LINEAR ALGEBRA

IFoS - 2011

(1)(b) Find the linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^3$ which has its range the subspace spanned by $(1,0,-1)$, $(1,2,2)$.

→ The standard basis of $\mathbb{R}^3$ is $S = \{e_1, e_2, e_3\}$ where $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ & $e_3 = (0,0,1)$.

Let T be the required linear transformation such that the range is spanned by $(1,0,-1)$ and $(1,2,2)$.

Let us assume that $T(e_1) = (1,0,-1)$
$T(e_2) = (1,2,2)$
$T(e_3) = (0,0,0)$

$$T(x,y,z) = T(xe_1 + ye_2 + ze_3) = xT(e_1) + yT(e_2) + zT(e_3)$$
$$T(x,y,z) = x(1,0,-1) + y(1,2,2) + z(0,0,0)$$
$$T(x,y,z) = (x+y, 2y, -x+2y)$$

which is the required linear transformation.

(1)(a) Let V be the vector space of 2×2 matrices over the field of real numbers $\mathbb{R}$. Let $W = \{A \in V / \text{Trace } A = 0\}$. Show that W is a subspace of V. Find a basis of W and dimension of W.

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Let $A_1 = \begin{bmatrix} x_1 & y_1 \\ z_1 & w_1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} x_2 & y_2 \\ z_2 & w_2 \end{bmatrix}$ be any two elts of W. Then $x_1 + w_1 = 0$ & $x_2 + w_2 = 0$

Let $a, b \in \mathbb{R}$.

$$aA_1 + bA_2 = \begin{bmatrix} ax_1 + bx_2 & ay_1 + by_2 \\ az_1 + cz_2 & aw_1 + bw_2 \end{bmatrix}$$ where $ax_1 + bx_2 + aw_1 + bw_2 = 0$ as $a(x_1 + w_1) + b(x_2 + w_2) = 0$.

$\therefore aA_1 + bA_2 \in W$.

$\therefore W$ is a subspace of V.

Now: Let $\begin{bmatrix} x & y \\ z & w \end{bmatrix} \in W$. Then $x + w = 0$. Therefore, there $x = -w$.

are three free variables.

Hence $\dim W = 3$. $\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} x & y \\ z & -x \end{bmatrix} = x\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} + y\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + z\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

Basis can be given by $S = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \right\}$

(2) Let $V = \{(x, y, z, u) \in \mathbb{R}^4 : y + z + u = 0\}$, $W = \{(x, y, z, u) \in \mathbb{R}^4 : x + y = 0, z = 2u\}$ be two subspaces of $\mathbb{R}^4$. Find bases for V, W, $V + W$ & $V \cap W$.

$\rightarrow$ V: $\{(x, y, z, u) \in \mathbb{R}^4 \mid y + z + u = 0\}$.

$\Rightarrow$ x can take any value and $y + z + u = 0$ for $\forall (x, y, z, u) \in V$

Now $y + z = -u$

$\therefore (x, y, z, u) = (x, y, z, -(y + z)) = x(1, 0, 0, 0) + y(0, 1, 0, -1) + z(0, 0, 1, -1)$

Clearly: $(1, 0, 0, 0)$, $(0, 1, 0, -1)$ and $(0, 0, 1, -1)$ are L.I. vectors.

$\therefore$ Basis of $V = \{(1, 0, 0, 0), (0, 1, 0, -1), (0, 0, 1, -1)\}$.

W: $\{(x, y, z, u) \in \mathbb{R}^4 \mid x + y = 0, z = 2u\}$.

$\forall (x, y, z, u) \in W$, $x = -y$, $z = 2u$.

$\therefore (x, y, z, u) = (x, -x, 2u, u) = x(1, -1, 0, 0) + u(0, 0, 2, 1)$

Clearly $(1, -1, 0, 0)$, $(0, 0, 2, 1)$ are linearly independent vectors.

$\therefore$ Basis of $W = \{(1, -1, 0, 0), (0, 0, 2, 1)\}$

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$V+W$: Consider $\begin{bmatrix}1&0&0&0\\0&1&0&-1\\0&0&1&-1\\1&-1&0&0\\0&0&2&1\end{bmatrix} \overset{R_4\to R_4-R_1}{\sim} \begin{bmatrix}1&0&0&0\\0&1&0&-1\\0&0&1&-1\\0&-1&0&0\\0&0&2&1\end{bmatrix} \overset{R_3\to R_3+R_5}{\sim} \begin{bmatrix}1&0&0&0\\0&1&0&-1\\0&0&3&0\\0&-1&0&0\\0&0&2&1\end{bmatrix}$

$R_5\to 3R_5-2R_3$ $\begin{bmatrix}1&0&0&0\\0&1&0&-1\\0&0&3&0\\0&-1&0&0\\0&0&0&3\end{bmatrix} \overset{R_2\to 3R_2+R_5}{\sim} \begin{bmatrix}1&0&0&0\\0&3&0&0\\0&0&3&0\\0&-1&0&0\\0&0&0&3\end{bmatrix} \overset{R_2\to R_2+3R_4}{\sim} \begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&3&0\\0&-1&0&0\\0&0&0&3\end{bmatrix}$

$R_2\leftrightarrow R_4$ $\begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&3&0\\0&0&0&0\\0&0&0&3\end{bmatrix} \overset{R_4\leftrightarrow R_5}{\sim} \begin{bmatrix}1&0&0&0\\0&-1&0&0\\0&0&3&0\\0&0&0&3\\0&0&0&0\end{bmatrix}$

Clearly the vectors $\{(1,0,0,0), (0,-1,0,0), (0,0,3,0), (0,0,0,3)\}$ are L.I. & hence form the basis of $V+W$.

$V\cap W$: $\{(x,y,z,u)\in\mathbb{R}^4 \mid y+z+u=0,\ x+y=0,\ z=2u\}$.

$\forall (x,y,z,u)\in V\cap W$, $x=-y$, $z=2u$ & $y+z+u=0$
$y+2u+u=0$
$y=-3u$.

$\therefore x=3u,\ y=-3u,\ z=2u,\ u=u,$

$\therefore (x,y,z,u) = (3u,-3u,2u,u) = u(3,-3,2,1)$

$\therefore$ Basis of $V\cap W = \{(3,-3,2,1)\}$

(3) Find the characteristic polynomial of $A=\begin{bmatrix}3&1&1\\2&4&2\\-1&-1&1\end{bmatrix}$ & hence compute A^{10}.

→ Char polynomial of A is given by $|A-\lambda I|=0$.

$$\begin{vmatrix}3-\lambda&1&1\\2&4-\lambda&2\\-1&-1&1-\lambda\end{vmatrix} = (3-\lambda)\left[(4-\lambda)(1-\lambda)+2\right]+1\left[-2-2(1-\lambda)\right]+1\left[-2+(4-\lambda)\right] = 0$$

$\Rightarrow (3-\lambda)\left[6-5\lambda+\lambda^2\right]-4+2\lambda+2-\lambda=0$

$\Rightarrow 18-6\lambda-15\lambda+5\lambda^2+3\lambda^2-\lambda^3+\lambda-2=0$

$\Rightarrow \lambda^3-8\lambda^2+20\lambda-16=0$ which is the reqd char. polynomial of A

By Cayley-Hamilton Theorem, we get

$A^3 - 8A^2 + 20A - 16I = 0$

$A^3 = 8A^2 - 20A + 16I.$

Premultiplying by A on both sides

$A^4 = 8A^3 - 20A^2 + 16A = 8(8A^2 - 20A + 16I) - 20A^2 + 16A = 44A^2 - 144A + 128I$

~~$A^5 = 8A^4 - 20A^3 + 16A = 8(44A^2 - 144A + 128I)$~~

$A^5 = 44A^3 - 144A^2 + 128A = 44(8A^2 - 20A + 16I) - 144A^2 + 128A$

$A^5 = 208A^2 - 752A + 704I$

$A^6 = 208A^3 - 752A^2 + 704A = 208(8A^2 - 20A + 16I) - 752A^2 + 704A$

$= 912A^2 - 3456A + 3328I$

$A^7 = 912A^3 - 3456A^2 + 3328A = 912(8A^2 - 20A + 16I) - 3456A^2 + 3328A$

$= 3840A^2 - 14912A + 14592I$

$A^8 = 3840A^3 - 14912A^2 + 14592A$

$A^8 = 3840(8A^2 - 20A + 16I) - 14912A^2 + 14592A$

$= 15808A^2 - 62208A + 61440I$

$A^9 = 15808A^3 - 62208A^2 + 61440A$

$= 15808(8A^2 - 20A + 16I) - 62208A^2 + 61440A$

$= 64256A^2 - 254720A + 252928I$

$A^{10} = 64256A^3 - 254720A^2 + 252928A$

$= 64256(8A^2 - 20A + 16I) - 254720A^2 + 252928A$

$A^{10} = 259328A^2 - 1032192A + 1028096I$

$$A^{10} = 259328\begin{bmatrix}3 & 1 & 1\\ 2 & 4 & 2\\ -1 & -1 & 1\end{bmatrix}\begin{bmatrix}3 & 1 & 1\\ 2 & 4 & 2\\ -1 & -1 & 1\end{bmatrix} - 1032192\begin{bmatrix}3 & 1 & 1\\ 2 & 4 & 2\\ -1 & -1 & 1\end{bmatrix} + 1028096\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$= \begin{bmatrix}524800 & 523776 & 523776\\ 1047552 & 1048576 & 1047552\\ -523776 & -523776 & -522752\end{bmatrix}$$

2(c) Let $A = \begin{bmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{bmatrix}$. Find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

$\longrightarrow$ Char. eq^n of A is given by $|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & -3 & 3 \\ 0 & -(5+\lambda) & 6 \\ 0 & -3 & 4-\lambda \end{vmatrix} = 0$

$\rightarrow (1-\lambda)\left[-(5+\lambda)(4-\lambda) + 18\right] = 0$

$\rightarrow (1-\lambda)\left[-2 + \lambda + \lambda^2\right] = 0 \Rightarrow (\lambda-1)(\lambda^2 + 2\lambda - \lambda - 2) = 0$

$\rightarrow (\lambda-1)^2(\lambda+2) = 0 \qquad \lambda = 1, 1, -2.$

Eigen values of A are $1, 1, -2$.

Eigen vectors of A corr. to eigen value

① $\lambda = 1$: $(A - I)X = 0$

$\begin{bmatrix} 0 & -3 & 3 \\ 0 & -6 & 6 \\ 0 & -3 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_2 \to R_2 - 2R_1,\ R_3 \to R_3 - R_1$

$\begin{bmatrix} 0 & -3 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-3y + 3z = 0 \Rightarrow y = z$.

$\therefore X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ y \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

$\therefore X_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},\ X_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.

② $\lambda = -2$: $(A + 2I)X = 0$

$\begin{bmatrix} 3 & -3 & 3 \\ 0 & -3 & 6 \\ 0 & -3 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$R_3 \to R_3 - R_2$ $\begin{bmatrix} 3 & -3 & 3 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$-3y + 6z = 0 \Rightarrow y = 2z$

$3x - 3y + 3z = 0$

$x = y - z = z$.

$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} z \\ 2z \\ z \end{bmatrix} = z\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

$\therefore X_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$

$\therefore$ Eigen vectors of A corr. to eigen value $\lambda = 1$ are $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ & $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and corr. to eigen value of $\lambda = -2$ ~~are~~ is $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$.

Since algebraic multiplicity of both roots equals to their geometric multiplicity, the matrix A is diagonalizable

Let $P = [X_1\ X_2\ X_3]$ & $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$

$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$.

Then, $D = P^{-1}AP$.