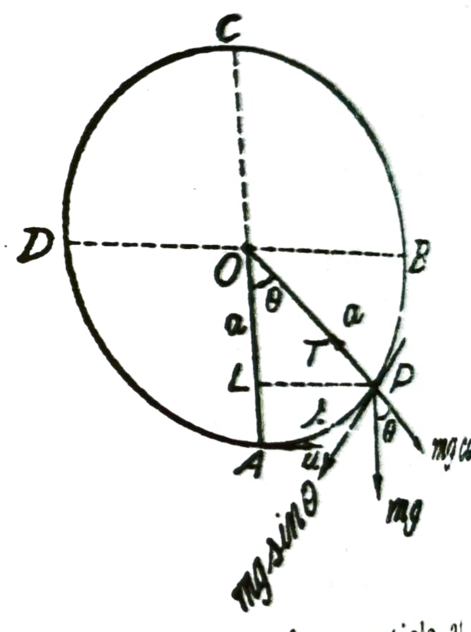
Constrained Motion

§ 1. **Introduction.** *The motion of a particle is called* **constrained motion,** *if it is compelled to move along a given curve or surface.*

Here in this chapter we shall consider the motion on smooth plane curves, vertical circle and cycloid only.

§ **2. Motion in a vertical circle.** *A heavy particle is tied to one end of a light inextensible string whose other end is attached to a fixed point. It is projected horizontally with a given velocity u from its vertical position of equilibrium; to discuss the subsequent motion.* **[Meerut 77, 79, 88; Agra 1976; Lucknow 79; Kanpur 81; Allahabad 78, 79; Rohilkhand 86**



Let one end of a string of length a be attached to the fixed point O and a particle of mass m be attached at the other end A. Let OA be the vertical position of equilibrium of the string. Let the particle be projected horizontally from A with velocity u. Since the string is inextensible the particle starts moving in a circle whose centre is O and radius a. If P is the position of the particle at time t such that $\angle AOP=\theta$ and arc $AP=s$, the forces acting on the particle at P are :

(i) weight mg of the particle acting vertically downwards, and (ii) tension T in the string acting along PO.

If v be the velocity of the particle at P, the tangential and normal accelerations of P are

$$\frac{d^2s}{dt^2} \text{ (in the direction of } s \text{ increasing)}$$

and $$\frac{v^2}{\rho} \text{ (along inwards drawn normal at } P).$$

$\therefore$ the equations of motion of the particle along the tangent and normal are

$$m\frac{d^2s}{dt^2}=-mg\sin\theta \qquad ...(1)$$

and
$$m\frac{v^2}{\rho}=T-mg\cos\theta. \qquad ...(2)$$

Also $s=\text{arc } AP=a\theta.$

$$\therefore\ v=\frac{ds}{dt}=a\frac{d\theta}{dt}$$

and
$$\frac{d^2s}{dt^2}=a\frac{d^2\theta}{dt^2} \qquad ...(3)$$

$\therefore$ from (1) and (3), we have

$$a\frac{d^2\theta}{dt^2}=-g\sin\theta.$$

Multiplying both sides by $2a\dfrac{d\theta}{dt}$ and integrating w.r.t. 't', we have

$$v^2=\left(a\frac{d\theta}{dt}\right)^2=2ag\cos\theta+A,$$

where A is constant of integration.

But initially at A, $\theta=0$, $v=u$.

$\therefore$ $A=u^2-2ag\cos 0=u^2-2ag.$

$\therefore$ $v^2=u^2-2ag+2ag\cos\theta.$ ...(4)

Now for a circle $\rho=a$ (radius).

$\therefore$ from (2), we have

$$T=\frac{m}{a}v^2+mg\cos\theta=\frac{m}{a}(v^2+ag\cos\theta).$$

Substituting the value of v^2 from (4), we have

$$T=\frac{m}{a}(u^2-2ag+3ag\cos\theta). \qquad ...(5)$$

If the velocity $v=0$ at $\theta=\theta_1$ then from (4), we have

$$0=u^2-2ag+2ag\cos\theta_1$$

or
$$\cos\theta_1=\frac{2ag-u^2}{2ag}. \qquad ...(6)$$

If h_1 is the height from the lowest point A of the point where the velocity vanishes, then

$$h_1=OA-a\cos\theta_1=a-a.\frac{2ag-u^2}{2ag}$$

or
$$h_1=\frac{u^2}{2g}. \qquad ...(7)$$

Again if the tension $T=0$, at $\theta=\theta_2$, then from (5), we have

$$0=u^2-2ag+3ag\cos\theta_2.$$

$$\therefore \quad \cos\theta_2=\frac{2ag-u^2}{3ag}. \quad \text{...(8)}$$

If h_2 is the height from the lowest point A of the point where the tension vanishes, then

$$h_2=OA-a\cos\theta_2=a-a.\frac{2ag-u^2}{3ag}$$

or $$h_2=\frac{u^2+ag}{3g} \quad \text{...(9)}$$

Now the following cases may arise here.

Case I. The velocity v vanishes before the tension T.

This is possible if and only if $h_1<h_2$

or $\frac{u^2}{2g}<\frac{u^2+ag}{3g}$ or $3u^2<2(u^2+ag)$

or $u^2<2ag$ or $u<\sqrt{(2ag)}$.

But when $u<\sqrt{(2ag)}$, we have from (6), $\cos\theta_1=+$ ive *i.e.*, θ_1 is an acute angle.

Thus if the particle is projected with the velocity $u<\sqrt{(2ag)}$, then it will oscillate about A and will not rise upto the horizontal diameter through O. **[Gorakhpur 1978]**

Case II. The velocity v and the tension T vanish simultaneously.

This is possible if and only if $h_1=h_2$

i.e., $\frac{u^2}{2g}=\frac{u^2+ag}{3g}$ *i.e.*, $u^2=2ag$ *i.e.*, $u=\sqrt{(2ag)}$.

Also when $u=\sqrt{(2ag)}$, we have from (6) and (8), $\theta_1=\pi/2=\theta_2$.

Thus if the particle is projected with the velocity $u=\sqrt{(2ag)}$, then it will rise upto the level of the horizontal diameter through O and will oscillate about A in the semi-circular arc BAD.

Case III. Condition for describing the complete circle. **[Meerut 1988]**

At the highest point C, we have $\theta=\pi$. Therefore from (4) and (5), we have at C, $v^2=u^2-4ag$

and $$T=\frac{m}{a}(u^2-5ag).$$

If $u^2>5ag$ *i.e.*, if $u>\sqrt{(5ag)}$, then neither the velocity v nor the tension T is zero at the highest point C, and so the particle will go on describing the complete circle.

And if $u^2=5ag$ *i.e.*, if $u=\sqrt{(5ag)}$, then at the highest point C the tension T vanishes whereas the velocity does not vanish. Hence in this case the string will become momentarily slack at C and the particle will go on describing the complete circle.

Thus the condition for describing the complete circle by the particle is that $u \geqslant \sqrt{(5ag)}$. **In other words the least velocity of projection for describing the complete circle is $\sqrt{(5ag)}$.**

[Meerut 1973; Gorakhpur 77]

Case IV. The tension T vanishes before the velocity v.

This is possible if and only if $h_1>h_2$

i.e., $$\frac{u^2}{2g}>\frac{u^2+ag}{3} \quad \textit{i.e.,}\quad u^2>2ag \quad \textit{i.e.,}\quad u>\sqrt{(2ag)}.$$

When $u>\sqrt{(2ag)}$, we have from (8), $\cos\theta_2=-$ive showing that θ_2 must be $>90°$.

Now at the point where the tension T is zero, the string becomes slack. Since the velocity v is not zero at that point, therefore the particle will leave the circular path and trace a parabolic path while moving freely under gravity.

Thus if the particle is projected with the velocity u such that $\sqrt{(2ag)}<u<\sqrt{(5ag)}$, then it will leave the circular path at a point somewhere between B and C and trace out a parabolic path.

[Meerut 1974; 79]

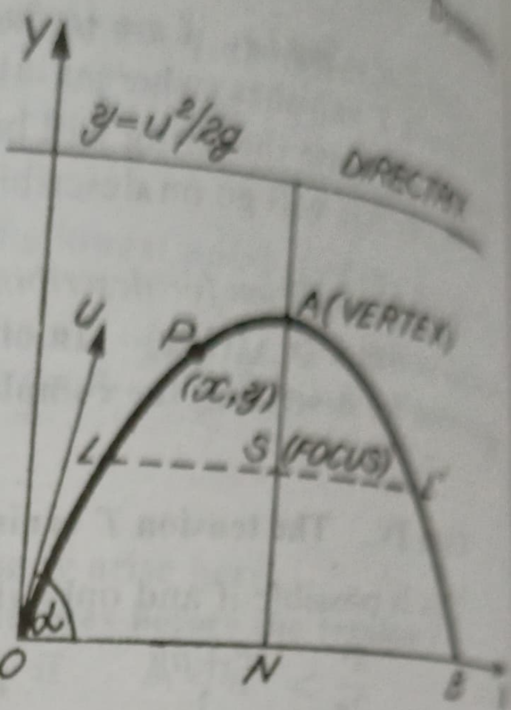
§ 3. *A particle is projected, along the inside of a smooth fixed hollow sphere (or circle) from is lowest point: to discuss the motion.*

The discussion is exactly the same as in § 2 with the difference that in this case the tension T is replaced by the reaction R between the particle and the sphere (or circle).

§ 4. **Some important results of the motion of a projectile to be used in this chapter.** Suppose a particle of mass m is projected in vaccum, in a vertical plane through the point of projection, with velocity u in a direction making an angle α with the horizontal. Then the path of the projectile is a **parabola**.

The following results about the motion of the projectile to be used in this chapter should be remembered.

Take the point of projection O as the origin, the horizontal line OX in the plane of projection as the x-axis and the vertical line OY as the y-axis. Then the initial horizontal velocity of the projectile is $u \cos \alpha$ and the initial vertical velocity is $u \sin \alpha$.



The equation of the trajectory *i.e.*, the equation of the parabolic path is

$$y = x \tan \alpha - \tfrac{1}{2} g \frac{x^2}{u^2 \cos^2 \alpha}.$$

The length of the **latus rectum** LSL' of the above parabolic path is

$$\frac{2}{g} u^2 \cos^2 \alpha = \frac{2}{g} \text{(horizontal velocity)}^2.$$

If H is the maximum height NA attained by the projectile above the point of projection O, then considering the vertical motion from O to A and using the formula $v^2 = u^2 + 2fh$, we have

$$0 = u^2 \sin^2 \alpha - 2gH$$

or
$$H = \frac{u^2 \sin^2 \alpha}{2g}$$

Thus the maximum height of the projectile above the point of projection is $\dfrac{u^2 \sin^2 \alpha}{2g}$.

Also remember that the velocity of a projectile at any point P of its path is that due to a fall from the directrix to that point.

Illustrative Examples

Ex. 1. *A heavy particle of weight W, attached to a fixed point by a light inextensible string, describes a circle in a vertical plane. The tension in the string has the values mW and nW respectively* *when the particle is at the highest and lowest point in the path. Show that $n=m+6$.* [Agra 1976, 79; Lucknow 80; Allahabad 77; Rohilkhand 86]

Sol. Let M be the mass of the particle. Then

$W=Mg$ i.e., $M=W/g$.

Proceeding as in § 2, the tension T in the string in any position is given by

$$T=\frac{M}{a}(u^2-2ag+3ag\cos\theta)$$ [See eqn. (5) of § 2 and deduce it here]

or $$T=\frac{W}{ag}(u^2-2ag+3ag\cos\theta). \quad ...(1)$$

Now mW is given to be the tension in the string at the highest point and nW that at the lowest point. Therefore $T=mW$ when $\theta=\pi$ and $T=nW$ when $\theta=0$. So from (1), we have

$$mW=\frac{W}{ag}(u^2-2ag+3ag\cos\pi) \text{ giving } m=\frac{1}{ag}(u^2-5ag) \quad ...(2)$$

and $$nW=\frac{W}{ag}(u^2-2ag+3ag\cos 0) \text{ giving } n=\frac{1}{ag}(u^2+ag). \quad ...(3)$$

Subtracting (2) from (3), we have

$n-m=6$ or $n=m+6$.

Ex. 2 (a). *A heavy particle hanging vertically from a point by a light inextensible string of length l is started so as to make a complete revolution in a vertical plane. Prove that the sum of the tensions at the end of any diameter is constant.*

[Rohilkhand 1977; Agra 80, 85; Meerut 76; Kanpur 83]

Sol. Proceeding as in § 2, the tension T in the string in any position is given by

$$T=\frac{m}{l}(u^2-2lg+3lg\cos\theta), \quad ...(1)$$

where θ is the angle which the string makes with OA.

Now take any diameter of the circle. If at one end of this diameter we have $\theta=\alpha$, then at the other end we shall have $\theta=\pi+\alpha$. Let T_1 and T_2 be the tensions at these ends i.e., $T=T_1$ when $\theta=\alpha$ and $T=T_2$ when $\theta=\pi+\alpha$. Then from (1), we have

$$T_1=\frac{m}{l}(u^2-2lg+3lg\cos\alpha) \quad ...(2)$$

and

$$T_2=\frac{m}{l}[u^2-2lg+3lg\cos(\pi+\alpha)]$$

or $$T_2=\frac{m}{l}(u^2-2lg-3lg\cos\alpha).$$

Adding (2) and (3), we have

$$T_1+T_2=2\frac{m}{l}(u^2-2lg)$$

which is constant, as it is independent of α.

Hence the sum of the tensions at the ends of any diameter is constant.

Ex. 2 (b). *A particle makes complete revolutions in a vertical circle. If ω_1, ω_2 be the greatest and least angular velocities and R_1, R_2 the greatest and least reactions, prove that when the particle projected from the lowest point of the circle makes an angle θ at the centre, its angular velocity is*

$$\sqrt{[\omega_1^2\cos^2\tfrac{1}{2}\theta+\omega_2^2\sin^2\tfrac{1}{2}\theta]}$$ **[Meerut 1975, 82]**

and that the reaction is

$$R_1\cos^2\tfrac{1}{2}\theta+R_2\sin^2\tfrac{1}{2}\theta.$$ **[Rohilkhand 1979]**

Sol. Proceed as in § 2. Replace the tension T by the reaction R.

Let u be the velocity of projection at the lowest point. For making complete circles, we must have $u^2\geqslant 5ag$.

If v be the velocity of the particle at any time t, then proceeding as in § 2, we have

$$v^2=\left(a\frac{d\theta}{dt}\right)^2=u^2-2ag+2ag\cos\theta, \quad \ldots(1)$$

and $$R=\frac{m}{a}(u^2-2ag+3ag\cos\theta). \quad \ldots(2)$$

If ω be the angular velocity of the particle at time t, then $\omega=d\theta/dt$. So from (1), we have

$$a^2\omega^2=u^2-2ag+2ag\cos\theta. \quad \ldots(3)$$

From the equation (3) we observe that the angular velocity ω is greatest when $\cos\theta=1$ i.e., $\theta=0$ and is least when $\cos\theta=-1$ i.e., $\theta=\pi$. So putting $\theta=0$, $\omega=\omega_1$ and $\theta=\pi$, $\omega=\omega_2$ in (3), we get

$$a^2\omega_1^2=u^2 \quad \text{and} \quad a^2\omega_2^2=u^2-4ag. \quad \ldots(4)$$

Now from (3), we have

$$a^2\omega^2=u^2-2ag(1-\cos\theta)=\tfrac{1}{2}[2u^2-4ag(1-\cos\theta)]$$
$$=\tfrac{1}{2}[2u^2-(u^2-a^2\omega_2^2)(1-\cos\theta)] \quad [\because \text{ from (4), } 4ag=u^2-a^2\omega_2^2]$$
$$=\tfrac{1}{2}[2u^2-u^2(1-\cos\theta)+a^2\omega_2^2(1-\cos\theta)]$$
$$=\tfrac{1}{2}[u^2(1+\cos\theta)+a^2\omega_2^2(1-\cos\theta)]$$

$=\frac{1}{2}[a^2\omega_1^2(1+\cos\theta)+a^2\omega_2^2(1-\cos\theta)]$ [$\because$ from (4), $u^2=a^2\omega_1^2$]
$=\frac{1}{2}[2a^2\omega_1^2\cos^2\frac{1}{2}\theta+2a^2\omega_2^2\sin^2\frac{1}{2}\theta]$.

$$\therefore \quad \omega^2=\omega_1^2\cos^2\tfrac{1}{2}\theta+\omega_2^2\sin^2\tfrac{1}{2}\theta$$

or
$$\omega=\sqrt{[\omega_1^2\cos^2\tfrac{1}{2}\theta+\omega_2^2\sin^2\tfrac{1}{2}\theta]}.$$

From the equation (2) we observe that the reaction R is greatest when $\cos\theta=1$ *i.e.*, $\theta=0$ and is least when $\cos\theta=-1$ *i.e.*, $\theta=\pi$. So putting $\theta=0$, $R=R_1$ and $\theta=\pi$, $R=R_2$ in (2), we get
$R_1=(m/a)(u^2+ag)$ and $R_2=(m/a)(u^2-5ag)$. ...(5)

Now from (2), we have

$R=(m/a)[u^2-2ag+3ag\cos\theta]$
$=\frac{1}{2}(m/a)[2u^2-4ag+6ag\cos\theta]$
$=\frac{1}{2}(m/a)[(u^2+ag)(1+\cos\theta)+(u^2-5ag)(1-\cos\theta)]$ **[Note]**
$=\frac{1}{2}[R_1(1+\cos\theta)+R_2(1-\cos\theta)]$ [From (5)]
$=\frac{1}{2}[2R_1\cos^2\frac{1}{2}\theta+2R_2\sin^2\frac{1}{2}\theta]=R_1\cos^2\frac{1}{2}\theta+R_2\sin^2\frac{1}{2}\theta$.

Ex. 3. *A heavy particle hangs from a fixed point O, by a string of length a. It is projected horizontally with a velocity $v^2=(2+\sqrt{3})ag$; show that the string becomes slack when it has described an angle $\cos^{-1}(-1/\sqrt{3})$.* **[Meerut 1973, 78, 82, 84]**

Sol. Refer fig. of § 2, page 156.

The equations of motion of the particle are

$$m\frac{d^2s}{dt^2}=-mg\sin\theta \quad ...(1)$$

and
$$m\frac{v^2}{a}=T-mg\cos\theta. \quad ...(2)$$

Also
$$s=a\theta. \quad ...(3)$$

From (1) and (3), we have $a\dfrac{d^2\theta}{dt^2}=-g\sin\theta$.

Multiplying both sides by $2a\,(d\theta/dt)$ and then integrating w.r.t. t, we have $v^2=\left(a\dfrac{d\theta}{dt}\right)^2=2ag\cos\theta+A$,

where A s the constant of integration.

But initially at A, $\theta=0$ and $v^2=(2+\sqrt{3})ag$.

$\therefore$ $(2+\sqrt{3})ag=2ag\cos 0+A$, giving $A=\sqrt{3}ag$.

$\therefore$ $v^2=2ag\cos\theta+\sqrt{3}ag$.

Substituting this value of v^2 in (2), we have

$$T=\frac{m}{a}[v^2+ag\cos\theta]$$
$$=\frac{m}{a}[3\sqrt{ag}+3ag\cos\theta]. \quad ...(4)$$

The string becomes slack when $T=0$.

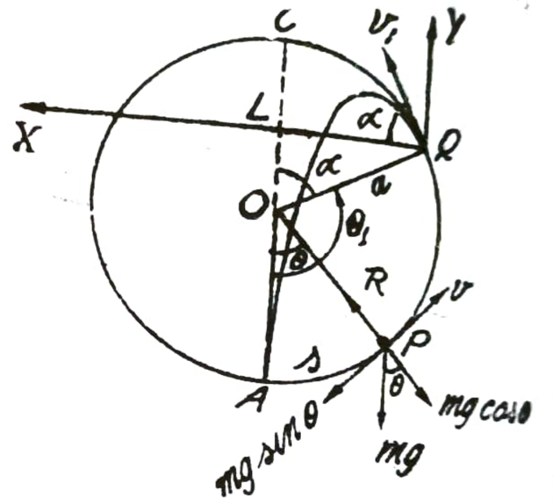
$\therefore$ from (4), we have

$$0=\frac{m}{a}\,[\sqrt{3}ag+3ag\cos\theta]$$

or $$\cos\theta=-1/\sqrt{3} \text{ or } \theta=\cos^{-1}(-1/\sqrt{3}).$$

Ex. 4. *A particle inside and at the lowest point of a fixed smooth hollow sphere of radius a is projected horizontally with velocity $\sqrt{(\frac{7}{2}ag)}$. Show that it will leave the sphere at a height $\frac{3}{2}a$ above the lowest point and its subsequent path meets the sphere again at the point of projection.* **[Meerut 1979; Kanpur 77]**

Sol. A particle is projected from the lowest point A of a sphere with velocity $u=\sqrt{(\frac{7}{2}ag)}$ to move along the inside of the sphere. Let P be the position of the particle at any time t where arc $AP=s$ and $\angle AOP=\theta$. If v be the velocity of the particle at P, the equations of motion along the tangent and normal are



$$m\frac{d^2s}{dt^2}=-mg\sin\theta \qquad ...(1)$$

and $$m\frac{v^2}{a}=R-mg\cos\theta. \qquad ...(2)$$

Also $$s=a\theta. \qquad ...(3)$$

From (1) and (3), we have $a\dfrac{d^2\theta}{dt^2}=-g\sin\theta.$

Multiplying both sides by $2a\dfrac{d\theta}{dt}$ and then integrating, we have

$$v^2=\left(a\frac{d\theta}{dt}\right)^2=2ag\cos\theta+A$$

But at the point A, $\theta=0$ and $v=u=\sqrt{(\frac{7}{2}ag)}$.

$\therefore$ $A=\frac{7}{2}ag-2ag=\frac{3}{2}ag.$

$\therefore$ $v^2=\frac{3}{2}ag+2ag\cos\theta. \qquad ...(4)$

Now from (2) and (4), we have

$$R=\frac{m}{a}[v^2+ag\cos\theta]=\frac{m}{a}\left[\frac{3}{2}ag+2ag\cos\theta+ag\cos\theta\right]$$
$$=3mg\,(\tfrac{1}{2}+\cos\theta).$$

If the particle leaves the sphere at the point Q, where $\theta=\theta_1$, then $0=3mg\left(\frac{1}{2}+\cos\theta_1\right)$ or $\cos\theta_1=-\frac{1}{2}$.

If $\angle COQ=\alpha$, then $\alpha=\pi-\theta_1$.

$$\therefore \quad \cos\alpha=\cos(\pi-\theta_1)=-\cos\theta_1=\tfrac{1}{2}. \qquad \ldots(5)$$

$$\therefore \quad AL=AO+OL=a+a\cos\alpha=a+\frac{a}{2}=\frac{3a}{2}$$

i.e., the particle leaves the sphere at a height $\frac{3}{2}a$ above the lowest point.

If v_1 is the velocity of the particle at the point Q, then putting $v=v_1$, $R=0$ and $\theta=\theta_1$ in (2), we get

$$v_1^2=-ag\cos\theta_1=-ag.\left(-\tfrac{1}{2}\right)=\tfrac{1}{2}ag.$$

$\therefore$ the particle leaves the sphere at the ~~point~~ Q with velocity $v_1=\sqrt{\left(\frac{1}{2}ag\right)}$ making an angle α with the horizontal and subsequently describes a parabolic path.

The equation of the parabolic trajectory w.r.t. QX and QY as co-ordinate axes is

$$y=x\tan\alpha-\frac{1}{2}\frac{gx^2}{v_1^2\cos^2\alpha}$$

or

$$y=x.\sqrt{3}-\frac{gx^2}{2.\frac{1}{2}ag.\frac{1}{4}}$$

[$\because$ $\cos\alpha=\frac{1}{2}$ and so $\sin\alpha=\sqrt{(1-\cos^2\alpha)}=\sqrt{3}/2$. Thus $\tan\alpha=\sqrt{3}$.]

or

$$y=\sqrt{3}x-\frac{4x^2}{a}. \qquad \ldots(6)$$

From the figure, for the point A, $x=QL=a\sin\alpha=a\sqrt{3}/2$ and

$$y=-LA=-\tfrac{3}{2}a.$$

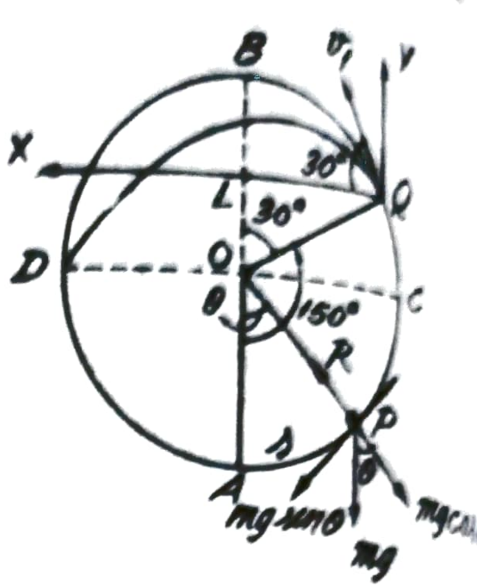
If we put $x=a\sqrt{3}/2$ in the equation (6), we get

$$y=a\frac{\sqrt{3}}{2}.\sqrt{3}-\frac{4}{a}.\frac{3a^2}{4}=\frac{3a}{2}-3a=-\frac{3}{2}a.$$

Thus the co-ordinates of the point A satisfy the equation (6). Hence the particle, after leaving the sphere at Q, describes a parabolic path which meets the sphere again at the point of projection A.

Ex. 5. *Find the velocity with which a particle must be projected along the interior of a smooth vertical hoop of radius a from the lowest point in order that it may leave the hoop at an angular distance of 30° from the vertical. Show that it will strike the hoop again at an extremity of the horizontal diameter.*

Sol. Let a particle of mass m be projected with velocity u from the lowest point A of a smooth circular hoop of radius a along the interior of the hoop. If P is its position at any time t such that $\angle AOP=\theta$ and arc $AP=s$, then the equations of motion along the tangent and normal are



$$m\frac{d^2s}{dt^2}=-mg\sin\theta \quad \ldots(1)$$

and $$m\frac{v^2}{a}=R-mg\cos\theta. \quad \ldots(2)$$

Also $$s=a\theta. \quad \ldots(3)$$

From (1) and (3), we have $a\dfrac{d^2\theta}{dt^2}=-g\sin\theta.$

Multiplying both sides by $2a\dfrac{d\theta}{dt}$ and then integrating, we have

$$v^2=\left(a\frac{d\theta}{dt}\right)^2=2ag\cos\theta+A.$$

But at the point A, $\theta=0$ and $v=u$. $\therefore\quad A=u^2-2ag.$

$$\therefore\quad v^2=u^2-2ag+2ag\cos\theta. \quad \ldots(4)$$

From (2) and (4), we have

$$-R=\frac{m}{a}(v^2+ag\cos\theta)$$

$$=\frac{m}{a}(u^2-2ag+3ag\cos\theta).$$

If the particle leaves the circular hoop at the point Q where $\theta=150°$, then

$$0=\frac{m}{a}(u^2-2ag+3ag\cos 150°)$$

or $$0=u^2-2ag-\frac{3\sqrt{3}}{2}ag.$$

$$u=[\tfrac{1}{2}ag(4+3\sqrt{3})]^{1/2}.$$

Hence the particle will leave the circular hoop at an angular distance of 30° from the vertical if the initial velocity of projection is $u=[\frac{1}{2}ag\,(4+3\sqrt{3})]^{1/2}$.

Again $OL=OQ\cos 30°=a(\sqrt{3}/2)$ and $QL=OQ\sin 30°=a/2$.

If v_1 is the velocity of the particle at the point Q, then $v=v_1$ when $\theta=150°$. Therefore from (4), we have

$$v_1^2=\tfrac{1}{2}ag\,(4+3\sqrt{3})-2ag+2ag\cos 150°=\tfrac{1}{2}ag\sqrt{3}$$

so that $v_1=(\frac{1}{2}ag\sqrt{3})^{1/2}$.

Thus the particle leaves the circular hoop at the point Q with velocity $v_1=(\frac{1}{2}\sqrt{3ag})^{1/2}$ at an angle 30° to the horizontal and subsequently it describes a parabolic path.

The equation of the parabolic trajectory w.r.t. QX and QY as co-ordinate axes is

$$y=x\tan 30°-\frac{gx^2}{2v_1^2\cos^2 30°}=\frac{x}{\sqrt{3}}-\frac{gx^2}{2.\frac{1}{2}\sqrt{3ag}.(\sqrt{3}/2)^2}$$

or
$$y=\frac{x}{\sqrt{3}}-\frac{4x^2}{3\sqrt{3a}}. \qquad \text{...(5)}$$

For the point D which is the extremity of the horizontal diameter CD, we have

$$x=QL+OD=\tfrac{1}{2}a+a=3a/2,\ y=-LO=-a\sqrt{3}/2.$$

Clearly the co-ordinates of the point D satisfy the equation (5). Hence the particle after leaving the circular hoop at Q, strikes the hoop again at an extremity of the horizontal diameter.

Ex. 6. *A particle is projected along the inner side of a smooth vertical circle of radius a, the velocity at the lowest point being u. Show that if $2ga<u^2<5ag$, the particle will leave the circle before arriving at the highest point and will describe a parabola whose latus rectum is*

$$\frac{2\,(u^2-2ag)^3}{27a^2g^3}. \qquad \textbf{(Meerut 1986S, 90P)}$$

Sol. For figure refer Ex. 5. Proceeding as in Ex. 5, the velocity v and the reaction R at any time t are given by

$$v^2=u^2-2ag+2ag\cos\theta \qquad \text{...(1)}$$

and
$$R=\frac{m}{a}\,(u^2-2ag+3ag\cos\theta). \qquad \text{...(2)}$$

If the particle leaves the circle at Q, where $\angle AOQ=\theta_1$, then from (2), we have

$$0=\frac{m}{a}\,(u^2-2ag+3ag\cos\theta_1)$$

or $$\cos\theta_1 = -\frac{u^2-2ag}{3ag}.$$

Since $2ag < u^2 < 5ag$, therefore $\cos\theta_1$ is negative and its absolute value is <1. Therefore θ_1 is real and $\frac{1}{2}\pi < \theta_1 < \pi$.

Thus the particle leaves the circle before arriving at the highest point. If v_1 is the velocity of the particle at the point Q then $v = v_1$ when $\theta = \theta_1$. Therefore from (1), we have

$$v_1^2 = u^2 - 2ag + 2ag\cos\theta_1$$
$$=(u^2-2ag) - 2ag\cdot\left(\frac{u^2-2ag}{3ag}\right)$$
$$=(u^2-2ag)\left(1-\tfrac{2}{3}\right) = \tfrac{1}{3}(u^2-2ag).$$

If $\angle BOQ = \alpha$, then $\alpha = \pi - \theta_1$.

$$\therefore \quad \cos\alpha = \cos(\pi-\theta_1) = -\cos\theta_1 = \frac{u^2-2ag}{3ag}.$$

Thus the particle leaves the circle at the point Q with velocity $v_1 = \sqrt{[\frac{1}{3}(u^2-2ag)]}$ at an angle α to the horizontal and subsequently it describes a parabolic path.

The latus rectum of the parabola

$$=\frac{2}{g}v_1^2\cos^2\alpha = \frac{2}{g}\cdot\frac{1}{3}(u^2-2ag)\cdot\left(\frac{u^2-2ag}{3ag}\right)^2 = \frac{2(u^2-2ag)^3}{27a^2g^3}$$

Ex. 7. *A heavy particle is attached to a fixed point by a fine string of length a; the particle is projected horizontally from the lowest point with velocity $\sqrt{[ag(2+3\sqrt{3}/2)]}$. Prove that the string would first become slack when inclined to the upward vertical at an angle of 30°, will become tight again when horizontal.*

[Meerut 1978]

Sol. Refer figure of Ex. 5 page 166. Taking $R = T$ (*i.e.*, the tension in the string), the equations of motion of the particle are

$$m\frac{d^2s}{dt^2} = -mg\sin\theta \quad \text{...(1)}$$

and $$m\frac{v^2}{a} = T - mg\cos\theta \quad \text{...(2)}$$

Also $$s = a\theta.$$

From (1) and (3), we have $a\dfrac{d^2\theta}{dt^2} = -g\sin\theta$.

Multiplying both sides by $2a\dfrac{d\theta}{dt}$ and integrating, we have

$$v^2 = \left(a\frac{d\theta}{dt}\right)^2 = 2ag\cos\theta + A.$$

But at the point A, $\theta=0$ and $v=\sqrt{[ag\,(2+3\sqrt{3}/2)]}$.

$\therefore\quad ag\,(2+3\sqrt{3}/2)=2ag+A \qquad \text{or} \qquad A=\frac{3}{2}\sqrt{3}ag.$

$\therefore\quad v^2=ag\,(2\cos\theta+\frac{3}{2}\sqrt{3}).$ ...(4)

From (2) and (4), we have

$$T=\frac{m}{a}\,(v^2+ag\cos\theta)=\frac{m}{a}\cdot\left[ag\,(2\cos\theta+\tfrac{3}{2}\sqrt{3})+ag\cos\theta\right]$$

$$=mg\,(3\cos\theta+\tfrac{3}{2}\sqrt{3}). \qquad ...(5)$$

If the string becomes slack at the point Q, where $\theta=\theta_1$, then at Q, $T=0=mg\,(3\cos\theta_1+\frac{3}{2}\sqrt{3})$

giving $\quad \cos\theta_1=-\sqrt{3}/2 \qquad i.e., \qquad \theta_1=150°.$

Hence the string becomes slack when inclined to the upward vertical at an angle of $180°-150°$ *i.e.*, $30°$.

If v_1 is the velocity of the particle at Q, then $v=v_1$ when $\theta=150°$. Therefore from (4), we have

$$v_1^2=ag\,(2\cos 150°+\tfrac{3}{2}\sqrt{3})=\tfrac{1}{2}\sqrt{3}ag.$$

Hence the particle leaves the circular path at the point Q with velocity $v_1=(\frac{1}{2}ag\sqrt{3})^{1/2}$ at an angle of $30°$ to the horizontal and subsequently it describes a parabolic path.

The equation of the parabolic trajectory w.r.t. QX and QY as coordinate axes is

$$y=x\tan 30°-\frac{gx^2}{2v_1^2\cos^2 30°}=\frac{x}{\sqrt{3}}-\frac{gx^2}{2.\frac{1}{2}\sqrt{3}ag.(\sqrt{3}/2)^2}$$

or $$y=\frac{x}{\sqrt{3}}-\frac{4x^2}{3\sqrt{3}a}. \qquad ...(6)$$

The co-ordinates of the point D, which is an extremity of the horizontal diameter CD, are given by

$x=QL+OD=\frac{1}{2}a+a=3a/2 \qquad \text{and} \qquad y=-LO=-a\sqrt{3}/2.$

Clearly the co-ordinates of the point D satisfy the equation (6) showing that the parabolic trajectory meets the circle again at D. When the particle is at D, the string again becomes tight because $OD=a=$ the length of the string.

Hence the string becomes slack when inclined to the upward vertical at an angle of $30°$ and becomes tight again when horizontal.

Ex. 8. *A heavy particle hanging vertically from a fixed point by a light inextensible cord of length l is struck by a horizontal blow which imparts it a velocity $2\sqrt{(gl)}$, prove that the cord becomes slack when the particle has risen to a height $\frac{2}{3}l$ above the fixed point.*

[Gorakhpur 1979; Meerut 77; 85S]

Also find the height of the highest point of the parabola subsequently described.

Sol. Refer figure of Ex. 4 page 164. Take $R=T$ (i.e., the tension in the string).

Let a particle tied to a cord OA of length l be struck by a horizontal blow which imparts it a velocity $2\sqrt{(gl)}$. If P is the position of the particle at time t such that $\angle AOP=\theta$, then the equations of motion are

$$m\frac{d^2s}{dt^2}=-mg\sin\theta \quad \ldots(1)$$

and
$$m\frac{v^2}{l}=T-mg\cos\theta. \quad \ldots(2)$$

Also
$$s=l\theta. \quad \ldots(3)$$

From (1) and (3), we have $l\dfrac{d^2\theta}{dt^2}=-g\sin\theta.$

Multiplying both sides by $2l\dfrac{d\theta}{dt}$ and integrating, we have

$$v^2=\left(l\frac{d\theta}{dt}\right)^2=2lg\cos\theta+A.$$

But at the point A, $\theta=0$ and $v=2\sqrt{(gl)}$.

$\therefore\quad 4gl=2lg+A$ so that $A=2gl$.

$\therefore\quad v^2=2lg(\cos\theta+1). \quad \ldots(4)$

From (2) and (4), we have

$$T=\frac{m}{l}(v^2+gl\cos\theta)=mg(3\cos\theta+2). \quad \ldots(5)$$

If the cord becomes slack at the point Q, where $\theta=\theta_1$, then from (5), we have

$$T=0=mg(3\cos\theta_1+2)$$

giving $\cos\theta_1=-2/3$.

If $\angle COQ=\alpha$, then $\alpha=\pi-\theta_1$ and $\cos\alpha=2/3$.

If v_1 is the velocity of the particle at Q, then $v=v_1$ where $\theta=\theta_1$. Therefore from (4), we have

$$v_1^2=2lg(1+\cos\theta_1)=2lg(1-\tfrac{2}{3})=2lg/3.$$

Now $OL=l\cos\alpha=\frac{2}{3}l$.

Thus the particle leaves the circular path at the point Q at a height $2l/3$ above the fixed point O with velocity $v_1=\sqrt{(2lg/3)}$ at an angle α to the horizontal and subsequently it describes a parabolic path.

Max. height H of the particle above Q

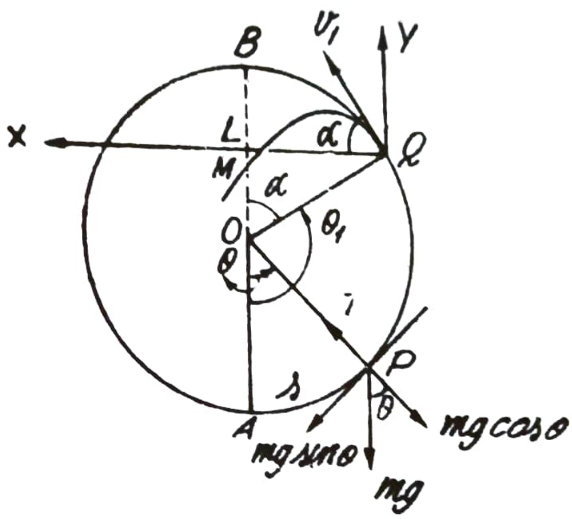
$$=\frac{v_1^2\sin^2\alpha}{2g}=\frac{v_1^2}{2g}(1-\cos^2\alpha)=\frac{\frac{2}{3}lg}{2g}\left(1-\frac{4}{9}\right)=\frac{5l}{27}.$$

∴ Height of the highest point of the parabolic path above the fixed point $O=OL+H=\frac{2}{3}l+\frac{5l}{27}=\frac{23l}{27}$.

Ex. 9. *A heavy particle hangs by an inextensible string of length a from a fixed point and is then projected horizontally with a velocity $\sqrt{(2gh)}$. If $\frac{5a}{2}>h>a$, prove that the circular motion ceases when the particle has reached the height $\frac{1}{3}(a+2h)$. Prove also that the greatest height ever reached by the particle above the point of projection is* $\frac{(4a-h)(a+2h)^2}{27a^2}$.

[Meerut 1984S]

Sol. Let a particle of mass m be attached to one end of a string of length a whose other end is fixed at O. The particle is projected horizontally with a velocity $u=\sqrt{(2gh)}$ from A. If P is the position of the particle at time t such that $\angle AOP=\theta$ and arc $AP=s$, then the equations of motion of the particle are



$$m\frac{d^2s}{dt^2}=-mg\sin\theta \qquad \text{...(1)}$$

and $$m\frac{v^2}{a}=T-mg\cos\theta \qquad \text{...(2)}$$

Also $$s=a\theta. \qquad \text{...(3)}$$

From (1) and (3), we have $a\frac{d^2\theta}{dt^2}=-g\sin\theta$.

Multiplying both sides by $2a\frac{d\theta}{dt}$ and integrating, we have

$$v^2=\left(a\frac{d\theta}{dt}\right)^2=2ag\cos\theta+A.$$

But at the point A, $\theta=0$, and $v=u=\sqrt{(2gh)}$.

∴ $A=2gh-2ag$.

$\therefore \quad v^2 = 2ag \cos\theta + 2gh - 2ag.$ ...(4)

From (2) and (4), we have

$$T = \frac{m}{a}(v^2 + ag\cos\theta) = \frac{m}{a}(3ag\cos\theta + 2gh - 2ag).$$

If the particle leaves the circular path at Q where $\theta = \theta_1$, then $T = 0$ when $\theta = \theta_1$.

$$\therefore \quad 0 = \frac{m}{a}(3ag\cos\theta_1 + 2gh - 2ag) \quad \text{or} \quad \cos\theta_1 = -\frac{2h-2a}{3a}.$$

Since $\frac{5}{2}a > h > a$ *i.e.*, $5a > 2h > 2a$, therefore $\cos\theta_1$ is negative and its absolute value is < 1. So θ_1 is real and $\frac{1}{2}\pi < \theta_1 < \pi$.

Thus the particle leaves the circular path at Q before arriving at the highest point.

Height of the point Q above A

$$= AL = AO + OL = a + a\cos(\pi - \theta_1) = a - a\cos\theta_1$$

$$= a + a.\frac{2h-2a}{3a} = \tfrac{1}{3}(a + 2h)$$

i.e., the particle leaves the circular path when it has reached a height $\frac{1}{3}(a+2h)$ above the point of projection.

If v_1 is the velocity of the particle at the point Q, then from (4), we have

$$v_1^2 = 2ag\cos\theta_1 + 2gh - 2ag$$

$$= -2ag.\frac{(2h-2a)}{3a} + 2g(h-a)$$

$$= 2g(h-a)(1 - \tfrac{2}{3}) = \tfrac{2}{3}g(h-a).$$

If $\angle LOQ = \alpha$, then $\alpha = \pi - \theta_1$.

$$\therefore \quad \cos\alpha = \cos(\pi - \theta_1) = -\cos\theta_1 = \frac{2(h-a)}{3a}.$$

Thus the particle leaves the circular path at the point Q with velocity $v_1 = \sqrt{\{\frac{2}{3}g(h-a)\}}$ at an angle $\alpha = \cos^{-1}\{2(h-a)/3a\}$ to the horizontal and will subsequently describe a parabolic path.

Maximum height of the particle above the point Q

$$= H = \frac{v_1^2\sin^2\alpha}{2g} = \frac{v_1^2}{2g}(1 - \cos^2\alpha) = \tfrac{1}{3}(h-a).\left[1 - \frac{4}{9a^2}(h-a)^2\right]$$

$$= \frac{1}{27a^2}(h-a)[9a^2 - 4(h^2 - 2ah + a^2)]$$

$$=\frac{(h-a)}{27a^2}\left[5a^2+8ah-4h^2\right]=\frac{1}{27a^2}(h-a)(a+2h)(5a-2h).$$

∴ Greatest height ever reached by the particle above the point of projection A

$$=AL+H=\tfrac{1}{3}(a+2h)+\frac{1}{27a^2}(h-a)(a+2h)(5a-2h)$$

$$=\frac{1}{27a^2}(a+2h)\left[9a^2+(h-a)(5a-2h)\right]$$

$$=\frac{1}{27a^2}(a+2h)\left[4a^2+7ah-2h^2\right]$$

$$=\frac{1}{27a^2}(a+2h)(a+2h)(4a-h)=\frac{1}{27a^2}(4a-h)(a+2h)^2.$$

Ex. 10. *A particle is projected along the inside of a smooth fixed sphere from its lowest point, with a velocity equal to that due to falling freely down the vertical diameter of the sphere. Show that the particle will leave the sphere and afterwards pass vertically over the point of projection at a distance equal to $\frac{25}{32}$ of the diameter.*

Sol. Refer figure of Ex. 9 page 171. Replace T by R (*i.e.*, reaction).

Here the velocity of projection $u=\sqrt{(2g.2a)}=\sqrt{(4ag)}$ *i.e.* the particle is projected from the lowest point A with velocity $u=2\sqrt{(ag)}$ inside a smooth sphere of radius a. If P is the position of the particle at time t such that $\angle AOP=\theta$, then the equations of motion are

$$m\frac{d^2s}{dt^2}=-mg\sin\theta \qquad \ldots(1)$$

and

$$m\frac{v^2}{a}=R-mg\cos\theta. \qquad \ldots(2)$$

Also

$$s=a\theta. \qquad \ldots(3)$$

From (1) and (3), we have $a\dfrac{d^2\theta}{dt^2}=-g\sin\theta.$

Multiplying both sides by $2a\dfrac{d\theta}{dt}$ and integrating, we have

$$v^2=\left(a\frac{d\theta}{dt}\right)^2=2ag\cos\theta+A.$$

But at the lowest point A, $\theta=0$ and $v=2\sqrt{(ag)}$.

∴ $A=4ag-2ag=2ag$

∴ $v^2=2ag\cos\theta+2ag. \qquad \ldots(4)$

From (2) and (4), we have

$$R=\frac{m}{a}(ag\cos\theta+v^2)$$

$$=\frac{m}{a}(3ag\cos\theta+2ag).$$

Here $2ag<u^2<5ag$, therefore the particle will leave the sphere at an angle θ_1 where $\pi/2<\theta_1<\pi$. ...(5)

If the particle leaves the sphere at the point Q, where $\theta=\theta_1$, then from (5), we have

$$R=0=\frac{m}{a}(3ag\cos\theta_1+2ag)\text{ giving }\cos\theta_1=-2/3.$$

If v_1 is the velocity of the particle at Q, then from (4), we have

$$v_1^2=2ag\cos\theta_1+2ag=2ag(\cos\theta_1+1)$$

or $$v_1^2=2ag(-\tfrac{2}{3}+1)=\tfrac{2}{3}ag.$$

If $\angle BOQ=\alpha$, then $\alpha=\pi-\theta_1$.

$\therefore\ \cos\alpha=\cos(\pi-\theta_1)=-\cos\theta_1=\frac{2}{3}$.

Hence the particle leaves the spere at the point Q with velocity $v_1=\sqrt{(\frac{2}{3}ag)}$ at an angle $\alpha=\cos^{-1}(\frac{2}{3})$ to the horizontal and subsequently it describes a parabolic path.

Equation of the trajectory described by the particle after leaving the sphere at Q w.r.t. QX and QY as co-ordinate axes is

$$y=x\tan\alpha-\frac{gx^2}{2v_1^2\cos^2\alpha}$$

or $$y=x.\frac{\sqrt{5}}{2}-\frac{gx^2}{2.\frac{2}{3}ag.\frac{4}{9}}$$

$[\because\ \cos\alpha=\frac{2}{3}.\ \therefore\ \sin\alpha=\sqrt{(1-\cos^2\alpha)}=\sqrt{5}/3$ and $\tan\alpha=\sin\alpha/\cos\alpha=\sqrt{5}/2]$

or $$y=\frac{\sqrt{5}}{2}x-\frac{27}{16a}x^2. \quad ...(6)$$

If the particle passes vertically over the point of projection A at the point M, then the x-co-ordinate of M is given by $x=QL=a\sin\alpha=a\sqrt{5}/3$. Let the y-coordinate of M be y_1.

The point M i.e., $(a\sqrt{5}/3, y_1)$ lies on the trajectory (6).

$$\therefore\ y_1=\frac{\sqrt{5}}{2}\cdot\frac{\sqrt{5}}{3}a-\frac{27}{16a}\cdot\frac{5a^2}{9}=\frac{5a}{6}-\frac{15a}{16}=-\frac{5a}{48}.$$

Since the y-coordinate of M is negative, therefore the point M is below the x-axis QX.

The required height $=AM=AO+OL+y_1=a+a\cos\alpha+y_1$

$$=a+\frac{2}{3}a-\frac{5a}{48}=\frac{25a}{16}=\frac{25}{32}(2a).$$

Hence the required height is equal to $\frac{25}{32}$ of the diameter of the sphere.

Ex. 11. *A particle is projected from the lowest point inside a smooth circle of radius a with a velocity due to a height h above the centre. Find the point where it leaves the circle and show that it will afterwards pass through*

(a) the centre if $h=\frac{1}{2}(a\sqrt{3})$,

and (b) the lowest point if $h=3a/4$. **[Rohilkhand 1985]**

Sol. Refer figure of Ex. 9 on page 171. Take $T=R$ (*i.e.*, reaction).

Here the velocity of projection u is equal to that due to a height h above the centre *i.e.*, due to a height $(h+a)$ above the lowest point A.

$\therefore\quad u=\sqrt{\{2g\,(h+a)\}}$.

Let the particle be projected from the lowest point A with velocity u along the inside of a smooth circle of radius a. If P is its position at time t such that $\angle AOP=\theta$ and arc $AP=s$, then the equations of motion along the tangent and normal are

$$m\frac{d^2s}{dt^2}=-mg\sin\theta \qquad \text{...(1)}$$

and

$$m\frac{v^2}{a}=R-mg\cos\theta. \qquad \text{...(2)}$$

Also

$$s=a\theta. \qquad \text{...(3)}$$

From (1) and (3), we have $a\dfrac{d^2\theta}{dt^2}=-g\sin\theta$.

Multiplying both sides by $2a(d\theta/dt)$ and integrating, we have

$$v^2=\left(a\frac{d\theta}{dt}\right)^2=2ag\cos\theta+A.$$

But at the point A, $\theta=0$ and $v^2=u^2=2g\,(h+a)$.

$\therefore\quad A=2g(h+a)-2ag=2gh$.

$\therefore\quad v^2=2ag\cos\theta+2gh. \qquad \text{...(4)}$

From (2), we have

$$R=\frac{m}{a}(v^2+ag\cos\theta)$$

$$=\frac{m}{a}(3\,ag\cos\theta+2gh). \qquad \text{...(5)}$$

If the particle leaves the circle at the point Q, where $\theta=\theta_1$, then from (5), we have

$$R=0=\frac{m}{a}\,(3ag\cos\theta_1+2h)$$

giving $\cos\theta_1=-\dfrac{2h}{3a}$.

If v_1 is the velocity of the particle at Q, then from (4), we have $v_1^2=2ag\cos\theta_1+2gh=2ag\left(\dfrac{-2h}{3a}\right)+2gh=\dfrac{2}{3}gh$.

If $\angle BOQ=\alpha$, then $\alpha=\pi-\theta_1$.

$\therefore\quad \cos\alpha=\cos(\pi-\theta_1)=-\cos\theta_1=(2h/3a)$,

and $\quad OL=a.\cos\alpha=2h/3$.

Hence the particle leaves the circle at the point Q at height $2h/3$ above the centre O with velocity $v_1=\sqrt{(2gh/3)}$ at an angle $\alpha=\cos^{-1}(2h/3a)$ to the horizontal and then it describes a parabolic path.

Equation of the trajectory of the parabola described by the particle after leaving the circle at Q w.r.t. QX and QY as co-ordinate axes is

$$y=x\tan\alpha-\frac{gx^2}{2v_1^2\cos^2\alpha}$$

or
$$y=x\tan\alpha-\frac{gx^2}{2.\frac{2}{3}gh\cos^2\alpha}$$

or
$$y=x\tan\alpha-\frac{3x^2}{4h\cos^2\alpha} \qquad \ldots(6)$$

(a) The co-ordinates of the centre O w.r.t. QX and QY as co-ordinate axes are given by

$$x=QL=a\sin\alpha \text{ and } y=-OL=-a\cos\alpha.$$

If the particle passes through the centre O *i.e.*, the point $(a\sin\alpha, -a\cos\alpha)$, then the point O will lie on the curve (6).

$$\therefore\quad -a\cos\alpha=a\sin\alpha.\tan\alpha-\frac{3a^2\sin^2\alpha}{4h\cos^2\alpha}$$

or
$$\frac{3a\sin^2\alpha}{4h\cos^2\alpha}=\frac{\sin^2\alpha}{\cos\alpha}+\cos\alpha=\frac{\sin^2\alpha+\cos^2\alpha}{\cos\alpha}=\frac{1}{\cos\alpha}$$

or
$$3a\sin^2\alpha=4h\cos\alpha$$

or
$$3a(1-\cos^2\alpha)=4h\cos\alpha$$

or
$$3a\left(1-\frac{4h^2}{9a^2}\right)=4h.\frac{2h}{3a} \qquad \left[\because\ \cos\alpha=\frac{2h}{3a}\right]$$

$$3a = \frac{h^2}{a}\left(\frac{8}{3}+\frac{4}{3}\right)=\frac{4h^2}{a}$$

or $\quad h^2=\frac{3}{4}a^2.$

or $\quad h=\frac{1}{2}(a\sqrt{3}).$

(b) The co-ordinates of the lowest point A w.r.t. QX and QY as co-ordinate axes are given by $x=QL=a \sin \alpha$

and $\quad y=-LA=-(LO+OA)$

$$=-(a \cos \alpha+a)=-a(\cos \alpha+1).$$

If the particle after leaving the circle at Q, passes through the lowest point A $[a \sin \alpha, -a(\cos \alpha+1)]$, then the point A will lie on (6).

$$\therefore \quad -a(\cos \alpha+1)=a \sin \alpha \tan \alpha-\frac{3a^2 \sin^2 \alpha}{4h \cos^2 \alpha}$$

or $$\frac{3a \sin^2 \alpha}{4h \cos^2 \alpha}=\frac{\sin^2 \alpha}{\cos \alpha}+\cos \alpha+1$$

$$=\frac{\sin^2 \alpha+\cos^2 \alpha+\cos \alpha}{\cos \alpha}=\frac{1+\cos \alpha}{\cos \alpha}$$

or $\quad 3a \sin^2 \alpha=4h \cos \alpha\,(1+\cos \alpha)$

or $\quad 3a(1-\cos^2 \alpha)=4h \cos \alpha\,(1+\cos \alpha)$

or $\quad 3a(1-\cos \alpha)(1+\cos \alpha)=4h \cos \alpha\,(1+\cos \alpha)$

or $\quad 3a(1-\cos \alpha)=4h \cos \alpha \qquad [\because\ 1+\cos \alpha \neq 0]$

or $\quad 3a\left(1-\dfrac{2h}{3a}\right)=4h.\dfrac{2h}{3a} \qquad \left[\because\ \cos \alpha=\dfrac{2h}{3a}\right]$

or $\quad 3a(3a-2h)=8h^2$ or $9a^2-6ah-8h^2=0$

or $\quad (3a+2h)(3a-4h)=0.$

$\therefore \quad 3a-4h=0 \qquad [\because\ 3a+2h \neq 0]$

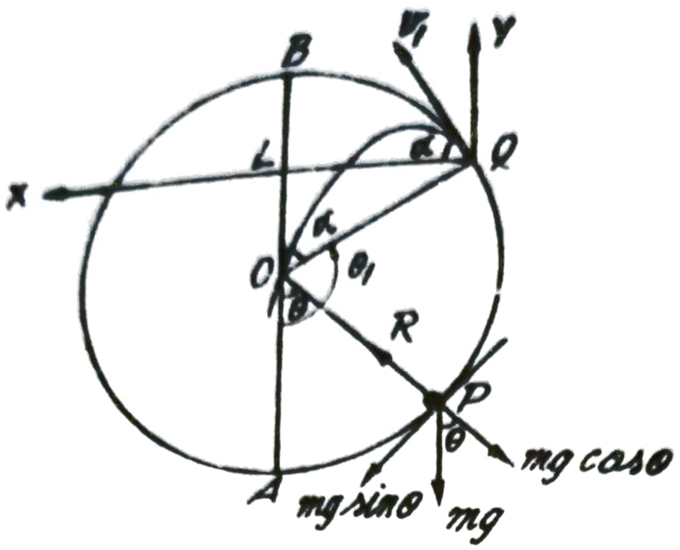
or $\quad h=3a/4.$

Ex. 12. *A particle is projected along the inside of a smooth vertical circle of radius a from the lowest point. Show that the velocity of projection required in order that after leaving the circle, the particle may pass through the centre is* $\sqrt{(\frac{1}{2}ag)}.(\sqrt{3}+1)$.

[Meerut 1988]

Sol. Let the particle be projected from the lowest point A along the inside of a smooth vertical circle of radius a, with velocity u. If P is the position of the particle at time t such that $\angle AOP=\theta$ and arc $AP=s$, the equations of motion of the particle along the tangent and normal are

$$m\frac{d^2s}{dt^2}=-mg \sin \theta, \qquad \ldots(1)$$



and $\quad m\dfrac{v^2}{a}=R-mg\cos\theta.$ ...(2)

Also $\quad s=a\theta.$ ...(3)

From (1) and (3), we have $a\dfrac{d^2\theta}{dt^2}=-g\sin\theta.$

Multiplying both sides by $2a\dfrac{d\theta}{dt}$ and integrating, we have

$$v^2=\left(a\frac{d\theta}{dt}\right)^2=2ag\cos\theta+A.$$

But at the lowest point A, $\theta=0$ and $v=u$. $\therefore \quad A=u^2-2ag.$

$\therefore \quad v^2=2ag\cos\theta+u^2-2ga.$ ...(4)

From (2) and (4), we have

$$R=\frac{m}{a}(v^2+ag\cos\theta)=\frac{m}{a}(u^2-2ag+3ag\cos\theta). \qquad ...(5)$$

If the particle leaves the circle at Q, where $\theta=\theta_1$, then from (5),

$$0=\frac{m}{a}(u^2-2ag+3ag\cos\theta_1)$$

or $$\cos\theta_1=-\left(\frac{u^2-2ag}{3ag}\right).$$

If $\angle BOQ=\alpha$, then $\alpha=\pi-\theta_1$.

$$\therefore \quad \cos\alpha=\cos(\pi-\theta_1)=-\cos\theta_1=\frac{u^2-2ag}{3ag}.$$

If v_1 is the velocity at Q, then putting $v=v_1$, $R=0$ and $\theta=\theta_1$ in (2), we have

$$v_1^2=-ag\cos\theta_1=-ag\cos(\pi-\alpha)=ag\cos\alpha.$$

Thus the particle leaves the circle at Q with velocity $v_1=\sqrt{(ag\cos\alpha)}$ at angle $\alpha=\cos^{-1}\left(\frac{u^2-2ag}{3ag}\right)$ to the horizontal and subsequently it describes a parabolic path.

The equation of the parabolic trajectory w.r.t QX and QY as co-ordinate axes is

$$y=x\tan\alpha-\frac{gx^2}{2v_1^2\cos^2\alpha}=x\tan\alpha-\frac{2x^2}{2ag\cos^3\alpha}. \quad \ldots(6)$$

$$[\because \; v_1^2=ag\cos\alpha]$$

The coordinates of the centre O w.r.t. QX and QY as coordinate axes are given by

$$x=QL=a\sin\alpha \text{ and } y=-LO=-a\cos\alpha.$$

If after leaving the circle at Q the particle passes through the centre O $(a\sin\alpha, -a\cos\alpha)$, then the point O lies on the curve (6).

$$\therefore \; -a\cos\alpha=a\sin\alpha.\tan\alpha-\frac{ga^2\sin^2\alpha}{2ag\cos^3\alpha}$$

or $\dfrac{\sin^2\alpha}{2\cos^3\alpha}=\dfrac{\sin^2\alpha}{\cos\alpha}+\cos\alpha=\dfrac{\sin^2\alpha+\cos^2\alpha}{\cos\alpha}=\dfrac{1}{\cos\alpha}$

or $\sin^2\alpha=2\cos^2\alpha$ or $1-\cos^2\alpha=2\cos^2\alpha$ or $3\cos^2\alpha=1$

or $\cos^2\alpha=1/3$ or $\cos\alpha=1/\sqrt{3}$.

$$\therefore \; \frac{u^2-2ag}{3ag}=\frac{1}{\sqrt{3}} \qquad \left[\because \; \cos\alpha=\frac{u^2-2ag}{3ag}\right]$$

or $\quad u^2-2ag=\sqrt{3}ag$

or $\quad u^2=(2+\sqrt{3})\,ag=\left(\dfrac{4+2\sqrt{3}}{2}\right)ag=\dfrac{ag}{2}(1+\sqrt{3})^2.$

$\therefore \; u=\sqrt{(\tfrac{1}{2}ag)}.(\sqrt{3}+1).$

Thus the particle will pass through the centre if the velocity of projection at the lowest point is $\sqrt{(\tfrac{1}{2}ag)}\,(\sqrt{3}+1)$.

Ex. 13. *A particle tied to a string of length a is projected from its lowest point, so that after leaving the circular path it describes a free path passing through the lowest point. Prove that the velocity of projection is* $\sqrt{(\tfrac{7}{2}ag)}$. **[Kanpur 1975]**

Sol. Refer figure of Ex. 12, page 178. Take $R=T$ (*i.e.*, the tension in the string).

Let a particle of mass m be attached to one end A of the string OA whose other end is fixed at O. Let the particle be projected from the lowest point A with velocity u. If the particle leaves the circular path at Q with velocity v_1 at an angle α to the horizontal, then proceed as in Ex. 12 to get

$$v_1=\sqrt{(ag\cos\alpha)}\quad\text{and}\quad \cos\alpha=\left(\frac{u^2-2ag}{3ag}\right).$$

After Q the particle describes a parabolic path whose equation w.r.t. the horizontal and vertical lines QX and QY as co-ordinate axes is

$$y=x\tan\alpha-\frac{gx^2}{v_1^2\cos^2\alpha}=x\tan\alpha-\frac{gx^2}{2ag\cos^3\alpha}. \quad ...(1)$$

$$[\because\ v_1^2=ag\cos\alpha]$$

The co-ordinates of the lowest point A w.r.t. QX and QY as co-ordinate axes are given by

$$x=QL=a\sin\alpha \quad\text{and}\quad y=-LA=-(LO+OA)$$
$$=-(a\cos\alpha+a)=-a(\cos\alpha+1).$$

If the particle passes through the lowest point $A\,[a\sin\alpha, -a(\cos\alpha+1)]$, then the point A lies on the curve (1).

$$\therefore\quad -a(\cos\alpha+1)=a\sin\alpha\tan\alpha-\frac{ga^2\sin^2\alpha}{2ag\cos^3\alpha}$$

or $$\frac{\sin^2\alpha}{2\cos^3\alpha}=\frac{\sin^2\alpha}{\cos\alpha}+\cos\alpha+1$$
$$=\frac{\sin^2\alpha+\cos^2\alpha+\cos\alpha}{\cos\alpha}=\frac{1+\cos\alpha}{\cos\alpha}$$

or $$\sin^2\alpha=2\cos^2\alpha\,(1+\cos\alpha)$$

or $$(1-\cos^2\alpha)=2\cos^2\alpha\,(1+\cos\alpha)$$

or $$(1-\cos\alpha)(1+\cos\alpha)=2\cos^2\alpha\,(1+\cos\alpha)$$

or $$1-\cos\alpha=2\cos^2\alpha \qquad [\because\ 1+\cos\alpha\neq 0]$$

or $$2\cos^2\alpha+\cos\alpha-1=0 \text{ or } (2\cos\alpha-1)(\cos\alpha+1)=0$$

or $$2\cos\alpha-1=0 \qquad [\because\ \cos\alpha+1\neq 0]$$

or $$\cos\alpha=\tfrac{1}{2}$$

or $$\frac{u^2-2ag}{3ag}=\tfrac{1}{2} \qquad \left[\because\ \cos\alpha=\frac{u^2-2ag}{3ag}\right]$$

or $$u^2=2ag+\frac{3}{2}ag=\frac{7}{2}ag \text{ or } u=\sqrt{\left(\frac{7}{2}ag\right)}.$$

Ex. 14. *Show that the greatest angle through which a person can oscillate on a swing, the ropes of which can support twice the person's weight at rest is 120°.*

If the ropes are strong enough and he can swing through 180° and if v is his speed at any point, prove that the tension in the rope at that point is $\frac{3mv^2}{2l}$ where m is the mass of the person and l the length of the rope.

Sol. Let u be the velocity of a person of mass m at the lowest point. If v is the velocity of the person and T the tension in the rope of length l at a point P at an angular distance θ from the lowest point, then proceed as in § 2 to get

$$v^2=u^2-2lg+2lg\cos\theta, \quad \text{...(1)}$$

and
$$T=\frac{m}{l}(u^2-2lg+3lg\cos\theta). \quad \text{...(2)}$$

Now according to the question the ropes can support twice the person's weight at rest. Therefore the maximum tension the rope can bear is $2mg$. So for the greatest angle through which the person can oscillate, the velocity u at the lowest point should be such that $T=2mg$ when $\theta=0$.

Then from (2), we have

$$2mg=\frac{m}{l}(u^2-2lg+3lg\cos 0)$$

or
$$2gl=u^2-2lg+3lg \quad \text{or} \quad u^2=lg.$$

Now from (1), we have

$$v^2=lg-2lg+2lg\cos\theta=2lg\cos\theta-lg=lg(2\cos\theta-1).$$

If $v=0$ at $\theta=\theta_1$, then $\quad 0=gl(2\cos\theta_1-1)$

or $\quad \cos\theta_1=\frac{1}{2}$. Therefore $\theta_1=60°$.

Thus the person can swing through an angle of 60° from the vertical on one side of the lowest point. Hence the person can oscillate through an angle of $60°+60°=120°$.

Second part. If the rope is strong enough and the person can swing through an angle of 180° *i.e.*, through an angle of 90° on one side of the lowest point, then $\quad v=0$, at $\theta=90°$.

$\therefore$ from (1), we have

$$0=u^2-2lg+2lg\cos 90° \quad \text{or} \quad u^2=2lg.$$

Thus if the person's velocity at the lowest point is $\sqrt{(2lg)}$, then he can swing through an angle of 180°.

Then from (1), we have $\quad v^2=2lg-2lg+2lg\cos\theta$

or
$$\cos\theta=\frac{v^2}{2lg}.$$

Therefore from (2), the tension in the rope at an angular distance θ where the velocity is v, is given by

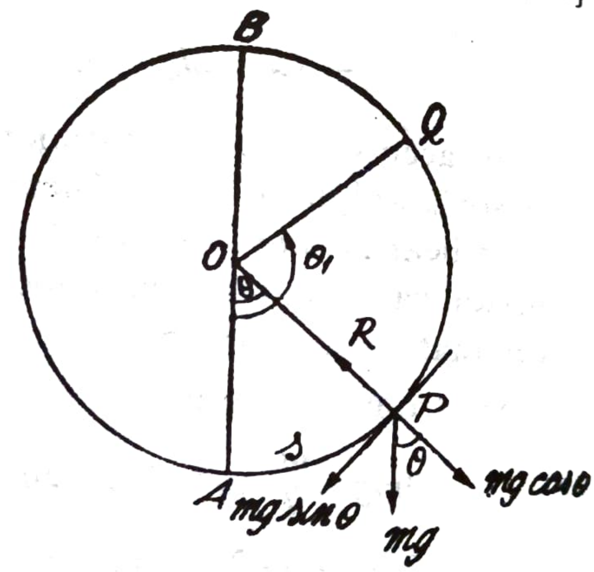
$$T=\frac{m}{l}\left[2lg-2lg+3lg\cdot\frac{v^2}{2lg}\right]=\frac{3mv^2}{2l}.$$

Ex. 15. *A particle is free to move on a smooth vertical circular wire of radius a. It is projected from the lowest point with velocity just sufficient to carry it to the highest point. Show that the reaction between the particle and the wire is zero after a time*

$$\sqrt{(a/g)}.\log(\sqrt{5}+\sqrt{6}).$$

[Agra 1980, 86; Kanpur 79, 81; Meerut 86P, 87P, 90]

Sol. Let a particle of mass m be projected from the lowest point A of a vertical circle of radius a with velocity u which is just sufficient to carry it to the highest point B.



If P is the position of the particle at any time t such that $\angle AOP=\theta$ and arc $AP=s$, then the equations of motion of the particle along the tangent and normal are

$$m\frac{d^2s}{dt^2}=-mg\sin\theta \quad ...(1)$$

and $$m\frac{v^2}{a}=R-mg\cos\theta. \quad ...(2)$$

Also $$s=a\theta. \quad ...(3)$$

From (1) and (3), we have $a\dfrac{d^2\theta}{dt^2}=-g\sin\theta.$

Multiplying both sides by $2a\,(d\theta/dt)$ and integrating, we have

$$v^2=\left(a\frac{d\theta}{dt}\right)^2=2ag\cos\theta+A.$$

But according to the question $v=0$ at the highest point B, where $\theta=\pi$. $\therefore$ $0=2ag\cos\pi+A$ or $A=2ag$.

$$\therefore\quad v^2=\left(a\frac{d\theta}{dt}\right)^2=2ag\cos\theta+2ag. \quad ...(4)$$

From (2) and (4), we have

$$R=\frac{m}{a}(v^2+ag\cos\theta)=\frac{m}{a}(2ag+3ag\cos\theta). \quad ...(5)$$

If the reaction $R=0$ at the point Q where $\theta=\theta_1$, then from (5), we have

$$0=\frac{m}{a}(2ag+3ag\cos\theta_1)$$

or $$\cos\theta_1=-2/3. \quad ...(6)$$

From (4), we have

$$\left(a\frac{d\theta}{dt}\right)^2=2ag(\cos\theta+1)=2ag.2\cos^2\tfrac{1}{2}\theta=2ag\cos^2\tfrac{1}{2}\theta.$$

$\therefore$ $\dfrac{d\theta}{dt}=2\sqrt{(g/a)}\cos\tfrac{1}{2}\theta$, the positive sign being taken before the radical sign because θ increases as t increases

or $$dt=\tfrac{1}{2}\sqrt{(a/g)}\sec\tfrac{1}{2}\theta\, d\theta.$$

Integrating from $\theta=0$ to $\theta=\theta_1$, the required time t is given by

$$t=\tfrac{1}{2}\sqrt{(a/g)}\int_{\theta=0}^{\theta_1}\sec\tfrac{1}{2}\theta\, d\theta$$

or $$t=\sqrt{(a/g)}\left[\log(\sec\tfrac{1}{2}\theta+\tan\tfrac{1}{2}\theta)\right]_0^{\theta_1}$$

or $$t=\sqrt{(a/g)}\log(\sec\tfrac{1}{2}\theta_1+\tan\tfrac{1}{2}\theta_1). \quad ...(7)$$

From (6), we have

$$2\cos^2\tfrac{1}{2}\theta_1-1=-\tfrac{2}{3}$$

or $$2\cos^2\tfrac{1}{2}\theta_1=1-\tfrac{2}{3}=\tfrac{1}{3}$$

or $$\cos^2\tfrac{1}{2}\theta_1=\tfrac{1}{6} \quad \text{or} \quad \sec^2\tfrac{1}{2}\theta_1=6.$$

$\therefore$ $$\sec\tfrac{1}{2}\theta_1=\sqrt{6}$$

and $$\tan\tfrac{1}{2}\theta_1=\sqrt{(\sec^2\tfrac{1}{2}\theta_1-1)}=\sqrt{(6-1)}=\sqrt{5}.$$

Substituting in (7), the required time is given by

$$t=\sqrt{(a/g)}\log(\sqrt{6}+\sqrt{5}).$$

Ex. 16. *A heavy bead slides on a smooth circular wire of radius a. It is projected from the lowest point with a velocity just sufficient to carry it to the heighest point, prove that the radius through the bead in time t will turn through an angle*

$$2\tan^{-1}[\sinh\{t\sqrt{(g/a)}\}]$$

and that the bead will take an infinite time to reach the highest point. **[Meerut 1972, 75, 84 P 85P, 87, 87S, 90S; Agra 88]**

Sol. Refer figure of Ex. 15 page 182.

The equations of motion of the bead are

$$m\frac{d^2s}{dt^2}=-mg\sin\theta, \qquad \ldots(1)$$

and $$m\frac{v^2}{a}=R-mg\cos\theta. \qquad \ldots(2)$$

Also $s=a\theta$. ...(3)

From (1) and (3), we have $a\dfrac{d^2\theta}{dt^2}=-g\sin\theta$.

Multiplying both sides by $2a(d\theta/dt)$ and integrating, we have

$$v^2=\left(a\frac{d\theta}{dt}\right)^2=2ag\cos\theta+A.$$

But according to the question at the highest point $v=0$

i.e., when $\theta=\pi$, $v=0$.

$\therefore$ $0=2ag\cos\pi+A$ or $A=2ag$.

$$\therefore\ v^2=\left(a\frac{d\theta}{dt}\right)^2=2ag+2ag\cos\theta=2ag(1+\cos\theta)$$

$$=2ag.2\cos^2\tfrac{1}{2}\theta$$

or $$a\frac{d\theta}{dt}=2\sqrt{(ag)}.\cos\tfrac{1}{2}\theta$$

or $$dt=\tfrac{1}{2}\sqrt{(a/g)}.\sec\tfrac{1}{2}\theta\, d\theta.$$

Integrating, the time t from A to P is given by

$$t=\tfrac{1}{2}\sqrt{(a/g)}.\int_0^\theta \sec\tfrac{1}{2}\theta\, d\theta$$

$$=\tfrac{1}{2}\sqrt{(a/g)}.2\Big[\log(\tan\tfrac{1}{4}\theta+\sec\tfrac{1}{2}\theta)\Big]_0^\theta$$

$$=\sqrt{(a/g)}\,[\log(\tan\tfrac{1}{2}\theta+\sec\tfrac{1}{2}\theta)-\log 1]$$

$$=\sqrt{(a/g)}.[\log\{\tan\tfrac{1}{2}\theta+\sqrt{(1+\tan^2\tfrac{1}{2}\theta)}\}]$$

$$=\sqrt{(a/g)}.\sinh^{-1}(\tan\tfrac{1}{2}\theta)$$

$$[\because\ \sinh^{-1}x=\log\{x+\sqrt{(1+x^2)}\}]$$

or $$t\sqrt{(g/a)}=\sinh^{-1}(\tan\tfrac{1}{2}\theta)$$

or $$\tan\tfrac{1}{2}\theta=\sinh\{t\sqrt{(g/a)}\}.$$

$$\therefore\ \theta=2\tan^{-1}[\sinh\{t\sqrt{(g/a)}\}].$$

Again the time to reach the highest point B while starting from A

$$=\tfrac{1}{2}\sqrt{(a/g)}\int_{\theta=0}^{\pi}\sec\tfrac{1}{2}\theta\, d\theta$$

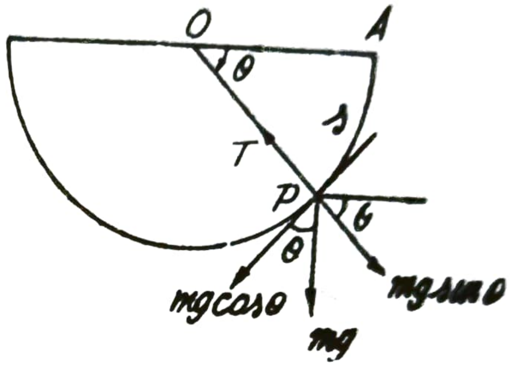
$$=\tfrac{1}{2}\sqrt{(a/g)}.2\Big[\log(\tan\tfrac{1}{4}\theta+\sec\tfrac{1}{2}\theta)\Big]_0^\pi$$

$$=\sqrt{(a/g)}.[\log(\tan\tfrac{1}{4}\pi+\sec\tfrac{1}{2}\pi)-\log(\tan 0+\sec 0)]$$

$=\sqrt{(a/g)}.[\log\infty-\log 1]=\infty.$

Therefore the bead takes an infinite time to reach the highest point.

Ex. 17. *A particle attached to a fixed peg O by a string of length l, is lifted up with the string horizontal and then let go. Prove that when the string makes an angle θ with the horizontal, the resultant acceleration is* $g\sqrt{(1+3\sin^2\theta)}$.



Sol. Let a particle of mass m be attached to a string of length l whose other end is attached to a fixed peg O. Initially let the string be horizontal in the position OA such that $OA=l$. The particle starts from A and moves in a circle whose centre is O and radius is l. Let P be the position of the particle at any time t such that $\angle AOP=\theta$ and arc $AP=s$. The forces acting on the particle at P are : (i) its weight mg acting vertically downwards and (ii) the tension T in the string along PO.

$\therefore$ the equations of motion of the particle along the tangent and normal at P are

$$m\frac{d^2s}{dt^2}=mg\cos\theta, \qquad \ldots(1)$$

and $$m\frac{v^2}{l}=T-mg\sin\theta. \qquad \ldots(2)$$

Also $$s=l\theta. \qquad \ldots(3)$$

From (1) and (3), we have $l\dfrac{d^2\theta}{dt^2}=g\cos\theta.$

Multiplying both sides by $2l(d\theta/dt)$ and integrating, we have

$$v^2=\left(l\frac{d\theta}{dt}\right)^2=2lg\sin\theta+A.$$

But initially at the point A, $\theta=0$, $v=0$. $\therefore$ $A=0$.

$\therefore$ $v^2=2lg\sin\theta.$ ...(4)

The resultant acceleration of the particle at P

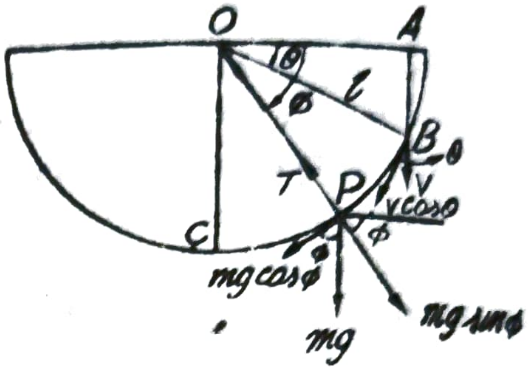
$=\sqrt{[(\text{Tangential accel.})^2+(\text{Normal accel.})^2]}$

$=\sqrt{\left[\left(\frac{d^2s}{dt^2}\right)^2+\left(\frac{v^2}{l}\right)^2\right]}$ $\left[\because \text{Normal accel}=\frac{v^2}{\rho}=\frac{v^2}{l}\right]$

$$=\sqrt{\left[(g\cos\theta)^2+\left(\frac{2lg\sin\theta}{l}\right)^2\right]}$$

$$=g\sqrt{[1-\sin^2\theta+4\sin^2\theta]}=g\sqrt{(1+3\sin^2\theta)}.$$

Ex. 18. *A particle attached to a fixed peg O by a string of length l, is let fall from a point in the horizontal line through O at a distance $l\cos\theta$ from O; show that its velocity when it is vertically below O is $\sqrt{[2gl(1-\sin^3\theta)]}$.*



Sol. Let a particle of mass m be attached to a string of length l whose other end is attached to a fixed peg O. Let the particle fall from a point A in the horizontal line through O such that $OA=l\cos\theta$. The particle will fall under gravity from A to B, where $OB=l$.

$\because$ $OA=l\cos\theta$ and $OB=l$, therefore $\therefore \angle AOB=\theta$ and $AB=l\sin\theta$.

$\therefore$ the velocity of the particle at B

$\therefore V=\sqrt{(2g.AB)}=\sqrt{(2gl\sin\theta)}$, vertically downwards.

As the particle reaches B, there is a jerk in the string and the impulsive tension in the string destroys the component of the velocity along OB and the component of the velocity along the tangent at B remains unaltered *i.e.*, the particle moves in the circular path with centre O and radius l with the tangential velocity $V\cos\theta$ at B.

[**Note :** In the figure write D at the end of the horizontal radius through O].

If P is the position of the particle at any time t such that $\angle DOP=\phi$ and arc $DP=s$, then the equations of motion of the particle along the tangent and normal are

$$m\frac{d^2s}{dt^2}=mg\cos\phi, \quad \text{...(1)}$$

and
$$\frac{mv^2}{l}=T-mg\sin\phi. \quad \text{...(2)}$$

Also
$$s=l\phi. \quad \text{...(3)}$$

From (1) and (3), we have $l\dfrac{d^2\phi}{dt^2}=g\cos\phi$.

Multipyling both sides by $2l\,(d\phi/dt)$ and integrating, we have

$$v^2=\left(l\frac{d\phi}{dt}\right)^2=2lg\sin\phi+A.$$

But at the point B, $\phi=\theta$ and $v=V\cos\theta$.

$\therefore\quad A=V^2\cos^2\theta-2lg\sin\theta=2gl\sin\theta.\cos^2\theta-2lg\sin\theta$

$=-2lg\sin\theta\,(1-\cos^2\theta)=-2lg\sin^3\theta.$

$\therefore\quad v^2=2lg\sin\phi-2lg\sin^3\theta.$

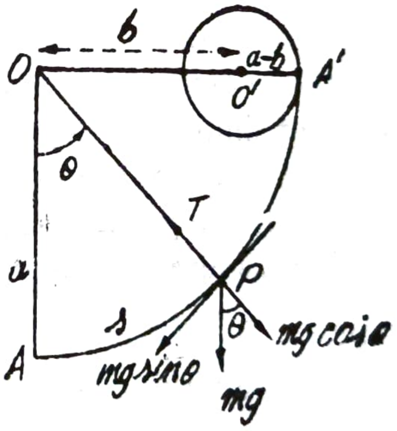
When the particle is at C vertically below O, we have at C $\phi=\pi/2$. Therefore the velocity v at C is given by

$$v^2=2lg\sin\tfrac{1}{2}\pi-2lg\sin^3\theta=2lg\,(1-\sin^3\theta).$$

$\therefore$ the required velocity $v=\sqrt{[2lg\,(1-\sin^3\theta)]}$.

Ex. 19. *A particle is hanging from a fixed point O by means of a string of length a. There is a small nail at O' in the same horizontal line with O at a distance $b\,(<a)$ from O. Find the minimum velocity with which the particle should be projected from its lowest point in order that it may make a complete revolution round the nail without the string becoming slack.* **[Meerut 1977]**

Sol. Let a particle of mass m hang from a fixed point O by means of a string OA of length a. Let O' be a nail in the same horizontal line with O at a distance $OO'=b\,(<a)$. Let the particle be projected from A with velocity u. It moves in a circle with centre at O and radius as a. If P is the position of the particle at any time t such that $\angle AOP=\theta$ and arc $AP=s$, then the equations of motion of the particle along the tangent and normal are



$$m\frac{d^2s}{dt^2}=-mg\sin\theta, \quad \ldots(1)$$

and

$$m\frac{v^2}{a}=T-mg\cos\theta. \quad \ldots(2)$$

Also

$$s=a\theta. \quad \ldots(3)$$

From (1) and (3), we have $a\dfrac{d^2\theta}{dt^2}=-g\sin\theta.$

Multiplying both sides by $2a\,(d\theta/dt)$ and integrating, we have

$$v^2=\left(a\frac{d\theta}{dt}\right)^2=2ag\cos\theta+A.$$

But initially at A, $\theta=0$ and $v=u$. $\therefore$ $A=u^2-2ag$.

$$\therefore\quad v^2=u^2-2ag+2ag\cos\theta. \quad ...(4)$$

At the point A', $\theta=\pi/2$. If v_1 is the velocity of A', then from (4), we have

$$v_1^2=u^2-2ag \quad \text{or} \quad v_1=\sqrt{(u^2-2ag)}.$$

Since there is a nail at O', the particle will describe a circle with centre at O' and radius as $O'A'=a-b$.

We know that if a particle is attached to a string of length l, the least velocity of projection from the lowest point in order to make a complete circle is $\sqrt{(5gl)}$. Also in this case, using the result (4), the velocity of the particle when it has described an angle θ from the lowest point is given by

$$v^2=5lg-2lg+2lg\cos\theta \quad [\because \text{ here } a=l \text{ and } u^2=5gl]$$
$$=3lg+2lg\cos\theta.$$

At $\theta=\pi/2$, if $v=v_2$, then $v_2=\sqrt{(3lg)}$. $[\because \cos\pi/2=0]$

Thus in order to describe a complete circle of radius l the minimum velocity of the particle at the end of the horizontal diameter should be $\sqrt{(3gl)}$. Therefore in order to describe a complete circle of radius $l=O'A'=a-b$ round O' the minimum velocity of the particle at A' should be $\sqrt{[3g\,(a-b)]}$.

But, as already found out, the velocity of the particle at A' is v_1.

$\therefore$ we must have $v_1\geqslant\sqrt{[3g\,(a-b)]}$

or $\quad \sqrt{(u^2-2ag)}\geqslant\sqrt{[3g\,(a-b)]}$

or $\quad u^2-2ag\geqslant 3g\,(a-b)$

or $\quad u^2\geqslant g\,(5a-3b)$

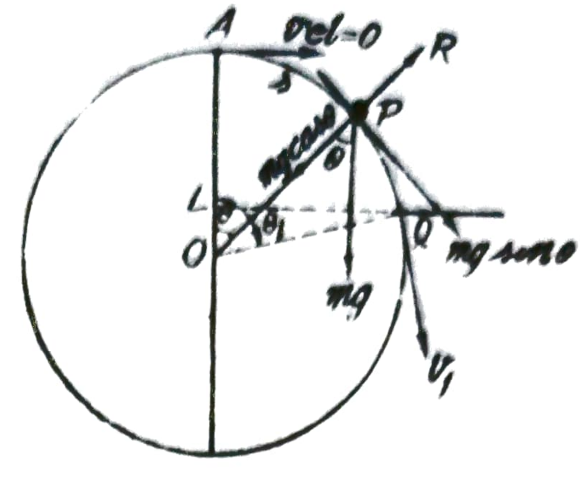
or $\quad u\geqslant\sqrt{[g(5a-3b)]}$.

Hence the required minimum velocity of projection of the particle at the lowest point is $\sqrt{[g\,(5a-3b)]}$.

§ 5. Motion on the outside of a smooth vertical circle. *A particle slides down the outside of a smooth vertical circle starting from rest at the highest point ; to discuss the motion.*

[Meerut 1974, 77, 81; Kanpur 76, 80; Agra 78]

Let a particle of mass m slide down the outside of a smooth vertical circle whose centre is O and radius a, starting from rest at the highest point A. Let P be the position of the particle at any time t such that $\angle AOP=\theta$ and arc $AP=s$. The forces acting on the particle at P are (i) weight mg acting vertically downwards and (ii) the reaction R acting along the outwards drawn normal OP. If v be the velocity of the particle at P the equations of motion of the particle along the tangent and normal are



$$m\frac{d^2s}{dt^2}=mg\sin\theta, \qquad \ldots(1)$$

(+ive sign is taken on the R. H. S. because $mg \sin\theta$ acts in the direction of s increasing)

and $$m\frac{v^2}{a}=mg\cos\theta-R. \qquad \ldots(2)$$

[Note that in equation (2) R has been taken with −ive sign because it is in the direction of outwards drawn normal and $mg\cos\theta$ with +ive sign because it is in the direction of inwards drawn normal.]

Also $$s=a\theta. \qquad \ldots(3)$$

From (1) and (3), we have $a\dfrac{d^2\theta}{dt^2}=g\sin\theta.$

Multiplying both sides by $2a\dfrac{d\theta}{dt}$ and integrating, we have

$$v^2=\left(a\frac{d\theta}{dt}\right)^2=-2ag\cos\theta+A.$$

But initially at A, $\theta=0$ and $v=0$ $\therefore$ $A=2ag$.

$$\therefore\ v^2=2ag-2ag\cos\theta=2ag(1-\cos\theta). \qquad \ldots(4)$$

From (2) and (4), we have

$$R=\frac{m}{a}\left[ag\cos\theta-v^2\right]=\frac{m}{a}\left[3ag\cos\theta-2ag\right]$$
$$=mg(3\cos\theta-2). \qquad \ldots(5)$$

If the particle leaves the circle at Q where $\angle AOQ=\theta_1$, then $R=0$ when $\theta=\theta_1$. Therefore from (5), we have

$$mg\,(3\cos\theta_1-2)=0 \quad \text{or} \quad \cos\theta_1=\tfrac{2}{3}.$$

Vertical depth of the point Q below A

$$=AL=OA-OL=a-a\cos\theta_1=a-\tfrac{2}{3}a=a/3.$$

Hence if a particle slides down the outside of a smooth vertical circle, starting from rest at the highest point, it will leave the circle after descending vertically a distance equal to one third of the radius of the circle. [Meerut 1981S]

If v_1 is the velocity of the particle at Q, then $v=v_1$ when $\theta=\theta_1$.

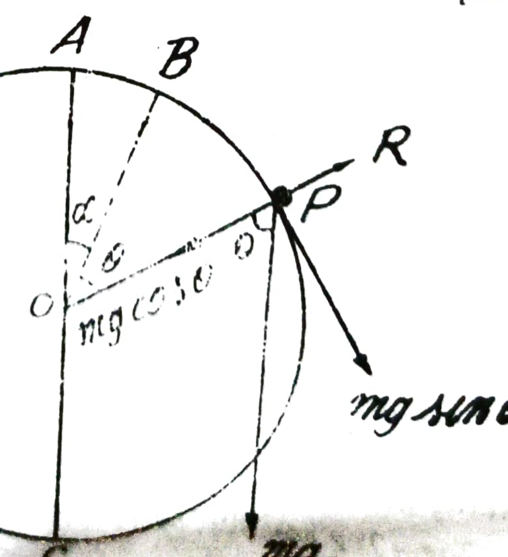
$\therefore$ from (4), we have

$$v_1^2=2ag\,(1-\cos\theta_1)=2ag\,(1-\tfrac{2}{3})=\tfrac{2}{3}ag.$$

The direction of the velocity v_1 is along the tangent to the circle at Q. Therefore the particle leaves the circle at Q with velocity $v_1=\sqrt{(\frac{2}{3}ag)}$ making an angle $\theta_1=\cos^{-1}(\frac{2}{3})$ below the horizontal line through Q. After leaving the circle at Q the particle will move freely under gravity and so it will describe a parabolic path.

Illustrative Examples

Ex. 20. *A particle is placed on the outside of a smooth vertical circle. If the particle starts from a point whose angular distance is α from the highest point of circle, show that it will fly off the curve when $\cos\theta=\frac{2}{3}\cos\alpha$.* [Rohilkhand 1988]



Sol. A particle slides down on the outside of the arc of a smooth vertical circle of radius a, starting from rest at a point B such that $\angle AOB=\alpha$. Let P be the position of the particle at any time t where arc $AP=s$ and $\angle POA=\theta$. The forces acting on the particle at P are : (i) weight mg acting vertically downwards and (ii) the reaction R along the outwards drawn normal OP.

If v be the velocity of the particle at P, the equations of motion of the particle along the tangent and normal are

$$m\frac{d^2s}{dt^2}=mg\sin\theta, \quad \ldots(1)$$

and
$$m\frac{v^2}{a}=mg\cos\theta-R. \quad \ldots(2)$$

Also
$$s=a\theta. \quad \ldots(3)$$

From (1) and (3), we have $a\dfrac{d^2\theta}{dt^2}=g\sin\theta.$

Multiplying both sides by $2a\,(d\theta/dt)$ and integrating, we have

$$v^2=\left(a\frac{d\theta}{dt}\right)^2=-2ag\cos\theta+A.$$

But initially at B, $\theta=\alpha$ and $v=0$. $\therefore$ $A=2ag\cos\alpha$.

$$\therefore\quad v^2=2ag\cos\alpha-2ag\cos\theta. \quad \ldots(4)$$

From (2) and (4), we have

$$R=\frac{m}{a}\left(-v^2+ag\cos\theta\right)=\frac{m}{a}\left(-2ag\cos\alpha+3ag\cos\theta\right)$$

$$=mg\left(-2\cos\alpha+3\cos\theta\right). \quad \ldots(5)$$

At the point where the particle flies off the circle, we have $R=0$.

$\therefore$ from (5), we have

$$0=mg\left(-2\cos\alpha+3\cos\theta\right)\quad\text{or}\quad\cos\theta=\tfrac{2}{3}\cos\alpha.$$

Ex. 21. *A particle is projected horizontally with a velocity $\sqrt{(ag/2)}$ from the highest point of the outside of a fixed smooth sphere of radius a. Show that it will leave the sphere at the point whose vertical distance below the point of projection is $a/6$.*

[Allahabad 1976]

Sol. Refer figure of § 5 on page 189.

Let a particle be projected horizontally with a velocity $\sqrt{(ag/2)}$ from the highest point A on the outside of a fixed smooth sphere of radius a. If P is the position of the particle at any time t such that $\angle AOP=\theta$ and arc $AP=s$, then the equations of motion along the tangent and normal are

$$m\frac{d^2s}{dt^2}=mg\sin\theta, \quad \ldots(1)$$

and $$m\frac{v^2}{a}=mg\cos\theta-R.$$ ...(2)

Here v is the velocity of the particle at P.

Also $s=a\theta$. ...(3)

From (1) and (3), we have $a\dfrac{d^2\theta}{dt^2}=g\sin\theta$.

Multiplying both sides by $2a\,(d\theta/dt)$ and integrating, we have

$$v^2=\left(a\frac{d\theta}{dt}\right)^2=-2ag\cos\theta+A.$$

But initially at A, $\theta=0$ and $v=\sqrt{(ag/2)}$.

$\therefore$ $ag/2=-2ag+A$ or $A=\frac{1}{2}ag+2ag=\frac{5}{2}ag$.

$\therefore$ $v^2=\frac{5}{2}ag-2ag\cos\theta$. ...(4)

From (2) and (4), we have

$$R=\frac{m}{a}(ag\cos\theta-v^2)=\frac{m}{a}(3ag\cos\theta-\tfrac{5}{2}ag)$$

or $$R=mg\,(3\cos\theta-\tfrac{5}{2}).$$ ...(5)

If the particle leaves the sphere at the point Q where $\theta=\theta_1$, then putting $R=0$ and $\theta=\theta_1$ in (5), we have

$$0=mg\,(3\cos\theta_1-\tfrac{5}{2})\quad\text{or}\quad\cos\theta_1=5/6.$$

Vertical depth of the point Q below the point of projection A

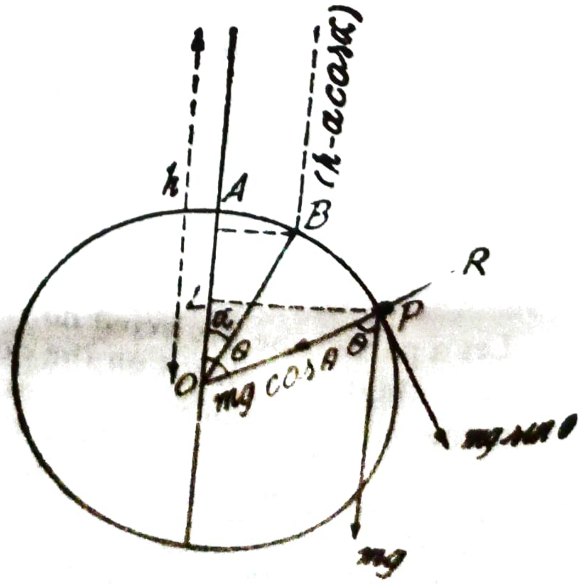
$$=AL=OA-OL=a-a\cos\theta_1=a-\tfrac{5}{6}a=\tfrac{1}{6}a.$$

Ex. 22. *A particle moves under gravity in a vertical circle sliding down the convex side of the smooth circular arc. If the initial velocity is that due to a fall to the starting point from a height h above the centre, show that it will fly off the circle when at a height $\frac{2}{3}h$ above the centre.* **[Gorakhpur 1981, Allahabad 87]**

Sol Let a particle start from the point B of a smooth vertical circle where $\angle AOB=\alpha$. The depth of the point B, from the point which is at a height h above the centre O, is

$$h-a\cos\alpha.$$

Therefore the initial velocity of the particle at B $=u=\sqrt{\{2g\,(h-a\cos\alpha)\}}$.



If P is the position of the particle at time t such that $\angle AOP=\theta$ and arc $AP=s$, the equations of motion along the tangent and normal are

$$m\frac{d^2s}{dt^2}=mg\sin\theta.$$

and
$$m\frac{v^2}{a}=mg\cos\theta-R. \qquad ...(2)$$

Also
$$s=a\theta. \qquad ...(3)$$

From (1) and (3), we have $a\dfrac{d^2\theta}{dt^2}=g\sin\theta.$

Multiplying both sides by $2a\,(d\theta/dt)$ and integrating, we have

$$v^2=\left(a\frac{d\theta}{dt}\right)^2=-2ag\cos\theta+A.$$

But initially at B, $\theta=\alpha$ and $v=\sqrt{\{2g\,(h-a\cos\alpha)\}}$.

$\therefore\quad 2g\,(h-a\cos\alpha)=-2ag\cos\alpha+A$ or $A=2gh.$

$$\therefore\quad v^2=-2ag\cos\theta+2gh. \qquad ...(4)$$

From (2) and (4), we have

$$R=\frac{m}{a}\,(ag\cos\theta-v^2)=\frac{m}{a}\,(3ag\cos\theta-2gh).$$

The particle will leave the sphere, where $R=0$ *i.e.*, where

$$\frac{m}{a}\,(3ag\cos\theta-2gh)=0 \quad\text{or}\quad \cos\theta=2h/3a.$$

Now the height of the point where the particle flies off the circle, above the centre $O=OL=a\cos\theta=2h/3$.

Ex. 23. *A particle is placed at the highest point of a smooth vertical circle of radius a and is allowed to slide down starting with a negligible velocity. Prove that it will leave the circle after describing vertically a distance equal to one third of the radius. Find the position of the directrix and the focus of the parabola subsequently described and show that its latus rectum is $\frac{16}{27}a$.*

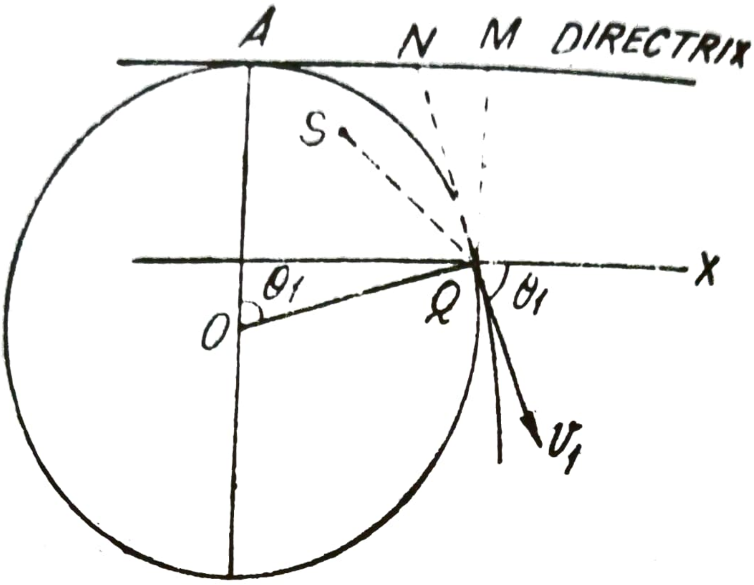
[Meerut 1976, 77, 78, 80, 81, 88P; Lucknow 76, 81; Agra 87]

Sol. For the first part see § 5 on page 189.

From § 5, the particle leaves the sphere at the point Q where $\angle AOQ=\theta_1$ and $\cos\theta_1=\frac{2}{3}$. The velocity v_1 at the point Q is $\sqrt{(2ag/3)}$; its direction is along the tangent to the circle at Q. After leaving the circle at the point Q, the particle describes a parabolic path with the velocity of projection $v_1=\sqrt{(2ag/3)}$ making an angle $\theta_1=\cos^{-1}(2/3)$ below the horizontal line through Q.

Latus rectum of the parabola subsequently described

$$=\frac{2v_1^2\cos^2\theta_1}{g}=\frac{2}{g}\cdot\frac{2ag}{3}\cdot\frac{4}{9}=\frac{16}{27}a.$$



To find the position of the directrix and the focus of the parabola. We know that in a parabolic path of a projectile the velocity at any point of its path is equal to that due to a fall from the directrix to that point.

Therefore if h is the height of the directrix above Q, then the velocity acquired in falling a distance h under gravity $=\sqrt{(2gh)}$.

$\therefore \quad v_1=\sqrt{(2ag/3)}=\sqrt{(2gh)}$

or $\quad h=a/3$ *i.e.*, the height of the directrix above Q is $a/3$.

Hence the directrix is the horizontal line through the highest point of the circle.

Let QM be the perpendicular from Q on the directrix and QN the tangent at Q. If S is the focus of the parabola subsequently described, we have by the geometrical properties of a parabola

$$QS=QM=a/3$$

and $\quad \angle SQN=\angle NQM.$

This gives the position of the focus S of the parabola.

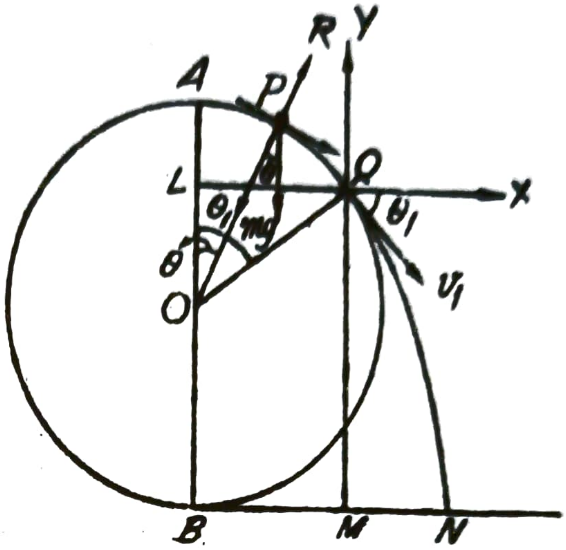
Ex. 24. *A heavy particle is allowed to slide down a smooth vertical circle of radius 27a from rest at the highest point. Show that on leaving the circle it moves in a parabola of latus rectum 16a.*

[Lucknow 1975; Kanpur 78, 80, 86]

Sol. Let us take the radius of the circle equal to b so that $b=27a$. Now proceed as in Ex. 23. We get

the latus rectum $=\frac{16b}{27}=\frac{16}{27}(27a)=16a.$

Ex 25. *A particle slides down the arc of a smooth vertical circle of radius a, being slightly displaced from rest at the highest point. Find where it will leave the circle and prove that it will strike a horizontal plane through the lowest point of the circle at a distance* $\frac{5}{27}(\sqrt{5}+4\sqrt{2})a$ *from the vertical diameter.*



Sol. Proceeding as in § 5, the particle leaves the circle at the point Q where $\angle AOQ=\theta_1$ and $\cos\theta_1=2/3$. The velocity v_1 of the particle at the point Q is $\sqrt{(2ag/3)}$ and is along the tangent to the circle at the point Q. After leaving the circle at the point Q the motion of the particle is that of a projectile and so it describes a parabolic path with the velocity of projection $v_1=\sqrt{(2ag/3)}$ making an angle $\theta_1=\cos^{-1}(2/3)$ below the horizontal line through Q.

Now the equation of the parabolic path of the particle w.r.t. the horizontal and vertical lines OX and OY as the coordinate axes is

$$y=x\tan(-\theta_1)-\frac{gx^2}{2v_1^2\cos^2(-\theta_1)}$$ [$\because$ for the motion of the projectile, the angle of projection $=-\theta_1$]

or $$y=-x\tan\theta_1-\frac{gx^2}{2v_1^2\cos^2\theta_1}$$

or $$y=-x\frac{\sqrt{5}}{2}-\frac{gx^2}{2.\frac{2}{3}ag.\frac{4}{9}}$$ [$\because$ $\cos\theta_1=\frac{2}{3}$ gives $\sin\theta_1=\sqrt{(1-\frac{4}{9})}=\sqrt{5}/3$ and $\tan\theta_1=\sqrt{5}/2$]

or $$y=-x\frac{\sqrt{5}}{2}-\frac{27}{16a}x^2. \qquad \ldots(1)$$

Let the particle strike the horizontal plane through the lowest point B at N. If (x_1, y_1) are the coordinates of the point N, then

$$x_1=MN \text{ and } y_1=-QM=-LB=-(LO+OB)$$
$$=-(a\cos\theta_1+a)=-(\tfrac{2}{3}a+a)=-5a/3.$$

The point $N(x_1, y_1)$ lies on the trajectory (1).

$\therefore$ $$y_1=-\frac{\sqrt{5}}{2}x_1-\frac{27}{16a}x_1^2$$

or $$\frac{-5a}{3}=-\frac{\sqrt{5}}{2}x_1-\frac{27}{16a}x_1^2$$

or $$81x_1^2+24\sqrt{5}ax_1-80a^2=0.$$

$$\therefore \quad x_1=\frac{-24\sqrt{5}a\pm\sqrt{(24\times 24\times 5a^2+4\times 81\times 80a^2)}}{2\times 81}$$

$$=\frac{-24\sqrt{5}a+120\sqrt{2}a}{162}$$ (leaving the $-$ive sign, since x_1 cannot be negative)

or $$x_1=MN=\frac{(-4\sqrt{5}+20\sqrt{2})a}{27}.$$

the required distance

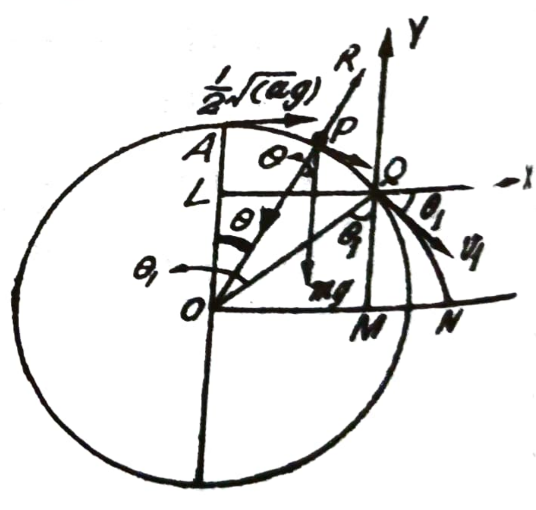
$$=BN=BM+MN=LQ+MN=a\sin\theta_1+MN$$

$$=a.\frac{\sqrt{5}}{3}+\frac{(-4\sqrt{5}+20\sqrt{2})a}{27} \qquad [\because \sin\theta_1=\sqrt{5}/3]$$

$$=\frac{5(\sqrt{5}+4\sqrt{2})a}{27}.$$

Ex. 26. *A body is projected, along the arc of a smooth circle of radius a and from the highest point with velocity* $\frac{1}{2}\sqrt{(ag)}$; *find where it will leave the circle and prove that it will strike a horizontal plane through the centre of the circle at a distance from the centre*

$$\frac{1}{64}\left[9\sqrt{(39)}+7\sqrt{7}\right]a.$$



Sol. Let a body be projected along the outside of a smooth vertical circle of radius a from the highest point A with velocity $\frac{1}{2}\sqrt{(ag)}$. If P is the position of the body at any time t, then the equations of motion of the body are $m\frac{d^2s}{dt^2}=mg\sin\theta,$ ...(1)

and $$m\frac{v^2}{a}=mg\cos\theta-R. \qquad ...(2)$$

Also $$s=a\theta. \qquad ...(3)$$

From (1) and (3), we have

$$a\frac{d^2\theta}{dt^2}=g\sin\theta.$$

Multiplying both sides by $2a\,(d\theta/dt)$ and integrating, we have

$$v^2=\left(a\frac{d\theta}{dt}\right)^2=-2ag\cos\theta+A.$$

But initially at A, $\theta=0$ and $v=\frac{1}{2}\sqrt{(ag)}$.

$\therefore$ $\frac{1}{4}ag=-2ag+A$ or $A=\frac{1}{4}ag+2ag=\frac{9}{4}ag.$

$\therefore$ $v^2=\frac{9}{4}ag-2ag\cos\theta=ag\left(\frac{9}{4}-2\cos\theta\right).$ ...(4)

From (2) and (4), we have

$$R=\frac{m}{a}(ag\cos\theta-v^2)=\frac{m}{a}\left(3ag\cos\theta-\tfrac{9}{4}ag\right)$$

$=3mg\left(\cos\theta-\frac{3}{4}\right).$ ...(5)

Suppose the body leaves the circle at the point Q, where $\theta=\theta_1$. Then putting $R=0$ and $\theta=\theta_1$ in (5), we have

$$0=3mg\left(\cos\theta_1-\tfrac{3}{4}\right) \quad \text{or} \quad \cos\theta_1=\tfrac{3}{4}.$$

If v_1 is the velocity of the body at Q, then from (4)

$$v_1^2=ag\left(\tfrac{9}{4}-2\cos\theta_1\right)=ag\left(\tfrac{9}{4}-\tfrac{3}{2}\right)=\tfrac{3}{4}ag.$$

Hence the body leaves the circle at the point Q with velocity $v_1=\frac{1}{2}\sqrt{(3ag)}$ at an angle $\theta_1=\cos^{-1}\left(\frac{3}{4}\right)$ below the horizontal line through Q and subsequently it describes a parabolic path. The equation of the parabolic trajectory of the body w.r.t. the horizontal and vertical lines QX and QY through Q as the coordinate axes is

$$y=x\tan(-\theta_1)-\frac{gx^2}{2v_1^2\cos^2(-\theta_1)}$$

or $$y=-x\tan\theta_1-\frac{gx^2}{2v_1^2\cos^2\theta_1}$$

or $$y=-x.\frac{\sqrt{7}}{3}-\frac{gx^2}{2.\frac{3}{4}ag.\frac{9}{16}}$$ $\left[\because\ \cos\theta_1=\frac{3}{4}\text{ gives }\sin\theta_1=\sqrt{(1-\frac{9}{16})}=\sqrt{7}/4 \text{ and } \tan\theta_1=\sqrt{7}/3\right]$

or $$y=-\frac{\sqrt{7}}{3}x-\frac{32}{27a}x^2.$$ ...(6)

Let the particle strike the horizontal plane through the centre O at N. If (x_1, y_1) are the coordinates of the point N, then

$x_1=MN$ and $y_1=-QM=-LO=-a\cos\theta_1=-\frac{3}{4}a.$

The point $N\,(x_1, y_1)$ lies on the trajectory (6).

$$\therefore\quad y_1=-\frac{\sqrt{7}}{3}x_1-\frac{33}{27a}x_1^2$$

or $$\frac{-3a}{4}=-\frac{\sqrt{7}}{3}x_1-\frac{32}{27a}x_1^2$$

or $$128x_1^2+36\sqrt{7}ax_1-81a^2=0.$$

$$\therefore \quad x_1 = \frac{-36\sqrt{7}a \pm \sqrt{(36 \times 36 \times 7a^2 + 4 \times 128 \times 81a^2)}}{2 \times 128}$$

$$= \frac{-36\sqrt{7}a + 36\sqrt{(39)}a}{256}$$ [neglecting the $-$ive sign because x_1 cannot be negative]

or $$x_1 = MN = \frac{9(\sqrt{39} - \sqrt{7})a}{64}.$$

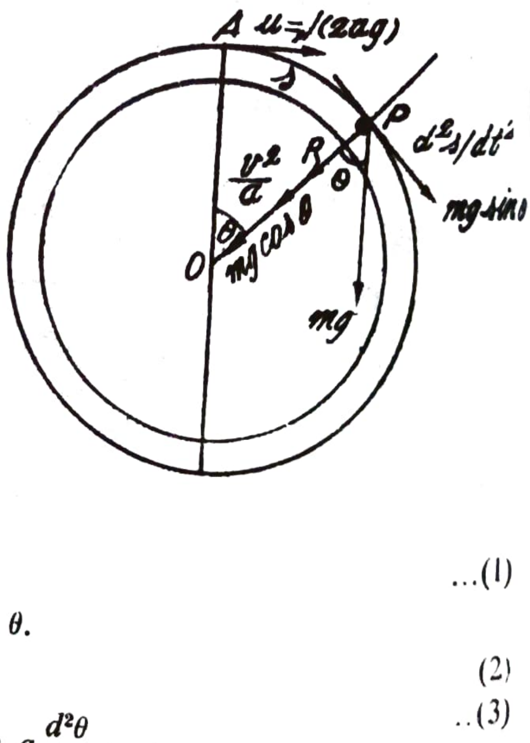
$\therefore$ the required distance $= ON = OM + MN = LQ + MN$

$= a \sin \theta_1 + MN$

$$= \frac{\sqrt{7}a}{4} + \frac{9(\sqrt{39} - \sqrt{7})a}{64} = \frac{1}{64}\left[9\sqrt{39} + 7\sqrt{7}\right]a.$$

Ex. 27. *A heavy particle slides under gravity down the inside of a smooth vertical tube held in a vertical plane. It starts from the highest point with velocity $\sqrt{(2ag)}$ where a is the radius of the circle. Prove that when in the subsequent motion the vertical component of the acceleration is maximum, the pressure on the curve is equal to twice the weight of the particle.*

[Gorakhpur 1978; Meerut 85]



Sol. Let P be the position of the particle at any time t such that $\angle AOP = \theta$ and arc $AP = s$. The forces acting on the particle at P are (i) weight mg acting vertically downwards and (ii) the reaction R along PO.

$\therefore$ the equations of motion of the particle are

$$m\frac{d^2s}{dt^2} = mg \sin\theta, \qquad \text{...(1)}$$

and $$m\frac{v^2}{a} = R + mg\cos\theta. \qquad \text{(2)}$$

Also $$s = a\theta. \qquad \text{...(3)}$$

From (1) and (3), we have $a\dfrac{d^2\theta}{dt^2} = g\sin\theta.$

Multiplying both sides by $2a(d\theta/dt)$ and integrating, we have

$$v^2=\left(a\frac{d\theta}{dt}\right)^2=-2ag\cos\theta+A.$$

But initially at A, $\theta=0$ and $v=\sqrt{(2ag)}$.

$\therefore\quad A=2ag+2ag=4ag.$

$\therefore\quad v^2=4ag-2ag\cos\theta.$ ...(4)

From (2) and (4), we have

$$R=\frac{m}{a}(v^2-ag\cos\theta)$$

or

$$R=mg\,(4-3\cos\theta).\qquad ...(5)$$

Now $\dfrac{d^2s}{dt^2}$ and $\dfrac{v^2}{a}$ are the accelerations at the point P along the tangent and inward drawn normal at P. Let f be the vertical component of acceleration at P. Then

$$f=\frac{d^2s}{dt^2}\sin\theta+\frac{v^2}{a}\cos\theta.$$

Substituting from (1) and (4), we have

$$f=g\sin\theta.\sin\theta+\frac{1}{a}(4ag-2ag\cos\theta)\cos\theta$$

$$=g\,(\sin^2\theta+4\cos\theta-2\cos^2\theta).$$

$$\therefore\quad \frac{df}{d\theta}=g\,(2\sin\theta\cos\theta-4\sin\theta+4\cos\theta\sin\theta)$$

$$=2g\sin\theta\,(3\cos\theta-2)$$

and

$$\frac{d^2f}{d\theta^2}=g\,[6\,(\cos^2\theta-\sin^2\theta)-4\cos\theta]$$

$$=g\,[6\,(2\cos^2\theta-1)-4\cos\theta].$$

For a maximum or a minimum of f, we have

$df/d\theta=0\quad$ *i.e.*, $\quad 2g\sin\theta\,(3\cos\theta-2)=0.$

$\therefore$ either $\sin\theta=0$ giving $\theta=0$

or $\quad 3\cos\theta-2=0$ giving $\cos\theta=\frac{2}{3}$.

But $\theta=0$ corresponds to the initial position A.

When $\cos\theta=\frac{2}{3}$, $\dfrac{d^2f}{d\theta^2}=g\,[6\,(2.\frac{4}{9}-1)-4.\frac{2}{3}]=-\frac{10}{3}g=-$ive.

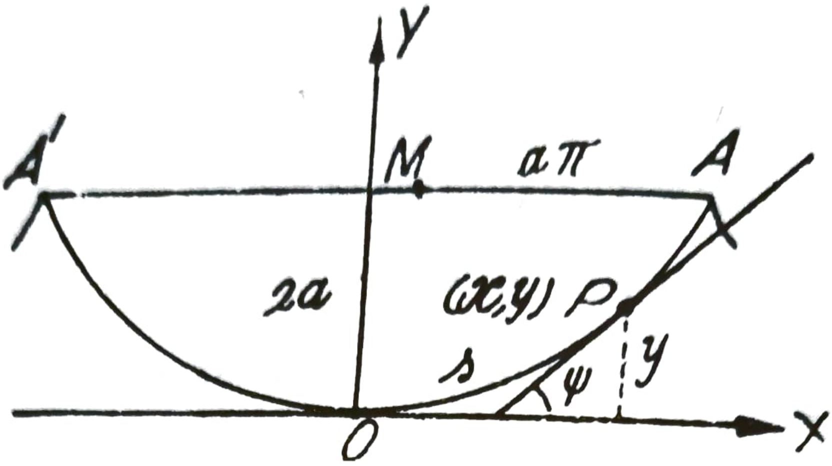
$\therefore$ f is maximum when $\cos\theta=\frac{2}{3}$.

Putting $\cos\theta=2/3$ in (5) the pressure on the curve is given by

$$R=mg\,(4-3.\tfrac{2}{3})=2mg=2.\ \text{(weight of the particle)}.$$

Cycloidal Motion

§ 6. Cycloid. *A cycloid is a curve which is traced out by a point on the circumference of a circle as the circle rolls along a fixed straight line.*



In the adjoining figure we have shown an inverted cycloid. The point O is called the vertex of the cycloid. The points A and A' are the cusps and straight line OY is the axis of the cycloid. The line AA' is called the base of the cycloid.

Let $P(x, y)$ be the coordinates of a point on the cycloid w.r.t. OX and OY as coordinate axes and ψ the angle which the tangent at P makes with OX. Then remember the following results :

(i) Parametric equations of the cycloid are given by

$$x=a(\theta+\sin\theta),\ y=a(1-\cos\theta),$$

where θ is the parameter and we have $\theta=2\psi$.

(ii) The intrinsic equation of cycloid is

$$s=4a\sin\psi, \text{ where arc } OP=s.$$

(iii) Arc $OA=4a$ and the height of the cycloid $=OM=2a$.

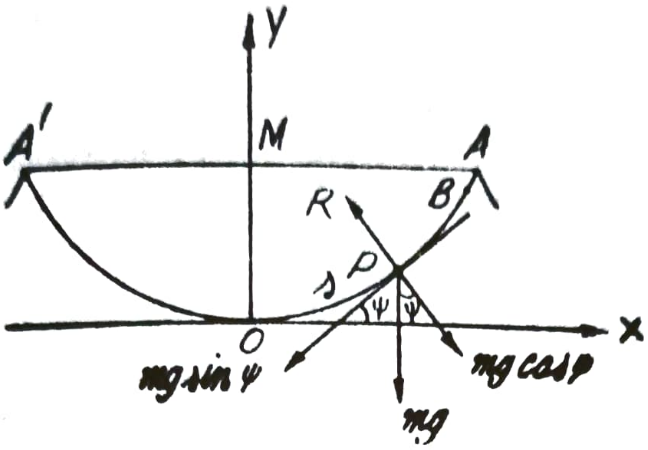
At the point O, $\psi=0$ and $s=0$ while at the cusp A, $\psi=\pi/2$ and $s=4a$.

(vi) For the above cycloid, the relation between s and y is

$$s^2=8ay.$$

§ 7. Motion on a cycloid. *A particle slides down the arc of a smooth cycloid whose axis is vertical and vertex downwards. To determine the motion.* **[Meerut 1974, 77, 88S; Rohilkhand 81, 88; Agra 76, 85; Kanpur 75, 76, 78; Lucknow 78; Gorakhpur 80; Allahabad 78]**

Let O be the vertex of a smooth cycloid and OM its axis. Suppose a particle of mass m slides down the arc of the cycloid starting at rest from a point B where arc $OB=b$. Let P be the position of the particle at any time t where arc $OP=s$ and ψ be the angle which the tangent at P to the cycloid makes with the



tangent at the vertex O. The forces acting on the particle at P are : (i) the weight mg acting vertically downwards and (ii) the normal reaction R acting along the inwards drawn normal at P. Resolving these forces along the tangent and normal at P, the tangential and normal equations of motion of P are

$$m\frac{d^2s}{dt^2}=-mg\sin\psi, \qquad \text{..(1)}$$

and $$m\frac{v^2}{\rho}=R-mg\cos\psi. \qquad \text{...(2)}$$

Here v is the velocity of the particle at P and is along the tangent at P.

[Note that the expression for the tangential acceleration is d^2s/dt^2 and it is positive in the direction of s increasing. In the equation (1) negative sign has been taken because $mg\sin\psi$ acts in the direction of s decreasing. Again the expression for normal acceleration is v^2/ρ and it is positive in the direction of inwards drawn normal. In the equation (2) we have taken R with +ive sign because it is in the direction of inwards drawn normal while negative sign has been fixed before $mg\cos\psi$ because it is in the direction of outwards drawn normal].

Now the intrinsic equation of the cycloid is

$$s=4a\sin\psi. \qquad \text{...(3)}$$

From (1) and (3), we have

$$\frac{d^2s}{dt^2}=-\frac{g}{4a}s, \qquad \text{..(4)}$$

which is the equation of a simple harmonic motion with centre at the points $s=0$ *i.e.*, at the point O. Thus the particle will oscillate in S.H.M. about the centre O. The time period T of this S.H.M. is given by

$$T=\frac{2\pi}{\sqrt{(g/4a)}}=\pi\sqrt{(a/g)},$$

which is independent of the amplitude (*i.e.*, the initial displacement b). *Thus from whatever point the particle may be allowed to slide down the arc of a smooth cycloid, the time period remains the same.* Such a motion is called *isochronous* motion.

[Meerut 1977; Rohilkhand 88; Agra 80]

Multiplying both sides of (4) by 2 (ds/dt) and then integrating w.r.t. 't', we get

$$v^2=\left(\frac{ds}{dt}\right)^2=-\frac{g}{4a}s^2+A.$$

But initially at the point B, $s=0$ and $v=0$.

Therefore $0=-(g/4a)\,b^2+A$ or $A=(g/4a)\,b^2$.

$$\therefore \quad v^2=\left(\frac{ds}{dt}\right)^2=-\frac{g}{4a}s^2+\frac{g}{4a}b^2=\frac{g}{4a}(b^2-s^2), \qquad \ldots(5)$$

which gives us the velocity of the particle at any position 's'. Substituting the value of v^2 in (2), we get R which gives us the pressure at any point on the cycloid.

Taking square root of (5), we get

$$\frac{ds}{dt}=-\sqrt{\left(\frac{g}{4a}\right)}\sqrt{(b^2-s^2)},$$

where the $-$ive sign has been taken because the particle is moving in the direction of s decreasing.

Separating the variables, we get

$$-\frac{ds}{\sqrt{(b^2-s^2)}}=\sqrt{\left(\frac{g}{4a}\right)}\,dt. \qquad \ldots(6)$$

Integrating, we have

$$\cos^{-1}(s/b)=\sqrt{(g/4a)}\,t+C.$$

But initially at B, $s=b$ and $t=0$. Therefore $\cos^{-1}1=0+C$ or $C=0$.

$$\therefore \quad \cos^{-1}(s/b)=\sqrt{(g/4a)}\,t,$$

or

$$s=b\cos\sqrt{(g/4a)}\,t,$$

which gives a relation between s and t.

If t_1 be the time from B to O, then integrating (6) from B to O, we have

$$-\int_b^0\frac{ds}{\sqrt{(b^2-s^2)}}=\int_0^{t_1}\sqrt{\left(\frac{g}{4a}\right)}\,dt$$

[Note that at B, $s=b$ and $t=0$ while at O, $s=0$ and $t=t_1$]

or $$\left[\cos^{-1}\frac{s}{b}\right]_b^0=\sqrt{\left(\frac{g}{4a}\right)}\Big[\,t\,\Big]_0^{t_1}$$

or $$\cos^{-1}0-\cos^{-1}1=\sqrt{\left(\frac{g}{4a}\right)}\,t_1$$

or $$\frac{\pi}{2}=\sqrt{\left(\frac{g}{4a}\right)}\,t_1$$

or $\qquad t_1=\pi\sqrt{(a/g)}$.

Thus time t_1 is independent of the initial displacement b of the particle. *Thus on a smooth cycloid the time of descent to the vertex is independent of the initial displacement of the particle.*

If T is time period of the particle *i.e.*, if T is the time for one complete oscillation, we have

$$T=4\times\text{time from } B \text{ to } O=4t_1=4\pi\sqrt{(a/g)}.$$

Illustrative Examples

Ex. 28. *A particle slides down a smooth cycloid whose axis is vertical and vertex downwards, strating from rest at the cusp. Find the velocity of the particle and the reaction on it at any point of the cycloid.* [Meerut 1975, 79]

Sol. Refer figure of § 7, on page 201.

Here the particle starts at rest from the cusp A.

The equations of motion of the particle along the tangent and normal are

$$m\frac{d^2s}{dt^2}=-mg\sin\psi \qquad \text{...(1)}$$

and $$m\frac{v^2}{\rho}=R-mg\cos\psi. \qquad \text{...(2)}$$

For the cycloid, $s=4a\sin\psi$. ..(3)

From (1) and (3), we have

$$\frac{d^2s}{dt^2}=-\frac{g}{4a}s.$$

Multiplying both sides by $2\dfrac{ds}{dt}$ and integrating, we have

$$v^2=\left(\frac{ds}{dt}\right)^2=-\frac{g}{4a}s^2+A.$$

But initially at the cusp A, $s=4a$ and $v=0$.

$$\therefore\quad A=\frac{g}{4a}\cdot(4a)^2=4ag.$$

$$\therefore\quad v^2=-\frac{g}{4a}s^2+4ag=-\frac{g}{4a}(4a\sin\psi)^2+4ag$$

$$=4ag\,(1-\sin^2\psi)$$

or $$v^2=4ag\cos^2\psi. \qquad \text{...(4)}$$

Differentiating (3), $\rho=ds/d\psi=4a\cos\psi$.

Substituting for v^2 and ρ in (2), we have

$$R=m\frac{v^2}{\rho}+mg\cos\psi=m.\frac{4ag\cos^2\psi}{4a\cos\psi}+mg\cos\psi$$

or $$R=2mg\cos\psi. \qquad \text{...(5)}$$

The equations (4) and (5) give the velocity and the reaction at any point of the cycloid.

Ex. 29. *A particle oscillates from cusp to cusp of a smooth cycloid whose axis is vertical and vertex lowest. Show that the velocity v at any point P is equal to the resolved part of the velocity V at the vertex along the tangent at P i.e., $v=V\cos\psi$.*

[Meerut 1975, 81, 82P, 82S; Rohilkhand 78, 86; Allahabad 78]

Sol. Proceed as in Ex. 28.

The velocity v of the particle at any point P of the cycloid is given by $\quad v=2\sqrt{(ag)}\cos\psi.$ [From equation (4)]

If V is the velocity of the particle at the vertex, where $\psi=0$, then $\quad V=2\sqrt{(ag)}\cos 0=2\sqrt{(ag)}.$

$\therefore\quad v=V\cos\psi=$ the resolved part of V along the tangent at P. Hence the velocity v at any point P is equal to the resolved part of the velocity V at the vertex along the tangent at P.

Ex. 30. *A heavy particle slides down a smooth cycloid starting from rest at the cusp, the axis being vertical and vertex downwards, prove that the magnitude of the acceleration is equal to g at every point of the path and the pressure when the particle arrives at the vertex is equal to twice the weight of the particle.*

[Meerut 1974, 75, 84S, 87, 90S; Agra 85, 87, 88; Lucknow 77; Gorakhpur 76; Kanpur 79, 86]

Sol. Refer figure of § 7 on page 201.

Here the particle starts at rest from the cusp A.

The equations of motion of the particle are

$$m\frac{d^2s}{dt^2}=-mg\sin\psi,$$

and

$$m\frac{v^2}{\rho}=R-mg\cos\psi. \qquad \ldots(1)$$

$\ldots(2)$

For the cycloid, $\quad s=4a\sin\psi. \qquad \ldots(3)$

From (1) and (3), we have $\dfrac{d^2s}{dt^2}=-\dfrac{g}{4a}s.$

Multiplying both sides by $2\,(ds/dt)$ and integrating, we have

$$v^2=\left(\frac{ds}{dt}\right)^2=-\frac{g}{4a}s^2+A.$$

But initially at the cusp A, $s=4a$ and $v=0$. $\quad\therefore\quad A=4ag.$

$$\therefore\quad v^2=-\frac{g}{4a}s^2+4ag=-\frac{g}{4a}(4a\sin\psi)^2+4ag=4ag\,(1-\sin^2\psi)$$

or $\qquad v^2=4ag\cos^2\psi$. ...(4)

Differentiating (3),

$$\rho=ds/d\psi=4a\cos\psi.$$

Now at the point P, tangential acceleration

$$=d^2s/dt^2=-g\sin\psi \qquad \text{[from (1)]}$$

and normal acceleration $=\dfrac{v^2}{\rho}=\dfrac{4ag\cos^2\psi}{4a\cos\psi}=g\cos\psi$.

$\therefore$ the resultant acceleration at any point P

$$=\sqrt{[(\text{tang. accel.})^2+(\text{normal accel.})^2]}$$
$$=\sqrt{[(-g\sin\psi)^2+(g\cos\psi)^2]}=g.$$

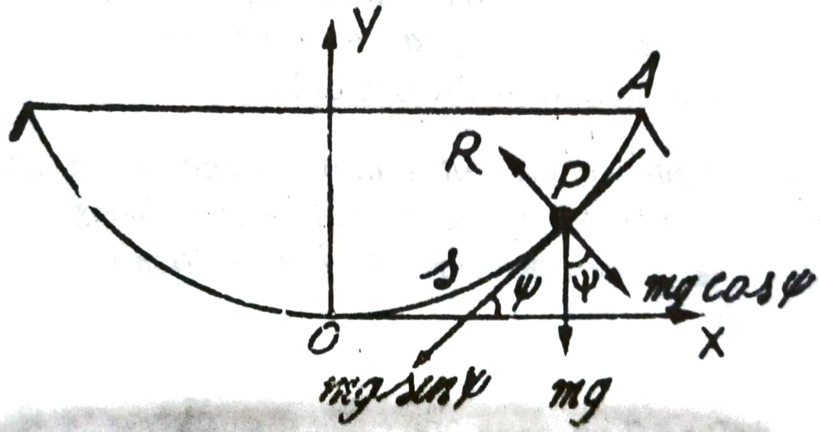
From (2) and (4), we have

$$R=m.\frac{4ag\cos^2\psi}{4a\cos\psi}+mg\cos\psi=2mg\cos\psi. \qquad ...(5)$$

At the vertex O, $\psi=0$. Therefore putting $\psi=0$ in (5), the pressure at the vertex $=2mg=$ twice the weight of the particle.

Ex. 31. *Prove that for a particle, sliding down the arc and starting from the cusp of a smooth cycloid whose vertex is lowest, the vertical velocity is maximum when it has described half the vertical height.*

[Meerut 1972, 88, 90; Agra 78; Kanpur 80, 85, 87; Allahabad 87]



Sol. Let a particle of mass m slide down the arc of a cycloid starting at rest from the cusp A. If P is the position of the particle at any time t, then the equations of motion of the particle along the tangent and normal are

$$m\frac{d^2s}{dt^2}=-mg\sin\psi \qquad ...(1)$$

and
$$m\frac{v^2}{\rho}=R-mg\cos\psi. \qquad ...(2)$$

For the cycloid, $\qquad s=4a\sin\psi$. ...(3)

From (1) and (3), we have $\dfrac{d^2s}{dt^2}=-\dfrac{g}{4a}s$.

Multipl ing both sides by 2 (ds/dt) and integrating, we have

$$v^2=\left(\frac{ds}{dt}\right)^2=-\frac{g}{4a}s^2+A.$$

But initially at the cusp A, $s=4a$ and $v=0$. $\therefore$ $A=4ag$.

$$\therefore \quad v^2=4ag-\frac{g}{4a}s^2=4ag-\frac{g}{4a}(4a\sin\psi)^2=4ag(1-\sin^2\psi)$$

$$=4ag\cos^2\psi$$

or $\qquad v=2\sqrt{(ag)}\cos\psi$, giving the velocity of the particle at the point P its direction being along the tangent at P. Let V be the vertical component of the velocity v at the point P. Then

$$V=v\cos(90°-\psi)=v\sin\psi=2\sqrt{(ag)}\cos\psi.\sin\psi$$

or $\qquad V=\sqrt{(ag)}\sin 2\psi$,

which is maximum when $\sin 2\psi=1$ *i.e.*, $2\psi=\pi/2$ *i.e.*, $\psi=\pi/4$.

When $\quad \psi=\pi/4, \quad s=4a\sin(\pi/4)=2\sqrt{2}a$.

Putting $s=2\sqrt{2}a$ in the relation $s^2=8ay$, we have

$$(2\sqrt{2}a)^2=8ay \quad \text{or} \quad y=8a^2/8a=a.$$

Thus at the point where the vertical velocity is maximum, we have $y=a$. The vertical depth fallen upto this point
$=$(the y-coordinate of A)$-a=2a-a=a=\frac{1}{2}(2a)$
$=$half the vertical height of the cycloid.

Ex. 32. *A particle oscillates in a cycloid ander gravity, the amplitude of the motion being b, and period being T. Show that its velocity at any time t measured from a position of rest is*

$$\frac{2\pi b}{T}\sin\left(\frac{2\pi t}{T}\right).$$

[Meerut 1977]

Sol. Refer § 7 on page 200.

The equations of motion of the particle are

$$m\frac{d^2s}{dt^2}=-mg\sin\psi \qquad \ldots(1)$$

and $$m\frac{v^2}{\rho}=R-mg\cos\psi. \qquad \ldots(2)$$

For the cycloid, $s=4a\sin\psi$. $\qquad \ldots(3)$

From (1) and (3), we have $\dfrac{d^2s}{dt^2}=-\dfrac{g}{4a}s$,

which represents a S. H. M. $\qquad \ldots(4)$

$\therefore$ the time period T of the particle is given by $T=2\pi/\sqrt{(g/4a)}$

$$T=4\pi\sqrt{(a/g)}. \quad ...(5)$$

or

Multiplying both sides of (4) by $2\frac{ds}{dt}$ and integrating, we have

$$v^2=\left(\frac{ds}{dt}\right)^2=-\frac{g}{4a}s^2+A. \quad ...(6)$$

But the amplitude of the motion is b. So the arcual distance of a position of rest from the vertex O is b *i.e.*, $v=0$ when $s=b$.

$\therefore$ from (6), we have

$$A=\frac{g}{4a}b^2.$$

Substituting this value of A in (6), we have

$$v^2=\left(\frac{ds}{dt}\right)^2=\frac{g}{4a}(b^2-s^2). \quad ...(7)$$

$$\because \quad \frac{ds}{dt}=-\frac{1}{2}\sqrt{\left(\frac{g}{a}\right)}\sqrt{(b^2-s^2)}$$

($-$ive sign is taken because the particle is moving in the direction of s decreasing)

or $$dt=-2\sqrt{(a/g)}\frac{ds}{\sqrt{(b^2-s^2)}}.$$

Integrating, $t=2\sqrt{(a/g)}.\cos^{-1}(s/b)+B.$
But $t=0$ when $s=b$. $\therefore$ $B=0$.
$\therefore$ $t=2\sqrt{(a/g)}\cos^{-1}(s/b)$

or $$s=b\cos\left\{\frac{t}{2}\sqrt{\left(\frac{g}{a}\right)}\right\}.$$

Substituting this value of s in (7), we have

$$v^2=\frac{g}{4a}\left[b^2-b^2\cos^2\left\{\frac{t}{2}\sqrt{(g/a)}\right\}\right]$$

$$=\frac{g}{4a}b^2\sin^2\left\{\frac{t}{2}\sqrt{(g/a)}\right\}$$

or $$v=\frac{b}{2}\sqrt{(g/a)}\sin\left\{\frac{t}{2}\sqrt{(g/a)}\right\}.$$

From (5), $\sqrt{(g/a)}=\frac{4\pi}{T}.$

$\therefore$ the velocity of the particle at any time t measured from the position of rest is given by

$$v=\frac{b}{2}\cdot\frac{4\pi}{T}\sin\left(\frac{t}{2}\cdot\frac{4\pi}{T}\right)=\left(\frac{2\pi b}{T}\right)\sin\left(\frac{2\pi t}{T}\right).$$

Ex. 33. *A particle starts from rest at the cusp of a smooth cycloid whose axis is vertical and vertex downwards. Prove that* *when it has fallen through half the distance measured along the arc to the vertex, two-thirds of the time of descent will have elapsed.*

[Meerut 1976, 83P; Rohilkhand 78; Agra 77, 79; Gorakhpur 77, 79, 81; Kanpur 88]

Sol. Refer figure of § 7 on page 201.

Let a particle of mass m start from rest from the cusp A of the cycloid. If P is the position of the particle after time t such that arc $OP=s$, the equations of motion along the tangent and normal are

$$m\frac{d^2s}{dt^2}=-mg\sin\psi, \quad \ldots(1)$$

and

$$m\frac{v^2}{\rho}=R-mg\cos\psi. \quad \ldots(2)$$

For the cycloid, $s=4a\sin\psi$. $\quad \ldots(3)$

From (1) and (3), we have $\dfrac{d^2s}{dt^2}=-\dfrac{g}{4a}s.$

Multiplying both sides by $2(ds/dt)$ and then integrating, we have

$$\left(\frac{ds}{dt}\right)^2=-\frac{g}{4a}s^2+A.$$

Initially at the cusp A, $s=4a$ and $\dfrac{ds}{dt}=0$.

$$\therefore \quad A=\frac{g}{4a}\cdot(4a)^2=4ag.$$

$$\therefore \quad \left(\frac{ds}{dt}\right)^2=-\frac{g}{4a}s^2+4ag=\frac{g}{4a}(16a^2-s^2)$$

or

$$ds/dt=-\tfrac{1}{2}\sqrt{(g/a)}\cdot\sqrt{(16a^2-s^2)}, \quad \ldots(4)$$

the $-$ive sign is taken because the particle is moving in the direction of s decreasing.

Separating the variables, we have

$$dt=-2\sqrt{(a/g)}\cdot\frac{ds}{\sqrt{(16a^2-s^2)}}. \quad \ldots(5)$$

If t_1 is the time from the cusp A (*i.e.*, $s=4a$) to the vertex O (*i.e.*, $s=0$), then integrating (5)

$$t_1=-2\sqrt{(a/g)}\cdot\int_{4a}^{0}\frac{ds}{\sqrt{(16a^2-s^2)}}$$

$$=2\sqrt{(a/g)}\left[\cos^{-1}\frac{s}{4a}\right]_{4a}^{0}=2\sqrt{(a/g)}\,\frac{\pi}{2}=\pi\sqrt{(a/g)}.$$

Again if t_2 is the time taken to move from the cusp A (*i.e.*, $s=4a$) to half the distance along the arc to the vertex *i.e.*, to $s=2a$, then integrating (5)

$$t_2=-2\sqrt{(a/g)}\int_{s=4a}^{2a}\frac{ds}{\sqrt{(16a^2-s^2)}}$$

$$=2\sqrt{(a/g)}\cdot\left[\cos^{-1}\frac{s}{4a}\right]_{4a}^{2a}$$

$$=2\sqrt{(a/g)}\,[\cos^{-1}\tfrac{1}{2}-\cos^{-1}1]=2\sqrt{(a/g)}.(\pi/3)=(2/3)\,t_1.$$

Ex. 34. *A particle slides down the arc of a smooth cycloid whose axis is vertical and vertex lowest, starting at rest from the cusp. Prove that the time occupied in falling down the first half of the vertical height is equal to the time of falling down the second half.* **[Meerut 1976, 83, 85S, 87P, 88P; Agra 76, 78; Lucknow 78, 80; Kanpur 79, 80, 85, 87]**

Sol. Let a particle start from rest from the cusp A of the cycloid. Proceeding as in the last example the velocity v of the particle at any point P, at time t, is given by

$$v^2=\left(\frac{ds}{dt}\right)^2=\frac{g}{4a}(16a^2-s^2),$$ [Refer equation (4) of the last example]

or $\dfrac{ds}{dt}=-\tfrac{1}{2}\sqrt{(g/a)}\,\sqrt{(16a^2-s^2)}$, the $-$ive sign is taken because the particle is moving in the direction of s decreasing.

$$\therefore\ dt=-2\sqrt{(a/g)}\frac{ds}{\sqrt{(16a^2-s^2)}}. \quad ...(1)$$

The vertical height of the cycloid is $2a$. At the point where the particle has fallen down the first half of the vertical height of the cycloid, we have $y=a$. Putting $y=a$ in the equation $s^2=8ay$, we get $s^2=8a^2$ or $s=2\sqrt{2}a$.

$\therefore$ integrating (1) from $s=4a$ to $s=2\sqrt{2}a$, the time t_1 taken in falling down the first half of the vertical height of the cycloid is given by

$$t_1=-2\sqrt{(a/g)}\int_{s=4a}^{2\sqrt{2}a}\frac{ds}{\sqrt{(16a^2-s^2)}}=2\sqrt{(a/g)}\left[\cos^{-1}(s/4a)\right]_{4a}^{2\sqrt{2}a}$$

$$=2\sqrt{(a/g)}\left[\cos^{-1}\frac{2\sqrt{2}a}{4a}-\cos^{-1}1\right]=2\sqrt{(a/g)}\left[\cos^{-1}\frac{1}{\sqrt{2}}-\cos^{-1}1\right]$$

$$=2\sqrt{(a/g)}\,[\tfrac{1}{4}\pi-0]=\tfrac{1}{2}\pi\sqrt{(a/g)}.$$

Again integrating (1) from $s=2\sqrt{2}a$ to $s=0$, the time t_2 taken in falling down the second half of the vertical height of the cycloid is given by

$$t_2=-2\sqrt{(a/g)}\int_{s=2\sqrt{2}a}^{0}\frac{ds}{\sqrt{(16a^2-s^2)}}$$

$$=2\sqrt{(a/g)}.\left[\cos^{-1}\left(\frac{s}{4a}\right)\right]_{2\sqrt{2}a}^{0}=2\sqrt{(a/g)}\left[\cos^{-1}0-\cos^{-1}\frac{1}{\sqrt{2}}\right]$$

$$=2\sqrt{(a/g)}\left[\tfrac{1}{2}\pi-\tfrac{1}{4}\pi\right]=\tfrac{1}{2}\pi\sqrt{(a/g)}.$$

Hence $t_1=t_2$ *i.e.*, the time occupied in falling down the first half of the vertical height is equal to the time of falling down the second half.

Ex. 35. *A particle is projected with velocity V from the cusp of a smooth inverted cycloid down the arc, show that the time of reaching the vertex is $2\sqrt{(a/g)}\tan^{-1}[\sqrt{(4ag)}/V]$.*

[Meerut 1971. 78, 81, 84, 85, 90P; Rohilkhand 80; Gorakhpur 76; Allahabad 76; Agra 86]

Sol. Refer figure of § 7 on page 201.

Let a particle be projected with velocity V from the cusp A of a smooth inverted cycloid down the arc. If P is the position of the particle at time t such that the tangent at P is inclined at an angle ψ to the horizontal and arc $OP=s$, then the equations of motion of the particle are

$$m\frac{d^2s}{dt^2}=-mg\sin\psi \quad ...(1)$$

and
$$m\frac{v^2}{\rho}=R-mg\cos\psi \quad ...(2)$$

For the cycloid, $s=4a\sin\psi$. ...(3)

From (1) and (3), we have $\dfrac{d^2s}{dt^2}=-\dfrac{g}{4a}s.$

Multiplying both sides by $2(ds/dt)$ and integrating, we have

$$v^2=\left(\frac{ds}{dt}\right)^2=-\frac{g}{4a}s^2+A.$$

But initially at the cusp A, $s=4a$ and $(ds/dt)^2=V^2$.

$\therefore$ $V^2=-(g/4a).16a^2+A$ or $A=V^2+4ag$.

$$\therefore\ v^2=\left(\frac{ds}{dt}\right)^2=V^2+4ag-\frac{g}{4a}s^2=\left(\frac{g}{4a}\right)\left[\frac{4a}{g}(V^2+4ag)-s^2\right]$$

or
$$\frac{ds}{dt}=-\tfrac{1}{2}\sqrt{(g/a)}\sqrt{\left[\frac{4a}{g}(V^2+4ag)-s^2\right]}$$

(–ive sign is taken because the particle is moving in the direction of s decreasing) or $$dt = -2\sqrt{(a/g)} \cdot \frac{ds}{\sqrt{[(4a/g)(V^2+4ag)-s^2]}}.$$

Integrating, the time t_1 from the cusp A to the vertex O is given by

$$t_1 = -2\sqrt{(a/g)} \int_{s=4a}^{0} \frac{ds}{\sqrt{[(4a/g)(V^2+4ag)-s^2]}}$$

$$=2\sqrt{(a/g)} \int_0^{4a} \frac{ds}{\sqrt{[(4a/g)(V^2+4ag)-s^2]}}$$

$$=2\sqrt{(a/g)} \left[\sin^{-1} \frac{s}{2\sqrt{(a/g)}\sqrt{(V^2+4ag)}}\right]_0^{4a}$$

$$=2\sqrt{(a/g)}.\sin^{-1}\left\{\frac{2\sqrt{(ag)}}{\sqrt{(V^2+4ag)}}\right\}$$

$$=2\sqrt{(a/g)}.\theta, \qquad \ldots(4)$$

where $\theta = \sin^{-1}\left\{\frac{2\sqrt{(ag)}}{\sqrt{(V^2+4ag)}}\right\}.$

We have $\sin\theta = \dfrac{2\sqrt{(ag)}}{\sqrt{(V^2+4ag)}}.$

$$\therefore \cos\theta = \sqrt{(1-\sin^2\theta)} = \sqrt{\left[1-\frac{4ag}{V^2+4ag}\right]} = \frac{V}{\sqrt{(V^2+4ag)}}.$$

$$\therefore \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{2\sqrt{(ag)}}{V} = \frac{\sqrt{(4ag)}}{V}$$

or $$\theta = \tan^{-1}[\sqrt{(4ag)}/V].$$

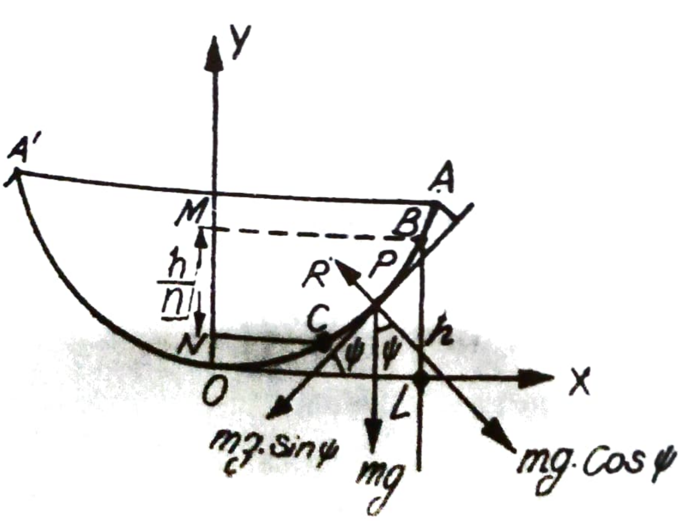
$\therefore$ from (4), the time of reaching the vertex is

$$=2\sqrt{(a/g)}.\tan^{-1}[\sqrt{(4ag)}/V].$$

Ex. 36 (a). *If a particle starts from rest at a given point of a cycloid with its axis vertical and vertex downwards, prove that it falls $1/n$ of the vertical distance to the lowest point in time*

$$2\sqrt{(a/g)}.\sin^{-1}(1/\sqrt{n}),$$

where a is the radius of the generating circle. **[Rohilkhand 1977]**



Sol. Let a particle start from rest at a given point B of a cycloid with its axis vertical and vertex downwards. Let h be the vertical height of the point B above the vertex O.

If arc $OB=s_1$, then from $s^2=8ay$, we have $s_1^2=8ah$.

If P is the position of the particle at time t such that the tangent at P is inclined at an angle ψ to the horizontal and arc $OP=s$, then the equations of motion along the tangent and normal at P are

$$m\frac{d^2s}{dt^2}=-mg\sin\psi \quad ...(1)$$

and $$m\frac{v^2}{\rho}=R-mg\cos\psi. \quad ...(2)$$

For the cycloid, $s=4a\sin\psi$. ...(3)

From (1) and (3), we have $\dfrac{d^2s}{dt^2}=-\dfrac{g}{4a}s.$

Multiplying both sides by $2\,(ds/dt)$ and integrating, we have

$$v^2=\left(\frac{ds}{dt}\right)^2=-\frac{g}{4a}s^2+A.$$

But at the point B, $s=s_1$ and $v=0$.

$$\therefore \quad 0=-\frac{g}{4a}.s_1^2+A \quad \text{or} \quad A=\frac{g}{4a}s_1^2.$$

$$\therefore \quad \left(\frac{ds}{dt}\right)^2=-\frac{g}{4a}s^2+\frac{g}{4a}s_1^2=\frac{g}{4a}(s_1^2-s^2)$$

or $$ds/dt=-\tfrac{1}{2}\sqrt{(g/a)}.\sqrt{(s_1^2-s^2)}$$

(negative sign is taken since the particle is moving in the direction of s decreasing)

or $$dt=-2\sqrt{(a/g)}\frac{ds}{\sqrt{(s_1^2-s^2)}}. \quad ...(4)$$

Integrating, we have

$$t=2\sqrt{(a/g)}\cos^{-1}(s/s_1)+A.$$

But at the point B, $s=s_1$ and $t=0$.

$$\therefore \quad 0=2\sqrt{(a/g)}\cos^{-1}1+A \quad \text{or} \quad A=0. \qquad [\because \cos^{-1}1=0]$$

$$\therefore \quad t=2\sqrt{(a/g)}\cos^{-1}(s/s_1)$$

$$=2\sqrt{\left(\frac{a}{g}\right)}\cos^{-1}\left[\frac{\sqrt{(8ay)}}{\sqrt{(8ah)}}\right] \qquad [\because s^2=8ay \text{ and } s_1^2=8ah]$$

$$=2\sqrt{(a/g)}\cos^{-1}\sqrt{(y/h)}. \quad ...(5)$$

Let C be the point at a vertical depth h/n below the point B. Then the height of C above $O=ON=h-(h/n)=h(1-1/n)$. Thus for the point C, we have $y=h(1-1/n)$.

If t_1 be the time taken by the particle from B to C, then putting $t=t_1$ and $y=h(1-1/n)$ in (5), we get

$$t_1=2\sqrt{(a/g)}\cos^{-1}\sqrt{[\{h(1-1/n)\}/h]}=2\sqrt{(a/g)}\cos^{-1}\sqrt{(1-1/n)}$$
$$=2\sqrt{(a/g)}\sin^{-1}\sqrt{\{1-(1-1/n)\}}\quad [\because \cos^{-1}x=\sin^{-1}\sqrt{(1-x^2)}]$$
$$=2\sqrt{(a/g)}\sin^{-1}(1/\sqrt{n}).$$

Ex. 36 (b). *A particle slides down the arc of a smooth cycloid whose axis is vertical and vertex lowest, starting from rest at a given point of the cycloid. Prove that the time occupied in falling down the first half of the vertical height to the lowest point is equal to the time of falling down the second half.*

Sol. Proceed as in Ex. 36 (a) by taking $n=2$.

Thus here if C be the point at a vertical depth $h/2$ below the point B, then at C, we have $y=h/2$. If t_1 be the time taken by the particle from B to C, then putting $t=t_1$ and $y=h/2$ in the result (5) of Ex. 36 (a), we get

$$t_1=2\sqrt{(a/g)}\cos^{-1}\sqrt{(\tfrac{1}{2}h/h)}=2\sqrt{(a/g)}\cos^{-1}(1/\sqrt{2})$$
$$=2\sqrt{(a/g)}\cdot\tfrac{1}{4}\pi=\tfrac{1}{2}\pi\sqrt{(a/g)}.$$

Again if t_2 be the time taken by the particle from B to O, then putting $t=t_2$ and $y=0$ in (5), we get

$$t_2=2\sqrt{(a/g)}\cos^{-1}0=2\sqrt{(a/g)}\cdot\tfrac{1}{2}\pi=\pi\sqrt{(a/g)}.$$

Since $t_2=2t_1$, therefore the time from B to C is equal to the time from C to O.

Ex. 37. *Two particles are let drop from the cusp of a cycloid down the curve at an interval of time t; prove that they will meet in time $2\pi\sqrt{(a/g)}+(t/2)$.* **[Kanpur 1981, 83; Rohilkhand 79; Lucknow 79; Gorakhpur 81; Meerut 85P]**

Sol. Refer the figure of § 7 on page 201.

Suppose a particle starts at rest from the cusp A. At any time T, the equation of motion of the particle along the tangent is given by

$$m\frac{d^2s}{dT^2}=-mg\sin\psi.$$

For the cycloid, $\quad s=4a\sin\psi.$

$$\therefore\quad \frac{d^2s}{dT^2}=-\frac{g}{4a}s.$$

Multiplying both sides by $2(ds/dT)$ and integrating, we have

$$v^2=\left(\frac{ds}{dT}\right)^2=-\frac{g}{4a}s^2+A.$$

The particle is dropped from the cusp. Therefore $v=0$ when $s=4a$.

$$\therefore\quad 0=-\frac{g}{4a}(4a)^2+A\quad\text{or}\quad A=4ag.$$

$$\therefore \quad \left(\frac{ds}{dT}\right)^2=-\frac{g}{4a}\, s^2+4ag=\frac{g}{4a}\,(16a^2-s^2)$$

or $$ds/dT=-\tfrac{1}{2}\sqrt{(g/a)}\,\sqrt{(16a^2-s^2)}$$

(−ive sign is taken because the particle is moving in the direction of s decreasing)

or $$dT=-2\sqrt{(a/g)}\,\frac{ds}{\sqrt{(16a^2-s^2)}}.$$

Integrating, $T=2\sqrt{(a/g)}\cos^{-1}\left(\dfrac{s}{4a}\right)+B.$

But at the cusp A, $T=0$, $s=4a$. $\therefore$ $B=0$.

$$\therefore \quad T=2\sqrt{(a/g)}\cos^{-1}(s/4a)$$

or $$\cos^{-1}(s/4a)=\tfrac{1}{2}T\sqrt{(g/a)}.$$

$$\therefore \quad s=4a\cos\left[\tfrac{1}{2}T\sqrt{(g/a)}\right]. \qquad \ldots(1)$$

Thus if a particle starts at rest from the cusp A, the equation (1) gives the arcural distance (*i.e.*, distance measured along the arc) of the particle from the vertex O at any time T measured from the instant the particle starts from the cusp A.

Let the two particles meet after time t_1 measured from the instant the first particle was dropped. Since the two particles are dropped at an interval of time t, therefore the second particle will be in motion for time (t_1-t) before it meets the first particle.

Let s_1 be the distance along the arc of the first particle at time t_1 measured from the instant it starts from the cusp A and s_2 that of the second particle at time t_1-t measured from the instant it starts from the cusp A. Then from (1), we have

$s_1=4a\cos\left[\tfrac{1}{2}t_1\sqrt{(g/a)}\right]$ and $s_2=4a\cos\left[\tfrac{1}{2}(t_1-t)\sqrt{(g/a)}\right]$.

But $s_1=s_2$, being the condition for the two particles to meet.

$$\therefore \quad 4a\cos\left[\tfrac{1}{2}t_1\sqrt{(g/a)}\right]=4a\cos\left[\tfrac{1}{2}(t_1-t)\sqrt{(g/a)}\right]$$

or $$\cos\left[\tfrac{1}{2}t_1\sqrt{(g/a)}\right]=\cos\left[\tfrac{1}{2}(t_1-t)\sqrt{(g/a)}\right]$$

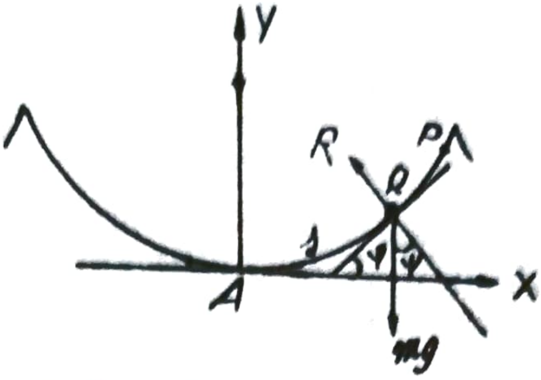
or $$\tfrac{1}{2}(t_1-t)\sqrt{(g/a)}=2\pi-\tfrac{1}{2}t_1\sqrt{(g/a)}$$

or $$t_1\sqrt{(g/a)}=2\pi+\tfrac{1}{2}t\sqrt{(g/a)} \qquad [\because \cos(2\pi-\alpha)=\cos\alpha]$$
or $t_1=2\pi\sqrt{(a/g)}+\tfrac{1}{2}t.$

Ex. 38. *A particle starts from rest at any point P in the arc of a smooth cycloid $s=4a\sin\psi$ whose axis is vertical and vertex A downwards; prove that the time of descent to the vertex is $\pi\sqrt{(a/g)}$.*

Show that if the particle is projected from P downwards along the curve with velocity equal to that with which it reaches A when starting from rest at P, it will now reach A in half the time taken in the preceding case.

[Meerut 1973, 82, 86, Rohilkhand 85]



Sol. A particle starts from rest at any point P in the arc of a smooth cycloid whose vertex is A. Let arc $AP=b$.

Let Q be the position of the particle at any time t where arc $AQ=s$ and let ψ be the angle which the tangent at Q to the cycloid makes with the tangent at the vertex A. The tangential equation of motion of the particle at Q is

$$m \frac{d^2 s}{dt^2} = -mg \sin \psi. \qquad \ldots(1)$$

But for the cycloid, $s=4a \sin \psi$.

$\therefore$ the equation (1) becomes $\dfrac{d^2 s}{dt^2} = -\dfrac{g}{4a} s$.

Multiplying both sides by $2\,(ds/dt)$ and integrating w.r.t. 't', we have

$$v^2 = \left(\frac{ds}{dt}\right)^2 = -\frac{g}{4a} s^2 + A. \qquad \ldots(2)$$

But initially at the point P, we have $s=b$ and $v=0$.

$$\therefore \quad v = -\frac{g}{4a} b^2 + A \quad \text{or} \quad A = \frac{g}{4a} b^2.$$

$$\therefore \quad v^2 = \left(\frac{ds}{dt}\right)^2 = -\frac{g}{4a} s^2 + \frac{g}{4a} b^2 = \frac{g}{4a} (b^2 - s^2). \qquad \ldots(3)$$

Taking square root of (3), we get

$$ds/dt = -\tfrac{1}{2}\sqrt{(g/a)}\,\sqrt{(b^2 - s^2)},$$

where the $-$ive sign has been taken because the particle is moving in the direction of s decreasing.

$$\therefore \quad dt = -2\sqrt{(a/g)}\, \frac{ds}{\sqrt{(b^2 - s^2)}}. \qquad \ldots(4)$$

Let t_1 be the time taken by the particle to reach the vertex A where $s=0$. Then integrating (4) from P to A, we have

$$\int_0^{t_1} dt = -2\sqrt{(a/g)} \int_b^0 \frac{ds}{\sqrt{(b^2 - s^2)}}.$$

$$\therefore \quad t_1 = 2\sqrt{(a/g)} \left[\cos^{-1} \frac{s}{b}\right]_b^0 = 2\sqrt{(a/g)}\, [\cos^{-1} 0 - \cos^{-1} 1]$$

$= 2\sqrt{(a/g)}\, [\tfrac{1}{2}\pi - 0] = \pi\sqrt{(a/g)}$, which proves the first result.

If v_1 is the velocity with which the particle reaches the vertex A, then at A, $v=v_1$ and $s=0$. So from (3), we have

$$v_1^2=\frac{g}{4a}(b^2-0^2)=\frac{g}{4a}b^2.$$

Second case. Now suppose the particle starts from P with velocity v_1 where $v_1^2=(g/4a)\,b^2$. Then applying the initial condition $s=b$ and $v=v_1$ in (2), we have

$$v_1^2=-\left(\frac{g}{4a}\right)b^2+A$$

or
$$A=v_1^2+\left(\frac{g}{4a}\right)b^2=\left(\frac{g}{4a}\right)b^2+\left(\frac{g}{4a}\right)b^2=\frac{g}{2a}b^2.$$

For this new value of A, (2) becomes

$$\left(\frac{ds}{dt}\right)^2=-\frac{g}{4a}s^2+\frac{g}{2a}b^2=\frac{g}{4a}(2b^2-s^2).$$

$$\therefore\ \frac{ds}{dt}=-\tfrac{1}{2}\sqrt{(g/a)}\sqrt{(2b^2-s^2)}$$

or
$$dt=-2\sqrt{(a/g)}\frac{ds}{\sqrt{(2b^2-s^2)}}. \qquad \ldots(5)$$

Let t_2 be the time taken by the particle to reach the vertex A in this case. Then integrating (5) from P to A, we have

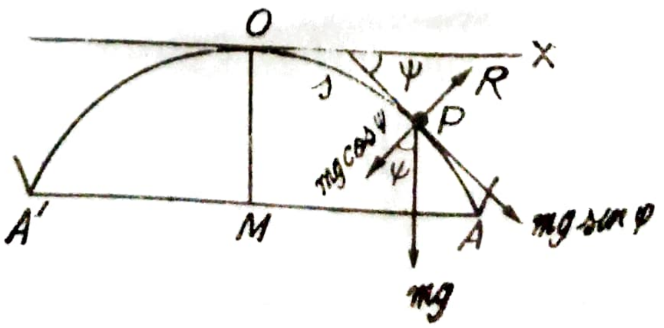
$$\int_0^{t_2}dt=-2\sqrt{(a/g)}\int_b^0\frac{ds}{\sqrt{(2b^2-s^2)}}.$$

$$\therefore\ t_2=2\sqrt{(a/g)}\left[\cos^{-1}\frac{s}{b\sqrt{2}}\right]_b^0$$

$=2\sqrt{(a/g)}\,[\cos^{-1}0-\cos^{-1}(1/\sqrt{2})]=2\sqrt{(a/g)}\,[\tfrac{1}{2}\pi-\tfrac{1}{4}\pi]$
$=2\sqrt{(a/g)}.\tfrac{1}{4}\pi=\tfrac{1}{2}\pi\sqrt{(a/g)}=\tfrac{1}{2}t_1$, which proves the second result.

§ 8. Motion on the outside of a smooth cycloid with its axis vertical and vertex upwards. *A particle is placed very close, to the vertex of a smooth cycloid whose axis is vertical and vertex upwards and is allowed to run down the curve, to discuss the motion.*

[Meerut 1979; Kanpur 77]



Let a particle of mass m, starting from rest at O, slide down the arc of a smooth cycloid whose axis OM is vertical and vertex O is upwards. Let P be the position of the particle at time t such that arc $OP=s$ and the tangent at P to the cycloid makes an angle ψ with the tangent at the vertex O. The forces acting on the particle at P are : (i) weight mg acting vertically downdards and (ii) the reaction R acting along the outwards drawn normal.

∴ The equations of motion along the tangent and normal are

$$m\frac{d^2s}{dt^2}=mg\sin\psi \quad ...(1)$$

and $$m\frac{v^2}{\rho}=mg\cos\psi-R. \quad ...(2)$$

Also for the cycloid, $s=4a\sin\psi$. ...(3)

From (1) and (3), we have $\dfrac{d^2s}{dt^2}=\dfrac{g}{4a}s$.

Multiplying both sides by $2(ds/dt)$ and integrating, we have

$$v^2=\left(\frac{ds}{dt}\right)^2=\frac{g}{4a}s^2+A.$$

Initially at O, $s=0$ and $v=0$, ∴ $A=0$.

$$\therefore \quad v^2=\frac{g}{4a}s^2=\frac{g}{4a}(4a\sin\psi)^2=4ag\sin^2\psi. \quad ...(4)$$

From (2) and (4), we have

$$R=mg\cos\psi-\frac{mv^2}{\rho}$$

$$=mg\cos\psi-\frac{m.4ag\sin^2\psi}{4a\cos\psi} \qquad \left[\because \rho=\frac{ds}{d\psi}=4a\cos\psi\right]$$

$$=\frac{mg}{\cos\psi}(\cos^2\psi-\sin^2\psi). \quad ...(5)$$

The equation (4) gives the velocity of the particle at any position and the equation (5) gives the reaction of the cycloid on the particle at any position. The pressure of the particle on the curve is equal and opposite to the reaction of the curve on the particle.

When the particle leaves the cycloid, we have $R=0$

i.e., $\dfrac{mg}{\cos\psi}(\cos^2\psi-\sin^2\psi)=0$

i.e., $\sin^2\psi=\cos^2\psi$ *i.e.*, $\tan^2\psi=1$

i.e., $\tan\psi=1$ *i.e.*, $\psi=45°$.

Hence the particle will leave the curve when it is moving in a direction making an angle 45° downwards with the horizontal.

[Meerut 1983]

Illustrative Examples

Ex. 39. *If a particle starts from the vertex of a cycloid whose axis is vertical and vertex upwards, prove that its velocity at any point varies as the distance of that point from the vertex measured along the arc.* **[Kanpur 1975]**

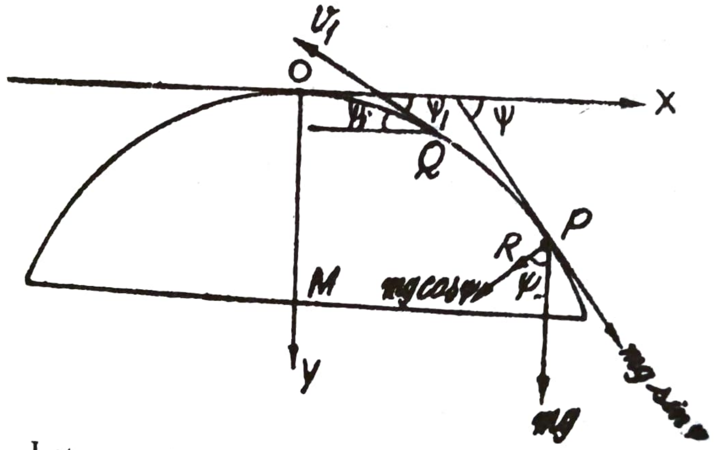
Sol. Proceed as in § 8. From the equation (4), the velocity v at any point P is given by

$$v^2 = (g/4a)\, s^2.$$

$$\therefore \quad v = \sqrt{(g/4a)}\,.\,s \quad \text{or} \quad v \propto s.$$

Hence the velocity varies as the distance measured along the arc.

Ex. 40. *A cycloid is placed with its axis vertical and vertex upwards and a heavy particle is projected from the cusp up the concave side of the curve with velocity $\sqrt{(2gh)}$; prove that the latus rectum of the parabola described after leaving the arc is $(h^2/2a)$, where a is the radius of the generating circle.* **[Rohilkhand 1987]**



Sol. Let a particle of mass m be projected with velocity $\sqrt{(2gh)}$ from the cusp A up the concave side of the cycloid. If P is the position of the particle after any time t such that arc $OP = s$, the equations of motion along the tangent and normal are

$$m\,(d^2s/dt^2) = mg \sin \psi, \quad \ldots(1)$$

and

$$m\,(v^2/\rho) = R + mg \cos \psi. \quad \ldots(2)$$

[Note that here the reaction R of the curve acts along the inwards drawn normal and the tangential component of mg acts in the direction of s increasing.]

For the cycloid, $\quad s = 4a \sin \psi.$

From (1) and (3), we have $\dfrac{d^2s}{dt^2} = \dfrac{g}{4a}\, s.$

Multiplying both sides by $2\,(ds/dt)$ and then integrating, we have

$$v^2 = (ds/dt)^2 = (g/4a)\, s^2 + A.$$

Initially at A, $s=4a$ and $v=\sqrt{(2gh)}$.

$\therefore$ $A=2gh-4ag$.

$$\therefore \quad v^2=\frac{g}{4a}s^2+2gh-4ag=\frac{g}{4a}(4a \sin \psi)^2+2gh-4ag$$
$$=4ag \sin^2 \psi+2gh-4ag=2gh-4ag(1-\sin^2 \psi)$$
$$=2gh-4ag \cos^2 \psi. \qquad \ldots(4)$$

From (2) and (4), we have

$$R=\frac{m}{4a \cos \psi}(2gh-4ag \cos^2 \psi)-mg \cos \psi$$

$$[\because \rho=ds/d\psi=4a \cos \psi]$$

$$=\frac{mg}{2a \cos \psi}(h-2a \cos^2 \psi)-mg \cos \psi$$
$$=\frac{mg}{2a \cos \psi}[h-2a \cos^2 \psi-2a \cos^2 \psi]$$
$$=\frac{mg}{2a \cos \psi}\left[h-4a \cos^2 \psi\right] \qquad \ldots(5)$$

Suppose the particle leaves the cycloid at the point Q where $\psi=\psi_1$. Then putting $\psi=\psi_1$ and $R=0$ in (5), we have

$$h-4a \cos^2 \psi_1=0$$
or
$$\cos^2 \psi_1=h/4a. \qquad \ldots(6)$$

If v_1 is the velocity at Q, then from (4), we have

$$v_1^2=2gh-4ag \cos^2 \psi_1=2gh-4ag.(h/4a)=gh.$$

$\therefore$ the particle leaves the cycloid at the point Q with velocity $v_1=\sqrt{(gh)}$ inclined at an angle ψ_1 to the horizontal given by (6). Subsequently it describes a parabolic path.

The latus rectum of the parabolic path described after Q
$=(2/g)$ (square of the horizontal velocity at Q)
$=(2/g)(v_1^2 \cos^2 \psi_1)=(2/g)(gh)(h/4a)=h^2/2a$.

Ex. 41. *A particle is placed very near the vertex of a smooth cycloid whose axis is vertical and vertex upwards, and is allowed to run down the curve. Prove that it will leave the curve when it has fallen through half the vertical height of the cycloid.*

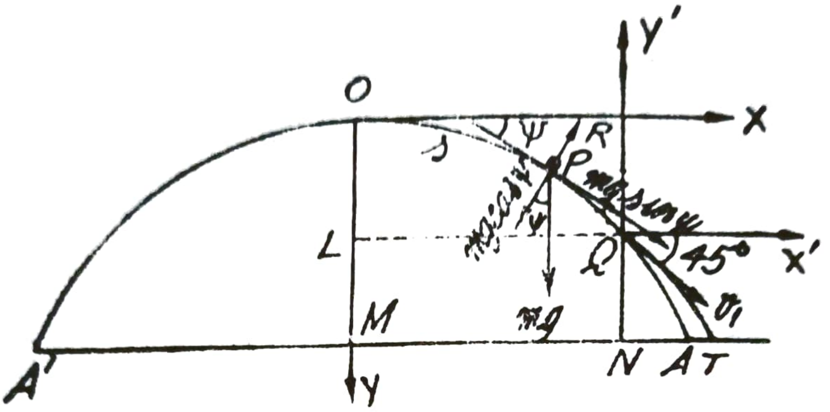
[Lucknow 1981; Gorakhpur 78; Allahabad 79; Meerut 77, 79, 80, 84P; Agra 86]

Also prove that the latus rectum of the parabola subsequently described is equal to the height of the cycloid. **[Kanpur 1973]**

Also show that it falls upon the base of the cycloid at a distance $(\frac{1}{2}\pi+\sqrt{3})\,a$ *from the centre of the base; a being the radius of the generating circle.*

[Kanpur 1984]

Sol. Let a particle of mass m, starting from rest at O, slide down the arc of a smooth cycloid whose axis OM is vertical and vertex O is upwards. Let P be the position of the particle at any time t such that arc $OP=s$. If the tangent at P makes an angle ψ with the horizontal, then the equations of motion of the particle along the tangent and normal at P are



$$m\frac{d^2s}{dt^2}=mg\sin\psi, \qquad \text{...(1)}$$

and $$m\frac{v^2}{\rho}=mg\cos\psi-R. \qquad \text{...(2)}$$

Also for the cycloid, $s=4a\sin\psi$. ...(3)

From (1) and (3), we have $\dfrac{d^2s}{dt^2}=\dfrac{g}{4a}s$.

Multiplying both sides by $2(ds/dt)$ and integrating, we have

$$v^2=\left(\frac{ds}{dt}\right)^2=\frac{g}{4a}s^2+A.$$

Initially at O, $s=0$ and $v=0$. $\therefore$ $A=0$.

$$\therefore \quad v^2=\frac{g}{4a}s^2=\frac{g}{4a}(4a\sin\psi)^2=4ag\sin^2\psi. \qquad \text{...(4)}$$

From (2) and (4), we have

$$R=mg\cos\psi-\frac{mv^2}{\rho}=mg\cos\psi-m.\frac{4ag\sin^2\psi}{4a\cos\psi}$$

$[\because \rho=ds/d\psi=4a\cos\psi]$

$$=\frac{mg}{\cos\psi}(\cos^2\psi-\sin^2\psi).$$

If the particle leaves the cycloid at the point Q, then at Q, $R=0$. From $R=0$, we have

$$\frac{mg}{\cos\psi}(\cos^2\psi-\sin^2\psi)=0$$

or $\sin^2\psi=\cos^2\psi$ or $\tan^2\psi=1$

or $\tan\psi=1$ or $\psi=45°$.

Thus at Q, we have $\psi=45°$. Putting $\psi=\frac{1}{4}\pi$ in $s=4a\sin\psi$, we have at Q, $s=4a\sin\frac{1}{4}\pi=4a.(1/\sqrt{2})=2\sqrt{2a}$. Again putting $s=2\sqrt{2a}$ in $s^2=8ay$, we have at Q, $y=s^2/8a=8a^2/8a=a$.

Thus $OL=a$. Therefore $LM=OM-OL=2a-a=a$. Hence the particle leaves the cycloid at the point Q, when it has fallen through half the vertical height of the cycloid.

Second part. If v_1 is the velocity of the particle at Q, then from (4), we have $\quad v_1^2=4ag\sin^2 45°=2ag$.

Hence the particle leaves the cycloid at Q with velocity $v_1=\sqrt{(2ag)}$ in a direction making an angle 45° downwards with the horizontal. After Q the particle will describe a parabolic path.

Latus rectum of the parabola described after Q

$$=\frac{2v_1^2\cos^2 45°}{g}=\frac{2.2ag.\frac{1}{2}}{g}=2a$$

i.e., the latus rectum of the parabola subsequently described is equal to the height of the cycloid.

Third part. The equation of the parabolic path described by the particle after leaving the cycloid at Q with respect to the horizontal and vertical lines QX' and QY' as the coordinate axes is

$$y=x\tan(-45°)-\frac{gx^2}{2v_1^2\cos^2(-45°)}$$ [Note that here the angle of projection for the motion of the projectile is $-45°$]

or $$y=-x-\frac{gx^2}{2.2ag.\frac{1}{2}}$$

or $$y=-x-\frac{x^2}{2a}. \quad ...(5)$$

Suppose after leaving the cycloid at Q the particle strikes the base of the cycloid at the point T. Let (x_1, y_1) be the coordinates of T with respect to QX' and QY' as the coordinate axes. Then

$$x_1=NT \quad \text{and} \quad y_1=-QN=-a.$$

But the point $T(x_1, -a)$ lies on the curve (5).

$$\therefore \quad -a=-x_1-\frac{x_1^2}{2a}$$

or $$x_1^2+2ax_1-2a^2=0.$$

$$\therefore \quad x_1=\frac{-2a\pm\sqrt{\{4a^2-4.1.(-2a^2)\}}}{2.1}.$$

Neglecting the $-$ive sign because x_1 cannot be negative, we have

$$x_1=NT=-a+a\sqrt{3}.$$

The parametric equations of the cycloid w.r.t. OX and OY as the coordinate axes are

$$x=a(\theta+\sin\theta),\ y=a(1-\cos\theta),$$

where θ is the parameter and $\theta=2\psi$.

$\therefore$ At the point Q, where $\psi=\frac{1}{4}\pi$, we have

$$x=LQ=a(2\psi+\sin 2\psi)=a[2.\tfrac{1}{4}\pi+\sin(2.\tfrac{1}{4}\pi)]=a(\tfrac{1}{2}\pi+1).$$

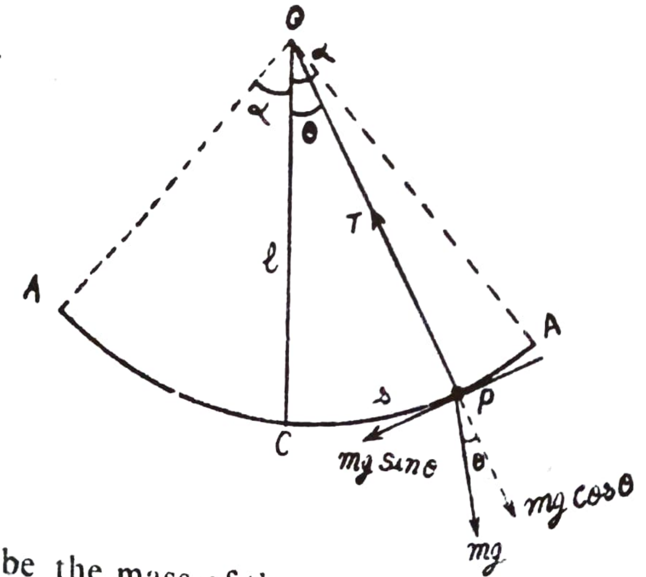
$\therefore$ the horizontal distance of the point T from the centre M of the base of the cycloid

$$=MT=MN+NT=LQ+NT$$
$$=a(\tfrac{1}{2}\pi+1)+(-a+a\sqrt{3})=(\tfrac{1}{2}\pi+\sqrt{3})\,a.$$

§ 9. Simple Pendulum.

Definition. *A light inextensible string and a heavy particle of negligible size tied to one end of the string whose other end is attached to a fixed point and oscillating in a vertical plane under gravity through a small angle, are said to form a simple pendulum.*

§ 10. Oscillations of a simple pendulum.



Let m be the mass of the particle and l the length of the string. Let P be the position of the particle and θ be the angle which the string makes with the vertical at any time t. Let OC be the vertical line through the fixed point O and arc $CP=s$.

$\therefore$ $s=l\theta$. ...(1)

The forces acting on the particle at time t are :

(i) its weight mg acting vertically downwards,

and (ii) the tension T in the string acting along PO.

$\therefore$ the equation of motion along the tangent at P is

$$m\frac{d^2s}{dt^2}=-mg\sin\theta$$

or $$d^2s/dt^2=-g\sin\theta$$

$$l\frac{d^2\theta}{dt^2}=-g\sin\theta.$$

or $\qquad$ [$\because$ $s=l\theta$, from (1)] $\qquad$ ...(2)

Case I. When the oscillations are small. **[Lucknow 1981]**

When the pendulum swings through a small angle on each side of the verticle OC, then θ is very small and hence we can take $\sin\theta=\theta$.

$\therefore$ from (2), we get

$$l\frac{d^2\theta}{dt^2}=-g\theta$$

or

$$\frac{d^2\theta}{dt^2}=-\frac{g}{l}\theta,$$

which shows that the motion of the particle is simple harmonic about C. The period T of a small complete oscillation is given by

$$T=2\pi\sqrt{(l/g)}. \qquad ...(3)$$

Case II. When the oscillations are not small.

Multiplying both sides of (2) by $2l\,(d\theta/dt)$ and integrating with respect to 't', we get

$$\left(l\frac{d\theta}{dt}\right)^2=2gl\cos\theta+A,$$

where A is a constant of integration.

If the pendulum oscillates through an angle α on each side of the vertical OC, then

$$v=l\,(d\theta/dt)=0, \text{ when } \theta=\alpha.$$

$\therefore$ $\quad 0=2gl\cos\alpha+A, \qquad$ or $\qquad A=-2gl\cos\alpha.$

$$\therefore \quad \left(l\frac{d\theta}{dt}\right)^2=2gl\cos\theta-2gl\cos\alpha$$

or

$$\frac{d\theta}{dt}=\sqrt{\left(\frac{2g}{l}\right)}\sqrt{(\cos\theta-\cos\alpha)}$$

or

$$dt=\sqrt{\left(\frac{l}{2g}\right)}\frac{d\theta}{\sqrt{(\cos\theta-\cos\alpha)}}. \qquad ...(4)$$

If t_1 is the time from C (*i.e.*, the lowest point) to the extreme position $\theta=\alpha$, then integrating (4), we get

$$t_1=\sqrt{\left(\frac{l}{2g}\right)}\int_{\theta=0}^{\alpha}\frac{d\theta}{\sqrt{(\cos\theta-\cos\alpha)}}$$

$$=\sqrt{\left(\frac{l}{2g}\right)}\int_0^{\alpha}\frac{d\theta}{\sqrt{\{(1-2\sin^2\frac{1}{2}\theta)-(1-2\sin^2\frac{1}{2}\alpha)\}}}$$

$$=\frac{1}{2}\sqrt{\left(\frac{l}{g}\right)}\int_0^{\alpha}\frac{d\theta}{\sqrt{(\sin^2\frac{1}{2}\alpha-\sin^2\frac{1}{2}\theta)}}$$

Substituting $\sin \frac{1}{2}\theta = \sin \frac{1}{2}\alpha \sin \phi$, so that
$\frac{1}{2} \cos \frac{1}{2}\theta \, d\theta = \sin \frac{1}{2}\alpha \cos \phi \, d\phi$, we get

$$t_1 = \frac{1}{2}\sqrt{\left(\frac{l}{g}\right)} \int_0^{\pi/2} \frac{2 \sin \frac{1}{2}\alpha \cos \phi \, d\phi}{\cos \frac{1}{2}\theta \sqrt{(\sin^2 \frac{1}{2}\alpha - \sin^2 \frac{1}{2}\alpha \sin^2 \phi)}}$$

$$= \sqrt{\left(\frac{l}{g}\right)} \int_0^{\pi/2} \frac{\sin \frac{1}{2}\alpha \cos \phi \, d\phi}{\cos \frac{1}{2}\theta \sin \frac{1}{2}\alpha \cos \phi} = \sqrt{\left(\frac{l}{g}\right)} \int_0^{\pi/2} \frac{d\phi}{\cos \frac{1}{2}\theta}$$

$$= \sqrt{\left(\frac{l}{g}\right)} \int_0^{\pi/2} \frac{d\phi}{\sqrt{(1 - \sin^2 \frac{1}{2}\theta)}} = \sqrt{\left(\frac{l}{g}\right)} \int_0^{\pi/2} \left(1 - \sin^2 \frac{1}{2}\alpha \sin^2 \phi\right)^{-1/2} d\phi$$

$$= \sqrt{\left(\frac{l}{g}\right)} \int_0^{\pi/2} \left[1 + \frac{1}{2} \sin^2 \frac{1}{2}\alpha \sin^2 \phi + \frac{1 \cdot 3}{2 \cdot 4} \sin^4 \frac{1}{2}\alpha \sin^4 \phi + \ldots\right] d\phi$$

$$= \sqrt{\left(\frac{l}{g}\right)} \left[\frac{1}{2}\pi + \frac{1}{2} \sin^2 \frac{1}{2}\alpha . \frac{1}{2} . \frac{1}{2}\pi + \frac{1 \cdot 3}{2 \cdot 4} \sin^4 \frac{1}{2}\alpha . \frac{3 \cdot 1}{4 \cdot 2} \frac{1}{2}\pi + \ldots\right]$$

[By Walli's formula]

$$= \sqrt{\left(\frac{l}{g}\right)} \left[\frac{1}{2}\pi + \frac{1}{2^2} . \frac{1}{2}\pi \sin^2 \frac{1}{2}\alpha + \left(\frac{1 \cdot 3}{2 \cdot 4}\right)^2 . \frac{1}{2}\pi \sin^4 \frac{1}{2}\alpha + \ldots\right]$$

$$= \frac{\pi}{2}\sqrt{\left(\frac{l}{g}\right)} \left[1 + \frac{1}{4} \sin^2 \frac{1}{2}\alpha + \frac{9}{64} \sin^4 \frac{1}{2}\alpha + \ldots\right]$$

$$= \frac{\pi}{2}\sqrt{\left(\frac{l}{g}\right)} \left(1 + \frac{1}{4} \sin^2 \frac{1}{2}\alpha\right). \qquad \ldots(5)$$

[Neglecting powers of $\sin \frac{1}{2}\alpha$ higher than 2]

Hence if T_1 is the time of one complete oscillation of a simple pendulum, then a second approximation to the period is given by

$$T_1 = 4t_1 = 2.\pi \sqrt{(l/g)} \left(1 + \frac{1}{4} \sin^2 \frac{1}{2}\alpha\right)$$

or
$$T_1 = T\left(1 + \frac{1}{4} \sin^2 \frac{1}{2}\alpha\right). \qquad [\because \; T = 2\pi \sqrt{(l/g)}, \text{ from (3)}]$$

Neglecting the powers of α higher than 2, we get

$$T_1 = 2\pi \sqrt{(l/g)} \left(1 + \tfrac{1}{16}\alpha^2\right) = T\left(1 + \tfrac{1}{16}\alpha^2\right).$$

§ 11. Beat of a pendulum. *A beat of a pendulum means its going from one extreme position of rest to the other position of rest i.e., half of the complete oscillation.*

$\therefore$ the time of a beat $= \frac{1}{2}T = \pi\sqrt{(l/g)}$.

§ 12. The Second's pendulum. *If a simple pendulum oscillates from rest in one second i.e., if the time of one beat of a simple pendulum is one second, then it is called a* **second's pendulum** *and such a clock is said to be a* **correct clock.**

Thus for a second's pendulum

$$1 = \pi\sqrt{(l/g)}$$

or
$$l = \frac{g}{\pi^2} = \frac{g}{(3 \cdot 1416)^2}.$$

In F. P. S. system $g = 32 \cdot 2$, then $l = \dfrac{32 \cdot 2}{(3 \cdot 1416)^2}$

$= 39 \cdot 14$ inches (appr.)

and in C. G. S. system $g=981$, then $l=\dfrac{981}{(3 \cdot 1416)^2}$

$=99 \cdot 4$ cm. (appr.).

§ 13. Gain or loss of beats (time) by a clock.

[Lucknow 1975, 77, 79]

The time t_0 of one beat of a clock is given by

$$t_0=\tfrac{1}{2}T=\pi\sqrt{(l/g)}.$$

Clearly t_0 depends upon the values of l and g. Thus there is a change in the time of a beat of a clock when l and g change, either one or both.

Thus if n is the number of beats in a given time t, then

$$t=n.\pi\sqrt{(l/g)}$$

or

$$n=\frac{t}{\pi}\sqrt{(g/l)}. \qquad \ldots(1)$$

Now we shall determine the loss or gain in the number of beats of a clock when l and g change, either one or both.

Taking log of both sides of (1), we get

$$\log n=\log t-\log \pi+\tfrac{1}{2}(\log g-\log l).$$

Differentiating, $\dfrac{1}{n}\,\delta n=\dfrac{1}{2g}\,\delta g-\dfrac{1}{2l}\,\delta l. \qquad \ldots(2)$

[$\because$ t and π are constants]

Now the following cases arise :

(a) When g remains constant. **[Lucknow 1975, 77, 79]**

If g remains constant, then $\delta g=0$. Therefore from (2), we get

$$\frac{1}{n}\,\delta n=-\frac{1}{2l}\,\delta l. \qquad \ldots(3)$$

$\therefore$ δn is positive or negative according as δl is negative or positive respectively.

Hence there is a gain or loss in the number of beats according as the length of the string is shortened or increased i.e., the clock becomes fast when the length of the pendulum is shortened and the clock becomes slow when the length of the pendulum is increased.

(b) When l remains constant. **[Lucknow 1975, 77, 79]**

If l remains constant, then $\delta l=0$. Therefore from (2), we get

$$\frac{1}{n}\,\delta n=\frac{1}{2g}\,\delta g. \qquad \ldots(4)$$

$\therefore$ there is a gain or loss in the number of beats according as g increases or decreases. *Hence the clock becomes fast when g increases and it becomes slow when g decreases.* In other words a clock becomes fast if it is taken to a place of more gravity and it becomes slow if it is taken to a place of less gravity.

Now we shall discuss the following two situations.

(i) **When the pendulum (or clock) is taken to the top of a mountain.**

We know that outside the surface of the earth the attraction varies inversely as the square of the distance from the centre of the earth.

Thus at a distance x from the centre of the earth, the attraction is given by μ/x^2.

On the surface of the earth where $x=r$ (*i.e.*, the radius of the earth) the attraction is g.

$$\therefore \quad g=\mu/r^2$$

or
$$\log g=\log \mu-2\log r.$$

Differentiating, we get

$$\frac{1}{g}\,\delta g=-\frac{2}{r}\,\delta r$$

$$[\because \mu \text{ is constant}]$$

or
$$\frac{1}{g}\,\delta g=-\frac{2}{r}\,h,$$

where $\delta r=h$ is the height of the mountain.

$\therefore$ from (4), we get $\dfrac{1}{n}\,\delta n=-\dfrac{1}{r}\,h$

or
$$\delta n=-\frac{n}{r}\,h. \qquad \text{...(5)}$$

The negative sign indicates that the number of beats are lost. *Hence the clock becomes slow when it is taken to the top of a mountain.*

(ii) **When the pendulum (or clock) is taken to the bottom of a mine.**

We know that inside the earth the attraction varies as the distance from the centre.

Thus at a distance x from the centre of the earth, the attraction is given by μx.

On the surface of the earth where $x=r$ (*i.e.*, the radius of the earth) the attraction is g.

$$\therefore \quad g=\mu r$$

or
$$\log g=\log \mu+\log r.$$

Differentiating, we get

$$\frac{1}{g}\,\delta g=\frac{1}{r}\,\delta r. \qquad [\because \mu \text{ is a constant}]$$

If the pendulum is taken to the bottom of a mine of depth d, then $\delta r=-d$.

$\therefore \quad \frac{1}{g}\,\delta g=-\frac{d}{r}$.

$\therefore$ from (4), we get $\frac{1}{n}\,\delta n=-\frac{1}{2r}\,d$

or $$\delta n=-\frac{n}{2r}\,d. \quad \ldots(6)$$

The negative sign indicates that the number of beats are lost. *Hence the clock becomes slow when it is taken to the bottom of a mine.*

Illustrative Examples

Ex. 42. *In a simple pendulum, show that the period T is given by*

$$T=2\pi\sqrt{\left(\frac{l}{g}\right)}\left[1+\frac{1}{4}\,k^2+\frac{9}{64}\,k^4+\ldots\right],$$

where $k=\sin\frac{1}{2}\alpha$ and α is the amplitude.

Sol. The time t_1 taken by the pendulum to swing from its lowest position to the position $\theta=\alpha$ (extreme position on one side), is given by

$$t_1=\frac{\pi}{2}\sqrt{\left(\frac{l}{g}\right)}\left[1+\frac{1}{4}\sin^2\frac{\alpha}{2}+\frac{9}{64}\sin^4\frac{\alpha}{2}+\ldots\right]$$

[see equation (5), § 19 on page 222]

$\therefore$ period T is given by

$$T=4t_1=2\pi\sqrt{\left(\frac{l}{g}\right)}\left[1+\frac{1}{4}\sin^2\frac{\alpha}{2}+\frac{9}{64}\sin^4\frac{\alpha}{2}+\ldots\right]$$

$$=2\pi\sqrt{\left(\frac{l}{g}\right)}\left[1+\frac{1}{4}\,k^2+\frac{9}{64}\,k^4+\ldots\right], \text{ where } k=\sin\tfrac{1}{2}\alpha.$$

Ex. 43. *A simple pendulum is started so as to make complete revolution in a vertical plane. Find the least velocity of projection.*

In the subsequent motion ω_1, ω_2 are the greatest and least angular velocities; and T_1, T_2 are the greatest and least tensions. Prove that when the pendulum makes an angle θ with the vertical, the angular velocity is

$$[\omega_1^2\cos^2\tfrac{1}{2}\theta+\omega_2^2\sin^2\tfrac{1}{2}\theta]^{1/2},$$

and that the tension is

$$T_1\cos^2\tfrac{1}{2}\theta+T_2\sin^2\tfrac{1}{2}\theta. \quad \text{[Agra 1985]}$$

Sol. [Refer fig. § 2 on page 156].

Let the string be inclined at an angle θ to the vertical at time t. The forces acting on the particle at P are : (i) The tension T in the string along PO and (ii) the weight mg of the particle acting vertically downwards.

Let l be the length of the string and let arc $AP=s$. The equations of motion of the particle along the tangent and normal at P are

arc $$m\frac{d^2s}{dt^2}=-mg\sin\theta, \quad ...(1)$$

and $$m\frac{v^2}{l}=T-mg\cos\theta. \quad ...(2)$$

Also $s=l\theta$, ...(3)

From (1) and (3), we get $l\dfrac{d^2\theta}{dt^2}=-g\sin\theta$.

Mu!utiplying both sides by $2l\,(d\theta/dt)$ and integrating w.r.t. 't', we get

$$v^2=\left(l\frac{d\theta}{dt}\right)^2=2gl\cos\theta+A,$$

where A is a constant of integration.

If the particle is projected with velocity u from the lowest point A, then $v=u$, when $\theta=0$.

$\therefore\quad u^2=2gl+A \quad$ or $\quad A=u^2-2gl.$

$$\therefore\quad v^2=\left(l\frac{d\theta}{dt}\right)^2=2gl\cos\theta+u^2-2gl. \quad ...(4)$$

Substituting in (2), we get

$$T=\frac{m}{l}\left[v^2+lg\cos\theta\right]$$

$$=\frac{m}{l}\left[u^2-2gl+3gl\cos\theta\right]. \quad ...(5)$$

The pendulum will make complete revolution if neither the velocity nor the tension vanishes before the particle reaches the highest point.

At the highest point $\theta=\pi$. So in order to make complete revolution we should have at the highest point

$$v^2=u^2-4gl\geq 0$$

and $$T=\frac{m}{l}\,(u^2-5gl)\geq 0.$$

$\therefore\quad u^2\geq 5gl \quad$ or $\quad u\geq\sqrt{(5gl)}.$

Hence for the particle to make a complete revolution in the vertical plane, the least velocity of projection $u=\sqrt{(5gl)}$.

For complete solution of the second and third parts of this question proceed as in Ex. 2 (b) on page 162.

Ex. 44. *Show that if the tension of the string when the bob is in its lowest position is k times the tension when the bob is in its highest position, the velocities in these positions being u_1 and u_2 respectively, then*

$$\frac{u_1^2}{u_2^2}=\frac{5k+1}{k+5}$$

Sol. Let the string be inclined at an angle θ to the vertical at time t. Let u be the velocity of projection of the bob from its lowest position. If v is the velocity and T the tension in the string at time t, then proceeding as in the preceding Ex. 43, we get

$$v^2=u^2-2gl+2gl\cos\theta \quad \text{...(1)}$$

and

$$T=\frac{m}{l}(u^2-2gl+3gl\cos\theta), \quad \text{...(2)}$$

where l is the length of the string.

If u_1, u_2 are the velocities and T_1, T_2 the tensions in the string in its lowest and highest positions respectively, we have

$$\theta=0,\ v=u_1,\ T=T_1$$

and

$$\theta=\pi,\ v=u_2,\ T=T_2.$$

Putting these values in (1) and (2), we have

$$u_1^2=u^2,\ u_2^2=u^2-4gl,$$

$$T_1=\frac{m}{l}(u^2+gl),\quad \text{and} \quad T_2=\frac{m}{l}(u^2-5gl).$$

$$\therefore\ u_1^2-u_2^2=4gl, \quad \text{or} \quad gl=\tfrac{1}{4}(u_1^2-u_2^2).$$

Also given that $T_1=kT_2$.

$$\therefore\ \frac{m}{l}(u^2+gl)=k\cdot\frac{m}{l}(u^2-5gl)$$

or

$$u^2+gl=ku^2-5kgl.$$

Substituting $u^2=u_1^2$ and $gl=\frac{1}{4}(u_1^2-u_2^2)$, we get

$$u_1^2+\tfrac{1}{4}(u_1^2-u_2^2)=ku_1^2-\tfrac{5}{4}k(u_1^2-u_2^2)$$

or

$$4u_1^2+u_1^2-u_2^2=4ku_1^2-5ku_1^2+5ku_2^2$$

or

$$(5+k)u_1^2=(5k+1)u_2^2$$

or

$$\frac{u_1^2}{u_2^2}=\frac{5k+1}{k+5}.$$

Ex. 45. *If a pendulum of length l makes n complete oscillations in a given time, show that if g is changed to $(g+g')$, the number of oscillations gained is $ng'/(2g)$* **[Meerut 1979]**

Sol. For a pendulum of length l, the time of one complete oscillation T is given by

$$T=2\pi\sqrt{(l/g)}.$$

$\therefore$ n = the number of complete oscillations in a given time

$$=\frac{t}{T}=\frac{t}{2\pi}\sqrt{\left(\frac{g}{l}\right)}.$$

$\therefore$ $\log n=\log\left(\frac{t}{2\pi}\right)+\frac{1}{2}\log g-\frac{1}{2}\log l.$

Differentiating,

$$\frac{1}{n}\,\delta n=\frac{1}{2g}\,\delta g-\frac{1}{2l}\,\delta l. \qquad \left[\because\ t/2\pi \text{ is constant}\right] \quad ...(1)$$

If l is fixed then $\delta l=0$ and if g is changed to $(g+g')$, then $\delta g=g'$.

$\therefore$ from (1), we get $\frac{1}{n}\,\delta n=\frac{1}{2g}\cdot g'$

or $\qquad \delta n=\frac{ng'}{2g}.$

Hence the number of oscillations gained

$$=\delta n=ng'/(2g).$$

Ex. 46. *If a pendulum beating seconds at the foot of a mountain, loses 9 seconds a day when taken to its summit, find the height of the mountain assuming the radius of the earth to be 4000 miles and neglecting the attraction of the mountain.*

Sol. For a pendulum beating seconds at the foot of a mountain, $\quad n$ = number of beats in a day $=24\times60\times60$.

If h is the height of the mountain and r the radius of the earth then the gain in the number of beats in a day at the top of the mountain is given by

$$\delta n=-\frac{n}{r}\,h. \qquad \text{[Refer equation (5) of § 13 on page 226]} \quad ...(1)$$

Here $\quad r=4000$ miles $=4000\times1760\times3$ ft.,

and $\quad \delta n=-9$.

$\therefore$ from (1), we have

$$-9=-\frac{24\times60\times60}{4000\times1760\times3}\,h,$$

or $\qquad h=2200$ ft.

Ex. 47. *Find approximately the height of a mountain at the top of which a pendulum which beats seconds at sea level, loses 8 seconds a day. The radius of earth may be taken 4000 miles.*

Sol. Proceed as in the preceding Ex. 46, height of mountain $=1955\cdot5$ ft.

Ex. 48. *A pendulum beats seconds accurately at a place where g is 32 ft./sec². Prove that it will gain 270 seconds per day, if it be taken to a place where g is 32·2 ft./sec².*

Sol. For a pendulum which beats seconds accurately, let the number of beats in a day be n. Then $n=24\times 60\times 60$.

When the length of the pendulum remains constant, from the equation (4) of § 13, the number of beats gained in a day is given by

$$\delta n=\frac{n}{2g}\,\delta g. \qquad \ldots(1)$$

Here $g=32$ ft./sec². and $\delta g=32\cdot 2-32=0\cdot 2$ ft./sec².

$\therefore$ from (1), we get

$$\delta n=\frac{24\times 60\times 60\times 0\cdot 2}{2\times 32}=270.$$

Hence the pendulum will gain 270 seconds per day.

Ex. 49. *If a pendulum of length l makes n complete oscillations in a given time, show that, if the length be changed to $l+l'$, the number of oscillations lost is $nl'/(2l)$.* **[Meerut 1979]**

Sol. For a pendulum of length l, the time of one complete oscillation T is given by $T=2\pi\sqrt{(l/g)}$.

$\therefore$ $n=$the number of complete oscillations in a given time t

$$=\frac{t}{T}=\frac{t}{2\pi}\sqrt{(g/l)}.$$

$\therefore$ $\log n=\log(t/2\pi)+\frac{1}{2}\log g-\frac{1}{2}\log l.$

Differentiating, $\dfrac{1}{n}\,\delta n=\dfrac{1}{2g}\,\delta g-\dfrac{1}{2l}\,\delta l.$ ...(1)

[$\because$ $t/2\pi$ is constant]

If g is fixed then $\delta g=0$ and if l is changed to $l+l'$, then $\delta l=l'$.

$\therefore$ from (1), we get

$$\frac{1}{n}\,\delta n=0-\frac{1}{2l}\cdot l', \quad \text{or} \quad \delta n=-\frac{n}{2l}\,l'.$$

Hence the number of oscillations lost in time $t=-\delta n=nl'/(2l)$.

Ex. 50. *A pendulum is carried to the top of a mountain 2640 feet high. How many seconds will it lose per day? By how much its present length be shortened so that it may beat seconds at the summit of the mountain?* **[Lucknow 1975, 80]**

Sol. For a second's pendulum, let the number of beats in a day be n. Then $n=24\times 60\times 60$.

If r is the radius of the earth then the gain in the number of beats in a day at the top of a mountain of height h is given by

$$\delta n = -\frac{n}{r} h.$$

Here $h=2640$ ft. and $r=4000\times1760\times3$ ft. ...(1)

$\therefore$ from (1), we get

$$\delta n = -\frac{24\times60\times60}{4000\times1760\times3}\times2640 = -10\cdot8$$

i.e., the pendulum will lose $10\cdot8$ seconds per day.

IInd part. For a second's pendulum, if n be the number of beats in a given time t, we have

$$n = \frac{t}{\pi}\sqrt{(g/l)}.$$

$\therefore$ $\log n = \log(t/\pi) + \frac{1}{2}\log g - \frac{1}{2}\log l.$

Differentiating, $\dfrac{\delta n}{n} = \dfrac{\delta g}{2g} - \dfrac{1}{2l}\,\delta l.$ ...(2)

The pendulum will give correct time at the top of the mountain if there is neither increase nor decrease in the number of beats there *i.e.*, if $\delta n=0$.

$\therefore$ putting $\delta n=0$ in (2), we get

$$0 = \frac{1}{2g}\,\delta g - \frac{1}{2l}\,\delta l$$

or $$\frac{\delta g}{g} = \frac{\delta l}{l}.$$ ...(3)

Now on the surface of the earth, attraction$=\mu/r^2=g$.

$\therefore$ $\log g = \log\mu - 2\log r.$

Differentiating, $\dfrac{1}{g}\,\delta g = -\dfrac{2}{r}\,\delta r.$

$\therefore$ from (3), we get, $-\dfrac{2}{r}\,\delta r = \dfrac{\delta l}{l}.$

But at the top of a mountain of height h, $\delta r=h$.

$$\therefore \quad \frac{\delta l}{l} = -\frac{2}{r} h$$

or $$\delta l = -\frac{2l}{r} h = -\frac{2\times2640}{4000\times1760\times3}\, l = -\frac{l}{4000}.$$

Hence the pendulum should be shortened by (1/4000) of its present length.

Ans.