx = dt$$

or $$\frac{(1-x^2)^{1/2} + [x^2/(1-x^2)^{1/2}]}{1-x^2}\,dx = dt \qquad \text{or} \qquad \frac{1}{(1-x^2)^{3/2}}\,dx = dt. \quad \text{... (6)}$$

Re-writing (4), we have

$$\int Q\,(\text{I.F.})\,dx = \int \frac{[x/(1-x^2)^{1/2}]+1}{(1-x^2)^{3/2}}\,e^{x/(1-x^2)^{1/2}}\,dx = \int (t+1)\,e^t\,dt, \text{ using (5) and (6)}$$

$$\therefore \quad \int Q\,(\text{I.F.})\,dx = (t+1)\,e^t - \int e^t\,dt = (t+1)\,e^t - e^t = te^t = \left\{x/(1-x^2)^{1/2}\right\}\times e^{x/(1-x^2)^{1/2}} \quad \text{...(7)}$$

Using (2) and (7) in (3), the required solution is

$$ye^{x/(1-x^2)^{1/2}} = \frac{x}{(1-x^2)^{1/2}}\,e^{x/(1-x^2)^{1/2}} + c \qquad \text{or} \qquad y = \frac{x}{(1-x^2)^{1/2}} + c\,e^{-x/(1-x^2)^{1/2}}$$

Ex. 8. *Solve* $x\,(1-x^2)\,dy + (2x^2y - y - ax^3)\,dx = 0.$

Sol. Re-writing the given equation, we have

$$x\,(1-x^2)\,\frac{dy}{dx} + y\,(2x^2-1) = ax^3 \qquad \text{or} \qquad \frac{dy}{dx} + \frac{2x^2-1}{x\,(1-x^2)}\,y = \frac{ax^2}{1-x^2}. \quad \text{... (1)}$$

Comparing (1) with $(dy/dx) + Py = Q$, we have

$$P = \frac{2x^2-1}{x\,(1-x^2)} = -\frac{1}{x} - \frac{1}{2\,(x+1)} - \frac{1}{2\,(x-1)} \qquad \text{and} \qquad Q = \frac{ax^2}{1-x^2} \quad \text{... (2)}$$

$$\int P\,dx = -\int\left[\frac{1}{x} + \frac{1}{2\,(x+1)} + \frac{1}{2\,(x-1)}\right]dx = -[\log x + \frac{1}{2}\times\log(x+1) + \frac{1}{2}\times\log(x-1)]$$

$$= -\left[\log x + (1/2)\times\log(x^2-1)\right] = -\log[x(x^2-1)^{1/2}] = \log[x(x^2-1)^{1/2}]^{-1}$$

$$\therefore \text{ Integrating factor } = e^{\int P\,dx} = e^{\log\{x(x^2-1)^{1/2}\}^{-1}} = \{x(x^2-1)^{1/2}\}^{-1} = 1/\{x(x^2-1)^{1/2}\}$$

Solution of (1) is $y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\,dx + c$, c being an arbitrary constant

or $$\frac{y}{x(x^2-1)^{1/2}} = \int \frac{ax^2}{1-x^2}\times\frac{1}{x(x^2-1)^{1/2}}\,dx + c = c - a\int\frac{x\,dx}{(x^2-1)^{3/2}}$$

$$\frac{y}{x(x^2-1)^{1/2}} = c - \frac{a}{2}\int\frac{dt}{t^{3/2}}, \text{ putting } x^2-1 = t \text{ and } 2x\,dx = dt$$

or $$\frac{y}{x(x^2-1)^{1/2}} = c - \frac{a}{2}\left[\frac{t^{-1/2}}{-(1/2)}\right] = c + \frac{a}{\sqrt{t}} = c + \frac{a}{(x^2-1)^{1/2}} \qquad \text{or} \qquad y = ax + cx\,(x^2-1)^{1/2}.$$

Ex. 11. *Solve dy/dx + y cos x = (1/2) × sin 2x* **[I.A.S. 2004]**

Sol. Integrating factor of the given equation = $e^{\int \cos x\, dx} = e^{\sin x}$ and solution is

$$ye^{\sin x} = c + \int (1/2) \times \left(\sin 2x\, e^{\sin x}\right) dx = c + \int \sin x\, e^{\sin x} \cos x\, dx$$

$$= c + \int t\, e^t\, dt, \text{ on putting } \sin x = t \text{ and } \cos x\, dx = dt,$$

$$= c + t\, e^t - \int e^t\, dt = c + e^t\,(t - 1)$$

or $\quad ye^{\sin x} = c + e^{\sin x}(\sin x - 1) \quad$ or $\quad y = ce^{-\sin x} + \sin x - 1.$

Ex. 1. *Solve $(dy/dx) + x \sin 2y = x^3 \cos^2 y$.*

[I.A.S. (Prel.) 2005; I.A.S. 1994; Calcutta 1995; Kanpur 1997; Lucknow 1996]

Sol. Dividing by $\cos^2 y$, $\qquad \sec^2 y\,(dy/dx) + 2x\,(\tan y) = x^3. \qquad$ ... (1)

Put $\tan y = v$ so that $\sec^2 y\,(dy/dx) = dv/dx$ Hence the above eqn. becomes $dv/dx + 2xv = x^3$, which is linear in v and x. Hence its I.F. $= e^{\int 2x\, dx} = e^{x^2}$ and its solution is given by

$$v \cdot e^{x^2} = \int x^3\, e^{x^2}\, dx + c, \; c \text{ being an arbitrary constant}$$

$$ve^{x^2} = (1/2) \times \int t\, e^t\, dt + c, \text{ putting } x^2 = t \text{ and } 2x\, dx = dt$$

$$= (1/2) \times [t \times e^t - \int (1 \times e^t)\, dt] + c = (1/2) \times (t\, e^t - e^t) + c$$

or $\quad \tan y \cdot e^{x^2} = (1/2) \times e^{x^2}\,(x^2 - 1) + c, \text{ as } \quad v = \tan y \quad$ and $\quad t = x^2$

or $\quad \tan y = (1/2) \times (x^2 - 1) + ce^{-x^2}$, dividing by e^{x^2}

Ex. 3. *Solve* $\dfrac{dz}{dx} + \dfrac{z}{x} \log z = \dfrac{z}{x^2} \cdot (\log z)^2$. **[I.A.S. 2001; Calcutta 1994]**

Sol. Here we have z in place of y and so the method of solution will remain similar. Dividing by $z\,(\log z)^2$, we get

$$\frac{1}{z\,(\log z)^2}\frac{dz}{dx} + \frac{1}{x}\frac{1}{(\log z)} = \frac{1}{x^2}. \qquad \text{... (1)}$$

Putting $\dfrac{1}{\log z} = v \quad$ so that $\quad \dfrac{(-1)}{(\log z)^2\, z}\dfrac{dz}{dx} = \dfrac{dv}{dx}, \quad$ (1) becomes

$$-\frac{dv}{dx} + \frac{1}{x}v = \frac{1}{x^2} \qquad \text{or} \qquad \frac{dv}{dx} - \frac{1}{x}v = -\frac{1}{x^2}, \qquad \text{... (2)}$$

whose I.F. $= e^{-\int (1/x)\, dx} = e^{-\log x} = 1/x$ and so solution is

$$\frac{v}{x} = \int \left(-\frac{1}{x^3}\right) dx + c = \frac{1}{2x^2} + c \qquad \text{or} \qquad \frac{1}{x\,(\log z)} = \frac{1}{2x^2} + c.$$

Ex. 5. *Solve* $(x^2 - 2x + 2y^2)\,dx + 2xy\,dy = 0.$ **[I.A.S. 1991]**

Sol. Re-writing the given equation, we have

$$2xy\frac{dy}{dx} + x^2 - 2x + 2y^2 = 0 \qquad \text{or} \qquad 2y\frac{dy}{dx} + \frac{x^2 - 2x}{x} + \frac{2y^2}{x} = 0$$

or $$2y\frac{dy}{dx} + \frac{2}{x}y^2 = \frac{2x - x^2}{x}. \qquad \dots (1)$$

Putting $y^2 = v$ so that $2y\,(dy/dx) = dv/dx$ ... (2)

Using (2), (1) gives $$\frac{dv}{dx} + \frac{2}{x}v = \frac{2x - x^2}{x} \qquad \dots (3)$$

Comparing (3) with $(dv/dx) + Pv = Q$, we have $P = 2/x$ and $Q = (2x - x^2)/x$ ...(4)

$\therefore$ Since $\int P\,dx = \int (2/x)\,dx = 2\log x = \log x^2$, hence I.F. of (3) $= e^{\int P\,dx} = e^{\log x^2} = x^2$.

and solution of (3) is $y\,(\text{I.F.}) = \int Q \cdot (\text{I.F.})\,dx + c$, c being an arbitrary constant.

or $$y^2x^2 = \int \left(\frac{2x - x^2}{x}\right) x^2\,dx = \int (2x^2 - x^3)\,dx + c \qquad \text{or} \qquad y^2x^2 = \frac{2x^3}{3} - \frac{x^4}{4} + c$$

Ex. 9. Solve $(xy^2 + e^{-1/x^3})\,dx - x^2y\,dy = 0.$ **[I.A.S. 2006]**

Sol. Re-writing the given equation, we have

$$x^2y\frac{dy}{dx} = xy^2 + e^{-1/x^3} \qquad \text{or} \qquad 2y\frac{dy}{dx} - \frac{2}{x}y^2 = \frac{2}{x^2}e^{-1/x^3} \qquad \dots (1)$$

Putting $y^2 = v$ and $2y\,(dy/dx) = dv/dx$, (1) reduces to

$$dv/dx - (2/x)\,v = (2/x^2)\,e^{-1/x^3}, \text{ which is linear equation} \qquad \dots (2)$$

It I.F. $= e^{\int(-2/x)\,dx} = e^{-2\log x} = e^{\log x^{-2}} = x^{-2}$ and solution is

$$vx^{-2} = \int (x^{-2}) \times (2x^{-2}e^{-x^{-3}})\,dx + c \qquad \text{or} \qquad vx^{-2} = 2\int x^{-4}e^{-x^{-3}}\,dx + c \qquad \dots (3)$$

Putting $-x^{-3} = u$ so that $3x^{-4}\,dx = du$ or $x^{-4}\,dx = (1/3) \times du$

$\therefore$ (3) reduces to $vx^{-2} = (2/3) \times \int e^u\,du + c$ or $vx^{-2} = (2/3) \times e^u + c$

or $y^2x^{-2} = (2/3) \times (-x^{-3}) + c$, as $v = y^2$ and $u = -x^{-3}$

or $y^2/x^2 = (2/3) \times e^{-1/x^3} + c$, c being an arbitrary constant.

Ex. 4. *Solve* $x\,(dx/dy) + 3y = x^3y^2.$ **[I.A.S. 2002]**

Sol. Dividing by xy^2, the given equation reduces to

$$(1/y^2)\,(dy/dx) + (3/x)\,(1/y) = x^2 \qquad \dots (1)$$

Putting $1/y = v$ and $(-1/y^2)\,(dy/dx) = dv/dx$, (1) reduces to

$$-\frac{dv}{dx} + \frac{3}{x}v = x^2 \qquad \text{or} \qquad \frac{dv}{dx} - \frac{3}{x}v = -x^2, \text{ which is linear equation}$$

Its I.F. $= e^{\int(-3/x)\,dx} = e^{-3\log x} = x^{-3}$ and its solution is

$$x^{-3}v = \int (-x^2)\,(x^{-3})\,dx + c \qquad \text{or} \qquad x^{-3}y^{-1} = -\log x + c.$$

Ex. 1. *Solve the ordinary differential equation (cos 3x) × (dy/dx) – 3y sin 3x = (1/2) × sin 6x + sin² 3x, 0 < x < π/2* **[I.A.S. 2007]**

Sol. Re-writing the given equation, we have

$$(dy/dx) - (3 \tan 3x)y = \sec 3x \{(1/2) \times \sin 6x + \sin^2 3x\} \qquad \dots (1)$$

which is linear whose I.F. $= e^{\int(-3\tan 3x)dx} = e^{\log \cos 3x} = \cos 3x$ and hence its solution is

$$y \cos 3x = \int \cos 3x \sec 3x \{(1/2) \times \sin 6x + \sin^2 3x\}\, dx + c$$

or $$y \cos 3x = \int \{(1/2) \times \sin 6x + (1/2) \times (1 - \cos 6x)\}\, dx + c$$

or $$y \cos 3x = -(1/12) \times \cos 6x + x/2 - (1/12) \times \sin 3x + c$$

or $y \cos 3x = (1/12) \times (6x - \cos 6x - \sin 6x) + c$, c being an arbitrary constant

Ex. 2. *Find the solution of the equation (1/y)dy + xy²dx = – 4x dx* **[I.A.S. 2007]**

Sol. Re-writing the given equation, we have

$$\frac{dy}{y} + x(4 + y^2)\, dx = 0 \qquad \text{or} \qquad \frac{dy}{y(y^2+4)} + x dx = 0$$

or $$\frac{1}{4}\left(\frac{1}{y} - \frac{y}{y^2+4}\right) dy + x\, dx = 0 \qquad \text{or} \qquad \left(\frac{2y}{y^2+4} - \frac{2}{y}\right) dy = 8x\, dx$$

Integrating, $\log (y^2 + 4) - 2 \log y - \log c = 4x^2$, c being an arbitrary constant

or $\log \{(y^2 + 4)/cy^2\} = 4x^2$ or $y^2 + 4 = cy^2 e^{4x^2}$

Ex. 12. *Find the orthogonal trajectories of the family of curves $x^2/(a^2 + \lambda) + y^2/(b^2 + \lambda) = 1$, where λ is a parameter.* **[Gorakhpur 1996; Kumaun 1995]**

or *Show that the system of confocal conics $\{x^2/(a^2 + \lambda)\} + \{y^2/(b^2 + \lambda)\} = 1$ is self orthogonal.*

[I.A.S. 1993; Bilaspur 1995; Kumaun 1997; Purvanchal 1998, 2007; Meerut 1998]

Sol. Given $$x^2/(a^2 + \lambda) + y^2(b^2 + \lambda) = 1. \qquad \dots (1)$$

Differentiating (1), $$\frac{2x}{a^2+\lambda} + \frac{2y}{b^2+\lambda}\frac{dy}{dx} = 0 \qquad \text{or} \qquad \frac{x}{a^2+\lambda} + \frac{y}{b^2+\lambda}\frac{dy}{dx} = 0$$

or $$x(b^2+\lambda) + y(a^2+\lambda)\frac{dy}{dx} = 0 \qquad \text{or} \qquad \lambda\left(x + y\frac{dy}{dx}\right) = -\left(b^2 x + a^2 y\frac{dy}{dx}\right).$$

$$\therefore \qquad \lambda = -\{b^2x + a^2y\,(dy/dx)\}/\{x + y\,(dy/dx)\}$$

$$\therefore \qquad a^2 + \lambda = a^2 - \frac{b^2x + a^2y\,(dy/dx)}{x + y\,(dy/dx)} = \frac{(a^2 - b^2)\,x}{x + y\,(dy/dx)}$$

and $$b^2 + \lambda = b^2 - \frac{b^2x + a^2y\,(dy/dx)}{x + y\,(dy/dx)} = \frac{-(a^2 - b^2)\,y\,(dy/dx)}{x + y\,(dy/dx)}.$$

Putting the above values of $(a^2 + \lambda)$ and $(b^2 + \lambda)$ in (1), we have

$$\frac{x^2\{x + y\,(dy/dx)\}}{(a^2 - b^2)\,x} - \frac{y^2\{x + y\,(dy/dx)\}}{(a^2 - b^2)\,y\,(dy/dx)} = 1$$

or $$\{x + y\,(dy/dx)\}\,\{x - y\,(dx/dy)\} = a^2 - b^2, \qquad \dots (2)$$

which is the differential equation of the family (1). Replacing dy/dx by $(-dx/dy)$ in (2), the differential equation of the required orthogonal trajectories is

$\{x + y\,(-dx/dy)\}\,\{x - y\,(-dy/dx)\} = a^2 - b^2$ or $\{x + y\,(dy/dx)\}\,\{x - y\,(dx/dy)\} = a^2 - b^2$, ... (3)

which is the same as the differential equation (2) of the given family of curves (1). Hence, the system of given curves (1) is self orthogonal, *i.e.*, each member of the given family of curves intersects its own members orthogonally.

Ex. 1. *Find the family of curves whose tangents form the angle of $\pi/4$ with the hyperbola $xy = c$.*

[I.A.S. 1994, 2006]

Sol. The given family of curves is $xy = c$, where c is a parameter ... (1)

Differentiating (1), $y + x\,(dy/dx) = 0$ or $y + xp = 0$, where $p = dy/dx$. ... (2)

(2) is the differential equation of given family (1).

Replacing p by $\dfrac{p + \tan(\pi/4)}{1 - p\tan(\pi/4)}$, *i.e.*, $\dfrac{p+1}{1-p}$ in (2) the differential equation of the desired family of curves is

$$y + \frac{p+1}{1-p}\,x = 0 \quad \text{or} \quad p = \frac{y+x}{y-x} \quad \text{or} \quad \frac{dy}{dx} = \frac{(y/x)+1}{(y/x)-1}. \qquad \text{... (3)}$$

Let $y/x = v$, *i.e.*, $y = xv$ so that $dy/dx = v + x\,(dv/dx)$. ... (4)

$\therefore$ From (3), $$v + x\frac{dv}{dx} = \frac{v+1}{v-1} \quad \text{or} \quad x\frac{dv}{dx} = -\frac{v^2 - 2v - 1}{v-1}$$

or $$(2/x)\,dx = -\{2\,(v-1)/(v^2 - 2v - 1)\}\,dv.$$

Integrating, $2\log x = -\log(v^2 - 2v - 1) + \log c$, c being an arbitrary constant

or $\log x^2 + \log(v^2 - 2v - 1) = \log c$ or $x^2(v^2 - 2v - 1) = c$

or $x^2(y^2/x^2 - 2y/x - 1) = c$ or $y^2 - 2xy - x^2 = c$.

Ex. 9. *Solve* $p^2y^2\cos^2\alpha - 2pxy\sin^2\alpha + y^2 - x^2\sin^2\alpha = 0$.

Sol. Given $(py)^2 - 2\,(py)\,x\tan^2\alpha + (y^2\sec^2\alpha - x^2\tan^2\alpha) = 0$.

$$\therefore \qquad py = \frac{2x\tan^2\alpha \pm \sqrt{\{4x^2\tan^4\alpha - 4\,(y^2\sec^2\alpha - x^2\tan^2\alpha)\}}}{2}$$

or $$py = x\tan^2\alpha \pm \sqrt{\{x^2\tan^2\alpha\,(\tan^2\alpha + 1) - y^2\sec^2\alpha\}}$$

or $$y\,(dy/dx) = x\tan^2\alpha \pm \sec\alpha\sqrt{(x^2\tan^2\alpha - y^2)}$$

or $$y\,dy - x\tan^2\alpha\,dx = \pm\sec\alpha\sqrt{(x^2\tan^2\alpha - y^2)}\,dx$$

or $$\pm\frac{x\tan^2 dx - y\,dy}{\sqrt{(x^2\tan^2\alpha - y^2)}} = -\sec\alpha\,dx.$$

Integrating, $\pm\sqrt{(x^2\tan^2\alpha - y^2)} = c - x\sec\alpha.$, c being an arbitrary constant

Squaring, $x^2\tan^2\alpha - y^2 = c^2 - 2cx\sec\alpha + x^2\sec^2\alpha$ or $x^2 + y^2 - 2c\,x\sec\alpha + c^2 = 0$.

Ex. 10. *Solve* $p^2 + 2py \cot x = y^2$. **[Andhra 2003; Kanpur 1997 Srivenkateshwara 2003; Kanpur 2008; Gulbarga 2005; Delhi Math (G) 1994]**

(*b*) *If the curve whose differential equation is* $p^2 + 2py \cot x = y^2$ *passes through* $(\pi/2, 1)$, *show that the equation of the curve is given by* $(2y - \sec^2 x/2)(2y - \operatorname{cosec}^2 x/2) = 0$.

Sol. (*a*) Given $p^2 + (2y \cot x)\, p - y^2 = 0$. Solving it for p, we get

$$p = [-2y \cot x \pm \sqrt{(4y^2 \cot^2 x + 4y^2)}]/2$$

or $$p = -y \cot x \pm y\,(\cot^2 x + 1)^{1/2} = -y\,(\cot x \pm \operatorname{cosec} x).$$

Its component equations are $dy/dx = -y\,(\cot x + \operatorname{cosec} x)$ ... (1)

and $dy/dx = -y\,(\cot x - \operatorname{cosec} x)$. ... (2)

By (1), $$\frac{dy}{dx} = -y\left(\frac{\cos x}{\sin x} + \frac{1}{\sin x}\right) = -\frac{1+\cos x}{\sin x}\,y = -\frac{2y\cos^2(x/2)}{2\sin(x/2)\cos(x/2)}$$

or $$(1/y)\,dy = -\cot(x/2)\,dx.$$

Integrating, $\log y = \log c - 2\log\sin(x/2)$ or $y = c\operatorname{cosec}^2(x/2)$. ... (3)

By (2), $$\frac{dy}{dx} = -y\left(\frac{\cos x}{\sin x} - \frac{1}{\sin x}\right) = \frac{1-\cos x}{\sin x}\,y = \frac{2y\sin^2(x/2)}{2\sin(x/2)\cos(x/2)}$$

or $$(1/y)\,dy = \tan(x/2)\,dx.$$

Integrating, $\log y = \log c - 2\log\cos(x/2)$ or $y = c\sec^2(x/2)$. ... (4)

$\therefore$ From (3) and (4), the combined solution is $(y - c\sec^2 x/2)(y - c\operatorname{cosec}^2 x/2) = 0$.

(*b*) As in part (*a*), the general equation of the curve is

$$(y - c\sec^2 x/2)(y - c\operatorname{cosec}^2 x/2) = 0. \qquad \text{... (5)}$$

Since (1) is to pass through $(\pi/2, 1)$, (5) $\Rightarrow$ $(1-2c)^2 = 0$ $\Rightarrow$ $c = 1/2$.

Putting $c = 1/2$ in (5), the equation of the required curve is

$$(2y - \sec^2 x/2)(2y - \operatorname{cosec}^2 x/2) = 0.$$

(*c*) Solving the given equation for x, we have $2x = b + (y/p) - ayp$. ... (1)

Differentiating (1) w.r.t. 'y', we have

$$\frac{2}{p} = \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - ap - ay\frac{dp}{dy} \quad \text{or} \quad \left(\frac{1}{p} + ap\right)\left(1 + \frac{y}{p}\frac{dp}{dy}\right) = 0$$

Neglecting the first factor which does not involve dp/dy, we get

$1 + (y/p)(dp/dy) = 0$ or $(1/p)\,dp + (1/y)\,dy = 0$ so that $py = c$.

Putting $p = c/y$ in the given equation, the required solution is

$(a\,c^2/y) + (2x - b)(c/y) - y = 0$ or $ac^2 + (2x - b)\,c - y^2 = 0$.

(*e*) Given that $y = 3\,px + 4\,p^2$. ... (1)

Differentiating (1) w.r.t. 'x', $p = 3\,p + 3\,x\,(dp/dx) + 8\,p\,(dp/dx)$

or $(3x + 8p)\dfrac{dp}{dx} = -2p$ or $\dfrac{dx}{dp} = \dfrac{3x + 8p}{-2p} = -\dfrac{3}{2p}x - 4$

or $(dx/dp) + (3/2\,p)\,x = -4$, which is linear equation ... (2)

Its I.F. $= e^{\int (3/2p)\,dp} = e^{(3/2)\log p} = e^{\log p^{3/2}} = p^{3/2}$ and solution is

$$xp^{3/2} = -4\int p^{3/2}\,dp + c = -4\,\frac{p^{5/2}}{(5/2)} + C = -\frac{8}{5}p^{5/2} + c$$

or $x = -(8/5)\,p + c\,p^{-3/2}$, c being an arbitrary constant ... (3)

Substituting the above value of x in (1), we get

$$y = 3\,p\,\{(-8/5)\,p + cp^{-3/2}\} + 4\,p^2 = 3\,cp^{-1/2} - (4/5)\,p^2 \quad \text{... (4)}$$

The required solution is given by (3) and (4) in parametric form, p being the parameter.

Ex. 1. (*a*) *Solve* $x^2\,(y - px) = yp^2$ *or* $yp^2 + x^3p - x^2y = 0$. **[Allahabad 1994, Delhi Maths (G) 1994, Kumaun 1998, Agra 1995, I.A.S. 1996, Lucknow 1995, S.V. Univ. (A.P.) 1997]**

Sol. (*a*) Given $x^2\,(y - px) = yp^2$ or $y - px = (yp^2)/x^2$

or $y^2 = pxy + (py/x)^2$ or $y^2 = (py/x)\,x^2 + (py/x)^2$. ... (1)

Putting $x^2 = u$ and $y^2 = v$ so that $2x\,dx = du$ and $2y\,dy = dv$, ... (2)

we get $\dfrac{2y\,dy}{2x\,dx} = \dfrac{dv}{du}$ or $\dfrac{Py}{x} = P$, where $P = \dfrac{dv}{du}$. ... (3)

Using (2) and (3), (1) becomes $v = Pu + P^2$, which is in Clairaut's form. So replacing P by arbitrary constant c, the required general solution is

$v = cu + c^2$ or $y^2 = cx^2 + c^2$, c being an arbitrary constant

Ex. 11. *Solve* $(x^2 + y^2)\,(1 + p)^2 - 2\,(x + y)\,(1 + p)\,(x + yp) + (x + yp)^2 = 0$ **[I.A.S. 2005]**

Sol. Let $x^2 + y^2 = v$ and $x + y = u$ ... (1)

Differentiating (1), $2\,(x\,dx + y\,dy) = dv$ and $dx + dy = du$

$\therefore$ $\dfrac{dv}{du} = \dfrac{2(x\,dx + y\,dy)}{dx + dy} = \dfrac{2\{x + y\,(dy/dx)\}}{1 + (dy/dx)}$ or $P = \dfrac{2(x + y\,p)}{1 + p}$, ... (2)

where $p = dv/du$ and $p = dy/dx$. Re-writing the given equation, we get

$(x^2 + y^2) - 2(x + y)\left(\dfrac{x + yp}{1 + p}\right) + \left(\dfrac{x + yp}{1 + p}\right)^2 = 0$ or $v - 2u \times \dfrac{P}{2} + \left(\dfrac{P}{2}\right)^2 = 0$, using (1) and (2)

or $v = uP - (P^2/4)$, which is in Clairaut's, form. Hence its general solution is

$v = uc - (c^2/4)$ or $x^2 + y^2 = C\,(x + y) - (c^2/4)$, by (1)

Ex. 3. *Find general and singular solutions of $8ap^3 = 27y$* **[I.A.S. 1993]**

Sol. Solving for p, $p = dy/dx = (3/2)\,(1/a^{1/3})\,y^{1/3}$

or $dx = (2/3)\,a^{1/3}\,y^{-1/3}\,dy$ so that $x + c = a^{1/3}\,y^{2/3}$

or $(x + c)^3 = ay^2$, on cubing both sides. ... (1)

Now, differenting (2) partially w.r.t. 'c', we get

$3\,(x + c)^2 = 0$ or $x + c = 0$ or $c = -\,x$. ... (2)

Eliminating c between (1) and (2), the c-discriminant relation is

$0 = ay^2$ or $y = 0$.

Now $y = 0$ gives $p = dy/dx = 0$. Substitution of $y = 0$ and $p = 0$ in the given differential equation satisfy it. Hence $y = 0$ is the singular solution.

Ex. 5. *Find the general and singular solution of $p^2 + y^2 = 1$.*

Sol. Given equation is $p^2 + 0{\cdot}p + (y^2 - 1) = 0$ ... (1)

Solving for p, $p = dy/dx = \pm\,(y^2 - 1)^{1/2}$ or $dx = \pm\,[1/(1 - y^2)^{1/2}]\,dy$.

Integrating, $x + c = \pm\,\cos^{-1} y$ or $\cos^{-1} y = \pm\,(x + c)$ or $y = \cos\,(x + c)$.

From (1), the p-discriminant relation is $B^2 - 4\,AC = 0$, *i.e.*,

$0 - 4 \cdot 1 \cdot (y^2 - 1) = 0$ or $y^2 - 1 = 0$ or $(y - 1)\,(y + 1) = 0$.

Now, $y - 1 = 0$ gives $p = dy/dx = 0$. Substitution of $y = 1$ and $p = 0$ in (1) satisfies it. Hence $y = 1$ is a singular solution. Similarly we see that $y = -\,1$ is also a singular solution.

Hence $y = \cos\,(x + c)$ is general solution and $y = \pm\,1$ are singular solutions.

Ex. 6. *Find the general and singular solution of $y^2\,(1 + p^2) = r^2$ or $y^2\,\{1 + (dy/dx^2\} = r^2$.*
[I.A.S. (Prel.) 2000, 01, 02, 06]

Sol. Re-writing the given equation, we have

$p = dy/dx = \pm\,(r^2 - y^2)^{1/2}/y$ or $dx = \pm\,(1/2) \times (r^2 - y^2)^{-1/2}\,(-\,2y)\,dy$

Integrating, $x + c = \pm\,(r^2 - y^2)^{1/2}$ or $(x + c)^2 + y^2 = r^2$, ... (1)

which is the general solution of the given differential equation.

Now differentiating (1) partially w.r.t. 'c', we get

$2\,(x + c) + 0 = 0$ so that $c = -\,x$. ... (2)

Eliminating c between (1) and (2), we get $y^2 = r^2$ or $(y - r)\,(y + r) = 0$, which is the c-discriminant relation. Since $y = r$ and $y = -\,r$ both satisfy the given differential equation and hence form the singular solutions.

Ex. 8. *Find the general and singular solution of $p^2y^2 \cos^2 \alpha - 2pxy \sin^2 \alpha + y^2 - x^2 \sin^2 \alpha = 0$*

Sol. For general solution refer Ex. 9 of Art. 4.3. Then we have

$$c^2 + 2\,(x \sec \alpha)\,x + x^2 + y^2 = 0,\ c \text{ being an arbitrary constant}$$

This is quadratic in c. So here c-discriminant relation is

$4x^2 \sec^2 \alpha - 4{\cdot}1{\cdot}\,(x^2 + y^2) = 0$ or $x^2\,(\sec^2 \alpha - 1) - y^2 = 0$

or $y^2 - x^2 \tan^2 \alpha = 0$ or $(y - x \tan \alpha)\,(y + x \tan \alpha) = 0$.

Now, $y = x \tan \alpha$ gives $p = dy/dx = \tan \alpha$. Substitution of $p = \tan \alpha$ and $y = x \tan \alpha$ in the given equation satisfies it. Hence $y = x \tan \alpha$ is a singular solution. Similarly, we easily verify that $y = -\,x \tan \alpha$ is also a singular solution.

Ex. 11. *Find the solution of the differential equation $y = 2xp - yp^2$ where $p = dy/dx$. Also find the singular solution.* **[Guwahati 1996]**

Sol. Given $y = 2xp - yp^2$ ... (1)

Solving (1) for x, $x = y/2p + yp/2$ ... (2)

Diff. (2) w.r.t. y and noting that $dx/dy = 1/p$, we get $$\frac{1}{p} = \frac{1}{2p} - \frac{y}{2p^2}\frac{dp}{dy} + \frac{p}{2} + \frac{y}{2}\frac{dp}{dy}$$

or $$\frac{y}{2}\frac{dp}{dy}\left(1 - \frac{1}{p^2}\right) + \frac{p}{2}\left(1 - \frac{1}{p^2}\right) = 0 \quad \text{or} \quad \frac{1}{2}\left(1 - \frac{1}{p^2}\right)\left(y\frac{dp}{dy} + p\right) = 0.$$

Omitting the first factor, for general solution we have

$y\,(dp/dy) + p = 0$ or $(1/p)\,dp + (1/y)\,dy = 0$

Integrating, $\log p + \log y = \log c$ or $py = c$ or $p = c/y$. ... (3)

Eliminating p from (1) and (3), the general solution is

$y = (2xc)/y - (yxc^2)/y^2$ or $y^2 = 2cx - c^2$ .... (4)

The p-disc. relation from (1) *i.e.* $yp^2 - 2xp + y = 0$ is given by

$4x^2 - 4y^2 = 0$ or $x^2 - y^2 = 0$ or $(x - y)(x + y) = 0$.

The c-disc. relation from (4) *i.e.* $c^2 - 2cx + y^2 = 0$ is given by

$4x^2 - 4y^2 = 0$ or $x^2 - y^2 = 0$ or $(x - y)(x + y) = 0$.

Hence $x - y = 0$ and $x + y = 0$ are singular solutions because these appear once in both the descriminants and also satisfy (1).

Ex. 13. *Find general and singular solutions of $3xy = 2px^2 - 2p^2$ or $y = (2x/3)\,p - (2/3x)\,p^2$* **[I.A.S. Prel. 1995, 2001]**

Sol. Given equation is $3xy = 2px^2 - 2p^2$. ... (1)

Solving (1) for y, $y = (2/3)\,px - (2/3)\,p^2 x^{-1}$. ... (2)

Differentiating (2) w.r.t. 'x' and writing p for dy/dx, we get

$$p = \frac{2p}{3} + \frac{2x}{3}\frac{dp}{dx} - \frac{2}{3}\left[2p\frac{dp}{dx}x^{-1} - x^{-2}p^2\right] \quad \text{or} \quad 3p - 2p - \frac{2p^2}{x^2} - 2x\frac{dp}{dx} + \frac{4p}{x}\frac{dp}{dx} = 0$$

or $$p - \frac{2p^2}{x^2} - 2\frac{dp}{dx}\left(x - \frac{2p}{x}\right) = 0 \quad \text{or} \quad p\left(1 - \frac{2p}{x^2}\right) - 2x\frac{dp}{dx}\left(1 - \frac{2p}{x^2}\right) = 0$$

or $\{1 - (2\,p/x^2)\}\,\{p - 2x\,(dp/dx)\} = 0$. ... (3)

Omitting the first factor which does not involve dp/dx, we get

$p - 2x\,(dp/dx) = 0$ or $(2/p)\,dp = (1/x)\,dx$

Integrating, $2\log p = \log x + \log c$ or $p^2 = xc$ or $p = \pm (xc)^{1/2}$.

Putting this value of p in (2), the required general solution is

$3y = \pm 2x\,(xc)^{1/2} - 2c$ or $3y + 2c = \pm 2x\,(xc)^{1/2}$

or $(3y + 2c)^2 = 4cx^3$ or $4c^2 + 4c\,(3y - x^3) + 9y^2 = 0$. ... (4)

From (4), the c-discriminant relation is $B^2 - 4AC = 0$, *i.e.*,

$16\,(3y - x^3)^2 - 4 \times 4 \times 9y^2 = 0$ or $x^3(x^3 - 6y) = 0$.

Now $x^3 = 0$ gives $x = 0$ and $dx/dy = 1/p = 0$ and these values do not satisfy (1). So $x = 0$ is not a singular solution.

Ex. 15. *Solve the differential equation $y = x - 2ap + ap^2$. Find the singular solution and interpret it geometrically.* **[I.A.S. 2000]**

Sol. Given that $y = x - 2ap + ap^2$, where $p = dy/dx$ ... (1)

Differentiating (1) w.r.t. 'x', $p = 1 - 2\ a\ (dp/dx) + 2\ ap\ (dp/dx)$

or $p - 1 = 2a\ (p - 1)\ (dp/dx)$ or $(p - 1)\ \{2a\ (dp/dx) - 1\} = 0$

Omitting the first factor since it does not involve dp/dx, we get

$2a\ (dp/dx) - 1 = 0$ or $dx = 2\ adp.$

Integrating, $x = 2ap + c$ so that $p = (x - c)/2a$ ... (2)

Substituting the value of p from (2) in (1), general solution of (1) is

$y = x - (x - c) + (1/4a)\ (x - c)^2$ or $4a\ (y - c) = x^2 + c^2 - 2xc$

or $c^2 - 2xc + 4ac + x^2 - 4ay = 0$ or $c^2 - 2c\ (x - 2a) + (x^2 - 4ay) = 0,$ ... (3)

which is a quadratic equation in parameter c. So the c-discriminant relation is

$4\ (x - 2a)^2 - 4\ (x^2 - 4ay) = 0$ or $y - x + a = 0$ ... (4)

Again, re-writing (1), $ap^2 - 2ap + (x - y) = 0,$ ... (5)

which is a quadratic in parameter p. Hence the p-discriminant relation is

$4a^2 - 4a\ (x - y) = 0$ or $y - x + a = 0$ ... (6)

From (4) and (6), we find that $y - x + a = 0$ is present in both p and c discriminant relations. Further $y - x + a = 0$ gives $y = x - a$ and $p = dy/dx = 1$. These satisfy (1). Hence $y - x + a = 0$ is singular solution of (1).

Geometrical interpretation of singular solution y − x + a = 0.

Re-writing (3), $(x - c)^2 = 4a\ (y - c),$ ... (7)

which represents a family of parabolas all of which touch the line $y - x + a = 0$, which is the envelope of this family of parabolas.

(c) The given equation is $(xp - y)^2 = p^2 - 1.$ ... (1)

Re-writing (1), $xp - y = \pm\ (p^2 - 1)^{1/2}$ or $y = px \pm (p^2 - 1)^{1/2}$ ... (2)

either of which is in Clairaut's form. So replacing p by the arbitrary constant c,

the required complete primitive is $y = cx \pm (c^2 - 1)^{1/2}$ ... (3)

$(y - cx)^2 = c^2 - 1$ or $c^2\ (x^2 - 1) - 2xyc + (y^2 + 1) = 0.$ ... (4)

From (4), the c-discriminant relation is $B^2 - 4AC = 0$, *i.e.*,

$(-\ 2xy)^2 - 4\ (x^2 - 1)\ (y^2 + 1) = 0$ or $x^2 - y^2 = 1.$ ... (5)

This relation satisfies (1) and hence it is the singular solution.

Geometrical interpresation. The complete primitive gives by (3) represents a family of straight lines, each member of which touches the rectangular hyperbola $x^2 - y^2 = 1$, which is the envelope of this family of straight lines.

Ex. 22. *Solve and examine for singular solution of $x^2\ (y - xp) = yp^2$.*

Sol. The given equation is $x^2\ (y - xp) = yp^2.$ ... (1)

The general solution of (1) is [Refer Ex. 1(a) of Art. 4.10]

$y^2 = cx^2 + c^2$ or $c^2 + cx^2 - y^2 = 0,$ ... (2)

which is a quadratic equation in c. Its c-discriminant relation is

$(x^2)^2 - 4 \cdot 1 \cdot (-\ y^2) = 0$ or $x^4 + 4y^2 = 0.$ ... (3)

Since (3) satisfies (1), so $x^4 + 4y^2 = 0$ is the required singular solution.

Ex. 24. *Reduce the equation $xyp^2 - p(x^2 + y^2 - 1) + xy = 0$ to Clairaut's form by the substitutions $x^2 = u$ and $y^2 = v$. Hence show that the equation represents a family of conics touching the four sides of a square.* **[I.A.S. 2004]**

Sol. The given equation is $\quad xyp^2 - p\,(x^2 + y^2 - 1) + xy = 0. \quad \ldots (1)$

The general solution of (1) is [Refer Ex. 1(*e*) of Art. 4.10]

$y^2 = cx^2 - c/(c - 1) \quad$ or $\quad c^2x^2 - c\,(x^2 + y^2 - 1) + y^2 = 0, \quad \ldots (2)$

which represents a family of conics. Since (2) is a quadratic equation is c so c-discriminant relation is given by $\quad (x^2 + y^2 - 1)^2 - 4x^2y^2 = 0 \quad$ or $\quad (x^2 + y^2 - 1)^2 - (2xy)^2 = 0$

or $\quad (x^2 + y^2 - 1 + 2xy)(x^2 + y^2 - 1 - 2\,xy) = 0 \quad$ or $\quad \{(x + y)^2 - 1^2\}\{(x - y)^2 - 1^2\} = 0$

or $\quad (x + y + 1)\,(x + y - 1)\,(x - y + 1)\,(x - y - 1) = 0 \quad \ldots (3)$

Now $x + y + 1 = 0$ gives $y = -x - 1$ and $p = dy/dx = -1$. These values satisfy (1). Hence $x + y + 1 = 0$ is a singular solution. Similarly $x + y - 1 = 0$, $x - y + 1 = 0$ and $x - y - 1 = 0$ are singular solutions. Clearly $x + y + 1 = 0$, $x + y - 1 = 0$, $x - y + 1 = 0$ and $x - y - 1 = 0$ form a square.

Geometrical interpretation. General solution (2) represents a family of conics all of which touch the straight lines $x + y + 1 = 0$, $x + y - 1 = 0$, $x - y + 1 = 0$ and $x - y - 1 = 0$ (forming a square) which are the envelopes of family of conics.

Ex. 28. *Find the general and singular solution of $y^2\,(y - xp) = x^4p^2$.*

Sol. The given equation is $\quad y^2\,(y - xp) = x^4p^2. \quad \ldots (1)$

The general solution of (1) is [Refer Ex. 4 of Art. 4.10]

$x = cy + c^2xy \quad$ or $\quad xyc^2 + yc - x = 0, \quad \ldots (2)$

which is a quadratic equation in c and so its c-discriminant relation is

$y^2 - 4\,(xy)\,(-x) = 0 \quad$ or $\quad y\,(y + 4x^2) = 0.$

Now, $y = 0$ gives $p = dy/dx = 0$. These values satisfy (1). So $y = 0$ is a singular solution. Again $y = -4x^2$ gives $p = dy/dx = -8x$. These values satisfy (1). Hence $y + 4x^2 = 0$ is also singular solution.

Ex. 30. *Reduce the equation $x^2p^2 + py\,(2x + y) + y^2 = 0$ where $p = dy/dx$ to Clairaut's form by putting $u = y$ and $v = xy$ and find its complete primitive and its singular solution.*

[I.A.S. 2006, Kumaun 1995]

Sol. The given equation is $\quad x^2p^2 + py\,(2x + y) + y^2 = 0 \quad \ldots (1)$

The complete primitive of (1) is [Refer Ex 6 of Art. 4.10]

$xy = cy + c^2 \quad$ or $\quad c^2 + cy - xy = 0$

which is a quadratic equation in c and hence c-discriminant relation is

$y^2 - 4\cdot 1\cdot(-xy) = 0 \quad$ or $\quad y\,(y + 4x) = 0.$

Since $y = 0$ and $y + 4x = 0$ both satisfy (1), so these are both singular solutions.

Ex. 31. *Solve $(px^2 + y^2)\,(px + y) = (p + 1)^2$ by reducing it to Clairaut's form and find its singular solution.*

Sol. Given $\quad (px^2 + y^2)\,(px + y) = (p + 1)^2. \quad \ldots (1)$

The general solution of (1) is [Refer Ex 10 of Art. 4.10] $\quad c^2\,(x + y) - xyc - 1 = 0.$

Its c-discriminant relation is $\quad B^2 - 4AC = 0$, *i.e.*,

$(xy)^2 - 4\,(x + y) \times (-1) = 0 \quad$ or $\quad x^2y^2 + 4\,(x + y) = 0.$

This relation satisfies (1), and hence it is the singular solution.

Ex. 1. *Obtain the complete primitive and singular solution of the following equations, explaining the geometrical significance of the irrelevent factors that present themselves.*

(i) $4\, xp^2 = (3x - a)^2$.

(ii) $xp^3 = (x - a)^2$. **[Ravishankar 1996; Vikram 1993]**

Sol. (*i*) The given differential equation is $\quad 4xp^2 - (3x - a)^2 = 0. \quad \ldots (1)$

The general solution of (1) is [Refer Ex. 2 (*i*) of Art. 4.3] $\quad (y + c)^2 = x\,(x - a)^2. \quad \ldots (2)$

Rewriting (2) as quadratic in c, we have $\quad c^2 + 2cy + y^2 - x\,(x - a)^2 = 0. \quad \ldots (3)$

Now from (1), the p-discriminant relation is

$0 - 4 \cdot 4x\,\{-(3x - a)^2\} = 0 \quad$ or $\quad x\,(3x - a)^2 = 0. \quad \ldots (4)$

Similarly from (3), the c-discriminant relation is given by

$4y^2 - 4\,[y^2 - x\,(x - a)^2] = 0 \quad$ or $\quad x\,(x - a)^2 = 0. \quad \ldots (5)$

Here $x = 0$ appears once in both the discriminants. Again (1) may be re-written as $4x - (3x - a)^2 / p^2 = 0$ which is satisfied because $x = 0$ gives $dx/dy = 0$ *i.e.* $1/p = 0$. Thus, by definition $x = 0$ is a singular solution.

$3x - a = 0$ is a tac-locus since it appears squared in the p-discriminant relation (4), does not occur in the c-discriminant relation (5), and does not satisfy the differential equation (1).

$x - a = 0$ is a node-locus since it appears squared in the c-discriminant relation (5), does not occur in the p-discriminant relation (4), and does not satisfy the differential equation (1).

(*ii*) Proceed as above. **Ans.** $x = 0$ is singular solution, $x - a = 0$ is tac locus, $x - 3a = 0$ in node-locus.

(*ii*) The given equation is $\quad x^4p^2 - xp - y = 0. \quad \ldots (1)$

Its general solution [Refer Ex. 2(*a*) is of Art. 4.7]

$$xc^2 - c - xy = 0. \quad \ldots (2)$$

As usual, the p-disc. relation is $x^2\,(4x^2y + 1) = 0$ and the c-disc. relation is $4x^2y + 1 = 0$.

$\therefore\ 4x^2y + 1 = 0$ is singular solution and $x = 0$ is a tac-locus.

Ex. 4. *The singular solution/solutions of $x\,(dy/dx)^2 - 2y\,(dy/dx) + 4x = 0$, $(x > 0)$ is/are*

(a) $y = \pm x^2$ *(b)* $y = 2x + 3$ *(c)* $y = x^2 - 2x$ *(d)* $y = \pm 2x$. **[I.A.S. Prel. 1994]**

Sol. Ans. (*d*). Given $\quad xp^2 - 2\,yp + 4x = 0 \quad \ldots (1)$

Solving for y, $\quad (1) \Rightarrow y = (1/2)\,xp - (2x)/p. \quad \ldots (2)$

Diff. (2) w.r.t. 'x', $\quad$ or $\quad \left(x\dfrac{dp}{dx} - p\right)\left(\dfrac{1}{2} + \dfrac{2}{p^2}\right) = 0,$

For general solution, we take only first factor.

$\therefore\ x\dfrac{dp}{dx} - p \quad$ or $\quad \dfrac{dx}{x} = \dfrac{dp}{p} \quad$ so that $\quad \log p = \log c + \log x \quad$ or $\quad p = cx.$

Putting this value of p in (1), the general solution of (1) is

$$c^2x^3 - 2y\,cx + 4x = 0 \quad \text{or} \quad c^2x^2 - 2cy + 4 = 0. \quad \ldots (3)$$

Both the p-discriminant and c-discriminant relations are same and are given by

$4y^2 - 16x^2 = 0 \quad$ or $\quad y^2 - 4x^2 = 0. \quad \ldots (4)$

From (4), $\quad y^2 = 4x^2 \quad$ so that $\quad 2\,yp = 8x \quad$ or $\quad p = (4x)/y.$

Putting this value of p in (1), we have

$x\,(16x^2/y^2) - 2y\,(4x/y) + 4x = 0 \quad$ or $\quad (4x^3)/y^2 - x = 0$

or $\quad 4x^3 - xy^2 = 0 \quad$ or $\quad 4x^3 - x\,(4x^2) = 0$, by (4), showing that (4) satisfies the given diff. equation. Now, $y^2 - 4x^2 = 0$ appears both in p and c discriminant relations and satisfy the given differential equation. So $y^2 - 4x^2 = 0$ or $y = \pm 2x$ are two singular solutions.

Ex. 1. *Solve* $(D^2 + a^2)\, y = \cot ax.$ **[Delhi Maths. (G) 2005]**

Sol. Here the auxiliary equation is $D^2 + a^2 = 0$ so that $D = 0 \pm ia.$

$\therefore$ C.F. $= e^{0x}(c_1 \cos ax + c_2 \sin ax) = c_1 \cos ax + c_2 \sin ax$, c_1, c_2 being arbitrary constants

Now, $\text{P.I.} = \dfrac{1}{D^2 + a^2} \cot ax = \dfrac{1}{(D+ai)(D-ia)} \cot ax$

$[\because D^2 + a^2 = D^2 - (ia)^2 = (D + ai)\,(D - ai)]$

$= \dfrac{1}{2ia}\left[\dfrac{1}{D - ia} + \dfrac{1}{D + ia}\right] \cot ax,$ on resolving into partial fractions

Now, $\dfrac{1}{D - ia} \cot ax = e^{iax}\displaystyle\int e^{-iax} \cot ax\, dx = e^{iax}\int (\cos ax - i \sin ax)\frac{\cos ax}{\sin ax}\, dx$

$[\because$ by Euler's theorem, $e^{-iax} = \cos ax - i \sin ax]$

$= e^{iax}\displaystyle\int\left(\frac{\cos^2 ax}{\sin ax} - i \cos ax\right) dx = e^{iax}\int\left(\frac{1 - \sin^2 ax}{\sin ax} - i \cos ax\right) dx$

$= e^{iax}\displaystyle\int (\operatorname{cosec} ax - \sin ax - i \cos ax)\, dx = e^{iax}\, [(1/a) \log \tan (ax/2) + (1/a) \cos ax - (i/a) \sin ax]$

$= e^{iax}\, [(1/a) \log \tan (ax/2) + (1/a)(\cos ax - i \sin ax)]$

$= e^{iax}\, [(1/a) \log \tan (ax/2) + (1/a)\, e^{-iax}]$, by Euler's theorem

$$\therefore \qquad \frac{1}{D - ia} \cot ax = \frac{1}{a}\left[e^{iax} \log \tan \frac{ax}{2} + 1\right]. \qquad \text{... (2)}$$

Replacing i by $-i$ in (2), $\dfrac{1}{D - ia} \cot ax = \dfrac{1}{a}\left[e^{-iax} \log \tan \dfrac{ax}{2} + 1\right]$ ... (3)

Using (2) and (3), (1) reduces to

$$\text{P.I.} = \frac{1}{2ia}\left[\frac{1}{a}\{e^{iax} \log \tan \frac{ax}{2} + 1\} - \frac{1}{a}\{e^{-iax} \log \tan \frac{ax}{2} + 1\}\right]$$

$$= \frac{1}{a^2} \cdot \frac{e^{iax} - e^{-iax}}{2i} \log \tan \frac{ax}{2} = \frac{1}{a^2} \sin ax \log \tan \frac{ax}{2}.$$

Hence the required general solution is $y = \text{C.F.} + \text{P.I.}$, i.e.,

$y = c_1 \cos ax + c_2 \sin ax + (1/a^2) \sin ax \log \tan (ax/2)$, where c_1 and c_2 are arbitrary constants.

(b) $(D^2 - 3D + 2)\, y = \cosh x.$ **[I.A.S. Prel. 2005]**

(c) $(D^3 - 5D^2 + 7D - 3)\, y = e^{2x} \cosh x$

(d) $(d^3y/dx^3) - y = (e^x + 1)^2.$ **[Delhi Maths (H) 1993, 1996]**

Sol. (*a*) Here the auxiliary equation is $D^3 - 3D + 2 = 0$ so that $D = 1, 2.$

$\therefore$ C.F. $= c_1 e^x + c_2 e^{2x}$, c_1, c_2 being arbitrary constants.

$$\text{and P.I.} = \frac{1}{D^2 - 3D + 2}(e^x + e^{2x}) = \frac{1}{(D-1)(D-2)}e^x + \frac{1}{(D-2)(D-1)}e^{2x}$$

$$= \frac{1}{D-1}\,\frac{1}{1-2}e^x + \frac{1}{D-2}\,\frac{1}{2-1}e^{2x} = -\frac{x}{1!}e^x + \frac{x}{1!}e^{2x}$$

Hence the general solution is $y = c_1 e^x + c_2 e^{2x} - xe^x + xe^{2x}.$

(*b*) Here auxiliary equation is $D^2 - 3D + 2 = 0$ so that $D = 1, 2$

$\therefore$ C.F. $= c_1 e^x + c_2 e^{2x}$, c_1, c_2 being arbitrary constants

$$\text{and} \quad \text{P.I} = \frac{1}{D^2 - 3D + 2}\cosh x = \frac{1}{(D-1)(D-2)}\,\frac{(e^x + e^{-x})}{2}$$

$$= \frac{1}{2}\,\frac{1}{(D-1)(D-2)}e^x + \frac{1}{2}\,\frac{1}{(D-1)(D-2)}e^{-x}$$

$$= \frac{1}{2}\,\frac{1}{D-1}\,\frac{1}{1-2}e^x + \frac{1}{2}\,\frac{1}{(-2)\times(-3)}e^{-x} = -\frac{1}{2}\,\frac{1}{D-1}e^x + \frac{1}{2}\times\frac{1}{6}e^{-x}$$

$$= -\frac{1}{2}\times\frac{x}{1!}e^x + \frac{1}{12}e^{-x} = -\frac{x}{2}e^x + \frac{1}{12}e^{-x}$$

$\therefore$ the required solution is $y = c_1 e^x + c_2 e^{2x} - (x/2) \times e^x + (1/12) \times e^{-x}$

Ex. 7. *If $(d^2x/dt^2) + (g/b)(x - a) = 0$, ($a$, b and g being constants) and $x = a'$ and $dx/dt = 0$ when $t = 0$, show that $x = a + (a' - a) \cos t\sqrt{(g/b)}$.* **[I.A.S. 1994, Kurushetra 1994]**

Sol. With $D \equiv d/dt$, given equation is $\{D^2 + (g/b)\}\, x = ga/b.$ ... (1)

Here auxiliary equation $D^2 + g/b = 0$ gives $D = 0 \pm i\sqrt{(g/b)}$.

$\therefore$ C.F. $= c_1 \cos t\sqrt{(g/b)} + c_2 \sin t\sqrt{(g/b)}$, c_1, c_2, being arbitrary constants

$$\text{P.I.} = \frac{1}{D^2 + (g/b)}\,\frac{ga}{b} = \frac{ga}{b}\,\frac{1}{D^2 + (g/b)}e^{0.t} = \frac{ga}{b}\,\frac{1}{0 + (g/b)}e^{0.t} = a.$$

So general solution is $x = c_1 \cos t\sqrt{(g/b)} + c_2 \sin t\sqrt{(g/b)} + a$. ...(2)

From (2), $dx/dt = -\, c_1\sqrt{(g/b)}\, \sin t\sqrt{(g/b)} + c_2\sqrt{(g/b)}\cos t\sqrt{(g/b)}$ ...(3)

Given $x = a'$ when $t = 0$. So (2) $\Rightarrow a' = c_1 + a$ or $c_1 = a' - a.$ ...(4)

Given $dx/dt = 0$ when $t = 0$. So (3) $\Rightarrow 0 = c_2\sqrt{g/b}$ or $c_2 = 0.$ ...(5)

Substituting the values of c_1 and c_2 in (2), the required solution is

$$x = (a' - a) \cos t\sqrt{g/b} + a \text{, as required.}$$

Ex. 4. *Solve the following differential equations:*

(a) $(D^3 + a^2 D)\, y = \sin ax$ **[I.A.S. Prel. 2006, Rajsthan 2010, Purvanchal 1999]**

(b) $(D^3 + 9D)\, y = \sin 3x.$

(c) $(d^3x/dy^3) + b^2(dx/dy) = \sin by.$

Sol. (*a*) Here auxiliary equation is $D^3 + a^2 D = 0$ so that $D = 0, 0 \pm ia.$

$\therefore$ C.F. $= c_1 e^{0.x} + e^{0.x}(c_2 \cos ax + c_3 \sin ax) = c_1 + c_2 \cos ax + c_3 \sin ax,$

where c_1, c_2 and c_3 arbitrary constants.

$$\text{P.I.} = \frac{1}{D^3 + a^2 D}\sin ax = \frac{1}{D^2 + a^2}\frac{1}{D}\sin ax = \frac{1}{D^2 + a^2}\left(-\frac{1}{a}\cos ax\right)$$

$$= -\frac{1}{a}\left[\text{Real part of } \frac{1}{D^2 + a^2}(\cos ax + i \sin ax)\right]$$

$$= -\frac{1}{a}\left[\text{Real part of } \frac{1}{D^2 + a^2} e^{iax}\right], \text{ by Euler's theorem}$$

$$= -\frac{1}{a}\left[\text{Real part of } \left(\frac{x}{2a}\sin ax - \frac{ix}{2a}\text{ as } ax\right)\right]$$

$$\left[\text{As in Ex. 3. } (a), \text{ prove that } \frac{1}{D^2 + a^2}e^{iax} = \frac{x}{2a}\sin ax - \frac{ix}{2a}\cos ax\right]$$

$= -(1/a)\,(x/2a)\sin ax = -(x/2a^2)\sin ax.$

Hence the general solution is $y = c_1 + c_2 \cos ax + c_3 \sin ax - (x/2a^2)\sin ax.$

Ex. 6. *Find a complete integral of $px + qy = pq$.* **[Kurukshetra 2006 Rajasthan 2000, 01, Gulbarga 2005; Meerut 2002; Kanpur 2004; Jiwaji 2004; Rewa 2001; Vikram 2000, 03, 04; Bhopal 2010]**

Sol. Here given equation is $f(x, y, z, p, q) \equiv px + qy - pq = 0.$ ...(1)

Charpit's auxiliary equations are $$\frac{dp}{f_x + p f_z} = \frac{dq}{f_y + q f_z} = \frac{dz}{-p f_p - q f_q} = \frac{dx}{-f_p} = \frac{dq}{-f_q}$$

or $$\frac{dp}{-(x-q)} = \frac{dy}{-(y-q)} = \frac{dz}{-p(x-q) - q(y-p)} = \frac{dp}{p + p.0} = \frac{dq}{q + q.0}, \text{ by (1)} \quad ...(2)$$

Taking the last two fractions of (2), $(1/p)dp = (1/q)dq.$

Integrating, $\log p = \log q + \log a$ or $p = aq.$ ...(3)

Substituting this value of p in (1), we have

$aqx + qy - aq^2 = 0$ or $aq = ax + y,$ as $q \neq 0$ ...(4)

$\therefore$ From (3) and (4), $q = (ax + y)/a$ and $p = ax + y.$ ...(5)

Putting these values of p and q in $dz = pdx + qdy$, we get

$dz = (ax + y)dx + [(ax + y)/a]\, dy$ or $adz = (ax + y)(adx + dy)$

or $adz = (ax + y)\, d(ax + y) = udu$, where $u = ax + y.$

Integrating, $az = u^2/2 + b = (ax + y)^2/2 + b,$

which is a complete integral, a and b being arbitrary constants.

Ex. 5. *Prove that* $L\left\{\frac{\sin t}{t}\right\} = \tan^{-1}\frac{1}{p}$ *and hence find* $L\left\{\frac{\sin at}{t}\right\}\cdot$ *Does the Laplace transform of* $\frac{\cos at}{t}$ *exist?*

(Meerut 1989, 91; Agra 81, 83, 85; Rohilkhand 2002)

Sol. Let $F(t) = \sin t$.

Now $\lim_{t \to 0} \frac{F(t)}{t} = \lim_{t \to 0} \frac{\sin t}{t} = 1$.

We have $L\{\sin t\} = \frac{1}{p^2+1} = f(p)$, say.

$$\therefore \quad L\left\{\frac{\sin t}{t}\right\} = \int_p^\infty f(x)\, dx = \int_p^\infty \frac{dx}{x^2+1} = (\tan^{-1} x)_p^\infty = \frac{\pi}{2} - \tan^{-1} p$$

$$= \cot^{-1} p = \tan^{-1}(1/p).$$

Now $L\left\{\frac{\sin at}{t}\right\} = aL\left\{\frac{\sin at}{at}\right\} = a \cdot \frac{1}{a}\tan^{-1}\frac{1}{(p/a)}$,

since $L\{F(at)\} = \frac{1}{p} f\left(\frac{p}{a}\right) = \tan^{-1}(a/p)$.

Again, since $L\{\cos at\} = \frac{p}{p^2+a^2} = f(p)$, we have

$$L\left\{\frac{\cos at}{t}\right\} = \int_p^\infty \frac{x}{x^2+a^2}\, dx = \left[\frac{1}{2}\log(x^2+a^2)\right]_p^\infty$$

$$= \frac{1}{2}\lim_{x \to \infty} \log(x^2+a^2) - \frac{1}{2}\log(p^2+a^2),$$

which does not exist since $\lim_{x \to \infty} \log(x^2+a^2)$ is infinite.

Hence $L\left\{\frac{\cos at}{t}\right\}$ does not exist.

Ex. 1. *Evaluate* $\int_0^\infty \frac{(e^{-at} - e^{-bt})}{t}\, dt.$ **(Meerut 1983)**

Sol. Let $F(t) = e^{-at} - e^{-bt}$.

Then $L\{F(t)\} = L\{e^{-at}\} - L\{e^{-bt}\} = \frac{1}{p+a} - \frac{1}{p+b} = f(p)$, say.

$$\therefore \quad L\left\{\frac{F(t)}{t}\right\} = \int_p^\infty f(x)\, dx = \int_p^\infty \left(\frac{1}{x+a} - \frac{1}{x+b}\right) dx = \left[\log \frac{(x+a)}{(x+b)}\right]_p^\infty$$

$$= \lim_{x \to \infty} \log \frac{x+a}{x+b} - \log \frac{p+a}{p+b}$$

$$= \lim_{x \to \infty} \log \frac{1+(a/x)}{1+(b/x)} - \log \frac{p+a}{p+b}$$

$$= 0 - \log \frac{p+a}{p+b} = \log \frac{p+b}{p+a}.$$

Thus $$L\left\{\frac{F(t)}{t}\right\} = \int_0^\infty e^{-pt} \cdot \frac{e^{-at} - e^{-bt}}{t}\, dt = \log \frac{p+b}{p+a}.$$

$\therefore$ Taking limit as $p \to 0$, we have

$$\int_0^\infty \frac{e^{-at} - e^{-bt}}{t}\, dt = \log \frac{b}{a}.$$

Ex. 2. *Prove that* $\int_0^\infty t^3 e^{-t} \sin t\, dt = 0.$ **(Meerut 1987)**

Sol. Let $F(t) = t^3 \sin t$.

$$\therefore \quad L\{F(t)\} = L\{t^3 \sin t\} = (-1)^3 \frac{d^3}{dp^3} L\{\sin t\}$$

$$= -\frac{d^3}{dp^3}\left\{\frac{1}{p^2+1}\right\} = -\frac{d^2}{dp^2}\left\{-\frac{2p}{(p^2+1)^2}\right\} = \frac{d}{dp}\left\{\frac{2-6p^2}{(p^2+1)^3}\right\}$$

or $$\int_0^\infty e^{-pt} \cdot t^3 \sin t\, dt = \frac{24(p^2-1)p}{(p^2+1)^4}.$$

Taking limit as $p \to 1$, we have

$$\int_0^\infty t^3 e^{-t} \sin t\, dt = 0.$$

Ex. 2. *Prove that*

$$L^{-1}\left\{\frac{5}{p^2} + \left(\frac{\sqrt{p}-1}{p}\right)^2 - \frac{7}{3p+2}\right\} = 1 + 6t - 4\sqrt{\left(\frac{t}{\pi}\right)} - \frac{7}{3} e^{-2t/3}.$$

(Meerut 1996)

Sol. We have $L^{-1}\left\{\frac{5}{p^2} + \left(\frac{\sqrt{p}-1}{p}\right)^2 - \frac{7}{3p+2}\right\}$

$$= L^{-1}\left\{\frac{5}{p^2} + \frac{p - 2\sqrt{p} + 1}{p^2} - \frac{7}{3}\cdot\frac{1}{p + (2/3)}\right\}$$

$$= 6\,L^{-1}\left\{\frac{1}{p^2}\right\} + L^{-1}\left\{\frac{1}{p}\right\} - 2L^{-1}\left\{\frac{1}{p^{3/2}}\right\} - \frac{7}{3}\,L^{-1}\left\{\frac{1}{p + (2/3)}\right\}$$

$$= 6\,\frac{t^{2-1}}{1\,!} + 1 - 2\cdot\frac{t^{3/2-1}}{\Gamma(3/2)} - \frac{7}{3}\,e^{-2t/3}$$

$$= 6t + 1 - 4\sqrt{(t/\pi)} - (7/3)\,e^{-2t/3}.$$

Ex. 3. *Show that*

$$L^{-1}\left\{\frac{1}{p}\,sin\,\frac{1}{p}\right\} = t - \frac{t^3}{(3\,!)^2} + \frac{t^5}{(5\,!)^2} - \frac{t^7}{(7\,!)^2} + \dots$$

Sol. $$L^{-1}\left\{\frac{1}{p}\sin\frac{1}{p}\right\} = L^{-1}\left\{\frac{1}{p}\left(\frac{1}{p} - \frac{(1/p)^3}{3\,!} + \frac{(1/p)^5}{5\,!} - \frac{(1/p)^7}{7\,!} + \dots\right)\right\}$$

$$= L^{-1}\left\{\frac{1}{p^2}\right\} - \frac{1}{3\,!}\,L^{-1}\left\{\frac{1}{p^4}\right\} + \frac{1}{5\,!}\,L^{-1}\left\{\frac{1}{p^6}\right\} - \frac{1}{7\,!}\,L^{-1}\left\{\frac{1}{p^8}\right\} + \dots$$

$$= t - \frac{t^3}{(3\,!)^2} + \frac{t^5}{(5\,!)^2} - \frac{t^7}{(7\,!)^2} + \dots .$$

§ 9. CHANGE OF SCALE PROPERTY.

Theorem. *If $L^{-1}\{f(p)\} = F(t)$, then $L^{-1}\{f(ap)\} = (1/a)\,F(t/a)$.*

(Meerut 1989; Agra 81, 83; Kanpur 85; Rohilkhand 96)

Proof. We have, $f(p) = \int_0^\infty e^{-pt}\,F(t)\,dt.$

$\therefore$ $$f(ap) = \int_0^\infty e^{-apt}\,F(t)\,dt = \frac{1}{a}\int_0^\infty e^{-px}\,F\left(\frac{x}{a}\right)dx,$$

putting $at = x$, so that $dt = (1/a)\,dx$

$$= \frac{1}{a}\int_0^\infty e^{-pt}\,F\left(\frac{t}{a}\right)dt = \frac{1}{a}\,L\left\{F\left(\frac{t}{a}\right)\right\} = L\left\{\frac{1}{a}\,F\left(\frac{t}{a}\right)\right\}.$$

Hence $L^{-1}\{f(ap)\} = (1/a)\,F(t/a).$ **Proved.**

Ex. 2. *If* $L^{-1}\left\{\frac{e^{-1/p}}{p^{1/2}}\right\}=\frac{\cos 2\sqrt{t}}{\sqrt{(\pi t)}}$, *find* $L^{-1}\left\{\frac{e^{-a/p}}{p^{1/2}}\right\}$, *where* $a>0$.

Sol. Since $L^{-1}\left\{\frac{e^{-1/p}}{p^{1/2}}\right\}=\frac{\cos 2\sqrt{t}}{\sqrt{(\pi t)}}$,

$$\therefore \qquad L^{-1}\left\{\frac{e^{-1/pk}}{(pk)^{1/2}}\right\}=\frac{1}{k}\frac{\cos 2\sqrt{(t/k)}}{\sqrt{(\pi t/k)}}$$

or

$$L^{-1}\left\{\frac{e^{-1/pk}}{p^{1/2}}\right\}=\frac{\cos 2\sqrt{(t/k)}}{\sqrt{(\pi t)}}.$$

Taking $k=1/a$, we have $L^{-1}\left\{\frac{e^{-a/p}}{p^{1/2}}\right\}=\frac{\cos 2\sqrt{(at)}}{\sqrt{(\pi t)}}$.

Ex. 10. *Find* $L^{-1}\left\{\frac{(p+1)\,e^{-\pi p}}{p^2+p+1}\right\}$.

Sol. We have $L^{-1}\left\{\frac{p+1}{p^2+p+1}\right\}=L^{-1}\left\{\frac{(p+\frac{1}{2})+\frac{1}{2}}{(p+\frac{1}{2})^2+\frac{3}{4}}\right\}$

$$=e^{-t/2}\,L^{-1}\left\{\frac{p+\frac{1}{2}}{p^2+\frac{3}{4}}\right\}$$

$$=e^{-t/2}\,L^{-1}\left\{\frac{p}{p^2+(\sqrt{3}/2)^2}\right\}+\frac{1}{2}e^{-t/2}\,L^{-1}\left\{\frac{1}{p^2+(\sqrt{3}/2)^2}\right\}$$

$$=e^{-t/2}\cos(\sqrt{3}t/2)+\frac{1}{2}e^{-t/2}.\,(2/\sqrt{3})\sin(\sqrt{3}t/2)$$

$$=\frac{e^{-t/2}}{\sqrt{3}}\left[\sqrt{3}\cos(\sqrt{3}t/2)+\sin(\sqrt{3}t/2)\right].$$

$$\therefore \qquad L^{-1}\left\{\frac{(p+1)\,e^{-\pi p}}{p^2+p+1}\right\}=\begin{cases}\frac{e^{-(t-\pi)/2}}{\sqrt{3}}\left[\sqrt{3}\cos\frac{\sqrt{3}}{2}(t-\pi)+\sin\frac{\sqrt{3}}{2}(t-\pi)\right], & t>\pi\\ 0 & ,t<\pi\end{cases}$$

$$=\frac{e^{-(t-\pi)/2}}{\sqrt{3}}\left[\sqrt{3}\cos\frac{\sqrt{3}}{2}(t-\pi)+\sin\frac{\sqrt{3}}{2}(t-\pi)\right]H(t-\pi).$$

§ 11. INVERSE LAPLACE TRANSFORM OF DERIVATIVES.

Theorem. *If* $L^{-1}\{f(p)\} = F(t)$, *then*

$$L^{-1}\{f^n(p)\} = L^{-1}\left[\frac{d^n}{dp^n} f(p)\right] = (-1)^n t^n F(t)$$

$$= (-1)^n t^n L^{-1}\{f(p)\},\ n = 1, 2, 3, \ldots \qquad \textbf{(Meerut 1991)}$$

Proof. Since, we have

$$L\{t^n F(t)\} = (-1)^n \frac{d^n}{dp^n} f(p) = (-1)^n f^n(p),$$

$$\therefore \quad L^{-1}\{f^n(p)\} = L^{-1}\left[\frac{d^n}{dp^n} f(p)\right] = (-1)^n t^n F(t) = (-1)^n t^n L^{-1}\{f(p)\}.$$

§ 12. INVERSE LAPLACE TRANSFORM OF INTEGRALS.

Theorem. *If* $L^{-1}\{f(p)\} = F(t)$, *then* $L^{-1}\left[\int_p^\infty f(x)\, dx\right] = \dfrac{F(t)}{t}$.

Proof. Since, we have

$$L\left[\frac{1}{t} F(t)\right] = \int_p^\infty f(x)\, dx, \text{ provided } \lim_{t \to 0}\left[\frac{F(t)}{t}\right] \text{ exists.}$$

$$\therefore \quad L^{-1}\left[\int_p^\infty f(x)\, dx\right] = \frac{F(t)}{t}.$$

Ex. 1. *Find* $L^{-1}\left\{\dfrac{p}{(p^2+a^2)^2}\right\}$. **(Meerut 1986, 91; Rohilkhand 95)**

Sol. $L^{-1}\left\{\dfrac{p}{(p^2+a^2)^2}\right\} = L^{-1}\left\{-\dfrac{1}{2} \cdot \dfrac{d}{dp}\left(\dfrac{1}{p^2+a^2}\right)\right\} = -\dfrac{1}{2} L^{-1}\left\{\dfrac{d}{dp}\left(\dfrac{1}{p^2+a^2}\right)\right\}$

$$= -\frac{1}{2} t \,.\, (-1)^1 L^{-1}\left\{\frac{1}{p^2+a^2}\right\} = \frac{t}{2a} \sin at.$$

Ex. 4. *Find* (i) $L^{-1}\left\{\log\left(1+\frac{1}{p^2}\right)\right\}$ **(Meerut 1987; Agra 83)**

(ii) $L^{-1}\left\{\frac{1}{p}\log\left(1+\frac{1}{p^2}\right)\right\}$. **(Garhwal 1996)**

Sol. (i) Let

$$f(p)=\log\left(1+\frac{1}{p^2}\right)=-\log\left(\frac{p^2}{p^2+1}\right)=-2\log p+\log(p^2+1).$$

$$\therefore \qquad f'(p)=-\frac{2}{p}+\frac{2p}{p^2+1}.$$

$$\therefore \qquad L^{-1}\{f'(p)\}=-2+2\cos t$$

or $$-t\,L^{-1}\{f(p)\}=-2(1-\cos t)$$

or $$L^{-1}\{\log(1+1/p^2)\}=2(1-\cos t)/t.$$

(ii) From part (i), we have

$$F(t)=L^{-1}\{\log(1+1/p^2)\}=2(1-\cos t)/t.$$

$$\therefore \qquad L^{-1}\left\{\frac{1}{p}\log\left(1+\frac{1}{p^2}\right)\right\}=L^{-1}\left\{\frac{1}{p}f(p)\right\}=\int_0^t F(x)\,dx$$

$$=\int_0^t \frac{2}{x}(1-\cos x)\,dx.$$

Ex. 1. *Use the convolution theorem to find* $L^{-1}\left\{\dfrac{p^2}{(p^2+a^2)^2}\right\}$.

Sol. We have $L^{-1}\left\{\dfrac{p}{p^2+a^2}\right\} = \cos at.$

$\therefore$ By the convolution theorem, we have

$$L^{-1}\left\{\frac{p^2}{(p^2+a^2)^2}\right\} = L^{-1}\left\{\frac{p}{p^2+a^2}\cdot\frac{p}{p^2+a^2}\right\} = \int_0^t \cos ax \cos a\,(t-x)\,dx$$

$$= \int_0^t \cos ax\,(\cos at \cos ax + \sin at \sin ax)\,dx$$

$$= \cos at \int_0^t \cos^2 ax\,dx + \sin at \int_0^t \cos ax \sin ax\,dx$$

$$= \frac{1}{2}\cos at \int_0^t (1+\cos 2ax)\,dx + \frac{1}{2}\sin at \int_0^t \sin 2ax\,dx$$

$$= \cos at \int_0^t \cos^2 ax\,dx + \sin at \int_0^t \cos ax \sin ax\,dx$$

$$= \frac{1}{2}\cos at \int_0^t (1+\cos 2ax)\,dx + \frac{1}{2}\sin at \int_0^t \sin 2ax\,dx$$

$$= \frac{1}{2}\cos at\left[x + \frac{1}{2a}\sin 2ax\right]_0^t + \frac{1}{2}\sin at.\left[-\frac{1}{2a}\cos 2ax\right]_0^t$$

$$= \frac{1}{2}\cos at\left[t + \frac{1}{2a}\sin 2at\right] + \frac{1}{4a}\sin at\,(1-\cos 2at)$$

$$= \frac{1}{2}t\cos at + \frac{1}{4a}\sin at + \frac{1}{4a}(\sin 2at \cos at - \sin at \cos 2at)$$

$$= \frac{1}{2}t\cos at + \frac{1}{4a}[\sin at + \sin(2at - at)] = \frac{1}{2a}[at\cos at + \sin at].$$

15. *Reduce the differential equation $x^2p^2 + yp(2x+y) + y^2 = 0$, where $p = \frac{dy}{dx}$, to Clairaut's form by the substitutions $u = y$, $v = xy$. Hence or otherwise solve it. Also prove that $y + 4x = 0$ is singular solution of the given equation.* (KU 2003, 2003 C)

Sol. We have $\quad x^2p^2 + yp(2x+y) + y^2 = 0 \qquad \ldots(1)$

Let $\quad u = y \quad$ and $\quad v = xy$

$\therefore \quad \frac{du}{dx} = \frac{dy}{dx} = p \quad$ and $\quad \frac{dv}{dx} = x\frac{dy}{dx} + y \,.\, 1 = xp + y$

$\therefore \quad \frac{dv}{du} = \frac{dv}{dx}\Big/\frac{du}{dx} = \frac{xp+y}{p}$

Let $\quad P = \frac{dv}{du} \quad \therefore \quad P = \frac{xp+y}{p} \quad$ or $\quad p = \frac{y}{P-x}$

$\therefore \quad (1) \Rightarrow \quad x^2 \,.\, \frac{y^2}{(P-x)^2} + y \,.\, \frac{y}{P-x}(2x+y) + y^2 = 0$

$\Rightarrow \quad x^2y^2 + y^2(2x+y)(P-x) + y^2(P-x)^2 = 0$

$\Rightarrow \quad x^2 + (2xP - 2x^2 + yP - xy) + (P^2 + x^2 - 2Px) = 0$

$\Rightarrow \quad P^2 + yP - xy = 0 \quad \Rightarrow \quad P^2 + uP - v = 0$

$\therefore \quad v = Pu + P^2 \qquad \ldots(2)$

This is a Clairaut's equation.

Replacing P by c, the solution of (2) is $v = cu + c^2$

$\Rightarrow \quad \mathbf{xy = cy + c^2}.$

This is the general solution of (1).

Let $\quad f(x, y, p) = x^2p^2 + yp(2x+y) + y^2$

This is quadratic in p

$\therefore \quad$ Disc. $= 0 \quad \Rightarrow \quad y^2(2x+y)^2 - 4.x^2.\, y^2 = 0$

$\Rightarrow \quad y^2(4x^2 + y^2 + 4xy - 4x^2) = 0 \quad \Rightarrow \quad y^3(y + 4x) = 0 \qquad \ldots(3)$

Let $\quad \phi(x, y, c) = c^2 + cy - xy$

This is quadratic in c

$\therefore \quad$ Disc. $= 0 \quad \Rightarrow \quad y^2 - 4.1.(-xy) = 0$

$\Rightarrow \quad y(y + 4x) = 0 \qquad \ldots(4)$

Using (3), the p-discriminant relation ($ET^2C = 0$) can be written as

$$y(y+4x)\,.\,y^2\,.\,1 = 0$$

Using (4), the c-discriminant relation ($EN^2C^3 = 0$) can be written as

$$y(y+4x).\ 1^2\,.\,1^3 = 0$$

$\therefore \quad E = 0 \quad \Rightarrow \quad y(y+4x) = 0 \qquad \ldots(5)$

$(5) \Rightarrow \quad y = 0 \quad$ or $\quad y + 4x = 0$

$y + 4x = 0 \quad \Rightarrow \quad \frac{dy}{dx} + 4 = 0$

$p = -4$

$\therefore \quad (1) \quad \Rightarrow \quad x^2(-4)^2 + (-4x)(-4)(2x - 4x) + (-4x)^2 = 0$

$\Rightarrow \quad 16x^2 - 32x^2 + 16x^2 = 0$, which is true.

$\therefore \quad \mathbf{y + 4x = 0}$ is singular solution of the given equation.

Remark. $y = 0$ is also a singular solution.

13. *Reduce the equation $y^2(y - xp) = x^4 p^2$ to Clairaut's form by using the substitutions $x = \dfrac{1}{u}$, $y = \dfrac{1}{v}$ and hence find its singular solution and equations of extraneous loci.*

Sol. We have $\quad y^2(y - xp) = x^4p^2 \quad$ ...(1)

Let $\quad x = \dfrac{1}{u} \quad$ and $\quad y = \dfrac{1}{v}$

$\therefore \quad u = \dfrac{1}{x} \quad$ and $\quad v = \dfrac{1}{y}$

$\therefore \quad \dfrac{du}{dx} = -\dfrac{1}{x^2} \quad$ and $\quad \dfrac{dv}{dx} = -\dfrac{1}{y^2}\dfrac{dy}{dx} = -\dfrac{p}{y^2}$

$\therefore \quad \dfrac{dv}{du} = \dfrac{dv}{dx}\Big/\dfrac{du}{dx} = \dfrac{-p/y^2}{-1/x^2} = \dfrac{px^2}{y^2}$

Let $\quad P = \dfrac{dv}{du} \quad \therefore \quad P = p\dfrac{x^2}{y^2} \quad$ or $\quad p = P\dfrac{y^2}{x^2} = P\dfrac{u^2}{v^2}$

$\therefore \quad (1) \Rightarrow \quad \dfrac{1}{v^2}\left(\dfrac{1}{v} - \dfrac{1}{u}.P\dfrac{u^2}{v^2}\right) = \dfrac{1}{u^4}.P^2\dfrac{u^4}{v^4}$

$\Rightarrow \quad \dfrac{1}{v} - \dfrac{Pu}{v^2} = \dfrac{P^2}{v^2} \quad \Rightarrow \quad v - Pu = P^2$

$\therefore \quad v = Pu + P^2 \quad$ ...(2)

(2) is a Clairaut's equation. Replacing P by c, the solution of (2) is $v = cu + c^2$

$\Rightarrow \quad \dfrac{1}{y} = c.\dfrac{1}{x} + c^2 \quad \Rightarrow \quad x = cy + c^2xy$

This is the general solution of (1).

Let $\quad f(x, y, p) = y^2(y - xp) - x^4p^2$

$\Rightarrow \quad f(x, y, p) = -x^4p^2 - xy^2p + y^3$

This is quadratic in p

$\therefore \quad$ Disc. = 0. $\quad \Rightarrow \quad x^2y^4 - 4.-x^4.y^3 = 0$

$\Rightarrow \quad x^2y^3(y + 4x^2) = 0 \quad$ ...(3)

Let $\quad \phi(x, y, c) = c^2xy + cy - x$

This is quadratic in c

$\therefore \quad$ Disc. = 0 $\quad \Rightarrow \quad y^2 - 4.xy.-x = 0$

$\Rightarrow \quad y(y + 4x^2) = 0 \quad$ ...(4)

Using (3), the p-discriminant relation ($ET^2C = 0$) can be written as

$$y(y + 4x^2) \,.\, (xy)^2 \,.\, 1 = 0$$

Using (4), the c-discriminant relation ($EN^2C^3 = 0$) can be written as

$$y(y + 4x)^2 . 1^2 \,.\, 1^3 = 0$$

$\therefore$ $E = 0$ $\Rightarrow$ $y(y + 4x^2) = 0$ ...(5)

(5) $\Rightarrow$ $y = 0$ or $y + 4x^2 = 0$

$y = 0$ $\Rightarrow$ $\dfrac{dy}{dx} = 0$ *i.e.,* $p = 0$.

$y = 0, p = 0$ satisfies the given equation.

$y + 4x^2 = 0$ $\Rightarrow$ $\dfrac{dy}{dx} = -8x$ *i.e.,* $p = -8x$

$\therefore$ (1) $\Rightarrow$ $(-4x^2)^2(-4x^2 - x \,.\, -8x) = x^4 \,.\, 64x^2$

$\Rightarrow$ $-16x^4 \,.\, 4x^2 = 64x^6$, which is true.

$\therefore$ $y + 8x^2 = 0$ satisfies the given equation.

$\therefore$ **$y(y+ 8x^2) = 0$** is the singular solution of the given equation.

Also $T = xy$ $\therefore$ The equation of the Tac – locus is **$xy = 0$.**

5. If y_1 and y_2 are a fundamental set of solutions of

$$t^2y'' - 2y' + (3 + t)y = 0$$

and if $W(y_1, y_2)(2) = 3$, find the value of $W(y_1, y_2)(6)$.
[§3.2 #35]

Sol. Rewrite the equation as the form

$$y'' - \frac{2}{t^2}y' + \frac{3+t}{t^2}y = 0$$

Then by Abel's formula, the Wronskian of two solutions is

$$W(y_1, y_2)(t) = c\exp\left(\int \frac{2}{t^2}dt\right) = c\exp\left(-\frac{2}{t}\right) = ce^{-\frac{2}{t}}$$

for some constant c. Since $W(y_1, y_2)(2) = 3$, we have that $ce^{-1} = 3 \Rightarrow c = 3e$. That is, the Wronskian of y_1 and y_2 is

$$W(y_1, y_2)(t) = 3e^{\frac{t-2}{t}}$$

23. Find a second independent solution of the equation

$$(x-1)y'' - xy' + y = 0, \qquad x > 1;,\ y_1(x) = e^x$$

[§3.4 #34]

Sol. Rewrite the equation as the form $y'' - \frac{x}{x-1}y' + \frac{1}{x-1}y = 0$. According to above problem, the second independent solution $y_2(x)$ is

$$y_2(x) = y_1(x)\int^x \frac{\exp\left(\int^s \frac{\tau}{\tau-1}d\tau\right)}{y_1(s)^2}ds = e^x\int^x \frac{\exp\left(\int^s \frac{\tau-1+1}{\tau-1}d\tau\right)}{e^{2s}}ds$$
$$= e^x\int^x \frac{\exp\left(\int^s \left(1+\frac{1}{\tau-1}\right)d\tau\right)}{e^{2s}}ds = e^x\int^x \frac{\exp\left(s+\ln(s-1)+c_0\right)}{e^{2s}}ds$$
$$= e^x\int^x \frac{c_1(s-1)e^s}{e^{2s}}ds = c_1e^x\int^x (s-1)e^{-s}ds, \text{ where } c_1 = e^{c_0}$$
$$= c_1e^x\left(-xe^{-x}+c_2\right) = -c_1x + c_1c_2e^x$$

By taking $c_1 = -1,\ c_2 = 0$, we get $y_2(x) = x$. □

20. Use the method of reduction of order to find a second solution of the differential equation

$$(x-1)y'' - xy' + y = 0, \qquad x > 1;,\ y_1(x) = e^x$$

[§3.4 #29]

Sol. Let $y_2(x) = v(x)e^x = vy_1$. Substituting into the equation, we get

$$(x-1)(v''y_1 + 2v'y_1' + vy_1'') - x(v'y_1 + vy_1') + vy_1 = 0$$

That is,

$$(x-1)y_1v'' + (2(x-1)y_1' - xy_1)v' = 0$$

Let $w = v'$, then we have that $w' = \left(\frac{x}{x-1} - \frac{2y_1'}{y_1}\right)w$, which is a separable equation. Thus

$$\int\frac{dw}{w} = \int\left(\frac{x}{x-1} - \frac{2y_1'}{y_1}\right)dx \Rightarrow \ln|w| = x + \ln(x-1) - 2\ln|y_1| + c_0$$

which implies that

$$w(x) = v'(x) = \frac{c_1(x-1)e^x}{y_1^2} = c_1(x-1)e^{-x}, \text{ where } c_1 = e^{c_0}$$

By integrating $w(x)$, we get

$$v(x) = c_1\int(x-1)e^{-x}dx = -c_1xe^{-x} + c_2$$

for some constants $c_1,\ c_2$. Thus, $y_2(t) = -c_1x + c_2e^x$. By setting $c_1 = -1,\ c_2 = 0$, we get $y_2(t) = x$. □

19. Use the method of reduction of order to find a second solution of the differential equation

$$xy'' - y' + 4x^3y = 0, \qquad x > 0;,\ y_1(x) = \sin x^2$$

[§3.4 #27]

Sol. Let $y_2(x) = v(x)\sin x^2 = vy_1$. Substituting into the equation, we get

$$x(v''y_1 + 2v'y_1' + vy_1'') - (v'y_1 + vy_1') + 4x^3vy_1 = 0$$

That is,

$$xy_1v''+2xy_1'v'-y_1v'+(xy_1''-y_1'+4x^3y_1)v = xy_1v''+(2xy_1'-y_1)v' = 0$$

Let $w = v'$, then we have that $w' = \left(\frac{1}{x} - \frac{2y_1'}{y_1}\right)w$, which is a separable equation. Thus

$$\int \frac{dw}{w} = \int \left(\frac{1}{x} - \frac{2y_1'}{y_1}\right)dx \Rightarrow \ln|w| = \ln x - 2\ln|y_1| + c_0$$

which implies that

$$w(x) = v'(x) = \frac{c_1x}{y_1^2} = \frac{c_1x}{\sin^2 x^2}, \text{ where } c_1 = e^{c_0}$$

By integrating $w(x)$, we get

$$v(x) = c_1\int \frac{xdx}{\sin^2 x^2} = \frac{c_1}{2}\int \csc^2 u du = -\frac{c_1}{2}\cot u + c_2$$
$$= -\frac{c_1}{2}\frac{\cos x^2}{\sin x^2} + c_2$$

for some constants $c_1,\ c_2$. Thus, $y_2(t) = -\frac{c_1}{2}\cos x^2 + c_2\sin x^2$. By setting $c_1 = -2,\ c_2 = 0$, we get $y_2(t) = \cos x^2$. $\square$

Ex. 2. *Solve* $p^2 + 2py \cot x = y^2$.

Sol. The given differential equation is

$$p^2 + 2py \cot x - y^2 = 0.$$

Solving for p, we get

$$p = \frac{dy}{dx} = \frac{-2y\cot x \pm \sqrt{(4y^2 \cot^2 x + 4y^2)}}{2}$$

$$= -y\cot x \pm y \operatorname{cosec} x = y(-\cot x \pm \operatorname{cosec} x).$$

Thus the component equations are

$$\frac{dy}{dx} = y(-\cot x + \operatorname{cosec} x), \qquad ...(1)$$

and

$$\frac{dy}{dx} = -y(\cot x + \operatorname{cosec} x). \qquad ...(2)$$

In each of the above differential equations, the variables are separable.

From (1), separating the variables, we have

$$\frac{dy}{y} = (-\cot x + \operatorname{cosec} x)\, dx.$$

Integrating, we get

$$\log y - \log c = -\log \sin x + \log \tan \tfrac{1}{2}x$$

or

$$\log\left(\frac{y}{c}\right) = \log\left\{\frac{\tan \frac{1}{2}x}{\sin x}\right\} = \log\left\{\frac{\sin \frac{1}{2}x / \cos \frac{1}{2}x}{2 \sin \frac{1}{2}x \cos \frac{1}{2}x}\right\}$$

$$= \log\left\{\frac{1}{2\cos^2 \frac{1}{2}x}\right\} = \log\left\{\frac{1}{1 + \cos x}\right\}.$$

$\therefore \quad y/c = 1/(1 + \cos x) \quad$ or $\quad y = c/(1 + \cos x). \qquad ...(3)$

From (2), separating the variables, we have

$$\frac{dy}{y} = -(\cot x + \operatorname{cosec} x)\, dx.$$

Integrating, we get

$$\log y - \log c = -(\log \sin x + \log \tan \tfrac{1}{2}x)$$

or

$$\log(y/c) = -\log\{(\sin x)(\tan \tfrac{1}{2}x)\}$$

$$= -\log\{2 \sin \tfrac{1}{2}x \cos \tfrac{1}{2}x \cdot (\sin \tfrac{1}{2}x / \cos \tfrac{1}{2}x)\} = -\log(2\sin^2 \tfrac{1}{2}x)$$

$$= -\log(1 - \cos x) = \log(1 - \cos x)^{-1} = \log\{1/(1 - \cos x)\}.$$

$\therefore \quad y/c = 1/(1 - \cos x) \quad$ or $\quad y = c/(1 - \cos x). \qquad ...(4)$

Thus the solutions of the given differential equation are given by (3) and (4). The single combined solution is

$$\left(y - \frac{c}{1 + \cos x}\right)\left(y - \frac{c}{1 - \cos x}\right) = 0.$$

5. Working Rules for Solving $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$

Step I. Find a particular integral of the equation $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=0$. Call this function as u. Sometimes it is given with the problem itself.

Step II. Take $y = uv$ and change the dependent variable y to v. The resultant equation will be a linear equation of first order with dependent variable $\frac{dv}{dx}$.

Step III. Put $p=\frac{dv}{dx}$. Solve the equation and find p. Put $p=\frac{dv}{dx}$ and integrate to find the value of v.

Step IV. Find $y = uv$. This gives the general solution of the given equation.

Example 1. *Solve*

$$x\frac{d^2y}{dx^2}-(2x-1)\frac{dy}{dx}+(x-1)y=0.$$

Sol. We have

$$x\frac{d^2y}{dx^2}-(2x-1)\frac{dy}{dx}+(x-1)y=0.$$

$$\Rightarrow \qquad \frac{d^2y}{dx^2}-\frac{2x-1}{x}\frac{dy}{dx}+\frac{x-1}{x}y=0 \qquad \ldots(1)$$

Comparing (1) with

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R, \text{ we get}$$

$$P=-\frac{2x-1}{x},\quad Q=\frac{x-1}{x},\quad R=0$$

Here $\qquad 1+P+Q=1-\frac{2x-1}{x}+\frac{x-1}{x}=0$

$\therefore$ e^x is a particular integral of (1), whose right member is already zero.

Let $y=e^x v$.

$$\therefore \qquad \frac{dy}{dx}=e^x\frac{dv}{dx}+ve^x=e^x\left(\frac{dv}{dx}+v\right)$$

and

$$\frac{d^2y}{dx^2}=e^x\left(\frac{d^2v}{dx^2}+\frac{dv}{dx}\right)+e^x\left(\frac{dv}{dx}+v\right)=e^x\left(\frac{d^2v}{dx^2}+2\frac{dv}{dx}+v\right).$$

$\therefore$ (1) becomes

$$e^x\left(\frac{d^2v}{dx^2}+2\frac{dv}{dx}+v\right)-\frac{2x-1}{x}e^x\left(\frac{dv}{dx}+v\right)+\frac{x-1}{x}e^x v=0$$

$$\Rightarrow \quad e^x\frac{d^2v}{dx^2}+\left(2e^x-\frac{2x-1}{x}e^x\right)\frac{dv}{dx}+\left(e^x-\frac{2x-1}{x}e^x+\frac{x-1}{x}e^x\right)v=0$$

$$\Rightarrow \qquad e^x\frac{d^2v}{dx^2}+\frac{e^x}{x}\frac{dv}{dx}=0 \quad\Rightarrow\quad \frac{d^2v}{dx^2}+\frac{1}{x}\frac{dv}{dx}=0$$

$$\Rightarrow \qquad \frac{dp}{dx}+\frac{1}{x}p=0 \text{ where } p=\frac{dv}{dx}$$

$\Rightarrow \qquad \dfrac{dp}{p} + \dfrac{dx}{x} = 0 \qquad \Rightarrow \quad \log p + \log x = \log c_1$

$\Rightarrow \qquad px = c_1 \qquad \Rightarrow \quad \dfrac{dv}{dx} x = c_1$

$\Rightarrow \qquad dv = c_1 \dfrac{dx}{x} \qquad \Rightarrow \quad v = c_1 \log x + c_2.$

$\therefore \qquad y = e^x v = e^x (c_1 \log x + c_2)$

$\therefore$ The general solution of (1) is $\mathbf{y = c_1 e^x \log x + c_2 e^x}$.

Example 2. *Solve* $(x+2)\dfrac{d^2y}{dx^2} - (2x+5)\dfrac{dy}{dx} + 2y = (x+1)e^x.$ (DLU 2004)

Sol. Dividing by $x + 2$, the given equation in standard form is

$$\frac{d^2y}{dx^2} - \frac{2x+5}{x+2}\frac{dy}{dx} + \frac{2}{x+2} y = \frac{x+1}{x+2} e^x . \qquad ...(1)$$

Comparing (1) with $\dfrac{d^2y}{dx^2} + P\dfrac{dy}{dx} + Qy = R$, we get

$$P = -\frac{2x+5}{x+2}, \quad Q = \frac{2}{x+2}, \quad R = \frac{x+1}{x+2} e^x$$

Here $\quad 4 + 2P + Q = 4 + 2\left(-\dfrac{2x+5}{x+2}\right) + \dfrac{2}{x+2} = 0$

$\therefore$ e^{2x} is a particular integral of (1) with its right member replaced by zero.

Let $\qquad y = e^{2x} v.$

$\therefore \qquad \dfrac{dy}{dx} = e^{2x}\left(\dfrac{dv}{dx} + 2v\right)$ and $\dfrac{d^2y}{dx^2} = e^{2x}\left(\dfrac{d^2v}{dx^2} + 4\dfrac{dv}{dx} + 4v\right).$

Putting the values of y, $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in (1), we get

$$e^{2x}\left(\frac{d^2v}{dx^2} + 4\frac{dv}{dx} + 4v\right) - \frac{2x+5}{x+2} e^{2x}\left(\frac{dv}{dx} + 2v\right) + \frac{2}{x+2} e^{2x} v = \frac{x+1}{x+2} e^x$$

$$\Rightarrow \qquad \frac{d^2v}{dx^2} + 4\frac{dv}{dx} + 4v - \frac{2x+5}{x+2}\left(\frac{dv}{dx} + 2v\right) + \frac{2}{x+2} v = \frac{x+1}{x+2} e^{-x}$$

$$\Rightarrow \qquad \frac{d^2v}{dx^2} + \frac{2x+3}{x+2}\frac{dv}{dx} = \frac{x+1}{x+2} e^{-x}$$

$$\Rightarrow \qquad \frac{dp}{dx} + \frac{2x+3}{x+2} p = \frac{x+1}{x+2} e^{-x} \qquad ...(2)$$

where $\quad p = \dfrac{dv}{dx}$.

(2) is a linear differential equation of the first order.

$$\text{I.F.} = e^{\int \frac{2x+3}{x+2}dx} = e^{\int\left(2-\frac{1}{x+2}\right)dx} = e^{2x-\log(x+2)} = e^{2x} + e^{\log(x+2)} = \frac{e^{2x}}{x+2}.$$

$\therefore$ The solution of (2) is

$$p\,\frac{e^{2x}}{x+2} = \int \frac{x+1}{x+2}e^{-x}\cdot\frac{e^{2x}}{x+2}\,dx + c_1$$

$$\Rightarrow \quad p\frac{e^{2x}}{x+2} = \int\frac{x+1}{(x+2)^2}e^x\,dx + c_1 = \int\left(\frac{1}{x+2}-\frac{1}{(x+2)^2}\right)e^x\,dx + c_1$$

$$= \frac{e^x}{x+2} + c_1$$

$$\therefore \quad p = e^{-x} + c_1 e^{-2x}(x+2) \quad \text{or} \quad \frac{dv}{dx} = e^{-x} + c_1 e^{-2x}(x+2)$$

$$\Rightarrow \quad \int dv = \int (e^{-x} + c_1 e^{-2x}(x+2))\,dx + c_2$$

$$\therefore \quad v = -e^{-x} + c_1\int (x+2)e^{-2x}\,dx + c_2$$

$$= -e^{-x} + c_1\left[(x+2)\cdot\frac{e^{-2x}}{-2} - \int 1\cdot\frac{e^{-2x}}{-2}\,dx\right] + c_2$$

$$= -e^{-x} - \frac{c_1}{2}(x+2)e^{-2x} - \frac{c_1}{4}e^{-2x} + c_2$$

$$\therefore \quad v = -e^{-x} - \frac{c_1}{4}(2x+5)e^{-2x} + c_2$$

$$\therefore \quad y = e^{2x}v = -e^x - \frac{c_1}{4}(2x+5) + c_2 e^{2x}$$

$\therefore$ The general solution of (1) is

$$\mathbf{y = -e^x - \frac{c_1}{4}(2x+5) + c_2 e^{2x}.}$$

Example 3. *Solve* $x^2\dfrac{d^2y}{dx^2} - 2x(1+x)\dfrac{dy}{dx} + 2(1+x)y = x^3$.

Sol. Dividing by x^2, the given equation in standard form is

$$\frac{d^2y}{dx^2} - \frac{2(1+x)}{x}\frac{dy}{dx} + \frac{2(1+x)}{x^2}y = x. \qquad \text{...(1)}$$

Comparing (1) with $\dfrac{d^2y}{dx^2} + P\dfrac{dy}{dx} + Qy = R$, we get

$$P = -\frac{2(1+x)}{x},\ Q = \frac{2(1+x)}{x^2},\ R = x.$$

Here $P + xQ = -\dfrac{2(1+x)}{x} + \dfrac{2(1+x)}{x} = 0.$

$\therefore$ x is a particular integral of (1) with its right member replaced by zero.

Let $y = xv$.

$$\therefore \quad \frac{dy}{dx} = x\frac{dv}{dx} + v \quad \text{and} \quad \frac{d^2y}{dx^2} = \left(x\frac{d^2v}{dx^2} + 1.\frac{dv}{dx}\right) + \frac{dv}{dx} = x\frac{d^2v}{dx^2} + 2\frac{dv}{dx}$$

Putting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in (1), we get

$$x\frac{d^2v}{dx^2} + 2\frac{dv}{dx} - \frac{2(1+x)}{x}\left(x\frac{dv}{dx} + v\right) + \frac{2(1+x)}{x^2}xv = x$$

$$\Rightarrow \quad x\frac{d^2y}{dx^2} - 2x\frac{dv}{dx} = x \quad \Rightarrow \quad \frac{d^2y}{dx} - 2\frac{dv}{dx} = 1 \quad \Rightarrow \quad \frac{dp}{dx} - 2p = 1 \qquad \text{...(2)}$$

where $p = \frac{dv}{dx}$

(2) is a linear differential equation of the first order.

$$\text{I.F.} = e^{\int -2\,dx} = e^{-2x}$$

$\therefore$ The solution of (2) is

$$p \,.\, e^{-2x} = \int e^{-2x}\,dx + c_1 \qquad \Rightarrow \quad pe^{-2x} = \frac{e^{-2x}}{-2} + c_1$$

$$\Rightarrow \quad \frac{dv}{dx} = -\frac{1}{2} + c_1 e^{2x} \qquad \Rightarrow \quad \int dv = \int\left(-\frac{1}{2} + c_1 e^{2x}\right)dx + c_2$$

Example 6. *Solve* $\sin^2 x \dfrac{d^2y}{dx^2} = 2y$ *given that* $y = \cot x$ *is a solution.*

Sol. We have

$$\sin^2 x \frac{d^2y}{dx^2} - 2y = 0 \quad \text{or} \quad \frac{d^2y}{dx^2} - \frac{2}{\sin^2 x} y = 0 \qquad ...(1)$$

Let $\quad y = v \cot x.$

$$\therefore \qquad \frac{dy}{dx} = -v \operatorname{cosec}^2 x + \cot x \frac{dv}{dx}$$

and

$$\frac{d^2y}{dx^2} = v \,.\, 2 \operatorname{cosec}^2 x \cot x - \operatorname{cosec}^2 x \frac{dv}{dx} + \cot x \frac{d^2v}{dx^2} - \operatorname{cosec}^2 x \frac{dv}{dx}$$

$$= 2v \operatorname{cosec}^2 x \cot x - 2 \operatorname{cosec}^2 x \frac{dv}{dx} + \cot x \frac{d^2v}{dx^2}$$

Putting the values of y, $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in (1), we get

$$2v \operatorname{cosec}^2 x \cot x - 2 \operatorname{cosec}^2 x \frac{dv}{dx} + \cot x \frac{d^2v}{dx^2} - \frac{2}{\sin^2 x} v \cot x = 0$$

$$\Rightarrow \qquad \cot x \frac{d^2v}{dx^2} - 2 \operatorname{cosec}^2 x \frac{dv}{dx} = 0$$

$$\Rightarrow \qquad \cot x \frac{dp}{dx} - 2p \operatorname{cosec}^2 x = 0 \qquad ...(2)$$

where $p = \dfrac{dv}{dx}$

$$\Rightarrow \qquad \frac{dp}{dx} - \frac{2 \operatorname{cosec}^2 x}{\cot x} dx = 0.$$

Integrating, we get

$$\log p + 2 \log \cot x = \log c_1$$

$$\Rightarrow \qquad p \cot^2 x = c_1 \;\Rightarrow\; \frac{dv}{dx} = c_1 \tan^2 x \;\Rightarrow\; dv = c_1 (\sec^2 x - 1)\, dx$$

Integrating, we get

$$v = c_1 (\tan x - x) + c_2$$

$$\therefore \qquad y = v \cot x = c_1 (1 - x \cot x) + c_2 \cot x$$

$\therefore$ **The general solution of (1) is** $\mathbf{y = c_1 (1 - x \cot x) + c_2 \cot x.}$

Example 3. *Solve* $\dfrac{d}{dx}\left(\cos^2 x \dfrac{dy}{dx}\right) + y\cos^2 x = 0.$ (KU 2004)

Sol. We have $\dfrac{d}{dx}\left(\cos^2 x \dfrac{dy}{dx}\right) + y\cos^2 x = 0.$

$\Rightarrow \quad \cos^2 x \dfrac{d^2y}{dx^2} + (-2\cos x \sin x)\dfrac{dy}{dx} + y\cos^2 x = 0$

Dividing by $\cos^2 x$, we get

$$\frac{d^2y}{dx^2} - 2\tan x \frac{dy}{dx} + y = 0 \qquad ...(1)$$

Here $\quad P = -2\tan x,\ Q = 1,\ R = 0.$

Now $\quad Q_1 = Q - \dfrac{1}{4}P^2 - \dfrac{1}{2}\dfrac{dP}{dx} = 1 - \dfrac{1}{4}(4\tan^2 x) - \dfrac{1}{2}(-2\sec^2 x)$

$= 2$, which is a constant.

Now $\quad \int P\,dx = \int -2\tan x\,dx = -2\log\sec x$

Let $\quad u = e^{-\frac{1}{2}\int P\,dx}$. $\quad \therefore \quad u = e^{-\frac{1}{2}\int(-2\log\sec x)} = \sec x.$

Let $\quad y = uv = (\sec x)\,v$

$\therefore$ (1) reduces to $\dfrac{d^2v}{dx^2} + Q_1 v = R_1 \qquad ...(2)$

where $\quad R_1 = \dfrac{R}{u} = \dfrac{0}{\sec x} = 0.$

$\therefore \quad (2) \Rightarrow \dfrac{d^2v}{dx^2} + 2v = 0 \Rightarrow (D^2 + 2)v = 0 \qquad ...(3)$

$\therefore$ The A.E. of (3) is $D^2 + 2 = 0. \quad \therefore \quad D = \pm\sqrt{2}\,i$

$\therefore \quad v = c_1\cos\sqrt{2}\,x + c_2\sin\sqrt{2}\,x.$

$\therefore \quad y = uv$ implies

$\mathbf{y = \sec x\,[c_1\cos\sqrt{2}\,x + c_2\sin\sqrt{2}\,x]}$

This is the general solution of the given equation.

Example 4. *Solve* $\dfrac{d^2y}{dx^2}+2x\dfrac{dy}{dx}+(x^2+1)y=x^3+3x.$ (MDU 2005)

Sol. We have

$$\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+(x^2+1)y=x^3+3x. \qquad ...(1)$$

Here $\quad P=2x,\ Q=x^2+1,\ R=x^3+3x.$

Now $\quad Q_1=Q-\frac{1}{4}P^2-\frac{1}{2}\frac{dP}{dx}=x^2+1-\frac{1}{4}\cdot 4x^2-\frac{1}{2}\cdot 2=0$, a constant.

Now $\quad \int P\,dx=\int 2x\,dx=x^2$

Let $\quad u=e^{-\frac{1}{2}\int P\,dx}.\ \therefore\ u=e^{-\frac{1}{2}x^2}$

Let $\quad y=uv=e^{-\frac{1}{2}x^2}.v$

$\therefore$ (1) reduces to $\dfrac{d^2v}{dx^2}+Q_1v=R_1 \qquad ...(2)$

where $\quad R_1=\dfrac{R}{u}=(x^3+3x)e^{x^2/2}.$

$\therefore$ (2) $\Rightarrow \dfrac{d^2v}{dx^2}+0.v=(x^3+3x)\,e^{x^2/2} \Rightarrow \dfrac{d^2v}{dx^2}=(x^3+3x)e^{x^2/2}.$

Integrating, we get

$$\frac{dv}{dx}=\int(x^3+3x)e^{x^2/2}\,dx+c_1=\int(x^2+3)\,x\,e^{x^2/2}\,dx+c_1$$

$$=\int(2t+3)\,e^t\,dt+c_1 \quad \text{where } t=x^2/2$$

$$=(2t+3)e^t-\int 2.e^t\,dt+c_1=(2t+3)e^t-2e^t+c_1=(2t+1)\,e^t+c_1$$

$$=(x^2+1)\,e^{x^2/2}+c_1$$

Integrating again we get

$$v=\int(x^2+1)\,e^{x^2/2}\,dx+c_1x+c_2.$$

Now, $\quad \int x^2\,e^{x^2/2}\,dx=\int x\,(x\,e^{x^2/2})\,dx=x.e^{x^2/2}-\int 1.e^{x^2/2}\,dx$

$\therefore \quad \int(x^2+1)e^{x^2/2}\,dx=xe^{x^2/2}.$

$\therefore \quad v=xe^{x^2/2}+c_1x+c_2$

$\therefore\ y=uv$ implies $\quad y=e^{-x^2/2}\,[xe^{x^2/2}+c_1x+c_2]$

or $\quad \mathbf{y=x+(c_1x+c_2)\ e^{-x^2/2}.}$

This is the general solution of the given equation.

Example 5. *Solve*

$$x^2(\log x)^2\frac{d^2y}{dx^2}-2x\log x\frac{dy}{dx}+(2+\log x)-2(\log x)^2\ y=x^2(\log x)^3.$$

Sol. Dividing the given equation by $x^2(\log x)^2$, we get

$$\frac{d^2y}{dx^2}-\frac{2}{x\log x}\frac{dy}{dx}+\frac{2+\log x-2(\log x)^2}{x^2(\log x)^2}y=\log x \qquad ...(1)$$

Here $\quad P=-\dfrac{2}{x\log x},\ Q=\dfrac{2+\log x-2(\log x)^2}{x^2(\log x)^2},\ R=\log x$

Now, $\quad Q_1=Q-\dfrac{1}{4}P^2-\dfrac{1}{2}\dfrac{dP}{dx}$

$$=\frac{2+\log x-2(\log x)^2}{x^2(\log x)^2}-\frac{1}{4}.\frac{4}{x^2(\log x)^2}$$

$$-\frac{1}{2}.(-2)\left[\frac{1}{x}.(-1)(\log x)^{-2}.\frac{1}{x}+\frac{1}{\log x}\left(\frac{-1}{x^2}\right)\right]$$

$$=\frac{2+\log x-2(\log x)^2}{x^2(\log x)^2}-\frac{1}{x^2(\log x)^2}-\frac{1}{x^2(\log x)^2}-\frac{1}{x^2\log x}$$

$$=-\frac{2}{x^2},\text{ which is of the form }\frac{\lambda}{x^2}.$$

Now $\quad \int P\,dx=\int-\dfrac{2}{x\log x}dx=-2\log\log x$

Let $\quad u=e^{-\frac{1}{2}\int P dx}.\quad\therefore\quad u=e^{-\frac{1}{2}.-2\log\log x}=e^{\log\log x}=\log x$

Let $\quad y=uv=(\log x)\,v.$

$\therefore$ (1) reduces to $\quad \dfrac{d^2v}{dx^2}+Q_1 v=R_1 \qquad ...(2)$

where $\quad R_1=\dfrac{R}{u}=\dfrac{\log x}{\log x}=1.$

$\therefore$ (2) $\Rightarrow\quad \dfrac{d^2v}{dx^2}-\dfrac{2}{x^2}v=1\quad\Rightarrow\quad x^2\dfrac{d^2v}{dx^2}-2v=x^2 \qquad ...(3)$

Let $\quad z=\log x.\quad\therefore\quad x=e^z$

$\therefore\quad x^2\dfrac{d^2}{dx^2}=D(D-1),\quad$ where $D=\dfrac{d}{dz}.$

$\therefore$ (3) $\Rightarrow\quad (D(D-1)-2)v=e^{2z}\quad\Rightarrow\quad(D^2-D-2)\,v=e^{2z} \qquad ...(4)$

$\therefore$ The A.E. of (4) is $\quad D^2-D-2=0.\quad\therefore\quad D=-1,2$

$\therefore\quad$ C.F. $=c_1e^{-z}+c_2e^{2z}=c_1(e^z)^{-1}+c_2(e^z)^2=c_1x^{-1}+c_2x^2$

$$\text{P.I.}=\frac{1}{D^2-D-2}e^{2z}=z.\frac{1}{2D-1}e^{2z}=z.\frac{1}{2(2)-1}e^{2z}=\frac{z}{3}e^{2z}=\frac{\log x}{3}.x^2$$

$\therefore\quad v=\text{C.F.}+\text{P.I.}=c_1x^{-1}+c_2x^2+\dfrac{x^2\log x}{3}$

$\therefore\quad y=uv$ implies $\mathbf{y=\log x\left(\dfrac{c_1}{x}+c_2x^2+\dfrac{x^2\log x}{3}\right)}.$

This is the general solution of the given equation.

Example 6. *Solve*

$$\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 1)\,y = -3e^{x^2}\,(\sin 2x + 5e^{-2x} + 6).$$

Sol. We have

$$\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 1)\,y = -3e^{x^2}\,(\sin 2x + 5e^{-2x} + 6). \qquad ...(1)$$

Here $\quad P = -4x,\ Q = 4x^2 - 1,\ R = -3\,e^{x^2}\,(\sin 2x + 5e^{-2x} + 6)$

Now $\quad Q_1 = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx} = 4x^2 - 1 - \frac{1}{4}.16x^2 - \frac{1}{2}(-4)$

$= 4x^2 - 1 - 4x^2 + 2 = 1$, which is a constant.

Now $\quad \int P\,dx = \int -4x\,dx = -2x^2$

Let $\quad u = e^{-\frac{1}{2}\int P\,dx}. \quad \therefore \quad u = e^{-\frac{1}{2}.-2x^2} = e^{x^2}.$

Let $\quad y = uv = e^{x^2}\,v.$

$\therefore \quad$ (1) reduces to $\dfrac{d^2v}{dx^2} + Q_1\,v = R_1 \qquad ...(2)$

where $\quad R_1 = \dfrac{R}{u} = \dfrac{-3e^{x^2}\,(\sin 2x + 5e^{-2x} + 6)}{e^{x^2}} = -3(\sin 2x + 5e^{-2x} + 6)$

$\therefore \quad (2) \Rightarrow \quad \dfrac{d^2v}{dx^2} + 1.v = -3(\sin 2x + 5e^{-2x} + 6)$

where $\quad R_1 = \dfrac{R}{u} = \dfrac{-3e^{x^2}\,(\sin 2x + 5e^{-2x} + 6)}{e^{x^2}} = -3(\sin 2x + 5e^{-2x} + 6)$

$\therefore \quad (2) \Rightarrow \quad \dfrac{d^2v}{dx^2} + 1.v = -3(\sin 2x + 5e^{-2x} + 6)$

$\Rightarrow \quad (D^2 + 1)v = -3\sin 2x - 15\,e^{-2x} - 18. \qquad ...(3)$

$\therefore \quad$ The A.E. of (3) is $\quad D^2 + 1 = 0. \quad \therefore \quad D = \pm i$

$\therefore \quad$ C.F. $= c_1\cos x + c_2\sin x$

(3+2sinx+cosx)dy=(1+2siny+cosy)dxthe question is from equation of first order and first degree

$$\frac{dy}{1+2\sin y+\cos y}=\frac{dx}{3+2\sin x+\cos x}$$

Use integrals to calculate $\frac{dy}{1+2\sin y+\cos y}$ and $\frac{dx}{3+2\sin x+\cos x}$.

Use the formula $\sin y=2\frac{\tan\left(\frac{y}{2}\right)}{1+\tan^2\left(\frac{y}{2}\right)}$ $\cos y=\frac{1-\tan^2\left(\frac{y}{2}\right)}{1+\tan^2\left(\frac{y}{2}\right)}$

Substitute $\tan\left(\frac{y}{2}\right)=u$

$$\frac{1+\tan^2\left(\frac{y}{2}\right)}{2}dy=du=>dy=\frac{2du}{1+u^2}$$

$$\int\frac{dy}{1+2\sin y+\cos y}=\int\frac{\frac{2du}{1+u^2}}{1+4\frac{u}{1+u^2}+\frac{1-u^2}{1+u^2}}$$

$$\int\frac{2du}{1+u^2+4u+1-u^2}=\int\frac{2du}{2+4u}=\int\frac{du}{1+2u}$$

Substitute $1+2u=v=>2du=dv=>du=\frac{dv}{2}$

$$\int\frac{du}{1+2u}=\int\frac{dv}{2}v=\left(\frac{1}{2}\right)\cdot\ln v+c=\frac{\ln(1+2u)}{2}+c=\left(\frac{\ln\left(1+2\tan\left(\frac{y}{2}\right)\right)}{2}+c\right.$$

$$\int\frac{dy}{1+2\sin y+\cos y}=\frac{\ln\left(1+2\tan\left(\frac{y}{2}\right)\right)}{2}+c$$

Calculate integral of $\frac{dx}{3+2\sin x+\cos x}$.

Substitute $\tan\left(\frac{x}{2}\right)=p=>dx=\frac{2dp}{1+p^2}$

$$\int\frac{dx}{3+2\sin x+\cos x}=\int\frac{\frac{2dp}{1+p^2}}{3+4\frac{p}{1+p^2}+\frac{1-p^2}{1+p^2}}$$

$$\int\frac{2dp}{3+3p^2+4p+1-p^2}=\int\frac{2dp}{2p^2+4p+4}=\int\frac{dp}{p^2+2p+2}$$

Create arctan function

$$\int\frac{dp}{(p^2+2p+1)-1+2}=\int\frac{dp}{(p+1)^2+1}=\arctan(p+1)+c$$

$$\int\frac{dx}{3+2\sin x+\cos x}=\arctan\left(\tan\left(\frac{x}{2}\right)+1\right)+c$$

ANSWER: $\frac{\ln\left(1+2\tan\left(\frac{y}{2}\right)\right)}{2}=\arctan\left(\tan\left(\frac{x}{2}\right)+1\right)+c$

Example 5. **Find the orthogonal trajectories of the system of circles touching a given straight line at a given point.**

Solution. Taking the given point as the origin and the given line as y-axis. Then the equation of the given family of circle is

$$x^2 + y^2 - 2ax = 0 \qquad \text{where } a \text{ is the parameter} \qquad \ldots(1)$$

Differentiating (1) with respect to x we get

$$2x + 2y \cdot \frac{dy}{dx} - 2a = 0$$

or
$$a = x + y\frac{dy}{dx} \qquad \ldots(2)$$

Eliminating a between (1) and (2), we get

$$y^2 - x^2 - 2xy\frac{dy}{dx} = 0 \qquad \ldots(3)$$

which is the differential equation of the family of circles.

Putting $-\frac{dx}{dy}$ for $\frac{dy}{dx}$ in (3) the differential equation of the required family of trajectories is

$$y^2 - x^2 + 2xy\frac{dx}{dy} = 0$$

or
$$y^2 - x^2 = -2xy \cdot \frac{dx}{dy}$$

or
$$\frac{dy}{dx} = \frac{-2xy}{y^2 - x^2}$$

or
$$\frac{dy}{dx} = \frac{2xy}{x^2 - y^2}$$

or
$$v + x \cdot \frac{dv}{dx} = \frac{2vx^2}{x^2\,[1 - v^2]}, \qquad \text{where } y = vx$$

or
$$x\frac{dv}{dx} = \frac{2v}{1 - v^2} - v$$

or
$$x\frac{dv}{dx} = \frac{2v - v + v^3}{1 - v^2}$$

or
$$x\frac{dv}{dx} = \frac{v + v^3}{1 - v^2}$$

or
$$\frac{1 - v^2}{v + v^3}\,dv = \frac{dx}{x}$$

or
$$\frac{1 + 3v^2}{v + v^3} - 2\left(\frac{2v}{1 + v^2}\right)dv = \frac{dx}{x} \qquad \text{(Note)}$$

Integrating, $\log(v + v^3) - 2\log(1 + v^2) = \log x + \log c$

or $$\frac{v+v^3}{(1+v^2)^2} = cx$$

or $$\frac{\left(\frac{y}{x}+\frac{y^3}{x^3}\right)}{\left(1+\frac{y^2}{x^2}\right)^2} = cx \qquad \left(\because v = \frac{y}{x}\right)$$

or $$\frac{\frac{(yx^2+y^3)}{x^3}}{\frac{(x^2+y^2)^2}{x^4}} = cx$$

or $$\frac{(yx^2+y^3)}{x^3} \times \frac{x^4}{(x^2+y^2)^2} = cx$$

or $$(yx^2+y^3) = c\,(x^2+y^2)^2$$

or $$y = c\,(x^2+y^2)$$ **Ans.**

Ex. 13. *Prove that the orthogonal trajectories of the family of conics $y^2 - x^2 + 4xy - 2cx = 0$ consists of a family of cubics with the common asymptote $x + y = 0$.* **[Meerut 2009]**

Sol. Given $\quad y^2 - x^2 + 4xy - 2cx = 0$, c being a parameter ... (1)

Differentating (1), $\quad 2y\,(dy/dx) - 2x + 4\,[x\,(dy/dx) + y] = 2c.$ ... (2)

From (1), $\quad 2c = (y^2 - x^2 + 4xy)/x.$ ... (3)

Eliminating c between (2) and (3), $2y\,(dy/dx) - 2x + 4\,\{x\,(dy/dx) + y\} = (y^2 - x^2 + 4xy)/x$

or $\quad 2xy\,(dy/dx) - 2x^2 + 4x^2\,(dy/dx) + 4xy = y^2 - x^2 + 4xy$

or $\quad 2x\,(y + 2x)\,(dy/dx) = x^2 + y^2,$ ... (4)

which is the differential equation of (1). Replacing dy/dx by $(-\,dx/dy)$ in (4), the differential equation of the required orthogonal trajectories is

$$2x\,(y + 2x)\,(-\,dx/dy) = x^2 + y^2 \quad \text{or} \quad \frac{dy}{dx} = -\frac{2x\,(y + 2x)}{x^2 + y^2} = -\frac{2\,(y/x) + 4}{1 + (y/x)^2} \quad \text{... (5)}$$

Putting $y/x = v$ or $y = xv$ so that $dy/dx = v + x\,(dv/dx)$, (5) gives

$$v + x\frac{dv}{dx} = -\frac{2v + 4}{1 + v^2} \quad \text{or} \quad x\frac{dv}{dx} = -v - \frac{2v + 4}{1 + v^2} = -\frac{4 + 3v + v^3}{1 + v^2}$$

or $$\frac{1 + v^2}{4 + 3v + v^3}\,dv = -\frac{dx}{x} \quad \text{or} \quad 3\frac{dx}{x} + \frac{3v^2 + 3}{4 + 3v + v^3}\,dv = 0.$$

Integrating, $\quad 3\log x + \log(4 + 3v + v^3) = \log c$, c being an arbitrary constant

or $\quad \log x^3 + \log(4 + 3v + v^3) = \log c \quad$ or $\quad x^3\,(4 + 3v + v^3) = c$

or $\quad x^3\,\{4 + 3\,(y/x) + (y/x)^3\} = c \quad$ or $\quad y^3 + 3x^2y + 4x^3 = \text{c}.$ ... (6)

Re-writing (6), $\quad (y + x)\,(y^2 - xy + 4x^2) = c.$ ... (7)

The asymptote of (7) corresponding to the factor $(x + y)$ is given by

$$x + y = \lim_{x\to\infty,\, y/x\to -1}\left(\frac{c}{y^2 - xy + 4x^2}\right)$$

$$= \lim_{x\to\infty,\, y/x\to -1}\left[\frac{(c/x^2)}{(y/x)^2 - (y/x) + y}\right] = \lim_{x\to\infty}\frac{(c/x^2)}{(-1)^2 - (-1) + 4} = 0.$$

Hence $x + y = 0$ is the asymptote common to the family of cubics (6).

Ex. 4. *Find the orthogonal trajectories of the system of circles touching a given straight line at a given point.* **[Purvanchal 1993]**

OR

Find the orthogonal trajectories of $x^2 + y^2 = 2ax$. **[GATE 2003]**

Sol. Let the given point be O (0, 0) and the given straight line be y-axis. Now, if a be the radius, then equation family of given circles is

$(x - a)^2 + (y - 0)^2 = a^2$ or $x^2 + y^2 = 2ax$, where a is a parameter. ... (1)

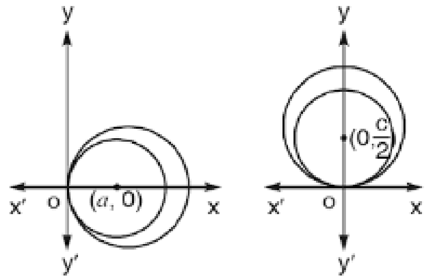
Differentiating (1) with respect to x, $2x + 2y\,(dy/dx) = 2a$ or $a = x + y\,(dy/dx)$ ... (2)

Eliminating a from (1) and (2), we get

$$x^2 + y^2 = 2x\left(x + y\frac{dy}{dx}\right)$$

or $2xy\,(dy/dx) = y^2 - x^2$, ... (3)

which is the differential equation of the given family of circles (1). Replacing dy/dx by $-\,dx/dy$, the differential equation of the required orthogonal trajectories is



Given system of circles

Orthogonal trajectories

$$-2xy\frac{dx}{dy} = y^2 - x^2 \quad \text{or} \quad \frac{dy}{dx} = \frac{2xy}{x^2 - y^2} = \frac{2\,(y/x)}{1 - (y/x)^2}, \qquad \text{... (3)}$$

which is a homogeneous differential equation.

Put $y/x = v$ or $y = xv$ so that $dy/dx = v + x\,(dv/dx)$

$\therefore$ (3) gives $$v + x\frac{dv}{dx} = \frac{2v}{1 - v^2} \quad \text{or} \quad x\frac{dv}{dx} = \frac{2v}{1 - v^2} - v$$

or $$x\frac{dv}{dx} = \frac{v + v^3}{1 - v^2} \quad \text{or} \quad \frac{dx}{x} = \frac{1 - v^2}{v\,(1 + v^2)}\,dv$$

or $$\frac{dx}{x} = \left(\frac{1}{v} - \frac{2v}{1 + v^2}\right) dv, \text{ on resolving into partial fractions}$$

Integrating, $\log x = \log v - \log(1 + v^2) + \log c$ or $x = (cv)/(1 + v^2)$

or $x\,(1 + v^2) = cv$ or $x\,(1 + y^2/x^2) = c\,(y/x)$, as $v = y/x$

$\therefore$ $x^2 + y^2 = cy$, c being parameter. ... (4)

Note: Here the orthogonal trajectories (4) again represents a family of circles touching x-axis at O (0, 0) and having variable radius ($c/2$).

Ex. 5. *Show that the Wronskian of the functions x^2, $x^2 \log x$ is non–zero. Can these functions be independent solutions of an ordinary differential equation; if so determine this equation.* **[Meerut 81, 83, 85, 87(R), 88]**

Sol. Let $y_1 = x^2$ and $y_2 = x^2 \log x$.

Wronskian of these functions is

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^2 & x^2 \log x \\ 2x & 2x \log x + x \end{vmatrix} = x^3 \neq 0$$

$\therefore$ These functions y_1 and y_2 are linearly independent solutions of an ordinary differential equation.

To determine the equation

Let $\quad y = c_1 y_1 + c_2 y_2 = c_1 x^2 + c_2 x^2 \log x$

so that $\quad dy/dx = 2c_1 x + 2c_2 . x \log x + c_2 x$

and $\quad d^2y/dx^2 = 2c_1 + 2c_2 \log x + 3c_2$

Eliminating c_1, c_2 from the above three relations, we have

$$x^2\, d^2y/dx^2 - 3x\,(dy/dx) + 4y = 0$$

which is the required differential equation. **Ans.**

(a) Show that the differential equation $(3y^2 - x) + 2y(y^2 - 3x)y' = 0$ admists an integrating factor which is a function of $(x + y^2)$. Hence solve the equation. ***(10)***

$$\boxed{(3y^2 - x)dx + 2y(y^2 - 3x)dy = 0}$$

$t = y^2 + x$

$(3(t-x) - x)dx + ((t-x) - 3x)(dt - dx) = 0$

$(3t - 4x)dx + (t - 4x)(dt - dx) = 0$

$2t dx + (t - 4x)dt = 0$

Suppose that an integrating factor of the form $f(x + y^2) = f(t)$ exists

$$2t f(t) dx + (t - 4x) f(t) dt = 0 = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial t} = dF$$

$$\begin{cases} \dfrac{\partial F}{\partial x} = 2t\, f(t) \\ \dfrac{\partial F}{\partial t} = (t - 4x) f(t) \end{cases} \rightarrow \quad \frac{\partial^2 F}{\partial x \partial t} = 2f + 2t \frac{df}{dt} = -4f$$

$$6f + 2t\frac{df}{dt} = 0 \quad \rightarrow \quad f(t) = \frac{c}{t^3} = \frac{c}{(y^2 + x)^3}$$

a integrating factor is $\dfrac{1}{t^3} = \dfrac{1}{(y^2 + x)^3}$

$$\frac{2t}{t^3} dx + \frac{t - 4x}{t^3} dt = 0 \quad \rightarrow \quad d\left(\frac{2x - t}{t^2}\right) = 0 \quad \rightarrow \quad \frac{2x - t}{t^2} = \text{constant}$$

The solution on implicit form is : $2x - t + c\,t^2 = 0$

$$\boxed{2x - (x + y^2) + c(x + y^2)^2 = 0}$$