IAS MATHEMATICS (OPT.)

PAPER - I : CALCULUS (2007 to 2000)

IAS-2007

12 M / 2007-I

1(c). Let $f(x) - [x \in (-\pi, \pi)]$ be defined by $f(x) = \sin|x|$. Is f continuous on $(-\pi, \pi)$? If it is continuous, then is it differentiable on $(-\pi, \pi)$?

Soln: Given that

$$f(x) = \sin|x|, \quad x \in (-\pi, \pi)$$

$$\text{i.e., } f(x) = \begin{cases} \sin(-x) & \text{if } x \in (-\pi, 0) \\ \sin x & \text{if } x \in [0, \pi) \end{cases}$$

Clearly $f(x)$ is continuous and differentiable over each subinterval.

The only doubtful point is the breaking point $x=0$.

At $x=0$, $f(x)=0$

Now LHL: $\lim_{x \to 0^-} f(x) = \sin(-x) = 0$

RHL: $\lim_{x \to 0^+} f(x) = \sin x = 0$

$\therefore \lim_{x \to 0} f(x) = 0 = \lim_{x \to 0^+} f(x)$

$\therefore f$ is continuous at $x=0$.

Now RHD:

$$Rf'(0) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x \to 0^+} \frac{\sin x - 0}{x}$$

$$= \lim_{x \to 0^+} \frac{\sin x}{x} = 1.$$

LHD:

$$Lf'(0) = \lim_{x\to 0-} \frac{f(x)-f(0)}{x-0}$$

$$= \lim_{x\to 0-} \frac{\sin(-x)-0}{x-0}$$

$$= \lim_{x\to 0} \frac{-\sin x}{x}$$

$$= -1$$

$\therefore\ Lf'(0) \neq Rf'(0)$

$\therefore$ $f(x)$ is not differentiable at $x=0$.

Hence f is continuous on $(-\pi, \pi)$.

Also f is differentiable on $(-\pi, \pi)$ except at $x=0$

IAS-2006

12M 2006-I

1(c). Find a and b so that $f'(2)$ exists, where

$$f(x)=\begin{cases}\frac{1}{|x|} & \text{if } |x|>2\\ a+bx^2 & \text{if } |x|\le 2.\end{cases}$$

Sol^n: Given fn.

$$f(x)=\begin{cases}\frac{1}{|x|} & \text{if } |x|>2\\ a+bx^2 & \text{if } |x|\le 2\end{cases}$$

$$=\begin{cases}\frac{1}{-x} & \text{if } x<-2\\ a+bx^2 & \text{if } -2\le x\le 2\\ \frac{1}{x} & \text{if } x>2\end{cases}$$

$\because |x|>2$
$\Rightarrow -|x|<-2$
$\Rightarrow -x<-2$ or $+x<-2$
$\Rightarrow x>2$ or $x<-2$

Since $f'(2)$ exists

$\Rightarrow f(x)$ is differentiable at $x=2$

$\Rightarrow f(x)$ is continuous at $x=2$

LHL: $\lim\limits_{x\to 2+} f(x) = \lim\limits_{x\to 2+} \frac{1}{x}$

$= \frac{1}{2}$

RHL: $\lim\limits_{x\to 2-} f(x) = \lim\limits_{x\to 2-} a+bx^2$

$= a+b(2)^2$

$= a+4b$

Also $f(2) = a+4b$.

Since f is continuous at $x=2$

$\therefore \lim\limits_{x\to 2-} f(x) = \lim\limits_{x\to 2+} f(x) = f(2)$

$\Rightarrow a+4b = \frac{1}{2} = a+4b \Rightarrow \boxed{a+4b=\frac{1}{2}}$ — (1)

Now LHD:

$$Lf'(2) = \lim_{x\to 2^-} \frac{f(x)-f(2)}{x-2}$$

$$= \lim_{x\to 2^-} \frac{a+bx^2-(a+4b)}{x-2}$$

$$= \lim_{x\to 2^-} \frac{bx^2-4b}{x-2}$$

$$= \lim_{x\to 2^-} \frac{b(x^2-4)}{x-2}$$

$$= \lim_{x\to 2^-} b(x+2)$$

$$= b(2+2)$$

$$= 4b.$$

RHD:

$$Rf'(2) = \lim_{x\to 2^+} \frac{f(x)-f(2)}{x-2}$$

$$= \lim_{x\to 2^+} \frac{\frac{1}{x}-(a+4b)}{x-2}$$

$$= \lim_{x\to 2^+} \frac{\frac{1}{x}-\frac{1}{2}}{x-2} \quad (\text{from } ①)$$

$$= \lim_{x\to 2^+} \frac{2-x}{2x(x-2)}$$

$$= \lim_{x\to 2^+} -\frac{1}{2x}$$

$$= -\frac{1}{4}$$

Since f is differentiable at $x=0$

$$\therefore Lf'(2) = Rf'(2)$$

$$\Rightarrow 4b = -\frac{1}{4}$$

$$\Rightarrow \boxed{b = -\frac{1}{16}}$$

From ①, $a+4b = 1/2$

$$\Rightarrow a+4\left(-\frac{1}{16}\right) = 1/2$$

$$\Rightarrow a = \frac{1}{2}+\frac{1}{4} = 3/4$$

$$\therefore \boxed{a=\frac{3}{4}} \text{ and } \boxed{b = -\frac{1}{16}}$$

P-I 2006 15M 7.

1(d). Express $\int_0^1 x^m(1-x^n)^p\,dx$ interms of Gamma function. and hence evaluate the integral. $\int_0^1 x^6\sqrt{(1-x^2)}\,dx$.

Sol'n: Put $x^n = z$ or $x = z^{1/n}$

so that $dx = \frac{1}{n} z^{\frac{1}{n}-1}\,dz$

$= \frac{1}{n} z^{\frac{1-n}{n}}\,dz$

when $x=0,\ z=0$ and when $x=1,\ z=1$

$$\therefore \int_0^1 x^m(1-x^n)^p\,dx = \int_0^1 z^{\frac{m}{n}}(1-z)^p \cdot \frac{1}{n} z^{\frac{1-n}{n}}\,dz$$

$$= \frac{1}{n}\int_0^1 z^{\frac{m+1-n}{n}}(1-z)^p\,dz$$

$$= \frac{1}{n} B\left(\frac{m+1-n}{n}+1,\ p+1\right)$$

$$= \frac{1}{n} B\left(\frac{m+1}{n},\ p+1\right)$$

$$= \frac{1}{n}\frac{\Gamma\left(\frac{m+1}{n}\right)\Gamma(p+1)}{\Gamma\left(\frac{m+1}{n}+p+1\right)} \quad \text{——— (1)}$$

Comparing $\int_0^1 x^6(1-x^2)^{1/2}\,dx$ with $\int_0^1 x^m(1-x^n)^p\,dx$,

we have $m=6,\ n=2,\ p=1/2$.

$\therefore$ from (1),

$$\int_0^1 x^6(1-x^2)^{1/2}\,dx = \frac{1}{2}\,\frac{\Gamma\left(\frac{7}{2}\right)\Gamma(3/2)}{\Gamma(5)}$$

$\Gamma\left(\frac{m+1}{n}+p+1\right) = \Gamma\left(\frac{7}{2}+\frac{3}{2}\right) = \Gamma 5$

$$= \frac{1}{2}\,\frac{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\frac{1}{2}\ \frac{1}{2}\Gamma\frac{1}{2}}{4!}$$

$$= \frac{15\pi}{32\times 4!} = \frac{5\pi}{256} \text{ Ans}$$

15M 2006 P-I
3(a). Find the values of a and b such that

$$\lim_{x\to 0} \frac{a\sin^2 x + b\log\cos x}{x^4} = \frac{1}{2}$$

Sol'n: $\underset{x\to 0}{Lt} \frac{a\sin^2 x + b\log\cos x}{x^4}$ | form $\frac{0}{0}$

$= \underset{x\to 0}{Lt} \frac{a(2\sin x\cos x) + \frac{b}{\cos x}(-\sin x)}{4x^3}$ | by L'Hospital's Rule

$= \underset{x\to 0}{Lt} \frac{a\sin 2x - b\tan x}{4x^3}$ | $\frac{0}{0}$ form

$= \underset{x\to 0}{Lt} \frac{2a\cos 2x - b\sec^2 x}{12x^2}$ ——— (1) | by L' Hospital's Rule

The denominator of (1) $\to 0$ as $x \to 0$ but (1) $\to$ a finite limit value $\frac{1}{2}$

$\therefore$ The numerator of (1) must be zero as $x \to 0$

$\therefore$ (1) $\equiv 2a\cos(0) - b\sec^2(0) = 0$

$\Rightarrow \boxed{2a - b = 0}$ ——— (2)

with this form (1)

$\underset{x\to 0}{Lt} \frac{2a\cos 2x - b\sec^2 x}{12x^2}$ | $\frac{0}{0}$ form

$= \underset{x\to 0}{Lt} \frac{-4a\sin 2x - b[2\sec^2 x\tan x]}{24x}$ | $\frac{0}{0}$ form

$= \underset{x\to 0}{Lt} \left[\frac{-4a\sin 2x}{24x} - \frac{2b\sec^2 x\tan x}{24x}\right]$

$= -\frac{a}{3} \underset{2x\to 0}{Lt} \frac{\sin 2x}{2x} - \frac{b}{12} \underset{x\to 0}{Lt} \sec^2 x \underset{x\to 0}{Lt} \frac{\tan x}{x}$

$$= -\frac{a}{3}(1) - \frac{b}{12}(1)(1)$$

$$= -\frac{a}{3} - \frac{b}{12}$$

$$= \frac{-4a-b}{12}$$

but limit of ① is equal to $\frac{1}{2}$

$$\therefore \frac{-4a-b}{12} = \frac{1}{2}$$

$$\Rightarrow -4a-b = 6 \quad \text{——} \quad ③$$

$$② - ③ \equiv 6a = 6$$

$$\Rightarrow \boxed{a=1}$$

$$② \equiv 2(1) - b = 0$$

$$\Rightarrow 2 = b$$

$$\Rightarrow \boxed{b=2}$$

$$\therefore a=1,\ b=2$$

2006

10M → If $z = x f(y/x) + g(y/x)$, show that

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = 0.$$

Solⁿ: Given $z = x f(y/x) + g(y/x)$ —— ①

Differentiating ① w.r.t x, partially

we get

$$\frac{\partial z}{\partial x} = f(y/x) + x f'(y/x)\left(-\frac{y}{x^2}\right) + g'(y/x)\left(-\frac{y}{x^2}\right)$$

$$= f(y/x) - \frac{y}{x} f'(y/x) - \frac{y}{x^2} g'(y/x)$$

Again differentiating w.r.t x, partially

$$\frac{\partial^2 z}{\partial x^2} = f'(y/x)\left(-\frac{y}{x^2}\right) - \left[-\frac{y}{x^2} f'(y/x) + \frac{y}{x} f''(y/x)\left(-\frac{y}{x^2}\right)\right]$$
$$- \left[\frac{y}{x^2} g''(y/x)\left(-\frac{y}{x^2}\right) + \left(-\frac{2y}{x^3}\right) g'(y/x)\right]$$

$$= \cancel{-\frac{y}{x^2} f'(y/x)} + \cancel{\frac{y}{x^2} f'(y/x)} + \frac{y^2}{x^3} f''(y/x)$$
$$+ \frac{y^2}{x^4} g''(y/x) + \frac{2y}{x^3} g'(y/x)$$

$$\Rightarrow x^2 \frac{\partial^2 z}{\partial x^2} = \frac{y^2}{x} f''(y/x) + \frac{y^2}{x^2} g''(y/x) + \frac{2y}{x} g'(y/x) \quad \text{—— ②}$$

Differentiating ① w.r.t y, partially, we get

$$\frac{\partial z}{\partial y} = x f'(y/x)\left(\frac{1}{x}\right) + g'(y/x)\left(\frac{1}{x}\right)$$

$$= f'(y/x) + g'(y/x)\left(\frac{1}{x}\right) \quad \text{—— ③}$$

Again Differentiating w.r.t y, we get

$$\frac{\partial^2 z}{\partial y^2} = f''(y/x)\left(\frac{1}{x}\right) + g''(y/x)\left(\frac{1}{x^2}\right)$$

$$\Rightarrow y^2 \frac{\partial^2 z}{\partial y^2} = \frac{y^2}{x} f''(y/x) + \frac{y^2}{x^2} g''(y/x) \quad \text{—— ④}$$

Differentiating (3) w.r.t x, partially,

we get

$$\frac{\partial^2 z}{\partial x \partial y} = f''(y/x)\left(-\frac{y}{x^2}\right) - \frac{1}{x^2} g'(y/x) + \frac{1}{x} g''(y/x)\left(-\frac{y}{x^2}\right)$$

$$= -\frac{y}{x^2} f''(y/x) - \frac{1}{x^2} g'(y/x) - \frac{y}{x^3} g''(y/x) \quad \text{--- (4)}$$

$$\Rightarrow 2xy \frac{\partial^2 z}{\partial x \partial y} = 2xy\left[-\frac{y}{x^2} f''(y/x) - \frac{1}{x^2} g'(y/x) - \frac{y}{x^3} g''(y/x)\right]$$

$$= -\frac{2y^2}{x} f''(y/x) - \frac{2y}{x} g'(y/x) - \frac{2y^2}{x^2} g''(y/x) \quad \text{--- (5)}$$

from (2), (4) & (5),

we have

$$x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2}$$

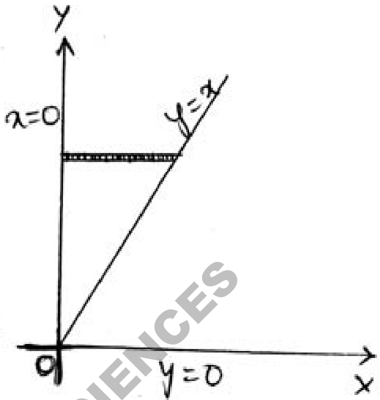
$$= 0$$

$$\therefore x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = 0$$

P-I 2006 15M

3(c). Change the order of integration in $\int_0^\infty\int_x^\infty \frac{e^{-y}}{y}\,dy\,dx$ and hence evaluate it.

Sol'n: The limits of integration are given by the straight lines $y=x$, $y=\infty$, $x=0$ and $x=\infty$ i.e, the region of integration is bounded by $x=0$, $y=x$ and an infinite boundary.



Hence taking the strips parallel to the x-axis.

The limits for y are from 0 to ∞. Hence changing the order of integration. we have

$$\int_0^\infty\int_x^\infty \frac{e^{-y}}{y}\,dx\,dy = \int_0^\infty\int_0^y \frac{e^{-y}}{y}\,dy\,dx$$

$$= \int_0^\infty \frac{e^{-y}}{y}\left[x\right]_0^y dy$$

$$= \int_0^\infty \frac{e^{-y}}{y}\,y\,dy$$

$$= \int_0^\infty e^{-y}\,dy$$

$$= \left[\frac{e^{-y}}{-1}\right]_0^\infty$$

$$=1$$

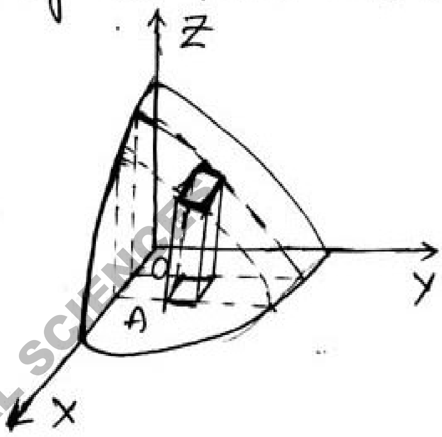
2006 P-I 15M

→ Find the volume of the uniform ellipsoid.

Sol'n: The required volume lies between the ellipsoid $z = c\sqrt{\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)}$ and the plane XOY,

and is bounded on the sides by the planes $x=0, y=0$.

The given ellipsoid cuts XOY plane in the ellipse.



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ z=0$$

Therefore the region A,

above which the required volume lies, is bounded by curves

$$y=0,\ y=b\sqrt{1-x^2/a^2},$$

$x=0$ and $x=a$.

Hence, the required volume

$$=\iint_A z\,dA$$

$$=\int_0^a\int_0^{b\sqrt{1-x^2/a^2}} c\sqrt{\left(1-\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)}\ dxdy$$

$$=c\int_0^a\int_0^{Y}\sqrt{\frac{Y^2}{b^2}-\frac{y^2}{b^2}}\ dxdy \text{ on putting } \sqrt{1-\frac{x^2}{a^2}}=\frac{Y}{b}$$

$$= \frac{c}{b}\int_0^a \left[\frac{1}{2}y\sqrt{Y^2-y^2} + \frac{1}{2}Y^2 \sin^{-1}\frac{y}{Y}\right]_0^Y dx$$

$$= \frac{c}{b}\int_0^a \frac{1}{2}Y^2 \frac{\pi}{2}\,dx$$

$$= \frac{\pi c}{4b}\int_0^a Y^2\,dx$$

$$= \frac{\pi c}{4b}\int_0^a b^2\left(1-\frac{x^2}{a^2}\right)dx \quad \left(\because Y = b\sqrt{1-\frac{x^2}{a^2}}\right)$$

$$= \frac{1}{4}\pi bc\left[x - \frac{x^3}{3a^2}\right]_0^a$$

$$= \frac{1}{4}\pi bc\left[a - \frac{a^3}{3a^2}\right]$$

$$= \frac{1}{4}\pi bc\,\frac{2a}{3}$$

$$= \frac{1}{6}\pi abc$$

IAS-2005

1(c). Show that the function given below is not Continuous at the origin. $f(x,y) = \begin{cases} 0 & \text{if } xy = 0 \\ 1 & \text{if } xy \neq 0 \end{cases}$

Sol^n: Let $(x,y) \to (0,0)$ along the co-ordinate axes.

$$\underset{x\to 0}{Lt}\, f(x,0) = \underset{x\to 0}{Lt}\, 0 = 0$$

and $$\underset{y\to 0}{Lt}\, f(0,y) = \underset{y\to 0}{Lt}\, 0 = 0$$

$$\therefore \underset{x\to 0}{Lt}\, f(x,0) = 0 = \underset{y\to 0}{Lt}\, f(0,y)$$

$\therefore f(x,y) \to 0$ as $(x,y) \to (0,0)$ along the co-ordinate axes

Let $(x,y) \to (0,0)$ along any other path

$$\underset{(x,y)\to(0,0)}{Lt}\, f(x,y) = 1$$

Since the two methods of approach to the limiting point give different limiting values

$\therefore \underset{(x,y)\to(0,0)}{Lt}\, f(x,y)$ does not exist

$\therefore f(x,y)$ is not continuous at $(0,0)$

1(d). Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined as

$$f(x,y) = \frac{xy}{\sqrt{x^2+y^2}}, \quad (x,y) \neq (0,0)$$
$$f(0,0) = 0 \qquad (x,y) = (0,0)$$

prove that f_x and f_y exist at $(0,0)$, but f is not differentiable at $(0,0)$.

Sol^n: Given that

$$f(x,y) = \begin{cases} \dfrac{xy}{\sqrt{x^2+y^2}}, & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0). \end{cases}$$

Now $f_x(0,0) = \lim_{h\to 0} \dfrac{f(0+h,0) - f(0,0)}{h-0}$

$= \lim_{h\to 0} \dfrac{0-0}{h} = 0$

$f_y(0,0) = \lim_{k\to 0} \dfrac{f(0,0+k) - f(0,0)}{k-0}$

$= \lim_{k\to 0} \dfrac{0-0}{k} = 0.$

$\therefore$ f possesses partial derivatives at $(0,0)$

Now we prove that f is not differentiable at $(0,0)$.

we have $f(0+h, 0+k) = \dfrac{hk}{\sqrt{h^2+k^2}}$

If possible suppose that f is differentiable at $(0,0)$

then $f(0+h,0+k) - f(0,0) = hf_x(0,0) + kf_y + \sqrt{h^2+k^2}\,\phi(h,k)$

where $\phi(h,k) \to 0$ as $(h,k) \to (0,0)$

where $A = f_x(0,0) = 0$

and $B = f_y(0,0) = 0$

$\Rightarrow f(h,k) = 0 - 0 + \sqrt{h^2+k^2}\,\phi(h,k)$

$\Rightarrow \phi(h,k) = \dfrac{f(h,k)}{\sqrt{h^2+k^2}}$ where $\phi(h,k) \to 0$ as $(h,k) \to (0,0)$

i.e. $\lim_{(h,k)\to(0,0)} \phi(h,k) = \lim_{(h,k)\to(0,0)} \dfrac{f(h,k)}{\sqrt{h^2+k^2}} = 0$ —— (1)

Now if $h = r\cos\theta$, $k = r\sin\theta$

then

$$\frac{f(h,k)}{\sqrt{h^2+k^2}} = \frac{r^2 \sin\theta\cos\theta}{\sqrt{r^2(\sin^2\theta+\cos^2\theta)}} \Big/ \sqrt{r^2(\sin^2\theta+\cos^2\theta)}$$

$$= \frac{r^2 \sin\theta\cos\theta}{r^2}$$

$$= \sin\theta\cos\theta.$$

$$\therefore \lim_{(h,k)\to(0,0)} \frac{f(h,k)}{\sqrt{h^2+k^2}} = \lim_{r\to 0} \sin\theta\cos\theta = 0 \quad (\text{from } (1)) \quad \text{--- } (2)$$

Since the expression $\sin\theta\cos\theta$ is independent of r and (2) implies that $\sin\theta\cos\theta = 0 \ \forall\theta$.

which is impossible for arbitrary θ.

$\therefore$ our assumption that f is differentiable is wrong.

$\therefore$ f is not differentiable at $(0,0)$

3(a).

P-I 2001 → If $u = x+y+z$, $uv = y+z$ and $uvw = z$, then find $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$

Sol'n: Since
$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

Given that
$$x+y+z = u \quad \text{—} (1)$$
$$y+z = uv \quad \text{—} (2)$$
$$z = uvw \quad \text{—} (3)$$

from (1) & (2)

$$x + uv = u$$
$$\Rightarrow \boxed{x = u - uv}$$

from (2) & (3)

$$y + uvw = uv$$
$$\Rightarrow \boxed{y = uv - uvw}$$

and $\boxed{z = uvw}$

$$\therefore \frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 1-v & -u & 0 \\ v-vw & u-uw & -uv \\ vw & uw & uv \end{vmatrix}$$

$$= uv\left[u(1-v) + uv\right]$$

$$= uv\left[u - uv + uv\right]$$

$$= u^2 v$$

3(b).

2008 P-I
15M → Evaluate $\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx$ in terms of Beta function. 12

Sol'n: $B(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

$$= \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx \quad \text{——} (1)$$

In the second integral on R.H.S of (1), Put $x = 1/t$, So that

$$dx = -\frac{1}{t^2}\,dt$$

when $x=1$, $t=1$; when $x\to\infty$, $t=0$

$$\therefore \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx = \int_1^0 \frac{\left(\frac{1}{t}\right)^{m-1}}{\left(1+\frac{1}{t}\right)^{m+n}}\left(\frac{-1}{t^2}\right)dt$$

$$= \int_0^1 \frac{t^{1-m}\,t^{m+n}}{(1+t)^{m+n}}\,\frac{1}{t^2}\,dt$$

$$= \int_0^1 \frac{t^{n-1}}{(1+t)^{m+n}}\,dt$$

$$= \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$\left[\because \int_a^b f(x)\,dx = \int_a^b f(z)\,dz\right]$$

$\therefore$ from (1)

$$B(m,n) = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_0^1 \frac{x^{n-1}}{(1+x)^{m+n}}\,dx$$

$$= \int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx$$

IAS-2004

P-I 2004 12M

1(d). Show that $x-\frac{x^2}{2}<\log(1+x)<x-\frac{x^2}{2(1+x)}$; $x>0$ 19.

Sol'n: Let $f(x)=x-\frac{x^2}{2}-\log(1+x)$

$$f'(x)=1-x-\frac{1}{1+x}$$

$$=\frac{1-x^2-1}{1+x}$$

$$=\frac{-x^2}{1+x}<0 \text{ for } x>0$$

$\therefore f'(x)<0$ for $x>0$

$\therefore f(x)$ is a decreasing function.

for $x>0$

$\therefore f(0)>f(x)$

Now $f(0)=0-0-\log 1$

$=0$

$\therefore f(x)<0$

$\Rightarrow x-\frac{x^2}{2}-\log(1+x)<0$

$\Rightarrow x-\frac{x^2}{2}<\log(1+x)$ —— (1)

Now let $g(x)=\log(1+x)-x+\frac{x^2}{2(1+x)}$

$$\Rightarrow g'(x)=\frac{1}{1+x}-1+\frac{1}{2}\left[\frac{(1+x)2x-x^2(1)}{(1+x)^2}\right]$$

$$=\frac{1}{1+x}-1+\frac{1}{2}\left[\frac{2x+x^2}{(1+x)^2}\right]$$

$$=\frac{1}{1+x}-1+\frac{1}{2}\,\frac{2x+x^2}{(1+x)^2}$$

$$= \frac{2(1+x) - 2(1+x)^2 + 2x + x^2}{2(1+x)^2}$$

$$= \frac{-x^2}{2(1+x)^2} < 0 \text{ for } x > 0$$

$\therefore g'(x) < 0$ for $x \geq 0$

$\therefore g(x)$ is a decreasing function for $x > 0$

$\therefore g(0) > g(x)$

But $g(0) = 0$

$\therefore g(x) < 0$

$$\Rightarrow \log(1+x) - x + \frac{x^2}{2(1+x)} < 0$$

$$\Rightarrow \log(1+x) < x - \frac{x^2}{2(1+x)} \quad \text{——— (2)}$$

Combining (1) & (2)

$$x - \frac{x^2}{2} < \log(1+x) < x - \frac{x^2}{2(1+x)}$$

P-I 15M 2006 2007

3(a). Let the roots of the equation in λ $(\lambda-x)^3+(\lambda-y)^3+(\lambda-z)^3=0$ be u,v,w. Prove that

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = -2\frac{(y-z)(z-x)(x-y)}{(u-v)(v-w)(w-u)}$$

Sol'n: Given that

$$(\lambda-x)^3+(\lambda-y)^3+(\lambda-z)^3=0$$

$$\Rightarrow 3\lambda^3-3\lambda^2(x+y+z)+3\lambda(x^2+y^2+z^2)-(x^3+y^3+z^3)=0$$

If the roots of the above equation be u,v,w then

$$u+v+w = x+y+z$$

$$uv+vw+wu = x^2+y^2+z^2$$

$$uvw = \frac{x^3+y^3+z^3}{3}$$

[$\because \alpha,\beta,\gamma$ are the roots of the equation $ax^3+bx^2+cx+d=0$ then

$\alpha+\beta+\gamma = -b/a$

$\alpha\beta+\beta\gamma+\gamma\alpha = c/a$

$\alpha\beta\gamma = -d/a$].

Now these relations can be written as

$$f_1 = u+v+w-x-y-z=0$$

$$f_2 = uv+vw+wu-x^2-y^2-z^2=0$$

$$f_3 = uvw-\tfrac{1}{3}(x^3+y^3+z^3)=0$$

15.

since $\dfrac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} = (-1)^3 \dfrac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)} \cdot \dfrac{\partial(u,v,w)}{\partial(x,y,z)}$.

$$\Rightarrow \frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} \Bigg/ \frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)} \quad \text{———} (1)$$

Now $$\frac{\partial(f_1,f_2,f_3)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial z} \end{vmatrix}$$

$$= \begin{vmatrix} -1 & -1 & -1 \\ -2x & -2y & -2z \\ -x^2 & -y^2 & -z^2 \end{vmatrix}$$

$$= (-1)^3 \begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ x^2 & y^2 & z^2 \end{vmatrix}$$

$$= (-1)^3(2) \begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ x^2 & y^2-x^2 & z^2-x^2 \end{vmatrix}$$

$$= (-1)^3(2) \begin{vmatrix} 1 & 0 & 0 \\ x & y-x & z-x \\ x^2 & y^2-x^2 & z^2-x^2 \end{vmatrix} \quad \begin{matrix} C_2 \sim C_2 - C_1 \\ C_3 \sim C_3 - C_1 \end{matrix}$$

$$= (-2)(y-x)(z-x) \begin{vmatrix} 1 & 0 & 0 \\ x & 1 & 1 \\ x^2 & y+x & z+x \end{vmatrix}$$

$$= (-2)(y-x)(z-x)\left[1(z+x)-(y+x)\right]$$

$$= (-2)(y-x)(z-x)(z-y)$$

$$= (-2)(x-y)(y-z)(z-x) \quad \text{———} (2)$$

Let $$\frac{\partial(f_1,f_2,f_3)}{\partial(u,v,w)} = \begin{vmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} & \frac{\partial f_1}{\partial w} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} & \frac{\partial f_2}{\partial w} \\ \frac{\partial f_3}{\partial u} & \frac{\partial f_3}{\partial v} & \frac{\partial f_3}{\partial w} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ v+w & w+u & u+v \\ vw & wu & uv \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ v+w & u-v & u-w \\ vw & w(u-v) & v(u-w) \end{vmatrix} \quad \begin{matrix} C_2 \sim C_2 - C_1 \\ C_3 \sim C_3 - C_1 \end{matrix}$$

$$= (u-v)(u-w)\begin{vmatrix} 1 & 0 & 0 \\ v+w & 1 & 1 \\ vw & w & v \end{vmatrix}$$

$$= (u-v)(u-w)(v-w)$$

$$= -(u-v)(v-w)(w-u) \quad \text{——— (3)}$$

$\therefore$ Substituting (2) & (3) in (1)

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{(-2)(x-y)(y-z)(z-x)}{-(u-v)(v-w)(w-u)}$$

$$= -2\,\frac{(x-y)(y-z)(z-x)}{(u-v)(v-w)(w-u)}$$

2000 / 15M 16.

3(b). Prove that an equation of the form $x^n = \alpha$, where $n \in N$ and $\alpha > 0$ is a real number, has a positive root.

Sol'n: Let $f(x) = x^n - \alpha$

then $f'(x) = nx^{n-1}$

Since $f'(x) > 0$ for $x > 0$

hence $f(x)$ is increasing on $(0, \infty)$

Let $x_1, x_2 \in (0, \infty)$ and $0 < x_1 < r < x_2$ such that $f(r) = 0$

then $f(x_1) < f(r) < f(x_2)$ (or)

$f(x_1) < 0 < f(x_2)$

$\therefore$ This shows that if $x \neq r$, $f(x) \neq 0$ on $(0, \infty)$

i.e. $x^n - \alpha = 0$ has at most one real +ve root.

P-I 2004 12M 14.

3(d). If the function f is defined by

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & ,(x,y)\neq(0,0) \\ 0 & ,(x,y)=(0,0) \end{cases}$$

then show that f possesses both the partial derivatives at $(0,0)$ but it is not continuous thereat.

Sol^n^:

$$f_x(0,0) = \underset{h\to 0}{Lt}\ \frac{f(0+h,0)-f(0,0)}{h}$$

$$= \underset{h\to 0}{Lt}\ \frac{0-0}{h}$$

$$= 0$$

$$f_y(0,0) = \underset{k\to 0}{Lt}\ \frac{f(0,0+k)-f(0,0)}{k}$$

$$= \underset{k\to 0}{Lt}\ \frac{0-0}{k}$$

$$= 0$$

$\therefore f$ possesses both the partial derivatives at $(0,0)$

Now let $(x,y)\to(0,0)$ along the straight line

$$y = mx$$

$$\underset{x\to 0}{Lt}\ f(x,mx) = \underset{x\to 0}{Lt}\ \frac{mx^2}{x^2(1+m^2)}$$

$$= \frac{m}{1+m^2}$$

which depends upon m.

$\therefore \lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

$\therefore f(x,y)$ is not continuous at $(0,0)$

IAS-2003

12M 2003-I

1(d). For all real numbers x, $f(x)$ is given as

$$f(x) = \begin{cases} e^x + a\sin x & x<0 \\ b(x-1)^2 + x-2 & x \geq 0 \end{cases}$$

find the values of a and b for which f is differentiable at $x=0$.

Sol^n: Given that

$$f(x) = \begin{cases} e^x + a\sin x, & x<0 \\ b(x-1)^2 + x-2, & x \geq 0. \end{cases}$$

Since f is differentiable at $x=0$

$\Rightarrow$ f must be continuous at $x=0$

and $f(0) = b(-1)^2 + 0 - 2 = b-2$

Now

LHL:

$$\underset{x\to 0^-}{Lt} f(x) = \underset{x\to 0^-}{Lt} e^x + a\sin x$$

$$= 1$$

RHL:

$$\underset{x\to 0^+}{Lt} f(x) = \underset{x\to 0^+}{Lt} b(x-1)^2 + x-2$$

$$= b-2$$

$$\therefore \underset{x\to 0^-}{Lt} f(x) = \underset{x\to 0^+}{Lt} f(x) = f(0)$$

$$\Rightarrow b-2=1$$

$$\Rightarrow \boxed{b=3}$$

Now LHD:

$$Lf'(0) = \lim_{x\to 0^-} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x\to 0^-} \frac{e^x + a\sin x - (b-2)}{x-0}$$

$$= \lim_{x\to 0^-} \frac{e^x + a\sin x - 1}{x} \quad (\because b=3)$$

$\left(\frac{0}{0} \text{ form}\right)$

$$= \lim_{x\to 0^-} \frac{e^x + a\cos x}{1} \quad (\text{by L' Hospital rule})$$

$$= 1 + a$$

RHD:

$$Rf'(0) = \lim_{x\to 0^+} \frac{f(x) - f(0)}{x - 0}$$

$$= \lim_{x\to 0^+} \frac{b(x-1)^2 + x - 2 - (b-2)}{x}$$

$$= \lim_{x\to 0^+} \frac{3(x-1)^2 + x - 2 - 1}{x} \quad (\because b=3)$$

$\left(\frac{0}{0} \text{ form}\right)$

$$= \lim_{x\to 0^+} \frac{6(x-1)+1}{1} \quad (\text{by L' Hospital rule})$$

$$= -5.$$

Since $f(x)$ is differentiable at $x=0$

$$\therefore L'f(0) = Rf'(0)$$

$$\Rightarrow a + 1 = -5$$

$$\Rightarrow \boxed{a = -6}$$

$$\therefore \boxed{a = -6} \text{ and } \boxed{b = 3}$$

P-I 2003 15M 18

3(b). Test the convergence of the integrals:

(i) $\int_0^1 \frac{dx}{x^{1/3}(1+x^2)}$ (ii) $\int_0^\infty \frac{\sin^2 x}{x^2}\,dx$

Sol^n: (i) Here $f(x) = \frac{1}{x^{1/3}(1+x^2)}$

and '0' is the only point of infinite discontinuity of f on $[0,1]$.

Take $g(x) = \frac{1}{x^{1/3}}$, then

$\underset{x\to 0+}{Lt} \frac{f(x)}{g(x)} = \underset{x\to 0+}{Lt} \frac{1}{1+x^2} = 1$ which is non-zero and finite.

$\therefore$ By comparison test $\int_0^1 f(x)\,dx$ and $\int g(x)\,dx$ converge or diverge together.

But $\int_0^1 g(x)\,dx = \int_0^1 \frac{dx}{x^{1/3}}$ is convergent. $(\because n=\frac{1}{3}<1)$

$\therefore \int_0^1 f(x)\,dx = \int_0^1 \frac{dx}{x^{1/3}(1+x^2)}$ is convergent.

(ii) Since $\lim_{x\to 0+} \frac{\sin^2 x}{x^2} = \lim_{x\to 0+}\left(\frac{\sin x}{x}\right)^2 = 1$,

therefore, 0 is not a point of infinite discontinuity.

Now $\int_0^\infty \frac{\sin^2 x}{x^2}\,dx = \int_0^1 \frac{\sin^2 x}{x^2}\,dx + \int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ —— (1)

The first integral on right is a proper integral and, therefore, Convergent.

Now we test the second integral on right for Convergence at ∞.

Since $\frac{\sin^2 x}{x^2} \leq \frac{1}{x^2}$ and $\int_1^\infty \frac{dx}{x^2}$ is Convergent. $(\because n = 2 > 1)$

$\therefore \int_1^\infty \frac{\sin^2 x}{x^2} dx$ is also Convergent.

Hence, from ①,

$$\int_0^\infty \frac{\sin^2 x}{x^2} dx \text{ is Convergent.}$$

IAS-2002

P-I 2002 12M

1(c). Show that $\frac{b-a}{\sqrt{1-a^2}} \le \sin^{-1}b - \sin^{-1}a \le \frac{b-a}{\sqrt{1-b^2}}$ for $0<a<b<1$ 19.

Sol'n : Let $f(x) = \sin^{-1}x \;\forall\; x \in [a,b]$

Where $a>0$; $b<1$

i.e $0<a<b<1$

$f(x)$ is continuous & derivable in $[a,b]$ and

$$f'(x) = \frac{1}{\sqrt{1-x^2}} \;\forall\; x \in (a,b)$$

$\therefore$ By Lagrange's Mean value theorem, $\exists\; c \in (a,b)$ Such that

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

$$\Rightarrow \frac{1}{\sqrt{1-c^2}} = \frac{\sin^{-1}b - \sin^{-1}a}{b-a} \quad \text{--- (1)}$$

Since $c \in (a,b)$

$\Rightarrow a<c<b$

$\Rightarrow a^2<c^2<b^2$

$\Rightarrow -a^2>-c^2>-b^2$

$\Rightarrow 1-a^2>1-c^2>1-b^2$

$\Rightarrow \sqrt{1-a^2}>\sqrt{1-c^2}>\sqrt{1-b^2}$

$\Rightarrow \frac{1}{\sqrt{1-a^2}} < \frac{1}{\sqrt{1-c^2}} < \frac{1}{\sqrt{1-b^2}}$

$$\Rightarrow \frac{1}{\sqrt{1-a^2}} < \frac{\sin^{-1}b - \sin^{-1}a}{b-a} < \frac{1}{\sqrt{1-b^2}}$$

$$\Rightarrow \frac{b-a}{\sqrt{1-a^2}} < \sin^{-1}b - \sin^{-1}a < \frac{b-a}{\sqrt{1-b^2}}$$

P-I 15M 2002

3(a). Let $f(x)=\begin{cases} x^p \sin\frac{1}{x}, & x\neq 0 \\ 0, & x=0 \end{cases}$ obtain condition on P

Such that (i) f is continuous at $x=0$ and

(ii) f is differentiable at $x=0$.

Sol'n: (i) At $x=0$

$f(0)=0$

LHL $\underset{x\to 0-}{dt} f(x) = \underset{x\to 0-}{dt}\ x^P(\sin 1/x)$ —— ①

RHL $\underset{x\to 0+}{Lt} f(x) = \underset{x\to 0+}{dt}\ x^P(\sin 1/x)$ —— ②

f is continuous at $x=0$

If the limits ① & ② both must be zero. this is possible only when $P>0$.

∴ The required condition for continuity of f at $x=0$ is $P>0$.

(ii) LHD $Lf'(0) = \underset{x\to 0-}{dt}\ \frac{f(x)-f(0)}{x-0}$

$= \underset{x\to 0-}{dt}\ \frac{x^P \sin 1/x - 0}{x}$

$= \underset{x\to 0-}{dt}\ x^{(P-1)}\sin 1/x$ —— ③

RHD $Rf'(0) = \underset{x\to 0+}{dt}\ \frac{f(x)-f(0)}{x-0}$

$= \underset{x\to 0+}{dt}\ \frac{x^P \sin 1/x - 0}{x}$

$= \underset{x\to 0+}{dt}\ x^{(P-1)}\sin 1/x$ —— ④

f is differentiable at $x=0$ if the limits ③ & ④ both must be zero.

This is possible only when $(P-1)>0$.

$\therefore$ The required condition for differentiability of f at $x=0$ is $P>1$

IAS-2001

P-I 15m 2001

3(b). Find the maximum and minimum radii vectors of section of the surface

$(x^2+y^2+z^2)^2 = a^2x^2 + b^2y^2 + c^2z^2$ by the plane $lx+my+nz=0$

Sol^n: we have to find the stationary values of $r^2 = x^2+y^2+z^2$ subject to the conditions

$$(x^2+y^2+z^2)^2 = a^2x^2+b^2y^2+c^2z^2 \quad \text{——— (1)}$$

and $$lx+my+nz=0 \quad \text{——— (2)}$$

Let us consider the function

$$F = (x^2+y^2+z^2) + \lambda_1\{(x^2+y^2+z^2)^2 - (a^2x^2+b^2y^2+c^2z^2)\} + \lambda_2(lx+my+nz)$$

$\therefore dF = F_x\,dx + F_y\,dy + F_z\,dz$, where

$F_x = [2x + \lambda_1\{4x(x^2+y^2+z^2) - 2a^2x\} + \lambda_2 l]$,

$F_y = [2y + \lambda_1\{4y(x^2+y^2+z^2) - 2b^2y\} + \lambda_2 m]$,

$F_z = [2z + \lambda_1\{4z(x^2+y^2+z^2) - 2c^2z\} + \lambda_2 n]$.

At stationary points, $F_x = F_y = F_z = 0$

or $$xF_x + yF_y + zF_z = 0$$

or $2(x^2+y^2+z^2) + 2\lambda_1[2(x^2+y^2+z^2)^2 - (a^2x^2+b^2y^2+c^2z^2)] + \lambda_2(lx+my+nz) = 0$

or $(x^2+y^2+z^2) + \lambda_1(x^2+y^2+z^2)^2 = 0$; using (1) and (2)

$\therefore r^2 + \lambda_1 r^4 = 0 \Rightarrow \lambda_1 = -\dfrac{1}{r^2}$.

Now $F_x = 0 \Rightarrow 2x - \frac{1}{r^2}(4r^2x - 2a^2x) = -\lambda_2 l$.

$\therefore x = \dfrac{-\lambda_2 l r^2}{2(a^2-r^2)}$.

Similarly $y = \dfrac{-\lambda_2 m r^2}{2(b^2-r^2)}$, $z = -\dfrac{\lambda_2 n r^2}{2(c^2-r^2)}$

Substituting these values in $lx+my+nz=0$,

We get $\dfrac{l^2r^2}{a^2-r^2} + \dfrac{m^2r^2}{b^2-r^2} + \dfrac{n^2r^2}{c^2-r^2} = 0$

IAS-2000

P-I 2000 12M

1(c). Use the Mean value theorem to prove that

$$\frac{2}{7} < \log(1.4) < \frac{2}{5}$$

27.

Sol'n : Let $f(t) = \log(1+t) \ \forall \ t \in [0, x]$ where $x > 0$

$f(t)$ is continuous & differentiable on $[0, x]$

and $f'(t) = \frac{1}{1+t} \ \forall \ t \in (0, x)$

By Lagrange's Mean value theorem, $\exists \, c \in (0, x)$

Such that $f'(c) = \frac{f(x) - f(0)}{x - 0}$

$$\Rightarrow \frac{1}{1+c} = \frac{\log(1+x) - \log 1}{x}$$

$$\Rightarrow \frac{1}{1+c} = \frac{\log(1+x)}{x} \quad \text{———} \ (1)$$

Since $c \in (0, x)$

$\Rightarrow 0 < c < x$

$\Rightarrow 1 < 1+c < 1+x$

$\Rightarrow 1 > \frac{1}{1+c} > \frac{1}{1+x}$

$\Rightarrow 1 > \frac{\log(1+x)}{x} > \frac{1}{1+x}$ (by (1))

$\Rightarrow \frac{1}{1+x} < \frac{\log(1+x)}{x} < 1$

$\Rightarrow \frac{x}{1+x} < \log(1+x) < x \quad (\because x > 0)$

putting $x = 2/5$, we get

$$\frac{2/5}{1+2/5} < \log(1+2/5) < 2/5$$

$$\Rightarrow \frac{2}{5} \times \frac{5}{7} < \log(7/5) < 2/5$$

$$\Rightarrow \frac{2}{7} < \log(1.4) < 2/5$$

2000 P-I

15M

3(b). Let $f(x) = \begin{cases} 0, x \text{ is irrational} \\ 1, x \text{ is rational} \end{cases}$ show that f is not Riemann-integrable on $[a,b]$. 23

Sol'n: Let f be defined on $[a,b]$ by

$$f(x) = \begin{cases} 0 & \text{when } x \text{ is rational} \\ 1 & \text{when } x \text{ is irrational} \end{cases}$$

Clearly $f(x)$ is bounded on $[a,b]$ because $0 \le f(x) \le 1 \; \forall x \in [a,b]$

Let $P = \{a = x_0, x_1, \ldots, x_n = b\}$ be a partition of $[a,b]$

Let $I_r = [x_{r-1}, x_r]$; $r = 1, 2, \ldots n$ be r^{th} subinterval of $[a,b]$.

$\therefore M_r = 1$; $m_r = 0$

$$U(P,f) = \sum_{r=1}^{n} M_r \delta_r = \sum_{r=1}^{n} 1 . \delta_r$$

$$= b - a$$

and $$L(P,f) = \sum_{r=1}^{n} m_r \delta_r = \sum_{r=1}^{n} 0 . \delta_r$$

$$= 0$$

Now $$\underline{\int_a^b} f(x)\,dx = \text{lub}\{L(P,f)\}_{P \in P[a,b]}$$

$$= 0$$

and $$\overline{\int_a^b} f(x)\,dx = \text{glb}\{U(P,f)\}_{P \in P[a,b]}$$

$$= b - a$$

$\therefore \underline{\int_a^b} f(x)\,dx \neq \overline{\int_a^b} f(x)\,dx$

$\therefore f$ is not Riemann integrable on $[a,b]$

$\therefore$ Every bounded function need not be a Riemann integrable.