6. Show that the lines

$$L_1 : \frac{x-4}{2} = \frac{y+5}{4} = \frac{z-1}{-3}$$

$$L_2 : \frac{x-2}{1} = \frac{y+1}{3} = \frac{z}{2}$$

are skew.

Solution:

Write the equation in parametric form.

$$L1 : x = 2t + 4, \quad y = 4t - 5, \quad z = -3t + 1$$

$$L2 : x = s + 2, \quad y = 3s - 1, \quad z = 2s$$

The lines are not parallel since the vectors $\vec{v}_1 = \langle 2, 4, -3 \rangle$ and $\vec{v}_2 = \langle 1, 3, 2 \rangle$ are not parallel. Next we try to find intersection point by equating x, y, and z.

$$\begin{array}{ll} (1) & 2t + 4 = s + 2 \\ (2) & 4t - 5 = 3s - 1 \\ (3) & -3t + 1 = 2s \end{array}$$

(1) gives $s = 2t + 2$. Substituting into (2) gives $4t - 5 = 3(2t + 2) - 1 \Rightarrow t = -5$. Then $s = -8$. However, this contradicts with (3). So there is no solution for s and t. Since the two lines are neither parallel nor intersecting, they are skew lines.

17.3 To find the equations of the two generating lines through any point ($a \cos \theta$, $b \sin \theta$, 0) of the principal elliptic section

$$x^2/a^2 + y^2/b^2 = 1, z = 0.$$

of the hyperboloid by the plane $z = 0$.

Let

$$\frac{x - a\cos\theta}{l} = \frac{y - b\sin\theta}{m} = \frac{z - 0}{n}$$

be a generator through the point ($a \cos \theta$, $b \sin \theta$, 0)

The point

$$(lr + a \cos \theta, mr + b \sin \theta, nr)$$

on the generator is point of the hyperboloid for all values of r so that the equation

$$\frac{(lr + a\cos\theta)^2}{a^2} + \frac{(mr + b\sin\theta)^2}{b^2} - \frac{n^2r^2}{c^2} = 1$$

or
$$\left[\frac{l^2}{a^2} + \frac{m^2}{b^2} - \frac{n^2}{c^2}\right]r^2 + 2r\left[\frac{l\cos\theta}{a} + \frac{m\sin\theta}{b}\right] = 0$$

is true for all values of r. Therefore,

$$\frac{l^2}{a^2} + \frac{m^2}{b^2} - \frac{n^2}{c^2} = 0 \text{ and } \frac{l\cos\theta}{a} + \frac{m\sin\theta}{b} = 0$$

Hence

$$\frac{l}{a\sin\theta} = \frac{m}{-b\cos\theta} = \frac{z}{\pm c}$$

Thus, we get

$$\frac{x - a\cos\theta}{a\sin\theta} = \frac{y - b\sin\theta}{-b\cos\theta} = \frac{z}{\pm c} \qquad ...(1)$$

as the two required generators.

Remark. Since every generator of either system meets the plane $z = 0$ at a point of the principal elliptic section, we see that the two systems of lines obtained from (1) as θ varies from 0 to 2π are the two systems of generators of the hyperboloid. The form (1) of the equations of two systems of generators is often found more useful than the forms (A) and (B) obtained earlier.

25. Show that the lines

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4} \text{ and } \frac{x-2}{3}=\frac{y-3}{4}=\frac{z+4}{5}$$

are coplanar, and also find the equation of the plane containing them.

Hint: We note that

$$\begin{vmatrix} x_2-x_1 & y_2-y_1 & z_2-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = \begin{vmatrix} 2-1 & 3-2 & 4-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix} = 0$$

Therefore, the given lines intersect and so, are coplanar. Further,

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = \begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ 3 & 4 & 5 \end{vmatrix}$$

$= 0$ yields

$(x-1)(-1)+(y-2)(2)+(z-3)(-1)=0$ or $x-2y+z=0$, which is the equation of the plane containing the given lines.

Example 3. Show that the lines drawn from the origin parallel to the normals to

$$ax^2 + by^2 + cz^2 = 1$$

at its points of intersection with the planes $lx + my + nz = p$, generate the cone

$$p^2\left(\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}\right) = \left(\frac{lx}{a}+\frac{my}{b}+\frac{nz}{c}\right)^2.$$

Solution. Let (α, β, γ) be any point, on the curve of intersection of

$$ax^2 + by^2 + cz^2 = 1$$

and $$lx + my + nz = p$$

$\therefore$ $$a\alpha^2 + b\beta^2 + c\gamma^2 = 1 \qquad ...(1)$$

and $$l\alpha + m\beta + n\gamma = p \qquad ...(2)$$

The equations of the normal to the given conicoid at (α, β, γ) are

$$\frac{x-\alpha}{a\alpha} = \frac{y-\beta}{b\beta} = \frac{z-\gamma}{c\gamma} \qquad ...(3)$$

Therefore equations of the lines through the origin and parallel to the normal (3) are

$$\frac{x}{a\alpha} = \frac{y}{b\beta} = \frac{z}{c\gamma} \qquad ...(4)$$

Now, from equations (1) and (2), we have

$$a\alpha^2 + b\beta^2 + c\gamma^2 = \left(\frac{l\alpha + m\beta + n\gamma}{p}\right)^2 \qquad ...(5)$$

The required locus is obtained by eliminating α, β, γ between (4) and (5). Thus, we get

$$\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c} = \frac{1}{p^2}\left(\frac{lx}{a}+\frac{my}{b}+\frac{nz}{c}\right)^2$$

or $$p^2\left(\frac{x^2}{a}+\frac{y^2}{b}+\frac{z^2}{c}\right) = \left(\frac{lx}{a}+\frac{my}{b}+\frac{nz}{c}\right)^2$$

which is the required locus.

Example 3:

Show that three mutually perpendicular tangent lines can be drawn to the sphere $x^2 + y^2 + z^2 = r^2$ from any point on the sphere $2(x^2 + y^2 + z^2) = 3r^2$.

Solution:

Let P (α, β, γ) be any point on the sphere

$$2(x^2 + y^2 + z^2) = 3r^2.$$

$$\therefore \quad 2(\alpha^2 + \beta^2 + \gamma^2) = 3r^2. \qquad ...(1)$$

The equation of the enveloping cone drawn from the point P (α, β, γ) to the sphere $x^2 + y^2 + z^2 = r^2$ is $SS_1 = T^2$.

$$\Rightarrow \quad (x^2 + y^2 + z^2 - r^2)(\alpha^2 + \beta^2 + \gamma^2 - r^2) = (\alpha x + \beta y + \gamma z - r^2)^2.$$

This cone will have three mutually perpendicular tangent lines if

$$a + b + c = 0.$$

Now $\quad a + b + c = (\beta^2 + \gamma^2 - r^2) + (\alpha^2 + \gamma^2 - r^2) + (\alpha^2 + \beta^2 - \gamma^2)$

$$= 2(\alpha^2 + \beta^2 + \gamma^2) - 3r^2 = 0, \text{ by } (1).$$

Example 5:

If $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ *represents one of a set of three mutually perpendicular generators of cone 5yz – 8zx – 3xy = 0, find the equations of the other two.*

Solution:

Since the sum of the coefficients of x^2, y^2 and z^2 in the given cone is zero, so the cone has three mutually perpendicular generators.

The equation of the plane through the origin and perpendicular to the given line $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ is

$$1(x-0)+2(y-0)+3(z-0)=0 \text{ or } x+2y+3z=0. \quad \ldots(1)$$

Let $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ be one of the two lines in which the plane (1) cuts the cone $5yz-8zx-3xy=0$.

$$\therefore \qquad l+2m+3n=0,\ 5mn-8nl-3lm=0$$

From these equations, we obtain

$$5mn+8n(2m+3n)+3m(2m+3n)=0$$

$$\Rightarrow \qquad m^2+5mn+4n^2=0 \text{ or } \frac{m^2}{n^2}+5\frac{m}{n}+4=0$$

$$\Rightarrow \qquad \left(\frac{m}{n}+4\right)\left(\frac{m}{n}+1\right)=0. \qquad \therefore \frac{m}{n}=-1 \text{ or } -4.$$

Now $\quad l+2m+3n=0 \Rightarrow \frac{l}{n}=-3-2\frac{m}{n}$.

If $\quad \frac{m}{n}=-1$, then $\frac{l}{n}=-1$ and so $\frac{l}{n}=\frac{m}{1}=\frac{n}{-1}$.

If $\quad \frac{m}{n}=-4$, then $\frac{l}{n}=5$ and so $\frac{l}{5}=\frac{m}{1}=\frac{n}{1}$.

Hence the other two perpendicular generators are

$$\frac{x}{1}=\frac{y}{1}=\frac{z}{-1};\ \frac{x}{5}=\frac{y}{-4}=\frac{z}{1}.$$

Ex. 3. *Show that the normals from (x', y', z') to the paraboloid $ax^2+by^2=2cz$ lie on the cone*

$$\frac{x'}{x-x'}-\frac{y'}{y-y'}+c\,\frac{(1/a-1/b)}{z-z'}=0.$$

Or

Find the equation of the cone through the five feet of the normals drawn from (x', y', z') to the paraboloid $ax^2+by^2=2cz$.

Solution. The equation of the paraboloid is

$$ax^2+by^2=2cz. \qquad \text{.. (1)}$$

If (α, β, γ) be a point on (1), then the equations of the normal at (α, β, γ) are

$$\frac{x-\alpha}{a\alpha}=\frac{y-\beta}{b\beta}=\frac{z-\gamma}{-c}.$$

If it passes through (x', y', z'), we get

$$\frac{x'-\alpha}{a\alpha}=\frac{y'-\beta}{b\beta}=\frac{z'-\gamma}{-c}=\lambda \text{ (say)}.$$

$$\therefore \quad \alpha=x'/(1+a\lambda),\ \beta=y'/(1+b\lambda),\ \gamma=z'+c\lambda. \qquad \text{...(2)}$$

There are five points like (α, β, γ) the normals at which pass through (α, β, γ). [See § 4 (B)]

Now the equations of any line through (x', y', z') are

$$\frac{x-x'}{l}=\frac{y-y'}{m}=\frac{z-z'}{n}. \qquad \text{...(3)}$$

If (3) is a normal to the paraboloid (1) at the point (α, β, γ) then we clearly have $l=a\alpha,\ m=b\beta,\ n=-c$

or $$l=\frac{ax'}{1+a\lambda},\ m=\frac{by'}{1+b\lambda},\ n=-c \text{ [using (2)]}$$

or $$\frac{ax'}{l}=1+a\lambda,\ \frac{by'}{m}=1+b\lambda,\ \frac{c}{n}=-1.$$

To eliminate λ, multiplying these relations by b, $-a$ and $(b-a)$ respectively and adding, we have

$$\frac{abx'}{l}-\frac{aby'}{m}+\frac{c\,(b-a)}{n}=b\,(1+a\lambda)-a\,(1+b\lambda)-(b-a)$$

or $$\frac{abx'}{l}-\frac{aby'}{m}+\frac{c\,(b-a)}{n}=0.$$

Dividing by ab, we get

$$\frac{x'}{l}-\frac{y'}{m}+\frac{c\,(1/a-1/b)}{n}=0. \qquad \text{...(4)}$$

Eliminating l, m, n between (3) and (4), we get the locus of the normal (3). Hence the normal (3) lies on the cone

$$\frac{x'}{x-x'}-\frac{y'}{y-y'}+c.\frac{(1/a-1/b)}{z-z'}=0. \qquad \text{...(5)}$$

Ex. 8. *Show that the points of intersection P, Q of the generators of opposite systems drawn through the point $A(a\cos\alpha, b\sin\alpha, 0)$, $B(a\cos\beta, b\sin\beta, 0)$ of the principal elliptic section of the hyperboloid* $x^2/a^2+y^2/b^2-z^2/c^2=1$ *are*

$$\left(\frac{a\cos\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)}, \frac{b\sin\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)}, \pm\frac{c\sin\frac{1}{2}(\alpha-\beta)}{\cos\frac{1}{2}(\alpha-\beta)}\right).$$

Hence show that if A and B are extremities of semi-conjugate diameters, the loci of the points P and Q are the ellipses

$$x^2/a^2+y^2/b^2=2, \; z=\pm c.$$

Solution. Let the co-ordinates of one of the two points of intersection of the generators, say of the point P be (x_1, y_1, z_1). The equation of the tangent plane to the hyperboloid

$$x^2/a^2+y^2/b^2-z^2/c^2=1$$

at the point (x_1, y_1, z_1) on it is

$$xx_1/a^2+yy_1/b^2-zz_1/c^2=1. \qquad (1)$$

The tangent plane (1) meets the plane $z=0$ in the line which is given by

$$xx_1/a^2+yy_1/b^2=1, \; z=0. \qquad ...(2)$$

The equations of the line joining the points $A\,(a\cos\alpha, b\sin\alpha, 0)$ and $B\,(a\cos\beta, b\sin\beta, 0)$ are

$$(x/a)\cos\tfrac{1}{2}(\alpha+\beta)+(y/b)\sin\tfrac{1}{2}(\alpha+\beta)=\cos\tfrac{1}{2}(\alpha-\beta), \; z=0. \qquad ..(3)$$

The lines given by the equations (2) and (3) are the same. Hence comparing these equations, we have

$$\frac{x_1/a}{\cos\frac{1}{2}(\alpha+\beta)}=\frac{y_1/b}{\sin\frac{1}{2}(\alpha+\beta)}=\frac{1}{\cos\frac{1}{2}(\alpha-\beta)}. \qquad ...(4)$$

Since the point $P\,(x_1, y_1, z_1)$ lies on the given hyperboloid, we have

$$x_1^2/a^2+y_1^2/b^2-z_1^2/c^2=1. \qquad ...(5)$$

Putting the values of x_1/a and y_1/b from (4) in (5), we get

$$\frac{\cos^2\frac{1}{2}(\alpha+\beta)}{\cos^2\frac{1}{2}(\alpha-\beta)}+\frac{\sin^2\frac{1}{2}(\alpha+\beta)}{\cos^2\frac{1}{2}(\alpha-\beta)}-\frac{z_1^2}{c^2}=1$$

or

$$\frac{1}{\cos^2\frac{1}{2}(\alpha-\beta)}-\frac{z_1^2}{c^2}=1$$

or $\qquad z_1^2/c^2=\sec^2 \frac{1}{2} (\alpha-\beta)-1 \quad$ or $\quad z_1^2/c^2=\tan^2 \frac{1}{2} (\alpha-\beta)$

or $\qquad z_1/c=\pm\sin \frac{1}{2} (\alpha-\beta)/\cos \frac{1}{2} (\alpha-\beta). \qquad$ ...(6)

Hence from (4) and (6) the co-ordinates of the required points of intersection P and Q are

$$\left(\frac{a \cos \frac{1}{2} (\alpha+\beta)}{\cos \frac{1}{2} (\alpha-\beta)}, \frac{b \sin \frac{1}{2} (\alpha+\beta)}{\cos \frac{1}{2} (\alpha-\beta)}, \pm\frac{c \sin \frac{1}{2} (\alpha-\beta)}{\cos \frac{1}{2} (\alpha-\beta)}\right). \qquad \text{...(7)}$$

Second part. If the points A and B are the extremities of the semi-conjugate diameters, then we have $\alpha-\beta=\frac{1}{2}\pi$.

$\therefore \quad x_1^2/a^2+y_1^2/b^2=1/\cos^2 \frac{1}{2} (\alpha-\beta),\ z_1=\pm c \tan \frac{1}{2} (\alpha-\beta)$

or $\qquad x_1^2/a^2+y_1^2/b^2=1/\cos^2 \frac{1}{4}\pi,\ z_1=\pm c \tan \frac{1}{4}\pi$

or $\qquad x_1^2/a^2+y_1^2/b^2=2,\ z_1=\pm c.$

Therefore, the loci of the points P and Q are

$$x^2/a^2+y^2/b^2=2,\ z=\pm c. \qquad \textbf{Proved.}$$

Ex. 9. *The generators through P of $x^2/a^2+y^2/b^2-z^2/c^2=1$ meet the principal elliptic section at A and B. If the median of the triangle APB through P is parallel to the fix d plane $lx+my+nz=0$, show that P lies on the surface $z\,(lx+my)+n\,(c^2+z^2)=0$.*

Solution. The principal elliptic section of the hyperboloid $x^2/a^2+y^2/b^2-z^2/c^2=1$ by the plane $z=0$ (say) is $x^2/a^2+y^2/b^2=1$, $z=0$. The co-ordinates, of the points A and B on this section are $A\,(a \cos \alpha, b \sin \alpha, 0)$ and $B\,(a \cos \beta, b \sin \beta, 0)$. Let P be (x_1, y_1, z_1).

Proceeding as in Ex. 8 above, the values of x_1, y_1, z_1 are given by (4) and (6) of Ex. 8.

Now if M is the mid. point of AB then

$$M\equiv(\tfrac{1}{2}a\,(\cos \alpha+\cos \beta), \tfrac{1}{2}b\,(\sin \alpha+\sin \beta), 0)$$

i.e. $\qquad M\equiv(a \cos \frac{1}{2} (\alpha+\beta) \cos \frac{1}{2} (\alpha-\beta),$

$\qquad\qquad b \sin \frac{1}{2} (\alpha+\beta) \cos \frac{1}{2} (\alpha-\beta), 0).$

$\therefore$ The direction ratios of the median PM are

$x_1-a \cos \frac{1}{2} (\alpha+\beta) \cos \frac{1}{2} (\alpha-\beta),$

$\qquad y_1-b \sin \frac{1}{2} (\alpha+\beta) \cos \frac{1}{2} (\alpha-\beta),\ z_1-0$

or $\qquad x_1-x_1 \cos^2 \frac{1}{2} (\alpha-\beta),\ y_1-y_1 \cos^2 \frac{1}{2} (\alpha-\beta),\ z_1$

[using (4) of Ex. 8]

or $\qquad x_1 \sin^2 \frac{1}{2} (\alpha-\beta),\ y_1 \sin^2 \frac{1}{2} (\alpha-\beta),\ z_1$

or $\qquad x_1, y_1, z_1 \operatorname{cosec}^2 \frac{1}{2} (\alpha-\beta)$, or $x_1, y_1, z_1 \{1+\cot^2 \frac{1}{2} (\alpha-\beta)\}$

or $\qquad x_1, y_1, z_1 \{1+c^2/z_1^2\}, \qquad$ [using (6) of Ex. 8]

It is given that the median PM is parallel to the plane $lx+my+nz=0$. Hence we have

$$lx_1+my_1+nz_1\,(1+c^2/z_1^2)=0.$$

$\therefore$ The locus of $P(x_1, y_1, z_1)$ is the surface

$$lx+my+nz\,(1+c^2/z^2)=0.$$

Ex. 7. *The generators of the paraboloid* $x^2/a-y^2/b=4z$ *are drawn through the point* $(\alpha, 0, \gamma)$. *Prove that the angle between them is* $\cos^{-1}\left(\dfrac{a-b+\gamma}{a+b+\gamma}\right)$

Solution. Let the equations of any line through $(\alpha, 0, \gamma)$ be

$$\frac{x-\alpha}{l}=\frac{y-0}{m}=\frac{z-\gamma}{n}=r \text{ (say)}. \qquad \ldots(1)$$

Any point on (1) is $(\alpha+lr, mr, \gamma+nr)$. If it lies on the given paraboloid $x^2/a-y^2/b=4z$, we have

$$(\alpha+lr)^2/a-(mr)^2/b=4(\gamma+nr)$$

or $$r^2\,(l^2/a-m^2/b)+2r\,(\alpha l/a-2n)+(\alpha^2/a-4\gamma)=0. \qquad \ldots(2)$$

If the line (1) is a generator, it wholly lies on the given paraboloid so that (2) is an identity in r and consequently we have

$$l^2/a-m^2/b=0,\ \alpha l/a-2n=0,\ \alpha^2/a=4\gamma. \qquad \ldots(3)$$

From the first and the second of the results (3),

$$\frac{l}{\sqrt{a}}=\frac{m}{\pm\sqrt{b}}=\frac{2n\sqrt{a}}{\alpha}$$

or $$\frac{l}{2a}=\frac{m}{\pm 2\sqrt{(ab)}}=\frac{n}{\alpha}.$$

Hence the d.c.'s of the two generators through $(\alpha, 0, \gamma)$ are proportional to $2a, 2\sqrt{(ab)}, \alpha$ and $2a, -2\sqrt{(ab)}, \alpha$.

If θ is the angle between the generators, we have

$$\cos\theta=\frac{4a^2-4ab+\alpha^2}{4a^2+4ab+\alpha^2}, \quad \text{or} \quad \cos\theta=\frac{4a^2-4ab+4a\gamma}{4a^2+4ab+4a\gamma}, \text{ using (3)}$$

or $$\cos\theta=\frac{a-b+\gamma}{a+b+\gamma} \quad \text{or} \quad \theta=\cos^{-1}\left(\frac{a-b+\gamma}{a+b+\gamma}\right).$$

Example 1:

A straight line is drawn through a variable point on the ellipse $(x^2/a^2) + (y^2/b^2) = 1$, $z = 0$ to meet two fixed lines $y = mx$, $z = c$ and $y = -mx$, $z = -c$. Find the locus of the straight line.

Solution:

Given fixed lines are $y - mx = 0,\ z - c = 0;$ ...(i)

$$y + mx = 0,\ z + c = 0 \quad ...(ii)$$

and the ellipse is $(x^2/a^2) + (y^2/b^2) = 1,\ z = 0$ ...(iii)

And line intersecting lines (i) and (ii) is

$$(y - mx) + k_1 (z - c) = 0,$$

$$(y + mx) + k_2 (z + c) = 0. \quad ...(iv)$$

If it meets the ellipse (iii), then we are to eliminate k_1, k_2 from (iii) and (iv).

Putting $z = 0$ in (iv), we get $y - mx - k_1c = 0$, $y + mx + k_2c = 0$

$\Rightarrow$ $$mx - y + k_1c = 0,\ mx + y + k_2c = 0$$

Adding and subtracting these, we get

$$x = -\frac{(k_1 + k_2)\ c}{2m},\ y = \frac{(k_1 - k_2)\ c}{2}$$

Substituting these values of x and y in (iii), we get

$$\frac{(k_1 + k_2)^2\ c^2}{4a^2m^2} + \frac{(k_1 - k_2)^2\ c^2}{4b^2} = 1$$

$\Rightarrow$ $(k_1 + k_2)^2\ c^2b^2 + (k_1 - k_2)^2\ c^2\ a^2\ m^2 = 4a^2b^2m^2$

$\Rightarrow$ $$\left\{\left(\frac{mx - y}{z - c}\right) + \left(-\frac{mx + y}{z + c}\right)\right\}^2 c^2\ b^2 + \left\{\frac{mx - y}{z - c} + \frac{mx + y}{z + c}\right\}^2 c^2a^2m^2$$

$$= 4a^2b^2m^2m \text{ from (iv)}$$

$\Rightarrow$ $\{(mx - y)\ (z + c) - (mx + y)\ (z - c)\}^2\ c^2b^2$

$+\{(mx - y)\ (z+c) + (mx+y)\ (z - c)\}^2\ c^2a^2m^2 = 4a^2b^2m^2\ (z^2 - c^2)^2$

$\Rightarrow$ $(cmx - yz)^2\ c^2b^2 + (mxz - cy)^2\ c^2a^2m^2 = a^2b^2m^2\ (z^2 - c^2)^2,$

which is the required locus. **Ans.**

Example 13:

The plane $lx + my = 0$ is rotated about the line of intersection with the plane $z = 0$ through an angle α. Prove that the equation to the plane in its new position is

$$lx + my \pm z\sqrt{(l^2 + m^2)}\ \tan\alpha = 0.$$

Solution:

The equation of any plane through the line of intersection of the planes $lx + my = 0$ and $z = 0$ is given by $(lx + my) = \lambda z = 0$ ...(i)

If (i) represents the planes obtained by rotating the given plane

$$lx + my = 0 \qquad \text{...(ii)}$$

through an angle α about the line of intersection of (ii) and the plane $z = 0$, then the angle between the planes (i) and (ii) is α and thus we have

$$\cos\alpha = \frac{l.l + m.m + \lambda.0}{\sqrt{(l^2 + m^2 + \lambda^2)}.\sqrt{(l^2 + m^2 + 0^2)}}$$

$\Rightarrow$ $(l^2 + m^2 + \lambda^2)(l^2 + m^2)\cos^2\alpha = (l^2 + m^2)^2$, squaring and cross multiplying

$\Rightarrow$ $(l^2 + m^2 + \lambda^2)\cos^2\alpha = (l^2 + m^2)$ or $l^2 + m^2 + \lambda^2 = (l^2 + m^2)\sec^2\alpha$

$\Rightarrow$ $\lambda^2 = (l^2 + m^2)\sec^2\alpha - (l^2 + m^2) = (l^2 + m^2)\tan^2\alpha$

$\Rightarrow$ $\lambda = \pm\sqrt{(l^2 + m^2)}\ \tan\alpha$

$\therefore$ From (i) the required equation is

$$lx + my \pm z\sqrt{(l^2 + m^2)}\ \tan\alpha = 0.$$ **Ans.**

Example 1:

Find the incentre of the tetrahedron formed by the planes $x = 0$, $y = 0$, $z = 0$ and $x + y + z = a$.

Solution:

Evidently the planes $x = 0$, $y = 0$ and $z = 0$ meet in $(0, 0, 0)$. Hence the incentre lies on the perpendicular from $(0, 0, 0)$ to the plane $x + y + z = a$ and divides it in the ratio 3: 1 [3 from the vertex $(0, 0, 0)$ and 1 from the plane $x + y + z = a$].

The equations of the perpendicular from $(0, 0, 0)$ to the plane $x + y + z = a$ is

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{1} = r \text{ (say)}$$

Any point on this perpendicular is (r, r, r). If it lies on the plane $x + y + z = a$, then we have $r + r + r = a$ or $r = a/3$.

$\therefore$ The perpendicular from $(0, 0, 0)$ meets the plane $x + y + z = a$ in (r, r, r) *i.e.*, $\left(\frac{1}{3}a, \frac{1}{3}a, \frac{1}{3}a\right)$. Also the incentre divides the join of $(0, 0, 0)$ and $\left(\frac{1}{3}a, \frac{1}{3}a, \frac{1}{3}a\right)$ in the ratio 3: 1, therefore if (x_1, y_1, z_1) be the required incentre,

Then we have $x_1 = \dfrac{3.\frac{1}{3}a + 1.0}{3+1} = \dfrac{1}{4}a$.

Similarly $\quad y_1 = \dfrac{1}{4}a = z.$

$\therefore$ The required incentre is $\left(\frac{1}{4}a, \frac{1}{4}a, \frac{1}{4}a\right)$. **Ans.**

Example 36:

P is a point on the plane lx + my + nz = p. A point Q is taJ›n on the line OP such that OP.OQ = P^2, prove that the locus of Q is

$$p\,(lx + my + nz) = x^2 + y^2 + z^2.$$

Solution:

Let Q be the point (α, β, γ) and **OQ = R. Then the direction ratios of the** line OQ are α − 0, β − 0, γ − 0, ***i.e.,* α, β, γ.**

∴ The direction cosines of OQ **are α/R, β/R, γ/R,** where

$$R = OQ = \sqrt{(\alpha^2 + \beta^2 + \gamma^2)} \qquad \text{...(i)}$$

∴ The equations of the line OQ are $\dfrac{x-0}{\alpha/R} = \dfrac{y-0}{\beta/R} = \dfrac{z-0}{\gamma/R} = r$ (say),

where r is the distance of any point from (0, 0, 0)

Let OP = r, then the co-ordinates of P are $\left(\dfrac{\alpha r}{R}, \dfrac{\beta r}{R}, \dfrac{\gamma r}{R}\right)$

But it is given that P is a point on the plane *l*x + my + nz = p. Then we have

$$\therefore \qquad l\frac{\alpha r}{R} + m\frac{\beta r}{R} + n\frac{\gamma r}{R} = P$$

or $$\frac{r}{R}\,(l\alpha + m\beta + n\gamma) = p, \qquad \text{...(ii)}$$

It is given that OP.OQ = p^2.

⇒ $r \,.\, R = p^2$, ∴ OP = r, OQ = R

⇒ $r = p^2/R$.

∴ from (ii) we get $(p^2/R^2)\,(l\alpha + m\beta + n\gamma) = p$

⇒ $p\,(l\alpha + m\beta + n\gamma) = R^2 = \alpha^2 + \beta^2 + \gamma^2$, from (i).

∴ The locus of Q (α, β, γ) is $p\,(lx + my + nz) = x^2 + y^2 + z^2$.

Hence proved.

Example 38:

Find the distance of the point (1, −2, 3) from the plane $x - y + z = 5$ measured parallel to the line $\frac{1}{2}x = \frac{1}{3}y = -\frac{1}{6}z$.

Solution:

We have to find the distance of this point from the given plane measured parallel to the given line whose direction ratios are 2, 3, − 6.

Now the equations of the line through (1, − 2, 3) and parallel to the line whose d.c.'s are 2, 3, − 6 are $\frac{x-1}{2} = \frac{y+2}{3} = \frac{z-3}{-6}$ ...(i)

Any point on this line (i) is $(1 + 2r, - 2 + 3r, 3 - 6r)$

If it is lies on the given plane $x - y + z = 5$, then we have

$$(1 + 2r) - (- 2 + 3r) + (3 - 6r) = 5$$

$\Rightarrow$ $1 - 7r = 0$ or $r = (1/7)$. Substituting this value of r in (i), the point is [(9/7), (−11/7), (15/7)] and therefore the required distance of this point from the given point (1, − 2, 3)

$$= \left[\left(\frac{9}{7}-1\right)^2 + \left(-\frac{11}{7}+2\right)^2 + \left(\frac{15}{7}-3\right)^2\right] = \sqrt{\left[\frac{4}{49}+\frac{9}{49}+\frac{36}{49}\right]} = 1 \text{. } \textbf{Ans.}$$

Example 75:

Show that the S.D. between the line $x + a = 2y = -12z$ and $x = y + 2a = 6z - 6a$ is $2a$.

Solution:

The equation of given line

$$\frac{x+a}{2} = \frac{y}{1} = \frac{z}{-1/6}$$

and $$\frac{x}{1} = \frac{y+2a}{1} = \frac{z-a}{1/6}$$

$\Rightarrow$ $$\frac{x+a}{12} = \frac{y}{6} = \frac{z}{-1} \quad \text{...(i)}$$

and $$\frac{x}{6} = \frac{y+2a}{6} = \frac{z-a}{1} \quad \text{...(ii)}$$

Let l, m, n be the d.c.'s of the line S.D. to the given lines, then S.D. being perpendicular to both the lines (i) and (ii) we have

$$12l + 6m - n = 0 \quad \text{and } 6l + 6m + n = 0.$$

Solving these equations we have

$$\frac{l}{2} = \frac{m}{-3} = \frac{n}{6} = \frac{\sqrt{(l^2 + m^2 + n^2)}}{\sqrt{(2^2 + 3^2 + 6^2)}} = \frac{1}{7}$$

$\therefore$ d.c.'s of S.D. are 2/7, – 3/7, 6/7. ...(iii)

Also A (– a, 0, 0) is point on the line (i) and B (0, – 2a, a) is a point on the line (ii).

$\therefore$ Required S.D. = projection of AB on a line whose d.c.'s are given by (iii)

$= \frac{2}{7}\ [0 + a] - \frac{3}{7}\ [-2a - 0] + \frac{6}{7}\ [a - 0] = 2a$ **Hence proved.**

Exercise 1.64: Find the equation of the right circular cylinder described on the circle through the points $A(a, 0, 0)$, $B(0, a, 0)$, $C(0, 0, a)$ as the guiding curve.

Solution: The equation of the plane through the points $A(a, 0, 0)$, $B(0, a, 0)$, $C(0, 0, a)$ is $x/a + y/a + z/a = 1$, or $x + y + z = a$.

It is obvious that the given points A, B, C are equidistant from the origin $O(0, 0, 0)$. Therefore, a sphere with a centre O and radius OA = 'a', passes through A, B and C. Hence the guiding curve is a circle given by

$$x^2 + y^2 + z^2 = a^2 \text{ and } x + y + z = a. \qquad ...(1.153)$$

The perpendicular from the centre $O(0, 0, 0)$ of the sphere on the plane $x + y + z = a$ passes through the centre of this circle. Therefore, it is the axis of the cylinder with direction ratios $<1, 1, 1>$. Also the length of the perpendicular from $O(0, 0, 0)$ to the plane $x + y + z = a$ is $\frac{a}{\sqrt{3}}$.

From Fig. 1.30, radius 'r' of the circle (1. 153)

= radius of the cylinder,

which is given by $r^2 = a^2 - \frac{a^2}{3} = \frac{2a^2}{3}$ or, $r = a\sqrt{2/3}$.

Now if $P(x, y, z)$ is any point on the cylinder, then

$OP^2 = ON^2 + NP^2$, which gives

$$x^2 + y^2 + z^2 = \left(\frac{1.x + 1.y + 1.z}{\sqrt{1+1+1}}\right)^2 + \frac{2}{3}a^2,$$

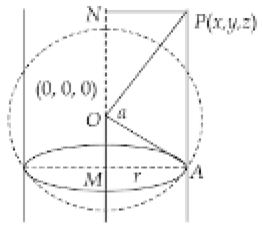


Fig. 1.30

since ON is the projection of OP on the axis with $d.r$'s $<1, 1, 1>$, of the cylinder.

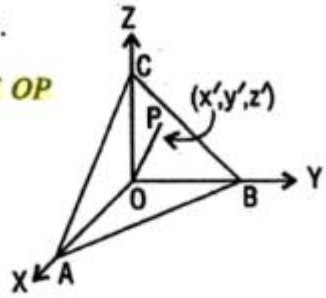
Simplifying we obtain

$$x^2 + y^2 + z^2 - yz - zx - xy = a^2,$$

as the required equation of the cylinder.

Ex. 43. *From a point $P(x', y', z')$, a plane is drawn at right angle to OP to meet the co-ordinate axes at A, B, C; prove that the area of the triangle ABC is $\frac{r^5}{2x'y'z'}$ where r is the measure of OP.*

Sol. Direction ratios of OP are $x'-0, y'-0, z'-0$. *i.e.* x', y', z'.



Direction cosines of OP are

$$\frac{x'}{\sqrt{(x'^2+y'^2+z'^2)}}, \frac{y'}{\sqrt{(x'^2+y'^2+z'^2)}}, \frac{z'}{\sqrt{(x'^2+y'^2+z'^2)}} \text{ i.e. } \frac{x'}{r}, \frac{y'}{r}, \frac{z'}{r}.$$

Equation of the plane ABC is

$$\frac{x'}{r}x+\frac{y'}{r}y+\frac{z'}{r}z=r \qquad [lx+my+nz=p]$$

$\Rightarrow x'x+y'y+z'z=r^2$ This plane meets the axes in

$$A\left(\frac{r^2}{x'}, 0, 0\right), B\left(0, \frac{r^2}{y'}, 0\right), C\left(0, 0, \frac{r^2}{z'}\right)$$

If $\Delta_x, \Delta_y, \Delta_z$ are the areas of projection of the triangle ABC on the yz-plane, zx-plane and xy-plane

then $\Delta_z = \text{triangle } OAB = \frac{1}{2}\cdot\frac{r^2}{x'}\cdot\frac{r^2}{y'} = \frac{1}{2}\cdot\frac{r^4}{x'y'}$

$\Delta_x = \text{triangle } OBC = \frac{1}{2}\cdot\frac{r^2}{y'}\cdot\frac{r^2}{z'} = \frac{1}{2}\cdot\frac{r^4}{y'z'}$

$\Delta_y = \text{triangle } OAC = \frac{1}{2}\cdot\frac{r^2}{x'}\cdot\frac{r^2}{z'} = \frac{1}{2}\cdot\frac{r^4}{x'z'}$

If the area of the triangle ABC is Δ, then

$$\Delta = \sqrt{(\Delta_x^2+\Delta_y^2+\Delta_z^2)}$$

$$= \frac{1}{2}r^4\left[\frac{1}{(x'y')^2}+\frac{1}{(y'z')^2}+\frac{1}{(z'x')^2}\right]^{1/2}$$

$$= \frac{1}{2}r^4\left[\frac{x'^2+y'^2+z'^2}{x'^2y'^2z'^2}\right]^{1/2} = \frac{1}{2}r^4\cdot\frac{r}{x'y'z'}$$

$$= \frac{1}{2}\frac{r^5}{x'y'z'}.$$

Example 1. Prove that the locus of the poles of the tangent planes of the conicoid $ax^2 + by^2 + cz^2 = 1$ with respect to the conicoid $\alpha x^2 + \beta y^2 + \gamma z^2 = 1$, is the conicoid

$$\frac{\alpha^2 x^2}{a} + \frac{\beta^2 y^2}{b} + \frac{\gamma^2 z^2}{c} = 1.$$

Solution. Any tangent plane to the conicoid $ax^2 + by^2 + cz^2 = 1$ is

$$lx + my + nz = p, \tag{1}$$

where
$$p^2 = \frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c}. \tag{2}$$

Let (x', y', z') be the pole of (1). Then its polar plane with respect to the conicoid $\alpha x^2 + \beta y^2 + \gamma z^2 = 1$ is

$$\alpha xx' + \beta yy' + \gamma zz' = 1. \tag{3}$$

Since (1) and (3) are identical, on comparing, we get

$$\frac{\alpha x'}{l} = \frac{\beta y'}{m} = \frac{\gamma z'}{n} = \frac{1}{p}. \tag{4}$$

Eliminating l, m, n from (2) and (4), we get

$$p^2 = \frac{p^2\alpha^2 x'^2}{a} + \frac{p^2\beta^2 y'^2}{b} + \frac{p^2\gamma^2 z'^2}{c}$$

$$\Rightarrow \quad \frac{\alpha^2 x'^2}{a} + \frac{\beta^2 y'^2}{b} + \frac{\gamma^2 z'^2}{c} = 1.$$

Hence the locus of (x', y', z') is

$$\frac{\alpha^2 x^2}{a} + \frac{\beta^2 y^2}{b} + \frac{\gamma^2 z^2}{c} = 1.$$

Example 16:

If the line of intersection of perpendicular tangent planes to the ellipsoid whose equation referred to rectangular axes is $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ passes through the fixed point (0, 0, k), show that it lies on the cone

$$x^2 (b^2 + c^2 - k^2) + y^2 (c^2 + a^2 - k^2) + (z - k)^2 (a^2 + b^2) = 0.$$

Solution:

The equation of any plane through (0, 0, k) is

$$l(x - 0) + m(y - 0) + n(z - k) = 0$$

$$\Rightarrow \quad lx + my + nz = nk. \qquad ...(i)$$

If the plane (i) is a tangent plane to the given ellipsoid, then

$$a^2l^2 + m^2b^2 + c^2n^2 = n^2k^2$$

$$a^2l^2 + m^2b^2 + n^2 (c^2 - k^2) = 0. \qquad ...(ii)$$

Again the equation of any line through (0, 0, k) is

$$\frac{x}{\lambda} = \frac{y}{\mu} = \frac{z - k}{\nu} \qquad ...(iii)$$

Since this line (iii) lines on the plane (i), so we have

$$l\lambda + m\mu + n\nu = 0. \qquad ...(iv)$$

From (ii) and (iv) we find that there are two sets of values of the d.c.'s l, m, n of the normal to (i) and hence there will be two tangent planes to the given ellipsoid.

Eliminating n between (ii) and (iv) we get

$$a^2l^2 + b^2m^2 + (c^2 - k^2)\left(\frac{-l\lambda + m\mu}{\nu}\right)^2 = 0$$

$$\Rightarrow l^2(a^2\nu^2 + \lambda^2c^2 - \lambda^2k^2) + m^2(b^2\nu^2 + \mu^2c^2 - \mu^2k^2) + 2lm\lambda\mu\,(c^2 - k^2) = 0$$

$$\Rightarrow (a^2\nu^2 + \lambda^2c^2 - \lambda^2k^2)(l/m^2)^2 + \lambda\mu\,(c^2 - k^2)(l/m) + (b^2\nu^2 + \mu^2c^2 - \mu^2k^2) = 0$$

If its roots are l_1/m_1 and l_2/m_2, then

$$\frac{l_1l_2}{m_1m_2} = \text{product of the roots} = \frac{b^2\nu^2 + \mu^2c^2 - \mu^2k^2}{a^2\nu^2 + \lambda^2c^2 - \lambda^2k^2}$$

$$\Rightarrow \frac{l_1l_2}{b^2\nu^2 + \mu^2\left(c^2 - k^2\right)} = \frac{m_1m_2}{a^2\nu^2 + \lambda^2\left(c^2 - k^2\right)} = \frac{n_1n_2}{b^2\lambda^2 + a^2\mu^2}, \quad ...(v)$$

by symmetry. **(Note)**

Also as the planes are perpendicular, so we have

$$l_1l_1 + m_1m_1 + n_1n_1 = 0$$

$\Rightarrow [b^2\nu^2 + \mu^2(c^2 - k^2)] + [a^2\nu^2 + \lambda^2(c^2 - k^2)] + [b^2\lambda^2 + a^2\mu^2] = 0,$ from (v)

$$\Rightarrow \lambda^2(b^2 + c^2 - k^2) + \mu^2(a^2 + c^2 - k^2) + \nu^2(a^2 + b^2) = 0. \quad ...(vi)$$

Eliminating λ, μ, ν between (iii) and (vi); we find that the cone generated by the line (ii) is

$$x^2(b^2 - c^2 - k^2) + y^2(a^2 + c^2 - k^2) + (z - k)^2(a^2 + b^2) = 0$$

Hence proved.

Example 3. Show that the feet of the normals from the point (α, β, γ) to the paraboloid $x^2 + y^2 = 2\,az$ lie on the sphere

$$x^2+y^2+z^2-y\left(\frac{a^2+b^2}{2\beta}\right)-z\,(a+\gamma)=0.$$

Solution. Normal at any point (x', y', z') to the given paraboloid is given by the equations

$$\frac{x-x'}{x'} = \frac{y-y'}{y'} = \frac{z-z'}{-a} = \lambda \text{ (say)}$$

If it passes through (α, β, γ), then

$$\frac{\alpha-x'}{x'} = \frac{\beta-y'}{y'} = \frac{\gamma-z'}{-a} = \lambda$$

$$\therefore \qquad x' = \frac{\alpha}{1+\lambda}, y' = \frac{\beta}{1+\lambda}, z' = \gamma + a\,\lambda$$

i.e. the feet of the normals are

$$\left(\frac{\alpha}{1+\lambda}, \frac{\beta}{1+\lambda}\,\gamma + a\lambda\right)$$

It lies on the paraboloid, therefore

$$\left(\frac{\alpha}{1+\lambda}\right)^2 + \left(\frac{\beta}{1+\lambda}\right)^2 = 2\,a(\gamma + a\,\lambda)$$

or $$(\alpha^2+\beta^2) = 2\,a(\gamma+a\,\lambda)(1+\lambda)^2 \qquad ...(1)$$

If it lies on the sphere then

$$\left(\frac{\alpha}{1+\lambda}\right)^2 + \left(\frac{\beta}{1+\lambda}\right)^2 + (\gamma + a\,\lambda)^2$$

$$-\frac{\beta}{1+\lambda}\left(\frac{\alpha^2+\beta^2}{2\beta}\right) - (\gamma + a\lambda)(a+\gamma) = 0$$

or $$\frac{\alpha^2+\beta^2}{(1+\lambda)^2} - \frac{\alpha^2+\beta^2}{2\,(1+\lambda)} + (\gamma+a\lambda)(\gamma+a\lambda-a-\gamma) = 0$$

or $$\frac{\alpha^2+\beta^2}{2\,(1+\lambda)^2}(2-1-\lambda) - a(\gamma+a\lambda)(1-\lambda) = 0$$

or $$\frac{1-\lambda}{2\,(1+\lambda)^2}[(\alpha^2+\beta^2) - 2a(\gamma+a\lambda)(1+\lambda)^2] = 0$$

Therefore $\alpha^2 + \beta^2 - 2\,a\,(\gamma + a\,\lambda)\,(1+\lambda)^2 = 0$, since $\lambda \neq 1$, which is the same as (1).

Hence the result.

Example 7. From a point P (x', y', z') a plane is drawn at right angles to OP to meet the coordinate axes at A, B, C ; prove that the area of triangle ABC is $\dfrac{r^5}{2x'y'z'}$, where r is the measure of OP.

Solution. The direction cosines of OP are proportional to x', y', z'. Hence equation of any plane perpendicular to OP is

$$x'x + y'y + z'z = d$$

If it passes through P (x', y', z'), we have

$$d = x'^2 + y'^2 + z'^2 = r^2$$

Hence the plane through P (x', y', z') and perpendicular to OP is

$$xx' + yy' + zz' = r^2$$

Therefore Coordinates of A, B, C are

$$\left(\frac{r^2}{x'}, 0, 0\right), \left(0, \frac{r^2}{y'}, 0\right) \text{ and } \left(0, 0, \frac{r^2}{z'}\right).$$

Now the projection of ΔABC on the plane XOY is ΔOAB.

$$\therefore \qquad \Delta ABC \cos\gamma = \Delta OAB = \frac{1}{2}\frac{r^2}{x'}\cdot\frac{r^2}{y'} = \frac{1}{2}\frac{r^4}{x'y'}$$

where γ is the angle between the planes ABC and OAB *i.e.* it is the angle between z-axis and the normal to the plane. But *d.c.*'s of the normal are proportional to x', y', z'.

$$\therefore \qquad \cos\gamma = \frac{z'}{\sqrt{(x'^2+y'^2+z'^2)}} = \frac{z'}{r}$$

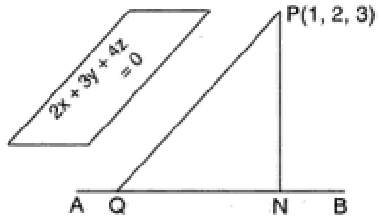
$$\text{Hence} \qquad \Delta ABC = \frac{1}{2}\,\frac{r^5}{x'y'z'}.$$

Example 5. *The equations of AB are* $\frac{x}{1}=\frac{y}{-2}=\frac{z}{3}$. *Through a point P(1, 2, 3), PN is drawn perpendicular to AB, and PQ is drawn parallel to the plane 2x + 3y + 4z = 0 to meet AB in Q. Find the equations of PN and PQ and the coordinates of N and Q.*

Sol. The given line AB is $\frac{x}{1}=\frac{y}{-2}=\frac{z}{3}=r$ (say)

Any point on this line is $(r, -2r, 3r)$ ...(1)

(a) Let this be the point N. Then, the d.r.'s of PN are $r-1, -2r-2, 3r-3$ and the d.r.'s of AB are $1, -2, 3$.



Since PN ⊥ AB,

$\therefore$ $1(r-1)+(-2)(-2r-2)+3(3r-3)=0$

or $14r-6=0$ or $r=\frac{3}{7}$.

$\therefore$ From (1), N is $\left(\frac{3}{7}, -\frac{6}{7}, \frac{9}{7}\right)$

Also the equations of PN are $\dfrac{x-1}{1-\frac{3}{7}}=\dfrac{y-2}{2+\frac{6}{7}}=\dfrac{z-3}{3-\frac{9}{7}}$ | Two point form

or $\dfrac{x-1}{1}=\dfrac{y-2}{5}=\dfrac{z-3}{3}$.

(b) Let the point given by (1) be the point Q, where PQ is ∥ to the plane.

The d.r.'s of PQ are $r-1, -2r-2, 3r-3$.

Also coeffs. of x, y, z in the plane are 2, 3, 4.

Since the line PQ is ∥ to the plane,

$\therefore$ $2(r-1)+3(-2r-2)+4(3r-3)=0$ $[al+bm+cn=0]$

or $8r-20=0$ or $r=\frac{5}{2}$.

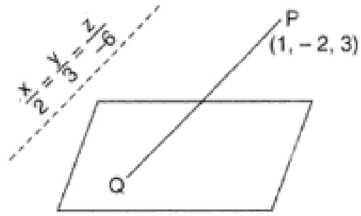
$\therefore$ From (1), the point Q is $\left(\frac{5}{2}, -5, \frac{15}{2}\right)$

Now equations of PQ are $\dfrac{x-1}{\frac{5}{2}-1}=\dfrac{y-2}{-5-2}=\dfrac{z-3}{\frac{15}{2}-3}$ | Two point form

or $\dfrac{x-1}{3}=\dfrac{y-2}{-14}=\dfrac{z-3}{9}$.

Example 2. *Find the distance of the point (1, – 2, 3) from the plane x – y + z = 5 measured parallel to the line* $\frac{x}{2} = \frac{y}{3} = \frac{z}{-6}$ *, the axes being rectangular.*

Sol. The given line is $\quad \frac{x}{2} = \frac{y}{3} = \frac{z}{-6} \qquad ...(i)$



Its d.r.'s are 2, 3, – 6.

Dividing each by $\sqrt{4+9+36} = 7$, the d.c.'s of the line (i) are $\frac{2}{7}, \frac{3}{7}, -\frac{6}{7}$.

Now equations of the line through P(1, – 2, 3) and ∥ to (i) are

$$\frac{x-1}{\frac{2}{7}} = \frac{y+2}{\frac{3}{7}} = \frac{z-3}{-\frac{6}{7}} = r \text{ (say)} \qquad ...(ii)$$

Since we have taken the actual d.c.'s in this case, r will give us the actual distance of any point from the given point P(1, – 2, 3). | **Note**

Now any point on the line (ii) is $Q\left(\frac{2}{7}r + 1, \frac{3}{7}r - 2, -\frac{6}{7}r + 3\right)$.

If this lies on the plane $\qquad x - y + z = 5$

then $\qquad \frac{2}{7}r + 1 - \left(\frac{3}{7}r - 2\right) + \left(-\frac{6}{7}r + 3\right) = 5 \quad \text{or} \quad -r + 1 = 0 \quad \therefore \quad r = 1$

which gives the required distance.

12.5. DIRECTOR SPHERE OF AN ELLIPSOID

The locus of the point of intersection of three mutually perpendicular tangent planes to an ellipsoid is called the **Director sphere.**

12.6. CONDITION THAT THE PLANE lx + my + nz = p, MAY TOUCH

(1) The ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, is $a^2l^2 + b^2m^2 + c^2n^2 = p^2$

(2) The hyperboloid of one sheet $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$, is $a^2 l^2 + b^2 m^2 - c^2n^2 = p^2$

(3) The hyperboloid of two sheets $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$, is $a^2 l^2 - b^2 m^2 - c^2n^2 = p^2$

(4) The paraboloid $\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = \frac{2z}{c}$ is $a^2l^2 \pm b^2m^2 + 2pn = 0$.

Q. 8. *Prove that the feet of the six normals from (α, β, γ) the ellipsoid.*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$ *on the curve of intersection of the ellipsoid and the cone*

$$\frac{a^2(b^2-c^2)\alpha}{x}+\frac{b^2(c^2-a^2)\beta}{y}+\frac{c^2(a^2-b^2)\gamma}{z}=0.$$

Sol. The ellipsoid is $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

Equation of the normal at (x_1, y_1, z_1), are

$$\frac{x-x_1}{\frac{x_1}{a^2}}=\frac{y-y_1}{\frac{y_1}{b^2}}=\frac{z-z_1}{\frac{z_1}{c^2}}$$

If it passes through (α, β, γ), then

$$\frac{\alpha-x_1}{\frac{x_1}{a^2}}=\frac{\beta-y_1}{\frac{y_1}{b^2}}=\frac{\gamma-z_1}{\frac{z_1}{c^2}}=\lambda \text{ (say)}$$

Then six feet of the normals from (α, β, γ) are given by

$$x_1=\frac{a^2\alpha}{a^2+\lambda},\ y_1=\frac{b^2\beta}{b^2+\lambda},\ z_1=\frac{c^2\gamma}{c^2+\lambda}$$

These give

$$\therefore \quad a^2+\lambda=\frac{a^2\alpha}{x_1},\ b^2+\lambda=\frac{b^2\beta}{y_1},\ c^2+\lambda=\frac{c^2\gamma}{z_1}$$

Multiplying these equations by b^2-c^2, c^2-a^2, a^2-b^2 and adding, we get

| **Note this step**

$$\frac{a^2\alpha(b^2-c^2)}{x_1}+\frac{b^2\beta(c^2-a^2)}{y_1}+\frac{c^2\gamma(a^2-b^2)}{z_1}$$
$$=(a^2+\lambda)(b^2-c^2)+(b^2+\lambda)(c^2-a^2)+(c^2+\lambda)(a^2-b^2)$$
$$=a^2(b^2-c^2)+b^2(c^2-a^2)+c^2(a^2-b^2)+\lambda(b^2-c^2)+\lambda(c^2-a^2)+\lambda(a^2-b^2)$$
$$=0+\lambda(0)=0$$

$\therefore$ (x_1, y_1, z_1) *i.e.*, the feet of the normals, lie on the cone

$$\frac{a^2\alpha(b^2-c^2)}{x}+\frac{b^2(c^2-a^2)\beta}{y}+\frac{c^2(a^2-b^2)\gamma}{z}=0$$

Also the feet (x_1, y_1, z_1) of the normals lie on the ellipsoid (1). Thus the feet of the six normals lie on the curve of intersection of the ellipsoid and the above cone.

Note. In the equation of the cone through the feet of six normals from a point to an ellipsoid.
Co-eff. of $x^2 = 0$, co-eff. of $y^2 = 0$, co-eff. of $z^2 = 0$, constant term = 0.

Q. 9. *Prove that the normals from (α, β, γ) to the paraboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}=2z$ lie on the cone*

$$\frac{\alpha}{x-\alpha}-\frac{\beta}{y-\beta}+\frac{a^2-b^2}{z-\gamma}=0.$$

Sol. The given paraboloid is

$$\frac{x^2}{a^2}+\frac{y^2}{b^4}=2z \qquad \text{...(1)}$$

Let any line through (α, β, γ) be

$$\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{n} \qquad \text{...(2)}$$

be the normal at (x_1, y_1, z_1) to (1).

The equation of the tangent plane at (x_1, y_1, z_1) to (1) is

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}-(z+z_1)=0 \qquad \text{...(3)}$$

Since (2) is normal to (3) $\therefore$ it is $\parallel$ to the normal to (3)

$$\therefore \qquad \frac{l}{\frac{x_1}{a^2}}=\frac{m}{\frac{y_1}{b^2}}=\frac{n}{-1}=k \text{ (say)} \qquad \text{...(4)}$$

Again if the normal at (x_1, y_1, z_1) to (1) passes through (α, β, γ),

Then $$x_1=\frac{a^2\alpha}{a^2+\lambda}, y_1=\frac{b^2\beta}{b^2+\lambda}, z_1=\gamma+\lambda. \qquad \text{| See Q . 10 ...(5)}$$

From (4), $$l=k\,\frac{x_1}{a^2}=\frac{k}{a^2}\cdot\frac{a^2\alpha}{a^2+\lambda}=\frac{k\alpha}{a^2+\lambda} \qquad \text{| Using (5)}$$

or $$a^2+\lambda=\frac{k\alpha}{l} \qquad \text{...(6)}$$

$$m=k\,\frac{y_1}{b^2}=\frac{k}{b^2}\cdot\frac{b^2\beta}{b^2+\lambda}=\frac{k\beta}{b^2+\lambda} \quad \text{or} \quad b^2+\lambda=\frac{k\beta}{m} \qquad \text{...(7)}$$

$$n=-k \qquad \text{...(8)}$$

Subtracting (7) from (6), we get

$$a^2-b^2=k\left(\frac{\alpha}{l}-\frac{\beta}{m}\right)=-n\left(\frac{\alpha}{l}-\frac{\beta}{m}\right) \qquad \text{...(9) | Using (8)}$$

To find the locus, we have to eliminate l, m, n from (2) and (9).

Putting the value of l, m, n from (2) in (9), we have

$$a^2-b^2=-(z-\gamma)\left(\frac{\alpha}{x-\alpha}-\frac{\beta}{y-\beta}\right)$$

or $$\frac{a^2-b^2}{z-\gamma}=-\frac{\alpha}{x-\alpha}+\frac{\beta}{y-\beta} \quad \text{or} \quad \frac{\alpha}{x-\alpha}-\frac{\beta}{y-\beta}+\frac{a^2-b^2}{z-\gamma}=0$$

which is the required result.

Example 1:

Find the equation of the sphere which cuts orthogonally each of the four spheres $x^2 + y^2 + z^2 + 2ax = a^2$; $x^2 + y^2 + z^2 + 2by = b^2$; $x^2 + y^2 + z^2 + 2cz = c^2$ and $x^2 + y^2 + z^2 = a^2 + b^2 + c^2$.

Solution:

Let the required equation of the sphere be given as

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \qquad ...(A)$$

If this sphere cuts the first of the given spheres orthogonally, then we have

$$2ua = d - a^2 \qquad ...(i),$$

using "$2uu' + 2vv' + 2ww' = d + d'$".

Similarly if its cuts the second and third of the given spheres orthogonally, then we have

$$2bv = d - b^2 \qquad ...(ii)$$

and $$2wc = d - c^2 \qquad ...(iii)$$

Also if this sphere cuts the last of the given spheres orthogonally. then we have

$$2u.0 + 2v.0 + 2w.0 = d - a^2 - b^2 - c^2$$

$\Rightarrow$ $d = a^2 + b^2 + c^2$ ...(iv)

From (i), (ii), (iii), with the help of (iv), we have

$$2u = (b^2 + c^2)/a,\ 2v = (c^2 + a^2)/b,\ 2w = (a^2 + b^2)/c.$$

Hence the required sphere from (A) is

$$x^2 + y^2 + z^2 + \left(\frac{b^2 + c^2}{a}\right)x + \left(\frac{c^2 + a^2}{b}\right)y + \left(\frac{a^2 + b^2}{c}\right)z + (a^2 + b^2 + c^2) = 0$$

Ex. 2. To find the condition that the section of the conicoid
ax* + by 2 + cz* + 2fyz + 2gzx + 9,hxy = 1

by the plane Ix + my + nz = may be a rectangular hyperbola.

The square of the reciprocal of the semi-diameter whose direction-cosines are X, /i, v is given by

^ = \2 + 1^ + CV 2 + 2//tv + 20?X + 27tX/*.

Take any three perpendicular diameters ; then we have by addition
111

Now, if r lt r 2 be the lengths of any two perpendicular semi-diameters of a rectangular hyperbola, r 1 2 + r 2 3 = 0.

Hence for any semi-diameter of the conicoid which is perpendicular to the plane of a section which is a rectangular hyperbola, we have

1

The required condition is therefore

al 2 + bm 2 + cri* + 2/mn + 2gnl + 2hlm = a + b + c = (a + b + c) (Z 2 + m? + n 2).

Ex. 3. Shew that the two lines given by the equations ax 2 + by z + cz 2 = Q t
lx + my+nz = Q will be at right angles, if

P (b + c)+ m 2 (c + a)+ ri 2 (a + 6) - 0.

The lines are the asymptotes of the section of the conicoid ax 2 + by* + cz 2 = 1
by the plane Ix + my + nz = 0.

Example 10. The limiting point of the coaxial system of sphere determind by

$$x^2 + y^2 + z^2 + 3x - 3y + 6 = 0$$

and $x^2 + y^2 + z^2 - 6y - 6z + 6 = 0$ are—

Solution : The equation of the coaxial system of sphere is

$$(x^2 + y^2 + z^2 + 3x - 3y + 6) + \lambda(x^2 + y^2 + z^2 - 6y - 6z + 6) = 0$$

or, $$x^2 + y^2 + z^2 + \left(\frac{3}{1+\lambda}\right)x - \left(\frac{3+6\lambda}{1+\lambda}\right)y - \left(\frac{6\lambda}{1+\lambda}\right)z + 6 = 0$$

Its centre is $\left(\frac{-3}{2(1+\lambda)}, \frac{3+6\lambda}{2(1+\lambda)}, \frac{6\lambda}{2(1+\lambda)}\right)$

Equating its radius to zero, we get

$$\frac{9}{4(1+\lambda)^2} + \frac{(3+6\lambda)^2}{4(1+\lambda)^2} + \frac{36\lambda^2}{4(1+\lambda)^2} - 6 = 0$$

or, $$9 + (3+6\lambda)^2 + 36\lambda^2 = 24(\lambda+1)^2$$

or, $$8\lambda^2 - 2\lambda - 1 = 0$$

$$\Rightarrow \qquad \lambda = \frac{1}{2}, -\frac{1}{4}$$

Substituting these values of λ in the coordinates of centre the required limiting points are $(-1, 2, 1)$ and $(-2, 1, -1)$.

Example 8. A variable plane makes intercepts on the coordinate axes, the sum of whose squares is constant and equal to k^2. Show that the locus of the foot of the perpendicular from the origin to the plane is

$$(x^{-2} + y^{-2} + z^{-2})^2 (x^2 + y^2 + z^2) = k^2.$$

Solution. Let the variable plane be

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c} = 1 \qquad ...(1)$$

Since the sum of the squares of intercepts on axes is k^2.

$$\therefore \quad a^2 + b^2 + c^2 = k^2 \qquad ...(2)$$

Now *d.c.*'s of the normal to the plane are proportional to $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$. Hence the equation of the normal from the origin to the plane (1) is

$$\frac{x}{1/a} = \frac{y}{1/b} = \frac{z}{1/c} = r \text{ (say).}$$

Any point of this line is $\left(\frac{r}{a}, \frac{r}{b}, \frac{r}{c}\right)$. Let this be the point (x_1, y_1, z_1) where the normal meets the plane.

$$\therefore \quad x_1 = \frac{r}{a}, y_1 = \frac{r}{b}, z_1 = \frac{r}{c} \qquad ...(3)$$

But (x_1, y_1, z_1) lies on the plane (1). Hence

$$\frac{x_1}{a}+\frac{y_1}{b}+\frac{z_1}{c} = 1$$

$$\therefore \quad \frac{x_1^2}{r}+\frac{y_1^2}{r}+\frac{z_1^2}{r} = 1 \qquad \text{[From (3)]}$$

i.e.

$$r = x_1^2 + y_1^2 + z_1^2 \qquad ...(4)$$

Also from (2), we have

$$\frac{r^2}{x_1^2}+\frac{r^2}{y_1^2}+\frac{r^2}{z_1^2} = k^2 \quad i.e. \quad r^2\left(\frac{1}{x_1^2}+\frac{1}{y_1^2}+\frac{1}{z_1^2}\right) = k^2.$$

Putting the value of r, we have

$$(x_1^2 + y_1^2 + z_1^2)^2 \left(\frac{1}{x_1^2}+\frac{1}{y_1^2}+\frac{1}{z_1^2}\right) = k^2$$

Therefore the locus of (x_1, y_1, z_1) is

$$(x^{-2} + y^{-2} + z^{-2}) (x^2 + y^2 + z^2)^2 = k^2.$$

Hence the result.

10. *Show that the spheres which cut two given spheres along great circle all pass through two fixed points.*

Sol. We take the yz-plane as the common radical plane and x-axis as the line of centres ;

The equation of the given spheres can be put in the form

$$x^2 + y^2 + z^2 + 2u_1x + d = 0 \qquad ...(1)$$

and

$$x^2 + y^2 + z^2 + 2u_2x + d = 0 \qquad ...(2)$$

Let the equation of another sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + c = 0 \qquad ...(3)$$

where u, v, w, c are different for different spheres.

If the sphere (3) cuts (1) along a great circle, then the centre $(-u, 0, 0)$ of (1) must lie on the radical plane *i.e.*, the plane of (1) and (3) *i.e.*, the plane

$$2(u - u_1)x + 2vy + 2wz + c - d = 0$$

$$\therefore \quad 2(u - u_1)(-u_1) + 2v(0) + 2w(0) + c - d = 0$$

or

$$2uu_1 - 2u_1^2 - c + d = 0 \qquad ...(4)$$

Similarly the sphere (3) cuts (2) in a great circle if

$$2uu_2 - 2u_2^2 - c + d = 0 \qquad ...(5)$$

Subtracting (5) from (4) $2u(u_1 - u_2) - 2(u_1^2 - u_2^2) = 0$ or $u = u_1 + u_2$

$\therefore$ From (4), $2u_1(u_1 - u_2) - 2u_1^2 - c + d = 0$ or $c = 2u_1u_2 + d.$

So that u and c are constant and depend on only u_1, u_2, d, the given quantities.

Now we have to prove that sphere (3) passes through two fixed points.

The sphere (3) meets x-axis where putting $y = 0$, $z = 0$ in (3)

We have $x^2 + 2ux + c = 0$

The roots of this equation are constants depending upon the constants u and c only.

Hence every sphere (3) cuts the x-axis in the same two points.

Hence the result.

Example 2. *Show that the S.D. between any two opposite edges of the tetrahedron formed by the planes*

$$y + z = 0,\ z + x = 0,\ x + y = 0,\ x + y + z = a \text{ is } \frac{2a}{\sqrt{6}}$$

and that three lines of shortest distance intersect in the point $x = y = z = -a$.

Sol. The equations to one of the pairs of opposite edges are

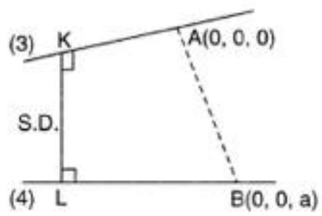
$$y + z = 0, \qquad z + x = 0 \qquad \text{...(1)}$$

and $$x + y = 0,\ \ x + y + z = a \qquad \text{...(2)}$$

These equations can be written in symmetrical form as

$$\frac{x}{1} = \frac{y}{1} = \frac{z}{-1} \qquad \text{...(3)}$$

and $$\frac{x}{1} = \frac{y}{-1} = \frac{z-a}{0} \qquad \text{...(4)}$$



Let l, m, n be the d.c.'s of KL, the line of S.D. Since the S.D. is $\perp$ to both the lines (3) and (4),

$$\therefore \qquad l + m - n = 0$$
$$l - m + 0\,.\,n = 0$$

Now by cross-multiplication,

$$\frac{l}{0-1} = \frac{m}{-1-0} = \frac{n}{-1-1} \quad \text{or} \quad \frac{l}{1} = \frac{m}{1} = \frac{n}{2}.$$

Thus, the d.r.'s of the S.D. are 1, 1, 2.

Dividing each by $\sqrt{1+1+4} = \sqrt{6}$,

the actual d.c.'s of the S.D. are $\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}$.

Let the two points (0, 0, 0), (0, 0, a) on the two lines (3) and (4) be denoted by A, B respectively.

$\therefore$ The length of S.D. = projection of AB on KL, the line of S.D.

$$= \frac{1}{\sqrt{6}}(0-0) + \frac{1}{\sqrt{6}}(0-0) + \frac{2}{\sqrt{6}}(a-0) = \frac{2a}{\sqrt{6}}.$$

Similarly, in case of other pairs of opposite edges.

The equations of KL, the line of intersection of the planes AKL and BLK, are

$$\begin{vmatrix} x & y & z \\ 1 & 1 & -1 \\ 1 & 1 & 2 \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} x & y & z-a \\ 1 & -1 & 0 \\ 1 & 1 & 2 \end{vmatrix} = 0$$

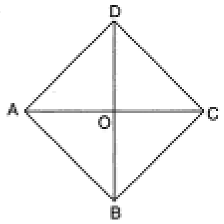
or $$x - y = 0 \quad \text{and} \quad x + y - z + a = 0$$

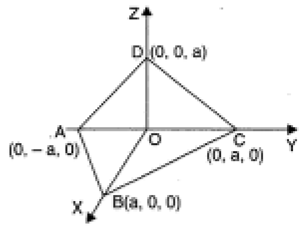
which clearly pass through the point $x = y = z = -a$, *i.e.*, the point $(-a, -a, -a)$. Similarly it can be shown that the lines of S.D. between other opposite edges pass through the same point.

Hence the three lines of S.D. intersect at the point $x = y = z = -a$.

Example 3. *A square ABCD of diagonal 2a is folded along the diagonal AC, so that the planes DAC, BAC are at right angles. Show that the shortest distance between DC and AB is* $\frac{2a}{\sqrt{3}}$.

Sol. Let O, the center of the square be taken as the origin and the coordinate axes be taken along OB, OC and OD, in the folded position of the square. The four vertices of the square are A$(0, -a, 0)$, B$(a, 0, 0)$, C$(0, a, 0)$ and D$(0, 0, a)$ respectively.





The equations of DC are $\frac{x-0}{0-0} = \frac{y-a}{0-a} = \frac{z-0}{a-0}$ or $\frac{x}{0} = \frac{y-a}{-1} = \frac{z}{1}$ ...(1)

The equations of AB are $\frac{x-a}{0-a} = \frac{y-0}{-a-0} = \frac{z-0}{0-0}$ or $\frac{x-a}{1} = \frac{y}{1} = \frac{z}{0}$ ...(2)

The equation of the plane passing through (1) and parallel to (2) is

$$\begin{vmatrix} x & y-a & z \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 0 \quad \text{or} \quad x - y - z + a = 0 \qquad ...(3)$$

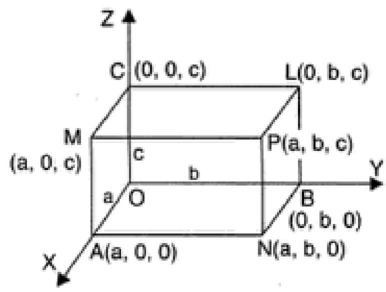
A point on the line (2) is $(a, 0, 0)$.

S.D. = length of perpendicular from the point $(a, 0, 0)$ to the plane (3)

$$= \frac{a - 0 - 0 + a}{\sqrt{1+1+1}} = \frac{2a}{\sqrt{3}}.$$

Example 4. *Prove that the shortest distance between a diagonal of a rectangular parallelopiped and the edge not meeting it is* $\frac{bc}{\sqrt{b^2+c^2}}, \frac{ca}{\sqrt{c^2+a^2}}, \frac{ab}{\sqrt{a^2+b^2}}$ *where a, b, c are the lengths of the edges.*

Sol. Let the co-terminous edges OA, OB and OC be taken as the axes of coordinates. Then the coordinates of the various corners are as shown in the figure.



Let us find the S.D. between the diagonal OP and the edge MC which does not meet OP.

The equations of OP are $\frac{x-0}{a-0}=\frac{y-0}{b-0}=\frac{z-0}{c-0}$ or $\frac{x}{a}=\frac{y}{b}=\frac{z}{c}$ ...(1)

and the equations of MC are $\frac{x-0}{a-0}=\frac{y-0}{0-0}=\frac{z-c}{c-c}$ or $\frac{x}{1}=\frac{y}{0}=\frac{z-c}{0}$ ...(2)

The equation of plane passing through (1) and parallel to (2) is

$$\begin{vmatrix} x & y & z \\ a & b & c \\ 1 & 0 & 0 \end{vmatrix} = 0 \quad \text{or} \quad cy - bz = 0 \qquad ...(3)$$

A point on the line (2) is $(0, 0, c)$

S.D. = length of perpendicular from $(0, 0, c)$ on (3)

$$= \frac{0-bc}{\sqrt{0+c^2+b^2}} = \frac{bc}{\sqrt{b^2+c^2}} \text{ (numerically)}$$

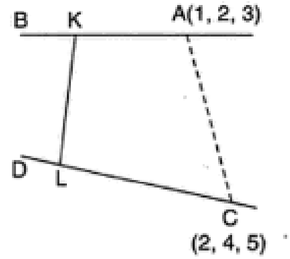
Similarly the S.D. between OP and the other edges not meeting it (*i.e.*, AN and AM) are

$$\frac{ca}{\sqrt{c^2+a^2}} \quad \text{and} \quad \frac{ab}{\sqrt{a^2+b^2}}.$$

Example 1. *Find the length and the equations of the shortest distance between the lines*

$$\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4} \text{ and } \frac{x-2}{3}=\frac{y-4}{4}=\frac{z-5}{5}.$$

Sol. [Method of Projection]



The given lines are $\frac{x-1}{2}=\frac{y-2}{3}=\frac{z-3}{4}$ ...(1)

and $\frac{x-2}{3}=\frac{y-4}{4}=\frac{z-5}{5}$ ...(2)

Let A, C be the points (1, 2, 3) and (2, 4, 5) on the lines (1) and (2).

Let l, m, n be the d.c.'s of the S.D., then since the **S.D. is $\perp$ to both the lines (1) and (2).**

$\therefore$ $2l + 3m + 4n = 0$; $3l + 4m + 5n = 0$

whence by cross-multiplication,

$$\frac{l}{15-16}=\frac{m}{12-10}=\frac{n}{8-9} \quad \text{or} \quad \frac{l}{1}=\frac{m}{-2}=\frac{n}{1} \quad \textit{i.e.}, \text{ d.r.'s of S.D. are } 1, -2, 1.$$

Dividing each by $\sqrt{(1)^2+(-2)^2+(1)^2}=\sqrt{6}$, the d.c.'s of shortest distance KL are $\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}$.

Now length of S.D. = **projection of AC on the line KL of S.D.**

$$= \frac{\mathbf{1}}{\sqrt{\mathbf{6}}}(\mathbf{2-1}) + \left(\frac{\mathbf{-2}}{\sqrt{\mathbf{6}}}\right)(\mathbf{4-2}) + \frac{\mathbf{1}}{\sqrt{\mathbf{6}}}(\mathbf{5-3}) \mid \text{Using } l(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1)$$

$$= \frac{1}{\sqrt{6}} - \frac{4}{\sqrt{6}} + \frac{2}{\sqrt{6}} = \frac{-1}{\sqrt{6}} = \frac{1}{\sqrt{6}} \text{ (in magnitude)}$$

The line of S.D. is the line of intersection of the plane through (1) and the S.D., and the plane through (2) and the S.D.

Now the equation of the plane through (1) and the S.D. is

$$\begin{vmatrix} x-1 & y-2 & z-3 \\ 2 & 3 & 4 \\ 1 & -2 & 1 \end{vmatrix} = 0 \quad \text{or} \quad 11x + 2y - 7z + 6 = 0 \qquad ...(3)$$

and the equation of the plane through (2) and the S.D. is

$$\begin{vmatrix} x-2 & y-4 & z-5 \\ 3 & 4 & 5 \\ 1 & -2 & 1 \end{vmatrix} = 0 \quad \text{or} \quad 7x + y - 5z + 7 = 0 \qquad ...(4)$$

Thus from (3) and (4), the equations of the S.D. are

$$11x + 2y - 7z + 6 = 0, \; 7x + y - 5z + 7 = 0.$$

Example 7. *Find the S.D. between the axis of z and the line*

$$ax + by + cz + d = 0,\ a'x + b'y + c'z + d' = 0.$$

Sol. The given lines are the z-axis *i.e.*, $\dfrac{x}{0} = \dfrac{y}{0} = \dfrac{z}{1}$ ...(1)

and $$ax + by + cz + d = 0 = a'x + b'y + c'z + d' \quad ...(2)$$

Any plane through the line (2) is $ax + by + cz + d + k(a'x + b'y + c'z + d') = 0$ ...(3)

This is $\parallel$ to the line (1) if $(a + ka') \,.\, 0 + (b + kb') \,.\, 0 + (c + kc') \,.\, 1 = 0$

or $$c + kc' = 0 \quad \therefore \quad k = \frac{-c}{c'}.$$

Substituting this value of k in (3), we get

$$ax + by + cz + d - \frac{c}{c'}(a'x + b'y + c'z + d') = 0$$

or $$(ac' - a'c)x + (bc' - b'c)y + (dc' - d'c) = 0 \quad ...(4)$$

Now (4) is the plane through the line (2) and $\parallel$ to (1).

$\therefore$ S.D. = $\perp$ distance of any point on line (1), say (0, 0, 0), from the plane (4)

$$= \frac{0 + 0 + dc' - d'c}{\sqrt{(ac' - a'c)^2 + (bc' - b'c)^2}} = \frac{dc' - d'c}{\sqrt{(ac' - a'c)^2 + (bc' - b'c)^2}}.$$

Ex. 2. ***Find the equations to the generating lines of the hyperboloid $x^2/4+y^2/9-z^2/16=1$ which pass through the points* (2, 3, —4) *and* (2, —1, 4/3).**

Solution. The equation of the hyperboloid is

$$x^2/4+y^2/9-z^2/16=1. \quad ...(1)$$

Let l, m, n be the d.c.'s of the generator. Hence the equations of the generator through the point (2, 3, —4) are given by

$$\frac{x-2}{l}=\frac{y-3}{m}=\frac{z+4}{n}=r \text{ (say)}. \quad \ldots(2)$$

Any point on (2) is $(lr+2,\ mr+3,\ nr-4)$. If it lies on (1), we get

$$\frac{1}{4}(lr+2)^2+\frac{1}{9}(mr+3)^2-\frac{1}{16}(nr-4)^2=1$$

or $$r^2\left(\frac{l^2}{4}+\frac{m^2}{9}-\frac{n^2}{16}\right)+2\left(\frac{l}{2}+\frac{m}{3}+\frac{n}{4}\right)=0. \quad \ldots(3)$$

If the line (2) is a generator then (3) is an identity in r, the conditions for which are

$$\frac{l^2}{4}+\frac{m^2}{9}-\frac{n^2}{16}=0 \quad \ldots(4)$$

and $$\frac{l}{2}+\frac{m}{3}+\frac{n}{4}=0. \quad \ldots(5)$$

Eliminating n between (4) and (5), we get

$$\frac{l^2}{4}+\frac{m^2}{9}-\left(\frac{l}{2}+\frac{m}{3}\right)^2=0 \quad \text{or} \quad lm=0.$$

Hence either $l=0$ or $m=0$.

If $l=0$, then from (5)

$$m/3+n/4=0 \quad \text{or} \quad m/3=n/-4.$$

Hence the d.c.'s of one generator are given by

$$l/0=m/3=n/-4. \quad \ldots(6)$$

If $m=0$, then from (5) $\frac{l}{2}+\frac{n}{4}=0$ or $\frac{l}{1}=\frac{n}{-2}$.

Hence the d.c.'s of the other generator are given by

$$l/1=m/0=n/-2. \quad \ldots(7)$$

Thus the equations of the two generators through the point $(2, 3, -4)$ are obtained by putting the values of l, m, n from (6) and (7) in (2) and are given by

$$\frac{x-2}{0}=\frac{y-3}{3}=\frac{z+4}{-4} \text{ and } \frac{x-2}{1}=\frac{y-3}{0}=\frac{z+4}{-2}.$$

Proceeding similarly as above the equations of the two generators through the point $(2, -1, 4/3)$ are given by

$$\frac{x-2}{3}=\frac{y+1}{6}=\frac{z-\frac{4}{3}}{10} \text{ and } \frac{x-2}{0}=\frac{y+1}{3}=\frac{z-\frac{4}{3}}{-4}.$$

EXAMPLE 1.22

Find the equation of the right circular cylinder whose guiding curve is

$$x^2 + y^2 + z^2 = 9,\ x - y + z = 3.$$

Second Method: The direction ratios of the plane $x - y + z = 3$ are $1, -1$, and 1. Since the axis of the cylinder is perpendicular to the plane, its direction cosines are proportional to $1, -1$, and 1.

Let (x_1, y_1, z_1) be any point on the cylinder. The equation of the generators through this point is

$$\frac{x - x_1}{1} = \frac{y - y_1}{-1} = \frac{z - z_1}{1} = \lambda, \text{ say.}$$

Any point on the generator is $(\lambda + x_1, -\lambda + y_1, \lambda + z_1)$. This point lies on the guiding curve if

$$(\lambda + x_1)^2 + (-\lambda + y_1)^2 + (\lambda + z_1)^2 = 9,$$
$$(\lambda + x_1) - (-\lambda + y_1) + (\lambda + z_1) = 3$$

or

$$x_1^2 + y_1^2 + z_1^2 + 2\lambda(x_1 - y_1 + z_1) + 3\lambda^2 = 9,$$
$$x_1 - y_1 + z_1 + 3\lambda = 3.$$

From second member, we have $\lambda = \dfrac{x_1 - y_1 + z_1 - 3}{-3}$. Substituting this value of λ in the first member, we get

$$x_1^2 + y_1^2 + z_1^2 - \frac{2(x_1 - y_1 + z_1 - 3)(x_1 - y_1 + z_1)}{3} + \frac{3(x_1 - y_1 + z_1 - 3)^2}{9} = 9$$

or

$$x_1^2 + y_1^2 + z_1^2 + y_1 z_1 - z_1 x_1 + x_1 y_1 - 9 = 0.$$

Hence, the locus of (x_1, y_1, z_1) is

$$x^2 + y^2 + z^2 + yz - zx + xy - 9 = 0,$$

which is the required equation of the right circular cylinder.

EXAMPLE 1.21

Find the equation of the right circular cylinder whose guiding curve is the circle passing through the points (1, 0, 0), (0, 1, 0), and (0, 0, 1).

Solution. The equation of the plane passing through A(1, 0, 0), B(0, 1, 0), and C(0, 0, 1) is

$$\begin{vmatrix} x-1 & y & z \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = 0 \quad \text{or}$$

$$x+y+z=1 \quad \text{or} \quad \frac{x}{1}+\frac{y}{1}+\frac{z}{1}=1.$$

Let O be the origin. Then OA = OB = OC = 1. Therefore, a sphere with center O(0, 0, 0) and radius unity passes through A, B, and C. Thus, the guiding curve is the circle

$$x^2+y^2+z^2=1, \; x+y+z=1.$$

The equation of the axis of the cylinder (which passes through (0, 0, 0) and is perpendicular to the plane $x + y + z = 1$) is

$$\frac{x-0}{1}=\frac{y-0}{1}=\frac{z-0}{1} \quad \text{or} \quad \frac{x}{1}=\frac{y}{1}=\frac{z}{1}.$$

The perpendicular distance from the center of the sphere $x^2+y^2+z^2=1$ to the plane $x + y + z = 1$ is

$$p=\frac{1}{\sqrt{1+1+1}}=\frac{1}{\sqrt{3}}.$$

Therefore, the radius of the circle is

$$r=\sqrt{(\text{radius of the sphere})^2-p^2} = \sqrt{1-\frac{1}{3}}$$

$$=\sqrt{\frac{2}{3}}.$$

Therefore, the equation of the right circular cylinder is

$$(l^2+m^2+n^2)(x^2+y^2+z^2-r^2)$$
$$=(lx+my+nz)^2$$

or

$$(1+1+1)\left(x^2+y^2+z^2-\frac{2}{3}\right)=(x+y+z)^2$$

or

$$3x^2+3y^2+3z^2-2=x^2+y^2+z^2+2xy+2yz \\ +2xz$$

or

$$x^2+y^2+z^2-xy-yz-zx-1=0.$$

Ex. 12. *Prove that the locus of the point of intersection of three tangent planes to $x^2/a^2+y^2/b^2+z^2/c^2=1$ which are parallel to conjugate diametral planes of $x^2/\alpha^2+y^2/\beta^2+z^2/\gamma^2=1$ is*

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}.$$

Solution. The equations of the ellipsoids are given as

$$x^2/a^2+y^2/b^2+z^2/c^2=1 \qquad \ldots(1)$$

and

$$x^2/\alpha^2+y^2/\beta^2+z^2/\gamma^2=1. \qquad \ldots(2)$$

Let OP, OQ and OR be the conjugate semi-diameters of the ellipsoid (2). Let (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) be the co-ordinates of the extremities P, Q and R respectively.

The equation of the diametral plane of OP with respect to the ellipsoid (2) is

$$xx_1/\alpha^2+yy_1/\beta^2+zz_1/\gamma^2=0. \qquad \ldots(3)$$

The equation of any plane parallel to the plane (3) is

$$xx_1/\alpha^2+yy_1/\beta^2+zz_1/\gamma^2=p_1, \text{ (say)} \qquad \ldots(4)$$

If the plane (4) be a tangent plane to the ellipsoid (1), then

$$p_1^2=a^2\,(x_1^2/\alpha^4)+b^2\,(y_1^2/\beta^4)+c^2\,(z_1^2/\gamma^4) \qquad \ldots(5)$$

[using the condition for tangency $p^2=a^2l^2+b^2m^2+c^2n^2$].

Similarly the equations of the other two tangent planes are

$$xx_2/\alpha^2+yy_2/\beta^2+zz_2/\gamma^2=p_2 \qquad \ldots(6)$$

and $$xx_3/\alpha^2+yy_3/\beta^2+zz_3/\gamma^2=p_3 \qquad \ldots(7)$$

where $$p_2^2=a^2\,(x_2^2/\alpha^4)+b^2\,(y_2^2/\beta^4)+c^2\,(z_2^2/\gamma^4) \qquad \ldots(8)$$

and $$p_3^2=a^2\,(x_3^2/\alpha^4)+b^2\,(y_3^2/\beta^4)+c^2\,(z_3^2/\gamma^4). \qquad \ldots(9)$$

Now to find the locus of the point of intersection of the tangent planes (4), (6) and (7), squaring and adding both sides of the equations (4), (6) and (7), we get

$$\sum\left\{\frac{x^2}{\alpha^4}\,(x_1^2+x_2^2+x_3^2)\right\}+2\sum\left\{\frac{xy}{\alpha^2\beta^2}\,(x_1y_1+x_2y_2+x_3y_3)\right\}$$
$$=\sum\left\{\frac{a^2}{\alpha^4}\,(x_1^2+x_2^2+x_3^2)\right\}$$

or $$\sum\left\{\frac{x^2}{\alpha^4}\cdot\alpha^2\right\}+0=\sum\left\{\frac{a^2}{\alpha^4}\cdot\alpha^2\right\}\quad [\because\ \Sigma\, x_1^2=\alpha^2,\ \Sigma\, x_1y_1=0]$$

or $$x^2/\alpha^2+y^2/\beta^2+z^2/\gamma^2=a^2/\alpha^2+b^2/\beta^2+c^2/\gamma^2.$$

This is the equation of the required locus.

Remark. In case $\alpha=\beta=\gamma$, the equation of the required locus becomes $x^2+y^2+z^2=a^2+b^2+c^2$ which is the equation of the **director sphere.**

Ex. 8. *Show that the points of intersection P, Q of the generators of opposite systems drawn through the point $A(a\cos\alpha, b\sin\alpha, 0)$, $B(a\cos\beta, b\sin\beta, 0)$ of the principal elliptic section of the hyperboloid* $x^2/a^2+y^2/b^2-z^2/c^2=1$ *are*

$$\left(\frac{a\cos\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)}, \frac{b\sin\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)}, \pm\frac{c\sin\frac{1}{2}(\alpha-\beta)}{\cos\frac{1}{2}(\alpha-\beta)}\right)$$

Hence show that if A and B are extremities of semi-conjugate diameters, the loci of the points P and Q are the ellipses

$$x^2/a^2+y^2/b^2=2,\ z=\pm c.$$

Solution. Let the co-ordinates of one of the two points of intersection of the generators, say of the point P be (x_1, y_1, z_1). The equation of the tangent plane to the hyperboloid

$$x^2/a^2+y^2/b^2-z^2/c^2=1$$

at the point (x_1, y_1, z_1) on it is

$$xx_1/a^2+yy_1/b^2-zz_1/c^2=1. \qquad (1)$$

The tangent plane (1) meets the plane $z=0$ in the line which is given by

$$xx_1/a^2+yy_1/b^2=1,\ z=0. \qquad ...(2)$$

The equations of the line joining the points $A\,(a\cos\alpha, b\sin\alpha, 0)$ and $B\,(a\cos\beta, b\sin\beta, 0)$ are

$$(x/a)\cos\tfrac{1}{2}(\alpha+\beta)+(y/b)\sin\tfrac{1}{2}(\alpha+\beta)=\cos\tfrac{1}{2}(\alpha-\beta),\ z=0. \qquad ..(3)$$

The lines given by the equations (2) and (3) are the same. Hence comparing these equations, we have

$$\frac{x_1/a}{\cos\frac{1}{2}(\alpha+\beta)}=\frac{y_1/b}{\sin\frac{1}{2}(\alpha+\beta)}=\frac{1}{\cos\frac{1}{2}(\alpha-\beta)}. \qquad ...(4)$$

Since the point $P\,(x_1, y_1, z_1)$ lies on the given hyperboloid, we have

$$x_1^2/a^2+y_1^2/b^2-z_1^2/c^2=1. \qquad ...(5)$$

Putting the values of x_1/a and y_1/b from (4) in (5), we get

$$\frac{\cos^2\frac{1}{2}(\alpha+\beta)}{\cos^2\frac{1}{2}(\alpha-\beta)}+\frac{\sin^2\frac{1}{2}(\alpha+\beta)}{\cos^2\frac{1}{2}(\alpha-\beta)}-\frac{z_1^2}{c^2}=1$$

or
$$\frac{1}{\cos^2\frac{1}{2}(\alpha-\beta)}-\frac{z_1^2}{c^2}=1$$

or $\qquad z_1^2/c^2=\sec^2 \frac{1}{2}(\alpha-\beta)-1$ or $z_1^2/c^2=\tan^2 \frac{1}{2}(\alpha-\beta)$

or $\qquad z_1/c=\pm\sin \frac{1}{2}(\alpha-\beta)/\cos \frac{1}{2}(\alpha-\beta)$. ...(6)

Hence from (4) and (6) the co-ordinates of the required points of intersection P and Q are

$$\left(\frac{a\cos \frac{1}{2}(\alpha+\beta)}{\cos \frac{1}{2}(\alpha-\beta)}, \frac{b\sin \frac{1}{2}(\alpha+\beta)}{\cos \frac{1}{2}(\alpha-\beta)}, \pm\frac{c\sin \frac{1}{2}(\alpha-\beta)}{\cos \frac{1}{2}(\alpha-\beta)}\right). \qquad ...(7)$$

Second part. If the points A and B are the extremities of the semi-conjugate diameters, then we have $\alpha-\beta=\frac{1}{2}\pi$.

$\therefore \quad x_1^2/a^2+y_1^2/b^2=1/\cos^2 \frac{1}{2}(\alpha-\beta),\ z_1=\pm c\tan \frac{1}{2}(\alpha-\beta)$

or $\quad x_1^2/a^2+y_1^2/b^2=1/\cos^2 \frac{1}{4}\pi,\ z_1=\pm c\tan \frac{1}{4}\pi$

or $\quad x_1^2/a^2+y_1^2/b^2=2,\ z_1=\pm c$.

Therefore, the loci of the points P and Q are

$$x^2/a^2+y^2/b^2=2,\ z=\pm c.$$ **Proved.**

Ex. 7. *Show that the perpendiculars from the origin on the generator of the hyperboloid* $x^2/a^2+y^2/b^2-z^2/c^2=1$ *lie on the cone*

$$\frac{a^2(b^2+c^2)^2}{x^2}+\frac{b^2(c^2+a^2)^2}{y^2}-\frac{c^2(a^2-b^2)^2}{z^2}=0.$$

Solution. The equation of the hyperboloid is

$$x^2/a^2+y^2/b^2-z^2/c^2=1. \qquad ...(1)$$

The equations of the generator of (1) belonging to one system and passing through any point $(a\cos\alpha, b\sin\alpha, 0)$ on the principal elliptic section $x^2/a^2+y^2/b^2=1$, $z=0$ are given by

$$\frac{x-a\cos\alpha}{a\sin\alpha}=\frac{y-b\sin\alpha}{-b\cos\alpha}=\frac{z}{c}. \qquad ...(2)$$

[See Ex. 1(a) above]

The equations to any line through the origin are

$$x/l=y/m=z/n. \qquad ...(3)$$

If (2) and (3) are perpendicular, then we have

$$al\sin\alpha-bm\cos\alpha+cn=0. \qquad ...(4)$$

The lines (2) and (3) will intersect *i.e.* they will be coplanar if

$$\begin{vmatrix} a\cos\alpha & b\sin\alpha & 0 \\ a\sin\alpha & -b\cos\alpha & c \\ l & m & n \end{vmatrix}=0$$

or $$a\cos\alpha\,(-bn\cos\alpha-cm)-b\sin\alpha\,(an\sin\alpha-cl)=0$$

or $$-abn\,(\cos^2\alpha+\sin^2\alpha)-acm\cos\alpha+bcl\sin\alpha=0$$

or $$lbc\sin\alpha-mac\cos\alpha-nab=0. \qquad \ldots(5)$$

Solving the relations (4) and (5) for $\sin\alpha$ and $\cos\alpha$, we have

$$\frac{\sin\alpha}{mnab^2+mnac^2}=\frac{\cos\alpha}{nlbc^2+nlba^2}=\frac{1}{-lmca^2+lmcb^2}$$

$$\therefore \quad \sin\alpha=-\frac{na\,(b^2+c^2)}{lc\,(a^2-b^2)},\ \cos\alpha=-\frac{nb\,(c^2+a^2)}{mc\,(a^2-b^2)}.$$

Squaring and adding *i.e.* eliminating α, we have

$$\frac{n^2a^2\,(b^2+c^2)^2}{l^2c^2\,(a^2-b^2)^2}+\frac{n^2b^2\,(c^2+a^2)^2}{m^2c^2\,(a^2-b^2)^2}=1$$

or $$\frac{a^2}{l^2}\,(b^2+c^2)^2+\frac{b^2}{m^2}\,(c^2+a^2)^2=\frac{c^2}{n^2}\,(a^2-b^2)^2.$$

Hence the line (3) lies on the cone

$$\frac{a^2\,(b^2+c^2)^2}{x^2}+\frac{b^2\,(c^2+a^2)^2}{y^2}-\frac{c^2\,(a^2-b^2)^2}{z^2}=0.$$

A similar result can be proved by taking generators of the other system.

Example 36:

If A and A' are the extremities of the major axis of the principal elliptic section and any generator meets two generators of the same system through A and A' in P and P' respectively, then prove that $AP.A'P' = b^2 + c^2$.

Solution:

We know that the points of intersection of a generator of λ-system with a generator of μ-system for the hyperboloid $(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1$ are given by

$$x = \frac{a\,(1 + \lambda\mu)}{\lambda + \mu},\ \frac{y = b(\lambda - \mu)}{\lambda + \mu},\ z = \frac{c(1 - \lambda\mu)}{\lambda + \mu} \qquad \ldots(i)$$

The extremities of the major axis of the principal elliptic section are A (a, 0, 0) and A' (–a, 0, 0)

$\therefore$ At A and A' from (i) we have $\lambda - \mu = 0,\ 1 - \lambda\mu = 0$

$\Rightarrow\ \lambda = \mu$ and $1 - \lambda^2 = 0 \ \Rightarrow\ \lambda = \pm 1$

Now consider the generator through A (a, 0, 0) corresponding to $\lambda = +1$ and then its point of intersection P with a generator of μ-system is obtained from (i) by putting $\lambda = +1$ and is

$$\left(a, \frac{b(1-\mu)}{1+\mu}, \frac{c(1-\mu)}{1+\mu}\right) \Rightarrow (a, bt, ct), \text{ where } t = \frac{1-\mu}{1+\mu}$$

$$\therefore \qquad AP^2 = (a - a)^2 + (bt - 0)^2 + (ct - 0)^2 = (b^2 + c^2)\,t^2 \quad \ldots(ii)$$

Again the generator through A'(–a, 0, 0) corresponding to $\lambda = -1$ meets the generator of μ-system at P', whose coordinates are obtained from (i) by putting $\lambda = -1$ and is

$$\left(-a, \frac{b(1-\mu)}{1+\mu}, \frac{c(1-\mu)}{1+\mu}\right) \Rightarrow \left(-a, \frac{b}{t}, -\frac{c}{t}\right), \text{ where } t = \frac{1-\mu}{1+\mu}$$

$$\therefore \quad (A'\,P')^2 = (-a - 1)^2 + \left(\frac{b}{t} - 0\right)^2 + \left(-\frac{c}{t} - 0\right)^2 = \frac{b^2 + c^2}{t^2} \quad \ldots(iii)$$

$\therefore$ From (ii) and (iii) we get

$AP^2 \,.\, (A'P')^2 = (b^2 + c^2)\, t^2 \,.\, [(b^2 + c^2)/t^2]$

$\Rightarrow\ AP^2 \,.\, (A'P')^2 = (b^2 + c^2)^2$

$\Rightarrow\ AP \,.\, A'P' = b^2 + c^2.$ **Proved.**

Ex. 13. *Show that in general two generators of the hyperboloid $x^2/a^2+y^2/b^2-z^2/c^2=1$ can be drawn to cut a given generator at right angles. Also show that if they meet the plane $z=0$ in P and Q, PQ touches the ellipse*

$$\frac{x^2}{a^6}+\frac{y^2}{b^6}=\frac{c^4}{a^4b^4}.$$

Solution. Proceeding as in § 3, the d.c.'s of a generator of the λ-system are proportional to

$$a(1-\lambda^2),\ 2b\lambda,\ c(1+\lambda^2)$$

and those of a generator of the μ-system are proportional to

$$a(1-\mu^2),\ -2b\mu,\ c(1+\mu^2).$$

If these generators are at right angles, then we have

$$a^2(1-\mu^2)(1-\lambda^2)-4b^2\mu\lambda+c^2(1+\mu^2)(1+\lambda^2)=0. \qquad \text{...(1)}$$

Now suppose the μ-generator is given so that μ is a constant and hence the equation (1) is a quadratic in λ. This shows that there are two generators of the λ-system which are at right angles to a generator of the μ-system. Let the generators of the λ-system cut the plane $z=0$ in the points $P(a\cos\alpha,\ b\sin\alpha,\ 0)$ and $Q(a\cos\beta,\ b\sin\beta,\ 0)$. The equations of these generators belonging to the λ system through the points P and Q are given by [See Ex. 1(a) above]

$$\frac{x-a\cos\alpha}{a\sin\alpha}=\frac{y-b\sin\alpha}{-b\cos\alpha}=\frac{z}{c}, \qquad \text{...(2)}$$

and

$$\frac{x-a\cos\beta}{a\sin\beta}=\frac{y-b\sin\beta}{-b\cos\beta}=\frac{z}{c}. \qquad \text{...(3)}$$

Also the equations of the generator belonging to the μ-system and passing through the point, say $(a \cos \theta, b \sin \theta, 0)$, are

$$\frac{x-a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{-b\cos\theta}=\frac{z}{-c}. \quad \ldots (4)$$

The generators (2) and (3) both intersect the generator given by (4) at right angles and hence we have

$$a^2 \sin\alpha \sin\theta+b^2 \cos\alpha \cos\theta-c^2=0,$$

and $$a^2 \sin\beta \sin\theta+b^2 \cos\beta \cos\theta-c^2=0.$$

Solving these last two relations, we have

$$\frac{a^2\sin\theta}{-\cos\alpha+\cos\beta}=\frac{b^2\sin\theta}{-\sin\beta+\sin\alpha}=\frac{c^2}{\sin\alpha\cos\beta-\cos\alpha\sin\beta}$$

or $$\frac{a^2\sin\theta}{2\sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)}=\frac{b^2\cos\theta}{2\cos\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)}$$

$$=\frac{c^2}{2\sin\frac{1}{2}(\alpha-\beta)\cos\frac{1}{2}(\alpha-\beta)}$$

or $$\frac{a^2\sin\theta}{\sin\frac{1}{2}(\alpha+\beta)}=\frac{b^2\cos\theta}{\cos\frac{1}{2}(\alpha+\beta)}=\frac{c^2}{\cos\frac{1}{2}(\alpha-\beta)}=\frac{1}{k} \text{ (say).} \quad \ldots (5)$$

Now the equations of the line joining the points P and Q are
$(x/a)\cos\frac{1}{2}(\alpha+\beta)+(y/b)\sin\frac{1}{2}(\alpha+\beta)=\cos\frac{1}{2}(\alpha-\beta)$, $z=0$.

Putting the values from (5), the above equations to the line PQ become

$$(x/a)\, kb^2\cos\theta+(y/b)\, ka^2\sin\theta=kc^2,\ z=0$$

or $$(x/a^3)\cos\theta+(y/b^3)\sin\theta=(c^2/a^2b^2),\ z=0. \quad \ldots (6)$$

$[\because k\neq 0]$

Now it is required to find the envelope of (6) and so differentiating (6) w.r.t. 'θ', we get

$$-(x/a^3)\sin\theta+(y/b^3)\cos\theta=0,\ z=0. \quad \ldots (7)$$

Squaring (6) and (7) and then adding, the equations of the envelope of the line PQ [or in other words the curve which the line PQ alwaye touches] are given by

$$\frac{x^2}{a^6}+\frac{y^2}{b^6}=\frac{c^4}{a^4b^4},\ z=0$$

which are the equations of an ellipse.

Example 73:

Show that the planes which cut $ax^2 + by^2 + cz^2 = 0$ perpendicular generators, touch the cone

$$S\,[x^2/(b + c)] = 0.$$

Solution:

We can prove that if the plane $ux + vy + wz = 0$ cuts the cone $ax^2 + by^2 + cz^2 = 0$ in perpendicular generators then

$$(b + c)\,u^2 + (c + a)\,v^2 + (a + b)\,w^2 = 0. \qquad ...(i)$$

Again if the plane $ux + vy + wz = 0$ touches the cone

$$\frac{x^2}{(b+c)} + \frac{y^2}{(c+a)} + \frac{z^2}{(a+b)} = 0$$

then we must have $\dfrac{u^2}{1/(b+c)} + \dfrac{v^2}{1/(c+a)} + \dfrac{w^2}{1/(a+b)} = 0$ **(Note)**

$\Rightarrow \quad (b + c)\,u^2 + (c + a)\,v^2 + (a + b)\,w^2 = 0.$

which is the same as (i) **Hence proved.**

Example 74:

Prove that the cones $ax^2 + by^2 + cz^2 = 0$ and $x^2/a + y^2/b + z^2/c = 0$ are reciprocal to each other.

Solution:

Let the cone reciprocal to $ax^2 + by^2 + cz^2 = 0$ ..(i)

be $Ax^2 + By^2 + Cz^2 + 2Fyz + 2Gzx + 2Hxy = 0,$ ...(ii)

where $A = \text{"}bc - f^2\text{"} = bc$, here $f = 0$. Similarly $B = ca$ and $C = ab$.

Also $F = \text{"}gh - af\text{"} = 0,\ G = 0,\ H = 0.$

$\therefore$ From (ii) the equation of the cone reciprocal to (i) is

$bcx^2 + cay + abz^2 = 0$

$\Rightarrow \quad x^2/a + y^2/b + z^2/c = 0.$ **Hence proved.**

8. *Tangent planes are drawn to the conicoid*

$$ax^2 + by^2 + cz^2 = 1$$

through the point (α, β, γ). *Prove that the perpendiculars to them from the origin generate the cone.*

$$(\alpha x + \beta y + \gamma z)^2 = \frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c}.$$

Sol. The equation of any plane through (α, β, γ) is

$$l(x - \alpha) + m(y - \beta) + n(z - \gamma) = 0$$

or $$lx + my + nz = l\alpha + m\beta + n\gamma \qquad ...(1)$$

| Form $lx + my + nz = p$

If it is the tangent plane to the conicoid

$$ax^2 + by^2 + cz^2 = 1,$$

then $$\frac{l^2}{a} + \frac{m^2}{b} + \frac{n^2}{c} = (l\alpha + m\beta + n\gamma)^2 \qquad ...(2)$$

Now d.c.'s of the normal to plane (1) are proportional to l, m, n

$\therefore$ Equation of $\perp$ to (1) through (0, 0, 0) are

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \qquad ...(3)$$

To find the locus of line (3), we are to eliminate l, m, n from (3) and (2). Putting the values of l, m, n from (3) in (2), we have

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} = (\alpha x + \beta y + \gamma z)^2$$

Hence the result.

Example 4. *Prove that the sum of squares of the intercepts made by a given sphere on any three mutually perpendicular lines through a fixed point is constant.* **[Imp.**

Sol. Let the given sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \quad ...(1)$$

and the three mutually $\perp$ lines through the fixed point (0, 0, 0) say, be

$$\frac{x}{l_1} = \frac{y}{m_1} = \frac{z}{n_1} \quad ...(2), \qquad \frac{x}{l_2} = \frac{y}{m_2} = \frac{z}{n_2} \quad ...(3)$$

and $$\frac{x}{l_3} = \frac{y}{m_3} = \frac{z}{n_3} \quad ...(4)$$

where l_1, m_1, n_1 ; etc. are the actual d.c.'s, so that

$$\left.\begin{array}{ll} l_1^2 + m_1^2 + n_1^2 = 1 \text{ etc.,} & l_1^2 + l_2^2 + l_3^2 = 1 \text{ etc.} \\ l_1l_2 + m_1m_2 + n_1n_2 = 0 \text{ etc.,} & l_1m_1 + l_2m_2 + l_3m_3 = 0 \end{array}\right\} \quad ...(5)$$

To find the intercept on line (2) :

Any point on line (2) is (l_1r, m_1r, n_1r).

If it lies on the sphere (1), then

$$r^2(l_1^2 + m_1^2 + n_1^2) + 2r(l_1u + m_1v + n_1w) + d = 0$$

or $$r^2 + 2r(l_1u + m_1v + n_1w) + d = 0 \qquad | \textit{ Using } (5)$$

It is a quadratic in r; let the two roots be r_1, r_2, which are the distances from O of the two points of intersection say A_1, A_2 of the line and the sphere.

$\therefore$ If L_1 is the length of intercept on the first line, then

$$L_1 = A_1A_2 = OA_2 - OA_1 = r_2 - r_1$$

$$\therefore \quad L_1^2 = (r_2 - r_1)^2 = (r_1 + r_2)^2 - 4r_1r_2$$

$$4\,(ul_1 + vm_1 + wn_1)^2 - 4d$$

$$= 4(u^2l_1^2 + v^2m_1^2 + w^2n_1^2 + 2uvl_1m_1 + 2vwm_1n_1 + 2wul_1n_1) - 4d$$

Similarly

$$L_2^2 = 4(u^2l_2^2 + v^2m_2^2 + w^2n_2^2 + 2uvl_2m_2 + 2vwm_2n_2 + 2wun_2l_2) - 4d$$

$$L_3^2 = 4(u^2l_3^2 + v^2m_3^2 + w^2n_3^2 + 2uvl_3m_3 + 2vwm_3n_3 + 2wun_3l_3) - 4d$$

Adding, the sum of squares of the intercepts $= L_1^2 + L_2^2 + L_3^2$

$$= 4[u^2(l_1^2 + l_2^2 + l_3^2) + v^2(m_1^2 + m_2^2 + m_3^2) + w^2(n_1^2 + n_2^2 + n_3^2)$$
$$+ 2uv(l_1m_1 + l_2m_2 + l_3m_3) + 2vw(m_1n_1 + m_2n_2 + m_3n_3)$$
$$+ 2wu(l_1n_1 + l_2n_2 + l_3n_3)] - 12d$$

$$= 4[u^2(1) + v^2(1) + w^2(1) + 2uv(0) + 2vw(0) + 2wu(0)] - 12d$$

$$= 4(u^2 + v^2 + w^2) - 12d \qquad | \textit{ Using } (5)$$

which is free from l_1, m_1, n_1 etc. and is therefore constant for any set of lines. Hence the result.

8. *P is a variable point on a given line and A, B, C are the projections on the axes. Show that the sphere OABC passes through a fixed circle.*

Sol. Let the given line be $\frac{x-x_1}{l}=\frac{y-y_1}{m}=\frac{z-z_1}{n}$

Any point on this line is (x_1+rl, y_1+rm, z_1+nr)

Its projection on the x-axis $[y=0, z=0]$ is A$(x+lr, 0, 0)$

Its projection on the y-axis $[z=0, x=0]$ is B$(0, y+mr, 0)$

Its projection on the z-axis $[x=0, y=0]$ is C$(0, 0, z+nr)$

Let the equation of sphere OABC through origin (0, 0, 0) be

$$x^2+y^2+z^2+2ux+2vy+2wz=0 \qquad \text{...(1)}$$

Since it passes through A$(x_1+lr, 0, 0)$

$$\therefore \qquad (x_1+lr)^2+0+0+2u(x_1+lr)+0+0=0$$

or $$(x_1+lr)^2+2u(x_1+lr)=0 \quad \therefore \quad 2u=-(x_1+lr)$$

Similarly $2v=-(y_1+mr)$ and $2w=-(z_1+nr)$

Putting these values of $2u$, $2v$, $2w$ in (1), we have

$$x^2+y^2+z^2-(x_1+lr)x-(y_1+mr)-(z_1+nr)z=0$$

or $$x^2+y^2+z^2-xx_1-yy_1-zz_1-r(lx+my+nz)=0$$

This is a sphere which passes through the fixed circle for all values of r

$$x^2+y^2+z^2-xx_1-yy_1-zz_1=0$$
$$lx+my+nz=0.$$

9. *Show that the centres of all sections of the sphere $x^2+y^2+z^2=r^2$ by planes through a point (x', y', z') be on the sphere $x(x-x')+y(y-y')+z(z-z')=0$.*

Sol. Let (x_1, y_1, z_1) be the centre of one of the sections. Centre of the given sphere is (0, 0, 0)

Then the line joining (0, 0, 0) and (x_1, y_1, z_1) is the normal to the plane and its direction cosine are $< x_1, y_1, z_1 >$.

Hence the equation of the plane cutting the sphere in a circle with centre (x_1, y_1, z_1) is

$$x_1(x-x_1)+y_1(y-y_1)+z_1(z-z_1) \qquad \text{...(1)}$$

If it passes through the point (x', y', z'), we have

$$x_1(x'-x_1)+y_1(y'-y_1)+z_1(z'-z_1)=0$$

Hence the locus of (x_1, y_1, z_1) is

$$x(x'-x)+y(y'-y)+z(z'-z)=0$$

or $$x(x-x')+y(y-y')+z(z-z')=0$$

Hence the result.

Example 63:

Find the locus of the centres of spheres of constant radius which pass through a given point and touch a given line.

Solution:

Take x-axis as the given line and (0, 0, a) as the given point. **(Note)**

Let the equation of the sphere by given as

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz = 0 \quad \ldots(i)$$

As it passes through the given point (0, 0, a) so $a^2 + 2wa + d = 0$ ...(ii)

Also the radius of the sphere (i) is given as constant k (say).

Then we have $\quad u^2 + v^2 + w^2 - d = k^2 \quad$ ...(iii)

The sphere (i) touches the given line which we have chosen as x-axis *i.e.*, $y = 0 = z$ at the points given by $x^2 + 2ux + d = 0$ ...(iv)

Since the sphere (i) touches the line $y = 0 = z$, so the roots of (iv) must be equal and therefore using '$b^2 = 4ac$' we have

$$(2u)^2 = 4 \,.\, 1 \,.\, d \quad \text{or} \quad u^2 = d \quad \ldots(v)$$

Eliminating d from (ii), (iii) and (v) we get

$$a^2 + 2wa + u^2 = 0,\ v^2 + w^2 = k^2$$

$\therefore$ The required locus of the centre $(-u, -v, -w)$ of the sphere (i) is given by the equations $a^2 + 2(-z)\, a + (-x)^2 = 0\ (-y)^2 + (-z)^2 = k^2$

$$\Rightarrow \quad x^2 - 2az + a^2 = 0,\ y^2 + z^2 = k^2$$

which is the curve of intersection of two quadratic surfaces

$$x^2 - 2az + a^2 = 0 \quad \text{and} \quad y^2 + z^2 = k^2$$ **Ans.**

Example 64:

Prove that the sphere

$S \equiv x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ *cuts the sphere*

$S' \equiv x^2 + y^2 + z^2 + 2u'x + 2v'y + 2w'z + d' = 0$ *in the great circle if*

$$2(u'^2 + v'^2 + w'^2) - d' = 2(uu' + vv' + ww') - d$$

or If $2(uu' + vv' + ww') = 2r'^2 + d + d'$, *where* r' *is the radius of the second sphere.*

Solution:

The equation of the plane through the circle of intersection of the given sphere is $S - S' = 0$

i.e., $2(u - u')x + 2(v - v')y + 2(w - w')z + (d - d') = 0$ ...(i)

If the sphere $S = 0$ cuts the sphere $S' = 0$ in a great circle, then the centre $(-u', -v', -w')$ of the sphere $S' = 0$ should lie on the plane (i) **(Note)**

$\therefore\ -2(u - u')u' - 2(v - v')v' - 2(w - w')w' + (d - d') = 0$

$\Rightarrow\ 2(u'^2 + v'^2 + w'^2) - d' = 2(uu' + vv' + ww') - d$ ...(ii)

Hence proved.

$\Rightarrow\ 2(r'^2 + d') - d' = 2(uu' + vv' + ww') - d$, where r' is the radius of the second sphere and so $r'^2 + u'^2 + v'^2 + w'^2 - d'$

$\Rightarrow\ 2r'^2 + d' = 2(uu' + vv' + ww')$ **Hence proved.**

Example 20. *Prove that the centres of the spheres which touch the lines $y = mx,\ z = c;\ y = -mx,\ z = -c$ lie upon the conicoid $mxy + cz(1 + m^2) = 0$.* **[V. Imp.]**

(Kanpur 1988; Gauhati 86; MDU 84; Rohil. 84, 85)

Sol. The given lines are $y = mz, \quad z = c$ ...(1)

and $y = -mx, \quad z = -c$ ...(2)

Let the sphere be given by the equation

$$x^2 + y^2 + z^2 - 2ux - 2vy - 2wz + d = 0 \quad ...(3)$$

The line (1) meets the sphere (3) where putting $y = mx$, $z = c$ in (3) we get

$$x^2 + m^2x^2 + c^2 - 2ux - 2vmx - 2wc + d = 0$$

or $$x^2 (1 + m^2) - 2(u + mv)x + (c^2 - 2cw + d) = 0 \quad ...(4)$$

Since the line (1) touches the sphere, so the two values of x given by (4) must be coincident *i.e.*, the discriminant of (4) is zero.

or $$4(u + mv)^2 - 4(1 + m^2) (c^2 - 2cw + d) = 0$$

or $$(u + mv)^2 = (1 + m)^2 (c^2 - 2cw + d) \quad ...(5)$$

Similarly the line (2) touches the sphere (3) if

[*change m to − m and c to −c in* (5)

$$(u - mv)^2 = (1 + m^2) (c^2 + 2cw + d) \quad ...(6)$$

Subtracting (6) from (5), (*to eliminate d*), we get

$$(u + mv)^2 - (u - mv)^2 = (1 + m^2) [(c^2 - 2cw + d) - (c^2 + 2cw + d)]$$

or $$4muv = (1 + m)^2 (-4cw)$$

or $$muv + (1 + m)^2cw = 0$$

$\therefore$ Locus of the centre (u, v, w) of the sphere (3) is

$$mxy + cz(1 + m^2) = 0.$$

Example 66:

Find the equation of a sphere touching the three co-ordinate planes. How many such spheres can be drawn?

Solution:

Let the equation of the sphere be given as

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0. \qquad \ldots(i)$$

If the sphere touches the yz-plane *i.e.,*, x = 0, then the length of the perpendicular from its centre (–u, –v, –w) to the plane x = 0 must be equal to its radius = $(u^2 + v^2 + w^2 - d)$.

i.e., $\quad \dfrac{-u}{2} = \sqrt{(u^2 + v^2 + w^2 - d)}$

$\Rightarrow \quad u^2 = u^2 + v^2 + w^2 - d$

$\Rightarrow \quad v^2 + w^2 = d. \qquad \ldots(ii)$

Similarly if the sphere (i) touches zx and xy-planes then we shall have

$$w^2 + u^2 = d \qquad \ldots(iii)$$

and $\quad u^2 + v^2 = d \qquad \ldots(iv)$

Adding (ii), (iii) and (iv) we get $2(u^2 + v^2 + w^2) = 3d$

$\Rightarrow \quad u^2 + v^2 + w^2 = \dfrac{3}{2}d$

$\Rightarrow \quad u^2 = \dfrac{1}{2}d$, from (ii)

Similarly from (iii), (iv) and (v),

we get $v^2 = \dfrac{1}{2}d = w^2$

$\therefore \quad u^2 = \dfrac{1}{2}d = v^2 = w^2 = \lambda^2$ (sy)

$\Rightarrow \quad u = \pm\lambda = v = w.$

Hence from (i) the required equation is

$$x^2 + y^2 + z^2 \pm 2\lambda(x + y + z) + 2\lambda^2 = 0. \quad \textbf{Ans.}$$

Since λ can take an infinite number of values, so an infinite number of such spheres can be drawn but if the radius of the sphere is given then λ can be expressed in terms of the given radius and then only eight such spheres can be possible as the sets of values of u, v and w can be taken in eight different ways.

Example 69:

Two spheres of radii r_1 and r_2 cut orthogonally. Prove that the radius of the common circle is $r_1 r_2/\sqrt{(r_1^2 + r_2^2)}$.

Solution:

Let the common circle be $x^2 + y^2 = a^2$, $z = 0$. Its radius is a. Any sphere through this circle is

$$x^2 + y^2 + z^2 - a^2 + 2kz = 0.$$

We choose the two spheres through the circle as

$$x^2 + y^2 + z^2 + 2k_1z - a^2 = 0,\ x^2 + y^2 + z^2 + 2k_2z - a^2 = 0.$$

These spheres will cut orthogonally if $2k_1k_2 = -a^2 - a^2$

i.e., $k_1^2\, k_2^2 = a^4.$ ...(i)

The radii of the above two spheres are given by

$$r_1^2 = k_1^2 + a^2 \text{ and } r_2^2 = k_2^2 + a^2.$$

From (i) and (ii) we obtain

$$(r_1^2 - a^2)(r_2^2 - a^2) = a^4 \Rightarrow r_1^2\, r_2^2 - r^2(r_1^2 + r_2^2) = 0.$$

Hence $a = \dfrac{r_1 r_2}{\sqrt{r_1^2 + r_2^2}}$.

10. *Show that the locus of the point of intersection of three mutually perpendicular tangent plane to the conicoid $a^2x^2 + b^2y^2 + c^2z^2 = 1$ is the sphere*

$$x^2 + y^2 + z^2 = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}.$$

Sol. The given conicoid is

$$a^2x^2 + b^2y^2 + c^2z^2 = 1 \qquad ...(1)$$

Let $l_1x + m_1y + n_1z = \sqrt{\dfrac{l_1^2}{a^2} + \dfrac{m_1^2}{b^2} + \dfrac{n_1^2}{c^2}}$ ...(2)

$l_2x + m_2y + n_2z = \sqrt{\dfrac{l_2^2}{a^2} + \dfrac{m_2^2}{b^2} + \dfrac{n_3^2}{c^2}}$ ...(3)

and $l_3x + m_3y + n_3z = \sqrt{\dfrac{l_3^2}{a^2} + \dfrac{m_3^2}{b^2} + \dfrac{n_3^2}{c^2}}$ ...(4)

be three mutually ⊥ tangent planes to (1)

where l_1, m_1, n_1 etc. are the actual direction cosine so that

$$l_1^2 + m_1^2 + n_1^2 = 1 \text{ etc.} \quad \text{and} \quad l_1^2 + l_2^2 + l_3^2 = 1 \text{ etc.}$$

$$l_1l_2 + m_1m_2 + n_1n_2 = 0 \text{ etc.} \quad \text{and} \quad l_1m_1 + l_2m_2 + l_3m_3 = 0 \text{ etc.} \qquad ...(5)$$

The co-ordinates of the point of intersection satisfy the three equations (2), (3), (4) and its locus is therefore obtained by eliminating $l_1, m_1, n_1; l_2, m_2, n_2; l_3, m_3, n_3$ from the equations.

∴ Squaring and adding (2), (3), (4), we have

6. *Show that the feet of the normals from the point* (α, β, γ) *to the paraboloid* $x^2 + y^2 = 2az$ *lie on the sphere*

$$x^2 + y^2 + z^2 - 2(a + \gamma) - \frac{y}{2\beta}(\alpha^2 + \beta^2) = 0.$$

Sol. The given paraboloid is $x^2 + y^2 = 2az$

or $$\frac{x^2}{a} + \frac{y^2}{a} = 2z \qquad ...(1)$$

Let (x_1, y_1, z_1) be the foot of normal through (α, β, γ) to (1) the equation of the normal is

$$\frac{x - x_1}{x_1/a} = \frac{y - y_1}{y_1/a} = \frac{z - z_1}{-1} \qquad ...(2)$$

If it passes through the given point (α, β, γ), then

$$\frac{\alpha - x_1}{x_1/a} = \frac{\beta - y_1}{y_1/a} = \frac{\gamma - z_1}{-1} = \lambda \quad \text{(say)}$$

From these, we get $\quad x_1 = \dfrac{a\alpha}{a + \lambda}, \quad y_1 = \dfrac{a\beta}{a + \lambda}, \quad z_1 = \gamma + \lambda \qquad ...(3)$

But (x_1, y_1, z_1) will lie on the paraboloid (1), so we have

$$\frac{1}{a} \cdot \frac{a^2\alpha^2}{(a + \lambda)^2} + \frac{1}{a} \frac{a^2\beta^2}{(a + \lambda)^2} = 2(\gamma + \lambda) \quad \text{or} \quad \frac{a(\alpha^2 + \beta^2)}{(a + \lambda)^2} = 2(\gamma + \lambda) \qquad ...(4)$$

The feet of the normals will lie on the given sphere if (x_1, y_1, z_1) satisfy the equation of the sphere

$$x^2 + y^2 + z^2 - z(a + \gamma) - \frac{y}{2\beta}(\alpha^2 + \beta^2) = 0$$

or if $$\frac{a^2\alpha^2}{(a + \lambda)^2} + \frac{a^2\beta^2}{(a + \lambda)^2} + (\gamma + \lambda)^2 - (\gamma + \lambda)(a + \lambda) - \frac{a}{2(a + \lambda)}(\alpha^2 + \beta^2) = 0$$

or if $$\frac{a(\alpha^2 + \beta^2)}{2(a + \lambda)^2}[2a - (a + \lambda)] + (\gamma + \lambda)[\gamma + \lambda - a - \gamma] = 0$$

or if $$\frac{a(\alpha^2 + \beta^2)}{2(a + \lambda)^2}(a - \lambda) + (\gamma + \lambda)(\lambda - a) = 0$$

or if $$(a - \lambda)\left[\frac{a(\alpha^2 + \beta^2)}{2(a + \lambda)^2} - \gamma - \lambda\right] = 0 \quad \text{or if} \quad \frac{a(\alpha^2 + \beta^2)}{2(a + \lambda)^2} = \gamma + \lambda \quad [\because \quad a - \lambda \neq 0]$$

which is true in view of (5). Hence the result.

1. Equations of the two systems of generating lines of hyperboloid of one sheet $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ are

(I) $\frac{x}{a}-\frac{z}{c}=\lambda\left(1-\frac{y}{b}\right),\ 1+\frac{y}{b}=\lambda\left(\frac{x}{a}+\frac{z}{c}\right)$

(II) $\frac{x}{a}-\frac{z}{c}=\mu\left(1+\frac{y}{b}\right),\ 1-\frac{y}{b}=\mu\left(\frac{x}{a}+\frac{z}{c}\right)$.

2. Equation of a generating line through any point of the hyperboloid of one sheet $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ can be written in the form

$$\frac{x-a\cos\alpha}{a\sin\alpha}=\frac{y-b\sin\alpha}{-b\cos\alpha}=\frac{z}{\pm c}.$$

3. The co-ordinates of the point of intersection of any two generating lines of two different systems of hyperboloid of one sheet (given by formula 1 above) are $x=a\,\frac{(1+\lambda\mu)}{\lambda+\mu}$, $y=b\,\frac{(\lambda-\mu)}{\lambda+\mu}$, $z=c\,\frac{(1-\lambda\mu)}{\lambda+\mu}$.

These equations are also called **Parametric Equations** of the hyperboloid ; λ, μ being the two parameters.

4. A point (θ, ϕ) on the hyperboloid $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ is the point whose co-ordinates are $(a\cos\theta\sec\phi,\ b\sin\theta\sec\phi,\ c\tan\phi)$.

5. Equations of the two systems of generators of the paraboloid $\frac{x^2}{a^2}-\frac{y^2}{b^2}=2z$ are

(I) $\frac{x}{a}-\frac{y}{b}=\lambda z,\ 2=\lambda\left(\frac{x}{a}+\frac{y}{b}\right)$

(II) $\frac{x}{a}-\frac{y}{b}=2\mu,\ z=\mu\left(\frac{x}{a}+\frac{y}{b}\right)$.

Ex. 3. *Show that the projections of the generators of a hyperboloid on co-ordinate planes are tangents to the sections of the hyperboloid by these respective planes.*

Sol. Let the equation of the hyperboloid be

$$x^2/a^2 + y^2/b^2 - z^2/c^2 = 1. \qquad \ldots(1)$$

The equations of a generator of the hyperboloid (1) are given by [See Ex. 1 (a) above]

$$\frac{x - a\cos\alpha}{a\sin\alpha} = \frac{y - b\sin\alpha}{-b\cos\alpha} = \frac{z}{c}. \qquad \ldots(2)$$

Now consider the co-ordinate plane XOY *i.e.*, the plane $z = 0$.

The section of the hyperboloid (1) by the plane $z = 0$ is

$$x^2/a^2 + y^2/b^2 = 1,\ z = 0. \qquad \ldots(3)$$

Again the projection of the generator (2) on the $z = 0$ plane is

$$\frac{x - a\cos\alpha}{a\sin\alpha} = \frac{y - b\sin\alpha}{-b\cos\alpha},\ z = 0$$

or $$\frac{x}{a\sin\alpha} - \frac{\cos\alpha}{\sin\alpha} = -\frac{y}{b\cos\alpha} + \frac{\sin\alpha}{\cos\alpha},\ z = 0$$

or $$\frac{x}{a\sin\alpha} + \frac{y}{b\cos\alpha} = \frac{1}{\sin\alpha\cos\alpha},\ z = 0$$

or $$(x/a)\cos\alpha + (y/b)\sin\alpha = 1,\ z = 0. \qquad \ldots(4)$$

These are clearly the equations of a tangent line to the section (3) of the hyperboloid at the point $(a\cos\alpha,\ b\sin\alpha,\ 0)$. Note that the first of the equations (4) *i.e.*, $(x/a)\cos\alpha + (y/b)\sin\alpha = 1$ is the equation of the plane through the generator and perpendicular to the plane $z = 0$.

Again consider the co-ordinate plane ZOX *i.e.*, the plane $y = 0$. The section of hyperboloid by the plane $y = 0$ is

$$x^2/a^2 - z^2/c^2 = 1,\ y = 0. \qquad \ldots(5)$$

The projection of the generator (2) on the plane $y = 0$ is

$$\frac{x - a\cos\alpha}{a\sin\alpha} = \frac{z}{c},\ y = 0$$

or $$\frac{x}{a}\sec\alpha - \frac{z}{c}\tan\alpha = 1,\ y = 0. \qquad \ldots(6)$$

The equation $(x/a)\sec\alpha - (z/c)\tan\alpha = 1$ represents the plane through the generator perpendicular to the plane $y = 0$. Clearly the equations (6) represent the straight line which is tangent to the section (5) of the hyperboloid at the point

$$(a\sec\alpha, 0, c\tan\alpha).$$

In a similar manner the result follows for the co-ordinate plane $x = 0$.

Ex. 6. *Show that the shortest distances between generators of the same system drawn at the ends of diameters of principal elliptic section of the hyperboloid*

$$x^2/a^2 + y^2/b^2 - z^2/c^2 = 1$$

lie on the surfaces whose equations are

$$\frac{cxy}{x^2 + y^2} = \pm \frac{abz}{a^2 - b^2}.$$

Sol. The equation of the hyperboloid is

$$x^2/a^2 + y^2/b^2 - z^2/c^2 = 1. \qquad \ldots(1)$$

The principal elliptic section of (1) by the plane $z = 0$ is

$$x^2/a^2 + y^2/b^2 = 1,\ z = 0. \qquad \ldots(2)$$

Let PQ be any diameter of (2), so that if the co-ordinates of the extremity P are $(a \cos \alpha, b \sin \alpha, 0)$ then those of Q are $(a \cos (\pi + \alpha), b \sin (\pi + \alpha), 0)$.

The equations of the generator through P are

$$\frac{x - a\cos\alpha}{a\sin\alpha} = \frac{y - b\sin\alpha}{-b\cos\alpha} = \frac{z}{c}. \qquad \ldots(3)$$

Replacing α by $\alpha + \pi$ in (3), the equations of the generator belonging to the system through Q are

$$\frac{x + a\cos\alpha}{-a\sin\alpha} = \frac{y + b\sin\alpha}{b\cos\alpha} = \frac{z}{c} \qquad \ldots(4)$$

Let l, m, n be the d.c.'s of the line of shortest distance between the generators (3) and (4). Since the line of shortest distance is perpendicular to both (3) and (4), we have

$$la\sin\alpha - mb\cos\alpha + cn = 0,$$

and $$-la\sin\alpha + mb\cos\alpha + cn = 0.$$

Solving these relations, we have

$$\frac{l}{2bc\cos\alpha} = \frac{m}{2ac\sin\alpha} = \frac{n}{0} \quad \text{or} \quad \frac{l}{b\cos\alpha} = \frac{m}{a\sin\alpha} = \frac{n}{0}. \qquad \ldots(5)$$

Now the equations of the line of shortest distance are

$$\begin{vmatrix} x - a\cos\alpha & y - b\sin\alpha & z \\ a\sin\alpha & -b\cos\alpha & c \\ b\cos\alpha & a\sin\alpha & 0 \end{vmatrix} = 0$$

and $$\begin{vmatrix} x + a\cos\alpha & y + b\sin\alpha & z \\ -a\sin\alpha & b\cos\alpha & c \\ b\cos\alpha & a\sin\alpha & 0 \end{vmatrix} = 0.$$

Expanding both the determinants with respect to the last column, we have

$$z(a^2\sin^2\alpha + b^2\cos^2\alpha) - \{(x - a\cos\alpha)\, a\sin\alpha - (y - b\sin\alpha)\, b\cos\alpha\} = 0 \qquad \ldots(6)$$

and $$z(-a^2\sin^2\alpha - b^2\cos^2\alpha) - c\,\{(x + a\cos\alpha)\, a\sin\alpha - (y + b\sin\alpha)\, b\cos\alpha\} = 0$$

i.e., $$z(a^2\sin^2\alpha + b^2\cos^2\alpha) + c\,\{(x + a\cos\alpha)\, a\sin\alpha - (y + b\sin\alpha)\, b\cos\alpha\} = 0 \qquad \ldots(7)$$

Thus the line of shortest distance is given by the planes (6) and (7). To find the locus of the line of shortest distance we are to eliminate the parameter 'α' between (6) and (7). For this we proceed as follows :

Subtracting (6) from (7), we have

$$c\,\{2ax\sin\alpha - 2by\cos\alpha\} = 0$$

or $$\tan\alpha = (by/ax). \qquad \ldots(8)$$

Adding (6) and (7), we get

$$2z\,(a^2\sin^2\alpha + b^2\cos^2\alpha) + c\,.\,\{2a^2\sin\alpha\cos\alpha - 2b^2\sin\alpha\cos\alpha\} = 0$$

or $$z\,(a^2\tan^2\alpha + b^2) + c\,(a^2 - b^2)\tan\alpha = 0, \text{ dividing by } 2\cos^2\alpha$$

or $$2\left(\frac{b^2y^2}{x^2} + b^2\right) + c\,(a^2 - b^2)\cdot\frac{by}{ax} = 0, \text{ substituting for } \tan\alpha \text{ from (8)}$$

or $$b^2z\,(x^2 + y^2)/x^2 + bc\,(a^2 - b^2)\,y/(ax) = 0$$

or $$\frac{cxy}{x^2 + y^2} = -\frac{abz}{a^2 - b^2}. \qquad \ldots(9)$$

In a similar manner if we consider the shortest distance between the generators of the other system, the locus of shortest distance is given by

$$\frac{cxy}{x^2 + y^2} = \frac{abz}{a^2 - b^2} \quad \text{[Simply replace } c \text{ by } -c]. \qquad \ldots(10)$$

From (9) and (10), the required locus is

$$\frac{cxy}{x^2 + y^2} = \pm\frac{abz}{a^2 - b^2}.$$

Example 1. Find the lengths of the sides of the skew quadrilateral formed by the generators of the hyperboloid.

$$x^2/4 + y^2 - z^2 = 49,$$

which pass through the two points (10, 5, 1), (14, 2, – 2).

Solution. Rewriting the given equation in the form

$$\left(\frac{x}{2}-z\right)\left(\frac{x}{2}+z\right) = (7-y)\,(7+y)$$

we see that the equations of the two systems of generating lines of the hyperboloid are

$$\frac{x}{2} - z = \lambda\,(7-y),\ \lambda\left[\frac{x}{2}+z\right] = 7+y \qquad ...(1)$$

$$\frac{x}{2} - z = \mu\,(7+y),\ \mu\left[\frac{x}{2}+z\right] = 7-y \qquad ...(2)$$

The generators (1) and (2) pass through the points (10, 5, 1) and (14, 2, – 2)

for $\lambda = 2,\ \mu = \frac{1}{3}$ and $l = \frac{1}{3},\ \mu = 1$ respectively.

The two pairs of generators through the two points, therefore, are

$$\begin{cases} \frac{x}{2}-z=(7-y), 2\left[\frac{x}{2}+z\right]=7+y & ...(4) \\ \frac{x}{2}-z=\frac{1}{3}(7+y), \frac{1}{3}\left[\frac{x}{2}+z\right]=7-y & ...(5) \end{cases}$$

$$\begin{cases} \frac{x}{2}-z=\frac{9}{5}(7-y), \frac{9}{5}\left[\frac{x}{2}+z\right]=7+y & ...(5) \\ \frac{x}{2}-z=7+y, \qquad \frac{x}{2}+z=7-y & ...(6) \end{cases}$$

Solving (3), and (6) as well as (4) and (5), we find that the two other vertices of the skew quadrilateral formed by the four generators are

$$\left(14, \frac{7}{3}, -\frac{7}{3}\right), \left(\frac{12}{2}, \frac{77}{16}, \frac{21}{16}\right)$$

The lengths of the sides are

$$\sqrt{(98)/16}, \sqrt{(308/3)}, \sqrt{2/3}, \sqrt{(7970)/16}.$$

Example 8:

Find the equation of a sphere which passes through the origin and intercepts lengths a, b, c on the x, y and z-axes respectively.

Or

Find the equation of the sphere which passes through (a, 0, 0) and (0, b, 0), (0, 0, c) and (0, 0, 0).

Or

Find its centre and radius.

Solution:

The sphere intercepts a length a on x-axis so it passes through the point (a, 0, 0). Similarly it passes through the points (0, b, 0) and (0, 0, c). Also it passes through the origin *i.e.,* (0, 0, 0).

Let the equation of the sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \qquad \ldots(i)$$

If it passes through (0, 0, 0) then from (i) we have $d = 0$ ...(ii)

If (i) passes through (a, 0, 0), then we get $a^2 + 2ua + d = 0$

$\Rightarrow$ from (ii), $a^2 + 2ua + 0 = 0$ or $u = -\frac{1}{2} a$, as $a \neq 0$

Similarly as (i) passes through (0, b, 0) and (0, 0, c) we get

$$y = -\tfrac{1}{2} b \quad \text{and} \quad w = -\tfrac{1}{2} c$$

Example 9:

A point moves so that the sum of the squares of its distances from the six faces of a cube is constant, show that its locus is a sphere.

Solution:

Take any vertex of the cube as origin and the edges terminating at that vertex as coordinate axes. Then if a be the length of each edge of the cube, then the equations of its six faces are $x = 0$, $y = 0$, $z = 0$, $x = a$, $y = a$ and $z = a$. **(Note)**

Let $P(x_1, y_1, z_1)$ be the moving point, then according to the given problem we have the sum of squares of its distances from the faces of the cube is constant k (say).

$$i.e., \left(\frac{x_1}{1}\right)^2 + \left(\frac{y_1}{1}\right)^2 + \left(\frac{z_1}{1}\right)^2 + \left(\frac{x_1 - a}{1}\right)^2 + \left(\frac{y_1 - a}{1}\right)^2 + \left(\frac{z_1 - a}{1}\right)^2 = k.$$

$$\Rightarrow x_1^2 + y_1^2 + z_1^2 + (x_1^2 - 2ax_1 + a^2) + (y_1^2 - 2ay_1 + a^2) + (z_1^2 - 2az_1 + a^2 = k)$$

$$\Rightarrow 2(x_1^2 + y_1^2 + z_1^2) - 2a(x_1 + y_1 + z_1) + (3a^2 - k) = 0$$

$$\Rightarrow x_1^2 + y_1^2 + z_1^2 - a(x_1 + y_1 + z_1) + d = 0, \text{ where } d = (3a^2 - k)/2$$

$\therefore$ The locus of $P(x_1 + y_1 + z_1)$ is $x^2 + y^2 + z^2 - ax - ay - az + d = 0$ which evidently represents a sphere as it is second degree equation in x, y, z; coefficients of x^2, y^2, z^2 are equal and terms containing product terms xy, yz, zx are absent. **Hence proved.**

Example 11:

Find the equation of the sphere circumscribing the tetrahedron whose faces are $y/b + z/c = 0$, $z/c + x/a = 0$, $x/a + y/b = 0$, $x/a + y/b + z/c = 1$.

Solution:

Taking three planes at a time and solving their equations, we get the coordinates of the vertices of this tetrahedron as (0, 0, 0), (a, b, –c), (a, –b, c) and (–a, b, c).

$2u = -(a^2 + b^2 + c^2)/a,$

$2v = -(a^2 + b^2 + c^2)/b$, $2w = -(a^2 + b^2 + c^2)/c$ and $d = 0$

$\therefore$ The required equation is

$$x^2 + \text{iojh} \quad \frac{x^2 + y^2 + z^2}{a^2 + b^2 + c^2} - \frac{x}{a} - \frac{y}{b} - \frac{z}{c} = 0$$ **Ans.**

Example 52:

Tangent plane at any point of the sphere $x^2 + y^2 + z^2 = r^2$ meets the coordinate axes at A, B, C. Show that the locus of the point of intersection of planes drawn parallel to the coordinate planes through A, B, C is the surface $x^{-2} + y^{-2} + z^{-2} = r^{-2}$.

Solution:

The equation of the tangent plane to $x^2 + y^2 + z^2 = r^2$ at any point (x_1, y_1, z_1) is $xx_1 + yy_1 + zz_1 = r^2$. ...(i)

The plane (i) meets the coordinate axes at $A(r^2/x_1, 0, 0)$, $B(0, r^2/y_1, 0)$, $C(0, 0, r^2/z_1)$. The equations of the planes parallel to the coordinate planes through A, B, C are

$$x = \frac{r^2}{x_1},\ y = \frac{r^2}{y_1},\ z = \frac{r^2}{z_1} \text{ respectively,}$$

$$\Rightarrow \qquad x^{-2} + y^{-2} + z^{-2} = \frac{1}{r^2}(x_1^2 + y_1^2 + z_1^2) = \frac{r^2}{r^4},$$

[since (x_1, y_1, z_1) lies on $x^2 + y^2 + z^2 = r^2$]

Hence $x^{-2} + y^{-2} + z^{-2} = r^{-2}$.

Example 71:

Find the equation of the sphere which touches the plane $3x + 2y - z + 2 = 0$ at the point $(1, -2, 1)$ and cuts orthogonally the sphere $x^2 + y^2 + z^2 - 4x + 6y + 4 = 0$.

Solution:

Let the equation of the required sphere be

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0. \qquad \ldots(i)$$

The sphere (i) cuts the given sphere orthogonally.

$\therefore \quad 2u(-2) + 2v(3) + 2w.0 = d + 4$

$\Rightarrow \quad 4u - 6v + d + 4 = 0. \qquad \ldots(ii)$

The tangent plane to (i) at the point $(1, -2, 1)$ is

$$1.x - 2y + 1.z + u(x + 1) + v(y - 2) + w(z + 1) + d = 0$$

$\Rightarrow \quad x(1 + u) + y(v - 2) + z(1 + w) + (u - 2v + 2w + d) = 0. \qquad \ldots(iii)$

The equation (iii) must be identical with $3x + 2y - z + 2 = 0$.

$$\therefore \quad \underset{(i)}{\frac{1+u}{3}} = \underset{(ii)}{\frac{v-2}{2}} = \underset{(iii)}{\frac{1+w}{-1}} = \underset{(iv)}{\frac{u-2v+2w+d}{2}}.$$

Now (i) = (ii) $\Rightarrow$ $2u - 3v + 8 = 0$, (iv)

(ii) = (iii) $\Rightarrow$ $v + 2w = 0$, $\Rightarrow w = -v/2$, ...(v)

(iii) = (iv) $\Rightarrow$ $u - 2v + 3w + d + 2 = 0$. ...(vi)

Subtracting (vi) from (ii), we get $3u - 4v - 3w + 2 = 0$

$\Rightarrow \quad 6u - 5v + 4 = 0$, by (v) $\qquad \ldots(vii)$

Solving (iv) and (vii), we get $u = 7/2$, $v = 5$.

Now from (v), (vi), we obtain $w = -5/2$ and $d = 12$.

Substituting the values of u, v, w and d in (i),

$x^2 + y^2 + z^2 + 7x + 10y - 5z + 12 = 0$ is the required sphere.

Example 2:

Find the locus of a point from which three mutually perpendicular lines can be drawn to intersect the central conic

$$ax^2 + by^2 = 1, \ z = 0.$$

Solution:

Let $P(\alpha, \beta, \gamma)$ be a point from which three mutually perpendicular lines can be drawn to intersect the given conic. The equation of the cone with vertex $P(\alpha, \beta, \gamma)$ and intersecting conic. $ax^2 + by^2 = 1, z = 0$ is

$$a(\alpha x - \gamma x)^2 + b\,(\beta z - \gamma y)^2 - (z - \gamma)^2 = 0.$$

It will have three mutually perpendicular lines if the sum of the coefficients of x^2, y^2 and z^2 is zero.

$\therefore \quad a\alpha^2 + a\gamma^2 + b\beta^2 + b\gamma^2 - 1 = 0$ or $a(\alpha^2 + \gamma^2) + b\,(\beta^2 + \gamma^2) = 1.$

Hence, the locus of $P\,(\alpha, \beta, \gamma)$ is $a(x^2 + z^2) + b\,(y^2 + z^2) = 1.$

Example 6:

Prove that the plane $lx + my + n = 0$ *cuts the cone*

$$(b - c)\,x^2 + (c - a)\,y^2 + (a - b)\,z^2 + 2fyz + 2gzx + 2hxy = 0$$

in perpendicular lines if

$$(b - c)\,l^2 + (c - a)\,m^2 + (a - b)\,n^2 + 2fm + 2gnl + 2hlm = 0.$$

Solution:

The sum of the coefficients of x^2, y^2 and z^2 in the equation of the given cone

$$= (b - c) + (c - a) + (a - b) = 0.$$

Thus the given cone has three mutually perpendicular generators. Thus given plane will cut the given cone is perpendicular lines if the normal to the plane through the vertex O viz.

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \text{ is a generator of the given cone.}$$

Hence $(b - c)\,l^2 + (c - a)\,m^2 + (a - b)\,n^2 + 2fmn + 2gnl + 2hlm = 0.$

Example 5.17 : *Prove that the plane $ax + by + cz = 0$ cuts the cone $xy + yz + zx = 0$ in perpendicular lines if $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$.*

Solution : Let $\frac{x-0}{l} = \frac{y-0}{m} = \frac{z-0}{n}$ be the line of section.

$\therefore l$, m, n satisfies equation of cone and plane.

$$\therefore \quad lm + mn + ln = 0$$

$$\text{and} \quad al + bm + cn = 0 \qquad \dots (1)$$

$$\Rightarrow \quad n = \frac{al + bm}{-c}$$

Substituting we get

$$lm - \frac{m}{c}(al + bm) - \frac{l}{c}(al + bm) = 0$$

$$\therefore \quad al^2 + lm\,(a + b - c) + bm^2 = 0$$

$$\therefore \quad a\left(\frac{l}{m}\right)^2 + (a + b - c)\left(\frac{l}{m}\right) + b = 0$$

which is a quadratic in $\frac{l}{m}$

Let $\frac{l_1}{m_1}$ and $\frac{l_2}{m_2}$ be the two roots of this equation.

$\therefore$ product of the roots $= \frac{b}{a}$

$$\text{i.e.} \quad \frac{l_1}{m_1} \cdot \frac{l_2}{m_2} = \frac{b}{a}$$

$$\therefore \quad \frac{l_1 l_2}{1/a} = \frac{m_1 m_2}{1/b} = \frac{n_1 n_2}{1/c} \text{ by symmetry}$$

Now two lines will be at right angle

$$\text{if} \quad l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

$$\text{i.e.} \quad \frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 0$$

Example 24:

Show that the perpendicular from the origin on the generator of the hyperboloid $(x^2/a^2) + (y^2/b^2) - (z^2/c^2) = 1$ lie on the curve

$$\frac{a^2 (b^2 + c^2)^2}{x^2} + \frac{b^2 (c^2 + a^2)^2}{y^2} = \frac{c^2 (a^2 - b^2)^2}{z^2}$$

Solution:

We know that the equations to a generator of the hyperboloid through any point of the principal elliptic section $(x^2/a^2) + (y^2/b^2) = 1$, $z = 0$ is

$$\frac{x - a\cos\theta}{a\sin\theta} = \frac{y - b\sin\theta}{-b\cos\theta} = \frac{z - 0}{+c} \qquad \text{...(i)}$$

Equation of any line through the origin is

$$\frac{x - 0}{l} = \frac{y - o}{m} = \frac{z - 0}{n} \qquad \text{...(ii)}$$

If the line (ii) is perpendicular to the generator (i), then

$$al\sin\theta - bm\cos\theta + cn = 0 \qquad \text{...(iii)}$$

Also if (i) and (ii) are complainer, then

$$\begin{vmatrix} a\cos\theta & b\sin\theta & 0 \\ a\sin\theta & -b\cos\theta & +c \\ l & m & n \end{vmatrix} = 0 \qquad \textbf{(Note)}$$

$\Rightarrow$ $a\cos\theta(-nb\cos\theta - mc) - b\sin\theta(an\sin\theta - lc) = 0$

$\Rightarrow$ $-anb(\cos^2\theta + \sin^2\theta) - amc\cos\theta + lbc\sin\theta = 0$

$\Rightarrow$ $bcl\sin\theta - acm\cos\theta - abn = 0$ ...(iv)

Solving (iii) and (iv) simultaneously for $\sin\theta$ and $\cos\theta$, we get

$$\frac{\sin\theta}{ab^2\ mn + ac^2\ mn} = \frac{\cos\theta}{bc^2\ nl + a^2b\ ln} = \frac{1}{-a^2c\ lm + b^2c\ lm}$$

$$\Rightarrow \qquad \frac{\sin\theta}{amn\,(b^2 + c^2)} = \frac{\cos\theta}{bnl\,(c^2 + a^2)} = \frac{1}{-c\ lm\,(a^2 + b^2)}$$

$$\Rightarrow \qquad \sin\theta = \frac{an\,(b^2 + c^2)}{-cl\,(a^2 - b^2)},\ \cos\theta = \frac{bn\,(c^2 + a^2)}{-cm\,(a^2 - b^2)}$$

$$\Rightarrow \qquad \left[\frac{an\,(b^2 + c^2)}{-cl\,(a^2 - b^2)}\right]^2 + \left[\frac{bn\,(c^2 + a^2)}{-cm\,(a^2 - b^2)}\right]^2 = 1$$

$$\Rightarrow \quad \frac{a^2\,(b^2 + c^2)^2}{l^2} + \frac{b^2\,(c^2 + a^2)^2}{m^2} = \frac{c^2}{n^2}$$

This shows that the line (ii) lies on the curve

$$\frac{a^2\,(b^2 + c^2)^2}{x^2} + \frac{b^2\,(c^2 + a^2)^2}{y^2} = \frac{c^2\,(a^2 - b^2)^2}{z^2}$$

Proved.

Example 12:

Prove that the lines drawn through the origin at right angles to the normal planes of the cone $ax^2 + by^2 + cz^2 = 0$ generates the cone

$$\frac{a(b - c)^2}{x^2} + \frac{b(c - a)^2}{y^2} + \frac{c(a - b)^2}{z^2} = 0$$

Solution:

Let $\frac{x}{l} = \frac{y}{m} = \frac{z}{n}$ be a generator of the given cone.

Then $\quad al^2 + bm^2 + cn^2 = 0 \qquad$ ...(i)

We know that the normal plane of the given cone which passes through the above generator is

$$mn\,(c - b)\,x + nl\,(a - c)\,y + lm\,(b - a)\,z = 0$$

$$\Rightarrow \qquad \frac{(b - c)\,x}{l} + \frac{(c - a)\,y}{m} + \frac{(a - b)}{n} = 0. \qquad \text{...(ii)}$$

Any line through origin at right angles to the plane (ii) is

$$\frac{x}{(b-c)/l} = \frac{y}{(c-a)/m} = \frac{z}{(a-b)/n} = 0. \qquad \text{...(iii)} \textbf{ (Note)}$$

Required locus of this line (iii) is obtained by eliminating l, m, n, between (i) and (ii) and is

$$a\left(\frac{b-c}{x}\right)^2 + b\left(\frac{c-a}{y}\right)^2 + c\left(\frac{a-b}{z}\right)^2 = 0$$

$$\Rightarrow \qquad \frac{a\,(b-c)^2}{x^2} + \frac{b\,(c-a)^2}{y^2} + \frac{c\,(a-b)^2}{z^2} = 0. \quad \textbf{Hence proved.}$$

Example 24:

Prove that all plane sections of the conicoid $ax^2 + by^2 + cz^2 = 1$ which are rectangular hyperbolas and which pass through the point (α, β, γ) touch the cone $\frac{(x-\alpha)^2}{b+c} + \frac{(y-\beta)^2}{c+a} + \frac{(z-\gamma)^2}{a+b} = 0.$

Solution:

The condition for the section to be a rectangular hyperbola is

$$(b+c)\,l^2 + (c+a)\,m^2 + (a+b) + n^2 = 0 \qquad \text{...(i)}$$

The plane through (α, β, γ) is

$$l\,(x-\alpha) + m\,(y-\beta) + n\,(z-\gamma) = 0$$

Shifting the origin to the point (α, β, γ), then above equation of the plane reduces to $lx + my + nz = 0$ ...(ii)

Relations (i) and (ii) show that normal to the plane (ii) is a generator of the cone $(b+c)\,x^2 + (c+a)\,y^2 + (a+b)\,z^2 = 0$

or the plane (ii) touches the reciprocal cone

$$\frac{x^2}{b+c} + \frac{y^2}{c+a} + \frac{z^2}{a+b} = 0.$$

Shifting the origin back, we get

$$\frac{(x-\alpha)^2}{b+c} + \frac{(y-\beta)^2}{c+a} + \frac{(z-\gamma)^2}{a+b} = 0. \quad \textbf{Hence proved.}$$

Example 31:

Prove that the section of the praboloid $ax^2 + by^2 = 2cz$ by a tangent plane to the given cone $(x^2/b) + (y^2/a) + \{z^2/(a + b)\} = 0$ is a rectangular hyperbola.

Solution:

If $lx + my + nz = 0$ be a tangent plane to the given cone, then its normal $x/l = y/m = z/n$ is a generator of the reciprocal cone

$$bx^2 + ay^2 + (a + b) z^2 = 0$$

$$\therefore \quad bl^2 + am^2 + (a + b) n^2 = 0 \qquad \text{...(i)}$$

Now if the section be a rectangular hyperbola, then the sum of the squares of the semi-axes is zero *i.e.*, $r_1^2 + r_2^2 = 0$,

where r_1^2 and r_2^2 are the roots of

$$abn^6r^4 - n^2p_0^2 [(a + b) n^2 + am^2 + bl^2] r^2 + p_0^4 (l^2 + m^2 + n^2) = 0$$

$\therefore$ $r_1^2 + r_2^2 = 0$ gives $(a + b) n^2 + am^2 + bl^2 = 0$, which is true by virtue of (i). **Hence proved.**

Ex. 12. ***Prove that the locus of the point of intersection of three tangent planes to $x^2/a^2+y^2/b^2+z^2/c^2=1$ which are parallel to conjugate diametral planes of $x^2/\alpha^2+y^2/\beta^2+z^2/\gamma^2=1$ is***

$$\frac{x^2}{\alpha^2}+\frac{y^2}{\beta^2}+\frac{z^2}{\gamma^2}=\frac{a^2}{\alpha^2}+\frac{b^2}{\beta^2}+\frac{c^2}{\gamma^2}.$$

Solution. The equations of the ellipsoids are given as

$$x^2/a^2+y^2/b^2+z^2/c^2=1 \qquad \text{...(1)}$$

and

$$x^2/\alpha^2+y^2/\beta^2+z^2/\gamma^2=1. \qquad \text{...(2)}$$

Let OP, OQ and OR be the conjugate semi-diameters of the ellipsoid (2). Let (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3) be the co-ordinates of the extremities P, Q and R respectively.

The equation of the diametral plane of OP with respect to the ellipsoid (2) is

$$xx_1/\alpha^2+yy_1/\beta^2+zz_1/\gamma^2=0. \qquad \text{...(3)}$$

The equation of any plane parallel to the plane (3) is

$$xx_1/\alpha^2+yy_1/\beta^2+zz_1/\gamma^2=p_1, \text{ (say)} \qquad \text{...(4)}$$

If the plane (4) be a tangent plane to the ellipsoid (1), then

$$p_1^2=a^2\ (x_1^2/\alpha^4)+b^2\ (y_1^2/\beta^4)+c^2\ (z_1^2/\gamma^4) \qquad \ldots(5)$$

[using the condition for tangency $p^2=a^2l^2+b^2m^2+c^2n^2$].

Similarly the equations of the other two tangent planes are

$$xx_2/\alpha^2+yy_2/\beta^2+zz_2/\gamma^2=p_2 \qquad \ldots(6)$$

and

$$xx_3/\alpha^2+yy_3/\beta^2+zz_3/\gamma^2=p_3 \qquad \ldots(7)$$

where $p_2^2=a^2\ (x_2^2/\alpha^4)+b^2\ (y_2^2/\beta^4)+c^2\ (z_2^2/\gamma^4)$ ...(8)

and $p_3^2=a^2\ (x_3^2/\alpha^4)+b^2\ (y_3^2/\beta^4)+c^2\ (z_3^2/\gamma^4)$. ...(9)

Now to find the locus of the point of intersection of the tangent planes (4), (6) and (7), squaring and adding both sides of the equations (4), (6) and (7), we get

$$\sum\left\{\frac{x^2}{\alpha^4}\ (x_1^2+x_2^2+x_3^2)\right\}+2\sum\left\{\frac{xy}{\alpha^2\beta^2}\ (x_1y_1+x_2y_2+x_3y_3)\right\}$$
$$=\sum\left\{\frac{a^2}{\alpha^4}\ (x_1^2+x_2^2+x_3^2)\right\}$$

or $\sum\left\{\frac{x^2}{\alpha^4}.\alpha^2\right\}+0=\sum\left\{\frac{a^2}{\alpha^4}.\alpha^2\right\}$ [$\because\ \Sigma\ x_1^2=\alpha^2,\ \Sigma\ x_1y_1=0$]

or $$x^2/\alpha^2+y^2/\beta^2+z^2/\gamma^2=a^2/\alpha^2+b^2/\beta^2+c^2/\gamma^2.$$

This is the equation of the required locus.

Remark. In case $\alpha=\beta=\gamma$, the equation of the required locus becomes $x^2+y^2+z^2=a^2+b^2+c^2$ which is the equation of the **director sphere.**

Ex.9. *Two perpendicular tangent planes to the paraboloid $x^2/a^2+y^2/b^2=2z$ intersect in a line lying on the plane* $x=0$. *Show that the line touches the parabola*

$$y^2=(a+b)\,(2z+a),\ x=0.$$

Solution. The equation of the paraboloid is

$$x^2/a+y^2/b=2z. \qquad \text{.. (1)}$$

Let the two tangent planes intersect in a line lying on the plane $x=0$ given by

$$my+nz=p,\ x=0. \qquad \text{.. (2)}$$

The equation of any plane through the line (2) is given by

$$my+nz-p+\lambda x=0 \text{ or } \lambda x+my+nz=p. \qquad \text{...(3)}$$

The plane (3) must be tangent plane to the parabo oid (1) and hence we have [By equation (4) of § 3 (iii)]

$$a\lambda^2+bm^2+2np=0. \qquad \text{...(4)}$$

This being a quadratic equation in λ gives two values of λ, say, λ_1 and λ_2.

$$\therefore\quad \lambda_1\lambda_2=(bm^2+2np)/a.$$

Hence the equation (3) gives two tangent planes for two values of λ. The d.r.'s of the normals of these two tangent planes are λ_1, m, n and λ_2, m, n. If these tangent planes be perpendicular, then

$$\lambda_1\lambda_2+m.m+n.n=0$$

or
$$(bm^2+2np)/a+m^2+n^2=0$$

or
$$(a+b)\,m^2+an^2+2np=0. \qquad \text{...(5)}$$

Now in order to prove that the line (2) touches a parabola, we should show that the envelope of (2) under condition (5) is the parabola in question.

Putting the value of p from (2) in (5), the equations of the line (2) are

$$(a+b)\,m^2+an^2+2n\,(my+nz)=0,\ x=0$$

or
$$(a+b)\,(m/n)^2+2y\,(m/n)+(a+2z)=0,\ x=0. \qquad \text{...(6)}$$

The first of these two equations being a quadratic in the parameter m/n, the equation of the envelope is given by the discriminant of (6) equated to zero

i.e. by
$$4y^2-4.(a+b)\,(a+2z)=0,\ x=0$$

i.e. by
$$y^2=(a+b)\,(a+2z),\ x=0. \qquad \textbf{Proved.}$$

Example 10. *POP' is a variable diameter of the ellipse*

$$z = 0, \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

and a circle is described in the plane PP'ZZ' on PP' as diameter. Prove that as PP' varies, the circle generates the surface,

$$(x^2 + y^2 + z^2)\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right) = x^2 + y^2. \qquad \textbf{[Imp.]}$$

Sol. The given ellipse is

$$z = 0, \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad ...(1)$$

Since POP′ is the diameter of the ellipse (1), then if P is (a cos θ, b sin θ, 0), P′ is (– a cos θ, – b sin θ, 0)

The required circle on PP′ as diameter is the intersaction of the sphere on PP′ as diameter and the plane PP′ZZ′.

Now equation of the sphere on PP′ as diameter is

$$(x - a\cos\theta)(x + a\cos\theta) + (y - b\sin\theta)(y + b\sin\theta) + (z - 0)(z - 0) = 0$$

| *Using* $(x - x_1)(x - x_2) + (y - y_1)(y - y_2) + (z - z_1)(z - z_2) = 0$

or $\quad x^2 - a^2\cos^2\theta + y^2 - b^2\sin^2\theta + z^2 = 0$

or $\quad x^2 + y^2 + z^2 = a^2\cos^2\theta + b^2\sin^2\theta \qquad ...(2)$

Now any plane thro' ZZ′, *i.e.*, z-axis or $x = 0$, $y = 0$ is

$$x + ky = 0 \qquad ...(3)$$

Since it passes thro' P (a cos θ, b sin θ, 0)

$\therefore \quad a\cos\theta + k\, b\sin\theta = 0$

or
$$k = \frac{-a\cos\theta}{b\sin\theta}$$

Putting this value of k in (3), the plane PP′ZZ′ becomes,

$$x - \frac{a\cos\theta}{b\sin\theta}\, y = 0$$

or
$$\frac{x}{a\cos\theta} = \frac{y}{b\sin\theta} \qquad ...(4)$$

To find the locus as PP′ varies, we have to eliminate θ, the unknown variable from (2) and (4), which taken together represent the circle on PP′ as diameter in the plane PP′ZZ′.

Now from (4),

$$\frac{\frac{x}{a}}{\cos\theta} = \frac{\frac{y}{b}}{\sin\theta} = \frac{\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}}{\sqrt{\cos^2\theta+\sin^2\theta}} = \sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}$$

$$\therefore \quad \cos\theta = \frac{\frac{x}{a}}{\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}},\ \sin\theta = -\frac{\frac{y}{b}}{\sqrt{\frac{x^2}{a^2}+\frac{y^2}{b^2}}}$$

Putting these values of sin θ, cos θ in (2), we get

$$x^2+y^2+z^2 = a^2\frac{x^2}{a^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)} + b^2.\frac{y^2}{b^2\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)}$$

or $\quad (x^2+y^2+z^2)\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right) = x^2+y^2$

which is the required locus.

Example 4:

Prove that the locus of the section of the ellipsoid $\Sigma (x^2/a^2) = 1$ by the plane PQR is the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1/3$.

Solution:

As we can prove that the equation of the plane PQR is

$$\left(\frac{x_1+x_2+x_3}{a^2}\right)x + \left(\frac{y_1+y_2+y_3}{b^2}\right)y + \left(\frac{z_1+z_2+z_3}{c^2}\right)z = 1. \quad \text{...(i)}$$

If (α, β, γ) be the centre of the section of the given ellipsoid by the plane PQR then the equation of PQR can be written as

"$T = S_1$" $\qquad$ i.e. $\frac{\alpha x}{a^2}+\frac{\beta y}{b^2}+\frac{\gamma z}{c^2} = \frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}$ $\qquad$ ...(ii)

∴ Equations (i) and (ii) represent the same plane, therefore comparing them we get

$$\frac{x_1+x_2+x_3}{\alpha} = \frac{y_1+y_2+y_3}{\beta} = \frac{z_1+z_2+z_3}{\gamma} = \frac{1}{\alpha^2/a^2+\beta^2/b^2+\gamma^2/c^2}$$

where $\quad \frac{\alpha}{a} = \left(\frac{x_1+x_2+x_3}{a}\right)\left(\frac{\alpha^2}{a^2}+\frac{\beta^2}{b^2}+\frac{\gamma^2}{c^2}\right)$ $\qquad$ **(Note)**

Similarly $\frac{\beta}{b} = \left(\frac{y_1 + y_2 + y_3}{b}\right)\left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2}\right)$

$$\frac{\gamma}{c} = \left(\frac{z_1 + z_2 + z_3}{c}\right)\left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2}\right)$$

Squaring and adding we get

$$\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} = \left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2}\right)^2$$

$$\left[\left(\frac{x_1 + x_2 + x_3}{a^2}\right)^2 + \left(\frac{y_1 + y_2 + y_3}{b^2}\right)^2 + \left(\frac{z_1 + z_2 + z_3}{c^2}\right)^2\right]$$

$$\Rightarrow \quad \frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2} = 3\left(\frac{\alpha^2}{a^2} + \frac{\beta^2}{b^2} + \frac{\gamma^2}{c^2}\right)^2$$

$\Rightarrow \quad (\alpha^2/a^2) + (\beta^2/b^2) + (\gamma^2/c^2) = 1/3.$

$\therefore$ The required locus of (α, β, γ) is

$(x^2/a^2) + (y^2/b^2) + (z^2/c^2) = 1/3.$ **Hence proved.**

Example 7

The plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ *meets the coordinate axes in A, B, C. Find the equation of the cone whose vertex is at the origin and the guiding curve is the circle through the points A, B, C.*

Solution

The equation of the plane is

$$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1 \qquad \ldots(1)$$

The plane meets the coordinate axes in A, B, C. The coordinates of A, B, C are $(a, 0, 0)$, $(0, b, 0)$, $(0, 0, c)$ respectively.

The guiding curve is the circle through the points A, B, C, which is the intersection of the plane through A, B, C and any sphere through the points A, B, C. The equation of the guiding curve is

$$x^2+y^2+z^2-ax-by-cz=0 \qquad \ldots(2)$$

Making Eq. (2) homogeneous using Eq. (1), the equation of the cone is

$$x^2+y^2+z^2-(ax+by+cz)(1)=0$$

$$x^2+y^2+z^2-(ax+by+cz)\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)=0$$

$$x^2+y^2+z^2-x^2-\frac{axy}{b}-\frac{azx}{c}-\frac{bxy}{a}-y^2-\frac{byz}{c}-\frac{czx}{a}-\frac{cyz}{b}-z^2=0$$

$$xy\left(\frac{a}{b}+\frac{b}{a}\right)+yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)=0$$

Example 79:

Prove that the locus of the line of intersection of tangent planes to the cone $ax^2 + by^2 + cz^2 = 0$ which touch along perpendicular generators is the cone

$$a^2 (b + c) x^2 + b^2 (c + a) y^2 + c^2 (a + b) z^2 + = 0.$$

Solution:

We know that the tangent plane to a cone at any point touches it along the generator through that point. Let $x/l = y/m = z/n$ be the line of intersection of two tangent planes which touch the cone along perpendicular generators.

$\therefore$ The equations of the plane containing these two perpendicular generators is $\quad ax \,.\, l + by \,.\, m + cz \,.\, n = 0 \quad$ ...(i)

Also the equation of the cone is $ax^2 + by^2 + cz^2 = 0$. ...(ii)

Now we can prove that the plane

$$ux + vy + wz = 0$$

cuts the cone $ax^2 + by^2 + cz^2 = 0$ in perpendicular generators if

$$(b + c)\, u^2\, (c + a)\, v^2 + (a + b)\, w^2 = 0.$$

$\therefore$ The plane (i) will cut the cone (ii) in perpendicular generators if

$$(b + c)\,(al)^2 + (c + a)\,(bm)^2 + (a + b)\,(cn)^2 = 0 \qquad \textbf{(Note)}$$

$\therefore$ The locus of the line $x/l = y/m = z/n$ is

$$(b + c)\, a^2x^2 + (c + a)\, b^2y^2 + (a + b)c^2z^2 = 0.$$ **Hence proved.**

Ex. 4. *CP, CQ are any two conjugate semi-diameters of the ellipse $x^2/a^2+y^2/b^2=1$, $z=c$. CP', CQ' are the conjugate semi-diameters of the ellipse $x^2/a^2+y^2/b^2=1$, $z=-c$, drawn in the same directions as CP and CQ. Prove that the hyperboloid $\dfrac{2x^2}{a^2}+\dfrac{2y^2}{b^2}-\dfrac{z^2}{c^2}=1$ is generated by either PQ' or $P'Q$.*

Solution. Since CP and CQ are the conjugate semi-diameters of the ellipse $x^2/a^2+y^2/b^2=1$, $z=c$, so that the co-ordinates of the extremities P and Q are

$P\equiv(a\cos\alpha, b\sin\alpha, c)$ and $Q\equiv(-a\sin\alpha, b\cos\alpha, c)$.

Similarly the co-ordinates of the extremities P' and Q' on $x^2/a^2+y^2/b^2=1$, $z=-c$ are

$P'\equiv(a\cos\alpha, b\sin\alpha, -c)$ and $Q'\equiv(-a\sin\alpha, b\cos\alpha, -c)$.

The equations to the line joining the points P and Q' *i.e.* of the line PQ' are

$$\frac{x-a\cos\alpha}{-a\sin\alpha-a\cos\alpha}=\frac{y-b\sin\alpha}{b\cos\alpha-b\sin\alpha}=\frac{z-c}{-c-c}$$

or $$\frac{(x/a)-\cos\alpha}{-(\sin\alpha+\cos\alpha)}=\frac{(y/b)-\sin\alpha}{\cos\alpha-\sin\alpha}=\frac{(z/c)-1}{-2}=r, \text{ (say).}$$

From these, we have

$$x/a=\cos\alpha-r(\cos\alpha+\sin\alpha), \quad \text{...(1)}$$

$$y/b=\sin\alpha-r(\cos\alpha-\sin\alpha), \quad \text{...(2)}$$

and $$z/c=1-2r. \quad \text{...(3)}$$

Squaring (1) and (2) and adding, we have

$$x^2/a^2+y^2/b^2=1+r^2(1+1)-2r(1)$$

[using $\cos^2\alpha+\sin^2\alpha=1$]

or $$2(x^2/a^2+y^2/b^2)=4r^2-4r+2$$
$$=(1-2r)^2+1$$
$$=z^2/c^2+1, \text{ using (3).}$$

$\therefore$ the line PQ' generates the hyperboloid

$$2x^2/a^2+2y^2/b^2-z^2/c^2=1.$$

Similarly proceeding as above, we can prove that the line $P'Q$ also generates the given hyperboloid.

Ex. 13. *Show that in general two generators of the hyperboloid $x^2/a^2+y^2/b^2-z^2/c^2=1$ can be drawn to cut a given generator at right angles. Also show that if they meet the plane $z=0$ in P and Q, PQ touches the ellipse*

$$\frac{x^2}{a^6}+\frac{y^2}{b^6}=\frac{c^4}{a^4b^4}.$$

Solution. Proceeding as in § 3, the d.c.'s of a generator of the λ-system are proportional to

$$a(1-\lambda^2),\ 2b\lambda,\ c(1+\lambda^2)$$

and those of a generator of the μ-system are proportional to

$$a(1-\mu^2),\ -2b\mu,\ c(1+\mu^2).$$

If these generators are at right angles, then we have

$$a^2(1-\mu^2)(1-\lambda^2)-4b^2\mu\lambda+c^2(1+\mu^2)(1+\lambda^2)=0. \qquad \ldots(1)$$

Now suppose the μ-generator is given so that μ is a constant and hence the equation (1) is a quadratic in λ. This shows that there are two generators of the λ-system which are at right angles to a generator of the μ-system. Let the generators of the λ-system cut the plane $z=0$ in the points $P(a\cos\alpha,\ b\sin\alpha,\ 0)$ and $Q(a\cos\beta,\ b\sin\beta,\ 0)$. The equations of these generators belonging to the λ system through the points P and Q are given by [See Ex. 1(a) above]

$$\frac{x-a\cos\alpha}{a\sin\alpha}=\frac{y-b\sin\alpha}{-b\cos\alpha}=\frac{z}{c}, \qquad \ldots(2)$$

and

$$\frac{x-a\cos\beta}{a\sin\beta}=\frac{y-b\sin\beta}{-b\cos\beta}=\frac{z}{c}. \qquad \ldots(3)$$

Also the equations of the generator belonging to the μ-system and passing through the point, say $(a \cos \theta, b \sin \theta, 0)$, are

$$\frac{x-a\cos\theta}{a\sin\theta}=\frac{y-b\sin\theta}{-b\cos\theta}=\frac{z}{-c}. \quad \text{.. (4)}$$

The generators (2) and (3) both intersect the generator given by (4) at right angles and hence we have

$$a^2 \sin\alpha \sin\theta + b^2 \cos\alpha \cos\theta - c^2 = 0,$$

and $$a^2 \sin\beta \sin\theta + b^2 \cos\beta \cos\theta - c^2 = 0.$$

Solving these last two relations, we have

$$\frac{a^2 \sin\theta}{-\cos\alpha+\cos\beta}=\frac{b^2\sin\theta}{-\sin\beta+\sin\alpha}=\frac{c^2}{\sin\alpha\cos\beta-\cos\alpha\sin\beta}$$

or $$\frac{a^2\sin\theta}{2\sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)}=\frac{b^2\cos\theta}{2\cos\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)}$$

$$=\frac{c^2}{2\sin\frac{1}{2}(\alpha-\beta)\cos\frac{1}{2}(\alpha-\beta)}$$

or $$\frac{a^2\sin\theta}{\sin\frac{1}{2}(\alpha+\beta)}=\frac{b^2\cos\theta}{\cos\frac{1}{2}(\alpha+\beta)}=\frac{c^2}{\cos\frac{1}{2}(\alpha-\beta)}=\frac{1}{k}\ \text{(say)}. \quad \text{...(5)}$$

Now the equations of the line joining the points P and Q are

$$(x/a)\cos\tfrac{1}{2}(\alpha+\beta)+(y/b)\sin\tfrac{1}{2}(\alpha+\beta)=\cos\tfrac{1}{2}(\alpha-\beta),\ z=0.$$

Putting the values from (5), the above equations to the line PQ become

$$(x/a)\,kb^2\cos\theta+(y/b)\,ka^2\sin\theta=kc^2,\ z=0$$

or $$(x/a^3)\cos\theta+(y/b^3)\sin\theta=(c^2/a^2b^2),\ z=0. \quad \text{...(6)}$$

$[\because k\neq 0]$

Now it is required to find the envelope of (6) and so differentiating (6) w.r.t. 'θ', we get

$$-(x/a^3)\sin\theta+(y/b^3)\cos\theta=0,\ z=0. \quad \text{...(7)}$$

Squaring (6) and (7) and then adding, the equations of the envelope of the line PQ [or in other words the curve which the line PQ alwaye touches] are given by

$$\frac{x^2}{a^6}+\frac{y^2}{b^6}=\frac{c^4}{a^4b^4},\ z=0$$

which are the equations of an ellipse.

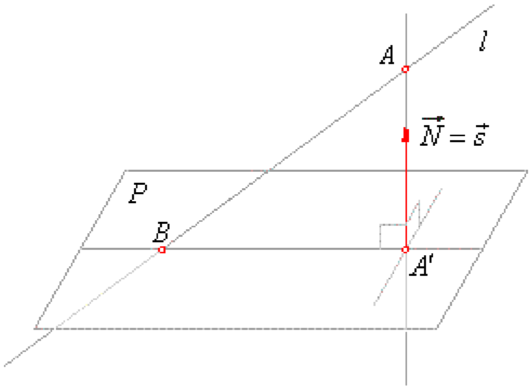
Projection of a line onto a plane

Orthogonal projection of a line onto a plane is a line or a point. If a given line is perpendicular to a plane, its projection is a point, that is the intersection point with the plane, and its direction vector s is coincident with the normal vector N of the plane.

If a line is parallel with a plane then it is also parallel with its projection onto the plane and orthogonal to the normal vector of the plane that is

$$s \perp N \quad \Rightarrow \quad s \cdot N = 0.$$

Projection of a line which is not parallel nor perpendicular to a plane, passes through their intersection B and through the projection A' of any point A of the line onto the plane, as shows the right figure.



Example: Determine projection of the line $\frac{x-15}{15} = \frac{y+12}{-15} = \frac{z-17}{11}$ onto the plane

$13x - 9y + 16z - 69 = 0.$

Solution: First determine coordinates of the intersection point of the line and the plane,

$$\frac{x-15}{15} = t \Rightarrow x = 15t + 15, \quad \frac{y+12}{-15} = t \Rightarrow y = -15t - 12 \text{ and } \frac{z-17}{11} = t \Rightarrow z = 11t + 17$$

plug these variable coordinates of the line into the plane

$$x = 15t + 15, \quad y = -15t - 12 \text{ and } z = 11t + 17 \quad \Rightarrow \quad 13x - 9y + 16z - 69 = 0,$$

that is, $13 \cdot (15t + 15) - 9 \cdot (-15t - 12) + 16 \cdot (11t + 17) - 69 = 0 \quad \Rightarrow \quad t = -1$ thus,

thus, $x = 15t + 15 = 15 \cdot (-1) + 15 = 0, \quad y = -15t - 12 = -15 \cdot (-1) - 12 = 3$

and $z = 11t + 17 = 11 \cdot (-1) + 17 = 6$ therefore, the intersection $B(0, 3, 6)$.

Then, find the projection A' of a point $A(15, -12, 17)$ of the given line, onto the plane, as the intersection of the normal through the point A, and the plane.

So, write the equation of the normal

$$\left.\begin{array}{l} A(15, -12, 17) \\ \vec{s} = N = 13\vec{i} - 9\vec{j} + 16\vec{k} \end{array}\right\} \Rightarrow \frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}, \quad \frac{x-15}{13} = \frac{y+12}{-9} = \frac{z-17}{16}.$$

Repeat the same procedure to find the projection A' as for the intersection B, that is

$$\frac{x-15}{13} = t \Rightarrow x = 13t + 15, \quad \frac{y+12}{-9} = t \Rightarrow y = -9t - 12 \text{ and } \frac{z-17}{16} = t \Rightarrow z = 16t + 17$$

plug these variable coordinates of the normal into the equation of the given plane to find the projection A', so

$$x = 13t + 15,\quad y = -9t - 12 \text{ and } z = 16t + 17 \quad \Rightarrow \quad 13x - 9y + 16z - 69 = 0,$$

$$13(13t + 15) - 9(-9t - 12) + 16(16t + 17) - 69 = 0,\quad t = -1.$$

Thus, $x = 13 \cdot (-1) + 15 = 2$, $y = -9 \cdot (-1) - 12 = -3$ and $z = 16 \cdot (-1) + 17 = 1$, $A'(2, -3, 1)$.

Finally, as the projection of the given line onto the given plane passes through the intersection B and the projection A' then, by plugging their coordinates into the equation of the line through two points

$$\left.\begin{matrix} B(0, 3, 6) \\ A'(2, -3, 1) \end{matrix}\right\} \Rightarrow \frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1} \Rightarrow \frac{x}{2} = \frac{y - 3}{-6} = \frac{z - 6}{-5}$$

obtained is the equation of the projection.

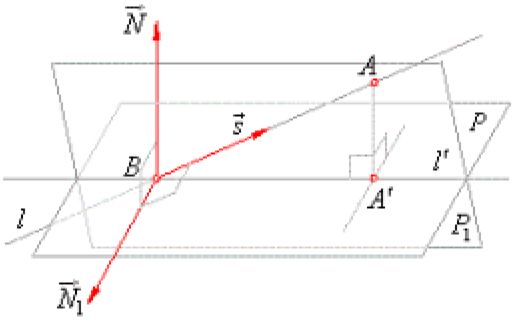
Example: Projection of the line $\dfrac{x - 15}{15} = \dfrac{y + 12}{-15} = \dfrac{z - 17}{11}$ onto the plane $13x - 9y + 16z - 69 = 0$,

the same as in the above example, can be calculated applying simpler method.

Solution: Intersection of the given plane and the orthogonal plane through the given line, that is, the plane through three points, intersection point B, the point A of the given line and its projection A' onto the plane, is at the same time projection of the given line onto the given plane, as shows the below figure.

The direction vector N_1, of the plane determined by three points A, B and A', is the result of the vector product of the normal vector of the given plane and the direction vector s of the given line, that is

$$\vec{N}_1 = \vec{N} \times \vec{s} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 13 & -9 & 16 \\ 15 & -15 & 11 \end{vmatrix} = 141\vec{i} + 97\vec{j} - 60\vec{k}.$$



By plugging the point $A(15, -12, 17)$ into the equation of the plane,

$$A(15, -12, 17) \Rightarrow 141x + 97y - 60z + D = 0 \Rightarrow 141 \cdot 15 + 97 \cdot (-12) - 60 \cdot 17 + D = 0,\ D = 69$$

obtained is the equation of the plane $P_1 :: 141x + 97y - 60z + 69 = 0$.

Finally, the line of the intersection l' of the given plane

$$P :: 13x - 9y + 16z - 69 = 0 \text{ and the plane } P_1 :: 141x + 97y - 60z + 69 = 0$$

is at the same time the projection of the given line onto the given plane.

To check the obtained result, write the vector product of normal vectors of planes P and P_1,

$$\vec{N} \times \vec{N}_1 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 13 & -9 & 16 \\ 141 & 97 & -60 \end{vmatrix} = -1012\vec{i} + 3036\vec{j} + 2530\vec{k} = -506 \cdot (2\vec{i} - 6\vec{j} - 5\vec{k}).$$

therefore, $N \times N_1 = \lambda s$ the vector product is collinear with the direction vector of the intersection line, what proves the result.

Ex. 6. *Prove that if two generators of the hyperboloid $x^2/a^2+y^2/b^2-z^2/c^2=1$ through the points P $(a\cos\alpha, b\sin\alpha, 0)$ and $Q(a\cos\beta, b\sin\beta, 0)$ intersect at right angles, their projections on the plane $z=0$ intersect at an angle θ, where*

$$\tan\theta=(ab/c^2)\sin(\alpha-\beta).$$

Solution. The equation of the hyperboloid is

$$x^2/a^2+y^2/b^2-z^2/c^2=1. \quad \text{...(1)}$$

The equations of the generator of (1) through the point $P(a\cos\alpha, b\sin\alpha, 0)$ belonging to the one system of generators [See Ex. 1(a) above] are

$$\frac{x-a\cos\alpha}{a\sin\alpha}=\frac{y-b\sin\alpha}{-b\cos\alpha}=\frac{z}{c}. \quad \text{...(2)}$$

Also the equations of the generator of (1) through the point $Q(a\cos\beta, b\sin\beta, 0)$ belonging to the other system of generators are

$$\frac{x-a\cos\beta}{a\sin\beta}=\frac{y-b\sin\beta}{-b\cos\beta}=\frac{z}{-c}. \quad \text{...(3)}$$

If the generators (2) and (3) are at right angles, then using the condition $l_1l_2+m_1m_2+n_1n_2=0$, we have

$$a^2\sin\alpha\sin\beta+b^2\cos\alpha\cos\beta-c^2=0. \quad \text{...(4)}$$

Now we know that the projections of the generators (2) and (3) on the plane $z=0$ are tangents to the ellipse $x^2/a^2+y^2/b^2=1$, $z=0$ [*i.e.* to the section of (1) by the plane $z=0$] at the points P and Q. Therefore the projections of (2) and (3) on the plane $z=0$ are respectively given by

$$(x/a)\cos\alpha+(y/b)\sin\alpha=1,\ z=0 \quad \text{...(5)}$$

and
$$(x/a)\cos\beta+(y/b)\sin\beta=1,\ z=0. \quad \text{...(6)}$$

The slopes m_1 and m_2 of the lines (5) and (6) lying in the plane $z=0$ are given by

$m_1=-(b\cos\alpha)/(a\sin\alpha)$ and $m_2=-(b\cos\beta)/(a\sin\beta)$.

The angle θ between (5) and (6) is given by

$$\tan\theta=\frac{m_1-m_2}{1+m_1m_2} \quad \text{or} \quad \tan\theta=\frac{-\dfrac{b}{a}\left\{\dfrac{\cos\alpha}{\sin\alpha}-\dfrac{\cos\beta}{\sin\beta}\right\}}{1+\dfrac{b^2}{a^2}\cdot\dfrac{\cos\alpha}{\sin\alpha}\cdot\dfrac{\cos\beta}{\sin\beta}}$$

or $$\tan\theta = \frac{-ab\,[\sin\beta\cos\alpha - \cos\beta\sin\alpha]}{a^2\sin\alpha\sin\beta + b^2\cos\alpha\cos\beta}$$

$$= \frac{ab\sin(\alpha-\beta)}{c^2} \quad \text{[using (4)].}$$

Proved.

§ 3. Perpendicular generators. *To find the locus of the point of intersection of two perpendicular generators of hyperboloid of one sheet.*

The equation of the hyperboloid is

$$x^2/a^2 + y^2/b^2 - z^2/c^2 = 1. \qquad \ldots(1)$$

The equations of any generator of the λ-system are

$$\frac{x}{a} - \frac{z}{c} = \lambda\left(1 - \frac{y}{b}\right),\ \frac{x}{a} + \frac{z}{c} = \frac{1}{\lambda}\left(1 + \frac{y}{b}\right)$$

or $$\frac{x}{a} + \frac{\lambda}{b}\,y - \frac{z}{c} = \lambda,\ \frac{x}{a} - \frac{1}{\lambda b}\,y + \frac{z}{c} = \frac{1}{\lambda}$$

or $$\frac{x}{a} + \frac{\lambda y}{b} - \frac{z}{c} = \lambda,\ \frac{\lambda x}{a} - \frac{1}{b}\,y + \frac{\lambda z}{c} = 1.$$

Let l_1, m_1, n_1 be the d.c.'s of this generator, so that we have

$$\frac{l_1}{a} + \lambda\,\frac{m_1}{b} - \frac{n_1}{c} = 0,\ \lambda\,\frac{l_1}{a} - \frac{m_1}{b} + \lambda\,\frac{n_1}{c} = 0.$$

Solving, we get $$\frac{l_1/a}{1-\lambda^2} = \frac{m_1/b}{2\lambda} = \frac{n_1/c}{1+\lambda^2} \qquad \ldots(2)$$

The equations of any generator of the μ-system are

$$\frac{x}{a}-\frac{z}{c}=\mu\left(1+\frac{y}{b}\right), \frac{x}{a}+\frac{z}{c}=\frac{1}{\mu}\left(1-\frac{y}{b}\right)$$

If l_2, m_2, n_2 be the d.c.'s of this generator, then proceeding as above, we get

$$\frac{l_2/a}{1-\mu^2}=\frac{m_2/b}{-2\mu}=\frac{n_2/c}{1+\mu^2} \quad ...(3)$$

If the two generators given above are perpendicular, then we have

$$l_1l_2+m_1m_2+n_1n_2=0. \quad ...(4)$$

Putting proportionate values from (2) and (3) in (4), we have

$$a^2(1-\lambda^2)(1-\mu^2)-4b^2\lambda\mu+c^2(1+\lambda^2)(1+\mu^2)=0$$

or $\quad a^2(1+\lambda^2\mu^2-\lambda^2-\mu^2)-b^2(4\lambda\mu)+c^2(1+\lambda^2\mu^2+\lambda^2+\mu^2)=0$

or $\quad a^2\{(1+\lambda\mu)^2-(\lambda^2+\mu^2+2\lambda\mu)\}-b^2\{(\lambda+\mu)^2-(\lambda-\mu)^2\}$
$\quad +c^2\{(1-\lambda\mu)^2+(\lambda^2+\mu^2+2\lambda\mu)\}=0$

or $\quad a^2(1+\lambda\mu)^2-a^2(\lambda+\mu)^2+b^2(\lambda-\mu)^2-b^2(\lambda+\mu)^2$
$\quad +c^2(1-\lambda\mu)^2+c^2(\lambda+\mu)^2=0$

or $\quad a^2(1+\lambda\mu)^2+b^2(\lambda-\mu)^2+c^2(1-\lambda\mu)^2=(a^2+b^2-c^2)(\lambda+\mu)^2$

or $\quad a^2\left\{\frac{1+\lambda\mu}{\lambda+\mu}\right\}^2+b^2\left\{\frac{\lambda-\mu}{\lambda+\mu}\right\}^2+c^2\left\{\frac{1-\lambda\mu}{\lambda+\mu}\right\}^2=a^2+b^2-c^2.$

Hence the locus of the point of intersection of two perpendicular generators *i.e.* of the point

$$\left(a.\frac{1+\lambda\mu}{\lambda+\mu}, b.\frac{\lambda-\mu}{\lambda+\mu}, c.\frac{1-\lambda\mu}{\lambda+\mu}\right)$$

[See result (7) of § 2 (B), property III]

is given by

$$x^2+y^2+z^2=a^2+b^2-c^2.$$

This is called the director sphere.

Hence the required locus is the curve of intersection of the hyperboloid and the director sphere.

Ex. 4. *CP, CQ are any two conjugate semi-diameters of the ellipse $x^2/a^2+y^2/b^2=1,\ z=c$. CP', CQ' are the conjugate semi-diameters of the ellipse $x^2/a^2+y^2/b^2=1,\ z=-c$, drawn in the same directions as CP and CQ. Prove that the hyperboloid $\dfrac{2x^2}{a^2}+\dfrac{2y^2}{b^2}-\dfrac{z^2}{c^2}=1$ is generated by either PQ' or P'Q.*

Solution. Since CP and CQ are the conjugate semi-diameters of the ellipse $x^2/a^2+y^2/b^2=1,\ z=c$, so that the co-ordinates of the extremities P and Q are

$P\equiv(a\cos\alpha,\ b\sin\alpha,\ c)$ and $Q\equiv(-a\sin\alpha,\ b\cos\alpha,\ c)$.

Similarly the co-ordinates of the extremities P' and Q' on $x^2/a^2+y^2/b^2=1,\ z=-c$ are

$P'\equiv(a\cos\alpha,\ b\sin\alpha,\ -c)$ and $Q'\equiv(-a\sin\alpha,\ b\cos\alpha,\ -c)$.

The equations to the line joining the points P and Q' *i.e.* of the line PQ' are

$$\frac{x-a\cos\alpha}{-a\sin\alpha-a\cos\alpha}=\frac{y-b\sin\alpha}{b\cos\alpha-b\sin\alpha}=\frac{z-c}{-c-c}$$

or
$$\frac{(x/a)-\cos\alpha}{-(\sin\alpha+\cos\alpha)}=\frac{(y/b)-\sin\alpha}{\cos\alpha-\sin\alpha}=\frac{(z/c)-1}{-2}=r,\ \text{(say)}.$$

From these, we have

$$x/a=\cos\alpha-r(\cos\alpha+\sin\alpha), \qquad \ldots(1)$$

$$y/b=\sin\alpha-r(\cos\alpha-\sin\alpha), \qquad \ldots(2)$$

and
$$z/c=1-2r. \qquad \ldots(3)$$

Squaring (1) and (2) and adding, we have

$$x^2/a^2+y^2/b^2=1+r^2(1+1)-2r(1)$$

[using $\cos^2\alpha+\sin^2\alpha=1$]

or
$$2(x^2/a^2+y^2/b^2)=4r^2-4r+2$$
$$=(1-2r)^2+1$$
$$=z^2/c^2+1,\quad \text{using (3)}.$$

$\therefore$ the line PQ' generates the hyperboloid

$$2x^2/a^2+2y^2/b^2-z^2/c^2=1.$$

Similarly proceeding as above, we can prove that the line $P'Q$ also generates the given hyperboloid.

Ex. 7. *Show that the perpendiculars from the origin on the generator of the hyperboloid* $x^2/a^2+y^2/b^2-z^2/c^2=1$ *lie on the cone*

$$\frac{a^2(b^2+c^2)^2}{x^2}+\frac{b^2(c^2+a^2)^2}{y^2}-\frac{c^2(a^2-b^2)^2}{z^2}=0.$$

Solution. The equation of the hyperboloid is

$$x^2/a^2+y^2/b^2-z^2/c^2=1. \qquad \ldots(1)$$

The equations of the generator of (1) belonging to one system and passing through any point $(a\cos\alpha, b\sin\alpha, 0)$ on the principal elliptic section $x^2/a^2+y^2/b^2=1$, $z=0$ are given by

$$\frac{x-a\cos\alpha}{a\sin\alpha}=\frac{y-b\sin\alpha}{-b\cos\alpha}=\frac{z}{c}. \qquad \ldots(2)$$

[See Ex. 1(a) above]

The equations to any line through the origin are

$$x/l=y/m=z/n. \qquad \ldots(3)$$

If (2) and (3) are perpendicular, then we have

$$al\sin\alpha-bm\cos\alpha+cn=0. \qquad \ldots(4)$$

The lines (2) and (3) will intersect *i.e.* they will be coplanar if

$$\begin{vmatrix} a\cos\alpha & b\sin\alpha & 0 \\ a\sin\alpha & -b\cos\alpha & c \\ l & m & n \end{vmatrix}=0$$

or $\quad a\cos\alpha(-bn\cos\alpha-cm)-b\sin\alpha(an\sin\alpha-cl)=0$

or $\quad -abn(\cos^2\alpha+\sin^2\alpha)-acm\cos\alpha+bcl\sin\alpha=0$

or $\quad lbc\sin\alpha-mac\cos\alpha-nab=0. \qquad \ldots(5)$

Solving the relations (4) and (5) for $\sin\alpha$ and $\cos\alpha$, we have

$$\frac{\sin\alpha}{mnab^2+mnac^2}=\frac{\cos\alpha}{nlbc^2+nlba^2}=\frac{1}{-lmca^2+lmcb^2}.$$

$$\therefore \quad \sin\alpha=-\frac{na(b^2+c^2)}{lc(a^2-b^2)},\ \cos\alpha=-\frac{nb(c^2+a^2)}{mc(a^2-b^2)}.$$

Squaring and adding *i.e.* eliminating α, we have

$$\frac{n^2a^2(b^2+c^2)^2}{l^2c^2(a^2-b^2)^2}+\frac{n^2b^2(c^2+a^2)^2}{m^2c^2(a^2-b^2)^2}=1$$

or $$\frac{a^2}{l^2}(b^2+c^2)^2+\frac{b^2}{m^2}(c^2+a^2)^2=\frac{c^2}{n^2}(a^2-b^2)^2.$$

Hence the line (3) lies on the cone

$$\frac{a^2(b^2+c^2)^2}{x^2}+\frac{b^2(c^2+a^2)^2}{y^2}-\frac{c^2(a^2-b^2)^2}{z^2}=0.$$

A similar result can be proved by taking generators of the other system.

Ex. 8. *Show that the points of intersection P, Q of the generators of opposite systems drawn through the point $A(a\cos\alpha, b\sin\alpha, 0)$, $B(a\cos\beta, b\sin\beta, 0)$ of the principal elliptic section of the hyperboloid* $x^2/a^2+y^2/b^2-z^2/c^2=1$ *are*

$$\left(\frac{a\cos\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)}, \frac{b\sin\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)}, \pm\frac{c\sin\frac{1}{2}(\alpha-\beta)}{\cos\frac{1}{2}(\alpha-\beta)}\right)$$

Hence show that if A and B are extremities of semi-conjugate diameters, the loci of the points P and Q are the ellipses

$$x^2/a^2+y^2/b^2=2,\ z=\pm c.$$

Solution. Let the co-ordinates of one of the two points of intersection of the generators, say of the point P be (x_1, y_1, z_1). The equation of the tangent plane to the hyperboloid

$$x^2/a^2+y^2/b^2-z^2/c^2=1$$

at the point (x_1, y_1, z_1) on it is

$$xx_1/a^2+yy_1/b^2-zz_1/c^2=1. \quad (1)$$

The tangent plane (1) meets the plane $z=0$ in the line which is given by

$$xx_1/a^2+yy_1/b^2=1,\ z=0. \quad ...(2)$$

The equations of the line joining the points $A(a\cos\alpha, b\sin\alpha, 0)$ and $B(a\cos\beta, b\sin\beta, 0)$ are

$$(x/a)\cos\tfrac{1}{2}(\alpha+\beta)+(y/b)\sin\tfrac{1}{2}(\alpha+\beta)=\cos\tfrac{1}{2}(\alpha-\beta),\ z=0. \quad ..(3)$$

The lines given by the equations (2) and (3) are the same. Hence comparing these equations, we have

$$\frac{x_1/a}{\cos\frac{1}{2}(\alpha+\beta)}=\frac{y_1/b}{\sin\frac{1}{2}(\alpha+\beta)}=\frac{1}{\cos\frac{1}{2}(\alpha-\beta)}. \quad ...(4)$$

Since the point $P(x_1, y_1, z_1)$ lies on the given hyperboloid, we have

$$x_1^2/a^2+y_1^2/b^2-z_1^2/c^2=1. \quad ...(5)$$

Putting the values of x_1/a and y_1/b from (4) in (5), we get

$$\frac{\cos^2\frac{1}{2}(\alpha+\beta)}{\cos^2\frac{1}{2}(\alpha-\beta)}+\frac{\sin^2\frac{1}{2}(\alpha+\beta)}{\cos^2\frac{1}{2}(\alpha-\beta)}-\frac{z_1^2}{c^2}=1$$

or

$$\frac{1}{\cos^2\frac{1}{2}(\alpha-\beta)}-\frac{z_1^2}{c^2}=1$$

or $\qquad z_1^2/c^2=\sec^2 \frac{1}{2}(\alpha-\beta)-1$ or $z_1^2/c^2=\tan^2\frac{1}{2}(\alpha-\beta)$

or $\qquad z_1/c=\pm\sin\frac{1}{2}(\alpha-\beta)/\cos\frac{1}{2}(\alpha-\beta)$. ...(6)

Hence from (4) and (6) the co-ordinates of the required points of intersection P and Q are

$$\left(\frac{a\cos\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)}, \frac{b\sin\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)}, \pm\frac{c\sin\frac{1}{2}(\alpha-\beta)}{\cos\frac{1}{2}(\alpha-\beta)}\right). \quad ...(7)$$

Second part. If the points A and B are the extremities of the semi-conjugate diameters, then we have $\alpha-\beta=\frac{1}{2}\pi$.

$\therefore \quad x_1^2/a^2+y_1^2/b^2=1/\cos^2\frac{1}{2}(\alpha-\beta),\ z_1=\pm c\tan\frac{1}{2}(\alpha-\beta)$

or $\qquad x_1^2/a^2+y_1^2/b^2=1/\cos^2\frac{1}{4}\pi,\ z_1=\pm c\tan\frac{1}{4}\pi$

or $\qquad x_1^2/a^2+y_1^2/b^2=2,\ z_1=\pm c$.

Therefore, the loci of the points P and Q are

$$x^2/a^2+y^2/b^2=2,\ z=\pm c. \qquad \textbf{Proved.}$$

Example 51:

Show that the plane $x + y - z = 0$ cuts the conicoid $4x^2 + 2y^2 + z^2 + 3yz + zx - 1 = 0$ in a circle. Find also the radius of this circle.

Solution:

The given equation of the conicoid can be rewritten as

$$[(4x^2 + 2y^2 + z^2 + 3yz + zx) - \lambda(x^2 + y^2 + z^2)] + \lambda[x^2 + y^2 + z^2 - (1/\lambda)] = 0 \quad ...(i)$$

Choose λ such that $(4x^2 + 2y^2 + z^2 + 3yz + zx - \lambda(x^2 + y^2 + z^2) = 0$

$\Rightarrow (4-\lambda)x^2 + (2-\lambda)y^2 + (1-\lambda)z^2 + 3yz + zx = 0 \quad ...(ii)$

represents a pair of planes, the condition for the same is

$$\begin{vmatrix} 4-\lambda & 0 & 1/2 \\ 0 & 2-\lambda & 3/2 \\ 1/2 & 3/2 & 1-\lambda \end{vmatrix} = 0 \qquad \textbf{(Note)}$$

$\Rightarrow (4-\lambda)[(2-\lambda)(1-\lambda)-(3/2)(3/2)] = (1/2)[0-(1/2)(2-\lambda)] = 0$

$\Rightarrow 4(4-\lambda)(2-\lambda)(1-\lambda) - 9(4-\lambda) - (2-\lambda) = 0$

$\Rightarrow \quad -4\lambda^3 + 28\lambda^2 - 46\lambda - 6 = 0$, on simplifying

$\Rightarrow \quad \lambda = 3$ (real)

$\therefore$ Substituting 3 for λ in (ii), we get

$$[(4x^2 + 2y^2 + z^2 + 3yz + zx - 3(x^2 + y^2 + z^2)] = 0$$

$\Rightarrow$ $x^2 - y^2 - 2z^2 + 3yz + zx = 0$

$\Rightarrow$ $(x + y - z)(x - y + 2x) = 0$

$\therefore$ Then plane $x + y - z = 0$ cuts the given conicoid in a circle.

Also as the plane $x + y - z = 0$ passes through the center of the sphere $x^2 + y^2 + z^2 - (1/\lambda) = 0$, where $\lambda = 3$

i.e., the sphere $x^2 + y^2 + z^2 = 1/3$.

$\therefore$ The circle is a great circle whose radius is the same as that of the sphere *i.e.,* $1/\sqrt{3}$. **Ans.**

Ex. 1 (a). ***Prove that the feet of the six normals drawn to the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ from any point (x_1, y_1, z_1) lie on the curve of intersection of the ellipsoid and the cone***

$$\frac{a^2(b^2-c^2)x_1}{x}+\frac{b^2(c^2-a^2)y_1}{y}+\frac{c^2(a^2-b^2)z_1}{z}=1.$$

(Rohilkhand 1980; Madras 77; Gurunanakdev 75; Lucknow 74)

Solution. The equation of the ellipsoid is

$$x^2/a^2+y^2/b^2+z^2/c^2=1 \qquad ...(1)$$

Proceeding as in § 12, the co-ordinates (α, β, γ) of the six feet of six normals drawn to the ellipsoid (1) from the given point (x_1, y_1, z_1) are given by

$\alpha=\dfrac{a^2x_1}{a^2+\lambda}$, $\beta=\dfrac{b^2y_1}{b^2+\lambda}$, $\gamma=\dfrac{c^2z_1}{c^2+\lambda}$ where λ is a parameter and is given by an equation of sixth degree.

Solving each of the above relations for λ, we get

$$\lambda=\frac{a^2x_1}{\alpha}-a^2,\ \lambda=\frac{b^2y_1}{\beta}-b^2,\ \lambda=\frac{c^2z_1}{\gamma}-c^2.$$

Now we clearly have

$$\left(\frac{a^2x_1}{\alpha}-a^2\right)(b^2-c^2)+\left(\frac{b^2y_1}{\beta}-b^2\right)(c^2-a^2)+\left(\frac{c^2z_1}{\gamma}-c^2\right)(a^2-b^2)$$
$$=\lambda(b^2-c^2)+\lambda(c^2-a^2)+\lambda(a^2-b^2)=0$$

or $$\frac{a^2x_1(b^2-c^2)}{\alpha}+\frac{b^2y_1(c^2-a^2)}{\beta}+\frac{c^2z_1(a^2-b^2)}{\gamma}$$
$$=a^2(b^2-c^2)+b^2(c^2-a^2)+c^2(a^2-b^2)$$
$$=0.$$

$\therefore$ The six feet (α, β, γ) of the normals to (1) drawn from (x_1, y_1, z_1) lie on the surface

$$\frac{a^2x_1 (b^2-c^2)}{x}+\frac{b^2y_1 (c^2-a^2)}{y}+\frac{c^2z_1 (a^2-b^2)}{z}=0. \quad ...(2)$$

The equation (2) is a homogeneous equation of second degree and hence it represents a cone. But since the six feet (α, β, γ) of these normals lie on the given ellipsoid (1), therefore they lie on the curve of intersection of the ellipsoid (1) and the cone (2).

Ex. 1 (b). *Prove that the feet of the six normals drawn to the central conicoid $ax^2+by^2+cz^2=1$ from any point (x_1, y_1, z_1) lie on the curve of intersection of the given ellipsoid and the cone*

$$\frac{a (b-c) x_1}{x}+\frac{b (c-a) y_1}{y}+\frac{c (a-b) z_1}{z}=0.$$

(Rohilkhand 1977)

Solution. Replacing a^2 by $1/a$, b^2 by $1/b$ and c^2 by $1/c$ in Ex. 1 (a) above and proceeding exactly in the same way we get the required result.

Ex. 2. *If A, B, C; A'. B', C' are the feet of the six normals from a given point to the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ and the plane ABC is given by $lx+my+nz=p$, prove that the plane A' B' C' is given by $(x/a^2l)+(y/b^2m)+(z/c^2n)+(1/p)=0$.* **(Punjab 1977)**

Solution. The equation of the given ellipsoid is

$$x^2/a^2+y^2/b^2+z^2/c^2=1. \quad ...(1)$$

Let the co-ordinates of the given point be $(x_1. y_1. z_1)$. Now the co-ordinates (α, β, γ) of the feet of six normals from $(x_1, y_1. z_1)$ to (1) are given by

$$\alpha=a^2x_1/(a^2+\lambda), \ \beta=b^2y_1/(b^2+\lambda), \ \gamma=c^2z_1/(c^2+\lambda) \quad ..(2)$$

where λ is a parameter and its six values are given by the equation

$$\frac{a^2x_1^2}{(a^2+\lambda)^2}+\frac{b^2y_1^2}{(b^2+\lambda)^2}+\frac{c^2z_1^2}{(c^2+\lambda)^2}=1. \quad ..(3)$$

Now the equation of the plane ABC is given to be

$$lx+my+nz=p$$

and three of the six feet of normals given by (2) lie on this plane, so that we have

$$\frac{la^2x_1}{a^2+\lambda}+\frac{mb^2y_1}{b^2+\lambda}+\frac{nc^2z_1}{c^2+\lambda}-p=0 \quad ...(4)$$

This being a cubic equation in λ gives us three values of λ. Again let the equation of the plane $A'\ B'\ C'$ be

$$l'x+m'y+n'z-p'=0.$$

According to the question the remaining three of the six feet of normals given by (2) lie on this plane and so we have

$$\frac{l'a^2x_1}{(a^2+\lambda)}+\frac{m'b^2y_1}{(b^2+\lambda)}+\frac{n'c^2z_1}{(c^2+\lambda)}-p'=0. \quad \ldots(5)$$

This equation is of third degree in λ and hence gives us the remaining three values of λ.

Clearly the product of the equations (4) and (5) must give the same equation as is equation (3). Hence comparing the like terms, we get

$$\frac{ll' \cdot a^4x_1^2}{(a^2+\lambda)^2}=\frac{a^2x_1^2}{(a^2+\lambda)^2} \quad \text{or} \quad l'=\frac{1}{a^2l}$$

Similarly, we have

$$m'=\frac{1}{b^2m},\ n'=\frac{1}{c^2n},\ p'=-\frac{1}{p}.$$

Putting these values in $l'x+m'y+n'z-p'=0$, the required equation of the plane $A'\ B'\ C'$ is given by

$$\frac{x}{a^2l}+\frac{y}{b^2m}+\frac{z}{c^2n}+\frac{1}{p}=0.$$ **Proved.**

Ex. 3. *If the feet of the three normals from P to the ellipsoid $x^2/a^2+y^2/b^2+z^2/c^2=1$ lie in the plane $x/a+y/b+z/c=1$, prove that the feet of the other three lie in the plane $x/a+y/b+z/c \quad 1=0$ and P lies on the line*

$$a\,(b^2-c^2)\,x=b\,(c^2-a^2)\,y=c\,(a^2-b^2)\,z.$$

(Allahabad 1979)

Solution. The equation of the ellipsoid is

$$x^2/a^2+y^2/b^2+z^2/c^2=1. \quad (1)$$

The first part is exactly similar to Ex. 2 above.

Now clearly the six feet of the normals lie on

$$(x/a+y/b+z/c-1)\,(x/a+y/b+z/c+1)=0$$

or $$(x/a+y/b+z/c)^2-1=0$$

or $$(x^2/a^2+y^2/b^2+z^2/c^2-1)+2\,\{(yz/bc)+(zx/ca)+(xy/ab)\}=0 \quad (2)$$

Since these six feet lie on the ellipsoid (1), therefore, using (1) the equation (2) becomes

$$\frac{yz}{bc}+\frac{zx}{ca}+\frac{xy}{ab}=0 \text{ or } ayz+bzx+cxy=0, \quad (3)$$

Example 3:

Prove that the tangent planes to the hyperboloid $(x^2/a^2) + (y^2/b^2) - (z^2/c^2) = 1$ which are parallel to tangent planes to the cone

$$\frac{b^2c^2x^2}{c^2 - b^2} + \frac{c^2a^2y^2}{c^2 - a^2} + \frac{a^2b^2z^2}{a^2 + b^2} = 0$$ *cut the surface in perpendicular generators.*

Solution:

We know that the equation of the cone reciprocal to the cone

$ax^2 + by^2 + cz^2 = 0$ is $(x^2/a) + (y^2/b) + (z^2/c) = 0$.

$\therefore$ The equation of the cone reciprocal to the given cone

$$\frac{c^2 - b^2}{b^2c^2}x^2 + \frac{c^2 - a^2}{c^2a^2}y^2 + \frac{a^2 + b^2}{a^2b^2}z^2 = 0. \quad \text{...(i)}$$

Let $lx + my + nz = 0$ be a tangent plane to the given cone so that by definition its normal with d. ratios l, m, n is a generator of its reciprocal cone (i).

$\therefore$ We have $$\frac{c^2 - b^2}{b^2c^2}l^2 + \frac{c^2 - a^2}{c^2a^2}m^2 + \frac{a^2 + b^2}{a^2b^2}n^2 = 0 \quad \text{...(ii)}$$

Let any plane parallel to the tangent plane to the given cone be

$$lx + my + nz = p \quad \text{...(iii)}$$

If it is a tangent plane to the given hyperboloid, then

$$p^2 = a^2l^2 + b^2m^2 - c^2n^2 \quad \text{...(iv)}$$

Again if it is a tangent plane at the point (x_1, y_1, z_1) then its equation is

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} - \frac{zz_1}{c^2} = 1 \quad \text{...(v)}$$

Comparing (iii) and (v), we get

$$\frac{x_1/a^2}{l} = \frac{y_1/b^2}{m} = \frac{z_1/c^2}{-n} = \frac{1}{p}$$

$$\Rightarrow \quad \frac{x_1}{la^2} = \frac{y_1}{mb^2} = \frac{z_1}{-nc^2} = \frac{1}{p} \quad \text{...(vi)}$$

Also the plane (iii) cuts the given hyperboloid in perpendicular generators if (x_1, y_1, z_1) lies on the director sphere

$$x^2 + y^2 + z^2 = a^2 + b^2 - c^2$$

$\therefore$ $$x^2 + y_1^2 + z_1^2 = a^2 + b^2 - c^2$$

$\Rightarrow$ $$\left(\frac{a^2 l}{p}\right)^2 + \left(\frac{b^2 m}{p}\right)^2 + \left(\frac{-c^2 n}{p}\right)^2 = a^2 + b^2 - c^2, \text{ from (iv)}$$

$\Rightarrow$ $$a^2l^2 + b^2m^2 + c^2n^2 = (a^2 + b^2 + c^2)\, p^2$$

$$= (a^2 + b^2 + c^2)(a^2l^2 + b^2m^2 - c^2n^2), \text{ from (iv)}$$

$\Rightarrow$ $$a^2l^2 (b^2 - c^2) + b^2m^2 (a^2 - c^2) - c^2n^2 (a^2 + b^2) = 0$$

$\Rightarrow$ $$\frac{l^2 (c^2 - b^2)}{b^2c^2} + \frac{m^2 (c^2 - a^2)}{c^2a^2} + \frac{n^2 (a^2 + b^2)}{a^2b^2} = 0,$$

dividing each term by $-a^2b^2c^2$ which is true by virtue of (ii).

Example 6. *Any plane whose normal lies on the cone*

$$(b + c)\, x^2 + (c + a)\, y^2 + (a + b)\, z^2 = 0$$

cuts the surface

$$ax^2 + by^2 + cz^2 = 1$$

in a rectangular hyperbola. **[Imp.]**

Sol. Let the plane be $ux + vy + wz = 0$ ...(1)

This cuts the surface $ax^2 + by^2 + cz^2 = 1$

in rectangular hyperbola.

Let the asymptote of this hyperbola be

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \quad ...(2)$$

The asymptote (2) lies on plane (1)

$\therefore$ $$ul + vm + wn = 0 \quad ...(3)$$

Also any point on (2) is (lr, mr, nr). This point will lie on the surface $ax^2 + by^2 + cz^2 = 1$, if

$$r^2(al^2 + bm^2 + cn^2) = 1 \text{ or } al^2 + bm^2 + cn^2 = \frac{1}{r^2}.$$

But $r \to \infty$, as the asymptote cuts the surface at ∞

$\therefore$ We have $$al^2 + bm^2 + cn^2 = 0 \quad ...(4)$$

The asymptote of a rectangular hyperbola are $\perp$. Thus the two lines given by (3) and (4) are $\perp$.

$\therefore$ $u^2 (b + c) + v^2 (c + a) + w^2 (a + b) = 0$

| *Refer Ex.* 18 (*i*), *page* 35

This shows that the normal $\frac{x}{u} = \frac{y}{v} = \frac{z}{w}$ to plane (1) lies on the cone $(b + c)\, x^2 + (c + a)\, y^2 + (a + b)\, z^2 = 0$.

Example 5:

If the generators through P a point on the hyperboloid $(x^2/a^2) + (y^2/b^2) - (z^2/c^2) = 1$ whose centre is at O, meet the plane $z = 0$ in A and B and the volume of the tetrahedron OAPB is constant and equal to abc/6, prove that P lies on one of the planes $z = \pm c$.

Solution:

Let the coordinates of P, A and B be (x_1, y_1, z_1), $(a \cos \alpha, b \sin \alpha, 0)$ and $(a \cos \beta, b \sin \beta, 0)$ respectively. The values of x_1, y_1, z_1. Also O is the origin.

$\therefore$ The volume of the tetrahedron OAPB

$$= " \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}", \text{ as O is origin}$$

$$= ' \frac{1}{6}\begin{vmatrix} x_1 & y_1 & z_1 \\ a\cos\alpha & b\sin\alpha & 0 \\ a\cos\beta & b\sin\beta & 0 \end{vmatrix}',$$

$$= \frac{1}{6} z_1 \, ab \, (\sin\beta\cos\alpha - \sin\alpha\cos\beta) = \frac{1}{6} \, ab \, z_1 \sin(\beta - \alpha)$$

$$= \frac{1}{6} ab \left[c \tan \frac{\alpha - \beta}{2}\right] \sin(\beta - \alpha), \text{ from (iv)}$$

$$= \frac{1}{6} \, abc \tan \frac{\alpha - \beta}{2} \sin(\alpha - \beta), \text{ numerically}$$

$$= \frac{1}{6} \, abc, \text{ given}$$

$$\therefore \tan \frac{\alpha - \beta}{2} \sin(\alpha - \beta) = 1$$

$$\Rightarrow \tan \frac{\alpha - \beta}{2}, \frac{2 \tan \{(\alpha - \beta)/2\}}{1 + \tan^2 \{(\alpha - \beta)/2\}} = 1$$

Example 94:

Prove that the locus of a variable line which intersects the three given lines $y = mx$, $z = c$; $y = -mx$, $z = -c$; $y = z$, $mx = -c$ is the surface $y^2 - m^2x^2 = z^2 - c^2$.

Solution :

The equation of the plane through the line $y = mx$, $z = c$ is

$$(y - mx) + \lambda_1 (z - c) = 0 \qquad ...(i)$$

The equation of the plane through the line $y = -mx$, $z = -c$ is

$$(y + mx) + \lambda_2 (z + c) = 0 \qquad ...(ii)$$

Now any line intersecting the first two given lines is given by plane (i) and (ii). The above two planes intersect in a line and as it meets the third line $y = z$, $mx = -c$, so putting $mx = -c$ and $z = y$ in (i) and (ii) we get

$(y + c) + \lambda_1 (y - c) = 0$ and $(y - c) + \lambda_2 (y + c) = 0$

$$\Rightarrow \quad \left(\frac{y+c}{y-c}\right) = -\lambda_1 \text{ and } \left(\frac{y+c}{y-c}\right) = -\frac{1}{\lambda_2} \qquad \textbf{(Note)}$$

$$\therefore -\lambda_1 = -1/\lambda_2 \text{ or } \lambda_1\lambda_2 = 1 \qquad ...(iii)$$

Eliminating λ_1 and λ_2 between (i), (ii) and (iii) we get

$$\left(\frac{y-mx}{z-c}\right)\left(\frac{y+mx}{z+c}\right) = 1 \qquad \textbf{(Note)}$$

$$\Rightarrow \quad y^2 - m^2x^2 = z^2 - c^2. \qquad \textbf{Hence proved.}$$

Example 95:

Prove that the locus of a line which meets the lines $y = \pm mx$, $z = \pm c$ and the circle $x^2 + y^2 = a^2$, $z = 0$ is

$$c^2m^2 (cy - mxz)^2 + c^2 (yz - cmx)^2 = a^2.m^2 (z^2 - c^2)^2.$$

Solution :

Given lines are $y - mx = 0$, $z - c = 0$; ...(i)

$y + mx = 0$, $z + c = 0$ ...(ii)

and the circle is $x^2 + y^2 = a^2$, $z = 0$ ...(iii)

Any line intersecting (i) and (ii) is

$$(y - mx) + k (z - c) = 0, \; (y + mx) + k_2 (z + c) = 0 \qquad ...(iv)$$

If it meets the circle (iii), then we are to eliminate k_1, k_2 from (iii) and (iv).

Putting z = 0 in (iv) we get $(y - mx) - k_1c = 0$, $(y + mx) + k_2c = 0$

$$\Rightarrow \qquad mx - y + k_1c = 0,\ mx + y + k_2c = 0$$

Adding and subtracting these, we get

$$x = -\frac{(k_1 + k_2)\,c}{2m},\ y = \frac{c\,(k_1 - k_2)}{2}$$

Substituting these values of x and y in (iii), we get

$$\frac{(k_1 + k_2)\,c^2}{4m^2} + \frac{(k_1 - k_2)^2\ c^2}{4} = a^2$$

$$\Rightarrow \qquad [(k_1 + k_2)^2 + m^2\,(k_1 - k_2)^2]\,c^2 = 4a^2m^2. \qquad \text{...(v)}$$

$$\Rightarrow \left[\left\{\left(\frac{mx - y}{z - c}\right) + \left(-\frac{mx + y}{z + c}\right)\right\}^2 + m^2\left\{\left(\frac{mx - y}{z - c}\right) - \left(-\frac{mx + y}{z + c}\right)\right\}^2\right] c^2$$

$$= 4a^2m^2 \text{ from (iv)}$$

Now simplify and get the result.

Example 35:

A sphere passes through the circle $z = 0$, $x^2 + y^2 = a^2$. Prove that the locus of the extremities of its diameter parallel to x-axis is the rectangular hyperbola $y = 0$, $x^2 = z^2 = a^2$.

Solution:

Any sphere through the given circle $x^2 + y^2 + z^2 = a^2$, $z = 0$ is $x^2 + y^2 + z^2 - a^2 + lz = 0$. ...(1)

The centre of this sphere is $(0, 0, -\lambda/2)$. So the equations of its diameter parallel to the x-axis are

$$y = 0,\ z = -\lambda/2 \text{ (or } \lambda = -2z). \qquad \text{...(2)}$$

Eliminating λ between (1) and (2), we obtain

$$y = 0,\ x^2 + z^2 - a^2 - 2z^2 = 0.$$

Hence $\qquad y = 0,\ x^2 - z^2 = a^2.$

Example 36:

A variable plane is drawn parallel to a given plane x/a + y/b + z/c = 0 and meets the coordinate axes in A, B, C. Prove that the circle ABC lies on the cone

$$yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0$$

Solution:

Any plane parallel to the plane x/a + y/b + z/c = 0 is $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=k$, k being a constant. ...(1)

The plane (1) meets the coordinate axes at A(ka, 0, 0), B(0, kb, 0), C(0, 0, kc).

Now the equation of the sphere through O, A, B, C is

$$x^2 + y^2 + z^2 - k(ax + by + cz) = 0. \quad ...(2)$$

The equations (1) and (2) represent the circle ABC.

Eliminating k between (1) and (2), we obtain

$$x^2+y^2+z^2-\left(\frac{x}{a}+\frac{y}{b}+\frac{z}{c}\right)(ax+by+cz)=0$$

$$\Rightarrow \quad -\frac{x}{a}(by+cz)-\frac{y}{b}(ax+cz)-\frac{z}{c}(ax+by)=0$$

Hence yz $yz\left(\frac{b}{c}+\frac{c}{b}\right)+zx\left(\frac{c}{a}+\frac{a}{c}\right)+xy\left(\frac{a}{b}+\frac{b}{a}\right)=0.$

Example 1:

Show that the cone whose vertex is the origin and which passes through the curve of intersection of the sphere $x^2 + y^2 + z^2 = 3a^2$ and any plane at a distance a, from the origin has three mutually perpendicular generators.

Solution:

Any plane at a distance 'a' from the origin is

$$lx + my + nz = a. \quad ...(1)$$

(Hence l, m, n are the direction cosines of the normal to the plane). The equation of the cone whose vertex is the origin and which passes through the intersection of the given sphere and the plane (1) is

$$x^2 + y^2 + z^2 = 3a^2 = 3(lx + my + nz)^2$$

$$\Rightarrow \quad (1-3l^2)x^2 + (1-3m^2)y^2 + (1-3m^2)z^2 - 6lmxy - 6mnyz - 6nlxy = 0. \quad ...(2)$$

Here $a + b + c = (1-3l^2) + (1-3m^2) + (1-3n^2)$

$$= 3 - 3\ (l^2 + m^2 + n^2) = 3 - 3,\ 1 = 0.$$

Hence the cone (2) has three mutually perpendicular generators.

Example 38:

Spheres are described to contain the circle $z = 0$. $x^2 + y^2 = a^2$. Prove that the locus of the extremities of their diameters which are parallel to the x-axis is the rectangular hyperbola $x^2 - z^2 = a^2$, $y = 0$

Solution:

The equation of the sphere through the given circle

$x^2 + y^2 = a^2$, $z = 0$ is given as $(x^2 + y^2 = a^2) + \lambda z = 0$ ...(1)

Its centre is $(0, 0, -\lambda/2)$ and radius $= \sqrt{[(-l/2)^2 - (-a^2)]}$

$$= [\sqrt{(\lambda^2 + 4a^2)}]/2$$

Now the equation of the diameter of the sphere (1) and parallel to x-axis *i.e.*, the line through the enter $(0, 0 -\lambda/2)$ and parallel to the line with d.c.'s 1, 0, 0 are $\dfrac{x-0}{1} = \dfrac{y-0}{0} = \dfrac{z+(\lambda/2)}{0}$

The coordinates of any point on it at a distance r from the centre $(0, 0 -\lambda/2)$ of the sphere (1) are $(r, 0 -\lambda/2)$.

If we take $r = \pm \dfrac{1}{2}\sqrt{(\lambda^2 + 4a^2)}$ = ± radius of the sphere then we find that the coordinates of the extremities of the diameter parallel to x-axis are given

by $x = \pm \dfrac{1}{2}\sqrt{(\lambda^2 + 4a^2)}$, $y = 0$, $z = -\lambda/2$ ...(2)

Required locus is obtained by eliminating λ from (2).

From (2) we have $4x^2 = \lambda^2 + 4a^2$, $y = 0$, $2z = -\lambda$

$\Rightarrow$ $4x^2 = (-2z)^2 + 4a^2$, $y = 0$ on eliminating λ

$\Rightarrow$ $x^2 - z^2 = a^2$, $y = 0$ which is the required locus and is a rectangular hyperbola on the plane $y = 0$. **Hence proved.**

Example 90:

Prove that the centres of the spheres which touch the lines $y = mx$, $z = c$; $y = -mx$, $z = -c$ lie upon the conicoid $mxy + cz (l + m^2) = 0$.

Solution:

Let the equation of the sphere which touches the lines be given as

$$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0 \qquad \ldots(i)$$

If this sphere touches the line $y = mx$, $z = c$, then we get

$$x^2 + m^2x^2 + c^2 + 2ux + 2vmx + 2wc + d = 0.$$

putting $y = mx$, $z = c$ in (i)

$$\Rightarrow \quad x^2 (1 + m^2) + 2 (u + vm) x + (c^2 + 2wc + d) = 0 \quad \ldots(ii)$$

If the line $y = mx$, $z = c$ touches the sphere (i) then the roots of (ii) must be coincident and the condition for the same is

$$[2(u + vm)]^2 = 4 (1 + m^2) (c^2 + 2wc + d) \qquad \ldots \text{“}b^2 = 4ac\text{”}$$

$$\Rightarrow \quad (u + vm)^2 = (1 + m^2) (c^2 + 2wc + d). \qquad \ldots(iii)$$

Similarly if the line $y = -mx$, $z = -c$ touches the sphere (i), then we shall have

$$(u - vm)^2 = (1 + m^2) (c^2 - 2wc + d). \qquad \ldots(iv)$$

putting $-m$ for m and $-c$ for c in (iii)

Subtracting (iv) from (iii), we get $4uvm = (l + m^2) (4wc)$

$$(-u) (-v) m + (1 + m^2) (-w) c = 0. \qquad \textbf{(Note)}$$

$\therefore$ The locus of the centre $(-u, -v, -w)$ of the sphere is

$$xym + (1 + m^2) zc = 0 \qquad \textbf{Hence proved.}$$

Ex. 1. *Show that the feet of the normals from the point (x', y', z') to the paraboloid $x^2 + y^2 = 2az$ lie on the sphere*

$$x^2 + y^2 + z^2 - z(a + z') - \frac{1}{2}(y/y')(x'^2 + y'^2) = 0.$$

Sol. The equation of the paraboloid is

$$x^2 + y^2 = 2az. \qquad \ldots(1)$$

Let (α, β, γ) be a point on (1) so that

$$\alpha^2 + \beta^2 = 2a\gamma. \qquad \ldots(2)$$

The equations of the normal at (α, β, γ) to (1) are

$$\frac{x - \alpha}{\alpha} = \frac{y - \beta}{\beta} = \frac{z - \gamma}{-a}. \qquad \ldots(3)$$

If the normal (3) passes through (x', y', z'), we get

$$\frac{x' - \alpha}{\alpha} = \frac{y' - \beta}{\beta} = \frac{z' - \gamma}{-a} = \lambda \text{ (say)}.$$

From these, we get

$$\alpha = x'/(1 + \lambda), \beta = y'/(1 + \lambda), \gamma = z' + a\lambda. \qquad \ldots(4)$$

Therefore the equations of the cubic curve on which lie the feet of the normals from (x', y', z') are given by

$$x = \frac{x'}{1 + \lambda}, y = \frac{y'}{1 + \lambda}, z = z' + a\lambda, \qquad \ldots(5)$$

where λ is the parameter. The values of the parameter λ giving the feet of the normals are obtained by putting the values of α, β, γ from (4) in (2) and so are given by

Example 6:

Determine completely the surface represented by

$$2y2 = 2yz + 2zx - 2xy - x - 2y = 3z = 2 = 0.$$

Solution:

Here 'a' = 0 'b' = 2, 'c' = 0, 'f' = –1, 'g' = 1, 'h' = –1, 'u' = –1/2, 'v' = –1, 'w' = 3/2, 'd' = –2.

$\therefore$ The discriminating cubic is

$$\begin{vmatrix} a-\lambda & h & g \\ h & b-\lambda & f \\ g & f & c-\lambda \end{vmatrix} = 0 \text{ or } \begin{vmatrix} -\lambda & -1 & 1 \\ -1 & 2-\lambda & -1 \\ 1 & -1 & -\lambda \end{vmatrix} = 0$$

$\Rightarrow \quad -1\,[-(2 - \lambda)\lambda - 1] + [\lambda + 1] + [1 - (2 - \lambda)] = 0$

$\Rightarrow \quad -\lambda^2 + 2\lambda^2 + \lambda + \lambda + 1 + 1 - 2 + \lambda = 0$ or $\lambda^2 - 2\lambda^2 - 3\lambda = 0$

$\Rightarrow \quad \lambda\,(^2 - 2\lambda - 3) = 0$ or $\lambda(\lambda + 1)(\lambda - 3) = 0$ or $\lambda = 0, -1, 3$

$\Rightarrow \quad \lambda_1 = 3, \lambda_2 = -1, \lambda_3 = 0$

Now putting $\lambda = 0$ in the determinant give by (i) and assaulting each row with λ_2, m_3, n_3, we have

$$-m_3 + n_3 = 0, -l_3 + 2m_3 - n_3 = 0, l_3 - m_3 = 0$$

From these on solving we get $l_3 = m_3 = n_3 = 1/\sqrt{3}$ **(Note)**

Further the line of centres is given by any two of

$$\partial F \partial x = 0, \partial F \partial y = 0, \partial F \partial z = 0,$$

$$\text{Now } \frac{\partial F}{\partial x} = 0 \Rightarrow 2z - 2v - 1 = 0;$$

$$\frac{\partial F}{\partial y} = 0 \Rightarrow 4y - 2z - 2x - 2 = 0 \text{ or } x - 2y + z + 1 = 0$$

$$\frac{\partial F}{\partial z} = 0 \Rightarrow -2y + 2x + 3 = 0 \text{ or } 2x - 2y + 3 = 0$$

Let (α, β, γ) be any point on the line of centres. Choosing $\gamma = 0$, $\beta = 1/2$, $\alpha = -2$ we find that $(-2, -1/2, 0)$ is a point on the line of centres.

Now $k = u\alpha + v\beta + w\gamma + d$

$$= \left(-\frac{1}{2}\right)(-2) + (-1)\left(-\frac{1}{2}\right) + \left(\frac{3}{2}\right)(0) - 2 = -\frac{1}{2} \neq 0$$

Hence the given surface reduces to $\lambda_1 x^2 + \lambda_2 x^2 + k = 0$

i.e., $3x^2 - y^2 = 1/2 = 0.$

which represents a hyperbolic cylinder as λ_1, λ_2 are of different signs.

Also the equation of the axis of the cylinder is

$$\frac{x - (-2)}{l_2} = \frac{y - (-1/2)}{m_2} = \frac{z - 0}{n_3} \text{ or } \frac{x + 2}{1} = \frac{y + (1/2)}{1} = \frac{z}{1} \quad \textbf{Ans.}$$

Ex. 14. *Prove that all plane sections of $ax^2+by^2+cz^2=1$ which are rectangular hyperbolas and which pass through the point (α, β, γ) touch the cone*

$$\frac{(x-\alpha)^2}{b+c}+\frac{(y-\beta)^2}{c+a}+\frac{(z-\gamma)^2}{a+b}=0.$$

Solution. If the central plane section of $ax^2+by^2+cz^2=1$ by the plane $lx+my+nz=0$ is a rectangular hyperbola then (as proved in Cor. 3, § 3), we have

$$(b+c)\ l^2+(c+a)\ m^2+(a+b)\ n^2=0. \qquad ...(1)$$

Again the equation of the plane parallel to the plane $lx+my+nz=0$ and passing through the point (α, β, γ) is

$$l\ (x-\alpha)+m\ (y-\beta)+n\ (z-\gamma)=0. \qquad ...(2)$$

Since the plane (2) is parallel to the plane $lx+my+nz=0$, therefore (1) is also the condition for the section of the given conicoid by the plane (2) to be a rectangular hyperbola.

Shifting the origin to the point (α, β, γ), the equation (2) becomes

$$lx+my+nz=0. \qquad (3)$$

$\therefore$ (1) shows that the plane (3) envelopes the cone

$$x^2/(b+c)+y^2/(c+a)+z^2/(a+b)=0. \qquad ...(4)$$

Again shifting the origin back to (α, β, γ), the plane (2) envelopes the cone

$$(x-\alpha)^2/(b+c)+(y-\beta)^2/(c+a)+(z-\gamma)^2/(a+b)=0.$$

Ex. 15. *Find the condition that the section of the conicoid*

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=1$$

by the plane $lx+my+nz=0$ may be a rectangular hyperbola.

Solution. Let λ, μ, ν be the d. c.'s of the semi-diameter of length r of the given conicoid. The extremity $(\lambda r, \mu r, \nu r)$ of this semi-diameter will lie on the given conicoid, so that we have

$$(a\lambda^2+b\mu^2+c\nu^2+2f\mu\nu+2g\nu\lambda+2h\lambda\mu)\ r^2=1$$

or $$1/r^2=a\lambda^2+b\mu^2+c\nu^2+2f\mu\nu+2g\nu\lambda+2h\lambda\mu. \qquad ...(1)$$

Now consider any three perpendicular semi-diameters of lengths r_1, r_2, r_3 with their corresponding d.c.'s as λ_1, μ_1, ν_1; λ_2, μ_2, ν_2; λ_3, μ_3, ν_3 so that $\lambda_1^2+\lambda_2^2+\lambda_3^2=1$ etc. and $\lambda_1\mu_1+\lambda_2\mu_2+\lambda_3\mu_3=0$, etc.

Writing the relation (1) for these three mutually perpendicular semi-diameters of lengths r_1, r_2, r_3 and then adding the resulting relations, we get

$$1/r_1^2+1/r_2^2+1/r_3^2=a\Sigma\lambda_1^2+b\Sigma\mu_1^2+c\Sigma\nu_1^2+2f\Sigma\mu_1\nu_1$$
$$+2g\Sigma\nu_1\lambda_1+2h\Sigma\lambda_1\mu_1$$

or $$(r_1^2+r_2^2)/r_1^2r_2^2+1/r_3^2=a+b+c. \qquad \text{...(2)}$$

Now if the section of the conicoid by the plane $lx+my+nz=0$ is a rectangular hyperbola having as its principal semi-axes the two perpendicular semi-diameters of lengths r_1 and r_2 then $r_1^2+r_2^2=0$. Hence (2) gives

$$1/r_3^2=a+b+c. \qquad \text{...(3)}$$

Hence for any semi-diameter of length r of the conicoid which is perpendicular to the plane of the section which is a rectangular hyperbola, we have

$$1/r^2=a+b+c. \qquad \text{...(4)}$$

Also the d.c.'s λ, μ, ν of this semi-diameter become the same as the d.c.'s l, m, n of the normal to the plane $lx+my+nz=0$.

Hence from (1) and (4), the required condition is given by

$$al^2+bm^2+cn^2+2fmn+2gnl+2hlm=a+b+c$$
$$=(a+b+c)\,(l^2+m^2+n^2)$$

or $$(b+c)\,l^2+(c+a)\,m^2+(a+b)\,n^2-2fmn-2gnl \quad 2hlm=0.$$