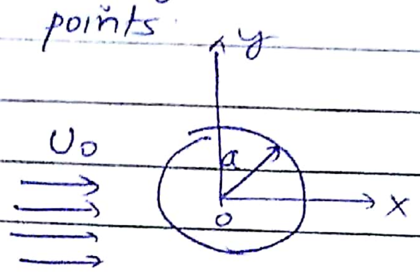
Q. 5 (d) Consider a uniform flow U_0 in the positive x direction. A cylinder of radius a is located at the origin. Find the stream function and the velocity potential. Find also the stagnation points.

Ans-



Complex potential $= w = U_0 \left(z + \frac{a^2}{z}\right)$

at $z = re^{i\theta}$

$$w = U_0 \left(re^{i\theta} + \frac{a^2}{r} e^{-i\theta} \right)$$

$$w = U_0 \left[r\cos\theta + ir\sin\theta + \frac{a^2}{r}\cos\theta - \frac{ia^2}{r}\sin\theta \right]$$

$$w = U_0 \left[r\cos\theta + \frac{a^2}{r}\cos\theta \right] + iU_0 \left[r\sin\theta - \frac{a^2}{r}\sin\theta \right]$$

as $w = \phi + i\psi$

$$\boxed{\phi = U_0 \left[r\cos\theta + \frac{a^2}{r}\cos\theta \right]} \quad \text{and} \quad \boxed{\psi = U_0 \left[r\sin\theta - \frac{a^2}{r}\sin\theta \right]}$$

and $q = \left| \frac{dw}{dz} \right| = U_0 \left| 1 - \frac{a^2}{z^2} \right|$

$$= U_0 \left| 1 - \frac{a^2}{r^2} e^{-i2\theta} \right| \qquad \left[\because r = a \right]$$

$$q = U_0 \left| 1 - e^{-i2\theta} \right|$$

at stagnation point $q = 0$

$\therefore \quad 1 - e^{-i2\theta} = 0$

$1 = e^{-i2\theta} \Rightarrow \cos 2\theta - i\sin 2\theta = 1$

$\cos 2\theta = 1 \qquad \sin 2\theta = 0$

or $2\theta = 0 \qquad 2\theta = 2\pi$

$$\boxed{\theta = 0 \text{ and } \theta = \pi}$$

Q. 8 (c)

Q. In an axis-symmetric motion, show that stream function exists due to equation of continuity. Express the velocity components in terms of the stream function. Find the equation satisfied by the stream function if the flow is irrotational.

Sol- Consider the fluid motion in cylindrical ~~coordiate~~ coordinates,

equation of continuity is,

$$\nabla\cdot(\rho\vec{q}) + \frac{\partial\rho}{\partial t} = 0$$

For incompressible fluid and steady flow,

$$\nabla\cdot(\vec{q}) = 0$$

$$\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}(rq_r) + \frac{1}{r}\frac{\partial}{\partial\theta}(q_\theta) + \frac{\partial}{\partial z}(q_z) = 0$$

For axis-symmetric flow $\frac{\partial}{\partial\theta} = 0$

$$\Rightarrow \frac{1}{r}\frac{\partial}{\partial r}(rq_r) + \frac{\partial}{\partial z}(q_z) = 0$$

$$\Rightarrow \frac{\partial}{\partial r}(rq_r) + r\frac{\partial}{\partial z}(q_z) = 0$$

Now the condition that $rq_r\,dz - rq_z\,dr$ may be an exact differential, let it be equal to $d\psi$

$$\Rightarrow rq_r\,dz - rq_z\,dr = d\psi = \frac{\partial\psi}{\partial r}dr + \frac{\partial\psi}{\partial z}dz$$

$$\Rightarrow rq_r = \frac{\partial\psi}{\partial z} \qquad -rq_z = \frac{\partial\psi}{\partial r}$$

$$\boxed{q_r = \frac{1}{r}\frac{\partial\psi}{\partial z}} \qquad \boxed{q_z = \frac{-1}{r}\frac{\partial\psi}{\partial r}}$$

which satisfy continuity equation.

~~eqn of cont~~

and streamlines are given by,

$$\frac{dr}{q_r} = \frac{dz}{q_z}$$

or $\quad r q_r\, dz - r q_z\, dr = 0$

or $\quad d\psi = 0$

or $\quad \boxed{\psi = \text{constant}}$

$\Rightarrow$ ψ exists, due to equation of continuity.

If flow is irrotational, then ϕ (potential) exists, such that $\vec{q} = -\nabla\phi$

$$q_r = -\frac{\partial \phi}{\partial r} \qquad q_z = -\frac{\partial \phi}{\partial z}$$

Also, $\quad q_r = \frac{1}{r}\frac{\partial \psi}{\partial z} \qquad q_z = -\frac{1}{r}\frac{\partial \psi}{\partial r}$

Also, $\quad \frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial r}\right) = \frac{\partial}{\partial r}\left(\frac{\partial \phi}{\partial z}\right)$

$$\frac{\partial}{\partial z}\left(-\frac{1}{r}\frac{\partial \psi}{\partial z}\right) = \frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \psi}{\partial r}\right)$$

$$-\frac{1}{r}\frac{\partial^2 \psi}{\partial z^2} = \frac{1}{r}\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r^2}\frac{\partial \psi}{\partial r}$$

or $\quad \boxed{\frac{\partial^2 \psi}{\partial z^2} - \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{\partial^2 \psi}{\partial r^2} = 0}$

CSE-2015

Mechanics

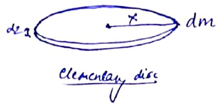
Raj Aryan

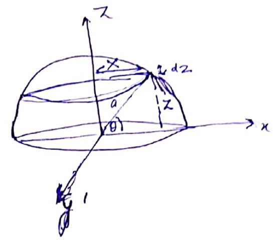
5 (e) Calculate the moment of inertia of a solid uniform hemisphere $x^2+y^2+z^2=a^2$, $z \geq 0$ with mass m about OZ axis. (10)

Sol:

Let ρ be the density of solid hemisphere.

Let dz be the width of elementary disc





$$dV = \pi x^2 dz$$

$$dm = \rho dV = \rho \pi x^2 dz$$

$$\rho = \frac{M}{V} = \frac{M}{\frac{2}{3}\pi a^3} = \frac{3M}{2\pi a^3} \quad \text{--- (B)}$$

$$x^2+z^2=a^2$$

$$x^2 = a^2-z^2$$

$$x^4 = (a^2-z^2)^2 \quad \text{--- (A)}$$

Now,

$$dI = \frac{1}{2} dm\, x^2 = \frac{1}{2}\rho\pi x^2 dz\, x^2$$

$$dI = \frac{1}{2}\rho\pi x^4 dz$$

$$I = \int_0^a dI = \frac{1}{2}\rho\pi\int_0^a x^4 dz$$

using (A)

$$= \frac{1}{2}\rho\pi\int_0^a (a^4 - 2a^2z^2 + z^4)\,dz$$

$$= \frac{1}{2}\rho\pi\left[a^4 z - \frac{2a^2z^3}{3} + \frac{z^5}{5}\right]_0^a$$

$$= \frac{1}{2}\rho\pi\left[a^5 - \frac{2}{3}a^5 + \frac{a^5}{5}\right]$$

$$= \frac{1}{2}\rho\pi\left[\frac{15a^5 - 10a^5 + 3a^5}{15}\right]$$

$$= \frac{1}{2}\,\frac{3M}{2\pi a^3}\,\pi\left[\frac{8a^5}{15}\right]$$

$$= \frac{2}{5}Ma^2$$

Hence, moment of inertia required $= \frac{2}{5}Ma^2$

4.11 APPLICATIONS OF HAMILTONIAN DYNAMICS

1. Motion of a Simple Pendulum

Consider a simple pendulum having the bob of mass m and length l as shown in fig. 4.2. In the equilibrium position A of the bob, the string OA is vertical. Let us displace the bob to the position B so that the string takes the position OB.

$$A\hat{O}B = \theta \text{ (say)}$$

The polar coordinates of the bob at the displaced position are i (a constant) and θ. With respect to the coordinate frame XY as indicated in the figure, let (x, y) be the cartesian coordinates of the bob at the position B. We then have

$$x = l \cos\theta, \quad y = l - l \sin\theta \tag{117}$$

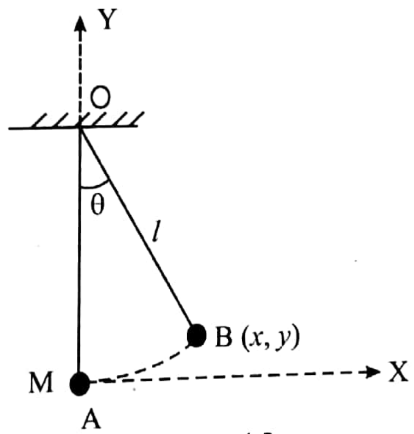


Figure 4.2

The kinetic energy of the bob at the position B is

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m\left[(-l\sin\theta\,\dot{\theta})^2 + (-l\cos\theta\,\dot{\theta})^2\right]$$

or

$$T = \frac{1}{2}ml^2\dot{\theta}^2 \tag{118}$$

With respect to the horizontal drawn from A as the reference zero of potential energy, the potential energy of the bob at the position B is

$$V = mgl\,(1 - \cos\theta) \tag{119}$$

We thus have the Lagrangian of the pendulum as

$$L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos\theta) \tag{120}$$

The momentum conjugate to the coordinate θ is by definiton

$$p_\theta = \frac{\partial L}{\partial \dot{\theta}} = ml^2\dot{\theta} \tag{121}$$

The Hamiltonian for the pendulum is then by definition given by

$$H = p_\theta\dot{\theta} - L$$

or

$$H = ml^2\dot{\theta}^2 - \frac{1}{2}ml^2\dot{\theta}^2 + mgl(1-\cos\theta)$$

Substituting for $\dot{\theta}$ from Eq. (121) we obtain

$$H = \frac{1}{2}ml^2\frac{p_\theta^2}{m^2l^4} + mgl(1-\cos\theta)$$

or

$$H = \frac{1}{2}\frac{p_\theta^2}{ml^2} + mgl(1-\cos\theta) \quad (12$$

The Hamilt...

7(c) A Hamiltonian of a system with one degree of freedom has the form

$$H = \frac{p^2}{2\alpha} - bqpe^{-\alpha t} + \frac{b\alpha}{2} q^2 e^{-\alpha t}(\alpha + be^{-\alpha t}) + \frac{K}{2} q^2$$

where α, b, K are constants, q is the generalized coordinate and p is the corresponding generalised momentum.

(i) find a Lagrangian corresponding to this Hamiltonian

(ii) find an equivalent Lagrangian that is not explicitly dependent on time (20)

Soln:

Given,

$$H = \frac{p^2}{2\alpha} - bqpe^{-\alpha t} + \frac{b\alpha}{2} q^2 e^{-\alpha t}(\alpha + be^{-\alpha t}) + \frac{K}{2} q^2 \quad \text{—①}$$

As the system has only one degree of freedom

$$\therefore \; H = p\dot{q} - L$$

$$\Rightarrow \; L = p\dot{q} - H \quad \text{—②}$$

Also

$$\frac{\partial H}{\partial p} = \dot{q}$$

$$\Rightarrow \; \frac{p}{\alpha} - bqe^{-\alpha t} = \dot{q} \quad \text{—③ (from ①)}$$

$$\Rightarrow \; p = \alpha(\dot{q} + bqe^{-\alpha t}) \quad \text{—④}$$

Now, from ②

$$L = p\left[\frac{p}{\alpha} - bqe^{-\alpha t}\right] - \frac{p^2}{2\alpha} + bqpe^{-\alpha t} - \frac{b\alpha}{2} q^2 e^{-\alpha t}(\alpha + be^{-\alpha t}) - \frac{K}{2} q^2$$

[using ① and ③]

$$= \frac{p^2}{\alpha} - \cancel{bqpe^{-\alpha t}} - \frac{p^2}{2\alpha} + \cancel{bqpe^{-\alpha t}} - \frac{b\alpha}{2} q^2 e^{-\alpha t}(\alpha + be^{-\alpha t}) - \frac{K}{2} q^2$$

$$= \frac{p^2}{2\alpha} - \frac{b\alpha}{2} q^2 e^{-\alpha t}(\alpha + be^{-\alpha t}) - \frac{K}{2} q^2$$

$$= \frac{\alpha^2[\dot{q} + bqe^{-\alpha t}]^2}{2\alpha} - \frac{b\alpha}{2} q^2 e^{-\alpha t}(\alpha + be^{-\alpha t}) - \frac{K}{2} q^2 \quad \text{[using ④]}$$

$$= \frac{1}{2}\left[(\alpha\dot{q}^2 + \alpha b^2 q^2 e^{-2\alpha t} + 2\alpha\dot{q}bqe^{-\alpha t}) - (b\alpha^2 q^2 e^{-\alpha t} + b^2 \alpha q^2 e^{-2\alpha t}) - Kq^2\right]$$

$$= \frac{1}{2}\left[(\alpha\dot{q}^2 - Kq^2) + 2\alpha\dot{q}bqe^{-\alpha t} - b\alpha^2 q^2 e^{-\alpha t}\right]$$

(as α, b are constants)

$$= \frac{1}{2}\left[(\alpha\dot{q}^2 - Kq^2) + \frac{d}{dt}\{\alpha b q^2 e^{-\alpha t}\}\right]$$

(i) $L = L_0 + \frac{1}{2}\frac{d}{dt}\{\alpha b q^2 e^{-\alpha t}\}$ where $L_0 = \frac{1}{2}\{(\alpha\dot{q}^2 - Kq^2)\}$

(ii) $L = L_0$ is an equivalent Lagrangian i.e, not explicitly dependent on time

As Lagrangian which differ by a total time derivative are equations that lead to same eqn of motion.