CSE 2012

1. $f_n(x) = \begin{cases} 0 & x < \frac{1}{n+1} \\ \sin \pi/x & \frac{1}{n+1} \le x \le 1/n \\ 0 & x > 1/n. \end{cases}$

At $x = \frac{1}{n+1}$ $\quad f_n(x) = 0 = L.H.L$

$x = \frac{1}{n}$ $\quad f_n(x) = 0 = R.H.L$

$f_n(x)$ is continuous in $\frac{1}{n+1} \le x \le \frac{1}{n}$

for $x > 1/n$ $\quad f_n(x) \to 0$

$\therefore$ $f_n(x)$ converges to $f(x) = 0$

Uniform Convergence

By M_n test

$$\left| f_n(x) - f(x) \right| = \left| \sin \frac{\pi}{x} \right| = 1 \text{ (supremum)} = M_n$$

$$\lim_{n\to\infty} M_n \neq 0$$

$\therefore$ $f_n(x)$ is not uniformly continuous.

2. $\sum u_n = \sum \left(\frac{n}{n+1}\right)^n n^6$

$$u_n = \left(\frac{n}{n+1}\right)^n n^6$$

$$u_{n+1} = \left(\frac{n}{n+1}\right)^{n+1} (n+1)^6$$

$$\frac{u_n}{u_{n+1}} = \frac{n+1}{n} \left(\frac{1}{1+\frac{1}{n}}\right)^6$$

$$\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{n+1}{n} > 1$$

$\therefore$ By Ratio test

$\sum u_n$ is convergent.

3. $f(x,y) = \begin{cases} \frac{(x+y)^2}{x^2+y^2} & (x,y) \neq (0,0) \\ 1 & (x,y) = (0,0) \end{cases}$

At $(0,0)$ $f(x,y) = 1$

$$\frac{\partial f}{\partial x}(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{(h+0)^2}{h^2+0^2} - 1}{h} = 0$$

$$\frac{\partial f}{\partial y}(0,0) = \lim_{h\to 0} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{(0+h)^2}{0^2+h^2} - 1}{h}$$

$$= 0$$

$\therefore \frac{\partial f}{\partial x}(0,0) = 0, \frac{\partial f}{\partial y}(0,0) = 0.$

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~~4.(i) Given planes 3x+4y~~

Continuity of $f(x,y)$ at $(0,0)$.

$$f(x,y) = \frac{(x+y)^2}{x^2+y^2} \qquad (x,y) \neq (0,0)$$

Take $y = mx$ for approaching $(0,0)$

$$\therefore f(x, mx) = \frac{(1+m)^2}{1+m^2}$$

Since $f(x,y)$ is dependent on m at $(0,0)$

$\therefore$ It is not continuous at $(0,0)$

4. Given planes $3x+4y+5z = 7$ — (1)

$x - z = 9$ — (2)

Distance of (x,y,z) from origin $= \sqrt{x^2+y^2+z^2}$

$\therefore R^2 = x^2+y^2+z^2$ — (3)

Let $f = x^2+y^2+z^2 + \lambda_1(3x+4y+5z-7) + \lambda_2(x-z-9)$

where λ_1, λ_2 are Lagrange's multiplier

$$\frac{\partial f}{\partial x} = 2x + 3\lambda_1 + \lambda_2 = 0$$

$$\frac{\partial f}{\partial y} = 2y + 4\lambda_1 = 0$$

$$\frac{\partial f}{\partial z} = 2z + 5\lambda_1 - \lambda_2 = 0$$

$\frac{\frac{(0+h)^2}{0^2+h^2} - 1}{h}$

0

These equations give us

$$x_0 = -\left(\frac{3\lambda_1 + \lambda_2}{2}\right),\ y_0 = -(2\lambda_1),\ z_0 = -\left(\frac{5\lambda_1 - \lambda_2}{2}\right)$$

Use these values in (1) and (2)

$$-\frac{3}{2}(3\lambda_1 + \lambda_2) - 8\lambda_1 - \frac{5}{2}(5\lambda_1 - \lambda_2) = 7$$

and $$-\left(\frac{3\lambda_1 + \lambda_2}{2}\right) + \left(\frac{5\lambda_1 - \lambda_2}{2}\right) = 9$$

$\therefore -50\lambda_1 + 2\lambda_2 = 14$ $\quad \therefore \lambda_1 = -\frac{2}{3}$

and $2\lambda_1 - 2\lambda_2 = 18$ $\quad \lambda_2 = -\frac{29}{3}$

$$\therefore x = -\frac{\left(-\frac{6}{3} - \frac{29}{3}\right)}{2},\ \boxed{y = \frac{4}{3}},\ \boxed{z = -\frac{19}{6}}$$

$$\boxed{x = \frac{35}{6}}$$

$f_{xx} = 2 > 0$, $f_{xy} = 0$, $f_{yy} = 2$, $f_{yz} = 0$, $f_{zz} = 2$, $f_{zx} = 0$

$$\begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = 4 > 0,\quad \begin{vmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{vmatrix} = 8 > 0$$

Since all > 0

$\therefore$ these (x, y, z) will give minimum value of $x^2 + y^2 + z^2$

$$\therefore R^2 = \frac{275}{6} \quad \therefore \boxed{R = \frac{5\sqrt{66}}{6}}$$

5. Given $f(x)$ differentiable on $[0,1]$

$f(1) = f(0) = 0$ and $\int_0^1 f^2(x)\,dx = 1$

Since $f(x)$ is differentiable $f'(x)$ exists.

and it is defined on $[0,1]$.

$\therefore$ On closed interval it is bounded

and therefore integrable.

Now $\int_0^1 x\, f(x) f'(x)\,dx = \left[x \int_0^1 f(x) f'(x)\,dx \right]_0^1 - \int_0^1 \int f(x) f'(x)\,dx$

$$= \left[\frac{1 \times f^2(1)}{2} - \frac{1 \times f^2(0)}{2} \right] - \int_0^1 \frac{f^2(x)}{2}$$

$$= -\int_0^1 \frac{f^2(x)}{2}$$

$$= -\tfrac{1}{2} \quad \left(\text{Given } \int_0^1 f^2(x)\,dx = 1\right)$$

6. let $f(x) = \begin{cases} 1 & , x \text{ is rational} \\ -1 & , x \text{ is irrational.} \end{cases}$

This function has infinite points of discontinuity. Also set of limit point of points of discontinuity are infinite.

$\therefore$ It is not riemann integrable.

In a interval $[a, b]$

Let us divide it into a partition 'P'

of n intervals of length $\frac{b-a}{n}$.

$\therefore x_0 = a,\ x_1 = x_0 + h$ - ———— $x_n = a + nh$.

$U(P,f) = \sum_{r=1}^{n} M_r \times \delta_r$ $M_r \rightarrow$ supremum in interval

$\delta_r \rightarrow$ length of interval

Now $M_r = 1$, $\delta_r = \frac{b-a}{n}$

$\therefore U(P,f) = \frac{b-a}{n} \times n = b-a$

$$\boxed{\lim_{n\to\infty} U(P,f) = \inf.\ U(P,f) = b-a} = \overline{\int_a^b} f\,dx$$

$L(P,f) = \sum_{r=1}^{n} m_r \delta_r$ m_r - infimum in the interval

δ_r - length of interval

$m_r = -1$, $\delta_r = \frac{b-a}{n}$

$\therefore L(P,f) = \sum_{r=1}^{n} (-1)\left(\frac{b-a}{n}\right) = -(b-a)$

$\sup L(P,f) = -(b-a) = \underline{\int_a^b} f\,dx$

$\underline{\int_a^b} f\,dx \neq \overline{\int_a^b} f\,dx$ $\therefore f$ is not Reimann integrable

Now $|f(x)| = 1$; x is rational and irrational

Now for $|f|$.

$$U(P,f) = \sum M_r \delta_r = 1\left(\frac{b-a}{n}\right)(n)$$
$$= b-a$$

$$\therefore \overline{\int_a^b} |f|\,dx = \inf U(P,f) = b-a$$

$$L(P,f) = \sum m_r \delta_r = 1\left(\frac{b-a}{n}\right)n$$
$$= (b-a)$$

$$\underline{\int_a^b} |f|\,dx = \sup L(P,f) = (b-a)$$

$$\therefore \underline{\int_a^b} |f|\,dx = \overline{\int_a^b} |f|\,dx = \int_a^b |f|\,dx$$

$\therefore$ $|f(x)|$ is Reimann integrable.

2013 - CSE

1. $f(x) = \begin{cases} \frac{x^2}{2} + 4 & x \geq 0 \\ -\frac{x^2}{2} + 2 & x < 0 \end{cases}$

At $x=0$ Left hand limit

$$\lim_{h \to 0} f(0-h)$$

$$= \lim_{h \to 0} \left(-\frac{(-h)^2}{2} + 2 \right) = 2$$

Right hand limit

$$\lim_{h \to 0} f(0+h) = \lim_{h \to 0} \left(\frac{h^2}{2} + 4 \right) = 4$$

$\therefore$ function is not continuous at $x=0$.

Since function has finite number of points of discontinuity

$\therefore$ $\boxed{\text{It is Reimann integral}}$

$\therefore$ $g(x) = \int_0^x f(x)\,dx$

~~However~~, $g'(x)$ does not exists.

as for $g'(x)$ to exists $f(x)$ has to continuous on the interval.

2.. Given $\sum f_n(x) = \sum \frac{(-1)^{n-1}}{n+x^2}$

Now, $f_n(x) = \frac{(-1)^{n-1}}{n+x^2} \leq \frac{(-1)^{n-1}}{n} = M_n$

Now, Series $\sum M_n$ is Convergent

because by Leibnitz test

1) $U_n > U_{n+1}$

ii) $\lim_{n\to\infty} U_n \to 0$

$\therefore$ $\sum f_n(x) \leq \sum M_n$ is uniformly convergent by M- test for series.

For Absolute Convergence

$\sum |f_n(x)| = \sum \frac{1}{n+x^2}$

$\therefore |f_n(x)| = \frac{1}{n+x^2}$ ~~.....~~

~~now $\sum M_n$ is not convergent~~

$$U_n = \frac{1}{n+x^2}, \quad U_{n+1} = \frac{1}{(n+1)+x^2}$$

Now

$$\frac{U_n}{U_{n+1}} = \frac{\frac{1}{n+x^2}}{\frac{1}{(n+1)+x^2}} = \frac{n+1+x^2}{n+x^2} = \left(1+\frac{1}{n+x^2}\right)$$

$$= \frac{(n+x^2+1)}{n+x^2} = \left(\frac{n+1+x^2}{n}\right)\left(1+\frac{x^2}{n}\right)^{-1}$$

$$= \left(1+\frac{1+x^2}{n} - \frac{x^2}{n} - - - - \right)$$

$$\frac{U_n}{U_{n+1}} = \left(1+\frac{1}{n} + f(n^2)\right)$$

$$\lim_{n\to\infty} \frac{U_n}{U_{n+1}} = 1$$ So by gauss test $\lambda = 1$ $\therefore$ It is divergent

So $\sum f_n(x)$ is not convergent.

3: Given $f(x) = [x]^2 + 3$ in $[-1, 2]$

now $[x]$ is discontinuous at integer point $x=0$, $x=1$

$f(x)$ is also discontinuous at $x=0$, $x=1$

Since these are finite number of points

$\therefore$ $f(x)$ is Reimann integrable

$$I = \int_{-1}^{2} ([x]^2 + 3)\, dx$$

$$= \int_{-1}^{0} ((-1)^2 + 3)\, dx + \int_{0}^{1} ((0)^2 + 3)\, dx + \int_{1}^{2} ((1)^2 + 3)\, dx$$

$$= \int_{-1}^{0} 4\, dx + \int_{0}^{1} 3\, dx + \int_{1}^{2} 4\, dx$$

$$= 4 + 3 + 4$$

$$\boxed{I = 11}$$