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Maxima and Minima

(Of Functions of Several Independent Variables)

§ 1. Definition.

Let $f(x, y, z, ...)$ be any function of several independent variables $x, y, z, ...$ supposed to be continuous for all values of these variables in the neighbourhood of their values $a, b, c, ...$ respectively. Then $f(a, b, c, ...)$ is said to be a **maximum** or a **minimum** value of $f(x, y, z, ...)$ according as $f(a+h, b+k, c+l, ...)$ is less or greater than $f(a, b, c, ...)$ for all sufficiently small independent values of $h, k, l, ...$ positive or negative, provided they are not all zero.

§ 2. Necessary Conditions for the Existence of Maxima or Minima.

From the definition it is obvious that we shall have a maximum or a minimum of $f(x, y, z, ...)$ for those values of $x, y, z, ...$ for which the expression $f(x+h, y+k, z+l, ...) - f(x, y, z, ...)$ is of **invariable sign** for all sufficiently small independent values of $h, k, l, ...$ provided they are not all equal to zero. There will be a maximum or a minimum according as this sign is negative or positive.

Expanding by Taylor's theorem for several variables, we have

$$f(x+h, y+k, z+l, ...)$$

$$= \left[1 + \frac{1}{1!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} + l\frac{\partial}{\partial z} + ...\right) + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y} + l\frac{\partial}{\partial z} + ...\right)^2 + ...\right] f(x, y, z, ...).$$

$$\therefore \quad f(x+h, y+k, z+l, ...) - f(x, y, z)$$

$$= \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + l\frac{\partial f}{\partial z} + ...\right) + \text{terms of the second and higher orders in } h, k, l, ... \quad ...(1)$$

Now by taking $h, k, l, ...$ sufficiently small, the first degree terms in $h, k, l, ...$ can be made to govern the sign of the right hand side and therefore of the left hand side of (1). But if $h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + l\frac{\partial f}{\partial z} + ...$ is not equal to zero, the sign of this expression will change by changing the sign of each of $h, k, l, ...$ Hence as a necessary condition for the occurrence of a maximum or a minimum of $f(x, y, z, ...)$, we must have

$$h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + l\frac{\partial f}{\partial z} + ... = 0. \quad ...(2)$$

Since (2) is true whatever be the values of $h, k, l, ...$ independent of each other, we must have as a necessary consequence

$$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial z} = 0, ...$$

If there are n independent vairables, we have then obtained n simultaneous equations to give us the values $a, b, c, ...$ of the n variables $x, y, z, ...$ for which $f(x, y, z, ...)$ may have a maximum or minimum value.

The conditions $\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial z} = 0, ...$ *are necessary but not sufficient for the existence of maxima and minima.*

§ 3. Stationary and Extreme points.

A point $(a_1, a_2, ..., a_n)$ is called a **stationary point**, if all the first order partial derivatives of the function $f(x_1, x_2, ..., x_n)$ vanish at that point. Also then the value of the function $f(x_1, x_2, ..., x_n)$ is said to be stationary at that point. A stationary point which is either a maximum or a minimum is called an **extreme point** and the value of the function at that point is called an **extreme value.** A stationary point is not necessarily an extreme point. Thus a stationary value may be a maximum or a minimum or neither of these two. To decide whether a stationary point is really an extreme point, a further investigation is required.

§ 4. Lagrange's necessary and sufficient condition for the maxima or minima of a function of three independent variables.

Necessary Conditions. Let $f(x, y, z)$ be a function of three independent variables x, y and z. Then as derived in § 2, for $f(x, y, z)$ to be a maximum or a minimum at any point (a, b, c), it is necessary that

$$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0 \text{ and } \frac{\partial f}{\partial z} = 0$$

at that point.

Hence the points where the value of the function $f(x, y, z)$ is stationary (*i.e.*, may be a maximum or a minimum) are obtained by solving the simultaneous equations

$$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial z} = 0.$$

Sufficient Conditions. Before deriving the sufficient conditions for the existence of a maximum or a minimum of a function of three independent variables, we obtain the following two algebraic lemmas regarding the signs of quadratic expressions.

Lemma 1. Let $I_2 = ax^2 + 2hxy + by^2$ be a quadratic expression in two variables x and y. We can write

$$I_2 = \frac{1}{a}[a^2x^2 + 2ahxy + aby^2], \text{ if } a \neq 0$$

$$= \frac{1}{a}[(ax + hy)^2 + (ab - h^2)y^2].$$

The expression within the square brackets will be positive if $ab - h^2$ is positive and in that case the sign of the expression I_2 will be the same as that of a.

In case $ab - h^2$ is not positive, we can say nothing about the sign of the expression within the square brackets and hence nothing about the sign of the given quadratic expression I_2.

Lemma 2. In three variables x, y and z,

$$I_3 \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$

$$= \frac{1}{a}[a^2x^2 + aby^2 + acz^2 + 2fayz + 2gazx + 2haxy], \text{ if } a \neq 0$$

$$= \frac{1}{a}[a^2x^2 + 2ax(gz + hy) + aby^2 + acz^2 + 2fayz]$$

$$= \frac{1}{a}[(ax + hy + gz)^2 + aby^2 + acz^2 + 2fayz - (gz + hy)^2]$$

$$= \frac{1}{a}[(ax + hy + gz)^2 + (ab - h^2)y^2 + 2yz(fa - gh) + (ac - g^2)z^2]$$

Now I_3 will be of the same sign as a provided the expression within the square brackets is positive which will of course be so if

$ab - h^2$ and $\{(ab - h^2)(ac - g^2) - (fa - gh)^2\}$

are both positive *i.e.*, if

$$ab - h^2 \text{ and } a(abc + 2fgh - af^2 - bg^2 - ch^2)$$

are both positive.

Thus I_3 will be positive if

$$a, \begin{vmatrix} a & h \\ h & b \end{vmatrix}, \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

be all positive and will be negative if these three expressions are alternately negative and positive.

Now we are in a position to derive Lagrange's sufficient conditions for the existence of a maximum or a minimum of a function of three independent variables at a stationary point.

Let a set of the values of x, y, z obtained by solving the equations

$$\frac{\partial f}{\partial x} = \frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} = 0$$

be a, b, c.

Let the values of the six second order partial derivatives

$$\frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial y^2}, \frac{\partial^2 f}{\partial z^2}, \frac{\partial^2 f}{\partial y\, \partial z}, \frac{\partial^2 f}{\partial z\, \partial x} \text{ and } \frac{\partial^2 f}{\partial x\, \partial y}$$

at the point (a, b, c) be denoted by A, B, C, F, G and H respectively.

Then, we have

$$f(a + h, b + k, c + l) - f(a, b, c)$$

$$= \frac{1}{2!}(Ah^2 + Bk^2 + Cl^2 + 2Fkl + 2Glh + 2Hhk) + R_3 \quad \ldots(1)$$

where R_3 consists of terms of third and higher orders of small quantities h, k and l. By taking h, k and l sufficiently small, the second degree terms in h, k and l can be made to govern the sign of the right hand side and therefore of the left hand side of (1). If this group of terms forms an expression of invariable sign for all such values of h, k and l, we shall have a maximum or a minimum value of $f(x, y, z)$ at (a, b, c) according as that sign is negative or positive.

Hence by our lemma 2, if the expressions

$$A, \begin{vmatrix} A & H \\ H & B \end{vmatrix}, \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$$

be all positive, we shall have a minimum of $f(x, y, z)$ at (a, b, c) and if these expressions be alternately negative and positive, we shall have a maximum of $f(x, y, z)$ at (a, b, c), whilst if these conditions are not satisfied, we shall in general have neither a maximum nor a minimum of $f(x, y, z)$ at (a, b, c).

§ 5. Working Rule for finding the maxima and minima of a function of three independent variables.

Suppose $f(x, y, z)$ is a given function of three independent variables x, y and z. Find $\partial f/\partial x$, $\partial f/\partial y$ and $\partial f/\partial z$ and solve the simultaneous equations $\partial f/\partial x = 0$, $\partial f/\partial y = 0$ and $\partial f/\partial z = 0$. All the triads (a, b, c) of the values of x, y and z obtained on solving these equations will give the stationary values of $f(x, y, z)$ *i.e.*, will give the points at which the function $f(x, y, z)$ may be a maximum or a minimum.

To discuss the maximum or minimum of $f(x, y, z)$ at any point (a, b, c) obtained on solving the equations $\partial f/\partial x = 0$, $\partial f/\partial y = 0$ and $\partial f/\partial z = 0$, we find the values at this point of the six partial derivatives of second order of $f(x, y, z)$ symbolically denoted as follows:

$$A = \frac{\partial^2 f}{\partial x^2}, B = \frac{\partial^2 f}{\partial y^2}, C = \frac{\partial^2 f}{\partial z^2}, F = \frac{\partial^2 f}{\partial y\, \partial z}, G = \frac{\partial^2 f}{\partial z\, \partial x} \text{ and } H = \frac{\partial^2 f}{\partial x\, \partial y}.$$

Similarly $\dfrac{\partial u}{\partial y} = \dfrac{(xz - y^2)\, u}{y\,(x+y)\,(y+z)}$

and $\dfrac{\partial u}{\partial z} = \dfrac{(by - z^2)\, u}{z\,(y+z)\,(z+b)}$.

Now for a maximum or a minimum of u, we must have

$\dfrac{\partial u}{\partial x} = 0$ *i.e.*, $ay - x^2 = 0$

$\dfrac{\partial u}{\partial y} = 0$ *i.e.*, $xz - y^2 = 0$

and $\dfrac{\partial u}{\partial z} = 0$ *i.e.*, $by - z^2 = 0$.

From the above equations, it follows that a, x, y, z and b are in geometrical progression. Let r be the common ratio of this geometrical progression. Then

$$ar^4 = b \quad \text{or} \quad r = (b/a)^{1/4}.$$

Also $x = ar, y = ar^2, z = ar^3$.

Substituting these values, we get

$$u = \frac{ar \cdot ar^2 \cdot ar^3}{a\,(1+r) \cdot ar\,(1+r) \cdot ar^2\,(1+r) \cdot ar^3\,(1+r)}$$

$$= \frac{1}{a\,(1+r)^4} = \frac{1}{a\,[1 + (b/a)^{1/4}]^4} = \frac{1}{(a^{1/4} + b^{1/4})^4}.$$

This gives a stationary value of u. To decide whether this value of u is a maximum or a minimum we proceed to find the second order partial derivatives of u.

We have

$$\frac{\partial^2 u}{\partial x^2} = \frac{-2xu}{x\,(a+x)\,(x+y)} + (ay - x^2)\,\frac{\partial}{\partial x}\left[\frac{u}{x\,(a+x)\,(x+y)}\right].$$

$\therefore$ when $x = ar, y = ar^2, z = ar^3$, we have

$$A = \frac{\partial^2 u}{\partial x^2} = \frac{-2 \cdot ar \cdot u}{ar \cdot a\,(1+r) \cdot ar\,(1+r)} = \frac{-2u}{a^2 r\,(1+r)^2},$$

which is negative.

Hence the above stationary value of u is a maximum.

Ans. Maximum value of $u = \dfrac{1}{(a^{1/4} + b^{1/4})^4}$.

Note. In the complicated problems in order to find whether a stationary value of u is a maximum or a minimum, it is sufficient to find the value of a second partial differential coefficient of u with respect to any of the independent variables. The value of u will be maximum or minimum according as the value of this second partial derivative at the stationary point under consideration is –ive or +ive.

Ex. 4. *Show that* $u = (x+y+z)^3 - 3\,(x+y+z) - 24xyz + a^3$ *has minimum at* $(1, 1, 1)$ *and maximum at* $(-1, -1, -1)$.

Sol. For a maximum or a minimum of u, we must have

$$\frac{\partial u}{\partial x} = 3\,(x+y+z)^2 - 3 - 24yz = 0, \quad \text{...(1)}$$

$$\frac{\partial u}{\partial y} = 3\,(x+y+z)^2 - 3 - 24zx = 0 \quad \text{...(2)}$$

and $$\frac{\partial u}{\partial z} = 3\,(x+y+z)^2 - 3 - 24xy = 0. \quad \text{...(3)}$$

Subtracting (2) from (1), we get

$$24z\,(x - y) = 0$$

which has $x = y$ for one of its solutions.

Similarly subtracting (3) from (1), we get

$$24y\,(x - z) = 0$$

which has $x = z$ for one of its solutions.

Thus the equations (1), (2) and (3) are satisfied when

$$x = y = z.$$

Putting $y = x$ and $z = x$ (1), we get

$$27x^2 - 3 - 24x^2 = 0 \text{ or } 3x^2 = 3 \text{ or } x^2 = 1 \text{ or } x = \pm 1.$$

$\therefore$ $x = y = z = 1$ and $x = y = z = -1$ are solutions of the equations (1), (2) and (3).

Hence u is stationary at the points $(1, 1, 1)$ and $(-1, -1, -1)$.

Now $A = \dfrac{\partial^2 u}{\partial x^2} = 6\,(x+y+z)$, $B = \dfrac{\partial^2 u}{\partial y^2} = 6\,(x+y+z)$,

$C = \dfrac{\partial^2 u}{\partial z^2} = 6\,(x+y+z)$, $F = \dfrac{\partial^2 u}{\partial y\,\partial z} = 6\,(x+y+z) - 24x$,

$G = \dfrac{\partial^2 u}{\partial z\,\partial x} = 6\,(x+y+z) - 24y$, $H = \dfrac{\partial^2 u}{\partial x\,\partial y} = 6\,(x+y+z) - 24z$.

Nature of u at (1, 1, 1). At the stationary point $(1, 1, 1)$, we have $A = 18, B = 18, C = 18, F = -6, G = -6, H = -6$.

$\therefore$ at the point $(1, 1, 1)$, we have

$$A = 18, \begin{vmatrix} A & H \\ H & B \end{vmatrix} = \begin{vmatrix} 18 & -6 \\ -6 & 18 \end{vmatrix} = 288$$

and $$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = \begin{vmatrix} 18 & -6 & -6 \\ -6 & 18 & -6 \\ -6 & -6 & 18 \end{vmatrix}$$

$$= 6^3 \cdot \begin{vmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{vmatrix}$$

If the expressions.

$$A, \begin{vmatrix} A & H \\ H & B \end{vmatrix}, \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$$

be all positive, we shall have a minimum of $f(x,y,z)$ at (a,b,c) and if these expressions be alternately negative and positive, we shall have a maximum of $f(x,y,z)$ at (a,b,c), whilst if these conditions are not satisfied, we shall in general have neither a maximum nor a minimum of $f(x,y,z)$ at (a,b,c).

Solved Examples

Ex. 1. *Discuss the maximum or minimum values of u where* $u = x^2 + y^2 + z^2 + x - 2z - xy$.

Sol. For a maximum or a minimum of u, we must have

$$\frac{\partial u}{\partial x} = 2x - y + 1 = 0,$$

$$\frac{\partial u}{\partial y} = -x + 2y = 0,$$

and $$\frac{\partial u}{\partial z} = 2z - 2 = 0.$$

These equations give $x = -2/3, y = -1/3, z = 1$.

$\therefore$ $(-2/3, -1/3, 1)$ is the only point at which u is stationary *i.e.*, at which u may have a maximum or a minimum.

Now $$\frac{\partial^2 u}{\partial x^2} = 2, \frac{\partial^2 u}{\partial y^2} = 2, \frac{\partial^2 u}{\partial z^2} = 2,$$

$$\frac{\partial^2 u}{\partial y\,\partial z} = 0, \frac{\partial^2 u}{\partial z\,\partial x} = 0 \text{ and } \frac{\partial^2 u}{\partial x\,\partial y} = -1.$$

If A, B, C, F, G and H denote the respective values of these six partial derivatives of second order at the point $(-2/3, -1/3, 1)$, then

$$A = 2, B = 2, C = 2, F = 0, G = 0, H = -1.$$

Now we have

$$A = 2, \begin{vmatrix} A & H \\ H & B \end{vmatrix} = \begin{vmatrix} 2 & -1 \\ -1 & 2 \end{vmatrix} = 3$$

and $$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & 0 & 2 \end{vmatrix} = 6.$$

Since these three expressions are all positive, we have a minimum of u when $x = -2/3, y = -1/3, z = 1$.

Ex. 2. *Show that the point such that the sum of the squares of its distances from n given points shall be minimum, is the centre of the mean position of the given points.*

Sol. Let the n given points be $(a_1, b_1, c_1), (a_2, b_2, c_2), \ldots\ldots, (a_n, b_n, c_n)$ and let (x, y, z) be the coordinates of the required point.

If u denotes the sum of the squares of the distances of (x,y,z) from the n given points, then

$$u = \Sigma\,[(x-a_1)^2 + (y-b_1)^2 + (z-c_1)^2]$$
$$= \Sigma\,(x-a_1)^2 + \Sigma\,(y-b_1)^2 + \Sigma\,(z-c_1)^2.$$

For a maximum or a minimum of u, we must have

$$\frac{\partial u}{\partial x} = 2\,\Sigma\,(x-a_1) = 2nx - 2\,\Sigma\,a_1 = 0,$$

$$\frac{\partial u}{\partial y} = 2\,\Sigma\,(y-b_1) = 2ny - 2\,\Sigma\,b_1 = 0,$$

and $$\frac{\partial u}{\partial z} = 2\,\Sigma\,(z-c_1) = 2nz - 2\,\Sigma\,c_1 = 0.$$

Solving these equations, we get

$$x = \frac{\Sigma\,a_1}{n}, y = \frac{\Sigma\,b_1}{n}, z = \frac{\Sigma\,c_1}{n}.$$

Now $$A = \frac{\partial^2 u}{\partial x^2} = 2n, B = \frac{\partial^2 u}{\partial y^2} = 2n, C = \frac{\partial^2 u}{\partial z^2} = 2n,$$

$$F = \frac{\partial^2 u}{\partial y\,\partial z} = 0, G = \frac{\partial^2 u}{\partial z\,\partial x} = 0, H = \frac{\partial^2 u}{\partial x\,\partial y} = 0.$$

We have $A = 2n$, $$\begin{vmatrix} A & H \\ H & B \end{vmatrix} = \begin{vmatrix} 2n & 0 \\ 0 & 2n \end{vmatrix} = 4n^2,$$

and $$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = \begin{vmatrix} 2n & 0 & 0 \\ 0 & 2n & 0 \\ 0 & 0 & 2n \end{vmatrix} = 8n^3.$$

Since these three expressions are all positive, u is minimum when

$$x = \frac{\Sigma\,a_1}{n}, y = \frac{\Sigma\,b_1}{n} \text{ and } z = \frac{\Sigma\,c_1}{n}$$

i.e., u is minimum when the point (x,y,z) is the centre of the mean position of the n given points.

Ex. 3. *Find the maximum value of u where*

$$u = \frac{xyz}{(a+x)(x+y)(y+z)(z+b)}.$$ **(Meerut 1998)**

Sol. We have

$$\log u = \log x + \log y + \log z - \log(a+x) - \log(x+y) - \log(y+z) - \log(z+b).$$

$$\therefore \quad \frac{1}{u}\frac{\partial u}{\partial x} = \frac{1}{x} - \frac{1}{a+x} - \frac{1}{x+y} = \frac{ay - x^2}{x(a+x)(x+y)}$$

or $$\frac{\partial u}{\partial x} = \frac{(ay - x^2)\,u}{x(a+x)(x+y)}.$$

$$= 6^3 \cdot \begin{vmatrix} 1 & 1 & 1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{vmatrix}, \text{ by } R_1 + R_2 + R_3$$

$$= 6^3 \cdot \begin{vmatrix} 1 & 0 & 0 \\ -1 & 4 & 0 \\ -1 & 0 & 4 \end{vmatrix}, \text{ by } R_2 - R_1 \text{ and } R_3 - R_1$$

$$= 6^3 . 16.$$

Since these three expressions are all positive, we have a minimum of u at the point $(1, 1, 1)$.

Nature of u at the stationary point $(-1, -1, -1)$.

At the stationary point $(-1, -1, -1)$, we have

$$A = -18 = B = C, F = 6 = G = H.$$

$\therefore$ at the point $(-1, -1, -1)$, we have

$$A = -18, \begin{vmatrix} A & H \\ H & B \end{vmatrix} = \begin{vmatrix} -18 & 6 \\ 6 & -18 \end{vmatrix} = 288$$

and
$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = \begin{vmatrix} -18 & 6 & 6 \\ 6 & -18 & 6 \\ 6 & 6 & -18 \end{vmatrix} = -6^3 . 16.$$

Since these three expressions are alternately negative and positive, we have a maximum of u at the point $(-1, -1, -1)$.

Ex. 5. *Find the maximum or minimum values of u where*

$$u = axy^2z^3 - x^2y^2z^3 - xy^3z^3 - xy^2z^4.$$

Sol. We have $\partial u/\partial x = ay^2z^3 - 2xy^2z^3 - y^3z^3 - y^2z^4$

$$= y^2z^3 (a - 2x - y - z),$$

$$\partial u/\partial y = 2axyz^3 - 2x^2yz^3 - 3xy^2z^3 - 2xyz^4$$

$$= xyz^3 (2a - 2x - 3y - 2z)$$

and $\partial u/\partial z = 3axy^2z^2 - 3x^2y^2z^2 - 3xy^3z^2 - 4xy^2z^3$

$$= xy^2z^2 (3a - 3x - 3y - 4z).$$

For a maximum or a minimum of u, we must have

$$\partial u/\partial x = 0, \partial u/\partial y = 0, \partial u/\partial z = 0.$$

Now one solution of the equations $\partial u/\partial x = 0$, $\partial u/\partial y = 0$, $\partial u/\partial z = 0$ is given by the equations

$$2x + y + z = a, 2x + 3y + 2z = 2a, 3x + 3y + 4z = 3a.$$

Solving these equations, we get $x = a/7, y = 2a/7, z = 3a/7$.

$\therefore$ u is stationary at the point $(a/7, 2a/7, 3a/7)$.

Now $A = \dfrac{\partial^2 u}{\partial x^2} = y^2z^3 . (-2)$.

At the stationary point $(a/7, 2a/7, 3a/7)$, the value of A is – ive.

$\therefore$ u has a maximum value at the point $(a/7, 2a/7, 3a/7)$.

Putting $x = a/7, y = 2a/7, z = 3a/7$ in the value of u, the maximum value of u at the point $(a/7, 2a/7, 3a/7) = 108a^7/7$.

Ex. 6. *Find the maximum value of*

$$(ax + by + cz)\, e^{-(\alpha^2x^2 + \beta^2y^2 + \gamma^2z^2)}.$$

Sol. Let $u = (ax + by + cz)\, e^{-(\alpha^2x^2 + \beta^2y^2 + \gamma^2z^2)}$.

Then $\log u = \log (ax + by + cz) - (\alpha^2x^2 + \beta^2y^2 + \gamma^2z^2)$.

$$\therefore \quad \frac{1}{u}\frac{\partial u}{\partial x} = \frac{a}{ax + by + cz} - 2\alpha^2x, \frac{1}{u}\frac{\partial u}{\partial y} = \frac{b}{ax + by + cz} - 2\beta^2y,$$

$$\frac{1}{u}\frac{\partial u}{\partial z} = \frac{c}{ax + by + cz} - 2\gamma^2z.$$

For a maximum or a minimum of u, we must have

$$\partial u/\partial x = 0, \partial u/\partial y = 0, \partial u/\partial z = 0.$$

These give
$$x\,(ax + by + cz) = \frac{a}{2\alpha^2} \quad \text{–(1)}$$

$$y\,(ax + by + cz) = \frac{b}{2\beta^2} \quad \text{–(2)}$$

and
$$z\,(ax + by + cz) = \frac{c}{2\gamma^2}. \quad \text{–(3)}$$

Multiplying (1), (2), (3) by a, b, c respectively and adding, we get

$$(ax + by + cz)^2 = \tfrac{1}{2}\left(\frac{a^2}{\alpha^2} + \frac{b^2}{\beta^2} + \frac{c^2}{\gamma^2}\right),$$

so that
$$(ax + by + cz) = \sqrt{\left\{\frac{1}{2}\left(\frac{a^2}{\alpha^2} + \frac{b^2}{\beta^2} + \frac{c^2}{\gamma^2}\right)\right\}} = R, \text{say}.$$

Then $x = \dfrac{a}{2\alpha^2 R}, y = \dfrac{b}{2\beta^2 R}, z = \dfrac{c}{2\gamma^2 R}$.

$\therefore$ u is stationary when $x = \dfrac{a}{2\alpha^2 R}, y = \dfrac{b}{2\beta^2 R}, z = \dfrac{c}{2\gamma^2 R}$.

Again
$$\frac{1}{u}\frac{\partial^2 u}{\partial x^2} - \frac{1}{u^2}\left(\frac{\partial u}{\partial x}\right)^2 = -\frac{a^2}{(ax + by + cz)^2} - 2\alpha^2.$$

Now at a stationary point, we have $\partial u/\partial x = 0$.

$\therefore$ at the stationary point found above, we have

$$\frac{\partial^2 u}{\partial x^2} = -u\left[\frac{a^2}{(ax + by + cz)^2} + 2\alpha^2\right],$$ which is –ive because u is positive for the values of x, y, z found above.

$\therefore$ u is maximum at the stationary point found above.

Also putting the values of x, y, z found above in the value of u, the maximum value of u

$$= R \cdot e^{-\frac{1}{4R^2}\left(\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}\right)} = R \cdot e^{-\frac{1}{4R^2} \cdot 2R^2}$$

$$= R \cdot e^{-1/2} = \frac{R}{\sqrt{e}} = \frac{1}{\sqrt{e}} \cdot \sqrt{\left\{\frac{1}{2}\left(\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}\right)\right\}}$$

$$= \sqrt{\left\{\frac{1}{2e}\left(\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}\right)\right\}}.$$

§ 6. Lagrange's method of undetermined multipliers.

Let $u = f(x_1, x_2, \ldots, x_n)$

be a function of n variables $x_1, x_2, \ldots, x_n$. Let these variables be connected by m equations

$$\phi_1(x_1, x_2, \ldots, x_n) = 0,\ \phi_2(x_1, x_2, \ldots, x_n) = 0, \ldots,$$
$$\phi_m(x_1, x_2, \ldots, x_n) = 0$$

so that only $n - m$ of the n variables are independent.

For a maximum or a minimum of u, we have

$$du = \frac{\partial u}{\partial x_1} dx_1 + \frac{\partial u}{\partial x_2} dx_2 + \frac{\partial u}{\partial x_3} dx_3 + \ldots + \frac{\partial u}{\partial x_n} dx_n = 0.$$

Also differentiating the m given equations connecting the variables, we get

$$d\phi_1 = \frac{\partial \phi_1}{\partial x_1} dx_1 + \frac{\partial \phi_1}{\partial x_2} dx_2 + \frac{\partial \phi_1}{\partial x_3} dx_3 + \ldots + \frac{\partial \phi_1}{\partial x_n} dx_n = 0,$$

$$d\phi_2 = \frac{\partial \phi_2}{\partial x_1} dx_1 + \frac{\partial \phi_2}{\partial x_2} dx_2 + \frac{\partial \phi_2}{\partial x_3} dx_3 + \ldots + \frac{\partial \phi_2}{\partial x_n} dx_n = 0,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$
$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$d\phi_m = \frac{\partial \phi_m}{\partial x_1} dx_1 + \frac{\partial \phi_m}{\partial x_2} dx_2 + \frac{\partial \phi_m}{\partial x_3} dx_3 + \ldots + \frac{\partial \phi_m}{\partial x_n} dx_n = 0.$$

Multiplying the above $m + 1$ equations obtained on differentiation by $1, \lambda_1, \lambda_2, \ldots, \lambda_m$ respectively and adding, we get an equation which may be written as ...(1)

$$P_1 dx_1 + P_2 dx_2 + P_3 dx_3 + \ldots + P_n dx_n = 0,$$

where $P_r = \frac{\partial u}{\partial x_r} + \lambda_1 \frac{\partial \phi_1}{\partial x_r} + \lambda_2 \frac{\partial \phi_2}{\partial x_r} + \ldots + \lambda_m \frac{\partial \phi_m}{\partial x_r}$.

Now the m multipliers $\lambda_1, \lambda_2, \ldots, \lambda_m$ are at our choice. We choose them such that they satisfy the m linear equations

$$P_1 = 0, P_2 = 0, \ldots, P_m = 0.$$

Then the equation (1) reduces to ...(2)

$$P_{m+1} dx_{m+1} + P_{m+2} dx_{m+2} + \ldots + P_n dx_n = 0.$$

It is immaterial which of the $n - m$ of the n variables $x_1, x_2, \ldots, x_n$ are regarded as independent. Let us regard the $n - m$ variables $x_{m+1}, x_{m+2}, \ldots, x_n$ as independent. Then since the $n - m$ quantities $dx_{m+1}, dx_{m+2}, \ldots, dx_n$ are all independent of one another, their coefficients must be separately zero in the relation (2). Hence we must have

$$P_{m+1} = 0, P_{m+2} = 0, \ldots, P_n = 0.$$

Thus we get $m + n$ equations

$$P_1 = 0, P_2 = 0, \ldots, P_n = 0$$

and $$\phi_1 = 0, \phi_2 = 0, \ldots, \phi_m = 0,$$

which together with the relation $u = f(x_1, x_2, \ldots, x_n)$ determine the m multipliers $\lambda_1, \lambda_2, \ldots, \lambda_m$, the values of $x_1, x_2, \ldots, x_n$ and u at the stationary point. This method is known as **Lagrange's method of undetermined multipliers.** It is very convenient to apply and it often gives us the maximum or minimum values of u without actually determining the values of the multipliers $\lambda_1, \ldots, \lambda_m$. The only drawback of this method is that it does not determine the nature of the stationary point.

Solved Examples

Ex. 1 (a). *If* $u = x^2 + y^2 + z^2$, *where* $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 1$, *find the maximum or minimum values of u.* **(Meerut 1991 S)**

Sol. We have $u = x^2 + y^2 + z^2$, ...(1)

where the variables x, y, z are connected by the relation

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 1. \quad \ldots(2)$$

For a maximum or a minimum of u, we have

$$du = 0$$
$$\Rightarrow \quad 2x\,dx + 2y\,dy + 2z\,dz = 0$$
$$\Rightarrow \quad x\,dx + y\,dy + z\,dz = 0. \quad \ldots(3)$$

Also differentiating the given relation (2), we get

$$2ax\,dx + 2by\,dy + 2cz\,dz + 2fy\,dz + 2fz\,dy + 2gz\,dx + 2gx\,dz + 2hx\,dy + 2hy\,dx = 0$$

or $$(ax + hy + gz)\,dx + (hx + by + fz)\,dy + (gx + fy + cz)\,dz = 0. \quad \ldots(4)$$

Multiplying (3) by 1, (4) by λ and adding, and then equating the coefficients of dx, dy, dz to zero, we have

$$x + \lambda(ax + hy + gz) = 0, \quad \ldots(5)$$
$$y + \lambda(hx + by + fz) = 0, \quad \ldots(6)$$
and $$z + \lambda(gx + fy + cz) = 0. \quad \ldots(7)$$

Multiplying (5) by x, (6) by y, (7) by z and adding, we get

$$x^2 + y^2 + z^2 + \lambda\,(ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy) = 0$$

or $\quad u + \lambda\,.1 = 0$, using (1) and (2).

$\therefore \quad \lambda = -u$.

Hence from (5), we have

$$x - u\,(ax + hy + gz) = 0$$

or $\quad x\,(1 - au) - huy - guz = 0$

or $\quad \left(a - \dfrac{1}{u}\right) x + hy + gz = 0.$ ...(8)

Similarly from (6) and (7), we have

$$hx + (b - 1/u)\,y + fz = 0, \qquad \text{...(9)}$$

and $\quad gx + fy + (c - 1/u)\,z = 0.$ ...(10)

Eliminating x, y, z from (8), (9), (10), we get

$$\begin{vmatrix} a - (1/u) & h & g \\ h & b - (1/u) & f \\ g & f & c - (1/u) \end{vmatrix} = 0. \qquad \text{...(11)}$$

Hence the required maximum or minimum values of u are the roots of the equation (11).

Ex. 1(b). *Find the maxima and minima of $x^2 + y^2$ subject to the condition*

$$ax^2 + 2hxy + by^2 = 1. \qquad \textbf{(Meerut 1996)}$$

Sol. Let $\quad u = x^2 + y^2,$ ...(1)

where the variables x and y are connected by the relation

$$ax^2 + 2hxy + by^2 = 1. \qquad \text{...(2)}$$

For a maximum or a minimum of u, we have $du = 0$

$\Rightarrow \quad 2x\,dx + 2y\,dy = 0 \quad \Rightarrow \quad x\,dx + y\,dy = 0.$ ...(3)

Also differentiating the given relation (2), get

$$2ax\,dx + 2hx\,dy + 2hy\,dx + 2by\,dy = 0$$

or $\quad (ax + hy)\,dx + (hx + by)\,dy = 0$ ...(4)

Multiplying (3) by 1, (4) by λ and adding, and then equating the coefficients of dx, dy to zero, we have

$$x + \lambda\,(ax + hy) = 0 \qquad \text{...(5)}$$

and $\quad y + \lambda\,(hx + by) = 0$ ...(6)

Multiplying (5) by x, (6) by y and adding, we get

$$x^2 + y^2 + \lambda\,(ax^2 + 2hxy + by^2) = 0$$

or $\quad u + \lambda\,.1 = 0,$ using (1) and (2).

$\therefore \quad \lambda = -u.$

Hence from (5), we have

$$x - u\,(ax + hy) = 0 \quad \text{or} \quad x\,(1 - au) - huy = 0$$

or $\quad \left(a - \dfrac{1}{u}\right) x + hy = 0$ ...(7)

Similarly from (6), we have

$$hx + \left(b - \frac{1}{u}\right) y = 0 \qquad \text{...(8)}$$

Eliminating x and y from (7) and (8), we get

$$\begin{vmatrix} a - \dfrac{1}{u} & h \\ h & b - \dfrac{1}{u} \end{vmatrix} = 0 \quad \text{or} \quad \left(a - \frac{1}{u}\right)\left(b - \frac{1}{u}\right) = h^2 \qquad \text{...(9)}$$

Hence the required maximum or minimum values of $u = x^2 + y^2$ are the roots of the equation (9).

Ex. 2. *Find the stationary values of $x^2 + y^2 + z^2$ subject to the conditions*

$$ax^2 + by^2 + cz^2 = 1 \text{ and } lx + my + nz = 0.$$

Interpret the result geometrically. **(Meerut 1991)**

Sol. Let $\quad u = x^2 + y^2 + z^2,$ ...(1)

where the variables x, y and z are connected by the relations

$$ax^2 + by^2 + cz^2 = 1, \qquad \text{...(2)}$$

and $\quad lx + my + nz = 0.$ ...(3)

For a stationary value of u, we have

$$du = 0$$

$\Rightarrow \quad 2x\,dx + 2y\,dy + 2z\,dz = 0$

$\Rightarrow \quad x\,dx + y\,dy + z\,dz = 0.$ ...(4)

Also differentiating the given relations (2) and (3), we get

$$2ax\,dx + 2by\,dy + 2cz\,dz = 0$$

i.e., $\quad ax\,dx + by\,dy + cz\,dz = 0$ ...(5)

and $\quad l\,dx + m\,dy + n\,dz = 0.$ ...(6)

Multilpying (4) by 1, (5) by λ and (6) by μ and adding, and then equating the coefficients of dx, dy, dz to zero, we get

$$x + \lambda ax + \mu l = 0, \qquad \text{...(7)}$$

$$y + \lambda by + \mu m = 0, \qquad \text{...(8)}$$

and $\quad z + \lambda cz + \mu n = 0.$ ...(9)

Multiplying the equations (7), (8) and (9) by x, y and z respectively and adding, we get

$$x^2 + y^2 + z^2 + \lambda\,(ax^2 + by^2 + cz^2) + \mu\,(lx + my + nz) = 0,$$

or $\quad u + \lambda\,.1 + \mu\,.0 = 0$, using (1), (2) and (3)

or $\quad \lambda = -u.$

Substituting for λ in the equations (7), (8) and (9), we get

$$x = \frac{\mu l}{au - 1},\ y = \frac{\mu m}{bu - 1},\ z = \frac{\mu n}{cu - 1}.$$

Substituting these values of x, y, z in (3), we get

$$\frac{\mu l^2}{au - 1} + \frac{\mu m^2}{bu - 1} + \frac{\mu n^2}{cu - 1} = 0$$

or
$$\frac{l^2}{au-1}+\frac{m^2}{bu-1}+\frac{n^2}{cu-1}=0. \quad ...(10)$$

Hence the stationary (*i.e.*, maximum or minimum) values of u are given by the equation (10). The equation (10) is a quadratic in u and so it gives two stationary values of u.

Geometrical interpretation. The surface $ax^2+by^2+cz^2=1$ represents an ellipsoid (or a hyperboloid) whose centre is origin, and $lx+my+nz=0$ is a plane passing through the origin. Therefore the point (x,y,z) satisfying both the conditions (2) and (3) lies on the conic in which (2) and (3) intersect. Also $x^2+y^2+z^2$ gives the square of the distance of (x,y,z) from the origin which is also the centre of the conic of intersection. The maximum and minimum values of this distance are the major and minor semi-axes of the conic. So the equation (10) gives the squares of the lengths of the semi-axes of the conic of intersection.

Ex. 3. *Find the maximum and minimum values of*
$$\frac{x^2}{a^4}+\frac{y^2}{b^4}+\frac{z^2}{c^4},$$
when $lx+my+nz=0$ *and* $x^2/a^2+y^2/b^2+z^2/c^2=1$.

Interpret the result geometrically.

Sol. Let $u=x^2/a^4+y^2/b^4+z^2/c^4$. Then for a maximum or a minimum of u, we have
$$du=0$$
$$\Rightarrow \quad \frac{x}{a^4}dx+\frac{y}{b^4}dy+\frac{z}{c^4}dz=0 \quad ...(1)$$

Also differentiating the two given equations connecting the variables x,y and z, we get
$$l\,dx+m\,dy+n\,dz=0, \quad ...(2)$$
and
$$\frac{x}{a^2}dx+\frac{y}{b^2}dy+\frac{z}{c^2}dz=0. \quad ...(3)$$

Multiplying (1), (2) and (3) by $1,\lambda$ and μ respectively and adding, and then equating to zero the coefficients of dx, dy and dz, we get
$$\frac{x}{a^4}+\lambda l+\mu\frac{x}{a^2}=0, \quad ...(4)$$
$$\frac{y}{b^4}+\lambda m+\mu\frac{y}{b^2}=0, \quad ...(5)$$
and
$$\frac{z}{c^4}+\lambda n+\mu\frac{z}{c^2}=0. \quad ...(6)$$

Multiplying the equations (4), (5) and (6) by x,y and z respectively and adding, we get
$$u+\lambda.0+\mu.1=0 \quad \text{or} \quad \mu=-u.$$
Putting $\mu=-u$ in (4), we get

$$\frac{x}{a^4}+\lambda l-\frac{xu}{a^2}=0, \text{ or } \frac{x}{a^2}\left\{u-\frac{1}{a^2}\right\}=\lambda l, \text{ or } x=\frac{\lambda la^4}{a^2u-1}.$$

Similarly from (5) and (6), we get
$$y=\frac{\lambda mb^4}{b^2u-1} \quad \text{and} \quad z=\frac{\lambda nc^4}{c^2u-1}.$$

Substituting these values of x,y,z in $lx+my+nz=0$, we get
$$\frac{l^2a^4}{a^2u-1}+\frac{m^2b^4}{b^2u-1}+\frac{n^2c^4}{c^2u-1}=0. \quad ...(7)$$

The equation (7) gives the required maximum or minimum values of u.

Geometrical interpretation. The equation of the tangent plane to the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ at any point (x,y,z) on it is
$$\frac{Xx}{a^2}+\frac{Yy}{b^2}+\frac{Zz}{c^2}=1. \quad ...(8)$$

If p be the length of the perpendicular from origin which is also the centre of the ellipsoid to the tangent plane (1), then
$$p^2=\frac{1}{x^2/a^4+y^2/b^4+z^2/c^4}.$$

If the point (x,y,z) on the ellipsoid also lies on the given plane $lx+my+nz=0$, the problem consists of finding out the maximum or minimum values of the perpendicular distance from the origin to the tangent planes to the ellipsoid at the points common to the plane $lx+my+nz=0$ and the ellipsoid.

Ex. 4. *Find the maximum and minimum values of*
$$u=a^2x^2+b^2y^2+c^2z^2,$$
where $x^2+y^2+z^2=1$ *and* $lx+my+nz=0$.

Sol. We have $u=a^2x^2+b^2y^2+c^2z^2$, ...(1)

where the variables x,y,z are connected by relations
$$x^2+y^2+z^2=1 \quad ...(2)$$
and
$$lx+my+nz=0 \quad ...(3)$$

For a maximum or a minimum of u, we have
$$du=0$$
$$\Rightarrow \quad 2a^2x\,dx+2b^2y\,dy+2c^2z\,dz=0 \quad ...(4)$$
$$\Rightarrow \quad a^2x\,dx+b^2y\,dy+c^2z\,dz=0.$$

Also differentiating the two given equations (2) and (3) connecting the variables x,y and z, we get
$$2x\,dx+2y\,dy+2z\,dz=0 \quad ...(5)$$
i.e.,
$$x\,dx+y\,dy+z\,dz=0 \quad ...(6)$$
and
$$l\,dx+m\,dy+n\,dz=0$$

Multiplying (4), (5) and (6) by $1,\lambda$ and μ respectively and adding, and then equating to zero the coefficients of dx, dy and dz, we get

$$a^2x + \lambda x + \mu l = 0, \qquad ...(7)$$
$$b^2y + \lambda y + \mu m = 0, \qquad ...(8)$$
and $$c^2z + \lambda z + \mu n = 0. \qquad ...(9)$$

Multiplying the equations (7), (8) and (9) by x, y and z respectively and adding, we get

$$u + \lambda .1 + \mu .0 = 0 \text{ or } \lambda = -u.$$

Putting $\lambda = -u$ in (7), we get

$$a^2x - ux + \mu l = 0$$

or $$x = \frac{\mu l}{u - a^2}.$$

Similarly from (8) and (9), we get

$$y = \frac{\mu m}{u - b^2} \quad \text{and} \quad z = \frac{\mu n}{u - c^2}.$$

Substituting these values of x, y, z in $lx + my + nz = 0$, we get

$$\frac{\mu l^2}{u - a^2} + \frac{\mu m^2}{u - b^2} + \frac{\mu n^2}{u - c^2} = 0$$

or $$\frac{l^2}{u - a^2} + \frac{m^2}{u - b^2} + \frac{n^2}{u - c^2} = 0. \qquad ...(10)$$

The equation (10) gives the required maximum or minimum values of u.

Ex. 5. *Find the maximum and minimum values of u^2 when*

$$u^2 = a^2x^2 + b^2y^2 + c^2z^2$$

while $x^2 + y^2 + z^2 = 1$ and $lx + my + nz = 0$.

Sol. Proceed exactly as in solved example 4 taking the function as u^2 in place of u. The required maximum or minimum values of u^2 are the roots of the equation

$$\frac{l^2}{u^2 - a^2} + \frac{m^2}{u^2 - b^2} + \frac{n^2}{u^2 - c^2} = 0.$$

Ex. 6. *Show that the maximum and minimum values of*

$$u = x^2 + y^2 + z^2$$

subject to the conditions

$$px + qy + rz = 0 \text{ and } x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$$

are given by the equation

$$\frac{a^2p^2}{u - a^2} + \frac{b^2q^2}{u - b^2} + \frac{c^2r^2}{u - c^2} = 0.$$

Sol. Do yourself. Proceed exactly as in solved examples 2 and 4.

Ex. 7. *Find the maximum value of $x^m y^n z^p$ subject to the condition $x + y + z = a$.* **(Meerut 1994P, 95BP)**

Sol. Let $u = x^m y^n z^p$, ...(1)

where the variables x, y, z are connected by the relation

$$x + y + z = a. \qquad ...(2)$$

From (1), $\log u = m \log x + n \log y + p \log z$.

$$\therefore \quad \frac{1}{u} du = \frac{m}{x} dx + \frac{n}{y} dy + \frac{p}{z} dz.$$

For a maximum or a minimum of u, we have

$$du = 0$$

$$\Rightarrow \quad \frac{m}{x} dx + \frac{n}{y} dy + \frac{p}{z} dz = 0. \qquad ...(3)$$

Differentiating the given equation (2) connecting the variables x, y and z, we get

$$dx + dy + dz = 0. \qquad ...(4)$$

Multiplying (3) by 1 and (4) by λ, and adding and then equating the coefficients of dx, dy, dz to zero, we get

$$\frac{m}{x} + \lambda = 0, \frac{n}{y} + \lambda = 0, \frac{p}{z} + \lambda = 0.$$

From these, we get $x = -m/\lambda, y = -n/\lambda, z = -p/\lambda$.

Putting these values of x, y, z in $x + y + z = a$, we get

$$-\left(\frac{m}{\lambda} + \frac{n}{\lambda} + \frac{p}{\lambda}\right) = a \quad \text{or} \quad -\frac{1}{\lambda}(m + n + p) = a$$

or $$-\frac{1}{\lambda} = \frac{a}{m + n + p}.$$

$\therefore$ u is stationary when

$$x = \frac{am}{m + n + p}, y = \frac{an}{m + n + p}, z = \frac{ap}{m + n + p}.$$

Let us now find the nature of this stationary value of u.

Since the variables x, y and z are connected by the relation (2), only two of them may be regarded as independent.

Let us regard x and y as independent variables and z as a function of x and y given by (2).

From (1), we have

$$\log u = m \log x + n \log y + p \log z.$$

$$\therefore \quad \frac{1}{u}\frac{\partial u}{\partial x} = \frac{m}{x} + \frac{p}{z}\frac{\partial z}{\partial x}.$$

Differentiating (2) partially w.r.t. x taking y as constant, we get

$$1 + (\partial z/\partial x) = 0 \quad \text{or} \quad \partial z/\partial x = -1.$$

$$\therefore \quad \frac{1}{u}\frac{\partial u}{\partial x} = \frac{m}{x} - \frac{p}{z},$$

so that $$\frac{1}{u}\frac{\partial^2 u}{\partial x^2} - \frac{1}{u^2}\left(\frac{\partial u}{\partial x}\right)^2 = -\frac{m}{x^2} + \frac{p}{z^2}\frac{\partial z}{\partial x} = -\frac{m}{x^2} - \frac{p}{z^2}.$$

But at the stationary point, we have $\partial u/\partial x = 0$.

$\therefore$ at the stationary point found above, we have

$$\frac{\partial^2 u}{\partial x^2} = -u\left[\frac{m}{x^2}+\frac{p}{z^2}\right] = -x^m y^n z^p\left[\frac{m}{x^2}+\frac{p}{z^2}\right],$$

which is –ive for the values of x, y, z found above.

Hence at the stationary point found above the value of u is maximum and this maximum value

$$=\left(\frac{am}{m+n+p}\right)^m\left(\frac{an}{m+n+p}\right)^n\left(\frac{ap}{m+n+p}\right)^p = \frac{a^{m+n+p}\, m^m n^n p^p}{(m+n+p)^{m+n+p}}.$$

Ex. 8. *Find the maximum or minimum value of $x^p y^q z^r$ subject to the condition*

$$ax+by+cz=p+q+r.$$

Sol. For complete solution of this question proceed as in solved example 7.

Let $\quad u = x^p y^q z^r, \quad$ ...(1)

where the variables x, y, z are connected by the relation

$$ax+by+cz=p+q+r. \quad ...(2)$$

From (1), $\log u = p\log x + q\log y + r\log z$.

$$\therefore \quad \frac{1}{u}du = \frac{p}{x}dx+\frac{q}{y}dy+\frac{r}{z}dz.$$

For a maximum or a minimum of u, we have $du=0$

$$\Rightarrow \quad \frac{p}{x}dx+\frac{q}{y}dy+\frac{r}{z}dz=0. \quad ...(3)$$

Also differentiating the given equation (2), we get

$$a\,dx+b\,dy+c\,dz=0. \quad ...(4)$$

Multiplying (3) by 1 and (4) by λ, and adding and then equating the coefficients of dx, dy, dz to zero, we get

$$\frac{p}{x}+\lambda a=0, \frac{q}{y}+\lambda b=0, \frac{r}{z}+\lambda c=0.$$

From these, we get $x=-p/\lambda a, y=-q/\lambda b, z=-r/\lambda c$.

Putting these values of x, y, z in (2), we get

$$-\left(\frac{p}{\lambda}+\frac{q}{\lambda}+\frac{r}{\lambda}\right)=p+q+r \quad \text{or} \quad -\frac{1}{\lambda}(p+q+r)=p+q+r$$

or $\quad \lambda=-1$.

$\therefore$ u is stationary when $x=p/a, y=q/b, z=r/c$.

Now regard x and y as independent variables and z as a function of x and y given by (2).

From (1), we have

$$\log u = p\log x + q\log y + r\log z.$$

$$\therefore \quad \frac{1}{u}\frac{\partial u}{\partial x}=\frac{p}{x}+\frac{r}{z}\frac{\partial z}{\partial x}.$$

Differentiating (2) partially w.r.t. x taking y as constant, we get

$$a+c(\partial z/\partial x)=0 \quad \text{or} \quad \partial z/\partial x=-a/c.$$

$$\therefore \quad \frac{1}{u}\frac{\partial u}{\partial x}=\frac{p}{x}-\frac{r}{z}\cdot\frac{a}{c},$$

so that $\quad \dfrac{1}{u}\dfrac{\partial^2 u}{\partial x^2}-\dfrac{1}{u^2}\left(\dfrac{\partial u}{\partial x}\right)^2=-\dfrac{p}{x^2}+\dfrac{a}{c}\cdot\dfrac{r}{z^2}\dfrac{\partial z}{\partial x}=-\dfrac{p}{x^2}-\dfrac{a^2}{c^2}\cdot\dfrac{r}{z^2}.$

But at the stationary point, we have $\partial u/\partial x=0$.

$\therefore$ at the stationary point found above, we have

$$\frac{\partial^2 u}{\partial x^2}=-u\left[\frac{p}{x^2}+\frac{a^2}{c^2}\cdot\frac{r}{z^2}\right]=-x^p y^q z^r\left[\frac{p}{x^2}+\frac{a^2}{c^2}\cdot\frac{r}{z^2}\right],$$

which is –ive for the values of x, y, z found above.

Hence u is maximum at the stationary point found above and this maximum value of $u = (p/a)^p.(q/b)^q.(r/c)^r$.

Ex. 9. *Find the minimum value of $x+y+z$, subject to the condition*

$$(a/x)+(b/y)+(c/z)=1.$$

Sol. Let $\quad u=x+y+z, \quad$ ...(1)

where the variables x, y, z are connected by the relation

$$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=1. \quad ...(2)$$

For a maximum or a minimum of u, we have

$$du=0 \Rightarrow dx+dy+dz=0. \quad ...(3)$$

Also differentiating the given equation (2), we get

$$-\frac{a}{x^2}dx-\frac{b}{y^2}dy-\frac{c}{z^2}dz=0. \quad ...(4)$$

Multiplying (3) by 1 and (4) by λ, and adding and then equating the coefficients of dx, dy, dz to zero, we get

$$1-\frac{\lambda a}{x^2}=0, 1-\frac{\lambda b}{y^2}=0, 1-\frac{\lambda c}{z^2}=0.$$

From these, we get $x=\sqrt{(\lambda a)}, y=\sqrt{(\lambda b)}, z=\sqrt{(\lambda c)}$.

Putting these values of x, y, z in (2), we get

$$\frac{a}{\sqrt{(\lambda a)}}+\frac{b}{\sqrt{(\lambda b)}}+\frac{c}{\sqrt{(\lambda c)}}=1 \text{ or } \frac{1}{\sqrt{\lambda}}(\sqrt{a}+\sqrt{b}+\sqrt{c})=1$$

or $\quad \sqrt{\lambda}=\sqrt{a}+\sqrt{b}+\sqrt{c}$.

$\therefore$ u is stationary when

$$x=\sqrt{a}(\sqrt{a}+\sqrt{b}+\sqrt{c}), y=\sqrt{b}(\sqrt{a}+\sqrt{b}+\sqrt{c}),$$

$$z=\sqrt{c}(\sqrt{a}+\sqrt{b}+\sqrt{c}).$$

Let us regard x and y as independent variables and z as a function of x and y given by (2).

From (1), we have

$$\frac{\partial u}{\partial x} = 1 + \frac{\partial z}{\partial x}.$$

Differentiating (2) partially w.r.t. x taking y as constant, we get

$$-\frac{a}{x^2} - \frac{c}{z^2}\frac{\partial z}{\partial x} = 0 \quad \text{or} \quad \frac{\partial z}{\partial x} = -\frac{az^2}{cx^2}.$$

$$\therefore \quad \frac{\partial u}{\partial x} = 1 - \frac{az^2}{cx^2},$$

so that $$\frac{\partial^2 u}{\partial x^2} = \frac{2az^2}{cx^3} - \frac{2az}{cx^2}\frac{\partial z}{\partial x} = \frac{2az^2}{cx^3} + \frac{2az}{cx^2}\cdot\frac{az^2}{cx^2},$$

which is positive for the values of x, y, z found above.

Hence u is minimum at the stationary point found above and the minimum value of u

$$= \sqrt{a}\,(\sqrt{a} + \sqrt{b} + \sqrt{c}) + \sqrt{b}\,(\sqrt{a} + \sqrt{b} + \sqrt{c}) + \sqrt{c}\,(\sqrt{a} + \sqrt{b} + \sqrt{c})$$

$$= (\sqrt{a} + \sqrt{b} + \sqrt{c})^2.$$

Ex. 10. *Find the minimum value of $x^2 + y^2 + z^2$, given that*

$$ax + by + cz = p. \qquad \textbf{(Meerut 1991 P)}$$

Sol. Do yourself. Proceed as in solved example 9. The required minimum value of u is $p^2/(a^2 + b^2 + c^2)$.

Ex. 11. *Find the maximum or minimum value of $x^p y^q z^r$ subject to the condition*

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 1.$$

Sol. Let $u = x^p y^q z^r$, ...(1)

where the variables x, y, z are connected by the relation

$$(a/x) + (b/y) + (c/z) = 1. \qquad ...(2)$$

From (1), $\log u = p \log x + q \log y + r \log z$.

$$\therefore \quad \frac{1}{u}\,du = \frac{p}{x}\,dx + \frac{q}{y}\,dy + \frac{r}{z}\,dz.$$

For a maximum or a minimum of u, we have $du = 0$.

$$\therefore \quad \frac{p}{x}\,dx + \frac{q}{y}\,dy + \frac{r}{z}\,dz = 0. \qquad ...(3)$$

Also differentiating the given equation (2), we get

$$-\frac{a}{x^2}\,dx - \frac{b}{y^2}\,dy - \frac{c}{z^2}\,dz = 0. \qquad ...(4)$$

Multiplying (3) by 1 and (4) by λ, and adding and then equating the coefficients of dx, dy, dz to zero, we get

$$\frac{p}{x} - \frac{\lambda a}{x^2} = 0,\ \frac{q}{y} - \frac{\lambda b}{y^2} = 0,\ \frac{r}{z} - \frac{\lambda c}{z^2} = 0.$$

From these, we get $x = a\lambda/p,\ y = b\lambda/q,\ z = c\lambda/r$.

Putting these values of x, y, z in (2), we get

$$\frac{p}{\lambda} + \frac{q}{\lambda} + \frac{r}{\lambda} = 1 \text{ or } \frac{1}{\lambda}(p + q + r) = 1 \text{ or } \lambda = p + q + r.$$

$\therefore$ u is stationary when

$$\frac{px}{a} = \frac{qy}{b} = \frac{rz}{c} = p + q + r.$$

Now regard x and y as independent variables and z as a function of x and y given by (2).

From (1), we have

$$\log u = p \log x + q \log y + r \log z.$$

$$\therefore \quad \frac{1}{u}\frac{\partial u}{\partial x} = \frac{p}{x} + \frac{r}{z}\frac{\partial z}{\partial x}.$$

Differentiating (2) partially w.r.t. x taking y as constant, we get

$$-\frac{a}{x^2} - \frac{c}{z^2}\frac{\partial z}{\partial x} = 0 \quad \text{or} \quad \frac{\partial z}{\partial x} = -\frac{az^2}{cx^2}.$$

$$\therefore \quad \frac{1}{u}\frac{\partial u}{\partial x} = \frac{p}{x} - \frac{r}{z}\cdot\frac{az^2}{cx^2} = \frac{p}{x} - \frac{arz}{cx^2},$$

so that $$\frac{1}{u}\frac{\partial^2 u}{\partial x^2} - \frac{1}{u^2}\left(\frac{\partial u}{\partial x}\right)^2 = -\frac{p}{x^2} + \frac{2arz}{cx^3} - \frac{ar}{cx^2}\frac{\partial z}{\partial x}$$

$$= -\frac{p}{x^2} + \frac{2arz}{cx^3} + \frac{ar}{cx^2}\cdot\frac{az^2}{cx^2}.$$

But at the stationary point, we have $\partial u/\partial x = 0$.

$\therefore$ at the stationary point found above, we have

$$\frac{\partial^2 u}{\partial x^2} = u\left[-\frac{p}{x^2} + \frac{2arz}{cx^3} + \frac{a^2rz^2}{c^2x^4}\right]$$

$$= x^p y^q z^r\left[-p\cdot\frac{p^2}{a^2(p+q+r)^2} + \frac{2ar}{c}\cdot\frac{c(p+q+r)}{r}\cdot\frac{p^3}{a^2(p+q+r)^3} + \frac{a^2r}{c^2}\cdot\frac{c^2(p+q+r)^2}{r^2}\cdot\frac{p^4}{a^4(p+q+r)^4}\right]$$

$$= x^p y^q z^r\left[-\frac{p^3}{a^2(p+q+r)^2} + \frac{2p^2}{a^2(p+q+r)^2} + \frac{p^4}{ra^2(p+q+r)^2}\right]$$

$$= x^p y^q z^r\left[\frac{p^3}{a^2(p+q+r)^2} + \frac{p^4}{ra^2(p+q+r)^2}\right],$$

which is +ive for the values of x, y, z found above.

Hence u is minimum at the stationary point given by

$$\frac{px}{a} = \frac{qy}{b} = \frac{rz}{c} = p + q + r.$$

Also the minimum value of u

$$= \left[\frac{a(p+q+r)}{p}\right]^p \left[\frac{b(p+q+r)}{q}\right]^q \left[\frac{c(p+q+r)}{r}\right]^r$$

$$= \frac{a^p b^q c^r}{p^p q^q r^r}(p+q+r)^{p+q+r}.$$

Ex. 12. *Find the minimum value of $x^4 + y^4 + z^4$, where $xyz = c^3$.*

Sol. Let $u = x^4 + y^4 + z^4$, ...(1)

where the variables x, y, z are connected by the relation

$xyz = c^3$. ...(2)

For a maximum or a minimum of u, we have

$du = 0 \Rightarrow 4x^3\,dx + 4y^3\,dy + 4z^3\,dz = 0$

$\Rightarrow x^3\,dx + y^3\,dy + z^3\,dz = 0.$ ...(3)

Also from the given relation (2), we have

$\log x + \log y + \log z = \log c^3.$

Differentiating this, we get

$(1/x)\,dx + (1/y)\,dy + (1/z)\,dz = 0.$ ...(4)

Multiplying (3) by 1 and (4) by λ, and adding and then equating the coefficients of dx, dy, dz to zero, we get

$$x^3 + \frac{\lambda}{x} = 0,\ y^3 + \frac{\lambda}{y} = 0,\ z^3 + \frac{\lambda}{z} = 0.$$

From these, we get $x^4 = y^4 = z^4 = -\lambda$.

Now from (2), $x^4y^4z^4 = c^{12}$.

$\therefore\ -\lambda^3 = c^{12}$ or $\lambda = -c^4$.

$\therefore$ u is sationary when $x^4 = y^4 = z^4 = c^4$ *i.e.*, when $x = y = z = c$.

Now regard x and y as independent variables and z as a function of x and y given by (2).

From (1), we have $\dfrac{\partial u}{\partial x} = 4x^3 + 4z^3\dfrac{\partial z}{\partial x}$.

Now from (2), we have $\log x + \log y + \log z = \log c^3$.

Differentiating this partially w.r.t. x taking y as constant, we get

$$\frac{1}{x} + \frac{1}{z}\frac{\partial z}{\partial x} = 0 \quad \text{or} \quad \frac{\partial z}{\partial x} = -\frac{z}{x}.$$

$$\therefore\ \frac{\partial u}{\partial x} = 4x^3 - 4z^3 \cdot \frac{z}{x} = 4x^3 - 4\frac{z^4}{x},$$

so that $$\frac{\partial^2 u}{\partial x^2} = 12x^2 + \frac{4z^4}{x^2} - \frac{16}{x}z^3\frac{\partial z}{\partial x}$$

$$= 12x^2 + \frac{4z^4}{x^2} - \frac{16z^3}{x}\left(-\frac{z}{x}\right) = 12x^2 + \frac{4z^4}{x^2} + \frac{16z^4}{x^2}.$$

At the stationary point (c, c, c) found above, we have

$$\frac{\partial^2 u}{\partial x^2} = 12c^2 + 4c^2 + 16c^2 = 32c^2, \text{ which is +ive.}$$

$\therefore$ u is minimum at the point $x = y = z = c$ and the minimum value of $u = c^4 + c^4 + c^4 = 3c^4$.

Ex. 13. *Find the maximum value of u, when $u = x^2y^3z^4$ and $2x + 3y + 4z = a$.*

Sol. Let $u = x^2y^3z^4$,

where the variables x, y, z are connected by the relation ...(1)

$2x + 3y + 4z = a$..

From (1), $\log u = 2\log x + 3\log y + 4\log z$. ...(2)

$$\therefore\ \frac{1}{u}du = \frac{2}{x}dx + \frac{3}{y}dy + \frac{4}{z}dz.$$

For a maximum or a minimum of u, we have

$du = 0 \Rightarrow (2/x)\,dx + (3/y)\,dy + (4/z)\,dz = 0.$ ...(3)

Differentiating the given equation (2), we have

$2\,dx + 3\,dy + 4\,dz = 0.$ ...(4)

Multiplying (3) by 1 and (4) by λ, and adding and then equating the coefficients of dx, dy, dz to zero, we get

$$\frac{2}{x} + 2\lambda = 0,\ \frac{3}{y} + 3\lambda = 0,\ \frac{4}{z} + 4\lambda = 0.$$

From these, get $x = -1/\lambda, y = -1/\lambda, z = -1/\lambda$.

Putting these values of x, y, z in (2), we get

$$-\frac{2}{\lambda} - \frac{3}{\lambda} - \frac{4}{\lambda} = a \text{ or } -\frac{9}{\lambda} = a \text{ or } \lambda = -\frac{9}{a}.$$

$\therefore$ u is stationary when $x = y = z = a/9$.

Now regard x and y as independent variables and z as a function of x and y given by (2).

From (1), we have

$\log u = 2\log x + 3\log y + 4\log z$.

$$\therefore\ \frac{1}{u}\frac{\partial u}{\partial x} = \frac{2}{x} + \frac{4}{z}\frac{\partial z}{\partial x}.$$

Differentiating (2) partially w.r.t. x taking y as constant, we get

$2 + 4(\partial z/\partial x) = 0$ or $\partial z/\partial x = -1/2$.

$$\therefore\ \frac{1}{u}\frac{\partial u}{\partial x} = \frac{2}{x} + \frac{4}{z}\cdot\left(-\frac{1}{2}\right) = \frac{2}{x} - \frac{2}{z},$$

so that $$\frac{1}{u}\frac{\partial^2 u}{\partial x^2} - \frac{1}{u^2}\left(\frac{\partial u}{\partial x}\right)^2 = -\frac{2}{x^2} + \frac{2}{z^2}\frac{\partial z}{\partial x} = -\frac{2}{x^2} - \frac{1}{z^2}.$$

But at the stationary point, we have $\partial u/\partial x = 0$.

$\therefore$ at the stationary point found above, we have

$$\frac{\partial^2 u}{\partial x^2} = -u\left(\frac{2}{x^2} + \frac{1}{z^2}\right) = -x^2y^3z^4\left(\frac{2}{x^2} + \frac{1}{z^2}\right),$$

which is –ive for $x = y = z = a/9$.

Hence at the stationary point $x = y = z = a/9$, u is maximum and the maximum value of u

$$= (a/9)^2 (a/9)^3 (a/9)^4 = (a/9)^9.$$

Ex. 14. *Given $u = 5xyz/(x + 2y + 4z)$. Find the values of x, y, z for which u is maximum subject to the condition $xyz = 8$.* **(Meerut 1994)**

Sol. We have $\quad u = 5xyz/(x + 2y + 4z)$, ...(1)

where the variables x, y, z are connected by the relation

$$xyz = 8. \qquad ...(2)$$

From (1) and (2), we have $u = 40/(x + 2y + 4z)$.

$$\therefore \quad du = \frac{-40}{(x + 2y + 4z)^2}(dx + 2\,dy + 4\,dz).$$

For a maximum or a minimum of u, we have $du = 0$

$$\Rightarrow \quad dx + 2\,dy + 4\,dz = 0. \qquad ...(3)$$

From (2), $\log x + \log y + \log z = \log 8$.

Differentiating this, we get

$$(1/x)\,dx + (1/y)\,dy + (1/z)\,dz = 0. \qquad ...(4)$$

Multiplying (3) by 1 and (4) by λ, and adding and then equating to zero the coefficients of dx, dy and dz, we get

$$1 + (\lambda/x) = 0,\ 2 + (\lambda/y) = 0,\ 4 + (\lambda/z) = 0.$$

From these, we get $x = -\lambda, y = -\lambda/2, z = -\lambda/4$.

Putting these values of x, y, z in (2), we get

$$-\lambda^3/8 = 8 \text{ or } \lambda^3 = -64 \text{ or } \lambda = -4.$$

$\therefore$ u is stationary at the point given by $x = 4, y = 2, z = 1$.

Now regard x and y as independent variables and z as a function of x and y given by (2).

We have $u = 40/(x + 2y + 4z)$.

$$\therefore \quad \frac{\partial u}{\partial x} = -\frac{40}{(x + 2y + 4z)^2}\left[1 + 4\frac{\partial z}{\partial x}\right].$$

From (2), $\log x + \log y + \log z = \log 8$.

$\therefore$ $(1/x) + (1/z)(\partial z/\partial x) = 0 \quad$ or $\quad \partial z/\partial x = -z/x$.

$$\therefore \quad \frac{\partial u}{\partial x} = -\frac{40}{(x + 2y + 4z)^2}\left[1 - 4\frac{z}{x}\right],$$

so that
$$\frac{\partial^2 u}{\partial x^2} = \frac{80}{(x + 2y + 4z)^3}\left[1 + 4\frac{\partial z}{\partial x}\right]\left[1 - 4\frac{z}{x}\right] - \frac{40}{(x + 2y + 4z)^2}\left[\frac{4z}{x^2} - \frac{4}{x}\frac{\partial z}{\partial x}\right]$$

$$= \frac{80}{(x + 2y + 4z)^3}\left[1 - \frac{4z}{x}\right]^2 - \frac{40}{(x + 2y + 4z)^2}\left[\frac{4z}{x^2} + \frac{4z}{x^2}\right].$$

$\therefore$ at the stationary point (4, 2, 1) found above, we have

$$\frac{\partial^2 u}{\partial x^2} = \frac{80}{12^3}[1 - 1]^2 - \frac{40}{144}\left[\frac{1}{4} + \frac{1}{4}\right], \text{ which is –ive.}$$

$\therefore$ u is maximum at the point given by $x = 4, y = 2, z = 1$.

Ex. 15. *Divide a number a into three parts such that their product will be maximum.*

Sol. Let $\quad u = xyz$,

where the variables x, y and z are connected by the relation ...(1)

$$x + y + z = a. \qquad ...(2)$$

From (1), $\log u = \log x + \log y + \log z$.

$\therefore$ $(1/u)\,du = (1/x)\,dx + (1/y)\,dy + (1/z)\,dz$.

For a maximum or a minimum of u, we have $du = 0$

$$\Rightarrow \quad (1/x)\,dx + (1/y)\,dy + (1/z)\,dz = 0. \qquad ...(3)$$

Also differentiating the equation (2), we have

$$dx + dy + dz = 0. \qquad ...(4)$$

Multiplying (3) by 1 and (4) by λ, and adding and then equating the coefficients of dx, dy, dz to zero, we get

$$\frac{1}{x} + \lambda = 0,\ \frac{1}{y} + \lambda = 0,\ \frac{1}{z} + \lambda = 0.$$

From these, we get $x = y = z = -1/\lambda$.

Putting these values of x, y, z in (2), we get

$$-3/\lambda = a \quad \text{or} \quad \lambda = -3/a.$$

$\therefore$ u is stationary at the point given by

$$x = y = z = a/3.$$

Now regard x and y as independent variables and z as a function of x and y given by (2).

From (1), $\log u = \log x + \log y + \log z$.

$$\therefore \quad \frac{1}{u}\frac{\partial u}{\partial x} = \frac{1}{x} + \frac{1}{z}\frac{\partial z}{\partial x}.$$

But from (2), $1 + (\partial z/\partial x) = 0$ or $\partial z/\partial x = -1$.

$$\therefore \quad \frac{1}{u}\frac{\partial u}{\partial x} = \frac{1}{x} - \frac{1}{z},$$

so that
$$\frac{1}{u}\frac{\partial^2 u}{\partial x^2} - \frac{1}{u^2}\left(\frac{\partial u}{\partial x}\right)^2 = -\frac{1}{x^2} + \frac{1}{z^2}\frac{\partial z}{\partial x} = -\frac{1}{x^2} - \frac{1}{z^2}.$$

But at the stationary point, we have $\partial u/\partial x = 0$.

$\therefore$ at the stationary point $(a/3, a/3, a/3)$, we have

$$\frac{\partial^2 u}{\partial x^2} = -u\left[\frac{1}{x^2} + \frac{1}{z^2}\right] = -xyz\left[\frac{1}{x^2} + \frac{1}{z^2}\right]$$

which is negative for $x = y = z = a/3$.

Hence u is maximum when $x = y = z = a/3$ and the maximum value of $u = (a/3)^3$.

Ans. The required three parts of a are $a/3, a/3, a/3$ and the maximum value of the product is $(a/3)^3$.

Ex. 16. *Show that the maximum and minimum of the radii vectors of the sections of the surface*

$$(x^2 + y^2 + z^2)^2 = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}$$

by the plane $\lambda x + \mu y + \nu z = 0$

are given by the equation

$$\frac{a^2\lambda^2}{1 - a^2r^2} + \frac{b^2\mu^2}{1 - b^2r^2} + \frac{c^2\nu^2}{1 - c^2r^2} = 0.$$

Sol. We have to find the maximum and minimum values of r, where

$$r^2 = x^2 + y^2 + z^2. \qquad ...(1)$$

Also the variables x, y, z are connected by the relations

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = (x^2 + y^2 + z^2)^2 = r^4 \qquad ...(2)$$

and

$$\lambda x + \mu y + \nu z = 0. \qquad ...(3)$$

From (1), $2r\,dr = 2x\,dx + 2y\,dy + 2z\,dz$.

For a maximum or a minimum of r, we have

$$dr = 0 \Rightarrow x\,dx + y\,dy + z\,dz = 0. \qquad ...(4)$$

Differentiating (2), we get

$$\frac{2x}{a^2}dx + \frac{2y}{b^2}dy + \frac{2z}{c^2}dz = 4r^3\,dr.$$

But for a maximum or a minimum of r, we have $dr = 0$.

$$\therefore \quad \frac{x}{a^2}dx + \frac{y}{b^2}dy + \frac{z}{c^2}dz = 0. \qquad ...(5)$$

Also differentiating (3), we get

$$\lambda dx + \mu dy + \nu dz = 0. \qquad ...(6)$$

Multiplying (4) by 1, (5) by λ_1 and (6) by λ_2, and adding and then equating to zero the coefficients of dx, dy, dz, we get

$$x + \frac{x}{a^2}\lambda_1 + \lambda\lambda_2 = 0 \qquad ...(7)$$

$$y + \frac{y}{b^2}\lambda_1 + \mu\lambda_2 = 0 \qquad ...(8)$$

$$z + \frac{z}{c^2}\lambda_1 + \nu\lambda_2 = 0. \qquad ...(9)$$

Multiplying (7), (8), (9) by x, y, z respectively and adding, we get

$$r^2 + r^4.\lambda_1 + 0.\lambda_2 = 0 \quad \text{or} \quad \lambda_1 = -1/r^2.$$

$\therefore$ from (7), we have $x - \dfrac{x}{a^2} \cdot \dfrac{1}{r^2} + \lambda\lambda_2 = 0$

or

$$x = \frac{a^2r^2\lambda\lambda_2}{1 - a^2r^2}.$$

Similarly from (8) and (9), we have

$$y = \frac{b^2r^2\mu\lambda_2}{1 - b^2r^2} \quad \text{and} \quad z = \frac{c^2r^2\nu\lambda_2}{1 - c^2r^2}.$$

Substituting these values of x, y, z in $\lambda x + \mu y + \nu z = 0$, we get

$$\frac{a^2r^2\lambda^2\lambda_2}{1 - a^2r^2} + \frac{b^2r^2\mu^2\lambda_2}{1 - b^2r^2} + \frac{c^2r^2\nu^2\lambda_2}{1 - c^2r^2} = 0$$

or

$$\frac{a^2\lambda^2}{1 - a^2r^2} + \frac{b^2\mu^2}{1 - b^2r^2} + \frac{c^2\nu^2}{1 - c^2r^2} = 0. \qquad ...(10)$$

The equation (10) gives the maximum and minimum values of r.

Ex. 17. *Find the points where*

$$u = ax^p + by^q + cz^r$$

has extreme values subject to the condition

$$x^l + y^m + z^n = k.$$

Sol. We have $u = ax^p + by^q + cz^r$, ...(1)

where the variables x, y, z are connected by the relation

$$x^l + y^m + z^n = k. \qquad ...(2)$$

For a maximum or a minimum of u, we have

$$du = 0 \Rightarrow apx^{p-1}\,dx + bqy^{q-1}\,dy + crz^{r-1}\,dz = 0. \qquad ...(3)$$

Also differentiating (2), we get

$$lx^{l-1}\,dx + my^{m-1}\,dy + nz^{n-1}\,dz = 0. \qquad ...(4)$$

Multiplying (3) by 1 and (4) by λ, and adding and then equating to zero the coefficients of dx, dy, dz, we get

$$ap\,x^{p-1} + \lambda l\,x^{l-1} = 0, \qquad ...(5)$$

$$bq\,y^{q-1} + \lambda m\,y^{m-1} = 0, \qquad ...(6)$$

and

$$cr\,z^{r-1} + \lambda n\,z^{n-1} = 0. \qquad ...(7)$$

From (5), we have

$$ap\,x^{p-1} = -\lambda l\,x^{l-1} \quad \text{or} \quad ap\,x^{p-l} = -\lambda l$$

or

$$\frac{x^{p-l}}{l/pa} = -\lambda.$$

Similarly from (6) and (7), we have

$$\frac{y^{q-m}}{m/qb} = -\lambda \quad \text{and} \quad \frac{z^{r-n}}{n/rc} = -\lambda.$$

Hence the values of x, y, z for which u has extreme values are given by $\dfrac{x^{p-l}}{l/pa} = \dfrac{y^{q-m}}{m/qb} = \dfrac{z^{r-n}}{n/rc}$.

Ex. 18. *If two variables x and y are connected by the relation $ax^2 + by^2 = ab$, show that the maximum and minimum values of the function $u = x^2 + y^2 + xy$ will be the roots of the equation*

$$4(u-a)(u-b) = ab.$$

Sol. We have $\quad u = x^2 + y^2 + xy, \quad$ ...(1)

where the variables x and y are connected by the relation

$$ax^2 + by^2 = ab. \quad ...(2)$$

For a maximum or a minimum of u, we have

$$du = 0 \Rightarrow 2x\,dx + 2y\,dy + y\,dx + x\,dy = 0$$

$\Rightarrow$ $(2x+y)\,dx + (2y+x)\,dy = 0.$ ...(3)

Also differentiating (2), we have

$$2ax\,dx + 2by\,dy = 0 \text{ or } ax\,dx + by\,dy = 0. \quad ...(4)$$

Multiplying (3) by 1 and (4) by λ, and adding and then equating to zero the coefficients of dx and dy, we get

$$(2x+y) + \lambda\, ax = 0 \quad ...(5)$$

and $\quad (2y+x) + \lambda\, by = 0. \quad$ ...(6)

Multiplying (5) by x and (6) by y and adding, we get

$$2(x^2 + y^2 + xy) + \lambda(ax^2 + by^2) = 0$$

or $\quad 2u + \lambda\, ab = 0$ or $\lambda = -2u/ab$.

Putting $\lambda = -2u/ab$ in (5), we get

$$(2x+y) - \frac{2u}{b}x = 0$$

or $\quad 2(b-u)x + by = 0. \quad$ ...(7)

Similarly putting $\lambda = -2u/ab$ in (6), we get

$$(2y+x) - \frac{2u}{a}y = 0$$

or $\quad ax + 2(a-u)y = 0. \quad$ ...(8)

Eliminating x and y from (7) and (8), we get

$$\begin{vmatrix} 2(b-u) & b \\ a & 2(a-u) \end{vmatrix} = 0$$

or $\quad 4(b-u)(a-u) - ab = 0$

or $\quad 4(u-a)(u-b) = ab$, which gives the maximum and minimum values of u.

Ex. 19. *Show that the maximum and minimum values of $u = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$ subject to the conditions $lx + my + nz = 0$ and $x^2 + y^2 + z^2 = 1$ are given by the equation*

$$\begin{vmatrix} a-u & h & g & l \\ h & b-u & f & m \\ g & f & c-u & n \\ l & m & n & 0 \end{vmatrix} = 0.$$

Sol. We have $u = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$, ...(1)

where the variables x, y and z are connected by the relations

$$lx + my + nz = 0 \quad ...(2)$$

and $\quad x^2 + y^2 + z^2 = 1. \quad$ ...(3)

For a maximum or a minimum of u, we have $du = 0$

$\Rightarrow$ $2(ax + gz + hy)\,dx + 2(by + fz + hx)\,dy + 2(cz + fy + gx)\,dz = 0$

$\Rightarrow$ $(ax + hy + gz)\,dx + (hx + by + fz)\,dy + (gx + fy + cz)\,dz = 0.$ ...(4)

Also differentiating (2) and (3), we get

$$l\,dx + m\,dy + n\,dz = 0 \quad ...(5)$$

and $\quad x\,dx + y\,dy + z\,dz = 0. \quad$ ...(6)

Multiplying (4) by 1, (5) by λ and (6) by μ, and adding and then equating to zero the coefficients of dx, dy and dz, we get

$$(ax + hy + gz) + \lambda l + \mu x = 0 \quad ...(7)$$

$$(hx + by + fz) + \lambda m + \mu y = 0 \quad ...(8)$$

and $\quad (gx + fy + cz) + \lambda n + \mu z = 0. \quad$ ...(9)

Multiplying (7), (8) and (9) by x, y and z respectively and adding, we get

$$u + \lambda.0 + \mu.1 = 0 \text{ or } \mu = -u.$$

Putting $\mu = -u$ in (7), (8) and (9), we get

$$(a-u)x + hy + gz + \lambda l = 0, \quad ...(10)$$

$$hx + (b-u)y + fz + \lambda m = 0, \quad ...(11)$$

and $\quad gx + fy + (c-u)z + \lambda n = 0. \quad$ ...(12)

Also the relation (2) can be written as

$$lx + my + nz + \lambda.0 = 0. \quad ...(13)$$

Eliminating x, y, z and λ from (10), (11), (12) and (13), we get

$$\begin{vmatrix} a-u & h & g & l \\ h & b-u & f & m \\ g & f & c-u & n \\ l & m & n & 0 \end{vmatrix} = 0,$$

which gives the maximum and minimum values of u.

Ex. 20. *Prove that if $x + y + z = 1$, $ayz + bzx + cxy$ has an extreme value equal to*

$$\frac{abc}{2bc + 2ca + 2ab - a^2 - b^2 - c^2}.$$

Prove also that if a, b, c are all positive and c lies between $a + b - 2\sqrt{(ab)}$ and $a + b + 2\sqrt{(ab)}$ this value is true maximum and that if a, b, c are all negative and c lies between $a + b \pm 2\sqrt{(ab)}$, it is true minimum.

Sol. Let $\quad u = ayz + bzx + cxy, \quad$ ...(1)

where the variables x, y and z are connected by the relation

$$x + y + z = 1. \quad ...(2)$$

For a maximum or a minimum of u, we have $du = 0$

$\Rightarrow$ $(bz + cy)\,dx + (cx + az)\,dy + (ay + bx)\,dz = 0.$ ...(3)

Also differentiating (2), we get

$$dx + dy + dz = 0.$$ ...(4)

Multiplying (3) by 1 and (4) by λ, and adding and then equating to zero the coefficients of dx, dy and dz, we get

$$bz + cy + \lambda = 0,\ cx + az + \lambda = 0,\ ay + bx + \lambda = 0.$$

$$\therefore\quad -\lambda = bz + cy = cx + az = ay + bx.$$

From these, we have

$$z = \frac{ay + bx - cx}{a} = \frac{ay + bx - cy}{b}.$$

$$\therefore\quad bx\,(a + c - b) = ay\,(b + c - a)$$

$$\text{or}\quad \frac{x}{a\,(b + c - a)} = \frac{y}{b\,(a + c - b)} = \frac{z}{c\,(a + b - c)},\quad \text{(by symmetry)}$$

$$= \frac{x + y + z}{2\Sigma bc - \Sigma a^2} = \frac{1}{2\Sigma bc - \Sigma a^2}.$$ ...(5)

Hence u is stationary for the values of x, y and z given by (5).

Also the stationary value of u

$$= ayz + bzx + cxy$$

$$= \frac{abc\,(2bc + 2ca + 2ab - a^2 - b^2 - c^2)}{(2bc + 2ca + 2ab - a^2 - b^2 - c^2)^2}$$

$$= \frac{abc}{2bc + 2ca + 2ab - a^2 - b^2 - c^2}.$$

Now let us regard x and y as independent variables and z as a function of x and y given by (2).

From (1), $\dfrac{\partial u}{\partial x} = (bz + cy) + (ay + bx)\dfrac{\partial z}{\partial x}$, regarding y as a constant.

Also from (2), $1 + (\partial z/\partial x) = 0$ or $\partial z/\partial x = -1$.

$$\therefore\quad \frac{\partial u}{\partial x} = bz + cy - ay - bx,$$

so that $r = \dfrac{\partial^2 u}{\partial x^2} = b\dfrac{\partial z}{\partial x} - b = -b - b = -2b,$

$s = \dfrac{\partial^2 u}{\partial x\,\partial y} = b\dfrac{\partial z}{\partial y} + c - a = c - a - b.$ [$\because$ from (2), $\partial z/\partial y = -1$]

Similarly $t = \dfrac{\partial^2 u}{\partial y^2} = -2a.$

$$\therefore\quad rt - s^2 = 4ab - (c - a - b)^2$$

$$= \{2\sqrt{(ab)} + c - a - b\}\,\{2\sqrt{(ab)} - c + a + b\}$$

$$= [c - \{a + b - 2\sqrt{(ab)}\}]\,[\{a + b + 2\sqrt{(ab)}\} - c].$$

Hence $rt - s^2$ will be positive when $c > a + b - 2\sqrt{(ab)}$ and $c < a + b + 2\sqrt{(ab)}$ whether a, b, c are all +ive or -ive.

But when a, b, c are all +ive, r is -ive and so the stationary value is a true maximum in this case. Also when a, b, c are all -ive, r is +ive and so the stationary value is a true minimum in this case.

Ex. 21. *Find the maximum or minimum value of* $x^2 + y^2 + z^2$, *subject to the conditions*

$$lx + my + nz = 1,\ l'x + m'y + n'z = 1.$$

Sol. Let $u = x^2 + y^2 + z^2$, ...(1)

where the variables x, y and z are connected by the relations

$$lx + my + nz = 1,$$ ...(2)

and

$$l'x + m'y + n'z = 1.$$ ...(3)

For a maximum or a minimum of u, we have $du = 0$

$$\Rightarrow\quad 2x\,dx + 2y\,dy + 2z\,dz = 0$$

$$\Rightarrow\quad x\,dx + y\,dy + z\,dz = 0.$$ ...(4)

Also differentiating (2) and (3), we get

$$l\,dx + m\,dy + n\,dz = 0$$ ...(5)

and

$$l'dx + m'dy + n'dz = 0.$$ ...(6)

Multiplying (4) by 1, (5) by λ and (6) μ, and adding and then equating to zero the coefficients of dx, dy and dz, we get

$$x + l\lambda + l'\mu = 0,$$ ...(7)

$$y + m\lambda + m'\mu = 0,$$ ...(8)

and

$$z + n\lambda + n'\mu = 0.$$ ...(9)

Multiplying the equations (7), (8) and (9) by x, y and z respectively and adding, we get

$$u + \lambda.1 + \mu.1 = 0.$$ ...(10)

Again multiplying the equations (7), (8) and (9) by l, m and n respectively and adding, we get

$$1 + \lambda\Sigma l^2 + \mu\Sigma ll' = 0.$$ ...(11)

Next multiplying the equations (7), (8) and (9) by l', m' and n' respectively and adding, we get

$$1 + \lambda\,\Sigma ll' + \mu\,\Sigma l'^2 = 0.$$ ...(12)

Now eliminating λ and μ from (10), (11) and (12), we get

$$\begin{vmatrix} u & 1 & 1 \\ 1 & \Sigma l^2 & \Sigma ll' \\ 1 & \Sigma ll' & \Sigma l'^2 \end{vmatrix} = 0.$$

The above equation gives the maximum or minimum value of u.

Note. If we wish to find an explicit expression for the extreme value of u and also wish to say whether it is maximum or minimum we proceed as follows :

Solving the equations (11) and (12) for λ and μ, we get

$$\frac{\lambda}{\Sigma ll' - \Sigma l'^2} = \frac{\mu}{\Sigma ll' - \Sigma l^2} = \frac{1}{\Sigma l^2\,\Sigma l'^2 - (\Sigma ll')^2}$$

$$\lambda = \frac{\Sigma ll' - \Sigma l'^2}{\Sigma (mn' - m'n)^2} \quad \text{and} \quad \mu = \frac{\Sigma ll' - \Sigma l^2}{\Sigma (mn' - m'n)^2}$$

or [$\because \Sigma l^2 \Sigma l'^2 - (\Sigma ll')^2 = \Sigma (mn' - m'n)^2$, by Lagrange's identity]

Putting these values of λ and μ in (10), the maximum or minimum value of u is given by

$$u = -\lambda - \mu = \frac{\Sigma l^2 + \Sigma l'^2 - 2\Sigma ll'}{\Sigma (mn' - m'n)^2} = \frac{\Sigma (l - l')^2}{\Sigma (mn' - m'n)^2}.$$

To find the nature of this stationary value of u.

Since there are two relations amongst the variables x, y and z, therefore only one variable will be independent. Let it be x. Then

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dx} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dx}$$

$$= 2x + 2y \cdot \frac{dy}{dx} + 2z \cdot \frac{dz}{dx}.$$

Now differentiating (2) and (3) w.r.t. x, we get

$$l + m\frac{dy}{dx} + n\frac{dz}{dx} = 0$$

and

$$l' + m'\frac{dy}{dx} + n'\frac{dz}{dx} = 0.$$

Solving these, we get

$$\frac{dy/dx}{nl' - n'l} = \frac{dz/dx}{m'l - l'm} = \frac{1}{mn' - m'n}.$$

$$\therefore \quad \frac{dy}{dx} = \frac{nl' - n'l}{mn' - m'n} \text{ and } \frac{dz}{dx} = \frac{m'l - l'm}{mn' - m'n}.$$

$$\therefore \quad \frac{du}{dx} = 2x + 2y \cdot \frac{nl' - n'l}{mn' - m'n} + 2z \cdot \frac{m'l - l'm}{mn' - m'n}$$

and so

$$\frac{d^2u}{dx^2} = 2 + 2\frac{dy}{dx} \cdot \frac{nl' - n'l}{mn' - m'n} + 2\frac{dz}{dx} \cdot \frac{m'l - l'm}{mn' - m'n}$$

$$= 2 + 2\left(\frac{nl' - n'l}{mn' - m'n}\right)^2 + 2\left(\frac{m'l - l'm}{mn' - m'n}\right)^2,$$

which is +ive.

Hence the stationary value of u found above is the minimum value.

Ex. 22. *Prove that of all rectangular parallelopipeds of the same volume, the cube has the least surface.*

Sol. Let x, y, z be the dimensions of the rectangular parallelopiped, V be its volume and S be its surface. Then

$$S = 2xy + 2yz + 2zx \quad \text{...(1)}$$

and

$$xyz = V = \text{some constant}. \quad \text{...(2)}$$

For a maximum or minimum of S, we have

$$dS = 2(y + z)\,dx + 2(z + x)\,dy + 2(x + y)\,dz = 0$$

i.e.

$$(y + z)\,dx + (z + x)\,dy + (x + y)\,dz = 0. \quad \text{...(3)}$$

Also differentiating (2) and observing that V is constant, we have

$$yz\,dx + zx\,dy + xy\,dz = 0. \quad \text{...(4)}$$

Multiplying (3) by 1 and (4) by λ, and adding and then equating to zero the coefficients of dx, dy and dz, we get

$$(y + z) + \lambda yz = 0,$$

$$(z + x) + \lambda zx = 0, \quad \text{...(5)}$$

and

$$(x + y) + \lambda xy = 0. \quad \text{...(6)}$$

These give $-\lambda = \frac{1}{y} + \frac{1}{z} = \frac{1}{z} + \frac{1}{x} = \frac{1}{x} + \frac{1}{y}$. ...(7)

$$\therefore \quad \frac{1}{y} - \frac{1}{x} = 0 \quad \text{or} \quad x = y.$$

Similarly $y = z$.

Hence for a stationary value of S, we have

$$x = y = z = V^{1/3}, \text{ from (2)}.$$

Thus S is stationary when the rectangular parallelopiped is a cube.

Let us now find the nature of this stationary value of S.

Here S is a function of three variables x, y, z which are connected by the relation (2) so that only two variables are independent. Let us regard x and y as independent variables and z to be dependent on them.

Then from (1), $\frac{\partial S}{\partial x} = 2y + 2y\frac{\partial z}{\partial x} + 2z + 2x\frac{\partial z}{\partial x}$.

Also from (2), $yz + xy\frac{\partial z}{\partial x} = 0$ *i.e.*, $\frac{\partial z}{\partial x} = -\frac{z}{x}$.

$$\therefore \quad \frac{\partial S}{\partial x} = 2y - \frac{2yz}{x} + 2z - 2z = 2y - \frac{2yz}{x}$$

and

$$\frac{\partial^2 S}{\partial x^2} = \frac{2yz}{x^2} - \frac{2y}{x} \cdot \frac{\partial z}{\partial x} = \frac{2yz}{x^2} + \frac{2yz}{x^2} = \frac{4yz}{x^2} = 4 \text{ at } x = y = z.$$

Similarly by symmetry $\frac{\partial^2 S}{\partial y^2} = 4$ at $x = y = z$.

Also $\frac{\partial^2 S}{\partial x\,\partial y} = 2 - \frac{2z}{x} - \frac{2y}{x}\frac{\partial z}{\partial y}$.

But differentiating (2) partially w.r.t. y taking x as constant, we get

$$xz + xy\frac{\partial z}{\partial y} = 0 \quad \text{or} \quad \frac{\partial z}{\partial y} = -\frac{z}{y}.$$

$$\therefore \quad \frac{\partial^2 S}{\partial x\,\partial y} = 2 - \frac{2z}{x} - \frac{2y}{x}\left(-\frac{z}{y}\right) = 2 - \frac{2z}{x} + \frac{2z}{x} = 2.$$

Thus at the stationary point $x = y = z = V^{1/3}$, we have

$$r = \frac{\partial^2 S}{\partial x^2} = 4, \quad s = \frac{\partial^2 S}{\partial x\,\partial y} = 2 \text{ and } t = \frac{\partial^2 S}{\partial y^2} = 4.$$

$\therefore \quad rt - s^2 = 4 \times 4 - 2^2 = 12$ which is > 0.

Also $r = 4$ which is > 0.

Hence the stationary value of S given by $x = y = z = V^{1/3}$ is a minimum.

Thus of all rectangular paralleopipeds of the same volume, the cube has the least surface.

Ex. 23. *Discuss the maxima and minima of the function*

$$u = \sin x \sin y \sin z,$$

where x, y, z are the angles of a triangle.

Sol. We have $u = \sin x \sin y \sin z$, ...(1)

where $x + y + z = \pi$. ...(2)

For a maximum or a minimum of u, we must have

$$du = \cos x \sin y \sin z \, dx + \sin x \cos y \sin z \, dy + \sin x \sin y \cos z \, dz = 0 \quad ...(3)$$

Also form (2), we have

$$dx + dy + dz = 0. \quad ...(4)$$

Multiplying (3) by 1 and (4) by λ and adding and then equating to zero the coefficients of dx, dy, dz, we get

$$\cos x \sin y \sin z + \lambda = 0,$$
$$\sin x \cos y \sin z + \lambda = 0,$$

and
$$\sin x \sin y \cos z + \lambda = 0.$$

From these, we get

$$-\lambda = \cos x \sin y \sin z = \sin x \cos y \sin z = \sin x \sin y \cos z$$

or $\quad \cot x = \cot y = \cot z$ (dividing by $\sin x \sin y \sin z$)

i.e., $\quad x = y = z = \pi/3$, from (2).

Thus u is stationary when $x = y = z = \pi/3$.

Let us now find the nature of this stationary value of u.

Since variables x, y and z are connected by the relation (2), only two of them may be regarded as independent.

Let us regard x and y as independent and z to be dependent on them by the relation (2).

Then from (1),

$$\frac{\partial u}{\partial x} = \sin y \sin z \cos x + \sin x \sin y \cos z \frac{\partial z}{\partial x}.$$

Also form (2),

$$1 + \frac{\partial z}{\partial x} = 0 \quad \text{or} \quad \frac{\partial z}{\partial x} = -1.$$

$$\therefore \quad \frac{\partial u}{\partial x} = \sin y \sin z \cos x - \sin x \sin y \cos z$$

and
$$\frac{\partial^2 u}{\partial x^2} = -\sin y \sin z \sin x + \sin y \cos x \cos z \frac{\partial z}{\partial x} - \cos x \sin y \cos z + \sin x \sin y \sin z \frac{\partial z}{\partial x}$$

$$= -2 \sin x \sin y \sin z - 2 \sin y \cos x \cos z.$$

Also
$$\frac{\partial^2 u}{\partial x \partial y} = \cos y \sin z \cos x + \sin y \cos x \cos z \frac{\partial z}{\partial y} - \sin x \cos y \cos z + \sin x \sin y \sin z \frac{\partial z}{\partial y}$$

$$= \cos y \sin z \cos x - \sin y \cos x \cos z - \sin x \cos y \cos z - \sin x \sin y \sin z.$$

Hence putting $x = y = z = \pi/3$, we get

$$r = \frac{\partial^2 u}{\partial x^2} = -2 \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} - 2 \cdot \frac{\sqrt{3}}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$$

$$= -\frac{3\sqrt{3}}{4} - \frac{\sqrt{3}}{4} = -\sqrt{3},$$

$$s = \frac{\partial^2 u}{\partial x \partial y} = \frac{\sqrt{3}}{8} - \frac{\sqrt{3}}{8} - \frac{\sqrt{3}}{8} - \frac{3\sqrt{3}}{8} = -\frac{\sqrt{3}}{2}.$$

Also by symmetry, $t = \dfrac{\partial^2 u}{\partial y^2} = -\sqrt{3}$.

$\therefore$ at the stationary point $x = y = z = \pi/3$, we have

$$rt - s^2 = 3 - (3/4) = 9/4 \text{ which is } > 0$$

and $r = -\sqrt{3}$ which is < 0.

$\therefore$ at the stationary point $x = y = z = \pi/3$, u is maximum.

Hence u is maximum when $x = y = z = \pi/3$ and the maximum value of $u = \left(\dfrac{\sqrt{3}}{2}\right)^3 = \dfrac{3\sqrt{3}}{8}$.

Ex. 24. *In a plane triangle ABC, find the maximum value of $u = \cos A \cos B \cos C$.* (Meerut 1994P, 95BP)

Sol. We have $\quad u = \cos A \cos B \cos C$, ...(1)

where the variables A, B and C are connected by the relation

$$A + B + C = \pi. \quad ...(2)$$

From (1), $\log u = \log \cos A + \log \cos B + \log \cos C$.

$$\therefore \quad \frac{1}{u} du = -\tan A \, dA - \tan B \, dB - \tan C \, dC.$$

For a maximum or a minimum of u, we must have $du = 0$

$$\tan A \, dA + \tan B \, dB + \tan C \, dC = 0. \quad ...(3)$$

Also differentiating (2), we get

$$dA + dB + dC = 0. \quad ...(4)$$

Multiplying (3) by 1 and (4) by λ, and adding and then equating to zero the coefficients of dA, dB and dC, we get

$$\tan A + \lambda = 0, \; \tan B + \lambda = 0, \; \tan C + \lambda = 0$$

or $\quad -\lambda = \tan A = \tan B = \tan C$

or $\quad A = B = C = \pi/3$, from (2).

Thus u is stationary when $A = B = C = \pi/3$ i.e., the triangle is equilateral.

Now to show that the stationary value of u given by
$$A = B = C = \pi/3 \text{ is maximum.}$$

Let us regard A and B as independent variables and C as a function of A and B given by (2).

From (1), $\log u = \log \cos A + \log \cos B + \log \cos C$.

$$\therefore \quad \frac{1}{u}\frac{\partial u}{\partial A} = -\tan A - \tan C \cdot \frac{\partial C}{\partial A}.$$

Also differentiating (2) partially w.r.t. A taking B as constant, we get
$$1 + (\partial C/\partial A) = 0 \text{ or } \partial C/\partial A = -1.$$

$$\therefore \quad \frac{1}{u}\frac{\partial u}{\partial A} = -\tan A + \tan C,$$

so that
$$\frac{1}{u}\frac{\partial^2 u}{\partial^2 A} - \frac{1}{u^2}\left(\frac{\partial u}{\partial A}\right)^2 = -\sec^2 A + \sec^2 C \cdot \frac{\partial C}{\partial A}$$
$$= -(\sec^2 A + \sec^2 C).$$

But at the stationary point $\partial u/\partial A = 0$.

$\therefore$ at the stationary point found above, we have
$$\frac{\partial^2 u}{\partial A^2} = -u(\sec^2 A + \sec^2 C)$$
$$= -\cos A \cos B \cos C(\sec^2 A + \sec^2 A),$$

which is –ive for $A = B = C = \pi/3$.

Hence u is maximum when $A = B = C = \pi/3$ and the maximum value of $u = \left(\cos\frac{\pi}{3}\right)^3 = \left(\frac{1}{2}\right)^3 = 1/8$.

Ex. 25. *Find a plane triangle ABC such that*
$$u = \sin^m A \sin^n B \sin^p C$$
has maximum value.

Sol. We have $u = \sin^m A \sin^n B \sin^p C$, ...(1)

where the variables A, B and C are connected by the relation
$$A + B + C = \pi. \quad \text{...(2)}$$

From (1), $\log u = m \log \sin A + n \log \sin B + p \log \sin C$.

$$\therefore \quad \frac{1}{u}du = m \cot A\, dA + n \cot B\, dB + p \cot C\, dC. \quad \text{...(3)}$$

For a maximum or a minimum of u, we must have $du = 0$

$$\Rightarrow \quad m \cot A\, dA + n \cot B\, dB + p \cot C\, dC = 0. \quad \text{...(4)}$$

Also differentiating (2), we get
$$dA + dB + dC = 0.$$

Multiplying (3) by 1 and (4) by λ, and adding and then equating coefficients of dA, dB and dC, we get

$$m \cot A + \lambda = 0, n \cot B + \lambda = 0, p \cot C + \lambda = 0.$$
$$\therefore \quad -\lambda = m \cot A = n \cot B = p \cot C.$$

Hence u is stationary when A, B, C are given by
$$m \cot A = n \cot B = p \cot C$$

or
$$\frac{\tan A}{m} = \frac{\tan B}{n} = \frac{\tan C}{p}.$$

Now to show that the above stationary value of u is maximum.

Let us regard A and B as independent variables and C as a function of A and B given by (2).

From (1), $\log u = m \log \sin A + n \log \sin B + p \log \sin C$.

$$\therefore \quad \frac{1}{u}\frac{\partial u}{\partial A} = m \cot A + p \cot C \cdot \frac{\partial C}{\partial A}.$$

But from (2), $1 + (\partial C/\partial A) = 0$ or $\partial C/\partial A = -1$.

$$\therefore \quad \frac{1}{u}\frac{\partial u}{\partial A} = m \cot A - p \cot C,$$

so that
$$\frac{1}{u}\frac{\partial^2 u}{\partial A^2} - \frac{1}{u^2}\left(\frac{\partial u}{\partial A}\right)^2 = -m \operatorname{cosec}^2 A + p \operatorname{cosec}^2 C \cdot \frac{\partial C}{\partial A}$$
$$= -(m \operatorname{cosec}^2 A + p \operatorname{cosec}^2 C).$$

But at the stationary point found above, we have
$$\frac{\partial^2 u}{\partial A^2} = -u(m \operatorname{cosec}^2 A + p \operatorname{cosec}^2 C)$$
$$= -\sin^m A \sin^n B \sin^p C(m \operatorname{cosec}^2 A + p \operatorname{cosec}^2 C),$$

which is obviously –ive if A, B, C are the angles of a triangle.

Hence u is maximum when A, B, C are given by
$$\frac{\tan A}{m} = \frac{\tan B}{n} = \frac{\tan C}{p}.$$

Ex. 26. *Show that if the perimeter of a triangle is constant, its area is a maximum when it is equilateral.*

Sol. Let a, b, c denote the sides of a triangle, $2s$ its constant perimeter and u its area.

Then
$$u^2 = s(s-a)(s-b)(s-c), \quad \text{...(1)}$$

where the variables a, b, c are connected by the relation
$$a + b + c = 2s. \quad \text{...(2)}$$

From (1), $2 \log u = \log s + \log(s-a) + \log(s-b) + \log(s-c)$.

$$\therefore \quad \frac{2}{u}du = -\frac{1}{s-a}da - \frac{1}{s-b}db - \frac{1}{s-c}dc.$$

For a maximum or a minimum of u, we must have $du = 0$
$$\frac{da}{s-a} + \frac{db}{s-b} + \frac{dc}{s-c} = 0. \quad \text{...(3)}$$

Also differentiating (2), we have
$$da + db + dc = 0. \quad \text{...(4)}$$

Multiplying (3) by 1 and (4) by λ, and adding and then equating to zero the coefficients of da, db and dc, we get

$$\frac{1}{s-a}+\lambda=0, \frac{1}{s-b}+\lambda=0, \frac{1}{s-c}+\lambda=0.$$

$$\therefore \quad -\lambda=\frac{1}{s-a}=\frac{1}{s-b}=\frac{1}{s-c}$$

or $$s-a=s-b=s-c \text{ or } a=b=c.$$

Hence u is stationary when $a=b=c$ *i.e.*, the triangle is equilateral.

Now to show that the stationary value of u given by $a=b=c$ is maximum.

Let us regard a and b as independent variables and c as a function of a and b given by (2).

From (1), differentiating logarithmically, we have

$$\frac{2}{u}\frac{\partial u}{\partial a}=-\frac{1}{s-a}-\frac{1}{s-c}\frac{\partial c}{\partial a}.$$

But from (2), $1+(\partial c/\partial a)=0$ or $\partial c/\partial a=-1$.

$$\therefore \quad \frac{2}{u}\frac{\partial u}{\partial a}=-\frac{1}{s-a}+\frac{1}{s-c},$$

so that $$\frac{2}{u}\frac{\partial^2 u}{\partial a^2}-\frac{2}{u^2}\left(\frac{\partial u}{\partial a}\right)^2=-\frac{1}{(s-a)^2}+\frac{1}{(s-c)^2}\cdot\frac{\partial c}{\partial a}$$

$$=-\left[\frac{1}{(s-a)^2}+\frac{1}{(s-c)^2}\right].$$

But at the stationary point, we have $\partial u/\partial a=0$.

$\therefore$ at the stationary point found above, we have

$$\frac{2}{u}\frac{\partial^2 u}{\partial a^2}=-\left[\frac{1}{(s-a)^2}+\frac{1}{(s-c)^2}\right]$$

or $$\frac{\partial^2 u}{\partial a^2}=-\frac{u}{2}\left[\frac{1}{(s-a)^2}+\frac{1}{(s-c)^2}\right], \text{ which is -ive.}$$

Hence u is maximum when $a=b=c$ *i.e.*, the area of the triangle is maximum when it is equilateral.

Ex. 27. *Find the triangle of maximum area inscribed in a circle.*

Sol. Let x, y, z denote the angles of a triangle inscribed in a given circle of radius k. If u be the area of the triangle, then

$$u=\tfrac{1}{2}k^2(\sin 2x+\sin 2y+\sin 2z), \quad \ldots(1)$$

where the variables x, y, z are connected by the relation

$$x+y+z=\pi. \quad \ldots(2)$$

From (1), $du=k^2(\cos 2x\,dx+\cos 2y\,dy+\cos 2z\,dz)$.

For a maximum or a minimum of u, we must have $du=0$

$$\Rightarrow \quad \cos 2x\,dx+\cos 2y\,dy+\cos 2z\,dz=0. \quad \ldots(3)$$

Also differentiating (2), we have

$$dx+dy+dz=0. \quad \ldots(4)$$

Multiplying (3) by 1 and (4) by λ, and adding and then equating to zero the coefficients of dx, dy and dz, we get

$$\cos 2x+\lambda=0, \cos 2y+\lambda=0, \cos 2z+\lambda=0$$

$$-\lambda=\cos 2x=\cos 2y=\cos 2z$$

or $$1-2\sin^2 x=1-2\sin^2 y=1-2\sin^2 z$$

or $$\sin^2 x=\sin^2 y=\sin^2 z$$

or $$\sin x=\sin y=\sin z$$

or $$x=y=z=\pi/3, \text{ from (2).}$$

or Thus u is stationary when $x=y=z=\pi/3$ *i.e.*, the triangle is equilateral.

Now to show that the stationary value of u given by $x=y=z=\pi/3$ is maximum.

Let us regard x and y as independent variables and z as a function of x and y given by (2).

From (1), $\dfrac{\partial u}{\partial x}=k^2\left(\cos 2x+\cos 2z\dfrac{\partial z}{\partial x}\right)$.

Also differentiating (2) partially w.r.t. x taking y as a constant, we get

$$1+(\partial z/\partial x)=0 \quad \text{or} \quad \partial z/\partial x=-1.$$

$$\therefore \quad \frac{\partial u}{\partial x}=k^2(\cos 2x-\cos 2z),$$

so that $$\frac{\partial^2 u}{\partial x^2}=k^2\left[-2\sin 2x+2\sin 2z\cdot\frac{\partial z}{\partial x}\right]$$

$$=-2k^2(\sin 2x+\sin 2z).$$

Also $$\frac{\partial^2 u}{\partial x\,\partial y}=k^2\left(2\sin 2z\cdot\frac{\partial z}{\partial y}\right)=-2k^2\sin 2z.$$

Hence putting $x=y=z=\pi/3$, we get

$$r=\frac{\partial^2 u}{\partial x^2}=-2k^2\left(\frac{\sqrt{3}}{2}+\frac{\sqrt{3}}{2}\right)=-2k^2\sqrt{3},$$

$$s=\frac{\partial^2 u}{\partial x\,\partial y}=-2k^2\cdot\frac{\sqrt{3}}{2}=-k^2\sqrt{3}.$$

Also by symmetry, $t=\dfrac{\partial^2 u}{\partial y^2}=-2k^2\sqrt{3}$.

$\therefore$ at the stationary point $x=y=z=\pi/3$, we have

$$rt-s^2=12k^4-3k^4=9k^4 \text{ which is } >0$$

and $$r=-2k^2\sqrt{3} \text{ which is -ive.}$$

$\therefore$ at the stationary point $x=y=z=\pi/3$, u is maximum.

Hence the triangle of maximum area inscribed in a circle is equilateral.

Ex. 28. *Show that the volume of the greatest rectangular parallelopiped that can be inscribed in the ellipsoid*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

is $8abc/3\sqrt{3}$.

or

Find the maximum value of u : (Meerut 1996)

$$u = 8xyz, \text{ given } \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1.$$

(Meerut 1995)

Sol. Let (x, y, z) denote the coordinates of the vertex of the rectangular parallelopiped which lies in the positive octant and let V denote its volume. Then, we have to find the maximum value of

$$V = 8xyz \quad \text{...(1)}$$

subject to the condition

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1. \quad \text{...(2)}$$

For a maximum or a minimum of V, we have

$$dV = 8yz\,dx + 8zx\,dy + 8xy\,dz = 0$$

i.e.,

$$yz\,dx + zx\,dy + xy\,dz = 0. \quad \text{...(3)}$$

Also differentiating (2), we get

$$\frac{2x}{a^2}dx+\frac{2y}{b^2}dy+\frac{2z}{c^2}dz=0$$

i.e.,

$$\frac{x}{a^2}dx+\frac{y}{b^2}dy+\frac{z}{c^2}dz=0. \quad \text{...(4)}$$

Multiplying (3) by 1 and (4) by λ, and adding and then equating the coefficients of dx, dy, dz to zero, we get

$$yz+\frac{\lambda x}{a^2}=0,\ zx+\frac{\lambda y}{b^2}=0 \text{ and } xy+\frac{\lambda z}{c^2}=0.$$

From these, we get

$$\frac{x}{a^2}=-\frac{yz}{\lambda},\ \frac{y}{b^2}=-\frac{zx}{\lambda},\ \frac{z}{c^2}=-\frac{xy}{\lambda}$$

or

$$\frac{x^2}{a^2}=-\frac{xyz}{\lambda},\ \frac{y^2}{b^2}=-\frac{xyz}{\lambda},\ \frac{z^2}{c^2}=-\frac{xyz}{\lambda}$$

or

$$\frac{x^2}{a^2}=\frac{y^2}{b^2}=\frac{z^2}{c^2}=\frac{x^2/a^2+y^2/b^2+z^2/c^2}{3}=\frac{1}{3}, \text{ using (2)}$$

or

$$x=a/\sqrt{3},\ y=b/\sqrt{3},\ z=c/\sqrt{3}.$$

Thus V is stationary when $x=a/\sqrt{3}, y=b/\sqrt{3}, z=c/\sqrt{3}$.

Now regard x and y as independent variables and z as a function of x and y given by (2).

Then from (1), $\dfrac{\partial V}{\partial x}=8yz+8xy\dfrac{\partial z}{\partial x}$.

Differentiating (2) partially w.r.t. x taking y as constant, we get

$$\frac{2x}{a^2}+\frac{2z}{c^2}\frac{\partial z}{\partial x}=0 \quad\text{or}\quad \frac{\partial z}{\partial x}=-\frac{c^2x}{a^2z}.$$

$$\therefore\quad \frac{\partial V}{\partial x}=8yz+8xy\cdot\left(-\frac{c^2x}{a^2z}\right)=8yz-\frac{8c^2x^2y}{a^2z}$$

and so

$$\frac{\partial^2 V}{\partial x^2}=8y\frac{\partial z}{\partial x}-\frac{16c^2xy}{a^2z}+\frac{8c^2x^2y}{a^2z^2}\cdot\frac{\partial z}{\partial x}$$

$$=8y\cdot\left(\frac{-c^2x}{a^2z}\right)-\frac{16c^2xy}{a^2z}-\frac{8c^2x^2y}{a^2z^2}\cdot\frac{c^2x}{a^2z},$$

which is –ive when $x=a/\sqrt{3}, y=b/\sqrt{3}, z=c/\sqrt{3}$.

Hence V is maximum when $x=a/\sqrt{3}, y=b/\sqrt{3}, z=c/\sqrt{3}$ and the maximum value of $V=\dfrac{8abc}{3\sqrt{3}}$.

Note. In complicated problems to show that whether the stationary value of a function is maximum or minimum, it will be sufficient to see whether the second partial differential coefficient of the function w.r.t. any of the independent variables is negative or positive.

Ex. 29. *Prove that the rectangular solid of maximum volume which can be inscribed in a sphere is a cube.*

Sol. Referred to the centre as origin, let the equation of the sphere be $x^2+y^2+z^2=a^2$.

Let (x, y, z) denote the coordinates of that vertex of the rectangular parallelopiped inscribed in the sphere which lies in the positive octant and let V denote the volume of the rectangular parallelopiped. Then, we have to find the maximum value of

$$V = 8xyz \quad \text{...(1)}$$

subject to the condition

$$x^2+y^2+z^2=a^2. \quad \text{...(2)}$$

From (1), $\log V=\log 8+\log x+\log y+\log z$.

$$\therefore\quad \frac{1}{V}dV=\frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz.$$

For a maximum or a minimum of V, we must have $dV=0$

$$\Rightarrow\quad \frac{1}{x}dx+\frac{1}{y}dy+\frac{1}{z}dz=0. \quad \text{...(3)}$$

Also differentiating (2), we have

$$x\,dx+y\,dy+z\,dz=0. \quad \text{...(4)}$$

Multiplying (3) by 1 and (4) by λ, and adding and then equating to zero the coefficients of dx, dy, dz, we get

$$\frac{1}{x} + \lambda x = 0, \frac{1}{y} + \lambda y = 0, \frac{1}{z} + \lambda z = 0$$

or $$-1/\lambda = x^2 = y^2 = z^2 \quad \text{or} \quad x = y = z.$$

Thus V is stationary when $x = y = z = a/\sqrt{3}$, from (2).

The lengths of the edges of the rectangular parallelopiped are $2x, 2y, 2z$. So V is stationary when the rectangular parallelopiped is a cube.

Now regard x and y as independent variables and z as a function of x and y given by (2).

From (1), $\log V = \log 8 + \log x + \log y + \log z$.

$$\therefore \quad \frac{1}{V}\frac{\partial V}{\partial x} = \frac{1}{x} + \frac{1}{z}\cdot\frac{\partial z}{\partial x}.$$

Differentiating (2) partially w.r.t. x taking y as constant, we get

$$2x + 2z\,(\partial z/\partial x) = 0 \quad \text{or} \quad \partial z/\partial x = -x/z.$$

$$\therefore \quad \frac{1}{V}\frac{\partial V}{\partial x} = \frac{1}{x} + \frac{1}{z}\cdot\frac{-x}{z} = \frac{1}{x} - \frac{x}{z^2},$$

so that $$\frac{1}{V}\frac{\partial^2 V}{\partial x^2} - \frac{1}{V^2}\left(\frac{\partial V}{\partial x}\right)^2 = -\frac{1}{x^2} - \frac{1}{z^2} + \frac{2x}{z^3}\frac{\partial z}{\partial x} = -\frac{1}{x^2} - \frac{1}{z^2} - \frac{2x^2}{z^4}.$$

But at the stationary point, we have $\partial V/\partial x = 0$.

$\therefore$ at the stationary point found above, we have

$$\frac{\partial^2 V}{\partial x^2} = -V\left[\frac{1}{x^2} + \frac{1}{z^2} + \frac{2x^2}{z^4}\right] = -8xyz\left[\frac{1}{x^2} + \frac{1}{z^2} + \frac{2x^2}{z^4}\right],$$

which is –ive when $x = y = z = a/\sqrt{3}$.

Thus V is maximum when $x = y = z = a/\sqrt{3}$.

Hence the rectangular solid of maximum volume inscribed in a sphere is a cube.

Ex. 30. *A rectangular box open at the top is to have a given capacity. Find the dimensions of the box requiring least material for its construction.*

Sol. Let the given capacity of the box be V, its three edges be x, y, z and its surface be S. Then

$$S = xy + 2yz + 2zx \qquad \text{...(1)}$$

and $$xyz = V. \qquad \text{...(2)}$$

For a maximum or a minimum of S, we have

$$dS = (y + 2z)\,dx + (x + 2z)\,dy + 2\,(x + y)\,dz = 0. \qquad \text{...(3)}$$

Also from (2), since V is constant, we have

$$yz\,dx + zx\,dy + xy\,dz = 0. \qquad \text{...(4)}$$

Multiplying (3) by 1 and (4) by λ, and adding and then equating to zero the coefficients of dx, dy, dz, we get

$$(y + 2z) + \lambda yz = 0, \qquad \text{...(5)}$$

$$(x + 2z) + \lambda zx = 0, \qquad \text{...(6)}$$

and $$2x + 2y + \lambda xy = 0. \qquad \text{...(7)}$$

Multiplying (5) by x, (6) by y and subtracting, we get

$$2zx - 2zy = 0 \text{ or } 2z\,(x - y) = 0, \text{ or } x = y.$$

[The root $z = 0$ is inadmissible because the depth of the box cannot be zero.]

Similarly, from the equations (6) and (7), we get $y = 2z$.

Hence the dimensions of the box for a stationary value of S are

$$x = y = 2z = (2V)^{1/3}, \text{ from (2)}.$$

Let us now find the nature of this stationary value of S.

Regard x and y as independent variables and z as a function of x and y given by (2).

Then from (1), $$\frac{\partial S}{\partial x} = y + 2y\frac{\partial z}{\partial x} + 2z + 2x\frac{\partial z}{\partial x}.$$

Differentiating (2) partially w.r.t. x taking y as constant, we get

$$yz + xy\frac{\partial z}{\partial x} = 0 \text{ i.e., } \frac{\partial z}{\partial x} = -\frac{z}{x}.$$

$$\therefore \quad \frac{\partial S}{\partial x} = y - \frac{2yz}{x} + 2z - 2z = y - \frac{2yz}{x}$$

and so $$\frac{\partial^2 S}{\partial x^2} = \frac{2yz}{x^2} - \frac{2y}{x}\cdot\frac{\partial z}{\partial x} = \frac{2yz}{x^2} + \frac{2yz}{x^2} = \frac{4yz}{x^2} = 2 \text{ at } x = y = 2z.$$

Thus at the stationary point $x = y = 2z = (2V)^{1/3}$, we have

$$r = \frac{\partial^2 S}{\partial x^2} = 2, \text{ which is positive.}$$

Similarly we can find $s = \dfrac{\partial^2 S}{\partial x\,\partial y}$ and $t = \dfrac{\partial^2 S}{\partial y^2}$ at the stationary point $x = y = 2z = (2V)^{1/3}$ and can show that $rt - s^2$ is positive.

Since at the stationary point $x = y = 2z = (2V)^{1/3}$, $rt - s^2 > 0$ and $r > 0$, therefore the stationary value of S at this point is a minimum.

Hence the dimensions of the box requiring least meterial for its construction are given by $x = y = 2z = (2V)^{1/3}$.

□