IMS

MATHS

BOOK-18

KINEMATICS (Equations of Continuity) | SET - I

Definitions and Basic Concepts

1. Hydrodynamics : Hydrodynamics is that branch of mathematics which deals with the motion of fluids or that of bodies in fluids.

2. Fluid : By fluid we mean a substance which is capable of flowing. Actual fluids are divided into two categories : (i) liquids, (ii) gases. We regard liquids as incompressible fluids for all practical purposes and gases as compressible fluids. Actual fluids have five physical properties : density, volume, temperature, pressure and viscosity.

3. Shearing stress : Two types of forces act on a fluid element. One of them is *body force* and the other is *surface force*. The body force is proportional to the mass of the body on which it acts while the surface force acts on the boundary of the body and so it is proportional to the surface area.

Suppose F is a surface force acting on an elementary surface area dS at the point P of surface S. Let F_1 and F_2 be resolved parts of F in the directions of tangent and normal at P. The normal force per unit area is called *normal stress* and is also called *pressure*. The tangential force per unit area is called *shearing stress*. Hence F_1 is a kind of shearing stress and F_2 is a normal stress.

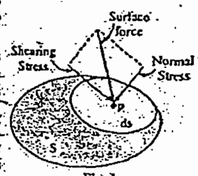


Fig. 1

4. Perfect Fluid : A fluid is said to be perfect if it does not exert any shearing stress, however small. the following have the same meaning : perfect, frictionless, inviscous, nonviscous and ideal.

From the definition of shearing stress and body force it is clear that body force per unit area at every point of surface of a perfect fluid acts along the normal to the surface at that point.

5. Difference between Perfect fluid and Real fluid : Actual fluid or real fluid is viscous and compressible. The main difference between real fluid and perfect fluid is that stress across any plane surface of perfect fluid is always normal to the surface, while it is not true in case of real fluid. In case of viscous fluid, both shearing stress and normal stress exist.

6. Viscosity : Viscosity is that property of real fluid as a result of which they offer some resistance to shearing, *i.e.*, sliding movement of one particle past or near another particle. Viscosity is also known as internal friction of fluid. All known fluids have this property in varying degree. Viscosity of glycerine and oil is large in comparison to viscosity of water or gases.

7. Velocity : Let a fluid particle be at P at any time t s. $\vec{OP} = \mathbf{r}$ and at time $t + \delta t$, let it be at Q, where $\vec{OQ} = \mathbf{r} + \delta\mathbf{r}$.

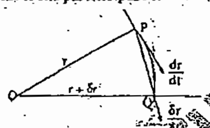


Fig. 2

$$\vec{PQ} = \vec{OQ} - \vec{OP} = \mathbf{r} + \delta\mathbf{r} - \mathbf{r} = \delta\mathbf{r}.$$

Thus δt seconds produce increment $\vec{PQ} = \delta\mathbf{r}$ in $\mathbf{r}$. If $\delta t \to 0$, $\delta\mathbf{r} \to 0$, then $Q \to P$ and the chord PQ coincides with the tangent at P to the curve.

We define $\quad \dfrac{d\mathbf{r}}{dt} = \underset{\delta t \to 0}{\mathrm{Lt}} \dfrac{\delta \mathbf{r}}{\delta t}$

The vector $\dfrac{d\mathbf{r}}{dt}$ is defined as velocity $\mathbf{q}$ of the particle at P.

Thus $\quad \mathbf{q} = \dfrac{d\mathbf{r}}{dt} = f(\mathbf{r}, t).$

8. Flux (Flow) across any surface

The rate of flow, *i.e.*, flux across any surface S is defined as the integral

$$\int_S \rho\,(\mathbf{q} \cdot \mathbf{n})\, ds.$$

We also define

Flux = density . normal velocity . area of the surface.

$\mathbf{n}$ being unit outward normal vector of any point P.

The fluid motion may be studied by two different methods.

(1) Lagrangian method. (2) Eulerian method.

1. Lagrangian method : In this method, any particle of the fluid is selected and its motion is studied. Hence we determine the history of every fluid particle.

Let a fluid particle be initially at the point (a, b, c). After a lapse of time t, let the same fluid particle be at (x, y, z). It is obvious that x, y, z are functions of t. But since particles which have initially different positions occupy different positions after the motion is allowed, hence the coordinates of final position, *i.e.*, (x, y, z) depend on (a, b, c) also. Thus,

$$x = f_1(a, b, c, t), \qquad y = f_2(a, b, c, t), \qquad z = f_3(a, b, c, t).$$

If the motion is everywhere continuous, then f_1, f_2, f_3 are continuous functions so that we can assume that first and second order partial derivatives w.r.t. a, b, c, t exist. Components of acceleration of a fluid particle are $\ddot{x}, \ddot{y}, \ddot{z}$, where

$$\ddot{x} = \frac{\partial^2 f_1}{\partial t^2}, \quad \ddot{y} = \frac{\partial^2 f_2}{\partial t^2}, \quad \ddot{z} = \frac{\partial^2 f_3}{\partial t^2}.$$

2. Eulerian method : In this method, any point fixed in the space occupied by a fluid is selected and we observe the change in the state of the fluid as the fluid passes through this point. Since the point is fixed and so x, y, z, t are independent variables. Hence the symbols $\dot{x}, \dot{y}, \dot{z}, \ddot{x}, \ddot{y}, \ddot{z}$ having no meaning.

Remark : In Eulerian method we study the motion of every fluid particle at a fixed point whereas in Lagrangian method we study the motion of every fluid particle at every point. Hence Euler's method corresponds to Local time rate of change and Lagrangian method corresponds to individual time rate of change.

Ex. *Explain the difference between Eulerian and Lagrangian methods in hydrodynamics.*

Local and Individual time rate of change

Consider a fluid motion associated with scalar point function $\phi(\mathbf{r}, t)$. Keeping the point $P(\mathbf{r})$ fixed, the change in ϕ is

$$\phi(\mathbf{r}, t + \delta t) - \phi(\mathbf{r}, t).$$

and its rate of change is

$$\lim_{\delta t \to 0} \frac{\phi(\mathbf{r}, t + \delta t) - \phi(\mathbf{r}, t)}{\delta t} = \frac{\partial \phi}{\partial t}.$$

Since $P(\mathbf{r})$ is fixed hence $\dfrac{\partial \phi}{\partial t}$ is called *local time rate of change*.

Keeping the particle fixed, change in ϕ is

$$\phi(\mathbf{r} + \delta\mathbf{r}, t + \delta t) - \phi(\mathbf{r}, t)$$

and its rate of change is

$$\lim_{\delta t \to 0} \frac{\phi(\mathbf{r} + \delta\mathbf{r}, t + \delta t) - \phi(\mathbf{r}, t)}{\delta t} = \frac{d\phi}{dt}.$$

$\dfrac{d\phi}{dt}$ is called *individual time rate of change*.

Since $\quad \phi = \phi(\mathbf{r}, t) = \phi(x, y, z, t)$

$$\therefore \quad d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz + \frac{\partial \phi}{\partial t}dt.$$

Dividing by dt,

$$\frac{d\phi}{dt} = \frac{\partial \phi}{\partial x}\frac{dx}{dt} + \frac{\partial \phi}{\partial y}\frac{dy}{dt} + \frac{\partial \phi}{\partial z}\frac{dz}{dt} + \frac{\partial \phi}{\partial t} = u\frac{\partial \phi}{\partial x} + v\frac{\partial \phi}{\partial y} + w\frac{\partial \phi}{\partial z} + \frac{\partial \phi}{\partial t}$$

$$\left[\text{For} \quad \frac{d\mathbf{r}}{dt} = \mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k} = \mathbf{i}\frac{dx}{dt} + \mathbf{j}\frac{dy}{dt} + \mathbf{k}\frac{dz}{dt}\right]$$

or, $\quad \dfrac{d\phi}{dt} = \dfrac{\partial \phi}{\partial t} + \left[\left(\mathbf{i}\dfrac{\partial}{\partial x} + \mathbf{j}\dfrac{\partial}{\partial y} + \mathbf{k}\dfrac{\partial}{\partial z}\right)\phi\right] \cdot (\mathbf{i}u + \mathbf{j}v + \mathbf{k}w)$

$$= \frac{\partial \phi}{\partial t} + (\nabla\phi) \cdot \mathbf{q}$$

or $\quad \dfrac{d\phi}{dt} = \dfrac{\partial \phi}{\partial t} + \mathbf{q} \cdot \nabla\phi.$

This is the relation between the two time rates.

Note : Similarly, for a vector function, it can be proved that

$$\frac{d\mathbf{f}}{dt} = \frac{\partial \mathbf{f}}{\partial t} + (\mathbf{q} \cdot \nabla)\mathbf{f}.$$

Acceleration

To explain the method of differentiation following the fluid and to obtain an expression for acceleration.

Consider a scalar function $\phi(\mathbf{r}, t)$ associated with fluid motion. Then $\phi(\mathbf{r}, t) = \phi(x, y, z, t)$.

$$\therefore \quad d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz + \frac{\partial \phi}{\partial t}dt.$$

Dividing by dt and taking

$$\dot{x} = \frac{dx}{dt} = u, \quad \frac{dy}{dt} = v, \quad \frac{dz}{dt} = w,$$

we obtain $\quad \dfrac{d\phi}{dt} = \dfrac{\partial \phi}{\partial x}u + \dfrac{\partial \phi}{\partial y}v + \dfrac{\partial \phi}{\partial z}w + \dfrac{\partial \phi}{\partial t}$

Taking $\quad \mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}, \quad \dfrac{d\phi}{dt} = \dfrac{\partial \phi}{\partial t} + \mathbf{q} \cdot \nabla\phi$

or $\quad \dfrac{d\phi}{dt} = \left[\dfrac{\partial}{\partial t} + \mathbf{q} \cdot \nabla\right]\phi.$

This $\Rightarrow \quad \dfrac{d}{dt} = \dfrac{\partial}{\partial t} + \mathbf{q} \cdot \nabla$

The operator $\dfrac{d}{dt}$ is called '*Differentiation following the fluid*'.

Sometimes we also write $\dfrac{D}{Dt}$ in place of $\dfrac{d}{dt}$. Acceleration $\mathbf{a}$ is defined as total derivative (Material derivative) of $\mathbf{q}$ w.r.t. t. Then

$$\mathbf{a} = \frac{d\mathbf{q}}{dt} = \left[\frac{\partial}{\partial t} + \mathbf{q} \cdot \nabla\right]\mathbf{q} = \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)\mathbf{q}.$$

Equating the coefficients of $\mathbf{i}, \mathbf{j}, \mathbf{k}$ from both sides,

$$a_1 = \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)u,$$

$$a_2 = \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)v,$$

$$a_3 = \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)w.$$

where a_1, a_2, a_3 are components of the acceleration along the axis.

Kinds of Motion

1. Stream line (Laminar) motion : A fluid motion is said to be stream line motion if the tracks of a fluid particle form parts of regular curves.

2. Turbulent motion : A fluid motion is said to be turbulent if the paths are widely irregular.

3. Steady motion : A fluid motion is said to be steady if the condition at any point in the fluid at any time remains the same for all time. That is to say, a fluid motion is said to be steady if

$$\frac{\partial \rho}{\partial t} = 0, \quad \frac{\partial p}{\partial t} = 0, \quad \frac{\partial \mathbf{q}}{\partial t} = 0,$$

where $\rho, p, \mathbf{q}$ denote density, pressure, velocity respectively.

Rotational motion : A fluid motion is said to be rotational if $W = \text{curl } \mathbf{q} \neq 0$ at every time and at every point.

5. Irrotational motion : A fluid motion is said to be irrotational if $\mathbf{W} = \text{curl } \mathbf{q} = 0$ at every point and at every time.

Definitions of some curves

1. Stream line

A stream line or line of flow is a curve s.t. the tangent at any point if it; at any instant of time, coincides with the direction of the motion of the fluid at that point. It means that direction of tangent and direction of velocity are parallel, *i.e.*, $\mathbf{q}$ is parallel to $d\mathbf{r}$ and so $\mathbf{q} \times d\mathbf{r} = 0$.

This $\Rightarrow \dfrac{dx}{u} = \dfrac{dy}{v} = \dfrac{dz}{w}$ or $\dfrac{dr}{-\frac{\partial \phi}{\partial r}} = \dfrac{r\,d\theta}{-\frac{1}{r}\frac{\partial \phi}{\partial \theta}} = \dfrac{r \sin\theta\, d\omega}{-\frac{1}{r \sin\theta}\frac{d\phi}{d\omega}}$

These are the required differential equations of a stream line. Stream lines form doubly infinite set at any time t. Here

$$\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}.$$

2. Stream tube : The stream lines drawn though each point of a closed curve enclose a tubular surface in the fluid which is called stream tube or tube of flow. A tube of flow of infinitesimal cross section is called *stream filament*.

3. Path lines

A path line is a curve which a particular fluid particle describes during its motion. The differential equations of path lines are

$$\frac{d\mathbf{r}}{dt} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}, \quad i.e.,$$

$$\frac{dx}{dt} = u, \quad \frac{dy}{dt} = v, \quad \frac{dz}{dt} = w.$$

Path lines form a triply infinite set.

4. Difference between stream lines and path lines

The tangents to the stream lines give the directions of velocities of fluid particles at various points at a given time, while tangents to the path lines give the directions of velocities of a given fluid particle at various times. That is to say, stream lines show how each fluid particle is moving at a given instant whereas the path lines show how a given fluid particle is moving at each instant. In steady flow, stream lines do not vary with time and coincide with path lines.

Streak lines : A streak line is a line on which lie all those fluid elements that at some earlier instant passed through a particular point in space.

A streak line is defined as the locus of different particles passing through fixed point.

Velocity potential

Suppose $\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ is velocity at any point $P(x, y, z)$. Also suppose the expression $u\,dx + v\,dy + w\,dz$ is an exact differential, say $-d\phi$.

Then $$-d\phi = u\,dx + v\,dy + w\,dz$$

or $$-\left(\frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z} + \frac{\partial \phi}{\partial t}dt\right) = u\,dx + v\,dy + w\,dz$$

where $$\phi = \phi(x, y, z; t).$$

This $\Rightarrow$ $u = -\dfrac{\partial \phi}{\partial x}, \; v = -\dfrac{\partial \phi}{\partial y}, \; w = -\dfrac{\partial \phi}{\partial z}, \; 0 = -\dfrac{\partial \phi}{\partial t}$

Hence $$\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$$
$$= -\left(\mathbf{i}\frac{\partial \phi}{\partial x} + \mathbf{j}\frac{\partial \phi}{\partial y} + \mathbf{k}\frac{\partial \phi}{\partial z}\right) = -\nabla\phi.$$

Integration of $\dfrac{\partial \phi}{\partial t} = 0$ declares that $\phi = 0 + f_1(x, y, z)$

where $f_1(x, y, z)$ is a constant of integration. This equation also declares that $\phi = \phi(x, y, z)$.

$\mathbf{q} = -\nabla\phi$ is the required relation. Here ϕ is defined as *velocity potential* or *velocity function*. The negative sign in the equation $\mathbf{q} = -\nabla\phi$ is a convention. It ensures that flow takes from higher to lower potentials.

Theorem 1. To show that surfaces exist which cut stream lines orthogonally if the velocity potential exists.

Proof : The differential equations of stream lines are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \qquad \text{...(1)}$$

The surfaces which cut (1) orthogonally are given by

$$u\,dx + v\,dy + w\,dz = 0. \qquad \text{...(2)}$$

The necessary and sufficient condition for the existence of (2) is that (2) must admit a solution of the type

$$f(x, y, z) = c \qquad \text{...(3)}$$

c being constant of integration.

The necessary and sufficient condition for the existence of (3) is that

$$u\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right) + v\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) + w\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) = 0. \qquad \text{...(4)}$$

If we show that (4) is satisfied whenever velocity potential exists, *i.e.*, when $u = -\dfrac{\partial \phi}{\partial x}, v = -\dfrac{\partial \phi}{\partial y}, w = -\dfrac{\partial \phi}{\partial z}$, the result will follows.

$$\text{L.H.S. of (4)} = u\left(-\frac{\partial^2 \phi}{\partial z\,\partial y} + \frac{\partial^2 \phi}{\partial y\,\partial z}\right) + v\left(-\frac{\partial^2 \phi}{\partial x\,\partial z} + \frac{\partial^2 \phi}{\partial z\,\partial x}\right) + w\left(-\frac{\partial^2 \phi}{\partial x\,\partial y} + \frac{\partial^2 \phi}{\partial x\,\partial y}\right)$$
$$= u \cdot 0 + v \cdot 0 + w \cdot 0 = 0 = \text{R.H.S. of (4)}$$

Hence (4) is satisfied.

Some definitions

1. Vorticity Vector : If $\mathbf{q}$ be the velocity vector, then $\mathbf{W} = \text{curl } \mathbf{q}$ or rot $\mathbf{q}$ is called vorticity vector. The mathematicians Lamb, Milne-Thompson, Rutherford, Goldstein etc. follow the definition $\mathbf{W} = \text{curl } \mathbf{q}$ whereas Birkhoff, Robertson etc. follow the definition $\mathbf{W} = \frac{1}{2} \text{curl } \mathbf{q}$. If we write $\mathbf{W} = \mathbf{W}(\xi, \eta, \zeta)$ then $\mathbf{W} = \text{curl } \mathbf{q}$ gives

$$\xi\mathbf{i} + \eta\mathbf{j} + \zeta\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

Equating the coefficients of $\mathbf{i}, \mathbf{j}, \mathbf{k}$ on both sides,

$$\xi = \left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \quad \eta = \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right), \quad \zeta = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right).$$

A fluid motion is said to be irrotational if $\xi = 0, \eta = 0, \zeta = 0$ otherwise rotational.

2. Vortex line : Vortex line is a curve such that tangent at each point of it at any instant of time is in the direction of vorticity vector at that point. It means that $\mathbf{W}$ is parallel to $d\mathbf{r}$. This $\Rightarrow \mathbf{W} \times d\mathbf{r} = 0 \Rightarrow \dfrac{dx}{\xi} = \dfrac{dy}{\eta} = \dfrac{dz}{\zeta}$, these are the differential equations of a vortex line.

3. Vortex tube : The vortex lines drawn through each point of a closed curve enclose a tubular space in the fluid known as vortex tube. A vortex tube of infinitesimal cross section is called a *vortex filament* or simply *vortex*.

4. Beltranic flow : A fluid motion is said to be Beltranic flow if $\mathbf{q}$ is parallel to $\mathbf{W}$, *i.e.*, if $\mathbf{q} \times \mathbf{W} = 0$. In this case $\mathbf{q}$ is called Beltranic vector.

Boundary surface

The contact between the fluid and the surface will be maintained if the fluid and surface have the same velocity along the normal to the surface. Let P be an arbitrary point on the boundary surface $F(\mathbf{r}, t) = 0$ where the fluid velocity is $\mathbf{q}$ and velocity of the surface is $\mathbf{u}$. Since normal component of velocity of fluid = normal component of the velocity of surface.

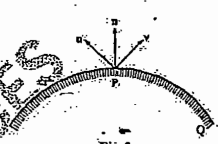


Fig. 3

This $\Rightarrow \quad \mathbf{q} \cdot \mathbf{n} = \mathbf{u} \cdot \mathbf{n} \qquad \text{...(1)}$

Since ∇F is normal to the surface $F(\mathbf{r}, t) = 0$. Hence $\mathbf{n}$ and ∇F both are parallel vectors. Now (1) takes the form

$$\mathbf{q} \cdot \nabla F = \mathbf{u} \cdot \nabla F. \qquad \text{...(2)}$$

Let $P(\mathbf{r}, t)$ move to a point $Q(\mathbf{r} + \delta\mathbf{r}, t + \delta t)$ in time δt. Since Q also lies on $F(\mathbf{r}, t) = 0$. Hence $F(\mathbf{r} + \delta\mathbf{r}, t + \delta t) = 0$. By Taylor's theorem, namely

$$f(x + h, y + k) = f(x, y) + \left(h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y}\right) + \ldots\ldots$$

We get $F(\mathbf{r}, t) + \left(\delta\mathbf{r} \cdot \nabla F + \delta t \dfrac{\partial F}{\partial t}\right) = 0. \qquad$ Also $F(\mathbf{r}, t) = 0.$

Hence $\dfrac{\delta\mathbf{r}}{\delta t} \cdot \nabla F + \dfrac{\partial F}{\partial t} = 0$ or $\dfrac{\partial F}{\partial t} = -\dfrac{d\mathbf{r}}{dt} \cdot \nabla F = -\mathbf{u} \cdot \nabla F.$

For $\mathbf{u}$ is the velocity of the surface.

Now (2) becomes

$$\mathbf{q} \cdot \nabla F = -\frac{\partial F}{\partial t} \quad \text{or} \quad \left[\frac{\partial}{\partial t} + \mathbf{q} \cdot \nabla\right] F = 0 \quad \text{or} \quad \frac{dF}{dt} = 0$$

or equivalently $\left(\dfrac{\partial}{\partial t} + u\dfrac{\partial}{\partial x} + v\dfrac{\partial}{\partial y} + w\dfrac{\partial}{\partial z}\right) F = 0.$

This is the required condition for the surface to be a possible form of boundary surface. If the surface is a rigid surface, then the condition becomes

$$u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} = 0.$$

Remark : Normal component of velocity for the boundary

$$= \mathbf{q} \cdot \mathbf{n} = \mathbf{q} \cdot \frac{\nabla F}{|\nabla F|} = -\frac{\partial F/\partial t}{|\nabla F|} \qquad \because \frac{\partial F}{\partial t} + \mathbf{q} \cdot \nabla F = 0$$
$$= -\frac{\partial F/\partial t}{[F_x^2 + F_y^2 + F_z^2]^{1/2}} \text{ where } F_x = \frac{\partial F}{\partial x} \text{ etc.}$$

Equation of continuity

The rate of generation of mass within a given volume must be balanced by an equal net outward flow of mass from the volume. This amounts to saying that matter is neither created nor destroyed.

Equation of Continuity by Euler's method

Or Determine equation of continuity by vector approach for a non-homogeneous incompressible fluid.

Consider a fixed surface S, enclosing a volume V in the region occupied by a moving fluid. Let $\mathbf{n}$ be a unit outward normal vector drawn on the surface element dS, where fluid velocity is $\mathbf{q}$ and fluid density is ρ. Inward normal velocity is $-\mathbf{n} \cdot \mathbf{q}$. Mass of the fluid entering across the surface S in unit time is

$$\int_S \rho(-\mathbf{n} \cdot \mathbf{q})\,dS = -\int_S \mathbf{n} \cdot \rho\mathbf{q}\,dS = -\int_V \nabla \cdot (\rho\mathbf{q})\,dV \qquad \text{...(1)}$$

The mass of the fluid within the volume V is $\int_V \rho\,dV$.

Rate of generation of the fluid within the volume is

$$\frac{\partial}{\partial t}\int_V \rho\,dV = \int_V \frac{\partial \rho}{\partial t}dV \qquad \text{...(2)}$$

[For $\rho\dfrac{\partial}{\partial t}(dV) = \rho\, d\left(\dfrac{\partial V}{\partial t}\right) = \rho \cdot 0 = 0$, as volume is constant w.r.t. time]. Here local time rate of change has been taken because the surface is stationary. Equation of continuity gives

$$\int_V \frac{\partial \rho}{\partial t}dV = -\int_V \nabla \cdot (\rho\mathbf{q})\,dv \qquad \text{[on equating (1) to (2)]}$$

or $$\int_V \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{q})\right] dv = 0$$

Since S is arbitrary and so V is arbitrary. Hence integrand of the last integral vanishes.

$$\therefore \quad \frac{\partial \rho}{\partial t}+\nabla\cdot(\rho \mathbf{q})=0. \qquad \dots(3)$$

This is Eulerian equation of continuity.

By (3), $\quad \frac{\partial \rho}{\partial t}+\mathbf{q}\cdot\nabla\rho+\rho\nabla\cdot\mathbf{q}=0$

or $\quad \left[\frac{\partial}{\partial t}+\mathbf{q}\cdot\nabla\right]\rho+\rho\nabla\cdot\mathbf{q}=0$

or $$\frac{d\rho}{dt}+\rho\,\nabla\cdot\mathbf{q}=0. \qquad \dots(4)$$

This is an alternate form of (3).

[Equation (3) is also called equation of mass of conservation].

Deductions : (i) To prove $\frac{d}{dt}(\log\rho)+\nabla.\mathbf{q}=0.$

Dividing (4) by ρ and writing

$$\frac{1}{\rho}\frac{d\rho}{dt}=\frac{d}{dt}(\log\rho),$$

we get the required result.

(ii) To write cartesian form of the equation of continuity. We know

$$\frac{d}{dt}=\frac{\partial}{\partial t}+\mathbf{q}\cdot\nabla=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}.$$

Now (4) is reduced to

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}\right)\rho+\rho\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=0.$$

This is the cartesian form.

(iii) Suppose the fluid is incompressible so that

$$\frac{d\rho}{dt}=0. \text{ Then (4)} \Rightarrow \rho\nabla\cdot\mathbf{q}=0 \Rightarrow \nabla.\mathbf{q}=0$$

$$\Rightarrow \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0.$$

This is the equation of continuity in this case.

Note : In this case $\mathbf{q}$ is solenoidal vector. For a vector $\mathbf{f}$ is said to be solenoidal vector if $\nabla.\mathbf{f}=0$.

(iv) Let the motion be irrotational and incompressible. Then there exists velocity potential ϕ s.t. $\mathbf{q}=-\nabla\phi$.

Here also $\frac{d\rho}{dt}=0$. Now (4) becomes

$$0+\rho\nabla\cdot(-\nabla\phi)=0 \quad \text{or} \quad \nabla^2\phi=0$$

or $$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0.$$

This is the equation of continuity in this case.

Note : This deduction can also be expressed as : *Show that the equation of continuity reduces to Laplace's equation when the liquid is incompressible and irrotational.*

(v) Suppose the motion is symmetrical.

In this case velocity has only one component, say u.

Then we have $\frac{d}{dt}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}$ as $\mathbf{q}=u$, $\nabla=\frac{\partial}{\partial x}$.

Now (4) becomes

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right)\rho+\rho\frac{\partial u}{\partial x}=0.$$

(vi) For steady motion : In this case $\frac{\partial\rho}{\partial t}=0$. Now equation (3) becomes

$$\nabla\cdot(\rho\mathbf{q})=0.$$

or equivalently, $\quad \frac{\partial(\rho u)}{\partial x}+\frac{\partial(\rho v)}{\partial y}+\frac{\partial(\rho w)}{\partial z}=0.$

This is Euler's equation of continuity for steady motion.

Problem. *Write full form for the operator used for differentiation following the fluid motion and give equation of continuity.*

Solution : $\quad \frac{d}{dt}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}+v\frac{\partial}{\partial y}+w\frac{\partial}{\partial z}$

$\frac{d}{dt}$ = operator of differentiation following fluid motion.

Equation of continuity is

$$\frac{d\rho}{dt}+\rho\nabla\cdot\mathbf{q}=0.$$

Equation of continuity by Lagrange's method

Let initially a fluid particle be at (a,b,c) at time $t=t_0$; when its volume is dV_0 and density is ρ_0. After a lapse of time t, let the same fluid particle be at (x,y,z) when its volume is dV and density ρ. Since the mass of fluid element remains invariant during its motion. Hence

$$\rho_0\,dV_0=\rho dV \quad \text{or} \quad \rho_0\, da\,db\,dc=\rho\,dx\,dy\,dz$$

or $$\rho_0\,da\,db\,dc=\rho\frac{\partial(x,y,z)}{\partial(a,b,c)}da\,db\,dc$$

or $$\rho J=\rho_0 \quad \dots(1) \qquad \text{where } J=\frac{\partial(x,y,z)}{\partial(a,b,c)}$$

(1) is the required equation of continuity.

Remark : This article can also be expressed as : *By considering the constancy of mass of a finite volume of the fluid, obtain the equation of continuity.*

Equivalence between Eulerian and Lagrangian forms of equations of continuity

Let initially a fluid particle be at (a,b,c) at time $t=t_0$, when its volume is dV_0 and density is ρ_0. After a lapse of time t, let the same fluid particle be at (x,y,z) when its volume is dV and density is ρ. The velocity components in the two systems are connected by the equations :

$$u=\dot{x},\ v=\dot{y},\ w=\dot{z},\ \mathbf{q}=u\mathbf{i}+v\mathbf{j}+w\mathbf{k}.$$

Also $\quad x=x(a,b,c,t),\ y=y(a,b,c,t),\ z=z(a,b,c,t)$

$$\frac{\partial u}{\partial a}=\frac{\partial}{\partial a}\left(\frac{dx}{dt}\right)=\frac{d}{dt}\left\{\frac{\partial x}{\partial a}\right\}. \text{ Similarly, } \frac{\partial v}{\partial a}=\frac{d}{dt}\left(\frac{\partial y}{\partial a}\right) \text{ etc.}$$

Firstly, we shall determine $\frac{dJ}{dt}$.

$$J=\frac{\partial(x,y,z)}{\partial(a,b,c)}=\begin{vmatrix}\frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} & \frac{\partial x}{\partial c}\\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} & \frac{\partial y}{\partial c}\\ \frac{\partial z}{\partial a} & \frac{\partial z}{\partial b} & \frac{\partial z}{\partial c}\end{vmatrix}$$

$$\therefore \quad \frac{dJ}{dt}=\begin{vmatrix}\frac{\partial u}{\partial a} & \frac{\partial u}{\partial b} & \frac{\partial u}{\partial c}\\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} & \frac{\partial y}{\partial c}\\ \frac{\partial z}{\partial a} & \frac{\partial z}{\partial b} & \frac{\partial z}{\partial c}\end{vmatrix}+\begin{vmatrix}\frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} & \frac{\partial x}{\partial c}\\ \frac{\partial v}{\partial a} & \frac{\partial v}{\partial b} & \frac{\partial v}{\partial c}\\ \frac{\partial z}{\partial a} & \frac{\partial z}{\partial b} & \frac{\partial z}{\partial c}\end{vmatrix}+\begin{vmatrix}\frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} & \frac{\partial x}{\partial c}\\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} & \frac{\partial y}{\partial c}\\ \frac{\partial w}{\partial a} & \frac{\partial w}{\partial b} & \frac{\partial w}{\partial c}\end{vmatrix}$$

or $$\frac{dJ}{dt}=J_1+J_2+J_3 \qquad \dots(1), \text{ say}$$

Now J_1 is expressible as

$$J_1=\begin{vmatrix}\frac{\partial u}{\partial a} & \frac{\partial y}{\partial a} & \frac{\partial z}{\partial a}\\ \frac{\partial u}{\partial b} & \frac{\partial y}{\partial b} & \frac{\partial z}{\partial b}\\ \frac{\partial u}{\partial c} & \frac{\partial y}{\partial c} & \frac{\partial z}{\partial c}\end{vmatrix}$$

$$=\begin{vmatrix}\frac{\partial u}{\partial x}\frac{\partial x}{\partial a}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial a}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial a} & \frac{\partial y}{\partial a} & \frac{\partial z}{\partial a}\\ \frac{\partial u}{\partial x}\frac{\partial x}{\partial b}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial b}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial b} & \frac{\partial y}{\partial b} & \frac{\partial z}{\partial b}\\ \frac{\partial u}{\partial x}\frac{\partial x}{\partial c}+\frac{\partial u}{\partial y}\frac{\partial y}{\partial c}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial c} & \frac{\partial y}{\partial c} & \frac{\partial z}{\partial c}\end{vmatrix}$$

$$=\frac{\partial u}{\partial x}\frac{\partial(x,y,z)}{\partial(a,b,c)}+\frac{\partial u}{\partial y}\frac{\partial(y,y,z)}{\partial(a,b,c)}+\frac{\partial u}{\partial z}\frac{\partial(z,y,z)}{\partial(a,b,c)}$$

$$=J\frac{\partial u}{\partial x}+\frac{\partial u}{\partial y}\cdot 0+\frac{\partial u}{\partial z}\cdot 0=J\frac{\partial u}{\partial x}$$

$\therefore \quad J_1=J\frac{\partial u}{\partial x}$ [For a determinant vanishes if any two of its columns are identical]

Similarly, $\quad J_2=J\frac{\partial v}{\partial y},\quad J_3=J\frac{\partial w}{\partial z}.$

Now (1) becomes $\quad \frac{dJ}{dt}=J\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)=J\,\nabla\cdot\mathbf{q}$

$$\therefore \quad \frac{dJ}{dt}=J\,\nabla.\mathbf{q} \qquad \dots(2)$$

Step I. Lagrangian equation of continuity

$$\Rightarrow \rho J=\rho_0 \Rightarrow \frac{d}{dt}(\rho J)=0 \Rightarrow \frac{d\rho}{dt}J+\rho\frac{dJ}{dt}=0$$

$$\Rightarrow J\frac{d\rho}{dt}+\rho J\,\nabla.\mathbf{q}=0, \text{ by (2)}$$

Dividing by J, $\quad \frac{d\rho}{dt}+\rho\,\nabla.\mathbf{q}=0.$

$\Rightarrow$ Eulerian equation of continuity.

Step II. Eulerian equation of continuity

$$\Rightarrow \frac{d\rho}{dt}+\rho\nabla.\mathbf{q}=0 \Rightarrow \frac{d\rho}{dt}+\rho\frac{1}{J}\frac{dJ}{dt}=0, \text{ by (2)}$$

$$\Rightarrow J\frac{d\rho}{dt}+\rho\frac{dJ}{dt}=0 \Rightarrow \frac{d}{dt}(\rho J)=0.$$

integrating we get $\rho J=\rho_0$, say.

$\Rightarrow$ Lagrangian equation of continuity.

Generalised Orthogonal curvilinear co-ordinates

Suppose $f_1(x,y,z)=a_1, f_2(x,y,z)=a_2, f_3(x,y,z)=a_3$, are the three independent orthogonal families of surfaces, where (x,y,z) are cartesian co-ordinates of a point. the surfaces a_1 = const., a_2 = const., a_3 = const. form an orthogonal system in which (a_1,a_2,a_3) may be used as the *orthogonal curvilinear co-ordinates* of a point in the space. The relation between the two co-ordinates (x,y,z) and (a_1,a_2,a_3) can also be expressed by the relations :

$$x=x(a_1,a_2,a_3),\ y=y(a_1,a_2,a_3),\ z=z(a_1,a_2,a_3).$$

$$dx=\frac{\partial x}{\partial a_1}da_1+\frac{\partial x}{\partial a_2}da_2+\frac{\partial x}{\partial a_3}da_3$$

$$dy=\frac{\partial y}{\partial a_1}da_1+\frac{\partial y}{\partial a_2}da_2+\frac{\partial y}{\partial a_3}da_3$$

$$dz=\frac{\partial z}{\partial a_1}da_1+\frac{\partial z}{\partial a_2}da_2+\frac{\partial z}{\partial a_3}da_3$$

Squaring and adding these equations column-wise, we obtain.

$$dx^2+dy^2+dz^2=(h_1da_1)^2+(h_2da_2)^2+(h_3da_3)^2+\text{coeff. of } da_1da_2 + \text{coeff. of } da_2da_3+\text{coeff. of } da_3da_1$$

where $$h_1^2=\left(\frac{\partial x}{\partial a_1}\right)^2+\left(\frac{\partial y}{\partial a_1}\right)^2+\left(\frac{\partial z}{\partial a_1}\right)^2 \text{ etc.}$$

By orthogonal property, the terms containing da_1da_2, da_2da_3, da_3da_1 vanish.

Hence

$$dx^2+dy^2+dz^2=(h_1da_1)^2+(h_2da_2^2)+(h_3da_3)^2.$$

Using the fact that the line element in cartesian co-ordinates is given by

$ds^2 = dx^2 + dy^2 + dz^2$, we get

$$ds^2 = (h_1\,da_1)^2 + (h_2\,da_2)^2 + (h_3\,da_3)^2.$$

Equation of continuity in generalised orthogonal curvilinear co-ordinates

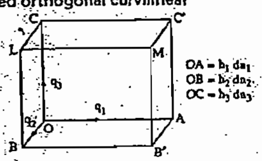


Fig. 4

Let ρ be the fluid density at a curvilinear point $P(a_1, a_2, a_3)$ enclosed by a small parallelopiped with edges of lengths $h_1\,da_1$, $h_2\,da_2$, $h_3\,da_3$. Let q_1, q_2, q_3 be the velocity components along OA, OB, OC respectively. Mass of the fluid that passes in unit time across the face $OBLC$

$$= \text{density . area . normal velocity}$$

$$= \rho\,(h_2\,da_2 \,.\, h_3\,da_3) \,.\, q_1$$

$$= \rho\, q_1\, h_2\, h_3\, da_2\, da_3$$

$$= f(a_1, a_2, a_3), \text{ say.}$$

Mass of the fluid that passes in unit time across the face

$$C'MB'A' = f(a_1 + \delta a_1, a_2, a_3)$$

$$= f(a_1, a_2, a_3) + \delta a_1 \,.\, \frac{\partial f}{\partial a_1}$$

Now the excess of flow in overflow out from the faces $OBLC$ and $MB'AC'$ in unit time

$$= f - \left(f + \delta a_1 \frac{\partial f}{\partial a_1}\right)$$

$$= -\delta a_1 \,.\, \frac{\partial f}{\partial a_1}$$

$$= -\delta a_1 \,.\, \frac{\partial}{\partial a_1}(\rho q_1 h_2 h_3)\, da_2 \,.\, da_3$$

$$= -\frac{\partial}{\partial a_1}(\rho q_1 h_2 h_3)\, da_1 \,.\, da_2 \,.\, da_3$$

Similarly, the excess of flow in over flow out from the faces $CLMC'$ and $OBB'A$; $OCC'A$ and $LMB'B$ are respectively

$$-\frac{\partial}{\partial a_3}(\rho q_3 h_1 h_2)\, da_1\, da_2\, da_3 \text{ and } -\frac{\partial}{\partial a_2}(\rho q_2 h_1 h_3)\, da_1\, da_2\, da_3.$$

Rate of increment in mass of the fluid within the parallelopiped

$$= \frac{\partial}{\partial t}(\rho h_1\, da_1 \,.\, h_2\, da_2 \,.\, h_3\, da_3)$$

$$= \frac{\partial \rho}{\partial t} \,.\, h_1 h_2 h_3\, da_1 \,.\, da_2 \,.\, da_3 \,.$$

Equation of continuity says that

Increase in mass = total excess of flow in over flow out

i.e.,
$$\frac{\partial \rho}{\partial t} h_1 h_2 h_3\, da_1\, da_2\, da_3 = -\left[\frac{\partial}{\partial a_1}(\rho q_1 h_2 h_3) + \frac{\partial}{\partial a_2}(\rho q_2 h_1 h_3) + \frac{\partial}{\partial a_3}(\rho q_3 h_1 h_2)\right] da_1\, da_2\, da_3$$

or
$$\frac{\partial \rho}{\partial t} + \left[\frac{\partial}{\partial a_1}(\rho q_1 h_2 h_3) + \frac{\partial}{\partial a_2}(\rho q_2 h_1 h_3) + \frac{\partial}{\partial a_3}(\rho q_3 h_1 h_2)\right] \frac{1}{h_1 h_2 h_3} = 0$$

This is the required equation of continuity.

Deductions : (i) Rectangular cartesian co-ordinates :

$$ds^2 = dx^2 + dy^2 + dz^2 = (h_1\, da_1)^2 + (h_2\, da_2)^2 + (h_3\, da_3)^2.$$

Hence $\quad h_1 = h_2 = h_3 = 1,\ a_1 = x,\ a_2 = y,\ a_3 = z.$

In this case the equation of continuity becomes

$$\frac{\partial \rho}{\partial t} + \left[\frac{\partial}{\partial x}(\rho q_1) + \frac{\partial}{\partial y}(\rho q_2) + \frac{\partial}{\partial z}(\rho q_3)\right] = 0$$

(ii) Spherical co-ordinates :

Here $\quad ds^2 = (dr)^2 + (r\,d\theta)^2 + (r \sin\theta\, d\omega)^2.$

Then $\quad h_1 = 1,\ h_2 = r,\ h_3 = r\sin\theta,\ a_1 = r,\ a_2 = \theta,\ a_3 = \omega.$

In this case the equation of continuity becomes

$$\frac{\partial \rho}{\partial t} + \frac{1}{1 \,.\, r \,.\, r\sin\theta}\left[\frac{\partial}{\partial r}(\rho q_1 \,.\, r \,.\, r\sin\theta) + \frac{\partial}{\partial \theta}(\rho q_2 \,.\, 1 \,.\, r\sin\theta) + \frac{\partial}{\partial \omega}(\rho q_3 \,.\, 1 \,.\, r)\right] = 0$$

or
$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(\rho q_1 r^2) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(\rho q_2 \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \omega}(\rho q_3) = 0$$

(iii) Cylindrical co-ordinates. Here we have

$$ds^2 = (dr)^2 + (r\,d\theta)^2 + (dz)^2.$$

Then $h_1 = 1, h_2 = r, h_3 = 1, a_1 = r, a_2 = \theta, a_3 = z.$

The equation of continuity is

$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\left[\frac{\partial}{\partial r}(\rho q_1 r) + \frac{\partial}{\partial \theta}(\rho q_2) + \frac{\partial}{\partial z}(\rho q_3 r)\right] = 0$$

or
$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(\rho q_1 r) + \frac{1}{r}\frac{\partial}{\partial \theta}(\rho q_2) + \frac{\partial}{\partial z}(\rho q_2) = 0$$

Equation of continuity in cartesian co-ordinates

Let ρ denote fluid density at $P(x, y, z)$ enclosed by a small parallelopiped with edges of lengths $\delta x, \delta y, \delta z$. Let u, v, w be velocity components along AA', AP, AB respectively. Mass of the fluid that passes in unit time across the face $APCB$

$$= \text{density . area . normal velocity}$$

$$= \rho \,.\, \delta y \,.\, \delta z \,.\, u = f(x, y, z), \text{ say.}$$

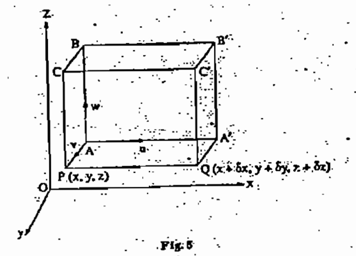


Fig. 5

Mass of the fluid that passes in unit time across the face $QA'B'C'$

$$= f(x + \delta x, y, z) = f + \delta x \,.\, \frac{\partial f}{\partial x}$$

Now the excess of flow in flow out from the face $APCB$ and $QA'B'C'$ in unit time

$$= f - \left(f + \delta x \,.\, \frac{\partial f}{\partial x}\right) = -\delta x \,.\, \frac{\partial f}{\partial x} = -\delta x \,.\, \frac{\partial}{\partial x}(\rho u\, \delta y \,.\, \delta z)$$

$$= -\frac{\partial (\rho u)}{\partial x} \,.\, \delta x\, \delta y\, \delta z.$$

Similarly, the excess of flow in over flow out from the faces $CC'B'B$, $PQA'A$ and $AA'B'B$, $CC'QP$ is respectively

$$-\frac{\partial}{\partial z}(\rho w) \,.\, \delta x\, \delta y\, \delta z \quad \text{and} \quad -\frac{\partial}{\partial y}(\rho v)\, \delta x\, \delta y\, \delta z.$$

Rate of increment in mass of the fluid within the parallelopiped

$$= \frac{\partial}{\partial t}(\rho\, \delta x \,.\, \delta y \,.\, \delta z) = \frac{\partial \rho}{\partial t} \,.\, \delta x \,.\, \delta y \,.\, \delta z$$

Equation of continuity says that

Increase in mass = total excess of flow in over flow out i.e.,

$$\frac{\partial \rho}{\partial t}\, \delta x\, \delta y\, \delta z = -\delta x\, \delta y\, \delta z\left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right]$$

or
$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0. \qquad \ldots(1)$$

This is the required equation of continuity.

Deductions : (i) If the fluid is incompressible, then (1) becomes

$$0 + \rho\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right] = 0$$

or
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

This is the equation of continuity in this case.

(ii) The equation (1) is also expressible as

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)\rho + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

(iii) If velocity has one component u, say, then (1) becomes

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho u}{\partial x} = 0.$$

This equation is very important for further study.

Equation of continuity in spherical polar co-ordinates

To derive the equation of conservation of mass in spherical co-ordinates.

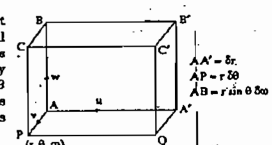


Fig. 6

Let ρ denote fluid density at a point $P(r, \theta, \omega)$ enclosed by a small parallelopiped with edges of lengths $\delta r, r\,\delta\theta, r\sin\theta\,\delta\omega$. Let u, v, w be velocity components along AA', AP, AB respectively. Mass of the fluid that passes in unit time across the face $APCB$ is density . area . normal velocity

$$= \rho \,.\, (r\,\delta\theta \,.\, r\sin\theta\,\delta\omega) \,.\, u$$

$$= \rho\, r^2 u \sin\theta\, \delta\theta . \delta\omega = f(r, \theta, \omega), \text{ say.}$$

Mass of the fluid that passes in unit time across the face $A'QC'B'$ is

$$f(r + \delta r, \theta, \omega) = f + \delta r \,.\, \frac{\partial f}{\partial r} \,.$$

Now excess of flow in over flow out from the faces $APCB$, $A'QC'B'$ in unit time

$$= f - \left(f + \delta r \,.\, \frac{\partial f}{\partial r}\right) = -\delta r \,.\, \frac{\partial f}{\partial r}$$

$$= -\delta r \,.\, \frac{\partial}{\partial r}(\rho\, r^2 u \sin\theta\, \delta\theta . \delta\omega)$$

$$= -\delta r \,.\, \frac{\partial}{\partial r}(\rho u \,.\, r\,\delta\theta \,.\, r\sin\theta\,\delta\omega).$$

Similarly, the excess of flow in over flow out from the faces $APQA'$, $CC'B'B$ and $AA'B'B$, $PQC'C$ are, respectively

$$-r\sin\theta\,\delta\omega \,.\, \frac{\partial}{r\sin\theta\,\partial\omega}(\rho w \,.\, r\,\delta\theta \,.\, \delta r)$$

and
$$-r\,\delta\theta \,.\, \frac{\partial}{r\,\partial\theta}(\rho v \,.\, \delta r \,.\, r\sin\theta\,\delta\omega)$$

Total excess of flow in over flow out

$$= -\delta r \,.\, \frac{\partial}{\partial r}(\rho u r^2 \sin\theta\, \delta\theta . \delta\omega) - \delta\omega\frac{\partial}{\partial \omega}(\rho w\, r\,\delta\theta\, \delta r) - \delta\theta \,.\, \frac{\partial}{\partial \theta}(\rho v\, r\sin\theta\, \delta r\, \delta\omega)$$

$$= -\left[\frac{\partial}{\partial r}(\rho u r^2) \,.\, \sin\theta + r\frac{\partial}{\partial \theta}(\rho v \sin\theta) + r\frac{\partial}{\partial \omega}(\rho w)\right] \,.\, \delta r \,.\, \delta\theta \,.\, \delta\omega$$

Rate of increment in mass of the fluid within the parallelopiped

$$= \frac{\partial}{\partial t}(\rho\, \delta r \,.\, r\,\delta\theta \,.\, r\sin\theta\,\delta\omega)$$

$$=\frac{\partial \rho}{\partial t}\cdot r^2 \sin\theta\,\delta r\,.\,\delta\theta\,.\,\delta\omega.$$

By equation of continuity

$$\frac{\partial \rho}{\partial t} r^2 \sin\theta\,\delta r\,\delta\theta\,\delta\omega = -\left[\frac{\partial}{\partial r}(\rho u r^2)\,.\,\sin\theta + r\frac{\partial}{\partial\theta}(\rho v \sin\theta) + r\frac{\partial(\rho\omega)}{\partial\omega}\right]\delta r\,\delta\theta\,\delta\omega$$

Simplifying this we get

$$\frac{\partial \rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(\rho u r^2)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\rho v\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\omega}(\rho\omega)=0.$$

This is the required equation of continuity.

Problem 1. *Each particle of a mass of liquid moves in a plane through axis of z; find the equation of continuity.*

Solution : Prove as in above Article 1.20 that

$$\frac{\partial \rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(\rho u r^2)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\rho v\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\omega}(\rho w)=0$$

Fluid particles move along the axis of z and hence $w=0$.

Equation of continuity is

$$\frac{\partial \rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(\rho u r^2)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\rho v\sin\theta)=0.$$

Problem 2. *Homogeneous liquid moves so that the path of any particle P lies in the plane POX, where OX is fixed axis.*

Prove that if $OP=r$, $\angle POX=\theta$, $\mu=\cos\theta$, the equation of continuity is

$$\frac{\partial}{\partial r}(r^2 q_r)-\frac{\partial}{\partial\mu}(rq_\theta\sin\theta)=0,$$

where q_r, q_θ are the components of velocity along and prependicular to OP in the plane POX.

Solution : Here motion lies in xy-plane.

Hence $w=0$. Prove as a Article 1.20 that

$$\frac{\partial \rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(\rho u r^2)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\rho v\sin\theta)+\frac{1}{r\sin\theta}\frac{\partial(\rho\omega)}{\partial\omega}=0$$

Put $\quad w=0,\quad \rho=\text{const. so that } \frac{\partial\rho}{\partial t}=0.$

we get

$$\frac{1}{r^2}\frac{\partial}{\partial r}(\rho u r^2)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\rho v\sin\theta)=0$$

$$\frac{\partial}{\partial r}(ur^2)+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(rv\sin\theta)=0 \qquad \dots(1)$$

But $\quad \mu=\cos\theta \Rightarrow d\mu=-\sin\theta\,d\theta$

$$\Rightarrow \frac{\partial}{\partial\mu}=-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}$$

Also $u=q_r$, $v=q_\theta$. With these values (1) becomes

$$\frac{\partial}{\partial r}(r^2q_r)-\frac{\partial}{\partial\mu}(rq_\theta\sin\theta)=0.$$

Equation of continuity in cylindrical co-ordinates

Let ρ denote fluid density at a point $P(r,\theta,z)$ enclosed by a small parallelopiped with edges of lengths $\delta r, r\delta\theta, \delta z$. Let u, v, w be velocity components along AA', AP, AB, respectively. Mass of the fluid that passes in unit time across the face $APCB$ is

density . area . normal velocity

$$=\rho\,.\,r\delta\theta\,\delta z\,u$$

$$=f(r,\theta,z),\text{ say.}$$

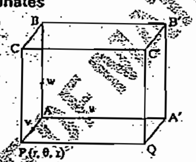


Fig. 7

Mass of the fluid that passes in unit time from the face $A'QC'B'$ is

$$f(r+\delta r,\theta,z)=f+\delta r\,\frac{\partial f}{\partial r}.$$

Now excess of flow in over flow out from the faces $APCB$ and $A'QC'B'$ in unit time

$$=f-\left(f+\delta r\,\frac{\partial f}{\partial r}\right)=-\delta r\,\frac{\partial f}{\partial r}=-\delta r\frac{\partial}{\partial r}(\rho u\,r\delta\theta\,.\,\delta z).$$

Similarly, the excess of flow in over flow out from the faces $AA'B'B$, $PQC'C$ and $PAA'Q$, $CC'B'B$ are, respectively

$$-r\,\delta\theta\,.\,\frac{\partial}{r\,\delta\theta}(\rho v\,.\,\delta r\,.\,\delta z)\quad\text{and}\quad -\delta z\frac{\partial}{\partial z}(\rho w\,\delta r\,.\,r\,\delta\theta).$$

Hence total excess of flow in over flow out

$$=-\left[\delta r\,.\,\frac{\partial}{\partial r}(\rho u\,r\,\delta\theta\,.\,\delta z)+\delta\theta\,.\,\frac{\partial}{\partial\theta}(\rho v\,\delta r\,.\,\delta z)+\delta z\frac{\partial}{\partial z}(\rho w\,\delta r\,.\,r\,\delta\theta)\right]$$

$$=-\left[\frac{\partial}{\partial r}(\rho ur)+\frac{\partial}{\partial\theta}(\rho v)+\frac{\partial}{\partial z}(\rho w)\,.\,r\right]\delta r\,.\,\delta\theta\,.\,\delta z.$$

Rate of increment in mass of the fluid within the parallelopiped

$$=\frac{\partial}{\partial t}(\rho\,\delta r\,.\,r\,\delta\theta\,.\,\delta z)$$

$$=\frac{\partial\rho}{\partial t}\,.\,r\,\delta r\,\delta\theta\,\delta z.$$

By equation of continuity,

$$\frac{\partial\rho}{\partial t}r\,\delta r\,\delta\theta\,\delta z=-\left[\frac{\partial}{\partial r}(\rho u r)+\frac{\partial}{\partial\theta}(\rho v)+\frac{\partial}{\partial z}(\rho w)\,.\,r\right]\delta r\,\delta\theta\,\delta z$$

or $$\frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(\rho u r)+\frac{1}{r}\frac{\partial}{\partial\theta}(\rho v)+\frac{\partial}{\partial z}(\rho w)=0.$$

This is the required equation of continuity.

Certain Symmetrical forms of equations of continuity

1. Spherical Symmetry

The motion is symmetrical about the centre of the sphere and velocity q has only one component along the radius r. Also $q=q(r,t)$. We consider two consecutive spheres of radii r and $r+\delta r$. Mass of the fluid which passes in unit time across the inner sphere is density . area . normal velocity

$$=\rho\,.\,4\pi r^2\,.\,q=f(r,t),\text{ say.}$$

Mass of the fluid that passes across the outer sphere in unit time

$$=f(r+\delta r,t)=f+\delta r\,.\,\frac{\partial f}{\partial r}.$$

The excess of flow in over flow out from these two faces

$$=f-\left(f+\delta r\,.\,\frac{\partial f}{\partial r}\right)=-\delta r\,.\,\frac{\partial f}{\partial r}$$

$$=-\delta r\,.\,\frac{\partial}{\partial r}(\rho\,4\pi r^2 q)=-\frac{\partial}{\partial r}(\rho r^2 q)\,4\pi\,\delta r.$$

Rate of increment in the mass of the fluid within the spheres

$$=\frac{\partial}{\partial t}(4\pi r^2\,\delta r\,.\,\rho)$$

$$=\frac{\partial\rho}{\partial t}\,.\,4\pi r^2\,\delta r.$$

By the def. of equation of continuity

$$\frac{\partial\rho}{\partial t}4\pi r^2\,\delta r=-4\pi\,\delta r\,.\,\frac{\partial}{\partial r}(\rho q r^2)$$

or $$\frac{\partial\rho}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(\rho q r^2)=0 \qquad \dots(1)$$

This is the required equation of continuity.

Deductions : (i). If the fluid is incompressible, then the last becomes

$$0+\frac{\rho}{r^2}\frac{\partial}{\partial r}(r^2 q)=0\quad\text{or}\quad\frac{\partial}{\partial r}(r^2q)=0$$

Integrating, $\quad r^2q=\text{const.}=f(t)\quad$ or $\quad r^2q=f(t).$

(ii) Problem : *The particles of fluid move symmetrically in space with regard to fixed sphere, show that equation of continuity is*

$$\frac{\partial\rho}{\partial t}+u\frac{\partial\rho}{\partial r}+\frac{\rho}{r^2}\frac{\partial}{\partial r}(r^2u)=0.$$

This follows from equation (1) and there replace q by u.

2. Cylindrical symmetry : In this case velocity q at any point is perpendicular to a fixed axis and is a function of r and t only, where r is perpendicular distance of the point from the axis. Consider two consecutive cylinders of radii r and $r+\delta r$ bounded by the planes at unit distance apart. Flow across the inner surface

$$=\rho\,.\,2\pi r\,.\,q=f(r,t)\text{ say.}$$

Flow across outer surface

$$=f(r+\delta r,t)=f+\delta r\,.\,\frac{\partial f}{\partial r}.$$

Excess of flow in over flow out

$$=f-\left(f+\delta r\frac{\partial f}{\partial r}\right)=-\delta r\,.\,\frac{\partial f}{\partial r}=-2\pi\,\delta r\,.\,\frac{\partial}{\partial r}(\rho rq)$$

Rate of increment in the mass of the fluid contained in the cylinders

$$=\frac{\partial}{\partial t}(\rho\,.\,2\pi r\,.\,\delta r)=\frac{\partial\rho}{\partial t}\,.\,2\pi r\,\delta r$$

By the def. of equation of continuity,

$$\frac{\partial\rho}{\partial t}\,.\,2\pi r\,\delta r=-2\pi\,\delta r\,.\,\frac{\partial}{\partial r}(\rho rq)$$

or $$\frac{\partial\rho}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(\rho rq)=0$$

This is the required equation of continuity.

Deduction : When ρ is constant, then the last gives

$$0+\frac{1}{r}\,.\,\rho\frac{\partial}{\partial r}(qr)=0\quad\text{or}\quad\frac{\partial}{\partial r}(rq)=0$$

Integrating $\quad rq=\text{const.}=f(t)\quad$ or $\quad rq=f(t).$

Solved problems related to stream lines and possible liquid motion :

Problem 1. *Find the stream lines and paths of the particles for the two dimensional velocity field :*

$$u=\frac{x}{1+t},\quad v=y,\quad w=0.$$

Solution : We have

$$u=\frac{x}{1+t},\quad v=y,\quad w=0.$$

Step I. To determine stream lines.

Stream lines are the solution of

$$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w}$$

Putting the value $\frac{(1+t)}{x}dx=\frac{dy}{y}=\frac{dz}{0}.$

This $\quad\Rightarrow\left(\frac{1+t}{x}\right)dx=\frac{dy}{y},\ \frac{dy}{y}=\frac{dz}{0}$

$$\Rightarrow (1+t)\log x=\log y+\log a,\ dz=0$$

$$\Rightarrow x^{1+t}=ay,\quad z=b$$

These two equations represent stream lines.

Step II. To determine path lines.

Path lines are the solutions of

$$\frac{dx}{dt}=\frac{x}{1+t},\quad\frac{dy}{dt}=y,\quad\frac{dz}{dt}=0$$

H.O.: 105-106, Top Floor, Mukherjee Tower, Dr. Mukherjee Nagar, Delhi-9. B.O.: 25/8, Old Rajender Nagar Market, Delhi-60
Ph:. 011-45629987, 09999329111, 09999197625 || Email: ims4ims2010@gmail.com, www.ims4maths.com

This $\Rightarrow \dfrac{dx}{x} = \dfrac{dt}{1+t}, \dfrac{dy}{y} = dt, dz = 0.$

Integrating, $\log x = \log(1+t) + \log a,$

$$\log y = t - \log b, z = c.$$

or $$x = a(1+t), (y/b) = e^t, z = c$$

or $$y = be^{[(x/a)-1]}, z = c.$$

These two equations represent path lines.

Problem 2. *Determine the streamlines and the path of the particles*

$$u = x/(1+t), v = y/(1+t), w = z/(1+t).$$ (IAS-1994)

Solution : The equation of the streamlines are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

or $$\frac{dx}{x/(1+t)} = \frac{dy}{y/(1+t)} = \frac{dz}{z/(1+t)}$$

or $$dx/x = dy/y = dz/z$$
(i) (ii) (iii)

By integrating (i) and (ii), we have

$\log x = \log y + \log A$, A is integration constant.

$\Rightarrow$ $x = Ay$ ... (1)

By integrating (i) and (iii), we have

$\log x = \log z + \log B$, B is an integration constant.

$\Rightarrow$ $x = Bz.$ ... (2)

Hence the streamlines are given by the intersection of (1) and (2). The differential equation of path lines is given by

$$\mathbf{q} = \frac{d\mathbf{r}}{dt}$$

This $$\Rightarrow \frac{dx}{dt} = \frac{x}{1+t}, \frac{dy}{dt} = \frac{y}{1+t}, \frac{dz}{dt} = \frac{z}{1+t}$$

$$\Rightarrow \frac{dx}{x} = \frac{dt}{1+t}, \frac{dy}{y} = \frac{dt}{1+t}, \frac{dz}{z} = \frac{dt}{1+t}$$

Integrating, we get

$$\log x = \log(1+t) + \log a$$
$$\log y = \log(1+t) + \log b$$
$$\log z = \log(1+t) + \log c$$

$\Rightarrow$ $x = a(1+t), y = b(1+t), z = c(1+t)$

These give required path lines.

Problem 3. *The velocity* $\mathbf{q}$ *in three-dimensional flow field for an incompressible fluid is given by*

$$\mathbf{q} = 2x\mathbf{i} - y\mathbf{j} - z\mathbf{k}$$

Determine the equations of the stream lines passing through the point (1, 1, 1).

Solution : The equations of stream lines are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \Rightarrow \frac{dx}{2x} = \frac{dy}{-y} = \frac{dz}{-z}$$
(i) (ii) (iii)

From (i) and (ii), we have

$$\frac{dx}{2x} = \frac{dy}{-y} \Rightarrow \frac{dx}{x} + \frac{2dy}{y} = 0$$

By integrating, we obtain

$$\log x + 2\log y = \log A$$

or $xy^2 = A$, where A is an integration constant.

From (i) and (iii), we have

$$\frac{dx}{2x} = \frac{dz}{-z} \Rightarrow \frac{dx}{x} + \frac{2dz}{z} = 0$$

By integrating, we have

$xz^2 = B$, where B is an integration constant.

At the point $(1, 1, 1)$ $A = 1 = B$

Hence the required stream lines are

$$xy^2 = 1 \quad \text{and} \quad xz^2 = 1.$$

Problem 4. *Find the equation of the stream lines for the flow*

$$\mathbf{q} = -\mathbf{i}(3y^2) - \mathbf{j}(6x)$$

at the point (1, 1).

Solution : The equations of streamline are given by

$$\frac{dx}{u} = \frac{dy}{v}$$

Here $\mathbf{q} = -\mathbf{i}(3y^2) - \mathbf{j}(6x) \Rightarrow u = -3y^2, v = -6x$

or $$\frac{dx}{-3y^2} = \frac{dy}{-6x} \Rightarrow \frac{2dx}{y^2} = \frac{dy}{x} \quad \text{or} \quad 2x\,dx = y^2\,dy$$

By integrating, we have

$x^2 = \frac{1}{3}y^3 + c$, where c is an integration constant.

At the point $(1, 1)$, $c = \frac{2}{3} \Rightarrow 3x^2 = y^3 + 2.$

which determines the equation of the stream lines for the flow field.

Problem 5. *The velocity field at a point in fluid is given as*

$$\mathbf{q} = (x/t, y, 0).$$

Obtain path lines and streak lines.

Solution : Here $\mathbf{q} = (x/t, y, 0).$

The differential equations of path lines are given by

$$\mathbf{q} = \frac{d\mathbf{r}}{dt} = \frac{dx}{dt}\mathbf{i} + \frac{dy}{dt}\mathbf{j} + \frac{dz}{dt}\mathbf{k} = \frac{x}{t}\mathbf{i} + y\mathbf{j}$$

$\Rightarrow$ $$\frac{dx}{dt} = \frac{x}{t}, \frac{dy}{dt} = y, \frac{dz}{dt} = 0.$$ ... (1, 2, 3)

By integrating (1), we have

$$\frac{dx}{dt} = \frac{x}{t} \Rightarrow \log x = \log t + \log A \Rightarrow x = At.$$ ... (4)

Let (x_0, y_0, z_0) be the coordinates of the chosen fluid particle at time $t = t_0$, then

$$x_0 = At_0 \Rightarrow A = \frac{x_0}{t_0}.$$

From (4), we have $x = \frac{x_0}{t_0}t.$

By integrating (2), we have

$$\frac{dy}{y} = dt$$

or $$\log y = t + \log B \Rightarrow y = Be^t$$

At $y = y_0, t = t_0 \Rightarrow B = y_0 e^{-t_0}$ ... (5)

From (5), we have

$$y = y_0 e^{t - t_0}$$

By integrating (3), we have

$\dfrac{dz}{dt} = 0 \Rightarrow z = c$ i.e., z is independent of $t \Rightarrow z = z_0.$

Hence the path lines are given by

$$x = (x_0/t_0)t, y = y_0 e^{t-t_0}, z = z_0$$ ... (6)

Let the fluid particle (x_0, y_0, z_0) pass through a fixed point (x_1, y_1, z_1) at an instant of time $t = T$, where $t_0 \le T \le t$. Then the relation (6) reduces to

$$x_1 = (x_0/t_0)T, y_1 = y_0 e^{T-t_0}, z_1 = z_0$$

or $x_0 = (x_1/T)t_0, y_0 = y_1 e^{t_0 - T}, z_0 = z_1$ ... (7)

where T is the parameter. Substituting the relation (7) into (6), we have

$$x = (x_1/T)t, y = y_1 e^{t-T}, z = z_1,$$

which gives the equation of streak lines passing through the point (x_1, y_1, z_1).

Problem 6. *The velocity components in a two-dimensional flow field for an incompressible fluid are given by*

$$u = e^x \cosh y \quad \text{and} \quad v = -e^x \sinh y.$$

Determine the equation of the stream lines for this flow.

Solution : The equations of the stream lines are given by

$$\frac{dx}{u} = \frac{dy}{v} \Rightarrow \frac{dx}{e^x \cosh y} = \frac{dy}{e^x \sinh y} \quad \text{or} \quad dx + \coth y\,dy = 0$$

By integrating, we have

$$x + \log \sinh y = \log c \Rightarrow \sinh y = ce^{-x}$$

where $\log c$ is an integration constant.

Problem 7. *Obtain the stream lines of a flow*

$$u = x, \quad v = -y.$$

Or, If the velocity $\mathbf{q}$ *is given by*

$$\mathbf{q} = x\mathbf{i} - y\mathbf{j},$$

determine the equation of the stream lines.

Solution : $\mathbf{q} = \mathbf{i}u + \mathbf{j}v + w\mathbf{k}$

Here we have $u = x, \quad v = -y, \quad w = 0.$

Stream lines are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

or $$\frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{0}$$

$\Rightarrow$ $$\frac{dx}{x} = \frac{dy}{-y}, \frac{dx}{x} = \frac{dz}{0}$$ (i.e., $dz = 0$)

Integrating these equations,

$$\log x + \log y = \log c, \quad z = c_1$$

or $$xy = c, \quad z = c_1$$

Stream lines are given by $xy = c, \quad z = c_1.$

Problem 8. *Consider the velocity field given by*

$$\mathbf{q} = (1 + At)\mathbf{i} + x\mathbf{j}.$$

Find the equation of stream line at $t = t_0$ passing through the point (x_0, y_0). Also obtain the equation of path line of a fluid element which comes to (x_0, y_0) at $t = t_0$. Show that, if $A = 0$ (i.e., steady flow), the stream lines and path lines coincide.

Solution : $\mathbf{q} = (1 + At)\mathbf{i} + x\mathbf{j}$

and $\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$

This $\Rightarrow$ $u = 1 + At, \quad v = x, \quad w = 0.$

I. To determine stream lines.

These lines are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

Stream lines at time $t = t_0$ are given by

$$\frac{dx}{1 + At_0} = \frac{dy}{x}$$

in two dimensional motion.

or $$x\,dx = (1 + At_0)\,dy$$

Integrating $\frac{x^2}{2} = (1 + At_0)\, y + \frac{c}{2}$

or $$x^2 = 2(1 + At_0)\, y + c \quad \ldots (1)$$

$$x_0^2 = 2(1 + At_0)\, y_0 + c \quad \ldots (2)$$

(1) – (2) gives

$$x^2 - x_0^2 = 2(1 + At_0)(y - y_0)$$

II. To find path lines which pass through (x_0, y_0) at time $t = t_0$.

Equations of path lines are $\dot{x} = u, \quad \dot{y} = v$.

or $$\frac{dx}{dt} = 1 + At, \quad \frac{dy}{dt} = x$$

$\Rightarrow$ $$dx = (1 + At)\, dt \quad \ldots (3)$$

$$dy = x\, dt \quad \ldots (4)$$

Integrating (3), we get

$$x = t + \frac{A}{2} t^2 + c_1 \quad \ldots (5)$$

Put $x = x_0, \quad t = t_0$

$$x_0 = t_0 + \frac{A}{2} t_0^2 + c_1 \quad \ldots (6)$$

(5) – (6) gives

$$x - x_0 = (t - t_0) + \frac{A}{2}(t^2 - t_0^2) \quad \ldots (7)$$

Using (7) in (4),

$$dy = \left[x_0 + (t - t_0) + \frac{A}{2}(t^2 - t_0^2)\right] dt$$

Integrating, $$y = x_0 t + \frac{t^2}{2} - t_0 t + \frac{A}{2}\left(\frac{t^3}{3} - t_0^2 t\right) + c_2 \quad \ldots (8)$$

Putting, $y = y_0, \quad t = t_0$, we get

$$y_0 = x_0 t_0 + \frac{t_0^2}{2} - t_0^2 + \frac{A}{2}\left(\frac{t_0^3}{3} - t_0^3\right) + c_2 \quad \ldots (9)$$

(8) – (9) gives

$$y - y_0 = x_0 (t - t_0) + \frac{1}{2}(t^2 - t_0^2) - t_0 (t - t_0) + \frac{A}{2}\left[\left(\frac{t^3 - t_0^3}{3}\right) - t_0^2 (t - t_0)\right]$$

or $$y - y_0 = (t - t_0)\left[x_0 + \frac{1}{2}(t + t_0) - t_0 + \frac{A}{2}\left\{\left(\frac{t_0^2 + t^2 + tt_0}{3}\right) - t_0^2\right\}\right]$$

or $$y - y_0 = (t - t_0)\left[x_0 + \frac{1}{2}(t - t_0) + \frac{A}{6}\{t^2 + tt_0 - 2t_0^2\}\right] \quad \ldots (10)$$

Required path lines are given by (7) and (10).

III. Let $A = 0$.

To show that path lines and stream lines are coincident.

By step I, stream lines

$$x^2 - x_0^2 = 2(1 + At_0)(y - y_0)$$

or $$x^2 - x_0^2 = 2(y - y_0) \quad \ldots (11)$$

By step II, path lines are given by

$$y - y_0 = (t - t_0)\left[x_0 + \frac{1}{2}(t - t_0) + \frac{A}{6}\{t^2 + tt_0 - 2t_0^2\}\right] \quad \text{by (10)}$$

and $$x - x_0 = (t - t_0) + \frac{A}{2}(t^2 - t_0^2), \quad \text{by (7)}$$

This $\Rightarrow$ $$y - y_0 = (t - t_0)\left[x_0 + \frac{1}{2}(t - t_0)\right]$$

and $$x - x_0 = t - t_0$$

Eliminating $t - t_0$ from the last two equations,

$$y - y_0 = (x - x_0)\left[x_0 + \frac{1}{2}(x - x_0)\right]$$

or $$2(y - y_0) = x^2 - x_0^2,$$

which is the same as equation (11). Hence stream lines and path lines are coincident.

Problem 9. *Prove that liquid motion is possible when velocity at (x, y, z) is given by*

$$u = \frac{3x^2 - r^2}{r^5}, \quad v = \frac{3xy}{r^5}, \quad w = \frac{3xz}{r^5}, \quad \text{where } r^2 = x^2 + y^2 + z^2$$

and the stream lines are the intersection of the surfaces, $(x^2 + y^2 + z^2)^3 = c(y^2 + z^2)^2$, *by the planes passing through Ox.*

Solution : Step I. To prove that the liquid motion is possible. For this we have to show that the equation of continuity namely

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \ldots (1)$$

is satisfied.

$$r^2 = x^2 + y^2 + z^2 \quad \Rightarrow \frac{\partial r}{\partial x} = \frac{x}{r}, \frac{\partial r}{\partial y} = \frac{y}{r}, \frac{\partial z}{\partial r} = \frac{z}{r}$$

$$\frac{\partial u}{\partial x} = \frac{(6x - 2x) r^5 - 5r^3 x (3x^2 - r^2)}{r^{10}}, \quad \frac{\partial v}{\partial y} = \frac{3x}{r^{10}}(r^5 - 5r^3 y^2),$$

$$\frac{\partial w}{\partial z} = \frac{3x}{r^{10}}(r^5 - 5r^3 z^2)$$

This $\Rightarrow$ $$\frac{\partial u}{\partial x} = \frac{3x}{r^7}(3r^2 - 5x^2), \quad \frac{\partial v}{\partial y} = \frac{3x}{r^7}(r^2 - 5y^2),$$

$$\frac{\partial w}{\partial z} = \frac{3x}{r^7}(r^2 - 5z^2).$$

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$ Hence the result.

Step II. To determine stream lines.

Stream lines are the solutions of $\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$. Putting the values,

$$\frac{dx}{3x^2 - r^2} = \frac{dy}{3xy} = \frac{dz}{3xz} = \frac{x\, dx + y\, dy + z\, dz}{x(3r^2 - r^2)} = \frac{y\, dy + z\, dz}{3x(y^2 + z^2)}.$$

This $\Rightarrow$ $$\frac{dy}{3xy} = \frac{dz}{3xz} \quad \ldots (2)$$

and $$\frac{x\, dx + y\, dy + z\, dz}{2(x^2 + y^2 + z^2)} = \frac{y\, dy + z\, dz}{3(y^2 + z^2)} \quad \ldots (3)$$

(2) $\Rightarrow \frac{dy}{y} - \frac{dz}{z} = 0$, integrating this $\log \frac{y}{z} = \log a$

or $y = az$ ... (4), this is a plane through Ox.

Integrating (3), we get

$$\frac{1}{2}\log(x^2 + y^2 + z^2) = \frac{1}{3}\log(y^2 + z^2) + \frac{1}{6}\log b$$

or $$(x^2 + y^2 + z^2)^3 = b(y^2 + z^2)^2. \quad \ldots (5)$$

Problem 10. *If the velocity of an incompressible fluid at the point (x, y, z) is given by*

$$\left(\frac{3xz}{r^5}, \frac{3yz}{r^5}, \frac{3z^2 - r^2}{r^5}\right)$$

prove that the liquid motion is possible and the velocity potential is $\cos\theta / r^2$. *Also determine the stream lines.* (IFoS-2009)

Solution : Given $u = \frac{3xz}{r^5}, \quad v = \frac{3yz}{r^5}, \quad w = \frac{3z^2 - r^2}{r^5}$.

Since $r^2 = x^2 + y^2 + z^2$ hence $\frac{\partial r}{\partial x} = \frac{x}{r}$ etc.

Step I. To prove that the liquid motion is possible. For this we have to provo that the equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

is satisfied.

$$\frac{\partial u}{\partial x} = \frac{3z}{r^{10}}(r^5 - 5r^3 x^2); \quad \frac{\partial v}{\partial y} = \frac{3z}{r^{10}}(r^5 - 5r^3 y^2),$$

$$\frac{\partial w}{\partial z} = \frac{1}{r^{10}}[(6z - 2z) r^5 - 5r^3 (3z^2 - r^2) z]$$

This $\Rightarrow$ $$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{3z}{r^{10}}[2r^5 - 5r^3 (r^2 - z^2)] + \frac{1}{r^{10}}(9zr^5 - 15r^3 z^3] = 0.$$

Hence the result.

Step II. The show that $\phi = \cos\frac{\theta}{r^2}$.

$$\partial\phi = \frac{\partial\phi}{\partial x} dx + \frac{\partial\phi}{\partial y} dy + \frac{\partial\phi}{\partial z} dz = -u\, dx - v\, dy - w\, dz$$

$$= -\frac{1}{r^5}[3xz\, dx + 3yz\, dy + (3z^2 - r^2)\, dz]$$

$$= -\frac{1}{r^5}[3z(x\, dx + y\, dy + z\, dz) - r^2\, dz]$$

$$= -\frac{1}{r^5}\left[3z\, d\left(\frac{r^2}{2}\right) - r^2\, dz\right]$$

$$= -\frac{3z}{r^4} dr + \frac{dz}{r^3} = d\left(\frac{z}{r^3}\right).$$

Integrating, $\phi = \frac{z}{r^3} = \frac{r\cos\theta}{r^3} = \frac{\cos\theta}{r^2}$, neglecting constant of integration.

Aliter. $$\frac{\partial\phi}{\partial x} = -u = \frac{-3xz}{r^5}$$

Integrating w.r.t. x,

$$\phi = -\frac{3z}{2}\int (2x)(x^2 + y^2 + z^2)^{-5/2}\, dx$$

$$= \left(\frac{-3z}{2}\right)\left(\frac{-2}{3}\right)(x^2 + y^2 + z^2)^{-3/2}$$

or $$\phi = \frac{z}{(x^2 + y^2 + z^2)^{3/2}} = \frac{z}{r^3} = \frac{r\cos\theta}{r^3} = \frac{\cos\theta}{r^2},$$

on neglecting constant of integration.

Step III. Stream lines are the solutions of

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}.$$

Putting the values of respective terms,

$$\underset{(1)}{\frac{dx}{3xz}} = \underset{(2)}{\frac{dy}{3yz}} = \underset{(3)}{\frac{dz}{3z^2 - r^2}} = \underset{(4)}{\frac{x\, dx + y\, dy + z\, dz}{3z(x^2 + y^2 + z^2) - r^2 z}}.$$

Taking the ratios (1) and (2), $\frac{dx}{x} = \frac{dy}{y}$.

Integration yields the result

$$\log x = \log y + \log a \quad \text{or} \quad x = ay. \quad \ldots (5)$$

By (1) and (4),

$$\frac{dx}{3x} = \frac{x dx + y dy + z dz}{2r^2}$$

or $$\frac{4dx}{x} = 3\left(\frac{2x\, dx + 2y\, dy + 2z\, dz}{x^2 + y^2 + z^2}\right).$$

Integrating, $4\log x = 3\log(x^2 + y^2 + z^2) + \log b$.

or $$x^4 = b\,(x^2+y^2+z^2)^3. \qquad \dots (6)$$

The (5) and (6) equations represent stream lines.

Problem 11. *Show that if velocity potential of an irrotational fluid motion is equal to $A\,(x^2+y^2+z^2)^{-3/2}\,z\tan^{-1}(y/x)$, the lines of flow lie on the series of the surfaces.*

$$x^2+y^2+z^2 = K^{2/3}(x^2+y^2)^{2/3}. \qquad \text{(IAS–2008 model)}$$

Solution : Spherical co-ordinates are

$$x = r\sin\theta\cos\omega,\ y = r\sin\theta\sin\omega,\ z = r\cos\theta.$$

$$\phi = A\,(x^2+y^2+z^2)^{-3/2}\,z\tan^{-1}\frac{y}{x}$$

$$= A\,r^{-3}\,.\,r\cos\theta\tan^{-1}(\tan\omega)$$

or $\phi = A\,r^{-2}\,\omega\,.\cos\theta$. Lines of flow are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

or equivalently, $$\frac{dr}{-\frac{\partial\phi}{\partial r}} = \frac{r\,d\theta}{-\frac{1}{r}\frac{\partial\phi}{\partial\theta}} = \frac{r\sin\theta\,d\omega}{-\frac{1}{r\sin\theta}\frac{\partial\phi}{\partial\omega}}$$

or $$\frac{dr}{\frac{2A\,\omega\cos\theta}{r^3}} = \frac{r\,d\theta}{\frac{1}{r}\frac{A}{r^2}\omega\sin\theta} = \frac{r\sin\theta\,d\omega}{-\frac{1}{r\sin\theta}\cdot\frac{A\cos\theta}{r^2}}$$

or $$\underset{(1)}{\frac{dr}{2\omega\cos\theta}} = \underset{(2)}{\frac{r\,d\theta}{\omega\sin\theta}} = \underset{(3)}{\frac{r\sin^2\theta\,d\omega}{-\cos\theta}}$$

By (1) and (2), $$\frac{dr}{r} = \frac{2\cos\theta}{\sin\theta}\,d\theta.$$

Integrating, $$\log r = 2\log\sin\theta + \log K$$

or $$r = K\sin^2\theta = K\left(\frac{x^2+y^2}{r^2}\right)$$

or $$r^3 = K\,(x^2+y^2)$$

or $$(x^2+y^2+z^2)^{3/2} = K\,(x^2+y^2)$$

or $$x^2+y^2+z^2 = K^{2/3}(x^2+y^2)^{2/3}.$$

Stream lines lie on this surface.

Problem 12. *Given $u = -c^2y/r^2$, $v = c^2x/r^2$, $w = 0$, where r denotes distance from z axis. Find the surfaces which are orthogonal to stream lines, the liquid being homogeneous.* (IAS–2003)

Solution : Step I : To show that liquid motion is possible, we have to show that the equation of continuity $\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0\right)$ is satisfied.

Here $$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = \frac{2c^2y}{r^3}\cdot\frac{x}{r} - \frac{2c^2x}{r^3}\cdot\frac{y}{r} + 0 = 0$$

as $r^2 = x^2+y^2$. Hence result I.

Step II : The surfaces orthogonal to stream lines are the solutions of

$$u\,dx + v\,dy + w\,dz = 0$$

i.e., $$-\frac{c^2y}{r^2}dx + \frac{c^2x}{r^2}dy + 0\,dz = 0$$

or $-\frac{dx}{x} + \frac{dy}{y} = 0$, integrating this $\log\frac{y}{x} = \log a$

or $\frac{y}{x} = a$ or $y = ax$.

This surface is orthogonal to stream lines.

Problem 13. *Show that*

$$u = -\frac{2xyz}{(x^2+y^2)^2},\quad v = \frac{(x^2-y^2)\,z}{(x^2+y^2)^2},\quad w = \frac{y}{x^2+y^2}$$

are the velocity components of a possible liquid motion. Is this motion irrotational ?

Solution : Step I : To show that the motion is possible, we have to show that the equation of continuity $\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0\right)$ is satisfied.

Here $$\frac{\partial u}{\partial x} = -\frac{2yz}{(x^2+y^2)^4}\left[(x^2+y^2)^2 - 2\,(x^2+y^2)\,2x^2\right]$$

$$= \frac{2yz}{(x^2+y^2)^3}(y^2-3x^2).$$

$$\frac{\partial v}{\partial y} = \frac{z}{(x^2+y^2)^4}\left[-2y\,(x^2+y^2)^2 - (x^2-y^2)\,2\,(x^2+y^2)\,2y\right]$$

$$= -\frac{2yz}{(x^2+y^2)^3}(3x^2-y^2)$$

$\therefore$ $$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = \frac{2yz}{(x^2+y^2)^3}\left[(3x^2-y^2)+(y^2-3x^2)+0\right] = 0.$$

Hence the result I.

Step II. To test the nature of the motion. The motion will be irrotational if

$$\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} = 0,\quad \frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} = 0,\quad \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} = 0$$

$$\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = -\frac{2xz\,(x^2-3y^2)}{(x^2+y^2)^2} - \frac{2xz\,(3y^2-x^2)}{(x^2+y^2)^2} = 0$$

$$\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} = \frac{x^2-y^2}{(x^2+y^2)^2} - \frac{x^2-y^2}{(x^2+y^2)^2} = 0$$

$$\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} = -\frac{2xy}{(x^2+y^2)^2} + \frac{2xy}{(x^2+y^2)^2} = 0.$$

Hence the motion is irrotational.

Problem 14. *Given $u = -\omega y$, $v = \omega x$, $w = 0$; show that the surfaces intersecting the stream lines orthogonally exist and are the planes through z-axis, although the velocity potential does not exist.*

Solution : Step I. To show that liquid motion is possible, we have to show that the equation of continuity $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$ is satisfied.

Here $$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0+0+0 = 0.$$ Hence the result I.

Step II. To show that the surfaces orthogonal to stream lines are planes through z-axis.

The required surfaces are solutions of

$$u\,dx + v\,dy + w\,dz = 0,\quad i.e.,$$

$$-\omega y\,dx + \omega x\,dy + o\,dz = 0$$

or $$\frac{dx}{x} - \frac{dy}{y} = 0,$$

integrating $\log\frac{x}{y} = \log a$ or $\frac{x}{y} = a$ or $x = ay$,

which is a plane through z-axis.

Step III. To show that velocity potential ϕ does not exist,

By def., $$d\phi = -(u\,dx + v\,dy + w\,dz)$$

$$= -\left[-\omega y\,dx + \omega x\,dy + o\,dz\right].$$

or $$d\phi = \omega y\,dx - \omega x\,dy = M\,dx + N\,dy,\ \text{say}.$$

Here $\frac{\partial M}{\partial y} = \omega$, $\frac{\partial N}{\partial x} = -\omega$. Hence $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$.

Therefore the equation is not exact so that $d\phi = \omega y\,dx - \omega x\,dy$ can not be integrated so that ϕ does not exist.

Problem 15. *In the steady motion of homogeneous liquid if the surfaces $f_1 = a_1$, $f_2 = a_2$ define the stream lines, prove that the most general values of the velocity components u, v, w are*

$$F(f_1,f_2)\frac{\partial(f_1,f_2)}{\partial(y,z)},\ F(f_1,f_2)\frac{\partial(f_1,f_2)}{\partial(z,x)},\ F(f_1,f_2)\frac{\partial(f_1,f_2)}{\partial(x,y)}.$$

Solution : Since the motion is steady, hence stream lines are independent of t. Therefore f_1 and f_2 are functions of x, y, z only.

$$f_1 = a_1,\ f_2 = a_2 \Rightarrow df_1 = 0,\ df_2 = 0 \Rightarrow$$

$$\frac{\partial f_1}{\partial x}dx + \frac{\partial f_1}{\partial y}dy + \frac{\partial f_1}{\partial z}dz = 0$$

$$\frac{\partial f_2}{\partial x}dx + \frac{\partial f_2}{\partial y}dy + \frac{\partial f_2}{\partial z}dz = 0$$

$$\frac{dx}{\frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial z} - \frac{\partial f_1}{\partial z}\frac{\partial f_2}{\partial y}} = \frac{dy}{\frac{\partial f_2}{\partial z}\frac{\partial f_1}{\partial x} - \frac{\partial f_2}{\partial x}\frac{\partial f_1}{\partial z}} = \frac{dz}{\frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial y} - \frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial x}}$$

or $$\frac{dx}{J_1} = \frac{dy}{J_2} = \frac{dz}{J_3} \qquad \dots (1)$$

where $$J_1 = \frac{\partial(f_1,f_2)}{\partial(y,z)},\ J_2 = \frac{\partial(f_1,f_2)}{\partial(z,x)},\ J_3 = \frac{\partial(f_1,f_2)}{\partial(x,y)}.$$

But the stream lines are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \qquad \dots (2)$$

On comparing (1) and (2), $\frac{u}{J_1} = \frac{v}{J_2} = \frac{w}{J_3} = F$, say.

$\therefore$ $$u = J_1F,\ v = J_2F,\ w = J_3F. \qquad \dots (3)$$

To determine the nature of F.

In order to make the liquid motion possible, the velocity components must satisfy the equation of continuity, namely.

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z} = 0.$$

This $\Rightarrow$ $$F\left(\frac{\partial J_1}{\partial x}+\frac{\partial J_2}{\partial y}+\frac{\partial J_3}{\partial z}\right) + \left(J_1\frac{\partial F}{\partial x} + J_2\frac{\partial F}{\partial y} + J_3\frac{\partial F}{\partial z}\right) = 0$$

By the property of Jacobian, $\frac{\partial J_1}{\partial x}+\frac{\partial J_2}{\partial y}+\frac{\partial J_3}{\partial z} = 0.$

Hence $$\frac{\partial(f_1,f_2)}{\partial(y,z)}\frac{\partial F}{\partial x} + \frac{\partial(f_1,f_2)}{\partial(z,x)}\frac{\partial F}{\partial y} + \frac{\partial(f_1,f_2)}{\partial(x,y)}\frac{\partial F}{\partial z} = 0.$$

or $$\begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} & \frac{\partial F}{\partial z} \\ \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{vmatrix} = 0 \quad \text{or} \quad \frac{\partial(F,f_1,f_2)}{\partial(x,y,z)} = 0.$$

This proves that F, f_1, f_2 are not independent.

Therefore $F = F(f_1, f_2)$. Now (3) proves the required result.

Solved problems related to boundary surface.

Problem 16. *Show that the variable ellipsoid*

$$\frac{x^2}{a^2k^2t^4} + kt^2\left[\left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2\right] = 1$$

is a possible form for the boundary surface of a liquid at any time t.

Solution : Let

$$F(x,y,z,t) = \frac{x^2}{a^2k^2t^4} + kt^2\left[\left(\frac{y}{b}\right)^2 + \left(\frac{z}{c}\right)^2\right] - 1 = 0 \qquad \dots (1)$$

To show that $F = 0$ is a possible form of boundary surface, it is enough to prove that

$$u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} + \frac{\partial F}{\partial t} = 0. \qquad \dots (2)$$

Putting the values of respective terms;

$$\frac{u\,2x}{a^2k^2t^4}+vkt^2\frac{2y}{b^2}+wkt^2\frac{2z}{c^2}-\frac{4x^2}{a^2k^2t^5}+2kt\left[\left(\frac{y}{b}\right)^2+\left(\frac{z}{c}\right)^2\right]=0$$

or $$\frac{2x}{a^2k^2t^4}\left(u-\frac{2x}{t}\right)+\frac{2k}{b^2}t^2y\left(v+\frac{y}{t}\right)+\frac{2k}{c^2}t^2z\left(w+\frac{z}{t}\right)=0$$

Hence (2) is satisfied if we take

$$u-\frac{2x}{t}=0,\quad v+\frac{y}{t}=0,\quad w+\frac{z}{t}=0,$$

i.e., if $$u=\frac{2x}{t},\quad v=-\frac{y}{t},\quad w=-\frac{z}{t}$$

It will be a justificable step if the equation of continuity

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

is satisfied.

Here $$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=\frac{2}{t}-\frac{1}{t}-\frac{1}{t}=0.$$

Hence (1) is a possible form of boundary surface.

Similar Problem : *Show that the ellipsoid*

$$\frac{x^2}{a^2k^2t^{2n}}+kt^n\left(\frac{y^2}{b^2}+\frac{z^2}{c^2}\right)=1$$

is a possible form of boundary surface.

Problem 17. Show that $\frac{x^2}{a^2}\tan^2 t+\frac{y^2}{b^2}\cot^2 t-1=0$ ... (1)

is a possible form of boundary surface and find an expression for normal velocity.

Solution : To show that $F=0$ is a possible form of boundary surface, we have to show that

$$u\frac{\partial F}{\partial x}+v\frac{\partial F}{\partial y}+w\frac{\partial F}{\partial z}+\frac{\partial F}{\partial t}=0 \quad \text{... (2)}$$

Putting the values of various terms, we get

$$u\frac{2x}{a^2}\tan^2 t+v\cdot\frac{2y}{b^2}\cot^2 t+w.0+\left(\frac{2x^2}{a^2}\tan t\sec^2 t-\frac{2y^2}{b^2}\cot t\ \operatorname{cosec}^2 t\right)=0$$

or $$\frac{2x}{a^2}\tan^2 t\left(u+\frac{x\sec^2 t}{\tan t}\right)+\frac{2y}{b^2}\cot^2 t\left(v-\frac{y\operatorname{cosec}^2 t}{\cot t}\right)=0.$$

Thus (2) will be satisfied if we take

$$u+\frac{x\sec^2 t}{\tan t}=0,\quad v-y\frac{\operatorname{cosec}^2 t}{\cot t}=0,$$

i.e., $$u=\frac{-x}{\sin t\cos t},\quad v=\frac{y}{\sin t\cos t}.$$

This will be a justifiable step if the equation of continuity, namely

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0 \text{ is satisfied.}$$

Now

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=-\frac{1}{\sin t\cos t}+\frac{1}{\sin t\cos t}+0=0.$$

Hence (1) is a possible form of boundary surface.

Second Part. Normal velocity $=\frac{-\partial F/\partial t}{|\nabla F|}$

$$=\frac{-\left(\frac{2x}{a^2}\tan t\sec^2 t-\frac{2y}{b^2}\cot t\operatorname{cosec}^2 t\right)}{\left[\left(\frac{2x}{a^2}\tan^2 t\right)^2+\left(\frac{2y}{b^2}\cot^2 t\right)^2\right]^{1/2}}$$

$$=-\frac{(b^2x\tan t\sec^2 t-a^2t\cot t\operatorname{cosec}^2 t)}{(b^4x^2\tan^4 t+a^4y^2\cot^4 t)^{1/2}}$$

Problem 18. *Determine the restriction on f_1, f_2, f_3 if*

$$\frac{x^2}{a^2}f_1(t)+\frac{y^2}{b^2}f_2(t)+\frac{z^2}{c^2}f_3(t)=1$$

is a possible form of boundary surface of a liquid.

Solution : Let $F=\frac{x^2}{a^2}f_1(t)+\frac{y^2}{b^2}f_2(t)+\frac{z^2}{c^2}f_3(t)-1=0$ ... (1)

To show that $F=0$ is a possible form of boundary surface, we have to prove that $(F=0)$ satisfies the condition

$$\frac{\partial F}{\partial t}+u\frac{\partial F}{\partial x}+v\frac{\partial F}{\partial x}+w\frac{\partial F}{\partial x}=0 \quad \text{... (2)}$$

Putting the values of respective terms,

$$\frac{x^2}{a^2}f_1'+\frac{y^2}{a^2}f_2'+\frac{z^2}{c^2}f_3'+u\frac{2x}{a^2}f_1+v\frac{2y}{b^2}f_2+w\frac{2z}{c^2}f_3=0$$

or $$\frac{2x}{a^2}f_1\left(u+\frac{xf_1'}{2f_1}\right)+\frac{2y}{b^2}f_2\left(v+\frac{yf_2'}{2f_2}\right)+\frac{2z}{c^2}f_3\left(w+\frac{zf_3'}{2f_3}\right)=0.$$

If we take $u+x\frac{f_1'}{2f_1}=0,\ v+y\frac{f_2'}{2f_2}=0,\ w+\frac{z}{2f_2}f_3'=0$, then (2) is satisfied. This will be a justifiable step if the values of u, v, w satisfy the equation of continuity.

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0.$$

Putting the values,

$$-\frac{1}{2}\left[\frac{f_1'}{f_1}+\frac{f_2'}{f_2}+\frac{f_3'}{f_3}\right]=0$$

Integrating, $\log f_1f_2f_3=\log c$ or $f_1f_2f_3=c.$

Problem 19. *Show that all necessary and sufficient conditions can be satisfied by a velocity potential of the form $\phi=\alpha x^2+\beta y^2+\gamma z^2$, and the bounding surface of the form*

$$F=ax^4+by^4+cz^4-X(t)=0,$$

where $X(t)$ is a given function of time and $\alpha, \beta, \gamma, a, b, c$ are suitable functions of the time.

Solution : Let $\phi=\alpha x^2+\beta y^2+\gamma z^2$ ... (1)

and $F(x,y,z,t)=ax^4+by^4+cz^4-X(t)=0.$ ... (2)

Step I. To prove that ϕ satisfies all the necessary conditions (*i.e.*, equation of continuity)

or $$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}=0$$

Putting the values of respective terms,

$$2\alpha+2\beta+2\gamma=0 \quad\text{or}\quad \alpha+\beta+\gamma=0.$$

The velocity potential ϕ has to satisfy this condition.

Step II. To prove $F=0$ satisfies the condition of boundary surface. We know that

$$u=-\frac{\partial\phi}{\partial x},\quad v=-\frac{\partial\phi}{\partial y},\quad w=-\frac{\partial\phi}{\partial z}$$

This $\Rightarrow$ $u=-2\alpha x,\quad v=-2\beta y;\quad w=-2\gamma z.$

$F=0$ will be a boundary surface if it satisfies the condition,

$$u\frac{\partial F}{\partial x}+v\frac{\partial F}{\partial y}+w\frac{\partial F}{\partial z}+\frac{\partial F}{\partial t}=0.$$

Putting the values of respective terms,

$$-2\alpha x\cdot 4ax^3-2\beta y\cdot 4by^3-2\gamma z.4cz^3+x^4a'+y^4b'+z^4c'-X'(t)=0$$

or $$x^4(a'-8\alpha a)+y^4(b'-8\beta b)+z^4(c'-8\gamma c)-X'(t)=0 \quad\text{... (3)}$$

Since (2) and (3) both have to hold for all points (x, y, z) on the surface hence they should be identical. Comparing, we get

$$\underset{(4)}{\frac{a'-8\alpha a}{a}}=\underset{(5)}{\frac{b'-8\beta b}{b}}=\underset{(6)}{\frac{c'-8\gamma c}{c}}=\underset{(7)}{\frac{X'(t)}{X(t)}}$$

By (4) and (7), $\frac{da}{dt}-8a\alpha=\frac{a}{X}\cdot\frac{dX}{dt}$

or $$\frac{da}{a}=8\alpha dt+\frac{dX}{X}$$

Integrating, $$\log a=\log X+\int 8\alpha dt.$$

Similarly, $$\log b=\log X+\int 8\beta\, dt,\quad \text{by (5) and (7)}$$

and $$\log c=\log X+\int 8\gamma\, dt,\quad \text{by (6) and (7).}$$

$$\text{s.t. } \alpha+\beta+\gamma=0$$

The surface $F=0$ will have to satisfy these conditions for the possible form of boundary surface.

Problem 20. *Prove that a surface of the form*

$$ax^4+by^4+cz^4-X(t)=0$$

is a possible form of boundary surface of a homogeneous liquid at time t, the velocity potential of the liquid motion being

$$\phi=(\beta-\gamma)x^2+(\gamma-\alpha)y^2+(\alpha-\beta)z^2$$

where X, α, β, γ are given functions of time and a, b, c are suitable functions of time.

Solution : Proceed as above.

Here equation of continuity $\Rightarrow$ $(\beta-\gamma)+\gamma-\alpha+\alpha-\beta=0$

Condition of boundary surface $\Rightarrow$

$$\log a=8\int(\beta-\gamma)\,dt+\log X$$

$$\log b=8\int(\gamma-\alpha)\,dt+\log X$$

$$\log c=8\int(\alpha-\beta)\,dt+\log X$$

Problem 21. *Show that*

$$\frac{x^2}{a^2}f(t)+\frac{y^2}{b^2}\phi(t)=1,$$

where $f(t)\,\phi(t)=$ const. is a possible form of the boundary surface of a liquid. (IAS – 2007)

Solution : Let $F=\frac{x^2}{a^2}f(t)+\frac{y^2}{b^2}\phi(t)-1=0.$ ... (1)

To prove $F=0$ is a possible form of boundary surface. For this we have to prove that

$$u\frac{\partial F}{\partial x}+v\frac{\partial F}{\partial y}+w\frac{\partial F}{\partial z}+\frac{\partial F}{\partial t}=0. \quad\text{... (2)}$$

Putting the values,

$$u\frac{2x}{a^2}f+v\frac{2y}{b^2}\phi+w.0+\frac{x^2}{a^2}f'+\frac{y^2}{b^2}\phi'=0$$

or $$\frac{2x}{a^2}f\left(u+\frac{x}{2}\frac{f'}{f}\right)+\frac{2y\phi}{b^2}\left(v+\frac{y\phi'}{2\phi}\right)=0.$$

If we take $u+\frac{x}{2}\frac{f'}{f}=0,\ v+\frac{y}{2}\cdot\frac{\phi'}{\phi}=0$, then the condition (2) will be satisfied.

Here we get

$$u = -\frac{x}{2} \cdot \frac{f'}{f}, \quad v = -\frac{y}{2} \frac{\phi'}{\phi}$$

This will be a justifiable step if the equation of continuity

$$\frac{\partial u}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial v}{\partial z} = 0$$

is satisfied. Putting the values,

$$-\frac{1}{2} \cdot \frac{f'}{f} - \frac{1}{2} \cdot \frac{\phi'}{\phi} + 0 = 0$$

or $$\frac{df}{f} + \frac{d\phi}{\phi} = 0$$

Integrating, $\log f\phi = \log$ const. or $f\phi$ = const. which is given. Hence (1) is a possible form of boundary surface.

Solved Problems related to equation of continuity :

Problem 22. *A mass of fluid is in motion so that the lines of motion lie on the surface of co-axial cylinders; show that the equation of continuity is*

$$\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial \theta} (\rho v_\theta) + \frac{\partial}{\partial z} (\rho v_z) = 0,$$

where v_θ, v_z are velocities perpendicular and parallel to z.

Solution : Consider a point P whose cylindrical co-ordinates are (r, θ, z). With P as centre, construct a parallelopiped with edges of lengths $dr, r\,d\theta, dz$. Since lines of motion lie on the surface of the cylinders hence the fluid lies on the surface of the cylinders. It means that there is no velocity in the direction of dr. Equation of continuity gives

$$\frac{\partial}{\partial t} (\rho\, dr \,.\, r\,d\theta \,.\, dz)$$

$$= -\left[dr \frac{\partial}{\partial r} (\rho \,.\, 0 \,.\, r\,d\theta \,.\, dz) + r\,d\theta \frac{\partial}{r\,\partial \theta} (\rho v_\theta \,.\, dr \,.\, dz) + dz \frac{\partial}{\partial z} (\rho v_z\, dr \,.\, r \,.\, d\theta) \right]$$

or $$\frac{\partial \rho}{\partial t} + \frac{1}{r}\left[\frac{\partial}{\partial \theta} (\rho v_\theta) + r \frac{\partial}{\partial z} (\rho v_z) \right] = 0$$

or $$\frac{\partial \rho}{\partial t} + \frac{1}{r} \frac{\partial}{\partial \theta} (\rho v_\theta) + \frac{\partial}{\partial z} (\rho v_z) = 0.$$

Problem 23. *If every particle moves on the surface of a sphere, prove that the equation of continuity is*

$$\frac{\partial \rho}{\partial t} \cos \theta + \frac{\partial}{\partial \theta} (\rho \omega \cos \theta) + \frac{\partial}{\partial \phi} (\rho \omega' \cos \theta) = 0,$$

ρ being the density, θ, ϕ the latitude and longitude respectively of an element and ω, ω' the angular velocities of any element in latitude and longitude respectively.

Solution : Step I. To determine the equation of continuity in spherical co-ordinates. Consider an arbitrary point whose polar co-ordinates are (r, θ, ϕ). With P as centre, construct a parallelopiped with edges of lengths $dr, r\,d\theta, r \sin \theta\, d\phi$.

Let q_1, q_2, q_3 be velocity components at P along $dr, r, d\theta, r \sin \theta\, d\phi$, respectively. The equation of continuity gives

$$\frac{\partial}{\partial t} (\rho\, dr \,.\, r\,d\theta \,.\, r \sin \theta\, d\phi)$$

$$= -\left[dr \frac{\partial}{\partial r} (\rho q_1 \,.\, r\,d\theta \,.\, r \sin \theta\, d\phi) + r\,d\theta \frac{\partial}{r\,\partial \theta} (\rho q_2 \,.\, dr \,.\, r \sin \theta\, d\phi) + r \sin \theta\, d\phi \frac{\partial}{r \sin \theta\, \partial \phi} (\rho q_3 \,.\, dr \,.\, r\,d\theta) \right]$$

Simplifying, we get

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2 \sin \theta}\left[\sin \theta \frac{\partial}{\partial r} (\rho q_1 r^2) + r \frac{\partial}{\partial \theta} (\rho q_2 \,.\, \sin \theta) + r \frac{\partial}{\partial \phi} (\rho q_3) \right] = 0$$

or $$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (\rho q_1 r^2) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} (\rho q_2 \,.\, \sin \theta) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} (\rho q_3) = 0 \quad \ldots (1)$$

This is the equation of continuity in spherical co-ordinates.

Step II. To determine the equation of continuity in the required case.

It is given that fluid particles move on the surface of sphere, hence $q_1 = 0$.

To get the equation of continuity in present case, we have to replace θ by $90 - \theta$ in equation (1) and $d\theta$ by $d(90 - \theta) = -d\theta$.

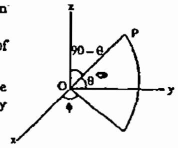


Fig. 8

For OP line makes an angle $90 - \theta$ with z-axis.

$$\dot{\theta} = \omega, \quad \dot{\phi} = \omega'$$

"$q_2 = r\dot{\theta}$" gives $q_2 = r \frac{d}{dt} (90 - \theta) = -r\dot{\theta} = -r\omega$

"$q_3 = r \sin \theta\, \dot{\phi}$" gives

$$q_3 = r \sin (90 - \theta)\, \omega' = (r \cos \theta)\, \omega'$$

Putting these values in (1),

$$\frac{\partial \rho}{\partial t} + 0 + \frac{1}{r \sin (90 - \theta)} \left(-\frac{\partial}{\partial \theta} \left\{ \rho (-r\omega) \cos \theta \right\} \right) + \frac{1}{r \sin (90 - \theta)} \frac{\partial}{\partial \phi} (\rho r \cos \theta\, \omega') = 0$$

or $$\frac{\partial \rho}{\partial t} + \frac{1}{\cos \theta} \frac{\partial}{\partial \theta} (\rho \omega \cos \theta) + \frac{1}{\cos \theta} \frac{\partial}{\partial \phi} (\rho \cos \theta\, \omega') = 0$$

or $$\frac{\partial \rho}{\partial t} \cos \theta + \frac{\partial}{\partial \theta} (\rho \omega \cos \theta) + \frac{\partial}{\partial \phi} (\rho \cos \theta\, \omega') = 0$$

This is the required equation of continuity.

Problem 24. *If the lines of motion are curves on the surface of cones having their vertices at the origin and the axis of z for common axis, prove that the equation of continuity is*

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r} (\rho q_r) + \frac{2 \rho q_r}{r} + \frac{\operatorname{cosec} \theta}{r} \frac{\partial}{\partial \omega} (\rho q_\omega) = 0.$$

Solution : Step I. To derive the equation of continuity in spherical co-ordinates. (Here write Step I of Problem 23).

Step II. To determine the equation of continuity in the required case. It is given that lines of flow lie on the surfaces of cones and hence velocity perpendicular to the surface is zero so that $q_2 = 0$. Now (1) becomes

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (\rho q_1 r^2) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \phi} (\rho q_3) = 0.$$

Replacing q_1 by q_r, q_3 by q_ω and ϕ by ω,

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (\rho r^2 q_r) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \omega} (\rho q_\omega) = 0$$

or $$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial r} (\rho q_r) + \frac{2}{r} \rho q_r + \frac{\operatorname{cosec} \theta}{r} \frac{\partial}{\partial \omega} (\rho q_\omega) = 0.$$

Problem 25. *If the lines of motion are curves on the surfaces of spheres all touching the xy-plane at the origin O, the equation of continuity is*

$$r \sin \theta \frac{\partial \rho}{\partial t} + \frac{\partial \rho v}{\partial \phi} + \sin \theta \frac{\partial (\rho u)}{\partial \theta} + \rho u (1 + 2 \cos \theta) = 0$$

where r is the radius CP of one of the spheres, θ the angle PCO, u the velocity in the plane PCO, v the perpendicular velocity, and φ the inclination of the plane PCO to a fixed plane through z-axis.

Solution : We consider any two consecutive spheres with centres C and C'.

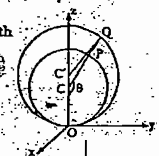


Fig. 9

Let $CP = r$, $C'Q = r + \delta r$, $\angle PCO = \theta$.

Then $CC' = \delta r$, $CQ = CP + PQ = r + PQ$

Since $$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$\therefore$ $$a^2 = b^2 + c^2 - 2bc \cos A$$

Applying this formula in $\Delta C'CQ$,

$$C'Q^2 = CC'^2 + CQ^2 - 2CC'.CQ \cos (\pi - \theta)$$

or $$(r + \delta r)^2 = (\delta r)^2 + (r + PQ)^2 + 2\delta r\, (r + PQ) \cos \theta$$

Neglecting PQ^2,

$$r\,\delta r - r\,\delta r \cos \theta = PQ\, (r + \delta r \cos \theta)$$

or $$PQ = r\, \delta r\, (1 - \cos \theta)\, (r + \delta r \cos \theta)^{-1}$$

$$= \delta r\, (1 - \cos \theta) \left(1 + \frac{\delta r}{r} \cos \theta \right)^{-1}$$

$$= \delta r\, (1 - \cos \theta) \left(1 - \frac{\delta r}{r} \cos \theta \right)$$

$$= \delta r\, (1 - \cos \theta)$$

neglecting δr^2 and its higher powers.

$$PQ = (1 - \cos \theta)\, \delta r.$$

Since the lines of flow lie on the surfaces of the spheres, hence velocity along PQ is zero. Now we consider a parallelopiped with edges of lengths $(1 - \cos \theta)\, \delta r$, $r\,\delta\theta$, $r \sin \delta\phi$, the velocities along these elements are o, u, v respectively. The equation of continuity gives

$$\frac{\partial}{\partial t} [\rho (1 - \cos \theta)\, dr \,.\, r\,d\theta \,.\, r \sin \theta\, d\phi]$$

$$= -\left[(1 - \cos \theta)\, dr \,.\, \frac{\partial}{(1 - \cos \theta)\, \partial r} (\rho \,.\, o \,.\, r\,d\theta \,.\, r \sin \theta\, d\phi) + r\,d\theta \frac{\partial}{r\,\partial \theta} [\rho u\, (1 - \cos \theta)\, dr \,.\, r \sin \theta \,.\, d\phi] + r \sin \theta\, d\phi \frac{\partial}{r \sin \theta\, \partial \phi} [\rho v \,.\, (1 - \cos \theta)\, dr \,.\, r\,d\theta] \right]$$

or $$\frac{\partial \rho}{\partial t} + \frac{1}{r^2 \sin \theta\, (1 - \cos \theta)} \left[r \frac{\partial}{\partial \theta} [\rho u\, (1 - \cos \theta) \sin \theta] + r\, (1 - \cos \theta) \frac{\partial}{\partial \phi} (\rho v) \right] = 0$$

or $$r \sin \theta \frac{\partial \rho}{\partial t} + \frac{1}{(1 - \cos \theta)} \frac{\partial}{\partial \theta} [\rho u\, (1 - \cos \theta) \sin \theta] + \frac{\partial}{\partial \phi} (\rho v) = 0$$

or $$r \sin \theta \frac{\partial \rho}{\partial t} + \sin \theta \frac{\partial}{\partial \theta} (\rho u) + \frac{\partial}{\partial \phi} (\rho v) + \rho u\, (1 + 2 \cos \theta) = 0.$$

For $(1 - \cos \theta) \cos \theta + \sin^2 \theta = (1 - \cos \theta) [\cos \theta + 1 + \cos \theta]$.

Problem 26. *The particles of a fluid move symmetrically in space with regard to a fixed centre; prove that the equation of continuity is*

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial r} + \frac{\rho}{r^2} \frac{\partial}{\partial r} (r^2 u) = 0.$$

where u is the velocity at a distance r.

Solution : Here first prove :

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (\rho\, q r^2) = 0 \quad \ldots (1)$$

Put $q = u$ in (1), then

$$\frac{\partial \rho}{\partial t} + \frac{1}{r^2} \frac{\partial}{\partial r} (\rho\, u r^2) = 0.$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + \frac{1}{r^2} \left[(r^2 u) \frac{\partial \rho}{\partial r} + \rho \frac{\partial}{\partial r} (u r^2) \right] = 0$$

$$\Rightarrow \frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial r} + \frac{\rho}{r^2} \frac{\partial}{\partial r} (u r^2) = 0.$$

Problem 27. *If ω is the area of cross section of a stream filament, prove that the equation of continuity is*

$$\frac{\partial}{\partial t} (\rho \omega) + \frac{\partial}{\partial s} (\rho \omega q) = 0$$

Solution : Consider a volume bounded by the cross-sections through points P and Q where Q is at a distance ds from P. Mass of the fluid within the volume $= \rho\omega\, ds$. By def. of continuity, rate of generation of mass

= excess of flow in over flow out through this volume.

i.e., $$\frac{\partial}{\partial t}(\rho\omega\, ds) = -ds\frac{\partial}{\partial s}(\rho q\,\omega)$$

or $$\frac{\partial(\rho\omega)}{\partial t} + \frac{\partial}{\partial s}(\rho\omega\, q) = 0$$

Fig. 10

Problem 28. *A mass of fluid moves in such a way that each particle describes a circle in one plane about a fixed axis; show that the equation of continuity is*

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial\theta}(\rho\omega) = 0$$

where ω is the angular velocity of a particle whose azimuthal angle is θ at time t.

Solution : Consider a point P whose polar co-ordinates are (r, θ). Let there be an elementary area $r\,\delta\theta\,\delta r$, when this area is revolved about O, then it describes a circle so that velocity OP vanishes. By equation of continuity,

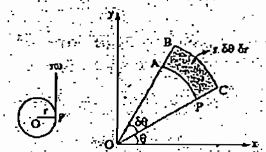


Fig. 11

$$\frac{\partial}{\partial t}(\rho\, r\,\delta\theta\,\delta r) = -\left[\delta r\frac{\partial}{\partial r}(\rho\,.\,0\,.\,r\,\delta\theta) + r\,\delta\theta\frac{\partial}{r\,\partial\theta}(\rho\, q\,\delta r)\right]$$

or $$\frac{\partial\rho}{\partial t} + \frac{1}{r}\left[0 + \frac{\partial}{\partial\theta}(\rho\omega r)\right] = 0. \quad \text{For } q = r\omega$$

or $$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial\theta}(\rho\omega) = 0.$$

Problem 29. *Show that in the motion of a fluid in two dimensions if the current co-ordinates (x, y) are expressible in terms of initial co-ordinates (a, b) and the time, then the motion is irrotational if*

$$\frac{\partial(\dot{x}, x)}{\partial(a, b)} + \frac{\partial(\dot{y}, y)}{\partial(a, b)} = 0.$$

Solution : Let u, v be velocity components parallel to the axis of x and y, respectively. Then

$$\dot{x} = u,\ \dot{y} = v,\ \frac{\partial u}{\partial a} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial a}$$

Observe that

$$\frac{\partial(\dot{x}, x)}{\partial(a, b)} + \frac{\partial(\dot{y}, y)}{\partial(a, b)} = \frac{\partial(u, x)}{\partial(a, b)} + \frac{\partial(v, y)}{\partial(a, b)} = \begin{vmatrix} \frac{\partial u}{\partial a} & \frac{\partial u}{\partial b} \\ \frac{\partial x}{\partial a} & \frac{\partial x}{\partial b} \end{vmatrix} + \begin{vmatrix} \frac{\partial v}{\partial a} & \frac{\partial v}{\partial b} \\ \frac{\partial y}{\partial a} & \frac{\partial y}{\partial b} \end{vmatrix}$$

$$= \frac{\partial u}{\partial a}\frac{\partial x}{\partial b} - \frac{\partial x}{\partial a}\frac{\partial u}{\partial b} + \frac{\partial v}{\partial a}\frac{\partial y}{\partial b} - \frac{\partial y}{\partial a}\frac{\partial v}{\partial b}$$

$$= \frac{\partial x}{\partial b}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial a}\right) - \frac{\partial x}{\partial a}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial b} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial b}\right)$$
$$+ \frac{\partial y}{\partial b}\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial a}\right) - \frac{\partial y}{\partial a}\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial b} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial b}\right)$$

$$= \frac{\partial u}{\partial y}\left(\frac{\partial x}{\partial b}\frac{\partial y}{\partial a} - \frac{\partial x}{\partial a}\frac{\partial y}{\partial b}\right) + \frac{\partial v}{\partial x}\left(\frac{\partial y}{\partial b}\frac{\partial x}{\partial a} - \frac{\partial y}{\partial a}\frac{\partial x}{\partial b}\right)$$

$$= \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\frac{\partial(x, y)}{\partial(a, b)}$$

$$= 0 \text{ iff } \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

or iff motion is irrotational.

Problem 30. *If q is the resultant velocity at any point of a fluid which is moving irrotationally in two dimensions, prove that*

$$\left(\frac{\partial q}{\partial x}\right)^2 + \left(\frac{\partial q}{\partial y}\right)^2 = q\nabla^2 q$$

Solution : Since motion is irrotational, therefore ϕ exists. Equation of continuity is

$$\nabla^2\phi = 0 \quad \text{or} \quad \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \qquad \text{... (1)}$$

$$\mathbf{q} = -\nabla\phi \text{ gives } \quad q^2 = \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 \qquad \text{... (2)}$$

Differentiating (2) partially w.r.t. x and y respectively,

$$q\frac{\partial q}{\partial x} = \frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x^2} + \frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial x\,\partial y} \qquad \text{... (3)}$$

$$q\frac{\partial q}{\partial y} = \frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial y\,\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial y^2} \qquad \text{... (4)}$$

Again differentiating (3) w.r.t. x and (4) w.r.t. y, we get

$$\left(\frac{\partial q}{\partial x}\right)^2 + q\frac{\partial^2 q}{\partial x^2} = \left(\frac{\partial^2\phi}{\partial x^2}\right)^2 + \frac{\partial\phi}{\partial x}\frac{\partial^3\phi}{\partial x^3} + \left(\frac{\partial^2\phi}{\partial x\,\partial y}\right)^2 + \frac{\partial\phi}{\partial y}\frac{\partial^3\phi}{\partial x^2\,\partial y} \qquad \text{... (5)}$$

$$\left(\frac{\partial q}{\partial y}\right)^2 + q\frac{\partial^2 q}{\partial y^2} = \left(\frac{\partial^2\phi}{\partial x\,\partial y}\right)^2 + \frac{\partial\phi}{\partial x}\frac{\partial^3\phi}{\partial y^2\,\partial x} + \left(\frac{\partial^2\phi}{\partial y^2}\right)^2 + \frac{\partial\phi}{\partial y}\frac{\partial^3\phi}{\partial y^3} \qquad \text{... (6)}$$

Adding (5) and (6),

$$\left(\frac{\partial q}{\partial x}\right)^2 + \left(\frac{\partial q}{\partial y}\right)^2 + q\nabla^2 q = \frac{\partial\phi}{\partial y}\frac{\partial}{\partial y}\left\{\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right\} + \frac{\partial\phi}{\partial x}\frac{\partial}{\partial x}\left\{\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right\}$$
$$+ \left\{\left(\frac{\partial^2\phi}{\partial x^2}\right)^2 + \left(\frac{\partial^2\phi}{\partial y^2}\right)^2\right\} + 2\frac{\partial^2\phi}{\partial x\,\partial y}$$

Using (1) and noting that $\frac{\partial^2\phi}{\partial x^2} = -\frac{\partial^2\phi}{\partial y^2}$, we get

$$\left(\frac{\partial q}{\partial x}\right)^2 + \left(\frac{\partial q}{\partial y}\right)^2 + q\nabla^2 q = \frac{\partial\phi}{\partial y}\frac{\partial}{\partial y}(0) + \frac{\partial\phi}{\partial x}\frac{\partial}{\partial x}(0) + 2\left[\left(\frac{\partial^2\phi}{\partial x\,\partial y}\right)^2 + \left(\frac{\partial^2\phi}{\partial x^2}\right)^2\right]$$

$$= 2\left[\left(\frac{\partial^2\phi}{\partial x\,\partial y}\right)^2 + \left(\frac{\partial^2\phi}{\partial x^2}\right)^2\right] \qquad \text{... (7)}$$

Squaring and adding (3) and (4),

$$q^2\left[\left(\frac{\partial q}{\partial x}\right)^2 + \left(\frac{\partial q}{\partial y}\right)^2\right] = \left(\frac{\partial\phi}{\partial x}\right)^2\left[\left(\frac{\partial^2\phi}{\partial x^2}\right)^2 + \left(\frac{\partial^2\phi}{\partial y\,\partial x}\right)^2\right] + \left(\frac{\partial\phi}{\partial y}\right)^2\left[\left(\frac{\partial^2\phi}{\partial x\,\partial y}\right)^2 + \left(\frac{\partial^2\phi}{\partial y^2}\right)^2\right]$$
$$+ 2\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y}\left[\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2\phi}{\partial x\,\partial y} + \frac{\partial^2\phi}{\partial y^2}\frac{\partial^2\phi}{\partial x\,\partial y}\right]$$

But $\frac{\partial^2\phi}{\partial x^2} = -\frac{\partial^2\phi}{\partial y^2}$. Hence the last gives

$$q^2\left[\left(\frac{\partial q}{\partial x}\right)^2 + \left(\frac{\partial q}{\partial y}\right)^2\right] = \left(\frac{\partial\phi}{\partial x}\right)^2\left[\left(\frac{\partial^2\phi}{\partial x^2}\right)^2 + \left(\frac{\partial^2\phi}{\partial y\,\partial x}\right)^2\right] + \left(\frac{\partial\phi}{\partial y}\right)^2\left[\left(\frac{\partial^2\phi}{\partial x\,\partial y}\right)^2 + \left(\frac{\partial^2\phi}{\partial x^2}\right)^2\right]$$
$$+ 2\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y}\left[\frac{\partial^2\phi}{\partial x^2}\frac{\partial^2\phi}{\partial x\,\partial y} - \frac{\partial^2\phi}{\partial x^2}\frac{\partial^2\phi}{\partial x\,\partial y}\right]$$

$$= \left[\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2\right]\left[\left(\frac{\partial^2\phi}{\partial x^2}\right)^2 + \left(\frac{\partial^2 y}{\partial x\,\partial y}\right)^2\right] + 0$$

Using (2),

$$q^2\left[\left(\frac{\partial q}{\partial x}\right)^2 + \left(\frac{\partial q}{\partial y}\right)^2\right] = q^2\left[\left(\frac{\partial^2\phi}{\partial x^2}\right)^2 + \left(\frac{\partial^2\phi}{\partial y\,\partial x}\right)^2\right]$$

or $$\left(\frac{\partial q}{\partial x}\right)^2 + \left(\frac{\partial q}{\partial y}\right)^2 = \left(\frac{\partial^2\phi}{\partial x^2}\right)^2 + \left(\frac{\partial^2\phi}{\partial y\,\partial x}\right)^2$$

Using this in (7),

$$\left(\frac{\partial q}{\partial x}\right) + \left(\frac{\partial q}{\partial y}\right)^2 + q\nabla^2 q = 2\left[\left(\frac{\partial q}{\partial x}\right)^2 + \left(\frac{\partial q}{\partial y}\right)^2\right]$$

or $$q\nabla^2 q = \left(\frac{\partial q}{\partial x}\right)^2 + \left(\frac{\partial q}{\partial y}\right)^2$$

Problem 31. *The velocity components for a two-dimensional fluid system can be given in the Eulerian system by $u = 2x + 2y + 3t$, $v = x + y + \frac{1}{2}t$. Find the displacement of a fluid particle in the Lagrangian system.*

Solution :
$$u = 2x + 2y + 3t \qquad \text{... (1)}$$
$$v = x + y + \frac{1}{2}t \qquad \text{... (2)}$$

$$u = Dx = \dot{x} = \frac{dx}{dt}, \quad v = \dot{y} = Dy = \frac{dy}{dt}, \ \frac{d}{dt} = D.$$

By (1) and (2), $$(D - 2)x - 2y = 3t \qquad \text{... (3)}$$
$$(D - 1)y - x = \frac{1}{2}t \qquad \text{... (4)}$$

Operationg (4) by $D - 2$,

$$(D - 2)(D - 1)y - (D - 2)x = \frac{1}{2}(D - 2)t$$

or $$(D^2 - 3D + 2)y - (D - 2)x = \frac{1}{2}(1 - 2t) \qquad \text{... (5)}$$

(3) + (5) gives

$$(D^2 - 3D + 2)y - 2y = \frac{1}{2} + 2t$$

or $$(D^2 - 3D)y = \frac{1}{2} + 2t$$

Auxiliary equation is given by

$$m^2 - 3m = 0, \quad \text{this} \Rightarrow m = 0, 3$$

$$\text{C.F.} = c_1 e^{0t} + c_2 e^{3t} = c_1 + c_2 e^{3t}$$

$$\text{P.I.} = \frac{1}{D^2 - 3D}\left(\frac{1}{2} + 2t\right) = -\frac{1}{3D}\left(1 - \frac{D}{3}\right)^{-1}\left(\frac{1}{2} + 2t\right)$$

$$= -\frac{1}{3D}\left(1 + \frac{D}{3} \ldots\ldots\right)\left(\frac{1}{2} + 2t\right)$$

$$= -\frac{1}{3D}\left[\left(\frac{1}{2} + 2t\right) + \frac{1}{3}(2)\right] = -\frac{1}{3D}\left[\frac{7}{6} + 2t\right]$$

$$= -\frac{1}{3}\left(\frac{7}{6}t + t^2\right)$$

y = C.F. + P.I. gives

$$y = c_1 + c_2 e^{3t} - \frac{1}{3}\left(\frac{7}{6}t + t^2\right) \qquad \text{... (6)}$$

$$Dy = 3c_2 e^{3t} - \frac{1}{3}\left(\frac{7}{6} + 2t\right) \qquad \text{... (6')}$$

By (4), $$Dy - y - x = \frac{1}{2}t$$

or $$x = Dy - y - \frac{1}{2}t$$

Using (6) and (6'),

$$x = 3c_2 e^{3t} - \frac{1}{3}\left(\frac{7}{6} + 2t\right) - \left(c_1 + c_2 e^{3t} - \frac{7t}{18} - \frac{1}{3}t^2\right) - \frac{1}{2}t$$

or $$x = -c_1 + 2c_2 e^{3t} - \frac{7}{18} - \frac{7}{9}t + \frac{1}{3}t^2 \qquad \text{... (7)}$$

By (6), $$y = c_1 + c_2 e^{3t} - \frac{7}{18}t - \frac{t^2}{3} \qquad \text{... (8)}$$

Initial conditions are $x = x_0, y = y_0$ at $t = 0$.

Putting in (7) and (8), we get

$$x_0 = -c_1 + 2c_2 - \frac{7}{18}, \quad y_0 = c_1 + c_2$$

Solving these two, we get

$$c_1 = \left(\frac{2y_0 - x_0}{3}\right) - \frac{7}{54}$$

$$c_2 = \left(\frac{x_0 + y_0}{3}\right) + \frac{7}{54}.$$

Putting these values in (7) and (8), we get the required expressions :

$$x = \frac{1}{3}(x_0 - 2y_0) + \left[\frac{2}{3}(x_0 + y_0) + \frac{7}{27}\right]e^{3t} - \frac{7}{27} - \frac{7}{9}t + \frac{t^2}{3}$$

$$y = \frac{1}{3}(2y_0 + x_0) + \left[\left(\frac{x_0 + y_0}{3} + \frac{7}{54}\right)\right]e^{2t} - \frac{7}{18}t - \frac{t^2}{3} - \frac{7}{54}$$

Problem 32. *The velocities at a point in a fluid in the Eulerian system are given by $u = x + y + z + t$, $v = 2(x + y + z) + t$, $w = 3(x + y + z) + t$. Find the displacement of a fluid particle in the Lagrangian system. Also determine the velocity of the fluid particle at $(x_0 y_0 z_0)$.*

Solution : The velocity components may be expressed in terms of the displacement as

$$u = \frac{dx}{dt} = x + y + z + t, \quad \ldots (1)$$

$$v = \frac{dy}{dt} = 2(x + y + z) + t, \quad \ldots (2)$$

$$w = \frac{dz}{dt} = 3(x + y + z) + t. \quad \ldots (3)$$

The differential equations can be written in form of operator as

$$(D - 1)x - y = z + t \quad \ldots (4)$$

$$-2x + (D - 2)y = 2z + t \quad \ldots (5)$$

$$-3x - 3y + (D - 3)z = t \quad \ldots (6)$$

Multiplying (4) by $(D - 2)$ and adding to (5), we have

$$[(D - 1)(D - 2) - 2]x = (D - 2)z + 2z + (D - 2)t + t$$

$$(D^2 - 3D)x = Dz + 1 - t. \quad \ldots (7)$$

Multiplying (4) by 2 and (5) by $(D - 1)$ and adding, we have

$$[(D - 1)(D - 2) - 2]y = (D - 1)(2z + t) + 2z + 2t$$

or

$$(D^2 - 3D)y = 2Dz + 1 + t. \quad \ldots (8)$$

Multiplying (6) by $(D^2 - 3D)$, we have

$$(D^2 - 3D)(D - 3)z = 3(D^2 - 3D)x + 3(D^2 - 3D)y + (D^2 - 3D)t$$

From (7) and (8), we have

$$(D^2 - 3D)(D - 3)z = 3(Dz + 1 - t) + 3(2Dz + 1 + t) + (D^2 - 3D)t$$

$$D^2(D - 6)z = 3. \quad \ldots (9)$$

The solution of the differential equation (9) is given by

$$z = A + Bt + Ce^{6t} - \frac{1}{4}t^2. \quad \ldots (10)$$

From the equations (5) and (6), we have

$$(D - 2)y - 2z = 2x + t,$$

$$-3y + (D - 3)z = 3x + t.$$

Solving the equations, we have

$$(D^2 - 5D)y = 2Dx + 1 - t \quad \ldots (11)$$

$$(D^2 - 5D)z = 3Dx + 1 + t. \quad \ldots (12)$$

From (1), we have

$$(D - 1)x = y + z + t$$

or

$$(D - 1)(D^2 - 5D)x = (D^2 - 5D)y + (D^2 - 5D)z + (D^2 - 5D)t$$

or

$$(D - 1)(D^2 - 5D)x = 2Dx + 1 - t + 3Dx + 1 + t - 5$$

or

$$(D^3 - 6D^2)x = -3 \quad \ldots (13)$$

The solution of the differential equation becomes

$$x = A_1 + B_1 t + C_1 e^{6t} + \frac{1}{4}t^2. \quad \ldots (14)$$

Proceeding in the same manner, we have

$$y = A_2 + B_2 t + C_2 e^{6t} \quad \ldots (15)$$

Thus the equations (10), (14) and (15) determine the displacement of a fluid particle.

Let $x = x_0$, $y = y_0$, $z = z_0$ when $t = t_0 = 0$

The relations (14), (15) and (10) give

$$x_0 = A_1 + C_1, \quad y_0 = A_2 + C_2, \quad z_0 = A + C$$

Thus

$$x = x_0 - C_1 + B_1 t + C_1 e^{6t} + \frac{1}{4}t^2, \quad \ldots (16)$$

$$y = y_0 - C_2 + B_2 t + C_2 e^{6t}, \quad \ldots (17)$$

$$z = z_0 - C + Bt + Ce^{6t} - \frac{1}{4}t^2. \quad \ldots (18)$$

Substituting these values in (1), (2) and (3), we obtain the following identities

$$B_1 + 6C_1 e^{6t} + \frac{1}{2}t = x_0 + y_0 + z_0 - (C_1 + C_2 + C) + (B_1 + B_2 + B)t + (C_1 + C_2 + C)e^{6t} + t \quad \ldots (19)$$

$$B_2 + 6C_2 e^{6t} = 2(x_0 + y_0 + z_0) - 2(C_1 + C_2 + C) + 2(B_1 + B_2 + B)t + 2(C_1 + C_2 + C)e^{6t} + t, \quad \ldots (20)$$

$$B + 6Ce^{6t} - \frac{1}{2}t = 3(x_0 + y_0 + z_0) - 3(C_1 + C_2 + C) + 3(B_1 + B_2 + B)t + 3(C_1 + C_2 + C)e^{6t} + t. \quad \ldots (21)$$

Equating the coefficients of t, e^{6t} and the constant term, we have

$$\left.\begin{aligned} x_0 + y_0 + z_0 - (C_1 + C_2 + C) &= B_1, \\ C_1 + C_2 + C &= 6C_1, \\ B_1 + B_2 + B + 1 &= \frac{1}{2}. \end{aligned}\right\} \quad \ldots (22)$$

$$\left.\begin{aligned} 2(x_0 + y_0 + z_0) - (C_1 + C_2 + C) &= B_2, \\ 2(C_1 + C_2 + C) &= 6C_2, \\ 2(B_1 + B_2 + B) + 1 &= 0. \end{aligned}\right\} \quad \ldots (23)$$

$$\left.\begin{aligned} 3(x_0 + y_0 + z_0) - 3(C_1 + C_2 + C) &= B, \\ 3(C_1 + C_2 + C) &= 6C, \\ 3(B_1 + B_2 + B) + 1 &= -\frac{1}{2}. \end{aligned}\right\} \quad \ldots (24)$$

From these three sets, we obtain

$$C_1 = \frac{1}{6}\left(x_0 + y_0 + z_0 + \frac{1}{12}\right), \quad C_2 = \frac{1}{3}\left(x_0 + y_0 + z_0 + \frac{1}{12}\right),$$

$$C = \frac{1}{2}\left(x_0 + y_0 + z_0 + \frac{1}{12}\right).$$

Also $\quad B_1 = -\frac{1}{12}, \quad B_2 = -\frac{1}{6}, \quad B = -\frac{1}{4}.$

Substituting these values in the relations (16), (17), and (18) and simplifying, we get

$$x = \frac{5}{6}x_0 - \frac{1}{6}y_0 - \frac{1}{6}z_0 + \frac{1}{6}\left(x_0 + y_0 + z_0 + \frac{1}{12}\right)e^{6t} - \frac{1}{12}t + \frac{1}{4}t^2 - \frac{1}{72},$$

$$y = -\frac{1}{3}x_0 + \frac{2}{3}y_0 - \frac{1}{3}z_0 + \frac{1}{3}\left(x_0 + y_0 + z_0 + \frac{1}{12}\right)e^{6t} - \frac{1}{6}t - \frac{1}{36},$$

$$z = -\frac{1}{2}x_0 - \frac{1}{2}y_0 + \frac{1}{2}z_0 + \frac{1}{2}\left(x_0 + y_0 + z_0 + \frac{1}{12}\right)e^{6t} - \frac{1}{4}t - \frac{1}{4}t^2 - \frac{1}{24},$$

determines the displacement of a fluid particle in Lagrangian description.

Let u_1, v_1, w_1 be the components of the velocity in the Lagrangian description. We have

$$u_1 = \frac{\partial x}{\partial t} = \left(x_0 + y_0 + z_0 + \frac{1}{12}\right)e^{6t} - \frac{1}{12} + \frac{1}{2}t,$$

$$v_1 = \frac{\partial y}{\partial t} = 2\left(x_0 + y_0 + z_0 + \frac{1}{12}\right)e^{6t} - \frac{1}{6},$$

$$w_1 = \frac{\partial z}{\partial t} = 3\left(x_0 + y_0 + z_0 + \frac{1}{12}\right) - \frac{1}{4} - \frac{1}{2}t.$$

Thus the velocity of the fluid particle is given by

$$\mathbf{q}_1 = u_1\mathbf{i} + v_1\mathbf{j} + w_1\mathbf{k}.$$

Problem 33. *The velocity components of flow in cylindrical co-ordinates are $(r^2 z\cos\theta, rz\sin\theta, z^2 t)$. Determine the components of acceleration of a fluid particle.*

Solution : Let u, v, w be velocity components in cylindrical co-ordinates (r, θ, z). We know that

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial r}, \quad \frac{\partial}{\partial y} = \frac{1}{r}\frac{\partial}{\partial \theta}$$

Given $\quad u = r^2 z\cos\theta, \quad v = rz\sin\theta, \quad w = z^2 t$

Let a_1, a_2, a_3 be components of acceleration.

$$\mathbf{a} = \mathbf{i}a_1 + \mathbf{j}a_2 + \mathbf{k}a_3$$

$$\mathbf{a} = \frac{d}{dt}\mathbf{q} = \left(\frac{\partial}{\partial t} + \mathbf{q}.\nabla\right)\mathbf{q}$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{q}.\nabla) = \frac{\partial}{\partial t} + u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial \theta} + w\frac{\partial}{\partial z}$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + r^2 z\cos\theta\frac{\partial}{\partial r} + z\sin\theta\frac{\partial}{\partial \theta} + z^2 t\frac{\partial}{\partial z} \quad \ldots (1)$$

$$a_1 = \frac{du}{dt} - \frac{v^2}{r}, \quad a_2 = \frac{dv}{dt} + \frac{vu}{r}, \quad a_3 = \frac{dw}{dt}$$

$$a_1 = \left(\frac{\partial}{\partial t} + r^2 z\cos\theta\frac{\partial}{\partial r} + z\sin\theta\frac{\partial}{\partial \theta} + z^2 t\frac{\partial}{\partial z}\right)(r^2 z\cos\theta) - \frac{v^2}{r}$$

$$= 0 + (r^2 z\cos\theta)(2rz\cos\theta) + (z\sin\theta)(-r^2 z\sin\theta) + (z^2 t)(r^2\cos\theta) - rz^2\sin^2\theta$$

$$= rz^2[2r^2\cos^2\theta - r\sin^2\theta - \sin^2\theta + rt\cos\theta]$$

$$a_2 = \frac{dv}{dt} + \frac{uv}{r}$$

$$= \left(\frac{\partial}{\partial t} + r^2 z\cos\theta\frac{\partial}{\partial r} + z\sin\theta\frac{\partial}{\partial \theta} + z^2 t\frac{\partial}{\partial z}\right)rz\sin\theta + z^2 r^2\sin\theta\cos\theta$$

$$= 0 + (r^2 z\cos\theta)(z\sin\theta) + (z\sin\theta)(rz\cos\theta) + (z^2 t)r\sin\theta + z^2 r^2\sin\theta\cos\theta$$

$$= z^2\sin\theta[2r^2\cos\theta + r\cos\theta + rt].$$

$$a_3 = \frac{dw}{dt} = \left(\frac{\partial}{\partial t} + r^2 z\cos\theta\frac{\partial}{\partial r} + z\sin\theta\frac{\partial}{\partial \theta} + z^2 t\frac{\partial}{\partial z}\right)(z^2 t)$$

$$= z^2 + 0 + 0 + z^2 t(2zt) = z^2[1 + 2zt^2]$$

Finally,

$$\text{Ans.} \quad \begin{cases} a_1 = rz^2[2r^2\cos^2\theta - r\sin^2\theta - \sin^2\theta + rt\cos\theta] \\ a_2 = z^2\sin\theta[2r^2\cos\theta + r\cos\theta + rt] \\ a_3 = z^2[1 + 2zt^2] \end{cases}$$

Problem 34. *Give examples of irrotational and rotational flows.*

Solution. I. Cconsider fluid motion given by $u = kx$, $v = 0$, $w = 0$, $(k \neq 0)$

Then $\quad \mathbf{q} = \mathbf{i}kx$

$$\text{curl}\,\mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ kx & 0 & 0 \end{vmatrix}$$

$$\text{curl } \mathbf{q} = \mathbf{i}\,(0) - \mathbf{j}\,(0) + \mathbf{k}\,(0) = 0$$

$\therefore$ Motion is irrotational.

II. Consider fluid motion given by

$$u = \alpha y,\ v = 0,\ w = 0,\ (\alpha \neq 0)$$

$$\text{curl } \mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \alpha y & 0 & 0 \end{vmatrix}$$

$$= \mathbf{i}\,(0) - \mathbf{j}\,(0) + \mathbf{k}\,(0 - \alpha)$$

$$\therefore \quad \text{curl } \mathbf{q} = -\alpha \mathbf{k} \neq 0.$$

Hence motion is not irrotational.

Consequently motion is rotational.

Problem 35. *If velocity distribution is*

$$\mathbf{q} = \mathbf{i}\,(Ax^2yt) + \mathbf{j}\,(By^2zt) + \mathbf{k}\,(Czt^2)$$

where A, B, C are constants, then find acceleration and vorticity components.

Solution : Let $\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$.

Then $u = Ax^2yt,\quad v = By^2zt,\quad w = Czt^2$

I. Let $\mathbf{a} = \mathbf{i}a_1 + \mathbf{j}a_2 + \mathbf{k}a_3$ denote acceleration. Then

$$\mathbf{a} = \frac{d\mathbf{q}}{dt},\quad a_1 = \frac{du}{dt} \text{ etc.}$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{q}\cdot\nabla = \frac{\partial}{\partial t} + u\frac{\partial}{\partial y} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + Ax^2yt\frac{\partial}{\partial x} + By^2zt\frac{\partial}{\partial y} + Czt^2\frac{\partial}{\partial z} \quad \ldots (1)$$

$a_1 = \dfrac{du}{dt}$ gives

$$a_1 = \left(\frac{\partial}{\partial t} + Ax^2yt\frac{\partial}{\partial x} + By^2zt\frac{\partial}{\partial t} + Czt^2\frac{\partial}{\partial z}\right)(Ax^2yt)$$

$$= Ax^2y + (Ax^2yt)\,(2Axyt) + (By^2zt)\,(Axt) + (Czt^2)\,(0)$$

$$= Ax^2y\,[1 + 2Axyt^2 + Byzt^2] \quad \ldots (2)$$

$a_2 = \dfrac{dv}{dt}$ with $\dfrac{d}{dt}$ given by (1)

$$a_2 = \left(\frac{\partial}{\partial t} + Ax^2yt\frac{\partial}{\partial x} + By^2zt\frac{\partial}{\partial y} + Czt^2\frac{\partial}{\partial z}\right)(By^2zt)$$

$$= By^2z + (Ax^2yt)\,(0) + (By^2zt)\,(2Byzt) + (Czt^2)\,(By^2t)$$

$$= By^2z\,[1 + 2Byzt^2 + Ct^3] \quad \ldots (3)$$

$a_3 = \dfrac{dw}{dt}$ with $\dfrac{d}{dt}$ given by (1),

$$= \left(\frac{\partial}{\partial t} + Ax^2yt\frac{\partial}{\partial x} + By^2zt\frac{\partial}{\partial y} + Czt^2\frac{\partial}{\partial z}\right)Czt^2$$

$$a_3 = 2Czt + (Ax^2y)\,(0) + (By^2zt)\,(0) + (Czt^2)\,(Ct^2)$$

$$= Czt\,[2 + Ct^2] \quad \ldots (4)$$

Acceleration components are given by (2), (3) and (4).

II. Let $\mathbf{W} = \text{curl } \mathbf{q}$. Then $\mathbf{W}$ is vorticity vector.

$$\mathbf{W} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Ax^2yt & By^2zt & Czt^2 \end{vmatrix}$$

$$= \mathbf{i}\,(0 - By^2t) - \mathbf{j}\,(0 - 0) + \mathbf{k}\,(0 - Ax^2t)$$

Vorticity components are

$$-By^2t,\ 0,\ -Ax^2t.$$

Problem 36. *Test whether the motion specified by*

$$\mathbf{q} = \frac{k^2(x\mathbf{j} - y\mathbf{i})}{x^2 + y^2} \quad (k = \text{const.})$$

is a possible motion for an incompressible fluid. If so, determine the equations of stream lines. Also tell whether the motion is of the potential kind and if it determines the velocity potential. (IFoS-2011)

Solution : Here $u = \dfrac{-k^2y}{x^2 + y^2},\quad v = \dfrac{k^2x}{x^2 + y^2},\quad w = 0.$

I. Equation of continuity for incompressible fluid is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

But

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{2k^2xy}{(x^2 + y^2)^2} - \frac{2k^2xy}{(x^2 + y^2)^2} + 0$$

$$= 0$$

Hence equation of continuity is satisfied.

II. Stream lines are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{0}$$

or

$$\frac{dx\,(x^2 + y^2)}{-k^2y} = \frac{(x^2 + y^2)\,dy}{k^2x} = \frac{dz}{0}$$

$$\Rightarrow \quad x\,dx + y\,dy = 0,\quad dz = 0$$

$$\Rightarrow \quad x^2 + y^2 = a^2,\quad z = b$$

Hence stream lines are circles whose centres lie on z-axis.

III. To test the existence of velocity potential.

$$-d\phi = u\,dx + v\,dy + w\,dz$$

$$= -k^2y\,\frac{dx}{x^2 + y^2} + k^2x\cdot\frac{dy}{x^2 + y^2}$$

$$d\phi = k^2\left[\frac{y\,dx}{x^2 + y^2} - \frac{x}{x^2 + y^2}dx\right]$$

$$= k^2\,(M\,dx + N\,dy), \text{ say}$$

$$\frac{\partial M}{\partial y} = \frac{1}{x^2 + y^2} + y\left[\frac{-2y}{(x^2 + y^2)^2}\right] = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$$\frac{\partial N}{\partial x} = -\left[\frac{(x^2 + y^2) - 2x^2}{(x^2 + y^2)^2}\right] = \frac{x^2 - y^2}{(x^2 + y^2)^2} = \frac{\partial M}{\partial y}$$

Hence $M\,dx + N\,dy$ is exact. Therefore its solution of given by

$$\phi = \int \frac{k^2y\,dx}{x^2 + y^2} + \int 0\,dy + C = \frac{k^2y}{y}\tan^{-1}\left(\frac{x}{y}\right) + C$$

Hence ϕ exists and is given by

$$\phi = k^2\tan^{-1}\left(\frac{x}{y}\right) + C$$

Problem 37. *The velocity vector in the flow field is given by*

$$\mathbf{q} = \mathbf{i}\,(Az - By) + \mathbf{j}\,(Bx - Cz) + \mathbf{k}\,(Cy - Ax)$$

where A, B, C are non-zero constants.

Determine the equations of the vortex lines.

Solution : Let $\mathbf{W} = \mathbf{i}\xi + \mathbf{j}\eta + \mathbf{k}\zeta$ be the vorticity vector. Then $\mathbf{W} = \text{curl } \mathbf{q}$

or

$$\mathbf{W} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ Az - By & Bx - Cz & Cy - Ax \end{vmatrix}$$

$$= \mathbf{i}\,(C + C) - \mathbf{j}\,(-A - A) + \mathbf{k}\,(B + B)$$

This $\Rightarrow \xi = 2C,\quad \eta = 2A,\quad \zeta = 2B$

Vortex lines are given by

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dz}{\zeta}$$

Putting the values,

$$\frac{dx}{2C} = \frac{dy}{2A} = \frac{dz}{2B}$$

or

$$\frac{dx}{C} = \frac{dy}{A} = \frac{dz}{B}$$

$$\Rightarrow \quad A\,dx - C\,dy = 0,\quad B\,dy - A\,dz = 0$$

Integrating,

$$Ax - Cy = c_1,\quad By - Az = c_2$$

Vortex lines are given by these equations.

Problem 38. *Show that $\phi = (x - t)(y - t)$ represents the velocity potential of an incompressible two dimensional fluid. Show that the stream lines at time t are the curves*

$$(x - t)^2 - (y - t)^2 = constant$$

and that the paths of fluid particles have the equations (IFoS-2010)

$$\log\,(x - y) = \frac{1}{2}\,[(x + y) - a\,(x - y)^{-1} + b,$$

where a and b are constants.

Solution : Given $\phi = (x - t)\,(y - t) \quad \ldots (1)$

I. To show that the liquid motion is possible,

$$\frac{\partial \phi}{\partial x} = y - t,\quad \frac{\partial \phi}{\partial y} = x - t$$

$$\Rightarrow \quad \frac{\partial^2 \phi}{\partial x^2} = 0,\ \frac{\partial^2 \phi}{\partial y^2} = 0 \quad \Rightarrow \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$$

$\Rightarrow \quad \nabla^2\phi = 0$, which is the equation of continuity.

Hence (1) represents velocity potential of an incompressible two dimensional fluid.

II. To determine stream lines.

$$u = -\frac{\partial \phi}{\partial x} = -(y - t).$$

$$v = -\frac{\partial \phi}{\partial y} = -(x - t)$$

Stream lines are given by

$$\frac{dx}{u} = \frac{dy}{v}$$

$$\frac{dx}{-(y - t)} = \frac{dy}{-(x - t)}$$

or $\quad (x - t)\,dx = (y - t)\,dy$

Integrating, $\quad \dfrac{x^2}{2} - tx = \dfrac{y^2}{2} - ty + \text{const.}$

or $\quad x^2 - 2tx = y^2 - 2ty + \text{const.}$

or $\quad (x - t)^2 = (y - t)^2 + \text{const.}$

or $\quad (x - t)^2 - (y - t)^2 = \text{const.}$

which represents stream lines.

III. To determine path lines.

$$\frac{dx}{dt} = u = -\frac{\partial \phi}{\partial x} = -(y - t)$$

$$\frac{dy}{dt} = v = -\frac{\partial \phi}{\partial y} = -(x - t)$$

$$\Rightarrow \quad dx = (t - y)\,dt \quad \ldots (2)$$

$$dy = (t - x)\,dt \quad \ldots (3)$$

Upon subtraction, $\quad dx - dy = (x - y)\,dt$

or $\quad \dfrac{dx - dy}{x - y} = dt$

Integrating, $\log(x-y) = t + \log c$

or $x - y = c e^{t}$ ...(4)

$(2)+(3) \Rightarrow dx + dy = [2t - (x+y)]\,dt$ ...(5)

Put $x + y = u,\ dx + dy = du$, then (5) gives

$$\frac{du}{dt} + u = 2t \quad \text{...(6)}$$

It is of the type $\frac{dy}{dx} + Py = Q$ whose solution is

$$y e^{\int P\,dx} = c + \int Q e^{\int P\,dx}\,dx$$

Hence solution of (6) is

$$u e^{t} = k + \int 2t e^{t}\,dt$$

$$u e^{t} = k + 2(t-1)e^{t}$$

or $u = k e^{-t} + 2(t-1)$

or $(x+y) = \frac{kc}{x-y} + 2\log\left(\frac{x-y}{c}\right) - 2$, by (4)

or $\log(x-y) = \frac{1}{2}[(x+y) - kc(x-y)^{-1}] + 1 + \log c$

Taking $1 + \log c = b,\ \frac{kc}{2} = a$, we get

$$\log(x-y) = \frac{1}{2}[(x+y) - a(x-y)^{-1}] + b$$

This represents path lines.

Problem 39: *Determine whether the motion specified by*

$$q = \frac{A(x\mathbf{j} - y\mathbf{i})}{x^2+y^2},\ (A = \text{const.})$$

is a possible motion for an incompressible fluid. If so, determine the equations of the streamlines. Also, show that the motion is of potential kind. Find the velocity potential.

Solution. We know that

$$\nabla \cdot q = 0. \quad \text{...(1)}$$

or $A\left[-\frac{\partial}{\partial x}\left(\frac{y}{x^2+y^2}\right) + \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right)\right] = 0,$

or $A\left\{\frac{2xy}{(x^2+y^2)^2} - \frac{2xy}{(x^2+y^2)^2}\right\} = 0,$

which is evident. Thus the equation of continuity for an incompressible fluid is satisfied and hence it is a possible motion for an incompressible fluid.

The equation of the streamlines are

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

or $\frac{dx}{-Ay/(x^2+y^2)} = \frac{dy}{Ax/(x^2+y^2)} = \frac{dz}{0}$

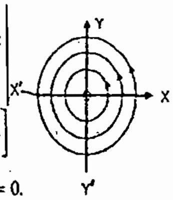
or $x\,dx + y\,dy = 0,\ dz = 0$

By integrating, we have

$$x^2 + y^2 = \text{constant},\ z = \text{constant}. \quad \text{...(2, 3)}$$

Thus the streamlines are circles whose centres are on Z-axis, their planes being perpendicular to the axis.

Again $\nabla \times q = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ -\frac{Ay}{(x^2+y^2)} & \frac{Ax}{(x^2+y^2)} & 0 \end{vmatrix}$



or $\nabla \times q = \mathbf{k}\left[\frac{\partial}{\partial x}\left\{\frac{Ax}{x^2+y^2}\right\} + \frac{\partial}{\partial y}\left\{\frac{Ay}{x^2+y^2}\right\}\right]$

or $\nabla \times q = \mathbf{k}A\left[\frac{y^2-x^2}{(x^2+y^2)^2} + \frac{x^2-y^2}{(x^2+y^2)^2}\right] = 0.$

Thus the flow is of potential kind, so we can determine $\phi(x,y,z)$ such that

$$q = -\nabla\phi$$

or, $\frac{\partial\phi}{\partial x} = -u = \frac{Ay}{x^2+y^2},\ \frac{\partial\phi}{\partial y} = -v = -\frac{Ax}{x^2+y^2},$

$\frac{\partial\phi}{\partial z} = -w = 0,$ ...(4, 5, 6)

which shows that ϕ is independent of z, hence

$$\phi = \phi(x,y).$$

Integrating the relation (4), we have

$$\phi(x,y) = A\tan^{-1}(x/y) + f(y)$$

or $\frac{\partial\phi}{\partial y} = f'(y) - Ax/(x^2+y^2).$

Using the relation (5), we get

$$f'(y) = 0 \Rightarrow f(y) = \text{constant}.$$

Therefore $\phi(x,y) = A\tan^{-1}(x/y).$

Problem 40: *Show that the velocity potential*

$$\phi = \frac{1}{2}a(x^2 + y^2 - 2z^2)$$

satisfies the Laplace equation. Also determine the streamlines. (IAS-2002)

Solution. Let ϕ be the velocity potential for the velocity field q then

$$q = -\nabla\phi = -\tfrac{1}{2}a\nabla(x^2+y^2-2z^2)$$

$$q = -\tfrac{1}{2}a(2x\mathbf{i} + 2y\mathbf{j} - 4z\mathbf{k}).$$

Taking divergence of both the sides, we have

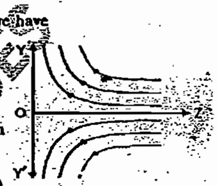
$\nabla^2\phi = -\nabla \cdot q$

or $\nabla^2\phi = -\tfrac{1}{2}a\,\nabla\cdot(2x\mathbf{i} + 2y\mathbf{j} - 4z\mathbf{k}) = 0$

or $\nabla^2\phi = -\tfrac{1}{2}a(2+2-4) = 0$

Hence Laplace equation is satisfied.

The equation of streamlines are given by



$dx/u = dy/v = dz/w$

or $dx/(-ax) = dy/(-ay) = dz/(2az)$

(i) (ii) (iii)

From (ii) and (iii), we have

$$\log y = \tfrac{1}{2}\log z - \log C,$$

where C is an integration constant.

or $y^2 z = C,$

which represents a cubic hyperbola.

Problem 41: *Show that*

$$u = -\frac{2xyz}{(x^2+y^2)^2},\ v = \frac{(x^2-y^2)z}{(x^2+y^2)^2},\ w = \frac{y}{x^2+y^2},$$

are the velocity components of a possible liquid motion. Is this motion irrotational ?

Solution. The condition for the possible liquid motion is given by

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

$\Rightarrow 2yz\cdot\frac{3x^2-y^2}{(x^2+y^2)^3} + 2yz\cdot\frac{y^2-3x^2}{(x^2+y^2)^3} + 0 = 0,$

which is an identity. Hence (u, v, w) are the velocity components of a possible liquid motion.

Again the condition for irrotational motion is

$$\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} = 0,\ \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} = 0 \text{ and } \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0$$

So $\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y} = \frac{x^2-y^2}{(x^2+y^2)^2} - \frac{x^2-y^2}{(x^2+y^2)^2} = 0,$

$\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} = -\frac{2xy}{(x^2+y^2)^2} + \frac{2xy}{(x^2+y^2)^2} = 0,$

and $\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = \frac{2xz(3y^2-x^2)}{(x^2+y^2)^2} - \frac{2xz(3y^2-x^2)}{(x^2+y^2)^2} = 0.$

Thus $\nabla \times q = 0, \Rightarrow$ that the motion is irrotational.

Problem 42: *Find the necessary and sufficient condition that vortex lines may be at right angles to the streamlines.* (IAS-2005)

Solution. The equations of the streamlines and the vortex lines are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w},$$

and $\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dz}{\zeta}.$ ...(1, 2)

The equation (1) and (2) are at right angles. It follows that

$$u\xi + v\eta + w\zeta = 0$$

$\Rightarrow u\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + v\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + w\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0.$

In order that $u\,dx + v\,dy + w\,dz$ may be a perfect differential, we have

$$u\,dx + v\,dy + w\,dz = \lambda\,d\phi = \lambda\left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right)$$

$\Rightarrow u = \lambda\frac{\partial\phi}{\partial x},\ v = \lambda\frac{\partial\phi}{\partial y},\ w = \lambda\frac{\partial\phi}{\partial z},$

which determines the necessary and sufficient condition.

Problem 43: *In an incompressible fluid the vorticity at every point is constant in magnitude and direction; prove that the components of velocity u, v, w are the solutions of Laplace equation.* (IAS-2004)

Solution. Let Ω be the vorticity at any point in an incompressible fluid then

$$\Omega = \xi\mathbf{i} + \eta\mathbf{j} + \zeta\mathbf{k}$$

where $\xi = \dfrac{\partial w}{\partial y} - \dfrac{\partial v}{\partial z}, \eta = \dfrac{\partial u}{\partial z} - \dfrac{\partial w}{\partial x}, \zeta = \dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial y} = 0.$

The magnitude and direction cosines of its direction are given by

$$\Omega = \sqrt{\xi^2 + \eta^2 + \zeta^2} \quad \text{and} \quad \frac{\xi}{\Omega}, \frac{\eta}{\Omega}, \frac{\zeta}{\Omega}$$

Differentiating η partially with regard to z and ζ with regard to y and subtracting, we have

$$\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0 \Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0.$$

Hence the velocity components satisfy Laplace Equation.

Problem 44: *Find the vorticity components of a fluid-particle when velocity distribution is*

$$\mathbf{q} = \mathbf{i}\,(k_1 x^2 yt) + \mathbf{j}\,(k_2 y^2 zt) + \mathbf{k}\,(k_3 z t^2),$$

where k_1, k_2, k_3 are constants.

Solution. The vorticity components ξ, η, ζ are given by

$$\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = -k_2 y^2 t,$$

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0$$

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -k_1 x^2 t.$$

Problem 45: *Determine the equations of the vortex lines when the velocity vector of the flow field is given by*

$$\mathbf{q} = \mathbf{i}\,(Az - By) + \mathbf{j}\,(Bx - Cz) + \mathbf{k}\,(Cy - Ax),$$

where A, B, C are constants.

Solution. The vorticity components are given by

$$\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = C + C = 2C,$$

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = A + A = 2A,$$

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = B + B = 2B.$$

The equations of the vortex lines are

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dz}{\zeta}$$

$$\Rightarrow \quad \frac{dx}{2C} = \frac{dy}{2A} = \frac{dz}{2B}$$

(i) (ii) (iii)

From (i) and (ii), we have

$Ax - Cy = k_1,$ ...(1)

From (ii) and (iii), we have

$By - Az = k_2,$ where k_1 and k_2 are integration constants. ...(2)

Hence the vortex lines (1) and (2) are the straightlines.

Problem 46: *Investigate the nature of the liquid motion given by*

$$u = \frac{ax - by}{x^2 + y^2}, v = \frac{ay + bx}{x^2 + y^2}, w = 0.$$

Also, determine the velocity potential.

Solution. Here $u = \dfrac{ax - by}{x^2 + y^2}, v = \dfrac{ay + bx}{x^2 + y^2}, w = 0,$

$$\frac{\partial u}{\partial x} = \frac{a\,(x^2 + y^2) - 2x\,(ax - by)}{(x^2 + y^2)^2} = \frac{a\,(y^2 - x^2) + 2bxy}{(x^2 + y^2)^2},$$

$$\frac{\partial v}{\partial y} = \frac{a\,(x^2 + y^2) - 2y\,(ay + bx)}{(x^2 + y^2)^2} = \frac{a\,(x^2 - y^2) - 2bxy}{(x^2 + y^2)^2}$$

$$\Rightarrow \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

Thus the liquid motion satisfies the continuity equation hence it is a possible motion.

Let Ω be the vorticity then

$$\Omega = \mathbf{i}\xi + \mathbf{j}\eta + \mathbf{k}\zeta,$$

where $\xi = \dfrac{\partial w}{\partial y} - \dfrac{\partial v}{\partial z} = 0,$

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} = 0,$$

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0.$$

It follows that the nature of the liquid motion is irrotationl.

Let ϕ be the velocity potential, then

$$d\phi = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = -u\,dx - v\,dy$$

$$\Rightarrow \quad d\phi = -\left\{\frac{ax - by}{x^2 + y^2}dx + \frac{ay + bx}{x^2 + y^2}dy\right\}$$

$$\Rightarrow \quad d\phi = -\left\{\frac{a\,(x\,dx + y\,dy)}{x^2 + y^2} + \frac{b\,(x\,dy - y\,dx)}{x^2 + y^2}\right\}$$

$$\Rightarrow \quad \phi = -\frac{1}{2}a \log (x^2 + y^2) + b \tan^{-1}\left(\frac{y}{x}\right).$$

Problem 47: *If $u\,dx + v\,dy + w\,dz = d\theta + \lambda\,d\mu$ where λ, θ, μ are functions of x, y, z and t, prove that the vortex lines at any time are the lines of intersection of the surfaces $\lambda = const.$ and $\mu = const.$*

Solution. We know that

$$u\,dx + v\,dy + w\,dz = d\theta + \lambda\,d\mu$$

or
$$u\,dx + v\,dy + w\,dz = \left(\frac{\partial \theta}{\partial x}dx + \frac{\partial \theta}{\partial y}dy + \frac{\partial \theta}{\partial z}dz + \frac{\partial \theta}{\partial t}dt\right) + \lambda\left(\frac{\partial \mu}{\partial x}dx + \frac{\partial \mu}{\partial y}dy + \frac{\partial \mu}{\partial z}dz + \frac{\partial \mu}{\partial t}dt\right)$$

Equating coefficient of dx, dy, dz and dt, we have

$$u = \frac{\partial \theta}{\partial x} + \lambda\frac{\partial \mu}{\partial x}, v = \frac{\partial \theta}{\partial y} + \lambda\frac{\partial \mu}{\partial y},$$

$$w = \frac{\partial \theta}{\partial z} + \lambda\frac{\partial \mu}{\partial z}, O = \frac{\partial \theta}{\partial t} + \lambda\frac{\partial \mu}{\partial t}.$$

The components of spin are

$$2\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = \frac{\partial}{\partial y}\left(\frac{\partial \theta}{\partial z} + \lambda\frac{\partial \mu}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial \theta}{\partial y} + \lambda\frac{\partial \mu}{\partial y}\right)$$

$$\Rightarrow \quad 2\xi = \lambda\frac{\partial^2 \mu}{\partial y\,\partial z} + \frac{\partial \lambda}{\partial y}\frac{\partial \mu}{\partial z} - \lambda\frac{\partial^2 \mu}{\partial y\,\partial z} - \frac{\partial \lambda}{\partial z}\frac{\partial \mu}{\partial y}$$

$$\Rightarrow \quad 2\xi = \frac{\partial \lambda}{\partial y}\frac{\partial \mu}{\partial z} - \frac{\partial \lambda}{\partial z}\frac{\partial \mu}{\partial y}$$

$$\Rightarrow \quad 2\xi = \begin{vmatrix} \dfrac{\partial \lambda}{\partial y} & \dfrac{\partial \lambda}{\partial z} \\ \dfrac{\partial \mu}{\partial y} & \dfrac{\partial \mu}{\partial z} \end{vmatrix}$$

Similarly
$$2\eta = \begin{vmatrix} \dfrac{\partial \lambda}{\partial z} & \dfrac{\partial \lambda}{\partial x} \\ \dfrac{\partial \mu}{\partial z} & \dfrac{\partial \mu}{\partial x} \end{vmatrix} \quad \text{and} \quad 2\zeta = \begin{vmatrix} \dfrac{\partial \lambda}{\partial x} & \dfrac{\partial \lambda}{\partial y} \\ \dfrac{\partial \mu}{\partial x} & \dfrac{\partial \mu}{\partial y} \end{vmatrix}.$$

Therefore
$$2\left(\xi\frac{\partial \lambda}{\partial x} + \eta\frac{\partial \lambda}{\partial y} + \zeta\frac{\partial \lambda}{\partial z}\right) = \begin{vmatrix} \lambda_x & \lambda_y & \lambda_z \\ \lambda_x & \lambda_y & \lambda_z \\ \mu_x & \mu_y & \mu_z \end{vmatrix} = 0$$

$$\Rightarrow \quad \xi\lambda_x + \eta\lambda_y + \zeta\lambda_z = 0$$

Similarly
$$\xi\mu_x + \eta\mu_y + \xi\mu_z = 0$$

It follows that the vortex lines lie on the surfaces

$$\lambda = \text{const. and } \mu = \text{const.}$$

Problem 48: *If the velocity of an incompressible fluid at the point (x, y, z) is given by $3xz/r^5, 3yz/r^5, (3z^2 - r^2)/r^5$, prove that the liquid motion is possible and that the velocity potential is $\cos\theta/r^2$. Also, determine the stream lines.* (IFoS-2009)

Solution. The condition for the possible liquid motion is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

$$u = \frac{3xz}{r^5} \Rightarrow \frac{\partial u}{\partial x} = \frac{3z}{r^5} - \frac{15xz}{r^6}\cdot\frac{\partial r}{\partial x} = \frac{3z}{r^5} - \frac{15x^2 z}{r^7}$$

or
$$\frac{3z}{r^5} - \frac{15x^2 z}{r^7} + \frac{3z}{r^5} - \frac{15y^2 z}{r^7} + \frac{6z}{r^5} - \frac{15z^3}{r^5} + \frac{3z}{r^5} = 0$$

or
$$\frac{15z}{r^5} - \frac{15z\,(x^2 + y^2 + z^2)}{r^7} = 0 \Rightarrow \frac{15z}{r^5} - \frac{15z}{r^5} = 0,$$

which is an identity. Hence (u, v, w) are the velocity components of a possible liquid motion.

If ϕ be the velocity potential, then

$$d\phi = (\partial\phi/\partial x)\,dx + (\partial\phi/\partial y)\,dy + (\partial\phi/\partial z)\,dz$$

or
$$d\phi = -(u\,dx + v\,dy + w\,dz)$$

or
$$d\phi = -\frac{1}{r^5}\{3xz\,dx + 3yz\,dy + (3z^2 - r^2)\,dz\}$$

or
$$d\phi = -\frac{1}{r^5}\{3z\,(x\,dx + y\,dy + z\,dz) - r^2\,dz\}$$

or
$$d\phi = -\frac{3z}{2}\frac{d\,(x^2 + y^2 + z^2)}{r^5} + \frac{dz}{r^3}$$

or
$$d\phi = -\frac{3z}{2}\frac{d\,(r^2)}{r^5} + \frac{dz}{r^3} = -\frac{3z}{2}\cdot\frac{2r\,dr}{r^5} + \frac{dz}{r^3} = d\left(\frac{z}{r^3}\right)$$

By integrating, we have

$$\phi = \frac{z}{r^3} = \frac{r\cos\theta}{r^3} = \frac{\cos\theta}{r^2},$$

constant of integration vanishes.

The equations to the streamlines are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

or
$$\frac{dx}{3xz/r^5} = \frac{dy}{3yz/r^5} = \frac{dz}{(3z^2 - r^2)/r^5}$$

or
$$\frac{dx}{3xz} = \frac{dy}{3yz} = \frac{dz}{3z^2 - (x^2 + y^2 + z^2)} = \frac{x\,dx + y\,dy + z\,dz}{2z\,(x^2 + y^2 + z^2)}$$

(i) (ii) (iii) (iv)

From (i) and (ii), we have

$$\frac{dx}{x} = \frac{dy}{y} \Rightarrow \log x = \log y + \log c \Rightarrow x = cy. \qquad ...(1)$$

From (i) and (iv), we have

$$\frac{dx}{3x} = \frac{x\,dx + y\,dy + z\,dz}{2\,(x^2 + y^2 + z^2)}$$

By integrating, we have

$$\frac{2}{3}\log x = \frac{1}{2}\log (x^2 + y^2 + z^2) + \log D,$$

where D is an arbitrary constant.

$$x^{2/3} = D\,(x^2 + y^2 + z^2)^{1/2}. \qquad ...(2)$$

Thus the equation (1) and (2) represents the stream lines.

Problem 49: *For an incompressible fluid $u = -\omega y, v = \omega x, w = 0$, show that the surfaces intersecting the streamlines orthogonally exist and are the* *planes through Z-axis, although the velocity potential does not exist. Discuss the nature of flow.* (IAS-2003)

Solution. The motion will be possible if it satisfies the equation of continuity, that is,

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0,$$

which is true from the given relation. Hence the motion is a possible one.

The differential equation to the lines of flow are

$$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w}\Rightarrow\frac{dx}{-\omega y}=\frac{dy}{\omega x}=\frac{dz}{0}$$

or $\quad x\,dx+y\,dy=0$ and $dz=0$

By integrating, we have

$$x^2+y^2=\text{const., and } z=\text{const.}$$

The surfaces which cut the stream lines orthogonally are

$$u\,dx+v\,dy+w\,dz=0$$

or $$-\omega y\,dx+\omega x\,dy=0$$

By integrating, we have

$$dx/x-dy/y=0\Rightarrow\log(x/y)=\log c,$$

where c is an arbitrary constant.

Therefore $x=cy$, which represents a plane through Z-axis and cuts the stream line orthogonally.

The velocity potential will exist if $u\,dx+v\,dy+w\,dz$ is a perfect differential. But $u\,dx+v\,dy+w\,dz$ is not a perfect differential, therefore, the surfaces intersecting streamlines orthogonally exist and are the planes through Z-axis, although the velocity potential does not exist. Further

$$\nabla\times\mathbf{q}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ -\omega y & \omega x & 0\end{vmatrix}=2\omega\mathbf{k}.$$

Hence the flow is not of the potential kind. It shows that a rigid body rotating about Z-axis with constant angular velocity $\omega\mathbf{k}$ gives the same type of motion.

EQUATION OF MOTION | SET - II

Theorem 1. Euler's equation of motion : *To derive Euler's Dynamical equations.*

Proof : Let a closed surface S enclosing a volume V of a non-viscous fluid be moving with the fluid so that S contains the same number of fluid particles at any time t. Consider a point P inside S. Let ρ be the fluid density, q the fluid velocity and dV the elementary volume enclosing P. Since the mass $\rho\, dV$ remains unchanged throughout the motion so that

$$\frac{d}{dt}(\rho\, dV) = 0 \qquad \ldots (1)$$

The entire momentum M of the volume V is

$$\mathbf{M} = \int_V \mathbf{q}\,\rho\, dV$$

or momentum = mass × velocity

$$\therefore \quad \frac{d\mathbf{M}}{dt} = \int \left[\frac{d\mathbf{q}}{dt}\rho\, dV + \frac{d}{dt}(\rho\, dV)\,\mathbf{q}\right]$$

using (1),

$$\frac{d\mathbf{M}}{dt} = \int \frac{d\mathbf{q}}{dt}\rho\, dV. \qquad \ldots (2)$$

Let n be the unit outward normal vector on the surface element dS. Suppose F is the external force per unit mass acting on the fluid and p the pressure at any point on the element ds. Total surface force is

$$\int_V \mathbf{F}\rho\, dV + \int_S p(-\mathbf{n})\, dS$$

[For pressure acts along inward normal]

$$= \int_V \mathbf{F}\rho\, dV + \int_V -\nabla p\, dV. \qquad \text{by Gauss Theorem}$$

$$= \int_V (\mathbf{F}\rho - \nabla p)\, dV. \qquad \ldots (3)$$

By Newton's second law of motion,

rate of change of momentum = total applied force

i.e.,

$$\int \frac{d\mathbf{q}}{dt}\rho\, dV = \int (\mathbf{F}\rho - \nabla p)\, dV, \qquad \text{by (2) and (3)}$$

or

$$\int \left[\frac{d\mathbf{q}}{dt}\rho - \mathbf{F}\rho + \nabla p\right] dV = 0$$

Since S is arbitrary and so V is arbitrary so that the integrand of the last integral vanishes,

i.e.,

$$\frac{d\mathbf{q}}{dt}\rho - \mathbf{F}\rho + \nabla p = 0$$

or

$$\frac{d\mathbf{q}}{dt} = \mathbf{F} - \frac{1}{\rho}\nabla p \qquad \ldots (4)$$

This equation is known as Euler's equation of motion. If we write

$$\mathbf{q} = \mathbf{q}(u, v, w), \quad \mathbf{F} = \mathbf{F}(X, Y, Z)$$

then the cartesian equivalent of (4) is

$$\frac{d}{dt}(u\mathbf{i} + v\mathbf{j} + w\mathbf{k}) = (\mathbf{i}X + \mathbf{j}Y + \mathbf{k}Z) - \frac{1}{\rho}\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)p$$

This

$$\Rightarrow \frac{du}{dt} = X - \frac{1}{\rho}\frac{\partial p}{\partial x}, \quad \frac{dv}{dt} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y}, \quad \frac{dw}{dt} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

with

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}.$$

Deduction : (i) To derive symmetrical form.

Here we have $\mathbf{q} = u$, $\nabla = \frac{\partial}{\partial r}$,

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial r}, \quad \mathbf{F} = F.$$

Now (4) becomes

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial r}\right)u = F - \frac{1}{\rho}\frac{\partial p}{\partial r}.$$

(ii) To derive Lamb's hydrodynamical equation, By (4),

$$\left(\frac{\partial}{\partial t} + \mathbf{q}\cdot\nabla\right)\mathbf{q} = \mathbf{F} - \frac{1}{\rho}\nabla p \qquad \ldots (5)$$

or

$$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q}\cdot\nabla)\mathbf{q} = \mathbf{F} - \frac{1}{\rho}\nabla p$$

$$\because \quad \nabla(\mathbf{q}\cdot\mathbf{q}) = 2\,[\mathbf{q}\times \text{curl}\,\mathbf{q} + (\mathbf{q}\cdot\nabla)\mathbf{q}]$$

using this in (5),

$$\frac{\partial \mathbf{q}}{\partial t} + \nabla\left(\frac{1}{2}q^2\right) - \mathbf{q}\times\text{curl}\,\mathbf{q} = \mathbf{F} - \frac{1}{\rho}\nabla p$$

writing $\mathbf{W} = \text{curl}\,\mathbf{q}$, we obtain

$$\frac{\partial \mathbf{q}}{\partial t} + \nabla\left(\frac{1}{2}q^2\right) + \mathbf{W}\times\mathbf{q} = \mathbf{F} - \frac{1}{\rho}\nabla p$$

This is known as Lamb's hydrodynamical equation. The chief advantage of the is that it is invariant under a change of co-ordinate system.

(iii) Euler's equation in cylindrical co-ordinates.

Euler's equation of motion is

$$\frac{d\mathbf{q}}{dt} = \frac{D\mathbf{q}}{dt} = \mathbf{F} - \frac{1}{\rho}\nabla p \qquad \ldots (1)$$

Let (q_r, q_θ, q_z) be the velocity components and (F_r, F_θ, F_z) be the components of external force in r, θ, z directions. Then we know that

$$\frac{D\mathbf{q}}{Dt} = \left(\frac{Dq_r}{Dt} - \frac{q_\theta^2}{r}, \frac{Dq_\theta}{Dt} + \frac{q_r q_\theta}{r}, \frac{Dq_z}{Dt}\right)$$

$$\mathbf{F} = (F_r, F_\theta, F_z), \quad \nabla p = \left(\frac{\partial p}{\partial r}, \frac{1}{r}\frac{\partial p}{\partial \theta}, \frac{\partial p}{\partial z}\right)$$

Subsituting in (1) and equating the coefficient of i, j, k, we obtain Euler's equations of motion in cylindrical coordinates as :

$$\left.\begin{aligned} \frac{Dq_r}{Dt} - \frac{q_\theta^2}{r} &= F_r - \frac{1}{\rho}\frac{\partial p}{\partial r} \\ \frac{Dq_\theta}{Dt} + \frac{q_r q_\theta}{r} &= F_\theta - \frac{1}{\rho}\frac{\partial p}{r\partial \theta} \\ \frac{Dq_z}{Dt} &= F_z - \frac{1}{\rho}\frac{\partial p}{\partial z} \end{aligned}\right\} \qquad \ldots (2)$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + q_r\frac{\partial}{\partial r} + \frac{q_\theta}{r}\frac{\partial}{\partial \theta} + q_z\frac{\partial}{\partial z}. \qquad \ldots (3)$$

(iv) Euler's equations of motion in spherical polar coordinates :

Euler's equation of motion is

$$\frac{d\mathbf{q}}{dt} = \frac{D\mathbf{q}}{Dt} = \mathbf{F} - \frac{1}{\rho}\nabla p \qquad \ldots (1)$$

Let (q_r, q_θ, q_ϕ) be the velocity components and (F_r, F_θ, F_ϕ) be the components of external force in r, θ, ϕ directions. Then we know that

$$\frac{D\mathbf{q}}{Dt} = \left(\frac{Dq_r}{Dt} - \frac{q_\theta^2 + q_\phi^2}{r}, \frac{Dq_\theta}{Dt} - \frac{q_\phi^2\cot\theta}{r} + \frac{q_r q_\theta}{r}, \frac{Dq_\phi}{Dt} + \frac{q_\theta q_\phi\cot\theta}{r}\right)$$

$$\mathbf{F} = (F_r, F_\theta, F_\phi), \quad \nabla p = \left(\frac{\partial p}{\partial r}, \frac{1}{r}\frac{\partial p}{\partial r}, \frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi}\right).$$

Substituting in (1) and equating the coefficients of i, j, k we obtain Euler's equations of motion in spherical polar coordinates as :

$$\left.\begin{aligned} \frac{Dq_r}{Dt} - \frac{q_\theta^2 + q_\phi^2}{r} &= F_r - \frac{1}{\rho}\frac{\partial p}{\partial r} \\ \frac{Dq_\theta}{Dt} - \frac{q_\phi^2\cot\theta}{r} + \frac{q_r q_\theta}{r} &= F_\theta - \frac{1}{\rho}\frac{\partial p}{r\partial \theta} \\ \frac{Dq_\phi}{Dt} + \frac{1}{r}q_\theta q_\phi\cot\theta &= F_\phi - \frac{1}{\rho}\cdot\frac{1}{r\sin\theta}\cdot\frac{\partial p}{\partial \phi} \end{aligned}\right\} \qquad \ldots (2)$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + q_r\frac{\partial}{\partial r} + \frac{q_\theta}{r}\frac{\partial}{\partial \theta} + \frac{q_\phi}{r\sin\theta}\frac{\partial}{\partial \phi}$$

Definition

The velocity q is called Beltrani vector if q is parallel to W, *i.e.*, if $\mathbf{q}\times\mathbf{W} = 0$.

Def. A fluid is said to be barotropic if $p = f(\rho)$.

Def. Conservative field of force :

In a conservative field of force, the work done by a force F in taking a unit mass from a point A to a point B is independent of the path, *i.e.*,

$$\int_{ACB} \mathbf{F}\cdot d\mathbf{r} = \int_{ADB} \mathbf{F}\cdot d\mathbf{r} = -\underset{\text{say}}{\Omega},$$

Here Ω is a scalar function and is known as potential function. It can be proved that

$$\mathbf{F} = -\nabla\Omega.$$

Theorem 2. Pressure equation (Bernoulli's equation for unsteady motion). When velocity potential exists and forces are conservative and derivable from a potential Ω, the equations of motion can always be integrated and the solution is

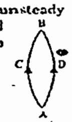


Fig. 2.1

$$\int \frac{dp}{\rho} - \frac{\partial \phi}{\partial t} + \frac{1}{2}q^2 + \Omega = F(t).$$

Proof : Existence of velocity potential ⇒ the motion is irrotational and $\mathbf{q} = -\nabla\phi$.

Forces are conservative ⇒ $\mathbf{F} = -\nabla\Omega$.

Let

$$P = \int_0^p \frac{dp}{\rho}, \text{ then } \frac{dP}{dp} = \frac{1}{\rho} \text{ so that } \nabla p = \Sigma\frac{i\,\partial P}{\partial x}$$

or

$$\nabla P = \Sigma i\frac{dP}{dp}\frac{\partial p}{\partial x} = \Sigma\frac{i}{\rho}\frac{\partial p}{\partial x} = \frac{1}{\rho}\nabla p \quad \text{or} \quad \nabla P = \frac{1}{\rho}\nabla p$$

By Euler's equation,

$$\frac{d\mathbf{q}}{dt} = \mathbf{F} - \frac{1}{\rho}\nabla p \quad \text{or} \quad \frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q}\cdot\nabla)\mathbf{q} = -\nabla\Omega - \nabla P$$

or

$$\frac{\partial}{\partial t}(-\nabla\phi) + \nabla(\Omega + P) + (\mathbf{q}\cdot\nabla)\mathbf{q} = 0$$

or

$$\nabla\left(-\frac{\partial \phi}{\partial t} + \Omega + P\right) + \frac{1}{2}\nabla q^2 - \mathbf{q}\times\text{curl}\,\mathbf{q} = 0$$

For

$$\nabla(\mathbf{q}\cdot\mathbf{q}) = 2\,[\mathbf{q}\times\text{curl}\,\mathbf{q} + (\mathbf{q}\cdot\nabla)\mathbf{q}]$$

or

$$\nabla\left(\Omega + p + \frac{1}{2}q^2 - \frac{\partial \phi}{\partial t}\right) = 0$$

[For $\text{curl}\,\mathbf{q} = \nabla\times\mathbf{q} = \nabla\times(-\nabla\phi) = -\text{curl grad}\,\phi = 0$].

Multiplying scalarly by $d\mathbf{r}$ and noting the $d\mathbf{r}\cdot\nabla = d$, we get

$$d\left(\Omega + P + \frac{1}{2}q^2 - \frac{\partial \phi}{\partial t}\right) = 0$$

Integrating, $\quad \Omega + P + \frac{1}{2}q^2 - \frac{\partial\phi}{\partial t} = F(t)$

where $F(t)$ is a constant of integration.

or $$\Omega + \int \frac{dp}{\rho} + \frac{1}{2}q^2 - \frac{\partial\phi}{\partial t} = F(t) \qquad \text{... (1)}$$

The equation is known as *Bernoulli's equation for unsteady irrotational motion.* This is also known as pressure equation.

If fluid is incompressible then (1) $\Rightarrow$

$$\Omega + \frac{p}{\rho} + \frac{1}{2}q^2 - \frac{\partial\phi}{\partial t} = F(t). \text{ For } \int \frac{dp}{\rho} = \frac{1}{\rho}\int dp = \frac{p}{\rho}.$$

Deduction : Suppose the motion is steady.

Then $\frac{\partial\phi}{\partial t} = 0$. Now (1) becomes

$$\Omega + \int \frac{dp}{\rho} + \frac{1}{2}q^2 = F(t) = C = \text{absolute constant}$$

or $$\Omega + \int \frac{dp}{\rho} + \frac{1}{2}q^2 = C.$$

This is known as *Bernoulli's equation for steady motion.*

If ρ = constant, then

$$\Omega + \frac{p}{\rho} + \frac{1}{2}q^2 = \text{const.}$$

Ex. *Derive Bernoulli's equation for unsteady motion of an incompressible fluid and hence derive expression for steady motion.*

Solution : Here write the above proof and its deduction complete.

Problem 1. *Show that the velocity field*

$$u(x, y) = \frac{B(x^2 - y^2)}{(x^2 + y^2)^2}, \; v(x, y) = \frac{2Bxy}{(x^2 + y^2)^2}, \; w = 0$$

satisfies the equation of motion for an inviscid incompressible flow. Determine the pressure associated with this velocity field. B is constant. (IFoS-2012, IAS-2006 model)

Solution : Euler's equation of motion in absence of external forces is

$$\frac{d\mathbf{q}}{dt} = -\frac{1}{\rho}\nabla p$$

or, $$\left(\frac{\partial}{\partial t} + \mathbf{q}\cdot\nabla\right)\mathbf{q} = -\frac{1}{\rho}\nabla p$$

But motion is two dimensional as $w = 0$ and $\mathbf{q} = u\mathbf{i} + v\mathbf{j}$

$$\therefore \quad \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\mathbf{q} = -\frac{1}{\rho}\left(\mathbf{i}\frac{\partial p}{\partial x} + \mathbf{j}\frac{\partial p}{\partial y}\right)$$

Putting the values,

$$\left[\frac{\partial}{\partial t} + \frac{B(x^2 - y^2)}{(x^2 + y^2)^2}\frac{\partial}{\partial x} + \frac{2Bxy}{(x^2 + y^2)^2}\frac{\partial}{\partial y}\right](u\mathbf{i} + v\mathbf{j}) = -\frac{1}{\rho}\left(\mathbf{i}\frac{\partial p}{\partial x} + \mathbf{j}\frac{\partial p}{\partial y}\right)$$

As u, v are independent of t, by assumption.

$\therefore \quad \frac{\partial u}{\partial t} = 0 = \frac{\partial v}{\partial t}$. Hence the last gives

$$\frac{B}{(x^2 + y^2)^2}\left[(x^2 - y^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}\right](u\mathbf{i} + v\mathbf{j}) = -\frac{1}{\rho}\left(\mathbf{i}\frac{\partial p}{\partial x} + \mathbf{j}\frac{\partial p}{\partial y}\right)$$

This $\Rightarrow$ $$-\frac{1}{\rho}\frac{\partial p}{\partial x} = \frac{B}{(x^2 + y^2)^2}\left[(x^2 - y^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}\right]\frac{B(x^2 - y^2)}{(x^2 + y^2)^2} \qquad \text{... (1)}$$

and $$-\frac{1}{\rho}\frac{\partial p}{\partial y} = \frac{B}{(x^2 + y^2)^2}\left[(x^2 - y^2)\frac{\partial}{\partial x} + 2xy\frac{\partial}{\partial y}\right]\frac{2Bxy}{(x^2 + y^2)^2} \qquad \text{... (2)}$$

But $$\frac{\partial}{\partial x}\left\{\frac{x^2 - y^2}{(x^2 + y^2)^2}\right\} = \frac{2x(3y^2 - x^2)}{(x^2 + y^2)^3} \qquad \text{... (3)}$$

$$\frac{\partial}{\partial y}\left\{\frac{x^2 - y^2}{(x^2 + y^2)^2}\right\} = -\frac{2y(3x^2 - y^2)}{(x^2 + y^2)^3} \qquad \text{... (4)}$$

$$\frac{\partial}{\partial x}\left\{\frac{2xy}{(x^2 + y^2)^2}\right\} = \frac{2y(y^2 - x^2)}{(x^2 + y^2)^3} \qquad \text{... (5)}$$

$$\frac{\partial}{\partial y}\left\{\frac{2xy}{(x^2 + y^2)^2}\right\} = \frac{2x(x^2 - y^2)}{(x^2 + y^2)^3} \qquad \text{... (6)}$$

Writing (1) with the help of (3) and (4),

$$\frac{\partial p}{\partial x} = \frac{-2\rho B^2}{(x^2 + y^2)^5}[(x^2 - y^2)x(3y^2 - x^2) - 2xy^2(3x^2 - y^2)]$$

or $$\frac{\partial p}{\partial x} = \frac{2\rho B^2 x}{(x^2 + y^2)^3} \qquad \text{... (7)}$$

Writing (2) with the help of (5) and (6),

$$\frac{\partial p}{\partial y} = \frac{2\rho B^2}{(x^2 + y^2)^5}[(x^2 - y^2)y(y^2 - x^2) + 2x^2y(x^2 - y^2)]$$

or $$\frac{\partial p}{\partial y} = \frac{2B^2 y(x^2 - y^2)}{(x^2 + y^2)^4} \qquad \text{... (8)}$$

Differentiating (7) and (8) partially w.r.t. y and x we find that

$$\frac{\partial^2 p}{\partial y\,\partial x} = \frac{\partial^2 p}{\partial x\,\partial y} \text{ (Prove it)}$$

This proves that velocity field satisfies the equation of motion.

$$dp = -\frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial y}dy$$

Using (7) and (8),

$$dp = 2\rho B^2\left[\frac{x\,dx}{(x^2 + y^2)^3} - \frac{y(x^2 - y^2)}{(x^2 + y^2)^4}dy\right]$$
$$= 2\rho B^2[M\,dx + N\,dy], \text{ say} \qquad \text{... (9)}$$

$$\frac{\partial M}{\partial y} = -\frac{6xy}{(x^2 + y^2)^4} = \frac{\partial N}{\partial x}$$

$\therefore \quad M\,dx + N\,dy$ is exact.

$$\int (M\,dx + N\,(dy)) = \int \frac{x\,dx}{(x^2 + y^2)^3} + \int 0\,dy$$

$$= \frac{1}{2}\int 2x(x^2 + y^2)^{-3}dx + c = -\frac{1}{4(x^2 + y^2)^2} + c$$

In view of this (9) becomes,

$$p = -\frac{2\rho B^2}{4(x^2 + y^2)^2} + c_1$$

or, $$p = -\frac{\rho B^2}{2(x^2 + y^2)^2} + c_1$$

This is the required expression for pressure.

Problem 2. *The particle velocity for a fluid motion referred to rectangular axis is given by the components*

$$u = A\cos\frac{\pi x}{2a}\cos\frac{\pi z}{2a}, \; v = 0, \; w = A\sin\frac{\pi x}{2a}\sin\frac{\pi z}{2a},$$

where A is a constant. Show that this is a possible motion of an incompressible fluid under no body forces in an infinite fixed rigid tube, $-a \le x \le a$, $0 \le z \le 2a$. Also, find the pressure associated with this velocity field.

Solution : The equations of motion for a two-dimensional steady, inviscid, incompressible flow under no body force, in cartesian coordinates, are given by

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x},$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial y},$$

$$u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial z}. \qquad \text{... (1, 2, 3)}$$

Here $$u = A\cos\frac{\pi x}{2a}\cos\frac{\pi z}{2a}, \; v = 0, \; w = A\sin\frac{\pi x}{2a}\sin\frac{\pi z}{2a}. \qquad \text{... (4)}$$

From equation (2), it follows that the pressure p is independent of y *i.e.*, $p = p(x, z)$.

Using (4) into (1) and (3), we have

$$\left(A\cos\frac{\pi x}{2a}\cos\frac{\pi z}{2a}\right)\left(-\frac{\pi A}{2a}\sin\frac{\pi x}{2a}\cos\frac{\pi z}{2a}\right) + \left(A\sin\frac{\pi x}{2a}\sin\frac{\pi z}{2a}\right)$$
$$\times\left(-\frac{\pi A}{2a}\cos\frac{\pi x}{2a}\sin\frac{\pi z}{2a}\right) = -\frac{1}{\rho}\frac{\partial p}{\partial x}.$$

or $$\frac{\pi A^2}{2a}\left[\cos\frac{\pi x}{2a}\sin\frac{\pi x}{2a}\cos^2\frac{\pi z}{2a} + \cos\frac{\pi x}{2a}\sin\frac{\pi x}{2a}\sin^2\frac{\pi z}{2a}\right] = \frac{1}{\rho}\frac{\partial p}{\partial x}$$

or $$\frac{\pi A^2}{2a}\cos\frac{\pi x}{2a}\sin\frac{\pi x}{2a} = \frac{1}{\rho}\frac{\partial p}{\partial x}. \qquad \text{... (5)}$$

and $$\left(A\cos\frac{\pi x}{2a}\cos\frac{\pi z}{2a}\right)\left(\frac{\pi A}{2a}\cos\frac{\pi x}{2a}\sin\frac{\pi z}{2a}\right) + \left(A\sin\frac{\pi x}{2a}\sin\frac{\pi z}{2a}\right)$$
$$\times\left(\frac{\pi A}{2a}\sin\frac{\pi x}{2a}\cos\frac{\pi z}{2a}\right) = -\frac{1}{\rho}\frac{\partial p}{\partial z}.$$

or $$\frac{\pi A^2}{2a}\left[\cos\frac{\pi z}{2a}\sin\frac{\pi z}{2a}\cos^2\frac{\pi x}{2a} + \cos\frac{\pi z}{2a}\sin\frac{\pi z}{2a}\sin^2\frac{\pi x}{2a}\right] = -\frac{1}{\rho}\frac{\partial p}{\partial z}.$$

or $$\frac{\pi A^2}{2a}\cos\frac{\pi z}{2a}\sin\frac{\pi z}{2a} = -\frac{1}{\rho}\frac{\partial p}{\partial z}. \qquad \text{... (6)}$$

The equations (5) and (6) show that the velocity components satisfy the equations of motion.

Again, $$dp = \frac{\partial p}{\partial x}dx + \frac{\partial p}{\partial z}dz$$

or $$dp = \frac{\pi\rho A^2}{2a}\left[\cos\frac{\pi x}{2a}\sin\frac{\pi x}{2a}dx - \cos\frac{\pi z}{2a}\sin\frac{\pi z}{2a}dz\right]$$

By integrating, we have

$$p = \frac{1}{2}\rho A^2\left[\cos^2\frac{\pi z}{2a} - \cos^2\frac{\pi x}{2a}\right] + C,$$

where C is an integration constant. This gives the reqruied pressure distribution.

Problem 3. *Determine the pressure, if the velocity field $q_r = 0$, $q_\theta = Ar + B$, $q_z = 0$, satisfies the equation of motion $\rho\frac{q_\theta^2}{r} = \frac{dp}{dr}$, where A and B are arbitrary constants.*

Solution : $$\frac{dp}{dr} = \rho\frac{1}{r}\left(Ar + \frac{B}{r}\right)^2$$

or $$\frac{dp}{dr} = \rho\left(A^2 r + \frac{B^2}{r^3} + 2AB\frac{1}{r}\right)$$

By integrating, we have

$$p = \rho\left(\frac{1}{2}A^2r^2 - \frac{B^2}{2r^2} + 2AB\log r\right) + C,$$

where C is an integration constant.

Initially, *i.e.*, at $t = 0$, $x = a$, $y = b$, $z = c$ so that

$$\frac{\partial x}{\partial a} = 1,\ \frac{\partial y}{\partial b} = 1,\ \frac{\partial z}{\partial c} = 1,\ \frac{\partial x}{\partial b} = 0,\ \frac{\partial y}{\partial c} = 0 \text{ etc.}$$

Subjecting (6) to this condition,

$$(Q - 0)_0 + \left(0 - \frac{\partial v}{\partial c} \cdot 1\right)_0 + \left(\frac{\partial w}{\partial b} \cdot 1 - 0\right)_0 = c$$

or $$c = \left(\frac{\partial w}{\partial b}\right)_0 - \left(\frac{\partial v}{\partial c}\right)_0 = \xi_0$$

where $$W = \text{curl } q = \xi i + \eta j + \zeta k$$

$$\left(\frac{\partial u}{\partial b}\frac{\partial x}{\partial c} - \frac{\partial u}{\partial c}\frac{\partial x}{\partial b}\right) + \left(\frac{\partial v}{\partial b}\frac{\partial y}{\partial c} - \frac{\partial v}{\partial c}\frac{\partial y}{\partial b}\right) + \left(\frac{\partial w}{\partial b}\frac{\partial z}{\partial c} - \frac{\partial w}{\partial c}\frac{\partial z}{\partial b}\right) = \xi_0$$

or $$\left[\frac{\partial x}{\partial c}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial b} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial b} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial b}\right) - \frac{\partial x}{\partial b}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial c} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial c} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial c}\right)\right]$$

$$+ \left[\frac{\partial y}{\partial c}(\ldots) - \frac{\partial y}{\partial b}(\ldots)\right] + \left[\frac{\partial z}{\partial c}(\ldots) - \frac{\partial z}{\partial b}(\ldots)\right] = \xi_0$$

or $$\frac{\partial (y, z)}{\partial (b, a)}\xi + \frac{\partial (z, x)}{\partial (b, c)}\eta + \frac{\partial (x, y)}{\partial (b, c)}\zeta = \xi_0 \qquad \ldots (7)$$

Similarly $$\frac{\partial (y, z)}{\partial (c, a)}\xi + \frac{\partial (z, x)}{\partial (c, a)}\eta + \frac{\partial (x, y)}{\partial (c, a)}\zeta = \eta_0 \qquad \ldots (8)$$

$$\frac{\partial (y, z)}{\partial (a, b)}\xi + \frac{\partial (y, z)}{\partial (a, b)}\eta + \frac{\partial (y, z)}{\partial (a, b)}\zeta = \zeta_0 \qquad \ldots (9)$$

Multiplying (7), (8), (9) by

$$\frac{\partial x}{\partial a},\ \frac{\partial x}{\partial b},\ \frac{\partial x}{\partial c}$$

respectively and adding columnwise,

$$\xi \frac{\partial (x, y, z)}{\partial (a, b, c)} + \eta \frac{\partial (z, x, z)}{\partial (a, b, c)} + \zeta \frac{\partial (x, y, z)}{\partial (a, b, c)} = \xi_0 \frac{\partial x}{\partial a} + \eta_0 \frac{\partial x}{\partial b} + \zeta_0 \frac{\partial x}{\partial c}.$$

But $$\rho \frac{\partial (x, y, z)}{(a, b, c)} = \rho J = \rho_0$$

Hence $$\xi \frac{\rho_0}{\rho} = \xi_0 \frac{\partial x}{\partial a} + \eta_0 \frac{\partial x}{\partial b} + \zeta_0 \frac{\partial x}{\partial c}$$

or $$\frac{\xi}{\rho} = \frac{\xi_0}{\rho_0}\frac{\partial x}{\partial a} + \frac{\eta_0}{\rho_0}\frac{\partial x}{\partial b} + \frac{\zeta_0}{\rho_0}\frac{\partial x}{\partial c} \qquad \ldots (10)$$

Similarly $$\frac{\eta}{\rho} = \frac{\xi_0}{\rho_0}\frac{\partial y}{\partial a} + \frac{\eta_0}{\rho_0}\frac{\partial y}{\partial b} + \frac{\zeta_0}{\rho_0}\frac{\partial y}{\partial c} \qquad \ldots (11)$$

and $$\frac{\zeta}{\rho} = \frac{\xi_0}{\rho_0}\frac{\partial z}{\partial a} + \frac{\eta_0}{\rho_0}\frac{\partial z}{\partial b} + \frac{\zeta_0}{\rho_0}\frac{\partial z}{\partial c}. \qquad \ldots (12)$$

The euations (10), (11) and (12) are called *Cauchy integrals*. The vector form of these equations is

$$\frac{W}{\rho} = \left(\frac{\xi_0}{\rho_0}\frac{\partial}{\partial a} + \frac{\eta_0}{\rho_0} + \frac{\zeta_0}{\rho_0}\frac{\partial}{\partial c}\right) r$$

or $$\frac{W}{\rho} = \left(\frac{W}{\rho} . \nabla\right)_0 r.$$

Deduction 1. *To prove the principle of permanence of irrotational motion.*

Proof : If $W_0 = 0$, *i.e.* if $\xi_0 = \eta_0 = \zeta_0 = 0$, then (10), (11), (12)

$$\Rightarrow \quad \xi = \eta = \zeta = 0$$

i.e., $\quad W_0 = 0 \Rightarrow W = 0$

This proves that if the motion be irrotational initially, then it is always irrotational for all time. This establishes the principle of irrotational motion for all time t.

Deduction 2. *To prove Cauchy's integrals are the integrals of Helmholtz vorticity equations.*

Or *To prove Helmholtz equations with the help of Cauchy's integrals.*

Proof : $(10) \times \frac{\partial u}{\partial x} + (11) \times \frac{\partial u}{\partial y} + (12) \times \frac{\partial u}{\partial z}$ gives

$$\frac{1}{\rho}\left(\xi \frac{\partial u}{\partial x} + \eta \frac{\partial u}{\partial y} + \zeta \frac{\partial u}{\partial z}\right) = \frac{\xi_0}{\rho_0}\left[\frac{\partial u}{\partial x}\frac{\partial x}{\partial a} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial a} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial a}\right) + \ldots + \ldots$$

$$= \frac{\xi_0}{\rho_0}\frac{\partial u}{\partial a} + \frac{\eta_0}{\rho_0}\frac{\partial u}{\partial b} + \frac{\zeta_0}{\rho_0}\frac{\partial u}{\partial c} = \frac{d}{dt}\left(\frac{\xi}{\rho}\right), \text{ according to (10).}$$

$\therefore$ $$\frac{1}{\rho}\left(\xi \frac{\partial u}{\partial x} + \eta \frac{\partial u}{\partial y} + \zeta \frac{\partial u}{\partial z}\right) = \frac{d}{\partial t}\left(\frac{\xi}{\rho}\right)$$

Similarly $$\frac{1}{\rho}\left(\xi \frac{\partial v}{\partial x} + \eta \frac{\partial v}{\partial y} + \zeta \frac{\partial v}{\partial z}\right) = \frac{d}{\partial t}\left(\frac{\eta}{\rho}\right)$$

and $$\frac{1}{\rho}\left(\xi \frac{\partial w}{\partial x} + \eta \frac{\partial w}{\partial y} + \zeta \frac{\partial w}{\partial z}\right) = \frac{d}{\partial t}\left(\frac{\zeta}{\rho}\right).$$

This is equivalent to single vector equation;

$$\frac{1}{\rho}\left(\xi \frac{\partial q}{\partial x} + \eta \frac{\partial q}{\partial y} + \zeta \frac{\partial q}{\partial z}\right) = \frac{d}{dt}\left(\frac{W}{\rho}\right)$$

or $$\frac{1}{\rho}(W . \nabla) q = \frac{d}{dt}\left(\frac{W}{\rho}\right)$$

or $$\frac{d}{dt}\left(\frac{W}{\rho}\right) = \left(\frac{W}{\rho} . \nabla\right) q.$$

This is known as Helmholtz vorticity equation.

Theorem 8. Equations for Impulsive Action : To obtain general equations of motion for impulsive action.

Proof : Consider an arbitrary closed surface S moving with a non-viscous fluid such that it encloses a volume V. Let q_1 and q_2 be fluid velocities at P within S just before the impulse and just after the impulse. Let ρ be fluid density at P. Suppose I is the external impulse per unit mass and $\bar{\omega}$ the impulse pressure on a surface element dS. Also let n be unit outward normal vecotr.

Change of momentum = Total impulsive forces

i.e., $$\int \rho (q_2 - q_1)\, dV = \int I\rho\, dV + \int -n\bar{\omega}\, dS$$

[For ω acts along inward normal]

By Gauss theorem the last gives

$$\int [\rho (q_2 - q_1) - I\rho + \nabla\bar{\omega}]\, dV = 0.$$

Since the surface S is arbitrary and hence the integrand of the last integral vanishes.

$\therefore$ $$\rho (q_2 - q_1) - I\rho + \nabla\bar{\omega} = 0$$

or $$q_2 - q_1 = I - \frac{1}{\rho}\nabla\bar{\omega} \qquad \ldots (1)$$

This is the required equation for impulsive action. If

$$I = I(X, Y, Z),\quad q_2 = q_2(u, v, w),\quad q_1 = q_1(u_0, v_0, w_0),$$

then the cartesian equivalent of (1) is

$$u - u_0 = X - \frac{1}{\rho}\frac{\partial\bar{\omega}}{\partial x};\quad v - v_0 = Y - \frac{1}{\rho}\frac{\partial\bar{\omega}}{\partial y},\quad w - w_0 = Z - \frac{\partial\bar{\omega}}{\partial z}.$$

Deduction (i). Vorticity in a non-vicous incompressible fluid is never generated by impulses if the extenal forces are conservative.

Proof : External impulses are conservative $\Rightarrow I = -\nabla\Omega$.

Fluid is incompressible $\Rightarrow \rho$ is constant.

By (1), $$q_2 - q_1 = -\nabla\left(\Omega + \frac{\bar{\omega}}{\rho}\right)$$

or $$\nabla \times (q_2 - q_1) = 0 \text{ as } \nabla \times \nabla = \text{curl grad} = 0$$

or $$\text{curl } q_2 = \text{curl } q_1 \text{ or } W_2 = W_1.$$

From this the required result follows.

(ii) To prove $\nabla^2\bar{\omega} = 0$ under suitable conditions.

Proof : Let the external impulse be absent so that $I = 0$. Also let ρ be constant. Then (1) gives

$$q_2 - q_1 = -\nabla\left(\frac{\bar{\omega}}{\rho}\right) \qquad \ldots (2)$$

or $$\nabla . (q_2 - q_1) = -\nabla . \nabla\left(\frac{\bar{\omega}}{\rho}\right) = -\nabla^2\left(\frac{\bar{\omega}}{\rho}\right)$$

or $$\nabla^2\bar{\omega} = \rho[-\nabla . q_2 + \nabla . q_1] = \rho[-0 + 0] \quad \text{or} \quad \nabla^2\bar{\omega} = 0.$$

For $\nabla . q_1 = 0 = \nabla . q_2$ is the equation of continuity.

Remark : If the motion is irrotational, then

$$-\nabla\phi_2 + \nabla\phi_1 = -\nabla\left(\frac{\bar{\omega}}{\rho}\right), \text{ by (2)}$$

or $$\nabla[\rho(\phi_2 - \phi_1) - \bar{\omega}] = 0$$

Integrating, $\rho(\phi_2 - \phi_1) - \bar{\omega} = 0$, neglecting constant of integration

or $$\bar{\omega} = \rho(\phi_2 - \phi_1).$$

(iii) To prove $\bar{\omega} = \rho\phi$ under suitable conditions. Let the external impulse be absent so that $I = 0$. Also let ρ be constant and motion starts from rest. Then (1) gives

$$q_2 - 0 = 0 - \frac{1}{\rho}\nabla\bar{\omega}.$$

Since the motion starts from rest by the application of impulsive pressure hence it must be irrotational. Then $q = -\nabla\phi$.

$\therefore$ $$-\nabla\phi = -\frac{1}{\rho}\nabla\bar{\omega} \quad \text{or} \quad \nabla(\rho\phi - \bar{\omega}) = 0.$$

Integrating it, $\rho\phi - \bar{\omega} = 0$, neglecting constant of integration.

or $$\bar{\omega} = \rho\phi.$$

If $$\rho = 1, \text{ then } \bar{\omega} = \phi.$$

Remark : If $I = 0$, $q_2 = 0$, then

$$(1) \Rightarrow -q_1 = -\frac{1}{\rho}\nabla\bar{\omega}.$$

Further, if velocity has one component, then this gives

$$u = \frac{1}{\rho}\frac{\partial\bar{\omega}}{\partial x} = \frac{1}{\rho}\frac{d\bar{\omega}}{dx}$$

or $$d\bar{\omega} = \rho u\, dx.$$

This equation is very important for further study.

Def. Flow : Consider any two points A and B in a fluid. The value of the integral

$$\int_A^B (u\, dx + v\, dy + w\, dz) = \int_A^B q.dr$$

taken along any path in the fluid, is called flow from A to B along that path. If the motion is irrotational, then the flow is

$$\int_A^B q.dr = \int_A^B -\nabla\phi.dr = -\int_A^B d\phi = \phi_A - \phi_B$$

where ϕ_A and ϕ_B denote velocity potentials at A and B, respectively.

Def. Circulation :

Flow along a closed path c is defined as circulation.

$$\text{circulation} = \int_c q.dr.$$

If the motion is irrotational, then circulation $= \phi_A - \phi_B = \phi_A - \phi_A = 0$.

For a closed path, points A and B coincide.

Theorem 9. Kelvin's Circulation theorem : The circulation along any closed path moving with the fluid is constant for all times if the external forces are conservative and density ρ is function of pressure p only.

Proof : Let c be a closed path and cir denotes circulation. Then

$$\text{cir} = \int_c \mathbf{q}.d\mathbf{r}.$$

$$\therefore \quad \frac{d}{dt}(\text{cir}) = \int_c \left[\frac{d\mathbf{q}}{dt}.d\mathbf{r} + \mathbf{q}.\frac{d}{dt}(d\mathbf{r})\right]$$

$$= \int_c \left[\frac{d\mathbf{q}}{dt}.d\mathbf{r} + \mathbf{q}.d\mathbf{q}\right] \left[\text{For } \mathbf{q}.\frac{d}{dt}(d\mathbf{r}) = \mathbf{q}.d\left(\frac{d\mathbf{r}}{dt}\right)\right]$$

$$= \int_c \left[\left(\mathbf{F} - \frac{1}{\rho}\nabla p\right).d\mathbf{r} + d\left(\frac{1}{2}q^2\right)\right],$$

by Euler's equation.

$$= \int_c \left(-\nabla\Omega - \frac{1}{\rho}\nabla p\right).d\mathbf{r} + d\left(\frac{1}{2}q^2\right)$$

$$= \int_c \left[\left(-\frac{dp}{\rho} - d\Omega\right) + d\left(\frac{1}{2}q^2\right)\right] \text{ as } d\mathbf{r}.\nabla = d$$

$$= -\left[\Omega - \frac{1}{2}q^2 + \int \frac{dp}{\rho}\right] = 0. \qquad \dots(1)$$

For, on R.H.S. of (1), the quantities involved are single valued and on passing once round the circuit, the change expressed in (1) is zero. Thus $\frac{d}{dt}(\text{cir}) = 0$.

This ⇒ circulation is constant along c for all times.

Theorem 10. Permanence of irrotational motion : *If the motion of a non viscous fluid is once irrotational, it remains irrotational even afterwards provided the external forces are conservative and density ρ is a function of pressure p only.*

Proof : Let c denote a closed path moving with the fluid and cir denotes circulation.

Then $\text{cir} = \int_c \mathbf{q}.d\mathbf{r} = \int_S \mathbf{n}.\text{curl}\,\mathbf{q}\,dS$, by Stoke's theorem.

Suppose motion is once irrotational. Then cir along c is zero. By Kelvin's theorem cir is constant for all times along c. Consequently cir along c is zero for all times,

i.e., $\quad \text{cir} = 0 \quad \forall\, t$ along c.

Then $\int_S \mathbf{n}.\text{curl}\,\mathbf{q}\,dS = 0$. Also S is arbitrary.

Hence n.curl q = 0 or curl q = 0; this ⇒ motion is irrotational for all times. Hence motion is permanently irrotational.

Theorem 11. *To obtain equation of energy.*

Proof : Consider an arbitrary closed surface S moving with a non-viscous fluid s.t. it encloses a volume V. Let n be the unit inward drawn normal vector on an element dS. Let the force be conservative so that $\mathbf{F} = -\nabla\Omega$. Since force potential Ω is supposed to be independent of time, so that

$$\frac{\partial\Omega}{\partial t} = 0. \text{ Further } \frac{d}{dt} = \frac{\partial}{\partial t} + (\mathbf{q}.\nabla)$$

Hence $$\frac{d\Omega}{dt} = \frac{\partial\Omega}{\partial t} + (\mathbf{q}.\nabla)\,\Omega = (\mathbf{q}.\nabla)\,\Omega \qquad \dots(1)$$

Let T, W, I denote kinetic energy, potential energy and intrinsic energy respectively. Since Ω is force potential per unit mass hence

$$W = \int \Omega\, dm = \int \Omega\rho\, dV$$

$$T = \int \frac{1}{2}\rho q^2\, dV = \frac{1}{2}\int q^2 \rho\, dV$$

Since elementary mass remains invariant throughout the motion hence

$$\frac{d}{dt}(\rho\, dV) = 0.$$

Now $$\frac{dT}{dt} = \frac{1}{2}\int \frac{dq^2}{dt}\rho\, dV + \frac{1}{2}\int q^2.0 = \int \mathbf{q}.\frac{d\mathbf{q}}{dt}\rho\, dV + 0$$

$$[\text{as } q^2 = \mathbf{q}^2 = \mathbf{q}.\mathbf{q}]$$

$$\frac{dW}{dt} = \int \frac{d\Omega}{dt}\rho\, dV + \int \Omega.0 = \int \frac{d\Omega}{dt}\rho\, dV + 0$$

Intrinsic energy E per unit mass of the fluid is defined as the work done by the unit mass of the fluid against external pressure p under the supposed relation between pressure and density from its actual state to some standard state in which pressure and density are p_0 and ρ_0 respectively. Then

$$I = \int E\rho\, dV, \quad E = \int_V^{V_0} p\, dV \text{ where } V\rho = 1$$

$$= \int_\rho^{\rho_0} p\, d\left(\frac{1}{\rho}\right) = -\int_\rho^{\rho_0} \frac{p}{\rho^2}\, d\rho$$

or $$E = \int_{\rho_0}^{\rho} \frac{p}{\rho^2}\, d\rho. \quad \text{Hence } dE = \frac{p}{\rho^2}\, d\rho$$

$$\frac{dI}{dt} = \int \left[\frac{dE}{dt}\rho\, dV + E\frac{d}{dt}(\rho\, dV)\right] = \int \frac{dE}{dt}\rho\, dV + 0$$

$$= \int \frac{dE}{d\rho}\frac{d\rho}{dt}\rho\, dV = \int \frac{p}{\rho^2}\frac{d\rho}{dt}\rho\, dV = \int \frac{p}{\rho}\frac{d\rho}{dt}\, dV$$

$$= \int \frac{p}{\rho}(-\rho\,\nabla.\mathbf{q})\, dV \qquad \text{as } \frac{d\rho}{dt} + \rho\,\nabla.\mathbf{q} = 0$$

is the equation of continuity.

or $$\frac{dI}{dt} = -\int p\,(\nabla.\mathbf{q})\, dV$$

Finally, $$\frac{dT}{dt} = \int \mathbf{q}.\frac{d\mathbf{q}}{dt}\rho\, dv;\ \frac{dW}{dt} = \int \frac{d\Omega}{dt}\rho\, dV, \qquad \dots(2)$$

$$\frac{dI}{dt} = -\int p\,(\nabla.\mathbf{q})\, dV \qquad \dots(3)$$

By Euler's equation, $\frac{d\mathbf{q}}{dt} = -\nabla\Omega - \frac{1}{\rho}\nabla p$

$$\therefore \quad \mathbf{q}.\frac{d\mathbf{q}}{dt}\rho\, dV = -[(\mathbf{q}.\nabla)\,\Omega]\,\rho\, dV - (\mathbf{q}.\nabla p)\, dV$$

Integrating over V and using (2),

$$\frac{dT}{dt} + \int (\mathbf{q}.\nabla\Omega)\rho\, dV + \int (\mathbf{q}.\nabla p)\, dV = 0$$

or $$\frac{dT}{dt} + \int \frac{d\Omega}{dt}\rho\, dV + \int (\mathbf{q}.\nabla p)\, dV = 0, \text{ by (1)}$$

or $$\frac{dT}{dt} + \frac{dW}{dt} + \int (\mathbf{q}.\nabla p)\, dV = 0, \qquad \dots(4), \text{ by (2)}$$

But $$\nabla.(p\mathbf{q}) = p\,\nabla.\mathbf{q} + \mathbf{q}.\nabla p$$

or $$\int \nabla.(p\mathbf{q})\, dV - \int p\,\nabla.\mathbf{q}\, dV = \int (\mathbf{q}.\nabla p)\, dV$$

or $$\int -\hat{\mathbf{n}}.(p\mathbf{q})\, dS + \frac{dI}{dt} = \int (\mathbf{q}.\nabla p)\, dV, \text{ by (3),}$$

as $\hat{\mathbf{n}}$ is inward normal.

Now (4) becomes $$\frac{d}{dt}(T + W + I) - \int \mathbf{n}.(p\mathbf{q})\, dS = 0.$$

or $$\frac{d}{dt}(T + W + I) = \int_S \mathbf{n}.(p\mathbf{q})\, dS$$

This is energy equation. This proves that : rate of change of total energy (K.E. + Potential + Intrinsic) of a portion of fluid is equal to the work done by external pressure on the boundary provided the external forces are conservative.

Corollary. Principle of energy for incompressible fluids. In the present case $I = 0$. Hence the rate of change of total energy (K.E. + P.E.) is equal to the work done by the pressure on the boundary.

WORKING RULES

In order to solve the equations of motion, we adopt the following techniques :

(1) Equation of motion is $\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = F - \frac{1}{\rho}\frac{\partial p}{\partial x}$

where $$F = -\frac{\partial\Omega}{\partial x}.$$

(2) Equation of continuity (i) $x^2 v = F(t)$ for spherical symmetry if ρ = const.

(ii) $xv = F(t)$ for cylindrical symmetry if ρ = const.

(iii) $\frac{\partial\rho}{\partial t} + \frac{\partial \rho v}{\partial x} = 0$ (geneal case)

(3) Generally the fluid is assumed to be at rest at infinity,

i.e., $x = \infty$, $v = 0$, $p = \Pi$, say.

(4) If r be the radius of cavity (or hollow sphere), then $x = r$, $v = \dot{r}$, $p = 0$.

(5) When $r = a$, $v = 0$ so that $F(t) = 0$.

(6) Boyle's law : $p_1 V_1 = p_2 V_2$ = const. Its alternate form is $p = k\rho$.

(7) Flux = cross sectional area . normal velocity . density.

(8) Equation of impulsive action is $d\bar{\omega} = \rho v\, dx = \rho v'\, dr'$

$$\int_0^{\pi/2} \sin^p\theta\cos^q\theta\, d\theta = \Gamma\left(\frac{p+1}{2}\right).\Gamma\left(\frac{q+1}{2}\right)\Big/2\Gamma\left(\frac{p+q+2}{2}\right)$$

$$\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi} \text{ and } \Gamma(n)\,\Gamma\left(n+\frac{1}{2}\right) = \frac{\sqrt{\pi}\,\Gamma(2n)}{2^{2n-1}}$$

(9) K.E. of the liquid = work done = $\int -p\,dV$.

(10) If a sphere of radius a is annihilated, then when $x = a$, $p = 0$ so that $v = \dot{x} = 0$.

(11) If a problem contains external and internal radii, *i.e.*, R and r, then subject the result (which is obtained from the integration of the equation of motion) to the two boundary conditions for R and r. In this way we obtain an equation free from constant of integration C and pressure p. Again we integrate this equation to obtain the required result.

SOLVED EXAMPLES

Problem 1. *A sphere is at rest in an infinite mass of homogeneous liquid of density, ρ, the pressure at infinity being Π, show that, if the radius R of the sphere varies in any manner, the pressure at the surface of the sphere at any time is*

$$\Pi + \frac{1}{2}\rho\left[\frac{d^2R^2}{dt^2} + \left(\frac{dR}{dt}\right)^2\right].$$

If $R = a\,(2 + \cos nt)$, show that, to prevent cavitation in the fluid, Π must not be less than $3\rho\, a^2 n^2$.

Problem 4. *For an inviscid incompressible, steady flow with negligible body forces, velocity components in spherical polar coordinates are given by*

$$u_r = V\left(1-\frac{R^3}{r^3}\right)\cos\theta,\quad u_\theta = -V\left(1+\frac{R^3}{2r^3}\right)\sin\theta,$$

$u_\phi = 0$. *Show that it is a possible solution of momentum equations (i.e., equations of motion) R and V are constants.*

Solution : Write $u_r = u,\ u_\theta = v,\ u_\phi = w$. Then

$$u = V\left(1-\frac{R^3}{r^3}\right)\cos\theta,\quad v = -V\left(1+\frac{R^3}{2r^3}\right)\sin\theta,\quad w = 0.$$

To show that velocity components satisfy equation of momentum, we have to show that the velocity components satisfy Euler's equation of motion.

$$\frac{d\mathbf{q}}{dt} = \mathbf{F} - \frac{1}{\rho}\nabla p$$

or $$\left[\frac{\partial}{\partial t} + \mathbf{q}\cdot\nabla\right]\mathbf{q} = \mathbf{F} - \frac{1}{\rho}\nabla p$$

By assumption, $\mathbf{q}$ is independent of t, $\frac{\partial \mathbf{q}}{\partial t} = 0$ and body force $\mathbf{F}$ is negligible.

$$\therefore \quad (\mathbf{q}\cdot\nabla)\mathbf{q} = -\frac{1}{\rho}\nabla p \quad \ldots (1)$$

Write $$D = u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial\theta} + \frac{w}{r\sin\theta}\frac{\partial}{\partial\phi}$$

Putting the values of u, v, w,

$$D = V\left[\left(1-\frac{R^3}{r^3}\right)\cos\theta\frac{\partial}{\partial r} - \frac{1}{r}\left(1+\frac{R^3}{2r^3}\right)\sin\theta\frac{\partial}{\partial\theta}\right] \quad \ldots (2)$$

Spherical polar equivalent of (1) is

$$Du - \frac{(v^2+w^2)}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r}$$

$$Dv + \frac{uv}{r} - \frac{w\cot\theta}{r} = -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial\theta}$$

$$Dw + \frac{uw}{r} + \frac{vw}{r}\cot\theta = -\frac{1}{\rho}\frac{1}{r\sin\theta}\frac{\partial p}{\partial\phi}$$

Since $w = 0$, hence the above equations become

$$Du - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \quad \ldots (3)$$

$$Dv + \frac{uv}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial\theta} \quad \ldots (4)$$

$$0 = -\frac{1}{\rho}\frac{1}{r\sin\theta}\frac{\partial p}{\partial\phi} \quad \ldots (5)$$

$(5) \Rightarrow \frac{\partial p}{\partial\phi} = 0 \Rightarrow p = f(r, \theta)$.

By (3), $\frac{\partial p}{\partial r} = \frac{\rho v^2}{r} - \rho Du$

Putting the values

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = \frac{1}{r}\left\{V\left(1+\frac{R^3}{2r^3}\right)\sin\theta\right\}^2 + VD\left(1-\frac{R^3}{r^3}\right)\cos\theta$$

With D given by (2), simplifying, we get

$$\frac{1}{\rho}\frac{\partial p}{\partial r} = \frac{3V^2R^3}{2r^4}\left(1+\frac{R^3}{2r^3}\right)\sin^2\theta - \frac{3V^2R^3}{r^4}\left(1-\frac{R^3}{r^3}\right)\cos^2\theta \quad \ldots (6)$$

Similarly (4) gives

$$\frac{1}{\rho}\frac{\partial p}{\partial\theta} = \frac{3V^2R^3}{2r^3}\left(1-\frac{R^3}{r^3}\right)\sin\theta\cos\theta + \frac{3V^2R^3}{2r^3}\left(1+\frac{R^3}{2r^3}\right)\sin\theta\cos\theta \quad \ldots (7)$$

(Calculate it)

Differentiating (6) partially w.r.t. θ and simplifying, we get

$$\frac{1}{\rho}\frac{\partial^2 p}{\partial\theta\,\partial r} = \left(\frac{9V^2R^3}{r^4} - \frac{9V^2R^6}{2r^7}\right)\sin\theta\cos\theta \quad \ldots (8)$$

Differentiating (7) partially w.r.t. r and simplifying, we get

$$\frac{1}{\rho}\frac{\partial^2 p}{\partial r\,\partial\theta} = \left(\frac{9V^2R^3}{r^4} - \frac{9V^2R^6}{2r^7}\right)\sin\theta\cos\theta \quad \ldots (9)$$

Since (8) and (9) are identical hence equation of motion is satisfied.

Problem 5. *The velocity components*

$$u_r(r,\theta) = -V\left(1-\frac{a^2}{r^2}\right)\cos\theta,$$

$$u_\theta(r,\theta) = V\left(1+\frac{a^2}{r^2}\right)\sin\theta$$

satisfy equations of motion for a two dimensional inviscid incompressible flow. Find the pressure associated with velocity field. V and a are constants.

Solution : Euler's equation of motion in absence of external forces is

$$\frac{d\mathbf{q}}{dt} = -\frac{1}{\rho}\nabla p \quad \ldots (1)$$

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{q}\cdot\nabla$$

But $u_r = u,\ v_\theta = v$ are independent of t.

$$\therefore \quad \frac{\partial \mathbf{q}}{\partial t} = 0.$$

Now (1) becomes

$$(\mathbf{q}\cdot\nabla)\mathbf{q} = -\frac{1}{\rho}\nabla p \quad \ldots (2)$$

Write $u_r = u,\ u_\theta = v,\ u_z = w$. Then

$$u = -V\left(1-\frac{a^2}{r^2}\right)\cos\theta,\quad v = V\left(1+\frac{a^2}{r^2}\right)\sin\theta,\ w = 0.$$

Write $$D = u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial\theta} + w\frac{\partial}{\partial z}.\quad \text{But } w = 0$$

$$\therefore \quad D = u\frac{\partial}{\partial r} + \frac{v}{r}\frac{\partial}{\partial\theta}.$$

Putting the values of u, v, we get

$$D = V\left[-\left(1-\frac{a^2}{r^2}\right)\cos\theta\frac{\partial}{\partial r} + \left(\frac{1}{r}+\frac{a^2}{r^2}\right)\sin\theta\frac{\partial}{\partial\theta}\right] \quad \ldots (3)$$

Cylindrical equivalent of (2) is

$$Du - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} \quad \ldots (4)$$

$$Dv + \frac{uv}{r} = -\frac{1}{\rho r}\frac{\partial p}{\partial\theta} \quad \ldots (5)$$

$$Dw = -\frac{1}{\rho}\frac{\partial p}{\partial z} \quad \ldots (6)$$

But $w = 0 \Rightarrow Dw = 0 \Rightarrow \frac{\partial p}{\partial z} = 0 \Rightarrow p = p(r, \theta)$

Putting the values in (4) and (5),

$$-VD\left(1-\frac{a^2}{r^2}\right)\cos\theta - \frac{V^2}{r}\left(1+\frac{a^2}{r^2}\right)^2\sin^2\theta = -\frac{1}{\rho}\frac{\partial p}{\partial r} \quad \ldots (7)$$

$$VD\left(1+\frac{a^2}{r^2}\right)\sin\theta - \frac{V^2}{r}\left(1-\frac{a^2}{r^4}\right)\sin\theta\cos\theta = -\frac{1}{\rho}\frac{1}{r}\frac{\partial p}{\partial\theta} \quad \ldots (8)$$

Simplifying (7) with the help of (3),

$$-\frac{1}{\rho}\frac{\partial p}{\partial r} = \frac{2V^2a^2}{r^3}\left(1-\frac{a^2}{r^2}\right)\cos^2\theta - \frac{2V^2a^2}{r^3}\left(1+\frac{a^2}{r^2}\right)\sin^2\theta \quad \ldots (9)$$

Simplifying (8) with the help of (3),

$$-\frac{1}{\rho}\frac{\partial p}{\partial\theta} = \frac{2a^2V^2}{r^2}\left(1-\frac{a^2}{r^2}\right)\sin\theta\cos\theta + \frac{2a^2V^2}{r^3}\left(1+\frac{a^2}{r^2}\right)\sin\theta\cos\theta \quad \ldots (10)$$

Differentiating (9) w.r.t. θ and simplifying,

$$-\frac{1}{\rho}\frac{\partial^2 p}{\partial\theta\,\partial r} = \frac{8V^2a^2}{r^3}\sin\theta\cos\theta \quad \ldots (11)$$

Differentiating (10) partially w.r.t. r,

$$\frac{\partial^2 p}{\partial r\,\partial\theta} = \frac{8}{r^3}V^2a^2\sin\theta\cos\theta \quad \ldots (12)$$

Evidently R.H.S. of (11) and (12) are equal. This proves that the given velocity components satisfy equations of motion.

II. To find pressure p.

$$dp = \frac{\partial p}{\partial r}dr + \frac{\partial p}{\partial\theta}d\theta$$

Putting the values from (9) and (10),

$$-\frac{\rho\,dp}{2V^2a^2} = \left[\frac{1}{r^2}\left(1-\frac{a^2}{r^2}\right)\cos^2\theta - \frac{1}{r^3}\left(1+\frac{a^2}{r^2}\right)\sin^2\theta\right]dr + \left\{\frac{1}{r^2}\left(1-\frac{a^2}{r^2}\right) + \frac{1}{r^3}\left(1+\frac{a^2}{r^2}\right)\sin\theta\cos\theta\right\}d\theta$$

$$= M\,dr + N\,d\theta \quad \ldots (13)$$

It can be seen that $\frac{\partial M}{\partial\theta} = \frac{\partial N}{\partial r}$. (Prove it)

Hence $M\,dr + N\,d\theta$ is exact. Solution of (13) is given by

$$\frac{-\rho}{2V^2a^2}p = \int\left[\left(\frac{1}{r^3}-\frac{a^2}{r^5}\right)\cos^2\theta - \left(\frac{1}{r^2}+\frac{a^2}{r^5}\right)\sin^2\theta\right]dr$$

$$= \cos^2\theta\left(-\frac{1}{2r^2}+\frac{a^2}{4r^4}\right) - \sin^2\theta\left(-\frac{1}{2r^2}-\frac{a^2}{4r^4}\right)$$

$$p = -\frac{2V^2a^2}{\rho}\left[-\frac{1}{2r^2}\cos\theta + \frac{a^2}{4r^4}\right] + c$$

Bernoulli's Theorem 3. Bernoulli's equation for steady motion : If (i) the forces are conservative (ii) motion is steady (iii) density ρ is a function of pressure p only, then the equation of motion is

$$\int\frac{dp}{\rho} + \frac{1}{2}q^2 + \Omega = C,\ C \textit{ being absolute constant.}$$

Proof : Step I. Forces are conservative $\Rightarrow \mathbf{F} = -\nabla\Omega$. Motion is steady $\Rightarrow \frac{\partial \mathbf{q}}{\partial t} = 0$, density is a function of pressure p only $\Rightarrow$ there exists a relation of the type $P = \int_c^p \frac{dp}{\rho}$ so that $\nabla P = \frac{1}{\rho}\nabla p$.

By Euler's equation, $\frac{d\mathbf{q}}{dt} = -\nabla\Omega - \nabla P$

or $$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q}\cdot\nabla)\mathbf{q} = -\nabla(\Omega+P) \quad \text{or} \quad \nabla(\Omega+P) + (\mathbf{q}\cdot\nabla)\mathbf{q} = 0.$$

But $$\nabla(\mathbf{q}\cdot\mathbf{q}) = 2[\mathbf{q}\times\text{curl}\,\mathbf{q} + (\mathbf{q}\cdot\nabla)\mathbf{q}]$$

$$\therefore \quad \nabla(\Omega+P) + \frac{1}{2}\nabla q^2 - \mathbf{q}\times\text{curl}\,\mathbf{q} = 0$$

or $$\nabla\left(\Omega + P + \frac{1}{2}q^2\right) = \mathbf{q}\times\text{curl}\,\mathbf{q}. \quad \ldots (1)$$

Step II. Multiplying (1) scalarly by $\mathbf{q}$ and noting that

$$\mathbf{q}\cdot(\mathbf{q}\times\text{curl}\,\mathbf{q}) = (\mathbf{q}\times\mathbf{q})\cdot\text{curl}\,\mathbf{q} = 0.$$

For $\mathbf{q}\times\mathbf{q} = 0$, we obtain $\mathbf{q}\cdot\nabla\left(\Omega + P + \frac{1}{2}q^2\right) = 0.$

The solution of this is $\Omega + P + \frac{1}{4}q^2 = \text{const.} = C$

or $$\Omega + \int \frac{dp}{\rho} + \frac{1}{2}q^2 = C.$$

Theorem 4. If the motion of an ideal fluid, for which density is a function of pressure p only, is steady and the external forces are conservative, then there exists a family of surfaces which contain the stream lines and vortex lines.

Proof : Step I. $$\nabla\left(\Omega + P + \frac{1}{2}q^2\right) = \mathbf{q} \times \text{curl}\, \mathbf{q}. \quad \ldots (1)$$

Here write step I of Theorem 3.

Step II. Write $\mathbf{W} = \text{curl}\ \mathbf{q}$ = vorticity vector.

Then $$\nabla\left(\Omega + P + \frac{1}{2}q^2\right) = \mathbf{q} \times \mathbf{W}. \quad \ldots (2)$$

Write $$\nabla\left(\Omega + P + \frac{1}{2}q^2\right) = \mathbf{N}.$$

Then $$\mathbf{N} = \mathbf{q} \times \mathbf{W}.$$

Thus $\Rightarrow$ $$\mathbf{N} \cdot \mathbf{q} = 0 = \mathbf{N} \cdot \mathbf{W}.$$ [For $\mathbf{a}\cdot(\mathbf{b} \times \mathbf{c}) = 0$, if any two of $\mathbf{a}, \mathbf{b}, \mathbf{c}$ are equal]

$\Rightarrow$ N is perpendicular to both q and W.

Also N perpendicular to the family of surfaces
$$\Omega + P + \frac{1}{2}q^2 = \text{const.} = C.$$
[For ∇f is perpendicular everywhere to $f = \text{const.}$]

This leads to the conclusion that q and W both are tangential to the surface
$$\Omega + P + \frac{1}{2}q^2 = C.$$

It means that the surfaces $\Omega + P + \frac{1}{2}q^2 = C$ contains stream lines and vortex lines.

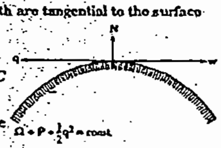


Fig. 2.2

Remark : The above theorem can also be restated as follows :

To prove that for steady motion of an inviscid isotropic fluid
$$(\rho = f(p)),\ \int \frac{dp}{\rho} + \frac{1}{2}q^2 + \Omega = \text{const.}$$
over a surface containing the stream lines and vortex lines. Comment on the nature of this constant.

Theorem 5. Lagrange's equation of motion. To obtain Lagrange's equation of motion.

Proof : Let initially a fluid particle be at (a, b, c) at time $t = t_0$, when its volume is dV_0 and density is ρ_0. After a lapse of time t, let he same fluid particle be at (x, y, z) when its volume is dV and density is ρ. The equation of continuity is
$$\rho J = \rho_0 \quad \ldots (1)$$
where $J = \frac{\partial(x, y, z)}{\partial(a, b, c)}$. The components of acceleration are
$$\ddot{x} = \frac{\partial^2 x}{\partial t^2},\quad \ddot{y} = \frac{\partial^2 y}{\partial t^2},\quad \ddot{z} = \frac{\partial t^2}{\partial t^2}.$$

Let the external forces be conservative so that $\mathbf{F} = -\nabla\Omega$.

But Euler's equation of motion,
$$\frac{d\mathbf{q}}{dt} = \mathbf{F} - \frac{1}{\rho}\nabla p = -\nabla\Omega - \frac{1}{\rho}\nabla p.$$

Its cartesian equivalent is
$$\frac{\partial^2 x}{\partial t^2} = -\frac{\partial\Omega}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x}$$
$$\frac{\partial^2 y}{\partial t^2} = -\frac{\partial\Omega}{\partial y} - \frac{1}{\rho}\frac{\partial p}{\partial y}$$
$$\frac{\partial^2 z}{\partial t^2} = -\frac{\partial\Omega}{\partial z} - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

Multiplying these equations by
$$\frac{\partial x}{\partial a},\ \frac{\partial y}{\partial a},\ \frac{\partial z}{\partial a}$$
respectively and then adding columnwise
$$\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial a} + \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial a} + \frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial a} = -\frac{\partial\Omega}{\partial a} - \frac{1}{\rho}\cdot\frac{\partial p}{\partial a} \quad \ldots (2)$$

Replacing a by b and c respectively, we get two more equations
$$\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial b} + \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial b} + \frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial b} = -\frac{\partial\Omega}{\partial b} - \frac{1}{\rho}\frac{\partial p}{\partial b} \quad \ldots (3)$$
$$\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial c} + \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial c} + \frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial c} = -\frac{\partial\Omega}{\partial c} - \frac{1}{\rho}\frac{\partial p}{\partial c} \quad \ldots (4)$$

The equations (1), (2), (3) and (4) together represent Lagrange's hydrodynamical equations.

Theorem 6. Helmholtz vorticity equation. If the external forces are conservative and density is a function of pressure p only, then
$$\frac{d}{dt}\left(\frac{\mathbf{W}}{\rho}\right) = \left(\frac{\mathbf{W}}{\rho}\cdot\nabla\right)\mathbf{q}.$$

Proof : F is conservative $\Rightarrow$ $\mathbf{F} = -\nabla\Omega$

ρ is a function of p only $\Rightarrow$ there exists a relation of the type
$$P = \int_p^p \frac{dp}{\rho}.$$
$$\Rightarrow \quad \nabla P = \Sigma i\frac{\partial P}{\partial x} = \Sigma i\frac{dP}{dp}\frac{\partial p}{\partial x} = \Sigma\frac{1}{\rho} i\frac{\partial p}{\partial x}$$
$$\Rightarrow \quad \nabla P = \frac{1}{\rho}\nabla p.$$

By Euler's equations of motion,
$$\frac{d\mathbf{q}}{dt} = \mathbf{F} - \frac{1}{\rho}\nabla p.$$

or $$\frac{\partial \mathbf{q}}{\partial t} + (\mathbf{q}\cdot\nabla)\mathbf{q} = -\nabla\Omega - \nabla P.$$

But $$\nabla(\mathbf{q}\cdot\mathbf{q}) = 2[\mathbf{q} \times \text{curl}\,\mathbf{q} + (\mathbf{q}\cdot\nabla)\mathbf{q}]$$

$\therefore$ $$\frac{\partial \mathbf{q}}{\partial t} + \frac{1}{2}\nabla q^2 - \mathbf{q} \times \text{curl}\,\mathbf{q} = -\nabla(\Omega + P)$$

or $$\frac{\partial \mathbf{q}}{\partial t} + \nabla\left(\Omega + P + \frac{1}{2}q^2\right) = \mathbf{q} \times \mathbf{W}.$$

Taking curl of both sides and noting that curl grad = 0, we obtain
$$\text{curl}\,\frac{\partial \mathbf{q}}{\partial t} = \frac{\partial}{\partial t}\text{curl}\,\mathbf{q} = \frac{\partial \mathbf{W}}{\partial t} = \text{curl}(\mathbf{q} \times \mathbf{W})$$

or $$\frac{\partial \mathbf{W}}{\partial t} = \mathbf{q}(\nabla\cdot\mathbf{W}) - \mathbf{W}(\nabla\cdot\mathbf{q}) + (\mathbf{W}\cdot\nabla)\mathbf{q} - (\mathbf{q}\cdot\nabla)\mathbf{W}.$$

But $\nabla\cdot\mathbf{W} = \text{div curl}\,\mathbf{q} = 0$ and equation of continuity is
$$\frac{d\rho}{dt} + \rho(\nabla\cdot\mathbf{q}) = 0$$

Hence $$\frac{\partial \mathbf{W}}{\partial t} = 0 + \frac{\mathbf{W}}{\rho}\frac{d\rho}{dt} + (\mathbf{W}\cdot\nabla)\mathbf{q} - (\mathbf{q}\cdot\nabla)\mathbf{W}$$

or $$\left[\frac{\partial}{\partial t} + \mathbf{q}\cdot\nabla\right]\mathbf{W} = \frac{\mathbf{W}}{\rho}\frac{d\rho}{dt} + (\mathbf{W}\cdot\nabla)\mathbf{q}$$

or $$\frac{d\mathbf{W}}{dt} = \frac{\mathbf{W}}{\rho}\frac{d\rho}{dt} + (\mathbf{W}\cdot\nabla)\mathbf{q}$$

or $$\frac{1}{\rho}\frac{d\mathbf{W}}{dt} - \frac{\mathbf{W}}{\rho^2}\frac{d\rho}{dt} = (\mathbf{W}\cdot\nabla)\mathbf{q}\frac{1}{\rho}$$

or $$\frac{d}{dt}\left(\frac{\mathbf{W}}{\rho}\right) = \left(\frac{\mathbf{W}}{\rho}\cdot\nabla\right)\mathbf{q} \quad \ldots (1)$$

This is called Helmholtz vorticity equation. If we write
$$\mathbf{W} = \xi i + \eta j + \zeta k,\quad \mathbf{q} = ui + vj + wk,$$
then the cartesian form of (1) is
$$\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} + \frac{\partial}{\partial t}\right)\left(\frac{\xi}{\rho}\right) = \frac{1}{\rho}\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial z}\right)u$$
$$\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} + \frac{\partial}{\partial t}\right)\left(\frac{\eta}{\rho}\right) = \frac{1}{\rho}\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial z}\right)v$$
$$\left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z} + \frac{\partial}{\partial t}\right)\left(\frac{\zeta}{\rho}\right) = \frac{1}{\rho}\left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial z}\right)w.$$

Remark. For ρ = const. (1) *was originally given by Stoke and Helmholtz and later on exended to the above form by Nanson.*

Theorem 7. Cauchy's integrals : Lagrange's hydrodynamical equations are
$$\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial a} + \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial a} + \frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial a} = -\frac{\partial\Omega}{\partial a} - \frac{1}{\rho}\cdot\frac{\partial p}{\partial a}$$
with two similar equations.

If we write $Q = \Omega + \int_c^p \frac{dp}{\rho}$, then the last becomes
$$\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial a} + \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial a} + \frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial a} = -\frac{\partial Q}{\partial a} \quad \ldots (1)$$

Similarly we have
$$\frac{\partial^2 x}{\partial t^2}\frac{\partial x}{\partial b} + \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial b} + \frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial b} = -\frac{\partial Q}{\partial b} \quad \ldots (2)$$
$$\frac{\partial^2 y}{\partial t^2}\frac{\partial x}{\partial c} + \frac{\partial^2 y}{\partial t^2}\frac{\partial y}{\partial c} + \frac{\partial^2 z}{\partial t^2}\frac{\partial z}{\partial c} = -\frac{\partial Q}{\partial c} \quad \ldots (3)$$

Put $$\dot{x} = \frac{\partial x}{\partial t} = u,\ \dot{y} = \frac{\partial y}{\partial t} = v,\ \dot{z} = \frac{\partial y}{\partial t} = w.$$

Now (2) and (3) are expressible as
$$\frac{\partial u}{\partial t}\frac{\partial x}{\partial b} + \frac{\partial v}{\partial t}\frac{\partial y}{\partial b} + \frac{\partial w}{\partial t}\frac{\partial z}{\partial b} = -\frac{\partial Q}{\partial b} \quad \ldots (4)$$
$$\frac{\partial u}{\partial t}\frac{\partial x}{\partial c} + \frac{\partial v}{\partial t}\frac{\partial y}{\partial c} + \frac{\partial w}{\partial t}\frac{\partial z}{\partial c} = -\frac{\partial Q}{\partial c} \quad \ldots (5)$$

Eliminating Q between (4) and (5), we have
$$\frac{\partial}{\partial c}\text{ L.H.S. of (4)} = \frac{\partial}{\partial b}\text{ L.H.S. of (5)}.$$

i.e. $$\frac{\partial^2 u}{\partial c\,\partial t}\frac{\partial x}{\partial b} + \frac{\partial u}{\partial t}\frac{\partial^2 x}{\partial b\,\partial c} + \frac{\partial^2 v}{\partial c\,\partial t}\frac{\partial y}{\partial b} + \frac{\partial v}{\partial t}\frac{\partial^2 y}{\partial b\,\partial c} + \frac{\partial^2 w}{\partial t\,\partial c}\frac{\partial z}{\partial b} + \frac{\partial w}{\partial t}\frac{\partial^2 z}{\partial b\,\partial c}$$
$$= \frac{\partial^2 u}{\partial b\,\partial t}\frac{\partial x}{\partial c} + \frac{\partial u}{\partial t}\frac{\partial^2 x}{\partial b\,\partial c} + \frac{\partial^2 v}{\partial b\,\partial t}\frac{\partial y}{\partial c} + \frac{\partial v}{\partial t}\frac{\partial^2 y}{\partial c\,\partial b} + \frac{\partial^2 w}{\partial b\,\partial t}\frac{\partial z}{\partial c} + \frac{\partial w}{\partial t}\frac{\partial^2 z}{\partial b\,\partial c}$$

or $$\left(\frac{\partial^2 u}{\partial b\,\partial t} - \frac{\partial^2 u}{\partial c\,\partial t}\frac{\partial x}{\partial b}\right) + \left(\frac{\partial^2 v}{\partial b\,\partial t}\frac{\partial y}{\partial c} - \frac{\partial^2 v}{\partial c\,\partial t}\frac{\partial y}{\partial b}\right) + \left(\frac{\partial^2 w}{\partial b\,\partial t}\frac{\partial z}{\partial c} - \frac{\partial^2 w}{\partial c\,\partial t}\frac{\partial z}{\partial b}\right) = 0$$

or $$\left[\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial b}\frac{\partial x}{\partial c} - \frac{\partial u}{\partial c}\frac{\partial x}{\partial b}\right) - \frac{\partial u}{\partial b}\frac{\partial^2 x}{\partial t\,\partial c} + \frac{\partial u}{\partial c}\frac{\partial^2 x}{\partial t\,\partial b}\right]$$
$$+ \left[\frac{\partial}{\partial t}\left(\frac{\partial v}{\partial b}\frac{\partial y}{\partial c} - \frac{\partial v}{\partial c}\frac{\partial y}{\partial b}\right) - \frac{\partial v}{\partial b}\frac{\partial^2 y}{\partial t\,\partial c} + \frac{\partial v}{\partial c}\frac{\partial^2 y}{\partial t\,\partial b}\right]$$
$$+ \left[\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial b}\frac{\partial z}{\partial c} - \frac{\partial w}{\partial c}\frac{\partial z}{\partial b}\right) - \frac{\partial w}{\partial b}\frac{\partial^2 z}{\partial t\,\partial c} + \frac{\partial w}{\partial c}\frac{\partial^2 z}{\partial t\,\partial b}\right] = 0$$

But $$\frac{\partial u}{\partial b}\frac{\partial^2 x}{\partial t\,\partial c} = \frac{\partial u}{\partial b}\frac{\partial u}{\partial c},\quad \frac{\partial u}{\partial c}\frac{\partial^2 x}{\partial t\,\partial b} = \frac{\partial u}{\partial c}\frac{\partial u}{\partial b}.$$

Hence terms outside the brackets cancel so that.
$$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial b}\frac{\partial x}{\partial c} - \frac{\partial u}{\partial c}\frac{\partial x}{\partial b}\right) + \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial b}\frac{\partial y}{\partial c} - \frac{\partial v}{\partial c}\frac{\partial y}{\partial b}\right) + \frac{\partial}{\partial t}\left(\frac{\partial w}{\partial t}\frac{\partial z}{\partial c} - \frac{\partial w}{\partial c}\frac{\partial z}{\partial b}\right) = 0.$$

Integrating w.r.t. t.
$$\left(\frac{\partial u}{\partial b}\frac{\partial x}{\partial c} - \frac{\partial u}{\partial c}\frac{\partial x}{\partial b}\right) + \left(\frac{\partial v}{\partial b}\frac{\partial y}{\partial c} - \frac{\partial v}{\partial c}\frac{\partial y}{\partial b}\right) + \left(\frac{\partial w}{\partial b}\frac{\partial z}{\partial c} - \frac{\partial w}{\partial c}\frac{\partial z}{\partial b}\right) = c \quad \ldots (6)$$

Solution : Equation of motion is $\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=0-\frac{1}{\rho}\frac{\partial p}{\partial x}$ and equation of continuity is $x^2v=F(t)$ so that $\frac{\partial v}{\partial t}=\frac{F'(t)}{x^2}$.

Hence $\frac{F'(t)}{x^2}+\frac{\partial}{\partial x}\left(\frac{1}{2}v^2\right)=-\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right)$ as ρ is constant.

Integrating w.r.t. x, $$\frac{-F'(t)}{x}+\frac{1}{2}v^2=-\frac{p}{\rho}+C. \qquad \ldots(1)$$

Boundary conditions are

(2) when $x=\infty$, $p=\Pi$, $v=0$, ...(2)

(3) When $x=R$, $p=p$, $v=\dot{R}$. ...(3)

Also $x^2v=F(t)=R^2\dot{R}$

$$\therefore \quad F'(t)=2R(\dot{R})^2+R^2\ddot{R}.$$

Subjecting (1) to the conditions (2) and (3),

$$0+0=-\frac{\Pi}{\rho}+C \text{ and}$$

$$-\frac{F'(t)}{R}+\frac{1}{2}(\dot{R})^2=-\frac{p}{\rho}+C=-\frac{p}{\rho}+\frac{\Pi}{\rho}$$

or $$\frac{p}{\rho}=\frac{\Pi}{\rho}-\frac{1}{2}(\dot{R})^2+\frac{1}{R}[2R(\dot{R})^2+R^2\ddot{R}]$$

or $$p=\Pi+\frac{1}{2}\rho[3(\dot{R})^2+2R\ddot{R}] \qquad \ldots(4)$$

Now $$\frac{d^2R^2}{dt^2}+(\dot{R})^2=\frac{d}{dt}(2R\dot{R})+\dot{R}^2=2\dot{R}^2+2R\ddot{R}+\dot{R}^2$$

Now (4) becomes $$p=\Pi+\frac{1}{2}\rho\left[\frac{d^2R^2}{dt^2}+\dot{R}^2\right] \qquad \ldots(5)$$

Second part : Let $R=a(2+\cos nt)$ ... (6). Let there be no cavitation in the fluid everywhere on the surface so that $p>0$. Then we have to prove that $\Pi>3\rho a^2n^2$.

We have $\dot{R}=-an\sin nt$, $\ddot{R}=-an^2\cos nt$.

Observe that $2R\ddot{R}+3\dot{R}^2=2a(2+\cos nt)(-an^2\cos nt)+3a^2n^2\sin^2 nt$

$$=a^2n^2\{-4\cos nt-2\cos^2 nt+3\sin^2 nt\}$$

$$=a^2n^2\{-4\cos nt-2+5\sin^2 nt\}$$

using this in (4)

$$p=\Pi+\frac{1}{2}\rho a^2n^2(-4\cos nt-2+5\sin^2 nt). \qquad \ldots(7)$$

As $\cos nt$ varies from -1 and 1 and so R varies from a to $3a$, by (6). Thus sphere shrinks from $R=3a$ to $R=a$ and so there is a possibility of cavitation. Also p is minimum when $nt=0$ or $2m\pi$.

$$p_{min}=\Pi+\frac{1}{2}\rho a^2n^2(-4-2+0), \text{ by (7)} =\Pi-3\rho a^2n^2$$

$$p>0 \Rightarrow p_{min}>0 \Rightarrow \Pi-3\rho a^2n^2>0 \Rightarrow \Pi>3\rho a^2n^2$$

Problem 2. *An infinite mass of fluid acted on by a force $\mu r^{-3/2}$ per unit mass is directed to the origin. If initially the fluid is a rest and there is a cavity in the form of the sphere $r=c$ in it, show that the cavity will be filled up after an interval of time $(2/5\mu)^{1/2}c^{5/4}$.* (IAS-2009)

Solution : Let v be the velocity, p the pressure at a distance x from the origin, then the equations of motion and continuity are respectively.

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=-\mu x^{-3/2}-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

and $$x^2v=F(t) \text{ so that } v=\frac{F(t)}{x^2}, \frac{\partial v}{\partial t}=\frac{F'(t)}{x^2}$$

$$\therefore \quad \frac{F'(t)}{x^2}+\frac{\partial}{\partial x}\left(\frac{1}{2}v\right)^2=-\mu x^{-3/2}-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

Integrating, $$\frac{F'(t)}{x}+\frac{1}{2}v^2=\frac{2\mu}{\sqrt{x}}-\frac{p}{\rho}+C. \qquad \ldots(1)$$

Boundary conditions are

2. When $x=\infty$, $v=0$, $p=0$
3. When $x=r$, (radius of cavity), $p=0$, $v=\dot{r}$
4. When $r=c$, $v=0$ so that $F(t)=0$.
5. Let T be the required time of filling the cavity.

Subjecting (1) to the conditions (2) and (3),

$$0+0=0-0+C \quad\text{and}\quad \frac{-F'(t)}{r}+\frac{1}{2}(\dot{r}^2)=\frac{2\mu}{\sqrt{r}}-0+C$$

or $$\frac{-F'(t)}{r}+\frac{1}{2}\dot{r}^2=\frac{2\mu}{\sqrt{r}}.$$

Since $$r^2(\dot{r})=F(t),\ r^2dr=F(t)dt.$$

Multiplying by $2F(t)dt$ or $2r^2dr$.

$$\frac{-2F'(t)F(t)dt}{r}+\frac{F^2(t)}{r^4}.r^2dr=\frac{4\mu}{\sqrt{r}}.r^2dr$$

or $$d\left[\frac{-F^2(t)}{r}\right]=4\mu r^{3/2}dr$$

Integrating, $$\frac{-F^2(t)}{r}=4\mu.\frac{2}{5}r^{5/2}+A. \qquad \ldots(6)$$

Subjecting (6) to (4), $$0=\frac{8\mu}{5}c^{5/2}+A$$

Now (6) $$\Rightarrow \frac{-(r^2\dot{r})^2}{r}=\frac{8\mu}{5}(r^{5/2}-c^{5/2})$$

$$\Rightarrow \frac{dr}{dt}=-\left[\frac{8\mu}{5r^3}\cdot(c^{5/2}-r^{5/2})\right]^{1/2}$$

[negative sign is taken as velocity increases when r decreases]

$$-\int_c^0\frac{r^{3/2}}{(c^{5/2}-r^{5/2})^{1/2}}dr=\int_0^T\left(\frac{8\mu}{5}\right)^{1/2}dt$$

or $$T=\left(\frac{5}{8\mu}\right)^{1/2}\int_0^c\frac{r^{3/2}dr}{(c^{5/2}-r^{5/2})^{1/2}}. \qquad \ldots(7)$$

Put $r^{5/2}=c^{5/2}\sin^2\theta$, $\frac{5}{2}r^{3/2}dr=c^{5/2}.2\sin\theta\cos\theta\, d\theta$.

$$T=\left(\frac{5}{8\mu}\right)^{1/2}\int_0^{\pi/2}\frac{4}{5}c^{5/2}.\frac{\sin\theta\cos\theta\, d\theta}{c^{5/4}\cos\theta}=\left(\frac{5}{8\mu}\right)^{1/2}.\frac{4}{5}c^{5/4}(-\cos\theta)_0^{\pi/2}$$

or $$T=\left(\frac{2}{5\mu}\right)^{1/2}.c^{5/4}.$$

Aliter: Equation of continuity is $x^2u=r^2v$ ... (1) where u is velocity at distance x and v is velocity at a distance r. K.E., T of liquid when radius of cavity is r :

$$T=\int_r^\infty\frac{1}{2}(4\pi x^2dx.\rho)u^2$$

$$=2\pi\rho\int_r^\infty x^2\left(\frac{r^2v}{x^2}\right)^2dx$$

$$=2\pi\rho v^2r^4\int_r^\infty\frac{dx}{x^2}=2\pi\rho v^2r^4\left(-\frac{1}{x}\right)_r^\infty$$

$$=2\pi\rho v^2r^3.$$

If Ω is force potential due to external forces, then

$$-\frac{\partial\Omega}{\partial r}=-\frac{\mu}{x^{3/2}} \text{ as } F=-\nabla\Omega$$

Integrating $$\Omega=\frac{2\mu}{x^{1/2}}$$

Work done by external forces

$$=\int_r^c\Omega\, dm=\int_r^c\frac{2\mu}{\sqrt{x}}(4\pi x^2dx.\rho)$$

$$=8\pi\rho\mu\int_r^c x^{3/2}dx$$

$$=\frac{16}{5}\pi\rho\mu(c^{5/2}-r^{5/2})$$

By principle of energy, work done = K.E.

or $$\frac{16}{5}\pi\rho\mu(c^{5/2}-r^{5/2})=2\pi\rho v^2r^3$$

$$v=\frac{dr}{dt}=-\left(\frac{8\mu}{5}\right)^{1/2}\left[\frac{c^{5/2}-r^{5/2}}{r^3}\right]^{1/2}$$

$$\text{Time}=T=-\left(\frac{5}{8\mu}\right)^{1/2}\int_c^\infty\frac{r^{3/2}dr}{(c^{5/2}-r^{5/2})^{1/2}}$$

$$=\left(\frac{2}{5\mu}\right)^{1/2}c^{5/4}.$$

Problem 3. *Steam is rushing from a boiler through a conical pipe, the diameters of the ends of which are D and d; if V and v be the corresponding velocities of the stream, and if the motion be supposed to be that of divergence from the vertex of the cone, prove that*

$$\frac{v}{V}=\frac{D^2}{d^2}e^{(v^2-V^2)/2k}$$

where k is the pressure divided by the density and supposed to be constant.

Solution : Let u be the velocity at a distance x from the end A, the equation of motion is

$$u\frac{\partial u}{\partial x}=0-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

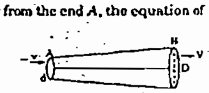


Fig. 2.3

(Since the motion is steady)

or $$\frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right)=-\frac{k}{\rho}\frac{\partial\rho}{\partial x} \qquad \text{as } p=k\rho$$

Integrating, $$\frac{1}{2}u^2=-k\log\rho+c$$

or $$\log\rho-\log A_1=-\frac{u^2}{2k} \quad\text{or}\quad \rho=A_1e^{-u^2/2k}. \qquad \ldots(1)$$

Boundary conditions are

(i) $\rho=\rho_1$ when $u=v$

(ii) $\rho=\rho_2$ when $u=V$.

Subjecting (1) to (i) and (ii) we obtain $\rho_1=A_1e^{-v^2/2k}$ and $\rho_2=A_1e^{-V^2/2k}$

This $$\Rightarrow \frac{\rho_1}{\rho_2}=e^{(V^2-v^2)/2k} \qquad \ldots(2)$$

By the equation of continuity

Flux at A = Flux at B

$$\pi\left(\frac{d}{2}\right)^2v.\rho_1=\pi\left(\frac{D}{2}\right)^2.V.\rho_2$$

or $$\frac{\rho_1}{\rho_2}=\frac{V}{v}.\frac{D^2}{d^2}.$$

Now (2) becomes $\dfrac{V}{v}\cdot\dfrac{D^2}{d^2}=e^{(V^2-v^2)/2k}$

or $$\frac{v}{V}=\frac{D^2}{d^2}e^{(v^2-V^2)/2k}$$

Problem 4. *A mass of homogeneous liquid is moving so that the velocity at any point is proportional to the time, and that the pressure is given by*

$$\frac{p}{\rho}=\mu xyz+\frac{1}{2}t^2(y^2z^2+z^2x^2+x^2y^2)$$

prove that this motion may have been generated from rest by finite natural forces independent of time; and show that if the direction of motion at every point coincides with the direction of acting forces, each particle of the liquid describes a curve which is the intersection of two hyperbolic cylinders.

Solution : Velocity is proportional to time, *i.e.*, $q=\lambda t$ ... (1).

Also $$\frac{p}{\rho}=\mu xyz-\frac{1}{2}t^2(y^2z^2+z^2x^2+x^2y^2) \quad \text{... (2)}$$

Step I. Let the motion be generated from rest by finite natural force $\mathbf{F}$ (conservative force), then there exists velocity potential ϕ s.t. $\mathbf{q}=-\nabla\phi$. To prove that $\mathbf{F}$ is independent of time.

By pressure equation, $\dfrac{p}{\rho}+\dfrac{1}{2}q^2+\Omega-\dfrac{\partial\phi}{\partial t}=F(t)$

$$\therefore \quad \frac{p}{\rho}=\frac{\partial\phi}{\partial t}-\Omega-\frac{1}{2}\lambda^2t^2+F(t). \quad \text{... (3)}$$

$$q=\lambda t,\quad \mathbf{q}=-\nabla\phi \Rightarrow \phi=t\,f(x,y,z).$$

Write $\dfrac{\partial f}{\partial x}=f_x$ etc.; (3) is expressible as

$$\frac{p}{\rho}=f-\Omega-\frac{1}{2}\lambda^2t^2+F(t). \quad \text{... (4)}$$

Comparing (2) and (4), $f-\Omega=\mu xyz,\ \lambda^2=\Sigma y^2z^2,\ F(t)=0$

Now $\lambda^2t^2=q^2=(\nabla\phi)^2=t^2(\nabla f)^2=t^2(f_x^2+f_y^2+f_z^2)$

or $$\lambda^2=f_x^2+f_y^2+f_z^2 \quad \text{or} \quad \Sigma f_x^2=\Sigma y^2z^2$$

or $$\Sigma(f_x^2-y^2z^2)=0$$

This $\Rightarrow f_x^2-y^2z^2=0,\ f_y^2-z^2x^2=0,\ f_z^2=\Sigma y^2z^2=0$

$\Rightarrow f=xyz$

We have seen that $f-\Omega=\mu xyz$, this $\Rightarrow$

$$xyz-\Omega=\mu xyz \quad \text{or} \quad \mathbf{F}=-\nabla\Omega=\nabla(\mu-1)xyz$$

or $$\mathbf{F}=(\mu-1)\nabla(xyz). \quad \text{... (5)}$$

This $\Rightarrow$ $\mathbf{F}$ is independent of t.

Step II. Let the direction of motion coincide with the direction of acting force so that

$$\frac{u}{F_1}=\frac{v}{F_2}=\frac{w}{F_3}. \quad \text{... (6)}$$

To prove that stream lines are the intersection of two hyperbolic cylinders. Equations of stream lines are

$$\frac{dx}{u}=\frac{dy}{v}=\frac{dz}{w}.$$

Using (6), $$\frac{dx}{F_1}=\frac{dy}{F_2}=\frac{dz}{F_3}.$$

By (5), $$\frac{dx}{(\mu-1)yz}=\frac{dy}{(\mu-1)zx}=\frac{dz}{(\mu-1)xy}$$

or $$\frac{dx}{yz}=\frac{dy}{zx}=\frac{dz}{xy} \quad \text{or} \quad x\,dx=y\,dy=z\,dz.$$

This $\Rightarrow$ $x\,dx=y\,dy,\ x\,dx=z\,dz.$

Integration yields the result $x^2-y^2=a^2,\ x^2-z^2=b^2$. This represents two distinct hyperbolic cylinders. Hence the result.

Problem 5. *Air, obeying Boyle's law, is in motion in a uniform tube of small section, prove that if ρ be the density and v the velocity at a distance x from a fixed point at time t.*

$$\frac{\partial^2\rho}{\partial t^2}=\frac{\partial^2}{\partial x^2}\{(v^2+k)\rho\},\quad \textit{where } k=\frac{p}{\rho}.$$

Solution : Equation of continuity is

$$\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho v)=0. \quad \text{... (1)}$$

Equation of motion is

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x}. \quad \text{... (2)}$$

By Boyle's law, pr. vol. = const.

But vol. density = mass.

Hence pr. vol. = const, vol. = $\dfrac{\text{mass}}{\rho}$.

$$\therefore \quad \text{pr.}\frac{\text{mass}}{\rho}=\text{const.}$$

This $\Rightarrow \dfrac{p}{\rho}=\text{const.}=k$, say $\Rightarrow p=k\rho$.

By (2), $$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=-\frac{k}{\rho}\frac{\partial\rho}{\partial x}. \quad \text{... (3)}$$

To determine $\dfrac{\partial^2\rho}{\partial t^2}$.

By (1), $$\frac{\partial^2\rho}{\partial t^2}=\frac{\partial}{\partial t}\left[-\frac{\partial}{\partial x}(\rho v)\right]=-\frac{\partial}{\partial x}\left[\frac{\partial}{\partial t}(\rho v)\right]$$

$$\frac{\partial^2\rho}{\partial t^2}=-\frac{\partial}{\partial x}\left[v\frac{\partial\rho}{\partial t}+\rho\frac{\partial v}{\partial t}\right]$$

$$=-\frac{\partial}{\partial x}\left[v\left(-\frac{\partial\rho v}{\partial x}\right)+\rho\left(-\frac{k}{\rho}\frac{\partial\rho}{\partial x}-v\frac{\partial v}{\partial x}\right)\right]$$

$$=\frac{\partial}{\partial x}\left[v\frac{\partial\rho v}{\partial x}+k\frac{\partial\rho}{\partial x}+\rho v\frac{\partial v}{\partial x}\right]$$

$$=\frac{\partial}{\partial x}\left[\frac{\partial}{\partial x}(\rho v.v)+\frac{\partial}{\partial x}(k\rho)\right]=\frac{\partial^2}{\partial x^2}(\rho v^2+k\rho)$$

or $$\frac{\partial^2\rho}{\partial t^2}=\frac{\partial^2}{\partial x^2}\{\rho(v^2+k)\}$$

Problem 6. *An elastic fluid, the weight of which is neglected obeying Boyle's is in motion in a uniform straight tube; show that on the hypothesis of parallel sections the velocity at any time t at a distance r from a fixed point in the tube is defined by the equation*

$$\frac{\partial^2v}{\partial t^2}+\frac{\partial}{\partial r}\left(2v\frac{\partial v}{\partial t}+v^2\frac{\partial v}{\partial r}\right)=k\frac{\partial^2v}{\partial r^2}.$$

Solution : Boyle's law is $\dfrac{p}{\rho}=k$ as volume $=\dfrac{1}{\rho}$.

Equations of continuity and motion are

$$\frac{\partial\rho}{\partial t}+\frac{\partial\rho v}{\partial r}=0 \quad \text{... (1)}$$

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial r}=-\frac{1}{\rho}\frac{\partial p}{\partial r}.$$

i.e., $$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial r}=-\frac{k}{\rho}\frac{\partial\rho}{\partial r} \quad \text{... (2)}$$

To determine $\dfrac{\partial^2v}{\partial t^2}$. By (2), we get

$$-\frac{\partial^2v}{\partial t^2}=\frac{\partial}{\partial t}\left[v\frac{\partial v}{\partial r}+\frac{k}{\rho}\frac{\partial\rho}{\partial r}\right]=\frac{\partial}{\partial r}\left(v\frac{\partial v}{\partial t}+\frac{k}{\rho}\frac{\partial\rho}{\partial t}\right)$$

$$=\frac{\partial}{\partial r}\left[v\left(-v\frac{\partial v}{\partial r}-\frac{k}{\rho}\frac{\partial\rho}{\partial r}\right)+\frac{k}{\rho}\left(-\frac{\partial\rho v}{\partial r}\right)\right], \quad \text{by (1), (2).}$$

$$\frac{\partial^2v}{\partial t^2}=\frac{\partial}{\partial r}\left[v^2\frac{\partial v}{\partial r}+\frac{vk}{\rho}\frac{\partial\rho}{\partial r}+\frac{kv}{\rho}\frac{\partial\rho}{\partial r}+k\frac{\partial v}{\partial r}\right]$$

$$\frac{\partial}{\partial t}\left[v\frac{\partial v}{\partial r}+\frac{k}{\rho}\frac{\partial\rho}{\partial r}\right]=\frac{\partial}{\partial t}\left[\frac{\partial}{\partial r}\left(\frac{1}{2}v^2\right)+\frac{\partial}{\partial r}(k\log\rho)\right]$$
$$=\frac{\partial^2}{\partial t\,\partial r}\left(\frac{1}{2}v^2+k\log\rho\right)$$
$$=\frac{\partial}{\partial r}\frac{\partial}{\partial t}\left(\frac{1}{2}v^2+k\log\rho\right)$$
$$=\frac{\partial}{\partial r}\left[v\frac{\partial v}{\partial t}+\frac{k}{\rho}\frac{\partial\rho}{\partial t}\right]$$

$$\frac{\partial^2v}{\partial t^2}=\frac{\partial}{\partial r}\left(v^2\frac{\partial v}{\partial r}+\frac{2kv}{\rho}\frac{\partial\rho}{\partial r}\right)+k\frac{\partial^2v}{\partial r^2}$$

$$=\frac{\partial}{\partial r}\left[v^2\frac{\partial v}{\partial r}+2v\left(-\frac{\partial v}{\partial t}-v\frac{\partial v}{\partial r}\right)\right]+k\frac{\partial^2v}{\partial r^2}, \quad \text{by (2).}$$

$$=\frac{\partial}{\partial r}\left[-v^2\frac{\partial v}{\partial r}-2v\frac{\partial v}{\partial t}\right]+k\frac{\partial^2v}{\partial r^2}$$

or $$\frac{\partial^2v}{\partial t^2}+\frac{\partial}{\partial r}\left(v^2\frac{\partial v}{\partial r}+2v\frac{\partial v}{\partial t}\right)=k\frac{\partial^2v}{\partial r^2}.$$

Problem 7. *A mass of liquid surrounds a solid sphere of radius a, and its outer surface, which is a concentric sphere of radius b, is subject to a given constant pressure Π no other forces being action on the liquid. Then solid sphere suddenly shrinks into a concentric sphere; it is required to determine the subsequent motion and the impulsive action on the sphere.*

Solution : Equations of motion and continuity are

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x} \quad \text{... (1)}$$

$$x^2v=F(t). \quad \text{... (2)}$$

Hence $$\frac{F'(t)}{x^2}+\frac{\partial}{\partial x}\left(\frac{1}{2}v^2\right)=-\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right).$$

Integrating w.r.t. x, we get

$$-\frac{F'(t)}{x}+\frac{1}{2}v^2=-\frac{p}{\rho}+C. \quad \text{... (3)}$$

Since the liquid is contained between two spheres $r=a,\ r=b$, so we suppose that r and R are internal and external radii at any time t and the corresponding velocities are u and U, respectively. Boundary conditions are

when $$x=r,\ v=u=\dot r,\ p=0. \quad \text{... (4)}$$

(Since pressure vanishes on the internal boundary)

When $$x=R,\ v=\dot R=U,\ p=\Pi. \quad \text{... (5)}$$

(Since outer surface is subjected to constant pressure Π).

$$r=a,\ v=\dot r=0 \text{ so that } F(t)=0. \quad \text{... (6)}$$

Subjecting (3) to the conditions (4) and (5),

$$-\frac{F'(t)}{r}+\frac{1}{2}u^2=0+C$$

$$-\frac{F'(t)}{R}+\frac{1}{2}U^2=-\frac{\Pi}{\rho}+C.$$

Also $r^2u=F(t)=R^2U$ upon subtraction,

$$F'(t)\left\{\frac{1}{R}-\frac{1}{r}\right\}+\frac{1}{2}F^2\left\{\frac{1}{r^4}-\frac{1}{R^4}\right\}=\frac{\Pi}{\rho}. \quad \text{... (7)}$$

Since $r^2u=F(t)=R^2U$ i.e., $r^2\,dr=F(t)\,dt=R^2dR$.

Multiplying (7) by $2F(t)\,dt=2r^2\,dr=2R^2\,dR$, we get

$$2FF'\left\{\frac{1}{R}-\frac{1}{r}\right\}dt+F^2\left\{\frac{dr}{r^2}-\frac{dR}{R^2}\right\}=\frac{\Pi}{\rho}\cdot 2r^2\,dr$$

or $$d\left[\left(\frac{1}{R}-\frac{1}{r}\right)F^2\right]=\frac{\Pi}{\rho}2r^2\,dr.$$

Integrating, $\left(\frac{1}{R}-\frac{1}{r}\right)F^2(t)=\frac{2}{3}\frac{r^3}{\rho}.\Pi+A.$

Subjecting this to (6), $0=\frac{2}{3}\frac{a^3}{\rho}.\Pi+A.$

Subtracting, we get

$$\left(\frac{1}{R}-\frac{1}{r}\right)F^2(t)=\frac{2\Pi}{3\rho}(r^3-a^3)$$

$$\left(\frac{R-r}{rR}\right)(r^2u)^2=\frac{2\Pi}{3\rho}(a^3-r^3)$$

or $$r^3u^2\left(\frac{R-r}{R}\right)=\frac{2\Pi}{3\rho}(a^3-r^3) \quad \ldots(8)$$

with $R^3-r^3=b^3-a^3$.

For total mass of liquid is constant

⇒ volume of liquid at any time t = volume of liquid initially.

$$\Rightarrow \frac{4}{3}\pi R^3-\frac{4}{3}\pi r^3=\frac{4}{3}\pi b^3-\frac{4}{3}\pi a^3.$$

To determine the equation of impulsive action. Equation of impulsive action is

$$d\varpi=\rho v\,dx=\frac{\rho F}{x^2}dx$$

$$\int_0^{\varpi}d\varpi=\int_r^R\frac{\rho F}{x^2}dx=-\rho F\left\{\frac{1}{R}-\frac{1}{r}\right\}=\rho r^2u\left[\frac{1}{r}-\frac{1}{R}\right]$$

or $$\varpi=\rho r^2u\left(\frac{1}{r}-\frac{1}{R}\right)$$

The whole impulse on the surface of the sphere is

$$4\pi r^2\varpi=4\pi r^2\rho r^2u\left(\frac{1}{r}-\frac{1}{R}\right)=4\pi r^3\rho u\left(\frac{R-r}{R}\right) \quad \ldots(9)$$

(8) and (9) are the required equations.

Problem 8. *An infinite fluid in which a spherical hollow shell of radius a is initially at rest under the action of no forces. If a constant pressure Π is applied at infinity, show that the time of filling up the cavity is $\pi^2a\left(\frac{\rho}{\Pi}\right)^{1/2}.2^{5/6}.[\Gamma(1/3)]^{-3}$.*

Solution : The equations of motion and continuity are

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x} \quad \ldots(1)$$

$$x^2v=F(t).$$

Hence $$\frac{F'(t)}{x^2}+v\frac{\partial v}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x}.$$

Integrating w.r.t. x,

$$-\frac{F'(t)}{x}+\frac{1}{2}v^2=-\frac{p}{\rho}+C. \quad \ldots(2)$$

Let T be the time of filling up the cavity. Boundary conditions are :

(i) when $x=\infty$, $v=0$, $p=\Pi$, (since constant pressure Π is applied at infinity)

(ii) when $x=r$ = radius of cavity, $v=u=\dot{r}$; $p=0$.

[The pressure vanishes on the surface of cavity]

(iii) When $r=a$, so that $\dot{r}=u=0$, $F(t)=0$.

Subjecting (2) to the condition (i),

$$0=-\frac{\Pi}{\rho}+C \quad \text{or} \quad C=\frac{\Pi}{\rho}.$$

Subjecting (2) to (ii),

$$\frac{-F'(t)}{r}+\frac{1}{2}u^2=0+C$$

or $$\frac{-F'(t)}{r}+\frac{1}{2}u^2=\frac{\Pi}{\rho} \quad \text{s.t.} \quad r^2u=F(t)=r^2\dot{r}$$

or $$\frac{-F'(t)}{r}+\frac{1}{2}\frac{F^2}{r^4}=\frac{\Pi}{\rho}.$$

Multiplying this by $2F\,dt=2r^2\,dr$,

$$\frac{2FF'}{r}dt-\frac{F^2}{r^2}dr=-\frac{\Pi}{\rho}2r^2\,dr$$

or $$d\left(\frac{F^2}{r}\right)=-\frac{\Pi}{\rho}.2r^2\,dr.$$

Integration yields $$\frac{F^2}{r}=-\frac{2\Pi}{3\rho}r^3+A.$$

Subjecting this to (iii),

$$0=-\frac{2\Pi}{3\rho}a^3+A$$

Hence. $$\frac{F^2}{r}=\frac{2\Pi}{3\rho}(a^3-r^3) \quad \text{or} \quad \frac{r^4\dot{r}^2}{r}=\frac{2\Pi}{3\rho}(a^3-r^3)$$

or $$\frac{dr}{dt}=-\left[\frac{2\Pi}{3\rho}\cdot\frac{a^3-r^3}{r^3}\right]^{1/2}$$

[Negative sign is taken as velocity increases when r decreases).

$$\int_0^T dt=-\int_a^0\left[\frac{3\rho}{2\Pi}\cdot\frac{r^3}{a^3-r^3}\right]^{1/2}dr$$

or $$T=\left(\frac{3\rho}{2\Pi}\right)^{1/2}I. \quad \ldots(3)$$

where $$I=\int_0^a\left(\frac{r^3}{a^3-r^3}\right)^{1/2}dr.$$

Put $r^3=a^3\sin^2\theta$, $3r^2\,dr=2a^3\sin\theta\cos\theta\,d\theta$

$$I=\int_0^{\pi/2}\frac{a^{3/2}\sin\theta}{a^{3/2}\cos\theta}\cdot\frac{2a^3\sin\theta\cos\theta\,d\theta}{3r^2}$$

$$=\int_0^{\pi/2}\frac{2a^3\sin^2\theta\,d\theta}{3(a\sin^{2/3}\theta)^2}$$

$$=\frac{2a}{3}\int_0^{\pi/2}(\sin\theta)^{2/3}(\cos\theta)^0\,d\theta$$

$$=\frac{2a}{3}\cdot\frac{\Gamma\left(\frac{5}{6}\right)\cdot\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{5}{6}+\frac{1}{2}\right)}=\frac{a}{3}\sqrt{\pi}\frac{\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(1+\frac{1}{2}\right)}$$

$$=\frac{a}{3}\sqrt{\pi}\cdot\frac{\Gamma\left(\frac{5}{6}\right)}{\left(\frac{1}{3}\right)\Gamma\left(\frac{1}{3}\right)}=\frac{a\sqrt{\pi}}{\Gamma\left(\frac{1}{3}\right)}.\Gamma\left(\frac{5}{6}\right) \quad \ldots(4).$$

Recall that

$$\Gamma(n)\Gamma(1-n)=\frac{\pi}{\sin n\pi}, \quad \Gamma(n)\Gamma\left(n+\frac{1}{2}\right)=\sqrt{\pi}\frac{\Gamma(2n)}{2^{2n-1}}$$

For $n=\frac{1}{3}$, $\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)\Gamma\left(\frac{2}{3}\right)=\frac{\pi}{\sin(\pi/3)}=\frac{2\pi}{\sqrt{3}}$

and $$\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{5}{6}\right)=\frac{\sqrt{\pi}\,\Gamma\left(\frac{2}{3}\right)}{2^{-1/3}}$$

Hence $$\Gamma\left(\frac{5}{6}\right)=\frac{\sqrt{\pi}.2^{1/3}}{\Gamma\left(\frac{1}{3}\right)}\cdot\frac{2\pi}{\sqrt{3}\,\Gamma\left(\frac{1}{3}\right)}=\frac{2^{4/3}}{\sqrt{3}}.\pi^{3/2}.\frac{1}{\left[\Gamma\left(\frac{1}{3}\right)\right]^2}$$

using this in (4), $$I=a\pi^2.\frac{2^{4/3}}{\sqrt{3}}.\frac{1}{\left[\Gamma\left(\frac{1}{3}\right)\right]^2}$$

Now (3) is reduced to

$$T=\pi^2a\left(\frac{\rho}{\Pi}\right)^{1/2}.\frac{1}{\left[\Gamma\left(\frac{1}{3}\right)\right]^3}.2^{5/6}$$

Alter : Let v be velocity when radius of cavity is r. Similarly u is the velocity when radius is x. Equation of continuity is

$$x^2u=r^2v \quad \ldots(1)$$

$$\text{K.E.}=\int\frac{1}{2}(4\pi x^2\,dx.\rho)u^2$$

$$=2\pi\rho\int_r^{\infty}x^2\left(\frac{r^2v}{x^2}\right)^2dx=2\pi\rho r^4v^2\int_r^{\infty}\frac{dx}{x^2}$$

$$\text{K.E.}=2\pi\rho r^4v^2\left(-\frac{1}{x}\right)_r^{\infty}=2\pi\rho r^3v^2$$

Work done by outer pressure

$$=\int\Pi(4\pi x^2\,dx)=4\pi\Pi\int_r^a x^2\,dx$$

$$=\frac{4\pi}{3}\Pi(a^3-r^3)$$

By principle of energy,

$$2\pi\rho r^3v^2=\frac{4\pi}{3}\Pi(a^3-r^3)$$

or $$v=-\frac{dr}{dt}=\left(\frac{2\Pi}{3\rho}\right)^{1/2}\left(\frac{a^3-r^3}{r^3}\right)^{1/2}$$

$$\int_0^t dt=-\int_a^0\left(\frac{3\rho}{2\Pi}\right)^{1/2}\frac{r^{3/2}\,dr}{(a^3-r^3)^{1/2}}$$

From this the required result follows.

Problem 9. *A pulse travelling along a fine straight uniform tube filled with gas causes the density at time t and distance x from the origin where the velocity is u_0 to become $\rho_0\phi(vt-x)$. Prove that the velocity u (at time t and distance x from the origin) is given by* $v+\frac{(u_0-v)\phi(vt)}{\phi(vt-x)}$.

Solution : Equation of continuity is

$$\frac{\partial\rho}{\partial t}+\frac{\partial\rho u}{\partial x}=0. \quad \ldots(1)$$

we have to prove $$u=v+\frac{(u_0-v)\phi(vt)}{\phi(vt-x)}$$

Given $$\rho=\rho_0\phi(vt-x) \quad \ldots(2)$$

and $$u=u_0 \text{ when } x=0. \quad \ldots(3)$$

$$(2)\Rightarrow\frac{\partial\rho}{\partial t}=\rho_0v\phi'(vt-x),\quad\frac{\partial\rho}{\partial x}=-\rho_0\phi'(vt-x)$$

Putting these values in (1), we get

$$\rho_0v\phi'(vt+x)+u\{-\rho_0\phi'(vt-x)\}+\rho_0\phi(vt-x)\frac{\partial u}{\partial x}=0$$

or $$(v-u)\phi'+\phi\frac{\partial u}{\partial x}=0 \quad \text{or} \quad \frac{du}{v-u}+\frac{\phi'(vt-x)}{\phi(vt-x)}dx=0$$

Integrating, $-\log(v-u)-\log\phi(vt-x)=-\log A$

or $$(v-u)\phi(vt-x)=A \quad \ldots(4)$$

In view of (3), this ⇒ $(v-u_0)\phi(vt)=A$

∴ $$(v-u)\phi(vt-x)=(v-u_0)\phi(vt)$$

or $$v - u = \frac{(v - u_0)\,\phi(vt)}{\phi(vt - x)}$$

or $$u = v + \frac{(u_0 - v)\,\phi(vt)}{\phi(vt - x)}$$

Problem 10. *A stream in a horizontal pipe after passing a contraction in the pipe at which its sectional area is A, is delivered at atmospheric pressure at a place where the sectional area is B. Show that if a side tube is connected with the pipe at the former place, water will be sucked up through it into the pipe from a reservoir at a depth $\frac{s^2}{2g}\left(\frac{1}{A^2} - \frac{1}{B^2}\right)$ below the pipe, s being the delivery per second.*

Solution : Let v and V be the the velocity of the stream at two cross-sections. The equation of continuity is given by

flux at the first cross section = flux at the second cross section

i.e., $Av\rho = BV\rho = s$, (given)

[For flux = cross section area × density · normal velo.]

Also $\rho = 1$ for stream.

Hence $v = \frac{s}{A}$, $V = \frac{s}{B}$.

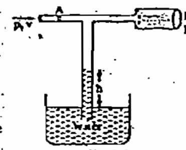


Fig. 2.4

The equation of motion is $u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$ as the motion is steady,

or $$\frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = -\frac{\partial p}{\partial x} \text{ as } \rho = 1.$$

Integrating, $$\frac{1}{2}u^2 = -p + C \qquad \dots (1)$$

Boundary conditions are :

(i) $u = V$, $p = \Pi$.

(Since stream is deliverd at atmospheric pressure $p = \Pi$, say at a place where cross-sectional area is B).

(ii) $u = v, p = p$.

In view of (i) and (ii), (1) gives $\frac{1}{2}V^2 = -\Pi + C$

$$\frac{1}{2}v^2 = -p + C.$$

Upon subtraction, $\Pi - p = \frac{1}{2}(v^2 - V^2) = \frac{1}{2}\left(\frac{s^2}{A^2} - \frac{s^2}{B^2}\right)$

or $$\Pi - p = \frac{s^2}{2}\left(\frac{1}{A^2} - \frac{1}{B^2}\right). \qquad \dots (2)$$

Let h be the height of water column in the side tube which is sucked from a reservoir, then $\Pi - p$ = difference of pressure $= \rho gh = gh$ as $\rho = 1$.

Now (2) $\Rightarrow gh = \frac{s^2}{2}\left(\frac{1}{A^2} - \frac{1}{B^2}\right)$ or $h = \frac{s^2}{2g}\left(\frac{1}{A^2} - \frac{1}{B^2}\right)$.

This concludes the problem.

Problem 11. *Show that the rate per unit of time at which work is done by the internal pressure between the parts of a compressible fluid obeying Boyle's law is*

$$\iiint p\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)dx\,dy\,dz,$$

where p is the pressure and (u, v, w) the velocity at any point, and the integration extends through the volume of the fluid.

Solution : Let W denote work done, then rate of work done is $\frac{dW}{dt}$. Let $\mathbf{q} = u\mathbf{i} + v\mathbf{j} + w\mathbf{k}$ and $dV = dx\,dy\,dz$.

Then we have to prove that $$\frac{dW}{dt} = \int p(\nabla.\mathbf{q})\,dV \qquad \dots (1)$$

We know that $W = \int -p\,dV$

and $$\frac{d\rho}{dt} + \rho(\nabla.\mathbf{q}) = 0 \text{ (equation of continuity)} \qquad \dots (2)$$

Hence $$\frac{dW}{dt} = \int -\frac{dp}{dt}dV = -\int\frac{d\,k\rho}{dt}dV \text{ as } p = \rho k \qquad \text{(Boyle's law)}$$

or $$\frac{dW}{dt} = -k\int\frac{d\rho}{dt}dV = -k\int -\rho(\nabla.\mathbf{q})\,dV, \text{ by (2)}$$

$$= \int k\rho(\nabla.\mathbf{q})\,dV = \int p(\nabla.\mathbf{q})\,dV$$

Hence the result (1).

Problem 12. *A spherical mass of fluid of radius b has a concentric spherical cavity of radius a, which contains gas at pressure p whose mass may be neglected; at every point of the external boundary of the liquid an impulsive pressure $\overline{\omega}$ per unit area is applied. Assuming that the gas obeys Boyle's law, show that when the liquid first comes to rest, the radius of the internal spherical surface will be $a\exp\{-\overline{\omega}^2/2p\rho a^2(b-a)\}$ where ρ is the density of the liquid.*

Solution ; Equation of impulsive action is $d\overline{\omega} = \rho v\,dx$ and equation of continuity is

$$x^2 v = F(t).$$

$\therefore$ $$d\overline{\omega} = \rho F(t)\,.\frac{dx}{x^2}$$

This $$\Rightarrow \int_0^{\overline{\omega}} d\overline{\omega} = \int_a^b \rho F\frac{dx}{x^2}$$

or $$\overline{\omega} = \rho F\left[-\frac{1}{x}\right]_a^b = \left(\frac{b-a}{ab}\right)\rho F(t). \qquad \dots (1)$$

Let r be the radius of internal spherical cavity and p_1 the pressure there. Since gas obeys Boyle's law hence

$$\frac{4}{3}\pi r^3 p_1 = \frac{4}{3}\pi a^3 p \text{ or } p_1 = \frac{a^3 p}{r^3}.$$

[Internal cavity of radius a contains gas at pressure p].

Finally, the liquid is at rest.

$$\text{Gain in K.E.} = \int_a^b \frac{1}{2}(4\pi x^2\,dx\,.\,\rho)v^2 = 2\pi\rho\int_a^b x^2\frac{F^2}{x^4}dx \quad \text{as } x^2 v = F(t)$$

$$= 2\pi\rho\left(-\frac{1}{x}\right)_a^b F^2 = 2\pi\rho F^2\left\{\frac{b-a}{ab}\right\}$$

$$= 2\pi\rho\left(\frac{b-a}{ab}\right)\cdot\frac{\overline{\omega}^2 a^2 b^2}{\rho^2(b-a)^2}$$

$$= 2\pi\,ab\,\overline{\omega}^2/\rho(b-a).$$

Work done in compressing the gas from radius a to radius r is $\int_a^r -p\,dV$ in usual notation

$$= -\int_a^r \frac{4\pi r^2\,dr}{r^3}\,a^3 p = -4\pi p a^3\log\left(\frac{r}{a}\right).$$

But gain in K.E. = work done.

$\because$ $$\frac{2\pi ab\overline{\omega}^2}{\rho(b-a)} = -4\pi p a^3\log\left(\frac{r}{a}\right)$$

or $$\log\left(\frac{r}{a}\right) = -\frac{\overline{\omega}^2 b}{2a^2 p\rho(b-a)}$$

Hence $$r = a\exp\left\{-\frac{\overline{\omega}^2 b}{2a^2 p\rho(b-a)}\right\}.$$

Problem 13. *Two equal closed cylinders, of height c, with their bases in the same horizontal plane, are filled, one with water, and the other with air of such a density as to support a column h of water, h < c. If a communication be open between them at their bases, the height x, to which the water rises, is given by the equation*

$$cx - x^2 + ch\log\left(\frac{c-x}{c}\right) = 0.$$

Solution : Suppose that the cylinders A and B are filled with water and gas respectively. Let k be the cross section of each cylinder. The water and gas both are at rest before and after the communication is allowed between the cylinders. Hence initial and final both K.E. are zero. Change in K.E. = 0.

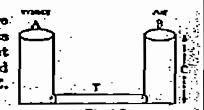


Fig. 2.5

This $\Rightarrow$ Total work done = change in K.E.

$\Rightarrow$ Total work done = 0.

Initial potential energy due to water in A

= 'Mgh' in usual notation.

$$= \int_0^c (kz\rho)\,g\,dz = \frac{1}{2}kg\rho c^2$$

and final potential energy due to watr of height $c - x$ in A and height x in B is

$$\int_0^{c-x}(kz\rho)\,g\,dz + \int_0^x (kz\rho)\,g\,dz = \frac{1}{2}kg\rho\,[(c-x)^2 + x^2]$$

Now work done by gravity = loss in potential energy

= Initial P.E. – Final P.E.

$$= \frac{1}{2}kg\rho\,[c^2 - (c-x)^2 - x^2] = kg\rho\,(cx - x^2)$$

$\therefore$ Work done by gravity $= kg\rho\,(cx - x^2)$. $\qquad \dots (2)$

Also some work is done against the compression of air in B. Let p be the pressure of the gas when the height of water level in B is y. By Boyle's law,

$$P_1V_1 = P_2V_2 \quad \text{or} \quad pk(c-y) = h\rho g\,.\,kc.$$

This $\Rightarrow p = \frac{h\rho gc}{c-y}$, ρ being density of water.

[For pressure = hdg = height . density . g and initial pressure of the gas in B is equal to pressure due to a column h of water (given)].

Work done against the compression of gas in B

$$= \int_0^x -p\,dV, \text{ in usual notation.}$$

$$= \int_0^x -\left(hg\rho\frac{c}{c-y}\right)k\,dy, \quad dV = k\,dy$$

$$= hg\rho ck\log\left(\frac{c-k}{c}\right). \qquad \dots (3)$$

Equating the sum of (2) and (3) to (1),

$$kg\rho\,(cx - x^2) + hg\rho ck\log\left(\frac{c-x}{c}\right) = 0$$

$$cx - x^2 + ch\log\left(\frac{c-x}{c}\right) = 0$$

Problem 13 (a). *Water oscillates in a bent uniform tube in a vertical plane. If O be the lowest point of the tube, AB the equilibrium level of water, α, β the inclinations of the tube to the horizontal at A, B and OA = a, OB = b, the period of oscillation is given by*

$$2\pi\left\{\frac{(a+b)}{g(\sin\alpha+\sin\beta)}\right\}^{1/2}$$

Solution : Suppose O is the lowest point of the tube, AB the equilibrium level of water, h the height of AB above O, α, β the inclinations of the tube to the horizontal at A and B respectively and θ, the inclination at a distance s from O. Let $OA = a$,

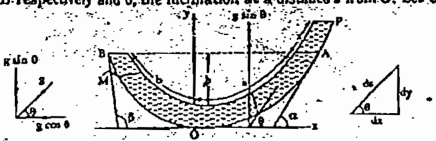


Fig. 2.6

$OB = b$, $AP = x$. Let water in the tube be displaced at small distance x from its equilibrium position so that $AP = x$. After displacement

(i) $p = \Pi$, $y = h + x\sin\alpha$, $s = OP = a + x$ at P.

(ii) $p = \Pi$, $y = h - x\sin\beta$, $s = OM = -(b - x)$ at M.

Let u denote velocity. Equation of continuity is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}=0$$

or $$\frac{\partial u}{\partial s}=0 \qquad \ldots(1)$$

This $\Rightarrow u$ is independent of s.

Equation of motion is

$$\frac{\partial u}{\partial t}+\frac{\partial u}{\partial s}=-g\sin\theta-\frac{1}{\rho}\frac{\partial p}{\partial s}$$

But $$\frac{\partial u}{\partial s}=0,\quad \sin\theta=\frac{\partial y}{\partial s}.$$

Hence we have

$$\frac{\partial u}{\partial t}=-g\frac{\partial y}{\partial s}-\frac{\partial}{\partial s}\left(\frac{p}{\rho}\right).$$

Integrating w.r.t. s,

$$\int\frac{\partial u}{\partial t}\,ds=-gy-\frac{p}{\rho}+f(t)$$

or $$s\frac{\partial u}{\partial t}=-gy-\frac{p}{\rho}+f(t)$$

($f(t)$ is constant of integration)

Applying (i) and (ii),

$$(a+x)\frac{\partial u}{\partial t}=-g(h+x\sin\alpha)-\frac{\Pi}{\rho}+f(t)$$

$$-(b-x)\frac{\partial u}{\partial t}=-g(h-x\sin\beta)-\frac{\Pi}{\rho}+f(t).$$

Upon subtraction,

$$(a+b)\frac{\partial u}{\partial t}=-gx(\sin\alpha+\sin\beta)$$

Since $$u=\dot{x},\quad \dot{u}=\ddot{x}$$

$\therefore$ $$(a+b)\ddot{x}=-gx(\sin\alpha+\sin\beta)$$

or $$\ddot{x}=-\mu x \qquad \ldots(1),\ \text{where } \mu=\frac{g(\sin\alpha+\sin\beta)}{(a+b)}$$

(1) represents S.H.M. Its time period T is given by

$$T=\frac{2\pi}{\sqrt{\mu}}=2\pi\left[\frac{(a+b)}{g(\sin\alpha+\sin\beta)}\right]^{1/2}$$

Problem 14. *A given quantity of liquid moves, under no forces, in a smooth conical tube having a small vertical angle, and the distances of its nearer and farther extremities for the vertex at time t are r and r′, show that*

$$2r\frac{d^2r}{dt^2}+\left(\frac{dr}{dt}\right)^2\left[3-\frac{r}{r'}-\frac{r'^2}{r^2}-\frac{r^3}{r'^3}\right]=0,$$

the pressure at the two surfaces being equal. Show also that the preceding equation results from supposing the vis viva of the mass of liquid to be constant; and that the velocity of inner surface is given by the equation

$$V^2=\frac{Cr'}{r^3(r'-r)},\quad r'^3-r^3=c^3,$$

C and c being constants.

Solution : At any time t, let p be the pressure at a distance x from the vertex and v the velocity there. The equation of motion is

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=0-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

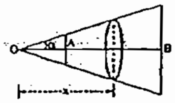


Fig. 2.7

and the equation of continuity is

$$(x\tan\alpha)^2 v=F(t),$$

i.e., $$x^2v=f(t)$$

where $f(t)=\dfrac{F(t)}{\tan^2\alpha}$. $\left(\text{Here } \dfrac{r}{h}=\dfrac{r}{x}=\tan\alpha\right)$

Hence, $$\frac{f'(t)}{x}+\frac{\partial}{\partial x}\left(\frac{1}{2}v^2\right)=-\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right)$$

Integrating w.r.t. x,

$$-\frac{f'(t)}{x}+\frac{1}{2}v^2=-\frac{p}{\rho}+C. \qquad \ldots(1)$$

Boundary conditions are

(i) when $x = r$, $p = p$, $v = \dot{r} = u$, say

(ii) when $x = r'$, $v = U = \dot{r}'$, $p = p$.

[Since the pressure at the two ends is equal].

Subjecting (1) to the conditions (i) and (ii),

$$-\frac{f'(t)}{r}+\frac{1}{2}u^2=-\frac{p}{\rho}+C$$

$$-\frac{f'(t)}{r'}+\frac{1}{2}U^2=-\frac{p}{\rho}+C$$

Upon subtraction

$$\left(\frac{1}{r'}-\frac{1}{r}\right)f'(t)+\frac{1}{2}(u^2-U^2)=0.$$

But $$r^2u=f(t)=r'^2U$$

$\therefore$ $$\left(\frac{1}{r'}-\frac{1}{r}\right)\frac{d}{dt}(r^2u)+\frac{1}{2}u^2\left(1-\frac{r^4}{(r')^4}\right)=0,\ u=\dot{r}$$

or $$\left(\frac{r-r'}{rr'}\right)\left[2r\left(\frac{dr}{dt}\right)^2+r^2\frac{d^2r}{dt^2}\right]+\frac{1}{2}\left(\frac{dr}{dt}\right)^2\left[\frac{r'^4-r^4}{r'^4}\right]=0$$

Dividing by $(r-r')/2r'$, we get

$$\frac{2}{r}\left[2r\left(\frac{dr}{dt}\right)^2+r^2\frac{d^2r}{dt^2}\right]-\left(\frac{dr}{dt}\right)^2\frac{(r+r')(r^2+r'^2)}{r'^3}=0$$

or $$2r\frac{d^2r}{dt^2}+\left(\frac{dr}{dt}\right)^2\left[3-\frac{r}{r'}-\frac{r^3}{r'^3}-\frac{r^2}{r'^2}\right]=0$$

This proves the first required result.

Second Part : The vis-via = 2K.E.

$$=2\int_r^{r'}\frac{1}{2}(\pi x^2\tan^2\alpha)\,dx\,\rho v^2=\pi\rho\tan^2\alpha\int_r^{r'}\frac{x^2}{x^4}f^2(t)\,dx$$

$$=\pi\rho\tan^2\alpha\, f^2(t)\left(\frac{1}{r}-\frac{1}{r'}\right).$$

By the principle of conservation of vis-via,

$$\pi\rho\tan^2\alpha\, f^2(t)\cdot\left(\frac{1}{r}-\frac{1}{r'}\right)=\text{const.}=C_1$$

or $$\left(\frac{1}{r}-\frac{1}{r'}\right)(r^2u)^2=C_2$$

or $$\left(\frac{r'-r}{rr'}\right)r^4u^2=C_2$$

or $$u^2=\frac{r'C_2}{(r'-r)r^3}$$

Replacing u by V and C_2 by C, we get

$$V^2=Cr'/r^3(r'-r).$$

Again, since mass is constant and so is volume.

This $\Rightarrow \frac{1}{3}(\pi r'^2\tan\alpha.r'-\pi r^2\tan^2\alpha.r)=\text{const.}$

or $r'^3-r^3=\text{const.}=c^3$, say. For volume $=\frac{\pi}{3}(\text{radius})^2h$

This concludes the problem.

Problem 16. *A portion of homogeneous fluid is confined between two concentric spheres of radii A and a, and is attracted towards their centre by a force varying inversely as the square of the distance the inner spherical surface is suddenly annihilated, and when the radii of the inner and outer surfaces of the fluid are r and R, the fluid impinges on a solid ball concentric with their surfaces; prove that the impulsive pressure at any point of the ball for different values of R and r varies as*

$$\left[(a^2-r^2-A^2+R^2)\left(\frac{1}{r}-\frac{1}{R}\right)\right]^{1/2}$$

Solution : The equation of continuity is $x^2v=F(t)$ so that $\dfrac{\partial v}{\partial t}=\dfrac{F'(t)}{x^2}$. Equation of motion is

$$\frac{\partial v}{\partial t}+\frac{\partial v}{\partial x}=F-\frac{1}{\rho}\frac{\partial p}{\partial x}=-\frac{\mu}{x^2}-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

[as μ/x^2 is a force towards the centre].

or $$\frac{F'(t)}{x^2}+\frac{\partial}{\partial x}\left(\frac{1}{2}v^2\right)=-\frac{\mu}{x^2}-\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right)$$

Integrating w.r.t. x, $$\frac{-F'(t)}{x}+\frac{1}{2}v^2=\frac{\mu}{x}-\frac{p}{\rho}+C. \qquad \ldots(1)$$

Let r and R be internal and external radii at any time t. Boundary conditions are

(i) when $x = r$, $v = \dot{r} = u$ say $p = 0$.
(Since pressure vanishes on the internal surface).

(ii) when $x = R$, $v = \dot{R} = U$ say, $p = 0$.
(Since pressure vanishes on the surface of a annihilated sphere).

(iii) when $r = a$, $R = A$, the velocity is zero so that $F(t) = 0$.

Subjecting (1) to the conditions (i), and (ii),

$$\frac{-F'(t)}{r}+\frac{1}{2}u^2=\frac{\mu}{r}+C$$

$$\frac{-F'(t)}{R}+\frac{1}{2}U^2=\frac{\mu}{R}+C$$

Upon subtraction

$$\left\{\frac{1}{R}-\frac{1}{r}\right\}F'(t)+\frac{1}{2}(u^2-U^2)=\mu\left(\frac{1}{r}-\frac{1}{R}\right)$$

or $$\left(\frac{1}{R}-\frac{1}{r}\right)F'(t)+\frac{1}{2}F^2\left\{\frac{1}{r^4}-\frac{1}{R^4}\right\}=\mu\left\{\frac{1}{r}-\frac{1}{R}\right\}.$$

[For $r^2u=F(t)=R^2U$]

Multiplying by $2F dt$ or equivalently by $2R^2 dR = 2r^2 dr$, we obtain

$$2FF'\left\{\frac{1}{R}-\frac{1}{r}\right\}dt + F^2\left\{\frac{dr}{r^2}-\frac{dR}{R^2}\right\} = 2\mu\,(r\,dr - R\,dR)$$

or
$$d\left[\left\{\frac{1}{R}-\frac{1}{r}\right\}F^2\right] = 2\mu\,(r\,dr - R\,dR).$$

Integrating. $\left(\frac{1}{R}-\frac{1}{r}\right)F^2(t) = \mu\,(r^2 - R^2) + C_1$.

Subjecting this to (iii), $0 = \mu\,(a^2 - A^2) + C_1$

$$\therefore \quad \left(\frac{1}{R}-\frac{1}{r}\right)F^2 = \mu\,(r^2 - R^2 - a^2 + A^2). \qquad \dots (2)$$

The equation of impulsive action is

$$d\overline{\omega} = \rho\, v\, dx = \frac{\rho F(t)}{x^2}dx$$

This $\Rightarrow$
$$\int_0^{\overline{\omega}} d\overline{\omega} = \int_r^R \rho\,(F)\frac{dx}{x^2} \Rightarrow \overline{\omega} = \left[\frac{1}{r}-\frac{1}{R}\right]\rho F(t)$$

Putting the values of $F(t)$ from (2) in this equation,

$$\overline{\omega} = \left(\frac{1}{r}-\frac{1}{R}\right)\rho\left[\frac{\mu\,(r^2 - R^2 - a^2 + A^2)}{\left(\frac{1}{R}-\frac{1}{r}\right)}\right]^{1/2}$$

or
$$\overline{\omega} = \rho\left[\mu\,(a^2 - r^2 - A^2 + R^2)\left(\frac{1}{r}-\frac{1}{R}\right)\right]^{1/2}$$

or
$$\overline{\omega} \text{ various as } \left[(a^2 - r^2 - A^2 + R^2)\left(\frac{1}{r}-\frac{1}{R}\right)\right]^{1/2}$$

Problem 16. *A sphere of radius a is surrounded by infinite liquid of density ρ, the pressure at infinity being Π. The sphere is suddenly annihilated. Show that pressure at a distance r from the centre immediately falls to $\Pi\left(1-\frac{a}{r}\right)$.*

Show further that if the liquid is brought to rest by impinging on a concentric sphere of radius $a/2$, the impulsive pressure sustained by the surface of this sphere is $\left[\frac{7}{6}\Pi\rho a^2\right]^{1/2}$.

Solution : The equation of motion is $x^2 v = F(t)$ so that $\frac{\partial v}{\partial t} = \frac{F'(t)}{x^2}$. Equation of motion is

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

or
$$\frac{F'(t)}{x^2} + \frac{\partial}{\partial x}\left(\frac{1}{2}v^2\right) = -\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right)$$

Integrating w.r.t. x, $\quad \frac{-F'(t)}{x} + \frac{1}{2}v^2 = -\frac{p}{\rho} + C \qquad \dots (1)$

Boundary conditions are

(i) when $x = \infty, p = \Pi, v = 0$.

(ii) when $x = a, v = \dot{x} = 0, p = 0, t = 0$

[Since the sphere of radius a is annihilated and pressure vanishes on the annihilated sphere]

Immediately after annihilation, the liquid has no time to move. So we suppose

(iii) when $t = 0, x = r, v = 0, p = p_0$, where $r > a$.

We want to prove $\quad p_0 = \Pi\left(1-\frac{a}{r}\right)$.

Subjecting (1) to (i) and (ii), $\quad 0 = -\frac{\Pi}{\rho} + C$

and
$$-\frac{F'(0)}{a} + 0 = 0 + C.$$

This $\Rightarrow$
$$\frac{-F'(0)}{a} = C = \frac{\Pi}{\rho} \qquad \dots (2)$$

In view of (iii), (1) gives $\frac{-F'(0)}{r} = \frac{-p_0}{\rho} + C$

or
$$\frac{a\Pi}{\rho}\cdot\frac{1}{r} = \frac{-p_0}{\rho} + \frac{\Pi}{\rho}, \text{ by (2)}$$

or
$$p_0 = \Pi\left(1-\frac{a}{r}\right).$$

Second Part : Let $\overline{\omega}$ be the required impulsive pressure. Then we have to prove that $\overline{\omega} = \left[\frac{7}{6}\Pi\rho a^2\right]^{1/2}$

First we shall determine velocity on the inner surface. Let r be the radius of inner surface. Then

(iv) when $x = r, v = \dot{r} = u$ say, $p = 0$ when $r < a$

[Note the difference of (ii) and (iv).]

Since pressure vanishes on the inner surface. In view of the above condition, (1) gives

$$\frac{-F'(t)}{r} + \frac{1}{2}u^2 = C = \frac{\Pi}{\rho}$$

or
$$\frac{-F'(t)}{r} + \frac{1}{2}\cdot\frac{F^2}{r^4} = \frac{\Pi}{\rho} \quad \text{as } r^2 u = F(t)$$

Multiplying by $2F(t)\,dt$ or equivalently by $2r^2 dr$, we obtain

$$\frac{-2FF'dt}{r} + \frac{F^2 dr}{r^2} = \frac{\Pi}{\rho}2r^2 dr.$$

or
$$d\left(\frac{-F^2}{r}\right) = \frac{\Pi}{\rho}2r^2 dr.$$

Integrating $\quad \frac{-F^2}{r} = \frac{2}{3}\frac{\Pi}{\rho}r^3 + C_1$

or
$$-r^3 u^2 = \frac{2}{3}\frac{\Pi}{\rho}r^3 + C_1.$$

In view of (ii), this

$$\Rightarrow \quad 0 = \frac{2}{3}\frac{\Pi}{\rho}a^3 + C_1$$

$$\therefore \quad -r^3u^2 = \frac{2}{3}\frac{\Pi}{\rho}r^3 - \frac{2\Pi}{3\rho}a^3$$

or
$$r^3u^2 = \frac{2}{3}\frac{\Pi}{\rho}(a^3 - r^3) \text{ or } u^2 = \frac{2}{3}\frac{\Pi}{\rho}\left(\frac{a^3}{r^3}-1\right)$$

$$\therefore \quad [(u^2)_{r=a/2}] = \frac{2}{3}\frac{\Pi}{\rho}.(8-1)$$

or
$$[(u)_{r=a/2}] = \left[\frac{14\Pi}{3\rho}\right]^{1/2}$$

Equation of impulsive action is $d\overline{\omega} = \rho\, v\, dx$.

This $\Rightarrow d\overline{\omega} = \rho\, u\, dr \Rightarrow \int_0^{\overline{\omega}} d\overline{\omega} = \rho\,(u)_{r=a/2}\int_0^{a/2} dr$

$$\Rightarrow \overline{\omega} = \rho\left[\frac{14\Pi}{3\rho}\right]^{1/2}\cdot\frac{a}{2} = \left[\frac{7}{6}\rho\Pi a^2\right]^{1/2}.$$

Problem 17. *A sphere whose radius at time t is $b + a\cos nt$ is surrounded by liquid extending to infinity under no forces. Prove that the pressure at a distance r from the centre is less than the pressure at an infinite distance by*

$$\frac{na^2}{r}(b + a\cos nt)\left[a\,(1 - 3\sin^2 nt) + b\cos nt + \frac{a}{2r^3}(b + a\cos nt)^3\sin^2 nt\right].$$

Solution : Let Π be the pressure at infinity and p_0 at a distance r. Then we have to prove that

$$\frac{\Pi - p_0}{\rho} = \frac{na^2}{r}(b + a\cos nt)\left[a\,(1 - 3\sin^2 nt) + b\cos nt + \frac{a}{2r^3}(b + a\cos nt)^3\sin^2 nt\right] \qquad \dots (1)$$

Equation of continuity is $x^2 v = F(t)$ so that $\frac{\partial v}{\partial t} = \frac{F'(t)}{x^2}$.

Equation of motion is

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

or
$$\frac{F'(t)}{x^2} + \frac{\partial}{\partial x}\left(\frac{1}{2}v^2\right) = -\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right).$$

Integrating, $\quad \frac{-F'(t)}{x} + \frac{1}{2}v^2 = -\frac{p}{\rho} + C \qquad \dots (2)$

Boundary conditions are

(i) when $x = \infty, v = 0, p = \Pi$

(ii) when $x = r, v = \dot{r} = u$ say, $p = p_0$

Subjecting (2) to (i), $\quad 0 = -\frac{\Pi}{\rho} + C \;$ or $\; C = \frac{\Pi}{\rho}$.

$$\therefore \quad \frac{-F'(t)}{x} + \frac{1}{2}v^2 = \frac{\Pi - p}{\rho}$$

Subjecting this to (ii),

$$\frac{-F'(t)}{r} + \frac{1}{2}u^2 = \frac{\Pi - p_0}{\rho}$$

$$\frac{-F'(t)}{r} + \frac{1}{2}\frac{F^2}{r^4} = \frac{\Pi - p_0}{\rho} \qquad \dots (3)$$

Let R be the radius at any time t. Then

$R = b + a\cos nt$. Also let $U = \dot{R}$. We have

$r^2\dot{r} = r^2 u = F(t) = R^2\dot{R}, \dot{R} = -na\sin nt.$

$F(t) = R^2\dot{R} = (b + a\cos nt)^2(-na\sin nt)$

$F'(t) = 2(b + a\cos nt)n^2a^2\sin^2 nt - an^2\cos nt\,(b + a\cos nt)^2.$

Putting these in (3), we get

$$\frac{\Pi - p_0}{\rho} = -\frac{1}{r}[2(b + a\cos nt)n^2a^2\sin^2 nt - (b + a\cos nt)^2 n^2 a\cos nt] + \frac{1}{2}\cdot\frac{(b + a\cos nt)^4}{r^4}\cdot n^2a^2\sin^2 nt$$

$$= (b + a\cos nt)\frac{n^2a}{r}\left[-2a\sin^2 nt + (b + a\cos nt)\cos nt + (b + a\cos nt)^3\cdot\frac{a}{2r^3}\sin^2 nt\right]$$

$$= (b + a\cos nt)\frac{n^2a}{r}\left[a\,(1 - 3\sin^2 nt) + b\cos nt + \frac{1}{2}\cdot\frac{a}{r^3}(b + a\cos nt)^3\sin^2 nt\right]$$

This proves the required result.

Problem 18. *A mass of liquid of density ρ whose external surface is a long circular cylinder of radius a, which is subject to a constant pressure Π, surrounds a coaxial long circular cylinder of radius b. The internal cylinder is suddenly destroyed. Show that if v is the velocity at the internal surface when its radius is r, then*

$$v^2 = \frac{2\Pi(b^2 - r^2)}{\rho r^2\log[(r^2 + a^2 - b^2)/r^2]}$$

Solution : Equation of continuity is $xu = F(t)$ and equation of motion is

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \quad \text{and} \quad u = \dot{x}.$$

or
$$\frac{F'(t)}{x} + \frac{\partial}{\partial x}\left(\frac{1}{2}u^2\right) = -\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right)$$

Integrating, $F'(t)\log x + \frac{1}{2}u^2 = -\frac{p}{\rho} + C$

or $F'(t)\log x + \frac{1}{2}\frac{F^2}{x^2} = -\frac{p}{\rho} + C$ ...(1)

$ux = F \Rightarrow x\dot{x} = F \Rightarrow x\,dx = F(t)\,dt$.

Let R and r be external and internal radii at any time t. Since total mass of liquid is constant. Hence mass of the liquid at anytime t = mass of the liquid at time $t = 0$.

i.e., $(\pi R^2 h - \pi r^2 h)\rho = (\pi a^2 h - \pi b^2 h)\rho$

or $R^2 - r^2 = a^2 - b^2$ or $R^2 = r^2 + a^2 - b^2$ ...(2)

Boundary conditions are

(i) when $x = R, u = \dot{R}, p = \Pi$.

[For external boundary is subjected to a constant pressure Π].

(ii) when $x = r, u = \dot{r} = v, p = 0$.

(For pressure vanishes on the internal boundary).

(iii) when $r = b, u = b = 0$, i.e., $F(t) = 0, p = 0$.

Subjecting (1) to (i) and (ii),

$$F'(t)\log R + \frac{1}{2}\cdot\frac{F^2}{R^2} = -\frac{\Pi}{\rho} + C$$

$$F'(t)\log r + \frac{1}{2}\cdot\frac{F^2}{r^2} = 0 + C$$

upon subtracting, $(\log R - \log r)F' + \frac{1}{2}F^2\left\{\frac{1}{R^2} - \frac{1}{r^2}\right\} = \frac{-\Pi}{\rho}$ ...(3)

Multiplying (3) by $2F\,dt = 2r\,dr = 2R\,dR$,

$$2FF'\,dt.(\log R - \log r) + F^2\left\{\frac{dR}{R} - \frac{dr}{r}\right\} = -\frac{\Pi}{\rho}.2r\,dr$$

or $d[(\log R - \log r)F^2] = -2r\frac{\Pi}{\rho}.dr$

Integrating, $(\log R - \log r)F^2 = -\frac{\Pi}{\rho}r^2 + C_1$

By (2), this $\Rightarrow \left[\log\frac{(r^2 + a^2 - b^2)^{1/2}}{r}\right]F^2 = -\frac{r^2\Pi}{\rho} + C_1$

In view of (iii), this $\Rightarrow 0 = -b^2\frac{\Pi}{\rho} + C_1$

$\therefore \left[\log\frac{(r^2 + a^2 - b^2)^{1/2}}{r}\right]F^2 = (b^2 - r^2)\frac{\Pi}{\rho}$

or $\left[\log\left(\frac{r^2 + a^2 - b^2}{r^2}\right)\right](rv)^2 = 2(b^2 - r^2)\frac{\Pi}{\rho}$

or $v^2 = \frac{2\Pi(b^2 - r^2)}{\rho r^2}\cdot\frac{1}{\log[(r^2 + a^2 - b^2)/r^2]}$

Alternate method: Equation of continuity is $xu = F(t)$, where $\dot{x} = u$. Let R and r be external and internal radii. Since total mass of the liquid is constant hence

$(\pi R^2 h - \pi r^2 h)\rho = (\pi a^2 h - \pi b^2 h)\rho$

or $R^2 - r^2 = a^2 - b^2$ or $R^2 = r^2 + a^2 - b^2$. ...(1)

K.E. of the liquid $= \frac{1}{2}\int_r^R (2\pi x\,dx)\rho u^2 = \pi\rho\int_r^R x\,dx\frac{F^2}{x^2}$

$= \pi\rho F^2\log\left(\frac{R}{r}\right)$

Work done by outer pressure $= \int_a^R -p\,dV = \int_a^R -2\pi x\,dx\,\Pi$

$= \pi\Pi(a^2 - R^2) = \pi\Pi(b^2 - r^2)$, by (1)

$\therefore$ Work done = K.E.

$\pi\Pi(b^2 - r^2) = \pi\rho F^2\log\left(\frac{R}{r}\right) = \left(\frac{\pi}{2}\right)\rho r^2 v^2\log\left(\frac{R^2}{r^2}\right)$

or $v^2 = \frac{2\Pi(b^2 - r^2)}{\rho r^2}\cdot\frac{1}{\log[(r^2 + a^2 - b^2)/r^2]}$

Problem 19. *Liquid is contained between two parallel planes; the free surface is a circular cylinder of radius a whose axis is perpendicular to the planes. All the liquid within a concentric circular cylinder of radius b is suddenly annihilated. Prove that if Π be the pressure at the outer surface, the initial pressure at any point of the liquid, distant r from the centre, is* (IAS-2006)

$$\Pi\left[\frac{\log r - \log b}{\log a - \log b}\right].$$

Solution: The equation of continuity is $xv = F(t)$ and equation of motion is

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$\therefore \frac{F'(t)}{x} + \frac{\partial}{\partial x}\left(\frac{1}{2}v^2\right) = \frac{\partial}{\partial x}\left(-\frac{p}{\rho}\right)$.

Integrating $F'(t)\log x + \frac{1}{2}v^2 = -\frac{p}{\rho} + C$ ...(1)

Note that initially (i.e., at $t = 0$) the liquid is at rest.

Boundary conditions are

(i) when $x = a, v = \dot{x} = 0, p = \Pi, t = 0$.

[Since the outer surface is subjected to a constant pressure Π].

(ii) when $x = b, v = \dot{x} = 0, p = 0, t = 0$.

[Since pressure vanishes on the surface of annihilated sphere].

(iii) when $x = r, t = 0, p = p_0$ say.

We have to prove that $p_0 = \Pi\left[\frac{\log r - \log b}{\log a - \log b}\right]$.

Subjecting (1) to (i) and (ii),

$F'(0)\log a = -\frac{\Pi}{\rho} + C$

and $F'(0)\log b = 0 + C$.

This $\Rightarrow F'(0)\log a = -\frac{\Pi}{\rho} + F'(0)\log b$

or $F'(0)\log\left(\frac{a}{b}\right) = \left(-\frac{\Pi}{\rho}\right)$. ...(2)

Also, by (2), $C = F'(0)\log b = -\frac{\Pi}{\rho}\frac{\log b}{\log(a/b)}$ ...(3)

In view of (iii), (1) gives $F'(0)\log r = -\frac{p_0}{\rho} + C$

or $-\frac{\Pi}{\rho}\frac{\log r}{\log(a/b)} = -\frac{p_0}{\rho} - \frac{\Pi}{\rho}\frac{\log b}{\log(a/b)}$, by (2)

or $p_0 = \frac{\Pi(\log r - \log b)}{\log(a/b)} = \Pi\left(\frac{\log r - \log b}{\log a - \log b}\right)$.

Problem 20. *An infinite mass of homogeneous incompressible fluid is at rest subject to a uniform pressure Π and contains a spherical cavity of radius a, filled with a gas at a pressure $m\Pi$; prove that if the inertia of the gas be neglected, and Boyle's law be supposed to hold throughout the ensuing motion, the radius of the sphere will oscillate between the value a and na, where n is determined by the equation*

$$1 + 3m\log n - n^3 = 0.$$

If m be nearly equal to 1, the time of an oscillation will be $2\pi\left(\frac{a^2\rho}{3\Pi}\right)^{1/2}$, ρ being the density of the fluid.

Solution: Equation of continuity is $x^2 v = F(t)$ so that $\dot{x} = v$, $\frac{\partial v}{\partial t} = \frac{F'(t)}{x^2}$.

Equation of motion is

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

or $\frac{F'(t)}{x^2} + \frac{\partial}{\partial x}\left(\frac{1}{2}v^2\right) = \frac{\partial}{\partial x}\left(-\frac{p}{\rho}\right)$

Integrating w.r.t. x,

$-\frac{F'(t)}{x} + \frac{1}{2}v^2 = -\frac{p}{\rho} + C$. ...(1)

Boundary conditions are

(i) when $x = \infty, p = \Pi, v = 0$.

[Since the infinite mass is at rest subjected to a constant pressure Π]

Let r be the radius of cavity at any time t, then $r < a$ and p_1 the pressure there.

Since the gas within cavity obeys Boyle's law

$P_1V_1 = P_2V_2$ = const., i.e., $\frac{4}{3}\pi r^3 p_1 = m\Pi.\frac{4}{3}\pi a^3$

$p_1 = m\Pi\frac{a^3}{r^3}$. ...(2)

(ii) when $x = r, p = p_1, v = \dot{r} = u$ say.

(iii) when $r = a, p = m\Pi, v = \dot{r} = 0$.

Subjecting (1) to (i),

$$0 = -\frac{\Pi}{\rho} + C$$

Now (1) becomes

$$-\frac{F'(t)}{x} + \frac{1}{2}\cdot\frac{F^2}{x^4} = \frac{\Pi - p}{\rho}.$$

Multiplying by $2F\,dt = 2x^2\,dx$ [as $x^2\dot{x} = F(t)$], we get

$-\frac{2FF'}{x}dt + \frac{F^2}{x^2}dx = \frac{\Pi - p}{\rho}.2x^2\,dx$ ...(3)

Now we can not integrate this equation w.r.t. x as p is not constant due to the fact that cavity contains gas at varying pressure. So we subject this equation to the condition (ii) and using (2),

$$-\frac{2FF'}{r}dt + \frac{F^2}{r^2}dr = \frac{1}{\rho}\left(\Pi - m\Pi\frac{a^3}{r^3}\right)2r^2\,dr$$

or $d\left(-\frac{F^2}{r}\right) = \frac{2\Pi}{\rho}\left(r^2 - \frac{ma^3}{r}\right)dr$

Integrating, $-\frac{F^2}{r} = \frac{2\Pi}{\rho}\left(\frac{1}{3}r^3 - ma^3\log r\right) + C_2$

or $r^3u^2 = \frac{2\Pi}{\rho}\left(ma^3\log r - \frac{r^3}{3}\right) + C_3$.

By (iii), this $\Rightarrow 0 = \frac{2\Pi}{\rho}\left[ma^3\log a - \frac{a^3}{3}\right] + C_3$.

Upon subtraction, we get

$\therefore r^3u^2 = \frac{2\Pi}{\rho}\left[ma^3\log\left(\frac{r}{a}\right) - \left(\frac{r^3 - a^3}{3}\right)\right]$ ...(4)

Since radius oscillates between a and na hence we put $r = na$, $u = \dot{r} = 0$. Hence we get

$$0 = \frac{2\Pi}{\rho}\left[ma^3\log n + \frac{1}{3}(a^3 - n^3a^3)\right]$$

or $3m\log n + 1 - n^3 = 0$

or $1 + 3m\log n - n^3 = 0$.

Second part: When $m = 1$ (approximately).

Let $r = a + y$, y being small $u = \dot{r} = \dot{y}$.

Now (4) gives

$$(a+y)^3 \dot{y}^2 = \frac{2\Pi}{\rho}\left[a^3 \log\left(\frac{a+y}{a}\right) + \frac{1}{3}\{a^3 - (a+y)^3\}\right]$$

or $$\dot{y}^2 = \frac{2\Pi}{3\rho}\left[3\log\left(1+\frac{y}{a}\right) + 1 - \left(1+\frac{y}{a}\right)^3\right]\left(1+\frac{y}{a}\right)^{-3}$$

Expanding upto second degree terms,

$$\dot{y}^2 = \frac{2\Pi}{3\rho}\left(1 - \frac{3y}{a} + \ldots\right)\left[3\left(\frac{y}{a} - \frac{y^2}{2a^2}\right) + 1 - \left(1 + \frac{3y}{a} + \frac{3.2}{2}.\frac{y^2}{a^2}\right)\right]$$

$$= \frac{2\Pi}{\rho}\left(1 - \frac{3y}{a} + \ldots\right)\left(-\frac{9y^2}{2a^2}\right) = \frac{2\Pi}{3\rho}\left(-\frac{9y^2}{2a^2}\right)$$

$$= -\frac{3\Pi}{\rho a^2} y^2.$$

Differentiating w.r.t. t, $2\dot{y}\ddot{y} = -\frac{3\Pi}{\rho a^2} 2y\dot{y}$

or $\ddot{y} = -\mu y$ where $\mu = \frac{3\Pi}{\rho a^2}$.

It is the equation for S.H.M.

Hence time period $= T = \frac{2\pi}{\sqrt{\mu}} = 2\pi\left(\frac{a^2\rho}{3\Pi}\right)^{1/2}$.

Problem 21. *A solid sphere of radius a is surrounded by a mass of liquid whose volume is $\frac{4}{3}\pi c^3$, and its centre is attracted by a force μx^3. If the solid sphere be suddenly annihilated, show that velocity of inner surface, when its radius is x, is given by*

$$\dot{x}^2 x^2 [(x^3 + c^3)^{1/3} - x] = \left(\frac{2\Pi}{3\rho} + \frac{2\mu c^3}{9}\right)(a^3 - x^3)(c^3 + x^3)^{1/3}$$

where ρ is the density. Π the external pressure and μ the distance.

Solution : The force $F = -\mu x^2$ as μx^2 is a force directed towards the origin, *i.e.*, in the negative direction. Equation of continuity is $x^2 v = F(t)$ so that

$$\frac{\partial v}{\partial t} = \frac{F'(t)}{x^2}$$ Equation of motion is

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\mu x^2 - \frac{1}{2}\frac{\partial p}{\partial x}$$

$$\frac{F'(t)}{x^2} + v\frac{\partial v}{\partial x} = -\mu x^2 - \frac{\partial}{\partial x}\left(\frac{p}{\rho}\right)$$

Integrating w.r.t. x, $-\frac{F'(t)}{x} + \frac{1}{2}v^2 = -\frac{\mu x^3}{3} - \frac{p}{\rho} + C$ ... (1)

Let r and R be internal and external radii respectively at any time t. Since the total mass of the liquid is constant hence

$$\left(\frac{4}{3}\pi R^3 - \frac{4}{3}\pi r^3\right)\rho = \frac{4}{3}\pi c^3 \rho$$

or $$R^3 - r^3 = c^3 \quad \ldots (2)$$

First we shall prove that

$$\dot{r}^2 r^2 [(r^3 + c^3)^{1/3} - r] = \left(\frac{2\Pi}{3\rho} + \frac{2}{9}\mu c^3\right)(a^3 - r^3)(c^3 + r^3)^{1/3} \quad \ldots (3)$$

This equation is obtained by putting $r = x$ in the given result. Boundary conditions are

(i) when $x = R$, $v = \dot{R} = U$ say, $p = \Pi$.
(ii) when $x = r$, $v = \dot{r} = u$ say, $p = 0$.
Since pressure vanishes on the surface of inner sphere.
(iii) when $r = a$, $v = 0$ so that $F(t) = 0$.
Here also we have $x^2 v = R^2 U = r^2 u = F(t)$.
Subjecting (1) to (i) and (ii),

$$\frac{-F'(t)}{R} + \frac{1}{2}U^2 = -\frac{\mu}{3}R^3 - \frac{\Pi}{\rho} + C$$

$$\frac{-F'(t)}{r} + \frac{1}{2}u^2 = -\frac{\mu}{3}r^3 + C$$

Upon subtraction,

$$\left\{\frac{1}{r} - \frac{1}{R}\right\}F'(t) + \frac{1}{2}.F^2\left\{\frac{1}{R^4} - \frac{1}{r^4}\right\} = \frac{\mu}{3}(r^3 - R^3) - \frac{\Pi}{\rho}$$

[as $r^2 u = F(t) = R^2 U$]

or $$\left\{\frac{1}{r} - \frac{1}{R}\right\}F'(t) + \frac{1}{2}F^2\left\{\frac{1}{R^4} - \frac{1}{r^4}\right\} = -\frac{\mu}{3}c^3 - \frac{\Pi}{\rho}.$$

Multiplying by $2F\,dt = 2R^2\,dR = 2r^2\,dr$,

$$\left\{\frac{1}{r} - \frac{1}{R}\right\}2FF'\,dt + F^2\left\{\frac{dR}{R^2} - \frac{dr}{r^2}\right\} = -\frac{\mu c^3}{3}.2r^2\,dr - 2r^2\,dr.\frac{\Pi}{\rho}$$

or $$d\left[\left(\frac{1}{r} - \frac{1}{R}\right)F^2\right] = \left(-\frac{\mu c^3}{3} - \frac{\Pi}{\rho}\right)2r^2\,dr.$$

Integrating, $$\left(\frac{1}{r} - \frac{1}{R}\right)F^2 = \left(-\frac{\mu c^3}{3} - \frac{\Pi}{\rho}\right)\frac{2}{3}r^3 + C_2. \quad \ldots (4)$$

By (iii), this gives $$0 = -\frac{\mu c^3}{9}.2a^3 - \frac{2\Pi}{3\rho}.a^3 + C_2 \quad \ldots (5)$$

(4) − (5) gives $$\left(\frac{1}{r} - \frac{1}{R}\right)(r^2 u)^2 = \frac{2\mu c^3}{9}(a^3 - r^3) + \frac{2\Pi}{3\rho}(a^3 - r^3)$$

or $$(R - r)r^3 u^2 = \left(\frac{2\mu c^3}{9} + \frac{2\Pi}{3\rho}\right)(a^3 - r^3)R$$

or $$[(c^3 + r^3)^{1/3} - r]r^3 \dot{r}^2 = \left(\frac{2\mu c^3}{9} + \frac{2\Pi}{3\rho}\right)(a^3 - r^3)(c^2 + r^3)^{1/3}$$

Replacing r by x, we get the required result to be established.

Problem 22. *A mass of liquid of density ρ and volume $\frac{4}{3}\pi c^3$, is in the form of a spherical shell; a constant pressure Π is exerted on the external surface of the shell; there is no pressure on the internal surface and no other forces act on the liquid; initially the liquid is at rest and the internal radius of the shell is 2c; prove that the velocity of the internal surface, when its radius is c, is*

$$\left(\frac{14\Pi}{3\rho}.\frac{2^{1/3}}{2^{1/3} - 1}\right)^{1/2}.$$

Solution : Equations of continuity and motion are

$$x^2 v = F(t)$$

and $$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}.$$

Hence $$\frac{F'(t)}{x^2} + \frac{\partial}{\partial x}\left(\frac{1}{2}v^2\right) = -\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right)$$

Integrating $$\frac{-F'(t)}{x} + \frac{1}{2}v^2 = -\frac{p}{\rho} + C. \quad \ldots (1)$$

Let r and R be internal and external radii respectively. Since the total mass of the liquid is constant therefore

$$\frac{4}{3}\pi R^3 - \frac{4}{3}\pi r^3 = \frac{4}{3}\pi c^3 \text{ or } R^3 - r^3 = c^3. \quad \ldots (2)$$

Boundary conditions are
(i) when $x = R$, $v = \dot{R} = U$, $p = \Pi$.
Since external surface is subjected to a constant pressure Π.
(ii) when $x = r$, $v = \dot{r} = u$ say $p = 0$.
Since there is no pressure on the internal surface.
(iii) when $t = 0$ and $r = 2c$, $v = 0$ so that $F(t) = 0$.
For internal radius of the shell is $2c$.
We want to prove that

$$(u)_{r=c} = \left[\frac{14\Pi}{3\rho}.\frac{2^{1/3}}{2^{1/3} - 1}\right]^{1/2}.$$

Subjecting (1) to the conditions (i) and (ii),

$$\frac{-F'(t)}{R} + \frac{1}{2}U^2 = -\frac{\Pi}{\rho} + C$$

$$\frac{-F'(t)}{r} + \frac{1}{2}u^2 = 0 + C$$

upon subtraction $$F'(t)\left\{\frac{1}{r} - \frac{1}{R}\right\} + \frac{1}{2}(U^2 - u^2) = -\frac{\Pi}{\rho}$$

or $$\left(\frac{1}{r} - \frac{1}{R}\right)F'(t) + \frac{F^2}{2}\left\{\frac{1}{R^4} - \frac{1}{r^4}\right\} = -\frac{\Pi}{\rho}.$$

[For $R^2 U = F(t) = r^2 u$].

Multiply by $2F\,dt$ or its equivalent $2R^2\,dR = 2r^2\,dr$,

$$\left(\frac{1}{r} - \frac{1}{R}\right)2FF'\,dt + F^2\left[\frac{dR}{R^2} - \frac{dr}{r^2}\right] = -\frac{\Pi}{\rho}.2r^2\,dr.$$

or $$d\left[\left(\frac{1}{r} - \frac{1}{R}\right)F^2\right] = -\frac{\Pi}{\rho}.2r^2\,dr.$$

Integrating, $$\left(\frac{1}{r} - \frac{1}{R}\right)F^2 = -\frac{2\Pi}{3\rho}r^3 + C_1. \quad \ldots (3)$$

In view of (iii), $$0 = -\frac{2\Pi}{3\rho}.8c^3 + C_1 \quad \ldots (4)$$

(3) − (4) $\Rightarrow$ $$\left(\frac{1}{r} - \frac{1}{R}\right)F^2(t) = -\frac{2\Pi}{3\rho}(r^3 - 8c^3)$$

or $$\left[\frac{1}{r} - \frac{1}{(c^3 + r^3)^{1/3}}\right](r^2 u)^2 = \frac{2\Pi}{3\rho}(8c^3 - r^3), \quad \text{using (2)}$$

Putting $r = c$, $$\left[\frac{1}{c} - \frac{1}{c\,2^{1/3}}\right]c^4 (u^2)_{r=c} = \frac{2\Pi}{3\rho}(8c^3 - c^3)$$

or $$(u)_{r=c} = \left[\frac{14\Pi}{\rho}.\frac{2^{1/3}}{2^{1/3} - 1}\right]^{1/2}$$

Problem 23. *A mass of gravitating fluid is at rest under its own attraction only, the free surface being a sphere of radius b and the inner surface a rigid concentric shell of radius a. Show that if this shell suddenly disappears, the initial pressure at any point of the fluid at distance r from the centre is*

$$\frac{2}{3}\pi\rho^2(b - r)(r - a)\left(\frac{a + b}{r} + 1\right).$$

Solution : Let r be the radius of inner surface at any time t. The force F of attraction at a distance x from the centre of the liquid is

$$\frac{4}{3}\pi\rho\frac{\gamma(x^3 - r^3)}{x^2}. \quad \left[\text{For } F = \frac{\gamma m_1 m_2}{d^2}\right].$$

Equations of continuity and motion are

$$x^2 v = F(t) \text{ and } \frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} = -\frac{4}{3}\pi\rho\gamma\frac{(x^3 - r^3)}{x^2} - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

[as the force is directed towards the origin]

or $$\frac{F'(t)}{x^2} + \frac{\partial}{\partial x}\left(\frac{1}{2}v^2\right) = -\frac{4}{3}\pi\rho\gamma\left(x - \frac{r^3}{x^2}\right) = \frac{\partial}{\partial x}\left(\frac{p}{\rho}\right)$$

Integrating, $$\frac{-F'(t)}{x} + \frac{1}{2}v^2 = -\frac{4}{3}\pi\rho\gamma\left(\frac{x^2}{2} + \frac{r^3}{x}\right) - \frac{p}{\rho} + C. \quad \ldots (1)$$

Boundary conditions are
(i) when $t = 0$, $v = 0$, $r = a$, $p = p_0$ are
[For initially the radius of the inner surface is a and also this surface contains gravitating mass and so there will be pressure on it].
(ii) when $t = 0$, $x = a$, $v = 0$, $p_0 = 0$.

For pressure vanishes on the annihilated surface.

(iii) when $t = 0, x = b, p_0 = 0, v = 0$.

[Since there exists no outer pressure].

We want to determine the value of initial pressure.

Subjecting (1) to (i), $$\frac{-F'(0)}{x} = -\frac{4}{3}\pi\rho\gamma\left(\frac{x^2}{2}+\frac{a^3}{x}\right)-\frac{p_0}{\rho}+C. \quad \text{... (2)}$$

Subjecting this to (ii) and (iii),

$$\frac{-F'(0)}{a} = -\frac{4}{3}\pi\rho\gamma\left(\frac{a^2}{2}+\frac{a^3}{a}\right)+C \quad \text{... (3)}$$

$$\frac{-F'(0)}{b} = -\frac{4}{3}\pi\rho\gamma\left(\frac{b^2}{2}+\frac{a^3}{b}\right)+C$$

Upon subtraction,

$$\left\{\frac{1}{b}-\frac{1}{a}\right\}F'(0) = -\frac{4}{3}\pi\rho\gamma\left[\frac{a^2-b^2}{2}+a^3\left(\frac{1}{a}-\frac{1}{b}\right)\right]$$

or $$F'(0) = -\frac{4}{3}\pi\rho\gamma ab\left[\frac{a+b}{2}-\frac{a^2}{b}\right]$$

or $$F'(0) = -\frac{2}{3}\pi\rho\gamma a\,[b(a+b)-2a^2] \quad \text{... (4)}$$

(2) – (3) gives

$$F(0)\left[\frac{1}{a}-\frac{1}{x}\right] = -\frac{4}{3}\pi\rho\gamma\left[\frac{x^2-a^2}{2}+a^3\left\{\frac{1}{x}-\frac{1}{a}\right\}\right]-\frac{p_0}{\rho}$$

or $$p_0 = -\frac{4}{3}\pi\rho^2\gamma(x-a)\left[\frac{x+a}{2}-\frac{a^2}{x}\right]-F(0)\frac{(x-a)}{xa}\rho$$

$$= -\frac{2}{3}\pi\rho^2\gamma(x-a)\left[2\left(\frac{x+a}{2}-\frac{a^2}{x}\right)+\frac{F(0)}{xa\,.\,(2/3)\pi\rho\gamma}\right]$$

$$= -\frac{2}{3}\pi\rho^2\gamma(x-a)\left[\frac{(x+a)x-2a^2}{x}-\frac{a}{xa}\,.\,\{b(a+b)-2a^2\}\right], \text{ by (4).}$$

$$= -\frac{2}{3}\pi\rho^2\gamma(x-a)\left[\frac{x^2-b^2+ax-ab}{x}\right]$$

$$= \frac{2}{3}\pi\rho^2\gamma(x-a)(b-x)\left[1+\frac{a+b}{x}\right]$$

Replacing x by r we get the required result.

Problem 24. *A volume $\frac{4}{3}\pi c^3$ of gravitating liquid, of density ρ, is initially in the form of a spherical shell of infinitely great radius. If the liquid shell contracts under the influence of its own attraction, there being no external or internal pressure, show that when the radius of the inner spherical surface is x, its velocity will be given by*

$$V^2 = \frac{4\pi\rho\gamma z}{15x^2}[2z^4+2z^3x+2z^2x^2-3zx^3-3x^4]$$

where γ is constant of gravitation and $z^3 = x^3 + c^3$.

Solution : ~~Let r be the~~ radius of inner surface at any time t. The force F of attraction at a distance x from the centre of the liquid is

$$\frac{4}{3}\pi\rho\gamma\frac{(x^3-r^3)}{x^2} \qquad \left[\text{For } F = \frac{m_1 m_2}{d^2}\right]$$

Equations of continuity and motion are

$$x^2 v = F(t) \quad \text{and} \quad \frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x} = -\frac{4}{3}\pi\rho\gamma\frac{(x^3-r^3)}{x^2}-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

$\therefore$ $$\frac{F'(t)}{x^2}+\frac{\partial}{\partial x}\left(\frac{1}{2}v^2\right) = -\frac{4}{3}\pi\rho\gamma\left(x-\frac{r^3}{x^2}\right)-\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right)$$

Integrating, $$-\frac{F'(t)}{x}+\frac{1}{2}v^2 = -\frac{4}{3}\pi\rho\gamma\left(\frac{x^2}{2}+\frac{r^3}{x}\right)-\frac{p}{\rho}+C \quad \text{... (1)}$$

Let R be the external radius at any time t. Since the total mass of the liquid is constant,

$$\frac{4}{3}\pi R^3\rho-\frac{4}{3}\pi r^3\rho = \frac{4}{3}\pi c^3\rho \quad \text{or} \quad R^3-r^3 = c^3. \quad \text{... (2)}$$

Boundary conditionsa are

(i) when $x = R, v = \dot{R} = U$ say, $p = 0$.

Since there being no external pressure.

(ii) when $x = r, v = \dot{r} = u$ say $p = 0$.

Subjecting (1) and (i) and (ii),

$$-\frac{F'(t)}{R}+\frac{1}{2}U^2 = -\frac{4}{3}\pi\rho\gamma\left(\frac{R^2}{2}+\frac{r^3}{R}\right)+C$$

$$-\frac{F'(t)}{r}+\frac{1}{2}u^2 = -\frac{4}{3}\pi\rho\gamma\left(\frac{r^2}{2}+\frac{r^3}{r}\right)+C$$

Upon subtracting,

$$\left\{\frac{1}{r}-\frac{1}{R}\right\}F'(t)+\frac{1}{2}(U^2-u^2) = -\frac{4}{3}\pi\rho\gamma\left[\frac{R^2-r^2}{2}+r^3\left(\frac{1}{R}-\frac{1}{r}\right)\right]$$

Since $$r^2u = F = R^2U.$$

Hence,

$$F'(t)\left\{\frac{1}{r}-\frac{1}{R}\right\}+\frac{F^2}{2}\left\{\frac{1}{R^4}-\frac{1}{r^4}\right\} = -\frac{4}{3}\pi\rho\gamma\left[\frac{R^2-r^2}{2}+r^3\left(\frac{1}{R}-\frac{1}{r}\right)\right]$$

Multiplying by $2F\,dt = 2r^2\,dr = R^2\,dR$,

$$2FF'\left\{\frac{1}{r}-\frac{1}{R}\right\}+\left\{\frac{dR}{R^2}-\frac{dr}{r^2}\right\}F^2 = -\frac{4}{3}\pi\rho\gamma\,[R^4\,dR-r^4\,dr+r^2\,2R\,dR-2r^4\,dr]$$

or $$d\left[\left(\frac{1}{r}-\frac{1}{R}\right)F^2\right] = -\frac{4}{3}\pi\rho\gamma\,\{(R^4\,dR-r^4\,dr)+2r^3(R\,dR-r\,dr)\}$$

$$= -\frac{4}{3}\pi\rho\gamma\,[(R^4dR-r^4dr)+2(R^3-c^3)R\,dR-2r^4dr].$$

Integrating, we get

$$\left(\frac{1}{r}-\frac{1}{R}\right)F^2 = -\frac{4}{3}\pi\rho\gamma\left[\frac{R^5-r^5-2r^5}{5}+\frac{2R^5}{5}-\frac{2c^3R^2}{2}\right].$$

neglecting constant of integration. But $r^2u = F(t)$.

$\therefore$ $$u^2 = -\frac{4}{15}\pi\rho\gamma\,[3(R^5-r^5)-5c^3R^2]\,.\,\frac{R}{r^3(R-r)}$$

$$= \frac{4}{15}\pi\rho\gamma\,.\,\frac{R}{r^3}\left[\frac{3(r^5-R^5)+5R^2(R^3-r^3)}{R-r}\right], \text{ by (2)}$$

$$= \frac{4}{15}\pi\rho\gamma\frac{R}{r^3}[2R^4+2R^3r^2+2R^2r^2-3Rr^3-3r^4].$$

Replacing R by z, r by x and u by V, we get the required result.

Problem 25. *A mass of perfectly incompressible fluid of density ρ, is bounded by concentric surfaces. The outer surface is contained by a flexible envelope which exerts continuously a uniform pressure Π and contracts from radius R_1 to radius R_2. The hollow is filled with a gas obeying Boyle's law, its radius contracts c_1 to c_2 and the pressure of the gas is initially p_1. Initially the whole mass is at rest. Prove that, neglecting the mass of the gas, the velocity v of the inner surface when the configuration (R_2, c_2) is reached, is given by*

$$\frac{1}{2}v^2 = \frac{c_1^3}{c_2^3}\left[\frac{1}{3}\left(1-\frac{c_2^3}{c_1^3}\right)\frac{\Pi}{\rho}-\frac{p_1}{\rho}\log\frac{c_1}{c_2}\right]\Big/\left(1-\frac{c_2}{R_2}\right).$$

Solution : The equations of continuity and motion are

$$x^2v = F(t) \quad \text{and} \quad \frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$$

Hence $$\frac{F'(t)}{x^2}+\frac{\partial}{\partial x}\left(\frac{1}{2}v^2\right) = -\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right)$$

Integrating,

$$-\frac{F'(t)}{x}+\frac{1}{2}v^2 = -\frac{p}{\rho}+C$$

or $$-\frac{F'(t)}{x}+\frac{1}{2}\frac{F^2}{x^4} = -\frac{p}{\rho}+C \quad \text{... (1)}$$

Let r and R be internal and external radii at any time t. Let P be the pressure at a distance r as cavity contains gas. By Boyle's law,

$$\frac{4}{3}\pi r^3\,.\,P = \frac{4}{3}\pi c_1^3 p_1 \quad \text{or} \quad P = \frac{c_1^3}{r^3}\,.\,p_1.$$

Boundary conditions are

(i) when $x = R, v = \dot{R} = U$, say, $p = \Pi$.

For the outer surface exerts a uniform pressure Π.

(ii) when $x = r, v = \dot{r} = u$, say, $p = P$.

(iii) when $r = c_1, v = 0$ so that $F(t) = 0$.

Here $r^2u = F(t) = R^2U$ so that $U^2 = \frac{F^2}{R^4}$, $u^2 = \frac{F^2}{r^4}$.

Subject (1) to (i) and (ii),

$$-\frac{F'(t)}{R}+\frac{1}{2}\frac{F^2(t)}{R^4} = -\frac{\Pi}{\rho}+C$$

$$-\frac{F'(t)}{r}+\frac{1}{2}\frac{F^2(t)}{r^4} = -\frac{P}{\rho}+C.$$

Upon subtraction,

$$\left\{\frac{1}{r}-\frac{1}{R}\right)F'(t)+\left(\frac{1}{R^4}-\frac{1}{r^4}\right)\frac{F^2}{2} = -\frac{\Pi}{\rho}+\frac{c_1^3}{r^3}\,.\,\frac{p_1}{\rho}$$

Multiplying by $2F\,dt = 2r^2\,dr = 2R^2\,dR$, we get

$$2FF'\left\{\frac{1}{r}-\frac{1}{R}\right\}+F^2\left\{\frac{dR}{R^2}-\frac{dr}{r^2}\right\} = \frac{1}{\rho}\left[\frac{c_1^3p_1}{r^3}-\Pi\right]2r^2\,dr$$

or $$d\left[\left\{\frac{1}{r}-\frac{1}{R}\right\}F^2\right] = \frac{1}{\rho}\left[\frac{c_1^3p_1}{r^3}-\Pi\right]2r^2\,dr.$$

Integrating,

$$\left(\frac{1}{r}-\frac{1}{R}\right)F^2(t) = \frac{2}{\rho}\left[c_1^3p_1\log r-\frac{\Pi}{3}r^3\right]+A$$

In view of (iii), this $\Rightarrow$

$$0 = \frac{2}{\rho}\left[c_1^3p_1\log c_1-\frac{\Pi}{3}c_1^3\right]+A$$

Upon subtraction,

$$u^2 = \frac{2}{\rho}\,.\,\frac{R}{(R-r)r^3}\left[c_1^3p_1\log\left(\frac{r}{c_1}\right)-\frac{\Pi}{3}(r^3-c_1^3)\right]$$

For configuration (R_2, c_2), i.e., when $R = R_2$, $r = c_2$, the velocity v is given by

$$\frac{1}{2}v^2 = \frac{1}{2}(u^2)_{(R_2,c_2)} = \frac{1}{\rho}\,.\,\frac{R_2}{(R_2-c_2)c_2^3}\left[c_1^3p_1\log\left(\frac{c_2}{c_1}\right)-\frac{\Pi}{3}(c_2^3-c_1^3)\right]$$

or $$\frac{1}{2}v^2 = \frac{c_1^3}{c_2^3}\left[-\frac{p_1}{\rho}\log\left(\frac{c_1}{c_2}\right)+\frac{\Pi}{3\rho}\left(1-\frac{c_2^3}{c_1^3}\right)\right]\Big/\left(1-\frac{c_2}{R_2}\right).$$

Problem 26. *A sphere of radius a is alone in an unbounded liquid which is at rest at a great distance from the sphere and is subject to no external force. The sphere is forced to vibrate radially keeping its spherical shape, the radius r at any time being given by $r = a + b\cos nt$. Show that if Π is the pressure in the liuid at a great distance from the sphere, the least pressure (assumed positive at the surface of the sphere during the motion) is $\Pi - n^2\rho b(a+b)$.*

Solution : Equations of continuity and motion are

$$x^2v = F(t) \quad \text{and} \quad \frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x} = -\frac{1}{\rho}\frac{\partial p}{\partial x}.$$

Hence $\dfrac{F'(t)}{x^2}+\dfrac{\partial}{\partial x}\left(\dfrac{1}{2}v^2\right)=-\dfrac{\partial}{\partial x}\left(\dfrac{p}{\rho}\right)$.

Integrating,

$$-\frac{F'(t)}{x}+\frac{1}{2}v^2=-\frac{p}{\rho}+C, \qquad \text{... (1)}$$

Subjecting it to the boundary conditon,

when $x=\infty$, $v=0$, $p=\Pi$, we get $0=-\dfrac{\Pi}{\rho}+C$.

$$\therefore \qquad -\frac{F'(t)}{x}+\frac{1}{2}v^2=\frac{\Pi-p}{\rho}$$

when $x=r$, let $p=p_1$. Then $v=u=\dot{r}$ so that

$$-\frac{F'(t)}{r}+\frac{1}{2}u^2=\frac{\Pi-p_1}{\rho}. \qquad \text{... (2)}$$

Given $\quad r=a+b\cos nt$.

Hence $\quad \dot{r}=u=-bn\sin nt$

$$F(t)=r^2u=(a+b\cos nt)^2(-bn\sin nt)$$

$$F'(t)=2(a+b\cos nt)(b^2n^2\sin^2 nt)-bn^2\cos nt\,(a+b\cos nt)^2$$

or $$\frac{F'(t)}{r}=n^2b\,[2b\sin^2 nt-\cos nt\,(a+b\cos nt)].$$

This $\Rightarrow -\dfrac{2F'(t)}{r}+u^2=2n^2b\,[-2b\sin^2 nt+(a+b\cos nt)\cos nt]+b^2n^2\sin^2 nt$

$$=n^2b\,[-3b\sin^2 nt+2b\cos^2 nt+2a\cos nt]$$

Using this in (2),

$$2(p_1-\Pi)=n^2\rho b\,[3b\sin^2 nt-2b\cos^2 nt-2a\cos nt]. \qquad \text{... (3)}$$

In order that p_1 is least, we must have $t=0$.

$$\therefore \qquad 2(p_1-\Pi)=n^2\rho b\,[-2b-2a]$$

or $$p_1=\Pi-n^2\rho b\,(a+b).$$

Problem 27. *A centre of force attracting inversely as the square of the distance is at the centre of a spherical cavity within an infinite mass of incompressible fluid, the pressure which at an infinite distance is Π, and is such that the work done by this pressure on a unit area through a unit length is one half the work done by the attractive force on a unit volume of the fluid from infinity to the initial boundary of the cavity; prove that the time of filling up the cavity will be:*

$$\pi a\left(\frac{\rho}{\Pi}\right)^{1/2}\left[2-\left(\frac{3}{2}\right)^{3/2}\right].$$

a being the initial radius of the cavity and ρ the density of the fluid.

Solution : Equation of continuity is $x^2v=F(t)$ and equation of motion is

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=-\frac{\mu}{x^2}-\frac{1}{\rho}\frac{\partial p}{\partial x}.$$

This $\Rightarrow \dfrac{F'(t)}{x^2}+\dfrac{\partial}{\partial x}\left(\dfrac{1}{2}v^2\right)=-\dfrac{\mu}{x^2}-\dfrac{\partial}{\partial x}\left(\dfrac{p}{\rho}\right)$.

Integrating, $\quad -\dfrac{F'(t)}{x}+\dfrac{1}{2}v^2=\dfrac{\mu}{x}-\dfrac{p}{\rho}+C. \qquad \text{... (1)}$

Boundary conditions are

(i) when $x=\infty$, $v=0$, $p=\Pi$.

Let r be the radius of cavity at any time t. Then

(ii) when $x=r$, $v=\dot{r}=u$ say, $p=0$.

Since pressure vanishes on the surface of cavity.

(iii) when $r=a$, $v=u=0$ so that $F(t)=0$.

Subjecting (1) to (i) and (ii), $0=-\dfrac{\Pi}{\rho}+C$

and $$-\frac{F'(t)}{r}+\frac{1}{2}u^2=\frac{\mu}{r}+C$$

These two equations $\Rightarrow -\dfrac{F'(t)}{r}+\dfrac{1}{2}\dfrac{F^2}{r^4}=\dfrac{\mu}{r}+\dfrac{\Pi}{\rho}$ as $r^2u=F(t)$

Multiply by $\quad 2F\,dt=2r^2\,dr$,

$$-\frac{2FF'}{r}dt+\frac{F^2}{r^2}dr=\left(\frac{\mu}{r}+\frac{\Pi}{\rho}\right)2r^2\,dr$$

or $$d\left[-\frac{F^2}{r}\right]=2\mu r\,dr+\frac{\Pi}{\rho}\cdot 2r^2\,dr$$

Integration yields, $\quad -\dfrac{F^2}{r}=\mu r^2+\dfrac{2}{3}\dfrac{\Pi}{\rho}r^3+A \qquad \text{... (2)}$

In view of (iii), this $\Rightarrow \quad 0=\mu a^2+\dfrac{2}{3}\dfrac{\Pi}{\rho}a^3+A$

Upon subtraction,

$$-\frac{F^2}{r}=\mu(r^2-a^2)+\frac{2}{3}\frac{\Pi}{\rho}\cdot(r^3-a^3)$$

or $$r^3u^2=\mu(a^2-r^2)+\frac{2}{3}\frac{\Pi}{\rho}(a^3-r^3). \qquad \text{... (3)}$$

It is given that

Work done by Π on unit area through a unit length

$=\dfrac{1}{2}$. work done by $-\dfrac{\mu}{x^2}$ on a unit volume of fluid from $x=\infty$ to $x=a$.

Hence $$\Pi.1.1=\frac{1}{2}\int_{\infty}^{a}-\frac{\mu}{x^2}x\,dx=\frac{\mu\rho}{2a}.$$

This $\Rightarrow$ $$\mu=2a\frac{\Pi}{\rho}$$

Now (3) becomes

$$r^3u^2=\frac{2a\Pi}{\rho}(a^2-r^2)+\frac{2}{3}\frac{\Pi}{\rho}(a^3-r^3)$$

$$\frac{dr}{dt}=-\left(\frac{2\Pi}{3\rho}\right)^{1/2}\left[\frac{3a(a^2-r^2)+(a^3-r^3)}{r^3}\right]^{1/2}$$

[Negative sign is taken before the radical sign because r decreases when t increases].

Let T be the required time. Then

$$\int_0^T dt=-\left(\frac{3\rho}{2\Pi}\right)^{1/2}\int_a^0\frac{r^{3/2}\,dr}{[3a(a^2-r^2)+(a^3-r^3)]^{1/2}}$$

or $$T=\left(\frac{3\rho}{2\Pi}\right)^{1/2}\int_0^a\frac{r^{3/2}\,dr}{[(a-r)(2a+r)^2]^{1/2}}$$

$$=\left(\frac{3\rho}{2\Pi}\right)^{1/2}\int_0^a\frac{r^{3/2}\,dr}{(a-r)^{1/2}(2a+r)}$$

Put $\quad r=a\sin^2\theta$.

$$T=\left(\frac{3\rho}{2\Pi}\right)^{1/2}\int_0^{\pi/2}\frac{a^{3/2}\sin^3\theta\,.\,2a\sin\theta\cos\theta\,d\theta}{a^{1/2}\cos\theta\,.\,a\,(2+\sin^2\theta)}$$

$$=2a\left(\frac{3\rho}{2\Pi}\right)^{1/2}\int_0^{\pi/2}\left(\sin^2\theta-2+\frac{4}{2+\sin^2\theta}\right)d\theta$$

$$=2a\left(\frac{3\rho}{2\Pi}\right)^{1/2}\left[\frac{\pi}{4}-2\cdot\frac{\pi}{2}+4\int_0^{\pi/2}\frac{\sec^2\theta\,d\theta}{2+3\tan^2\theta}\right]$$

$$=2a\left(\frac{3\rho}{2\Pi}\right)^{1/2}\left[-\frac{3\pi}{4}+4\int_0^{\infty}\frac{du}{2+3u^2}\right],\quad \tan\theta=u$$

$$=2a\left(\frac{3\rho}{2\Pi}\right)^{1/2}\left[-\frac{3\pi}{4}+\frac{4}{3}\cdot\frac{\pi}{2}\cdot\sqrt{\frac{3}{2}}\right]$$

$$=\pi a\left(\frac{\rho}{\Pi}\right)^{1/2}\left[2-\left(\frac{3}{2}\right)^{3/2}\right]$$

For integral $=\dfrac{4}{3}\cdot\left[\dfrac{1}{\sqrt{2/3}}\tan^{-1}\dfrac{u}{\sqrt{(2/3)}}\right]_0^{\infty}=\dfrac{4}{3}\cdot\dfrac{\pi}{2}\left(\dfrac{3}{2}\right)^{1/2}=\pi\left(\dfrac{2}{3}\right)^{1/2}$

Problem 28. *A spherical hollow of radius a initially exists in an infinite fluid subject to a constant pressure at infinity. Show that the pressure at distance r from the centre when the radius of the cavity is x, is to the pressure at infinity as*

$$3x^2r^4+(a^3-4x^3)r^3-(a^3-x^3)x^3 : 3x^2r^4.$$

Solution : Equation of continuity is $x^2v=F(t)$ and equation of motion is

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x}$$

This $\Rightarrow \dfrac{F'(t)}{x^2}+\dfrac{\partial}{\partial x}\left(\dfrac{1}{2}v^2\right)=-\dfrac{\partial}{\partial x}\left(\dfrac{p}{\rho}\right)$.

Integrating, $\quad \dfrac{-F'(t)}{x}+\dfrac{1}{2}v^2=-\dfrac{p}{\rho}+C. \qquad \text{... (1)}$

Let r be the radius of cavity at any time t. Boundary conditions are

(i) when $x=\infty$, $v=0$, $p=\Pi$.

(ii) when $x=r$, $v=u=\dot{r}$, $p=0$.

Since pressure vanishes on the surface of cavity.

(iii) when $r=a$, $v=u=0$ so that $F(t)=0$.

Subjecting (1) to (i) and (ii),

$$0=-\frac{\Pi}{\rho}+C \quad\text{and}\quad \frac{-F'(t)}{r}+\frac{1}{2}u^2=0+C.$$

This $\Rightarrow \dfrac{-F'(t)}{r}+\dfrac{1}{2}\dfrac{F^2}{r^4}=\dfrac{\Pi}{\rho} \qquad \text{... (2)} \qquad [\text{as } r^2u=F(t)]$

Multiply by $2F\,dt$ $(=2r^2\,dr)$, we get

$$\frac{-2FF'\,dt}{r}+\frac{F^2}{r^4}\cdot r^2\,dr=\frac{\Pi}{\rho}\cdot 2r^2\,dr$$

or $$d\left[\frac{-F^2}{r}\right]=\frac{2\Pi}{\rho}\cdot r^2\,dr.$$

Integrating, $\quad \dfrac{-F^2}{r}=\dfrac{2}{3}\dfrac{\Pi}{\rho}r^3+A$

In view of (iii), this gives $0=\dfrac{2}{3}\dfrac{\Pi}{\rho}a^3+A$

$$\therefore \qquad \frac{-F^2}{r}=\frac{2}{3}\frac{\Pi}{\rho}(r^3-a^3)$$

or $$F^2(t)=\frac{2}{3}\frac{\Pi r}{\rho}(a^3-r^3) \qquad \text{... (3)}$$

Using this in (2), we get

$$-\frac{F'(t)}{r}=\frac{\Pi}{\rho}-\frac{1}{2}\frac{F^2}{r^4}=\frac{\Pi}{\rho}-\frac{\Pi}{3\rho r^3}(a^3-r^3)$$

or $$F'(t)=\frac{\Pi}{3\rho r^2}(a^3-4r^3) \qquad \text{... (4)}$$

Writing (1) with the help of (3) and (4),

$$-\frac{\Pi}{3\rho}\cdot\frac{1}{r^2}\cdot\frac{(a^3-4r^3)}{x}+\frac{1}{2}\cdot\frac{1}{x^4}\cdot\frac{2}{3}\frac{\Pi r}{\rho}(a^3-r^3)=-\frac{p}{\rho}+\frac{\Pi}{\rho}$$

or $$\frac{p}{\Pi}=1+\left(\frac{a^3-4r^3}{3r^2x}\right)-\frac{r(a^3-r^3)}{3x^4}$$

$$=\frac{3x^4r^2+(a^3-4r^3)x^3-r^3(a^3-r^3)}{3x^4r^2}$$

This gives the pressure at a distance x, when r is the radius of cavity. In order to get the pressure at a distance r when the radius of cavity is x, we replace r by x and x by r. Thus

$$\frac{p}{\Pi}=\frac{3r^4x^2+(a^3-4x^3)r^2-x^3(a^3-x^3)}{3r^4x^2}$$

Problem 29. *A spehre is at rest in an infinite mass of homogeneous liquid of density ρ, the pressure at infinity being P. If the radius R of the sphere varies in such a way that $R = a + b\cos nt$, where $b < a$, show that pressure at the surface of the sphere at any time is $P+\frac{bn^2\rho}{4}(b-4a\cos nt-5b\cos nt)$.*

Solution : For the sake of convenience we write $P = \Pi$.

Prove as in problem 26 that

$$2(p_1-\Pi)=n^2\rho b\,[3b\sin^2 nt-2b\cos^2 nt-2a\cos nt].$$

[This is the equation (3) of Problem 26].

$$=n^2\rho b\left[\frac{b}{2}\{3(1-\cos 2nt)-2(1+\cos 2nt)\}-2a\cos nt\right]$$

$$=\frac{n^2\rho b}{2}[b-4a\cos nt-5b\cos^2 nt]$$

or $$p_1=\Pi+\frac{n^2\rho b}{4}[b-4a\cos nt-5b\cos^2 nt].$$

Problem 30. *A mass of uniform liquid is in the form of a thick spherical shell bounded by concentric spheres of radii a and b ($a < b$). The cavity is filled with gas, the pressure of which varies according to Boyle's law, and is intially equal to atmospheric pressure Π and the mass of which may be neglected. The outer surface of the shell is exposed to atmospheric pressure. Prove that if the system is symmetrically disturbed, so that particle moves along a line joining it to the centre, the time of small oscillation is $2\pi a\left[\rho\cdot\frac{b-a}{3\Pi b}\right]^{1/2}$, where ρ is the density of the liquid.*

Solution : Equation of continuity is $x^2v = F(t)$ and equation of motion is

$$\frac{\partial v}{\partial t}+v\frac{\partial v}{\partial x}=-\frac{1}{\rho}\frac{\partial p}{\partial x}.$$

This $\Rightarrow \frac{F'(t)}{x^2}+\frac{\partial}{\partial x}\left(\frac{1}{2}v^2\right)=-\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right).$

Integrating, $$\frac{-F'(t)}{x}+\frac{1}{2}v^2=-\frac{p}{\rho}+C. \quad \text{...(1)}$$

Let r and R be internal and external radii of the shell at any time t. Since the shell contains gas hence there will be pressure on the inner surface. Let $p = p_1$ when $x = r$. Since the total mass of the liquid is constant.

$$\left(\frac{4}{3}\pi R^3-\frac{4}{3}\pi r^3\right)\rho=\left(\frac{4}{3}\pi b^3-\frac{4}{3}\pi a^3\right)\rho$$

or $$R^3-r^3=b^3-a^3 \quad \text{... (2)}$$

By Boyle's law, $\frac{4}{3}\pi r^3p_1=\frac{4}{3}\pi a^3.\Pi$

[$\because$ the initial pressure of the gas is equal to atmospheric pressure Π].

Boundary conditions are

(i) when $x = R$, $v = \dot{R} = U$ say, $p = \Pi$

(Since the outer surface is exposed to atmospheric pressure Π).

(ii) when $x = r$, $v = \dot{r} = u$ say, $p = p_1 = \frac{a^3\Pi}{r^3}$.

We want to determine an equation of form $\ddot{x} = -\mu x$.

Subjecting (1) to (i) and (ii),

$$\frac{-F'(t)}{R}+\frac{1}{2}U^2=-\frac{\Pi}{\rho}+C$$

$$\frac{-F'(t)}{r}+\frac{1}{2}u^2=-\frac{a^3\Pi}{\rho r^3}+C$$

Upon subtraction, $\left\{\frac{1}{r}-\frac{1}{R}\right\}F'(t)+\frac{1}{2}(U^2-u^2)=\frac{\Pi}{\rho}\left(\frac{a^3}{r^3}-1\right).$

For small oscillations, U^2 and u^2 are small quantities and hence neglected.

$\therefore$ $$F'(t)=\frac{\Pi}{\rho}\left(\frac{a^3-r^3}{r^2}\right)\frac{R}{R-r}$$

$F(t)=r^2u \Rightarrow F'(t)=2ru^2+r^2\dot{u}=r^2\dot{u}$ as u^2 is neglected

$\therefore$ $$r^2\dot{u}=r^2\ddot{r}=\frac{\Pi}{\rho}\left(\frac{a^3-r^3}{r^2}\right)\cdot\frac{R}{R-r}.$$

Since the displacement is small, let $r = a + x$, $R = b + x'$. Then

$$(a+x)^2\ddot{x}=\frac{\Pi}{\rho}\frac{[a^3-(a+x)^3]}{(a+x)^2}\cdot\frac{b+x'}{b+x'-a-x}$$

or $$\ddot{x}=\frac{\Pi}{a\rho}\frac{[1-(1+x/a)^3](b+x')}{\left(1+\frac{x}{a}\right)^4(x'-x+b-a)}$$

$$=\frac{\Pi}{a\rho}\frac{(-3x/a)(b+x')}{(1+4x/a)(x'-x+b-a)}\text{ approx.} \quad \text{... (3)}$$

(2) $\Rightarrow (b+x')^3-(a+x)^3=b^3-a^3$

$\Rightarrow b^3\left(1+\frac{3x'}{b}\right)-a^3\left(1+\frac{3x}{a}\right)=b^3-a^3$

$\Rightarrow x'=\frac{a^2x}{b^2}.$

N^r of (3) $=\left(-\frac{3x}{a}\right)(b+x')=-\frac{3x}{a}\left(b+\frac{a^2x}{b^2}\right)=-\frac{3xb}{a}$, x^2 is neglected.

or $$\left(-\frac{3x}{a}\right)(b+x')=-\frac{3xb}{a}$$

D^r of (3) $=\left(1+\frac{4x}{a}\right)(x'-x+b-a)=\left(1+\frac{4x}{a}\right)\left(\frac{a^2x}{b^2}-x+b-a\right)$

$$=\frac{a^2x}{b^2}-x+b-a+\frac{4xb}{a}-4x$$

$$=x\left(\frac{a^2}{b^2}-5+\frac{4b}{a}\right)+b-a$$

Therefore $\left(-\frac{3x}{a}\right)(b+x')\Big/\left[\left(1+\frac{4x}{a}\right)(x'-x+b-a)\right.$

$$=-\frac{3xb}{a}\cdot\frac{1}{b-a}\left[1+\left(\frac{a^2}{b^2}-5+\frac{4b}{a}\right)\frac{x}{b-a}\right]^{-1}$$

$$=-\frac{3xb}{a(b-a)}, \quad \text{using this in (3)}$$

$$\ddot{x}=-\frac{3xb}{a(b-a)}\cdot\frac{\Pi}{a\rho}=-\mu x, \quad \text{when } \mu=\frac{3b\Pi}{a^2\rho(b-a)}$$

Time of small oscillation is

$$\frac{2\pi}{\sqrt{\mu}}=2\pi a\left[\frac{\rho(b-a)}{3b\Pi}\right]^{1/2}$$

Problem 30. *A velocity field is given by*

$$\mathbf{q}=\frac{(-\mathbf{i}y+\mathbf{j}x)}{x^2+y^2}.$$

Determine whether the flow is irrotational. Calculate the circulation round (a) square with corners at (1, 0), (2, 0), (2, 1), (1, 1); (b) unit circle with centre at the origin.

Solution : $$\mathbf{q}=\frac{-\mathbf{i}y+\mathbf{j}x}{(x^2+y^2)}=u\mathbf{i}+v\mathbf{j}$$

(i) To determine the nature of motion

$$\text{Curl }\mathbf{q}=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & 0\end{vmatrix}$$

$$=\mathbf{i}(0)+\mathbf{j}(0)+\mathbf{k}\left[\frac{\partial}{\partial x}\left\{\frac{x}{x^2+y^2}\right\}+\frac{\partial}{\partial y}\left\{\frac{y}{x^2+y^2}\right\}\right]$$

$$=\mathbf{k}\left[\frac{(y^2-x^2)}{(x^2+y^2)}+\frac{(x^2-y^2)}{(x^2+y^2)^2}\right]=0$$

Motion is irrotational.

(ii) Let Γ denote circulation. Then

$\Gamma=\int_c \mathbf{q}\cdot d\mathbf{r}$, where c is closed path.

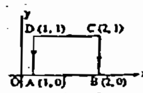


Fig. 2.8

Applying Stoke's theorem

$$\left[\int_c \mathbf{F}\cdot d\mathbf{r}=\int_S \text{curl }\mathbf{F}.\hat{\mathbf{n}}\,dS\right]$$

we got $$\Gamma=\int_S \text{curl }\mathbf{q}\,.\,\hat{\mathbf{n}}\,dS. \quad \text{...(1)}$$

Hence q must be continuous differentiable over S. In present case q is not continuous at the origin but origin does not lie inside the rectangle so that Stoke's theorem is applicable. By part (i), curl q = 0.

Now (1) gives $\Gamma = 0$

(b) Equation of path c is $x^2 + y^2 = 1$.

This circle c contains origin, the point of singularity. Hence Stoke's theorem is not applicable.

$$\Gamma=\int_d \mathbf{q}\cdot d\mathbf{r}=\int_c\left(\frac{-y}{x^3-y^2}dx+\frac{xdy}{x^3+y^3}\right)$$

$$=\int_c (Mdx+Ndy),\text{ say.} \quad \text{...(2)}$$

$$\frac{\partial M}{\partial y}=\frac{y^2-x^2}{(x^2+y^2)^2}=\frac{\partial N}{\partial x}$$

$\therefore$ $Mdx + Ndy$ is exact.

$\therefore$ $$\int(Mdx+Ndy)=\int\frac{-y}{x^2+y^2}dx+\int 0\,dy=-y\int\frac{dx}{x^2+y^2}$$

$$=-\frac{y}{y}\tan^{-1}\left(\frac{x}{y}\right)=-\tan^{-1}\left(\frac{x}{y}\right)$$

Now (2) becomes

$$\Gamma=\int_c \mathbf{q}\,.\,d\mathbf{r}=-\left[\tan^{-1}\frac{x}{y}\right]_c=-\left[\tan^{-1}\left(\frac{r\cos\theta}{r\sin\theta}\right)\right]_c$$

$$=-[\tan^{-1}(\cot\theta)]_c=-\left[\tan^{-1}\left\{\tan\left(\frac{\pi}{2}-\theta\right)\right\}\right]_c$$

$$=-\left[\left(\frac{\pi}{2}-\theta\right)\right]_0^{2\pi}=-\left[\left(\frac{\pi}{2}-2\pi\right)-\left(\frac{\pi}{2}-0\right)\right]$$

$\Gamma = 2\pi.$

Problem 31. *Show that if $\phi=-\frac{1}{2}(ax^2+by^2+cz^2)$, $V=\frac{1}{2}(lx^2+my^2+nz^2)$, where a, b, c, l, m, n are functions of time and $a + b + c = 0$, irrotational motion is possible with a free surface of equipressure if $(l+a^2+\dot{a})e^{2\int adt}$, $(m+b^2+\dot{b})e^{2\int bdt}$, $(n+c^2+\dot{c})e^{2\int cdt}$ are constants.*

Solution : $$\phi=-\frac{1}{2}(ax^2+by^2+cz^2)$$

(i) Motion is irrotational if $\nabla^2\phi = 0$

$$0 = \nabla^2\phi = \Sigma\frac{\partial^2\phi}{\partial x^2} = \Sigma\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial x}\right) = \Sigma\frac{\partial}{\partial x}(-ax)$$

or $\Sigma a = 0$

or $a + b + c = 0$

(ii) Bernoulli's pressure equation for unsteady motion is

$$\frac{p}{\rho} + \frac{1}{2}q^2 - \frac{\partial\phi}{\partial t} + V = F(t) \quad \text{... (2)}$$

$$(1) \Rightarrow \frac{\partial\phi}{\partial t} = -\Sigma\frac{1}{2}\dot{a}x^2 \quad \text{... (3)}$$

$$q^2 = (\nabla\phi).(\nabla\phi) = (\nabla\phi)^2 = \Sigma\left(\frac{\partial\phi}{\partial x}\right)^2 = \Sigma(ax)^2$$

Putting the values in (2),

$$\frac{p}{\rho} + \frac{1}{2}\Sigma a^2x^2 + \frac{1}{2}\Sigma\dot{a}x^2 + \frac{1}{2}\Sigma lx^2 = F(t)$$

$$\frac{-2p}{\rho} = \Sigma x^2(l + a^2) + \Sigma\dot{a}x^2 - 2F(t) \quad \text{... (3)}$$

For a free surface of equipressure :

$$\frac{dp}{dt} = 0$$

or, $$\frac{\partial p}{\partial t} + u\frac{\partial p}{\partial x} + v\frac{\partial p}{\partial y} + w\frac{\partial p}{\partial z} = 0$$

or, $$\frac{\partial p}{\partial t} + \Sigma u\frac{\partial p}{\partial x} = 0 \quad \text{... (4)}$$

By (3), $$\frac{-2}{\rho}\frac{\partial p}{\partial t} = \Sigma x^2(\dot{l} + 2a\dot{a}) + \Sigma\ddot{a}x^2 - 2F'(t)$$

or $$\frac{-2}{\rho}\frac{\partial p}{\partial t} = \Sigma x^2(\dot{l} + 2a\dot{a} + \ddot{a}) - 2F'(t)$$

By (3), $$\frac{-2}{\rho}\frac{\partial p}{\partial x} = \Sigma 2x(l + a^2) + \Sigma 2\dot{a}x$$

or $$\frac{-2}{\rho}\frac{\partial p}{\partial x} = 2\Sigma[l + a^2 + \dot{a}]x$$

$$u = -\frac{\partial\phi}{\partial x} = ax$$

Putting these values in (4),

$$\Sigma x^2(\dot{l} + 2a\dot{a} + \ddot{a}) - 2F'(t) + \Sigma 2ax^2(l + a^2 + \dot{a}) = 0$$

or $$\Sigma x^2[(\dot{l} + 2\dot{a}a + \ddot{a}) + 2a(l + a^2 + \dot{a})] - 2F'(t) = 0$$

It is identity. Hence each coefficient of x^2, y^2, z^2 vanishes identically.

$$(\dot{l} + 2\dot{a}a + \ddot{a}) + 2a(l + a^2 + \dot{a}) = 0 \text{ etc.} \quad \text{... (5)}$$

and $$F'(t) = 0 \quad \text{... (6)}$$

Integrating (6), we get $F(t) = c$ = constant.

By (5), $$\int\left(\frac{\dot{l} + 2a\dot{a} + \ddot{a}}{l + a^2 + \dot{a}}\right)dt + \int 2a\,dt = 0$$

or $$\log(l + a^2 + \dot{a}) + 2\int a\,dt = \log c_1$$

or $$(l + a^2 + \dot{a})e^{2\int a\,dt} = c_1$$

Similarly $$(m + b^2 + \dot{b})e^{2\int b\,dt} = c_2$$

$$(n + c^2 + \dot{c})e^{2\int c\,dt} = c_3$$

Problem 32. *Fluid is coming out from a small hole of cross-section σ_1 in a tank. if the minimum cross-section of the stream coming out of the hole is σ_2, then show that*

$$\frac{\sigma_2}{\sigma_1} = \frac{1}{2}.$$

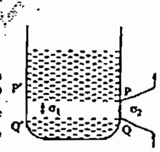


Fig. 2.9

Solution : Let PQ be the hole and $P'Q'$ be its image on the opposite wall of the tank. Let p_1 be the pressure at PQ when the hole is closed. Let p_2 be the pressure and q_2 be the velocity at the minimum cross-section. The velocity of the fluid coming out from minimum cross-section is at right angles to the hole and the direction of velocity will be horizontal there.

Equation of motion is

$$\sigma_1(p_1 - p_2) = \sigma_2\rho q_2^2$$

$$\Rightarrow \quad (p_1 - p_2) = \frac{\sigma_2}{\sigma_1}\rho q_2^2 \quad \text{... (1)}$$

Bernoulli's equation for the steam line connecting a point of $P'Q'$ and a point of minimum cross-section of the jet, becomes

$$\frac{p_1}{\rho} = \frac{p_2}{\rho} + \frac{1}{2}q_2^2 \Rightarrow p_1 - p_2 = \frac{1}{2}\rho q_2^2 \quad \text{... (2)}$$

From (1) and (2), we have

$$\frac{\sigma_2}{\sigma_1} = \frac{1}{2}.$$

Problem 33. *A horizontal straight pipe gradually reduces in diameter from 24 in. to 12 in. Determine the total longitudinal thrust exerted on the pipe if the pressure at the larger end is 50 lbf/in² and the velocity of the water is 8 ft/sec.*

Solution : Let A_1 and A_2 be the cross-section of the larger and the smaller end. Let q_1 and q_2 be the velocity and p_1 and p_2 be the pressure at the larger and the smaller end of the pipe. From the equation of continuity, we have

$$A_1q_1 = A_2q_2$$

or $$\pi(12)^2q_1 = \pi(6)^2q_2 \Rightarrow 4q_1 = q_2 \quad \text{... (1)}$$

$$q_1 = 8 \text{ ft/second} = 8 \times 12 \text{ inches/second}$$
$$= 96 \text{ inches/second}$$

By Bernoulli's equation, we have

$$\frac{p_1}{\rho} + \frac{1}{2}q_1^2 = \frac{p_2}{\rho} + \frac{1}{2}q_2^2$$

or $$p_1 - p_2 = \frac{1}{2}\rho(q_2^2 - q_1^2) = \frac{1}{2}\rho \times 15 \times (96)^2, \text{ by (1).} \quad \text{... (2)}$$

To longitudinal thrust exerted on the pipe

$$= p_1A_1 - p_2A_2$$
$$= \pi(12)^2p_1 - \pi(6)^2p_2$$
$$= 36\pi(4p_1 - p_2) \quad \text{... (3)}$$

From (2), we have

$$p_2 = p_1 - \frac{1}{2}\rho \times 15 \times (96)^2$$

From (3) and (4), we have

$$\text{Total thrust} = 36\pi\left(p_1 + \frac{1}{2}\rho \times 15 \times 96 \times 96\right)$$
$$= 36\pi\left(150 + \frac{1}{2} \times \frac{62{\cdot}4 \times 15 \times 96 \times 96}{12 \times 12 \times 12}\right)$$
$$= 36 \times 2640\pi$$

Problem 34. *Liquid is discharged at the rate of 3.86 ft³/sec from a siphon in the reservoir. The siphon has a diameter of 6 in. Find the elevation x and the fluid pressure at the top of the siphon.*

Solution : Bernoulli's equation for the three points on the same streamline can be written as

$$\frac{q_0^2}{2g} + \frac{p_0}{\rho g} + z_0 = \frac{q_1^2}{2g} + \frac{p_1}{\rho g} + z_1 = \frac{q_2^2}{2g} + \frac{p_2}{\rho g} + z_2$$

Here $q_0 = 0, p_1 = p_2, z = z_0 - z_2$ (let)

$$q_2 = \frac{3{\cdot}86}{\pi\left(\frac{1}{4}\right)^2}$$
$$= \frac{3{\cdot}86 \times 16 \times 7}{22}$$
$$= 19{\cdot}62 \text{ ft./sec.}$$

and $$q_2^2 = 2gz$$

$$z = \frac{19{\cdot}62 \times 19{\cdot}62}{2 \times 32}$$
$$= 6 \text{ ft. (app.)}$$

Since the velocity at the top is the same as that at the bottom. Bernoulli's equation written between these two levels gives

$$\frac{p_1}{\rho g} = -8 \text{ ft. of liquid.}$$

i.e., below the atmospheric pressure.

Problem 35. *A conical pipe has diameters of 10 cm. and 15 cm. at the two ends. If the velocity at the smaller end is 2 m/sec, what is the velocity at the other end and the discharge through the pipe?*

Solution : Let q_1 and q_2 be the velocity at the smaller and larger end. From continuity equation, we have

$$q_1A_1 = q_2A_2.$$

Here $q_2 = 2$ m/sec. $A_1 = (\pi/4)(0{\cdot}1)^2$, $A_2 = (\pi/4)(0{\cdot}15)^2$.

$$\therefore \quad q_2 = q_1\frac{A_1}{A_2}$$
$$= 2\frac{(0{\cdot}1)^2}{(0{\cdot}15)^2}$$
$$= 0{\cdot}89 \text{ m/sec.}$$

Discharge through the pipe

$$Q = q_1A_1$$
$$= 2\left(\frac{\pi}{4}\right)(0{\cdot}1)^2$$
$$= 0{\cdot}0157 \text{ m}^2\text{/sec.}$$

Problem 36. *A horizontal conical pipe has diameter 25 cm and 40 cm at the two ends. (a) Calculate the pressure at the larger end if the pressure at the smaller end is 5 m. of water and rate of flow is 0.3 m²/sec. (b) Calculate the discharge through the pipe if the manometer connected between the two ends reads 10 cm. of mercury.*

Solution : Let q_1, q_2 be the velocities and p_1, p_2 be the pressure at the larger and smaller ends of the conical pipe. Let Q be the discharge through the pipe, then

$$Q = A_1q_1$$

$$q_1 = \frac{Q}{A_1} = \frac{0{\cdot}3}{(\pi/4)(0{\cdot}4)^2} = 2{\cdot}38 \text{ m/sec.} \quad \text{... (1)}$$

From the continuity equation, we have

$$A_1q_1 = A_2q_2$$

or $$q_2 = \frac{A_1}{A_2}q_1 = \frac{(0{\cdot}4)^2}{(0{\cdot}25)^2} \times 2{\cdot}38 = 6{\cdot}10 \text{ m/sec.} \quad \text{... (2)}$$

(a) Using Bernoulli's equation, we have

$$\frac{p_1}{\rho} + \frac{q_1^2}{2g} = \frac{p_2}{\rho} + \frac{q_2^2}{2g}. \quad (\text{Here } z_1 = z_2)$$

or
$$5+\frac{(2\cdot38)^2}{2\times9\cdot81}=\frac{p_2}{\rho}+\frac{(6\cdot10)^2}{2\times9\cdot81}$$

or
$$\frac{p_2}{\rho}=5+\frac{(2\cdot38)^2-(6\cdot10)^2}{2\times9\cdot81}$$
$$=3\cdot4\text{ m}=0\cdot34\text{ kg/cm}^2$$

Pressure at the larger end $=0\cdot34$ kg/cm^2.

(b) From manometer, we have
$$\frac{p_1}{\rho}-\frac{p_2}{\rho}=10\,(13\cdot6-1)=126\text{ cm.}=1\cdot26\text{ m.}$$

From continuity equation, we have
$$A_1q_1=A_2q_2$$
$$\Rightarrow\quad q_2=\frac{A_1}{A_2}q_1=\frac{(0\cdot4)^2}{(0\cdot25)^2}q_1=2\cdot56q_1.$$

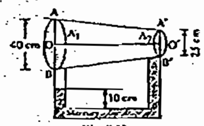


Fig. 2.11

Using Bernoulli's equation, we have
$$\frac{p_1}{\rho}+\frac{q_1^2}{2g}=\frac{p_2}{\rho}+\frac{q_2^2}{2g}$$

or
$$\frac{q_1^2}{2g}\left(\frac{q_2^2}{q_1^2}-1\right)=\frac{p_1-p_2}{\rho}$$
$$\Rightarrow\quad \frac{q_1^2}{2g}\,[(2\cdot56)^2-1]\;1\cdot26$$
$$q_1=\sqrt{1\cdot26\times2\times\frac{9\cdot81}{5\cdot55}}=2\cdot11\text{ m/sec.}$$

Hence discharge through the pipe is
$$Q=A_1q_1$$
$$Q=\frac{\pi}{4}\times(0\cdot4)^2\times2\cdot11=2\cdot65\text{ m}^3/\text{sec.}$$

Problem 37. *A pipe of 10 cm diameter is suddenly enlarged to 20 cm diameter. Find the loss of head when 50 litre/sec of water is flowing.*

Solution : Let q_1 and q_2 be the velocities at the smaller and larger end of the pipe, then
$$Q=A_1q_1=A_2q_2$$

or
$$q_1=\frac{Q}{A_1},\quad q_2=\frac{Q}{A_2}$$

or
$$q_1=\frac{0\cdot05}{(\pi/4)\,(0\cdot1)},\quad q_2=\frac{0\cdot05}{(\pi/4)\,(0\cdot2)^2}$$

or
$$q_1=6\cdot36\text{ m/sec.},\quad q_2=1\cdot59\text{ m/sec.}$$

Loss of head due to sudden enlargement
$$=\frac{(q_1-q_2)^2}{2g}=\frac{(6\cdot36-1\cdot59)^2}{2\times9\cdot81}=1\cdot16\text{ m}$$

EXERCISES

1. Assuming that the earth is a fluid sphere of radius a, of constant density ρ, and without rotations, show that the pressure at a distance r from the centre is $\frac{1}{2}g a\rho\left(1-\frac{r^2}{a^2}\right)$.
2. A mass of fluid of density ρ and volume $(4/3)\pi c^3$ is is in the form of a spherical shell. There is a consant pressure p on the external surface, and zero pressure on the internal surface. Initially the fluid is at rest and the external radius is $2nc$. Show that when the external radius becomes nc the velocity U of the external surface is given by
$$U^2=\frac{14p}{3\rho}\cdot\frac{(n^3-1)^{1/3}}{n-(n^3-1)^{1/3}}$$
3. Prove that if
$$\lambda=\frac{\partial u}{\partial t}-v\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)+w\left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial x}\right)$$
and μ, ν are two similar expressions, then $\lambda\,dx+\mu\,dy+\nu\,dz$ is a perfect differential, if the forces are conservative and the density is constant.
4. If q is the resultant velocity at any point of a fluid which is moving irrotationally in two dimensions, prove, that
$$\left(\frac{\partial q}{\partial x}\right)^2+\left(\frac{\partial q}{\partial y}\right)^2=q\,\nabla^2q.$$
5. Show that in the motion of a fluid in two dimensions if the current co-ordinates (x, y) are expressible in terms of initial co-ordinates (a, b) and the time, then the motion is irrotational if
$$\frac{\partial(\dot{x},x)}{\partial(a,b)}+\frac{\partial(\dot{y},y)}{\partial(a,b)}=0$$
6. Prove that for steady motion of an inviscid isotropic fluid $\rho=f(p)$.
$$\int\frac{dp}{\rho}+\frac{1}{2}q^2+\Omega=\text{const.}$$
over a surface containing the stream lines and vortex lines. Comment on the nature of this constant.
7. (a) When velocity potential exists and the forces are conservative, show that Euler's Dynamical equations can always be integrated in the form
$$\int\frac{dp}{\rho}+\frac{1}{2}q^2-\frac{\partial\phi}{\partial t}+V=f(t)$$
where the symbols have their usual meanings.

   (b) Air, obeying Boyle's law, is in motion in a uniform tube of small section. Prove that if ρ be the density and u the velocity at a distance x from a fixed point at time t,
$$\frac{\partial^2\rho}{\partial t^2}=\frac{\partial^2}{\partial x^2}[(u^2+k)\,\rho],$$
where k is the pressure divided by the density and supposed constant.
8. An infinite fluid, in which there is a spherical hollow of radius a, is initially at rest under the action of no forces. If a constant pressure Π is applied at infinity, show that the time of filling up the cavity is $2^{5/3}\pi^2a\left(\frac{\rho}{\Pi}\right)^{1/2}\cdot\left\{\Gamma\left(\frac{1}{3}\right)\right\}^{-3}$.
9. Prove that the ciculation in any closed path moving with the fluid is constant for all time giving the conditions under which it holds. Hence deduce the theorem of the permanence of irrotational motion.
10. A sphere of radius a alone is an unbounded liquid which is at a great distance from the sphere and is subjected to no external forces. The sphere is forced to vibrate radially keeping its spherical shape, the radius r at any time being given by $r=a+b\cos nt$. Show that if Π is the pressure in the liquid at a great distance from the sphere, the least pressure (assumed) positive at the surface of the sphere during the motion is $\Pi-n^2\rho b\,(a+b)$.
11. If $\bar{\omega}$ is the impulsive pressure; ϕ, ϕ' the velocity potential immediately before and after an impulse acts, V the potential of impulses, prove that
$$\bar{\omega}+\rho V+\rho\,(\phi-\phi')=\text{const.},$$
where ρ is the density of the fluid.

SOURCES, SINKS & DOUBLETS
(Motion in two Dimensions)

SET - III

3.1. Motion in two dimensions :

If the lines of motion are parallel to a fixed plane (say, xy plane), and if the velocity at corresponding points of all planes has the same magnitude and direction, then motion is said to be two dimensional. Evidently, in this case $w = 0$ and

$$u = u(x, y, t),\ v = v(x, y, t).$$

In the diagram, a normal is drawn through P which meets $x'y'$ plane in P'. The points P and P' are *corresponding points*.

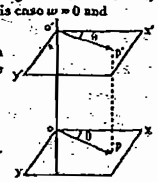


Fig. 3.1.

3.2. Lagrange's stream function :

(*i.e. current function*).

Suppose the motion is two-dimensional so that $w = 0$. The differential equations of stream lines are given by

$$\frac{dx}{u} = \frac{dy}{v},\ i.e.,$$

$$v\,dx - u\,dy = 0\ (= M dx + N dy) \qquad \ldots (1)$$

The equation of continuity for incompressible fluid in two dimensions is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\text{This} \Rightarrow \frac{\partial(-u)}{\partial x} = \frac{\partial v}{\partial y}\ \left(\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}\right)$$

This declares that (1) is an exact differential say $d\psi$, *i.e.*

$$v\,dx - u\,dy = d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy.$$

This $\Rightarrow v = \dfrac{\partial \psi}{\partial x},\ -u = \dfrac{\partial \psi}{\partial y}.$

Now (1) is expressible as $d\psi = 0$.

Integrating it, $\psi =$ const. $\qquad \ldots (2)$

This function ψ is called the *stream function or current function.*

The stream lines are given by (1), *i.e.*, $\psi =$ const. It follows that stream function is constant along a stream line.

Remark. (1) It is clear that the existence of a stream function is a consequence of stream lines and equation of continuity for incompressible fliud. (2) Stream function exists for all types of two dimensional motion—rotational or irrotational. (3) The necessary conditions for the existance of ψ are

(i) the flow must be continuous

(ii) the flow must be incompressible.

3.3. The difference of the values of ψ at the two points represents the flux of a fluid across any curve joining the two points.

Proof : Suppose ds is a line element at a point $P(x, y)$ of a curve AB. Let the tangent PT make an angle θ with x-axis. Let PN be normal at P and (u, v) the velocity components of the fluid at P. Direction cosines of the normal PN are

$$\cos(90 + \theta),\ \cos\theta,\ \cos 90,$$

i.e., $\qquad -\sin\theta,\ \cos\theta,\ 0.$

For PN makes angles $90 + \theta, \theta, 90$ with x, y, z axes respectively

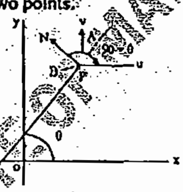


Fig. 3.2.

Inward normal velocity $= \hat{n}.q$, in usual notation

$$= u(-\sin\theta) + v(\cos\theta) + (0).0$$

$$= -u\sin\theta + v\cos\theta$$

Flux across the curve AB from right to left

= density. normal velo. area of the cross section

$$= \int_{AB} \rho(\hat{n}.q)\,ds = \int_{AB} \rho(-u\sin\theta + v\cos\theta)\,ds$$

$$= \rho\int_{AB}\left[-u\frac{dy}{ds} + v\frac{dx}{ds}\right]ds \ \text{ as } \tan\theta = \frac{dy}{dx}$$

$$= \rho\int_{AB}\left[\left(\frac{\partial \psi}{\partial y}\right)dy + \left(\frac{\partial \psi}{\partial x}\right)dx\right] = \rho\int_{AB} d\psi = \rho\{\psi_2 - \psi_1\}$$

where ψ_1 and ψ_2 are the values of ψ at A and B respectively.

Flux across AB is $\rho\{\psi_2 - \psi_1\}$.

This proves the required result.

3.4. Irrotational motion in two dimensions :

To show that in two-dimensional irrotational motion, stream function and velocity potential both satisfy Laplace's equation.

Proof : Let the fluid motion be irrotational so that $\exists$ velocity potential ϕ s.t. $q = -\nabla\phi$, this $\Rightarrow$

$$u = -\frac{\partial \phi}{\partial x},\ v = -\frac{\partial \phi}{\partial y}. \qquad \ldots (1)$$

(Here the component w does not exist).

If ψ is a stream function, then

$$u = -\frac{\partial \psi}{\partial y},\ v = \frac{\partial \psi}{\partial x}. \qquad \ldots (2)$$

Step I : From (1) and (2),

$$-\frac{\partial \phi}{\partial x} = -\frac{\partial \psi}{\partial y},\ -\frac{\partial \phi}{\partial y} = \frac{\partial \psi}{\partial x}.$$

This $\Rightarrow \dfrac{\partial^2 \psi}{\partial x^2} + \dfrac{\partial^2 \psi}{\partial y^2}$

$$= \frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right) = 0.$$

This $\Rightarrow \nabla^2\psi = 0$. Hence the result.

Step II : To show that ϕ satisfies Laplace's equation.

Solution : We know that

$$u = -\frac{\partial \phi}{\partial x},\ v = -\frac{\partial \phi}{\partial y}$$

By equation of continuity,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0,$$

i.e., $$\frac{\partial}{\partial x}\left(-\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \phi}{\partial y}\right) = 0$$

or $$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \ \text{ or } \ \nabla^2\phi = 0.$$

Hence the result.

Note the following points :

(1) The stream function ψ exists whether the motion is irrotational or not.
(2) The velocity potential ϕ exists only when the motion is irrotational.
(3) When motion is irrotational, ϕ exists.
(4) ϕ and ψ both satisfy Laplace' equation, *i.e.*,

$$\nabla^2\phi = 0 = \nabla^2\psi$$

Also $\qquad \phi_x = \psi_y,\ \phi_y = -\psi_x$

i.e., $$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y},\ \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}.$$

Problem 1. *To show that the family of curves $\phi(x, y) =$ const. and $\psi(x, y) =$ const cut orthogonally at their point of intersection.*

Or,

To show that the curves of constant potential and constant stream functions cut orthogonally at their point of intersection.

Solution. Curve of constant potential is given by

$$\phi = \text{const, this} \Rightarrow d\phi = 0 \Rightarrow \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy = 0,$$

or $$-\frac{\phi_x}{\phi_y} = \frac{dy}{dx} = m_1, \text{ say, where } \phi_x = \frac{\partial \phi}{\partial x}, \phi_y = \frac{\partial \phi}{\partial y}.$$

Here m_1 is the gradient of tangent to the curve $\phi =$ const.

Similarly, $\psi = \text{const} \Rightarrow -\dfrac{\psi_x}{\psi_y} = \dfrac{dy}{dx} = m_2$, say

$$\text{Then } m_1 m_2 = \left(-\frac{\phi_x}{\phi_y}\right)\left(-\frac{\psi_x}{\psi_y}\right) = \frac{\phi_x \cdot \psi_x}{\phi_y \cdot \psi_y} = \frac{(-u)v}{(-v)(-u)} = -1$$

or $\quad m_1 m_2 = -1$, this proves the required result.

Problem 2. *If $\phi = A(x^2 - y^2)$ represents a possible flow phenomenon, determine the stream function.*

Solution : Here $\phi = A(x^2 - y^2)$.

Since $$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y} \Rightarrow \frac{\partial \psi}{\partial y} = 2Ax$$

$$\Rightarrow \psi = 2Axy + C,$$

where C is an integration constant, which is the required stream function.

Problem 3. *The velocity potentials $\phi_1 = x^2 - y^2$ and $\phi_2 = \sqrt{r}\cos(\theta/2)$ are solutions of the Laplace equation. Prove that the velocity potential $\phi_3 = (x^2 - y^2) + \sqrt{r}\cos(\theta/2)$ satisfies $\nabla^2\phi_3 = 0$.*

Solution : Here $\phi_1 = x^2 - y^2$ and $\phi_2 = \sqrt{r}\cos(\theta/2)$.

The Laplace's equation in cartesian coordinates and cylindrical polar coordinates is given as

$$\nabla^2\phi_1 = \frac{\partial^2\phi_1}{\partial x^2} + \frac{\partial^2\phi_1}{\partial y^2} = 2 - 2 = 0.$$

and $$\nabla^2\phi_2 = \frac{\partial^2\phi_2}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\phi_2}{\partial\theta^2} + \frac{1}{r}\frac{\partial\phi_2}{\partial r}$$

or $$\nabla^2\phi_2 = -\frac{1}{4r^{3/2}}\cos(\theta/2) - \frac{1}{4r^{3/2}}\cos(\theta/2) + \frac{1}{2r^{3/2}}\cos(\theta/2) = 0$$

$\Rightarrow$ that ϕ_1 and ϕ_2 satisfy Laplace's equation.

Thus $\qquad \nabla^2\phi_1 = 0,\ \nabla^2\phi_2 = 0$

Adding $\qquad \nabla^2(\phi_1 + \phi_2) = 0$

But $\qquad \phi_1 + \phi_2 = \phi_3$

$\therefore \qquad \nabla^2\phi_3 = 0.$

Problem 4. *Show that $u = 2Axy$, $v = A(a^2 + x^2 - y^2)$ are the velocity components of a possible fluid motion. Determine the stream function.*

Solution : Here $u = 2Axy$, $v = A(a^2 + x^2 - y^2)$.

This will be a possible fluid motion if it satisfies the equation of continuity *i.e.*,

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \Rightarrow 2Ay-2Ay=0,$$

which is true. Therefore, the given velocity components constitute a possible fluid motion.

We know that $u=-\frac{\partial \psi}{\partial y}$ and $v=\frac{\partial \psi}{\partial x}$.

So $\frac{\partial \psi}{\partial y}=-2Axy$, and $\frac{\partial \psi}{\partial x}=A(a^2+x^2-y^2)$. ... (1)

By integrating, we have

$$\psi=-Axy^2+f(x,t). \qquad ...(2)$$

Differentiating (2), we have

$$\frac{\partial \psi}{\partial x}=-Ay^2+\frac{\partial f}{\partial x}. \qquad ...(3)$$

From (1) and (3), we have

$$-Ay^2+\frac{\partial f}{\partial x}=A(a^2+x^2-y^2) \Rightarrow \frac{\partial f}{\partial x}=A(a^2+x^2)$$

By integrating, we have

$$f(x,t)=A\left(a^2x+\frac{1}{3}x^3\right)+g(t).$$

Substituting the value of $f(x,t)$ in (2), we have

$$\psi=A\left(a^2x-xy^2+\frac{1}{3}x^3\right)+g(t),$$

which is the required stream function.

Problem 5. *Find the stream function ψ for the given velocity potential $\phi=cx$, where c is constant. Also, draw a set of streamlines and equipotential lines.* (IFoS-2010 model)

Solution : The velocity potential $\phi=cx$ represents fluid flow because it satisfies Laplace equation $\nabla^2\phi=0$.

Since $-\frac{\partial \phi}{\partial x}=-c=u$ and $u=-\frac{\partial \psi}{\partial y}$.

Therefore $\frac{\partial \psi}{\partial y}=c \Rightarrow \psi=cy+f(x)$. ... (1)

Differentiating with regard to x, we have

$$\frac{\partial \psi}{\partial x}=f'(x)$$

But $\frac{\partial \psi}{\partial x}=v=-\frac{\partial \phi}{\partial y} \Rightarrow \frac{\partial \psi}{\partial x}=0,$ as $\frac{\partial \phi}{\partial y}=0$

$\Rightarrow$ $f'(x)=0, \Rightarrow f(x)=$ const. ... (2)

The stream function ψ is given as

$$\psi=\text{const.}+cy,$$

which represents parallel flow in which stream lines are parallel to X-axis.

The corresponding stream lines and equipotential lines are represented as follows (Fig. 3.3):

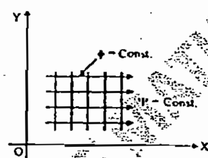


Fig. 3.3.

Problem 6. *A velocity field is given by $\mathbf{q}=-x\mathbf{i}+(y+t)\mathbf{j}$. Find the stream function and the stream lines for this field at $t=2$.*

Solution. Here $\mathbf{q}=u\mathbf{i}+v\mathbf{j}=-x\mathbf{i}+(y+t)\mathbf{j}$

$\Rightarrow$ $u=-x, v=y+t$

We know that

$$-\frac{\partial \psi}{\partial y}=u=-x \text{ and } \frac{\partial \psi}{\partial x}=v=y+t. \qquad ...(1, 2)$$

By integrating (1) with regard to y, we have

$$\psi=xy+f(x,t), \qquad ...(3)$$

where $f(x,t)$ is an integration constant.

or $$\frac{\partial \psi}{\partial x}=y+\frac{\partial f}{\partial x} \qquad ...(4)$$

From (2) and (4), we have

$$y+\frac{\partial f}{\partial x}=y+t \Rightarrow \frac{\partial f}{\partial x}=t$$

$\Rightarrow$ $f(x,t)=xt+g(t)$. ... (5)

From (3) and (5), we have

$$\psi=xy+xt+g(t).$$

At $t=2$, $\psi=x(y+2)+g(2)$.

The stream lines are given by $\psi=$ const., therefore

$$x(y+2)=\text{const.},$$

which represent rectangular hyperbolas.

Problem 7. *Prove that for the complex potential $\tan^{-1}z$ the stream lines and equi-potentials are circles. Find the velocity at any point and examine the singularities at $z=\pm i$.*

Solution. The complex potential is given by

$$w=\phi+i\psi=\tan^{-1}z. \qquad ...(1)$$

Also $\bar{w}=\phi-i\psi=\tan^{-1}\bar{z}$. ... (2)

By subtracting (1) and (2), we have

$$2i\psi=\tan^{-1}z-\tan^{-1}\bar{z}=\tan^{-1}\frac{z-\bar{z}}{1+z\bar{z}}$$

or $$\tan 2i\psi=\frac{2iy}{1+x^2+y^2} \Rightarrow x^2+y^2+1=2y\coth 2\psi.$$

The stream lines $\psi=$ constant represent the circles

$$x^2+y^2+1=2y\coth 2\psi. \qquad (3)$$

Similarly, by adding (1) and (2), we have

$$2\phi=\tan^{-1}z+\tan^{-1}\bar{z}=\tan^{-1}\frac{z+\bar{z}}{1-z\bar{z}}=\tan^{-1}\left(\frac{2x}{1-x^2-y^2}\right)$$

or $1-x^2-y^2=2x\cot 2\phi$. ... (4)

The equi-potentials $\phi=$ const. also represent circles which are orthogonal to the streamlines $\psi=$ const. and form a co-axial system with limit points at $z=\pm i$. The velocity component (u, v) is given by

$$\frac{dw}{dz}=-u+iv=\frac{1}{z^2+1}, \text{ by (1)} \qquad ...(5)$$

the denominator vanishes at $z=\pm i$, therefore, it represents the singularities at these points.

At $z=+i$, substitute $z=i+z_1$, where $|z_1|$ is very small

$$-u+iv=\frac{dw}{dz}=\frac{dw}{dz_1}=\frac{1}{1+(-1+2iz_1)}=\frac{1}{2iz_1}$$

by integrating, we have

$$w=-\frac{1}{2}i\log z_1$$

$\Rightarrow$ that the singularity at $z=i$ is a vortex of strength $k=-\frac{1}{2}$ with circulation $-\pi k$.

Similarly, the singularity at $z=-i$ is a vortex of strength $k=\frac{1}{2}$ with cirulation πk.

3.5. Complex Potential.

Suppose ϕ and ψ represent velocity potential and stream function of a two dimensional irrotational motion of a prefect fluid. Let $w=\phi+i\psi$. Then w is defined as complex potential of the fluid motion. Since $\phi=\phi(x,y)$, $\psi=\psi(x,y)$ and so $w=\phi+i\psi$ can be expressed as function of z. Hence $w=f(z)=\phi+i\psi$ where $z=x+iy$.

Also we know that

$$-\frac{\partial \phi}{\partial x}=u=-\frac{\partial \psi}{\partial y}, -\frac{\partial \phi}{\partial y}=v=\frac{\partial \psi}{\partial x}$$

i.e., $$\frac{\partial \phi}{\partial x}=\frac{\partial \psi}{\partial y}, \frac{\partial \phi}{\partial y}=-\frac{\partial \psi}{\partial x}$$

which are Cauchy-Riemann equations. Thus Cauchy-Riemann equations are satisfied so that w is analytic function of z.

Conversely, if w is analytic function, then its real and imaginary, *i.e.*, ϕ and ψ give the velocity potential and stream function for a possible two dimensional irrotational fluid motion.

Theorem 1. To prove that any relation of the form $w=f(z)$ where $w=\phi+i\psi$ and $z=x+iy$, represents a two dimensional irrotational motion, in which the magnitude of velocity is given by

$$\left|\frac{dw}{dz}\right|.$$

Proof. $w=\phi+i\psi$, $w=f(z)$.

Differentiating w.r.t. x,

$$\frac{dw}{dz}\cdot\frac{\partial z}{\partial x}=\frac{\partial \phi}{\partial x}+i\frac{\partial \psi}{\partial x}=-u+iv$$

or $$\frac{dw}{dz}=-u+iv \text{ as } z=x+iy \Rightarrow \partial z/\partial x=1.$$

This $\Rightarrow \left|\frac{dw}{dz}\right|=\sqrt{(u^2+v^2)}=$ magnitude of velocity.

Hence $\left|\frac{dw}{dz}\right|$ represents magnitude of velocity.

Remark : The points, where velocity is zero, are called *stagnation points*.

Thus for stagnation points, $\frac{dw}{dz}=0$.

3.6. Cauchy-Riemann equations in polar form.

$$w=f(z), w=\phi+i\psi, \quad z=re^{i\theta}$$

Hence $\phi+i\psi=f(re^{i\theta})$

Differentiating w.r.t. r and θ, respectively,

$$\frac{\partial \phi}{\partial r}+i\frac{\partial \psi}{\partial r}=f'(re^{i\theta}).e^{i\theta}$$

$$\frac{\partial \phi}{\partial \theta}+i\frac{\partial \psi}{\partial \theta}=f'(re^{i\theta}).ri e^{i\theta}$$

Combining these two equations,

$$ri\left[\left(\frac{\partial \phi}{\partial r}\right)+i\frac{\partial \psi}{\partial r}\right]=\frac{\partial \phi}{\partial \theta}+i\frac{\partial \psi}{\partial \theta}$$

Equating real and imaginary parts,

$$-r\frac{\partial \psi}{\partial r}=\frac{\partial \phi}{\partial \theta}, r\frac{\partial \phi}{\partial r}=\frac{\partial \psi}{\partial \theta}$$

This $\Rightarrow \frac{\partial \phi}{\partial r}=\frac{1}{r}\frac{\partial \psi}{\partial \theta}, \frac{1}{r}\frac{\partial \phi}{\partial \theta}=-\frac{\partial \psi}{\partial r}$

These two equations are known as polar form of Cauchy-Riemann equations.

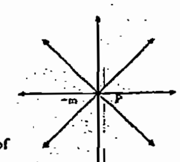


Fig. 3.4.

3.7. Two dimensional sources, sinks.

(i) Sources :

A source (two dimensional simple source) is a point from which liquid is emitted radially and symmetrically in all directions in xy-plane.

(ii) Sink : A point to which fluid is flowing in symmetrically and radially in all directions is called sink. This sink is a negative of source.

Difference between source and sink.

Source is a point at which liquid is continuously created and sink is a point at which liquid is continuously annihilated. Really speaking, source and sink are purely abstract conceptions which do not occur in nature.

(iii) Strength : Strength of a source is defined as total volume of flow per unit time from it.

Thus, if $2\pi m$ is the total volume of flow across any small circle surrounding the source, then m is called strength of the source. Sink is a source of strength $-m$.

3.8. Complex potential due to a source :

To find the complex potential for a two dimensional source of strength m placed at the origin.

Proof : Consider a source of strength m at the origin. We are required to determine complex potential w at a point $P(r, \theta)$ due to this source. The velocity at P due to the source is purely radial, let this velocity be q_r. Flux across a circle of radius r surrounding the source at O is $2\pi r q_r$. By definition of strength,

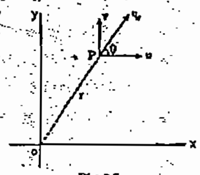


Fig. 3.5.

$2\pi r q_r = 2\pi m$, hence $q_r = m/r$.

Then $\quad u = q_r \cos\theta = \dfrac{m}{r}\cos\theta$

$\quad v = q_r \sin\theta = \dfrac{m}{r}\sin\theta$.

We know that

$$\frac{dw}{dz} = -u + iv$$

$$= \frac{m}{r}[-\cos\theta + i\sin\theta]$$

or $\quad \dfrac{dw}{dz} = -\dfrac{m}{r}e^{-i\theta} = -\dfrac{m}{re^{i\theta}}$

$$= -\frac{m}{z} \quad \text{or} \quad dw = -m\frac{dz}{z}$$

Integrating, $\quad w = -m \log z \qquad \ldots (1)$

(neglecting constant of integration)

(1) is the required expression.

Deductions : (i) If the source $+m$ is at a point $z = z_1$ in place of $z = 0$, then by shifting the origin,

we have $\quad w = -m\log(z - z_1)$.

This is the required expression for w in this case.

(ii) *To find the complex potential w at any point z due to sources of strength $m_1, m_2, m_3, \ldots$ situated at $a_1, a_2, a_3, \ldots$*

Proof : *Step I.* To determine w due to a source $+m$ at the point $z = 0$. (Here prove as in § 3.8 that $w = -m \log z$.

Step II. If a source of strength $+m_1$ is at $z = z_1$, then

$w = -m_1 \log(z - z_1)$, by step I.

The required complex potential is given by

$$w = -m_1\log(z - a_1) - m_2\log(z - a_2) - \ldots$$

$$= -\sum_{n=1}^{\infty} m_n \log(z - a_n).$$

Hence $\quad \phi = -\sum_{n=1}^{\infty} m_n \log r_n, \quad \psi = -\sum_{n=1}^{\infty} m_n \theta_n$

where $\quad z - a_n = r_n e^{i\theta_n}$.

3.9. Two dimensional doublet.

A doublet is defined as a combination of source $+m$ and sink $-m$ at a small distance δs apart s.t. the product $m\delta s$ is finite. (Sink $-m$ means sink of strength $-m$).

Strength of doublet. If $m\delta s = \mu =$ finite where $m \to \infty$, $\delta s \to 0$, then μ is called strength of the doublet and line δs is called the axis of the doublet and its direction is taken from sink to source.

3.10. Complex potential for a doublet :

Let a doublet AB of strength μ be formed by a sink $-m$ at A $(z = a)$ and source $+m$ at B $(z = a + \delta a)$.

Then $\quad \mu = m.AB$.

$\quad \delta a = AB\, e^{i\alpha}$. $\qquad \ldots (1)$

[For $z = re^{i\theta}$]

as α is the inclination of the axis of the doublet with x-axis. The complex potential w due to this doublet at any point $P(z)$ is given by

$$w = +m\log(z - a) - m\log[z - (a + \delta a)]$$

$$= -m\log\left(\frac{z - a - \delta a}{z - a}\right)$$

$$= -m\log\left(1 - \frac{\delta a}{z - a}\right)$$

$$= m.\left(\frac{\delta a}{z - a}\right) \text{ upto first approximation.}$$

as $\quad \left| -\log(1 - z) = z + \dfrac{z^2}{2} + \dfrac{z^3}{3} + \ldots \right.$

or, $\quad w = \dfrac{m.AB\, e^{i\alpha}}{z - a}$, by (1) $= \dfrac{\mu e^{i\alpha}}{z - a}$

$\therefore \quad w = \dfrac{\mu e^{i\alpha}}{z - a}$ is the required expression.

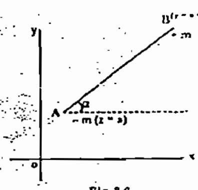


Fig. 3.6.

Deductions (i) If the axis of doublet is along x-axis, then

$\alpha = 0$ so that $w = \dfrac{\mu e^{i.0}}{z - a} = \dfrac{\mu}{z - a}$.

(ii) If the axis of doublet is along x-axis and the doublet is at the origin, then $\alpha = 0$, $a = 0$ so that

$w = \dfrac{\mu}{z}$.

(iii) If a system consists of doublets of strength $\mu_1, \mu_2, \ldots$ placed at $z = a_1, a_2, \ldots$, then w due to this system is given by

$$w = \sum_{n=1}^{\infty} \frac{\mu_n e^{i\alpha_n}}{z - a_n}$$

where α_n is the inclination of the axis of the doublet of strength μ_n with x-axis.

3.11. Image.

If there exists a curve C in the xy-plane in a fluid s.t. there is no flow across it, then the system of sources, sinks and doublets on one side of C is said to be the images of the sources, sinks and doublets on the other side of C.

Significance of Image

A two dimensional irrotational motion when confined to rigid boundaries is regarded to have been caused by the presence of sources and sinks. If we take the set of sources and sinks (imagining) to be on either side of the rigid boundaries, the velocity normal to these boundaries will be zero. As sch these boundaries can be taken as stream lines. This is due to the property of stream lines that the velocity prependicular to strenm lines is zero. This set of sources and sinks on either side is called the image. Thus the motion is no longer constrained by boundaries so that it is possible to predict the nature of the velocity and pressure at each point of the fluid.

3.12. *To find the image of a simple source w.r.t a plane (straight line) and show that the image of a doublet w.r.t. a plane is an equal doublet symmetrically placed.*

Proof : (i) *To find the image of a source w.r.t a straight line (plane).* We are to determine the image of a source $+m$ at $A(a, 0)$ w.r.t the straight line OY. Place a source $+m$ at $B(-a, 0)$. The complex potential at P due to this system is given by

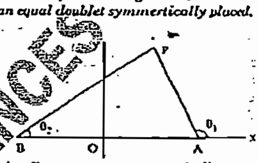


Fig. 3.7.

$$w = -m\log(z - a) - m\log(z + a)$$

$$= -m\log(z - a)(z + a)$$

$$= -m\log(r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2})$$

$$= -m\log[r_1 r_2 e^{i(\theta_1 + \theta_2)}] \quad [\text{where } PA = r_1, PB = r_2]$$

or $\quad \phi + i\psi = -m\,[\log(r_1 r_2) + i(\theta_1 + \theta_2)]$

This $\Rightarrow \psi = -m(\theta_1 + \theta_2)$. $\qquad \ldots (1)$

If P lies on y-axis, then $PA = PB$ so that $\angle PAB = \angle PBA$.

i.e., $\quad \pi - \theta_1 = \theta_2$, $\pi = \theta_1 + \theta_2$. ... (2)

By (1) and (2),

$\psi = -m\pi$ or $\psi =$ const.

It means that y-axis is stream line. Hence the image of a source $+m$ at $A(a, 0)$ is a source $+m$ at $B(-a, 0)$. That is to say, image of a source w.r.t. a line is a source of the same strength situated on the opposite side of the line at an equal distance.

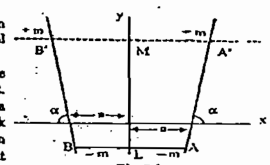


Fig. 3.8.

(ii) *Image of a doublet w.r.t. a plane.* We are to find the image of the doublet AA' w.r.t. y-axes. Treat the doublet AA' as a combination of source $+m$ at A' and sink $-m$ at A with its axis AA' inclined at an angle α with x-axis. The images of $-m$ at A and $+m$ at A' w.r.t. y-axis are respectively $-m$ at B and $+m$ at B' s.t. $BL = LA$, $B'M = MA'$. Hence the image is a doublet BB' of the same strength with its axis anti-parallel to AA'.

3.13 Image of a source in a circle.

We are required to find the image of a source $+m$ at A w.r.t the circle whose centre is O. Let B be the inverse point of A w.r.t the circle. Let P be any current point on the circle at which ψ is to be determined.

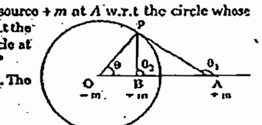


Fig. 3.9.

Place a source $+m$ at B and sink $-m$ at O. The value of ψ due to this system is given by

$\psi = -m\theta_1 - m\theta_2 + m\theta$

or $\quad \psi = -m(\theta_1 + \theta_2 - \theta)$. $\qquad \ldots (1)$

Since B is the inverse point of A,

$OB \cdot OA = (\text{radius})^2 = OP^2$

or $\quad \dfrac{OB}{OP} = \dfrac{OP}{OA}$ also $\angle BOP = \angle POA$.

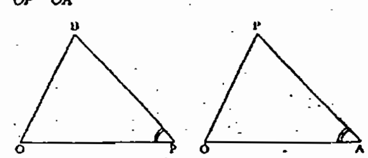


Fig. 3.10.

Hence ΔOPB and ΔOPA are similar. Therefore

$\angle OPB = \angle OAP$, i.e., $\theta_2 - \theta = \pi - \theta_1$ or $\theta_2 + \theta_1 - \theta = \pi$.

Now (1) becomes $\psi = -m\pi$ or $\psi =$ const.

This declares that circle is a stream line so that there exists no flux across the boundary. It means that :

Image of source $+m$ at A is a source $+m$ at B; the inverse point of A and sink $-m$ at the centre.

3.14. Image of a doublet relative to a circle

We are required to find the image of the doublet AA' w.r.t the circle whose centre is O. The axis of the doublet is inclined at an angle α with Ox. Let $OA = f$ and μ the strength of the doublet. Treat this doublet as a combination of sink $-m$ at A and source $+m$ at A' so that $\mu = m . AA'$ where $m \to \infty$, and length of $AA' \to 0$. The image of sink $-m$ at A' at A is a sink $-m$ at B, the inverse point of A and source $+m$ at O. The image of source $+m$ at A' is a source $+m$ at B'; the inverse point of A' and sink $-m$ at O. Compounding these, we find that source $+m$ and sink $-m$ both at O cancel each other and there remains a doublet of strength $\mu = m . BB'$ at B, the inverse of A.

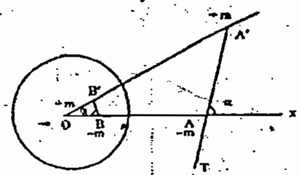


Fig. 3.11.

For $A' \to A \Rightarrow B' \to B$.

Here we have $OB . OA = a^2 = OB' . OA'$,

a = radius of the circle.

Hence $\dfrac{OB}{OB'} = \dfrac{OA'}{OA}$. Also $\angle SOB' = \angle A'OA$.

This $\Rightarrow \Delta OBB'$ and $\Delta OAA'$ are similar.

Hence $\dfrac{BB'}{A'A} = \dfrac{OB}{OA'}$

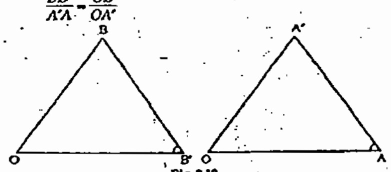


Fig. 3.12.

This $\Rightarrow m . BB' = m . AA' . \dfrac{OB}{OA'}$

$m . BB' = (m AA') \dfrac{OB . OA}{OA' . OA}$

Taking limits as $A' \to A$, so that

$B' \to B$, we get

$$\mu' = \mu \frac{a^2}{f^2}$$

Also, by similarity of triangles,

$\angle OBB' = \angle OA'A = \angle OAT$ (in limit) $= \alpha$.

Thus the image of a doublet of strength μ at A (where $OA = f$) relative to a circle is a doublet of strength $\mu' = \mu a^2/f^2$ at B, the inverse point of A, the axis of the doublet makes supplementary angle, i.e., $\pi - \alpha$ with the radius OA.

3.15. Circle Theorem of Milne-Thomson

Suppose $f(z)$ is the complex potential of a two dimensional irrotational motion of an incompressible liquid with no rigid boundaries. Then, if a circular cylinder $|z| = a$ is inserted in the flow field, the complex potential of the resulting motion is given by

$$w = f(z) + \bar{f}(a^2/z) \text{ for } |z| \geq a$$

provided $f(z)$ has no singularity inside $|z| = a$.

Proof : Let C be the cross-section of the circular cylinder $|z| = a$. Then on C, $z\bar{z} = a^2$ or $\bar{z} = a^2/z$ so that $f(\bar{z}) = f(a^2/z)$ and so $\bar{f}(\bar{z}) = \bar{f}(a^2/z)$.

This $\Rightarrow w = f(z) + \bar{f}(a^2/z) = f(z) + \bar{f}(\bar{z}) = f(z) + \overline{f(z)}$

$\Rightarrow w =$ real quantity.

Equating imaginary parts on both sides, $\psi = 0$ or $\psi =$ const. for 0 is also a constant. Thus ψ is constant along the boundary C. It means that C is a stream line in the new flow.

We know that if z lies outside C, then the point a^2/z is inside C. Also it is given that $f(z)$ has singularities outside C. Consequently, $f(a^2/z)$ and therefore $\bar{f}(a^2/z)$ has singularities inside C. It means that the additional term $\bar{f}(a^2/z)$ introduces no new singularity outside C. In particular $\bar{f}(a^2/z)$ has no singularity at $z = \infty$ as $f(z)$ has no singularity at $z = 0$.

Since the motion is irrotational and fluid is incompressible, the function $f(z)$ will satisfy Laplace's equation and therefore w will satisfy Laplace's equation for two dimensional irrotational flow of liquid with C inserted as does the function $f(z)$ in the absence of C.

Remark : The Milne-Thomson circle theorem provides a conventional method for finding the image system of a given two dimensional system which lies outside a circular boundary. For, if $w = f(z)$ *represents the given system* in the presence of the circular boundary $|z| = a$, then $w = f(a^2/z)$ *represents the image system.*

3.16. Image of source w.r.t. a circle of radius a. (i.e. alternative method of 3.13).

Consider a source of strength $+m$ at $z = f$ so that the complex potential due to this source is

$$f(z) = -m \log(z - f).$$

Let a circular cylinder $|z| = a$ (where $a < f$) be inserted, then by circle theorem, the complex potential is given by

$$w = f(z) + \bar{f}\left(\frac{a^2}{z}\right) = -m \log(z - f) - m \log\left(\frac{a^2}{z} - f\right)$$

$$= -m \log(z - f) - m \log\left(\frac{a^2 - fz}{z}\right)$$

$$= -m \log(z - f) - m \log\left(\frac{-f}{z}\right)\left(z - \frac{a^2}{f}\right)$$

$$= -m \log(z - f) - m \log\left(z - \frac{a^2}{f}\right) - m \log(-f) + m \log z$$

Ignoring the constant term $-m \log(-f)$, we get

$$w = -m \log(z - f) + m \log z - m \log\left(z - \frac{a^2}{f}\right) \quad \text{...(1)}$$

This is the complex potential due to

(i) source $+m$ at $z = f$,

(ii) sink $-m$ at $z = 0$,

(iii) source $+m$ at $z = a^2/f$.

For this complex potential, circle is a stream line and hence the image system for a source $+m$ outside the circle consists of a source $+m$ at the inverse point and sink $-m$ at the origin, the centre of the centre. Since f and a^2/f both are inverse points w.r.t the circle $|z| = a$.

3.17. Alternative method for the image of a doublet relative to a circle.

The complex potential $f(z)$ due to a doublet of strength μ at $z = f$ with its axis inclined at an angle α, is given by

$$f(z) = \frac{\mu e^{i\alpha}}{z - f}$$

When a circular cylinder $|z| = a$ where $a < f$, is inserted in the flow of motion, then the complex potential is given by

$$w = f(z) + \bar{f}(a^2/z), \text{ by circle theorem}$$

$$= \frac{\mu e^{i\alpha}}{z - f} + \left[\overline{\left(\frac{\mu e^{i\alpha}}{(a^2/z) - f}\right)}\right] = \frac{\mu e^{i\alpha}}{z - f} + \frac{\mu e^{-i\alpha}}{(a^2/z) - f}$$

$$= \frac{\mu e^{i\alpha}}{z - f} - \frac{\mu z\, e^{i(\pi - \alpha)}}{a^2 - fz} = \frac{\mu e^{i\alpha}}{z - f} + \frac{\mu z\, e^{i(\pi - \alpha)}}{f[z - (a^2/f)]}$$

$$= \frac{\mu e^{i\alpha}}{z - f} + \frac{\mu e^{i(\pi - \alpha)}\left(z - \frac{a^2}{f} + \frac{a^2}{f}\right)}{f[z - (a^2/f)]}$$

$$= \frac{\mu e^{i\alpha}}{z - f} + \frac{\mu}{f} e^{i(\pi - \alpha)} + \frac{\mu a^2}{f^2} \cdot \frac{e^{i(\pi - \alpha)}}{\left(z - \frac{a^2}{f}\right)}$$

Ignoring the constant term $\mu e^{i(\pi - \alpha)}/f$, we get

$$w = \frac{\mu e^{i\alpha}}{z - f} + \frac{\mu a^2}{f^2} \cdot \frac{e^{i(\pi - \alpha)}}{\left(z - \frac{a^2}{f}\right)}$$

This is the complex potential due to

(i) doublet of strength μ at $z = f$ with its axis inclined at an angle α.

(ii) doublet of strength $\mu a^2/f^2$ at $z = a^2/f$, the inverse point of $z = f$, its axis is inclined $\pi - \alpha$.

For this complex potential circle is a stream line and hence the image system for a doublet of strength μ at $z = f$ (outside the circle) is a doublet of strength $\mu' = \mu a^2/f^2$ and its axis inclined at an angle $\pi - \alpha$.

3.18. Blasius Theorem :

In steady two dimensional motion given by the complex potential $w = f(z) = \phi + i\psi$, if the pressure thrusts on the fixed cylinder of any shape are represented by a force (X, Y) and a couple of moment N about the origin of coordinates, then neglecting external forces.

$$X - iY = \frac{i\rho}{2} \int_c \left(\frac{dw}{dz}\right)^2 dz,$$

$$\text{and } n = \text{ real part of} \left[-\frac{1}{2} \rho \int_c \left(\frac{dw}{dz}\right)^2 z\, dz\right]$$

where ρ is the density and integrals are taken round the contour c of the cylinder.

Proof: Consider an element ds of arc surrounding the point $P(x, y)$ of the fixed cylinder. c denotes the boundary of the cylinder of any shape and size. Let the tangent at P make an angle θ with X-axis, so that the inward normal at P make angle $90° + \theta$ with X-axis. The thrust pds at P acts along inward normal, its components along x and y axes are respectively

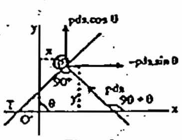


Fig. 3.13.

$$pds \cos(90° + \theta), pds . \sin(90° + \theta)$$

i.e., $\quad -pds . \sin\theta, pds . \cos\theta.$

Hence $\quad X = \int_c -pds . \sin\theta, \; Y = \int_c pds . \cos\theta$

This $\quad \Rightarrow X - iY = \int_c p(-\sin\theta - i\cos\theta)\, ds$

$$= -i \int_c p(\cos\theta - i\sin\theta)\, ds.$$

Bernoulli's equation for steady motion gives

$$\frac{p}{\rho} + \frac{1}{2} q^2 = A = \text{const.}$$

or $\quad p = \left(A - \frac{1}{2} q^2\right)\rho. \quad \text{...(1)}$

$\therefore \quad X - iY = -i\rho \int_c \left(A - \frac{1}{2} q^2\right)(\cos\theta - i\sin\theta)\, ds$

$$= \frac{i\rho}{2}\int_c q^2 e^{-i\theta}\,ds - i\rho A\int_c (\cos\theta - i\sin\theta)\,ds$$

But $\quad \dfrac{dx}{ds} = \cos\theta,\ \dfrac{dy}{ds} = \sin\theta$ as $\tan\theta = dy/dx$.

$\therefore \quad X - iY = \dfrac{i\rho}{2}\displaystyle\int_c q^2 e^{-i\theta}\,ds - i\rho A\int_c (dx - i\,dy).$

But $\displaystyle\int_c (dx - i\,dy) = \int_c dz = 0$, by Cauchy's theorem.

Hence $\quad X - iY = \dfrac{i\rho}{2}\displaystyle\int_c q^2 e^{-i\theta}\,ds.$ ... (2)

Let u and v be velocity components. Then we know that

$$\frac{dw}{dz} = -u + iv = -q\cos\theta + iq\sin\theta = -q(\cos\theta - i\sin\theta)$$

or $\quad \dfrac{dw}{dz} = -qe^{-i\theta}$, or $\left(\dfrac{dw}{dz}\right)^2 dz = q^2 e^{-2i\theta}\,.(dx + i\,dy)$

or $\quad \left(\dfrac{dw}{dz}\right)^2 dz = q^2 e^{-2i\theta}\,.(\cos\theta + i\sin\theta)\,ds = q^2 e^{-i\theta}\,ds.$ ... (3)

Using this in (2) we get the first required result, namely

$$X - iY = \frac{i\rho}{2}\int_c \left(\frac{dw}{dz}\right)^2 dz.$$

we consider anticlockwise moments as positive.

The moment of the thrust $p\,ds$ about the origin is

$$N = \int_c [-(-p\,ds\sin\theta)y + (p\,ds\cos\theta)x]$$

$$= \int_c p\,(y\sin\theta + x\cos\theta)\,ds = \int_c\left(A - \frac{1}{2}q^2\right)\rho\,(y\sin\theta + x\cos\theta)\,ds$$

$$= A\rho\int_c (y\sin\theta + x\cos\theta)\,ds - \frac{\rho}{2}\int_c q^2(y\sin\theta + x\cos\theta)\,ds$$

$$= A\rho\int_c (y\,dy + x\,dx) - \frac{\rho}{2}\int_c q^2(y\sin\theta + x\cos\theta)\,ds$$

But $A\rho\displaystyle\int_c (y\,dy + x\,dy) = A\rho\int_c y\,dy + A\rho\int_c x\,dx = 0 + 0$, by Cauchy's theorem.

Hence $\quad N = $ Real part of $\left[-\dfrac{\rho}{2}\displaystyle\int_c q^2 z\,e^{-i\theta}\,ds\right]$

$= $ Real part of $\left[-\dfrac{\rho}{2}\displaystyle\int_c z\left(\dfrac{dw}{dz}\right)^2 dz\right]$, by (3).

This proves the second required result.

Solved Problems

Problem 1. *A line source is in the presence of an infinite plane on which is placed a semi-circular cylindrical boss, the direction of the source is parallel to the axis of boss, the source is at a distance c from the plane and the axis of boss, whose radius is a. Show that the radius to the point on the boss at which the velocity is a maximum makes an angle θ with the radius to the source, where*

$$\theta = \cos^{-1}\frac{a^2 + c^2}{[2(a^4 + c^4)]^{1/2}}$$

Or

If the axis of y and the circle $x^2 + y^2 = a^2$, are fixed boundaries and there is a two-dimensional source at the point (c, 0), where c > a, show that the radius drawn from the origin to the point on the circle, where the velocity is a maximum, makes with the axis of x an angle

$$\cos^{-1}\left[\frac{a^2 + c^2}{[2(a^4 + c^4)]^{1/2}}\right]$$

When c = 2a, show that the required angle is $\cos^{-1}[5/\sqrt{(34)}]$.

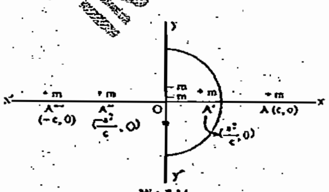


Fig. 2.14.

Solution: The object system consists of source $+m$ at A (c, 0) with semi-circular boundary and parts of y-axis lying outside. Image system consists of (a) (i) source $+m$ at A', the inverse point of A so that $OA \cdot OA' = a^2$ or $OA' = a^2/c$

(ii) sink $-m$ at O, the centre (origin). It is due to circle.

(b) Above system now gives its own images as

(i) source $+m$ at A'' ($z = -a^2/c$)

(This is the image of A' relative to y-axis)

(ii) source $+m$ at A''' ($z = -c$)

This is the image of A relative to y-axis.

(iii) sink $-m$ at O.

(This is the image of $-m$ at O relative to y-axis).

The complex potential due to object system with rigid boundary is equivalent to object system and its image system without rigid boundary. Now complex potential is given by

$$w = -m\log(z - c) - m\log\left(z - \frac{a^2}{c}\right) + 2m\log(z - 0)$$
$$- m\log(z + c) - m\log\left(z + \frac{a^2}{c}\right)$$

$$w = -m\log(z^2 - c^2) - m\log\left(z^2 - \frac{a^4}{c^2}\right) + 2m\log z$$

$$\frac{dw}{dz} = -2m\left[\frac{z}{z^2 - c^2} + \frac{z}{z^2 - \frac{a^4}{c^2}} - \frac{1}{z}\right]$$

or $\quad \dfrac{dw}{dz} = -\dfrac{2m\,(z^4 - a^4)}{z\,(z^2 - c^2)\left(z^2 - \dfrac{a^4}{c^2}\right)}$

If q is velocity at $z = ae^{i\theta}$, then

$$q = \left|\frac{dw}{dz}\right| = \frac{2m\,|a^4 e^{4i\theta} - a^4|}{|ae^{i\theta}|\,.\,|a^2 e^{2i\theta} - c^2|\,.\,\left|a^2 e^{2i\theta} - \dfrac{a^4}{c^2}\right|}$$

or $\quad q = \dfrac{2mac^2\,|e^{4i\theta} - 1|}{|a^2 e^{2i\theta} - c^2|\,.\,|c^2 e^{2i\theta} - a^2|}$ ... (1)

But $\quad |e^{4i\theta} - 1|^2 = (\cos 4\theta - 1)^2 + \sin^2 4\theta$

$= 2 - 2\cos 4\theta = 4(\sin 2\theta)^2$

$|e^{4i\theta} - 1| = 2\sin 2\theta$ ... (2)

$|c^2 e^{2i\theta} - a^2|^2 = (c^2\cos 2\theta - a^2)^2 + (c^2\sin 2\theta)^2$

$= c^4 + a^4 - 2a^2c^2\cos 2\theta$ ... (3)

$|a^2 e^{2i\theta} - c^2|^2 = (a^2\cos 2\theta - c^2)^2 + (a^2\sin 2\theta)^2$

$= a^4 + c^4 - 2a^2c^2\cos 2\theta$ ... (4)

Writing (1) with the help of (2), (3), (4),

$$q = \frac{4mac^2\sin 2\theta}{(a^4 + c^4 - 2a^2c^2\cos 2\theta)} \quad \text{... (5)}$$

q is maximum if $\dfrac{d}{d\theta}\left[\dfrac{\sin 2\theta}{a^4 + c^4 - 2a^2c^2\cos 2\theta}\right] = 0$

This gives,

$$2\cos 2\theta\,(a^4 + c^4 - 2a^2c^2\cos 2\theta) - \sin 2\theta\,(4a^2c^2\sin 2\theta) = 0$$

or $\quad 2(a^4 + c^4)\cos 2\theta - 4a^2c^2 = 0$

$$\cos 2\theta = \frac{2a^2c^2}{a^4 + c^4} \quad \text{... (6)}$$

or $\quad \theta = \dfrac{1}{2}\cos^{-1}\left[\dfrac{2c^2c^2}{a^4 + c^4}\right]$

This gives the position of the point where pressure is minimum as :

$$\frac{p}{\rho} + \frac{1}{2}q^2 = c$$

suggests that p is minimum if q is maximum.

By (6). $\quad 2\cos^2\theta - 1 = \dfrac{2a^2c^2}{a^4 + c^4}$

or $\quad 2\cos^2\theta = \dfrac{(a^2 + c^2)^2}{a^4 + c^4}$ or $\cos^2\theta = \dfrac{(a^2 + c^2)^2}{2(a^4 + c^4)}$

or $\quad \cos\theta = \dfrac{a^2 + c^2}{[2(a^4 + c^4)]^{1/2}}$ ... (7)

If $\quad c = 2a, \quad \cos\theta = \dfrac{(1 + 4)a^2}{[2(17a^4)]^{1/2}} = \dfrac{5}{\sqrt{(34)}}$

Similar Problem : *In a two dimensional motion of an infinite liquid there is a rigid boundary consisting of that part of the circle $x^2 + y^2 = a^2$ which lies in the first and fourth quadrants and the parts of y-axis which lie outside the circle. A simple source of strength m is placed at the point (f, 0) where f > a. Prove that the speed of the fluid at the point (a cos θ, a sin θ) of the semicircular boundary is $4amf^2\sin 2\theta/[a^4 + f^4 - 2a^2f^2\cos 2\theta]$.*

Find at what speed of the boundary the pressure is least.

Hint : Put $c = f$ in the above problem and refer equations (5) and (7).

Problem 2. *A region is bounded by a fixed quadrantal arc and its radii, with a source and an equal sink at the ends of one of the bounding radii. Show that the motion is given by*

$$w = -m\log\left(\frac{z^2 - a^2}{z}\right)$$

and prove that the stream line leaving either the source or the sink at an angle α with the radius is $r^2\sin(\alpha + \theta) = a^2\sin(\alpha - \theta)$.

Solution: The object system and its image system consists of (i) source $+m$ at A ($z = a$), (ii) sink $-m$ at $z = 0$, (iii) source $+m$ at A' ($z = -a$).

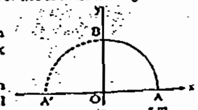


Fig. 2.16.

The complex potential due to object system with rigid boundary is equivalent to the complex potential due to object system and its image system with no rigid boundary, hence complex potential is given by

$$w = -m\log(z + a) + m\log(z - 0) - m\log(z - a)$$

or $$w = -m\log\frac{(z+a)(z-a)}{z} = -m\log\frac{z^2-a^2}{z}$$

or $$w = -m\log\left(\frac{z^2-a^2}{z}\right)$$

Second Part: We have $w = -m\log\left(\frac{z^2-a^2}{z}\right)$

or $$\phi + i\psi = -m\log(r^2 e^{i2\theta} - a^2) + m\log re^{i\theta}.$$

Equating imaginary parts,

$$\psi = -m\tan^{-1}\left(\frac{r^2\sin 2\theta}{r^2\cos 2\theta - a^2}\right) + m\tan^{-1}\left(\frac{r\sin\theta}{r\cos\theta}\right)$$

$$= -m\left[\tan^{-1}\left(\frac{r^2\sin 2\theta}{r^2\cos 2\theta - a^2}\right) - \tan^{-1}\left(\frac{\sin\theta}{\cos\theta}\right)\right]$$

$$= -m\tan^{-1}\left[\frac{r^2(\sin 2\theta\cos\theta - \sin\theta\cos 2\theta) + a^2\sin\theta}{(r^2\cos 2\theta - a^2)\cos\theta + r^2\sin 2\theta\sin\theta}\right]$$

$$\left[\begin{array}{l}\text{For } \tan^{-1}a - \tan^{-1}b = \tan^{-1}\dfrac{a-b}{1+ab} \\ \text{and } \log(x+iy) = \dfrac{1}{2}\log(x^2+y^2) + i\tan^{-1}\dfrac{y}{x}\end{array}\right]$$

or $$\psi = -m\tan^{-1}\left[\frac{r^2\sin(2\theta-\theta) + a^2\sin\theta}{r^2\cos(2\theta-\theta) - a^2\cos\theta}\right]$$

or $$\psi = -m\tan^{-1}\left[\frac{(r^2+a^2)\sin\theta}{(r^2-a^2)\cos\theta}\right] \quad \text{... (1)}$$

$\psi = -m(\pi - \alpha)$ gives the stream lines which make angle α at A. By (1) and (2),

$$-m(\pi - \alpha) = -m\tan^{-1}\left[\frac{(r^2+a^2)\sin\theta}{(r^2-a^2)\cos\theta}\right]$$

or $$-\tan\alpha = \frac{(r^2+a^2)\sin\theta}{(r^2-a^2)\cos\theta}$$

or $$-\sin\alpha\,.\cos\theta\,.(r^2-a^2) = (r^2+a^2)\sin\theta\,.\cos\alpha$$

or $$r^2\sin(\alpha+\theta) = a^2\sin(\alpha-\theta).$$

Remark: To justify the image system of the above problem :

Let OA be a bounding radius. Consider a source $+m$ at A, sink $-m$ at O. Take an image source $+m$ at A' s.t.

$OA = OA' = a$. Then complex potential W is given by,

$w = -m\log(z-a) + m\log(z-a) - m\log(z-a)$

this gives

$$\psi = -m\tan^{-1}\left[\frac{(r^2+a^2)\sin\theta}{(r^2-a^2)\cos\theta}\right] \text{[By equation (1) of the above solution]}$$

or, $$\psi = -m\tan^{-1}\left[\frac{(r^2+a^2)}{(r^2-a^2)}\tan\theta\right] \quad \text{... (1)}$$

By (1), at $r = a$, $\psi = -m\pi/2 =$ const.

and at $\theta = \pi/2$, $\psi = -m\pi/2 =$ const

Also when $\theta = 0$, $\psi = 0 =$ const.

OA is stream line when $\theta = 0$

OB is stream line when $\theta = \pi/2$

and arc AB is stream line when $r = a$

Thus the image system for the fluid motion bounded by quandrantal arc $OABO$ due to sink $-m$ at O, source $+m$ at A would be a source $+m$ at A'.

Problem 3. *Within a circular boundary of radius a there is two dimensional liquid motion due to a source producing liquid at the rate m, at a distance f from the centre and an equal sink at the centre. Find the velocity potential and show that the resultant of the pressure on the boundary is* $\rho m^2 f^3/[2a^2\pi(a^2-f^2)]$. *Deduce as a limit, the velocity potential due to a doublet at the centre.*

Solution: Liquid is generated due to a source at the rate m at the point A where $OA = f$. Let k be the strength of the source, then by def. $2\pi k = m$ or $k = m/2\pi$. the object system consists of (i) a source $+k$ at A (ii) sink $-k$ at O. The image system consists of (i) source $+k$ at A', the inverse of A so that $OA.OA' = a^2$ or $OA' = f' = a^2/f$ and a sink $-k$ at O.

(ii) sink k at infinity, the inverse point O and a source $+k$ at O.

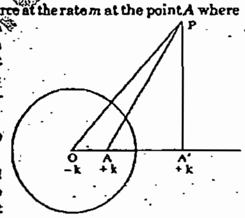


Fig. 3.16.

Source $+k$ and sink $-k$ both at O cancel each other. Finally, the object and its image system consists of source $+k$ at A, source $+k$ at A', sink $-k$ at O. Sink at infinity is neglected, since it has no effect on fluid motion.

The complex potential due to object system with rigid boundary is equivalent to complex potential due to object sysem and its image system with no rigid boundary. Hence w is given by

$$w = -k\log(z-f) - k\log(z-f') + k\log z. \quad \text{... (1)}$$

Equating real parts from both sides,

$$\phi = -k\log|z-f| - k\log|z-f'| + k\log|z|$$

$$= -k\log AP - k\log A'P + k\log OP.$$

or $$\phi = -k\log\frac{AP\,.\,A'P}{OP.}$$

Second Part: By (1), $\dfrac{dw}{dz} = -\dfrac{k}{z-f} - \dfrac{k}{z-f'} + \dfrac{k}{z}$

$$\frac{1}{k^2}\left(\frac{dw}{dz}\right)^2 = \frac{1}{(z-f)^2} + \frac{1}{(z-f')^2} + \frac{1}{z^2} + 2\left[\frac{1}{(z-f)(z-f')} - \frac{1}{z(z-f)} - \frac{1}{z(z-f')}\right]$$

The poles inside the boundary c of the circle are $z = 0$ and $z = f$. Hence the sum of the residues of the function

$$\frac{1}{k^2}\left(\frac{dw}{dz}\right) \text{ at } z = 0$$

and $z = f$ is obtained by adding the coefficients of $\frac{1}{z}$ and $\frac{1}{z-f}$.

$$\text{Sum of residues} = \frac{2}{f-f'} - \frac{2}{f} + \frac{2}{f} + \frac{2}{f} + \frac{2}{f'} = \frac{2f}{(f-f')f'}$$

By Cauchy's residues theorem,

$$\int_c \frac{1}{k^2}\left(\frac{dw}{dz}\right)^2 dz = 2\pi i\,[\text{Sum of residues}]$$

or $$\int_c\left(\frac{dw}{dz}\right)^2 dz = \frac{4\pi f k^2 i}{(f-f')f'}$$

By Blausius theorem,

$$X - iY = \frac{i\rho}{2}\int_c\left(\frac{dw}{dz}\right)^2 dz = \frac{i\rho}{2}\cdot\frac{4\pi f k^2 i}{(f-f')f'}$$

$$= \frac{2\pi\rho f k^2}{\frac{a^2}{f}\left(\frac{a^2}{f} - f\right)} = \frac{2\pi\rho f^3}{a^2(a^2-f^2)}\cdot\frac{m^2}{4\pi^2}$$

or $$X - iY = \rho f^3 m^2/2a^2\pi(a^2-f^2).$$

Equating real and imaginary parts, we get,

$$X = \rho f^3 m^2/2\pi a^2(a^2-f^2), Y = 0.$$

Resultant pressure on the boundary

$$= (X^2+Y^2)^{1/2} = \rho m^2 f^3/2\pi a^2(a^2-f^2).$$

Third Part: To deduce velocity potential due to a doublet at O as a limit.

If we take limit as $f \to \infty$, then $A \to \infty$ and hence neglected. Also A' comes near the point O. We have already a sink $-k$ at O and we have brought a source near it. This combination forms a doublet of strength μ where $\mu = k.(a^2/f)$ as $f \to \infty$.

Now w becomes

$$w = -k\log(z-f') + k\log z \text{ as source } +k \text{ at } A \text{ is neglected.}$$

or $$w = -k\log\frac{1}{z}\left(z - \frac{a^2}{f}\right) = -k\log\left(1 - \frac{a^2}{fz}\right)$$

$$= k\left[\frac{a^2}{fz} + \frac{1}{2}\left(\frac{a^2}{fz}\right)^2 + \ldots\right] \text{For } -\log(1-x) = x + \frac{x^2}{2} + \ldots$$

$$= \frac{ka^2}{fz} \text{ neglecting higher degree terms}$$

or $$\phi + i\psi = \frac{ma^2}{2\pi f r e^{i\theta}} = \frac{ma^2 e^{-i\theta}}{2\pi r f}$$

This $\Rightarrow \phi = \dfrac{ma^2}{2\pi f r}\cos\theta.$

This is the required velocity potential.

Remark. By (1),

or $$w = -k\log\left(1 - \frac{f}{z}\right) - k\log\left(z - \frac{a^2}{f}\right)$$

$$= -k\log\left(1 - \frac{f}{z}\right) - k\log\left(-\frac{f}{a^2}\right)\left(1 - \frac{fz}{a^2}\right)$$

$$= -k\log\left(1 - \frac{f}{z}\right) - k\log\left(1 - \frac{fz}{a^2}\right) \quad \text{neglecting constant.}$$

$$= k\left[-\log\left(1 - \frac{f}{z}\right) - \log\left(1 - \frac{fz}{a^2}\right)\right]$$

$$= k\left[\left(\frac{f}{z} + \ldots\right) + \left(\frac{fz}{a^2} + \ldots\right)\right]$$

or $$w = k\left[\frac{f}{z} + \frac{fz}{a^2}\right]$$

If we make $f \to 0$ so that $\dfrac{a^2}{f} \to \infty$, then we get a doublet at the centre and its strength $\mu = kf$. Then $w = \dfrac{\mu}{z} + \dfrac{\mu z}{a^2}$.

Equating real parts, $\phi = \mu\left(\dfrac{1}{r} + \dfrac{r}{a^2}\right)\cos\theta.$

Thus we get two answers for the two limits namely $f \to 0$ and $f \to \infty$.

Problem 4. *A source of fluid situated in space of two dimensions is of such strength that $2\pi\rho\mu$ represents the mass of fluid of density ρ emitted per unit of time. Show that the force necessory to hold a circular disc at rest in the plane of source is*

$$2\pi\rho\mu^2 a^2/r(r^2-a^2),$$

where a is the radius of the disc and r the distance of the source from its centre. In what direction is the disc urged by the pressure?

Solution: Let X and Y be the components of the required force. Then we have to prove that

$$\sqrt{(X^2+Y^2)} = \frac{2\pi\rho\mu^2 a^2}{r(r^2-a^2)}.$$

This $\Rightarrow r > a$. By Blausius theorem,

$$X - iY = \frac{i\rho}{2}\int_c\left(\frac{dw}{dz}\right)^2 dz,$$

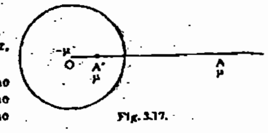


Fig. 3.17.

where c represents the boundary of the disc. Since $2\pi\rho\mu$ represents the mass of the fluid emitted at A hence strength of the sources is μ. The image of source $+\mu$ at A $(OA = r)$ is a source $+\mu$ at the inverse point A' s.t. $OA.OA' = a^2$ and sink $-\mu$ at O.

Then $\quad OA' = a^2/r = r'$, say.

The complex potential due to object system with rigid boundary is equivalent to the complex potential due to the object system and its image system with no rigid boundary. Hence

$$w = -\mu \log(z-r) - \mu\log(z-r') + \mu\log(z-0)$$

$$\frac{dw}{dz} = -\mu\left[\frac{1}{z-r} + \frac{1}{z-r'} - \frac{1}{z}\right]$$

$$\frac{1}{\mu^2}\left(\frac{dw}{dz}\right)^2 = \frac{1}{(z-r)^2} + \frac{1}{(z-r')^2} + \frac{1}{z^2} + \frac{2}{(z-r)(z-r')} - \frac{2}{z(z-r')} - \frac{2}{z(z-r)}$$

The function $\frac{1}{\mu^2}\left(\frac{dw}{dz}\right)^2$ has poles $z = 0$ and $z = r'$ within c. Residue at $z = 0$ is the sum of coefficients of $\frac{1}{z}$ which is equal to

$$\left[-\frac{2}{z-r'} - \frac{2}{z-r}\right]_{z=0} = 2\left(\frac{1}{r'} + \frac{1}{r}\right)$$

Residue at $z = r'$ = sum of coefficients of $1/(z-r')$

$$= \left[\frac{2}{z-r} - \frac{2}{z}\right]_{z=r'} = \frac{2}{r'-r} - \frac{2}{r'}.$$

Sum of residues at $z = 0$ and $z = r'$

$$= \frac{2}{r'-r} - \frac{2}{r'} + \frac{2}{r'} + \frac{2}{r} = \frac{2r'}{(r'-r)r} = \frac{2a^2}{(a^2-r^2)r}.$$

By Cauchy's residues theorem,

$$\int_c \frac{1}{\mu^2}\left(\frac{dw}{dz}\right)^2 dz = 2\pi i.\ \text{Sum of residues within } C$$

$$= 2\pi i.\frac{2a^2}{(a^2-r^2)r}.$$

We have seen that

$$X - iY = \frac{i\rho}{2}\int_c\left(\frac{dw}{dz}\right)^2 dz$$

$$= \frac{i\rho}{2}.\frac{2\pi i\, 2a^2}{(a^2-r^2)}\frac{\mu^2}{r} = \frac{2a^2\pi\mu^2\rho}{r(r^2-a^2)}.$$

This $\Rightarrow X = \dfrac{2a^2\pi\mu^2\rho}{r(r^2-a^2)}, Y = 0$

$$\Rightarrow \quad \sqrt{(X^2+Y^2)} = \frac{2\pi a^2\mu^2\rho}{r(r^2-a^2)}$$

This also declares that the force is purely along $\overrightarrow{OA}$, the disc will be urged to move along OA. Also the cylinder is attracted towards the source, and sketch of the stream lines reveals that the pressure is greater on the opposite side of the disc than that of the source.

Remark: The above problem can be expressed as:

Show that the force per unit length exerted on a circular cylinder, radius a, due to a source of strength m, at a distance c from the axis is

$$2\pi\rho m^2 a^2/c(c^2-a^2).$$

Problem 5. *What arrangement of sources and sinks will give rise to the function $w = \log\left(z - \frac{a^2}{z}\right)$? Draw a rough sketch of the stream lines in this case and prove that two of them sub-divide into the circle $r = a$ and axis of y.*

Solution: Given $w = \log\left(z - \dfrac{a^2}{z}\right)$

This $\Rightarrow w = \log\left(\dfrac{z^2-a^2}{z}\right) = \log\dfrac{(z-a)(z+a)}{z}$

or $\quad w = \log(z-a) + \log(z+a) - \log(z-0)$ ... (1)

This shows that the given arrangement consists of two sinks each of strength -1 at $z = a$ and $z = -a$, and a source of strength $+1$ at the origin.

Second Part: To determine stream lines.

By (1),

$$\phi + i\psi = \log(x - a + iy) + \log(x + a + iy) - \log(x + iy).$$

Equating imaginary parts,

$$\psi = \tan^{-1}\frac{y}{x-a} + \tan^{-1}\frac{y}{x+a} - \tan^{-1}\frac{y}{x}$$

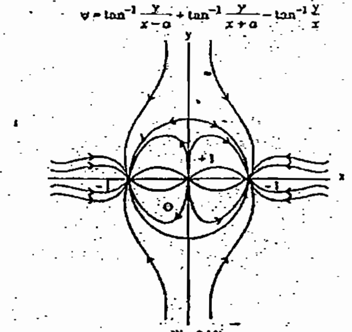


Fig. 3.18.

$$= \tan^{-1}\left[\frac{y/(x-a) + y/(x+a)}{1 - y/(x-a).y/(x+a)}\right] - \tan^{-1}\frac{y}{x}$$

$$= \tan^{-1}\frac{2xy}{x^2-a^2-y^2} - \tan^{-1}\frac{y}{x}$$

$$= \tan^{-1}\left[\frac{2xy/(x^2-a^2-y^2) - (y/x)}{1 + (y/x).2xy/(x^2-a^2-y^2)}\right]$$

$$= \tan^{-1}\frac{y(x^2+y^2+a^2)}{x(x^2+y^2-a^2)}$$

Stream lines are given by ψ = const., *i.e.*,

$$\tan^{-1}\frac{y(x^2+y^2+a^2)}{x(x^2+y^2-a^2)} = \text{const.},$$

or $\quad \dfrac{y(x^2+y^2+a^2)}{x(x^2+y^2-a^2)} = \text{const.},$ ... (2)

If const. = 0, then (2) $\Rightarrow y(x^2+y^2+a^2) = 0$

$\Rightarrow y = 0$, for $x^2+y^2+a^2 \neq 0$.

If const. = ∞, then (2) $\Rightarrow x(x^2+y^2-a^2) = 0 \Rightarrow x = 0, r^2 = a^2$

$\Rightarrow x = 0, r = a$

But $x = 0$ represents y-axis and $r = a$ represents circle with radius a and centre at the origin. Thus we see that particular stream lines are y-axis and the circle $r = a$.

A rough sketch of the stream lines is as given in figure 3.18.

Similar Problem. *What arrangement of sources and sinks will give rise to the function $w = \log\left(z - \frac{1}{z}\right)$? Draw a rough sketch of stream lines in this case and prove that two of them subdivide into the circle $r = 1$ and axis of y.*

Hint. On replacing a by 1 in the above problem, we get this problem.

Problem 6. *In the case of two dimensional fluid motion produced by a source of strength μ placed at a point S outside a rigid circular disc of radius a whose centre is O, show that velocity of slip of the fluid in contact with the disc is greatest at the points where the lines joining S to the ends of the diameter at right angles to OS cut the circle, and prove that its magnitude at these points is* $2\mu r/(r^2-a^2)$, where $OS = r$.

Solution. Let S' be the inverse point of S w.r.t. the circle so that $OS.OS' = a^2$ or $OS' = a^2/r = r'$.

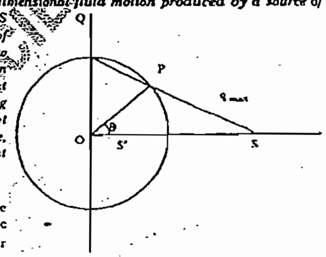


Fig. 3.19.

The image system consists of source $+\mu$ at S' and sink $-\mu$ at O. Take O as origin and OS as real axis, then the equation of complex potential is given by

$$w = -\mu\log(z-r) - \mu\log(z-r') + \mu\log(z-0)$$

$$\frac{dw}{dz} = -\frac{\mu}{z-r} - \frac{\mu}{z-r'} + \frac{\mu}{z}$$

$$q = \left|\frac{dw}{dz}\right| = \left|\frac{-\mu}{z-r} - \frac{\mu}{z-r'} + \frac{\mu}{z}\right| = \mu\left|\frac{z^2-r'r}{(z-r)(z-r')z}\right|$$

$$= \mu\left|\frac{z^2-a^2}{(z-r)\{z-(a^2/r)\}z}\right|.$$

In order to determine velocity at any point on the boundary of the disc, we shall put $z = a\,e^{i\theta}$.

Then $\quad q = \mu\left|\dfrac{a^2e^{2i\theta} - a^2}{(ae^{i\theta}-r)\,[(ae^{i\theta} - (a^2/r)]\,a\,e^{i\theta}}\right|$

or $\quad q = \mu r\left\{\dfrac{(\cos 2\theta - 1)^2 + (\sin 2\theta)^2}{\{(a\cos\theta - r)^2 + a^2\sin^2\theta\}\,\{(r\cos\theta - a)^2 + r^2\sin^2\theta\}}\right\}^{1/2}$

or $\quad q = \mu\dfrac{2r\sin\theta}{(a^2+r^2-2ar\cos\theta)^{1/2}(a^2+r^2-2ra\cos\theta)^{1/2}}$

or $\quad q = \dfrac{2\mu r\sin\theta}{a^2+r^2-2ar\cos\theta}$. ... (1)

For q to be maximum, $\dfrac{dq}{d\theta} = 0$, this $\Rightarrow$

$$2\mu r\left\{\frac{\cos\theta(a^2+r^2-2ar\cos\theta) - 2ar\sin^2\theta}{(a^2+r^2-2ar\cos\theta)^2}\right\} = 0$$

or $\quad (a^2+r^2-2ar\cos\theta)\cos\theta - 2ar\sin^2\theta = 0$

or $\quad \cos\theta = 2ar/(a^2+r^2)$ ... (2)

The value of θ, given by (2), gives maximum velocity.

$$(2) \Rightarrow \sin\theta = (r^2-a^2)/(r^2+a^2)$$

By (1), $q_{max} = 2\mu r\left\{\dfrac{(r^2-a^2)/(r^2+a^2)}{a^2+r^2-2ar\,\{2ar/(a^2+r^2)\}}\right\}$

$$= \frac{2\mu r(r^2-a^2)}{(r^2-a^2)^2} = \frac{2\mu r}{r^2-a^2}$$

or $\quad q_{max} = 2\mu r/(r^2-a^2)$.

The velocity will be along the direction of tangent to the boundary and will be equal to the velocity of slip as the boundary of the disc is a stream line.

Remark. This result is also expressible as

$$q_{max} = \frac{2\mu.OS}{OS^2-a^2}.$$

Problem 7. *Between the fixed boundaries $\theta = \pi/4$ and $\theta = -\pi/4$, there is a two-dimensional fluid motion due to a source of strength m at the point $(r = a, \theta = 0)$ and an equal sink at the $(r = b, \theta = 0)$. Show that the stream function is*

$$-m \tan^{-1}\left[\frac{r^4 (a^4 - b^4) \sin 4\theta}{r^8 - r^4 (a^4 - b^4) \cos 4\theta + a^4 b^4}\right]$$

and that the velocity at (r, θ) is

$$\frac{4m (a^4 - b^4) r^3}{(r^8 - 2a^4 r^4 \cos 4\theta + a^8)^{1/2} (r^8 - 2b^4 r^4 \cos 4\theta + b^8)^{1/2}}.$$

Solution. Consider the transformation $\zeta = z^2$ which maps points from z-plane to ζ-plane. Let $z = r e^{i\theta}$ and $\zeta = R e^{i\delta}$, then

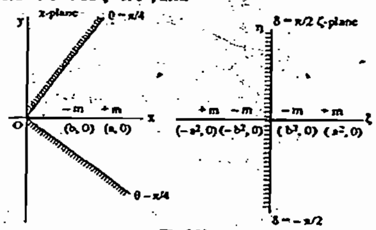


Fig. 3.20.

$$\zeta = z^2 \Rightarrow R e^{i\delta} = r^2 e^{i2\theta} \Rightarrow R = r^2, \delta = 2\theta$$

Also $\theta = \pm \pi/4$ so that $\delta = \pm \pi/2$, *i.e.*, η-axis.

By this transformation points $(a, 0)$ and $(b, 0)$ in z-plane are mapped on $(a^2, 0)$ and $(b^2, 0)$ in ζ-plane. The images of $+m$ at $(a^2, 0)$ and $-m$ at $(b^2, 0)$ in ζ-plane w.r.t. η-axis are $+m$ at $(-a^2, 0)$ and $-m$ at $(-b^2, 0)$, respectively.

The complex potential due to object system with rigid boundary is equivalent to the complex potential due to object system and its image system without rigid boundary. This $\Rightarrow$

$$w = -m \log (\zeta - a^2) - m \log (\zeta + a^2) + m \log (\zeta - b^2) + m \log (\zeta + b^2)$$

$$= -m \log (\zeta^2 - a^4) + m \log (\zeta^2 - b^4)$$

or $$w = -\log (z^4 - a^4) + m \log (z^4 - b^4) \qquad \ldots (1)$$

$$= -m \log (r^4 e^{i4\theta} - a^4) + m \log (r^4 e^{i4\theta} - b^4)$$

Equating imaginary parts,

$$\psi = -m \left[\tan^{-1}\left(\frac{r^4 \sin 4\theta}{r^4 \cos 4\theta - a^4}\right) - \tan^{-1}\left(\frac{r^4 \sin 4\theta}{r^4 \cos 4\theta - b^4}\right)\right]$$

Since $\quad \tan^{-1} x - \tan^{-1} y = \tan^{-1} (x - y)/(1 + xy)$

Hence $$\psi = -m \tan^{-1}\left[\frac{r^4 (a^4 - b^4) \sin 4\theta}{r^8 - r^4 (a^4 + b^4) \cos 4\theta + a^4 b^4}\right]$$

This completes the first part.

By (1),

$$\frac{dw}{dz} = -\frac{m (4z^3)}{z^4 - a^4} + \frac{4mz^3}{z^4 - b^4}$$

$$= -4mz^3 \left[\frac{a^4 - b^4}{(z^4 - a^4)(z^4 - b^4)}\right]$$

$$q = \left|\frac{dw}{dz}\right| = \frac{4mr^3 (a^4 - b^4)}{|(r^4 e^{i4\theta} - a^4)(r^4 e^{i4\theta} - b^4)|}$$

or $$q = \frac{4m r^3 (a^4 - b^4)}{[(r^8 + a^8 - 2a^4 r^4 \cos 4\theta)(r^8 + b^8 - 2b^4 r^4 \cos 4\theta)]^{1/2}}$$

This completes the problem.

Problem 8. *Between the fixed boundaries $\theta = \pi/6$ and $\theta = -\pi/6$, there is a two dimensional liquid motion due to a source at the point $r = c$, $\theta = \alpha$, and a sink at the origin, absorbing water at the same rate as the source produces it. Find the stream function and show that one of the stream lines is a part of the curve*

$$r^3 \sin 3\alpha = c^3 \sin 3\theta.$$

Solution. Consider the map $\zeta = z^3$ from z-plane to so ζ-plane. Let $z = r e^{i\theta}$, $\zeta = R e^{i\beta}$. Then $R e^{i\beta} = r^3 e^{i3\theta}$, this $\Rightarrow$

$$R = r^3 \quad \ldots (1), \qquad \beta = 3\theta \quad \ldots (2)$$

By this map the boundaries $\theta = \pm \pi/6$ are mapped on the boundaries $\beta = \pm \pi/2$, *i.e.*, η-axis is the new boundary in ζ-plane. The points (c, α) and $(0, 0)$ in z-plane are mapped respectively on the points $(c^3, 3\alpha)$ and $(0, 0)$ by virtue of (1) and (2). The object system consists of (i) source $+m$ at $(c^3, 3\alpha)$ and (ii) $-m$ at $(0, 0)$. The image system consists of (i) source $+m$ at $(c^3, \pi - 3\alpha)$ and (ii) sink $-m$ at $(0, 0)$ w.r.t. η-axis. The complex potential is given by

$$w = -m \log (\zeta - c^3 e^{i3\alpha}) + m \log (\zeta - 0) - m \log (\zeta - c^3 e^{i(\pi - 3\alpha)}) + m \log (\zeta - 0)$$

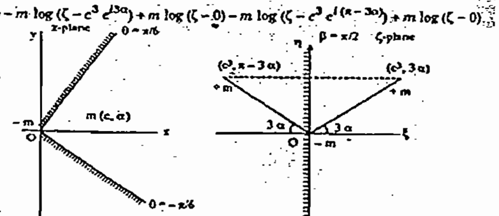


Fig. 3.21.

$$= 2m \log \zeta - m \log (\zeta - c^3 e^{i3\alpha}) - m \log (\zeta + c^3 e^{-i3\alpha})$$

Putting $\quad \zeta = z^3$,

$$w = 2m \log z^3 - m \log (z^3 - c^3 e^{i3\alpha})(z^3 + c^3 e^{-i3\alpha})$$

or $$w = 6m \log z - m \log (z^6 - c^6 - 2i z^3 c^3 \sin 3\alpha)$$

$$= 6m \log r e^{i\theta} - m \log (r^6 e^{i6\theta} - c^6 - 2i c^3 \sin 3\alpha \,.\, r^3 e^{i3\theta})$$

$$= 6m \log r e^{i\theta} - m \log [(r^6 \cos 6\theta - c^6 + 2c^3 r^3 \sin 3\alpha \,.\, \sin 3\theta$$
$$+ i (r^6 \sin 6\theta - 2r^3 c^3 \sin 3\alpha \,.\, \cos 3\theta)]$$

Equating imaginary parts on both sides,

$$\psi = 6m \tan^{-1}\left(\frac{r \sin \theta}{r \cos \theta}\right)$$
$$- m \tan^{-1}\left(\frac{r^6 \sin 6\theta - 2r^3 c^3 \sin 3\alpha \,.\, \cos 3\theta}{r^6 \cos 6\theta - c^6 + 2c^3 r^3 \sin 3\alpha \,.\, \sin 3\theta}\right)$$

Stream lines are given by $\quad \psi = $ const., *i.e.*,

$$6m\theta - m \tan^{-1}\left(\frac{r^6 \sin 6\theta - 2r^3 c^3 \sin 3\alpha \,.\, \cos 3\theta}{r^6 \cos 6\theta - c^6 + 2c^3 r^3 \sin 3\alpha \,.\, \sin 3\theta}\right) = \text{const.}$$

Taking const. = 0, we get particular stream lines as

$$6m\theta - m \tan^{-1}\left(\frac{r^6 \sin 6\theta - 2r^3 c^3 \sin 3\alpha \,.\, \cos 3\theta}{r^6 \cos 6\theta - c^6 + 2c^3 r^3 \sin 3\alpha \,.\, \sin 3\theta}\right) = 0$$

or $$6\theta = \tan^{-1}\left(\frac{r^6 \sin 6\theta - 2r^3 c^3 \sin 3\alpha \,.\, \cos 3\theta}{r^6 \cos 6\theta - c^6 + 2c^3 r^3 \sin 3\alpha \,.\, \sin 3\theta}\right)$$

or $$\sin 6\theta \,.\, (r^6 \cos 6\theta - c^6 + 2c^3 r^3 \sin 3\alpha \,.\, \sin 3\theta)$$
$$= \cos 6\theta \,(r^6 \sin 6\theta - 2c^3 r^3 \sin 3\alpha \,.\, \cos 3\theta)$$

or $$-c^6 \sin 6\theta + 2c^3 r^3 \sin 3\alpha \,.\, \cos (6\theta - 3\theta) = 0$$

or $$2r^3 \sin 3\alpha \,.\, \cos 3\theta - c^3 \sin 6\theta = 0$$

or $$2\cos 3\theta \,[r^3 \sin 3\alpha - c^3 \sin 3\theta] = 0$$

This $$\Rightarrow \cos 3\theta = 0, \qquad \ldots (3)$$

$$r^3 \sin 3\alpha = c^3 \sin 3\theta.$$

By (3), $\theta = \pm \pi/6$ which gives no new stream lines as these are the given stream lines. The other stream line is a part of the curve

$$r^3 \sin 3\alpha = c^3 \sin \theta.$$

Problem 9. *In the case of motion of liquid in a part of a plane bounded by a straight line due to a source in the plane prove that if $m\rho$ is the mass of the liquid (of density ρ) generated at the source per unit of time, the pressure on the length $2l$ of the boundary immediately opposite to the source is less than that on an equal length at a great distance by*

$$\frac{1}{2} \frac{m^2 \rho}{\pi^2}\left(\frac{1}{c} \tan^{-1} \frac{l}{c} - \frac{l}{l^2 + c^2}\right)$$

where c is the distance of the source from the boundary.

Solution. Suppose μ is the strength of the source at P where $OP = c$. Then by def. of strength

$$2\pi \mu \rho = m\rho$$

$$\therefore \quad m/2\pi = \mu.$$

The boundary is y-axis. The image of a source $+\frac{m}{2\pi}$ at $P(c, 0)$ is a source $+m/2\pi$ at $P'(-c, 0)$.

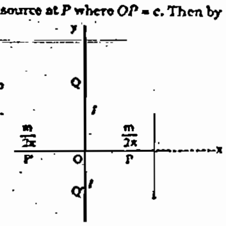


Fig. 3.22.

Now the complex potential is

$$w = -\frac{m}{2\pi} \log (z - c) - \frac{m}{2\pi} \log (z + c)$$

$$= -\frac{m}{2\pi} \log (z^2 - c^2)$$

$$\frac{dw}{dz} = -\frac{m}{2\pi} \cdot \frac{2z}{z^2 - c^2}$$

$$q = \left|\frac{dw}{dz}\right| = \left|\frac{m}{\pi} \cdot \frac{z}{z^2 - c^2}\right|$$

For any point on y-axis, $z = iy$, so that

$$q = \frac{m}{\pi}\left|\frac{z}{z^2 - c^2}\right| = \frac{m}{\pi}\left|\frac{iy}{-y^2 - c^2}\right| = \frac{my}{\pi (y^2 + c^2)}$$

This is the expression for velocity at any point on y-axis. By Bernoulli's equation for steady motion,

$$\frac{p}{\rho} + \frac{1}{2} q^2 = A.$$

Subjecting this to the condition

$$p = p_0 \text{ when } y = \infty, q = 0, \text{ we get } A = p_0/\rho.$$

(Since velocity is negligible at great distance).

Hence $$\frac{p}{\rho} + \frac{1}{2} q^2 = \frac{p_0}{\rho}.$$

Pressure on QQ' $$= \int_{-l}^{l} p \, dy.$$

But $$p - p_0 = -\frac{1}{2} \rho q^2 \Rightarrow \int_{-l}^{l} (p - p_0) \, dy = -\frac{1}{2} \rho \int_{-l}^{l} q^2 \, dy$$

Required difference of pressure

$$= \int_{-l}^{l} (p_0 - p) \, dy = \frac{1}{2} \rho \int_{-l}^{l} \frac{m^2}{\pi^2} \frac{y^2 \, dy}{(y^2 + c^2)^2}.$$

$$= \frac{m^2}{\pi^2}\rho\int_0^l \frac{y^2}{(y^2+c^2)^2}\,dy.$$

[Put $y = c\tan\theta,\ dy = c\sec^2\theta\, d\theta$]

$$= \frac{m^2\rho}{\pi^2}\int_0^{\theta_1}\frac{c^2\tan^2\theta \cdot c}{c^4\sec^4\theta}\sec^2\theta\, d\theta = \frac{m^2\rho}{\pi^2 c}\int_0^{\theta_1}\sin^2\theta\, d\theta, \text{ where } \tan\theta_1 = \frac{l}{c}.$$

$$= \frac{m^2\rho}{2\pi^2 c}\int_0^{\theta_1}(1-\cos 2\theta)\,d\theta = \frac{m^2\rho}{2\pi^2 c}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\theta_1}$$

$$= \frac{m^2\rho}{2\pi^2 c}[\theta_1 - \sin\theta_1\cos\theta_1] = \frac{m^2\rho}{2\pi^2 c}\left[\tan^{-1}\frac{l}{c} - \frac{lc}{l^2+c^2}\right].$$

$$= \frac{m^2\rho}{2\pi^2}\left[\frac{1}{c}\tan^{-1}\frac{l}{c} - \frac{l}{l^2+c^2}\right]$$

Problem 10. *Within a rigid boundary in the form of the circle $(x+a)^2+(y-4a)^2 = 8a^2$, there is liquid motion due to doublet of strength μ at the point $(0, 3a)$ with its axis along the axis of y. Show that velocity potential is*

$$\mu\left[\frac{4(x-3a)}{(x-3a)^2+y^2} + \frac{y-3a}{x^2+(y-3a)^2}\right]$$

Solution. The rigid boundary is a circle given by

$$(x+a)^2+(y-4a)^2 = 8a^2.$$

The centre is $(-a, 4a)$ and radius $= \sqrt{(8a^2)}$.

Object doublet is at $P(0, 3a)$ with its axis along y-axis, CM and PN are perpendiculars on x-axis and CM respectively. Produce CP to meet x-axis at Q. Evidently, $CN = NP = a$ so that $\angle NPC = 45°$ and therefore $\angle CQM = 45°$ so that

$CQ = \sqrt{[(4a)^2 + (4a)^2]} = 4a\sqrt{2}$.

Hence $CM = MQ = 4a$.

Observe that

$CP \cdot CQ = a\sqrt{2} \cdot 4a\sqrt{2}$

$= 8a^2 = (\text{radius})^2$.

Hence Q is the inverse point of P w.r.t the circle. The image of the doublet μ at $P(0, 3a)$ w.r.t circle is a doublet μ' at the inverse point $Q(3a, 0)$ with its axis along x-axis. For object and image doublets make supplementary angles with the line CQ.

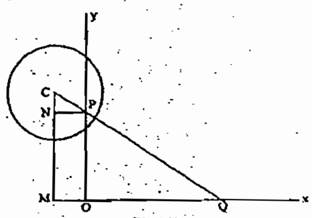


Fig. 3.23.

Here

$$\mu' = \frac{\mu a^2}{f^2} = \mu\frac{8a^2}{CP^2} = \frac{\mu\, 8a^2}{2a^2} = 4\mu.$$

Thus
$$w = \frac{\mu e^{i\pi/2}}{z - i\,3a} + \frac{4\mu e^{i.0}}{z-3a}$$

$$= \frac{\mu i}{x + i(y-3a)} + \frac{4\mu}{(x-3a)+iy}$$

$$= \frac{\mu i\,[x - i(y-3a)]}{x^2+(y-3a)^2} + \frac{4\mu\,[(x-3a)-iy]}{(x-3a)^2+y^2}$$

Equating real parts on both sides,

$$\phi = \mu\left[\frac{y-3x}{x^2+(y-3a)^2} + \frac{4(x-3a)}{(x-3a)^2+y^2}\right]$$

This concludes the problem.

Problem 11. *In the part of an infinite plane bounded by a circular quadrant AB and the production of the radii OA, OB, there is a two dimensional motion due to the production of liquid at A, and its absorption at B, at the uniform rate m. Find the velocity potential of the motion; and show that the fluids which issue from A in the direction making an angle μ with OA follows the path whose polar equation is*

$$r = a\sin^{1/2}2\theta\,[\cot\mu + \sqrt{(\cot^2\mu + \operatorname{cosec}^2 2\theta)}]^{1/2},$$

the positive sign being taken for all the square roots.

Solution. The object system consists of source $+m/2\pi$ at A and $-m/2\pi$ at B. The image of $+(m/2\pi)$ at A w.r.t. circular boundary is a source $+m/2\pi$ at A, the inverse point of A and sink $-m/2\pi$ at O. The image of sink $-m/2\pi$ at B. w.r.t. circle is a sink $-m/2\pi$ at B, the inverse point of B and source $+m/2\pi$ at O. The source $+m/2\pi$ and sink $-m/2\pi$ both at O cancel each other.

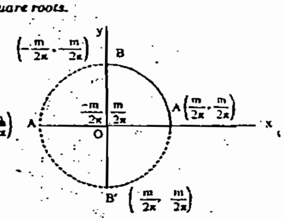


Fig. 3.24.

Image w.r.t. bounding plane.

The image of source $+m/2\pi$ at A is a source $+m/2\pi$ at A' w.r.t. line BB' and image of sink $-m/2\pi$ at B w.r.t. the line AA' is a sink $-m/2\pi$ at B'. Also the images at A and B have their images $+m/2\pi$ and $-m/2\pi$ at A' and B' respectively.

The object and its image system consists of 2 sources of strength $+m/2\pi$ at A, 2 sinks of strength $-m/2\pi$ at B, two sources $+m/2\pi$ at A', two sinks $-m/2\pi$ at B'.

The complex potential due to object system with rigid boundary is equivalent to the complex potential due to object system and its image systems with no rigid boundary. Thus

$$w = -\frac{2m}{2\pi}\log(z-a) - \frac{2m}{2\pi}\log(z+a) + \frac{2m}{2\pi}\log(z-ia) + \frac{2m}{2\pi}\log(z+ia).$$

or
$$w = -\frac{m}{\pi}\log(z-a) - \frac{m}{\pi}\log(z+a) + \frac{m}{\pi}\log(z-ia) + \frac{m}{\pi}\log(z+ia) \quad \ldots(1)$$

Equating real part on both sides.

$$\phi = -\frac{m}{\pi}[\log|z-a| + \log|z+a| - \log|z-ia| - \log|z+ia|]$$

$$= -\frac{m}{\pi}[\log PA + \log PA' - \log PB - \log PB']$$

or
$$\phi = -\frac{m}{\pi}\log\frac{PA \cdot PA'}{PB \cdot PB'}$$

This is the required expression for velocity potential. Again by (1),

$$\phi + i\psi = -\frac{m}{\pi}\log(z^2-a^2) + \frac{m}{\pi}\log(z^2+a^2)$$

or
$$\phi + i\psi = -\frac{m}{\pi}[\log(r^2e^{i2\theta}-a^2) - \log(r^2e^{i2\theta}+a^2)].$$

Equating imaginary parts.

$$\psi = -\frac{m}{\pi}\left[\tan^{-1}\left(\frac{r^2\sin 2\theta}{r^2\cos 2\theta - a^2}\right) - \tan^{-1}\left(\frac{r^2\sin 2\theta}{r^2\cos 2\theta + a^2}\right)\right]$$

$$= -\frac{m}{\pi}\tan^{-1}\left[\frac{2a^2r^2\sin 2\theta}{r^4\cos^2 2\theta - a^4 + r^4\sin^2 2\theta}\right] \quad \ldots(2)$$

For $\tan^{-1}x - \tan^{-1}y = \tan^{-1}[(x-y)/(1+xy)]$

For a particular streamline which leaves A at an angle μ,

$$\psi = -\frac{m}{\pi}\mu. \quad \ldots(3)$$

By (2) and (3),

$$-\frac{m}{\pi}\mu = -\frac{m}{\pi}\tan^{-1}\frac{2a^2r^2\sin 2\theta}{r^4-a^4}$$

This $\Rightarrow \tan\mu = \dfrac{2a^2r^2\sin 2\theta}{r^4-a^4}$

$$\Rightarrow (r^2)^2 - 2a^2r^2\sin 2\theta\cot\mu - a^4 = 0.$$

This is quadratic in r^2.

Hence $r^2 = \dfrac{2a^2\sin 2\theta\cot\mu \pm \sqrt{(4a^4\sin^2 2\theta\cot^2\mu + 4a^4)}}{2}$

Taking positive radical sign,

$$r = [a^2\sin 2\theta\cot\mu + a^2\sqrt{(\sin^2 2\theta\cot^2\mu + 1)}]^{1/2}.$$

or
$$r = a(\sin 2\theta)^{1/2}[\cot\mu + \sqrt{(\cot^2\mu + \operatorname{cosec}^2 2\theta)}]^{1/2}.$$

This is the required path.

Problem 12. *Two sources, each of strength m, are placed at the points $(-a, 0)$ and $(a, 0)$ and a sink of strength 2m is placed at the origin. Show that the stream lines are curves* (IAS-2009)

$$(x^2+y^2)^2 = a^2(x^2-y^2+\lambda xy), \text{ where } \lambda \text{ is a parameter.}$$

Show also that the fluid speed at any point is $2ma^2/r_1r_2r_3$ where r_1, r_2, r_3 are respectively the distances of the point from the source and the sink.

Solution. The complex potential at any point $P(z)$ is given by

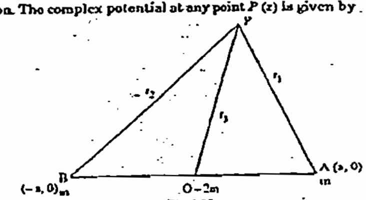


Fig. 3.25.

$$w = -m\log(z-a) - m\log(z+a) + 2m\log(z-0)$$

or
$$w = -m\log(z^2-a^2) + m\log z^2 \quad \ldots(1)$$

or
$$\phi + i\psi = -m\log(x^2-a^2-y^2+2ixy) + m\log(x^2-y^2+2ixy).$$

Equating imaginary parts,

$$\psi = -m\tan^{-1}\frac{2xy}{x^2-a^2-y^2} + m\tan^{-1}\frac{2xy}{x^2-y^2}$$

$$= -m\tan^{-1}\frac{2a^2xy}{(x^2-y^2)(x^2-a^2-y^2)+4x^2y^2}$$

Stream lines are given by ψ = const., *i.e.*,

$$-m\tan^{-1}\frac{2a^2xy}{(x^2-y^2)(x^2-a^2-y^2)+4x^2y^2} = -m\tan^{-1}\left(\frac{2}{\lambda}\right) \text{ say}$$

or
$$\lambda a^2xy = (x^2-y^2)^2 - a^2(x^2-y^2) + 4x^2y^2$$

or
$$\lambda a^2xy = (x^2+y^2)^2 - a^2(x^2-y^2)$$

or
$$(x^2+y^2)^2 = a^2(x^2-y^2+\lambda xy) \quad \text{where } \lambda \text{ is a variable parameter.}$$

This completes the first part of the problem.

$$\text{Flow speed} = \left|\frac{dw}{dz}\right| = \left|-\frac{2mz}{z^2-a^2} + \frac{2mz}{z^2}\right| = \frac{2ma^2}{|z(z^2-a^2)|}$$

$$= \frac{2ma^2}{|z|\,.\,|z-a|\,.\,|z+a|} = \frac{2ma^2}{r_1r_2r_3}.$$

This concludes the problem.

Problem 13. *The space on one side of an infinite plane wall $y = 0$ is filled with inviscid, incompressible fluid, moving at infinity with velocity U in the direction of x-axis. The motion of the fluid is wholly two dimensional in xy-plane. A doublet of strength μ is at a distance a from the wall and the points in the negative direction of x-axis. Show that if $\mu < 4a^2 U$, the pressure of the fluid on the wall is maximum at points distant $a\sqrt{3}$ from O, the foot of the perpendicular from the doublet on the wall and is a minimum also.*

If $\mu = 4a^2 U$, find points where the velocity of the fluid is zero and show that stream lines include the circle.

$$x^2 + (y-a)^2 = 4a^2.$$

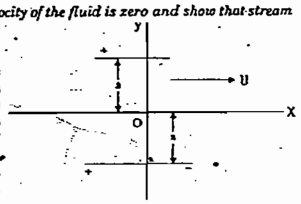


Fig. 3.26.

Solution. Since the points of the doublet are in the negative direction of x-axis so that the doublet makes an angle π with x-axis. Image of the given doublet is an equal doublet similarly oriented at $z = -ia$.

The system consists object doublet, image doublet and stream with velocity U parallel to x-axis. Hence

$$w = \frac{\mu e^{i\pi}}{z - ia} + \frac{\mu e^{i\pi}}{z + ia} - Uz$$

$$= -\frac{\mu}{z - ia} - \frac{\mu}{z + ia} - Uz$$

$$= -\frac{2\mu z}{z^2 + a^2} - Uz$$

$$-\frac{dw}{dz} = U + \frac{2\mu}{(z^2+a^2)^2}[z^2 + a^2 - 2z^2].$$

or $$-\frac{dw}{dz} = U + \frac{2\mu(a^2 - z^2)}{(z^2+a^2)^2} \qquad \text{... (1)}$$

$$\left| -\frac{dw}{dz} \right| = q = \left| U + \frac{2\mu(a^2 - z^2)}{(a^2 + z^2)^2} \right|.$$

For any point on the wall, $z = x$ so that,

$$q = U + \frac{2\mu(a^2 - x^2)}{(x^2 + a^2)^2}.$$

This $\Rightarrow q^2 - U^2 = \frac{4\mu^2(a^2 - x^2)^2}{(x^2+a^2)^4} + \frac{4\mu U(a^2 - x^2)}{(x^2+a^2)^2}. \qquad \text{... (2)}$

To determine pressure at any point on the wall. By Bernoulli's equation for steady motion, $\frac{p}{\rho} + \frac{1}{2}q^2 = C$. Subjecting this to the condition $p = \Pi, q = U$ where $z \to \infty$, so that $\frac{\Pi}{\rho} + \frac{1}{2}U^2 = C$.

Thus $\frac{p}{\rho} + \frac{1}{2}q^2 = \frac{\Pi}{\rho} + \frac{1}{2}U^2$ or $\frac{1}{2}(q^2 - U^2) = \frac{\Pi - p}{\rho}$.

or $$\frac{\Pi - p}{\rho} = \frac{2\mu^2(a^2 - x^2)^2}{(a^2 + x^2)^4} + \frac{2\mu U(a^2 - x^2)}{(a^2+x^2)^2}, \text{ using (2).}$$

$$-\frac{1}{\rho}\frac{dp}{dx} = -\frac{8\mu^2 x(a^2 - x^2)}{(a^2+x^2)^4} - \frac{16\mu^2(a^2 - x^2)^2 \cdot x}{(a^2+x^2)^5} + 2\mu U\left[\frac{-2x}{(a^2+x^2)^2} - \frac{4x(a^2 - x^2)}{(a^2+x^2)^3}\right]$$

$$= \frac{8\mu^2 x(a^2 - x^2)}{(a^2+x^2)^5}\, a^2 + x^2 + 2(a^2 - x^2) - \frac{4\mu U x}{(a^2+x^2)^3}\, a^2 + x^2 + 2(a^2 - x^2)$$

or $$\frac{1}{\rho}\frac{dp}{dx} = \frac{4\mu x(3a^2 - x^2)}{(a^2+x^2)^5}\left[2\mu(a^2 - x^2) + U(a^2 + x^2)^2\right]$$

For extremum values of p, $\frac{dp}{dx} = 0$, this $\Rightarrow x(3a^2 - x^2) = 0$ so that $x = 0, \pm a\sqrt{3}$.

Thus, if $\mu < 4a^2 U$, then $\frac{d^2p}{dx^2} < 0$ so that p is maximum where $z = a\sqrt{3}$. Again, if $\mu < 4a^2 U$, then $d^2p/dx^2 > 0$ where $z = 0$ so that p is minimum. Consider the case in which $\mu = 4a^2 U$.

Let the fluid velocity $= 0$, so that $\frac{dw}{dz} = 0$, then (1) $\Rightarrow$

$$U + \frac{2 \cdot 4a^2 U(a^2 - z^2)}{(a^2 + z^2)^2} = 0$$

or $$(z^2 + a^2)^2 + 8a^2(a^2 - z^2) = 0.$$

On the wall this becomes,

$$(x^2 + a^2)^2 + 8a^2(a^2 - x^2) = 0$$

or $$x^4 - 6a^2x^2 + 9a^4 = 0 \text{ or } (x^2 - 3a^2)^2 = 0 \text{ or } x = \pm a\sqrt{3}$$

Ans. $(\pm a\sqrt{3}, 0)$ are the points where velocity vanishes.

To determine stream lines.

We have $w = -\frac{2\mu z}{z^2 + a^2} - Uz$

or $$\phi + i\psi = -\frac{2 \cdot 4a^2 U(x+iy)(x^2 + a^2 - y^2 - 2ixy)}{(x^2 + a^2 - y^2)^2 + 4x^2y^2} - Uz$$

or $$-\psi = \frac{8a^2 U[-2x^2y + y(x^2 + a^2 - y^2)]}{(x^2 + a^2 - y^2)^2 + 4x^2y^2} + Uy$$

Stream lines are given by $\psi =$ const. Take const. $= 0$ Then stream lines are given by $\psi = 0$, i.e. $\frac{8a^2y(-2x^2 + x^2 + a^2 - y^2)}{(x^2 + a^2 - y^2)^2 + 4x^2y^2} + y = 0$

or $$8a^2[a^2 - (x^2 + y^2)] + (x^2 - y^2)^2 + a^4 + 2a^2(x^2 - y^2) + 4x^2y^2 = 0$$

or $$9a^4 - 8a^2(x^2 + y^2) + (x^2 + y^2)^2 + 2a^2(x^2 - y^2) = 0$$

or $$(x^2 + y^2)^2 - 6a^2x^2 - 10a^2y^2 + 9a^4 = 0$$

or $$(x^2 + y^2 - 3a^2)^2 - 4a^2y^2 = 0$$

or $$(x^2 + y^2 - 3a^2 - 2ay)(x^2 + y^2 - 3a^2 + 2ay) = 0.$$

This includes the circle $x^2 + y^2 - 2ay - 3a^2 = 0$.

i.e., $$x^2 + (y-a)^2 = 4a^2.$$

Problem 14. *Find the lines of flow in two dimensional fluid motion given by*

$$\phi + i\psi = -\frac{n}{2}(x + iy)^2 e^{2int}.$$

Prove or verify that the paths of the particles of the fluid (in polar co-ordinates) may be obtained by eliminating t from the equations

$$r\cos(nt + \theta) - x_0 = r\sin(nt + \theta) - y_0 = nt(x_0 - y_0).$$

Solution. Write $x + iy = re^{i\theta}$, it is given that

$$\phi + i\psi = -\frac{n}{2}(x + iy)^2 e^{2int} = -\frac{n}{2}r^2 e^{i2\theta} \cdot e^{i2nt}$$

or $$\phi + i\psi = -\frac{1}{2}nr^2 e^{i2(nt+\theta)}.$$

This $\Rightarrow \phi = -\frac{nr^2}{2}\cos 2(nt + \theta)$, $\psi = -\frac{nr^2}{2}\sin 2(nt + \theta)$.

Lines of low are given by $\psi =$ const., i.e.,

$$-\frac{nr^2}{2}\sin 2(nt + \theta) = \text{const.}$$

i.e., $$r^2 \sin 2(nt + \theta) = \text{const}$$

By def., $\dot{r} = -\frac{\partial\phi}{\partial r} = nr\cos 2(nt + \theta)$

$$r\dot{\theta} = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{1}{r}\left[\frac{nr^2}{2}\sin 2(nt + \theta) \cdot 2\right]$$

or $$\dot{\theta} = -n r \sin 2(nt + \theta)$$

$$\frac{d}{dt}[r\cos(nt + \theta)] = \dot{r}\cos(nt + \theta) - (n + \dot{\theta})r\sin(nt + \theta)$$

$$= nr\cos 2(nt + \theta)\cos(nt + \theta)[n - n\sin 2(nt + \theta)]r\sin(nt + \theta)$$

$$= nr[\cos 2\{(nt + \theta) - (nt + \theta)\} - \sin(nt + \theta)]$$

$$= nr[\cos(nt + \theta) - \sin(nt + \theta)].$$

Similarly we can show that

$$\frac{d}{dt}[r\sin(nt + \theta)] = nr[\cos(nt + \theta) - \sin(nt + \theta)].$$

Hence we see that

$$\frac{d}{dt}[r\cos(nt + \theta)] = \frac{d}{dt}[r\sin(nt + \theta)]$$

$$= nr[\cos(nt + \theta) - \sin(nt + \theta)] \qquad \text{... (1)}$$

This $\Rightarrow \frac{d}{dt}[r\cos(nt + \theta)] = \frac{d}{dt}[r\sin(nt + \theta)]$

Integrating, $r\cos(nt + \theta) - r\sin(nt + \theta) = A$. $\qquad$ ...(2)

Subjecting this to initial condition, when $t = 0$

$$r\cos\theta = x = x_0,\ r\sin\theta = y = y_0. \qquad \text{... (3)}$$

we get $\quad A = x_0 - y_0$

$\therefore \quad r\cos(nt + \theta) - r\sin(nt + \theta) = x_0 - y_0$

Hence, by (1), $\frac{d}{dt}[\cos(nt + \theta)] = \frac{d}{dt}[r\sin(nt + \theta)] = n(x_0 - y_0)$.

Integrating, $r\cos(nt + \theta) = r\sin(nt + \theta) = nt(x_0 - y_0) + B$... (4)

This $\Rightarrow r\cos(nt + \theta) = nt(x_0 - y_0) + B$.

Subjecting this to (3), $\quad x_0 = 0 + B$.

$\therefore \quad r\cos(nt + \theta) = nt(x_0 - y_0) + x_0$

or $\quad r\cos(nt + \theta) - x_0 = nt(x_0 - y_0)$.

Again, (4) gives $r\sin(nt + \theta) - y_0 = nt(x_0 - y_0)$

Combining the last two equations,

$$r\cos(nt + \theta) - x_0 = r\sin(nt + \theta) - y_0 = nt(x_0 - y_0)$$

This concludes the problem.

Problem 15. *Use the method of images to prove that if there be a source m at the point z_0 in a fluid bounded by the lines $\theta = 0$ and $\theta = \pi/3$, the solution is*

$$\phi + i\psi = -m\log[(z^3 - z_0^3)(z^3 - z_0'^3)]. \qquad \text{(IFoS-2008)}$$

where $\quad z_0 = x_0 + iy_0,\ z'_0 = x_0 - iy_0$.

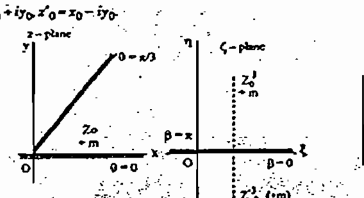


Fig. 3.27.

Solution. Consider the map $\zeta = z^3$ from z-plane to ζ-plane, where $z = re^{i\theta}$, $\zeta = Re^{i\beta}$ so that $Re^{i\beta} = r^3 e^{i3\theta}$. This $\Rightarrow R = r^3$, $\beta = 3\theta$. hence the source $+m$ at z_0 in z– plane is mapped on the source $+m$ at z_0^3 in ζ-plane. Also the boundaries $\theta = 0$, $\theta = \pi/3$ in z-plane become $\beta = 0$, $\beta = \pi$, i.e. ζ-axis.

The image of source $+m$ at z_0^3 w.r.t. ζ-axis is a source $+m$ at $\bar{z}_0^3$, where $z_0' = x_0 - iy_0$.

The complex potential due to object system with rigid boundaries is equivalent to the object and its image system without rigid boundaries. Hence w is given by

$$w = -m \log(\zeta - z_0^3) - m \log(\zeta - \bar{z}_0^3),$$

or $$\phi + i\psi = -m \log(z^3 - z_0^3)(z^3 - \bar{z}_0^3).$$

Problem 16. *A source S and sink T of equal strength m are situated within the space bounded by a circle whose centre is O. If S and T are at equal distance from O on opposite sides of it and on the same diameter AOB, show that velocity of the liquid at any point P is*

$$2m \cdot \frac{OS^2 + OA^2}{OS} \cdot \frac{PA \cdot PB}{PS \cdot PS' \cdot PT \cdot PT'}$$

where S' and T' are inverse points of S and T w.r.t. the circle.

Solution. Take O as origin and OA as x-axis.

Let $OS = OT = c$, $OA = OB = a$. Then

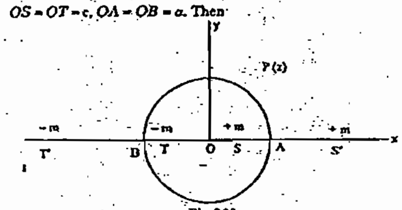


Fig. 3.28.

$$OS \cdot OS' = a^2, OT \cdot OT' = a^2.$$

Hence $OS' = OT' = a^2/c$.

The object system consists of

(i) source $+m$ at $S(c, 0)$,

(ii) sink $-m$ at $T(-c, 0)$,

The image system consists of

(i)' source $+m$ at $S'(a^2/c, 0)$ and sink $-m$ at O.

(ii)' sink $-m$ at $T'(-a^2/c, 0)$ and source $+m$ at O.

Source and sink both at O cancel each other.

Hence

$$w = -m \log(z - c) + m \log(z + c) - m \log(z - a^2/c) + m \log(z + a^2/c)).$$

$$-\frac{dw}{dz} = m\left[\frac{1}{z-c} - \frac{1}{z+c} + \frac{1}{z-c'} - \frac{1}{z+c'}\right] \text{ where } c' = a^2/c.$$

$$= m\left[\frac{2c}{z^2-c^2} + \frac{2(a^2/c)}{z^2-(a^4/c^2)}\right] = 2m\,\frac{(z^2-a^2)(c+a^2/c)}{(z^2-c^2)(z^2-c'^2)}$$

$$= 2m \cdot \frac{a^2+c^2}{c} \cdot \frac{(z-a)(z+a)}{(z-c)(z+c)(z-a^2/c)\left(z+\frac{a^2}{c}\right)}$$

Taking modulus of both sides and noting that fluid velocity $= \left|-\frac{dw}{dz}\right|$, we get

$$\text{velo.} = 2m \cdot \frac{OA^2 + OS^2}{OS} \cdot \frac{PA \cdot PB}{PS \cdot PT \cdot PS' \cdot PT'}.$$

Problem 17. *Prove that $u = -\omega y$, $v = -\omega x$, $w = 0$ represents a possible motion of inviscid fluid. Find the stream function and sketch stream lines. What is the basic difference between this motion and one represented by the potential $\phi = A \log r$, $r = (x^2+y^2)^{1/2}$.*

Solution. I. Consider the motion defined by

$$u = -\omega y, v = \omega x, w = 0.$$

Evidently it is two dimensional motion.

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 + 0 + 0 = 0.$$

This declares that the liquid motion is possible.

$$d\psi = \frac{\partial \psi}{\partial x} dx + \frac{\partial \psi}{\partial y} dy.$$

But $$-\frac{\partial \phi}{\partial x} = u = -\frac{\partial \psi}{\partial y}, \quad -\frac{\partial \phi}{\partial y} = v = \frac{\partial \psi}{\partial x}.$$

$\therefore$ $$d\psi = v\,dx - u\,dy = \omega x\,dx + \omega y\,dy = d\left[\frac{\omega}{2}(x^2+y^2)\right].$$

Integrating, $\psi = \frac{\omega}{2}(x^2+y^2) + a$

This gives the required stream function.

Stream lines are given by ψ = const. $= b$, say, so that

$$x^2 + y^2 = \frac{2(b-a)}{\omega} = c \text{ or } x^2 + y^2 = c$$

It means that stream lines are concentric circles with their centres at the origin.

II. Next, we consider the motion defined by

$$\phi = A \log r = \frac{A}{2}\log(x^2+y^2).$$

This $\Rightarrow \frac{\partial \phi}{\partial x} = \frac{Ax}{x^2+y^2} \Rightarrow \frac{\partial^2 \phi}{\partial x^2} = \frac{A(y^2-x^2)}{(x^2+y^2)^2}$

$$\Rightarrow \nabla^2 \phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = \frac{A(y^2-x^2) + A(x^2-y^2)}{(x^2+y^2)^2} = 0.$$

Hence liquid motion is possible.

III. Difference. The basic difference in these two motions is that velocity potential does not exist in the first case whereas in the second case it exists.

Problem 18. *A two dimensional flow field is given by $\psi = xy$. Show that the flow is irrotational. Find velocity potential, stream lines.*

Solution. $\psi = xy$

$$u = -\frac{\partial \psi}{\partial y} = -x, \qquad v = \frac{\partial \psi}{\partial x} = y$$

$$\mathbf{q} = u\mathbf{i} + v\mathbf{j} = -x\mathbf{i} + y\mathbf{j}$$

$$\text{curl}\,\mathbf{q} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -x & y & 0 \end{vmatrix}$$

$$= \mathbf{i}(0) - \mathbf{j}(0) + \mathbf{k}(0) = 0$$

$\therefore$ Motion is irrotational.

(ii) $$d\phi = \frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy = -u\,dx - v\,dy$$

$$= x\,dx - y\,dy = M\,dx + N\,dy, \text{ say.}$$

$$\frac{\partial M}{\partial y} = 0 = \frac{\partial N}{\partial x}.$$

$\therefore$ $M\,dx + N\,dy$ is exact. Solution is

$$\int d\phi = \int x\,dx + \int -y\,dy = \frac{x^2-y^2}{2} + c$$

or $$\phi = \frac{x^2-y^2}{2} + c.$$

This is the expression for velocity potential.

(iii) Stream lines are given by

$$\psi = \text{const.} \qquad \text{But } \psi = xy$$

$\therefore$ $$xy = c$$

gives stream lines.

Problem 19. *Show that velocity potential*

$$\frac{1}{2}\log\left[\frac{(x+a)^2+y^2}{(x-a)^2+y^2}\right]$$ IAS-2014 (20M)

gives a possible motion. Determine the form of stream lines and the curves of equal speed.

Solution. Given, $\phi = \frac{1}{2}\log[(x+a)^2+y^2] - \frac{1}{2}\log[(x-a)^2+y^2]$. ...(1)

$$\frac{\partial \phi}{\partial x} = \frac{x+a}{(x+a)^2+y^2} - \frac{(x-a)}{(x-a)^2+y^2}$$

$$\frac{\partial^2 \phi}{\partial x^2} = \frac{[(x+a)^2+y^2] - 2(x+a)^2}{[(x+a)^2+y^2]^2} - \frac{[(x-a)^2+y^2] - 2(x-a)^2}{[(x-a)^2+y^2]^2}$$

or $$\frac{\partial^2 \phi}{\partial x^2} = \frac{y^2-(x+a)^2}{[(x+a)^2+y^2]^2} - \frac{y^2-(x-a)^2}{[(x-a)^2+y^2]^2} \quad ...(2)$$

By (1), $$\frac{\partial \phi}{\partial y} = \frac{y}{(x+a)^2+y^2} - \frac{y}{(x-a)^2+y^2}$$

$$\frac{\partial^2 \phi}{\partial y^2} = \frac{(x+a)^2+y^2-2y^2}{[(x+a)^2+y^2]^2} - \frac{(x-a)^2+y^2-2y^2}{[(x-a)^2+y^2]^2} \quad ...(3)$$

Adding (2) and (3), $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ or $\nabla^2 \phi = 0$.

Thus the equation of continuity is satisfied and so (1) gives a possible liquid motion.

Second Part. To determine stream lines.

$$-\frac{\partial \phi}{\partial x} = u = -\frac{\partial \psi}{\partial y}, \quad -\frac{\partial \phi}{\partial y} = v = \frac{\partial \psi}{\partial x}.$$

Hence $$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad \frac{\partial \phi}{\partial y} = -\frac{\partial \psi}{\partial x}.$$

Now $$\frac{\partial \psi}{\partial y} = \frac{x+a}{(x+a)^2+y^2} - \frac{x-a}{(x-a)^2+y^2}$$

Integrating w.r.t. y,

$$\psi = \tan^{-1}\frac{y}{x+a} - \tan^{-1}\frac{y}{x-a} + F(x) \ldots \quad ...(4)$$

where $F(x)$ is constant of integration. To determine $F(x)$,

$$\frac{\partial \psi}{\partial x} = -\frac{\partial \phi}{\partial y} = -\frac{y}{(x+a)^2+y^2} + \frac{y}{(x-a)^2+y^2} \quad ...(5)$$

By (4), $$\frac{\partial \psi}{\partial x} = -\frac{y}{(x+a)^2+y^2} + \frac{y}{(x-a)^2+y^2} + F'(x) \quad ...(6)$$

Equating (5) to (6), $F'(x) = 0$. Integrating this

$F(x) =$ absolute const. and hence neglected.

Since it has no effect on the fluid motion.

Now (4) becomes

$$\psi = \tan^{-1}\frac{y}{x+a} - \tan^{-1}\frac{y}{x-a} \quad ...(7)$$

$$= \tan^{-1}\frac{-2ay}{x^2-a^2+y^2}$$

Stream lines are given by ψ = const., *i.e.*,

$$\tan^{-1}\left[\frac{-2ay}{x^2-a^2+y^2}\right] = \text{const. or } \frac{y}{x^2-a^2+y^2} = \text{const.}$$

If we take const. $= 0$, then we get $y = 0$, *i.e.*, x-axis.

If we take conts. $= \infty$, then we get circle $x^2 - a^2 + y^2 = 0$.

i.e., $x^2+y^2=a^2$.

Thus stream lines include x axis and circle.

Third Part. To determine curves of equal speed.

By (1) and (7), we obtain

$$w=\phi+i\psi=\frac{1}{2}\log[(x+a)^2+y^2]-\frac{1}{2}\log[(x-a)^2+y^2]$$
$$+i\tan^{-1}\frac{y}{x+a}-i\tan^{-1}\frac{y}{x-a}$$
$$=\log[(x+a)+iy]-\log[(x-a)+iy]$$
$$=\log(z+a)-\log(z-a)$$
$$\frac{dw}{dz}=\frac{1}{z+a}-\frac{1}{z-a}=\frac{-2a}{(z-a)(z+0)}$$
$$\left|\frac{dw}{dz}\right|=q=\frac{2a}{|z-a|\,.\,|z+a|}$$

Write $|z-a|=r$, $|z+a|=r'$. Then speed $=\dfrac{2a}{rr'}$.

The curves of equal speed are given by

$\dfrac{2a}{rr'}=$ const., *i.e.*, $rr'=$ const. which are Cassini ovals.

Problem 20. *Parallel line sources (perpendicular to the xyplane) of equal strength m are placed at the points $z=nia$, where $n=\ldots,-2,-1,0,1,2,3,\ldots$, prove that the complex potential is*

$$w=-m\log\sinh(\pi z/a).$$

Hence show that the complex potential for two dimensional doublets (line doublets), with their axes parallel to the x-axis, of strength μ at the same points, is given by

$$w=\mu\coth(\pi z/a).$$

Solution. Sources of equal strength m are placed at $z=\pm nia$ where $n=0,1,2,3,\ldots$ The complex potential due to this system at any point z is given by

$$w=-m\log(z-0)-\sum_{n=1}^{\infty}m\log(z-nia)-\sum_{n=1}^{\infty}m\log(z+nia)$$
$$=-m\log z-\sum_{n=1}^{\infty}m\log(z^2+n^2a^2)$$
$$=-\sum_{n=1}^{\infty}m\log\left(1+\frac{z^2}{n^2a^2}\right).n^2a^2.z$$
$$=-\sum_{n=1}^{\infty}m\log\left(1+\frac{z^2}{n^2a^2}\right)\frac{\pi z}{a}-\sum_{n=1}^{\infty}m\log\left(n^2a^2\frac{a}{\pi}\right)$$

Neglecting constant, $w=-\sum_{n=1}^{\infty}m\log\frac{\pi z}{a}\left(1+\frac{z^2}{n^2\pi^2}\right)$

Putting $\dfrac{\theta}{\pi}=\dfrac{z}{a}$, we get

$$w=-\sum_{n=1}^{\infty}m\log\theta\left(1+\frac{\theta^2}{n^2\pi^2}\right)$$

or
$$w=-m\log\theta.\left(1+\frac{\theta^2}{\pi^2}\right)\left(1+\frac{\theta^2}{2^2\pi^2}\right)\left(1+\frac{\theta^2}{3^2\pi^2}\right)$$
$$=-m\log\sinh\theta=-m\log\sinh\left(\frac{\pi z}{a}\right)$$

or
$$w=-m\log\sinh(\pi z/a). \qquad \ldots(1)$$

This proves the first required result.

Note that $w=-m\log(z-a)$ due to source $+m$ at $z=a$ and $w=m/(z-a)$ due to doublet $+m$ at $z=a$ with its axis along x-axis. *i.e.* $w=\dfrac{d}{dz}[m\log(z-a)]$ for a doublet $+m$ at $z=a$ with its axis along x-axis.

Therefore the complex potential for the doublets of strength m at these points is negative derivative of (1), so that

$$w=\frac{d}{dz}[m\log\sinh(\pi z/a)]$$

i.e.,
$$w=\frac{m\pi}{a}\coth\left(\frac{\pi z}{a}\right)=\mu\coth\left(\frac{\pi z}{a}\right)$$

This proves the second required result.

Miscellaneous Problems

Problem 21. *An area A is bounded by that part of the x-axis for which $x>a$ and by that branch of $x^2-y^2=a^2$ which is in the positive quadrant. There is a two dimensional unit sink at $(a,0)$ which sends out liquid uniformly in all directions. Show by means of the transformation $w=\log(z^2-a^2)$ that in steady motion the stream lines of the liquid within the area A are portions of rectangular hyperbolas. Draw the stream lines corresponding to $\psi=0,\pi/4$ and $\pi/2$. If ρ_1 and ρ_2 are the distances of a point P within the fluid from the points $(\pm a,0)$, show that the velocity of the fluid at P is measured by $2PO/\rho_1\rho_2$, O being the origin.*

Solution. *Step I.* $w=\log(z^2-a^2)$ is expressible as

$$\phi+i\psi=\log(x^2-y^2-a^2+2ixy)$$

This $\Rightarrow \psi=\tan^{-1}\left(\dfrac{2xy}{x^2-y^2-a^2}\right)$. $\qquad\ldots(1)$

Stream lines are given by $\psi=$ const. $=k$, say, then

$$\tan^{-1}\left(\frac{2xy}{x^2-y^2-a^2}\right)=k$$

or $\tan k=2xy/(x^2-y^2-a^2)$ $\qquad\ldots(2)$

If $k=0$, then (2) $\Rightarrow 2xy=0\Rightarrow x=0, y=0$.

If $k=\pi/2$, then (2) $\Rightarrow x^2-y^2-a^2=0\Rightarrow x^2-y^2=a^2$.

Thus stream lines are parts of the curves $x^2-y^2=a^2$, $x=0$, $y=0$. Hence liquid flows in an area A bounded by $x=0$, $y=0$, $x^2-y^2=a^2$ in the positive quadrant.

Step II. $w=\log(z^2-a^2)$ is expressible as
$$w=\log(z-a)+\log(z+a).$$

This proves that the liquid motion is generated by two sinks of strength unity at $(a,0)$ and $(-a,0)$. Consequently, the image of sink -1 at $(a,0)$ is an equal sink at $(-a,0)$, relative to y-axis *i.e.*, relative to the area A.

Step III. To show that velocity $q=2.OP/\rho_1\rho_2$

We have $w=\log(z^2-a^2)$.

Hence $\dfrac{dw}{dz}=\dfrac{2z}{z^2-a^2}$.

This $\Rightarrow q=\left|\dfrac{dw}{dz}\right|=\dfrac{2|z|}{|z-a|\,.\,|z+a|}$.

Let P be a point within the fluid. Then $|z|=|z-0|=OP$.

$\rho_1=|z-a|=$ distance between P and $(a,0)$,

$\rho_2=|z+a|$ distance between P and $(-a,0)$

Thus $q=\dfrac{2OP}{\rho_1\rho_2}$.

Step IV. To determine stream lines corresponding to

$$\psi=0,\frac{\pi}{4},\frac{\pi}{2}.$$

By (1), $\tan\psi=\dfrac{2xy}{x^2-y^2-a^2}$

Putting $\psi=0,\frac{\pi}{4},\frac{\pi}{2}$, we obtain

$$\frac{2xy}{x^2-y^2-a^2}=\tan 0=0,\ \frac{2xy}{x^2-y^2-a^2}=\tan\frac{\pi}{4}=1,$$
$$\frac{2xy}{x^2-y^2-a^2}=\infty$$

i.e. $xy=0;\ x^2-y^2-a^2=2xy;\ x^2-y^2-a^2=0$

i.e. $x=0\ y=0;\ x^2-y^2-2xy-a^2=0;\ x^2-y^2=a^2$.

Thus stream lines lie along

(i) x and y-axes — relative to $\psi=0$

(ii) the curve $x^2-y^2-2xy-a^2=0$ — relative to $\psi=\pi/4$

(iii) rectangular hyperbola $x^2-y^2=a^2$ — relative to $\psi=\pi/2$.

Problem 22. *Show that the velocity vector* **q** *is everywhere tangent to the lines in xy-plane along which $\psi(x,y)=$ const.*

Solution. Given $\psi(x,y)=$ const. $\qquad\ldots(1)$

(1) $\Rightarrow d\psi=0\Rightarrow \dfrac{\partial\psi}{\partial x}dx+\dfrac{\partial\psi}{\partial y}dy=0$

But $u=-\dfrac{\partial\psi}{\partial y}$, $v=\dfrac{\partial\psi}{\partial x}$

Now the last gives $v\,dx-u\,dy=0$,

or $\dfrac{dy}{dx}=\dfrac{v}{u}$

slope m_1 of the tangent to the curve (1) is $m_1=v/u$.

But slope of direction of velocity **q** is $\dfrac{v}{u}$.

Consequently, direction of velocity is tangent to $\psi=$ const.

Problem 23. *A velocity field is given by* $\mathbf{q}=-x\mathbf{i}+(y+t)\mathbf{j}$. *Find the stream function and steam lines for this field at $t=2$.*

Solution. $\mathbf{q}=u\mathbf{i}+v\mathbf{j}=-x\mathbf{i}+\mathbf{j}(x+t)$

$$u=-x \qquad v=y+t$$

But $u=-\dfrac{\partial\psi}{\partial y}$, $v=\dfrac{\partial\psi}{\partial x}$

$\therefore$
$$\frac{\partial\psi}{\partial y}=x,\ \frac{\partial\psi}{\partial x}=y+t$$
$$d\psi=\frac{\partial\psi}{\partial x}dx+\frac{\partial\psi}{\partial y}dy$$
$$d\psi=(y+t)\,dx+x\,dy \qquad\ldots(1)$$
$$=Mdx+Ndy,\ \text{say}$$
$$\frac{\partial M}{\partial y}=1=\frac{\partial N}{\partial x}$$

$\therefore$ $Mdx+Ndy$ is exact.

Solution of (1) is given by

$$\psi=\int(y+t)\,dx+\int 0\,dy=x(y+t)+c$$

or $\psi=x(y+t)+c$ $\qquad\ldots(2)$

This is the required expression for stream function. Stream lines at $t=2$ are given by $(\psi)_{t=2}=$ const.

or $x(y+2)+c=$ const.

or $x(y+2)=a$.

Problem 24. *Prove that in the two dimensional liquid motion due to any number of sources at different points on a circle, the circle is a stream line provided that there is no boundary and that the algebraic sum of strengths of the sources is zero.*

Show that the same is true if the region of flow is bounded by a circle in which cuts orthogonally the circle in question.

Solution. Suppose $A_1, A_2, A_3, \ldots$ are the positions of the sources of strengths $m_1, m_2, m_3, \ldots$

Take any point P on the circle and the diameter through it as the initial line.

Let $\angle A_1PA = \theta$, $\angle A_2PA_1 = \alpha_1$, $\angle A_3PA_2 = \alpha_2$, ... etc.

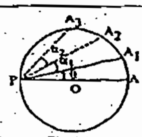


Fig. 3.29.

Then stream line is given by

$$\psi = -m_1\theta - m_2(\theta + \alpha_1) - m_3(\theta + \alpha_1 + \alpha_2) \ldots$$

$$= -(m_1 + m_2 + m_3 + \ldots)\theta - [m_2\alpha_1 + m_3(\alpha_1 + \alpha_2) + \ldots]$$

or $\psi = -\theta\Sigma m - \text{const.}$

For whatever be the position of P, $\alpha_1, \alpha_2, \ldots$ etc. do not change. Since the angle subtended at the circumference by an arc is always the same.

If the algebraic sum of the strengths is zero, *i.e.*, if $\Sigma m = 0$, then ψ = const., meaning thereby circle is a steam line. Hence the result.

Second Part. Let O' be the centre of a circle which cuts the given circle orthogonally. Join O' to A_1. If $O'A_1$ cuts the original at A', then A' is the inverse point of A_1 w.r.t. circle whose centre is O'.

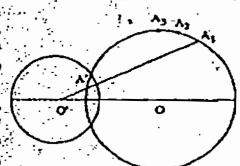


Fig. 3.30.

Relative to the circle whose centre is O', the image of source $+m_1$ at A_1 is a source $+m_1$ at A' and sink $-m_1$ at O'. If the barriers are omitted, we are left with system $2\Sigma m$ on the original circle and $-\Sigma m$ at O' and as $\Sigma m = 0$, we again get the same result, *i.e.*, ϕ = const.

Problem 25. *Find the velocity potential when there is a source and an equal sink inside a circular cavity and show that one of the stream lines is an arc of the circle which passes through the source and sink and cuts orthogonally the boundary of the cavity.* **(Kanpur 90)**

Solution. Consider a source $+m$ at A and a sink $-m$ at B respectively inside an circular cavity whose centre is O and radius is a. Let $OA = b$, $OB = c$, $\angle BOA = \alpha$. Let A' and B', be respectively inverse points of A and B respectively. Then

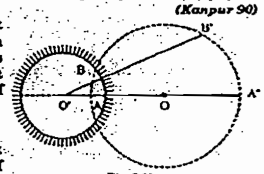


Fig. 3.31

$$OA \cdot OA' = a^2 = OB \cdot OB'$$

The image of source $+m$ at A is a sink $+m$ at A' and a sink $-m$ at O. The image of sink $-m$ at B is a sink $-m$ at B' and a source $+m$ at O. The source $+m$ and sink $-m$ both at O cancel each other. Thus w is given by

$$w = -m\log(z - b) - m\log\left(z - \frac{a^2}{b}\right) + m\log(z - ce^{i\alpha}) + m\log\left(z - \frac{a^2}{c}e^{i\alpha}\right)$$

Equating real and imaginary parts from both sides, we can easily get velocity potential and stream function, respectively.

Since $OA \cdot OA' = OB \cdot OB' = a^2$.

Hence points A, A', B, B' are concyclic. Let the circle through these points meet the cavity in C and C'. Then $OA\,OA' = OC^2$. Hence OC is tangent at C to the circle through B, B', A, A'. It declares the fact that the two circles intersect orthogonally. Also the circle through A, A', B, B' passes through A and B, *i.e.*, the same source and sink, hence it must be a stream line.

Problem 26. *Prove that for liquid circulating irrotationally in part of the plane between two non-intersecting circles the curves of constant velocity are Cassini's ovals.*

Solution. Suppose CC' is the line of centres. Take two points A and B s.t. they are inverse points w.r.t. both the circles. Consider a point P on one of the circles.

Write $PA = r$, $PC = r_1$.

Since $CA \cdot CB = CP^2$, hence ΔCPA and ΔCPB are similar so that

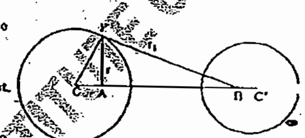


Fig. 3.32.

$$\frac{CP}{CB} = \frac{PA}{PB}\ i.e., \frac{CP}{CB} = \frac{r}{r_1}\ \text{or}\ \frac{r}{r_1} = \text{const.}$$

It means the equations of the two circles can be written as

$$\frac{r}{r_1} = k_1, \frac{r}{r_2} = k_2 \text{ say.}$$

Also these two circles are stream lines, hence ψ must be of the form $\psi = f(r/r_1)$, but $f(r/r_1)$ is plane harmonic. Consequently $\psi = f(r/r_1) = A\log(r/r_1)$ as $\log r$ is the only function of r which is plane harmonic. Hence

$$\phi = -A(\theta - \theta_1) \text{ as } \frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{\partial\psi}{\partial r}$$

$$w = \phi + i\psi = -A(\theta - \theta_1) + iA\log(r/r_1)$$

$$= iA\,[\log(r/r_1) + i(\theta - \theta_1)]$$

$$= iA\log\left[\frac{r}{r_1}e^{i(\theta - \theta_1)}\right] = iA\log\left[\frac{re^{i\theta}}{r_1e^{i\theta_1}}\right]$$

Choosing A to be $(-a, 0)$ and B to be $(a, 0)$, then

$$w = iA\log\left(\frac{z + a}{z - a}\right) \text{ This} \Rightarrow$$

$$q = \left|\frac{dw}{dz}\right| = |iA| \cdot \left|\frac{1}{z + a} - \frac{1}{z - a}\right| = \frac{2Aa}{|z + a| \cdot |z - a|}$$

or $$q = \frac{2Aa}{rr_1}$$

Curves of constant velocity are given by

$$q = \text{const., } i.e., \frac{2Aa}{rr_1} = \text{const. } i.e.,$$

rr_1 = const. which are clearly Cassini's ovals.

Problem 27. *If a homogeneous liquid is acted on by a repulsive force from the origin, the magnitude of which at distance r from the origin is μr pre unit mass, shew that it is possible for the liquid to move steadily, without being constrained by any boundaries, in the space between one branch of the hyperbola $x^2 - y^2 = a^2$ and the asymptotes and find the velocity potential.*

Solution. The liquid moves steadily between the space given by one branch of $x^2 - y^2 = a^2$ ...(1) and its asymptotes given by

$$x^2 - y^2 = 0. \quad \ldots(2)$$

(1) and (2) are clearly stream lines. For $x^2 - y^2$ is a harmonic function as it satisfies Laplace's equation. Thus

$$\psi = A(x^2 - y^2) = Ar^2(\cos^2\theta - \sin^2\theta) = Ar^2\cos 2\theta$$

or $$\psi = Ar^2\sin\left(\frac{\pi}{2} + 2\theta\right) \quad A \text{ being a constant.}$$

Using $$\frac{\partial\phi}{\partial r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \text{ we get } \phi = Ar^2\cos\left(\frac{\pi}{2} + 2\theta\right)$$

$$w = \phi + i\psi = Ar^2\left[\cos\left(\frac{\pi}{2} + 2\theta\right) + i\sin\left(\frac{\pi}{2} + 2\theta\right)\right]$$

or $$w = Ar^2 e^{i(\pi/2) + 2i\theta} = Ar^2e^{2i\theta} = A(re^{i\theta})^2$$

For $e^{i(\pi/2)} = 1$.

Hence $w = Az^2$. Hence $q = \left|\frac{dw}{dz}\right| = 2A|z| = 2Ar$

In case of steady motion, the equation of motion is

$$\frac{p}{\rho} + \frac{1}{2}q^2 + \Omega = \text{const.} \quad \ldots(1)$$

Given $$-\frac{\partial\Omega}{\partial r} = \mu r \quad [F = -\nabla\Omega]$$

This $\Rightarrow$ $\Omega = -\frac{\mu}{2}r^2$, neglecting constant.

Putting the values in (1)

$$\frac{p}{\rho} + 2A^2r^2 - \frac{\mu}{2}r^2 = \text{const.}$$

Subjecting this to the condition p = const. on free surface, we get

$$2A^2r^2 - \frac{\mu}{2}r^2 = 0 \text{ or } A = \sqrt{\mu}/2.$$

Hence $q = 2Ar = \frac{2\sqrt{\mu}}{2} \cdot r$ or $q = r\sqrt{\mu}$.

velocity potential $= \phi = Ar^2\cos\left(\frac{\pi}{2} + 2\theta\right) = -Ar^2\sin 2\theta$.

or $$\phi = -\frac{\sqrt{\mu}}{2}r^2\sin 2\theta$$

Problem 28. *If the fluid fills the region of space on the positive side of x-axis, is a rigid boundary, and if there be a source + m at the point (0, a) and an equal sink at (0, b), and if the pressure on the negative side of the boundary be the same as the pressure of the fluid at infinity, show that the resultant pressure on the boundary is $\pi\rho m^2(a - b)^2/ab(a + b)$, where ρ is the density of the fluid.* **IAS-2008**

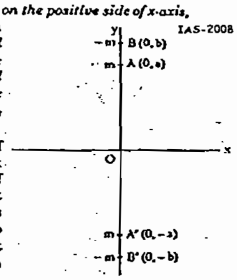


Fig. 3.33.

Solution. The object system consists of source $+m$ at $A(0, a)$, *i.e.*, at $z = ia$ and sink $-m$ at $z = ib$. The image system consists of source $+m$ at $A'(z = -ia)$ and sink $-m$ at $B'(z = -ib)$ w.r.t. the positive line OX which is rigid boundary. The complex potential due to object system with rigid boundary is equivalent to the object system and its image system with no rigid boundary

$\therefore$ $w = -m\log(z - ia) + m\log(z - ib)$
$-m\log(z + ia) + m\log(z + ib)$

or $$w = -m\log(z^2 + a^2) + m\log(z^2 + b^2)$$

$$\frac{dw}{dz} = -2mz\left[\frac{1}{z^2 + a^2} - \frac{1}{z^2 + b^2}\right] = \frac{2mz(a^2 - b^2)}{(z^2 + a^2)(z^2 + b^2)}$$

$\therefore$ $$q = \left|\frac{dw}{dz}\right| = \frac{2m(a^2 - b^2)|z|}{|z^2 + a^2|\,|z^2 + b^2|}$$

For any point on x-axis, we have $z = x$ so that

$$q = \frac{2mx(a^2 - b^2)}{(x^2 + a^2)(x^2 + b^2)}$$

This is expression for velocity at any point on x-axis. Let p_0 be the pressure at $x = \infty$. By Bernoulli's equation for steady motion.

$\therefore$ $$\frac{p}{\rho} + \frac{1}{2}q^2 = C.$$

In view of $p = p_0$, $q = 0$ when $x = \infty$, we get $C = p_0/\rho$.

$$\frac{p_0 - p}{\rho} = \frac{1}{2}q^2.$$

Required pressure P on boundary is given by

$$P = \int_{-\infty}^{\infty}(p_0 - p)\,dx = \int_{-\infty}^{\infty}\frac{1}{2}\rho q^2\,dx$$

$$= \frac{1}{2}\rho \int_{-\infty}^{\infty} \frac{4m^2 x^2 (a^2 - b^2)^2}{(x^2+a^2)^2 (x^2+b^2)^2}\, dx$$

$$= 4\rho m^2 (a^2 - b^2)^2 \int_0^{\infty} \frac{x^2\, dx}{(x^2+a^2)^2 (x^2+b^2)^2}$$

$$= 4m^2 \rho \int_0^{\infty} \left[\frac{a^2+b^2}{a^2-b^2}\left\{\frac{1}{x^2+b^2} - \frac{1}{x^2+a^2}\right\} - \frac{a^2}{(x^2+a^2)^2} - \frac{b^2}{(x^2+b^2)^2}\right] dx$$

$$= 4m^2 \rho \left[\frac{a^2+b^2}{a^2-b^2}\left\{\frac{\pi}{2b} - \frac{\pi}{2a}\right\} - \frac{\pi}{4a} - \frac{\pi}{4b}\right]$$

$$= \frac{\pi \rho\, m^2 (a-b)^2}{ab\,(a+b)}.$$

For $\displaystyle\int_0^{\infty} \frac{dx}{x^2+a^2} = \left[\frac{1}{a}\tan^{-1}\frac{x}{a}\right]_0^{\infty} = \frac{\pi}{2a}$

$$\int_0^{\infty} \frac{dx}{(x^2+a^2)^2} = \frac{1}{a^3}\int_0^{\pi/2} \cos^2\theta\, d\theta,\ x = a\tan\theta$$

$$= \frac{1}{2a^3}\int_0^{\pi/2} (1+\cos 2\theta)\, d\theta = \frac{\pi}{2}\cdot\frac{1}{2a^3} = \frac{\pi}{4a^3}.$$

Problem 29. *An infinite mass of liquid is moving irrotationally and steadily under the influence of a source of strength μ and an equal sink at a distance $2a$ from it. Prove that the kinetic energy of the liquid which passes in unit time across the plane which bisects at right angles in the line joining the source and sink is $\frac{8\pi}{7a^4}\pi\rho\mu^3$, ρ being the density of the liquid.*

Solution. Consider a source $+\mu$ at $A\,(a, 0, 0)$ and sink $-\mu$ at $B\,(-a, 0, 0)$ s.t. $AB = 2a$. Consider a point $P\,(0, y, 0)$ on Y-axis. Consider a circular strip bounded by the radius y and $y + \delta y$. Mass of the liquid passing through this strip is

$\rho\,(2\pi y \cdot \delta y)\, q = 2\pi\rho\, qy \cdot \delta y$ per unit time $= \delta m$, say.

Recall that $r^2 v =$ const. so that $v =$ const$/r^2$ is the equation of continuity in case of spherical symmetry.

Hence velocity at P due to source at $A = \dfrac{\mu}{AP^2}$ along AP

velocity at P due to sink at $B = \dfrac{\mu}{BP^2}$ along PB

$= \dfrac{\mu}{AP^2}$ along PB

$AP = PB$. Let $\angle PAO = \theta$.

Then resultant velocity at P along AB

$$= \frac{2\mu\cos\theta}{AP^2} = \frac{2\mu\cos\theta}{a^2+y^2} = \frac{2\mu a}{(a^2+y^2)^{3/2}} = q.$$

Required K.E. $= \dfrac{1}{2}\displaystyle\int_0^{\infty} \delta m.q^2 = \frac{1}{2}\int_0^{\infty} 2\pi\rho q y\, q^2\, dy$

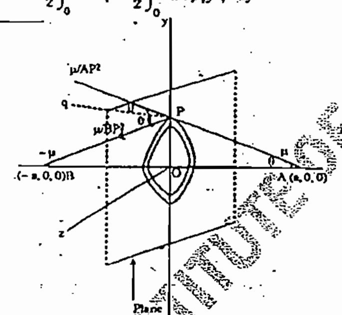


Fig. 3.34

$$= \pi\rho\int_0^{\infty}\left[\frac{2\mu a}{(a^2+y^2)^{3/2}}\right]^3 y\, dy = 8\pi\rho\,\mu^3 a^3 \int_0^{\infty} \frac{y\, dy}{(a^2+y^2)^{9/2}}$$

$$= \frac{8\pi\rho\mu^3}{a^4}\int_0^{\pi/2} \sin t.\cos^6 t\, dt,\ \text{put } y = a\tan t$$

$$\frac{8\pi\rho\mu^3}{a^4}\left[-\frac{1}{7}\cos^7 t\right]_0^{\pi/2} = \frac{8}{7a^4}\pi\rho\mu^3.$$

Problem 30. *A single source is placed in an infinite perfectly elastic fluid, which is also a perfect conductor of heat; show that if the motion be steady, the velocity at distance r from the source satisfies the equation*

$$\left(V - \frac{k}{V}\right)\frac{\partial V}{\partial r} = \frac{2k}{r}$$

and hence that $\qquad r = \dfrac{1}{\sqrt{V}}\, e^{V^2/4k}$.

Solution. Since the motion is steady and is due to a single source, hence the flow is purely radial. Equation of motion is

$$\frac{dq}{dt} = F - \frac{1}{\rho}\nabla p.$$

Here we have

$$\frac{d}{dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial r},\ \nabla = \frac{\partial}{\partial r},\ F = 0 \text{ as external forces are absent.}$$

Hence $\left(\dfrac{\partial}{\partial t} + u\dfrac{\partial}{\partial r}\right)u = -\dfrac{1}{\rho}\nabla p.$

Motion is steady, $\dfrac{\partial u}{\partial t} = 0$, $p = k\rho$ (Boyle's law)

$$u\frac{\partial u}{\partial r} = -\frac{k}{\rho}\frac{\partial \rho}{\partial r}. \qquad \ldots(1)$$

Motion has spherical symmetry and hence equation of continuity

is $\qquad \dfrac{\partial \rho}{\partial t} + \dfrac{1}{r^2}\dfrac{\partial}{\partial r}(\rho u r^2) = 0.$

But $\qquad \dfrac{\partial \rho}{\partial t} = 0$ as the motion is steady.

Hence $\dfrac{\partial}{\partial r}(\rho u r^2) = 0$ or $ur^2\dfrac{\partial \rho}{\partial r} + \rho r^2\dfrac{\partial u}{\partial r} + \rho u.2r = 0$

or $$u\frac{\partial \rho}{\partial r} + \rho\frac{\partial u}{\partial r} + 2\frac{\rho u}{r} = 0. \qquad \ldots(2)$$

Eliminating $\dfrac{\partial \rho}{\partial r}$ from (1) and (2), we get

$$u\left(-\frac{\rho u}{k}\frac{\partial u}{\partial r}\right) + \rho\frac{\partial u}{\partial r} + \frac{2\rho u}{r} = 0$$

or $$\frac{2k}{r} = \left(u - \frac{k}{u}\right)\frac{\partial u}{\partial r} \qquad \ldots(3)$$

This $\Rightarrow \dfrac{2k}{r} = \left(u - \dfrac{k}{u}\right)\dfrac{du}{dr} \qquad$ as $u = u(r)$

or $\qquad \left(u - \dfrac{k}{u}\right)du = \dfrac{2k}{r}\,dr$

Integrating, $\dfrac{u^2}{2} - k\log u = 2k\log r + \log A$

or $\qquad \dfrac{u^2}{2} = k\log(r^2 A_1 . u)$, where $k\log A_1 = \log A$.

or $\qquad r^2 u A_1 = e^{u^2/2k}$. Take $A_1 = 1$, we get

$$r = \frac{1}{\sqrt{u}}\, e^{u^2/4k} \qquad \ldots(4)$$

Replacing u by V in (3) and (4), we et the two required results.

Problem 31. *In two dimensional irrotational fluid motion, show that if the stream lines are confocal ellipses,*

$$\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1.$$

$$\psi = A\log\left[\sqrt{(a^2+\lambda)} + \sqrt{(b^2+\lambda)}\right] + B$$

and the velocity at any point is inversely proportional to the square root of the rectangle under the focal radii of the point.

Solution. The conformal transformation $z = c\cos w \qquad \ldots(1)$

is known to yield the given type of confocal ellipses.

$(1) \Rightarrow x + iy = c\cos(\phi + i\psi) \Rightarrow x = c\cos\phi\cosh\psi,\ y = c\sin\phi\sinh\psi.$

Eliminating ϕ, we get

$$\frac{x}{c^2\cosh^2\phi} + \frac{y^2}{c^2\sinh^2\psi} = 1. \qquad \ldots(2)$$

Stream lines are given by $\psi =$ const. By virtue of this, (2) declares that stream lines are confocal ellipses. Comparing (2) with the equation,

$$\frac{x^2}{a^2+\lambda} + \frac{y}{b^2+\lambda} = 1, \text{ we get } c\cosh\psi = \sqrt{(a^2+\lambda)},\ c\sinh\psi = \sqrt{(b^2+\lambda)} \qquad \ldots(3)$$

This $\Rightarrow c(\cosh\psi + \sinh\psi) = \sqrt{(a^2+\lambda)} + \sqrt{(b^2+\lambda)}$

or $\qquad ce^{\psi} = \sqrt{(a^2+\lambda)} + \sqrt{(b^2+\lambda)}$

or $$\psi = \log\left[(a^2+\lambda) + \sqrt{(b^2+\lambda)}\right] - \log c \qquad \ldots(4)$$

If $w = \phi + i\psi$ is the complex potential of some fluid motion, then so is Aw. Hence (4) gives

$$\psi = A\log\left[\sqrt{(a^2+\lambda)} + \sqrt{(b^2+\lambda)}\right] - B.$$

Velocity. (1) $\Rightarrow dz/dw = -c\sin w = -c\sqrt{[1 - (z^2/c^2)]}$

$$q^{-1} = \frac{1}{q} = \left|-\frac{dz}{dw}\right| = \sqrt{|c^2 - z^2|} = \sqrt{[|c - z| \cdot |c + z|]} \qquad \ldots(5)$$

By (3), $c^2(\cosh^2\psi - \sinh^2\psi) = (a^2+\lambda) - (b^2+\lambda) = a^2 - b^2$

or $\qquad c^2 = a^2 - a^2(1 - e^2)$. For $b^2 = a^2(1 - e^2)$

or $\qquad c = ae.$

Now (5) becomes $q^{-1} = \sqrt{[|z - ae| \cdot |z + ae|]} \qquad \ldots(6)$

$(\pm ae, 0)$ are co-ordinates of foci, denoted by S and S'. P is a point z. Then $r_1 = SP = |z - ae|$,

$r_2 = S'P = |z + ae|$.

Now (6) is expressible as

$$q^{-1} = \sqrt{(r_1 r_2)} \text{ or } q = \frac{1}{\sqrt{(r_1 r_2)}}$$

From this the reqired result follows.

Problem 32. *λ denoting a variable parameter, and f a given function, find the condition that $f(x, y, \lambda) = 0$ should be a possible system of stream lines for steady irrotational motion in two dimensions.*

Solution. Suppose $f(x, y, \lambda) = 0$ represents stream lines for different values of λ. Solving this equation, we get

$$\lambda = F(x, y). \qquad \ldots(1)$$

We also know that $\psi =$ const. represents stream lines. So we can suppose that (1) and $\phi = c$ both represent the same stream lines. It means that.

$\psi = \psi(\lambda)$. Now $\dfrac{\partial \psi}{\partial x} = \dfrac{d\psi}{d\lambda}\dfrac{\partial \lambda}{\partial x}$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{d^2 \psi}{d\lambda^2}\left(\frac{\partial \lambda}{\partial x}\right)^2 + \frac{d\psi}{d\lambda}\cdot\frac{\partial^2 \lambda}{\partial x^2} \quad \dots (3)$$

But the motion is irrotational and so $\nabla^2 \psi = 0$.

i.e.,
$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0.$$

In view of (3), this becomes

$$\left[\frac{d^2 \psi}{d\lambda^2}\left(\frac{\partial \lambda}{\partial x}\right)^2 + \frac{d\psi}{d\lambda}\frac{\partial^2 \lambda}{\partial x^2}\right] + \left[\frac{d^2 \psi}{d\lambda^2}\left(\frac{\partial \lambda}{\partial y}\right)^2 + \frac{d\psi}{d\lambda}\frac{\partial^2 \lambda}{\partial y^2}\right] = 0$$

or
$$\frac{d^2 \psi}{d\lambda^2}\left[\left(\frac{\partial \lambda}{\partial x}\right)^2 + \left(\frac{\partial \lambda}{\partial y}\right)^2\right] + \frac{d\psi}{d\lambda}\left[\frac{\partial^2 \lambda}{\partial x^2} + \frac{\partial^2 \lambda}{\partial y^2}\right] = 0$$

This is the required condition.

EXERCISE 3

1. Prove that any relation of the form $w = f(z)$, where $w = \phi + i\psi$ and $z = x + iy$ represents a two dimensional irrotational motion in which magnitude of velocity is $|(dw/dz)|$.
2. Show the $U = 2cxy$, $V = (a^2 + x^2 - y^2)$ are the velocity components of a possible fluid motion. Determine stream function and sketch the stream lines.
3. Show that ϕ and ψ satisfy Laplace's equation.
4. Between the fixed boundaries $\theta = \pi/6$ and $\theta = -\pi/6$, there is a two dimentional liquid motion due to a source at the point ($r = c$, $\theta = \alpha$) and a sink at the origin, absorbing water at the same rate as the source produces it. Find the stream function, and show that one of the stream lines is a part of the curve

$$r^3 \sin 3\alpha = c^3 \sin 3\theta.$$

6. The internal boundary of a liquid is composed of the two orthogonal circles

$$x^2 + y^2 + 2y = 1,\; x^2 + y^2 - 2y = 1.$$

A source producing liquid at the rate m is placed at one of the points of intersection, $z = 1$. Show that the complex potential of the fluid motion is.

$$\frac{m}{2\pi}\log\left[\frac{z(z^2+1)}{(z-1)^4}\right]$$

6. State and prove the theorem of Blasius for fluid thrust and couple on a fixed cylinder placed in steady, two dimensional, irrotational flow.
7. Prove that $(u = -\omega)$, $v = \omega x$, $w = 0$ represents a possible motion of inviscid fluid. Find the stream function and sketch stream lines. What is the basic difference between that motion and one represented by the potential $\phi = A \log r$, $r = (x^2 + y^2)^{1/2}$.
8. If $W = f(z)$ is the complex potential for a two dimensional motion having no rigid boundaries and no singularities of flow within the circle $|z| = a$, show that on introducing the rigid boundary $|z| = a$ into the flow, the new complex potential is given by

$$W = f(z) + \bar{f}(a^2/z) \text{ for } |z| \geq a.$$

VORTEX MOTION | SET - IV

Vorticity

If $\mathbf{q}(u, v, w)$ be the velocity of a fluid particle, then $\mathbf{W} = \frac{1}{2}\,\text{curl}\ \mathbf{q}$ is called vorticity vector or simply vorticity. As a matter of fact, vorticity is the angular velocity (velocity of rotation) of an infinitesimal fluid element. If $\mathbf{W}(\xi, \eta, \zeta)$, then

$$\xi\mathbf{i} + \eta\mathbf{j} + \zeta\mathbf{k} = \frac{1}{2}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$$

This $\Rightarrow \xi = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \eta = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right), \zeta = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$

The components ξ, η, ζ are called components of spin.

Note. Some authors use ξ, η, ζ for $\Omega_x, \Omega_y, \Omega_z$ and define

$\Omega = \xi\mathbf{i} + \eta\mathbf{j} + \zeta\mathbf{k} = \frac{1}{2}$ curl q. Thus, we have

$$\Omega_x = \xi = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right), \Omega_y = \eta = \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right), \Omega_z = \zeta = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

Remark 1. In the two-dimensional cartesian coordinates, the vorticity is given by

$$\Omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$$

Remark 2. In the two-dimensional polar coordinates the vorticity is given by

$$\Omega_z = \frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}$$

Remark 3. The vorticity components in cylindrical polar coordinates (r, θ, z) are given by

$$\Omega_r = \frac{1}{r}\frac{\partial v_z}{\partial \theta} - \frac{\partial v_\theta}{\partial z}, \Omega_\theta = \frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}, \Omega_z = \frac{v_\theta}{r} + \frac{\partial v_\theta}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}$$

Remark 4. The vorticity components in spherical polar coordinates (r, θ, ϕ) are given by

$$\Omega_r = \frac{1}{r}\frac{\partial v_\phi}{\partial \theta} - \frac{1}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi} + \frac{v_\phi}{r}\cot\theta$$

$$\Omega_\theta = \frac{1}{r\sin\theta}\frac{\partial v_r}{\partial \phi} - \frac{\partial v_\phi}{\partial r} - \frac{v_\phi}{r}$$

$$\Omega_\phi = \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} - \frac{1}{r}\frac{\partial v_r}{\partial \theta}$$

2. Irrotational motion

A fluid motion is said to be irrotational if $W = 0$.

3. Vortex lines

A vortex line is a curve drawn in the fluid s.t. the tangent to it is in the direction of the vorticity vector at that point at that instant. The differential equations of vortex lines are

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dz}{\zeta}.$$

4. Vortex tube. Vortex filament

If vortex lines are drawn through every point of a small closed curve, then the space bounded by these lines is called vortex tube. The fluid within this tube is known as vortex filament or simply vortex. Thus we can also say that boundary of a vortex filament is a vortex tube.

Properties of Vortex filament

1. *The product of the cross section and vorticity at any point on a vortex filament is constant along the filament.*

Let S_1 and S_2 be cross sectional areas at the end points P and Q of a vortex filament. Let $\hat{n}$ be unit out ward normal vector on S. Then, by Gauss divergence theorem

$$\int_V \mathbf{W}.\mathbf{n}\,dS = \int_V \nabla.\mathbf{W}\,dV = \int \nabla.\left(\frac{1}{2}\nabla\times\mathbf{q}\right)dV$$

$$= \frac{1}{2}\int_V (\text{div curl } \mathbf{q})\,dV = 0 \text{ as div curl} = 0.$$

$$\therefore \int_s \mathbf{W}.\mathbf{n}\,dS = 0,$$

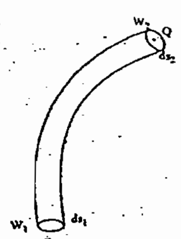


Fig. 8.1.

where S is the closed surface enclosing volume of V of the fluid in the vortex tube

or $\int_{S_1}\mathbf{W}.\mathbf{n}\,dS + \int_{S_2}\mathbf{W}.\mathbf{n}\,dS + \int_{walls}\mathbf{W}.\mathbf{n}\,dS = 0$

But on the walls of the tube, W is along the tube and so

$$\int_{walls}\mathbf{W}.\mathbf{n}\,dS = 0 \text{ as } \mathbf{W}.\mathbf{n} = 0 \text{ on the walls.}$$

Hence $\int_{S_1}\mathbf{W}.\mathbf{n}\,dS + \int_{S_2}\mathbf{W}.\mathbf{n}\,dS = 0$

or $\int_{S_1}\mathbf{W}.\mathbf{n}_1\,dS = \int_{S_2}\mathbf{W}.\mathbf{n}_2\,dS$

where $\mathbf{n}_1$ and n_2 are unit outward normals on the surfaces S_1 and S_2 drawn in the same sense.

This $\Rightarrow \int_S \mathbf{W}.\mathbf{n}\,dS = \text{const.}$

This proves the required result.

2. *Vortex lines move with the fluid, i.e., vortex filaments are composed of the same elements of the fluid.*

Circulation $= \int_c \mathbf{q}.d\mathbf{r} = \int_s \mathbf{n}.\text{curl } \mathbf{q}\,dS = \int_s (l\xi + m\eta + n\zeta)\,dS$, by Stoke's Theorem.

circulation $= 0 \Leftrightarrow l\xi + m\eta + n\zeta = 0.$

In brief we write cir. in place of circulation.

Suppose cir. $= 0$ along any closed path c over the surface S.

Then $l\xi + m\eta + n\zeta = 0$ at any point of S, by (1). It means that vortex lines all lie on S, *i.e.*, S is composed of the vortex lines. The particles which lie on S at any time t, lie on S' at time $t + \delta t$. Similarly c' is taken at time $t + \delta t$ in place of the path c at t. Hence circulation along c = circulation along c' so that circulation along $c' = 0$. It means that S' is composed of vortex lines.

Hence any surface composed of vortex lines continues to be composed of vortex lines as it moves with the fluid but the intersection of two such surfaces must be vortex so that vortex lines move with the fluid.

Problem 1. *Find the necessary and sufficient condition that vortex lines may be at right angles to the streamlines.* (IAS-2005)

Solution. The equations of the streamlines and the vortex lines are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}, \quad \ldots(1)$$

and

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dz}{\zeta}. \quad \ldots(2)$$

The equations (1) and (2) are at right angles. It follows that

$$u\xi + v\eta + w\zeta = 0$$

$$u\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + v\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + w\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0.$$

In order that $u\,dx + v\,dy + w\,dz$ may be a perfect differential, we have

$$u\,dx + v\,dy + w\,dz = \lambda\,d\phi = \lambda\left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right)$$

$$\Rightarrow \quad u = \lambda\frac{\partial\phi}{\partial x}, v = \lambda\frac{\partial\phi}{\partial y}, w = \lambda\frac{\partial\phi}{\partial z},$$

which determines the necessary and sufficient condition.

Problem 2. *In an incompressible fluid the vorticity at every point is constant in magnitude and direction; prove that the components of velocity u, v, w are the solutions of Laplace equation.* (IAS-2004 & 2010)

Solution. Let $\underline{\Omega}$ be the vorticity at any point in an incompressible fluid, then

$$\underline{\Omega} = \xi\mathbf{i} + \eta\mathbf{j} + \zeta\mathbf{k}$$

where $\xi = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$

The magnitude and direction cosines of its direction are given by

$$\Omega = \sqrt{\xi^2 + \eta^2 + \zeta^2} \text{ and } \frac{\xi}{\Omega}, \frac{\eta}{\Omega}, \frac{\zeta}{\Omega}$$

Differentiating η partially with regard to z and ζ with regard to y and subtracting, we have

$$\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

$$\Rightarrow \quad \frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0. \quad \ldots(1)$$

By equation of continuity,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$$

or $\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = \frac{-\partial u}{\partial x}$. Putting this in (1),

we get $\frac{\partial^2 u}{\partial z^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x^2} = 0$

or $\nabla^2 u = 0$

This proves that velocity components satisfy Laplace's equation.

Strength of vortex

If k be the circulation round a closed curve c which encloses a vortex tube, then by Stoke's theorem,

$$k = \int_c \mathbf{q}.d\mathbf{r} = \int_s \text{curl } \mathbf{q}.\hat{n}\,dS = \int_s 2\mathbf{W}.\hat{n}\,dS$$

$$= 2\int_s (l\xi + m\eta + n\zeta)\,dS \text{ as } \hat{n} = \hat{n}(\xi, \eta, \zeta)$$

If πa^2 be the cross section of the vortex tube (supposed small), then

$$k = 2(l\xi + m\eta + n\zeta)\int_s dS = 2\pi a^2(l\xi + m\eta + n\zeta) = \text{constant.}$$

This quantity k is also called strength of the vortex. This strength is taken to be positive when the circulation round the vortex is anticlockwise.

Def. Rectilinear Vortices

The liquid within the cylinder, of which the circle is a cross section, is said to form a rectilinear or columnar vortex.

To determine the complex potential due to a rectilinear vortex of strength k.

Consider a cylindrical vortex tube in xy-plane. Let the tube be surrounded by incompressible irrotational fluid. The motion is purely two dimensional so that

$$w=0,\ u=u(x,y),\ v=v(x,y).$$

Consequently $\xi=0,\ \eta=0,\ \zeta=\frac{1}{2}\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)$

But $\quad u=-\frac{\partial\phi}{\partial x}=-\frac{\partial\psi}{\partial y},\ v=-\frac{\partial\phi}{\partial y}=\frac{\partial\psi}{\partial x}$

$\therefore \quad 2\zeta=\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=\frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}$ (in polar co-ordinates)

Let $\zeta=\omega=$ const. within the cylinder and $\zeta=0$ outside the cylinder.

Symmetry exists about the origin and so ψ must be function of r only. Then

$$\frac{d^2\psi}{dr^2}+\frac{1}{r}\frac{d\psi}{dr}=\frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right)=\begin{cases}2\zeta \text{ within vortex}\\ 0 \text{ outside}\end{cases}$$

This $\Rightarrow r\frac{d\psi}{dr}=\begin{cases}\zeta r^2+A \text{ within vortex}\\ B \text{ outside}\end{cases}$

This $\Rightarrow \quad \psi=c\log r$ outside the vortex.

But $v=-\frac{\partial\phi}{\partial y}=\frac{\partial\psi}{\partial x}$ i.e., $v=-\frac{\partial\phi}{r\partial\theta}=\frac{\partial\psi}{\partial r}$

or $\quad \frac{\partial\phi}{\partial\theta}=(-r).\frac{c}{r}$ or $\phi=-c\theta$

$$W=\phi+i\psi=-c\theta+ic\log r=ic\,[\log r+i\theta]=ic\log re^{i\theta}$$

or $\quad W=ic\log z.$

Let k be the strength of vortex. Then

$$k=\int_0^{2\pi}\left(-\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right)r\,d\theta=\int_0^{2\pi}c\,d\theta=2\pi c.$$

$\therefore \quad W=\frac{ik}{2\pi}\log z.$

If the vortex be at $z=z_0$, then the complex potential at any point z outside the vortex is given by

$$W=\frac{ik}{2\pi}\log(z-z_0).$$

If there be a large number of vortices of strengths $k_1,k_2,\ldots,k_n$ at $z=z_1,z_2,\ldots,z_n$ respectively, then

$$W=\frac{i}{2\pi}\sum_{m=1}^{n}k_m\log(z-z_m).$$

Deduction. If the vortex has a circular cross section, then also $W=\frac{ik}{2\pi}\log(z-z_0)$ for vortex at $z=z_0$. The same proof will be given for this also.

Remark. Article 8.4 can be restated as follows:

(i) To discuss the motion set up by a vortex filament of given strength located at a given point in a liquid at rest at infinity.

(ii) To find velocity component of a single vortex.

$$u=-\frac{k}{2\pi}\left(\frac{y-y_0}{r^2}\right),\ v=\frac{k}{2\pi}\left(\frac{x-x_0}{r^2}\right)$$

where $\quad r^2=(x-x_0)^2+(y-y_0)^2$

Motion due to m vortices

To find velocity at a point $P(z)$ due to m vortices of strengths k_n at the points z_n $(n=1,2,\ldots,m)$.

The complex potential due to a vortex of strength k_1 at $z=z_1$ is $W=\frac{ik_1}{2\pi}\log(z-z_1)$.

$$\frac{dW}{dz}=\frac{ik_1}{2\pi}\cdot\frac{1}{z-z_1}.\ \text{But } \frac{dW}{dz}=-u_1+iv_1.$$

$$\therefore\quad -u_1+iv_1=\frac{ik_1}{2\pi}\cdot\frac{1}{z-z_1}=\frac{ik_1}{2\pi}\cdot\frac{1}{(x-x_1)+i(y-y_1)}$$

Take $\quad r_1^2=(x-x_1)^2+(y-y_1)^2.$

Then $\quad -u_1+iv_1=\frac{ik_1}{2\pi}\cdot\frac{(x-x_1)-i(y-y_1)}{r_1^2}$

This $\Rightarrow -u_1=\frac{k_1}{2\pi}\cdot\frac{(y-y_1)}{r_1^2},\ v_1=\frac{k_1}{2\pi}\cdot\left(\frac{x-x_1}{r_1^2}\right)$

u_1,v_1 are velocity components at P due to the vortex of strength k_1. Similarly if u_n,v_n be velocity components due to vortex of strength k_n, then

$$u_n=-\frac{k_n}{2\pi}\left(\frac{y-y_n}{r_n^2}\right),\ v_n=\frac{k_n}{2\pi}\cdot\left(\frac{x-x_n}{r_n^2}\right)$$

If u,v are velocity components along x and y axes at $P(z)$ due to vortices of strengths $k_1,k_2,\ldots k_m$ at points $z_1,z_2,\ldots,z_m$, then

$$u=\sum_{n=1}^{m}u_n=-\sum_{n=1}^{m}\frac{k_n}{2\pi}\left(\frac{y-y_n}{r_n^2}\right)$$

$$v=\sum_{n=1}^{m}v_n=\sum_{n=1}^{m}\frac{k_n}{2\pi}\left(\frac{x-x_n}{r_n^2}\right)$$

Also $-u+iv=\sum_n\frac{ik_n}{2\pi}\frac{1}{z-z_n},\ W=\sum_n\frac{ik_n}{2\pi}\log(z-z_n)$

Single vortex in the field of several vortices.

The value of ψ at any point inside of a circular vortex tube is given by

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{d\psi}{dr}\right]=2\zeta.$$

The solution of this is $\psi=\frac{1}{2}\zeta r^2+c\log r+\alpha$, where c and α are constants of integration. Velocity at right angle to the radius vector

$$=\frac{d\psi}{dr}=\zeta r+(c/r).$$

But velocity at the origin is finite so we take $c=0$.

$\therefore\quad \frac{d\psi}{dr}=\zeta r$, so that $\left(\frac{d\psi}{dr}\right)_{r=0}=0$

This $\Rightarrow$ velocity at the origin due to a single vortex is zero.

Thus the vortex filament (vortex) induces no velocity at its centre.

From this discussion we can say that if a vortex is in the field of several vortices, then its motion in the liquid will be on account of other vortices.

Remark. The above article 8.6 can be expressed as:

Show that a single rectilinear vortex in an unlimited mass of liquid remains stationary, and when such a vortex is in the presence of other vortices it has no tendency to move of itself but its motion through the liquid is entirely due to the velocities caused by the other vortices.

Centre of vortices.

Consider fluid motion due to n vortices of strength k_s at points z_s $(s=1,2,\ldots,n)$. The complex potential at $P(z)$ outside the filament is

$$W=\sum_{s=1}^{n}\frac{ik_s}{2\pi}\log(z-z_s).\ \text{Hence } -u+iv=\frac{dW}{dz}=\sum_s\frac{ik_s}{2\pi}\cdot\frac{1}{z-z_s}.$$

The velocity components u_r,v_r of the vortex of strength k_r at z_r are produced by other vortices and so

$$-u_r+iv_r=\sum_{s=1}^{n}\frac{ik_s}{2\pi}\cdot\frac{1}{z_r-z_s},\qquad \ldots(1)$$

where s takes all values from 1 to n except r.

By (1), $\sum_r k_r(-u_r+iv_r)=\sum_{r=1}\sum_{s=1,\ s\neq r}\frac{ik_rk_s}{2\pi}\cdot\frac{1}{z_r-z_s}$

or $\quad \sum_r k_r(-u_r+iv_r)=0.\qquad \ldots(2)$

Since the terms $\frac{ik_r\,k_s}{2\pi(z_r-z_s)}$ and $\frac{ik_s\,k_r}{2\pi(z_s-z_r)}$ cancel each other.

$(2)\Rightarrow \sum k_r u_r=0,\ \sum k_r v_r=0.\qquad \ldots(3)$

The equation (3) shows that if we regard k_s as a mass, the centre of gravity G of the vortex filament remains stationary during their motion about one another. *The point G is called the centre of the system of vortices.*

Two vortex filaments.

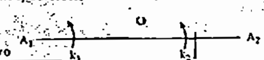


Fig. 8.2.

Consider fluid motion due to two vortices of strengths k_1,k_2 at $z=z_1,z_2$ respectively. Then the complex potential W is given by

$$W=\frac{ik_1}{2\pi}\log(z-z_1)+\frac{ik_2}{2\pi}\log(z-z_2)\qquad \ldots(1)$$

$$\Rightarrow -u+iv=\frac{dW}{dz}=\frac{ik_1}{2\pi}\cdot\frac{1}{z-z_1}+\frac{ik_2}{2\pi}\cdot\frac{1}{z-z_2}\qquad \ldots(2)$$

The velocity of A_1 is due to the vortex A_2 and so

$$-u_1+iv_1=\left(\frac{ik_2}{2\pi}\cdot\frac{1}{z-z_2}\right)_{z=z_1},\ \text{by (2)}$$

or $\quad -u_1+iv_1=\frac{ik_2}{2\pi}\cdot\frac{1}{z_1-z_2}\qquad \ldots(3)$

$\therefore\quad q_1=|-u_1+iv_1|=\frac{k_2}{2\pi\,.\,A_1A_2}\qquad \ldots(4)$

The velocity of vortex at A_2 is due to vortex at A_1 and so

$$-u_2+iv_2=\frac{ik_1}{2\pi}\left(\frac{1}{z-z_1}\right)_{z=z_2},\ \text{by (2)}$$

or $\quad -u_2+iv_2=\frac{ik_1}{2\pi}\cdot\frac{1}{z_2-z_1}\qquad \ldots(5)$

or $\quad q_2=\frac{k_1}{2\pi\,.\,A_1A_2}\qquad \ldots(6)$

By (3) and (5), $k_1(-u_1+iv_1)+k_2(-u_2+iv_2)=0$

or $\quad -(k_1u_1+k_2u_2)+i(k_1v_1+k_2v_2)=0$

or $\quad k_1u_1+k_2u_2=0,\quad k_1v_1+k_2v_2=0\qquad \ldots(7)$

This $\Rightarrow (k_1u_1+k_2u_2)+i(k_1v_1+k_2v_2)=0$

$\Rightarrow k_1(u_1+iv_1)+k_2(u_2+iv_2)=0\Rightarrow k_1\dot{z}_1+k_2\dot{z}_2=0$

or $\quad \frac{k_1\dot{z}_1+k_2\dot{z}_2}{k_1+k_2}=0$, if $k_1+k_2\neq 0$

or $\quad \frac{d}{dt}\left(\frac{k_1z_1+k_2z_2}{k_1+k_2}\right)=0$ or $\frac{k_1z_1+k_2z_2}{k_1+k_2}=$ const. $\qquad \ldots(8)$

The point $(k_1z_1+k_2z_2)/(k_1+k_2)$ may be called the centre G of vortices by analogy with the centre of gravity, the strengths are regarded as masses. The point G is fixed, by (8).

Since $\quad k_1\,.\,A_1G=k_2\,.\,A_2O.$

Hence $\frac{A_2G}{A_1G}=\frac{k_1}{k_2}$ or $\frac{A_1G+A_2G}{A_1G}=\frac{k_1+k_2}{k_2}$

or $A_1G=A_1A_2\cdot\frac{k_1}{k_1+k_2}$

By (4), $q_1=\frac{k_2}{k_1+k_2}\cdot A_1A_2\cdot\frac{k_1+k_2}{2\pi(A_1A_2)^2}=A_1G\cdot\omega$ as $\frac{dr}{dt}=\omega\times r$

where $\omega=\frac{k_1+k_2}{2\pi(A_1A_2)^2}$.

Similarly, $q_2=A_2G\cdot\omega$.

Hence vortex A_1 moves with velocity $A_1G\cdot\omega$ perpendicular to A_1A_2 whereas vortex A_2 moves with velocity $A_2G\cdot\omega$ perpendicular to A_1A_2 in opposite direction. If $k_1>k_2$, then G will be on A_1A_2 produced and if $k_2>k_1$, it will be on the line A_2A_1 produced.

This shows that the line A_1A_2 rotates about G with angular velocity ω. According to (7), the system has no velocity along A_1A_2. Hence A_1A_2 remains constant in length.

Deduction Vortex pair

If the vortices are of equal and opposite strengths, *i.e.*, $k_1=-k_2=k$ say. In this case both the vortices have a velocity $k/2\pi.A_1A_2$ at right angle to A_1A_2 and so they move in parallel paths through the liquid. Such an arrangement of vortices is called vortex pair.

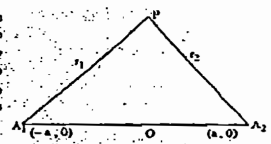


Fig. 8.3.

To determine stream function

$$W=\frac{ik}{2\pi}\log(z-z_1)-\frac{ik}{2\pi}\log(z-z_2)$$

$$=\frac{ik}{2\pi}\log\left(\frac{z-z_1}{z-z_2}\right)=\frac{ik}{2\pi}\log\left(\frac{r_1e^{i\theta_1}}{r_2e^{i\theta_2}}\right)$$

This $\Rightarrow \psi=\frac{k}{2\pi}\log\left(\frac{r_1}{r_2}\right)$

Stream lines are given by $\psi=$ const, *i.e.*,

$$\frac{k}{2\pi}\log\left(\frac{r_1}{r_2}\right)=\text{const, or } \frac{r_1}{r_2}=\text{const.}=b \text{ or } \frac{(x+a)^2+y^2}{(x-a)^2+y^2}=b^2$$

or $$x^2+y^2+2a\left(\frac{1+b^2}{1-b^2}\right)x+a^2=0$$

This shows that the stream lines are co-axial circles with A_1 and A_2 as limiting points.

Image of vortex w.r.t. a plane.

The image of a vortex filament of strength k is a vortex of strength $-k$ placed at the optical image of the given vortex. (Agra 2002)

Proof. Let the vortices of strength $+k$ and $-k$ be at z_1 and z_2 respectively. Then the complex potential at any point

$$P(z) \text{ is } W=\frac{ik}{2\pi}\log(z-z_1)-\frac{ik}{2\pi}\log(z-z_2)$$

or $$\phi+i\psi=\frac{ik}{2\pi}\log\left(\frac{z-z_1}{z-z_2}\right)$$

or $$\psi=\frac{k}{2\pi}\log\left|\frac{z-z_1}{z-z_2}\right|$$

or $$\psi=\frac{k}{2\pi}\log\frac{r_1}{r_2}$$

where $|z-z_1|=r_1$, $|z-z_2|=r_2$.

If $r_1=r_2$, then $OA=OB$, *i.e.*, B is the optical image of A w.r.t. y-axis. Now $\psi=0$ so that there is no flow across the plane. Now the required result follows.

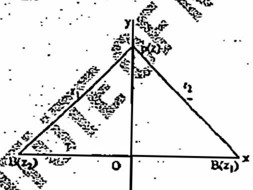


Fig. 8.4.

Image of a vortex outside a circular cylinder

(i) *The image of a vortex +k outside a circular cylinder is a vortex −k at the inverse point and vortex +k at the centre of the cylinder.*

(ii) *The image of a vortex +k inside a circular cylinder is a vortex −k at the inverse point.*

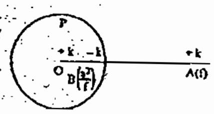


Fig. 8.5.

(i) is proved as follows:

To determine the image of a vortex $+k$ at A $(OA=f)$ w.r.t. the circular cylinder whose centre is at O and radius is a.

The complex potential due to vortex $+k$ at A is

$$W=\frac{ik}{2\pi}\log(z-f).$$

When we insert a circular cylinder of radius $|z|=a$, the complex potential is

$$W=\frac{ik}{2\pi}\log(z-f)-\frac{ik}{2\pi}\log\left(\frac{a^2}{z}-f\right), \text{ by circle theorem.}$$

$[W=f(z)+\bar{f}(a^2/z)]$

Addition of a constant term $ik\log(-f)$ does not change the nature of W. Then

$$W=\frac{ik}{2\pi}\log(z-f)-\frac{ik}{2\pi}\log\left(\frac{a^2}{z}-f\right)+ik\log(-f)$$

$$=\frac{ik}{2\pi}\log(z-f)-\frac{ik}{2\pi}\log\left(1-\frac{a^2}{zf}\right)$$

But $\left(1-\frac{a^2}{zf}\right)=\frac{1}{z}\left(z-\frac{a^2}{f}\right)$. Hence

$$W=\frac{ik}{2\pi}\log(z-f)-\frac{ik}{2\pi}\log\left(z-\frac{a^2}{f}\right)+\frac{ik}{2\pi}\log(z-0) \quad \ldots(1)$$

This is the complex potential of

(i) vortex $+k$ at A $(z=f)$, (ii) vortex $-k$ at B $(z=a^2/f)$,

(iii) vortex $+k$ at O $(z=0)$.

Since $OB=a^2/f$. Hence A and B are inverse points. If we put $z=ae^{i\theta}$ in (1), then $\psi=$ const.

For $$\psi=\frac{k}{2\pi}\log\frac{|z-f|\cdot|z|}{\left|z-\frac{a^2}{f}\right|}=\frac{k}{2\pi}\log\frac{PA.PO}{PB}$$

$$=\frac{k}{2\pi}\log\frac{aPA}{PB}=\frac{k}{2\pi}\log\left(\frac{af}{a}\right)=\text{const.}$$

Hence circle is a stream line.

Thus the image of a vortex $+k$ at A outside the cylinder is a vortex $-k$ at B, the inverse point of A and $+k$ at O, the centre of the circle.

Similarly we can prove the result (ii).

Vortex Doublet

Vortex doublet is a combination of a vortex of strength +k and a vortex of strength −k at a small distance δs apart s.t. $\lim_{\delta s\to 0}\frac{k}{2\pi}\cdot\delta s=$ *finite quantity* $=\mu$, *say, where* $k\to\infty$.

Then μ is called strength of the doublet and the axis of doublet is in the sense from $-k$ to k.

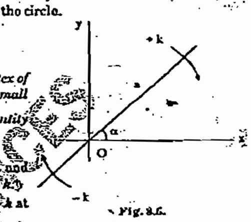


Fig. 8.6.

Consider vortices of strengths $+k$ and $-k$ at $z=ae^{i\alpha}$ and $z=-ae^{i\alpha}$ respectively so that the axis of doublet is inclined at an angle α with x-axis. The complex potential at any point $P(z)$ is given by

$$W=\frac{ik}{2\pi}\log(z-ae^{i\alpha})-\frac{ik}{2\pi}\log(z+ae^{i\alpha})$$

$$=-\frac{ik}{2\pi}\left[\log z\left(1+\frac{ae^{i\alpha}}{z}\right)-\log z\left(1-\frac{ae^{i\alpha}}{z}\right)\right]$$

$$=-\frac{ik}{2\pi}\left[\log\left(1+\frac{ae^{i\alpha}}{z}\right)-\log\left(1-\frac{ae^{i\alpha}}{z}\right)\right]$$

$$=-\frac{ik}{2\pi}\left[\left\{\frac{ae^{i\alpha}}{z}-\frac{1}{2}\left(\frac{ae^{i\alpha}}{z}\right)^2+\frac{1}{3}\left(\frac{ae^{i\alpha}}{z}\right)^3-\ldots\right\}+\left\{\frac{ae^{i\alpha}}{z}+\frac{1}{2}\left(\frac{ae^{i\alpha}}{z}\right)^2+\frac{1}{3}\left(\frac{ae^{i\alpha}}{z}\right)^3+\ldots\right\}\right]$$

$$=-\frac{2ik}{2\pi}\left[\frac{ae^{i\alpha}}{z}+\frac{1}{3}\left(\frac{ae^{i\alpha}}{z}\right)^3+\ldots\right] \text{ But } \frac{k}{2\pi}\cdot 2a=\mu \text{ as } k\to\infty, a\to 0.$$

$$=-\frac{ika}{\pi}\left[\frac{e^{i\alpha}}{z}+\frac{1}{3}\left(\frac{e^{i\alpha}}{z}\right)^3\cdot a^2+\ldots\right] \quad \ldots(1)$$

$$=-\mu i\left[\frac{e^{i\alpha}}{z}+0+0+\ldots\right]$$

or $$W=-i\mu\frac{e^{i\alpha}}{z} \quad \ldots(2)$$

This is the required complex potential due to vortex doublet with its axis inclined at an angle α with x-axis.

Deduction. By (2), $W=-\frac{i\mu}{r}e^{i(\alpha-\theta)}$.

This $\Rightarrow \psi=-\frac{\mu}{r}\cos(\alpha-\theta)$.

Take $\mu=Ub^2/2$, $\alpha=\pi/2$.

Then $\mu=-\frac{Ub^2}{r}\sin\theta$.

This shows that the motion due to a vortex at the centre with its axis perpendicular to the axis of motion is the same as the motion due to a circular cylinder of radius b moving with velocity U along x-axis.

Spiral vortex

The combination of a source and a vortex is known as spiral vortex.

Let there be a source of strength $+m$ and vortex $+k$ both at the origin. Then

$$W=-m\log z+\frac{ik}{2\pi}\log z=\left(\frac{ik}{2\pi}-m\right)\log z.$$

To find pressure due to spiral vortex.

$$\frac{dW}{dz}=(-m+ik)\cdot\frac{1}{z}.$$

or $$q^2=\frac{dW}{dz}\cdot\frac{d\overline{W}}{d\bar{z}}=\left(\frac{-m-ik}{z}\right)\left(\frac{-m+ik}{\bar{z}}\right)=\frac{m^2+k^2}{|z|^2}=\frac{m^2+k^2}{r^2}$$

For steady flow the pressure equation is given by

$$\frac{p}{\rho}+\frac{1}{2}q^2+\Omega=C$$

$$\therefore \quad \frac{p}{\rho}=C-\Omega-\frac{1}{2}\frac{m_1^2}{r^2} \text{ where } m_1^2=m^2+k^2.$$

This gives the pressure at any point z due to the given system.

Rectilinear vortices with circular section

Consider a straight circular vortex tube of radius a in a liquid in a direction perpendicular to xy-plane. The motion will be purely two dimensional and there is only one component ζ of vorticity in the direction of ζ-axis, its value is constant within the tube and zero outside. Thus

$$u = u(x, y),\ v = v(x, y),\ w = 0.$$

Also $\xi = 0, \eta = 0, \zeta = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$

But $u = -\frac{\partial \phi}{\partial x} = -\frac{\partial \psi}{\partial y},\ v = -\frac{\partial \phi}{\partial y} = \frac{\partial \psi}{\partial x}$

$\therefore$ $2\zeta = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = \frac{\partial^2 \psi}{\partial r^2} + \frac{1}{r}\frac{\partial \psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \psi}{\partial \theta^2}$ (in polar co-ordinates)

Let ζ = const. within the tube

and $\zeta = 0$ outside the tube.

Symmetry exists about the origin and so ψ must be a function of r only. Hence

$$\frac{d^2\psi}{dr^2} + \frac{1}{r}\frac{d\psi}{dr} = \frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right) = \begin{cases} 2\zeta & \text{within tube} \\ 0 & \text{outside tube} \end{cases}$$

Again integrating,

$$\psi_1 = \zeta\frac{r^2}{2} + A\log r + C \quad \text{for } r < a \qquad \ldots (1)$$

$$\psi_2 = B\log r + D \quad \text{for } r > a \qquad \ldots (2)$$

We now require that

(i) ψ is finite at the origin so that $A = 0$.

(ii) Solutions for $r < a$ and $r > a$ be coincident at the boundary of vortex, which requires that ψ and $d\psi/dr$ be continuous at $r = a$.

This $\Rightarrow$ $\zeta\frac{a^2}{2} + 0 + C = B\log a + D$ as $A = 0$ ... (3)

and $\zeta a = \frac{B}{a}$ ... (4)

For a particular solution of (2), we can take $D = 0$.

Now (3) $\Rightarrow \zeta\frac{a^2}{2} + C = \zeta a^2 \log a \Rightarrow C = \zeta a^2\left(\log a - \zeta\frac{a^2}{2}\right)$

Now (1) and (2) become,

$$\psi_1 = \frac{\zeta}{2}(r^2 - a^2) + \zeta a^2 \log a,\ r < a \qquad \ldots (5)$$

$$\psi_2 = \zeta a^2 \log r,\ r > a. \qquad \ldots (6)$$

The motion outside the cylindrical vortex is irrotational and ϕ_2 exists s.t.

$$v = -\frac{1}{r}\frac{\partial \phi_2}{\partial \theta} = -\frac{\partial \phi_2}{\partial y} = \frac{\partial \psi_2}{\partial x} = \frac{\partial \psi_2}{\partial r}.$$

or $-\frac{1}{r}\frac{\partial \phi_2}{\partial \theta} = \frac{\zeta a^2}{r}$. Then $\phi_2 = -\zeta a^2 \theta$.

Let k be the strength of the vortex so that k is a circlation round the tube. Then

$$k = 2\pi a^2 \zeta \text{ and so } \psi_1 = \frac{k}{4\pi a^2}(r^2 - a^2) + \frac{k}{2\pi}\log a$$

$$\psi_2 = \frac{k}{2\pi}\log r,\ \phi_2 = -\frac{k\theta}{2\pi}.$$

$$W_2 = \phi_2 + i\psi_2 = \frac{ik}{2\pi}[i\theta + \log r] = \frac{ik}{2\pi}\log re^{i\theta}$$

or $W_2 = \frac{ik}{2\pi}\log z.$

Finally, $\psi = \frac{k}{4\pi a^2}(r^2 - a^2) + \frac{k}{2\pi}\log a,\ r < a$ ... (7)

$$W = \frac{ik}{2\pi}\log z,\quad r > a. \qquad \ldots (8)$$

Note that $q = \left|\frac{dW}{dz}\right| = \left|\frac{ik}{2\pi z}\right| = \frac{k}{2\pi r},\ r > a$

Remember these results. The applications of these results are very wide.

Problem. *Find the velocity of a vortex placed inside an infinite circular cylinder*

Rankine combined vortex

Rankine combined vortex consists of a circular vortex with axis vertical in a mass of liquid which is moving irrotationally under the action of gravity only, the upper surface being exposed to atmospheric pressure Π. The external forces are derivable from the potential Ω, the potential energy function per unit mass, *i.e.*,

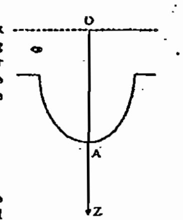


Fig. 8.7

$$\Omega = mgh.$$

Here $\Omega = 1gh = -gz$

where $OA = h = -z$.

We take the origin in the axis of the vortex and in the level of the liquid at infinity. We measure z in downward direction.

We know that

$$q = \frac{k}{2\pi r},\ r > a \text{ and } q = \frac{kr}{2\pi a^2},\ r < a, \text{(See Note of 8.13).}$$

By pressure equation for steady motion

$$\frac{p}{\rho} + \frac{1}{2}q^2 + \Omega = C$$

$\therefore$ $\frac{p}{\rho} = A - \frac{k^2}{8\pi^2 r^2} + gz$ for $r > a$ ... (1)

$$\frac{p}{\rho} = B + \frac{k^2 r^2}{8\pi^2 a^4} + gz \text{ for } r < a \qquad \ldots (2)$$

In order to preserve the continuity, p is the same on $r = a$

so that

$$A - \frac{k}{8\pi^2 a^2} + gz = B + \frac{k^2 a^2}{8\pi^2 a^4} + gz$$

or $A = B + \frac{k^2}{4\pi a^2}$

Hence $\frac{p}{\rho} = B + \frac{k^2}{4\pi^2 a^2} - \frac{k^2}{8\pi^2 r^2} + gz,\ r > a$ ... (3)

$$\frac{p}{\rho} = B + \frac{k^2 r^2}{8\pi^2 a^4} + gz,\ r < a. \qquad \ldots (4)$$

Considering the origin in the general level of the free surface when $r > a$ s.t. $z = 0, p = \Pi$ when $r \to \infty$.

Then $\frac{\Pi}{\rho} = B + \frac{k^2}{4\pi^2 a^2}$, by (3).

Now (3) and (4) take the form

$$\frac{p - \Pi}{\rho} = -\frac{k^2}{8\pi^2 r^2} + gz,\ r > a$$

$$\frac{p - \Pi}{\rho} = -\frac{k^2}{4\pi^2 a^2} + \frac{k^2 r^2}{8\pi^2 a^4} + gz,\ r < a.$$

On the free surface, $p = \Pi$, hence free surface is given by

$$gz = \frac{k^2}{8\pi^2 r^2},\ r > a$$

$$gz = \frac{k^2}{4\pi^2 a^2} - \frac{k^2 r^2}{8\pi^2 a^2},\ r < a.$$

To obtain the depth of depression at A, we put $r = 0$ in the second result so that

$$gz = \frac{k^2}{4\pi^2 a^2} \text{ or } z = \frac{k^2}{4\pi^2 a^2 g}$$

Rectilinear vortex with elliptic section

To show that a rectilinear vortex whose cross-section is an ellipse and whose spin is constant can maintain its form rotating as if it were a solid cylinder in an infinite mass of liquid.

Let the cross-section of rectilinear vortex be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad \ldots (1)$$

which is defined in terms of elliptic co-ordinates by $\xi = \alpha$ where $x + iy = c\cosh(\xi + i\eta)$ so that $a = c\cosh\alpha, b = c\sinh\alpha$ so that $a + b = ce^{\alpha}, a - b = ce^{-\alpha}$. Spin ζ is uniform within (1). The stream function satisfies

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 2\zeta \text{ within the vortex} \qquad \ldots (2)$$

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 0 \text{ outside the vortex.} \qquad \ldots (3)$$

Suppose the motion is due to rotation of an elliptic cylinder about the axis with uniform angular velocity ω. The complex potential for irrotational motion with circulation

k at an outside point is given by

$$W = \frac{i\omega}{4}(a + b)^2 e^{-2(\xi + i\eta)} + \frac{ik}{2\pi}(\xi + i\eta)$$

This $\Rightarrow \psi = \frac{\omega}{4}(a + b)^2 e^{-2\xi}\cos 2\eta + \frac{k\xi}{2\pi}$ outside vortex ... (4)

Circulation k is the strength of vortex so that

k = 2. angular velocity of vortex.area of cross section of vortex = $2\zeta . \pi ab$.

Note that ζ is velocity of rotation of liquid inside the vortex tube and ω is velocity of rotation of elliptic vortex when it is treated as a solid.

Let $\psi = Ax^2 + By^2$ within vortex. ... (5)

Then $\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} = 2A + 2B.$

Using (2), we get $A + B = \zeta$ ... (6)

Let $F = \frac{x^2}{a^2} + \frac{y^2}{b^2} - 1 = 0$. Then ∇F is parallel to the unit normal at any point of (1). If l, m be the direction cosines of the unit normal, then

$$il + jm = \frac{\nabla F}{|\nabla F|}. \text{ Then } l = \frac{px}{a^2}, m = \frac{py}{b^2}, p = 1\Big/\left(\frac{x^2}{a^4} + \frac{y^2}{b^4}\right)^{1/2}.$$

The boundary condition is :

Normal component of velocity of liquid = Normal component of velocity of boundary

i.e., $p\frac{x}{a^2}u + p\frac{y}{b^2}v = p\frac{x}{a^2}(-\omega y) + p\frac{y}{b^2}(\omega x)$

$$\left[\text{For } \dot{x} = \frac{d}{dt}(r\cos\theta) = -(r\sin\theta)\dot{\theta} = -y\omega,\ \frac{dy}{dt} = \frac{d}{dt}(r\sin\theta) = r\cos\theta\dot{\theta} = x\omega\right].$$

or $\frac{x}{a^2}\left(-\frac{\partial \psi}{\partial y}\right) + \frac{y}{a^2}\left(\frac{\partial \psi}{\partial x}\right) = -\frac{x}{a^2}\omega y + \frac{y}{b^2}\omega x$, where ψ given by (5)

or $\frac{x}{a^2}(-2By) + \frac{y}{b^2}(2Ax) = \omega xy\left[\frac{1}{b^2} - \frac{1}{a^2}\right]$

or $2(a^2 A - b^2 B) = \omega(a^2 - b^2).$ ... (7)

Further, tangential velocity $\partial\psi/\partial\xi$ is continuous on the boundary $\xi = \alpha$. Within the vortex $\psi = Ax^2 + By^2$,

$$x = c\cosh\xi\cos\eta,\ y = c\sinh\xi\sin\eta.$$

$$\frac{\partial \psi}{\partial \xi} = \frac{\partial \psi}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial \psi}{\partial y}\frac{\partial y}{\partial \xi}$$

$$= 2Ax \,.\, c\sinh\xi \,.\, \cos\eta + 2By \,.\, c\cosh\xi\sin\eta$$

$$\left(\frac{\partial\psi}{\partial\xi}\right)_{\xi=\alpha}=2\,[Ac^2\cosh\xi\sinh\xi\cos^2\eta+Bc^2\sinh\xi\cosh\xi\sin^2\eta]\ \xi=\alpha$$

$$=2\,[Aab\cos^2\eta+Bab\sin^2\eta]$$

or $$\left(\frac{\partial\psi}{\partial\xi}\right)_{\xi=\alpha}=ab\,[(A+B)+(A-B)\cos 2\eta] \quad \text{... (8)}$$

Outside the boundary,

$$\psi=\frac{\omega}{4}(a+b)^2 e^{-2\xi}\cos 2\eta+\frac{k\xi}{2\pi}$$

$$\frac{\partial\xi}{\partial\xi}=\frac{\omega}{4}(a+b)^2(-2)e^{-2\xi}\cos 2\eta+\frac{k}{2\pi}$$

$$\left(\frac{\partial\psi}{\partial\xi}\right)_{\xi=\alpha}=-\frac{\omega}{2}(a+b)^2\left(\frac{a-b}{a+b}\right)\cos 2\eta+\frac{k}{2\pi}$$

$$=\frac{k}{2\pi}-\frac{\omega}{2}(a^2-b^2)\cos 2\eta \quad \text{... (9)}$$

Equating (8) to (9),

$$ab\,[(A+B)+(A-B)\cos 2\eta]=\frac{k}{2\pi}-\frac{\omega}{2}(a^2-b^2)\cos 2\eta.$$

This $\Rightarrow A+B=\dfrac{k}{2\pi ab},\ A-B=-\dfrac{\omega}{2ab}(a^2-b^2)$ ... (10)

Also $A+B=\zeta,\ a^2A-b^2B=(\omega/2)(a^2-b^2)$

$\Rightarrow A+B=(k/2\pi ab)=\zeta \quad (a^2A-b^2B)/(A-B)=-ab$

$\Rightarrow A+B=(k/2\pi ab)=\zeta\ Aa(a+b)=Bb(a+b)$ or $Aa=Bb$

$\Rightarrow \dfrac{k}{2\pi ab}=\zeta,\ Aa=Bb=\dfrac{\zeta ab}{a+b}$ ... (11)

Putting these values in (10),

$$-\frac{\omega}{2ab}(a^2-b^2)=\frac{\zeta}{a+b}(b-a)$$

or $$\omega=\frac{2ab\zeta}{(a+b)^2}=\frac{2\zeta(1-e^2)^{1/2}}{[1+(1-e^2)^{1/2}]^2}\text{ as } b^2=a^2(1-e^2)$$

or $$\omega=k\zeta,\text{ where } K=\frac{2(1-e^2)^{1/2}}{[1+(1-e^2)^{1/2}]^2}=\text{const.}$$

It means that the vortex maintains its form rotating as if it were moving as solid cylinder in an infinite liquid.

To obtain the path of any particle in the vortex. Let (x, y) be the co-ordinates of any fluid particle in the vortex referred to the axis of section. Then,

$$\dot{x}-\omega y=u=-\frac{\partial\psi}{\partial y}=-2By=-2y\cdot\frac{\zeta a}{a+b}$$

$$\dot{y}+\omega x=v=\frac{\partial\psi}{\partial x}=2Ax=2x\cdot\frac{\zeta b}{a+b}$$

or $$\dot{x}=y\left[\omega-\frac{2\zeta a}{a+b}\right],\ \dot{y}=x\left[\frac{2\zeta b}{a+b}-\omega\right]$$

Now $$\omega-\frac{2\zeta a}{a+b}=\frac{2ab\zeta}{(a+b)^2}-\frac{2a\zeta}{a+b}=\frac{2a\zeta(-a)}{(a+b)(a+b)}=-\frac{a}{b}\omega$$

$$\frac{2\zeta b}{a+b}-\omega=\frac{2\zeta b}{a+b}-\frac{2\zeta ab}{(a+b)^2}=\frac{b}{a}\omega.$$

Hence $\dot{x}=-\dfrac{a}{b}\omega y,\ \dot{y}=\dfrac{b}{a}\omega x$

This $\Rightarrow \ddot{x}=-\dfrac{a}{b}\omega\dot{y},\ \ddot{y}=\dfrac{b}{a}\omega\dot{x}$

$\Rightarrow \ddot{x}=\dfrac{a}{b}\omega\left(\dfrac{b}{a}\omega x\right),\ \ddot{y}=\dfrac{b}{a}\omega\left(-\dfrac{a}{b}\omega y\right)$

$\Rightarrow \ddot{x}=-\omega^2x,\ \ddot{y}=-\omega^2y.$

Integrating these we get

$$x=a\,L\cos(t\sqrt{(\omega^2)}+\varepsilon);\ y=b\,L\sin(t\sqrt{(\omega^2)}+\varepsilon)$$

or $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=L^2$, showing thereby that path of the particle is a similar ellipse.

Periodic time $=T=\dfrac{2\pi}{\omega}=\dfrac{2\pi(a+b)^2}{2\zeta ab}=\dfrac{\pi(a+b)^2}{\zeta ab}$

[For, if $\ddot{x}=-\mu x$, then $T=2\pi/\sqrt{\mu}$]

Working Rule

If there exist two vortices of strengths k_1, k_2 at A_1, A_2 respectively, then the line A_1A_2 rotates with angular velocity

$$\omega=\frac{k_1+k_2}{2\pi(A_1A_2)^2}$$

about O, whereas the points A_1, A_2 move with velocity

$$\frac{k_2}{2\pi A_1A_2},\ \frac{k_1}{2\pi A_1A_2}$$

along a line (in opposite directions) respectively perpendicular to the line A_1A_2.

Problem 1. *An elliptic cylinder is filled with liquid which has molecular rotation ω at every point and whose particles move in a plane perpendicular to the axis; prove that the stream lines are similar ellipses described in periodic time*

$$\frac{\pi}{\omega}\cdot\frac{a^2+b^2}{ab}$$

Solution. There exists no liquid outside the cylinder. The stream function ψ satisfies the condition

$$\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}=2\omega. \quad \text{... (1)}$$

The cross section is given by

$$(x^2/a^2)+(y^2/b^2)-1=F(x,y)=0$$

Assume $\psi=Ax^2+By^2$. Subjecting ψ to (1), $A+B=\omega$.

The boundary condition is

normal component of velocity of liquid at boundary

= normal component of velocity of the surface.

This $\Rightarrow\ u\dfrac{2x}{a^2}+v\dfrac{2y}{b^2}=0$ as boundary is fixed.

[For ∇F is parallel to the normal on the boundary]

or $$\frac{\partial\psi}{\partial y}\frac{x}{a^2}+\frac{\partial\psi}{\partial x}\frac{y}{b^2}=0 \text{ or } -2By\frac{x}{a^2}+2Ax\frac{y}{b^2}=0$$

or $$a^2A-B^2B=0 \text{ or } \frac{A}{b^2}=\frac{B}{a^2}=\frac{A+B}{a^2+b^2}=\frac{\omega}{a^2+b^2}$$

Hence $$\psi=\frac{\omega}{a^2+b^2}[b^2x^2+a^2y^2]=\frac{\omega a^2b^2}{a^2+b^2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)$$

Stream lines are given by ψ = const., *i.e.*,

$$\frac{\omega a^2b^2}{a^2+b^2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=\text{const. or } \frac{x^2}{a^2}+\frac{y^2}{b^2}=\text{const.},$$

which are similar ellipses. Since the cylinder is fixed so that

$$\dot{x}=u=-\frac{\partial\psi}{\partial y}=-\frac{\omega a^2b^2}{a^2+b^2}\left(\frac{2y}{b^2}\right)$$

$$\dot{y}=v=\frac{\partial\psi}{\partial x}=\frac{\omega a^2b^2}{a^2+b^2}\left(\frac{2x}{a^2}\right)$$

This $\Rightarrow$ $$\ddot{x}=-\frac{\omega a^2b^2}{a^2+b^2}\cdot\frac{2}{b^2}\dot{y}=-\left(\frac{\omega a^2b^2}{a^2-b^2}\right)^2\frac{2}{b^2}\frac{2x}{a^2}$$

and $$\ddot{y}=\frac{\omega a^2b^2}{a^2+b^2}\frac{2\dot{x}}{a^2}=-\left(\frac{\omega a^2b^2}{a^2+b^2}\right)^2\frac{2}{a^2}\frac{2y}{b^2}$$

$$\Rightarrow \ddot{x}=-\mu^2x,\quad y=-\mu^2y,\quad \mu=\frac{2\omega ab}{a^2-b^2}$$

Integration of these gives

$$x=aL\cos(\mu t+\varepsilon),\ y=bL\sin(\mu t+\varepsilon).$$

This $\Rightarrow \dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=L^2$ which is a similar ellipse.

Periodic time $=T=\dfrac{2\pi}{\sqrt{\mu^2}}=\dfrac{2\pi}{\mu}=\dfrac{\pi(a^2+b^2)}{\omega ab}.$

Problem 2. *When an infinite liquid contains two parallel and equal vortices of the same strength at a distance 2b apart and the spin is in the same sense in both, show that the relative stream lines are given by*

$$\log(r^4+b^4-2b^2r^2\cos 2\theta)-\frac{r^2}{2b^2}=\text{const.}$$

θ being measured from the join of vortices, the origin being its middle point.

Solution. The figure 8.7 is self explanatory. Since

$$\cos A=(b^2+c^2-a^2)/2bc \text{ or } a^2=b^2+c^2-2bc\cos A.$$

In view of this we have

$$r_1^2=r^2+b^2-2rb\cos\theta$$

and $$r_2^2=r^2+b^2-2rb\cos(\pi-\theta)$$

i.e., $$r_2^2=r^2+b^2+2rb\cos\theta$$

$$r_1^2r_2^2=(r^2+b^2)-4r^2b^2\cos^2\theta$$

$$=r^4+b^4-2r^2b^2\cos 2\theta.$$

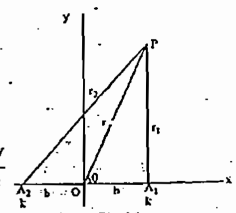


Fig. 8.8.

The stream function ψ at $P(r,\theta)$ is given by

$$\psi=\frac{k}{2\pi}\log r_1+\frac{k}{2\pi}\log r_2=\frac{k}{2\pi}\log r_1r_2$$

$$=\frac{k}{4\pi}\log r_1^2r_2^2$$

or $$\psi=\frac{k}{4\pi}\log(r^4+b^4-2r^2b^2\cos 2\theta) \quad \text{... (1)}$$

The line A_1A_2 revolves about O with angular velocity

$$\omega=\frac{k_1+k_2}{2\pi(A_1A_2)^2}=\frac{2k}{2\pi(2b)^2}=\frac{k}{4\pi b^2}$$

so that the velocity at any point P due to this motion is

$$\omega r=\frac{kr}{4\pi b^2}\quad as\quad v=r\omega.$$

To reduce the vortex system to rest we impose a velocity $-\dfrac{kr}{4\pi b^2}$ and if ψ' be the stream function due to this addition, then

$$\frac{\partial\psi'}{\partial r}=-\frac{1}{r}\frac{\partial\phi'}{\partial\theta}=-\frac{kr}{4\pi b^2}\quad\therefore\quad\psi'=-\frac{kr^2}{8\pi b^2}.$$

Hence the stream lines relative to the vortices are given by

$$\psi=\frac{k}{4\pi}\log(r^4+b^4-2r^2b^2\cos 2\theta)-\frac{kr^2}{8\pi b^2}=\text{const.}$$

or $$\log(r^4+b^4-2r^2b^2\cos 2\theta)-\frac{r^2}{2b^2}=\text{const.}$$

Problem 3. *When an infinite liquid contains two parallel equal and opposite vortices at a distance 2b, prove that the stream lines relative to the vortices are given by the equation*

$$\log\left[\frac{x^2+(y-b)^2}{x^2+(y+b)^2}\right]+\frac{y}{b}=c,$$

the origin being the middle point of the join which is taken for the axis of y.

or $\quad k\left[\frac{y}{2b}+\log\left(\frac{r_1}{r_2}\right)\right]=\text{const.}$

Solution. Suppose there are two vortices of strengths $k, -k$ at A_1, A_2, respectively s.t. origin O is the middle point of $A_1A_2 = 2b$ and A_1A_2 lie along y-axis. Both vortices will move along a line parallel to x-axis with velocity.

$$q=\frac{k}{2\pi(A_1A_2)}=\frac{k}{2\pi\,.\,2b}=\frac{k}{4\pi b}.$$

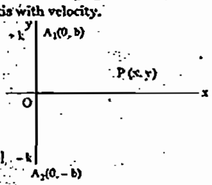


Fig. 5.9.

The complex potential W at P due to these two vortices is given by

$$W=\frac{ki}{2\pi}\log(z-ib)-\frac{ki}{2\pi}\log(z+ib)$$

$$=\frac{ki}{2\pi}\log[x+i(y-b)]-\frac{ki}{4\pi}\log[x+i(y+b)],$$

Equating imaginary parts from both sides,

$$\psi=\frac{k}{4\pi}\log[x^2+(y-b)^2]-\frac{k}{4\pi}\log[x^2+(y+b)^2]$$

or $\quad \psi=\frac{k}{4\pi}\log\left[\frac{x^2+(y-b)^2}{x^2+(y+b)^2}\right]$

To reduce the vortex system to rest, we superimpose a velocity $\frac{k}{4\pi b}$ along x-axis to the system. Let ψ' be the stream function due to this addition, then

$$-\frac{\partial\psi'}{\partial y}=-\frac{\partial\phi'}{\partial x}=-\frac{k}{4\pi b}.\quad\therefore\quad \psi'=\frac{ky}{4\pi b}.$$

Hence the stream lines relative to vortices are given by

$$\psi=\frac{k}{4\pi}\log\left[\frac{x^2+(y-b)^2}{x^2+(y+b)^2}\right]+\frac{ky}{4\pi b}=\text{const.}$$

or $\quad \log\left[\frac{x^2+(y-b)^2}{x^2+(y+b)^2}\right]+\frac{y}{b}=c.$

If we take $PA_1=r_1=[x^2+(y-b)^2]^{1/2}$,

and $\quad PA_2=r_2=[x^2+(y+b)^2]^{1/2}$, then the last gives

$$\log\frac{r_1^2}{r_2^2}+\frac{y}{b}=\text{const.}\quad\text{or}\quad \log\frac{r_1}{r_2}+\frac{y}{2b}=c$$

Problem 4. *If n rectilinear vortices of the same strength k are symmetrically arranged along generators of a circular cylinder of radius a in an infinite liquid, prove that the vortices will move round the cylinder uniformly in time $\frac{8\pi^2a^2}{(n-1)k}$, and find the velocity at any point of the liquid.*

Solution. The figure is self explanatory. The n vortices are at.

$$A_0, A_1, A_2, \ldots, A_{n-1} \text{ s.t.}$$

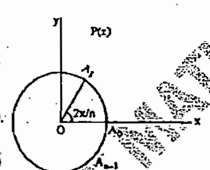


Fig. 5.10.

$$\angle A_0OA_1=\angle A_1OA_2=\ldots=\angle A_{n-1}OA_1=\frac{2\pi}{n}.$$

The co-ordinates of the points A_r are given by

$z=z_r=ae^{(2\pi/n)ir}$ where $r=0,1,2,\ldots,n-1$.

These are n roots of the equation $z^n-a^n=0$.

[For $z^n-a^n=0\Rightarrow z^n=a^ne^{2\pi ri}$]

Hence $\quad z^n-a^n=(z-z_0)(z-z_1)\ldots(z-z_{n-1})$

The complex potential due to n vortices at P is given by

$$W=\frac{ik}{2\pi}[\log(z-z_0)+\log(z-z_1)+\ldots+\log(z-z_{n-1})]$$

$$=\frac{ik}{2\pi}\log(z-z_0)(z-z_1)\ldots(z-z_{n-1})=\frac{ik}{2\pi}\log(z^n-a^n)\qquad\ldots(1)$$

For the point A_0, $z=a$ so that $r=a$, $\theta=0$.

If W' is the complex potential at A_0, then

$$W'=W-\frac{ik}{2\pi}\log(z-a)=\frac{ik}{2\pi}[\log(z^n-a^n)-\log(z-a)]$$

$$\phi'+i\psi'=\frac{ik}{2\pi}[\log(r^ne^{in\theta}-a^n)-\log(re^{i\theta}-a)]$$

$$\therefore\quad \psi'=\frac{k}{4\pi}[\log(r^{2n}+a^{2n}-2r^na^n\cos n\theta)-\log(r^2+a^2-2ra\cos\theta)]$$

$$\frac{\partial\psi'}{\partial r}=\frac{k}{4\pi}\left[\frac{2nr^{2n-1}-2nr^{n-1}a^n\cos n\theta}{r^{2n}+a^{2n}-2r^na^n\cos n\theta}-\frac{2r-2a\cos\theta}{r^2+a^2-2ra\cos\theta}\right]$$

$$\frac{\partial\psi'}{\partial\theta}=\frac{k}{4\pi}\left[\frac{2nr^na^n\sin n\theta}{r^{2n}-2r^na^n\cos n\theta+a^{2n}}-\frac{(2ra\sin\theta)}{r^2+a^2-2ra\cos\theta}\right]$$

$$\left(\frac{\partial\psi'}{\partial r}\right)_{r=a}=\frac{k}{4\pi a}\left[n\left(\frac{1-\cos n\theta}{1-\cos n\theta}\right)-\left(\frac{1-\cos\theta}{1-\cos\theta}\right)\right]=\frac{k}{4\pi a}(n-1)$$

$$\left(\frac{\partial\psi'}{\partial\theta}\right)_{r=a}=\frac{k}{4\pi}\left[\frac{n\sin n\theta}{1-\cos n\theta}-\frac{\sin\theta}{1-\cos\theta}\right]$$

Since $\quad \lim_{x\to0}\frac{F(x)}{G(x)}=\lim_{x\to0}\frac{F'(x)}{G'(x)}=\lim_{x\to0}\frac{F''(x)}{G''(x)}\left[\text{form }\frac{0}{0}\right]$

$$\left(\frac{\partial\psi'}{\partial\theta}\right)_{r=a}=\frac{k}{4\pi}\left[\frac{n^2\cos n\theta}{n\sin n\theta}-\frac{\cos\theta}{\sin\theta}\right]\text{ as }\theta\to0$$

$$=\frac{k}{4\pi}\left[\frac{-n^3\sin n\theta}{n^2\cos n\theta}-\frac{(-\sin\theta)}{\cos\theta}\right]\text{ as }\theta\to0$$

$$=\frac{k}{4\pi}[0+0]=0.$$

Finally, $\quad \frac{\partial\psi'}{\partial\theta}=\frac{k}{4\pi a}(n-1),\ \frac{\partial\psi'}{\partial\theta}=0$ as $r\to a$, $\theta\to0$.

Consequently, the velocity q_0 of the vortex A_0 is given by

$$q_0=\left[\left(\frac{\partial\psi'}{\partial r}\right)^2+\left(\frac{1}{r}\frac{\partial\psi'}{\partial\theta}\right)^2\right]^{1/2}=\frac{k(n-1)}{4\pi a}$$

This proves that the whole of velocity is along the tangent and there is no velocity along the normal to the circle. Hence the vortices will move round the cylinder with uniform velocity $k(n-1)/4\pi a$. The time of one complete revolution

$$=\frac{\text{distance}}{\text{velocity}}=\frac{2\pi a}{k(n-1)/4\pi a}=\frac{8\pi^2a^2}{(n-1)k}$$

Remark. Putting $z=re^{i\theta}$ in (1).

$$\phi+i\psi=\frac{ik}{2\pi}\log(r^ne^{in\theta}-a^n)=\frac{ik}{4\pi}\log(r^n\cos n\theta-a^n+ir^n\sin n\theta)$$

This $\Rightarrow$ $\psi=\frac{k}{4\pi}\log[(r^n\cos n\theta-a^n)^2+r^{2n}\sin^2 n\theta]$

or $\quad \psi=\frac{k}{4\pi}\log[r^{2n}-2a^nr^n\cos n\theta+a^{2n}]$.

Problem 5. *If (r_1,θ_1), (r_2,θ_2), ... be polar co-ordinates at time t of a system of rectilinear vortices of strengths $k_1, k_2, \ldots$ prove that $\Sigma kr^2=\text{const.}$ and $\Sigma kr^2\dot\theta=\frac{1}{2\pi}\Sigma k_1k_2$.*

Solution. Write $z=r_1e^{i\theta_1}=x_1+iy_1$, $z_2=r_2e^{i\theta_2}=x_2+iy_2$, ... The complex potential due to the vortices $k_1, k_2, \ldots$ at $z_1, z_2\ldots$ is given by

$$W=\sum_{n=1}^{\infty}\frac{ik_n}{2\pi}\log(z-z_n)$$

If u_μ, v_μ are the velocity components of the vortex k_μ, then since the motion of k_μ is due to the other vortices only, the complex potential for this motion is given by

$$W_\mu=W-\frac{ik_\mu}{2\pi}\log(z-z_\mu)=\sum_{n=1}^{\infty}\frac{ik_n}{2\pi}\log(z-z_n)\text{ except }n=\mu.$$

$$u_\mu-iv_\mu=-\left(\frac{dW_\mu}{dz}\right)_{z=z_\mu}=\sum_{n=1}^{\infty}\frac{ik_n}{2\pi}\frac{1}{z_\mu-z_n}\text{ except }n=\mu$$

Notice that

$$\sum_{\mu=1}^{\infty}k_\mu z_\mu\left(\frac{dW_\mu}{dz}\right)_{z=z_\mu}=\sum_{\mu=1}^{\infty}k_\mu z_\mu\sum_{n=1}^{\infty}\frac{ik_n}{2\pi}\cdot\frac{1}{z_\mu-z_n}\text{ except }n=\mu$$

$$=\frac{i}{2\pi}\sum_{\mu=1}^{\infty}\sum_{\substack{n=1\\n\neq\mu}}^{\infty}\frac{k_\mu k_nz_\mu}{z_\mu-z_n}$$

On the right hand side, every product term $k_\mu k_nz_\mu$ occurs twice and the two terms containing this are

$$\frac{k_\mu k_nz_\mu}{z_\mu-z_n}+\frac{k_nk_\mu z_n}{z_n-z_\mu}=\frac{k_\mu k_n}{z_\mu-z_n}(z_\mu-z_n)=k_\mu k_n$$

Hence $\quad \sum_{\mu=1}^{\infty}k_\mu z_\mu(-u_\mu+iv_\mu)=\frac{i}{2\pi}\sum_{\mu=1}^{\infty}\sum_{\substack{n=1\\n\neq\mu}}^{\infty}k_\mu k_n$

Equating real and imaginary parts

$$\sum_{\mu=1}^{\infty}k_\mu(-x_\mu u_\mu-y_\mu v_\mu)=0$$

and $\quad \sum_{\mu=1}^{\infty}k_\mu(x_\mu v_\mu-y_\mu u_\mu)=\frac{1}{2\pi}\sum_{\mu=1}^{\infty}\sum_{\substack{n=1\\n\neq\mu}}^{\infty}k_\mu k_n$

or $\quad \sum_{\mu=1}^{\infty}k_\mu\left(x_\mu\frac{dx_\mu}{dt}+y_\mu\frac{dy_\mu}{dt}\right)=0\qquad\ldots(1)$

and $\quad \sum_{\mu=1}^{\infty}k_\mu\left(x_\mu\frac{dy_\mu}{dt}-y_\mu\frac{dx_\mu}{dt}\right)=\frac{1}{2\pi}\sum_{\mu=1}^{\infty}\sum_{\substack{n=1\\n\neq\mu}}^{\infty}k_\mu k_n\qquad\ldots(2)$

(1) $\Rightarrow$ $\sum_{n=1}^{\infty}k_n(x_ndx_n+y_ndy_n)=0$

Integrating, $\sum_{n=1}^{\infty}k_n\frac{(x_n^2+y_n^2)}{2}=\text{const}$ or $\sum_n k_nr_n^2=\text{const.}$

It is also expressible as $\Sigma kr^2=\text{const.}$

This proves the first required result.

In polar co-ordinates, $\tan\theta_n=y_n/x_n$.

Differentiating w.r.t. t,

$$\sec^2\theta_n\cdot\dot\theta_n=\frac{\dot y_nx_n-\dot x_ny_n}{x_n^2}$$

$$x_n\dot y_n-\dot x_ny_n=x_n^2\sec^2\theta_n\cdot\dot\theta_n=r_n^2\cos^2\theta_n\cdot\sec^2\theta_n\cdot\dot\theta_n=r_n^2\dot\theta_n$$

Using this in (2), we get

$$\sum_{\mu=1}^{\infty}k_\mu r_\mu^2\dot\theta_\mu=\frac{1}{2\pi}\sum_{\mu=1}^{\infty}\sum_{\substack{n=1\\n\neq\mu}}^{\infty}k_\mu k_n$$

It is also expressible as $\Sigma kr^2\dot\theta=\frac{1}{2\pi}\Sigma k_1k_2$

This proves the second required result.

Problem 6. *An infinitely long line vortex of strength m, parallel to the axis of z, is situated in infinite liquid bounded by a rigid wall in the plane $y=0$. Prove that, if there be no field of force, the surfaces of equal pressure are given by*

$$[(x-a)^2+(y-b)^2][(x-a)^2+(y+b)^2]=C(y^2+b^2-(x-a)^2)$$

where (a,b) are the co-ordinates of the vortex and C is a parametric constant.

Solution. The image of vortex of strength m at $A_1(a, b)$ relative to the wall $y = 0$, *i.e.*, x-axis is a vortex of strength $-m$ at $A_2(a, -b)$. These two vortices form a vortex pair. The complex potential due to this system is given by

$$W = \frac{im}{2\pi}\log[z-(a+ib)] - \frac{im}{2\pi}\log[z-(a-ib)]$$

or $$W = \frac{im}{2\pi}\log[(x-a)+i(y-b)] - \frac{im}{2\pi}\log[(x-a)+i(y+b)]$$

$$-\frac{dW}{dz} = -\frac{im}{2\pi}\frac{1}{[z-(a+ib)]} + \frac{im}{2\pi}\frac{1}{[z-(a-ib)]}$$

or $$u - iv = -\frac{im}{2\pi}\frac{1}{[(x-a)+i(y-b)]} + \frac{im}{2\pi}\frac{1}{[(x-a)+i(y+b)]} \quad \ldots (1)$$

Writing $r_1^2 = (x-a)^2 + (y-b)^2,\ r_2^2 = (x-a)^2 + (y+b)^2,\ r_2^2 - r_1^2 = 4yb \quad \ldots (2)$

$$u - iv = -\frac{im}{2\pi r_1^2}[(x-a) - i(y-b)] + \frac{im}{2\pi}\frac{[(x-a)-i(y+b)]}{r_2^2}$$

Equating real and imaginary parts,

$$u = \frac{m}{2\pi}\left[-\frac{(y-b)}{r_1^2} + \frac{(y+b)}{r_2^2}\right]$$

$$-v = \frac{m}{2\pi}\left[-\frac{(x-a)}{r_1^2} + \frac{(x-a)}{r_2^2}\right]$$

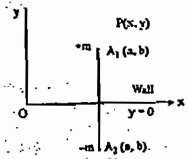


Fig. 8.11.

Each of these vortices will move with velocity

$$q = \frac{m}{2\pi(A_1A_2)} = \frac{m}{2\pi.2b} = \frac{m}{4\pi b}$$

along a line perpendicular to A_1A_2, *i.e.*, along x-axis.

To reduce the vortex system to rest, we superimpose a velocity $-m/4\pi b$ along x-axis to the system. In this case,

$$\left.\begin{aligned} u &= -\frac{m}{2\pi}\left[\frac{y-b}{r_1^2} - \frac{y+b}{r_2^2}\right] - \frac{m}{4\pi b} \\ v &= \frac{m}{2\pi}(x-a)\left[\frac{1}{r_1^2} - \frac{1}{r_2^2}\right]\end{aligned}\right\} \quad \ldots (3)$$

Now the motion is steady. By Bernoulli's equation for steady motion,

$$\frac{p}{\rho} + \frac{1}{2}q^2 = \text{const.} \quad \ldots (4)$$

Surfaces of equal pressure are given by p = const. $\quad \ldots (5)$

By (4) and (5), q^2 = const. or $u^2 + v^2$ = const. $= c$

or $$\frac{m^2}{4\pi^2}\left[\frac{y-b}{r_1^2} - \frac{y+b}{r_2^2}\right]^2 + \frac{m^2}{16\pi^2 b^2} + \frac{2m^2}{8\pi^2 b}\left[\frac{y-b}{r_1^2} - \frac{y+b}{r_2^2}\right] + \frac{m^2}{4\pi}(x-a)^2\left[\frac{1}{r_1^2} - \frac{1}{r_2^2}\right] = c$$

or $$\frac{(y-b)^2}{r_1^4} + \frac{(y+b)^2}{r_2^4} - \frac{2(y^2-b^2)}{r_1^2 r_2^2} + \frac{1}{b}\left[\frac{y-b}{r_1^2} - \frac{y+b}{r_2^2}\right] + (x-a)^2\left[\frac{1}{r_1^4} + \frac{1}{r_2^4} - \frac{2}{r_1^2 r_2^2}\right] = \frac{4\pi^2 c}{m^2} - \frac{1}{4b^2} = c_1$$

or $$\frac{(x-a)^2 + (y-b)^2}{r_1^4} + \frac{(x-a)^2 + (y+b)^2}{r_2^4} - \frac{2}{r_1^2 r_2^2}[(x-a)^2 + y^2 - b^2] + \frac{1}{b}\left[\frac{y-b}{r_1^2} - \frac{y+b}{r_2^2}\right] = c_1$$

or $$\frac{1}{r_1^2} + \frac{1}{r_2^2} - \frac{2[(x-a)^2 + y^2 - b^2]}{r_1^2 r_2^2} + \frac{[(r_2^2 - r_1^2)y - b(r_2^2 + r_1^2)]}{b r_1^2 r_2^2} = c_1$$

Multiplying by $r_1^2 r_2^2$,

$$r_2^2 + r_1^2 - 2[(x-a)^2 + y^2 - b^2] + \frac{1}{b}(4yb)y - (r_2^2 + r_1^2) = c_1 r_1^2 r_2^2$$

or $$r_1^2 r_2^2 = -\frac{2}{c_1}[(x-a)^2 + y^2 - b^2 - 2y^2]$$

or $$r_1^2 r_2^2 = c_3[y^2 + b^2 - (x-a)^2] \text{ where } -\frac{2}{c_1} = c_3.$$

From this the required result follows.

Problem 7. *A long fixed cylinder of radius a is surrounded by infinite incompressible liquid, and there is in the liquid a vortex filament of strength k which is parallel to the axis of the cylinder at a distance c ($c > a$) from this axis. Given that there is no circulation round any circuit enclosing the cylinder but not the filament, show that the speed q of the fluid at the surface of the cylinder is*

$$\frac{k}{2\pi a}\left[1 - \left(\frac{c^2 - a^2}{r^2}\right)\right],$$

r being the distance of the point considered from the filament (Meerut 98)

Solution. The image of vortex of strength $+k$ at A outside the cylinder is a vortex $-k$ at B, inverse of A and vortex $+k$ at O, the centre.

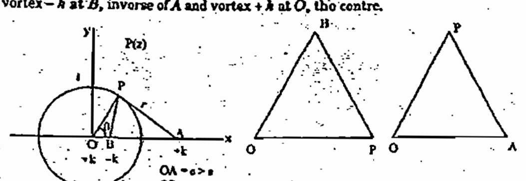


Fig. 8.12.

$\therefore$ $OB \cdot OA = (\text{radius})^2$.

$\therefore$ $f = OB = \dfrac{a^2}{OA} = \dfrac{a^2}{c}$ so that $cf = a^2$.

$$a^2 + c^2 - r^2 = 2ac\cos\theta \quad \ldots (1)$$

For $\cos A = (b^2 + c^2 - a^2)/2bc$.

The complex potential at P ($z = ae^{i\theta}$) due to this system is given by

$$W = \frac{ik}{2\pi}\log(z-c) - \frac{ik}{2\pi}\log(z-f) + \frac{ik}{2\pi}\log(z-0)$$

$$\frac{dW}{dz} = \frac{ik}{2\pi}\left[\frac{1}{z-c} + \frac{1}{z-f} - \frac{1}{z}\right] = \frac{ik}{2\pi}\cdot\frac{z^2 - 2fz + a^2}{z(z-c)(z-f)}$$

$$q = \left|\frac{dW}{dz}\right| = \frac{k}{2\pi}\cdot\frac{|z^2 - 2fz + a^2|}{|z| \cdot |z-c| \cdot |z-f|} = \frac{k}{2\pi}\cdot\frac{|z^2 - 2fz + a^2|}{OP \cdot BP \cdot AP}$$

or $$q = \frac{k}{2\pi}\cdot\frac{|z^2 - 2fz + a^2|}{a\,r\,BP} \quad \ldots (2)$$

$\therefore$ $OB \cdot OA = a^2 = OP^2$

$\therefore$ $\dfrac{OB}{OP} = \dfrac{OP}{OA}$. Also $\angle BOP = \angle POA$.

Hence the $\Delta s\ OBP$ and OPA are similar.

This $\Rightarrow$ $\dfrac{OB}{OP} = \dfrac{OP}{OA} = \dfrac{BP}{AP}$

$$\Rightarrow BP = \frac{AP \cdot OP}{OA} = \frac{ra}{c}$$

$$\Rightarrow a\,r\,BP = ar\frac{ra}{c} = \frac{a^2r^2}{c} \quad \ldots (3)$$

$$z^2 - 2fz + a^2 = (z-f)^2 + a^2 - f^2 = (ae^{i\theta} - f)^2 + a^2 - f^2$$
$$= (a\cos\theta - f + i\,a\sin\theta)^2 + a^2 - f^2$$
$$= (a\cos\theta - f)^2 - a^2\sin^2\theta + 2ia\sin\theta(a\cos\theta - f) + a^2 - f^2$$
$$= 2a^2\cos^2\theta - 2af\cos\theta + 2ia\sin\theta(a\cos\theta - f)$$
$$= 2a[\cos\theta(a\cos\theta - f) + i\sin\theta(a\cos\theta - f)]$$
$$= 2a(a\cos\theta - f)(\cos\theta + i\sin\theta)$$

$$|z^2 - 2fz + a^2| = a(f - a\cos\theta) = 2\left(\frac{a^2}{c} - a\cos\theta\right)a$$

$$= 2\frac{a^2}{c}(a - c\cos\theta) = \frac{a}{c}[2a^2 - a^2 - c^2 + r^2], \text{ by (1)}$$

$$= \frac{a}{c}[a^2 - c^2 + r^2] = \frac{a}{c}r^2\left[1 - \left(\frac{c^2 - a^2}{r^2}\right)\right] \quad \ldots (4)$$

Writing (2) with the help of (3) and (4),

$$q = \frac{k}{2\pi}\cdot\frac{ar^2}{c}\left[1 - \left(\frac{c^2 - a^2}{r^2}\right)\right]\cdot\frac{c}{a^2r^2} = \frac{k}{2\pi a}\left[1 - \left(\frac{c^2 - a^2}{r^2}\right)\right]$$

Problem 8: *Find the motion of a straight vortex filament in an infinite region plane wall to which the filament is parallel, and prove that the pressure defect at any point of the wall due to the filament is proportional to $\cos^2\theta\cos 2\theta$, where θ is the inclination of the plane through the filament and the point to the plane through the filament perpendicular to the wall.*

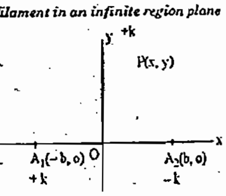


Fig. 8.13.

Solution. $z_1 = x_1 + iy_1,\ \bar{z}_1 = x_1 - iy_1$. The image of vortex $+k$ at A_1 ($z = z_1$) w.r.t. the wall, *i.e.*, x-axis is a vortex $-k$ at A_2 ($z = \bar{z}_1$). These two vortices together form a vortex pair which will move parallel to x-axis with velocity

$$\frac{k}{2\pi(A_1A_2)} = \frac{k}{4\pi y_1}.$$

Each vortex moves with this velocity parallel to x-axis so that $x_1 = x_1(t)$ but y_1 remains constant.

Hence $\dot{x}_1 = k/4\pi y_1$.

The complex potential due to this vortex pair at $P(z)$ is given by

$$W = \frac{ik}{2\pi}\log(z - z_1) - \frac{ik}{2\pi}\log(z - \bar{z}_1) \quad \ldots (1)$$

$$\frac{dW}{dz} = \frac{ik}{2\pi}\left[\frac{1}{z - z_1} - \frac{1}{z - \bar{z}_1}\right]$$

$$u_0 - iv_0 = -\left(\frac{dW}{dz}\right)_{z=0} = \frac{ik}{2\pi}\left[\frac{1}{z_1} - \frac{1}{\bar{z}_1}\right]$$

where $u_0 - iv_0$ is complex velocity at O. We shall study the pressure at some point say O on x-axis. Let $OA_1 = r$, $\angle OA_1A_2 = \theta$, then $y_1 = r\cos\theta$, $x_1 = r\sin\theta$, $x_1^2 + y_1^2 = r^2$.

$$u_0 - iv_0 = \frac{ik}{2\pi}\left[\frac{x_1 - iy_1}{x_1^2 + y_1^2} - \frac{x_1 + iy_1}{x_1^2 + y_1^2}\right] = \frac{ky_1}{\pi r^2} = \frac{k}{\pi r}\cos\theta.$$

This $\Rightarrow$ $u_0 = \dfrac{k}{\pi r}\cos\theta,\ v_0 = 0 \Rightarrow q^2 = u_0^2 + v_0^2 \Rightarrow q = u_0 = \dfrac{k\cos\theta}{\pi r}$

where q is fluid velocity at O.

By (1), $(W)_{z=0} = \dfrac{ik}{2\pi}[\log(-z_1) - \log(-\bar{z}_1)]$

$$= \frac{ik}{2\pi}[\log(-x_1 - iy_1) - \log(-x_1 + iy_1)]$$

Equating real parts,

$$(\phi)_{x=0} = -\frac{k}{2\pi}\left[\tan^{-1}\left(\frac{-y_1}{-x_1}\right) - \tan^{-1}\left(\frac{y_1}{-x_1}\right)\right]$$

$$= -\frac{k}{2\pi}\left[\tan^{-1}\left(\frac{y_1}{x_1}\right) + \tan^{-1}\left(\frac{y_1}{x_1}\right)\right] \text{as } \tan^{-1}(-0) = -\tan^{-1} 0$$

$$= -\frac{k}{\pi}\tan^{-1}\left(\frac{y_1}{x_1}\right)$$

$$\left(\frac{\partial\phi}{\partial t}\right)_{x=0} = -\frac{k}{\pi}\cdot\frac{1}{1+(y_1/x_1)^2}\cdot\left(\frac{-y_1}{x_1^2}\right)\cdot\dot{x}_1$$

$$= \frac{k}{\pi}\cdot\frac{y_1}{x_1^2+y_1^2}\left(\frac{k}{4\pi y_1}\right) = \frac{k^2}{4\pi^2 r^2}$$

By pressure equation, $\frac{p}{\rho} + \frac{1}{2}q^2 - \frac{\partial\phi}{\partial t} = c$ ... (2)

Let $p = p_0$ when the vortex is not present, *i.e.*, when

$$q = 0, \frac{\partial\phi}{\partial t} = 0. \quad \text{Now (2)} \Rightarrow \frac{p_0}{\rho} = c.$$

Hence $\frac{p - p_0}{\rho} = \frac{\partial\phi}{\partial t} - \frac{1}{2}q^2 = \frac{k^2}{4\pi^2 r^2} - \frac{1}{2}\cdot\frac{k^2\cos^2\theta}{\pi^2 r^2}$

$$= \frac{k^2}{4\pi^2 r^2}(1 - 2\cos^2\theta) = \frac{k^2}{4\pi^2 r^2}\cdot(-\cos 2\theta)$$

or $$p - p_0 = \frac{-k^2\rho}{4\pi^2 y_1^2}\cos^2\theta\cos 2\theta$$

This $\Rightarrow p_0 - p = \frac{k^2\rho}{4\pi^2 y_1^2}\cos^2\theta\cos 2\theta$

$\Rightarrow$ $p_0 - p$ is proportional to $\cos^2\theta \cdot \cos 2\theta$.

This proves the required result.

Problem 9. *An infinite liquid contains two parallel, equal and opposite rectilinear vortex filaments at a distance 2b. Show that the paths of the fluid particles relative to the vortices can be represented by the equation*

$$\log\left(\frac{r^2 + b^2 - 2rb\cos\theta}{r^2 + b^2 + 2rb\cos\theta}\right) + \frac{r\cos\theta}{b} = \text{const.}$$

or $$\log\left[\frac{(x-b)^2 + y^2}{(x+b)^2 + y^2}\right] + \frac{x}{b} = c.$$

O is the middle point of the join which is taken as x-axis.

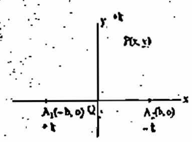


Fig. 8.14.

Solution. Let the vortices of strengths $+k, -k$ be at $A_1(-b, 0), A_2(b, 0)$ s.t. A_1A_2 is along x-axis. The complex potential due to this vortex pair at $P(x, y)$ is

$$W = \frac{ik}{2\pi}\log(z+b) - \frac{ik}{2\pi}\log(z-b)$$

or $$\phi + i\psi = \frac{ik}{2\pi}[\log(x+b+iy) - \log(x-b+iy)].$$

Equating imaginary parts from both sides,

$$\psi = \frac{k}{4\pi}[\log\{(x+b)^2 + y^2\} - \log\{(x-b)^2 + y^2\}] \quad ... (1)$$

The vortex pair will move along a line parallel to y-axis with velocity

$$\frac{k}{2\pi(A_1A_2)} = \frac{k}{2\pi(2b)} = \frac{k}{4\pi b}.$$

To reduce the system to rest, we have to superimpose a velocity $(-k/4\pi b)$ parallel to y-axis. If ψ' be the stream function due to this disturbance, then

$$\frac{-k}{4\pi b} = v = -\frac{\partial\phi'}{\partial y} = \frac{\partial\psi'}{\partial x} \Rightarrow \psi' = -\frac{kx}{4\pi b}$$

The stream lines relative to the vortex system are given by ψ = const., i.e.,

$$\frac{k}{4\pi}[\log\{(x+b)^2 + y^2\} - \log\{(x-b)^2 + y^2\}] - \frac{kx}{4\pi b} = \text{const.}$$

or $$-\log\{(x+b)^2 + y^2\} + \log\{(x-b)^2 + y^2\} + \frac{x}{b} = \text{const.} \quad ... (2)$$

Changing into polar co-ordinates,

or $$\log\left\{\frac{(x-b)^2 + y^2}{(x+b)^2 + y^2}\right\} + \frac{x}{b} = \text{const.}$$

$$\log\left\{\frac{(r\cos\theta - b)^2 + r^2\sin^2\theta}{(r\cos\theta + b)^2 + r^2\sin^2\theta}\right\} + \frac{r\cos\theta}{b} = \text{const.}$$

or $$\log\left\{\frac{r^2 + b^2 - 2rb\cos\theta}{r^2 + b^2 + 2rb\cos\theta}\right\} + \frac{r\cos\theta}{b} = \text{const.}$$

Deduction. For the second statement take A_1A_2 as y-axis and OY as x-axis, make the corresponding changes everywhere, *i.e.*, in place of x write y, in place of y write x. The result at once follows from (2).

Problem 10. *When a pair of equal and opposite rectilinear vortices are situated in a long circular cylinder at equal distance from its axis, show that path of each vortex is given by the equation.* (IAS-2010)

$$(r^2\sin^2\theta - b^2)(r^2 - a^2)^2 = 4a^2b^2r^2\sin^2\theta,$$

θ being measured from the line through the centre perpendicular to the join of the vortices.

Solution. Let x-axis be the axis of the cylinder. Consider the vortices $+k$ at $A(r, \theta)$ and $-k$ at $B(r, -\theta)$ inside the cylinder s.t. distances of A and B from the axis are equal. Evidently, AB is perpendicular to x-axis.

The image of vortex $+k$ at A w.r.t. the cylinder is a vortex $-k$ at A', the inverse point of A. Similarly the image of vortex $-k$ at B is a vortex $+k$ at B'.

$$OB \cdot OB' = a^2 = OA \cdot OA',$$

where a is the radius of the cylinder. Then

$$OB' = \frac{a^2}{r} = OA' \text{ as } OB = OA = r.$$

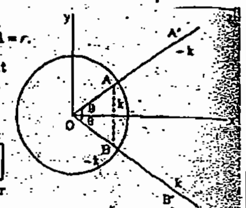


Fig. 8.15.

The complex potential due to this system at $P(z)$ is

$$W = \frac{ik}{2\pi}\left[\log(z - re^{i\theta}) - \log\left(z - \frac{a^2}{r}e^{i\theta}\right) - \log(z - re^{-i\theta}) + \frac{ik}{2\pi}\log\left(z - \frac{a^2}{r}e^{-i\theta}\right)\right]$$

The motion of the vortex at A is due to other vortices.

If W' be the complex potential for the motion of A, then

$$W' = W - \frac{ik}{2\pi}\log(z - re^{i\theta}) \text{ at } z = re^{i\theta}$$

$$= \frac{ik}{2\pi}\left[-\log\left(z - \frac{a^2e^{i\theta}}{r}\right) - \log(z - re^{-i\theta}) - \log\left(z - \frac{a^2}{r}e^{-i\theta}\right)\right] \text{ at } z = re^{i\theta}$$

$$W' = -\frac{ik}{2\pi}\left[\log\left(re^{i\theta} - \frac{a^2}{r}e^{i\theta}\right) + \log(re^{i\theta} - re^{-i\theta}) - \log\left(re^{i\theta} - \frac{a^2}{r}e^{-i\theta}\right)\right]$$

$$= -\frac{ik}{2\pi}[\log(r^2 - a^2)e^{i\theta} - \log r + \log(2ir\sin\theta)$$

$$-[\log\{(r^2 - a^2)\cos\theta + i\sin\theta(r^2 + a^2)\} + \log r]$$

$\therefore$ $$\psi = -\frac{k}{2\pi}[\log|(r^2 - a^2)e^{i\theta}| + \log|2ir\sin\theta|$$

$$-\log|\{(r^2 - a^2)\cos\theta + i\sin\theta(r^2 + a^2)\}|]$$

$$= -\frac{k}{2\pi}[\log(r^2 - a^2) + \log 2r\sin\theta$$

$$-\frac{1}{2}\log\{(r^2 - a^2)^2\cos^2\theta + \sin^2\theta(r^2 + a^2)^2\}]$$

Stream lines are given by ψ = const., *i.e.*,

$$\log\left\{\frac{(r^2 - a^2)^2(2r\sin\theta)^2}{(r^2 - a^2)^2\cos^2\theta + (r^2 + a^2)^2\sin^2\theta}\right\} = \text{const.} = \log 4b^2$$

or $$(r^2 - a^2)^2 r^2\sin^2\theta = b^2[r^4 + a^4 - 2r^2a^2\cos 2\theta]$$

or $$(r^2 - a^2)^2 r^2\sin^2\theta = b^2[(r^2 - a^2)^2 + 2r^2a^2 \cdot 2\sin^2\theta]$$

or $$(r^2 - a^2)^2[r^2\sin^2\theta - b^2] = 4r^2a^2\sin^2\theta.$$

This completes the proof.

Problem 11. *In an incompressible fluid, the vorticity at every point is constant in magnitude and direction. Show that the components of velocity u, v, w are solutions of Laplace's equation.* (IAS-2010)

Solution. Let $W = \xi i + \eta j + \zeta k$, $\quad q = ui + vj + wk$.

Vorticity is constant in magnitude and direction

$\Rightarrow$ ξ, η, ζ are constant

$$\Rightarrow \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) = \xi = \text{const.}, \frac{1}{2}\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) = \eta = \text{const.},$$

$$\frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \zeta = \text{const.}$$

$\therefore$ $\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} = \text{const.}$ ... (1), $\quad \frac{\partial w}{\partial x} - \frac{\partial u}{\partial z} = \text{const.}$ ... (2)

$$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \text{const.} \quad ... (3)$$

Differentiation of (2) and (3) w.r.t. z and y gives

$$\frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 w}{\partial x\partial z}, \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x\partial y}$$

Equation of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0.$$

Observe that

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y\partial x} + \frac{\partial^2 w}{\partial x\partial z}$$

$$= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = \frac{\partial}{\partial x}(0) = 0.$$

$\therefore$ $\nabla^2 u = 0$. Similarly we can prove $\nabla^2 v = 0, \nabla^2 w = 0$.

It means that components of velocity are solutions of Laplace's equation.

Problem 12. *Three parallel rectilinear vortices of the same strength k and in the same sense meet any plane perpendicular to them in an equilateral triangle of side a. Show that the vortices all move round the same cylinder with uniform speed $2\pi a^2/3k$.*

Solution. The Figure 8.16 is self explanatory. Let r be the radius of the circumcircle of equilateral ΔABC, then $OA = OB = OC, AB = a$.

$$\cos\left(\frac{\pi}{6}\right) = \frac{1}{2}a/r \quad \text{or} \quad r = a/\sqrt{3}.$$

The complex potential of the motion is given by

$$W = ik[\log(z - z_A) + \log(z - z_B) + \log(z - z_c)]$$

$$= ik\log(z - r)(z - re^{2\pi i/3})(z - re^{4\pi i/3}) = ik\log(z^3 - r^3)$$

For the motion of the vortex at A,

$$W_1 = W - ik\log(z - z_A)$$

$$= ik\log\left(\frac{z^3 - r^3}{z - r}\right)$$

$$= ik\log(z^2 + zr + r^2)$$

$$u_A - iv_A = -\left(\frac{dW_1}{dz}\right)_{z=r}$$

$$= -ik\left(\frac{2z + r}{z^2 + zr + r^2}\right)_{z=r}$$

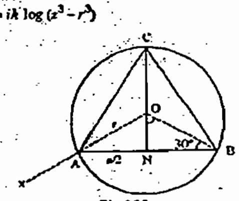


Fig. 8.16.

$$= -ik \cdot \frac{1}{r}.$$

or $\quad q_A = |u_A - iv_A| = k/r.$

If T be the time during which the vortex A moves round the cylinder, then we have $2\pi \cdot OA = T \cdot q_A$

or $\quad T = \frac{2\pi r}{q_A} = \frac{2\pi r}{k/r} = \frac{2\pi}{k} \cdot r^2 = \frac{2\pi}{k} \cdot \frac{a^2}{3} = \frac{2\pi a^2}{3k}.$

Remark. In the above solution, ik has been taken in place of $ik/2\pi$ in order to get the required result.

Problem 13. *If a vortex pair is situated within a cylinder show that it will remain at rest if the distance of either from the centre is given by $a(\sqrt{5}-2)^{1/2}$, where a is the radius of the cylinder.*

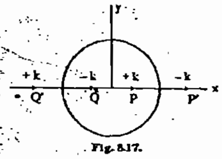


Fig. 8.17.

Solution. The vortex pair PQ consists of vortex $+k$ at P and vortex $-k$ at Q. The image of vortex $+k$ at P is a vortex $-k$ at P', the inverse point of P.

Similarly, the image of vortex $-k$ at Q is a vortex $+k$ at Q'. Let $OP = OQ = r$. Then $OP \cdot OP' = a^2 = OQ \cdot OQ'$.

Hence $\quad OP' = \frac{a^2}{r} = OQ'$. Thus $z_P = r, z_Q = -r$

$$z_{Q'} = -a^2/r, \qquad z_{P'} = a^2/r.$$

The complex potential for this motion is

$$W = \frac{ik}{2\pi}[\log(z - z_P) - \log(z - z_{P'}) - \log(z - z_Q) + \log(z - z_{Q'})]$$

The motion of P is due to other vortices.

For the motion of P,

$$W_1 = W - \frac{ik}{2\pi}\log(z - z_P).$$

$$\frac{dW_1}{dz} = -\frac{ik}{2\pi}\left[\frac{1}{z - z_{P'}} + \frac{1}{z - z_{Q'}} - \frac{1}{z - z_{Q'}}\right]$$

$$u_P - iv_P = \left(\frac{-dW_1}{dz}\right)_{z = z_P} = \frac{ik}{2\pi}\left[\frac{1}{z_P - z_{P'}} + \frac{1}{z_P - z_Q} - \frac{1}{z_P - z_{Q'}}\right]$$

$$= \frac{ik}{2\pi}\left[\frac{r}{r^2 - a^2} + \frac{1}{2r} - \frac{r}{r^2 + a^2}\right]$$

This $\Rightarrow u_P = 0,\ v_P = \frac{-k}{2\pi}\left[\frac{r}{a^2 - r^2} - \frac{1}{2r} + \frac{r}{r^2 + a^2}\right]$

The vortex at P will be at rest if $q_P = 0$, *i.e.*, $\sqrt{(u_P^2 + v_P^2)} = 0$,

or $\quad v_P = 0 \quad$ or $\quad \frac{r}{a^2 - r^2} - \frac{1}{2r} + \frac{r}{r^2 + a^2} = 0$

or $\quad 2r^2(r^2 + a^2) - (a^2 - r^2)(a^2 + r^2) + 2r^2(a^2 - r^2) = 0$

or $\quad r^4 + 4r^2a^2 - a^4 = 0 \quad$ or $\quad \left(\frac{r^2}{a^2}\right)^2 + 4\left(\frac{r^2}{a^2}\right) - 1 = 0$

or $\quad \frac{r^2}{a^2} = \frac{-4 \pm \sqrt{(20)}}{2} = -2 \pm \sqrt{5}$

or $\quad r = (-2 \pm \sqrt{5})^{1/2} a.$

The value $(-2 - \sqrt{5})^{1/2} a$ is not admissible because this root gives imaginary value of r.

Hence $\quad r = a(-2 + \sqrt{5})^{1/2}.$

Problem 14. *Two point vortices each of strength k are situated at $(\pm a, 0)$ and a point vortex of strength $-k/2$ is situated at the origin. Show that the fluid motion is stationary and find the equations of stream lines. Show that the stream line which passes through the stagnation points meet the x-axis at $(\pm b, 0)$ where*

$$3\sqrt{3}(b^2 - a^2)^2 = 16a^3b.$$

Solution. The complex potential of the fluid motion is given by

$$W = \frac{ik}{2\pi}\log(z - a) + \frac{ik}{2\pi}\log(z + a) - \frac{ik}{4\pi}\log z$$

or $\quad W = \frac{ik}{2\pi}\left[\log(z^2 - a^2) - \frac{1}{2}\log z\right] \qquad \ldots(1)$

For the motion of vortex at A, the complex potential is

$$W' = W - \frac{ik}{2\pi}\log(z - a)$$

$$= \frac{ik}{2\pi}\left[\log(z + a) - \frac{1}{2}\log z\right]$$

$$\left(-\frac{dW'}{dz}\right)_{z=a} = \frac{-ik}{2\pi}\left[\frac{1}{z + a} - \frac{1}{2z}\right]_{z=a} = 0$$

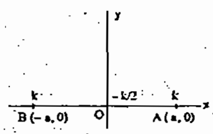


Fig. 8.18.

This $\Rightarrow u_A - iv_A = 0 \Rightarrow$ vortex at A is at rest.

The same fact is true at O, B also. Hence the fluid motion is stationary. This proves the first part of the problem.

To determine stream lines.

$$\phi + i\psi = \frac{ik}{2\pi}\left[\log(z^2 - a^2) - \frac{1}{2}\log z\right] \quad \text{by (1)},$$

$$= \frac{ik}{2\pi}\left[\log(x^2 - y^2 - a^2 + 2ixy) - \frac{1}{2}\log(x + iy)\right]$$

$$\psi = \frac{k}{4\pi}\left[\log\{(x^2 - y^2 - a^2)^2 + 4x^2y^2\} - \frac{1}{2}\log(x^2 + y^2)\right]$$

Stream lines are given by ψ = const., *i.e.*,

$$\log\left[\frac{(x^2 - y^2 - a^2)^2 + 4x^2y^2}{\sqrt{(x^2 + y^2)}}\right] = \log c$$

or $\quad (x^2 - y^2 - a^2)^2 + 4x^2y^2 = c(x^2 + y^2)^{1/2}$

or $\quad (x^2 - y^2)^2 + a^4 - 2a^2(x^2 - y^2) + 4x^2y^2 = c(x^2 + y^2)^{1/2}$

or $\quad (x^2 + y^2)^2 - 2a^2(x^2 - y^2) + a^4 = c(x^2 + y^2)^{1/2}. \qquad \ldots(2)$

These are the required stream lines.

Third Part. The stagnation points are given by

$$\frac{dW}{dz} = 0, \text{ i.e., } \frac{2z}{z^2 - a^2} - \frac{1}{2z} = 0, \text{ by (1)}$$

or $\quad 3z^2 + a^2 = 0 \quad$ or $\quad z = \pm ia/\sqrt{3}.$

Stagnation points are at $\left(0, \frac{a}{\sqrt{3}}\right), \left(0, -\frac{a}{\sqrt{3}}\right)$

The stream lines, given by (2), will pass through stagnation points only if

$$\left(\frac{a^2}{3}\right)^2 - 2a^2\left(0 - \frac{a^2}{3}\right) + a^4 = c\left(0 + \frac{a^2}{3}\right)^{1/2}$$

or $\quad c = 16a^3/3\sqrt{3}.$

The stream lines (2) will pass through $(\pm b, 0)$ if

$$b^4 - 2a^2(b^2 - 0) + a^4 = c(b^2 + 0)^{\frac{1}{2}} = b \cdot 16a^3/3\sqrt{3}$$

or $\quad b^4 - 2a^2b^2 + a^4 = (16a^3b/3\sqrt{3})$

or $\quad 3\sqrt{3}(b^2 - a^2)^2 = 16a^3b.$

This concludes the problem.

Problem 15. *A fixed cylinder of radius a is surrounded by incompressible homogeneous fluid extending to infinity. Symmetrically arranged round it are generators on a cylinder of radius $c (c > a)$ co-axial with the given one are n straight parallel vortex filaments each of strength k. Show that the filaments will remain on this cylinder throughout the motion and revolve round its axis with angular velocity*

$$\frac{k}{4\pi c^2} \cdot \frac{(n+1)c^{2n} + (n-1)a^{2n}}{c^{2n} - a^{2n}} \text{ where } a^2 = bc.$$

Find also the velocity at any point of the fluid.

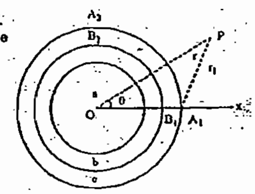


Fig. 8.19.

Solution. Consider vortices of the same strength

$$+k \text{ at } A_1, A_2, \ldots, A_n,$$

symmetrically arranged round a circle of radius $c (> a)$. Then

$$\angle A_1OA_2 = 2\pi/n.$$

Let $A_iP = r_i$ for $i = 1, 2, \ldots, n$

and $\quad OP = r, \angle POX = \theta$

using the formula $\cos A = (b^2 + c^2 - a^2)/2bc$, we get

$$\cos\theta = \frac{r^2 + c^2 - r_1^2}{2rc}$$

or $\quad r_1^2 = r^2 + c^2 - 2rc\cos\theta.$

The image of vortex $+k$ at A_i is a vortex $-k$ at B_i, the inverse point of A_i so that

$$OA_i \cdot OB_i = a^2.$$

Take $OB_i = b$ for $i = 1, \ldots, n$

Then $\quad a^2 = bc.$

Thus there are vortices of the same strength $-k$ placed at $B_1, B_2, \ldots, B_n$ on a circle of radius b.

Let $B_iP = h_i$ for $i = 1, 2, \ldots, n$. Then $h_1^2 = r^2 + b^2 - 2rb\cos\theta.$

The stream function ψ at P due to this system is given by

$$\psi = \frac{k}{2\pi}[\log r_1 + \log r_2 + \ldots + \log r_n] - \frac{k}{2\pi}[\log h_1 + \log h_2 + \ldots + \log h_n]$$

or $\quad \psi = \frac{k}{4\pi}[\log r_1^2 r_2^2 \ldots r_n^2 - \log h_1^2 h_2^2 \ldots h_n^2]$

$$= \frac{k}{4\pi}\left[\log(r^2 + c^2 - 2rc\cos\theta)\left\{r^2 + c^2 - 2rc\cos\left(\theta + \frac{2\pi}{n}\right)\right\}\ldots\right.$$
$$\left. - \log\{r^2 + b^2 - 2rb\cos\theta\}\left\{r^2 + b^2 - 2rb\cos\left(\theta + \frac{2\pi}{n}\right)\right\}\ldots\right]$$

$$= \frac{k}{4\pi}\log\left(\frac{r^{2n} + c^{2n} - 2r^n c^n\cos n\theta}{r^{2n} + b^{2n} - 2r^n b^n\cos n\theta}\right)$$

The motion of A_1 is due to other vortices.

For the motion of A_1,

$$\psi_1 = \psi - \frac{k}{4\pi}\log(r^2 + c^2 - 2rc\cos\theta)$$

or $\quad \psi_1 = \frac{k}{4\pi}(r^{2n} + c^{2n} - 2r^n c^n\cos n\theta) - \frac{k}{4\pi}\log(r^{2n} + b^{2n} - 2r^n b^n\cos n\theta)$

$$- \frac{k}{4\pi}\log(r^2 + c^2 - 2rc\cos\theta) \qquad \ldots(1)$$

Write $\frac{\partial\psi_1}{\partial c} = \frac{\partial\psi_1}{\partial r}$ when $r = c, \theta = 0$, and $\frac{1}{c}\frac{\partial\psi_1}{\partial\theta} = \left(\frac{1}{r}\frac{\partial\psi_1}{\partial\theta}\right)_{\substack{r=0\\ \theta=0}}$

$$\frac{\partial\psi_1}{\partial r} = \frac{k}{4\pi}\left[\frac{2n r^{2n-1} - 2n r^{n-1} c^n\cos n\theta}{r^{2n} + c^{2n} - 2r^n c^n\cos n\theta} - \frac{2n r^{2n-1} - 2n r^{n-1} b^n\cos n\theta}{r^{2n} + b^{2n} - 2r^n b^n\cos n\theta}\right]$$
$$- \frac{k}{4\pi} \cdot \frac{2r - 2c\cos\theta}{r^2 + c^2 - 2rc\cos\theta}$$

$$\left(\frac{\partial\psi_1}{\partial r}\right)_{r=c} = \frac{k}{4\pi}\left[\frac{n}{c} - \frac{2nc^{n-1}(c^n - b^n\cos n\theta)}{c^{2n} + b^{2n} - 2c^n b^n\cos n\theta}\right] - \frac{k}{4\pi c}$$

$$\left(\frac{\partial\psi}{\partial r}\right)_{r=c,\theta=0} = \frac{\partial\psi_1}{\partial c} = \frac{k}{4\pi}\left[\left(\frac{n-1}{c}\right)\frac{2n c^{n-1}}{c^n - b^n}\right]$$

$$\frac{1}{r}\frac{\partial\psi_1}{\partial\theta}=\frac{k}{4\pi r}\left[\frac{2n\,r^n c^n\sin n\theta}{r^{2n}+c^{2n}-2r^n c^n\cos n\theta}-\frac{2n\,r^n b^n\sin n\theta}{r^{2n}+b^{2n}-2r^n b^n\cos n\theta}\right]$$

$$-\frac{k}{4\pi r}\cdot\frac{2rc\sin\theta}{r^2+c^2-2rc\cos\theta}$$

$$\left(\frac{1}{r}\frac{\partial\psi_1}{\partial\theta}\right)_{r=c}=\frac{k}{4\pi c}\left[\frac{n\sin n\theta}{1-\cos n\theta}-\frac{n\sin n\theta}{1-\cos n\theta}-\frac{\sin\theta}{1-\cos\theta}\right]=-\frac{k\sin\theta}{4\pi c\,(1-\cos\theta)}$$

Note that

$$\lim\frac{F(x)}{G(x)}=\lim\frac{F'(x)}{G'(x)}=\lim\frac{F''(x)}{G''(x)},\left[\text{form }\frac{0}{0}\right]$$

$$\therefore\quad\left(\frac{1}{r}\frac{\partial\psi_1}{\partial\theta}\right)_{\substack{r=c\\ \theta=0}}=-\frac{k}{4\pi c}\left[\frac{\cos\theta}{\sin\theta}\right]=-\frac{k}{4\pi c}\cdot\left[\frac{-\sin\theta}{\cos\theta}\right]\text{as }\theta\to 0$$

or $$\frac{1}{c}\frac{\partial\psi_1}{\partial\theta}=0.$$

Now $$q_{A_1}^2=\left(\frac{\partial\psi_1}{\partial c}\right)^2+\left(\frac{1}{c}\frac{\partial\psi_1}{\partial\theta}\right)^2=\left(\frac{\partial\psi_1}{\partial c}\right)^2\Rightarrow q_{A_1}=\left(\frac{\partial\psi_1}{\partial r}\right)_{\substack{r=c\\ \theta=0}}$$

Angular velocity of A_1

$$=q_{A_1}\cdot\frac{1}{c}=\frac{k}{4\pi c^2}\left[\frac{(n-1)(c^n-b^n)-2n\,c^n}{c^n-b^n}\right]\cdot\frac{c^n}{c^n}$$

$$=\frac{k}{4\pi c^2}\cdot\frac{(n+1)\,c^{2n}+(n-1)\,a^{2n}}{c^{2n}-a^{2n}}\quad\text{as } a^2=bc\text{ and so } a^{2n}=b^n c^2$$

Problem 16. Four Vortices. *A rectilinear vortex filament of strength k is infinite liquid bounded by two perpendicular infinite plane walls whose line of intersection is parallel to the filament. Show that the filament will retrace out a curve (in a plane at right angles to the walls) $r\sin 2\theta=$ const. where r is the distance of the vortex from the line of intersection of the walls, and θ the angle between one of the walls and plane containing the filament and line of intersection.*

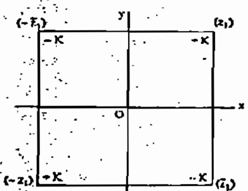


Fig. 8.20

Solution. Let there be vortex $+k$ at z_1. Here we have two rigid boundaries at right angles to each other, say x-axis and y-axis. The image of $+k$ at z_1 w.r.t. y-axis is $-k$ at $-\bar{z}_1$. Now the images of $+k$ at z_1 and $-k$ at $-\bar{z}_1$ w.r.t. $x-$axis are $-k$ at $\bar{z}_1$ and $+k$ at $-z_1$. The complex potential due to this system at $P(z)$ is given by

$$W=\frac{ik}{2\pi}\log(z-z_1)(z+z_1)-\frac{ik}{2\pi}\log(z+\bar{z}_1)(z-\bar{z}_1)$$

For the motion of vortex $+k$ at z_1,

$$W'=W-\frac{ik}{2\pi}\log(z-z_1)=\frac{ik}{2\pi}\log\frac{(z+z_1)}{(z+\bar{z}_1)(z-\bar{z}_1)}$$

$$=\frac{ik}{2\pi}[\log(z+z_1)-\log(z^2-\bar{z}_1^2)]$$

$$u_1-iv_1=\left(-\frac{dW'}{dz}\right)_{z=z_1}=-\frac{ik}{2\pi}\left[\frac{1}{z+z_1}-\frac{2z}{z^2-\bar{z}_1^2}\right]_{z=z_1}$$

$$=-\frac{k}{4\pi}\left[\frac{ix_1+y_1}{x_1^2+y_1^2}-\frac{x_1+iy_1}{x_1y_1}\right]$$

$$u_1=\frac{dx_1}{dt}=-\frac{k}{4\pi}\left[\frac{y_1}{x_1^2+y_2^2}-\frac{1}{y_1}\right]=-\frac{k}{4\pi}\frac{(-x_1^2)}{(x_1^2+y_1^2)y_1}\qquad\text{... (1)}$$

$$v_1=\frac{dy_1}{dt}=\frac{k}{4\pi}\left[\frac{x_1}{x_1^2+y_1^2}-\frac{1}{x_1}\right]=\frac{-ky_1^2}{4\pi(x_1^2+y_1^2)x_1}\qquad\text{... (2)}$$

Dividing, $\dfrac{dx_1}{dy_1}=-\dfrac{x_1^3}{y_1^3}$ or $\dfrac{dx_1}{x_1^3}+\dfrac{dy_1}{y_1^3}=0$

Integrating, $-\dfrac{1}{2}\left(\dfrac{1}{x_1^2}+\dfrac{1}{y_1^2}\right)=-a$ or $x_1^2+y_1^2=2ax_1^2y_1^2$

or $r_1^2=2ar_1^4\sin^2\theta_1\cos^2\theta_1$ or $(r_1^2\sin 2\theta_1)^2=2r_1^2/a$.

This $\Rightarrow r_1^2\sin^2 2\theta_1=b^2\Rightarrow r_1\sin 2\theta_1=b$.

Hence the required path is $r_1\sin 2\theta_1=b$, which represents a cote of spiral.

Problem 17. *To find the paths of particles due to a vortex in a liquid filling the space between two parallel planes, the vortex being midway between them.*

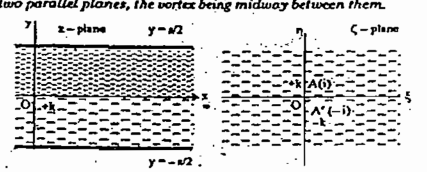


Fig. 8.31.

Solution. Let there be a vortex $+k$ at the origin O which fills the space between the parallel planes $y=a/2$, $y=-a/2$. Consider the transformation $\zeta=i\,e^{\pi z/a}$.

Then $\zeta+i\eta=ie^{(\pi/a)(x+iy)}=ie^{\pi x/a}\cdot e^{i\pi y/a}$

This $\Rightarrow\xi=-e^{\pi x/a}\sin\dfrac{\pi y}{a},\ \eta=e^{\pi x/a}\cos\dfrac{\pi y}{a}$

$$\Rightarrow\xi^2+\eta^2=e^{\frac{2\pi x}{a}}$$

Planes $y=\pm a/2\Rightarrow\xi=\mp e^{\pi x/a},\ \eta=0.$

Points $\left(x,\pm\dfrac{a}{2}\right)$ in z-plane become $(\xi,0)$ in ξ-plane where

$$\xi=\mp e^{\pi x/a}.$$

Also $z=0\Rightarrow\xi=0,\ \eta=1\Rightarrow\zeta=i$

Thus the space between the planes $y=\pm a/2$ in z-plane corresponds to entire ξ-axis in ζ-plane. That is to say, the space in z-plane corresponds to upper half of ζ-plane. The point O in z-plane corresponds to the point A in ζ-plane. The image of $+k$ at A relative to ξ-axis is a vortex $-k$ at A'. The complex potential in ζ-plane is

$$W=\frac{ik}{2\pi}\log(\zeta-i)-\frac{ik}{2\pi}\log(\zeta+i)$$

or $$\phi+i\psi=\frac{ik}{2\pi}\log\left[\frac{\xi+i(\eta-1)}{\xi+i(\eta+1)}\right]$$

This $\Rightarrow$ $$\psi=\frac{k}{4\pi}\log\left[\frac{\xi^2+(\eta-1)^2}{\xi^2+(\eta+1)^2}\right]$$

Paths of fluid particles are given by $\psi=$ *const.* $=\dfrac{k}{4\pi}\log b$, say

Then $$\xi^2+(\eta-1)^2=b\,[\xi^2+(\eta+1)^2].$$

or $$(\xi^2+\eta^2)(b-1)+2\eta(b+1)+(b-1)=0.$$

Dividing by $b-1$ and writing $c=2(b+1)/(b-1)$,

$$\xi^2+\eta^2+\eta c+1=0$$

Putting the values of ξ,η we obtain

$$e^{2\pi x/a}+ce^{\pi x/a}\cos\frac{\pi y}{a}+1=0,$$

or $$e^{\pi x/a}+c\cos\frac{\pi y}{a}+e^{-\pi x/a}=0$$

or $$2\cosh\frac{\pi x}{a}+c\cos\frac{\pi y}{a}=0$$

or $$\cosh\frac{\pi x}{a}=\lambda\cos\frac{\pi y}{a},\text{ where }\lambda\text{ is constant.}$$

This is the required path.

Routh's Theorem

Consider a vortex $+k$ at P in a domain in z-plane. By the transformation $\zeta=f(z)$, the point P, domain C_1 are transformed onto point Q and domain C_2 in ζ-plane. If P be z_1 and Q be ζ_1, the complex potential W in ζ-plane at any point ζ is given by

$$W=W_\zeta+\frac{ik}{2\pi}\log(\zeta-\zeta_1),$$

where W_ζ is the complex potential excluding the term due to vortex $+k$ at ζ_1. Similarly

$$W=W_z+\frac{ik}{2\pi}\log(z-z_1).$$

Hence $$W_\zeta+\frac{ik}{2\pi}\log(\zeta-\zeta_1)=W_z+\frac{ik}{2\pi}\log(z-z_1)$$

or $$W_z=W_\zeta+\frac{ik}{2\pi}\log\left(\frac{\zeta-\zeta_1}{z-z_1}\right)$$

The velocity of vortex ζ_1 can be obtained from

$$u-iv=-\left(\frac{dW_\zeta}{d\zeta}\right)_{\zeta=\zeta_1}$$

Hence for the motion of vortex at z_1,

$$W_{z_1}=W_{\zeta_1}+\frac{ik}{2\pi}\left[\log\left(\frac{\zeta-\zeta_1}{z-z_1}\right)\right]_{\substack{z=z_1\\ \zeta=\zeta_1}}$$

If $\psi_1(\xi_1,\eta_1)$ and $\psi_2(x_1,y_1)$ be stream functions corresponding to W_{ζ_1}, W_{z_1} respectively, then

$$\psi_2(x_1,y_1)=\psi_1(\xi_1,\eta_1)+\psi.\qquad\text{... (1)}$$

where $\psi=$ Imaginary part of $\dfrac{ik}{2\pi}\log\left(\dfrac{\zeta-\zeta_1}{z-z_1}\right)$ at $\zeta=\zeta_1, z=z_1$

$$=\text{Real Part (i.e. R.P.)}\lim_{\substack{\zeta\to\zeta_1\\ z\to z_1}}\frac{k}{2\pi}\log\left(\frac{\zeta-\zeta_1}{z-z_1}\right)$$

$$\frac{\partial\psi}{\partial y_1}=\lim\text{R.P.}\frac{\partial}{\partial y}\left[\frac{k}{2\pi}\log\left(\frac{\zeta-\zeta_1}{z-z_1}\right)\right]$$

or $$\frac{\partial\psi}{\partial y_1}=\lim\text{R.P.}\frac{d}{dz}\left[\frac{ik}{2\pi}\log\left(\frac{\zeta-\zeta_1}{z-z_1}\right)\right]_{z=z_1}\text{as }\frac{\partial}{\partial y}=i\frac{d}{dz}\qquad\text{... (2)}$$

Expanding $\zeta-\zeta_1$ in terms of $z-z_1$

$$\zeta-\zeta_1=(z-z_1)\left(\frac{d\zeta}{dz}\right)_1+\frac{1}{2}(z-z_1)^2\left(\frac{d^2\zeta}{dz^2}\right)_1$$

or $$\frac{\zeta-\zeta_1}{z-z_1}=\left(\frac{d\zeta}{dz}\right)_1+\frac{1}{2}(z-z_1)\left(\frac{d\zeta}{dz}\right)_1$$

$$\therefore\quad\text{R.P. of }\frac{ik}{2\pi}\frac{d}{dz}\left[\log\left(\frac{\zeta-\zeta_1}{z-z_1}\right)\right]$$

$$=\text{R.P. of }\frac{ik}{2\pi}\frac{d}{dz}\left[\log\left\{\left(\frac{d\zeta}{dz}\right)_1+\frac{1}{2}(z-z_1)\left(\frac{d^2\zeta}{dz^2}\right)_1+\dots\right\}\right]$$

$$=\text{R.P. of }\frac{ik}{2\pi}\cdot\frac{0+\frac{1}{2}\cdot\left(\frac{d^2\zeta}{dz^2}\right)_1+\dots}{\left(\frac{d\zeta}{dz}\right)_1+\frac{1}{2}(z-z_1)\left(\frac{d^2\zeta}{dz^2}\right)_1+\dots}$$

$$= \text{R.P. of } \frac{ik}{2\pi} \cdot \frac{\frac{1}{2}\left(\frac{d^2\zeta}{dz^2}\right)_1}{\left(\frac{d\zeta}{dz}\right)_1} \text{ as } z \to z_1, \zeta \to \zeta_1$$

$$= \text{R.P. of } \frac{ik}{4\pi}\frac{d}{dz}\log\left(\frac{d\zeta}{dz}\right)_1 = \text{R.P. } \frac{k}{4\pi}\frac{\partial}{\partial y}\log\left(\frac{\partial\zeta}{dz}\right)_1$$

$$= \frac{k}{4\pi}\frac{\partial}{\partial y}\log\left|\left(\frac{d\zeta}{dz}\right)_1\right|$$

$$\therefore \quad \psi = \frac{k}{4\pi}\log\left|\frac{d\zeta}{dz}\right|_1, \text{ according to (2).}$$

Now (1) becomes

$$\psi_2(x,y) = \psi_1(\xi_1, \eta_1) + \frac{k}{4\pi}\log\left|\frac{d\zeta}{dz}\right|_1$$

This result is known as Routh's theorem.

Problem 18. *The space enclosed between the planes $x = 0$, $x = a$, $y = 0$ on the positive side of $y = 0$ is filled with uniform incompressible fluid. A rectilinear vortex parallel to the axis of z has co-ordinates (x_1, y_1). Determine the velocity at any point of the liquid and show that the path of vortex is given by*

$$\cot^2\frac{\pi x}{a} + \coth^2\frac{\pi y}{a} = \text{const.}$$

Solution. Consider the mapping $\zeta = -\cos(\pi z/a)$ from z-plane into ζ-plane. We have $\xi + i\eta = -\cos\frac{\pi}{a}(x + iy)$

$$\text{This} \Rightarrow \xi = -\cos\frac{\pi x}{a}\cosh\frac{\pi y}{a}, \quad \eta = -\sin\frac{\pi x}{a}\cdot\sinh\frac{\pi y}{a}$$

$$y = 0 \Rightarrow \xi = -\cos\left(\frac{\pi x}{a}\right), \eta = 0$$

$$x = 0 \Rightarrow \xi = -\cosh\left(\frac{\pi y}{a}\right), \eta = 0$$

$$x = a \Rightarrow \xi = \cosh\left(\frac{\pi y}{a}\right), \eta = 0$$

All these points lie on ξ-axis. Hence the semi-infinite strip in z-plane corresponds to upper half of ζ-plane. Let $A(z = z_1)$ in z-plane be mapped onto $\zeta = \zeta_1$ in ζ-plane. The image of vortex $+k$ at $B(\zeta_1)$ relative to the boundary ξ-axis is a vortex $-k$ at $B'(\bar{\zeta}_1)$. The complex potential at any point P (not occupied by the vortex) is given by

$$W = \frac{ik}{2\pi}\log(\zeta - \zeta_1) - \frac{ik}{2\pi}\log(\zeta - \bar{\zeta}_1)$$

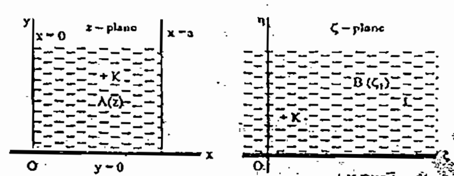


Fig. 8.21.

$$= \frac{ik}{2\pi}\left[\log\left(-\cos\frac{\pi z}{a} + \cos\frac{\pi z_1}{a}\right) - \log\left(-\cos\frac{\pi z}{a} + \cos\frac{\pi \bar{z}_1}{a}\right)\right]$$

$$\frac{dW}{dz_1} = \frac{ik}{2\pi}\left[\frac{\sin(\pi z/a)}{-\cos(\pi z/a) + \cos(\pi z_1/a)} - \frac{\sin(\pi z/a)}{-\cos(\pi z/a) + \cos(\pi \bar{z}_1/a)}\right]\cdot\frac{\pi}{a}$$

$$= \frac{ik}{2a}\frac{\sin(\pi z/a)\left[\cos\frac{\pi \bar{z}_1}{a} - \cos\frac{\pi z_1}{a}\right]}{\left(-\cos\frac{\pi z}{a} + \cos\frac{\pi z_1}{a}\right)\left(-\cos\frac{\pi z}{a} + \cos\frac{\pi \bar{z}_1}{a}\right)}$$

Take $\lambda = \pi/2a$

$$\frac{dW}{dz} = \frac{ik}{2a}\cdot\frac{\sin(\pi z/a)\sin\lambda(z_1 + \bar{z}_1)\cdot\sin\lambda(z_1 - \bar{z}_1)}{\sin\lambda(z + z_1)\cdot\sin\lambda(z - z_1)\cdot\sin\lambda(z + \bar{z}_1)\cdot\sin\lambda(z - \bar{z}_1)}$$

$$= \frac{ik}{2a}\cdot\frac{\sin(\pi z/a)\sin 2\lambda x_1\cdot\sin\lambda(2iy_1)}{\sin\lambda(z + z_1)\sin\lambda(z - z_1)\cdot\sin\lambda(z + \bar{z}_1)\cdot\sin\lambda(z - \bar{z}_1)}$$

$$\left|\frac{dW}{dz}\right| = \frac{k}{2a}\cdot\frac{\sin(2\lambda x_1)\cdot\sinh(2\lambda y_1)\cdot|\sin(\pi z/a)|}{|\sin\lambda(z + z_1)\cdot\sin\lambda(z - z_1)\sin\lambda(z + \bar{z}_1)\cdot\sin\lambda(z - \bar{z}_1)|}$$

This gives velocity at any point.

Second Part.

$$\psi_2(x_1, y_1) = \psi_1(\xi_1, \eta_1) + \frac{k}{4\pi}\log\left|\frac{d\zeta}{dz}\right|_1 \quad \dots (3)$$

$$\psi_1(\xi_1, \eta_1) = -\frac{k}{4\pi}\log\eta_1 = -\frac{k}{4\pi}\log\sin ux_1\cdot\sinh uy_1 \text{ where } u = \pi/a.$$

[This result has been proved later on in § 8.16].

$$\left|\frac{d\zeta}{dz}\right|_1 = u\,|\sin uz_1| = u\,[(\sin ux_1\cdot\cosh uy_1)^2 + (\cos ux_1\cdot\sinh uy_1)^2]^{1/2}$$

Putting the values in (3),

$$\psi_2(x_1, y_1) = \frac{k}{4\pi}\log u\left[\frac{(\sin ux_1\cdot\cosh uy_1)^2 + (\cos ux_1\cdot\sinh uy_1)^2}{(\sin ux_1\cdot\sinh uy_1)^2}\right]^{1/2}$$

$$= \frac{k}{4\pi}\log u\,[\cot^2 ux_1 + \coth^2 uy_1]^{1/2}$$

Paths of vortex $A(x_1, y_1)$ are given by

$$\psi_2(x_1, y_1) = \text{const.}$$

$$\text{This} \Rightarrow \left(\coth\frac{\pi y_1}{a}\right)^2 + \left(\cot\frac{\pi x_1}{a}\right)^2 = \text{const} = c.$$

$\therefore$ Required path is given by

$$\cot^2\left(\frac{\pi x}{a}\right) + \coth^2\left(\frac{\pi y}{a}\right) = c$$

Motion of any vortex

Let there be single vortex $+k$ at (x_1, y_1), *i.e.*, $z = z_1$ in front of a fixed wall $y = 0$. The image of vortex $+k$ at z_1 w.r.t. x-axis is a vortex $-k$ at $\bar{z}_1$. The complex potential due to this sysem is

$$W = \frac{ik}{2\pi}\log(z - z_1) - \frac{ik}{2\pi}\log(z - \bar{z}_1)$$

or $$\phi + i\psi = \frac{ik}{2\pi}\left[\log\{(x - x_1) + i(y - y_1)\} - \log\{(x - x_1) + i(y + y_1)\}\right]$$

$$\psi = \frac{k}{4\pi}\left[\log\{(x - x_1)^2 + (y - y_1)^2\} - \log\{(x - x_1)^2 + (y + y_1)^2\}\right]$$

For the motion of vortex $+k$ at z_1,

$$\psi' = -\frac{k}{4\pi}\log\{(x - x_1)^2 + (y + y_1)^2\}$$

Since the motion is due to vortex $-k$ at $\bar{z}_1$.

If $$\frac{\partial\chi}{\partial y_1} = \left(\frac{\partial\psi'}{\partial y}\right)_{\substack{y = y_1\\ x = x_1}} = \left(\frac{\partial\psi'}{\partial y}\right)_1$$

then $$\frac{\partial\chi}{\partial y_1} = -\frac{k}{4\pi}\cdot\left[\frac{2(y + y_1)}{(x - x_1)^2 + (y + y_1)^2}\right]_{z = z_1} = -\frac{k}{4\pi}\cdot\frac{1}{y_1}$$

$$\frac{\partial\chi}{\partial x_1} = \frac{-k}{4\pi}\left[\frac{2(x - x_1)}{(x - x_1)^2 + (y + y_1)^2}\right]_{z = z_1} = 0$$

$$d\chi = \frac{\partial\chi}{\partial x_1}dx_1 + \frac{\partial\chi}{\partial y_1}dy_1 = \frac{-k}{4\pi y_1}dy_1$$

$$\text{This} \Rightarrow \chi = -\frac{k}{4\pi}\log y_1$$

$$\chi = \text{const.} \Rightarrow y_1 = \text{const.}$$

This proves that the path of a vortex is a streamline.

Remark. Remember the value of χ for further study.

Problem 19. *A vortex in an infinite liquid occupying the upper half of the z-plane bounded by a circle of radius a, centre O and parts of x-axis outside the circle.*

Solution. By the transformation $\zeta = z + \frac{a^2}{z}$, the portion occupied by liquid in z-plane is transformed onto the upper half of ζ-plane. Let $A(z_1)$ correspond to $B(\zeta_1)$. The image of $B(\zeta_1)$ w.r.t. ξ-axis is a vortex $-k$ at an equal distance on either side of ξ-axis, *i.e.*, at $B_1(\bar{\zeta}_1)$. The stream function due to the vortex $+k$ at B is

$$\psi_1(\xi_1, \eta_1) = \frac{-k}{4\pi}\log\eta_1.$$

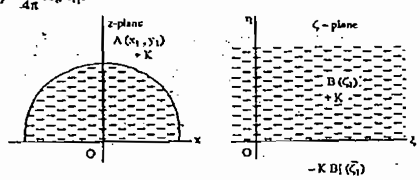


Fig. 8.23.

By Routh's theorem 8.15,

$$\psi_2(x_1, y_1) = \psi_1(\xi_1, \eta_1) + \log\left|\frac{\partial\zeta}{\partial z}\right|_1 \quad \dots (1)$$

$$\frac{d\zeta}{dz} = 1 - \frac{a^2}{z^2} = \frac{z^2 - a^2}{z^2} = \frac{x^2 - y^2 - a^2 + 2ixy}{x^2 - y^2 + 2ixy}$$

$$\left|\frac{d\zeta}{dz}\right| = \frac{\{(x^2 - a^2 - y^2)^2 + 4x^2y^2\}^{1/2}}{\{(x^2 - y^2)^2 + 4x^2y^2\}^{1/2}}$$

$$= \frac{\{(x^2 + y^2 - a^2)^2 + 4a^2y^2\}^{1/2}}{x^2 + y^2}$$

Putting this in (1),

$$\psi_2(x_1, y_1) = -\frac{k}{4\pi}\log\eta_1 + \frac{k}{4\pi}\log\left\{\frac{(x_1^2 + y_1^2 - a^2)^2 + 4a^2y_1^2}{(x_1^2 + y_1^2)^2}\right\}^{1/2}$$

$$= \frac{k}{4\pi}\log\left|\frac{(x_1^2 + y_1^2 - a^2)^2 + 4a^2y_1^2}{(x_1^2 + y_1^2)^2\eta_1^2}\right|^{1/2}$$

The path of vortex at (x_1, y_1) is $\psi_2(x_1, y_1) = \text{const.}$

i.e., $$\frac{(x_1^2 + y_1^2 - a^2)^2 + 4a^2y_1^2}{(x_1^2 + y_1^2)^2\eta_1^2} = \text{const.}$$

Hence the required path is given by

$$(x^2 + y^2 - a^2)^2 + 4a^2y^2 = c\,[(x^2 + y^2)^2\eta^2] \quad \dots (2)$$

$$\zeta = z + \frac{a^2}{z} \Rightarrow \xi + i\eta = x + iy + \frac{a^2(x - iy)}{x^2 + y^2}$$

$$\Rightarrow \eta = y - \frac{ay}{x^2+y^2} = \frac{y(x^2+y^2-a^2)}{x^2+y^2}$$

$$\Rightarrow \eta^2(x^2+y^2) = y^2(x^2+y^2-a^2)^2$$

Using this in (2),

$$(x^2+y^2-a^2)^2 + 4a^2y^2 = cy^2(x^2+y^2-a^2)^2$$

This is the required path.

Problem 20. *To find the path of a vortex in the angle between two planes to which it is parallel.*

Solution. Let the two planes in z-plane be inclined at an angle π/n. By the transformation $\zeta = cz^n$,

i.e., $$Re^{i\delta} = cr^n e^{in\theta}. \quad \ldots (1)$$

these two boundaries of z-plane are transformed onto ξ-axis of ζ-plane

For $\delta = n\theta$, $\theta = 0, \pi/n \Rightarrow \delta = 0, \pi \Rightarrow \xi$-axis.

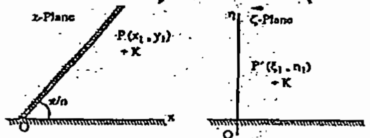


Fig. 8.24.

By this map the vortex $+k$ at $P(x_1, y_1)$ is transformed onto vortex $+k$ at $P'(\zeta_1)$. The image of $+k$ at P' w.r.t. ξ-axis is a vortex $-k$ at $Q(\bar{\zeta}_1)$. The complex potential W at any point ζ in ζ-plane is given by

$$W = \frac{ik}{2\pi}\log(\zeta-\zeta_1) - \frac{ik}{2\pi}\log(\zeta-\bar{\zeta}_1)$$

$$= \frac{ik}{2\pi}\log\left[\frac{(\xi-\xi_1)+i(\eta-\eta_1)}{(\xi-\xi_1)+i(\eta+\eta_1)}\right]$$

$$\therefore \quad \psi = \frac{k}{4\pi}\log\left[\frac{(\xi-\xi_1)^2+(\eta-\eta_1)^2}{(\xi-\xi_1)^2+(\eta+\eta_1)^2}\right]$$

The stream function $\psi_1(\xi_1, \eta_1)$ at P' is given by

$$\psi_1(\xi_1, \eta_1) = -\frac{k}{4\pi}\log \eta_1 \text{ (Refer 8.16)}$$

By Routh's theorem 8.15,

$$\psi_2(x_1, y_1) = \psi_1(\xi_1, \eta_1) + \frac{k}{4\pi}\log\left|\frac{d\zeta}{dz}\right|_1$$

$$= -\frac{k}{4\pi}\log\eta_1 + \log|ncz^{n-1}| = \frac{k}{4\pi}\log\frac{ncr^{n-1}}{\eta_1}.$$

Path of vortex at P is given by

$$\psi_2(x_1, y_1) = \text{const.}$$

i.e., $$\frac{cnr^{n-1}}{\eta_1} = \text{const.} \quad \text{or} \quad r^{n-1} = c\eta_1 = ca\, r_1^n \sin n\theta_1$$

or $$r_1 \sin n\theta_1 = b, \text{ where } b = 1/ac.$$

Hence the required path is $r \sin n\theta = b$.

Problem 21. *Prove that the necessary and sufficient condition that the vortex lines may be at right angles to the stream lines are*

$$(u, v, w) = \mu\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$$

Solution. The differential equations of stream lines and vortex lines are respectively

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \quad \ldots (1)$$

and $$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dz}{\zeta} \quad \ldots (2)$$

(1) and (2) will intersect orthogonally iff

$$u\xi + v\eta + w\zeta = 0$$

or $$\text{iff } u\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) + v\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + w\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0$$

But this is the condition that

$u\,dx + v\,dy + w\,dz$ is perfect differential

or $$u\,dx + v\,dy + w\,dz = \mu\,d\phi = \mu\left(\frac{\partial\phi}{\partial x}dx + \frac{\partial\phi}{\partial y}dy + \frac{\partial\phi}{\partial z}dz\right)$$

This $\Rightarrow$ $u, v, w = \mu\left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$

Exercises

1. Show that for a vortex-pair the relative stream lines are given by $k\left(\frac{y}{2a} + \log\frac{r_1}{r_2}\right) = \text{const.}$ where $2a$ is the distance between the vortices and r_1, r_2 are the distances of any point from them.
2. Prove that, if n rectilinear vortices of equal strength k are symmetrically arranged as generators of a right circular cylinder of radius a and infinite length in an incompressible liquid, then the two-dimensional motion of the liquid is given by $W = \frac{ik}{2\pi}\log(z^n - a^n)$, the origin of co-ordinates being the centre of the cross-section of the cylinder. Show that the vortices move round the cylinder with speed $(n-1)k/4\pi a$.
3. Investigate the nature of the motion of the liquid
$$u = \frac{ax - by}{x^2+y^2}, v = \frac{ay + bx}{x^2+y^2}, w = 0.$$
Also determine the pressure at any point (x, y).
4. In problem 3, determine the velocity potential.
5. When an infinite liquid contains two parallel, equal and opposite vortices at a distance $2b$, prove that the stream lines relative to the vortices are given by the equation.
$$\log\left[\frac{x^2+(y-b)^2}{x^2+(y+b)^2}\right] + \frac{y}{b} = c$$
the origin being the middle point of the join, which is taken for the axis of y.
6. A long fixed cylinder of radius a is surrounded by infinite frictionless incompressible liquid, and there is in the liquid a vortex filament of strength k, which is parallel to the axis of the cylinder at a distance f ($f > a$) from this axis. Given that there is no circulation round any circuit enclosing cylinder but not the filament, show that the speed q of the fluid at the surface of the cylinder is
$$\frac{k}{a}\left[1 - \frac{f^2 - a^2}{r^2}\right]$$
r being the distance of the point considered from the filament.

Answers

3. Irrotational motion,
$$\frac{p}{\rho} = c - \frac{1}{2}\left(\frac{a+b^2}{x^2+y^2}\right)$$
4. $\phi = -\left[\frac{a}{2}\log(x^2+y^2) + b\tan^{-1}\frac{y}{x}\right]$

Vortex Sheet.

Let a fluid be moving irrotationally everywhere except in the domain which lies between the surfaces S_1 and S_3. Consider an element dS_2 of the surface S_2 s.t. P is the centre of gravity of dS_2 of the surface S_2 s.t. P is the centre of gravity of dS_2 and the points P_1 and P_2 are taken on normal at P s.t.

$$PP_1 = \varepsilon\, n/2, \; PP_2 = -\varepsilon n/2,$$

where n is a vector normal to S_2 at P and $\varepsilon > 0$ be very small number. As P describes the surface S_2, the points P_1 and P_2 describe S_1 and S_3 respectively parallel to S_2 which is midway between them. The normals at the boundaries of dS_1, dS_2, dS_3 enclose a cylindrical element of volume $dv = \varepsilon\, dS_2$. Let W be vorticity vector at P, then

$$W\,dv = W\varepsilon\,dS_2 = W'\,dS_2, \text{ where } W' = \varepsilon W.$$

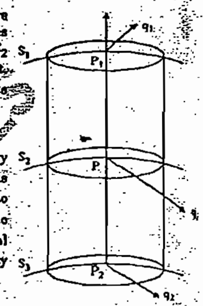


Fig. 8.25.

Now if $\varepsilon \to 0$ and $W \to \infty$ in such a way that W' remains unaltered. Now we define the surface S_2 as the vortex sheet of vorticity W' per unit area. It can be proved that the normal components of velocity are continuous across the vortex sheet.

Infinite single row of vortices of equal strength.

To study the motion induced in an infinite liquid by an infinite row of parallel rectilinear vortices of the same strength k at a distance apart. *(Agra 2000)*

Consider an infinite number of vortices each of strength k. These are placed at points $z = 0, \pm a, \pm 2a, \ldots$

Such arrangement is called vortex sheet. The complex potential at any point z is given by

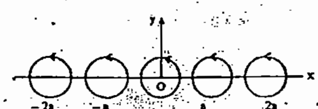


Fig. 8.26.

$$W = \frac{ik}{2\pi}[\log z + \log(z-a) + \log(z-2a) + \ldots] + \frac{ik}{2\pi}[\log(z+a) + \log(z+2a) + \ldots]$$

$$= \frac{ik}{2\pi}\log z(z^2-a^2)(z^2-2^2a^2)\ldots$$

$$= \frac{ik}{2\pi}\log\left[\frac{\pi z}{a}\left(1-\frac{z^2}{a^2}\right)\left(1-\frac{z^2}{a^2}\right)\left(1-\frac{z^2}{2^2a^2}\right)\ldots\right] + \text{const.}$$

$$= \frac{ik}{2\pi}\log\sin\left(\frac{\pi z}{a}\right) \ldots (1), \text{ ignoring constant}$$

as $$\sin\theta = \theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\ldots$$

To determine the motion of vortex at $z = 0$.

If W_1 be the complex potential at $z = 0$, then

$$W_1 = \frac{ik}{2\pi}\log\sin\frac{\pi z}{a} - \frac{ik}{2\pi}\log(z-0).$$

Since the motion of vortex at $z = 0$ is due to other vortices,

$$u_0 - iv_0 = -\left(\frac{dW_1}{dz}\right)_{z=0} = -\frac{ik}{2\pi}\left[\frac{\pi}{a}\cot\frac{\pi z}{a} - \frac{1}{z}\right]_{z=0} = 0.$$

Thus vortex at the origin is at rest. Similarly, all the other vortices are at rest. Thus we can say that the *vortex row induces no velocity in itself.*

Velocity at any point of the fluid. If u, v are velocity components at any point z,

$$u - iv = -\frac{dW}{dz} = -\frac{ik}{2\pi}\cdot\frac{\pi}{a}\cot\frac{\pi z}{a}, \text{ by (1).}$$

or $$u - iv = -\frac{ik}{2a}\frac{\cos b(x+iy)}{\sin b(x+iy)}\cdot\frac{\sin b(x-iy)}{\sin b(x-iy)}, \text{ where } b = \pi/a.$$

$$= -\frac{ik}{2\pi}\cdot\frac{\sin 2bx - i\sinh 2by}{\cosh 2by - \cos 2bx}$$

$$u = -\frac{k}{2a}\cdot\frac{\sinh\left(\frac{2\pi y}{a}\right)}{\cosh(2\pi y/a) - \cos(2\pi x/a)},$$

$$u = \frac{k}{2a}\frac{\sin(2\pi x/a)}{\cosh(2\pi y/a) - \cos(2\pi x/a)},$$

$$\phi + i\psi - (\phi - i\psi) = \frac{ik}{2\pi}\log\sin\frac{\pi z}{a} - \left(-\frac{ik}{2\pi}\log\sin\frac{\pi\bar{z}}{a}\right)$$

or $$2i\psi = \frac{ik}{2\pi}\log\sin\frac{\pi z}{a}\cdot\sin\frac{\pi\bar{z}}{a}$$

or $$\psi = \frac{k}{4\pi}\log\left[\cosh\frac{2\pi y}{a} - \cos\frac{2\pi x}{a}\right], \text{ neglecting const.}$$

Stream lines are given by ψ = const., *i.e.*,

$$\cosh\frac{2\pi y}{a} - \cos\frac{2\pi x}{a} = \text{const.}$$

8.19. Two infinite rows of vortices.

Two rows of vortices, one below the other, the upper of positive sign, the lower of negative sign.

The system consists of two infinite rows of vortices one above the other at a distance b. Let the upper and lower rows be taken as x-axis and $y = b$ respectively.

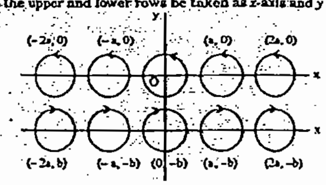


Fig. 8.27.

The vortices are placed in these rows in such a way that one vortex of the upper row is just above one of the lower row. Let the strengths of each of vortices in the upper row be $+k$ and $-k$ that of lower. Thus the system consists of

(i) vortices each of strength $+k$ at

$$z = 0, \pm a \pm 2a, \pm 3a, \ldots$$

(ii) vortices each of strength $-k$ at

$$z = \pm a - ib, \pm 2a - ib, \pm 3a - ib.$$

The complex poential W at point z is

$$W = W_1 + W_2 \qquad \ldots(1)$$

where $$W_1 = \frac{ik}{2\pi}\{\log z + \log(z-a) + \log(z+a) + \log(z-2a) + \log(z+2a) + \ldots\}$$

$$= \frac{ik}{2\pi}\log\{z(z^2-a^2)(z^2-2^2a^2)\ldots\}$$

$$= \frac{ik}{2\pi}\log\sin\left(\frac{\pi z}{a}\right)$$

Similarly, $W_2 = -\frac{ik}{2\pi}\log\sin\frac{\pi}{a}(z+ib)$

The components (u_0, v_0) of velocity of vortex $+k$ at $z = 0$ is

$$W_0 = W - \frac{ik}{2\pi}\log z$$

$$u_0 - iv_0 = -\left(\frac{dW_0}{dz}\right)_{z=0} = -\frac{ik}{2\pi}\left[\frac{\pi}{a}\cot\frac{\pi z}{a} - \frac{\pi}{a}\cot\frac{\pi}{a}(z+ib) - \frac{1}{z}\right]_{z=0}$$

$$= \frac{ik}{2a}\cot\left(\frac{\pi ib}{a}\right) = \frac{ik}{2a}(-i)\coth\left(\frac{\pi b}{a}\right) = \frac{k}{2a}\coth\left(\frac{\pi b}{a}\right)$$

$\therefore$ $$u_0 = \frac{k}{2a}\coth\left(\frac{\pi b}{a}\right), v_0 = 0$$

Thus the vortex system moves parallel to itself with velocity

$$\frac{k}{2a}\coth\frac{\pi b}{a}.$$

The velocity component u, v at any point z is given by

$$-\frac{dW}{dz} = u - iv = -\frac{ik}{2\pi}\left[\frac{\pi}{a}\cot\frac{\pi z}{a} - \frac{\pi}{a}\cot\pi\left(\frac{z+ib}{a}\right)\right]$$

or $$u - iv = \frac{ik}{2a}\left[\cot\frac{\pi z}{a} - \cot\frac{\pi(z+ib)}{a}\right]$$

Karman's vortex street.

An arrangement of two parallel rows of infinite vortices of the same spacing a such that each vortex k of one row is directly above the middle of the joining of two vortices of strengths − k of another row, is said to be Karman's vortex street.

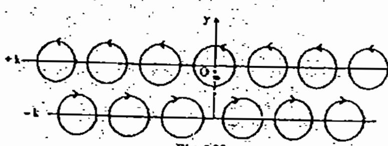


Fig. 8.28.

The sysem consists of vortices each of strength $+k$ at

$$(0, 0), (\pm a, 0), (\pm 2a, 0), \ldots$$

and vortices of strength $-k$ at

$$\left(\pm\frac{a}{2}, -b\right), \left(\pm\frac{3a}{2}, -b\right), \ldots$$

The complex potential W at any point z is given by

$$W = \frac{ik}{2\pi}[\{\log z + \log(z-a) + \log(z-2a)\} + \{\log(z+a) + \log(z+2a+\ldots\}] - \frac{ik}{2\pi}\left[\log\left(z-\frac{a}{2}+ib\right) + \log\left(z+\frac{a}{2}+ib\right) + \log\left(z-\frac{3a}{2}+ib\right)\left(z+\frac{3a}{2}+ib\right)+\ldots\right]$$

$$= \frac{ik}{2\pi}\left[\log\{z(z^2-a^2)(z^2-2^2a^2)\ldots\} - \log\left\{(z+ib)^2 - \left(\frac{a}{2}\right)^2\right\}\left\{(z+ib)^2 - \left(\frac{3a}{2}\right)^2\right\}\ldots\right]$$

or $$W = \frac{ik}{2\pi}\left[\log\sin\left(\frac{\pi z}{a}\right) - \log\sin\frac{\pi}{a}\left(z+\frac{a}{2}+ib\right)\right] \qquad \ldots(1)$$

$$-\frac{dW}{dz} = u - iv = -\frac{ik}{2\pi}\cdot\frac{\pi}{a}\left[\cot\frac{\pi z}{a} - \cot\frac{\pi}{a}\left(z+\frac{a}{2}+ib\right)\right]$$

Let u_0, v_0 be velocity components of the vortex $+k$ at the origin.

Then $$u_0 - iv_0 = -\frac{dW_0}{dz} = -\frac{ik}{2\pi}\left[\frac{\pi}{a}\left\{\cot\frac{\pi z}{a} - \cot\frac{\pi}{a}\left(z+\frac{a}{2}+ib\right)\right\} - \frac{1}{z}\right]_{z=0}$$

as $$W_0 = W - \frac{ik}{2\pi}\log z$$

$$u_0 - iv_0 = -\frac{ik}{2\pi}\left[-\frac{\pi}{a}\cot\frac{\pi}{a}\left(\frac{a}{2}+ib\right)\right] = \frac{ik}{2a}\cot\left(\frac{\pi}{2}+\frac{i\pi b}{a}\right)$$

or $$u_0 - iv_0 = -\frac{ik}{2a}\tan\left(\frac{i\pi b}{a}\right) = -\frac{ik}{2a}\cdot i\tanh\left(\frac{\pi b}{a}\right)$$

$$= \frac{k}{2a}\tanh\left(\frac{\pi b}{a}\right)$$

It means that the vortex at the origin moves with the velocity

$$\frac{k}{2a}\tanh\left(\frac{\pi b}{a}\right)$$

Problem 22. *An infinite row of equidistant rectilinear vortices is at a distance a apart. The vortices are of the same numerical strength k but they are alternately of opposite signs. Find the complex function that determines the velocity potential and the stream function. Show that the vortices remain at rest and draw stream lines. Show also that if α be the radius of a vortex, the amount of flow between any vortex and the next is* (IAS-2011)

$$\frac{k}{\pi}\log\cot\frac{\pi\alpha}{2a}$$

Solution. Let the vortices each of strength k be placed at $(0, 0), (\pm 2a, 0), (\pm 4a, 0), \ldots$ and vortices each of strength $-k$ be placed at $(\pm a, 0), (\pm 3a, 0), (\pm 5a, 0) \ldots$

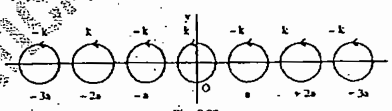


Fig. 8.20.

The complex potential at any point $P(z)$ is given by

$$W = \frac{ik}{2\pi}\log z + \frac{ik}{2\pi}\{\log(z-2a) + \log(z+2a) + \log(z-4a) + \log(z+4a)\ldots\} - \frac{ik}{2\pi}\{\log(z-a) + \log(z+a) + \log(z-3a) + \log(z+3a) + \ldots\}$$

$$= \frac{ik}{2\pi}\log\left[\frac{z(z^2-2^2a^2)(z^2-4^2a^2)\ldots}{(z^2-a^2)(z^2-3^2a^2)(z^2-5^2a^2)\ldots}\right]$$

$$= \frac{ik}{2\pi}\log\left[\frac{\frac{z}{2a}\left\{1-\left(\frac{z}{2a}\right)^2\right\}\left\{1-\left(\frac{z}{4a}\right)^2\right\}\ldots}{\left\{1-\left(\frac{z}{a}\right)^2\right\}\left\{1-\left(\frac{z}{3a}\right)^2\right\}\ldots}\right] + \text{const.}$$

$$= \frac{ik}{2\pi}\log\frac{\sin(\pi z/2a)}{\cos(\pi z/2a)} = \frac{ik}{2\pi}\log\tan\left(\frac{\pi z}{2a}\right)$$

$$W = \frac{ik}{2\pi}\log\tan\left(\frac{\pi z}{2a}\right) \qquad \ldots(1)$$

$$\phi + i\psi = \frac{ik}{2\pi}\log\tan\left(\frac{\pi z}{2a}\right) \qquad \ldots(2)$$

$$\phi - i\psi = -\frac{ik}{2\pi}\log\tan\left(\frac{\pi\bar{z}}{2a}\right) \qquad \ldots(3)$$

(2)—(3) gives, $$2i\psi = \frac{ik}{2\pi}\log\left(\tan\frac{\pi z}{2a}\right)\left(\tan\pi\frac{\bar{z}}{2a}\right)$$

or $$\psi = \frac{k}{4\pi}\log\left[\sin\left(\frac{\pi z}{2a}\right)\sin\left(\frac{\pi\bar{z}}{2a}\right)\Big/\cos\left(\frac{\pi z}{2a}\right)\cos\left(\frac{\pi\bar{z}}{2a}\right)\right]$$

or $$\psi = \frac{k}{4\pi}\log\left(\frac{\cosh\frac{\pi y}{a} - \cos\frac{\pi x}{a}}{\cosh\frac{\pi y}{a} + \cos\frac{\pi x}{a}}\right) \qquad \ldots(4)$$

Stream lines are given by ψ = const., *i.e.*,

$$\cosh\frac{\pi y}{a} = b\cos\frac{\pi x}{a}.$$

(2) + (3) gives $$2\phi = \frac{ik}{2\pi}\log\frac{\tan(\pi z/2a)}{\tan(\pi\bar{z}/2a)}$$

$$\phi = \frac{ik}{4\pi}\log\left[\frac{\sin(\pi z/2a)\cos(\pi\bar{z}/2a)}{\sin(\pi\bar{z}/2a)\cos(\pi z/2a)}\right]$$

$$= \frac{ik}{4\pi}\log\frac{\sin(\pi x/a) + i\sinh(\pi y/a)}{\sin(\pi x/a) - i\sinh(\pi y/a)}$$

or $$\phi = -\frac{k}{4\pi}\left[\tan^{-1}\frac{\sinh(\pi y/a)}{\sin(\pi x/a)} + \tan^{-1}\frac{\sinh(\pi y/2a)}{\sin(\pi x/a)}\right]$$

or $$\phi = -\frac{k}{2\pi}\tan^{-1}\frac{\sinh(\pi y/a)}{\sin(\pi x/a)} \qquad \ldots(5)$$

Required velocity potential and stream function are given by (4) and (5).

Second Part. Consider the motion of the vortex $+k$ at the origin. Then

$W_1 = W - \frac{ik}{2\pi}\log z.$

Let u_0, v_0 be the velocity components of the vortex at (0, 0).

$$u_0 - iv_0 = -\left(\frac{dW_1}{dz}\right)_{z=0} = \frac{ik}{2\pi}\left[\frac{\sec(\pi z/2a)}{\tan(\pi z/2a)}\cdot\frac{\pi}{2a} - \frac{1}{z}\right]_{z=0} = 0$$

This shows that the vortex at (0, 0) is at rest. Similarly we can prove that every vortex is at rest.

Third Part. To determine the flow, ψ at any point of x-axis *i.e.*, at $(x, 0)$ is

$$\psi = \frac{k}{4\pi}\log\left[\frac{1-\cos\left(\frac{\pi x}{a}\right)}{1+\cos\left(\frac{\pi x}{a}\right)}\right] = \frac{k}{4\pi}.2\log\tan\left(\frac{\pi x}{2a}\right), \text{ by (4).}$$

or $$\psi(x, 0) = \frac{k}{4\pi}\log\tan\left(\frac{\pi x}{2a}\right)$$

Flow between two consecutive vortices

$$= 2 \text{ flow across } (a-\alpha, 0) \text{ to } (\alpha, 0)$$
$$= \psi(a-\alpha, 0) - \psi(\alpha, 0)$$
$$= \frac{k}{2\pi}\log\left\{\tan\frac{\pi}{2a}(a-\alpha)\Big/\tan\frac{\pi\alpha}{2a}\right\}$$
$$= \frac{k}{2\pi}\log\left(\cot\frac{\pi\alpha}{2a}\right)^2 = \frac{k}{\pi}\log\cot\left(\frac{\pi\alpha}{2a}\right)$$

Problem 23. *Prove that a thin cylindrical vortex of strength σ, running parallel to a plane boundary at a distance a will travel with velocity $\sigma/4\pi a$; and show that a stream of fluid will flow past between the travelling vortex and the boundary of total amount:*

$$\frac{\sigma}{2\pi}\left[\log\left(\frac{2a}{c}\right) - \frac{1}{2}\right]$$

per unit length along the vortex. where c is the small radius of the cross section of the vortex.

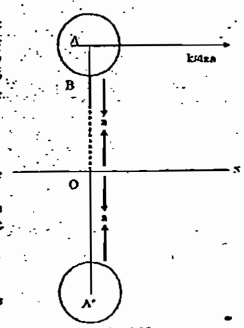


Fig. 8.28.

Solution. Let the plane boundary be x-axis. The image of cylindrical vortex $+\sigma$ at A ($z = ia$) is a vortex $-\sigma$ at A' ($z = -ia$).

$$\therefore \quad W = \frac{i\sigma}{2\pi}\log\left(\frac{z-ia}{z+ia}\right)$$

The velocity at A will be due to vortex $-\sigma$ at A' and is given by

$$\frac{dW}{dz} = \frac{d}{dz}\left[-\frac{i\sigma}{2\pi}\log(z+ia)\right]_{z=ia} = \frac{-i\sigma}{2\pi}\cdot\frac{1}{2ia} = \frac{-\sigma}{4\pi a}$$

$$-\frac{dW}{dz} = \frac{\sigma}{4\pi a}$$

To reduce the system at rest, we add a velocity $-\sigma/4\pi a$ to the system. In this case, ψ is given by

$$\psi = \frac{\sigma}{2\pi}\log\left|\frac{z-ia}{z+ia}\right| + \frac{\sigma y}{4\pi a} \qquad \left[\text{For } -\frac{\sigma}{4\pi a} = -\frac{\partial\phi}{\partial x} = -\frac{\partial\psi}{\partial y}\right]$$

or $$\psi = \frac{\sigma}{4\pi}\log\left[\frac{x^2+(y-a)^2}{(x^2+(y+a)^2}\right] + \frac{\sigma y}{4\pi a}$$

Total flow between the travelling vortex and the plane boundary

$$= \psi_0 - \psi_B = \psi(0, 0) - \psi(0, a-c)$$
$$= \frac{\sigma}{2\pi}\log 1 - \frac{\sigma}{4\pi}\log\frac{c^2}{(2a-c)^2} - \frac{\sigma(a-c)}{4\pi a}$$
$$= \frac{\sigma}{4\pi}\log\left(\frac{2a-c}{c}\right)^2 - \frac{\sigma(a-c)}{4\pi a} = \frac{\sigma}{2\pi}\left[\log\frac{2a}{c}\left(1-\frac{c}{2a}\right) - \frac{1}{2} + \frac{c}{2a}\right]$$
$$= \frac{\sigma}{2\pi}\left[\log\left(\frac{2a}{c}\right) - \frac{1}{2} + \frac{c}{2a} + \log\left(1-\frac{c}{2a}\right)\right]$$
$$= \frac{\sigma}{2\pi}\left[\log\left(\frac{2a}{c}\right) - \frac{1}{2} + \frac{c}{2a} + \left\{\frac{-c}{2a} - \left(\frac{c}{2a}\right)^2 \ldots\right\}\right]$$

$$\text{as } -\log(1-x) = x + \frac{x^2}{2} + \frac{x^3}{3} \ldots$$

$$= \frac{\sigma}{2\pi}\left[\log\left(\frac{2a}{c}\right) - \frac{1}{2}\right] \text{ neglecting } c^2 \text{ in the expansion.}$$

Problem 24. *If $u\,dx + v\,dy + w\,dz = d\theta + \lambda\,d\mu$ where θ, λ, μ are functions of x, y, z, t, prove that the vortex lines at any time are the lines of intersection of the surfaces λ = const. and μ = const.*

Solution. By what is given,

$$u\,dx + v\,dy + w\,dz = \frac{\partial\theta}{\partial x}dx + \frac{\partial\theta}{\partial y}dy + \frac{\partial\theta}{\partial z}dz + \frac{\partial\theta}{\partial t}dt + \lambda\left[\frac{\partial\mu}{\partial x}dx + \frac{\partial\mu}{\partial y}dy + \frac{\partial\mu}{\partial z}dz + \frac{\partial\mu}{\partial t}dt\right]$$

This $\Rightarrow u = \frac{\partial\theta}{\partial x} + \lambda\frac{\partial\mu}{\partial x}, v = \frac{\partial\theta}{\partial y} + \lambda\frac{\partial\mu}{\partial y}, w = \frac{\partial\theta}{\partial z} + \lambda\frac{\partial\mu}{\partial z}, 0 = \frac{\partial\theta}{\partial t} + \lambda\frac{\partial\mu}{\partial t}$

$$\xi\mathbf{i} + \eta\mathbf{j} + \zeta\mathbf{k} = \mathbf{W} = \frac{1}{2}\operatorname{curl}\mathbf{q} = \frac{1}{2}\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ u & v & w\end{vmatrix}$$

This $\Rightarrow \xi = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) \Rightarrow 2\xi = \frac{\partial}{\partial y}(\theta_z + \lambda\mu_z) - \frac{\partial}{\partial z}(\theta_y + \lambda\mu_y)$

$\Rightarrow 2\xi = \lambda_y\mu_z - \lambda_z\mu_y$, where $\theta_y = \partial\theta/\partial y$ etc.

or $$2\xi = \begin{vmatrix}\lambda_y & \lambda_z\\ \mu_y & \mu_z\end{vmatrix}. \text{ Similarly } 2\eta = \begin{vmatrix}\lambda_z & \lambda_x\\ \mu_z & \mu_x\end{vmatrix}$$

$$2\zeta = \begin{vmatrix}\lambda_x & \lambda_y\\ \mu_x & \mu_y\end{vmatrix}$$

$$\therefore \quad 2(\xi\lambda_x + \eta\lambda_y + \zeta\lambda_z) = \begin{vmatrix}\lambda_x & \lambda_y & \lambda_z\\ \lambda_x & \lambda_y & \lambda_z\\ \mu_x & \mu_y & \mu_z\end{vmatrix} = 0$$

Since a determinant vanishes if its any two rows are identical.

$\therefore \quad \xi\lambda_x + \eta\lambda_y + \zeta\lambda_z = 0.$

Similarly $\quad \xi\mu_x + \eta\mu_y + \zeta\mu_z = 0.$

The last two equations prove that the vortex lines lie on the surfac θ = const. and μ = const.

Cauchy's Integral

For this refer Theorem 7, Chapter 2.

Helmholtz vortricity equation.

For this refer Theorem 6, Chapter 2.

Steady motion.

By Euler's equation of motion,

$$\frac{d\mathbf{q}}{dt} = \mathbf{F} - \frac{1}{\rho}\nabla p$$

or $$\frac{\partial\mathbf{q}}{\partial t} + (\mathbf{q}.\nabla)\mathbf{q} = -\nabla V - \frac{1}{\rho}\nabla p$$

But motion is steady so that $\frac{\partial\mathbf{q}}{\partial t} = 0.$

Hence $(\mathbf{q}.\nabla)\mathbf{q} = -\nabla V - \frac{1}{\rho}\nabla p$

This $\Rightarrow \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right)u = -\frac{\partial V}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x}$ and two similar expansions] u and w.

or $$u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} + w\frac{\partial w}{\partial x} + v\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + w\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) + \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\partial V}{\partial x} = 0$$

or $$\frac{\partial}{\partial x}(u^2+v^2+w^2) + v(-2\zeta) + w(2\eta) + \frac{1}{\rho}\frac{\partial p}{\partial x} + \frac{\partial V}{\partial x} = 0$$

$$\text{as } \mathbf{W}(\xi, \eta, \zeta) = \frac{1}{2}\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ u & v & w\end{vmatrix}$$

Taking $\chi = \int\frac{dp}{\rho} + V + \frac{1}{2}q^2$, we get

$$\frac{\partial\chi}{\partial x} = 2(v\zeta - w\eta) \qquad \ldots(1)$$

Similarly, $$\frac{\partial\chi}{\partial y} = 2(w\xi - u\zeta) \qquad \ldots(2)$$

$$\frac{\partial\chi}{\partial z} = 2(u\eta - v\xi) \qquad \ldots(3)$$

Multiplying (1), (2), (3) by u, v, w and then adding,

$$u\frac{\partial\chi}{\partial x} + v\frac{\partial\chi}{\partial y} + w\frac{\partial\chi}{\partial z} = 0 \qquad \ldots(4)$$

Multiplying (1), (2) and (3) by ξ, η, ζ and then adding,

$$\xi\frac{\partial\chi}{\partial x} + \eta\frac{\partial\chi}{\partial y} + \zeta\frac{\partial\chi}{\partial z} = 0 \qquad \ldots(5)$$

(4) and (5) show that the surface χ = const. contains the stream lines (whose direction cosines ae parallel to (u, v, w) and vortex lines whose direction cosines are parallel to ξ, η, ζ.

When the motion is irrotational, *i.e.*, $\xi = \eta = \zeta = 0$, then χ = const. throughout the liquid.

8.24. To prove that in an infinite liquid at rest at infinity there can be only one type of motion when the components of spin are given.

Proof. If possible, let there be two values of q namely q_1 and q_2. Then we have

$$2\xi = \left(\frac{\partial w_1}{\partial y} - \frac{\partial v_1}{\partial z}\right) \text{ etc.}$$

and $$2\xi = \left(\frac{\partial w_2}{\partial y} - \frac{\partial v_2}{\partial z}\right) \text{ etc.}$$

This $\Rightarrow 0 = \frac{\partial w'}{\partial y} - \frac{\partial v'}{\partial z}$ etc. where $u' = u_1 - u_2, v' = v_1 - v_2, w' = w_1 - w_2.$

Thus we have

$$\frac{\partial w'}{\partial y} - \frac{\partial v'}{\partial z} = 0, \frac{\partial v'}{\partial z} - \frac{\partial w'}{\partial y} = 0, \frac{\partial v'}{\partial x} - \frac{\partial u'}{\partial y} = 0.$$

This proves that the motion determined by (u', v', w') is irrotational, but there cannot be any irrotational motion in a liquid which is at rest at infinity. Consequently

$$u' = 0, v' = 0, w' = 0.$$

so that $\quad q_1 = q_2 \Rightarrow u_1 = u_2, v_1 = v_2, w_1 = w_2.$

Hence there can be only one motion.

Vortex between parallel walls.

Consider a vortex $+k$ in the liquid midway between two parallel walls at a distance a apart. Let the vortex $+k$ be at the origin and the line through it be parallel to the walls as x-axis.

The images of $+k$ at $z = 0$ are $-k$ at $z = ia, -ia$ w.r.t. the walls

$$y = a/2, y = -a/2.$$

These vortices will have vortex images of strengths

$$+k \text{ at } z = 2ia, -2ia \text{ w.r.t.}$$

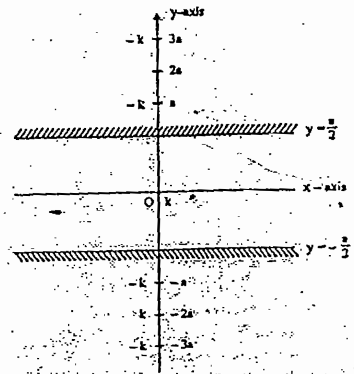


Fig. 8.31.

the two walls. It means that the image system is a row of vortices along y-axis, each at a distance a apart, alternating in signs.

Thus.

(i) $+k$ at $z = 0, \pm 2ia, \pm 4ia, \ldots, \pm 2nia$

(ii) $-k$ at $z = \pm ia, \pm 3ia, \pm 5ia, \ldots, \pm(2n-1)ia$.

The complex potential at any point z due to vortices of strength k is given by

$$\frac{ik}{2\pi}\left[\log(z-0) + \sum_{n=1}^{\infty}\{\log(z-2ina) + \log(z+2ina)\}\right]$$

$$= \frac{ik}{2\pi}\left[\log z + \sum_{n=1}^{\infty}\log(z^2 + 4n^2a^2)\right]$$

$$= \frac{ik}{2\pi}\log\left\{z\prod_1^{\infty}\left(1 + \frac{z^2}{4n^2a^2}\right)\right\}; \text{ ignoring the constants.}$$

Hence the complex potential for the whole row is given by

$$W = \frac{ik}{2\pi}\log\left[\frac{z\prod_1^{\infty}\left(1+\frac{z^2}{4n^2a^2}\right)}{\prod_1^{\infty}\left(1+\frac{z^2}{(2n-1)^2a^2}\right)}\right]$$

$$= \frac{ik}{2\pi}\log\frac{\sinh(\pi z/2a)}{\cosh(\pi z/2a)}$$

or $$W = \frac{ik}{2\pi}\log\tanh(\pi z/2a)$$

$$2i\psi = W - \overline{W} = \frac{ik}{2\pi}\log[\tanh(\pi z/2a)\tanh(\pi\bar{z}/2a)]$$

or $$\psi = \frac{k}{4\pi}\log\left[\frac{(-i)^2\sin\frac{i\pi}{2a}(x+iy)\sin\frac{i\pi}{2a}(x-iy)}{\cos\frac{i\pi}{2a}(x+iy)\cos\frac{i\pi}{2a}(x-iy)}\right]$$

or $$\psi = \frac{k}{4\pi}\log\left[\frac{\sin\frac{\pi}{2a}(ix-y)\sin\frac{\pi}{2a}(ix+y)}{\cos\frac{\pi}{2a}(ix-y)\cos\frac{\pi}{2a}(ix+y)}\right]$$

or $$\psi = \frac{k}{4\pi}\log\left[\frac{\cosh\frac{\pi x}{x} - \cos\frac{\pi y}{a}}{\cosh\left(\frac{\pi x}{a}\right) + \cos\left(\frac{\pi y}{a}\right)}\right]$$

Stream lines are given by ψ = const., which

$$\Rightarrow \cos\frac{\pi y}{a} = b\cosh\frac{\pi x}{a}, \text{ where } b \text{ is constant.}$$

The motion of the vortex $+k$ at O is given by $\left(\frac{dW_0}{dz}\right)_{z=0}$

s.t. $$u_0 - iv_0 = -\left(\frac{dW_0}{dz}\right)_{z=0} = -\frac{d}{dz}\left[W - \frac{ik}{2\pi}\log z\right]_{z=0}$$

since the motion is due to other vortices

or $$u_0 - iv_0 = -\frac{ik}{2\pi}\left[\frac{2\pi}{2a}\operatorname{cosech}\left(\frac{2\pi z}{2a}\right) - \frac{1}{z}\right]_{z=0} = 0$$

$$\left[\text{For } \frac{d}{dx}(\log\tanh bx) = \frac{1}{\tanh bx}\cdot b\,\operatorname{sech}^2 bx = \frac{2b}{2\sinh x\cosh x}\right]$$

Hence the vortex at O is at rest.

Problem 25. *Prove that in a steady motion of a liquid*

$$H = \frac{p}{\rho} + \frac{1}{2}q^2 + V = \textit{constant along a stream line.}$$

If this constant has the same value everywhere in the liquid, then prove that the motion must be either irrotational or the vortex lines must coincide with the stream lines.

In two dimensional motion of a liquid with constant vorticity ζ prove that

$$\Delta(H - 2\zeta\psi) = 0.$$

Show also that if the motion be steady the pressure is given by

$$\frac{p}{\rho} + \frac{1}{2}q^2 + V - 2\zeta\psi = \text{const.},$$

where Δ is Laplace's operator.

Solution. Prove as in 8.23 that

$$\frac{\partial H}{\partial x} = 2(v\zeta - w\eta) \qquad \ldots(1)$$

$$\frac{\partial H}{\partial y} = 2(w\xi - u\zeta), \qquad \ldots(2)$$

$$\frac{\partial H}{\partial z} = 2(u\eta - v\xi) \qquad \ldots(3)$$

where $$H = V + \frac{1}{2}q^2 + \int\frac{dp}{\rho} = \frac{p}{\rho} + \frac{1}{2}q^2 + V$$

Also $$u\frac{\partial H}{\partial x} + v\frac{\partial H}{\partial y} + w\frac{\partial H}{\partial z} = 0 \qquad \ldots(4)$$

$$\xi\frac{\partial H}{\partial x} + \eta\frac{\partial H}{\partial y} + \zeta\frac{\partial H}{\partial z} = 0 \qquad \ldots(5)$$

(4) and (5) show that the surface H = const. contains the stream lines and vortex lines.

If H has the value everywhere, then

$$\frac{\partial H}{\partial x} = 0, \frac{\partial H}{\partial y} = 0, \frac{\partial H}{\partial z} = 0$$

so that $v\zeta - w\eta = 0, w\xi - u\zeta = 0, u\eta - v\zeta = 0$.

This $\Rightarrow$ (i) $\frac{u}{\xi} = \frac{v}{\eta} = \frac{w}{\zeta}$ or (ii) $\xi = 0, \eta = 0, \zeta = 0$.

(iii) $\Rightarrow$ motion is irrotational

(iv) $\Rightarrow$ stream lines are given by

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$

coincide with vortex lines given by

$$\frac{dx}{\xi} = \frac{dy}{\eta} = \frac{dz}{\zeta}$$

Second Part. Consider two dimensional motion s.t. ζ = const. To prove that $\Delta(H - 2\zeta\psi) = 0$

$$2\zeta = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right), \frac{-\partial\phi}{\partial x} = u = \frac{-\partial\psi}{\partial y}, \frac{-\partial\phi}{\partial y} = v = \frac{\partial\psi}{\partial x}$$

$\therefore$ $$2\zeta = \frac{\partial^2\psi}{\partial x^2} - \left(\frac{-\partial^2\psi}{\partial y^2}\right) = \nabla^2\psi, \; 2\zeta = \nabla^2\psi$$

Equation of motion is

$$\frac{dq}{dt} = -\nabla V - \frac{1}{\rho}\nabla p$$

This $\Rightarrow \left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)u = \frac{-\partial V}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x}$ as $w = 0$

$$\Rightarrow \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x} + v\left(\frac{\partial u}{\partial y}0 - \frac{\partial v}{\partial x}\right) = \frac{-\partial V}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$q^2 = u^2 + v^2 \text{ so that } \frac{\partial q^2}{\partial x} = 2\left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}\right)$$

$\therefore$ $$\frac{\partial u}{\partial t} + \frac{1}{2}\frac{\partial q^2}{\partial x} + v(-2\zeta) = -\frac{\partial V}{\partial x} - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

or $$\frac{\partial u}{\partial t} - 2v\zeta = -\frac{\partial H}{\partial x} \qquad \ldots(6)$$

Similarly $$\frac{\partial v}{\partial t} + 2u\zeta = -\frac{\partial H}{\partial y} \qquad \ldots(7)$$

Differentiating (6) and (7) w.r.t. x and y respectively and then adding,

$$-\Delta H = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right) + 2\zeta\left(-\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

But $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ is the equation of continuity.

$\therefore$ $$-\Delta H = 2\zeta\left(-\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2}\right) = -2\zeta\Delta\psi$$

or $$\Delta(H - 2\zeta\psi) = 0 \ldots (8) \text{ as } \zeta = \text{const.}$$

This proves the second required result.

Third Part. Further, suppose that the motion is steady so that

$$\frac{\partial u}{\partial t} = 0 = \frac{\partial v}{\partial t}.$$

Integrating (8), $\quad H - 2\zeta\psi$ = const. = C, say

Putting the value of H,

Then $\frac{p}{\rho} + \frac{1}{2}q^2 + V - 2\zeta\psi = C$

This concludes the problem.

Problem 26. *A mass of liquid whose outer boundary is an infinitely long cylinder of radius b is in a state of cyclic irrotational motion is under the action of a uniform pressure P over the external surface. Prove that there must be a concentric cylindrical hollow whose radius a is given by*

$$8\pi^2a^2b^2P = Mk^2$$

where M is the mass of unit length of the liquid and k the circulation.

Solution. The complex potential is given by

$$W = \frac{ik}{2\pi}\log(z - z_0) = \frac{ik}{2\pi}\log re^{i\theta}$$

Hence $\psi = \frac{k}{2\pi}\log r, \phi = -\frac{k\theta}{2\pi}$.

$$u = -\frac{\partial\phi}{\partial r} = 0, v = -\frac{1}{r}\frac{\partial\phi}{\partial\theta} = -\frac{1}{r}\left(\frac{k}{-2\pi}\right) = \frac{k}{2\pi r}$$

$$q^2 = u^2 + v^2 = 0 + v^2 \text{ or } q = v = k/2\pi r$$

By Bernoulli's equation for steady motion,

$$\frac{p}{\rho} + \frac{1}{2}q^2 = c. \qquad \ldots(1)$$

Subjecting this to the boundary

$$p = P_0, r = b, q = \frac{k}{2\pi b},$$

$$\therefore \quad \frac{P}{\rho} + \frac{k^2}{8\pi^2 b^2} = c.$$

Now (1) becomes

$$\frac{p}{\rho} + \frac{1}{2}q^2 = \frac{P}{\rho} + \frac{k^2}{8\pi^2 b^2}$$

$$\therefore \quad p = P + \frac{k^2\rho}{8\pi^2}\left(\frac{1}{b^2} - \frac{1}{r^2}\right) \qquad \dots (2)$$

If r is very small, then $p < 0$. Hence there must be a cavity, say $r = a$. Then when $r = a, p = 0$, Since pressure vanishes on the surface of cavity. Then (2) $\Rightarrow$

$$P + \frac{k^2\rho}{8\pi^2}\left(\frac{1}{b^2} - \frac{1}{a^2}\right) = 0 \quad \text{or} \quad 8\pi^3 a^2 b^2 P = Mk^2, \text{ where}$$

$$M = \pi\rho\,(b^2 - a^2).$$

Problem 27. *Prove that, in the steady motion of an incompressible liquid, under the action of conservative forces, we have*

$$\xi\frac{\partial u}{\partial x} + \eta\frac{\partial u}{\partial y} + \zeta\frac{\partial u}{\partial z} = 0$$

and two more similar equations in v and w.

Solution. By Helmholtz vortricity equation,

$$\frac{d}{dt}\left(\frac{W}{\rho}\right) = \left(\frac{W \cdot \nabla}{\rho}\right)q$$

Since ρ is constant and so $\frac{d}{dt}(W) = (W \cdot \nabla)\,q$

Motion is steady $\Rightarrow \frac{dW}{dt} = 0 \Rightarrow (W \cdot \nabla)q = 0$

$$\Rightarrow \quad \left(\xi\frac{\partial}{\partial x} + \eta\frac{\partial}{\partial y} + \zeta\frac{\partial}{\partial z}\right)q = 0$$

$$\Rightarrow \quad \xi\frac{\partial u}{\partial x} + \eta\frac{\partial u}{\partial y} + \zeta\frac{\partial u}{\partial z} = 0$$

and two similar expressions for v and w.

MOMENTS AND PRODUCTS OF INERTIA | SET-I

1.1. Definitions :

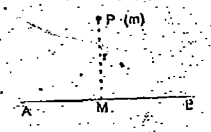
(a) Rigid Body. *A rigid body is a collection of particles such that the distance between any two particles of the body remains always the same.*

(b) Moment of Inertia of a particle.

The moment of inertia of a particle of mass m at the point P, about the line AB is defined by

$$I = mr^2,$$

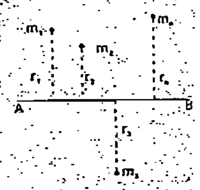
where r is the perpendicular distance of P from the line AB.



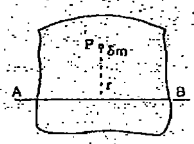
(c) Moment of Inertia of a system of particles. The moment of inertia of a system of particles of masses $m_1, m_2, \ldots, m_n$ at distances $r_1, r_2, \ldots, r_n$ respectively from the line AB, about the line AB is defined by

$$I = m_1 r_1^2 + m_2 r_2^2 + \ldots + m_n r_n^2$$

$$= \sum_{i=1}^{n} m_i r_i^2.$$



(d) Moment of Inertia of a boby. Let δm be the mass of an elementary portion of the body and r its distance from the line AB, then the moment of inertia of the mass δm about the line AB is $r^2 \delta m$.



$\therefore$ The moment of inertia of the body about the line AB is given by

$$I = \int r^2 dm,$$

where the integration is taken over the whole body.

(e) Radius of Gyration. The moment of inertia of a body about the line AB is given by

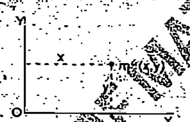
$$I = \int r^2 dm$$

If the total mass of the body is M and K a quantity such that

$$I = MK^2,$$

then K is called the *radius of gyration* of the body about the line AB.

(f) Product of Inertia. Let (x, y) be the coordinates of a mass m with respect to two mutually perpendicular lines OX and OY as axes. Then the product of inertia of mass m with respect to the lines OX and OY is defined by mxy.



If (x, y) be the coordinates of the mass m of an elementary portion of the body with respect to the perpendicular axes OX and OY then the product of inertia of the body about these axes OX and OY is defined by Σmxy.

1.2. Moment and Product of Inertia with respect to three mutually perpendicular axes.

Let (x, y, z) be the coordinates of the mass m of a body with respect to three mutually perpendicular axes OX, OY, OZ in space. Then we shall denote by A, B, C the moments of inertia of the body about the coordinate axes OX, OY, OZ respectively and by D, E, F the products of inertia about the axes OY, OZ; OZ, OX and OX, OY respectively. These moments and products of inertia are given by

$A = \Sigma m (y^2 + z^2)$, $\quad B = \Sigma m (z^2 + x^2)$, $\quad C = \Sigma m (x^2 + y^2)$

$D = \Sigma myz$, $\quad E = \Sigma mzx$, $\quad F = \Sigma mxy$.

1.3. Some Simple Propositions :

Prop. I. *If A, B, C denote the moments and D, E, F the products of inertia about three mutually perpendicular axes, the sum of any two of them is greater than the third.*

We have, $A = \Sigma m (y^2 + z^2)$, $B = \Sigma m (z^2 + x^2)$, $C = \Sigma m (x^2 + y^2)$

then $A + B - C = \Sigma m (y^2 + z^2) + \Sigma m (z^2 + x^2) - \Sigma m (x^2 + y^2)$

$$= 2\Sigma mz^2 = + ve.$$

$\therefore A + B > C$.

Prop. II. *The sum of the moments of inertia about any three rectangular axes meeting at a given point is always constant and is equal to twice the moment of inertia about that point.*

We have

$A + B + C = \Sigma m (y^2 + z^2) + \Sigma m (z^2 + x^2) + \Sigma m (x^2 + y^2)$

$= 2\Sigma m (x^2 + y^2 + z^2) = 2\Sigma mr^2$

$= 2$ (*M.I.* of the body about the given point)

$\because r = \sqrt{(x^2 + y^2 + z^2)} =$ distance of the mass m at (x, y, z) from the given point O as origin.

Thus the sum $A + B + C$ is independent of the directions of axes and is equal to twice the moment of inertia about the given point.

Prop. III. *The sum of the moments of inertia of a body with reference to any plane through a given point and its normal at that point is constant and is equal to the moment of inertia of the body with respect to the point.*

Let the given point O be taken as the origin and the plane as XY plane. If C' is the moment of inertia of the body about the XY plane and C the moment of inertia of the body about its normal at O which is Z-axis, then

$$C' = \Sigma mz^2 \text{ and } C = \Sigma m (x^2 + y^2).$$

$\therefore C' + C = \Sigma m (x^2 + y^2 + z^2) = \Sigma mr^2$

$= M.I.$ of the body about O.

Thus $C' + C$ is independent of the plane through O and is constant equal to the moment of inertia of the body about the point.

Note. By Prop. II, we have $A + B + C = 2\Sigma mr^2$

and by prop. III, we have $C + C' = \Sigma mr^2$

$\therefore C + C' = \frac{1}{2}(A + B + C)$ or $C' = \frac{1}{2}(A + B - C)$.

Thus if A', B', C' denote the moments of inertia of the body with respect to the planes YZ, ZX and XY respectively, then

$A' = \frac{1}{2}(B + C - A)$, $B' = \frac{1}{2}(C + A - B)$ and $C' = \frac{1}{2}(A + B - C)$

Prop. IV. $A > 2D$, $B > 2E$ and $C > 2F$.

we know that $A.M. > G.M.$

$\therefore \frac{y^2 + z^2}{2} > \sqrt{(y^2 \cdot z^2)}$ or $y^2 + z^2 > 2yz$

or $\Sigma m(y^2 + z^2) > 2\Sigma myz$

i.e. $A > 2D$.

Similarly $B > 2E$ and $C > 2F$.

MOMENTS OF INERTIA IN SOME SIMPLE CASES.

1.4. Moment of Inertia of a uniform rod of length 2a

(i) *About a line through an end and perpendicular to the rod.*

Let M be the mass of a rod AB of length $2a$, then mass of the rod per unit length $= \rho = M/2a$.

Consider an element PQ of breadth δx at a distance x from the end A.

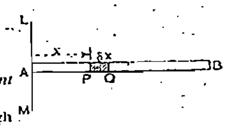
Mass of the element $PQ = \frac{M}{2a}\delta x = \delta m$.

M.I. of this element PQ about the line LM passing through the end A and perpendicular to the rod AB

$= x^2 \delta m = \frac{M}{2a} x^2 \delta x$.

$\therefore$ M.I. of the rod AB about LM

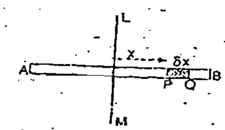
$$= \int_0^{2a} \frac{M}{2a} x^2 dx = \frac{M}{2a}\left[\frac{1}{3}x^3\right]_0^{2a} = \frac{4}{3}Ma^2$$



(ii) *About a line through the middle point and perpendicular to the rod.*

Let LM be the line passing through the middle point C and perpendicular to the rod AB.

Consider an element PQ of breadth δx at a distance x from the middle point C.



Mass of the element

$PQ = \frac{M}{2a}\delta x = \delta m$ $\qquad (\because \rho = M/2a)$

M.I. of the element PQ about the line LM

$= x^2 \delta m = x^2 \frac{M}{2a}\delta x$.

$\therefore$ M.I. of the rod AB about LM

$$= \int_{-a}^{a} \frac{M}{2a} x^2 dx = \frac{M}{2a}\left[\frac{1}{3}x^3\right]_{-a}^{a} = \frac{1}{3}Ma^2$$

1.5 Moment of Inertia of a rectangular lamina.

(i) *About a line through its centre and paralles to a side.*

Let M be the mass of a rectangular lamina $ABCD$ such that $AB = 2a$ and $BC = 2b$.

$\therefore$ Mass per unit area of the rectangle $= \rho = \frac{M}{4ab}$.

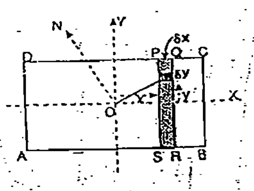
Let OX and OY be the lines parallel to the sides AB and BC of the rectangle through its centre C.

Consider an elementary strip $PQRS$ of breadth δx at a distance x from O and parallel to BC.

Mass of the strip

$PQRS = \rho \cdot 2b\delta x$

$= \frac{M}{4ab} 2b\delta x = \frac{M}{2a}\delta x = \delta m$.



M.I. of the strip about $OX = \frac{1}{3}b^2 \delta m$.

[see § 1.4 (ii)]

$= \frac{1}{3}b^2 \cdot \frac{M}{2a}\delta x = \frac{1}{6}\frac{Mb^2}{a}\delta x$.

$\therefore$ M.I. of the rectangle $ABCD$ about OX

$= \int_{-a}^{a} \frac{Mb^2}{6a}\, dx = \frac{Mb^2}{6a}[x]_{-a}^{a} = \frac{1}{3} Mb^2.$

Similarly M.I. of the rectangle $ABCD$ about $OY = \frac{1}{3} Ma^2$.

Aliter. Consider an elementary area $\delta x \delta y$ at a point (x, y) of the lamina.

Mass of the elementary area $= \rho \delta x \delta y = \frac{M}{4ab} \delta x \delta y = \delta m.$

M.I. of this elementary mass about $OX = y^2 \delta m = \frac{My^2}{4ab} \delta x \delta y.$

$\therefore$ M.I. of the rectangular lamina $ABCD$ about OX

$= \int_{-a}^{a} \int_{-b}^{b} \frac{M}{4ab} y^2\, dx dy = \frac{M}{4ab} \int_{-a}^{a} \left[\frac{1}{3} y^3\right]_{-b}^{b} dx$

$= \frac{M}{4ab} \cdot \frac{2}{3} b^3 \int_{-a}^{a} dx = \frac{M}{6a} b^2 [x]_{-a}^{a} = \frac{1}{3} Mb^2.$

(ii) *About a line through its centre and perpendicular to its plane.*

Let ON be the line through the centre O and perpendicular to the plane of the rectangular lamina $ABCD$.

Consider an elementary area $\delta x \delta y$ at a point (x, y) of the lamina.

Mass of the elementary area $= \rho \delta x \delta y = \frac{M}{4ab} \delta x \delta y = \delta m.$

Distance of this elementary area from $ON = \sqrt{(x^2 + y^2)}$.

$\therefore$ M.I. of this elementary mass about ON

$= ON^2\, \delta m = (x^2 + y^2) \cdot \frac{M}{4ab} \delta x \delta y.$

Hence M.I. of the rectangular lamina about ON

$= \int_{-a}^{a} \int_{-b}^{b} \frac{M}{4ab} (x^2 + y^2)\, dx dy$

$= \frac{M}{4ab} \int_{-a}^{a} \left[x^2 y + \frac{1}{3} y^3\right]_{-b}^{b} dx = \frac{M}{4ab} \int_{-a}^{a} 2\left(bx^2 + \frac{1}{3} b^3\right) dx$

$= \frac{M}{4ab}\left[2\left(\frac{b}{3} x^3 + \frac{1}{3} b^3 x\right)\right]_{-a}^{a} = \frac{M}{4ab} \cdot \frac{4}{3}(ba^3 + b^3 a).$

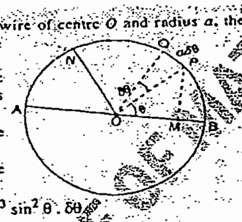
$= \frac{M}{3}(a^2 + b^2).$

Note. M.I. about $ON = \frac{M}{3}(a^2 + b^2) = \frac{1}{3} Ma^2 + \frac{1}{3} Mb^2$

$=$ M.I. about OY + M.I. about OX.

1.6. Moment of Inertia of a Circular wire.

(i) *About a diameter.*



Let M be the mass of the circular wire of centre O and radius a, then mass per unit lenth of the wire $= \rho = M/2\pi a$.

Consider an elementary arc $PQ = a\, \delta\theta$ of the wire, then its mass $= \rho a \delta\theta = \delta m$.

Distance of this element from the diameter $AB = PM = a \sin\theta$.

$\therefore$ M.I. of this element about the diameter AB.

$= (a \sin\theta)^2 \cdot \delta m = a^2 \sin^2\theta \cdot \rho a \delta\theta = \rho a^3 \sin^2\theta \cdot \delta\theta.$

Hence M.I. of the circular wire about the diameter AB

$= \int_0^{2\pi} = \rho a^3 \sin^2\theta\, d\theta = \frac{1}{2} \rho a^3 \int_0^{2\pi} (1 - \cos 2\theta)\, d\theta$

$= \frac{1}{2} \rho a^3 \left[\theta - \frac{1}{2} \sin 2\theta\right]_0^{2\pi} = \frac{1}{2} \cdot \frac{M}{2\pi a} \cdot a^3 \cdot 2\pi$ $\quad (\because \rho = M/2\pi a)$

$= \frac{1}{2} Ma^2.$

(ii) *About a line through the centre and perpendicular to its plane.*

Let ON be the line through the centre O and perpendicular to the plane of the circular wire.

$\therefore$ M.I. of the elementary arc PQ about ON

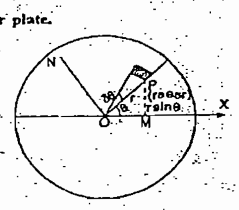
$= OP^2 \cdot \delta m = a^2 \cdot \rho a\, \delta\theta = \rho a^3\, \delta\theta.$

Hence M.I. of the wire about ON

$= \int_0^{2\pi} \rho a^3\, d\theta = \frac{M}{2\pi a} \cdot a^3 [\theta]_0^{2\pi} = Ma^2.$ $\quad (\because \rho = M/2\pi a)$

1.7. Moment of Inertia of a Circular plate.

(i) *About a diameter.*



Let M be the mass of a circular plate of centre O and radius a; then mass per unit area of the plate $= \rho = M/\pi a^2$.

Consider an elementary area $r\delta\theta\delta r$ at the point $P(r, \theta)$ of the plate referred to the centre O as the pole and OX as the initial line.

Mass of the element $= \rho \cdot r\delta\theta\delta r = \delta m.$

Distance of this element from $OX = OM = r \sin\theta$.

$\therefore$ M.I. of the element about OX.

$= (r \sin\theta)^2 \cdot \delta m = r^2 \sin^2\theta \cdot \rho r\, \delta\theta\delta r = \rho r^3 \sin^2\theta\, \delta\theta\delta r.$

Hence M.I. of the circular plate about OX

$= \int_0^{2\pi} \int_0^{a} \rho r^3 \sin^2\theta\, d\theta dr = \rho \int_0^{2\pi} \left[\frac{1}{4} r^4\right]_0^{a} \sin^2\theta\, d\theta$

$= \frac{1}{4} \rho a^4 \cdot \int_0^{2\pi} \frac{1}{2}(1 - \cos 2\theta)\, d\theta = \frac{1}{8} \rho a^4 \left[\theta - \frac{1}{2} \sin 2\theta\right]_0^{2\pi}$

$= \frac{1}{8} \rho a^4 \cdot 2\pi = \frac{1}{4} \cdot \frac{M}{\pi a^2} \cdot \pi a^4 = \frac{1}{4} Ma^2$ $\quad (\because \rho = M/\pi a^2)$

(ii) *About a line through the centre and perpendicular to its plane.*

Let ON be the line through the centre O and perpendicular to the plane of the plate.

$\therefore$ M.I. of the elementary area about ON

$= OP^2 \cdot \delta m = r^2 \cdot \rho r \delta\theta \delta r = \rho r^3\, \delta\theta\delta r.$

Hence M.I of the circular plate about O

$= \int_0^{2\pi} \int_0^{a} \rho r^3\, d\theta dr = \rho \int_0^{2\pi} \left[\frac{1}{4} r^4\right]_0^{a} d\theta$

$= \frac{1}{4} \rho a^4 [\theta]_0^{2\pi} = \frac{1}{4} \cdot \frac{M}{\pi a^2} \cdot a^4 \cdot 2\pi = \frac{1}{2} Ma^2$ $\quad (\because \rho = M/\pi a^2)$

1.8. Moment of Inertia of an elliptic disc. [Meerut TDC 95 (P)]

Let M be the mass of an elliptic disc of axes $2a$ and $2b$, then mass per unit area of the disc

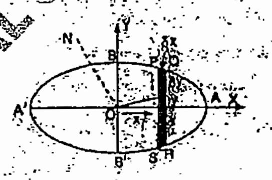
$$= \rho = \frac{M}{\pi ab}.$$

Consider an elementary area $\delta x \delta y$ at the point (x, y), then its mass $= \rho \delta x\, \delta y = \delta m$.

$\therefore$ M.I. of the elementary mass about OX

$= y^2\, \delta m = y^2\, \rho \delta x \delta y.$

Hence moment of inertia of the elliptic disc about OX



$= \int_{-a}^{a} \int_{-y}^{y} y^2\, \rho\, dx dy$

$= \rho \int_{-a}^{a} \left[\frac{1}{3} y^3\right] dx$

$= \frac{2}{3} \rho \int y^3 dx$

$= \frac{2}{3} \rho \int_{-a}^{a} b^3 \left(1 - \frac{x^2}{a^2}\right)^{3/2} dx$ $\quad \left(\because \text{Equation of the ellipse is } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1\right)$

$= \frac{2}{3} \rho b^3 \int_{-\pi/2}^{\pi/2} (1 - \sin^2\theta)^{3/2} a \cos\theta\, d\theta$, Putting $x = a \sin\theta$

$= \frac{2}{3} \rho b^3 a \int_{-\pi/2}^{\pi/2} \cos^4\theta\, d\theta = \frac{2}{3} \rho b^3 a \cdot 2 \int_0^{\pi/2} \cos^4\theta\, d\theta$

$= \frac{4}{3} \cdot \rho b^3 a \frac{\Gamma(\frac{5}{2})\, \Gamma(\frac{1}{2})}{2\Gamma(3)} = \frac{4}{3} \rho b^3 a \cdot \frac{3}{16} \pi = \frac{1}{4} \rho \pi b^3 a$

$= \frac{1}{4} \cdot \frac{M}{\pi ab} \cdot \pi b^3 a = \frac{1}{4} Mb^2$ $\quad (\because \rho = M/\pi ab)$

Similarly M. I. of the an elliptic disc about the minor axis $BB' = \frac{1}{4} Ma^2$.

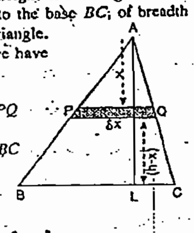
And M.I. of the disc about the line ON through the centre O and perpendicular to its plane

$=$ M.I. about OA + M.I. about OB

$= \frac{1}{4} Mb^2 + \frac{1}{4} Ma^2 = \frac{1}{4} M(a^2 + b^2).$

1.9. Moment of Inertia of a uniform triangular lamina about one side.

Let M be the mass and $h = AL$, the height of a triangular lamina ABC. Let PQ be an elementary strip parallel to the base BC, of breadth δx and at a distance x from the vertex A of the triangle.



From similar triangles APQ and ABC, we have

$x/AL = PQ/BC$.

$\therefore PQ = ax/h$, where $BC = a$.

$\therefore \delta m =$ mass of the elementary strip PQ

$= \rho PQ\, \delta x = \rho\, (ax/h)\, \delta x$

$\therefore$ M. I. of the elementary strip about BC

$= (x - h)^2\, \delta m = \frac{\rho a}{h}(h - x)^2 x \delta x.$

$\therefore$ M. I. of the triangle ABC about BC

$= \int_0^{h} \frac{\rho a}{h}(h - x)^2 x\, dx = (\rho a/h) \int_0^{h} (h^2 x - 2hx^2 + x^3)\, dx$

$= (\rho a/h)\left[\frac{1}{2} h^2 x^2 - \frac{2}{3} hx^3 + \frac{1}{4} x^4\right]_0^{h} = \frac{1}{12} \rho a h^3 = \frac{1}{6} Mh^2.$

$[\because M =$ mass of $\Delta ABC = \rho \cdot (\frac{1}{2} ah)]$

1.10 Moment of Inertia of a rectangular parallelopiped about an axis through its centre and parallel to one of its edges.

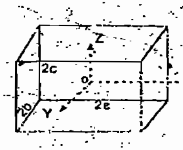
Let O be the centre and $2a, 2b, 2c$ the lengths of the edges of a rectangular parallelopiped. If M is the mass of the parallelopipod, the mass per unit volume

$$= \rho = \frac{M}{2a.2b.2c} = \frac{M}{8abc}$$

Let OX, OY, OZ be the axes through the centre and parallel to the edges of the rectangular parallelopiped.

Consider an elementary volume $\delta x\, \delta y\, \delta z$ of the parallelopiped at the point $P(x, y, z)$, then its mass

$= \rho\, \delta x\, \delta y\, \delta z = \delta m$.



Distance of the point $P(x, y, z)$ from OX is $\sqrt{(y^2+z^2)}$.

$\therefore$ M.I. of the elementary volume of mass δm at P about OX

$= \rho\,(y^2+z^2)\,\delta x\, \delta y\, \delta z$.

Hence M.I. of the rectangular parallelopiped about OX (which is parallel to $2a$).

$$= \int_{-a}^{a}\int_{-b}^{b}\int_{-c}^{c} \rho\,(y^2+z^2)\,dx\,dy\,dz$$

$$= \int_{-a}^{a}\int_{-b}^{b}\left[y^2 z + \tfrac{1}{3}z^3\right]_{-c}^{c} dx\,dy = \rho\int_{-a}^{a}\int_{-b}^{b} 2\left(y^2c + \tfrac{1}{3}c^3\right)dx\,dy$$

$$= 2\rho\int_{-a}^{a}\left[\tfrac{1}{3}y^3c + \tfrac{1}{3}c^3y\right]_{-b}^{b} dx = \tfrac{2}{3}\rho\int_{-a}^{a} 2\,(b^3c + c^3b)\,dx$$

$$= \frac{4\rho}{3}bc\,(b^2+c^2)\,[x]_{-a}^{a} = \frac{4\rho}{3}bc\,(b^2+c^2).\,2a$$

$$= \frac{4}{3}\cdot\frac{M}{8abc}\cdot bc\,(b^2+c^2).\,2a \quad \because\ \rho = \frac{M}{8abc}$$

$$= \tfrac{1}{3}M(b^2+c^2).$$

Similarly M.I. of the rectangular parallelopiped about the lines OY, OZ, through centre O and parallel to $2b$ and $2c$ are $\frac{1}{3}M(c^2+a^2)$ and $\frac{1}{3}M(a^2+b^2)$ respectively.

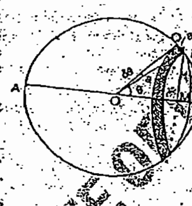
Note : For cube of side $2a$, $2b = 2c = 2a$.

$\therefore$ M.I. of a cube about a line through its centre and parallel to one edge $= \frac{2}{3}Ma^2$.

1.11. M.I of a spherical shell (i.e. hollow sphere) about diameter.

A spherical shell (*i.e.* hollow sphere) of radius a is formed by the revolution of a semi-circular arc of radius a about its diameter.

Consider an elementary arc $PQ = a\delta\theta$ at the point P of the semi-circular arc. A circular ring of radius $PM = a\sin\theta$ will be fromed by the revolution of this arc PQ about the diameter AB.



Mass of this elementary ring

$= \delta m = \rho\,.\,2\pi\, PM\,.\,a\delta\theta$.

$= \rho\,.\,2\pi a\sin\theta\,.\,a\,\delta\theta = \rho\, 2\pi a^2 \sin\theta\,\delta\theta$.

where $\rho = \dfrac{M}{4\pi a^2}$, M is the mass of the shell.

M.I. of this elementary ring about AB
(a line through the centre of the ring and perpendicular to its plane)

$= PM^2.\,\delta m = a^2\sin^2\theta.\,\rho\, 2\pi a^2 \sin\theta\,\delta\theta$. (see § 1.6)

$= 2\pi\rho\, a^4 \sin^3\theta\,\delta\theta$

$\therefore$ M.I. of the shell about the diameter AB

$$= \int_0^{\pi} 2\pi\rho a^4 \sin^3\theta\, d\theta = 2\pi\rho a^4\int_0^{\pi}(1-\cos^2\theta)\sin\theta\, d\theta$$

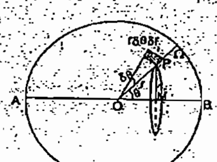
$$= -2\pi\rho a^4\int_1^{-1}(1-t^2)\,dt \quad \text{Putting } \cos\theta = t, \text{ so that } -\sin\theta\, d\theta = dt$$

$$= -2\pi.\frac{M}{4\pi a^2}a^4\left[t - \tfrac{1}{3}t^3\right]_1^{-1} = \tfrac{2}{3}Ma^2.$$

1.12. M.I. of a solid sphere about a diameter.

A solid sphere of radius a is formed by the revolution of a semi-circular area of radius a about its diameter.

Consider an elementary area $r\delta\theta\delta r$ at the point $P(r, \theta)$ of the semi-circular area. When this element is revolved about the diameter AB, a circular ring of radius $PM = r\sin\theta$ and cross-section $r\delta\theta\delta r$ is formed.



Mass of this elementary ring

$= \delta m = \rho.\,2\pi.\,r\sin\theta.\,r\delta\theta\delta r$

$= \rho 2\pi r^2 \sin\theta\,\delta\theta\delta r$

where $\rho = \dfrac{M}{\left(\frac{4}{3}\pi a^3\right)}$, M is the mass of the sphere. M.I. of this elementary ring about AB (a line through the centre of the ring and perpendicular to its plane)

$= PM^2.\,\delta m = r^2\sin^2\theta.\,\rho 2\pi r^2\sin\theta\,\delta\theta\delta r$

$= 2\pi\rho r^4\sin^3\theta\,\delta\theta\delta r$.

$\therefore$ M.I. of the sphere about the diameter AB

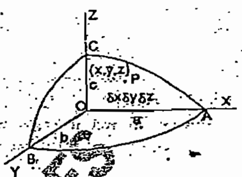
$$= \int_{\theta=0}^{\pi}\int_{r=0}^{a} 2\pi\rho r^4 \sin^3\theta\, d\theta\, dr = 2\pi\rho\,\tfrac{1}{5}a^5\int_0^{\pi}\sin^3\theta\, d\theta$$

$$= \frac{2\pi}{5}\cdot\frac{M}{\frac{4}{3}\pi a^3}\cdot a^5\cdot\left(\tfrac{4}{3}\right) = \tfrac{2}{5}Ma^2.$$

1.13. M.I. of an ellipsoid.

Let the equation of the ellipsoid be

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1.$$



Consider an elementary volume $\delta x\,\delta y\,\delta z$ at the point $P(x, y, z)$ of the ellipsoid in the positive octant.

$\therefore$ Mass of this element $= \rho\delta x\,\delta y\,\delta z$

where ρ = Mass per unit volume

$= \dfrac{M}{\frac{4}{3}\cdot\pi abc} = \dfrac{3M}{4\pi abc}$, M is the mass of the ellopsoid.

Distance of the point $P(x, y, z)$ from $OX = \sqrt{(y^2+z^2)}$.

$\therefore$ M.I. of this elementary volume about OX

$= (y^2+z^2)\,\rho\,\delta x\,\delta y\,\delta z$.

$\therefore$ M.I. of the ellipsoid about OX

$= 8\iiint (y^2+z^2)\,\rho\, dx\,dy\,dz$, where, $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}+\dfrac{z^2}{c^2}\le 1$

the integration being extended over positive octant of the ellipsoid.

Putting $\dfrac{x^2}{a^2} = u, \dfrac{y^2}{b^2} = v, \dfrac{z^2}{c^2} = w$

i.e. $x = au^{1/2}, y = bv^{1/2}, z = cw^{1/2}$

so that $dx = \frac{1}{2}au^{-1/2}\,du$, $dy = \frac{1}{2}bv^{-1/2}\,dv$, $dz = \frac{1}{2}cw^{-1/2}\,dw$, we have

M.I. of the ellipsoid about OX (*i.e.* the axis $2a$)

$$= 8\frac{abc}{8}\cdot\rho\iiint (b^2v + c^2w)\,u^{-1/2}v^{-1/2}w^{-1/2}\,du\,dv\,dw$$

$$= abc\,\rho\left[b^2\iiint u^{\frac{1}{2}-1}v^{\frac{3}{2}-1}w^{\frac{1}{2}-1}\,du\,dv\,dw + c^2\iiint u^{\frac{1}{2}-1}v^{\frac{1}{2}-1}w^{\frac{3}{2}-1}\,du\,dv\,dw\right]$$

where $u+v+w\le 1$

$$= abc\,\rho\left[\frac{b^2.\,\Gamma(\frac{1}{2})\,\Gamma(\frac{3}{2})\,\Gamma(\frac{1}{2})}{\Gamma(1+\frac{1}{2}+\frac{3}{2}+\frac{1}{2})} + c^2.\frac{\Gamma(\frac{1}{2})\,\Gamma(\frac{1}{2})\,\Gamma(\frac{3}{2})}{\Gamma(1+\frac{1}{2}+\frac{1}{2}+\frac{3}{2})}\right], \text{ By Dirichlet's theorem.}$$

$$= abc\,\frac{3M}{4\pi abc}(b^2+c^2).\,\frac{\sqrt{\pi}.\frac{1}{2}\sqrt{\pi}.\sqrt{\pi}}{\frac{5}{2}.\frac{3}{2}.\frac{1}{2}.\sqrt{\pi}} = \tfrac{1}{5}M(b^2+c^2).$$

1.14. Reference Table.

The moments of Inertis of some standard rigid bodies considered in § 1.4 to § 1.13 are given in the following table. The students are advised to remember all these as thoy will be used frequently.

Rigid body	M.I.
1. Uniform thin rod of length $2a$ and mass M.	
(i) About a line through the *middle point* and perpendicular to its length	$\frac{1}{3}Ma^2$
(ii) About a line through one *end* and perpendicular to its length	$\frac{4}{3}Ma^2$
2. Rectangular plate of sides $2a$, $2b$ and mass M.	
(i) About a line through the centre and parallel to the side $2a$	$\frac{1}{3}Mb^2$
(ii) About a line through the centre and parallel to the side $2b$	$\frac{1}{3}Ma^2$
(iii) About a line through the centre and perpendicular to the plate	$\frac{1}{3}M(a^2+b^2)$
3. Rectangular parallelopiped of edges $2a$, $2b$, $2c$ and mass M. About a line through its centre and parallel to the edge $2a$	$\frac{1}{3}M(b^2+c^2)$
4. Circular ring of radius a and mass M.	
(i) About its diameter	$\frac{1}{2}Ma^2$
(ii) About a line through the centre and perpendicular to the plane of the ring	Ma^2

5. Circular plate of radius a and mass M.	
(i) About its diameter	$\frac{1}{4}Ma^2$
(ii) About a line through the centre and perpendicular to its plane	$\frac{1}{2}Ma^2$
6. Elliptic disc of axes $2a$ and $2b$ and mass M.	
(i) About the axis $2a$	$\frac{1}{4}Mb^2$
(ii) About the axis $2b$	$\frac{1}{4}Ma^2$
(iii) About a line through the centre and perpendicular to its plane	$\frac{1}{4}M(a^2+b^2)$
7. Spherical shell of radius a and mass M. About a diameter	$\frac{2}{3}Ma^2$
8. Solid sphere of radius a and mass M. About a diameter	$\frac{2}{5}Ma^2$
9. Ellipsoid of axis $2a$, $2b$, $2c$ and mass M. About the axis $2a$.	$\frac{1}{5}M(b^2+c^2)$

Routh's Rule. All the above M.I. may be remembered with the help of the following Routh's Rule.

M.I. about an axis of symmetry

$$= \text{Mass} \times \frac{\text{Sum of squares of perpendicular axis}}{3, 4 \text{ or } 5}$$

The denominator is 3, 4 or 5 according as the body is rectangular (including rod), elliptical (including circular) or ellipsoid (including sphere).

EXAMPLES

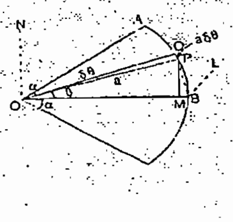
Ex. 1. *Find the M.I. of the arc of a circle about*

(i) the diameter bisecting the arc

(ii) an axis through the centre, perpendicular to its plane

(iii) an axis through its middle point perpendicular to its plane.

Sol. Let OB be the diameter bisecting the circular arc ABC subtending an angle 2α at the centre O. Let a be the radius of the arc.



Consider an elementary arc $PQ = a\,\delta\theta$ at the point P of the arc.

$\therefore$ Its Mass $\delta m = \rho a\,\delta\theta$

where ρ = mass per unit length of the arc $= \frac{M}{2\alpha a}$, M is the mass of the arc ABC.

(i) Distance of P from diameter $OB = PM = a\sin\theta$.

$\therefore$ M.I. of the elementary arc about OB

$= PM^2 . \delta m = (a\sin\theta)^2 \rho a\,\delta\theta$

$= \rho a^3 \sin^2\theta\,\delta\theta$.

$\therefore$ M.I. of the arc ABC about the diameter OB

$$= \int_{-\alpha}^{\alpha} \rho a^3 \sin^2\theta\, d\theta = \tfrac{1}{2}\rho a^3 \int_{-\alpha}^{\alpha} (1-\cos 2\theta)\, d\theta$$

$$= \tfrac{1}{2}\rho a^3 \Big[\theta - \tfrac{1}{2}\sin 2\theta\Big]_{-\alpha}^{\alpha} = \tfrac{1}{2}\frac{M}{2\alpha a}a^3[2\alpha - \sin 2\alpha]$$

$$= \frac{Ma^2}{2\alpha}(\alpha - \sin\alpha\cos\alpha).$$

(ii) Distance of the point P from ON, an axis through the centre and perpendicular to the plane of the arc $= OP = a$.

$\therefore$ M.I. of the elementary mass δm at P about ON

$= a^2 . \delta m = \rho a^3\,\delta\theta$

$$\therefore \text{ M.I. of the arc } ABC \text{ about } ON = \int_{-\alpha}^{\alpha} \rho a^3\, d\theta = \rho a^3 [\theta]_{-\alpha}^{\alpha}$$

$$= \frac{M}{2\alpha a}a^3 . 2\alpha = Ma^2.$$

(iii) Distance of the point P from BL, an axis through the middle point B of the arc ABC and perpendicular to its plane

$= PB = \sqrt{(OP^2 + OB^2 - 2OP . OB\cos\theta)} = \sqrt{(a^2 + a^2 - 2a^2\cos\theta)}$

$= a\sqrt{[2(1-\cos\theta)]} = a\sqrt{[2.2\sin^2\tfrac{1}{2}\theta]} = 2a\sin\tfrac{1}{2}\theta$

$\therefore$ M.I. of the elementary mass δm at P about $BL = PB^2 . \delta m$

$= (2a\sin\tfrac{1}{2}\theta)^2 \rho a\,\delta\theta = 4a^3 \rho \sin^2\tfrac{1}{2}\theta\,\delta\theta$.

$$\therefore \text{ M.I. of the arc } ABC \text{ about } BL = \int_{-\alpha}^{\alpha} 4a^3 \rho \sin^2\tfrac{1}{2}\theta\, d\theta$$

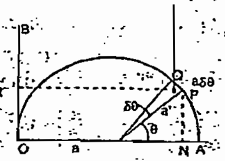
$$= 2a^3\rho \int_{-\alpha}^{\alpha} (1-\cos\theta)\, d\theta = 2a^3 . \frac{M}{2\alpha a}[\theta - \sin\theta]_{-\alpha}^{\alpha}$$

$$= \frac{2Ma^2}{\alpha}(\alpha - \sin\alpha).$$

Ex. 2. *Find the product of inertia of a semicircular wire about diameter and tangent at its extremity.*

Sol. Let M be the mass, a the radius and OA the diameter of a semi-circular arc. Let OB be the tangent at the extremity O.

Consider an elementary arc $PQ = a\,\delta\theta$ at the point P of the wire.



$\therefore$ Its mass $= \delta m = \rho a\,\delta\theta$

where ρ = mass per unit length $= \frac{M}{\pi a}$.

P.I. of this elementary mass about OA and $OB = PN . PL . \delta m$

$= a\sin\theta\,(a + a\cos\theta)\,\rho a\,\delta\theta$

$= \rho a^3 (\sin\theta + \sin\theta\cos\theta)\,\delta\theta$

$\therefore$ P.I. of the wire about OA and OB

$$= \int_0^{\pi} \rho a^3 (\sin\theta + \sin\theta\cos\theta)\, d\theta = \rho a^3\Big[-\cos\theta + \tfrac{1}{2}\sin^2\theta\Big]_0^{\pi}$$

$$= \frac{M}{\pi a}a^3[2] = \frac{2Ma^2}{\pi}.$$

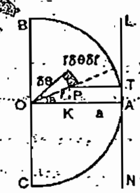
Ex. 3. *Show that the M.I. of a semi-circular lamina about a tangent parallel to the bounding diameter is $Ma^2\left(\frac{5}{4} - \frac{8}{3\pi}\right)$ where a is the radius and M is the mass of lamina.*

Sol. Let LN be the tangent parallel to the bounding diameter BC of a semi-circular lamina of radius a and mass M.

Consider an elementary area $r\,\delta\theta\,\delta r$ at the point P of the lamina, then its mass $\delta m = \rho r\,\delta\theta\,\delta r$.

Where ρ = Mass per unit area

$$= \frac{M}{\frac{1}{2}\pi a^2} = \frac{2M}{\pi a^2}$$



Distance of the point P from $LN = PT$

$= KA = OA - OK = a - r\cos\theta$

$\therefore$ M.I. of the elementary mass δm at P about LN

$= PT^2 . \delta m = (a - r\cos\theta)^2 . \rho r\,\delta\theta\,\delta r$

$\therefore$ M.I. of the lamina about LN

$$= \int_{\theta=-\pi/2}^{\pi/2} \int_{r=0}^{a} (a - r\cos\theta)^2 \rho r\, d\theta\, dr$$

$$= \rho \int_{-\pi/2}^{\pi/2} \int_0^a (a^2 r - 2ar^2\cos\theta + r^3\cos^2\theta)\, d\theta\, dr$$

$$= \rho \int_{-\pi/2}^{\pi/2} \left[\frac{a^2}{2}r^2 - \tfrac{2}{3}ar^3\cos\theta + \tfrac{1}{4}r^4\cos^2\theta\right]_{r=0}^{a} d\theta$$

$$= \rho \int_{-\pi/2}^{\pi/2} \left[\tfrac{1}{2}a^4 - \tfrac{2}{3}a^4\cos\theta + \tfrac{1}{4}a^4\cos^2\theta\right] d\theta$$

$$= 2\rho a^4 \int_0^{\pi/2} (\tfrac{1}{2} - \tfrac{2}{3}\cos\theta + \tfrac{1}{4}\cos^2\theta)\, d\theta$$

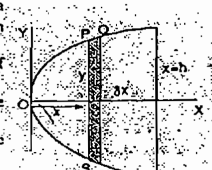
$$= 2\rho a^4 \left[\left\{\tfrac{1}{2}\theta - \tfrac{2}{3}\sin\theta\right\}_0^{\pi/2} + \tfrac{1}{4}\frac{\Gamma(\frac{3}{2})\Gamma(\frac{1}{2})}{2\Gamma(2)}\right]$$

$$= 2 . \frac{2M}{\pi a^2}a^4\left[\tfrac{1}{4}\pi - \tfrac{2}{3} + \tfrac{1}{4} . \frac{\pi}{4}\right] = Ma^2\left(\frac{5}{4} - \frac{8}{3}\pi\right)$$

Ex. 4. *Show that the M.I. of parabolic area (of latus rectum $4a$) cut off by an ordinate at distance h from the vertex is $\frac{3}{7}Mh^2$ about the tangent at the vertex and $\frac{4}{5}Mah$ about the axis.*

Sol. Let the equation of the parabola of latus rectum $4a$ be $y^2 = 4ax$.

Let OAB be the portion of the parabola cut off by an ordinate at a distance h from the vertex.



Consider an elementary strip $PQRS$ of width δx, parallel to Oy.

$\therefore$ Mass of the strip $\delta m = \rho . 2y\,\delta x$, where ρ is the mass per unit area.

$\therefore$ M = Mass of the portion $OABO$ of the parabola

$$= \int_0^h \rho\, 2y\, dx$$

$$= 2\rho \int_0^h 2\sqrt{(ax)}\, dx = 4\rho\sqrt{a} . \tfrac{2}{3}h^{3/2} = \tfrac{8}{3}\rho a^{1/2} h^{3/2}$$

Now, the distance of every point of the strip from Oy, the tangent at the vertex, is x.

$\therefore$ M.I. of the strip about $Oy = x^2\,\delta m = \rho\, 2x^2 y\,\delta x$.

M.I. of the whole area $OABO$ about $Oy = \int_0^h 2\rho\, x^2 y\, dx$

$$= 2\rho \int_0^h x^2\, 2\sqrt{(ax)}\, dx = 4\rho a^{1/2} \int_0^h x^{5/2}\, dx = \tfrac{8}{7}\rho a^{1/2} h^{7/2}$$

$$= \tfrac{3}{7}(\tfrac{8}{3}\rho a^{1/2} h^{3/2})\, h^2 = \tfrac{3}{7}Mh^2.$$

Again M.I. of the strip PQR about $OX = \frac{1}{3}y^2\,\delta m$

$= \frac{1}{3} y^2 . \rho . 2 y \delta x = \frac{2}{3} \rho y^3 \delta x$

$\therefore$ M.I. of the whole area OABO about $OX = \int_0^h \frac{2}{3} \rho y^3 \, dx$

$= \frac{2}{3} \rho \int_0^h (4ax)^{3/2} \, dx = \frac{16}{3} a^{3/2} \rho \frac{2}{5} h^{5/2} = \frac{4}{5} (\frac{8}{3} \rho a^{1/2} h^{3/2}) . ah = \frac{4}{5} Mah.$

Ex. 5. *Find the M.I. of the area of the lemniscate* $r^2 = a^2 \cos 2\theta$

(i) *about its axis*

(ii) *about a line through the origin in its plane and perpendicular to its axis.*

(iii) *about a line through the origin and perpendicular to its plane.*

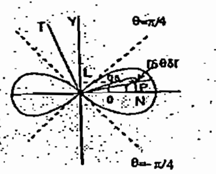
Sol. The loop of the lemniscate is formed between $\theta = -\pi/4$ and $\theta = \pi/4$. The curve is as shown in the fig.

Consider an elementary area $r \delta\theta \delta r$ at the point $P(r, \theta)$ of the curve, then its mass $\delta m = \rho r \delta\theta \delta r$

$\therefore$ The mass of the whole area is given by

$$M = 2\int_{\theta=-\pi/4}^{\pi/4} \int_{r=0}^{a\sqrt{(\cos 2\theta)}} \rho r \, d\theta \, dr = \rho \int_{-\pi/4}^{\pi/4} a^2 \cos 2\theta \, d\theta$$

$$= \rho a^2 \left[\tfrac{1}{2} \sin 2\theta \right]_{-\pi/4}^{\pi/4} = \rho a^2. \quad ...(1)$$



(i) M.I. of elementary mass δm at P about the axis OX

$= PN^2 . \delta m = (r \sin \theta)^2 \rho r \delta\theta \delta r$

$= \rho r^3 \sin^2 \theta \delta\theta \delta r.$

$\therefore$ M.I. of the lemniscate about OX

$$= 2\int_{\theta=-\pi/4}^{\pi/4} \int_{r=0}^{a\sqrt{(\cos 2\theta)}} \rho r^3 \sin^2 \theta \, d\theta \, dr$$

$$= 2\rho \int_{-\pi/4}^{\pi/4} \tfrac{1}{4} a^4 \cos^2 2\theta \sin^2 \theta \, d\theta$$

$$= 2 . \frac{2\rho a^4}{4} \int_0^{\pi/4} \tfrac{1}{2} \cos^2 2\theta \, (1 - \cos 2\theta) \, d\theta$$

$$= \tfrac{1}{4} \rho a^4 \int_0^{\pi/2} \cos^2 t \, (1 - \cos t) \, dt, \qquad \text{Putting } 2\theta = t, \text{ so that } d\theta = \tfrac{1}{2} dt$$

$$= \tfrac{1}{4} \rho a^4 \left[\int_0^{\pi/2} \cos^2 t \, dt - \int_0^{\pi/2} \cos^3 t \, dt \right]$$

$$= \tfrac{1}{4} \rho a^4 \left[\frac{\Gamma(\frac{3}{2}) \Gamma(\frac{1}{2})}{2\Gamma(2)} - \frac{\Gamma(2) \Gamma(\frac{1}{2})}{2\Gamma(\frac{5}{2})} \right] = \frac{1}{4} Ma^2 \left(\frac{\pi}{4} - \frac{2}{3} \right) \qquad \text{from (1)}$$

$$= \frac{Ma^2}{16} (\pi - 8/3).$$

(ii) Distance of the point $P(r, \theta)$ from OY a line through the origin in the plane of the lemniscate and perpendicular to its axis $= PL = r \cos \theta$.

$\therefore$ M.I. of δm at P about OY

$= PL^2 . \delta m = r^2 \cos^2 \theta \, \rho r \, \delta\theta \, \delta r = \rho r^3 \cos^2 \theta \, \delta\theta \, \delta r.$

$\therefore$ M.I of the lemniscate about OY

$$= 2\int_{\theta=-\pi/4}^{\pi/4} \int_{r=0}^{a\sqrt{(\cos 2\theta)}} \rho r^3 \cos^2 \theta \, d\theta \, dr$$

$$= \frac{2\rho}{4} \int_{-\pi/4}^{\pi/4} a^4 \cos^2 2\theta \cos^2 \theta \, d\theta$$

$$= \frac{\rho}{2} 2a^4 \int_0^{\pi/4} \tfrac{1}{2} \cos^2 2\theta \, (1 + \cos 2\theta) \, d\theta$$

$$= \tfrac{1}{2} Ma^2 . \tfrac{1}{2} \int_0^{\pi/2} \cos^2 t \, (1 + \cos t) \, dt, \qquad \text{Putting } 2\theta = t.$$

$$= \tfrac{1}{4} Ma^2 \left(\frac{\pi}{4} + \frac{2}{3} \right) \qquad \text{as in case (i)}$$

$$= \tfrac{1}{48} Ma^2 (3\pi + 8).$$

(iii) Let OT be the line through the origin and prependicular to the plane of the lemniscate.

Distance of δm at P from $OT = OP = r$

$\therefore$ M.I. of δm at P about $OT = OP^2 . \delta m = r^2 . \rho r \delta\theta \delta r = \rho r^3 \delta\theta \delta r$

$\therefore$ M.I. of the lemniscate about OT

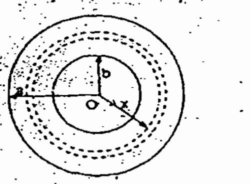
$$= 2\int_{\theta=-\pi/4}^{\pi/4} \int_{r=0}^{a\sqrt{(\cos 2\theta)}} \rho r^3 \, d\theta \, dr = \frac{2\rho}{4} \int_{-\pi/4}^{\pi/4} a^4 \cos^2 2\theta \, d\theta$$

$$= \tfrac{1}{2} \rho a^4 . 2\int_0^{\pi/4} \tfrac{1}{2} (1 + \cos 4\theta) \, d\theta = \tfrac{1}{2} Ma^2 \left[\theta + \tfrac{1}{4} \sin 4\theta \right]_0^{\pi/4} = \tfrac{1}{8} \pi Ma^2.$$

Ex. 6. *Find the M.I. of a hollow sphere about a diameter, its external and internal radii being a and b respectively.*

Sol. If M is the mass of the given hollow sphere, then mass per unit volume

$$\rho = \frac{M}{(\frac{4}{3}\pi a^3 - \frac{4}{3}\pi b^3)} = \frac{3M}{4\pi (a^3 - b^3)}$$



Consider a concentric spherical shell of radius x (s.t. $b < x < a$) and thickness δx

Mass of this elementary shell $= \delta m = \rho . 4\pi x^2 \delta x$

M.I. of this shell about a diameter

$= \frac{2}{3} x^2 . \delta m$

$= \frac{2}{3} x^2 . \rho 4\pi x^2 \delta x = \frac{8}{3} \rho \pi x^4 \delta x.$

$\therefore$ M.I. of the given hollow sphere about a diameter

$$= \int_b^a \tfrac{8}{3} \rho \pi x^4 dx = \tfrac{8}{3} \rho \pi \tfrac{1}{5} (a^5 - b^5)$$

$$= \tfrac{8}{15} \pi . \frac{3M}{4\pi (a^3 - b^3)} . (a^5 - b^5)$$

$$= \frac{2M}{5} . \frac{a^5 - b^5}{a^3 - b^3}.$$

Ex. 7. *Show that the M.I. of a paraboloid of revolution about its axis is* $M/3 \times$ *the square of the radius of its base.*

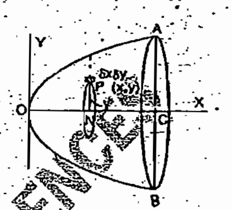
Sol. Let the praboloid of revolution be generated by the revolution of the area bounded by the parabola $y^2 = 4ax$, and x-axis about the axis OX. Let b be the radius of its base.

$\therefore$ for the point A,

$y = AC = b$

$\therefore$ from $y^2 = 4ax$,

$x = \dfrac{b^2}{4a} = OC.$



Consider an elementary area $\delta x \, \delta y$ at the point $P(x, y)$ of the area $OACO$.

By the revolution of this area $\delta x \, \delta y$ about OX, a circular ring of radius y and area of cross-section $\delta x \, \delta y$ is formed.

Mass of this elementary ring

$\delta m = \rho 2\pi y \, \delta x \, \delta y.$

where ρ is the mass per unit volume.

$\therefore$ Mass of the paraboloid of revolution

$$M = \int_{x=0}^{b^2/4a} \int_{y=0}^{\sqrt{(4ax)}} \rho 2\pi y \, dx \, dy = 2\pi\rho . \tfrac{1}{2} \int_0^{b^2/4a} \left[y^2 \right]_0^{\sqrt{(4ax)}} dx$$

$$= \pi\rho \int_0^{b^2/4a} 4ax \, dx = 4\pi\rho a . \left[\frac{1}{2} x^2 \right]_0^{b^2/4a} = \frac{\pi\rho b^4}{8a} \quad ...(1)$$

Now M.I. of the elementary ring of mass δm about OX (a line through its centre and perpendicular to its plane)

$= y^2 \delta m = y^2 . \rho . 2\pi y \, \delta x \, \delta y = 2\pi\rho \, y^3 \, \delta x \, \delta y$

$\therefore$ M.I. of the paraboloid of revolution about OX

$$= \int_{x=0}^{b^2/4a} \int_{y=0}^{\sqrt{(4ax)}} 2\pi\rho \, y^3 \, dx \, dy = \frac{2\pi\rho}{4} \int_0^{b^2/4a} 16a^2 x^2 \, dx$$

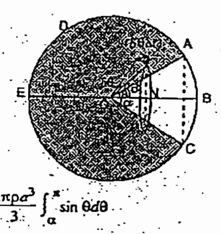
$$= 8\pi\rho a^2 . \frac{1}{3} \left(\frac{b^2}{4a} \right)^3 = \frac{\pi\rho b^6}{24a} = \frac{1}{3} \left(\frac{\pi\rho b^4}{8a} \right) b^2$$

$= \frac{1}{3} M$. (square of the radius of the base).

Ex. 8. *From a uniform sphere of radius a, spherical sector of vertical angle* 2α *is removed. Show that the M.I. of the remainder of mass M about the axis of symmetry is*

$$\tfrac{1}{5} Ma^2 (1 + \cos \alpha)(2 - \cos \alpha).$$

Sol. Let the spherical sector $OABCO$ of vertical angle 2α be removed from the sphere of radius a and centre O. This may be generated by the revolution of the area $OADEO$ of the circle of radius a and centre at O about the diameter EB.



Consider an elementary area $r\delta\theta\delta r$ at the point P of this area. By the revolution of this elementary area about EB, a circular ring of radius $PN = r \sin \theta$ and area of cross-section $r\delta\theta \, \delta r$ is formed.

Mass of this elementary ring,

$\delta m = \rho . 2\pi r \sin \theta . r\delta\theta\delta r$

$= 2\pi\rho r^2 \sin \theta \, \delta\theta\delta r.$

$\therefore$ Mass of the remainder

$$M = \int_{\theta=\alpha}^{\pi} \int_{r=0}^{a} 2\pi \rho r^2 \sin \theta d\theta dr = \frac{2\pi\rho a^3}{3} \int_\alpha^\pi \sin \theta d\theta$$

$$= \frac{2\pi\rho}{3} a^3 (1 + \cos \alpha) \quad \therefore \rho = \frac{3M}{2\pi a^3 (1 + \cos \alpha)} \quad ...(1)$$

Now M.I. of the elementary ring about EB, the line through the centre and perpendicular to its plane,

$= PN^2 . \delta m = r^2 \sin^2 \theta . 2\pi\rho r^2 \sin \theta \delta\theta\delta r = 2\pi\rho r^4 \sin^3 \theta \, \delta\theta \, \delta r$

$\therefore$ M.I of the remainder about EB (the exis of symmetry)

$$= \int_{\theta=\alpha}^{\pi} \int_{r=0}^{a} 2\pi \rho r^4 \sin^3 \theta \, d\theta dr = \tfrac{2}{5} \pi\rho a^5 \int_\alpha^\pi \sin^3 \theta d\theta$$

$$= \tfrac{2}{5} \pi\rho a^5 \int_\alpha^\pi \tfrac{1}{4} (3 \sin \theta - \sin 3\theta) \, d\theta$$

$$= \frac{\pi\rho a^5}{10}\left[-3\cos\theta + \frac{1}{3}\cos 3\theta\right]_{\alpha}^{\pi}$$

$$= \frac{\pi\rho a^5}{10}\left[\frac{8}{3} + 3\cos\alpha - \frac{1}{3}\cos 3\alpha\right]$$

$$= \frac{\pi a^5}{30}\cdot\rho\,[8 + 9\cos\alpha - (4\cos^3\alpha - 3\cos\alpha)]$$

$$= \frac{4\pi a^5}{30}\rho\,[2 + 3\cos\alpha - \cos^3\alpha]$$

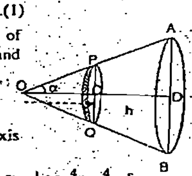
$$= \frac{2\pi a^5}{15}\cdot\frac{3M}{2\pi a^3(1+\cos\alpha)}\cdot(1+\cos\alpha)(2+\cos\alpha-\cos^2\alpha) \quad \text{[from (1)]}$$

$$= \tfrac{1}{5}\pi a^2(1+\cos\alpha)(2-\cos\alpha)$$

Ex. 9. *Find the M.I. of a right solid cone of mass M, height h and radius of whose base is a, about its axis.*

Sol. Let O be the vertex of the right solid cone of mass M, height h and radius of whose base is a. If α is the semi-vertical angle and ρ the density of the cone, then

$M = \frac{1}{3}\pi\rho h^3\tan^2\alpha.$ ...(1)



Consider an elementery disc PQ of thickness δx, parallel to the base AB and at a distance x from the vertex O.

∴ Mass of the disc,

$\delta m = \rho\pi x^2\tan^2\alpha\delta x.$

M.I. of this elementary disc about axis OD.

$= \frac{1}{2}\delta m CP^2 = \frac{1}{2}(\rho\pi x^2\tan^2\alpha\delta x)\,x^2\tan^2\alpha = \frac{1}{2}\rho\pi x^4\tan^4\alpha\delta x.$

∴ M.I. of the cone about axis OD.

$= \int_0^h \frac{1}{2}\rho\pi x^4\tan^4\alpha\,dx = \rho\frac{\pi}{10}h^5\tan^4\alpha = \frac{3}{10}Mh^2\tan^2\alpha.$ from (1)

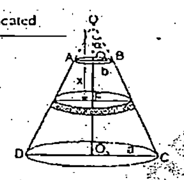
$= \frac{3}{10}Ma^2.$ $(\because \tan\alpha = a/h)$

Ex. 10. *Find the M.I. of a truncated cone about its axis, the radii of its ends being a and b.*

Sol. Let $ABCD$ be the truncated cone with the vertex at O and of semi-vertical angle α. Also let $O_1B = b$ and $O_2C = a$.

Consider an elementary strip perpendicular to the axis at a distance x from O and of thickness δx.

∴ Its Mass $= \delta m = \rho\pi(x\tan\alpha)^2\delta x.$



If M is the total mass of the truncated cone then

$M = \int_{x=b\cot\alpha}^{a\cot\alpha}\rho\pi x^2\tan^2\alpha\,dx$

$\because OO_1 = b\cot\alpha,\ OO_2 = a\cot\alpha$

$= \frac{1}{3}\rho\pi\tan^2\alpha\,(a^3-b^3)\cot^3\alpha$

$= \frac{1}{3}\rho\pi\cot\alpha\,(a^3-b^3)$

$\therefore\ \rho = \dfrac{3M\tan\alpha}{\pi(a^3-b^3)}$ ...(1)

Now M.I. of the elementary disc about O_1O_2, a line through the centre and perpendicular to its plane

$= \frac{1}{2}(x\tan\alpha)^2\cdot\delta m = \frac{1}{2}x^2\tan^2\alpha\cdot\rho\pi x^2\tan^2\alpha\delta x$

$= \frac{1}{2}\rho\pi x^4\tan^4\alpha\delta x.$

∴ M.I. of the truncated cone about its axis O_1O_2.

$M = \int_{x=b\cot\alpha}^{a\cot\alpha}\frac{1}{2}\rho\pi x^4\tan^4\alpha\,dx = \frac{1}{10}\rho\pi(a^5-b^5)\cot^5\alpha\cdot\tan^4\alpha$

$= \frac{1}{10}\cdot\frac{3M\tan\alpha}{\pi(a^3-b^3)}\pi(a^5-b^5)\cot\alpha = \frac{3M}{10}\cdot\frac{a^5-b^5}{a^3-b^3}$ from (1)

Ex. 11. (a) *Find the M.I. about the x-axis of the portion of ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$, which lies in the positive octant, supposing the law of volume density to be $\rho = \mu xyz$.*

Sol. (Refer fig of § 1.13 on page 11).

Consider an elemetary volume $\delta x\,\delta y\,\delta z$ at the point $P(x, y, z)$ where $\rho = \mu xyz$.

∴ Mass of this element $= \rho\delta x\,\delta y\,\delta z = \mu xyz\,\delta x\,\delta y\,\delta z.$

∴ M = Mass of the octant $= \iiint \mu xyz\,dx\,dy\,dz.$

where $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} \le 1$

the integration being extended over the positive octant.

Putting $\dfrac{x^2}{a^2} = u,\ \dfrac{y^2}{b^2} = v,\ \dfrac{z^2}{c^2} = w$

i.e. $x = au^{1/2}, y = bv^{1/2}, z = cw^{1/2}$, so that $dx = \frac{1}{2}au^{-1/2}du$ etc.

$\therefore M = \frac{\mu}{8}a^2b^2c^2\iiint u^{1/2}v^{1/2}w^{1/2}\cdot u^{-1/2}v^{-1/2}w^{-1/2}\,du\,dv\,dw,$

where $u + v + w \le 1.$

$= \frac{1}{8}\mu a^2b^2c^2\iiint u^{1-1}\cdot v^{1-1}\cdot w^{1-1}\,du\,dv\,dw,\ u+v+w \le 1.$

$= \frac{1}{8}\mu a^2b^2c^2\dfrac{\Gamma(1)\Gamma(1)\Gamma(1)}{\Gamma(1+1+1+1)}$, By Dirichlet's theorem

$= \frac{1}{48}\mu a^2b^2c^2.$ ...(1)

Now M.I. of the elementary mass δm at, P, about OX

$= (y^2+z^2)\cdot\delta m$

$\because$ Distance of $P(x, y, z)$ from, OX is $\sqrt{(y^2+z^2)}$

$= \mu xyz(y^2+z^2)\delta x\,\delta y\,\delta z$

∴ M.I. of the octant of the ellipsoid about OX

$= \iiint \mu xyz(y^2+z^2)\,dx\,dy\,dz,$ where $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} \le 1$

The intrgration being extended over the positive octant.

Putting $x = au^{1/2}, y = bv^{1/2}, z = cw^{1/2}$, so that $dx = \frac{1}{2}au^{-1/2}du$ etc.

$\therefore$ M.I. $= \frac{1}{8}\mu a^2b^2c^2\iiint u^{1/2}v^{1/2}w^{1/2}(b^2v + c^2w)\,u^{-1/2}v^{-1/2}w^{-1/2}\,du\,dv\,dw$

where $u + v + w \le 1$

$= \frac{1}{8}\mu a^2b^2c^2$

$\left[b^2\iiint u^{1-1}v^{2-1}w^{1-1}\,du\,dv\,dw + c^2\iiint u^{1-1}v^{1-1}w^{2-1}\,du\,dv\,dw\right]$

$= \frac{1}{8}\mu a^2b^2c^2\left[b^2\dfrac{\Gamma(1)\Gamma(2)\Gamma(1)}{\Gamma(1+1+2+1)} + c^2\dfrac{\Gamma(1)\Gamma(1)\Gamma(2)}{\Gamma(1+1+1+2)}\right]$

$= 6M(b^2+c^2)\cdot\frac{1}{24} = \frac{1}{4}M(b^2+c^2).$ By Dirichlet's theorem

(b) *Show that the M.I. of an ellipsoid of mass M and semi-axes a, b, c, with regard to a diametral plane whose direction-cosines referred to principal planes are (l, m, n) is $\frac{1}{5}M(a^2l^2 + b^2m^2 + c^2n^2)$.*

Sol. From § 1.12 on page (11), the moments of inertia of the ellipsoid with regard to the principal axes are

$\frac{1}{5}M(b^2+c^2), \frac{1}{5}M(c^2+a^2), \frac{1}{5}M(a^2+b^2).$

∴ By prop. II of § 1.3 on page (2), the moments of inertia with regard to principal planes are

$\frac{1}{5}Ma^2, \frac{1}{5}Mb^2, \frac{1}{5}Mc^2.$

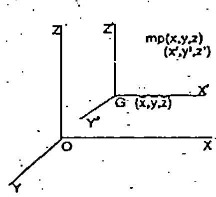
∴ M.I. of the ellipsoid about the diametral plane whose d.c's referred to principal planes are l, m, n is

$\frac{1}{5}Ma^2\cdot l^2 + \frac{1}{5}Mb^2\cdot m^2 + \frac{1}{5}Mc^2\cdot n^2 = \frac{1}{5}M(a^2l^2 + b^2m^2 + c^2n^2).$

1.15. Theorem of Parallel Axis :

The moments and products of inertia about axes through the centre of gravity are given, to find the moments and products of inertia about parallel axes.

Let $(\bar{x}, \bar{y}, \bar{z})$ be the coordinates of the centre of gravity G of the body referred to the rectangular axes OX, OY, OZ through a fixed point O. Let GX', GY', GZ' be the axes through G parallel to the axes OX, OY, OZ respectively.



If (x, y, z) and (x', y', z') are the coordinates of a particle of mass m at P referred to the coordinate axes OX, OY, OZ and parallel axes GX', GY', GZ' respectively, then

$x = \bar{x} + x',\ y = \bar{y} + y',\ z = \bar{z} + z'.$

M.I. of the body about OX

$= \Sigma m(y^2+z^2) = \Sigma m[(\bar{y}+y')^2 + (\bar{z}+z')^2]$

$= \Sigma m(y'^2 + z'^2) + (\bar{y}^2 + \bar{z}^2)\Sigma m + 2\bar{y}\cdot\Sigma my' + 2\bar{z}\Sigma mz'$ ...(1)

Now referred to GX', GY', GZ' as axes the coordinates of G are (0, 0, 0)

$\therefore \dfrac{\Sigma mx'}{\Sigma m}$ or $\Sigma mx' = 0$. Similarly $\Sigma my' = 0, \Sigma mz' = 0.$

∴ From (1), M.I. of the body alout OX

$= \Sigma m(y'^2 + z'^2) + M(\bar{y}^2 + \bar{z}^2)$

= *M.I. of the body about GX' + M.I. of the total mass M at G about OX.*

Also, Product of Intertia (P.I) of the body about OX and OY

$= \Sigma mxy = \Sigma m[(\bar{x}+x')(\bar{y}+y')]$

$= \Sigma mx'y' + \bar{x}\bar{y}\Sigma m + \bar{x}\Sigma my' + \bar{y}\Sigma mx'$

$= \Sigma mx'y' + M\bar{x}\bar{y}$

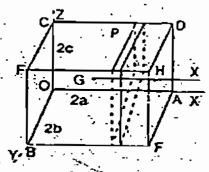
= P.I. about GX' and GY' + P.I. of the total mass M at G about OX and OY

EXAMPLES

Ex. 12. *Find the M.I. of a rectangular parallelopiped about an edge.*

Sol. Let $2a, 2b, 2c$ be the lengths of the edges of a rectangular parallelopiped of mass M.

$\therefore$ M.I. of the rectangular parallelopiped about the edge OA = M.I. of the rectangular parallelopiped about a parallel axis GX' through its C. G. 'G' + M.I. of total mass M at C. G. 'G' about OA.



$= \frac{M}{3}(b^2 + c^2)$

+ M (perpendicular distance of G from $OA)^2$

$= \frac{M}{3}(b^2 + c^2) + M(b^2 + c^2) = \frac{4}{3}M(b^2 + c^2).$

Alter: Consider an element $\delta x\, \delta y\, \delta z$ at the point P whose conordinates referred to the rectangular axes along edges OA, OB, OC are (x, y, z).

$\therefore$ M.I. of this element about OA

$= (\rho\delta x\, \delta y\, \delta z) . (y^2 + z^2).$

$\therefore$ M.I. of the rectangular parallelopiped about OA

$$= \int_{x=0}^{2a}\int_{y=0}^{2b}\int_{z=0}^{2c} \rho\,(y^2 + z^2)\,dx\,dy\,dz = \frac{4}{3}M(b^2 + c^2)$$

$\because \rho = \frac{M}{8abc}$

Ex. 13. *Find the M.I. of a right circular cylinder about*

(i) *its axis.*

(ii) *a staight line through its C. G. and perpendicular to its axis.*

Sol. Let a be the radius, h the height and M the mass of a right circular cylinder. If ρ is the density of the cylinder then $M = \rho\pi a^2 h$.

Consider an elementary disc, of breadth δx, perpendicular to the axis O_1O_2 and at a distance x from the centre of gravity O of cylinder.

$\therefore$ Mass of the disc $\delta m = \rho . \pi a^2 \delta x$.

M.I. of the disc about $O_1O_2 = \frac{1}{2}a^2\,\delta m = \frac{1}{2}a^2 . \rho\pi a^2 \delta x = \frac{1}{2}\rho\pi a^4 \delta x.$

$\therefore$ M.I. of the cylinder about O_1O_2

$$= \int_{-h/2}^{h/2} \frac{1}{2}\rho\pi a^4 dx = \frac{1}{2}\rho\pi a^4 h = \frac{1}{2}Ma^2$$

$(\because M = \rho\pi a^2 h)$

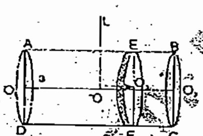
(ii) Let OL be the line through the C. G. 'O' and perpendicular to the axis of the cylinder.

M.I. of the elementary disc about OL

= M.I. of the disc about the parallel line EF through its C. G. O_3 + M.I. of the total M at O_3 about OL

$= \frac{1}{4}a^2\delta m + x^2\delta m = (\frac{1}{4}a^2 + x^2)\,\delta m$

$= (\frac{1}{4}a^2 + x^2)\,\rho\pi a^2 \delta x.$



$\therefore$ M.I. of the cylinder about OL

$$= \int_{-h/2}^{h/2} (\tfrac{1}{4}a^2 + x^2)\,\rho\pi a^2 dx$$

$$= \rho\pi a^2\left[\tfrac{1}{4}a^2 x + \tfrac{1}{3}x^3\right]_{-h/2}^{h/2}$$

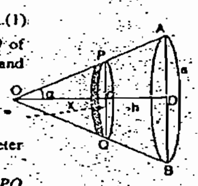
$$= \tfrac{1}{4}\rho\pi a^2 h\,(a^2 + \tfrac{1}{3}h^2) = \tfrac{1}{4}M(a^2 + \tfrac{1}{3}h^2).$$

Ex. 14. *Prove that the M.I. of a uniform right circular solid cone of mass M, height h and base-radius a, about a diameter of its base is* $\frac{M}{20}(3a^2 + 2h^2)$.

Sol. Let O be the vertex of a right circular cone of mass M, height h and base-radius a. If α is the semi-vertical angle and ρ the density of the cone, then

$M = \frac{1}{3}\pi h^3 \tan^2\alpha\rho.$ ...(1)

Cosider an elementary disc PQ of thickness δx, parallel to the base AB and at a distance x from the vertex O.



$\therefore$ Mass of the disc

$= \delta m = \rho\pi x^2 \tan^2\alpha\,\delta x.$

$\therefore$ M.I. of the disc about the diameter AB of the base of the cone

= Its M.I. about parallel diameter PQ of the disc + M.I. of the total mass δm at centre C about AB

$= \frac{1}{4}\delta m . CP^2 + \delta m . CD^2 = \rho\pi x^2 \tan^2\alpha\,[\frac{1}{4}x^2\tan^2\alpha + (h - x)^2]\,\delta x.$

$\therefore$ M.I. of the cone about the diameter of the base

$$= \int_0^h \rho\pi x^2 \tan^2\alpha\,[\tfrac{1}{4}x^2\tan^2\alpha + (h - x^2)^2]\,\delta x.$$

$$= \frac{1}{4}\rho\pi\tan^2\alpha\int_0^h (x^4\tan^2\alpha + 4h^2x^2 - 8hx^3 + 4x^4)\,dx$$

$$= \frac{1}{4}\rho\pi\tan^2\alpha\left[\frac{1}{5}h^5\tan^2\alpha + \frac{4}{3}h^5 - 2h^5 + \frac{4}{5}h^5\right]$$

$$= \frac{1}{60}\rho\pi h^5\tan^2\alpha\,(3\tan^2\alpha + 2) = \frac{1}{20}Mh^2\,(3\tan^2\alpha + 2).$$ [from (1)]

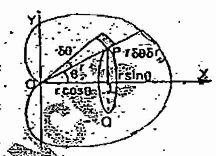
$$= \frac{1}{20}Mh^2\left(3 . \frac{a^2}{h^2} + 2\right) = \frac{M}{20}\left(3a^2 + 2h^2\right)$$

$(\because \tan\alpha = a/h).$

Ex. 15. *A solid body of density ρ is in the shape of the solid farmed by the revolution of the cardiod $r = a(1 + \cos\theta)$ about the initial line, show that its M.I. about a straight line through the pole and perpendicular to the initial line is $\frac{352}{105}\pi\rho a^5$.* (IAS-2003)

Sol. Let OX be the initial line (axis of the cardiod) and OY the line perpendicular to it through the pole O.

Consider an elementary area $r\delta\theta\delta r$ at the point $P(r, \theta)$. Then the mass of the elementary ring of radius PL obtained by the revolution of element $r\delta\theta\delta r$ about OX,



$\delta m = \rho . 2\pi\, PL . r\delta\theta\delta r$

$= 2\pi\rho r \sin\theta . r\delta\theta\delta r$

$= 2\pi\rho r^2 \sin\theta\,\delta\theta\delta r,$

where ρ is the mass per unit volume of the solid formed by the revolution of the cardiod about the initial line OX.

M.I. of this elementary ring about OY

= Its M.I. about the diameter PQ + M.I. of mass δm at centre L about OY

$= \frac{1}{2}\delta m . PL^2 + \delta m . OL^2 = (\frac{1}{2}PL^2 + OL^2)\,\delta m$

$= (\frac{1}{2}r^2\sin^2\theta + r^2\cos^2\theta)\,2\pi\rho r^2 \sin\theta\,\delta\theta\delta r$

$= \pi\rho\,(\sin^2\theta + 2\cos^2\theta)\,r^4 \sin\theta\,\delta\theta\delta r$

$= \pi\rho\,(1 + \cos^2\theta)\,r^4 \sin\theta\,\delta\theta\delta r.$

$\therefore$ M.I. of the soild of revolution about OY

$$= \int_{\theta=0}^{\pi}\int_{r=0}^{a(1+\cos\theta)} \pi\rho\,(1 + \cos^2\theta)\,r^4 \sin\theta\,d\theta\,dr$$

$$= \tfrac{1}{5}\pi\rho a^5\int_0^{\pi}(1 + \cos^2\theta)\,(1 + \cos\theta)^5 \sin\theta\,d\theta .$$

$$= -\tfrac{1}{5}\pi\rho a^5\int_2^0 \{1 + (t - 1)^2\} . t^5\,dt$$ Putting $1 + \cos\theta = t$

$$= \tfrac{1}{5}\pi\rho a^5\int_0^2 (2t^5 - 2t^6 + t^7)\,dt$$

$$= \tfrac{1}{5}\pi\rho a^5\left[\tfrac{1}{3}t^6 - \tfrac{2}{7}t^7 + \tfrac{1}{8}t^8\right]_0^2 = \tfrac{1}{5}\pi\rho a^5 . \left(\frac{352}{21}\right) = \frac{352}{105}\pi\rho a^5.$$

Ex. 16. *Find the M.I. of a triangle ABC about a perpendicular to the plane through A.*

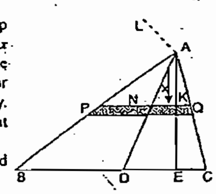
Sol. Let AL be a line through A and perpendicular to the plane of the triangle ABC of mass M and dentsity ρ.

Let the height of the triangle, $AE = h$.

$\therefore M = \frac{1}{2}\rho BC . AE$

$= \frac{1}{2}\rho ah$ ...(1)

Consider an elementary strip PQ of thickness δx at a distance x from A and parallel to BC. Let the median AD and the perpendicular AE meet PQ at N and K respectively. Clearly N will be the middle point of PQ.



From similar triangles APQ and ABC, we have

$$\frac{PQ}{BC} = \frac{AK}{AE} \text{ or } \frac{PQ}{a} = \frac{x}{h} \text{ or } PQ = \frac{ax}{h}$$

Also from similar triangle ANK and ADE, we have

$$\frac{AN}{AD} = \frac{AK}{AE} \text{ or } \frac{AN}{AD} = \frac{x}{h} \text{ or } AN = \frac{x}{h}AD$$

In ΔADE, we have

$AD^2 = AE^2 + DE^2 = AE^2 + (BE - BD)^2$

$= (AE^2 + BE^2) + BD^2 - 2\,BE . BD$

$= AB^2 + (\frac{1}{2}BC)^2 - 2 . AB\cos B . \frac{1}{2}BC$

$= c^2 + \frac{1}{4}a^2 - c . \frac{a^2 + c^2 - b^2}{2ac} . a$

or $AD^2 = \frac{1}{4}(2b^2 + 2c^2 - a^2)$ ...(2)

Now, mass of the elementary strip PQ,

$\delta m = \rho PQ\delta x = \rho\,\frac{ax}{h}\,\delta m.$

M.I. of strip PQ about the line AL

= M.I. of strip PQ about the line parallel to AL through its C. G. 'N' + M.I. of mass δm at N about AL.

$= \frac{1}{3}\delta m \cdot \left(\frac{1}{2}PQ\right)^2 + \delta m\, AN^2 = \left(\frac{1}{12}\frac{a^2x^2}{h^2} + \frac{x^2}{h^2}AD^2\right)\delta m$

$= \frac{x^2}{12h^2}\left(a^2 + 12AD^2\right)\frac{\rho a x}{h}\delta x$

$= \frac{\rho a}{12h^3}\left[a^2 + 12 \cdot \frac{1}{4}(2b^2 + 2c^2 - a^2)\right] \cdot x^3 \delta x$

$= \frac{\rho a}{6h^3}(3b^2 + 3c^2 - a^2)x^3\,\delta x.$

$\therefore$ M.I. of the ΔABC about AL

$= \int_0^h \frac{\rho a}{6h^3}(3b^2 + 3c^2 - a^2)x^3\,dx = \frac{\rho a h}{24}(3b^2 + 3c^2 - a^2)$

$= \frac{M}{12}(3b^2 + 3c^2 - a^2).$ [from (1)]

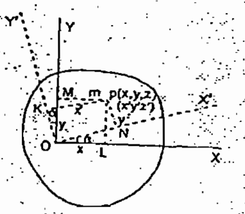
1.16. Moment and Product of Inertia of a Plane Lamina about a Line.

If the moments and products of inertia of a plane lamina about two perpendicular axes in its plane are given, to find the moment and product of inertia about any perpendicular lines through their point of intersection.

Let A and B be the moments of inertia and F the product of inertia of a plane lamina about the perpendicular axes OX and OY in its plane.

Consider an element of mass m of the lamina at the point P whose co-ordinates are (x, y) with reference to the axes OX and OY.

$\therefore A = \Sigma my^2, B = \Sigma mx^2$ and $F = \Sigma mxy$.



Let OX' and OY' be the perpendicular axes in the plane of the lamina and inclined at an angle α to OX and OY respectively. If (x', y') are the co-ordinates of the point P with reference to these axes, then

$x' = PK = x\cos\alpha + y\sin\alpha$

and $y' = PN = y\cos\alpha - x\sin\alpha$.

$\therefore$ M.I. of the lamina about OX'

$= \Sigma m PN^2 = \Sigma m y'^2 = \Sigma m (y\cos\alpha - x\sin\alpha)^2$

$= (\Sigma my^2)\cos^2\alpha + (\Sigma mx^2)\sin^2\alpha - 2(\Sigma mxy)\sin\alpha\cos\alpha$

$= A\cos^2\alpha + B\sin^2\alpha - F\sin 2\alpha.$ ...(1)

Also P.I. of the lamina about OX' and OY'

$= \Sigma m PN \cdot PK = \Sigma m y' x'$

$= \Sigma m (y\cos\alpha - x\sin\alpha)(x\cos\alpha + y\sin\alpha)$

$= (\Sigma my^2 - \Sigma mx^2)\sin\alpha\cos\alpha + (\Sigma mxy)(\cos^2\alpha - \sin^2\alpha)$

$= \frac{1}{2}(A - B)\sin 2\alpha + F\cos 2\alpha.$ ...(2)

1.17. M.I. of a Body about a Line.

Given the moments and products of inertia of a body about three mutually perpendicular axes, to find the M.I. about any line through their meeting point.

Let OX, OY, OZ be three mutually perpendicular axes. Consider an element of mass m' of the body at the point $P(x, y, z)$ then

A = M.I. about OX

$= \Sigma m'(y^2 + z^2)$

B = M.I. about OY

$= \Sigma m'(z^2 + x^2)$

C = M.I. about OZ

$= \Sigma m'(x^2 + y^2)$

D = P.I. about OY and OZ

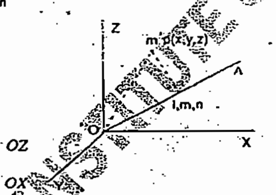
$= \Sigma m' yz$

E = P.I. about OZ and OX

$= \Sigma m' zx$

F = P.I. about OX and OY

$= \Sigma m' xy.$



Let OA be a line through the point O, (meeting point of the axes), and l, m, n its direction cosines.

If PL is the perpendicular from P on OA, then

$PL^2 = OP^2 - OL^2 = (x^2 + y^2 + z^2) - (lx + my + nz)^2$

$= x^2(1 - l^2) + y^2(1 - m^2) + z^2(1 - n^2) - 2mnyz - 2nlxz - 2lmxy$

$= x^2(m^2 + n^2) + y^2(n^2 + l^2) + z^2(l^2 + m^2) - 2mnyz - 2nlxz - 2lmxy$

($\because l^2 + m^2 + n^2 = 1$)

$= (y^2 + z^2)l^2 + (z^2 + x^2)m^2 + (x^2 + y^2)n^2 - 2mnyz - 2nlxz - 2lmxy$

$\therefore$ M.I. of the body about OA

$= \Sigma m' PL^2 = l^2\Sigma m'(y^2 + z^2) + m^2\Sigma m'(z^2 + x^2)$
$+ n^2\Sigma m'(x^2 + y^2) - 2mn\Sigma m'yz - 2nl\Sigma m'xz - 2lm\Sigma m'xy$

$= Al^2 + Bm^2 + Cn^2 - 2Dmn - 2Enl - 2Flm.$ ...(1)

Note. § 1.16. is a special case of § 1.17.

For a plane lamina $n = 0$, $l = \cos\alpha$ and $m = \cos(90° - \alpha) = \sin\alpha$.

Putting $n = 0$ in (1), we get the M.I. of the lamina about OA.

$= A\cos^2\alpha + \beta\sin^2\alpha - F\sin 2\alpha.$

EXAMPLES

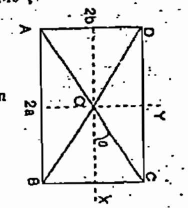
Ex. 17. *Show that M.I. of a rectangle of mass M and sides $2a, 2b$ about a diagonal is $\frac{2M}{3}\frac{a^2b^2}{a^2 + b^2}$.*

Deduce that in case of a square.

Sol. Let $ABCD$ be a rectangle of mass M and $AB = 2a, BC = 2b$

Then M.I. of rectangle about $OX = A = \frac{1}{3}Mb^2$,

and M.I. of rectangle about $OY = B = \frac{1}{3}Ma^2$.



P.I. of the rectangle about OX and $OY = F = 0$. (By symmetry)

If diagonal AC make an angle θ with AB, then

$\cos\theta = \frac{AB}{AC} = \frac{2a}{\sqrt{(4a^2 + 4b^2)}} = \frac{a}{\sqrt{(a^2 + b^2)}}$

and $\sin\theta = \frac{BC}{AB} = \frac{b}{\sqrt{(a^2 + b^2)}}$

$\therefore$ M.I. of the rectangle about AC

$= A\cos^2\theta + B\sin^2\theta - F\sin 2\theta$ (see equation (1), § 1.16)

$= \frac{1}{3}Mb^2 \cdot \frac{a^2}{a^2 + b^2} + \frac{1}{3}Ma^2 \cdot \frac{b^2}{a^2 + b^2} - 0 = \frac{2M}{3} \cdot \frac{a^2b^2}{a^2 + b^2}$.

Deduction. For a square, $2b = 2a$.

$\therefore$ M.I. of square about AC

$= \frac{2M}{3} \cdot \frac{a^4}{a^2 + a^2} = \frac{1}{3}Ma^2$.

Ex. 18. *Show that the M.I. of an elliptic area of mass M and semi-axes a and b about a diameter of length $2r$ is $\frac{1}{4}M\frac{a^2b^2}{r^2}$.*

Sol. Let PP' be the diameter of length $2r$ of an elliptic area of mass M and semi-axes a and b. Equation of the ellipse is

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$ ...(1)

If PP' make an angle θ with OX then co-ordinates of P are $(r\cos\theta, r\sin\theta)$

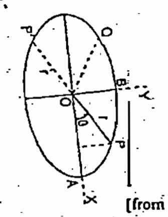
Since P lie on equation (1).

$\therefore (r^2/a^2)\cos^2\theta + (r^2/b^2)\sin^2\theta = 1$

or $b^2\cos^2\theta + a^2\sin^2\theta = \frac{a^2b^2}{r^2}$ ...(2)

Now, M.I. of the ellipse about $OX = A = \frac{1}{4}Mb^2$,

and M.I. of the ellipse about $OY = B = \frac{1}{4}Ma^2$.



Also P.I. of the ellipse about OX and OY

$= F = 0$. (By symmetry)

$\therefore$ M.I. of the ellipse about the diameter PP'

$= A\cos^2\theta + B\sin^2\theta - F\sin 2\theta.$

$= \frac{1}{4}Mb^2\cos^2\theta + \frac{1}{4}Ma^2\sin^2\theta - 0$

$= \frac{1}{4}M(b^2\cos^2\theta + a^2\sin^2\theta)$

$= \frac{M}{4} \cdot \frac{a^2b^2}{r^2}$ [from (2)]

Ex. 19. *If k_1 and k_2 be the radii of gyration of an elliptic lamina about two conjugate diameter, then*

$$\frac{1}{k_1^2} + \frac{1}{k_2^2} = 4\left(\frac{1}{a^2} + \frac{1}{b^2}\right)$$

Sol. Let $OP = r_1$ and $OQ = r_2$ be two conjugate semi-diameters of an elliptic lamina of mass M and semi-axes a, b.

M.I. of the ellipse about OP

$= Mk_1^2 = \frac{M}{4}\frac{a^2b^2}{r^2}$. (See Ex. 18)

$\therefore \frac{1}{k_1^2} = \frac{4r_1^2}{a^2b^2}$. Similarly, $\frac{1}{k_2^2} = \frac{4r_2^2}{a^2b^2}$.

$\therefore \frac{1}{k_1^2} + \frac{1}{k_2^2} = \frac{4}{a^2b^2}(r_1^2 + r_2^2) = \frac{4}{a^2b^2}(a^2 + b^2).$

($\because r_1^2 + r_2^2 = a^2 + b^2$. By property)

$= 4(1/a^2 + 1/b^2).$

Ex. 20. *Show that the M.I. of an elliptic area of mass M and equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, about a diameter parallel to the axis of x is $\frac{-aM\Delta}{4(ab - h^2)^2}$,*

where $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$.

Sol. Equation of the ellipse is

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$ ...(1)

Shifting the origin to the centre of the ellipse, the equation of the ellipse becomes

$$\left(\frac{4}{3}\pi abc\rho\right)\cdot\frac{b^2+c^2}{5}.$$

Let the ellipsoid decrease indefinitely small in size.

$\therefore$ M.I. of the enclosed ellipsoidal shell

$$= d\left\{\frac{4}{3}\pi abc\rho\cdot\frac{b^2+c^2}{5}\right\}. \quad ...(1)$$

Since the shell is bounded by similar and similarly situated concentric ellipsoids, therefore if a', b', c' are the semi-axes of the similar ellipsoid, then we have

$$\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}$$

$\therefore$ $b=\frac{b'}{a'}a=pa$ and $c=\frac{c'}{a'}a=qa$.

$\therefore$ From (1), M.I. of the ellipsoidal shell

$$= d\left[\frac{4}{3}\pi\rho pq\cdot\frac{p^2+q^2}{5}a^5\right]$$

$$= \tfrac{4}{3}\pi\rho pq\,.(p^2+q^2)\,a^4da. \quad ...(2)$$

But the mass of the ellipsoid $=\frac{4}{3}\pi abc\rho=\frac{4}{3}\pi\rho pqa^3$

$\therefore$ Mass of the ellipsoidal shell

$M=d(\frac{4}{3}\pi\rho pqa^3)=4\pi\rho pqa^2da.$

Hence from (2), we have

M.I. of the ellipsoidal shell

$$=\frac{M}{3}(p^2+q^2)\,a^2=\frac{M}{3}(b^2+c^2)$$

1.20. M.I. of Hetrogeneous Bodies.

The method of differentiation can be used in finding the M.I. of a hetrogenous body whose boundary is a surface of uniform density. For this proceed as follows :

(i) Find the M.I. of homogenous solid body of density ρ.

(ii) Express this M.I. in terms of a single parameter α (say) i.e. M.I. $=\rho\phi(\alpha)$.

(iii) Then by differentiation, the M.I. of a shell which is considered to be made of a layor of uniform density ρ
$=\rho\phi'(\alpha)\,d\alpha$.

(iv) Replace ρ by the variable density σ.

(v) Thus the M.I. of the given hetrogeneous body is given by
M.I. $=\int\rho\phi'(\alpha)\,d\alpha$.

For illustration see the following examples.

EXAMPLES

Ex. 28. *Show that the M.I. of a hetrogenous ellipsoid about the major axis is $\frac{2}{9}M(b^2+c^2)$, the strata of uniform density being similar concentric ellipsoids and the density along the major axis varying as the distance from the centre.*

Sol. (i) We know that the M.I. of an ellipsoid of density ρ and semi-axes a, b, c about x-axis is equal to

$$\left(\frac{4}{3}\pi abc\rho\right)\cdot\frac{b^2+c^2}{5}.$$

Also the mass of the ellipsoid $=\frac{4}{3}\pi abc\rho$.

(ii) Since the boundary surfaces are similar concentric ellipsoid, therefore, if a', b', c' are the semi-axes of the similar ellipsoid then we have

$$\frac{a}{a'}=\frac{b}{b'}=\frac{c}{c'}$$

i.e. $b=\frac{b'}{a'}a=pa$, and $c=\frac{c'}{a'}a=qa$.

$\therefore$ M.I. of the ellipsoid about x-axis

$$=\frac{4}{3}\pi abc\rho\cdot\frac{b^2+c^2}{5}=\frac{4}{3}\pi\rho pq\cdot\frac{p^2+q^2}{5}a^5.$$

(iii) Differentiating the above M.I., the M.I. of a shell of uniform density ρ

$$=d\left(\frac{4}{3}\pi\rho pq\cdot\frac{p^2+q^2}{5}a^5\right)$$

$$=\tfrac{4}{3}\pi\rho pq\,(p^2+q^2)\,a^4da.$$

(iv) Since the density varies as the distance from the centre.

$\therefore$ $\sigma=\lambda a$.

Replacing ρ by $\sigma=\lambda a$ and then integrating the M.I. of the hetrogenous ellipsoid about the major axis

$$=\int_0^a\frac{4}{3}\pi\lambda apq\,(p^2+q^2)\,a^4da$$

$$=\frac{4}{3}\pi\lambda pq\,(p^2+q^2)\int_0^a a^5da=\frac{2}{9}\pi\lambda pq\,(p^2+q^2)\,.\,a^6 \quad ...(1)$$

Also the mass of the ellipsoid $=\frac{4}{3}\pi abc\rho=\frac{4}{3}\pi\rho pqa^3$.

$\therefore$ Mass of the ellipsoidal shell $=d(\frac{4}{3}\pi\rho pqa^3)$

$=4\pi\rho pqa^2da.$

Replacing ρ by $\sigma=\lambda a$ and then integrating, the mass of the hetrogeneous ellipsoid is given by

$$M=\int_0^a 4\pi\lambda apqa^2da=\pi\lambda pqa^4.$$

Hence from (1), M.I. of the hetrogeneous ellipsoid

$=\frac{2}{9}M\,(p^2+q^2)\,a^2=\frac{2}{9}M\,(b^2+c^2).$

Ex. 29. *The M.I. of a hetrogeneous ellipse about minor axis, the strata of uniform density being confocal ellipses and density along minor axis, varying as the distance from the centre is*

$$\frac{3M}{20}\cdot\frac{4a^5+c^5-5a^3c^2}{2a^3+c^3-3ac^2}.$$

Sol. For confocal ellipses, we have

$a^2e^2=a^2-b^2=$ Constant.

$\therefore$ Taking $a^2-b^2=c^2$, the cuation of the confocal ellipse is

$$\frac{x^2}{b^2+c^2}+\frac{y^2}{b^2}=1,\ \text{where } a^2=b^2+c^2. \quad ...(1)$$

The M.I. of homogenous ellipse of uniform density ρ about minor axis is

$$(\rho\pi ba)\,.\,\frac{a^2}{4}=\rho\pi b\sqrt{(b^2+c^2)}\,.\,\frac{b^2+c^2}{4}=\tfrac{1}{4}\rho\pi b\,(b^2+c^2)^{3/2}.$$

Differentiating, the M.I. of an elliptic strafa of uniform density ρ

$$=d\,\{\tfrac{1}{4}\rho\pi b\,(b^2+c^2)^{3/2}\}$$

$$=\tfrac{1}{4}\rho\pi\,[1.(b^2+c^2)^{3/2}+b.\tfrac{3}{2}(b^2+c^2)^{1/2}.2b]\,db$$

$$=\tfrac{1}{4}\pi\rho\sqrt{(b^2+c^2)}.(4b^2+c^2)\,db.$$

Since the density varies as the distance from the centre, therefore replacing ρ by λb and integrating the M.I. of the hetrogeneous ellipse about minor axis

$$=\int_0^b\tfrac{1}{4}\pi\lambda b\sqrt{(b^2+c^2)}.(4b^2+c^2)\,db$$

$$=\tfrac{1}{4}\pi\lambda\left[\int_0^b 4(b^2+c^2)^{3/2}bdb-3\int_0^b c^2(b^2+c^2)^{1/2}\,bdb\right]$$

$$=\tfrac{1}{4}\pi\lambda\left[\tfrac{4}{5}(b^2+c^2)^{5/2}-c^2(b^2+c^2)^{3/2}\right]_0^b$$

$$=\tfrac{1}{4}\pi\lambda\,[\tfrac{4}{5}\{(b^2+c^2)^{5/2}-c^5\}-c^2\{(b^2+c^2)^{3/2}-c^3\}]$$

$$=\tfrac{1}{4}\pi\lambda\,[\tfrac{4}{5}(a^5-c^5)-c^2(a^3-c^3)]. \quad ...(2)$$

Also the mass of the ellipse $=\pi\rho ba=\rho\pi b\sqrt{(b^2+c^2)}$

$\therefore$ Mass of the elliptic strata of uniform density ρ

$$=d\,\{\rho\pi b.\sqrt{(b^2+c^2)}\}$$

$$=\rho\pi\,\{1.\sqrt{(b^2+c^2)}+b.\tfrac{1}{2}(b^2+c^2)^{-1/2}.2b\}\,db$$

$$=\rho\pi\cdot\frac{2b^2+c^2}{\sqrt{(b^2+c^2)}}\,db.$$

Replacing ρ by λb and integrating the mass of the hetrogeneous ellipse

$$M=\int_0^b\pi\lambda b\cdot\frac{2b^2+c^2}{\sqrt{(b^2+c^2)}}\,db$$

$$=\pi\lambda\left[\int_0^b 2\sqrt{(b^2+c^2)}\,.\,bdb-c^2\int_0^b\frac{bdb}{\sqrt{(b^2+c^2)}}\right]$$

$$=\pi\lambda\left[\tfrac{2}{3}(b^2+c^2)^{3/2}-c^2\sqrt{(b^2+c^2)}\right]_0^b$$

$$=\pi\lambda\,[\tfrac{2}{3}\{(b^2+c^2)^{3/2}-c^3\}-c^2\{(b^2+c^2)^{1/2}-c\}]$$

$$=\pi\lambda\,[\tfrac{2}{3}(a^3-c^3)-c^2(a-c)]. \qquad \because b^2+c^2=a^2.$$

Hence from (2), the M.I. of the hetrogeneous ellipse about the minor axis

$$=\frac{M}{4}\cdot\frac{\frac{4}{5}(a^5-c^5)-c^2(a^3-c^3)}{\frac{2}{3}(a^3-c^3)-c^2(a-c)}$$

$$=\frac{3M}{20}\cdot\frac{4a^5+c^5-5a^3c^2}{2a^3+c^3-3ac^2}.$$

1.21. Momental Ellipsoid.

The M.I. of a body about a line OQ whose direction cosines are l, m, n is given by

$Al^2+Bm^2+Cn^2-2Dmn-2Enl-2Flm,$

where A, B, C, D, E, F are the moments and products of inertia of the body about the axes.

Let P be a point on OQ such that the M.I. of the body about OQ may be inversely proportional to OP^2.

i.e. $Al^2+Bm^2+Cn^2-2Dmn-2Enl-2Flm\propto\dfrac{1}{OP^2}$

or $Al^2+Bm^2+Cn^2-2Dmn-2Enl-2Flm=\dfrac{Mk^4}{r^2}$,

where $OP=r$.

or $Al^2r^2+Bm^2r^2+Cn^2r^2-2Dmr.nr-2Enr.lr-2Flr.mr=Mk^4$

or $Ax^2+By^2+Cz^2-2Dyz-2Ezx-2Fxy=Mk^4$. ...(1)

Since A, B, C are essentially positive, therefore equation (1) represent an ellipsoid. *This is called the* **momental ellipsoid** *of the body at O.*

By solid geometry, we can find three mutually perpendicular diameters such that with these diameters as coordinate axes, the equation of the ellipsoid is transformed into the form

$A_1x^2 + B_1y^2 + C_1z^2 = Mk^4$. ...(2)

The product of inertia with respect to these new axes will vanish. *These three new axes are called the principal axes of the body at the point O. And a plane through any two of these axes is called a* **principal plane** *of the body.*

Hence for every body there exists at every point O, a set of three mutually perpendicular axes, which are the three principal diameters of the momental ellipsoid at O, such that the products of inertia of the body about them, taken two at a time vanish.

Note. When the three principal moments of inertia at any point O are the same, the ellipsoid becomes a sphere. In this case every diameter is a principal diameter and all radii vectors are the same.

1.22. Momental Ellipse.

Let OX and OY be two mutually perpendicular axes and OQ a line through O, all in the plane of a lamina. Then M.I. of the plane lamina, about OQ is given by

$$A\cos^2\theta - 2F\sin\theta\cos\theta + B\sin^2\theta,$$

where A, B denote the moments of inertia about OX, OY and F the product of inertia about OX and OY.

Let P be a point on OQ such that the M.I. of the lamina about OQ may be inversely proportional to OP^2.

i.e. $A\cos^2\theta - 2F\sin\theta\cos\theta + B\sin^2\theta \propto \dfrac{1}{OP^2}$

or $A\cos^2\theta - 2F\sin\theta\cos\theta + B\sin^2\theta = \dfrac{Mk^4}{r^2}$, where $OP = r$.

or $Ar^2\cos^2\theta - 2Fr\cos\theta\, r\sin\theta + Br^2\sin^2\theta = Mk^4$

or $Ax^2 - 2Fxy + By^2 = Mk^4$ ...(1)

Since A and B are essentially positive, therefore equation (1) represent an ellipse. *This is called a* momental *ellipse of the lamina at O.*

Note. The section of the momental ellipsoid at O by the plane of the lamina is the momental ellipse.

EXAMPLES

Ex. 30. *Find the momental ellipsoid at any point O of a material straight rod AB of mass M and length 2a.*

Sol. Let G be the centre of gravity of a material straight rod AB of mass M and length $2a$. Let O be a point on the rod s.t. $OG = c$.

Consider the axis OX along the rod and axis OY perpendicular to the rod.

$\therefore$ A = M.I. of the rod about $OX = 0$.

B = M.I. of the rod about OY = M.I. of the rod about parallel axis GY' + M.I. of mass M at G about OY

$= \frac{1}{3}Ma^2 + Mc^2 = M(\frac{1}{3}a^2 + c^2)$

Similarly C = M.I. of the rod about $OZ = M(\frac{1}{3}a^2 + c^2)$.

The coordinates of the C.G. 'G' of the rod are $(c, 0, 0)$

$\therefore$ $D = O = E = F$.

Hence equation of the momental ellipsoid at O is

$Ax^2 + By^2 + Cz^2 - 2Dyz - 2Ezx - 2Fxy = \text{Const}$

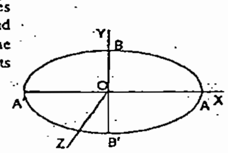
or $O + M(\frac{1}{3}a^2 + c^2)y^2 + M(\frac{1}{3}a^2 + c^2)z^2 = \text{Const}$.

or $M(\frac{1}{3}a^2 + c^2)(y^2 + z^2) = \text{Const}$.

or $y^2 + z^2 = \text{const}$

Ex. 31. *Show that the momental ellipsoid at the centre of an elliptic plate is* $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + z^2\left(\dfrac{1}{a^2} + \dfrac{1}{b^2}\right) = \text{const.}$

Sol. Let M be the mass of an elliptic plate of semi-axes a and b. Let the axes OX and OY be taken along the major and minor axes of the elliptic plate in its plane and the axes OZ perpendicular to its plane. Then



A = M.I. of the plate about OX

$= \frac{1}{4}Mb^2$

B = M.I. of the plate about $OY = \frac{1}{4}Ma^2$

C = M.I. of the plate about OZ

$= \frac{1}{4}M(a^2 + b^2)$

and since plate is symmetrical about OX and OY

$\therefore$ $D = O = E = F$.

$\therefore$ Equation of the momental ellipsoid at O is

$Ax^2 + By^2 + Cz^2 - 2Dmn - 2Enl - 2Flm = \text{Const}$.

or $\frac{1}{4}Mb^2x^2 + \frac{1}{4}Ma^2y^2 + \frac{1}{4}M(a^2 + b^2)z^2 = \text{Const}$.

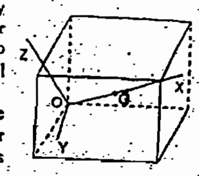
or $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + z^2\left(\dfrac{1}{a^2} + \dfrac{1}{b^2}\right) = \text{Const}$.

Ex. 32. *Show that the equation of the momental ellipsoid at the corner of a cube of side 2a referred to its principal axes is*

$$2x^2 + 11(y^2 + z^2) = C,$$

where C is constant.

Sol. Let G be the centre of gravity of a cube of side $2a$. Let O be a corner of the cube at which we have to determine the equation of the momental ellipsoid.



Take the line OX through G as the axis of x and two mutually perpendicular lines OY and OZ through O as the axis of y and z.

The coordinates of G referred to OX, OY, OZ as axis are $(a\sqrt{3}, 0, 0)$ and the products of inertia of the cube about any two mutually perpendicular lines through G is zero.

$\therefore$ the product of inertia about the axes OX, OY, OZ taken in pairs is zero. Thus OX, OY, OZ are the principal axes of the momental ellipsoid at O.

Since the M.I. of the cube about any axis (parallel to an edge) through $G = \frac{2}{3}Ma^2$

$\therefore$ A = M.I. about $OX = A'l^2 + B'm^2 + C'n^2 = \frac{2}{3}Ma^2$

$\because$ $A' = B' = C' = \frac{2}{3}Ma^2$

B = M.I. about OY = M.I. about parallel axis through G + M.I. of total mass M at G about OY

$= \frac{2}{3}Ma^2 + M.OG^2 = \frac{2}{3}Ma^2 + M(a\sqrt{3})^2 = \frac{11}{3}Ma^2$.

Similarly, C = M.I. about $OZ = \frac{11}{3}Ma^2$.

and $D = O = E = F$.

Hence equation of the momental ellipsoid at O is

$Ax^2 + By^2 + Cz^2 - 2Dmn - 2Enl - 2Flm = \text{Const}$.

or $\frac{2}{3}Ma^2x^2 + \frac{11}{3}Ma^2y^2 + \frac{11}{3}Ma^2z^2 = \text{Const}$.

or $2x^2 + 11(y^2 + z^2) = C$, where C is a constant.

Ex. 33. *Show that the momental ellipsoid at the centre of an ellipsoid is* $(b^2 + c^2)x^2 + (c^2 + a^2)y^2 + (a^2 + b^2)z^2 = \text{const.}$

Sol. The equation of an ellipsoid, referred to the principal axes is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

$\therefore$ A = M.I. about $OX = \frac{1}{5}M(b^2 + c^2)$

B = M.I. about $OY = \frac{1}{5}M(c^2 + a^2)$

C = M.I. about $OZ = \frac{1}{5}M(a^2 + b^2)$

and $D = O = E = F$.

Hene equation of the momental ellipsoid at the centre of the ellipsoid is

$Ax^2 + By^2 + Cz^2 - 2Dmn - 2Enl - 2Flm = \text{const}$.

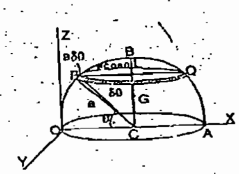
or $\frac{1}{5}M(b^2 + c^2)x^2 + \frac{1}{5}M(c^2 + a^2)y^2 + \frac{1}{5}M(a^2 + b^2)z^2 = \text{const}$.

or $(b^2 + c^2)x^2 + (c^2 + a^2)y^2 + (a^2 + b^2)z^2 = \text{const}$.

Ex. 34. *Show that the momental ellipsoid at a point on the edge of the circular base of a thin hemispherical shell is*

$$2x^2 + 5(y^2 + z^2) - 3zx = \text{const}.$$

Sol. Let O be a point on the the edge of the circular base of a thin hemisherical shell of radius a and mass M. Take the axis OX along the diameter OA of base of the shell, axis OY perpendicular to OX through O in the plane of the base and axis OZ perpendicular to the base. The thin hemispherical shell



radius a is obtained by the revolution of arc OB of the quadrant of a circle of radius a about the line CB which is parallel to OZ and at a distance a from it.

Consider an element of arc $a\,\delta\theta$ at P By the revolution of this arc about CB a circular ring of radius $PL = a\cos\theta$ and cross-section $a\,\delta\theta$ is obtained.

Mass of this elementary ring

$= \delta m = \rho \cdot 2\pi a\cos\theta \cdot a\delta\theta = 2\pi a^2\rho\cos\theta\,\delta\theta$.

M.I. of this elementary ring about OA

= Its M.I. about PQ + M.I. of mass δm at centre L about OA.

$= \int_0^{\pi/2} 2\pi\rho a^4 \sin^3\theta d\theta = 2\rho\pi a^4 \cdot \frac{\Gamma(2)\Gamma(\frac{1}{2})}{2\Gamma(\frac{5}{2})} = \frac{4}{3}\rho\pi a^4 = \frac{2}{3}Ma^2$, ... from (1)

(ii) M.I. of the elementary ring about OY

= Its M.I. about parallel diameter through L

+ M.I. of δm at L about OY

$= \frac{1}{2}\delta m\, PL^2 + \delta m\, OL^2 = [\frac{1}{2}a^2\sin^2\theta + (a - a\cos\theta)^2] \cdot 2\rho\pi a^2 \sin\theta\,\delta\theta$

$= \rho\pi a^4 [\sin^2\theta + 2(1-\cos\theta)^2]\sin\theta\delta\theta$

$= \rho\pi a^4 [\sin^2\theta + 2 + 2\cos^2\theta - 4\cos\theta]\sin\theta\delta\theta$

$= \rho\pi a^4 (3 + \cos^2\theta - 4\cos\theta)\sin\theta\delta\theta$

$\therefore B$ = M.I. of the shell about OY

$= \int_0^{\pi} \rho\pi a^4 (3 + \cos^2\theta - 4\cos\theta)\sin\theta d\theta$

$= -\rho\pi a^4 \int_1^{0}(3 + t^2 - 4t)\,dt$, Putting $\cos\theta = t$.

$= \frac{4}{3}\pi\rho a^4 = \frac{2}{3}Ma^2$, from (1).

And C = M.I. of the shell about $OZ = B = \frac{2}{3}Ma^2$. (By Summetry)

(iii) Since the coordinates of

$\therefore$ C.G. are $G(a/2, 0, 0)$

$\therefore D$ = P.I. of the shell about OY and OZ

= P.I. of the shell about lines through C. G., 'G' parallel to OY and OZ + P.I. of the total mass M at G about OY and OZ.

$= O + M \cdot O \cdot O = O$.

(Since shell is symmetrical about lines through G, parallel to OY and OZ).

Similarly $E = O = F$.

If l, m, n are the direction cosines of any line through the vertex O, then M.I. of the shell about this line

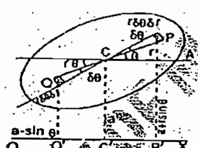
$= Al^2 + Bm^2 + Cn^2 - 2Dmn - 2Enl - 2Flm$

$= \frac{2}{3}Ma^2l^2 + \frac{2}{3}Ma^2 \cdot m^2 + \frac{2}{3}Ma^2n^2 = \frac{2}{3}Ma^2(l^2 + m^2 + n^2) = \frac{2}{3}Ma^2$.

1.18. Theorem I. *A closed curve revolves round any line OX in its own plane which does not intersect it. Show that the M.I. of the solid of revolution so formed about OX is equal to $M(a^2 + 3k^2)$, where M is the mass of the solid generated, a is the distance from OX of the centre C of the cruve, and k is the radius of gyration of the curve about a line through C parallel to OX.*

Prof. Let C be the centre of the closed curve which revolve round any line OX in its own plane which does not intersect it. Given that the distance of C from OX, $CC' = a$.

If M is the mass of the solid of revolution fromed about OX, then by Pappus Theorem, we have $M = 2\pi a\rho S$, where S is the area of the closed surface.



Consider an element $r\delta\theta\delta r$ at $P(r, \theta)$ taking C as the pole and the line CA parallel to OX as the initial line. For this element $r\delta\theta\delta r$ at P there will be an equal element for the same value of θ at Q in the opposite direction.

The distances of P and Q from OX are given by

$PP' = a + r\sin\theta$ and $QQ' = a - r\sin\theta$.

Now, the area of the closed curve

$S = 2\iint r d\theta dr$ ...(1)

the intergration being taken to cover the upper half of the area.

$\therefore$ M.I. of the area S about CA is Spk^2

and $S\rho k^2 = 2\iint (r\sin\theta)^2 \cdot \rho r d\theta dr$

the intrgration being taken to cover the upper half of the area.

$= 2\rho\iint r^3\sin^2\theta\, d\theta dr$ ...(2)

$\therefore$ M.I. of the solid of revolution about OX.

$= \iint [2\pi(a + r\sin\theta) \cdot (a + r\sin\theta)^2 + 2\pi(a - r\sin\theta) \cdot (a - r\sin\theta)^2] \cdot \rho r d\theta\, dr$

$= \iint 4\pi\rho\,(a^3 + 3ar^2\sin^2\theta)\, r d\theta dr$

$= 4\pi\rho a^3 \iint r\,d\theta\,dr + 12\pi\rho a \iint r^3\sin^2\theta\,d\theta dr$

$= 4\pi\rho a^3 \cdot S + 6\pi\rho a \cdot S\rho k^2$ [By (1) and (2)]

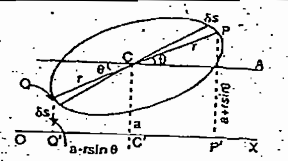
$= 2\pi\rho a S(a^2 + 3k^2) = M(a^2 + 3k^2)$. $\because M = 2\pi\rho a S$.

Theorem II. A closed curve revolves round any line OX in its own plane which does not intersect it. Show that the M.I. of the surface of revolution so formed about OX is equal to $M(a^2 + 3k^2)$, where M is the mass of the surface generated, a is the distance from OX of the centre C of the curve and k is the radius of gyration of the arc of the curve about a line through C parallel to OX.

Prof. Let l be the length of the arc of the closed curve, then

$l = 2\int ds$ ...(1)

the integration being taken to cover the upper half of the arc



By Pappus theroem, the mass M of the solid of revolution is given by

$M = 2\pi a\rho l$.

If k is the radius of gyration of the arc of the curve about OX then its M.I. about CA (a line parallel to OX) is ρk^2.

Consider an element δs at $P(r, \theta)$ of the arc taking C as centre and CA as initial line. For this element δs at $P(r, \theta)$ on the arc ther will be an equal arc δs for the same value of θ in opposite direction at Q on the arc.

we have $PP' = a + r\sin\theta$ and $QQ' = a - r\sin\theta$.

$\therefore$ M.I. of the arc of the curve about CA.

$l\rho k^2 = 2\int (r\sin\theta)^2 \cdot \rho ds$

$= 2\rho\int r^2\sin^2\theta\, ds$ ...(2)

the integration being taken to cover the upper half of the arc.

Now, M.I. of the surface of revolution about OX

$= \int [2\pi(a + r\sin\theta) \cdot (a + r\sin\theta)^2 + 2\pi(a - r\sin\theta) \cdot (a - r\sin\theta)^2] \cdot \rho ds$

$= \int 4\pi\rho(a^3 + 3ar^2\sin^2\theta)\, ds$

$= 4\pi\rho a^3 \int ds + 12\pi\rho a \int r^2\sin^2\theta\, ds$

$= 2\pi\rho a^3 l + 6\pi a\rho l k^2$ [By (1) and (2)]

$= 2\pi\rho a l(a^2 + 3k^2) = M(a^2 + 3k^2)$. $(\because M = 2\pi a\rho l)$

EXAMPLES

Ex. 25. *The M.I. a out its axis of a solid rubber tyre, of mass M and circular cross-section of radius a is $(M/4)(4b^2 + 3a^2)$, where b is the radius of the curve.*

Sol. Let OX be the axis of the solid tyre of mass M and circular cross-section of radius a. Solid tyre is obtained by the revolution of the circle of radius a and centre C about OX, where $CC' = b$.

Let CA be the line through C, parallel to OX.

Then M.I. of the circular area of mass M' (say) about CA

$M'k^2 = \frac{1}{4}M'a^2$ $\therefore k^2 = \frac{1}{4}a^2$.

From Theorem 1 of § 1.17, M.I. of the solid tyre about OX

$= M(b^2 + 3k^2)$ here a is equal to b

$= M(b^2 + \frac{3}{4}a^2) = (M/4)(4b^2 + 3a^2)$.

Ex. 26. *The M.I. about its axis of a hollow tyre, of mass M and circular corss-section of radius a is $(M/2)(2b^2 + 3a^2)$, where b is the radius of the curve.*

Sol. Refer figure of last Ex. 25.

Here the hollow tyre is obtained by the revolution of the arc of the circle of radius a and centre C about OX, where $CC' = b$.

M.I. of the arc of mass M' (say) of the circle about CA.

$M'k^2 = \frac{1}{2}M'a^2$. $\therefore k^2 = \frac{1}{2}a^2$.

$\therefore$ From Theorem II of § 1.18, M.I. of the hollow tyre about OX

$= M(b^2 + 3k^2)$, here a is equal to b

$= M(b^2 + \frac{3}{2}a^2) = (M/2)(2b^2 + 3a^2)$.

1.19. M.I. by the Method of Differentiation.

If y is a function of x and δx, δy are small increments in the values of x and y respectively, then we know that

$\lim_{\delta x \to 0} \frac{\delta y}{\delta x} = \frac{dy}{dx}$ i.e. $\frac{\delta y}{\delta x} = \frac{dy}{dx}$ approximately,

or $\delta y = \frac{dy}{dx}\delta x$.

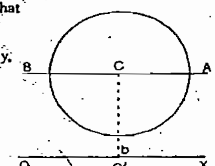


FIG. EX 25

For example :

(i) Area of a circle, $A = \pi r^2$, then

$\delta A = \left(\frac{d}{dr}A\right) \cdot \delta r = \frac{d}{dr}(\pi r^2) \cdot \delta r = (2\pi r)\,\delta r$

= (Circumference of a circle of radius r) × thickness δr.

(ii) Volume of sphere, $V = \frac{4}{3}\pi r^3$, then

$\delta V = \left(\frac{d}{dr}V\right) \cdot \delta r = \frac{d}{dr}(\frac{4}{3}\pi r^3) \cdot \delta r = (4\pi r^2)\,\delta r$

= (Surface of the spherical shell of radius r) × thickness δr.

This method of differentiation can be used in finding the moments of inertia in some cases. For this see the following examples.

EXAMPLES

Ex. 27. *Show that the M.I. of a thin homogeneous ellipsoidal shell (bounded by similar and similarly situated concentric ellipsoids) about an axes is $(M/3)(b + c^2)$, where M is the mass of the shell.*

Sol. We know that the M.I. of an ellipsoid of density ρ and semi-axis a, b, c about x-axis is equal to

$ax^2+2hxy+by^2+\frac{\Delta}{ab-h^2}=0,$ ...(2)

where $\Delta = abc+2fgh-af^2-bg^2-ch^2$. (By geometry)

Putting $y=0$ in (2), we have $x^2=-\frac{\Delta}{a(ab-h^2)}$.

∴ If r is the length of the semi-diameter of the ellipse parallel to the axis of x, then

$r^2=-\frac{\Delta}{a(ab-h^2)}$ ...(3)

Now, the equation (2) of the ellipse can be written as

$-\frac{a}{c'}x^2-\frac{2h}{c'}xy-\frac{b}{c'}y^2=1,$ ...(4)

where $c'=\Delta/(ab-h^2)$.

Which is of the standard form $Ax^2+2Hxy+By^2=1$.

The squares of the lengths of the semi-axes of the ellipse are given by the values R^2 in the equation

$\left(A-\frac{1}{R^2}\right)\left(B-\frac{1}{R^2}\right)=H^2$

or $\left(-\frac{a}{c'}-\frac{1}{R^2}\right)\left(-\frac{b}{c'}-\frac{1}{R^2}\right)=\left(-\frac{h}{c'}\right)^2$

or $\frac{1}{R^4}+\left(\frac{a+b}{c'}\right)\cdot\frac{1}{R^2}+\frac{ab-h^2}{c'^2}=0.$ ...(5)

If α and β are the lengths of semi-axes of ellipse then $1/\alpha^2$, $1/\beta^2$ are the roots of (5).

∴ $\frac{1}{\alpha^2}\cdot\frac{1}{\beta^2}=\frac{ab-h^2}{c'^2}$ or $\alpha^2\beta^2=\frac{c'^2}{ab-h^2}=\frac{\Delta^2}{(ab-h^2)^3}$

∴ From Ex. 18, M.I. of the ellipse about the diameter

$=\frac{M}{4}\frac{\alpha^2\beta^2}{r^2}=\frac{M}{4}\frac{\Delta^2}{(ab-h^2)^3}\cdot\left[-\frac{a(ab-h^2)}{\Delta}\right]=-\frac{aM\Delta}{4(ab-h^2)^2}$

Ex. 21. *Show that the M.I. of an ellipse of mass M and semi-axes a and b about a tangent is $\frac{5}{4}Mp^2$, where p is the perpendicular from the centre on the tangent.*

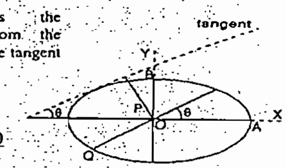
Sol. Let the equation of an ellipse be $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

∴ Equation of the tangent to the ellipse is

$y=mx+\sqrt{(a^2m^2+b^2)}$ ...(1)

where $m=\tan\theta$, if tangent is inclined at an angle θ to the axis of x.

If p is the perpendicular from the centre (0, 0) on the tangent (1), then



$p=\frac{\sqrt{(a^2m^2+b^2)}}{\sqrt{(1+m^2)}}$

$=\frac{\sqrt{(a^2\tan^2\theta+b^2)}}{\sqrt{(1+\tan^2\theta)}}$

$=\sqrt{(a^2\sin^2\theta+b^2\cos^2\theta)}.$ ...(2)

M.I. of the ellipse about the diameter PQ which is parallel to the tangent

$=A\cos^2\theta+B\sin^2\theta-F\sin 2\theta$

$=\frac{1}{4}Mb^2\cos^2\theta+\frac{1}{4}Ma^2\sin^2\theta-0$

$=\frac{1}{4}M(b^2\cos^2\theta+a^2\sin^2\theta)=\frac{1}{4}Mp^2$, from (2).

∴ M.I. of the ellipse about the tangent

= Its M.I. about the parallel line through C.G. 'O' + M.I. of mass M at O about the tangent.

$=\frac{1}{4}Mp^2+Mp^2=\frac{5}{4}Mp^2.$

Ex. 22. *Show that the sum of the moments of inertia of an elliptic area about any two perpendicular tangents is always the same.*

Sol. M.I. of an elliptic area about a tangent inclined at an angle θ to the major axis

$=\frac{5}{4}Mp^2$ (See last Ex. 21)

$=\frac{5}{4}M(a^2\sin^2\theta+b^2\cos^2\theta)$.

Replacing θ by $\theta+\pi/2$, the M.I of the elliptic area about a perpendicular tangent

$=\frac{5}{4}M(a^2\cos^2\theta+b^2\sin^2\theta)$

∴ Sum of the moments of inertia about any two perpendicular tangent

$=\frac{5}{4}M(a^2\sin^2\theta+b^2\cos^2\theta)+\frac{5}{4}M(a^2\cos^2\theta+b^2\sin^2\theta)$

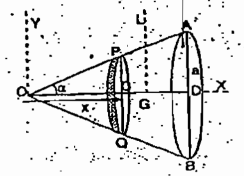
$=\frac{5}{4}M(a^2+b^2)$.

which is always the same as it is independent of θ.

Ex. 23. *Show that the M.I. of a right solid cone whose height is h and radius of whose base is a, is $\frac{3Ma^2}{20}\cdot\frac{6h^2+a^2}{h^2+a^2}$ about a slant side, and $\left(\frac{3M}{80}\right)(h^2+4a^2)$ about a line through the centre of gravity of the cone perpendicular to its axis.*

Sol. Let M be the mass of a right circular cone of height h and radius of whose base is a. If α is the semi-vertical angle and ρ the density of the cone, then

$M=\frac{1}{3}\rho\pi h^3\tan^2\alpha$ ...(1)



Take the vertex of the cone as the origin, x-axis along the axis OD of the cone and y-axis perpendicular to OD.

The slant side OA make an angle α with OX.

∴ M.I. of the cone about $OA=A\cos^2\alpha+B\sin^2\alpha-F\sin 2\alpha$. ...(2)

where, A = M.I. of the cone about OX, B = M.I. of the cone about OY and F = P.I. of the cone about OX and OY.

Now consider an elementary disc PQ parallel to the base AB of the cone, of thickness δx and at a distance x from O.

Mass of this elementary disc $=\delta m=\rho\pi x^2\tan^2\alpha\,\delta x$.

(i) M.I. of the elementary disc about

$OX=\frac{1}{2}\delta m\cdot CP^2=\frac{1}{2}\rho\pi x^4\tan^4\alpha\,\delta x$.

∴ A = M.I. of the cone about $OX=\int_0^h\frac{1}{2}\rho\pi x^4\tan^4\alpha\,dx$

$=\frac{1}{10}\rho\pi h^5\tan^4\alpha=\frac{3M}{10}h^2\tan^2\alpha$; from (1)

$=\frac{3}{10}Ma^2$, $\tan\alpha=\frac{a}{h}$

(ii) M.I. of the elementary disc about OY

= Its M.I. about parallel diameter PQ

+ M.I. of mass δm at C about OY

$=\frac{1}{4}\delta m\cdot CP^2+\delta m\cdot OC^2=\left(\frac{1}{4}x^2\tan^2\alpha+x^2\right)\cdot\rho\pi x^2\tan^2\alpha\,\delta x$

$=\frac{1}{4}(\tan^2\alpha+4)\rho\pi x^4\tan^2\alpha\,\delta x$

∴ B = M.I. of the cone about OY

$=\int_0^h\frac{1}{4}(\tan^2\alpha+4)\rho\pi x^4\tan^2\alpha\,dx$

$=\frac{1}{20}\rho\pi h^5(\tan^2\alpha+4)\tan^2\alpha=\frac{3}{20}Mh^2(\tan^2\alpha+4)$, from (1)

$=\frac{3}{20}M(a^2+4h^2)$, ∵ $\tan\alpha=\frac{a}{h}$.

(iii) F = P.I. of the cone about OX and $OY=0$. By symmetry about OX.

Also $\cos\alpha=\frac{OD}{OA}=\frac{OD}{\sqrt{(OD^2+AD^2)}}=\frac{h}{\sqrt{(h^2+a^2)}}$

and $\sin\alpha=\frac{AD}{OA}=\frac{a}{\sqrt{(h^2+a^2)}}$.

∴ from (2) M.I. of the cone about slant side

$=\frac{3}{10}Ma^2\cdot\frac{h^2}{h^2+a^2}+\frac{3}{20}M(a^2+4h^2)\frac{a^2}{h^2+a^2}=\frac{3Ma^2}{20}\cdot\frac{6h^2+a^2}{h^2+a^2}$

Second Part. Let GL be a line through the C.G. G of the cone and perpendicular to its axis OD. Then

M.I. of the cone about OY = M.I. of the cone about parallel line GL through C.G. G + M.I. of total mass M at G about OY.

∴ M.I. of the cone about the line GL

= M.I. of the cone about OY − M.I. of total mass M at G about OY.

$=\frac{3}{20}M(a^2+4h^2)-M\cdot OG^2=\frac{3}{20}M(a^2+4h^2)-M\cdot\left(\frac{3}{4}h\right)^2$, (∵ $OG=\frac{3}{4}h$)

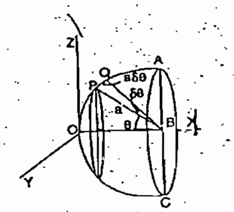
$=\frac{3M}{80}(h^2+4a^2)$.

Ex. 24. *Show that for a thin hemispherical shell of mass M and radius a, the M.I. about any line through the vertex is $\frac{2}{3}Ma^2$.*

Sol. A hemispherical shell with vertex at the origin O is generated by the revolution of the arc OA of quadrant OAB of the circle of radius a.

If ρ is the density of the shell, then

$M=2\pi\rho a^2$ ...(1)



Take the x-axis along the symmetrical radius OB of the shell and axes OY and OZ perpendicular to OX.

Consider an elementary arc $a\,\delta\theta$ at the point P of the arc OA.

The mass of the elementary ring obtained by the revolution of this elementary arc $a\,\delta\theta$ at P about OX.

∴ $\delta m=\rho\cdot 2\pi PL\cdot a\delta\theta=2\rho\pi a^2\sin\theta\,\delta\theta$, (∵ $PL=a\sin\theta$)

(i) M.I. of the elementary ring about OX

$=\delta m\cdot PL^2=2\rho\pi a^2\sin\theta\,\delta\theta\,a^2\sin^2\theta=2\rho\pi a^4\sin^3\theta\,\delta\theta$

∴ A = M.I. of the shell about OX

$= \frac{1}{4}PL^2\,\delta m + CL^2\,\delta m = (\frac{1}{4}a^2\cos^2\theta + a^2\sin^2\theta).2\pi a^2\rho\cos\theta\delta\theta$

$= \pi\rho a^4(\cos^2\theta + 2\sin^2\theta)\cos\theta\delta\theta = \pi\rho a^4(1+\sin^2\theta)\cos\theta\delta\theta.$

$\therefore A$ = M.I. of the hemispherical shell about OX

$= \int_0^{\pi/2}\pi\rho a^4(1+\sin^2\theta)\cos\theta d\theta.$

$= \pi\rho a^4\int_0^1(1+t^2)\,dt,$ Putting $\sin\theta = t$

$= \pi\rho a^4\left[t+\frac{1}{3}t^3\right]_0^1 = \frac{4}{3}\pi\rho a^4 = \frac{2}{3}Ma^2,$ $M = 2\pi a^2\rho$

B = M.I. of the hemi-spherical shell about OY

= Its M.I. about parallel diameter through C + M.I. of total mass M at C about OY.

$= \frac{2}{3}Ma^2 + Ma^2 = \frac{5}{3}Ma^2$

Also M.I. of the elementary ring about OZ

= Its M.I. about BC + M.I. of its mass δm at L about OZ

$= PL^2\,\delta m + OC^2\,\delta m = (a^2\cos^2\theta + a^2)\,2\pi a^2\rho\cos\theta\delta\theta$

$= 2\pi a^4\rho(\cos^3\theta + \cos\theta).\delta\theta.$

$\therefore C$ = M.I. of the hemispherical shell about OZ

$= \int_0^{\pi/2}2\pi a^4\rho(\cos^3\theta + \cos\theta)\,d\theta = 2\pi a^4\rho.\left[\frac{\Gamma(2)\Gamma(\frac{1}{2})}{2\Gamma(\frac{5}{2})} + (\sin\theta)_0^{\pi/2}\right]$

$= Ma^2(\frac{2}{3}+1) = \frac{5}{3}Ma^2.$

Coordinates of C.G. 'G' of the shell are $(a, 0, a/2)$

D = P.I. of the shell about OY, OZ

= P.I. of the shell about lines parallel to OY, OZ through C + P.I. of mass M at G about OY, OZ

$= O + M.O.a/2 = 0.$

Similarly E = P.I. of the shell about OZ, OX

$= O + M.a/2.a = \frac{1}{2}Ma^2$

and F = P.I. of the shell about OX and $OY = O + M.a.O = 0.$

Hence the equation of momental ellipsoid at O is

$Ax^2 + By^2 + Cz^2 - 2Dyz - 2Ezx - 2Fxy$ = const.

or $\frac{2}{3}Ma^2x^2 + \frac{5}{3}Ma^2y^2 + \frac{5}{3}Ma^2z^2 - O - 2.\frac{1}{2}Ma^2zx - O$ = const.

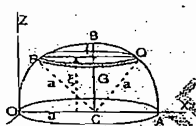
or $2x^2 + 5(y^2+z^2) - 3zx$ = const.

Ex. 35. *Show that the momental ellipsoid at a point on the rim of a hemisphere is* $2x^2 + 7(y^2+z^2) - \frac{15}{4}xz$ = *Const.*

Sol. Let O be a point on the rim of a hemisphere of radius a and mass M. If ρ is the density then

$M = \frac{2}{3}\pi a^3\rho.$

Take the axis OX along the diameter OA of the circular base axis OY perpendicular to OX through O in the plane of the base and axis OZ perpendicular to the base.



Consider an elementary strip PQ of thickness $\delta\xi$ parallel to the base and at a distance ξ from C then

Mass of this elementary disc, $\delta m = \rho\pi PL^2.\delta\xi = \rho\pi(a^2-\xi^2)\,\delta\xi.$

M.I. of the elementary disc about OX = Its M.I. about PQ + M.I. of mass δm at L about OX.

$= \frac{1}{4}PL^2\,\delta m + CL^2\delta m = [\frac{1}{4}(a^2-\xi^2)+\xi^2].\rho\pi(a^2-\xi^2)\,\delta\xi$

$= \frac{1}{4}\pi\rho(a^4 + 2a^2\xi^2 - 3\xi^4)\,\delta\xi.$

$\therefore A$ = M.I. of the hemisphere about OX.

$= \int_0^a\frac{1}{4}\pi\rho(a^4+2a^2\xi^2-3\xi^4)\,d\xi = \frac{4}{15}\pi\rho a^5 = \frac{2}{5}Ma^2.$

B = M.I. of the hemisphere about OY

= Its M.I. about the line through C (diameter of base) and parallel to OY + M.I. of total mass M at C about OY

$= \frac{2}{5}Ma^2 + Ma^2 = \frac{7}{5}Ma^2.$

Also M.I. of the elementary disc about OZ

= Its M.I. about CB + M.I. of its mass δm at L about OZ.

$= \frac{1}{2}PL^2\,\delta m + OC^2\,\delta m = [\frac{1}{2}(a^2-\xi^2)+a^2]\,\rho\pi(a^2-\xi^2)\,d\xi$

$= \frac{1}{2}\rho\pi(3a^4 - 4a^2\xi^2 + \xi^4)\,d\xi$

$\therefore C$ = M.I. of the hemisphere about OZ

$= \int_0^a\frac{1}{2}\rho\pi(3a^4-4a^2\xi^2+\xi^4)\,d\xi = \frac{14}{15}\rho\pi a^5 = \frac{7}{5}Ma^2.$

Coordinates of the C.G. 'G' of the hemisphere are $(a, 0, \frac{3}{8}a)$.

$\therefore D$ = P.I. of the hemisphere about OY and OZ

= Its P.I. about lines through G, parallel to OY and OZ + P.I. of mass M at G about OY and OZ

$= 0 + M.O.\frac{3}{8}a = 0$

Similarly E = P.I. of hemisphere about OZ and OX

$= 0 + M.\frac{3}{8}a.a = \frac{3}{8}Ma^2$

F = P.I. of hemisphere about OX and $OY = O + M.a.O = 0.$

Hence the equation of momental ellipsoid at O is

$Ax^2 + By^2 + Cz^2 - 2Dyz - 2Ezx - 2Fxy$ = const.

or $\frac{2}{5}Ma^2x^2 + \frac{7}{5}Ma^2y^2 + \frac{7}{5}Ma^2z^2 - 0 - 2.\frac{3}{8}Ma^2zx - 0$ = cost.

or $2x^2 + 7(y^2+z^2) - \frac{15}{4}xz$ = const.

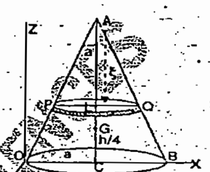
Ex. 36. *Prove that the equation of the momental ellipsoid at a point on the circular edge of a solid cone is*

$(3a^2+2h^2)x^2 + (23a^2+2h^2)y^2 - 26a^2z^2 - 10ahxz$ = *const,*

where h is the height and a the radius of the base.

Sol. Let O be a point on the circular edge of a solid cone of mass M, semi-vertical angle α, height h and radius of base a. If ρ is its density, then $M = \frac{1}{3}\pi\rho h^3\tan^2\alpha.$

Take the axis OX along the diameter OB of the base, axis OY perpendicular to OB in the plane of the base and OZ perpendicular to the base.



Consider an elementary disc PQ parallel to the base, at a distance ξ from the vertex A and of thickness $\delta\xi$.

$\therefore$ Mass of this elementary disc, $\delta m = \rho\pi PL^2\delta\xi$

$= \rho\pi\xi^2\tan^2\alpha\,d\xi.$

M.I. of this elementary disc about OX

= Its M.I. about PQ + M.I. of its mass δm at L about OX.

$= \frac{1}{4}PL^2\delta m + CL^2\delta m = [\frac{1}{4}\xi\tan^2\alpha + (h-\xi)^2]\,\rho\pi\xi^2\tan^2\alpha d\xi.$

$\therefore A$ = M.I. of the cone about OX

$= \int_0^h[\frac{1}{4}\xi^2\tan^2\alpha + (h-\xi)^2]\,\rho\pi\xi^2\tan^2\alpha d\xi$

$= \rho\pi\tan^2\alpha\int_0^h[\frac{1}{4}\xi^4\tan^2\alpha + h^2\xi^2 - 2h\xi^3 + \xi^4]\,d\xi$

$= \rho\pi h^5\tan^2\alpha\left[\frac{1}{20}\tan^2\alpha + \frac{1}{30}\right] = \frac{1}{60}\rho\pi h^5\tan^2\alpha(3\tan^2\alpha + 2)$

$= \frac{1}{20}Mh^2(3\tan^2\alpha + 2) = \frac{1}{20}M(3a^2+2h^2),$ $(\because \tan\alpha = a/h)$

B = M.I. of the cone about OY

= Its M.I. about line parallel to OY through C (*i.e.* diameter of base) M.I. of total mass M at C about OY

$= \frac{1}{20}M(3a^2+2h^2) + Mh^2 = \frac{1}{20}M(23a^2+2h^2).$

Now M.I. of the elementary disc about OZ

= Its M.I. about AC + M.I. of its mass δm at L about OZ

$= \frac{1}{2}PL^2\delta m + OC^2\delta m = (\frac{1}{2}\xi^2\tan^2\alpha + a^2)\,\rho\pi\xi^2\tan^2\alpha d\xi$

$= \rho\pi(\frac{1}{2}\xi^4\tan^2\alpha + a^2\xi^2)\,d\xi.$

$\therefore C$ = M.I. of the cone about OZ

$= \int_0^h\rho\pi(\frac{1}{2}\xi^4\tan^2\alpha + a^2\xi^2)\tan^2\alpha d\xi$

$= \rho\pi h^3\left(\frac{1}{10}h^2\tan^2\alpha + \frac{1}{3}a^2\right)\tan^2\alpha$

$= \frac{1}{10}M(3h^2\tan^2\alpha + 10a^2) = \frac{13}{10}Ma^2,$ $(\because \tan\alpha = a/h)$

The coordinates of C.G. G of the cone are $(a, 0, h/4)$.

$\therefore D$ = P.I. of the cone about OY and OZ

= P.I. of the cone about lines through G parallel to OY and OZ + P.I. of the mass M at G about OY and OZ

$= 0 + M.0.h/4 = 0.$

Similarly, E = P.I. of the cone about OZ and OX

$= 0 + M\cdot\frac{h}{4}\cdot a = \frac{1}{4}Mah$

and F = P.I. of the cone about OX and $OY = 0 + M.a.0 = 0$.

Hence the equation of the momental ellipsoid at O is

$Ax^2 + By^2 + Cz^2 - 2Dyz - 2Ezx - 2Fxy$ = constant.

or $\frac{1}{20}M(3a^2+2h^2)x^2 + \frac{1}{20}M(23a^2+2h^2)y^2$

$+\frac{13}{10}Ma^2z^2 - 0 - 2\cdot\frac{1}{4}Mahzx - 0$ = constant.

or $(3a^2+2h^2)x^2 + (23a^2+2h^2)y^2 + 26a^2z^2 - 10ahxz$ = const.

Ex. 37. *If $S \equiv Ax^2 + By^2 + Cz^2 - 2Dyz - 2Ezx - 2Fxy = \text{constant}$ be the equation of the momental ellipsoid at the centre of gravity O of a body referred to any rectangular axes through O, then prove that momental ellipsoid at the point (p, q, r) is*

$$S + M[(qz - ry)^2 + (rx - pz)^2 + (py - qx)^2] = \text{const.}$$

where M is the mass of the body.

Sol. Since $S \equiv Ax^2 + By^2 + Cz^2 - 2Dyz - 2Ezx - 2Fxy = \text{constant}$ is the equation of the momental ellipsoid at the centre of gravity O of the body referred to the rectangular axes at O, therefore A, B, C are the moments and D, E, F are the products of inertia of the body about the rectangular axes through O.

Let A', B', C' be the moments and D', E', F' the products of inertia of the body about the parallel rectangular axes through (p, q, r). If M is the mass of the body, then

A' = M.I. about x-axis through C.G. O + M.I. of mass M at O about the axis parallel to x-axis through (p, q, r)

$= A + M(q^2 + r^2)$.

Similarly, $B' = B + M(r^2 + p^2)$, $C' = C + M(p^2 + q^2)$

$D' = D + Mqr$, $E' = E + Mrp$, $F' = F + Mpq$.

Hence the equation of the momental ellipsoid at (p, q, r) is

$A'x^2 + B'y^2 + C'z^2 - 2D'yz - 2E'zx - 2F'xy = \text{const.}$

or $[A + M(q^2 + r^2)]x^2 + [B + M(r^2 + p^2)]y^2 + [C + M(p^2 + q^2)]z^2 - 2(D + Mqr)yz - 2(E + Mrp)zx - 2(F + Mpq)xy = \text{const.}$

or $(Ax^2 + By^2 + Cz^2 - 2Dyz - 2Ezx - 2Fxy) + M[(q^2z^2 + r^2y^2 - 2qryz) + (r^2x^2 + p^2z^2 - 2prxz) + (p^2y^2 + q^2x^2 - 2pqxy)] = \text{const.}$

or $S + M[(qz - ry)^2 + (rx - pz)^2 + (py - qx)^2] = \text{const.}$

1.23. Equimomental Bodies.

Two systems or bodies are said to be equimomental or kinetically (or dynamically) equivalent when moments and products of inertia of one system or body about all axes are each equal to the moments and products of inertia of the other system or body about the same axes.

The necessary and sufficient conditions for two systems to be equimomental are that :

(i) the centre of gravity of the two systems is the same point;

(ii) both the systems have the same mass; and

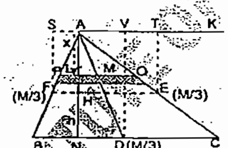
(iii) the two systems have the same principal axes and same principal moments about the centre of gravity.

1.24. *The moments and products of inertia of a uniform triangle about any lines are the same as the moments and products of inertia of three particles placed at the middle points of the sides, each equal to one-third of the mass of the triangle.*

Let AD be the median of a triangle ABC of mass M. Let AN be the perpendicular on BC from A, AK perpendicular to AN in the plane of the triangle ABC and $AN = h$.

$\therefore M = \frac{1}{2} BC \cdot AN \rho = \frac{1}{2} ah\rho$

Consider an elementary strip PQ parallel to BC of thickness δx and at a distance x from A.



From similar triangles APQ and ABC, we have

$\frac{PQ}{BC} = \frac{AL}{AN}$

$\therefore PQ = \frac{AL}{AN} \cdot BC = \frac{xa}{h}$

Now mass of the strip $PQ = \rho PQ \delta x = \frac{\rho a}{h} x \delta x$.

$\therefore$ M.I. of the strip about AK

= Its M.I. about PQ + M.I. of its mass δm at its C.G. (*i.e.* middle point of PQ) about AK

$= O + x^2 \delta m = \frac{\rho a}{h} x^3 \delta x.$

$\therefore$ M.I. of the ΔABC about $AK = \int_0^h \frac{\rho a}{h} x^3 dx$

$= \frac{1}{4} \rho a h^3 = \frac{1}{2} Mh^2.$ ...(1)

Also M.I. of the strip PQ about AN

= M.I. of the strip about parallel line through its C.G. M (middle point of BC) + M.I. of its mass δm at M about AN

$= \frac{1}{3}(\frac{1}{2} PQ)^2 \delta m + LM^2 \delta m = \left[\frac{1}{3}\left(\frac{ax}{2h}\right)^2 + LM^2\right] \cdot \frac{\rho a}{h} x \delta x.$

But from similar triangles ALM and AND, we have

$\frac{LM}{ND} = \frac{AL}{AN} = \frac{x}{h} \quad \therefore \quad LM = \frac{x}{h} ND.$

$\therefore$ M.I. of the strip PQ about AN

$= \left[\frac{1}{3} \cdot \frac{a^2x^2}{4h^2} + \frac{ND^2}{h^2} \cdot x^2\right] \frac{\rho a}{h} x \delta x$

$= \frac{\rho a}{12h^3}(a^2 + 12ND^2) x^3 \delta x.$

$\therefore$ M.I. of the triangle ABC about AN

$= \int_0^h \frac{\rho a}{12h^3}(a^2 + 12ND^2) x^3 dx$

$= \frac{\rho a h}{48}(a^2 + 12ND^2) = \frac{M}{24}[a^2 + 12(BD - BN)^2]$

$= \frac{M}{24}\left[a^2 + 12\left(\frac{a}{2} - c\cos B\right)^2\right]$

$= \frac{M}{24}\left[a^2 + 12\left(\frac{a}{2} - c\frac{a^2 + c^2 - b^2}{2ac}\right)^2\right]$

$= \frac{M}{24}\left[a^2 + \frac{3}{a^2}(b^2 - c^2)^2\right] = \frac{M}{24a^2}[a^4 + 3(b^2 - c^2)^2].$ ...(2)

and P.I. of the triangle ABC about AK and AN

$= \int_0^h (AL \cdot LM) \frac{\rho a}{h} x dx = \int_0^h \left(x \cdot \frac{x}{h} ND\right) \frac{\rho a}{h} x dx = \frac{1}{4} \rho a h^2 \cdot ND$

$= \frac{1}{2} Mh \cdot ND = \frac{1}{2} Mh(BD - BN) = \frac{1}{2} Mh(a/2 - c\cos B).$

$= \frac{1}{2} Mh\left(\frac{a}{2} - c \cdot \frac{a^2 + c^2 - b^2}{2ac}\right) = \frac{1}{4} \frac{Mh}{a}(b^2 - c^2)$ ...(3)

Now we shall consider a system of three particles each of mass $M/3$ placed at the middle points D, E, F of the sides of the ΔABC and find their moments and products of inertia about AK and AN.

M.I. of the three particles each of mass $M/3$ at D, E, F about AK

$\frac{M}{3} DV^2 + \frac{M}{3} ET^2 + \frac{1}{3} FS^2 = \frac{M}{3}\left[h^2 + \left(\frac{h}{2}\right)^2 + \left(\frac{h}{2}\right)^2\right] = \frac{1}{2} Mh^2.$ ...(4)

M.I. of the three particles each of mass $M/3$ at D, E, F about AN

$= \frac{M}{3} DN^2 + \frac{M}{3} EH^2 + \frac{M}{3} FH^2$

$= \frac{M}{3}[(BD - BN)^2 + (\frac{1}{2} CN)^2 + (\frac{1}{2} BN)^2]$

$= \frac{M}{3}\left[\left(\frac{a}{2} - c\cos B\right)^2 + \frac{1}{4}(b\cos C)^2 + \frac{1}{4}(c\cos B)^2\right]$

$= \frac{M}{12}[(a - 2c\cos B)^2 + b^2\cos^2 C + c^2\cos^2 B]$

$= \frac{M}{12}[(b\cos C + c\cos B - 2c\cos B)^2 + (b^2\cos^2 C + c^2\cos^2 B)]$

$= \frac{M}{12}[(b\cos C - c\cos B)^2 + (b\cos C - c\cos B)^2 + 2bc\cos B\cos C]$

$= \frac{M}{6}[(b\cos C - c\cos B)^2 + bc\cos B\cos C]$

$= \frac{M}{6}\left[\left(b \cdot \frac{a^2 + b^2 - c^2}{2ab} - c \cdot \frac{a^2 + c^2 - b^2}{2ac}\right)^2 + bc \cdot \frac{a^2 + c^2 - b^2}{2ac} \cdot \frac{a^2 + b^2 - c^2}{2ab}\right]$

$= \frac{M}{24a^2}\left[4(b^2 - c^2)^2 + a^4 - (b^2 - c^2)^2\right]$

$= \frac{M}{24a^2}\left[a^4 + 3(b^2 - c^2)^2\right]$ ...(5)

and P.I. of the three particles each of mass $M/3$ at D, E, F about AK and AN

$= \frac{M}{3} DN \cdot AN + \frac{M}{3} EH \cdot AH - \frac{M}{3} FH \cdot AH$

$= \frac{M}{3}\left[DN \cdot h + \frac{1}{2} CN \cdot \frac{h}{2} - \frac{1}{2} BN \cdot \frac{h}{2}\right] = \frac{1}{12} Mh[4DN + CN - BN]$

$= \frac{1}{12} Mh[4(BD - BN) + CN - BN] = \frac{1}{12} Mh\left[4 \cdot \frac{a}{2} + CN - 5BN\right]$

$= \frac{1}{12} Mh\left[4 \cdot \frac{a}{2} + b\cos C - 5 \cdot c\cos B\right]$

$= \frac{1}{12} Mh\left[4 \cdot \frac{a}{2} + b \cdot \frac{a^2 + b^2 - c^2}{2ab} - 5c \cdot \frac{a^2 + c^2 - b^2}{2ac}\right]$

$= \frac{Mh}{4a}(b^2 - c^2)$ ...(6)

From (1), (2), (3) and (4), (5), (6), it is clear that the moments and products of inertia of the ΔABC of mass M about AK and AN are the same as those of three particles each of mass $M/3$ placed at the middle points of the sides.

Note. Also the two systems have the same mass M and the same centre of gravity.

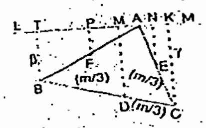
Hence the triangle of mass M is equimomental to three particles each of mass $M/3$ placed at the middle points of the sides.

EXAMPLES

Ex. 38. *Obtain the moment of inertia for a triangular lamina ABC about a straight line through A (or any vertex) in the plane of the triangle.*

Sol. Let m be the mass of the triangle ABC, then the triangle is equimomental to the three particles each of mass $m/3$ placed at the middle points D, E, F of its sides.

Let LM be any line through the vertex A and in the plane of the triangle ABC. Let β and γ be the distances of the vertices B and C from the line LM, *i.e.* $BT = \beta$ and $CK = \gamma$.



$\therefore$ Perpendicular distances of D, E, F from LM are as follows

$DM = \frac{1}{2}(\beta + \gamma)$, $EN + \frac{1}{2} CK = \frac{1}{2}\gamma$ and $FP = \frac{1}{2} BT = \frac{1}{2}\beta$.

$\therefore$ M.I. of the triangle ABC about LM

= Sum of M.I. of masses $m/3$ each at D, E, F about LM

$= \frac{m}{3} \cdot DM^2 + \frac{m}{3} \cdot EN^2 + \frac{m}{3} \cdot FP^2$

$= \frac{m}{3}\left[\frac{1}{4}(\beta + \gamma)^2 + \frac{1}{4}\gamma^2 + \frac{1}{4}\beta^2\right] = \frac{m}{6}(\beta^2 + \gamma^2 + \beta\gamma)$.

Ex. 39. *If α, β, γ be the distances of the vertices of a uniform triangular lamino of mass m from any line in its plane, prove that the M.I. about this line is $\frac{1}{6} m (\alpha^2 + \beta^2 + \gamma^2 + \beta\gamma + \gamma\alpha + \alpha\beta)$.*

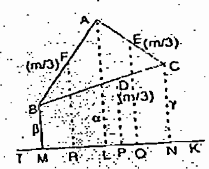
Hence deduce that if h be the distance of the centre of inertia of the triangle from the line, then M.I. about this line is

$\frac{1}{12} m(\alpha^2 + \beta^2 + \gamma^2 + 9h^2)$.

Sol. Let ABC be the triangular lamina of mass m and AL, BM, CN the perpendiculars from A, B, C on a line TK in its plane, then

$AL = \alpha$, $BM = \beta$, $CN = \gamma$.

If DP, EQ, FR are the perpendiculars from the middle points D, E, F of sides on TK, then



$DP = \frac{1}{2}(BM + CN) = \frac{1}{2}(\beta + \gamma)$

$EQ = \frac{1}{2}(AL + CN) = \frac{1}{2}(\alpha + \gamma)$,

$FR = \frac{1}{2}(AL + BM) = \frac{1}{2}(\alpha + \beta)$.

Since the triangle is equimomental to the three particles each of mass $m/3$ placed at the middle points D, E, F of the triangle,

$\therefore$ M.I. of the ΔABC about TK

= Sum of M.I. of masses $\frac{1}{3}m$ each at D, E, F about TK

$= \left(\frac{m}{3}\right)(DP)^2 + \left(\frac{m}{3}\right)(EQ)^2 + \left(\frac{m}{3}\right)(FR)^2$

$= \frac{m}{3} \cdot \frac{1}{4}(\beta + \gamma)^2 + \frac{m}{3} \cdot \frac{1}{4}(\alpha + \gamma)^2 + \frac{m}{3} \cdot \frac{1}{4}(\alpha + \beta)^2$

$\frac{1}{6}m(\alpha^2 + \beta^2 + \gamma^2 + \beta\gamma + \gamma\alpha + \alpha\beta)$ ...(1)

Deduction. If h is the distance of the centre of inertia of the ΔABC from TK, then $h = \frac{1}{3}(\alpha + \beta + \gamma)$.

$\therefore$ From (1), M.I. of the ΔABC about TK

$= \frac{1}{12}m(2\alpha^2 + 2\beta^2 + 2\gamma^2 + 2\beta\gamma + 2\gamma\alpha + 2\alpha\beta)$

$= \frac{1}{12}m[\alpha^2 + \beta^2 + \gamma^2 + (\alpha + \beta + \gamma)^2] = \frac{1}{12}m(\alpha^2 + \beta^2 + \gamma^2 + 9h^2)$.

Ex. 40. *Show that a uniform triangular lamina of moss m is equimomental with three particles, each of mass m/12 placed at the angular points and a particle of mass $\frac{3}{4}m$ placed at the centre of inertia of the triangle.*

Sol. (Refer fig of Ex. 39).

If α, β, γ are the distances of the vertices A, B, C of triangle ABC from a line TK in its plane, then

M.I. of the triangle ABC about TK

$= \frac{1}{6}m(\alpha^2 + \beta^2 + \gamma^2 + \beta\gamma + \gamma\alpha + \alpha\beta)$ (see Ex. 39)

The C.G. of the masses $m/12$ each at the points A, B, C and a particle of mass $\frac{3}{4}m$ placed at the centre of inertia of the triangle is the same point as the C.G. of the triangular lamina.

Also, sum of the masses of the four particles.

$= \frac{1}{12}m + \frac{1}{12}m + \frac{1}{12}m + \frac{3}{4}m$ = mass of the ΔABC.

and M.I. of the four particles about the line TK

$= \frac{1}{12}m \cdot AL^2 + \frac{1}{12}m \cdot BM^2 + \frac{1}{12}m \cdot CN^2 + \frac{3}{4}mh^2$

$= \frac{1}{12}m(\alpha^2 + \beta^2 + \gamma^2 + 9h^2)$

$= \frac{1}{6}m(\alpha^2 + \beta^2 + \gamma^2 + \beta\gamma + \gamma\alpha + \alpha\beta)$. $\because h = \frac{1}{3}(\alpha + \beta + \gamma)$

= M.I. of the ΔABC about the line TK.

Hence the triangular lamina and the four particles are equimomental.

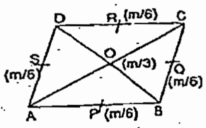
Ex. 41. *ABCD is a uniform parallelogram of mass M. At the middle points of the four sides are placed particles each of mass M/6 and at the intersection of the diagonals a particle of mass M/3, show that these five particles and the parallelogram are equimomental systems.*

Sol. Let $ABCD$ be a uniform parallelogram, of mass M, and P, Q, R, S the middle points of its sides.

Then mass of

ΔABD = mass of $\Delta BCD = M/2$.

Now the ΔABD is equimomental to three particles each of mass equal to one third the mass of the triangle ABD, *i.e.* ΔABD is equimomental to the three particle each of mass $\frac{1}{3}(\frac{1}{2}M) = \frac{1}{6}M$ at its the middle points P, O and S of its sides.

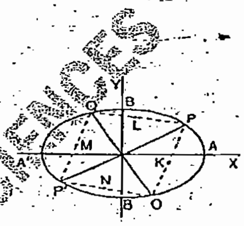


Similarly the ΔBCD is equimomental to three particles each of mass $\frac{1}{3}(\frac{1}{2}M) = \frac{1}{6}M$ at the middle points Q, R and O of its sides.

Hence the parallelogram $ABCD$ of mass M is equimomental to the particles each of mass $M/6$ at the middle points P, Q, R, S of the sides and particle of mass $\frac{1}{6}M + \frac{1}{6}M = \frac{1}{3}M$ at O (the point of intersection of the diagonals).

Ex. 42. *Particles each equal to one-quarter of the mass of an elliptic area are placed at the middle points of the chords joining the extremities of any pair of conjugate diameters. Prove that these four particles are equimomental to the elliptic area.*

Sol. Let POP' and QOQ' be the conjugate diameters of an elliptic area of mass m. If ϕ is the eccentric angle of P then eccentric angle of Q is $(\pi/2 + \phi)$.



$\therefore$ Coordinates of P are $(a\cos\phi, b\sin\phi)$ and coordinates of Q are $[a\cos(\phi + \pi/2), b\sin(\phi + \pi/2)]$ or $(-a\sin\phi, b\cos\phi)$.

$\therefore$ Coordinates of P' are $(-a\cos\phi, -b\sin\phi)$ and that of Q' are $(a\sin\phi, -b\cos\phi)$.

If $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)$ are the coordinates of the middle points L, M, N, K of Chords $PQ, QP', P'Q', Q'P$ respectively, then

$x_1 = \frac{1}{2}a(\cos\phi - \sin\phi)$, $y_1 = \frac{1}{2}b(\sin\phi + \cos\phi)$

$x_2 = -\frac{1}{2}a(\cos\phi + \sin\phi)$, $y^2 = \frac{1}{2}b(\cos\phi - \sin\phi)$

$x_3 = \frac{1}{2}a(\sin\phi - \cos\phi)$, $y_3 = -\frac{1}{2}b(\sin\phi + \cos\phi)$

and $x_4 = \frac{1}{2}a(\sin\phi + \cos\phi)$, $y_4 = \frac{1}{2}b(\sin\phi - \cos\phi)$.

If $(\bar{x}, \bar{y})$ are the coordinates of four particles each of mass $m/4$ at L, M, N, K then

$\bar{x} = \frac{1}{4}(x_1 = x_2 + x_3 + x_4) = 0$ and $\bar{y} = \frac{1}{4}(y_1 + y_2 + y_3 + y_4) = 0$

i.e. C.G. of the four particles is at O which is also the C.G. of the elliptic lamina.

Also M.I. of the four particles at L, M, N, K, about the major axis

$= \frac{m}{4}(y_1^2 + y_2^2 + y_3^2 + y_4^2)$

$= \frac{m}{4} \cdot \frac{1}{4} b^2 [(\sin\phi + \cos\phi)^2 + (\cos\phi - \sin\phi)^2 + (\sin\phi + \cos\phi)^2 + (\sin\phi - \cos\phi)^2]$

$= \frac{1}{4}mb^2$ = M.I. of the elliptic area about major axis.

Similarly M.I. of the four particles at L, M, N, K about the minor axis $= \frac{1}{4}ma^2$ = M.I. of the elliptic area about minor axis.

and P.I. of the four particles at L, M, N, K about OX, OY

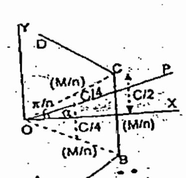
$= \frac{1}{4}m(x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4) = 0$

= P.I. of the elliptic area about OX and OY.

Thus the four particles each of mass $m/4$ at L, M, N, K have the same mass, same C.G. and the same principal moments as that of the elliptic area. Hence the particles are equimomental to the elliptic area.

Ex. 43. *Show that the M.I. of a regular polygon of n sides about any straight line through its centre is $\frac{Mc^2}{24} \cdot \frac{2 + \cos(2\pi/n)}{1 - \cos(2\pi/n)}$, where n is the number of sides and c is the length of each side.*

Sol. Let $ABCD.....A$ be a regular polygon of n sides each of length c. Let O be the centre of the polygon and lines OX (bisecting BC) and OY (perpendicular to OX) be taken in its plane as the axes of X and Y respectively.



If M is the mass of the polygon then it can be divided into n isosceles triangles each of mass M/n.

$\therefore$ mass of isoscles tirangle $OBC = M/n$.

Also $\angle BOX = \angle COX = \frac{1}{2}\angle BOC$

$= \frac{1}{2}(2\pi/n) = \pi/n$.

Now the triangle OBC is equimomental to three particles each of mass $\frac{1}{3}(M/n)$ at the middle points of its sides.

$\therefore$ M.I. of the triangle OBC about OX

$$= \frac{M}{3n} \cdot O + \frac{M}{3n} \cdot \left(\frac{c}{4}\right)^2 + \frac{M}{3n}\left(\frac{c}{4}\right)^2 = \frac{Mc^2}{24n} = A_1$$

$\therefore$ M.I. of the triangle OBC about OY

$$= \frac{M}{3n}\left(\tfrac{1}{2}c \cot\frac{\pi}{n}\right)^2 + \frac{M}{3n}\left(\tfrac{1}{4}c\cot\frac{\pi}{n}\right)^2 + \frac{M}{3n}\left(\tfrac{1}{4}c\cot\frac{\pi}{n}\right)^2$$

$$= \frac{Mc^2}{8n}\cot^2\frac{\pi}{n} = B_1 \qquad \because OL = \frac{c}{2}\cot\frac{\pi}{n}$$

and P.I. of the triangle OBC about OX and OY

$= O = F_1$ $\qquad$ ($\because$ ΔOBC is symmetrical about OX)

Let OP be a line inclined at an angle α to OX, then M.I. of ΔOBC about OP

$= A_1 \cos^2\alpha + B_1 \sin^2\alpha - 2F_1 \sin 2\alpha$

$$= \left(\frac{M}{24n}c^2\right)\cos^2\alpha + \left(\frac{Mc^2}{8n}\cot^2\frac{\pi}{n}\right)\sin^2\alpha \qquad ...(1)$$

The M.I. of the other triangles about OP are obtained by replacing α by $\alpha + 2\pi/n$, $\alpha + 4\pi/n$,...... in (1), successively, then

M.I. of the polygon about OP

$$= \frac{Mc^2}{24n}\left[\cos^2\alpha + \cos^2(\alpha + 2\pi/n) + \cos^2(\alpha + 4\pi/n) + ...n \text{ terms}\right]$$

$$+ \frac{Mc^2}{8n}\cot^2\frac{\pi}{n}\left[\sin^2\alpha + \sin^2\left(\alpha + \frac{2\pi}{n}\right) + \sin^2(\alpha + 4\pi/n) + ... n \text{ terms}\right]$$

$$= \frac{Mc^2}{24n}\cdot\frac{1}{2}\left[\{1 + \cos 2\alpha\} + \left\{1 + \cos\left(2\alpha + \frac{4\pi}{n}\right)\right\} + ... n \text{ terms}\right]$$

$$+ \frac{Mc^2}{8n}\cdot\cot^2\frac{\pi}{n}\cdot\frac{1}{2}\left[\{1 - \cos 2\alpha\} + \left\{1 - \cos\left(2\alpha + \frac{4\pi}{n}\right)\right\} + ... n \text{ terms}\right]$$

$$= \frac{Mc^2}{48n}\{n + S\} + \frac{Mc^2}{16n}\cot^2\frac{\pi}{n}\{n - S\} \qquad ...(2)$$

where $S = \cos 2\alpha + \cos(2\alpha + 4\pi/n) + \cos(2\alpha + 6\pi/n) + ...n$ terms

$$= \frac{\cos\{2\alpha + (n-1)2\pi/n\} \cdot \sin 2\pi}{\sin(2\pi/n)} = 0.$$

$\therefore$ M.I. of the polygon about OP

$$= \frac{Mc^2}{48n}\cdot n + \frac{Mc^2}{16n}\cdot\left(\cot^2\frac{\pi}{n}\right)\cdot n$$

$$= \frac{Mc^2}{48}\left[\frac{\sin^2(\pi/n) + 3\cos^2(\pi/n)}{\sin^2(\pi/n)}\right]$$

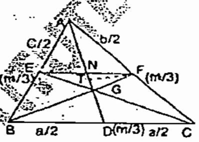
$$= \frac{Mc^2}{48}\left[\frac{(1 - \cos)(2\pi/n) + 3(1 + \cos)(2\pi/n)}{1 - \cos(2\pi/n)}\right]$$

$$= \frac{Mc^2}{24}\cdot\frac{2 + \cos(2\pi/n)}{1 - \cos(2\pi/n)}.$$

Ex. 44. *Show that there is a momental ellipse at the centre of inertia of a uniform triangle which touches the sides of the triangle at the middle points.*

Sol. Let ABC be a triangle of mass M. Let G be its C.G. and D, E, F the middle points of its sides.

Now the momental ellipse at the centre of inertia G will pass through D, E and F if the moments of interia of the triangle ABC about GD, GE and GF are equal to $\frac{Mk^4}{GD^2}, \frac{Mk^4}{GE^2}$ and $\frac{Mk^4}{GF^2}$ respectively.



Let the ΔABC be replaced by three particles each of mass $\frac{1}{3}M$ placed at the middle points D, E, F.

Then M.I. of the triangle ABC about AD

$= (M/3) \cdot EN^2 + (M/3)\, FT^2 = \frac{1}{3}M\,[(\frac{1}{2}c \sin BAD)^2 + (\frac{1}{2}b \sin CAD)^2]$

$= \frac{1}{12}M\,[c^2 \sin^2 BAD + b^2 \sin^2 CAD]$ $\qquad ...(1)$

But in triangles BAD and CAD, we have

$$\frac{\sin BAD}{a/2} = \frac{\sin B}{AD} \text{ and } \frac{\sin CAD}{a/2} = \frac{\sin C}{AD}$$

$$\therefore \sin BAD = \frac{a}{2}\cdot\frac{\sin B}{AD} \text{ and } \sin CAD = \frac{a}{2}\cdot\frac{\sin C}{AD}.$$

$\therefore$ from (1), we have

M.I. of the ΔABC about AD

$$= \frac{1}{12}M\left[\tfrac{1}{4}a^2c^2\sin^2 B + \tfrac{1}{4}a^2b^2\sin^2 C\right]\cdot\frac{1}{AD^2}$$

$$= \frac{1}{12}M(\Delta^2 + \Delta^2)\frac{1}{AD^2} = \left(\frac{M\Delta^2}{6}\right)\cdot\frac{1}{AD^2}.$$

$$= \left(\frac{M\Delta^2}{54}\right)\cdot\frac{1}{GD^2} \qquad \because GD = \tfrac{1}{3}AD$$

Similarly M.I. of the triangle about $GE = \left(\frac{M\Delta^2}{54}\right)\cdot\frac{1}{GE^2}$

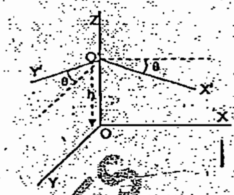
and about $GF = \left(\frac{M\Delta^2}{54}\right)\cdot\frac{1}{GF^2}$.

Thus the momental ellipse at G will pass through P, Q and R. Also GD is the diameter of the ellipse and bisect EF; $\therefore$ the tangent at P will be parallel to EF which is parallel to BC. Hence BC is tangent to the momental ellipse at P. Similarly the sides CA and AB are tangents to the momental ellipse at E and F respectively.

1.25. Principal Axes.

To find whether a given straight line is at any point of its length a principal axis of a material system. And if the line is a principal axis, then to determine the other two principal axes.

Let the given straight line OZ be taken as the axis of z and a point O on it as the origin. Let the two perpendicular lines OX and OY, perpendicular to OZ be taken as the axes of x and y respectively.



Now let the line OZ be the principal axis of the system at O' where $OO' = h$. Let $O'X'$ inclined at an angle θ to a line parallel to OX and $O'Y'$ be the other two principal axes.

Consider a particle of mass m at the point P of the material system. Let (x, y, z) and (x', y', z') be the coordinates of the point P with reference to the two sets of axes OX, OY, OZ and $O'X', O'Y', O'Z'$ respectively, then we have

$x' = x\cos\theta + y\sin\theta,\ y' = -x\sin\theta + y\cos\theta,\ z' = z - h$

We know that the necessary and sufficient conditions for the axes $O'X', O'Y', O'Z'$ to be the principal axes of the system are that the products of inertia of the system with reference to these axes taken two at a time vanish *i.e.*

$\Sigma my'z' = 0$, $\Sigma mz'x' = 0$ and $\Sigma mx'y' = 0$.

We have, $\Sigma my'z' = \Sigma m(-x\sin\theta + y\cos\theta)(z - h)$

$= (\Sigma myz)\cos\theta - (\Sigma mxz)\sin\theta + h(\Sigma mx)\sin\theta - h(\Sigma my)\cos\theta$

$= D\cos\theta - E\sin\theta + Mh(\bar{x}\sin\theta - \bar{y}\cos\theta)$ $\qquad ...(1)$

$\because \bar{x} = \frac{\Sigma mx}{\Sigma m} = \frac{\Sigma mx}{M}, \bar{y} = \frac{\Sigma my}{M}$.

$\Sigma mz'x' = \Sigma m(z - h)(x\cos\theta + y\sin\theta)$

$= (\Sigma yz)\sin\theta + (\Sigma mzx)\cos\theta - h(\Sigma mx)\cos\theta - h(\Sigma my)\sin\theta$

$= D\sin\theta + E\cos\theta - Mh(\bar{x}\cos\theta + \bar{y}\sin\theta)$ $\qquad ...(2)$

and $\Sigma mx'y' = \Sigma m(x\cos\theta + y\sin\theta)(-x\sin\theta + y\cos\theta)$

$= [(\Sigma my^2) - (\Sigma mx^2)]\sin\theta\cos\theta + (\Sigma mxy)(\cos^2\theta - \sin^2\theta)$

$= \frac{1}{2}[\Sigma m(y^2 + z^2) - \Sigma m(x^2 + y^2)]\sin 2\theta + (\Sigma mxy)\cos 2\theta$

$= \frac{1}{2}(A - B)\sin 2\theta + F\cos 2\theta$ $\qquad ...(3)$

Now $\Sigma mx'y' = 0$, if $\frac{1}{2}(A - B)\sin 2\theta + F\cos 2\theta = 0$

$$\text{or } \tan 2\theta = \frac{2F}{B - A} \text{ or } \theta = \frac{1}{2}\tan^{-1}\left(\frac{2F}{B - A}\right) \qquad ...(4)$$

Also $\Sigma my'z' = 0$, and $\Sigma mz'x' = 0$, if

$D\cos\theta - E\sin\theta + Mh(\bar{x}\sin\theta - \bar{y}\cos\theta) = 0$

and $D\sin\theta + E\cos\theta - Mh(\bar{x}\cos\theta + \bar{y}\sin\theta) = 0$

$$\therefore Mh = \frac{E\sin\theta - D\cos\theta}{\bar{x}\sin\theta - \bar{y}\cos\theta} = \frac{D\sin\theta + E\cos\theta}{\bar{x}\cos\theta + \bar{y}\sin\theta}$$

$$\text{Thus } Mh = \frac{E\sin\theta - D\cos\theta}{\bar{x}\sin\theta - \bar{y}\cos\theta} = \frac{D\sin\theta + E\cos\theta}{\bar{x}\cos\theta + \bar{y}\sin\theta}$$

$$= \frac{(E\sin\theta - D\cos\theta)\sin\theta + (D\sin\theta + E\cos\theta)\cos\theta}{(\bar{x}\sin\theta - \bar{y}\cos\cos\theta)\sin\theta + (\bar{x}\cos\theta + \bar{y}\sin\theta)\cos\theta} = \frac{E}{\bar{x}}$$

$$\text{Also } Mh = \frac{E\sin\theta - D\cos\theta}{\bar{x}\sin\theta - \bar{y}\cos\theta} = \frac{D\sin\theta + E\cos\theta}{\bar{x}\cos\theta + \bar{y}\sin\theta}$$

$$= \frac{(E\sin\theta - D\cos\theta)(-\cos\theta) + (D\sin\theta + E\cos\theta)\sin\theta}{(\bar{x}\sin\theta - \bar{y}\cos\theta)(-\cos\theta) + (\bar{x}\cos\theta + \bar{y}\sin\theta)\sin\theta} = \frac{D}{\bar{y}}$$

$$\therefore Mh = \frac{E}{\bar{x}} = \frac{D}{\bar{y}} \qquad ...(5)$$

Thus the condition that the axis OZ may be the principal axis of the system at some point of its length is that

$$\frac{E}{\bar{x}} = \frac{D}{\bar{y}} \qquad ...(6)$$

And if condition (6) is satisfied then the point O' where the line OZ is the principal axis is given by

$$OO' = h = \frac{E}{M\bar{x}} = \frac{D}{M\bar{y}} \qquad ...(7)$$

Cor. 1. *If an axis passes through the C.G. of a body and is a principal axis at any point of its lenght, then it is a principal axis at all points of its length.*

Let z axis be a principal axis at O, then $D = E = 0$. $\therefore$ from (7), we get $h = 0$. Which implies that there is no such other point as O'. But if z-axis is a principal axis at O and passes through the C.G. of the body then $\bar{x} = 0, \bar{y} = 0$ and $D = E = 0$, and from (7), we see that h becomes indeterminate.

Hence if an axis passes through the C.G. of a body and is a principal axis at any point of its length, then it is a principal axis at all points of its length.

Cor. 2. *Through each point in the plane of a lamina, there exist a pair of principal axes of the lamina.*

Let a line through any point O of the lamina and perpendicular to its plane be taken as the axis of z. In this case $\bar{z}$ (z coordinate of the C. G. of the body) $=0$, $\therefore D=0=E$. Thus eq. (6) is satisfied for every point O in the plane of the lamina. Also from (7), $h=0$.

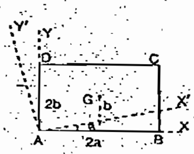
Thus z-axis (the line perpendicular to the plane of the lamina) is a principal axis of the lamina at the point O where it intersects the lamina and the other two principal axes will be the axes through O in the place of the lamina.

EXAMPLES

Ex. 45. (a). *The lengths AB and AD of the sides of a rectangle ABCD are 2a and 2b; show that the inclination to AB of one of the principal axes at A is* $\frac{1}{2}\tan^{-1}\dfrac{3ab}{2(a^2-b^2)}$.

(b) *Find the principal axes at a corner of a square.*

Sol. (a) Let AB and AD be taken as the axes of x and y respectively and z axis, a line through the corner A and perpendicular to the plane of the rectangle.



Then A = M.I. of the rectangle about AB

= M.I. of the rectangle about the axis parallel to AB through C.G. 'G'

+ M.I. of whole mass M at G about AB.

$=\frac{1}{3}Mb^2+Mb^2=\frac{4}{3}Mb^2$.

Similarly B = M.I. of the rectangle about AD

$=\frac{1}{3}Ma^2+Ma^2=\frac{4}{3}Ma^2$.

and F = P.I. of the rectangle about AB and AD

= P.I. of the rectangle about axes parallel to AX, AY through C.G. 'G'

+ P. I. of whole mass M at G about AB and AD.

$=0+M.a.b=Mab$

If the principal axis at A is inclind at an angle θ to AB, then

$$\tan 2\theta=\frac{2F}{B-A}=\frac{2Mab}{\frac{4}{3}M(a^2-b^2)}=\frac{3ab}{2(a^2-b^2)}$$

$$\therefore \theta=\tfrac{1}{2}\tan^{-1}\frac{3ab}{2(a^2-b^2)}$$

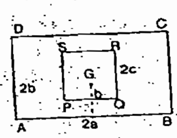
(b) Proceed as in (a). Here $2b=2a$

$\therefore \theta=\frac{1}{2}\tan^{-1}\infty=\frac{\pi}{4}$

Ex. 46. *A uniform rectangular plate whose sides are of lengths 2a, 2b has a portion cut out in the form of a square whose centre is the centre of the rectangle and whose mass is half the mass of the plate. Show that the axes of greatest and least M.I. at a corner of the rectangle make angles* θ, $\frac{1}{2}\pi+\theta$ *with a side, where*

$$\tan 2\theta=\frac{6}{5}\frac{ab}{a^2-b^2}$$

Sol. Let M be the mass of the rectangle $ABCD$ of sides $AB=2a$, $AD=2b$ and let $2c$ be the side of the square $PQRS$ cut out from it with its centre at the centre of the rectangle such that the mass of square $=\frac{1}{2}M$.



A = M.I. of the remaining portion about AB

= M.I. of the rectangle about AB − M.I. of the square about AB

$=(\frac{1}{3}Mb^2+Mb^2)-[\frac{1}{3}(\frac{1}{2}M)c^2+(\frac{1}{2}M)b^2]=\frac{1}{6}M(5b^2-c^2)$

Similarly,

B = M.I. of the remaining portion about $AD=\frac{1}{6}M(5a^2-c^2)$.

F = P.I. of the remaining portion about AB and AD

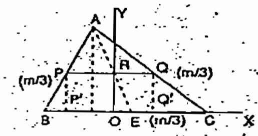
$=(0+Mab)-(0+\frac{1}{2}Mab)=\frac{1}{2}Mab$.

$\therefore$ If the principal axes in the plane of the rectangle at O make angles θ and $\frac{1}{2}\pi+\theta$ to the sides AB, then

$$\tan 2\theta=\frac{2F}{B-A}=\frac{Mab}{\frac{1}{6}M(5a^2-5b^2)}=\frac{6}{5}\frac{ab}{a^2-b^2}$$

Ex. 47. *ABC is a triangular area and AD is perpendicular to BC and AE is a median. O is the middle point of DE, show that BC is a principal axis of the triangle at O.*

Sol. Let O be the middle point of DE where AD and AE are the perpendiculars from A on BC and the median respectively. Let the lines OX and OY along and perpendicular to BC be taken as the axes of reference.



Let P and Q be the middle points of AB, and AC respectively the PQ is parallel to BC and is bisected at the point R where the median AE meets OY.

If m is the mass of the ΔABC then it can be replaced by three particles each of mass $m/3$ at the middle points $m/3$ at the middle points E, P, Q of the sides of the triangle.

P.I. of the ΔABC about OX and OY

= P.I. of masses $m/3$ each at E, P and Q about OX and OY

$=\frac{m}{3}OE.0+\frac{m}{3}OQ'.QQ'+\frac{m}{3}(-OP').PP'$

$=(m/3)\frac{1}{2}PQ(QQ'-PP')$ $\qquad \because OP'=OQ'=\frac{1}{2}PQ$

$=0$, $\qquad \because PP'=QQ'$.

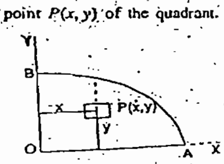
Thus the P.I. of the triangle vanishes about BC and perpendicular to BC at O. Hence BC is the principal axis of the triangle ABC at O.

Ex. 48. *Show that at the centre of a quadrant of an ellipse, the principle axis in its plane are inclined at an angle* $\frac{1}{2}\tan^{-1}\left(\frac{4}{\pi}\frac{ab}{a^2-b^2}\right)$ *to the axis.*

Sol. Let OAB be the quadrant of an ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Let $\delta x\,\delta y$ be an elementary area at the point $P(x,y)$ of the quadrant. Then A = M.I. of the quadrant about OX



$=\int_{x=0}^{a}\int_{y=0}^{(b/a)\sqrt{(a^2-x^2)}}\rho y^2\,dx\,dy$

$=\frac{\rho b^3}{3a^3}\int_0^a(a^2-x^2)^{3/2}\,dx$ (Put $x=a\sin\theta$)

$=\frac{\rho b^3}{3a^3}a^4\int_0^{\pi/2}\cos^4\theta\,d\theta=\frac{\rho b^3a^4}{3a^3}\frac{\Gamma(\frac{5}{2})\Gamma(\frac{1}{2})}{2\,\Gamma(3)}$

$=\frac{1}{16}\rho\pi b^3a=\frac{1}{4}Mb^2$. $\qquad \because M$ (mass of quadrant) $=\frac{\rho}{4}\pi ab$.

B = M.I. of the quadrant about OY

$=\int_{x=0}^{a}\int_{y=0}^{(b/a)\sqrt{(a^2-x^2)}}\rho x^2\,dx\,dy=\rho\frac{b}{a}\int_0^a x^2\sqrt{(a^2-x^2)}\,dx$

$=\frac{1}{4}Ma^2$. (Put $x=a\sin\theta$)

F = P.I. of the quadrant about OX and OY

$=\int_{x=0}^{a}\int_{y=0}^{(b/a)\sqrt{(a^2-x^2)}}\rho xy\,dx\,dy=\frac{1}{2}\rho\frac{b^2}{a^2}\int_0^a x(a^2-x^2)\,dx=\frac{Mab}{2\pi}$.

If the principal axes are inclined at an angle θ to OX and OY, then

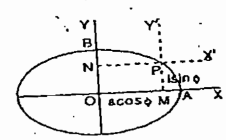
$\tan 2\theta=\frac{2F}{B-A}=\frac{4ab}{\pi(a^2-b^2)}$, $\therefore \theta=\frac{1}{2}\tan^{-1}\left(\frac{4}{\pi}\frac{ab}{a^2-b^2}\right)$.

Ex. 49. *Find the principal axes of an elliptic area at any point of its bounding arc.*

Sol. Let $P(a\cos\phi, b\sin\phi)$ be a point on the arc of an elliptic area bounded by the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Consider PX' and PY' axes parallel to the axes of the ellipse. Then



A = M.I. of the elliptic area about PX'

$=\frac{1}{4}Mb^2+M(PM)^2$

$=M(\frac{1}{4}b^2+b^2\sin^2\phi)$.

B = M.I. of the elliptic area about PY'

$=\frac{1}{4}Ma^2+M(PN)^2=M(\frac{1}{4}a^2+a^2\cos^2\theta)$.

and F = P.I. of the elliptic area about PX' and PY'

$=0+M.PM.PN=M\,ab\cos\phi\sin\phi$.

$\therefore$ If the principal axes at P make an angle θ with OX and OY then

$$\tan 2\theta=\frac{2F}{B-A}=\frac{2M\,ab\cos\phi\sin\phi}{M(\frac{1}{4}a^2+a^2\cos^2\phi)-M(\frac{1}{4}b^2+b^2\sin^2\phi)}$$

$$\therefore \theta=\tfrac{1}{2}\tan^{-1}\left[\frac{8ab\tan\phi}{(a^2-b^2)\sec^2\phi+4a^2-4b^2\tan^2\phi}\right]$$

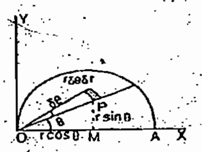
$=\frac{1}{2}\tan^{-1}\left[\frac{8ab\tan\phi}{(5a^2-b^2)+(a^2-5b^2)\tan^2\phi}\right]$.

Ex. 50. *Show that at an extremity of the bounding diameter of a semi-circular lamina the principal axis makes an angle $\frac{1}{2}\tan^{-1}(8/3\pi)$ to the diameter.*

Sol. Let the axis of x and y be taken along the diameter OA and perpendicular to OA at O in the palne of the lamina.

∴ Equation of the semi-circular lamina is $r=2a\cos\theta$.

Let $\rho r\delta\theta\delta r$ be the mass of an elementary area at P.



∴ A = M.I. of the lamina about OX

$=\int_{\theta=0}^{\pi/2}\int_{r=0}^{2a\cos\theta}(r\sin\theta)^2 . \rho r d\theta dr$

$=\frac{1}{4}(2a)^4\rho\int_0^{\pi/2}\sin^2\theta\cos^4\theta\, d\theta$

$=4\rho a^4\frac{\Gamma(\frac32)\Gamma(\frac52)}{2\Gamma(4)}=\frac{1}{8}\pi\rho a^4$

B = M.I. of the lamina about OY

$=\int_0^{\pi/2}\int_0^{2a\cos\theta}(r\cos\theta)^2\rho r d\theta dr=\frac14(2a)^4\rho\int_0^{\pi/2}\cos^6\theta\, d\theta$

$=4\rho a^4\cdot\frac{\Gamma(\frac12)\Gamma(\frac72)}{2\Gamma(4)}=\frac{5}{8}\pi\rho a^4$.

and F = P.I. of the lamina about OX and OY

$=\int_0^{\pi/2}\int_0^{2a\cos\theta}(r\cos\theta).(r\sin\theta).\rho r d\theta dr$

$=\frac14\rho(2a)^4\int_0^{\pi/2}\cos^5\theta\sin\theta\, d\theta=\frac23\rho a^4$.

∴ If the principal axis make an angle θ' to OX at O then

$\tan 2\theta'=\frac{2F}{B-A}=\frac{8}{3\pi}$

∴ $\theta'=\frac12\tan^{-1}\left(\frac{8}{3\pi}\right)$

Ex. 51. *Show that hte principal axes at the node of a half-loop of the lemniscate $r^2=a^2\cos 2\theta$ are inclined to the initial line at angles*

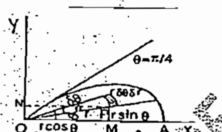
$$\frac12\tan^{-1}\frac12 \text{ and } \frac{\pi}{2}+\frac12\tan^{-1}\frac12.$$

Sol. The equation of the lemniscate is

$r^2=a^2\cos 2\theta$.

Consider an element of area $r\delta\theta\delta r$ at $P(r,\theta)$.

δm = Mass of the elementary area $=\rho r\delta\theta\delta r$.



∴ A = M.I. of half loop of the lemniscate about OX

$\int_{\theta=0}^{\pi/4}\int_{r=0}^{a\sqrt{(\cos 2\theta)}}PM^2 . \rho r d\theta dr=\int_{\theta=0}^{\pi/4}\int_{r=0}^{a\sqrt{(\cos 2\theta)}}r^2\sin^2\theta . \rho r d\theta dr$

$=\rho\int_0^{\pi/4}\left[\frac14 r^4\right]_0^{a\sqrt{(\cos 2\theta)}}\sin^2\theta\, d\theta=\frac14\rho a^4\int_0^{\pi/4}\cos^2 2\theta\sin^2\theta\, d\theta$

$=\frac18\rho a^4\int_0^{\pi/4}\cos^2 2\theta(1-\cos 2\theta)\, d\theta$

$=\frac1{16}\rho a^4\int_0^{\pi/2}(\cos^2 t-\cos^3 t)\, dt$ Putting $2\theta=t$

$=\frac1{16}\rho a^4\left[\frac{\Gamma(\frac32)\Gamma(\frac12)}{2\Gamma(2)}-\frac{\Gamma(2)\Gamma(\frac12)}{2\Gamma(\frac52)}\right]$

$=\frac1{16}\rho a^4\left(\frac\pi4-\frac23\right)=\frac{\rho a^4}{192}(3\pi-8)$

B = M.I. of half loop of the lemniscate about OY

$=\int_0^{\pi/4}\int_0^{a\sqrt{(\cos 2\theta)}}PN^2 . \rho r d\theta dr=\int_0^{\pi/4}\int_0^{a\sqrt{(\cos 2\theta)}}r^2\cos^2\theta . \rho r d\theta dr$

$=\frac14\rho a^4\int_0^{\pi/4}\cos^2 2\theta . \cos^2\theta\, d\theta=\frac18\rho a^4\int_0^{\pi/4}\cos^2 2\theta(1+\cos 2\theta)\, d\theta$

$=\frac{\rho a^4}{192}(3\pi+8)$. (As above)

and F = P.I. of half loop of the lemniscate about OX, OY

$=\int_0^{\pi/4}\int_0^{a\sqrt{(\cos 2\theta)}}PM . PN . \rho r d\theta dr$

$=\int_0^{\pi/4}\int_0^{a\sqrt{(\cos 2\theta)}}r\sin\theta . r\cos\theta . \rho r d\theta dr$

$=\frac14\rho a^4\int_0^{\pi/4}\cos^2 2\theta . \cos\theta\sin\theta\, d\theta=\frac18\rho a^4\int_0^{\pi/4}\cos^2 2\theta . \sin 2\theta\, d\theta$

$=\frac18\rho a^4\left[-\frac16\cos^3 2\theta\right]_0^{\pi/4}=\frac1{48}\rho a^4$.

∴ If the principal axis at O make an angle ϕ to OX then

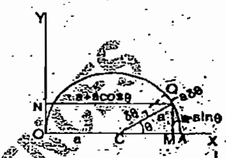
$\phi=\frac12\tan^{-1}\frac{2F}{B-A}=\frac12\tan^{-1}\left\{\frac{8}{(3\pi+8)-(3\pi-8)}\right\}=\frac12\tan^{-1}\frac12$

The other principal axis being at right angles to this principal axis will be inclined to OX at angle $\pi/2+\frac12\tan^{-1}\frac12$.

Ex. 52. *A wire is in the form of a semi-circle of radius a. Show that at an end of its diameter the principal axes in its plane are inclined to the diameter at angles*

$$\frac12\tan^{-1}\frac4\pi \text{ and } \frac\pi2+\frac12\tan^{-1}\frac4\pi.$$

Sol. Let C be the centre and OA the diameter of a semi-circular wire of radius a. Let the axis OX and OY be taken along and perpendicular to the diameter OA.



Consider an elementary arc $a\delta\theta$ at P, then its mass, $\delta m=\rho a\delta\theta$.

where $\rho=\frac{M}{(\pi a)}$

∴ A = M.I. of the wire about OX

$=\int_0^\pi PM^2 . \rho a d\theta=\int_0^\pi a^2\sin^2\theta . \rho a d\theta=\frac12\rho a^3\int_0^\pi(1-\cos 2\theta)\, d\theta$

$=\frac12\rho a^3\left[\theta-\frac12\sin 2\theta\right]_0^\pi=\frac12\pi\rho a^3=\frac12 Ma^2$.

B = M.I. of the wire about OY

$=\int_0^\pi PN^2 . \rho a d\theta=\int_0^\pi(a+a\cos\theta)^2 . \rho a d\theta$

$=\rho a^3\int_0^\pi(1+2\cos\theta+\cos^2\theta)\, d\theta$

$=\rho a^3\int_0^\pi\{1+2\cos\theta+\frac12(1+\cos 2\theta)\}\, d\theta$

$=\frac12\rho a^3\int_0^\pi(3+4\cos\theta+\cos 2\theta)\, d\theta$

$=\frac12\rho a^3\left[3\theta+4\sin\theta+\frac12\sin 2\theta\right]_0^\pi$

$=\frac32\pi\rho a^3=\frac32 Ma^2$.

and F = P.I. of the wire about OX and OY

$=\int_0^\pi PM . PN . \rho a d\theta=\int_0^\pi a\sin\theta . (a+a\cos\theta) . \rho a d\theta$

$=\rho a^3\int_0^\pi(\sin\theta+\frac12\sin 2\theta)\, d\theta=\rho a^3\left[-\cos\theta-\frac14\cos 2\theta\right]_0^\pi$

$=2\rho a^3=\frac2\pi Ma^2$.

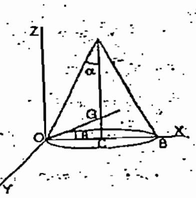
∴ If the principal axis at O make an angle θ to OX, then

$\theta=\frac12\tan^{-1}\frac{2F}{B-A}=\frac12\tan^{-1}\left\{\frac{\frac4\pi Ma^2}{(\frac32-\frac12)Ma^2}\right\}=\frac12\tan^{-1}\frac4\pi$.

The other principal axis being at right angles to this principal axis will be inclined to OX at angle $\frac\pi2+\frac12\tan^{-1}\frac4\pi$.

Ex. 53. *Find the principal axes of a right circular cone at a point on the circumference of the base, and show that one of them will pass through its C.G. if the vertical angle of the cone is $2\tan^{-1}\frac12$.*

Sol. Let O be a point on the circumference of the base of a right circular cone of mass M, height h and semi-vertical angle α. Take the axis OX along the diameter OB of the base, axis OY perpendicular to OB and in the plane of the base and axis OZ perpendicular to the base of the cone.



Then from Ex. 36 on page 51, we have

A = M.I. of the cone about OX

$=\frac{M}{20}(3a^2+2h^2)$

B = M.I. of the cone about $OY=\frac{M}{20}(23a^2+2h^2)$.

C = M.I. of the cone about $OZ=\frac{13}{10}Ma^2$

D = P.I. about $OY, OZ = 0$,
E = P.I. about $OZ, OX = \frac{1}{4} Mah$, and
F = P.I. about $OX, OY = 0$.
Here $D = 0$ and $F = 0$, therefore the axis OY will be the principal axis at O. Other two principal axes will be in the xz plane. If one of these principal axes is inclined at an angle θ to OX in xz plane, then

$$\tan 2\theta = \frac{2E}{C-A} = \frac{\frac{1}{2} Mah}{\frac{13}{10} Ma^2 - \frac{M}{20}(3a^2 + 2h^2)} = \frac{10ah}{23a^2 - 2h^2} \quad ...(1)$$

The other principal axis will be perpendicular to this principal axis in xz plane.
2nd Part. If one of the principal axis pass through the C.G. 'G' of the cone, then

$$\tan\theta = \frac{CG}{OC} = \frac{h}{4a}$$

$$\therefore \tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta} = \frac{8ah}{16a^2 - h^2} \quad ...(2)$$

$\therefore$ From (1) and (2), we have

$$\frac{10ah}{23a^2 - 2h^2} = \frac{8ah}{16a^2 - h^2}$$

or $5(16a^2 - h^2) = 4(23a^2 - 2h^2)$
or $3h^2 = 12a^2$ or $h = 2a$.

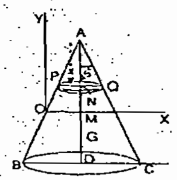
$$\therefore \tan\alpha = \frac{OC}{AC} = \frac{a}{h} = \frac{1}{2}.$$

i.e verticle angle of the cone $= 2\alpha = 2\tan^{-1}\frac{1}{2}$.

Ex. 54. *If the vertical angle of the cone is 90° the point at which a generator is a principal axis divides the generator in the ratio 3 : 7.*

Sol. Let h be the height of a cone of vertical angle 90°.

Let the generator AB be the principal axis of the cone at the point O. Consider the section of the cone through the generator AB and the axis AD. Take OX and OY, the axis of x and axis of y, perpendicular to AD and parallel to AD respectively in this section and OZ the z-axis perpendicular to this section of the cone.



Since the cone is symmetrical about OZ.
$\therefore D = O = E$.

$\therefore OZ$ is a principal axis at O. The other two principal axes at O are the generator AB and the line through O and perpendicular to generator AB in the above section of the cone.

Consider an elementary circular disc of width δx at a distance x from the vertex A and perpendicular to the axis AD. *i.e.* $AN = x$.
$\therefore$ Radius of the disc $= PN = x\tan 45° = x$.
Mass of the elementary disc, $\delta m = \rho\pi x^2\delta x$.
M.I. of this disc about $OX = \frac{1}{4} PN^2\,\delta m + MN^2\,\delta m$
$= \{\frac{1}{4}x^2 + (AM - x)^2\}\,\rho\pi x^2\delta x$.
$\therefore A$ = M.I. of the cone about OX

$$= \int_0^h \{\tfrac{1}{4}x^2 + (AM - x)^2\}\,\rho\pi x^2\,dx$$

$$= \pi\rho\int_0^h (\tfrac{5}{4}x^4 - 2AM\,.\,x^3 + AM^2\,.\,x^2)\,dx$$

$$= \pi\rho\,[\tfrac{5}{4}\,.\,\tfrac{1}{5}h^5 - 2AM\,.\,\tfrac{1}{4}h^4 + AM^2\,.\,\tfrac{1}{3}h^3]$$

$$= \frac{1}{12}\pi\rho h^3(3h^2 - 6h\,.\,AM + 4AM^2).$$

Also M.I. of the elementary disc about OY
$= \frac{1}{4} PN^2\,\delta m + OM^2\,\delta m = (\frac{1}{4}x^2 + AM^2)\,\pi\rho x^2\,\delta x$, $\because OM = AM$
$\therefore B$ = M.I. of the cone about OY

$$= \int_0^h (\tfrac{1}{4}x^2 + AM^2)\,\pi\rho x^2\,dx = \pi\rho\int_0^h (\tfrac{1}{4}x^4 + AM^2 x^2)\,dx$$

$$= \pi\rho\left(\frac{1}{10}h^4 + AM^2\frac{1}{3}h^3\right) = \frac{1}{30}\pi\rho h^3(3h^2 + 10AM^2).$$

Since the principal axis AB make an angle $AOX = 45°$ to OX.
$\therefore$ From $\tan 2\theta = \dfrac{2F}{B-A}$, we have

$$\tan 90° = \frac{2F}{B-A} \text{ or } \infty = \frac{2F}{B-A} \text{ or } B - A = 0 \text{ or } A = B.$$

$$\therefore \frac{1}{12}\pi\rho h^3(3h^2 - 6hAM + 4AM^2) = \frac{1}{30}\pi\rho h^3(3h^2 + 10AM^2).$$

or $\dfrac{1}{12}(3h^2 - 6hAM) = \dfrac{1}{30}\cdot 3h^2$

or $5(3h^2 - 6hAM) = 6h^2$ or $9h^2 = 30hAM$ or $AM = \dfrac{3}{10}h$.

From similar triangles AOM and ABD,

$$\frac{AO}{AB} = \frac{AM}{AD} = \frac{AM}{h} = \frac{3}{10}$$

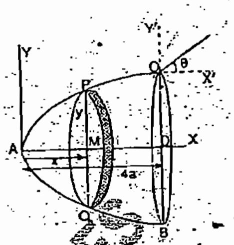
$\therefore AO = \frac{3}{10}AB$ and $OB = AB - AO = AB - \frac{3}{10}AB = \frac{7}{10}AB$.

$$\therefore \frac{AO}{OB} = \frac{3}{7}.$$

Ex. 55. *The length of the axis of a solid parabola of revolution is equal to the latus-rectum of the generating parabola. Prove that one principal axis at a point in the circular rim meets the axis of revolution at an angle* $\frac{1}{2}\tan^{-1}\frac{2}{9}$.

Sol. Let the length of L.R. of the parabola be $4a$.
$\therefore$ Length of the axis $AD = 4a$, and equation of the parabola is

$$y^2 = 4ax \quad ...(1)$$



Let O be a point in the circular rim and OX', OY' the axes parallel to AX and AY.

If the principal axis at O is inclined at an angle θ to OX' (*i.e.* to the axis of revolution AX), then

$$\theta = \frac{1}{2}\tan^{-1}\frac{2F}{B-A} \quad ...(i)$$

Consider an elementary strip PQ of width δx at a distance x from A and perpendicular to AX, then its mass
$\delta m = \rho\pi PM^2\delta x = \rho\pi y^2\,\delta x$,
where (x, y) are coordinates of the point P.
M.I. of this elementary disc about OX'
$= \frac{1}{4} PM^2\delta m + OD^2\delta m$
$= (\frac{1}{4}y^2 + OD^2)\,\rho\pi y^2\delta x$
$= \{\frac{1}{4}y^2 + (4a)^2\}\,\rho\pi y^2\delta x$
($\because$ at O, $x = AD = 4a$, $y = OD$ $\therefore OD^2 = 4a\,.\,4a$ or $OD = 4a$)
$= (2ax + 16a^2)\,4\rho\pi ax\delta x$ $\quad\because y^2 = 4ax$.
A = M.I. of the solid about OX'

$$= \int_0^{4a}(2ax + 16a^2)\,4\rho\pi\,.\,ax\,dx$$

$$= 4\pi\rho a\left[2a\frac{x^3}{3} + 16a^2\frac{x^2}{2}\right]_0^{4a}$$

$$= \frac{64 \times 32}{3}\rho\pi a^5.$$

Also M.I. of the elementary disc about OY'
$= \frac{1}{4}PM^2\delta m + MD^2\delta m + [\frac{1}{4}y^2 + (4a - x)^2]\,\rho\pi y^2\delta x$
$= [ax + (4a - x)^2]\,4\pi\rho ax\delta x = (16a^2 - 7ax + x^2)\,4\pi\rho ax\delta x$
$\therefore B$ = M.I. of the solid about OY'

$$= \int_0^{4a}(16a^2 - 7ax + x^2)\,4\pi\rho ax\,dx$$

$$= 4\pi\rho a\left[8a^2x^2 - \frac{7a}{3}x^3 + \frac{1}{4}x^4\right]_0^{4a} = \frac{1}{3} \times 64 \times 8\pi\rho a^5.$$

And P.I. of the elementary disc about OX', OY'
$= 0 + OD\,.\,MD\,.\,\delta m = 4a\,.\,(4a - x)\,\rho\pi y^2\delta x$
$= 4a\,.\,(4a - x)\,\rho\pi\,.\,4ax\delta x$.
$\therefore F$ = P.I. of the solid about OX', OY'

$$= \int_0^{4a}\rho\pi\,16a^2(4a - x)\,x\,dx$$

$$= 16\rho\pi a^2\left[2ax^2 - \frac{1}{3}x^3\right]_0^{4a} = \frac{1}{3} \times 16 \times 32\rho\pi a^5.$$

$\therefore$ From (1), we have

$$\theta = \frac{1}{2}\tan^{-1}\frac{\frac{2}{3} \times 16 \times 32\rho\pi a^5}{(\frac{1}{3} \times 64 \times 8 - \frac{1}{3} \times 64 \times 32)\,\rho\pi a^5} = \frac{1}{2}\tan^{-1}(\tfrac{2}{9}) \text{ numerically.}$$

Ex. 56. *A uniform lamina is bounded by a parabolic arc, of latus rectum 4a, and a double ordinate at a distance b from the vertex. If* $b = \frac{1}{3}a(7 + 4\sqrt{7})$, *show that two of the principal axes at the end of a latus rectum are the tangent and normal there.*

Sol. Let the equation of the parabola be

$$y^2 = 4ax \quad ...(1)$$

$\therefore$ Coordinates of the end L of L.R. LL' are $(a, 2a)$.
Differentiating (1) we, get $\dfrac{dy}{dx} = \dfrac{2a}{y}$.

$\therefore$ At $L(a, 2a)$, $\dfrac{dy}{dx} = \dfrac{2a}{2a} = 1$.

$\therefore$ Equation of the tangetn LT at L is

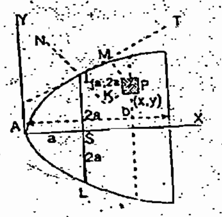
$$y - 2a = 1\,.\,(x - a) \text{ or } y - x - a = 0 \quad ...(2)$$

and the equation of the normal LN at L is

$y-2a=-\frac{1}{1}(x-a)$ ...(3)

or $y+x-3a=0$.

Consider an element $\delta x \delta y$ at the point $P(x, y)$ of the lamina, then



PM = length of perpendicular from P on tangent LT given by (2)

$\frac{y-x-a}{\sqrt{(1+1)}}=\frac{y-x-a}{\sqrt{2}}$

and PK = length of perpendicular from P on the normal LN given by (3)

$=\frac{y+x-3a}{\sqrt{2}}$.

P.I. of the element about LT and LN

$$PM \cdot PK \cdot \delta m=\left(\frac{y-x-a}{\sqrt{2}}\right)\left(\frac{y+x-3a}{\sqrt{2}}\right)\rho \delta x \delta y$$

If the tangent and normal at L are the principal axes, then the P.I. of the lamina about these will be zero.

i.e. P.I. of the lamina about LT and LN

$$=\int_{x=0}^{b}\int_{y=-2\sqrt{(ax)}}^{2\sqrt{(ax)}}\left(\frac{y-x-a}{\sqrt{2}}\right)\left(\frac{y+x-3a}{\sqrt{2}}\right)\rho\, dx dy=0$$

or $\frac{\rho}{2}\int_0^b\int_{-2\sqrt{(ax)}}^{2\sqrt{(ax)}}(y^2-4ay+(3a^2+2ax-x^2))\,dxdy=0$

or $\int_0^b\left\{\frac{1}{3}y^3-2ay^2+(3a^2+2ax-x^2)\,y\right\}_{-2\sqrt{(ax)}}^{2\sqrt{(ax)}}dx=0$

or $2\int_0^b\left\{\frac{8}{3}ax\sqrt{(ax)}+2(3a^2+2ax-x^2)\sqrt{(ax)}\right\}dx=0$

or $\int_0^b(\frac{8}{3}a^{3/2}x^{3/2}+6a^{5/2}x^{1/2}+4a^{3/2}x^{3/2}-2a^{1/2}x^{5/2})\,dx=0$

or $\left[\frac{16}{15}a^{3/2}b^{5/2}+4a^{5/2}b^{3/2}+\frac{8}{5}a^{3/2}b^{5/2}-\frac{4}{7}a^{1/2}b^{7/2}\right]=0$

or $\frac{16}{15}ab+4a^2+\frac{8}{5}ab-\frac{4}{7}b^2=0$

or $b^2-\frac{14}{3}ab-7a^2=0$

or $b=\frac{\frac{14}{3}a\pm\sqrt{\left\{\frac{196}{9}a^2+28a^2\right\}}}{2}=\frac{1}{2}\left(\frac{14}{3}\pm\frac{8}{3}\sqrt{7}\right)a$

or $b=\frac{a}{3}(7+4\sqrt{7})$. Leaving – ve sign, as b can not be negative.

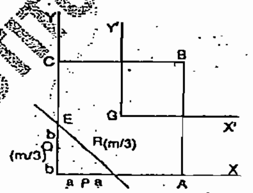
Hence if $b=\frac{a}{3}(7+4\sqrt{7})$,

then the principal axes at L are the tangents and normal there.

Ex. 57. *A uniform square lamina is bounded by the axes of x and y and the lines $x=2c, y=2c$, and a corner is cut off by the line $x/a+y/b=2$. Show that the principal axes at the centre of the square are inclined to the axis of x at angles given by*

$$\tan 2\theta=\frac{ab-2(a+b)c+3c^2}{(c-b)(a+b-2c)}$$

Sol. Let $OABC$ be the square lamina of mass M bounded by the axes and the lines $x=2c, y=2c$.



The line $\frac{x}{a}+\frac{y}{b}=2$ *i.e.* $\frac{x}{2a}+\frac{y}{2b}=1$ cut off intercepts $OD=2a$, and $OE=2b$ on the axes. Let m be the mass of the triangular lamina ODE cut off from the square. The triangle ODE can be replaced by three particles each of mass $m/3$ at the middle points P, Q, R of its sides.

Consider the lines GX', GY' through G and parallel the sides of the square as the new axes of reference. With reference to these new axes the coordinates of P are $[-(c-a), -c]$, Q are $[-c, -(c-b)]$, R are $[-(c-a), -(c-b)]$.

$\therefore$ A = M.I. of the remaining area about GX'

= M.I. of square $OABC$ about GX' – M.I. of ΔODE about GX'

= M.I. of square $OABC$ about GX'—(M.I. of three particles each of mass $m/3$ at P, Q and R)

$=\frac{1}{3}Mc^2-\frac{m}{3}\left[c^2+(c-b)^2+(c-b)^2\right]$

Similarly

B = M.I. of the remaining area about GY'

$=\frac{1}{3}Mc^2-\frac{m}{3}\left[(c-a)^2+c^2)^2+(c-a)^2\right]$

and F = P.I. of the remaining area about GX', GY'

= P.I. of the square $OABC$ about GX', GY'

—(P.I. of three particles each of mass $m/3$ at P, Q and R)

$=0-\frac{m}{3}[(c-a)c+c(c-b)+(c-a)(c-b)]$

$=-\frac{m}{3}[(ab-2(a+b)c+3c^2]$.

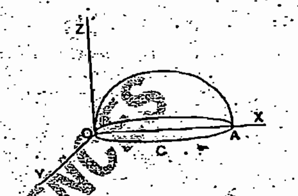
$\therefore$ If the principal axis at the centre G is inclined at an angle θ to the axis of x, then

$$\tan 2\theta=\frac{2F}{B-A}=\frac{-(m/3)[ab-2c(a+b)+3c^2]}{(m/3)[2(c-b)^2-2(c-a)^2]}$$

$$=\frac{ab-2c(a+b)+3c^2}{(a-b)(a+b-2c)}.$$

Ex. 58. *Show that one of the principal axes at a point on the circular rim of the solid hemisphere, is inclined at an angle $\tan^{-1}\frac{1}{3}$ to the radius through the point.*

Sol. Let C be the centre and OA the diameter of the circular rim of a hemisphere of radius a and mass M. Take OX and OY the axis of x and y along and perpendicular to OA in the plane of the circular rim of the hemisphere and OZ the z-axis perpendicular to this plane.



As in Ex. 35 on page 49 we have

A = M.I. of the hemisphere about $OX=\frac{2}{5}Ma^2, B=\frac{2}{5}Ma^2, C=\frac{7}{5}Ma^2$

D = P.I. about $OY, OZ=0, E=\frac{3}{8}Ma^2$ and $F=0$.

Since $D=O=F$. $\therefore$ y-axis OY is the principal axis at the point O and the other two principal axes at O lie in xz plane. If one of these principal axes make an angle θ to OX, then

$$\tan 2\theta=\frac{2E}{C-A}=\frac{\frac{3}{4}Ma^2}{(\frac{7}{5}-\frac{2}{5})Ma^2}=\frac{3}{4}$$

or $\frac{2\tan\theta}{1-\tan^2\theta}=\frac{3}{4}$

or $3\tan^2\theta+8\tan\theta-3=0$ or $(3\tan\theta-1)(\tan\theta+3)=0$

$\therefore$ $\tan\theta=\frac{1}{3}$ or $\theta=\tan^{-1}(\frac{1}{3})$ $\because$ $\tan\theta=-3\Rightarrow\theta>\pi/2$

which is inadmissible.

Ex. 59. *Show that one of the principal axes at any point on the edge of the circular base of a thin hemispherical shell is inclined at an angle $\pi/8$ to the radius through the point.*

Sol. Let OA be the diameter of the circular base of a thin hemispherical shell of radius a and mass M. Take OX, OY, OZ the axes of x, y and z as in the last Ex. 58.

As in Ex. 34 on page 48, we have

$A=\frac{2}{3}Ma^2, B=\frac{2}{3}Ma^2, C=\frac{5}{3}Ma^2, D=0, E=\frac{1}{2}Ma^2$ and $F=0$.

Since $D=O=F$, $\therefore$ OY is the principal axis at O and the other two principal axes at O will lie in xz plane. If one of these principal axes make an angle θ to OX, then

$$\theta=\frac{1}{2}\tan^{-1}\frac{2E}{C-A}=\frac{1}{2}\tan^{-1}\frac{Ma^2}{(\frac{5}{3}-\frac{2}{3})Ma^2}=\frac{1}{2}\tan^{-1}1=\frac{\pi}{8}$$

1.26. Principal Moments :

Moments of inertia of a body about its principal axes at any point are called its principal moments at that point.

The equation of the ellipsoid at any point is given by

$Ax^2+By^2+Cz^2-2Dyz-2Ezx-2Fxy=MK^4$. ...(1)

Taking the principal axes as the coordinate axes equation (1) reduces to the form

$$A'x^2+B'y^2+C'z^2=MK^4$$

Where A', B', C' are the principal moments and are the values of λ in the cubic equation

$$\begin{vmatrix} A-\lambda & H & G \\ H & B-\lambda & F \\ G & F & C-\lambda \end{vmatrix}=0.$$

This cubic equation in λ is called the reduction cubic.

EXAMPLES

Ex. 60. *If A and B be the moments of inertia of a uniform lamina about perpendicular axes OX and OY, lying in its plane, and F be the product of inertia of the lamina about these lines, show that the principal moments at O are equal to*

$$\frac{1}{2}[A+B\pm\sqrt{((A-B)^2+4F^2)}]$$

Sol. Here we consider the uniform lamina, so there will be momental ellipse at O whose equation is given by

Ax^2+By^2-2Fxy = Constant ...(1)

Taking the principal axes as the coordinate axes, equation (1) reduce to the form

$A'x^2 + B'y^2 =$ Constant. ...(2)

Equating the invariants* of (1) and (2) we have

$A' + B' = A + B$ ...(3)

and $A'B' = AB - F^2$ ...(4)

$\therefore A' - B' = \sqrt{\{(A'+B')^2 - 4A'B'\}} = \sqrt{\{(A+B)^2 - 4(AB - F^2)\}}$

or $A' - B' = \sqrt{\{(A-B)^2 + 4F^2\}}$ ...(5)

Adding and subtracting (3) and (5) we have

$A' = \frac{1}{2}[A + B + \sqrt{\{(A-B)^2 + 4F^2\}}]$

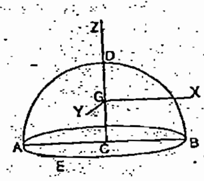
and $B' = \frac{1}{2}[A + B - \sqrt{\{(A-B)^2 + 4F^2\}}]$

i.e. the principal moments at O are equal to

$\frac{1}{2}[A + B \pm \sqrt{(A-B)^2 + 4F^2}]$

Ex. 61. *Show that for a thin hemispherical solid of radius a and mass M, the principal moments of inertia at the centre of gravity are* $\frac{83}{320}Ma^2, \frac{83}{320}Ma^2, \frac{2}{3}Ma^2$.

Sol. Let G be the centre of gravity of a hemispherical solid of radius a and mass M. If C is the centre and CD the central radius of the hemisphere, then $CG = 3a/8$.



Take GX and GY the axes through G and parallel to the plane base be taken as the axis of x and y respectively and GZ the central radius as the z-axis then

$A =$ M.I. about $GX =$ M.I. about $AB - M \cdot CG^2$

$$= \frac{2}{5}Ma^2 - M\left(\frac{3}{8}a\right)^2 = \frac{83}{320}Ma^2,$$

$$B = \text{M.I. about } GY = \frac{2}{5}Ma^2 - M\left(\frac{3a}{8}\right)^2 = \frac{83}{320}Ma^2,$$

$C =$ M.I. about $CZ = \frac{2}{5}Ma^2$.

Now coordinates of C are $(0, 0, -3a/8)$.

$D =$ P.I. about GY, GZ

$=$ P.I. about parallel lines $CB, CE -$ P.I. of M at C about GY, GZ

$= 0 - M.0.(-3a/8) = 0.$

Similarly, $E = 0, F = 0, D = 0, E = F$.

$\therefore$ GX, GY, GZ are the principal axes at G.

Hence $\frac{83}{320}Ma^2, \frac{83}{320}Ma^2, \frac{2}{5}Ma^2$ are the principal moments.

Ex. 62. *Show that for a thin hemispherical shell of radius a and mass M, the principal moments of inertia at the centre of gravity are* $\frac{5}{12}Ma^2, \frac{5}{12}Ma^2, \frac{2}{5}Ma^2$.

Sol. (Refer figure of Ex. 61).

Let G be the C.G. of the hemispherical shell of radius a and mass M. Here C.G. $= a/2$. Taking the axes of x, y, z as in Ex. 61, coordinates of C are $(0, 0, -a/2)$.

$$\therefore A = \frac{2}{3}Ma^2 - M.CG^2 = \frac{2}{3}Ma^2 - M\left(\frac{a}{2}\right)^2 = \frac{5}{12}Ma^2.$$

Similarly, $B = \frac{5}{12}Ma^2, C = \frac{2}{3}Ma^2$ and $D = 0 = E = F$.

$\because$ $D = 0 = E = F$, $\therefore$ the lines GX, GY, GZ are the principal axes at G. Thus the principal moments at G are

$\frac{5}{12}Ma^2, \frac{5}{12}Ma^2, \frac{2}{3}Ma^2$.

Ex. 63. *A uniform solid circular cone of semi-vertical angle α and height h is cut in half by a plane through its axis. Show that the principal moments of inertia at the vertex for one of the halves are* $\frac{3}{5}Mh^2(1+\frac{1}{4}\tan^2\alpha)$ *and* $\frac{3}{10}Mh^2\left(1+\frac{3}{4}\tan^2\alpha\right)$

$$\pm\frac{3}{10}Mh^2\sqrt{\left[\left(1-\frac{1}{4}\tan^2\alpha\right)^2+\left(\frac{64}{9\pi^2}\right)\tan^2\alpha\right]}$$

Sol. Let $OACBDO$ be the half cone of mass M, $ACBD$ its semi-circular base and OAB its triangular face. Take the z-axis OZ along OC, y-axis OY perpendicular to OC in the plane of the triangular face and x-axis OX perpendicular to this triangular face.

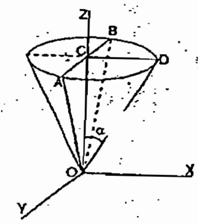
Since half cone is symmetrical about zx plane which is perpendicular to OY.

$\therefore$ $D =$ P.I. about $OY, OZ = 0$ and $F =$ P.I. about $OX, OY = 0$.

$\therefore$ OY is the principal axis at O.

$M =$ Mass of the half cone

$= \frac{1}{2}(\frac{1}{3}\rho\pi h^3\tan^2\alpha)$.



$B =$ Principal moment about OY

$$= \frac{1}{2}\left[\frac{1}{20}\rho\pi h^5(\tan^2\alpha + 4)\tan^2\alpha\right]$$

(see Ex. 23 on page 34)

$$= \frac{3}{20}Mh^2(4 + \tan^2\alpha)$$

$$= \frac{3}{5}Mh^2\left(1 + \frac{1}{4}\tan^2\alpha\right)$$

$A =$ M.I. about $OX =$ M.I. about

$$OY = \frac{3}{5}Mh^2\left(1 + \frac{1}{4}\tan^2\alpha\right)$$

$C =$ M.I. about

$$OZ = \frac{1}{2}\left[\frac{1}{10}\rho\pi h^5\tan^4\alpha\right] = \frac{3}{10}Mh^2\tan^2\alpha$$

$E =$ P.I. about OX, OZ

$$= 2\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\alpha}\int_{r=0}^{h\sec\theta}\rho r d\theta dr \cdot r\sin\theta d\theta \cdot r\cos\theta \cdot r\sin\theta\cos\phi$$

$$= 2\rho\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\alpha}\frac{1}{5}h^5\sec^5\theta \cdot \sin^2\theta\cos\theta\cos\phi d\phi d\theta$$

$$= \frac{2\rho}{5}h^5\int_{\phi=0}^{\pi/2}\int_{\theta=0}^{\alpha}\tan^2\theta\sec^2\theta\cos\phi d\phi d\theta$$

$$= \frac{2\rho}{5}h^5\int_{\phi=0}^{\pi/2}\left[\frac{1}{3}\tan^3\theta\right]_0^{\alpha}\cos\phi d\phi = \frac{2\rho}{15}h^5\tan^3\alpha \cdot [\sin\phi]_0^{\pi/2}$$

$$= \frac{4}{5\pi}Mh^2\tan\alpha$$

If the principal axis (other than OY) make an angle θ to OZ, then

$$\tan 2\theta = \frac{2E}{A-C} = \frac{(8/5\pi)Mh^2\tan\alpha}{\frac{3}{5}Mh^2(1+\frac{1}{4}\tan^2\alpha) - \frac{3}{10}Mh^2\tan^2\alpha} = \frac{(8/3\pi)\tan\alpha}{1-\frac{1}{4}\tan^2\alpha}$$

$$\sin 2\theta = \frac{(8/3\pi)\tan\alpha}{\sqrt{[(64/9\pi^2)\tan^2\alpha + (1-\frac{1}{4}\tan^2\alpha)^2]}}$$

and $$\cos 2\theta = \frac{1-(1/4)\tan^2\alpha}{\sqrt{[(64/9\pi^2)\tan^2\alpha + (1-\frac{1}{4}\tan^2\alpha)^2]}}$$

Hence the other principal moment

$= C\cos^2\theta + A\sin^2\theta - 2E\sin\theta\cos\theta$

$= \frac{1}{2}C(1+\cos 2\theta) + \frac{1}{2}A(1-\cos 2\theta) - E\sin 2\theta$

$$= \frac{3}{20}Mh^2\tan^2\alpha(1+\cos 2\theta) + \frac{3}{10}Mh^2(1+\frac{1}{4}\tan^2\alpha)(1-\cos 2\theta) - \frac{4}{5\pi}Mh^2\tan\alpha\sin 2\theta$$

$$= \frac{3}{10}Mh^2(1+\frac{3}{4}\tan^2\alpha) - \frac{3}{10}Mh^2(1-\frac{1}{4}\tan^2\alpha)\cos 2\theta - \frac{4}{5\pi}Mh^2\tan\alpha\sin 2\theta$$

$$= \frac{3}{10}Mh^2(1+\frac{3}{4}\tan^2\alpha) - \frac{3}{10}Mh^2(1-\frac{1}{4}\tan^2\alpha) \cdot \frac{(1-\frac{1}{4}\tan^2\alpha)}{\sqrt{[(64/9\pi^2)\tan^2\alpha + (1-\frac{1}{4}\tan^2\alpha)^2]}}$$

$$- \frac{4}{5\pi}Mh^2\tan\alpha \cdot \frac{(8/3\pi)\tan\alpha}{\sqrt{[(64/9\pi^2)\tan^2\alpha + (1-\frac{1}{4}\tan^2\alpha)^2]}}$$

$$= \frac{3}{10}Mh^2(1+\frac{3}{4}\tan^2\alpha) - \frac{3}{10}Mh^2\frac{(1-\frac{1}{4}\tan^2\alpha)^2 + (64/9\pi^2)\tan^2\alpha}{\sqrt{[(64/9\pi^2)\tan^2\alpha + (1-\frac{1}{4}\tan^2\alpha)^2]}}$$

$$= \frac{3}{10}Mh^2(1+\frac{3}{4}\tan^2\alpha) - \frac{3}{10}Mh^2\sqrt{[(1-\frac{1}{4}\tan^2\alpha)^2 + (64/9\pi^2)\tan^2\alpha]}.$$

Replacing θ by $\theta + \pi/2$, the other principal moment is

$= C\sin^2\theta + A\cos^2\theta + 2E\cos\theta\sin\theta$

$$= \frac{3}{10}Mh^2(1+\frac{3}{4}\tan^2\alpha) + \frac{3}{10}Mh^2\sqrt{[(1-\frac{1}{4}\tan^2\alpha) + (64/9\pi^2)\tan^2\alpha]}.$$

Ex. 64. *Prove that the principal radii of gyration at the C.G. of a triangle are the roots of the equation*

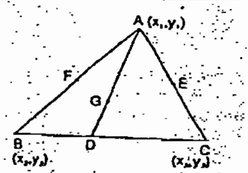
$$x^4 - \frac{a^2+b^2+c^2}{36}x^2 + \frac{\Delta^2}{108} = 0$$

where Δ is the area of the triangle.

Sol. Let ABC be the triangle of mass M. Taking the centre of gravity of the triangle G as the origin and the principal axes through G as axes. Let $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ be the coordinates of the vertices A, B, C respectively.

Since C.G. 'G' is taken as origin and

$\left(\frac{x_1+x_2+x_3}{3}, \frac{y_1+y_2+y_3}{3}\right)$



are the coordinates of C.G.,

$\therefore\ x_1+x_2+x_3=0$

and $y_1+y_2+y_3=0$ ...(1)

Thus $(x_1+x_2+x_3)^2=0$

or $x_1^2+x_2^2+x_3^2=-2(x_1x_2+x_2x_3+x_3x_1)$ ...(2)

Similarly $y_1^2+y_2^2+y_3^2=-2(y_1y_2+y_2y_3+y_3y_1)$ ...(3)

Now, $BC^2=a^2=(x_3-x_2)^2+(y_3-y_2)^2$

$CA^2=b^2=(x_1-x_3)^2+(y_1-y_3)^2$

and $AB^2=c^2=(x_2-x_1)^2+(y_2-y_1)^2$

$\therefore\ a^2+b^2+c^2=2(x_1^2+x_2^2+x_3^2)+2(y_1^2+y_2^2+y_3^2)$

$-2(x_1x_2+x_2x_3+x_3x_1)-2(y_1y_2+y_2y_3+y_3y_1)$

$=2(x_1^2+x_2^2+x_3^2)+2(y_1^2+y_2^2+y_3^2)+(x_1^2+x_2^2+x_3^2)+(y_1^2+y_2^2+y_3^2)$

or $a^2+b^2+c^2=3(x_1^2+x_2^2+x_3^2+y_1^2+y_2^2+y_3^2)$ ...(4)

The triangle ABC may be replaced by three particles each of mass $M/3$ placed at the middle points D, E, F of the sides whose coordinates are $\left(\frac{x_2+x_3}{2}, \frac{y_2+y_3}{2}\right)$, $\left(\frac{x_3+x_1}{2}, \frac{y_3+y_1}{2}\right)$ and $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$ respectively.

$\therefore\ A=$ Principal moment about x axis.

$=\frac{1}{3}M\left(\frac{x_2+x_3}{2}\right)^2+\frac{1}{3}M\left(\frac{x_3+x_1}{2}\right)^2+\frac{1}{3}M\left(\frac{x_1+x_2}{2}\right)^2$

$=\frac{M}{12}[2(x_1^2+x_2^2+x_3^2)+2(x_1x_2+x_2x_3+x_3x_1)]$

$=\frac{1}{12}M(x_1^2+x_2^2+x_3^2)$ Using (2)

Similarly $B=$ Principal moment about y-axis

$=\frac{1}{12}M(y_1^2+y_2^2+y_3^2)$

$\therefore\ A+B=\frac{1}{12}M(x_1^2+x_2^2+x_3^2+y_1^2+y_2^2+y_3^2)$

or $A+B=\frac{1}{36}M(a^2+b^2+c^2)$ Using (4)

Since x, y axes through G are principal axes.

$\therefore$ P.I. about x, y axes $=0$

or $\frac{M}{3}\left(\frac{x_2+x_3}{2}\right)\left(\frac{y_2+y_3}{2}\right)+\frac{M}{3}\left(\frac{x_3+x_1}{2}\right)\left(\frac{y_3+y_1}{2}\right)+\frac{M}{3}\left(\frac{x_1+x_2}{2}\right)\left(\frac{y_1+y_2}{2}\right)=0$

or $(x_2+x_3)(y_2+y_3)+(x^3+x^1)(y_3+y_1)+(x_1+x_2)(y_1+y_2)=0$

or $(-x_1)(-y_1)+(-x_2)(-y_2)+(-x_3)(-y_3)=0$ Using (1)

or $x_1y_1+x_2y_2+x_3y_3=0$ ...(5)

Also $AB=\frac{1}{144}M^2(x_1^2+x_2^2+x_3^3)(y_1^2+y_2^2+y_3^3)$

$=\frac{1}{144}M^2[(x_1y_1+x_2y_2+x_3y_3)^2+(x_1y_2-x_2y_1)^2+(x_2y_3-x_3y_2)^2+(x_3y_1-x_1y_3)^2]$

NNow $\Delta=$ area of the triangle ABC

$=\frac{1}{2}[x_1(y_2-y_3)+x_2(y_3-y_1)+x_3(y_1-y_2)]$

or $2\Delta=x_1(y_2+y_1+y_2)+x_2(-y_1-y_2-y_1)+(-x_1-x_2)(y_1-y_2)$ Using (1)

$=3(x_1y_2-x_2y_1)$ or $x_1y_2-x_2y_1=\frac{2}{3}\Delta$.

Similarly, $x_2y_3-x_3y_2=\frac{2}{3}\Delta$ and $x_3y_1-x_1y_3=\frac{2}{3}\Delta$.

$\therefore\ AB=\frac{1}{144}M^2[0+(\frac{2}{3}\Delta)^2+(\frac{2}{3}\Delta)^2+(\frac{2}{3}\Delta)^2]=\frac{M^2\Delta^2}{108}$ ...(6)

If k_1 and k_2 are the principal radii of gyration, then $A=Mk_1^2$ and $B=Mk_2^2$

$\therefore\ k_1^2+k_2^2=\frac{1}{M}(A+B)=\frac{1}{36}(a^2+b^2+c^2)$, [from (4)]

and $k_1^2 \cdot k_2^2=\frac{AB}{M^2}=\frac{\Delta^2}{108}$. [from (6)]

$\therefore\ k_1^2$ and k_2^2 are the roots of the equation

$x^4-(k_1^2+k_2^2)x^2+(k_1^2 \cdot k_2^2)=0$

or $x^4-\frac{1}{36}(a^2+b^2+c^2)+\frac{1}{108}\Delta^2=0$.

Ex. 65. *Three rods AB, BC, CD, each of mass m and length 2a are such that each is perpendicular to the other two. Show that the principal moments of inertia at the centre of mass are ma^2, $\frac{11}{3}ma^2$ and $4ma^2$.*

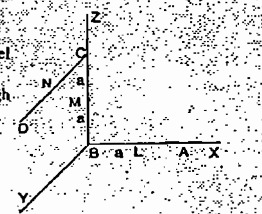
Sol. Let BY be a line parallel to CD. Taking BA, BY, BC as the axes of x, y, z respectively, the coordinates of middle points L, M, N of rods AB, BC, CD are $(a, 0, 0)$, $(0, 0, a)$ and $(0, a, 2a)$ respectively.

If $(\bar{x}, \bar{y}, \bar{z})$ are the coordinates of the C.G. 'G' of the rods AB, BC, CD each of mass m, then

$\bar{x}=\frac{m.a+m.0+m.0}{m+m+m}=\frac{1}{3}a, \bar{y}=\frac{m.0+m.0+m.a}{m+m+m}=\frac{1}{3}a$

and $\bar{z}=\frac{m.0+m.a+m.2a}{m+m+m}=a$,

i.e. coordinates of G are $(\frac{1}{3}a, \frac{1}{3}a, a)$.



Let GX', GY', GZ' be the axes parallel to BA, BY and BC.

In reference to these axes through G the coordinates of

L are $(a-a/3, 0-a/3, 0-a)$

i.e $(\frac{2}{3}a, -\frac{1}{3}a, -a)$,

M are $(0-\frac{1}{3}a, 0-\frac{1}{3}a, a-a)$

i.e. $(-\frac{1}{3}a, -\frac{1}{3}a, 0)$,

and N are $(0-\frac{1}{3}a, a-\frac{1}{3}a, 2a-a)$ i.e. $(-\frac{1}{3}a, \frac{2}{3}a, a)$

$\therefore\ A_1=$ M.I. of the three rods about GX'

$=$ M.I. of AB + M.I. of BC + M.I. of CD about GX'

$=[m\{(-\frac{1}{3}a)^2+(-a)^2\}]$

$+[\frac{1}{3}ma^2+m\{(-\frac{1}{3}a)^2+0^2\}]+[\frac{1}{3}ma^2+(\frac{2}{3}a)^2+a^2]]=\frac{10}{3}ma^2$,

$B_1=$ M.I. of the three rods about GY'

$=[\frac{1}{3}ma^2+m\{(-a)^2+(\frac{2}{3}a)^2\}+[\frac{1}{3}ma^2+m\{0+(-\frac{1}{3}a)^2\}]$

$+m[a^2+(-\frac{1}{3}a)^2]=\frac{10}{3}ma^2$

$C_1=$ M.I. of the three rods about GZ'

$=[\frac{1}{3}ma^2+m\{(\frac{2}{3}a)^2+(-\frac{1}{3}a)^2\}]+[m\{(-\frac{1}{3}a)^2+(-\frac{1}{3}a)^2\}]$

$+[\frac{1}{3}ma^2+m\{(-\frac{1}{3}a)^2+(\frac{2}{3}a)^2\}]=2ma^2$,

$D_1=$ P.I. about GY', $GZ'=\Sigma my_1z_1$

$=m(-\frac{1}{3}a)(-a)+m(-\frac{1}{3}a).0+m(\frac{2}{3}a)(a)=ma^2$

$E_1=$ P.I. about GZ', $GX'=\Sigma mz_1x_1$

$=m(-a)(\frac{2}{3}a)+m(0)(-\frac{1}{3}a)+m.a(-\frac{1}{3}a)=-ma^2$

and $F_1=$ P.I. about GX', $GY'=\Sigma mx_1y_1$

$=m(\frac{2}{3}a)(-\frac{1}{3}a)+m(-\frac{1}{3}a)(-\frac{1}{3}a)+m(-\frac{1}{3}a)(\frac{2}{3}a)=-\frac{1}{3}ma^2$.

Hence the momental ellipsoid at G is

$A_1x^2+B_1y^2+C_1z^2-2D_1yz-2E_1zx-2F_1xy=3mk^4$

or $\frac{10}{3}ma^2x^2+\frac{10}{3}ma^2y^2+2ma^2z^2-2ma^2yz+2ma^2zx+\frac{2}{3}ma^2xy=3mk^4$

or $\frac{1}{3}ma^2[10x^2+10y^2+6z^2-6yz+6zx+2xy]=3mk^4$. ...(1)

Reducing $10x^2+10y^2+6z^2-6yz+6zx+2xy$ by means of the discriminating cubic $\lambda^3-(a+b+c)\lambda^2+(ab+bc+ca-f^2-g^2-h^2)\lambda-(abc+2fgh-af^2-bg^2-ch^2)=0$, we have

$\lambda^3-26\lambda^2+201\lambda-396=0$

or $(\lambda-3)(\lambda-11)(\lambda-12)=0\ \therefore\ \lambda=3, 11, 12$.

Hence the equation of the momental ellipsoid (1) referred to the principal axes through G takes the form

$\frac{1}{3}ma^2(3x^2+11y^2+12z^2)=3mk^4$

or $ma^2x^2+\frac{11}{3}ma^2y^2+4ma^2z^2=3mk^4$.

Hence the principal moments at the centre of inertia are ma^2, $\frac{11}{3}ma^2$ and $4ma^2$.

EXERCISE

1. Show that the moment of inertia of the part of the area of parabola cut off by any ordinate at a distance x form the vertex is $(3/7)\ Mx^2$ about the tangent at the vertex, and $(1/5)\ My^2$ about the principal diameter where y is the ordinate corresponding to x [Hint. See Ex. 4 on page 15].

2. The principal axes at the centre of gravity being the axes of reference, prove that the momental ellipsoid at the point (p, q, r) is

$$(A/M+q^2+r^2)x^2+(B/M+r^2+p^2)y^2+(C/M+p^2+q^2)z^2-2qryz-2rpzx-2pqxy=\text{constant},$$

when referred to its centre of gravity as origin.

3. Show that a uniform rod, of mass m, is kinetically equivalent to three particles rigidly connected and situated one at each end of the rod and one at its middle point, the masses of the particles being $m/6, m/6, 2m/3$.

4. Show that any lamina is dynamically equivalent to the three particles, each one-third of the mass of the lamina, placed at the corners of a maximum triangle inscribed in the ellipse, whose equation referred to the principal axes at the centre of inertia is $x^2/B+y^2/A=2$, where mA and mB are the principal moments of inertia about OX and OY, and m is the mass.

5. Show that there is momental ellipse at an angular point of a triangular area which touches the oposite side at its middle point and bisects the adjacent sides.

6. Find the principal axes at a corner point of solid cube [Hint. In Ex. 32 on page 47, $D=E=F=0$. $\therefore$ OG is one principal axis at O. Other two principal axes pass through O and at right angles to OG].

7. Two particles, each of mass m are placed at the extremities of the minor axis of an elliptic area of mass M. Prove that principal axes at any point of the circumference of the ellipse will be the tangent and normal to the ellipse, if

$$\frac{m}{M}=\frac{5}{8}\cdot\frac{e^2}{1-2e^2}$$

8. A uniform lamina bounded by the ellipse $b^2x^2+a^2y^2=a^2b^2$ has an elliptic hole (semi-axes c, d) in it whose major axis lies in the line $x=y$, the centre being at a distance r from

origin, prove that if one of the principal axes at the point (x, y) makes angle θ with x-axis, then $\tan\theta = \dfrac{2abxy - cd\{4(x\sqrt{2}-r)(y\sqrt{2}-r) - (c^2-d^2)\}}{ab\{(x^2-y^2+a^2-b^2)\} - cd\{2(x\sqrt{2}-r)^2 - 2(y\sqrt{2}-r)^2\}}$

9. The principal axes at the centre of gravity being the axes of reference, obtain the equation of the ellipsoid at the point (p, q, r) and show that the principal moments of inertia at this point are roots of

$$\begin{vmatrix} (I-A)/M - q^2 - r^2 & pq & rp \\ pq & (I-B)/M - r^2 - p^2 & qr \\ rp & qr & (I-C)/M - p^2 - q^2 \end{vmatrix} = 0$$

where I, M, A, B have there usual meanings.

10. Find the M.I. of a quadrant of the elliptic arc $x^2/a^2 + y^2/b^2 = 1$ of mass M about line through its centre and perpendicular to its plane, the density at any point is proportional to xy.

11. Find the M.I. of the solid generation by the revolution of the parabola $y^2 = 4ax$ about the x-axis from $x = 0$ to $x = h$ about x-axis.

12. Find the M.I. of an ellipsoid about the axis of z.

$$\left[\text{Ans. } \frac{1}{5}M(a^2 \times b^2)\right]$$

13. Find the M.I. of the cardoid $r = a(1 + \cos\theta)$ of density ρ, about the initial line.

D' ALEMBERT'S PRINCIPLE | SET - II

§ 2.1. Motion of a Particle.

The motion of a particle is determined by Newton's second law of motion, which states that *'the rate of change of momentum in any direction is proportional to the applied force in that direction'*. From this law, we deduce the formula $P = mf$, where f is the acceleration of the particle of mass m in the direction of the applied force P.

If (x, y, z) be the coordinates of a moving particle of mass m, at any time t and X, Y, Z be the components of the forces parallel to the axes, then by Newton's second law of motion the equations of motion of the particle are

$$m\ddot{x} = X, m\ddot{y} = Y, m\ddot{z} = Z.$$

§ 2.2. Motion of a Rigid Body.

A rigid body is an assemblage of particles rigidly connected together such that the distance between any two constituent particles does not change on account of the effect of forces.

For a rigid body we assume that

(i) the action between its two particles act along the straight ine joining them,

(ii) the action an reaction between the two particles are qual and opposite.

In considering the motion of a rigid body, we write the equation of motion. of the particles of the body according to the equations in § 2.1. But here the external forces acting on a particle of the body include, together with the applied fores, the unknown inner forces acting due to the action of the rest of the body on it.

D' Alembert proposed a method which enables us to obtain all the necessary equations without writing down the equations of motion of all particles and without considering the unknown inner forces. This important principle is based on the forllowing rule which is a natural consequence of Newton's third law of motion.

The internal actions and reactions of any system of rigid bodies in motion are in equilibrium amongst themselves.

§ 2.3. Definitions.

Impressed forces.

The external forces acting on a body are called 'impressed forces'. For example, the weight of the body is the impressed force on the body.

In case a body is tied to a string then the tension in the string is also an impressed force on the body.

Effective forces.

The effective force on a particle is defined as the product of its mass m and its acceleration f. If a particle of mass m is situated at the point (x, y, z) at time t, then the effective forces on this particle at this time are $m\ddot{x}, m\ddot{y}, m\ddot{z}$ parallel to the axes.

§ 2.4. D' Alembert's Principle.

The reversed effective forces at each point of the body and the impressed (external) forces on the system are in equilibrium.

Let (x, y, z) be the coordinates of a particle of mass m, of a rigid body which is in motion, at any time t. If f is the resultant of component accelerations $\ddot{x}, \ddot{y}, \ddot{z}$ then the effective force on the particle is mf. Let F denote the resultant of the impressed forces and R the resultant of the internal forces (mutual actions) on the particle. Then by Newton's second law of forces, mf is the resultant of F and R. Thus $-mf$ (reversed effective force), F and R are in equilibrium. This holds good for every particle of the body.

Thus $\Sigma(-mf)$, ΣF and ΣR are in equilibrium, the summation extending to all the particles of the body.

But the internal actions and reactions of different particles of a body are in equilibrium i.e. $\Sigma R = 0$, therefore $\Sigma(-mf)$ and ΣF are in equilibrium.

Hence the reversed effective forces acting at each particle of the body and the impressed (external) forces on the system are in equilibrium.

Vector Method :

Consider a rigid body in motion. At time t, let $\mathbf{r}$ be the position vector of a particle of mass m and $\mathbf{F}$ and $\mathbf{R}$ the external and internal forces respectively acting on it.

$\therefore$ By newton's second law

$$m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F} + \mathbf{R}$$

or
$$\mathbf{F} + \mathbf{R} - m\frac{d^2\mathbf{r}}{dt^2} = 0$$

i.e. the forces $\mathbf{F}, \mathbf{R}, -m\dfrac{d^2\mathbf{r}}{dt^2}$ acting on a particle of mass m are in equilibrium.

Now applying the same argument of every particle of the rigid body, the forces $\Sigma\mathbf{F}, \Sigma\mathbf{R}$ and $\Sigma\left(-m\dfrac{d^2\mathbf{r}}{dt^2}\right)$ are in equilibrium, where the summation extends to all particles.

But the internal forces acting on the body form pairs of equal and opposite forces $\therefore$ $\Sigma\mathbf{R} = 0$.

Thus the forces $\Sigma\mathbf{F}$ and $\Sigma\left(-m\dfrac{d^2\mathbf{r}}{dt^2}\right)$ are in equilibrium,

i.e.
$$\Sigma\mathbf{F} + \Sigma\left(-m\frac{d^2\mathbf{r}}{dt^2}\right) = 0$$

Hence the reversed effective forces acting at each particle of the body and the impressed (external) forces on the system are in equilibrium.

Note. The above D' Alembert's principle reduces the probelm of dynamics to the problem of statics. Thus we mark all the external forces of the system and mark the effective forces in opposite directions and then solve this problem as a problem of statics by equating to zero the resolved parts of all these forces in two mutually perpendicular directions and taking moments about suitable points.

§ 2.5. General Equations of motion of a body.

To deduce the general equations of motion of a rigid body from D' Alembert's principle.

Let X, Y, Z be the components, parallel to the axes, of the external force acting on a particle of mass m whose coordinates are (x, y, z) at time t, referred to any set of rectangular axes. Then reversed effective forces parallel to the axes on the particle are $-m\ddot{x}, -m\ddot{y}, -m\ddot{z}$. Thus the resultant of external forces and the reversed effective forces acting on the particle m parallel to the axes are $X - m\ddot{x}$, $Y - m\ddot{y}$ and $Z - m\ddot{z}$ respectively.

By D' Alembert's principle the forces whose components are $X - m\ddot{x}, Y - m\ddot{y}, Z - m\ddot{z}$ acting at the particle m at (x, y, z) together with similar forces acting at each other particle of the body, form a system in equilibrium.

Hence, as in statics the six conditions of equilibrium are

$\Sigma(X - m\ddot{x}) = 0, \Sigma(Y - m\ddot{y}) = 0, \Sigma(Z - m\ddot{z}) = 0$

$\Sigma\{y(Z - m\ddot{z}) - z(Y - m\ddot{y})\} = 0, \Sigma\{z(X - m\ddot{x}) - x(Z - m\ddot{z})\} = 0$

and $\Sigma\{x(Y - m\ddot{y}) - y(X - m\ddot{x})\} = 0.$

where the summation is extended to all the particles of the body

These six equations can be written as

$\Sigma m\ddot{x} = \Sigma X$ ...(1) $\quad \Sigma m\ddot{y} = \Sigma Y$ ...(2)

$\Sigma m\ddot{z} = \Sigma Z$ ...(3) $\quad \Sigma m(y\ddot{z} - z\ddot{y}) = \Sigma(yZ - zY)$ ...(4)

$\Sigma m(z\ddot{x} - x\ddot{z}) = \Sigma(zX - xZ)$ ...(5)

and $\Sigma m(x\ddot{y} - y\ddot{x}) = \Sigma(xY - yX)$ ...(6)

These equations (1) to (6) are the general equations of motion of a body.

Equations (1), (2), (3) state that *the sums of the components, parallel to the coordinate axes, of the effective forces is respectively equal to the sums of the components parallel to the same axes of the external (impressed) forces.*

Equations (4), (5), (6) state that *the sums of the moments about the axes of coordinates of the effective forces are respectively equal to the sums of the moments about the same axes of the external (impressed) forces*

The equations (1), (2) and (3) can be written as $\dfrac{d}{dt}(\Sigma m\dot{x}) = \Sigma X$, $\dfrac{d}{dt}(\Sigma m\dot{y}) = \Sigma Y$ and $\dfrac{d}{dt}(\Sigma m\dot{z}) = \Sigma Z$.

Which shows that *the rate of change of linear momentum of the system in any direction is equal to the total external force in that direction.*

The equations (4), (5) and (6) can be written as

$\dfrac{d}{dt}\{\Sigma m(y\dot{z} - z\dot{y})\} = \Sigma(yZ - zY)$, $\dfrac{d}{dt}\{\Sigma m(z\dot{x} - x\dot{z})\} = \Sigma(zX - xZ)$

and $\dfrac{d}{dt}\{\Sigma m(x\dot{y} - y\dot{x})\} = \Sigma(xY - yX)$

Which shows that *the rate of change of angular momentum (moment of momentum) about any given axis is equal to the total moment of oll the external forces about the axis.*

Vector Method : Consider a rigid body in motion. At time t let $\mathbf{r}$ be the position vector of a particle of mass m and $\mathbf{F}$ the external force acting on it.

Then by D' Alembert's principle, we have

$$\Sigma\mathbf{F} + \Sigma\left(-m\frac{d^2\mathbf{r}}{dt^2}\right) = 0$$

or
$$\Sigma m\frac{d^2\mathbf{r}}{dt^2} = \mathbf{F}. \quad ...(1)$$

Taking cross product by $\mathbf{r}$, we have

$$\Sigma m\mathbf{r} \times \frac{d^2\mathbf{r}}{dt^2} = \Sigma\mathbf{r} \times \mathbf{F} \quad ...(2)$$

Equations (1) and (2) are in general vector equations of motion of a rigid body.

Deduction of general equations of motion in scaler form.

To deduce the general equations of motion of a rigid body, we substitute the following in (1), (2).

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \text{ and } \mathbf{F} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}$$

where (x, y, z) are the cartesian coordinates of the particle m and X, Y, Z are the components of force $\mathbf{F}$ parallel to the axes respectively.

Substituting in (1) and (2), we get

$$\Sigma m(\ddot{x}\mathbf{i}+\ddot{y}\mathbf{j}+\ddot{z}\mathbf{k})=\Sigma(X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k}) \quad ...(3)$$

and $\Sigma\{m(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\times(\ddot{x}\mathbf{i}+\ddot{y}\mathbf{j}+\ddot{z}\mathbf{k})\}=\Sigma(x\mathbf{i}+y\mathbf{j}+z\mathbf{k})\times(X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k})$

or $\Sigma m\{(y\ddot{z}-z\ddot{y})\mathbf{i}+(z\ddot{x}-x\ddot{z})\mathbf{j}+(x\ddot{y}-y\ddot{x})\mathbf{k}\}$

$$=\Sigma\{(yZ-zY)\mathbf{i}+(zX-xZ)\mathbf{j}+(xY-yX)\mathbf{k}\} \quad ...(4)$$

Equating coefficients of $\mathbf{i},\mathbf{j},\mathbf{k}$ on the two sides of equations (3) and (4), we get the six equations of motion of the rigid body in cartesian form.

§ 2.6. Linear Momentum.

The linear momentum in a given direction is equal to the product of the whole mass of the body and the resolved part of the velocity of its centre of gravity in that direction.

Let $(\bar{x},\bar{y},\bar{z})$ be the coordinates of the centre of gravity of a body of mass M, then we have

$$\bar{x}=\frac{\Sigma mx}{\Sigma m}=\frac{\Sigma mx}{M} \qquad \because \Sigma m=M.$$

$\therefore$ $\Sigma mx=M\bar{x}$. Similarly, $\Sigma my=M\bar{y}$ and $\Sigma mz=M\bar{z}$.

Differentiating these relation w.r.t. 't', we get

$\Sigma m\dot{x}=M\dot{\bar{x}}$, $\Sigma m\dot{y}=M\dot{\bar{y}}$, and $\Sigma m\dot{z}=M\dot{\bar{z}}$.

Hence the result.

§ 2.7. Motion of the Centre of Inertia.

To show that the centre of inertia of a body moves as if all the mass of the body were collected at it and if all the external forces acting on the body were acting on it in directions parallel to those in which they act.

If $(\bar{x},\bar{y},\bar{z})$ be the coordinates of the centre of inertia of a body of mass M, then as in § 2.6, we have

$$\Sigma mx=M\bar{x},\ \Sigma my=M\bar{y},\ \Sigma mz=M\bar{z}.$$

Differentiating twice w.r.t. 't', we get

$$\Sigma m\ddot{x}=M\ddot{\bar{x}},\ \Sigma m\ddot{y}=M\ddot{\bar{y}} \text{ and } \Sigma m\ddot{z}=M\ddot{\bar{z}}. \quad ...(1)$$

But from the general equations of motion of a body, we get (see § 2.5)

$$\Sigma m\ddot{x}=\Sigma X,\ \Sigma m\ddot{y}=\Sigma Y \text{ and } \Sigma m\ddot{z}=\Sigma Z. \quad ...(2)$$

$\therefore$ From (1) and (2), we get

$$M\ddot{\bar{x}}=\Sigma X,\ M\ddot{\bar{y}}=\Sigma Y \text{ and } M\ddot{\bar{z}}=\Sigma Z.$$

These are the equations of motion of a particle of mass M placed at the centre of inertia of the body, and acted on by forces ΣX, ΣY, ΣZ parallel to the original directions of the forces acting on the different points of the body.

This proves the theorem.

Vector method. Consider a rigid body in motion. At time t let $\mathbf{r}$ be the position vector of a particle m of the body and $\mathbf{F}$ the external force acting on it. Then the equation of motion of the body is

$$\Sigma m\frac{d^2\mathbf{r}}{dt^2}=-\mathbf{F}. \quad ...(1)$$

If $\bar{\mathbf{r}}$ is the position vector of the centre of inertia of the body, then we have

$$\bar{\mathbf{r}}=\frac{\Sigma m\mathbf{r}}{\Sigma m}=\frac{\Sigma m\mathbf{r}}{M} \text{ or } \Sigma m\mathbf{r}=M\bar{\mathbf{r}}$$

$$\therefore\ \Sigma m\frac{d^2\mathbf{r}}{dt^2}=M\frac{d^2\bar{\mathbf{r}}}{dt^2}. \quad ...(2)$$

From (1) and (2), we have

$$M\frac{d^2\bar{\mathbf{r}}}{dt^2}=\Sigma\mathbf{F}. \quad ...(3)$$

Which is the vector form of the equation of motion of a particle of mass M placed at the centre of inertia of the body and acted upon by the external forces $\Sigma\mathbf{F}$.

Deduction of the equations of motion of the centre of inertia in scalar form.

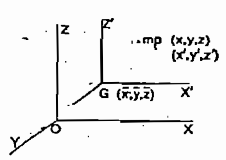
Substituting $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$, and $\mathbf{F}=X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k}$ in (3) and equating the coefficients of $\mathbf{i},\mathbf{j},\mathbf{k}$ from the two sides we can get the equations of motion of the centre of inertia in scalar form.

Note. The proposition discussed in § 2.7 is called *the principle of conservation of motion of translation.* From this it follows that the motion of C.G. is independent of rotation.

§ 2.8. Motion Relative to the Centre of Inertia.

To show that the motion of a body about its centre of inertia is the same as it would be if the centre of inertia were fixed and the same forces acted on the body.

Let $(\bar{x},\bar{y},\bar{z})$ be the coordinates of the centre of gravity (centre of inertia) G of the body referred to the rectangular axes OX, OY, OZ through a fixed point O. Let GX', GY', GZ' be the axes through G parallel to the axes OX, OY, OZ respectively.



If (x,y,z) and (x',y',z') are the coordinates of a particle of mass m at P referred to the coordinate axes OX, OY, OZ and parallel axes GX', GY', GZ' respectively, then

$x=\bar{x}+x'$, $y=\bar{y}+y'$, $z=\bar{z}+z'$.

$\therefore$ $\ddot{x}=\ddot{\bar{x}}+\ddot{x}'$, $\ddot{y}=\ddot{\bar{y}}+\ddot{y}'$, $\ddot{z}=\ddot{\bar{z}}+\ddot{z}'$.

Now consider the equation $\Sigma m(y\ddot{z}-z\ddot{y})=\Sigma(yZ-zY)$, which becomes

$\Sigma m\{(\bar{y}+y')(\ddot{\bar{z}}+\ddot{z}')-(\bar{z}+z')(\ddot{\bar{y}}+\ddot{y}')\}$

$$=\Sigma\{(\bar{y}+y')Z-(\bar{z}+z')Y\}$$

or $\Sigma m(y'\ddot{z}'-z'\ddot{y}')+\bar{y}\ddot{\bar{z}}\,\Sigma m+\bar{y}\,\Sigma m\ddot{z}'+\ddot{\bar{z}}\,\Sigma my'$

$$-\bar{z}\ddot{\bar{y}}\,\Sigma m-\bar{z}\,\Sigma m\ddot{y}'-\ddot{\bar{y}}\,\Sigma mz'$$

$$=\Sigma(y'Z-z'Y)+\bar{y}\,\Sigma Z-\bar{z}\,\Sigma Y. \quad ...(1)$$

Now referred to GX', GY', GZ' as axes the coordinates of G are $(0, 0, 0)$.

$\therefore$ $\dfrac{\Sigma mx'}{\Sigma m}=0$ or $\Sigma mx'=0$.

Similarly, $\Sigma my'=0$, $\Sigma mz'=0$.

$\therefore$ $\Sigma m\ddot{x}'=0$, $\Sigma m\ddot{y}'=0$, $\Sigma m\ddot{z}'=0$.

Also from § 2.7. we have $M\ddot{\bar{x}}=\Sigma X$, $M\ddot{\bar{y}}=\Sigma Y$, $M\ddot{\bar{z}}=\Sigma Z$.

Thus, from eqn. (1), we get

$\Sigma m(y'\ddot{z}'-z'\ddot{y}')+\bar{y}\ddot{\bar{z}}M-\bar{z}\ddot{\bar{y}}M=\Sigma(y'Z-z'Y)+\bar{y}\Sigma Z-\bar{z}\Sigma Y$

or $\Sigma m(y'\ddot{z}'-z'\ddot{y}')+\bar{y}\Sigma Z-\bar{z}\Sigma Y=\Sigma(y'Z-z'Y)+\bar{y}\Sigma Z-\bar{z}\Sigma Y$

or $\quad \Sigma m(y'\ddot{z}'-z'\ddot{y}')=\Sigma(y'Z-z'Y)$.

Similarly, we get the other two equations as

$$\Sigma m(z'\ddot{x}'-x'\ddot{z}')=\Sigma(z'X-x'Z)$$

and

$$\Sigma m(x'\ddot{y}'-y'\ddot{x}')=\Sigma(x'Y-y'X).$$

But these equations are the same as would have been obtained if we had regarded the centre of gravity as fixed point.

Hence the proposition.

Vector method. Consider a rigid body in motion. At time t, let $\bar{\mathbf{r}}$ be the position vector of the centre of inertia G of a rigid body of mass M. Let m be the mass of a particle of the body and $\mathbf{r}$ its position vector referred to the fixed origin O and $\mathbf{r}'$ its position vector referred to the centre of inertia $\bar{\mathbf{r}}$.

$\therefore$ $\mathbf{r}=\bar{\mathbf{r}}+\mathbf{r}'$, so that $\dfrac{d^2\mathbf{r}}{dt^2}=\dfrac{d^2\bar{\mathbf{r}}}{dt^2}+\dfrac{d^2\mathbf{r}'}{dt^2}$

The moment vector equation of the rigid body is

$$\Sigma m\mathbf{r}\times\frac{d^2\mathbf{r}}{dt^2}=\Sigma\mathbf{r}\times\mathbf{F};$$

or

$$\Sigma\left\{m(\bar{\mathbf{r}}+\mathbf{r}')\times\left(\frac{d^2\bar{\mathbf{r}}}{dt^2}+\frac{d^2\mathbf{r}'}{dt^2}\right)\right\}=\Sigma\{(\bar{\mathbf{r}}+\mathbf{r}')\times\mathbf{F}\}$$

or

$$\Sigma m\mathbf{r}'\times\frac{d^2\mathbf{r}'}{dt^2}+\bar{\mathbf{r}}\times\frac{d^2\bar{\mathbf{r}}}{dt^2}\Sigma m+\bar{\mathbf{r}}\,\Sigma m\frac{d^2\mathbf{r}'}{dt^2}+\frac{d^2\bar{\mathbf{r}}}{dt^2}\Sigma m\mathbf{r}'$$

$$=\bar{\mathbf{r}}\,\Sigma\mathbf{F}+\Sigma\mathbf{r}'\times\mathbf{F}. \quad ...(1)$$

Now position vector of the centre of inertia G of the body referred to G as origin is O.

$\therefore$ $\dfrac{\Sigma m\mathbf{r}'}{\Sigma m}=0$, *i.e.* $\Sigma m\mathbf{r}'=0$, so that $\Sigma m\dfrac{d^2\mathbf{r}'}{dt^2}=0$.

Also the equation of motion of the centre of inertia is

$$M\frac{d^2\bar{\mathbf{r}}}{dt^2}=\Sigma\mathbf{F}.$$

$\therefore$ From eqn. (1), we have

$$\Sigma m\mathbf{r}'\times\frac{d^2\mathbf{r}'}{dt^2}+\bar{\mathbf{r}}\times\left(\frac{d^2\bar{\mathbf{r}}}{dt^2}.M\right)+0+0=\bar{\mathbf{r}}\times\Sigma\mathbf{F}+\Sigma\mathbf{r}'\times\mathbf{F}.$$

or

$$\Sigma m\mathbf{r}'\times\frac{d^2\mathbf{r}'}{dt^2}+\bar{\mathbf{r}}\times\Sigma\mathbf{F}=\bar{\mathbf{r}}\times\Sigma\mathbf{F}+\Sigma\mathbf{r}'\times\mathbf{F}$$

or

$$\Sigma m\mathbf{r}'\times\frac{d^2\mathbf{r}'}{dt^2}=\Sigma\mathbf{r}'\times\mathbf{F}. \quad ...(2)$$

Which is the vector equation of motion of a rigid body when the centre of inertia is regarded as a fixed point.

Deduction of the corresponding equations in scalar form.

If (x,y,z) and (x',y',z') are the cartesian coordinates of the particles m referred to the rectangular axes through the fixed point O and the parallel axes through the centre of inertia G respectively, then we have

$$\bar{\mathbf{r}}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k} \text{ and } \mathbf{r}'=x'\mathbf{i}+y'\mathbf{j}+z'\mathbf{k}.$$

Let $(\bar{x},\bar{y},\bar{z})$ be the coordinates of G referred to the axes through O, then

$$\bar{\mathbf{r}}=\bar{x}\mathbf{i}+\bar{y}\mathbf{j}+\bar{z}\mathbf{k}.$$

Also if X, Y, Z are the components of external force $\mathbf{F}$ paralel to the axes, then

$$\mathbf{F}=X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k}.$$

Substituting in (2), we have

$\Sigma m\{(x'\mathbf{i}+y'\mathbf{j}+z'\mathbf{k})\times(\ddot{x}'\mathbf{i}+\ddot{y}'\mathbf{j}+\ddot{z}'\mathbf{k})\}$

$$=\Sigma\{(x'\mathbf{i}+y'\mathbf{j}+z'\mathbf{k})\times(X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k})\}$$

or $\Sigma m\{(y'\ddot{z}'-z'\ddot{y}')\mathbf{i}+(z'\ddot{x}'-x'\ddot{z}')\mathbf{j}+(x'\ddot{y}'-y'\ddot{x}')\mathbf{k}\}$

$$=\Sigma\{(y'Z-z'Y)\mathbf{i}+(z'X-x'Z)\mathbf{j}+(x'Y-y'X)\mathbf{k}\}.$$

Equating the coefficients of $\mathbf{i},\mathbf{j},\mathbf{k}$ from the two sides we shall get the equations of motion of the body in scalar form referred to the centre of inertia as fixed point.

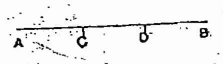
Note 1. The propositiion discussed in § 2.8 is called the *principle of conservation of motion of rotation.* From this it follows that the motion round the centre of inertia is independent of its motion of translation.

Note 2. The two propositions discussed in § 2.7 and 2.8 together prove the principle of the independence of the motion of translation and rotation.

EXAMPLES

Ex. 1. *A rod revolving on a smooth horizontal plane about one end, which is fixed, breaks into two parts, what is the subsequent motion of the two parts.*

Sol. Let the rod AB revolving about the end A on a smooth horizontal plane break into two parts AC and CB. Clearly the part AC will continue to rotate about A with the same angular velocity.

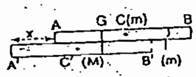


The part CB at the instant of breaking acquires the same angular velocity and its centre of gravity D has a linear velocity. Hence this part CB will fly off along the tangent line (the direction of linear velocity) at D to the circle with A as centre and AD as radius. Also, since the motion of a body about its centre of inertia is the same as if the centre of inertia was fixed and the same forces acted on the body, the part CB will continue rotating about D with the same angular velocity.

Hence the part CB will move along the tangent at D to the circle with A as centre and AD as radius with the velocity acquired by its centre of gravity at the instant of breaking and this part will also go on rotating about D with the same angular velocity.

Ex. 2. *A rough uniform board, of mass m and length $2a$, rests on a smooth horizontal plane and a man of mass M walks on it from one end to the other. Find the distance through which the board moves in this time.*

Sol. Here the external forces are (i) the weights of the board and the man acting vertically downwards and (ii) the reaction of the horizontal plane acting vertically upwards. Thus there are no external forces in the horizontal direction, therefore by D' Alembert's principle, the C.G. of the system will remain at rest. As a matter of fact as the man moves forward, the board slips backwards, keeping the poistion of C.G. of the system unchanged.



Let AB be the position of the board when the man of mass M is at A.

$\therefore$ Distance of C.G. of the system from A (towards B)

$$=\frac{M.0+m.AC}{M+m}=\frac{M.0+m.a}{M+m}=\frac{ma}{M+m}=x_1 \text{ (say)}. \quad (\because AG=BG=a)$$

Let $A'B'$ be the positioin of the board when the man reaches the other end B of the board. If the board slips through a distance $AA'=x$ (backwards) during the time the man walks from A to B, then in this position the distance of C.G. of the system from A (towards B)

$$=\frac{M.AB'+m.AC'}{M+m}=\frac{M.(2a-x)+m(a-x)}{M+m}=x_2 \text{ (say)}$$

Since the position of the C.G., 'G' of the system remains unchanged

$\therefore x_1=x_2$

or $\dfrac{ma}{M+m}=\dfrac{M(2a-x)+m(a-x)}{M+m}$

or $ma=2aM+ma-(M+m)x$ or $x=2aM/(m+M)$

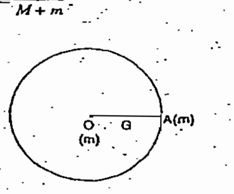
Which is the required distance.

Ex. 3. *A circular board is placed on a smooth horizontal plane and a boy runs round the edge of it at a uniform rate, what is the motion of the board.* (IAS-2008)

Sol. Let M be the mass and O the centre of the board. If initially the boy is at the point A on the edge of the board then the C.G. 'G' of the system will be on the radius OA, such that

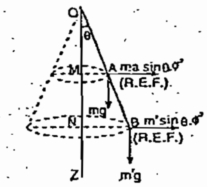
$$OG=\frac{M.0+m.a}{M+m}=\frac{ma}{M+m}$$

Since the external forces, weight of the board and the boy act vertically downwards and the reaction of the smooth horizontal plane act vertically upwards, therefore there is no external force in the horizontal direction during the motion. Thus by D' Alembert's principle the C.G. 'G' of the system will remain at rest. Hence as the boy runs round the edge of the board with uniform spead, the centre O of the board will describe a circle of radius $OG=ma/(M+m)$ round the centre at G.



Ex. 4. *Find the motion of the rod OAB, with two masses m and m' attached to it at A and B respectively, when it moves round the vertical as a conical pendulum with uniform angular velocity, the angle θ which the rod makes with the vertical being constant.*

Sol. Let OAB be the rod with two masses m and m' attahced at A and B respectively such that $OA=a$ and $OB=b$. When the rod OAB moves round the vertical as a conical pendulem with uniform angular velocity, making constant angle θ with the vertical the masses m and m' move in circles on horizontal planes with radii $a\sin\theta$ and $b\sin\theta$ and centres at M and N respectively. The motion about the vertical being with uniform angular velocity, the effective forces are entirely in wards. Let ϕ be the angle that the plane through OAB makes with a fixed vertical plane through OZ, then the only effective forces on the particles are $ma\sin\theta\,\dot\phi^2$ and $m'b\sin\theta\,\dot\phi^2$ along AM and BN respectively.



By D' Alembert's principle the external forces, weights $mg, m'g$ and the reaction at O, and the reversed effective forces $ma\sin\theta\,\dot\phi^2$ along MA and $m'b\sin\theta\,\dot\phi^2$ along NB will keep the rod in equilibrium.

To avoid reaction at O, taking moment about the point O, we get

$ma\sin\theta\,\dot\phi^2 \,.\, OM+m'b\sin\theta\,\dot\phi^2 \,.\, ON-mg \,.\, MA-m'g \,.\, NB=0$

or $(ma\sin\theta \,.\, a\cos\theta+m'b\sin\theta \,.\, b\cos\theta)\dot\phi^2=g(ma\sin\theta+m'b\sin\theta)$

or $\dot\phi^2=\dfrac{(ma+m'b)g}{(ma^2+m'b^2)\cos\theta}$ $(\because \sin\theta\neq 0)$

Which will determine the motion of the rod.

Ex. 5. *A uniform rod OA, of length $2a$, free to turn about its end O, revolves with uniform angular velocity ω about the vertical OZ through O, and is inclined at a constant angle α to OZ, show that the value of α is either zero or $\cos^{-1}(3g/4a\omega^2)$.*

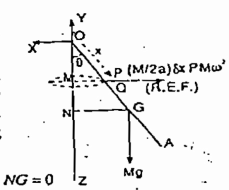
Sol. Let the rod OA of length $2a$ and mass M revolve with uniform angular velocity ω about the vertical OZ through O, making a constant angle α to OZ. Let $PQ=\delta x$ be an element of the rod at a distance x from O. The mass of the element PQ is $\dfrac{M}{2a}\delta x$.

This element PQ will make a circle in the horizontal plane with radius $PM(=x\sin\alpha)$ and centre at M. Since the rod revolve with uniform angular velocity, the only effective force on this element is $\dfrac{M}{2a}\delta x \,.\, PM \,.\, \omega^2$ along PM.

Thus the reversed effective force on the element PQ is

$\dfrac{M}{2a}\delta x \,.\, x\sin\alpha \,.\, \omega^2$ along MP.

Now by D' Alembert's principle all the reversed effective forces acting at different points of the rod, and the external forces, weight mg and rection at O are in equilibrium. To avoid reaction at O, taking moment about O, we get



$$\sum\left(\frac{M}{2a}\delta x \,.\, \omega^2 \,.\, \sin\alpha\right).OM-Mg\,.\,NG=0$$

or $\displaystyle\int_0^{2a}\frac{M}{2a}\omega^2x^2\sin\alpha\cos\alpha\,dx$

$-Mg\,.\,a\sin\alpha=0, \quad (\because OM=x\cos\alpha)$

or $\dfrac{M}{2a}\omega^2\,.\left\{\dfrac{1}{3}(2a)^3\right\}.\sin\alpha\cos\alpha-Mg\,a\sin\alpha=0$

or $Mg\,a\sin\alpha\left(\dfrac{4a}{3g}\omega^2\cos\alpha-1\right)=0$

$\therefore$ either $\sin\alpha=0$ i.e. $\alpha=0$

or $\dfrac{4a}{3g}\omega^2\cos\alpha-1=o.i.e.\ \cos\alpha=\dfrac{3g}{4a\omega^2}$

Hence, the rod is inclined at an angle zero or $\cos^{-1}\left(\dfrac{3g}{4a\omega^2}\right)$

Note. If $\omega^2<\dfrac{3g}{4a}$, then $\cos\alpha>1$. $\therefore$ in this case $\cos\alpha=\dfrac{3g}{4a\omega^2}$ gives an impossible value of α *i.e.* when $\omega^2<\dfrac{3g}{4a}$, then $\alpha=0$ is the only possible value of α.

Ex. 6 *A rod, of length $2a$, revolves with uniform angular velocity ω about a veritical axis through a smooth joint at one extremity of the rod so that it describes a cone of semi-vertical angle α, show that $\omega^2=3g/(4a\cos\alpha)$.*

Prove also that direction of reaction at the hinge makes with the vertical an angle $\tan^{-1}\left(\dfrac{3}{4}\tan\alpha\right)$

Sol. Refer figure of last Ex. 5.

Proceeding as in last Ex. 5, we get

$\cos\alpha=\dfrac{3g}{4a\omega^2}$, i.e. $\omega^2=\dfrac{3g}{4a\cos\alpha}$ ....(1)

Second Part :

If X and Y are the horizontal and vertical components of the reaction at the hinge O, as shown in the figure, then resolving the forces horizontally and vertically we get

$$X = \Sigma \frac{M}{2a} \delta x . PM . \omega^2 = \int_0^{2a} \frac{M}{2a} \omega^2 x \sin \alpha \, dx \qquad (\because PM = x \sin \alpha)$$

$$= \frac{M}{2a} \omega^2 \left\{ \tfrac{1}{2} (2a)^2 \right\} \sin \alpha = Ma\omega^2 \sin \alpha$$

and $Y = Mg$.

If the reaction at O make an angle θ with the vertical, then

$$\tan \theta = \frac{X}{Y} = \frac{Ma\omega^2 \sin \alpha}{Mg} = \frac{a}{g} \left(\frac{3g}{4a \cos \alpha} \right) \sin \alpha \qquad \text{[substituting from (1)]}$$

or $$\theta = \tan^{-1} \left(\frac{3}{4} \tan \alpha \right)$$

Ex. 7. *Two uniform spheres, each of mass M and radius a, are firmly fixed to the ends of two uniform thin rods, each of mass m and length l, and the other ends of the rods are freely hinged to a point O. The whole system revolves as in the Governor of a steam Engine, about a vertical line through O with the angular velocity ω. Show that when the motion is steady, the rods are inclined to the vertical at an angle θ, given by the equation*

$$\cos \theta = \frac{g}{\omega^2} \cdot \frac{M (l + a) + \frac{1}{2} ml}{M (l + a)^2 + \frac{1}{3} ml^2}$$

Sol. Let OA, OB be two rods, each of length l and mass M attached freely to a point O. Let C and D be the centres of two spheres each of mass M and radius a atteched to the other ends of the two rods. When the motion is steady let θ be the inclination of the rods to the vertical.
Consider the motion of one of the spheres, say the sphere with centre at C. Let δx be an element PQ of the rod at P such that $OP = x$, then mass of the element is $(m/l)\ \delta x$.
The reversed effective force at the element δx at P is

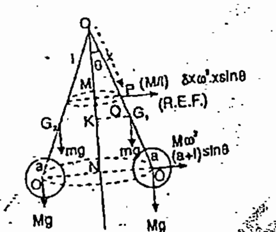
$$\frac{m}{l} \delta x . \omega^2 . PM = \frac{m}{l} \delta x \ \omega^2 x \sin \theta \ \text{ alone } MP.$$

And the reversed effective force on the sphere is

$$M\omega^2 CN = M\omega^2 (a + l) \sin \theta \ \text{ along } CN.$$

The external forces on the rod OA and sphere with centre at C are the weights mg and the Mg and reaction at O.
To avoid reaction at O, taking moment about O, we get

$$\Sigma \frac{m}{l} \delta x \ \omega^2 . \sin \theta . OM + M\omega^2 (a + l) \sin \theta . ON$$



$$-mg . KG_1 - Mg . NC = 0$$

or $$\int_0^l \frac{m}{l} \omega^2 x^2 \sin \theta \cos \theta \, dx + M\omega^2 (a + l)^2 \sin \theta \cos \theta$$

$$- mg \frac{l}{2} \sin \theta - Mg (a + l) \sin \theta = 0$$

$$[\omega^2 . \{ \tfrac{1}{2} ml^2 + M (a + l)^2 \} \cos \theta - g \{ \tfrac{1}{2} ml + M (a + l) \}] \sin \theta = 0$$

$\therefore$ Either $\sin \theta = 0$, *i.e.* $\theta = 0$ which is inadmissible.

$$\therefore \omega^2 \{ \tfrac{1}{3} ml^2 + M (a + l)^2 \} \cos \theta - g \{ \tfrac{1}{2} ml + M (a + l) \} = 0$$

or $$\cos \theta = \frac{g}{\omega^2} \cdot \frac{M (a + l) + \frac{1}{2} ml}{M (a + l)^2 + \frac{1}{3} ml^2}.$$

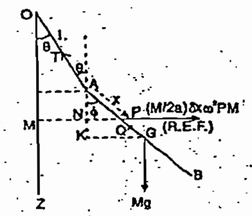
Ex. 8. *A rod of length $2a$, is suspended by a string of length l, attached to one end, if the string and rod revolve about the vertical with uniform angular velocity, and their inclinations to the vertical be θ and ϕ respectively, show that*

$$\frac{3l}{a} = \frac{(4 \tan \theta - 3 \tan \phi) \sin \phi}{(\tan \phi - \tan \theta) \sin \theta}$$

Sol. Let the rod AB of length $2a$ and mass m be suspended by a string OA of length l. Let θ and ϕ be the inclinations of the string and the rod to the vertical respectively.

Consider an element PQ ($= \delta x$) of the rod at a distance x from A, then mass of this element is $(M/2a)\ \delta x$.

As the rod revolve with uniform angular velocity ω, about the vertical OZ, the element δx will describe a circle of radius PM in the horizontal plane.



The reversed effective force on element δx is

$$\frac{M}{2a} \delta x . \omega^2 . PM = \frac{M}{2a} \delta x . \omega^2 . (l \sin \theta + x \sin \phi), \text{ along } MP.$$

The external forces acting on the rod are (i) tension T at A along AO, and (ii) its weight Mg acting vertically down wards at its middle point G.

Resolving horizantally and vertically the forces acting on the rod, we get

$$T \sin \theta = \Sigma \frac{m}{2a} \delta x \ \omega^2 (l \sin \theta + x \sin \phi) .$$

or $$T \sin \theta = \frac{M}{2a} \omega^2 \int_0^{2a} (l \sin \theta + x \sin \phi) \, dx$$

or $$T \sin \theta = \frac{M}{2a} \omega^2 \left[lx \sin \theta + \frac{1}{2} x^2 \sin \phi \right]$$

or $$T \sin \theta = M\omega^2 (l \sin \theta + a \sin \phi). \qquad ...(1)$$

and $$T \cos \theta = Mg. \qquad ...(2)$$

Now taking moment about A of all the forces acting on the rod AB, we get

$$-Mg . KG + \Sigma \frac{M}{2a} \delta x \omega^2 (l \sin \theta + x \sin \phi) . AN = 0$$

or $$Mga \sin \phi = \frac{M\omega^2}{2a} \int_0^{2a} (l \sin \theta + x \sin \phi) x \cos \phi \, d\phi$$

$$= \frac{M}{2a} \omega^2 \left[\frac{1}{2} lx^2 \sin \theta + \frac{1}{3} x^3 \sin \phi \right]_0^{2a} \cos \phi$$

$$= a M\omega^2 (l \sin \theta + \frac{4a}{3} \sin \phi) . \cos \theta$$

or $$g \tan \phi = \tfrac{1}{3} \omega^2 (3l \sin \theta + 4a \sin \phi). \quad ...(3)$$

Dividing (1) by (2), we get

$$\tan \theta = \frac{\omega^2}{g} (l \sin \theta + a \sin \phi).$$

or $$\omega^2 = g \tan \theta / (l \sin \theta + a \sin \phi).$$

Substituting in (3), we get

$$g \tan \phi = \frac{1}{3} \frac{g \tan \theta (3l \sin \theta + 4a \sin \phi)}{(l \sin \theta + a \sin \phi)}$$

or $$3 \tan \phi (l \sin \theta + a \sin \phi) = \tan \theta (3l \sin \theta + 4a \sin \phi)$$

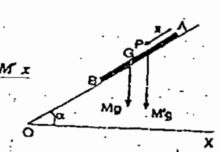
or $$3l \sin \theta (\tan \phi - \tan \theta) = \sin \phi (4a \tan \theta - 3a \tan \phi)$$

or $$\frac{3l}{a} = \frac{(4 \tan \theta - 3 \tan \phi) \sin \phi}{(\tan \phi - \tan \theta) \sin \theta}$$

Ex. 9. *A plank of mass M is initially at rest along a line of greatest slope of a smooth plane inclined at an angle α to the horizon, and a man of mass M', starting from the upper end, walks down the plank so that it does not move, show that he gets to the other end in time $\sqrt{\left\{ \frac{2M'a}{(M + M') g \sin \alpha} \right\}}$, where a is the length of the plane.* (IAS-2005)

Sol. Let the plank AB of mass M and length a rest along the line of greatest slope of a smooth plane inclined at an angle α to the horizon. A man of mass M' starts moving down the plank from the upper end A. Let the man move down the plank through a distance $AP = x$ in time t. Since the plank does not move, therefore if $\bar{x}$ is the distance of the C. G. of the plank and the man from A in this position, then

$$\bar{x} = \frac{M . AG + M' . AP}{M + M'} = \frac{M . (a/2) + M' x}{M + M'}$$



Differentiating twice w. r. t. 't', we get

$$\ddot{\bar{x}} = \frac{M'}{M + M'} \ddot{x}, \qquad ...(1)$$

Now the total weight $(M + M') g$ will act vertically downwards at the C. G. of the system.

$\therefore$ The equation of motion of the C. G. of the system is given by

$$(M+M')\ddot{x}=(M+M')g\sin\alpha. \quad ...(2)$$

$\therefore$ From (1) and (2), we get

$$M'\ddot{x}=(M+M')g\sin\alpha.$$

Integrating, we get $M'\dot{x}=(M+M')g\sin\alpha\; t+c_1.$

But initially when $t=0, \dot{x}=0 \quad \therefore c_1=0.$

$\therefore M'\dot{x}=(M+M')g\sin\alpha\,.\,t.$

Integrating again, we get $M'x=M+M'\; g\sin\alpha\,.\frac{1}{2}t^2+c_2.$

Initially when $t=0, x=0. \quad \therefore C_2=0.$

$\therefore M'x=(M+M')g\sin\alpha\,.\,\frac{1}{2}t^2.$

or $$t=\sqrt{\left\{\frac{2M'x}{(M+M')g\sin\alpha}\right\}}$$

Putting $x=AB=a$, the time to reach the other and B of the plank is given by

$$t=\sqrt{\left\{\frac{2M'x}{(M+M')g\sin\alpha}\right\}}$$

§ 2.9 Impulse of a Force.

The impulse of a force acting on a particle in any interval of time is defined to be the change in momentum produced.

Thus due to a force F, if the velocity of a particle of mass m changes from v_1 to v_2 in time t, then the impulse I is given by

$$I=mv_2-mv_1=m(v_2-v_1)$$

$$=m\int_{t_1}^{t_2}dv=\int_{t_1}^{t_2}m\frac{dv}{dt}dt$$

$$=\int_{t_1}^{t_2}F\,.\,dt \text{ since } F=m\frac{dv}{dt}$$

Thus the impulse of the force F is the time integral of the force.

Now let the force F increase indefinitely and the interval (t_2-t_1) decrease to a very small quantity such that the time integral $\int_{t_1}^{t_2}F\,dt$ remains finite.

Such a force is called impulsive force.

Note. The impulsive force can be measured by the change in momentum produced.

§ 2.10 An Important Rule.

The effect of an impulse on a body remains the same even if all the finite forces acting simultaneously on it are neglected.

let I be the impulse due to an impulsive force F which acts for a time T. If f is the finite force acting simultaneously on the body, then

$$m(v_2-v_1)=\int_0^T F\,dt+\int_0^T f\,dt=I+fT$$

Since $fT\to 0$ as $T\to 0 \quad \therefore I=m(v_2-v_1).$

Which shows that the finite force f acting on the body may be neglected in forming the equations.

§ 2.11 General Equations of Motion under Impulsive Forces.

To determine the general equations of motion of a system acted on by a number of impulses at a time.

Let u, v, w and u', v', w', be the velocities parallel to the axes respectively before and after the action of impulsive forces on the particle of mass m. If X', Y', Z', are the resolved parts of the total impulse on m parallel to the axes, then

$$\Sigma m(u'-u)=\Sigma\int_0^T X\,dt=\Sigma X'$$

or $$\Sigma mu'-\Sigma mu=\Sigma X' \quad ...(1)$$

Similarly $$\Sigma mv'-\Sigma mv=\Sigma Y' \quad ...(2)$$

and $$\Sigma mw'-\Sigma mw=\Sigma Z' \quad ...(3)$$

i.e. the change in momentum parallel to any of the axes is equal to the total impulse of the external forces parallel to the corresponding axis.

Hence the change in momentum parallel to any of the axes of the whole mass M, supposed collected at the centre of inertia and moving with it, is equal to the impulse of the external force parallel to the corresponding axis.

Again we have the equation

$$\Sigma m(y\ddot{z}-z\ddot{y})=\Sigma m(yZ-zY)$$

or $$\frac{d}{dt}\Sigma m(Y\dot{z}-z\dot{y})=\Sigma m(yZ-zY)$$

Integrating this, we have

$$\left[\Sigma m(y\dot{z}-z\dot{y})\right]_0^\tau=\Sigma\left[y\int_0^\tau Z\,dt-z\int_0^\tau Y\,dt\right]$$

Since the time interval τ is so small that the body has not moved during this interval, we may take x, y, z, as constants. Thus the above equation becomes

$$\Sigma m\{y(w'-w)-z(v'-v)\}=\Sigma(yZ'-zY')$$

or $$\Sigma m(yw'-zv')-\Sigma m(yw-zv)=\Sigma(yZ'-zY') \quad ...(4)$$

Similarly,

$$\Sigma m(zu'-xw')-\Sigma m(zu-xw)=\Sigma(zX'-xZ') \quad ...(5)$$

and $$\Sigma m(xv'-yu')-\Sigma m(xv-yu)=\Sigma(xY'-yX') \quad ...(6)$$

Hence the change in the moment of momentum about any of the axes is equal to the moment about that axis of the impulses of the external forces.

Vector method. Let $\mathbf{I}$ and $\mathbf{I}'$ be the resultant external and internal impulses acting on the particle of mass m at P. Also let the velocity of m change from $\mathbf{v}_1$ to $\mathbf{v}_2$ then

$\because$ Impulses = change in momentum

$$\therefore \mathbf{I}+\mathbf{I}'=m(\mathbf{v}_2-\mathbf{v}_1) \quad ...(1)$$

or $$\Sigma\mathbf{I}+\Sigma\mathbf{I}'-\Sigma m\mathbf{v}_2-\Sigma m\mathbf{v}_1$$

But $\Sigma\mathbf{I}'=0$, by Newton's third law

$\therefore$ we get, $\Sigma\mathbf{I}=\Sigma m\mathbf{v}_2-\Sigma m\mathbf{v}_1$

i. e. the total external impulse applied to the system of particles is equal to the change of linear momentum produced.

Now, let $\vec{OP}=\mathbf{r}$, then from (1), we get

$$\Sigma\mathbf{r}\times(\mathbf{I}+\mathbf{I}')=\Sigma\mathbf{r}\times m(\mathbf{v}_2-\mathbf{v}_1)$$

or $$\Sigma\mathbf{r}\times\mathbf{I}=\Sigma\mathbf{r}\times m\mathbf{v}_2-\Sigma\mathbf{r}\times m\mathbf{v}_1 \quad (\text{Since } \Sigma\mathbf{r}\times\mathbf{I}'=0)$$

i. e. the total vector sum of the moments of the external impulses about any point O is equal to the increase in the angular momentum produced about the same point.

EXAMPLES

Ex. 10. *Two persons are situated on a perfectly smooth horizontal plane at a distance a from each other. One of the persons, of mass M throws a ball of mass m towards the other, which reaches him in time t. prove that the first person will begin to slide along the plane with velocity ma/ (Mt).*

Sol. Let I be the impulse between the ball and the first person. If the first person throws a ball with the velocity u and begins to slide along the plane with velocity v,

then since, impulse = change in momentum

$I=M(v-0)$ (for the first person)

and $I=m(u-0)$ (for the ball)

$$mu=Mv \quad ...(1)$$

Since the ball reaches the second person in time t.

$$\therefore a=ut \quad ...(2)$$

From (2) $u=a/t$, Substituting in (1), we get

$$v=\frac{ma}{Mt}$$

Ex. 11. *A cannon of mass M, resting on a rough horizontal plane of coefficient of friction μ, is fired with such a charge that the relative velocity of the ball and cannon at the moment when it leaves the cannon is u. Show that the cannon will recoil a distance* (IFoS-2009)

$$\left(\frac{mu}{M+m}\right)^2\cdot\frac{1}{2\mu g}.$$

along the plane, m being the mass of the ball.

Sol. Let I be the impulse between the cannon and the ball. If v is the velocity of the ball and V the velocity of cannon in opposite direction, then the relative velocity of the ball and cannon at the moment the ball leaves the cannon is

$$v+V=u \text{ (given)}$$

Also since, impulse = change in momentum

$\therefore I=m(v-0)$ (for the ball)

and $I=M(V-0)$ (for the cannon)

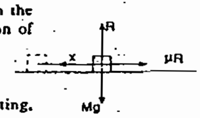
$$\therefore mv=MV \text{ or } v=\frac{MV}{m} \quad ...(2)$$

Substituting from (2), in (1), we get

$$\frac{MV}{m}+V=u \text{ or } V(M+m)=mu$$

or $$V=mu/(M+m) \quad ...(3)$$

If the cannon moves through a distance x in the direction opposite to the direction of motion of the ball in time t, then on the rough plane, for the cannon the equation of motion is



$$M\ddot{x}=-\mu R=-\mu Mg$$

or $$\ddot{x}=-\mu g$$

Multiplying both sides by $2\dot{x}$ and integrating, we get

$$\dot{x}^2=-2\mu gx+C$$

But initially when $x=0, \dot{x}=V$ (Starting velocity of the cannon)

$\therefore C=V^2.$

$\therefore \dot{x}^2=V^2-2\mu gx.$

When the cannon comes to rest $\dot{x}=0$,

$\therefore 0=V^2-2\mu gx$

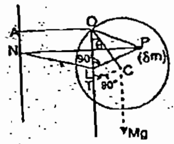
or $$x=\frac{V^2}{2\mu g}=\left(\frac{mu}{M+m}\right)^2\cdot\frac{1}{2\mu g} \quad [\text{Substituting from (3)}]$$

which is the required distance.

MISCELLANEOUS EXAMPLES

Ex. 12. *A thin circular disc of mass M and radius a, can turn freely about a thin axis OA, which is perpendicular to its plane and passes through a point O of its circumference. The axis OA is compelled to move in a horizontal plane with angular velocity ω about its end A. Show that the inclination θ to the vertical of the radius of the disc through O is $\cos^{-1}(g/a\omega^2)$, unless $\omega^2 < (g/a)$ and then θ is zero.* (IAS-2002)

Sol. Let C be the centre of the thin circular disc of mass M which can turn about a thin horizontal axis OA perpendicular to its plane and passing through a point O of its circumference. When the axis OA turns horizontally round A, the disc will be raised in its own vertical plane. Let the radius OC turn through an angle θ to the vertical in time t.



Consider an element of mass δm of the disc at P. Let PN and PL be the perpendiculars from P on the verticals through A and O respectively. Then obviously the ΔPNL will be in the horizontal plane and NL will be parallel cribe a circle of radius PN with constant angular velocity ω about the vertical through A. Thus the reversed affective force on the element δm at P along NP is $\delta m \cdot NP \cdot \omega^2$.

But $\overrightarrow{NP} = \overrightarrow{NL} + \overrightarrow{LP}$

$\therefore\ \delta m\,\omega^2\,\overrightarrow{NP} = \delta m\omega^2\,\overrightarrow{NL} + \delta m\,\omega^2\,\overrightarrow{LP}$.

Thus the reversed effective force $\delta m\omega^2 NP$ along NP is equivalent to forces $\delta m\omega^2 NL$ along NL and $\delta m\omega^2 LP$ along LP. The external forces on the disc are its weight Mg acting verticaly downwards at it centre C and the reaction at O.

By D' Alembert's principle, reversed effective forces and the external forces keep the system in equilibrium.

To avoid reaction at O, we take the moment about the axis OA.

The forces $\delta m\omega^2 NL$ along NL acts parallel to OA, hence its moment about OA vanishes.

$\therefore$ Taking moment of all the forces about OA, we have

$$Mg.CT = \Sigma\,\delta m\omega^2\,LP\,.\,OL + 0$$

or $Mg\,a\sin\theta = \omega^2\,\Sigma\,\delta m\,LP\,.\,OL$

$= \omega^2$.(P.I. of the disc about OL and the horizontal line through O)

$= \omega^2$.(P.I. of the disc about the parallel lines through C.G. 'C' + P.I. of whole mass M at C.G. 'C' about the horizontal and vertical lines through O)

$= \omega^2\,(O + M.CT.OT) = \omega^2\,Ma\sin\theta.a\cos\theta$

or $\sin\theta\,(g - a\omega^2\cos\theta) = 0$

which gives, either $\sin\theta = 0$ *i.e.* $\theta = 0$.

or $g - a\omega^2\cos\theta = 0$, *i.e.* $\cos\theta = g/a\omega^2$ or $\theta = \cos^{-1}(g/a\omega^2)$.

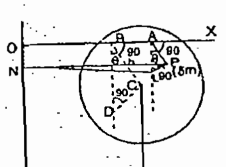
If $\omega^2 < (g/a)$, $\cos\theta > 1$, which is not possible and hence in this case $\theta = 0$ is the only possible value.

Ex. 13. *A thin heavy disc can turn freely about an axis in its own plane, and this axis revolves horizontally with a uniform angular velocity ω about a fixed point on itself. Show that the inclination θ of the plane of the disc to the vertical is given by $\cos\theta = (gh/k^2\omega^2)$ where h is the distance of the centre of inertia of the disc from the axis and k is the radius of gyration of the disc about the axis. If $\omega^2 < gh/k^2$, prove that the plane of the disc is vertical.*

Sol. Let C be the centre of a thin heavy disc of mass M which can turn about an axis OX in its own plane. When the axis revolves horizontally with a uniform angular velocity ω about a fixed point O on itself, the disc will turn about OX. Let θ be the inclination of the plane of the disc to the vertical at time t. If CB is the perpendicular from C on OX, then

$$CB = h \text{ (given)}$$

Consider an element of mass δm of the disc at P. Let PN be the perpendiculars from P on the vertical through O and PA perpendicular on OX. Let PL be the perpendicular from P on the vertical through A, then obviously the ΔPNL will be in a horizontal plane and NL will be parallel to OA. Also $\angle PAL = \theta = \angle CBD$, where CD is perpendicular from C on the vertical through B.



Now P will describe a circle of radius PN with constant angular velocity ω about the vertical through the fixed point O.

Thus the reversed effective force on the element δm at P along NP is $\delta m \cdot NP \cdot \omega^2$.

But $\overrightarrow{NP} = \overrightarrow{NL} + \overrightarrow{LP}$.

$\therefore\ \delta m\omega^2\,\overrightarrow{NP} = \delta m\omega^2\,\overrightarrow{NL} + \delta m\omega^2\,\overrightarrow{LP}$.

Thus the reversed effective force $\delta m\omega^2 NL$ along NP is equivalent to the forces $\delta m\omega^2 NL$ along NL and $\delta m\omega^2 LP$ along LP. The external forces on the disc are its weight Mg acting vertically down wards at its centre C and the reaction at the axis OX.

By D' Alembert's principle, reversed effective forces and the external forces keep the system in equilibrium.

To avoid reaction on the axis OX, we take the moment about the axis OX.

The force $\delta m\omega^2 NL$ along NL is parallel to OX, hence its moment about OX vanishes.

Therefore taking moment of all forces about OX, we have

$$Mg\,DC = \Sigma\,\delta m\omega^2 LP.AL + O$$

$$Mgh\sin\theta = \omega^2\,\Sigma\,\delta m\,.\,AP\sin\theta\,.\,AP\cos\theta\,,$$

$$= \omega^2\sin\theta\cos\theta\,\Sigma\,\delta m\,AP^2$$

$$= \omega^2\sin\theta\cos\theta\,.\ \text{(M.I. of the disc about } OX)$$

$$= \omega^2\sin\theta\cos\theta\,.\,Mk^2$$

or $\sin\theta\,(gh - \omega^2 k^2\cos\theta) = 0$,

which gives either $\sin\theta = 0$, *i.e.* $\theta = 0$, or $gh - \omega^2k^2\cos\theta = 0$, i.e. $\cos\theta = (gh/k^2\omega^2)$.

Now when $\omega^2 < gh/k^2$, $\cos\theta > 1$, which is not possible and hence in this case $\theta = 0$ is the only possible value, i.e. when $\omega^2 < (gh/k^2)$, the plane of the disc is vertical.

EXERCISE

1. State D' Alembert's principle and apply it to prove that the motions of translation and rotation of a rigid body can be regarded as independent of each other. [Hint. See § 2.7 and § 2.8]. [Raj. 80]
2. A light rod OAB can turn freely in a vertical plane about a smooth fixed hinge at O; two heavy particles of masses m and m' are attached to the rod at A and B oscillate with it. Find the motion.
3. A plank of mass M', and length $2a$ is initially at rest along a line of greatest slop of a smooth plane inclined at angle α to the horizon and a man, of mass M starting from the upper end walks down the plank, so that it does not move, show that he gets to the other end in time.

$$\left\{\frac{4Ma}{(M+M')\,g\sin\alpha}\right\}^{1/2}$$

[Hint. See Ex. 9 on page 105].

MOTION IN TWO DIMENSIONS | SET-III

4.1. Equations of Motion :

To determine the equations of motion in two dimensions when the forces acting on the body are finite.

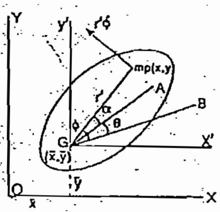
The motion of a rigid body consists of two independent motions :

(i) the motion of centre of gravity (centre of inertia), and

(ii) the motion about the centre of gravity (centre of inertia).

(i) Motion of centre of gravity *(Cartesian Method). Motion of the C.G. states that the motion of the C.G. is such that the total mass M of the rigid body is concentrated at the C.G. and all the external forces are transferred parallel to themselves and act at the C.G. of the body.*

Consider a particle of mass m at the point P whose co-ordinate with reference to two fixed axes OX and OY are (x, y).



The effective forces acting on the particle at P are $m\ddot{x}$ and $m\ddot{y}$ parallel to the axes. If X and Y are the components of the external forces acting at P, then by D' Alembert's principle the forces $X - m\ddot{x}$, $Y - m\ddot{y}$ together with similar forces acting on all other particles of the body form a system in equilibrium.

Therefore, we have

$\Sigma(X - m\ddot{x}) = 0$, $\Sigma(Y - m\ddot{y}) = 0$ and $\Sigma[x(Y - m\ddot{y}) - y(X - m\ddot{x})] = 0$.

i.e. $\Sigma m\ddot{x} = \Sigma X$, $\Sigma m\ddot{y} = \Sigma Y$ and $\Sigma m(x\ddot{y} - y\ddot{x}) = \Sigma(xY - yX)$ ...(1)

Let $(\bar{x}, \bar{y})$ be the coordinates of the C.G. G of the body. Let (x', y') be the coordinates of the point P with reference to the axes GX', GY' through G and parallel to OX and OY respectively.

$\therefore x = \bar{x} + x'$ and $y = \bar{y} + y'$.

Then $M\bar{x} = \Sigma mx$ and $M\bar{y} = \Sigma my$, where $M = \Sigma m$ = Mass of the body.

$\therefore M\ddot{\bar{x}} = \Sigma m\ddot{x}$ and $M\ddot{\bar{y}} = \Sigma m\ddot{y}$.

$\therefore$ From first two equations of (1), we have

$M\ddot{\bar{x}} = \Sigma X$ and $M\ddot{\bar{y}} = \Sigma Y$. ...(2)

which are the equations of motion of the centre of gravity.

Vector Method :

Let $\bar{\mathbf{r}}$ be the position vector of C.G. G and $\mathbf{F}$ the external force acting at any particle m of the body, then

$M\dfrac{d^2\bar{\mathbf{r}}}{dt^2} = \Sigma \mathbf{F}$ i.e. $M\ddot{\bar{\mathbf{r}}} = \Sigma \mathbf{F}$. ...(3)

Let $(\bar{x}, \bar{y})$ be the coordinates of C.G. G and X, Y the components of the force $\mathbf{F}$ parallel to the axes, then

$$\bar{\mathbf{r}} = \bar{x}\,\mathbf{i} + \bar{y}\,\mathbf{j} \text{ and } \mathbf{F} = X\,\mathbf{i} + Y\,\mathbf{j}.$$

$\therefore$ From (3), we have

$$M(\ddot{\bar{x}}\,\mathbf{i} + \ddot{\bar{y}}\,\mathbf{j}) = \Sigma(X\,\mathbf{i} + Y\,\mathbf{j}).$$

$\therefore M\ddot{\bar{x}} = \Sigma X$ and $M\ddot{\bar{y}} = \Sigma Y$

(ii) Motion about the centre of gravity. Motion about the centre of gravity states that the moments of the effective forces about the C.G. 'G' is equal to the sum of the moments of the external forces about G.

Substituting $x = \bar{x} + x'$, $y = \bar{y} + y'$ in the third equation of (1), we have

$\Sigma m[(\bar{x} + x')(\ddot{\bar{y}} + \ddot{y}') - (\bar{y} + y')(\ddot{\bar{x}} + \ddot{x}')]$

$$= \Sigma[(\bar{x} + x')Y - (\bar{y} + y')X]$$

or $(\bar{x}\ddot{\bar{y}} - \bar{y}\ddot{\bar{x}})\Sigma m + \bar{x}\Sigma m\ddot{y}' + \ddot{\bar{y}}\Sigma mx' - \bar{y}\Sigma m\ddot{x}' - \ddot{\bar{x}}\Sigma my'$

$+ \Sigma m(x'\ddot{y}' - y'\ddot{x}') = \bar{x}\Sigma Y - \bar{y}\Sigma X + \Sigma(x'Y - y'X)$ ...(4)

Now the coordinates of C.G. G with respect to the axes GX' and GY'

are $\dfrac{\Sigma mx'}{\Sigma m} = 0$ and $\dfrac{\Sigma my'}{\Sigma m} = 0$ [as coordinates of G, w.r.t. GX', GY' are (0, 0)]

or $\Sigma mx' = 0$ and $\Sigma my' = 0$.

$\therefore \Sigma m\ddot{x}' = 0$ and $\Sigma m\ddot{y}' = 0$.

Also $\Sigma m = M$ = Mass of the body.

$\therefore$ Substituting in (4), we have

$(\bar{x}\ddot{\bar{y}} - \bar{y}\ddot{\bar{x}})M + \Sigma m(x'\ddot{y}' - y'\ddot{x}') = \bar{x}\Sigma Y - \bar{y}\Sigma X + \Sigma(x'Y - y'X)$

$\therefore$ Using the equations in (2) we have

$\bar{x}\Sigma Y - \bar{y}\Sigma X + \Sigma m(x'\ddot{y}' - y'\ddot{x}') = \bar{x}\Sigma Y - \bar{y}\Sigma X + \Sigma(x'Y - y'X)$

or $\Sigma m(x'\ddot{y}' - y'\ddot{x}') = \Sigma(x'Y - y'X)$

or $\dfrac{d}{dt}\Sigma m(x'\dot{y}' - y'\dot{x}') = \Sigma(x'Y - y'X)$. ...(5)

Let θ be the angle which a line GA fixed in the body make with a line GB fixed in space. As the particle m and GA will move with the body, $\angle PGA$ will remain constant.

Let $\angle PGA = \alpha$ (constant).

$\therefore$ If $\angle PGB = \phi$, then $\phi = \theta + \alpha$ $\therefore \dot{\phi} = \dot{\theta}$ and $\ddot{\phi} = \ddot{\theta}$

Let $GP = r'$, therefore velocity of m at P is $r'\dot{\phi}$ perpendicular to GP in the plane AGP and its moment about G is $r'\dot{\phi} \cdot r' = r'^2\dot{\phi}$.

$\therefore \Sigma m(x'\dot{y}' - y'\dot{x}') = \Sigma mr'^2\dot{\phi}$

$$= \Sigma mr'^2\dot{\theta} = \dot{\theta}\,\Sigma mr'^2 = Mk^2\dot{\theta},$$

where k is the radius of gyration of the body about G.

Hence from (5), we have

$\dfrac{d}{dt}(Mk^2\dot{\theta}) = \Sigma(x'Y - y'X)$ or $Mk^2\ddot{\theta} = L$. ...(6)

where $L = \Sigma(x'Y - y'X)$ is the moment of the external forces about G.

Equation (6) is the equation of motion of the body relative to the centre of gravity.

Vector Method :

Let $\mathbf{r}'$ be the position vector of the particle m at P relative to the centre of gravity G and $\mathbf{F}$ the external force acting on it, then we have

$\Sigma \mathbf{r}' \times m\dfrac{d^2\mathbf{r}'}{dt^2} = \Sigma \mathbf{r}' \times \mathbf{F}$ or $\dfrac{d}{dt}\left(\Sigma m\mathbf{r}' \times \dfrac{d}{dt}\mathbf{r}'\right) = \Sigma \mathbf{r}' \times \mathbf{F}$. ...(7)

Let θ be the angle which a line GA fixed in the body make with a line GB fixed in space. As the particle m and GA will move with the body, $\angle PGA$ will remain constant.

Let $\angle PGA = \alpha$ (constant).

$\therefore$ If $\angle PGB = \phi$, then $\phi = \theta + \alpha$. $\therefore \dot{\phi} = \dot{\theta}$ and $\ddot{\phi} = \ddot{\theta}$.

Let $GP = |\mathbf{r}'| = r'$, therefore velocity of m relation to G is $r'\dot{\phi}$ perpendicular to GP in the plane AGP.

If $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$ are the unit vectors along and perpendicular to $\mathbf{r}'$ in the plane AGP, then

$\mathbf{r}' = r'\hat{\mathbf{e}}_1$ and $\dfrac{d\mathbf{r}'}{dt} = r'\dfrac{d\phi}{dt}\hat{\mathbf{e}}_2$.

$\therefore \Sigma m\mathbf{r}' \times \dfrac{d}{dt}\mathbf{r}' = \Sigma m(r'\hat{\mathbf{e}}_1) \times \left(r'\dfrac{d\phi}{dt}\hat{\mathbf{e}}_2\right) = \Sigma mr'^2\dot{\theta}(\hat{\mathbf{e}}_1 \times \hat{\mathbf{e}}_2)$

$= \dot{\theta}\,\Sigma mr'^2(\hat{\mathbf{n}})$ $(\because \dot{\phi} = \dot{\theta})$

where $\hat{\mathbf{n}}$ is the unit vector normal to the plane AGP

$= \dot{\theta}(\Sigma mr'^2)\,\hat{\mathbf{n}} = \dot{\theta}(Mk^2)\,\hat{\mathbf{n}}$

where k is the radius of gyration of the body about G.

Also $\Sigma \mathbf{r}' \times \mathbf{F} = \Sigma p'F\hat{\mathbf{n}}$, where p' is the length of the perpendicular from G upon the direction of $\mathbf{F}$.

Hence from (7), we have

$\dfrac{d}{dt}(Mk^2\dot{\theta})\,\hat{\mathbf{n}} = (\Sigma p'F)\,\hat{\mathbf{n}}$.

Equating coefficients of $\hat{\mathbf{n}}$, we have

$\dfrac{d}{dt}(Mk^2\dot{\theta}) = \Sigma p'F$ or $Mk^2\ddot{\theta} = \Sigma p'F$. ...(8)

If (x', y') are the coordinates of P relative to GX' and GY' as axes and X, Y the components of $\mathbf{F}$ in these directions, then scalar moment of the force $\mathbf{F}$ about G.

$= p'F = x'Y - y'X$.

$\therefore$ From (8), we have

$Mk^2\ddot{\theta} = \Sigma(x'Y - y'X)$.

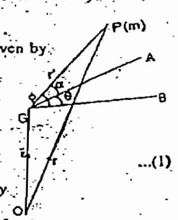
Hence the equations of motion of a rigid body moving in two dimensions are $M\ddot{\bar{x}} = \Sigma X$, $M\ddot{\bar{y}} = \Sigma Y$ and $Mk^2\ddot{\theta} = \Sigma(x'Y - y'X)$.

4.2. Kinetic Energy :

To express the kinetic energy in terms of the motion of the centre of gravity and the motion relative to the centre of gravity, when a body is moving in two dimensions (i.e. parallel to a plane).

At any time t, let $\mathbf{r}$ be the position vector (p.v.) of a particle of mass m, referred to the origin O. If $\bar{\mathbf{r}}$ is the p.v. of the C.G. G of the body, w.r.t. the origin O and $\mathbf{r}'$ the p.v. of m w.r.t. G, then

$$\mathbf{r} = \bar{\mathbf{r}} + \mathbf{r}'.$$



The kineic energy (K.E.), T of the body is given by

$T = \frac{1}{2}\Sigma m\dot{\mathbf{r}}^2$

$= \frac{1}{2}\Sigma m(\dot{\bar{\mathbf{r}}} + \dot{\mathbf{r}}')^2$

$= \frac{1}{2}\Sigma m\dot{\bar{\mathbf{r}}}^2 + \frac{1}{2}\Sigma m\dot{\mathbf{r}}'^2 + \Sigma m\dot{\bar{\mathbf{r}}} \cdot \dot{\mathbf{r}}'$

$= \frac{1}{2}\dot{\bar{\mathbf{r}}}^2\Sigma m + \frac{1}{2}\Sigma m\dot{\mathbf{r}}'^2 + \dot{\bar{\mathbf{r}}} \cdot \Sigma m\dot{\mathbf{r}}'$ ...(1)

Now the p.v. of G w.r.t. the origin at G is given by

$\dfrac{\Sigma m\mathbf{r}'}{\Sigma m}$, $\therefore \dfrac{\Sigma m\mathbf{r}'}{\Sigma m} = 0$, i.e. $\Sigma m\mathbf{r}' = 0$ and $\Sigma m\dot{\mathbf{r}}' = 0$.

Also $\Sigma m = M$. $\therefore$ From (1), we have

$T = \frac{1}{2}M\dot{\bar{\mathbf{r}}}^2 + \frac{1}{2}\Sigma m\dot{\mathbf{r}}'^2$. ...(2)

Another form. Let $\bar{\mathbf{v}}$ be the velocity of the centre of gravity G of the body. If $\hat{\mathbf{e}}$ is the unit vector perpendicular to the direction of $\mathbf{r}'$, then we have

$$\bar{\mathbf{v}} = \frac{d\bar{\mathbf{r}}}{dt} = \dot{\bar{\mathbf{r}}}$$

Let GA be a line fixed in the body and GB fixed in space.

Since m moves with the body. $\therefore AGm$ = Constant $= \alpha$ (say).

If $\angle AGB = \theta$ and $\angle mGB = \phi$, then $\phi = \theta + \alpha$. $\therefore \dot{\phi} = \dot{\theta}$.

$\therefore \dot{\mathbf{r}}'^2 = \left(r'\dfrac{d\phi}{dt}\hat{\mathbf{e}}\right)^2 = r'^2\dot{\theta}^2$. $(\because \dot{\phi} = \dot{\theta}$ and $\hat{\mathbf{e}}^2 = \hat{\mathbf{e}} \cdot \hat{\mathbf{e}} = 1)$

$\therefore$ From (2), we have

$$T=\frac{1}{2}M\dot{\bar{r}}^2+\frac{1}{2}\Sigma mr'^2\dot{\theta}^2=\frac{1}{2}M\dot{\bar{r}}^2+\frac{1}{2}\dot{\theta}^2\Sigma mr'^2$$

$$T=\frac{1}{2}Mv^2+\frac{1}{2}Mk^2\dot{\theta}^2. \quad ...(3)$$

where $|\dot{\bar{r}}|=v$ = velocity of the C.G. G, and k is the radius of gyration of the body about the centre of gravity.

The equation (3) shows that the kinetic energy of a body of mass M moving in two dimensions is equal to the K.E. of a particle of mass M placed at the C.G. and moving with it together with the K.E. of the body relative to the C.G.

i.e. K.E. of the body = K.E. due to translation + K.E. due to rotation.

4.3. Moment of Momentum (Angular Momentum) :

To find the moment of momentum of a body about the fixed origin O, when the body is moving in two dimensions.

At time t, let $\mathbf{r}$ and $\bar{\mathbf{r}}$ be the position vectors of a particle m, and the C.G. G of the body respectively w.r.t. the origin O. Also let $\mathbf{r}'$ be the position vector of the particle m w.r.t. the C.G. G.

Then $\mathbf{r}=\bar{\mathbf{r}}+\mathbf{r}'$.

Let $\mathbf{H}$ be the moment of momentum or angular momentum of the body about O, then we have

$$\mathbf{H}=\Sigma\mathbf{r}\times m\dot{\mathbf{r}}=\Sigma m\mathbf{r}\times\dot{\mathbf{r}}$$
$$=\Sigma m(\bar{\mathbf{r}}+\mathbf{r}')\times(\dot{\bar{\mathbf{r}}}+\dot{\mathbf{r}}')$$
$$=\bar{\mathbf{r}}\times\dot{\bar{\mathbf{r}}}\Sigma m+\bar{\mathbf{r}}\times\Sigma m\dot{\mathbf{r}}'+(\Sigma m\mathbf{r}')\times\dot{\bar{\mathbf{r}}}+\Sigma m\mathbf{r}'\times\dot{\mathbf{r}}'. \quad ...(1)$$

Now the p.v. of G, w.r.t. the origin at G is given by

$\frac{\Sigma m\mathbf{r}'}{\Sigma m}$, $\therefore$ $\frac{\Sigma m\mathbf{r}'}{\Sigma m}=0$, i.e. $m\mathbf{r}'=0$ and $\Sigma m\dot{\mathbf{r}}'=0$.

Also $\Sigma m=M$. $\therefore$ From (1), we have

$$\mathbf{H}=\bar{\mathbf{r}}\times M\dot{\bar{\mathbf{r}}}+\Sigma\mathbf{r}'\times m\dot{\mathbf{r}}'$$

or $\mathbf{H}=\bar{\mathbf{r}}\times M\bar{\mathbf{v}}+\Sigma\mathbf{r}'\times m\dot{\mathbf{r}}'$, $\quad ...(2)$

where $\bar{\mathbf{v}}$ is the velocity of the C.G.

Another form. Let $\hat{\mathbf{n}}$ be the unit vector parallel to H, then we have

$$\bar{\mathbf{r}}\times M\bar{\mathbf{v}}=M\bar{\mathbf{r}}\times\bar{\mathbf{v}}$$
$$=(M\bar{v}p)\,\hat{\mathbf{n}}$$

By the definition of moment, where p is the perpendicular from the origin O on the direction of the velocity $\bar{v}$ of the C.G. G.

Also $\Sigma\mathbf{r}'\times m\dot{\mathbf{r}}'=\dot{\theta}\,(Mk^2)\,\hat{\mathbf{n}}$, (see § 4.1 on p. 168) and $\mathbf{H}=H\hat{\mathbf{n}}$.

Therefore from (2), we have

$$H\hat{\mathbf{n}}=(M\bar{v}p+Mk^2\dot{\theta})\,\hat{\mathbf{n}}$$

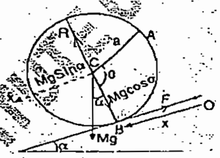
or $H=M\bar{v}p+Mk^2\dot{\theta}$.

Which shows that the moment of momentum (or angular momentum) of a rigid body about a point O is equal to the angular momentum about O of a single particle of mass M (equal to the mass of the body) at its C.G. and moving with the velocity of the centroid, together with the angular momentum of the body in its motion relative to the centroid.

i.e. Angular momentum of the rigid body = Angular momentum of the centre of gravity + Angular momentum relative to the centre of gravity.

4.4. *A uniform sphere rolls down an inclined plane, rough enough to prevent any sliding ; to discuss the motion.*

Initially, let the sphere be at rest with its points A in contact with the point O of the inclined plane. After time t, let the centre "C" of the sphere describe a distance x parallel to the inclined plane. Let CA be a line fixed in the body, make an angle θ with the normal to the plane, a line fixed in the space.



If F be the frictional force and R, the normal reaction at the point of contact B, then equations of motion of C.G. of the body are

$$M\ddot{x}=Mg\sin\alpha-F. \quad ...(1)$$

Since there is no motion perpendicular to the plane, we have

$$M\ddot{y}=0=Mg\cos\alpha-R \quad ...(2)$$

Also equation of motion about the centre of gravity is

$$Mk^2\ddot{\theta}=F\,.\,a \quad ...(3)$$

There is no sliding, $\therefore$ we have $OB=\text{arc }AB$

i.e., $x=a\theta$, $\dot{x}=a\dot{\theta}$ and $\ddot{x}=a\ddot{\theta}$. $\quad ...(4)$

From (4), $\ddot{\theta}=\ddot{x}/a$, $\therefore$ from (3), we have

$$F=\frac{1}{a}Mk^2\,.\,\frac{1}{a}\ddot{x}=\frac{Mk^2}{a^2}\frac{d^2x}{dt^2}$$

Substituting the value of F in (1), we get

$$M\ddot{x}=Mg\sin\alpha-\frac{Mk^2\ddot{x}}{a^2} \text{ or } \ddot{x}=\frac{a^2g\sin\alpha}{a^2+k^2} \quad ...(5)$$

which shows that the sphere rolls down with a constant acceleration

$$\frac{a^2g\sin\alpha}{a^2+k^2}$$

Integrating (5) we get $\dot{x}=\frac{a^2g\sin\alpha}{a^2+k^2}t+C$; and C, the constant of integration vanishes as t and $\dot{x}$ vanish together.

Integrating again, $x=\frac{1}{2}\frac{a^2g\sin\alpha}{a^2+k^2}t^2$;

because constant of integration again vanishes as x and t vanish simultaneously.

Various cases :

(i) For a solid sphere, $k^2=\frac{2}{5}a^2$ and then from (5) $\ddot{x}=\frac{5}{7}g\sin\alpha$.

(ii) For hollow sphere, $k^2=\frac{2}{3}a^2$ $\therefore$ $\ddot{x}=\frac{3}{5}g\sin\alpha$.

(iii) For a circular disc, $k^2=\frac{1}{2}a^2$ $\therefore$ $\ddot{x}=\frac{2}{3}g\sin\alpha$.

(iv) For a circular ring, $k^2=a^2$ $\therefore$ $\ddot{x}=\frac{1}{2}g\sin\alpha$.

Pure rolling : Eliminating $\ddot{x}$ from (5), and (1), we get

$$F=Mg\sin\alpha-\frac{5}{7}Mg\sin\alpha=\frac{2}{7}Mg\sin\alpha, \quad \left(\because k^2=\frac{2a^2}{5}\right)$$

Also from (2) $R=Mg\cos\alpha$.

In order that there may be no sliding, $\frac{F}{R}$ must be less than μ *i.e.* For pure rolling $F<\mu R$ *i.e.* $\mu>\frac{F}{R}=\frac{2}{7}\tan\alpha$.

EXAMPLES

Ex. 1. A *uniform solid cylinder is placed with its axis horizontal on a plane, whose inclination to the horizon is α, show that the least coefficient of friction between it and the plane so that it may roll and not slide, is* $\frac{1}{3}\tan\alpha$.

If the cylinder be hollow and of small thickness, the least value is $\frac{1}{2}\tan\alpha$.

Sol. Ref. fig § 4.4.

Let the cylinder roll down a distance x along the inclined plane in time t. If θ is the angle turned by the cylinder during this time t, then $x=a\theta$

($\because$ there is no sliding)

$\therefore$ $x=a\theta$ and $\ddot{x}=a\ddot{\theta}$

Let R be the reaction and F the frictional force.

$\therefore$ The equations of motion of C.G. of the cylinder are

$$M\ddot{x}=Mg\sin\alpha-F \quad ...(1)$$

$$\text{and } M\ddot{y}=0=Mg\cos\alpha-R. \quad ...(2)$$

Also taking moments about the axis through the centre of gravity 'G' of

$$Mk^2\ddot{\theta}=F\,.\,a \text{ or } \frac{Mk^2}{a}\ddot{x}=F\,.\,a \quad \because \ddot{x}=a\ddot{\theta}$$

$$\text{or } M\ddot{x}=\frac{a^2}{k^2}F. \quad ...(3)$$

$\therefore$ From (1) and (3), we have

$$\frac{a^2}{k^2}F=Mg\sin\alpha-F \text{ or } F=\left(\frac{a^2}{k^2}+1\right)=Mg\sin\alpha$$

$$\text{or } F=\frac{k^2}{a^2+k^2}Mg\sin\alpha. \quad ...(4)$$

From (2), $R=Mg\cos\alpha$. $\therefore$ $\frac{F}{R}=\frac{k^2}{a^2+k^2}\tan\alpha$

$\because$ For pure rolling, $\mu>\frac{F}{R}$ $\therefore$ $\mu\,.\,\frac{k^2}{a^2+k^2}\tan\alpha$

When the cylinder is solid, then $k^2=\frac{1}{2}a^2$

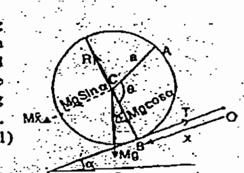
$\therefore$ for pure rolling $\mu>\frac{\frac{1}{2}a^2}{a^2+\frac{1}{2}a^2}\tan\alpha$ or $\mu>\frac{1}{3}\tan\alpha$.

In case of hollow cylinder, $k^2=a^2$,

$\therefore$ for pure rolling, $\mu>\frac{a^2}{a^2+a^2}\tan$ or $\mu>\frac{1}{2}\tan$.

Ex. 2. *A cylinder rolls down a smooth plane whose inclination to the horizontal is α, unwrapping, as it goes, a fine string fixed to the highest point of the plane; find its acceleration and tension of the string.*

Sol. Let T be the tension in the string when the cylinder has rolled down a distance x along the inclined plane, and in this time (say, t), let θ be the angle turned by the cylinder. Since the string is tight, so the motion is of pure rolling.



$$\therefore x=OB=\text{arc }AP=a\theta \quad ...(1)$$

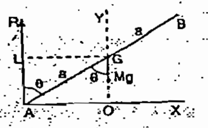
$\therefore$ $\dot{x}=a\dot{\theta}$ and $\ddot{x}=a\ddot{\theta}$

Equations of motion of the centre of gravity of the cylinder are

$$M\ddot{x}=Mg\sin\alpha-T \quad ...(2)$$

$$\text{and } M\ddot{y}=0=Mg\cos\alpha-R. \quad ...(3)$$

Also taking moments about the centre, we have *i.e.* if initially the rod will be in equilibrium in its vertical position with one end A in contact of the smooth floor at A then when it is displaced slightly, the end A will move on the horizontal floor such that the C.G. G move along the vertical line GO. At time t, let the rod be inclined at an angle θ to the vertical. Taking the point O as origin, horizontal and vertical lines through O as axes the coordinates of G are given by



$x = 0$ and $y = a\cos\theta \therefore \dot{y} = -a\sin\theta\dot{\theta}$ and $\ddot{y} = -a\cos\theta\dot{\theta}^2 - a\sin\theta\ddot{\theta}$

The equation of motion of the C.G. G is

$$M\ddot{y} = M(-a\cos\theta\dot{\theta}^2 - a\sin\theta\ddot{\theta}) = R \quad ...(1)$$

Taking moment about G, we have

$$Mk^2\ddot{\theta} = R.GL \text{ or } M(a^2/3)\ddot{\theta} = R.a\sin\theta \quad ...(2)$$

Also the energy equation gives

K.E. $= \frac{1}{2}M\dot{y}^2 + \frac{1}{2}Mk^2\dot{\theta}^2 =$ Work done.

or $\frac{1}{2}M(\dot{x}^2 + \dot{y}^2 + \frac{1}{3}a^2\dot{\theta}^2) = Mg(a - a\cos\theta)$

or $\frac{1}{2}(a^2\sin^2\theta + \frac{1}{3}a^2)\dot{\theta}^2 = ga(1 - \cos\theta)$

$$\therefore \dot{\theta}^2 = \frac{6g(1-\cos\theta)}{a(1 + 3\sin^2\theta)} \quad ...(3)$$

Differentiating (3) w.r.t. 't', we have

$$2\dot{\theta}\ddot{\theta} = \frac{6g}{a}\left[\frac{\sin\theta}{(1+3\sin^2\theta)} - \frac{6\sin\theta\cos\theta(1-\cos\theta)}{(1+3\sin^2\theta)^2}\right]\dot{\theta}$$

$$\text{or } \ddot{\theta} = \frac{3g}{a}\left[\frac{1 + 3\sin^2\theta - 6\cos\theta(1-\cos\theta)}{(1+3\sin^2\theta)^2}\right]\sin\theta$$

$$\text{or } \ddot{\theta} = \frac{3g}{a}\left[\frac{4 - 6\cos\theta + 3\cos^2\theta}{(1+3\sin^2\theta)^2}\right]\sin\theta \quad ...(4)$$

$\therefore$ From (2) and (4), we have

$$R = Mg\left[\frac{4 - 6\cos\theta + 3\cos^2\theta}{(1+3\sin^2\theta)^2}\right] = Mg\left[\frac{1 + 3(1-\cos\theta)^2}{(1+3\sin^2\theta)^2}\right] \quad ...(5)$$

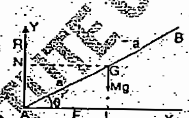
From (5) it is clear that R is always positive for any value of R. So our assumption throughout the motion that one end of the rod is always incontact with the floor is correct.

When $\theta = \pi/2$, *i.e.* just before the rod strikes the floor, $R = \frac{1}{4}Mg$.

Ex. 6. *A uniform rod is held at an inclination α to the horizon with one end in contact with a horizontal table whose coefficient of friction is μ. If it be then released show that it will commence to slide if*

$$\mu < \left(\frac{3\sin\alpha\cos\alpha}{1 + 3\sin^2\alpha}\right)$$

Sol. Let AB be the rod of mass M and length $2a$. Let F be the frictional force and R the normal reaction. Taking the point A as origin and the horizontal and vertical lines through A as axes, the coordinates of the C.G. G of the rod are given by



$x = a\cos\theta, y = a\sin\theta$

Equations of motion of C.G. are

$$M\ddot{x} = M[-a\cos\theta\,\dot{\theta}^2 - a\sin\theta\,\ddot{\theta}] = F \quad ...(1)$$

$$M\ddot{y} = M[-a\sin\theta\,\dot{\theta}^2 + a\sin\theta\,\ddot{\theta}] = R - Mg \quad ...(2)$$

Initially, when the rod was inclined at an angle α to the horizontal, the coodinates of G were $(a\cos\alpha, a\sin\alpha)$.

$\therefore$ Vertical downward displacement of $G = a(\sin\alpha - \sin\theta)$

$\therefore$ The equation of energy is

K.E. at time t = work done by the gravity.

$\therefore \frac{1}{2}M[(\dot{x}^2 + \dot{y}^2) + k^2\dot{\theta}^2] = Mg(a\sin\alpha - a\sin\theta)$

or $\frac{1}{2}M(a^2\dot{\theta}^2 + \frac{1}{3}a^2\dot{\theta}^2) = aMg(\sin\alpha - \sin\theta)$

$$\therefore \dot{\theta}^2 = \frac{3g}{2a}(\sin\alpha - \sin\theta) \quad ...(3)$$

Differentiating (3) w.r.t. t, we get

$$\ddot{\theta} = \frac{-3g}{4a}\cos\theta.$$

Putting the values of $\dot{\theta}^2$ and $\ddot{\theta}$ from equations (3) and (4) in (1) and (2), we have

$$F = M\left[-a\cos\theta.\frac{3g}{2a}(\sin\alpha - \sin\theta) - a\sin\theta\left(\frac{-3g}{4a}\cos\theta\right)\right]$$

$$= \frac{3}{4}Mg\cos\theta(3\sin\theta - 2\sin\alpha)$$

$$\text{and } R = Mg + M\left[-a\sin\theta.\frac{3g}{2a}(\sin\alpha - \sin\theta) + a\cos\theta\left(\frac{-3g}{4a}\cos\theta\right)\right]$$

$$= \frac{1}{4}Mg[4 - 6\sin\theta\sin\alpha + 6\sin^2\theta - 3\cos^2\theta]$$

When $\theta = \alpha$, $F = \frac{3}{4}Mg\cos\alpha\sin\alpha$ and $R = \frac{1}{4}Mg(4 - 3\cos^2\alpha)$

$$= \frac{1}{4}Mg[1 + 3(1 - \cos^2\alpha)] = \frac{1}{4}Mg(1 + 3\sin^2\alpha)$$

The end A will commence to slide if

$$\mu < \frac{F}{R} \text{ i.e. } \mu < \frac{3\sin\alpha\cos\alpha}{1 + 3\sin^2\alpha}$$

Ex. 7. *A uniform rod is held at an inclination 45° to the vertical with one end in contact with a horizontal table whose coefficient of friction is μ, if it is then released, show that it will commence to slide if $\mu < 3/5$.*

Sol. *Putting $\alpha = 45°$ in the last Ex. 6, we have*

$$\mu < \frac{3\sin 45°\cos 45°}{1 + 3\sin^2 45°} \text{ or } \mu < 3/5.$$

Ex. 8. The lower end of a uniform rod, inclined initially at an angle α to the horizon is placed on a smooth horizontal table. A horizontal force is applied to its lower end of such a magnitude that the rod rotates in vertical plane with constant angular velocity ω. Show that when the rod is inclined at an angle θ to the horizon the magnitude of the force is $Mg\cot\theta - Ma\omega^2\cos\theta$, where M is the mas of the rod.

Sol. (Refer fig. of Ex. 6 on p. 181).

Let AB be the rod of mass M and length $2a$, inclined, initially at an angle α to the horizontal. Let F be the horizontal force applied to the lower end A, so that the rod rotates in a vertical plane with angular velocity ω.

At any time t, let the rod make an angle θ to the horizontal. Since the rod rotates with a constant angular velocity ω in a vertical plane.

$\therefore \dot{\theta} = \omega$ (constant), so that $\ddot{\theta} = 0$

$\because GL = a\sin\alpha.$

$\therefore$ The equation of motion of the C.G. G along the vertical direction is

$$M\frac{d^2}{dt^2}(a\sin\theta) = Ma(-a\sin\theta\dot{\theta}^2 + \cos\theta\ddot{\theta}) = R - Mg$$

$$R = Mg - Ma\omega^2\sin\theta \quad ...(1)$$

$\dot{\theta} = \omega$ and $\ddot{\theta} = 0$.

The equation of motion of the C.G. in the horizontal direction is not written as the end A is not fixed.

Taking moment about G, we have

$Mk^2\ddot{\theta} = -R.GN + F.GL$

or $0 = -R.a\sin\theta + F.a\cos\theta.\ (\because \ddot{\theta} = 0)$

$\therefore F = R\cot\theta = (Mg - Ma\omega^2\sin\theta)\cot\theta$

or $F = Mg\cot\theta - Ma\omega^2\cos\theta.$

Ex. 9. *A uniform rod is held nearly vertically with one end resting on an imperfectly rough plane. It is released from rest and fall forwards. The inclination to the vertical at any instant is θ. Prove that :*

(i) If the coefficient of friction is less than a certain finite amount, the lower end of the rod will slip back ward before.

$$\sin^2(\theta/2) = (1/6).$$

(ii) However great the coefficient of friction may be, the lower end will begin to slip forward at a value of $\sin^2(\theta/2)$ between $\frac{1}{6}$ and $\frac{1}{3}$.

Sol. (i) Proceeding as in § 4.5, we have

$$F = \frac{3}{4}Mg\sin\theta(3\cos\theta - 2) \text{ and } R = \frac{1}{4}Mg(1 - 3\cos\theta)^2$$

Obviously, $F = 0$ if $\sin\theta = 0$ or $3\cos\theta - 2 = 0$

$\sin\theta = 0$, gives $\theta = 0$

$3\cos\theta - 2 = 0$, gives $1 - 2\sin^2(\theta/2) = \frac{2}{3}$ or $\sin^2(\theta/2) = \frac{1}{6}$

$\therefore F = 0$, when $\theta = 0$ or $\sin^2(\theta/2) = \frac{1}{6}$

The value of F is positive when θ takes all intermediate values between $\theta = 0$ and $\theta = \cos^{-1}\frac{2}{3}$ and is continuous function of θ, hence between these two values of θ, where F vanishes, F has a maximum value for some θ. Let F_1 be the maximum value. We observe that for $0 \le \theta < \cos^{-1}\frac{2}{3}$ the value $R \le Mg$.

Thus there is a finite value of μ for which $F_1 > \mu R$ and therefore for this value of μ, sliding will take place before $\cos^{-1}\frac{2}{3}$ *i.e.* before $\sin^2(\theta/2) = \frac{1}{6}$. Since F is positive (in the forward direction), hence the slipping will start in the backward direction.

(ii) We observe from the value of F that if $\cos\theta > 3/2$, F changes its sign, i.e. direction of the friction is reversed if

$F' = -F = \frac{3}{4}mg\sin\theta(2 - 3\cos\theta)$.

Now the slipping may start when $F' > \mu R$,

i.e. when $3\sin\theta(2 - 3\cos\theta) > \mu(1 - 3\cos\theta)^2$. $\quad$ ...(1)

As θ increases from $\cos^{-1}\frac{2}{3}$ to $\cos^{-1}\frac{1}{3}$, the term on the left hand side increase while the right hand side term decrease from 1 to 0. Therefore for some value of θ between $\cos^{-1}\frac{2}{3}$ and $\cos^{-1}\frac{1}{3}$ i.e. when $\sin^2(\theta/2)$ lies

$Mk^2\ddot{\theta} = T \cdot a$ or $M \cdot \frac{1}{2}a^2\ddot{\theta} = T \cdot a$

or $\frac{1}{2}M\ddot{x} = T$. $[\because \ddot{x} = a\ddot{\theta}]$ ...(5)

From (2) and (4), we have

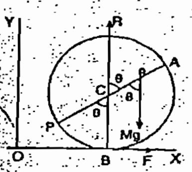
$Mg \sin \alpha = M\ddot{x} + T = M\ddot{x} + \frac{1}{2}M\ddot{x} = \frac{3}{2}M\ddot{x}$, i.e. $\ddot{x} = \frac{2}{3}g \sin \alpha$.

$\therefore$ From (5), $T = \frac{1}{2}M\ddot{x} = \frac{1}{3}Mg \sin \alpha$.

Ex. 3. *A circular cylinder, whose centre of inertia is at a distance c from axis, rolls on a horizontal plane. If it be just started from a position of unstable equilibrium. Show that the normal reaction of the plane when the centre of mass is in its lowest position is* $\left[1 + \frac{4c^2}{(a-c)^2 + k^2}\right]$ *times its weight, where k is the radius of gyration about an axis through the centre of mass.*

Sol. Initially let the point of contact P of the cylinder be at O when its centre of gravity G was vertically above the centre C of the cylinder.

In time t let the radius through G turn through an angle θ, and let B be the point of contact of the cylinder to the horizontal plane at this time t.



Taking O as origin and horizontal and vertical line as axes, the co-ordinates $(\bar{x}, \bar{y})$ of G are given by

$$\bar{x} = a\theta + c \sin \theta,\ \bar{y} = a + c \cos \theta.$$

$(\because CG = c$ and $Ob =$ Arc $BP = a\theta)$

Equations of motion of C.G. are

$$M\frac{d^2\bar{x}}{dt^2} = M\frac{d^2}{dt^2}(a\theta + c \sin \theta) = F \quad ...(1)$$

$$\text{and } M\frac{d^2\bar{y}}{dt^2} = M\frac{d^2}{dt^2}(a + c \cos \theta) = R - Mg. \quad ...(2)$$

Also energy equation gives

$\frac{1}{2}M[\dot{\bar{x}}^2 + \dot{\bar{y}}^2] + k^2\dot{\theta}^2] =$ work done by the forces.

i.e. $\frac{1}{2}M[(a\dot{\theta} + c \cos \theta\dot{\theta})^2 + (-c \sin \theta\dot{\theta})^2] + \frac{1}{2}Mk^2\dot{\theta}^2$

$$= Mg(c - c \cos \theta) \quad ...(3)$$

Let ω be the angular velocity when G is in its lowest position i.e. $\dot{\theta} = \omega$ when $\theta = \pi$;

$\therefore$ From (3), we have

$\frac{1}{2}M[(a-c)^2 + k^2]\omega^2 = 2mgc$ i.e. $\omega^2 = \frac{4gc}{k^2 + (a-c)^2}$

From (2), we have $R = Mg - Mc(\sin \theta\ddot{\theta} + \cos \theta\dot{\theta}^2)$

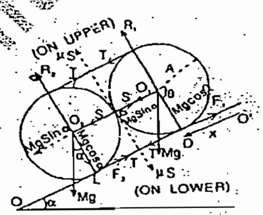
When the C.G. 'G' is in its lowest position, i.e. when $\theta = \pi$, then $\dot{\theta} = \omega$.

In this position $R = Mg - Mc \cos \pi\omega^2$.

$$= Mg + Mc\frac{4cg}{k^2 + (a-c)^2} = Mg\left[1 + \frac{4c^2}{k^2 + (a-c)^2}\right].$$

Ex. 4. *Two equal cylinders, each of mass M, are bound together by an elastic string, whose tension is T, and roll with their axes horizontal down a rough plane of inclination* α. *Show that their acceleration is* $\frac{2}{3}g \sin \alpha\left[1 - \frac{2\mu T}{Mg \sin \alpha}\right]$, *where* μ *is the coefficient of friction between the cylinder.*

Sol. Let the two equal cylinders, each of mass M and centres O_1 and O_2, bounded by an elastic string whose tension is T, roll down the inclined plane. Let R_1, F_1 be the normal reaction and friction on the upper cylinder and R_2, F_2 be the normal reaction and friction on the lower cylinder due to the plane. Let S be the normal reaction between the two cylinders at P. The frictional force μS between the two cylinders acts away from the plane for upper cylinder and towards the plane for the lower cylinder.



At any time t let the cylinders move through a distance x along the plane in downward direction and θ be the angle turned by them.

As there is no slipping, we have

$x = a\theta$, i.e. $\ddot{x} = a\ddot{\theta}$ ...(1)

Equations of motion of the upper cylinder are given by

$M\ddot{x} = Mg \sin \alpha + 2T - F_1 - S$ ...(2)

$M\ddot{y} = 0 = R_1 - Mg \cos \alpha + \mu S$ ...(3)

and $Mk^2\ddot{\theta} = F_1 \cdot a - \mu S \cdot a$ ...(4)

The equations of motion for the lower cylinder are given by

$M\ddot{x} = Mg \sin \alpha - 2T - F_2 + S$ ...(5)

$M\ddot{y} = 0 = R_2 - Mg \cos \alpha - \mu S$ ...(6)

and $Mk^2\ddot{\theta} = F_2 \cdot a - \mu S \cdot a$ ...(7)

Subtracting (7) from (4) and (5) from (2), we have

$F_1 = F_2$ and $S = 2T$. ...(8)

From (4), we have

$F_1 = \frac{Mk^2}{a}\ddot{\theta} + \mu S = Ma\ddot{\theta} + \mu S\left(\because k^2 = \frac{a^2}{2}\right)$

$= \frac{1}{2}M\ddot{x} + 2\mu T$ [From (1) and (8)]

$\therefore$ From (2), we have

$M\ddot{x} = Mg \sin \alpha + 2T - (\frac{1}{2}M\ddot{x} + 2\mu T) - 2T\ (\because S = 2T)$

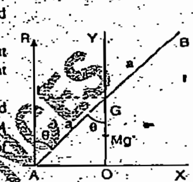
or $\ddot{x} = \frac{2}{3}g \sin \alpha\left[1 - \frac{2\mu T}{Mg \sin \alpha}\right]$.

4.5. Slipping of rods. (one end on a rough horizontal plane):

A uniform rod is held in a vertical position with one end resting upon a perfectly rough table and when released rotates about the end in contact with the table. To discuss the motion.

Let AB be the rod of mass M and length $2a$.

Let the rod which is rotating about A makes an angle θ with the vertical at any time t.



Let A be takes as the origin and horizontal and vertical lines through A as axes. Then the coordinates $(\bar{x}, \bar{y})$ of centre of gravity G are given by

$\bar{x} = a \sin \theta, \bar{y} = a \cos \theta$

$\therefore \dot{\bar{x}} = a \cos \theta\dot{\theta}, \dot{\bar{y}} = -a \sin \theta\dot{\theta}$

and $\ddot{\bar{x}} = -a \sin \theta\dot{\theta}^2 + a \cos \theta\ddot{\theta}, \ddot{\bar{y}} = -a \cos \theta\dot{\theta}^2 - a \sin \theta\ddot{\theta}$

Let F be the frictional force and R the normal reaction at A.

Then the equatons of motion of C.G. are

$M\ddot{x} = M[a \cos \theta\ddot{\theta} - a \sin \theta\dot{\theta}^2] = F$ ...(1)

and $M\ddot{y} = M[-a \sin \theta\ddot{\theta} - a \cos \theta\dot{\theta}^2] = R - Mg$ ...(2)

Taking moment about G we have

$Mk^2\ddot{\theta} = R \cdot GL - F \cdot GN$

or $M\frac{a^2}{3}\ddot{\theta} = Ra \sin \theta - Fa \cos \theta$

$= [Mg + M(-a \sin \theta\ddot{\theta} - a \cos \theta\dot{\theta}^2)]\, a \sin \theta - M(a \cos \theta\ddot{\theta}$

$- a \sin \theta\dot{\theta}^2]\, a \cos \theta$

$= Mg\, a \sin \theta - Ma^2\ddot{\theta}$ [From (1) and (2)]

or $\ddot{\theta} = \frac{3g}{4a} \sin \theta$ ...(3)

Multiplying by $2\dot{\theta}$ and integrating, we get $\dot{\theta}^2 = -\frac{3g}{2a} \cos \theta + C$

But when $\theta = 0, \dot{\theta} = 0$, $\therefore C = \frac{3g}{2a}$, $\therefore \dot{\theta}^2 = \frac{3g}{2a}(1 - \cos \theta)$ ...(4)

Substituting the values of $\dot{\theta}^2$ and $\ddot{\theta}$, from (1) and (2) we have

$F = M[a \cos \theta] \cdot \frac{3g}{4a} \sin \theta - a \sin \theta \cdot \frac{3g}{2a}(1 - \cos \theta)$

$= \frac{3}{4}Mg \sin \theta\,(3 \cos \theta - 2)$ ...(5)

and $R = Mg + M\left[-a \sin \theta \cdot \frac{3g}{4a} \sin \theta - a \cos \theta \cdot \frac{3g}{2a}(1 - \cos \theta)\right]$

$= \frac{1}{4}Mg[4 - 3(1 - \cos^2 \theta) - 6 \cos \theta(1 - \cos \theta)]$

$= \frac{1}{4}Mg(1 - 6 \cos \theta + 9 \cos^2 \theta) = \frac{1}{4}Mg(1 - 3 \cos \theta)^2$ ...(6)

From (6) it is clear that R does not change its sign and vanishes when $\cos \theta = \frac{1}{3}$, hence the end A does not leave the plane.

Also from (5) we see that F changes its sign as θ passes through the angle $\cos^{-1}\frac{2}{3}$; thus its direction is then reversed.

$R = 0$, when $\cos \theta = \frac{1}{3}$, hence the ratio F/R becomes infinite when $\cos \theta = \frac{1}{3}$ i.e. unless the plane be infinitely rough there will be sliding at this value of θ. In practive the end A of the rod begins to slip for some value of θ less than $\cos^{-1}\frac{1}{3}$. The end A will slip back wards or forward according as the slipping takes place before or after the inclination of the rod is $\cos^{-1}\frac{2}{3}$.

EXAMPLES

Ex. 5. *A uniform rod is placed in a vertical position with one end on a smooth horizontal floor. It is then let go, and it falls to the floor from rest in a vertical position. To find its angular velocity in any position. To find its angular velocity in any position and the pressure on the floor.*

Sol. The rod will be in equilibrium position when it is vertical with one end resting an the smooth floor. The rod will begin to move when it is displaced slightly from its vertical position. Since there is no horizontal force so the centre of gravity G of the rod will move in a vertical line between $\frac{1}{6}$ and $\frac{1}{2}$, the condition (I) is satisfied and the slipping will then start in the forward direction.

Ex. 10. *A uniform rod is placed with one end in contact with a horizontal table, and is then at an inclination α to the horizon and is allowed to fall. When it becomes horizontal, show that its angular velocity is $\sqrt{\left(\frac{3g}{2a}\sin\alpha\right)}$ whether the plane is perfectly smooth or perfectly rough. Show also that the end of the rod will not leave the plane in either case.*

Sol. (Refer fig. of Ex. 6 on p. 181)

Let AB be the rod of mass M and length $2a$ resting with end A on the horizontal table. Let the rod be allowed to fall at an inclination α to the horizontal.

Let at any instant t, the rod make an angle θ with the horizontal. Let R and F be normal reaction and frictional force at this instant. Taking O as origin and coordinate axes along the horizontal and vertical through A the coordinates of G are given by

$x = a\cos\theta, y = a\sin\theta.$

Case I. When plane is perfectly rough.

The energy equation gives

$\frac{1}{2}M(\dot{x}^2+\dot{y}^2)+\frac{1}{2}Mk^2\dot{\theta}^2 =$ work done by gravity

or $\frac{1}{2}M\left(a^2\dot{\theta}^2+\frac{1}{3}a^2\dot{\theta}^2\right) = Mga(\sin\alpha - \sin\theta)$

or $\dot{\theta}^2 = \frac{3g}{2a}(\sin\alpha - \sin\theta).$ ...(1)

When the rod becomes horizontal i.e. when $\theta = 0$, the angular velocity $\dot{\theta} = \omega$ (say) is given by

$\omega^2 = \frac{3g}{2a}\sin\alpha$ or $\omega = \sqrt{\left(\frac{3g}{2a}\sin\alpha\right)}.$

Differentiating (1) w.r.t. t, we have $\ddot{\theta} = \frac{-3g}{4a}\cos\theta.$ ...(2)

The equation of motion of C.G., in vertical direction is

$M\frac{d^2}{dt^2}(a\sin\theta) = Ma(-\sin\theta\dot{\theta}^2+\cos\theta\ddot{\theta}) = R - Mg.$

$\therefore R = Mg + Ma\left[-\sin\theta.\frac{3g}{2a}(\sin\alpha-\sin\theta)+\cos\theta\left(-\frac{3g}{4a}\cos\theta\right)\right].$

[Substituting the values of $\dot{\theta}^2$ and $\ddot{\theta}$ from (1) and (2)]

or $R = \frac{1}{4}Mg(4 - 6\sin\alpha\sin\theta + 6\sin^2\theta - 3\cos^2\theta)$

$= \frac{1}{4}Mg[(1-3\sin\alpha\sin\theta)^2 - 9\sin^2\alpha\sin^2\theta + 6\sin^2\theta + 3(1-\cos^2\theta)]$

$= \frac{1}{4}Mg[(1-3\sin\alpha\sin\theta)^2 + 9\sin^2\theta(1-\sin^2\alpha)]$

$= \frac{1}{4}Mg[(1-3\sin\alpha\sin\theta)^2 + 9\sin^2\theta\cos^2\alpha].$

This shows that R is always positive. Hence, the end A of the rod never leaves the plane.

Case II. When the plane is perfectly smooth.

In this case there is no horizontal forces, hence C.G. moves in a vertical line i.e. the velocity of G is only in the vertical direction.

$\because y = a\sin\theta, \therefore \dot{y} = \cos\theta\dot{\theta}.$

The energy equation gives

$\frac{1}{2}M\dot{y}^2+\frac{1}{2}Mk^2\dot{\theta}^2 =$ work done by gravity

i.e. $\frac{1}{2}M\left(a^2\cos^2\theta\dot{\theta}^2+\frac{1}{3}a^2\dot{\theta}^2\right) = Mg(a\sin\alpha - a\sin\theta)$

or $\dot{\theta}^2\left(\cos^2\theta+\frac{1}{3}\right) = \left(\frac{2g}{a}\right)(\sin\alpha-\sin\theta)$ ...(3)

when the rod becomes horizontal i.e. when $\theta = 0$, the angular velocity $\dot{\theta} = \omega$ (say) is given by

$\omega^2\left(1+\frac{1}{3}\right) = \frac{2g}{a}\sin\alpha$ or $\omega^2 = \frac{3g}{2a}\sin\alpha \therefore \omega = \sqrt{\left(\frac{3g}{2a}\sin\alpha\right)}.$

This gives the angular velocity, when the plane is perfectly smooth.

Differentiating (3) w.r.t. t, we have

$2\dot{\theta}\ddot{\theta}\left(\cos^2\theta+\frac{1}{3}\right) - 2\dot{\theta}^2\sin\theta\cos\theta\dot{\theta} = -2\left(\frac{g}{a}\right)\cos\theta\dot{\theta}$

or $\ddot{\theta}\left(\cos^2\theta+\frac{1}{3}\right) - \dot{\theta}^2\sin\theta\cos\theta = -\left(\frac{g}{a}\right)\cos\theta$

or $\ddot{\theta}\left(\cos^2\theta+\frac{1}{3}\right) - \sin\theta\cos\theta\left[\frac{(2g/a)(\sin\alpha-\sin\theta)}{\cos^2\theta+\frac{1}{3}}\right] = -\left(\frac{g}{a}\right)\cos\theta$

or $\ddot{\theta}\left(\cos^2\theta+\frac{1}{3}\right)^2 = \sin\theta\cos\theta\left[\frac{2g}{a}(\sin\alpha-\sin\theta)\right] - \frac{g}{a}\cos\theta\left(\cos^2\theta+\frac{1}{3}\right)$

$= -(g/a)\cos\theta\left[-2\sin\theta(\sin\alpha-\sin\theta)+\cos^2\theta+\frac{1}{3}\right]$

$= -(g/a)\cos\theta\left[\sin^2\theta - 2\sin\theta\sin\alpha + \sin^2\alpha\right.$
$\left. \qquad + \sin^2\theta + \cos^2\theta + \frac{1}{3} - \sin^2\alpha\right]$ ...(4)

$= -(g/a)\cos\theta\left[(\sin\theta-\sin\alpha)^2 + \frac{1}{3} + \cos^2\alpha\right]$

or $\ddot{\theta}(3\cos^2\theta+1)^2 = (-3g/a)\cos\theta\left[3(\sin\theta-\sin\alpha)^2+1+3\cos^2\alpha\right]$ ...(4)

Also taking moment about G, we have

$Mk^2\ddot{\theta} = R.GN$

or $M(a^2/3)\ddot{\theta} = -Ra\cos\theta$ or $R = -(M/3)a\sec\theta\ddot{\theta}.$

$\therefore$ Substituting from (4), we have

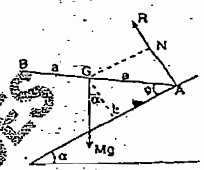
$R = Mg\left[\frac{3(\sin\theta-\sin\alpha)^2+1+3\cos^2\alpha}{(3\cos^2\theta+1)^2}\right].$

Clearly R is positive for every value of α and θ.

Hence the end A never leaves the plane.

Ex. 11. *A uniform rod of mass M is placed at right angles to a smooth plane of inclination α with one end in contact with it. The rod is then released. Show that when the inclination to the plane is ϕ, the reaction of the plane will be*

$$Mg\left[\frac{3(1-\sin\phi)^2+1}{(1+3\cos^2\phi)^2}\right]\cos\alpha$$



Sol. Let AB be the rod of mass M and length $2a$ placed at right angles to the smooth plane of inclination α with one end A in contact with it. As there is no force acting along the inclined plane, so initially there is no motion along this plane i.e. the C.G. G moves perpendicular to the plane. Let ϕ be the angle which the rod make with the inclined plane at time t.

$\therefore GL = a\sin\phi = y$ (say). Therefore the equation of motion of G, perpendicular to the inclined plane is

$M\frac{d^2}{dt^2}(y) = M\frac{d^2}{dt^2}(a\sin\phi) = R - Mg\cos\alpha.$

$\therefore R = Mg\cos\alpha + M(a\cos\phi\ddot{\phi} - a\sin\phi\dot{\phi}^2)$ ...(1)

Taking moment about G, we have

$Mk^2\ddot{\phi} = -R.GN$

or $M\frac{1}{3}a^2\ddot{\phi} = -R\,a\cos\phi.$ ...(2)

Also the energy equation gives

Sum of K.E. $= \frac{1}{2}Mv^2+\frac{1}{2}Mk^2\dot{\phi}^2 =$ Work done by gravity

or $\frac{1}{2}M(0+\dot{y}^2)+\frac{1}{2}M.\frac{1}{3}a^2\dot{\phi}^2 = Mg\cos\alpha(a - a\sin\phi)$

or $\frac{1}{2}M(a\cos\phi\dot{\phi})^2+\frac{1}{6}Ma^2\dot{\phi}^2 = Mg\,a\cos\alpha(1-\sin\phi)$

or $\dot{\phi}^2 = \frac{6g(1-\sin\phi)}{a(1+3\cos^2\phi)}\cos\alpha.$ ...(3)

Differentiating w.r.t. t, we have

$2\dot{\phi}\ddot{\phi} = \frac{6g\cos\alpha}{a}\left[\frac{-\cos\phi}{(1+3\cos^2\phi)}+\frac{6\cos\phi\sin\phi(1-\sin\phi)}{(1+3\cos^2\phi)^2}\right]\dot{\phi}$

or $\ddot{\phi} = \frac{3g}{a}\cos\alpha\left[\frac{-(1+3\cos^2\phi)+6\sin\phi(1-\sin\phi)}{(1+3\cos^2\phi)^2}\right]\cos\phi$

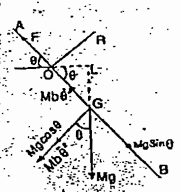
$= \frac{3g}{a}\cos\phi\cos\alpha\left[\frac{-1-3(1-\sin^2\phi)+6(\sin\phi-\sin^2\phi)}{(1+3\cos^2\phi)^2}\right]$

$= -\frac{3g}{a}\cos\phi\cos\alpha\left[\frac{1+3(1-\sin\phi)^2}{(1+3\cos^2\phi)^2}\right]$

$\therefore$ From (2), we have

$R = Mg\left[\frac{3(1-\sin\phi)^2+1}{(1+3\cos^2\phi)^2}\right].\cos\alpha.$

Ex. 12. *A rough uniform rod, of length $2a$, is placed on a rough table at right angles to its edge; if its centre of gravity be initially at distance b beyond the edge, show that the rod will begin to slide when it has turned through an angle $\frac{\mu a^2}{a^2+9b^2}$ where μ is the coefficient of friction.*



Sol. Let AB be the rod of mass M and length $2a$. Initially the rod was at right angles to the edge of the rough table.

In time t let the rod turn through an angle θ. Let there be no sliding when the rod has turned through this angle. Let F and R be the normal reaction and the force of friction on the rod. Accelerations of G along and perpendicular to GO are respectively $b\dot{\theta}^2$ and $b\ddot{\theta}$. Where $OG = b$. Equations of motion of centre of gravity G are

$Mb\ddot{\theta} = Mg\cos\theta - R$ ...(1)

and $Mb\dot{\theta}^2 = F - Mg\sin\theta$ ...(2)

Taking moments about O, the point of contact of the rod and table, we have

$Mk^2\ddot{\theta} = Mg.OL = Mg.b\cos\theta$

or $M\left(b^2+\frac{a^2}{3}\right)\ddot{\theta} = Mg\,b\cos\theta \therefore \ddot{\theta} = \frac{3gb}{a^2+3b^2}\cos\theta$ ...(3)

Multiplying (3) by $2\dot{\theta}$ and integrating, we have,

$$\dot{\theta}^2 = \frac{6gb}{a^2+3b^2}\sin\theta + C$$

Initially when $\theta = 0, \dot{\theta} = 0 \therefore C = 0$

$\therefore \dot{\theta}^2 = \frac{6gb}{a^2+3b^2}\sin\theta$ ...(4)

Putting the values of $\ddot{\theta}$ and $\dot{\theta}^2$ in (1) and (2), we have

$R = -Mb\cdot\frac{3gb}{a^2+3b^2}\cos\theta + Mg\cos\theta = \frac{Mga^2}{a^2+3b^2}\cos\theta$

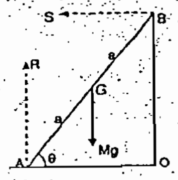
and $F = Mg\sin\theta + Mb\frac{6gb}{a^2+3b^2}\sin\theta = Mg\frac{a^2+9b^2}{a^2+3b^2}\sin\theta.$

The rod will begin to slide when $F = \mu R$,

i.e. when $Mg\frac{a^2+9b^2}{a^2+3b^2}\sin\theta = \mu\frac{Mga^2}{a^2+3b^2}\cos\theta$

or when $\tan\theta = \frac{\mu a^2}{a^2+9b^2}$.

4.6. *A uniform straight rod slides down in a vertical plane, its ends being in contact with two smooth planes, one horizontal and the other vertical. If it started from rest an angle α with the horizontal; to discuss the motion.*



Let AB be the rod of mass M and length $2a$ sliding down from rest at an angle α to the horizontal with its ends A and B on smooth horizontal and vertical planes respectively. At any instant t, let the rod make an angle θ to the horizontal. Let R and S be the reactions at the ends A and B of the rod AB.

Taking O as origin, horizontal and vertical lines through O as axes, the coordinates of G are given by.

$\bar{x} = a\cos\theta$ and $\bar{y} = a\sin\theta$

$\therefore \dot{\bar{x}} = -a\sin\theta\dot{\theta}$ and $\ddot{\bar{x}} = -a\cos\theta\dot{\theta}^2 - a\sin\theta\ddot{\theta}$

Also, $\dot{\bar{y}} = a\cos\theta\dot{\theta}$ and $\ddot{\bar{y}} = -a\sin\theta\dot{\theta}^2 + a\cos\theta\ddot{\theta}$

The equations of motion of the C.G. 'G' are given by

$M\ddot{\bar{x}} = M(-a\cos\theta\dot{\theta}^2 - a\sin\theta\ddot{\theta}) = S$ ...(1)

and $M\ddot{\bar{y}} = M(-a\sin\theta\dot{\theta}^2 + a\cos\theta\ddot{\theta}) = R - Mg$ ...(2)

Also the energy equation gives

$\frac{1}{2}M(\dot{\bar{x}}^2+\dot{\bar{y}}^2) + \frac{1}{2}Mk^2\dot{\theta}^2 =$ Work done by the gravity

or $\frac{1}{2}M(a^2\dot{\theta}^2 + \frac{1}{3}a^2\dot{\theta}^2) = Mg(a\sin\alpha - a\sin\theta)$

or $\dot{\theta}^2 = (3g/2a)(\sin\alpha - \sin\theta)$ ...(3)

Differentiating (3) w.r.t. 't' and dividing by $2\dot{\theta}$, we have

$\ddot{\theta} = -(3g/4a)\cos\theta.$ ...(4)

Putting the values of $\dot{\theta}^2$ and $\ddot{\theta}$ in (1) and (2), we have,

$S = M[-a\cos\theta(3g/2a)(\sin\alpha - \sin\theta) - a\sin\theta\{-(3g/4a)\cos\theta\}]$

$= \frac{3}{4}Mg\cos\theta(3\sin\theta - 2\sin\alpha)$ ...(5)

and $R = Mg + M[-a\sin\theta(3g/2a)(\sin\alpha - \sin\theta) + a\cos\theta\{-(3g/4a)\cos\theta\}]$

$= \frac{1}{4}Mg[4 - 6\sin\theta\sin\alpha + 6\sin^2\theta - 3\cos^2\theta]$

$= \frac{1}{4}Mg[1 - 6\sin\theta\sin\alpha + 9\sin^2\theta]$

$= \frac{1}{4}Mg[1 - \sin^2\alpha + \sin^2\alpha - 6\sin\theta\sin\alpha + 9\sin^2\theta]$

$= \frac{1}{4}Mg[(3\sin\theta - \sin\alpha)^2 + \cos^2\alpha]$ ...(6)

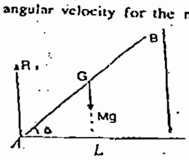
From (5), it is clear that $S = 0$ when $\sin\theta = \frac{2}{3}\sin\alpha$ and S will be negative for smaller values of θ. Hence the end B leaves the wall when $\sin\theta = \frac{2}{3}\sin\alpha$.

Also from (6) it is clear that R is always positive *i.e.* the end A never leaves the plane.

When the end B leaves the plane $\sin\theta = \frac{2}{3}\sin\alpha$, the equations of motion (1), (2), (3) and (4) cease to hold good for further motion.

Putting $\sin\theta = \frac{2}{3}\sin\alpha$ in (3), the angular velocity of the rod is $\sqrt{\left(\frac{g}{2a}\sin\alpha\right)}$. This will be the initial angular velocity for the next part of the motion.

Second part of the motion.



When the end B leaves the wall, let R_1 be the normal reaction at A. Let the rod have turned through an angle ϕ from the horizontal.

The equations of motion of C.G. G are

$M\ddot{x} = 0$ ...(7)

$M\ddot{y} = R_1 - Mg$ ...(8)

and $M\frac{a^2}{3}\ddot{\phi} = -R_1 a\cos\phi$ ...(9)

But $y = GL = a\sin\phi$. $\therefore \ddot{y} = -a\sin\phi\dot{\phi}^2 + a\cos\phi\ddot{\phi}$

$\therefore$ From (8), $R_1 = Mg + M(-a\sin\phi\dot{\phi}^2 + a\cos\phi\ddot{\phi})$

Putting in (9), we have

$\frac{1}{3}Ma^2\ddot{\phi} = -Ma(g - a\sin\phi\dot{\phi}^2 + a\cos\phi\ddot{\phi})\cos\phi$

or $(\frac{1}{3} + \cos^2\phi)\cdot\ddot{\phi} - \sin\phi\cos\phi\dot{\phi}^2 = -\frac{g}{a}\cos\phi.$ ...(10)

Integrating it, we get

$(\frac{1}{3} + \cos^2\phi)\cdot\dot{\phi}^2 = -\frac{2g}{a}\sin\phi + C$ ...(11)

when $\sin\phi = \frac{2}{3}\sin\alpha$, $\dot{\phi} = \sqrt{\left(\frac{g}{2a}\sin\alpha\right)}$,

$\therefore \frac{g\sin\alpha}{2a}[\frac{1}{3} + 1 - \frac{4}{9}\sin^2\alpha] = -\frac{2g}{a}\cdot\frac{2}{3}\sin\alpha + C$

or $C = \frac{2g\sin\alpha}{a}\left(1 - \frac{\sin^2\alpha}{9}\right)$

$\therefore$ From (11), we have

$(\frac{1}{3} + \cos^2\phi)\cdot\dot{\phi}^2 = \frac{2g\sin\alpha}{a}\left(1 - \frac{\sin^2\alpha}{9}\right) - \frac{2g}{a}\sin\phi.$ ...(12)

When $\phi = 0$ *i.e.* when rod becomes horizontal, its angular velocity Ω, is given by

$\Omega^2(\frac{1}{3} + 1) = \frac{2g\sin\alpha}{a}\left(1 - \frac{\sin^2\alpha}{9}\right)$

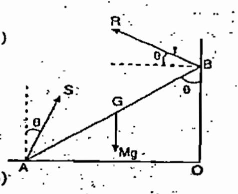
i.e. $\Omega^2 = \frac{3g}{a}\left(1 - \frac{\sin^2\alpha}{9}\right)\sin\alpha$ ...(13)

Ex. 13. *A heavy rod of length $2a$ is placed in a vertical plane with its ends in contact with a rough vertical wall and an equally rough horizontal plane ; the coefficient of friction being $\tan\varepsilon$. Show that it will begin to slip down if its initial inclination to the vertical is greater than 2ε. Prove also that the inclination θ of the rod to the vertical at any time is given by*

$$\ddot{\theta}(k^2 + a^2\cos 2\varepsilon) - a^2\dot{\theta}^2\sin 2\varepsilon = ag\sin(\theta - 2\varepsilon)$$

Sol. Let AB be the rod of mass M and length $2a$. When AB makes an angle θ with the vertical let R and S be the resultant reactions at B and A respectively.

Equations of motion of C.G. G are



$M\frac{d^2}{dt^2}(a\sin\theta) = -S\sin\varepsilon + R\cos\varepsilon$ ...(1)

and $M\frac{d^2}{dt^2}(a\cos\theta) = R\sin\varepsilon + S\cos\varepsilon - Mg$ ...(2)

Taking moments about G, we have

$Mk^2\ddot{\theta} = Sa\sin(\theta - \varepsilon) - Ra\cos(\theta - \varepsilon)$ ...(3)

From (1), we have

$Ma(\cos\theta\ddot{\theta} - \sin\theta\dot{\theta}^2) = R\cos\varepsilon - S\sin\varepsilon$ ...(4)

From (2), we have

$Ma(\sin\theta\ddot{\theta} + \cos\theta\dot{\theta}^2) = Mg - R\sin\varepsilon - S\cos\varepsilon$ ...(5)

Solving equation (4) and (5), we have

$R = Mg\sin\varepsilon + Ma\cos(\theta+\varepsilon)\ddot{\theta} - Ma\sin(\theta+\varepsilon)\dot{\theta}^2$ ...(6)

$S = Mg\cos\varepsilon - Ma\sin(\theta+\varepsilon)\ddot{\theta} - Ma\cos(\theta+\varepsilon)\dot{\theta}^2$ ...(7)

Putting the values of R and S in (3), we have

$Mk^2\ddot{\theta} = a\sin(\theta - \varepsilon)[Mg\cos\varepsilon - Ma\sin(\theta+\varepsilon)\ddot{\theta} - Ma\cos(\theta+\varepsilon)\dot{\theta}]$

$- a\cos(\theta - \varepsilon)[Mg\sin\varepsilon + Ma\cos(\theta+\varepsilon)\ddot{\theta} - Ma\sin(\theta+\varepsilon)\dot{\theta}^2]$

$= Mga\sin(\theta - 2\varepsilon) - Ma^2\ddot{\theta}\cos 2\varepsilon + Ma^2\dot{\theta}^2\sin 2\varepsilon$

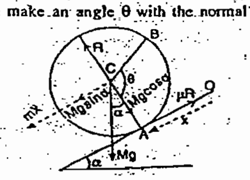
or $\ddot{\theta}(k^2 + a^2\cos 2\varepsilon) - a^2\dot{\theta}^2\sin 2\varepsilon = ag\sin(\theta - 2\varepsilon)$ which gives θ.

If $\theta > 2\varepsilon$ it is obvious that $\ddot{\theta}$ is positive and hence the rod starts slipping if $\theta > 2\varepsilon$.

4.7. When rolling and sliding are combined.

An imperfectly rough sphere moves from rest down a plane inclined at an angle α to the horizone; to determine the motion.

Let C be the centre of a sphere of radius a. In time t, let the sphere turn through an angle θ, *i.e.* let CB be a radius (a line fixed in the body) which was initially normal to the plane make an angle θ with the normal CA at time t.



If the friction is not sufficient to produce pure rolling then the sphere will slide as well as turn. So the maximum friction μR will act up the plane where μ is the coefficient of friction. Let x be the distance described by the centre of gravity C parallel to the inclined plane in time t.

There is no motion perpendicular to the plane, so the C.G. of the sphere always moves parallel to the plane.

$\therefore$ The equations of motion are

$M\ddot{x} = Mg\sin\alpha - \mu R$ ...(1)

$0 = R - Mg\cos\alpha$ ...(2)

and $M\frac{2}{5}a^2\ddot{\theta} = \mu Ra$ ...(3)

From (1) and (2), we have

$\ddot{x} = g(\sin\alpha - \mu\cos\alpha)$ ...(4)

integrating (4) w.r.t. 't' we have

$\dot{x} = g(\sin\alpha - \mu\cos\alpha)t$ ...(5)

The constant of integration vanishes as $\dot{x} = 0$ when $t = 0$.

Integrating (5) again, we have

$x = g(\sin\alpha - \mu\cos\alpha)\frac{t^2}{2}$ ...(6)

constants of integration vanish as $x = 0$ when $t = 0$.

$\therefore$ From (2) and (3), we get

$a\ddot{\theta} = \frac{5}{2}\mu g\cos\alpha$. Integrating it, we get $a\dot{\theta} = \frac{5}{2}\mu gt\cos\alpha$.

Constant of integration vanishes as $\dot{\theta} = 0$, where $t = 0$.

Integrating it again, we get $\theta = \frac{15\mu g}{4}t^2\cos\alpha$ ...(7)

The constant of integration vanish, as $\theta = 0$ when $t = 0$.

The velocity of the point of contact A down the plane

= velocity of C, the centre of sphere + velocity of A relative to C

$= \dot{x} - a\dot{\theta}$

$= g(\sin\alpha - \mu\cos\alpha)t - \frac{5}{2}\mu gt\cos\alpha$

$= \frac{1}{2}g(2\sin\alpha - 7\mu\cos\alpha)$ ...(8)

There are following three cases :

First case. If $2\sin\alpha > 7\mu\cos\alpha$ *i.e.* if $\mu < \frac{2}{7}\tan\alpha$.

In this case velocity of the point of contact is positive for all values of t *i.e.* it does not vanish, hence the point of contact always slides down and the maximum friction μR acts. The sphere never rolls. The equations of motion established above hold good throughout the entire motion.

Second case. If $2\sin\alpha = 7\mu\cos\alpha$ *i.e.* if $\mu = \frac{2}{7}\tan\alpha$

In this case velocity of the point of contact is zero for all values of t and therefore motion of the sphere is that of pure rolling throughout and the maximum friction μR is always exerted. The equations of motion established above hold good.

Third case. If $2\sin\alpha < 7\mu\cos\alpha$ *i.e.* $\mu > \frac{2}{7}\tan\alpha$

In this case velocity of the point of contact is negative *i.e.* if the maximum friction μR were allowed to act, the point of contact will slide up the plane which is impossible because the amount of friction will only act which is just sufficient to keep the point of contact at rest. Hence in this case the motion is of pure rolling from the very start and remains the same throughout and the maximum friction μR is not exerted. Therefore in this case the equations of motion established above do not hold good.

If now F is the frictional force, then the equations of motion are

$M\ddot{x} = Mg\sin\alpha - F$ ...(9)

$0 = R - Mg\cos\alpha$ ...(10)

and $M\cdot\frac{2}{5}a^2\ddot{\theta} = Fa$ ...(11)

Since the point of contact A is at rest.

$\therefore \dot{x} - a\dot{\theta} = 0$, *i.e.* $\dot{x} = a\dot{\theta}$ $\therefore$ $\ddot{x} = a\ddot{\theta}$

From (9) and (11), we have

$\frac{2}{5}Ma\ddot{\theta} = F = -M\ddot{x} + Mg\sin\alpha$

or $\frac{2}{5}M\ddot{x} + M\ddot{x} = Mg\sin\alpha$ $\therefore$ $\ddot{x} = a\ddot{\theta} = \frac{5}{7}g\sin\alpha$ ...(12)

Integrating (12), we have

$\dot{x} = a\dot{\theta} = \frac{5}{7}gt\sin\alpha$.

Constant of integration vanishes as $\dot{x} = 0$, when $t = 0$.

Integrating, again, we have

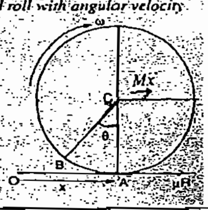
$x = a\theta = \frac{5}{14}gt^2\sin\alpha$ ...(13)

the constant of integration again vanishes as $x = 0$, where $t = 0$.

EXAMPLES

Ex. 14 *A homogeneous sphere of radius a, rotating with angular velocity ω about horizontal diameter is gently placed on a table whose coefficient of friction is μ. Show that there will be slipping at the point of contact for a time $(2a\omega/7\mu g)$, and that then the sphere will roll with angular velocity $(2\omega/7)$.*

Sol. As the sphere is gently placed on the table, so the initial velocity of the centre of the sphere is zero, while initial angular velocity is ω.



Initial velocity of the point of contact = Initial velocity of the centre C + Initial velocity of the point of contact with respect to the centre $C = 0 + a\omega$ in the direction from right to left, *i.e.* the point of contact will slip in the direction right to left, therefore full friction μR will act in the direction left to right.

Let x be the distance advanced by the centre C in the horizontal direction and θ be the angle through which the sphere turns in time t. Then at any time t the equations of motion are,

$M\ddot{x} = \mu R$, (where $R = Mg$). ...(1)

and $Mk^2\ddot{\theta} = M\frac{2a^2}{5}\ddot{\theta} = -\mu Ra$. ...(2)

From (1), we have $\ddot{x} = \mu g$ and from (2), we have $a\ddot{\theta} = -\frac{5}{2}\mu g$.

Integrating these equations, we have

$\dot{x} = \mu gt + C_1$ and $a\dot{\theta} = -\frac{5}{2}\mu gt + C_2$.

Since initially when $t = 0$, $\dot{x} = 0$, $\dot{\theta} = \omega$.

$\therefore$ $C_1 = 0$ and $C_2 = a\omega$.

$\therefore$ $\dot{x} = \mu gt$, ...(3)

and $a\dot{\theta} = -\frac{5}{2}\mu gt + a\omega$. ...(4)

Velocity of the point of contact $= \dot{x} - a\dot{\theta}$.

$\therefore$ The point of contact will come to rest when $\dot{x} - a\dot{\theta} = 0$, *i.e.* when $\mu gt - (-\frac{5}{2}\mu gt + a\omega) = 0$ or when $t = (2a\omega/7\mu g)$.

Therefore after time $(2a\omega/7\mu g)$ the slipping will stop and pure rolling will commence.

Putting this value of t in (4), we get $\dot{\theta} = (2\omega/7)$.

When rolling commences, let F be the frictional force. Therefore the equations of motion are

$M\ddot{x} = F$, ...(5)

$M\cdot\frac{2}{5}a^2\ddot{\theta} = -Fa$, ...(6)

and $\dot{x} - a\dot{\theta} = 0$. ...(7)

From (7) $\dot{x} = a\dot{\theta}$ and $\ddot{x} = a\ddot{\theta}$.

Now from (5) and (6), we get

$M\ddot{x} = F = -\frac{2}{5}Ma\ddot{\theta}$, or $a\ddot{\theta} = -\frac{2}{5}a\ddot{\theta}$ ($\because \ddot{x} = a\ddot{\theta}$)

or $\frac{7}{5}a\ddot{\theta} = 0$ or $\ddot{\theta} = 0$.

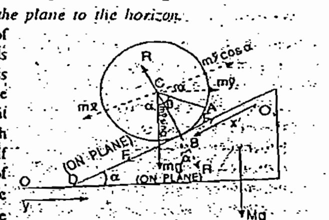
Integrating $\dot{\theta}$ = Constant $= \frac{2}{7}\omega$.

Ex. 15. *An inclined plane of mass M is capable of moving freely on a smooth horizontal plane. A perfectly rough sphere of mass m is placed on its inclined face and rolls down under the action of gravity. If y the horizontal distance advanced by the inclined plane and x the part of the plane rolled over by the sphere, prove that*

$$(M+m)y = mx\cos\alpha \text{ and } \frac{7}{5}x - y\cos\alpha = \frac{1}{2}gt^2\sin\alpha;$$

where α is the inclination of the plane to the horizon.

Sol. Let C be the centre of sphere of mass m which rolls down the inclined plane of mass M and inclination α to the horizontal. Initially let the point A of sphere be in contact with point O of the inclined plane. If at time t, the point of contact of the sphere and the inclined plane is B and during this time the sphere turns through an angle θ, then



$$\angle ACB = \theta.$$

If $O_1B = x$, then x = Arc $AB = a\theta$. $\therefore$ $\dot{x} = a\dot{\theta}$;

and $\ddot{x} = a\ddot{\theta}$. ...(1)

Let the inclined plane shift through a distance $OD = y$ in time t.

$\therefore$ The accelerations of the centre C of the sphere are $\ddot{x}$ down the plane and parallel to it, and $\ddot{y}$ horizontally as shown in the figure.

$\therefore$ The acceleration of the centre C, parallel to the inclined plane is $\ddot{x} - \ddot{y}\cos\alpha$ (downwards).

Let F be the frictional force up the plane.

$\therefore$ The equations of motion of the sphere are

$m(\ddot{x} - \ddot{y}\cos\alpha) = mg\sin\alpha - F$, ...(2)

$m\ddot{y}\sin\alpha = mg\cos\alpha - R$; ...(3)

and $m\cdot k^2\ddot{\theta} = m\cdot\frac{2}{5}a^2\ddot{\theta} = F\cdot a$ ...(4)

Also the equation of motion of the inclined plane is,

$M\ddot{y} = R\sin\alpha - F\cos\alpha$. ...(5)

From (1) and (4), we have $\frac{2}{5}m\ddot{x} = F$.

$\therefore$ Substituting in (2), we get

$\ddot{x} - \ddot{y}\cos\alpha = g\sin\alpha - \frac{2}{5}\ddot{x}$ or $\frac{7}{5}\ddot{x} = \ddot{y}\cos\alpha + g\sin\alpha$. ...(6)

Integrating, $\frac{7}{5}\dot{x} - \dot{y}\cos\alpha = gt\sin\alpha + C_1$.

But when $t = 0$, $\dot{x} = 0$, $\dot{y} = 0$. $\therefore$ $C_1 = 0$.

$\therefore \frac{7}{5}\dot{x} - \dot{y}\cos\alpha = gt\sin\alpha$.

Again integrating, $\frac{7}{5}x - y\cos\alpha = \frac{1}{2}gt^2\sin\alpha + C_2$

Also $C_2 = 0$, $\because$ when $t = 0, x = 0, y = 0$.

$\therefore$ We have $\frac{7}{5}x - y\cos\alpha = \frac{1}{2}gt^2\sin\alpha$ ...(7)

Substituting $R = m(-\ddot{y}\sin\alpha + g\cos\alpha)$ from (3) and $F = m[g\sin\alpha - (\ddot{x} - \ddot{y}\cos\alpha)]$ from (2) in (5), we have

$M\ddot{y} = m(-\ddot{y}\sin\alpha + g\cos\alpha)\sin\alpha - m[g\sin\alpha - (\ddot{x} - \ddot{y}\cos\alpha)]\cos\alpha$

$= -m\ddot{y}(\sin^2\alpha + \cos^2\alpha) + m\ddot{x}\cos\alpha = -m\ddot{y} + m\ddot{x}\cos\alpha$.

$\therefore (M+m)\ddot{y} = m\ddot{x}\cos\alpha$.

Integrating, $(M+m)\dot{y} = m\dot{x}\cos\alpha$, constant of integrating is zero as $\dot{y} = 0$ when $\dot{x} = 0$.

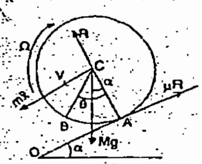
Integrating again, $(M+m)y = mx\cos\alpha$ ...(8)

The constant of integration vanishes again as when $x = 0, y = 0$

Equations (7) and (8) are the required equations.

Ex. 16. *A hoop is projected with velocity V down an inclined plane of inclination α, the coefficient of friction being μ (> tan α). It has initially such a backward spin Ω that after a time t_1 it start moving uphill and continues to do so for a time t_2 after which it once more descends. The motion being in a vertical plane at right angles to the give inclined plane, show that $(t_1 + t_2)g\sin\alpha = a\Omega - V$.*

Sol. Let C be the centre of the hoop projected with velocity V down an inclined plane of inclination α. At time t, let the radius CB (a line fixed in the body) make an angle θ with CA which is normal to the plane (CA is a line fixed in space). Initially CB was normal to the plane, *i.e.* initially B was in contact with the plane at A.



The velocity of the point of contact A down the plane, initially

= Velocity of centre C + Velocity of A relative to C,

$= V + a\Omega$, which is a positive quantity.

As the point of contact slides down the friction uR acts up the plane.

The equations of motion are

$M\ddot{x} = Mg\sin\alpha - \mu R$. ...(1)

$0 = R - Mg\cos\alpha$ ...(2)

and $Mk^2\ddot{\theta} = Ma^2\ddot{\theta} = -\mu Ra$. ...(3)

From (1) and (2), we have $\ddot{x} = g(\sin\alpha - \mu\cos\alpha)$

Integrating $\dot{x} = g(\sin\alpha - \mu\cos\alpha)t + C$.

But when $t = 0, \dot{x} = V$. $\therefore C = V$.

Therefore $\dot{x} = g(\sin\alpha - \mu\cos\alpha)t + V$. ...(4)

From (1) and (3), we get $a\ddot{\theta} = -\mu g\cos\alpha$.

Integrating $a\dot{\theta} = -\mu gt\cos\alpha + C_1$.

But, when $t = 0, \dot{\theta} = \Omega$. $\therefore C_1 = a\Omega$.

Therefore $a\dot{\theta} = -\mu gt\cos\alpha + a\Omega$. ...(5)

The hoop will cease to move downwards when $\dot{x} = 0$. $\therefore$ let $\dot{x} = 0$ when $t = t_1$, therefore from (4), we get

$0 = g(\sin\alpha - \mu\cos\alpha)t_1$

or $t_1 = \frac{V}{g(\mu\cos\alpha - \sin\alpha)}$ ...(6)

The velocity of the point of contact is $\dot{x} + a\dot{\theta}$. When $\dot{x} = 0$, for the hoop to move uphill $a\dot{\theta}$ should be positive. It follows that throughout the downward motion $\dot{x} + a\dot{\theta}$ is always positive. Therefore when moving downwards pure rolling does not take place. Thus the equations established above are true throughout the downward motion.

Putting the value of t_1 from (6) in (5), we have

$a\dot{\theta} = a\Omega - \frac{\mu V\cos\alpha}{(\mu\cos\alpha - \sin\alpha)}$

since $a\dot{\theta}$ is positive, the hoop begins to move uphill.

When the hoop starts moving uphill. The initial velocity of the centre is zero and $a\dot{\theta}$ is positive with the sense of the direction as ψ.

Initially velocity of the point of contact upto will $= 0 - a\dot{\theta}$, which is negative, *i.e.* initially the velocity of the point of contact is in the downwards direction hence the friction μR acts upwards.

$\therefore$ The equations of motions are

$M\ddot{y} = -Mg\sin\alpha + \mu R$, ...(7)

$0 = R - Mg\cos\alpha$, ...(8)

and $Ma^2\ddot{\phi} = -\mu Ra$. ...(9)

Substituting $R = Mg\cos\alpha$ from (8) in (7) and (9), we have

$\ddot{y} = g(\mu\cos\alpha - \sin\alpha)$ and $a\ddot{\phi} = -\mu g\cos\alpha$.

Integrating these two equations, and using the initial conditions, we get $\dot{y} = g(\mu\cos\alpha - \sin\alpha)t$ ...(10)

and $a\dot{\phi} = -\mu gt\cos\alpha + a\Omega - \frac{\mu V\cos\alpha}{\mu\cos\alpha - \sin\alpha}$ ...(11)

$\left[\because \text{when } t = 0, \dot{y} = 0, a\dot{\phi} = a\Omega - \frac{\mu V\cos\alpha}{\mu\cos\alpha - \sin\alpha}\right]$

The rolling commences when the velocity of the point of contact is zero *i.e.* $\dot{y} - a\dot{\phi} = 0$ or $\dot{y} = a\dot{\phi}$

If rolling commences at time $t = t'$, then from (10) and (11), we have

$g(\mu\cos\alpha - \sin\alpha)t' = -\mu gt'\cos\alpha + a\Omega - \frac{\mu V\cos\alpha}{\mu\cos\alpha - \sin\alpha}$

or $gt'(2\mu\cos\alpha - \sin\alpha) = a\Omega - \frac{\mu V\cos\alpha}{\mu\cos\alpha - \sin\alpha}$ ...(12)

Which gives the value of t'.

At this time from (10), $\dot{y} = gt'(\mu\cos\alpha - \sin\alpha)$.

When Rolling commence. Equations of motion are

$M\ddot{z} = F - Mg\sin\alpha$ ...(13)

$Ma\ddot{\psi} = -Fa$ ...(14)

and $\dot{z} - a\dot{\psi} = 0$ ...(15)

Solving these equations, we get $F = \frac{1}{2}Mg\sin\alpha$

Since $\mu > \tan\alpha$, $\therefore \mu R > \tan\alpha \cdot Mg\cos\alpha$ *i.e.* $\mu R > Mg\sin\alpha$.

$F = \frac{1}{2}Mg\sin\alpha < \mu R$. So the condition of pure rolling is satisfied, and hence the equations of motion hold good for the motion.

From (13) we have, $M\ddot{z} = F - Mg\sin\alpha = \frac{1}{2}Mg\sin\alpha - Mg\sin\alpha$

or $\ddot{z} = -\frac{1}{2}g\sin\alpha$.

Integrating, $\dot{z} = -\frac{1}{2}gt\sin\alpha + k$.

But when $t = 0, \dot{z} = \dot{y} = gt'(\mu\cos\alpha - \sin\alpha)$

$\therefore k = gt'(\mu\cos\alpha - \sin\alpha)$

$\dot{z} = -\frac{1}{2}gt\sin\alpha + gt'(\mu\cos\alpha - \sin\alpha)$ ...(16)

The hoop ceases to move up the hill if $\dot{z} = 0$. If this happens at time $t = t''$, then from (16), we have

$\therefore 0 = -\frac{1}{2}gt''\sin\alpha + gt'(\mu\cos\alpha - \sin\alpha)$

or $t'' = 2\frac{(\mu\cos\alpha - \sin\alpha)t'}{\sin\alpha}$

$\therefore t_2 = t' + t'' = t' + 2\frac{(\mu\cos\alpha - \sin\alpha)}{\sin\alpha}t' = \left(\frac{2\mu\cos\alpha - \sin\alpha}{\sin\alpha}\right)t'$

$= \left(\frac{2\mu\cos\alpha - \sin\alpha}{\sin\alpha}\right)\frac{1}{g(2\mu\cos\alpha - \sin\alpha)}\left(a\Omega - \frac{\mu V\cos\alpha}{\mu\cos\alpha - \sin\alpha}\right)$ (from 12)

$= \frac{1}{g\sin\alpha}\left(a\Omega - \frac{\mu V\cos\alpha}{\mu\cos\alpha - \sin\alpha}\right)$

$\therefore$ The total time $= t_1 + t_2$

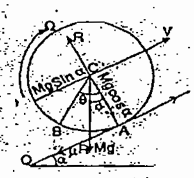
$= \frac{V}{g(\mu\cos\alpha - \sin\alpha)} + \frac{1}{g\sin\alpha}\left(a\Omega - \frac{\mu V\cos\alpha}{\mu\cos\alpha - \sin\alpha}\right)$

$= \frac{1}{g\sin\alpha}\left[a\Omega - \frac{V(\mu\cos\alpha - \sin\alpha)}{\mu\cos\alpha - \sin\alpha}\right] = \frac{1}{g\sin\alpha}(a\Omega - V)$

or $(t_1 + t_2)g\sin\alpha = a\Omega - V$.

Ex. 17. *A sphere, of radius a is projected up an inclined plane with velocity V and angular velocity Ω in the sense which would cause it to roll up, if $V > a\Omega$ and the coefficient of friction $\frac{2}{7}\tan\alpha$, show that the sphere will cease to ascend at the end of a time $\frac{5V + 2a\Omega}{5g\sin\alpha}$, where α is the inclination of the plane.* **(IAS-1995)**

Sol. Let C be the centre and M the mass of the sphere. Let the radius CB which is a line fixed in the body makes an angle θ after time t with CA normal to the plane (CA is a line fixed in space). Initially CB was normal to the plane. The initial velocity of the point of contact A up the plane = Velocity of the centre C + velocity of A relative to $C = V - a\Omega$, which is positive as $V > a\Omega$.



Hence the friction μR acts down the plane, implying, that the sphere slides as well as turns.

The equations of motion of the sphere are

$M\ddot{x} = -Mg\sin\alpha - \mu R$ ...(1)

$0 = R - Mg\cos\alpha$ ...(2)

and $Mk^2\ddot{\theta} = M\cdot\frac{2a^2}{5}\ddot{\theta} = \mu Ra$ ...(3)

From (1) and (2), we have $M\ddot{x} = -Mg\sin\alpha - \mu Mg\cos\alpha$

or $\ddot{x} = -g(\sin\alpha + \mu\cos\alpha)$

Integrating, $\dot{x} = -g(\sin\alpha + \mu\cos\alpha)t + C_1$

But when $t = 0, \dot{x} = V$. $\therefore C_1 = V$.

$\therefore \dot{x} = -g(\sin\alpha + \mu\cos\alpha)t + V$ ...(4)

From (2) and (3), we have $a\ddot{\theta} = \frac{5}{2}\mu g\cos\alpha$.

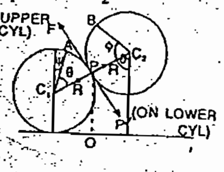
Integrating, $a\dot{\theta} = \frac{5}{2}\mu gt\cos\alpha + C_2$

But when $t = \theta$, $\dot{\theta} = \Omega$ $\therefore C_2 = a\Omega$

$\therefore a\dot{\theta} = \frac{5}{2}\mu gt\cos\alpha + a\Omega$ ...(5)

The velocity of the point of contact $= \dot{x} - a\dot{\theta}$. Rolling commences, say after time t_1 when $\dot{x} - a\dot{\theta} = 0$

Sol. Let C_1 be the centre, a the radius and m the mass of uniform cylinder resting initially on smooth horizontal plane. Let C_2 be the centre of equal cylinder placed on a the first touching along its highest generator. Consider the vertical section of the system by the vertical plane through centres C_1 and C_2. At time t, let the lines C_1a and C_2B fixed in the two cylinders make angles ψ and ϕ to the vertical respectively while initially C_1A, C_2B were vertical and B coincide with A. Let C_1C_2 make angle θ to the vertical at time t.



Since there is no slipping and spheres are equal

$\therefore \angle AC_1P = \angle BC_2P$ i.e. $\theta - \psi = \phi - \theta$ or $\psi + \phi = 2\theta$ ...(1)

Considering motion the two cylinders and taking moments about C_1 and C_2 we get

$m \cdot \frac{a^2}{2}\ddot{\psi} = Fa$ (For lower cly.) ...(2)

and $m\frac{a^2}{2}\ddot{\phi} = Fa$ (For upper cyl.) ...(3)

$\therefore \ddot{\psi} = \ddot{\phi}$. Integrating twice $\dot{\psi} = \dot{\phi}$ and $\psi = \phi$

The constants of integration vanish $\because \dot{\psi} = 0, \dot{\phi} = 0, \psi = 0, \phi = 0$

$\because \psi = \phi$, $\therefore$ from (1) we get $\psi = \phi = \theta$.

$\therefore$ A and B coincide with P, at time t. Hence the same generators remain in contact untill the cylinders separate.

$\because$ Cylinders are of same masses and $P_{c_1} = P_{c_2}$

$\therefore$ P, the point of contact of the two cylinders is the common centre of gravity of the system. Since there is no horizontal force on the system, therefore the common C.G. 'P' decends vertically. Thus if the vertical line through P cuts the horizontal plane at O, then O is the fixed point. Referred to O as origin, horizontal and vertical lines through O as axes, the coordinates (x_{c_1}, y_{c_1}) and (x_{c_2}, y_{c_2}) of C_1 and C_2 are given by

$x_{c_1} = -a\sin\theta, y_{c_1} = a; x_{c_2} = a\sin\theta, y_{c_2} = 2a\cos\theta.$

The energy equation gives

$$\left(\frac{1}{2}m\frac{a^2}{2}\dot{\psi}^2 + \frac{1}{2}mv_{c1}^2\right) + \left(\frac{1}{2}m\frac{a^2}{2}\dot{\phi}^2 + \frac{1}{2}mv_{c2}^2\right) = mg(2a - 2a\cos\theta)$$

or

$$\left[\frac{1}{4}ma^2\dot{\theta}^2 + \frac{1}{2}m(a^2\cos^2\theta\dot{\theta}^2)\right] + \left[\frac{1}{4}ma^2\dot{\theta}^2 + \frac{1}{2}m(a^2\cos^2\theta\dot{\theta}^2) + 4a^2\sin^2\theta\dot{\theta}^2\right]$$
$$= 2mga(1-\cos\theta) \quad \because \phi = \psi = \theta$$

or $a(1 + 2\cos^2\theta + 4\sin^2\theta)\dot{\theta}^2 = 4g(1-\cos\theta)$

or $a(5 - 2\cos^2\theta)\dot{\theta}^2 = 4g(1-\cos\theta)$ ...(4)

Differentiating and dividing by $2\dot{\theta}$, we get

$a(5 - 2\cos^2\theta)\ddot{\theta} + 2a\sin\theta\cos\theta\dot{\theta}^2 = 2g\sin\theta$ ...(5)

Now equation of horizontal motion of the upper cylinder is

$$R\sin\theta - F\cos\theta = m\ddot{x}_{c2} = m\frac{d^2}{dt^2}(a\sin\theta)$$

or $R\sin\theta - F\cos\theta = ma(\cos\theta\ddot{\theta} - \sin\theta\dot{\theta}^2)$ ...(6)

Eliminating F between (3) and (6), we get

$R\sin\theta - \frac{1}{2}ma\ddot{\phi}\cos\theta = ma(\cos\theta\ddot{\theta} - \sin\theta\dot{\theta}^2)$

or $R\sin\theta = \frac{1}{2}ma(3\cos\theta\ddot{\theta} - 2\sin\theta\dot{\theta}^2) \because \phi = \theta$ ...(7)

The cylinders will separate, when $R = 0$, $\therefore$ from (7),

$0 = \frac{1}{2}ma(3\cos\theta\ddot{\theta} - 2\sin\theta\dot{\theta}^2)$ or $\ddot{\theta} = \frac{2}{3}\tan\theta\dot{\theta}^2$ ...(8)

From (4), we get $\dot{\theta}^2 = \frac{4g}{a}\left(\frac{1-\cos\theta}{5-2\cos^2\theta}\right)$

$\therefore$ from (8), $\ddot{\theta} = \frac{8g}{3a}\tan\theta\left(\frac{1-\cos\theta}{5-2\cos^2\theta}\right)$

Substituting in (5), we get

$$a(5-2\cos^2\theta)\cdot\frac{8g}{3a}\tan\theta\left(\frac{1-\cos\theta}{5-2\cos^2\theta}\right) + 2a\sin\theta\cos\theta\cdot\frac{4g}{a}\left(\frac{1-\cos\theta}{5-2\cos^2\theta}\right) = 2g\sin\theta$$

or $4(5-2\cos^2\theta)(1-\cos\theta) + 12\cos^2\theta(1-\cos\theta) = 3(5-2\cos^2\theta)\cos\theta$

or $4(5 - 5\cos\theta - 2\cos^2\theta + 2\cos^3\theta) + 12\cos^2\theta - 12\cos^3\theta = 15\cos\theta - 6\cos^3\theta$

or $2\cos^3\theta + 4\cos^2\theta - 35\cos\theta + 20 = 0$,

which is the required result.

EXERCISE

- One end of a thread which is wound on a reel, is fixed and the reel falls in a vertical line, its axis being horizontal and the unwound part of the thread being vertical. If the reel be a solid cylinder of radius a and weight W, show that the acceleration of the centre of the reel is $\frac{2}{3}g$ and the tension of the thread is $\frac{1}{3}W$.
- A uniform beam lies on a rough horizontal table at right angles to the edge, and is held so that one-third of its lengths is in contact with the table. Prove that after it is released it will begin to slide over the edge of the table when it has tunred through an angle $\tan^{-1}(\frac{1}{2}\mu)$, μ being the coefficient of friction between the table and the beam.
  [Hint. See Ex. 12, on page 187, here $b = a/3$].
- If a sphere be projected up an inclined plane, for which $\mu = (1/7)\tan\alpha$, with velocity V and an initial angular velocity Ω (in the direction in which it would roll up), and if $V > a\Omega$, show that the friction acts downwards at first, and upwards afterwards, and prove that the whole time during which the sphere rises is
$$\frac{17V + 4a\Omega}{18g\sin\alpha}$$
- A sphere is projected with an underhand twist down a rough inclined plane; show that it will turn back in the course of its motion if $2a\omega(\mu - \tan\alpha) > 5\mu u$, where u, ω are the initial linear and angular velocities of the sphere, μ is the coefficient of friction, and α is the inclination of the plane.
- A homogeneous sphere, of mass M, is placed on an imperfectly rough table, and particle, of mass m, is attached to the end of a horizontal diameter. Show that the sphere will begin to roll or slide according as μ is greater or less than
$$\frac{5(M+m)m}{7M^2 + 17Mm + 5m^2}$$
- A uniform sphere is placed on the top of a fixed rough circular cylinder whose generators are horizontal. Show that if slightly displaced, it will roll on the cylinder untill it reaches a point where the inclination of the tangent plane to horizon is given by
$$2\sin\theta = \mu(17\cos\theta - 10)$$
  μ being the coefficient of friction.
- A uniform beam of mass M and length l stands upright on perfectly rough ground; on the top of it, which is flat, rests a weight of mass m, the coefficient of friction between the beam and the weight being μ. If the beam is allowed to fall to the ground, its inclination θ to the vertical when the weight slips is given by
$$\left(\frac{4}{3}M + 3m\right)\cos\theta - (M/6\mu)\sin\theta = M + 2m.$$
- A cylinder, of radius a, lies within a rough fixed cylinderical cavity of radius $2a$. The centre of gravity of the cylinder is at a distance c from the axis, and the initial state is that of stable equilibriom at the lowest poiont o fthe cavity. Show that the smallest angular velocity with which the cylinder must be started that it may roll right round the cavity is given by
$$\Omega^2(a+c) = g\left[1 + \frac{4(a+c)^2}{(a-c)^2 + k^2}\right]$$
  where k is the radius of gygration about the centre of gravity.
  Find also the normal reaction between the cylinders to any position.
- A circular cylinder of radius a and of radius of gyration k rolls without slippin inside a hollow cylinder of radius b which is free to move about its axis. Show that the plane through their axis will move like a simple circular pendulum of lenght
$$(b-a)(1+n), \text{ where } n = \frac{(k^2/a^2)}{1 + \frac{b^2}{a^2}\cdot\frac{mk^2}{MK^2}}$$
  where k and K are the radii of gyration, of the inner and outer cylinders respectively, about their axes and m and M their masses.
- A uniform rough ball is at rest within a hollow cylindrical gardan roller, and the roller is then drawn along a level path with uniform velocity V. If $V^2 < (27/7)g(b-a)$, show that the ball will roll completely round the inside of the roller; a, b being the radii of the ball and roller.

LAGRANGE'S EQUATIONS | SET-IV

8.1. Generalised Coordinates.

The independent quantities which determine the position of a dynamical system are called its **generalised coordinates.**

8.2. Degrees of Freedom.

The number of independent motions which a dynamical system can have is called its **degree of freedom.** But the number of independent motions is the same as the number of the generalised cordinates. Hence the number of degrees of freedom of the system is equal to the number of the generalised coordinates.

Exmamples :

1. The degree of freedom of a particle moving in space is 3. It is because three coordinates (x, y, z) are required to specify its position in space.
2. The degree of freedom of a rigid body which can move freely in space is 6.

A rigid body is fixed in space if any three non-collinear points of the body are fixed. Let these three point A, B, C have coordinates (x_1, y_1, z_1), (x_2, y_2, z_2) and (x_3, y_3, z_3). Then as the distance between every pair of particles of a rigid body is unaltered

$$\therefore AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 = \text{const.} \quad ...(1)$$

$$BC^2 = (x_3 - x_2)^2 + (y_3 - y_2)^2 + (z_3 - z_2)^2 = \text{const.} \quad ...(2)$$

$$CA^2 = (x_1 - x_3)^2 + (y_1 - y_3)^2 + (z_1 - z_3)^2 = \text{const.} \quad ...(3)$$

Here any three coordinates can be expressed in terms of the remaining six. Thus only six independent coordinates are required to describe the motion.

Hence the degree of freedom of the rigid body which can move freely in space is 6.

Note. In general the degree of freedom of a system containing n-particles moving freely in space is 3n as it requires 3n coordinates to specify its position.

8.3. Holonomic System and Non-Holonomic System.

Let $\theta, \phi, \psi ...$ be the generalised coordinates of a system, then the cartesian coordinates (x, y, z) of any point of it at any time t can be expressed as functions of $\theta, \phi, \psi, ...$ *i.e.*

$x = f_1(t, \theta, \phi, \psi, ...)$, $y = f_2(t, \theta, \phi, \psi, ...)$, $z = f_3(t, \theta, \phi, \psi, ...)$.

If these functions do not involve velocities i.e. $\dot{\theta}, \dot{\phi}$ etc. or any higher derivative with respect to t, then such a system is called a holonomic system otherwise it is said to be non-holonomic system.

8.4. Conservative and non-conservative System.

If the forces acting on a system are derivable from a potential function (or potential energy) V, then the forces are called *conservative* otherwise *non-conservative.*

8.5. Lagrange's Equations for finite forces.

Consider a holonomic dynamical system moving under the action of conservative forces.

Let (x, y, z) be the coordinates of any particle m of the system referred to any rectangular axes, and let them be expressed in terms of a certain number of generalised coordinates $\theta, \phi, \psi, ...$, so that if t is the time, then we have

$$x = f_1(t, \theta, \phi, ...), y = f_2(t, \theta, \phi, ...), z = f_3(t, \theta, \phi, ...) \quad (A)$$

Since the system is holonomic, so these functions do not contain $\dot{\theta}, \dot{\phi}, ...$ or any higher derivative with respect to t.

By D' Alembert's principle, the reversed effective forces and the external forces acting at each particle of a body form a system of forces in equilibrium. Thus if X, Y, Z are the components of the external force acting at the particle m at the point (x, y, z), then giving the system a virtual displacement consistent with the geometrical conditions, at time t, the equation of virtual work is

$$\Sigma[(X - m\ddot{x})\,\delta x + (Y - m\ddot{y})\,\delta y + (Z - m\ddot{z})\,\delta z] = 0 \quad ...(1)$$

or $\quad \Sigma m(\ddot{x}\,\delta x + \ddot{y}\,\delta y + \ddot{z}\,\delta z) = \Sigma(X\,\delta x + Y\,\delta y + Z\,\delta z)$

or $\quad \Sigma m(\ddot{x}\,\delta x + \ddot{y}\,\delta y + \ddot{z}\,\delta z) = -\delta V \quad ...(2)$

where V is the potential function.

Now $\delta x = \dfrac{\partial x}{\partial \theta}\delta\theta + \dfrac{\partial x}{\partial \phi}\delta\phi + ...$,

the term $\dfrac{\partial x}{\partial t}\delta t$ is not taken as δx is the variaton of x at time t.

Similarly $\delta y = \dfrac{\partial y}{\partial \theta}\delta\theta + \dfrac{\partial y}{\partial \phi}\delta\phi + ...$, and $\delta z = \dfrac{\partial z}{\partial \theta} + \dfrac{\partial z}{\partial \phi}\delta\phi + ...$

Also as V is a function of $\theta, \phi, \psi, ...$

$$\therefore \delta V = \frac{\partial V}{\partial \theta}\delta\theta + \frac{\partial V}{\partial \phi}\delta\phi + ...$$

Substituting in (2), we get

$$\Sigma m\left(\ddot{x}\frac{\partial x}{\partial \theta} + \ddot{y}\frac{\partial y}{\partial \theta} + \ddot{z}\frac{\partial z}{\partial \theta}\right)\delta\theta + \left(\ddot{x}\frac{\partial x}{\partial \phi} + \ddot{y}\frac{\partial y}{\partial \phi} + \ddot{z}\frac{\partial z}{\partial \phi}\right)\partial\phi + ... = -\left(\frac{\partial V}{\partial \theta}\delta\theta + \frac{\partial V}{\partial \phi}\delta\phi + ...\right)$$

Since $\delta\theta, \delta\phi, ...$, are all independent of each other, hence equating the coefficients of $\delta\theta, \delta\phi, ...$, on both the sides, we have

$$\Sigma m\left(\ddot{x}\frac{\partial x}{\partial \theta} + \ddot{y}\frac{\partial y}{\partial \theta} + \ddot{z}\frac{\partial z}{\partial \theta}\right) = -\frac{\partial V}{\partial \theta} \quad ...(3)$$

With similar expressions in $\phi, \psi, ...$

From (A), we have

$$\dot{x} = \frac{\partial x}{\partial t} + \frac{\partial x}{\partial \theta}\dot{\theta} + \frac{\partial x}{\partial \phi}\dot{\phi} + ...$$

$\therefore \dfrac{\partial \dot{x}}{\partial \dot{\theta}} = \dfrac{\partial x}{\partial \theta}$, (Since $\dot{\theta}, \dot{\phi}, ...$ all are independent of each other)

Similarly, $\dfrac{\partial \dot{y}}{\partial \dot{\theta}} = \dfrac{\partial y}{\partial \theta}, \dfrac{\partial \dot{z}}{\partial \dot{\theta}} = \dfrac{\partial z}{\partial \theta}$

Substituting in the L.H.S. of (3), we have

$$\Sigma m\left(\ddot{x}\frac{\partial x}{\partial \theta} + \ddot{y}\frac{\partial y}{\partial \theta} + \ddot{z}\frac{\partial z}{\partial \theta}\right) = \Sigma m\left(\ddot{x}\frac{\partial \dot{x}}{\partial \dot{\theta}} + \ddot{y}\frac{\partial \dot{y}}{\partial \dot{\theta}} + \ddot{z}\frac{\partial \dot{z}}{\partial \dot{\theta}}\right)$$

$$= \frac{d}{dt}\left[\Sigma m\left(\dot{x}\frac{\partial \dot{x}}{\partial \dot{\theta}} + \dot{y}\frac{\partial \dot{y}}{\partial \dot{\theta}} + \dot{z}\frac{\partial \dot{z}}{\partial \dot{\theta}}\right)\right] - \Sigma m\left[\dot{x}\frac{d}{dt}\left(\frac{\partial \dot{x}}{\partial \dot{\theta}}\right) + \dot{y}\frac{d}{dt}\left(\frac{\partial \dot{y}}{\partial \dot{\theta}}\right) + \dot{z}\frac{d}{dt}\left(\frac{\partial \dot{z}}{\partial \dot{\theta}}\right)\right]$$

$$= \frac{d}{dt}\left[\Sigma m\left(\dot{x}\frac{\partial \dot{x}}{\partial \dot{\theta}} + \dot{y}\frac{\partial \dot{y}}{\partial \dot{\theta}} + \dot{z}\frac{\partial \dot{z}}{\partial \dot{\theta}}\right)\right] - \Sigma m\left[\dot{x}\frac{d}{dt}\left(\frac{\partial x}{\partial \theta}\right) + \dot{y}\frac{d}{dt}\left(\frac{\partial y}{\partial \theta}\right) + \dot{z}\frac{d}{dt}\left(\frac{\partial z}{\partial \theta}\right)\right] \quad ...(4)$$

Now $\dfrac{d}{dt}\left(\dfrac{\partial x}{\partial \theta}\right) = \dfrac{\partial}{\partial t}\left(\dfrac{\partial x}{\partial \theta}\right) + \dfrac{\partial}{\partial \theta}\left(\dfrac{\partial x}{\partial \theta}\right)\dot{\theta} + \dfrac{\partial}{\partial \phi}\left(\dfrac{\partial x}{\partial \theta}\right)\dot{\phi} + ...$

$$= \frac{\partial^2 x}{\partial t\,\partial \theta} + \frac{\partial^2 x}{\partial \theta^2}\dot{\theta} + \frac{\partial^2 x}{\partial \phi\,\partial \theta}\dot{\phi} + ...$$

$$= \frac{\partial^2 x}{\partial \theta\,\partial t} + \frac{\partial^2 x}{\partial \theta^2}\dot{\theta} + \frac{\partial^2 x}{\partial \theta\,\partial \phi}\dot{\phi} + ...$$

$$= \frac{\partial}{\partial \theta}\left(\frac{\partial x}{\partial t} + \frac{\partial x}{\partial \theta}\dot{\theta} + \frac{\partial x}{\partial \phi}\dot{\phi} + ...\right) = \frac{\partial}{\partial \theta}(\dot{x})$$

i.e. $\dfrac{d}{dt}\left(\dfrac{\partial x}{\partial \theta}\right) = \dfrac{\partial \dot{x}}{\partial \theta}$, similarly $\dfrac{d}{dt}\left(\dfrac{\partial y}{\partial \theta}\right) = \dfrac{\partial \dot{y}}{\partial \theta}, \dfrac{d}{dt}\left(\dfrac{\partial z}{\partial \theta}\right) = \dfrac{\partial \dot{z}}{\partial \theta}$

Substituting in (4), we get

$$\Sigma m\left(\ddot{x}\frac{\partial x}{\partial \theta} + \ddot{y}\frac{\partial y}{\partial \theta} + \ddot{z}\frac{\partial z}{\partial \theta}\right) = \frac{d}{dt}\left[\Sigma m\left(\dot{x}\frac{\partial \dot{x}}{\partial \dot{\theta}} + \dot{y}\frac{\partial \dot{y}}{\partial \dot{\theta}} + \dot{z}\frac{\partial \dot{z}}{\partial \dot{\theta}}\right)\right] - \Sigma m\left(\dot{x}\frac{\partial \dot{x}}{\partial \theta} + \dot{y}\frac{\partial \dot{y}}{\partial \theta} + \dot{z}\frac{\partial \dot{z}}{\partial \theta}\right)$$

$$= \frac{d}{dt}\left[\frac{\partial}{\partial \dot{\theta}}\left\{\tfrac{1}{2}\Sigma m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)\right\}\right] - \frac{\partial}{\partial \theta}\left\{\tfrac{1}{2}\Sigma m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)\right\}$$

$$= \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta}$$

where T = K. E. of the system $= \frac{1}{2}\Sigma m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

Hence from (3), we have

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = -\frac{\partial V}{\partial \theta}$$

Similarly, $\quad \dfrac{d}{dt}\left(\dfrac{\partial T}{\partial \dot{\phi}}\right) - \dfrac{\partial T}{\partial \phi} = -\dfrac{\partial V}{\partial \phi}$

with similar expressions for each coordinate.

These equations are called **Lagrange's equations** for finite forces where V is the potential function and T the total kinetic energy.

If W is the work function then $W + V$ = Const.

i.e. $\dfrac{\partial W}{\partial \theta} = -\dfrac{\partial V}{\partial \theta}$, etc.

Thus using the work function W the Lagrange's equations are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = \frac{\partial W}{\partial \theta}, \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} = \frac{\partial W}{\partial \phi}, \text{ etc.}$$

Vector Method. At time t, let $\mathbf{F}$ be the external force acting on a particle of mass m whose position vector with regard to any origin O is $\mathbf{r}$ with generalised coordinates $\theta, \phi, \psi, ...$ Then

By D' Alembert's principle, the reversed effective forces and the external forces acting at each particle of a body form a system of forces in equilibrium.

Thus giving the system a virtual displacement consistent with the geometrical conditions, at time t, the equation of virtual work is

$$\Sigma(\mathbf{F} - m\ddot{\mathbf{r}}) \cdot \delta\mathbf{r} = 0 \text{ or } \Sigma m\ddot{\mathbf{r}} \cdot \delta\mathbf{r} = \Sigma\mathbf{F} \cdot \delta\mathbf{r} \quad ...(2)$$

Let δW denote the virtual workdone by the external forces, then

$$\delta W = \Sigma\mathbf{F} \cdot \delta\mathbf{r} \quad ...(3)$$

and δW is called the virtual work function.

From (2) and (3), we have $\delta W = \Sigma m\ddot{\mathbf{r}} \cdot \delta\mathbf{r}$. $\quad ...(4)$

Now from (1), we have

$$\delta\mathbf{r} = \frac{\partial\mathbf{r}}{\partial\theta}\delta\theta + \frac{\partial\mathbf{r}}{\partial\phi}\delta\phi + ... \text{ (Taking } t \text{ constant)}$$

Particularly, $\delta\mathbf{r} = \dfrac{\partial\mathbf{r}}{\partial\theta}\delta\theta$ if only θ is allowed to change.

And $\dot{\mathbf{r}} = \dfrac{d\mathbf{r}}{dt} = \dfrac{\partial\mathbf{r}}{\partial t} + \dfrac{\partial\mathbf{r}}{\partial\theta}\dot{\theta} + \dfrac{\partial\mathbf{r}}{\partial\phi}\dot{\phi} + ...$

$$\therefore \frac{\partial \dot{r}}{\partial \dot{\theta}} = \frac{\partial r}{\partial \theta} \qquad \text{...(5)}$$

($\because$ $\theta, \phi, \ldots$ are all independent of each other)

Thus from (4), we have

$\delta W = \Sigma m\ddot{r} \cdot \delta r = \Sigma m\ddot{r} \cdot \frac{\partial r}{\partial \theta}\delta\theta$, when only θ is allowed to change.

$$= \Sigma m\ddot{r} \cdot \frac{\partial \dot{r}}{\partial \dot{\theta}}\delta\theta \qquad \text{from (5)}$$

$$= \Sigma m\left[\frac{d}{dt}\left(\dot{r} \cdot \frac{\partial \dot{r}}{\partial \dot{\theta}}\right) - \dot{r} \cdot \frac{d}{dt}\left(\frac{\partial \dot{r}}{\partial \dot{\theta}}\right)\right]\delta\theta$$

$$= \Sigma m\left[\frac{d}{dt}\left(\dot{r} \cdot \frac{\partial \dot{r}}{\partial \dot{\theta}}\right) - \dot{r} \cdot \frac{d}{dt}\left(\frac{\partial r}{\partial \theta}\right)\right]\delta\theta \qquad \text{...(6)}$$

Now $\frac{d}{dt}\left(\frac{\partial r}{\partial \theta}\right) = \frac{\partial}{\partial t}\left(\frac{\partial r}{\partial \theta}\right) + \frac{\partial}{\partial \theta}\left(\frac{\partial r}{\partial \theta}\right)\dot{\theta} + \frac{\partial}{\partial \phi}\left(\frac{\partial r}{\partial \theta}\right)\dot{\phi} + \ldots$

$$= \frac{\partial}{\partial \theta}\left(\frac{\partial r}{\partial t} + \frac{\partial r}{\partial \theta}\dot{\theta} + \frac{\partial r}{\partial \phi}\dot{\phi} + \ldots\right) = \frac{\partial \dot{r}}{\partial \theta}$$

Substituting in (6), we have

$$\delta W = \Sigma m\left[\frac{d}{dt}\left(\dot{r} \cdot \frac{\partial \dot{r}}{\partial \dot{\theta}}\right) - \dot{r} \cdot \frac{\partial \dot{r}}{\partial \theta}\right]\delta\theta$$

$$= \Sigma\left[\frac{d}{dt}\frac{\partial}{\partial \dot{\theta}}\left(\frac{1}{2}m\dot{r}^2\right) - \frac{\partial}{\partial \theta}\left(\frac{1}{2}m\dot{r}^2\right)\right]\delta\theta$$

$$= \left[\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta}\right]\delta\theta,$$

where T = K.E. of the system $= \Sigma\frac{1}{2}m\dot{r}^2$

or $\frac{\partial W}{\partial \theta} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta}$

or $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = \frac{\partial W}{\partial \theta}$ ...(7)

when only θ is allowed to change.

Similarly when only ϕ is allowed to change, we have

$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} = \frac{\partial W}{\partial \phi}$ ...(8)

Similar equations corresponding to the variations in other generalised coordinates.

These equations (7), (8), ... are called the **Lagrange's equations** for finite forces where W is the work function and T the total kinetic energy.

8.6. Lagrangian Function.

When the forces are conservative and $(\theta, \phi, \psi, \ldots)$ are the generalised coordinates of a system, we can find the potential function V as the function of $(\theta, \phi, \psi, \ldots)$, such that W + V = const, where W is the work function.

$$\therefore \frac{\partial W}{\partial \theta} = -\frac{\partial V}{\partial \theta}, \quad \frac{\partial W}{\partial \phi} = -\frac{\partial V}{\partial \phi}, \text{ etc.}$$

Hence Lagrange's equations for a conservative holonomic dynamical system becomes

$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = -\frac{\partial V}{\partial \theta}, \ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} = -\frac{\partial V}{\partial \phi}$, etc.

or $\frac{d}{dt}\left\{\frac{\partial}{\partial \dot{\theta}}(T - V)\right\} - \frac{\partial}{\partial \theta}(T - V) = 0,$

$\frac{d}{dt}\left\{\frac{\partial}{\partial \dot{\phi}}(T - V)\right\} - \frac{\partial}{\partial \phi}(T - V) = 0,$ etc.

Since V does not contain $\dot{\theta}, \dot{\phi}, \ldots$

or $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0, \ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0,$ etc.

Then **$L = T - V$ is called the Lagrangian function or Lagrange's function or kinetic potential.**

8.7. Principle of Energy. *To deduce the principle of energy from the Lagrange's equations (Conservative field).*

If $(\theta, \phi, \psi, \ldots)$ are the generalised coordinates of a dynamical system, then the Lagrange's equations are

$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = -\frac{\partial V}{\partial \theta}, \ \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} = -\frac{\partial V}{\partial \phi}$, etc. ...(1)

If x, y, z do not contain t explicitly, we have

$$\dot{x} = \frac{\partial x}{\partial \theta}\dot{\theta} + \frac{\partial x}{\partial \phi}\dot{\phi} + \ldots$$

$\therefore$ K.E., $T = \frac{1}{2}\Sigma m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

$$= \frac{1}{2}\Sigma m\left[\left(\frac{\partial x}{\partial \theta}\dot{\theta} + \frac{\partial x}{\partial \phi}\dot{\phi} + \ldots\right)^2 + \left(\frac{\partial y}{\partial \theta}\dot{\theta} + \frac{\partial y}{\partial \phi}\dot{\phi} + \ldots\right)^2 + \left(\frac{\partial z}{\partial \theta}\dot{\theta} + \frac{\partial z}{\partial \phi}\dot{\phi} + \ldots\right)^2\right]$$

$$= A_{11}\dot{\theta}^2 + A_{22}\dot{\phi}^2 + A_{33}\dot{\psi}^2 + \ldots + 2A_{12}\dot{\theta}\dot{\phi} + 2A_{13}\dot{\theta}\dot{\psi} + \ldots$$

which is a homogeneous quadratic function of $\dot{\theta}, \dot{\phi}$ etc.

Hence by Euler's theorem

$\dot{\theta}\frac{\partial T}{\partial \dot{\theta}} + \dot{\phi}\frac{\partial T}{\partial \dot{\phi}} + \ldots = 2T$ ...(2)

Also $\frac{dT}{dt} = \frac{\partial T}{\partial \dot{\theta}}\ddot{\theta} + \frac{\partial T}{\partial \dot{\phi}}\ddot{\phi} + \ldots + \frac{\partial T}{\partial \theta}\dot{\theta} + \frac{\partial T}{\partial \phi}\dot{\phi} + \ldots$ ...(3)

Now multiplying the Lagrange's equations in (1) by $\dot{\theta}, \dot{\phi}, \ldots$ respectively and then adding, we get

$$\left[\dot{\theta}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) + \dot{\phi}\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) + \ldots\right] - \left(\dot{\theta}\frac{\partial T}{\partial \theta} + \dot{\phi}\frac{\partial T}{\partial \phi} + \ldots\right) = -\left(\dot{\theta}\frac{\partial V}{\partial \theta} + \dot{\phi}\frac{\partial V}{\partial \phi} + \ldots\right)$$

or $\frac{d}{dt}\left(\dot{\theta}\frac{\partial T}{\partial \dot{\theta}} + \dot{\phi}\frac{\partial T}{\partial \dot{\phi}} + \ldots\right) - \left\{\ddot{\theta}\frac{\partial T}{\partial \dot{\theta}} + \ddot{\phi}\frac{\partial T}{\partial \dot{\phi}} + \ldots\right\} - \left\{\dot{\theta}\frac{\partial T}{\partial \theta} + \dot{\phi}\frac{\partial T}{\partial \phi} + \ldots\right\} = -\left(\dot{\theta}\frac{\partial V}{\partial \theta} + \dot{\phi}\frac{\partial V}{\partial \phi} + \ldots\right)$

or $\frac{d}{dt}(2T) - \frac{dT}{dt} = -\frac{dV}{dt}$ [Using (2) and (3)]

or $2\frac{dT}{dt} - \frac{dT}{dt} = -\frac{dV}{dt}$

or $\frac{d}{dt}(T + V) = 0 \quad \therefore \quad T + V = \text{Constant.}$

I.e. the sum of the kinetic and potential energies of a system of conservative forces is constant throughout the motion.

8.8. Small Oscillations.

To explain how Lagrange's equations are used in case of small oscillations.

To investigate the theory of small oscillations by the use of Lagrange's equations about the position of equilibrium, the generalised coordinates* θ, ϕ, ψ must be chosen such that they vanish in the position of equilibrium.

Since the system makes small oscillations about the position of equilibrium, so θ, ϕ, ψ and $\dot{\theta}, \dot{\phi}, \dot{\psi}$ will remain small during the whole motion.

If x, y, z do not contain t explicitly, then the total K.E., T and the work function W are given by

$T = \Sigma\frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

$$= \Sigma\frac{1}{2}m\left[\left(\frac{\partial x}{\partial \theta}\dot{\theta} + \frac{\partial x}{\partial \phi}\dot{\phi} + \frac{\partial x}{\partial \psi}\dot{\psi}\right)^2 + \left(\frac{\partial y}{\partial \theta}\dot{\theta} + \frac{\partial y}{\partial \phi}\dot{\phi} + \frac{\partial y}{\partial \psi}\dot{\psi}\right)^2 + \left(\frac{\partial z}{\partial \theta}\dot{\theta} + \frac{\partial z}{\partial \phi}\dot{\phi} + \frac{\partial z}{\partial \psi}\dot{\psi}\right)^2\right]$$

$$= A_{11}\dot{\theta}^2 + A_{22}\dot{\phi}^2 + A_{33}\dot{\psi}^2 + 2A_{12}\dot{\theta}\dot{\phi} + 2A_{23}\dot{\phi}\dot{\psi} + 2A_{13}\dot{\theta}\dot{\psi} \qquad \text{...(1)}$$

and $W = C + B_1\theta + B_2\phi + B_3\psi + B_{11}\theta^2 + B_{22}\phi^2 + B_{33}\psi^2$ ...(2)

Now choosing X, Y, Z such that θ, ϕ, ψ can be expressed by the equations of the form

$$\theta = \lambda_1 X + \lambda_2 Y + \lambda_3 Z$$

$\phi = \mu_1 X + \mu_2 Y + \mu_3 Z$

$\psi = \nu_1 X + \nu_2 Y + \nu_3 Z$

Choosing $\lambda_1, \lambda_2, \lambda_3, \mu_1, \mu_2, \mu_3, \nu_1, \nu_2, \nu_3$ such that when the above values of θ, ϕ, ψ and their derivatives $\dot{\theta}, \dot{\phi}, \dot{\psi}$ are substituted in (1) and (2), then there is no term containing $\dot{X}\dot{Y}, \dot{Y}\dot{Z}, \dot{Z}\dot{X}$ in T and there is no term containing XY, YZ, ZX in W. Then X, Y, Z are called the **Principal or Normal Coordinates.**

Thus when X, Y, Z are principal coordinates, then from (1) and (2), we have

$$T = A'_{11}\dot{X}^2 + A'_{22}\dot{Y}^2 + A'_{33}\dot{Z}^2$$

and $W = C' + B'_1 X + B'_2 Y + B'_3 Z + B'_{11}X^2 + B'_{22}Y^2 + B'_{33}Z^2$.

Then the Lagrange's equations are

$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{X}}\right) - \frac{\partial T}{\partial X} = \frac{\partial W}{\partial X}$ etc.

or $\frac{d}{dt}(2A'_{11}\dot{X}) = (B'_1 + 2B'_{11}X)$, etc.

or $2A'_{11}\ddot{X} = B'_1 + 2B'_{11}X$, etc.

which can be put in the forms

$\ddot{X} = -n_1^2 X, \ \ddot{Y} = -n_2^2 Y, \ \ddot{Z} = -n_3^2 Z$

which represent S.H.M's giving the small oscillations about the position of equilibrium.

EXAMPLES

Ex. 1. *For a simple pendulum (i) find the Lagrangian function and (ii) obtain an equation describing its motion.* (IFoS-2011)

Sol. Let l be the length of the simple pendulum and θ the angle made by the string with the vertical at time t. Thus θ is the only generalised coordinate. Then the velocity of mass M at A will be $v = l\dot{\theta}$.

$\therefore$ Total K.E. $T = \frac{1}{2}Mv^2 = \frac{1}{2}Ml^2\dot{\theta}^2$

And the potential function

$V = Mg(A'B) = Mg(l - l\cos\theta)$

$= Mgl(1 - \cos\theta)$

$\therefore$ The equation of horizontal motion of the hemisphere is

$M\ddot{x} = -R \sin\theta$

or $M \frac{d}{dt}(\dot{x}) = -R \sin\theta$

The particle will leave the hemisphere if $R = 0$

i.e. if $\frac{d}{dt}(\dot{x}) = 0$ or $\frac{d}{dt}\left(\frac{-ma\cos\theta\,\dot{\theta}}{M+m}\right) = 0$

or $\cos\theta\,\ddot{\theta} = \sin\theta\,\dot{\theta}^2$ ...(3)

Differentiating w.r.t. 't' and dividing by $2a\dot{\theta}$, we get

$(M + m\sin^2\theta)\,\ddot{\theta} + m\sin\theta\cos\theta\,\dot{\theta}^2 = \frac{(M+m)g}{a}\sin\theta$

Substituting $\ddot{\theta} = \frac{\sin\theta}{\cos\theta}\dot{\theta}^2$ from (3), we get

$(M + m\sin^2\theta)\frac{\sin\theta}{\cos\theta}\dot{\theta}^2 + m\sin\theta\cos\theta\,\dot{\theta}^2 = (M+m)\frac{g}{a}\sin\theta$

or $(M + m\sin^2\theta + m\cos^2\theta)\,\dot{\theta}^2 = (M+m)\frac{g}{a}\cos\theta$

$\therefore \dot{\theta}^2 = \frac{g}{a}\cos\theta$ ...(4)

Substituting from (4) in (2), we get

$(M + m\sin^2\theta)\,g\cos\theta = 2g(M+m)(\cos\alpha - \cos\theta)$

or $(M + m - m\cos^2\theta)\cos\theta = 2(M+m)(\cos\alpha - \cos\theta)$

or $m\cos^3\theta - (M+m)(3\cos\theta - 2\cos\alpha) = 0$

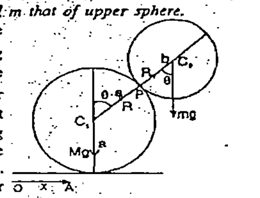
which is the required result.

Ex. 42. *Two unequal smooth spheres, one placed on the top of the other are in unstable equilibrium, the lower sphere resting on a smooth table. The system is slightly disturbed; show that the sphere will separate when the lines joining their centres make an angle θ with the vertical given by the equation*

$$m\cos^3\theta = (M+m)(3\cos\theta - 2)$$

where M is the mass of the lower, and m that of upper sphere.

Sol. Let C_1 be the centre and a the radius of the sphere of mass M resting on a horizontal smooth table. Let C_2 be the centre and b the radius of another sphere of mass m resting at the highest point of the first sphere of mass m resting at the highest point of the first sphere in the position of unstable equilibrium. When the system is disturbed then after



time t, let the lower sphere have moved through a distance $OA = x$ on the table and let the ling joining centres C_1C_2 turn through an angle θ to the vertical. Note that C_1C_2 was vertical initially.

Both the spheres and the horizontal plane are given to be smooth so there are no forces acting to turn either sphere about its centre. Hence there is no rotary motion.

Referred to the horizontal and vertical lines through O, as axes, the coordinates (x_{c_1}, y_{c_1}) of centre C_1 and (x_{c_2}, y_{c_2}) of centre C_2 are given by

$x_{c_1} = x,\ y_{c_1} = a\ ;\ x_{c_2} = x + c\sin\theta,\ y_{c_2} = a + c\cos\theta$, where $c = C_1C_2 = a + b$.

As the spheres and the horizontal plane are smooth, there is no horizontal force on the system.

$\therefore \frac{d}{dt}(M\dot{x}_{c_1} + m\dot{x}_{c_2}) = 0$

or $\frac{d}{dt}[M\dot{x} + m(\dot{x} + c\cos\theta\,\dot{\theta})] = 0$

Integrating, $M\dot{x} + m(\dot{x} + c\cos\theta\,\dot{\theta}) = C$

But initially $\dot{x} = 0, \dot{\theta} = 0 \therefore C = 0$

$\therefore M\dot{x} + m(\dot{x} + c\cos\theta\,\dot{\theta}) = 0$ or $\dot{x} = -\frac{mc}{M+m}\cos\theta\,\dot{\theta}$ ...(1)

Also the energy equation gives

$\frac{1}{2}Mv_{c1}^2 + \frac{1}{2}mv_{c2}^2 = mg(c - c\cos\theta)$

or $M\dot{x}^2 + m(\dot{x}^2 + c^2\dot{\theta}^2 + 2c\,\dot{\theta}\dot{x}\cos\theta) = 2mgc[1 - \cos\theta]$ ...(2)

Substituting the value of $\dot{x}$ from (1) in (2), we get

$(M+m)\frac{m^2c^2}{(M+m)^2}\cos^2\theta\,\dot{\theta}^2 + mc^2\dot{\theta}^2 + 2mc\dot{\theta}\cdot\left(\frac{-mc}{M+m}\cos\theta\,\dot{\theta}\right)\cos\theta$

$= 2mgc(1 - \cos\theta)$

or $\left[c - \frac{mc}{M+m}\cos^2\theta\right]\dot{\theta}^2 = 2g(1 - \cos\theta)$

or $c(M + m\sin^2\theta)\,\dot{\theta}^2 = 2g(M+m)(1 - \cos\theta)$ ...(3)

Differentiating w.r.t. 't' and dividing by $2\dot{\theta}$, we get

$(M + m\sin^2\theta)\,\ddot{\theta} + m\sin\theta\cos\theta\,\dot{\theta}^2 = (g/c)(M+m)\sin\theta$ ...(4)

If R is the reaction between the two spheres at the point of contact P, then considering the horizontal motion of the lower sphere, we get

$-R\sin\theta = M\ddot{x}_{c1} = M\ddot{x} = -\frac{Mmc}{M+m}(\cos\theta\,\ddot{\theta} - \sin\theta\,\dot{\theta}^2)$ ...(5)

When the two spheres separate, then $R = 0$, $\therefore$ from (5), we get

$0 = -\frac{Mmc}{M+m}(\cos\theta\,\ddot{\theta} - \sin\theta\,\dot{\theta}^2)$

or $\cos\theta\,\ddot{\theta} = \sin\theta\,\dot{\theta}^2$ or $\ddot{\theta}^2 = \cot\theta\,\dot{\theta}$ ...(6)

From (4) and (6), eliminating $\ddot{\theta}$, we get

$(M + m\sin^2\theta)\,\ddot{\theta} + m\sin\theta\cos\theta\cdot\cot\theta\,\dot{\theta} = (g/c)(M+m)\sin\theta$

or $(M+m)\,\ddot{\theta} = \frac{g}{c}(M+m)\sin\theta \therefore \ddot{\theta} = \frac{g}{c}\sin\theta$ ...(7)

$\therefore$ from (6), $\dot{\theta}^2 = \frac{g}{c}\cos\theta$ ...(8)

Substituting the value of $\dot{\theta}^2$ from (8) in (3), we get

$c(M + m\sin^2\theta)\cdot(g/c)\cos\theta = 2g(M+m)(1 - \cos\theta)$

or $(M + m - m\cos^2\theta)\cos\theta = 2(M+m)(1 - \cos\theta)$

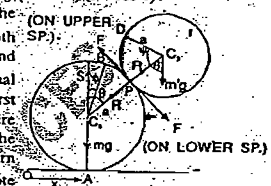
or $m\cos^3\theta = (M+m)(3\cos\theta - 2)$

which is the required result.

Ex. 43. *Two homogenous spheres of equal radii and masses m and m′ rest on a smooth horizontal plane with m′ on the highest point of m. If the system be disturbed show that the inclination θ of their common normal to the vertical is given by*

$$a\dot{\theta}^2(7m + 5m'\sin^2\theta) = 5g(m + m')(1 - \cos\theta).$$

Sol. Let C_1 be the centre and m the mass of the sphere resting on a smooth horizontal plane. Let C_2 be the centre and m' the mass of another sphre of equal radius resting on the highest of the first sphere. In time t, let the lower sphere move through a distance $OA = x$, on the table while the line of centres C_1C_2 turn through an angel θ with the vertical. Note that C_1C_2 was vertical, initially. During this time t, let the two spheres turn through angles ϕ and ψ to the vertical. Initially C_1B, C_1C_2 and C_2D coincided vertically.



Since there is no slipping between the two spheres

$\therefore$ Arc BP = Arc DP or $a(\theta - \phi) = a(\psi - \theta)$ or $\psi + \phi = 2\theta$ ...(1)

Considering the motion of the spheres and taking moments about their centres, we get

$m\cdot\frac{2}{5}a^2\ddot{\phi} = F\cdot a$ (For lower sphere) ...(2)

and $m'\cdot\frac{2}{5}a^2\ddot{\psi} = F\cdot a$ (For upper sphere) ...(3)

From (2) and (3), we get

Integrating $m\dot{\phi} = m'\dot{\psi}$, (Constant of integration is $0 \because$ when $\dot{\phi} = 0, \dot{\psi} = 0$)

or $\frac{\dot{\phi}}{m'} = \frac{\dot{\psi}}{m} = \frac{\dot{\phi} + \dot{\psi}}{m' + m} = \frac{2\dot{\theta}}{m' + m}$ from (1)

$\therefore \dot{\phi} = \frac{2m'\dot{\theta}}{(m + m')}$ and $\dot{\psi} = \frac{2m\dot{\theta}}{(m + m')}$ ...(4)

Referred to the horizontal and vertical lines through O as axes the coordinates (x_{c1}, y_{c1}) of centre C_1 and (x_{c2}, y_{c2}) of centre C_2 are given by

$x_{c1} = x, y_{c1} = a; x_{c2} = x + 2a\sin\theta, y_{c2} = a + 2a\cos\theta$

Since there is no horizontal force on the system,

$\therefore \frac{d}{dt}(m\dot{x}_{c1} + m'\dot{x}_{c2}) = \frac{d}{dt}[m\dot{x} + m'(\dot{x} + 2a\cos\theta\,\dot{\theta})] = 0$

Integrating, $m\dot{x} + m'(\dot{x} + 2a\cos\theta\,\dot{\theta}) = C$

Initially $\dot{x} = 0, \dot{\theta} = 0 \therefore C = 0$

$\therefore m\dot{x} + m'(\dot{x} + 2a\cos\theta\,\dot{\theta}) = 0$ or $\dot{x} = \frac{-2am'\cos\theta\,\dot{\theta}}{(m + m')}$ ...(5)

The energy equation gives,

$\left(\frac{1}{2}m\cdot\frac{2}{5}a^2\dot{\phi}^2 + \frac{1}{2}mv_{c1}^2\right) + \left(\frac{1}{2}m'\cdot\frac{2}{5}a^2\dot{\psi}^2 + \frac{1}{2}m'v_{c2}^2\right)$

$= m'g(2a - 2a\cos\theta)$

or $\frac{2}{5}ma^2\dot{\phi}^2 + m\dot{x}^2 + \frac{2}{5}m'a^2\dot{\psi}^2 + m'(\dot{x}^2 + 4a^2\dot{\theta}^2 + 4a\dot{\theta}\dot{x}\cos\theta)$

$= 4am'g(1 - \cos\theta)$

Substituting the values of $\dot{\phi}$, $\dot{\psi}$ and $\dot{x}$ from (4), (5)

$\frac{2}{5}ma^2\cdot\frac{4m'^2\dot{\theta}^2}{(m + m')^2} + \frac{2}{5}m'a^2\cdot\frac{4m^2\dot{\theta}^2}{(m + m')^2} + (m + m')\frac{4a^2m'^2}{(m + m')^2}\cos^2\theta\,\dot{\theta}^2$

$+ 4m'a^2\dot{\theta}^2 + 4am'\dot{\theta}\left(\frac{-2am'\cos\theta\,\dot{\theta}}{(m + m')}\right)\cos\theta = 4am'g(1 - \cos\theta)$

or $\left[\frac{8}{5}\cdot\frac{mm'a^2(m + m')}{(m + m')^2} - \frac{4a^2m'^2\cos^2\theta}{(m + m')} + 4m'a^2\right]\dot{\theta}^2 = 4am'g(1 - \cos\theta)$

or $[2m - 5m'\cos^2\theta + 5(m + m')]\,a\dot{\theta}^2 = 5(m + m')\,g(1 - \cos\theta)$

or $(7m + 5m'\sin^2\theta)\,a\dot{\theta}^2 = 5(m + m')\,g(1 - \cos\theta)$,

which is the required result.

Ex. 44. *A uniform solid cylinder rests on a smooth horizontal plane and on it placed a second equal cylinder, touching it along its highest generator, if there is no slipping between the cylinders and system moves from rest, show that the cylinders separate when the plane of either axes makes an angle θ with vertical given by the equation*

$$2\cos^3\theta + 4\cos^2\theta - 35\cos\theta + 20 = 0.$$

Also show that until the cylinders separate the same generators remain is contact.

CB and OA were vertical. In time t, let the cylinder turn through an angle ψ.

$\therefore \angle DOA = \psi$. Also let the plane through the axes make an angle θ to the vertical at time t. Since there is no slipping.

$\therefore$ Arc PA = Arc BP or $a(\theta - \psi) = b(\phi - \theta)$

or $a\psi + b\phi = (a+b)\theta$ i.e. $a\psi + b\phi = c\theta$, where $c = a + b$. ...(1)

Equations of motion for the lower cylinder, taking moment about O is

$M \cdot a^2\ddot{\psi} = Fa$, (here $k^2 = a^2$) ...(2)

Since the centre C describes a circle of radius $OC = a + b = c$, about O, therefore its accelerations along and perpendicular to CO are $mc\dot{\theta}^2$ and $mc\ddot{\theta}$ respectively.

$\therefore$ Equations of motion of the sphere are

$mc\dot{\theta}^2 = mg\cos\theta - R$, ...(3)

and $mc\ddot{\theta} = mg\sin\theta - F$. ...(4)

Also taking moment about O, we get

$m \cdot \frac{2}{5}b^2\ddot{\phi} = F \cdot b$. ...(5)

$\therefore$ From (2) and (5), we get

$$Ma\ddot{\psi} = \frac{2}{5}mb\ddot{\phi}.$$

Integrating, $Ma\dot{\psi} = \frac{2}{5}mb\dot{\phi}$, (initially $\dot{\phi} = 0, \dot{\psi} = 0$; $\therefore$ Constant of integration is also zero).

$\therefore \dfrac{a\dot{\psi}}{2m} = \dfrac{b\dot{\phi}}{5M} = \dfrac{a\dot{\psi} + b\dot{\phi}}{2m + 5M} = \dfrac{c\dot{\theta}}{2m + 5M}$ [from (1)]

$\therefore a\dot{\psi} = \dfrac{2mc\dot{\theta}}{2m + 5M}$ and $b\dot{\phi} = \dfrac{5Mc\dot{\theta}}{2m + 5M}$.

The coordinates of C referred to horizontal and vertical lines through O as axes are $(c\sin\theta, c\cos\theta)$.

$$\therefore v_c^2 = \dot{x}^2 + \dot{y}^2 = c^2\dot{\theta}^2.$$

Therefore energy equation gives

$\frac{1}{2}Ma^2\dot{\psi}^2 + (\frac{1}{2}m \cdot \frac{2}{5}b^2\dot{\phi}^2 + \frac{1}{2}mc^2\dot{\theta}^2) = mg(c - c\cos\theta)$

or $\frac{1}{2}M\left(\dfrac{2mc\dot{\theta}}{2m + 5M}\right)^2 + \frac{2}{5}m\left(\dfrac{5Mc\dot{\theta}}{2m + 5M}\right)^2 + mc^2\dot{\theta}^2 = 2mgc(1 - \cos\theta)$

or $\left[\dfrac{2mM(2m + 5M)}{(2m + 5M)^2} + m\right]c\dot{\theta}^2 = 2mg(1 - \cos\theta)$

or $\left[\dfrac{2M}{2m + 5M} + 1\right]c\dot{\theta}^2 = 2g(1 - \cos\theta)$

or $c\dot{\theta}^2 = 2\left(\dfrac{2m + 5M}{2m + 7M}\right)g(1 - \cos\theta)$. ...(6)

Differentiating w.r.t. t and dividing by $2\dot{\theta}$, we get

$c\ddot{\theta} = \left(\dfrac{2m + 5M}{2m + 7M}\right)g\sin\theta$. ...(7)

$\therefore$ From (3) and (4), using (6) and (7), we get

$R = mg\cos\theta - m\,2\left(\dfrac{2m + 5M}{2m + 7M}\right)g(1 - \cos\theta)$

$= \left(\dfrac{mg}{2m + 7M}\right)[(17M + 6m)\cos\theta - (10M + 4m)]$

and $F = mg\sin\theta - m\left(\dfrac{2m + 5M}{2m + 7M}\right)g\sin\theta = \dfrac{2m\,Mg\sin\theta}{(2m + 7M)}$

$\therefore \dfrac{F}{R} = \dfrac{2M\sin\theta}{[(17M + 6m)\cos\theta - (10M + 4m)]}$

$\therefore$ Slipping of the sphere will begin, when $F = \mu R$.

i.e. when $\mu = \dfrac{F}{R}$ i.e. $\mu = \dfrac{2M\sin\theta}{[(17M + 6m)\cos\theta - (10M + 4m)]}$

or $2M\sin\theta = \mu[(17M + 6m)\cos\theta - (10M + 4m)]$,

which gives the value of θ.

Also when slipping begins $\theta < \pi$.

Now $R = \dfrac{F}{\mu} = \dfrac{2mMg\sin\theta}{2m + 7M}$

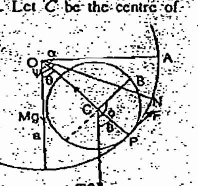
which is positive for all values of θ between 0 and π. Hence, the slipping begins before the sphere leaves the cylinder.

Ex. 40. *A thin hollow cylinder of radius a and mass M is free to turn about its axis which is horizontal and a smaller cylinder of radius b and mass m rolls inside it without slipping, the axes of the two cylinders being parallel. Show that when the plane of the two axes is inclined at an angle θ to the vertical, angular velocity of the larger cylinder is given by*

$$a^2(M + m)(2M + m)\omega^2 = 2gm^2(a - b)(\cos\theta - \cos\alpha)$$

provided both the cylinders are at rest when $\theta = \alpha$.

Sol. Let O be the centre of the hollow cylinder of radius a and mass M which is free to turn about its horizontal axis. Let C be the centre of the smaller cylinder of radius b and mass m which rolls inside the hollow cylinder. Consider the vertical cross-section of the two cylinders through O and C.



Let the line CB fixed in the smaller cylinder and ON the line fixed in the outer cylinder make angles ϕ and ψ to the vertical at time t. Initially CB and ON coincided with OA where $\angle DOA = \alpha$.

Since there is no slipping, $\therefore$ Arc MP = Arc BP

or $a(\psi - \theta) = b(\phi - \theta) \therefore b\phi = a\psi - c\theta$, where $c = a - b$ ...(1)

Considering the motion of two cylinders and taking moments about their centres O and C, we get

$Ma^2\ddot{\psi} = -Fa$ (For outer) ...(2)

and $mb^2\ddot{\phi} = Fb$ (For inner) ...(3)

From (2) and (3), we get

$$Ma\ddot{\psi} = -mb\ddot{\phi}$$

Integrating, $Ma\dot{\psi} = -mb\dot{\phi}$ ...(4)

($\because$ Initially when $\dot{\psi} = 0, \dot{\phi} = 0$, $\therefore$ const. of integration is 0)

or $Ma\dot{\psi} = -m(a\dot{\psi} - c\dot{\theta})$ From (1)

or $a(M + m)\dot{\psi} = mc\dot{\theta}$ ...(5)

$\therefore$ The coordinates of C referred to the horizontal and vertical lines through O as axes are $(c\sin\theta, c\cos\theta)$

$\therefore v_c^2 = \dot{x}^2 + \dot{y}^2 = c^2\dot{\theta}^2$

$\therefore$ Energy equation gives

$\frac{1}{2}Ma^2\dot{\psi}^2 + (\frac{1}{2}mb^2\dot{\phi}^2 + \frac{1}{2}mc^2\dot{\theta}^2) = mg(c\cos\alpha - c\cos\theta)$

Substituting the vlaues of $b\dot{\phi}$ and $c\dot{\theta}$ from (4) and (5), we get

$\frac{1}{2}Ma^2\dot{\psi}^2 + \frac{1}{2}m\left(-\dfrac{Ma\dot{\psi}}{m}\right)^2 + \frac{1}{2}m\left(\dfrac{a(m + M)\dot{\psi}}{m}\right)^2 = -mgc(\cos\alpha - \cos\theta)$

or $a^2\left[M + \dfrac{M^2}{m} + \dfrac{(M + m)^2}{m}\right]\dot{\psi}^2 = -2mgc(\cos\alpha - \cos\theta)$

or $a^2[(m + M)M + (M + m)^2]\dot{\psi}^2 = 2m^2gc(\cos\theta - \cos\alpha)$

or $a^2(M + m)(2M + m)\omega^2 = 2gm^2(a - b)(\cos\theta - \cos\alpha)$

$\because c = a - b$ and $\dot{\psi} = \omega$.

which is the requried result.

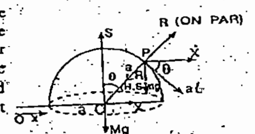
4.14. Motion of one body on another, when both bodies are free to turn.

EXAMPLES

Ex. 41. *A hemisphere of mass M is free to slide with its base on a smooth horizontal table. A particle of mass m is placed on the hemisphere at an angular distance α from the vertex, show that the radius to the point of contact at which the particle leaves the surface, makes with the vertical an angle θ given by the equation*

$$m\cos^3\theta - (M + m)(3\cos\theta - 2\cos\alpha) = 0$$

Sol. In time t let the hemisphere move through a distance x on the horizontal plane. At time t let the particle be at the point P at an angular distance θ which was initially at an angular distance α from the vertex. The velocities of the particle m at P are $\dot{x}$ and $a\dot{\theta}$ along horizontal and the tangent at P and angle between them is θ.



The coordinates (x_P, y_P) of P rerferred to the horizontal and vertical lines through O as axes are given by

$$x_P = x + a + a\sin\theta \text{ and } y_P = a\cos\theta$$

Since there is no horizontal force on the system.

$\therefore \dfrac{d}{dt}(M\dot{x} + m\dot{x}_P) = 0$

or $\dfrac{d}{dt}[M\dot{x} + m(\dot{x} + a\cos\theta\,\dot{\theta})] = 0$

Integrating, $M\dot{x} + m(\dot{x} + a\cos\theta\,\dot{\theta}) = C$.

But initially when $\dot{x} = 0, \dot{\theta} = 0$, $\therefore C = 0$

$\therefore M\dot{x} + m(\dot{x} + a\cos\theta\,\dot{\theta}) = 0$

$\therefore \dot{x} = -\dfrac{ma\cos\theta\,\dot{\theta}}{(M + m)}$ ...(1)

Now K.E. of the hemisphere $= \frac{1}{2}M\dot{x}^2$

and K.E. of the particle $= \frac{1}{2}m(\dot{x}_P^2 + \dot{y}_P^2)$

$$= \frac{1}{2}m(\dot{x}^2 + a^2\dot{\theta}^2 + 2a\dot{\theta}\dot{x}\cos\theta)$$

As there are no forces to turn the hemisphere, so there is no rotational energy. Hence the energy equation gives

$\frac{1}{2}M\dot{x}^2 + \frac{1}{2}m(\dot{x}^2 + a^2\dot{\theta}^2 + 2a\dot{\theta}\dot{x}\cos\theta) = mg(a\cos\alpha - a\cos\theta)$.

or $(M + m)\dot{x}^2 + ma^2\dot{\theta}^2 + 2am\dot{\theta}\dot{x}\cos\theta = 2mga(\cos\alpha - \cos\theta)$

or $(M + m)\dfrac{m^2a^2\cos^2\theta\dot{\theta}^2}{(M + m)^2} + ma^2\dot{\theta}^2 + 2am\dot{\theta}\left(\dfrac{-ma\cos\theta\dot{\theta}}{M + m}\right)\cos\theta$

$= 2mga(\cos\alpha - \cos\theta)$

[Substituting the value of $\dot{x}$ from (1)]

or $\left(1 - \dfrac{m}{M + m}\cos^2\theta\right)a^2\dot{\theta}^2 = 2ga(\cos\alpha - \cos\theta)$

or $[M + m(1 - \cos^2\theta)]a\dot{\theta}^2 = 2g(M + m)(\cos\alpha - \cos\theta)$

or $(M + m\sin^2\theta)a\dot{\theta}^2 = 2g(M + m)(\cos\alpha - \cos\theta)$ ...(2)

or $d\dot{\theta}^2 = \frac{v^2}{d} - \frac{10}{7} g (1 - \cos\theta)$ ...(4)

$\therefore$ from (2), $R = Mg \cos\theta + \frac{Mv^2}{d} - \frac{10}{7} Mg (1 - \cos\theta)$

The ball will leave the globe when $R = 0$.

i.e. when $Mg \cos\theta + \frac{Mv^2}{d} - \frac{10}{7} Mg (1 - \cos\theta) = 0$

or $\frac{17}{7} g \cos\theta = \frac{1}{7}\left(10g - \frac{7v^2}{d}\right)$ or $\cos\theta = -\frac{7v^2 - 10gd}{17\, gd}$ ...(5)

Since the ball may leave the globe when its centre rises above a horizontal line through O.

$\therefore$ θ is obtuse and hence $\cos\theta$ is negative.

From (5), we see that $\cos\theta$ is negative, if

$$7v^2 > 10gd \text{ i.e. if } v > \sqrt{\left(\tfrac{10}{7} gd\right)}.$$

Also numerically $\cos\theta$ must be less than 1

i.e. $\frac{7v^2 - 10gd}{17gd} < 1$ or $7v^2 - 10gd < 17gd$

or $7v^2 < 27gd$ or $v < \sqrt{\left(\frac{27}{7} gd\right)}$.

Hence, v lies between $\sqrt{\left(\frac{10}{7} gd\right)}$ and $\sqrt{\left(\frac{27}{7} gd\right)}$.

4.13. Motion of the Body of Another, when the Lower Body is free to Turn about its Axis :

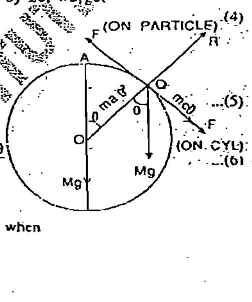
EXAMPLES

Ex. 37. *A rough cylinder of mass M is capable of motion about its axis which is horizontal; a particle of mass m is placed on it vertically above the axis and the system is slightly disturbed. Show that the particle will slip on the cylinder when it has moved through an angle θ given by* $\mu (M + 6m) \cos\theta - M \sin\theta = 4m\mu$, *where μ is the coefficient of friction.*

Sol. Let O be the centre and M the mass of the rough cylinder which is capable of motion about its axis which is horizontal. A particle of mass m is placed on the vertically above the axis. When the system is slighly displaced, let at time t the cylinder turn through an angle θ. Thus at time t the particle is at P and $\angle AOP = \theta$.

Since the particle m describes a circle of radius $OP = a$ about O,

$\therefore$ its accelerations along and perpendicular to PO are $ma\dot{\theta}^2$ & $ma\ddot{\theta}$.



$\therefore$ The equations of motion of the particle are

$$ma\dot{\theta}^2 = mg \cos\theta - R, \quad ...(1)$$

and $$ma\ddot{\theta} = mg \sin\theta - F. \quad ...(2)$$

The coordinates of P referred to the horizontal and vertical lines through O as axes are $(a \sin\theta, a \cos\theta)$.

$\therefore$ Energy equation, gives

$\frac{1}{2} Mk^2 \dot{\theta}^2 + \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) =$ work done by gravity (K.E. of cyl.) (K.E. of Particale)

$\frac{1}{2} M \cdot \frac{1}{2} a^2\dot{\theta}^2 + \frac{1}{2} m \cdot a^2\dot{\theta}^2 = mg (a - a \cos\theta)$

or $(M + 2m) a\dot{\theta}^2 = 4mg (1 - \cos\theta)$ ...(3)

Differentiating w.r.t. t and then dividing by $2\dot{\theta}$, we get

$(M + 2m) a\ddot{\theta} = amg \sin\theta$. ...(4)

$\therefore$ From (1) and (3), we get

$R = Mg \cos\theta - m \frac{4mg}{(M + 2m)} (1 - \cos\theta)$

$= \frac{mg}{M + 2m} [(M + 6m) \cos\theta - 4m]$ ...(5)

From (2) and (4), we get

$F = mg \sin\theta - m \cdot \frac{2mg \sin\theta}{(M + 2m)} = \frac{mMg \sin\theta}{M + 2m}$. ...(6)

$\therefore$ $\frac{F}{R} = \frac{M \sin\theta}{(M + 6m) \cos\theta - 4m}$

The particle slips off from the cylinder, when

$F = \mu R$ *i.e.* when $\mu = F/R$

i.e. when $\mu = \frac{M \sin\theta}{(M + 6m) \cos\theta - 4m}$

or $\mu (M + 6m) \cos\theta - 4m\mu = M \sin\theta$

or $\mu (M + 6m) \cos\theta - M \sin\theta = 4m\mu$.

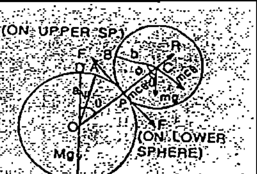
Ex. 38. *The mass of a sphere is* $\frac{1}{5}$ *of that of another sphere of the same material which is free to move about its centre as a fixed point, the first sphere rolls down the second from rest at the highest point, the coeffcient of friction being μ. Prove that sliding will begin when the angle θ which the line of centres makes with the vertical is given by*

$$\sin\theta = 2\mu (5 \cos\theta - 3).$$

Sol. Let M be the mass and a the radius of the sphere which can move about its centre O as a fixed point. Let C be the centre, b the radius and m the mass of the sphere which rolls down the first sphere starting from rest from its highest point. $\therefore$ $M = 5m$.

In time t, let the fixed sphere turn through an angle ψ,

i.e. $\angle DOA = \psi$.



During this time t, let the upper sphere roll to the point P such that the line CB fixed in this sphere make an angle ψ to the vertical.

Initially B coincide with A which was the highest point of the first sphere *i.e.* initially CB and OA were vertical.

Also let the line OC joining centres make an angle θ to the vertical at time t. Let F be the friction sufficient for pure rolling.

Since there is no slipping between the two spheres.

$\therefore$ Arc AP = Arc BP or $a(\theta - \psi) = b(\phi - \theta)$

$\therefore$ $a\psi + b\phi = (a + b)\theta$

and $a\dot{\phi} + b\dot{\phi} = c\dot{\theta}$, where $c = a + b$. ...(1)

Equations of motion for the lower sphere taking moment about O is

$M \cdot \frac{2}{5} a^2\ddot{\psi} = Fa$. ...(2)

Since the centre C describe a circle of radius $OC = a + b = c$, about O. Therefore its accelerations along and perpendicular to CO are $mc\dot{\theta}^2$ and $mc\ddot{\theta}$ respectively.

$\therefore$ Equations of motion of the upper sphere are

$mc\dot{\theta}^2 = mg \cos\theta - R$, ...(3)

and $mc\ddot{\theta} = mg \sin\theta - F$. ...(4)

Also taking moment about O, we get

$m \cdot \frac{2}{5} b^2\ddot{\phi} = Fb$. ...(5)

From (2) and (5), we get $Ma\ddot{\psi} = mb\ddot{\phi}$

Integrating, $Ma\dot{\psi} = mb\dot{\phi}$. (initially $\dot{\phi} = 0$, $\dot{\psi} = 0$ $\therefore$ Constant of integration is also zero).

$\therefore$ $\frac{a\dot{\psi}}{m} = \frac{b\dot{\phi}}{M} = \frac{a\dot{\psi} + b\dot{\phi}}{m + M} = \frac{c\dot{\theta}}{m + M}$ [from (1)] ...(6)

$\therefore$ $a\dot{\psi} = \frac{mc\dot{\theta}}{m + M}$ and $b\dot{\phi} = \frac{Mc\dot{\theta}}{m + M}$.

The coordinates of C referred to horizontal and vertical lines through O as axes are $(c \sin\theta, \cos\theta)$ $\therefore$ $v_c^2 = \dot{x}^2 + \dot{y}^2 = c^2\dot{\theta}^2$

$\therefore$ Energy equation, gives

$\frac{1}{2} M\frac{2}{5} a^2\dot{\psi}^2 + \left(\frac{1}{2} m \cdot \frac{2}{5} b^2\dot{\phi}^2 + \frac{1}{2} mc^2\dot{\theta}^2\right) = mg (c - c \cos\theta)$

or $\frac{2}{5} M\left(\frac{mc\dot{\theta}}{m + M}\right)^2 + \frac{2}{5} m\left(\frac{Mc\dot{\theta}}{m + M}\right)^2 + mc^2\dot{\theta}^2 = 2mgc (1 - \cos\theta)$

or $\left[\frac{2}{5} \frac{Mm (m + M)}{(m + M)^2} + m\right] c^2\dot{\theta}^2 = 2mgc (1 - \cos\theta)$

or $\left(\frac{2}{5} \frac{M}{m + M} + 1\right) c\dot{\theta}^2 = 2g (1 - \cos\theta)$

or $\left(\frac{2}{5} \frac{5m}{m + 5m} + 1\right) c\dot{\theta}^2 = 2g (1 - \cos\theta)$

$c\dot{\theta}^2 = \frac{3}{2} g (1 - \cos\theta)$ ...(7)

Differentiating (7) w.r.t. t and dividing by $2\dot{\theta}$, we get

$c\ddot{\theta} = \frac{3}{4} g \sin\theta$. ...(8)

$\therefore$ From (3) and (4), using (7) and (8), we get

$R = mg \cos\theta - m \cdot \frac{3}{2} g (1 - \cos\theta) = \frac{1}{2} mg (5 \cos\theta - 3)$

and $F = mg \sin\theta - m \cdot \frac{3}{4} g \sin\theta = \frac{1}{4} mg \sin\theta$.

$\therefore$ $\frac{F}{R} = \frac{\sin\theta}{2 (5 \cos\theta - 3)}$

$\therefore$ Sliding of the upper sphere begins, when $F = \mu R$.

i.e. when $\mu = \frac{F}{R}$ *i.e.* when $\mu = \frac{\sin\theta}{2 (5 \cos\theta - 3)}$

or when $\sin\theta = 2\mu (5 \cos\theta - 3)$.

Ex. 39. *A uniform circular cylinder of mass M is free to rotate about its axis which is smooth ond horizontal and about which its radius of gyration is equal to its radius. A uniform solid sphere of mass m is placed with its lowest point in contact with the highest generator of the cylinder, both sphere and cylinder being initially at rest. The sphere is then slightly disturbed and rolls down the cylinder. Show that the slipping takes place before the sphere leaves the cylinder, and begins when*

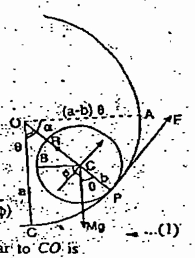
$$2M \sin\theta = \mu [(17M + 6m) \cos\theta - (10M + 4m)]$$

where θ is the inclination to the vertical of the plane through their axes and μ is the coefficient of friction.

Sol. (Refer to fig. of Ex. 38)

Let O be the centre, M the mass and a, the radius of the cylinder which is free to move about its axis which is fixed horizontally. Let C be the centre, m the mass and b the radius of the rolling sphere which is initially placed at rest at the highest point of the cylinder. In time t, let the sphere roll down to the point P of the cylinder such that the line CB fixed in this sphere make an angle ψ to the vertical. Initially B coincided with A which was the highest point of the cylinder, *i.e.* initially the inner circumference of the fixed circle. Let the point B of the plate be at the point A of the fixed circle initially such that $\angle AOC = \alpha$.

At time t, let the plate roll down to the point P s.t. $\angle COP = \theta$ and ϕ the angle that the line CB fixed in space make with the vertical.



Since there is no slipping between the two bodies

$\therefore$ Arc AP – Arc PB (upper side of the plate in the figure)

or $a(\alpha-\theta) = b[2\pi - (\theta+\phi)]$ or $-a\theta = -b(\theta+\phi)$

i.e. $b\phi = (a-b)\theta$. $\therefore$ $b\dot{\phi} = (a-b)\dot{\theta}$ ...(1)

Equation of motion of the plate perpendicular to CO is

$$m(a-b)\ddot{\theta} = F - Mg\sin\theta$$

Also for the motion relative to C,

$mk^2\ddot{\phi} = -Fb$

or $m\frac{1}{2}b^2\ddot{\phi} = -Fb$

or $\frac{1}{2}m.(a-b)\ddot{\theta} = -F$ [substituting from (1)]

or $m(a-b)\ddot{\theta} = -2F$

Substituting in (2), we get

$-2F = F - mg\sin\theta$ or $3F = mg\sin\theta$

$\therefore$ $F = \frac{1}{3}mg\sin\theta = (\frac{1}{3}\sin\theta)$ times the weight of the plate.

Ex. 34. *A solid homogeneous sphere is rolling on the inside of a fixed hollow sphere, the two centres being always in the same vertical plane. Show that the smaller sphere will make complete revolution if, when it is in its lowest position, the pressure on it is greater than $\frac{34}{7}$ times its own weight.* (IFoS-2008)

Sol. Refer fig. of § 4.12 on page 229.

Let O be the centre and a the radius of fixed hollow sphere. Let C be the centre, M the mass and b the radius of the sphere rolling inside this fixed sphere. A time t let the line CB fixed in moving sphere make an angle ϕ to the vertical and then let the line OC joining centres make an angle θ to the vertical where initially B coincided with A.

Since there is no slipping $\therefore$ Arc AP = Arc PB.

or $a\theta = b(\phi+\theta)$ $\therefore$ $b\phi = (a-b)\theta = c\theta$ ...(1)

where $c = a - b$.

Since C describe circle of radius $OC = a - b = c$ (say), about C.

$\therefore$ the equations of motion are

$Mc\dot{\theta}^2 = R - Mg\cos\theta$ ...(2)

and $Mc\ddot{\theta} = F - Mg\sin\theta$ ...(3)

The coordinates (x_c, y_c) of C referred to the horizontal and vertical lines through O as axes are given by

$x_c = c\sin\theta$ and $y_c = c\cos\theta$ $\therefore$ $v_c^2 = \dot{x}_c^2 + \dot{y}_c^2 = c^2\dot{\theta}^2$

At time t, K.E. of the moving sphere

$= \frac{1}{2}Mk^2\dot{\phi}^2 + \frac{1}{2}Mv_c^2 = \frac{1}{2}M.\frac{2}{5}b^2\dot{\phi}^2 + \frac{1}{2}Mc^2\dot{\theta}^2$

$= \frac{1}{5}Mc^2\dot{\theta}^2 + \frac{1}{2}Mc^2\dot{\theta}^2 = \frac{7}{10}Mc^2\dot{\theta}^2$ [from (1)]

If ω is the initial angular velocity *i.e.* $\dot{\theta} = \omega$, then the initial K.E. at $t = 0$ is $\frac{7}{10}Mc^2\omega^2$.

$\therefore$ The energy equation gives

Change in K.E. = work done by the gravity.

$\frac{7}{10}Mc^2\dot{\theta}^2 - \frac{7}{10}Mc^2\omega^2 = -Mg(c - c\cos\theta)$

or $c\dot{\theta}^2 = c\omega^2 - \frac{10}{7}g(1-\cos\theta)$ ...(4)

$\therefore$ From (2), $R = Mg\cos\theta + M[c\omega^2 - \frac{10}{7}g(1-\cos\theta)]$ ...(5)

The sphere will make complete revolution if $R = 0$ when $\theta = \pi$.

$\therefore$ from (5) $0 = Mg\cos\pi + M\{c\omega^2 - \frac{10}{7}g(1-\cos\pi)\}$

or $c\omega^2 = g\frac{20}{7}g$ or $\omega^2 = \frac{27g}{7c}$.

$\therefore$ $\omega = \sqrt{\left(\frac{27g}{7c}\right)}$ is the least velocity of ω to make the complete revolution.

Now at the lowest position when $\theta = 0$, $\dot{\theta} = \omega = \sqrt{\left(\frac{27g}{7c}\right)}$, then from (5),

$R = Mg + M\left[c.\frac{27g}{7c} - \frac{10}{7}g(1-1)\right] = \frac{34}{7}Mg$

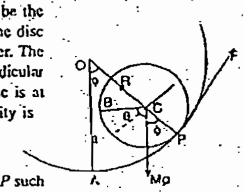
$\therefore$ When the sphere makes complete revolution, then reaction at the lowest position is greater than 34/7 times its own weight.

Ex. 35. *A disc rolls on the inside of a fixed hollow circular cylinder whose axis is horizontal, the plane of the disc being vertical and perpendicular to the axis of the cylinder, if when in the lowest position its centre is moving with a velocity $\sqrt{\left[\frac{8g}{3(a-b)}\right]}$, show that the centre of the disc will describe an angle ϕ about the centre of the cylinder in time*

$$\sqrt{\left[\frac{3(a-b)}{2g}\right]}\log\tan\left(\frac{\pi}{4}+\frac{\phi}{4}\right)$$

Sol. Let O be the centre and a the radius of the fixed hollows circular cylinder whose axis is horizontal. Let C be the centre, M the mass and b the radius of the disc which rolls inside the fixed hollow cylinder. The plane of the disc being vertical and perpendicular to the axis of the cylinder. When the disc is at the lowest point A then its angular velocity is

$$\sqrt{\left[\frac{8g}{3(a-b)}\right]}$$



In time t let the disc roll to the point P such that $\angle AOP = \phi$, and let the line CB fixed in the disc make an angle θ to the vertical at time

Since there is no slipping, $\therefore$ Arc AP = Arc PB

or $a\phi = b(\theta+\phi)$ or $b\theta = (a-b)\phi$ $\therefore$ $b\dot{\theta} = (a-b)\dot{\phi}$ ...(1)

where $a - b = c$ (say).

The coordinates (x_c, y_c) of the centre C refered to the horizontal and vertical lines through O as axes, are given by

$x_c = OC\sin\phi = c\sin\phi$ and $y_c = OC\cos\phi = c\cos\phi$

$\therefore$ $v_c^2 = \dot{x}_c^2 + \dot{y}_c^2 = c^2\dot{\phi}^2$. K.E. of the disc at time t.

$= \frac{1}{2}Mk^2\dot{\theta}^2 + \frac{1}{2}Mv_c^2 = \frac{1}{2}M.\frac{1}{2}b^2\dot{\theta}^2 + \frac{1}{2}Mc^2\dot{\phi}^2$

$= \frac{1}{4}Mc^2\dot{\phi}^2 + \frac{1}{2}Mc^2\dot{\phi}^2 + \frac{3}{4}Mc^2\dot{\phi}^2$ using (1)

Since initially when $t = 0$, $\dot{\phi} = \sqrt{\left[\frac{8g}{3(a-b)}\right]}$

$\therefore$ K.E. of the disc at time $t = 0$ is $\frac{3}{4}Mc^2.\frac{8g}{3(a-b)} = 2Mgc$,

$\because$ $a - b = c$,

$\therefore$ The energy equation gives

Change in K.E. = Work done by gravity.

$\frac{3}{4}Mc^2\dot{\phi}^2 = g(1+\cos\phi) = g.2\cos^2\frac{1}{2}\phi$

$\therefore$ $\dot{\phi} = \frac{d\phi}{dt} = \sqrt{\left(\frac{8g}{3c}\right)}\cos\frac{1}{2}\phi$ or $dt = \sqrt{\left(\frac{3c}{8g}\right)}\sec\frac{1}{2}\phi\,d\phi$.

Integrating, the angle described in time t is given by

$$t = \sqrt{\left(\frac{3c}{8g}\right)}\int_0^{\phi}\sec\frac{1}{2}\phi\,d\phi = \sqrt{\left(\frac{3c}{8g}\right)}\left[2\log\tan\left(\frac{\pi}{4}+\frac{\phi}{4}\right)\right]_0^{\phi}.$$

or $t = \sqrt{\left[\frac{3(a-b)}{2g}\right]}\log\tan\left(\frac{\pi}{4}+\frac{\phi}{4}\right)$

Ex. 36. *A solid spherical ball rests in equilibrium at the bottom of a fixed spherical globe whose inner surface is perfectly rough. The ball is struck a horizontal blow of such a magnitude that the initial speed of its centre is v, prove that if v, lies between*

$$\sqrt{\left(\frac{10}{7}gd\right)} \text{ and } \sqrt{\left(\frac{27}{7}gd\right)}$$

the ball would leave the globe, d being the difference between the radii of the ball and the globe.

Sol. (Ref. fig. of § 4.12 on page 229).

Let O be the centre and a the radius of the fixed spherical globe. Let a solid ball of radius b, centre C and mass M rest at the lowest point A of the globe. At time t, let the ball roll to the point C s.t. $\angle AOP = \theta$. At this time t let the line CB fixed in the ball make an angle ϕ to the vertical. The point B coincided with A at time $t = 0$. Since there is no slipping.

$\therefore$ Arc AP = Arc BP or $a\theta = b(\phi+\theta)$ $\therefore$ $b\phi = (a-b)\theta = d\theta$ ...(1)

Let R be the normal reaction and F the friction at the point P. The equations of motion of the ball along and perpendicular to CO are

$Md\dot{\theta}^2 = R - Mg\cos\theta$ ...(2)

and $Md\ddot{\theta} = F - Mg\sin\theta$. ...(3)

The coordinates (x_c, y_c) of the centre C referred to the horizontal and vertical lines through O as axes are given by

$x_c = d\sin\theta$ and $y_c = d\cos\theta$. $\therefore$ $v_c^2 = \dot{x}_c^2 + \dot{y}_c^2 = d^2\dot{\theta}^2$

$\therefore$ K.E. of the ball at time t

$= \frac{1}{2}Mk^2\dot{\phi}^2 + \frac{1}{2}Mv_c^2 = \frac{1}{2}M.\frac{2}{5}b^2\dot{\phi}^2 + \frac{1}{2}Md^2\dot{\theta}^2$

$= \frac{1}{2}Md^2\dot{\theta}^2 + \frac{1}{2}Md^2\dot{\theta}^2 = \frac{7}{10}Md^2\dot{\theta}^2$

Initially at time $t = 0$, velocity of the centre C of the ball is v

$\therefore$ $v = d\dot{\theta}$.

$\therefore$ K.E. of the ball at time $t = 0$ is $\frac{7}{10}Mv^2$

$\therefore$ Energy equation gives

Change in K.E. = Work done by gravity.

i.e. $\frac{7}{10}Md^2\dot{\theta}^2 - \frac{7}{10}Mv^2 = -Mg(d - d\cos\theta)$

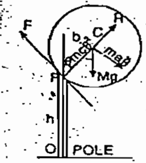
or $(1+4\mu^2)\cos\theta + 6\mu\sin\theta + 2(1-2\mu^2)\cos\theta = 2(1-2\mu^2)e^{2\mu\theta}$

or $3\cos\theta + 6\mu\sin\theta = 2(1-2\mu^2)e^{2\mu\theta}$

or $\cos\theta + 2\mu\sin\theta = Ae^{2\mu\theta}$, where $A = \frac{2}{3}(1-2\mu^2)$.

Ex. 31. *A uniform sphere of radius a is gently placed on the top of a thin vertical pole of height h ($> a$) and then allowed to fall over. Show that however rough the pole may be the sphere will slip on the pole before it finally falls off it.*

Sol. Let a sphere of mass M, radius a and centre C be placed on the top P of a thin vertical pole OP of height $h > a$ and allowed to fall over. Assuming that the friction is sufficient to keep the point of contact P at rest, let θ be the angle turned by the sphere is time t.



Since the centre C describe the circle of radius a with the centre at P, therefore its acceleration along and perpendicular to CP and $a\dot\theta^2$ and $a\ddot\theta$ respectively.

$\therefore$ The equations of motion of the sphere are

$Ma\dot\theta^2 = Mg\cos\theta - R$ ...(1)

and $Ma\ddot\theta = Mg\sin\theta - F$ ...(2)

Coordinates (x_c, y_c) of the centre C are given by

$x_c = a\sin\theta, y_c = a\cos\theta \therefore v_c^2 = \dot x_c^2 + \dot y_c^2 = a^2\dot\theta^2$

$\therefore$ The energy equation gives

$\frac{1}{2}Mk^2\dot\theta^2 + \frac{1}{2}Mv_c^2 = Mg(a - a\cos\theta)$

or $\frac{1}{2}M\cdot\frac{2}{5}a^2\dot\theta^2 + \frac{1}{2}Ma^2\dot\theta^2 = Mga(1-\cos\theta)$

$\therefore a\dot\theta^2 = \frac{10}{7}g(1-\cos\theta)$ ...(3)

Differentiating w.r.t. 't' and dividing by $2\dot\theta$, we get

$a\ddot\theta = \frac{5}{7}g\sin\theta$ ...(4)

From (1) and (3), we get

$R = Mg\cos\theta - M\cdot\frac{10}{7}g(1-\cos\theta) = \frac{1}{7}Mg(17\cos\theta - 10)$

And from (2) and (4), we get

$F = Mg\sin\theta - M\frac{5}{7}g\sin\theta = \frac{2}{7}Mg\sin\theta$.

The sphere will fall off, when $R = 0$

i.e. when $\frac{1}{7}Mg(17\cos\theta - 10) = 0$ *i.e.* when $\cos\theta = \frac{10}{17}$...

Also the sphere will slip, when $F \geq \mu R$ or when $\mu \leq F/R$.

or $\mu \leq 2\sin\theta/(17\cos\theta - 10)$ ...(5)

From (5) we observe that if μ is not negative then $\mu = 0$, when $\theta = 0$ *i.e.* when the motion begins

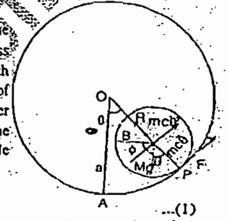
And $\mu = \infty$, when $\cos\theta = 10/17$, when the sphere falls off.

Thus the sphere will slip between $\theta = 0$ and $\theta = \cos^{-1}\left(\frac{10}{17}\right)$, if μ lies between O and ∞.

Hence, however rough the pole may be, the spehere will slip on the pole before it finally falls over.

§ 4.12. *A hollow cylinder, of radius a is fixed with its axis horizontal, inside it moves a solid cylinder, of radius b, whose velocity in its lowest position is gives, if the friction between the cylinders be sufficient to prevent any slidien, find the motion.*

Let O be the centre and a the radius of the fixed cylinder. Let C be the centre, M the mass and b the radius of the solid cylinder resting with its point B in contact with the lowest point A of the fixed cylinder. In time t let the inside cylinder roll to the point P such that $\angle AOP = \theta$ and the line CB fixed in moving cylinder make an angle ϕ to the vertical. Since there is pure rolling.



$\therefore$ Arc AP = Arc BP or $a\theta = b(\phi + \theta)$

$\therefore b\phi = (a-b)\theta$ ...(1)

Let R be the normal reaction and F the friction at the point P. Since the centre C describe a circle of radius $OC = a - b = c$ (say) about O.

$\therefore$ Its accelerations along and perpendicular to CO are $Mc\dot\theta^2$ and $Mc\ddot\theta$ respectively.

$\therefore$ The equations of motion of the moving cylinder are

$Mc\dot\theta^2 = R - Mg\cos\theta$ ...(2)

and $Mc\ddot\theta = F - Mg\sin\theta$ ...(3)

The coordinates (x_c, y_c), referred to the horizontal and vertical lines through O as axes are given by

$x_c = OC\sin\theta = c\sin\theta$ and $y_c = OC\cos\theta = c\cos\theta$

$\therefore v_c^2 = \dot x_c^2 + \dot y_c^2 = c^2\dot\theta^2$.

$\therefore$ K.E. of the moving cylinder at time t

$= \frac{1}{2}Mk^2\dot\phi^2 + \frac{1}{2}Mv_c^2 = \frac{1}{2}M\cdot\frac{1}{2}b^2\dot\phi^2 + \frac{1}{2}Mc^2\dot\theta^2$

$= \frac{1}{4}Mc^2\dot\theta^2 + \frac{1}{2}Mc^2\dot\theta^2 = \frac{3}{4}Mc^2\dot\theta^2$, [from (1)]

If $\dot\theta = \omega$, is the angular velocity initially at $t = 0$, then initial K.E. of the moving cylinder $= \frac{3}{4}Mc^2\omega^2$

$\therefore$ Energy equation gives

Change in K.E = Work done

or $\frac{3}{4}Mc^2\dot\theta^2 - \frac{3}{4}Mc^2\omega^2 = -Mg(c - c\cos\theta)$

or $c\dot\theta^2 = c\omega^2 - \frac{4}{3}g(1-\cos\theta)$ ...(4)

Differentiating w.r.t. 't' and dividing by $2\dot\theta$, we get

$c\ddot\theta = -\frac{2}{3}g\sin\theta$ ...(5)

From (2) and (4), we get

$R = Mg\cos\theta + M[c\omega^2 - \frac{4}{3}g(1-\cos\theta)] = Mc\omega^2 + \frac{1}{3}Mg(7\cos\theta - 4)$ ...(6)

And from (3) and (5), we get

$F = Mg\sin\theta + M\cdot(-\frac{2}{3}g\sin\theta) = \frac{1}{3}Mg\sin\theta$ ...(7)

Case I. In order that the cylinder may just make complete revolution, R should be zero at the highest point

i.e. $R = 0$ when $\theta = \pi$; $\therefore$ from (6), we have

$O = Mc\omega^2 + \frac{1}{3}Mg(7\cos\pi - 4)$ or $\omega^2 = \frac{11g}{3c}$

or $\omega = \sqrt{\left(\frac{11g}{3(a-b)}\right)}$ ($\because c = a - b$)

Case II. The moving cylinder will leave the fixed cylinder.

when $R = 0$ *i.e.* from (6), $0 = Mc\omega^2 + \frac{1}{3}Mg(7\cos\theta - 4)$

or $\cos\theta = \frac{1}{7g}[4g - 3(a-b)\omega^2]$, ($\because c = a - b$) ...(8)

i.e. the moving cylinder will leave the cylinder at an angle θ to the vertical given by (8).

Case III. Small oscillations. If the moving cylinder makes small oscillations about the lowest point of the fixed cylinder, then θ is always small. $\therefore$ From (5), we get

$\ddot\theta = -\frac{2g}{3(a-b)}\theta$, $c = a - b$.

$\therefore$ The time of small oscillation is

$$2\frac{\pi}{\sqrt{\left(\frac{2g}{3(a-b)}\right)}} = 2\pi\sqrt{\left(\frac{3(a-b)}{2g}\right)}$$

EXAMPLES

Ex. 32. *A circular cylinder of radius a and radius of gyration k rolls without slipping inside a fixed hollow cylinder of radius b. Show that the plane through their axes moves like a circular pendulum of length* $(b-a)(1 + k^2/a^2)$.

Sol. (Ref. fig. of § 4.12 on page 229).

Let P be the point of contact of the two cylinders at time t s.t. $\angle AOP = \theta$. Let ϕ the angle which the line CB fixed in moving cylinder make with the vertical at time t. Here radius of fixed cylinder is a and that of moving cylinder is a. Since there is pure rolling therefore

Arc AP = Arc BP.

or $b\theta = a(\phi + \theta)$ *i.e.* $a\phi = (b-a)\theta \therefore \dot\phi = c\dot\theta$ ...(1)

where $c = b - a$.

Let R be the normal reaction and F the friction at the point P.

$\therefore$ The centre C describes a circle of radius $OC = b - a = c$, therefore its accelerations along and perpendicular to CO are $c\dot\theta^2$ and $c\ddot\theta$ respectively.

$\therefore$ The equations of motion of the moving cylinder are

$Mc\dot\theta^2 = R - Mg\cos\theta$ ...(2)

and $Mc\ddot\theta = F - Mg\sin\theta$ ...(3)

Also for the motion relative to the centre of inertia C,

$Mk^2\ddot\phi$ = Moment of the forces alout $C = -Fa$ ...(4)

$Mk^2\cdot\frac{c}{a}\ddot\theta = -Fa$ *i.e.* $F = -Mk^2\frac{c}{a^2}\ddot\theta$.

Substituting in (3), we get

$Mc\ddot\theta = -Mk^2\frac{c}{a^2}\ddot\theta - Mg\sin\theta$

or $c(1 + k^2/a^2)\ddot\theta = -g\sin\theta$ or $\ddot\theta = -\frac{g}{c(1+k^2/a^2)}\theta = -\mu\theta$ (say)

$\because \theta$ is very small.

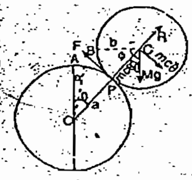
$\therefore$ Length of the simple equivalent pendulum is

$g/\mu = c(1 + k^2/a^2) = (b-a)(1 + k^2/a^2)$.

Ex. 33. *A circular plate rolls down the inner circumference of a rough circle under the action of gravity, the planes of both the plate and the circle being vertical. When the line joining their centres is inclined at an angle θ to the vertical, show that the friction between the bodies is $\frac{1}{3}\sin\theta$ times the weight of the plate.*

Sol. Let O be the centre of the fixed circle of radius a. Let C be the centre, m the mass and b the radius of the circular plate which rolls down

Sol. Let O be the centre and a the radius of the fixed sphere. Let C be the centre and b the radius of the sphere resting on the fixed sphere with its point B in contact to the point A of the fixed sphere such that OA make an angle α to the vertical. The upper sphere rolls and at time t, let P be the point of contact of the two spheres such that the common normal OC make an angle θ to the vertical. Let CB make an angle ϕ to the vertical at time t.



Since there is pure rolling.

$\therefore$ Arc AP = Arc BP

or $a(\theta-\alpha)=b(\phi-\theta)$

i.e. $b\phi=(a+b)\theta-a\alpha=c\theta-a\alpha$

$\therefore$ $b\phi=c\theta$, where $a+b=c$ (say). ...(1)

Let R be the normal reaction and F the friction acting on the upper sphere. Since the centre C of the upper sphere describe a circle of radius $CO=a+b=c$, so its acceleration along and perpendicular to CO are $c\dot{\theta}^2$ and $c\ddot{\theta}$ respectively.

$\therefore$ The equations of motion of the upper moving sphere along and perpendicular to CO are

$Mc\dot{\theta}^2=Mg\cos\theta-R$, ...(2)

and $Mc\ddot{\theta}=Mg\sin\theta-F$. ...(3)

The coordinates (x_c, y_c) of the centre C referred to the horizontal and vertical lines through O as axes are given by

$x_c=OC\sin\theta=c\sin\theta$ and $y_c=OC\cos\theta=c\cos\theta$.

$\therefore$ $v_c^2=\dot{x}_c^2+\dot{y}_c^2=c^2\dot{\theta}^2$.

$\therefore$ The energy equation, gives

$$\frac{1}{2}Mk^2\dot{\phi}^2+\frac{1}{2}Mc^2\dot{\theta}^2=Mg(c\cos\alpha-c\cos\theta)$$

$$\text{or } \frac{1}{2}Mk^2\left(\frac{c\dot{\theta}}{b}\right)^2+\frac{1}{2}Mc^2\dot{\theta}^2=Mgc(\cos\alpha-\cos\theta) \quad \text{[From (1)]}$$

$$\therefore \dot{\theta}^2=\frac{2b^2g}{c(k^2+b^2)}(\cos\alpha-\cos\theta). \quad ...(4)$$

Differentiating w.r.t. t and dividing by $2\dot{\theta}$, we get

$$\ddot{\theta}=\frac{gb^2\sin\theta}{c(k^2+b^2)} \quad ...(5)$$

Now from (2) and (4), we get

$$R=Mg\cos\theta-Mc\cdot\frac{2b^2g}{c(k^2+b^2)}(\cos\alpha-\cos\theta)$$

$$=\frac{Mg}{(k^2+b^2)}[(k^2+3b^2)\cos\theta-2b^2\cos\alpha]$$

And from (3) and (5), we get

$$F=Mg\sin\theta-M\cdot\frac{cgb^2\sin\theta}{c(k^2+b^2)}=\frac{Mgk^2\sin\theta}{(k^2+b^2)}$$

The sphere will slip, when $F=\mu R$

$$\text{or if } \frac{Mgk^2\sin\theta}{(k^2+b^2)}=\mu\cdot\frac{Mg}{(k^2+b^2)}[(k^2+3b^2)\cos\theta-2b^2\cos\alpha]$$

$$\text{or if } k^2\sin\theta=\mu[(k^2+3b^2)\cos\theta-2b^2\cos\alpha] \quad ...(6)$$

2nd Part. The upper sphere will leave the fixed sphere, if

$$R=0, \text{ i.e. if } \frac{Mg}{(k^2+b^2)}[(k^2+3b^2)\cos\theta-2b^2\cos\alpha]=0$$

$$\text{or if } \cos\theta=\frac{2b^2\cos\alpha}{k^2+3b^2} \text{ i.e. if } \theta=\cos^{-1}\left(\frac{2b^2\cos\alpha}{k^2+2b^2}\right)$$

Ex. 28. *A solid uniform sphere, resting on another fixed sphere is slightly displaced and begins to roll down. Show that it will slip when the common normal makes with the vertical an angle given by*

$$2\sin\theta=\mu(17\cos\theta-10\cos\alpha).$$

Sol. Put $k^2=\frac{2}{5}b^2$ in equation (6) of last Ex. 27.

Ex. 29. *A rough solid circular cylinder rolls down a second rough cylinder, which is fixed with its axis horizontal. If the plane through their axis makes an angle α with the vertical when first cylinder is at rest, show that the cylinders will separate when this angle of inclination is $\cos^{-1}\left(\frac{4}{7}\cos\alpha\right)$.*

Sol. (Refer fig. of Ex. 27 on p. 223).

Let O be the centre and a the radius of the fixed cylinder. Let C be the centre and b the radius of the cylinder resting on the fixed cylinder with its point B in contact to the point A of the fixed cylinder such that OA make an angle α to the vertical. The upper cylinder rolls and at time t, let P be the point of contact of the two cylinders such that the line OC joining centres make an angle θ to the vertical. Let CB make an angle ϕ to the vertical at time t. Since there is pure rolling.

$\therefore$ Arc AP = Arc BP or $a(\theta-\alpha)=b(\phi-\theta)$

i.e. $b\phi=(a+b)\theta-a\alpha$ i.e. $b\phi=c\theta$ ...(1)

where $a+b=c$ (say).

Let R be the normal reaction and F the friction acting on the upper sphere. Therefore the eqution of motion of the cylinder along CO is

$Mc\dot{\theta}^2=Mg\cos\theta-R$. ...(2)

The co-ordinates (x_c, y_c) of the centre C referred to the horizontal and vertical lines through O as axes are given by

$x_c=OC\sin\theta=c\sin\theta$ and $y_c=OC\cos\theta=c\cos\theta$.

$\therefore$ $v_c^2=\dot{x}_c^2+\dot{y}_c^2=c^2\dot{\theta}^2$.

$\therefore$ The energy equation, gives

$$\frac{1}{2}Mk^2\dot{\theta}^2+\frac{1}{2}Mv_c^2=Mg(c\cos\alpha-c\cos\theta)$$

$$\text{or } \frac{1}{2}M\cdot\frac{1}{2}b^2\cdot\dot{\phi}^2+Mc^2\dot{\theta}^2=Mgc(\cos\alpha-\cos\theta)$$

$$\text{or } \frac{1}{4}M(c\dot{\theta})^2+\frac{1}{2}Mc^2\dot{\theta}^2=Mgc(\cos\alpha-\cos\theta) \quad \text{[from (1)]}$$

$$\text{or } \dot{\theta}^2=\frac{4g}{3c}(\cos\alpha-\cos\theta).$$

Substituting in (2), we get

$$R=Mg\cos\theta-Mc\dot{\theta}^2=Mg\cos\theta-Mc\cdot\frac{4g}{3c}(\cos\alpha-\cos\theta)$$

$$\text{or } R=\frac{1}{3}Mg(7\cos-4\cos\alpha).$$

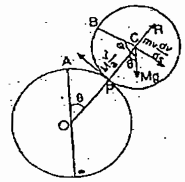
The cylinders will separate when $R=0$,

i.e. when $\frac{1}{3}Mg(7\cos\theta-4\cos\alpha)=0$.

$\therefore$ $\theta=\cos^{-1}\left(\frac{4}{7}\cos\alpha\right)$.

Ex. 30. *A homogeneous sphere rolls down an imperfectly rough fixed sphere, starting from rest at the highest point. If the spheres separate when the line joining their centres makes an angle θ with the vertical, prove that $\cos\theta+2\mu\sin\theta=Ae^{2\mu\theta}$.*

Sol. Let C be the centre, M the mass and a the radius of the sphere which rolls down an imperfectly rough fixed sphere. Since the fixed sphere is imperfectly rough, so the moving sphere will rolls as well as slide on it.



$\therefore$ The frictional force μR will act upwards.

$\therefore$ The equations of motion along and perpendicular to CO (line joining centres) are

$$M\frac{v^2}{a}=Mg\cos\theta-R \quad ...(1)$$

$$\text{and } Mv\frac{dv}{ds}=Mg\sin\theta-\mu R. \quad ...(2)$$

From (1), $R=Mg\cos\theta-Mv^2/a$, substituting in (2), we get

$$Mv\frac{dv}{ds}=Mg\sin\theta-\mu Mg\cos\theta-\mu M\frac{v^2}{a}$$

$$\text{or } \frac{1}{2}\frac{dv^2}{ds}-\frac{\mu}{a}v^2=g(\sin\theta-\cos\theta)$$

$$\text{or } \frac{1}{2}\frac{dv^2}{d\theta}\cdot\frac{d\theta}{ds}-\frac{\mu}{a}v^2=g(\sin\theta-\mu\cos\theta)$$

$$\text{or } \frac{1}{2}\frac{dv^2}{d\theta}\cdot\frac{1}{a}-\frac{\mu}{a}v^2=g(\sin\theta-\mu\cos\theta) \qquad \left(\because s=a\theta \therefore \frac{ds}{d\theta}=a\right)$$

$$\text{or } \frac{dv^2}{d\theta}-2\mu v^2=2ga(\sin\theta-\mu\cos\theta) \quad ...(3)$$

Which is a linear differential equation in v^2.

$\therefore$ I.F. $=e^{\int-2\mu d\theta}=e^{-2\mu\theta}$. $\therefore$ The solution of (3) is

$$v^2\cdot e^{-2\mu\theta}=C+2ag\int e^{-2\mu\theta}(\sin\theta-\mu\cos\theta)\,d\theta$$

$$=C+2ag\left[\int e^{-2\mu\theta}\sin\theta d\theta-\mu\int e^{-2\mu\theta}\cos\theta d\theta\right]$$

$$=C+2ag\cdot\frac{1}{4\mu^2+1}\cdot e^{-2\mu\theta}[(-2\mu\sin\theta-\cos\theta)-\mu(-2\mu\cos\theta+\sin\theta)]$$

$$=C+\frac{2ag}{4\mu^2+1}\cdot e^{-2\mu\theta}[-3\mu\sin\theta-(1-2\mu^2)\cos\theta]$$

But initially when $\theta=0, v=0$ $\therefore$ $C=\frac{2ag}{1+4\mu^2}(1-2\mu^2)$.

$$\therefore v^2e^{-2\mu\theta}=\frac{2ag}{1+4\mu^2}(1-2\mu^2)+\frac{2ag}{1+4\mu^2}e^{-2\mu\theta}[-\mu\sin\theta-(1-2\mu^2)\cos\theta] \quad ...(4)$$

The spheres will separate when $R=0$

$\therefore$ from (1), we have

$Mv^2/a=Mg\cos\theta$

or $v^2=ag\cos\theta$

Substituting in (4), we get

$$ag\cos\theta e^{-2\mu\theta}=\frac{2ag}{1+4\mu^2}(1-2\mu^2)+\frac{2ag}{1+4\mu^2}e^{-2\mu\theta}[-3\mu\sin\theta-(1-2\mu^2)\cos\theta]$$

$$\text{or } (1+4\mu^2)\cos\theta=2(1-2\mu^2)e^{2\mu\theta}+2[-3\mu\sin\theta-(1-2\mu^2)\cos\theta]$$

At time t, the coordinates (x_G, y_G) of the C.G. 'G' are given by,

$\therefore x_G = ON = OP + PN = x + CG\cos\theta = x + \frac{2a}{\pi}\cos\theta = a\theta + \frac{2a}{\pi}\cos\theta$

and $y_G = NG = PL = CP - CL = a - CG\sin\theta = a - \frac{2a}{\pi}\sin\theta$.

Assuming F, the force of friction sufficient for pure rolling, the equations of motion of the wire are

$$F = M\ddot{x}_G = M\frac{d^2}{dt^2}\left(a\theta + \frac{2a}{\pi}\cos\theta\right) = M\left(a\ddot\theta - \frac{2a}{\pi}\sin\theta\ddot\theta - \frac{2a}{p}\cos\theta\dot\theta^2\right) \quad ...(1)$$

$$R - Mg = M\ddot{y}_G = M\frac{d^2}{dt^2}\left(a - \frac{2a}{\pi}\sin\theta\right) = M\left(-\frac{2a}{\pi}\cos\theta\ddot\theta + \frac{2a}{\pi}\sin\theta\dot\theta^2\right) \quad ...(2)$$

and $Mk^2\ddot\theta = R\cdot GL - F\cdot GN = R\cdot\frac{2a}{\pi}\cos\theta - F\left(a - \frac{2a}{\pi}\sin\theta\right)$ ...(3)

Since we want only the initial motion when $\theta = 0, \dot\theta = 0$ but $\ddot\theta$ is not zero. $\therefore$ From (1), (2) and (3), we get

$F = Ma\ddot\theta$, $R = Mg - \frac{2a}{\pi}M\ddot\theta$ and $Mk^2\ddot\theta = \frac{2a}{\pi}R - aF$. ...(4)

From these equations we get the initial values of F, R and $\ddot\theta$.

Thus eliminating R and F from equations (4), we get

$Mk^2\ddot\theta = \frac{2a}{\pi}\left(Mg - \frac{2a}{\pi}M\ddot\theta\right) - a(Ma\ddot\theta)$

or $\left(k^2 + \frac{4a^2}{\pi^2} + a^2\right)\ddot\theta = \frac{2a}{\pi}g$. ...(5)

But $Mk^2 = Ma^2 - M(2a/\pi)^2$ i.e. $k^2 = a^2 - 4a^2/\pi^2$.

Substituting in (5), we get

$\left(a^2 - \frac{4a^2}{\pi^2} + \frac{4a^2}{\pi^2} + a^2\right)\ddot\theta = \frac{2ag}{\pi}$ $\therefore$ $\ddot\theta = \frac{g}{\pi a}$ ...(6)

$\therefore F = Ma\ddot\theta = Ma\cdot\frac{g}{\pi a} = \frac{Mg}{\pi}$

and $R = Mg - \frac{2a}{\pi}M\ddot\theta = Mg - \frac{2a}{\pi}M\cdot\frac{g}{\pi a} = Mg\cdot\frac{\pi^2 - 2}{\pi^2}$

i.e. $\frac{F}{R} = \frac{\pi}{\pi^2 - 2}$.

$\therefore$ The wire will roll or slide according as $F <$ or $> \mu R$

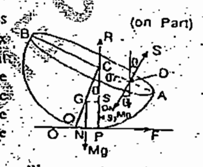
i.e. as $\mu >$ or $< \frac{F}{R}$ i.e. as $\mu >$ or $< \frac{\pi}{\pi^2 - 2}$.

When $\mu = \frac{\pi}{\pi^2 - 2}$ the wire will commence to roll if

$k^2 > a^2/3$ i.e. if $a^2 - 4a^2/\pi^2 > a^2/3$ i.e. if $\pi^2 > 6$ which is true.

Hence the wire rolls, when $\mu = \frac{\pi}{\pi^2 - 2}$.

Ex. 23. *A homogenous solid hemisphere of mass M and radius a rests with its vertex in contact with a rough horizontal plane, and a particle of mass m is placed on its base, which is smooth, at a distance c from the centre. Show that the hemisphere will commence to roll or slide according as the coefficient of friction μ is greater or less than* $\frac{25mac}{26(M+m)a^2 + 40mc^2}$.



Sol. Let a homogenous solid sphere of mass M, centre C and radius a rest with its vertex Q in contact with a rough horizontal plane at O. When the particle of mass m is placed on the base at the point D s.t. $CD = c$, then let the hemisphere roll. And when CG make an angle θ to the vertical, let the point of contact move through a distance $OP = x$.

Since the motion is assumed to be of pure rolling,

$\therefore x = OP = \text{arc } PQ = a\theta$ so that $\dot x = a\dot\theta$ and $\ddot x = a\ddot\theta$

The coordinates (x_G, y_G) of the C.G. 'G' referred to the horizontal and vertical lines through O as axes, are given by

$x_G = ON = OP - PN = x - \frac{3}{8}a\sin\theta$, $y_G = PL = CP - CL = \frac{3}{8}a\cos\theta$.

$\therefore$ The equations of motions of the sphere are

$F - S\sin\theta = M\ddot x_G = M\frac{d^2}{dt^2}\left(x - \frac{3}{8}a\sin\theta\right)$

or $F - S\sin\theta = M\left[a\ddot\theta - \frac{3}{8}a(\cos\theta\ddot\theta - \sin\theta\dot\theta^2)\right]$ ...(1)

$R - Mg - S\cos\theta = M\ddot y_G = M\frac{d^2}{dt^2}\left(a - \frac{3a}{8}\cos\theta\right)$

or $R - Mg - S\cos\theta = M\cdot\frac{3a}{8}(\sin\theta\ddot\theta + \cos\theta\dot\theta^2)$ ...(2)

and taking moment about G,

$Mk^2\ddot\theta = S\cdot CD - F\cdot GN - R\cdot GL$

or $Mk^2\ddot\theta = Sc - F\left(a - \frac{3}{8}a\cos\theta\right) - R\frac{3}{8}a\sin\theta$ ...(3)

The coordinates (x', y') of the particle of mass m, at D are

$x' = OP + LD = x + c\cos\theta = a\theta + c\cos\theta$, $y' = LP = a - c\sin\theta$

$\therefore$ The equation of motion of the particle is

$S\cos\theta - mg = m\ddot y' = m\frac{d^2}{dt^2}(a - c\sin\theta) = m(-c\cos\theta\ddot\theta + c\sin\theta\dot\theta^2)$ ...(4)

Since we want only the initial motion, when $\theta = 0, \dot\theta = 0$ but $\ddot\theta$ is not zero. $\therefore$ From (1), (2), (3) and (4), we get

$F = \frac{5}{8}Ma\ddot\theta$, $R = Mg + S$, $Mk^2\ddot\theta = Sc - \frac{5}{8}aF$ and $S = mg - mc\ddot\theta$ ...(5)

From these equations we get the initial values of F, R, S and $\ddot\theta$.

Substituting the values of F and S from first and fourth relations of (5) in the third, we get

$Mk^2\ddot\theta = (mg - mc\ddot\theta)c - \frac{5}{8}a\cdot\frac{5}{8}Ma\ddot\theta$

or $\left(Mk^2 + \frac{25}{64}Ma^2 + mc^2\right)\ddot\theta = mgc$ ...(6)

But $Mk^2 = \frac{2}{5}Ma^2 - M\left(\frac{3}{8}a\right)^2 = \frac{83}{320}Ma^2$.

$\therefore$ From (6), we get

$\left(\frac{83}{320}Ma^2 + \frac{25}{64}Ma^2 + mc^2\right)\ddot\theta = mgc$ or $\ddot\theta = \frac{20mgc}{13Ma^2 + 20mc^2}$

Thus $F = \frac{5}{8}Ma\ddot\theta = \frac{25mgcaM}{2(13Ma^2 + 20mc^2)}$

and $R = Mg + S = Mg + mg - mc\ddot\theta = \frac{13Ma^2(M + m) + 20Mmc^2}{(13Ma^2 + 20mc^2)}g$

$\therefore \frac{F}{R} = \frac{25mac}{26a^2(M + m) + 40mc^2}$

Thus the hemisphere will roll or slide according as $F <$ or $> \mu R$

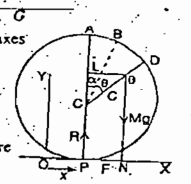
i.e. if $\mu >$ or $< \frac{F}{R}$

i.e. if $\mu >$ or $< \frac{25mac}{26(M + m)a^2 + 40mc^2}$

Ex. 24. *A sphere, of radius a, whose centre of gravity G is not at its centre C is placed on a rough horizontal table so that CG is inclined at an angle α to the upwards drawn vertical, show that it will commence to slide along the table, if the coefficient of friction μ be less than* $\frac{c\sin\alpha(a + c\cos\alpha)}{k^2 + (a + c\cos\alpha)^2}$ *where $CG = c$ and k is the radius of gyration about a horizontal axis through G.*

Sol. Let M be the mass, C the centre and G the C.G. of the sphere of radius a s.t. $CG = c$. Initially CG make an angle α to the vertical, assuming the motion to be pure rolling let C-G make an angle $\alpha + \theta$ to the vertical at time t. If the point of contact is shifted through a distance x, during this time, then $x = OP = a\theta$

$\therefore \dot x = a\dot\theta$ and $\ddot x = a\ddot\theta$.



The coordinates (x_G, y_G) of the C.G. G referred to the horizontal and vertical axes through O are given by

$x_G = ON = OP + PN = x + c\sin(\theta + \alpha)$

$= a\theta + c\sin(\theta + \alpha)$,

$y_G = NG = a + c\cos(\theta + \alpha)$.

$\therefore$ The equations of motion of the sphere are

$F = M\ddot x_G = M\frac{d^2}{dt^2}\{a\theta + c\sin(\theta + \alpha)\}$

or $F = M[a\ddot\theta + c\cos(\theta + \alpha)\ddot\theta - c\sin(\theta + \alpha)\dot\theta^2]$ ...(1)

$R - Mg = M\ddot y_G = M\frac{d^2}{dt^2}\{a + c\cos(\theta + \alpha)\}$

or $R - Mg = -Mc[\sin(\alpha + \theta)\ddot\theta + \cos(\theta + \alpha)\dot\theta^2]$ ...(2)

and taking moment about G, we get

$Mk^2\ddot\theta = R\cdot GL - F\cdot GN = Rc\sin(\theta + \alpha) - F\{a + c\cos(\theta + \alpha)\}$ ...(3)

Here we want only the initial motion, when $\theta = 0, \dot\theta = 0$, but $\ddot\theta$ is not zero $\therefore$ From (1), (2), (3), we get

$F = M(a + c\cos\alpha)\ddot\theta$; $R = Mg - Mc\sin\alpha\ddot\theta$

and $Mk^2\ddot\theta = Rc\sin\alpha - F(a + c\cos\alpha)$.

From these equations we get the initial values of F, R and $\ddot\theta$. Eliminating R and F from these equations, we get

$Mk^2\ddot\theta = (Mg - Mc\sin\alpha\ddot\theta)c\sin\alpha - (a + c\cos\alpha)\cdot M(a + c\cos\alpha)$

or $[k^2 + c^2\sin^2\alpha + (a + c\,cso\,\alpha)^2]\ddot\theta = gc\sin\alpha$

$\therefore \ddot\theta = \frac{gc\sin\alpha}{[k^2 + c^2\sin^2\alpha + (a + c\cos\alpha)^2]}$ ...(4)

Then substituting the value of $\ddot\theta$,

$F = \frac{M(a + c\cos\alpha)gc\sin\alpha}{[k^2 + c^2\sin^2\alpha + (a + c\cos\alpha)^2]}$

and $R = \frac{Mg - Mc\sin\alpha\cdot gc\sin\alpha}{[k^2 + c^2\sin^2\alpha + (a + c\cos\alpha)^2]}$

$= \frac{Mg[k^2 + (a + c\cos\alpha)^2]}{[k^2 + c^2\sin^2\alpha + (a + c\cos\alpha)^2]}$

$\therefore \frac{F}{R} = \frac{c\sin\alpha(a + c\cos\alpha)}{k^2 + (a + c\cos\alpha)^2}$

The sphere will commence to slide, if

$F > \mu R$ *i.e.* if $\mu < F/R$

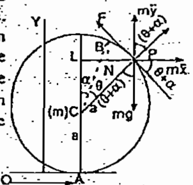
i.e. if $\mu < \dfrac{c \sin \alpha (a + c \cos \alpha)}{k^2 + (a + c \cos \alpha)^2}$.

Ex. 25. *A heavy uniform sphere, of mass M, is resting on a perfectly rough horizontal plane, and a particle, of mass M, is gently placed on it an angular distance α from its highest point. Show that the particle will at once slip on the sphere if*

$$\mu < \frac{\sin \alpha\,[7M + 5m(1 + \cos \alpha)]}{7M \cos \alpha + 5m (1 + \cos \alpha)^2}$$

where μ is the coeffcient of friction between the sphere and the particle.

Sol. Let the heavy sphere of mass M and radius a rest on a perfectly rough horizontal plane. Let the particle m be placed on the sphere at an angular distance α from the highest point. At time t when the sphere roll through a distance $OA = x$, let the particle m be at rest at P such that CP is inclined at an angle $(\theta + \alpha)$ to the vertical.



Since the sphere rolls, $\therefore\ x = a\theta$

i.e., $\dot{x} = a\dot{\theta}$ and $\ddot{x} = a\ddot{\theta}$.

The coordinates of the point P, referred to the horizontal and vertical lines OX and OY as axes are given by

$x' = OA + LP = x + a \sin(\theta + \alpha) = a\theta + a \sin(\theta + \alpha)$

and $y' = AL = AC + CL = a + a \cos(\theta + \alpha)$

$\therefore$ The equations of motion of the particle m along and perpendicular to CP are

$$R - mg \cos(\theta + \alpha) = m\ddot{x}' \sin(\theta + \alpha) + m\ddot{y}' \cos(\theta + \alpha)$$

or $R - mg \cos(\theta + \alpha) = m[a\ddot{\theta} + a \cos(\theta + \alpha)\ddot{\theta} - a \sin(\theta + \alpha)\dot{\theta}^2] \sin(\theta + \alpha) - ma[\sin(\theta + \alpha)\ddot{\theta} + \cos(\theta + \alpha)\dot{\theta}^2] \cos(\theta + \alpha)$

or $R - mg \cos(\theta + \alpha) = ma[\sin(\theta + \alpha)\ddot{\theta} - \dot{\theta}^2]$ ...(1)

and $F - mg \sin(\theta + \alpha) = m\ddot{y}' \sin(\theta + \alpha) - m\ddot{x}' \cos(\theta + \alpha)$

$= -ma[\sin(\theta + \alpha)\ddot{\theta} + \cos(\theta + \alpha)\dot{\theta}^2] \sin(\theta + \alpha) - m[a\ddot{\theta} + a \cos(\theta + \alpha)\ddot{\theta} - a \sin(\theta + \alpha)\dot{\theta}^2] \cos(\theta + \alpha)$

or $F - mg \sin(\theta + \alpha) = -ma[1 + \cos(\theta + \alpha)]\ddot{\theta}$. ...(2)

The energy equation gives

$[\frac{1}{2} Mk^2\dot{\theta}^2 + \frac{1}{2} Ma^2\dot{\theta}^2] + \frac{1}{2} m(\dot{x}'^2 + \dot{y}'^2) = mg\,BN$

K.E. of sphere K.E. of particle

or $\frac{1}{2} M \cdot \frac{2}{5} a^2\dot{\theta}^2 + \frac{1}{2} Ma^2\dot{\theta}^2 + \frac{1}{2} m[\{a\dot{\theta} + a \cos(\theta + \alpha)\dot{\theta}\}^2 + \{-a \sin(\theta + \alpha)\dot{\theta}\}^2] = mg\{a \cos \alpha - a \cos(\theta + \alpha)\}$

or $[7Ma + 10ma\{1 + \cos(\theta + \alpha)\}]\dot{\theta}^2 = 10mg\{\cos \alpha - \cos(\theta + \alpha)\}$

Differentiating w.r.t. t, we get

$[7Ma + 10ma\{1 + \cos(\theta + \alpha)\}]2\dot{\theta}\ddot{\theta} - 10ma \sin(\theta + \alpha)\dot{\theta}^3 = 10mg \sin(\theta + \alpha)\dot{\theta}$

or $[7Ma + 10ma\{1 + \cos(\theta + \alpha)\}]\ddot{\theta} - 5ma \sin(\theta + \alpha)\dot{\theta}^2 = 5mg \sin(\theta + \alpha)$. ...(3)

Here we want only the initial motion, when $\theta = 0$, $\dot{\theta} = 0$, but $\ddot{\theta}$ is zero. $\therefore$ From (1), (2) and (3), we get

$R = mg \cos \alpha + ma \sin \alpha\ddot{\theta}$; $F = mg \sin \alpha - ma(1 + \cos \alpha)\ddot{\theta}$

and $[7Ma + 10ma(1 + \cos \alpha)]\ddot{\theta} = 5mg \sin \alpha$.

or $\ddot{\theta} = 5mg \sin \alpha/[7Ma + 10ma(1 + \cos \alpha)]$

Substituting the value of $\ddot{\theta}$, we get

$\therefore\ R = mg \cos \alpha + ma \sin \alpha\, 5mg \sin \alpha/[7Ma + 10ma(1 + \cos \alpha)]$

$= [7M \cos \alpha + 5m(1 + \cos \alpha)^2]/[7M + 10m(1 + \cos \alpha)]$

and $F = \dfrac{mg \sin \alpha - ma(1 + \cos \alpha) \cdot 5mg \sin \alpha}{[7Ma + 10ma(1 + \cos \alpha)]}$

$= \dfrac{[7M + 5m(1 + \cos \alpha)]\, g \sin \alpha}{[7M + 10m(1 + \cos \alpha)]}$

$\therefore\ \dfrac{F}{R} = \dfrac{\sin \alpha\,[7M + 5m(1 + \cos \alpha)]}{7M \cos \alpha + 5m(1 + \cos \alpha)^2}$.

The particle will slide on the sphere, if $F > \mu R$

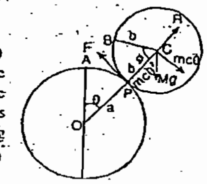
i.e. if $\mu < F/R$

or if $\mu < \dfrac{\sin \alpha\,[7M + 5m(1 + \cos \alpha)]}{7M + \cos \alpha + 5m(1 + \cos \alpha)^2}$

4.11. One of the Bodies Fixed :

A solid homogeneous sphere resting on the top of another fixed sphere is slightly displaced and begins to roll down it. Show that it will slip when the common normal makes with the vertical an angle θ given by the equation 2 sin (θ − λ) = 5 sin λ (3 cos θ − 2), where λ is the angle of friction.

Let O be the centre and a the radius of the fixed sphere. Let C be the center and b the radius of the solid homogeneous sphere resting on this fixed sphere at its highest point A. In time t, let the sphere roll to the point P, such that the common normal OC make an angle θ to the vertical. Let BC the line fixed in moving sphere make an angle ϕ to the vertical at time t. Initially B coincided with A.



Since there is pure rolling.

$\therefore$ Arc AP = Arc PB or $a\theta = b(\phi - \theta)$. ...(1)

Let R be the normal reaction and F the friction acting on the upper sphere at P. Since the centre C describe a circle of radius $OC = a + b = c$ (say), so its acceleration along and perpendicular to CO are $c\dot{\theta}^2$ and $c\ddot{\theta}$ respectively.

$\therefore$ The equations of motion of the upper sphere along and perpendicular to CO are

$Mc\dot{\theta}^2 = -R + Mg \cos \theta$, ...(2)

and $Mc\ddot{\theta} = Mg \sin \theta - F$. ...(3)

The co-ordinates (x_c, y_c) of the centre C referred to the horizontal and vertical lines through O as axes are given by

$x_c = OC \sin \theta = c \sin \theta$ and $y_c = OC \cos \theta = c \cos \theta$.

$\therefore\ v_c^2 = \dot{x}_c^2 + \dot{y}_c^2 = c^2\dot{\theta}^2$.

$\therefore$ The energy equation, gives

$\frac{1}{2} Mk^2\dot{\phi}^2 + \frac{1}{2} Mv_c^2$ = Work done by the C.G. C

or $\frac{1}{2} M \cdot \frac{2}{5} b^2\dot{\phi}^2 + \frac{1}{2} Mc^2\dot{\theta}^2 = Mg(c - c \cos \theta)$

or $\frac{1}{5} c^2\dot{\theta}^2 + \frac{1}{2} c^2\dot{\theta}^2 = gc(1 - \cos \theta)$

[$\because$ from (1), $b\dot{\phi} = (a + b)\dot{\theta} = c\dot{\theta}$]

or $c\dot{\theta}^2 = \frac{10}{7} g(1 - \cos \theta)$. ...(4)

Differentiating (4) w.r.t. t, we get

$2c\dot{\theta}\ddot{\theta} = \frac{10}{7} g \sin \theta\dot{\theta}$. $\therefore\ \ddot{\theta} = \left(\frac{5g}{7c}\right) \sin \theta$. ...(5)

From (2) and (4), we get

$R = -Mc\dot{\theta}^2 + Mg \cos \theta = -\frac{10}{7} Mg(1 - \cos \theta) + Mg \cos \theta$

$= \frac{1}{7} Mg(17 \cos \theta - 10)$. ...(6)

And from (3) and (5), we get

$F = -Mc\ddot{\theta} + Mg \sin \theta = -\frac{5}{7} Mg \sin \theta + Mg \sin \theta$

$= \frac{2}{7} Mg \sin \theta$. ...(7)

The sphere will slip if $F = \mu R$

or $\frac{2}{7} Mg \sin \theta = \mu \cdot \frac{1}{7} Mg(17 \cos \theta - 10)$

or $2 \sin \theta = \dfrac{\sin \lambda}{\cos \lambda}(17 \cos \theta - 10)$

where λ is the angle of friction

or $2 \sin \theta \cos \lambda = 17 \sin \lambda \cos \theta - 10 \sin \lambda$

or $2(\sin \theta \cos \lambda - \cos \theta \sin \lambda) = 5 \sin \lambda(3 \cos \theta - 2)$

or $2 \sin(\theta - \lambda) = 5 \sin \lambda(3 \cos \theta - 2)$. ...(8)

Remark 1. The upper sphere will leave the fixed sphere when $R = 0$, *i.e.* when $\frac{1}{7} Mg(17 \cos \theta - 10) = 0$ [from (6)]

i.e. when $\theta = \cos^{-1}\left(\frac{10}{17}\right)$.

Remark 2. When the two spheres are smooth. In this case $F = 0$

$\therefore$ The energy equation becomes

$\frac{1}{2} mc^2\dot{\theta}^2 = mg(c - c \cos \theta)$ *i.e.* $\dot{\theta}^2 = \frac{2g}{c}(1 - \cos \theta)$.

Since equation (2) remains unchanged

$\therefore\ R = Mg \cos \theta - Mc\dot{\theta}^2 = Mg \cos \theta - 2Mg(1 - \cos \theta)$

$= Mg(3 \cos \theta - 2)$.

$\therefore$ The upper sphere will leave the fixed sphere when $R = 0$, *i.e.* when $Mg(3 \cos \theta - 2) = 0$ *i.e.* when $\theta = \cos^{-1}\left(\frac{2}{3}\right)$.

EXAMPLES

Ex. 26. *A solid homogeneous sphere, resting on the top of another fixed sphere is slightly displaced and begins to roll down it. Show that it will slip when the common normal makes with the vertical an angle L given by the equation 4 sin (α − 30°) = 5 (3 cos L − 2), where 30° is the angle of friction.*

Sol. Put $\lambda = 30°$ and $\theta = L$ in § 4.11.

Ex. 27. *A solid uniform sphere, resting on the top of another fixed sphere is slightly displaced and begins to roll down. If the plane through their axes makes an angle α with the vertical when first sphere is at rest, show that it will slip when the common normal makes with the vertical an angle given by* $k^2 \sin \theta = \mu[(k^2 + 3b^2) \cos \theta - 2b^2 \cos \alpha]$, *where b is the radius of the moving sphere and k is the radius of gyration.*

The upper sphere will leave the fixed sphere if

$$\theta = \cos^{-1}\left(\frac{2b^2 \cos \alpha}{k^2 + 3b^2}\right).$$

which is negative, as $v > 5a\Omega$, hence the ring returns backwards to the point of projection.

The velocity of the point of contact

= Velocity of the centre + Velocity relative to the centre

$= \dot{y} - a\dot{\phi} = 0 - \frac{2}{9}(5a\Omega - v) = \frac{2}{9}(v - 5a\Omega) > 0$ as $v > 5a\Omega$.

$\therefore$ The velocity of the point of contact is up the plane, hence the friction μR acts downwards.

$\therefore$ The equations of motion are :

$M\ddot{z} = Mg \sin + \mu R = Mg \sin \alpha + \frac{1}{4} \tan \alpha . Mg \cos \alpha$

or $\ddot{z} = \frac{5}{4} g \sin \alpha$ ...(10)

and $Ma^2\ddot{\phi} = -\mu Ra = -\frac{1}{4} \tan \alpha . Mg \cos \alpha . a$

$\therefore a\ddot{\psi} = -\frac{1}{4} g \sin \alpha$. ...(11)

Integrating (10) and (11), and using the initial conditions, that when $t = 0, \dot{z} = 0$ and $a\dot{\psi} = \frac{2}{9}(v - 5a\Omega)$, we get

$\dot{z} = \frac{5}{4} gt \sin \alpha$ and $a\dot{\psi} = -\frac{1}{4} gt \sin \alpha + \frac{2}{9}(v - 5a\Omega)$.

$\therefore$ The velocity of the point of contact down the plane

$= \dot{z} - a\dot{\psi} = \frac{5}{4} gt \sin \alpha - [-\frac{1}{4} gt \sin \alpha - \frac{2}{9}(v - 5a\Omega)]$

$= \frac{3}{2} gt \sin \alpha + \frac{2}{9}(v - 5a\Omega)$, which is positive as $v > 5a\Omega$.

Hence the ring slides back to the point of contact.

4.9. Two Bodies In Contact :

When two bodies are in contact, then to determine whether the relative motion involves sliding at the point of contact.

Let a moving body be placed over the other body. Let P be the point of contact of the moving body and assume that its initial velocity is zero. To find the relative motion of the bodies which is either sliding or rolling we proceed as follows.

First assume that the body rolls and let F be the force of friction sufficient to keep the point of contact P at rest. Now write the equations of motion and the geometrical equation to express the condition that the tangential velocity of the point P is zero. Solve these equations and find F/R. Now there are two possibilities :

Case I. If $F/R < \mu$ In this case the necessary friction can be called into play to keep the point P at rest. Thus the moving body rolls and will keep rolling so long as $F/R \le \mu$.

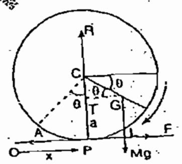
Case II. If $F/R > \mu$. In this case the point of contact will slide and the equations of motion discussed above will not hold good. Thus in this case proceed as follows :

Write the equations of motion, supposing that the point of contact slides. Hence the frictional force is μR instead of F and there is no geometrical equation in this case.

Solving these equations find the tangential velocity of the point of contact P. If this velocity is not zero and is in the direction opposite to the direction in which μR acts, μ has a proper sign and the body will slide at P and go on sliding so long as the velocity of the point P does not vanish. When velocity at P vanish, we again proceed to case I.

4.10. *A sphere of radius a whose centre of gravity G is at a distance c from its centre C, is placed on a rough plane so that CG, is horizontal. Show that it will begin to roll or slide according as the coefficient of friction*

$\mu >$ or $< \frac{ac}{k^2 + a^2}$, where k is the radius of gyration about a horizontal axis through G. If μ is equal to this value, what happens ?



Let M be the mass, C the centre and G the centre of gravity of the sphere of radius a. Initially CG is horizontal and at time t, let it be inclined at an angle θ to the horizontal. Assming that the sphere rolls, let $OP = x$ be the horizontal distance moved by the point of contact P from its initial position O, in time t. Since the motion is of pure rolling. $\therefore x = a\theta$; so that

$\dot{x} = a\dot{\theta}$ and $\ddot{x} = a\ddot{\theta}$. ...(1)

The co-ordinates (x_G, y_G) of the C.G. G referred to O as origin and the horizontal and vertical lines through O as origin are given by

$x_G = OL = OP + PL = x + c \cos \theta$, $y_G = LG = a - c \cos \theta$.

Assuming F, the force of friction sufficient for pure rolling, the equations of motion of the sphere are

$F = M\ddot{x}_G = M\frac{d^2}{dt^2}(x + c \cos \theta) = M(\ddot{x} - c \sin \theta\ddot{\theta} - c \cos \theta\dot{\theta}^2)$

or $F = M(a\ddot{\theta} - c \sin \theta\ddot{\theta} - c \cos \theta\dot{\theta}^2)$ ...(2)

$R - Mg = M\ddot{y}_G = M\frac{d^2}{dt^2}(a - c \cos \theta) = M(-c \cos \theta\ddot{\theta} + c \sin \theta\dot{\theta}^2)$ ...(3)

and $Mk^2\ddot{\theta} = R . GT - F . GL$

or $Mk^2\ddot{\theta} = R . c \cos \theta - F(a - c \sin \theta)$. ...(4)

$\therefore$ Here we want only the initial motion when $\theta = 0, \dot{\theta} = 0$, but $\ddot{\theta}$ is not zero.

$\therefore$ From (2), (3) and (4), we get

$F = Ma\ddot{\theta}$, $R = Mg - Mc\ddot{\theta}$ and $Mk^2\ddot{\theta} = Rc - aF$. ...(5)

From these equations we get the initial values of F, R and $\ddot{\theta}$.

Thus eliminating R and F, from equation (5), we get

$Mk^2\ddot{\theta} = (Mg - Mc\ddot{\theta})c - a(Ma\ddot{\theta})$ or $(k^2 + a^2 + c^2)\ddot{\theta} = gc$.

$\therefore \ddot{\theta} = gc/(k^2 + a^2 + c^2)$.

$\therefore F = \frac{Magc}{(k^2 + a^2 + c^2)}$ and $R = Mg - \frac{Mc . gc}{(k^2 + a^2 + c^2)} = \frac{Mg(k^2 + a^2)}{(k^2 + a^2 + c^2)}$

i.e. $\frac{F}{R} = \frac{ac}{k^2 + a^2}$.

Thus the sphere will begin to slide or roll according as

$F <$ or $> \mu R$ *i.e.* as $\mu >$ or $< F/R$

i.e. as $\mu >$ or $< \frac{ac}{k^2 + a^2}$.

When $\mu = \frac{ac}{k^2 + a^2}$. In this case we shall consider whether F/R is a little greater or little less than μR when θ is small but not absolutly zero.

$\therefore$ From equations (2), (3) and (4), we get

$Mk^2\ddot{\theta} = [Mg + M(-c \cos \theta\ddot{\theta} + c \sin \theta\dot{\theta}^2)] c \cos \theta$

$- M(a\ddot{\theta} - c \sin \theta\ddot{\theta} - c \cos \theta\dot{\theta}^2)(a - c \sin \theta)$

or $(k^2 + a^2 + c^2 - 2ac \sin \theta)\ddot{\theta} - ac \cos \theta\dot{\theta}^2 = gc \cos \theta$.

Integrating, we get

$(k^2 + a^2 + c^2 - 2ac \sin \theta)\dot{\theta}^2 = 2gc \sin \theta$. ...(6)

Since θ is small. Therefore taking θ for $\sin \theta$ and unity for $\cos \theta$ in (6), we get

$(k^2 + a^2 + c^2 - 2ac\,\theta)\dot{\theta}^2 = 2gc\,\theta$

or $(k^2 + a^2 + c^2)\dot{\theta}^2 = 2gc\,\theta$ ...(7)

Neglecting $\theta\dot{\theta}^2$.

Then from (4), we get

$(k^2 + a^2 + c^2 - 2ac\,\theta)\ddot{\theta} - ac . \frac{2gc\,\theta}{(k^2 + a^2 + c^2)} = gc1$. ($\because \cos \theta = 1, \sin \theta = \theta$)

or $(k^2 + a^2 + c^2 - 2ac\,\theta)\ddot{\theta} = gc\left(1 + \frac{2ac\,\theta}{k^2 + a^2 + c^2}\right)$

or $(k^2 + a^2 + c^2)\left(1 - \frac{2ac\,\theta}{k^2 + a^2 + c^2}\right)\ddot{\theta} = gc\left(1 + \frac{2ac\,\theta}{k^2 + a^2 + c^2}\right)$

or $(k^2 + a^2 + c^2)\ddot{\theta} = gc\left(1 + \frac{2ac\,\theta}{k^2 + a^2 + c^2}\right)\left(1 - \frac{2ac\,\theta}{k^2 + a^2 + c^2}\right)^{-1}$

$= gc\left(1 + \frac{2ac\,\theta}{k^2 + a^2 + c^2}\right)\left(1 + \frac{2ac\,\theta}{k^2 + a^2 + c^2}\right)$

neglecting higher powers of θ

$= gc\left(1 + \frac{4ac\,\theta}{k^2 + a^2 + c^2}\right)$ approximately. ...(8)

$\therefore$ From (2) and (3), we have

$\frac{F}{R} = \frac{(a - c \sin \theta)\ddot{\theta} - c \cos \theta\dot{\theta}^2}{g - c \cos \theta\ddot{\theta} + c \sin \theta\dot{\theta}^2} = \frac{(a - c\theta)\ddot{\theta} - c . \dot{\theta}^2}{g - c\ddot{\theta} + c\theta\dot{\theta}^2}$

$= \frac{(a - c\theta)\ddot{\theta} - c\dot{\theta}^2}{g - c\ddot{\theta}}$, neglecting $\theta\dot{\theta}^2$

$= \frac{ac}{k^2 + a^2}\left[1 - \frac{3c(k^2 - a^2/3)\theta}{a(k^2 + a^2)}\right]$.

[Substituting the value of $\dot{\theta}^2$ and $\ddot{\theta}$ from (7) and (8)]

If $k^2 > \frac{a^2}{3}$, then $\frac{F}{R} < \frac{ac}{k^2 + a^2}$ *i.e.* $\frac{F}{R} < \mu$ or $F < \mu R$.

$\therefore$ In this case the sphere rolls

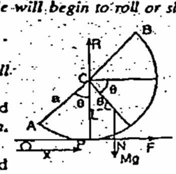
If $k^2 < \frac{a^2}{3}$, then $\frac{F}{R} > \frac{ac}{k^2 + a^2}$ *i.e.* $\frac{F}{R} > \mu$ or $F > \mu R$.

i.e. in this case the sphere slides.

EXAMPLES

Ex. 22. *If a uniform semi-circular wire be placed in a vertical plane with one extremity on a rough horizontal plane, and the diameter though the extremity vertical, show that the semi-circle will begin to roll or slide according as μ be greater or less than $\frac{\pi}{\pi^2 - 2}$.*

If μ has this value, prove that the wire will roll.



Sol. Let C be the centre, G the C.G. and M the mass of the circular wire of radius a. $\therefore CG = 2a/\pi$.

Assuming that the wire rolls, let CG be inclined at an angle θ to the horizontal at time t, which was initially horizontal.

Let the point of contact P move, through a distance $OP = x$ from its initial position O, in time t. Since the motion is of pure rolling $\therefore x = \text{Arc } AP = a\theta$, so that $\dot{x} = a\dot{\theta}$ and $\ddot{x} = a\ddot{\theta}$.

$x = -\frac{1}{2}\mu g t^2 + ut$ ...(5)

Putting $t = t_1 = u/\mu g$ in (5), the distance traversed by the ring during this time is given by

$x = -\frac{1}{2}\mu g\left(\frac{u^2}{\mu^2 g^2}\right) + u\left(\frac{u}{\mu g}\right) = \frac{u^2}{2\mu g}$ ...(6)

and from (5), at time $t = t_1 = u/\mu g$, $a\dot\theta = a\Omega - u$,

which is positive ∵ $u < a\Omega$.

∴ Hence the ring returns from right to left.

When the ring returns.

The initial velocity of the point of contact is in the direction from left to right, ∴ the friction μR will act in the direction from right to left.

If y is the distance traversed in time t from right to left and ϕ the angle turned by the ring in this time, then for this motion, the equations of motion are

$M\ddot y = \mu R = \mu M g$ i.e. $\ddot y = \mu g$ ...(7)

and $Mk^2\ddot\phi = Ma^2\ddot\phi = -\mu R a = -\mu M g a$ i.e. $a\ddot\phi = -\mu g$ ...(8)

Integrating (7) and (8) and using the initial conditions that when $t = 0$, $\dot y = 0$ and $a\dot\phi = a\Omega - u$, we get

$\dot y = \mu g t$ ...(9)

$a\dot\phi = -\mu g t + a\Omega - u$ ...(10)

These equations (9) and 10 holds until pure rolling commence i.e. the velocity of the point of contact $= \dot y - a\dot\phi = 0$.

If this occurs at time $t = t_2$, then from (9) and (10), we get

At $t = t_2$, $\dot y = a\dot\phi$ ∴ $\mu g t_2 = -\mu g t_2 + a\Omega - u$ or $t_2 = (a\Omega - u)/2\mu g$.

∴ At time t_2, $\dot y = \mu g t_2 = \frac{1}{2}(a\Omega - u)$

Integrating (9) we get $y = \frac{1}{2}\mu g t^2 + C$; $C = 0$ ∵ at $t = 0$, $y = 0$.

∴ $y = \frac{1}{2}\mu g t^2$

∴ distance traversed in time t_1,

$y = \frac{1}{2}\mu g t_2^2 = (a\Omega - u)^2/(8\mu g)$ ...(12)

When pure rolling commences, the equations of motions are

$M\ddot z = F$ and $Mk^2\ddot\phi = Ma^2\ddot\psi = -Fa$ ...(13)

Since there is no sliding. ∴ $\dot z = a\dot\psi$ i.e. $\ddot z = a\ddot\psi$

∴ from equations (13) we get

$-F = Ma\ddot\phi = M\ddot z = F$ or $2F = 0$ ∴ $F = 0$.

Thus no friction is required.

∴ $\ddot z = 0$. Integrating $\dot z = \text{constant} = \frac{1}{2}(a\Omega - u)$ ...(14)

∵ at $t = 0$, $\dot z = \dot y = \frac{1}{2}(a\Omega - u)$ from (11)

i.e. when pure rolling commences from right to left, the ring continues to move with constant velocity $\frac{1}{2}(a\Omega - u)$.

The distance of the point from where the pure rolling commences, from the starting point O

$s = x - y = \frac{u^2}{2\mu g} - \frac{(a\Omega - u)^2}{8\mu g}$

∴ The time taken by the ring to traverse this distance

$t_3 = \frac{s}{\dot z} = \frac{2}{a\Omega - u}\left[\frac{u^2}{2\mu g} - \frac{(a\Omega - u)^2}{8\mu g}\right]$

or $t_3 = \frac{u^2}{\mu g(a\Omega - u)} - \frac{a\Omega - u}{4\mu g}$

∴ The total time taken by the ring to return to the point of projection

$= t_1 + t_2 + t_3 = \frac{u}{\mu g} + \frac{a\Omega - u}{2\mu g} + \left[\frac{u^2}{\mu g(a\Omega - u)} - \frac{a\Omega - u}{4\mu g}\right]$

$= \frac{(a\Omega + u)^2}{4\mu g(a\Omega - u)}$

Second Part. *When $u > a\Omega$.*

Considering the motion in the forward direction discussed in the beginning. In this case the velocity of the point of contact

$= \dot x + a\dot\theta = (-\mu g t + u) + (-\mu g t + a\Omega)$, from (3), & (4)

$= -2\mu g t + a\Omega + u$.

Pure rolling will commence when $\dot x + a\dot\theta = 0$.

i.e. when $-2\mu g t + a\Omega + u = 0$. ∴ at $t = (a\Omega + u)/2\mu g$.

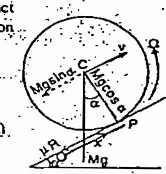
Also it has been proved that the forward motion ceases after time $t = u/\mu g$.

Thus the rolling will comence before the forward motion has ceased, if $\frac{a\Omega + u}{2\mu g} > \frac{u}{\mu g}$

i.e. if $u > a\Omega$.

Hence when $u > a\Omega$, the pure rolling will commence before the forward motion ceases.

Ex. 21. *A thin napkin ring, of radius a, is projected up a plane inclined at angle α to the horizontal with velocity v, and an initial angular velocity Ω in the sense which would cause the ring to move down the plane. If $v > 5a\Omega$ and $\mu = \frac{1}{4}\tan\alpha$, show that the ring will never roll and will cease to ascend at the end of a time $\frac{4(2v - a\Omega)}{9g\sin\alpha}$ and will slide back to the point of projection.*



Sol. Let C be the centre and M the mass of the ring. The initial velocity of the point of contact is $v + a\Omega$, which acts up the plane; ∴ the friction μR acts down the plane.

∴ The equations of motion are

$M\ddot x = -Mg\sin\alpha - \mu R$

$= -Mg\sin\alpha - \mu Mg\cos\alpha$ (∵ $R = Mg\cos\alpha$)

or $\ddot x = -g(\sin\alpha + \mu\cos\alpha)$

$= -g(\sin\alpha + \frac{1}{4}\tan\alpha\cos\alpha)$

or $\ddot x = -\frac{5}{4}g\sin\alpha$, ...(1)

and $Ma^2\ddot\theta = -\mu R a = -\frac{1}{4}\tan\alpha \,.\, Mg\cos\alpha \,.\, a = -\frac{1}{4}Mga\sin\alpha$

∴ $a\ddot\theta = -\frac{1}{4}g\sin\alpha$. ...(2)

Integrating (1) and (2), and using the initial conditions, that at $t = 0$, $\dot x = v$ and $\dot\theta = \Omega$, we get

$\dot x = -\frac{5}{4}g\sin\alpha \,.\, t + v$. ...(3)

and $a\dot\theta = -\frac{1}{4}g\sin\alpha \,.\, t + a\Omega$. ...(4)

From (3), the velocity of the centre is zero after time $t = t_1 = 4v/(5g\sin\alpha)$.

The velocity of the point of contact at any time t

$= \dot x + a\dot\theta = -\frac{5}{4}g\sin\alpha + v - \frac{1}{4}g\sin\alpha\, t + a\Omega$

[substituting from (3) and (4)]

$= v + a\Omega - \frac{3}{2}gt\sin\alpha$

∵ The point of contact will come to rest when $\dot x + a\dot\theta = 0$. If it happens at time $t = t_2$, then

$v + a\Omega - \frac{3}{2}gt_2\sin\alpha = 0$.

∴ $t_2 = 2(v + a\Omega)/(3g\sin\alpha)$.

Clearly, $\frac{2(v + a\Omega)}{3g\sin\alpha} < \frac{4v}{5g\sin\alpha}$ as $v > 5a\Omega$. i.e. $t_2 < t_1$.

∴ Therefore pure rolling may begin before the upward motion ceases if the friction is sufficient for pure rolling.

At this time t_2, $\dot x = -\frac{5}{4}g\sin\alpha\, t_2 + v = \frac{1}{6}(v - 5a\Omega)$

and $\dot\theta = \frac{1}{6a}(5a\Omega - v)$.

Clearly, $\dot x$ is positive and $\dot\theta$ is negative as $v > 5a\Omega$,

i.e. $\dot\theta = \frac{1}{6a}(v - 5a\Omega)$, in clockwise direction.

When pure rolling commences and rotation is in clockwise direction, the friction F acts in upwards direction.

The equations of motion are

$M\ddot y = -Mg\sin\alpha + F$ and $Ma^2\ddot\phi = -Fa$. ...(5)

Since there is pure rolling. ∴ $y = a\phi$ i.e $\dot y = a\dot\phi$ and $\ddot y = a\ddot\phi$.

Solving these equations, we get

$F = \frac{1}{2}Mg\sin\alpha$.

But $\mu R = \frac{1}{4}\tan\alpha \,.\, Mg\cos\alpha = \frac{1}{4}Mg\sin\alpha$ i.e. $F > \mu R$. Thus the friction is not sufficient for pure rolling. Hence the sliding persists and pure rolling is not possible. Therefore, the above equations of motion now become

$M\ddot y = -Mg\sin\alpha + \mu R = -Mg\sin\alpha + \frac{1}{4}\tan\alpha \,.\, Mg\cos\alpha$

$= -\frac{3}{4}Mg\sin\alpha$ or $\ddot y = -\frac{3}{4}g\sin\alpha$ ...(6)

and $Ma^2\ddot\phi = -\mu R a = -\frac{1}{4}\tan\alpha \,.\, Mg\cos\alpha \,.\, a = -\frac{1}{4}Mga\sin\alpha$

or $a\ddot\phi = -\frac{1}{4}g\sin\alpha$ ...(7)

Integrating (6) and (7), and using the initial conditions, that at $t = 0$, $\dot y = (v - 5a\Omega)/6$ and $a\dot\phi = (5a\Omega - v)/6$, we get

$\dot y = -\frac{3}{4}g\sin\alpha \,.\, t + \frac{1}{6}(v - 5a\Omega)$, ...(8)

and $a\dot\phi = -\frac{1}{4}g\sin\alpha \,.\, t + \frac{1}{6}(5a\Omega - v)$. ...(9)

Clearly, $\dot y = 0$ after time $t = t_3 = \frac{2(v - 5a\Omega)}{9g\sin\alpha}$.

Putting this value of time in (9), we get

$a\dot\phi = \frac{2}{9}(v - 5a\Omega)$.

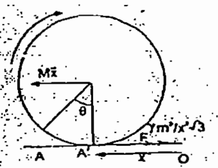
Hence, total time of upwards motion

$= t_2 + t_3 = 2.\frac{(v + a\Omega)}{3g\sin\alpha} + \frac{2(v - 5a\Omega)}{9g\sin\alpha} = \frac{4(2v - a\Omega)}{9g\sin\alpha}$

Again when the upward motion ceases, we have

$a\dot\phi = -\frac{1}{4}g\sin\alpha t_3 + \frac{1}{6}(5a\Omega - v) = \frac{2}{9}(5a\Omega - v)$.

As the plane is perfectly rough, there is pure rolling. Therefore the force of friction F at the point of contact A' acts in the direction opposite to the tendency of motion of the point of contact *i.e.* F acts towards O, in the direction of x decreasing.



$\therefore$ The equation of motion of the sphere at A' is

$$m\ddot{x} = -F - \frac{\gamma m^2}{x^2\sqrt{3}} \quad ...(1)$$

and $mk^2\ddot{\theta} = m.\frac{2}{5}a^2\ddot{\theta} = -Fa.$ ...(2)

As there is no slipping. Therefore the velocity of the point of contact $= \dot{x} + a\dot{\theta} = 0$.

or $\dot{x} = -a\dot{\theta} \therefore \ddot{x} = -a\ddot{\theta}$. ...(3)

From (2) and (3), we get $F = -\frac{2}{5}m.(-\ddot{x}) = \frac{2}{5}m\ddot{x}$.

Substituting in (1), we get $m\ddot{x} = -\frac{2}{5}m\ddot{x} = -\frac{\gamma m^2}{x^2\sqrt{3}}$

or $\frac{7}{5}m\ddot{x} = -\frac{\gamma m^2}{x^2\sqrt{2}} \therefore \ddot{x} = -\frac{5\gamma m}{7x^2\sqrt{3}}$.

Multiplying by $2\dot{x}$ and integrating, we get

$$\dot{x}^2 = \frac{10\gamma m}{7\sqrt{3}x} + C.$$

But initially when the sphere was at A,

$x = OA = \frac{1}{\sqrt{3}}AB = \frac{1}{\sqrt{3}}.4a, \dot{x} = 0$.

$\therefore C = -\frac{5\gamma m}{14a}. \therefore \dot{x}^2 = \frac{10\gamma m}{7\sqrt{3}x} - \frac{5\gamma m}{14a}$. ...(4)

When the spheres will collide, their points of contact with the plane will from an equilateral triangle of side $2a$ *i.e.* at this time $x = 2a/\sqrt{3}$.

$\therefore$ Putting $x = 2a/\sqrt{3}$ in (4), the velocity of the centres of the sphere when they collide is given by

$$(\text{Velocity})^2 = \frac{10\gamma m}{7\sqrt{3}}\frac{\sqrt{3}}{2a} - \frac{5\gamma m}{14a} = \frac{5\gamma m}{14a}$$

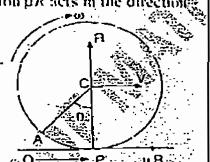
i.e. Velocity $= \sqrt{\left(\frac{5\gamma m}{14a}\right)}$

4.8. *A uniform circular disc is projected with its plane vertical along a rough horizontal plane with a velocity V of translation and an angular velocity ω about the centre. Find the motion.*

Case I. When V is from left of right, ω clockwise and V > aω.

In this case initial velocity of the point of contact P is given by $V - a\omega$, ~~which is positive~~, as $V > a\omega$. Thus the friction μR acts in the direction left to right.

In time t, let the centre move through a distance x and let the disc turn through an angle θ.



$\therefore$ The equations of motion are given by

$M\ddot{x} = -\mu R = -\mu mg$.

$\therefore \ddot{x} = -\mu g$, ...(1)

and $Mk^2\ddot{\theta} = M\frac{a^2}{2}\ddot{\theta} = \mu Ra = \mu Mga$

i.e. $a\ddot{\theta} = 2\mu g$. ...(2)

Integrating (1) and (2) and using the initial conditions, (when $t = 0, \dot{x} = V$ and $\dot{\theta} = \omega$), we have

$\dot{x} = -\mu gt + V$, ...(3)

and $a\dot{\theta} = 2\mu gt + a\omega$. ...(4)

The rolling begins when the velocity of the point of contact P, *i.e.* $\dot{x} - a\dot{\theta} = 0$. If this happen after time t_1, then

$\dot{x} - a\dot{\theta} = -\mu gt_1 + V - 2\mu gt_1 - a\omega = 0$.

$\therefore t_1 = (V - a\omega)/3\mu g$.

$\therefore$ From (3), at this time velocity of the centre is given by

$\dot{x} = -\mu gt_1 + V = -\frac{1}{3}(V - a\omega) + V = \frac{1}{3}(2V + a\omega)$. ...(5)

When rolling commences equations of motions are

$M\ddot{x} = -F$ and $Mk^2\ddot{\phi} = \frac{1}{2}Ma^2\ddot{\phi} = Fa$. ...(6)

As there is no sliding, so $\dot{x} = a\dot{\phi} \therefore \ddot{x} = a\ddot{\phi}$.

$\therefore$ From (6) $F = -M\ddot{x} = -Ma\ddot{\phi} = -2F$

or $3F = 0 \therefore F = 0$.

Thus no friction is required for rolling, throughout the motion. Hence the equations of motion (6) for pure rolling reduce to

$M\ddot{x} = 0$ and $\frac{1}{2}Ma^2\ddot{\phi} = 0$ *i.e.* $\ddot{x} = 0$ ...(7)

and $\ddot{\phi} = 0$. ...(8)

Integrating (7), we get $\dot{x} =$ constant $= \frac{1}{3}(2V + a\omega)$, from (5).

Hence the disc continues to roll with constant velocity

$$\frac{1}{3}(2V + a\omega).$$

Case II. When V is from left to right ω clockwise and V < aω.

In this case the initial velocity of the point of contact P

$$= V - a\omega,$$

which is negative, *i.e.* its direction is from right to left. Therefore the friction μR will act from left to right.

$\therefore$ The equations of motion are

$M\ddot{x} = \mu R = \mu Mg$. $\therefore \ddot{x} = \mu g$. ...(9)

and $Mk^2\ddot{\theta} = \frac{1}{2}Ma^2\ddot{\theta} = -\mu Ra = -\mu Mga$. $\therefore a\ddot{\theta} = -2\mu g$. ...(10)

Integrating (9), (10) and using the initial conditions (when $t = 0, \dot{x} = V$ and $\dot{\theta} = \omega$), we have

$\dot{x} = \mu gt + V$. ...(11)

and $a\dot{\theta} = -2\mu gt + a\omega$. ...(12)

Now, pure rolling begins, when the velocity of the point of contact P, *i.e.* $\dot{x} - a\dot{\theta} = 0$. If this happens after time t_2, then

$\dot{x} - a\dot{\theta} = \mu gt_2 + V - (-2\mu gt_2 + a\omega) = 0$.

$\therefore t_2 = (a\omega - V)/(3\mu g)$.

From (11), at this time the velocity of the centre is given by

$\dot{x} = \mu gt_2 + V = \frac{1}{3}(a\omega - V) + V = \frac{1}{3}(2V + a\omega)$. ...(13)

When rolling begins the equations of motions are the same [*i.e.* equation (6)] as in case I.

$\therefore$ As in case I, $F = 0$.

Hence the disc continues to roll with constant velocity

$$\frac{1}{2}(2V + a\omega).$$

Case III. When V is from left to right and ω anti-clockwise.

In this case the velocity of the point of contact $P = V + a\omega$, which is positive, *i.e.* its direction is from left to right. Therefore the friction μR will act from right to left.

$\therefore$ The equations of motion are

$M\ddot{x} = -\mu R = -\mu Mg$. $\therefore \ddot{x} = -\mu g$ ...(14)

and $Mk^2\ddot{\theta} = \frac{1}{2}Ma^2\ddot{\theta} = -\mu Ra$.

or $\frac{1}{2}Ma^2\ddot{\theta} = -\mu Mga$. $\therefore a\ddot{\theta} = -2\mu g$. ...(15)

Integrating (14), (15) and using the initial conditions, (when $t = 0, \dot{x} = V$ and $\dot{\theta} = \omega$), we have

$\dot{x} = -\mu g3 + V$, ...(16)

and $a\dot{\theta} = -2\mu gt + a\omega$. ...(17)

For pure rolling to commence, the velocity of the point of contact P, *i.e.* $\dot{x} + a\dot{\theta} = 0$ ($\dot{x}$ and $\dot{\theta}$ are in the same direction). If this happens after time t_3, then

$\dot{x} + a\dot{\theta} = -\mu gt_3 + V + (-2\mu gt_3 + a\omega) = 0$.

$\therefore t_3 = (V + a\omega)/3\mu g$.

From (16), at this time the velocity of the centre is given by

$\dot{x} = -\mu gt_3 + V = -\frac{1}{3}(V + a\omega) + V = \frac{1}{3}(2V - a\omega)$. ...(18)

If $2V > a\omega$, the velocity of the centre is from left to right. Hen clas in case I and II the notion will be of pure rolling with constant velocity $\frac{1}{3}(2V - a\omega)$.

If $2V < a\omega$, then $\dot{x}$ is negative, *i.e.* the direction of velocity of the centre is from right to left, *i.e.* backwards. Thus, when pure rolling begins, the disc rolls back towards the initial point.

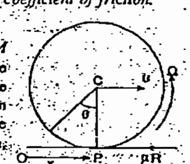
From (16), we see that $\dot{x} = 0$ when $t = V/\mu g$, and at this time from (17), $a\dot{\theta} = a\omega - 2V$, which is positive, $\because 2V < a\omega$.

Hence if $2V < a\omega$, the disc begins to move backwards before the pure rolling begins.

EXAMPLES

Ex. 20. *A napkin ring, of radius a, is projected forward on a rough horizontal table with a linear velocity u and a backward spin Ω which is > u/a. Find the motion and show that the ring will return to the point of projection in time* $\frac{(\mu + a\Omega)^2}{4\mu g(a\Omega - u)}$ *where μ is the coefficient of friction. What happens if u > aΩ.*

Sol. Let C be the centre of the ring and M its mass. Initially $\vec{u}$, Ω in anticlockwise direction, and $u < a\Omega$. The initial velocity of the point of contact is $u + a\Omega$ and is in the direction left to right. Hence the friction μR acts in the direction from right to left. For this motion, equations of motion are



$M\ddot{x} = -\mu R = -\mu Mg$ *i.e.* $\ddot{x} = -\mu g$ ...(1)

and $Mk^2\ddot{\theta} = Ma^2\ddot{\theta} = -\mu Ra = -\mu Mg.a$ *i.e.* $a\ddot{\theta} = -\mu g$ ...(2)

Integrating (1) and (2) and using the initial conditions that when $t = 0, \dot{x} = u$ and $\dot{\theta} = \Omega$, we get

$\dot{x} = -\mu gt + u$ ...(3)

and $a\dot{\theta} = -\mu gt + a\Omega$ ...(4)

The ring ceases to move forward if $\dot{x} = 0$. If this happens after time t_1, then from (3), we have,

$0 = -\mu gt_1 + u$

$\therefore t_1 = u/\mu g$.

Also integrating (4) and using the condition that when $x = 0, t = 0$, we get or $-g(\sin\alpha+\mu\cos\alpha)t_1+V-\frac{5\mu}{2}gt_1\cos\alpha-a\Omega=0$

$\therefore t_1=\frac{2V-2a\Omega}{g(7\mu\cos\alpha+2\sin\alpha)}$

Putting $t=t_1$ in (4), we get

$\dot{x}=V-g(\sin\alpha+\mu\cos\alpha)\left[\frac{2V-2a\Omega}{g(7\mu\cos\alpha+2\sin\alpha)}\right]$

$=\frac{5\mu\cos\alpha+2a\Omega(\sin\alpha+\mu\cos\alpha)}{7\mu\cos\alpha+2\sin\alpha}=V_1$ (say).

When rolling begins *i.e.* when the point of contact has been brough to rest, let F be the friction which is sufficient for pure rolling. Because the point of contact is at rest, so friction will try to keep it at rest if possible, hence the friction F acts upwards.

$\therefore$ Equations of motions are

$M\ddot{y}=-Mg\sin\alpha+F$ ...(6)

and $Mk^2\ddot{\phi}=M\frac{2a^2}{5}\ddot{\phi}=-Fa$ ...(7)

Since, throughout the motion the point of contact is at rest

$\therefore \dot{y}-a\dot{\phi}=0$ or $\dot{y}=a\dot{\phi}$ $\therefore \ddot{y}=a\ddot{\phi}$

Solving equations (6) and (7), we get $F=\frac{2}{7}mg\sin\alpha$ ...(8)

Again $\mu R=\mu.Mg\cos\alpha>\frac{2}{7}\tan\alpha.Mg\cos\alpha$ *i.e.* $\mu R>\frac{2}{7}Mg\sin\alpha$.

$\therefore$ The conditions $F<\mu R$ is satisfied.

Putting the value of F from (8) in (6), we have

$\ddot{y}=-\frac{5}{7}g\sin\alpha$.

Integrating, $\dot{y}=-\frac{5}{7}gt\sin\alpha+C$

But when $t=0$, $\dot{y}=V_1$, $\therefore C=V_1$

$\therefore \dot{y}=-\frac{5}{7}gt\sin\alpha+V_1$ ...(9)

Now the sphere will cease to ascend the plane when $\dot{y}=0$, if this happen after time t_2, then from (9), we have

$0=-\frac{5}{7}gt_2\sin\alpha+V_1$ or $t_2=\frac{7V_1}{5g\sin\alpha}$

$\therefore$ The total time of ascent $=t_1+t_2$

$=\frac{2V+2a\Omega}{g(7\mu\cos\alpha+2\sin\alpha)}+\frac{7}{5g\sin\alpha}\times\left\{\frac{5\mu V\cos\alpha+2a\Omega(\sin\alpha+\mu\cos\alpha)}{7\mu\cos\alpha+2\sin\alpha}\right\}$

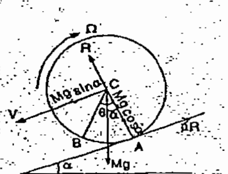
$=\frac{10(V-a\Omega)\sin\alpha+35\mu V\cos\alpha+14a\Omega(\sin\alpha+\mu\cos\alpha)}{5g\sin\alpha(7\mu\cos\alpha+2\sin\alpha)}$

$=\frac{5V(7\mu\cos\alpha+2\sin\alpha)+2a\Omega(7\mu\cos\alpha+2\sin x)}{5g\sin\alpha(7\mu\cos\alpha+2\sin\alpha)}$

$=\frac{5V+2a\Omega}{5g\sin\alpha}$

Ex. 18. *A uniform sphere, of radius a, is rotating about a horizontal diameter with angular velocity Ω and is gently placed on a rough plane which is inclined at angle α to the horizontal, the sense of rotation being such as to tend to cause the sphere to move up the plane along the line of greatest slope. Show that, if the coefficient of friction be $\tan\alpha$, the centre of the sphere will remain at rest for a time $\frac{2a\Omega}{5g\sin\alpha}$ and will then move downwards with acceleration $\frac{5}{7}\sin\alpha$.*

If the body be a thin circular hoop instead of sphere, show that the time is $\frac{a\Omega}{g\sin x}$ and the acceleration $\frac{1}{2}g\sin\alpha$.



Sol. Let the sphere of mass M rotating with an angular velocity Ω about the horizontal diamter be placed gently on the inclined plane. Hence the velocity of the centre is zero.

The sense of rotation at the time of placing the sphere on inclined plane is such that it tend to cause the sphere move up the plane, that means the sense of Ω is as shown in the figure.

$\therefore$ The initial velocity of the point of contact A down the plane

= Velocity of the centre C + velocity of A relative to C

$=0+a\Omega$ which is a positive quantity.

i.e. the initial velocity of the point of contact so down the plane, so the friction μR acts up the plane.

Equations of motion are

$M\ddot{x}=Mg\sin\alpha-\mu R$ ...(1)

$0=R-M\cos\alpha$ ...(2)

and $Mk^2\ddot{\theta}=-\mu Ra$ ...(3)

where $\mu=\tan\alpha$

From (1) and (2), we get

$M\ddot{x}=Mg\sin\alpha-\tan\alpha.Mg\cos\alpha=0$ or $\ddot{x}=0$

Integrating, $\dot{x}=C$. But when $t=0$, $\dot{x}=0$ $\therefore C=0$

$\therefore \dot{x}=0$

From (2) and (3), we get

$Mk^2\ddot{\theta}=-\tan\alpha(Mg\cos\alpha)a=-Mga\sin\alpha$

or $k^2\ddot{\theta}=-ga\sin\alpha$. Integrating $k^2\dot{\theta}=-gta\sin\alpha+C_1$

But when $t=0$, $\dot{\theta}=\Omega$ $\therefore C_1=k^2\Omega$, so $k^2\dot{\theta}=-gta\sin\alpha+k^2\Omega$ ...(5)

From equations (4) and (5), we observe that the centre of the sphere does not move at all, but the sphere goes on revolving.

Now the sphere will ceases to rotate when $\dot{\theta}=0$.

$\therefore$ From (5), we get $0=-gat\sin\alpha+k^2\Omega$

or $t=\frac{k^2\Omega}{ga\sin\alpha}$ ...(6)

For sphere $k^2=\frac{2}{5}a^2$. Putting in (6), we see that the sphere will remain at rest for a time $\frac{2}{5}\frac{a\Omega}{g\sin\alpha}$

Now when $\dot{x}$ and $a\dot{\theta}$ become zero, the velocity of the point of contact $(\dot{x}+a\dot{\theta})$ becomes zero, therefore pure rolling may commence provided the friction is sufficient for pure rolling. Let F be the friction sufficient for pure rolling. $\therefore$ The equations of motion are

$M\ddot{y}=Mg\sin\alpha-F$ ...(7)

$Mk^2\ddot{\phi}=Fa$ ...(8) and $\dot{y}-a\dot{\phi}=0$ ...(9)

From (9), we get $\dot{y}=a\dot{\phi}$ $\therefore \ddot{y}=a\ddot{\phi}$.

$\therefore$ From (9), $M(k^2/a)\ddot{y}=Fa$ or $M\ddot{y}=F(a^2/k^2)$

$\therefore$ From (7), we get $F=\frac{Mg\sin\alpha}{1+(a^2/k^2)}$, obviously $F<Mg\sin\alpha$.

But $\mu R=\tan\alpha.Mg\cos\alpha=Mg\sin\alpha$

$\therefore F<\mu R$, the rolling continues and the equations (7), (8) and (9) hold good.

$\therefore$ From (7) $M\ddot{y}=Mg\sin\alpha-\frac{Mg\sin\alpha}{1+(a^2/k^2)}$

or $\ddot{y}=\frac{ga^2\sin\alpha}{(a^2+k^2)}$ ...(10)

But $k^2=\frac{2}{5}a^2$, $\ddot{y}=\frac{5}{7}g\sin\alpha$.

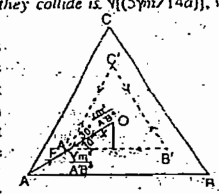
For circular hoop. If it was circular hoop instead of sphere then $k^2=a^2$.

$\therefore$ From (6) and (10) we get

$t=a\Omega/(g\sin\alpha)$ and $\ddot{y}=\frac{1}{2}g\sin\alpha$.

i.e. the hoop will remain at rest for a time $a\Omega/(g\sin\alpha)$ and then move downwards with an acceleration $\frac{1}{2}g\sin\alpha$.

Ex: 19. *Three uniform spheres, each of radius a and of mass m atract one another according to the law of the inverse square of the distance. Initially they are placed on a perfectly rough horizontal plane with their centres forming a triangle whose sides are each of lenght $4a$. Show that the velocity of their centres when they collide is $\sqrt{[(5\gamma m/14a)]}$, where γ is the constant of gravitation.*



Sol. Let A, B, C be the points of contact of the spheres with the horizontal plane, when they were at rest initially such that ABC is an equilateral triangle of side $4a$. Let O be the centre of the triangle ABC.

Due to the symmetry of the attraction, the spheres will move such that their points of contact with the horizontal plane always form an equilateral triangle.

At time t, let A', B', C' be the positions of the points of contact of the spheres with the horizontal plane such that $OA'=x$.

Now $A'L=OA'\cos 30^\circ$

or $\frac{1}{2}A'B'=x\sqrt{\left(\frac{3}{2}\right)}$. $\therefore x=\frac{1}{\sqrt{3}}A'B'$.

Force of attraction between spheres at A' and $B'=\gamma m^2/A'B'^2$ and that between spheres at A' and C' is $\gamma m^2/A'C'^2$.

Now the force of attraction on the sphere at A' due to the other two spheres at B' and C'

$=\left(\frac{\gamma m^2}{A'B'^2}\cos 30^\circ+\frac{\gamma m^2}{A'C'^2}\cos 30^\circ\right)$ in the direction $A'O$

$=\frac{\gamma m^2}{3x^2}.\frac{\sqrt{3}}{2}+\frac{\gamma m^2}{3x^2}.\frac{\sqrt{3}}{2}$ $(A'B'=A'C'=\sqrt{3}x)$

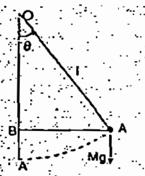
$=\frac{\gamma m^2}{\sqrt{3}x^2}$ in the direction $A'O$ (*i.e.* towards x-decreasing).

(i) $\therefore$ The Lagrangian function

$L = T - V = \frac{1}{2}Ml^2\dot{\theta}^2 - Mgl(1 - \cos\theta)$

(ii) Lagrange's θ-equation is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \text{ i.e. } \frac{d}{dt}(Ml^2\dot{\theta}) + Mgl\sin\theta = 0$$



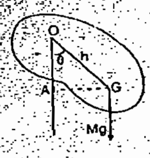
or $Ml^2\ddot{\theta} + Mgl\sin\theta = 0$ or

$\ddot{\theta} = -(g/l)\sin\theta$ or $\ddot{\theta} = -(g/l)\theta$, Since θ is small.

Which is the required equation of motion.

Ex. 2. *Use Lagrange's equations to find the equation of motion of the compound pendulum which oscillates in a vertical plane about a fixed horizontal axis.*

Sol. Let the vertical plane through the C.G. of the pendulum meet the horizontal axis of rotation at O. Let $OG = h$. Let OG make an angle θ to the vertical at time t. Thus θ is the only generalised coordinate. If k is the radius of gyration of the pendulum about the axis of rotation through O, then



K.E., $T = \frac{1}{2}Mk^2\dot{\theta}^2$

And the potential energy relative to the horizontal plane through O is

$V = -Mgh\cos\theta$.

$\therefore$ Lagrange's θ equation is

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = -\frac{\partial V}{\partial \theta} \text{ i.e. } \frac{d}{dt}(Mk^2\dot{\theta}) = -Mgh\sin\theta$$

or $Mk^2\ddot{\theta} = -Mgh\sin\theta$ or $\ddot{\theta} = -(gh/k^2)\theta$.

Since θ is small which is the required equation of motion.

Ex. 3. *A particle of mass m moves in a conservative forces field. Find the Lagrangian function and (ii) the equation of motion in cylindrical coordinates* (ρ, ϕ, z).

Sol. Let P be the position of the particle of mass m whose cylindrical coordinates referred to axes OX, OY, OZ are (ρ, ϕ, z).

$\therefore$ If (x, y, z) are its cartesian coordinates, then

$x = OA = \rho\cos\phi$.

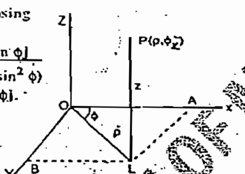
$y = OB = \rho\sin\phi$, $z = z$.

If $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are the unit vectors along OX, OY, OZ respectively, then

$$\vec{OP} = \mathbf{r} = \rho\cos\phi\,\mathbf{i} + \rho\sin\phi\,\mathbf{j} + z\mathbf{k}$$

If $\hat{\rho}_1$ and $\hat{\phi}_1$ are the unit vectors in the directions of ρ and ϕ increasing respectively then

$$\hat{\rho}_1 = \frac{\partial \mathbf{r}}{\partial \rho} \Big/ \left|\frac{\partial \mathbf{r}}{\partial \rho}\right| = \frac{\cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}}{\sqrt{(\cos^2\phi + \sin^2\phi)}} = \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}.$$



$$\hat{\phi}_1 = \frac{\partial \mathbf{r}}{\partial \phi} \Big/ \left|\frac{\partial \mathbf{r}}{\partial \phi}\right| = \frac{-\rho\sin\phi\,\mathbf{i} + \rho\cos\phi\,\mathbf{j}}{\sqrt{(\rho^2\sin^2\phi + \rho^2\cos^2\phi)}} = -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}.$$

Now $\mathbf{v} = \dot{\mathbf{r}} = (\dot{\rho}\cos\phi - \rho\sin\phi\dot{\phi})\,\mathbf{i} + (\dot{\rho}\sin\phi + \rho\cos\phi\dot{\phi})\,\mathbf{j} + \dot{z}\mathbf{k}$

$= \dot{\rho}(\cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}) + \rho\dot{\phi}(-\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}) + \dot{z}\mathbf{k}$

$= (\dot{\rho})\hat{\rho}_1 + (\rho\dot{\phi})\hat{\phi}_1 + \dot{z}\,\mathbf{k}$

$\therefore v^2 = \dot{\rho}^2 + (\rho\dot{\phi})^2 + \dot{z}^2$

Total K.E., $T = \frac{1}{2}mv^2 = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2)$

Let $V = V(\rho, \phi, z)$ be the potential function.

$\therefore$ (i) Lagrangian function, $L = T - V$

i.e. $L = \frac{1}{2}m(\dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2) - V(\rho, \phi, z)$.

(ii) Lagrange's ρ equation is $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\rho}}\right) - \frac{\partial L}{\partial \rho} = 0$

or $\frac{d}{dt}(m\dot{\rho}) - \left(m\rho\dot{\phi}^2 - \frac{\partial V}{\partial \rho}\right) = 0$

i.e. $m\ddot{\rho} - m\rho\dot{\phi}^2 = -\frac{\partial V}{\partial \rho}$ ...(1)

Lagrange's ϕ equation is $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0$

or $\frac{d}{dt}(m\rho^2\dot{\phi}) - \left(-\frac{\partial V}{\partial \phi}\right) = 0$ or $\frac{d}{dt}(m\rho^2\dot{\phi}) = -\frac{\partial V}{\partial \phi}$ ...(2)

and Lagrange's z equation is $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0$

or $\frac{d}{dt}(m\dot{z}) - \left(-\frac{\partial V}{\partial z}\right) = 0$ or $m\ddot{z} = -\frac{\partial V}{\partial z}$ ...(3)

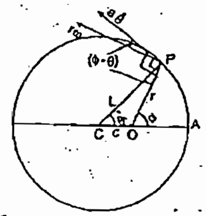
Ex. 4. *A particle P moves on a smooth horizontal circular wire of radius a which is free to rotate about a vertical axis through a point O, distance c from the centre C. If the* $\angle PCO = \theta$, *show that*

$$a\ddot{\theta} + \dot{\omega}(a - c\cos\theta) = c\omega^2\sin\theta,$$

Where ω is the angular velocity of the wire.

Sol. Let M be the mass of the particle moving on a smooth circular wire which is free to rotate about the vertical axis through O s.t. $CO = c$. At time t, let the particle be at P, s.t. $\angle PCO = \theta$.

Let $OP = r$ and $\angle POA = \phi$.



$\therefore$ In ΔOCP,

$OP^2 = CO^2 + CP^2 - 2CO.CP\cos\theta$

or $r^2 = c^2 + a^2 - 2ca\cos\theta$ ...(1)

Also $CP = CL + PL = c\cos\theta + r\cos(\phi - \theta)$, $\therefore \angle CPO = \phi - \theta$

$\therefore r\cos(\phi - \theta) = a - c\cos\theta$ $\because CP = a$.

The particle moves on the circle on account of which its velocity is $a\dot{\theta}$ along the tangent at P. Also as the circle revolves, P also revolve about the fixed point O, due to which its velocity is $r\omega$ perpendicular to OP. The angle between these two velocities of P is $\phi - \theta$. Thus if v_P is the resultant velocity of P, then

$v_P^2 = (a\dot{\theta})^2 + (r\omega)^2 + 2a\dot{\theta}.r\omega.\cos(\phi - \theta)$

$= a^2\dot{\theta}^2 + (c^2 + a^2 - 2ca\cos\theta)\,\omega^2 + 2a\omega\dot{\theta}(a - c\cos\theta)$

$\therefore$ Total K.E., $T = \frac{1}{2}M.v_P^2$

$$= \tfrac{1}{2}M\,[a^2\dot{\theta}^2 + (c^2 + a^2 - 2ac\cos\theta)\,\omega^2 + 2a\omega\dot{\theta}(a - c\cos\theta)]$$

and work function, $W = 0$, Since Mg acts vertically while particle moves in the horizontal plane.

$\therefore$ Lagrange's θ-equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = \frac{\partial W}{\partial \theta}$

i.e. $\frac{d}{dt}\left[\frac{1}{2}M\left\{2a^2\dot{\theta} + 2a\omega(a - c\cos\theta)\right\}\right]$

$$-\tfrac{1}{2}M(2ac\sin\theta\,\omega^2 + 2a\omega c\dot{\theta}\sin\theta) = 0$$

or $Ma\,[a\ddot{\theta} + \dot{\omega}(a - c\cos\theta) + \omega c\sin\theta\dot{\theta}]$

$$- Mac\sin\theta\,\omega^2 - Ma\omega c\dot{\theta}\sin\theta = 0$$

or $a\ddot{\theta} + \dot{\omega}(a - c\cos\theta) = c\omega^2\sin\theta$.

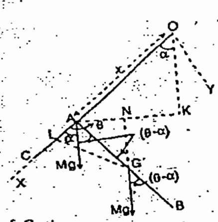
Ex. 5. *A bead of mass M, slides on a smooth fixed wire, whose inclination to the vertical is* α, *and has hinged to it a rod of mass m and length 2l, which can move freely in the vertical plane through the wire. If the system starts from rest with the rod hanging vertically, show that*

$$\{4M + m(1 + 3\cos^2\theta)\}\,l\dot{\theta}^2 = 6(M + m)g\sin\alpha(\sin\theta - \sin\alpha).$$

where θ *is the angle between the rod and the lower part of the wire.*

Sol. Let OC be the fixed wire whose inclination to the vertical is α. At time t, let the bead of mass M be at A and the rod AB of mass m inclined at angle θ to the lower part of the wire. Initially the bead was at O and the rod was hanging vertically. Let $OA = x$.

Taking O as origin wire OC as X-axis and the line OY perpendicular to OC as Y-axis, the coordinates (x_G, y_G) of the C.G. 'G' of the rod are given by



$x_G = OA + AL = x + l\cos\theta$

and $y_G = GL = l\sin\theta$. If v_G is vel. of G, then

$v_G^2 = \dot{x}_G^2 + \dot{y}_G^2 + (\dot{x} - l\sin\theta\dot{\theta})^2 + (l\cos\theta\dot{\theta})^2$

If T is the total kinetic energy and W the work function of the system, then

T = K.E. of the bead + K.E. of the rod

$= \frac{1}{2}M\dot{x}^2 + [\frac{1}{2}m\frac{1}{3}l^2\dot{\theta}^2 + \frac{1}{2}mv_G^2]$

$= \frac{1}{2}M\dot{x}^2 + \frac{1}{2}m\,[\frac{1}{3}l^2\dot{\theta}^2 + (\dot{x} - l\sin\theta\dot{\theta})^2 + (l\cos\theta\dot{\theta})^2]$

$= \frac{1}{2}(M + m)\dot{x}^2 - ml\dot{x}\dot{\theta}\sin\theta + \frac{2}{3}ml^2\dot{\theta}^2$

and $W = Mg.OK + mg(OK + NG - l)$

$= Mgx\cos\alpha + mg\,[x\cos\alpha + l\cos(\theta - \alpha) - l]$

$\because \angle AGN = \theta - \alpha$

$= (M + m)gx\cos\alpha + mgl\,[\cos(\theta - \alpha) - 1]$

$\therefore$ Lagrange's x-equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{x}}\right) - \frac{\partial T}{\partial x} = \frac{\partial W}{\partial x}$

i.e. $\frac{d}{dt}[(M + m)\dot{x} - ml\dot{\theta}\sin\theta] - 0 = (M + m)g\cos\alpha$

or $(M + m)\ddot{x} - ml\ddot{\theta}\sin\theta - ml\dot{\theta}^2\cos\theta = (M + m)g\cos\alpha$ ...(1)

And Lagrange's θ-equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = \frac{\partial W}{\partial \theta}$

i.e. $\frac{d}{dt}\left[-ml\dot{x}\sin\theta+\frac{4}{3}ml^2\dot{\theta}\right]+ml\dot{x}\dot{\theta}\cos\theta=-mgl\sin(\theta-\alpha)$.

or $-ml\ddot{x}\sin\theta-ml\dot{x}\cos\theta\dot{\theta}+\frac{4}{3}ml^2\ddot{\theta}+ml\dot{x}\dot{\theta}\cos\theta=-mgl\sin(\theta-\alpha)$

or $-\ddot{x}\sin\theta+\frac{4}{3}l\ddot{\theta}=-g\sin(\theta-\alpha)$ ...(2)

In order to eliminate $\ddot{x}$ between (1) and (2), multiplying (1) by sin θ and (2) by $(M+m)$ and adding, we get

$-ml\ddot{\theta}\sin^2\theta-ml\dot{\theta}^2\sin\theta\cos\theta+\frac{4}{3}l(M+m)\ddot{\theta}$

$=(M+m)g\cos\alpha\sin\theta-g(M+m)\sin(\theta-\alpha)$

or $(4M+4m-3m\sin^2\theta)l\ddot{\theta}-3ml\dot{\theta}^2\sin\theta\cos\theta=3(M+m)g\cos\theta\sin\alpha$

or $\frac{d}{dt}\{4M+m(1+3\cos^2\theta)l\dot{\theta}^2\}=6(M+m)g\cos\theta\sin\alpha.\dot{\theta}$

Integrating both sides w.r.t. 't' we get

$\{4M+m(1+3\cos^2\theta)\}l\dot{\theta}^2=6(M+m)g\sin\alpha\sin\theta+C$ ...(3)

But initially when the bead was at O, $\theta=\alpha$ and $\dot{\theta}=0$

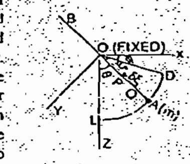
∴ $C=-6(M+m)g\sin^2\alpha$.

∴ Hence from (3), we get

$\{4M+m(1+3\cos^2\theta)\}l\dot{\theta}^2=6(M+m)g\sin\alpha(\sin\theta-\sin\alpha)$.

Ex. 6. A uniform rod, of mass 3m and length 2l, has its middle point fixed and a mass m attached at one extremity. The rod when in a horizontal position is set rotating about a vertical axis through its centre with an angular velocity equal to $\sqrt{(2ng/l)}$. Show that the heavy end of the rod will fall till the inclination of the rod to the vertical is $\cos^{-1}[\sqrt{(n^2+1)}-n]$, and will then rise again. **(IAS-2008 model)**

Sol. Let AB be the rod of mass $3m$ and length $2l$. The middle point O of the rod is fixed and a mass m attached at the extremity A. Initially let the rod rest along OX in the plane of the paper. Let a line OY perpendicular to the plane of the paper and a line OZ perpendicular to OX in the plane of the paper be taken as axes of Y and Z respectively. At time t, let the rod turn through an angle ϕ to OX i.e. the plane OAL containing the rod and Z axis make an angle ϕ with X-Z plane. And let θ be the inclination of the rod with OZ at this time t.



If P is a point of the rod at a distance $OP=\xi$, from O then coordinates of P are given by

$$x_p=\xi\sin\theta\cos\phi,\ y_p=\xi\sin\theta\sin\phi,\ z_p=\xi\cos\theta.$$

∴ If v_p and v_A are the velocities of the point P and A respectively, then

$v_p^2=\dot{x}_p^2+\dot{y}_p^2+\dot{z}_p^2=(\xi\cos\theta\cos\phi\dot{\theta}-\xi\sin\theta\sin\phi\dot{\phi})^2$

$+(\xi\cos\theta\sin\phi\dot{\theta}+\xi\sin\theta\cos\phi\dot{\phi})^2+(-\xi\sin\theta\dot{\theta})^2$

$=\xi^2(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta)$

∴ At A, $\xi=OA=l$, ∴ $v_A^2=l^2(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta)$.

Let $PQ=\delta\xi$ be an element of the rod at P, then mass of this element,

$\delta m=\frac{3m}{2l}\delta\xi$.

∴ K.E. of the element $PQ=\frac{1}{2}\delta m.v_p^2=\frac{1}{2}\xi^2(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta).\frac{3m}{2l}\delta\xi$

∴ K.E. of the rod $AB=\frac{3m}{4l}\int_{-l}^{l}\xi^2(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta)d\xi$

$=\frac{1}{2}ml^2(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta)$

and K.E. of mass m at $A=\frac{1}{2}mv_A^2=\frac{1}{2}ml^2(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta)$

∴ The total kinetic energy of the system

T = K.E. of the rod + K.E. of the particle

$=ml^2(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta)$

The work function $W=mg.z_A=mgl\cos\theta$.

∴ Lagrange's θ-equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right)-\frac{\partial T}{\partial\theta}=\frac{\partial W}{\partial\theta}$

i.e. $\frac{d}{dt}(2ml^2\dot{\theta})-2ml^2\dot{\phi}^2\sin\theta\cos\theta=-mgl\sin\theta$

or $2l\ddot{\theta}-2l\dot{\phi}^2\sin\theta\cos\theta=-g\sin\theta$ ...(1)

And Lagrange's φ-equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\phi}}\right)-\frac{\partial T}{\partial\phi}=\frac{\partial W}{\partial\phi}$

i.e. $\frac{d}{dt}(2ml^2\dot{\phi}\sin^2\theta)=0$ or $\frac{d}{dt}(\dot{\phi}\sin^2\theta)=0$ ...(2)

Integrating (2) we get $\dot{\phi}\sin^2\theta=C$ (Const.)

But initially when $\theta=(\pi/2)$ (∵ Rod was horizontal),

$$\dot{\phi}=\sqrt{(2ng/l)}$$

∴ $C=\sqrt{(2ng/l)}$ ∴ $\dot{\phi}\sin^2\theta=\sqrt{(2ng/l)}$

Substituting the value of $\dot{\phi}$ from (3) in (1), we get

$2l\ddot{\theta}-2l.\frac{2ng}{l\sin^4\theta}\sin\theta\cos\theta=-g\sin\theta$

or $2l\ddot{\theta}-4ng\cot\theta\operatorname{cosec}^2\theta=-g\sin\theta$ ...(4)

Multiplying both sides by $\dot{\theta}$ and integrating, we get

$l\dot{\theta}^2+2ng\cot^2\theta=g\cos\theta+D$.

But initially when $\theta=\pi/2$, $\dot{\theta}=0$ ∴ $D=0$

∴ $l\dot{\theta}^2+2ng\cot^2\theta=g\cos\theta$ ...(5)

The rod will fall till $\dot{\theta}=0$

i.e. $2ng\cot^2\theta=g\cos\theta$ or $2n\cos^2\theta-\cos\theta\sin^2\theta=0$

or $\cos\theta(2n\cos\theta-\sin^2\theta)=0$.

∴ either $\cos\theta=0$ i.e. $\theta=(\pi/2)$.

or $2n\cos\theta-\sin^2\theta=0$ i.e. $2n\cos\theta-(1-\cos^2\theta)=0$

or $\cos^2\theta+2n\cos\theta-1=0$, ∴ $\cos\theta=\frac{-2n\pm\sqrt{(4n^2+4)}}{2}$

or $\cos\theta=-n+\sqrt{(n^2+1)}$, Leaving negative sign.

∵ negative value of cos θ is inadmissible as θ can not be obtuse.

∴ $\theta=\cos^{-1}[\sqrt{(n^2+1)}-n]$.

From (4) we have $2l\ddot{\theta}=\frac{g(4n\cos\theta-\sin^4\theta)}{\sin^3\theta}$ ...(6)

When $\cos\theta=-n+\sqrt{(n^2+1)}$, $\cos^2\theta=2n^2+1-2n\sqrt{(n^2+1)}$

∴ $4n\cos\theta-\sin^4\theta=4n\cos\theta-(1-\cos^2\theta)^2$

$=4n[-n+\sqrt{(n^2+1)}]-[-2n^2+2n\sqrt{(n^2+1)}]^2$

$=-4n^2+4n\sqrt{(n^2+1)}-4n^4-4n^2(n^2+1)+8n^3\sqrt{(n^2+1)}$

$=-8n^2-8n^4+4n\sqrt{(n^2+1)}+8n^3\sqrt{(n^2+1)}$

$=4n\sqrt{(n^2+1)}[-2n\sqrt{(n^2+1)}+1+2n^2]$

$=4n\sqrt{(n^2+1)}[-n+\sqrt{(n^2+1)}]^2$, which is positive

∵ θ is acute angle ∴ $\sin^3\theta$ is also positive

∴ when $\theta=\cos^{-1}[\sqrt{(n^2+1)}-n]$, from (6), we see that $\ddot{\theta}$ is positive. Hence from this position the rod will rise again.

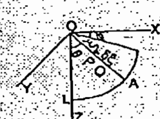
Ex. 7. A uniform rod, of length 2a, can turn freely about one end, which is fixed. Initially it is inclined at an angle α to the downward drawn vertical and it is set rotating about a vertical axis through its fixed end with angular velocity ω. Show that, during the motion the rod is always inclined to the vertical at an angle which is > or < α, according as $\omega^2<\text{or}>\frac{3g}{4a\cos\alpha)}$, and that in each case its check motion is included between the inclination α and

$\cos^{-1}[-n+\sqrt{(1-2n\cos\alpha+n^2)}]$, where $n=a\omega^2\sin^2\alpha/3g$.

If it be slightly disturbed when revolving steadily at a constant angle α, show that the time of a small oscillation is

$$2\pi\sqrt{\left[\frac{4a\cos\alpha}{(3g(1+3\cos^2\alpha))}\right]}$$

Sol. Let OA be the rod of length $2a$ and mass M which can turn freely about one end O which is fixed. Let the horizontal and vertical lines in the plane of paper be taken as the axis of x and axis of z respectively and the y axis OY perpendicular to the plane of the paper. Initially let the rod be in the X-Y plane inclined at an angle α to the vertical (i.e. with OZ). At time t let the rod make an angle θ to OZ and the plane AOZ through the rod and OZ make an angle φ to X-Z plane.



If P is a point of the rod at a distance $OP=\xi$, from O, then coordinates of P are given by

$x_P=\xi\sin\theta\cos\phi$, $y_P=\xi\sin\theta\sin\phi$, $z_P=\xi\cos\theta$.

∴ If v_P is the velocity of the point P, then

$v_P^2=\dot{x}_P^2+\dot{y}_P^2+\dot{z}_P^2=\xi^2(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta)$

Let $PQ=\delta\xi$ be an element of the rod at P, then mass of this element,

$\delta m=\frac{M}{2a}\delta\xi$

K.E. of this element $PQ=\frac{1}{2}\delta m.v_P^2=\frac{1}{2}\xi^2(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta)\frac{M}{2a}\delta\xi$

K.E. of the rod OA, $T=\frac{M}{4a}\int_0^{2a}\xi^2(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta)d\xi$

$=\frac{2}{3}Ma^2(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta)$

And the rork function $W=Mg(a\cos\theta-a\cos\alpha)$

Lagrange's θ-equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right)-\frac{\partial T}{\partial\theta}=\frac{\partial W}{\partial\theta}$

i.e. $\frac{d}{dt}\left(\frac{4}{3}Ma^2\dot{\theta}\right)-\frac{4}{3}Ma^2\dot{\phi}^2\sin\theta\cos\theta=-Mga\sin\theta$

or $4a\ddot{\theta}-4a\dot{\phi}^2\sin\theta\cos\theta=-3g\sin\theta$

And Lagrange's φ equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\phi}}\right)-\frac{\partial T}{\partial\phi}=\frac{\partial W}{\partial\phi}$

i.e. $\frac{d}{dt}\left(\frac{4}{3} Ma^2\dot{\phi}\sin^2\theta\right)=0$ or $\frac{d}{dt}(\dot{\phi}\sin^2\theta)=0$ ...(2)

Integrating (2), $\dot{\phi}\sin^2\theta = C$ (const.)

But initially when $\theta=\alpha$, $\dot{\phi}=\omega$ $\therefore$ $C=\omega\sin^2\alpha$

$\therefore$ $\dot{\phi}\sin^2\theta=\omega\sin^2\alpha$ ...(3)

Substituting the value of $\dot{\phi}$ from (3) in (1), we get

$$4a\ddot{\theta}-4a.\frac{\omega^2\sin^4\alpha}{\sin^4\theta}\sin\theta\cos\theta=-3g\sin\theta$$

or $4a\ddot{\theta}=4a\omega^2\sin^4\alpha\cot\theta\operatorname{cosec}^2\theta-3g\sin\theta$ ...(4)

Multiplying both sides by $\dot{\theta}$ and integrating, we get

$2a\dot{\theta}^2=4a\omega^2\sin^4\alpha(-\frac{1}{2}\operatorname{cosec}^2\theta)+3g\cos\theta+D$ ...(5)

But initially $\theta=\alpha$ and $\dot{\theta}=0$

$\therefore$ $D=2a\omega^2\sin^4\alpha\operatorname{cosec}^2\alpha-3g\cos\alpha=2a\omega^2\sin^2\alpha-3g\cos\alpha.$

$\therefore$ from (5) we get

$$2a\dot{\theta}^2=2a\omega^2\sin^2\alpha\left(1-\frac{\sin^2\alpha}{\sin^2\theta}\right)+3g(\cos\theta-\cos\alpha)$$

$$=2a\omega^2\frac{\sin^2\alpha}{\sin^2\theta}(\sin^2\theta-\sin^2\alpha)+3g(\cos\theta-\cos\alpha)$$

$$=2a\omega^2\frac{\sin^2\alpha}{\sin^2\theta}(\cos^2\alpha-\cos^2\theta)+3g(\cos\theta-\cos\alpha)$$

$$=\frac{3g(\cos\alpha-\cos\theta)}{\sin^2\theta}\left[\frac{2a\omega^2}{3g}\sin^2\alpha(\cos\alpha+\cos\theta)-\sin^2\theta\right]$$

$$=[2n(\cos\alpha+\cos\theta)-(1-\cos^2\theta)]\quad ...(6)$$

where $n=(a\omega^2/3g)\sin^2\alpha$.

From (6), we see that $\dot{\theta}=0$, when

$3g(\cos\alpha-\cos\theta)[2n(\cos\alpha+\cos\theta)-1+\cos^2\theta]=0$

$\therefore$ either $\cos\alpha-\cos\theta=0$

$\therefore$ $\theta=\alpha$, which is the initial position.

or $2n(\cos\alpha+\cos\theta)-1+\cos^2\theta=0$

i.e. $\cos^2\theta+2n\cos\theta+(2n\cos\alpha-1)=0$

$$\therefore\ \cos\theta=\frac{-2n\pm\sqrt{[4n^2-4.1.(2n\cos\alpha-1)]}}{2}$$

or $\cos\theta=-n+\sqrt{(1-2n\cos\alpha+n^2)}$

Leaving –ve sign, $\because$ negative value of $\cos\alpha$ is inadmissible as θ can not be obtuse.

$\therefore$ $\theta=\cos^{-1}[-n+\sqrt{(1-2n\cos\alpha+n^2)}]$ ...(7)

Thus the motion is included between $\theta=\alpha$ and $\theta=\theta_1$ given by (7)

The rod is always inclined to the vertical at an angle θ_1, such that $\theta_1 >$ or $<\alpha$

if $\cos\theta_1 >$ or $<\cos\alpha$

or if $-n+\sqrt{(1-2n\cos\alpha+n^2)} >$ or $<\cos\alpha$

or if $1-2n\cos\alpha+n^2 >$ or $<(n+\cos\alpha)^2$

or if $1-\cos^2\alpha >$ or $<4n\cos\alpha$

or if $\sin^2\alpha >$ or $<4.(a\omega^2/3g)\sin^2\alpha\cos\alpha$

or if $\omega^2 <$ or $> 3g/(4a\cos\alpha)$.

2nd Part. Small oscillation about the steady motion.

The motion will be steady if the rod goes round inclined at the same angle α with the vertical.

i.e. if $\theta=\alpha$, throughout the motion so that $\ddot{\theta}=0$

Putting $\theta=\alpha$ and $\ddot{\theta}=0$ in (4), we get

$4a\omega^2\sin^4\alpha\cot\alpha\operatorname{cosec}^2\alpha-3g\sin\alpha=0$

or $\omega^2=3g/(4a\cos\alpha)$

Now when $\omega^2=3g/(4a\cos\alpha)$ and there are small oscillations about the position $\theta=\alpha$, then putting $\theta=\alpha+\psi$ and $\omega^2=3g/(4a\cos\alpha)$ in (4), we get

$$4a\ddot{\psi}=4a\frac{3g}{4a\cos\alpha}\frac{\sin^4\alpha\cot(\alpha+\psi)}{\sin^3(\alpha+\psi)}-3g\sin(\alpha+\psi)$$

$$\text{or }\ddot{\psi}=\frac{3g}{4a}\left[\frac{\sin^4\alpha\cos(\alpha+\psi)}{\cos\alpha\sin^3(\alpha+\psi)}-\sin(\alpha+\psi)\right]$$

$$=\frac{3g}{4a}\left[\frac{\sin^4\alpha(\cos\alpha\cos\psi-\sin\alpha\sin\psi)}{\cos\alpha(\sin\alpha\cos\psi+\cos\alpha\sin\psi)^3}-(\sin\alpha\cos\psi+\cos\alpha\sin\psi)\right]$$

$$=\frac{3g}{4a}\left[\frac{\sin^4\alpha(\cos\alpha-\psi\sin\alpha)}{\cos\alpha(\sin\alpha+\psi\cos\alpha)^3}-(\sin\alpha+\psi\cos\alpha)\right]\quad(\because\ \psi\text{ is small})$$

$$=\frac{3g}{4a}\sin\alpha[(1-\psi\tan\alpha)(1+\psi\cot\alpha)^{-3}-(1+\psi\cot\alpha)]$$

$$=\frac{3g}{4a}\sin\alpha[(1-\psi\tan\alpha)(1-3\psi\cot\alpha)-(1+\psi\cot\alpha)]$$

Neglecting squares and higher powers of ψ.

$$=\frac{3g}{4a}\sin\alpha[1-(\tan\alpha+3\cot\alpha)\psi-1-\psi\cot\alpha]$$

$$=-\frac{3g}{4a}\sin\alpha[\tan\alpha+4\cot\alpha]\psi=-\frac{3g}{4a}\left(\frac{\sin^2\alpha+4\cos^2\alpha}{\cos\alpha}\right)\psi$$

$$\text{or }\ddot{\psi}=-\left\{\frac{3g(1+3\cos^2\alpha)}{4a\cos\alpha}\right\}\psi=-\mu\psi.$$

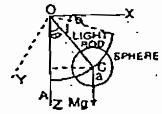
Hence time of small oscillation

$=2\pi/\sqrt{\mu}=2\pi\sqrt{[4a\cos\alpha/\{3g(1+3\cos^2\alpha)\}]}$.

Ex. 8. *A solid uniform sphere has a light rod rigidly attached to it which passes through its centre. This rod is joined to a fixed vertical axis such that the angle θ between the rod and the axis may alter but the rod must turn with the axis. If the vertical axis be forced to revolve constantly with uniform angular velocity, show that the equation of motion is of the form*

$$\dot{\theta}^2=n^2(\cos\theta-\cos\beta)(\cos\alpha-\cos\theta)$$

Show also that the total energy imparted to the sphere as θ increases from θ_1 to θ_2, varies as $\cos^2\theta_1-\cos^2\theta_2$



Sol. Let OA be the fixed vertical axis, OC the light rod of length say l, and C the centre of the sphere. Let a be the radius and M the mass of the sphere. The rod is weighless.

Let z-axis be taken along the vertical line OA, and x-axis perpendicular to it in the plane of the paper. Let OY be the y-axis perpendicular to the plane of the paper.

At time t, let the rod make an angle θ with the axis OZ and during this time let the plane COA turn through an angle ϕ with the XOY plane.

Since the vertical axis revolve with constant angular velocity,

$\therefore$ $\dot{\phi}=\omega$ (constant).

If (x_c, y_c, z_c) are the coordinates of the centre C of sphere, at time t, then

$x_c=l\sin\theta\cos\phi$, $y_c=l\sin\theta\sin\phi$, $z_c=l\cos\theta$.

$$\therefore\ v_c^2=\dot{x}_c^2+\dot{y}_c^2+\dot{z}_c^2$$

$$=(l\dot{\theta}\cos\theta\cos\phi-l\dot{\phi}\sin\theta\sin\phi)^2+(l\dot{\theta}\cos\theta\sin\phi+l\dot{\phi}\sin\theta\cos\phi)^2+(-l\sin\theta\dot{\theta})^2$$

$$=l^2(\dot{\theta}^2+\dot{\phi}^2\sin^2\theta)=l^2(\dot{\theta}^2+\omega^2\sin^2\theta)\quad\because\ \dot{\phi}=\omega$$

If T be the total K.E. and W the work function of the system, then

$$T=\tfrac{1}{2}M.\tfrac{2}{5}a^2\dot{\theta}^2+\tfrac{1}{2}Mv_c^2$$

$$=\tfrac{1}{2}M[(\tfrac{2}{5}a^2+l^2)\dot{\theta}^2+l^2\omega^2\sin^2\theta]\quad ...(1)$$

and $W=Mgl\cos\theta+C$ ...(2)

Lagrange's θ-equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right)-\frac{\partial T}{\partial\theta}=\frac{\partial W}{\partial\theta}$

$$\frac{d}{dt}\left[\tfrac{1}{2}M\left\{\left(\frac{2}{5}a^2+l^2\right)\right\}2\dot{\theta}\right]-\tfrac{1}{2}M.2l^2\omega^2\sin\theta\cos\theta=-Mgl\sin\theta$$

or $(\frac{2}{5}a^2+l^2)\ddot{\theta}=l^2\omega^2\sin\theta\cos\theta-gl\sin\theta$

Multiplying both sides by $2\dot{\theta}$ and integrating, we get

$(\frac{2}{5}a^2+l^2)\dot{\theta}^2=-l^2\omega^2\cos^2\theta+2gl\cos\theta+C_1$ ...(3)

If $\dot{\theta}=0$ when $\theta=\alpha$ and $\theta=\beta$, then from (3), we have

$0=-l^2\omega^2\cos^2\alpha+2gl\cos\alpha+C_1$ and $0=-l^2\omega^2\cos^2\beta+2gl\cos\beta+C_1$.

Subtracting, we get

$0=l^2\omega^2(\cos^2\alpha-\cos^2\beta)-2gl(\cos\alpha-\cos\beta)$

or $2g=l\omega^2(\cos\alpha+\cos\beta)$

$\therefore$ $C_1=l^2\omega^2\cos^2\alpha-2gl\cos\alpha$

$=l^2\omega^2\cos^2\alpha-l^2\omega^2(\cos\alpha+\cos\beta)\cos\alpha$

$=-l^2\omega^2\cos\alpha\cos\beta$.

$\therefore$ Substituting in (3), we get

$$(\tfrac{2}{5}a^2+l^2)\dot{\theta}^2=-l^2\omega^2\cos^2\theta+l\omega^2(\cos\alpha+\cos\beta).l\cos\theta-l^2\omega^2\cos\alpha\cos\beta$$

$$=l^2\omega^2[-\cos^2\theta+(\cos\alpha+\cos\beta)\cos\theta-\cos\alpha\cos\beta]$$

$$=l^2\omega^2(\cos\theta-\cos\beta)(\cos\alpha-\cos\theta)$$

or $\dot{\theta}^2=n^2(\cos\theta-\cos\beta)(\cos\alpha-\cos\theta)$

where $n^2=\dfrac{l^2\omega^2}{(\frac{2}{5}a^2+l^2)}$

2nd Part. If V is the potential energy of the system, then we know that $W=C_2-V$, where C_2 is a constant

$\therefore$ Total energy of the sphere

$=$ K.E. + Pot. energy $=T+V=T-W+C_2$

$=\frac{1}{2}M[(\frac{2}{5}a^2+l^2)\dot{\theta}^2+l^2\omega^2\sin^2\theta]-Mgl\cos\theta-C+C_2$

$=\frac{1}{2}M\{(-l^2\omega^2\cos^2\theta+2gl\cos\theta+C_1)+l^2\omega^2\sin^2\theta\}-Mgl\cos\theta-C+C_2$ [from (3)]

$=\frac{1}{2}Ml^2\omega^2(-\cos^2\theta+\sin^2\theta)+(\frac{1}{2}MC_1-C+C_2)$

$=\frac{1}{2}Ml^2\omega^2(-\cos^2\theta+1-\cos^2\theta)+(\frac{1}{2}MC_1-C+C_2)$

$=-Ml^2\omega^2\cos^2\theta+(\frac{1}{2}Ml^2\omega^2+\frac{1}{2}MC_1-C+C_2)$

$=-Ml^2\omega^2\cos^2\theta+A$ where A is a constant

$\therefore$ Total energy imparted, when θ increases from θ_1 to θ_2

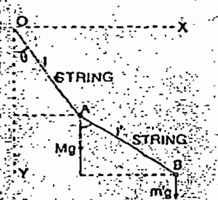
$$= \left[-Ml^2\omega^2\cos^2\theta + A\right]_{\theta_1}^{\theta_2} = Ml^2\omega^2(\cos^2\theta_1 - \cos^2\theta_2)$$

i.e. total energy imparted varies as $(\cos^2\theta_1 - \cos^2\theta_2)$.

Ex. 9. *A mass m hangs from a fixed point by a light string of length l, and a mass m′ hangs from m by a second string of length l′. For oscillations in a vertical plane, show that the period of the principal oscillations are the values of $2\pi/n$ where n is given by the equation*

$$n^4 - n^2\frac{m+m'}{m}\,g\left(\frac{1}{l}+\frac{1}{l'}\right) + g^2\frac{m+m'}{mll'} = 0.$$

Sol. Let OA and AB be the strings of lengths l and l' respectively. The mass at A is m and that at B is m'. At time t let the strings make angles θ and ϕ to the vertical.



Referred to the horizontal and vertical lines OX, OY through O as axes the coordinates of A and B are given by

$x_A = l\sin\theta,\ y_A = l\cos\theta$;

$x_B = l\sin\theta + l'\sin\phi$,

$y_B = l\cos\theta + l'\cos\phi$.

If v_A and v_B are the velicities of m and m' at A and B respectively, then

$v_A^2 = \dot{x}_A^2 + \dot{y}_A^2 = (l\cos\theta\dot\theta)^2 + (l\sin\theta\dot\theta)^2 = l^2\dot\theta^2$

and $v_B^2 = \dot{x}_B^2 + \dot{y}_B^2 = (l\cos\theta\dot\theta + l'\cos\phi\dot\phi)^2 + (-l\sin\theta\dot\theta - l'\sin\phi\dot\phi)^2$

$= l^2\dot\theta^2 + l'^2\dot\phi^2 + 2ll'\dot\theta\dot\phi\cos(\theta-\phi) = l^2\dot\theta^2 + l'^2\dot\phi^2 + 2ll'\dot\theta\dot\phi$.

(As θ and ϕ are small).

If T is the total kinetic energy and W the work function of the system, then

$T = \frac{1}{2}mv_A^2 + \frac{1}{2}ml^2\dot\theta^2 + \frac{1}{2}m'(l^2\dot\theta^2 + l'^2\dot\phi^2 + 2ll'\dot\theta\dot\phi)\ \frac{1}{2}m'v_B^2 =$

or $T = \frac{1}{2}[(m+m')l^2\dot\theta^2 + m'l'^2\dot\phi^2 + 2m'll'\dot\theta\dot\phi]$

and $W = mg\,y_A + m'g\,y_B + C = mgl\cos\theta + m'g(l\cos\theta + l'\cos\phi) + C$

or $W = gl(m+m')\cos\theta + m'gl'\cos\phi + C$.

Lagrange's θ-equation is, $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right) - \frac{\partial T}{\partial\theta} = \frac{\partial W}{\partial\theta}$

i.e. $\frac{d}{dt}[(m+m')l^2\dot\theta + m'll'\dot\phi] - 0 = -gl(m+m')\sin\theta$

or $(m+m')l\ddot\theta + m'l'\ddot\phi = -g(m+m')\theta$ $\quad\because\ \theta$ is small ...(1)

And Lagrange's ϕ-equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\phi}\right) - \frac{\partial T}{\partial\phi} = \frac{\partial W}{\partial\phi}$

i.e. $\frac{d}{dt}(m'l'^2\dot\phi + m'll'\dot\theta) - 0 = -m'gl'\sin\phi$

or $l'\ddot\phi + l\ddot\theta = -g\phi$. $\quad(\because\ \phi$ is small) ...(2)

Equations (1) and (2), can be written as

$(m+m')(lD^2+g)\theta + m'l'D^2\phi = 0$ ...(3)

$lD^2\theta + (l'D^2+g)\phi = 0$ ...(4)

Eliminating ϕ, between (3) and (4), we get

$[(m+m')(lD^2+g)(l'D^2+g) - m'll'D^4]\theta = 0$

or $[mll'D^4 + (m+m')(l+l')gD^2 + (m+m')g^2]\theta = 0$ ...(5)

If $(2\pi/n)$ is the period of principal oscillation, then solution of (5) must be

$\theta = A\cos(nt+B)$, $\therefore\ D\theta = -An\sin(nt+B)$,

$D^2\theta = -An^2\cos(nt+B) = -n^2\theta$, $D^4\theta = -n^2D^2\theta = n^4\theta$

Substituting in (5), we get

$[mll'n^4 - (m+m')(l+l')gn^2 + (m+m')g^2]\theta = 0$

or $n^4 - n^2\frac{m+m'}{m}g\left(\frac{1}{l}+\frac{1}{l'}\right) + g^2\frac{m+m'}{mll'} = 0$. $\quad\because\ \theta \neq 0$

Ex. 10. *A mass M hangs from a fixed point at the end of a very long string whose length is a; to M is suspended a mass m by a string whose length l is small compared with a; prove that the time of a small oscillation of m is* $2\pi\sqrt{\left(\frac{M}{M+m}\cdot\frac{l}{g}\right)}$

Sol. Proceed exactly as in Ex. 9.

Here $m = M$, $m' = m$, $l = a$ (very large), $l' = l$.

$\therefore$ From the result of Ex. 9, we get

$$n^4 - n^2\left(\frac{M+m}{M}\right)\left(\frac{1}{a}+\frac{1}{l}\right)g + \frac{g^2(M+m)}{Mal} = 0$$

$$\text{or } n^4 - n^2\left(\frac{M+m}{M}\right)\left(\frac{l}{a}+1\right)\cdot\frac{g}{l} + \frac{(M+m)g^2}{Ml^2}\cdot\frac{l}{a} = 0$$

Since a is large compaired to l $\therefore\ \frac{l}{a} \to 0$.

Hence taking $\frac{l}{a} = 0$, we get

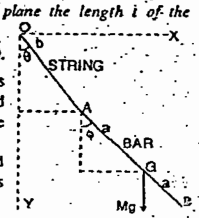
$$n^4 - n^2\left(\frac{M+m}{M}\right)1.\frac{g}{l} = 0 \text{ or } n^2 = \left(\frac{M+m}{M}\right)\frac{g}{l}$$

$\therefore$ Time of a small oscillation of m is

$$\frac{2\pi}{n} \text{ i.e. } 2\pi\sqrt{\left(\frac{M}{M+m}\cdot\frac{l}{g}\right)}$$

Ex. 11. *A uniform bar of length 2a is hung from a fixed point by a string of length b fastened to one end of the bar show that when the system makes small normal oscillations in a vertical plane the length l of the equivalent pendulum is a root of the quadratic.*

$$l^2 - (\tfrac{4}{3}a + b)l + \tfrac{1}{3}ab = 0.$$



Sol. Let AB be the bar of length $2a$ and mass M, and OA the string of length b and O the fixed point. At time t let the string and the bar make angles θ and ϕ to the vertical respectively.

Referred to O as origin, horizontal and vertical lines OX and OY as axes the coordinates of the C.G. 'G' of the rod are given by

$x_G = b\sin\theta + a\sin\phi$ and

$y_G = b\cos\theta + a\cos\phi$

$\therefore\ v_G^2 = \dot{x}_G^2 + \dot{y}_G^2 = (b\cos\theta\dot\theta + a\cos\phi\dot\phi)^2 + (-b\sin\theta\dot\theta - a\sin\phi\dot\phi)^2$

$= b^2\dot\theta^2 + a^2\dot\phi^2 + 2ab\dot\theta\dot\phi\cos(\theta-\phi)$

$= b^2\dot\theta^2 + a^2\dot\phi^2 + 2ab\dot\theta\dot\phi$. $\quad$ as θ and ϕ are small.

If T be the total K.E. and W the work function of the system, then

$T = \frac{1}{2}M\frac{1}{3}a^2\dot\phi^2 + \frac{1}{2}Mv_G^2 = \frac{1}{2}M(\frac{1}{3}a^2\dot\phi^2 + b^2\dot\theta^2 + a^2\dot\phi^2 + 2ab\dot\theta\dot\phi)$

or $T = \frac{1}{2}M(\frac{4}{3}a^2\dot\phi^2 + b^2\dot\theta^2 + 2ab\dot\theta\dot\phi)$

and $W = Mgy_G + C = Mg(b\cos\theta + a\cos\phi) + C$.

Lagrange's θ-equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right) - \frac{\partial T}{\partial\theta} = \frac{\partial W}{\partial\theta}$

i.e. $\frac{d}{dt}(Mb^2\dot\theta + Mab\dot\phi) - 0 = -Mgb\sin\theta$

or $b\ddot\theta + a\ddot\phi = -g\theta$ $\quad(\because\ \theta$ is small) ...(1)

And Lagrange's ϕ-equations $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\phi}\right) - \frac{\partial T}{\partial\phi} = \frac{\partial W}{\partial\phi}$

i.e. $\frac{d}{dt}\left(\frac{4}{3}Ma^2\dot\phi + Mab\dot\theta\right) - 0 = -Mga\sin p$ or $4a\ddot\phi + 3b\ddot\theta = -3g\phi$

$(\because\ \phi$ is small) ...(2)

Equations (1) and (2), can be written as

$(bD^2+g)\theta + aD^2\phi = 0$, and $3bD^2\theta + (4aD^2 + 3g0\phi = 0$

Eliminating ϕ, between these two equations, we have

$[(4aD^2+3g)(bD^2+g) - 3abD^4]\theta = 0$

or $[abD^4 + (4a+3b)gD^2 + 3g^2]\theta = 0$ ...(3)

If l is the length of the simple equivalent pendulum, then solution of (3) must be

$\theta = A\cos[\sqrt{(g/l)}\,t + B]$ $\therefore\ D\theta = -A\sqrt{(g/l)}\sin[\sqrt{(g/l)}\,t + B]$

or $D^2\theta = -A(g/l)\cos[\sqrt{(g/l)}\,t + B] = -(g/l)\theta$.

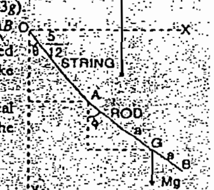
$D^4\theta = -(g/l)D^2\theta = (g^2/l^2)\theta$.

Substituting in (3), we get

$$\left[ab\frac{g^2}{l^2} - (4a+3b)\frac{g^2}{l} + 3g^2\right]\theta = 0$$

or $l^2 - (\frac{4}{3}a + b)l + \frac{1}{3}ab = 0$. $\quad\because\ \theta \neq 0$.

Ex. 12. *A uniform rod, of length 2a, which has one end attached to a fixed point by a light inextensible string of length 5a/12, is performing small oscillations in a vertical plane about its position of equilibrium. Find its position at any time, and show that the period of its principal oscillations are* $2\pi\sqrt{(5a/3g)}$ *and* $\pi\sqrt{(a/3g)}$.



Sol. Let OA be the string of length $\frac{5}{12}a$ and AB the rod of mass M and length $2a$. Let O be the fixed point. At time t, let the string and the rod make angles θ and ϕ, to the vertical respectively.

Referred to O as origin, horizontal and vertical lines OX and OY as axes, the coordinates of the C.G. 'G' of the rod one given by

$x_G = \frac{5}{12}a\sin\theta + a\sin\phi$,

$y_G = \frac{5}{12}a\cos\theta + a\cos\phi$

$\therefore\ v_G^2 = \dot{x}_G^2 + \dot{y}_G^2 = (\frac{5}{12}a\cos\theta\dot\theta + a\cos\phi\dot\phi)^2 + (-\frac{5}{12}a\sin\theta\dot\theta - a\sin\phi\dot\phi)^2$

$= \frac{25}{144}a^2\dot\theta^2 + a^2\dot\phi^2 + \frac{5}{6}a^2\dot\theta\dot\phi\cos(\theta-\phi)$

$= \frac{25}{144}a^2\dot\theta^2 + a^2\dot\phi^2 + \frac{5}{6}a^2\dot\theta\dot\phi$. $\quad\because\ \theta$ and ϕ are small.

If T be the total K.E. and W the work function of the system, then

$T = \frac{1}{2}M\frac{1}{3}a^2\dot\phi^2 + \frac{1}{2}Mv_G^2$

$$= \frac{1}{2}M\left[\frac{1}{3}a^2\dot\phi^2 + \frac{25}{144}a^2\dot\theta^2 + a^2\dot\phi^2 + \frac{5}{6}a^2\dot\theta\dot\phi\right]$$

or $T = \frac{1}{2} Ma^2 \left[\frac{25}{144} \dot{\theta}^2 + \frac{4}{3} \dot{\phi}^2 + \frac{5}{6} \dot{\theta}\dot{\phi} \right]$

and $W = Mg.y_G + C = Mga\left(\frac{5}{12}\cos\theta + \cos\phi\right) + C$

Lagrange's θ–equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = \frac{\partial W}{\partial \theta}$

i.e. $\frac{d}{dt}\left[\frac{1}{2} Ma^2 \left(\frac{25}{72}\dot{\theta} + \frac{5}{6}\dot{\phi}\right)\right] - 0 = -\frac{5}{12} Mga \sin\theta$

or $5\ddot{\theta} + 12\ddot{\phi} = -12c\theta$, ($\because$ θ is small) ...(1)

Taking $(g/a) = c$

And Lagrange's ϕ–equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} = \frac{\partial W}{\partial \phi}$

i.e. $\frac{d}{dt}\left[\frac{1}{2} Ma^2 \left(\frac{8}{3}\dot{\phi} + \frac{5}{6}\dot{\theta}\right)\right] - 0 = -Mga \sin\phi$

or $5\ddot{\theta} + 16\ddot{\phi} = -12c\phi$, ($\because$ ϕ is small and $(g/a) = c$) ...(2)

Equations (1) and (2), can be written as

$(5D^2 + 12c)\,\theta + 12D^2\phi = 0$, and $5D^2\theta + (16D^2 + 12c)\,\phi = 0$.

Eliminating ϕ between these two equations, we have

$[(5D^2 + 12c)(16D^2 + 12c) - 60D^4]\,\theta = 0$

or $(5D^4 + 63cD^2 + 36c^2)\,\theta = 0$ ...(3)

Let the solution of (3) be

$\theta = A\cos(pt + B)$. $\therefore$ $D^2\theta = -p^2\theta$ and $D^4\theta = p^4\theta$.

Substituting in (3), we get

$(5p^4 - 63cp^2 + 36c^2)\,\theta = 0$

or $5p^4 - 63cp^2 + 36c^2 = 0$ $\because$ $\theta \neq 0$

or $(5p^2 - 3c)(p^2 - 12c) = 0$

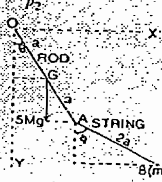
$\therefore$ $p_1^2 = \frac{3}{5}c = \frac{3g}{5a}$ and $p_2^2 = 12c = \frac{12g}{a}$ $\because$ $c = \frac{g}{a}$

Hence period of oscillations are $\frac{2\pi}{p_1}$ and $\frac{2\pi}{p_2}$

i.e. $2\pi\sqrt{(5a/3g)}$ and $2\pi\sqrt{(a/12g)}$

i.e. $2\pi\sqrt{(5a/3g)}$ and $\pi\sqrt{(a/3g)}$.

Ex. 13. *A uniform rod, of mass 5m and length 2a, turns freely about one end which is fixed, to its other extremity is attached one end of a light string, of length 2a, which carries at its other end a particle of mass m, show that the periods of the small oscillations in a vertical plane are the same as those of simple pendulum of lengths 2a/3 and 20a/7.*



Sol. Let OA be the rod of mass $5m$ and length $2a$ turning about the fixed end O, AB the string of length $2a$ and m the mass attached at the end B.

At time t let the rod and the string make angles θ and ϕ to the vertical respectively.

Referred to O as origin, horizontal and vertical lines OX and OY as axes, the coordinates of C.G. 'G' of the rod and that of the end B are given by

$x_G = a\sin\theta$, $y_G = a\cos\theta$; $x_B = 2a(\sin\theta + \sin\phi)$, $y_B = 2a(\cos\theta + \cos\phi)$

$\therefore$ If v_G and v_B are the velocities of G and m at B, then

$v_G^2 = \dot{x}_G^2 + \dot{y}_G^2 = (a\cos\theta\dot{\theta})^2 + (-a\sin\theta\dot{\theta})^2 = a^2\dot{\theta}^2$

$v_B^2 = \dot{x}_B^2 + \dot{y}_B^2 = [2a(\cos\theta\dot{\theta} + \cos\phi\dot{\phi})]^2 + [2a(-\sin\theta\dot{\theta} - \sin\phi\dot{\phi})]^2$

$= 4a^2[\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\cos(\theta - \phi)] = 4a^2(\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}]$.

$\because$ θ and ϕ are small.

If T be the K.E. and W the work function of the system, then

T = K.E. of the rod + K.E. of the particle

$= [\frac{1}{2}.5m.\frac{1}{3}a^2\dot{\theta}^2 + \frac{1}{2}.5m.v_G^2] + \frac{1}{2}mv_B^2$

$= \frac{1}{2}5m(\frac{1}{3}a^2\dot{\theta}^2 + a^2\dot{\theta}^2) + \frac{1}{2}m.4a^2(\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi})$

or $T = ma^2(\frac{16}{3}\dot{\theta}^2 + 2\dot{\phi}^2 + 4\dot{\theta}\dot{\phi})$

and $W = 5mg.y_G + mg.y_B + C = 5mga\cos\theta + mg.2a(\cos\theta + \cos\phi) + C$

$= mga(7\cos\theta + 2\cos\phi)$.

$\therefore$ Lagrange's θ–equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = \frac{\partial W}{\partial \theta}$

i.e. $\frac{d}{dt}\left[ma^2\left(\frac{32}{3}\dot{\theta} + 4\dot{\phi}\right)\right] - 0 = -7mga\sin\theta = -7mga\theta$. $\because$ θ is small

or $32\ddot{\theta} + 12\ddot{\phi} = -21c\theta$. (taking $g/a = c$). ...(1)

And Lagrange's ϕ–equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} = \frac{\partial W}{\partial \phi}$

i.e. $\frac{d}{dt}[ma^2(4\dot{\phi} + 4\dot{\theta})] - 0 = -2mga\sin\phi = -2mga\phi$.

$\because$ ϕ is small.

or $2\ddot{\theta} + 2\ddot{\phi} = -c\phi$ ...(2)

Equations (1) and (2) can be written as

$(32D^2 + 21c)\,\theta + 12D^2\phi = 0$ and $2D^2\theta + (2D^2 + c)\,\phi = 0$.

Eliminating ϕ between these two equations, we get

$[(2D^2 + c)(32D^2 + 21c) - 24D^4]\,\theta = 0$

or $(40D^4 + 74cD^2 + 21c^2)\,\theta = 0$. ...(3)

Let the solution of (3) be given by $\theta = A\cos(pt + B)$.

$\therefore$ $D^2\theta = -p^2\theta$ and $D^4\theta = p^4\theta$.

Substituting in (3), we get

$(40p^4 - 74cp^2 + 21c^2)\,\theta = 0$

or $(2p^2 - 3c)(20p^2 - 7c) = 0$ $\because$ $\theta \neq 0$.

$\therefore$ $p_1^2 = \frac{3c}{2} = \frac{3g}{2a}$ and $p_2^2 = \frac{7c}{20} = \frac{7g}{20a}$

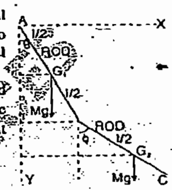
Hence the lengths* of simple equivalent pendulum are

$$\frac{g}{p_1^2} \text{ and } \frac{g}{p_2^2} \text{ i.e. } \frac{2a}{3} \text{ and } \frac{20a}{7}$$

Ex. 14. *Two equal rods AB and BC, each of length l smoothly joined at B are suspended from A and oscillate in a vertical plane through A. Show that the periods of normal oscillations are 2π/n, where*

$n^2 = \left(3 \pm \frac{6}{\sqrt{7}}\right)\frac{g}{l}$

Sol. Let AB and BC be the rods of equal length l and mass M. At time t, let the two rods make angles θ and ϕ to the vertical respectively.



Referred to A as origin horizontal and vertical lines AX and AY as axes, the coordinates of C. G. G_1 of rod AB and that of C. G. G_2 of rod BC are given by

$x_{G_1} = \frac{1}{2}l\sin\theta$, $y_{G_1} = \frac{1}{2}l\cos\theta$

$x_{G_2} = l\sin\theta = \frac{1}{2}l\sin\phi$, $y_{G_2} = l\cos\theta + \frac{1}{2}l\cos\phi$.

$\therefore$ If v_{G_1} and v_{G_2} are velocities of G_1 and G_2, then

$v_{G_1}^2 = \dot{x}_{G_1}^2 + \dot{y}_{G_1}^2 = (\frac{1}{2}l\cos\theta\dot{\theta})^2 + (-\frac{1}{2}l\sin\theta\dot{\theta})^2 = \frac{1}{4}l^2\dot{\theta}^2$

$v_{G_2}^2 = \dot{x}_{G_2}^2 + \dot{y}_{G_2}^2 = (l\cos\theta\dot{\theta} + \frac{1}{2}l\cos\phi\dot{\phi})^2 + (-l\sin\theta\dot{\theta} - \frac{1}{2}l\sin\phi\dot{\phi})^2$

$= l^2[\dot{\theta}^2 + \frac{1}{4}\dot{\phi}^2 + \dot{\theta}\dot{\phi}\cos(\theta - \phi)]$

$= l^2[\dot{\theta}^2 + \frac{1}{4}\dot{\phi}^2 + \dot{\theta}\dot{\phi}]$, ($\because$ θ, ϕ are small)

If T be the total kinetic energy and W the work function of the system, then

T = K.E. of rod AB + K.E. of rod BC

$= [\frac{1}{2}M.\frac{1}{3}(\frac{1}{2}l)^2\dot{\theta}^2 + \frac{1}{2}M.v_{G_1}^2] + [\frac{1}{2}m.\frac{1}{3}(\frac{1}{2}l)\dot{\phi}^2 + \frac{1}{2}M.v_{G_2}^2]$

$= \frac{1}{2}M[\frac{1}{12}l^2\dot{\theta}^2 + \frac{1}{4}l^2\dot{\theta}^2] + \frac{1}{2}M[\frac{1}{12}l^2\dot{\phi}^2 + l^2(\dot{\theta}^2 + \frac{1}{4}\dot{\phi}^2 + 2\dot{\theta}\dot{\phi})]$

$= \frac{1}{2}Ml^2(\frac{4}{3}\dot{\theta}^2 + \frac{1}{3}\dot{\phi}^2 + \dot{\theta}\dot{\phi})$

and $W = Mgy_{G_1} + Mgy_{G_2} + C = Mg[\frac{1}{2}l\cos\theta + l\cos\theta + \frac{1}{2}l\cos\phi] + C$

$= \frac{1}{2}Mgl(3\cos\theta + \cos\phi)$.

$\therefore$ Lagrange's θ–equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = \frac{\partial W}{\partial \theta}$

i.e. $\frac{d}{dt}[\frac{1}{2}Ml^2(\frac{8}{3}\dot{\theta} + \dot{\phi})] - 0 = \frac{1}{2}Mgl(-3\sin\theta) = -\frac{3}{2}Mgl\theta$ ($\because$ θ is small)

or $8\ddot{\theta} + 3\ddot{\phi} = -9c\theta$, (where $c = g/l$). ...(1)

Equations (1) and (2) can be written as

$(8D^2 + 9c)\,\theta + 3D^2\phi = 0$ and $3D^2 + \theta + (2D^2 + 3c)\,\phi = 0$.

Eliminating ϕ between these two equations, we get

$[(2D^2 + 3c)(8D^2 + 9c) - 9D^4] = 0$

or $(7D^4 + 42cD^2 + 27c^2)\,\theta = 0$ ...(3)

If the periods of normal oscillations are $2\pi/n$, then the solution of (3), must be

$$\theta = A\cos(nt + B). \quad \therefore D^2\theta = -n^2\theta \text{ and } D^4\theta = n^4\theta.$$

Substituting in (3), we get

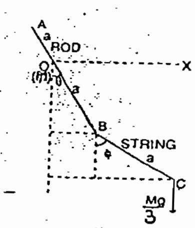
$(7n^4 - 42cn^2 + 27c^2)\,\theta = 0$

or $7n^4 - 42cn^2 + 27c^2 = 0$ $\because$ $\theta \neq 0$.

$\therefore$ $n^2 = \frac{42c + \sqrt{[(42c)^2 - 4.7.27c^2]}}{2.7}$

or

$n^2 = \left(3 \pm \frac{6}{\sqrt{7}}\right)c = \left(3 \pm \frac{6}{\sqrt{7}}\right)\frac{g}{l}$ ($\because$ $c = g/l$)



Ex. 15. *A uniform straight rod of length 2a is freely movable about its centre and a particle of mass one-third that of the rod is attached by a light inextensible string of*

*When $\theta = A\cos(pt + B)$, the period of oscillation is given by $T = (2\pi/p)$. But if l is the lengths of simple equivalent pendulum, then

$T = 2\pi\sqrt{\left(\frac{l}{g}\right)}$ $\therefore$ $\frac{2\pi}{p} = 2\pi\sqrt{\left(\frac{l}{g}\right)}$ $\therefore$ $l = \frac{g}{p^2}$

length a to one end of the rod; show tha one period of principal oscillation is $(\sqrt{5}+1)\pi\sqrt{(a/g)}$.

Sol. Let M be the mass of the rod AB of length $2a$, BC the string and $M/3$ the mass at C.

At time t, let the rod and the string make angles θ and ϕ to the vertical respectively.

Referred to the middle point O of the rod AB as origin, horizontal and vertical lines OX and OY through O as axes, the coordinates of C are given by $x_c = a(\sin\theta + \sin\phi)$,

$y_c = a(\cos\theta + \cos\phi)$.

$\therefore\ v_c^2 = \dot{x}_c^2 + \dot{y}_c^2$

$= a^2(\cos\theta\dot{\theta} + \cos\phi\dot{\phi})^2 + a^2(-\sin\theta\dot{\theta} - \sin\phi\dot{\phi})^2$

$= a^2[\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\cos(\theta-\phi)] = a^2(\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi})$ $(\because \theta, \phi$ are small$)$

If T be the total kinetic energy and W the work function of the system, then

$T =$ K.E. of the rod + K.E. of the particle at C

$= [\frac{1}{2}M.\frac{1}{3}a^2\dot{\theta}^2 + \frac{1}{2}Mv_0^2] + \frac{1}{2}(\frac{1}{3}M)v_c^2$

$= \frac{1}{6}Ma^2\dot{\theta}^2 + \frac{1}{6}Ma^2(\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}) = \frac{1}{6}Ma^2(2\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi})$ $(\because v_0 = 0)$

and $W = mg.0 = \frac{1}{3}Mg.y_c + C = \frac{1}{3}Mga(\cos\theta + \cos\phi) + C$

Lagrange's θ-equation is $\frac{d}{dt}\left(\frac{\partial T}{down30\ dedel_\theta}\right) - \frac{\partial T}{\partial\theta} = \frac{\partial W}{\partial\theta}$

i.e. $\frac{d}{dt}[\frac{1}{6}Ma^2(4\dot{\theta} + 2\dot{\phi})] - 0 = \frac{1}{3}Mga(-\sin\theta) = -\frac{1}{3}Mga\theta$, $\because \theta$ is small

or $2\ddot{\theta} + \ddot{\phi} = -c\theta$, (where $c = g/a$)

And Lagrnage's ϕ-equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\phi}}\right) - \frac{\partial T}{\partial\phi} = \frac{\partial W}{\partial\phi}$

i.e. $\frac{d}{dt}[\frac{1}{6}Ma^2(2\dot{\phi} + 2\dot{\theta})] - 0 = \frac{1}{3}Mga(-\cos\phi) = -\frac{1}{3}Mga\phi$ $\because \phi$ is small ...(2)

or $\ddot{\theta} + \ddot{\phi} = -c\phi$, where $c = g/a$

Equations (1) and (2) can be written as

$(2D^2 + c)\theta + D^2\phi = 0$ and $D^2\theta + (D^2 + c)\phi = 0$

Eliminating ϕ between these two equations, we get

$[(D^2 + c)(2D^2 + c) - D^4]\theta = 0$

or $(D^4 + 3cD^2 + c^2)\theta = 0$. ...(3)

Let the solution of (3) be given by $\theta = A\cos(pt + B)$.

$\therefore\ D^2\theta = -p^2\theta$ and $D^4\theta = p^4\theta$.

Substituting in (2), we get

$(p^4 - 3cp^2 + c^2)\theta = 0$ or $p^4 - 3cp^2 + c^2 = 0$ $\because \theta \neq 0$

$\therefore\ p^2 = \frac{3c \pm \sqrt{(9c^2 - 4c^2)}}{2} = \left(\frac{3\pm\sqrt{5}}{2}\right)c = \left(\frac{3\pm\sqrt{5}}{2}\right)\frac{g}{a}$

$\therefore$ One value of p^2 is $p_1^2 = \left(\frac{3-\sqrt{5}}{2}\right)\frac{g}{a}$

$\therefore$ One period of principal oscillation

$= \frac{2\pi}{p_1} = 2\pi\sqrt{\left[\frac{2}{3-\sqrt{5}}\cdot\frac{a}{g}\right]} = 2\pi\sqrt{\left[\frac{2(3+\sqrt{5})}{(3-\sqrt{5})(3+\sqrt{5})}\frac{a}{g}\right]}$

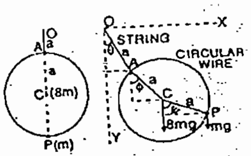
$= 2\pi\sqrt{\left[\frac{(6+2\sqrt{5})}{4}\cdot\frac{a}{g}\right]} = 2\pi\sqrt{\left[\left(\frac{\sqrt{5}+1}{2}\right)^2\frac{a}{g}\right]}$

$= (\sqrt{5}+1)\pi\sqrt{(a/g)}$.

Ex. 16. *A smooth circular wire of mass 8m and radius a, swings in a vertical plane being suspended by an enextansible string of length a attached to one point of it, a particle of mass m can slide on the wire. Prove that the periods of normal oscillations are*

$$2\pi\sqrt{\left(\frac{8a}{3g}\right)}\ 2\pi\sqrt{\left(\frac{a}{3g}\right)}\ 2\pi\sqrt{\left(\frac{8a}{9g}\right)}$$

Sol. Initially the particle is at the lowest point of the wire and the string hange vertically as shown in the figure on next page. At time t, let the string OA a the radius AC make angles θ and ϕ to the vertical. During



this time t, let the particle move to the position P's the radius CP make an angle ψ with the vertical.

Referred to O as origin, horizontal and vertical lines OX, OY through O as axes, the coordinates (x_c, y_c) of C and (x_P, y_P) of P are given by

$x_c = a\sin\theta + a\sin\phi$, $y_c = a\cos\theta + a\cos\phi$

$x_P = a\sin\theta + a\sin\phi + a\sin\psi$, $y_P = a\cos\theta + a\cos\phi + a\cos\psi$.

If v_c and v_P are velocities of C and m at P, then

$v_c^2 = \dot{x}_c^2 + \dot{y}_c^2 = a^2(\cos\theta\dot{\theta} + \cos\phi\dot{\phi})^2 + a^2(-\sin\theta\dot{\theta} - \sin\phi\dot{\phi})^2$

$= a^2[\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\cos(\theta-\phi)] = a^2(\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi})$, $\because \theta, \phi$ are small

and $v_P^2 = \dot{x}_P^2 + \dot{y}_P^2 = a^2(\cos\theta\dot{\theta} + \cos\phi\dot{\phi} + \cos\psi\dot{\psi})^2 + a^2(-\sin\theta\dot{\theta} - \sin\phi\dot{\phi} - \sin\psi\dot{\psi})^2$

$= a^2[\dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2 + 2\dot{\theta}\dot{\phi}\cos(\theta-\psi) + 2\dot{\phi}\dot{\psi}\cos(\phi-\psi) + 2\dot{\psi}\dot{\theta}\cos(\psi-\theta)]$

$= a^2(\dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2 + 2\dot{\theta}\dot{\phi} + 2\dot{\phi}\dot{\psi} + 2\dot{\psi}\dot{\theta})$, $\because \theta, \phi, \psi$ are small

If T be the total kinetic energy and W the work function of the system, then

$T =$ K.E of circular wire + K.E. of mass m.

$= [\frac{1}{2}8ma^2\dot{\phi}^2 + \frac{1}{2}8mv_c^2] + \frac{1}{2}mv_P^2$

$= \frac{1}{2}8ma^2(\dot{\phi}^2 + \dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}) + \frac{1}{2}ma^2(\dot{\theta}^2 + \dot{\phi}^2 + \dot{\psi}^2 + 2\dot{\theta}\dot{\phi} + 2\dot{\phi}\dot{\psi} + 2\dot{\psi}\dot{\theta})$

$= \frac{1}{2}ma^2(9\dot{\theta}^2 + 17\dot{\phi}^2 + \dot{\psi}^2 + 18\dot{\theta}\dot{\phi} + 2\dot{\phi}\dot{\psi} + 2\dot{\psi}\dot{\theta})$

and $W = 8mg.y_c + mg.y_P + C$

$= 8mga(\cos\theta + \cos\phi) + mga(\cos\theta + \cos\phi + \cos\psi) + C$

$= mga(9\cos\theta + 9\cos\phi + \cos\psi) + C$

$\therefore$ Lagrange's θ-equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right) - \frac{\partial T}{\partial\theta} = \frac{\partial W}{\partial\theta}$

i.e. $\frac{d}{dt}[\frac{1}{2}ma^2(18\dot{\theta} + 18\dot{\phi} + 2\dot{\psi})] - 0 = mga(-9\sin\theta) = -9mga\theta$

$\because \theta$ is small

or $9\ddot{\theta} + 9\ddot{\phi} + \ddot{\psi} = -9c\theta$, where $c = g/a$.

Lagrnage's ϕ-equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\phi}}\right) - \frac{\partial T}{\partial\phi} = \frac{\partial W}{\partial\phi}$

i.e. $\frac{d}{dt}[\frac{1}{2}ma^2(34\dot{\phi} + 18\dot{\theta} + 2\dot{\psi})] - 0 = mga(-9\sin\phi)$

$= -9mga\phi$, $\because \phi$ is small

or $9\ddot{\theta} + 17\ddot{\phi} + \ddot{\psi} = -9c\phi$. ...(2)

Lagrange's ψ-equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\psi}}\right) - \frac{\partial T}{\partial\psi} = \frac{\partial W}{\partial\psi}$

i.e. $\frac{d}{dt}[\frac{1}{2}ma^2(2\dot{\psi} + 2\dot{\phi} + 2\dot{\theta})] - 0 = mga(-\sin\psi) = -mga\psi$, $\because \psi$ is small

or $\ddot{\theta} + \ddot{\phi} + \ddot{\psi} = -c\psi$.

Equations (1), (2) and (3) can be written as

$(9D^2 + 9c)\theta + 9D^2\phi + D^2\psi = 0$

$9D^2\theta + (17D^2 + 9c)\phi + D^2\psi = 0$

$D^2\theta + D^2\phi + (D^2 + c)\psi = 0$.

Eliminating ϕ and ψ between these equations we get

$$\begin{vmatrix} 9D^2+9c & 9D^2 & D^2 \\ 9D^2 & 17D^2+9c & D^2 \\ D^2 & D^2 & D^2+c \end{vmatrix}\theta = 0$$

or $$\begin{vmatrix} 9c & 0 & D^2 \\ 0 & 8D^2+9c & D^2 \\ -8D^2-9c & 0 & 2D^2+c \end{vmatrix}\theta = 0$$

[subtracting 9 times of column 3 from column 1 and column 2]

or $$\begin{vmatrix} 9c & 0 & D^2 \\ 0 & 8D^2+9c & D^2 \\ -8D^2-9c & 0 & 2D^2+c \end{vmatrix}\theta = 0$$ [adding row 2 in row 3]

$\therefore$ Expanding w.r.t. column 2, we get

$$(8D^2+9c)\begin{vmatrix} 9c & D^2 \\ -8D^2-9c & 2D^2+c \end{vmatrix}\theta = 0$$

or $(8D^2 + 9c)[9c(2D^2 + c) - D^2(-8D^2 - 9c)]\theta = 0$

or $(8D^2 + 9c)(8D^4 + 27cD^2 + 9c^2)\theta = 0$

or $(8D^2 + 9c)(8D^2 + 3c)(D^2 + 3c)\theta = 0$. ...(4)

Let the solution of equation (4) be given by

$\theta = A\cos(p\theta + B)$, $D^2\theta = -p^2\theta$.

Substituting in (4), we get

$(9c - 8p^2)(3c - 8p^2)(3c - p^2)\theta = 0$

or $(8p^2 - 9c)(8p^2 - 3c)(p^2 - 3c) = 0$ $\because \theta \neq 0$.

$\therefore\ p_1^2 = \frac{9}{8}c = \frac{9g}{8a}, p_2^2 = \frac{3}{8}c = \frac{3g}{8a}, p_3^2 = 3c = \frac{3g}{a}$

Hence the periods of oscillations are $\frac{2\pi}{p_1}, \frac{2\pi}{p_2}, \frac{2\pi}{p_3}$

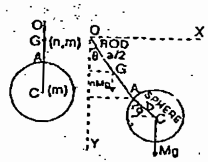
or $2\pi\sqrt{\left(\frac{8a}{9g}\right)}$ $2\pi\sqrt{\left(\frac{8a}{3g}\right)}$ $2\pi\sqrt{\left(\frac{a}{3g}\right)}$

Ex. 17. *To a point of a solid homogeneous sphere, of mass M, is freely hinged one end of a homogeneous rod, of mass nM, and the other end is freely hinged to a fixed point. If the system make small oscillations under gravity about the position of equilibrium, the centre of the sphere and the rod being always in a vertical plane passing through the fixed point, show that the periods of the principal oscillations are the values of $2\pi/p$ given by the equation*

$$2ab(6+7n)p^4 - p^2 g\{10a(3+n)+21b(2+n)\}+15g^2(2+n)=0$$

where a is the length of the rod and b is the radius of the sphere.

Sol. Initially the rod OA of length a and mass nM is vertical with the sphere of mass M and radius b attached at the end. A is such that AC is also vertical.



At time t, let the rod and the sphere turn through an angle θ and ϕ respectively to the vertical. That is at time t the rod OA make an angle θ and the radius AC an angle ϕ to the vertical.

Referred to the point O as origin, horizontal and vertical lines OX, OY as axes, the coordinates (x_G, y_G) of C.G. 'G' of the rod and (x_c, y_c) of the centre C of the sphere are given by

$x_G = \frac{1}{2}a\sin\theta$, $y_G = \frac{1}{2}a\cos\theta$; $x_c = a\sin\theta + b\sin\phi$,

$y_c = a\cos\theta + b\cos\phi$

$\therefore\ v_G^2 = \dot{x}_G^2 + \dot{y}_G^2 = (\frac{1}{2}a\cos\theta\dot{\theta})^2 + (-\frac{1}{2}a\sin\theta\dot{\theta})^2 = \frac{1}{4}a^2\dot{\theta}^2$

$v_c^2 = \dot{x}_c^2 + \dot{y}_c^2 = (a\cos\theta\dot{\theta} + b\cos\phi\dot{\phi})^2 + (-a\sin\theta\dot{\theta} - b\sin\phi\dot{\phi})^2$

$= a^2\dot{\theta}^2 + b^2\dot{\phi}^2 + 2ab\,\dot{\theta}\dot{\phi}\cos(\theta-\phi)$

$= a^2\dot{\theta}^2 + b^2\dot{\phi}^2 + 2ab\dot{\theta}\dot{\phi}$ $\quad\because\ \theta$ and ϕ are small.

If T be the total K.E. and W the work function of the system then we have

T = K.E. of the rod + K.E. of the sphere

$= [\frac{1}{2}nM.\frac{1}{3}(\frac{1}{2}a)^2\dot{\theta}^2 + \frac{1}{2}nM.v_G^2] + [\frac{1}{2}M.\frac{2}{5}b^2\dot{\phi}^2 + \frac{1}{2}M v_c^2]$

$= \frac{1}{2}nM(\frac{1}{12}a^2\dot{\theta}^2 + \frac{1}{4}a^2\dot{\theta}^2) + \frac{1}{2}M[\frac{2}{5}b^2\dot{\phi}^2 + a^2\dot{\theta}^2 + b^2\dot{\phi}^2 + 2ab\dot{\theta}\dot{\phi}]$

$= \frac{1}{6}a^2(n+3)M\dot{\theta}^2 + \frac{7}{10}Mb^2\dot{\phi}^2 + ab\,M\dot{\theta}\dot{\phi}$

and $W = nMg.y_G + Mg.y_c + C$

$= nMg\frac{1}{2}a\cos\theta + Mg(a\cos\theta + b\cos\phi) + C$

$= \frac{1}{2}a(n+2)Mg\cos\theta + b\,Mg\cos\phi + C$

$\therefore$ Lagrange's θ-equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right) - \frac{\partial T}{\partial\theta} = \frac{\partial W}{\partial\theta}$

$\therefore$ i.e. $\frac{d}{dt}[\frac{1}{3}a^2(n+3)M\dot{\theta} + abM\dot{\phi}] - 0 = -\frac{1}{2}a(n+2)Mg\sin\theta$

or $2a(n+3)\ddot{\theta} + 6b\ddot{\phi} = -3(n+2)g\theta$, $\quad(\because\ \theta$ is small) ...(1)

And Lagrange's ϕ-equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\phi}}\right) - \frac{\partial T}{\partial\phi} = \frac{\partial W}{\partial\phi}$

i.e. $\frac{d}{dt}(\frac{7}{5}Mb^2\dot{\phi} + ab\,M\dot{\theta}) - 0 = -b\,Mg\sin\phi$

or $5a\ddot{\theta} + 7b\ddot{\phi} = -5g\phi$, $\quad(\because\ \phi$ is small) ...(2)

Equations (1) and (2), can be written as

$[2a(n+3)D^2 + 3(n+2)g]\theta + 6bD^2\phi = 0$

and $\quad 5aD^2\theta + (7bD^2 + 5g)\phi = 0$

Eliminating ϕ between these two equations, we get

$[\{2a(n+3)D^2 + 3(n+2)g\}(7bD^2 + 5g) - 30abD^4]\theta = 0$

or $[2ab(7n+6)D^4 + \{10a(n+3) + 21b(n+2)\}gD^2$

$+ 15(n+2)g^2]\theta = 0$ ...(3)

If the periods of principal oscillations are the value of $\frac{2\pi}{p}$, then solution of (3) must be $\theta = A\cos(pt + B)$.

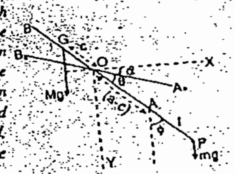
$\therefore\ D^2\theta = -p^2\theta$ and $D^4\theta = p^4\theta$.

Substituting in (3), we get

$2ab(6+7n)p^4 - p^2 g\{10a(3+n) + 21b(2+n)\} + 15g^2(2+n) = 0$

$\because\ \theta \neq 0$

Ex. 18. *A uniform rod AB, of length 2a, can turn freely about a point distance c from its centre, and is at rest at an angle α to the horizon when a particle is hung by a light string of length l from one end. If the particle be displaced slightly in the vertical plane of the rod, show that it will oscillate in the same time as a simple pendulum of length*



$$l.\frac{a^2 + 3ac\cos^2\alpha + 3c^2\sin^2\alpha}{a^2 + 3ac}$$

Sol. Let M be the mass, G the C.G. of the rod AB of length $2a$ which can turn freely about a point O, s.t. $OG = c$. When a particle of mass m is hung by a light string of length l from the end let A_0B_0 be the equilibrium position of the rod at an angle α with the horizontal. In this position the string will hang vertically. Since the system is at rest taking moment about O, we get

$Mc = m(a-c) = mb$ ...(1)

where $b = OA = a - c$.

If the particle be displaced slightly, let the rod AB make an angles $\theta + \alpha$ with the horizontal, and let the string AP make an angle ϕ to the vertical at time t.

Referred to O as origin, horizontal and vertical lines OX and OY as axes the coordinates (x_G, y_G) of C.G. 'G' and (x_P, y_P) of P are given by

$x_G = -c\cos(\theta+\alpha)$, $y_G = -c\sin(\theta+\alpha)$,

$x_P = b\cos(\theta+\alpha) + l\sin\phi$, $y_P = b\sin(\theta+\alpha) + l\cos\phi$

$\therefore\ v_G^2 = \dot{x}_G^2 + \dot{y}_G^2 = [c\sin(\theta+\alpha)\dot{\theta}]^2 + [c\cos(\theta+\alpha)\dot{\theta}]^2 = c^2\dot{\theta}^2$

and $v_P^2 = \dot{x}_P^2 + \dot{y}_P^2 = [-b\sin(\theta+\alpha)\dot{\theta} + l\cos\phi\dot{\phi}]^2$

$+ [b\cos(\theta+\alpha)\dot{\theta} - l\sin\phi\dot{\phi}]^2$

$= b^2\dot{\theta}^2 + l^2\dot{\phi}^2 - 2bl\dot{\theta}\dot{\phi}\sin(\theta+\phi+\alpha)$

$= b^2\dot{\theta}^2 + l^2\dot{\phi}^2 - 2bl\dot{\theta}\dot{\phi}[\sin(\theta+\phi)\cos\alpha + \cos(\theta+\phi)\sin\alpha]$

$= b^2\dot{\theta}^2 + l^2\dot{\phi}^2 - 2bl\dot{\theta}\dot{\phi}[(\theta+\phi)\cos\alpha + 1.\sin\alpha]$ ($\because\ \theta$ and ϕ are small)

$= b^2\dot{\theta}^2 + l^2\dot{\phi}^2 - 2bl\dot{\theta}\dot{\phi}\sin\alpha$

Neglecting small quantities of higher order.

If T be the total K.E. and W the work function of the system, then

T = K.E. of the rod + K.E. of the particle

$= \frac{1}{2}M\frac{1}{3}a^2\dot{\theta}^2 + \frac{1}{2}M.v_G^2 + \frac{1}{2}m v_P^2$

$= \frac{1}{2}M[\frac{1}{3}a^2\dot{\theta}^2 + c^2\dot{\theta}^2] + \frac{1}{2}m[b^2\dot{\theta}^2 + l^2\dot{\phi}^2 - 2bl\dot{\theta}\dot{\phi}\sin\alpha]$

$= \frac{1}{2}\{M(\frac{1}{3}a^2 + c^2) + mb^2\}\dot{\theta}^2 + \frac{1}{2}m(l^2\dot{\phi}^2 - 2bl\dot{\theta}\dot{\phi}\sin\alpha)$

and $W = Mg\,y_G + mg\,y_P + C$

$= -Mgc\sin(\theta+\alpha) + mg[b\sin(\theta+\alpha) + l\cos\phi] + C$

$= g(-Mc + mb)\sin(\theta+\alpha) + mgl\cos\phi + C$

$= mgl\cos\phi + C$. $\quad\because\ Mc = mb$ from (1)

Lagrange's θ-equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right) - \frac{\partial T}{\partial\theta} = \frac{\partial W}{\partial\theta}$

i.e. $\frac{d}{dt}[\{M(\frac{1}{3}a^2 + c^2) + mb^2\}\dot{\theta} - Mbl\dot{\phi}\sin\alpha] = 0$

or $\frac{d}{dt}[\{M(\frac{1}{3}c^2 + c^2) + Mcb\}\dot{\theta} - Mcl\dot{\phi}\sin\alpha] = 0$

$\because\ mb = Mc$ from (1)

or $\{(a^2 + 3c^2) + 3bc\}\ddot{\theta} - 3cl\ddot{\phi}\sin\alpha\ \} = 0$

or $\{a^2 + 3c^2 + 3(a-c)c\}\ddot{\theta} - 3cl\ddot{\phi}\sin\alpha = 0$ $\quad\because\ b = a - c$

or $(a^2 + 3ac)\ddot{\theta} - 3cl\ddot{\phi}\sin\alpha = 0$ ...(1)

And Lagrange's ϕ-equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\phi}}\right) - \frac{\partial T}{\partial\phi} = \frac{\partial W}{\partial\phi}$

i.e. $\frac{d}{dt}[m(l^2\dot{\phi} - 2bl\dot{\theta}\sin\alpha)] - 0 = -mgl\sin\phi = -mgl\phi$ $\quad\because\ \phi$ is small.

or $-b\ddot{\theta}\sin\alpha + l\ddot{\phi} = -g\phi$.

Eliminating θ between (1) and (2), we get

$-b.\frac{3cl\sin\alpha}{(a^2+3ac)}\ddot{\phi}\sin\alpha + l\ddot{\phi} = -g\phi$ ...(2)

or $\frac{-3cl(a-c)\sin^2\alpha + l(a^2+3ac)}{(a^2+3ac)}\ddot{\phi} = -g\phi$ $\quad\because\ b = a - c$

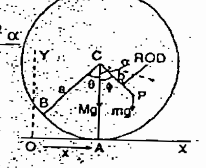
or $\ddot{\phi} = -\frac{g}{l}.\frac{a^2+3ac}{a^2 - 3ac\sin^2\alpha + 3c^2\sin^2\alpha + 3ac}\phi$

or $\ddot{\phi} = -\frac{g}{l}\frac{a^2+3ac}{a^2 + 3ac\cos^2\alpha + 3c^2\sin^2\alpha}\phi = -\mu\phi$. (say)

Hence the length of equivalent simple pendulum

$= \frac{g}{\mu} = l\frac{a^2 + 3ac\cos^2\alpha + 3c^2\sin^2\alpha}{a^2 + 3ac}$

Ex. 19. *A hollow circular roller of radius a and mass M has attached to the centre of its axis a particle of mass m by light rod of length b. The particle is free to swing about the axis in a plane perpendicular to the axis.*



The roller is at rest on a horizontal plane rough enough to prevent slipping, with rod held at angle α with the downward vertical. If the rod is then released prove that the centre of the roller will oscillate through a distance

$$\frac{2mba^2\sin\alpha}{[M(a^2+k^2)+ma^2]},$$

where k is the radius of gyration of the roller about the axis.

Sol. The figure is the vertical cross-section of the cylinder through the centre of its axis, and the particle.

Let C be the centre of the axis of the cylinder of radius a and mass M, CP be the light rod of length b and m the mass attached at P. Initially the point B of the roller is in contact of the horizontal plane at O i.e. initially CB is vertical and the rod is held at angle α with the downward vertical and then released.

In time t, let the roller roll through a distance x on the horizontal plane. At this time t let the radius CB of the roller and the rod make angles θ and ϕ to the vertical respectively.

If A is the new point of contact of the roller and the plane, then $OA = x$. Since there is no slipping,

$\therefore\ x = OA = \text{Arc } AB = a\theta.$

Reffered to O as origin, horizontal and vertical lines through O as axes, the coordinates (x_c, y_c) of C and (x_P, y_P) of P are given by

$x_c = x = a\theta$, $y_c = a$; $x_P = a\theta + b\sin\phi$, $y_P = a - b\cos\phi$.

$\therefore\ v_c^2 = \dot{x}_c^2 + \dot{y}_c^2 = a^2\dot{\theta}^2$

and $v^2 r = \dot{x}_P^2 + \dot{y}_P^2 = (a\dot{\theta} + b\cos\phi\dot{\phi})^2 + (b\sin\phi\dot{\phi})^2$

$= a^2\dot{\theta}^2 + b^2\dot{\phi}^2 + 2ab\dot{\theta}\dot{\phi}\cos\phi$

If T be the kinetic energy and W the work function of the system then, we have

T = K.E. of the roller + K.E. of the particle

$= [\frac{1}{2}Mk^2\dot{\theta}^2 + \frac{1}{2}M v_c^2] + [\frac{1}{2}m v_P^2]$

$= \frac{1}{2}M(k^2 + a^2)\dot{\theta}^2 + \frac{1}{2}m(a^2\dot{\theta}^2 + b^2\dot{\phi}^2 + 2ab\dot{\theta}\dot{\phi}\cos\phi)$

and $W = mg(b\cos\phi - b\cos\alpha)$

$\therefore$ Lagrange's θ–equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right) - \frac{\partial T}{\partial\theta} = \frac{\partial W}{\partial\theta}$

i.e. $\frac{d}{dt}[M(k^2 + a^2)\dot{\theta} + m(a^2\dot{\theta} + ab\dot{\phi}\cos\phi)] = 0$

Integrating, $\{M(k^2 + a^2) + ma^2\}\dot{\theta} + mab\dot{\phi}\cos\phi = C$

But initially, $\phi = \alpha$, $\dot{\theta} = 0$, i.e. $\dot{\phi} = 0$ $\therefore$ $C = 0$

$\therefore$ We have $\{M(k^2 + a^2) + ma^2\}\dot{\theta} + mab\dot{\phi}\cos\phi = 0$

or $\{M(k^2 + a^2) + ma^2\}\, d\theta = -mab\cos\phi\, d\phi$

The rod swings about a horizontal axis through, it falls from an angle α to the right of the vertical and rises through an equal angle α to the left of the vertical. If the roller turns through an angle β during this half oscillation, then integrating (1) between $\phi = \alpha$ to $\phi = -\alpha$, we get

$\{M(k^2 + a^2) + ma^2\}\ \beta = -[mab\sin\phi]_{\alpha}^{-\alpha}$

$\therefore\ \beta = \frac{2mab\sin\alpha}{\{M(a^2 + k^2) + ma^2\}}$

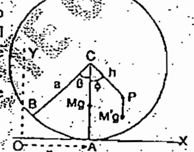
Hence the centre of the roller will move forward through a distance $x = a\beta$ (Putting $\theta = \beta$)

i.e. $\frac{2mb\,a^2\sin\alpha}{\{M(a^2 + k^2) + ma^2\}}$

When the rod oscillate from an angle α to the left of the vertical and rises through an equal angle α to the right of the vertical, then the centre of the roller move back to its original position.

Hence the centre of the roller will oscillate through a distance

$$\frac{2mba^2\sin\alpha}{[M(a^2 + k^2) + ma^2]}$$



Ex. 20. *A hollow cylindrical garden roller is fitted with a counterpoise which can turn on the axis of the cylinder, the system is placed on a rough horizontal plane and oscillates under gravity, if $2\pi/p$ be the time of a small oscillation; shaw that p is given by the equation $p^2[(2M + M')k^2 - M'h^2] = (2M + M')gh$, where M and M' are the masses of the roller and counterpoise, k is the radius of gyration of M' about the axis of the cylinder and h is the distance of its centre of mass from the axis.*

Sol. Let C be the centre of the axis of the cylinder of radius a and mass M. At time t, let the radius CB of the roller and the counterpoise CP make angles θ and ϕ to the vertical. Initially CB was vertical and B was in contact of the horizontal plane at O. Let A be the point of contact of the roller with the plane at time t. If $OA = x$ then $x = \text{Arc } AB = a\theta$

(since there is no slipping)

Referred to O as origin, horizontal and vertical lines through O as axes, the coordinates (x_c, y_c) of C and (x_P, y_P) of P are given by

$x_c = x = a\theta$, $y_c = a$; $x_P = a\theta + h\sin\phi$, $y_P = a - h\cos\phi$.

$\therefore\ v_c^2 = \dot{x}_c^2 + \dot{y}_c^2 = a^2\dot{\theta}^2$

and $v_P^2 = \dot{x}_P^2 + \dot{y}_P^2 = (a\dot{\theta} + h\cos\phi\dot{\phi})^2 + (h\sin\phi\dot{\phi})^2$

$= a^2\dot{\theta}^2 + h^2\dot{\phi}^2 + 2ah\dot{\theta}\dot{\phi}$ (neglecting higher power of ϕ which is small)

If T be the kinetic energy and W the work function of the system, then

T = K.E. of the roller + K.E. of the counterpoise

$= [\frac{1}{2}Ma^2\dot{\theta}^2 + \frac{1}{2}M v_c^2] + [\frac{1}{2}M'(k^2 - h^2)\dot{\phi}^2 + \frac{1}{2}M' v_P^2]$

[$\because$ If K is the radius of gyration of M' about the parallel axis through P, then $\frac{1}{2}Mk^2 = \frac{1}{2}MK^2 + \frac{1}{2}Mh^2$ i.e. $K^2 = k^2 - h^2$]

$= Ma^2\dot{\theta}^2 + \frac{1}{2}M'[(K^2 - h^2)\dot{\phi}^2 + a^2\dot{\theta}^2 + h^2\dot{\phi}^2 + 2ah\dot{\theta}\dot{\phi}]$

$= Ma^2\dot{\theta}^2 + \frac{1}{2}M'(k^2\dot{\phi}^2 + a^2\dot{\theta}^2 + 2ah\dot{\theta}\dot{\phi})$

and $W = -M'g(h - h\cos\phi) = M'gh\cos\phi + C$.

$\therefore$ Lagrange's θ–equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right) - \frac{\partial T}{\partial\theta} = \frac{\partial W}{\partial\theta}$

i.e. $\frac{d}{dt}[2Ma^2\dot{\theta} + M'(a^2\dot{\theta} + ah\dot{\phi})] = 0$

or $(2M + M')a\ddot{\theta} + M'h\ddot{\phi} = 0$. ...(1)

And Lagrange's ϕ-equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\phi}}\right) - \frac{\partial T}{\partial\phi} = \frac{\partial W}{\partial\phi}$

i.e. $\frac{d}{dt}(M'(k^2\dot{\phi} + ah\dot{\theta})) = -M'gb\sin\phi = -M'gh\phi$, $\because$ ϕ is small

or $ah\ddot{\theta} + k^2\ddot{\phi} = -gh\phi$. ...(2)

Eliminating $\ddot{\theta}$ between (1) and (2), we get

$[(2M + M')k^2 - M'h^2]\ddot{\phi} = -(2M + M')gh\phi$ ...(3)

If $2\pi/p$ is the time of oscillation, then solution of (3) must be $\phi = A\cos(pt + B)$, so that $\ddot{\phi} = -p^2\phi$.

Substituting in (3), we get

$[-\{(2M + M')k^2 - M'h^2\}p^2 + (2M + M')gh]\phi = 0$

or $p^2[(2M + M')k^2 - M'h^2] = (2M + M')gh$, ($\because$ $\phi \neq 0$)

Ex. 21. *A perfectly rough sphere lying inside a hollow cylinder which rests on a perfectly rough plane is slightly displaced from its position of equilibrium. Show that the time of a small oscillation is*

$$2\pi\sqrt{\left(\frac{a-b}{g}\cdot\frac{14M}{10M + 7m}\right)}$$

where a is the radius of the cylinder, b that of the sphere and M, m are the masses of the cylinder and sphere.

Sol. The figure is the vertical cross-section through the centres of the cylinder and the sphere.

Let C and C' be the centres of the cylinder and the sphere. Initially the point B of cylinder is in contact with the horizontal plane at O and the sphere rests in cylinger with its point D in contact with the point B.

At time t let the line CC' joining centres make an angle θ to the vertical and at this time let the radius $C'D$ of sphere and CB of cylinder make angles θ and ψ to the vertical respectively. Since there is no slipping.

$\therefore\ OA = \text{Arc } AB = a\psi$ and

Arc BP = Arc PD.

i.e. $a(\psi + \theta) = b(\theta + \phi)$

or $b\phi = (a - b)\theta + a\psi$,

so that $b\dot{\phi} = (a - b)\dot{\theta} + a\dot{\psi} = c\dot{\theta} + a\dot{\psi}$, ...(1)

where $c = a - b$.

Referred to O as origin, horizontal and vertical lines through O as axes, the coordinates (x_c, y_c) of C and (x_c', y_c') of C' are given by

$x_c = OA = a\psi$; $y_c = a$; $x_c' = a\psi + c\sin\theta$, $y_c' = a - c\cos\theta$, where $c = a - b$.

$\therefore\ v_c^2 = \dot{x}_c^2 + \dot{y}_c^2 = a^2\dot{\psi}^2$

and $v_c'^2 = \dot{x}_c'^2 + \dot{y}_c'^2 = (a\dot{\varphi} + c\cos\theta\dot{\theta}) + (c\sin\theta\dot{\theta})^2$

$= a^2\dot{\psi}^2 + c^2\dot{\theta}^2 + 2ac\dot{\psi}\dot{\theta}$, $\because$ θ is small

If T be the kinetic energy, and W the work function of the cylinder, then

T = K.E. of the cylinder + K.E. of sphere

$= [\frac{1}{2}M a^2\dot{\psi}^2 + \frac{1}{2}M v_c^2] + [\frac{1}{2}m\frac{2}{5}b^2\dot{\theta}^2 + \frac{1}{2}m v_c'^2]$

$= Ma^2\dot{\psi}^2 + \frac{1}{2}m(\frac{2}{5}b^2\dot{\phi}^2 + a^2\dot{\psi}^2 + c^2\dot{\theta}^2 + 2ac\dot{\psi}\dot{\theta})$

$= Ma^2\dot{\psi}^2 + \frac{1}{2}m[\frac{2}{5}(c\dot{\theta} + a\dot{\psi})^2 + a^2\dot{\psi}^2 + c^2\dot{\theta}^2 + 2ac\dot{\psi}\dot{\theta}]$

$= \frac{1}{10}(10M + 7m)a^2\dot{\psi}^2 + \frac{7}{10}m(c^2\dot{\theta}^2 + 2ac\dot{\psi}\dot{\phi})$

and $W = -mg - b(a - c\cos\theta) = mg(c\cos\theta - a) + mgb$

$\therefore$ Lagrange's θ–equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right) - \frac{\partial T}{\partial\theta} = \frac{\partial W}{\partial\theta}$

i.e. $\frac{d}{dt}[\frac{7}{10}m(2c^2\dot{\theta} + 2ac\dot{\psi})] = -mgc\sin\theta = -mgc\theta$, $\because$ θ is small

or $7c\ddot{\theta} + 7a\ddot{\psi} = -5g\theta$. ...(2)

And Lagrange's ψ–equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\psi}}\right) - \frac{\partial T}{\partial\psi} = \frac{\partial W}{\partial\psi}$

i.e. $\frac{d}{dt}\left[\frac{1}{5}(10M+7m)\,a^2\dot{\psi}+\frac{7}{5}mac\dot{\theta}\right]=0$

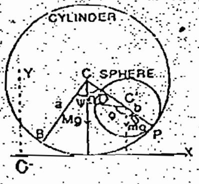
or $7mc\ddot{\theta}+(10M+7m)\,a\ddot{\psi}=0$ ...(3)

Eliminating $\ddot{\psi}$ between (2) and (3), we get

$[7ac\,(10M+7m)-49mac]\,\ddot{\theta}=-5ga\,(10M+7m)\theta$

or $\ddot{\theta}=-\left(\frac{10M+7m}{14M}\cdot\frac{g}{a-b}\right)\theta=-\mu\theta \quad \because c=a-b$

which represent S. H. M.



∴ The time of small oscillation is

$$\frac{2\pi}{\sqrt{\mu}}=2\pi\sqrt{\left(\frac{a-b}{g}\cdot\frac{14M}{10M+7m}\right)}$$

Ex. 22. *A perfectly rough sphere of mass m and radius b rests at the lowest point of a fixed spherical cavity of radius a. To the highest point of the movable sphere is attached a particle of mass m′ and the system is disturbed. Show that the oscillations are the same as those of a simple pendulum of length*

$$(a-b)\cdot\frac{4m'+7m/5}{m+m'(2-a/b)}$$ (IFoS-2009)

Sol. Let O be the centre of the fixed spherical cavity and C the centre of the sphere of mass m and radius b resting at the lowest point A of the cavity. A particle of mass m' is attached at the highest point D_0 of the sphere. In time t, let the line OC joining centres and the diameter B_0D_0 turn through angles θ and ϕ respectively from the verticals, i.e. at time t B and D correspond to the point B_0 and D_0 at time $t=0$.

Since there is no slipping between the sphere and cavity, therefore if P is their point of contact at time t, then

Arc AP = Arc PB i.e. $a\theta=b(\theta+\phi)$

or $b\phi=(a-b)\,\theta=c\theta$, where $a-b=c$ (say) $\therefore\ b\phi=c\theta$ ...(1)

Referred to centre O as origin, horizontal and vertical lines OX and OY as axes, the coordinates (x_c, y_c) of C and (x_D, y_D) of D respectively are

$x_c=c\sin\theta,\ y_c=c\cos\theta$;

$x_D=c\sin\theta+b\sin\phi,\ y_D=c\cos\theta-b\cos\phi$.

$\therefore\ v_c^2=\dot{x}_c^2+\dot{y}_c^2=(c\cos\theta\dot{\theta})^2+(-c\sin\theta\dot{\theta})^2=c^2\dot{\theta}^2$

and $v_D^2=\dot{x}_D^2+\dot{y}_D^2=(c\cos\theta\dot{\theta}+b\cos\phi\dot{\phi})^2$

$\qquad +(-c\sin\theta\dot{\theta}+b\sin\phi\dot{\phi})^2$

$=c^2\dot{\theta}^2+b^2\dot{\phi}^2+2bc\dot{\theta}\dot{\phi}\cos(\theta+\phi)=c^2\dot{\theta}^2+b^2\dot{\phi}^2+2bc\dot{\theta}\dot{\phi}$

($\because$ θ and ϕ are small)

If T be the kinetic energy and W the work function of the system, then we have

T = K.E. of the sphere + K.E. of the particle

$=[\frac{1}{2}m\frac{2}{5}b^2\dot{\phi}^2+\frac{1}{2}m\,v_c^2]+[\frac{1}{2}m'v_D^2]$

$=\frac{1}{2}m(\frac{2}{5}b^2\dot{\phi}^2+c^2\dot{\theta}^2)+\frac{1}{2}m'(c^2\dot{\theta}^2+b^2\dot{\phi}^2+2bc\dot{\theta}\dot{\phi})$

$=\frac{1}{2}m(\frac{2}{5}b^2\dot{\phi}^2+b^2\dot{\phi}^2)+\frac{1}{2}m'(b^2\dot{\phi}^2+b^2\dot{\phi}^2+2b.b\dot{\phi}.\dot{\phi})$ using (1)

$=\frac{1}{10}b^2(7m+20m')\dot{\phi}^2$

and $W=-mg\,(OC'-y_c)+m'g\,(y_D-OD_0)$

$=-mg\,(c-c\cos\theta)+m'g\,\{c\cos\theta-b\cos\phi-(a-2b)\}$

$=(m+m')\,cg\cos\theta-m'bg\cos\phi+C$

$=(m+m')\,cg\cos(b\phi/c)-m'bg\cos\phi+C. \quad \because c\theta=b\phi$

∴ Lagrange's ϕ–equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\phi}}\right)-\frac{\partial T}{\partial\phi}=\frac{\partial W}{\partial\phi}$

i.e. $\frac{d}{dt}[\frac{1}{5}b^2(7m+20m')\dot{\phi}]-0=-(m+m')\,cg\frac{b}{c}\sin\left(\frac{b}{c}\phi\right)+m'bg\sin\phi$

or $\frac{1}{5}b^2(7m+20m')\ddot{\phi}=-(m+m')\,bg\frac{b}{c}\phi+m'bg\phi. \quad \because \phi$ is small

or $b\,(\frac{7}{5}m+4m')\ddot{\phi}=-\frac{g}{c}\left[(m+m')-\frac{c}{b}m'\right]\phi$

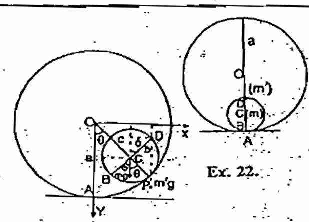
or $b\,(4m'+\frac{7}{5}m)\ddot{\phi}=-\frac{g}{a-b}\left[(m+m')-\frac{a-b}{b}m'\right]\phi. \quad \because c=a-b$

$\ddot{\phi}=-\frac{g}{a-b}\cdot\frac{m+(2-a/b)\,m'}{4m'+\frac{7}{5}m}\phi=-\mu\phi$ (say)

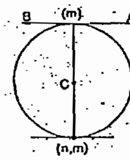
which represent S. H. M.

∴ The length of the simple equivalent pendulum is

$$\frac{g}{\mu}=(a-b)\cdot\frac{4m'+7m/5}{m+m'(2-a/b)}$$



Ex. 22.



Ex. 23.

Ex. 23. *A plank 2a feet long is placed symmetrically across a light cylinder of radius a which rests and is free to roll on a perfectly rough horizontal plane. A heavy particle whose mass is n times that of the plank is embedded in the cylinder at its lowest point. If the system is slightly displaced, show that its periods of oscillations are values of* $\frac{2\pi}{p}\sqrt{\left(\frac{a}{g}\right)}$ *given by the equation*

$$4p^4-(n+12)\,p^2+3\,(n-1)=0.$$

Sol. The figure is the vertical cross-section of the system through the centre of gravity of the plank.

Let G the centre of gravity of the plank of mass m and length $2a$ placed symmetrically across a light cylinder of radius a and centre at C. A mass nm is embedded at the lowest point of the cylinder. In time t let the cylinder turn through an angle θ to the vertical and during this time let the plank turn through an angle ϕ to the horizontal. Initially G coincided with E which was the highest point of the cylinder. And initially F coincided with O. If D is the point of contact of the cylinder and the horizontal plane at time t, then as there is no slipping.

$\therefore\ OD=\text{Arc }FD=a\theta$ and $PG=\text{Arc }PE=a\,(\phi-\theta)$.

Referred to O as origin, the horizontal and vertical lines through O as axes, the coordinates (x_F, y_F) of F and (x_G, y_G) of G are given by

$x_F=OD-FL=a\theta-a\sin\theta,\ y_F=CD-CL=a-a\cos\theta$

$x_G=OD+MN=OD+MP-PN=a\theta+a\sin\phi-a\,(\phi-\theta)\cos\phi$

$=a\theta+a\phi-a\,(\phi-\theta).1=2a\theta, \quad \because \theta$ and ϕ are small

and $y_G=CD+CM+NG=a+a\cos\phi+a\,(\phi-\theta)\sin\phi$

$=a+a\left(1-\frac{\phi^2}{2!}\right)+a\,(\phi-\theta)\,\phi=2a-a\theta\phi+\frac{a}{2}\phi^2$,

up to first approximation.

$\therefore\ v_F^2=\dot{x}_F^2+\dot{y}_F^2=(a\dot{\theta}-a\cos\theta\dot{\theta})^2+(a\sin\theta\dot{\theta})^2-2a^2\dot{\theta}^2\cos\theta$

$=a^2\dot{\theta}^2+a^2\dot{\theta}^2=2a^2(1-\cos\theta)\,\dot{\theta}^2$

and $v_G^2=\dot{x}_G^2+\dot{y}_G^2=(2a\dot{\theta})^2+(-a\dot{\phi}\theta-a\dot{\theta}\phi+a\phi\dot{\phi})^2=4a^2\dot{\theta}^2$

Neglecting higher powers of θ and ϕ as they are small.

If T be the K.E. and W the work function of the system, then, we have

T = K.E. of the plank + K.E. of nm

$=[\frac{1}{2}m.\frac{1}{3}a^2\dot{\phi}^2+\frac{1}{2}mv_G^2]+\frac{1}{2}nm.v_F{}^2$

$=\frac{1}{2}m.\frac{1}{3}a^2\dot{\phi}^2+4a^2\dot{\theta}^2+\frac{1}{2}nm.2a^2(1-\cos\theta)\,\dot{\theta}^2$

$=\frac{1}{2}ma^2(\frac{1}{3}\dot{\phi}^2+4a^2\dot{\theta}^2)$; the last term is zero as θ is small.

and $W=-mg.(y_G-2a)-nmg.y_F$

$=-mg\,(2a-a\theta\phi+\frac{1}{2}a\phi^2-2a)-nmg\,(a-a\cos\theta)$

$=mga\,(\theta\phi-\frac{1}{2}\phi^2)-nmg\left[a-a\left(1-\frac{\theta^2}{2!}\right)\right]$

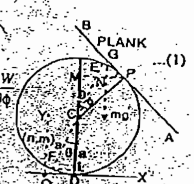
Neglecting higher powers of θ

$=mga\,(\theta\phi-\frac{1}{2}\phi^2)-\frac{1}{2}nmga\theta^2$

∴ Lagrange's θ-equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\theta}}\right)-\frac{\partial T}{\partial\theta}=\frac{\partial W}{\partial\theta}$

i.e. $\frac{d}{dt}(4ma^2\dot{\theta})-0=mga\,\phi-nmga\theta$

or $4\ddot{\theta}=-c\,(n\theta-\phi)$, where $c=(g/a)$. ...(1)



And Lagrange's ϕ-equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot{\phi}}\right)-\frac{\partial T}{\partial\phi}=\frac{\partial W}{\partial\phi}$

i.e. $\frac{d}{dt}(\frac{1}{3}ma^2\dot{\phi})=mga\,(\theta-\phi)$ or $\ddot{\phi}=-3c\,(\phi-\theta)$

Equations (1) and (2), can be written as –

$(4D^2+cn)\,\theta-c\phi=0$ and $3c\theta-(D^2+3c)\,\phi=0$.

Eliminating ϕ between these two equations, we get

$[(D^2+3c)(4D^2+cn)-3c^2]\,\theta=0$

or $[4D^4+c\,(n+12)\,D^2+3\,(n-1)\,c^2]\,\theta=0$. ...(3)

If the periods of oscillations are the values of

$\frac{2\pi}{p}\sqrt{\left(\frac{a}{g}\right)}$ i.e. values of $\frac{2\pi}{p\sqrt{c}}$ $\quad \because c=\frac{g}{a}$

then the solution of (3) must be

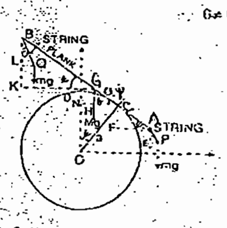
$\theta=A\cos[p\sqrt{c}t+B]$ so that $D^2\theta=-cp^2\theta$ and $D^4\theta=c^2p^4\theta$.

Substituting in (3), we get

$$\{4c^2p^4 - D(n+12)cp^2 + 3(n-1)c^2\}\theta = 0$$

or $4p^4 - (n+12)p^2 + 3(n-1)c^2 = 0$ $\quad \theta \neq 0.$

Ex. 24. *A plank, of mass M, radius of gyration k and length 2b, can swing like a see saw acoss a perfectly rough fixed cylinder of radius a. At its ends hang two particles each of mass m, by strings of length l. Show that, as the system swings, the lengths of its simple equivalent pendulum are l and* $\dfrac{Mk^2 + 2mb^2}{(M+2m)a}$



Sol. The figure is the cross-section of the system through the centre of gravity of the plank, and the strings.

AB is the plank of mass M and length $2b$; AP and BQ strings each of length l and O the centre of the cylinder of radius a. Initially the C.G. G of the plank is at the highest point D of the cylinder and the strings AP and BQ vertical.

At time t, let the string AP BQ make angles θ and ϕ respectively with the vertical. Let C be the point of contact of the plank with the cylinder at this time s.t. $\angle DOC = \psi$.

Since there is no slipping between the plank and the cylinder

$$\therefore CG = \text{Arc } CD = a\psi; \text{ So } AC = AG - CG = b - a\psi.$$

Referred to O as origin, horizontal and vertical lines OX and OY as axes the coordinates (x_G, y_G) of G, (x_P, y_P) of P and (x_Q, y_Q) of Q are given by

$$x_G = CN - CH = a\sin\psi - a\psi\cos\psi = a\psi - a\psi\left(1 - \frac{\psi^2}{2!}\right) = \tfrac{1}{2}a\psi^3$$

Neglecting higher powers of ψ as it is small.

$$y_G = ON + HG = a\cos\psi + a\psi\sin\psi = a\left(1 - \frac{\psi^2}{2!}\right) + a\psi.\psi = a + \tfrac{1}{2}a\psi^2$$

$$x_P = CN + AF + PE = a\sin\psi + (b - a\psi)\cos\psi + l\sin\theta$$

$$= a\psi + (b - a\psi)\left(1 - \frac{\psi^2}{2!}\right) + l\theta = b + l\theta - \tfrac{1}{2}b\psi^2$$

Neglecting higher powers of θ and ψ, which are small.

$$y_P = ON - CF - AE = a\cos\psi - (b - a\psi)\sin\psi - l\cos\theta$$

$$= a(1 - \psi^2/2!) - (b - a\psi).\psi - l(1 - \theta^2/2!)$$

$$= a - l - b\psi + \tfrac{1}{2}a\psi^2 + \tfrac{1}{2}l\theta^2.$$

Neglecting higher powers of ψ and θ

$$x_Q = -(GK - NH - LQ) = -b\cos\psi + (a\sin\psi - a\psi\cos\psi)\; l\sin\phi$$

$$= -b(1 - \psi^2/2!) + a\psi - a\psi(1 - \psi^2/2!) + l\phi = -b + l\phi + \tfrac{1}{2}b\psi^2$$

Neglecting higher powers of ψ and ϕ.

$$y_Q = ON + HG + (BK - BL) = a\cos\psi + a\psi\sin\psi + b\sin\psi - l\cos\phi$$

$$= a(1 - \psi^2/2!) + a\psi.\psi + b\psi - l(1 - \phi^2/2!)$$

$$= a - l + b\psi + \tfrac{1}{2}a\psi^2 + \tfrac{1}{2}l\phi^2.$$

Neglecting higher powers of ψ and ϕ.

$$\therefore v_G^2 = \dot{x}_G^2 + \dot{y}_G^2 = (\tfrac{3}{2}a\psi^2\dot{\psi})^2 + (a\psi\dot{\psi})^2 = a^2\psi^2\dot{\psi}^2$$

Neglecting smaller quantities.

$$v_P^2 = \dot{x}_P^2 + \dot{y}_P^2 + (l\dot{\theta} - b\psi\dot{\psi})^2 + (-b\dot{\psi} + a\psi\dot{\psi} + l\theta\dot{\theta})^2$$

$$= l^2\dot{\theta}^2 + b^2\dot{\psi}^2$$

Neglecting smaller quantities.

Similarly $v_Q^2 = l\dot{\phi}^2 + b^2\dot{\psi}^2$

If T be the K.E. and W the work function of the system then, we have

T = K.E of plank + K.E. of m at P + K.E. of m at Q

$$= (\tfrac{1}{2}Mk^2\dot{\psi}^2 + \tfrac{1}{2}M.v_G^2) + \tfrac{1}{2}m.v_P^2 + \tfrac{1}{2}m.v_Q^2$$

$$= \tfrac{1}{2}Mk^2\dot{\psi}^2 + \tfrac{1}{2}m(l^2\dot{\theta}^2 + b^2\dot{\psi}^2) + \tfrac{1}{2}m(l^2\dot{\phi}^2 + b^2\dot{\psi}^2)$$

Neglecting terms of degree higher than 2 of angles.

$$\text{and } W = -Mg.y_G - mg.y_P - mg.y_Q + C$$

$$= -Mg(a + \tfrac{1}{2}a\psi^2) - mg(a - l - b\psi + 2a\psi^2 + \tfrac{1}{2}l\theta^2)$$

$$- mg(a - l + b\psi + \tfrac{1}{2}a\psi^2 + \tfrac{1}{2}l\phi^2) + C$$

$$C_1 - \tfrac{1}{2}mgl(\theta^2 + \phi^2) - \tfrac{1}{2}ag(M + 2m)\psi^2$$

$\therefore$ Lagrange's θ–equation is $\dfrac{d}{dt}\left(\dfrac{\partial T}{\partial\dot{\theta}}\right) - \dfrac{\partial T}{\partial\theta} + \dfrac{\partial W}{\partial\theta}$

i.e. $\dfrac{d}{dt}(ml^2\dot{\theta}) - 0 = -mgl\theta$ or $\ddot{\theta} = -\dfrac{g}{l}\theta = -\mu_1\theta$

$\therefore$ Length of simple equivalent pendulum is $g/\mu_1 = l$.

The same length is obtained by forming ϕ-equation.

And Lagrange's ψ equation is $\dfrac{d}{dt}\left(\dfrac{\partial T}{\partial\dot{\psi}}\right) - \dfrac{\partial T}{\partial\psi} = \dfrac{\partial W}{\partial\psi}$

i.e. $\dfrac{d}{dt}(Mk^2\dot{\psi} + 2mb^2\dot{\psi}) - 0 = -\tfrac{1}{2}ag(M + 2m)\psi$

or $(Mk^2 + 2mb^2)\ddot{\psi} = -\tfrac{1}{2}ag(M + 2m)\psi$

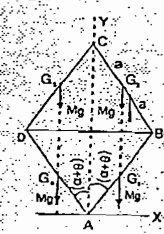
or $\ddot{\psi} = -\dfrac{(M + 2m)ag}{Mk^2 + 2mb^2}\psi = -\mu_2\psi$

$\therefore$ Length of simple equivalent pendulum is (g/μ_2)

$$= \frac{Mk^2 + 2mb^2}{(M + 2m)a}$$

Hence for the system the lengths of its simple equivalent pendulum are l and $\dfrac{Mk^2 + 2mb^2}{(M + 2m)a}$

Ex. 25. *Four uniform rods, each of length 2a are hinged at their ends so as to form a rhombus ABCD. The angles B and D are connected by an elastic string and the lowest end A rests on a horizontal plane while the end C slides on a smooth vertical wire passing through A; in the position of equilibrium the string is stretched to twice its natural length and the angle BAD is 2α. Show that the time of a small oscillation about this position is* $2\pi\left\{\dfrac{2a(1 + 3\sin^2\alpha)\cos\alpha}{3g\cos 2\alpha}\right\}^{1/2}$



Sol. Let M be the mass of each of the rods AB, BC, CD and DA. In the position of equilibrium the rods make angle α with the vertical. When the system is slightly displaced from the position of equilibrium, let the rods make angle $\alpha + \theta$ with the vertical, where θ is small angular displacement.

Referred to A as origin, the horizontal and vertical lines through A as axes, the coordinates (x_{G_1}, y_{G_1}) and (x_{G_2}, y_{G_2}) of G_1 and G_2 are given by

$$x_{G_1} = a\sin(\alpha + \theta),\; y_{G_1} = a\cos(\alpha + \theta),$$

$$x_{G_2} = a\sin(\alpha + \theta),\; y_{G_2} = a\cos(\alpha + \theta).$$

$$\therefore v_{G_1}^2 = \dot{x}_{G_1}^2 + \dot{y}_{G_1}^2 = a^2\dot{\theta}^2$$

$$\text{and } v_{G_2}^2 = \dot{x}_{G_2}^2 + \dot{y}_{G_2}^2 = a^2\{1 + 8\sin^2(\alpha + \theta)\}\dot{\theta}^2$$

Similarly, $v_{G_3}^2 = a^2\dot{\theta}^2$ and $v_{G_4}^2 = a^2\{1 + 8\sin^2(\alpha + \theta)\}\dot{\theta}^2$

If T be the kinetic energy and W the work function of the system, then we have

$$T = 2.[\tfrac{1}{2}M.\tfrac{1}{3}a^2\dot{\theta}^2 + v_{G_2}^2] + 2[\tfrac{1}{2}M.\tfrac{1}{3}a^2\dot{\theta}^2 + v_{G_2}^2]$$

$$= \tfrac{8}{3}Ma^2[1 + 3\sin^2(\alpha + \theta)]\dot{\theta}^2$$

$$\text{and } W = 2(-mg)y_{G_1} + 2(-mg)y_{G_2} - 2\int_0^{2a\sin(\alpha+\theta)} \lambda \cdot \frac{x - a\sin\alpha}{a\sin\alpha}\,dx$$

(where $4x$ is the stretched length of the string at time t and $4a\sin\alpha$ is its stretched length in equilibrium position)

$$= -8Mga\cos(\alpha + \theta) - \frac{\lambda a}{\sin\alpha}[2\sin(\alpha + \theta) - \sin\alpha]^2 + C$$

$\therefore$ Lagrange's θ-equation is $\dfrac{d}{dt}\left(\dfrac{\partial T}{\partial\dot{\theta}}\right) - \dfrac{\partial T}{\partial\theta} = \dfrac{\partial W}{\partial\theta}$

i.e. $\dfrac{d}{dt}[\tfrac{8}{3}Ma^2\{1 + 3\sin^2(\alpha + \theta).2\dot{\theta}\}] - 16Ma^2\sin(\alpha + \theta)\cos(\alpha + \theta)\dot{\theta}^2$

$$= 8Mga\sin(\alpha + \theta) - (4\lambda a/\sin\alpha).\{2\sin(\alpha + \theta) - \sin\alpha\}.\cos(\alpha + \theta).$$

or $\dfrac{16}{3}Ma^2\{1 + 3\sin^2(\alpha + \theta)\}\ddot{\theta}$

$$= 8Mga\sin(\alpha + \theta) - (4\lambda a/\sin\alpha)\{2\sin(\alpha + \theta) - \sin\alpha\}\cos(\alpha + \theta) \quad ...(1)$$

But initially when $\theta = 0, \dot{\theta} = 0, \ddot{\theta} = 0$. $\therefore$ from (1), we get

$$0 = 8Mga\sin\alpha - \frac{4\lambda a}{\sin\alpha}.\sin\alpha\cos\alpha \text{ i.e. } \lambda = 2Mg\tan\alpha.$$

Substituting $\lambda = 2Mg\tan\alpha$ in (1), we get

$$\frac{16}{3}Ma^2\{1 + 3\sin^2(\alpha + \theta)\}.\ddot{\theta}$$

$$= 8Mga\sin(\alpha + \theta) - \frac{8Mga}{\cos\alpha}\{2\sin(\alpha + \theta) - \sin\alpha\}.\cos(\alpha + \theta)$$

or $2a\{1 + 3(\sin\alpha\cos\theta + \cos\alpha\sin\theta)^2\}\ddot{\theta}$

$$= 3g(\sin\alpha\cos\theta + \cos\alpha\sin\theta) - \frac{3g}{\cos\alpha}.\{2(\sin\alpha\cos\theta$$

$$+ \cos\alpha\sin\theta) - \sin\alpha\}.(\cos\alpha\cos\theta - \sin\alpha\sin\theta)$$

or $2a(1 + 3\sin^2\alpha)\ddot{\theta}$

$$= 3g(\sin\alpha + \cos\alpha) - \frac{3g}{\cos\alpha}(2\sin\alpha + 2\theta\cos\alpha - \sin\alpha)(\cos\alpha - \theta\sin\alpha)$$

or $2a(1 + 3\sin^2\alpha)\ddot{\theta}$

$$= 3g(\sin\alpha + \theta\cos\alpha) - \frac{3g}{\cos\alpha}(\sin\alpha\cos\alpha - \theta\sin^2\alpha + 2\theta\cos^2\alpha)$$

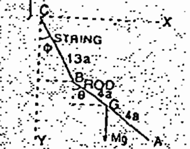
[neglecting higher powers of θ]

or $2a(1 + 3\sin^2\alpha)\ddot{\theta} = \dfrac{3g}{\cos\alpha}.[\sin^2\alpha - \cos^2\alpha]\theta$

or $\ddot{\theta} = \dfrac{3g\cos 2\alpha}{2a(1 + 3\sin^2\alpha)\cos\alpha}\theta = -\mu\theta$ (say)

which represent S.H.M. Therefore the time of small oscillation about the position of equilibrium is $(2\pi/\sqrt{\mu})$.

i.e. $2\pi\left\{\dfrac{2a(1+3\sin^2\alpha)\cos\alpha}{3g\cos 2\alpha}\right\}^{1/2}$



Ex. 26. *A uniform rod AB of length 8a is suspended from a fixed point C by means of a light inextensible string of length 13a attached to B. If the system is slightly displaced in a vertical plane, show that* $(\theta+3\phi)$ *and* $(12\theta-13\phi)$ *are principal coordinates, where θ and φ are the angles which the rod and string respectively make with the vertical. Also show that periods of small oscillations are*

$$2\pi\sqrt{\left(\frac{a}{g}\right)} \text{ and } 2\pi\sqrt{\left(\frac{52a}{3g}\right)}$$

Sol. Let M be the mass and G the centre of gravity of the rod AB. Referred to C as origin, the horizontal and vertical lines through C as axes the coordinates (x_G, y_G) of G are given by

$x_G = 13a\sin\phi + 4a\sin\theta$

and $y_G = 13a\cos\phi + 4a\cos\theta$.

$\therefore v_G^2 = \dot{x}_G^2 + \dot{y}_G^2$

$= (13a\cos\phi\dot{\phi} + 4a\cos\theta\dot{\theta})^2 + (-13a\sin\phi\dot{\phi} - 4a\sin\theta\dot{\theta})^2$

$= a^2[169\dot{\phi}^2 + 16\dot{\theta}^2 + 104\dot{\theta}\dot{\phi}\cos(\theta-\phi)]$

$= a^2(169\dot{\phi}^2 + 16\dot{\theta}^2 + 104\dot{\theta}\dot{\phi})$, $\because$ θ and φ are small

If T be the kinetic energy and W the work function of the system, then we have

$$T = \frac{1}{2}M\cdot\frac{1}{3}(4a)^2\dot{\theta}^2 + \frac{1}{2}M\cdot v_G^2 = \frac{1}{2}Ma^2\left(\frac{64}{3}\dot{\theta}^2 + 169\dot{\phi}^2 + 104\dot{\theta}\dot{\phi}\right)$$

and $W = Mg(13a\cos\phi + 4a\cos\theta) + C$.

$\therefore$ Lagrange's θ-equation is $\dfrac{d}{dt}\left(\dfrac{\partial T}{\partial\dot{\theta}}\right) - \dfrac{\partial T}{\partial\theta} = \dfrac{\partial W}{\partial\theta}$

i.e. $\dfrac{d}{dt}\left[\dfrac{1}{2}Ma^2\left(\dfrac{64}{3}\cdot 2\dot{\theta} + 104\dot{\phi}\right)\right] = -4aMg\sin\theta = -4aMg\theta$, $\because$ θ is small

or $16\ddot{\theta} + 39\ddot{\phi} = -3c\theta$, (where $c = g/a$).

And Lagrange's φ-equation is $\dfrac{d}{dt}\left(\dfrac{\partial T}{\partial\dot{\phi}}\right) - \dfrac{\partial T}{\partial\phi} = \dfrac{\partial W}{\partial\phi}$

i.e. $\dfrac{d}{dt}[\tfrac{1}{2}Ma^2(169\cdot 2\dot{\phi} + 104\dot{\theta})] = -13Mga\sin\phi = -13Mga\phi$, $\because$ φ is small

or $4\ddot{\theta} + 13\ddot{\phi} = -c\phi$, $(c = g/a)$. ...(2)

To find the principal coordinates.

Multiplying (2) by λ and adding to (1), we have

$(16+4\lambda)\ddot{\theta} + (39+13\lambda)\ddot{\phi} = -c(3\theta + \lambda\phi)$. ...(3)

Now choose λ such that

$\dfrac{16+4\lambda}{39+13\lambda} = \dfrac{3}{\lambda}$ or $(4\lambda+16)\lambda = 3(39+13\lambda)$

or $4\lambda^2 - 23\lambda - 117 = 0$ or $(\lambda-9)(4\lambda+13) = 0$

$\therefore \lambda = 9, -13/4$.

When $\lambda = 9$, (3) reduce to

$D^2(\theta+3\phi) = -\dfrac{3}{52}c(\theta+3\phi)$. ...(4)

And when $\lambda = -13/4$, (3) reduce to

$D^2(12\theta-13\phi) = -c(12\theta-13\phi)$ ...(5)

Putting $\theta+3\phi = X$ and $12\theta-13\phi = Y$ in (4) and (5), we have

$D^2X = -\dfrac{3g}{52a}X$ and $D^2Y = -\dfrac{g}{a}Y$ $\left[\because c = \dfrac{g}{a}\right]$

which represents two independent simple harmonic motions.

Thus the principal coordinates are X and Y i.e. $\theta+3\phi$ and $12\theta-13\phi$.

Also the periods of small oscillations are

$$2\pi\Big/\sqrt{\left(\frac{3g}{52a}\right)} \text{ and } 2\pi\Big/\sqrt{\left(\frac{g}{a}\right)}$$

i.e. $2\pi\sqrt{\left(\dfrac{52a}{3g}\right)}$ and $2\pi\sqrt{\left(\dfrac{a}{g}\right)}$

Ex. 27. *A ring slides on a smooth circular hoop of equal mass and of radius a which can turn a vertical plane about a fixed point O in its circumference. If θ and φ be the inclination to the vertical of the radius through O and of the radius through the ring, prove that the principal coordinates are* $(2\theta+\phi)$ *and* $(\phi-\theta)$, *and the periods of small oscillations are* $2\pi\sqrt{(a/2g)}$ *and* $2\pi\sqrt{(2a/g)}$.

Sol. Let M be the mass of each of the ring and the circular hoop of radius a and centre C, which can turn about the point O of its circumference. At time t, let the radius OC of the hoop make an angle θ with the vertical. At this time t, let the ring be at P, such that CP make an angle φ with the vertical. Initially the ring was at the end A of diameter OA which was vertical.

Referred to O as origin, the horizontal and vertical lines through O as axes, the coordinates (x_c, y_c) of the centre C and (x_P, y_P) of the point P are given by

$x_c = a\sin\theta$, $y_c = a\cos\theta$;

$x_P = a(\sin\theta + \sin\phi)$, $y_P = a(\cos\theta + \cos\phi)$

$\therefore v_c^2 = \dot{x}_c^2 + \dot{y}_c^2 = (a\cos\theta\dot{\theta})^2 + (-a\sin\theta\dot{\theta})^2 = a^2\dot{\theta}^2$

and $v_P^2 = \dot{x}_P^2 + \dot{y}_P^2 = a^2(\cos\theta\dot{\theta} + \cos\phi\dot{\phi})^2 + a^2(-\sin\theta\dot{\theta} - \sin\phi\dot{\phi})^2$

$= a^2\{\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}\cos(\theta-\phi)\} = a^2(\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi})$

$\because$ θ, φ are small

If T be the kinetic energy and W the work function of the system, then we have

T = K.E. of the hoop + K.E. of the ring

$= [\tfrac{1}{2}Mk^2\dot{\theta}^2 + \tfrac{1}{2}M\cdot v_c^2] + (\tfrac{1}{2}Mv_P^2)$

$= \tfrac{1}{2}M(a^2\dot{\theta}^2 + a^2\dot{\theta}^2) + \tfrac{1}{2}Ma^2(\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi}) = \tfrac{1}{2}Ma^2(3\dot{\theta}^2 + \dot{\phi}^2 + 2\dot{\theta}\dot{\phi})$

and $W = Mgy_c + Mgy_P + C = Mga(2\cos\theta + \cos\phi) + C$.

$\therefore$ Lagrange's θ-equation is $\dfrac{d}{dt}\left(\dfrac{\partial T}{\partial\dot{\theta}}\right) - \dfrac{\partial T}{\partial\theta} = \dfrac{\partial W}{\partial\theta}$

i.e. $\dfrac{d}{dt}[\tfrac{1}{2}Ma^2(6\dot{\theta} + 2\dot{\phi})] = -2Mga\sin\theta = -2Mga\theta$, $\because$ θ is small

or $3\ddot{\theta} + \ddot{\phi} = -2c\theta$, (where $c = g/a$). ...(1)

And Lagrange's φ-equation is $\dfrac{d}{dt}\left(\dfrac{\partial T}{\partial\dot{\phi}}\right) - \dfrac{\partial T}{\partial\phi} = \dfrac{\partial W}{\partial\phi}$

i.e. $\dfrac{d}{dt}[\tfrac{1}{2}Ma^2(2\dot{\phi} + 2\dot{\theta})] = -Mga\sin\phi = -Mga\phi$, $\because$ φ is small

or $\ddot{\theta} + \ddot{\phi} = -c\phi$, (where $c = g/a$). ...(2)

Multiplying (2) by λ and adding to (1), we get

$(3+\lambda)\ddot{\theta} + (1+\lambda)\ddot{\phi} = -c(2\theta + \lambda\phi)$. ...(3)

Now choose λ such that

$\dfrac{3+\lambda}{1+\lambda} = \dfrac{2}{\lambda}$ or $\lambda^2 + \lambda - 2 = 0$

or $(\lambda-1)(\lambda+2) = 0$ $\therefore \lambda = 1, -2$.

When $\lambda = 1$, (3) reduce to

$$D^2(2\theta+\phi) = -\tfrac{1}{2}c(2\theta+\phi). \quad ...(4)$$

And when $\lambda = -2$, (3) reduce to

$$D^2(\phi-\theta) = -2c(\phi-\theta). \quad ...(5)$$

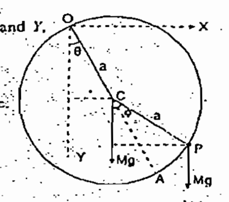
Putting $2\theta+\phi = X$ and $\phi-\theta = Y$ in (4) and (5), we have

$D^2X = -\dfrac{g}{2a}X$ and $D^2Y = -\dfrac{2g}{a}Y$, $\because c = \dfrac{g}{a}$

which represents two independent S.H.M.

Thus the principal coordinates are X and Y, i.e. $2\theta+\phi$ and $\phi-\theta$.

Also the periods of small oscillations are



$$2\pi\Big/\sqrt{\left(\frac{g}{2a}\right)} \text{ and } 2\pi\Big/\sqrt{\left(\frac{2g}{a}\right)} \text{ i.e.}$$

$$2\pi\sqrt{\left(\frac{2a}{g}\right)} \text{ and } 2\pi\sqrt{\left(\frac{a}{2g}\right)}$$

Ex. 28. *At the lowest point of a smooth circular tube, of mass M and radius a, is placed a particle of mass M′; the tube hangs in a vertical plane from its highest point, which is fixed, and can turn freely in its own plane about this point. If this system be slightly displaced show that the periods of the two independent oscillations of the system are* $2\pi\sqrt{\left(\dfrac{2a}{g}\right)}$ *and* $2\pi\sqrt{\left(\dfrac{M}{M+M'}\cdot\dfrac{a}{g}\right)}$.

And that for one principal mode of oscillation the particle remains at rest relative to the tube and for the other, the centre of gravity of the particle and the tube remains at rest.

Sol. Let C be the centre of the smooth circular tube of mass M and radius a which is hanged from its highest point O. At the lowest point A (note that initially diameter OA is vertical) is placed a particle of mass M'. At time t, let the radius OC make an angle θ with the vertical and at this time let the mass M' be at P such that CP make an angle φ with the vertical.

Referred to O as origin, horizontal and vertical lines through O as axes the coordinates (x_c, y_c) of the centre C and (x_P, y_P) of the point P are given by

$x_c = a\sin\theta$, $y_c = a\cos\theta$;

$x_P = a(\sin\theta + \sin\phi)$, $y_P = a(\cos\theta + \cos\phi)$

As in last Ex. 27, $v_c^2 = a^2\dot\theta^2$ and $v_P^2 = a^2(\dot\theta^2 + \dot\phi^2 + 2\dot\theta\dot\phi)$.

If T be the K.E. and W the work function of the system, then

T = K.E. of the circular table + K.E. of the particle

$= (\frac{1}{2}Mk^2\dot\theta^2 + \frac{1}{2}Mv_c^2) + \frac{1}{2}M'v_P^2$

$= \frac{1}{2}M(a^2\dot\theta^2 + a^2\dot\theta^2) + \frac{1}{2}M'a^2(\dot\theta^2 + \dot\phi^2 + 2\dot\theta\dot\phi)$

$= \frac{1}{2}(2M + M')a^2\dot\theta^2 + \frac{1}{2}M'a^2\dot\phi^2 + M'a^2\dot\theta\dot\phi$

and $W = Mg\,y_c + M'g\,y_P + C = (M + M')ga\cos\theta + M'ga\cos\phi + C$

$\therefore$ Lagrange's θ-equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right) - \frac{\partial T}{\partial\theta} = \frac{\partial W}{\partial\theta}$

i.e. $\frac{d}{dt}[(2M + M')a^2\dot\theta + M'a^2\dot\phi] = -(M + M')ga\sin\theta$

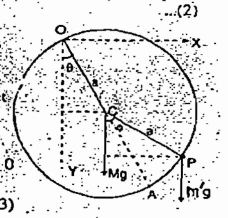
$= -(M + M')ga\theta$, $\because$ θ is small

or $(2M + M')\ddot\theta + M'\ddot\phi = -(M + M')c\theta$, (where $c = g/a$) ...(1)

And Lagrange's ϕ-equation is $\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\phi}\right) - \frac{\partial T}{\partial\phi} = \frac{\partial W}{\partial\phi}$ $\because$ ϕ is small.

i.e. $\frac{d}{dt}(M'a^2\dot\phi + M'a^2\dot\theta) = -M'ga\sin\phi = -M'ga\phi$, $\because$ ϕ is small.

or $\ddot\theta + \ddot\phi = -c\phi$, (where $c = g/a$) ...(2)



Equations (1) and (2) can be written as

$[(2M + M')D^2 + (M + M')c]\theta + M'D^2\phi = 0$

and $D^2\theta + (D^2 + c)\phi = 0$.

Eliminating ϕ between these equations, we get

$[(D^2 + c)\{(2M + M')D^2 + (M + M')c\} - M'D^2]\theta = 0$ or

$[2MD^4 + c(3M + 2M')D^2 + c^2(M + M')]\theta = 0$ ...(3)

Let $\theta = A\cos(pt + B)$ be the solution of (3)

$\therefore$ $D^2\theta = -p^2\theta$ and $D^4\theta = p^4\theta$. Substituting in (3), we get

$[2Mp^4 - c(3M + 2M')p^2 + c^2(M + M')]\theta = 0$

or $(2p^2 - c)[Mp^2 - c(M + M')] = 0$ $\because$ $\theta \neq 0$

$\therefore$ $p_1^2 = \frac{c}{2} = \frac{g}{2a}$ and $p_2^2 = \frac{c(M + M')}{M} = \frac{g(M + M')}{aM}$ $\because$ $c = \frac{g}{a}$

Hence the periods of two independent oscillations are

$\frac{2\pi}{p_1}$ and $\frac{2\pi}{p_2}$ i.e. $2\pi\sqrt{\left(\frac{2a}{g}\right)}$ and $2\pi\sqrt{\left(\frac{M}{M + M'}\frac{a}{g}\right)}$

2nd Part. Multiplying (2) by λ and adding to (1), we get

$\{(2M + M') + \lambda\}\ddot\theta + (M' + \lambda)\ddot\phi = -c[(M + M')\theta + \lambda\phi]$ ...(4)

Now choose, λ such that

$\frac{(2M + M') + \lambda}{M' + \lambda} = \frac{M + M'}{\lambda}$ or $\lambda^2 + M\lambda - (M + M')M' = 0$

or $(\lambda - M')\{\lambda + (M + M')\} = 0$ $\therefore$ $\lambda = M', -(M + M')$.

When $\lambda = M'$, (4), reduce to

$D^2\{(M + M')\theta + M'\phi\} = -\frac{1}{2}c\{(M + M')\theta + M'\phi\}$ ...(5)

And when $\lambda = -(M + M')$, (4) reduce to

$D^2(\phi - \theta) = -\frac{(M + M')}{M}c(\phi - \theta)$ ...(6)

Putting $(M + M')\theta + M'\phi = X$ and $\phi - \theta = Y$, in (5) and (6), we have

$D^2X = -\frac{g}{2a}X$ and $D^2Y = -\frac{(M + M')g}{Ma}Y$. $\because$ $c = \frac{g}{a}$

$\therefore$ The Principal Coordinates are X and Y

i.e. $(M + M')\theta + M'\phi$ and $\phi - \theta$.

For one principal mode $\phi - \theta = 0$ i.e. $\phi = \theta$ which shows that the particle remains at rest relative to the tube.

And for the second mode, $(M + M')\theta + M'\phi = 0$. Now x-coordinate of the C.G. of the particle and the tube

$= \frac{Mx_c + M'x_P}{M + M'} = \frac{Ma\sin\theta + M'a(\sin\theta + \sin\phi)}{M + M'}$

$= \frac{a}{M + M'}\{M\theta + M'(\theta + \phi)\}$, ($\because$ θ and ϕ are small)

$= \frac{a}{M + M'}\{(M + M')\theta + M'\phi\} = 0$, using (7)

shows that the common C.G. of the particle and the tube remains rest.

8.9. Lagrange's Equations of Motion for Impulsive Forces.

Let (x, y, z) be the coordinates of any particle m of the system referred to the rectangular axes, and let them be expressed in terms of the independent variables (generalized coordinates) $\theta, \phi, \psi, \ldots$ so that if t is the time, then we have

$x = f_1(t, \theta, \phi, \psi, \ldots)$, $y = f_2(t, \theta, \phi, \psi, \ldots)$, $z = f_3(t, \theta, \phi, \psi, \ldots)$. ...(1)

Since the change in momentum of a system is equal to the impulses of the forces acting on it, hence if X, Y, Z be the components of the applied impulses at (x, y, z), then giving the system a vertical displacement, the equation of virtual work is $\Sigma m[(u_1 - u_0)\delta x + (v_1 - v_0)\delta y + (w_1 - w_0)\delta z]$

$= \Sigma(X\delta x + Y\delta y + Z\delta z)$ ...(2)

where (u_0, v_0, w_0) and (u_1, v_1, w_1) are respectively, the velocities of m just before and just after the application of the impulses.

From equation (1), we have

$$\delta x = \frac{\partial x}{\partial\theta}\delta\theta + \frac{\partial x}{\partial\phi}\delta\phi + \ldots$$

Similar expressions for δy and δz.

Also $\dot x = \frac{\partial x}{\partial t} + \frac{\partial x}{\partial\theta}\dot\theta + \frac{\partial x}{\partial\phi}\dot\phi + \ldots$ $\therefore$ $\frac{\partial\dot x}{\partial\dot\theta} = \frac{\partial x}{\partial\theta}$

Similarly $\frac{\partial\dot y}{\partial\dot\theta} = \frac{\partial y}{\partial\theta}, \frac{\partial\dot z}{\partial\dot\theta} = \frac{\partial z}{\partial\theta}$ etc.

Substituting the values of $\delta x, \delta y, \delta z$, we have

$$\Sigma(X\delta x + Y\delta y + Z\delta z) = \Sigma\left[\left(X\frac{\partial x}{\partial\theta} + Y\frac{\partial y}{\partial\theta} + Z\frac{\partial z}{\partial\theta}\right)\delta\theta + \left(X\frac{\partial x}{\partial\phi} + Y\frac{\partial y}{\partial\phi} + Z\frac{\partial z}{\partial\phi}\right)\delta\phi + \ldots\right]$$

$= I_\theta\delta_\theta + I_\phi\delta\phi + \ldots = \delta U$ (say). ...(3)

Here $I_\theta\delta\theta, I_\phi\delta\phi, \ldots$ (where $I_\theta, I_\phi, \ldots$ are functions of $\theta, \phi, \ldots$) represents the vertical work (moment) of the applied impulses corresponding to the vertical displacementss $\delta\theta, \delta\phi, \ldots$.

In relation (3), $\delta\theta, \delta\phi, \ldots$ are called the *generalised virtual displacements* and $I_\theta, I_\phi, \ldots$ are called the *generalised components* of impulses.

Substituting for $\delta x, \delta y, \delta z$ the coefficient of $\delta\theta$ in L.H.S. of (2)

$$= \Sigma m\left[(u_1 - u_0)\frac{\partial x}{\partial\theta} + (v_1 - v_0)\frac{\partial y}{\partial\theta} + (w_1 - w_0)\frac{\partial z}{\partial\theta}\right]$$

$$= \Sigma m\left[\left(u_1\frac{\partial x}{\partial\theta} + v_1\frac{\partial y}{\partial\theta} + w_1\frac{\partial z}{\partial\theta}\right) - \left(u_0\frac{\partial x}{\partial\theta} + v_0\frac{\partial y}{\partial\theta} + w_0\frac{\partial z}{\partial\theta}\right)\right] \quad ...(4)$$

But $u_1 = \dot x, v_1 = \dot y, w_1 = \dot z$ and $u_0 = (\dot x)_0, v_0 = (\dot y)_0, w_0 = (\dot z)_0$.

Also $\frac{\partial x}{\partial\theta} = \left(\frac{\partial x}{\partial\theta}\right)_0$ since the coordinates do not change abruptly

$= \left(\frac{\partial\dot x}{\partial\dot\theta}\right)_0$ etc.

$\therefore$ From (4), the coefficient of $\delta\theta$ in L.H.S. of (2)

$$= \Sigma m\left[\left(\dot x\frac{\partial\dot x}{\partial\dot\theta} + \dot y\frac{\partial\dot y}{\partial\dot\theta} + \dot z\frac{\partial\dot z}{\partial\dot\theta}\right)_1 - \left(\dot x\frac{\partial\dot x}{\partial\dot\theta} + \dot y\frac{\partial\dot y}{\partial\dot\theta} + \dot z\frac{\partial\dot z}{\partial\dot\theta}\right)_0\right]$$

$= \left(\frac{\partial T}{\partial\dot\theta}\right)_1 - \left(\frac{\partial T}{\partial\dot\theta}\right)_0$, $\because$ $T = \Sigma\frac{1}{2}m(\dot x^2 + \dot y^2 + \dot z^2)$

Where $[(\partial T/\partial\dot\theta)_1]$ and $[(\partial T/\partial\dot\theta)_0]$ are the values of $(\partial T/\partial\dot\theta)$ just after and just before the impulse.

Hence from (2), we have

$$\left\{\left(\frac{\partial T}{\partial\dot\theta}\right)_1 - \left(\frac{\partial T}{\partial\dot\theta}\right)_0\right\}\delta\theta + \left\{\left(\frac{\partial T}{\partial\dot\phi}\right)_1 - \left(\frac{\partial T}{\partial\dot\phi}\right)_0\right\}\delta\phi + \ldots$$

$= \delta U = I_\theta\delta\theta + I_\phi\delta\phi + \ldots$.

Since $\theta, \phi, \psi, \ldots$ are arbitrary, so equating coefficients of $\delta\theta, \delta\phi, \ldots$, from the two sides, we get

$$\left(\frac{\partial T}{\partial\dot\theta}\right)_1 - \left(\frac{\partial T}{\partial\dot\theta}\right)_0 = I_\theta, \left(\frac{\partial T}{\partial\dot\phi}\right)_1 - \left(\frac{\partial T}{\partial\dot\phi}\right)_0 = I_\phi \text{ etc.} \quad ...(5)$$

These are called the **Lagrange's equations for impulsive forces.** Also $(\partial T/\partial\dot\theta)$ is called the *generalised momentum* associated with the generalised coordinate θ. Similarly for each of the other remaining coordinates.

Hence equations (5) state that

The change in the generalised component of momentum is equal to the generalised component of impulse.

8.10. Deduction of Lagrange's equations of motion under impulsive forces, from the Lagrange's equations of motion for a system under finite forces.

Let $F_\theta, F_\phi, F_\psi, \ldots$ be the generalised forces associated with the n generalised coordinates $\theta, \phi, \psi, \ldots$ etc. Then Lagrange's equations of motion for finite forces are

$$\frac{d}{dt}\left(\frac{\partial T}{\partial\dot\theta}\right) - \frac{\partial T}{\partial\theta} = F_\theta, \frac{d}{dt}\left(\frac{\partial T}{\partial\dot\phi}\right) - \frac{\partial T}{\partial\phi} = F_\phi \text{ etc.}$$

where $F_\theta = \frac{\partial U}{\partial\theta}$ etc.

Now multiplying Lagrange's θ-equation by dt and integrating from $t = 0$ to $t = \tau$, where τ is small, we have

$$\left[\frac{\partial T}{\partial\dot\theta}\right]_{t=0}^{\tau} - \int_0^\tau\frac{\partial T}{\partial\theta}dt = \int_0^\tau F_\theta\,d\theta. \quad ...(1)$$

Since the coordinates do not change abruptly, therefore

$$\lim_{\tau\to 0}\int_0^\tau\frac{\partial T}{\partial\theta}dt = 0.$$

Let $F_\theta \to \infty$ and $\tau \to 0$ in such a way that

$$\lim_{\tau \to 0} \int_0^\tau F_0 \, dt = I_0.$$

Then I_0 is called the *generalised impulsive* force associated with the coordinate θ.

The generalised impulsive forces are calculated easily by the formula $\delta U = I_\theta \delta\theta + I_\phi \delta\phi + \ldots$, where δU is the work which would be done in generalised displacement by the impulsive forces if they were ordinary forces.

Thus equation (1), reduces to $\left(\frac{\partial T}{\partial \dot\theta}\right)_{t=0}^{\tau} = I_0$ i.e. $\left(\frac{\partial T}{\partial \dot\theta}\right)_1 - \left(\frac{\partial T}{\partial \dot\theta}\right)_0 = I_0$

where $[(\partial T/\partial\dot\theta)_1]$ and $[(\partial T/\partial\dot\theta)_0]$ are the values of $(\partial T/\partial\dot\theta)$ just after and just before the impulse respectively.

Which is the *Lagrange's θ-equation under impulsive forces.*

Similarly the other Lagrange's equations can be deduced.

EXAMPLES

Ex. 29. *A heavy uniform rod of mass m and length $2a$ rotating in a vertical plane falls and strikes a smooth horizontal plane. If u and ω be its linear and angular velocities and θ the inclination of the rod to the vertical just before impact, prove that the impulse J is given by*

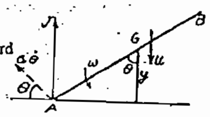
$$(1 + 3\sin^2\theta) J = m(u + a\omega \sin\theta).$$

Sol. Let G be the centre of gravity of the rod AB of mass m and length $2a$. Let y be the height of the centre of gravity G from the plane and θ the inclination of the rod with the vertical at any time t.

$\therefore$ Kinetic energy,

$T = \frac{1}{2}m(\frac{1}{3}a^2\dot\theta^2) + \frac{1}{2}m\dot y^2$

$= \frac{1}{2}m(\frac{1}{3}a^2\dot\theta^2 + \dot y^2)$



And virtual displacement of A in vertical upward direction

= Virtual displacement of C.G. G

+ Virtual displacement of A with regard to G

$= \delta y + a\delta\theta \sin\theta$. $\therefore \delta U = J(\delta y + a\sin\theta\delta\theta)$.

$\therefore$ Lagrange's equations are

$\left(\frac{\partial T}{\partial \dot y}\right)_1 - \left(\frac{\partial T}{\partial \dot y}\right)_0 =$ coeff. of δy in δU

or $m(\dot y - \dot y_0) = J$. ...(1)

And $\left(\frac{\partial T}{\partial \dot\theta}\right)_1 - \left(\frac{\partial T}{\partial \dot\theta}\right)_0 =$ Coefficient of $\delta\theta$ in δU

or $\frac{1}{3}ma^2(\dot\theta - \dot\theta_0) = Ja\sin\theta$. ...(2)

But $\dot y_0 = -u$ ($\because u$ is downwards and y is measured upwards)

and $\dot\theta_0 = -\omega$.

$\therefore$ From (1) and (2), we have $m(\dot y + u) = J$, ...(3)

and $ma(\dot\theta + \omega) = 3J\sin\theta$. ...(4)

Since the plane is inelastic, therefore the end A does not rise from the plane.

$\therefore$ Vertical velocities of the end A after impact $= 0$.

i.e. $\dot y + a\dot\theta\sin\theta = 0$. ...(5)

Substituting $\dot y = -u + \frac{3J}{m}$ and $a\dot\theta = -a\omega + \frac{J}{m}\sin\theta$ from (3) and (4) in (5),

we get $-u + \frac{J}{m} + \left(-a\omega + \frac{3J}{m}\sin\theta\right)\sin\theta = 0$

or $(1 + 3\sin^2\theta)J = m(u + a\omega\sin\theta)$.

Ex. 30. *Three equal uniform rods AB, BC, CD each of mass m and length 2a are at rest in a straight line smoothly joined at B and A. A blow I is given to the middle rod at a distance c from the centre O in a direction perpendicular to it; show that the initial velocity of O is 2I/3m, and that the initial angular velocities of the rods are*

$$\frac{(5a+9c)I}{10ma^2}, \frac{6cI}{5ma^2} \text{ and } \frac{(5a-9c)I}{10ma^2}$$

Sol. Let O be the middle point of the middle rod BC. A blow I is given to the middle rod perpendicular to it at the point P, such that $OP = c$. Let $\dot x$ and $\dot\theta$ be the linear and angular velocities of the rod BC just after the impulse. Also let $\dot\phi$ and $\dot\psi$ be the angular velocities of rods AB and CD respectively at time t.

Let u and v be the vertical velocities of centre of gravity G_1 of AB and that of G_2 of CD respectively.

$\because$ Vertical velocity of end B of rod AB

= Vertical velocity of end B of rod BC

$\therefore \dot x + a\dot\theta = v - a\dot\psi$ or $v = \dot x + a\dot\theta + a\dot\psi$.

The kinetic energy of the system is given by

$T =$ K.E. of rod AB + K.E. of rod BC + K.E. of rod CD

$= [\frac{1}{2}m.\frac{1}{3}a^2\dot\phi^2 + \frac{1}{2}mu^2] + [\frac{1}{2}m.\frac{1}{3}a^2\dot\theta^2 + \frac{1}{2}m\dot x^2] + [\frac{1}{2}m.\frac{1}{3}a^2\dot\psi^2 + \frac{1}{2}mv^2]$

$= \frac{1}{2}m[\frac{1}{3}a^2\dot\phi^2 + (\dot x - a\dot\theta + a\dot\phi)^2 + \frac{1}{3}a^2\dot\theta^2 + \dot x^2 + \frac{1}{3}a^2\dot\psi^2 + (\dot x + a\dot\theta + a\dot\psi)^2]$

$= \frac{1}{2}m[3\dot x^2 + \frac{7}{3}a^2\dot\theta^2 + \frac{4}{3}a^2\dot\psi^2 + \frac{4}{3}a^2\dot\psi^2 + 2a\dot x\dot\phi + 2a\dot x\dot\psi - 2a^2\dot\theta\dot\phi + 2a^2\dot\theta\dot\psi]$

Since before impulse the system was at rest.

$\therefore$ K.E. before impulse $= 0$. Also $\delta U = I(\delta x + c\delta\theta)$.

Lagrange's x-equation is $\left(\frac{\partial T}{\partial \dot x}\right)_1 - \left(\frac{\partial T}{\partial \dot x}\right)_0 =$ coeff. of δx in δU

i.e. $\frac{1}{2}m(6\dot x + 2a\dot\phi + 2a\dot\psi) - 0 = I$ or $3\dot x + a\dot\phi + a\dot\psi = I/m$ ...(1)

Lagrange's θ-equation is $\left(\frac{\partial T}{\partial \dot\theta}\right)_1 - \left(\frac{\partial T}{\partial \dot\theta}\right)_0 =$ coeff. of $\delta\theta$ in δU

i.e. $\frac{1}{2}m(\frac{14}{3}a^2\dot\theta - 2a^2\dot\phi + 2a^2\dot\psi) = Ic$ or $\frac{7}{3}a\dot\theta - a\dot\phi + a\dot\psi = Ic/am$ ...(2)

Lagrange's ϕ-equation is $\left(\frac{\partial T}{\partial \dot\phi}\right)_1 - \left(\frac{\partial T}{\partial \dot\phi}\right)_0 =$ coeff. of $\delta\phi$ in δU

i.e. $\frac{1}{2}m(\frac{8}{3}a^2\dot\phi + 2a\dot x - 2a^2\dot\theta) = 0$ or $\dot x - a\dot\theta + \frac{4}{3}a\dot\phi = 0$ ...(3)

And Lagrange's ψ-equation is $\left(\frac{\partial T}{\partial \dot\psi}\right)_1 - \left(\frac{\partial T}{\partial \dot\psi}\right)_0 =$ coeff. of $\delta\psi$ in δU

i.e. $\frac{1}{2}m(\frac{8}{3}a^2\dot\psi + 2a\dot x + 2a^2\dot\theta) = 0$ or $\dot x + a\dot\theta + \frac{4}{3}a\dot\psi = 0$ ...(4)

Adding equations (3) and (4), we get

$2\dot x + \frac{4}{3}a\dot\phi + \frac{4}{3}a\dot\psi = 0$ or $\frac{3}{2}\dot x + a\dot\psi + a\dot\phi = 0$ ...(5)

Subtracting (5) from (1), we get $\frac{3}{2}\dot x = I/m$, $\therefore \dot x = 2I/3m$.

Subtracting (3) from (4), we get

$2a\dot\theta - \frac{4}{3}a\dot\phi - \frac{4}{3}a\dot\phi = 0$ or $\frac{3}{2}a\dot\theta + a\dot\psi - a\dot\phi = 0$ ...(6)

Subtracting (6) from (2), we get

$\left(\frac{7}{3} - \frac{3}{2}\right)a\dot\theta = \frac{Ic}{am}$ $\therefore \dot\theta = \frac{6cI}{5ma^2}$

Substituting the values of $\dot x$ and $\dot\theta$ in (4), we get

$\frac{2I}{3m} + \frac{6cI}{5ma} + \frac{4}{3}a\dot\psi = 0$ $\therefore \dot\psi = -\frac{(5a+9c)I}{10ma^2}$

Substituting the values of $\dot x$ and $\dot\theta$ in (3), we get

$\frac{2I}{3m} - \frac{6cI}{5ma} + \frac{4}{3}a\dot\phi = 0$ $\therefore \dot\phi = -\frac{(5a-9c)I}{10ma^2}$

Hence the initial angular velocities of the rods are

$$\frac{(5a+9c)I}{10ma^2}, \frac{6cI}{5ma^2} \text{ and } \frac{(5a-9c)I}{10ma^2}$$

Ex. 31. *Three equal uniform rods AB, BC, CD are smoothly joined at B and C and the ends A and D as fastened to smooth fixed points whose distance apart is equal to the length of either rod. The frame being at rest in the form of the square, a blow J is given perpendicular to AB at its middle point and in the plane of the square. Show that the energy set up is $\frac{3J^2}{40m}$, where m is the mass of each rod. Find also the blows at the joints A and C.*

Sol. Let m be the mass of each of the rods AB, BC, CD each of length $2a$. A and D are fixed points s.t. $AD = 2a$. The blow J is given at the middle point G_1 of the rod AB and perpendicular to it.

The rods AB and CD will turn through the same angle say θ and BC will remain parallel to AD.

$\therefore$ Just after the impact, velocity of rod AB = Velocity of rod $CD = a\dot\theta$ and velocity of rod $BC = 2a\dot\theta$.

$\therefore T =$ K.E. of the system

$= 2.$(K.E. of rod AB) + K.E. of rod BC

$= 2.[\frac{1}{2}m.\frac{1}{3}a^2\dot\theta^2 + (a\dot\theta)^2] + \frac{1}{2}m(2a\dot\theta)^2$

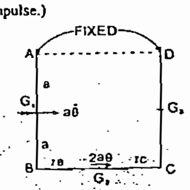
$= \frac{10}{3}ma^2\dot\theta^2$.

Initial K.E. of the system $= 0$.

And $\delta U = Ja\delta\theta$ (virtual work done by impulse.)

$\therefore$ Lagrange's θ-equation for the blow is

$\left(\frac{\partial T}{\partial \dot\theta}\right)_1 - \left(\frac{\partial T}{\partial \dot\theta}\right)_0 =$ coeff. of $\delta\theta$ in δU



i.e. $\frac{20}{3}Ma^2\dot\theta - 0 = Ja$ $\therefore \dot\theta = \frac{3J}{20ma}$

Hence the energy set up by blow

$= \frac{10}{3}ma^2\left(\frac{3J}{20ma}\right)^2 = \frac{3J^2}{40m}$

IInd part. Let I_B and I_C be the impulses at B and C as shown in the figure.

Considering the motion of rod AB and taking moments about A, we have

Change in angular momentum about the axis through A.

= Moments of the impulses about this axis

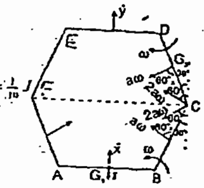
i.e. $m.\frac{4}{3}a^2\dot\theta = J.a - I_B.2a$ or $I_B = \frac{1}{2}J - \frac{2}{3}m\dot\theta$

or $I_B = \frac{1}{2}J - \frac{2}{3}m\left(\frac{3J}{20ma}\right) = \frac{1}{5}J.$

Again considering the motion of rod CD and taking moment about D, we get

$M.\frac{4}{3}a^2\dot{\theta} = I_C 2a$

$\therefore\ I_C = \frac{2}{3}ma\dot{\theta} = \frac{2}{3}ma\left(\frac{3J}{20ma}\right) = \frac{1}{10}J$



Ex. 32. *Six equal uniform rods from a regular hexagon loosely joined at the angular points, and rest on a smooth table, a blow is given perpendicular to one of them at its middle point; find the resulting motion and show that the opposite rod begins to move with one-tenth of the velocity of the rod that is struck.*

Sol. Let M be the mass of the six rods and I the impulse at the middle point G_1 of the rod AB. The motion of rods BC and EF; CD and AF will be symmetrical. Let ω be the angular velocity of each of the rod as all the rods make equal angle after the impulse. Let $\dot{x}$ and $\dot{y}$ be the linear velocities of AB and ED respectively perpendicular to themselves.

Now velocity of end C of rod BC perpendicular to AB

= velocity of end C of rod CD, perpendicular to AB

i.e. {*Vel. of B + vel. of C rel. to B*}

= *vel. of D + vel. of C rel. to D.*}

or $\dot{x} + 2a\omega\cos 60° = \dot{y} - 2a\omega\cos 60°$

$\therefore a\omega = \frac{1}{2}(\dot{y} - \dot{x})$ ...(1)

Now vel. of G_2 relative to B is $a\omega$, perpendicular to BC, its component parallel to AB is $a\omega\sin 60°$ and perpendicular to AB is $a\omega\cos 60°$

$\therefore$ {*Vel. of G_2 parallel to AB*

= (*Vel. of B + Vel. of G_2 rel. to B*)

parallel to

$AB = 0 + a\omega\sin 60° = \sqrt{3}a\omega/2$

And vel. G_2 perp. to AB = (Vel. of B + Vel. of G_2 rel. to B), perp. to

$AB = \dot{x} + a\omega\cos 60° = \dot{x} + a\omega/2$

$\therefore$ (Vel. of $G_2)^2 = (\sqrt{3}a\omega/2)^2 + (\dot{x} + a\omega/2)^2$

Also vel. of G_3 relative to D is $a\omega$, perpendicular to CD, its components parallel and perpendicular to ED are $a\omega\sin 60°$ and $a\omega\cos 60°$ respectively.

$\therefore$ Vel. of G_3, Parallel to $ED = 0 + a\omega\sin 60° = a\omega\sqrt{3}/2$

and vel. of G_3, perp. to $ED = \dot{y} - a\omega\cos 60° = \dot{y} - a\omega/2$.

$\therefore$ (vel. of $G_3)^2 = (a\omega\sqrt{3}/2)^2 + (\dot{y} - a\omega/2)^2$

Now K.E. of rod $AB = \frac{1}{2}M\dot{x} = T_1$ (say)

K.E. of rod BC = K.E. of rod $EF = \frac{1}{2}M.\frac{1}{3}a^2\omega^2 + \frac{1}{2}M(\text{vel. of } G_2)^2$

$= \frac{1}{2}M\left[\frac{1}{3}a^2\omega^2 + \frac{3}{4}a^2\omega^2 + (\dot{x} + \frac{1}{2}a\omega)^2\right]$

$= \frac{1}{6}M(4a^2\omega^2 + 3\dot{x}^2 + 3a\omega\dot{x})$

$= \frac{1}{6}M\left[(\dot{y} - \dot{x})^2 + 3\dot{x}^2 + \frac{3}{2}(\dot{y} - \dot{x})\dot{x}\right]$ [from (1)]

$= \frac{1}{12}M(5\dot{x}^2 + 2\dot{y}^2 - \dot{x}\dot{y}) = T_2$ (say)

And K.E. of rod CD = K.E. of rod AF

$= \frac{1}{2}M.\frac{1}{3}a^2\omega^2 + \frac{1}{2}M(\text{vel. of } G_3)^2$

$= \frac{1}{2}M\left[\frac{1}{3}a^2\omega^2 + \frac{3}{4}a^2\omega^2 + (\dot{y} - \frac{1}{2}a\omega)^2\right]$

$= \frac{1}{6}M(4a^2\omega^2 + 3\dot{y}^2 - 3a\omega\dot{y})]$

$= \frac{1}{6}M\left[(\dot{y} - \dot{x})^2 + 3\dot{y}^2 - \frac{3}{2}(\dot{y} - \dot{x})\dot{y}\right]$ [from (1)]

$= \frac{1}{12}M(2\dot{x}^2 + 5\dot{y}^2 - \dot{x}\dot{y}) = T_3$ (say)

And K.E. of rod $DE = \frac{1}{2}M\dot{y}^2 = T_4$ (say)

$\therefore T$ = K.E. of the system $= T_1 + 2T_2 + 2T_3 + T_4$

$= \frac{1}{2}M\dot{x}^2 + \frac{1}{6}M(5\dot{x}^2 + 2\dot{y}^2 - \dot{x}\dot{y}) + \frac{1}{6}M(2\dot{x}^2 + 5\dot{y}^2 - \dot{x}\dot{y}) + \frac{1}{2}M\dot{y}^2$

$= \frac{1}{6}M(5\dot{x}^2 + 5\dot{y}^2 - \dot{x}\dot{y})$.

Also K.E. before impulse = 0

Also $\delta U = I\,\delta x$

$\therefore$ Lagrange's x-equation is

$\left(\frac{\partial T}{\partial \dot{x}}\right)_1 - \left(\frac{\partial T}{\partial \dot{x}}\right)_0$ = coeff. of δx in δU

i.e. $\frac{1}{3}M(10\dot{x} - \dot{y}) - 0 = I$ or $10\dot{x} - \dot{y} = 3I/M$ ...(2)

And Lagrange's y-equation is

$\left(\frac{\partial T}{\partial \dot{y}}\right)_1 - \left(\frac{\partial T}{\partial \dot{y}}\right)_0$ = coeff. of δy in δU

i.e. $\frac{1}{3}M(10\dot{y} - \dot{x}) - 0 = 0$ or $10\dot{y} - \dot{x} = 0$ ...(3)

Solving (2) and (3), we get $\dot{x} = \frac{10I}{33M}$, $\dot{y} = \frac{I}{33M}$,

and $a\omega = \frac{1}{2}(\dot{y} - \dot{x}) = -\frac{3I}{22aM}$

EXERCISE

1. A homogeneous rod OA, of mass m_1, and length $2a$, is freely hinged at O to a fixed point; at its other end is freely attached another homogeneous rod AB, of mass m_2 and length $2b$; the system moves under gravity; find equations to determine motion.

Hint. $T = \frac{2}{3}m_1a^2\dot{\theta}^2 + \frac{1}{2}m_2\left[\frac{1}{3}b^2\dot{\phi}^2 + 4a^2\dot{\theta}^2 + b^2\dot{\phi}^2 + 4ab\dot{\theta}\dot{\phi}\cos(\phi - \theta)\right]$

$W = g(m_1 + 2m_2)a\cos\theta + m_2gb\cos\phi + C.$

Lag. θ and ϕ equations, which determine motion.

$(\frac{1}{3}m_1 + m_2)4a\ddot{\theta} + 2m_1b\ddot{\phi}\cos(\phi - \theta) - 2m_2b\dot{\phi}^2\sin(\phi - \theta) = -g(m_1 + 2m_2)\sin\theta$

And $\frac{4}{3}b\ddot{\phi} + 2a\ddot{\theta}\cos(\phi - \theta) + 2a\dot{\theta}^2\sin(\phi - \theta) = -g\sin\theta.$

2. A uniform straight rod, of length $2a$, is freely movable about its centre and a particle of mass one fourth that of the rod is attached by a light inextensible string of length $3a$, to one end of the rod. Find the period of principal oscillations.

[Hint. Proceed as in ex. 15 on page 380]

3. A perfectly rough sphere rests at the lowest point of a fixed spherical cavity of double its own radius. To the highest point of the movable sphere is attached a particle of mass 7/20 time that of the sphere and the system is disturbed. Show that the oscillations are the same as those of a simple pendulum of length 14/5 times the radius of the sphere.
[Hint. Proceed exactly as in ex 22 on page 393].
Here $a = 2b$ and $m' = 7m/20$.

4. A thin circular ring of radius a and mass M lies on a smooth horizontal plane and two light elastic strings are attached to it at opposite ends of a diameter, the other ends of the string being fastened to fixed points in the diameter produced. Show that for small oscillations in the plane of the ring the periods are the values of $2\pi/p$ given by

$$\frac{Mlp^2}{2T} - 1 = 0 \text{ or } \frac{b}{l - b} \text{ or } \frac{l}{a}$$

5. A uniform rod of mass $2m$ and length a can turn about one end which is fixed and to the other is smoothly hinged a uniform rod of mass m and length $2a$. If the system performs small oscillations in a vertical plane about its position of equilibrium, show that in one principal mode the inclinations to the vertical, measured in opposite directions, of the rods are equal. Determine the other principal mode and periods of small oscillations.
[Ans. Principal coordinates $2(\theta + \phi)$ and $2\theta - \phi$.
Periods $2\pi\sqrt{(1/3g)}$ and $2\pi\sqrt{(11a/6g)}$.

6. A square $ABCD$ formed of four freely jointed uniform rods; is rotating about its centre of gravity on smooth horizontal table with angular velocity ω. When the joint A is suddenly fixed, show that the initial angular velocities of the rods are equal and that 3/5 of the kinetic energy is immediately lost.

7. A rhombus of smoothly jointed equal uniform rods of lengths $2a$ lies on a smooth horizontal table. It is struck by a blow J perpendicular to one of its sides at a distance c from the middle point; show that the angular velocity of the side instantly becomess $3Jc/8ma^2$, where m is the mass of each side.

8. A frame work in the form of a regular hexagon $ABCDEF$ consists of uniform rods loosely jointed at the corners and rests on a smooth table; a string tied to the middle point of AB is jerked in the direction of AB. Find the resulting initial motion and show that the velocities along AB and DE of their middle points are in opposite directions and in the ratio 59 : 4.

9. Obtain Lagrange's equations of motion for a system under finite forces and hence deduce the same under impulsive forces.

CSAT TEST SERIES SCHEDULE – 2015

TEST NO.	DATE	DAY	SECTIONS TO BE COVERED	No. of Qns.	TOPICS TO BE COVERED
Test-1	8Mar2015	Sunday	BN GMA DS GC DM	23 21 05 26 05	BN- Chapters- 1, 2, 3, 4, 5, 6, 7 GMA- Chapters- 1, 2, 3, 4, 5, 6 Data Sufficiency General Comprehension Decision Making
Test-2	15Mar2015	Sunday	LR & AA DI GC DM & IPS	25 25 25 05	LR & AA- Chapters- 1, 2, 3, 4, 5, 6, 7, 8 DI- Table Chart & Pie Chart General Comprehension DM & IPS
Test-3	22Mar2015	Sunday	BN LR & AA DI GC	20 20 10 30	BN- Chapters- 8, 9, 10, 11, 12,13 LR & AA- Chapters- 9, 10, 11, 12, 13, 14 DI- Mixed Charts General Comprehension
Test-4	29Mar2015	Sunday	GMA DI DS GC	25 25 5 25	GMA- 7, 8, 9, 10 DI- Bar Chart & Line Chart Data Sufficiency General Comprehension
Test-5	5Apr2015	Sunday	BN GMA LR & AA GC	25 10 15 30	BN- Chapters- 1, 2, 4, 6 GMA- Chapters- 11, 12, 13, 14 LR & AA- Chapters- 15, 16, 17, 18, 19 General Comprehension
Test-6	12Apr2015	Sunday	GMA LR & AA GC DM & IPS	25 20 30 5	GMA- Chapters- 3, 4, 5, 11, 12 LR & AA- Chapters- 20, 21, 22, 23, 24 General Comprehension DM & IPS
Test-7	19Apr2015	Sunday	BN LR & AA DI GC	10 25 20 25	BN- Chapters- 7, 8, 9, 10 LR & AA- Chapters- 25, 26, 27, 28, 29, 30 DI- Pie Chart, Line Chart & Mixed Chart General Comprehension
Test-8	26Apr2015	Sunday	BN GMA GC DM & IPS	20 20 30 10	BN- Chapters- 3, 5, 11, 12 GMA- Chapters- 1, 2, 7, 8, 9 General Comprehension DM & IPS
Test-9	3May2015	Sunday	DI LR & AA GC DM & IPS	25 22 28 05	DI- Table Chart, Bar Chart & Mixed Chart LR & AA- Chapters- 3, 5, 6, 7, 11, 12, 14 General Comprehension DM & IPS
Test-10	10May2015	Sunday	BN GMA DI LR & AA GC DM & IPS	12 13 10 10 30 05	BN- Chapters- 2, 4, 8, 9, 10, 11 GMA- Chapters- 10, 11, 12, 13, 14 DI- Mixed Chart & Pie Chart LR & AA- Chapters- 15, 16, 17, 18, 20, 22 General Comprehension DM & IPS
Test-11	17May2015	Sunday	Full Syllabus	80	All Sections
Test-12	24May2015	Sunday	Full Syllabus	80	All Sections
Test-13	31May2015	Sunday	Full Syllabus	80	All Sections
Test-14	3 June 2015	Wednesday	Full Syllabus	80	All Sections
Test-15	7 June 2015	Sunday	Full Syllabus	80	All Sections
Test-16	14June2015	Sunday	Full Syllabus	80	All Sections
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Test-18	24June2015	Wednesday	Full Syllabus	80	All Sections
Test-19	28June2015	Sunday	Full Syllabus	80	All Sections
Test-20	5 July2015	Sunday	Full Syllabus	80	All Sections
Test-21	12July2015	Sunday	Full Syllabus	80	All Sections
Test-22	15July2015	Wednesday	Full Syllabus	80	All Sections
Test-23	19July2015	Sunday	Full Syllabus	80	All Sections
Test-24	26July2015	Sunday	Full Syllabus	80	All Sections
Test-25	2 Aug 2015	Sunday	Full Syllabus	80	All Sections

Test Timing
Delhi Centres:- Rajinder Nagar 10am, Mukherjee Nagar 2pm (Flexi Timing For Weekdays)
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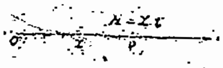
4. Organisational culture, organisational environment and other ecological factors that act as a constraint
5. To check different qualities of candidates as a Team Member and Leader in organisation
6. Intra and inter organisational aspect including public relations

Head Office: 105-106, Top Floor, Mukherjee Tower, Mukherjee Nagar, Delhi-9. Branch Office: 25/8, Old Rajender Nagar Market, Delhi-60. Ph.: 9999197625, 011-45629987. Regional Office: H.No. 1-10-237, 2nd Floor, Room No. 202 R.K'S-Kancham's Blue Sapphire Ashok Nagar Hyderabad-20. Ph. 9652351152.

RECTILINEAR MOTION. (S.H.M) SET-I

1. Introduction. When a point (or particle) moves along a straight line, its motion is said to be a rectilinear motion. Hence in this chapter we shall discuss the motion of a point (or particle), along a straight line which may be either horizontal or vertical.

2. Velocity and acceleration.



Suppose a particle moves along a straight line OX where O is a fixed point on the line. Let P be the position of the particle at time t, where $OP=x$. If $\mathbf{r}$ denotes the position vector of P and $\mathbf{i}$ denotes the unit vector along OX, then $\mathbf{r}=\vec{OP}=x\mathbf{i}$.

Let $\mathbf{v}$ be the velocity vector of the particle at P. Then

$$\mathbf{v}=\frac{d\mathbf{r}}{dt}=\frac{d}{dt}(x\mathbf{i})=\frac{dx}{dt}\mathbf{i}+x\frac{d\mathbf{i}}{dt}=\frac{dx}{dt}\mathbf{i},$$

because $\mathbf{i}$ is a constant vector. Obviously the vector $\mathbf{v}$ is collinear with the vector $\mathbf{i}$. Thus for a particle moving along a straight line the direction of velocity is always along the line itself. If at P the particle be moving in the direction of x increasing (*i.e.*, in the direction OX) and if the magnitude of its velocity *i.e.*, its *speed* be v, we have

$$\mathbf{v}=v\mathbf{i}=\frac{dx}{dt}\mathbf{i}. \text{ Therefore } \frac{dx}{dt}=v.$$

On the other hand if at P the particle be moving in the direction of x decreasing (*i.e.*, in the direction XO) and if the magnitude of its velocity be v, we have

$$\mathbf{v}=-v\mathbf{i}=\frac{dx}{dt}\mathbf{i}. \text{ Therefore, } \frac{dx}{dt}=-v.$$

Remember. *In the case of a rectilinear motion the velocity of a particle at time t is dx/dt along the line itself and is taken with positive or negative sign according as the particle is moving in the direction of x increasing or x decreasing.*

Now let $\mathbf{a}$ be the acceleration vector of the particle at P. Then

$$\mathbf{a}=\frac{d\mathbf{v}}{dt}=\frac{d}{dt}\left(\frac{dx}{dt}\mathbf{i}\right)=\frac{d^2x}{dt^2}\mathbf{i}.$$

Thus the vector $\mathbf{a}$ is collinear with $\mathbf{i}$ *i.e.*, the direction of acceleration is always along the line itself. If at P the acceleration be acting in the direction of x increasing and if its magnitude be f, we have $\mathbf{a}=f\mathbf{i}=\frac{d^2x}{dt^2}\mathbf{i}$. Therefore $\frac{d^2x}{dt^2}=f$. On the other hand if at P the acceleration be acting in the direction of x decreasing and if its magnitude be f, we have

$$\mathbf{a}=-f\mathbf{i}=\frac{d^2x}{dt^2}\mathbf{i}\text{ ; therefore } \frac{d^2x}{dt^2}=-f.$$

Remember. *In the case of a rectilinear motion the acceleration of a particle at time t is d^2x/dt^2 along the line itself and is taken with positive or negative sign according as it acts in the direction of x increasing or x decreasing.*

Since the acceleration is produced by the force, therefore while considering the sign of d^2x/dt^2 we must notice the direction of the acting force and not the direction in which the particle is moving. For example if the direction of the acting force is that of x increasing, then d^2x/dt^2 must be taken with positive sign whether the particle is moving in the direction of x increasing or in the direction of x decreasing.

Other Expressions for acceleration :

Let $v=\frac{dx}{dt}$. We can then write

$$\frac{d^2x}{dt^2}=\frac{d}{dt}\left(\frac{dx}{dt}\right)=\frac{dv}{dt}=\frac{dv}{dx}\frac{dx}{dt}=v\frac{dv}{dx}$$

Thus $\frac{d^2x}{dt^2}$, $\frac{dv}{dt}$ and $v\frac{dv}{dx}$ are three expressions for representing the acceleration and any one of them may be used to suit the convenience in working out the problems.

Note. Often we denote dx/dt by $\dot{x}$ and d^2x/dt^2 by $\ddot{x}$.

Illustrative Examples :

Ex. 1. *If at time t the displacement x of a particle moving away from the origin is given by $x=a\sin t+b\cos t$, find the velocity and acceleration of the particle.*

Sol. Given that $x=a\sin t+b\cos t$.

Differentiating w.r.t. 't', we have the velocity $v=dx/dt=a\cos t-b\sin t$.

Differentiating again, we have the acceleration $=dv/dt=-a\sin t-b\cos t=-x$.

Ex. 2. *A point moves in a straight line so that its distance s from a fixed point at any time t is proportional to t^n. If v be the velocity and f the acceleration at any time t, show that*

$$v^2=nfs/(n-1).$$

Sol. Here, distance $s\propto t^n$.

$\therefore$ let $s=k\,t^n$, ...(1)

where k is a constant of proportionality.

Differentiating (1), w.r.t. 't', we have the velocity $v=ds/dt=knt^{n-1}$. ...(2)

Again differentiating (2), the acceleration $f=dv/dt=kn(n-1)\,t^{n-2}$. ...(3)

$$\therefore\ v^2=(knt^{n-1})^2=k^2n^2t^{2n-2}$$
$$=\frac{n.\{kn(n-1)\,t^{n-2}\}.kt^n}{(n-1)}$$
$$=\frac{nfs}{(n-1)},\text{ substituting from (1) and (3).}$$

Ex. 3. *A particle moves along a straight line such that its displacement x, from a point on the line at time t, is given by*

$$x=t^3-9t^2+24t+6.$$

Determine (i) the instant when the acceleration becomes zero, (ii) the position of the particle at that instant and (iii) the velocity of the particle, then.

Sol. Here, $x=t^3-9t^2+24t+6$.

$\therefore$ the velocity $v=dx/dt=3t^2-18t+24$

and the acceleration $f=d^2x/dt^2=6t-18$.

(i) Now the acceleration $=0$, when $6t-18=0$ or $t=3$.

Thus the acceleration is zero when $t=3$ seconds.

(ii) When $t=3$, position of the particle is given by $x=3^3-9.3^2+24.3+6=24$ units.

(iii) When $t=3$, the velocity $v=3\cdot3^2-18\cdot3+24=-3$ units.

Thus when $t=3$, the velocity of the particle is 3 units in the direction of x decreasing.

Ex. 4. *A particle moves along a straight line and its distance from a fixed point on the line is given by $x=a\cos(\mu t+\epsilon)$. Show that its acceleration varies as the distance from the origin and is directed towards the origin.*

Sol. We have $x=a\cos(\mu t+\epsilon)$. ...(1)

Differentiating w.r.t. t, we get

$dx/dt=-a\mu\sin(\mu t+\epsilon)$,

and $d^2x/dt^2=-a\mu^2\cos(\mu t+\epsilon)=-\mu^2x$. from (1)

Hence the acceleration varies as the distance x from the origin. The negative sign indicates that it is in the negative sense of x-axis *i.e.*, towards the origin.

Ex. 5. *A particle moves along a straight line such that its distance x from a fixed point on it and the velocity v there are related by $v^2=\mu(a^2-x^2)$. Prove that the acceleration varies as the distance of the particle from the origin and is directed towards the origin.*

Sol. We have $v^2=\mu(a^2-x^2)$. ...(1)

Differentiating (1) w.r.t. x, we get

$$2v\frac{dv}{dx}=\mu(-2x).\quad \therefore\ \frac{d^2x}{dt^2}=v\frac{dv}{dx}=-\mu x.$$

Hence the acceleration varies as the distance x from the origin. The negative sign indicates that it is in the direction of x decreasing *i.e.*, towards the origin.

Ex. 6. *The velocity of a particle moving along a straight line, when at a distance x from the origin (centre of force) varies as $\sqrt{\{(a^2-x^2)/x^2\}}$. Find the law of acceleration.*

Sol. Let v be the velocity of the particle when it is at a distance x from the origin. Then according to the question, we have $v=\mu\sqrt{\{(a^2-x^2)/x^2\}}$, where μ is a constant.

$\therefore\ v^2=\mu^2(a^2-x^2)/x^2=\mu^2(a^2/x^2-1)$.

Differentiating w.r.t. x, we get

$$2v\frac{dv}{dx}=\mu^2\left(-\frac{2a^2}{x^3}\right).\quad \therefore\ v\frac{dv}{dx}=\frac{d^2x}{dt^2}=-\frac{\mu^2a^2}{x^3}.$$

Hence the acceleration varies inversely as the cube of the distance from the origin and is directed towards the centre of force.

Ex. 7. *The law of motion in a straight line being given by $s=\frac{1}{2}vt$, prove that the acceleration is constant.*

Sol. We have $s=\frac{1}{2}vt=\frac{1}{2}\frac{ds}{dt}t.\ \left[\because\ v=\frac{ds}{dt}\right]$

Differentiating w.r.t., 't' we get

$$\frac{ds}{dt}=\frac{1}{2}\frac{d^2s}{dt^2}t+\frac{1}{2}\frac{ds}{dt}\quad\text{or}\quad\frac{1}{2}\frac{ds}{dt}=\frac{1}{2}\frac{d^2s}{dt^2}t$$

or $\frac{ds}{dt}=\frac{d^2s}{dt^2}t.$

Differentiating again w.r.t. t, we get

$$\frac{d^3s}{dt^3}=\frac{d^3s}{dt^3}+\frac{d^3s}{dt^3}t \quad \text{or} \quad \frac{d^3s}{dt^3}t=0 \quad \text{or} \quad \frac{d^3s}{dt^3}=0$$

because $t \neq 0$.

Now $\frac{d^3s}{dt^3}=0 \Rightarrow \frac{d}{dt}\left(\frac{d^2s}{dt^2}\right)=0 \Rightarrow \frac{d^2s}{dt^2}=\text{constant}.$

Hence the acceleration is constant.

Ex. 8. *A point moves in a straight line so that its distance from a fixed point in that line is the square root of the quadratic function of the time; prove that its acceleration varies inversely as the cube of the distance from the fixed point.*

Sol. At any time t, let x be the distance of the particle from a fixed point on the line. Then according to the question, we have

$x=\sqrt{(at^2+2bt+c)}$, where a, b, c are constants.

$$\therefore \quad x^2=at^2+2bt+c. \qquad \text{...(1)}$$

Differentiating w.r.t. t, we get

$$2x\frac{dx}{dt}=2at+2b$$

or $$\frac{dx}{dt}=\frac{at+b}{x}. \qquad \text{...(2)}$$

Differentiating again w.r.t. 't', we have

$$\frac{d^2x}{dt^2}=\frac{ax-(at+b)(dx/dt)}{x^2}=\frac{ax-(at+b)\{(at+b)/x\}}{x^2}, \text{[from (2)]}$$

$$=\frac{ax^2-(at+b)^2}{x^3}=\frac{a(at^2+2bt+c)-(a^2t^2+2abt+b^2)}{x^3}$$

$$=\frac{ac-b^2}{x^3}=(\text{some constant}).\frac{1}{x^3}.$$

Hence the acceleration varies inversely as the cube of the distance x from the fixed point.

Ex. 9. *If a point moves in a straight line in such a manner that its retardation is proportional to its speed, prove that the space described in any time is proportional to the speed destroyed in that time.*

Sol. Here it is given that the retardation $\propto$ speed.

$$\therefore \quad -\frac{dv}{dt}=kv, \quad \text{where } k \text{ is a constant of proportionality}$$

or $$-v\frac{dv}{dx}=kv \quad \text{or} \quad dx=-\frac{1}{k}dv.$$

Integrating, $x=-(v/k)+A$,

where A is constant of integration.

Suppose the particle starts from the origin with velocity u. Then $v=u$, $x=0$.

$$\therefore \quad 0=-\frac{u}{k}+A \quad \text{or} \quad A=\frac{u}{k}.$$

$$\therefore \quad x=-\frac{v}{k}+\frac{u}{k}=\frac{1}{k}(u-v)$$

or $$(u-v)=kx. \qquad \text{...(1)}$$

Now the space described in time t is x and the speed destroyed in time $t=u-v$. Hence from (1), we conclude that the space described in any time is proportional to the speed destroyed in that time.

Ex. 10. *Prove that if a point moves with a velocity varying as any power (not less than unity) of its distance from a fixed point which it is approaching, it will never reach that point.*

Sol. If x is the distance of the particle from the fixed point O at any time t, then its speed v at that time is given by $v=kx^n$, where k is a constant and n is not less than 1.

Since the particle is moving towards the fixed point *i.e.*, in the direction of x decreasing, therefore

$dx/dt=-v$

or $$dx/dt=-kx^n. \qquad \text{...(1)}$$

Case I. If $n=1$, then from (1), we have

$$dx/dt=-kx$$

or $$dt=-\frac{1}{k}\frac{dx}{x}.$$

Integrating, $t=-(1/k)\log x+A$, where A is a constant.

Putting $x=0$, the time t to reach the fixed point O is given by

$$t=-(1/k)\log 0+A=\infty$$

i.e., the particle will never reach the fixed point O.

Case II. If $n>1$, then from (1), we have

$$dt=-\frac{1}{k}x^{-n}dx.$$

Integrating, $$t=-\frac{1}{k}\frac{x^{-n+1}}{-n+1}+B, \text{ where } B \text{ is a constant}$$

or $$t=\frac{1}{k(n-1)x^{n-1}}+B.$$

Putting $x=0$, the time t to reach the fixed point O is given by

$$t=\infty+B=\infty$$

i.e., the particle will never reach the fixed point O.

Hence if $n \geq 1$, the particle will never reach the fixed point it is approaching.

Ex. 11. *The velocity of a particle moving along a straight line is given by the relation $v^2=ax^2+2bx+c$. Prove that the acceleration varies as the distance from a fixed point in the line.*

Sol. Here given that $v^2=ax^2+2bx+c$.

Differentiating w.r.t. 'x', we have

$$2v\frac{dv}{dx}=2ax+2b$$

or $$f=v\frac{dv}{dx}=ax+b=a\left(x+\frac{b}{a}\right).$$

Let P be the position of the particle at time t.

If $x=-(b/a)$ is the fixed point O', then the distance of the particle at time t from O'

$$=O'P=x-\left(-\frac{b}{a}\right)=x+\frac{b}{a}.$$

$$\therefore \quad f=a.O'P \quad \text{or} \quad f \propto O'P.$$

Hence the acceleration varies as the distance from a fixed point $x=-(b/a)$ in the line.

Ex. 12. *If t be regarded as a function of velocity v, prove that the rate of decrease of acceleration is given by $f^3(d^2t/dv^2)$, f being the acceleration.*

Sol. Let f be the acceleration at time t. Then $f=dv/dt$.

Now the rate of decrease of acceleration $=-df/dt$

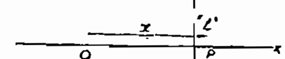
$$=-\frac{d}{dt}\left(\frac{dv}{dt}\right)=-\frac{d}{dt}\left(\frac{dt}{dv}\right)^{-1},$$ regarding t as a function of v

$$=-\left[\frac{d}{dv}\left(\frac{dt}{dv}\right)^{-1}\right]\frac{dv}{dt}=\left(\frac{dt}{dv}\right)^{-2}\frac{d^2t}{dv^2}\cdot\frac{dv}{dt}$$

$$=\left(\frac{dv}{dt}\right)^2\cdot\frac{dv}{dt}\cdot\frac{d^2t}{dv^2}=\left(\frac{dv}{dt}\right)^3\cdot\frac{d^2t}{dv^2}=f^3\frac{d^2t}{dv^2}.$$

2. Motion under constant acceleration. *A particle moves in a straight line with a constant acceleration f, the initial velocity being u, to discuss the motion.*

Suppose a particle moves in a straight line OX starting from O with velocity u. Take O as origin. Let P be the position of the particle at any time t, where $OP=x$. The acceleration of P is constant and is f. Therefore, the equation of motion of P is



$$\frac{d^2x}{dt^2}=f. \qquad \text{...(1)}$$

If v is the velocity of the particle at any time t, then $v=dx/dt$. So integrating (1) w.r.t. t, we get

$v=dx/dt=ft+A$, where A is constant of integration.

But initially at O, $v=u$ and $t=0$; therefore $A=u$. Thus we have

$$v=\frac{dx}{dt}=u+ft. \qquad \text{...(2)}$$

The equation (2) gives the velocity v of the particle at any time t.

Now integrating (2) w.r.t. 't', we get

$x=ut+\frac{1}{2}ft^2+B$, where B is a constant.

But at O, $t=0$ and $x=0$; therefore $B=0$. Thus we have

$$x=ut+\tfrac{1}{2}ft^2. \qquad \text{...(3)}$$

The equation (3) gives the position of the particle at any time t.

The equation of motion (1) can also be written as

$$v\frac{dv}{dx}=f \quad \text{or} \quad 2v\frac{dv}{dx}=2f.$$

Integrating it w.r.t. x, we get

$v^2=2fx+C$. But at O, $x=0$ and $v=u$; therefore $C=u^2$. Hence we have

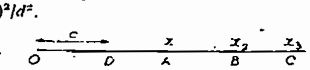
$$v^2=u^2+2fx. \qquad \text{...(4)}$$

Thus in equations (2), (3) and (4) we have obtained the three well known formulae of rectiliner motion with constant acceleration.

Illustrative Fxamples

Ex. 13. *A particle moves in a straight line with constant acceleration and its distances from the origin O on the line (not necessarily the position at time $t=0$) at times t_1, t_2, t_3 are x_1, x_2, x_3 respectively. Show that if t_1, t_2, t_3 form an A. P. whose common difference is d and x_1, x_2, x_3 are in G. P., then the acceleration is $(\sqrt{x_1}-\sqrt{x_3})^2/d^2$.*

Sol. Let O be the origin and D the position of start *i.e.*, the position at $t=0$.



Let $OD=c$. Suppose u is the initial velocity and f the constant acceleration. Let A, B, C be the positions of the particle at times t_1, t_2, t_3 respectively and let $OA=x_1$, $OB=x_2$ and $OC=x_3$. Then

$$x_1-c=ut_1+\tfrac{1}{2}ft_1^2, \; x_2-c=ut_2+\tfrac{1}{2}ft_2^2, \; x_3-c=ut_3+\tfrac{1}{2}ft_3^2.$$

These equations give

$$x_1+x_3-2x_2=u\,(t_1+t_3-2t_2)+\tfrac{1}{2}f\,(t_1^2+t_3^2-2t_2^2). \quad \ldots(1)$$

But x_1, x_2, x_3 are in G.P., so that $x_2=\sqrt{(x_1 x_3)}$. Also t_1, t_2, t_3 are in A.P. whose common difference is d. Therefore $t_1+t_3=2t_2$ and $t_3-t_1=2d$. Putting these values in (1), we get

$$x_1+x_3-2\sqrt{(x_1 x_3)}=u.0+\tfrac{1}{2}f\left[t_1^2+t_3^2-2\left(\frac{t_1+t_3}{2}\right)^2\right].$$

$$\therefore\ (\sqrt{x_1}-\sqrt{x_3})^2=\tfrac{1}{4}f\,[2t_1^2+2t_3^2-(t_1^2+t_3^2+2t_1t_3)]$$
$$=\tfrac{1}{4}f\,(t_3-t_1)^2=\tfrac{1}{4}f\,(2d)^2=fd^2.$$

Hence $f=(\sqrt{x_1}-\sqrt{x_3})^2/d^2$.

Ex. 14. *Two cars start off to race with velocities u and u' and travel in a straight line with uniform accelerations f and f' respectively. If the race ends in a dead heat, prove that the length of the course is*

$$\{2\,(u-u')\,(uf'-u'f)\}/(f-f')^2.$$

Sol. Let s be the length of the course. By dead heat we mean that each car moves the distance s in the same time, say t. Then considering the motion of the first car we have $s=ut+\frac{1}{2}ft^2$; and considering the motion of the second car, we have $s=u't+\frac{1}{2}f't^2$. These equations can be written as

$$\tfrac{1}{2}ft^2+ut-s=0, \quad \ldots(1)$$

and

$$\tfrac{1}{2}f't^2+u't-s=0. \quad \ldots(2)$$

By the method of cross multiplication, we get from (1) and (2)

$$\frac{t^2}{\begin{vmatrix} u & -s \\ u' & -s\end{vmatrix}}=\frac{t}{\begin{vmatrix} -s & \frac{1}{2}f \\ -s & \frac{1}{2}f'\end{vmatrix}}=\frac{1}{\begin{vmatrix} \frac{1}{2}f & u \\ \frac{1}{2}f' & u'\end{vmatrix}}$$

or

$$\frac{t^2}{(u'-u)\,s}=\frac{t}{\frac{1}{2}s\,(f-f')}=\frac{1}{\frac{1}{2}\,(fu'-f'u)}.$$

Eliminating t, we have

$$\frac{(u'-u)\,s}{\frac{1}{2}\,(fu'-f'u)}=\left[\frac{\frac{1}{2}s\,(f-f')}{\frac{1}{2}\,(fu'-f'u)}\right]^2=\frac{s^2\,(f-f')^2}{(fu'-f'u)^2}.$$

Since $s\neq 0$, therefore $s=\{2\,(u'-u)\,(fu'-f'u)\}/(f-f')^2$
$=\{2\,(u-u')\,(uf'-u'f)\}/(f-f')^2$.

Ex. 15. *Two particles P and Q move in a straight line AB. The particle P starts from A in the direction AB with velocity u and constant acceleration f, and at the same time Q starts from B in the direction BA with velocity u_1 and constant acceleration f_1: if they pass one another at the middle point of AB and arrive at the other ends of AB with equal velocities, prove that*

$$(u+u_1)\,(f-f_1)=8\,(fu_1-f_1u).$$

Sol. Let $AB=2s$. Let v be the velocity of either particle after moving the distance $AB=2s$. Then

$$v^2=u^2+2f\,(2s)=u_1^2+2f_1\,(2s).$$

$$\therefore\ s=\frac{u^2-u_1^2}{4\,(f_1-f)}.$$

Now let t be the time taken by each particle to reach the middle point of AB. Then each particle moves distance s in time t. Therefore

$$s=ut+\tfrac{1}{2}ft^2=u_1t+\tfrac{1}{2}f_1t^2. \quad \ldots(1)$$

Since $t\neq 0$, therefore from (1), we have $u+\frac{1}{2}ft=u_1+\frac{1}{2}f_1t$

or $t=2\,(u-u_1)/(f_1-f)$.

Now considering the motion of the particle P to cover the first half of the journey AB and using the formula $s=ut+\frac{1}{2}ft^2$, we get

$$\frac{u^2-u_1^2}{4\,(f_1-f)}=u.\frac{2\,(u-u_1)}{(f_1-f)}+\tfrac{1}{2}f.\frac{4\,(u-u_1)^2}{(f_1-f)^2}$$

or $(u+u_1)\,(f_1-f)=8u\,(f_1-f)+8f\,(u-u_1)$ $\quad[\because\ u-u_1\neq 0]$

or $(u+u_1)\,(f_1-f)=8\,(uf_1-fu_1)$

or $(u+u_1)\,(f-f_1)=8\,(fu_1-f_1u)$.

Ex. 16. *A train travels a distance s in t seconds. It starts from rest and ends at rest. In the first part of journey it moves with constant acceleration f and in the second part with constant retardation f'. Show that if s is the distance between the two stations, then*

$$t=\sqrt{[2s\,(1/f+1/f')]}.$$

Sol. Let v be the velocity at the end of the first part of the motion, or say in the beginning of the second part of the motion and t_1 and t_2 be the times for the two motions respectively. Then $t=t_1+t_2$.

Let x be the distance described in the first part. Then the distance described in the second part is $s-x$. Considering the first part of the motion with constant acceleration f, we have

$$\left.\begin{aligned} v&=0+ft_1=ft_1,\\ v^2&=0+2fx=2fx.\end{aligned}\right\} \quad \ldots(1)$$

Again considering the second part of the motion with constant retardation f', we have

$$\left.\begin{aligned} 0&=v-f't_2\ \text{ i.e., } v=f't_2,\\ 0&=v^2-2f'\,(s-x)\ \text{ i.e., } v^2=2f'\,(s-x).\end{aligned}\right\} \quad \ldots(2)$$

From (1) and (2), we have

$$(s-x)+x=\frac{v^2}{2f'}+\frac{v^2}{2f}\ \text{ or }\ s=\frac{v^2}{2}\left(\frac{1}{f}+\frac{1}{f'}\right). \quad \ldots(3)$$

Also $t_1+t_2=v/f+v/f'=v\,(1/f+1/f')$. $\quad \ldots(4)$

Substituting the value of v from (3) in (4), we get

$$t=t_1+t_2=\sqrt{\left\{\frac{2s}{(1/f+1/f')}\right\}}\cdot\left(\frac{1}{f}+\frac{1}{f'}\right)=\sqrt{\left[2s\left(\frac{1}{f}+\frac{1}{f'}\right)\right]}.$$

Ex. 17. *A point moving in a straight line with uniform acceleration describes distances a, b feet in successive intervals of t_1, t_2 seconds. Prove that the acceleration is $2\,(t_1b-t_2a)/[t_1t_2\,(t_1+t_2)]$.*

Sol. Let u be the initial velocity and f be the uniform acceleration of the particle. Then from $s=ut+\frac{1}{2}ft^2$, we have

$$a=ut_1+\tfrac{1}{2}ft_1^2 \quad \ldots(1)$$

and

$$a+b=u\,(t_1+t_2)+\tfrac{1}{2}f\,(t_1+t_2)^2. \quad \ldots(2)$$

Subtracting (1) from (2), we have

$$b=ut_2+\tfrac{1}{2}f\,(t_2^2+2t_1t_2). \quad \ldots(3)$$

Multiplying (3) by t_1 and (1) by t_2 and subtracting, we have

$$bt_1-at_2=\tfrac{1}{2}f\,(t_2^2+2t_1t_2)\,t_1-\tfrac{1}{2}f\,t_1^2t_2$$
$$=\tfrac{1}{2}f\,(t_2^2t_1+t_1^2t_2)=\tfrac{1}{2}f\,t_1t_2\,(t_2+t_1).$$

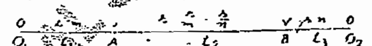
$$\therefore\ f=\frac{2\,(bt_1-at_2)}{t_1t_2\,(t_1+t_2)}.$$

Ex. 18. *For $1/m$ of the distance between two stations a train is uniformly accelerated and for $1/n$ of the distance it is uniformly retarded: it starts from rest at one station and comes to rest at the other. Prove that the ratio of its greatest velocity to its average velocity is $\left(1+\frac{1}{m}+\frac{1}{n}\right):1$.*

Sol. Let O_1 and O_2 be two stations at a distance s apart and A and B two points between O_1 and O_2 such that

$$O_1A=s/m \quad\text{and}\quad BO_2=s/n.$$

$$\therefore\ AB=s-s/m-s/n.$$



The train starts at rest from O_1 and moves with uniform acceleration f from O_1 to A. Let V be its velocity at the point A. It moves with constant velocity V from A to B and then moves with uniform retardation f' from B to O_2. The velocity at the station O_2 is zero.

Let t_1, t_2, t_3 be the times taken to travel the distances O_1A, AB and BO_2 respectively.

Now the greatest velocity of the train during its journey from O_1 to $O_2=V$ and the average velocity of the train$=s/(t_1+t_2+t_3)$.

$$\therefore\ \text{the required ratio}=\frac{\text{greatest velocity}}{\text{average velocity}}=\frac{V}{s/(t_1+t_2+t_3)}$$
$$=\frac{V\,(t_1+t_2+t_3)}{s}. \quad \ldots(1)$$

For motion from O_1 to A, using the formula $v=u+ft$, we have

$$V=0+ft_1. \quad \therefore\ f=\frac{V}{t_1}.$$

Now using the formula $s=ut+\frac{1}{2}ft^2$ for the same motion, we have

$$\frac{s}{m}=0+\tfrac{1}{2}\,\frac{V}{t_1}.t_1^2$$

or

$$t_1=\frac{2s}{Vm}. \quad \ldots(2)$$

For motion from A to B, $AB=V.t_2$.

$$\therefore\ t_2=\frac{AB}{V}=\frac{s-s/m-s/n}{V}. \quad \ldots(3)$$

For motion from B to O_2, using the formula $v=u+ft$, we have

$$0=V-f't_3, \quad \therefore\ f'=V/t_3.$$

Using the formula $s=ut+\frac{1}{2}ft^2$ for the same motion, we have

$$\frac{s}{n}=Vt_3-\tfrac{1}{2}.\frac{V}{t_3}.t_3^2=\frac{Vt_3}{2}$$

or

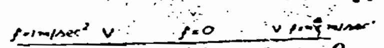
$$t_3=\frac{2s}{Vn}. \quad \ldots(4)$$

Substituting from (2), (3) and (4) in (1), the required ratio

$$=\frac{V\left\{\frac{2s}{Vm}+\frac{1}{V}\left(s-\frac{s}{m}-\frac{s}{n}\right)+\frac{2s}{Vn}\right\}}{s}=\frac{\frac{1}{m}+\frac{1}{n}+1}{1}.$$

Ex. 19. *The greatest possible acceleration of a train is 1 m/sec² and the greatest possible retardation is $\frac{5}{3}$ m/sec². Find the least time taken to run between two stations 12 km. apart if the maximum speed is 22 m/sec.*

Sol. Let a train start from the station O_1 and move with uniform acceleration 1 m/sec² upto A for time t_1 seconds.



Let the velocity of the train at A be $V=22$ m/sec. Then the train moves with constant velocity V from A to B for time t_2 seconds. In the last the train moves from B to the second station O_2 under constant retardation $\frac{4}{3}$ m/sec.^{2} for time t_3 seconds. Thus the least time to travel between the two stations O_1 and O_2 is $(t_1+t_2+t_3)$ seconds.

Also $O_1O_2=12$ km. $=12000$ meters.

Now using the formula $v=u+ft$ for the parts O_1A and BO_2 of the journey, we have

$$V=22=0+1.\,t_1 \text{ so that } t_1=22,$$

and $0=22-\frac{4}{3}t_3$ so that $t_3=\frac{33}{2}$.

Now $O_1A=$(Average velocity from O_1 to A)$\times t_1$

$$=\frac{0+22}{2}\times 22=242 \text{ meters,}$$

and $BO_2=\frac{22+0}{2}\times\frac{33}{2}=\frac{363}{2}$ meters.

$\therefore\ AB=O_1O_2-O_1A-BO_2=12000-242-\frac{363}{2}$

$$=\frac{23153}{2} \text{ meters.}$$

$\therefore\ t_2=\frac{AB}{V}=\frac{23153}{2\times 22}=\frac{23153}{44}$ seconds.

$\therefore$ the required time $=(t_1+t_2+t_3)$ seconds

$=\left(22+\frac{33}{2}+\frac{23153}{44}\right)$ seconds $=\frac{24847}{44}$ seconds

$=9$ minutes 25 seconds approximately.

Ex. 20. *Two points move in the same straight line starting at the same moment from the same point in the same direction. The first moves with constant velocity u and the second with constant acceleration f (its initial velocity being zero). Show that the greatest distance between the points before the second catches first is $u^2/2f$ at the end of the time u/f from the first.*

Sol. If s_1 and s_2 are the distances moved by the two particles in time t, then

$$s_1=ut \text{ and } s_2=0+\tfrac{1}{2}ft^2.$$

$\therefore$ the distance s between the two particles at time t is given by

$$s=s_1-s_2=ut-\tfrac{1}{2}ft^2=\frac{f}{2}\left(\frac{2u}{f}t-t^2\right)$$

or
$$s=\frac{f}{2}\left[\frac{u^2}{f^2}-\left(t-\frac{u}{f}\right)^2\right]. \quad ...(1)$$

Now s is greatest if $(t-u/f)^2=0$ i.e., if $t=u/f$.

Also the greatest value of $s=\frac{f}{2}\cdot\frac{u^2}{f^2}=\frac{u^2}{2f}$.

Ex. 21. *The speed of a train increases at constant rate α from zero to v, then remains constant for an interval and finally decreases to zero at a constant rate β. If l be the total distance described, prove that the total time occupied is $(l/v)+(v/2)(1/\alpha+1/\beta)$. Also find the least value of time when $\alpha=\beta$.*

Sol. Let t_1, t_2, t_3 be the times taken to cover the distances x, y, z of the first, second and last phase of the journey. Whole distance $l=x+y+z$.

Equations for the first and last part of the journey are

$$\left.\begin{aligned} v^2&=2\alpha x,\\ \text{and } v&=\alpha t_1\end{aligned}\right\} \ ...(1); \quad \text{and } \left.\begin{aligned} v^2&=2\beta z,\\ v&=\beta t_3\end{aligned}\right\} \quad (2)$$

From (1), on eliminating α we have $x=\frac{1}{2}vt_1$; and from (2), on eliminating β, we have $z=\frac{1}{2}vt_3$.

Also considering the motion for the middle part of the journey, we have $y=vt_2$.

Thus $x+y+z=v\left(\frac{1}{2}t_1+t_2+\frac{1}{2}t_3\right)$

i.e., $l=v\left[(t_1+t_2+t_3)-\frac{1}{2}(t_1+t_3)\right]$

or $\frac{l}{v}=(t_1+t_2+t_3)-\frac{1}{2}(t_1+t_3)$.

$\therefore$ the total time occupied i.e., $t_1+t_2+t_3=(l/v)+\frac{1}{2}(t_1+t_3)$

$$=\frac{l}{v}+\frac{1}{2}\left(\frac{v}{\alpha}+\frac{v}{\beta}\right). \quad \text{[from (1) and (2)]}$$

$$=\frac{l}{v}+\frac{1}{2}v\left(\frac{1}{\alpha}+\frac{1}{\beta}\right). \quad ...(3)$$

Let t denote the total time occupied when $\alpha=\beta$.

Then putting $\alpha=\beta$ in the above result (3), we have

$t=\frac{l}{v}+\frac{v}{\alpha}$. Therefore $\frac{dt}{dv}=-\frac{l}{v^2}+\frac{1}{\alpha}$

For least value of t, we have $dt/dv=0$, i.e., $-\frac{l}{v^2}+\frac{1}{\alpha}=0$

i.e., $\frac{l}{v}=\frac{v}{\alpha}$ i.e., $v=\sqrt{(l\alpha)}$.

Also then the time $=2\left(\frac{l}{v}\right)=\frac{2l}{\sqrt{(l\alpha)}}=2\sqrt{(l/\alpha)}$. This time is least because $d^2t/dv^2=2l/v^3$ which is positive for $v=\sqrt{(l\alpha)}$.

Ex. 22. *A lift ascends with constant acceleration f, then with constant velocity and finally stops under constant retardation f. If the total distance ascended is s and the total time occupied is t, show that the time during which the lift is ascending with constant velocity is $\sqrt{(t^2-(4s/f))}$.*

Sol. Let v be the maximum velocity produced during the ascent. Since this velocity is produced under a constant acceleration f during the first part of the ascent and destroyed under the same retardation f during the last part of the ascent, therefore, the distances as well as the times for these two ascents are equal. Let x be the distance and t_1 the time for each of these two parts. We have then

$$\left.\begin{aligned} v^2&=2fx,\\ \text{and } v&=ft_1\end{aligned}\right\} \quad ...(1)$$

for the first and last part of the motion.

Also considering the middle part of the motion, we have

$$v(t-2t_1)=s-2x.$$

From (1) and (2), on eliminating v and x, we have

$$ft_1(t-2t_1)=s-\frac{v^2}{f}=s-\frac{f^2t_1^2}{f}=s-ft_1^2.$$

$$\therefore\ ft_1^2-ftt_1+s=0.$$

Solving this as a quadratic in t_1, we get

$$t_1=\frac{ft\pm\sqrt{(f^2t^2-4fs)}}{2f}$$

or $2t_1=t\pm\sqrt{\left(t^2-\frac{4s}{f}\right)}$ or $t-2t_1=\sqrt{\left(t^2-\frac{4s}{f}\right)}$.

This gives the time of ascent with constant velocity.

Ex. 23. *Prove that the shortest time from rest to rest in which a steady load of P tons can lift a weight of W tons through a vertical distance h feet is $\sqrt{\{(2h/g).P/(P-W)\}}$ seconds.*

Sol. The time will be shortest if the load acts continuously during the first part of the ascent. Let f be the acceleration during the first part of the ascent. Then by Newton's second law of motion, f is given by

$$P-W=(W/g)f. \quad ...(1)$$

During the second part of the ascent, P ceases to act and W then moves only under gravity. Therefore the retardation is g.

Let x and y be the distances and t_1, t_2 the corresponding times for the two parts in the ascent.

If v be the velocity at the end of the first part of the ascent or at the beginning of the second part of the ascent, we have then

$$\left.\begin{aligned} v^2&=2fx,\\ v&=ft_1\end{aligned}\right\} \quad ...(2)$$

[Equations for the first part of the ascent]

and
$$\left.\begin{aligned} v^2&=2gy\\ v&=gt_2\end{aligned}\right\} \quad ...(3)$$

[Equations for the second part of the ascent]

Also $x+y=h$ (given).

From (2) and (3), we get

$$\frac{v^2}{2f}+\frac{v^2}{2g}=x+y$$

i.e.,
$$\frac{v^2}{2}\left(\frac{1}{f}+\frac{1}{g}\right)=h. \quad ...(4)$$

Also
$$\frac{v}{f}+\frac{v}{g}=t_1+t_2. \quad ...(5)$$

Now the total time of ascent

$$=t_1+t_2=\left(\frac{1}{f}+\frac{1}{g}\right)v \quad \text{[from (5)]}$$

$$=\left(\frac{1}{f}+\frac{1}{g}\right)\sqrt{\left[2h\Big/\left(\frac{1}{f}+\frac{1}{g}\right)\right]} \quad \text{[from (4)]}$$

$$=\sqrt{\left[2h\left(\frac{1}{f}+\frac{1}{g}\right)\right]}=\sqrt{\left[\frac{2h}{g}\left(\frac{g}{f}+1\right)\right]}$$

$$=\sqrt{\left[\frac{2h}{g}\left(\frac{W}{P-W}+1\right)\right]} \quad \text{[from (1)]}$$

$$=\sqrt{\left[\frac{2h}{g}\cdot\frac{P}{P-W}\right]}.$$

Ex. 24. *Prove that the mean kinetic energy of a particle of mass m moving under a constant force, in any interval of time is $\frac{1}{3}m(u_1^2+u_1u_2+u_2^2)$, where u_1 and u_2 are the initial and final velocities.*

Sol. Let the interval of time during which the particle moves be T. If the particle moves under a constant acceleration f and v be its velocity at any time t, we have $v=u_1+ft$.

Now the mean kinetic energy of the particle during the time T

$$=\frac{1}{T}\int_0^T \tfrac{1}{2}mv^2\,dt=\frac{m}{2T}\int_0^T (u_1+ft)^2\,dt=\frac{m}{2fT}\cdot\frac{1}{3}\Big[(u_1+ft)^3\Big]_0^T$$

$$=\frac{m}{6fT}\left[(u_1+fT)^3-u_1^3\right]=\frac{m}{6(u_2-u_1)}(u_2^3-u_1^3)$$

[$\because$ $u_2=u_1+fT$ and so $u_2-u_1=fT$]

$$=\tfrac{1}{3}m(u_1^2+u_1u_2+u_2^2).$$

Ex. 25. *A bullet fired into a target loses half its velocity after penetrating 3 cm. How much further will it penetrate?*

Sol. If u cm./sec. is the initial velocity of the bullet then its velocity after penetrating 3 cm. will be $\frac{1}{2}u$ cm./sec.
Let f cm./sec^2. be the retardation of the bullet.

Then from $v^2=u^2+2fs$, we have

$$(u/2)^2=u^2-2.f.3 \text{ giving } f=u^2/8.$$

If the bullet penetrates further by a cm, then from $v^2=u^2+2fs$, we have

$$0=(u/2)^2-2.(u^2/8).a.$$

$$\therefore \ a=1 \text{ cm.}$$

Ex. 26. *A load W is to be raised by a rope from rest to rest, through a height h; the greatest tension which the rope can safely bear is nW. Show that the least time in which the ascent can be made is $[2nh/(n-1)g]^{1/2}$.*

Sol. Obviously the time for ascent is least when the acceleration of the load is greatest. If m is the mass of the load, then $W=mg$ or $m=W/g$. Let f be the greatest acceleration of the load in the upward direction. Since the rope can bear the greatest tension nW, therefore when f is the greatest acceleration of the load, then the tension T in the rope is nW.

$\therefore$ by Newton's second law of motion $P=mf$, we have

$$T-W=nW-W=mf \text{ or } f=(n-1)(W/m)=(n-1)g. \quad \ldots(1)$$

Let the load W move upwards upto the height h_1 under the acceleration f. After that the tension in the rope ceases to act and therefore above the height h_1 the load will move under gravity which acts vertically downwards. If the load comes to rest after moving through a subsequent height h_2 above the height h_1, then according to the question

$$h_1+h_2=h. \quad \ldots(2)$$

If V is the maximum velocity of the load acquired at the end of the first part and t_1, t_2 are the times taken for describing the heights h_1 and h_2 respectively, then from $v=u+ft$, we have

$$V=0+ft_1 \quad \text{and} \quad 0=V-gt_2.$$

$$\therefore \ t_1=V/f \quad \text{and} \quad t_2=V/g.$$

Also from $v^2=u^2+2fs$, we have

$$V^2=0+2fh_1 \quad \text{and} \quad 0=V^2-2gh_2.$$

$$\therefore \ h_1=\frac{V^2}{2f} \quad \text{and} \quad h_2=\frac{V^2}{2g}.$$

Now from $h_1+h_2=h$, we have

$$\frac{V^2}{2f}+\frac{V^2}{2g}=h \text{ or } \frac{V^2}{2}\left(\frac{1}{f}+\frac{1}{g}\right)=h.$$

$$\therefore \ V=\sqrt{\{2h/(1/f+1/g)\}}. \quad \ldots(3)$$

$\therefore$ the least time of ascent

$$=t_1+t_2=\frac{V}{f}+\frac{V}{g}=V\left(\frac{1}{f}+\frac{1}{g}\right)$$

$$=\sqrt{\left\{\frac{2h}{(1/f+1/g)}\right\}}\cdot\left(\frac{1}{f}+\frac{1}{g}\right) \quad \text{[substituting for } V \text{ from (3)]}$$

$$=\sqrt{\left[2h\left(\frac{1}{f}+\frac{1}{g}\right)\right]}$$

$$=\sqrt{\left[2h\left\{\frac{1}{(n-1)g}+\frac{1}{g}\right\}\right]} \quad \text{[substituting for } f \text{ from (1)]}$$

$$=\left[\frac{2nh}{(n-1)g}\right]^{1/2}$$

3. Newton's Laws of Motion.

The Newton's laws of motion are as follows.

Law 1. *Every body continues in its state of rest, or of uniform motion in a straight line, unless it is compelled by some external force or forces to change its state.*

Law 2. *The rate of change of momentum of a body is proportional to the impressed force, and takes place in the direction in which the force acts.*

Law 3. *To every action there is an equal and opposite reaction.*

4. Equation of motion *of a particle moving in a straight line as deduced from the Newton's second law of motion.*

Let v be the velocity at time t of a particle of mass m moving in a straight line under the action of the impressed force P. Since from Newton's second law of motion the rate of change of momentum is proportional to the impressed force, therefore

$$P \propto \frac{d}{dt}(mv), \quad [\because \text{ by def.. momentum}=\text{mass}\times\text{velocity}]$$

or $\quad P=k\dfrac{d}{dt}(mv)$, where k is some constant

or $\quad P=km\dfrac{dv}{dt}$ provided m is constant

or $\quad P=kmf. \quad \ldots(1)$

$[\because f=\text{acceleration}=dv/dt]$.

Let us suppose that a unit force is that which produces a unit acceleration in a particle of unit mass. Then

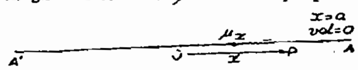
$P=1$, when $m=1$ and $f=1$.

$\therefore$ from (1), we have $k=1$.

Hence we have, $P=mf$, which is the required equation of motion of the particle.

5. Simple Harmonic Motion. (S.H.M.) Definition. *The kind of motion, in which a particle moves in a straight line in such a way that its acceleration is always directed towards a fixed point on the line (called the centre of force) and varies as the distance of the particle from the fixed point, is called simple harmonic motion.*

Let O be the centre of force taken as origin. Suppose the particle starts from rest from the point A where $OA=a$. It begins to move towards the centre of attraction O. Let P be the position of the particle after time t, where $OP=x$. By the definition of S.H.M. the magnitude of acceleration at P is proportional to x.



Let it be μx, where μ is a constant called the intensity of force. Also on account of a centre of attraction at O, the acceleration of P is towards O *i.e.*, in the direction of x decreasing. Therefore the equation of motion of P is

$$\frac{d^2x}{dt^2}=-\mu x \quad \ldots(1)$$

where the negative sign has been taken because the force acting on P is towards O *i.e.*, in the direction of x decreasing. The equation (1) gives the acceleration of the particle at any position.

Multiplying both sides of (1) by $2dx/dt$, we get

$$2\frac{dx}{dt}\frac{d^2x}{dt^2}=-2\mu x\frac{dx}{dt}.$$

Integrating with respect to t, we get

$$v^2=\left(\frac{dx}{dt}\right)^2=-\mu x^2+C,$$

where C is a constant of integration and v is the velocity at P.

Initially at the point A, $x=a$ and $v=0$; therefore $C=\mu a^2$.

Thus, we have

$$v^2=\left(\frac{dx}{dt}\right)^2=-\mu x^2+\mu a^2$$

or $$v^2=\mu(a^2-x^2). \quad \ldots(2)$$

The equation (2) gives the velocity at any point P. From (2) we observe that v^2 is maximum when $x^2=0$ or $x=0$. Thus in a S.H.M. the velocity is maximum at the centre of force O. Let this maximum velocity be v_1. Then at O, $x=0$, $v=v_1$. So from (2) we get $v_1^2=\mu a^2$ or $v_1=a\sqrt{\mu}$.

Also from (2) we observe that $v=0$ when $x^2=a^2$ *i.e.*, $x=\pm a$. Thus in a S.H.M. the velocity is zero at points equidistant from the centre of force.

Now from (2), on taking square root, we get $dx/dt=-\sqrt{\mu}\sqrt{(a^2-x^2)}$, where the $-$ive sign has been taken because at P the particle is moving in the direction of x decreasing.

Separating the variables, we get

$$-\frac{1}{\sqrt{\mu}}\frac{dx}{\sqrt{(a^2-x^2)}}=dt \quad \ldots(3)$$

Integrating both sides, we get

$$\frac{1}{\sqrt{\mu}}\cos^{-1}\frac{x}{a}=t+D, \text{ where } D \text{ is a constant.}$$

But initially at A, $x=a$ and $t=0$; therefore $D=0$.

Thus we have

$$\frac{1}{\sqrt{\mu}}\cos^{-1}\frac{x}{a}=t \text{ or } x=a\cos(\sqrt{\mu}t). \quad \ldots(4)$$

The equation (4) gives a relation between x and t, where t is the time measured from A. If t_1 be the time from A to O, then at O, we have $t=t_1$ and $x=0$. So from (4), we get $t_1=\dfrac{1}{\sqrt{\mu}}\cos^{-1}0$ $=\dfrac{1}{\sqrt{\mu}}\dfrac{\pi}{2}=\dfrac{\pi}{2\sqrt{\mu}}$, which is independent of the initial displacement a of the particle. *Thus in a S.H.M. the time of descent to the centre of force is independent of the initial displacement of the particle.*

Note. The time of descent t_1 from A to O can also be found from (3) with the help of the definite integrals $-\dfrac{1}{\sqrt{\mu}}\displaystyle\int_a^0\frac{dx}{\sqrt{(a^2-x^2)}}=\int_0^{t_1}dt$

For fixing the limits of integration, we observe that at A, $x=a$ and $t=0$ while at O, $x=0$ and $t=t_1$.

Nature of Motion. The particle starts from rest at A where its acceleration is maximum and is μa towards O. It begins to move towards the centre of attraction O and as it approaches the centre of force O, its velocity goes on increasing. When the particle reaches O its acceleration is zero and its velocity is maximum and is $a\sqrt{\mu}$ in the direction OA'. Due to this velocity gained at O the particle moves towards the left of O. But on account of the centre of attraction at O a force begins to act upon the particle against its direction of motion. So its velocity goes on decreasing and it comes to instantaneous rest at A' where $OA'=OA$. The rest at A' is only instantaneous. The particle at once begins to move towards the centre of attraction O and retracing its path it again comes to instantaneous rest at A. Thus the motion of the particle is oscillatory and it continues to oscillate between A and A'. To start from A and to come back to A is called one *complete oscillation*.

Few Important Definitions :

1. **Amplitude.** *In a S.H.M. the distance from the centre of force of the position of maximum displacement is called the amplitude of the motion.* Thus the amplitude is the distance of a position of instantaneous rest from the centre of force. In the formulae (2) and (4) of this article the amplitude is a.

2. **Time period.** *In a S.H.M. the time taken to make one complete oscillation is called time period or periodic time.* Thus if T is the time period of the S.H.M., then

$T=4.$ (time from A to O)$=4.\dfrac{\pi}{2\sqrt{\mu}}=\dfrac{2\pi}{\sqrt{\mu}}$, which is independent of the amplitude a.

3. **Frequency.** *The number of complete oscillations in one second is called the frequency of the motion.* Since the time taken to make one complete oscillation is $\dfrac{2\pi}{\sqrt{\mu}}$ seconds, therefore if n is the frequency, then $n.\dfrac{2\pi}{\sqrt{\mu}}=1$ or $n=\dfrac{\sqrt{\mu}}{2\pi}$.

Thus the frequency is the reciprocal of the periodic time.

Important Remark 1. In a S.H.M. if the centre of force is not at origin but is at the point $x=b$, then the equation of motion is $d^2x/dt^2=-\mu(x-b)$. Similarly $d^2x/dt^2=-\mu(x+b)$ is the equation of a S.H.M. in which the centre of force is at the point $x=-b$.

Important Remark 2. In the above article when after instantaneous rest at A' the particle begins to move towards A, we have

from (2) $$\frac{dx}{dt}=+\sqrt{\mu}\sqrt{(a^2-x^2)},$$

where the +ive sign has been taken because the particle is moving in the direction of x increasing.

Separating the variables, we have $\dfrac{dx}{\sqrt{(a^2-x^2)}}=\sqrt{\mu}\,dt.$

Integrating, we get $-\cos^{-1}(x/a)=\sqrt{\mu}t+B$. Now the time from A to A' is $\pi/\sqrt{\mu}$. Therefore at A', we have $t=\pi/\sqrt{\mu}$ and

$x=-a$. These give $-\cos^{-1}(-a/a)=\sqrt{\mu}.(\pi/\sqrt{\mu})+B$
or $-\cos^{-1}(-1)=\pi+B$ or $-\pi=\pi+B$ or $B=-2\pi$. Thus we have
$-\cos^{-1}(x/a)=\sqrt{\mu}\,t-2\pi$ or $\cos^{-1}(x/a)=2\pi-\sqrt{\mu}t$
or $x=a\cos(2\pi-\sqrt{\mu}t)$ or $x=a\cos\sqrt{\mu}t$. Thus in S.H.M. the equation $x=a\cos\sqrt{\mu}t$ is valid throughout the entire motion from A to A' and back from A' to A.

4. **Phase and Epoch.** From equation (1), we have
$$\frac{d^2x}{dt^2}+\mu x=0,$$
which is a linear differential equation with constant coefficients and its general solution is given by
$$x=a\cos(\sqrt{\mu}t+\epsilon). \quad ...(5)$$
The constant ϵ is called the **starting phase** or the epoch of the motion and the quantity $\sqrt{\mu}t+\epsilon$ is called the **argument** of the motion.

The **phase** *at any time t of a S.H.M. is the time that has elapsed since the particle passed through its extreme position in the positive direction.*

From (5), x is maximum when $\cos(\sqrt{\mu}t+\epsilon)$ is maximum *i.e.*, when $\cos(\sqrt{\mu}t+\epsilon)=1$.

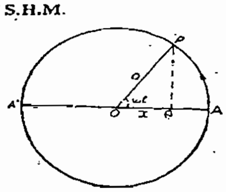
Therefore if t_1 is the time of reaching the extreme position in the positive direction, then
$$\cos(\sqrt{\mu}t_1+\epsilon)=1$$
or $$\sqrt{\mu}t_1+\epsilon=0 \quad \text{or} \quad t_1=-\frac{\epsilon}{\sqrt{\mu}}$$

$\therefore$ the phase at time $t=t-t_1=t+\dfrac{\epsilon}{\sqrt{\mu}}$.

5. **Periodic Motion.** *A point is said to have a periodic motion when it moves in such a manner that after a certain fixed interval of time called periodic time it acquires the same position and moves with the same velocity in the same direction. Thus S.H.M. is a periodic motion.*

6. **Geometrical representation of S.H.M.**

Let a particle move with a uniform angular velocity ω round the circumference of a circle of radius a. Suppose AA' is a fixed diameter of the circle. If the particle starts from A and P is its position at time t, then $\angle AOP=\omega t$.



Draw PQ perpendicular to the diameter AA'.

If $OQ=x$, then
$$x=a\cos\omega t. \quad ...(1)$$

As the particle P moves round the circumference, the foot Q of the perpendicular on the diameter AA' oscillates on AA' from A to A' and from A' to A back. Thus the motion of the point Q is periodic.

From (1), we have
$$\frac{dx}{dt}=-a\omega\sin\omega t \quad ...(2)$$
and $$\frac{d^2x}{dt^2}=-a\omega^2\cos\omega t=-\omega^2x. \quad ...(3)$$

The equations (2) and (3) give the velocity and acceleration of Q at any time t.

The equation (3) shows that Q executes a simple harmonic motion with centre at the origin O. From equation (1), we see that the amplitude of this S.H.M. is a because the maximum value of x is a.

The periodic time of Q = The time required by P to turn through an angle 2π with a uniform angular velocity ω
$$=\frac{2\pi}{\omega}.$$

Thus if a particle describes a circle with constant angular velocity, the foot of the perpendicular from it on any diameter executes a S.H.M.

§ 7. Important results about S.H.M.

We summarize the important relations of a S.H.M. as follows : (Remember them)

(i) Referred to the centre as origin the equation of S.H.M. is $\ddot{x}=-\mu x$,
or the equation $\ddot{x}=-\mu x$ represents a S.H.M. with centre at the origin.

(ii) The velocity v at a distance x from the centre and the distance x from the centre at time t are respectively given by
$$v^2=\mu(a^2-x^2) \text{ and } x=a\cos\sqrt{\mu}t,$$
where a is the amplitude and the time t has been measured from the extreme position in the positive direction.

(iii) Maximum acceleration $=\mu a$, (at extreme points)

(iv) Maximum velocity $=\sqrt{\mu}a$. (at the centre)

(v) Periodic time $T=\dfrac{2\pi}{\sqrt{\mu}}$.

(vi) *Frequency* $n=\dfrac{1}{T}=\dfrac{\sqrt{\mu}}{2\pi}$.

Illustrative Examples.

Ex. 27. *The maximum velocity of a body moving with S.H.M. is 2 ft./sec. and its period is $\frac{1}{5}$ sec. What is its amplitude ?*

Sol. Let the amplitude be a ft. Then the maximum velocity $=a\sqrt{\mu}$ ft./sec. $=2$ ft./sec. (given).

$\therefore \quad a\sqrt{\mu}=2. \quad ...(1)$

Also the time period $T=2\pi/\sqrt{\mu}$ seconds $=\frac{1}{5}$ seconds (given)

$\therefore \quad \dfrac{2\pi}{\sqrt{\mu}}=\dfrac{1}{5} \quad ...(2)$

Multiplying (1) and (2) to eliminate μ, we have

$2\pi a=\dfrac{2}{5}. \quad \therefore \quad a=\dfrac{1}{5\pi}.$

$\therefore$ the required amplitude $=\dfrac{1}{5\pi}$ ft. $=\cdot064$ ft. nearly.

Ex. 28. *At what distance from the centre the velocity in a S.H.M. will be half of the maximum ?*

Sol. Take the centre of the motion as origin. Let a be the amplitude. In a S.H.M., the velocity v of the particle at a distance x from the centre is given by
$$v^2=\mu(a^2-x^2). \quad ...(1)$$

From (1), v is max. when $x=0$. Therefore max velocity $=\sqrt{\mu}a$.

Let x_1 be the distance from the centre of the point where the velocity is half of the maximum *i.e.*, where the velocity is $\frac{1}{2}a\sqrt{\mu}$. Then putting $x=x_1$ and $v=\frac{1}{2}a\sqrt{\mu}$ in (1), we get

$\frac{1}{4}a^2\mu=\mu(a^2-x_1^2)$, or $\frac{1}{4}a^2=a^2-x_1^2$

or $x_1^2=\dfrac{3a^2}{4}$ or $x_1=\pm a\sqrt{3}/2$.

Thus there are two points, each at a distance $a\sqrt{3}/2$ from the centre, where the velocity is half of the maximum.

Ex. 29. *A particle moves in a straight line and its velocity at a distance x from the origin is $k\sqrt{(a^2-x^2)}$, where a and k are constants. Prove that the motion is simple harmonic and find the amplitude and the periodic time of the motion.*

Sol. We know that in a rectilinear motion the expression for velocity at a distance x from the origin is dx/dt. So according to the question, we have
$$\left(\frac{dx}{dt}\right)^2=k^2(a^2-x^2). \quad ...(1)$$

Differentiating (1) w.r.t. t, we get

$$2\frac{dx}{dt}\cdot\frac{d^2x}{dt^2}=k^2\left(-2x\frac{dx}{dt}\right).$$

$\therefore\quad \frac{d^2x}{dt^2}=-k^2x$, which is the equation of a S. H. M. with centre at the origin and $\mu=k^2$. Hence the given motion is simple harmonic.

The time period $T=2\pi/\sqrt{\mu}=2\pi/\sqrt{k^2}=2\pi/k$.

Now to find the amplitude we are to find the distance from the centre of a point where the velocity is zero. So putting $dx/dt=0$ in (1), we get $0=k^2(a^2-x^2)$ or $x=\pm a$. Since here the centre is at origin, therefore the amplitude $=a$.

Ex. 30. *Show that if the displacement of a particle in a straight line is expressed by the equation $x=a\cos nt+b\sin nt$, it describes a simple harmonic motion whose amplitude is $\sqrt{(a^2+b^2)}$ and period is $2\pi/n$.*

Sol. Given $x=a\cos nt+b\sin nt$. ...(1)

$\therefore\quad dx/dt=-an\sin nt+bn\cos nt$, ...(2)

and $d^2x/dt^2=-an^2\cos nt-bn^2\sin nt=-n^2(a\cos nt+b\sin nt)$
$=-n^2x$, from (1).

Now $d^2x/dt^2=-n^2x$ is the equation of a S. H. M. with centre at the origin and $\mu=n^2$. Hence the given motion is simple harmonic.

The time period $T=2\pi/\sqrt{\mu}=2\pi/\sqrt{n^2}=2\pi/n$. Also the amplitude is the distance from the centre of a point where the velocity is zero. Since here the centre is at origin, therefore the amplitude is the value of x when $dx/dt=0$. Putting $dx/dt=0$ in (2), we get

$0=-an\sin nt+bn\cos nt$ or $\tan nt=b/a$.

$\therefore\quad \sin nt=b/\sqrt{(a^2+b^2)}$ and $\cos nt=a/\sqrt{(a^2+b^2)}$.

Substituting these in (1), we have

$$\text{the amplitude}=a\,\frac{a}{\sqrt{(a^2+b^2)}}+b\cdot\frac{b}{\sqrt{(a^2+b^2)}}=\frac{a^2+b^2}{\sqrt{(a^2+b^2)}}=\sqrt{(a^2+b^2)}.$$

Ex. 31. *The speed v of a particle moving along the axis of x is given by the relation $v^2=n^2(8bx-x^2-12b^2)$. Show that the motion is simple harmonic with its centre at $x=4b$, and amplitude $=2b$.*

Sol. Given $v^2=(dx/dt)^2=n^2(8bx-x^2-12b^2)$. ...(1)

Differentiating (1) w.r.t. t, we get

$$2\frac{dx}{dt}\cdot\frac{d^2x}{dt^2}=n^2(8b-2x)\frac{dx}{dt}.$$

$\therefore\quad \frac{d^2x}{dt^2}=n^2(4b-x)=-n^2(x-4b)$, which is the equation of a S.H.M. with centre at the point $x-4b=0$ *i.e.*, at the point $x=4b$. [Note that centre is the point where the acceleration d^2x/dt^2 is zero.]

Now $v=0$ where $8bx-x^2-12b^2=0$ *i.e.*, $x^2-8bx+12b^2=0$ *i.e.*, $(x-6b)(x-2b)=0$ *i.e.*, $x=6b$ or $2b$. Thus the positions of instantaneous rest are given by $x=2b$ and $x=6b$. The distance of any of these two positions from the centre $x=4b$ is the amplitude.

Hence the amplitude is the distance of the point $x=6b$ from the point $x=4b$. Thus the amplitude $=6b-4b=2b$.

Ex. 32. *The speed v of the point P which moves in a line is given by the relation $v^2=a+2bx-cx^2$, where x is the distance of the point P from a fixed point on the path, and a, b, c are constants. Show that the motion is simple harmonic if c is positive; determine the period and the amplitude of the motion.*

Sol. Here given that, $v^2=a+2bx-cx^2$. ...(1)

Differentiating both sides of (1) w.r.t. x, we have

$$2v\frac{dv}{dx}=2b-2cx.$$

$$\therefore\quad \frac{d^2x}{dt^2}=v\frac{dv}{dx}=-c\left(x-\frac{b}{c}\right). \quad ...(2)$$

Since c is positive, therefore the equation (2) represents a S. H. M. with the centre of force at the point $x=b/c$.

Hence the relation (1) represents a S. H. M. of period

$$T=\frac{2\pi}{\sqrt{\mu}}=\frac{2\pi}{\sqrt{c}}, \text{ because in the equation (2), } \mu=c.$$

To determine the amplitude, putting $v=0$ in (1), we have

$$a+2bx-cx^2=0$$

or $\quad cx^2-2bx-a=0.$

$$\therefore\quad x=\frac{b\pm\sqrt{(b^2+ac)}}{c}$$

$\therefore$ the distances of the two positions of instantaneous rest A and A' from the fixed point O are given by

$$OA=\frac{b+\sqrt{(b^2+ac)}}{c} \quad\text{and}\quad OA'=\frac{b-\sqrt{(b^2+ac)}}{c}$$

The distance of any of these two positions from the centre $x=(b/c)$ is the amplitude of the motion.

$$\therefore\quad \text{the amplitude}=\frac{b+\sqrt{(b^2+ac)}}{c}-\frac{b}{c}=\frac{\sqrt{(b^2+ac)}}{c}.$$

Ex. 33. *In a S. H. M. of period $2\pi/\omega$ if the initial displacement be x_0 and the initial velocity u_0, prove that*

(i) *amplitude* $=\sqrt{\left(x_0^2+\frac{u_0^2}{\omega^2}\right)}$

(ii) *position at time* $t=\sqrt{\left(x_0^2+\frac{u_0^2}{\omega^2}\right)}\cdot\cos\left\{\omega t-\tan^{-1}\left(\frac{u_0}{\omega x_0}\right)\right\}$,

and (iii) *time to the position of rest* $=\frac{1}{\omega}\tan^{-1}\left(\frac{u_0}{\omega x_0}\right)$.

Sol. We know that in a S. H. M. the time period $=2\pi/\sqrt{(\mu)}$.

Since here the time period is $2\pi/\omega$, therefore $2\pi/\sqrt{(\mu)}=2\pi/\omega$ *i.e.*, $\mu=\omega^2$.

Now taking the centre of the motion as origin, the equation of the given S. H. M. is

$$\frac{d^2x}{dt^2}=-\omega^2x. \quad ...(1)$$

Multiplying (1) by $2(dx/dt)$ and integrating w.r.t. 't', we get

$$\left(\frac{dx}{dt}\right)^2=-\omega^2x^2+A, \text{ where } A \text{ is a constant.}$$

But initially at $x=x_0$, the velocity $\frac{dx}{dt}=u_0$.

Therefore $\quad u_0^2=-\omega^2x_0^2+A$ or $A=u_0^2+\omega^2x_0^2$.

Thus we have

$$\left(\frac{dx}{dt}\right)^2=-\omega^2x^2+u_0^2+\omega^2x_0^2=\omega^2\left(x_0^2+\frac{u_0^2}{\omega^2}-x^2\right) \quad ...(2)$$

(i) Now the amplitude is the distance from the centre of a point where the velocity is zero. Since here the centre is origin, therefore the amplitude is the value of x when velocity is zero. Putting $\frac{dx}{dt}=0$ in (2), we get $x=\pm\sqrt{\left(x_0^2+\frac{u_0^2}{\omega^2}\right)}$.

Here the required amplitude is $\sqrt{\left(x_0^2+\frac{u_0^2}{\omega^2}\right)}$.

(ii) Assuming that the particle is moving in the direction of x increasing, we have from (2)

$$\frac{dx}{dt}=\omega\sqrt{\left\{\left(x_0^2+\frac{u_0^2}{\omega^2}\right)-x^2\right\}}$$

or $\quad dt=\frac{1}{\omega}\,\frac{dx}{\sqrt{\{(x_0^2+u_0^2/\omega^2)-x^2\}}}.$

Integrating, $\quad t=-\frac{1}{\omega}\cos^{-1}\left\{\frac{x}{\sqrt{(x_0^2+u_0^2/\omega^2)}}\right\}+B,$

where B is a constant.

But initially, when $t=0$, $x=x_0$.

$$\therefore\quad B=\frac{1}{\omega}\cos^{-1}\left\{\frac{x_0}{\sqrt{(x_0^2+u_0^2/\omega^2)}}\right\}=\frac{1}{\omega}\tan^{-1}\left(\frac{u_0}{\omega x_0}\right).$$

$$\therefore\quad t=-\frac{1}{\omega}\cos^{-1}\left\{\frac{x}{\sqrt{(x_0^2+u_0^2/\omega^2)}}\right\}+\frac{1}{\omega}\tan^{-1}\left(\frac{u_0}{\omega x_0}\right)$$

or $\quad \cos^{-1}\left\{\frac{x}{\sqrt{(x_0^2+u_0^2/\omega^2)}}\right\}=-\left\{\omega t-\tan^{-1}\left(\frac{u_0}{\omega x_0}\right)\right\}$

or $\quad \frac{x}{\sqrt{(x_0^2+u_0^2/\omega^2)}}=\cos\left[-\left\{\omega t-\tan^{-1}\frac{u_0}{\omega x_0}\right\}\right]$

$$=\cos\left(\omega t-\tan^{-1}\frac{u_0}{\omega x_0}\right)$$

or $\quad x=\sqrt{\left(x_0^2+\frac{u_0^2}{\omega^2}\right)}\cos\left(\omega t-\tan^{-1}\frac{u_0}{\omega x_0}\right), \quad ...(3)$

which gives the position of the particle at time t.

(iii) Substituting the value of x from (3) in (2), we get

$$\left(\frac{dx}{dt}\right)^2=\omega^2\left(x_0^2+\frac{u_0^2}{\omega^2}\right)\cdot\sin^2\left\{\omega t-\tan^{-1}\left(\frac{u_0}{\omega x_0}\right)\right\}.$$

Putting $\frac{dx}{d}=0$, we get

$$0=\omega^2\left(x_0^2+\frac{u_0^2}{\omega^2}\right)\sin^2\left\{\omega t-\tan^{-1}\left(\frac{u_0}{\omega x_0}\right)\right\}$$

or $\quad \sin\left\{\omega t-\tan^{-1}\left(\frac{u_0}{\omega x_0}\right)\right\}=0$

or $\quad \omega t-\tan^{-1}\left(\frac{u_0}{\omega x_0}\right)=0$ or $t=\frac{1}{\omega}\tan^{-1}\left(\frac{u_0}{\omega x_0}\right)$.

Hence the time of the position of rest $=\frac{1}{\omega}\tan^{-1}\left(\frac{u_0}{\omega x_0}\right)$.

Ex. 34. *Show that in a simple harmonic motion of amplitude a and period 'T', the velocity v at a distance x from the centre is given by the relation $v^2T^2=4\pi^2(a^2-x^2)$.*

Find the new amplitude if the velocity were doubled when the particle is at a distance $\frac{1}{2}a$ from the centre ; the period remaining unaltered.

Sol. Let the equation of S. H. M. with centre as origin be $d^2x/dt^2=-\mu x$.

The time period $T=2\pi/\sqrt{\mu}$. ...(1)

Let a be the amplitude. Then the velocity v at a distance x from the centre is given by

$$v^2=\mu(a^2-x^2). \quad ...(2)$$

From (1), $\mu=4\pi^2/T^2$. Putting this value of μ in (2), we have

$$v^2=\frac{4\pi^2}{T^2}(a^2-x^2) \quad \text{or} \quad v^2T^2=4\pi^2(a^2-x^2). \qquad \ldots(3)$$

Let v_1 be the velocity at a distance $\frac{1}{2}a$ from the centre. Then putting $x=\frac{1}{2}a$ and $v=v_1$ in (3), we get

$$v_1^2T^2=4\pi^2(a^2-\tfrac{1}{4}a^2)=3\pi^2a^2. \qquad \ldots(4)$$

Let a_1 be the new amplitude when the velocity at the point $x=\frac{1}{2}a$ is doubled *i.e.*, when the velocity at the point $x=\frac{1}{2}a$ is any how made $2v_1$. Since the period remains unchanged, therefore putting $v=2v_1$, $a=a_1$ and $x=\frac{1}{2}a$ in (3), we get

$$4v_1^2T^2=4\pi^2(a_1^2-\tfrac{1}{4}a^2)$$

or $4\times3\pi^2a^2=4\pi^2(a_1^2-\frac{1}{4}a^2)$ $\quad [\because$ from (4), $v_1^2T^2=3\pi^2a^2]$

or $a_1^2=3a^2+\frac{1}{4}a^2=13a^2/4$. Hence the new amplitude $a_1=(a\sqrt{13})/2$.

Ex. 35. *Show that the particle executing S.H.M. requires one sixth of its period to move from the position of maximum displacement to one in which the displacement is half the amplitude.*

Sol. Let the equation of S.H.M. with centre as origin be $d^2x/dt^2=-\mu x$.

The time period $T=2\pi/\sqrt{\mu}$.

Let a be the amplitude of the motion. Then

$$(dx/dt)^2=\mu(a^2-x^2).$$

Suppose the particle is moving from the position of maximum displacement $x=a$ in the direction of x decreasing. Then

$$\frac{dx}{dt}=-\sqrt{\mu}\sqrt{(a^2-x^2)} \quad \text{or} \quad dt=-\frac{1}{\sqrt{\mu}\sqrt{(a^2-x^2)}}. \qquad \ldots(1)$$

Let t_1 be the time from the maximum displacement $x=a$ to the point $x=\frac{1}{2}a$. Then integrating (1), we get

$$\int_0^{t_1} dt=-\frac{1}{\sqrt{\mu}}\int_a^{a/2}\frac{dx}{\sqrt{(a^2-x^2)}}.$$

$$t_1=\frac{1}{\sqrt{\mu}}\left[\cos^{-1}\frac{x}{a}\right]_a^{a/2}=\frac{1}{\sqrt{\mu}}\left[\cos^{-1}\frac{1}{2}-\cos^{-1}1\right]$$

$$=\frac{1}{\sqrt{\mu}}\left[\frac{\pi}{3}-0\right]=\frac{1}{\sqrt{\mu}}\cdot\frac{\pi}{3}=\frac{1}{6}\left(\frac{2\pi}{\sqrt{\mu}}\right)=\frac{1}{6}\cdot\text{(time period } T).$$

Ex. 36. *A particle is performing a simple harmonic motion of period T about a centre O and it passes through a point P where $OP=b$ with velocity v in the direction OP; prove that the time which elapses before it returns to P is* IAS-2007

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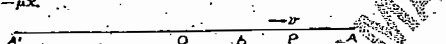
$$\frac{T}{\pi}\tan^{-1}\left(\frac{vT}{2\pi b}\right).$$

Sol. Let the equation of the S.H.M. with centre O as origin be $d^2x/dt^2=-\mu x$.



The time period $T=2\pi/\sqrt{\mu}$.

Let the amplitude be a. Then $(dx/dt)^2=\mu(a^2-x^2)$. $\qquad \ldots(1)$

When the particle passes through P its velocity is given to be v in the direction OP. Also $OP=b$. So putting $x=b$ and $dx/dt=v$ in (1), we get

$$v^2=\mu(a^2-b^2). \qquad \ldots(2)$$

Let A be an extremity of the motion. From P the particle comes to instantaneous rest at A and then returns back to P. In S.H.M. the time from P to A is equal to the time from A to P.

$\therefore$ the required time $=2\times$ time from A to P.

Now for the motion from A to P, we have

$$\frac{dx}{dt}=-\sqrt{\mu}\sqrt{(a^2-x^2)} \quad \text{or} \quad dt=-\frac{1}{\sqrt{\mu}}\frac{dx}{\sqrt{(a^2-x^2)}}$$

Let t_1 be the time from A to P. Then at A, $t=0$, $x=a$ and at P, $t=t_1$ and $x=b$. Therefore integrating (3), we get

$$\int_0^{t_1} dt=\frac{1}{\sqrt{\mu}}\int_a^b\frac{-dx}{\sqrt{(a^2-x^2)}}; \quad \text{or} \quad t_1=\frac{1}{\sqrt{\mu}}\left[\cos^{-1}\frac{x}{a}\right]_a^b$$

$$=\frac{1}{\sqrt{\mu}}\left[\cos^{-1}\frac{b}{a}-\cos^{-1}1\right]=\frac{1}{\sqrt{\mu}}\cos^{-1}\frac{b}{a}$$

Hence the required time $=2t_1=\dfrac{2}{\sqrt{\mu}}\cos^{-1}\dfrac{b}{a}$

$$=\frac{2}{\sqrt{\mu}}\tan^{-1}\left\{\frac{\sqrt{(a^2-b^2)}}{b}\right\}=\frac{2}{\sqrt{\mu}}\tan^{-1}\left(\frac{v}{b\sqrt{\mu}}\right)$$

$$\left[\because \text{ from (2), } \sqrt{(a^2-b^2)}=\frac{v}{\sqrt{\mu}}\right]$$

$$=\frac{2}{2\pi/T}\tan^{-1}\left\{\frac{v}{b(2\pi/T)}\right\} \quad [\because T=2\pi/\sqrt{\mu} \text{ so that } \sqrt{\mu}=2\pi/T]$$

$$=\frac{T}{\pi}\tan^{-1}\left(\frac{vT}{2\pi b}\right).$$

Ex. 37. *A point moving in a straight line with S.H.M. has velocities v_1 and v_2 when its distances from the centre are x_1 and x_2. Show that the period of motion is*

$$2\pi\sqrt{\left(\frac{x_1^2-x_2^2}{v_2^2-v_1^2}\right)}.$$

Sol. Let the equation of the S.H.M. with centre O as origin be $d^2x/dt^2=-\mu x$. Then the time period $T=2\pi/\sqrt{\mu}$.

If a be the amplitude of the motion, we have

$$v^2=\mu(a^2-x^2),$$

where v is the velocity at a distance x from the centre.

But when $x=x_1$, $v=v_1$ and when $x=x_2$, $v=v_2$.

Therefore from (1), we have

$$v_1^2=\mu(a^2-x_1^2) \text{ and } v_2^2=\mu(a^2-x_2^2).$$

These give $v_2^2-v_1^2=\mu\{(a^2-x_2^2)-(a^2-x_1^2)\}=\mu(x_1^2-x_2^2)$

i.e., $\mu=(v_2^2-v_1^2)/(x_1^2-x_2^2)$.

Hence the time period $T=2\pi/\sqrt{\mu}=2\pi\sqrt{\left(\dfrac{x_1^2-x_2^2}{v_2^2-v_1^2}\right)}$

Ex. 38. *A particle is moving with S.H.M. and while making an excursion from one position of rest to the other, its distances from the middle point of its path at three consecutive seconds are observed to be x_1, x_2, x_3; prove that the time of a complete oscillation is*

$$2\pi\Big/\cos^{-1}\left(\frac{x_1+x_3}{2x_2}\right).$$

Sol. Take the middle point of the path as origin. Let the equation of the S.H.M. be $d^2x/dt^2=-\mu x$. Then the time period $T=2\pi/\sqrt{\mu}$.

Let a be the amplitude of the motion. If the time t be measured from the position of instantaneous rest $x=a$, we have

$$x=a\cos\sqrt{\mu}t,$$

where x is the distance of the particle from the centre at time t.

Let x_1, x_2, x_3 be the distances of the particle from the centre at the ends of t_1^{th}, $(t_1+1)^{th}$ and $(t_1+2)^{th}$ seconds. Then from (1),

$$x_1=a\cos\sqrt{\mu}t_1, \qquad \ldots(2)$$

$$x_2=a\cos\sqrt{\mu}(t_1+1), \qquad \ldots(3)$$

and $$x_3=a\cos\sqrt{\mu}(t_1+2). \qquad \ldots(4)$$

$\therefore$ $x_1+x_3=a[\cos\sqrt{\mu}t_1+\cos\sqrt{\mu}(t_1+2)]$

$=2a\cos\sqrt{\mu}(t_1+1)\cos\sqrt{\mu}=2x_2\cos\sqrt{\mu}$, [from (3)].

$\therefore$ $\cos\sqrt{\mu}=(x_1+x_3)/2x_2$ or $\sqrt{\mu}=\cos^{-1}((x_1+x_3)/2x_2)$.

Hence the time period $T=\dfrac{2\pi}{\sqrt{\mu}}=\dfrac{2\pi}{\cos^{-1}\{(x_1+x_3)/2x_2\}}$

Ex. 39 (a). *At the ends of three successive seconds the distances of a point moving with S.H.M. from the mean position measured in the same direction are 1, 5 and 5. Show that the period of a complete oscillation is $2\pi/\theta$ where $\cos\theta=3/5$.*

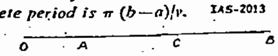
Sol. Proceed as in Ex. 38.

Ex. 39 (b). *At the end of three successive seconds, the distances of a point moving with simple harmonic motion from its mean position measured in the same direction are 1, 3 and 4. Show that the period of complete oscillation is*

$$\frac{2\pi}{\cos^{-1}(5/6)}.$$

Ex. 40. *A body moving in a straight line OAB with S.H.M. has zero velocity when at the points A and B whose distances from O are a and b respectively, and has velocity v when half way between them. Show that the complete period is $\pi(b-a)/v$.* IAS-2013

Sol. In the figure, A and B are the positions of instantaneous rest in a S.H.M. Let C be the middle point of AB. Then C is the centre of the motion. Also it is given that $OA=a$, $OB=b$.



The amplitude of the motion $=\frac{1}{2}AB=\frac{1}{2}(OB-OA)=\frac{1}{2}(b-a)$.

Now in a S.H.M. the velocity at the centre $=(\sqrt{\mu})\times$amplitude. Since in this case the velocity at the centre is given to be v,

therefore $\quad v=\frac{1}{2}(b-a)\cdot\sqrt{\mu}$ or $\sqrt{\mu}=2v/(b-a)$.

Hence time period $T=2\pi/\sqrt{\mu}=2\pi[(b-a)/2v]=\pi(b-a)/v$.

Ex. 41. *A point executes S.H.M. such that in two of its positions velocities are u, v and the two corresponding accelerations are α, β; show that the distance between the two positions is $(v^2-u^2)/(\alpha+\beta)$ and the amplitude of the motion is*

$$\frac{\{(v^2-u^2)(\alpha^2v^2-\beta^2u^2)\}^{1/2}}{\alpha^2-\beta^2}.$$

Sol. Let the equation of the S.H.M. with centre as origin be $d^2x/dt^2=-\mu x$.

If a be the amplitude of the motion, we have

$$(dx/dt)^2=\mu(a^2-x^2),$$

where dx/dt is the velocity at a distance x from the centre.

Let x_1 and x_2 be the distances from the centre of the two positions where u and v are the velocities and α and β are the accelerations respectively. Then

$$\alpha=\mu x_1, \quad \ldots(1)$$
$$\beta=\mu x_2, \quad \ldots(2)$$
$$u^2=\mu\,(a^2-x_1^2), \quad \ldots(3)$$
and
$$v^2=\mu\,(a^2-x_2^2). \quad \ldots(4)$$

Adding (1) and (2), we get $\alpha+\beta=\mu\,(x_1+x_2)$. ...(5)

Also subtracting (3) from (4), we get
$$v^2-u^2=\mu\,(x_1^2-x_2^2)=\mu\,(x_1-x_2)\,(x_1+x_2)=(\alpha+\beta)\,(x_1-x_2). \quad \text{[from (5)]}$$

$\therefore$ $(x_1-x_2)=(v^2-u^2)/(\alpha+\beta)$. This gives the distance between the two positions.

Now to get the amplitude a it is obvious that we have to eliminate x_1, x_2 and μ from the equations (1), (2), (3) and (4). Substituting for x_1 and x_2 from (1) and (2) in (3) and (4), we have

$$u^2=\mu\left(a^2-\frac{\alpha^2}{\mu^2}\right) \quad i.e., \quad a^2\mu^2-u^2\mu-\alpha^2=0 \quad \ldots(6)$$

and
$$v^2=\mu\left(a^2-\frac{\beta^2}{\mu^2}\right) \quad i.e., \quad a^2\mu^2-v^2\mu-\beta^2=0. \quad \ldots(7)$$

By the method of cross multiplication, we have from (6) and (7),

$$\frac{\mu^2}{u^2\beta^2-v^2\alpha^2}=\frac{\mu}{-a^2\alpha^2+a^2\beta^2}=\frac{1}{a^2u^2-a^2v^2}.$$

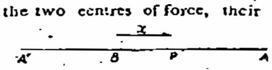
Equating the two values of μ^2 found from the above equations, we get

$$\frac{\alpha^2v^2-u^2\beta^2}{a^2\,(v^2-u^2)}=\left[\frac{a^2\,(\alpha^2-\beta^2)}{a^2\,(v^2-u^2)}\right]^2, \quad \text{or} \quad \frac{\alpha^2v^2-u^2\beta^2}{a^2\,(v^2-u^2)}=\frac{(\alpha^2-\beta^2)^2}{(v^2-u^2)^2}.$$

$$\therefore \quad a^2=\frac{(\alpha^2v^2-\beta^2u^2)\,(v^2-u^2)}{(\alpha^2-\beta^2)^2} \quad \text{or} \quad a=\frac{\{(v^2-u^2)\,(\alpha^2v^2-\beta^2u^2)\}^{1/2}}{(\alpha^2-\beta^2)}.$$

Ex. 42. *A particle rests in equilibrium under the attraction of two centres of force which attract directly as the distance, their intensities being μ and μ'; the particle is displaced slightly towards one of them, show that the time of a small oscillation is $2\pi/\sqrt{(\mu+\mu')}$.* (IFoS-2009)

Sol. Suppose A and A' are the two centres of force, their intensities being μ and μ' respectively. Let a particle of mass m be in equilibrium at



B under the attraction of these two centres. If $AB=a$ and $A'B=a'$, the forces of attraction at B due to the centres A and A' are $m\mu a$ and $m\mu' a'$ respectively in opposite directions. As these two forces balance, we have
$$m\mu a=m\mu' a'. \quad \ldots(1)$$

Now suppose the particle is slightly displaced towards A and then let go. Let P be the position of the particle after time t, where $BP=x$.

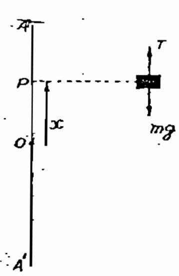
The attraction at P due to the centre A is $m\mu.AP$ or $m\mu(a-x)$ in the direction PA *i.e.*, in the direction of x increasing. Also the attraction at P due to the centre A' is $m\mu'.A'P$ or $m\mu'(a'+x)$ in the direction PA' *i.e.*, in the direction of x decreasing. Hence by Newton's second law of motion, the equation of motion of the particle at P is
$$m\,(d^2x/dt^2)=m\mu\,(a-x)-m\mu'\,(a'+x), \quad \ldots(2)$$
where the force in the direction of x increasing has been taken with +ive sign and the force in the direction of x decreasing has been taken with −ive sign.

Simplifying the equation (2), we get
$$m\,(d^2x/dt^2)=m\,(\mu a-\mu x-\mu' a'-\mu' x)$$
or
$$d^2x/dt^2=-(\mu+\mu')\,x. \quad [\because \text{ by (1)}, m\mu a=m\mu' a']$$

This is the equation of a S.H.M. with centre at the origin. Hence the motion of the particle is simple harmonic with centre at B and its time period is $2\pi/\sqrt{(\mu+\mu')}$.

Ex. 43. *A body is attached to one end of an inelastic string, and the other end moves in a vertical line with S.H.M. of amplitude a, making n oscillations per second. Show that the string will not remain tight during the motion unless $n^2<g/(4\pi^2a)$.*

Sol. Suppose the string remains tight during the motion so that the body also moves in an identical S.H.M. Let m be the mass of the body.



Let the body move in S.H.M. between A and A' and suppose O is the centre of the motion, where $OA=a$.

Since the body makes n oscillations per second, therefore its time period $\dfrac{2\pi}{\sqrt{\mu}}=\dfrac{1}{n}$.

This gives $\mu=4\pi^2n^2$.

At time t, let the body be in a position P, where $OP=x$. The impressed force acting on the body is $T-mg$ along OP. Here T is the tension of the string. By Newton's law, the equation of motion of the body is $m\,(d^2x/dt^2)=T-mg$.
$$\therefore \quad T=mg+m\,(d^2x/dt^2).$$

Obviously T is least when d^2x/dt^2 is least. But the least value of d^2x/dt^2 is $-\mu a$. Hence least $T=mg-m\mu a$.

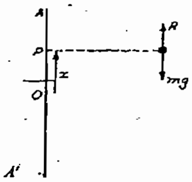
The string will remain tight if this least tension is positive *i.e.*, if $m\mu a<mg$

i.e., if $m4\pi^2n^2a<mg$ $\quad [\because \mu=4\pi^2n^2]$

i.e., if $n^2<g/(4\pi^2a)$. Hence the result.

Ex. 44. *A horizontal shelf is moved up and down with S. H. M. of period $\frac{1}{2}$ sec. What is the amplitude admissible in order that a weight placed on the shelf may not be jerked off?*

Sol. Let m be the mass of the body placed on the shelf. Suppose along with the shelf, the body moves in an identical S. H. M. between A and A'. Let O be the centre of the motion so that $OA=a$ is the amplitude.



The time period $2\pi/\sqrt{\mu}=\frac{1}{2}$; (given)

$\therefore \quad \mu=16\pi^2$.

Let P be the position of the body at time t, where $OP=x$. The impressed force acting on the body is $R-mg$ along OP. Here R is the reaction of the shelf. By Newton's law the equation of motion of the body is
$$m(d^2x/dt^2)=R-mg.$$
$$R=mg+m\,(d^2x/dt^2).$$

Obviously R is least when d^2x/dt^2 is least and the least value of d^2x/dt^2 is $-\mu a$. Hence least $R=mg-m\mu a$.

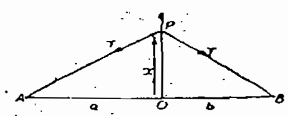
The body will not be jerked off if this least value of R remains non-negative *i.e.*, if $m\mu a\leq mg$

i.e., if $m16\pi^2a\leq mg$ $\quad [\because \mu=16\pi^2]$

i.e., if $a\leq g/(16\pi^2)$. Hence the greatest admissible value of the amplitude $a=g/(16\pi^2)$.

Ex. 45. *A particle of mass m is attached to a light wire which is stretched tightly between two fixed points with a tension T. If a, b be the distance of the particle from the two ends, prove that the period of small transverse oscillation of mass m is*
$$\frac{2\pi}{\sqrt{\left\{\dfrac{T\,(a+b)}{mab}\right\}}}$$

Sol. Let a light wire be stretched tightly between the fixed points A and B with a tension T. Let a particle of mass m be attached at the point O of the wire where $AO=a$ and $OB=b$.



Let the particle be displaced slightly perpendicular to AB (*i.e.*, in the transverse direction) and then let go. Let P be the position of the particle at any time t, where $OP=x$. Since the displacement is small, therefore the tension in the string in any displaced position can be taken as T which is the tension in the string in the original position. The equation of motion of the particle is

$$m\frac{d^2x}{dt^2}=-(T\cos\angle OPA+T\cos\angle OPB)$$
$$=-T\left(\frac{OP}{AP}+\frac{OP}{BP}\right)=-T\left(\frac{x}{\sqrt{(a^2+x^2)}}+\frac{x}{\sqrt{(b^2+x^2)}}\right)$$
$$=-T\left\{\frac{x}{a}\left(1+\frac{x^2}{a^2}\right)^{-1/2}+\frac{x}{b}\left(1+\frac{x^2}{b^2}\right)^{-1/2}\right\}$$
$$=-T\left[\frac{x}{a}\left(1-\frac{1}{2}\frac{x^2}{a^2}+\ldots\right)+\frac{x}{b}\left(1-\frac{1}{2}\frac{x^2}{b^2}+\ldots\right)\right]$$
$$=-T\left(\frac{x}{a}+\frac{x}{b}\right), \text{ neglecting higher powers of } x/a \text{ and } x/b \text{ which are very small}$$
$$=-T\left(\frac{a+b}{ab}\right)x.$$

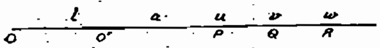
$$\therefore \quad \frac{d^2x}{dt^2}=-\frac{T\,(a+b)}{mab}\,x=-\mu x, \text{ where } \mu=\frac{T\,(a+b)}{mab}.$$

This is the standard equation of a S. H. M. with centre at the origin. The time period
$$T=\frac{2\pi}{\sqrt{\mu}}=2\pi\Big/\sqrt{\left\{\frac{T\,(a+b)}{mab}\right\}}=2\pi\sqrt{\left\{\frac{mab}{T\,(a+b)}\right\}}.$$

Ex. 46. *If in a S. H. M. u, v, w be the velocities at distances a, b, c from a fixed point on the straight line which is not the centre of force, show that the period T is given by the equation*
$$\frac{4\pi^2}{T^2}\,(a-b)\,(b-c)\,(c-a)=\begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}.$$

Sol. Let O and O' be the centre of force and the fixed point respectively on the line of motion and let

$OO' = l$. Let u, v, w be the velocities of the particle at P, Q, R respectively where
$$O'P = a,\ O'Q = b,\ O'R = c.$$



For a S.H.M. of amplitude A, the velocity V at a distance x from the centre of force is given by
$$V^2 = \mu\,(A^2 - x^2). \qquad \ldots(1)$$
At P, $x = OP = l + a$, $V = u$
at Q, $x = OQ = l + b$; $V = v$
and at R, $x = OR = l + c$, $V = w$.

$\therefore$ from (1), we have
$$u^2 = \mu\,\{A^2 - (l+a)^2\}$$
or
$$\frac{u^2}{\mu} = A^2 - l^2 - a^2 - 2al$$
or
$$\left(\frac{u^2}{\mu} + a^2\right) + 2l.a + (l^2 - A^2) = 0. \qquad \ldots(2)$$
Similarly,
$$\left(\frac{v^2}{\mu} + b^2\right) + 2l.b + (l^2 - A^2) = 0. \qquad \ldots(3)$$
and
$$\left(\frac{w^2}{\mu} + c^2\right) + 2l.c + (l^2 - A^2) = 0. \qquad \ldots(4)$$
Eliminating $2l$ and $(l^2 - A^2)$ from (2), (3) and (4), we have
$$\begin{vmatrix} \frac{u^2}{\mu} + a^2 & a & 1 \\ \frac{v^2}{\mu} + b^2 & b & 1 \\ \frac{w^2}{\mu} + c^2 & c & 1 \end{vmatrix} = 0$$
or
$$\begin{vmatrix} \frac{u^2}{\mu} & a & 1 \\ \frac{v^2}{\mu} & b & 1 \\ \frac{w^2}{\mu} & c & 1 \end{vmatrix} + \begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} = 0$$
or
$$-\begin{vmatrix} a^2 & a & 1 \\ b^2 & b & 1 \\ c^2 & c & 1 \end{vmatrix} = \frac{1}{\mu}\begin{vmatrix} u^2 & a & 1 \\ v^2 & b & 1 \\ w^2 & c & 1 \end{vmatrix}$$
or
$$\mu\begin{vmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{vmatrix} = \begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$
or
$$\mu\,(a-b)\,(b-c)\,(c-a) = \begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix} \qquad \ldots(5)$$

But the time period $T = \frac{2\pi}{\sqrt{\mu}}$, so that $\mu = \frac{4\pi^2}{T^2}$.

Hence from (5), we have
$$\frac{4\pi^2}{T^2}\,(a-b)\,(b-c)\,(c-a) = \begin{vmatrix} u^2 & v^2 & w^2 \\ a & b & c \\ 1 & 1 & 1 \end{vmatrix}$$

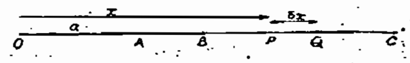
8. Hooke's Law :

Statement. *The tension of an elastic string is proportional to the extension of the string beyond its natural length.*

If x is the stretched length of a string of natural length l, then by Hooke's law the tension T in the string is given by $T = \lambda.\frac{x-l}{l}$, where λ is called **the modulus of elasticity** of the string. Remember that the direction of the tension is always opposite to the extension.

Theorem. *Prove that the work done against the tension in stretching a light elastic string, is equal to the product of its extension and the mean of its final and initial tensions.*

Proof. Let $OA = a$ be the natural length of a string whose one end is fixed at O. Let the string be stretched beyond its natural length.



Let B and C be the two positions of the free end A of the string during its any extension and let $OB = b$ and $OC = c$.

Then by Hooke's law,
$$\text{the tension at } B = T_B = \lambda\,\frac{b-a}{a}, \qquad \ldots(1)$$
and
$$\text{the tension at } C = T_C = \lambda\,\frac{c-a}{a}, \qquad \ldots(2)$$
where λ is the modulus of elasticity of the string.

Now we find the work done against the tension in stretching the string from B to C.

Let P be any position of the free end of the string during its extension from B to C and let $OP = x$.

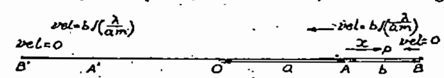
Then the tension at $P = T_P = \lambda.\,\frac{x-a}{a}$.

Now suppose the free end of the string is slightly stsetched from P to Q, where $PQ = \delta x$. Then the work done against the tension in stretching the string from P to Q
$$= T_P\,\delta x = \lambda\,\frac{(x-a)}{a}.\delta x.$$
$\therefore$ the work done against the tension in stretching the string from B to C
$$= \int_b^c \frac{\lambda}{a}(x-a)\,dx = \frac{\lambda}{2a}\Big[(x-a)^2\Big]_b^c$$
$$= \frac{\lambda}{2a}\left[(c-a)^2 - (b-a)^2\right] = \frac{\lambda}{2a}\left[\{(c-a)-(b-a)\}\{(c-a)+(b-a)\}\right]$$
$$= (c-b).\frac{1}{2}\left[\frac{\lambda}{a}(c-a) + \frac{\lambda}{a}(b-a)\right]$$
$$= (c-b).\tfrac{1}{2}\,[T_C + T_B], \qquad \text{[from (1) and (2)]}$$
$$= BC \times (\text{mean of the tension at } B \text{ and } C).$$
Hence, the work done against the tension in stretching the string is equal to the product of the extension and the mean of the initial and final tensions.

Now we shall discuss a few simple and interesting cases of simle harmonic motion.

9. Particle attached to one end of a horizontal elastic string. *A particle of mass m is attached to one end of a horizontal elastic string whose other end is fixed to a point on a smooth horizontal table. The particle is pulled to any distance in the direction of the string and then let go; to discuss the motion.*

Let a string OA of natural length a lie on a smooth horizontal table. The end O of the string is attached to a fixed point of the table and a particle of mass m is attached to the other end A. The mass m is pulled upto B, where $AB = b$, and then let go.



Let P be the position of the particle after time t, where $AP = x$. The table being smooth, the only horizontal force acting on the particle at P is the tension T in the string OP. Since the direction of tension is always opposite to the extension, therefore, the force T acts in the direction PA *i.e.*, in the direction of x decreasing. Also by Hooke's law $T = \lambda\,(x/a)$. Hence the equation of motion of the particle at P is
$$m\,\frac{d^2x}{dt^2} = -\lambda\,\frac{x}{a} \text{ or } \frac{d^2x}{dt^2} = -\frac{\lambda}{am}\,x. \qquad \ldots(1)$$
The equation (1) shows that the motion of the particle is simple harmonic with centre at the origin A. The equation of motion (1) holds good so long as the string is stretched. Since the string becomes slack just as the particle reaches A, therefore the equation (1) holds good for the motion of the particle from B to A.

Multiplying (1) by $2\,(dx/dt)$ and integrating, we get
$$\left(\frac{dx}{dt}\right)^2 = -\frac{\lambda}{am}\,x^2 + C, \text{ where } C \text{ is a constant.}$$
At the point B, $x = b$ and $dx/dt = 0$; $\therefore$ $C = (\lambda/am)\,b^2$.

Thus we have
$$\left(\frac{dx}{dt}\right)^2 = \frac{\lambda}{am}\,(b^2 - x^2). \qquad \ldots(2)$$
This equation gives velocity in any position from B to A. Putting $x = 0$ in (2), we have the velocity at $A = \sqrt{(\lambda/am)}\,b$, in the direction AO.

The time from B to A is $\frac{1}{4}$ of the complete time period of a S.H.M. whose equation is (1).

Character of the motion. The motion from B to A is simple harmonic. When the particle reaches A, the string becomes slack and the simple harmonic motion ceases. But due to the velocity gained at A the particle continues to move to the left of A. So long as the string is loose there is no force on the particle to change its velocity because the only force here is that of tension and the tension is zero so long as the string is loose. Thus the particle moves from A to A' with uniform velocity $\sqrt{(\lambda/am)}\, b$ gained by it at A. Here A' is a point on the other side of O such that $OA'=OA$. When the particle passes A' the string again becomes tight and begins to extend. The tension again comes into picture and the particle begins to move in S. H. M. But now the force of tension acts against the direction of motion of the particle. So the velocity of the particle starts decreasing and the particle comes to instantaneous rest at B', where $A'B'=AB$. The time from A' to B' is the same as that from B to A. At B' the particle at once begins to move towards A' because of the tension which attracts it towards A'. Retracing its path the particle again comes to instantaneous rest at B and thus it continues to oscillate between B and B'.

During one complete oscillation the particle covers the distance between A and B and also that between A' and B' twice while moving in S. H. M. Also it covers the distance between A and A' twice with uniform velocity $\sqrt{(\lambda/am)}\, b$. Hence the total time for one complete oscillation

=the complete time period of a S.H.M. whose equation is (1)

+the time taken to cover the distance $4a$ with uniform velocity $\sqrt{(\lambda/am)}\, b$

$$=\frac{2\pi}{\sqrt{(\lambda/am)}}+\frac{4a}{\sqrt{(\lambda/am)}\, b}=2\pi\sqrt{\left(\frac{am}{\lambda}\right)}+\frac{4a}{b}\sqrt{\left(\frac{am}{\lambda}\right)}$$

$$=2\left(\pi+\frac{2a}{b}\right)\sqrt{\left(\frac{am}{\lambda}\right)}.$$

Illustrative Examples :

Ex. 47. *One end of an elastic string (modulus of elasticity λ) whose natural length is a, is fixed to a point on a smooth horizontal table and the other is tied to a particle of mass m, which is lying on the table. The particle is pulled to a distance from the point of attachment of the string equal to twice its natural length and then let go. Show that the time of a complete oscillation is*

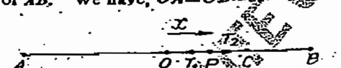
$$2(\pi+2)\sqrt{\left(\frac{am}{\lambda}\right)}.$$

Sol. Proceed exactly in the same way as in 9. Here, the particle is pulled to a distance from the point of attachment of the string equal to twice its natural length. Therefore initially the increase b in the length of the string is equal to $2a-a$ *i.e.*, a.

Now proceed as in § 9, taking $b=a$.

Ex. 48. *A light elastic string whose modulus of elasticity is λ is stretched to double its length and is tied to two fixed points distant $2a$ apart. A particle of mass m tied to its middle point is displaced in the line of the string through a distance equal to half its distance from the fixed points and released. Find the time of a complete oscillation and the maximum velocity acquired in the subsequent motion.*

Sol. Let an elastic string of natural length a be stretched between two fixed points A and B distant $2a$ apart, O being the middle point of AB. We have, $OA=OB=a$.



Natural length of the portions OA and OB each is $a/2$ (since the string is stretched to double its length). A particle of mass m attached to the middle point O is displaced towards B upto a point C, where $OC=a/2$ and then let go. Let P be the position of the particle after any time t, where $OP=x$. [Note that we have taken O as origin. The direction OP is that of x increasing and the direction PO is that of x decreasing]. At P there are two horizontal forces acting on the particle :

(i) The tension T_1 in the string AP acting in the direction PA *i.e.*, in the direction of x decreasing.

(ii) The tension T_2 in the string BP acting in the direction PB *i.e.*, in the direction of x increasing.

[Note that the string AP is extended in the direction AP and so the tension T_1 in it acts in the opposite direction PA].

By Hooke's law, $T_1=\lambda\dfrac{a+x-\frac{1}{2}a}{a/2}$ and $T_2=\lambda\dfrac{a-x-\frac{1}{2}a}{a/2}$.

Hence by Newton's second law of motion ($P=mf$), the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2}=T_2-T_1=\lambda\frac{a-x-a/2}{a/2}-\lambda\frac{a+x-a/2}{a/2}=-\frac{4\lambda x}{a}.$$

$$\therefore\quad \frac{d^2x}{dt^2}=-\frac{4\lambda}{am}x. \quad ...(1)$$

Thus the motion is S.H.M. with centre at the origin O. Since we have displaced the particle towards B only upto the point C so that the portion BC of the string is just in its natural length, therefore during the entire motion of the particle both the portions of the string remain taut and so the entire motion of the particle is governed by the above equation. Thus the particle makes oscillations in S.H.M. about O and the time period of one complete oscillation

=the time period of S.H.M. whose equation is (1)

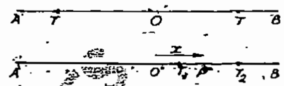
$$=2\pi\Big/\sqrt{\left(\frac{4\lambda}{am}\right)}=\pi\sqrt{\left(am/\lambda\right)}.$$

The amplitude (*i.e.*, the maximum displacement from the centre) of this S.H.M. is $a/2$.

$\therefore$ the maximum velocity$=(\sqrt{\mu})\times$amplitude

$$=\sqrt{(4\lambda/am)}\,.(a/2)=\sqrt{(a\lambda/m)}.$$

Ex. 49. *A particle of mass m executes simple harmonic motion in the line joining the points A and B on a smooth table and is connected with these points by elastic strings whose tensions in equilibrium are each T; show that the time of an oscillation is $2\pi\,[ml l'/T(l+l')]^{1/2}$, where l, l' are the extensions of the strings beyond their natural legths.*



Sol. A particle of mass m rests at O being pulled by two horizontal strings AO and BO whose other ends are connected to two fixed points A and B. Let a, a' be the natural lengths of the strings AO and BO whose extensions beyond their natural lengths are l and l' respectively. Let λ and λ' be the respective modulii of elasticity of the two strings AO and BO. At O the particle is in equilibrium under the tensions of the two strings. Therefore

$$\frac{\lambda l}{a}=\frac{\lambda' l'}{a'}=T \text{ (given)} \quad ...(1)$$

From (1), we have $\dfrac{T}{l}=\dfrac{\lambda}{a}$ and $\dfrac{T}{l'}=\dfrac{\lambda'}{a'}$ ...(2)

Now suppose the particle is slightly pulled towards B and then let go. It begins to move towards O. Let P be the position of the particle after any time t, where $OP=x$. [Note that we have taken O as origin. The direction OP is that of x increasing and the direction PO is that of x decreasing.]

At P there are two horizontal forces acting on the particle :

(i) The tension T_1 in the string AP acting in the direction PA, *i.e.*, in the direction of x decreasing.

(ii) The tension T_2 in the string BP acting in the direction PB, *i.e.*, in the direction of x increasing. [Note that the string AP is extended in the direction AP and so the tension T_1 in it acts in the opposite direction PA.]

By Hooke's law, $T_1=\lambda\dfrac{(l+x)}{a}$ and $T_2=\lambda'\dfrac{(l'-x)}{a'}$.

Hence by Newton's second law of motion ($P=mf$), the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2}=T_2-T_1=\frac{\lambda'(l'-x)}{a'}-\frac{\lambda(l+x)}{a}$$

$$=\frac{\lambda' x}{a'}-\frac{\lambda x}{a}. \qquad \left[\because \text{ by (1) } \frac{\lambda l'}{a'}=\frac{\lambda l}{a}\right]$$

$$=-x\left(\frac{\lambda'}{a'}+\frac{\lambda}{a}\right).$$

$$\frac{d^2x}{dt^2}=-\frac{x}{m}\left(\frac{\lambda'}{a'}+\frac{\lambda}{a}\right)=-\frac{x}{m}\left(\frac{T}{l'}+\frac{T}{l}\right), \quad \text{from (2)}$$

$$=-\frac{T(l+l')}{mll'}x, \quad ...(3)$$

showing that the motion of the particle is simple harmonic with centre at the origin O.

Since we have given only a slight displacement to the particle towards B, therefore during the entire motion of the particle both the strings remain taut and so the entire motion of the particle is governed by the equation (3). Thus the particle makes small oscillations in S.H.M. about O and the time period of one complete oscillation

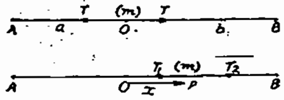
$$=\frac{2\pi}{\sqrt{\mu}}=\frac{2\pi}{\sqrt{\{T(l+l')/mll'\}}}=2\pi\left[\frac{mll'}{T(l+l')}\right]^{1/2}$$

Remark. In order that the entire motion of the particle should remain simple harmonic with centre at O, the particle must be pulled towards B only upto that distance which does not allow the string OB to become slack.

Ex. 50. *Two light elastic strings are fastened to a particle of mass m and their other ends to fixed points so that the strings are taut. The modulus of each is λ, the tension T, and length a and b. Show that the period of an oscillation along the line of the strings is*

$$2\pi\left[\frac{mab}{(T+\lambda)(a+b)}\right]^{1/2}.$$

Sol. Let the two light elastic strings be fastened to a particle of mass m at O and their other ends be attached to two fixed points A and B so that the strings are taut and $OA=a$, $OB=b$. If l and l' are the natural lengths of the strings OA and OB respectively, then in the position of equilibrium of the particle at O,



tension in the string OA = tension in the string $OB=T$. (as given).

Applying Hooke's law, we have

$$T=\lambda\frac{a-l}{l}=\lambda\frac{b-l'}{l'}. \qquad \text{...(1)}$$

From $T=\lambda\dfrac{a-l}{l}$, we have $Tl=\lambda a-\lambda l$

i.e., $\quad l(T+\lambda)=\lambda a$

i.e., $$\frac{\lambda}{l}=\frac{T+\lambda}{a}. \qquad \text{...(2)}$$

Similarly $$\frac{\lambda}{l'}=\frac{T+\lambda}{b}. \qquad \text{...(3)}$$

Now suppose the particle is slightly pulled towards B and then let go. It begins to move towards O. Let P be the position of the particle after any time t, where $OP=x$. The direction OP is that of x increasing and the direction PO is that of x decreasing.

At P there are two horizontal forces acting on the particle.

(i) The tension T_1 in the string AP acting in the direction PA i.e., in the direction of x decreasing.

(ii) The tension T_2 in the string BP acting in the direction PB i.e., in the direction of x increasing.

By Hooke's law, $T_1=\lambda\dfrac{a+x-l}{l}$, $T_2=\lambda\dfrac{b-x-l'}{l'}$.

Hence by Newton's second law of motion ($P=mf$), the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2}=T_2-T_1=\frac{\lambda(b-x-l')}{l'}-\frac{\lambda(a+x-l)}{l}$$

$$=-\frac{\lambda}{l'}x-\frac{\lambda}{l}x, \quad \left[\because \text{ from (1), } \frac{\lambda(b-l')}{l'}=\frac{\lambda(a-l)}{l}\right]$$

$$=-\left[\frac{T+\lambda}{b}+\frac{T+\lambda}{a}\right]x. \qquad \text{[from (2) and (3)]}$$

$$=-\frac{(T+\lambda)(a+b)}{ab}\,x.$$

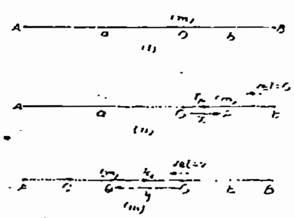
$$\therefore \quad \frac{d^2x}{dt^2}=-\frac{(T+\lambda)(a+b)}{mab}\,x, \qquad \text{...(4)}$$

showing that the motion of the particle is simple harmonic with centre at the origin O.

Since we have given only a slight displacement to the particle towards B, therefore during the entire motion of the particle both the strings remain taut and the entire motion of the particle is governed by the equation (4). Thus the particle makes small oscillations in S. H. M. about O and the time period of one complete oscillation

$$=\frac{2\pi}{\sqrt{\mu}}=\frac{2\pi}{\sqrt{\{(T+\lambda)(a+b)/mab\}}}=2\pi\left[\frac{mab}{(T+\lambda)(a+b)}\right]^{1/2}.$$

Ex. 51. *An elastic string of natural length $(a+b)$ where $a>b$ and modulus of elasticity λ has a particle of mass m attached to it at a distance a from one end, which is fixed to a point A of a smooth horizontal plane. The other end of the string is fixed to a point B so that the string is just unstretched. If the particle be held at B and then released, show that it will oscillate to and fro through a distance $\dfrac{b(\sqrt{a}+\sqrt{b})}{\sqrt{a}}$ in a periodic time $\pi(\sqrt{a}+\sqrt{b})\sqrt{(m/\lambda)}$.*



Sol. Let AB be an elastic string of natural length $a+b$ attached to two fixed points A and B distant $a+b$ apart. Let a particle of mass m be attached to the point O of the string such that $OA=a$, $OB=b$ and $a>b$.

When the particle is held at B, the portion AO of the string is stretched while the portion OB is slack and so when the particle is released from B, it moves towards O starting from rest at B.

If P is the position of the particle between O and B, [see fig. (ii)], at any time t after its release from B and $OP=x$, then the tension in the string AP is $T_P=\lambda\dfrac{x}{a}$ acting towards O and the tension in the string PB is zero because it is slack.

$\therefore$ the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2}=-T_P=-\frac{\lambda}{a}x$$

or $$\frac{d^2x}{dt^2}=-\frac{\lambda}{am}x. \qquad \text{...(1)}$$

which represents a S. H. M. with centre at O and amplitude OB.

If t_1 be the time from B to O, then

$t_1=\frac{1}{4}\times$ time period of the S. H. M. represented by (1)

$$=\frac{1}{4}\cdot\frac{2\pi}{\sqrt{(\lambda/am)}}=\frac{\pi}{2}\sqrt{\left(\frac{am}{\lambda}\right)}. \qquad \text{...(2)}$$

Now multiplying both sides of (1) by $2(dx/dt)$ and then integrating, we have

$$\left(\frac{dx}{dt}\right)^2=-\frac{\lambda}{am}.x^2+k, \text{ where } k \text{ is a constant.}$$

But at the point B, $x=OB$ and $dx/dt=0$.

$$\therefore \quad 0=-\frac{\lambda}{am}b^2+k \quad \text{or} \quad k=\frac{\lambda b^2}{am}.$$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2=\frac{\lambda}{am}(b^2-x^2). \qquad \text{...(3)}$$

If V is the velocity of the particle at O where $x=0$, then from (3), we have

$$V^2=\frac{\lambda}{am}.b^2 \quad \text{or} \quad V=\sqrt{\left(\frac{\lambda}{am}\right)}.b. \qquad \text{...(4)}$$

At the point O, the tension in either of the two portions of the string is zero and the velocity of the particle is V to the left of O, due to which the particle moves towards the left of O. As the particle moves to the left of O, the string OA becomes slack and the string OB is stretched.

If Q is the position of the particle between O and A, [see fig. (iii)], at any time t, since it starts moving from O to the left of it and $OQ=y$, then the tension in the string QB is $T_Q=\lambda\dfrac{y}{b}$ acting towards O and the tension in the string $QA=0$ because it is slack.

$\therefore$ The equation of motion of the particle at Q is

$$m\frac{d^2y}{dt^2}=-T_Q=-\frac{\lambda y}{b}$$

or $$\frac{d^2y}{dt^2}=-\frac{\lambda}{bm}y. \qquad \text{...(4)}$$

Multiplying both sides of (4) by $2(dy/dt)$ and then integrating, we have

$$\left(\frac{dy}{dt}\right)^2=-\frac{\lambda}{bm}y^2+D, \text{ where } D \text{ is a constant.}$$

But at O, $y=0$ and $\left(\dfrac{dy}{dt}\right)^2=V^2=\dfrac{\lambda}{am}.b^2$.

$$\therefore \quad \frac{\lambda}{am}.b^2=-\frac{\lambda}{bm}.0+D \quad \text{or} \quad D=\frac{\lambda}{am}b^2.$$

$$\therefore \quad \left(\frac{dy}{dt}\right)^2=\frac{\lambda}{m}\left(\frac{b^2}{a}-\frac{1}{b}y^2\right)$$

or $$\left(\frac{dy}{dt}\right)^2=\frac{\lambda}{bm}\left(\frac{b^3}{a}-y^2\right). \qquad \text{...(5)}$$

If the particle comes to instantaneous rest at the point C between O and A such that $OC=c$, then at C, $y=c$ and $dy/dt=0$.

$\therefore$ from (5), we have

$$0=\frac{\lambda}{bm}\left(\frac{b^3}{a}-c^2\right) \quad \text{or} \quad c=b\sqrt{\left(\frac{b}{a}\right)}.$$

From C the particle retraces its path and comes to instantaneous rest at B.

The particle thus oscillates to and fro through a distance

$$BC=BO+OC=b+c=b+b\sqrt{\left(\frac{b}{a}\right)}=\frac{b(\sqrt{a}+\sqrt{b})}{\sqrt{a}}.$$

The equation (4) represents a S. H. M. with centre at O, amplitude OC and time period $T'=2\pi\Big/\sqrt{\left(\frac{\lambda}{bm}\right)}=2\pi\sqrt{\left(\frac{bm}{\lambda}\right)}$.

If t_2 be the time from O to C, we have

$$t_2=\tfrac{1}{4}.(T')=\frac{\pi}{2}\sqrt{\left(\frac{bm}{\lambda}\right)}.$$

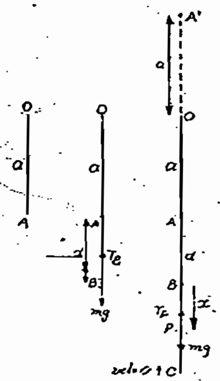
Hence the required periodic time for making a complete oscillation between B and C

$=2.$(time from B to C) $=2(t_1+t_2)$

$$=2\left[\frac{\pi}{2}\sqrt{\left(\frac{am}{\lambda}\right)}+\frac{\pi}{2}\sqrt{\left(\frac{bm}{\lambda}\right)}\right]=\pi(\sqrt{a}+\sqrt{b})\sqrt{\left(\frac{m}{\lambda}\right)}.$$

10. Particle suspended by an elastic string. *A particle of mass m is suspended from a fixed point by a light elastic string of natural length a and modulus of elasticity λ. The particle is pulled down a little in the line of the string and released; to discuss the motion.*

Let one end of the string OA of natural length a be attached to the fixed point O and a particle of mass m be attached to the other end A. Due to the weight mg of the particle the string OA is stretched and if B is the position of equilibrium of the particle such that $AB=d$, then the tension T_B in the string will balance the weight of the particle



i.e., $mg=T_B$

or $mg=\lambda\dfrac{AB}{OA}=\lambda\dfrac{d}{a}$. ...(1)

The particle is pulled down to a point C such that $BC=c$ and then released. At the point C, the tension in the string is greater than the weight of the particle and so the particle starts moving vertically upwards with velocity zero at C. Let P be the position of the particle at any time t, where $BP=x$. The tension in the string when the particle is at P is $T_P=\lambda\dfrac{d+x}{a}$, acting vertically upwards.

The resultant force acting on the particle at P in the vertically upwards direction$=T_P-mg=\lambda\left(\dfrac{d+x}{a}\right)-mg=\dfrac{\lambda d}{a}+\dfrac{\lambda x}{a}-mg$

$$=\frac{\lambda x}{a},\left[\because \frac{\lambda d}{a}=mg, \text{ from (1)}\right].$$

Also the acceleration of the particle at P is d^2x/dt^2 in the direction of x increasing i.e., in the vertically downwards direction.

$\therefore$ by Newton's law, the equation of motion of P is given by

$$m\frac{d^2x}{dt^2}=-\frac{\lambda x}{a} \quad\text{or}\quad \frac{d^2x}{dt^2}=-\frac{\lambda}{am}x. \quad ...(2)$$

This equation holds good so long as the tension operates i.e., when the string is extended beyond its natural length.

Equation (2) is the standard equation of a S.H.M. with centre at the origin B and the amplitude of the motion is $BC=c$.

The periodic time T of the S.H.M. represented by the equation (2) is given by

$$T=2\pi\Big/\sqrt{\left(\frac{\lambda}{am}\right)}=2\pi\sqrt{\left(\frac{am}{\lambda}\right)}. \quad ...(3)$$

The motion of the particle remains simple harmonic as long as there is tension in the string i.e., as long the particle remains in the region from C to A.

In case the string becomes slack during the motion of the particle, the particle will begin to move freely under gravity.

Now there are two cases.

Case I. If $BC\leq AB$ i.e., $c\leq d$. In this case the particle will not rise above A and it will come to instantaneous rest before or just reaching A. The whole motion will be S.H.M. with centre at B, amplitude BC and period T given by (3).

Case II. If $BC>AB$ i.e., $c>d$. In this case the particle will rise above A, and the motion will be simple harmonic upto A and above A the particle will move freely under gravity.

Multiplying both sides of (2) by $2\,(dx/dt)$ and then integrating, we have $\left(\dfrac{dx}{dt}\right)^2=-\dfrac{\lambda}{am}x^2+k$, where k is a constant.

But at C, $x=BC=c$ and $dx/dt=0$.

$$\therefore\quad 0=-\frac{\lambda}{am}c^2+k \quad\text{or}\quad k=\frac{\lambda}{am}c^2.$$

$$\therefore\quad \left(\frac{dx}{dt}\right)^2=\frac{\lambda}{am}(c^2-x^2). \quad ...(4)$$

Now if V is the velocity of the particle at A, where $x=-BA=-d$, then, from (4), we have

$$V^2=\frac{\lambda}{am}(c^2-d^2) \quad\text{or}\quad V=\sqrt{\left[\frac{\lambda}{am}(c^2-d^2)\right]}, \quad ...(5)$$

the direction of V being vertically upwards.

If h is the height to which the particle rises above A, then

$$h=\frac{V^2}{2g}=\frac{\lambda(c^2-d^2)}{2amg}, \quad ...(6)$$

provided $h\leq 2a$.

Also in this case the maximum height attained by the particle during its entire motion

$$=CB+BA+h,$$
$$=c+d+h. \quad ...(7)$$

If $h\leq 2a$ i.e., if $h\leq AA'$, then the particle, after coming to instantaneous rest, will retrace its path i.e., it will fall freely under gravity upto A and below A it will move in S.H.M. till it comes to instantaneous rest at C.

If $h=2a=AA'$, then the particle will just come to rest at A' and will then move downwards, retracing its path.

In this case the maximum height attained by the particle

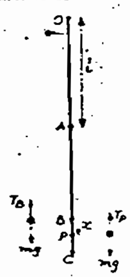
$$=c+d+2a. \quad ...(8)$$

If $h>2a$ i.e., if $h>AA'$, then the particle will rise above A' also and so the string will again become stretched and the particle will again begin to move in simple harmonic motion. After coming to instantaneous rest the particle will retrace its path.

Illustrative Examples

Ex. 52 (a). *An elastic string without weight of which the unstretched length is l and modulus of elasticity is the weight of n oz. is suspended by one end and a mass m oz. is attached to the other end. Show that the time of a small vertical oscillation is $2\pi\sqrt{(ml/ng)}$.*

Sol. $OA=l$ is the natural length of a string whose one end is fixed at O. B is the position of equilibrium of a particle of mass m oz. attached to the other end of the string. Considering the equilibrium of the particle at B, we have $mg=$the tension T_B in the string OB.



$$\therefore\quad mg=ng\frac{AB}{l}, \quad ...(1)$$

because modulus of elasticity of the string is given to be ng.

Now suppose the particle is pulled slightly upto C (so that $BC<AB$) and then let go. It starts moving vertically upwards with velocity zero at C. Let P be its position at any point t, where $BP=x$. The direction BP is that of x increasing and the direction PB is that of x decreasing. At P there are two forces acting on the particle:

(i) The weight mg acting vertically downwards i.e., in the direction of x increasing.

and (ii) the tension $T_P=ng\dfrac{AB+x}{l}$ in the string OP, acting vertically upwards i.e., in the direction of x decreasing.

Hence by Newton's second law of motion, the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2}=mg-ng\frac{AB+x}{l}=mg-ng\frac{AB}{l}-ng\frac{x}{l}$$

$$=-ng\frac{x}{l},\qquad\left[\because \text{ from (1), } mg=ng\frac{AB}{l}\right].$$

$$\therefore\quad \frac{d^2x}{dt^2}=-\frac{ng}{lm}x, \quad ...(2)$$

which is the equation of a simple harmonic motion with centre at the origin B and amplitude BC.

Since $BC<AB$, therefore during the entire motion of the particle the string will not become slack.

Thus the entire motion of the particle is governed by the equation (2) and the particle will make oscillations in simple harmonic motion about the centre B.

The time of one oscillation

$$=\frac{2\pi}{\sqrt{\mu}}=\frac{2\pi}{\sqrt{(ng/lm)}}=2\pi\sqrt{\left(\frac{lm}{ng}\right)}.$$

Ex. 52 (b). *A light elastic string of natural length l is hung by one end and to the other end are tied successively particles of masses m_1 and m_2. If t_1 and t_2 be the periods and c_1, c_2 the statical extensions corresponding to these two weights, prove that*

$$g\,(t_1^2-t_2^2)=4\pi^2(c_1-c_2).$$

Sol. One end of a string OA of natural length l is attached to a fixed point O. Let B be the position of equilibrium of a particle of mass m attached to the other end of the string. Then AB is the statical extension in the string corresponding to this particle of mass m. Let $AB=d$.

In the equilibrium position of the particle of mass m at B, the tension $T_B=\lambda\,(d/l)$ in the string OB balances the weight mg of the particle.

$$\therefore\quad \frac{\lambda d}{l}=mg \quad\text{or}\quad \frac{\lambda}{lm}=\frac{g}{d}. \quad ...(1)$$

Now suppose the particle at B is slightly pulled down upto C and then let go. Let P be the position of the particle at any time t where $BP=x$. When the particle is at P, the tension T_P in the string P is $\lambda\dfrac{d+x}{l}$, acting vertically upwards.

By Newton's second law of motion, the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2}=-\frac{\lambda(d+x)}{l}+mg.$$

[Note that the weight mg of the particle has been taken with the +ive sign because it is acting vertically downwards *i.e.*, in the direction of x increasing.]

or $$m\frac{d^2x}{dt^2}=-\frac{\lambda d}{l}-\frac{\lambda x}{l}+mg$$

$$=-\frac{\lambda x}{l}, \qquad \left[\because \frac{\lambda d}{l}=mg\right].$$

$$\therefore \frac{d^2x}{dt^2}=-\frac{\lambda}{lm}x=-\frac{g}{d}x, \text{ [from (1)]}.$$

Hence the motion of the particle is simple harmonic about the centre B and its period is $\dfrac{2\pi}{\sqrt{(g/d)}}$ *i.e.*, $2\pi\sqrt{\left(\dfrac{d}{g}\right)}$.

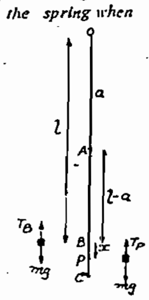
But according to the question, the period is t_1 when $d=c_1$ and the period is t_2 when $d=c_2$.

$\therefore \quad t_1=2\pi\sqrt{(c_1/g)}$ and $t_2=2\pi\sqrt{(c_2/g)}$.

so that $\quad t_1^2-t_2^2=(4\pi^2/g)(c_1-c_2)$

or $\quad g(t_1^2-t_2^2)=4\pi^2(c_1-c_2)$.

Ex. 53. *A mass m hangs from a light spring and is given a small vertical displacement. If l is the length of the spring when the system is in equilibrium and n the number of oscillations per second, show that the natural length of the spring is $l-(g/4\pi^2n^2)$.*



Sol. Let $OA=a$ be the natural length of the spring which extends to a length $OB=l$ when a particle of mass m hangs in equilibrium. In the position of equilibrium of the particle at B, the tension T_B in the spring is $\lambda\{(l-a)/a\}$ and it balances the weight mg of the particle.

$$\therefore \lambda[(l-a)/a]=mg. \qquad \ldots(1)$$

Now suppose the particle at B is slightly pulled down upto C and then let go. It moves towards B starting at rest from C. Let P be the position of the particle after any time t, where $BP=x$. When the particle is at P, the tension T_P in the spring OP is $\lambda\dfrac{l+x-a}{a}$ acting vertically upwards *i.e.*, in the direction of x decreasing.

By Newton's second law of motion, the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2}=mg-\lambda\frac{l+x-a}{a}=mg-\lambda\frac{l-a}{a}-\frac{\lambda x}{a}$$

$$=-\frac{\lambda x}{a}, \text{ [from (1)]}.$$

$$\therefore \frac{d^2x}{dt^2}=-\frac{\lambda}{am}x=-\frac{g}{l-a}x, \left[\because \text{ from (1)}, \frac{\lambda}{am}=\frac{g}{l-a}\right].$$

Hence the motion of the particle is simple harmonic with centre at the origin B and the time period T (*i.e.*, the time for one complete oscillation) $=2\pi\sqrt{\left(\dfrac{l-a}{g}\right)}$ seconds.

Since n is given to be the number of oscillations per second, therefore $n.T=1$ or $n^2T^2=1$

or $$n^2\frac{4\pi^2(l-a)}{g}=1 \text{ or } l-a=\frac{g}{4\pi^2n^2}$$

or $$a=l-\frac{g}{4\pi^2n^2}.$$

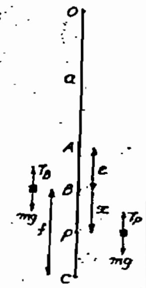
This gives the natural length a of the spring.

Ex. 54. *A heavy particle attached to a fixed point by an elastic string hangs freely, stretching the string by a quantity e. It is drawn down by an additional distance f and then let go: determine the height to which it will arise if $f^2-e^2=4ae$, e being the unstretched length of the string.*

Sol. Let $OA=a$ be the natural length of an elastic string whose one end is fixed at O. Let B be the position of equilibrium of a particle of mass m attached to the other end of the string. It is given that $AB=e$. In the position of equilibrium of the particle at B, the tension T_B in the string OB is $\lambda(e/a)$ and it balances the weight mg of the particle.

$$\therefore \quad mg=\lambda(e/a). \qquad \ldots(1)$$

Now suppose the particle is pulled down to a point C, such that $BC=f$, and then let go. It moves towards B starting with velocity zero at C. Let P be the position of the particle after any time t, where $BP=x$. Note that we have taken B as the origin. When the particle is at P, there are two forces acting upon it:



(i) the tension $T_P=\lambda\dfrac{OP-OA}{OA}=\lambda\dfrac{e+x}{a}$ in the string OP, acting vertically upwards *i.e.*, in the direction of x decreasing, and (ii) the weight mg of the particle acting vertically downwards *i.e.*, in the direction of x increasing.

Hence by Newton's second law of motion, the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2}=mg-\lambda\frac{e+x}{a}=mg-\frac{\lambda e}{a}-\frac{\lambda x}{a}$$

$$=-\frac{\lambda x}{a}, \qquad \left[\because \text{ from (1)}, mg=\frac{\lambda e}{a}\right].$$

$$\therefore \frac{d^2x}{dt^2}=-\frac{\lambda}{am}x=-\frac{g}{e}x, \quad \left[\because \text{ from (1)}, \frac{\lambda}{am}=\frac{g}{e}\right].$$

Thus the equation of motion of the particle is

$$\frac{d^2x}{dt^2}=-\frac{g}{e}x, \qquad \ldots(2)$$

which is the equation of a simple harmonic motion with centre at the origin B and amplitude BC. The equation (2) governs the motion of the particle so long as the string does not become slack.

Since $f^2-e^2=4ae=+$ive, therefore $f>e$ *i.e.*, $BC>AB$. So when the particle, while moving in simple harmonic motion, reaches the point A, its velocity is not zero. But at A the string becomes slack and so above A the particle will move freely under gravity.

Let us first find the velocity at A for the S.H.M. given by (2). Multiplying both sides of (2) by $2(dx/dt)$ and integrating w.r.t. 't', we get $\left(\dfrac{dx}{dt}\right)^2=-\dfrac{g}{e}x^2+k$, where k is a constant.

But at C, $x=BC=f$ and $\left(\dfrac{dx}{dt}\right)=0$. Therefore $0=-\left(\dfrac{g}{e}\right)f^2+k$

or $$k=\left(\frac{g}{e}\right)f^2.$$

$$\therefore \left(\frac{dx}{dt}\right)^2=-\frac{g}{e}x^2+\frac{g}{e}f^2=\frac{g}{e}(f^2-x^2). \qquad \ldots(3)$$

The equation (3) gives the velocity of the particle at any point from C to A. Let v_1 be the velocity of the particle at A. Then at A, $x=-e$ and $\left(\dfrac{dx}{dt}\right)^2=v_1^2$. Therefore, from (3), we have

$$v_1^2=\frac{g}{e}(f^2-e^2)=\frac{g}{e}4ae \qquad [\because f^2-e^2=4ae]$$

$=4ag$, the direction of v_1 being vertically upwards.

Above A the motion of the particle is freely under gravity. If the particle rises to a height h above A, we have

$0=v_1^2-2gh$, [using the formula $v^2=u^2+2fs$]

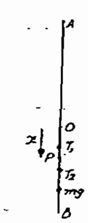
$=4ag-2gh$, $\quad [\because v_1^2=4ag]$.

$\therefore \quad 2gh=4ag$ or $h=2a$.

Hence the total height to which the particle rises above C

$$=CB+BA+h=f+e+2a.$$

Ex. 55. *A heavy particle is attached to one point of a uniform elastic string. The ends of the string are attached to two points in a vertical line. Show that the period of a vertical oscillation in which the string remains taut is $2\pi\sqrt{(mh/2\lambda)}$, where λ is the coefficient of elasticity of the string and h the harmonic mean of the unstretched lengths of the two parts of the string.*



Sol. Let a particle of mass m be attached to a point O of a string whose ends have been fastened to two fixed points A and B in a vertical line. The string is taut and the particle is in equilibrium at O. Let $OA=a$ and $OB=b$. Also let a_1 and b_1 be the natural lengths of the stretched portions OA and OB of the string.

Considering the equilibrium of the particle at O, we have the resultant upward force = the resultant downward force

i.e., the tension in OA = the tension in OB + the weight of the particle

i.e., $$\lambda\frac{(a-a_1)}{a_1}=\lambda\frac{(b-b_1)}{b_1}+mg. \qquad \ldots(1)$$

Now suppose the particle is slightly displaced towards B and then let go. During this slight displacement of the particle both the portions of the string remain taut. Let P be the position of the particle after any time t, where $OP=x$.

When the particle is at P, there are three forces acting upon it :

(i) The tension $T_1=\lambda\frac{a+x-a_1}{a_1}$ in the string AP acting in the direction PA *i.e.*, in the direction of x decreasing.

(ii) The tension $T_2=\lambda\frac{b-x-b_1}{b_1}$ in the string BP acting in the direction PB *i.e.*, in the direction of x increasing.

(iii) The weight mg. of the particle acting vertically downwards *i.e.*, in the direction of x increasing.

Hence by Newton's second law of motion the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2}=-\lambda\frac{a+x-a_1}{a_1}+\lambda\frac{b-x-b_1}{b_1}+mg$$

$$=-\lambda\frac{a-a_1}{a_1}+\lambda\frac{b-b_1}{b_1}+mg-\frac{\lambda x}{a_1}-\frac{\lambda x}{b_1}$$

$$=-\lambda\left(\frac{1}{a_1}+\frac{1}{b_1}\right)x \qquad \text{[by (1)]}$$

$$=-\lambda\left(\frac{a_1+b_1}{a_1b_1}\right)x.$$

$\therefore\ \frac{d^2x}{dt^2}=-\frac{\lambda}{m}\frac{(a_1+b_1)}{a_1b_1}x$, which is the equation of motion of a S.H.M. with centre at the origin O. This equation of motion holds good so long as both the portions of the string remain taut. But the initial displacement given to the particle below O being small, both the portions of the string must remain taut for ever. Hence this equation governs the entire motion of the particle. Thus the entire motion of the particle is simple harmonic about the centre O and the time period of one complete oscillation

$$=2\pi\sqrt{\left\{\frac{m\,a_1b_1}{\lambda\,(a_1+b_1)}\right\}}=\pi\sqrt{\left\{\frac{m\,(2a_1b_1)}{2\lambda\,(a_1+b_1)}\right\}}=2\pi\sqrt{\left(\frac{mh}{2\lambda}\right)},$$

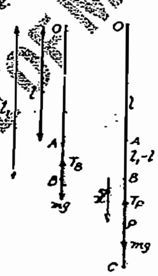
where $h=\frac{2a_1b_1}{a_1+b_1}$ is the harmonic mean between a_1 and b_1.

Ex. 56. *A light elastic string of natural length l has one extremity fixed at a point O and the other attached to a stone, the weight of which in equilibrium would extend the string to a length l_1. Show that if the stone be dropped from rest at O, it will come to instantaneous rest at a depth $\sqrt{(l_1^2-l^2)}$ below the equilibrium position.*

Sol. $OA=l$ is the natural length of a string whose one end is fixed at O. B is the position of equilibrium of a stone of mass m attached to the other end of the string and $OB=l_1$. When the stone rests at B, the tension T_B of the string balances the weight of the stone. Therefore

$$T_B=\frac{\lambda\,(l_1-l)}{l}=mg, \qquad \text{...(1)}$$

where λ is the modulus of elasticity of the string.



Now the stone is dropped from O. It falls the distance OA $(=l)$ freely under gravity. If v_1 be the velocity gained by the stone at A, we have $v_1=\sqrt{(2gl)}$ downwards. When the stone falls below A, the string begins to extend beyond its natural length and the tension begins to operate. During the fall from A to B, the force of tension acting vertically upwards remains less than the weight of the stone acting vertically downwards. Therefore during the fall from A to B the velocity of the stone goes on increasing. When the stone begins to fall below B its velocity goes on decreasing because now the force of tension exceeds the weight of the stone. Let the stone come to instantaneous rest at C, where $BC=a$.

During the motion of the stone below A, let P be its position after any time t, where $BP=x$. [Note that we have taken the position of equilibrium B of the stone as origin. The direction BP is that of x increasing and the direction PB is that of x decreasing].

When the stone is at P, there are two forces acting upon it :

(i) The tension $T_P=\lambda\frac{(l_1+x)-l}{l}$ in the string OP acting in the direction OP *i.e.*, in the direction of x decreasing.

(ii) The weight mg of the stone acting vertically downwards *i.e.*, in the direction of x increasing.

Hence by Newton's second law of motion $(P=mf)$, the equation of motion of the stone at P, is

$$m\frac{d^2x}{dt^2}=mg-\lambda\frac{(l_1+x)-l}{l}=mg-\lambda\frac{(l_1-l)}{l}-\frac{\lambda x}{l}$$

$$=-\frac{\lambda x}{l}, \qquad \text{[from (1)]}.$$

[Note that the force acting in the direction of x increasing has been taken with + ive sign and that in the direction of x decreasing with − ive sign].

Thus $\frac{d^2x}{dt^2}=-\frac{\lambda}{lm}x$, ...(2)

which is the equation of a S.H.M. with centre at the origin B. The equation (2) holds good so long as the string is stretched *i.e.*, for the motion of the stone between A and C.

Multiplying (2) by $2\,(dx/dt)$ and integrating w.r.t. 't', we get

$$\left(\frac{dx}{dt}\right)^2=-\frac{\lambda}{lm}x^2+D, \text{ where } D \text{ is a constant.}$$

At A, $x=-(l_1-l)$ and $dx/dt=\sqrt{(2gl)}$;

$\therefore\ 2gl=-\frac{\lambda}{lm}(l_1-l)^2+D$ or $D=2gl+\frac{\lambda}{lm}(l_1-l)^2$.

Thus, we have $\left(\frac{dx}{dt}\right)^2=-\frac{\lambda}{lm}x^2+2gl+\frac{\lambda}{lm}(l_1-l)^2$. ...(3)

The equation (3) gives velocity of the stone at any point between A and C. At C, $x=a$, $dx/dt=0$. Therefore (3) gives

$$0=-\frac{\lambda}{lm}a^2+2gl+\frac{\lambda}{lm}(l_1-l)^2$$

or $$-\frac{g}{(l_1-l)}a^2+2gl+\frac{g}{(l_1-l)}(l_1-l)^2=0$$

$$\left[\text{from (1), } \frac{\lambda}{lm}=\frac{g}{l_1-l}\right]$$

or $$\frac{a^2}{l_1-l}=2l+l_1-l=l_1+l$$

or $$a^2=(l_1-l)(l_1+l)=l_1^2-l^2.$$

$$\therefore\ a=\sqrt{(l_1^2-l^2)}.$$

Ex. 57. *A light elastic string whose natural length is a has one end fixed to a point O and to the other end is attached a weight which in equilibrium would produce an extension e. Show that if the weight be let fall from rest at O, it will come to stay instantaneously at a point distant $\sqrt{(2ae+e^2)}$ below the position of equilibrium.*

Sol. Proceed as in the preceding example 56. Take $l=a$, $l_1-l=e$ or $l_1=e+a$. Then the required distance $=\sqrt{(l_1^2-l^2)}$ $=\sqrt{[(e+a)^2-a^2]}=\sqrt{(2ae+e^2)}$.

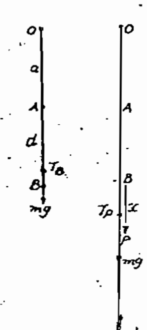
Ex. 58. *A light elastic string of natural length a has one extremity fixed at a point O and the other attached to a body of mass m. The equilibrium length of the string with the body attached is $a\sec\theta$. Show that if the body be dropped from rest at O it will come to instantaneous rest at a depth $a\tan\theta$ below the position of equilibrium.* (IAS-2006)

Sol. Proceed as in Example 56. Take $l=a$ and $l_1=a\sec\theta$. We have then, the required depth below the equilibrium position $=\sqrt{(a^2\sec^2\theta-a^2)}=a\sqrt{(\sec^2\theta-1)}=a\tan\theta$.

Ex. 59. *A heavy particle is attached to one end of an elastic string, the other end of which is fixed. The modulus of elasticity of the string is equal to the weight of the particle. The string is drawn vertically down till it is four times its natural length and then let go. Show that the particle will return to this point in time $\sqrt{\left(\frac{a}{g}\right)}\left[\frac{4\pi}{3}+2\sqrt{3}\right]$, where a is the natural length of the string.*

Sol. Let $OA=a$ be the natural length of an elastic string whose one end is fixed at O. Let B be the position of equilibrium of a particle of mass m attached to the other end of the string and $AB=d$. If T_B is the tension in the string OB, then by Hooke's law,

$$T_B=\lambda\frac{OB-OA}{OA}=\lambda\frac{d}{a},$$



where λ is the modulus of elasticity of the string. Considering the equilibrium of the particle at B, we have

$$mg=T_B=\lambda\frac{d}{a}=mg\frac{d}{a}. \qquad [\because\ \lambda=mg, \text{ as given}]$$

$$\therefore\ d=a.$$

Now the particle is pulled down to a point C such that $OC=4a$ and then let go. It starts moving towards B with velocity zero at C. Let P be the position of the particle at time t, where $BP=x$. [Note that we have taken the position of equilibrium B as origin. The direction BP is that of x increasing and the direction PB is that of x decreasing.]

When the particle is at P, there are two forces acting upon it.

(i) The tension $T_P=\lambda\frac{a+x}{a}=\frac{mg}{a}(a+x)$ in the string OP acting in the direction PO *i.e.*, in the direction of x decreasing.

(ii) The weight mg of the particle acting vertically downwards *i.e.*, in the direction of x increasing.

Hence by Newton's second law of motion ($P=mf$), the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2}=mg-\frac{mg}{a}(a+x)=-\frac{mgx}{a}.$$

Thus $$\frac{d^2x}{dt^2}=-\frac{g}{a}x, \quad \text{...(1)}$$

which is the equation of a S.H.M. with centre at the origin B and the amplitude $BC=2a$ which is greater than $AB=a$.

Multiplying both sides of (1) by $2(dx/dt)$ and integrating w.r.t. t, we have

$$\left(\frac{dx}{dt}\right)^2=-\frac{g}{a}x^2+k, \text{ where } k \text{ is a constant.}$$

At the point C, $x=BC=2a$, and the velocity $dx/dt=0$;

$$\therefore \quad k=\frac{g}{a}4a^2.$$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2=\frac{g}{a}(4a^2-x^2). \quad \text{...(2)}$$

Taking square root of (2), we have

$$\frac{dx}{dt}=-\sqrt{\left(\frac{g}{a}\right)}\sqrt{(4a^2-x^2)},$$

the $-$ive sign has been taken because the particle is moving in the direction of x decreasing.

Separating the variabes, we have

$$dt=-\sqrt{\left(\frac{a}{g}\right)}\frac{dx}{\sqrt{(4a^2-x^2)}} \quad \text{...(3)}$$

If t_1 be the time from C to A, then integrating (3) from C to A, we get

$$\int_0^{t_1}dt=-\sqrt{\left(\frac{a}{g}\right)}\int_{2a}^{-a}\frac{dx}{\sqrt{(4a^2-x^2)}}$$

or $$t_1=\sqrt{\left(\frac{a}{g}\right)}\left[\cos^{-1}\frac{x}{2a}\right]_{2a}^{-a}$$

$$=\sqrt{\left(\frac{a}{g}\right)}\left[\cos^{-1}\left(-\tfrac{1}{2}\right)-\cos^{-1}(1)\right]=\sqrt{\left(\frac{a}{g}\right)}\cdot\frac{2\pi}{3}$$

Let v_1 be the velocity of the particle at A. Then at A $x=-a$ and $(dx/dt)^2=v_1^2$.

So from (2), we have $v_1^2=(g/a)(4a^2-a^2)$

or $v_1=\sqrt{(3ag)}$, the direction of v_1 being vertically upwards.

Thus the velocity at A is $\sqrt{(3ag)}$ and is in the upwards direction so that the particle rises above A. Since the tension of the string vanishes at A, therefore at A the simple harmonic motion ceases and the particle when rising above A moves freely under gravity. Thus the particle rising from A with velocity $\sqrt{(3ag)}$ moves upwards till this velocity is destroyed. The time t_2 for this motion is given by

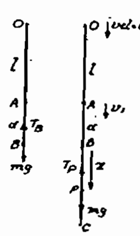
$$0=\sqrt{(3ag)}-gt_2, \text{ so that } t_2=\sqrt{\left(\frac{3a}{g}\right)}.$$

Conditions being the same, the equal time t_2 is taken by the particle in falling freely back to A. From A to C the particle will take the same time t_1 as it takes from C to A. Thus the whole time taken by the particle to return to $C=2(t_1+t_2)$

$$=2\left[\sqrt{\left(\frac{a}{g}\right)}\cdot\frac{2\pi}{3}+\sqrt{\left(\frac{3a}{g}\right)}\right]=\sqrt{\left(\frac{a}{g}\right)}\left[\frac{4\pi}{3}+2\sqrt{3}\right].$$

Ex. 60. *A heavy particle of mass m is attached to one end of an elastic string of natural length l, whose other end is fixed at O. The particle is then let fall from rest at O. Show that, part of the motion is simple harmonic, and that, if the greatest depth of the particle below O is l* $\cot^2(\theta/2)$, *the modulus of elasticity of the string is* $\frac{1}{2}mg\tan^2\theta$.

Sol. Let $OA=l$ be the natural length of an elastic string whose one end is fixed at O. Let B be the position of equilibrium of a particle of mass m attached to the other end of the string and let $AB=d$. In the equilibrium position at B, the tension T_B in the string OB balances the weight mg of the particle. Therefore,



$$T_B=\lambda\frac{d}{l}=mg, \quad \text{...(1)}$$

where λ is the modulus of elasticity of the string. Now the particle is dropped at rest from O. It falls the distance OA freely under gravity. If v_1 be the velocity gained by it at A, we have $v_1=\sqrt{(2gl)}$ in the downward direction. When the particle falls below A, the string begins to extend beyond its natural length and the tension begins to operate. During the fall from A to B the force of tension acting vertically upwards remains less than the weight of the particle acting vertically downwards. Therefore during the fall from A to B the velocity of the particle goes on increasing. When the particle begins to fall below B, its velocity goes on decreasing because now the force of tension exceeds the weight of the particle. Let the particle come to instantaneous rest at C, where $OC=l\cot^2\frac{1}{2}\theta$, as given.

During the motion of the particle below A, let P be its position after any time t, where $BP=x$. [Note that we have taken the position of equilibrium B of the particle as origin. The direction BP is that of x increasing and the direction PB is that of x decreasing.]

When the particle is at P, there are two forces acting upon it.

(i) The tension $T_P=\lambda\dfrac{d+x}{l}$ in the string OP, acting in the direction PO *i.e.*, in the direction of x decreasing.

(ii) The weight mg of the particle acting vertically downwards *i.e.*, in the direction of x increasing.

Hence by Newton's second law of motion, the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2}=mg-\lambda\frac{d+x}{l}$$

$$=mg-\frac{\lambda d}{l}-\frac{\lambda x}{l}=-\frac{\lambda x}{l}, \text{ by (1).}$$

$$\therefore \quad \frac{d^2x}{dt^2}=-\frac{\lambda}{lm}x=-\frac{g}{d}x. \quad \text{...(2)}$$

$$\left[\because \text{ from (1), } \frac{\lambda}{lm}=\frac{g}{d}\right]$$

The equation (2) represents a S. H. M. with centre at the point B and amplitude BC. Hence the motion of the particle below A is simple harmonic.

Multiplying (2) by $2(dx/dt)$ and integrating w.r.t. 't', we get

$$\left(\frac{dx}{dt}\right)^2=-\frac{g}{d}x^2+D, \text{ where } D \text{ is a constant.}$$

At the point A, $x=-d$ and the velocity $=dx/dt=\sqrt{(2gl)}$.

$$\therefore \quad D=2gl+gd.$$

$$\therefore \quad \text{we have, (velocity)}^2=\left(\frac{dx}{dt}\right)^2=-\frac{g}{d}x^2+2gl+gd. \quad \text{...(3)}$$

The above equation (3) gives the velocity of the particle at any point between A and C. At C, $x=BC=OC-OB=l\cot^2\frac{1}{2}\theta-(l+d)$ and $dx/dt=0$. Therefore (3) gives

$$0=-\frac{g}{d}[(l\cot^2\tfrac{1}{2}\theta-l)-d]^2+2gl+gd$$

$$=-\frac{g}{d}[(l\cot^2\tfrac{1}{2}\theta-l)^2+d^2-2ld(\cot^2\tfrac{1}{2}\theta-1)]+2gl+gd$$

$$=-\left[\frac{g}{d}(l\cot^2\tfrac{1}{2}\theta-l)^2-2gl\cot^2\tfrac{1}{2}\theta\right]$$

$$=-\left[\frac{\lambda}{ml}(l\cot^2\tfrac{1}{2}\theta-l)^2-2gl\cot^2\tfrac{1}{2}\theta\right], \left[\because \frac{g}{d}=\frac{\lambda}{ml} \text{ by (1)}\right].$$

$$\lambda=\frac{2mgl^2\cot^2\frac{1}{2}\theta}{(l\cot^2\frac{1}{2}\theta-l)^2}=\frac{2mg\cot^2\frac{1}{2}\theta}{(\cot^2\frac{1}{2}\theta-1)^2}$$

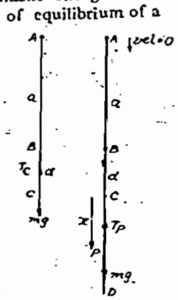
$$=\frac{2mg\cot^2\frac{1}{2}\theta}{(\cos^2\frac{1}{2}\theta-\sin^2\frac{1}{2}\theta)^2}\cdot\sin^4\tfrac{1}{2}\theta$$

$$=\frac{1}{2}\frac{mg.4\cos^2\frac{1}{2}\theta\sin^2\frac{1}{2}\theta}{\cos^2\theta}=\frac{1}{2}mg\cdot\frac{\sin^2\theta}{\cos^2\theta}=\frac{1}{2}mg\tan^2\theta.$$

Ex. 61. *One end of a light elastic string of natural length a and modulus of elasticity 2mg is attached to a fixed point A and the other end to a particle of mass m. The particle initially held at rest at A, is let fall. Show that the greatest extension of the string is* $\frac{1}{2}a(1+\sqrt{5})$ *during the motion and show that the particle will reach back A again after a time* $(\pi+2-\tan^{-1}2)\sqrt{(2a/g)}$. (IAS-2009)

Sol. $AB=a$ is the natural length of an elastic string whose one end is fixed at A. Let C be the position of equilibrium of a particle of mass m attached to the other end of the string and let $BC=d$. In the position of equilibrium at C, the tension $T_C=\lambda\dfrac{d}{a}=2mg\dfrac{d}{a}$ in the string AC balances the weight mg of the particle.

$$\therefore \quad mg=2mg(d/a) \text{ or } d=a/2. \quad \text{...(1)}$$



Now the particle is dropped at rest from A. It falls the distance AB freely under gravity. If v_1 be the velocity gained at B, we have $v_1=\sqrt{(2ga)}$ in the downward direction. When the particle falls below B, the string begins to extend beyond its natural length and the tension begins to operate. During the fall from B to C the velocity of the particle goes on increasing as the tension remains less than the weight of the particle and when the particle begins to fall below C, its velocity goes on decreasing because now the force of tension exceeds the weight of the particle. Let the particle come to instantaneous rest at D.

During the motion of the particle below B, let P be its position after any time t, where $CP=x$. If T_P be the tension in the string AP, we have $T_P=\lambda\dfrac{d+x}{a}=2mg\dfrac{\frac{1}{2}a+x}{a}$, acting vertically upwards.

By Newton's second law of motion, the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2}=mg-T_P=mg-2mg\frac{\frac{1}{2}a+x}{a}=-\frac{2mg}{a}x.$$

$$\therefore\quad \frac{d^2x}{dt^2}=-\frac{2g}{a}x, \qquad \ldots(2)$$

which is the equation of a S. H. M. with centre at the point C and amplitude CD.

Multiplying (2) by $2(dx/dt)$ and integrating w.r.t. 't', we get

$\left(\frac{dx}{dt}\right)^2=-\frac{2g}{a}.x^2+k$, where k is a constant.

At the point B, the velocity

$$=dx/dt=\sqrt{(2ga)} \text{ and } x=-d=-\frac{a}{2}.$$

$$\therefore\quad k=2ga+\frac{2g}{a}.\frac{a^2}{4}=2ga+\frac{2ga}{4}=\frac{5ag}{2}.$$

$$\therefore\quad \text{We have } \left(\frac{dx}{dt}\right)^2=-\frac{2g}{a}x^2+\frac{5ag}{2}. \qquad \ldots(3)$$

The equation (3) gives the velocity of the particle at any point between B and D. At D, $x=CD$ and $dx/dt=0$. So putting $dx/dt=0$ in (3), we have

$$0=-\frac{2g}{a}.x^2+\frac{5ag}{2} \quad\text{or}\quad x^2=\frac{5a^2}{4}$$

or $$x=\frac{a}{2}\sqrt{5}=CD.$$

$\therefore$ the greatest extension of the string

$$=BC+CD=\tfrac{1}{2}a+\tfrac{1}{2}a\sqrt{5}=\tfrac{1}{2}a\,(1+\sqrt{5}).$$

Now from (3), we have $\left(\frac{dx}{dt}\right)^2=\frac{2g}{a}\left[\frac{5}{4}a^2-x^2\right]$.

$\therefore\quad \frac{dx}{dt}=\sqrt{\left(\frac{2g}{a}\right)}\sqrt{\left[\frac{5}{4}a^2-x^2\right]}$, the +ive sign has been taken because the particle is moving in the direction of x increasing.

Separating the variables, we have $dt=\sqrt{\left(\frac{a}{2g}\right)}\frac{dx}{\sqrt{[\frac{5}{4}a^2-x^2]}}$

If t_1 is the time from B to D, then

$$\int_0^{t_1}dt=\sqrt{\left(\frac{a}{2g}\right)}\int_{-a/2}^{(a\sqrt{5})/2}\frac{dx}{\sqrt{[\frac{5}{4}a^2-x^2]}}$$

or $$t_1=\sqrt{\left(\frac{a}{2g}\right)}\left[\sin^{-1}\left\{\frac{x}{(a\sqrt{5})/2}\right\}\right]_{-a/2}^{(a\sqrt{5})/2}$$

$$=\sqrt{\left(\frac{a}{2g}\right)}\left[\sin^{-1}1+\sin^{-1}\frac{1}{\sqrt{5}}\right]=\sqrt{\left(\frac{a}{2g}\right)}\left(\frac{\pi}{2}+\tan^{-1}\tfrac{1}{2}\right)$$

$$=\sqrt{\left(\frac{a}{2g}\right)}\left(\frac{\pi}{2}+\cot^{-1}2\right)=\sqrt{\left(\frac{a}{2g}\right)}\left(\frac{\pi}{2}+\frac{\pi}{2}-\tan^{-1}2\right)$$

$$=\sqrt{\left(\frac{a}{2g}\right)}(\pi-\tan^{-1}2).$$

And if t_2 is the time from A to B, (while falling freely under gravity), then

$$a=0.t_2+\frac{1}{2}gt_2^2 \quad\text{or}\quad t_2=\sqrt{\left(\frac{2a}{g}\right)}.$$

$\therefore$ the total time to return back to $A=2$ (time from A to D)

$$=2(t_2+t_1)=2\left[\sqrt{\left(\frac{a}{2g}\right)}(\pi-\tan^{-1}2)+\sqrt{\left(\frac{2a}{g}\right)}\right]$$

$$=\sqrt{\left(\frac{2a}{g}\right)}[\pi-\tan^{-1}2+2].$$

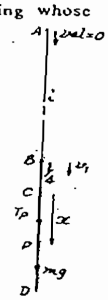
This proves the required result.

Ex. 62. *A light elastic string AB of length l is fixed at A and is such that if a weight w be attached to B, the string will be stretched to a length 2l. If a weight $\frac{1}{4}w$ be attached to B and let fall from the level of A prove that (i) the amplitude of the S.H.M. that ensues is 3l/4; (ii) the distance through which it falls is 2l; and (iii) the period of oscillation is*

$$\sqrt{\left(\frac{l}{4g}\right)}(4\sqrt{2}+\pi+2\sin^{-1}\tfrac{1}{3}).$$

Sol. $AB=l$ is the natural length of an elastic string whose one end is fixed at A. Let λ be the modulus of elasticity of the string. If a weight w be attached to the other end of the string, it extends the string to a length $2l$ while hanging in equilibrium. Therefore

$$w=\lambda\frac{2l-l}{l}=\lambda. \qquad \ldots(1)$$



Now in the actual problem a particle of weight $\frac{1}{4}w$ or mass $\frac{1}{4}(w/g)$ is attached to the free end of the string. Let C be the position of equilibrium of this weight $\frac{1}{4}w$. Then considering the equilibrium of this weight at C, we have

$$\tfrac{1}{4}w=\lambda\frac{BC}{l}=w\frac{BC}{l} \qquad [\because \text{ by (1), } \lambda=w]$$

$$\therefore\quad BC=\tfrac{1}{4}l.$$

Now the weight $\frac{1}{4}w$ is dropped from A. It falls the distance $AB\,(=l)$ freely under gravity. If v_1 be the velocity gained by this weight at B, we have $v_1=\sqrt{(2gl)}$ in the downward direction. When this weight falls below B, the string begins to extend beyond its natural length and the tension begins to operate. The velocity of the weight continues increasing upto C, after which it starts decreasing. Suppose the weight comes to instantaneous rest at D, where $CD=a$.

During the motion of the weight below B, let P be its position after any time t, where $CP=x$. [Note that we have taken C as origin and CP is the direction of x increasing]. If T_P be the tension in the string AP, we have $T_P=w\frac{\frac{1}{4}l+x}{l}$ acting vertically upwards.

The equation of motion of this weight $w/4$ at P is

$$\frac{1}{4}\frac{w}{g}\frac{d^2x}{dt^2}=\frac{1}{4}w-w\frac{\frac{1}{4}l+x}{l}=\frac{1}{4}w-\frac{1}{4}w-w\frac{x}{l}$$

or $$\frac{1}{4}\frac{w}{g}\frac{d^2x}{dt^2}=-w\frac{x}{l} \quad\text{or}\quad \frac{d^2x}{dt^2}=-\frac{4g}{l}x, \qquad \ldots(2)$$

which is the equation of a S.H.M. with centre at the origin C, and amplitude $CD\,(=a)$. The equation (2) holds good so long as the string is stretched *i.e.*, for the motion of the weight from B to D.

Multiplying (2) by $2(dx/dt)$ and integrating w.r.t. 't', we get

$\left(\frac{dx}{dt}\right)^2=-\frac{4g}{l}x^2+k$, where k is a constant.

At B, $x=-\frac{1}{4}l$ and $dx/dt=\sqrt{(2gl)}$;

$$\therefore\quad 2gl=-\frac{4g}{l}.\frac{1}{16}l^2+k \quad\text{or}\quad k=\frac{9}{4}gl.$$

Thus, we have $\left(\frac{dx}{dt}\right)^2=-\frac{4g}{l}x^2+\frac{9}{4}gl=\frac{4g}{l}\left(\frac{9}{16}l^2-x^2\right) \qquad \ldots(3)$

The equation (3) gives velocity at any point between B and D. At D, $x=a$, $dx/dt=0$. Therefore (3) gives

$$0=\frac{4g}{l}\left(\frac{9}{16}l^2-a^2\right) \quad\text{or}\quad a=\tfrac{3}{4}l.$$

Hence the amplitude a of the S.H.M. that ensues is $\frac{3}{4}l$.

Also the total distance through which the weight falls $=AB+BC+CD=l+\frac{1}{4}l+\frac{3}{4}l=2l$.

Now let t_1 be the time taken by the weight to fall freely under gravity from A to B.

Then using the formula $v=u+ft$, we get

$$\sqrt{(2gl)}=0+gt_1 \quad\text{or}\quad t_1=\sqrt{(2l/g)}.$$

Again let t_2 be the time taken by the weight to fall from B to D while moving in S.H.M. From (3), on taking square root, we get

$$\frac{dx}{dt}=+\sqrt{\left(\frac{4g}{l}\right)}\sqrt{\left(\frac{9}{16}l^2-x^2\right)},$$

where the +ive sign has been taken because the weight is moving in the direction of x increasing. Separating the variables, we get

$$\sqrt{\left(\frac{l}{4g}\right)}\frac{dx}{\sqrt{\left(\frac{9l^2}{16}-x^2\right)}}=dt.$$

Integrating from B to D, we get

$$\int_0^{t_2}dt=\sqrt{\left(\frac{l}{4g}\right)}\int_{-l/4}^{3l/4}\frac{dx}{\sqrt{(\frac{9}{16}l^2-x^2)}}.$$

$$\therefore\quad t_2=\sqrt{\left(\frac{l}{4g}\right)}\left[\sin^{-1}\frac{x}{\frac{3}{4}l}\right]_{-l/4}^{3l/4}=\sqrt{\left(\frac{l}{4g}\right)}\left[\sin^{-1}1-\sin^{-1}(-\tfrac{1}{3})\right]$$

$$=\sqrt{\left(\frac{l}{4g}\right)}\left[\tfrac{1}{2}\pi+\sin^{-1}\tfrac{1}{3}\right].$$

Hence the total time taken to fall from A to $D=t_1+t_2$

$$=\sqrt{\left(\frac{2l}{g}\right)}+\sqrt{\left(\frac{l}{4g}\right)}\left[\tfrac{1}{2}\pi+\sin^{-1}\tfrac{1}{3}\right]$$

$$=\sqrt{\left(\frac{l}{4g}\right)}\left[\frac{\pi}{2}+\sin^{-1}\tfrac{1}{3}+2\sqrt{2}\right].$$

Now after instantaneous rest at D, the weight begins to move upwards. From D to B it moves in S.H.M. whose equation is (2). At B the string becomes slack and S.H.M. ceases. The velocity of the weight at B is $\sqrt{(2gl)}$ upwards. Above B the weight rises freely under gravity and comes to instantaneous rest at A. Thus it oscillates again and again between A and D.

The time period of one complete oscillation $=2$ time from A to $D=2(t_1+t_2)=\sqrt{\left(\frac{l}{4g}\right)}\left\{\pi+4\sqrt{2}+2\sin^{-1}\tfrac{1}{3}\right\}$.

Ex. 63. *A heavy particle of mass m is attached to one end of an elastic string of natural length l ft., whose modulus of elasticity is equal to the weight of the particle and the other end is fixed at O. The particle is let fall from O. Show that a part of the motion is simple harmonic and that the greatest depth of the particle below O is $(2+\sqrt{3})\,l$ ft. Show that this depth is attained in time*

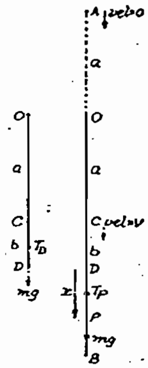
$$[\sqrt{2}+\pi-\cos^{-1}(1/\sqrt{3})]\sqrt{(l/g)} \text{ seconds.}$$

Sol. Proceed as in the preceding example.

Ex. 64. *A particle of mass m is attached to one end of an elastic string of natural length a and modulus of elasticity 2mg, whose other end is fixed at O. The particle is let fall from A, when A is* *vertically above O and $OA=a$. Show that its velocity will be zero at B, where $OB=3a$.*

Calculate also the time from A to B.

Sol. Let $OC=a$, be the natural length of an elastic string suspended from the fixed point O. The modulus of elasticity λ of the string is given to be equal to $2mg$, where m is the mass of the particle attached to the other end of the string.



If D is the position of equilibrium of the particle such that $CD=b$, then at D the tension T_D in the string OD balances the weight of the particle.

$$\therefore \quad mg=T_D=\lambda \cdot \frac{b}{a}=2mg.\frac{b}{a}$$

or $\qquad b=a/2.$

The particle is let fall from A where $OA=a$. Then the motion from A to C will be freely under gravity.

If V is the velocity of the particle gained at the point C, then

$$V^2=0+2g.2a \quad \text{or} \quad V=2\sqrt{(ag)}, \qquad \ldots(1)$$

in the downward direction.

As the particle moves below C, the string begins to extend beyond its natural length and the tension begins to operate. The velocity of the particle continues increasing upto D after which it starts decreasing. Suppose that the particle comes to instantaneous rest at B. During the motion below C, let P be the position of the particle at any time t, where $DP=x$. If T_P is the tension in the string OP, we have

$$T_P=\lambda\ \frac{b+x}{a}, \text{ acting vertically upwards.}$$

$\therefore$ The equation of motion of the particle at P is

$$m\ \frac{d^2x}{dt^2}=mg-T_P=mg-\lambda.\frac{b+x}{a}$$

$$=mg-2mg.\frac{\frac{1}{2}a+x}{a}=-\frac{2mg}{a}\ x$$

or $$\frac{d^2x}{dt^2}=-\frac{2g}{a}\ x, \qquad \ldots(2)$$

which represents a S. H. M. with centre at D and holds good for the motion from C to B.

Multiplying both sides of (2) by $2\ (dx/dt)$ and then integrating, we have

$$\left(\frac{dx}{dt}\right)^2=-\frac{2g}{a}\ x^2+k, \quad \text{where } k \text{ is a constant.}$$

But at C, $x=-DC=-b=-a/2$ and $(dx/dt)^2=V^2=4ag$.

$$4ag=-\frac{2g}{a}\cdot\frac{a^2}{4}+k \qquad \text{or} \qquad k=\tfrac{9}{2}ag.$$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2=-\frac{2g}{a}\ x^2+\frac{9}{2}\ ag$$

or $$\left(\frac{dx}{dt}\right)^2=\frac{2g}{a}\left(\frac{9}{4}\ a^2-x^2\right) \qquad \ldots(3)$$

If the particle comes to instantaneous rest at B where $DB=x_1$, (say), then

at B, $x=x_1$ and $dx/dt=0$. Therefore from (3), we have

$$0=\frac{2g}{a}\left(\frac{9}{4}\ a^2-x_1^2\right), \text{ giving } x_1=\frac{3}{2}\ a.$$

Now $OB=OC+CD+DB=a+\frac{1}{2}a+\frac{3}{2}a=3a$, which proves the first part of the question.

To find the time from A to B.

If t_1 is the time from A to C, then from $s=ut+\frac{1}{2}ft^2$,

$$2a=0+\tfrac{1}{2}gt_1^2. \qquad \therefore \quad t_1=2\sqrt{(a/g)}. \qquad \ldots(4)$$

Now from (3), we have

$$\frac{dx}{dt}=\sqrt{\left(\frac{2g}{a}\right)}\sqrt{\left(\frac{9}{4}\ a^2-x^2\right)}.$$

the +ive sign has been taken because the particle is moving in the direction of x increasing

or $$dt=\sqrt{\left(\frac{a}{2g}\right)}\cdot\frac{dx}{\sqrt{(\frac{9}{4}a^2-x^2)}}$$

Integrating from C to B, the time t_2 from C to B is given by

$$t_2=\sqrt{\left(\frac{a}{2g}\right)}\int_{x=-a/2}^{3a/2}\frac{dx}{\sqrt{(\frac{9}{4}a^2-x^2)}}$$

$$=\sqrt{\left(\frac{a}{2g}\right)}\cdot\left[\sin^{-1}\left(\frac{x}{3a/2}\right)\right]_{-a/2}^{3a/2}$$

$$=\sqrt{\left(\frac{a}{2g}\right)}\cdot\left[\sin^{-1}1-\sin^{-1}\left(-\frac{1}{3}\right)\right]$$

$$=\sqrt{\left(\frac{a}{2g}\right)}\cdot\left[\frac{\pi}{2}+\sin^{-1}\left(\frac{1}{3}\right)\right].$$

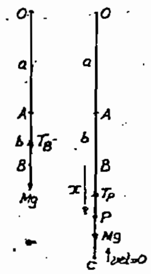
$\therefore$ the time from A to $B=t_1+t_2$

$$=2\sqrt{(a/g)}+\sqrt{(a/2g)}.[\pi/2+\sin^{-1}(1/3)]$$

$$=\tfrac{1}{2}\sqrt{(a/2g)}\ [4\sqrt{2}+\pi+2\sin^{-1}(1/3)].$$

Ex. 65. *Two bodies of masses M and M', are attached to the lower end of an elastic string whose upper end is fixed and hang at rest; M' falls off; show that the distance of M from the upper end of the string at time t is $a+b+c\cos\{\sqrt{(g/b)}\ t\}$, where a is the unstretched length of the string, b and c the distances by which it would be stretched when supporting M and M' respectively.*

Sol. Let $OA=a$ be the natural length of an elastic string suspended from the fixed point O. If B is the position of equilibrium of the particle of mass M attached to the lower end of the string and $AB=b$, then



$$Mg=\lambda\ \frac{AB}{a}=\lambda\cdot\frac{b}{a}. \qquad \ldots(1)$$

Similarly $\quad M'g=\lambda\ \frac{c}{a}. \qquad \ldots(2)$

Adding (1) and (2), we have

$$(M+M')\ g=\lambda\ \frac{b+c}{a}.$$

Thus the string will be stretched by the distance $b+c$ when supporting both the masses M and M' at the lower end. Let OC be the stretched length of the string when both the masses M and M' are attached to its lower end. Then

$$AC=b+c \text{ and so } BC=AC-AB=b+c-b=c.$$

Now when M' falls off at C, the mass M will begin to move towards B starting with velocity zero at C. Let P be the position of the particle of mass M at any time t, where $BP=x$.

If T_P be the tension in the string OP, then

$$T_P=\lambda\ \frac{b+x}{a}, \text{ acting vertically upwards.}$$

$\therefore$ the equation of motion of the particle of mass M at P is

$$M\frac{d^2x}{dt^2}=Mg-T_P=Mg-\lambda\ \frac{b+x}{a}$$

$$=Mg-\lambda\ \frac{b}{a}-\frac{\lambda x}{a}$$

$$=Mg-Mg-\frac{Mg}{b}\ x, \quad \left[\because \text{ from (1), } Mg=\frac{\lambda b}{a}\right]$$

$$=-\frac{Mg}{b}\ x.$$

$$\therefore \quad \frac{d^2x}{dt^2}=-\frac{g}{b}\ x, \qquad \ldots(3)$$

which represents a S. H. M. with centre at B and amplitude BC.

Multiplying both sides of (3) by $2\ (dx/dt)$ and then integrating w.r.t. 't', we have

$$\left(\frac{dx}{dt}\right)^2=-\frac{g}{b}\ x^2+k, \quad \text{where } k \text{ is a constant.}$$

But at the point C, $x=BC=c$ and $dx/dt=0$.

$$\therefore \quad 0=-(g/b)\ c^2+k \quad \text{or} \quad k=(g/b)\ c^2.$$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2=\frac{g}{b}\ (c^2-x^2)$$

or $$\frac{dx}{dt}=-\sqrt{\left(\frac{g}{b}\right)}\sqrt{\left(c^2-x^2\right)},$$

the $-$ive sign has been taken since the particle is moving in the direction of x decreasing.

$$\therefore \quad dt=-\sqrt{\left(\frac{b}{g}\right)}\frac{dx}{\sqrt{(c^2-x^2)}}, \text{ separating the variables.}$$

Integrating, $\quad t=\sqrt{(b/g)}\cos^{-1}(x/c)+D$, where D is a constant.

But at C, $t=0$ and $x=c$; $\quad \therefore \quad D=0.$

$$\therefore \quad t=\sqrt{(b/g)}\cos^{-1}(x/c)$$

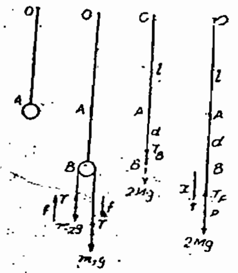
or $$x=BP=c\cos\{\sqrt{(g/b)}\ t\}.$$

$\therefore$ the required distance of the particle of mass M at time t from the point O

$$=OP=OA+AB+BP=a+b+c\cos\{\sqrt{(g/b)}\ t\}.$$

Ex. 66. *A smooth light pulley is suspended from a fixed point by a spring of natural length l and modulus of elasticity mg. If masses m_1 and m_2 hang at the ends of a light inextensible string passing round the pulley, show that the pulley executes simple harmonic motion about a centre whose depth below the point of suspension is $l\{1+(2M/m)\}$, where M is the harmonic mean between m_1 and m_2.*

Sol. Let a smooth light pulley be suspended from a fixed point O by a spring OA of natural length l and modulus of elasticity $\lambda = ng$. Let B be the position of equilibrium of the pulley when masses m_1 and m_2 hang at the ends of a light inextensible string passing round the pulley. Let T be the tension in the inextensible string passing round the pulley. Let us first find the value of T.



Let f be the common acceleration of the particles m_1, m_2 which hang at the ends of a light inextensible string passing round the pulley. If $m_1 > m_2$, then the equations of motion of m_1, m_2 are

$$m_1 g - T = m_1 f \quad \text{and} \quad T - m_2 g = m_2 f.$$

Solving, we get $T = \dfrac{2m_1 m_2}{(m_1+m_2)} g = Mg$,

where $M = \dfrac{2m_1 m_2}{m_1+m_2}$ = the harmonic mean between m_1 and m_2.

Now the pressure on the pulley $= 2T = 2Mg$ and therefore the pulley, which itself is light, behaves like a particle of mass $2M$.

Now the problem reduces to the vertical motion of a mass $2M$ attached to the end A of the string OA whose other end is fixed at O. If B is the equilibrium position of the mass $2M$ and $AB = d$, then the tension T_B in the spring OB is $\lambda(d/l)$, acting vertically upwards.

For equilibrium of the pulley of mass $2M$ at the point B,

we have $$2Mg = T_B = \lambda \frac{d}{l} = ng\frac{d}{l}$$

or $$d = \frac{2Ml}{n}. \quad \ldots(1)$$

Now let the particle of mass $2M$ be slightly pulled down and then let go. If P is the position of this particle at time t such that $BP = x$, then the tension in the spring OP

$$= T_P = \lambda\frac{d+x}{l} = ng\cdot\frac{d+x}{l}, \text{ acting vertically upwards.}$$

$\therefore$ The equation of motion of the pulley is given by

$$2M\cdot\frac{d^2x}{dt^2} = 2Mg - T_P$$

$$= 2Mg - ng\frac{d+x}{l} = 2Mg - ng\frac{d}{l} - \frac{ng}{l}x = -\frac{ng}{l}x, \quad [\text{by (1)}]$$

$$\therefore \quad \frac{d^2x}{dt^2} = -\frac{ng}{2Ml}x,$$

which represents a simple harmonic motion about the centre B.

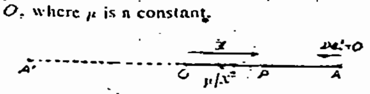
Hence the pulley executes simple harmonic motion with centre at the point B whose depth below the point of suspension O is given by

$$OB = OA + AB = l + d$$
$$= l + \frac{2Ml}{n} = l\left(1 + \frac{2M}{n}\right).$$

11. Motion under inverse square law.

A particle moves in a straight line under an attraction towards a fixed point on the line, which varies inversely as the square of the distance from the fixed point. If the particle was initially at rest, to investigate the motion.

Let a particle start from rest from a point A such that $OA = a$, where O is the fixed point (*i.e.*, the centre of force) on the line and is taken as origin. Let P be the position of the particle at any time t, such that $OP = x$. Then the acceleration at $P = \mu/x^2$, towards O, where μ is a constant.



$\therefore$ the equation of motion of the particle at P is

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x^2}. \quad \ldots(1)$$

[$-$ive sign has been taken because d^2x/dt^2 is positive in the direction of x increasing while here μ/x^2 acts in the direction of x decreasing].

Multiplying both sides of (1) by $2(dx/dt)$ and then integrating w.r.t. 't', we have $\left(\dfrac{dx}{dt}\right)^2 = \dfrac{2\mu}{x} + A$, where A is constant of integration.

But at A, $x = OA = a$ and $dx/dt = 0$.

$$\therefore \quad 0 = \frac{2\mu}{a} + A \quad \text{or} \quad A = -\frac{2\mu}{a}.$$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2 = 2\mu\left(\frac{1}{x} - \frac{1}{a}\right). \quad \ldots(2)$$

which gives the velocity of the particle at any distance x from the centre of force O.

From (2), we have on taking square root

$$\frac{dx}{dt} = -\sqrt{\left(\frac{2\mu}{a}\right)}\cdot\sqrt{\left(\frac{a-x}{x}\right)}.$$

[Here $-$ive sign is taken since the particle is moving in the direction of x decreasing].

Separating the variables, we get

$$dt = -\sqrt{\left(\frac{a}{2\mu}\right)}\cdot\sqrt{\left(\frac{x}{a-x}\right)}\,dx.$$

Integrating, $t = -\sqrt{\left(\dfrac{a}{2\mu}\right)}\displaystyle\int\sqrt{\left(\frac{x}{a-x}\right)}\,dx + B$, where B is constant of integration.

Putting $x = a\cos^2\theta$, so that $dx = -2a\cos\theta\sin\theta\,d\theta$, we have

$$t = \sqrt{\left(\frac{a}{2\mu}\right)}\int\sqrt{\left(\frac{a\cos^2\theta}{a - a\cos^2\theta}\right)}\cdot 2a\sin\theta\cos\theta\,d\theta + B$$

$$= a\sqrt{\left(\frac{a}{2\mu}\right)}\cdot\int 2\cos^2\theta\,d\theta + B = a\sqrt{\left(\frac{a}{2\mu}\right)}\cdot\int(1+\cos 2\theta)\,d\theta + B$$

$$= a\sqrt{\left(\frac{a}{2\mu}\right)}\cdot\left(\theta + \frac{\sin 2\theta}{2}\right) + B = a\sqrt{\left(\frac{a}{2\mu}\right)}(\theta + \sin\theta\cos\theta) + B$$

$$= a\sqrt{\left(\frac{a}{2\mu}\right)}[\theta + \sqrt{(1-\cos^2\theta)}\cos\theta] + B.$$

But $x = a\cos^2\theta$ means $\cos\theta = \sqrt{(x/a)}$ and $\theta = \cos^{-1}\sqrt{(x/a)}$.

$$\therefore \quad t = a\sqrt{\left(\frac{a}{2\mu}\right)}\cdot\left[\cos^{-1}\sqrt{\left(\frac{x}{a}\right)} + \sqrt{\left(1-\frac{x}{a}\right)}\cdot\sqrt{\left(\frac{x}{a}\right)}\right] + B.$$

But initially at A, $t = 0$ and $x = OA = a$.

$$\therefore \quad 0 = a\sqrt{\left(\frac{a}{2\mu}\right)}[0+0] + B \quad \text{or} \quad B = 0.$$

$$\therefore \quad t = a\sqrt{\left(\frac{a}{2\mu}\right)}\left[\cos^{-1}\sqrt{\left(\frac{x}{a}\right)} + \sqrt{\left(1-\frac{x}{a}\right)}\cdot\sqrt{\left(\frac{x}{a}\right)}\right]. \quad \ldots(3)$$

which gives the time from the initial position A to any point distant x from the centre of force.

Putting $x = 0$ in (3), the time t_1 taken by the particle from A to O is given by

$$t_1 = a\sqrt{\left(\frac{a}{2\mu}\right)}\cdot\left[\frac{\pi}{2} + 0\right] = \frac{\pi}{2}\sqrt{\left(\frac{a^3}{2\mu}\right)}. \quad \ldots(4)$$

Putting $x = 0$ in (2), we see that the velocity at O is infinite and therefore the particle moves to the left of O. But the acceleration on the particle is towards O, so the particle moves to the left of O under retardation which is inversely proportional to the square of the distance from O. The particle will come to instantaneous rest at A', where $OA' = OA = a$, and then retrace its path. Thus, the particle will oscillate between A and A'.

Time of one complete oscillation $= 4 \times$ (time from A to O)
$$= 4t_1 = 2\pi\sqrt{(a^3/2\mu)}.$$

12. Motion of a particle under the attraction of the earth.

Newton's law of gravitation. When a particle moves under the attraction of the earth, the acceleration acting on it towards the centre of the earth will be as follows :

1. When the particle moves (upwards or downwards) outside the surface of the earth, the acceleration varies inversely as the square of the distance of the particle from the centre of the earth.
2. When the particle moves inside the earth through a hole made in the earth, the acceleration varies directly as the distance of the particle from the centre of the earth.
3. The value of the acceleration at the surface of the earth is g.

Illustrative Examples :

Ex. 67. *Show that the time occupied by a body, under the acceleration K/x^2 towards the origin, to fall from rest at distance a to distance x from the attracting centre can be put in the form*

$$\sqrt{\left(\frac{a^3}{2K}\right)}\left[\cos^{-1}\sqrt{\left(\frac{x}{a}\right)} + \sqrt{\left\{\frac{x}{a}\left(1-\frac{x}{a}\right)\right\}}\right].$$

Prove also that the time occupied from $x = 3a/4$ to $a/4$ is one-third of the whole time of descent from a to 0.

Sol. For the first part see equation (3) of § 11. (Deduce this equation here).

Thus the time t measured from the initial position $x = a$ to any point at a distance x from the centre O is given by

$$t = \sqrt{\left(\frac{a^3}{2K}\right)}\left[\cos^{-1}\sqrt{\left(\frac{x}{a}\right)} + \sqrt{\left\{\frac{x}{a}\left(1-\frac{x}{a}\right)\right\}}\right]. \quad \ldots(1)$$

[Note that here $\mu = K$].

Let t_1 be the whole time of descent from $x=a$ to $x=0$. Then at O, $x=0$, $t=t_1$. Putting these values in the relation (1) connecting x and t, we have

$$t_1=\sqrt{\left(\frac{a^3}{2K}\right)}\left[\cos^{-1} 0+0\right]=\frac{\pi}{2}\sqrt{\left(\frac{a^3}{2K}\right)}. \quad \ldots(2)$$

Now let t_2 be the time from $x=a$ to $x=3a/4$. Then putting $x=3a/4$ and $t=t_2$ in (1), we get

$$t_2=\sqrt{\left(\frac{a^3}{2K}\right)}\left[\cos^{-1}\left(\frac{\sqrt{3}}{2}\right)+\sqrt{\left(\frac{3}{4}\cdot\frac{1}{4}\right)}\right]=\sqrt{\left(\frac{a^3}{2K}\right)}\left[\frac{\pi}{6}+\frac{\sqrt{3}}{4}\right].$$

Again let t_3 be the time from $x=a$ to $x=a/4$. Then putting $x=a/4$ and $t=t_3$ in (1), we get

$$t_3=\sqrt{\left(\frac{a^3}{2K}\right)}\left[\cos^{-1}\tfrac{1}{2}+\sqrt{\left(\frac{1}{4}\cdot\frac{3}{4}\right)}\right]=\sqrt{\left(\frac{a^3}{2K}\right)}\left[\frac{\pi}{3}+\frac{\sqrt{3}}{4}\right].$$

Therefore if t_4 be the time from $x=3a/4$ to $x=a/4$, we have

$$t_4=t_3-t_2=\sqrt{\left(\frac{a^3}{2K}\right)}\left[\frac{\pi}{3}-\frac{\pi}{6}\right]=\frac{\pi}{6}\sqrt{\left(\frac{a^3}{2K}\right)}$$

$$=\tfrac{1}{3}\left[\frac{\pi}{2}\sqrt{\left(\frac{a^3}{2K}\right)}\right]=\tfrac{1}{3}t_1, \text{ from (2).}$$

Hence the time from $x=3a/4$ to $x=a/4$ is one-third of the whole time of descent from $x=a$ to $x=0$.

Note. To find the time from $x=3a/4$ to $x=a/4$, we have first found the times from $x=a$ to $x=3a/4$ and from $x=a$ to $x=a/4$ because in the relation (1) connecting x and t the time t has been measured from the point $x=a$.

Ex. 68. *Show that the time of descent to the centre of force, varying inversely as the square of the distance from the centre, through first half of its initial distance is to that through the last half as $(\pi+2):(\pi-2)$.*

Sol. Let the particle start from rest from the point A at a distance a from the centre of force O. If x is the distance of the particle from the centre of force at time t, then the equation of motion of the particle at time t is

$$\frac{d^2x}{dt^2}=-\frac{\mu}{x^2}.$$

Now proceeding as in § 11, page 126, we find that the time t measured from the initial position $x=a$ to any point distant x from the centre O is given by the equation

$$t=\sqrt{\left(\frac{a^3}{2\mu}\right)}\left[\cos^{-1}\sqrt{\left(\frac{x}{a}\right)}+\sqrt{\left\{\frac{x}{a}\left(1-\frac{x}{a}\right)\right\}}\right] \quad \ldots(1)$$

[Give the complete proof for deducing this equation here].

Now let B be the middle point of OA. Then at B, $x=a/2$. Let t_1 be the time from A to B i.e., the time to cover the first half of the initial displacement. Then at B, $x=a/2$ and $t=t_1$. So putting $x=a/2$ and $t=t_1$ in (1), we get

$$t_1=\sqrt{\left(\frac{a^3}{2\mu}\right)}\left[\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)+\tfrac{1}{2}\right]=\sqrt{\left(\frac{a^3}{2\mu}\right)}\left[\frac{\pi}{4}+\frac{1}{2}\right].$$

Again let t_2 be the time from A to O. Then at O, $x=0$ and $t=t_2$. So putting $x=0$ and $t=t_2$ in (1), we get

$$t_2=\sqrt{\left(\frac{a^3}{2\mu}\right)}\left[\cos^{-1} 0+0\right]=\sqrt{\left(\frac{a^3}{2\mu}\right)}\cdot\frac{\pi}{2}.$$

Now if t_3 be the time from B to O (i.e., the time to cover the last half of the initial displacement), then

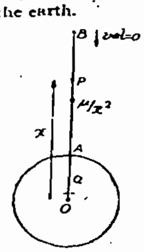
$$t_3=t_2-t_1=\sqrt{\left(\frac{a^3}{2\mu}\right)}\cdot\left[\frac{\pi}{4}-\frac{1}{2}\right].$$

We have $\dfrac{t_1}{t_3}=\dfrac{\frac{1}{4}\pi+\frac{1}{2}}{\frac{1}{4}\pi-\frac{1}{2}}=\dfrac{\pi+2}{\pi-2}$, which proves the required result.

Ex. 69. *If the earth's attraction vary inversely as the square of the distance from its centre and g be its magnitude at the surface, the time of falling from a height h above the surface to the surface is $\sqrt{\left(\frac{a+h}{2g}\right)}\left[\sqrt{\left(\frac{h}{a}\right)}+\frac{a+h}{a}\sin^{-1}\sqrt{\left(\frac{h}{a+h}\right)}\right]$, where a is the radius of the earth.*

Sol. Let O be the centre of the earth taken as origin. Let OB be the vertical line through O which meets the surface of the earth at A and let $AB=h$; $OA=a$ is the radius of the earth.

A particle falls from rest from B towards the surface of the earth. Let P be the position of the particle at any time t, where $OP=x$. [Note that O is the origin and OP is the direction of x increasing]. According to the Newton's law of gravitation the acceleration of the particle at P is μ/x^2 directed towards O i.e., in the direction of x decreasing. Hence the equation of motion of the particle at P is



$$\frac{d^2x}{dt^2}=-\frac{\mu}{x^2} \quad \ldots(1)$$

The equation (1) holds good for the motion of the particle from B to A. At A (i.e., on the surface of the earth $x=a$ and $d^2x/dt^2=-g$. Therefore $-g=-\mu/a^2$ or $\mu=a^2g$. Thus the equation (1) becomes

$$\frac{d^2x}{dt^2}=-\frac{a^2g}{x^2}.$$

Integrating, we get

$$\left(\frac{dx}{dt}\right)^2=\frac{2a^2g}{x}+C. \text{ At } B,\ x=OB=a+h,\ \frac{dx}{dt}=0.$$

$$\therefore\quad 0=\frac{2a^2g}{a+h}+C \quad\text{or}\quad C=-\frac{2a^2g}{a+h}.$$

Thus, we have

$$\left(\frac{dx}{dt}\right)^2=\frac{2a^2g}{x}-\frac{2a^2g}{a+h}=2a^2g\left(\frac{1}{x}-\frac{1}{a+h}\right).$$

For the sake of convenience let us put $a+h=b$. Then

$$\left(\frac{dx}{dt}\right)^2=2a^2g\left(\frac{1}{x}-\frac{1}{b}\right)=\frac{2a^2g}{b}\left(\frac{b-x}{x}\right). \quad \ldots(2)$$

The equation (2) gives velocity at any point from B to A. From (2) on taking square root, we get

$$\frac{dx}{dt}=-a\sqrt{\left(\frac{2g}{b}\right)}\sqrt{\left(\frac{b-x}{x}\right)},$$

where the negative sign has been taken because the particle is moving in the direction of x decreasing.

$$\therefore\quad dt=-\frac{1}{a}\sqrt{\left(\frac{b}{2g}\right)}\sqrt{\left(\frac{x}{b-x}\right)}\,dx. \quad \ldots(3)$$

Let t_1 be the time from B to A. Then integrating (3) from B to A, we get

$$\int_0^{t_1} dt=-\frac{1}{a}\sqrt{\left(\frac{b}{2g}\right)}\int_b^a\sqrt{\left(\frac{x}{b-x}\right)}\,dx.$$

$$\therefore\quad t_1=-\frac{1}{a}\sqrt{\left(\frac{b}{2g}\right)}\int_b^a\sqrt{\left(\frac{x}{b-x}\right)}\,dx.$$

Put $x=b\cos^2\theta$; so that $dx=-2b\cos\theta\sin\theta\,d\theta$.

$$\therefore\quad t_1=\frac{1}{a}\sqrt{\left(\frac{b}{2g}\right)}\int_0^{\cos^{-1}\sqrt{(a/b)}}\frac{\cos\theta}{\sin\theta}\,2b\cos\theta\sin\theta\,d\theta$$

$$=\sqrt{\left(\frac{b}{2g}\right)}\frac{b}{a}\int_0^{\cos^{-1}\sqrt{(a/b)}}2\cos^2\theta\,d\theta$$

$$=\sqrt{\left(\frac{b}{2g}\right)}\frac{b}{a}\int_0^{\cos^{-1}\sqrt{(a/b)}}(1+\cos 2\theta)\,d\theta$$

$$=\sqrt{\left(\frac{b}{2g}\right)}\frac{b}{a}\left[\theta+\tfrac{1}{2}\sin 2\theta\right]_0^{\cos^{-1}\sqrt{(a/b)}}$$

$$=\sqrt{\left(\frac{b}{2g}\right)}\frac{b}{a}\left[\theta+\sin\theta\cos\theta\right]_0^{\cos^{-1}\sqrt{(a/b)}}$$

$$=\sqrt{\left(\frac{b}{2g}\right)}\frac{b}{a}\left[\theta+\cos\theta\sqrt{(1-\cos^2\theta)}\right]_0^{\cos^{-1}\sqrt{(a/b)}}$$

$$=\sqrt{\left(\frac{b}{2g}\right)}\frac{b}{a}\left[\cos^{-1}\sqrt{\left(\frac{a}{b}\right)}+\sqrt{\left(\frac{a}{b}\right)}\sqrt{\left(1-\frac{a}{b}\right)}\right]$$

$$=\sqrt{\left(\frac{b}{2g}\right)}\left[\frac{b}{a}\cos^{-1}\sqrt{\left(\frac{a}{b}\right)}+\sqrt{\left(\frac{b}{a}\right)}\sqrt{\left(1-\frac{a}{b}\right)}\right]$$

$$=\sqrt{\left(\frac{a+h}{2g}\right)}\left[\frac{a+h}{a}\cos^{-1}\sqrt{\left(\frac{a}{a+h}\right)}+\sqrt{\left(\frac{a+h}{a}\right)}\sqrt{\left(1-\frac{a}{a+h}\right)}\right]$$

[replacing b by $a+h$]

$$=\sqrt{\left(\frac{a+h}{2g}\right)}\left[\frac{a+h}{a}\sin^{-1}\sqrt{\left(1-\frac{a}{a+h}\right)}+\sqrt{\left(\frac{a+h}{a}\right)}\sqrt{\left(\frac{h}{a+h}\right)}\right]$$

$$=\sqrt{\left(\frac{a+h}{2g}\right)}\left[\frac{a+h}{a}\sin^{-1}\sqrt{\left(\frac{h}{a+h}\right)}+\sqrt{\left(\frac{h}{a}\right)}\right].$$

Ex. 70. *A particle falls towards the earth form infinity; show that its velocity on reaching the surface of the earth is the same as that which it would have acquired in falling with constant acceleration g through a distance equal to the earth's radius.*

Sol. Let a be the radius of the earth and O be the centre of the earth taken as origin. Let the vertical line through O meet the earth's surface at A. [Draw figure as in Ex. 69].

A particle falls from rest from infinity towards the earth. Let P be the position of the particle at any time t, where $OP=x$. [Note that O is the origin and OP is the direction of x increasing.] According to Newton's law of gravitation the acceleration of the particle at P is μ/x^2 towards O i.e., in the direction of x decreasing. Hence the equation of motion of the particle at P is

$$\frac{d^2x}{dt^2}=-\frac{\mu}{x^2}. \quad \ldots(1)$$

The equation (1) holds good for the motion of the particle upto A. At A (i.e., on the surface of the earth),

$$x=a \text{ and } \frac{d^2x}{dt^2}=-g$$

$\therefore\ -g=-\mu/a^2$ or $\mu=a^2g$. Thus the equation (1) becomes

$$\frac{d^2x}{dt^2}=-\frac{a^2g}{x^2}.$$

Multiplying both sides by $2\,(dx/dt)$ and integrating w.r.t. 't', we get

$$\left(\frac{dx}{dt}\right)^2=\frac{2a^2g}{x}+C.$$

But initially when $x=\infty$, the velocity $dx/dt=0$. Therefore $C=0$.

$$\therefore \left(\frac{dx}{dt}\right)^2=\frac{2a^2g}{x} \quad \ldots(2)$$

Putting $x=a$ in (2), the velocity V at the earth's surface is given by

$$V^2=2a^2g/a=2ag \quad \text{or} \quad V=\sqrt{(2ag)}. \quad \ldots(3)$$

If v_1 is the velocity acquired by the particle in falling a distance equal to the earth's radius a with constant acceleration g, then

$$v_1^2=0+2ag \quad \text{or} \quad v_1=\sqrt{(2ag)}. \quad \ldots(4)$$

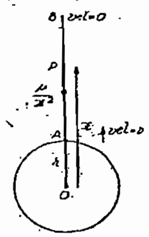
From (3) and (4), we have $V=v_1$, which proves the required result.

Ex. 71. *If h be the height due to the velocity v at the earth's surface supposing its attraction constant and H the corresponding height when the variation of gravity is taken into account, prove that $\frac{1}{h}-\frac{1}{H}=\frac{1}{r}$, where r is the radius of the earth.*

Sol. If h is the height of the particle due to the velocity v at the earth's surface, supposing its attraction constant (*i.e.*, taking the acceleration due to gravity as constant and equal to g), then from the formula $v^2=u^2+2fs$, we have

$$0^2=v^2-2gh.$$

$$\therefore v^2=2gh. \quad \ldots(1)$$



When the variation of gravity is taken into account, let P be the position of the particle at any time t measured from the instant the particle is projected vertically upwards from the earth's surface with velocity v, and let $OP=x$.

The acceleration of the particle at P is μ/x^2 directed towards O.

$\therefore$ the equation of motion of the particle at P is

$$\frac{d^2x}{dt^2}=-\frac{\mu}{x^2}. \quad \ldots(2)$$

[Here the $-$ive sign is taken since the acceleration acts in the direction of x decreasing.]

But at A *i.e.*, on the surface of the earth,

$$x=OA=r \text{ and } \frac{d^2x}{dt^2}=-g.$$

$\therefore$ from (2), we have $-g=-\mu/r^2$ or $\mu=gr^2$.

Substituting in (2), we have

$$\frac{d^2x}{dt^2}=-\frac{gr^2}{x^2}. \quad \ldots(3)$$

Multiplying both sides of (3) by $2(dx/dt)$ and then integrating w.r.t. 't', we have $\left(\frac{dx}{dt}\right)^2=\frac{2gr^2}{x}+A$, where A is a constant of integration.

But at the point A, $x=OA=r$ and $dx/dt=v$, which is the velocity of projection at A.

$$\therefore v^2=\frac{2gr^2}{r}+A \quad \text{or} \quad A=v^2-2gr.$$

$$\therefore \left(\frac{dx}{dt}\right)^2=\frac{2gr^2}{x}+v^2-2gr. \quad \ldots(4)$$

Suppose the particle in this case rises upto the point B, where $AB=H$. Then at the point B, $x=OB=OA+AB=r+H$ and $dx/dt=0$.

$\therefore$ from (4), we have $0=\frac{2gr^2}{r+H}+v^2-2gr$

or $$v^2=2gr-\frac{2gr^2}{r+H}=\frac{2grH}{r+H} \quad \ldots(5)$$

Equating the values of v^2 from (1) and (5), we have

$$2gh=\frac{2grH}{r+H} \quad \text{or} \quad \frac{1}{h}=\frac{r+H}{rH}$$

or $$\frac{1}{h}=\frac{1}{H}+\frac{1}{r} \quad \text{or} \quad \frac{1}{h}-\frac{1}{H}=\frac{1}{r}.$$

Ex. 72. *A particle is shot upwards from the earth's surface with a velocity of one mile per second. Considering variations in gravity, find roughly in miles the greatest height attained.*

Sol. [Refer fig. of Ex. 71].

Let r be the radius of the earth. Suppose the particle is projected vertically upwards from the surface of the earth with velocity u and it rises to a height H above the surface of the earth. Let P be the position of the particle at any time t and x the distance of P from the centre of the earth. Since P is outside the surface of the earth, therefore the equation of motion of P is

$$\frac{d^2x}{dt^2}=-\frac{\mu}{x^2}.$$

But on the surface of the earth, $x=r$ and $d^2x/dt^2=-g$. Therefore $-g=-(\mu/r^2)$ or $\mu=gr^2$.

$\therefore$ the equation of motion of P becomes

$$\frac{d^2x}{dt^2}=-\frac{gr^2}{x^2}. \quad \ldots(1)$$

Multiplying both sides of (1) by $2(dx/dt)$ and integrating w.r.t. 't', we get $\left(\frac{dx}{dt}\right)^2=\frac{2gr^2}{x}+C$, where C is constant of integration.

When $x=r$, $dx/dt=u$. Therefore $u^2=2gr+C$ or $C=u^2-2gr$.

$$\therefore \left(\frac{dx}{dt}\right)^2=\frac{2gr^2}{x}+u^2-2gr. \quad \ldots(2)$$

Since the particle rises to a height H above the surface of the earth, therefore $dx/dt=0$ when $x=r+H$.

Putting these values in (2), we get

$$0=\frac{2gr^2}{r+H}+u^2-2gr$$

or $$0=2gr^2+u^2(r+H)-2gr(r+H)$$

or $$u^2r+u^2H-2grH=0$$

or $$H(2gr-u^2)=u^2r.$$

$$H=\frac{u^2r}{(2gr-u^2)}$$

But according to the question, $u=1$ mile/second. Also $r=$ the radius of the earth $=4000$ miles, and

$$g=32 \text{ ft./second}^2=\frac{32}{3\times 1760} \text{ miles/sec}^2.$$

Hence, $$H=\frac{4000}{\frac{2\times 32\times 4000}{3\times 1760}-1} \text{ miles}=\frac{1}{\frac{2}{165}-\frac{1}{4000}} \text{ miles}$$

$$=\frac{165}{2}\left[1-\frac{165}{8000}\right]^{-1} \text{ miles}=\frac{165}{2}\left[1+\frac{165}{8000}\right] \text{ miles approximately,}$$

[expanding by binomial theorem and neglecting higher powers]

$$=\left[\frac{165}{2}+\frac{(165)^2}{16000}\right] \text{ miles}=82\cdot5 \text{ miles}+1\cdot5 \text{ miles nearly}$$

$=84$ miles approximately.

Remark. If the particle is projected from the surface of the earth with a velocity 1 kilometre per second, then for the calculation work we shall take $r=6380$ km. and $g=9\cdot8$ metre/sec^2 $=10^{-3}\times 9\cdot8$ km./sec^2. The answer in this case is $51\cdot43$ km approximately.

Ex. 73. *A particle is projected vertically upwards from the surface of earth with a velocity just sufficient to carry it to the infinity. Prove that the time it takes to reach a height h is*

$$\frac{1}{3}\sqrt{\left(\frac{2a}{g}\right)}\left[\left(1+\frac{h}{a}\right)^{3/2}-1\right]. \quad \text{(IFoS-2012)}$$

where a is the radius of the earth.

Sol. [Refer fig. of Ex. 71]

Let O be the centre of the earth and A the point of projection on the earth's surface.

If P is the position of the particle at any time t, such that $OP=x$, then the acceleration at $P=\mu/x^2$ directed towards O.

$\therefore$ the equation of motion of the particle at P is

$$\frac{d^2x}{dt^2}=-\frac{\mu}{x^2} \quad \ldots(1)$$

But at the point A, on the surface of the earth, $x=a$ and $d^2x/dt^2=-g$.

$$\therefore -g=-\mu/a^2 \quad \text{or} \quad \mu=a^2g.$$

$$\therefore \frac{d^2x}{dt^2}=-\frac{a^2g}{x^2}.$$

Multiplying by $2(dx/dt)$ and intergating w.r.t. 't', we get

$$\left(\frac{dx}{dt}\right)^2=\frac{2a^2g}{x}+C, \text{ where } C \text{ is a constant.}$$

But when $x\to\infty$, $dx/dt\to 0$. $\therefore$ $C=0$.

$$\therefore \left(\frac{dx}{dt}\right)^2=\frac{2a^2g}{x} \quad \text{or} \quad \frac{dx}{dt}=\frac{a\sqrt{(2g)}}{\sqrt{x}} \quad \ldots(2)$$

[Here $+$ive sign is taken because the particle is moving in the direction of x increasing.]

Separating the variables, we have

$$dt=\frac{1}{a\sqrt{(2g)}}\sqrt{(x)}\,dx.$$

Integrating between the limits $x=a$ to $x=a+h$, the required time t to reach a height h is given by

$$t=\frac{1}{a\sqrt{(2g)}}\int_a^{a+h}\sqrt{(x)}\,dx=\frac{1}{a\sqrt{(2g)}}\left[\frac{2}{3}x^{3/2}\right]_a^{a+h}$$

$$=\frac{1}{3a}\sqrt{\left(\frac{2}{g}\right)}\left[(a+h)^{3/2}-a^{3/2}\right]=\frac{1}{3}\sqrt{\left(\frac{2a}{g}\right)}\left[\left(1+\frac{h}{a}\right)^{3/2}-1\right].$$

Ex. 74. *Calculate in miles per second the least velocity which will carry the particle from earth's surface to infinity.*

Sol. The least velocity of projection from the earth's surface to carry the particle to infinity is that for which the velocity of the particle tends to zero as the distance of the particle from the earth's surface tends to infinity. Now proceed as in Ex. 73.

The velocity at a distance x from the centre of the earth is given by $\left(\frac{dx}{dt}\right)^2=\frac{2a^2g}{x}$.

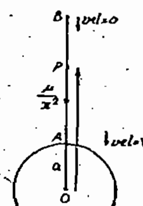
$\therefore$ putting $x=a$, the least velocity V at the earth's surface which will carry the particle to infinity is given by $V=\sqrt{(2ag)}$.

But $a=4000$ miles $=4000\times3\times1760$ ft. and $g=32$ ft/sec².

$$\therefore \quad V=\sqrt{[2\times4000\times3\times1760\times32]} \text{ ft./sec.}$$
$$=8\times200\times4\sqrt{(33)} \text{ ft./sec.}$$
$$=\frac{8\times200\times4\times\sqrt{33}}{3\times1760} \text{ miles/sec.}$$
$$=7 \text{ miles/sec. approximately.}$$

Ex. 75. *Assuming that a particle falling freely under gravity can penetrate the earth without meeting any resistance, show that a particle falling from rest at a distance b $(b>a)$ from the centre of the earth would on reaching the centre acquire a velocity $\sqrt{[ga(3b-2a)/b]}$ and the time to travel from the surface to the centre of the earth is $\sqrt{\left(\frac{a}{g}\right)}\sin^{-1}\sqrt{\left[\frac{b}{(3b-2a)}\right]}$, where a is the radius of the earth and g is the acceleration due to gravity on the earth's surface.*

Sol. Let the particle fall from rest from the point B such that $OB=b$, where O is the centre of the earth. Let P be the position of the particle at any time t measured from the instant it starts falling from B and let $OP=x$.



Acceleration at $P=\mu/x^2$ towards O. The equation of motion of P is

$$\frac{d^2x}{dt^2}=-\frac{\mu}{x^2},$$

which holds good for the motion from B to A *i.e.*, outside the surface of the earth.

But at the point A (on one earth's surface) $x=a$ and $d^2x/dt^2=-g$.

$$\therefore \quad -g=-\mu/a^2 \quad \text{or} \quad \mu=a^2g.$$

$$\therefore \quad \frac{d^2x}{dt^2}=-\frac{a^2g}{x^2}. \qquad \ldots(1)$$

Multiplying both sides of (1) by $2(dx/dt)$ and then integrating w.r.t. 't', we have $\left(\frac{dx}{dt}\right)^2=\frac{2a^2g}{x}+A$, where A is a constant.

But at B, $x=OB=b$ and $dx/dt=0$.

$$\therefore \quad 0=\frac{2a^2g}{b}+A \text{ or } A=-\frac{2a^2g}{b}.$$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2=2a^2g\left(\frac{1}{x}-\frac{1}{b}\right). \qquad \ldots(2)$$

If V is the velocity of the particle at the point A, then at A, $x=OA=a$ and $(dx/dt)^2=V^2$.

$$\therefore \quad V^2=2a^2g\left(\frac{1}{a}-\frac{1}{b}\right). \qquad \ldots(3)$$

Now the particle starts moving through a hole from A to O with velocity V at A.

Let x, $(x<a)$, be the distance of the particle from the centre of the earth at any time t measured from the instant the particle starts penetrating the earth at A. The acceleration at this point will be λx towards O, where λ is a constant.

The equation of motion (inside the earth) is $\frac{d^2x}{dt^2}=-\lambda x$, which holds good for the motion from A to O.

At A, $x=a$ and $d^2x/dt^2=-g$. $\therefore$ $\lambda=g/a$.

$$\therefore \quad \frac{d^2x}{dt^2}=-\frac{g}{a}x.$$

Multiplying both sides by $2(dx/dt)$ and then integrating w.r.t. 't', we have

$$\left(\frac{dx}{dt}\right)^2=-\frac{g}{a}x^2+B, \text{ where } B \text{ is a constant.} \qquad \ldots(4)$$

But at A, $x=OA=a$ and $\left(\frac{dx}{dt}\right)^2=V^2=2a^2g\left(\frac{1}{a}-\frac{1}{b}\right)$, from (3).

$$\therefore \quad 2a^2g\left(\frac{1}{a}-\frac{1}{b}\right)=-\frac{g}{a}a^2+B \quad \text{or} \quad B=ag\left(\frac{3b-2a}{b}\right).$$

Substituting the value of B in (4), we have

$$\left(\frac{dx}{dt}\right)^2=ag\left(\frac{3b-2a}{b}\right)-\frac{g}{a}x^2. \qquad \ldots(5)$$

Putting $x=0$ in (5), we get the velocity on reaching the centre of the earth as $\sqrt{[ga(3b-2a)/b]}$.

Again from (5), we have

$$\left(\frac{dx}{dt}\right)^2=\frac{g}{a}\left[a^2\frac{(3b-2a)}{b}-x^2\right],$$
$$=\frac{g}{a}(c^2-x^2), \text{ where } c^2=\frac{a^2}{b}(3b-2a).$$

$\therefore$ $\frac{dx}{dt}=-\sqrt{\left(\frac{g}{a}\right)}\cdot\sqrt{(c^2-x^2)}$, the $-$ive sign being taken because the particle is moving in the direction of x decreasing

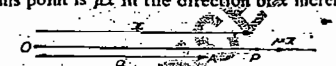
or $\quad dt=-\sqrt{\left(\frac{a}{g}\right)}\cdot\frac{dx}{\sqrt{(c^2-x^2)}}$, separating the variables.

Integrating from A to O, the required time t is given by

$$t=-\sqrt{\left(\frac{a}{g}\right)}\int_{x=a}^{0}\frac{dx}{\sqrt{(c^2-x^2)}}$$
$$=\sqrt{\left(\frac{a}{g}\right)}\int_0^a\frac{dx}{\sqrt{(c^2-x^2)}}=\sqrt{\left(\frac{a}{g}\right)}\left[\sin^{-1}\frac{x}{c}\right]_0^a$$
$$=\sqrt{\left(\frac{a}{g}\right)}\sin^{-1}\left(\frac{a}{c}\right)=\sqrt{\left(\frac{a}{g}\right)}\sin^{-1}\left[\frac{a}{a\sqrt{\{(3b-2a)/b\}}}\right], \text{ substituting for } c$$
$$=\sqrt{\left(\frac{a}{g}\right)}\sin^{-1}\sqrt{\left(\frac{b}{3b-2a}\right)}$$

§ 13. *A particle moves under an acceleration varying as the distance and directed away from a fixed point, to investigate the motion.*

Sol. Let O be the fixed point and x, the distance of the particle from O, at any time t. Then the acceleration of the particle at this point is μx in the direction of x increasing.



$\therefore$ the equation of motion of the particle is $\frac{d^2x}{dt^2}=\mu x$, $\qquad \ldots(1)$

where the $+$ive sign has been taken since the acceleration acts in the direction of x increasing.

Multiplying both sides of (1) by $2(dx/dt)$ and then integrating w.r.t. 't', we have

$$(dx/dt)^2=\mu x^2+A, \text{ where } A \text{ is a constant.}$$

Suppose the particle starts from rest at a distance a from O, *i.e.*, $dx/dt=0$ at $x=a$. Then $0=\mu a^2+A$, or $A=-\mu a^2$.

$$\therefore \quad (dx/dt)^2=\mu(x^2-a^2), \qquad \ldots(2)$$

which gives the velocity at any distance x from O.

From (2), on extracting square root, we have

$$dx/dt=\sqrt{\mu}\sqrt{(x^2-a^2)}$$

[$+$ive sign being taken because the particle moves in the direction of x increasing].

or $\quad dt=\frac{1}{\sqrt{\mu}}\cdot\frac{dx}{\sqrt{(x^2-a^2)}}$.

Integrating, $t=\frac{1}{\sqrt{\mu}}\cosh^{-1}\frac{x}{a}+B$.

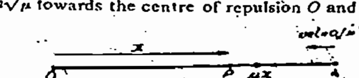
But when $t=0$, $x=a$. $\therefore$ $B=0$.

$$\therefore \quad t=\frac{1}{\sqrt{\mu}}\cosh^{-1}\frac{x}{a} \quad \text{or} \quad x=a\cosh(\sqrt{\mu}t), \qquad \ldots(3)$$

which gives the position of the particle at time t.

Ex. 76. *If a particle is projected towards the centre of repulsion, varying as the distance from the centre, from a distance a from it with a velocity $a\sqrt{\mu}$; prove that the particle will approach the centre but will never reach it.*

Sol. Let the particle be projected from the point A with velocity $a\sqrt{\mu}$ towards the centre of repulsion O and let $OA=a$.



If P is the position of the particle at time t such that $OP=x$, then at P, the acceleration on the particle is μx in the direction PA.

$\therefore$ the equation of motion of the particle is

$\frac{d^2x}{dt^2}=\mu x$. [$+$ive sign is taken because the acceleration is in the direction of x increasing],

Multiplying by $2(dx/dt)$ and integrating w.r.t. 't', we have

$$(dx/dt)^2=\mu x^2+C, \text{ where } C \text{ is a constant.}$$

But at A, $x=a$ and $(dx/dt)^2=a^2\mu$. $\therefore$ $C=0$.

$$\therefore \quad (dx/dt)^2=\mu x^2 \quad \text{or} \quad dx/dt=-\sqrt{\mu}x. \qquad \ldots(1)$$

[$-$ive sign is taken because the particle is moving in the direction of x decreasing].

The equation (1) shows that the velocity of the particle will be zero when $x=0$ and not before it and so the particle will approach the centre O.

From (1), we have $dt=-\frac{1}{\sqrt{\mu}}\frac{dx}{x}$.

Integrating between the limits $x=a$ to $x=0$, the time t_1 from A to O is given by

$$t_1=-\frac{1}{\sqrt{\mu}}\int_a^0\frac{dx}{x}=\frac{1}{\sqrt{\mu}}\left[\log x\right]_0^a=\frac{1}{\sqrt{\mu}}(\log a-\log 0)$$
$$=\infty. \qquad [\because \log 0=-\infty].$$

Hence the particle will take an infinite time to reach the centre O or in other words it will never reach the centre O.

14. *A particle moves in such a way that its acceleration varies inversely as the cube of the distance from a fixed point and is directed towards the fixed point; discuss the motion.*

Let O be the fixed point and x the distance of the particle from O, at any time t. Then the equation of motion of the particle is $\dfrac{d^2x}{dt^2}=-\dfrac{\mu}{x^3}$.

[The $-$ive sign has been taken because the force is given to be attractive.]

Multiplying both sides of (1) by $2(dx/dt)$ and then integrating w.r.t. 't', we have

$$\left(\frac{dx}{dt}\right)^2=\frac{\mu}{x^2}+A.$$

Suppose the particle starts from rest at a distance a from O, i.e., $dx/dt=0$ at $x=a$.

Then $0=\dfrac{\mu}{a^2}+A$ or $A=-\dfrac{\mu}{a^2}$.

$$\therefore\ \left(\frac{dx}{dt}\right)^2=\mu\left(\frac{1}{x^2}-\frac{1}{a^2}\right), \quad ...(2)$$

which gives the velocity at any distance x from the centre of force O.

From (2), we have $\dfrac{dx}{dt}=-\dfrac{\sqrt{\mu}}{a}\dfrac{\sqrt{(a^2-x^2)}}{x}$

[the $-$ive sign has been taken since the particle is moving in the direction of x decreasing.]

or $\quad dt=-\dfrac{a}{\sqrt{\mu}}\cdot\dfrac{x\,dx}{\sqrt{(a^2-x^2)}}$, separating the variables

$$=\frac{a}{2\sqrt{\mu}}\cdot(a^2-x^2)^{-1/2}\,(-2x)\,dx.$$

Integrating, $t=\dfrac{a}{\sqrt{\mu}}\cdot\sqrt{(a^2-x^2)}+B$.

But initially when $t=0$, $x=a$. $\therefore\ B=0$.

$$\therefore\ t=\frac{a}{\sqrt{\mu}}\sqrt{(a^2-x^2)}. \quad ...(3)$$

which gives the position of the particle at any time t.

Ex. 77. *A particle moves in a straight line towards a centre of force $\mu/(\text{distance})^3$ starting from rest at a distance a from the centre of force; show that the time of reaching a point distant b from the centre of force is $a\sqrt{\left(\dfrac{a^2-b^2}{\mu}\right)}$ and that its velocity then is $\sqrt{[\mu(a^2-b^2)]}/ab$. Also show that the time to reach the centre is $a^2/\sqrt{\mu}$.*

Sol. Let the particle start at rest from A and at time t let it be at P, where $OP=x$; O being the centre of force.

Given that the acceleration at P is μ/x^3 towards O, we have

$$\frac{d^2x}{dt^2}=-\frac{\mu}{x^3}. \quad ...(1)$$

Multiplying both sides of (1) by $2(dx/dt)$ and integrating w.r.t. 't', we have $\left(\dfrac{dx}{dt}\right)^2=\dfrac{\mu}{x^2}+C$.

When $x=a$, $dx/dt=0$, so that $C=-\mu/a^2$.

Hence $\left(\dfrac{dx}{dt}\right)^2=\mu\left(\dfrac{1}{x^2}-\dfrac{1}{a^2}\right)=\mu\left(\dfrac{a^2-x^2}{a^2x^2}\right)$.

$$\therefore\ \frac{dx}{dt}=-\frac{\sqrt{[\mu(a^2-x^2)]}}{ax} \quad ...(2)$$

the negative sign being taken because the particle is moving towards x decreasing.

Putting $x=b$ in (2), the velocity at $x=b$ is $\sqrt{[\mu(a^2-b^2)]}/ab$, in magnitude. This proves the second result.

If t_1 is the time from $x=a$ to $x=b$, then integrating (2) after seperating the variables, we get

$$t_1=-\frac{a}{\sqrt{\mu}}\int_a^b\frac{x}{\sqrt{(a^2-x^2)}}\,dx=\frac{a}{2\sqrt{\mu}}\int_a^b\frac{-2x}{\sqrt{(a^2-x^2)}}\,dx$$

$$=\frac{a}{2\sqrt{\mu}}\left[2\sqrt{(a^2-x^2)}\right]_a^b=\frac{a\sqrt{(a^2-b^2)}}{\sqrt{\mu}}.$$

This proves the first result.

And if T be the time to reach the centre O, where $x=0$, then

$$T=\frac{a}{2\sqrt{\mu}}\int_a^0\frac{-2x}{\sqrt{(a^2-x^2)}}\,dx=\frac{a}{2\sqrt{\mu}}\left[2\sqrt{(a^2-x^2)}\right]_a^0=\frac{a^2}{\sqrt{\mu}}.$$

15. Motion under miscellaneous laws of forces.

Now we shall give a few examples in which the particle moves under different laws of acceleration.

Ex. 78. *A particle whose mass is m is acted upon by a force $m\mu\left[x+\dfrac{a^4}{x^3}\right]$ towards origin; if it starts from rest at a distance a, show that it will arrive at origin in time $\pi/(4\sqrt{\mu})$.* (IAS-2006, 2012)

Sol. Given $\dfrac{d^2x}{dt^2}=-\mu\left[x+\dfrac{a^4}{x^3}\right]$, $\quad ...(1)$

the $-$ive sign being taken because the force is attractive.

Integrating it after multiplying throughout by $2(dx/dt)$, we get

$$\left(\frac{dx}{dt}\right)^2=\mu\left[-x^2+\frac{a^4}{x^2}\right]+C.$$

When $x=a$, $dx/dt=0$, so that $C=0$.

$$\therefore\ \left(\frac{dx}{dt}\right)^2=\mu\left[\frac{a^4-x^4}{x^2}\right]$$

or
$$\frac{dx}{dt}=-\frac{\sqrt{\mu}\sqrt{(a^4-x^4)}}{x}, \quad ...(2)$$

the $-$ive sign is taken because the particle is moving in the direction of x decreasing.

If t_1 be the time taken to reach the origin, then integrating (2), we get

$$t_1=-\frac{1}{\sqrt{\mu}}\int_a^0\frac{x}{\sqrt{(a^4-x^4)}}\,dx=\frac{1}{\sqrt{\mu}}\int_0^a\frac{x\,dx}{\sqrt{(a^4-x^4)}}.$$

Put $x^2=a^2\sin\theta$ so that $2x\,dx=a^2\cos\theta\,d\theta$. When $x=0$, $\theta=0$ and when $x=a$, $\theta=\pi/2$.

$$\therefore\ t_1=\frac{1}{\sqrt{\mu}}\int_0^{\pi/2}\frac{\frac{1}{2}a^2\cos\theta\,d\theta}{a^2\cos\theta}=\frac{1}{2\sqrt{\mu}}\int_0^{\pi/2}d\theta=\frac{1}{2\sqrt{\mu}}\Big[\theta\Big]_0^{\pi/2}$$

$$=\frac{1}{2\sqrt{\mu}}\cdot\frac{\pi}{2}=\frac{\pi}{4\sqrt{\mu}}.$$

Ex. 79. *A particle moves in a straight line with an acceleration towards a fixed point in the straight line, which is equal to $\mu/x^2-\lambda/x^3$ at a distance x from the given point; the particle starts from rest at a distance a: show that it oscillates between this distance and the distance $\dfrac{\lambda a}{(2\mu a-\lambda)}$ and the periodic time is $\dfrac{2\pi\mu a^3}{(2a\mu-\lambda)^{3/2}}$.*

Sol. Let O be the fixed point taken as origin and A the starting point such that $OA=a$. At any time t let P be the position of the particle, where $OP=x$. Equation of motion of the particle is

$$\frac{d^2x}{dt^2}=-\left(\frac{\mu}{x^2}-\frac{\lambda}{x^3}\right). \text{ [given]} \quad ...(1)$$

Integrating, we get $\left(\dfrac{dx}{dt}\right)^2=\dfrac{2\mu}{x}-\dfrac{\lambda}{x^2}+C$.

When $x=a$, $\dfrac{dx}{dt}=0$, so that $C=-\dfrac{2\mu}{a}+\dfrac{\lambda}{a^2}$.

$$\left(\frac{dx}{dt}\right)^2=2\mu\left(\frac{1}{x}-\frac{1}{a}\right)-\lambda\left(\frac{1}{x^2}-\frac{1}{a^2}\right)$$

$$=\left(\frac{1}{x}-\frac{1}{a}\right)\left(2\mu-\frac{\lambda}{x}-\frac{\lambda}{a}\right)$$

$$=\left(\frac{1}{x}-\frac{1}{a}\right)\left(\frac{2a\mu-\lambda}{a}-\frac{\lambda}{x}\right)$$

$$=\lambda\left(\frac{1}{x}-\frac{1}{a}\right)\left(\frac{2a\mu-\lambda}{\lambda a}-\frac{1}{x}\right). \quad ...(2)$$

The particle comes to rest where $dx/dt=0$, i.e., where

$$\left(\frac{1}{x}-\frac{1}{a}\right)\left(\frac{2a\mu-\lambda}{\mu a}-\frac{1}{x}\right)=0.$$

One solution of this equation is $\dfrac{1}{x}-\dfrac{1}{a}=0$ i.e., $x=a$, which gives the initial position. Another solution is $\dfrac{2a\mu-\lambda}{\lambda a}-\dfrac{1}{x}=0$ i.e., $x=\dfrac{\lambda a}{(2a\mu-\lambda)}$ which gives the other position of instantaneous rest.

Hence the particle oscillates between $x=a$ and $x=\dfrac{\lambda a}{(2a\mu-\lambda)}$.

This proves one result. To prove the other result, put $\dfrac{\lambda a}{2a\mu-\lambda}=b$, so that the equation (2) becomes

$$\left(\frac{dx}{dt}\right)^2=\lambda\left(\frac{1}{x}-\frac{1}{a}\right)\left(\frac{1}{b}-\frac{1}{x}\right)=\frac{\lambda}{ab}\frac{(a-x)(x-b)}{x^2}$$

or
$$\frac{dx}{dt}=-\sqrt{\left(\frac{\lambda}{ab}\right)}\cdot\frac{\sqrt{\{(a-x)(x-b)\}}}{x}$$

[the $-$ive sign is taken because the particle is moving in the direction of x decreasing.]

or
$$dt=-\sqrt{\left(\frac{ab}{\lambda}\right)}\cdot\frac{x\,dx}{\sqrt{\{(a-x)(x-b)\}}}.$$

Integrating between the limits $x=a$ to $x=b$, the time t_1 from one position of rest to the other position of rest is given by

$$t_1=-\sqrt{\left(\frac{ab}{\lambda}\right)}\int_a^b\frac{x\,dx}{\sqrt{\{(a-x)(x-b)\}}}$$

$$=\sqrt{\left(\frac{ab}{\lambda}\right)}\int_b^a\frac{x\,dx}{\sqrt{\{-ab-\{x^2-(a+b)x\}\}}}$$

$$=\sqrt{\left(\frac{ab}{\lambda}\right)}\int_b^a\frac{x\,dx}{\sqrt{\{\frac{1}{4}(a-b)^2-\{x-\frac{1}{2}(a+b)\}^2\}}}$$

$$=\sqrt{\left(\frac{ab}{\lambda}\right)}\int_{-(a-b)/2}^{(a-b)/2}\frac{\{\frac{1}{2}(a+b)+y\}\,dy}{\sqrt{\{\frac{1}{4}(a-b)^2-y^2\}}}$$

putting $x-\frac{1}{2}(a+b)=y$ so that $dx=dy$

$$= \sqrt{\left(\frac{ab}{\lambda}\right)} \int_{-(a-b)/2}^{(a-b)/2} \frac{\frac{1}{2}(a+b)}{\sqrt{\{\frac{1}{4}(a-b)^2-y^2\}}}\,dy + \sqrt{\left(\frac{ab}{\lambda}\right)} \int_{-(a-b)/2}^{(a-b)/2} \frac{y}{\sqrt{\{\frac{1}{4}(a-b)^2-y^2\}}}\,dy$$

$$= 2\sqrt{\left(\frac{ab}{\lambda}\right)} \cdot \frac{1}{2}(a+b) \int_0^{(a-b)/2} \frac{y}{\sqrt{\{\frac{1}{4}(a-b)^2-y^2\}}},$$

the second integral vanishes because the integrand is an odd function of y

$$= (a+b)\sqrt{\left(\frac{ab}{\lambda}\right)}\left[\sin^{-1}\left\{\frac{y}{\frac{1}{2}(a-b)}\right\}\right]_0^{(a-b)/2}$$

$$= (a+b)\sqrt{\left(\frac{ab}{\lambda}\right)}\left[\sin^{-1} 1 - \sin^{-1} 0\right] = \frac{\pi}{2}(a+b)\sqrt{\left(\frac{ab}{\lambda}\right)}.$$

$\therefore$ the periodic time of one complete oscillation

$$= 2t_1 = 2.\frac{\pi}{2}(a+b)\sqrt{\left(\frac{ab}{\lambda}\right)}$$

$$= \pi\left(a + \frac{\lambda a}{2a\mu-\lambda}\right)\sqrt{\left\{\frac{a}{\lambda}\cdot\frac{\lambda a}{2a\mu-\lambda}\right\}}$$

$$= \pi\frac{2a^2\mu}{(2a\mu-\lambda)}\cdot\frac{a}{\sqrt{(2a\mu-\lambda)}} = \frac{2\pi\mu a^4}{(2a\mu-\lambda)^{3/2}}.$$

Remark. To evaluate the integral giving the time t_1, we can also make the substitution $x = a\cos^2\theta + b\sin^2\theta$, so that $dx = -2(a-b)\sin\theta\cos\theta\,d\theta$. Also $\theta = 0$ when $x = a$ and $\theta = \pi/2$ when $x = b$.

Ex. 80. *A particle moves in a straight line under a force to a point in it, varying as (distance)$^{-4/3}$. Show that the velocity in falling from rest at infinity to a distance a is equal to that acquired in falling from rest at a distance a to a distance $a/8$.*

Sol. If x is the distance of the particle from the fixed point at time t, then the equation of motion of the particle is

$$\frac{d^2x}{dt^2} = -\mu x^{-4/3}. \quad ...(1)$$

Multiplying both sides of (1) by $2(dx/dt)$ and then integrating w.r.t. t, we have

$$\left(\frac{dx}{dt}\right)^2 = \frac{6\mu}{x^{1/3}} + A. \quad ...(2)$$

If the particle falls from rest at infinity, *i.e.*, $dx/dt = 0$ when $x = \infty$, we have from (2), $A = 0$.

$$\therefore \quad (dx/dt)^2 = 6\mu/x^{1/3}.$$

If v_1 is the velocity of the particle at $x = a$, then

$$v_1^2 = 6\mu/a^{1/3}. \quad ...(3)$$

Again if the particle falls from rest at a distance a, *i.e.*, if $dx/dt = 0$ when $x = a$, we have, from (2)

$$0 = \frac{6\mu}{a^{1/3}} + A \quad \text{or} \quad A = -\frac{6\mu}{a^{1/3}}.$$

$$\therefore \quad \left(\frac{dx}{dt}\right)^2 = 6\mu\left(\frac{1}{x^{1/3}} - \frac{1}{a^{1/3}}\right).$$

If in this case v_2 is the velocity of the particle at $x = a/8$, then

$$v_2^2 = 6\mu\left[\left(\frac{8}{a}\right)^{1/3} - \frac{1}{a^{1/3}}\right] = 6\mu\left(\frac{2}{a^{1/3}} - \frac{1}{a^{1/3}}\right) = \frac{6\mu}{a^{1/3}}. \quad ...(4)$$

From (3) and (4), we observe that $v_1 = v_2$, which proves the required result.

Ex. 81. *Find the time of descent to the centre of force, when the force varies as (distance)$^{-5/3}$, and show that the velocity at the centre is infinite.*

Sol. Let O be the centre of force taken as the origin. Suppose a particle starts at rest from A, where $OA = a$. The particle moves towards O on account of a centre of attraction at O. Let P be the position of the particle at any time t, where $OP = x$. The acceleration of the particle at P is $\mu x^{-5/3}$ directed towards O. Therefore the equation of motion of the particle at P is

$$\frac{d^2x}{dt^2} = -\mu x^{-5/3}. \quad ...(1)$$

Multiplying both sides of (1) by $2\,(dx/dt)$ and integrating w.r.t. t, we have

$$\left(\frac{dx}{dt}\right)^2 = -\frac{2\mu x^{-2/3}}{-2/3} + k = \frac{3\mu}{x^{2/3}} + k, \text{ where } k \text{ is a constant.}$$

At A, $x = a$ and $dx/dt = 0$, so that $(3\mu/a^{2/3}) + k = 0$

or $\quad k = -3\mu/a^{2/3}$.

$$\therefore \quad \left(\frac{dx}{dt}\right)^2 = \frac{3\mu}{x^{2/3}} - \frac{3\mu}{a^{2/3}} = \frac{3\mu\,(a^{2/3} - x^{2/3})}{a^{2/3}\,x^{2/3}}. \quad ...(2)$$

which gives the velocity of the particle at any distance x from the centre of force O. Putting $x = 0$ in (2), we see that at O, $(dx/dt)^2 = \infty$. Therefore the velocity of the particle at the centre is infinite.

Taking square root of (2), we get

$$\frac{dx}{dt} = -\sqrt{(3\mu)}\sqrt{\left(\frac{a^{2/3} - x^{2/3}}{a^{2/3}\,x^{2/3}}\right)},$$ where the $-$ive sign has been taken because the particle is moving in the direction of x decreasing.

Separating the variables, we get

$$dt = -\frac{a^{1/3}}{\sqrt{(3\mu)}}\frac{x^{1/3}}{\sqrt{(a^{2/3} - x^{2/3})}}\,dx. \quad ...(3)$$

Let t_1 be the time from A to O. Then at A, $t = 0$ and $x = a$ while at O, $t = t_1$ and $x = 0$. So integrating (3) from A to O, we have

$$\int_0^{t_1} dt = -\frac{a^{1/3}}{\sqrt{(3\mu)}}\int_a^0 \frac{x^{1/3}}{\sqrt{(a^{2/3} - x^{2/3})}}\,dx = \frac{a^{1/3}}{\sqrt{(3\mu)}}\int_0^a \frac{x^{1/3}\,dx}{\sqrt{(a^{2/3} - x^{2/3})}}.$$

Putting $x = a\sin^3\theta$, so that $dx = 3a\sin^2\theta\cos\theta\,d\theta$. When $x = 0$, $\theta = 0$ and when $x = a$, $\theta = \pi/2$.

$$\therefore \quad t_1 = \frac{a^{1/3}}{\sqrt{(3\mu)}}\int_0^{\pi/2} \frac{a^{1/3}\sin\theta}{a^{1/3}\cos\theta}\,3a\sin^2\theta\cos\theta\,d\theta$$

$$= \frac{3a^{4/3}}{\sqrt{(3\mu)}}\int_0^{\pi/2}\sin^3\theta\,d\theta = \frac{3a^{4/3}}{\sqrt{(3\mu)}}\cdot\frac{2}{3.1} = \frac{2a^{4/3}}{\sqrt{(3\mu)}}.$$

Hence the time of descent to the centre of force is $2a^{4/3}/\sqrt{(3\mu)}$.

Ex. 82. *A particle starts from rest at a distance a from the centre of force which attracts inversely as the distance. Prove that the time of arriving at the centre is $a\sqrt{(\pi/2\mu)}$.*

Sol. If x is the distance of the particle from the centre of force at time t, then the equation of motion is

$$\frac{d^2x}{dt^2} = -\frac{\mu}{x}.$$

Multiplying both sides by $2(dx/dt)$ and then integrating w.r.t. 't', we have $(dx/dt)^2 = -2\mu\log x + A$, where A is a constant.

But initially at $x = a$, $dx/dt = 0$.

$$\therefore \quad 0 = -2\mu\log a + A \quad \text{or} \quad A = 2\mu\log a.$$

$$\therefore \quad (dx/dt)^2 = 2\mu(\log a - \log x) = 2\mu\log(a/x)$$

or $\quad dx/dt = -\sqrt{(2\mu)}\sqrt{\{\log(a/x)\}}$,

where the $-$ive sign has been taken since the particle is moving in the direction of x decreasing.

Separating the variables, we have

$$dt = -\frac{1}{\sqrt{(2\mu)}}\frac{dx}{\sqrt{\{\log(a/x)\}}}.$$

Integrating from $x = a$ to $x = 0$, the required time t_1 to reach the centre is given by

$$t_1 = -\frac{1}{\sqrt{(2\mu)}}\int_{x=a}^0 \frac{dx}{\sqrt{\{\log(a/x)\}}}.$$

Put $\log\left(\frac{a}{x}\right) = u^2$ *i.e.*, $x = ae^{-u^2}$, so that $dx = -2ae^{-u^2}u\,du$.

When $x = a$, $u = 0$ and when $x \to 0$, $u \to \infty$.

$$\therefore \quad t_1 = \frac{2}{\sqrt{(2\mu)}}\int_0^\infty e^{-u^2}\,du. \text{ But } \int_0^\infty e^{-u^2}\,du = \frac{\sqrt{\pi}}{2} \text{ (Remember)}$$

$$\therefore \quad t_1 = \frac{2a}{\sqrt{(2\mu)}}\cdot\frac{\sqrt{\pi}}{2} = a\sqrt{\left(\frac{\pi}{2\mu}\right)}.$$

Ex. 83. *A particle moves in a straight line, its acceleration directed towards a fixed point O in the line and is always equal to $\mu(a^5/x^2)^{1/3}$ when it is at a distance x from O. If it starts from rest at a distance a from O, show that it will arrive at O with a velocity $a\sqrt{(6\mu)}$ after time $\frac{8}{15}\sqrt{\left(\frac{6}{\mu}\right)}$.*

Sol. Take the centre of force O as origin. Suppose a particle starts from rest at A, where $OA = a$. It moves towards O because of a centre of attraction at O. Let P be the position of the particle after any time t, where $OP = x$. The acceleration of the particle at P is $\mu a^{5/3}x^{-2/3}$ directed towards O. Therefore the equation of motion of the particle is

$$\frac{d^2x}{dt^2} = -\mu a^{5/3}x^{-2/3}. \quad ...(1)$$

Multiplying both sides of (1) by $2(dx/dt)$ and integrating w.r.t. 't', we have

$$\left(\frac{dx}{dt}\right)^2 = -\frac{2\mu a^{5/3}x^{1/3}}{1/3} + k = -6\mu a^{5/3}x^{1/3} + k,$$

where k is a constant.

At A, $x = a$ and $dx/dt = 0$, so that

$$-6\mu a^{5/3}a^{1/3} + k = 0 \quad \text{or} \quad k = 6\mu a^2.$$

$$\therefore \quad (dx/dt)^2 = -6\mu a^{5/3}x^{1/3} + 6\mu a^2 = 6\mu a^{5/3}(a^{1/3} - x^{1/3}). \quad ...(2)$$

which gives the velocity of the particle at any distance x from the centre of force. Suppose the particle arrives at O with the velocity v_1. Then at O, $x = 0$ and $(dx/dt)^2 = v_1^2$. So from (2), we have

$$v_1^2 = 6\mu a^{5/3}(a^{1/3} - 0) = 6\mu a^2 \quad \text{or} \quad v_1 = a\sqrt{(6\mu)}.$$

Now taking square root of (2), we get

$$dx/dt = -\sqrt{(6\mu a^{5/3})}\sqrt{(a^{1/3} - x^{1/3})},$$

where the $-$ive sign has been taken because the particle moves in the direction of x decreasing.

Separating the variables, we get

$$dt = -\frac{1}{\sqrt{(6\mu a^{5/3})}}\cdot\frac{dx}{\sqrt{(a^{1/3} - x^{1/3})}}. \quad ...(3)$$

Let t_1 be the time from A to O. Then integrating (3) from A to O, we have

$$\int_0^{t_1} dt = -\frac{1}{\sqrt{(6\mu a^{5/3})}}\int_a^0 \frac{dx}{\sqrt{(a^{1/3} - x^{1/3})}}$$

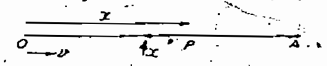
$$= \frac{1}{\sqrt{(6\mu a^{5/3})}}\int_0^a \frac{dx}{\sqrt{(a^{1/3} - x^{1/3})}}$$

Put $x=a\sin^6\theta$, so that $dx=6a\sin^5\theta\cos\theta\,d\theta$. When $x=0$, $\theta=0$ and when $x=a$, $\theta=\pi/2$.

$$\therefore\quad t_1=\frac{1}{\sqrt{(6\mu a^{5/6})}}\int_0^{\pi/2}\frac{6a\sin^5\theta\cos\theta\,d\theta}{a^{1/6}\cos\theta}$$

$$=\sqrt{\left(\frac{6}{\mu}\right)}\int_0^{\pi/2}\sin^5\theta\,d\theta=\sqrt{\left(\frac{6}{\mu}\right)}\cdot\frac{4.2}{5.3.1}=\frac{8}{15}\sqrt{\left(\frac{6}{\mu}\right)}.$$

Ex. 84. *A particle starts with a given velocity v and moves under a retardation equal to k times the space described. Show that the distance traversed before it comes to rest is $v/\sqrt{k}$.*



Sol. Suppose the particle starts from O with velocity v and moves in the straight line OA. Let P be the position of the particle after any time t, where $OP=x$. Then the retardation of the particle at P is kx *i.e.*, the acceleration of the particle at P is kx and is directed towards O *i.e.*, in the direction of x decreasing. Therefore the equation of motion of the particle at P is

$$d^2x/dt^2=-kx. \qquad \ldots(1)$$

Multiplying both sides of (1) by $2(dx/dt)$ and integrating w.r.t. t, we have $(dx/dt)^2=-kx^2+C$, where C is a constant.

At O, $x=0$ and $dx/dt=v$, so that $v^2=C$.

$$\therefore\quad (dx/dt)^2=v^2-kx^2, \qquad \ldots(2)$$

which gives the velocity of the particle at a distance x from O.

From (2), $dx/dt=0$ when $v^2-kx^2=0$ *i.e.*, when $x=v/\sqrt{k}$.

Hence the distance traversed before the particle comes to rest is $v/\sqrt{k}$.

Ex. 85. *Assuming that at a distance x from a centre of force, the speed v of a particle, moving in a straight line is given by the equation $x=ae^{bv^2}$, where a and b are constants. Find the law and the nature of the force.*

Sol. Given, $x=ae^{bv^2}$. Therefore $e^{bv^2}=x/a$

or $\quad bv^2=\log(x/a)=\log x-\log a. \qquad \ldots(1)$

Differentiating both sides of (1) w.r.t. x, we get

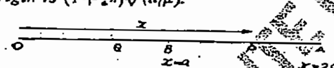
$$2bv\frac{dv}{dx}=\frac{1}{x}\quad\text{or}\quad v\frac{dv}{dx}=\frac{1}{2b}\frac{1}{x}.$$

$\therefore$ the equation of motion of the particle is

$$\frac{d^2x}{dt^2}=\frac{1}{2b}\cdot\frac{1}{x}\qquad\left[\text{Note that } v\frac{dv}{dx}=\frac{d^2x}{dt^2}\right]$$

Hence the acceleration varies inversely as the distance of the particle from the centre of force. Also the force is repulsive or attractive according as b is positive or negative.

Ex. 86. *A particle of mass m moving in a straight line is acted upon by an attractive force which is expressed by the formula $m\mu a^3/x^2$ for values of $x\geqslant a$, and by the formula $m\mu x/a$ for $x\leqslant a$, where x is the distance from a fixed origin in the line. If the particle starts at a distance 2a from the origin, prove that it will reach the origin with velocity $(2\mu a)^{1/2}$. Prove further that the time taken to reach the origin is $(1+\frac{3}{4}\pi)\sqrt{(a/\mu)}$.*



Sol. Let O be the origin and A the point from which the particle starts. We have $OA=2a$ and let $OB=a$, so that B is the middle point of OA.

Motion from A to B. The particle starts from rest at A and it moves towards B. Let P be its position at any time t, where $OP=x$. According to the question the acceleration of P is $\mu a^3/x^2$ and is directed towards O *i.e.* in the direction of x decreasing. Therefore the equation of motion of P is

$$\frac{d^2x}{dt^2}=-\frac{\mu a^3}{x^2}. \qquad \ldots(1)$$

Multiplying (1) by $2(dx/dt)$ and integrating w.r.t. t, we have

$$\left(\frac{dx}{dt}\right)^2=\frac{2\mu a^3}{x}+C.$$

When $x=2a$, $dx/dt=0$, so that $C=-2\mu a^3/2a$.

$$\therefore\quad\left(\frac{dx}{dt}\right)^2=\frac{2\mu a^3}{x}-\frac{2\mu a^3}{2a}=2a^3\mu\left[\frac{1}{x}-\frac{1}{2a}\right]=a\mu\,\frac{2a-x}{x}, \qquad \ldots(2)$$

which gives the velocity of the particle at any position between A and B. Suppose the particle reaches B with the velocity v_1. Then at B, $x=a$ and $(dx/dt)^2=v_1^2$. So from (2), we get

$$v_1^2=a\mu\,\frac{2a-a}{a}=a\mu\ \text{or}\ v_1=\sqrt{(a\mu)},$$ its direction being towards the origin O.

Now taking square root of (2), we get

$$\frac{dx}{dt}=-\sqrt{(a\mu)}\sqrt{\left(\frac{2a-x}{x}\right)},$$ where the $-$ive sign has been taken because the particle is moving in the direction of x decreasing.

Separating the variables, we get

$$dt=-\frac{1}{\sqrt{(a\mu)}}\sqrt{\left(\frac{x}{2a-x}\right)}\,dx. \qquad \ldots(3)$$

Let t_1 be the time from A to B. Then at A, $x=2a$ and $t=0$, while at B, $x=a$ and $t=t_1$. So integrating (3) from A to B, we get

$$\int_0^{t_1}dt=-\frac{1}{\sqrt{(a\mu)}}\int_{2a}^{a}\sqrt{\left(\frac{x}{2a-x}\right)}\,dx.$$

Put $x=2a\cos^2\theta$, so that $dx=-4a\cos\theta\sin\theta\,d\theta$. When $x=2a$, $\theta=0$ and when $x=a$, $\theta=\pi/4$.

$$\therefore\quad t_1=-\frac{1}{\sqrt{(a\mu)}}\int_0^{\pi/4}\frac{\cos\theta}{\sin\theta}\cdot(-4a\cos\theta\sin\theta)\,d\theta$$

$$=\sqrt{\left(\frac{a}{\mu}\right)}\int_0^{\pi/4}2\cos^2\theta\,d\theta=2\sqrt{\left(\frac{a}{\mu}\right)}\int_0^{\pi/4}(1+\cos 2\theta)\,d\theta$$

$$=2\sqrt{\left(\frac{a}{\mu}\right)}\left[\theta+\tfrac{1}{2}\sin 2\theta\right]_0^{\pi/4}=2\sqrt{\left(\frac{a}{\mu}\right)}\left[\frac{\pi}{4}+\frac{1}{2}\right]=\sqrt{\left(\frac{a}{\mu}\right)}\left[\frac{\pi}{2}+1\right].$$

Motion from B to O. Now the particle starts from B towards O with velocity $\sqrt{(a\mu)}$ gained by it during its motion from A to B. Let Q be its position after time t since it starts from B and let $OQ=x$. Now according to the question the acceleration of Q is $\mu x/a$ directed towards O. Therefore the equation of motion of Q is

$$\frac{d^2x}{dt^2}=-\frac{\mu x}{a}. \qquad \ldots(4)$$

Multiplying both sides of (4) by $2(dx/dt)$ and integrating w.r.t. t, we have

$$\left(\frac{dx}{dt}\right)^2=-\frac{\mu}{a}x^2+D.$$

At B, $x=a$ and $(dx/dt)^2=v_1^2=a\mu$, so that $a\mu=-a\mu+D$

or $\quad D=2a\mu.$

$$\therefore\quad\left(\frac{dx}{dt}\right)^2=-\frac{\mu}{a}x^2+2a\mu=\frac{\mu}{a}(2a^2-x^2), \qquad \ldots(5)$$

which gives the velocity of the particle at any position between B and O. Let v_2 be the velocity of the particle at O. Then putting $x=0$ and $(dx/dt)^2=v_2^2$ in (5), we get

$$v_2^2=\frac{\mu}{a}(2a^2-0)=2a\mu\quad\text{or}\quad v_2=\sqrt{(2a\mu)}.$$

Hence the particle reaches the origin with the velocity $\sqrt{(2a\mu)}$.

Now taking square root of (5), we get

$$\frac{dx}{dt}=-\sqrt{\left(\frac{\mu}{a}\right)}\sqrt{(2a^2-x^2)},$$ where the $-$ive sign has been taken because the particle is moving in the direction of x decreasing.

Separating the variables, we have

$$dt=-\sqrt{\left(\frac{a}{\mu}\right)}\frac{dx}{\sqrt{(2a^2-x^2)}}. \qquad \ldots(6)$$

Let t_2 be the time from B to O. Then at B, $t=0$ and $x=a$ while at O, $x=0$ and $t=t_2$. So integrating (6) from B to O, we get

$$\int_0^{t_2}dt=-\sqrt{\left(\frac{a}{\mu}\right)}\int_a^0\frac{dx}{\sqrt{(2a^2-x^2)}}$$

i.e.,
$$t_2=\sqrt{\left(\frac{a}{\mu}\right)}\left[\cos^{-1}\frac{x}{a\sqrt{2}}\right]_a^0$$

$$=\sqrt{\left(\frac{a}{\mu}\right)}\left[\frac{\pi}{2}-\frac{\pi}{4}\right]=\sqrt{\left(\frac{a}{\mu}\right)}\frac{\pi}{4}.$$

Hence the whole time taken to reach the origin $O=t_1+t_2$

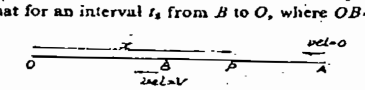
$$=\sqrt{\left(\frac{a}{\mu}\right)}\left[\frac{\pi}{2}+1\right]+\sqrt{\left(\frac{a}{\mu}\right)}\frac{\pi}{4}=\sqrt{\left(\frac{a}{\mu}\right)}\left[\frac{3\pi}{4}+1\right].$$

Ex. 87. *A particle moves along the axis of x starting from rest at $x=a$. For an interval t_1 from the beginning of the motion the acceleration is $-\mu x$, for a subsequent time t_2 the acceleration is μx, and at the end of this interval the particle is at the origin; prove that*

$$\tan(\sqrt{\mu}t_1).\ \tanh(\sqrt{\mu}t_2)=1.$$

Sol. Let the particle moving along the axis of x start from rest at A such that $OA=a$.

Let $-\mu x$ be the acceleration for an interval t_1 from A to B and μx that for an interval t_2 from B to O, where $OB=b$.



For motion from A to B, the equation of motion is

$$\frac{d^2x}{dt^2}=-\mu x. \qquad \ldots(1)$$

Multiplying both sides by $2(dx/dt)$ and then integrating w.r.t. 't', we have

$$(dx/dt)^2=-\mu x^2+A,$$ where A is a constant.

But at $x=a$, $dx/dt=0$. $\therefore\ 0=-\mu a^2+A$ or $A=\mu a^2$.

$$\therefore\quad (dx/dt)^2=\mu(a^2-x^2) \qquad \ldots(2)$$

or $$dx/dt = -\sqrt{\mu}\sqrt{(a^2-x^2)}$$

[the $-$ive sign is taken because the particle is moving in the direction of x decreasing.]

or $$dt = -\frac{1}{\sqrt{\mu}} \cdot \frac{dx}{\sqrt{(a^2-x^2)}}, \quad \text{[separating the variables].}$$

Integrating between the limits $x=a$ to $x=b$, the time t_1 from A to B is given by

$$t_1 = -\frac{1}{\sqrt{\mu}} \int_{x=a}^{b} \frac{dx}{\sqrt{(a^2-x^2)}} = \frac{1}{\sqrt{\mu}} \left[\cos^{-1}\frac{x}{a}\right]_{x=a}^{b} = \frac{1}{\sqrt{\mu}} \cos^{-1}\frac{b}{a}$$

$\therefore$ $\cos(\sqrt{\mu} t_1) = b/a$ and $\sin(\sqrt{\mu} t_1) = \sqrt{[1-\cos^2(\sqrt{\mu} t_1)]}$

$$= \sqrt{\left(1-\frac{b^2}{a^2}\right)} = \frac{\sqrt{(a^2-b^2)}}{a}$$

Dividing, $\tan(\sqrt{\mu} t_1) = \dfrac{\sqrt{(a^2-b^2)}}{b}$. ...(3)

If V is the velocity at B where $x=b$, then from (2),

$$V^2 = \mu(a^2-b^2). \quad \text{...(4)}$$

For motion from B to O, the velocity at B is V and the particle moves towards O under the acceleration μx.

$\therefore$ the equation of motion is $\dfrac{d^2x}{dt^2} = \mu x$. ...(5)

Integrating, $(dx/dt)^2 = \mu x^2 + B$, where B is a constant.

But at the point B, $x=b$ and $(dx/dt)^2 = V^2 = \mu(a^2-b^2)$.

$\therefore$ $\mu(a^2-b^2) = \mu b^2 + B$ or $B = \mu(a^2-2b^2)$.

$\therefore$ $\left(\dfrac{dx}{dt}\right)^2 = \mu[x^2+(a^2-2b^2)]$ or $\dfrac{dx}{dt} = -\sqrt{\mu}\sqrt{[x^2+(a^2-2b^2)]}$

or $$dt = -\frac{1}{\sqrt{\mu}} \frac{dx}{\sqrt{[x^2+(a^2-2b^2)]}}$$

Integrating between the limits $x=b$ to $x=0$, the time t_2 from B to O is given by

$$t_2 = -\frac{1}{\sqrt{\mu}} \int_{x=b}^{0} \frac{dx}{\sqrt{[x^2+(a^2-2b^2)]}}$$

$$= -\frac{1}{\sqrt{\mu}} \left[\sinh^{-1}\frac{x}{\sqrt{(a^2-2b^2)}}\right]_b^0 = \frac{1}{\sqrt{\mu}} \sinh^{-1}\frac{b}{\sqrt{(a^2-2b^2)}}$$

$\therefore$ $\sinh(\sqrt{\mu} t_2) = \dfrac{b}{\sqrt{(a^2-2b^2)}}$ so that

$\cosh(\sqrt{\mu} t_2) = \sqrt{\{1+\sinh^2(\sqrt{\mu} t_2)\}}$

$$= \sqrt{\left(1+\frac{b^2}{a^2-2b^2}\right)} = \sqrt{\left(\frac{a^2-b^2}{a^2-2b^2}\right)}.$$

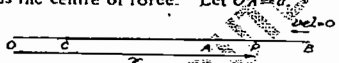
Dividing, $\tanh(\sqrt{\mu} t_2) = \dfrac{b}{\sqrt{(a^2-b^2)}}$. ...(6)

Multiplying (3) and (6), we have

$$\tan(\sqrt{\mu} t_1) \cdot \tanh(\sqrt{\mu} t_2) = 1.$$

Ex. 88. *A particle starts from rest at a distance b from a fixed point, under the action of a force through the fixed point, the law of which at a distance x is $\mu\left[1-\dfrac{a}{x}\right]$ towards the point when $x>a$ but $\mu\left[\dfrac{a^2}{x^2}-\dfrac{a}{x}\right]$ from the same point when $x<a$; prove that particle will oscillate through a space $\left[\dfrac{b^2-a^2}{b}\right]$.*

Sol. Let the particle start from rest at B, where $OB=b$, and move towards the centre of force. Let $OA=a$.



Motion from B to A *i.e.*, when $x>a$.

Since the law of force, when $x>a$, is $\mu(1-a/x)$ towards O, therefore the equation of motion is

$$\frac{d^2x}{dt^2} = -\mu\left(1-\frac{a}{x}\right).$$

Multiplying both sides by $2(dx/dt)$ and integrating w.r.t. 't', we have $\left(\dfrac{dx}{dt}\right)^2 = -2\mu(x-a\log x)+C$, where C is a constant.

But at B, $x=OB=b$ and $dx/dt=0$. $\therefore$ $C=2\mu(b-a\log b)$.

$$\therefore \left(\frac{dx}{dt}\right)^2 = 2\mu(b-a\log b-x+a\log x). \quad \text{...(1)}$$

If V is the velocity at the point A where $x=OA=a$, then from (1), we have $V^2 = 2\mu(b-a-a\log b+a\log a)$. ...(2)

Motion from A towards O *i.e.*, when $x<a$.

The velocity of the particle at A is V and it moves towards O under the law of force $\mu\left(\dfrac{a^2}{x^2}-\dfrac{a}{x}\right)$ at the distance x from the fixed point O.

$\therefore$ the equation of motion is $\dfrac{d^2x}{dt^2} = \mu\left[\dfrac{a^2}{x^2}-\dfrac{a}{x}\right]$.

Multiplying both sides by $2(dx/dt)$ and integrating, we have $\left(\dfrac{dx}{dt}\right)^2 = 2\mu\left(-\dfrac{a^2}{x}-a\log x\right)+D$, where D is a constant.

But at the point A,

$x=a$ and $(dx/dt)^2 = V^2 = 2\mu(b-a-a\log b+a\log a)$.

$\therefore$ $D = 2\mu(b-a-a\log b+a\log a)+2\mu(a+a\log a)$

$$= 2\mu(b-a\log b+2a\log a) = 2\mu\{b+a\log(a^2/b)\}.$$

$$\therefore \left(\frac{dx}{dt}\right)^2 = -2\mu\left(\frac{a^2}{x}+a\log x\right)+2\mu\left\{b+a\log\left(\frac{a^2}{b}\right)\right\} \quad \text{...(3)}$$

If the particle comes to rest at the point C, where $x=c$, then putting $x=c$ and $dx/dt=0$ in (3), we get

$$2\mu\left(\frac{a^2}{c}+a\log c\right) = 2\mu\left\{b+a\log\left(\frac{a^2}{b}\right)\right\}$$

or $$\frac{a^2}{c}+a\log c = \frac{a^2}{(a^2/b)}+a\log\left(\frac{a^2}{b}\right).$$

$$\therefore c = a^2/b \text{ i.e., } OC = a^2/b.$$

Since B and C are the positions of instantaneous rest of the particle, therefore the particle oscillates through the space BC.

We have $BC = OB-OC = b-\dfrac{a^2}{b} = \dfrac{b^2-a^2}{b}$,

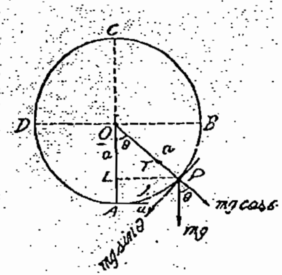
which proves the required result.

CONSTRAINED MOTION | SET-II

1. Introduction. *The motion of a particle is called constrained motion, if it is compelled to move along a given curve or surface.*

Here in this chapter we shall consider the motion on smooth plane curves, vertical circle and cycloid only.

2. Motion in a vertical circle. *A heavy particle is tied to one end of a light inextensible string whose other end is attached to a fixed point. It is projected horizontally with a given velocity u from its vertical position of equilibrium; to discuss the subsequent motion.*



Let one end of a string of length a be attached to the fixed point O and a particle of mass m be attached at the other end A. Let OA be the vertical position of equilibrium of the string. Let the particle be projected horizontally from A with velocity u. Since the string is inextensible the particle starts moving in a circle whose centre is O and radius a. If P is the position of the particle at time t such that $\angle AOP=\theta$ and arc $AP=s$, the forces acting on the particle at P are :

(i) weight mg of the particle acting vertically downwards,

and (ii) tension T in the string acting along PO.

If v be the velocity of the particle at P, the tangential and normal accelerations of P are

$\dfrac{d^2s}{dt^2}$ (in the direction of s increasing)

and $\dfrac{v^2}{\rho}$ (along inwards drawn normal at P).

$\therefore$ the equations of motion of the particle along the tangent and normal are

$$m\frac{d^2s}{dt^2}=-mg\sin\theta \qquad \ldots(1)$$

and
$$m\frac{v^2}{\rho}=T-mg\cos\theta. \qquad \ldots(2)$$

Also $s=\text{arc } AP=a\theta.$

$\therefore$ $v=\dfrac{ds}{dt}=a\dfrac{d\theta}{dt}$

and
$$\frac{d^2s}{dt^2}=a\frac{d^2\theta}{dt^2} \qquad \ldots(3)$$

$\therefore$ from (1) and (3), we have

$$a\frac{d^2\theta}{dt^2}=-g\sin\theta.$$

Multiplying both sides by $2a\dfrac{d\theta}{dt}$ and integrating w.r.t. 't', we have

$$v^2=\left(a\frac{d\theta}{dt}\right)^2=2ag\cos\theta+A,$$

where A is constant of integration.

But initially at A, $\theta=0$, $v=u$.

$\therefore$ $A=u^2-2ag\cos 0=u^2-2ag.$

$$\therefore\quad v^2=u^2-2ag+2ag\cos\theta. \qquad \ldots(4)$$

Now for a circle $\rho=a$ (radius).

$\therefore$ from (2), we have

$$T=\frac{m}{a}v^2+mg\cos\theta=\frac{m}{a}(v^2+ag\cos\theta).$$

Substituting the value of v^2 from (4), we have

$$T=\frac{m}{a}(u^2-2ag+3ag\cos\theta). \qquad \ldots(5)$$

If the velocity $v=0$ at $\theta=\theta_1$ then from (4), we have

$$0=u^2-2ag+2ag\cos\theta_1$$

or
$$\cos\theta_1=\frac{2ag-u^2}{2ag}. \qquad \ldots(6)$$

If h_1 is the height from the lowest point A of the point where the velocity vanishes, then

$$h_1=OA-a\cos\theta_1=a-a\cdot\frac{2ag-u^2}{2ag}$$

or
$$h_1=\frac{u^2}{2g}. \qquad \ldots(7)$$

Again if the tension $T=0$, at $\theta=\theta_2$, then from (5), we have

$$0=u^2-2ag+3ag\cos\theta_2.$$

$$\therefore\quad \cos\theta_2=\frac{2ag-u^2}{3ag} \qquad \ldots(8)$$

If h_2 is the height from the lowest point A of the point where the tension vanishes, then

$$h_2=OA-a\cos\theta_2=a-a\cdot\frac{2ag-u^2}{3ag}$$

or
$$h_2=\frac{u^2+ag}{3g} \qquad \ldots(9)$$

Now the following cases may arise here.

Case I. The velocity v vanishes before the tension T.

This is possible if and only if $h_1<h_2$

or $\dfrac{u^2}{2g}<\dfrac{u^2+ag}{3g}$ or $3u^2<2(u^2+ag)$

or $u^2<2ag$ or $u<\sqrt{(2ag)}$.

But when $u<\sqrt{(2ag)}$, we have from (6), $\cos\theta_1=+$ ive *i.e.*, θ_1 is an acute angle.

Thus if the particle is projected with the velocity $u<\sqrt{(2ag)}$, then it will oscillate about A and will not rise upto the horizontal diameter through O.

Case II. The velocity v and the tension T vanish simultaneously.

This is possible if and only if $h_1=h_2$

i.e., $\dfrac{u^2}{2g}=\dfrac{u^2+ag}{3g}$ *i.e.*, $u^2=2ag$ *i.e.*, $u=\sqrt{(2ag)}$.

Also when $u=\sqrt{(2ag)}$, we have from (6) and (8), $\theta_1=\pi/2=\theta_2$.

Thus if the particle is projected with the velocity $u=\sqrt{(2ag)}$, then it will rise upto the level of the horizontal diameter through O and will oscillate about A in the semi-circular arc BAD.

Case III. Condition for describing the complete circle.

At the highest point C, we have $\theta=\pi$. Therefore from (4) and (5), we have at C, $v^2=u^2-4ag$

and
$$T=\frac{m}{a}(u^2-5ag).$$

If $u^2>5ag$ *i.e.*, if $u>\sqrt{(5ag)}$, then neither the velocity v nor the tension T is zero at the highest point C, and so the particle will go on describing the complete circle.

And if $u^2=5ag$ *i.e.*, if $u=\sqrt{(5ag)}$, then at the highest point C the tension T vanishes whereas the velocity does not vanish. Hence in this case the string will become momentarily slack at C and the particle will go on describing the complete circle.

Thus the condition for describing the complete circle by the particle is that $u\geq\sqrt{(5ag)}$. **In other words the least velocity of projection for describing the complete circle is $\sqrt{(5ag)}$.**

Case IV. The tension T vanishes before the velocity v.

This is possible if and only if $h_1>h_2$

i.e., $\dfrac{u^2}{2g}>\dfrac{u^2+ag}{3g}$ *i.e.*, $u^2>2ag$, *i.e.*, $u>\sqrt{(2ag)}$.

When $u>\sqrt{(2ag)}$, we have from (8), $\cos\theta_2=-$ ive, showing that θ_2 must be $>90°$.

Now at the point where the tension T is zero, the string becomes slack. Since the velocity v is not zero at that point, therefore the particle will leave the circular path and trace a parabolic path while moving freely under gravity.

Thus if the particle is projected with the velocity u such that $\sqrt{(2ag)}<u<\sqrt{(5ag)}$, then it will leave the circular path at a point somewhere between B and C and trace out a parabolic path.

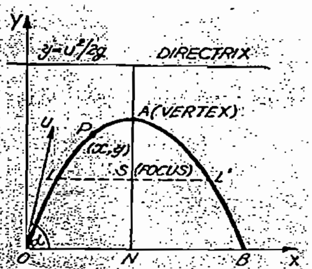
3. *A particle is projected along the inside of a smooth fixed hollow sphere (or circle) from its lowest point; to discuss the motion.*

The discussion is exactly the same as in § 2 with the difference that in this case the tension T is replaced by the reaction R between the particle and the sphere (or circle).

4. Some important results of the motion of a projectile to be used in this chapter. Suppose a particle of mass m is projected in vaccum, in a vertical plane through the point of projection, with velocity u in a direction making an angle α with the horizontal. Then the path of the projectile is a parabola.

The following results about the motion of the projectile to be used in this chapter should be remembered.

Take the point of projection O as the origin, the horizontal line OX in the plane of projection as the x-axis and the vertical line OY as the y-axis. Then the initial horizontal velocity of the projectile is $u \cos \alpha$ and the initial vertical velocity is $u \sin \alpha$.



The equation of the trajectory *i.e.*, the equation of the parabolic path is

$$y = x \tan \alpha - \frac{1}{2} g \frac{x^2}{u^2 \cos^2 \alpha}.$$

The length of the latus rectum LSL' of the above parabolic path is

$$\frac{2}{g} u^2 \cos^2 \alpha = \frac{2}{g} (\text{horizontal velocity})^2.$$

If H is the maximum height NA attained by the projectile above the point of projection O, then considering the vertical motion from O to A and using the formula $v^2 = u^2 + 2fs$, we have

$$0 = u^2 \sin^2 \alpha - 2gH$$

or $$H = \frac{u^2 \sin^2 \alpha}{2g}.$$

Thus the maximum height of the projectile above the point of projection is $\frac{u^2 \sin^2 \alpha}{2g}$.

Also remember that the velocity of a projectile at any point P of its path is that due to a fall from the directrix to that point.

Illustrative Examples

Ex. 1. *A heavy particle of weight W, attached to a fixed point by a light inextensible string, describes a circle in a vertical plane. The tension in the string has the values mW and nW respectively when the particle is at the highest and lowest point in the path. Show that $n = m + 6$.*

Sol. Let M be the mass of the particle. Then
$W = Mg$ *i.e.*, $M = W/g$.

Proceeding as in § 2, the tension T in the string in any position is given by

$$T = \frac{M}{a}(u^2 - 2ag + 3ag \cos \theta)$$ [See eqn. (5) of § 2 and deduce it here]

or $$T = \frac{W}{ag}(u^2 - 2ag + 3ag \cos \theta). \quad \text{...(1)}$$

Now mW is given to be the tension in the string at the highest point and nW that at the lowest point. Therefore $T = mW$ when $\theta = \pi$ and $T = nW$ when $\theta = 0$. So from (1), we have

$$mW = \frac{W}{ag}(u^2 - 2ag + 3ag \cos \pi) \text{ giving } m = \frac{1}{ag}(u^2 - 5ag) \quad \text{...(2)}$$

and $$nW = \frac{W}{ag}(u^2 - 2ag + 3ag \cos 0) \text{ giving } n = \frac{1}{ag}(u^2 + ag). \quad \text{...(3)}$$

Subtracting (2) from (3), we have
$n - m = 6$ or $n = m + 6$.

Ex. 2 (a). *A heavy particle hanging vertically from a point by a light inextensible string of length l is started so as to make a complete revolution in a vertical plane. Prove that the sum of the tensions at the end of any diameter is constant.*

Sol. Proceeding as in § 2, the tension T in the string in any position is given by

$$T = \frac{m}{l}(u^2 - 2lg + 3lg \cos \theta), \quad \text{...(1)}$$

where θ is the angle which the string makes with OA.

Now take any diameter of the circle. If at one end of this diameter we have $\theta = \alpha$, then at the other end we shall have $\theta = \pi + \alpha$. Let T_1 and T_2 be the tensions at these ends *i.e.*, $T = T_1$ when $\theta = \alpha$ and $T = T_2$ when $\theta = \pi + \alpha$. Then from (1), we have

$$T_1 = \frac{m}{l}(u^2 - 2lg + 3lg \cos \alpha) \quad \text{...(2)}$$

and $$T_2 = \frac{m}{l}[u^2 - 2lg + 3lg \cos(\pi + \alpha)]$$

or $$T_2 = \frac{m}{l}(u^2 - 2lg - 3lg \cos \alpha). \quad \text{...(3)}$$

Adding (2) and (3), we have

$$T_1 + T_2 = 2\frac{m}{l}(u^2 - 2lg),$$

which is constant, as it is independent of α.

Hence the sum of the tensions at the ends of any diameter is constant.

Ex. 2 (b). *A particle makes complete revolutions in a vertical circle. If ω_1, ω_2 be the greatest and least angular velocities and R_1, R_2 the greatest and least reactions, prove that when the particle projected from the lowest point of the circle makes an angle θ at the centre its angular velocity is*

$$\sqrt{[\omega_1^2 \cos^2 \tfrac{1}{2}\theta + \omega_2^2 \sin^2 \tfrac{1}{2}\theta]}$$

and that the reaction is

$$R_1 \cos^2 \tfrac{1}{2}\theta + R_2 \sin^2 \tfrac{1}{2}\theta.$$

Sol. Proceed as in § 2. Replace the tension T by the reaction R.

Let u be the velocity of projection at the lowest point. For making complete circles, we must have $u^2 \geq 5ag$.

If v be the velocity of the particle at any time t, then proceeding as in § 2, we have

$$v^2 = \left(a \frac{d\theta}{dt}\right)^2 = u^2 - 2ag + 2ag \cos \theta. \quad \text{...(1)}$$

and $$R = \frac{m}{a}(u^2 - 2ag + 3ag \cos \theta). \quad \text{...(2)}$$

If ω be the angular velocity of the particle at time t, then $\omega = d\theta/dt$. So from (1), we have

$$a^2\omega^2 = u^2 - 2ag + 2ag \cos \theta. \quad \text{...(3)}$$

From the equation (3) we observe that the angular velocity ω is greatest when $\cos \theta = 1$ *i.e.*, $\theta = 0$ and is least when $\cos \theta = -1$ *i.e.*, $\theta = \pi$. So putting $\theta = 0$, $\omega = \omega_1$ and $\theta = \pi$, $\omega = \omega_2$ in (3), we get $a^2\omega_1^2 = u^2$ and $a^2\omega_2^2 = u^2 - 4ag$. ...(4)

Now from (3), we have

$a^2\omega^2 = u^2 - 2ag(1 - \cos \theta) = \frac{1}{2}[2u^2 - 4ag(1 - \cos \theta)]$

$= \frac{1}{2}[2u^2 - (u^2 - a^2\omega_2^2)(1 - \cos \theta)]$ [∵ from (4), $4ag = u^2 - a^2\omega_2^2$]

$= \frac{1}{2}[2u^2 - u^2(1 - \cos \theta) + a^2\omega_2^2(1 - \cos \theta)]$

$= \frac{1}{2}[u^2(1 + \cos \theta) + a^2\omega_2^2(1 - \cos \theta)]$

$= \frac{1}{2}[a^2\omega_1^2(1 + \cos \theta) + a^2\omega_2^2(1 - \cos \theta)]$ [∵ from (4), $u^2 = a^2\omega_1^2$]

$= \frac{1}{2}[2a^2\omega_1^2 \cos^2 \frac{1}{2}\theta + 2a^2\omega_2^2 \sin^2 \frac{1}{2}\theta]$.

∴ $\omega^2 = \omega_1^2 \cos^2 \frac{1}{2}\theta + \omega_2^2 \sin^2 \frac{1}{2}\theta$

or $\omega = \sqrt{[\omega_1^2 \cos^2 \frac{1}{2}\theta + \omega_2^2 \sin^2 \frac{1}{2}\theta]}$.

From the equation (2) we observe that the reaction R is greatest when $\cos \theta = 1$ *i.e.*, $\theta = 0$ and is least when $\cos \theta = -1$ *i.e.*, $\theta = \pi$. So putting $\theta = 0$, $R = R_1$ and $\theta = \pi$, $R = R_2$ in (2), we get $R_1 = (m/a)(u^2 + ag)$ and $R_2 = (m/a)(u^2 - 5ag)$. ...(5)

Now from (2), we have

$R = (m/a)[u^2 - 2ag + 3ag \cos \theta]$

$= \frac{1}{2}(m/a)[2u^2 - 4ag + 6ag \cos \theta]$

$= \frac{1}{2}(m/a)[(u^2 + ag)(1 + \cos \theta) + (u^2 - 5ag)(1 - \cos \theta)]$

$= \frac{1}{2}[R_1(1 + \cos \theta) + R_2(1 - \cos \theta)]$ [From (5)]

$= \frac{1}{2}[2R_1 \cos^2 \frac{1}{2}\theta + 2R_2 \sin^2 \frac{1}{2}\theta] = R_1 \cos^2 \frac{1}{2}\theta + R_2 \sin^2 \frac{1}{2}\theta$.

Ex. 3. *A heavy particle hangs from a fixed point O, by a string of length a. It is projected horizontally with a velocity $v^2 = (2 + \sqrt{3})\, ag$; show that the string becomes slack when it has described an angle $\cos^{-1}(-1/\sqrt{3})$.*

Sol. Refer fig. of § 2, page 156.

The equations of motion of the particle are

$$m \frac{d^2s}{dt^2} = -mg \sin \theta \quad \text{...(1)}$$

and $$m \frac{v^2}{a} = T - mg \cos \theta. \quad \text{...(2)}$$

Also $$s = a\theta. \quad \text{...(3)}$$

From (1) and (3), we have $a \frac{d^2\theta}{dt^2} = -g \sin \theta$.

Multiplying both sides by $2a\,(d\theta/dt)$ and then integrating w.r.t. t, we have $v^2 = \left(a \frac{d\theta}{dt}\right)^2 = 2ag \cos \theta + A$,

where A is the constant of integration.

But initially at A, $\theta = 0$ and $v^2 = (2 + \sqrt{3})\, ag$.

∴ $(2 + \sqrt{3})\, ag = 2ag \cos 0 + A$, giving $A = \sqrt{3}ag$.

∴ $v^2 = 2ag \cos \theta + \sqrt{3}ag$.

Substituting this value of v^2 in (2), we have

$$T = \frac{m}{a}[v^2 + ag \cos \theta]$$

$$= \frac{m}{a}[\sqrt{3}ag + 3ag \cos \theta]. \quad \text{...(4)}$$

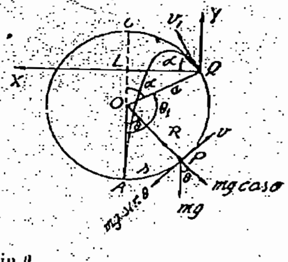
The string becomes slack when $T=0$.

$\therefore$ from (4), we have

$$0=\frac{m}{a}\left[\sqrt{3ag}+3ag\cos\theta\right]$$

or $\cos\theta=-1/\sqrt{3}$ or $\theta=\cos^{-1}(-1/\sqrt{3})$.

Ex. 4. *A particle inside and at the lowest point of a fixed smooth hollow sphere of radius a is projected horizontally with velocity $\sqrt{(\frac{7}{2}ag)}$. Show that it will leave the sphere at a height $\frac{3}{2}a$ above the lowest point and its subsequent path meets the sphere again at the point of projection.*



Sol. A particle is projected from the lowest point A of a sphere with velocity $u=\sqrt{(\frac{7}{2}ag)}$ to move along the inside of the sphere. Let P be the position of the particle at any time t where arc $AP=s$ and $\angle AOP=\theta$. If v be the velocity of the particle at P, the equations of motion along the tangent and normal are

$$m\frac{d^2s}{dt^2}=-mg\sin\theta \qquad \ldots(1)$$

and $$m\frac{v^2}{a}=R-mg\cos\theta. \qquad \ldots(2)$$

Also $$s=a\theta. \qquad \ldots(3)$$

From (1) and (3), we have $a\dfrac{d^2\theta}{dt^2}=-g\sin\theta$.

Multiplying both sides by $2a\dfrac{d\theta}{dt}$ and then integrating, we have

$$v^2=\left(a\frac{d\theta}{dt}\right)^2=2ag\cos\theta+A$$

But at the point A, $\theta=0$ and $v=u=\sqrt{(\frac{7}{2}ag)}$.

$\therefore$ $A=\frac{7}{2}ag-2ag=\frac{3}{2}ag$.

$\therefore$ $v^2=\frac{3}{2}ag+2ag\cos\theta$. $\qquad \ldots(4)$

Now from (2) and (4), we have

$$R=\frac{m}{a}\left[v^2+ag\cos\theta\right]=\frac{m}{a}\left[\frac{3}{2}ag+2ag\cos\theta+ag\cos\theta\right]$$

$$=3mg\left(\tfrac{1}{2}+\cos\theta\right).$$

If the particle leaves the sphere at the point Q, where $\theta=\theta_1$, then $0=3mg\left(\frac{1}{2}+\cos\theta_1\right)$ or $\cos\theta_1=-\frac{1}{2}$.

If $\angle COQ=\alpha$, then $\alpha=\pi-\theta_1$.

$$\therefore \cos\alpha=\cos(\pi-\theta_1)=-\cos\theta_1=\tfrac{1}{2}. \qquad \ldots(5)$$

$$\therefore AL=AO+OL=a+a\cos\alpha=a+\frac{a}{2}=\frac{3a}{2}$$

i.e., the particle leaves the sphere at a height $\frac{3}{2}a$ above the lowest point.

If v_1 is the velocity of the particle at the point Q, then putting $v=v_1$, $R=0$ and $\theta=\theta_1$ in (2), we get

$$v_1^2=-ag\cos\theta_1=-ag\left(-\tfrac{1}{2}\right)=\tfrac{1}{2}ag.$$

$\therefore$ the particle leaves the sphere at the point Q with velocity $v_1=\sqrt{(\frac{1}{2}ag)}$ making an angle α with the horizontal and subsequently describes a parabolic path.

The equation of the parabolic trajectory w.r.t. QX and QY as co-ordinate axes is

$$y=x\tan\alpha-\frac{1}{2}\frac{gx^2}{v_1^2\cos^2\alpha}$$

or $$y=x\sqrt{3}-\frac{gx^2}{2\cdot\frac{1}{2}ag\cdot\frac{1}{4}}$$

$\left[\because \cos\alpha=\frac{1}{2}\right.$ and so $\sin\alpha=\sqrt{(1-\cos^2\alpha)}=\sqrt{3}/2$. Thus $\left.\tan\alpha=\sqrt{3}.\right]$

or $$y=\sqrt{3}x-\frac{4x^2}{a}. \qquad \ldots(6)$$

From the figure, for the point A, $x=QL=a\sin\alpha=a\sqrt{3}/2$

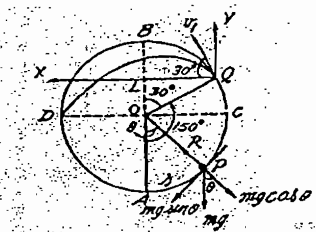
and $$y=-LA=-\tfrac{3}{2}a.$$

If we put $x=a\sqrt{3}/2$ in the equation (6), we get

$$y=a\frac{\sqrt{3}}{2}\cdot\sqrt{3}-\frac{4}{a}\cdot\frac{3a^2}{4}=\frac{3a}{2}-3a=-\frac{3}{2}a.$$

Thus the co-ordinates of the point A satisfy the equation (6). Hence the particle, after leaving the sphere at Q, describes a parabolic path which meets the sphere again at the point of projection A.

Ex. 5. *Find the velocity with which a particle must be projected along the interior of a smooth vertical hoop of radius a from the lowest point in order that it may leave the hoop at an angular distance of 30° from the vertical. Show that it will strike the hoop again at an extremity of the horizontal diameter.*



Sol. Let a particle of mass m be projected with velocity u from the lowest point A of a smooth circular hoop of radius a along the interior of the hoop. If P is its position at any time t such that $\angle AOP=\theta$ and arc $AP=s$, then the equations of motion along the tangent and normal are

$$m\frac{d^2s}{dt^2}=-mg\sin\theta \qquad \ldots(1)$$

and $$m\frac{v^2}{a}=R-mg\cos\theta \qquad \ldots(2)$$

Also $$s=a\theta. \qquad \ldots(3)$$

From (1) and (3), we have $a\dfrac{d^2\theta}{dt^2}=-g\sin\theta$.

Multiplying both sides by $2a\dfrac{d\theta}{dt}$ and then integrating, we have

$$v^2=\left(a\frac{d\theta}{dt}\right)^2=2ag\cos\theta+A.$$

But at the point A, $\theta=0$ and $v=u$. $\therefore$ $A=u^2-2ag$.

$$\therefore v^2=u^2-2ag+2ag\cos\theta. \qquad \ldots(4)$$

From (2) and (4), we have

$$R=\frac{m}{a}(v^2+ag\cos\theta)$$

$$=\frac{m}{a}(u^2-2ag+3ag\cos\theta).$$

If the particle leaves the circular hoop at the point Q where $\theta=150°$, then

$$0=\frac{m}{a}(u^2-2ag+3ag\cos 150°)$$

or $$0=u^2-2ag-\frac{3\sqrt{3}}{2}ag.$$

$$u=\left[\tfrac{1}{2}ag\,(4+3\sqrt{3})\right]^{1/2}.$$

Hence the particle will leave the circular hoop at an angular distance of 30° from the vertical if the initial velocity of projection is $u=\left[\frac{1}{2}ag\,(4+3\sqrt{3})\right]^{1/2}$.

Again $OL=OQ\cos 30°=a(\sqrt{3}/2)$ and $QL=OQ\sin 30°=a/2$.

If v_1 is the velocity of the particle at the point Q, then $v=v_1$ when $\theta=150°$. Therefore from (4), we have

$$v_1^2=\tfrac{1}{2}ag\,(4+3\sqrt{3})-2ag+2ag\cos 150°=\tfrac{1}{2}ag\sqrt{3}$$

so that $$v_1=\left(\tfrac{1}{2}ag\sqrt{3}\right)^{1/2}.$$

Thus the particle leaves the circular hoop at the point Q with velocity $v_1=\left(\frac{1}{2}\sqrt{3}ag\right)^{1/2}$ at an angle 30° to the horizontal and subsequently it describes a parabolic path.

The equation of the parabolic trajectory w.r.t. QX and QY as co-ordinate axes is

$$y=x\tan 30°-\frac{gx^2}{2v_1^2\cos^2 30°}=\frac{x}{\sqrt{3}}-\frac{gx^2}{2\cdot\frac{1}{2}\sqrt{3}ag\cdot(\sqrt{3}/2)^2}$$

or $$y=\frac{x}{\sqrt{3}}-\frac{4x^2}{3\sqrt{3}a} \qquad \ldots(5)$$

For the point D which is the extremity of the horizontal diameter CD, we have

$$x=QL+OD=\tfrac{1}{2}a+a=3a/2,\quad y=-LO=-a\sqrt{3}/2.$$

Clearly the co-ordinates of the point D satisfy the equation (5). Hence the particle after leaving the circular hoop at Q strikes the hoop again at an extremity of the horizontal diameter.

Ex. 6. *A particle is projected along the inner side of a smooth vertical circle of radius a, the velocity at the lowest point being u. Show that if $2ga<u^2<5ag$, the particle will leave the circle before arriving at the highest point, and will describe a parabola whose latus rectum is*

$$\frac{2\,(u^2-2ag)^3}{27a^2g^2}.$$

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Sol. For figure refer Ex. 5. Proceeding as in Ex. 5, the velocity v and the reaction R at any time t are given by

$$v^2=u^2-2ag+2ag\cos\theta, \qquad \ldots(1)$$

and $$R=\frac{m}{a}(u^2-2ag+3ag\cos\theta). \qquad \ldots(2)$$

If the particle leaves the circle at Q, where $\angle AOQ=\theta_1$, then from (2), we have

$$0=\frac{m}{a}(u^2-2ag+3ag\cos\theta_1).$$

or $\qquad \cos\theta_1 = -\dfrac{u^2-2ag}{3ag}$.

Since $2ag < u^2 < 5ag$, therefore $\cos\theta_1$ is negative and its absolute value is <1. Therefore θ_1 is real and $\frac{1}{2}\pi < \theta_1 < \pi$.

Thus the particle leaves the circle before arriving at the highest point. If v_1 is the velocity of the particle at the point Q, then $v = v_1$ when $\theta = \theta_1$. Therefore from (1), we have

$$v_1^2 = u^2 - 2ag + 2ag\cos\theta_1$$
$$= (u^2-2ag) - 2ag\left(\frac{u^2-2ag}{3ag}\right)$$
$$= (u^2-2ag)\left(1-\tfrac{2}{3}\right) = \tfrac{1}{3}(u^2-2ag).$$

If $\angle BOQ = \alpha$, then $\alpha = \pi - \theta_1$.

$$\therefore \cos\alpha = \cos(\pi-\theta_1) = -\cos\theta_1 = \frac{u^2-2ag}{3ag}.$$

Thus the particle leaves the circle at the point Q with velocity $v_1 = \sqrt{[\frac{1}{3}(u^2-2ag)]}$ at an angle α to the horizontal and subsequently it describes a parabolic path.

The latus rectum of the parabola

$$= \frac{2}{g} v_1^2 \cos^2\alpha = \frac{2}{g} \cdot \frac{1}{3}(u^2-2ag)\left(\frac{u^2-2ag}{3ag}\right)^2 = \frac{2(u^2-2ag)^3}{27a^2g^3}.$$

Ex. 7. *A heavy particle is attached to a fixed point by a fine string of length a; the particle is projected horizontally from the lowest point with velocity $\sqrt{[ag(2+3\sqrt{3}/2)]}$. Prove that the string would first become slack when inclined to the upward vertical at an angle of 30°, will become tight again when horizontal.*

Sol. Refer figure of Ex. 5 page 166. Taking $R = T$ (*i.e.*, the tension in the string), the equations of motion of the particle are

$$m\frac{d^2s}{dt^2} = -mg\sin\theta \qquad \text{...(1)}$$

and $\qquad m\dfrac{v^2}{a} = T - mg\cos\theta \qquad \text{...(2)}$

Also $\qquad s = a\theta$.

From (1) and (3), we have $a\dfrac{d^2\theta}{dt^2} = -g\sin\theta$.

Multiplying both sides by $2a\dfrac{d\theta}{dt}$ and integrating, we have

$$v^2 = \left(a\frac{d\theta}{dt}\right)^2 = 2ag\cos\theta + A.$$

But at the point A, $\theta = 0$ and $v = \sqrt{[ag(2+3\sqrt{3}/2)]}$.

$\therefore \quad ag(2+3\sqrt{3}/2) = 2ag + A \quad$ or $\quad A = \frac{3}{2}\sqrt{3}ag$.

$\therefore \quad v^2 = ag(2\cos\theta + \frac{3}{2}\sqrt{3}) \qquad \text{...(4)}$

From (2) and (4), we have

$$T = \frac{m}{a}(v^2 + ag\cos\theta) = \frac{m}{a}\left[ag(2\cos\theta + \tfrac{3}{2}\sqrt{3}) + ag\cos\theta\right]$$
$$= mg(3\cos\theta + \tfrac{3}{2}\sqrt{3}). \qquad \text{...(5)}$$

If the string becomes slack at the point Q, where $\theta = \theta_1$, then at Q, $T = 0 = mg(3\cos\theta_1 + \frac{3}{2}\sqrt{3})$

giving $\qquad \cos\theta_1 = -\sqrt{3}/2 \quad$ *i.e.*, $\quad \theta_1 = 150°$.

Hence the string becomes slack when inclined to the upward vertical at an angle of $180° - 150°$ *i.e.*, $30°$.

If v_1 is the velocity of the particle at Q, then $v = v_1$ when $\theta = 150°$. Therefore from (4), we have

$$v_1^2 = ag(2\cos 150° + \tfrac{3}{2}\sqrt{3}) = \tfrac{1}{2}\sqrt{3}ag.$$

Hence the particle leaves the circular path at the point Q with velocity $v_1 = (\frac{1}{2}ag\sqrt{3})^{1/2}$ at an angle of 30° to the horizontal and subsequently it describesia parabolic path.

The equation of the parabolic trajectory w.r.t. QX and QY as coordinate axes is

$$y = x\tan 30° - \frac{gx^2}{2v_1^2\cos^2 30°} = \frac{x}{\sqrt{3}} - \frac{gx^2}{2\cdot\frac{1}{2}\sqrt{3}ag\cdot(\sqrt{3}/2)^2}$$

or $\qquad y = \dfrac{x}{\sqrt{3}} - \dfrac{4x^2}{3\sqrt{3}a}. \qquad \text{...(6)}$

The co-ordinates of the point D, which is an extremity of the horizontal diameter CD, are given by

$x = QL - OD = \frac{1}{2}a + a = 3a/2 \quad$ and $\quad y = -LO = -a\sqrt{3}/2$.

Clearly the co-ordinates of the point D satisfy the equation (6) showing that the parabolic trajectory meets the circle again at D. When the particle is at D, the string again becomes tight because $OD = a$ = the length of the string.

Hence the string becomes slack when inclined to the upward vertical at an angle of 30° and becomes tight again when horizontal.

Ex. 8. *A heavy particle hanging vertically from a fixed point by a light inextensible cord of length l is struck by a horizontal blow which imparts it a velocity $2\sqrt{(gl)}$, prove that the cord becomes slack when the particle has risen to a height $\frac{2}{3}l$ above the fixed point. Also find the height of the highest point of the parabola subsequently described.*

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Sol. Refer figure of Ex. 4 page 164. Take $R = T$ (*i.e.*, the tension in the string).

Let a particle tied to a cord OA of length l be struck by a horizontal blow which imparts it a velocity $2\sqrt{(gl)}$. If P is the position of the particle at time t such that $\angle AOP = \theta$, then the equations of motion are

$$m\frac{d^2s}{dt^2} = -mg\sin\theta \qquad \text{...(1)}$$

and $\qquad m\dfrac{v^2}{l} = T - mg\cos\theta. \qquad \text{...(2)}$

Also $\qquad s = l\theta. \qquad \text{...(3)}$

From (1) and (3), we have $l\dfrac{d^2\theta}{dt^2} = -g\sin\theta$.

Multiplying both sides by $2l\dfrac{d\theta}{dt}$ and integrating, we have

$$v^2 = \left(l\frac{d\theta}{dt}\right)^2 = 2lg\cos\theta + A.$$

But at the point A, $\theta = 0$ and $v = 2\sqrt{(gl)}$.

$\therefore \quad 4gl = 2lg + A$ so that $A = 2gl$.

$\therefore \quad v^2 = 2lg(\cos\theta + 1). \qquad \text{...(4)}$

From (2) and (4), we have

$$T = \frac{m}{l}(v^2 + gl\cos\theta) = mg(3\cos\theta + 2). \qquad \text{...(5)}$$

If the cord becomes slack at the point Q, where $\theta = \theta_1$, then from (5), we have

$$T = 0 = mg(3\cos\theta_1 + 2)$$

giving $\cos\theta_1 = -2/3$.

If $\angle COQ = \alpha$, then $\alpha = \pi - \theta_1$ and $\cos\alpha = 2/3$.

If v_1 is the velocity of the particle at Q, then $v = v_1$ where $\theta = \theta_1$. Therefore from (4), we have

$$v_1^2 = 2lg(1 + \cos\theta_1) = 2lg(1 - \tfrac{2}{3}) = 2lg/3.$$

Now $\quad OL = l\cos\alpha = \frac{2}{3}l$.

Thus the particle leaves the circular path at the point Q at a height $2l/3$ above the fixed point O with velocity $v_1 = \sqrt{(2lg/3)}$ at an angle α to the horizontal and subsequently it describes a parabolic path.

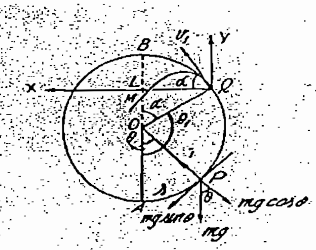
Max. height H of the particle above Q

$$= \frac{v_1^2\sin^2\alpha}{2g} = \frac{v_1^2}{2g}(1-\cos^2\alpha) = \frac{\frac{2}{3}lg}{2g}\left(1 - \frac{4}{9}\right) = \frac{5l}{27}.$$

$\therefore$ Height of the highest point of the parabolic path above the fixed point $O = OL + H = \dfrac{2}{3}l + \dfrac{5l}{27} = \dfrac{23l}{27}$.

Ex. 9. *A heavy particle hangs by an inextensible string of length a from a fixed point and is then projected horizontally with a velocity $\sqrt{(2gh)}$. If $\dfrac{5a}{2} > h > a$, prove that the circular motion ceases when the particle has reached the height $\frac{1}{3}(a+2h)$. Prove also that the greatest height ever reached by the particle above the point of projection is $\dfrac{(4a-h)(a+2h)^2}{27a^2}$.*

Sol. Let a particle of mass m be attached to one end of a string of length a whose other end is fixed at O. The particle is projected horizontally with a velocity $u = \sqrt{(2gh)}$ from A. If P is the position of the particle at time t such that $\angle AOP = \theta$ and arc $AP = s$, then the equations of motion of the particle are



$$m\frac{d^2s}{dt^2} = -mg\sin\theta \qquad \text{...(1)}$$

and $\qquad m\dfrac{v^2}{a} = T - mg\cos\theta \qquad \text{...(2)}$

Also $\qquad s = a\theta. \qquad \text{...(3)}$

From (1) and (3), we have $a\dfrac{d^2\theta}{dt^2} = -g\sin\theta$.

Multiplying both sides by $2a\dfrac{d\theta}{dt}$ and integrating, we have

$$v^2 = \left(a\frac{d\theta}{dt}\right)^2 = 2ag\cos\theta + A.$$

But at the point A, $\theta = 0$, and $v = u = \sqrt{(2gh)}$.

$\therefore \quad A = 2gh - 2ag$.

$\therefore \quad v^2 = 2ag\cos\theta + 2gh - 2ag.$ ...(4)

From (2) and (4), we have

$$T = \frac{m}{a}(v^2 + ag\cos\theta) = \frac{m}{a}(3ag\cos\theta + 2gh - 2ag).$$

If the particle leaves the circular path at Q where $\theta = \theta_1$, then $T = 0$ when $\theta = \theta_1$.

$$\therefore \quad 0 = \frac{m}{a}(3ag\cos\theta_1 + 2gh - 2ag) \quad \text{or} \quad \cos\theta_1 = -\frac{2h-2a}{3a}.$$

Since $\frac{5}{2}a > h > a$ *i.e.*, $5a > 2h > 2a$, therefore $\cos\theta_1$ is negative and its absolute value is < 1. So θ_1 is real and $\frac{1}{2}\pi < \theta_1 < \pi$.

Thus the particle leaves the circular path at Q before arriving at the highest point.

Height of the point Q above A

$$= AL = AO + OL = a + a\cos(\pi - \theta_1) = a - a\cos\theta_1$$

$$= a + a\cdot\frac{2h-2a}{3a} = \tfrac{1}{3}(a + 2h)$$

i.e., the particle leaves the circular path when it has reached a height $\frac{1}{3}(a+2h)$ above the point of projection.

If v_1 is the velocity of the particle at the point Q, then from (4), we have

$$v_1^2 = 2ag\cos\theta_1 + 2gh - 2ag$$

$$= -2ag\cdot\frac{(2h-2a)}{3a} + 2g(h-a)$$

$$= 2g(h-a)(1 - \tfrac{2}{3}) = \tfrac{2}{3}g(h-a).$$

If $\angle LOQ = \alpha$, then $\alpha = \pi - \theta_1$.

$$\therefore \quad \cos\alpha = \cos(\pi - \theta_1) = -\cos\theta_1 = \frac{2(h-a)}{3a}.$$

Thus the particle leaves the circular path at the point Q with velocity $v_1 = \sqrt{\{\frac{2}{3}g(h-a)\}}$ at an angle $\alpha = \cos^{-1}\{2(h-a)/3a\}$ to the horizontal and will subsequently describe a parabolic path.

Maximum height of the particle above the point Q

$$= H = \frac{v_1^2\sin^2\alpha}{2g} = \frac{v_1^2}{2g}(1 - \cos^2\alpha) = \tfrac{1}{3}(h-a)\left[1 - \frac{4}{9a^2}(h-a)^2\right]$$

$$= \frac{1}{27a^2}(h-a)[9a^2 - 4(h^2 - 2ah + a^2)]$$

$$= \frac{(h-a)}{27a^2}\left[5a^2 + 8ah - 4h^2\right] = \frac{1}{27a^2}(h-a)(a+2h)(5a-2h).$$

$\therefore$ Greatest height ever reached by the particle above the point of projection A

$$= AL + H = \tfrac{1}{3}(a+2h) + \frac{1}{27a^2}(h-a)(a+2h)(5a-2h)$$

$$= \frac{1}{27a^2}(a+2h)[9a^2 + (h-a)(5a-2h)]$$

$$= \frac{1}{27a^2}(a+2h)[4a^2 + 7ah - 2h^2]$$

$$= \frac{1}{27a^2}(a+2h)(a+2h)(4a-h) = \frac{1}{27a^2}(4a-h)(a+2h)^2.$$

Ex. 10. *A particle is projected along the inside of a smooth fixed sphere, from its lowest point, with a velocity equal to that due to falling freely down the vertical diameter of the sphere. Show that the particle will leave the sphere and afterwards pass vertically over the point of projection at a distance equal to $\frac{25}{32}$ of the diameter.*

Sol. Refer figure of Ex. 9, page 171. Replace T by R (*i.e.*, reaction).

Here the velocity of projection $u = \sqrt{(2g\cdot 2a)} = \sqrt{(4ag)}$ *i.e.*, the particle is projected from the lowest point A with velocity $u = 2\sqrt{(ag)}$ inside a smooth sphere of radius a. If P is the position of the particle at time t such that $\angle AOP = \theta$, then the equations of motion are

$$m\frac{d^2s}{dt^2} = -mg\sin\theta \qquad ...(1)$$

and
$$m\frac{v^2}{a} = R - mg\cos\theta. \qquad ...(2)$$

Also $\quad s = a\theta.$ ...(3)

From (1) and (3), we have $a\dfrac{d^2\theta}{dt^2} = -g\sin\theta.$

Multiplying both sides by $2a\dfrac{d\theta}{dt}$ and integrating, we have

$$v^2 = \left(a\frac{d\theta}{dt}\right)^2 = 2ag\cos\theta + A.$$

But at the lowest point A, $\theta = 0$ and $v = 2\sqrt{(ag)}$.

$\therefore \quad A = 4ag - 2ag = 2ag.$

$\therefore \quad v^2 = 2ag\cos\theta + 2ag.$ ...(4)

From (2) and (4), we have

$$R = \frac{m}{a}(ag\cos\theta + v^2)$$

$$= \frac{m}{a}(3ag\cos\theta + 2ag). \qquad ...(5)$$

Here $2ag < u^2 < 5ag$, therefore the particle will leave the sphere at an angle θ_1 where $\pi/2 < \theta_1 < \pi$.

If the particle leaves the sphere at the point Q, where $\theta = \theta_1$, then from (5), we have

$$R = 0 = \frac{m}{a}(3ag\cos\theta_1 + 2ag) \text{ giving } \cos\theta_1 = -2/3.$$

If v_1 is the velocity of the particle at Q, then from (4), we have

$$v_1^2 = 2ag\cos\theta_1 + 2ag = 2ag(\cos\theta_1 + 1)$$

or $\quad v_1^2 = 2ag(-\frac{2}{3} + 1) = \frac{2}{3}ag.$

If $\angle BOQ = \alpha$, then $\alpha = \pi - \theta_1$.

$\therefore \quad \cos\alpha = \cos(\pi - \theta_1) = -\cos\theta_1 = \frac{2}{3}.$

Hence the particle leaves the spere at the point Q with velocity $v_1 = \sqrt{(\frac{2}{3}ag)}$ at an angle $\alpha = \cos^{-1}(\frac{2}{3})$ to the horizontal and subsequently it describes a parabolic path.

Equation of the trajectory described by the particle after leaving the sphere at Q w.r.t. QX and QY as co-ordinate axes is

$$y = x\tan\alpha - \frac{gx^2}{2v_1^2\cos^2\alpha}$$

or
$$y = x\cdot\frac{\sqrt{5}}{2} - \frac{gx^2}{2\cdot\frac{2}{3}ag\cdot\frac{4}{9}}$$

[$\because \cos\alpha = \frac{2}{3}$, $\sin\alpha = \sqrt{(1 - \cos^2\alpha)} = \sqrt{5}/3$ and $\tan\alpha = \sin\alpha/\cos\alpha = \sqrt{5}/2$]

or
$$y = \frac{\sqrt{5}}{2}x - \frac{27}{16a}x^2. \qquad ...(6)$$

If the particle passes vertically over the point of projection A at the point M, then the x-co-ordinate of M is given by $x = QL = a\sin\alpha = a\sqrt{5}/3$. Let the y-coordinate of M be y_1.

The point M *i.e.*, $(a\sqrt{5}/3, y_1)$ lies on the trajectory (6).

$$\therefore \quad y_1 = \frac{\sqrt{5}}{2}\cdot\frac{\sqrt{5}}{3}a - \frac{27}{16a}\cdot\frac{5a^2}{9} = \frac{5a}{6} - \frac{15a}{16} = -\frac{5a}{48}.$$

Since the y-coordinate of M is negative, therefore the point M is below the x-axis QX.

The required height $= AM = AO + OL + y_1 = a + a\cos\alpha + y_1$

$$= a + \frac{2}{3}a - \frac{5a}{48} = \frac{25a}{16} = \frac{25}{32}(2a).$$

Hence the required height is equal to $\frac{25}{32}$ of the diameter of the sphere.

Ex. 11. *A particle is projected from the lowest point inside a smooth circle of radius a with a velocity due to a height h above the centre. Find the point where it leaves the circle and show that it will afterwards pass through*

(a) the centre if $h = \frac{1}{2}(a\sqrt{3})$.

and (b) the lowest point if $h = 3a/4$.

Sol. Refer figure of Ex. 9 on page 171. Take $T = R$ (*i.e.*, reaction).

Here the velocity of projection u is equal to that due to a height h above the centre *i.e.*, due to a height $(h + a)$ above the lowest point A.

$\therefore \quad u = \sqrt{\{2g(h + a)\}}.$

Let the particle be projected from the lowest point A with velocity u along the inside of a smooth circle of radius a. If P is its position at time t such that $\angle AOP = \theta$ and arc $AP = s$, then the equations of motion along the tangent and normal are

$$m\frac{d^2s}{dt^2} = -mg\sin\theta. \qquad ...(1)$$

and
$$m\frac{v^2}{a} = R - mg\cos\theta. \qquad ...(2)$$

Also $\quad s = a\theta.$ ...(3)

From (1) and (3), we have $a\dfrac{d^2\theta}{dt^2} = -g\sin\theta.$

Multiplying both sides by $2a(d\theta/dt)$ and integrating, we have

$$v^2 = \left(a\frac{d\theta}{dt}\right)^2 = 2ag\cos\theta + A.$$

But at the point A, $\theta = 0$ and $v^2 = u^2 = 2g(h + a)$.

$\therefore \quad A = 2g(h + a) - 2ag = 2gh.$

$\therefore \quad v^2 = 2ag\cos\theta + 2gh.$ ...(4)

From (2), we have

$$R = \frac{m}{a}(v^2 + ag\cos\theta)$$

$$= \frac{m}{a}(3ag\cos\theta + 2gh). \qquad ...(5)$$

If the particle leaves the circle at the point Q, where $\theta=\theta_1$, then from (5), we have

$$R=0=\frac{m}{a}(3ag\cos\theta_1+2h)$$

giving $\cos\theta_1=-\frac{2h}{3a}$.

If v_1 is the velocity of the particle at Q, then from (4), we have $v_1^2=2ag\cos\theta_1+2gh=2ag\left(\frac{-2h}{3a}\right)+2gh=\frac{2}{3}gh$.

If $\angle BOQ=\alpha$, then $\alpha=\pi-\theta_1$,

$\therefore$ $\cos\alpha=\cos(\pi-\theta_1)=-\cos\theta_1=(2h/3a)$,

and $OL=a\cos\alpha=2h/3$.

Hence the particle leaves the circle at the point Q at height $2h/3$ above the centre O with velocity $v_1=\sqrt{(2gh/3)}$ at an angle $\alpha=\cos^{-1}(2h/3a)$ to the horizontal and then it describes a parabolic path.

Equation of the trajectory of the parabola described by the particle after leaving the circle at Q w.r.t. QX and QY as co-ordinate axes is

$$y=x\tan\alpha-\frac{gx^2}{2v_1^2\cos^2\alpha}$$

or $$y=x\tan\alpha-\frac{gx^2}{2.\frac{2}{3}gh\cos^2\alpha}$$

or $$y=x\tan\alpha-\frac{3x^2}{4h\cos^2\alpha} \qquad ...(6)$$

(a) The co-ordinates of the centre O w.r.t. QX and QY as co-ordinate axes are given by

$$x=QL=a\sin\alpha \text{ and } y=-OL=-a\cos\alpha.$$

If the particle passes through the centre O i.e., the point $(a\sin\alpha, -a\cos\alpha)$, then the point O will lie on the curve (6).

$$\therefore\quad -a\cos\alpha=a\sin\alpha.\tan\alpha-\frac{3a^2\sin^2\alpha}{4h\cos^2\alpha}$$

or $$\frac{3a\sin^2\alpha}{4h\cos^2\alpha}=\frac{\sin^2\alpha}{\cos\alpha}+\cos\alpha=\frac{\sin^2\alpha+\cos^2\alpha}{\cos\alpha}=\frac{1}{\cos\alpha}$$

or $$3a\sin^2\alpha=4h\cos\alpha$$

or $$3a(1-\cos^2\alpha)=4h\cos\alpha$$

or $$3a\left(1-\frac{4h^2}{9a^2}\right)=4h.\frac{2h}{3a} \qquad \left[\because \cos\alpha=\frac{2h}{3a}\right]$$

or $$3a=\frac{h^2}{a}\left(\frac{8}{3}+\frac{4}{3}\right)=\frac{4h^2}{a}$$

or $$h^2=\tfrac{3}{4}a^2.$$

$\therefore$ $h=\frac{1}{2}(a\sqrt{3})$.

(b) The co-ordinates of the lowest point A w.r.t. QX and QY as co-ordinate axes are given by $x=QL=a\sin\alpha$

and $y=-LA=-(LO+OA)$
$=-(a\cos\alpha+a)=-a(\cos\alpha+1)$.

If the particle after leaving the circle at Q passes through the lowest point $A[a\sin\alpha, -a(\cos\alpha+1)]$, then the point A will lie on (6).

$$\therefore\quad -a(\cos\alpha+1)=a\sin\alpha\tan\alpha-\frac{3a^2\sin^2\alpha}{4h\cos^2\alpha}$$

or $$\frac{3a\sin^2\alpha}{4h\cos^2\alpha}=\frac{\sin^2\alpha}{\cos\alpha}+\cos\alpha+1$$
$$=\frac{\sin^2\alpha+\cos^2\alpha+\cos\alpha}{\cos\alpha}=\frac{1+\cos\alpha}{\cos\alpha}$$

or $3a\sin^2\alpha=4h\cos\alpha(1+\cos\alpha)$

or $3a(1-\cos^2\alpha)=4h\cos\alpha(1+\cos\alpha)$

or $3a(1-\cos\alpha)(1+\cos\alpha)=4h\cos\alpha(1+\cos\alpha)$

or $3a(1-\cos\alpha)=4h\cos\alpha \qquad [\because 1+\cos\alpha\neq 0]$

or $$3a\left(1-\frac{2h}{3a}\right)=4h\frac{2h}{3a} \qquad \left[\because \cos\alpha=\frac{2h}{3a}\right]$$

or $3a(3a-2h)=8h^2$ or $9a^2-6ah-8h^2=0$

or $(3a+2h)(3a-4h)=0$.

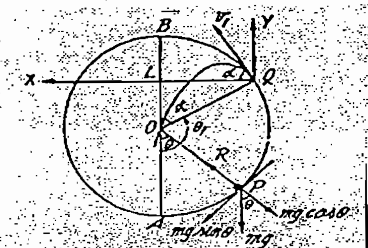
$\therefore$ $3a-4h=0$. $\qquad [\because 3a+2h\neq 0]$

or $h=3a/4$.

Ex. 12 *A particle is projected along the inside of a smooth vertical circle of radius a from the lowest point. Show that the velocity of projection required in order that after leaving the circle, the particle may pass through the centre is* $\sqrt{(\frac{1}{2}ag)}.(\sqrt{3}+1)$.

Sol. Let the particle be projected from the lowest point A along the inside of a smooth vertical circle of radius a, with velocity u. If P is the position of the particle at time t such that $\angle AOP=\theta$ and arc $AP=s$, the equations of motion of the particle along the tangent and normal are

$$m\frac{d^2s}{dt^2}=-mg\sin\theta, \qquad ...(1)$$



and $$m\frac{v^2}{a}=R-mg\cos\theta. \qquad ...(2)$$

Also $s=a\theta$. $\qquad ...(3)$

From (1) and (3), we have $a\frac{d^2\theta}{dt^2}=-g\sin\theta$.

Multiplying both sides by $2a\frac{d\theta}{dt}$ and integrating, we have

$$v^2=\left(a\frac{d\theta}{dt}\right)^2=2ag\cos\theta+A.$$

But at the lowest point A, $\theta=0$ and $v=u$ $\therefore$ $A=u^2-2ag$.

$\therefore$ $v^2=2ag\cos\theta+u^2-2ga$. $\qquad ...(4)$

From (2) and (4), we have

$$R=\frac{m}{a}(v^2+ag\cos\theta)=\frac{m}{a}(u^2-2ag+3ag\cos\theta). \qquad ...(5)$$

If the particle leaves the circle at Q, where $\theta=\theta_1$, then from (5),

$$0=\frac{m}{a}(u^2-2ag+3ag\cos\theta_1)$$

or $$\cos\theta_1=-\left(\frac{u^2-2ag}{3ag}\right).$$

If $\angle BOQ=\alpha$, then $\alpha=\pi-\theta_1$.

$$\therefore\quad \cos\alpha=\cos(\pi-\theta_1)=-\cos\theta_1=\frac{u^2-2ag}{3ag}$$

If v_1 is the velocity at Q, then putting $v=v_1$, $R=0$ and $\theta=\theta_1$ in (2), we have

$$v_1^2=-ag\cos\theta_1=-ag\cos(\pi-\alpha)=ag\cos\alpha.$$

Thus the particle leaves the circle at Q with velocity $v_1=\sqrt{(ag\cos\alpha)}$ at angle $\alpha=\cos^{-1}\left(\frac{u^2-2ag}{3ag}\right)$ to the horizontal and subsequently it describes a parabolic path.

The equation of the parabolic trajectory w.r.t QX and QY as co-ordinate axes is

$$y=x\tan\alpha-\frac{gx^2}{2v_1^2\cos^2\alpha}=x\tan\alpha-\frac{gx^2}{2ag\cos^3\alpha} \qquad ...(6)$$
$$[\because v_1^2=ag\cos\alpha]$$

The coordinates of the centre O w.r.t. QX and QY as coordinate axes are given by

$$x=QL=a\sin\alpha \text{ and } y=-LO=-a\cos\alpha.$$

If after leaving the circle at Q the particle passes through the centre O $(a\sin\alpha, -a\cos\alpha)$, then the point O lies on the curve (6).

$$\therefore\quad -a\cos\alpha=a\sin\alpha\tan\alpha-\frac{ga^2\sin^2\alpha}{2ag\cos^3\alpha}$$

or $$\frac{\sin^2\alpha}{2\cos^3\alpha}=\frac{\sin^2\alpha}{\cos\alpha}+\cos\alpha=\frac{\sin^2\alpha+\cos^2\alpha}{\cos\alpha}=\frac{1}{\cos\alpha}$$

or $\sin^2\alpha=2\cos^2\alpha$ or $1-\cos^2\alpha=2\cos^2\alpha$ or $3\cos^2\alpha=1$

or $\cos^2\alpha=1/3$ or $\cos\alpha=1/\sqrt{3}$.

$$\therefore\quad \frac{u^2-2ag}{3ag}=\frac{1}{\sqrt{3}} \qquad \left[\because \cos\alpha=\frac{u^2-2ag}{3ag}\right]$$

or $u^2-2ag=\sqrt{3}ag$

or $$u^2=(2+\sqrt{3})ag=\left(\frac{4+2\sqrt{3}}{2}\right)ag=\frac{ag}{2}(1+\sqrt{3})^2.$$

$\therefore$ $u=\sqrt{(\frac{1}{2}ag)}(\sqrt{3}+1)$.

Thus the particle will pass through the centre if the velocity of projection at the lowest point is $\sqrt{(\frac{1}{2}ag)}(\sqrt{3}+1)$.

Ex. 13 *A particle tied to a string of length a is projected from its lowest point, so that after leaving the circular path it describes a free path passing through the lowest point. Prove that the velocity of projection is* $\sqrt{(\frac{7}{2}ag)}$.

Sol. Refer figure of Ex. 12, page 178. Take $R=T$ (i.e., the tension in the string).

Let a particle of mass m be attached to one end A of the string OA whose other end is fixed at O. Let the particle be projected from the lowest point A with velocity u. If the particle leaves the circular path at Q with velocity v_1 at an angle α to the horizontal, then proceed as in Ex. 12 to get

$$v_1=\sqrt{(ag\cos\alpha)} \quad \text{and} \quad \cos\alpha=\left(\frac{u^2-2ag}{3ag}\right).$$

After Q the particle describes a parabolic path whose equation w.r.t. the horizontal and vertical lines QX and QY as co-ordinate axes is

$$y=x\tan\alpha-\frac{gx^2}{v_1^2\cos^2\alpha}=x\tan\alpha-\frac{gx^2}{2ag\cos^3\alpha} \quad ...(1)$$
$$[\because \; v_1^2=ag\cos\alpha]$$

The co-ordinates of the lowest point A w.r.t. QX and QY as co-ordinate axes are given by

$x=QL=a\sin\alpha$ and $y=-LA=-(LO+OA)$
$=-(a\cos\alpha+a)=-a(\cos\alpha+1).$

If the particle passes through the lowest point $A[a\sin\alpha, -a(\cos\alpha+1)]$, then the point A lies on the curve (1).

$$\therefore \; -a(\cos\alpha+1)=a\sin\alpha\tan\alpha-\frac{ga^2\sin^2\alpha}{2ag\cos^3\alpha}$$

or $\quad \dfrac{\sin^2\alpha}{2\cos^3\alpha}=\dfrac{\sin^2\alpha}{\cos\alpha}+\cos\alpha+1$

$$=\frac{\sin^2\alpha+\cos^2\alpha+\cos\alpha}{\cos\alpha}=\frac{1+\cos\alpha}{\cos\alpha}$$

or $\quad \sin^2\alpha=2\cos^2\alpha(1+\cos\alpha)$

or $\quad (1-\cos^2\alpha)=2\cos^2\alpha(1+\cos\alpha)$

or $\quad (1-\cos\alpha)(1+\cos\alpha)=2\cos^2\alpha(1+\cos\alpha)$

or $\quad 1-\cos\alpha=2\cos^2\alpha \quad [\because \; 1+\cos\alpha\neq0]$

or $\quad 2\cos^2\alpha+\cos\alpha-1=0$ or $(2\cos\alpha-1)(\cos\alpha+1)=0$

or $\quad 2\cos\alpha-1=0 \quad [\because \; \cos\alpha+1\neq0]$

or $\quad \cos\alpha=\frac{1}{2}$

or $\quad \dfrac{u^2-2ag}{3ag}=\dfrac{1}{2} \quad \left[\because \; \cos\alpha=\dfrac{u^2-2ag}{3ag}\right]$

or $\quad u^2=2ag+\dfrac{3}{2}ag=\dfrac{7}{2}ag$ or $u=\sqrt{\left(\dfrac{7}{2}ag\right)}.$

Ex. 14. *Show that the greatest angle through which a person can oscillate on a swing, the ropes of which can support twice the person's weight at rest is 120°.*

If the ropes are strong enough and he can swing through 180° and if v is his speed at any point, prove that the tension in the rope at that point is $\frac{3mv^2}{2l}$ where m is the mass of the person and l the length of the rope.

Sol. Let u be the velocity of a person of mass m at the lowest point. If v is the velocity of the person and T the tension in the rope of length l at a point P at an angular distance θ from the lowest point, then proceed as in § 2 to get

$$v^2=u^2-2lg+2lg\cos\theta \quad ...(1)$$

and $\quad T=\dfrac{m}{l}(u^2-2lg+3lg\cos\theta). \quad ...(2)$

Now according to the question, the ropes can support twice the person's weight at rest. Therefore the maximum tension the rope can bear is $2mg$. So for the greatest angle through which the person can oscillate, the velocity u at the lowest point should be such that $T=2mg$ when $\theta=0$.

Then from (2), we have

$$2mg=\frac{m}{l}(u^2-2lg+3lg\cos0)$$

or $\quad 2gl=u^2-2lg+3lg \quad$ or $\quad u^2=lg.$

Now from (1), we have

$$v^2=lg-2lg+2lg\cos\theta=2lg\cos\theta-lg=lg(2\cos\theta-1).$$

If $v=0$ at $\theta=\theta_1$, then $\quad 0=gl(2\cos\theta_1-1)$

or $\quad \cos\theta_1=\frac{1}{2}$. Therefore $\theta_1=60°$.

Thus the person can swing through an angle of 60° from the vertical on one side of the lowest point. Hence the person can oscillate through an angle of $60°+60°=120°$.

Second part. If the rope is strong enough and the person can swing through an angle of 180° *i.e.*, through an angle of 90° on one side of the lowest point, then $\quad v=0$, at $\theta=90°$.

$\therefore$ from (1), we have

$$0=u^2-2lg+2lg\cos90° \quad \text{or} \quad u^2=2lg.$$

Thus if the person's velocity at the lowest point is $\sqrt{(2lg)}$, then he can swing through an angle of 180°.

Then from (1), we have $\quad v^2=2lg-2lg+2lg\cos\theta$

or $\quad \cos\theta=\dfrac{v^2}{2lg}.$

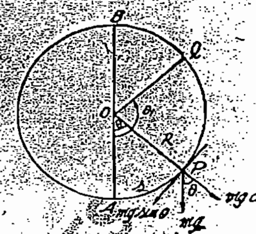
Therefore from (2), the tension in the rope at an angular distance θ where the velocity is v, is given by

$$T=\frac{m}{l}\left[2lg-2lg+3lg\cdot\frac{v^2}{2lg}\right]=\frac{3mv^2}{2l}.$$

Ex. 15. *A particle is free to move on a smooth vertical circular wire of radius a. It is projected from the lowest point with velocity just sufficient to carry it to the highest point. Show that the reaction between the particle and the wire is zero after a time* IAS-2006

$$\sqrt{(a/g)}\log(\sqrt{5}+\sqrt{6}).$$

Sol. Let a particle of mass m be projected from the lowest point A of a vertical circle of radius a with velocity u which is just sufficient to carry it to the highest point B.



If P is the position of the particle at any time t such that $\angle AOP=\theta$ and arc $AP=s$, then the equations of motion of the particle along the tangent and normal are

$$m\frac{d^2s}{dt^2}=-mg\sin\theta \quad ...(1)$$

and $\quad m\dfrac{v^2}{a}=R-mg\cos\theta \quad ...(2)$

Also $\quad s=a\theta. \quad ...(3)$

From (1) and (3), we have $a\dfrac{d^2\theta}{dt^2}=-g\sin\theta.$

Multiplying both sides by $2a(d\theta/dt)$ and integrating, we have

$$v^2=\left(a\frac{d\theta}{dt}\right)^2=2ag\cos\theta+A.$$

But according to the question $v=0$ at the highest point B, where $\theta=\pi$. $\therefore \; 0=2ag\cos\pi+A \quad$ or $\quad A=2ag.$

$$\therefore \; v^2=\left(a\frac{d\theta}{dt}\right)^2=2ag\cos\theta+2ag. \quad ...(4)$$

From (2) and (4), we have

$$R=\frac{m}{a}(v^2+ag\cos\theta)=\frac{m}{a}(2ag+3ag\cos\theta). \quad ...(5)$$

If the reaction $R=0$ at the point Q where $\theta=\theta_1$, then from (5), we have

$$0=\frac{m}{a}(2ag+3ag\cos\theta_1)$$

or $\quad \cos\theta_1=-2/3. \quad ...(6)$

From (4), we have

$$\left(a\frac{d\theta}{dt}\right)^2=2ag(\cos\theta+1)=2ag\cdot2\cos^2\tfrac{1}{2}\theta=4ag\cos^2\tfrac{1}{2}\theta.$$

$\therefore \; \dfrac{d\theta}{dt}=2\sqrt{(g/a)}\cos\frac{1}{2}\theta$, the positive sign being taken before the radical sign because θ increases as t increases

or $\quad dt=\frac{1}{2}\sqrt{(a/g)}\sec\frac{1}{2}\theta\, d\theta.$

Integrating from $\theta=0$ to $\theta=\theta_1$, the required time t is given by

$$t=\tfrac{1}{2}\sqrt{(a/g)}\int_{\theta=0}^{\theta_1}\sec\tfrac{1}{2}\theta\, d\theta$$

or $\quad t=\sqrt{(a/g)}\left[\log(\sec\frac{1}{2}\theta+\tan\frac{1}{2}\theta)\right]_0^{\theta_1}$

or $\quad t=\sqrt{(a/g)}\log(\sec\frac{1}{2}\theta_1+\tan\frac{1}{2}\theta_1). \quad ...(7)$

From (6), we have

$$2\cos^2\tfrac{1}{2}\theta_1-1=-\tfrac{2}{3}$$

or $\quad 2\cos^2\frac{1}{2}\theta_1=1-\frac{2}{3}=\frac{1}{3}$

or $\quad \cos^2\frac{1}{2}\theta_1=\frac{1}{6}$ or $\sec^2\frac{1}{2}\theta_1=6.$

$\therefore \; \sec\frac{1}{2}\theta_1=\sqrt{6}$

and $\quad \tan\frac{1}{2}\theta_1=\sqrt{(\sec^2\frac{1}{2}\theta_1-1)}=\sqrt{(6-1)}=\sqrt{5}.$

Substituting in (7), the required time is given by

$$t=\sqrt{(a/g)}\log(\sqrt{6}+\sqrt{5}).$$

Ex. 16. *A heavy bead slides on a smooth circular wire of radius a. It is projected from the lowest point with a velocity just sufficient to carry it to the heighest point, prove that the radius through the bead in time t will turn through an angle*

$$2\tan^{-1}[\sinh\{t\sqrt{(g/a)}\}]$$

and that the bead will take an infinite time to reach the highest point.

Sol. Refer figure of Ex. 15 page 182.

$\therefore$ The equations of motion of the bead are

$$m\frac{d^2s}{dt^2}=-mg\sin\theta, \quad ...(1)$$

and $$m\frac{v^2}{a}=R-mg\cos\theta. \quad ...(2)$$

Also $s=a\theta$. ...(3)

From (1) and (3), we have $a\dfrac{d^2\theta}{dt^2}=-g\sin\theta$.

Multiplying both sides by $2a(d\theta/dt)$ and integrating, we have

$$v^2=\left(a\frac{d\theta}{dt}\right)^2=2ag\cos\theta+A.$$

But according to the question at the highest point $v=0$.

i.e., when $\theta=\pi$, $v=0$.

$\therefore$ $0=2ag\cos\pi+A$, or $A=2ag$.

$$\therefore v^2=\left(a\frac{d\theta}{dt}\right)^2=2ag+2ag\cos\theta=2ag(1+\cos\theta)$$

$$=2ag.2\cos^2\tfrac{1}{2}\theta$$

or $$a\frac{d\theta}{dt}=2\sqrt{(ag)}\cos\tfrac{1}{2}\theta$$

or $$dt=\tfrac{1}{2}\sqrt{(a/g)}\sec\tfrac{1}{2}\theta\,d\theta.$$

Integrating, the time t from A to P is given by

$$t=\tfrac{1}{2}\sqrt{(a/g)}\int_0^{\theta}\sec\tfrac{1}{2}\theta\,d\theta$$

$$=\tfrac{1}{2}\sqrt{(a/g)}.2\left[\log(\tan\tfrac{1}{2}\theta+\sec\tfrac{1}{2}\theta)\right]_0^{\theta}$$

$$=\sqrt{(a/g)}\,[\log(\tan\tfrac{1}{2}\theta+\sec\tfrac{1}{2}\theta)-\log 1]$$

$$=\sqrt{(a/g)}\,[\log\{\tan\tfrac{1}{2}\theta+\sqrt{(1+\tan^2\tfrac{1}{2}\theta)}\}]$$

$$=\sqrt{(a/g)}.\sinh^{-1}(\tan\tfrac{1}{2}\theta)$$

$$[\because \sinh^{-1}x=\log\{x+\sqrt{(1+x^2)}\}]$$

or $$t\sqrt{(g/a)}=\sinh^{-1}(\tan\tfrac{1}{2}\theta)$$

or $$\tan\tfrac{1}{2}\theta=\sinh\{t\sqrt{(g/a)}\}.$$

$\therefore$ $\theta=2\tan^{-1}[\sinh\{t\sqrt{(g/a)}\}]$.

Again the time to reach the highest point B while starting from A

$$=\tfrac{1}{2}\sqrt{(a/g)}\int_{\theta=0}^{\pi}\sec\tfrac{1}{2}\theta\,d\theta$$

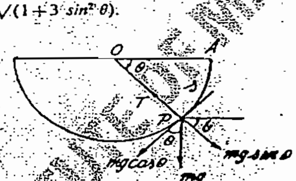
$$=\tfrac{1}{2}\sqrt{(a/g)}.2\left[\log(\tan\tfrac{1}{2}\theta+\sec\tfrac{1}{2}\theta)\right]_0^{\pi}$$

$$=\sqrt{(a/g)}.[\log(\tan\tfrac{1}{2}\pi+\sec\tfrac{1}{2}\pi)-\log(\tan 0+\sec 0)]$$

$$=\sqrt{(a/g)}.[\log\infty-\log 1]=\infty.$$

Therefore the bead takes an infinite time to reach the highest point.

Ex. 17 *A particle attached to a fixed peg O by a string of length l, is lifted up with the string horizontal and then let go. Prove that when the string makes an angle θ with the horizontal, the resultant acceleration is* $g\sqrt{(1+3\sin^2\theta)}$.



Sol. Let a particle of mass m be attached to a string of length l whose other end is attached to a fixed peg O. Initially let the string be horizontal in the position OA such that $OA=l$. The particle starts from A and moves in a circle whose centre is O and radius is l. Let P be the position of the particle at any time t such that $\angle AOP=\theta$ and arc $AP=s$. The forces acting on the particle at P are : (i) its weight mg acting vertically downwards and (ii) the tension T in the string along PO.

$\therefore$ the equations of motion of the particle along the tangent and normal at P are

$$m\frac{d^2s}{dt^2}=mg\cos\theta, \quad ...(1)$$

and $$m\frac{v^2}{l}=T-mg\sin\theta. \quad ...(2)$$

Also $s=l\theta$. ...(3)

From (1) and (3), we have $l\dfrac{d^2\theta}{dt^2}=g\cos\theta$.

Multiplying both sides by $2l(d\theta/dt)$ and integrating, we have

$$v^2=\left(l\frac{d\theta}{dt}\right)^2=2lg\sin\theta+A.$$

But initially at the point A, $\theta=0$, $v=0$. $\therefore$ $A=0$.

$\therefore$ $v^2=2lg\sin\theta$. ...(4)

The resultant acceleration of the particle at P

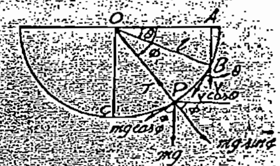
$$=\sqrt{[(\text{Tangential accel.})^2+(\text{Normal accel.})^2]}$$

$$=\sqrt{\left[\left(\frac{d^2s}{dt^2}\right)^2+\left(\frac{v^2}{l}\right)^2\right]} \quad \left[\because \text{Normal accel}=\frac{v^2}{\rho}=\frac{v^2}{l}\right]$$

$$=\sqrt{\left[(g\cos\theta)^2+\left(\frac{2lg\sin\theta}{l}\right)^2\right]}$$

$$=g\sqrt{(1-\sin^2\theta+4\sin^2\theta)}=g\sqrt{(1+3\sin^2\theta)}.$$

Ex. 18. *A particle attached to a fixed peg O by a string of length l, is let fall from a point in the horizontal line through O at a distance l cos θ from O; show that its velocity when it is vertically below O is* $\sqrt{[2gl(1-\sin^3\theta)]}$.



Sol. Let a particle of mass m be attached to a string of length l whose other end is attached to a fixed peg O. Let the particle fall from a point A in the horizontal line through O such that $OA=l\cos\theta$. The particle will fall under gravity from A to B, where $OB=l$.

$\because$ $OA=l\cos\theta$ and $OB=l$, therefore $\angle AOB=\theta$ and $AB=l\sin\theta$.

$\therefore$ the velocity of the particle at B

$=V=\sqrt{(2g.AB)}=\sqrt{(2gl\sin\theta)}$, vertically downwards.

As the particle reaches B, there is a jerk in the string and the impulsive tension in the string destroys the component of the velocity along OB and the component of the velocity along the tangent at B remains unaltered *i.e.*, the particle moves in the circular path with centre O and radius l with the tangential velocity $V\cos\theta$ at B.

[Note : In the figure write D at the end of the horizontal radius through O].

If P is the position of the particle at any time t such that $\angle DOP=\phi$ and arc $DP=s$, then the equations of motion of the particle along the tangent and normal are

$$m\frac{d^2s}{dt^2}=mg\cos\phi, \quad ...(1)$$

and $$m\frac{v^2}{l}=T-mg\sin\phi. \quad ...(2)$$

Also $s=l\phi$. ...(3)

From (1) and (3), we have $l\dfrac{d^2\phi}{dt^2}=g\cos\phi$.

Multiplying both sides by $2l(d\phi/dt)$ and integrating, we have

$$v^2=\left(l\frac{d\phi}{dt}\right)^2=2lg\sin\phi+A.$$

But at the point B, $\phi=\theta$ and $v=V\cos\theta$.

$\therefore$ $A=V^2\cos^2\theta-2lg\sin\theta=2gl\sin\theta.\cos^2\theta-2lg\sin\theta$

$=-2lg\sin\theta(1-\cos^2\theta)=-2lg\sin^3\theta$.

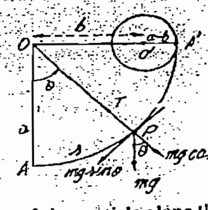
$\therefore$ $v^2=2lg\sin\phi-2lg\sin^3\theta$.

When the particle is at C vertically below O, we have at C $\phi=\pi/2$. Therefore the velocity v at C is given by

$v^2=2lg\sin\tfrac{1}{2}\pi-2lg\sin^3\theta=2lg(1-\sin^3\theta)$.

$\therefore$ the required velocity $v=\sqrt{[2lg(1-\sin^3\theta)]}$.

Ex. 19. *A particle is hanging from a fixed point O by means of a string of length a. There is a small nail at O' in the same horizontal line with O at a distance b (<a) from O. Find the minimum velocity with which the particle should be projected from its lowest point in order that it may make a complete revolution round the nail without the string becoming slack.*



Sol. Let a particle of mass m hang from a fixed point O by means of a string OA of length a. Let O' be a nail in the same horizontal line with O at a distance $OO'=b$ $(<a)$. Let the particle be projected from A with velocity u. It moves in a circle with centre at O and radius as a. If P is the position of the particle at any time t such that $\angle AOP=\theta$ and arc $AP=s$, then the equations of motion of the particle along the tangent and normal are

$$m\frac{d^2s}{dt^2}=-mg\sin\theta, \quad ...(1)$$

and $$m\frac{v^2}{a}=T-mg\cos\theta. \quad ...(2)$$

Also $s=a\theta$. ...(3)

From (1) and (3), we have $a\dfrac{d^2\theta}{dt^2}=-g\sin\theta$.

Multiplying both sides by $2a\,(d\theta/dt)$ and integrating, we have

$$v^2=\left(a\frac{d\theta}{dt}\right)^2=2ag\cos\theta+A.$$

But initially at A, $\theta=0$ and $v=u$. $\therefore$ $A=u^2-2ag$.

$$\therefore\ v^2=u^2-2ag+2ag\cos\theta. \qquad ...(4)$$

At the point A', $\theta=\pi/2$. If v_1 is the velocity of A', then from (4), we have

$$v_1^2=u^2-2ag \quad \text{or} \quad v_1=\sqrt{(u^2-2ag)}.$$

Since there is a nail at O', the particle will describe a circle with centre at O' and radius as $O'A'=a-b$.

We know that if a particle is attached to a string of length l, the least velocity of projection from the lowest point, in order to make a complete circle is $\sqrt{(5gl)}$. Also in this case, using the result (4), the velocity of the particle when it has described an angle θ from the lowest point is given by

$$v^2=5lg-2lg+2lg\cos\theta \quad [\because \text{ here } a=l \text{ and } u^2=5gl]$$
$$=3lg+2lg\cos\theta.$$

At $\theta=\pi/2$, if $v=v_2$, then $v_2=\sqrt{(3lg)}$. $\quad [\because \cos\pi/2=0]$

Thus in order to describe a complete circle of radius l the minimum velocity of the particle at the end of the horizontal diameter should be $\sqrt{(3gl)}$. Therefore in order to describe a complete circle of radius $l=O'A'=a-b$ round O' the minimum velocity of the particle at A' should be $\sqrt{[3g(a-b)]}$.

But, as already found out, the velocity of the particle at A' is v_1.

$\therefore$ we must have $v_1\geq\sqrt{[3g(a-b)]}$

or $\quad \sqrt{(u^2-2ag)}\geq\sqrt{[3g(a-b)]}$

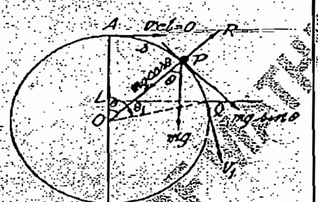
or $\quad u^2-2ag\geq 3g(a-b)$

or $\quad u^2\geq g(5a-3b)$

or $\quad u\geq\sqrt{[g(5a-3b)]}$.

Hence the required minimum velocity of projection of the particle at the lowest point is $\sqrt{[g(5a-3b)]}$.

5. Motion on the outside of a smooth vertical circle. *A particle slides down the outside of a smooth vertical circle starting from rest at the highest point; to discuss the motion.*



Let a particle of mass m slide down the outside of a smooth vertical circle whose centre is O and radius a, starting from rest at the highest point A. Let P be the position of the particle at any time t such that $\angle AOP=\theta$ and arc $AP=s$. The forces acting on the particle at P are (i) weight mg acting vertically downwards and (ii) the reaction R acting along the outwards drawn normal OP. If v be the velocity of the particle at P the equations of motion of the particle along the tangent and normal are

$$m\frac{d^2s}{dt^2}=mg\sin\theta, \qquad ...(1)$$

(+ive sign is taken on the R. H. S. because $mg\sin\theta$ acts in the direction of s increasing)

and $$m\frac{v^2}{a}=mg\cos\theta-R. \qquad ...(2)$$

[Note that in equation (2) R has been taken with $-$ive sign because it is in the direction of outwards drawn normal and $mg\cos\theta$ with +ive sign because it is in the direction of inwards drawn normal.]

Also $\quad s=a\theta. \qquad ...(3)$

From (1) and (3), we have $a\dfrac{d^2\theta}{dt^2}=g\sin\theta$.

Multiplying both sides by $2a\dfrac{d\theta}{dt}$ and integrating, we have

$$v^2=\left(a\frac{d\theta}{dt}\right)^2=-2ag\cos\theta+A.$$

But initially at A, $\theta=0$ and $v=0$. $\therefore$ $A=2ag$.

$$\therefore\ v^2=2ag-2ag\cos\theta=2ag(1-\cos\theta). \qquad ...(4)$$

From (2) and (4), we have

$$R=\frac{m}{a}\left[ag\cos\theta-v^2\right]=\frac{m}{a}\left[3ag\cos\theta-2ag\right]$$
$$=mg(3\cos\theta-2). \qquad ...(5)$$

If the particle leaves the circle at Q where $\angle AOQ=\theta_1$, then $R=0$ when $\theta=\theta_1$. Therefore from (5), we have

$$mg(3\cos\theta_1-2)=0 \quad \text{or} \quad \cos\theta_1=\tfrac{2}{3}.$$

Vertical depth of the point Q below A

$=AL=OA-OL=a-a\cos\theta_1=a-\frac{2}{3}a=a/3$.

Hence if a particle slides down the outside of a smooth vertical circle, starting from rest at the highest point, it will leave the circle after descending vertically a distance equal to one third of the radius of the circle.

If v_1 is the velocity of the particle at Q, then $v=v_1$ when $\theta=\theta_1$.

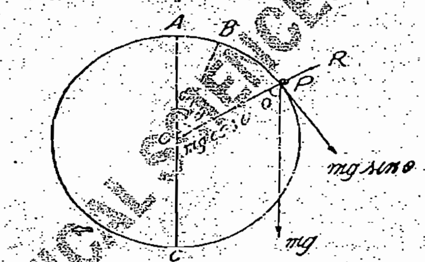
$\therefore$ from (4), we have

$$v_1^2=2ag(1-\cos\theta_1)=2ag(1-\tfrac{2}{3})=\tfrac{2}{3}ag.$$

The direction of the velocity v_1 is along the tangent to the circle at Q. Therefore the particle leaves the circle at Q with velocity $v_1=\sqrt{(\frac{2}{3}ag)}$ making an angle $\theta_1=\cos^{-1}(\frac{2}{3})$ below the horizontal line through Q. After leaving the circle at Q the particle will move freely under gravity and so it will describe a parabolic path.

Illustrative Examples

Ex. 20. *A particle is placed on the outside of a smooth vertical circle. If the particle starts from a point whose angular distance is α from the highest point of circle, show that it will fly off the curve when $\cos\theta=\frac{2}{3}\cos\alpha$.*



Sol. A particle slides down on the outside of the arc of a smooth vertical circle of radius a, starting from rest at a point B such that $\angle AOB=\alpha$. Let P be the position of the particle at any time t, where arc $AP=s$ and $\angle POA=\theta$. The forces acting on the particle at P are : (i) weight mg acting vertically downwards and (ii) the reaction R along the outwards drawn normal OP.

If v be the velocity of the particle at P, the equations of motion of the particle along the tangent and normal are

$$m\frac{d^2s}{dt^2}=mg\sin\theta, \qquad ...(1)$$

and $$m\frac{v^2}{a}=mg\cos\theta-R. \qquad ...(2)$$

Also $\quad s=a\theta. \qquad ...(3)$

From (1) and (3), we have $a\dfrac{d^2\theta}{dt^2}=g\sin\theta$.

Multiplying both sides by $2a\,(d\theta/dt)$ and integrating, we have

$$v^2=\left(a\frac{d\theta}{dt}\right)^2=-2ag\cos\theta+A.$$

But initially at B, $\theta=\alpha$ and $v=0$. $\therefore$ $A=2ag\cos\alpha$.

$$\therefore\ v^2=2ag\cos\alpha-2ag\cos\theta. \qquad ...(4)$$

From (2) and (4), we have

$$R=\frac{m}{a}(-v^2+ag\cos\theta)=\frac{m}{a}(-2ag\cos\alpha+3ag\cos\theta)$$
$$=mg(-2\cos\alpha+3\cos\theta). \qquad ...(5)$$

At the point where the particle flies off the circle, we have $R=0$.

$\therefore$ from (5), we have

$$0=mg(-2\cos\alpha+3\cos\theta) \quad \text{or} \quad \cos\theta=\tfrac{2}{3}\cos\alpha.$$

Ex. 21. *A particle is projected horizontally with a velocity $\sqrt{(ag/2)}$ from the highest point of the outside of a fixed smooth sphere of radius a. Show that it will leave the sphere at the point whose vertical distance below the point of projection is $a/6$.*

Sol. Refer figure of § 5 on page 189.

Let a particle be projected horizontally with a velocity $\sqrt{(ag/2)}$ from the highest point A on the outside of a fixed smooth sphere of radius a. If P is the position of the particle at any time t such that $\angle AOP=\theta$ and arc $AP=s$, then the equations of motion along the tangent and normal are

$$m\frac{d^2s}{dt^2}=mg\sin\theta, \qquad ...(1)$$

and $\quad m\dfrac{v^2}{a}=mg\cos\theta-R.$ ...(2)

Here v is the velocity of the particle at P.

Also $\quad s=a\theta.$ ...(3)

From (1) and (3), we have $a\dfrac{d^2\theta}{dt^2}=g\sin\theta.$

Multiplying both sides by $2a\,(d\theta/dt)$ and integrating, we have

$$v^2=\left(a\frac{d\theta}{dt}\right)^2=-2ag\cos\theta+A.$$

But initially at A, $\theta=0$ and $v=\sqrt{(ag/2)}$.

$\therefore\quad ag/2=-2ag+A\quad$ or $\quad A=\tfrac{1}{2}ag+2ag=\tfrac{5}{2}ag.$

$\therefore\quad v^2=\tfrac{5}{2}ag-2ag\cos\theta.$ ...(4)

From (2) and (4), we have

$$R=\frac{m}{a}(ag\cos\theta-v^2)=\frac{m}{a}(3ag\cos\theta-\tfrac{5}{2}ag)$$

or $\quad R=mg\,(3\cos\theta-\tfrac{5}{2}).$ ...(5)

If the particle leaves the sphere at the point Q where $\theta=\theta_1$, then putting $R=0$ and $\theta=\theta_1$ in (5), we have

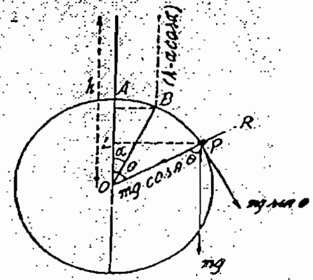
$$0=mg\,(3\cos\theta_1-\tfrac{5}{2})\quad\text{or}\quad\cos\theta_1=5/6.$$

Vertical depth of the point Q below the point of projection A

$$=AL=OA-OL=a-a\cos\theta_1=a-\tfrac{5}{6}a=\tfrac{1}{6}a.$$

Ex. 22. *A particle moves under gravity in a vertical circle sliding down the convex side of the smooth circular arc. If the initial velocity is that due to a fall to the starting point from a height h above the centre, show that it will fly off the circle when at a height $\tfrac{2}{3}h$ above the centre.*

Sol. Let a particle start from the point B of a smooth vertical circle where $\angle AOB=\alpha$. The depth of the point B, from the point which is at a height h above the centre O, is $h-a\cos\alpha$. Therefore the initial velocity of the particle at B $=u=\sqrt{\{2g\,(h-a\cos\alpha)\}}$.



If P is the position of the particle at time t such that $\angle AOP=\theta$ and arc $AP=s$, the equations of motion along the tangent and normal are

$$m\frac{d^2s}{dt^2}=mg\sin\theta.$$

and $\quad m\dfrac{v^2}{a}=mg\cos\theta-R.$ ...(2)

Also $\quad s=a\theta.$ ...(3)

From (1) and (3), we have $a\dfrac{d^2\theta}{dt^2}=g\sin\theta.$

Multiplying both sides by $2a\,(d\theta/dt)$ and integrating, we have

$$v^2=\left(a\frac{d\theta}{dt}\right)^2=-2ag\cos\theta+A.$$

But initially at B, $\theta=\alpha$ and $v=\sqrt{\{2g\,(h-a\cos\alpha)\}}$.

$\therefore\quad 2g\,(h-a\cos\alpha)=-2ag\cos\alpha+A\quad$ or $\quad A=2gh.$

$\therefore\quad v^2=-2ag\cos\theta+2gh.$ ...(4)

From (2) and (4), we have

$$R=\frac{m}{a}(ag\cos\theta-v^2)=\frac{m}{a}(3ag\cos\theta-2gh).$$

The particle will leave the sphere, where $R=0$ *i.e.*, where

$$\frac{m}{a}(3ag\cos\theta-2gh)=0\quad\text{or}\quad\cos\theta=2h/3a.$$

Now the height of the point where the particle flies off the circle, above the centre $O=OL=a\cos\theta=2h/3.$

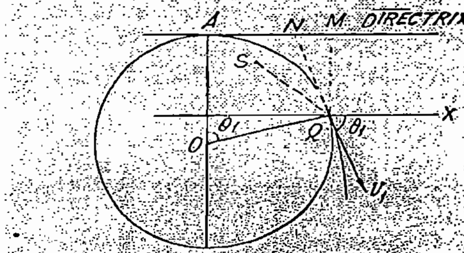
Ex. 23. *A particle is placed at the highest point of a smooth vertical circle of radius a and is allowed to slide down starting with a negligible velocity. Prove that it will leave the circle after describing vertically a distance equal to one third of the radius. Find the position of the directrix and the focus of the parabola subsequently described and show that its latus rectum is $\tfrac{16}{27}a$.*

Sol. For the first part see § 5 on page 189.

From § 5, the particle leaves the sphere at the point Q where $\angle AOQ=\theta_1$ and $\cos\theta_1=\tfrac{2}{3}$. The velocity v_1 at the point Q is $\sqrt{(2ag/3)}$; its direction is along the tangent to the circle at Q. After leaving the circle at the point Q, the particle describes a parabolic path with the velocity of projection $v_1=\sqrt{(2ag/3)}$ making an angle $\theta_1=\cos^{-1}(2/3)$ below the horizontal line through Q.

Latus rectum of the parabola subsequently described

$$=\frac{2v_1^2\cos^2\theta_1}{g}=\frac{2}{g}\cdot\frac{2ag}{3}\cdot\frac{4}{9}=\frac{16}{27}a.$$



To find the position of the directrix and the focus of the parabola. We know that in a parabolic path of a projectile the velocity at any point of its path is equal to that due to a fall from the directrix to that point.

Therefore if h is the height of the directrix above Q, then the velocity acquired in falling a distance h under gravity $=\sqrt{(2gh)}$.

$\therefore\quad v_1=\sqrt{(2ag/3)}=\sqrt{(2gh)}$

or $\quad h=a/3$ *i.e.*, the height of the directrix above Q is $a/3$.

Hence the directrix is the horizontal line through the highest point of the circle.

Let QM be the perpendicular from Q on the directrix and QN the tangent at Q. If S is the focus of the parabola subsequently described, we have by the geometrical properties of a parabola

$$QS=QM=a/3$$

and $\quad \angle SQN=\angle NQM.$

This gives the position of the focus S of the parabola.

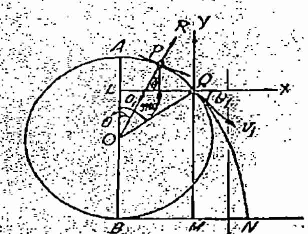
Ex. 24. *A heavy particle is allowed to slide down a smooth vertical circle of radius $27a$ from rest at the highest point. Show that on leaving the circle it moves in a parabola of latus rectum $16a$.*

Sol. Let us take the radius of the circle equal to b so that $b=27a$. Now proceed as in Ex. 23. We get

the latus rectum $=\dfrac{16b}{27}=\dfrac{16}{27}(27a)=16a.$

Ex. 25. *A particle slides down the arc of a smooth vertical circle of radius a, being slightly displaced from rest at the highest point. Find where it will leave the circle and prove that it will strike a horizontal plane through the lowest point of the circle at a distance $\tfrac{5}{27}(\sqrt{5}+4\sqrt{2})a$ from the vertical diameter.*

Sol. Proceeding as in § 5, the particle leaves the circle at the point Q where $\angle AOQ=\theta_1$ and $\cos\theta_1=2/3$. The velocity v_1 of the particle at the point Q is $\sqrt{(2ag/3)}$ and is along the tangent to the circle at the point Q. After leaving the circle at the point Q the motion of the particle is that of a projectile and so it describes a parabolic path with the velocity of projection $v_1=\sqrt{(2ag/3)}$ making an angle $\theta_1=\cos^{-1}(2/3)$ below the horizontal line through Q.



Now the equation of the parabolic path of the particle w.r.t. the horizontal and vertical lines OX and OY as the coordinate axes is

$$y=x\tan(-\theta_1)-\frac{gx^2}{2v_1^2\cos^2(-\theta_1)}$$ [$\because$ for the motion of the projectile, the angle of projection $=-\theta_1$]

or $\quad y=-x\tan\theta_1-\dfrac{gx^2}{2v_1^2\cos^2\theta_1}$

or $\quad y=-x\dfrac{\sqrt{5}}{2}-\dfrac{gx^2}{2\cdot\frac{2}{3}ag\cdot\frac{4}{9}}$ [$\because\ \cos\theta_1=\tfrac{2}{3}$ gives $\sin\theta_1=\sqrt{(1-\tfrac{4}{9})}=\sqrt{5}/3$ and $\tan\theta_1=\sqrt{5}/2$]

or $\quad y=-x\dfrac{\sqrt{5}}{2}-\dfrac{27}{16a}x^2.$ ...(1)

Let the particle strike the horizontal plane through the lowest point B at N. If (x_1, y_1) are the coordinates of the point N, then

$$x_1=MN\text{ and }y_1=-QM=-LB=-(LO+OB)$$
$$=-(a\cos\theta_1+a)=-(\tfrac{2}{3}a+a)=-5a/3.$$

The point $N\,(x_1, y_1)$ lies on the trajectory (1).

$$\therefore\quad y_1=-\frac{\sqrt{5}}{2}x_1-\frac{27}{16a}x_1^2.$$

or $\frac{-5a}{3} = -\frac{\sqrt{5}}{2}x_1 - \frac{27}{16a}x_1^2$

or $81x_1^2 + 24\sqrt{5}ax_1 - 80a^2 = 0$

∴ $x_1 = \frac{-24\sqrt{5}a \pm \sqrt{(24\times 24\times 5a^2 + 4\times 81\times 80a^2)}}{2\times 81}$

$= \frac{-24\sqrt{5}a + 120\sqrt{2}a}{162}$ (leaving the −ive sign, since x_1 cannot be negative)

or $x_1 = MN = \frac{(-4\sqrt{5}+20\sqrt{2})a}{27}$

∴ the required distance

$= BN = BM + MN = LQ + MN = a\sin\theta_1 + MN$

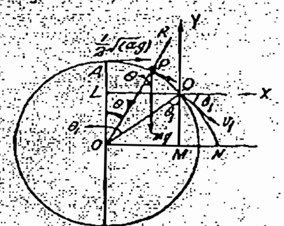
$= a.\frac{\sqrt{5}}{3} + \frac{(-4\sqrt{5}+20\sqrt{2})a}{27}$ [∵ $\sin\theta_1 = \sqrt{5}/3$]

$= \frac{5(\sqrt{5}+4\sqrt{2})a}{27}$.

Ex. 26. *A body is projected along the arc of a smooth circle of radius a and from the highest point with velocity* $\frac{1}{2}\sqrt{(ag)}$; *find where it will leave the circle and prove that it will strike a horizontal plane through the centre of the circle at a distance from the centre*

$$\frac{1}{64}\left[9\sqrt{(39)}+7\sqrt{7}\right]a.$$

Sol. Let a body be projected along the outside of a smooth vertical circle of radius a from the highest point A with velocity $\frac{1}{2}\sqrt{(ag)}$. If P is the position of the body at any time t, then the equations of motion of the body are $m\frac{d^2s}{dt^2} = mg\sin\theta$, ...(1)



and $m\frac{v^2}{a} = mg\cos\theta - R$. ...(2)

Also $s = a\theta$. ...(3)

From (1) and (3), we have

$a\frac{d^2\theta}{dt^2} = g\sin\theta$.

Multiplying both sides by $2a(d\theta/dt)$ and integrating, we have

$v^2 = \left(a\frac{d\theta}{dt}\right)^2 = -2ag\cos\theta + A$.

But initially at A, $\theta = 0$ and $v = \frac{1}{2}\sqrt{(ag)}$.

∴ $\frac{1}{4}ag = -2ag + A$ or $A = \frac{1}{4}ag + 2ag = \frac{9}{4}ag$.

∴ $v^2 = \frac{9}{4}ag - 2ag\cos\theta = ag(\frac{9}{4} - 2\cos\theta)$. ...(4)

From (2) and (4), we have

$R = \frac{m}{a}(ag\cos\theta - v^2) = \frac{m}{a}(3ag\cos\theta - \frac{9}{4}ag)$

$= 3mg(\cos\theta - \frac{3}{4})$. ...(5)

Suppose the body leaves the circle at the point Q, where $\theta = \theta_1$. Then putting $R = 0$ and $\theta = \theta_1$ in (5), we have

$0 = 3mg(\cos\theta_1 - \frac{3}{4})$ or $\cos\theta_1 = \frac{3}{4}$.

If v_1 is the velocity of the body at Q, then from (4)

$v_1^2 = ag(\frac{9}{4} - 2\cos\theta_1) = ag(\frac{9}{4} - \frac{3}{2}) = \frac{3}{4}ag$.

Hence the body leaves the circle at the point Q with velocity $v_1 = \frac{1}{2}\sqrt{(3ag)}$ at an angle $\theta_1 = \cos^{-1}(\frac{3}{4})$ below the horizontal line through Q and subsequently it describes a parabolic path. The equation of the parabolic trajectory of the body w.r.t. the horizontal and vertical lines QX and QY through Q as the coordinate axes is

$y = x\tan(-\theta_1) - \frac{gx^2}{2v_1^2\cos^2(-\theta_1)}$

or $y = -x\tan\theta_1 - \frac{gx^2}{2v_1^2\cos^2\theta_1}$

or $y = -x.\frac{\sqrt{7}}{3} - \frac{gx^2}{2.\frac{3}{4}ag.\frac{9}{16}}$ [∵ $\cos\theta_1 = \frac{3}{4}$ gives $\sin\theta_1 = \sqrt{(1-\frac{9}{16})} = \sqrt{7}/4$ and $\tan\theta_1 = \sqrt{7}/3$]

or $y = -\frac{\sqrt{7}}{3}x - \frac{32}{27a}x^2$. ...(6)

Let the particle strike the horizontal plane through the centre O at N. If (x_1, y_1) are the coordinates of the point N, then

$x_1 = MN$ and $y_1 = -QM = -LO = -a\cos\theta_1 = -\frac{3}{4}a$.

The point $N(x_1, y_1)$ lies on the trajectory (6).

∴ $y_1 = -\frac{\sqrt{7}}{3}x_1 - \frac{33}{27a}x_1^2$

or $\frac{-3a}{4} = -\frac{\sqrt{7}}{3}x_1 - \frac{32}{27a}x_1^2$

or $128x_1^2 + 36\sqrt{7}ax_1 - 81a^2 = 0$.

∴ $x_1 = \frac{-36\sqrt{7}a \pm \sqrt{(36\times 36\times 7a^2 + 4\times 128\times 81a^2)}}{2\times 128}$

$= \frac{-36\sqrt{7}a + 36\sqrt{(39)}a}{256}$ [neglecting the −ive sign because x_1 cannot be negative]

or $x_1 = MN = \frac{9(\sqrt{39}-\sqrt{7})a}{64}$.

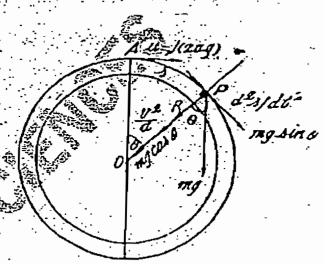
∴ the required distance $= ON = OM + MN = LQ + MN$

$= a\sin\theta_1 + MN$

$= \frac{\sqrt{7}a}{4} + \frac{9(\sqrt{39}-\sqrt{7})a}{64} = \frac{1}{64}\left[9\sqrt{39} + 7\sqrt{7}\right]a$.

Ex. 27. *A heavy particle slides under gravity down the inside of a smooth vertical tube held in a vertical plane. It starts from the highest point with velocity* $\sqrt{(2ag)}$, *where a is the radius of the circle. Prove that when, in the subsequent motion, the vertical component of the acceleration is maximum, the pressure on the curve is equal to twice the weight of the particle.*

Sol. Let P be the position of the particle at any time t such that $\angle AOP = \theta$ and arc $AP = s$. The forces acting on the particle at P are (i) weight mg acting vertically downwards and (ii) the reaction R along PO.



∴ the equations of motion of the particle are

$m\frac{d^2s}{dt^2} = mg\sin\theta$, ...(1)

and $m\frac{v^2}{a} = R + mg\cos\theta$. ...(2)

Also $s = a\theta$. ...(3)

From (1) and (3), we have $a\frac{d^2\theta}{dt^2} = g\sin\theta$.

Multiplying both sides by $2a(d\theta/dt)$ and integrating, we have

$v^2 = \left(a\frac{d\theta}{dt}\right)^2 = -2ag\cos\theta + A$.

But initially at A, $\theta = 0$ and $v = \sqrt{(2ag)}$.

∴ $A = 2ag + 2ag = 4ag$.

∴ $v^2 = 4ag - 2ag\cos\theta$. ...(4)

From (2) and (4), we have

$R = \frac{m}{a}(v^2 - ag\cos\theta)$

or $R = mg(4 - 3\cos\theta)$. ...(5)

Now $\frac{d^2s}{dt^2}$ and $\frac{v^2}{a}$ are the accelerations at the point P along the tangent and inward drawn normal at P. Let f be the vertical component of acceleration at P. Then

$f = \frac{d^2s}{dt^2}\sin\theta + \frac{v^2}{a}\cos\theta$.

Substituting from (1) and (4), we have

$f = g\sin\theta.\sin\theta + \frac{1}{a}(4ag - 2ag\cos\theta)\cos\theta$

$= g(\sin^2\theta + 4\cos\theta - 2\cos^2\theta)$.

∴ $\frac{df}{d\theta} = g(2\sin\theta\cos\theta - 4\sin\theta + 4\cos\theta\sin\theta)$

$= 2g\sin\theta(3\cos\theta - 2)$

and $\frac{d^2f}{d\theta^2} = g[6(\cos^2\theta - \sin^2\theta) - 4\cos\theta]$

$= g[6(2\cos^2\theta - 1) - 4\cos\theta]$.

For a maximum or a minimum of f, we have

$df/d\theta = 0$ i.e., $2g\sin\theta(3\cos\theta - 2) = 0$.

∴ either $\sin\theta = 0$ giving $\theta = 0$

or $3\cos\theta - 2 = 0$ giving $\cos\theta = \frac{2}{3}$.

But $\theta = 0$ corresponds to the initial position A.

When $\cos\theta = \frac{2}{3}$, $\frac{d^2f}{d\theta^2} = g[6(2.\frac{4}{9} - 1) - 4.\frac{2}{3}] = -\frac{10}{3}g = -$ ive.

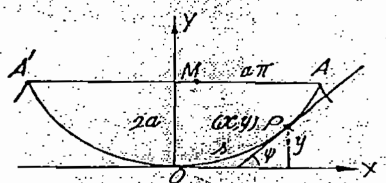
∴ f is maximum when $\cos\theta = \frac{2}{3}$.

Putting $\cos\theta = 2/3$ in (5) the pressure on the curve is given by

$R = mg(4 - 3.\frac{2}{3}) = 2mg = 2.$ (weight of the particle).

Cycloidal Motion

6. Cycloid. *A cycloid is a curve which is traced out by a point on the circumference of a circle as the circle rolls along a fixed straight line.*



In the adjoining figure we have shown an inverted cycloid. The point O is called the vertex of the cycloid. The points A and A' are the cusps and straight line OY is the axis of the cycloid. The line AA' is called the base of the cycloid.

Let $P(x, y)$ be the coordinates of a point on the cycloid w.r.t. OX and OY as coordinate axes and ψ the angle which the tangent at P makes with OX. Then remember the following results :

(i) Parametric equations of the cycloid are given by
$$x=a(\theta+\sin\theta),\ y=a(1-\cos\theta),$$
where θ is the parameter and we have $\theta=2\psi$.

(ii) The intrinsic equation of cycloid is
$$s=4a\sin\psi, \text{ where arc } OP=s.$$

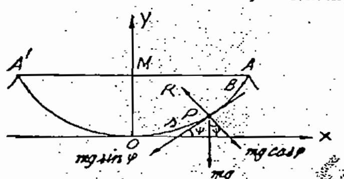
(iii) Arc $OA=4a$ and the height of the cycloid $=OM=2a$.

At the point O, $\psi=0$ and $s=0$ while at the cusp A, $\psi=\pi/2$ and $s=4a$.

(vi) For the above cycloid, the relation between s and y is
$$s^2=8ay.$$

7. Motion on a cycloid. *A particle slides down the arc of a smooth cycloid whose axis is vertical and vertex downwards. To determine the motion.*

Let O be the vertex of a smooth cycloid and OM its axis. Suppose a particle of mass m slides down the arc of the cycloid starting at rest from a point B where arc $OB=b$. Let P be the position of the particle at any time t where arc $OP=s$ and ψ be the angle which the tangent at P to the cycloid makes with the



tangent at the vertex O. The forces acting on the particle at P are : (i) the weight mg acting vertically downwards and (ii) the normal reaction R acting along the inwards drawn normal at P. Resolving these forces along the tangent and normal at P, the tangential and normal equations of motion of P are
$$m\frac{d^2s}{dt^2}=-mg\sin\psi, \quad ...(1)$$
and
$$m\frac{v^2}{\rho}=R-mg\cos\psi \quad ...(2)$$

Here v is the velocity of the particle at P and is along the tangent at P.

[Note that the expression for the tangential acceleration is d^2s/dt^2 and it is positive in the direction of s increasing. In the equation (1) negative sign has been taken because $mg\sin\psi$ acts in the direction of s decreasing. Again the expression for normal acceleration is v^2/ρ and it is positive in the direction of inwards drawn normal. In the equation (2) we have taken R with +ive sign because it is in the direction of inwards drawn normal while negative sign has been fixed before $mg\cos\psi$ because it is in the direction of outwards drawn normal].

Now the intrinsic equation of the cycloid is
$$s=4a\sin\psi. \quad ...(3)$$

From (1) and (3), we have
$$\frac{d^2s}{dt^2}=-\frac{g}{4a}s, \quad ...(4)$$
which is the equation of a simple harmonic motion with centre at the points $s=0$ *i.e.*, at the point O. Thus the particle will oscillate in S.H.M. about the centre O. The time period T of this S.H.M. is given by
$$T=\frac{2\pi}{\sqrt{(g/4a)}}=\pi\sqrt{(a/g)},$$
which is independent of the amplitude (*i.e.*, the initial displacement b). *Thus from whatever point the particle may be allowed to slide down the arc of a smooth cycloid, the time period remains the same.* Such a motion is called *isochronous* motion.

Multiplying both sides of (4) by $2(ds/dt)$ and then integrating w.r.t. 't', we get
$$v^2=\left(\frac{ds}{dt}\right)^2=-\frac{g}{4a}s^2+A.$$

But initially at the point B, $s=0$ and $v=0$.

Therefore $0=-(g/4a)b^2+A$ or $A=(g/4a)b^2$.
$$\therefore\ v^2=\left(\frac{ds}{dt}\right)^2=-\frac{g}{4a}s^2+\frac{g}{4a}b^2=\frac{g}{4a}(b^2-s^2), \quad ...(5)$$
which gives us the velocity of the particle at any position 's'. Substituting the value of v^2 in (2), we get R which gives us the pressure at any point on the cycloid.

Taking square root of (5), we get
$$\frac{ds}{dt}=-\sqrt{\left(\frac{g}{4a}\right)}\sqrt{(b^2-s^2)},$$
where the —ive sign has been taken because the particle is moving in the direction of s decreasing.

Separating the variables, we get
$$-\frac{ds}{\sqrt{(b^2-s^2)}}=\sqrt{\left(\frac{g}{4a}\right)}dt. \quad ...(6)$$

Integrating, we have
$$\cos^{-1}(s/b)=\sqrt{(g/4a)}\,t+C.$$

But initially at B, $s=b$ and $t=0$. Therefore $\cos^{-1}1=0+C$ or $C=0$.
$$\therefore\ \cos^{-1}(s/b)=\sqrt{(g/4a)}\,t,$$
or
$$s=b\cos\sqrt{(g/4a)}\,t,$$
which gives a relation between s and t.

If t_1 be the time from B to O, then integrating (6) from B to O, we have
$$-\int_b^0\frac{ds}{\sqrt{(b^2-s^2)}}=\int_0^{t_1}\sqrt{\left(\frac{g}{4a}\right)}dt$$
[Note that at B, $s=b$ and $t=0$ while at O, $s=0$ and $t=t_1$]

or $\left[\cos^{-1}\frac{s}{b}\right]_b^0=\sqrt{\left(\frac{g}{4a}\right)}\left[t\right]_0^{t_1}$

or $\cos^{-1}0-\cos^{-1}1=\sqrt{\left(\frac{g}{4a}\right)}t_1$

or $\frac{\pi}{2}=\sqrt{\left(\frac{g}{4a}\right)}t_1$

or $t_1=\pi\sqrt{(a/g)}$.

Thus time t_1 is independent of the initial displacement b of the particle. *Thus on a smooth cycloid the time of descent to the vertex is independent of the initial displacement of the particle.*

If T is time period of the particle *i.e.*, if T is the time for one complete oscillation, we have
$$T=4\times\text{time from } B \text{ to } O=4t_1=4\pi\sqrt{(a/g)}.$$

Illustrative Examples

Ex. 28. *A particle slides down a smooth cycloid whose axis is vertical and vertex downwards, starting from rest at the cusp. Find the velocity of the particle and the reaction on it at any point of the cycloid.*

Sol. Refer figure of § 7, on page 201.

Here the particle starts at rest from the cusp A.

The equations of motion of the particle along the tangent and normal are
$$m\frac{d^2s}{dt^2}=-mg\sin\psi \quad ...(1)$$
and
$$m\frac{v^2}{\rho}=R-mg\cos\psi. \quad ...(2)$$

For the cycloid, $s=4a\sin\psi$. ...(3)

From (1) and (3), we have
$$\frac{d^2s}{dt^2}=-\frac{g}{4a}s.$$

Multiplying both sides by $2\frac{ds}{dt}$ and integrating, we have
$$v^2=\left(\frac{ds}{dt}\right)^2=-\frac{g}{4a}s^2+A.$$

But initially at the cusp A, $s=4a$ and $v=0$.
$$\therefore\ A=\frac{g}{4a}\cdot(4a)^2=4ag.$$
$$\therefore\ v^2=-\frac{g}{4a}s^2+4ag=-\frac{g}{4a}(4a\sin\psi)^2+4ag$$
$$=4ag(1-\sin^2\psi)$$
or
$$v^2=4ag\cos^2\psi. \quad ...(4)$$

Differentiating (3), $\rho=ds/d\psi=4a\cos\psi$.

Substituting for v^2 and ρ in (2), we have
$$R=m\frac{v^2}{\rho}+mg\cos\psi=m\cdot\frac{4ag\cos^2\psi}{4a\cos\psi}+mg\cos\psi$$
or
$$R=2mg\cos\psi. \quad ...(5)$$

when it has fallen through half the distance measured along the arc to the vertex, two-thirds of the time of descent will have elapsed.

Sol. Refer figure of § 7 on page 201.

Let a particle of mass m start from rest from the cusp A of the cycloid. If P is the position of the particle after time t such that arc $OP=s$, the equations of motion along the tangent and normal are

$$m\frac{d^2s}{dt^2}=-mg\sin\psi \qquad \text{...(1)}$$

and $$m\frac{v^2}{\rho}=R-mg\cos\psi \qquad \text{...(2)}$$

For the cycloid, $s=4a\sin\psi$. ...(3)

From (1) and (3), we have $\dfrac{d^2s}{dt^2}=-\dfrac{g}{4a}s$.

Multiplying both sides by $2(ds/dt)$ and then integrating, we have

$$\left(\frac{ds}{dt}\right)^2=-\frac{g}{4a}s^2+A.$$

Initially at the cusp A, $s=4a$ and $\dfrac{ds}{dt}=0$.

$$\therefore \quad A=\frac{g}{4a}\cdot(4a)^2=4ag.$$

$$\therefore \quad \left(\frac{ds}{dt}\right)^2=-\frac{g}{4a}s^2+4ag=\frac{g}{4a}(16a^2-s^2) \qquad \text{...(4)}$$

or $$ds/dt=-\tfrac{1}{2}\sqrt{(g/a)}\cdot\sqrt{(16a^2-s^2)},$$

the $-$ive sign is taken because the particle is moving in the direction of s decreasing.

Separating the variables, we have

$$dt=-2\sqrt{(a/g)}\cdot\frac{ds}{\sqrt{(16a^2-s^2)}} \qquad \text{...(5)}$$

If t_1 is the time from the cusp A (*i.e.*, $s=4a$) to the vertex O (*i.e.*, $s=0$), then integrating (5),

$$t_1=-2\sqrt{(a/g)}\cdot\int_{4a}^{0}\frac{ds}{\sqrt{(16a^2-s^2)}}$$

$$=2\sqrt{(a/g)}\left[\cos^{-1}\frac{s}{4a}\right]_{4a}^{0}=2\sqrt{(a/g)}\,\frac{\pi}{2}=\pi\sqrt{(a/g)}.$$

Again if t_2 is the time taken to move from the cusp A (*i.e.* $s=4a$) to half the distance along the arc to the vertex *i.e.*, to $s=2a$, then integrating (5)

$$t_2=-2\sqrt{(a/g)}\int_{s=4a}^{2a}\frac{ds}{\sqrt{(16a^2-s^2)}}$$

$$=2\sqrt{(a/g)}\cdot\left[\cos^{-1}\frac{s}{4a}\right]_{4a}^{2a}$$

$$=2\sqrt{(a/g)}\,[\cos^{-1}\tfrac{1}{2}-\cos^{-1}1]=2\sqrt{(a/g)}\cdot(\pi/3)=(2/3)\,t_1.$$

Ex. 34. *A particle slides down the arc of a smooth cycloid whose axis is vertical and vertex lowest, starting at rest from the cusp. Prove that the time occupied in falling down the first half of the vertical height is equal to the time of falling down the second half.* IAS-2010

Sol. Let a particle start from rest from the cusp A of the cycloid. Proceeding as in the last example the velocity v of the particle at any point P, at time t, is given by

$$v^2=\left(\frac{ds}{dt}\right)^2=\frac{g}{4a}(16a^2-s^2). \quad \text{[Refer equation (4) of the last example]}$$

or $\dfrac{ds}{dt}=-\tfrac{1}{2}\sqrt{(g/a)}\,\sqrt{(16a^2-s^2)}$, the $-$ive sign is taken because the particle is moving in the direction of s decreasing.

$$\therefore \quad dt=-2\sqrt{(a/g)}\,\frac{ds}{\sqrt{(16a^2-s^2)}} \qquad \text{...(1)}$$

The vertical height of the cycloid is $2a$. At the point where the particle has fallen down the first half of the vertical height of the cycloid, we have $y=a$. Putting $y=a$ in the equation $s^2=8ay$, we get $s^2=8a^2$ or $s=2\sqrt{2}a$.

$\therefore$ integrating (1) from $s=4a$ to $s=2\sqrt{2}a$, the time t_1 taken in falling down the first half of the vertical height of the cycloid is given by

$$t_1=-2\sqrt{(a/g)}\int_{s=4a}^{2\sqrt{2}a}\frac{ds}{\sqrt{(16a^2-s^2)}}=2\sqrt{(a/g)}\left[\cos^{-1}(s/4a)\right]_{4a}^{2\sqrt{2}a}$$

$$=2\sqrt{(a/g)}\left[\cos^{-1}\frac{2\sqrt{2}a}{4a}-\cos^{-1}1\right]=2\sqrt{(a/g)}\left[\cos^{-1}\frac{1}{\sqrt{2}}-\cos^{-1}1\right]$$

$$=2\sqrt{(a/g)}\,[\tfrac{1}{4}\pi-0]=\tfrac{1}{2}\pi\sqrt{(a/g)}.$$

Again integrating (1) from $s=2\sqrt{2}a$ to $s=0$, the time t_2 taken in falling down the second half of the vertical height of the cycloid is given by

$$t_2=-2\sqrt{(a/g)}\int_{s=2\sqrt{2}a}^{0}\frac{ds}{\sqrt{(16a^2-s^2)}}$$

$$=2\sqrt{(a/g)}\cdot\left[\cos^{-1}\left(\frac{s}{4a}\right)\right]_{2\sqrt{2}a}^{0}=2\sqrt{(a/g)}\left[\cos^{-1}0-\cos^{-1}\frac{1}{\sqrt{2}}\right]$$

$$=2\sqrt{(a/g)}\,[\tfrac{1}{2}\pi-\tfrac{1}{4}\pi]=\tfrac{1}{2}\pi\sqrt{(a/g)}.$$

Hence $t_1=t_2$ *i.e.*, the time occupied in falling down the first half of the vertical height is equal to the time of falling down the second half.

Ex. 35. *A particle is projected with velocity V from the cusp of a smooth inverted cycloid down the arc, show that the time of reaching the vertex is $2\sqrt{(a/g)}\tan^{-1}[\sqrt{(4ag)}/V]$.* IAS-2009

Sol. Refer figure of § 7 on page 201.

Let a particle be projected with velocity V from the cusp A of a smooth inverted cycloid down the arc. If P is the position of the particle at time t such that the tangent at P is inclined at an angle ψ to the horizontal and arc $OP=s$, then the equations of motion of the particle are

$$m\frac{d^2s}{dt^2}=-mg\sin\psi \qquad \text{...(1)}$$

and $$m\frac{v^2}{\rho}=R-mg\cos\psi \qquad \text{...(2)}$$

For the cycloid, $s=4a\sin\psi$. ...(3)

From (1) and (3), we have $\dfrac{d^2s}{dt^2}=-\dfrac{g}{4a}s$.

Multiplying both sides by $2(ds/dt)$ and integrating, we have

$$v^2=\left(\frac{ds}{dt}\right)^2=-\frac{g}{4a}s^2+A.$$

But initially at the cusp A, $s=4a$ and $(ds/dt)^2=V^2$.

$$\therefore \quad V^2=-(g/4a)\cdot 16a^2+A \quad \text{or} \quad A=V^2+4ag.$$

$$\therefore \quad v^2=\left(\frac{ds}{dt}\right)^2=V^2+4ag-\frac{g}{4a}s^2=\left(\frac{g}{4a}\right)\left[\frac{4a}{g}(V^2+4ag)-s^2\right]$$

or $$\frac{ds}{dt}=-\tfrac{1}{2}\sqrt{(g/a)}\sqrt{\left[\frac{4a}{g}(V^2+4ag)-s^2\right]}$$

($-$ive sign is taken because the particle is moving in the direction of s decreasing)

or $$dt=-2\sqrt{(a/g)}\cdot\frac{ds}{\sqrt{[(4a/g)(V^2+4ag)-s^2]}}.$$

Integrating, the time t_1 from the cusp A to the vertex O is given by

$$t_1=-2\sqrt{(a/g)}\int_{s=4a}^{0}\frac{ds}{\sqrt{[(4a/g)(V^2+4ag)-s^2]}}$$

$$=2\sqrt{(a/g)}\int_{0}^{4a}\frac{ds}{\sqrt{[(4a/g)(V^2+4ag)-s^2]}}$$

$$=2\sqrt{(a/g)}\left[\sin^{-1}\frac{s}{2\sqrt{(a/g)}\sqrt{(V^2+4ag)}}\right]_{0}^{4a}$$

$$=2\sqrt{(a/g)}\cdot\sin^{-1}\left\{\frac{2\sqrt{(ag)}}{\sqrt{(V^2+4ag)}}\right\}$$

$$=2\sqrt{(a/g)}\cdot\theta, \qquad \text{...(4)}$$

where $\theta=\sin^{-1}\left\{\dfrac{2\sqrt{(ag)}}{\sqrt{(V^2+4ag)}}\right\}$.

We have $\sin\theta=\dfrac{2\sqrt{(ag)}}{\sqrt{(V^2+4ag)}}$.

$$\therefore \quad \cos\theta=\sqrt{(1-\sin^2\theta)}=\sqrt{\left[1-\frac{4ag}{V^2+4ag}\right]}=\frac{V}{\sqrt{(V^2+4ag)}}.$$

$$\therefore \quad \tan\theta=\frac{\sin\theta}{\cos\theta}=\frac{2\sqrt{(ag)}}{V}=\frac{\sqrt{(4ag)}}{V}$$

or $$\theta=\tan^{-1}[\sqrt{(4ag)}/V].$$

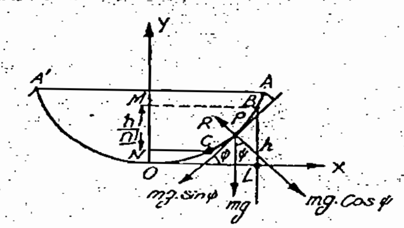
$\therefore$ from (4), the time of reaching the vertex is

$$=2\sqrt{(a/g)}\cdot\tan^{-1}[\sqrt{(4ag)}/V].$$

Ex. 36 (a). *If a particle starts from rest at a given point of a cycloid with its axis vertical and vertex downwards, prove that it falls $1/n$ of the vertical distance to the lowest point in time*

$$2\sqrt{(a/g)}\cdot\sin^{-1}(1/\sqrt{n}),$$

where a is the radius of the generating circle.



The equations (4) and (5) give the velocity and the reaction at any point of the cycloid.

Ex. 29. *A particle oscillates from cusp to cusp of a smooth cycloid whose axis is vertical and vertex lowest. Show that the velocity v at any point P is equal to the resolved part of the velocity V at the vertex along the tangent at P i.e., $v = V \cos\psi$.*

Sol. Proceed as in Ex. 28.

The velocity v of the particle at any point P of the cycloid is given by $v = 2\sqrt{(ag)} \cos\psi$. [From equation (4)]

If V is the velocity of the particle at the vertex, where $\psi = 0$, then $V = 2\sqrt{(ag)} \cos 0 = 2\sqrt{(ag)}$.

$\therefore$ $v = V \cos\psi$ = the resolved part of V along the tangent at P. Hence the velocity v at any point P is equal to the resolved part of the velocity V at the vertex along the tangent at P.

Ex. 30. *A heavy particle slides down a smooth cycloid starting from rest at the cusp, the axis being vertical and vertex downwards, prove that the magnitude of the acceleration is equal to g at every point of the path and the pressure when the particle arrives at the vertex is equal to twice the weight of the particle.*

Sol. Refer figure of § 7 on page 201.

Here the particle starts at rest from the cusp A.

The equations of motion of the particle are

$$m\frac{d^2s}{dt^2} = -mg\sin\psi \qquad \text{...(1)}$$

and $$m\frac{v^2}{\rho} = R - mg\cos\psi. \qquad \text{...(2)}$$

For the cycloid, $s = 4a\sin\psi$. ...(3)

From (1) and (3), we have $\frac{d^2s}{dt^2} = -\frac{g}{4a}s$.

Multiplying both sides by $2(ds/dt)$ and integrating, we have

$$v^2 = \left(\frac{ds}{dt}\right)^2 = -\frac{g}{4a}s^2 + A.$$

But initially at the cusp A, $s = 4a$ and $v = 0$. $\therefore$ $A = 4ag$.

$$\therefore \quad v^2 = -\frac{g}{4a}s^2 + 4ag = -\frac{g}{4a}(4a\sin\psi)^2 + 4ag = 4ag(1-\sin^2\psi)$$

or $v^2 = 4ag\cos^2\psi$.

Differentiating (3),

$$\rho = ds/d\psi = 4a\cos\psi.$$

Now at the point P, tangential acceleration

$= d^2s/dt^2 = -g\sin\psi$ [from (1)]

and normal acceleration $= \frac{v^2}{\rho} = \frac{4ag\cos^2\psi}{4a\cos\psi} = g\cos\psi$.

$\therefore$ the resultant acceleration at any point P

$= \sqrt{[(\text{tang. accel.})^2 + (\text{normal accel.})^2]}$

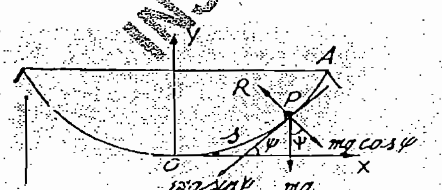
$= \sqrt{[(-g\sin\psi)^2 + (g\cos\psi)^2]} = g$.

From (2) and (4), we have

$$R = m.\frac{4ag\cos^2\psi}{4a\cos\psi} + mg\cos\psi = 2mg\cos\psi. \qquad \text{...(5)}$$

At the vertex O, $\psi = 0$. Therefore putting $\psi = 0$ in (5), the pressure at the vertex $= 2mg$ = twice the weight of the particle.

Ex. 31. *Prove that for a particle sliding down the arc and starting from the cusp of a smooth cycloid whose vertex is lowest, the vertical velocity is maximum when it has described half the vertical height.*



Sol. Let a particle of mass m slide down the arc of a cycloid starting at rest from the cusp A. If P is the position of the particle at any time t, then the equations of motion of the particle along the tangent and normal are

$$m\frac{d^2s}{dt^2} = -mg\sin\psi \qquad \text{...(1)}$$

and $$m\frac{v^2}{\rho} = R - mg\cos\psi. \qquad \text{...(2)}$$

For the cycloid, $s = 4a\sin\psi$. ...(3)

From (1) and (3), we have $\frac{d^2s}{dt^2} = -\frac{g}{4a}s$.

Multipl ing both sides by $2(ds/dt)$ and integrating, we have

$$v^2 = \left(\frac{ds}{dt}\right)^2 = -\frac{g}{4a}s^2 + A.$$

But initially at the cusp A, $s = 4a$ and $v = 0$. $\therefore$ $A = 4ag$.

$$\therefore \quad v^2 = 4ag - \frac{g}{4a}s^2 = 4ag - \frac{g}{4a}(4a\sin\psi)^2 = 4ag(1-\sin^2\psi)$$

$= 4ag\cos^2\psi$

or $v = 2\sqrt{(ag)}\cos\psi$, giving the velocity of the particle at the point P its direction being along the tangent at P. Let V be the vertical component of the velocity v at the point P. Then

$V = v\cos(90° - \psi) = v\sin\psi = 2\sqrt{(ag)}\cos\psi.\sin\psi$

or $V = \sqrt{(ag)}\sin 2\psi$,

which is maximum when $\sin 2\psi = 1$ *i.e.*, $2\psi = \pi/2$ *i.e.*, $\psi = \pi/4$.

When $\psi = \pi/4$, $s = 4a\sin(\pi/4) = 2\sqrt{2}a$.

Putting $s = 2\sqrt{2}a$ in the relation $s^2 = 8ay$, we have

$(2\sqrt{2}a)^2 = 8ay$ or $y = 8a^2/8a = a$.

Thus at the point where the vertical velocity is maximum, we have $y = a$. The vertical depth fallen upto this point $=$ (the y-coordinate of A) $- a = 2a - a = a = \frac{1}{2}(2a)$ = half the vertical height of the cycloid.

Ex. 32. *A particle oscillates in a cycloid under gravity, the amplitude of the motion being b and period being T. Show that its velocity at any time t measured from a position of rest is*

$$\frac{2\pi b}{T}\sin\left(\frac{2\pi t}{T}\right).$$

Sol. Refer § 7 on page 200.

The equations of motion of the particle are

$$m\frac{d^2s}{dt^2} = -mg\sin\psi \qquad \text{...(1)}$$

and $$m\frac{v^2}{\rho} = R - mg\cos\psi; \qquad \text{...(2)}$$

For the cycloid, $s = 4a\sin\psi$. ...(3)

From (1) and (3), we have $\frac{d^2s}{dt^2} = -\frac{g}{4a}s$, ...(4)

which represents a S. H. M.

$\therefore$ the time period T of the particle is given by $T = 2\pi/\sqrt{(g/4a)}$

or $T = 4\pi\sqrt{(a/g)}$. ...(5)

Multiplying both sides of (4) by $2\frac{ds}{dt}$ and integrating, we have

$$v^2 = \left(\frac{ds}{dt}\right)^2 = -\frac{g}{4a}s^2 + A. \qquad \text{...(6)}$$

But the amplitude of the motion is b. So the arcual distance of a position of rest from the vertex O is b i.e., $v = 0$ when $s = b$.

$\therefore$ from (6), we have

$$A = \frac{g}{4a}b^2.$$

Substituting this value of A in (6), we have

$$v^2 = \left(\frac{ds}{dt}\right)^2 = \frac{g}{4a}(b^2 - s^2). \qquad \text{...(7)}$$

$$\frac{ds}{dt} = -\frac{1}{2}\sqrt{\left(\frac{g}{a}\right)}\sqrt{(b^2 - s^2)}$$

($-$ive sign is taken because the particle is moving in the direction of s decreasing)

or $$dt = -2\sqrt{(a/g)}\frac{ds}{\sqrt{(b^2 - s^2)}}.$$

Integrating, $t = 2\sqrt{(a/g)}\cos^{-1}(s/b) + B$.

But $t = 0$ when $s = b$. $\therefore$ $B = 0$.

$\therefore$ $t = 2\sqrt{(a/g)}\cos^{-1}(s/b)$

or $$s = b\cos\left\{\frac{t}{2}\sqrt{\left(\frac{g}{a}\right)}\right\}.$$

Substituting this value of s in (7), we have

$$v^2 = \frac{g}{4a}\left[b^2 - b^2\cos^2\left\{\frac{t}{2}\sqrt{(g/a)}\right\}\right]$$

$$= \frac{g}{4a}b^2\sin^2\left\{\frac{t}{2}\sqrt{(g/a)}\right\}$$

or $$v = \frac{b}{2}\sqrt{(g/a)}\sin\left\{\frac{t}{2}\sqrt{(g/a)}\right\}.$$

From (5), $\sqrt{(g/a)} = \frac{4\pi}{T}$.

$\therefore$ the velocity of the particle at any time t measured from the position of rest is given by

$$v = \frac{b}{2}\cdot\frac{4\pi}{T}\sin\left(\frac{t}{2}\cdot\frac{4\pi}{T}\right) = \left(\frac{2\pi b}{T}\right)\sin\left(\frac{2\pi t}{T}\right).$$

Ex. 33. *A particle starts from rest at the cusp of a smooth cycloid whose axis is vertical and vertex downwards. Prove that*

Sol. Let a particle start from rest at a given point B of a cycloid with its axis vertical and vertex downwards. Let h be the vertical height of the point B above the vertex O.

If arc $OB=s_1$, then from $s^2=8ay$, we have $s_1^2=8ah$.

If P is the position of the particle at time t such that the tangent at P is inclined at an angle ψ to the horizontal and arc $OP=s$, then the equations of motion along the tangent and normal at P are

$$m\frac{d^2s}{dt^2}=-mg\sin\psi \quad ...(1)$$

and $$m\frac{v^2}{\rho}=R-mg\cos\psi. \quad ...(2)$$

For the cycloid, $s=4a\sin\psi$. ...(3)

From (1) and (3), we have $\dfrac{d^2s}{dt^2}=-\dfrac{g}{4a}s$.

Multiplying both sides by $2(ds/dt)$ and integrating, we have

$$v^2=\left(\frac{ds}{dt}\right)^2=-\frac{g}{4a}s^2+A.$$

But at the point B, $s=s_1$ and $v=0$.

$$\therefore \quad 0=-\frac{g}{4a}s_1^2+A \quad \text{or} \quad A=\frac{g}{4a}s_1^2.$$

$$\therefore \quad \left(\frac{ds}{dt}\right)^2=-\frac{g}{4a}s^2+\frac{g}{4a}s_1^2=\frac{g}{4a}(s_1^2-s^2)$$

or $$ds/dt=-\tfrac{1}{2}\sqrt{(g/a)}\cdot\sqrt{(s_1^2-s^2)}$$

(negative sign is taken since the particle is moving in the direction of s decreasing)

or $$dt=-2\sqrt{(a/g)}\frac{ds}{\sqrt{(s_1^2-s^2)}}. \quad ...(4)$$

Integrating, we have

$$t=2\sqrt{(a/g)}\cos^{-1}(s/s_1)+A.$$

But at the point B, $s=s_1$ and $t=0$.

$\therefore \quad 0=2\sqrt{(a/g)}\cos^{-1}1+A$ or $A=0$. $[\because \cos^{-1}1=0]$

$\therefore \quad t=2\sqrt{(a/g)}\cos^{-1}(s/s_1)$

$$=2\sqrt{\left(\frac{a}{g}\right)}\cos^{-1}\left[\frac{\sqrt{(8ay)}}{\sqrt{(8ah)}}\right] \quad [\because s^2=8ay \text{ and } s_1^2=8ah]$$

$$=2\sqrt{(a/g)}\cos^{-1}\sqrt{(y/h)}. \quad ...(5)$$

Let C be the point at a vertical depth h/n below the point B. Then the height of C above $O=ON=h-(h/n)=h(1-1/n)$. Thus for the point C, we have $y=h(1-1/n)$.

If t_1 be the time taken by the particle from B to C, then putting $t=t_1$ and $y=h(1-1/n)$ in (5), we get

$$t_1=2\sqrt{(a/g)}\cos^{-1}\sqrt{[\{h(1-1/n)\}/h]}=2\sqrt{(a/g)}\cos^{-1}\sqrt{(1-1/n)}$$
$$=2\sqrt{(a/g)}\sin^{-1}\sqrt{\{1-(1-1/n)\}} \quad [\because \cos^{-1}x=\sin^{-1}\sqrt{(1-x^2)}]$$
$$=2\sqrt{(a/g)}\sin^{-1}(1/\sqrt{n}).$$

Ex. 36 (b). *A particle slides down the arc of a smooth cycloid whose axis is vertical and vertex lowest, starting from rest at a given point of the cycloid. Prove that the time occupied in falling down the first half of the vertical height to the lowest point is equal to the time of falling down the second half.*

Sol. Proceed as in Ex. 36 (a) by taking $n=2$.

Thus here if C be the point at a vertical depth $h/2$ below the point B, then at C, we have $y=h/2$. If t_1 be the time taken by the particle from B to C, then putting $t=t_1$ and $y=h/2$ in the result (5) of Ex. 36 (a), we get

$$t_1=2\sqrt{(a/g)}\cos^{-1}\sqrt{(\tfrac{1}{2}h/h)}=2\sqrt{(a/g)}\cos^{-1}(1/\sqrt{2})$$
$$=2\sqrt{(a/g)}\cdot\tfrac{1}{4}\pi=\tfrac{1}{2}\pi\sqrt{(a/g)}.$$

Again if t_2 be the time taken by the particle from B to O, then putting $t=t_2$ and $y=0$ in (5), we get

$$t_2=2\sqrt{(a/g)}\cos^{-1}0=2\sqrt{(a/g)}\cdot\tfrac{1}{2}\pi=\pi\sqrt{(a/g)}.$$

Since $t_2=2t_1$, therefore the time from B to C is equal to the time from C to O.

Ex. 37. *Two particles are let drop from the cusp of a cycloid down the curve at an interval of time t; prove that they will meet in time $2\pi\sqrt{(a/g)}+(t/2)$.*

Sol. Refer the figure of §7 on page 201.

Suppose a particle starts at rest from the cusp A. At any time T, the equation of motion of the particle along the tangent is given by

$$m\frac{d^2s}{dT^2}=-mg\sin\psi.$$

For the cycloid, $s=4a\sin\psi$.

$$\therefore \quad \frac{d^2s}{dT^2}=-\frac{g}{4a}s.$$

Multiplying both sides by $2(ds/dT)$ and integrating, we have

$$v^2=\left(\frac{ds}{dT}\right)^2=-\frac{g}{4a}s^2+A.$$

The particle is dropped from the cusp. Therefore $v=0$ when $s=4a$.

$$\therefore \quad 0=-\frac{g}{4a}(4a)^2+A \quad \text{or} \quad A=4ag.$$

$$\therefore \quad \left(\frac{ds}{dT}\right)^2=-\frac{g}{4a}s^2+4ag=\frac{g}{4a}(16a^2-s^2).$$

or $$ds/dT=-\tfrac{1}{2}\sqrt{(g/a)}\cdot\sqrt{(16a^2-s^2)}$$

(–ive sign is taken because the particle is moving in the direction of s decreasing)

or $$dT=-2\sqrt{(a/g)}\cdot\frac{ds}{\sqrt{(16a^2-s^2)}}.$$

Integrating, $T=2\sqrt{(a/g)}\cos^{-1}\left(\dfrac{s}{4a}\right)+B.$

But at the cusp A, $T=0$, $s=4a$. $\therefore B=0$.

$$\therefore \quad T=2\sqrt{(a/g)}\cos^{-1}(s/4a)$$

or $$\cos^{-1}(s/4a)=\tfrac{1}{2}T\sqrt{(g/a)}.$$

$$\therefore \quad s=4a\cos[\tfrac{1}{2}T\sqrt{(g/a)}]. \quad ...(1)$$

Thus if a particle starts at rest from the cusp A, the equation (1) gives the arcural distance (*i.e.*, distance measured along the arc) of the particle from the vertex O at any time T measured from the instant the particle starts from the cusp A.

Let the two particles meet after time t_1 measured from the instant the first particle was dropped. Since the two particles are dropped at an interval of time t, therefore the second particle will be in motion for time (t_1-t) before it meets the first particle.

Let s_1 be the distance along the arc of the first particle at time t_1 measured from the instant it starts from the cusp A and s_2 that of the second particle at time t_1-t measured from the instant it starts from the cusp A. Then from (1), we have

$$s_1=4a\cos[\tfrac{1}{2}t_1\sqrt{(g/a)}] \text{ and } s_2=4a\cos[\tfrac{1}{2}(t_1-t)\sqrt{(g/a)}].$$

But $s_1=s_2$, being the condition for the two particles to meet.

$$\therefore \quad 4a\cos[\tfrac{1}{2}t_1\sqrt{(g/a)}]=4a\cos[\tfrac{1}{2}(t_1-t)\sqrt{(g/a)}]$$

or $\cos[\tfrac{1}{2}t_1\sqrt{(g/a)}]=\cos[\tfrac{1}{2}(t_1-t)\sqrt{(g/a)}]$

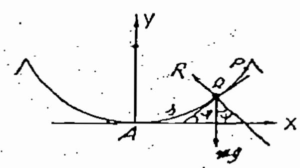
or $\tfrac{1}{2}(t_1-t)\sqrt{(g/a)}=2\pi-\tfrac{1}{2}t_1\sqrt{(g/a)}$ $\quad [\because \cos(2\pi-\alpha)=\cos\alpha]$

or $t_1\sqrt{(g/a)}=2\pi+\tfrac{1}{2}t\sqrt{(g/a)}$ or $t_1=2\pi\sqrt{(a/g)}+\tfrac{1}{2}t.$

Ex. 38. *A particle starts from rest at any point P in the arc of a smooth cycloid $s=4a\sin\psi$ whose axis is vertical and vertex A downwards; prove that the time of descent to the vertex is $\pi\sqrt{(a/g)}$.*

Show that if the particle is projected from P downwards along the curve with velocity equal to that with which it reaches A when starting from rest at P, it will now reach A in half the time taken in the preceding case.

Sol. A particle starts from rest at any point P in the arc of a smooth cycloid whose vertex is A. Let arc $AP=b$.



Let Q be the position of the particle at any time t where arc $AQ=s$ and let ψ be the angle which the tangent at Q to the cycloid makes with the tangent at the vertex A. The tangential equation of motion of the particle at Q is

$$m\frac{d^2s}{dt^2}=-mg\sin\psi. \quad ...(1)$$

But for the cycloid, $s=4a\sin\psi$.

$\therefore$ the equation (1) becomes $\dfrac{d^2s}{dt^2}=-\dfrac{g}{4a}s.$

Multiplying both sides by $2(ds/dt)$ and integrating w.r.t. 't', we have

$$v^2=\left(\frac{ds}{dt}\right)^2=-\frac{g}{4a}s^2+A. \quad ...(2)$$

But initially at the point P, we have $s=b$ and $v=0$.

$$\therefore \quad v=-\frac{g}{4a}b^2+A \quad \text{or} \quad A=\frac{g}{4a}b^2.$$

$$\therefore \quad v^2=\left(\frac{ds}{dt}\right)^2=-\frac{g}{4a}s^2+\frac{g}{4a}b^2=\frac{g}{4a}(b^2-s^2). \quad ...(3)$$

Taking square root of (3), we get

$$ds/dt=-\tfrac{1}{2}\sqrt{(g/a)}\sqrt{(b^2-s^2)},$$

where the –ive sign has been taken because the particle is moving in the direction of s decreasing.

$$\therefore \quad dt=-2\sqrt{(a/g)}\frac{ds}{\sqrt{(b^2-s^2)}}. \quad ...(4)$$

Let t_1 be the time taken by the particle to reach the vertex A where $s=0$. Then integrating (4) from P to A, we have

$$\int_0^{t_1}dt=-2\sqrt{(a/g)}\int_b^0\frac{ds}{\sqrt{(b^2-s^2)}}.$$

$$\therefore \quad t_1=2\sqrt{(a/g)}\left[\cos^{-1}\frac{s}{b}\right]_b^0=2\sqrt{(a/g)}\,[\cos^{-1}0-\cos^{-1}1]$$
$$=2\sqrt{(a/g)}\,[\tfrac{1}{2}\pi-0]=\pi\sqrt{(a/g)}, \text{ which proves the first result.}$$

If v_1 is the velocity with which the particle reaches the vertex A, then at A, $v=v_1$ and $s=0$. So from (3), we have

$$v_1^2=\frac{g}{4a}(b^2-0^2)=\frac{g}{4a}b^2.$$

Second case. Now suppose the particle starts from P with velocity v_1 where $v_1^2=(g/4a)\,b^2$. Then applying the initial condition $s=b$ and $v=v_1$ in (2), we have

$$v_1^2=-\left(\frac{g}{4a}\right)b^2+A$$

or
$$A=v_1^2+\left(\frac{g}{4a}\right)b^2=\left(\frac{g}{4a}\right)b^2+\left(\frac{g}{4a}\right)b^2=\frac{g}{2a}b^2.$$

For this new value of A, (2) becomes

$$\left(\frac{ds}{dt}\right)^2=-\frac{g}{4a}s^2+\frac{g}{2a}b^2=\frac{g}{4a}(2b^2-s^2).$$

$$\therefore\ \frac{ds}{dt}=-\tfrac{1}{2}\sqrt{(g/a)}\sqrt{(2b^2-s^2)}.$$

or
$$dt=-2\sqrt{(a/g)}\frac{ds}{\sqrt{(2b^2-s^2)}}. \qquad \text{...(5)}$$

Let t_2 be the time taken by the particle to reach the vertex A in this case. Then integrating (5) from P to A, we have

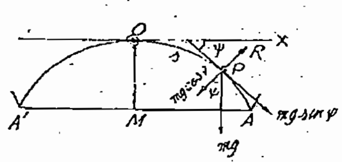
$$\int_0^{t_2}dt=-2\sqrt{(a/g)}\int_b^0\frac{ds}{\sqrt{(2b^2-s^2)}}$$

$$\therefore\ t_2=2\sqrt{(a/g)}\left[\cos^{-1}\frac{s}{b\sqrt{2}}\right]_b^0$$

$=2\sqrt{(a/g)}\,[\cos^{-1}0-\cos^{-1}(1/\sqrt{2})]=2\sqrt{(a/g)}\,[\tfrac12\pi-\tfrac14\pi]$

$=2\sqrt{(a/g)}.\tfrac14\pi=\tfrac12\pi\sqrt{(a/g)}=\tfrac12 t_1$, which proves the second result.

§ 8. Motion on the outside of a smooth cycloid with its axis vertical and vertex upwards. *A particle is placed very close, to the vertex of a smooth cycloid whose axis is vertical and vertex upwards and is allowed to run down the curve, to discuss the motion.*



Let a particle of mass m, starting from rest at O, slide down the arc of a smooth cycloid whose axis OM is vertical and vertex O is upwards. Let P be the position of the particle at time t such that arc $OP=s$ and the tangent at P to the cycloid makes an angle ψ with the tangent at the vertex O. The forces acting on the particle at P are : (i) weight mg acting vertically downdards and (ii) the reaction R acting along the outwards drawn normal.

∴ The equations of motion along the tangent and normal are

$$m\frac{d^2s}{dt^2}=mg\sin\psi \qquad \text{...(1)}$$

and
$$m\frac{v^2}{\rho}=mg\cos\psi-R. \qquad \text{...(2)}$$

Also for the cycloid, $s=4a\sin\psi$. ...(3)

From (1) and (3), we have $\dfrac{d^2s}{dt^2}=\dfrac{g}{4a}s.$

Multiplying both sides by $2(ds/dt)$ and integrating, we have

$$v^2=\left(\frac{ds}{dt}\right)^2=\frac{g}{4a}s^2+A.$$

Initially at O, $s=0$ and $v=0$; ∴ $A=0$.

$$\therefore\ v^2=\frac{g}{4a}s^2=\frac{g}{4a}(4a\sin\psi)^2=4ag\sin^2\psi. \qquad \text{...(4)}$$

From (2) and (4), we have

$$R=mg\cos\psi-\frac{mv^2}{\rho}$$

$$=mg\cos\psi-\frac{m.4ag\sin^2\psi}{4a\cos\psi} \qquad \left[\because\ \rho=\frac{ds}{d\psi}=4a\cos\psi\right]$$

$$=\frac{mg}{\cos\psi}(\cos^2\psi-\sin^2\psi). \qquad \text{...(5)}$$

The equation (4) gives the velocity of the particle at any position and the equation (5) gives the reaction of the cycloid on the particle at any position. The pressure of the particle on the curve is equal and opposite to the reaction of the curve on the particle.

When the particle leaves the cycloid, we have $R=0$

i.e., $\dfrac{mg}{\cos\psi}(\cos^2\psi-\sin^2\psi)=0$

i.e., $\sin^2\psi=\cos^2\psi$ *i.e.*, $\tan^2\psi=1$

i.e., $\tan\psi=1$ *i.e.*, $\psi=45°$.

Hence the particle will leave the curve when it is moving in a direction making an angle 45° downwards with the horizontal.

Illustrative Examples

Ex. 39. *If a particle starts from the vertex of a cycloid whose axis is vertical and vertex upwards, prove that its velocity at any point varies as the distance of that point from the vertex measured along the arc.*

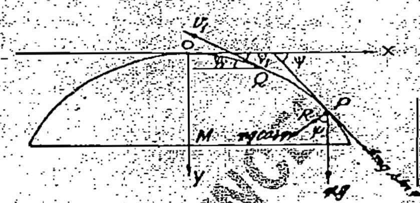
Sol. Proceed as in § 8. From the equation (4), the velocity v at any point P is given by

$$v^2=(g/4a)\,s^2.$$

$$\therefore\ v=\sqrt{(g/4a)}.s \quad \text{or} \quad v\propto s.$$

Hence the velocity varies as the distance measured along the arc.

Ex. 40. *A cycloid is placed with its axis vertical and vertex upwards and a heavy particle is projected from the cusp up the concave side of the curve with velocity $\sqrt{(2gh)}$; prove that the latus rectum of the parabola described after leaving the arc is $(h^2/2a)$, where a is the radius of the generating circle.*



Sol. Let a particle of mass m be projected with velocity $\sqrt{(2gh)}$ from the cusp A up the concave side of the cycloid. If P is the position of the particle after any time t such that arc $OP=s$, the equations of motion along the tangent and normal are

$$m\,(d^2s/dt^2)=mg\sin\psi, \qquad \text{...(1)}$$

and
$$m\,(v^2/\rho)=R+mg\cos\psi. \qquad \text{...(2)}$$

[Note that here the reaction R of the curve acts along the inwards drawn normal and the tangential component of mg acts in the direction of s increasing.]

For the cycloid, $s=4a\sin\psi$.

From (1) and (3), we have $\dfrac{d^2s}{dt^2}=\dfrac{g}{4a}s$.

Multiplying both sides by $2\,(ds/dt)$ and then integrating, we have $v^2=(ds/dt)^2=(g/4a)\,s^2+A$.

Initially at A, $s=4a$ and $v=\sqrt{(2gh)}$.

$\therefore\ A=2gh-4ag.$

$$\therefore\ v^2=\frac{g}{4a}s^2+2gh-4ag=\frac{g}{4a}(4a\sin\psi)^2+2gh-4ag$$

$$=4ag\sin^2\psi+2gh-4ag=2gh-4ag\,(1-\sin^2\psi)$$

$$=2gh-4ag\cos^2\psi. \qquad \text{...(4)}$$

From (2) and (4), we have

$$R=\frac{m}{4a\cos\psi}(2gh-4ag\cos^2\psi)-mg\cos\psi$$

$$[\because\ \rho=ds/d\psi=4a\cos\psi]$$

$$=\frac{mg}{2a\cos\psi}(h-2a\cos^2\psi)-mg\cos\psi$$

$$=\frac{mg}{2a\cos\psi}[h-2a\cos^2\psi-2a\cos^2\psi]$$

$$=\frac{mg}{2a\cos\psi}\left[h-4a\cos^2\psi\right] \qquad \text{...(5)}$$

Suppose the particle leaves the cycloid at the point Q where $\psi=\psi_1$. Then putting $\psi=\psi_1$ and $R=0$ in (5), we have

$$h-4a\cos^2\psi_1=0$$

or
$$\cos^2\psi_1=h/4a. \qquad \text{...(6)}$$

If v_1 is the velocity at Q, then from (4), we have

$$v_1^2=2gh-4ag\cos^2\psi_1=2gh-4ag.(h/4a)=gh.$$

∴ the particle leaves the cycloid at the point Q with velocity $v_1=\sqrt{(gh)}$ inclined at an angle ψ_1 to the horizontal given by (6). Subsequently it describes a parabolic path.

The latus rectum of the parabolic path described after Q

$=(2/g)$ (square of the horizontal velocity at Q)

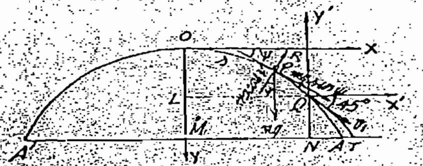
$=(2/g)\,(v_1^2\cos^2\psi_1)=(2/g)\,(gh).\,(h/4a)=h^2/2a.$

Ex. 41. *A particle is placed very near the vertex of a smooth cycloid whose axis is vertical and vertex upwards, and is allowed to run down the curve. Prove that it will leave the curve when it has fallen through half the vertical height of the cycloid.*

Also prove that the latus rectum of the parabola subsequently described is equal to the height of the cycloid.

Also show that it falls upon the base of the cycloid at a distance $(\frac12\pi+\sqrt{3})\,a$ from the centre of the base, a being the radius of the generating circle.

Sol. Let a particle of mass m, starting from rest at O, slide down the arc of a smooth cycloid whose axis OM is vertical and vertex O is upwards. Let P be the position of the particle at any time t such that arc $OP = s$. If the tangent at P makes an angle ψ with the horizontal, then the equations of motion of the particle along the tangent and normal at P are



$$m\frac{d^2s}{dt^2} = mg\sin\psi, \qquad \text{...(1)}$$

and $$m\frac{v^2}{\rho} = mg\cos\psi - R. \qquad \text{...(2)}$$

Also for the cycloid, $s = 4a\sin\psi$. ...(3)

From (1) and (3), we have $\dfrac{d^2s}{dt^2} = -\dfrac{g}{4a}s$.

Multiplying both sides by $2\,(ds/dt)$ and integrating, we have

$$v^2 = \left(\frac{ds}{dt}\right)^2 = \frac{g}{4a}s^2 + A.$$

Initially at O, $s = 0$ and $v = 0$. $\therefore$ $A = 0$.

$$\therefore \quad v^2 = \frac{g}{4a}s^2 = \frac{g}{4a}(4a\sin\psi)^2 = 4ag\sin^2\psi. \qquad \text{...(4)}$$

From (2) and (4), we have

$$R = mg\cos\psi - \frac{mv^2}{\rho} = mg\cos\psi - m.\frac{4ag\sin^2\psi}{4a\cos\psi} \qquad [\because \rho = ds/d\psi = 4a\cos\psi]$$

$$= \frac{mg}{\cos\psi}(\cos^2\psi - \sin^2\psi).$$

If the particle leaves the cycloid at the point Q, then at Q, $R = 0$. From $R = 0$, we have

$$\frac{mg}{\cos\psi}(\cos^2\psi - \sin^2\psi) = 0$$

or $\sin^2\psi = \cos^2\psi$ or $\tan^2\psi = 1$,

or $\tan\psi = 1$ or $\psi = 45°$.

Thus at Q, we have $\psi = 45°$. Putting $\psi = \frac{1}{4}\pi$ in $s = 4a\sin\psi$, we have at Q, $s = 4a\sin\frac{1}{4}\pi = 4a.(1/\sqrt{2}) = 2\sqrt{2}a$. Again, putting $s = 2\sqrt{2}a$ in $s^2 = 8ay$, we have at Q, $y = s^2/8a = 8a^2/8a = a$.

Thus $OL = a$. Therefore $LM = OM - OL = 2a - a = a$. Hence the particle leaves the cycloid at the point Q, when it has fallen through half the vertical height of the cycloid.

Second part. If v_1 is the velocity of the particle at Q, then from (4), we have $v_1^2 = 4ag\sin^2 45° = 2ag$.

Hence the particle leaves the cycloid at Q with velocity $v_1 = \sqrt{(2ag)}$ in a direction making an angle $45°$ downwards with the horizontal. After Q the particle will describe a parabolic path.

Latus rectum of the parabola described after Q

$$= \frac{2v_1^2\cos^2 45°}{g} = \frac{2.2ag.\frac{1}{2}}{g} = 2a,$$

i.e., the latus rectum of the parabola subsequently described is equal to the height of the cycloid.

Third part. The equation of the parabolic path described by the particle after leaving the cycloid at Q with respect to the horizontal and vertical lines QX' and QY' as the coordinate axes is

$$y = x\tan(-45°) - \frac{gx^2}{2v_1^2\cos^2(-45°)}$$ [Note that here the angle of projection for the motion of the projectile is $-45°$]

or $$y = -x - \frac{gx^2}{2.2ag.\frac{1}{2}}$$

or $$y = -x - \frac{x^2}{2a}. \qquad \text{...(5)}$$

Suppose after leaving the cycloid at Q the particle strikes the base of the cycloid at the point T. Let (x_1, y_1) be the coordinates of T with respect to QX' and QY' as the coordinate axes. Then

$$x_1 = NT \quad \text{and} \quad y_1 = -QN = -a.$$

But the point $T\,(x_1, -a)$ lies on the curve (5).

$$\therefore \quad -a = -x_1 - \frac{x_1^2}{2a}$$

or $$x_1^2 + 2ax_1 - 2a^2 = 0.$$

$$\therefore \quad x_1 = \frac{-2a \pm \sqrt{\{4a^2 - 4.1.(-2a^2)\}}}{2.1}.$$

Neglecting the $-$ive sign because x_1 cannot be negative, we have $x_1 = NT = -a + a\sqrt{3}$.

The parametric equations of the cycloid w.r.t. OX and OY as the coordinate axes are

$$x = a(\theta + \sin\theta),\ y = a(1 - \cos\theta),$$

where θ is the parameter and $\theta = 2\psi$.

$\therefore$ At the point Q, where $\psi = \frac{1}{4}\pi$, we have

$$x = LQ = a(2\psi + \sin 2\psi) = a[2.\tfrac{1}{4}\pi + \sin(2.\tfrac{1}{4}\pi)] = a(\tfrac{1}{2}\pi + 1).$$

$\therefore$ the horizontal distance of the point T from the centre M of the base of the cycloid

$$= MT = MN + NT = LQ + NT$$
$$= a(\tfrac{1}{2}\pi + 1) + (-a + a\sqrt{3}) = (\tfrac{1}{2}\pi + \sqrt{3})\,a.$$

CENTRAL ORBITS | SET-III

§ 1. Definitions.

1. Central force. *A force whose line of action always passes through a fixed point is called a central force. The fixed point is known as the centre of force.*

2. Central orbit. *A central orbit is the path described by a particle moving under the action of a central force.* The motion of a planet about the sun is an important example of a central orbit.

Theorem. *A central orbit is always a plane curve.*

Proof. Take the centre of force O as the origin of vectors. Let P be the position of a particle moving in a central orbit at any time t and let $\vec{OP} = \mathbf{r}$. Then $\frac{d^2\mathbf{r}}{dt^2}$ is the expression for the acceleration vector of the particle at the point P. Since the particle moves under the action of a central force with centre at O, therefore the only force acting on the particle at P is along the line OP or PO. So the acceleration vector of P is parallel to the vector $\vec{OP}$.

$\therefore \frac{d^2\mathbf{r}}{dt^2}$ is parallel to $\mathbf{r} \Rightarrow \frac{d^2\mathbf{r}}{dt^2} \times \mathbf{r} = 0$

$$\Rightarrow \frac{d^2\mathbf{r}}{dt^2} \times \mathbf{r} + \frac{d\mathbf{r}}{dt} \times \frac{d\mathbf{r}}{dt} = 0 \quad \left[\because \frac{d\mathbf{r}}{dt} \times \frac{d\mathbf{r}}{dt} = 0\right]$$

$$\Rightarrow \frac{d}{dt}\left(\frac{d\mathbf{r}}{dt} \times \mathbf{r}\right) = 0$$

$$\Rightarrow \frac{d\mathbf{r}}{dt} \times \mathbf{r} = \text{a constant vector} = \mathbf{h}, \text{ say.} \quad ...(1)$$

Taking dot product of both sides of (1) with the vector $\mathbf{r}$, we get

$$\mathbf{r} \cdot \left(\frac{d\mathbf{r}}{dt} \times \mathbf{r}\right) = \mathbf{r} \cdot \mathbf{h}.$$

But the left hand member is a scalar triple product involving two equal vectors, and so it vanishes.

$$\therefore \mathbf{r} \cdot \mathbf{h} = 0,$$

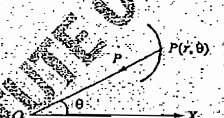
which shows that $\mathbf{r}$ is always perpendicular to a constant vector $\mathbf{h}$.

Thus the radius vector OP is always perpendicular to a fixed direction and hence lies in a plane. Therefore the path of P is a plane curve.

§ 2. Differential equation of a central orbit.

A particle moves in a plane with an acceleration which is always directed to a fixed point O in the plane ; to obtain the differential equation of the path.

Let a particle move in a plane with an acceleration P which is always directed to a fixed point O in the plane. Take the centre of force O as the pole. Let OX be the initial line and (r, θ) the polar co-ordinates of the position P of the moving particle at any instant t.



Since the acceleration of the particle is always directed towards the pole O, therefore the particle has only the radial acceleration and the transverse component of the acceleration of the particle is always zero. So the equations of motion of the particle at the point P are

the radial acceleration *i.e.*, $\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -P$, ...(1)

(the -ive sign has been taken because the radial acceleration P is in the direction of r decreasing)

and the transverse acceleration *i.e.*, $\frac{1}{r}\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0$. ...(2)

From (2), we have $\frac{d}{dt}\left(r^2\frac{d\theta}{dt}\right) = 0$.

Integrating, we get $r^2\frac{d\theta}{dt} = \text{constant} = h$, say. ...(3)

Let $r = 1/u$.

Now from (3), we have

$$d\theta/dt = h/r^2 = hu^2.$$

Also $\frac{dr}{dt} = -\frac{1}{u^2}\frac{du}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}\frac{d\theta}{dt} = -\frac{1}{u^2}\frac{du}{d\theta}u^2h = -h\frac{du}{d\theta}$,

and $\frac{d^2r}{dt^2} = -h\frac{d^2u}{d\theta^2}\frac{d\theta}{dt} = -h\frac{d^2u}{d\theta^2}(u^2h) = -h^2u^2\frac{d^2u}{d\theta^2}$.

Substituting in (1), we have

$$-h^2u^2\frac{d^2u}{d\theta^2} - \frac{1}{u}(u^2h)^2 = -P \quad \text{or} \quad h^2u^2\frac{d^2u}{d\theta^2} + h^2u^3 = P$$

or $$\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2}, \quad ...(4)$$

which is the differential equation of a central orbit in **polar form** referred to the centre of force as the pole.

Pedal form. If p is the length of the perpendicular drawn from the origin upon the tangent at the point P, we have

$$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2.$$

But $u = \frac{1}{r}$. Therefore $\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$

i.e., $$\left(\frac{du}{d\theta}\right)^2 = \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2.$$

So $$\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2. \quad ...(5)$$

Differentiating both sides of (5) w.r.t. 'θ', we have

$$-\frac{2}{p^3}\frac{dp}{d\theta} = 2u\frac{du}{d\theta} + 2\frac{du}{d\theta}\frac{d^2u}{d\theta^2} = 2\frac{du}{d\theta}\left(u + \frac{d^2u}{d\theta^2}\right)$$

or $$-\frac{1}{p^3}\frac{dp}{d\theta} = \frac{du}{d\theta}\cdot\frac{P}{h^2u^2} \quad \text{[From (4)]}$$

or $$-\frac{1}{p^3}\frac{dp}{dr}\cdot\frac{dr}{d\theta} = \left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)\left(\frac{P}{h^2u^2}\right) \quad \left[\because \frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}\right]$$

or $$\frac{1}{p^3}\frac{dp}{dr} = \frac{1}{r^2}\cdot\frac{P}{h^2u^2} = u^2\cdot\frac{P}{h^2u^2} = \frac{P}{h^2}$$

or $$P = \frac{h^2}{p^3}\frac{dp}{dr} \quad ...(6)$$

which is the differential equation of a central orbit in **pedal form.**

Angular momentum or moment of momentum. The expression $r^2(d\theta/dt)$ is called the angular momentum or the moment of momentum about the pole O of a particle of unit mass moving in a plane curve. Since in a central orbit $r^2(d\theta/dt) = \text{constant}$, therefore *in a central orbit the angular momentum is conserved.*

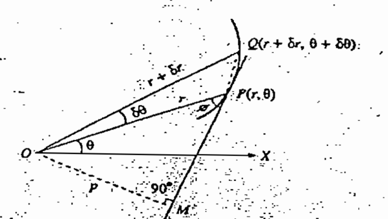
§ 3. Rate of description of the sectorial area.

In every central orbit, the sectorial area traced out by the radius vector to the centre of force increases uniformly per unit of time, and the linear velocity varies inversely as the perpendicular from the centre upon the tangent to the path.

Take the centre of force O as the pole and OX as the initial line. Let $P(r, \theta)$ and $Q(r + \delta r, \theta + \delta\theta)$ be the positions of a particle moving in a central orbit at times t and $t + \delta t$ respectively.

Sectorial area OPQ described by the particle in time δt

= area of the ΔOPQ



[$\because$ the point Q is very close to P and ultimately we have to take limit as $Q \to P$]

$= \frac{1}{2}OP.OQ \sin \angle POQ = \frac{1}{2}r(r + \delta r)\sin\delta\theta$.

$\therefore$ rate of description of the sectorial area

$$= \lim_{\delta t \to 0}\frac{\text{sectorial area } OPQ}{\delta t} = \lim_{\delta t \to 0}\frac{\frac{1}{2}(r + \delta r)\sin\delta\theta}{\delta t}$$

$$= \lim_{\delta t \to 0}\frac{1}{2}r(r + \delta r)\cdot\frac{\sin\delta\theta}{\delta\theta}\cdot\frac{\delta\theta}{\delta t} = \frac{1}{2}r^2\frac{d\theta}{dt} = \frac{1}{2}h. \quad ...(1)$$

[$\because r^2(d\theta/dt) = h$]

Thus the rate of description of the sectorial area is constant and is equal to $h/2$.

The rate of description of the sectorial area is also called the *areal velocity* of the particle about the fixed point O.

Again for a central orbit, we have $r^2\frac{d\theta}{dt} = h$.

$$\therefore r^2\frac{d\theta}{ds}\frac{ds}{dt} = h \quad \text{or} \quad r^2\frac{d\theta}{ds}\cdot v = h. \quad ...(2)$$

[$\because ds/dt = v$ (*i.e.*, the linear velocity)]

But from differential calculus, we have $r\frac{d\theta}{ds} = \sin\phi$, where ϕ is the angle between the radius vector and the tangent.

$\therefore r^2\frac{d\theta}{ds} = r\sin\phi = p$, where p is the length of the perpendicular drawn from the pole O on the tangent at P.

Putting $r^2(d\theta/ds) = p$ in (2), we get $vp = h$.

or $$v = \frac{h}{p} \qquad ...(3)$$

$\therefore v \propto 1/p$.

i.e., the linear velocity at P varies inversely as the perpendicular from the fixed point upon the tangent to the path.

From (3), we have $v^2 = \frac{h^2}{p^2}$.

But $$\frac{1}{p^2} = \frac{1}{r^2} + \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = u^2 + \left(\frac{du}{d\theta}\right)^2$$

$$\therefore v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] \qquad ...(4)$$

The equation (4) gives the linear velocity at any point of the path of a central orbit.

§ 4. Elliptic orbit. (Focus as the centre of force).

A particle moves in an ellipse under a force which is always directed towards its focus; to find

(i) the law of force,

(ii) the velocity at any point of its path

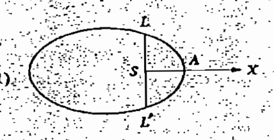
and (iii) the periodic time.

We know that the polar equation of an ellipse referred to its focus S as pole is

$$\frac{l}{r} = 1 + e\cos\theta$$

or $$u = \frac{1}{l} + \frac{e}{l}\cos\theta, \qquad ...(1)$$

where $u = 1/r$.



Differentiating, we have

$$\frac{du}{d\theta} = -\frac{e}{l}\sin\theta \text{ and } \frac{d^2u}{d\theta^2} = -\frac{e}{l}\cos\theta.$$

(i) **Law of force.** We know that the differential equation of a central orbit referred to the centre of force as pole is

$$\frac{P}{h^2u^2} = u + \frac{d^2u}{d\theta^2}$$

where P is the central acceleration assumed to be attractive.

Now here $$P = h^2u^2\left[u + \frac{d^2u}{d\theta^2}\right]$$

$$= h^2u^2\left[\frac{1}{l} + \frac{e}{l}\cos\theta - \frac{e}{l}\cos\theta\right],$$

substituting for u and $d^2u/d\theta^2$

$$= \frac{h^2u^2}{l} = \frac{h^2/l}{r^2} = \frac{\mu}{r^2}, \qquad ...(2)$$

where $$\mu = h^2/l \text{ or } h^2 = \mu l. \qquad ...(3)$$

$$\therefore P \propto \frac{1}{r^2}.$$

Hence the acceleration varies inversely as the square of the distance of the particle from the focus. Also the force is attractive because the value of P is positive.

(ii) **Velocity.** We know that the velocity in a central orbit is given by

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right].$$

$\therefore$ here, $$v^2 = h^2\left[\left(\frac{1}{l} + \frac{e}{l}\cos\theta\right)^2 + \left(-\frac{e}{l}\sin\theta\right)^2\right]$$

$$= h^2\left[\frac{1}{l^2} + \frac{2e}{l^2}\cos\theta + \frac{e^2}{l^2}\right] = \frac{h^2}{l}\left[\frac{1+e^2}{l} + 2\frac{e\cos\theta}{l}\right]$$

$$= \mu\left[\frac{1+e^2}{l} + 2\left(u - \frac{1}{l}\right)\right] \qquad \text{[from (1) and (3)]}$$

$$= \mu\left[2u - \frac{1-e^2}{l}\right] = \mu\left[\frac{2}{r} - \frac{1-e^2}{l}\right].$$

If $2a$ and $2b$ are the lengths of the major and the minor axes of the ellipse, we have

$$l = \text{the semi latus rectum} = \frac{b^2}{a} = \frac{a^2(1-e^2)}{a} = a(1-e^2).$$

$$\therefore \frac{1-e^2}{l} = \frac{1}{a} \quad \therefore v^2 = \mu\left(\frac{2}{r} - \frac{1}{a}\right), \qquad ...(4)$$

which gives the velocity of the particle at any point of its path.

Equation (4) *shows that the magnitude of the velocity at any point of the path depends only on the distance from the focus and that it is independent of the direction of the motion. Also* $v^2 < 2\mu/r$.

(iii) **Periodic time.** We know that in a central orbit the rate of description of the sectorial area is constant and is equal to $h/2$. Let T be the time period for one complete revolution *i.e.,* the time taken by the particle in describing the whole of the ellipse. The sectorial area traced in describing the whole arc of the ellipse is equal to the whole area of the ellipse.

$\therefore T(h/2) =$ the whole area of the ellipse $= \pi ab$.

$$\therefore T = \frac{2\pi ab}{h} = \frac{2\pi ab}{\sqrt{(\mu l)}} \qquad [\because h^2 = \mu l]$$

or $$T = \frac{2\pi ab}{\sqrt{(\mu(b^2/a))}} \qquad [\because l = b^2/a]$$

or $$T = \frac{2\pi a^{3/2}}{\sqrt{\mu}}. \qquad ...(5)$$

i.e., the time period for one complete revolution is proportional to $a^{3/2}$, a being semi-major axis.

§ 5. Hyperbolic and parabolic orbits.

(Centre of force being the focus).

(i) **Hyperbolic orbit.** In the case of hyperbola, we have $e > 1$.

Also $l = \frac{b^2}{a} = \frac{a^2(e^2-1)}{a} = a(e^2-1)$.

Proceeding as in § 4, we have $P = \mu/r^2$, where $h^2 = \mu l$.

[Note that this result does not depend upon the value of e].

Also proceeding as in establishing the result (4) of § 4, we have here

$$v^2 = \mu\left[\frac{2}{r} + \frac{e^2-1}{l}\right] \quad (\because e > 1)$$

or $$v^2 = \mu\left[\frac{2}{r} + \frac{1}{a}\right]. \quad \text{Note that here } v^2 > 2\mu/r.$$

(ii) **Parabolic orbit.** In this case $e = 1$.

Proceeding as in § 4, we have here $P = \mu/r^2$ and $v^2 = 2\mu/r$.

§ 6. Velocity from infinity.

In connection with the central orbits by the phrase '*velocity from infinity at any point*' we mean the velocity that a particle would acquire if it moved from rest at infinity in a straight line to that point under the action of an attractive force in accordance with the law associated with the orbit.

Suppose a particle falls from rest from infinity in a straight line under the action of a central attractive acceleration P directed towards the centre of force O.

Let Q be the position of the particle at any time t, where $OQ = r$.

Suppose v is the velocity of the particle at Q. The expression for acceleration at the point Q is $v\,(dv/dr)$.

The equation of motion of the particle at the point Q is

$v\frac{dv}{dr} = -P$, [–ive sign has been taken because the acceleration P is in the direction of r decreasing]

or $$v\,dv = -P\,dr. \qquad ...(1)$$

Let V be the velocity acquired in falling from rest at infinity to a point distant a from the centre of force O. Then integrating (1) from infinity to the point $r = a$, we get

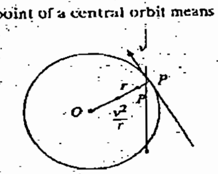
$$\int_0^V v\,dv = -\int_\infty^a P\,dr$$

or $$\frac{1}{2}V^2 = -\int_\infty^a P\,dr \text{ or } V^2 = -2\int_\infty^a P\,dr,$$

which gives the velocity from infinity at a distance a from the centre of force while moving under the central acceleration P associated with the orbit.

§ 7. Velocity in a circle.

The phrase '*velocity in a circle*' at any point of a central orbit means the velocity necessary to describe a circle, passing through that point and with centre at the centre of force, while moving under the action of the prescribed force associated with the orbit.



Take the centre of force O as the pole. Let P be the central acceleration, directed towards O, at any point P of the orbit, where $OP = r$. Suppose v is the velocity in a circle at P. Then v is the velocity at the point P of a particle which moves, under the same central acceleration P, in a circle with centre at O. But for a circle with centre at the pole O, the radius vector OP is also normal to the circle at P. Therefore,

the central radial acceleration P

= the inward normal acceleration v^2/ρ

i.e., $P = v^2/r$. [$\because$ for the circle, $\rho = r$]

$\therefore v^2 = rP$.

Thus while moving under a central attractive acceleration P, the velocity V in a circle at a distance a from the centre of force is given by

$$V^2 = a\left[P\right]_{r=a}.$$

§ 8. **Given the central orbit, to find the law of force.**

Case I. *The equation of the orbit being given in the polar form* (r, θ).

We know that referred to the centre of force as pole, the differential equation of a central orbit is

$$\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2}, \quad \text{...(1)}$$

where P is the central acceleration assumed to be attractive.

From the given equation of the orbit we can easily calculate u and $d^2u/d\theta^2$ and substituting their values in (1) we can determine P. Thus we find the law of force. If the value of P is positive, the force is attractive and if the value of P is negative, the force is repulsive.

Case II. *The equation of the orbit being given in the pedal form* (p, r).

The differential equation of a central orbit in (p, r) form is

$$\frac{h^2}{p^3}\frac{dp}{dr} = P. \quad \text{...(2)}$$

From the given equation of the orbit in (p, r) form, we can find out dp/dr and then substituting its value in (2) we can determine P.

Solved Examples

Ex. 1. *Find the law of force towards the pole under which the following curves are described.*

(i) $au = e^{n\theta}$ and (ii) $r = ae^{\theta \cot \alpha}$.

Sol. (i) We have $au = e^{n\theta}$. ...(1)

Differentiating w.r.t. 'θ', we have

$$\frac{du}{d\theta} = \frac{n}{a}e^{n\theta} = nu \quad \text{and} \quad \frac{d^2u}{d\theta^2} = n\frac{du}{d\theta} = n \cdot nu = n^2u.$$

Referred to the centre of force as pole, the differential equation of a central orbit is

$$\frac{P}{h^2u^2} = u + \frac{d^2u}{d\theta^2},$$

where P is the central acceleration assumed to be attractive.

$$\therefore \quad P = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right) = h^2u^2(u + n^2u) = h^2(1 + n^2)u^3$$

$$= \frac{h^2(1+n^2)}{r^3}. \quad [\because u = 1/r]$$

$\therefore$ $P \propto 1/r^3$ *i.e.*, the force varies inversely as the cube of the distance from the pole. Also the positive value of P indicates that the force is attractive *i.e.*, is directed towards the pole.

(ii) We have $r = ae^{\theta \cot \alpha}$

or $\dfrac{1}{u} = ae^{\theta \cot \alpha}$, $[\because r = 1/u]$.

$$\therefore \quad u = \frac{1}{a}e^{-\theta \cot \alpha}.$$

Differentiating w.r.t. 'θ', we have

$$\frac{du}{d\theta} = -\frac{\cot \alpha}{a}e^{-\theta \cot \alpha} = -u \cot \alpha$$

and $$\frac{d^2u}{d\theta^2} = -\frac{du}{d\theta}\cot \alpha = -(-u\cot\alpha)\cot\alpha = u\cot^2\alpha.$$

The differential equation of the central orbit is

$$\frac{P}{h^2u^2} = u + \frac{d^2u}{d\theta^2}$$

$$\therefore \quad P = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right) = h^2u^2(u + u\cot^2\alpha) = h^2(1 + \cot^2\alpha)u^3$$

$$= \frac{h^2 \operatorname{cosec}^2\alpha}{r^3}.$$

$\therefore$ $P \propto 1/r^3$ *i.e.*, the force varies inversely as the cube of the distance from the pole. Also the positive value of P indicates that the force is attractive.

Ex. 2. *A particle describes the curve $r^n = a^n \cos n\theta$ under a force to the pole. Find the law of force.*

Hence obtain the law of force under which a cardioid can be described.

Sol. The equation of the curve is $r^n = a^n \cos n\theta$.

Putting $r = 1/u$, we have

$$\frac{1}{u^n} = a^n \cos n\theta \quad \text{or} \quad a^nu^n = \sec n\theta. \quad \text{...(1)}$$

Taking logarithm of both sides of (1), we have

$$n \log a + n \log u = \log \sec n\theta.$$

Differentiating w.r.t. 'θ', we have

$$\frac{n}{u}\frac{du}{d\theta} = \frac{1}{\sec n\theta}n \sec n\theta \tan n\theta \quad \text{or} \quad \frac{du}{d\theta} = u \tan n\theta.$$

Differentiating again w.r.t. 'θ', we have

$$\frac{d^2u}{d\theta^2} = \frac{du}{d\theta}\tan n\theta + u(\sec^2 n\theta) \cdot n$$

$$= u \tan n\theta \cdot \tan n\theta + un \sec^2 n\theta \quad [\because du/d\theta = u \tan n\theta]$$

$$= u \tan^2 n\theta + un \sec^2 n\theta. \quad \text{...(2)}$$

The differential equation of the central orbit is

$$\frac{P}{h^2u^2} = u + \frac{d^2u}{d\theta^2}.$$

$$\therefore \quad P = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right) = h^2u^2(u + u\tan^2 n\theta + un\sec^2 n\theta)$$

[putting the value of $d^2u/d\theta^2$ from (2)]

$$= h^2u^3(\sec^2 n\theta + n\sec^2 n\theta) = h^2u^3(1+n)\sec^2 n\theta$$

$$= h^2(1+n)u^3 \cdot (a^nu^n)^2$$

$$= h^2a^{2n}(1+n)u^{2n+3} = \frac{h^2a^{2n}(1+n)}{r^{2n+3}}.$$

$\therefore$ $P \propto 1/r^{2n+3}$ *i.e.*, the force varies inversely as the $(2n+3)$th power of the distance from the pole.

Second part. Putting $n = 1/2$ in the equation of the path, we get

$$r^{1/2} = a^{1/2}\cos\tfrac{1}{2}\theta$$

or $$r = a\cos^2\tfrac{1}{2}\theta$$

or $r = \frac{1}{2}a \cdot 2\cos^2\frac{1}{2}\theta = \frac{1}{2}a(1 + \cos\theta)$, which is the equation of a cardioid.

Now putting $n = \frac{1}{2}$ in the value of P, we get

$$P \propto \frac{1}{r^{1+3}} \quad i.e., \quad P \propto \frac{1}{r^4}.$$

Ex. 3. *A particle describes the curve $r^2 = a^2 \cos 2\theta$ under a force to the pole. Find the law of force.*

Sol. The equation of the curve is $r^2 = a^2 \cos 2\theta$.

Proceed as in Ex. 2. Replacing n by 2 in the preceding exercise 2, we have

$$P = \frac{3h^2a^4}{r^7}. \quad \text{Therefore } P \propto \frac{1}{r^7}$$

i.e., the force varies inversely as the seventh power of the distance from the pole.

Ex. 4. *Find the law of force towards the pole under which the curve $r^n \cos n\theta = a^n$ is described.*

Sol. The equation of the curve is $r^n \cos n\theta = a^n$.

Replacing r by $1/u$, we have

$$\frac{1}{u^n}\cos n\theta = a^n$$

or $$a^nu^n = \cos n\theta. \quad \text{...(1)}$$

Taking logarithm of both sides of (1), we have

$$n \log a + n \log u = \log \cos n\theta.$$

Differentiating w.r.t. 'θ', we have

$$\frac{n}{u}\frac{du}{d\theta} = \frac{1}{\cos n\theta}\cdot(-n\sin n\theta)$$

or $$\frac{du}{d\theta} = -u\tan n\theta. \quad \text{...(2)}$$

Differentiating again w.r.t. 'θ', we have

$$\frac{d^2u}{d\theta^2} = -\frac{du}{d\theta}\tan n\theta - un\sec^2 n\theta = u\tan^2 n\theta - un\sec^2 n\theta$$

[Substituting for $du/d\theta$ from (2)]

The differential equation of the central orbit is

$$\frac{P}{h^2u^2} = u + \frac{d^2u}{d\theta^2}.$$

$$\therefore \quad P = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right) = h^2u^2(u + u\tan^2 n\theta - un\sec^2 n\theta)$$

$$= h^2u^3(\sec^2 n\theta - n\sec^2 n\theta) = h^2u^3(1-n)\sec^2 n\theta$$

$$= h^2u^3(1-n)\cdot\left(\frac{1}{a^nu^n}\right)^2 = \frac{h^2}{a^{2n}}\frac{(1-n)}{u^{2n-3}} = \frac{h^2(1-n)}{a^{2n}}\cdot r^{2n-3}.$$

$\therefore$ $P \propto r^{2n-3}$ *i.e.*, the force is proportional to the $(2n-3)^{th}$ power of the distance from the pole.

Ex. 5. *A particle describes the curve $r^n = A\cos n\theta + B\sin n\theta$ under a force to the pole. Find the law of force.*

Sol. Here $r^n = A\cos n\theta + B\sin n\theta$.

Let $A = k\cos\alpha$ and $B = k\sin\alpha$, where k and α are constants.

Replacing r by $1/u$, we have

$$r^n = u^{-n} = k\cos(n\theta - \alpha). \quad \text{...(1)}$$

$$\therefore \quad -n\log u = \log k + \log\cos(n\theta - \alpha).$$

Differentiating both sides w.r.t. 'θ', we have

$$\frac{-n}{u}\frac{du}{d\theta} = -n\tan(n\theta - \alpha) \quad \text{or} \quad \frac{du}{d\theta} = u\tan(n\theta - \alpha).$$

$$\therefore \quad \frac{d^2u}{d\theta^2} = \frac{du}{d\theta}\cdot\tan(n\theta - \alpha) + un\sec^2(n\theta - \alpha)$$

$$= u\tan^2(n\theta - \alpha) + un\sec^2(n\theta - \alpha).$$

The differential equation of the path is

$$\frac{P}{h^2u^2} = u + \frac{d^2u}{d\theta^2}.$$

$$\therefore\ P = h^2u^2 [u + u\tan^2(n\theta - \alpha) + un\sec^2(n\theta - \alpha)]$$
$$= h^2u^3 [\sec^2(n\theta - \alpha) + n\sec^2(n\theta - \alpha)]$$
$$= (1+n) h^2u^3 \sec^2(n\theta - \alpha)$$
$$= (1+n) h^2u^3 (ku^n)^2 \quad [\because \text{ from (1), } \sec(n\theta - \alpha) = ku^n]$$
$$= \frac{(1+n) h^2k^2}{r^{2n+3}}.$$

Thus $P \propto \frac{1}{r^{2n+3}}$, *i.e.*, the force is inversely proportional to the $(2n+3)$th power of the distance from the pole.

Ex. 6. *A particle describes a circle, pole on its circumference, under a force P to the pole. Find the law of force.*

Or

A particle describes the curve $r = 2a\cos\theta$ under the force P to the pole. Find the law of force.

Sol. Let a be the radius of the circle. If we take pole on the circumference of the circle and the diameter through the pole as the initial line, the equation of the circle is

$$r = 2a\cos\theta$$

or $$1/u = 2a\cos\theta. \quad ...(1)$$

$$\therefore\ -\log u = \log(2a) + \log\cos\theta.$$

Differentiating w.r.t. 'θ', we have

$$-\frac{1}{u}\frac{du}{d\theta} = -\tan\theta \quad \text{or} \quad \frac{du}{d\theta} = u\tan\theta.$$

and $$\frac{d^2u}{d\theta^2} = u\sec^2\theta + \frac{du}{d\theta}\tan\theta$$
$$= u\sec^2\theta + u\tan\theta\,.\tan\theta = u\sec^2\theta + u\tan^2\theta.$$

The differential equation of the path is

$$\frac{P}{h^2u^2} = u + \frac{d^2u}{d\theta^2}.$$

$$\therefore\ P = h^2u^2 [u + u\sec^2\theta + u\tan^2\theta] = h^2u^3 [(1+\tan^2\theta) + \sec^2\theta]$$
$$= 2h^2u^3\sec^2\theta$$
$$= 2h^2u^3 (2au)^2$$
$$= \frac{8a^2h^2}{r^5}.$$

$\therefore$ $P \propto 1/r^5$ *i.e.*, the force varies inversely as the fifth power of the distance from the pole. Also the positive value of P indicates that the force is attractive.

Ex. 7. *Find the law of force towards the pole under which the following curves are described.*

(i) $a = r\cosh n\theta$ and (ii) $a = r\tanh(\theta/\sqrt{2})$.

Sol. (i) The equation of the curve is

$$a = r\cosh n\theta = (1/u)\cosh n\theta$$

or $$u = (1/a)\cosh n\theta. \quad ...(1)$$

Differentiating, $\frac{du}{d\theta} = \frac{n}{a}\sinh n\theta$ and $\frac{d^2u}{d\theta^2} = \frac{n^2}{a}\cosh n\theta.$

The differential equation of the central orbit is

$$\frac{P}{h^2u^2} = u + \frac{d^2u}{d\theta^2}.$$

$$\therefore\ P = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right) = h^2u^2\left(u + \frac{n^2}{a}\cosh n\theta\right) = h^2u^2(u + n^2u)$$

$$= h^2(1+n^2)u^3 = \frac{h^2(1+n^2)}{r^3}.$$

$\therefore$ $P \propto 1/r^3$ *i.e.*, the force varies inversely as the cube of the distance from the pole.

(ii) The equation of the curve is

$$a = r\tanh(\theta/\sqrt{2}) = (1/u)\tanh(\theta/\sqrt{2})$$

or $$u = (1/a)\tanh(\theta/\sqrt{2}). \quad ...(1)$$

Differentiating, $\frac{du}{d\theta} = \frac{1}{a\sqrt{2}}\operatorname{sech}^2(\theta/\sqrt{2})$

and $$\frac{d^2u}{d\theta^2} = \frac{1}{a\sqrt{2}} \cdot 2\operatorname{sech}(\theta/\sqrt{2}) \cdot \left\{-\frac{1}{\sqrt{2}}\operatorname{sech}(\theta/\sqrt{2})\tanh(\theta/\sqrt{2})\right\}$$
$$= -\frac{1}{a}\operatorname{sech}^2(\theta/\sqrt{2})\tanh(\theta/\sqrt{2}) = -u\operatorname{sech}^2(\theta/\sqrt{2}).$$

The differential equation of the central orbit is

$$\frac{P}{h^2u^2} = u + \frac{d^2u}{d\theta^2}.$$

$$\therefore\ P = h^2u^2\left[u + \frac{d^2u}{d\theta^2}\right]$$
$$= h^2u^2[u - u\operatorname{sech}^2(\theta/\sqrt{2})] = h^2u^3[1 - \operatorname{sech}^2(\theta/\sqrt{2})]$$
$$= h^2u^3\tanh^2(\theta/\sqrt{2}) \quad [\because\ \operatorname{sech}^2\theta = 1 - \tanh^2\theta]$$
$$= h^2u^3(au)^2 \quad [\text{from (1)}]$$
$$= h^2a^2u^5 = \frac{h^2a^2}{r^5}.$$

$\therefore$ $P \propto 1/r^5$ *i.e.*, the force varies inversely as the $5th$ power of the distance from the pole.

Ex. 8. *A particle describes the curve $r = a\sin n\theta$ under a force P to the pole. Find the law of force.*

Sol. The equation of the curve is

$$r = a\sin n\theta$$

or $$u = \frac{1}{r} = \frac{1}{a}\operatorname{cosec} n\theta. \quad ...(1)$$

Differentiating, $\frac{du}{d\theta} = -\frac{n}{a}\operatorname{cosec} n\theta\cot n\theta = -nu\cot n\theta,$

and $$\frac{d^2u}{d\theta^2} = n^2u\operatorname{cosec}^2 n\theta - n\frac{du}{d\theta}\cot n\theta$$
$$= n^2u\operatorname{cosec}^2 n\theta - n\,.(-nu\cot n\theta)\cot n\theta$$
$$= n^2u^2\operatorname{cosec}^2 n\theta + n^2u\cot^2 n\theta.$$

The differential equation of the central orbit is

$$\frac{P}{h^2u^2} = u + \frac{d^2u}{d\theta^2}.$$

$$\therefore\ P = h^2u^2\left(u + \frac{d^2u}{d\theta^2}\right) = h^2u^2(u + n^2u\operatorname{cosec}^2 n\theta + n^2u\cot^2 n\theta)$$
$$= h^2u^3[1 + n^2\operatorname{cosec}^2 n\theta + n^2(\operatorname{cosec}^2 n\theta - 1)]$$
$$= h^2u^3[2n^2\operatorname{cosec}^2 n\theta - (n^2 - 1)]$$
$$= h^2u^3[2n^2(au)^2 - (n^2 - 1)]$$
$$= h^2[2n^2a^2u^5 - (n^2-1)u^3]$$
$$= h^2\left[\frac{2n^2a^2}{r^5} - \frac{(n^2-1)}{r^3}\right].$$

$$\therefore\ P \propto \left[\frac{2n^2a^2}{r^5} - \frac{(n^2-1)}{r^3}\right].$$

Ex. 9. *Find the law of force towards the pole under which the following curves are described.*

(i) $r^2 = 2ap$, (ii) $p^2 = ar$ and (iii) $b^2/p^2 = (2a/r) - 1$.

Sol. (i) The equation of the curve is $r^2 = 2ap$.

$$\therefore\ \frac{1}{p} = \frac{2a}{r^2} \quad \text{or} \quad \frac{1}{p^2} = \frac{4a^2}{r^4}.$$

Differentiating w.r.t. 'r', we have

$$-\frac{2}{p^3}\frac{dp}{dr} = -\frac{16a^2}{r^5}.$$

$$\frac{h^2}{p^3}\frac{dp}{dr} = \frac{8a^2h^2}{r^5} \quad ...(1)$$

Now from the pedal equation of a central orbit, we have

$$P = \frac{h^2}{p^3}\frac{dp}{dr} = \frac{8a^2h^2}{r^5}. \quad [\text{from (1)}]$$

$\therefore$ $P \propto 1/r^5$ *i.e.*, the force varies inversely as the fifth power of the distance from the pole.

(ii) The equation of the curve is $p^2 = ar$, which is the pedal equation of a parabola referred to the focus as the pole.

$$\therefore\ \frac{1}{p^2} = \frac{1}{a}\frac{1}{r}.$$

Differentiating w.r.t. 'r', we get

$$-\frac{2}{p^3}\frac{dp}{dr} = -\frac{1}{a}\frac{1}{r^2}.$$

$$\frac{h^2}{p^3}\frac{dp}{dr} = \frac{h^2}{2a}\frac{1}{r^2}. \quad ...(1)$$

From the pedal equation of a central orbit, we have

$$P = \frac{h^2}{p^3}\frac{dp}{dr} = \frac{h^2}{2a}\frac{1}{r^2}. \quad [\text{from (1)}]$$

$\therefore$ $P \propto 1/r^2$ *i.e.*, the force varies inversely as the square of the distance from the pole.

(iii) The equation of the given central orbit is

$$\frac{b^2}{p^2} = \frac{2a}{r} - 1. \quad ...(1)$$

(1) is the pedal equation of an ellipse referred to the focus as pole.

Differentiating both sides of (1) w.r.t. 'r', we get

$$-\frac{2b^2}{p^3}\frac{dp}{dr} = -\frac{2a}{r^2}, \quad \text{or} \quad \frac{h^2}{p^3}\frac{dp}{dr} = \frac{a}{b^2}\frac{h^2}{r^2}.$$

$$\therefore\ P = \frac{h^2}{p^3}\frac{dp}{dr} = \frac{ah^2}{b^2}\frac{1}{r^2}.$$

Thus $P \propto 1/r^2$ *i.e.*, the acceleration varies inversely as the square of the distance from the focus of the ellipse.

Ex. 10. *A particle describes the curve $r^2 = a^2\cos^2\theta + b^2\sin^2\theta$ under an attraction to the origin, prove that the attraction at a distance r is*

$$h^2[2(a^2+b^2)r^2 - 3a^2b^2]\,.\,r^{-7}.$$

Sol. The equation of the given curve is

$$r^2 = a^2\cos^2\theta + b^2\sin^2\theta$$

or $$\frac{1}{u^2} = \frac{a^2}{2}(1+\cos 2\theta) + \frac{b^2}{2}(1 - \cos 2\theta)$$

or $$\frac{1}{u^2} = \tfrac{1}{2}(a^2+b^2) + \tfrac{1}{2}(a^2 - b^2)\cos 2\theta. \quad ...(1)$$

Differentiating w.r.t. 'θ', we have

$$-\frac{2}{u^3}\frac{du}{d\theta} = -(a^2-b^2)\sin 2\theta$$

or $$\frac{du}{d\theta} = \tfrac{1}{2}(a^2-b^2)u^3\sin 2\theta.$$

Differentiating again w.r.t. 'θ', we have

$$\frac{d^2u}{d\theta^2} = \frac{3}{2}(a^2-b^2)u^2\cdot\frac{du}{d\theta}\sin 2\theta + (a^2-b^2)u^3\cos 2\theta$$
$$= \tfrac{3}{2}(a^2-b^2)u^2\cdot\tfrac{1}{2}(a^2-b^2)u^3\sin 2\theta\cdot\sin 2\theta + (a^2-b^2)u^3\cos 2\theta$$
$$= \tfrac{3}{4}u^5(a^2-b^2)^2\sin^2 2\theta + (a^2-b^2)u^3\cos 2\theta$$
$$= \tfrac{3}{4}u^5(a^2-b^2)^2(1-\cos^2 2\theta) + u^3(a^2-b^2)\cos 2\theta$$
$$= \tfrac{3}{4}u^5(a^2-b^2)^2 - \tfrac{3}{4}u^5\cdot\{(a^2-b^2)\cos 2\theta\}^2 + u^3(a^2-b^2)\cos 2\theta.$$

Now from (1), $(a^2-b^2)\cos 2\theta = \dfrac{2}{u^2}-(a^2+b^2)$.

$$\therefore\ \frac{d^2u}{d\theta^2} = \tfrac{3}{4}u^5(a^2-b^2)^2 - \tfrac{3}{4}u^5\left\{\frac{2}{u^2}-(a^2+b^2)\right\}^2 + u^3\cdot\left\{\frac{2}{u^2}-(a^2+b^2)\right\}$$
$$= \tfrac{3}{4}u^5(a^2-b^2)^2 - \tfrac{3}{4}u^5\left\{\frac{4}{u^4}-\frac{4}{u^2}(a^2+b^2)+(a^2+b^2)^2\right\} + 2u-(a^2+b^2)u^3$$
$$= \tfrac{3}{4}u^5(a^2-b^2)^2 - 3u + 3u^3(a^2+b^2) - \tfrac{3}{4}u^5(a^2+b^2)^2 + 2u-(a^2+b^2)u^3$$
$$= \tfrac{3}{4}u^5\{(a^2-b^2)^2-(a^2+b^2)^2\} + 2u^3(a^2+b^2) - u$$
$$= 2(a^2+b^2)u^3 - 3a^2b^2u^5 - u.$$

The differential equation of the central orbit is

$$\frac{P}{h^2u^2} = u + \frac{d^2u}{d\theta^2}.$$

$$\therefore\ P = h^2u^2\left[u+\frac{d^2u}{d\theta^2}\right] = h^2u^2[u+2(a^2+b^2)u^3-3a^2b^2u^5-u]$$
$$= h^2u^7[2(a^2+b^2)r^2-3a^2b^2] = h^2r^{-7}[2(a^2+b^2)r^2-3a^2b^2].$$

Ex. 11. *Show that the only law for a central attraction for which the velocity in a circle at any distance is equal to the velocity acquired in falling from infinity to the distance is that of inverse cube.*

Sol. Let the central acceleration P be given by

$$P = f'(r). \qquad \text{...(1)}$$

The equation of motion of the particle falling from infinity under the cental acceleration given by (1) is

$$v\frac{dv}{dr} = -P = -f'(r).$$

[Refer § 6 of this chapter on page 8]

or $\quad 2v\,dv = -2f'(r)\,dr.$

Integrating, $\quad v^2 = -2\int f'(r)\,dr + A,$

where A is constant of integration.

or $\quad v^2 = -2f(r) + A. \qquad \text{...(2)}$

Thus the velocity v at a distance r acquired in falling from infinity is given by (2). Again let V be the velocity of the particle moving in a circle under the same central acceleration P at a distance r from the centre of the circle. For a circle with centre at the centre of force pole, we have

the central radial attractive acceleration P = the inward normal acceleration V^2/ρ.

$\therefore\quad P = V^2/r \qquad [\because$ for the circle $\rho = r]$

or $\quad V^2 = rP = rf'(r).$

But according to the question

$V = v \quad$ or $\quad V^2 = v^2.$

$\therefore\quad rf'(r) = -2f(r) + A$

or $\quad r^2f'(r) + 2rf(r) = Ar$

or $\quad \dfrac{d}{dr}\{r^2f(r)\} = Ar.$

Integrating both sides w.r.t. 'r', we have

$r^2f(r) = \tfrac{1}{2}Ar^2 + B$, where B is a constant

or $\quad f(r) = \dfrac{A}{2} + \dfrac{B}{r^2}.$

Differentiating both sides w.r.t. 'r', we have

$$f'(r) = \frac{-2B}{r^3}$$

so that $\quad P = -\dfrac{2B}{r^3} \qquad [\because P = f'(r)]$

$\therefore\ P \propto 1/r^3$ *i.e.*, the law of force is that of inverse cube.

Ex. 12. *In a central orbit described under a force to a centre, the velocity at any point is inversely proportional to the distance of the point from the centre of force. Show that the path is an equiangular spiral.*

Sol. If v is the velocity of the particle at any point at a distance r from the centre of force, then according to the question

$$v \propto \frac{1}{r} \quad\text{or}\quad v = \frac{k}{r}, \qquad \text{...(1)}$$

where k is a constant.

But in a central orbit $\qquad v = h/p. \qquad \text{...(2)}$

where p is the length of the perpendicular from the pole on the tangent at any point of the path.

From (1) and (2), we have

$$\frac{k}{r} = \frac{h}{p} \quad\text{or}\quad p = \frac{h}{k}r$$

or $\quad p = ar$, where $a = h/k$ = a constant.

This is the pedal equation of an equiangular spiral. Hence the path is an equiangular spiral.

Ex. 13. *The velocity at any point of a central orbit is $(1/n)$th of what it would be for a circular orbit at the same distance. Show that the central force varies as $\dfrac{1}{r^{(2n^2+1)}}$ and that the equation of the orbit is*

$$r^{n^2-1} = a^{n^2-1}\cos(n^2-1)\theta.$$

Sol. Under the same central force P, let v and V be the velocities at a distance r from the centre of force in the central orbit and the circular orbit respectively. Then according to the question, we have $v = V/n$.

or $\qquad v^2 = V^2/n^2 \qquad \text{...(1)}$

But $\qquad V^2/r = P$

or $\qquad V^2 = Pr = P/u. \qquad \text{...(2)}$

$\therefore$ from (1) and (2), we have

$$v^2 = \frac{P}{n^2u} \quad\text{or}\quad h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right] = \frac{P}{n^2u} \qquad \text{...(3)}$$

$[\because$ for a central orbit, $v^2 = h^2\{u^2+(du/d\theta)^2\}]$.

Differentiating both sides of (3) w.r.t. 'θ', we have

$$h^2\left[2u\frac{du}{d\theta}+2\frac{du}{d\theta}\frac{d^2u}{d\theta^2}\right] = \frac{1}{n^2}\left[\frac{1}{u}\frac{dP}{d\theta}-\frac{P}{u^2}\frac{du}{d\theta}\right]$$
$$= \frac{1}{n^2}\left[\frac{1}{u}\frac{dP}{du}\frac{du}{d\theta}-\frac{P}{u^2}\frac{du}{d\theta}\right].$$

$$\therefore\ 2h^2\frac{du}{d\theta}\cdot\left[u+\frac{d^2u}{d\theta^2}\right] = \frac{1}{n^2}\frac{du}{d\theta}\left[\frac{1}{u}\frac{dP}{du}-\frac{P}{u^2}\right].$$

Dividing out by $du/d\theta$, we get

$$2h^2\left[u+\frac{d^2u}{d\theta^2}\right] = \frac{1}{n^2}\left[\frac{1}{u}\frac{dP}{du}-\frac{P}{u^2}\right]$$

or $$2\cdot\frac{P}{u^2} = \frac{1}{n^2}\left[\frac{1}{u}\frac{dP}{du}-\frac{P}{u^2}\right] \qquad \left[\because \frac{P}{h^2u^2} = u+\frac{d^2u}{d\theta^2}\right]$$

or $$2n^2\cdot\frac{P}{u^2} = \left[\frac{1}{u}\frac{dP}{du}-\frac{P}{u^2}\right] \quad\text{or}\quad (2n^2+1)\frac{P}{u^2} = \frac{1}{u}\frac{dP}{du}$$

or $$\frac{dP}{P} = (2n^2+1)\cdot\frac{du}{u}.$$

Integrating, $\log P = (2n^2+1)\log u + \log A$.

$$\therefore\ P = Au^{2n^2+1} = \frac{A}{r^{2n^2+1}}.$$

$\therefore\ P \propto \dfrac{1}{r^{2n^2+1}}$, which proves the first result.

Substituting $P = Au^{2n^2+1}$ in (3), we have

$$h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right] = \frac{Au^{2n^2+1}}{n^2u} = \frac{A}{n^2}u^{2n^2}.$$

Putting $u = \dfrac{1}{r}$ so that $\dfrac{du}{d\theta} = -\dfrac{1}{r^2}\dfrac{dr}{d\theta}$, we have

$$\frac{1}{r^2}+\left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 = \frac{A}{n^2h^2r^{2n^2}}$$

or $$r^{2n^2-2}+r^{2n^2-4}\left(\frac{dr}{d\theta}\right)^2 = \frac{A}{n^2h^2}$$

or $$r^{2n^2-4}\left(\frac{dr}{d\theta}\right)^2 = \frac{A}{n^2h^2}-r^{2n^2-2}$$

or $$\left(r^{n^2-2}\right)^2\left(\frac{dr}{d\theta}\right)^2 = a^{2n^2-2}-r^{2n^2-2},$$

setting $A/n^2h^2 = a^{2n^2-2}$ to get the required form of the answer.

$$\therefore\ \frac{dr}{d\theta} = \frac{\sqrt{\{a^{2n^2-2}-r^{2n^2-2}\}}}{r^{n^2-2}}$$

or $$\frac{r^{n^2-2}\,dr}{\sqrt{\left\{\left(a^{n^2-1}\right)^2-\left(r^{n^2-1}\right)^2\right\}}} = d\theta.$$

Putting $r^{n^2-1} = z$ so that $(n^2-1)r^{n^2-2}\,dr = dz$, we have

$$\frac{dz}{\sqrt{\left\{\left(a^{n^2-1}\right)^2-z^2\right\}}} = (n^2-1)\,d\theta.$$

Integrating, $\sin^{-1}\left(\frac{z}{a^{n^2-1}}\right) = (n^2-1)\theta + B$

or $$\sin^{-1}\left(\frac{r^{n^2-1}}{a^{n^2-1}}\right) = (n^2-1)\theta + B.$$

Initially when $\theta = 0$, let $r = a$. Then $B = \sin^{-1} 1 = \pi/2$.

$$\therefore \sin^{-1}\left(\frac{r^{n^2-1}}{a^{n^2-1}}\right) = (n^2-1)\theta + \frac{1}{2}\pi$$

or $$\frac{r^{n^2-1}}{a^{n^2-1}} = \sin\{(n^2-1)\theta + \tfrac{1}{2}\pi\} = \cos(n^2-1)\theta$$

or $$r^{n^2-1} = a^{(n^2-1)}\cos(n^2-1)\theta,$$

which is the required equation of the orbit.

Ex. 14. *A particle moves with a central acceleration μ/(distance)^2; it is projected with velocity V at a distance R. Show that its path is a rectangular hyperbola if the angle of projection is* IFoS-2010

$$\sin^{-1}\left[\frac{\mu}{VR\left(V^2-\frac{2\mu}{R}\right)^{1/2}}\right].$$

Sol. If the particle describes a hyperbola under the central acceleration μ/(distance)^2, then the velocity v of the particle at a distance r from the centre of force is given by

$$v^2 = \mu\left(\frac{2}{r}+\frac{1}{a}\right), \quad ...(1)$$

where $2a$ is the transverse axis of the hyperbola.

Since the particle is projected with velocity V at a distance R, therefore from (1), we have

$$V^2 = \mu\left(\frac{2}{R}+\frac{1}{a}\right) \text{ or } \frac{\mu}{a} = V^2 - \frac{2\mu}{R}. \quad ...(2)$$

If α is the required angle of projection to describe a rectangular hyperbola, then at the point of projection from the relation $h = vp$, we have

$$h = Vp = VR\sin\alpha \quad ...(3)$$

[$\because p = r\sin\phi$ and initially $r = R, \phi = \alpha$]

Also $h = \sqrt{(\mu l)} = \sqrt{\{\mu \cdot (b^2/a)\}} = \sqrt{(\mu a)}$. ...(4)

[$\because b = a$ for a rectangular hyperbola]

From (3) and (4), we have

$$VR\sin\alpha = \sqrt{(\mu a)}$$

or $$\sin\alpha = \frac{\sqrt{(\mu a)}}{VR} = \frac{\mu\sqrt{a}}{VR\sqrt{\mu}} = \frac{\mu}{VR\sqrt{(\mu/a)}}.$$

Substituting for μ/a from (2), we have

$$\sin\alpha = \mu / \{VR\sqrt{(V^2-2\mu/R)}\}$$

or $$\alpha = \sin^{-1}[\mu/\{VR\sqrt{(V^2-2\mu/R)}\}],$$

which is the required angle of projection.

Ex. 15. *A particle of unit mass describes an equiangular spiral of angle α, under a force which is always in the direction perpendicular to the straight line joining the particle to the pole of the spiral; show that the force is $\mu r^{2\sec^2\alpha-3}$ and that the rate of description of sectorial area about the pole is*

$$\tfrac{1}{2}\sqrt{(\mu\sin\alpha\cos\alpha)}\, r^{\sec^2\alpha}.$$

Sol. Here the particle is moving under a force which is always in the direction perpendicular to the straight line joining the particle to the pole of the spiral.

$\therefore$ the central radial acceleration $= \ddot{r} - r\dot{\theta}^2 = 0$ ...(1)

If F is the force on the particle of unit mass, perpendicular to the line joining the particle to the pole, then

$$F = \text{transverse acceleration}$$

i.e., $$F = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) \quad ...(2)$$

The equation of the equiangular spiral is

$$r = ae^{\theta\cot\alpha}. \quad ...(3)$$

Differentiating (3) w.r.t. 't', we have

$$\dot{r} = ae^{\theta\cot\alpha}\dot{\theta}\cot\alpha = r\dot{\theta}\cot\alpha$$

or $$\dot{\theta} = \frac{\dot{r}}{r}\tan\alpha. \quad ...(4)$$

$\therefore$ from (1) and (4), we have

$$\ddot{r} = r\left(\frac{\dot{r}}{r}\tan\alpha\right)^2$$

or $$\frac{\ddot{r}}{\dot{r}} = \frac{\dot{r}}{r}\tan^2\alpha.$$

Integrating, we have

$$\log\dot{r} = (\tan^2\alpha)\log r + \log A,$$

where A is a constant of integration

or $$\log\dot{r} = \log\left(Ar^{\tan^2\alpha}\right)$$

or $$\dot{r} = Ar^{\tan^2\alpha}. \quad ...(5)$$

Substituting the value of $\dot{r}$ from (5) in (4), we have

$$\dot{\theta} = \frac{1}{r}\tan\alpha \,.\, Ar^{\tan^2\alpha}$$

or $$\dot{\theta} = A\tan\alpha \,.\, r^{\tan^2\alpha-1} \quad ...(6)$$

$\therefore$ from (2), we have

$$F = \frac{1}{r}\frac{d}{dt}\left(r^2A\tan\alpha \,.\, r^{\tan^2\alpha-1}\right) = \frac{A\tan\alpha}{r}\frac{d}{dt}\left(r^{\tan^2\alpha+1}\right)$$

$$= \frac{A\tan\alpha}{r}\frac{d}{dt}\left(r^{\sec^2\alpha}\right) = \frac{A\tan\alpha}{r}\cdot\sec^2\alpha\, r^{\sec^2\alpha-1}\dot{r}$$

$$= A\tan\alpha\sec^2\alpha \,.\, r^{\sec^2\alpha-2} \,.\, Ar^{\tan^2\alpha}$$

$$= A^2\tan\alpha\sec^2\alpha\, r^{\sec^2\alpha-2+\tan^2\alpha}$$

$$= \mu r^{\sec^2\alpha-2+\sec^2\alpha-1}, \text{ where } \mu = A^2\tan\alpha\sec^2\alpha. \quad ...(7)$$

Thus $F = \mu r^{2\sec^2\alpha-3}$, which proves the first part.

Second Part. The rate of description of the sectorial area

$$= \tfrac{1}{2}r^2\dot{\theta}$$

$$= \tfrac{1}{2}r^2 A\tan\alpha\, r^{\tan^2\alpha-1}$$

$$= \tfrac{1}{2}A\tan\alpha\, r^{2+\tan^2\alpha-1}$$

$$= \tfrac{1}{2}\sqrt{(\mu\cot\alpha\cos^2\alpha)}\tan\alpha\, r^{\tan^2\alpha+1}$$

[substituting $A = \sqrt{(\mu\cot\alpha\cos^2\alpha)}$ from (7)]

$$= \tfrac{1}{2}\sqrt{(\mu\cot\alpha\cos^2\alpha\tan^2\alpha)}\, r^{\sec^2\alpha} = \tfrac{1}{2}\sqrt{(\mu\sin\alpha\cos\alpha)}\, r^{\sec^2\alpha}.$$

§ 9. Apse and Apsidal distance.

1. Apse. Definition. *An apse is a point on the central orbit at which the radius vector from the centre of force to the point has a maximum or minimum value.*

2. Apsidal distance. *The length of the radius vector at an apse is called an apsidal distance.*

3. Apsidal angle. *The angle between two consecutive apsidal distances is called an apsidal angle.*

Theorem. *At an apse the radius vector is perpendicular to the tangent i.e., at an apse the particle moves at right angles to the radius vector.*

From the definition of an apse, r is maximum or minimum at an apse i.e., $u = 1/r$ is minimum or maximum at an apse.

$\therefore$ at an apse, $du/d\theta = 0$.

But we know that $\frac{1}{p^2} = u^2 + \left(\frac{du}{d\theta}\right)^2$.

$\therefore$ at an apse, $\frac{1}{p^2} = u^2 = \frac{1}{r^2}$

or $p = r$ or $r\sin\phi = r$

or $\sin\phi = 1$ or $\phi = 90°$.

This proves that at an apse the radius vector is perpendicular to the tangent or in other words at an apse the particle moves at right angles to the radius vector.

Remember. At an apse $dr/d\theta = 0, du/d\theta = 0, \phi = 90°, p = r$ and the direction of motion is at right angles to the radius vector.

§ 10. Property of the apse-line. Theorem.

If the central acceleration P is a single-valued function of the distance, every apse-line divides the orbit into equal and symmetrical portions, and thus there can only be two apsidal distances.

Proof. Since the central acceleration P is a single valued function of r, therefore the acceleration of the particle is the same at the same distance r.

The differential equation of a central orbit is

$$\frac{d^2u}{d\theta^2} + u = \frac{P}{h^2u^2} \text{ or } h^2\left[\frac{d^2u}{d\theta^2}+u\right] = \frac{P}{u^2}.$$

Multiplying both sides by $2(du/d\theta)$ and integrating w.r.t. 'θ', we have

$$v^2 = h^2\left[\left(\frac{du}{d\theta}\right)^2 + u^2\right] = 2\int\frac{P}{u^2}du + C,$$

or $$v^2 = C - 2\int P\,dr. \quad ...(1)$$

$$\left[\because \frac{1}{u} = r \Rightarrow -\frac{1}{u^2}du = dr\right]$$

The equation (1) shows that if P is a single valued function of the distance r, then the velocity of the particle is the same at the same distance r and is independent of the direction of motion.

Thus we observe that both velocity and acceleration are the same at the same distance from the centre. Therefore if at an apse the direction of velocity is reversed, the particle will describe symmetrical orbit on both sides of the apse-line.

Now when the particle comes to a second apse, the path for the same reasons, is symmetrical about this second apsidal distance also. But this is possible only when the next (third) apsidal distance is equal to the one (first) before it and the angle between the first and the second apsidal distances is the same as the angle between the second and the third apsidal distances. Therefore if the central acceleration is a single valued function of the distance r, there are only two different apsidal distances. Also the angle between any two consecutive apsidal distances always remains the same and is called the *apsidal angle.*

§ 11. *To prove analytically that when the central acceleration varies as some integral power of the distance, there are at most two apsidal distances.*

Let the central acceleration P be given by

$P = \mu r^n$, where n is an integer.

Thus $P = \mu u^{-n}$ because $r = 1/u = u^{-1}$.

$\therefore$ the differential equation of the path is

$$h^2 \left\{u + \frac{d^2u}{d\theta^2}\right\} = \frac{P}{u^2} = \frac{\mu u^{-n}}{u^2} = \mu u^{-(n+2)}.$$

Multiplying both sides by $2\,(du/d\theta)$ and then integrating, we have

$$h^2 \left\{u^2 + \left(\frac{du}{d\theta}\right)^2\right\} = \frac{\mu u^{-(n+1)}}{-(n+1)} + A. \quad ...(1)$$

But at an apse $du/d\theta = 0$. So putting $du/d\theta = 0$ in (1), we have

$$h^2 u^2 = -\frac{\mu}{n+1} u^{-(n+1)} + A$$

or $$r^{n+3} - \frac{(n+1)}{\mu} Ar^2 + \frac{(n+1)}{\mu} h^2 = 0.$$

Whatever be the values of n or A this equation cannot have more than two changes of sign. Therefore by Descarte's rule of signs it cannot have more than two positive roots. Hence there are at most two positive values of r *i.e.*, at most two apsidal distances.

§ 12. Given the law of force, to find the orbit.

This problem is converse to that given in § 8 on page 9. For solving such a problem we substitute the given expression for P in the differential equation

$$h^2 \left[\frac{d^2u}{d\theta^2} + u\right] = \frac{P}{u^2} \quad ...(1)$$

or $$\frac{h^2}{p^3}\frac{dp}{dr} = P, \quad ...(2)$$

whichever is convenient. In case the force is repulsive, we take the value of P with negative sign.

Then integrating the resulting differential equation of the central orbit with the help of the given initial conditions, we get the (r, θ) or (p, r) equation of the orbit.

Illustrative Examples

Ex. 16 (a). *A particle moves with a central acceleration $\mu\,(r + a^4/r^3)$ being projected from an apse at a distance 'a' with a velocity $2a\sqrt{\mu}$. Prove that it describes the curve $r^2\,(2 + \cos\sqrt{3}\theta) = 3a^2$.*

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Sol. Here, the central acceleration,

$P = \mu\,(r + a^4/r^3) = \mu\,\{(1/u) + a^4u^3\}$, where $u = 1/r$.

$\therefore$ the differential equation of the path is

$$h^2 \left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{\mu}{u^2}\left(\frac{1}{u} + a^4u^3\right)$$

or $$h^2 \left[u + \frac{d^2u}{d\theta^2}\right] = \mu\left(\frac{1}{u^3} + a^4u\right).$$

Multiplying both sides by $2\,(du/d\theta)$ and integrating w.r.t. 'θ', we have

$$h^2 \left[2\cdot\frac{u^2}{2} + \left(\frac{du}{d\theta}\right)^2\right] = 2\mu\left(-\frac{1}{2u^2} + \frac{a^4u^2}{2}\right) + A$$

or $$v^2 = h^2 \left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\left(-\frac{1}{u^2} + a^4u^2\right) + A, \quad ...(1)$$

where A is a constant.

Now initially the particle has been projected from an apse (say, the point A) at a distance a with velocity $2\sqrt{\mu a}$. Therefore when $r = a$ *i.e.*, $u = 1/a$, $du/d\theta = 0$ (at an apse) and $v = 2\sqrt{\mu a}$.

$\therefore$ from (1), we have

$$4\mu a^2 = h^2\left[\frac{1}{a^2}\right] = \mu\left(-a^2 + a^4\cdot\frac{1}{a^2}\right) + A.$$

(i) (ii) (iii)

From (i) and (ii), we have $h^2 = 4\mu a^4$ and from (i) and (iii), we have

$4\mu a^2 = 0 + A$ *i.e.*, $A = 4\mu a^2$.

Substituting the values of h^2 and A in (1), we have

$$4\mu a^4 \left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\left(-\frac{1}{u^2} + a^4u^2\right) + 4\mu a^2$$

or $$4a^4\left(\frac{du}{d\theta}\right)^2 = -4a^4u^2 - \frac{1}{u^2} + a^4u^2 + 4a^2$$

or $$4a^4u^2\left(\frac{du}{d\theta}\right)^2 = (-1 - 3a^4u^4 + 4a^2u^2) \quad ...(2)$$

or $$2a^2u\frac{du}{d\theta} = \sqrt{[-1 - 3a^4u^4 + 4a^2u^2]} \quad \text{[taking square root]}$$

or $$d\theta = \frac{2a^2u\,du}{\sqrt{[-1 - 3a^4u^4 + 4a^2u^2]}}$$

$$= \frac{2a^2u\,du}{\sqrt{3}\,.\sqrt{[-\frac{1}{3} - (a^4u^4 - \frac{4}{3}a^2u^2)]}}$$

$$= \frac{2a^2u\,du}{\sqrt{3}\,.\sqrt{[-\frac{1}{3} - (a^2u^2 - \frac{2}{3})^2 + \frac{4}{9}]}}$$

$$= \frac{2a^2u\,du}{\sqrt{3}\,.\sqrt{[(\frac{1}{3})^2 - (a^2u^2 - \frac{2}{3})^2]}}$$

or $$\sqrt{3}\,d\theta = \frac{2a^2u\,du}{\sqrt{[(\frac{1}{3})^2 - (a^2u^2 - \frac{2}{3})^2]}}.$$

Substituting $a^2u^2 - \frac{2}{3} = z$, so that $2a^2u\,du = dz$, we have

$$\sqrt{3}\,d\theta = \frac{dz}{\sqrt{[(\frac{1}{3})^2 - z^2]}}.$$

Integrating, $\sqrt{3}\theta + B = \sin^{-1}(3z)$ where B is a constant

or $$\sqrt{3}\theta + B = \sin^{-1}(3a^2u^2 - 2). \quad ...(3)$$

Now take the aspe-line OA as the initial line. Then initially

$r = a, u = 1/a$ and $\theta = 0$.

$\therefore$ from (3), $0 + B = \sin^{-1} 1$ or $B = \frac{1}{2}\pi$.

Putting $B = \frac{1}{2}\pi$ in (3), we have

$$\sqrt{3}\theta + \tfrac{1}{2}\pi = \sin^{-1}(3a^2u^2 - 2)$$

or $$3a^2u^2 - 2 = \sin(\tfrac{1}{2}\pi + \sqrt{3}\theta) = \cos(\sqrt{3}\theta)$$

or $$\frac{3a^2}{r^2} - 2 = \cos(\sqrt{3}\theta) \quad \text{or} \quad 3a^2 - 2r^2 = r^2\cos(\sqrt{3}\theta).$$

$$\therefore \quad 3a^2 = r^2\,[2 + \cos(\sqrt{3}\theta)],$$

which is the equation of the required curve.

Remarks. We know that a central orbit is symmetrical about an apse-line. So if we take an apse-line as the initial line, then while extracting the square root of the equation (2) we can keep either the positive sign or the negative sign. In both the cases we shall get the same result. The students can verify it by solving the above problem while keeping the negative sign on extracting the square root of (2).

After extracting the square root of the equation (2) and then separating the variables, we should first try to integrate with respect to u. If we find any difficulty in integrating w.r.t. 'u', we should change u to r by putting $u = 1/r$.

Ex. 16 (b). *A particle subject to the central acceleration $(\mu/r^3) + f$ is projected from an apse at a distance 'a' with the velocity $\sqrt{\mu}/a$; prove that at any subsequent time t, $r = a - \frac{1}{2}ft^2$.*

Sol. Here the central acceleration

$$P = \frac{\mu}{r^3} + f = \mu u^3 + f, \quad \text{where } \frac{1}{r} = u.$$

$\therefore$ the differential equation of the path is

$$h^2 \left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{1}{u^2}(\mu u^3 + f)$$

or $$h^2 \left[u + \frac{d^2u}{d\theta^2}\right] = \mu u + \frac{f}{u^2}.$$

Multiplying both sides by $2\,(du/d\theta)$ and integrating, we have

$$v^2 = h^2 \left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu u^2 - \frac{2f}{u} + A, \quad ...(1)$$

where A is a constant.

But initially when $r = a$ *i.e.*, $u = 1/a$, $du/d\theta = 0$ (at an apse) and $v = \sqrt{\mu}/a$.

$$\therefore \text{ from (1), we have } \frac{\mu}{a^2} = h^2\left(\frac{1}{a^2}\right) = \frac{\mu}{a^2} - 2fa + A.$$

$\therefore$ $h^2 = \mu$ and $A = 2fa$.

Substituting the values of h^2 and A in (1), we have

$$\mu\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu u^2 - \frac{2f}{u} + 2fa$$

or $$\mu\left(\frac{du}{d\theta}\right)^2 = 2fa - \frac{2f}{u}. \quad ...(2)$$

Now $u = 1/r$, so that $\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$. Therefore, from (2), we have

$$\mu\left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 = 2fa - 2fr = 2f(a - r)$$

or $$\left(\frac{dr}{d\theta}\right)^2 = \frac{2fr^4}{\mu}(a - r) \quad \text{or} \quad \frac{dr}{d\theta} = -\sqrt{(2f/\mu)}\,.\,r^2\sqrt{(a - r)}.$$

Also $$h = r^2\frac{d\theta}{dt} = r^2\frac{d\theta}{dr}\cdot\frac{dr}{dt}.$$

$$\therefore \quad \sqrt{\mu} = r^2\,.\sqrt{\left(\frac{\mu}{2f}\right)}\cdot\frac{(-1)}{r^2\sqrt{(a - r)}}\cdot\frac{dr}{dt}$$

[substituting for h and $dr/d\theta$]

or $$dt = \frac{-1}{\sqrt{(2f)}}\cdot(a - r)^{-1/2}\,dr.$$

Integrating, $$t = \frac{1}{\sqrt{(2f)}}\cdot 2\,(a - r)^{1/2} + B,$$

where B is a constant.

But initially when $t = 0$, $r = a$; $\therefore$ $B = 0$.

$\therefore$ $t = \sqrt{(2/f)}\,.\,(a - r)^{1/2}$

or $t^2 = (2/f)(a - r)$

or $$a - r = \tfrac{1}{2}ft^2. \quad \therefore \quad r = a - \tfrac{1}{2}ft^2.$$

Ex. 17. *A particle moves under a force*
$$m\mu\,\{3au^4 - 2(a^2 - b^2)u^5\},\ a > b$$
and is projected from an apse at a distance $(a+b)$ with velocity $\sqrt{\mu}/(a+b)$. Show that the equation of its path is $r = a + b\cos\theta$.

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Sol. Here the central acceleration
$$P = \mu\,\{3au^4 - 2(a^2 - b^2)u^5\}.$$
∴ the differential equation of the path is
$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{\mu}{u^2}\{3au^4 - 2(a^2 - b^2)u^5\}$$
or
$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \mu\,\{3au^2 - 2(a^2 - b^2)u^3\}.$$
Multiplying both sides by $2(du/d\theta)$ and integrating, we have
$$h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = 2\mu\left\{au^3 - 2(a^2 - b^2)\frac{u^4}{4}\right\} + A$$
or
$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\,\{2au^3 - (a^2 - b^2)u^4\} + A, \quad ...(1)$$
where A is a constant.

But initially at an apse, $r = a + b$, $u = 1/(a+b)$, $du/d\theta = 0$ and $v = \sqrt{\mu}/(a+b)$.

∴ from (1), we have
$$\frac{\mu}{(a+b)^2} = h^2\left[\frac{1}{(a+b)^2}\right] = \mu\left[\frac{2a}{(a+b)^3} - \frac{(a^2 - b^2)}{(a+b)^4}\right] + A.$$
∴ $h^2 = \mu$ and $A = 0$.

Substituting the values of h^2 and A in (1), we have
$$\mu\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\,\{2au^3 - (a^2 - b^2)u^4\}$$
or
$$\left(\frac{du}{d\theta}\right)^2 = -u^2 + 2au^3 - (a^2 - b^2)u^4. \quad ...(2)$$
But $u = \dfrac{1}{r}$, so that $\dfrac{du}{d\theta} = -\dfrac{1}{r^2}\dfrac{dr}{d\theta}$.

Substituting in (2), we have
$$\left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 = -\frac{1}{r^2} + \frac{2a}{r^3} - \frac{(a^2 - b^2)}{r^4}$$
or
$$\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{1}{r^4}[-r^2 + 2ar - (a^2 - b^2)]$$
or
$$\left(\frac{dr}{d\theta}\right)^2 = -r^2 + 2ar - a^2 + b^2 = b^2 - (r^2 - 2ar + a^2)$$
$$= b^2 - (r - a)^2.$$
∴ $\dfrac{dr}{d\theta} = \sqrt{b^2 - (r-a)^2}$ or $d\theta = \dfrac{dr}{\sqrt{b^2 - (r-a)^2}}$.

Integrating, $\theta + B = \sin^{-1}\left(\dfrac{r-a}{b}\right)$. ...(3)

But initially when $r = a + b$, let us take $\theta = 0$. Then from (3),
$B = \sin^{-1}(1) = \pi/2$.

Substituting in (3), we have
$$\theta + \tfrac{1}{2}\pi = \sin^{-1}\left(\frac{r-a}{b}\right) \quad \text{or} \quad r - a = b\sin(\tfrac{1}{2}\pi + \theta)$$
or $r = a + b\cos\theta$, which is the required equation of the path.

Ex. 18. *A particle moves under a repulsive force $m\mu/(\text{distance})^3$ and is projected from an apse at a distance a with a velocity V; show that the equation to the path is $r\cos p\theta = a$, and that the angle θ described in time t is $(1/p)\tan^{-1}(pVt/a)$, where*
$$p^2 = (\mu + a^2V^2)/(a^2V^2).$$

Sol. Since the particle moves under a repulsive force
$$\frac{m\mu}{(\text{distance})^3} = \frac{m\mu}{r^3},$$
∴ the central acceleration $P = -\dfrac{\mu}{r^3} = -\mu u^3$.

∴ the differential equation of the path is
$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{-\mu u^3}{u^2} = -\mu u.$$
Multiplying both sides by $2(du/d\theta)$ and integrating, we have
$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = -\mu u^2 + A, \quad ...(1)$$
where A is a constant.

But initially at an apse, $r = a$, $u = 1/a$, $du/d\theta = 0$ and $v = V$.

∴ from (1), we have
$$V^2 = h^2\left[\frac{1}{a^2}\right] = -\frac{\mu}{a^2} + A.$$
∴ $h^2 = a^2V^2$ and $A = V^2 + (\mu/a^2)$. ...(2)

Substituting the values of h^2 and A in (1), we have
$$a^2V^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = -\mu u^2 + V^2 + \frac{\mu}{a^2}$$
or
$$a^2V^2\left(\frac{du}{d\theta}\right)^2 = -(a^2V^2 + \mu)u^2 + \frac{(a^2V^2 + \mu)}{a^2}$$
or
$$a^2\left(\frac{du}{d\theta}\right)^2 = \frac{(a^2V^2 + \mu)}{a^2V^2}(1 - a^2u^2)$$
or
$$a^2\left(\frac{du}{d\theta}\right)^2 = p^2(1 - a^2u^2), \quad \text{where } p^2 = \frac{\mu + a^2V^2}{a^2V^2}$$
or
$$a\frac{du}{d\theta} = p\sqrt{(1 - a^2u^2)} \quad \text{or} \quad p\,d\theta = \frac{a\,du}{\sqrt{(1 - a^2u^2)}}.$$
Integrating, $p\theta + B = \sin^{-1}(au)$, where B is a constant.

But initially when $u = 1/a$, let $\theta = 0$. Then $B = \sin^{-1} 1 = \frac{1}{2}\pi$.
$$\therefore \quad p\theta + \tfrac{1}{2}\pi = \sin^{-1}(au)$$
or $au = \sin(\frac{1}{2}\pi + p\theta)$

or $a/r = \cos p\theta$

or $r\cos p\theta = a$, ...(3)

which is the equation of the path.

Second part. We have
$$h = r^2\frac{d\theta}{dt}.$$
or $aV = a^2\sec^2 p\theta\,\dfrac{d\theta}{dt}$,

or $dt = (a/V)\sec^2 p\theta\,d\theta$.

Integrating, $t + C = \dfrac{a}{pV}\tan p\theta$.

But initially $t = 0$ and $\theta = 0$. Therefore $C = 0$.

∴ $t = \dfrac{a}{pV}\tan p\theta$ or $\tan p\theta = pVt/a$.

∴ $\theta = (1/p)\tan^{-1}(pVt/a)$,

which gives the angle θ described in time t.

Ex. 19. *A particle moves under a central force*
$$m\lambda\,(3a^3u^4 + 8au^2).$$
It is projected from an apse at a distance a from the centre of force with velocity $\sqrt{(10\lambda)}$. Show that the second apsidal distance is half of the first and that the equation to the path is
$$2r = a[1 + \text{sech}\,(\theta/\sqrt{5})].$$

Sol. Here the particle moves under the central force $m\lambda(3a^3u^4 + 8au^2)$. Therefore the central acceleration P is given by
$$P = \lambda\,(3a^3u^4 + 8au^2).$$
∴ the differential equation of the path is
$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{\lambda}{u^2}(3a^3u^4 + 8au^2)$$
or
$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \lambda\,(3a^3u^2 + 8a).$$
Multiplying both sides by $2(du/d\theta)$ and integrating, we have
$$h^2\left[2\cdot\frac{u^2}{2} + \left(\frac{du}{d\theta}\right)^2\right] = 2\lambda\cdot(a^3u^3 + 8au) + A$$
or
$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \lambda\,(2a^3u^3 + 16au) + A, \quad ...(1)$$
where A is a constant.

But initially at an apse, $r = a$, $u = 1/a$, $du/d\theta = 0$ and $v = \sqrt{(10\lambda)}$.

∴ from (1), we have
$$10\lambda = h^2\left[\frac{1}{a^2}\right] = \lambda\left(2a^3\cdot\frac{1}{a^3} + 16a\cdot\frac{1}{a}\right) + A.$$
∴ $h^2 = 10a^2\lambda$ and $A = 10\lambda - 18\lambda = -8\lambda$.

Substituting the values of h^2 and A in (1), we have
$$10a^2\lambda\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \lambda\,(2a^3u^3 + 16au) - 8\lambda$$
or
$$10a^2\left(\frac{du}{d\theta}\right)^2 = 2a^3u^3 - 10a^2u^2 + 16au - 8$$
or
$$5a^2\left(\frac{du}{d\theta}\right)^2 = [a^3u^3 - 5a^2u^2 + 8au - 4]$$
$$= a^2u^2(au - 1) - 4au(au - 1) + 4(au - 1)$$
$$= (au - 1)(a^2u^2 - 4au + 4)$$
$$= (au - 1)(au - 2)^2. \quad ...(2)$$
To find the second apsidal distance. At an apse, we have $du/d\theta = 0$.

∴ from (2), $0 = (au - 1)(au - 2)^2$

or $u = 1/a$ and $2/a$ or $r = a$ and $a/2$.

But $r = a$ is the first apsidal distance. Therefore the second apsidal distance is $a/2$ which is half of the first.

To find the equation of the path. From equation (2), we have
$$\sqrt{5}\,a\frac{du}{d\theta} = -(au - 2)\sqrt{(au - 1)}.$$
$$\therefore \quad \frac{d\theta}{\sqrt{5}} = \frac{-a\,du}{(au - 2)\sqrt{au - 1}}.$$
Substituting $au - 1 = z^2$, so that $a\,du = 2z\,dz$, we have
$$\frac{d\theta}{\sqrt{5}} = \frac{-2z\,dz}{(z^2 - 1)z}$$
or
$$\frac{d\theta}{2\sqrt{5}} = \frac{dz}{1 - z^2}.$$

Integrating, $\frac{\theta}{2\sqrt{5}}+B=\tanh^{-1}z$, where B is a constant

or $$\frac{\theta}{2\sqrt{5}}+B=\tanh^{-1}\sqrt{(au-1)}. \qquad ...(3)$$

But initially, when $u=1/a, \theta=0$.

$\therefore$ from (3), $B=0$.

Putting $B=0$ in (3), we get

$$\frac{\theta}{2\sqrt{5}}=\tanh^{-1}\sqrt{(au-1)}$$

or $$\tanh\left(\frac{\theta}{2\sqrt{5}}\right)=\sqrt{(au-1)}.$$

Now $$\cosh 2A=\frac{1+\tanh^2 A}{1-\tanh^2 A} \quad \text{(Remember)}$$

$$\therefore \cosh\left(\frac{\theta}{\sqrt{5}}\right)=\frac{1+\tanh^2(\theta/2\sqrt{5})}{1-\tanh^2(\theta/2\sqrt{5})}=\frac{1+(au-1)}{1-(au-1)}=\frac{au}{2-au}$$

or $2-au=au\,\text{sech}\,(\theta/\sqrt{5})$

or $2=au\,[1+\text{sech}\,(\theta/\sqrt{5})]=(a/r)\,[1+\text{sech}\,(\theta/\sqrt{5})]$

or $2r=a\,[1+\text{sech}\,(\theta/\sqrt{5})]$,

which is the required equation of the path.

Ex. 20. *A particle subject to a central force per unit of mass equal to $\mu\,\{2(a^2+b^2)u^5-3a^2b^2u^7\}$ is projected at the distance a with velocity $\sqrt{\mu}/a$ in a direction at right angles to the initial distance; show that the path is the curve*

$$r^2=a^2\cos^2\theta+b^2\sin^2\theta.$$

Sol: Here, the central acceleration

$$P=\mu\,\{2(a^2+b^2)u^5-3a^2b^2u^7\}.$$

$\therefore$ the differential equation of the path is

$$h^2\left[u+\frac{d^2u}{d\theta^2}\right]=\frac{P}{u^2}=\frac{\mu}{u^2}\{2(a^2+b^2)u^5-3a^2b^2u^7\}$$

or $$h^2\left[u+\frac{d^2u}{d\theta^2}\right]=\mu\,\{2(a^2+b^2)u^3-3a^2b^2u^5\}.$$

Multiplying both sides by $2\,(du/d\theta)$ and integrating, we have

$$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu\,\{(a^2+b^2)u^4-a^2b^2u^6\}+A, \qquad ...(1)$$

where A is a constant.

Now at the point of projection the direction of velocity is perpendicular to the radius vector. So the point of projection is an apse. Therefore initially when $r=a, u=1/a, du/d\theta=0$ and $v=\sqrt{\mu}/a$.

$\therefore$ from (1), we have

$$\frac{\mu}{a^2}=h^2\left[\frac{1}{a^2}\right]=\mu\left\{\frac{(a^2+b^2)}{a^4}-\frac{a^2b^2}{a^6}\right\}+A.$$

$\therefore \quad h^2=\mu$ and $A=0$.

Substituting the values of h^2 and A in (1), we have

$$\mu\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu\,\{(a^2+b^2)u^4-a^2b^2u^6\}$$

or $$\left(\frac{du}{d\theta}\right)^2=-u^2+(a^2+b^2)u^4-a^2b^2u^6.$$

Putting $u=\frac{1}{r}$, so that $\frac{du}{d\theta}=-\frac{1}{r^2}\frac{dr}{d\theta}$, we have

$$\left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)^2=-\frac{1}{r^2}+(a^2+b^2)\frac{1}{r^4}-a^2b^2\frac{1}{r^6}$$

or $$\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2=\frac{1}{r^6}\left[-r^4+(a^2+b^2)r^2-a^2b^2\right]$$

or $$\left(\frac{dr}{d\theta}\right)^2=\frac{1}{r^2}\left[-a^2b^2-r^4+(a^2+b^2)r^2\right]$$

$$=\frac{1}{r^2}\left[-a^2b^2-\{r^2-\tfrac{1}{2}(a^2+b^2)\}^2+\tfrac{1}{4}(a^2+b^2)^2\right]$$

$$=\frac{1}{r^2}\left[\tfrac{1}{4}(a^2-b^2)^2-\{r^2-\tfrac{1}{2}(a^2+b^2)\}^2\right]$$

$$\therefore \quad \frac{dr}{d\theta}=-\frac{1}{r}\sqrt{\left[\tfrac{1}{4}(a^2-b^2)^2-\{r^2-\tfrac{1}{2}(a^2+b^2)\}^2\right]}$$

or $$d\theta=\frac{-r\,dr}{\sqrt{[\frac{1}{4}(a^2-b^2)^2-\{r^2-\frac{1}{2}(a^2+b^2)\}^2]}}.$$

Putting $r^2-\frac{1}{2}(a^2+b^2)=z$, so that $2r\,dr=dz$, we have

$$d\theta=\frac{-\frac{1}{2}dz}{\sqrt{[\frac{1}{4}(a^2-b^2)^2-z^2]}}$$

Integrating, we get

$$\theta+B=\tfrac{1}{2}\cos^{-1}\left\{\frac{z}{\frac{1}{2}(a^2-b^2)}\right\}=\tfrac{1}{2}\cos^{-1}\left\{\frac{r^2-\frac{1}{2}(a^2+b^2)}{\frac{1}{2}(a^2-b^2)}\right\}$$

where B is a constant.

Initially when $r=a, \theta=0$.

$$\therefore \quad B=\tfrac{1}{2}\cos^{-1}\left\{\frac{a^2-\frac{1}{2}(a^2+b^2)}{\frac{1}{2}(a^2-b^2)}\right\}=\tfrac{1}{2}\cos^{-1}\left\{\frac{\frac{1}{2}(a^2-b^2)}{\frac{1}{2}(a^2-b^2)}\right\}$$

$$=\tfrac{1}{2}\cos^{-1}1=0.$$

Hence $$\theta=\tfrac{1}{2}\cos^{-1}\left\{\frac{r^2-\frac{1}{2}(a^2+b^2)}{\frac{1}{2}(a^2-b^2)}\right\}$$

or $$2\theta=\cos^{-1}\left\{\frac{r^2-\frac{1}{2}(a^2+b^2)}{\frac{1}{2}(a^2-b^2)}\right\}$$

or $$\cos 2\theta=\frac{r^2-\frac{1}{2}(a^2+b^2)}{\frac{1}{2}(a^2-b^2)}$$

or $r^2-\frac{1}{2}(a^2+b^2)=\frac{1}{2}(a^2-b^2)\cos 2\theta$

or $r^2=\frac{1}{2}(a^2+b^2)+\frac{1}{2}(a^2-b^2)\cos 2\theta$

$=\frac{1}{2}a^2(1+\cos 2\theta)+\frac{1}{2}b^2(1-\cos 2\theta)$

or $r^2=a^2\cos^2\theta+b^2\sin^2\theta$,

which is the required equation of the path.

Ex. 21. *A particle moves with a central acceleration $\lambda^2(8au^2+a^4u^5)$; it is projected with velocity 9λ from an apse at a distance $a/3$ from the origin; show that the equation to its path is*

$$\frac{1}{\sqrt{3}}\sqrt{\left(\frac{au+5}{au-3}\right)}=\cot(\theta/\sqrt{6}).$$

Sol: Here the central acceleration $P=\lambda^2(8au^2+a^4u^5)$.

$\therefore$ the differential equation of the path is

$$h^2\left[u+\frac{d^2u}{d\theta^2}\right]=\frac{P}{u^2}=\frac{\lambda^2}{u^2}(8au^2+a^4u^5)$$

or $$h^2\left[u+\frac{d^2u}{d\theta^2}\right]=\lambda^2(8a+a^4u^3).$$

Multiplying both sides by $2\,(du/d\theta)$ and integrating, we have

$$h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=2\lambda^2\left(8au+\frac{a^4u^4}{4}\right)+A$$

or $$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\lambda^2\left(16au+\frac{a^4u^4}{2}\right)+A. \qquad ...(1)$$

where A is a constant.

But initially when $r=a/3$ *i.e.*, $u=3/a, du/d\theta=0$ (at an apse) and $v=9\lambda$.

$\therefore$ from (1), we have

$$81\lambda^2=h^2\left[\frac{9}{a^2}\right]=\lambda^2\left[16a\cdot\frac{3}{a}+\frac{a^4}{2}\cdot\frac{81}{a^4}\right]+A.$$

$\therefore \quad h^2=9a^2\lambda^2$ and $A=\frac{-15}{2}\lambda^2$.

Substituting the values of h^2 and A in (1), we have

$$9a^2\lambda^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\lambda^2\left(16au+\frac{a^4u^4}{2}\right)-\frac{15}{2}\lambda^2$$

or $$9a^2\left(\frac{du}{d\theta}\right)^2=\frac{a^4u^4}{2}-9a^2u^2+16au-\frac{15}{2}$$

or $$18a^2\left(\frac{du}{d\theta}\right)^2=a^4u^4-18a^2u^2+32au-15$$

$=a^3u^3(au-1)+a^2u^2(au-1)-17au(au-1)+15(au-1)$

$=(au-1)(a^3u^3+a^2u^2-17au+15)$

$=(au-1)\{a^2u^2(au-1)+2au(au-1)-15(au-1)\}$

$=(au-1)^2(a^2u^2+2au-15)$

$=(au-1)^2(au-3)(au+5)$.

$$\therefore \quad 3\sqrt{2}a\frac{du}{d\theta}=(au-1)\sqrt{[(au-3)(au+5)]}$$

or $$\frac{d\theta}{3\sqrt{2}}=\frac{a\,du}{(au-1)\sqrt{[(au-3)(au+5)]}}.$$

Substituting $au+5=(au-3)z^2$, so that $au=\frac{3z^2+5}{z^2-1}$

and $$a\,du=\frac{6z(z^2-1)-(3z^2+5)\cdot 2z}{(z^2-1)^2}dz=\frac{-16z\,dz}{(z^2-1)^2}, \text{ we have}$$

$$\frac{d\theta}{3\sqrt{2}}=\frac{-\frac{16z\,dz}{(z^2-1)^2}}{\left(\frac{3z^2+5}{z^2-1}-1\right)\left(\frac{3z^2+5}{z^2-1}-3\right)\cdot z}$$

or $$\frac{d\theta}{3\sqrt{2}}=-\frac{dz}{z^2+3}.$$

Integrating, $\frac{\theta}{3\sqrt{2}}+B=\frac{1}{\sqrt{3}}\cot^{-1}(z/\sqrt{3})$, where B is a constant

or $$\frac{\theta}{3\sqrt{2}}+B=\frac{1}{\sqrt{3}}\cot^{-1}\left\{\sqrt{\left(\frac{au+5}{au-3}\right)}\cdot\frac{1}{\sqrt{3}}\right\}.$$

But initially, $u=3/a$ and $\theta=0$.

$\therefore \quad 0+B=\frac{1}{\sqrt{3}}\cot^{-1}\infty=0$ or $B=0$.

$$\therefore \quad \frac{\theta}{3\sqrt{2}}=\frac{1}{\sqrt{3}}\cot^{-1}\left\{\frac{1}{\sqrt{3}}\sqrt{\left(\frac{au+5}{au-3}\right)}\right\}$$

or $$\cot^{-1}\left\{\frac{1}{\sqrt{3}}\sqrt{\left(\frac{au+5}{au-3}\right)}\right\}=\frac{\theta}{\sqrt{6}}$$

or $$\frac{1}{\sqrt{3}}\sqrt{\left(\frac{au+5}{au-3}\right)}=\cot(\theta/\sqrt{6}),$$

which is the required equation of the path.

Ex. 22. *A particle moving with a central acceleration $\mu/(\text{distance})^3$ is projected from an apse at a distance a with a velocity V; show that the path is*

$$r\cosh\left\{\frac{\sqrt{(\mu-a^2V^2)}}{aV}\theta\right\}=a \text{ or } r\cos\left\{\frac{\sqrt{(a^2V^2-\mu)}}{aV}\theta\right\}=a$$

according as V is $<$ or $>$ the velocity from infinity.

Sol. Here, the central acceleration P

$$=\frac{\mu}{(\text{distance})^3}=\frac{\mu}{r^3}=\mu u^3.$$

The differential equation of the path is

$$h^2\left[u+\frac{d^2u}{d\theta^2}\right]=\frac{P}{u^2}=\frac{\mu u^3}{u^2}=\mu u.$$

Multiplying both sides by $2(du/d\theta)$ and integrating, we have

$$v^2=h^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu u^2+A, \quad \text{...(1)}$$

where A is a constant.

But initially when $r=a$ i.e., $u=\frac{1}{a}$, $\frac{du}{d\theta}=0$ (at an apse) and $v=V$.

$\therefore$ from (1), $V^2=h^2\left[\frac{1}{a^2}\right]=\frac{\mu}{a^2}+A.$

$\therefore$ $h^2=a^2V^2$ and $A=V^2-\frac{\mu}{a^2}=\frac{(V^2a^2-\mu)}{a^2}.$

Substituting the values of h^2 and A in (1), we have

$$a^2V^2\left[u^2+\left(\frac{du}{d\theta}\right)^2\right]=\mu u^2+\frac{(V^2a^2-\mu)}{a^2}$$

or $$a^2V^2\left(\frac{du}{d\theta}\right)^2=-a^2V^2u^2+\mu u^2+\frac{(V^2a^2-\mu)}{a^2}$$

$$=-(a^2V^2-\mu)u^2+(a^2V^2-\mu)/a^2$$

$$=(a^2V^2-\mu)(-u^2+1/a^2)$$

or $$a^4V^2\left(\frac{du}{d\theta}\right)^2=(a^2V^2-\mu)(1-a^2u^2). \quad \text{...(2)}$$

If V_1 is the velocity acquired by the particle in falling form infinity to the distance a, then

$$V_1^2=-2\int_\infty^a P\,dr=-2\int_\infty^a\frac{\mu}{r^3}dr=-2\left[-\frac{\mu}{2r^2}\right]_\infty^a=\frac{\mu}{a^2}.$$

Case I. When $V<V_1$ (velocity from infinity), we have $V^2<V_1^2$ or $V^2<\mu/a^2$ or $a^2V^2<\mu$ or $\mu-a^2V^2>0$.

$\therefore$ from (2), we have

$$a^4V^2\left(\frac{du}{d\theta}\right)^2=(\mu-a^2V^2)(a^2u^2-1)$$

or $$a^2V\frac{du}{d\theta}=\sqrt{(\mu-a^2V^2)}\,.\sqrt{(a^2u^2-1)}$$

or $$\frac{\sqrt{(\mu-a^2V^2)}}{aV}d\theta=\frac{a\,du}{\sqrt{(a^2u^2-1)}}.$$

Substituting $au=z$, so that $a\,du=dz$, we have

$$\frac{\sqrt{(\mu-a^2V^2)}}{aV}d\theta=\frac{dz}{\sqrt{(z^2-1)}}$$

Integrating, $\frac{\sqrt{(\mu-a^2V^2)}}{aV}\theta+B=\cosh^{-1}z$

or $\frac{\sqrt{(\mu-a^2V^2)}}{aV}\theta+B=\cosh^{-1}(au).$

But initially when $u=1/a$, $\theta=0$.

$\therefore$ $0+B=\cosh^{-1}1=0$ or $B=0$.

$\therefore$ $\frac{\sqrt{(\mu-a^2V^2)}}{aV}\theta=\cosh^{-1}(au)$

or $$au=\frac{a}{r}=\cosh\left\{\frac{\sqrt{(\mu-a^2V^2)}}{aV}\theta\right\}$$

or $$r\cosh\left\{\frac{\sqrt{(\mu-a^2V^2)}}{aV}\theta\right\}=a.$$

Case II. When $V>V_1$ (velocity from infinity), we have $V^2>V_1^2$ or $V^2>\mu/a^2$ or $a^2V^2-\mu>0$.

$\therefore$ from (2), we have

$$a^4V^2\left(\frac{du}{d\theta}\right)^2=(a^2V^2-\mu)(1-a^2u^2)$$

or $$a^2V\left(\frac{du}{d\theta}\right)=\sqrt{(a^2V^2-\mu)}\,.\sqrt{(1-a^2u^2)}$$

or $$\frac{\sqrt{(a^2V^2-\mu)}}{aV}d\theta=\frac{a\,du}{\sqrt{(1-a^2u^2)}}.$$

Integrating, $\frac{\sqrt{(a^2V^2-\mu)}}{aV}\theta+C=\sin^{-1}(au).$

But initially when $u=1/a$, $\theta=0$.

$\therefore$ $0+C=\sin^{-1}1$ or $C=\pi/2$.

$\therefore$ $\frac{\sqrt{(a^2V^2-\mu)}}{aV}\theta+\frac{\pi}{2}=\sin^{-1}(au)$

or $$au=\frac{a}{r}=\sin\left\{\frac{\sqrt{(a^2V^2-\mu)}}{aV}\theta+\frac{\pi}{2}\right\}$$

or $$a=r\cos\left\{\frac{\sqrt{(a^2V^2-\mu)}}{aV}\theta\right\}.$$

Ex. 23. *A particle, acted on by a repulsive central force $\mu r/(r^2-9c^2)^2$, is projected from an apse at a distance c with velocity $\sqrt{(\mu/8c^2)}$. Find the equation of its path and show that the time to the cusp is $\frac{4}{3}\pi c^2\sqrt{(2/\mu)}$.*

Sol. Considering the particle of unit mass, the central acceleration

$$P=\frac{-\mu r}{(r^2-9c^2)^2}$$

(Negative sign is taken because the force is repulsive).

The differential equation of the path in pedal form is

$$\frac{h^2}{p^3}\frac{dp}{dr}=P=-\frac{\mu r}{(r^2-9c^2)^2}$$

or $$-\frac{2h^2}{p^3}dp=\frac{2\mu r\,dr}{(r^2-9c^2)^2}=2\mu r(r^2-9c^2)^{-2}dr.$$

Integrating, $$v^2=\frac{h^2}{p^2}=-\frac{\mu}{(r^2-9c^2)}+A \quad \text{...(1)}$$

where A is a constant.

But the particle is projected from an apse at a distance c. Also at an apse, $p=r$. Therefore initially $p=r=c$ and $v=\sqrt{(\mu/8c^2)}$.

$\therefore$ from (1), we have

$$\frac{\mu}{8c^2}=\frac{h^2}{c^2}=-\frac{\mu}{(c^2-9c^2)}+A.$$

$\therefore$ $h^2=\mu/8$, and $A=\frac{\mu}{8c^2}-\frac{\mu}{8c^2}=0.$

Substituting the values of h^2 and A in (1), we have

$$\frac{\mu}{8p^2}=-\frac{\mu}{(r^2-9c^2)} \text{ or } 8p^2=9c^2-r^2, \quad \text{...(2)}$$

which is the pedal equation of the path and is a three-cusped hypocycloid.

Second part. Now we are to find the time to reach the cusp. At the cusp, we have $p=0$. So it is required to find the time from $p=c$ to $p=0$.

We know that in a central orbit,

$$v=\frac{ds}{dt}=\frac{h}{p}.$$

$\therefore$ $h\,dt=p\,ds$ or $h\,dt=p\frac{ds}{dr}dr.$

But $dr/ds=\cos\phi$.

$$\therefore h\,dt=p\cdot\frac{1}{\cos\phi}dr=\frac{p\,dr}{\sqrt{(1-\sin^2\phi)}}=\frac{p\,dr}{\sqrt{(1-(p^2/r^2))}}$$

$$=\frac{pr\,dr}{\sqrt{(r^2-p^2)}}=\frac{p(-8p)\,dp}{\sqrt{(9c^2-8p^2-p^2)}}$$

[$\because$ from (2), $-r\,dr=8p\,dp$]

$$=\frac{-8p^2\,dp}{3\sqrt{(c^2-p^2)}}.$$

Let t_1 be the required time to the cusp. Then integrating from $p=c$ to $p=0$, we get

$$ht_1=-\frac{1}{3}\int_c^0\frac{8p^2\,dp}{\sqrt{(c^2-p^2)}}=\frac{8}{3}\int_0^c\frac{p^2\,dp}{\sqrt{(c^2-p^2)}}$$

$$=\frac{8}{3}\int_0^{\pi/2}\frac{c^2\sin^2z}{c\cos z}c\cos z\,dz$$

[putting $p=c\sin z$, so that $dp=c\cos z\,dz$]

$$=\frac{8}{3}c^2\int_0^{\pi/2}\sin^2z\,dz=\frac{8}{3}c^2\cdot\frac{1}{2}\times\frac{\pi}{2}=\frac{2\pi c^2}{3}.$$

$$\therefore t_1=\frac{2\pi c^2}{3h}=\frac{2\pi c^2}{3}\cdot\sqrt{\left(\frac{8}{\mu}\right)} \quad [\because h^2=\mu/8]$$

$$=\frac{4\pi c^2}{3}\sqrt{\left(\frac{2}{\mu}\right)}.$$

Ex. 24. *A particle is moving with central acceleration $\mu(r^5-c^4r)$ being projected from an apse at a distance c with velocity $c^3\sqrt{(2\mu/3)}$, show that its path is the curve $x^4+y^4=c^4$.* IFoS-2012 IAS-2010 model

Sol. Here the central acceleration

$$P=\mu(r^5-c^4r)=\mu\left(\frac{1}{u^5}-\frac{c^4}{u}\right).$$

$\therefore$ The differential equation of the path is

$$h^2\left[u+\frac{d^2u}{d\theta^2}\right]=\frac{P}{u^2}=\frac{\mu}{u^2}\left(\frac{1}{u^5}-\frac{c^4}{u}\right)=\mu\left(\frac{1}{u^7}-\frac{c^4}{u^3}\right).$$

Multiplying both sides by 2 $(du/d\theta)$ and then integrating, we have

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\left(-\frac{1}{3u^6} + \frac{c^4}{u^2}\right) + A, \quad \ldots(1)$$

where A is a constant.

But initially, when $r = c$ *i.e.*, $u = 1/c$, $du/d\theta = 0$ (at an apse) and $v = c^3\sqrt{(2\mu/3)}$.

$\therefore$ from (1), we have $\dfrac{2\mu c^6}{3} = h^2 \cdot \dfrac{1}{c^2} = \mu\left(-\dfrac{c^6}{3} + c^6\right) + A$.

$\therefore$ $h^2 = \frac{2}{3}\mu c^8$, $A = 0$.

Substituting the values of h^2 and A in (1), we have

$$\tfrac{2}{3}\mu c^8\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\left(-\frac{1}{3u^6} + \frac{c^4}{u^2}\right)$$

or $$c^8\left(\frac{du}{d\theta}\right)^2 = -\frac{1}{2u^6} + \frac{3c^4}{2u^2} - c^8u^2 = \frac{1}{u^6}\left[-\tfrac{1}{2} + \tfrac{3}{2}c^4u^4 - c^8u^8\right]$$

$$= \frac{1}{u^6}\left[-\tfrac{1}{2} - (c^8u^8 - \tfrac{3}{2}c^4u^4)\right] = \frac{1}{u^6}\left[-\tfrac{1}{2} - (c^4u^4 - \tfrac{3}{4})^2 + \tfrac{9}{16}\right]$$

$$= \frac{1}{u^6}\left[(\tfrac{1}{4})^2 - (c^4u^4 - \tfrac{3}{4})^2\right].$$

$\therefore$ $$c^4u^3\frac{du}{d\theta} = \sqrt{[(\tfrac{1}{4})^2 - (c^4u^4 - \tfrac{3}{4})^2]}$$

or $$d\theta = \frac{c^4u^3\,du}{\sqrt{[(\frac{1}{4})^2 - (c^4u^4 - \frac{3}{4})^2]}}.$$

Putting $c^4u^4 - \frac{3}{4} = z$, so that $4c^4u^3\,du = dz$, we have

$$4\,d\theta = \frac{dz}{\sqrt{[(\frac{1}{4})^2 - z^2]}}.$$

Integrating, $4\theta + B = \sin^{-1}\left(\dfrac{z}{\frac{1}{4}}\right) = \sin^{-1}(4z)$,

where B is a constant

or $4\theta + B = \sin^{-1}(4c^4u^4 - 3)$.

But initially when $u = 1/c$, $\theta = 0$, $\therefore$ $B = \sin^{-1} 1 = \pi/2$.

$\therefore$ $4\theta + \frac{1}{2}\pi = \sin^{-1}(4c^4u^4 - 3)$

or $\sin(\frac{1}{2}\pi + 4\theta) = 4c^4u^4 - 3$

or $\cos 4\theta = 4c^4u^4 - 3$.

or $4c^4u^4 = 3 + \cos 4\theta$

or $4c^4/r^4 = [3 + \cos 4\theta]$

or $4c^4 = r^4[3 + (2\cos^2 2\theta - 1)] = 2r^4[1 + \cos^2 2\theta]$

$= 2r^4[(\cos^2\theta + \sin^2\theta)^2 + (\cos^2\theta - \sin^2\theta)^2]$

$= 4r^4(\cos^4\theta + \sin^4\theta)$.

$\therefore$ $c^4 = (r\cos\theta)^4 + (r\sin\theta)^4$

or $c^4 = x^4 + y^4$, $[\because x = r\cos\theta$ and $y = r\sin\theta]$

which is the required equation of the path.

Ex. 25. *If the law of force be $\mu(u^4 - \frac{10}{9}au^5)$ and the particle be projected from an apse at a distance $5a$ with a velocity equal to $\sqrt{(5/7)}$ of that in a circle at the same distance, show that the orbit is the limacon $r = a(3 + 2\cos\theta)$.*

Sol. Here the central acceleration

$$P = \mu\left(u^4 - \frac{10}{9}au^5\right) = \mu\left(\frac{1}{r^4} - \frac{10a}{9r^5}\right).$$

If V is the velocity for a circle at a distance $5a$ then

$$\frac{V^2}{5a} = [P]_{r=5a} = \mu\left[\frac{1}{(5a)^4} - \frac{10a}{9(5a)^5}\right] = \frac{7\mu}{9(5a)^4}$$

$\therefore$ $$V = \sqrt{\left[\frac{7\mu}{9\cdot(5a)^3}\right]}.$$

If v_1 is the velocity of projection of the particle, then

$$v_1 = \sqrt{\left(\frac{5}{7}\right)}V = \sqrt{\left(\frac{5}{7}\right)}\sqrt{\left(\frac{7\mu}{9(5a)^3}\right)} = \sqrt{\left(\frac{\mu}{225a^3}\right)}$$

The differential equation of the path is

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{\mu}{u^2}\left(u^4 - \frac{10}{9}au^5\right) = \mu\left(u^2 - \frac{10}{9}au^3\right).$$

Multiplying both sides by 2 $(du/d\theta)$ and then integrating, we have

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\left(\tfrac{2}{3}u^3 - \tfrac{5}{9}au^4\right) + A. \quad \ldots(1)$$

where A is a constant.

But initially, when $r = 5a$ *i.e.*, $u = \dfrac{1}{5a}$, $\dfrac{du}{d\theta} = 0$ and $v^2 = \dfrac{\mu}{225a^3}$.

$\therefore$ from (1), we have

$$\frac{\mu}{225a^3} = h^2\left(\frac{1}{5a}\right)^2 = \mu\left[\frac{2}{3}\left(\frac{1}{5a}\right)^3 - \frac{5a}{9}\left(\frac{1}{5a}\right)^4\right] + A.$$

$\therefore$ $h^2 = \dfrac{\mu}{9a}$, $A = 0$.

Substituting the values of h^2 and A in (1), we have

$$\frac{\mu}{9a}\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\left(\frac{2}{3}u^3 - \frac{5a}{9}u^4\right)$$

or $$\left(\frac{du}{d\theta}\right)^2 = 6au^3 - 5a^2u^4 - u^2.$$

Putting $u = \dfrac{1}{r}$, so that $\dfrac{du}{d\theta} = -\dfrac{1}{r^2}\dfrac{dr}{d\theta}$, we have

$$\left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 = \frac{6a}{r^3} - \frac{5a^2}{r^4} - \frac{1}{r^2}$$

or $$\left(\frac{dr}{d\theta}\right)^2 = 6ar - 5a^2 - r^2 = -5a^2 - (r^2 - 6ar)$$

$$= -5a^2 - (r - 3a)^2 + 9a^2 = 4a^2 - (r - 3a)^2.$$

$\therefore$ $$\frac{dr}{d\theta} = \sqrt{[(2a)^2 - (r - 3a)^2]}$$

or $$d\theta = \frac{dr}{\sqrt{[(2a)^2 - (r - 3a)^2]}}.$$

Integrating, $\theta + B = \sin^{-1}\left(\dfrac{r - 3a}{2a}\right)$, where B is a constant.

But initially when $r = 5a$, $\theta = 0$. $\therefore$ $B = \sin^{-1} 1 = \pi/2$.

$\therefore$ $\theta + \frac{1}{2}\pi = \sin^{-1}\left(\dfrac{r - 3a}{2a}\right)$ or $\sin(\frac{1}{2}\pi + \theta) = \dfrac{r - 3a}{2a}$

or $r - 3a = 2a\cos\theta$ or $r = a(3 + 2\cos\theta)$,

which is the required equation of the orbit.

Ex. 26. *A particle is projected from an apse at a distance a with the velocity from infinity under the action of a central acceleration μ/r^{2n+3}. Prove that the equation of the path is $r^n = a^n\cos n\theta$.*

Sol. Here, the central acceleration $P = \mu/r^{2n+3} = \mu u^{2n+3}$.

If V is the velocity of the particle at a distance a acquired in falling from rest from infinity under the same acceleration, then as in § 6, page 8,

$$V^2 = -2\int_\infty^a P\,dr = -2\int_\infty^a \frac{\mu}{r^{2n+3}}\,dr = -2\int_\infty^a \mu r^{-2n-3}\,dr$$

$$= -2\mu\left[\frac{r^{-2n-2}}{-2n-2}\right]_\infty^a = \frac{\mu}{(n+1)}\left[\frac{1}{r^{2n+2}}\right]_\infty^a = \frac{\mu}{(n+1)a^{2n+2}}.$$

The differential equation of the path is

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{1}{u^2}\cdot\mu u^{2n+3} = \mu u^{2n+1}.$$

Multiplying both sides by 2 $(du/d\theta)$ and integrating, we get

$$h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \frac{2\mu u^{2n+2}}{2(n+1)} + A, \text{ where } A \text{ is a constant}$$

or $$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \frac{\mu}{(n+1)}\cdot u^{2n+2} + A. \quad \ldots(1)$$

But initially when $r = a$, *i.e.*, $u = 1/a$, $du/d\theta = 0$ (at an apse) and $v = V = \sqrt{\mu}/\sqrt{[(n+1)a^{2n+2}]}$.

$\therefore$ from (1) we have

$$\frac{\mu}{(n+1)a^{2n+2}} = h^2\left[\frac{1}{a^2}\right] = \frac{\mu}{(n+1)}\cdot\frac{1}{a^{2n+2}} + A.$$

$\therefore$ $h^2 = \dfrac{\mu}{(n+1)a^{2n}}$ and $A = 0$.

Substituting the values of h^2 and A in (1), we have

$$\frac{\mu}{(n+1)a^{2n}}\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \frac{\mu}{(n+1)}u^{2n+2}$$

or $$\left(\frac{du}{d\theta}\right)^2 = a^{2n}\cdot u^{2n+2} - u^2.$$

Putting $u = \dfrac{1}{r}$, so that $\dfrac{du}{d\theta} = -\dfrac{1}{r^2}\dfrac{dr}{d\theta}$, we have

$$\left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 = \frac{a^{2n}}{r^{2n+2}} - \frac{1}{r^2}$$

or $$\frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 = \frac{a^{2n}}{r^{2n+2}} - \frac{1}{r^2} = \frac{a^{2n} - r^{2n}}{r^{2n+2}}$$

or $$\left(\frac{dr}{d\theta}\right)^2 = \frac{a^{2n} - r^{2n}}{r^{2n-2}} \quad\text{or}\quad \frac{dr}{d\theta} = \frac{\sqrt{(a^{2n} - r^{2n})}}{r^{n-1}}$$

or $$d\theta = \frac{r^{n-1}\,dr}{\sqrt{[(a^{2n} - r^{2n})]}}.$$

Substituting $r^n = z$, so that $nr^{n-1}\,dr = dz$, we have

$$n\,d\theta = \frac{dz}{\sqrt{[(a^n)^2 - z^2]}}.$$

Integrating, $n\theta + B = \sin^{-1}(z/a^n)$, where B is a constant

or $n\theta + B = \sin^{-1}(r^n/a^n)$.

But initially when $r = a$, $\theta = 0$. $\therefore$ $B = \sin^{-1}(1) = \pi/2$.

$\therefore$ $n\theta + \frac{1}{2}\pi = \sin^{-1}(r^n/a^n)$

or $r^n/a^n = \sin(\frac{1}{2}\pi + n\theta) = \cos n\theta$ or $r^n = a^n\cos n\theta$,

which is the required equation of the path.

Ex. 27. (a) *A particle is projected from an apse at a distance a with the velocity from infinity, the acceleration being μu^7; show that the equation to its path is* $r^2 = a^2\cos 2\theta$.

Sol. Proceed as in Ex. 26. Here $n = 2$.

(b) *A particle is projected from an apse at a distance a with velocity of projection $\sqrt{\mu}/(a^2\sqrt{2})$ under the action of a central force μu^5. Prove that the path is the circle $r = a\cos\theta$.*

Sol. Proceed as in Ex. 26. Here $n = 1$.

(c) *If the central force varies as the cube of the distance from a fixed point then find the orbit.*

Sol. We know that referred to the centre of force as pole the differential equation of a central orbit in pedal form is

$$\frac{h^2}{p^3}\frac{dp}{dr} = P, \quad \text{...(1)}$$

where P is the central acceleration assumed to be attractive.

Here $P = \mu r^3$. Putting $P = \mu r^3$ in (1), we get

$$\frac{h^2}{p^3}\frac{dp}{dr} = \mu r^3$$

or $$\frac{h^2}{p^3}dp = \mu r^3 dr$$

or $$-2\frac{h^2}{p^3}dp = -2\mu r^3 dr.$$

Integrating both sides, we get

$$v^2 = \frac{h^2}{p^2} = -\frac{\mu r^4}{2} + C \quad \text{...(2)}$$

Let $v = v_0$ when $r = r_0$.

Then $$v_0^2 = -\frac{\mu r_0^4}{2} + C$$

or $$C = v_0^2 + \frac{\mu r_0^4}{2}.$$

Putting this value of C in (2), the pedal equation of the central orbit is

$$\frac{h^2}{p^2} = -\frac{\mu r^4}{2} + v_0^2 + \frac{\mu r_0^4}{2}.$$

Ex. 28. *A particle moves with a central acceleration which varies inversely as the cube of the distance. If it be projected from an apse at a distance a from the origin with a velocity which is $\sqrt{2}$ times the velocity for a circle of radius a, show that the equation to its path is $r\cos(\theta/\sqrt{2}) = a$.*

Sol. Here the central acceleration varies inversely as the cube of the distance *i.e.*, $P = \mu/r^3 = \mu u^3$, where μ is a constant.

If V is the velocity for a circle of radius a, then

$$\frac{V^2}{a} = \left[P\right]_{r=a} = \frac{\mu}{a^3}$$

or $$V = \sqrt{(\mu/a^2)}.$$

$\therefore$ the velocity of projection $v_1 = \sqrt{2}V = \sqrt{(2\mu/a^2)}$.

The differential equation of the path is

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{\mu u^3}{u^2} = \mu u.$$

Multiplying both sides by $2\,(du/d\theta)$ and integrating, we have

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu u^2 + A, \quad \text{...(1)}$$

where A is a constant.

But initially when $r = a$ *i.e.*, $u = 1/a$, $du/d\theta = 0$ (at an apse), and $v = v_1 = \sqrt{(2\mu/a^2)}$.

$\therefore$ from (1), we have

$$\frac{2\mu}{a^2} = h^2\left[\frac{1}{a^2}\right] = \frac{\mu}{a^2} + A$$

$\therefore$ $h^2 = 2\mu$ and $A = \mu/a^2$.

Substituting the values of h^2 and A in (1), we have

$$2\mu\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu u^2 + \frac{\mu}{a^2}$$

or $$2\left(\frac{du}{d\theta}\right)^2 = \frac{1}{a^2} + u^2 - 2u^2 = \frac{1 - a^2u^2}{a^2}$$

or $$\sqrt{2}\,a\frac{du}{d\theta} = \sqrt{(1 - a^2u^2)} \quad \text{or} \quad \frac{d\theta}{\sqrt{2}} = \frac{a\,du}{\sqrt{(1 - a^2u^2)}}.$$

Integrating, $(\theta/\sqrt{2}) + B = \sin^{-1}(au)$, where B is a constant.

But initially, when $u = 1/a$, $\theta = 0$. $\therefore$ $B = \sin^{-1}1 = \frac{1}{2}\pi$.

$\therefore$ $(\theta/\sqrt{2}) + \frac{1}{2}\pi = \frac{1}{2}\sin^{-1}(au)$ or $au = a/r = \sin\{\frac{1}{2}\pi + (\theta/\sqrt{2})\}$

or $a = r\cos(\theta/\sqrt{2})$, which is the required equation of the path.

Ex. 29. *A particle moving under a constant force from a centre is projected at a distance a from the centre in a direction perpendicular to the radius vector with velocity acquired in falling to the point of projection from the centre, show that its path is $(a/r)^3 = \cos^2(\frac{3}{2}\theta)$.*

Also show that the particle will ultimately move in a straight line through the origin in the same way as if its path had always been this line.

If the velocity of projection be double that in the previous case show that the path is

$$\frac{\theta}{2} = \tan^{-1}\sqrt{\left[\frac{r-a}{a}\right]} - \frac{1}{\sqrt{3}}\tan^{-1}\sqrt{\left[\frac{r-a}{3a}\right]}.$$

Sol. Since the particle moves under a constant force directed away from a centre, therefore the central acceleration $P = -f$, where f is a constant.

While falling in a straight line from the centre of force to the point of projection, if v is the velocity of the particle at a distance r from the centre of force, then

$$v\frac{dv}{dr} = f \quad \text{or} \quad v\,dv = f\,dr.$$

Let V be the velocity of the particle acquired in falling from the centre to a distance a. Then

$$\int_0^V v\,dv = \int_0^a f\,dr \quad \text{or} \quad \frac{V^2}{2} = af \quad \text{or} \quad V = \sqrt{(2af)}.$$

Therefore the particle is projected from a distance a with velocity $\sqrt{(2af)}$ in a direction perpendicular to the radius vector.

The differential equation of the path is

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = -\frac{f}{u^2}.$$

Multiplying both sides by $2\,(du/d\theta)$ and integrating, we have

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \frac{2f}{u} + A, \quad \text{...(1)}$$

where A is a constant.

But initially, when $r = a$ *i.e.*, $u = 1/a$, $du/d\theta = 0$ (since the particle is projected perpendicular to the radius vector),

and $v = V = \sqrt{(2af)}$.

$\therefore$ from (1), $2af = h^2\left[\frac{1}{a^2}\right] = 2fa + A$.

$\therefore$ $h^2 = 2fa^3$ and $A = 0$.

Substituting the values of h^2 and A in (1), we have

$$2fa^3\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \frac{2f}{u}$$

or $$a^3\left(\frac{du}{d\theta}\right)^2 = -a^3u^2 + \frac{1}{u} = \frac{1 - a^3u^3}{u}$$

or $$a^{3/2}\frac{du}{d\theta} = \frac{\sqrt{(1 - a^3u^3)}}{u^{1/2}}$$

or $$d\theta = \frac{a^{3/2}u^{1/2}\,du}{\sqrt{(1 - a^3u^3)}}.$$

Substituting $a^{3/2}u^{3/2} = z$, so that $\frac{3}{2}a^{3/2}u^{1/2}\,du = dr$, we have

$$\frac{3}{2}d\theta = \frac{dz}{\sqrt{(1 - z^2)}}.$$

Integrating $\frac{3}{2}\theta + B = \sin^{-1}(z) = \sin^{-1}(a^{3/2}u^{3/2})$,

where B is a constant.

But initially when $u = 1/a, \theta = 0$, $\therefore$ $B = \sin^{-1}1 = \frac{1}{2}\pi$.

$\therefore$ $$\frac{3}{2}\theta + \frac{1}{2}\pi = \sin^{-1}(a^{3/2}u^{3/2})$$

or $$a^{3/2}u^{3/2} = \sin(\tfrac{1}{2}\pi + \tfrac{3}{2}\theta) = \cos\tfrac{3}{2}\theta$$

or $$a^{3/2}/r^{3/2} = \cos(\tfrac{3}{2}\theta)$$

or $$(a/r)^3 = \cos^2(\tfrac{3}{2}\theta). \quad \text{...(2)}$$

This is the required equation of the path.

Second part. Now as $r \to \infty$, $\cos(\frac{3}{2}\theta) \to 0$ *i.e.*, $\frac{3}{2}\theta \to \frac{1}{2}\pi$

i.e., $\theta \to \pi/3$.

Hence the particle ultimately moves in a straight line through the origin, inclined at an angle $\theta = \pi/3$, in the same way as if its path had always been this line.

Third part. If the velocity of projection of the particle is double of that in the previous case, then the initial conditions are:

$r = a$, $u = 1/a$, $du/d\theta = 0$ and $v = 2V = 2\sqrt{(2af)}$.

$\therefore$ from (1), we have $8af = h^2\left[\frac{1}{a^2}\right] = 2fa + A$.

$\therefore$ $h^2 = 8a^3f$ and $A = 6af$.

Substituting these values of h^2 and A in (1), we have

$$8a^3f\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \frac{2f}{u} + 6af$$

or. $$4a^3\left(\frac{du}{d\theta}\right)^2 = -4a^3u^2 + \frac{1}{u} + 3a.$$

Putting $u = \frac{1}{r}$, so that $\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta}$, we have

$$4a^3\left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 = -\frac{4a^3}{r^2} + r + 3a$$

or $$4a^3(dr/d\theta)^2 = r^5 + 3ar^4 - 4a^3r^2 = r^2(r^3 + 3ar^2 - 4a^3)$$
$$= r^2[r^2(r-a) + 4ar(r-a) + 4a^2(r-a)]$$
$$= r^2(r-a)(r^2 + 4ar + 4a^2) = r^2(r-a)(r+2a)^2$$

or $$2a^{3/2}(dr/d\theta) = r(r+2a)\sqrt{(r-a)}$$

or $$\frac{d\theta}{2} = \frac{a^{3/2}\,dr}{r(r+2a)\sqrt{(r-a)}}.$$

Substituting $r - a = z^2$, so that $dr = 2z\,dz$, we have

$$\frac{d\theta}{2} = \frac{2a^{3/2}z\,dz}{(z^2+a)(z^2+3a)\cdot z}$$

or $\quad \dfrac{d\theta}{2} = \sqrt{a}\left[\dfrac{1}{z^2+a} - \dfrac{1}{z^2+3a}\right]dz$.

Integrating, $\quad \dfrac{\theta}{2} + B = \sqrt{a}\left[\dfrac{1}{\sqrt{a}}\tan^{-1}\dfrac{z}{\sqrt{a}} - \dfrac{1}{\sqrt{(3a)}}\tan^{-1}\dfrac{z}{\sqrt{(3a)}}\right]$, where B is a constant

or $\quad \dfrac{\theta}{2} + B = \tan^{-1}\sqrt{\left(\dfrac{r-a}{a}\right)} - \dfrac{1}{\sqrt{3}}\tan^{-1}\sqrt{\left(\dfrac{r-a}{3a}\right)}$.

But initially when $r=a, \theta=0$. $\quad \therefore B=0$.

$\therefore \quad \dfrac{\theta}{2} = \tan^{-1}\sqrt{\left(\dfrac{r-a}{a}\right)} - \dfrac{1}{\sqrt{3}}\tan^{-1}\sqrt{\left(\dfrac{r-a}{3a}\right)}$,

which is the required equation of the path.

Ex. 30. *A particle moves with a central acceleration $\mu/(\text{distance})^5$ and projected from the apse at a distance a with a velocity equal to n times that which would be acquired in falling from infinity; show that the other apsidal distance is $a/\sqrt{(n^2-1)}$.*

If $n=1$ and particle be projected in any direction, show that the path is a circle passing through the centre of force.

Sol. Here, the central acceleration

$$P = \frac{\mu}{(\text{distance})^5} = \frac{\mu}{r^5} = \mu u^5.$$

Let V be the velocity from infinity to a distance a from the centre under the same acceleration. Then as in § 6 of this chapter on page 7,

$$V^2 = -2\int_\infty^a P\,dr = -2\int_\infty^a \frac{\mu}{r^5}dr = -2\left[\frac{\mu}{-4r^4}\right]_\infty^a = \frac{\mu}{2a^4}.$$

$\therefore \quad V = \sqrt{(\mu/2a^4)}$.

The differential equation of the path is

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{\mu u^5}{u^2} = \mu u^3.$$

Multiplying both sides by $2\,(du/d\theta)$ and integrating, we have

$$h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \frac{2\mu u^4}{4} + A, \text{ where } A \text{ is a constant}$$

or $$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \frac{\mu u^4}{2} + A. \quad ...(1)$$

But initially, when $r=a$ *i.e.*, $u=1/a$, $du/d\theta=0$ (at an apse) and $v = nV = n\sqrt{(\mu/2a^4)}$.

$\therefore$ from (1), we have $\dfrac{n^2\mu}{2a^4} = h^2\left[\dfrac{1}{a^2}\right] = \dfrac{\mu}{2a^4} + A$.

$\therefore \quad h^2 = \dfrac{n^2\mu}{2a^2}$ and $A = \dfrac{(n^2-1)\mu}{2a^4}$.

Substituting the values of h^2 and A in (1), we have

$$\frac{n^2\mu}{2a^2}\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \frac{\mu u^4}{2} + \frac{(n^2-1)\mu}{2a^4}$$

or $$\left(\frac{du}{d\theta}\right)^2 = \frac{1}{n^2a^2}\left[a^4u^4 - a^2n^2u^2 + (n^2-1)\right].$$

At an apse, we have $du/d\theta = 0$. Therefore the apsidal distances are given by

$0 = (1/n^2a^2)\left[a^4u^4 - a^2n^2u^2 + (n^2-1)\right]$

or $\quad a^4u^4 - a^2n^2u^2 + (n^2-1) = 0$

or $\quad \dfrac{a^4}{r^4} - \dfrac{a^2n^2}{r^2} + (n^2-1) = 0 \quad \left[\because u = \dfrac{1}{r}\right]$

or $\quad (n^2-1)r^4 - a^2n^2r^2 + a^4 = 0$,

which is a quadratic equation in r^2.

If r_1^2 and r_2^2 are its roots, then $r_1^2r_2^2 = a^4/(n^2-1)$.

or $\quad r_1r_2 = a^2/\sqrt{(n^2-1)} \quad ...(2)$

But the first apsidal distance, say r_1, is a.

$\therefore$ from (2), $ar_2 = a^2/\sqrt{(n^2-1)}$

i.e., the second apsidal distance $r_2 = a/\sqrt{(n^2-1)}$.

Second part. When $n=1$ and the particle is projected in any direction, say at an angle α to the radius vector, then at the point of projection, we have $\phi=\alpha$, $p = r\sin\phi = a\sin\alpha$

and so $\quad \dfrac{1}{p^2} = u^2 + \left(\dfrac{du}{d\theta}\right)^2 = \dfrac{1}{(a\sin\alpha)^2}$.

Thus in this case initially when $r=a$ *i.e.*, $u=1/a$, we have $v = V = \sqrt{(\mu/2a^4)}$ and $u^2 + (du/d\theta)^2 = 1/(a^2\sin^2\alpha)$.

$\therefore$ from (1), we have $\dfrac{\mu}{2a^4} = \dfrac{h^2}{(a^2\sin^2\alpha)} = \dfrac{\mu}{2a^4} + A$.

$\therefore \quad h^2 = (\mu\sin^2\alpha)/(2a^2)$ and $A=0$.

Substituting the values of h^2 and A in (1), we have

$$\frac{(\mu\sin^2\alpha)}{2a^2}\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \frac{\mu u^4}{2}$$

or $\quad u^2 + \left(\dfrac{du}{d\theta}\right)^2 = \dfrac{a^2u^4}{\sin^2\alpha} \quad$ or $\quad \left(\dfrac{du}{d\theta}\right)^2 = \dfrac{a^2u^4}{\sin^2\alpha} - u^2$.

Putting $u = \dfrac{1}{r}$, so that $\dfrac{du}{d\theta} = -\dfrac{1}{r^2}\dfrac{dr}{d\theta}$, we have

$$\left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 = \frac{a^2}{r^4\sin^2\alpha} - \frac{1}{r^2}$$

or $\quad \left(\dfrac{dr}{d\theta}\right)^2 = a^2\operatorname{cosec}^2\alpha - r^2 \quad$ or $\quad \dfrac{dr}{d\theta} = \sqrt{(a^2\operatorname{cosec}^2\alpha - r^2)}$

or $\quad d\theta = \dfrac{dr}{\sqrt{(a^2\operatorname{cosec}^2\alpha - r^2)}}$.

Integrating, $\theta + B = \sin^{-1}\left(\dfrac{r}{a\operatorname{cosec}\alpha}\right)$, where B is a constant.

Initially when $r=a$, let $\theta=0$. Then $B = \sin^{-1}(\sin\alpha) = \alpha$.

$\therefore \quad \theta + \alpha = \sin^{-1}\{r/(a\operatorname{cosec}\alpha)\}$

or $\quad r = (a\operatorname{cosec}\alpha)\sin(\theta+\alpha)$

or $\quad r = (a\operatorname{cosec}\alpha)\cos\{\tfrac{1}{2}\pi - (\theta+\alpha)\}$

or $\quad r = (a\operatorname{cosec}\alpha)\cos\{(\theta+\alpha) - \tfrac{1}{2}\pi\}$

or $\quad r = (a\operatorname{cosec}\alpha)\cos\{\theta - (\tfrac{1}{2}\pi - \alpha)\}$

or $\quad r = (a\operatorname{cosec}\alpha)\cos(\theta-\beta)$, where $\beta = \tfrac{1}{2}\pi - \alpha$.

This represents a circle of diameter $a\operatorname{cosec}\alpha$ and pole on its circumference. Hence the path of the particle is a circle through the centre of force.

Ex. 31. *If the acceleration at a distance r is μ/r^5 and the particle is projected at a distance a from the centre of force with velocity $\sqrt{(\mu/2a^4)}$, prove that the orbit is a circle through O of diameter $a\operatorname{cosec}\alpha$, where α is the inclination of the direction of projection to the radius vector.*

Sol. This is Ex. 30, part II. Do yourself.

Ex. 32. *A particle describes an orbit with a central acceleration $\mu u^3 - \lambda u^5$ being projected from an apse at a distance a with velocity equal to that from infinity. Show that its path is $r = a\cosh(\theta/n)$, where $n^2+1 = 2\mu a^2/\lambda$.*

Prove also that it will be at a distance r at the end of time

$$\sqrt{\left(\frac{a^2}{2\lambda}\right)}\left[a^2\log\left\{\frac{r+\sqrt{(r^2-a^2)}}{a}\right\} + r\sqrt{(r^2-a^2)}\right].$$

Sol. Here, the central acceleration

$$P = \mu u^3 - \lambda u^5 = \frac{\mu}{r^3} - \frac{\lambda}{r^5}.$$

Let V be the velocity from infinity at the distance a under the same acceleration. Then

$$V^2 = -2\int_\infty^a P\,dr = -2\int_\infty^a\left(\frac{\mu}{r^3} - \frac{\lambda}{r^5}\right)dr$$

$$= -2\left[-\frac{\mu}{2r^2} + \frac{\lambda}{4r^4}\right]_\infty^a = \frac{\mu}{a^2} - \frac{\lambda}{2a^4}$$

$$= \frac{\lambda}{2a^4}\left(\frac{2\mu a^2}{\lambda} - 1\right) = \frac{\lambda n^2}{2a^4} \quad \left[\because n^2+1 = \frac{2\mu a^2}{\lambda}\right].$$

$\therefore \quad V = (n/a^2)\sqrt{(\lambda/2)}$.

The differential equation of the path is

$$h^2\left\{u + \frac{d^2u}{d\theta^2}\right\} = \frac{P}{u^2} = \frac{\mu u^3 - \lambda u^5}{u^2} = \mu u - \lambda u^3.$$

Multiplying both sides by $2\,(du/d\theta)$ and integrating, we have

$$h^2\left\{u^2 + \left(\frac{du}{d\theta}\right)^2\right\} = 2\left(\frac{\mu u^2}{2} - \frac{\lambda u^4}{4}\right) + A, \quad \text{where } A \text{ is a constant}$$

or $$v^2 = h^2\left\{u^2 + \left(\frac{du}{d\theta}\right)^2\right\} = \mu u^2 - \frac{\lambda u^4}{2} + A. \quad ...(1)$$

But initially when $r=a$ *i.e.*, $u=1/a$, $du/d\theta=0$ (at an apse) and $v = V = (n/a^2)\sqrt{(\lambda/2)}$. Therefore from (1), we have

$$\frac{\lambda n^2}{2a^4} = h^2\left\{\frac{1}{a^2}\right\} = \frac{\mu}{a^2} - \frac{\lambda}{2a^4} + A.$$

$\therefore \quad h^2 = \dfrac{\lambda n^2}{2a^2}$ and $A = \dfrac{\lambda n^2}{2a^4} - \left(\dfrac{\mu}{a^2} - \dfrac{\lambda}{2a^4}\right) = \dfrac{\lambda}{2a^4}(n^2+1) - \dfrac{\mu}{a^2}$

$$= \frac{\lambda}{2a^4}\cdot\left(\frac{2\mu a^2}{\lambda}\right) - \frac{\mu}{a^2} = 0. \quad \left[\because n^2+1 = \frac{2\mu a^2}{\lambda}\right]$$

Substituting the values of h^2 and A in (1), we have

$$\frac{\lambda n^2}{2a^2}\left\{u^2 + \left(\frac{du}{d\theta}\right)^2\right\} = \mu u^2 - \frac{\lambda u^4}{2}$$

$$= \frac{\lambda}{2a^2}(n^2+1)u^2 - \frac{\lambda u^4}{2} \quad \left[\because n^2+1 = \frac{2\mu a^2}{\lambda}\right]$$

or $\quad n^2u^2 + n^2\left(\dfrac{du}{d\theta}\right)^2 = (n^2+1)u^2 - a^2u^4$

or $\quad n^2\left(\dfrac{du}{d\theta}\right)^2 = u^2 - a^2u^4$.

Putting $u = \dfrac{1}{r}$ so that $\dfrac{du}{d\theta} = -\dfrac{1}{r^2}\dfrac{dr}{d\theta}$, we have

$$n^2\left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 = \frac{1}{r^2} - \frac{a^2}{r^4} \quad \text{or} \quad n^2\left(\frac{dr}{d\theta}\right)^2 = r^2 - a^2$$

or $$\frac{dr}{d\theta} = \frac{\sqrt{(r^2-a^2)}}{n} \quad ...(2)$$

or $\quad \dfrac{d\theta}{n} = \dfrac{dr}{\sqrt{(r^2-a^2)}}$.

Integrating, $\theta/n + B = \cosh^{-1}(r/a)$, where B is a constant.

But initially when $r=a$, $\theta=0$ (say). Then $B = \cosh^{-1}(1) = 0$.

$\therefore \quad \theta/n = \cosh^{-1}(r/a)$ or $r = a\cosh(\theta/n)$,

which is the required equation of the path.

Second part. We know that

$$h = r^2 \frac{d\theta}{dt}$$

or $$h = r^2 \frac{d\theta}{dr} \cdot \frac{dr}{dt}.$$

Substituting for h and $dr/d\theta$, we have

$$\frac{n}{a}\sqrt{\left(\frac{\lambda}{2}\right)} = r^2 \cdot \frac{n}{\sqrt{(r^2 - a^2)}} \frac{dr}{dt}$$

or $$dt = a\sqrt{\left(\frac{2}{\lambda}\right)} \frac{r^2\, dr}{\sqrt{(r^2 - a^2)}}.$$

Integrating, the time t from the distance a to the distance r is given by

$$t = a\sqrt{(2/\lambda)} \int_{r=a}^{r} \frac{r^2\, dr}{\sqrt{(r^2 - a^2)}} = a\sqrt{(2/\lambda)} \int_a^r \frac{(r^2 - a^2) + a^2}{\sqrt{(r^2 - a^2)}}\, dr$$

$$= a\sqrt{(2/\lambda)} \int_a^r \left\{\sqrt{(r^2 - a^2)} + \frac{a^2}{\sqrt{(r^2 - a^2)}}\right\} dr$$

$$= a\sqrt{(2/\lambda)} \left[\frac{r}{2}\sqrt{(r^2 - a^2)} - \frac{a^2}{2}\log\{r + \sqrt{(r^2 - a^2)}\} + a^2 \log\{r + \sqrt{(r^2 - a^2)}\}\right]_a^r$$

$$= a\sqrt{(2/\lambda)} \left[\frac{r}{2}\sqrt{(r^2 - a^2)} + \frac{a^2}{2}\log\{r + \sqrt{(r^2 - a^2)}\}\right]_a^r$$

$$= a\sqrt{(2/\lambda)} \left[\frac{r}{2}\sqrt{(r^2 - a^2)} + \frac{a^2}{2}\log\{r + \sqrt{(r^2 - a^2)}\} - \frac{a^2}{2}\log a\right]$$

$$= a\sqrt{(2/\lambda)} \left[\frac{r}{2}\sqrt{(r^2 - a^2)} + \frac{a^2}{2}\log\left\{\frac{r + \sqrt{(r^2 - a^2)}}{a}\right\}\right]$$

$$= \sqrt{(a^2/2\lambda)} \left[r\sqrt{(r^2 - a^2)} + a^2 \log\left\{\frac{r + \sqrt{(r^2 - a^2)}}{a}\right\}\right].$$

Ex. 33. *A particle is acted on by a central repulsive force which varies as the nth power of the distance. If the velocity at any point be equal to that which would be acquired in falling from the centre to the point, show that the equation to the path is of the form*

$$r^{(n+3)/2} \cos \tfrac{1}{2}(n+3)\theta = constant.$$

Sol. Since the particle is acted on by a central repulsive force which varies as the n^{th} power of the distance, therefore the central acceleration

$$P = -\mu\,(\text{distance})^n = -\mu r^n = -\mu/u^n.$$

While falling in a straight line from rest from the centre of force if v is the velocity of the particle at a distance x from the centre, then

$$v\frac{dv}{dx} = \mu x^n \quad \text{or} \quad v\,dv = \mu x^n\, dx.$$

Let V be the velocity of the particle acquired in falling from the centre to a distance r. Then

$$\int_0^V v\,dv = \int_0^r \mu x^n\, dx$$

or $$\tfrac{1}{2}V^2 = \mu\left[\frac{x^{n+1}}{n+1}\right]_0^r = \frac{\mu}{n+1} r^{n+1}$$

or $$V^2 = \{2\mu/(n+1)\} r^{n+1}. \quad \text{...(1)}$$

The differential equation of the central orbit is

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = -\frac{\mu/u^n}{u^2} = -\mu u^{-n-2}.$$

Multiplying both sides by $2\,(du/d\theta)$ and integrating, we have

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \frac{-2\mu u^{-n-1}}{(-n-1)} + A = \frac{2\mu}{(n+1)\,u^{n+1}} + A, \quad \text{...(2)}$$

where A is a constant and v is the velocity of the particle in the orbit at a distance r from the centre.

But according to the question, we have

$$v^2 = V^2 \quad i.e., \quad \frac{2\mu}{(n+1)\,u^{n+1}} + A = \frac{2\mu}{n+1} r^{n+1}.$$

$\therefore \quad A = 0. \quad [\because u = 1/r].$

Substituting the value of A in (2), we have

$$h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \frac{2\mu}{(n+1)\,u^{n+1}}$$

or $$u^2 + \left(\frac{du}{d\theta}\right)^2 = \frac{2\mu}{(n+1)\,h^2 u^{n+1}} = \frac{\lambda^2}{u^{n+1}},$$

where $\lambda^2 = \dfrac{2\mu}{(n+1)\,h^2}$ = constant

or $$\left(\frac{du}{d\theta}\right)^2 = \frac{\lambda^2}{u^{n+1}} - u^2 = \frac{\lambda^2 - u^{n+3}}{u^{n+1}}$$

or $$\frac{du}{d\theta} = -\frac{\sqrt{(\lambda^2 - u^{n+3})}}{u^{(n+1)/2}} \quad \text{or} \quad d\theta = \frac{-u^{(n+1)/2}\, du}{\sqrt{(\lambda^2 - u^{n+3})}}.$$

Substituting $u^{(n+3)/2} = z$, so that $\frac{1}{2}(n+3)\, u^{(n+1)/2}\, du = dz$, we have

$$d\theta = \frac{-2\,dz}{(n+3)\sqrt{(\lambda^2 - z^2)}}$$

or $$\tfrac{1}{2}(n+3)\, d\theta = -\frac{dz}{\sqrt{(\lambda^2 - z^2)}}.$$

Integrating, $\frac{1}{2}(n+3)\,\theta + B = \cos^{-1}(z/\lambda) = \cos^{-1}\{u^{(n+3)/2}/\lambda\}$. ...(3).

Now choose λ such that when $u = 1/a, \theta = 0$; $(1/\lambda)(1/a)^{(n+3)/2} = 1$.

Then from (3), $0 + B = \cos^{-1} 1 = 0$. Therefore $B = 0$.

Putting $B = 0$ in (3), we have

$$\tfrac{1}{2}(n+3)\,\theta = \cos^{-1}\{u^{(n+3)/2}/\lambda\}$$

or $$u^{(n+3)/2} = \lambda \cos\{\tfrac{1}{2}(n+3)\,\theta\}$$

or $$r^{(n+3)/2} \cos\{\tfrac{1}{2}(n+3)\,\theta\} = 1/\lambda = \text{constant}.$$

This gives the required equation to the path.

Ex. 34. *A particle subject to a force producing an acceleration $\mu(r + 2a)/r^5$ towards the origin is projected from the point $(a, 0)$ with a velocity equal to the velocity from infinity at an angle $\cot^{-1} 2$ with the initial line; show that the equation to the path is*

$$r = a(1 + 2\sin\theta).$$

Sol. Here, the central acceleration

$$P = \frac{\mu(r + 2a)}{r^5} = \mu\left(\frac{1}{r^4} + \frac{2a}{r^5}\right) = \mu(u^4 + 2au^5).$$

Let V be the velocity of the particle acquired in falling from rest from infinity under the same acceleration to the point of projection which is at a distance a from the centre. Then

$$V^2 = -2\int_\infty^a P\,dr = -2\int_\infty^a \mu\left(\frac{1}{r^4} + \frac{2a}{r^5}\right) dr$$

$$= -2\mu\left[-\frac{1}{3r^3} - \frac{2a}{4r^4}\right]_\infty^a = 2\mu\left[\frac{1}{3a^3} + \frac{1}{2a^3}\right] = \frac{5\mu}{3a^3}$$

or $$V = \sqrt{(5\mu/3a^3)}.$$

According to the question the velocity of projection of the particle is equal to V i.e., $\sqrt{(5\mu/3a^3)}$.

Now the differential equation of the path is

$$h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2} = \frac{\mu}{u^2}(u^4 + 2au^5) = \mu(u^2 + 2au^3).$$

Multiplying both sides by $2\,(du/d\theta)$ and integrating, we have

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\left(\frac{2u^3}{3} + au^4\right) + A, \quad \text{...(1)}$$

where A is a constant.

Initially when $r = a$ i.e., $u = 1/a$, $v = \sqrt{(5\mu/3a^3)}$.

Also initially $\phi = \cot^{-1} 2$ or $\cot\phi = 2$ or $\sin\phi = 1/\sqrt{5}$.

But $p = r\sin\phi$. Therefore initially $p = a\,(1/\sqrt{5}) = a/\sqrt{5}$

or $$1/p^2 = 5/a^2.$$

But $1/p^2 = u^2 + (du/d\theta)^2$. Therefore initially, when $r = a$, we have $u^2 + (du/d\theta)^2 = 5/a^2$.

Applying the above initial conditions in (1), we have

$$\frac{5\mu}{3a^3} = h^2 \frac{5}{a^2} = \mu\left(\frac{2}{3a^3} + \frac{a}{a^4}\right) + A. \quad \therefore h^2 = \mu/3a, A = 0.$$

Substituting the values of h^2 and A in (1), we have

$$\frac{\mu}{3a}\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right] = \mu\left(\frac{2}{3}u^3 + au^4\right)$$

or $$\left(\frac{du}{d\theta}\right)^2 = 2au^3 + 3a^2u^4 - u^2.$$

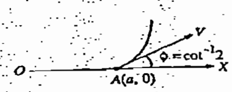
Putting $u = \dfrac{1}{r}$, so that $\dfrac{du}{d\theta} = -\dfrac{1}{r^2}\dfrac{dr}{d\theta}$, we have

$$\left(-\frac{1}{r^2}\frac{dr}{d\theta}\right)^2 = \frac{2a}{r^3} + \frac{3a^2}{r^4} - \frac{1}{r^2}$$

or $$(dr/d\theta)^2 = 2ar + 3a^2 - r^2 = 3a^2 - (r^2 - 2ar)$$
$$= 3a^2 - (r - a)^2 + a^2 = 4a^2 - (r - a)^2$$

or $$dr/d\theta = \sqrt{[(2a)^2 - (r - a)^2]}$$

[Note that as the particle starts moving from A, r increases as θ increases. So we have taken $dr/d\theta$ with +ive sign.]



or $$d\theta = \frac{dr}{\sqrt{[(2a)^2 - (r - a)^2]}}.$$

Integrating, $\theta + B = \sin^{-1}\left(\dfrac{r - a}{2a}\right)$.

But initially when $r = a$, $\theta = 0$. $\therefore B = \sin^{-1} 0 = 0$.

$\therefore \theta = \sin^{-1}\left(\dfrac{r - a}{2a}\right)$ or $\sin\theta = \dfrac{r - a}{2a}$

or $r = a(1 + 2\sin\theta)$, which is the required equation of the path.

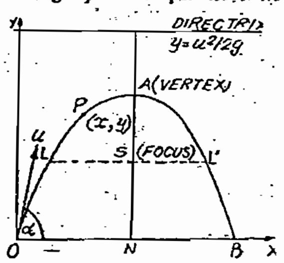
PROJECTILES | SET-IV

1. Introduction. If we throw a ball into the air (not vertically upwards), it describes a curved path. The body so projected is called a projectile and the curved path described by the body is called its trajectory. In this chapter, we shall study the motion of a projectile in a vertical plane through the point of projection, assuming that air offers no resistance and that the acceleration due to the attraction of the earth is constant and is equal to g *i.e.*, its value on the surface of the earth.

2. The Motion of a Projectile and its Trajectory. *A particle of mass m is projected, in a vertical plane through the point of projection, with velocity u in a direction making an angle α with the horizontal; to show that the path of the projectile in vaccum is a parabola.*

Take the point of projection O as the origin, the horizontal line OX in the plane of projection as the x-axis and the vertical line OY as the y-axis. Let $P(x, y)$ be the position of the particle at any time t.

There is no force acting upon the particle in the direction of x-axis. The only external force acting upon the particle is its weight mg acting vertically downwards *i.e.*, parallel to the y-axis in the direction of y-decreasing. Therefore the equations of motion of the particle at P are



$$d^2x/dt^2=0 \qquad \ldots(1)$$

and $$d^2y/dt^2=-g \qquad \ldots(2)$$

Integrating (1), we get $dx/dt=$constant.

But initially at the point of projection O, we have $dx/dt=$the horizontal component of the velocity at $O=u\cos\alpha$.

$\therefore$ throughout the motion of the projectile, we have

$$dx/dt=u\cos\alpha. \qquad \ldots(3)$$

Thus the horizontal velocity of a projectile remains constant *i.e.*, $u\cos\alpha$ throughout the motion.

Integrating (3), we get

$x=(u\cos\alpha).t+A$, where A is a constant.

But at the point O, we have $x=0$ and $t=0$. $\therefore A=0$.

$$\therefore \quad x=(u\cos\alpha).t. \qquad \ldots(4)$$

The equation (4) gives the horizontal displacement of the particle in time t.

Again integrating (2), we get

$dy/dt=-gt+C$, where C is a constant.

But initially at O, $t=0$ and $dy/dt=$the vertical component of the velocity at $O=u\sin\alpha$.

$\therefore$ $u\sin\alpha=0+C$ or $C=u\sin\alpha$.

$$\therefore \quad dy/dt=u\sin\alpha-gt. \qquad \ldots(5)$$

The equation (5) gives the vertical component of the velocity of the projectile at any time t.

Now integrating (5), we get

$y=(u\sin\alpha)t-\frac{1}{2}gt^2+B$, where B is a constant.

But initially at the point O, $y=0$ and $t=0$ so that $B=0$.

$$\therefore \quad y=(u\sin\alpha)t-\tfrac{1}{2}gt^2. \qquad \ldots(6)$$

The equation (6) gives the vertical displacement of the projectile from the point of projection in time t.

For a given value of y, say h, the equation (6) is a quadratic in t and will give two values of t. If the values of t are real and distinct, the smaller value of t will give the time for the projectile to be at a height h while rising upwards and the larger value will give the time for the projectile to be at a height h while falling downwards.

The equations (3), (4), (5) and (6) determine completely the motion of the projectile.

The equations (4) and (6) may be looked upon as the equations of the trajectory in parametric form, the parameter being t.

Eliminating t between (4) and (6), we get

$$y=(u\sin\alpha).\frac{x}{u\cos\alpha}-\tfrac{1}{2}g\left(\frac{x}{u\cos\alpha}\right)^2$$

or $$y=x\tan\alpha-\tfrac{1}{2}g\frac{x^2}{u^2\cos^2\alpha}, \qquad \ldots(7)$$

as the cartesian form of the equation of the trajectory. The equation (7) is a second degree equation in x and y in which the second degree terms are in a perfect square and hence it represents a parabola.

If v is the resultant velocity of the projectile at P at time t, we have

$$v=\sqrt{[(dx/dt)^2+(dy/dt)^2]}$$
$$=\sqrt{\{(u\cos\alpha)^2+(u\sin\alpha-gt)^2\}}$$
$$=\sqrt{(u^2-2ugt\sin\alpha+g^2t^2)}.$$

The direction of the velocity v is along the tangent to the trajectory at the point P. If this direction makes an angle θ with the horizontal, we have

$$\tan\theta=\frac{dy/dt}{dx/dt}=\frac{u\sin\alpha-gt}{u\cos\alpha}.$$

3. Latus Rectum, Vertex, Focus and Directrix of the Trajectory.

As found in the preceding article, referred to OX and OY as the coordinate axes, the eqution of the trajectory is

$$y=x\tan\alpha-\tfrac{1}{2}g\frac{x^2}{u^2\cos^2\alpha}. \qquad \ldots(1)$$

The equation (1) can be put in the form

$$\tfrac{1}{2}g\frac{x^2}{u^2\cos^2\alpha}-x\tan\alpha=-y$$

or $$x^2-\frac{2u^2\cos^2\alpha\tan\alpha}{g}x=-\frac{2u^2\cos^2\alpha}{g}y$$

or $$x^2-\frac{2u^2\cos\alpha\sin\alpha}{g}x=-\frac{2u^2\cos^2\alpha}{g}y$$

or $$\left(x-\frac{u^2\cos\alpha\sin\alpha}{g}\right)^2=-\frac{2u^2\cos^2\alpha}{g}y+\frac{u^4\cos^2\alpha\sin^2\alpha}{g^2}$$

or $$\left(x-\frac{u^2\cos\alpha\sin\alpha}{g}\right)^2=-\frac{2u^2\cos^2\alpha}{g}\left(y-\frac{u^2\sin^2\alpha}{2g}\right). \qquad \ldots(2)$$

If we shift the origin to the point $\left(\frac{u^2\cos\alpha\sin\alpha}{g}, \frac{u^2\sin^2\alpha}{2g}\right)$, the coordinate axes remaining parallel to their original directions, the equation (2) becomes

$$x^2=-\frac{2u^2\cos^2\alpha}{g}y. \qquad \ldots(3)$$

This is the standard equation of the parabola in the form $x^2=-4ay$ with its vertex at the new origin and its axis AN along the negative direction of the new y-axis. From the equation (3) it is clear that

the latus rectum of the trajectory

$$=\frac{2}{g}u^2\cos^2\alpha=\frac{2}{g}\text{(horizontal velocity)}^2.$$

Vertex. If A is the vertex of the trajectory, then A is the new origin. So referred to the original coordinate axes OX and OY, the coordinates of the vertex of the parabola (1) are

$$\left(\frac{u^2\sin\alpha\cos\alpha}{g}, \frac{u^2\sin^2\alpha}{2g}\right).$$

Focus. Let S be the focus of the trajectory. Then S is a point on the axis of the parabola. We shall find the coordinates of S with respect to the original coordinate axes OX and OY.

Obviously the x-coordinate of $S=$the x-coordinate of A

$$=\frac{u^2\sin\alpha\cos\alpha}{g}=\frac{u^2\sin 2\alpha}{2g}.$$

Again the y-coordinate of S

$=$the y-coordinate of $A-\frac{1}{4}$ latus rectum

$$=\frac{u^2\sin^2\alpha}{2g}-\frac{1}{4}.\frac{2u^2\cos^2\alpha}{g}=\frac{u^2\sin^2\alpha}{2g}-\frac{u^2\cos^2\alpha}{2g}$$

$$=-\frac{u^2}{2g}(\cos^2\alpha-\sin^2\alpha)=-\frac{u^2}{2g}\cos 2\alpha.$$

$\therefore$ the coordinates of the focus of the parabola (1) are

$$\left(\frac{u^2}{2g}\sin 2\alpha, \frac{-u^2}{2g}\cos 2\alpha\right)$$

We observe that the y-coordinate $(-u^2/2g)\cos 2\alpha$ of the focus is positive, zero or negative according as

$$2\alpha > \text{or} = \text{or} < \tfrac{1}{2}\pi$$

i.e., $$\alpha > \text{or} = \text{or} < \tfrac{1}{4}\pi.$$

If $\alpha=\frac{1}{4}\pi$, the y-coordinate of the focus becomes zero and then the focus is in the horizontal line OX.

Directrix. The directrix of the trajectory is a line perpendicular to the axis of the parabola and so it is a horizontal line.

The height of the directrix above the point of projection O

$=$the height of the vertex A above $O+\frac{1}{4}$ latus rectum

$$=\frac{u^2\sin^2\alpha}{2g}+\frac{1}{4}.\frac{2u^2\cos^2\alpha}{g}=\frac{u^2}{2g}(\cos^2\alpha+\sin^2\alpha)=\frac{u^2}{2g}.$$

Therefore the equation of the directrix of the parabola (1) is

$$y=u^2/2g.$$

We observe that the equation of the directrix is independent of the angle of projection α.

Therefore *the trajectories of all the particles projected in the same vertical plane from the same point with the same velocity in different directions have the same directrix.*

4. Time of flight, Horizontal range and Maximum height.

Time of flight. *The time taken by the particle from the point of projection to reach the horizontal plane through the point of projection again is called the time of flight.* It is usually denoted by T. In the figure of § 2, the time of flight T is the time from O to B.

Initial vertical velocity at O is $u \sin \alpha$ in the upward direction and the acceleration in the vertical direction is g acting vertically downwards. When the particle strikes the horizontal plane through O at the point B, its vertical displacement from O is zero. So considering the vertical motion from O to B and using the formula $s = ut + \frac{1}{2} ft^2$, we have

$$0 = (u \sin \alpha)\, T - \tfrac{1}{2} g T^2 \quad \text{or} \quad T\,[u \sin \alpha - \tfrac{1}{2} g T] = 0$$

or
$$T = \frac{2u \sin \alpha}{g}. \qquad [\because T \neq 0]$$

This gives the time of flight.

Horizontal range. *If B is the point where the projectile after projection from O, strikes the ground again, then OB is called the horizontal range.* The horizontal range is usually denoted by R.

To find the horizontal range R we consider the horizontal motion from O to B. The horizontal velocity remains constant and equal to $u \cos \alpha$ during the motion from O to B. Also the time from O to B is T. Therefore

$$R = (u \cos \alpha) . T = u \cos \alpha . \frac{2u \sin \alpha}{g}.$$

Thus
$$R = \frac{2u^2 \sin \alpha \cos \alpha}{g} = \frac{u^2 \sin 2\alpha}{g}. \qquad \ldots(1)$$

Maximum horizontal range. It is the greatest horizontal range for a given velocity of projection, say u. If u is given, then from (1), we see that R depends upon the angle of projection α. Obviously R is maximum when $\sin 2\alpha$ is maximum

i.e., when $\sin 2\alpha = 1$ or $2\alpha = \frac{1}{2}\pi$ or $\alpha = \frac{1}{4}\pi$.

Thus for a given velocity of projection the horizontal range is maximum when the angle of projection is 45°. Also the maximum horizontal range $= u^2/g$.

For the maximum horizontal range, the angle of projection $\alpha = \pi/4$. So in the case of maximum horizontal range, the y-coordinate of the focus of the trajectory

$$= \frac{-u^2 \cos 2\alpha}{2g} = \frac{-u^2 \cos \frac{1}{2}\pi}{2g} = 0$$ *i.e.*, the focus lies on the horizontal line OX.

Thus in the case of maximum horizontal range the focus lies in the range itself.

Again from (1), we observe that the expression for the range remains unchanged if we replace α by $\frac{1}{2}\pi - \alpha$. Therefore to obtain a given horizontal range for a given velocity of projection, there are two possible directions of projection. The inclinations, say α_1 and α_2, of these two directions of projection to the horizontal are complementary angles. Thus

$$\alpha_1 + \alpha_2 = \tfrac{1}{2}\pi = \tfrac{1}{4}\pi + \tfrac{1}{4}\pi \quad \text{or} \quad \alpha_2 - \tfrac{1}{4}\pi = \tfrac{1}{4}\pi - \alpha_1,$$

showing that *the two possible directions of projection for a given range are equally inclined to the direction of projection for the maximum range.*

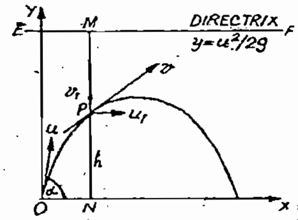
Greatest height. *The greatest vertical height reached by the projectile during its motion is called the greatest height.* It is usually denoted by H. If A is the highest point of the trajectory, then at A the vertical component of the velocity is zero. Let H be the height of A above the point of projection O. Considering the vertical motion from O to A and using the formula $v^2 = u^2 + 2fs$, we have

$$0 = u^2 \sin^2 \alpha - 2gH$$

or
$$H = \frac{u^2 \sin^2 \alpha}{2g}.$$

5. Velocity at any point of the trajectory. *The velocity of a projectile at any point of its path is that due to a fall from the directrix to that point.*

Suppose a particle is projected from O with velocity u at an angle α to the horizontal. Take O as origin, the horizontal line OX in the plane of motion as x-axis and the vertical line OY as y-axis.



Let v be the velocity of the projectile at any point P of its path. Let the height PN of P above O be h. Suppose u_1 and v_1 are the horizontal and vertical components of v. We have

$$u_1 = u \cos \alpha.$$

Also considering the vertical motion from O to P and using the formula $v^2 = u^2 + 2fs$, we have

$$v_1^2 = u^2 \sin^2 \alpha - 2gh.$$

Now $v^2 = u_1^2 + v_1^2 = u^2 \cos^2 \alpha + u^2 \sin^2 \alpha - 2gh = u^2 - 2gh$. Thus

$$v^2 = u^2 - 2gh. \qquad \ldots(1)$$

The relation (1) gives the velocity of the projectile at a height h above the point of projection.

The equation of the directrix EF of the trajectory is $y = u^2/2g$. The depth of P below the directrix $= MP = (u^2/2g) - h$. If a particle falls freely under gravity from M to P, let V be the velocity gained by it at P. Then

$$V^2 = 0 + 2g . MP = 2g\, [(u^2/2g) - h] = u^2 - 2gh. \qquad \ldots(2)$$

From (1) and (2), we observe that $v = V$. Hence the velocity of a projectile at any point of its path is that due to a fall from the directrix to that point.

6. Locus of the focus and vertex of the trajectory. *Particles are projected in the same verticle plane from the same point with the same velocity in different directions. To find the locus of the foci and also that of the vertices of their paths.*

Refer the figure of § 2.

Take the point of projection O as origin, the horizontal line OX lying in the plane of projection as the x-axis and the vertical line OY as the y-axis. Let u be the velocity of projection for each trajectory. Let S be the focus and A be the vertex of any trajectory for which α is the angle of projection. Here α is a parameter and we are to find the loci of the points S and A for varying values of α.

Locus of the focus. Let (x_1, y_1) be the co-ordinates of the focus S. Then

$$x_1 = \frac{u^2 \sin 2\alpha}{2g}, \quad y_1 = -\frac{u^2 \cos 2\alpha}{2g}.$$

Eliminating α between these two relations, we get

$$x_1^2 + y_1^2 = u^4/4g^2. \qquad \ldots(1)$$

Generalising (1) to get the locus of the point (x_1, y_1), we have
$$x^2 + y^2 = u^4/4g^2.$$

This is the locus of the foci and is obviously a circle whose centre is the point of projection O and radius is $u^2/2g$.

Locus of the vertex. Let (h, k) be the co-ordinates of the vertex A. Then
$$h = \frac{u^2 \sin \alpha \cos \alpha}{g} \qquad \ldots(2)$$

and
$$k = \frac{u^2 \sin^2 \alpha}{2g}. \qquad \ldots(3)$$

To find the locus of the point (h, k) for varying values of α, we have to eliminate α between (2) and (3).

From (3), $\sin^2 \alpha = 2gk/u^2$.

Squaring both sides of (2), we get

$$h^2 = \frac{u^4 \sin^2 \alpha \cos^2 \alpha}{g^2} = \frac{u^4 \sin^2 \alpha}{g^2}(1 - \sin^2 \alpha) \qquad \ldots(4)$$

Putting $\sin^2 \alpha = 2gk/u^2$ in (4), we get

$$h^2 = \frac{u^4}{g^2} \cdot \frac{2gk}{u^2}\left(1 - \frac{2gk}{u^2}\right) = \frac{2u^2 k}{g}\left(1 - \frac{2gk}{u^2}\right)$$

or $gh^2 = 2u^2 k - 4gk^2$ or $g\,(h^2 + 4k^2) = 2u^2 k$

or $h^2 + 4k^2 = 2u^2 k/g$.

Generalising (h, k), we get the locus of the vertex as the ellipse
$$x^2 + 4y^2 = 2u^2 y/g.$$

7. Some geometrical properties of a parabola. The following geometrical properties of a parabola will be often used while solving the problems on projectiles.

1. The distance of any point on a parabola from its focus is equal to its distance from the directrix.

2. The tangents at the extremities of any focal chord of a parabola intersect at right angles on the directrix.

3. The tangent at any point on a parabola bisects the angle between the focal distance of the point and the perpendicular drawn from the point to the directrix.

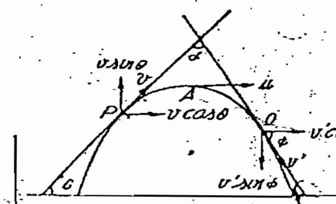
4. The line joining the point of intersection of the tangents at the extremities of any chord of a parabola to the middle point of the chord is parallel to the axis of the parabola.

Illustrative Examples

Ex. 1. *If α be the angle between the tangents at the extremities of any arc of a parabolic path, v and v' the velocities at these extremities and u the velocity at the vertex of the path, show that the time for describing the arc is* $(vv' \sin \alpha)/(gu)$.

Sol. PQ is an arc of a parabolic path and A is its vertex. Suppose the tangents at the points P and Q to the parabola make angles θ and ϕ respectively with the horizontal as shown in the figure. Since α is given to be the angle between the tangents at P and Q, therefore

$$\theta+\phi+\alpha=\pi \quad \text{or} \quad \theta+\phi=\pi-\alpha.$$



The velocity of the particle at P is v and is along the tangent at P. The velocity at Q is v' and is along the tangent at Q. The velocity at the vertex A is u and is along the tangent at A which is a horizontal line.

Since the horizontal velocity of a projectile remains constant throughout the motion, therefore

$$v \cos \theta=u=v' \cos \phi. \qquad \ldots(1)$$

The vertical velocity at $P=v \sin \theta$, vertically upwards and the vertical velocity at $Q=v' \sin \phi$, vertically downwards.

Let t be the time from P to Q. Considering the vertical motion from P to Q and using the formula $v=u+ft$, we have

$$-v' \sin \phi=v \sin \theta-gt, \text{ or } gt=v \sin \theta+v' \sin \phi.$$

$$\therefore \quad t=\frac{v \sin \theta+v' \sin \phi}{g}=\frac{uv \sin \theta+uv' \sin \phi}{gu}$$

[multiplying the Nr. and the Dr. by u]

$$=\frac{vv' \sin \theta \cos \phi+vv' \cos \theta \sin \phi}{gu}$$

[substituting suitably for u from (1)]

$$=\frac{vv' \sin (\theta+\phi)}{gu}=\frac{vv' \sin (\pi-\alpha)}{gu}=\frac{vv' \sin \alpha}{gu}.$$

Ex. 2. *If at any instant the velocity of a projectile be u, and its direction of motion θ to the horizontal, then show that it will be moving at right angles to this direction after time (u/g) cosec θ.*

Sol. Draw figure as in Ex. 1 by taking $\alpha=\pi/2$ and $\phi=\frac{1}{2}\pi-\theta$. The velocity of the projectile at the point P is u and its direction makes an angle θ with the horizontal. Let v be the velocity of the projectile at the point Q when it is moving at right angles to its direction at P. Obviously the tangent at Q to the path makes an angle $\frac{1}{2}\pi-\theta$ with the horizontal.

Since the horizontal velocity of a projectile remains constant throughout the motion, therefore

$$u \cos \theta=v \cos (\tfrac{1}{2}\pi-\theta)=v \sin \theta. \qquad \ldots(1)$$

The vertical velocity at P is $u \sin \theta$, vertically upwards and the vertical velocity at Q is $v \sin (\frac{1}{2}\pi-\theta)$ i.e., $v \cos \theta$, vertically downwards. Let t be the time from P to Q. Considering the vertical motion from P to Q and using the formula $v=u+ft$, we have

$$-v \cos \theta=u \sin \theta-gt \quad \text{or} \quad gt=u \sin \theta+v \cos \theta.$$

$$\therefore \quad t=\frac{1}{g}(u \sin \theta+v \cos \theta)=\frac{u}{g}\left(\sin \theta+\frac{\cos \theta}{\sin \theta} \cos \theta\right)$$

[substituting for v from (1)]

$$=\frac{u}{g \sin \theta}(\sin^2 \theta+\cos^2 \theta)=\frac{u}{g \sin \theta}=\frac{u}{g} \operatorname{cosec} \theta.$$

Ex. 3. *If v_1, v_2 be the velocities at the ends of a focal chord of a projectile's path and u the velocity at the vertex of the path, then show that $\frac{1}{v_1^2}+\frac{1}{v_2^2}=\frac{1}{u^2}$.*

Sol. Let P and Q be the extremities of a focal chord PSQ of a projectile's path. [Draw the figure as in Ex. 1]. Suppose the tangent at P to the path makes an angle θ with the horizontal. Since the tangents at the extremities of a focal chord cut at right angles, therefore the tangent at Q to the path makes an angle $\frac{1}{2}\pi-\theta$ with the horizontal.

The velocity at P is v_1 and is along the tangent at P. The velocity at Q is v_2 and is along the tangent at Q. The velocity at the vertex of the path is u and is in a horizontal direction. Since the horizontal velocity of a projectile remains constant throughout the motion, therefore

$$v_1 \cos \theta=u=v_2 \cos (\tfrac{1}{2}\pi-\theta)=v_2 \sin \theta.$$

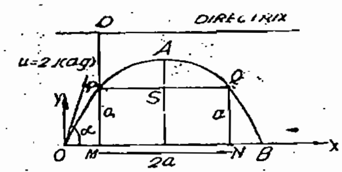
$$\therefore \quad \cos \theta=\frac{u}{v_1} \text{ and } \sin \theta=\frac{u}{v_2}.$$

Squaring and adding, we get

$$\frac{u^2}{v_1^2}+\frac{u^2}{v_2^2}=1 \quad \text{or} \quad u^2\left(\frac{1}{v_1^2}+\frac{1}{v_2^2}\right)=1 \quad \text{or} \quad \frac{1}{v_1^2}+\frac{1}{v_2^2}=\frac{1}{u^2}.$$

Ex. 4. *A particle is projected with a velocity $2\sqrt{(ga)}$ so that it just clears two walls of equal height a which are at a distance 2a from each other. Show that the latus rectum of the path is equal to 2a, and that the time of passing between the walls is $2\sqrt{(a/g)}$.*

Sol. PM and QN are two vertical walls each of height a and $MN=PQ=2a$. A particle is projected from O with velocity $u=2\sqrt{(ga)}$ at an angle, say α. The particle just clears the walls PM and QN.



Let S be the middle point of PQ. The chord PSQ is perpendicular to the axis of the parabola and the point S is on the axis. Also $PS=\frac{1}{2}PQ=a$.

The height of the directrix of the trajectory above the point of projection $O=DM=\frac{u^2}{2g}=\frac{4ga}{2g}=2a$.

$\therefore$ the perpendicular distance of P from the directrix

$$=PD=DM-PM=2a-a=a=PS.$$

Thus S is a point on the axis of the parabola such that $PD=PS$. Therefore S is the focus of the trajectory and consequently PSQ is the latus rectum of the path.

$\therefore$ the length of the latus rectum of the path $=PQ=2a$.

But the length of the latus rectum of the path $=\frac{2}{g}(u \cos \alpha)^2$.

$$\therefore \quad \frac{2}{g}(u \cos \alpha)^2=2a \text{ or } (u \cos \alpha)^2=ag \text{ or } u \cos \alpha=\sqrt{(ag)}.$$

Thus the horizontal velocity of the particle is $\sqrt{(ag)}$.

Let t_1 be the time of passing between the walls *i.e.*, the time from P to Q. Since the horizontal velocity of a projectile remains constant throughout the motion, therefore considering the horizontal motion from P to Q, we have $2a=(u \cos \alpha)\, t_1$.

$$\therefore \quad t_1=\frac{2a}{u \cos \alpha}=\frac{2a}{\sqrt{(ag)}}=2\sqrt{(a/g)}.$$

Ex. 5. *A body is projected at an angle α to the horizontal so as to clear two walls of equal height a at a distance 2a from each other. Show that the range is equal to 2a cot $\frac{1}{2}\alpha$.*

Sol. Draw figure as in Ex. 4.

Take the point of projection O as the origin, the horizontal line through O in the plane of motion as the x-axis and the vertical line through O as the y-axis. Let u be the velocity of projection and α be the angle of projection.

The equation of the trajectory is

$$y=x \tan \alpha-\frac{1}{2} g \frac{x^2}{u^2 \cos^2 \alpha}. \qquad \ldots(1)$$

The particle just clears two walls PM and QN each of height a and at a distance $2a$ from each other.

The y-co-ordinate of each of the points P and Q is a. Putting $y=a$ in (1), we get

$$a=x \tan \alpha-\frac{1}{2} \frac{gx^2}{u^2 \cos^2 \alpha}$$

or $$gx^2-2u^2 x \sin \alpha \cos \alpha+2au^2 \cos^2 \alpha=0. \qquad \ldots(2)$$

Let x_1 and x_2 be the x-co-ordinates of the points P and Q respectively. Then $x_1=OM$ and $x_2=ON$.

Let R be the range of the particle *i.e.*, let $OB=R$. From the symmetry of the path about the axis of the parabola, we have

$$NB=OM=x_1.$$

Now $R=OB=ON+NB=x_2+x_1$.

Obviously x_1 and x_2 are the roots of the quadratic (2) in x.

We have $$x_1+x_2=\frac{2u^2 \sin \alpha \cos \alpha}{g}=R$$

i.e., $$u^2=\frac{gR}{2 \sin \alpha \cos \alpha} \qquad \ldots(3)$$

and $$x_1 x_2=\frac{2au^2 \cos^2 \alpha}{g}.$$

But the distance between the walls $=2a=x_2-x_1$.

$$\therefore \quad 4a^2=(x_2-x_1)^2=(x_2+x_1)^2-4x_1x_2$$

$$=R^2-\frac{8au^2 \cos^2 \alpha}{g}=R^2-\frac{8a \cos^2 \alpha}{g} \cdot \frac{gR}{2 \sin \alpha \cos \alpha}$$

[substituting for u^2 from (3)]

or $$R^2-(4a \cot \alpha) R-4a^2=0.$$

$$\therefore \quad R=\frac{4a \cot \alpha \pm \sqrt{(16a^2 \cot^2 \alpha+16a^2)}}{2}.$$

Neglecting the $-$ive sign because R cannot be $-$ive, we have $R=2a \cot \alpha+2a \operatorname{cosec} \alpha=2a(\cot \alpha+\operatorname{cosec} \alpha)$

$$=2a \frac{\cos \alpha+1}{\sin \alpha}=2a \frac{2 \cos^2 \frac{1}{2}\alpha}{2 \sin \frac{1}{2}\alpha \cos \frac{1}{2}\alpha}=2a \cot \tfrac{1}{2}\alpha.$$

Ex. 6. *Two bodies are projected from the same point in directions making angles α_1 and α_2 with the horizontal and strike at the same point in the horizontal plane through the point of projection. If t_1, t_2 be their times of flight, show that*

$$\frac{t_1^2-t_2^2}{t_1^2+t_2^2}=\frac{\sin(\alpha_1-\alpha_2)}{\sin(\alpha_1+\alpha_2)}.$$

Sol. Let u_1 be the velocity of projection of the body projected at an angle α_1 and u_2 be that of the body projected at an angle α_2. Since the horizontal ranges in the two cases are given to be equal, therefore

$$\frac{2u_1^2\sin\alpha_1\cos\alpha_1}{g}=\frac{2u_2^2\sin\alpha_2\cos\alpha_2}{g}$$

or $$\frac{u_1^2}{u_2^2}=\frac{\sin\alpha_2\cos\alpha_2}{\sin\alpha_1\cos\alpha_1} \quad \text{...(1)}$$

Also $t_1=2u_1\sin\alpha_1/g$ and $t_2=2u_2\sin\alpha_2/g$.

$$\therefore \frac{t_1^2}{t_2^2}=\frac{u_1^2\sin^2\alpha_1}{u_2^2\sin^2\alpha_2}=\frac{\sin\alpha_2\cos\alpha_2\sin^2\alpha_1}{\sin\alpha_1\cos\alpha_1\sin^2\alpha_2} \quad \text{[from (1)]}$$

i.e., $$\frac{t_1^2}{t_2^2}=\frac{\sin\alpha_1\cos\alpha_2}{\cos\alpha_1\sin\alpha_2}.$$

Applying componendo and dividendo, we have

$$\frac{t_1^2-t_2^2}{t_1^2+t_2^2}=\frac{\sin\alpha_1\cos\alpha_2-\cos\alpha_1\sin\alpha_2}{\sin\alpha_1\cos\alpha_2+\cos\alpha_1\sin\alpha_2}=\frac{\sin(\alpha_1-\alpha_2)}{\sin(\alpha_1+\alpha_2)}.$$

Ex. 7. *If R be the range of a projectile on a horizontal plane and h its maximum height for a given angle of projection, show that the maximum horizontal range with the same velocity of projection is $2h+(R^2/8h)$.*

Sol. Let u be the velocity of projection and α be the angle of of projection. Then

$$h=\frac{u^2\sin^2\alpha}{2g} \text{ and } R=\frac{2u^2\sin\alpha\cos\alpha}{g}.$$

We have $$2h+\frac{R^2}{8h}=\frac{2u^2\sin^2\alpha}{2g}+\frac{4u^4\sin^2\alpha\cos^2\alpha}{g^2}\cdot\frac{1}{8}\cdot\frac{2g}{u^2\sin^2\alpha}$$

$$=\frac{u^2\sin^2\alpha}{g}+\frac{u^2\cos^2\alpha}{g}=\frac{u^2}{g}(\sin^2\alpha+\cos^2\alpha)=\frac{u^2}{g}$$

= the maximum horizontal range for the velocity of projection u.

Ex. 8. *If R be the horizontal range and h the greatest height of a projectile, prove that the initial velocity is*

$$\left[2g\left(h+\frac{R^2}{16h}\right)\right]^{1/2}.$$

Sol. Let u be the velocity of projection and α the angle of projection. Then

$$R=\frac{2u^2\sin\alpha\cos\alpha}{g} \quad \text{...(1)}$$

and $$h=\frac{u^2\sin^2\alpha}{2g} \quad \text{...(2)}$$

To obtain the required values of u we have to eliminate α between (1) and (2). Squaring both sides of (1), we get

$$R^2=\frac{4u^4\sin^2\alpha\cos^2\alpha}{g^2}=\frac{4u^4\sin^2\alpha\,(1-\sin^2\alpha)}{g^2}.$$

Substituting for $\sin^2\alpha$ from (2), we have

$$R^2=\frac{4u^4}{g^2}\cdot\frac{2gh}{u^2}\left(1-\frac{2gh}{u^2}\right)=\frac{8u^2h}{g}\left(1-\frac{2gh}{u^2}\right)=\frac{8u^2h}{g}-16h^2.$$

$$\therefore (8u^2h)/g=16h^2+R^2$$

or $$u^2=\frac{g}{8h}(16h^2+R^2)=\frac{g\cdot 16h}{8h}\left(h+\frac{R^2}{16h}\right)=2g\left(h+\frac{R^2}{16h}\right).$$

$$\therefore u=\left[2g\left(h+\frac{R^2}{16h}\right)\right]^{1/2}.$$

Ex. 9. *A number of particles start simultaneously from the same point in all directions in a vertical plane, with the same speed u. Show that after time t, they will all lie on a circle of radius ut. Show also that the centre of the circle descends with acceleration g.*

Sol. The common velocity of projection for all the particles is given to be u. If α be the angle of projection for a particle which after time t has co-ordinates (x, y), then

$$x=(u\cos\alpha).t \text{ and } y=(u\sin\alpha).t-\tfrac{1}{2}gt^2.$$

Eliminating α, all the particles at time t lie on the curve

$$x^2+(y+\tfrac{1}{2}gt^2)^2=u^2t^2, \quad \text{...(1)}$$

which is a circle of radius ut.

The co-ordinates of the centre of the circle (1) are $(0, -\frac{1}{2}gt^2)$.

If (X, Y) be the centre of the circle (1), we have

$$X=0,\ Y=-\tfrac{1}{2}gt^2. \quad \text{...(2)}$$

To find the acceleration of the centre, we differentiate the equations (2) with respect to t. Thus, we have

$$dX/dt=0 \text{ and } d^2X/dt^2=0.$$

Also $dY/dt=-\frac{1}{2}g.2t=-gt$ and $d^2Y/dt^2=-g$.

The x-component of the acceleration of the centre of the circle is 0 and the y-component is $-g$. So the centre of the circle descends with acceleration g.

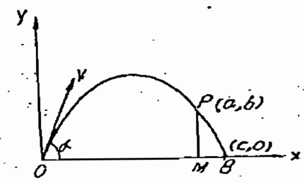
Ex. 10. *A particle just clears a wall of height b at a distance a and strikes the ground at a distance c from the point of projection. Prove that the angle of projection is*

$$\tan^{-1}\left\{\frac{bc}{a(c-a)}\right\}.$$

and the velocity of projection V is given by

$$\frac{2V^2}{g}=\frac{a^2(c-a)^2+b^2c^2}{ab(c-a)}.$$

Sol. Let the particle be projected from O with a velocity V at an angle α to the horizontal. Take the horizontal and vertical lines OX and OY in the plane of projection as the co-ordinate axes.



The equation of the trajectory is

$$y=x\tan\alpha-\tfrac{1}{2}g\frac{x^2}{V^2\cos^2\alpha} \quad \text{...(1)}$$

The particle just clears the wall PM of height b at a distance a from O and strikes the ground at the point B at a distance c from O. Thus both the points (a, b) and $(c, 0)$ lie on the curve (1).

Therefore $$b=a\tan\alpha-\tfrac{1}{2}g\frac{a^2}{V^2\cos^2\alpha} \quad \text{...(2)}$$

and $$0=c\tan\alpha-\tfrac{1}{2}g\frac{c^2}{V^2\cos^2\alpha} \quad \text{...(3)}$$

To eliminate V^2, we multiply (2) by c^2 and (3) by a^2 and subtract. Thus we get

$$bc^2=ac^2\tan\alpha-a^2c\tan\alpha \quad \text{or} \quad bc^2=ac\tan\alpha\,(c-a).$$

$$\tan\alpha=\frac{bc}{a(c-a)} \quad \text{...(4)}$$

Now from (3), $$\frac{2V^2}{g}=\frac{c\sec^2\alpha}{\tan\alpha}=\frac{c(1+\tan^2\alpha)}{\tan\alpha}.$$

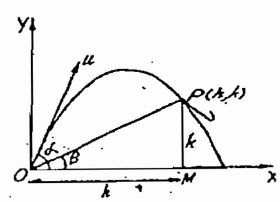
Substituting the value of $\tan\alpha$ from (4), we have

$$\frac{2V^2}{g}=\frac{c\,[1+\{b^2c^2/a^2(c-a)^2\}]}{bc/\{a(c-a)\}}=\frac{a^2(c-a)^2+b^2c^2}{ab(c-a)}.$$

Ex. 11. *A particle is projected from O at an elevation α and after t seconds it appears to have an elevation β as seen from the point of projection. Prove that the initial velocity was*

$$\frac{gt\cos\beta}{2\sin(\alpha-\beta)}.$$

Sol. Let u be the velocity of projection at O. Take the horizontal and vertical lines OX and OY in the plane of projection as the co-ordinate axes. Let $P(h, k)$ be the position of the particle after time t. Considering the motion from O to P in the horizontal and vertical directions, we have



$$h=(u\cos\alpha).t$$

and $$k=(u\sin\alpha).t-\tfrac{1}{2}gt^2.$$

Now according to the question, $\angle POX=\beta$.

$$\therefore \tan\beta=\frac{k}{h} \quad \text{or} \quad \frac{\sin\beta}{\cos\beta}=\frac{(u\sin\alpha)t-\frac{1}{2}gt^2}{(u\cos\alpha).t}$$

or $$\frac{\sin\beta}{\cos\beta}=\frac{u\sin\alpha-\frac{1}{2}gt}{u\cos\alpha} \quad [\because t\neq 0]$$

or $$u\cos\alpha\sin\beta=u\sin\alpha\cos\beta-\tfrac{1}{2}gt\cos\beta$$

or $$u(\sin\alpha\cos\beta-\cos\alpha\sin\beta)=\tfrac{1}{2}gt\cos\beta$$

or $$u\sin(\alpha-\beta)=\tfrac{1}{2}gt\cos\beta$$

or $$u=\frac{gt\cos\beta}{2\sin(\alpha-\beta)}.$$

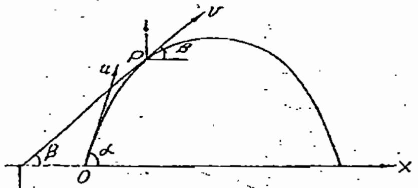
Ex. 12. *A particle is projected under gravity with a velocity u in a direction making an angle α with the horizontal. Show that the amount of deviation D in the direction of motion of the particle is given by*

$$\tan D=\frac{gt\cos\alpha}{u-gt\sin\alpha}.$$

Sol. Let O be the point of projection, u the velocity of projection and α the angle of projection. Let P be the position of the particle at any time t. Suppose v is the velocity of the particle at P. Let β be the inclination to the horizontal of the direction of the velocity at P.

Since the horizontal component of the velocity remains constant throughout the motion, therefore

$$v\cos\beta=u\cos\alpha. \quad \text{...(1)}$$



Considering the vertical motion from O to P, we have

$$v \sin \beta = u \sin \alpha - gt. \quad \ldots(2)$$

Dividing (2) by (1), we get

$$\tan \beta = \frac{u \sin \alpha - gt}{u \cos \alpha} = \tan \alpha - \frac{gt}{u \cos \alpha} \quad \ldots(3)$$

i.e., $$\tan \alpha - \tan \beta = \frac{gt}{u \cos \alpha}. \quad \ldots(4)$$

Now the deviation D = the angle between the tangents at the points O and $P = \alpha - \beta$.

We have $\tan D = \tan(\alpha - \beta) = \dfrac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$

$$= \frac{gt/(u \cos \alpha)}{1 + \tan \alpha \left(\tan \alpha - \dfrac{gt}{u \cos \alpha}\right)} \quad \text{[from (3) and (4)]}$$

$$= \frac{gt/(u \cos \alpha)}{1 + \tan^2 \alpha - \dfrac{gt \sin \alpha}{u \cos^2 \alpha}} = \frac{gt/(u \cos \alpha)}{\sec^2 \alpha - \dfrac{gt \sec^2 \alpha \sin \alpha}{u}}$$

$$= \frac{gt/(u \cos \alpha)}{\dfrac{\sec^2 \alpha}{u}(u - gt \sin \alpha)} = \frac{gt \cos \alpha}{u - gt \sin \alpha}.$$

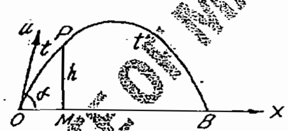
Ex. 13. *At any instant a projectile is moving with a velocity u in a direction making an angle α to the horizontal. After an interval of time t, the direction of its path makes an angle β with the horizontal. Prove that*

$$u \cos \alpha = \frac{gt}{\tan \alpha - \tan \beta}$$

Sol. Proceed as in Ex. 12. Here the velocity of the projectile at the point O is u and its direction makes an angle α with the horizontal. After an interval of time t the projectile is at the point P.

Ex. 14. *If t be the time in which a particle reaches a point P in its path and t' the time from P till it reaches the horizontal plane through the point of projection, show that the height of P above the horizontal plane is $\frac{1}{2}gtt'$.*

Sol. Let O be the point of projection, u the velocity of projection and α the angle of projection. Let OB be the horizontal range. Let P be a point on the path such that the time from O to P is t and the time from P to B is t'. Obviously $t + t'$ is the time of flight.



$$\therefore \quad t + t' = \frac{2u \sin \alpha}{g} \quad \ldots(1)$$

Let h be the height of P above the horizontal plane through O. Considering the vertical motion from O to P and using the formula $s = ut + \frac{1}{2}ft^2$, we get

$$h = (u \sin \alpha)\, t - \tfrac{1}{2}gt^2$$

$$= \tfrac{1}{2}g(t + t')\, t - \tfrac{1}{2}gt^2 \quad \text{[substituting for } u \sin \alpha \text{ from (1)]}$$

$$= \tfrac{1}{2}gt^2 + \tfrac{1}{2}gtt' - \tfrac{1}{2}gt^2 = \tfrac{1}{2}gtt'.$$

Ex. 15. *Two particles are projected from the same point in the same vertical plane with equal velocities. If t_1 and t_2 be the times taken to reach the common point of their paths and T_1 and T_2 the times for their highest points, then prove that $(t_1T_1 + t_2T_2)$ is independent of the directions of projection.*

Sol. Let O be the common point of projection and u the common velocity of projection. Take the horizontal and vertical lines OX and OY in the plane of projection as the co-ordinate axes. Let $P(h, k)$ be the other common point of the two paths.

Let α_1, α_2 be the directions of projection of the two particles. Let T_1, T_2 be the respective times to reach their greatest heights and t_1, t_2 be the respective times to reach the common point P.

We have $$T_1 = \frac{u \sin \alpha_1}{g} \text{ and } T_2 = \frac{u \sin \alpha_2}{g} \quad \ldots(1)$$

Considering the horizontal motion of the two particles from O to P, we have

$$h = (u \cos \alpha_1)\, t_1 = (u \cos \alpha_2)\, t_2.$$

$$\therefore \quad t_1 = \frac{h}{u \cos \alpha_1} \text{ and } t_2 = \frac{h}{u \cos \alpha_2} \quad \ldots(2)$$

From (1) and (2), we have

$$t_1T_1 + t_2T_2 = \frac{h}{u \cos \alpha_1} \cdot \frac{u \sin \alpha_1}{g} + \frac{h}{u \cos \alpha_2} \cdot \frac{u \sin \alpha_2}{g}$$

$$= \frac{h}{g}(\tan \alpha_1 + \tan \alpha_2). \quad \ldots(3)$$

Since the point (h, k) lies on both the trajectories, therefore

$$k = h \tan \alpha_1 - \frac{1}{2}\frac{gh^2}{u^2 \cos^2 \alpha_1} \quad \ldots(4)$$

and $$k = h \tan \alpha_2 - \frac{1}{2}\frac{gh^2}{u^2 \cos^2 \alpha_2}. \quad \ldots(5)$$

Subtracting (5) from (4), we get

$$h(\tan \alpha_1 - \tan \alpha_2) - \frac{1}{2}\frac{gh^2}{u^2}(\sec^2 \alpha_1 - \sec^2 \alpha_2) = 0$$

or $(\tan \alpha_1 - \tan \alpha_2) - \frac{1}{2}g\frac{h}{u^2}(\tan^2 \alpha_1 - \tan^2 \alpha_2) = 0$ $[\because h \neq 0]$

or $(\tan \alpha_1 - \tan \alpha_2)\,[1 - \frac{1}{2}g(h/u^2)(\tan \alpha_1 + \tan \alpha_2)] = 0$

or $1 - \frac{1}{2}g(h/u^2)(\tan \alpha_1 + \tan \alpha_2) = 0$ $[\because \tan \alpha_1 \neq \tan \alpha_2]$

or $(\tan \alpha_1 + \tan \alpha_2) = 2u^2/gh$.

Substituting this value of $(\tan \alpha_1 + \tan \alpha_2)$ in (3), we have

$t_1T_1 + t_2T_2 = \dfrac{h}{g} \cdot \dfrac{2u^2}{gh} = \dfrac{2u^2}{g^2}$, which is independent of α_1 and α_2.

Ex. 16. *Obtain the equation of the path of a projectile and show that it may be written in the form*

$$\frac{yR}{xR - x^2} = \tan \alpha,$$

where R is the horizontal range and α the angle of projection.

Sol. For the first part refer § 2. Thus referred to the point of projection O as origin, the horizontal and vertical lines OX and OY in the plane of projection as the co-ordinate axes, the equation of the path of a projectile is

$$y = x \tan \alpha - \frac{1}{2}g\frac{x^2}{u^2 \cos^2 \alpha} \quad \ldots(1)$$

where u is the velocity of projection.

If R is the horizontal range, then

$$R = \frac{2u^2 \sin \alpha \cos \alpha}{g}. \quad \ldots(2)$$

Substituting for u^2 from (2) in (1), we have

$$y = x \tan \alpha - \frac{1}{2}g\frac{x^2}{\cos^2 \alpha} \cdot \frac{2 \sin \alpha \cos \alpha}{Rg}$$

$$= x \tan \alpha - \frac{x^2}{R}\tan \alpha = \tan \alpha \left(x - \frac{x^2}{R}\right)$$

$\therefore \quad \dfrac{y}{x - (x^2/R)} = \tan \alpha$ or $\dfrac{yR}{xR - x^2} = \tan \alpha$, is the equation of the path in the required form.

Ex. 17. *A particle is projected in a direction making an angle θ with the horizontal. If it passes through the points (x_1, y_1) and (x_2, y_2) referred to horizontal and vertical axes through the point of projection, then prove that*

$$\tan \theta = \frac{x_2^2 y_1 - x_1^2 y_2}{x_1 x_2 (x_2 - x_1)}$$

Sol. The equation of the trajectory is

$$y = x \tan \theta - \frac{1}{2}\frac{gx^2}{u^2 \cos^2 \theta}. \quad \ldots(1)$$

Since the curve (1) passes through the points (x_1, y_1) and (x_2, y_2), therefore

$$y_1 = x_1 \tan \theta - \tfrac{1}{2}g\, x_1^2/(u^2 \cos^2 \theta) \quad \ldots(2)$$

and $$y_2 = x_2 \tan \theta - \tfrac{1}{2}g\, x_2^2/(u^2 \cos^2 \theta). \quad \ldots(3)$$

Multiplying (2) by x_2^2 and (3) by x_1^2 and subtracting, we get

$$y_1x_2^2 - y_2x_1^2 = x_1x_2^2 \tan \theta - x_2x_1^2 \tan \theta = x_1x_2(x_2 - x_1)\tan \theta.$$

$$\therefore \quad \tan \theta = \frac{y_1x_2^2 - y_2x_1^2}{x_1x_2(x_2 - x_1)}.$$

Ex. 18. *A gun is firing from the sea level out to sea. It is then mounted in a battery h feet higher up and fired at the same elevation α. Show that the range is increased by*

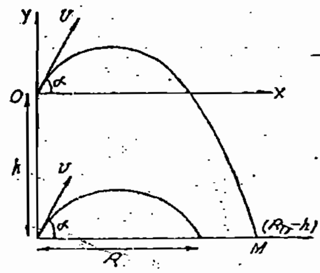
$$\frac{1}{2}\left\{\left(1 + \frac{2gh}{v^2 \sin^2 \alpha}\right)^{1/2} - 1\right\}$$

of itself, v being the velocity of projection.

Sol. Let R be the original range. Then

$$R = \frac{2v^2 \sin \alpha \cos \alpha}{g}. \quad \ldots(1)$$

Let O be a point at a height h above the water level. Let R_1 be the range on the sea when the shot is fired from O.



Referred to the horizontal and vertical lines OX and OY in the plane of projection as the co-ordinate axes, the co-ordinates of the point M where the shot strikes the water are $(R_1, -h)$.

The point $(R_1, -h)$ lies on the curve

$$y = x\tan\alpha - \frac{1}{2}\frac{gx^2}{v^2\cos^2\alpha}.$$

$$\therefore\quad -h = R_1\tan\alpha - \frac{1}{2}\frac{gR_1^2}{v^2\cos^2\alpha}$$

or $$R_1^2 - \frac{2}{g}v^2\sin\alpha\cos\alpha\, R_1 - \frac{2}{g}v^2h\cos^2\alpha = 0$$

or $$R_1^2 - RR_1 - \frac{2}{g}v^2h\cos^2\alpha = 0 \text{ or } R_1^2 - RR_1 = \frac{2}{g}v^2h\cos^2\alpha$$

or $$\left(R_1 - \tfrac{1}{2}R\right)^2 = \frac{1}{4}R^2 + \frac{2}{g}v^2h\cos^2\alpha = \frac{R^2}{4}\left[1 + \frac{1}{R^2}\cdot\frac{8}{g}v^2h\cos^2\alpha\right]$$

$$= \frac{R^2}{4}\left[1 + \frac{g^2}{4v^4\sin^2\alpha\cos^2\alpha}\cdot\frac{8}{g}v^2h\cos^2\alpha\right], \quad \text{[by (1)]}$$

$$= \frac{R^2}{4}\left[1 + \frac{2gh}{v^2\sin^2\alpha}\right].$$

$$\therefore\quad R_1 - \tfrac{1}{2}R = \tfrac{1}{2}R\left(1 + \frac{2gh}{v^2\sin^2\alpha}\right)^{1/2}$$

so that $$R_1 - R = \tfrac{1}{2}R\left(1 + \frac{2gh}{v^2\sin^2\alpha}\right)^{1/2} - \tfrac{1}{2}R$$

$$= \tfrac{1}{2}\left\{\left(1 + \frac{2gh}{v^2\sin^2\alpha}\right)^{1/2} - 1\right\}R.$$

Hence the range is increased by $\frac{1}{2}\left\{\left(1 + \frac{2gh}{v^2\sin^2\alpha}\right)^{1/2} - 1\right\}$ of its former value.

Ex. 19. *A projectile aimed at a mark which is in a horizontal plane through the point of projection, falls a metres short of it when the elevation is α and goes b metres too far when the elevation is β. Show that, if the velocity of projection be the same in all cases, the proper elevation is* $\frac{1}{2}\sin^{-1}\frac{a\sin 2\beta + b\sin 2\alpha}{a+b}$. IAS-2011

Sol. Let O be the point of projection and v the velocity of projection in all the cases. Let P be the point in the horizontal plane through O required to be hit from O. Let θ be the correct angle of projection to hit P from O. Then

OP = the range for the angle of projection $\theta = \dfrac{v^2\sin 2\theta}{g}$.

When the angle of projection is α, the particle falls at A and when the angle of projection is β, it falls at B. We have

$$OA = \frac{v^2\sin 2\alpha}{g} \text{ and } OB = \frac{v^2\sin 2\beta}{g}.$$

According to the question,

$$AP = OP - OA = a \quad\text{and}\quad PB = OB - OP = b.$$

$$\therefore\quad a = \frac{v^2\sin 2\theta}{g} - \frac{v^2\sin 2\alpha}{g} = \frac{v^2}{g}(\sin 2\theta - \sin 2\alpha), \quad \ldots(1)$$

and $$b = \frac{v^2\sin 2\beta}{g} - \frac{v^2\sin 2\theta}{g} = \frac{v^2}{g}(\sin 2\beta - \sin 2\theta). \quad \ldots(2)$$

Dividing (1) by (2), we get

$$\frac{a}{b} = \frac{\sin 2\theta - \sin 2\alpha}{\sin 2\beta - \sin 2\theta}$$

or $$a\sin 2\beta - a\sin 2\theta = b\sin 2\theta - b\sin 2\alpha$$

or $$(a+b)\sin 2\theta = a\sin 2\beta + b\sin 2\alpha$$

or $$\sin 2\theta = \frac{a\sin 2\beta + b\sin 2\alpha}{a+b}.$$

$$\therefore\quad 2\theta = \sin^{-1}\frac{a\sin 2\beta + b\sin 2\alpha}{a+b} \text{ or } \theta = \frac{1}{2}\sin^{-1}\frac{a\sin 2\beta + b\sin 2\alpha}{a+b}.$$

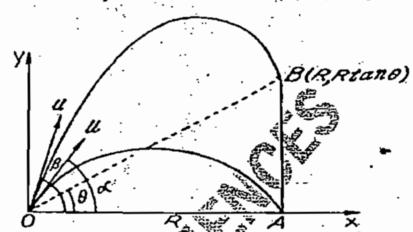
Ex. 20. *A shot fired at an elevation α is observed to strike the foot of a tower which rises above a horizontal plane through the point of projection. If θ be the angle subtended by the tower at this point, show that the elevation required to make the shot strike the top of the tower is* $\frac{1}{2}[\theta + \sin^{-1}(\sin\theta + \sin 2\alpha\cos\theta)]$.

Sol. Let AB be the tower and O the point of projection. It is given that $\angle AOB = \theta$.

Let u be the velocity of projection of the shot. When the shot is fired at an elevation α from O, it strikes the foot A of the tower AB. Let $OA = R$.

Then $$R = \frac{u^2\sin 2\alpha}{g}.$$

Referred to the horizontal and vertical lines OX and OY lying in the plane of motion as the co-ordinate axes, the co-ordinates of the top B of the tower are $(R, R\tan\theta)$.



If β be the angle of projection to hit B from O, then the point B lies on the trajectory whose equation is

$$y = x\tan\beta - \tfrac{1}{2}g\frac{x^2}{u^2\cos^2\beta}$$

$$\therefore\quad R\tan\theta = R\tan\beta - \tfrac{1}{2}g\frac{R^2}{u^2\cos^2\beta}$$

or $$\tan\theta = \tan\beta - \tfrac{1}{2}g\frac{R}{u^2\cos^2\beta} \qquad [\because\ R \neq 0]$$

Substituting the value of R from (1), we get

$$\tan\theta = \tan\beta - \tfrac{1}{2}g\frac{u^2\sin 2\alpha}{g}\cdot\frac{1}{u^2\cos^2\beta}$$

or $$\tan\theta = \tan\beta - \frac{\sin 2\alpha}{2\cos^2\beta}$$

or $$\frac{\sin\theta}{\cos\theta} = \frac{\sin\beta}{\cos\beta} - \frac{\sin 2\alpha}{2\cos^2\beta}.$$

Multiplying both sides by $2\cos^2\beta\cos\theta$, we get

$$2\cos^2\beta\sin\theta = 2\sin\beta\cos\beta\cos\theta - \cos\theta\sin 2\alpha$$

or $$(1+\cos 2\beta)\sin\theta = \sin 2\beta\cos\theta - \cos\theta\sin 2\alpha$$

or $$\sin 2\beta\cos\theta - \cos 2\beta\sin\theta = \sin\theta + \cos\theta\sin 2\alpha,$$

or $$\sin(2\beta - \theta) = \sin\theta + \cos\theta\sin 2\alpha$$

or $$2\beta - \theta = \sin^{-1}(\sin\theta + \cos\theta\sin 2\alpha)$$

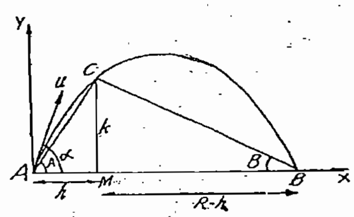
or $$2\beta = \theta + \sin^{-1}(\sin\theta + \cos\theta\sin 2\alpha)$$

or $$\beta = \tfrac{1}{2}[\theta + \sin^{-1}(\sin\theta + \sin 2\alpha\cos\theta)].$$

Ex. 21. *A particle is thrown over a triangle from one end of a horizontal base and grazing over the vertex falls on the other end of the base. If A, B be the base angles of the triangle and α the angle of projection, prove that* IFoS-2010

$$\tan\alpha = \tan A + \tan B.$$

Sol. Let A be the point of projection, u the velocity of projection and α the angle of projection.



The particle while grazing over the vertex C falls at the point B. If $AB = R$, then $R = \dfrac{2u^2\sin\alpha\cos\alpha}{g}$. ...(1)

Take the horizontal line AB as the x-axis and the vertical line AY as the y-axis. Let the co-ordinates of the vertex C be (h, k). Then the point (h, k) lies on the trajectory whose equation is

$$y = x\tan\alpha - \tfrac{1}{2}g\frac{x^2}{u^2\cos^2\alpha}$$

$$\therefore\quad k = h\tan\alpha - \tfrac{1}{2}g\frac{h^2}{u^2\cos^2\alpha} = h\tan\alpha\left[1 - \frac{gh}{2u^2\sin\alpha\cos\alpha}\right]$$

$= h\tan\alpha\left[1-\frac{h}{R}\right]$ [by (1)]

$\therefore\ \frac{k}{h}=\tan\alpha\left(\frac{R-h}{R}\right)$

or $\tan A=\tan\alpha\left(\frac{R-h}{R}\right)$. $\left[\because \text{ from } \triangle CAM,\ \tan A=\frac{k}{h}\right]$

$\therefore\ \tan\alpha=\tan A\left(\frac{R}{R-h}\right)=\tan A\left[\frac{(R-h)+h}{R-h}\right]$

$=\tan A\left[1+\frac{h}{R-h}\right]=\tan A+\tan A\,\frac{h}{R-h}$

$=\tan A+\frac{k}{h}\cdot\frac{h}{R-h}$ $\left[\because\ \tan A=\frac{k}{h}\right]$

$=(\tan A)+k/(R-h)$.

But from the $\triangle CMB$, $\tan B=k/(R-h)$.

$\therefore\ \tan\alpha=\tan A+\tan B$.

Ex. 22. *Two particles are projected simultaneously in the same vertical plane from the same point with velocities u and v at angles α and β to the horizontal. Prove that,*

(i) *The line joining them moves parallel to itself.*

(ii) *The time that elapses when their velocities are parallel, is*

$$\frac{uv\sin(\alpha-\beta)}{g\,(v\cos\beta-u\cos\alpha)}.$$

(iii) *The interval between their transits through the other common point to their paths is*

$$\frac{2uv\sin(\alpha-\beta)}{g\,(u\cos\alpha+v\cos\beta)}.$$

Sol. Take the common point of projection O as the origin and the horizontal and the vertical lines OX and OY in the plane of motion as the co-ordinate axes.

(i) Let $P\,(x_1, y_1)$ and $Q\,(x_2, y_2)$ be the respective positions of the two particles after time t. Then

$x_1=(u\cos\alpha)\,t,\ y_1=(u\sin\alpha)\,t-\frac12 gt^2$

and $x_2=(v\cos\beta)\,t,\ y_2=(v\sin\beta)\,t-\frac12 gt^2$.

The gradient (slope) of the line PQ

$$=\frac{y_2-y_1}{x_2-x_1}=\frac{(v\sin\beta-u\sin\alpha)\,t}{(v\cos\beta-u\cos\alpha)\,t}=\frac{v\sin\beta-u\sin\alpha}{v\cos\beta-u\cos\beta},$$

which is independent of the time t. Hence the line PQ moves parallel to itself.

(ii) Let θ_1 and θ_2 be the respective directions of motion of the two particles at time t. Then

$$\tan\theta_1=\frac{u\sin\alpha-gt}{u\cos\alpha}\quad\text{and}\quad\tan\theta_2=\frac{v\sin\beta-gt}{v\cos\beta}.$$

The two directions of motion will be parallel if

$\theta_1=\theta_2$ i.e., if $\frac{u\sin\alpha-gt}{u\cos\alpha}=\frac{v\sin\beta-gt}{v\cos\beta}$,

i.e., if $uv\sin\alpha\cos\beta-tgv\cos\beta=uv\cos\alpha\sin\beta-tgu\cos\alpha$

i.e., if $tg\,(v\cos\beta-u\cos\alpha)=uv\,(\sin\alpha\cos\beta-\cos\alpha\sin\beta)$

i.e., if $t=\frac{uv\sin(\alpha-\beta)}{g\,(v\cos\beta-u\cos\alpha)}$.

(iii) Let (h, k) be the co-ordinates of the other common point, say C, of their paths.

Let t_1 and t_2 be the respective times taken by the two particles to reach the common point (h, k). Considering their horizontal motion from O to C (horizontal distance for both is h), we have

$h=(u\cos\alpha)\,t_1=(v\cos\beta)\,t_2$.

$\therefore\ t_1=\frac{h}{u\cos\alpha}$ and $t_2=\frac{h}{v\cos\beta}$.

$\therefore\ t_1-t_2=h\left(\frac{1}{u\cos\alpha}-\frac{1}{v\cos\beta}\right)$. ...(1)

Since the point (h, k) lies on both the paths, therefore

$k=h\tan\alpha-\frac12\,\frac{gh^2}{u^2\cos^2\alpha}$

and $k=h\tan\beta-\frac12\,\frac{gh^2}{v^2\cos^2\beta}$.

Subtracting these two equations, we have

$$\tfrac12 gh\left(\frac{1}{u^2\cos^2\alpha}-\frac{1}{v^2\cos^2\beta}\right)=\tan\alpha-\tan\beta$$

or $\tfrac12 gh\left(\frac{1}{u\cos\alpha}-\frac{1}{v\cos\beta}\right)\left(\frac{1}{u\cos\alpha}+\frac{1}{v\cos\beta}\right)=\frac{\sin\alpha}{\cos\alpha}-\frac{\sin\beta}{\cos\beta}$

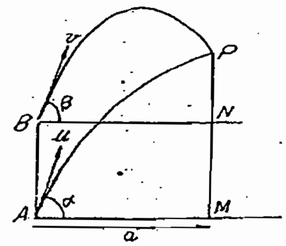
or $h\left(\frac{1}{u\cos\alpha}-\frac{1}{v\cos\beta}\right)=\frac{2\,uv\sin(\alpha-\beta)}{g\,(u\cos\alpha+v\cos\beta)}$. ...(2)

From (1) and (2), we have

$$t_1-t_2=\frac{2uv\sin(\alpha-\beta)}{g\,(u\cos\alpha+v\cos\beta)}.$$

Ex. 23. *Shots fired simultaneously from the bottom and top of a vertical cliff with elevations α and β respectively, strike an object simultaneously. Show that if a be the horizontal distance of the object from the cliff, the height of the cliff is*

$a\,(\tan\alpha-\tan\beta)$.



Sol. AB is a vertical cliff. A shot is fired from A, say with velocity u, at an elevation α. At the same time a shot is fired from B, say with velocity v, at an elevation β. The two shots strike an object P simultaneously. Let t be the time taken by each shot to reach P. The horizontal distance of P from the cliff AB is a.

Considering the horizontal motion of each shot from its point of projection upto the point P, we have

$a=(u\cos\alpha)\,t=(v\cos\beta)\,t$. ...(1)

Considering the vertical motion of each shot from its point of projection upto the point P, we have

$MP=(u\sin\alpha)\,t-\frac12 gt^2$

and $NP=(v\sin\beta)\,t-\frac12 gt^2$.

The height of the cliff. $AB=MN=MP-NP$

$=(u\sin\alpha)\,t-(v\sin\beta)\,t=(ut)\sin\alpha-(vt)\sin\beta$

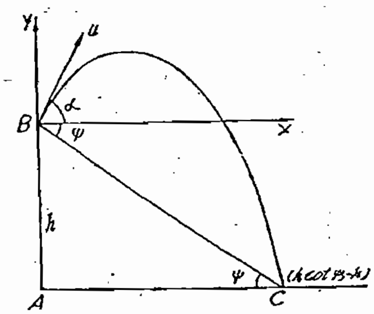
$=\frac{a}{\cos\alpha}\sin\alpha-\frac{a}{\cos\beta}\sin\beta$

$=a\,(\tan\alpha-\tan\beta)$.

Ex. 24. *From a tower an object was observed on the ground at a depression ϕ below the horizon. A gun was fired at an elevation α, but the shot missing the object struck the ground at a point whose depression was ψ. Prove that the correct elevation θ of the gun is given by*

$$\frac{\cos\theta\sin(\theta+\phi)}{\cos\alpha\sin(\alpha+\psi)}=\frac{\cos^2\phi\sin\psi}{\cos^2\psi\sin\phi}.$$

Sol. Let AB be a tower of height h. Let u be the velocity of projection of the shot. When projected at any elevation α from B, suppose the shot strikes the ground at C whose depression is ψ. Take the horizontal and vertical lines BX and BY as the co-



ordinate axes. The co-ordinates of the point C are $(h\cot\psi, -h)$. The point C lies on the trajectory whose equation is

$$y=x\tan\alpha-\frac12\,\frac{gx^2}{u^2\cos^2\alpha}.$$

$\therefore\ -h=h\cot\psi\tan\alpha-\frac12\,\frac{gh^2\cot^2\psi}{u^2\cos^2\alpha}$

or $1+\cot\psi\tan\alpha=\frac12 g\,\frac{h\cot^2\psi}{u^2\cos^2\alpha}$ $[\because\ h\neq 0]$

or $1+\frac{\cos\psi\sin\alpha}{\sin\psi\cos\alpha}=\frac12 g\,\frac{h\cos^2\psi}{u^2\sin^2\psi\cos^2\alpha}$

or $\frac{\sin(\alpha+\psi)}{\cos\alpha\sin\psi}=\frac12 g\,\frac{h\cos^2\psi}{u^2\sin^2\psi\cos^2\alpha}$

or $\cos\alpha\sin(\alpha+\psi)=\frac12 g\,\frac{h\cos^2\psi}{u^2\sin\psi}$. ...(1)

Again when θ is the angle of projection, the shot strikes the point on the ground whose depression is ϕ. Therefore replacing α be θ and ψ by ϕ, we have from (1)

$\cos\theta\sin(\theta+\phi)=\frac12 g\,\frac{h\cos^2\phi}{u^2\sin\phi}$. ...(2)

Dividing (2) by (1), we have

$$\frac{\cos\theta\sin(\theta+\phi)}{\cos\alpha\sin(\alpha+\psi)}=\frac{\cos^2\phi\sin\psi}{\cos^2\psi\sin\phi}.$$

Ex. 25. *If v_1, v_2, v_3 are the velocities at three points P, Q, R of the path of projectile where the inclinations to the horizon are α, $\alpha-\beta$, $\alpha-2\beta$ and if t_1, t_2 be the times of describing the arcs PQ, QR respectively, prove that* $v_3 t_1 = v_1 t_2$ **IAS-2010**

and $$\frac{1}{v_1}+\frac{1}{v_3}=\frac{2\cos\beta}{v_2}.$$

Sol. Since the horizontal velocity of a projectile remains constant throughout the motion, therefore

$$v_1 \cos\alpha = v_2 \cos(\alpha-\beta) = v_3 \cos(\alpha-2\beta). \quad \ldots(1)$$

Considering the vertical motion from P to Q and then from Q to R and using the formula $v=u+ft$, we get

$$v_2 \sin(\alpha-\beta) = v_1 \sin\alpha - g t_1 \quad \ldots(2)$$

and $$v_3 \sin(\alpha-2\beta) = v_2 \sin(\alpha-\beta) - g t_2. \quad \ldots(3)$$

From (2) and (3), we have

$$\frac{t_1}{t_2} = \frac{v_1 \sin\alpha - v_2 \sin(\alpha-\beta)}{v_2 \sin(\alpha-\beta) - v_3 \sin(\alpha-2\beta)}$$

$$= \frac{v_1 \sin\alpha - \dfrac{v_1\cos\alpha}{\cos(\alpha-\beta)} \sin(\alpha-\beta)}{\dfrac{v_3\cos(\alpha-2\beta)}{\cos(\alpha-\beta)} \sin(\alpha-\beta) - v_3 \sin(\alpha-2\beta)}$$

[substituting suitably for v_2 from (1)]

$$= \frac{v_1 [\sin\alpha \cos(\alpha-\beta) - \cos\alpha \sin(\alpha-\beta)]}{v_3 [\sin(\alpha-\beta)\cos(\alpha-2\beta) - \cos(\alpha-\beta)\sin(\alpha-2\beta)]}$$

$$= \frac{v_1 \sin\{\alpha-(\alpha-\beta)\}}{v_3 \sin\{(\alpha-\beta)-(\alpha-2\beta)\}} = \frac{v_1 \sin\beta}{v_3 \sin\beta} = \frac{v_1}{v_3}.$$

$\therefore$ $v_3 t_1 = v_1 t_2$. This proves the first result.

Again from (1), we have

$$\frac{1}{v_1} = \frac{1}{v_2}\frac{\cos\alpha}{\cos(\alpha-\beta)} \quad \text{and} \quad \frac{1}{v_3} = \frac{1}{v_2}\frac{\cos(\alpha-2\beta)}{\cos(\alpha-\beta)}$$

$$\therefore \quad \frac{1}{v_1}+\frac{1}{v_3} = \frac{1}{v_2}\frac{\cos\alpha + \cos(\alpha-2\beta)}{\cos(\alpha-\beta)}$$

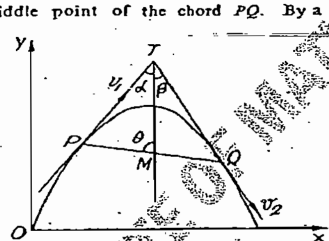
$$= \frac{1}{v_2}\frac{2\cos(\alpha-\beta)\cos\beta}{\cos(\alpha-\beta)} = \frac{2\cos\beta}{v_2}.$$

This proves the second result.

Ex. 26. *If v_1 and v_2 are the velocities at two points P and Q on a parabolic trajectory, and PT and QT the corresponding tangents, prove that*

$$\frac{v_1}{v_2} = \frac{PT}{QT}.$$

Sol. Let M be the middle point of the chord PQ. By a geometrical property of a parabola, the line TM is parallel to the axis of the parabola. But the axis of the parabola is a vertical line and so the line TM must also be vertical.



Let the tangents PT and QT make angles α and β respectively with the vertical line TM and let $\angle TMP = \theta$.

Since the horizontal velocity of projectile remains constant throughout the motion, therefore

$$v_1 \sin\alpha = v_2 \sin\beta$$

or $$\frac{v_1}{v_2} = \frac{\sin\beta}{\sin\alpha}. \quad \ldots(1)$$

In $\triangle TPM$, we have

$$\frac{PT}{\sin\theta} = \frac{PM}{\sin\alpha}$$

$\therefore$ $PT \sin\alpha = PM \sin\theta$. $\quad \ldots(2)$

Again in $\triangle TQM$, we have

$$\frac{QT}{\sin(\pi-\theta)} = \frac{QM}{\sin\beta}$$

$\therefore$ $QT \sin\beta = QM \sin\theta$. $\quad \ldots(3)$

But $PM = QM$, M being the middle point of PQ. Therefore from (2) and (3), we have

$$PT \sin\alpha = QT \sin\beta$$

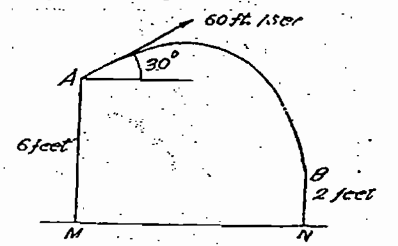
or $$\frac{\sin\beta}{\sin\alpha} = \frac{PT}{QT} \quad \ldots(4)$$

Now from (1) and (4), we have

$$\frac{v_1}{v_2} = \frac{PT}{QT}.$$

Ex. 27. *A cricket ball is thrown from a height of 6 feet at an angle of 30° to horizontal, with a speed of 60 feet/sec. It is caught by another fieldsman at a height of 2 feet from the ground. How far apart were the two men?*

Sol. Suppose a cricket ball is thrown from the point A with velocity 60 feet/sec. at an angle of 30°. The height AM of A above



the ground is 6 feet. The ball is caught at B whose height BN above the ground is 2 feet. The horizontal distance between A and B is MN.

Let T be the time taken by the ball to travel from A to B. Considering the vertical motion from A to B and using the formula $s=ut+\frac{1}{2}ft^2$, we get

$$-4 = (60 \sin 30°)\, T - \tfrac{1}{2}\times 32 \times T^2$$

or $$-4 = 30T - 16T^2 \quad \text{or} \quad 8T^2 - 15T - 2 = 0$$

or $$(8T+1)(T-2) = 0$$

$$\therefore \quad T = -\tfrac{1}{8}, 2.$$

Neglecting the negative value of T, we have $T = 2$ seconds.

Now considering the horizontal motion from A to B, we get $MN = (60 \cos 30°)\, T$ feet $= 60.(\sqrt{3}/2).2$ feet $= 60\sqrt{3}$ feet.

Hence the two fieldsmen were at a distance $60\sqrt{3}$ feet apart.

Ex. 28. *If the maximum horizontal range of a particle is R, show that the greatest height attained is $\frac{1}{4}R$.*

A boy can throw a ball 60 metres. How long is the ball in the air and what height does it attain?

Sol. If the ball is projected with velocity u at an angle α with the horizontal, then

$$\text{the horizontal range} = \frac{2u^2 \sin\alpha\cos\alpha}{g}$$

and $$\text{the greatest height} = \frac{u^2 \sin^2\alpha}{2g}.$$

The horizontal range is maximum when $\alpha = \frac{1}{4}\pi$. Thus

$$R = u^2/g.$$

If H be the greatest height attained in this case, then

$$H = \frac{u^2 \sin^2 \frac{1}{4}\pi}{2g} = \frac{u^2}{4g} = \frac{1}{4}\left(\frac{u^2}{g}\right) = \frac{1}{4}R.$$

This proves the first part of the problem. In the second part of the problem it is given that a boy can throw a ball 60 metres. So in this case $R = 60$ metres.

$\therefore$ in this case greatest height attained

$$H = \tfrac{1}{4}R = \tfrac{1}{4}\times 60 \text{ metres} = 15 \text{ metres}.$$

Hence the ball attains a height of 15 metres.

Also in the case of a projectile, the time of flight T is given by

$$T = \frac{2u \sin\alpha}{g}.$$

Since in this particular case $\alpha = \frac{1}{4}\pi$, therefore

$$T = (u/g)\sqrt{2}$$

or $$T^2 = \frac{2u^2}{g^2} = \frac{2}{g}\left(\frac{u^2}{g}\right) = \frac{2}{g}\times 60 \quad \left[\because \frac{u^2}{g} = R = 60 \text{ metres}\right]$$

$$= \frac{2\times 60}{9\cdot 8}.$$

$$\therefore \quad T = \sqrt{\left(\frac{2\times 60}{9\cdot 8}\right)} \text{ seconds} = 3\cdot 5 \text{ seconds approximately.}$$

Hence the ball remains in the air for about 3·5 seconds.

Ex. 29. *Three particles are projected from the same point in the same vertical plane with velocities u, v, w at elevations α, β, γ respectively. Prove that the foci of their paths will lie on a straight line if*

$$\frac{\sin 2(\beta-\gamma)}{u^2} + \frac{\sin 2(\gamma-\alpha)}{v^2} + \frac{\sin 2(\alpha-\beta)}{w^2} = 0.$$

Sol. Take the point of projection as the origin and the horizontal and the vertical lines in the plane of motion as the co-ordinate axes. Co-ordinates of the foci of the three trajectories are

$$\left(\frac{u^2 \sin 2\alpha}{2g}, \frac{-u^2\cos 2\alpha}{2g}\right), \left(\frac{v^2 \sin 2\beta}{2g}, \frac{-v^2\cos 2\beta}{2g}\right),$$

and $$\left(\frac{w^2 \sin 2\gamma}{2g}, \frac{-w^2\cos 2\gamma}{2g}\right).$$

These points will lie on a straight line if

$$\begin{vmatrix} \dfrac{u^2 \sin 2\alpha}{2g} & \dfrac{-u^2\cos 2\alpha}{2g} & 1 \\ \dfrac{v^2 \sin 2\beta}{2g} & \dfrac{-v^2\cos 2\beta}{2g} & 1 \\ \dfrac{w^2 \sin 2\gamma}{2g} & \dfrac{-w^2\cos 2\gamma}{2g} & 1 \end{vmatrix} = 0$$

i.e., if $\begin{vmatrix} u^2 \sin 2\alpha & u^2 \cos 2\alpha & 1 \\ v^2 \sin 2\beta & v^2 \cos 2\beta & 1 \\ w^2 \sin 2\gamma & w^2 \cos 2\gamma & 1 \end{vmatrix} = 0.$

Expanding the determinant in terms of the third column, we get $v^2w^2 \sin 2(\beta-\gamma)+w^2u^2 \sin 2(\gamma-\alpha)+u^2v^2 \sin 2(\alpha-\beta)=0.$

Dividing throughout by $u^2v^2w^2$, we get

$$\frac{\sin 2(\beta-\gamma)}{u^2}+\frac{\sin 2(\gamma-\alpha)}{v^2}+\frac{\sin 2(\alpha-\beta)}{w^2}=0,$$

as the required condition.

Ex. 30. *Particles are projected from the same point in a vertical plane with velocities which vary as*

$$1/\sqrt{(\sin \theta)},$$

θ being the angle of projection; find the locus of the vertices of the parabolas described.

Sol. Take the common point of projection O as the origin and the horizontal and vertical lines OX and OY in the plane of motion as the co-ordinate axes. Let (x_1, y_1) be the co-ordinates of the vertex of a trajectory for which the velocity of projection is u and the angle of projection is θ. Then

$$x_1=\frac{u^2 \sin \theta \cos \theta}{g} \quad \text{...(1)}$$

and $$y_1=\frac{u^2 \sin^2 \theta}{2g}. \quad \text{...(2)}$$

We are to find the locus of the point (x_1, y_1), for varying values of u and θ subject to the condition

$$u=\frac{\lambda}{\sqrt{(\sin \theta)}}, \quad \text{where } \lambda \text{ is some constant.}$$

Putting $u=\lambda/\sqrt{(\sin \theta)}$ in (1) and (2), we get

$$x_1=\frac{\lambda^2}{\sin \theta}\cdot\frac{\sin \theta \cos \theta}{g}=\frac{\lambda^2}{g}\cos \theta \quad \text{...(3)}$$

and $$y_1=\frac{\lambda^2}{\sin \theta}\cdot\frac{\sin^2 \theta}{2g}=\frac{\lambda^2}{2g}\sin \theta. \quad \text{...(4)}$$

Now we shall eliminate θ between (3) and (4). We have

$$\cos \theta=\frac{x_1}{\lambda^2/g} \quad \text{and} \quad \sin \theta=\frac{y_1}{\lambda^2/2g}.$$

Squaring and adding, we get

$$\frac{x_1^2}{\lambda^4/g^2}+\frac{y_1^2}{\lambda^4/4g^2}=\cos^2 \theta+\sin^2 \theta=1.$$

Generalising (x_1, y_1), we get the required locus as

$$\frac{x^2}{\lambda^4/g^2}+\frac{y^2}{\lambda^4/4g^2}=1, \quad \text{which is an ellipse.}$$

Ex. 31. *Particles are projected from the same point in a vertical plane with velocity $\sqrt{(2gk)}$; prove that locus of the vertices of their paths i.e. the ellipse $x^2+4y(y-k)=0$.*

Sol. Take the common point of projection O as the origin and the horizontal and the vertical lines OX and OY in the plane of projection as the co-ordinate axes. Here the velocity of projection u for each trajectory is $\sqrt{(2gk)}$. Let (x_1, y_1) be the co-ordinates of the vertex of a trajectory for which the angle of projection is α. Then $x_1=\frac{u^2 \sin \alpha \cos \alpha}{g}=\frac{2gk \sin \alpha \cos \alpha}{g}=2k \sin \alpha \cos \alpha$, ...(1)

and $$y_1=\frac{u^2 \sin^2 \alpha}{2g}=\frac{2gk \sin^2 \alpha}{2g}=k \sin^2 \alpha. \quad \text{...(2)}$$

We are to find the locus of the point (x_1, y_1) for varying values of α. For this we have to eliminate α between (1) and (2). Squaring both sides of (1), we get

$$x_1^2=4k^2 \sin^2 \alpha \cos^2 \alpha=4k^2 \sin^2 \alpha\,(1-\sin^2 \alpha). \quad \text{...(3)}$$

From (2), $\sin^2 \alpha=y_1/k$. Putting this value of $\sin^2 \alpha$ in (3), we get $x_1^2=4k^2 \cdot \frac{y_1}{k}\left(1-\frac{y_1}{k}\right)=4ky_1-4y_1^2$

or $x_1^2+4y_1^2-4ky_1=0$ or $x_1^2+4y_1(y_1-k)=0.$

$\therefore$ the locus of the point (x_1, y_1) is

$x^2+4y(y-k)=0$, which is an ellipse.

Ex. 32. *Particles are projected simultaneously in the same vertical plane from the same point. Show that the locus of the foci of all the trajectories is a parabola when for each trajectory there is the same*

(i) horizontal velocity (ii) initial vertical velocity (iii) time of flight.

Sol. Take the common point of projection O as the origin and the horizontal and the vertical lines OX and OY in the plane of projection as the co-ordinate axes.

Let (x_1, y_1) be the co-ordinates of the focus of the trajectory for which the velocity of projection is u and the angle of projection is α. Then

$$x_1=\frac{u^2 \sin 2\alpha}{2g}, \quad \text{...(1)}$$

and $$y_1=-\frac{u^2 \cos 2\alpha}{2g}. \quad \text{...(2)}$$

We are to find the locus of the point (x_1, y_1) for varying values of u and α subject to the three different given conditions.

(i) When the horizontal velocity for each trajectory is constant *i.e.*, when $u \cos \alpha=c$ (constant). ...(3)

We have to eliminate u and α between (1), (2) and (3).

From (1), $x_1=\frac{u^2 \sin \alpha \cos \alpha}{g}$.

Putting $u \cos \alpha=c$ in this relation, we get

$$x_1=\frac{cu \sin \alpha}{g} \quad \text{or} \quad u \sin \alpha=\frac{x_1g}{c}.$$

Now from (2), we have

$$y_1=-\frac{u^2}{2g}(\cos^2 \alpha-\sin^2 \alpha)=-\frac{1}{2g}(u^2 \cos^2 \alpha-u^2 \sin^2 \alpha).$$

Putting $u \cos \alpha=c$ and $u \sin \alpha=x_1g/c$ in this relation, we get

$$y_1=-\frac{1}{2g}\left(c^2-\frac{x_1^2g^2}{c^2}\right)=-\frac{c^2}{2g}+\frac{x_1^2g}{2c^2}$$

or $2gc^2y_1+c^4=x_1^2g^2$.

$\therefore$ the locus of the point (x_1, y_1) is $x^2g^2=2gc^2y+c^4$

or $x^2g^2=2gc^2\left(y+\frac{c^2}{2g}\right)$ or $x^2=\frac{2c^2}{g}\left(y+\frac{c^2}{2g}\right)$,

which is obviously a parabola.

(ii) When the initial vertical velocity for each trajectory is constant *i.e.*, when $u \sin \alpha=c$ (constant). ...(4)

We have to eliminate u and α between (1), (2) and (4).

From (1), $x_1=\frac{u^2 \sin \alpha \cos \alpha}{g}$. Putting $u \sin \alpha=c$ in this relation, we get

$$x_1=\frac{cu \cos \alpha}{g} \quad \text{or} \quad u \cos \alpha=\frac{x_1g}{c}.$$

From (2), we have $y_1=-\frac{1}{2g}(u^2 \cos^2 \alpha-u^2 \sin^2 \alpha)$.

Putting $u \sin \alpha=c$ and $u \cos \alpha=x_1g/c$ in this relation, we get

$$y_1=-\frac{1}{2g}\left(\frac{x_1^2g^2}{c^2}-c^2\right)=-\frac{x_1^2g}{2c^2}+\frac{c^2}{2g}.$$

$\therefore$ the locus of the point (x_1, y_1) is

$$y=-\frac{x^2g}{2c^2}+\frac{c^2}{2g} \quad \text{or} \quad \frac{x^2g}{2c^2}=-y+\frac{c^2}{2g}$$

or $x^2=-\frac{2c^2}{g}\left(y-\frac{c^2}{2g}\right)$, which is obviously a parabola.

(iii) When the time of flight T for each trajectory is constant *i.e.*, when

$$\frac{2u \sin \alpha}{g}=\text{constant} \quad \left[\because T=\frac{2u \sin \alpha}{g}\right]$$

or $u \sin \alpha=\text{constant}=c$, say. ...(5)

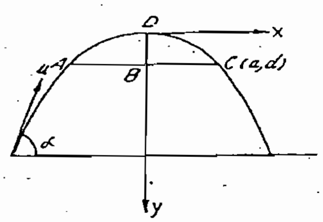
Now this part becomes exactly the same as part (ii).

Ex. 33. *A particle is to be projected so as just to pass through three equal rings, of diameter d, placed in parallel vertical planes at distances a apart, with their highest points in a horizontal straight line at a height h above the point of projection. Projection that the elevation must be $\tan^{-1}\{2\sqrt{(hd)}/a\}$.*

Sol. Let O be the point of projection, u the velocity of projection and α the angle of projection. Let A, B, C be the lowest points of the three rings and D the highest point of the middle ring.

According to the question the height of D above the point of projection O is h. Also $DB=d$ and $AB=BC=a$.

Now the particle just passes through the three rings. From the location of the rings it is obvious that the particle grazes over the lowest points A and C of the two side rings and just passes under the highest point D of the middle ring. Thus the particle is moving horizontally at D and the point D is the vertex of the parabolic path of the particle.



$\therefore$ $h=$ the height of the vertex D above the point of projection O

$=$ the greatest height attained by the particle

$=\frac{u^2 \sin^2 \alpha}{2g}$.

$\therefore$ $u^2 \sin^2 \alpha=2gh.$ ...(1)

The latus rectum of the parabolic trajectory $=\frac{2}{g}u^2 \cos^2 \alpha$.

$\therefore$ referred to the vertex D as origin and the horizontal and vertical lines DX and DY as the co-ordinate axes, the equation of the parabolic trajectory is

$x^2 = (\text{latus rectum})\, y$

i.e., $x^2 = \left(\dfrac{2}{g} u^2 \cos^2 \alpha\right) y$. ...(2)

The point C whose co-ordinates are (a, d) lies on the curve (2).

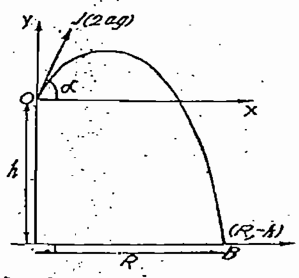
$$\therefore \quad a^2 = \left(\frac{2}{g} u^2 \cos^2 \alpha\right) d$$

or $u^2 \cos^2 \alpha = \dfrac{a^2 g}{2d}$. ...(3)

Dividing (1) by (3), we get $\tan^2 \alpha = 2gh \cdot \dfrac{2d}{a^2 g} = \dfrac{4hd}{a^2}$.

$\therefore$ $\tan \alpha = \dfrac{2\sqrt{(hd)}}{a}$ or $\alpha = \tan^{-1} \left\{\dfrac{2\sqrt{(hd)}}{a}\right\}$.

Ex. 34. *A particle is projected under gravity with velocity $\sqrt{(2ag)}$ from a point at a height h above a level plane. Show that the angle of projection α for the maximum range on the plane is given by $\tan^2 \alpha = a/(a+h)$, and that the maximum range is* $2\sqrt{\{a(a+h)\}}$.



Sol. Take the point of projection O as the origin and the horizontal and the vertical lines OX and OY as the co-ordinate axes. The velocity of projection u is given to be $\sqrt{(2ag)}$.

When the particle is projected at an angle α suppose it hits the ground at the point B whose horizontal distance from the point of projection O is R. Then R is the range on the horizontal plane for the angle of projection α. The point $B(R, -h)$ lies on the trajectory whose equation is

$$y = x \tan \alpha - \frac{1}{2} \frac{gx^2}{2ag \cos^2 \alpha}. \qquad [\because u^2 = 2ag]$$

$$\therefore \quad -h = R \tan \alpha - \frac{1}{4a} R^2 \sec^2 \alpha. \qquad ...(1)$$

Now R is a function of α given by the equation (1). For R to be maximum we must have $dR/d\alpha = 0$.

Differentiating both sides of (1) w.r.t. 'α', we get

$$0 = \frac{dR}{d\alpha} \tan \alpha + R \sec^2 \alpha - \frac{1}{4a} 2R \frac{dR}{d\alpha} \sec^2 \alpha - \frac{R^2}{4a} \cdot 2 \sec^2 \alpha \tan \alpha. \quad ...(2)$$

Putting $dR/d\alpha = 0$ in (2), we see that for a maximum value of R we must have

$$R \sec^2 \alpha - \frac{R^2}{2a} \sec^2 \alpha \tan \alpha = 0$$

or $\tan \alpha = \dfrac{2a}{R}$ as $\sec \alpha \neq 0$.

Putting this value of $\tan \alpha$ in (1), we have

$$-h = R \frac{2a}{R} - \frac{R^2}{4a}\left(1 + \frac{4a^2}{R^2}\right)$$

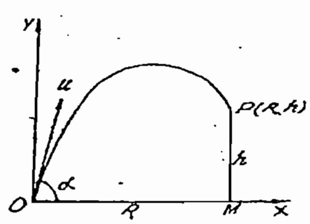
or $-h = 2a - \dfrac{R^2}{4a} - a$ or $\dfrac{R^2}{4a} = a + h$ or $R^2 = 4a(a+h)$

or $R = 2\sqrt{\{a(a+h)\}}$, which gives the maximum value of the range R.

Also for this value of R, $\tan^2 \alpha = \dfrac{4a^2}{R^2} = \dfrac{4a^2}{4a(a+h)} = \dfrac{a}{a+h}$.

Ex. 35. *A gun fires a shell with muzzle velocity u. Show that the farthest horizontal distance at which an aeroplane at a height h can be hit is $(u/g)\sqrt{(u^2-2gh)}$, and the gun's elevation then is*

$$\tan^{-1} \frac{u}{\sqrt{(u^2-2gh)}}.$$



Sol. Take the point of projection O as the origin and the horizontal and the vertical lines OX and OY as the co-ordinate axes. When the shell is projected at an angle α suppose it hits an aeroplane, which is at a height h above O, at the point P whose horizontal distance from O is R. The point $P(R, h)$ lies on the trajectory whose equation is

$$y = x \tan \alpha - \frac{1}{2} g \frac{x^2}{u^2 \cos^2 \alpha}.$$

$$\therefore \quad h = R \tan \alpha - \frac{1}{2} g \frac{R^2}{u^2} \sec^2 \alpha$$

or $2u^2 h = 2u^2 R \tan \alpha - gR^2 (1 + \tan^2 \alpha)$

or $gR^2 \tan^2 \alpha - 2u^2 R \tan \alpha + gR^2 + 2u^2 h = 0$. ...(1)

The equation (1) is a quadratic in $\tan \alpha$. Its roots will be real if $4u^4 R^2 - 4gR^2 (gR^2 + 2u^2 h) \geqslant 0$

i.e., if $u^4 - g(gR^2 + 2u^2 h) \geqslant 0$ *i.e.,* if $g^2 R^2 \leqslant u^4 - 2u^2 gh$.

i.e., if $R^2 \leqslant \dfrac{u^2}{g^2}(u^2 - 2gh)$ *i.e.,* if $R \leqslant \dfrac{u}{g}\sqrt{(u^2-2gh)}$.

Hence the maximum value of R is $(u/g)\sqrt{(u^2-2gh)}$. For this value of R the equation (1) gives

$$g \cdot \frac{u^2}{g^2}(u^2-2gh) \tan^2 \alpha - 2u^2 \cdot \frac{u}{g}\sqrt{(u^2-2gh)} \tan \alpha + g \cdot \frac{u^2}{g^2}(u^2-2gh) + 2u^2 h = 0$$

or $\dfrac{u^2}{g}(u^2-2gh) \tan^2 \alpha - 2u^2 \cdot \dfrac{u}{g} \sqrt{(u^2-2gh)} \tan \alpha + \dfrac{u^4}{g} = 0$

or $\left[\dfrac{u\sqrt{(u^2-2gh)}}{\sqrt{g}} \tan \alpha - \dfrac{u^2}{\sqrt{g}}\right]^2 = 0$.

$\therefore$ $\tan \alpha = \dfrac{u^2}{\sqrt{g}} \cdot \dfrac{\sqrt{g}}{u\sqrt{(u^2-2gh)}} = \dfrac{u}{\sqrt{(u^2-2gh)}}$

or $\alpha = \tan^{-1} \dfrac{u}{\sqrt{(u^2-2gh)}}$.

Note. The above question can also be solved by the method given in Ex. 34.

Ex. 36. *A shell is fired vertically upwards. It bursts at a height a above the point of projection. Show that the fragments on reaching the ground, lie within a circle of radius $(v/g)\sqrt{(v^2+2ag)}$, assuming that the fragments start with the same velocity v.*

Sol. Suppose the shell bursts at the point O whose height above the ground is a. All the fragments start from O with velocity v but at different elevations. We have to find the maximum range of a fragment on the ground.

Now proceed as in Ex. 34.

Ex. 37. *A gun is fired from a moving platform and the ranges of the shot are observed to be R and S when the platform is moving forward and backward respectively with velocity v. Prove that the elevation of the gun is*

$$\tan^{-1}\left\{\frac{g}{4v^2} \frac{(R-S)^2}{R+S}\right\}.$$

Sol. Let α be the elevation of the gun. Then the angle of projection of the shot relative to the gun is also α. Let u be the velocity of projection of the shot relative to the gun.

Since at time of projection of the ~~shot the gun~~ moves horizontally, therefore the initial actual horizontal velocity of the shot is affected by the motion of the gun while the initial actual vertical velocity of the shot remains unaffected. Thus the initial actual vertical velocity of the shot is $u \sin \alpha$.

Now first consider the case when the gun moves forward. In this case the actual horizontal velocity of the shot becomes $u \cos \alpha + v$.

$$\therefore \quad \text{the range } R = \frac{2}{g}(u \cos \alpha + v)\, u \sin \alpha.$$

Next consider the case when the gun moves backward. In this case the actual horizontal velocity of the shot becomes $u \cos \alpha - v$.

$$\therefore \quad \text{the range } S = \frac{2}{g}(u \cos \alpha - v)\, u \sin \alpha. \qquad ...(2)$$

From (1) and (2), we have

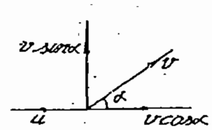
$R + S = (4/g)\, u^2 \cos \alpha \sin \alpha$ and $R - S = (4/g)\, uv \sin \alpha$.

$$\therefore \quad \frac{(R-S)^2}{R+S} = \frac{4v^2}{g} \tan \alpha$$

$$\tan \alpha = \frac{g}{4v^2} \frac{(R-S)^2}{(R+S)} \quad \text{or} \quad \alpha = \tan^{-1}\left\{\frac{g(R-S)^2}{4v^2(R+S)}\right\}.$$

Ex. 38. *A battleship is steaming ahead with velocity u. A gun is mounted on the ship so as to point straight backwards and is set at an angle of elevation α. If v be the velocity of projection relative to the gun, show that the range is $(2v/g) \sin \alpha\, (v \cos \alpha - u)$, and the angle of elevation for maximum range is*

$$\cos^{-1}\left[\frac{u + \sqrt{(u^2+8v^2)}}{4v}\right].$$



Sol. Since the ship is moving horizontally with a velocity u in a direction opposite that of the projection, therefore the initial actual horizontal velocity of the shot $= v \cos \alpha - u$.

Also the initial actual vertical velocity of the shot $= v \sin \alpha$.

$\therefore$ the range $R = \dfrac{2}{g}$ (horizontal vel.) (initial vertical vel.)

$$= \frac{2}{g}(v \cos \alpha - u)\, v \sin \alpha = \frac{2v}{g} \sin \alpha\, (v \cos \alpha - u).$$

Now R is a function of α. So R will be maximum when $dR/d\alpha=0$

i.e., when $\frac{2v}{g}(v\cos^2\alpha-v\sin^2\alpha-u\cos\alpha)=0$

i.e., when $2v\cos^2\alpha-u\cos\alpha-v=0$

i.e., when $\cos\alpha=\frac{u\pm\sqrt{(u^2+8v^2)}}{4v}$.

The negative sign before the radical is not admissible because it makes the value of $\cos\alpha$ negative or α obtuse.

$\therefore$ the angle of elevation for maximum range is

$$\cos^{-1}\left[\frac{u+\sqrt{(u^2+8v^2)}}{4v}\right].$$

Ex. 39. *A shot fired with velocity V at an elevation θ strikes point P on the horizontal plane through the point of projection. If the point P is receding from the gun with velocity v, show that the elevation must be changed to ϕ, where* IAS 2009

$$\sin 2\phi=\sin 2\theta+\frac{2v}{V}\sin\phi.$$

Sol. Let O be the point of projection. When the point P is stationary, then the original range $OP=\frac{V^2\sin 2\theta}{g}$.

When the point P recedes from O *i.e.*, moves away from O in the direction of motion of the shot, then to hit at P the angle of projection is changed to ϕ.

$\therefore$ the new range$=\frac{V^2\sin 2\phi}{g}$.

Also in this case the time of flight $T=\frac{2V\sin\phi}{g}$.

During this time P moves away from its original position a distance$=v.\frac{2V\sin\phi}{g}$.

In order to hit P, we should have

the new range=the original range+the distance moved by P in time T

i.e., $\frac{V^2\sin 2\phi}{g}=\frac{V^2\sin 2\theta}{g}+v\frac{2V\sin\phi}{g}$

i.e., $\sin 2\phi=\sin 2\theta+(2v/V)\sin\phi$.

Alternative solution. Let O be the point of projection. When the point P is stationary, then the original range $OP=\frac{V^2\sin 2\theta}{g}$.

When the point P recedes from O with velocity v, then to hit at P the angle of projection is changed to ϕ. In this case the initial horizontal velocity of the shot relative to P is $V\cos\phi-v$ and the initial vertical velocity of the shot relative to P is $V\sin\phi$. Therefore in this case the range of the shot relative to P

$=(2/g)(V\cos\phi-v)V\sin\phi$.

To hit P, we must have

$$\frac{2}{g}(V\cos\phi-v)V\sin\phi=\frac{V^2\sin 2\theta}{g}$$

i.e., $\frac{V^2\sin 2\phi}{g}-\frac{2}{g}vV\sin\phi=\frac{V^2}{g}\sin 2\theta$

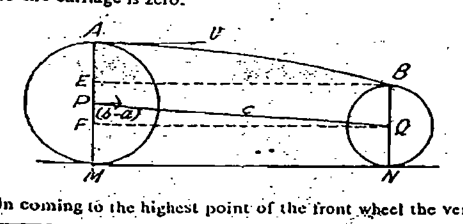
i.e., $\frac{V^2\sin 2\phi}{g}=\frac{V^2}{g}\sin 2\theta+\frac{2v}{g}.V\sin\phi$

i.e., $\sin 2\phi=\sin 2\theta+(2v/V)\sin\phi$.

Ex. 40. *The distance between the axle-trees of the front and hind wheels of a carriage of radii a and b respectively is c. A particle of mud driven from the highest point of the hind wheel alights on the highest point of the front wheel. Show that the velocity of the carriage is*

$$\left[\frac{g(c+b-a)(c+a-b)}{4(b-a)}\right]^{1/2}.$$

Sol. Let v be the velocity of the carriage. Then the velocity of the highest point A of the hind wheel is $2v$ horizontally. Therefore the actual velocity of the mud particle while driven from the highest point of the hind wheel is $2v$ horizontally. But the carriage is also moving horizontally with velocity v. Therefore the horizontal velocity of the mud particle relative to the carriage is $2v-v$ *i.e.*, v. The initial vertical velocity of the mud particle relative to the carriage is zero.



In coming to the highest point of the front wheel the vertical distance travelled by the particle$=AM-BN=2b-2a=2(b-a)$.

In the figure P and Q are the centres of the hind and front wheels and $PQ=c$ is the distance between the axle trees.

Let T be the time taken by the mud particle to travel from the highest point of the hind wheel to the highest point of the front wheel. Then considering the vertical motion of the particle, we have $2(b-a)=0.T+\frac{1}{2}gT^2$ [using the formula $s=ut+\frac{1}{2}ft^2$]

or $T=2\sqrt{\left(\frac{b-a}{g}\right)}$.

Also the horizontal distance moved by the particle relative to the carriage in time $T=EB=FQ=\sqrt{(PQ^2-PF^2)}=\sqrt{\{c^2-(b-a)^2\}}$.

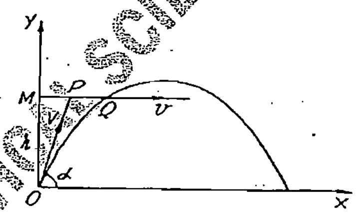
Considering the horizontal motion of the particle relative to the carriage, we have $\sqrt{\{c^2-(b-a)^2\}}=vT$.

$$\therefore\quad v=\frac{\sqrt{\{c^2-(b-a)^2\}}}{T}=\frac{\sqrt{\{c^2-(b-a)^2\}}}{2}.\sqrt{\left(\frac{g}{b-a}\right)}$$

$$=\sqrt{\left\{\frac{g(c+b-a)(c+a-b)}{4(b-a)}\right\}}.$$

Ex. 41. *An aeroplane is flying with constant velocity v at a constant height h. Show that, if a gun is fired point blank at the aeroplane after it has passed directly over the gun and when the angle of elevation as seen from the gun is α, the shell will hit the aeroplane, provided* $2(V\cos\alpha-v)v\tan^2\alpha=gh$.

Sol. Let O be the point of projection and OX and OY the horizontal and vertical lines through O in the plane of motion. Let P be the position of the aeroplane when the shot was fired from O. The gun is fired 'point blank' means that the initial velocity of the shot was along OP.



The path of the aeroplane is along the horizontal line PQ at a height h from O, and the path of the shot is the parabolic arc OQ. The point Q is common to the two paths. The shot can hit the aeroplane if both reach the point Q at the same time. Suppose this happens after a time t from the moment of projection of the shot. The distance PQ moved by the aeroplane in time t is vt.

Considering the horizontal motion of the shot from O to Q, we have $MQ=(V\cos\alpha)t$.

But $MQ=MP+PQ=h\cot\alpha+vt$.

$\therefore\quad h\cot\alpha+vt=(V\cos\alpha)t$. ...(1)

Considering the vertical motion of the shot from O to Q, we have $h=(V\sin\alpha)t-\frac{1}{2}gt^2$

or $h=t[V\sin\alpha-\frac{1}{2}gt]$. ...(2)

From (1), $t=\frac{h\cot\alpha}{V\cos\alpha-v}$.

Putting this value of t in (2), we get

$$h=\frac{h\cot\alpha}{V\cos\alpha-v}\left[V\sin\alpha-\frac{1}{2}g.\frac{h\cot\alpha}{V\cos\alpha-v}\right]$$

or $2(V\cos\alpha-v)^2=\cot\alpha[2V\sin\alpha(V\cos\alpha-v)-gh\cot\alpha]$

or $2(V\cos\alpha-v)^2=2V\cos\alpha(V\cos\alpha-v)-gh\cot^2\alpha$

or $2(V\cos\alpha-v)\{V\cos\alpha-(V\cos\alpha-v)\}=gh\cot^2\alpha$

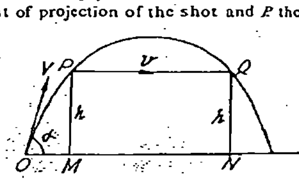
or $2(V\cos\alpha-v)v=gh\cot^2\alpha$

or $2(V\cos\alpha-v)v\tan^2\alpha=gh$.

Ex. 42. *A shot is fired with velocity V at an elevation α so as to hit a bird sitting at the top of a pole of height h. However the bird immediately starts flying horizontally away from the gun with velocity v. Show that it will not escape being hit if*

$$(2V\cos\alpha-v)(V^2\sin^2\alpha-2gh)^{1/2}=vV\sin\alpha.$$

Sol. Let O be the point of projection of the shot and P the position of the bird at the top of a pole PM of height h. As the bird could be hit if it remained sitting at the top P, therefore the trajectory of the shot passes through P.



If the bird starts flying horizontally away from P and is hit at another position Q of the trajectory, it is necessary that the bird and the shot should reach Q at the same time.

Suppose the shot is at a height h after a time t of its projection from O. Then $h=(V\sin\alpha)t-\frac{1}{2}gt^2$

or $\quad gt^2-2V\sin\alpha . t+2h=0.$

$$\therefore \quad t=\frac{2V\sin\alpha\pm\sqrt{(4V^2\sin^2\alpha-8gh)}}{2g}$$

$$=\frac{V\sin\alpha\pm\sqrt{(V^2\sin^2\alpha-2gh)}}{g}.$$

Let t_1 and t_2 be the two values of t. Then

$$t_1=\frac{V\sin\alpha-\sqrt{(V^2\sin^2\alpha-2gh)}}{g}$$

and $$t_2=\frac{V\sin\alpha+\sqrt{(V^2\sin^2\alpha-2gh)}}{g}.$$

Obviously t_1 is the time for the shot from O to P and t_2 is the time from O to Q. The horizontal distance $PQ=V\cos\alpha.(t_2-t_1)$.

Also the distance PQ is travelled by the bird in time t_2 with uniform velocity v.

$\therefore \quad PQ=vt_2.$

Hence $\quad V\cos\alpha.(t_2-t_1)=vt_2.$

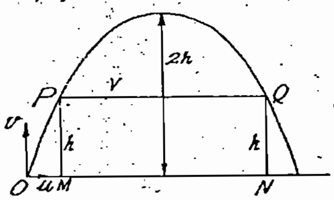
Substituting the values of t_2 and t_1 in this relation, we get

$$V\cos\alpha.\frac{2\sqrt{(V^2\sin^2\alpha-2gh)}}{g}=v\left[\frac{V\sin\alpha+\sqrt{(V^2\sin^2\alpha-2gh)}}{g}\right]$$

or $\quad (2V\cos\alpha-v)\sqrt{(V^2\sin^2\alpha-2gh)}=vV\sin\alpha.$

Ex. 43. *A stone is thrown in such a manner that it would just hit a bird at the top of a tree and afterwards reach a height double that of the tree. If at the moment of throwing the stone the bird flies away horizontally, show that, not withstanding this, the stone will hit the bird if its horizontal velocity be to that of the bird as* $(\sqrt{2}+1):2$.

Sol. Let O be the point of projection of the stone and P the top of the tree whose height above O is, say, h. Then according to the question the greatest height ever reached by the stone should be $2h$.



Let u and v be the initial horizontal and vertical components of the velocity of the stone and V the velocity of the bird which moves in the horizontal line PQ.

Since $2h$ is the greatest height of the trajectory, therefore

$$2h=v^2/2g$$

or $$v^2=4gh. \quad ...(1)$$

As the bird could be hit if it remained sitting at the top P, therefore the trajectory of the stone passes through P. If the bird starts flying horizontally away from P and is hit at another position Q of the trajectory, it is necessary that the bird and the stone should reach Q at the same time.

Suppose the stone is at a height h after a time t of its projection from O. Then

$$h=vt-\tfrac{1}{2}gt^2 \text{ or } gt^2-2vt+2h=0.$$

$$\therefore \quad t=\frac{2v\pm\sqrt{(4v^2-8gh)}}{2g}=\frac{v\pm\sqrt{(v^2-2gh)}}{g}$$

$$=\frac{2\sqrt{(gh)}\pm\sqrt{(2gh)}}{g}, \text{ substituting for } v \text{ from (1)}$$

$$=(2\pm\sqrt{2})\sqrt{(h/g)}.$$

Let t_1 and t_2 be the two values of t. Then

$$t_1=(2-\sqrt{2})\sqrt{(h/g)} \quad \text{and} \quad t_2=(2+\sqrt{2})\sqrt{(h/g)}.$$

Obviously t_1 is the time for the stone from O to P and t_2 is the time from O to Q.

The horizontal distance PQ is travelled by the stone in time t_2-t_1 with constant horizontal velocity u. Therefore $PQ=u(t_2-t_1)$.

Also the distance PQ is travelled by the bird in time t_2 with uniform velocity V.

$\therefore \quad PQ=Vt_2.$

Hence $(t_2-t_1)\,u=Vt_2$

or $$\frac{u}{V}=\frac{t_2}{t_2-t_1}=\frac{(2+\sqrt{2})\sqrt{(h/g)}}{(2\sqrt{2})\sqrt{(h/g)}}$$

$$=\frac{2+\sqrt{2}}{2\sqrt{2}}=\frac{\sqrt{2}(\sqrt{2}+1)}{2\sqrt{2}}=\frac{\sqrt{2}+1}{2}.$$

Ex. 44. *Prove that when a shot is projected from a gun at any angle of elevation the shot as seen from the point of projection will appear to descend past a vertical target with uniform velocity.*

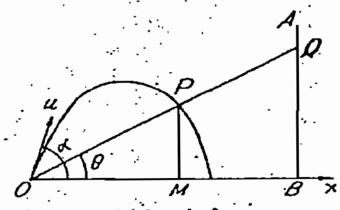
Sol. Let the shot be projected from O with velocity u at an angle α and let AB be the fixed vertical target at a given distance c from O i.e., $OB=c$.

Let P be the position of the shot at any time t. Join OP and produce it to meet the target in a point Q. From the point of projection O the corresponding point on the vertical target as seen from O in the straight line OP is Q. In the question, we have to find the vertical velocity of Q.

Let $QB=y$ and $\angle QOB=\theta$. Let M be the foot of the perpendicular from P on the horizontal line OX. Considering the horizontal and vertical motion of the shot from O to P, we have

$$OM=(u\cos\alpha)\,t,$$

and $$PM=(u\sin\alpha)\,t-\tfrac{1}{2}gt^2.$$



$$\therefore \quad \tan\theta=\frac{PM}{OM}=\frac{(u\sin\alpha)\,t-\frac{1}{2}gt^2}{(u\cos\alpha)\,t}=\tan\alpha-\frac{1}{2}\frac{gt}{u\cos\alpha}.$$

Now $y=QB=OB\tan\theta=c\tan\theta=c\left[\tan\alpha-\frac{1}{2}\frac{gt}{u\cos\alpha}\right]$.

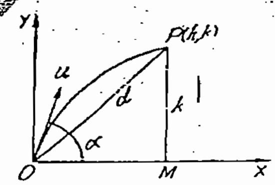
$\therefore \quad \dfrac{dy}{dt}=$ vertical velocity of $Q=-\dfrac{gc}{2u\cos\alpha}=$ a negative constant.

Since the value of dy/dt is a negative constant, therefore the shot as seen from the point of projection will appear to descend past a vertical target with uniform velocity.

§ 8. Projections to hit a given point.

(a) Two directions of projections to hit the given point.

Let a particle be projected from a given point O with a given velocity u so as to hit a given point P.



Referred to the horizontal and vertical lines OX and OY in the plane of motion as the co-ordinate axes, let the co-ordinates of the point P be (h, k). If the angle of projection is α, the equation of the trajectory is

$$y=x\tan\alpha-\tfrac{1}{2}g\frac{x^2}{u^2\cos^2\alpha}. \quad ...(1)$$

Since the point (h, k) lies on (1), therefore

$$k=h\tan\alpha-\tfrac{1}{2}g\,\frac{h^2}{u^2}\sec^2\alpha$$

or $$k=h\tan\alpha-\frac{gh^2}{2u^2}(1+\tan^2\alpha)$$

or $$\frac{2u^2}{gh^2}k=\frac{2u^2}{gh^2}.h\tan\alpha-(1+\tan^2\alpha)$$

or $$\tan^2\alpha-\frac{2u^2}{gh}\tan\alpha+\left(1+\frac{2u^2k}{gh^2}\right)=0. \quad ...(2)$$

The equation (2) is a quadratic in $\tan\alpha$. Therefore it gives in general two values of $\tan\alpha$ or two values for the angle α. *Thus there are in general two directions in which a particle may be projected from a given point O with a given velocity u so as to pass through a given point P.*

(b) Least velocity of projection to hit the given point.

In order to be able to hit the given point P from the given point O with the given velocity u, the two directions of projection given by equation (2) must be real.

The roots of the quadratic (2) in $\tan\alpha$ are real if its discriminant is $\geqslant 0$

i.e., if $\quad \dfrac{4u^4}{g^2h^2}-4\left(1+\dfrac{2u^2k}{gh^2}\right)\geqslant 0$

or $\quad u^4-g^2h^2\left(1+\dfrac{2u^2k}{gh^2}\right)\geqslant 0$

or $\quad u^4-g^2h^2-2u^2gk\geqslant 0$ or $u^4-2u^2gk\geqslant g^2h^2$

or $\quad (u^2-gk)^2\geqslant g^2h^2+g^2k^2$ or $(u^2-gk)^2\geqslant g^2(h^2+k^2)$

or $\quad u^2-gk\geqslant g\sqrt{(h^2+k^2)}$ or $u^2\geqslant gk+g\sqrt{(h^2+k^2)}$

or $\quad u^2\geqslant g\,\{k+\sqrt{(h^2+k^2)}\}$

or $\quad u\geqslant [g\,\{k+\sqrt{(h^2+k^2)}\}]^{1/2}.$

Hence the least value of $u=[g\,\{k+\sqrt{(h^2+k^2)}\}]^{1/2}$

$=\sqrt{\{g\,(k+d)\}}$, where $d=OP=\sqrt{(h^2+k^2)}$.

Thus remember that the least velocity of projection to hit P from O is $\sqrt{\{g(k+OP)\}}$, where k is the vertical height of P above O.

If the point P to be hit lies below the point of projection O, then replacing k by $-k$ in the above result, we see that the least velocity of projection to hit the point P is $\sqrt{\{g\,(OP-k)\}}$ where k is the vertical depth of P below O.

(c) Two times of flight to hit a given point.

Let a particle be projected from a given point O with a given velocity u, say at an angle α, so as to hit a given point P whose co-ordinates are (h, k). Since there can be two values of α to hit P, therefore α is a variable. If t is the time of flight from O to P, then considering the horizontal and vertical motions of the particle from O to P, we have $h=(u \cos \alpha) t$, ...(1)

and $k=(u \sin \alpha) t-\frac{1}{2}gt^2$. ...(2)

Eliminating α between (1) and (2), we have

$$h^2+(k+\tfrac{1}{2}gt^2)^2=u^2t^2$$

or $4h^2+(2k+gt^2)^2=4u^2t^2$

or $4h^2+4k^2+4gkt^2+g^2t^4=4u^2t^2$

or $g^2t^4+4(gk-u^2)t^2+4(h^2+k^2)=0$

or $$t^4+4\left(\frac{k}{g}-\frac{u^2}{g^2}\right)t^2+\frac{4}{g^2}(h^2+k^2)=0. \quad ...(3)$$

The equation (3) is a quadratic in t^2 and thus gives two values of t^2 and consequently two possible values of t to hit the given point. If corresponding to the two directions of projection to hit P the two possible times of flight are t_1 and t_2 then t_1^2 and t_2^2 are the roots of the quadratic (3) in t^2. From the theory of equations, we have

$$t_1^2+t_2^2=-4\left(\frac{k}{g}-\frac{u^2}{g^2}\right)$$

and $t_1^2t_2^2=\frac{4}{g^2}(h^2+k^2)=\frac{4}{g^2}OP^2$, so that $t_1t_2=\frac{2}{g}.OP$.

Illustrative Examples

Ex. 45. *If α, β are two possible directions to hit a given point (a, b), then show that $\tan(\alpha+\beta)=-a/b$.*

Sol. Let a particle be projected from a given point O with a given velocity u so as to hit a given point (a, b). If the angle of projection is θ, the equation of the trajectory is

$$y=x \tan \theta-\tfrac{1}{2}g\frac{x^2}{u^2 \cos^2 \theta} \quad ...(1)$$

Since the point (a, b) lies on (1), therefore

$$b=a \tan \theta-\tfrac{1}{2}g\frac{a^2}{u^2}\sec^2 \theta$$

or $b=a \tan \theta-\frac{1}{2}g\frac{a^2}{u^2}(1+\tan^2 \theta)$

or $$\tan^2 \theta-\frac{2u^2}{ga}\tan \theta+\left(1+\frac{2u^2b}{ga^2}\right)=0. \quad ...(2)$$

The equation (2) is a quadratic in $\tan \theta$ showing that there are in general two directions of projection to hit the given point (a, b). If α, β are the two possible directions of projection, then $\tan \alpha$, $\tan \beta$ are the roots of the quadratic (2) in $\tan \theta$.

$\therefore$ $\tan \alpha+\tan \beta=\frac{2u^2}{ga}$ and $\tan \alpha \tan \beta=1+\frac{2u^2b}{ga^2}$.

We have $$\tan(\alpha+\beta)=\frac{\tan \alpha+\tan \beta}{1-\tan \alpha \tan \beta}=\frac{2u^2/ga}{1-1-(2u^2b/ga^2)}$$

$$=-\frac{2u^2}{ga}\cdot\frac{ga^2}{2u^2b}=-\frac{a}{b}.$$

Ex. 46. *A particle is projected under gravity from A so as to pass through B; show that for a given velocity of projection there are two paths. Show that if B has horizontal and vertical co-ordinates x, y referred to A and the velocity of projection is $\sqrt{(2gh)}$, the angle between the two paths at B is a right angle if B lies on the ellipse $x^2+2y^2=2hy$.*

Sol. Let u be the velocity and α the angle of projection. Since the trajectory passes through the point $B(x, y)$, therefore

$$y=x \tan \alpha-\tfrac{1}{2}g\frac{x^2}{u^2 \cos^2 \alpha}$$

or $y=x \tan \alpha-\frac{1}{2}g(x^2/u^2)\sec^2 \alpha$ ...(1)

or $y=x \tan \alpha-\frac{1}{2}g(x^2/u^2)(1+\tan^2 \alpha)$. ...(2)

The equation (2) is a quadratic in $\tan \alpha$ showing that there are in general two directions of projection to hit B from A. Thus for a given velocity there are two paths from A to B.

Putting $u^2=2gh$ in (1), we have

$$y=x \tan \alpha-(x^2/4h)\sec^2 \alpha. \quad ...(3)$$

From (3), $\frac{dy}{dx}=\tan \alpha-\frac{x}{2h}\sec^2 \alpha=m$ (say). ...(4)

Then m is the gradient of the trajectory for the angle of projection α and the velocity of projection $\sqrt{(2gh)}$ at the point (x, y).

The equations (3) and (4) can be rearranged as

$x^2 \sec^2 \alpha-4hx \tan \alpha+4hy=0$, ...(5)

and $x \sec^2 \alpha-2h \tan \alpha+2mh=0$. ...(6)

Solving (5) and (6) for $\sec^2 \alpha$ and $\tan \alpha$, we have

$$\frac{\sec^2 \alpha}{-8mh^2x+8h^2y}=\frac{\tan \alpha}{4hxy-2mhx^2}=\frac{1}{-2hx^2+4hx^2}.$$

$\therefore$ $\sec^2 \alpha=\frac{8h^2(y-mx)}{2hx^2}=\frac{4h}{x^2}(y-mx)$

and $\tan \alpha=\frac{2hx(2y-mx)}{2hx^2}=\frac{2y-mx}{x}$.

Now the two paths depend upon the angle of projection α. So eliminating α from these, we get

$$\frac{4h}{x^2}(y-mx)=1+\frac{(2y-mx)^2}{x^2}$$

or $4h(y-mx)=x^2+(2y-mx)^2$

or $m^2x^2-4mx(y-h)+x^2+4y^2-4hy=0$. ...(7)

The equation (7) is a quadratic in m and so it gives us two values of m, say m_1 and m_2. Then m_1 and m_2 are the gradients of the two paths at B. Since m_1 and m_2 are the roots of the quadratic (7) in m, therefore

$$m_1m_2=\frac{x^2+4y^2-4hy}{x^2}.$$

The two paths at B are at right angles if $m_1m_2=-1$

i.e., if $\frac{x^2+4y^2-4hy}{x^2}=-1$

i.e., if $x^2+4y^2-4hy=-x^2$ i.e., if $2x^2+4y^2=4hy$

i.e., if $x^2+2y^2=2hy$.

Hence the two paths at B are at right angles if B lies on the ellipse $x^2+2y^2=2hy$.

Ex. 47. *A stone is projected with velocity u from a height h to hit a point in the level at a horizontal distance R from the point of projection. Show that the angle of projection is given by*

$$R^2 \tan^2 \alpha-\frac{2u^2}{g}R \tan \alpha+R^2-\frac{2hu^2}{g}=0.$$

Hence deduce that the maximum range on the level for this velocity is $\sqrt{\left(\frac{u^4}{g^2}+\frac{2hu^2}{g}\right)}$ *and that if R' is this maximum range and α the angle of projection to give the maximum range, then*

$\tan \alpha=u^2/gR'$ and $\tan 2\alpha=R'/h$.

Sol. Referred to the point of projection O as the origin, the equation of the trajectory for the angle of projection α is

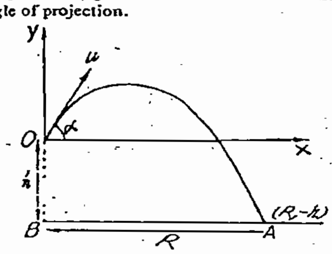
$$y=x \tan \alpha-\tfrac{1}{2}g\frac{x^2}{u^2 \cos^2 \alpha}. \quad ...(1)$$

Suppose the stone hits the ground at the point A whose co-ordinates are $(R, -h)$. Then the point $(R, -h)$ lies on the curve (1). Therefore

$$-h=R \tan \alpha-\tfrac{1}{2}g\frac{R^2}{u^2}(1+\tan^2 \alpha)$$

or $$R^2 \tan^2 \alpha-\frac{2u^2}{g}R \tan \alpha+R^2-\frac{2hu^2}{g}=0. \quad ...(2)$$

The equation (2) gives the values of $\tan \alpha$ and so the values of the angle of projection.



Now if u is given, then R is a function of α given by the equation (2). For R to be maximum we must have $dR/d\alpha=0$.

Differentiating both sides of (2) w.r.t. 'α', we get

$$2R\frac{dR}{d\alpha}\tan^2 \alpha+2R^2 \tan \alpha \sec^2 \alpha-\frac{2u^2}{g}\frac{dR}{d\alpha}\tan \alpha-\frac{2u^2}{g}R \sec^2 \alpha$$

$$+2R\frac{dR}{d\alpha}=0. \quad ...(3)$$

Putting $dR/d\alpha=0$ in (3), we have

$$2R^2 \tan \alpha \sec^2 \alpha-\frac{2u^2}{g}R \sec^2 \alpha=0$$

or $2R\left(R \tan \alpha-\frac{u^2}{g}\right)\sec^2 \alpha=0$

or $R \tan \alpha-(u^2/g)=0$ $[\because \sec \alpha \neq 0]$

or $\tan \alpha=u^2/gR$. ...(4)

The equation (4) gives the relation between the angle of projection and the maximum range. If R' is the maximum range, then replacing R by R' in (4), we have

$$\tan \alpha=u^2/gR'. \quad ...(5)$$

Putting $\tan \alpha=u^2/gR'$ and $R=R'$ in (2), the maximum range R' is given by

$$R'^2\frac{u^4}{g^2R'^2}-\frac{2u^2}{g}R'.\frac{u^2}{gR'}+R'^2-\frac{2hu^2}{g}=0$$

or $$\frac{u^4}{g^2}-\frac{2u^4}{g^2}+R'^2-\frac{2hu^2}{g}=0$$

or $$R'^2=\frac{u^4}{g^2}+\frac{2hu^2}{g} \qquad ...(6)$$

or $$R'=\sqrt{\left(\frac{u^4}{g^2}+\frac{2hu^2}{g}\right)}.$$

Now $$\tan 2\alpha=\frac{2\tan\alpha}{1-\tan^2\alpha}=\frac{2u^2/gR'}{1-u^4/g^2R'^2} \qquad \text{[from (5)]}$$

$$=\frac{2u^2gR'}{g^2R'^2-u^4}$$

$$=\frac{2u^2gR'}{g^2\left(\frac{u^4}{g^2}+\frac{2hu^2}{g}\right)-u^4} \qquad \text{[from (6)]}$$

$$=\frac{2u^2gR'}{2hgu^2}=\frac{R'}{h}.$$

Ex. 48. *Determine the least velocity with which a ball can be thrown to reach the top of a cliff 40 metres high and $40\sqrt{3}$ metres away from the point of projection.*

Sol. We know that the least velocity of projection u to hit a point P from a point O is given by

$$u=[g\{k+\sqrt{(h^2+k^2)}\}]^{1/2}, \qquad ...(1)$$

where h and k are respectively the horizontal and vertical distances of P from O. [Refer § 8, part (b), page 48]

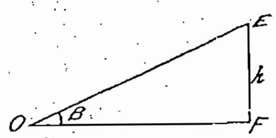
Here $h=40\sqrt{3}$ meters and $k=40$ metres. Substituting these values of h and k in (1) and putting $g=9{\cdot}8$ metres/sec², the required least velocity of projection

$=[9{\cdot}8\,\{40+\sqrt{(4800+1600)}\}]^{1/2}$ metres/sec.

$=[9{\cdot}8\times120]^{1/2}$ meters/sec. $=\sqrt{(1176)}$ metres/sec.

$=14\sqrt{6}$ metres/sec.

Ex. 49. *The angular elevation of an enemy's position on a hill h metres high is β. Show that in order to shell it, the initial velocity of the projectile must not be less than $\sqrt{[gh\,(1+\operatorname{cosec}\beta)]}$.*



Sol. In the figure FE is a hill h metres high and E is the position of the enemy. If O is the point from which the enemy's position is to be shelled, then according to the question $\angle EOF=\beta$. Let u be the least velocity of projection to hit E from O. Then

$u=\sqrt{\{g\,(OE+EF)\}}$. [Refer § 8, part (b), page 48]

$=\sqrt{\{g\,(h\operatorname{cosec}\beta+h)\}}$ [$\because$ $OE=h\operatorname{cosec}\beta$]

$=\sqrt{\{gh\,(1+\operatorname{cosec}\beta)\}}$.

Ex. 50. *A boy can throw a ball vertically upwards to a height h_1. Show that he cannot clear a wall of height h_2 distant d from him if* $2h_1<h_2+\sqrt{(h_2^2+d^2)}$.

Sol. Since the boy can throw a ball vertically upwards to a height h_1, therefore if u is the maximum velocity with which the boy can throw the ball, we have

$0=u^2-2gh_1$ [using the formula $v^2=u^2+2fs$]

or $u^2=2gh_1$ or $u=\sqrt{(2gh_1)}$.

Now the vertical height of the top of the wall from the point of projection is h_2 and its horizontal distance from the point of projection is d. To hit the top of the wall from the point of projection with velocity u, we must have

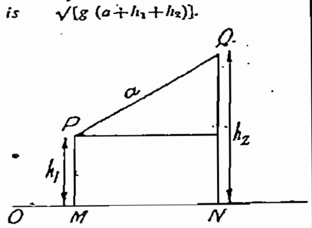
$u\geq[g\,\{h_2+\sqrt{(d^2+h_2^2)}\}]^{1/2}$ [by the formula for the least velocity of projection]

or $u^2\geq g\{h_2+\sqrt{(d^2+h_2^2)}\}$

or $2gh_1\geq g\,\{h_2+\sqrt{(d^2+h_2^2)}\}$ or $2h_1\geq h_2+\sqrt{(d^2+h_2^2)}$.

Therefore if $2h_1<h_2+\sqrt{(d^2+h_2^2)}$, the boy cannot clear the wall.

Ex. 51. *Two points P and Q are at a distance a apart, their heights above the ground being h_1 and h_2. Prove that the least velocity with which a particle can be thrown from the ground level, so as to pass through both the points, is* $\sqrt{[g\,(a+h_1+h_2)]}$.



Sol. Let O be the point of projection on the ground and u be the velocity of projection at O.

We have $PQ=a$ (given). Also the vertical height of Q above $P=h_2-h_1$.

If v be the least velocity of the projectile at P so as to hit Q, we must have

$v=[g\,\{PQ+(h_2-h_1)\}]^{1/2}=[g\,(a+h_2-h_1)]^{1/2}$

or $v^2=g\,(a+h_2-h_1)$. ...(1)

Now if a particle is projected from O with velocity u and its velocity at P is v, we have

$v^2=u^2-2gh_1$

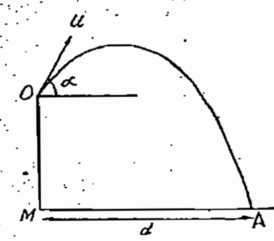
or $u^2=v^2+2gh_1$. ...(2)

From (2) it is clear that u is least when v is least. So putting for v^2 from (1) in (2), the least value of u is given by

$u^2=g\,(a+h_2-h_1)+2gh_1=g\,(a+h_1+h_2)$

or $u=\sqrt{[g\,(a+h_1+h_2)]}$.

Ex. 52. *A shot is fired with velocity u from the top of a cliff of height h and strikes the sea at a distance d from the foot of the cliff. Show that the possible times of flight are the roots of the equation* $\frac14g^2t^4-(gh+u^2)\,t^2+d^2+h^2=0$.



Sol. Let OM be a cliff of height h. A shot is fired from O with velocity u, say at an angle α. It strikes the sea at the point A whose distance from the foot of the cliff is h. Let t be the time of flight from O to A. Then considering the horizontal and vertical motions of the shot from O to A, we have

$d=(u\cos\alpha)\,t$, ...(1)

and $-h=(u\sin\alpha)\,t-\frac12gt^2$

i.e., $\frac12gt^2-h=(u\sin\alpha)\,t$. ...(2)

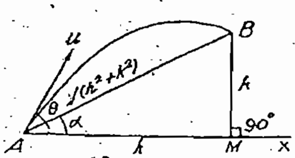
To eliminate α, squaring and adding (1) and (2), we get

$d^2+(\frac12gt^2-h)^2=u^2t^2$

or $\frac14g^2t^4-(gh+u^2)\,t^2+d^2+h^2=0$. ...(3)

Hence the possible times of flight are the roots of the equation (3).

Ex. 53. *If t_1 and t_2 be the times of flight from A to B and α the inclination of AB to the horizontal, prove that* $t_1^2+2t_1t_2\sin\alpha+t_2^2$ *is independent of α.*



Sol. Let u be the velocity at A, its direction making an angle θ with the horizontal AX. Let t be the time of flight from A to B. It is given that $\angle BAM=\alpha$. Let $AM=h$ and $BM=k$; then $\sin\alpha=k/\sqrt{(h^2+k^2)}$.

Considering the horizontal and vertical motions of the particle from A to B, we have

$h=(u\cos\theta)\,t$ ...(1)

and $k=(u\sin\theta)\,t-\frac12gt^2$

i.e., $k+\frac12gt^2=(u\sin\theta)\,t$. ...(2)

Squaring and adding (1) and (2), we get

$h^2+(k+\frac12gt^2)^2=u^2t^2$

or $h^2+k^2+\frac14g^2t^4+kgt^2-u^2t^2=0$

or $g^2t^4-4\,(u^2-kg)\,t^2+4\,(h^2+k^2)=0$. ...(3)

If t_1 and t_2 are the two possible times of flight from A to B, then t_1^2 and t_2^2 are the roots of the quadratic (3) in t^2. We have

$t_1^2+t_2^2=\frac{4\,(u^2-kg)}{g^2}$ and $t_1^2t_2^2=\frac{4\,(h^2+k^2)}{g^2}$.

Now $t_1^2+2t_1t_2\sin\alpha+t_2^2=(t_1^2+t_2^2)+2t_1t_2\sin\alpha$

$=\frac{4\,(u^2-kg)}{g^2}+2.\frac{2}{g}\sqrt{(h^2+k^2)}.\frac{k}{\sqrt{(h^2+k^2)}}$

$=\frac{4u^2}{g^2}-\frac{4k}{g}+\frac{4k}{g}=\frac{4u^2}{g^2}$,

which is independent of h, k and is therefore independent of α.

Ex. 54. *Show that the product of the two times of flight from P to Q with a given velocity of projection is $(2PQ)/g$.*

Sol. Let u be the velocity of projection at P and θ be the angle of projection. Let t be the time of flight from P to Q. Suppose h and k are respectively the horizontal and vertical distances of Q from P. Then proceeding as in Ex. 53, we have

$g^2t^4-4\,(u^2-kg)\,t^2+4\,(h^2+k^2)=0$.

If t_1 and t_2 are the two possible times of flight from P to Q, then t_1^2 and t_2^2 are the roots of the above quadratic in t^2. We have

$t_1^2t_2^2=\frac{4\,(h^2+k^2)}{g^2}=\frac{4}{g^2}\,(PQ)^2$ [$\because$ $PQ^2=h^2+k^2$]

so that $t_1t_2=(2/g)\,PQ$.

Ex. 55. *A shell bursts at a horizontal distance a from the foot of a hill of height h. Fragments of the shell fly in all directions with a velocity upto V. Find how long a man on the top of the hill will be in danger.*

Sol. Let u be the velocity and α the angle of projection for a fragment reaching the man. According to the question the greatest value of u can be V. If t be the time taken by this fragment to reach the man, then considering the horizontal and vertical motions of the fragment, we have $a=(u\cos\alpha)\,t$ ...(1)

and $\qquad h=(u \sin\alpha)\, t-\frac{1}{2}gt^2$

i.e., $\qquad h+\frac{1}{2}gt^2=(u \sin\alpha)\, t. \qquad \ldots(2)$

Squaring and adding (1) and (2), we get

$$a^2+(h+\tfrac{1}{2}gt^2)^2=u^2t^2$$

or $\qquad a^2+h^2+\frac{1}{4}g^2t^4+ght^2=u^2t^2$

or $\qquad g^2t^4-4(u^2-gh)\,t^2+4(a^2+h^2)=0. \qquad \ldots(3)$

If t_1 and t_2 are two possible times of flight of the fragment to reach the man, then t_1^2 and t_2^2 are the roots of the quadratic (3) in t^2. We have

$$t_1^2+t_2^2=\frac{4(u^2-gh)}{g^2} \quad \text{and} \quad t_1^2t_2^2=\frac{4(a^2+h^2)}{g^2}.$$

The period in which the man will be in danger on account of this fragment

$$=t_1-t_2=\sqrt{(t_1-t_2)^2}=\sqrt{\{(t_1^2+t_2^2)-2t_1t_2\}}$$

$$=\sqrt{\left\{\frac{4(u^2-gh)}{g^2}-2.\frac{2\sqrt{(a^2+h^2)}}{g}\right\}}$$

$$=\frac{2}{g}\{u^2-gh-g\sqrt{(a^2+h^2)}\}^{1/2}. \qquad \ldots(4)$$

From the result (4), we observe that the period t_1-t_2 increases as u increases. But the greatest value taken by u is V. Hence the man on the top of the hill will be in danger for a period

$$(2/g)\,\{V^2-gh-g\sqrt{(a^2+h^2)}\}^{1/2}.$$

Ex. 56. *A shell bursts on contact with the ground and pieces from it fly in all directions all with velocities upto 80 feet/sec. Show that a man 100 feet away is in danger for $\frac{5}{2}\sqrt{2}$ seconds.*

Sol. Proceed as in Ex. 55 by taking $a=100$ feet, $h=0$ and $V=80$ feet/sec. Note that here the man is on the ground and so $h=0$. The required period in which the man is in danger is

$$=\frac{2}{g}\{80^2-100g\}^{1/2} \text{ seconds}$$

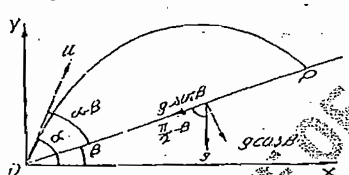
$$=\tfrac{2}{32}\{6400-100\times32\}^{1/2} \text{ seconds}=\tfrac{1}{16}\sqrt{(3200)} \text{ seconds}$$

$$=\tfrac{1}{16}\times10\times4\times\sqrt{2} \text{ seconds}=\tfrac{5}{2}\sqrt{2} \text{ seconds}.$$

9. Range and time of flight on an inclined plane.

A particle is projected with velocity u at an angle α to the horizontal from a point O on an inclined plane of inclination β to the horizontal. The particle is projected up the inclined plane to move in the vertical plane through the line of greatest slope. If the particle strikes the inclined plane, to determine the range and the time of flight.

Let O be the point of projection and u the velocity of projection making an angle α with the horizontal OX.



Suppose the particle strikes the inclined plane at P, where $OP=R$. Then R is the range up the inclined plane. Let T be the time of flight from O to P.

Initial velocity at O along the inclined plane

$$=u\cos(\alpha-\beta), \text{ up the plane}$$

and initial velocity at O perpendicular to the inclined plane

$$=u\sin(\alpha-\beta),$$

along the upward normal to the plane.

The resolved part of the acceleration g along the inclined plane $=g\sin\beta$, down the plane and the resolved part of g perpendicular to the inclined plane $=g\cos\beta$, along the downward normal to the plane.

While moving from O to P the displacement of the particle perpendicular to the inclined plane is zero. So considering the motion of the particle from O to P perpendicular to the inclined plane and using the formula $s=ut+\frac{1}{2}ft^2$, we get

$$0=u\sin(\alpha-\beta).T-\tfrac{1}{2}g\cos\beta.T^2.$$

But $T=0$ gives the time from O to O. Therefore the time of flight T from O to P is given by

$$T=\frac{2u\sin(\alpha-\beta)}{g\cos\beta}. \qquad \ldots(1)$$

Now considering the motion of the particle from O to P along the inclined plane and using the formula $s=ut+\frac{1}{2}ft^2$, we get

$$R=u\cos(\alpha-\beta).T-\tfrac{1}{2}g\sin\beta T^2$$

$$=T\,[u\cos(\alpha-\beta)-\tfrac{1}{2}g\sin\beta T]$$

$$=\frac{2u\sin(\alpha-\beta)}{g\cos\beta}\left[u\cos(\alpha-\beta)-\tfrac{1}{2}g\sin\beta\,\frac{2u\sin(\alpha-\beta)}{g\cos\beta}\right]$$

[substituting for T from (1)]

$$=\frac{2u\sin(\alpha-\beta)}{g\cos\beta}.\frac{u\{\cos(\alpha-\beta)\cos\beta-\sin(\alpha-\beta)\sin\beta\}}{\cos\beta}$$

$$=\frac{2u^2\sin(\alpha-\beta)\cos\{(\alpha-\beta)+\beta\}}{g\cos^2\beta}$$

i.e., $\qquad R=\dfrac{2u^2\sin(\alpha-\beta)\cos\alpha}{g\cos^2\beta} \qquad \ldots(2)$

This gives range up the inclined plane.

Maximum range up the inclined plane. From the formula (2), we observe that if u and β are given, then the range R depends upon the angle of projection α. We can write

$$R=\frac{u^2}{g\cos^2\beta}[\sin(\alpha-\beta+\alpha)+\sin(\alpha-\beta-\alpha)]$$

$$=\frac{u^2}{g\cos^2\beta}[\sin(2\alpha-\beta)-\sin\beta].$$

Obviously for given u and β, R is maximum when $\sin(2\alpha-\beta)$ is maximum i.e., when $\sin(2\alpha-\beta)=1$

i.e., $\quad$ when $2\alpha-\beta=\frac{1}{2}\pi, \qquad \ldots(3)$

i.e., $\quad$ when $\alpha=\frac{1}{4}\pi+\frac{1}{2}\beta.$

Also the maximum range

$$=\frac{u^2(1-\sin\beta)}{g\cos^2\beta}=\frac{u^2}{g}.\frac{1-\sin\beta}{1-\sin^2\beta}$$

$$=\frac{u^2}{g}\frac{(1-\sin\beta)}{(1+\sin\beta)(1-\sin\beta)}=\frac{u^2}{g(1+\sin\beta)}.$$

Thus the maximum range up the inclined plane

$$=\frac{u^2}{g(1+\sin\beta)}.$$

From (3) we observe that the range on the inclined plane is maximum when

$$2\alpha-\beta=\tfrac{1}{2}\pi$$

i.e., when $\qquad \alpha-\beta=\frac{1}{2}\pi-\alpha$

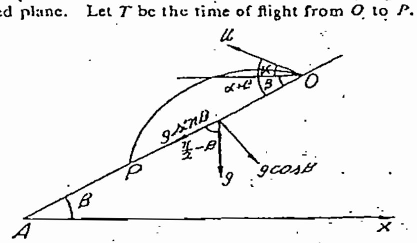
i.e., when the angle between the direction of projection and the inclined plane is the same as the angle between the direction of projection and the vertical.

Hence in the case of maximum range on the inclined plane the direction of projection bisects the angle between the vertical and the inclined plane.

Now the direction of projection at O is along the tangent to the parabolic path at O. Also the vertical through O is perpendicular to the directrix of the path. In the case of a parabola the tangent at any point bisects the angle between the focal distance of that point and the perpendicular from that point to the directrix. Therefore in the case of maximum range on the inclined plane the range OP coincides with the line joining O to the focus of the parabola. *Hence in the case of maximum range on an inclined plane the focus of the path lies in the range itself.*

10. Range and time of flight down an inclined plane.

Let O be a point on an inclined plane whose inclination to the horizontal is β. Suppose a particle is projected from O down the inclined plane. Let u be the velocity of projection making an angle α with the horizontal through O. Suppose the particle strikes the inclined plane at P, where $OP=R$. Then R is the range down the inclined plane. Let T be the time of flight from O to P.



Initial velocity at O along the inclined plane $=u\cos(\alpha+\beta)$ down the plane

and initial velocity at O perpendicular to the inclined plane

$$=u\sin(\alpha+\beta),$$

along the upward normal to the plane.

Resolved part of the acceleration g along the inclined plane is $g\sin\beta$ and perpendicular to the inclined plane is $g\cos\beta$ as shown in the figure.

While moving from O to P in time T the displacement of the particle perpendicular to the inclined plane is zero. So considering the motion of the particle from O to P perpendicular to the inclined plane and using the formula $s=ut+\frac{1}{2}ft^2$, we have

$$0=u\sin(\alpha+\beta)\,T-\tfrac{1}{2}g\cos\beta T^2$$

i.e., $\qquad T=\dfrac{2u\sin(\alpha+\beta)}{g\cos\beta} \qquad \ldots(1)$

Now considering the motion of the particle from O to P along the inclined plane and using the formula $s=ut+\frac{1}{2}ft^2$, we have

$$R=u\cos(\alpha+\beta).T+\tfrac{1}{2}g\sin\beta.T^2.$$

$= T\,[u\cos(\alpha+\beta)+\frac{1}{2}g\sin\beta\,.\,T]$

$= \frac{2u\sin(\alpha+\beta)}{g\cos\beta}\left[u\cos(\alpha+\beta)+\frac{1}{2}g\sin\beta\,.\,\frac{2u\sin(\alpha+\beta)}{g\cos\beta}\right]$

$= \frac{2u^2\sin(\alpha+\beta)\cos\alpha}{g\cos^2\beta}$ ...(2)

To find the maximum range down the inclined plane, we can write (2) as $R=\frac{u^2[\sin(2\alpha+\beta)+\sin\beta]}{g\cos^2\beta}$

$\therefore$ for given u and β, R is maximum when $\sin(2\alpha+\beta)=1$.

Also the maximum value of R

$$=\frac{u^2(1+\sin\beta)}{g\cos^2\beta}=\frac{u^2(1+\sin\beta)}{g(1-\sin^2\beta)}=\frac{u^2}{g(1-\sin\beta)}$$

Thus for motion down the inclined plane,

time of flight $=\frac{2u\sin(\alpha+\beta)}{g\cos\beta}$, range $=\frac{2u^2\sin(\alpha+\beta)\cos\alpha}{g\cos^2\beta}$,

and maximum range $=\frac{u^2}{g(1-\sin\beta)}$.

We observe that if we replace β by $-\beta$ in the results for motion up the inclined plane, we get the corresponding results for motion down the inclined plane.

Illustrative Examples

Ex. 57. *A particle is projected with velocity u from a point on a plane inclined at an angle β to the horizontal. If r and r′ are its maximum ranges up and down the plane, prove that 1/r+1/r′ is independent of the inclination of the plane.*

Sol. Here the inclination of the inclined plane to the horizontal is β.

$\therefore$ r = the maximum range up the inclined plane

$$=\frac{u^2}{g(1+\sin\beta)}$$

and r' = the maximum range down the inclined plane

$$=\frac{u^2}{g(1-\sin\beta)}.$$

Now $\frac{1}{r}+\frac{1}{r'}=\frac{g}{u^2}[(1+\sin\beta)+(1-\sin\beta)]=\frac{2g}{u^2}$, which is independent of the inclination β of the plane.

Ex. 58. *For a given velocity of projection the maximum range down an inclined plane is three times the maximum range up the inclined plane; show that the inclination of the plane to the horizontal is 30°*

Sol. Let u be the velocity of projection and β the inclination of the inclined plane to the horizontal. Then the maximum ranges up and down the inclined plane are respectively

$$\frac{u^2}{g(1+\sin\beta)} \text{ and } \frac{u^2}{g(1-\sin\beta)}.$$

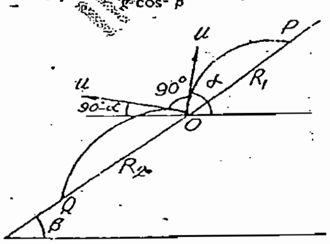
According to the question, $\frac{u^2}{g(1-\sin\beta)}=3\,\frac{u^2}{g(1+\sin\beta)}$.

$\therefore$ $1+\sin\beta=3-3\sin\beta$ or $4\sin\beta=2$ or $\sin\beta=1/2$ or $\beta=30°$.

Ex. 59. *If from a point on the side of a hill two bodies are projected in the vertical plane through the line of greatest slope with the same velocity but in directions at right angles to each other, show that the difference of their ranges is independent of their angles of projection.*

Sol. Let β be the inclination of the hill to the horizontal and O the point of projection. Suppose a particle is projected from O up the hill with velocity u making an angle α with the horizontal through O. If R_1 is the range of this particle on the hill, then using the formula for the range up an inclined plane, we have

$$R_1=\frac{2u^2\cos\alpha\sin(\alpha-\beta)}{g\cos^2\beta}. \quad ...(1)$$



Now the other particle is projected from O with velocity u in a direction at right angles to the direction of projection of the first particle. Therefore this particle moves down the hill and its direction of projection makes an angle $\frac{1}{2}\pi-\alpha$ with the horizontal through O. If R_2 be the range of this particle on the hill, then using the formula for the range down an inclined plane, we have

$$R_2=\frac{2u^2\cos(\frac{1}{2}\pi-\alpha)\sin\{(\frac{1}{2}\pi-\alpha)+\beta\}}{g\cos^2\beta}$$

$$=\frac{2u^2\sin\alpha\sin\{\frac{1}{2}\pi-(\alpha-\beta)\}}{g\cos^2\beta}$$

$$=\frac{2u^2\sin\alpha\cos(\alpha-\beta)}{g\cos^2\beta}. \quad ...(2)$$

From (1) and (2), we have

$$R_2-R_1=\frac{2u^2}{g\cos^2\beta}\left[\sin\alpha\cos(\alpha-\beta)-\cos\alpha\sin(\alpha-\beta)\right]$$

$$=\frac{2u^2}{g\cos^2\beta}\sin\{\alpha-(\alpha-\beta)\}=\frac{2u^2\sin\beta}{g\cos^2\beta},$$

which is independent of the angle of projection α.

Ex. 60. *Show that if a gun be situated on an inclined plane, the maximum range in a direction at right angles to the line of greatest slope is a harmonic mean between the maximum ranges up and down the plane respectively.*

Sol. Let β be the inclination of the inclined plane to the horizontal, O the point of projection and u the velocity of projection.

If R_1 and R_2 are the maximum ranges up and down the inclined plane respectively, then

$$R_1=\frac{u^2}{g(1+\sin\beta)} \text{ and } R_2=\frac{u^2}{g(1-\sin\beta)}.$$

Now the line of greatest slope through O is the line lying in the inclined plane and at right angles to the line in which the inclined plane meets the horizontal. Therefore the direction through O at right angles to the line of greatest slope is a horizontal direction. If R_3 be the maximum range in this direction, then

R_3 = the maximum range in a horizontal direction with velocity of projection $u=u^2/g$.

Now $\frac{1}{2}\left[\frac{1}{R_1}+\frac{1}{R_2}\right]=\frac{1}{2}\frac{g}{u^2}[(1+\sin\beta)+(1-\sin\beta)]$

$=g/u^2=1/R_3$.

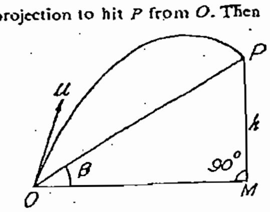
$\therefore$ $1/R_3$ is the arithmetic mean of $1/R_1$ and $1/R_2$

i.e., R_3 is the harmonic mean of R_1 and R_2.

Ex. 61. *The angular elevation of an enemy's position on a hill h metres high is β. Show that in order to shell it, the initial velocity of the projectile must not be less than* $\sqrt{[gh(1+\operatorname{cosec}\beta)]}$.

Sol. Let O be the point of projection and P the enemy's position. Then as given in the question, $PM=h$ metres and $\angle POM=\beta$.

Let u be the least velocity of projection to hit P from O. Then for the velocity of projection u at O, OP is the maximum range up the inclined plane OP.



$\therefore$ $OP=\frac{u^2}{g(1+\sin\beta)}$. ...(1)

But from $\triangle PMO$, we have

$OP=PM\operatorname{cosec}\beta=h\operatorname{cosec}\beta$. ...(2)

From (1) and (2), we have

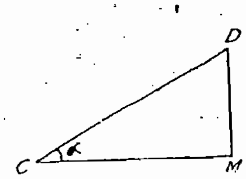
$$\frac{u^2}{g(1+\sin\beta)}=h\operatorname{cosec}\beta$$

i.e., $u^2=gh\operatorname{cosec}\beta(1+\sin\beta)=gh(\operatorname{cosec}\beta+1)$

i.e., $u=\sqrt{[gh(1+\operatorname{cosec}\beta)]}$.

Ex. 62. *The line joining C to D is inclined at an angle α to the horizontal. Show that the least velocity required to shoot from C to D is tan (¼π+½α) times the least velocity required to shoot from D to C.*

Sol. Let u be the least velocity of projection to hit D from C. Then for the velocity of projection u at C, CD is the maximum range up the inclined plane CD.



$\therefore$ $CD=\frac{u^2}{g(1+\sin\alpha)}$ ...(1)

Again let v be the least velocity to shoot C from D. Then for the velocity of projection v at D, DC is the maximum range down the inclined plane DC.

$$\therefore\ DC=\frac{v^2}{g(1-\sin\alpha)} \quad ...(2)$$

From (1) and (2), we have

$$\frac{u^2}{g(1+\sin\alpha)}=\frac{v^2}{g(1-\sin\alpha)}.$$

$$\therefore\ \frac{u^2}{v^2}=\frac{1+\sin\alpha}{1-\sin\alpha}=\frac{1-\cos(\frac{1}{2}\pi+\alpha)}{1+\cos(\frac{1}{2}\pi+\alpha)}=\frac{2\sin^2(\frac{1}{4}\pi+\frac{1}{2}\alpha)}{2\cos^2(\frac{1}{4}\pi+\frac{1}{2}\alpha)}$$

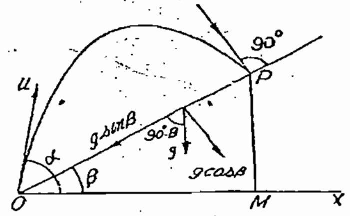
$$=\tan^2(\tfrac{1}{4}\pi+\tfrac{1}{2}\alpha).$$

$\therefore$ $u/v=\tan(\frac{1}{4}\pi+\frac{1}{2}\alpha)$ or $u=v\tan(\frac{1}{4}\pi+\frac{1}{2}\alpha)$, as was to be

Ex. 63. *A particle is projected at an angle α with the horizontal from the foot of the plane, whose inclination to the horizontal is β. Show that it will strike the plane at right angles if*

$$\cot\beta = 2\tan(\alpha-\beta).$$

Sol. Let O be the point of projection, u the velocity of projection and P the point where the particle strikes the plane.



Let T be the time of flight from O to P. Then by the usual formula for time of flight on an inclined plane, we have

$$T=\frac{2u\sin(\alpha-\beta)}{g\cos\beta} \quad ...(1)$$

Since in this question the particle strikes the inclined plane at right angles at P, therefore the direction of the velocity of the particle at P is perpendicular to the inclined plane. Consequently the resolved part of the velocity of the particle at P along the inclined plane is zero. Also the resolved part of the velocity of the particle at O along the inclined plane is $u\cos(\alpha-\beta)$ upwards and the resolved part of the acceleration g along the inclined plane is $g\sin\beta$ downwards. So considering the motion of the particle from O to P along the inclined plane and using the formula $v=u+ft$, we have $0=u\cos(\alpha-\beta)-g\sin\beta\, T$

i.e., $$T=\frac{u\cos(\alpha-\beta)}{g\sin\beta}. \quad ...(2)$$

Equating the values of T from (1) and (2), we have

$$\frac{2u\sin(\alpha-\beta)}{g\cos\beta}=\frac{u\cos(\alpha-\beta)}{g\sin\beta}$$

or $$\frac{2\sin(\alpha-\beta)}{\cos(\alpha-\beta)}=\frac{\cos\beta}{\sin\beta}$$

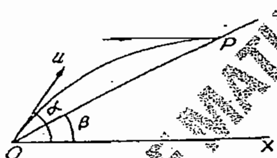
or $$2\tan(\alpha-\beta)=\cot\beta.$$

Ex. 64. *A shot is fired at an angle α to the horizontal up an hill of inclination β to the horizontal. Show that it strikes the hill:*

(a) *horizontally if $\tan\alpha=2\tan\beta$,*

(b) *normally if $\tan\alpha=2\tan\beta+\cot\beta$.*

Sol. (a). Let O be the point of projection, u the velocity of projection and P the point where the shot strikes the plane. Let T be the time of flight from O to P. Then by the usual formula for the time of flight on an inclined plane, we have



$$T=\frac{2u\sin(\alpha-\beta)}{g\cos\beta}. \quad ...(1)$$

Now according to the question the particle strikes the inclined plane horizontally at P i.e., the direction of the velocity of the particle at P is horizontal. So the vertical velocity of the particle at P is zero. Also the vertical velocity of the particle at O is $u\sin\alpha$ upwards and the acceleration in the vertical direction is g downwards. So considering the vertical motion of the particle from O to P and using the formula $v=u+ft$, we have

$$0=u\sin\alpha-gT$$

i.e. $$T=\frac{u\sin\alpha}{g}. \quad ...(2)$$

Equating the values of T from (1) and (2), we have

$$\frac{2u\sin(\alpha-\beta)}{g\cos\beta}=\frac{u\sin\alpha}{g}$$

or $2\sin(\alpha-\beta)=\sin\alpha\cos\beta$

or $2\sin\alpha\cos\beta-2\cos\alpha\sin\beta=\sin\alpha\cos\beta$

or $\sin\alpha\cos\beta=2\cos\alpha\sin\beta$

or $\dfrac{\sin\alpha}{\cos\alpha}=\dfrac{2\sin\beta}{\cos\beta}$

or $\tan\alpha=2\tan\beta.$

(b) Proceeding as in Ex. 63, we get the condition for striking the inclined plane normally at P as

$$\cot\beta=2\tan(\alpha-\beta).$$

$$\therefore\ \cot\beta=\frac{2(\tan\alpha-\tan\beta)}{1+\tan\alpha\tan\beta}$$

or $\cot\beta(1+\tan\alpha\tan\beta)=2\tan\alpha-2\tan\beta$

or $\cot\beta+\tan\alpha=2\tan\alpha-2\tan\beta$

or $\tan\alpha=2\tan\beta+\cot\beta.$

Ex. 65. *A particle is projected with a velocity u from a point on an inclined plane whose inclination to the horizontal is β, and strikes it at right angles. Show that*

(i) *the time of flight is $\dfrac{2u}{g\sqrt{(1+3\sin^2\beta)}}$,*

(ii) *the range on the inclined plane is $\dfrac{2u^2}{g}\cdot\dfrac{\sin\beta}{1+3\sin^2\beta}$,*

and (iii) the vertical height of the point struck, above the point of projection is $\dfrac{2u^2\sin^2\beta}{g(1+3\sin^2\beta)}$. IFoS 2012

Sol. Refer figure of Ex. 63, page 65.

Let O be the point of projection, u the velocity of projection, α the angle of projection and P the point where the particle strikes the plane at right angles.

Let T be the time of flight from O to P. Then by the formula for the time of flight on an inclined plane, we have

$$T=\frac{2u\sin(\alpha-\beta)}{g\cos\beta}. \quad ...(1)$$

Since the particle strikes the inclined plane at right angles at P, therefore the velocity of the particle at P along the inclined plane is zero. Also the resolved part of the velocity of the particle at O along the inclined plane is $u\cos(\alpha-\beta)$ upwards and the resolved part of the acceleration g along the inclined plane is $g\sin\beta$ downwards. So considering the motion of the particle from O to P along the inclined plane and using the formula $v=u+ft$, we have $0=u\cos(\alpha-\beta)-g\sin\beta T$

or $$T=\frac{u\cos(\alpha-\beta)}{g\sin\beta} \quad ...(2)$$

Equating the values of T from (1) and (2), we have

$$\frac{2u\sin(\alpha-\beta)}{g\cos\beta}=\frac{u\cos(\alpha-\beta)}{g\sin\beta}$$

or $$\tan(\alpha-\beta)=\tfrac{1}{2}\cot\beta, \quad ...(3)$$

as the condition for striking the plane at right angles.

(i) From (2),

$$T=\frac{u}{g\sin\beta\sec(\alpha-\beta)}=\frac{u}{g\sin\beta\sqrt{\{1+\tan^2(\alpha-\beta)\}}}$$

$$=\frac{u}{g\sin\beta\sqrt{\{1+\frac{1}{4}\cot^2\beta\}}}, \text{ substituting for } \tan(\alpha-\beta) \text{ from (3)}$$

$$=\frac{2u\sin\beta}{g\sin\beta\sqrt{(4\sin^2\beta+\cos^2\beta)}}=\frac{2u}{g\sqrt{(\sin^2\beta+\cos^2\beta+3\sin^2\beta)}}$$

$$=\frac{2u}{g\sqrt{(1+3\sin^2\beta)}}.$$

(ii) Let R be the range on the inclined plane; then $R=OP$. Considering the motion from O to P along the inclined plane and using the formula $v^2=u^2+2fs$, we have

$$0=u^2\cos^2(\alpha-\beta)-2g\sin\beta R$$

or $$R=\frac{u^2\cos^2(\alpha-\beta)}{2g\sin\beta}=\frac{u^2}{2g\sin\beta\sec^2(\alpha-\beta)}$$

$$=\frac{u^2}{2g\sin\beta\{1+\tan^2(\alpha-\beta)\}}$$

$$=\frac{u^2}{2g\sin\beta\{1+\frac{1}{4}\cot^2\beta\}} \quad \text{[from (3)]}$$

$$=\frac{4u^2\sin^2\beta}{2g\sin\beta(4\sin^2\beta+\cos^2\beta)}=\frac{2u^2\sin\beta}{g(1+3\sin^2\beta)}.$$

(iii) The vertical height of P above $O=PM$

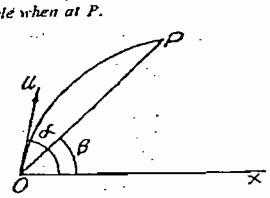
$$=OP\sin\beta=R\sin\beta=\frac{2u^2\sin^2\beta}{g(1+3\sin^2\beta)}.$$

Ex. 66. *Prove that if a particle is projected from O at an elevation α and after time t the particle is at P, then*

$$2\tan\beta=\tan\alpha+\tan\theta,$$

where β and θ are the inclinations to the horizontal of OP and of the direction of motion of the particle when at P.

Sol. Let O be the point of projection, u the velocity of projection and t the time of flight from O to P. It is given that $\angle POX=\beta$, where OX is the horizontal through O in the plane of motion. We can regard t as the time of flight on the inclined pane OP whose inclination to the horizontal is β.



$$\therefore\ t=\frac{2u\sin(\alpha-\beta)}{g\cos\beta}. \quad ...(1)$$

Since θ is the inclination to the horizontal of the direction of motion at P, therefore

$$\tan\theta=\frac{\text{vertical velocity at } P}{\text{horizontal velocity at } P}$$

$$=\frac{u\sin\alpha-gt}{u\cos\alpha}=\tan\alpha-\frac{g}{u\cos\alpha}t$$

$$=\tan\alpha-\frac{g}{u\cos\alpha}\cdot\frac{2u\sin(\alpha-\beta)}{g\cos\beta}, \text{ substituting for } t \text{ from (1)}$$

$$=\tan\alpha-\frac{2\sin(\alpha-\beta)}{\cos\alpha\cos\beta}=\tan\alpha-\frac{2(\sin\alpha\cos\beta-\cos\alpha\sin\beta)}{\cos\alpha\cos\beta}$$

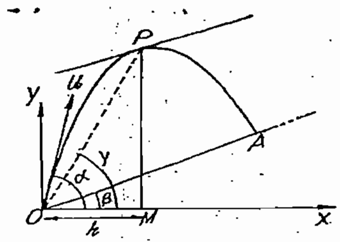
$$=\tan\alpha-2(\tan\alpha-\tan\beta)=\tan\alpha-2\tan\alpha+2\tan\beta$$

$$=2\tan\beta-\tan\alpha.$$

$$\therefore\ 2\tan\beta=\tan\alpha+\tan\theta.$$

Ex. 67. *A stone is thrown at an angle α with the horizontal from a point in a plane whose inclination to the horizontal is β, the trajectory lying in the vertical plane containing the line of greatest slope. Show that if γ be the elevation of that point of the path which is most distant from the inclined plane, then*

$$2\tan\gamma=\tan\alpha+\tan\beta.$$



Sol. Let O be the point of projection, u the velocity of projection and α the angle of projection. Let P be the point of the trajectory which is most distant from the inclined plane. Then the tangent at P to the trajectory is parallel to the line OA. Referred to the horizontal and vertical lines OX and OY in the plane of motion as the coordinate axes, let the coordinates of P be (h, k). It is given that $\angle POM=\gamma$. Therefore

$$\tan\gamma=k/h. \qquad ...(1)$$

The equation of the trajectory is

$$y=x\tan\alpha-\tfrac{1}{2}g\frac{x^2}{u^2\cos^2\alpha}.$$

$\therefore\ \dfrac{dy}{dx}=\tan\alpha-\dfrac{gx}{u^2\cos^2\alpha}$, which gives the slope of the tangent to the horizontal at any point (x, y) of the trajectory.

Since the tangent to the trajectory at the point $P(h, k)$ makes an angle β with the horizontal line OX, therefore

$$\left(\frac{dy}{dx}\right)_{(h,\,k)}=\tan\beta$$

i.e.,
$$\tan\alpha-\frac{gh}{u^2\cos^2\alpha}=\tan\beta. \qquad ...(2)$$

Alos the point ~~(h, k) lies on~~ the trajectory. Therefore, ~~we have~~

$$k=h\tan\alpha-\tfrac{1}{2}\frac{gh^2}{u^2\cos^2\alpha} \quad\text{or}\quad k=h\left[\tan\alpha-\tfrac{1}{2}\frac{gh}{u^2\cos^2\alpha}\right]$$

or
$$\frac{k}{h}=\tan\alpha-\tfrac{1}{2}\frac{gh}{u^2\cos^2\alpha}. \qquad ...(3)$$

But from (1), $\dfrac{k}{h}=\tan\gamma$ and from (2), $\dfrac{gh}{u^2\cos^2\alpha}=\tan\alpha-\tan\beta$.

Substituting these in (3), we get

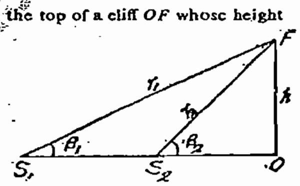
$$\tan\gamma=\tan\alpha-\tfrac{1}{2}(\tan\alpha-\tan\beta)$$

or
$$2\tan\gamma=2\tan\alpha-\tan\alpha+\tan\beta$$

or
$$2\tan\gamma=\tan\alpha+\tan\beta,$$

which proves the required result.

Ex. 68. *A fort is on the edge of a cliff of height h. Show that there is an annular region of area $8\pi hk$, in which the fort is out of range of the ship, but the ship is not out of range of the fort, where $\sqrt{(2gk)}$ is the velocity of the shells used by both.*



Sol. Let F be the fort on the top of a cliff OF whose height is h. Let S_1 be the farthest position of the ship where it can be hit from the fort with velocity of projection $\sqrt{(2gk)}$. Then $\sqrt{(2gk)}$ is the least velocity of projection to hit S_1 from F and consequently for the velocity of projection $\sqrt{(2gk)}$ at F, FS_1 is the maximum range down the inclined plane FS_1. Let $\angle FS_1O=\beta_1$ and $FS_1=r_1$. By the formula for the maximum range down an inclined plane, we have

$$r_1=FS_1=\frac{u^2}{g(1-\sin\beta_1)}, \text{ where } u \text{ is the velocity of projection}$$

$$=\frac{2gk}{g(1-\sin\beta_1)} \qquad [\because\ u^2=2gk]$$

$$=\frac{2k}{1-\sin\beta_1}.$$

$\therefore\ r_1(1-\sin\beta_1)=2k \quad\text{or}\quad r_1-r_1\sin\beta_1=2k$

or $\quad r_1-h=2k \qquad [\because$ from $\Delta FS_1O,\ h=r_1\sin\beta_1]$

or $\quad r_1=2k+h. \qquad ...(1)$

Again let S_2 be the farthest position of the ship from where the fort can be hit with velocity of projection $\sqrt{(2gk)}$. Then for the velocity of projection $\sqrt{(2gk)}$ at S_2, S_2F is the maximum range up the inclined plane S_2F. Let $\angle FS_2O=\beta_2$ and $FS_2=r_2$. By the formula for the maximum range up an inclined plane, we have

$$r_2=S_2F=\frac{u^2}{g(1+\sin\beta_2)}, \ u \text{ being the velocity of projection}$$

$$=\frac{2gk}{g(1+\sin\beta_2)} \qquad [\because\ u^2=2gk]$$

$$=\frac{2k}{1+\sin\beta_2}.$$

$\therefore\ r_2(1+\sin\beta_2)=2k \quad\text{or}\quad r_2+r_2\sin\beta_2=2k$

or $\quad r_2+h=2k \qquad [\because$ from $\Delta FS_2O,\ h=r_2\sin\beta_2]$

or $\quad r_2=2k-h. \qquad ...(2)$

Now if the ship is anywhere between S_1 and S_2, then the fort cannot be shelled from the ship while the ship can be shelled from the fort. If the line OS_1 revolves about O, then there is an annular region bounded by the concentric circles with centre at O and radii as OS_1 and OS_2 in which the fort is out of range of the ship while the ship is not out of range of the fort. The area of this annular region $=\pi(OS_1^2-OS_2^2)=\pi[(r_1^2-h^2)-(r_2^2-h^2)]$

$[\because$ from ΔFS_1O, $OS_1^2=r_1^2-h^2$, etc.$]$

$$=\pi(r_1^2-r_2^2)$$
$$=\pi[(2k+h)^2-(2k-h)^2] \qquad [\text{from (1) and (2)}]$$
$$=8\pi hk.$$

Ex. 69. *A fort and a ship are both armed with guns which give their projectiles a muzzle velocity $\sqrt{(2gh)}$ and guns in the fort are at a height k above the ship. If d_1 and d_2 are greatest horizontal ranges at which the fort and ship, respectively, can engage, prove that* $\dfrac{d_1}{d_2}=\sqrt{\left(\dfrac{h+k}{h-k}\right)}$.

Sol. Proceed exactly in the same way as in Ex. 68. Here $OF=k$. Thus replacing h by k and k by h in the results of Ex. 68, we get

$$r_1=2h+k \quad\text{and}\quad r_2=2h-k.$$

According to this question $\quad OS_1=d_1$ and $OS_2=d_2$

From ΔOS_1F, $OS_1=\sqrt{(FS_1^2-OF^2)}=\sqrt{(r_1^2-k^2)}$.

$\therefore\ d_1=\sqrt{\{(2h+k)^2-k^2\}}=\sqrt{(4h^2+4hk)}=2\sqrt{h}\sqrt{(h+k)}$.

Again from ΔOS_2F, $OS_2=\sqrt{(FS_2^2-OF^2)}=\sqrt{(r_2^2-k^2)}$.

$\therefore\ d_2=\sqrt{\{(2h-k)^2-k^2\}}=\sqrt{(4h^2-4hk)}=2\sqrt{h}\sqrt{(h-k)}$.

$$\therefore\ \frac{d_1}{d_2}=\frac{2\sqrt{h}\sqrt{(h+k)}}{2\sqrt{h}\sqrt{(h-k)}}=\sqrt{\left(\frac{h+k}{h-k}\right)}.$$

Ex. 70. *If u be the velocity of projection and v_1 the velocity of striking the plane when projected so that range up the plane is maximum and v_2 the velocity of striking the plane when projected so that range down the plane is maximum, prove that* $u^2=v_1v_2$.

Sol. Let β be the inclination of the plane to the horizontal.

For the velocity of projection u, the maximum range up the inclined plane $=\dfrac{u^2}{g(1+\sin\beta)}$

$\therefore$ the height of the point of striking the plane above the point of projection $=\dfrac{u^2}{g(1+\sin\beta)}\cdot\sin\beta=h_1$ (say).

Since the velocity of the projectile at this vertical height h_1 above the point of projection is given to be v_1, therefore

$$v_1^2=u^2-2gh_1 \qquad [\text{Refer § 5, page 7}]$$

$$=u^2-2g\cdot\frac{u^2\sin\beta}{g(1+\sin\beta)}=u^2\frac{1-\sin\beta}{1+\sin\beta}. \qquad ...(1)$$

Again for the velocity of projection u, the maximum range down the inclined plane $=\dfrac{u^2}{g(1-\sin\beta)}$.

$\therefore$ the depth of the point of striking the plane below the point of projection $=\dfrac{u^2}{g(1-\sin\beta)}\cdot\sin\beta=h_2$ (say).

Since the velocity of the projectile at this vertical depth h_2 below the point of projection is given to be v_2, therefore

$$v_2^2=u^2+2gh_2=u^2+2g\cdot\frac{u^2\sin\beta}{g(1-\sin\beta)}$$

$$=u^2\frac{1+\sin\beta}{1-\sin\beta}. \qquad ...(2)$$

From (1) and (2), we have

$$v_1^2v_2^2=u^4 \quad i.e.\quad v_1v_2=u^2.$$

Ex. 71. *Show that the greatest range up an inclined plane through the point of projection is equal to the distance through which a particle could fall freely during the corresponding time of flight.*

Sol. Let β be the inclination of the plane to the horizontal. If α is the angle of projection, then for maximum range up the plane,

$$\alpha=\tfrac{1}{4}\pi+\tfrac{1}{2}\beta.$$

Time of flight up the inclined plane is

$$T=\frac{2u\sin(\alpha-\beta)}{g\cos\beta}.$$

When $\alpha=\frac{1}{4}\pi+\frac{1}{2}\beta$, $T=\frac{2u\sin\{(\frac{1}{4}\pi+\frac{1}{2}\beta)-\beta\}}{g\cos\beta}$

$$=\frac{2u\sin(\frac{1}{4}\pi-\frac{1}{2}\beta)}{g\cos\beta}$$

The vertical distance fallen freely under gravity by a particle during this time T

$$=0.T+\frac{1}{2}gT^2=\frac{1}{2}g.\frac{4u^2\sin^2(\frac{1}{4}\pi-\frac{1}{2}\beta)}{g^2\cos^2\beta}$$

$$=\frac{2u^2\sin^2(\frac{1}{4}\pi-\frac{1}{2}\beta)}{g\cos^2\beta}=\frac{u^2\{1-\cos(\frac{1}{2}\pi-\beta)\}}{g\cos^2\beta}=\frac{u^2(1-\sin\beta)}{g(1-\sin^2\beta)}$$

$$=\frac{u^2}{g(1+\sin\beta)}=\text{the maximum range up the inclined plane.}$$

Ex. 72. *Two inclined planes intersect in a horizontal plane, their inclinations to the horizon being α and β; if a particle is projected at right angles to the former from a point in it so as to strike the other at right angles, the velocity of projection is*

$$\sin\beta\left\{\frac{2ag}{\sin\alpha-\sin\beta\cos(\alpha+\beta)}\right\}^{1/2}$$

a being the distance of the point of projection from the intersection of the planes.

Sol. Let OA and OB be the two inclined planes and P the point of projection so that $OP=a$. The particle is projected from P at right angles to OA, say, with velocity u. Let PN be perpendicular from P to BO produced and PM be drawn parallel to OB.

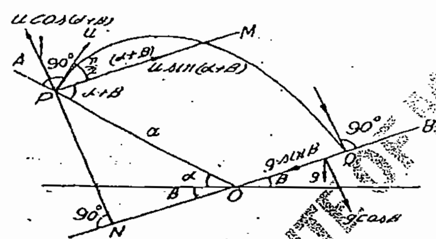
We have $PN=a\sin(\alpha+\beta)$. Also $\angle MPO=\angle PON=\alpha+\beta$, being the alternate angles. Thus the velocity of projection u makes an angle $\frac{1}{2}\pi-(\alpha+\beta)$ with PM.

The resolved part of the velocity at P along PM *i.e.*, parallel to OB $=u\cos\{\frac{1}{2}\pi-(\alpha+\beta)\}=u\sin(\alpha+\beta)$

and the resolved part of the velocity at P along NP *i.e.*, perpendicular to OB $=u\cos(\alpha+\beta)$.

The resolved parts of the acceleration g due to gravity along and perpendicular to OB are $g\sin\beta$ and $g\cos\beta$ as shown in the figure.

Let t be the time of flight from P to Q. Since the particle strikes the inclined plane OB at right angles at Q, therefore the velocity of the particle at Q along OB is zero. So considering the motion of the particle from P to Q parallel to OB and using



the formula $v=u+ft$, we have

$$0=u\sin(\alpha+\beta)-g\sin\beta.t$$

or $$t=\frac{u\sin(\alpha+\beta)}{g\sin\beta}. \quad ...(1)$$

Again the displacement from P to Q perpendicular to OB is $PN=a\sin(\alpha+\beta)$, in the downward direction. So considering the motion from P to Q perpendicular to OB and using the formula $s=ut+\frac{1}{2}ft^2$, we have

$$-a\sin(\alpha+\beta)=u\cos(\alpha+\beta).t-\frac{1}{2}g\cos\beta.t^2$$

$$=t\{u\cos(\alpha+\beta)-\frac{1}{2}g\cos\beta.t\}$$

$$=\frac{u\sin(\alpha+\beta)}{g\sin\beta}\left\{u\cos(\alpha+\beta)-\frac{1}{2}g\cos\beta.\frac{u\sin(\alpha+\beta)}{g\sin\beta}\right\}$$

[substituting for t from (1)]

$$=\frac{u^2\sin(\alpha+\beta)}{2g\sin^2\beta}\left\{2\cos(\alpha+\beta)\sin\beta-\sin(\alpha+\beta)\cos\beta\right\}$$

i.e.,

$$a=\frac{u^2}{2g\sin^2\beta}\left\{\sin(\alpha+\beta)\cos\beta-\cos(\alpha+\beta)\sin\beta-\cos(\alpha+\beta)\sin\beta\right\}$$

$$=\frac{u^2}{2g\sin^2\beta}\left[\sin\{(\alpha+\beta)-\beta\}-\sin\beta\cos(\alpha+\beta)\right]$$

$$=\frac{u^2}{2g\sin^2\beta}\left[\sin\alpha-\sin\beta\cos(\alpha+\beta)\right]$$

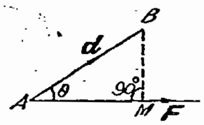
$$\therefore\quad u^2=\frac{2ag\sin^2\beta}{[\sin\alpha-\sin\beta\cos(\alpha+\beta)]}$$

or $$u=\sin\beta\left\{\frac{2ag}{\sin\alpha-\sin\beta\cos(\alpha+\beta)}\right\}^{1/2}.$$

WORK, ENERGY AND IMPULSE SET-V

1. The concept of work. We know that a force, when applied to a particle or body, often causes a change in its position. A force is said to do *work* when its point of application is displaced.

2. Work done by a constant force. Definition.



Suppose a constant force represented by the vector $\mathbf{F}$ *acts at the point* A. *Let the point* A *be displaced to the point* B, *where* $\overrightarrow{AB}=\mathbf{d}$. *Then the work* W *done by the constant force* $\mathbf{F}$ *during the displacement* $\mathbf{d}$ *of its point of application is defined as*

$$W=\mathbf{F}\cdot\mathbf{d}, \qquad \ldots(1)$$

where $\mathbf{F}\cdot\mathbf{d}$ *is the scalar product of the vectors* $\mathbf{F}$ *and* $\mathbf{d}$.

Let θ be the angle between the vectors $\mathbf{F}$ and $\mathbf{d}$. If $F=|\mathbf{F}|$ and $d=|\mathbf{d}|=AB$, then using the definition of the scalar product of two vectors, the equation (1) defining the work may be written as

$$W=Fd\cos\theta. \qquad \ldots(2)$$

Obviously $d\cos\theta$ is the displacement of the point of application of the force $\mathbf{F}$ in the direction of the force. *Hence the work done by a constant force is equal to the magnitude of the force multiplied by the displacement of the point of application of the force in the direction of the force.*

From the equation (2) we make the following observations :

(i) If $\theta=\frac{1}{2}\pi$ *i.e.*, if the displacement of the point of application of the force is perpendicular to the direction of the force, then $W=0$.

(ii) If $0\leqslant\theta<\frac{1}{2}\pi$ *i.e.*, if the displacement of the point of application of the force parallel to the line of action of the force is in the direction of the force, then W is positive.

(iii) If $\frac{1}{2}\pi<\theta\leqslant\pi$ *i.e.*, if the displacement of the point of application of the force parallel to the line of action of the force is opposite to the direction of the force, then W is negative.

Example. If a particle of mass m is displaced on a horizontal plane through a distance h, then during this displacement the work done by the weight mg of the particle is zero.

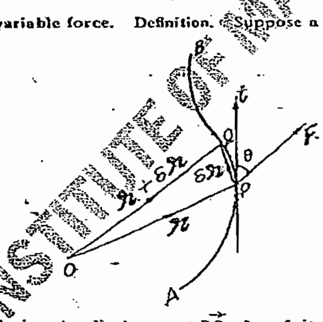
If a particle of mass m is raised through a vertical height h, then during this displacement the work done by the weight mg of the particle is $-mgh$.

Again if a particle of mass m falls through a vertical depth h, then during this displacement the work done by the weight mg of the particle is mgh.

3. Work done by a variable force. Definition. Suppose a variable force $\mathbf{F}$ acts on a particle which moves along an arc AB from A to B. Let P, Q be neighbouring points on this curve such that

$\overrightarrow{OP}=\mathbf{r}$, $\overrightarrow{OQ}=\mathbf{r}+\delta\mathbf{r}$,

$\overrightarrow{PQ}=\overrightarrow{OQ}-\overrightarrow{OP}=\delta\mathbf{r}$.



During the small displacement $\delta\mathbf{r}$ of its point of application the force $\mathbf{F}$ may be regarded as a constant force. So the work done by the force $\mathbf{F}$ during the displacement $\overrightarrow{PQ}=\delta\mathbf{r}$ of its point of application is equal to $\mathbf{F}\cdot\delta\mathbf{r}$. The work W done by the force $\mathbf{F}$ in displacing its point of application from A to B along the given arc AB is defined as the limiting sum of the elemental expressions $\mathbf{F}\cdot\delta\mathbf{r}$ as the point of application of the force moves from A to B along the given arc AB. Thus

$$W=\int_A^B \mathbf{F}\cdot d\mathbf{r}, \qquad \ldots(1)$$

where the integration is to be performed along the arc AB.

Referred to some frame of rectangular co-ordinate axes OX, OY and OZ let (x, y, z) be the co-ordinates of the point P. Then $\mathbf{r}=x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ so that $d\mathbf{r}=dx\mathbf{i}+dy\mathbf{j}+dz\mathbf{k}$. Also let $\mathbf{F}=X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k}$ where X, Y, Z are the components of the force $\mathbf{F}$ along OX, OY, OZ respectively. We have

$$\mathbf{F}\cdot d\mathbf{r}=(X\mathbf{i}+Y\mathbf{j}+Z\mathbf{k})\cdot(dx\mathbf{i}+dy\mathbf{j}+dz\mathbf{k})$$
$$=X\,dx+Y\,dy+Z\,dz.$$

$\therefore$ the equation (1) defining the work W done by the force $\mathbf{F}$ may be written as

$$W=\int_A^B (X\,dx+Y\,dy+Z\,dz). \qquad \ldots(2)$$

Again if s denotes the arc length of the curve AB measured from some fixed point on the curve to any other point P whose position vector is $\mathbf{r}$, then $d\mathbf{r}/ds=\mathbf{t}$, where $\mathbf{t}$ is the unit vector along the tangent at P to the curve in the sense of s increasing. We may write the equation (1) as

$$W=\int_A^B\left(\mathbf{F}\cdot\frac{d\mathbf{r}}{ds}\right)ds=\int_A^B \mathbf{F}\cdot\mathbf{t}\,ds$$
$$=\int_A^B F\cos\theta\,ds, \qquad \ldots(3)$$

where $F=|\mathbf{F}|$ and θ is the angle which the direction of the force $\mathbf{F}$ makes with the direction of the tangent of the curve in the sense of s increasing.

The integration on the right hand side of the above equations (1), (2) and (3) is to be performed along the arc AB of the given path of the particle.

4. Units of work. In the Centimeter Gram Second (C. G. S.) system the absolute unit of work is called an *erg*. It is the work done by a force of one dyne in displacing its point of application through 1 centimeter in its direction. Also in this system the gravitational (or practical) unit of work is one *gram-cm*. It is the work done by a force of one gram weight as its point of application is displaced through 1 cm. in its direction. The two units are related as follows : *one gm.-cm.* $=g$ *ergs* $=981$ *ergs.*

In the Meter Kilogram Second (M. K. S.) system the absolute unit of work is called a *joule*. It is the work done by a force of one newton in displacing its point of application through 1 meter in its direction. Also in this system the gravitational (or practical) unit of work is one *kilogram-meter*. It is the work done by a force of one kg. wt. as its point of application is displaced through 1 meter in its direction. We have

one kg.-m. $=g$ *joules* $=9{\cdot}8$ *joules.*

In the Foot Pound Second (F. P. S.) system the absolute unit of work is called a *foot-poundal*. It is the work done by a force of one poundal in displacing its point of application through 1 foot in its direction. Also in this system the gravitational unit of work is one *foot-pound*. It is the work done by a force of one pound weight as its point of application is displaced through 1 foot in its direction. We have

one ft.-lb. $=g$ *ft.-poundals* $=32$ *ft.-poundals.*

5. Power. The amount of work done by a force depends upon the time as well. *The power of an agent supplying the force is defined as the rate of doing the work.* Thus the power of an agent is the amount of work done by the agent in a unit time. The units of power may be taken as the *units of work per second.*

In the British system *i.e.*, in the F. P. S. system the unit of power used in engineering practice is one *Horse power* while in the M.K.S. system the unit of power used in engineering practice is one *watt*. We have

one Horse power (H. P.) $=550$ *ft.-lbs./sec.*

and *one Watt* $=$ *one joule/sec.* $=10^7$ *ergs/sec.*

Thus an engine is said to be of one H. P. if the work done by it per second is 550 foot-pounds or 550×32 foot-poundals.

Also remember that 1 H. P. $=746$ watts.

Illustrative Examples

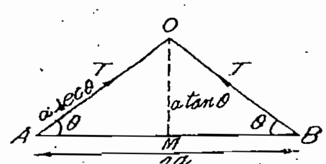
Ex. 1. *Prove that the work done against the tension in stretching a light elastic string, is equal to the product of its extension and the mean of its final and initial tensions.*

Sol. For the complete solution of this problem refer § 8, chapter 2, page 90.

Ex. 2. *If a light elastic string, whose natural length is that of a uniform rod be attached to the rod at both the ends and suspended by the middle point, show that the rod will descend until each of the two portions of the string is inclined to the horizon at an angle* θ, *given by the equation*

$$\cot^3\tfrac{1}{2}\theta-\cot\tfrac{1}{2}\theta=2n,$$

the modulus of elasticity of the string being n *times the weight of the rod.*



Sol. Let $2a$ be the length of the rod AB, O the middle point of the string AOB whose natural length is also $2a$. The string is suspended at the fixed point O. Initially the rod is held at rest in the level of O and then released. Due to the weight of the rod the string is stretched and the rod moves down. Let θ be the inclination of each of the two portions of the string to the horizontal when the rod again comes to rest.

The vertical distance moved by the centre of gravity of the rod $= OM = a \tan \theta$.

$\therefore$ the work done by the weight of the rod

$$= mg\, a \tan \theta, \quad \ldots(1)$$

where m is the mass of the rod.

In the initial position the tension in the string is zero because then there is no extension.

In the final position the extension in the length of the string

$$= 2a \sec \theta - 2a.$$

$\therefore$ in the final position, by Hooke's law, the tension T in the string $= \dfrac{\lambda\,(2a \sec \theta - 2a)}{2a}$, where λ is the modulus of elasticity

$$= \frac{nmg\,(2a \sec \theta - 2a)}{2a} \quad [\because \lambda = nmg]$$

$$= nmg\,(\sec \theta - 1).$$

We know that the work done in stretching an elastic string $=$ (mean of the initial and final tensions) $\times$ (the extension).

$\therefore$ the work done in stretching the string in question

$$= \tfrac{1}{2}\,(0 + T) \times (2a \sec \theta - 2a)$$

$$= \tfrac{1}{2}\,nmg\,(\sec \theta - 1) \cdot 2a\,(\sec \theta - 1)$$

$$= nmga\,(\sec \theta - 1)^2. \quad \ldots(2)$$

Since the works (1) and (2) are equal, therefore

$$mga \tan \theta = nmga\,(\sec \theta - 1)^2$$

or $\quad \tan \theta = n\,(\sec \theta - 1)^2$

or $\quad \sin \theta \cos \theta = n\,(1 - \cos \theta)^2$

or $\quad 2 \sin \tfrac{1}{2}\theta \cos \tfrac{1}{2}\theta\,(\cos^2 \tfrac{1}{2}\theta - \sin^2 \tfrac{1}{2}\theta) = 4n \sin^4 \tfrac{1}{2}\theta$

or $\quad \cot^3 \tfrac{1}{2}\theta - \cot \tfrac{1}{2}\theta = 2n,$

which proves the required result.

Ex. 3. *A spider hangs from the ceiling by a thread of modulus of elasticity equal to its weight. Show that it can climb to the ceiling with an expenditure of work equal to only three quarters of what would be required if the thread were inelastic.*

Sol. Let l be the natural length of the thread and l_1 its length when the spider hangs in equilibrium. In this position of equilibrium we should have

the weight of the spider $=$ the tension in the thread.

$\therefore \quad mg = \lambda \dfrac{l_1 - l}{l}$, where m is the mass of the spider and λ is the modulus of elasticity of the thread.

But $\quad \lambda = mg.$

$\therefore \quad mg = mg\,\dfrac{l_1 - l}{l}$

i.e., $\quad l_1 - l = l \quad$ *i.e.*, $\quad l_1 = 2l.$

Thus the spider hangs in equilibrium by the free end of the thread at a depth $2l$ below the ceiling.

If the length $2l$ were inelastic, the work that the spider does against its weight in climbing to the ceiling $= mg \cdot 2l = 2mgl$.

In case the thread is elastic the work done in stretching it to a length $2l$

$$= \tfrac{1}{2}\,(\text{initial tension} + \text{final tension}) \times \text{extension}$$

$$= \tfrac{1}{2}\,(0 + mg)\,(2l - l) = \tfrac{1}{2} mgl.$$

In this case when the spider reaches the ceiling, the thread reverts from its stretched to natural length, so the work done against the tension is the same as above but negative.

Therefore when the thread is elastic the total work done in climbing to the ceiling

$$= 2mgl - \tfrac{1}{2} mgl = \tfrac{3}{2} mgl = \tfrac{3}{4}\,(2mgl)$$

$= \tfrac{3}{4}$ of the work if the thread were inelastic.

Ex. 4. *A cylindrical cork of length l and radius r is slowly extracted from the neck of a bottle. If the normal pressure per unit of area between the bottle and unextracted part of the cork at any instant be constant and equal to P, show that the work done in extracting it is $\pi \mu r l^2 P$, where μ is the coefficient of friction.*

Sol. At any instant, if x is the length of the cork in contact with the bottle, then the area of the surface of the cork in contact with the bottle is equal to $2\pi r x$.

The normal pressure on this surface $= 2\pi r x P$.

$\therefore$ the force of friction on the cork when it moves rubbing the bottle $= \mu 2\pi r x P$.

$\therefore$ work done against this friction in extracting a length δx

$$= \mu 2\pi r x P \delta x$$

$$= 2\pi \mu r P x \delta x.$$

Hence the total work done in extracting the whole length of the cork $= \displaystyle\int_0^l 2\pi \mu r P x\,dx = 2\pi \mu r P \int_0^l x\,dx$

$$= 2\pi \mu r P \left[\frac{x^2}{2}\right]_0^l = 2\pi \mu r P \cdot \frac{l^2}{2} = \pi \mu r P l^2.$$

Ex. 5. *Prove that the work done in stretching an elastic string AB, of natural length l and modulus λ, from tension T_1 to tension T_2 is*

$$(l/2\lambda)\,(T_2^2 - T_1^2).$$

Sol. Let l_1 be the stretched length of the string in the state of tension T_1 and l_2 the stretched length in the state of tension T_2. Then by Hooke's law, we have

$$T_1 = \lambda \frac{l_1 - l}{l} \quad \ldots(1)$$

and

$$T_2 = \lambda \frac{l_2 - l}{l}. \quad \ldots(2)$$

Let W be the work done in stretching the string from tension T_1 to tension T_2. Then

$$W = \tfrac{1}{2}\,(\text{initial tension} + \text{final tension}) \times \text{extension}$$

$$= \tfrac{1}{2}\,(T_1 + T_2)\,(l_2 - l_1). \quad \ldots(3)$$

Subtracting (1) from (2), we have

$$T_2 - T_1 = \frac{\lambda}{l}\,(l_2 - l_1). \quad \ldots(4)$$

Substituting for $l_2 - l_1$ from (4) in (3), we have

$$W = \tfrac{1}{2}\,(T_1 + T_2) \cdot \frac{l}{\lambda}\,(T_2 - T_1) = \frac{l}{2\lambda}\,(T_2^2 - T_1^2).$$

Ex. 6. *A motor car weighing 10 quintals and travelling at 12 meters/sec. is brought to rest in 18 meters by the application of its brakes. Find the work done by the force of resistance due to brakes.*

Sol. Assuming that the resistance is uniform, let the retardation due to this resistance be r m./sec^2.

Here the initial velocity $u = 12$ m./sec., final velocity $v = 0$ m./sec. and the distance travelled $s = 18$ metres. Therefore using the formula $v^2 = u^2 + 2fs$, we have

$$0 = 12^2 - 2r \times 18 \quad \textit{i.e.},\ r = \frac{144}{36} = 4.$$

Now mass of the car $= 1000$ kg.

$\therefore$ using the formula $P = mf$, the force of resistance

$= 1000 \times 4$ newtons $= 4000$ newtons.

Hence the required work done $= 4000 \times 18$ joules

$$= 72000 \text{ joules} = \frac{72000}{9.8} \text{ kg.-meters}$$

$$= 7347 \text{ kg.-meters (approx.)}.$$

Ex. 7. *A train of total mass 250 tons is drawn by an engine working at 560 H.P. If at a certain instant the total resistance is 16 lbs. wt. per ton the weight of the train, and the velocity 30 miles an hour, what is the train's acceleration, measured in miles per hour per second.*

Sol. The velocity of 30 miles per hour $= 44$ ft. per sec.

Let P lbs. wt. be the pull of the engine when the velocity is 44 ft./sec. Then the rate at which the engine works

$$= P \times 44 \text{ ft.-lbs/sec.}$$

But the engine is working at 560 H.P.

i.e., at the rate of 560×550 ft.-lbs/sec.

$\therefore \quad P \times 44 = 560 \times 550 \quad$ or $\quad P = 7000.$

Total resistance $= (16 \times 250)$ lbs. wt. $= 4000$ lbs. wt.

$\therefore$ the net force in the direction of motion

$= (7000 - 4000)$ lbs. wt. $= 3000$ lbs. wt.

$= 3000 \times 32$ poundals $= 96000$ poundals.

If f ft./sec^2 be the acceleration of the train, we have by Newton's second law of motion

$$96000 = 250 \times 2240\, f \quad [1 \text{ ton} = 2240 \text{ lbs.}]$$

or $\quad f = \dfrac{96000}{250 \times 2240} = \dfrac{6}{35}.$

$\therefore$ the acceleration of the train $= \dfrac{6}{35}$ ft. per sec. per sec.

$$= \frac{6 \times 60 \times 60}{35 \times 1760 \times 3} \text{ miles per hour per second}$$

$$= \frac{9}{77} \text{ miles per hour per second.}$$

6. Kinetic energy. The capacity of a body for doing work is known as energy of the body. The kinetic energy (K.E.) of a body is the energy which the body possesses on account of being in motion. We can define it precisely as follows:

Kinetic energy. Definition. *The kinetic energy of a body is the amount of work which the body can perform against some resistance till reduced to rest.* [Rohilkhand 1977]

Since the K.E. has been defined to be equal to work done in some way, the units for measuring K.E. are the same as those for work.

Calculation of kinetic energy. *If at any instant a body of mass m be moving with velocity u, then the kinetic energy of the body at that instant is equal to $\tfrac{1}{2} mu^2$.*

At any instant let a body of mass m be moving with velocity u. Then, by definition, the K.E. of the body at that instant is equal to the amount of work which the body can perform against some resistance, say P, till reduced to rest. Suppose the body moves from the point A to the point B while the resistance P reduces its velocity from u to 0. The direction of the force of resistance is against the direction of motion. If v be the velocity of the body at any point between the above two positions A and B, we should have

$$mv \frac{dv}{ds} = -P, \quad \ldots(1)$$

assuming that the body is moving in the direction of s increasing so that the resistance P acts in the sense of s decreasing.

From (1), we have

$$Pds = -mvdv. \qquad \ldots(2)$$

Let $s=0$ at A and $s=b$ at B. Then integrating (2) from A to B, we have

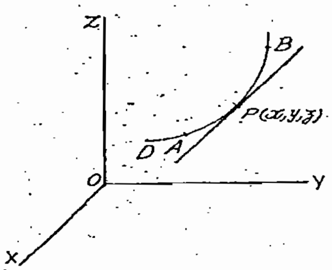
$$\int_0^b Pds = \int_u^0 -mvdv = -m\left[\tfrac{1}{2}v^2\right]_u^0 = \tfrac{1}{2}mu^2.$$

But $\int_0^b Pds$ is the work done against the resistance P while the body moves from A to B and so equal to the K.E. of the body at A.

Hence the K.E. of a body of mass m moving with velocity $u = \frac{1}{2}mu^2$.

Remark. If the mass m is measured in gms and the velocity u in cm./sec., then the K.E. is in ergs. If the mass m is measured in kgs. and the velocity u in m./sec., the K.E. is in joules.

§ 7. The work-energy principle. *The change in the kinetic energy of a particle during its motion from a position A to a position B is equal to the work done by the forces acting on the particle during that motion.*



Suppose a particle of mass m moves along any path under the action of any system of forces. Let X, Y, Z be the components of these forces along any three mutually perpendicular lines OX, OY, OZ taken as the co-ordinate axes. Suppose the velocity of the particle changes from v_1 to v_2 when it moves from A to B. Let $P(x, y, z)$ be the position of the particle at any time t where are $DP = s$, D being some fixed point on the path. The direction cosines of the tangent at P to the path in the sense of s increasing are dx/ds, dy/ds, dz/ds. Let v be the velocity of the particle at P. Then the expression for the tangential acceleration of the particle at P is $v\,(dv/ds)$, +ive in the direction of s increasing. Resolving the forces acting on the particle at P along the tangent at P, the tangential equation of motion of P is

$$mv\frac{dv}{ds} = X\frac{dx}{ds} + Y\frac{dy}{ds} + Z\frac{dz}{ds}$$

[By Newton's second law of motion]

$$\therefore\ mvdv = Xdx + Ydy + Zdz. \qquad \ldots(1)$$

Integrating both sides of (1) from A to B, we have

$$\int_{v_1}^{v_2} mv\,dv = \int_A^B (Xdx + Ydy + Zdz) \qquad \ldots(2)$$

Now $\int_{v_1}^{v_2} mv\,dv = m\left[\frac{v^2}{2}\right]_{v_1}^{v_2} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$

= K.E. of the particle at B − K.E. of the particle at A

= change in K.E. of the particle in moving from A to B.

Also $\int_A^B (Xdx + Ydy + Zdz)$ is the work done by the forces acting on the particle during its motion from A to B.

Hence from (2) we conclude that

the change in the K.E. = the work done by the forces.

This is known as the *principle of energy* or *the principle of work and energy.*

Remark. If a particle of mass m starts from rest and has velocity v after any time t, then by the principle of work and energy

$\frac{1}{2}mv^2 - \frac{1}{2}m.0^2$ = the work done by the forces acting on the particle during that time t.

Thus we can say that *the kinetic energy of a moving particle is the amount of work done by the forces acting on it giving it that motion, starting from rest.*

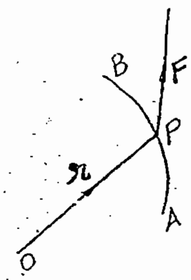
8. Conservative and non-conservative forces.

Conservative forces. Definition. *A force, is said to be conservative if the work done by it in displacing its point of application from one given point to another depends upon these points only and not upon the path followed.*

If a variable force P displaces its point of application from a point A to a point B, along a curve C, then the work W done by the force is given by

$$W = \int_A^B \mathbf{F}\cdot d\mathbf{r},$$

where the integration is to be performed along the curve C. The force $\mathbf{F}$ is conservative if and only if the value of the above integral does not depend upon the curve C.



Non-conservative forces. *The forces which are not conservative are called non-conservative.*

We shall give below (without proof) two characteristic properties of conservative forces and any of these can be taken as an equivalent definition of a system of conservative forces.

(i) *A force is conservative if and only if the work done by it on a particle as it makes a complete circuit (i.e., comes to the position that it started from) is zero.*

(ii) *A force* $\mathbf{F} = X\mathbf{i} + Y\mathbf{j} + Z\mathbf{k}$ *is conservative if and only if there exists a single valued function* $f(x, y, z)$ *such that*

$$\frac{\partial f}{\partial x} = X,\ \frac{\partial f}{\partial y} = Y,\ \frac{\partial f}{\partial z} = Z.$$

The function $f(x, y, z)$ *is called the* **potential function** *of the force* $\mathbf{F}$.

If a particle is displaced from the point $A(x_1, y_1, z_1)$ to the point $B(x_2, y_2, z_2)$ under such a force $\mathbf{F}$ along any curve C, then the work W done by $\mathbf{F}$ is given by

$$W = \int_A^B (Xdx + Ydy + Zdz)$$

$$= \int_A^B \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz\right)$$

$$= \int_A^B df = \Big[f(x, y, z)\Big]_A^B$$

$$= f(x_2, y_2, z_2) - f(x_1, y_1, z_1),$$

which obviously depends upon the points A and B and not upon the curve C.

Conservative forces do not change their character on account of any restraint while non-conservative forces change their character on account of extraneous circumstances. A few examples of the conservative forces are force of gravity, tension and normal reaction while a few examples of the non-conservative forces are force of friction and resistance of the air. Remember that a constant force $\mathbf{F}$ is always a conservative force and a central force $\mathbf{F}$ is also always a conservative force.

For instance suppose a particle is projected vertically upwards from a point O and after reaching a height h it comes back to the point of projection. Then the work done by gravity when the particle completes this circuit $= -mgh + mgh = 0$. Thus gravity is a conservative force.

Again consider a body put on a rough horizontal table. Let the frictional force be F. If the body is moved in a straight line from A to B, the work done by the force of friction F is $-F.AB$. Now if the body is moved back from B to A, the work done by the force of friction is $-F.AB$.

Thus the total work done in completing the circuit

$= -F.AB + (-F.AB)$

$= -2F.AB$ which is not zero.

Therefore frictional force is not conservative.

9. Potential Energy (P. E.). The potential energy of a body acted upon by a conservative system of forces, is the capacity of the body for doing work on account of its position. We may define it precisely as follows :

If a body is acted upon by a conservative system of forces, then its potential energy in any position is the amount of the work done by those forces in bringing the body from that position to some standard position.

For example the potential energy of a body of mass m placed at a height h above the ground is the amount of the work which its weight mg does when the body moves from this position to the ground which is usually supposed to be the standard position. Thus *for a body of mass m placed at a height h, potential energy* $= mgh$.

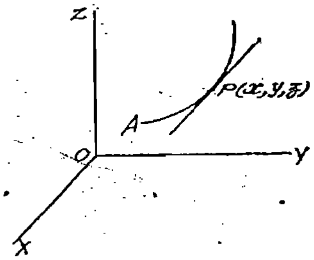
10. The principle of conservation of energy. *If a particle acted upon by a conservative system of forces moves along any path, the sum of its kinetic and potential energies remains constant.*

Suppose a particle of mass m moves along any path under the action of a system of conservative forces whose potential function is, say, $f(x, y, z)$.

Then $\dfrac{\partial f}{\partial x}=X,\ \dfrac{\partial f}{\partial y}=Y,\ \dfrac{\partial f}{\partial z}=Z,$ ...(1)

where X, Y, Z are the components of the forces along the co-ordinate axes OX, OY, OZ respectively.

Let $P(x, y, z)$ be the position of the particle at any time t, where arc $AP=s$, A being some fixed point on the path. The direction cosines of the tangent at P to the path in the sense of s increasing are $dx/ds,\ dy/ds,\ dz/ds$. Let v be the velocity of the particle at P. Then the tangential equation of motion of the particle at P is



$$mv\frac{dv}{ds}=X\frac{dx}{ds}+Y\frac{dy}{ds}+Z\frac{dz}{ds}$$

or $$mv\,dv=X\,dx+Y\,dy+Z\,dz.$$

Integrating both sides, we get

$\frac{1}{2}mv^2=\int(X\,dx+Y\,dy+Z\,dz)+C$, where C is a constant

$$=\int\left(\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz\right)+C \quad \text{[from (1)]}$$
$$=\int df+C=f(x, y, z)+C.$$

[Note that if $f(x, y, z)$ is a function of x, y, z, then from partial differentiation, we have

$$df=\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy+\frac{\partial f}{\partial z}dz.]$$

Now $\frac{1}{2}mv^2$ is the K.E. of the particle at the point P.

Thus the K.E. of the particle at $P=f(x, y, z)+C.$ ...(2)

Again the potential energy of the particle at $P(x, y, z)$ is equal to the work done by the conservative forces in moving the particle from P to some standard position, say, (x_1, y_1, z_1).

$$\therefore\ \text{P.E. at } P=\int_{(x,y,z)}^{(x_1,y_1,z_1)}(X\,dx+Y\,dy+Z\,dz)$$
$$=\int_{(x,y,z)}^{(x_1,y_1,z_1)}df=\Big[f(x, y, z)\Big]_{(x,y,z)}^{(x_1,y_1,z_1)}$$
$$=f(x_1, y_1, z_1)-f(x, y, z). \quad ...(3)$$

Adding (2) and (3), we have

K.E. at P+P.E. at $P=f(x_1, y_1, z_1)+C$,

which is constant because (x_1, y_1, z_1) is a fixed point.

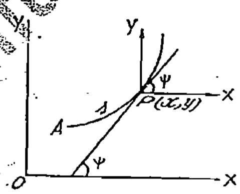
This proves the principle of conservation of energy.

11. The principle of conservation of energy for the motion in a plane. We have established the principle of work and energy and the principle of conservation of energy for the general motion in three dimensions. These principles can be similarly established, as special case, for the motion in two dimensions. We shall here establish the principle of conservation of energy for the motion in a plane.

If a particle acted upon by a conservative system of forces moves in a plane along any path, the sum of its kinetic and potential energies remains constant.

Suppose a particle of mass m moves in the plane XOY along any path under the action of a system of conservative forces whose potential function is, say, $f(x, y)$. Then

$$\frac{\partial f}{\partial x}=X,\ \frac{\partial f}{\partial y}=Y, \quad ...(1)$$



where X, Y are the components of the forces along the co-ordinate axes OX, OY respectively.

Let $P(x, y)$ be the position of the particle at any time t, where arc $AP=s$, A being some fixed point on the path. If the tangent at P to the path makes an angle ψ with OX, we have

$\cos\psi=dx/ds$ and $\sin\psi=dy/ds$.

Let v be the velocity of the particle at P. The tangential equation of motion of P is

$$mv\frac{dv}{ds}=X\cos\psi+Y\sin\psi=X\frac{dx}{ds}+Y\frac{dy}{ds}.$$

$\therefore\ mv\,dv=X\,dx+Y\,dy.$

Integrating both sides, we have

$\frac{1}{2}mv^2=\int(X\,dx+Y\,dy)+C$, where C is a constant

$$=\int\left(\frac{\partial f}{\partial x}dx+\frac{\partial f}{\partial y}dy\right)+C$$
$$=\int df+C=f(x, y)+C.$$

But $\frac{1}{2}mv^2$ is the kinetic energy of the particle at P.

$\therefore$ K.E. of the particle at $P=f(x, y)+C.$ ...(2)

Again the potential energy of the particle at $P(x, y)$ is equal to the work done by the conservative forces acting on the particle in moving it from $P(x, y)$ to some standard position, say, (x_1, y_1).

$$\therefore\ \text{P.E. at } P=\int_{(x,y)}^{(x_1,y_1)}(X\,dx+Y\,dy)$$
$$=\Big[f(x, y)\Big]_{(x,y)}^{(x_1,y_1)}$$
$$=f(x_1, y_1)-f(x, y). \quad ...(3)$$

Adding (2) and (3), we have

K.E. at P+P.E. at $P=f(x_1, y_1)+C$,

which is constant because (x_1, y_1) is a fixed point.

§ 12. The principle of conservation of linear momentum.

Momentum. Definition *If at any instant a particle of mass m moves with velocity $\mathbf{v}$, then the vector $m\mathbf{v}$ is called the momentum of the particle at that instant.* The direction of the momentum vector is obviously the same as that of the velocity vector.

If a particle of mass m grams moves in a straight line and its velocity at any instant is v cm./sec., then its momentum at that instant is mv gm.-cm./sec. and is in the direction of v.

The principle of conservation of linear momentum for a particle. *If the sum of the resolved parts of the forces acting on a particle in motion in any given direction is zero, then the resolved part of the momentum of the particle in that direction remains constant.*

Suppose a particle of mass m moves under the action of a force $\mathbf{F}$ whose resolved part in a given direction is zero. If $\mathbf{a}$ is the unit vector in the given direction, then the resolved part of $\mathbf{F}$ in the direction of $\mathbf{a}$ is $\mathbf{F}\cdot\mathbf{a}$. Thus it is given that $\mathbf{F}\cdot\mathbf{a}=0$.

Let $\mathbf{v}$ be the velocity of the particle at any time t. Then the momentum of the particle at that instant $=m\mathbf{v}$. The resolved part of $m\mathbf{v}$ in the direction of $\mathbf{a}$ is $m\mathbf{v}\cdot\mathbf{a}$. We have

$$\frac{d}{dt}(m\mathbf{v}\cdot\mathbf{a})=m\frac{d}{dt}(\mathbf{v}\cdot\mathbf{a}),\ \text{if } m \text{ is constant}$$
$$=m\left(\frac{d\mathbf{v}}{dt}\cdot\mathbf{a}+\mathbf{v}\cdot 0\right) \quad [\because \mathbf{a} \text{ is a constant vector}]$$
$$=\left(m\frac{d\mathbf{v}}{dt}\right)\cdot\mathbf{a}$$
$$=\mathbf{F}\cdot\mathbf{a} \quad [\because \text{ by Newton's second law of motion, } m\,(d\mathbf{v}/dt)=\mathbf{F}]$$
$$=0. \quad [\because \mathbf{F}\cdot\mathbf{a}=0]$$

Thus $\dfrac{d}{dt}(m\mathbf{v}\cdot\mathbf{a})=0$ and so $m\mathbf{v}\cdot\mathbf{a}$ is constant.

Remark. The principle of conservation of linear momentum also holds good for a system of particles. *Thus if the sum of the resolved parts of the forces acting on a system of particles in any given direction is zero, then the resolved part of the total momentum of the system in that direction remains constant.*

13. Impulse. Definition. When the force is constant. *If a constant force $\mathbf{F}$ acts on a particle during the time interval (t_0, t_1), the vector $\mathbf{I}=(t_1-t_0)\,\mathbf{F}$ is called the impulse of the force $\mathbf{F}$ during the interval (t_0, t_1).* Obviously, here direction of the impulse vector $\mathbf{I}$ is the same as that of the force $\mathbf{F}$.

When the force is variable. *If a variable force $\mathbf{F}(t)$ acts on a particle during the time interval (t_0, t_1), then the vector*

$$\mathbf{I}=\int_{t_0}^{t_1}\mathbf{F}(t)\,dt$$

is called the impulse of the force $\mathbf{F}(t)$ during the interval (t_0, t_1). Here the direction of the vector $\mathbf{I}$ is that of the time average of $\mathbf{F}$ over the interval (t_0, t_1).

Impulse-Momentum principle for a particle. *The change of momentum vector of a particle during a time interval is equal to the net impulse vector of the external forces during this interval.*

Let a particle of mass m move under the action of an external force $\mathbf{F}$. Let $\mathbf{v}$ be the velocity of the particle at the beginning of the time interval (t_0, t_1) and $\mathbf{v}_1$ be the velocity at the end of this time interval. If $\mathbf{v}$ is the velocity of the particle at any time t, then by Newton's second law of motion

$$m\frac{d\mathbf{v}}{dt}=\mathbf{F}. \quad ...(1)$$

If $\mathbf{I}$ is the impulse vector of the force $\mathbf{F}$ during the time interval (t_0, t_1), then

$$\mathbf{I}=\int_{t_0}^{t_1}\mathbf{F}\,dt=\int_{t_0}^{t_1}m\frac{d\mathbf{v}}{dt}\,dt \quad \text{[from (1)]}$$
$$=m\int_{t_0}^{t_1}d\mathbf{v}=m\Big[\mathbf{v}\Big]_{t_0}^{t_1}=m\,(\mathbf{v}_1-\mathbf{v}_0)$$
$$=m\mathbf{v}_1-m\mathbf{v}_0$$

= change in the momentum vector in the interval (t_0, t_1).

The equation $\mathbf{I}=m\mathbf{v}_1-m\mathbf{v}_0$ is known as the impulse-momentum principle. It gives us an exact relation between the impulse of a force and the change in motion produced.

Rectilinear motion with constant acceleration. Suppose a particle of mass m moves in a straight line under the action of a constant force F producing a constant acceleration f. Let u be the initial velocity of the particle and v be its velocity after time t.

The impulse of the force F during the time t

= the product of the force F and the time t

$= Ft = mft$ [$\because$ $F = mf$, by Newton's second law of motion]

$= m(v - u)$ [$\because$ $v = u + ft$]

$= mv - mu$

= the change of the momentum of the particle in time t.

If the interval t is indefinitely small, but u, v are finite, *i.e.*, change in momentum is finite, then certainly the force F must be indefinitely large. Such a force is called an impulsive force.

Thus a very large force acting for a very short period of time is called an impulsive force. For example, the blow by a hammer on a peg is an impulsive force. An impulsive force is measured by the change in the momentum of the body produced by it. The students should distinguish carefully between *impulse* and *impulsive force*.

Units of Impulse. The equivalence of impulse and the change in momentum enables us to adopt the same units for impulse as those used for momentum. Thus the absolute units of impulse are :

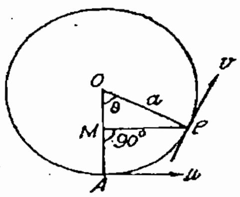
In C. G. S. system, gm. cm./sec.

In M. K. S. system, kg. m./sec.

In F. P. S. system, lb.-ft./sec.

Illustrative Examples

Ex. 8. *A bead of mass m is projected with velocity u along the inside of a smooth fixed vertical circle of radius a from the lowest point A. Use the principle of work and energy to find the velocity of the bead when it is at B, where $\angle AOB = \theta$, O being the centre of the circle.*



Sol. Let v be the velocity of the bead when it is at B. Then the change in the K. E. of the bead in moving from A to B

$= \frac{1}{2}mv^2 - \frac{1}{2}mu^2$.

The only force that does work in this displacement is the weight mg. The work done by the weight mg of the bead during its displacement from A to $B = -mg.AM = -mg.(OA - OM)$

$= -mg.(a - a\cos\theta)$

$= -mga(1 - \cos\theta)$.

Now by the principle of work and energy, the change in the kinetic energy = work done by the forces.

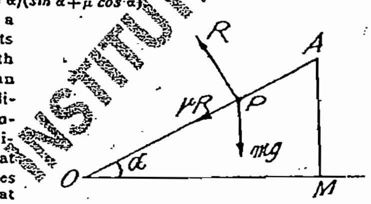
$\therefore$ $\frac{1}{2}mv^2 - \frac{1}{2}mu^2 = -mga(1 - \cos\theta)$

or $v^2 - u^2 = -2ga(1 - \cos\theta)$

or $v^2 = u^2 - 2ag(1 - \cos\theta)$,

which gives the velocity of the bead at B.

Ex. 9. *A particle is set moving with kinetic energy E straight up an inclined plane of inclination α and coefficient of friction μ. Prove that the work done against friction before the particle comes to rest is* $E\mu\cos\alpha/(\sin\alpha + \mu\cos\alpha)$.



Sol. Suppose a particle of mass m starts moving from O with kinetic energy E up an inclined plane of inclination α to the horizontal. Let P be the position of the particle at any time t. The forces acting on the particle at P are (i) its weight mg, which has component $mg\sin\alpha$ down the plane and $mg\cos\alpha$ perpendicular to the plane, (ii) the normal reaction R of the plane and (iii) the force of friction μR acting down the plane because its direction is opposite to the direction of motion.

Since there is no motion of the particle perpendicular to the inclined plane, therefore $R = mg\cos\alpha$.

$\therefore$ the force of friction $= \mu R = \mu mg\cos\alpha$.

Suppose the particle comes to rest at A where $OA = x$.

The only forces which work during the displacement of the particle from O to A are its weight and the force of friction. The work done by the weight $= -mg.AM = -mgx\sin\alpha$.

The work done by the force of friction

$= -(\mu mg\cos\alpha).x = -\mu mgx\cos\alpha$. ...(1)

Since the kinetic energy of the particle at O is E and at A is zero, therefore by the principle of work and energy during the motion of the particle from O to A

change in K. E. = work done by the forces

i.e., $0 - E = -mgx\sin\alpha - \mu mgx\cos\alpha$

or $E = xmg(\sin\alpha + \mu\cos\alpha)$

or $x = \dfrac{E}{mg(\sin\alpha + \mu\cos\alpha)}$.

Putting this value of x in (1), the work done by the force of friction

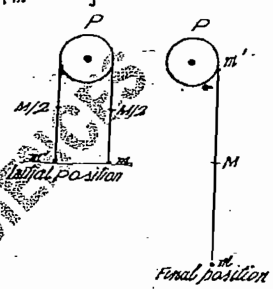
$$= -\mu mg\cos\alpha \cdot \frac{E}{mg(\sin\alpha + \mu\cos\alpha)}$$

$$= \frac{-E\mu\cos\alpha}{\sin\alpha + \mu\cos\alpha}.$$

Hence the work done against friction $= \dfrac{E\mu\cos\alpha}{\sin\alpha + \mu\cos\alpha}$.

Ex. 10. *A uniform string of mass M and length $2a$ is placed symmetrically over a smooth peg and has particles of masses m and m' attached to its ends ($m > m'$). Show that the string runs off the peg when its velocity is*

$$\sqrt{\left[\frac{M + 2(m - m')}{M + m + m'}\,ag\right]}.$$



Sol. In the initial position depth of the centre of gravity of the system from the peg P

$$= \frac{M.\frac{1}{2}a + ma + m'a}{M + m + m'}$$

$$= \frac{a}{2} \cdot \frac{M + 2m + 2m'}{M + m + m'}.$$

In the final position depth of the centre of gravity of the system from the peg P

$$= \frac{M.a + m.2a + m'.0}{M + m + m'}$$

$$= a\,\frac{M + 2m}{M + m + m'}$$

$\therefore$ displacement in the position of centre of gravity

$$= a\,\frac{M + 2m}{M + m + m'} - \frac{a}{2}\,\frac{M + 2m + 2m'}{M + m + m'} = \frac{a}{2}\,\frac{M + 2m - 2m'}{M + m + m'}.$$

The initial velocity of the system is zero; and let the final velocity be v.

By the principle of work and energy, we have

change in K. E. = work done by the forces.

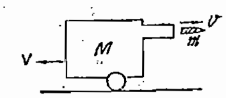
i.e., $\frac{1}{2}(M + m + m')v^2 - 0 = (M + m + m')\,g.\frac{a}{2}\,\frac{M + 2(m - m')}{M + m + m'}$

or $v^2 = \dfrac{M + 2(m - m')}{M + m + m'}\,ag$

or $v = \sqrt{\left[\dfrac{M + 2(m - m')}{M + m + m'}\,ag\right]}$.

This gives the velocity of the string when it runs off the peg.

Ex. 11. *A shot of mass m is fired horizontally from a gun of mass M with velocity u relative to the gun ; show that the actual velocities of the shot and the gun are $\dfrac{Mu}{M+m}$ and $\dfrac{mu}{M+m}$ respectively, and that their kinetic energies are inversely proportional to their masses.*



Sol. Let v be the actual velocity of the shot and V be the actual velocity with which the gun recoils.

Then the velocity of the shot relative to the gun $= v + V$.

But according to the question the velocity of the shot relative to the gun is u.

$\therefore$ $u = v + V$. ...(1)

Since in the horizontal direction no external force acts on the system, therefore by the principle of conservation of linear momentum applied in the horizontal direction

momentum before firing = momentum after firing

i.e., $0 = mv - MV$

i.e., $mv = MV$. ...(2)

From (2), $v = \dfrac{M}{m}V$. Substituting this value of v in (1), we have

$$u = \frac{M}{m}V + V = \left(\frac{M}{m} + 1\right)V = \frac{M + m}{m}V.$$

$\therefore$ $V = \dfrac{mu}{M + m}$ and so $v = \dfrac{M}{m}V = \dfrac{M}{m}\cdot\dfrac{mu}{M + m} = \dfrac{Mu}{M + m}$

$\therefore$ the actual velocity of the shot $= v = \dfrac{Mu}{M + m}$,

and the actual velocity of the gun $= V = \dfrac{mu}{M + m}$.

Again $\dfrac{\text{the K. E. of the shot}}{\text{the K. E. of the gun}} = \dfrac{\frac{1}{2}mv^2}{\frac{1}{2}MV^2} = \dfrac{m \cdot v^2}{MV^2}$

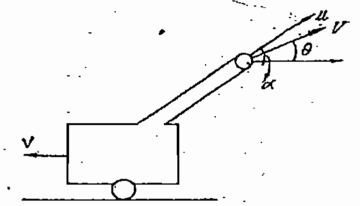
$$= \frac{m}{M}\cdot\frac{M^2}{m^2} \quad \left[\because \text{ from (2), } \frac{v}{V}=\frac{M}{m}\right]$$

$$= \frac{M}{m} = \frac{\text{the mass of the gun}}{\text{the mass of the shot}}.$$

Hence their kinetic energies are inversely proportional to their masses.

Ex. 12. *A gun is mounted on a gun carriage, movable on a smooth horizontal plane, and the gun is elevated at an angle α to the horizon. A shot is fired and leaves the gun in a direction inclined at an angle θ to the horizon. If the mass of the gun and its carriage be n times that of the shot, prove that*

$$\tan\theta = \left(1+\frac{1}{n}\right)\tan\alpha.$$



Sol. Let v be the actual velocity with which the shot leaves the gun and V the actual velocity with which the gun carriage recoils horizontally. According to the question the direction of v makes an angle θ with the horizontal.

The velocity of the shot relative to the gun in the horizontal direction $= v\cos\theta + V$

and the velocity of the shot relative to the gun in the vertical direction $= v\sin\theta$.

If u be the velocity of the shot relative to the gun, then the direction of u makes an angle α with the horizontal.

$$\therefore \quad \tan\alpha = \frac{\text{vertical component of } u}{\text{horizontal component of } u} = \frac{v\sin\theta}{v\cos\theta + V} \quad ...(1)$$

Now if the mass of the shot is m, then the mass of the gun and the carriage is nm.

Since in the horizontal direction no external force acts on the system, therefore applying the principle of conservation of linear momentum in the horizontal direction, we have

$$mv\cos\theta - nmV = 0$$

i.e., $\quad V = (v\cos\theta)/n.$

Substituting this value of V in (1), we have

$$\tan\alpha = \frac{v\sin\theta}{v\cos\theta + (v\cos\theta)/n} = \frac{v\sin\theta}{v\cos\theta\,(1+1/n)}$$

$$= \frac{\tan\theta}{1+1/n}.$$

$$\therefore \quad \tan\theta = \left(1+\frac{1}{n}\right)\tan\alpha.$$

Ex. 13. *A shell of mass m is fired from a gun of mass M which can recoil freely on a horizontal base, and the elevation of the gun is α. Prove that the inclination of the path of the shell to the horizon at the time of projection is*

$$\tan^{-1}\left\{\left(1+\frac{m}{M}\right)\tan\alpha\right\}.$$

Prove also that the energy of the shell on leaving the gun is to that of the gun as $[M^2+(m+M)^2\tan^2\alpha] : mM$, assuming that none of the energy of the explosion is lost.

Sol. Let v be the actual velocity and θ the actual elevation of the shell on leaving the gun. Suppose V is the actual velocity with which the gun recoils horizontally.

The velocity of the shell relative to the gun in the horizontal direction $= v\cos\theta + V$

and the velocity of the shell relative to the gun in the vertical direction $= v\sin\theta$.

Since the inclination of the velocity of the shell relative to the gun to the horizontal is equal to the elevation α of the gun, therefore

$$\tan\alpha = \frac{v\sin\theta}{v\cos\theta + V}. \quad ...(1)$$

Applying the principle of conservation of linear momentum in the horizontal direction, we have

momentum after firing = momentum before firing

i.e., $\quad mv\cos\theta - MV = 0$

i.e., $\quad mv\cos\theta = MV. \quad ...(2)$

Substituting the value of V from (2) in (1), we have

$$\tan\alpha = \frac{v\sin\theta}{v\cos\theta + (mv\cos\theta)/M} = \frac{v\sin\theta}{v\cos\theta\,(1+m/M)}$$

$$= \frac{\tan\theta}{1+m/M}.$$

$$\therefore \quad \tan\theta = \left(1+\frac{m}{M}\right)\tan\alpha$$

or $\quad \theta = \tan^{-1}\left\{\left(1+\frac{m}{M}\right)\tan\alpha\right\},$

which proves the first result.

Squaring both sides of (2), we have

$$m^2v^2\cos^2\theta = M^2V^2.$$

$$\therefore \quad \frac{mv^2}{MV^2} = \frac{M}{m}\sec^2\theta$$

or $\quad \dfrac{\frac{1}{2}mv^2}{\frac{1}{2}MV^2} = \dfrac{M}{m}(1+\tan^2\theta) = \dfrac{M}{m}\left\{1+\left(1+\dfrac{m}{M}\right)^2\tan^2\alpha\right\}.$

Hence on leaving the gun, we have

$$\frac{\text{kinetic energy of the shell}}{\text{kinetic energy of the gun}} = \frac{\frac{1}{2}mv^2}{\frac{1}{2}MV^2}$$

$$= \frac{M}{mM^2}\left\{M^2+(m+M)^2\tan^2\alpha\right\} = \frac{M^2+(m+M)^2\tan^2\alpha}{mM},$$

which proves the second result.

Ex. 14. *Assuming that in a canon the force on the ball depends only on the volume of gas generated by the gun powder, show that the ratio of the final velocity of the ball when the gun is free to recoil to its velocity when the gun is fixed is $\sqrt{\left(\frac{M}{M+m}\right)}$, where M and m are the masses of the gun and the ball respectively.*

Sol. Let E be the energy released by the explosion.

When the gun is free to recoil let v be the velocity of the ball and u the velocity with which the gun recoils. In this case the energy released is $\quad E = \frac{1}{2}mv^2 + \frac{1}{2}Mu^2. \quad ...(1)$

Also by the principle of conservation of linear momentum, we have $\quad mv - Mu = 0 \quad$ *i.e.,* $\quad mv = Mu. \quad ...(2)$

Again when the gun is fixed, let V be the velocity of the ball. The energy released is then

$$E = \tfrac{1}{2}mV^2. \quad ...(3)$$

From (1) and (3), on eliminating E, we get

$$mv^2 + Mu^2 = mV^2$$

or $\quad mv^2 + M\dfrac{m^2}{M^2}v^2 = mV^2$

or $\quad mv^2\left(1+\dfrac{m}{M}\right) = mV^2$

or $\quad v^2\left(\dfrac{M+m}{M}\right) = V^2$

or $\quad \dfrac{v^2}{V^2} = \dfrac{M}{M+m} \quad$ or $\quad \dfrac{v}{V} = \sqrt{\left(\dfrac{M}{M+m}\right)}.$

Ex. 15. *A gun of mass M fires a shell of mass m horizontally, and the energy of explosion is such as would be sufficient to project the shot vertically to a height h. Show that the velocity of recoil of the gun is* $\left[\dfrac{2m^2gh}{M(M+m)}\right]^{1/2}.$

Sol. Let E be the energy of the explosion. Since E is just sufficient to project a mass m vertically to a height h, therefore $E = \frac{1}{2}mu^2$; where u is the vertical velocity of projection just sufficient to raise a particle to a height h.

But for such a velocity of projection u, we have

$0 = u^2 - 2gh \quad$ *i.e.,* $\quad u^2 = 2gh.$

$\therefore \quad E = \frac{1}{2}m.2gh = mgh. \quad ...(1)$

When the shell is fired horizontally from the gun, let v be the velocity of the shell and V the velocity with which the gun recoils. We then have $\quad E = \frac{1}{2}mv^2 + \frac{1}{2}MV^2. \quad ...(2)$

Also by the principle of conservation of linear momentum, we have $\quad mv - MV = 0 \quad$ *i.e.,* $\quad mv = MV. \quad ...(3)$

From (1) and (2), we have equating the two values of E

$$mgh = \tfrac{1}{2}mv^2 + \tfrac{1}{2}MV^2$$

$$= \tfrac{1}{2}m.\frac{M^2V^2}{m^2} + \tfrac{1}{2}MV^2 \quad \text{[substituting for } v \text{ from (3)]}$$

$$= \tfrac{1}{2}MV^2\left(\frac{M}{m}+1\right) = \tfrac{1}{2}MV^2\,\frac{M+m}{m}.$$

$$\therefore \quad V^2 = \frac{2m^2gh}{M(M+m)} \quad \text{or} \quad V = \left[\frac{2m^2gh}{M(M+m)}\right]^{1/2}.$$

Ex. 16. *A shell of mass m is projected from a gun of mass M by an explosion which generates kinetic energy E. Prove that the initial velocity of the shell is $\sqrt{\left[\frac{2EM}{m(M+m)}\right]}$, it being assumed that at the instant of explosion the gun is free to recoil.*

Sol. Let u be the velocity of the shell while leaving the gun and v the velocity with which the gun recoils. Then we have

$$E = \tfrac{1}{2}mu^2 + \tfrac{1}{2}Mv^2. \quad ...(1)$$

Also by the principle of conservation of linear momentum, we have $\quad mu - Mv = 0 \quad$ *i.e.,* $\quad mu = Mv. \quad ...(2)$

To find u we have to eliminate v from (1) and (2).

From (2), we have $v = mu/M$. Putting this value of v in (1), we get

$$E = \tfrac{1}{2}mu^2 + \tfrac{1}{2}M.\frac{m^2u^2}{M^2} = \tfrac{1}{2}mu^2\left(1+\frac{m}{M}\right) = \frac{mu^2(M+m)}{2M}$$

$\therefore\quad u^2=\frac{2EM}{m(M+m)}$ or $u=\sqrt{\left[\frac{2EM}{m(M+m)}\right]}$.

Ex. 17. *A body of mass (m_1+m_2) moving in a straight line is split into two parts of masses m_1 and m_2 by an internal explosion which generates kinetic energy E. Show that if after the explosion the two parts move in the same line as before, their relative speed is*

$$\sqrt{\left[\frac{2E(m_1+m_2)}{m_1m_2}\right]}.$$

Sol. Let u be the velocity of the body of mass (m_1+m_2) before explosion and u_1 and u_2 the velocities of parts m_1 and m_2 after explosion. Then by the principle of conservation of linear momentum, we have

$$m_1u_1+m_2u_2=(m_1+m_2)u. \qquad \ldots(1)$$

Also K.E. after splitting = K.E. before splitting.

$$\therefore\quad \tfrac{1}{2}m_1u_1^2+\tfrac{1}{2}m_2u_2^2=E+\tfrac{1}{2}(m_1+m_2)u^2$$

or $\quad m_1u_1^2+m_2u_2^2=2E+(m_1+m_2)u^2. \qquad \ldots(2)$

We are to find the relative velocity which is equal to the difference of u_1 and u_2.

Multiplying (2) by (m_1+m_2) and then subtracting from it the square of (1), we get

$$(m_1+m_2)(m_1u_1^2+m_2u_2^2)-(m_1u_1+m_2u_2)^2=2E(m_1+m_2)$$

or $\quad m_1m_2(u_1^2+u_2^2-2u_1u_2)=2E(m_1+m_2)$

or $\quad m_1m_2(u_1-u_2)^2=2E(m_1+m_2)$

or $\quad (u_1-u_2)^2=\frac{2E(m_1+m_2)}{m_1m_2}$.

Hence $\quad u_1-u_2=\sqrt{\left[\frac{2E(m_1+m_2)}{m_1m_2}\right]}$.

It gives the relative velocity of m_1 with respect to m_2 after explosion.

Ex. 18. *A shell lying in a straight smooth horizontal tube suddenly breaks into two portions of masses m_1 and m_2. If s is the distance apart, in the tube, of the masses after a time t, show that the work done by the explosion is*

$$\tfrac{1}{2}\,\frac{m_1m_2}{m_1+m_2}\cdot\frac{s^2}{t^2}.$$

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Sol. Since the shell is lying in the tube, its velocity before explosion is zero. Let u_1 and u_2 be the velocities, of the masses m_1 and m_2 respectively after explosion. Then the relative velocity of the masses after explosion is u_1+u_2. Since the tube is smooth and horizontal, u_1+u_2 will remain constant.

$$\therefore\quad (u_1+u_2)\,t=s. \qquad \ldots(1)$$

Also by the principle of conservation of linear momentum, we have

$$m_1u_1-m_2u_2=0$$

i.e., $\quad m_1u_1=m_2u_2. \qquad \ldots(2)$

Substituting for u_2 from (2) in (1), we get

$$\left(u_1+\frac{m_1u_1}{m_2}\right)t=s$$

or $\quad u_1\left(\frac{m_1+m_2}{m_2}\right)t=s$

or $\quad u_1=\frac{m_2s}{(m_1+m_2)\,t}$.

$$\therefore\quad u_2=\frac{m_1}{m_2}u_1=\frac{m_1}{m_2}\cdot\frac{m_2s}{(m_1+m_2)\,t}=\frac{m_1s}{(m_1+m_2)\,t}.$$

Now the work done by the explosion

= the kinetic energy released due to the explosion

$=\tfrac{1}{2}m_1u_1^2+\tfrac{1}{2}m_2u_2^2$

$=\tfrac{1}{2}m_1\cdot\frac{m_2^2s^2}{(m_1+m_2)^2t^2}+\tfrac{1}{2}m_2\frac{m_1^2s^2}{(m_1+m_2)^2t^2}$

$=\tfrac{1}{2}\frac{s^2}{t^2}\cdot\frac{1}{(m_1+m_2)^2}(m_1m_2^2+m_2m_1^2)$

$=\tfrac{1}{2}\frac{s^2}{t^2}\cdot\frac{m_1m_2(m_1+m_2)}{(m_1+m_2)^2}=\tfrac{1}{2}\,\frac{m_1m_2}{m_1+m_2}\cdot\frac{s^2}{t^2}$.

Ex. 19. *A shell is moving with velocity u in the line AB. An internal explosion, which generates an energy E, breaks it into two fragments of masses m_1 and m_2 which move in the line AB. Show that their velocities are*

$$u+\sqrt{\left[\frac{2Em_2}{m_1(m_1+m_2)}\right]}\ \text{ and }\ u-\sqrt{\left[\frac{2Em_1}{m_2(m_1+m_2)}\right]}.$$

Sol. Let u_1 and u_2 be the velocities of the masses m_1 and m_2 respectively after the explosion. By the principle of conservation of linear momentum, we have

$$(m_1+m_2)\,u=m_1u_1+m_2u_2. \qquad \ldots(1)$$

Now the energy before explosion is $\tfrac{1}{2}(m_1+m_2)\,u^2$ and E is the energy due to explosion. Also the total energy after explosion is, $(\tfrac{1}{2}m_1u_1^2+\tfrac{1}{2}m_2u_2^2)$.

Since there has been no dissipation of energy, therefore by the principle of conservation of mechanical energy, we have

$$\tfrac{1}{2}(m_1+m_2)\,u^2+E=\tfrac{1}{2}m_1u_1^2+\tfrac{1}{2}m_2u_2^2. \qquad \ldots(2)$$

It is easy to observe that for all values of x

$$u_1=u+\frac{x}{m_1}\ \text{ and }\ u_2=u-\frac{x}{m_2} \qquad \ldots(3)$$

satisfy the equation (1). In order that these values of u_1 and u_2 may also satisfy the equation (2), we should have

$$\tfrac{1}{2}(m_1+m_2)\,u^2+E=\tfrac{1}{2}m_1\left(u+\frac{x}{m_1}\right)^2+\tfrac{1}{2}m_2\left(u-\frac{x}{m_2}\right)^2.$$

or $\quad (m_1+m_2)\,u^2+2E=m_1\left(u^2+\frac{2xu}{m_1}+\frac{x^2}{m_1^2}\right)+m_2\left(u^2-\frac{2xu}{m_2}+\frac{x^2}{m_2^2}\right)$

or $\quad 2E=x^2\left(\frac{1}{m_1}+\frac{1}{m_2}\right)$, the other terms cancelling one another

or $\quad 2E=x^2\frac{(m_1+m_2)}{m_1m_2}$ or $x^2=\frac{2Em_1m_2}{m_1+m_2}$

or $\quad x=\sqrt{\left(\frac{2Em_1m_2}{m_1+m_2}\right)}$.

Putting this value of x in (3), we get

$$u_1=u+\sqrt{\left[\frac{2Em_2}{m_1(m_1+m_2)}\right]}\ \text{ and }\ u_2=u-\sqrt{\left[\frac{2Em_1}{m_2(m_1+m_2)}\right]}.$$

Ex. 20. *A shell of mass M is moving with velocity V. An internal explosion generates an amount of energy E and breaks the shell into two portions whose masses are in the ratio $m_1 : m_2$. The fragments continue to move in the original line of motion of the shell. Show that their velocities are*

$$V+\sqrt{\left(\frac{2m_2E}{m_1M}\right)}\ \text{ and }\ V-\sqrt{\left(\frac{2m_1E}{m_2M}\right)}.$$

[Lucknow 1980; Rohilkhand 80]

Sol. Since the whole mass M is divided in the ratio $m_1 : m_2$, therefore masses of the fragments are

$$\frac{m_1M}{m_1+m_2}\ \text{ and }\ \frac{m_2M}{m_1+m_2}.$$

Now proceed as in Ex. 19.

Ex. 21. *A shot of mass m fired horizontally penetrates a thickness s of a fixed plate of mass M; prove that if M is free to move the thickness penetrated is $Ms/(M+m)$.*

Sol. Let u be the striking velocity of the shot and P be the force of resistance offered by the plate assumed to be uniform.

When the plate is fixed the velocity of the shot reduces to zero after penetrating a thickness s. During the motion of the shot the change in the K.E. of the shot $=0-\tfrac{1}{2}mu^2=-\tfrac{1}{2}mu^2$ and the work done by the force of resistance $=-Ps$. By the principle of work and energy, we have

change in K.E. = work done by the forces.

$$\therefore\quad -\tfrac{1}{2}mu^2=-Ps\ \text{ or }\ \tfrac{1}{2}mu^2=Ps. \qquad \ldots(1)$$

Again consider the case when the plate is free to move. In this case let x be the thickness penetrated and V be the common velocity of the shot and the plate when the penetration ceases. By the principle of work and energy applied to the shot and the plate considered together as one system, we have

$$\tfrac{1}{2}(m+M)\,V^2-\tfrac{1}{2}mu^2=-Px$$

or $\quad \tfrac{1}{2}mu^2-\tfrac{1}{2}(m+M)\,V^2=Px. \qquad \ldots(2)$

Also in this case during the time of impact the resultant horizontal force on the whole system is zero because the mutual impulsive action and reaction between the shot and the plate are equal and opposite. Therefore by the principle of conservation of linear momentum, we have

momentum before impact = momentum after impact

i.e., $\quad mu=(m+M)\,V. \qquad \ldots(3)$

Dividing (2) by (1), we get

$$\frac{x}{s}=\frac{mu^2-(m+M)\,V^2}{mu^2}$$

$$=\frac{mu^2-(m+M)\cdot\dfrac{m^2u^2}{(m+M)^2}}{mu^2}, \text{ substituting for } V \text{ from (3)}$$

$$=\frac{mu^2-\dfrac{m^2u^2}{m+M}}{mu^2}=1-\frac{m}{m+M}=\frac{M}{m+M}.$$

$\therefore\quad x=\frac{Ms}{M+m}$, which proves the required result.

Ex. 22. *If a shot of mass m striking a fixed metal plate with velocity u, penetrates it through a distance a, show that it will completely pierce through a plate free to move, of mass M and thickness b, if $b<\frac{Ma}{m+M}$, the resistance being supposed uniform.*

Sol. When the plate is free to move let x be the distance penetrated. Then proceeding as in Ex. 21, we have

$$x=\frac{Ma}{M+m}.$$

Since the thickness of the plate is b, therefore the shot will completely pierce through if

$$b<\frac{Ma}{M+m}.$$

Ex. 23. *A block of mass M rests on a smooth horizontal table and a bullet of mass m is fired into it. The penetration of the bullet is opposed by a constant resisting force. If the experiment is repeated with the block firmly fixed, show that the depth of penetration of the bullet and the time which elapses before the bullet is at rest relatively to the block are in each case increased in the ratio* $\left(1+\frac{m}{M}\right):1.$

Sol. Let u be the striking velocity of the bullet and P be the force of resistance offered by the block assumed to be uniform.

Case I. When the block is fixed. In this case let s be the thickness penetrated and t the time that elapses when the penetration stops.

By the principle of work and energy, we have

$$0-\tfrac{1}{2}mu^2=-Ps \quad i.e., \quad \tfrac{1}{2}mu^2=Ps. \qquad \ldots(1)$$

Also by the impulse-momentum principle, we have

$$0-mu=-Pt \quad i.e. \quad mu=Pt. \qquad \ldots(2)$$

Case II. When the block is free to move. In this case let s' be the thickness penetrated, t' the time taken when the penetration ceases and V the common velocity of the bullet and the block at the end of the penetration. In this case, we have

$$(m+M)V=mu, \qquad \ldots(3)$$

$$\tfrac{1}{2}mu^2-\tfrac{1}{2}(m+M)V^2=Ps', \qquad \ldots(4)$$

and $$mu-mV=Pt'. \qquad \ldots(5)$$

The equation (3) has been written by applying the principle of conservation of momentum to the impact of the bullet and the block, the equation (4) has been obtained by applying the work-energy principle to the motion of the bullet and block considered together and the equation (5) has been obtained by applying the impulse-momentum principle to the motion of the bullet only.

Dividing (1) by (4), we get

$$\frac{s}{s'}=\frac{mu^2}{mu^2-(m+M)V^2}=\frac{mu^2}{mu^2-(m+M)\cdot\frac{m^2u^2}{(m+M)^2}} \quad \text{[from (3)]}$$

$$=\frac{1}{1-\frac{m}{m+M}}, \text{ dividing the Nr. and Dr. each by } mu^2$$

$$=\frac{m+M}{M}=1+\frac{m}{M}.$$

Thus $s:s'::\left(1+\frac{m}{M}\right):1$. This proves one result.

Again dividing (2) by (5), we have

$$\frac{t}{t'}=\frac{mu}{mu-mV}=\frac{mu}{mu-m\cdot\frac{mu}{m+M}}, \text{ substituting for } V \text{ from (3)}$$

$$=\frac{1}{1-\frac{m}{m+M}}=\frac{m+M}{M}=1+\frac{m}{M}.$$

Thus $t:t'::\left(1+\frac{m}{M}\right):1$. This proves the other result.

Ex. 24. *A bullet of mass m moving with a velocity u strikes a block of mass M, which is free to move in the direction of the motion of the bullet, and is embedded in it. Show that a portion $M/(M+m)$ of the kinetic energy is lost. If the block is afterwards struck by an equal bullet moving in the same direction with the same velocity, show that there is a further loss of kinetic energy equal to*

$$\frac{M^2mu^2}{2(M+2m)(M+m)}.$$

Sol. Let v be the velocity of the block after the first bullet strikes it and is embedded in it. Then by the principle of conservation of momentum, we have

$$(m+M)v=mu. \qquad \ldots(1)$$

Loss of K.E. $=\frac{1}{2}mu^2-\frac{1}{2}(m+M)v^2$

$$=\tfrac{1}{2}mu^2-\tfrac{1}{2}(m+M)\cdot\frac{m^2u^2}{(m+M)^2}, \text{ substituting for } v \text{ from (1)}$$

$$=\tfrac{1}{2}mu^2\cdot\left[1-\frac{m}{m+M}\right]=\tfrac{1}{2}mu^2\cdot\frac{M}{m+M}$$

$$=\frac{M}{m+M}\cdot\text{(K.E. before striking).}$$

Thus the fraction of K.E. lost $=\frac{M}{m+M}$.

Again let V be the velocity of the block after the second bullet strikes it. Then

$$(2m+M)V=(m+M)v+mu=2mu. \qquad \ldots(2)$$

[∵ from (1), $(m+M)v=mu$]

∴ further loss of K.E. $=\frac{1}{2}(m+M)v^2+\frac{1}{2}mu^2-\frac{1}{2}(2m+M)V^2$

$$=\tfrac{1}{2}(m+M)\cdot\frac{m^2u^2}{(m+M)^2}+\tfrac{1}{2}mu^2-\tfrac{1}{2}(2m+M)\cdot\frac{4m^2u^2}{(2m+M)^2}$$

[substituting for v and V from (1) and (2)]

$$=\frac{M^2mu^2}{2(M+2m)(M+m)}.$$

Ex. 25. *A hammer of mass M lbs. falls freely from a height h feet on the top of an inelastic pile of mass m lbs. which is driven into the ground a distance a feet. Assuming that the resistance of the ground is constant, find its value and show that the time during which the pile is in motion is given by* $\frac{a(M+m)}{M}\left(\frac{2}{gh}\right)^{1/2}$. *Find also what fraction of kinetic energy is lost by impact.*

Sol. Let u ft./sec. be the velocity of the hammer just before impact with the pile. Then

$$u=\sqrt{(2gh)}. \qquad \ldots(1)$$

Since the pile is inelastic, therefore after impact the hammer will not rebound and the hammer and the pile will begin to move together as one body, say, with velocity v ft./sec.

By the principle of conservation of momentum, we have

$$Mu+m.0=(M+m)v,$$

$$\therefore \quad v=\frac{Mu}{M+m}. \qquad \ldots(2)$$

Suppose the resistance of the ground is R poundals and the retardation produced by it is f ft./sec^2.

Since the velocity v becomes zero after penetrating a distance a feet in the ground, therefore

$$0=v^2-2fa \quad \text{or} \quad f=\frac{v^2}{2a}. \qquad \ldots(3)$$

By Newton's second law of motion, $P=mf$, we have

$$R-(m+M)g=(m+M)f.$$

$$\therefore \quad R=(m+M)g+(m+M)f$$

$$=(m+M)g+(m+M)\cdot\frac{v^2}{2a}\left[\because \text{ from (3), } f=\frac{v^2}{2a}\right]$$

$$=(m+M)g+\frac{(m+M)}{2a}\cdot\frac{M^2u^2}{(m+M)^2} \quad \text{[from (2)]}$$

$$=(m+M)g+\frac{M^2}{2a}\cdot\frac{2gh}{m+M} \quad [\because \text{ from (1), } u^2=2gh]$$

$$=(m+M)g+\frac{M^2gh}{a(m+M)}.$$

Hence the resistance of the ground $=\left\{(m+M)+\frac{M^2h}{a(m+M)}\right\}$ lbs. wt.

Let t seconds be the time during which the pile is in motion. Then $0=v-ft$.

$$\therefore \quad t=\frac{v}{f}=\frac{v}{v^2/2a} \quad \text{[from (3)]}$$

$$=\frac{2a}{v}=\frac{2a}{Mu/(m+M)} \quad \text{[from (2)]}$$

$$=\frac{2a(m+M)}{Mu}=\frac{2a(m+M)}{M\sqrt{(2gh)}}=\frac{a(m+M)}{M}\cdot\sqrt{\left(\frac{2}{gh}\right)}.$$

Loss of K.E. by impact $=\frac{1}{2}Mu^2-\frac{1}{2}(M+m)v^2$

$$=\tfrac{1}{2}Mu^2-\tfrac{1}{2}(M+m)\cdot\frac{M^2u^2}{(m+M)^2} \quad \text{[from (2)]}$$

$$=\tfrac{1}{2}Mu^2\left[1-\frac{M}{m+M}\right]=\tfrac{1}{2}Mu^2\cdot\frac{m}{m+M} \text{ units of energy.}$$

$$\therefore \text{ fraction of K.E. lost}=\frac{\frac{1}{2}Mu^2\cdot\frac{m}{m+M}}{\frac{1}{2}Mu^2}=\frac{m}{m+M}.$$

Ex. 26. *Prove that if a hammer weighing W lbs. striking a nail weighing w lbs. with velocity V feet per second, drives it a feet into a fixed block of wood, the average resistance of the wood in pounds to the penetration of the nail is*

$$\frac{W^2}{W+w}\cdot\frac{V^2}{2ga}.$$

If, however, the block is free to recoil and weighs M lbs., the resistance obtained would be

$$\frac{MW^2}{(M+W+w)(W+w)}\cdot\frac{V^2}{2ag}.$$

It is to be noted that motion in the case of a nail being driven is in the horizontal direction.

Sol. When the block is fixed. Let u be the common velocity of the nail and the hammer immediately after striking. By the principle of conservation of momentum, we have

$$(W+w)u=WV$$

or $$u=WV/(W+w). \qquad \ldots(1)$$

Let P pounds weight be the average resistance of the wood to the penetration of the nail. Then by the principle of work and energy, we have $\frac{1}{2}(W+w)u^2=Pg.a.$ ...(2)

Putting the value of u from (1) in (2), we have

$$\frac{1}{2}(W+w).\frac{W^2V^2}{(W+w)^2}=Pag$$

or $P=\frac{W^2}{W+w}.\frac{V^2}{2ag}$. This proves the first result.

When the block is free to recoil. In this case let u_1 be the common velocity of the hammer, nail and the block when the penetration ceases. By the principle of conservation of momentum, we have $(M+W+w)u_1=WV$

or $u_1=WV/(M+W+w)$. ...(3)

If R pounds weight be the resistance in this case, then by the work-energy principle, we have

$$Rg.a=\frac{1}{2}(W+w)u^2-\frac{1}{2}(M+W+w)u_1^2. \quad ...(4)$$

Substituting the values of u and u_1 from (1) and (3) in (4), we get

$$R=\frac{MW^2}{(M+W+w)(W+w)}\cdot\frac{V^2}{2ag},$$

which proves the second result.

Ex. 27. *A hammer head of mass W kg. moving horizontally with velocity u m/sec. strikes an inelastic nail of mass m kg. fixed in a block of mass M kg. which is free to move. Prove that if the mean resistance of the block to penetration by the nail is a force P kg. wt., then the nail will penetrate with each blow a distance*

$$\frac{MW^2u^2}{2gP(W+m)(W+m+M)} \text{ metres.}$$

Sol. First consider the impulsive action between the hammer and the nail. Since the nail is inelastic, therefore immediately after striking, the hammer and the nail will begin to move as one body, say, with velocity v m./sec. By the principle of conservation of momentum, we have $Wu=(W+m)v$. ...(1)

Now the nail penetrates the block and let V m./sec. be the common velocity when penetration ceases. Then again by the principle of conservation of momentum, we have

$$(W+m+M)V=Wu. \quad ...(2)$$

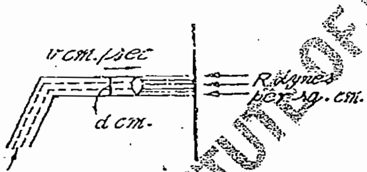
If x metres is the distance penetrated, then by the principle of work and energy, we have

$$-Pgx=\frac{1}{2}(W+m+M)V^2-\frac{1}{2}(W+m)v^2$$

or $2Pgx=(W+m)v^2-(W+m+M)V^2$. ...(3)

Substituting the values of v and V found from (1) and (2) in (3), we get $x=\frac{MW^2u^2}{2gP(W+m)(W+m+M)}$.

Ex. 28. *Water issuing from a nozzle of diameter d cms. with a velocity v cm./sec. impinges on a vertical wall, the jet being at right angles to the wall. If there is no splash, find the pressure exerted on the wall.*



Sol. As the jet strikes the wall, the wall exerts a force on it and destroys its momentum perpendicular to the wall. Let the force exerted by the wall on the jet be R dynes per sq. cm.

The impulse of the force exerted by the wall on the jet over the period of 1 second $=\frac{1}{4}\pi d^2R\times 1=\frac{1}{4}\pi d^2R$ gm.-cm./sec.

Mass of water that reaches the wall in 1 second
= volume of water coming out of jet in 1 sec. × density of water
$=\frac{1}{4}\pi d^2v.1=\frac{1}{4}\pi d^2v$ gms.
[∵ density of water = 1 gm. per cubic cm]

Change in the momentum of this mass of water on striking the wall $=\frac{1}{4}\pi d^2v[0-(-v)]$
$=\frac{1}{4}\pi d^2v^2$ gm.-cm./sec.

By the impulse-momentum principle, we have impulse of the force for any time = change in the momentum of the mass during that time.

$\therefore \frac{1}{4}\pi d^2R=\frac{1}{4}\pi d^2v^2$ or $R=v^2$.

By Newton's third law, action and reaction being equal in magnitude, pressure on the wall
$=v^2$ dynes per sq. cm.

Ex. 29. *A jet of water issues vertically at a speed of 30 feet per second from a nozzle of 0·1 square inch section. A ball weighing 1 lb. is balanced in the air by the impact of water on its underside. Show that the height of the ball above the level of the jet is 4·6 feet approximately.*

Sol. Let the height of the ball above the level of the jet be h feet. Suppose v ft./sec. is the velocity of the water at the time of striking the ball. Then

$$v^2=30^2-2gh \quad \text{or} \quad v=(900-2gh)^{1/2}. \quad ...(1)$$

Since the ball is balanced in the air by the impact of the water on its underside, therefore the force exerted by the water on the ball is equal and opposite to the weight of the ball. Hence the force exerted by the ball on the water is equal to the weight of the ball. Thus the force exerted by the ball on the water is equal to $1\times g$ i.e., g poundals in the vertically downwards direction.

The impulse of this force over the period of 1 second $=g\times 1$ $=g$ lb.-ft./sec.

Cross-section of the nozzle = 0·1 square inch
$=\frac{1}{10\times 12\times 12}$ sq. ft.

Density of water = 62·5 lbs. per cubic foot. Mass of water coming out of the nozzle per second
= volume of water coming out of jet in 1 second × density of water
$=\frac{1}{1440}\times 30\times 62\cdot 5$ lbs.

This mass of water strikes the ball with velocity v ft./sec. and is reduced to rest.

Change in the momentum of this mass of water on striking the ball $=\frac{1}{1440}\times 30\times 62\cdot 5\times v$ lb.-ft./sec.

By the impulse-momentum principle, we have impulse of the force for any time = change in the momentum of the mass during that time.

$$\therefore \quad g=\frac{1}{1440}\times 30\times 62\cdot 5\times v \text{ or } v=\frac{1440g\times 10}{30\times 625}=\frac{96g}{125}. \quad ...(2)$$

Equating the values of v from (1) and (2), we get

$$(900-2gh)^{1/2}=\tfrac{96}{125}g$$

or $900-2gh=\left(\frac{96}{125}\right)^2g^2$ or $h=\frac{900}{2g}-\left(\frac{96}{125}\right)\cdot\frac{g}{2}$

or $h=\frac{900}{64}-(\frac{96}{125})^2\times 16=14\cdot 006-9\cdot 4=4\cdot 6$ approx.

Ex. 30. *Two men, each of mass M, stand on two inelastic platforms each of mass m, hanging over a smooth pulley. One of the men leaping from the ground could raise his centre of gravity through a height h. Show that if he leaps with the same energy from the platform, his centre of gravity will rise a height* $\frac{M+2m}{2(M+m)}h$. *Initially the platforms hang at rest.*

Sol. Let u be the velocity of the man at the time he leaps up from the platform and v the common velocity of the remaining system.

If I be the impulsive force on the man due to which he leaps up with velocity u, we have

I = change in the momentum of the man $=Mu$. ...(1)

Considering the motion of the platform from which the man leaps up and assuming that the impulsive tension is I' in the string, we have $I-I'=mv$. ...(2)

Also considering the motion of the man and the platform at the other end of the string, we have

$$I'=(M+m)v. \quad ...(3)$$

Now the energy with which the man jumps up is given equal to Mgh. Since an equal energy is imparted to the system by the sudden pressing of the platform due to the jumping of the man, therefore $\frac{1}{2}mv^2+\frac{1}{2}(m+M)v^2+\frac{1}{2}Mu^2=Mgh$

or $(2m+M)v^2+Mu^2=Mgh$. ...(4)

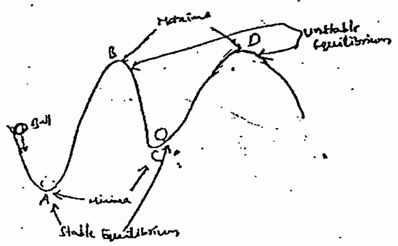
Eliminating I, I' and v between (1), (2), (3) and (4), we get

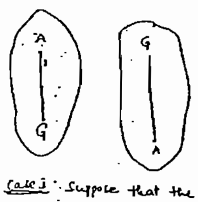
$$\frac{u^2}{2g}=\frac{2m+M}{2(m+M)}h.$$

Now the height through which the man rises while leaping up from the platform with velocity u

$$=\frac{u^2}{2g}=\frac{2m+M}{2(m+M)}h.$$

STABLE AND UNSTABLE EQUILIBRIUM | SET-V





Consider the equilibrium of a rigid body fixed at one point say A. For equilibrium of the body the centre of gravity G of the body must lie on the vertical line through the point of support A.

Case 1: Suppose that the centre of gravity G lies below the point of support A. In this case if the body be slightly displaced from its position of equilibrium, its centre of gravity will be raised. If the body be then let free, the force of gravity will bring the body back to its original position of equilibrium. In this case body is said to be in stable equilibrium.

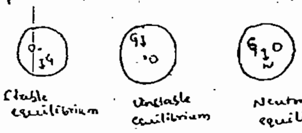
Case 2: Suppose that centre of gravity G lies above the point of support A. In this case if the body be slightly displaced from its position of equilibrium, its centre of gravity will be lowered. If the body be then let free, the force of gravity will still further move away the body from its original position of equilibrium. In this case we say that the body is in a state of unstable equilibrium.

Case 3: If the centre of gravity G is at the point of support A, the body will still be in equilibrium when displaced. In this case we say that the body is in a state of neutral equilibrium.

Stable Equilibrium: A body is said to be in stable equilibrium when slightly displaced from its position of equilibrium, the forces acting on the body tend to make it return towards its position of equilibrium.

Unstable Equilibrium: The equilibrium of a body is said to be unstable if when slightly displaced from its position of equilibrium, the forces acting on the body tend to move the body further away from its position of equilibrium.

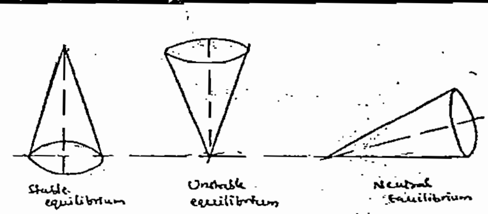
Neutral Equilibrium: A body is said to be in neutral equilibrium if the forces acting on it are such that they keep the body in equilibrium in any slightly displaced position.



(Sphere)

O → Geometric Centre

G → Centre of Gravity.

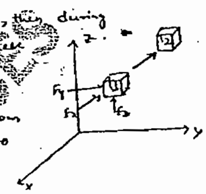


(Right Circular Cone)

The work function is work done by the forces in displacing the body from standard position to any position.

Let forces F_x, F_y, F_z parallel to the axes of co-ordinates are acting on a system, then during small displacement workdone by these forces is

$$dW = F_x\,dx + F_y\,dy + F_z\,dz$$



Integrating the above equation from some standard position (x_0, y_0, z_0) to any position (x, y, z) we have

$$W = \int_{(x_0,y_0,z_0)}^{(x,y,z)} (F_x\,dx + F_y\,dy + F_z\,dz)$$

↓ work function

→ If W_A and W_B are the values of the work function at two positions A and B, then

$W_B - W_A$ = workdone by forces in displacing the body from A to B.

→ If $F_x\,dx + F_y\,dy + F_z\,dz$ is an exact differential then the forces are called conservative forces.

Work function test for the nature of Stability of Equilibrium:

Let A be the position of equilibrium of a rigid body under the action of a given system of forces and let W be the work-function of the system in this position A. Suppose the body undergoes a small displacement and takes a position B, near to the position of equilibrium A.

∴ work function for position B = $W + dW$.

∴ work done by the forces in displacing the body from equilibrium position A to the near by position B = dW.

Since the body is in equilibrium in the position A, therefore by the principle of virtual work, we have

$$dW = 0$$

Hence the work function is stationary (maximum/minimum) in the position of equilibrium.

(i) Let W is maximum at the equilibrium position A, and let the body is slightly displaced to a position B.

Let W' = work function at B.

$$\therefore W' - W < 0 \quad (\because W \text{ is maximum})$$

It means that in displacing the body from A to B the workdone by the forces is negative i.e., the work is done against the forces, and hence the forces will have a tendency to bring the body back to the original position of equilibrium A.

Hence the equilibrium at A is stable.

Let W is minimum at the equilibrium position A.

If W' = value of the work function in a slightly displaced position B of the body.

$\therefore W' - W > 0$ (since W is minimum).

It means that in displacing the body from A to B the workdone by the forces is positive i.e, the work has been done by the forces and so the forces will have a tendency to move the body further away from the position of equilibrium. Hence in this case the equilibrium at A is unstable.

→ In the position of equilibrium of the body,

work function (W) = maximum or minimum

if W is maximum ⟶ stable equilibrium

and if W is minimum ⟶ unstable equilibrium

* <u>potential energy test for the nature of stability of equilibrium:</u>

potential energy of a body acted upon by a conservative system of forces, is defined as its capacity to do work by virtue of the position it has acquired.

If W: work function of the body in any position referred to some standard position.

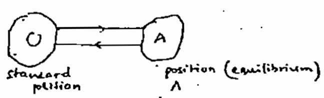
V = potential energy of the body in that position referred to the same standard position.

(measured by the amount of work it can do in passing from the position (present) to some standard position i.e, $V = -W$.

if V_A = potential energy at A

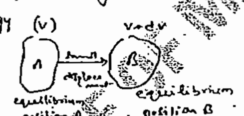
V_B = potential energy at B.

then $V_A - V_B$ = work done by the forces in displacing the body from A to B



Let at equilibrium position A, under given forces

V = potential energy (V)



$\therefore$ workdone by forces in displacing the body from the equilibrium position A to the near by position B

$= V - (V + dV)$

$= -dV$

• Since body is in equilibrium in the position A,

$\therefore$ By the principle of virtual work, we have

$-dV = 0 \Rightarrow dV = 0$

$\therefore$ potential energy (V) is stationary i.e, maximum or minimum in the position of equilibrium.

(i) Let V is minimum at the equilibrium position.

Let the body is slightly displaced to position B and potential energy there V'.

$\therefore V - V' < 0$

i.e, workdone by the forces acting on the body is negative i.e, work is done against the forces and so the forces will have a tendency to bring the body back to the original position of equilibrium.

Hence the equilibrium at A is stable.

(ii) Let V is maximum at the equilibrium position A. and V' = P.E. at equilibrium position B.

$\therefore V - V' > 0$

i.e, displacing the body from A to B the workdone by the forces is positive i.e, work is done by the forces and so the forces will tend to move the body further away from the position of equilibrium.

Hence the equilibrium at A is unstable

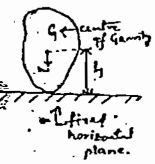
If $\left.\begin{array}{l} V = \text{minimum} \\ W = \text{maximum} \end{array}\right\}$ — stable equilibrium.

$\left.\begin{array}{l} V = \text{maximum} \\ W = \text{minimum} \end{array}\right\}$ — unstable equilibrium

* If only gravitational energy is involved, then

$h_{minimum}$ = for stable equilibrium

$h_{maximum}$ = for unstable equilibrium



* <u>Z-test for the nature of Stability:</u>

Let a body is in equilibrium under its weight only. i.e, the force of gravity is the only external force acting on the body.

Let Z = height of the C.G. of the body above a fixed horizontal plane.

Express $Z = f(\theta)$ i.e, some variable θ.

By the principle of virtual work, for the equilibrium of the body, we must have

$-W\,\delta Z = 0$ where W = Weight of the body

$\Rightarrow \delta Z = 0$

$\Rightarrow \frac{dZ}{d\theta}\delta\theta = 0$

$\Rightarrow \frac{dZ}{d\theta} = 0$

$\therefore$ The equilibrium positions of the body are given by the equation $\frac{dZ}{d\theta} = 0$

Thus in the position of equilibrium, the height of the centre of gravity of the body above a fixed level must be either maximum or minimum.

Let $\frac{dZ}{d\theta} = 0$

on solving, we have $\theta = \alpha, \beta, \gamma$ as the positions of equilibrium.

To test the nature of equilibrium at the position $\theta = \alpha$

first find $\left.\frac{d^2Z}{d\theta^2}\right|_{\theta=\alpha}$

if $\left(\frac{d^2Z}{d\theta^2}\right)_{\theta=\alpha} > 0$, then Z is minimum for $\theta = \alpha$.

So if we give a slight displacement to the body, the height of its centre of gravity will be raised and then on being set free the body will tend to come back to its original position of equilibrium.

$\therefore$ In this case the equilibrium is stable

Again if $\left(\frac{d^2Z}{d\theta^2}\right)_{\theta=\alpha} < 0$, then Z is maximum.

So if we give a slight displacement to the body, the height of its centre of gravity will be lowered and then on being let free the force of gravity will still displace the body further away from its original position of equilibrium. Therefore in this case the equilibrium is unstable.

If $\left(\frac{d^2Z}{d\theta^2}\right)_{\theta=\alpha} = 0$, then consider $\frac{d^3Z}{d\theta^3}$ and $\frac{d^4Z}{d\theta^4}$

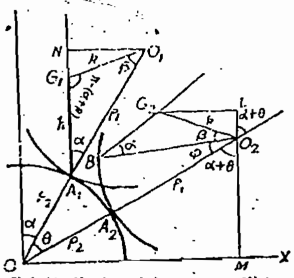
then for the position of equilibrium $\left(\frac{d^3Z}{d\theta^3}\right)_{\theta=\alpha} = 0$

and if $\frac{d^4Z}{d\theta^4} < 0$ stable (or) $\left(\frac{d^4Z}{d\theta^4}\right)_{\theta=\alpha} > 0$ unstable.

§ 7. Stability of a body resting on a fixed rough surface.

Theorem. *A body rests in equilibrium upon another fixed body, the portions of the two bodies in contact have radii of curvatures ρ_1 and ρ_2 respectively. The centre of gravity of the first body is at a height h above the point of contact and the common normal makes an angle α with the vertical; it is required to prove that the equilibrium is stable or unstable according as $h <$ or $> \dfrac{\rho_1\rho_2}{\rho_1+\rho_2}\cos\alpha$.*

Let O and O_1 be the centres of curvature of the lower and upper bodies in the position of rest and A_1 be their point of contact. In this position of equilibrium the common normal OA_1O_1 makes an angle α with the vertical OY. If G_1 is the centre of gravity of the upper body, then for equilibrium the line A_1G_1 must be vertical. It is given that $A_1G_1=h$.



Let $O_1G_1=k$

and $\angle OO_1G_1=\beta$.

Suppose the upper body is slightly displaced by pure rolling over the lower body which is fixed. Let A_2 be the new point of contact. O_2 is the new position of O_1 and the point A_1 of the upper body rolls up to the position B so that O_2B is the new position of the original normal O_1A_1. Also G_2 is the new position of G_1 so that $O_2G_2=O_1G_1=k$.

Suppose the common normal at A_2 makes angles θ and ϕ with the original normals OA_1 and O_2B.

We have $O_1A_1=\rho_1$ and $OA_1=\rho_2$. Also $O_2A_2=\rho_1$ and $OA_2=\rho_2$. Since the upper body rolls on the lower body without slipping therefore arc $A_1A_2=$ arc A_2B *i.e.,* $\rho_2\theta=\rho_1\phi$.

$$\therefore \frac{d\phi}{d\theta}=\frac{\rho_2}{\rho_1}. \qquad \ldots(1)$$

Let z be the height of G_2 above the fixed horizontal line OX. Then

$$z=LM=LO_2+O_2M$$
$$=O_2G_2\cos\angle G_2O_2L+OO_2\cos(\alpha+\theta)$$
$$=k\cos\{\pi-(\alpha+\theta+\phi+\beta)\}+(\rho_1+\rho_2)\cos(\alpha+\theta)$$
$$[\because O_2G_2=O_1G_1=k]$$
$$=(\rho_1+\rho_2)\cos(\alpha+\theta)-k\cos(\alpha+\theta+\phi+\beta).$$

$$\therefore \frac{dz}{d\theta}=-(\rho_1+\rho_2)\sin(\alpha+\theta)+k\sin(\alpha+\theta+\phi+\beta)\left(1+\frac{d\phi}{d\theta}\right)$$
[$\because$ α, β are constants and θ, ϕ are the only variables]
$$=-(\rho_1+\rho_2)\sin(\alpha+\theta)+k\sin(\alpha+\theta+\phi+\beta)\left(1+\frac{\rho_2}{\rho_1}\right) \quad \text{[from (1)]}$$
$$=\frac{\rho_1+\rho_2}{\rho_1}[-\rho_1\sin(\alpha+\theta)+k\sin(\alpha+\theta+\phi+\beta)]$$

and
$$\frac{d^2z}{d\theta^2}=\frac{\rho_1+\rho_2}{\rho_1}\left[-\rho_1\cos(\alpha+\theta)+k\cos(\alpha+\theta+\phi+\beta)\left(1+\frac{d\phi}{d\theta}\right)\right]$$
$$=\frac{\rho_1+\rho_2}{\rho_1}\left[-\rho_1\cos(\alpha+\theta)+k\cos(\alpha+\theta+\phi+\beta)\left(1+\frac{\rho_2}{\rho_1}\right)\right]$$
$$=\frac{\rho_1+\rho_2}{\rho_1^2}[-\rho_1^2\cos(\alpha+\theta)+k(\rho_1+\rho_2)\cos(\alpha+\theta+\phi+\beta)].$$

In the position of equilibrium $\theta=0$ and $\phi=0$.

Thus the equilibrium is stable or unstable according as $d^2z/d\theta^2$ is positive or negative for $\theta=\phi=0$,

i.e., according as $k(\rho_1+\rho_2)\cos(\alpha+\beta)>$ or $<\rho_1^2\cos\alpha$.

But from the $\triangle A_1G_1O_1$, we have

$h=A_1G_1=A_1N-G_1N=A_1O_1\cos\alpha-O_1G_1\cos\angle O_1G_1N$
$=\rho_1\cos\alpha-k\cos(\alpha+\beta)$.

$$\therefore k\cos(\alpha+\beta)=\rho_1\cos\alpha-h.$$

Hence the equilibrium is stable or unstable according as

$$(\rho_1+\rho_2)(\rho_1\cos\alpha-h)>\text{ or }<\rho_1^2\cos\alpha$$

i.e., $\quad(\rho_1+\rho_2)\rho_1\cos\alpha-(\rho_1+\rho_2)h>$ or $<\rho_1^2\cos^2\alpha$

i.e., $\quad(\rho_1+\rho_2)h<$ or $>(\rho_1+\rho_2)\rho_1\cos\alpha-\rho_1^2\cos^2\alpha$

i.e., $\quad(\rho_1+\rho_2)h<$ or $>\rho_1\rho_2\cos\alpha$

i.e., $$h<\text{ or }>\frac{\rho_1\rho_2}{\rho_1+\rho_2}\cos\alpha.$$

Cor. If $\alpha=0$, the above conditions give that the equilibrium is stable or unstable according as

$$h<\text{ or }>\frac{\rho_1\rho_2}{\rho_1+\rho_2} \quad i.e., \quad \frac{1}{h}>\text{ or }<\frac{\rho_1+\rho_2}{\rho_1\rho_2}$$

or $$\frac{1}{h}>\text{ or }<\frac{1}{\rho_1}+\frac{1}{\rho_2}.$$

Thus suppose that a body rests in equilibrium upon another body which is fixed and the portions of the two bodies in contact have radii of curvatures ρ_1 and ρ_2 respectively. The C.G. of the first body is at a height h above the point of contact and the common normal coincides with the vertical. Then the equilibrium is stable or unstable according as

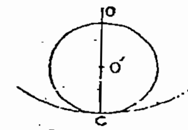
$$\frac{1}{h}>\text{ or }<\frac{1}{\rho_1}+\frac{1}{\rho_2}.$$

If the portions of the bodies in contact are spheres of radii r_1 and r_2, then in the above condition we put $\rho_1=r_1$ and $\rho_2=r_2$. Thus the equilibrium is stable or unstable according as

$$\frac{1}{h}>\text{ or }<\frac{1}{r_1}+\frac{1}{r_2}.$$

If the surface of the upper body at the point of contact is plane, then $\rho_1=\infty$ and if the surface of the lower body at the point of contact is plane, then $\rho_2=\infty$.

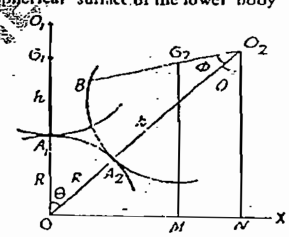
If the surface of the lower body at the point of contact instead of being convex is concave, then ρ_2 is to be taken with negative sign.



On account of its importance we shall now give an independent proof in case the surfaces in contact are spherical.

§ 8. *A body rests in equilibrium upon another fixed body, the portions of the two bodies in contact being spheres of radii r and R respectively and the straight line joining the centres of the spheres being vertical; if the first body be slightly displaced, to find whether the equilibrium is stable or unstable, the bodies being rough enough to prevent any sliding.* [Lucknow 76]

Let O be the centre of the spherical surface of the lower body which is fixed and O_1 that of the upper body which rests on the lower body, A_1 being their point of contact and the line OO_1 being vertical. If G_1 is the centre of gravity of the upper body, then for the equilibrium of the upper body, the line A_1G_1 must be vertical; let A_1G_1 be h. The figure is a section of the bodies by a vertical plane through G_1.



Suppose the upper body is slightly displaced by pure rolling over the lower body. Let A_2 be the new point of contact, O_2 is the new position of O_1 and the point A_1 of the upper body rolls up to the position B so that O_2B is the new position of O_1A_1. Also G_2 is the new position of G_1 so that $BG_2=A_1G_1=h$.

Let $\angle A_1OA_2=\theta$ and $\angle BO_2A_2=\phi$;

so that $\angle G_2O_2N=\theta+\phi$.

We have $O_1A_1=r$ and $OA_1=R$. Also $O_2A_2=O_2B=r$ and $OA_2=R$. Since the upper body rolls on the lower body without slipping, therefore

arc $A_1A_2=$ arc A_2B *i.e.,* $R\theta=r\phi$ *i.e.,* $\phi=(R/r)\theta$.

Now in order to find the nature of equilibrium, we should find the height z of the centre of gravity G_2 in the new position above the fixed horizontal line OX. We have

$$z=G_2M=O_2N=O_2G_2\cos(\theta+\phi)$$
$$=OO_2\cos\theta-(O_2B-BG_2)\cos(\theta+\phi)$$
$$=(R+r)\cos\theta-(r-h)\cos(\theta+\phi)$$
$$=(R+r)\cos\theta-(r-h)\cos\{\theta+(R/r)\theta\} \qquad [\because \phi=(R/r)\theta]$$
$$=(R+r)\cos\theta-(r-h)\cos\left\{\frac{\theta(r+R)}{r}\right\}.$$

For equilibrium, we have $dz/d\theta=0$

i.e., $$-(R+r)\sin\theta+(r-h)\sin\left\{\frac{\theta(r+R)}{r}\right\}\frac{r+R}{r}=0.$$

This is satisfied by $\theta=0$.

Now $$\frac{d^2z}{d\theta^2}=-(R+r)\cos\theta+(r-h)\cos\left\{\frac{\theta(r+R)}{r}\right\}\cdot\left(\frac{r+R}{r}\right)^2.$$

$$\therefore \left(\frac{d^2z}{d\theta^2}\right)_{\theta=0}=-(R+r)+(r-h)\left(\frac{r+R}{r}\right)^2$$
$$=\left(\frac{r+R}{r}\right)^2\left\{(r-h)-\frac{r^2}{R+r}\right\}=\left(\frac{r+R}{r}\right)^2\left\{r-\frac{r^2}{R+r}-h\right\}$$
$$=\left(\frac{r+R}{r}\right)^2\left\{\frac{rR}{R+r}-h\right\}.$$

This will be positive if

$$\frac{rR}{R+r}>h \text{ i.e., } \frac{1}{h}>\frac{R+r}{rR} \text{ i.e., } \frac{1}{h}>\frac{1}{r}+\frac{1}{R}$$

and negative, if $\dfrac{rR}{R+r}<h$ *i.e.,* $\dfrac{1}{h}<\dfrac{1}{r}+\dfrac{1}{R}$.

Hence the equilibrium is stable or unstable according as

$$\frac{1}{h}>\frac{1}{r}+\frac{1}{R} \quad\text{or}\quad \frac{1}{h}<\frac{1}{r}+\frac{1}{R}.$$

Here R is the radius of the lower body and r that of the upper body and h is the height of the C.G. of the upper body above the point of contact.

Now it remains to discuss the case when

$$1/h=1/r+1/R \quad i.e., \quad h=rR/(R+r).$$

In this case $d^2z/d\theta^2=0$. Hence we find $d^3z/d\theta^3$ and $d^4z/d\theta^4$

We have

$$\frac{d^3z}{d\theta^3}=(R+r)\sin\theta-(r-h)\sin\left\{\frac{\theta(r+R)}{r}\right\}.\left(\frac{r+R}{r}\right)^3,$$

and $$\frac{d^4z}{d\theta^4}=(R+r)\cos\theta-(r-h)\cos\left\{\frac{\theta(r+R)}{r}\right\}.\left(\frac{r+R}{r}\right)^4.$$

Obviously $\left(\frac{d^3z}{d\theta^3}\right)_{\theta=0}=0.$

Also $$\left(\frac{d^4z}{d\theta^4}\right)_{\theta=0}=(R+r)-(r-h)\left(\frac{r+R}{r}\right)^4$$

$$=(R+r)\left\{1-\frac{r-h}{r}\left(\frac{r+R}{r}\right)^3\right\}$$

$$=(R+r)\left\{1-\frac{r-h}{r}.\frac{R+r}{r}.\left(\frac{R+r}{r}\right)^2\right\}$$

$$=(R+r)\left\{1-\left(r-\frac{rR}{R+r}\right).\frac{R+r}{r^2}.\left(\frac{R+r}{r}\right)^2\right\} \qquad \left[\because \quad h=\frac{rR}{r+R}\right]$$

$$=(R+r)\left\{1-\frac{r^2}{R+r}.\frac{R+r}{r^2}.\left(\frac{R+r}{r}\right)^2\right\}$$

$$=(R+r)\left\{1-\left(\frac{R+r}{r}\right)^2\right\}$$

$$=(R+r)\left\{1-\left(1+\frac{R}{r}\right)^2\right\},$$ which is negative.

This shows that z is maximum and so in this case the equilibrium is unstable.

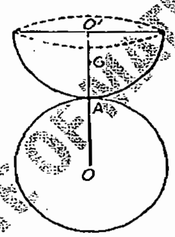
Hence if $\frac{1}{h}>\frac{1}{r}+\frac{1}{R}$, then equilibrium is stable

and if $\frac{1}{h}\leq\frac{1}{r}+\frac{1}{R}$, the equilibrium is unstable.

Remark. If the upper body has a plane face in contact with the lower body of radius R, then obviously $r=\infty$. And if the lower body be plane, then $R=\infty$.

Illustrative Examples

Ex. 1. *A hemisphere rests in equilibrium on a sphere of equal radius; show that the equilibrium is unstable when the curved, and stable when the flat surface of the hemisphere rests on the sphere.*



Sol. (i) When the curved surface of the hemisphere rests on the sphere. A hemisphere of centre O' rests on a sphere of centre O with its curved surface in contact with the sphere. The point of contact is A and $OA=O'A=a$ (say). Also the line OAO' is vertical.

If G is the centre of gravity of the hemisphere, then G lies on $O'A$ and $O'G=\frac{3}{8}a$.

Here $\rho_1=$the radius of curvature of the upper body at the point of contact $=$the radius of the hemisphere$=a$,

and $\rho_2=$the radius of curvature of the lower body at the point of contact$=a$.

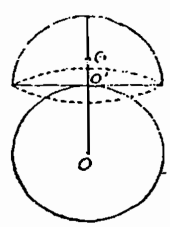
Also $h=$the height of the centre of gravity of the upper body above the point of contact A

$$=AG=O'A-O'G=a-\tfrac{3}{8}a=\tfrac{5}{8}a.$$

We have $$\frac{1}{h}=\frac{1}{5a/8}=\frac{8}{5a}.$$

and $$\frac{1}{\rho_1}+\frac{1}{\rho_2}=\frac{1}{a}+\frac{1}{a}=\frac{2}{a}=\frac{10}{5a}.$$

Thus $\frac{1}{h}<\frac{1}{\rho_1}+\frac{1}{\rho_2}$. Hence the equilibrium is unstable in this case.



(ii) When the flat surface of the hemisphere rests on the sphere. In this case a hemisphere of centre O' rests on a sphere of centre O and equal radius a with its flat surface (*i.e.* the plane base) in contact with the sphere. The point of contact is O' and G is the C.G. of the hemisphere.

Here $\rho_1=$the radius of curvature of the upper body at the point of contact$=\infty$.

[Note that the base of the hemisphere touches the sphere along a straight line]

and $\rho_2=$the radius of curvature of the lower body at the point of contact$=$the radius of the sphere$=a$.

Also $h=$the height of the C.G. of the hemisphere above the point of contact $O'=O'G=\frac{3}{8}a$.

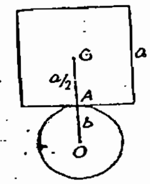
We have $$\frac{1}{h}=\frac{1}{3a/8}=\frac{8}{3a}.$$

and $$\frac{1}{\rho_1}+\frac{1}{\rho_2}=\frac{1}{\infty}+\frac{1}{a}=0+\frac{1}{a}=\frac{1}{a}=\frac{3}{3a}.$$

Obviously $\frac{1}{h}>\frac{1}{\rho_1}+\frac{1}{\rho_2}$. Hence in this case the equilibrium is stable.

Remark. Remember that for a straight line the radius of curvature at any point is infinity, and for a circle the radius of curvature at any point is equal to the radius of the circle.

Ex. 2. *A uniform cubical box of edge a is placed on the top of a fixed sphere, the centre of the face of the cube being in contact with the highest point of the sphere. What is the least radius of the sphere for which the equilibrium will be stable?*



Sol. A uniform cubical box of edge a is placed on the top of a fixed sphere of centre O. The point of contact is A. If G is the C.G. of the box, then for equilibrium the line OAG must be vertical. Let the radius of the sphere be b.

The figure shows the vertical section of the bodies through the point of contact A.

Here $\rho_1=$the radius of curvature of the upper body at the point of contact$=\infty$.

and $\rho_2=$the radius of curvature of the lower body at the point of contact$=b$.

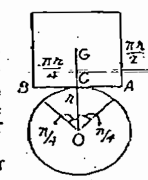
Also $h=$the height of the C.G. of the box above the point of contact $A=$half the edge of the box$=\frac{1}{2}a$.

The equilibrium will be stable, if

$\frac{1}{h}>\frac{1}{\rho_1}+\frac{1}{\rho_2}$ i.e., $\frac{1}{\frac{1}{2}a}>\frac{1}{\infty}+\frac{1}{b}$ i.e., $\frac{2}{a}>\frac{1}{b}$ i.e., $b>\frac{1}{2}a$.

Hence the least value of b for the equilibrium to be stable is

Ex. 3. *A heavy uniform cube balances on the highest point of a sphere whose radius is r. If the sphere is rough enough to prevent sliding and if the side of the cube be $\pi r/2$, show that the cube can rock through a right angle without falling.*



Sol. A heavy uniform cube balances on the highest point C of a sphere whose centre is O and radius r. The length of a side of the cube is $\pi r/2$. If G is the C.G. of the cube, then for equilibrium the line OCG must be vertical. In the figure we have shown a cross section of the bodies by a vertical plane through the point of contact C.

First we shall show that the equilibrium of the cube is stable.

Here $\rho_1=$the radius of curvature of the upper body at the point of contact $C=\infty$,

and $\rho_2=$the radius of curvature of the lower body at the point of contact$=r$.

Also $h=$the height of the centre of gravity G of the upper body above the point of contact $C=$half the edge of the cube$=\pi r/4$.

The equilibrium will be stable if

$$\frac{1}{h}>\frac{1}{\rho_1}+\frac{1}{\rho_2} \text{ i.e., } \frac{1}{\pi r/4}>\frac{1}{\infty}+\frac{1}{r}$$

i.e., $$\frac{4}{\pi r}>\frac{1}{r} \text{ i.e., } \frac{4}{\pi}>1 \text{ i.e., } 4>\pi$$

which is so because the value of π lies between 3 and 4.

Hence the equilibrium is stable. So if the cube is slightly displaced, it will tend to come back to its original position of equilibrium. During a swing to the right, the cube will not fall down till the right hand corner A of the lowest edge comes in contact with the sphere.

If θ is the angle through which the cube turns when the right hand corner A of the lowest edge comes in contact with the sphere, we have

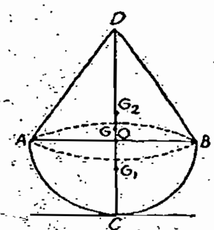
$$r\theta=\text{half the edge of the cube}=\pi r/4,$$

so that $\theta=\pi/4$.

Similarly the cube can turn through an angle $\pi/4$ to the left side on the sphere. Hence the total angle through which the cube can swing (or rock) without falling is $2.\frac{1}{4}\pi$ i.e., $\frac{1}{2}\pi$.

Ex. 4. *A body, consisting of a cone and a hemisphere on the same base, rests on a rough horizontal table the hemisphere being in contact with the table; show that the greatest height of the cone so that the equilibrium may be stable, is $\sqrt{3}$ times the radius of the hemisphere.*

Sol. AB is the common base of the hemisphere and the cone and COD is their common axis which must be vertical for equilibrium. The hemisphere touches the table at C.



Let H be the height OD of the cone and r be the radius OA or OC of the hemisphere. Let G_1 and G_2 be the centres of gravity of the hemisphere and the cone respectively. Then

$OG_1 = 3r/8$ and $OG_2 = H/4$.

If h be the height of the centre of gravity of the combined body composed of the hemisphere and the cone above the point of contact C, then using the formula $x = \frac{w_1x_1+w_2x_2}{w_1+w_2}$, we have

$$h = \frac{\frac{1}{3}\pi r^2 H.CG_2 + \frac{2}{3}\pi r^3.CG_1}{\frac{1}{3}\pi r^2 H + \frac{2}{3}\pi r^3} = \frac{\frac{1}{3}\pi r^2 H\left(r+\frac{1}{4}H\right)+\frac{2}{3}\pi r^3.\frac{5}{8}r}{\frac{1}{3}\pi r^2 H+\frac{2}{3}\pi r^3}$$

$$= \frac{H\left(r+\frac{1}{4}H\right)+\frac{5}{4}r^2}{H+2r}$$

Here ρ_1 = the radius of curvature at the point of contact C of the upper body which is spherical $= r$,

and ρ_2 = the radius of curvature of the lower body at the point of contact $= \infty$.

$\therefore$ the equilibrium will be stable if

$$\frac{1}{h} > \frac{1}{\rho_1}+\frac{1}{\rho_2} \text{ i.e., } \frac{1}{h} > \frac{1}{r}+\frac{1}{\infty} \text{ i.e., } \frac{1}{h} > \frac{1}{r}$$

i.e., $h < r$

i.e., $\frac{H\left(r+\frac{1}{4}H\right)+\frac{5}{4}r^2}{H+2r} < r$ i.e., $Hr+\frac{1}{4}H^2+\frac{5}{4}r^2 < Hr+2r^2$

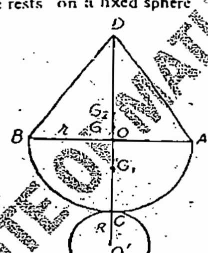
i.e., $\frac{1}{4}H^2 < \frac{3}{4}r^2$ i.e., $H^2 < 3r^2$ i.e., $H < r\sqrt{3}$.

Hence the greatest height of the cone consistent with the stable equilibrium of the body is $\sqrt{3}$ times the radius of the hemisphere.

Ex. *A solid homogeneous hemisphere of radius r has a solid right circular cone of the same substance constructed on the base; the hemisphere rests on the convex side of the fixed sphere of radius R. Show that the length of the axis of the cone consistent with stability for a small rolling displacement is*

$$\frac{r}{R+r}\left[\sqrt{\{(3R+r)(R-r)\}}-2r\right].$$

Sol. Let O be the centre of the common base AB of the hemisphere and the cone. The hemisphere rests on a fixed sphere of radius R and centre O', their point of contact being C. For equilibrium the line $O'COD$ must be vertical. Let H be the length of the axis OD of the cone. It is given that $OB = OC = r$ = the radius of the hemisphere.



If G_1 and G_2 are the centres of gravity of the hemisphere and the cone respectively, then

$OG_1 = 3r/8$ and $OG_2 = H/4$.

Let G be the centre of gravity of the combined body composed of the hemisphere and the cone. If h be the height of G above the point of contact C, then

$$h = \frac{\frac{2}{3}\pi r^3.\frac{5}{8}r+\frac{1}{3}\pi r^2 H\left(r+\frac{1}{4}H\right)}{\frac{2}{3}\pi r^3+\frac{1}{3}\pi r^2 H} = \frac{H\left(r+\frac{1}{4}H\right)+\frac{5}{4}r^2}{H+2r}.$$

Here ρ_1 = the radius of curvature at the point of contact C of the upper body $= r$

and ρ_2 = the radius of curvature at C of the lower body $= R$.

The equilibrium will be stable if

$$\frac{1}{h} > \frac{1}{\rho_1}+\frac{1}{\rho_2} \text{ i.e., } \frac{1}{h} > \frac{1}{r}+\frac{1}{R}$$

i.e., $\frac{H+2r}{H\left(r+\frac{1}{4}H\right)+\frac{5}{4}r^2} > \frac{R+r}{rR}$

i.e., $(R+r)\left(Hr+\frac{1}{4}H^2+\frac{5}{4}r^2\right)-rR(H+2r) < 0$

i.e., $\frac{1}{4}H^2(R+r)+H\{(R+r)r-rR\}+\frac{5}{4}r^2(R+r)-2r^2R < 0$

i.e., $H^2(R+r)+4r^2H+5r^3-3r^2R < 0$

i.e., $H^2(R+r)+4r^2H-r^2(3R-5r) < 0$

i.e., $H^2+\frac{4r^2}{R+r}H-\frac{r^2(3R-5r)}{R+r} < 0$

i.e., $\left(H+\frac{2r^2}{R+r}\right)^2-\frac{4r^4}{(R+r)^2}-\frac{r^2(3R-5r)}{R+r} < 0$

i.e., $\left(H+\frac{2r^2}{R+r}\right)^2-\frac{4r^4+r^2(3R-5r)(R+r)}{(R+r)^2} < 0$

i.e., $\left(H+\frac{2r^2}{R+r}\right)^2-\frac{r^2[4r^2+3R^2-2rR-5r^2]}{(R+r)^2} < 0$

i.e., $\left(H+\frac{2r^2}{R+r}\right)^2-\frac{r^2(3R^2-2rR-r^2)}{(R+r)^2} < 0$

i.e., $\left(H+\frac{2r^2}{R+r}\right)^2 < \frac{r^2(3R+r)(R-r)}{(R+r)^2}$

i.e., $H+\frac{2r^2}{R+r} < \frac{r}{R+r}\sqrt{\{(3R+r)(R-r)\}}$

i.e., $H < \frac{r}{R+r}\sqrt{\{(3R+r)(R-r)\}}-\frac{2r^2}{R+r}$

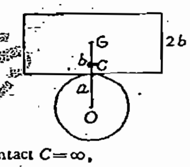
i.e., $H < \frac{r}{R+r}\left[\sqrt{\{(3R+r)(R-r)\}}-2r\right]$.

Therefore the greatest value of H consistent with the stability of equilibrium is

$$\frac{r}{R+r}\left[\sqrt{\{(3R+r)(R-r)\}}-2r\right].$$

Ex. 6. *A uniform beam, of thickness $2b$, rests symmetrically on a perfectly rough horizontal cylinder of radius a; show that the equilibrium of the beam will be stable or unstable according as b is less or greater than a.*

Sol. C is the point of contact of the beam and the cylinder and G is the centre of gravity of the beam. The figure shows the cross section of the bodies by a vertical plane through C. For equilibrium the line OCG is vertical.



Here ρ_1 = radius of curvature of the upper body at the point of contact $C = \infty$,

ρ_2 = radius of curvature of the lower body at $C = a$.

Also h = the height of C.G. of the beam above the point of contact $C = \frac{1}{2}$(thickness of the beam) $= \frac{1}{2}.2b = b$.

The equilibrium is stable or unstable according as

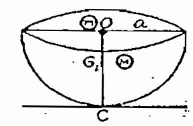
$$\frac{1}{h} > \text{ or } < \frac{1}{\rho_1}+\frac{1}{\rho_2} \text{ i.e., } \frac{1}{h} > \text{ or } < \frac{1}{\infty}+\frac{1}{a}$$

i.e., $\frac{1}{h} > \text{ or } < \frac{1}{a}$

i.e., $h < \text{ or } > a$ i.e., $b < \text{ or } > a$.

Ex. 7 (a). *A uniform solid hemisphere rests in equilibrium upon a rough horizontal plane with its curved surface in contact with the plane and a particle of mass m is fixed at the centre of the plane face. Show that for any value of m, the equilibrium is stable.*

Sol. C is the point of contact of the hemisphere and the plane and O is the centre of the base of the hemisphere. Let M be the mass of the hemisphere and a be its radius. A particle of mass m is placed at O. The mass M of the hemisphere acts at G_1 where $OG_1 = 3a/8$.



If h be the height of the centre of gravity of the combined body consisting of the hemisphere and the mass m above the point of contact C, then

$$h = \frac{M.\frac{5}{8}a+m.a}{M+m}$$

Here ρ_1 = the radius of curvature of the upper body at the point of contact $C = a$,

and ρ_2 = the radius of curvature of the lower body at the point of contact $C = \infty$.

The equilibrium will be stable if

$$\frac{1}{h} > \frac{1}{\rho_1}+\frac{1}{\rho_2} \text{ i.e., } \frac{1}{h} > \frac{1}{a}+\frac{1}{\infty} \text{ i.e., } \frac{1}{h} > \frac{1}{a} \text{ i.e., } h < a$$

i.e., $\frac{\frac{5}{8}aM+am}{M+m} < a$ i.e., $\frac{5}{8}aM+am < aM+am$

i.e., $\frac{5}{8}aM < aM$

i.e., $\frac{5}{8}a < a$, which is so whatever may be the value of m.

Hence for any value of m, the equilibrium is stable.

Ex. 7 (b). *A uniform hemisphere rests in equilibrium with its base upwards on the top of a sphere of double its radius. Show that the greatest weight which can be placed at the centre of the plane face without rendering the equilibrium unstable is one-eighth of the weight of the hemisphere.*

Sol. Draw figure yourself. Here a hemisphere rests on the top of a sphere. The base of the hemisphere is upwards. Let $2r$ be the radius of the sphere and r that of the hemisphere.

If W be the weight of the hemisphere and w be the weight placed at the centre of the base of the hemisphere, then

$$h = \frac{W.\frac{5}{8}r+w.r}{W+w}.$$

Here $\rho_1 = r$ and $\rho_2 = 2r$. The equilibrium will be stable if

$$\frac{1}{h} > \frac{1}{r}+\frac{1}{2r} \text{ i.e., } \frac{1}{h} > \frac{3}{2r} \text{ i.e., } \frac{W+w}{\frac{5}{8}rW+wr} > \frac{3}{2r}$$

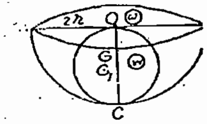
i.e., $2W+2w > \frac{15}{8}W+3w$ i.e., $\frac{1}{8}W > w$

i.e., $w < \frac{1}{2}W$,

which proves the required result.

Ex. 8 (a). *A solid sphere rests inside a fixed rough hemispherical bowl of twice its radius. Show that, however large a weight is attached to the highest point of the sphere, the equilibrium is stable.*

Sol. Let r be the radius of the solid sphere which rests inside a fixed rough hemispherical bowl of radius $2r$. Their point of contact is C and O is the highest point of the sphere so that $OC = 2r$. Let W and w be weights of the sphere and the weight attached to the highest point of the sphere. The weight W of the sphere acts at the middle point G_1 of its diameter OC.



If h is the height of the centre of gravity of the combined body consisting of the sphere and the weight w attached to O, then

$$h = \frac{W.r + w.2r}{W+w}.$$

Here ρ_1 = the radius of curvature of the upper body at the point of contact C = the radius of the sphere $= r$,

and ρ_2 = the radius of curvature of the lower body at the point of contact $C = -2r$, the negative sign is taken because the surface of the lower fixed body *i.e.*, the bowl at C is concave.

The equilibrium will be stable if

$$\frac{1}{h} > \frac{1}{\rho_1} + \frac{1}{\rho_2} \text{ i.e., } \frac{1}{h} > \frac{1}{r} - \frac{1}{2r} \text{ i.e., } \frac{1}{h} > \frac{1}{2r} \text{ i.e., } h < 2r$$

i.e., $$\frac{Wr + 2wr}{W+w} < 2r$$

i.e., $Wr + 2wr < 2Wr + 2wr$ *i.e.,* $Wr < 2Wr$,

which is so whatever be the value of w.

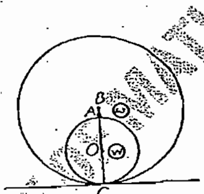
Hence, however large a weight is attached to the highest point of the sphere, the equilibrium is stable.

Ex. 8 (b). *A solid sphere rests inside a fixed rough hemispherical bowl of thrice its radius. Find the conditions and nature of equilibrium if a large weight is attached to the highest point of the sphere.*

Sol. Proceed exactly as in part (a). Equilibrium will be stable if weight of the sphere > weight attached.

Ex. 9. *A sphere of weight W and radius a lies within a fixed spherical shell of radius b, and a particle of weight w is fixed to the upper end of the vertical diameter. Prove that the equilibrium is stable if* $\frac{W}{w} > \frac{b-2a}{a}$.

Sol. C is the point of contact of the sphere and the spherical shell, O is the centre of the sphere, CA is the vertical diameter of the sphere and B is the centre of the spherical shell. We have $OC = a$ and $BC = b$.



The weight W of the sphere acts at O and a particle of weight w is attached to A. If h be the height of the centre of gravity of the combined body consisting of the sphere and the weight w attached at A, then

$$h = \frac{W.a + w.2a}{W+w} = \frac{W+2w}{W+w}a.$$

Here $\rho_1 = a$ and $\rho_2 = -b$.

The equilibrium will be stable if

$$\frac{1}{h} > \frac{1}{\rho_1} + \frac{1}{\rho_2} \text{ i.e., } \frac{1}{h} > \frac{1}{a} - \frac{1}{b} \text{ i.e., } \frac{W+w}{a(W+2w)} > \frac{b-a}{ab}$$

i.e., $(W+w)\, ab > a\,(b-a)(W+2w)$

i.e., $(W+w)\, b > (b-a)(W+2w)$

i.e., $W\{b-(b-a)\} > w\{2(b-a)-b\}$ *i.e.,* $Wa > w\,(b-2a)$

i.e., $$\frac{W}{w} > \frac{b-2a}{a}.$$

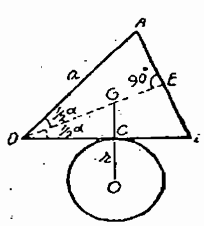
Ex. 10. *A lamina in the form of an isosceles triangle, whose vertical angle is α, is placed on a sphere, of radius r, so that its plane is vertical and one of its equal sides is in contact with the sphere; show that, if the triangle be slightly displaced in its own plane, the equilibrium is stable if $\sin\alpha < 3r/a$, where a is one of the equal sides of the triangle.*

Sol. DAB is an isosceles triangular lamina in which

$DA = DB = a$ and $\angle ADB = \alpha$.

The centre of gravity G of the lamina lies on its median DE which is perpendicular to AB and also bisects the angle ADB. We have

$DG = \frac{2}{3}DE = \frac{2}{3}a\cos\frac{1}{2}\alpha$.



The lamina rests on a fixed sphere whose centre is O and radius r. Their point of contact is C. For equilibrium the line OCG must be vertical.

If h be the height of the C.G. of the lamina above the point of contact C, then

$$h = GC = DG\sin\tfrac{1}{2}\alpha = \tfrac{2}{3}a\cos\tfrac{1}{2}\alpha\sin\tfrac{1}{2}\alpha = \tfrac{1}{3}a\sin\alpha.$$

Here ρ_1 = the radius of curvature of the upper body at the point of contact $C = \infty$;

and ρ_2 = the radius of curvature of the lower fixed body at the point $C = r$.

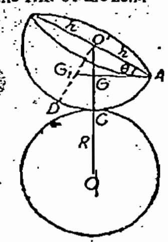
The equilibrium will be stable if

$$\frac{1}{h} > \frac{1}{\rho_1} + \frac{1}{\rho_2} \text{ i.e., } \frac{1}{h} > \frac{1}{\infty} + \frac{1}{r} \text{ i.e., } \frac{1}{h} > \frac{1}{r}$$

i.e., $h < r$ *i.e.,* $\frac{1}{3}a\sin\alpha < r$ *i.e.,* $\sin\alpha < 3r/a$.

Ex. 11. *A heavy hemispherical shell of radius r has a particle attached to a point on the rim, and rests with the curved surface in contact with a rough sphere of radius R at the highest point. Prove that if $R/r > \sqrt{5}-1$, the equilibrium is stable, whatever be the weight of the particle.* IAS-2012

Sol. Let O' be the centre of the base of the hemispherical shell of radius r. Let a weight be attached to the rim of the hemispherical shell at A. The centre of gravity G_1 of the hemispherical shell is on its symmetrical radius $O'D$ and $O'G_1 = \frac{1}{2}O'D = \frac{1}{2}r$.



Let G be the centre of gravity of the combined body consisting of the hemispherical shell and the weight at A. Then G lies on the line AG_1.

The hemispherical shell rests with its curved surface in contact with a rough sphere of radius R and centre O at the highest point C. For equilibrium the line $OCGO'$ must be vertical but AG_1 need not be horizontal.

Let $CG = h$. Also here $\rho_1 = r$ and $\rho_2 = R$.

The equilibrium will be stable if

$$\frac{1}{h} > \frac{1}{\rho_1} + \frac{1}{\rho_2} \text{ i.e., } \frac{1}{h} > \frac{1}{r} + \frac{1}{R} \text{ i.e., } \frac{1}{h} > \frac{R+r}{rR}$$

i.e., $$h < \frac{rR}{R+r}. \quad \ldots(1)$$

The value of h depends on the weight of the particle attached at A. So, the equilibrium will be stable, whatever be the weight of the particle attached at A, if the relation (1) holds even for the maximum value of h.

Now h will be maximum if $O'G$ is minimum *i.e.*, if $O'G$ is perpendicular to AG_1 or if $\triangle AO'G$ is right angled.

Let $\angle O'AG = \theta$. Then from right angled $\triangle AO'G_1$,

$$\tan\theta = \frac{O'G_1}{O'A} = \frac{\frac{1}{2}r}{r} = \frac{1}{2}. \quad \therefore \sin\theta = \frac{1}{\sqrt{5}}.$$

$\therefore$ the minimum value of $O'G$

$= O'A\sin\theta = r\,(1/\sqrt{5}) = r/\sqrt{5}$.

$\therefore$ the maximum value of $h = r -$ the minimum value of $O'G$

$$= r - \frac{r}{\sqrt{5}} = \frac{r\,(\sqrt{5}-1)}{\sqrt{5}}.$$

Hence the equilibrium will be stable, whatever be the weight of the particle at A, if

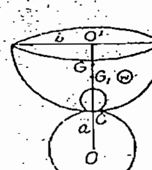
$$\frac{r\,(\sqrt{5}-1)}{\sqrt{5}} < \frac{rR}{R+r} \text{ i.e., if } \frac{\sqrt{5}-1}{\sqrt{5}} < \frac{R}{R+r}$$

i.e., if $(\sqrt{5}-1)R + (\sqrt{5}-1)r < R\sqrt{5}$

i.e., if $(\sqrt{5}-1)r < R$ *i.e.,* if $R/r > \sqrt{5}-1$.

Ex. 12. *A thin hemispherical bowl, of radius b and weight W rests in equilibrium on the highest point of a fixed sphere, of radius a, which is rough enough to prevent any sliding. Inside the bowl is placed a small smooth sphere of weight w; show that the equilibrium is not stable unless* $w < W\frac{a-b}{2b}$.

Sol. O is the centre, a the radius and C the highest point of the fixed sphere. A hemispherical bowl of radius b and weight W rests on the highest point C of this sphere and inside the bowl is placed a small smooth sphere of weight w. The weight W of the bowl acts at G_1 where $O'G_1 = \frac{1}{2}O'C$.



First we want to find out the height of the C.G. of the combined body consisting of the hemispherical bowl of weight W and sphere of weight w above the point of contact C. If the upper bowl be slightly displaced, the small smooth sphere placed inside it moves in such a way that the line of action of its weight w always passes through O', the centre of the base of the bowl. Hence so far as the question of the stability of the bowl is concerned the weight w of the small sphere may be taken to act at the centre O' of the bowl. If h be the height of the centre of gravity G of the combined body (*i.e.*, hemispherical shell of weight W and sphere of weight w) above the point of contact C, then

$$h=\frac{W.\frac{1}{2}b+w.b}{W+w}=\frac{(W+2w)\,b}{2\,(W+w)}.$$

Here $\rho_1=b$ and $\rho_2=a$. Hence the equilibrium will be stable if

$$\frac{1}{h}>\frac{1}{\rho_1}+\frac{1}{\rho_2}\ \text{i.e.,}\ \frac{1}{h}>\frac{1}{b}+\frac{1}{a}\ \text{i.e.,}\ \frac{1}{h}>\frac{a+b}{ab}\ \text{i.e.,}\ h<\frac{ab}{a+b}$$

i.e., $\dfrac{(W+2w)\,b}{2\,(W+w)}<\dfrac{ab}{a+b}$ i.e., $\dfrac{W+2w}{2\,(W+w)}<\dfrac{a}{a+b}$

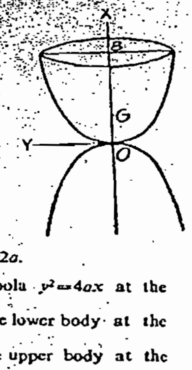
i.e., $(a+b)\,(W+2w)<2a\,(W+w)$

i.e., $w\,(2a+2b-2a)<W\,(2a-a-b)$

i.e., $2wb<W\,(a-b)$ i.e., $w<\dfrac{W\,(a-b)}{2b}$.

Ex. 13. *A solid frustum of a paraboloid of revolution of height h and latus rectum $4a$, rests with its vertex on the vertex of a paraboloid of revolution, whose latus rectum is $4b$; show that the equilibrium is stable if $h<\dfrac{3ab}{a+b}$.*

Sol. The point of contact of the two bodies is O and $OB=h$.



Let the equation of the generating parabola of the upper paraboloid be

$$y^2=4ax.$$

The parabola $y^2=4ax$ passes through the origin and the y-axis is tangent at the origin. If ρ be the radius of curvature of this parabola at the origin, then by Newton's formula for the radius of curvature at the origin, we have

$$\rho=\lim_{x\to 0}\frac{y^2}{2x}=\lim_{x\to 0}\frac{4ax}{2x}=\lim_{x\to 0}2a=2a.$$

$\therefore$ the radius of curvature of the parabola $y^2=4ax$ at the vertex (*i.e.*, at the origin) is $2a$.

So here, $\rho_1=$ the radius of curvature of the lower body at the point of contact $=2a$,

and $\rho_2=$ the radius of curvature of the upper body at the point of contact $=2b$.

If H be the height of the centre of gravity G of the upper body above the point of contact O, then

$$H=OG=\bar{x}=\frac{\int x\,dm}{\int dm}=\frac{\int_0^h x\,\pi y^2\,dx}{\int_0^h \pi y^2\,dx}$$

$$=\frac{\int_0^h x.4ax\,dx}{\int_0^h 4ax\,dx}=\frac{\int_0^h x^2\,dx}{\int_0^h x\,dx}=\frac{\left[\frac{x^3}{3}\right]_0^h}{\left[\frac{x^2}{2}\right]_0^h}=\frac{\frac{h^3}{3}}{\frac{h^2}{2}}=\tfrac{2}{3}h.$$

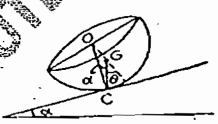
Now the equilibrium will be stable if

$$\frac{1}{H}>\frac{1}{\rho_1}+\frac{1}{\rho_2}\ \text{i.e.,}\ \frac{3}{2h}>\frac{1}{2a}+\frac{1}{2b}$$

i.e., $\dfrac{3}{h}>\dfrac{a+b}{ab}$ i.e., $\dfrac{h}{3}<\dfrac{ab}{a+b}$, i.e., $h<\dfrac{3ab}{a+b}$.

Ex. 14. *A solid hemisphere rests on a plane inclined to the horizon at an angle $\alpha<\sin^{-1}\frac{3}{8}$, and the plane is rough enough to prevent any sliding. Find the position of equilibrium and show that it is stable.*

Sol. Let O be the centre of the base of the hemisphere and r be its radius. If C is the point of contact of the hemisphere and the inclined plane, then $OC=r$. Let G be the centre of gravity of the hemisphere.



Then $OG=3r/8$. In the position of equilibrium the line CG must be vertical.

Since OC is perpendicular to the inclined plane and CG is perpendicular to the horizontal, therefore $\angle OCG=\alpha$. Suppose in equilibrium the axis of the hemisphere makes an angle θ with the vertical. From $\triangle OGC$, we have

$$\frac{OG}{\sin\alpha}=\frac{OC}{\sin\theta}\ \text{i.e.,}\ \frac{3r/8}{\sin\alpha}=\frac{r}{\sin\theta}.$$

$\therefore$ $\sin\theta=\frac{8}{3}\sin\alpha$, or $\theta=\sin^{-1}(\frac{8}{3}\sin\alpha)$,

giving the position of equilibrium of the hemisphere.

Since $\sin\theta<1$, therefore $\frac{8}{3}\sin\alpha<1$

i.e., $\sin\alpha<\frac{3}{8}$ i.e., $\alpha<\sin^{-1}\frac{3}{8}$.

Thus for the equilibrium to exist, we must have

$$\alpha<\sin^{-1}\tfrac{3}{8}.$$

Now let $CG=h$. Then

$$\frac{h}{\sin(\theta-\alpha)}=\frac{3r/8}{\sin\alpha},\ \text{so that}\ h=\frac{3r\sin(\theta-\alpha)}{8\sin\alpha}.$$

Here $\rho_1=r$ and $\rho_2=\infty$.

The equilibrium will be stable if

$$h<\frac{\rho_1\rho_2\cos\alpha}{\rho_1+\rho_2}\qquad\text{[See § 7]}$$

i.e., $\dfrac{1}{h}>\dfrac{\rho_1+\rho_2}{\rho_1\rho_2}\sec\alpha$ i.e., $\dfrac{1}{h}>\left(\dfrac{1}{\rho_1}+\dfrac{1}{\rho_2}\right)\sec\alpha$

i.e., $\dfrac{1}{h}>\dfrac{1}{r}\sec\alpha$ $\qquad[\because\ \rho_1=r,\ \rho_2=\infty]$

i.e., $h<r\cos\alpha$

i.e., $\dfrac{3r\sin(\theta-\alpha)}{8\sin\alpha}<r\cos\alpha$ $\qquad$[substituting for h]

or, $3\sin(\theta-\alpha)<8\sin\alpha\cos\alpha$

or, $3\sin\theta\cos\alpha-3\cos\theta\sin\alpha<8\sin\alpha\cos\alpha$

or, $8\sin\alpha\cos\alpha-3\sin\alpha\sqrt{(1-\frac{64}{9}\sin^2\alpha)}<8\sin\alpha\cos\alpha$ $\qquad[\because\ \sin\theta=\frac{8}{3}\sin\alpha]$

or, $-\sin\alpha\sqrt{(9-64\sin^2\alpha)}<0$

or, $\sin\alpha\sqrt{(9-64\sin^2\alpha)}>0.$ $\qquad$...(2)

But from (1),

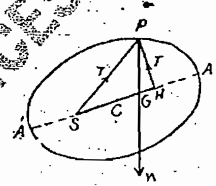
$$\sin\alpha<\tfrac{3}{8}\ \text{i.e.,}\ 64\sin^2\alpha<9\ \text{i.e.,}\ \sqrt{(9-64\sin^2\alpha)}$$

is a positive real number. Therefore the relation (2) is true. Hence the equilibrium is stable.

Ex. 15. *A rod SH, of length $2c$ and whose centre of gravity G is at a distance d from its centre, has a string, of length $2c\sec\alpha$, tied to its two ends and the string is then slung over a small smooth peg P; find the position of equilibrium and show that the position which is not vertical is stable.*

Sol. We have

$SP+PH=$ the length of the string $=2c\sec\alpha,$



as is given. The middle point of the rod SH is C and its centre of gravity is G such that $CG=d$.

Since in an ellipse the sum of the focal distances of any point on it is constant and is equal to the length $2a$ of its major axis, therefore the peg P must lie on an ellipse whose foci are S and H and for which the length of the major axis $2a=2c\sec\alpha$, so that

$$a=c\sec\alpha.$$

Now $SH=2c$ (given) and so $CH=c$. But $CH=ae$, where e is the eccentricity of this ellipse.

$\therefore$ $ae=c.$

If b be the length of the semi minor axis of this ellipse, then

$$b^2=a^2(1-e^2)=a^2-a^2e^2=c^2\sec^2\alpha-c^2=c^2\tan^2\alpha.$$

Hence the equation of this ellipse with C as origin and CH as x-axis is

$$\frac{x^2}{c^2\sec^2\alpha}+\frac{y^2}{c^2\tan^2\alpha}=1,$$

or $\qquad x^2\sin^2\alpha+y^2=c^2\tan^2\alpha.$

Shifting the origin to the point G $(d,0)$, it becomes

$$(x+d)^2\sin^2\alpha+y^2=c^2\tan^2\alpha.$$

Changing to polar coordinates, it becomes

$$(r\cos\theta+d)^2\sin^2\alpha+r^2\sin^2\theta=c^2\tan^2\alpha\qquad ...(1)$$

where G is the pole and GH is the initial line so that for the point P, $GP=r$ and $\angle PGH=\theta$.

If we find the value of θ for which r is maximum or minimum and regard the corresponding point P of the ellipse for the position of the peg and make PA vertical, we shall find the inclined position of equilibrium.

From (1),

$$r^2\cos^2\theta\sin^2\alpha+2rd\cos\theta\sin^2\alpha+d^2\sin^2\alpha+r^2-r^2\cos^2\theta=c^2\tan^2\alpha$$

or $r^2\cos^2\theta\cos^2\alpha-2rd\cos\theta\sin^2\alpha+(c^2\tan^2\alpha-r^2-d^2\sin^2\alpha)=0.$

This is a quadratic in $\cos\theta$. Therefore

$$\cos\theta=\frac{2rd\sin^2\alpha\pm\sqrt{[4r^2d^2\sin^4\alpha-4r^2\cos^2\alpha\,(c^2\tan^2\alpha-r^2-d^2\sin^2\alpha)]}}{2r^2\cos^2\alpha}$$

$$=\frac{d\sin^2\alpha\pm\sqrt{[d^2\sin^4\alpha-c^2\sin^2\alpha+r^2\cos^2\alpha+d^2\sin^2\alpha\cos^2\alpha]}}{r\cos^2\alpha}$$

$$=\frac{d\sin^2\alpha\pm\sqrt{[r^2\cos^2\alpha-(c^2-d^2)\sin^2\alpha]}}{r\cos^2\alpha}.$$

For real values of $\cos\theta$, we must have

$$r^2\cos^2\alpha\geq(c^2-d^2)\sin^2\alpha\ \text{i.e.,}\ r^2\geq(c^2-d^2)\tan^2\alpha.$$

Therefore the least value of r is $\sqrt{(c^2-d^2)}\tan\alpha$ and in that case $\cos\theta=\dfrac{d\sin^2\alpha}{r\cos^2\alpha}=\dfrac{d\sin^2\alpha}{\sqrt{(c^2-d^2)}\tan\alpha\cos^2\alpha}=\dfrac{d\tan\alpha}{\sqrt{(c^2-d^2)}}.$

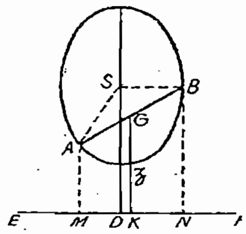
This gives the position of equilibrium in which the rod is not vertical. Since in this case r, the depth of the C.G. of the rod below the peg, is minimum, therefore the equilibrium is unstable.

The other two positions of equilibrium are when P is at A or A' *i.e.*, when the rod is vertical.

Ex. 16. *A smooth ellipse is fixed with its axis vertical and in it is placed a beam with its ends resting on the arc of the ellipse; if the length of the beam be not less than the latus rectum of the ellipse, show that when it is in stable equilibrium, it will pass through the focus.*

Sol. Let S be a focus and EF be the corresponding directrix of the ellipse. Referred to S as pole and the perpendicular SD from the focus to the directrix as the initial line, the polar equation of the ellipse is

$$l/r = 1 + e\cos\theta. \qquad ...(1)$$



Let AB be the beam and G its middle point *i.e.*, its centre of gravity. Let z be the height of G above the fixed line EF. Then

$$z = GK = \tfrac{1}{2}(AM + BN).$$

But by the definition of the ellipse,

$$\frac{AS}{AM} = e \text{ and } \frac{BS}{BN} = e, \text{ so that } AM = \frac{1}{e}AS \text{ and } BN = \frac{1}{e}BS.$$

$$\therefore \quad z = \tfrac{1}{2}\left[\frac{AS}{e} + \frac{BS}{e}\right] = \frac{1}{2e}(AS + BS). \qquad ...(2)$$

Now z will be minimum if $AS + BS$ is minimum *i.e.*, if A, S and B lie on the same straight line *i.e.*, if the beam AB passes through the focus S. But z is minimum implies that the equilibrium of the beam is stable. Hence the equilibrium of the beam is stable when it passes through the focus S.

In this case when the beam passes through the focus S, we have

$$AB = AS + BS$$

$$= \frac{l}{1 + e\cos\theta} + \frac{l}{1 + e\cos(\pi + \theta)}, \qquad \text{by (1).}$$

[Note that if the vectorial angle of B is θ then that of A is $\pi + \theta$]

$$= \frac{l}{1 + e\cos\theta} + \frac{l}{1 - e\cos\theta} = \frac{2l}{1 - e^2\cos^2\theta}.$$

$\therefore$ the length of the beam AB will be least when $1 - e^2\cos^2\theta$ is greatest *i.e.*, when $\cos\theta = 0$ or $\theta = \frac{1}{2}\pi$.

Then $AB = 2l =$ length of the latus rectum of the ellipse.

Therefore the least length of the beam is equal to the length of the latus rectum of the ellipse.

Problems based upon z-test

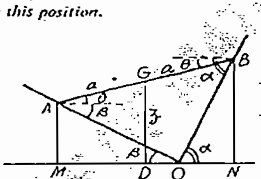
Ex. 17. *A uniform beam of length $2a$ rests with its ends on two smooth planes which intersect in a horizontal line. If the inclinations of the planes to the horizontal are α and β ($\alpha > \beta$), show that the inclination θ of the beam to the horizontal in one of the equilibrium positions is given by*

$$\tan\theta = \tfrac{1}{2}(\cot\beta - \cot\alpha)$$

and show that the beam is unstable in this position. IAS-2007

Sol. Let AB be a uniform beam of length $2a$ resting with its ends A and B on two smooth inclined planes OA and OB. Suppose the beam makes an angle θ with the horizontal. We have

$$\angle AOM = \beta \text{ and } \angle BON = \alpha.$$



The centre of gravity of the beam AB is its middle point G. Let z be the height of G above the fixed horizontal line MN. We shall express z as a function of θ.

We have, $z = GD = \frac{1}{2}(AM + BN)$

$$= \tfrac{1}{2}(OA\sin\beta + OB\sin\alpha). \qquad ...(1)$$

Now in the triangle OAB, $\angle OAB = \beta + \theta$, $\angle OBA = \alpha - \theta$ and $\angle AOB = \pi - (\alpha + \beta)$. Applying the sine theorem for the $\triangle OAB$, we have

$$\frac{OA}{\sin(\alpha - \theta)} = \frac{OB}{\sin(\beta + \theta)} = \frac{AB}{\sin[\pi - (\alpha + \beta)]} = \frac{2a}{\sin(\alpha + \beta)}.$$

$$\therefore \quad OA = \frac{2a\sin(\alpha - \theta)}{\sin(\alpha + \beta)}, \; OB = \frac{2a\sin(\beta + \theta)}{\sin(\alpha + \beta)}.$$

Substituting for OA and OB in (1), we have

$$z = \tfrac{1}{2}\left[\frac{2a\sin(\alpha - \theta)}{\sin(\alpha + \beta)}\sin\beta + \frac{2a\sin(\beta + \theta)}{\sin(\alpha + \beta)}\sin\alpha\right]$$

$$= \frac{a}{\sin(\alpha + \beta)}\left[\sin(\alpha - \theta)\sin\beta + \sin(\beta + \theta)\sin\alpha\right]$$

$$= \frac{a}{\sin(\alpha + \beta)}\Big[(\sin\alpha\cos\theta - \cos\alpha\sin\theta)\sin\beta + (\sin\beta\cos\theta + \cos\beta\sin\theta)\sin\alpha\Big]$$

$$= \frac{a}{\sin(\alpha + \beta)}\left[\sin\theta(\sin\alpha\cos\beta - \cos\alpha\sin\beta) + 2\cos\theta\sin\alpha\sin\beta\right].$$

$$\therefore \quad \frac{dz}{d\theta} = \frac{a}{\sin(\alpha + \beta)}\left[\cos\theta(\sin\alpha\cos\beta - \cos\alpha\sin\beta) - 2\sin\theta\sin\alpha\sin\beta\right] \qquad ...(2)$$

For equilibrium of the beam, we have $\dfrac{dz}{d\theta} = 0$

i.e., $\cos\theta(\sin\alpha\cos\beta - \cos\alpha\sin\beta) - 2\sin\theta\sin\alpha\sin\beta = 0$

i.e., $2\sin\theta\sin\alpha\sin\beta = \cos\theta(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$

or $\dfrac{\sin\theta}{\cos\theta} = \dfrac{1}{2}\left(\dfrac{\sin\alpha\cos\beta - \cos\alpha\sin\beta}{\sin\alpha\sin\beta}\right)$

or $\tan\theta = \frac{1}{2}(\cot\beta - \cot\alpha)$. ...(3)

This gives the required position of equilibrium of the beam.

Differentiating (2), we have

$$\frac{d^2z}{d\theta^2} = \frac{a}{\sin(\alpha + \beta)}\left[-\sin\theta(\sin\alpha\cos\beta - \cos\alpha\sin\beta) - 2\cos\theta\sin\alpha\sin\beta\right]$$

$$= \frac{-2a\sin\alpha\sin\beta\cos\theta}{\sin(\alpha + \beta)}\left[\tfrac{1}{2}\tan\theta(\cot\beta - \cot\alpha) + 1\right]$$

$$= \frac{-2a\sin\alpha\sin\beta\cos\theta}{\sin(\alpha + \beta)}\left[\tan^2\theta + 1\right] \qquad \text{[by (3)]}$$

$=$ a negative quantity because θ, α and β are all acute angles and $\alpha + \beta < \pi$.

Thus in the position of equilibrium $d^2z/d\theta^2$ is negative *i.e.*, z is maximum. Hence the equilibrium is unstable.

Ex. 18. *A uniform heavy beam rests between two smooth planes, each inclined at an angle $\frac{1}{4}\pi$ to the horizontal, so that the beam is in a vertical plane perpendicular to the line of action of the planes. Show that the equilibrium is unstable when the beam is horizontal.*

Sol. Draw figure as in Ex. 17, taking $\alpha = \beta = \frac{1}{4}\pi$. If the beam makes an angle θ with the horizontal and z be the height of the C.G. of the beam above the fixed horizontal line MN, then proceeding as in Ex. 17, we have

$$z = \frac{a}{\sin\frac{1}{2}\pi}\left[\sin\left(\tfrac{1}{4}\pi - \theta\right)\sin\tfrac{1}{4}\pi + \sin\left(\tfrac{1}{4}\pi + \theta\right)\sin\tfrac{1}{4}\pi\right]$$

$$= a\left[\left(\frac{1}{\sqrt{2}}\cos\theta - \frac{1}{\sqrt{2}}\sin\theta\right)\frac{1}{\sqrt{2}} + \left(\frac{1}{\sqrt{2}}\cos\theta + \frac{1}{\sqrt{2}}\sin\theta\right)\frac{1}{\sqrt{2}}\right]$$

$$= a\cos\theta.$$

$\therefore \quad dz/d\theta = -a\sin\theta.$

For equilibrium of the beam, we have $dz/d\theta = 0$ *i.e.*, $\sin\theta = 0$ *i.e.*, $\theta = 0$.

$\therefore$ the beam rests in a horizontal position.

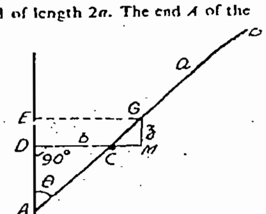
Now $d^2z/d\theta^2 = -a\cos\theta$.

When $\theta = 0$, $d^2z/d\theta^2 = -a\cos 0 = -a$, which is negative.

Thus in the position of equilibrium $d^2z/d\theta^2$ is negative *i.e.*, z is maximum. Hence the equilibrium is unstable.

Ex. 19. *A heavy uniform rod rests with one end against a smooth vertical wall and with a point in its length resting on a smooth peg; find the position of equilibrium and show that it is unstable.* IFoS-2013

Sol. Let AB be a uniform rod of length $2a$. The end A of the rod rests against a smooth vertical wall and the rod rests on a smooth peg C whose distance from the wall is say b *i.e.*, $CD = b$.



Suppose the rod makes an angle θ with the wall. The centre of gravity of the rod is at its middle point G. Let z be the height of G above the fixed peg C *i.e.*, $GM = z$. We shall express z in terms of θ. We have

$$z = GM = ED = AE - AD$$

$$= AG\cos\theta - CD\cot\theta$$

$$= a\cos\theta - b\cot\theta.$$

$\therefore \quad dz/d\theta = -a\sin\theta + b\,\text{cosec}^2\,\theta$

and $\quad d^2z/d\theta^2 = -a\cos\theta - 2b\,\text{cosec}^2\,\theta\cot\theta.$

For equilibrium of the rod, we have $dz/d\theta = 0$

i.e., $\quad -a\sin\theta + b\,\text{cosec}^2\,\theta = 0$

or $\quad a\sin\theta = b\,\text{cosec}^2\,\theta$, or $\sin^3\theta = b/a$

or $\quad \sin\theta = (b/a)^{1/3}$, or $\theta = \sin^{-1}(b/a)^{1/3}$.

This gives the position of equilibrium of the rod.

Again $d^2z/d\theta^2 = -(a\cos\theta + 2b\,\text{cosec}^2\,\theta\cot\theta)$

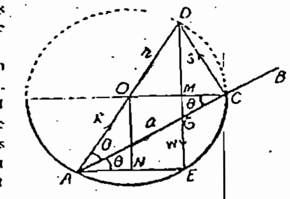
$=$ negative for all acute values of θ.

Thus $d^2z/d\theta^2$ is negative in the position of equilibrium and so z is maximum. Hence the equilibrium is unstable.

Ex. 20. *A heavy uniform rod, length $2a$, rests partly within and partly without a fixed smooth hemispherical bowl of radius r, the rim of the bowl is horizontal, and one point of the rod is in contact with the rim; if θ be the inclination of the rod to the horizon, show that $2r\cos 2\theta = a\cos\theta$.*

Show also that the equilibrium of the rod is stable.

Sol. Let AB be the rod of length $2a$ with its centre of gravity at G. A point C of its length is in contact with the rim of the bowl of radius r and centre O.



The rod is in equilibrium under the action of three forces. The reaction R of the bowl at A is along the normal AO and the reaction S of the rim at C is perpendicular to the rod. Let these reactions meet in a point

D. Since the line AOD passes through the centre O of the bowl and $\angle ACD$ is a right angle, therefore AOD is a diameter of the sphere of which the bowl is a part.

The third force on the rod is its weight W acting vertically downwards through its middle point G. Since the three forces must be concurrent, therefore the line DG is vertical.

Suppose the line DG meets the surface of the bowl at the point E. Join AE; then AE is horizontal because $\angle AED = 90°$, being the angle in a semi-circle.

We have $\therefore \angle DAE = \theta = \angle ACO$ $\quad [\because AE$ is parallel to $OC]$
$= \angle OAC. \quad [\because OA = OC]$

$\therefore \angle DAE = 2\theta$.

Suppose z is the depth of the centre of gravity G of the rod below the fixed horizontal line OC. Then

$$z = MG = ME - GE = ON - GE$$
$$= OA \sin 2\theta - AG \sin \theta = r \sin 2\theta - a \sin \theta.$$

$\therefore dz/d\theta = 2r \cos 2\theta - a \cos \theta$.

For the equilibrium of the rod, we must have $dz/d\theta = 0$
i.e., $2r \cos 2\theta - a \cos \theta = 0$ *i.e.*, $2r \cos 2\theta = a \cos \theta$.

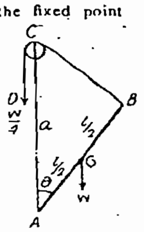
This gives the position of equilibrium of the rod.

Again $d^2z/d\theta^2 = -4r \sin 2\theta + a \sin \theta$
$= -2(2r \sin 2\theta) + a \sin \theta$
$= -2.DE - GE$, which is negative because $DE > GE$.

Thus the depth z of the C.G. of the rod below a fixed horizontal line is maximum. Hence the equilibrium is stable.

Ex. 21. *One end A of a uniform rod AE of weight W and length l is smoothly hinged at a fixed point, while B is tied to a light string which passes over a small smooth pulley at a distance a vertically above A and carries a weight $W/4$. If $l < a < 2l$, show that the system is in stable equilibrium when AB is vertically upwards, and that there is also a configuration of equilibrium in which the rod is at a certain angle to the vertical.*

Sol. Let AB be the rod of length l hinged at the fixed point A. The weight W of the rod acts through its middle point G. Let b be the length of the string BCD which is attached to E and passes over a smooth pulley at C, AC being vertical and equal to a. The string carries a weight $W/4$ at its other end D. Let $\angle BAC = \theta$.



From $\triangle BAC$,
$BC = \sqrt{(AB^2 + AC^2 - 2AB.AC.\cos\theta)}$
$= \sqrt{(l^2 + a^2 - 2la \cos\theta)}$.

$\therefore$ the length of the portion CD of the string hanging vertically

$= b - BC = b - \sqrt{(l^2 + a^2 - 2la \cos\theta)}$.

The weight W acts at the point G whose height above the fixed point A is $AG \cos\theta$ *i.e.*, $\frac{1}{2}l \cos\theta$. The weight $W/4$ acts at D whose height above A is $a - b + \sqrt{(l^2 + a^2 - 2la \cos\theta)}$.

Hence if z be the height, above the fixed point A, of the centre of gravity of the system consisting of the weight W and $W/4$, then

$(W + \frac{1}{4}W) z = W.\frac{1}{2}l \cos\theta + \frac{1}{4}W\{a - b + \sqrt{(l^2 + a^2 - 2la \cos\theta)}\}$

i.e., $5z = 2l \cos\theta + a - b + \sqrt{(l^2 + a^2 - 2la \cos\theta)}$.

$$\therefore \quad 5\frac{dz}{d\theta} = -2l \sin\theta + \frac{al \sin\theta}{\sqrt{(l^2 + a^2 - 2la \cos\theta)}}$$

and
$$5\frac{d^2z}{d\theta^2} = -2l \cos\theta + \frac{al \cos\theta}{\sqrt{(l^2 + a^2 - 2la \cos\theta)}} - \frac{a^2 l^2 \sin^2\theta}{(l^2 + a^2 - 2la \cos\theta)^{3/2}}$$

For the equilibrium of the system, we must have $dz/d\theta = 0$. Obviously $dz/d\theta$ vanishes when $\sin\theta = 0$ *i.e.*, $\theta = 0$ *i.e.*, the rod AB is vertically upwards. Thus the system is in equilibrium when the rod AB is vertically upwards.

For $\theta = 0$, we have $5\frac{d^2z}{d\theta^2} = -2l + \frac{al}{\sqrt{(l^2 + a^2 - 2la)}}$

$$= -2l + \frac{al}{a - l}, \text{ if } a > l$$
$$= l\frac{(a - 2l)}{a - l},$$

which is positive if $l < a < 2l$.

Thus if $l < a < 2l$, then for $\theta = 0$, $d^2z/d\theta^2$ is positive *i.e.*, z is minimum. Hence this is a stable position of equilibrium.

Again $dz/d\theta$ also vanishes when

$$-2 + \frac{a}{\sqrt{(l^2 + a^2 - 2l\, a \cos\theta)}} = 0 \quad \text{or} \quad 4 = \frac{a^2}{(l^2 + a^2 - 2l\, a \cos\theta)}$$

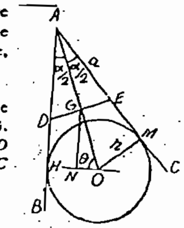
or $4l^2 + 4a^2 - 8\, la \cos\theta = a^2$

or $\cos\theta = \dfrac{3a^2 + 4l^2}{8la}$, which gives a real value of θ when $l < a < 2l$.

So there is also a configuration of equilibrium in which the rod is inclined to the vertical.

Ex. 22. *Two equal uniform rods are firmly jointed at one end so that the angle between them is α, and they rest in a vertical plane on a smooth sphere of radius r. Show that they are in a stable or unstable equilibrium according as the length of the rod is $>$ or $<$ $4r$ cosec α.*

Sol. Let AB and AC be two rods jointed at A and placed in a vertical plane on a smooth sphere of centre O and radius r. We have $\angle BAC = \alpha$. Since the rods are tangential to the sphere, therefore $\angle BAO = \angle CAO = \frac{1}{2}\alpha$.



Suppose $AB = AC = 2a$.

If D and E are the middle points of the rods AB and AC, then the combined C.G. of the rods is at the middle point G of ED which must be on AO. Suppose the rod AC touches the sphere at M. We have,

$OM = r$, $AE = a$, $\angle AMO = 90°$,
$\angle AGE = 90°$.

Suppose AO makes an angle θ with the horizontal line OH through the fixed point O. Let z be the height of the C.G. of the system above the horizontal through O. Then

$z = GN = OG \sin\theta = (AO - AG) \sin\theta$
$= (r \text{ cosec } \frac{1}{2}\alpha - a \cos\frac{1}{2}\alpha) \sin\theta$.

$\therefore dz/d\theta = (r \text{ cosec } \frac{1}{2}\alpha - a \cos\frac{1}{2}\alpha) \cos\theta$.

For the equilibrium of the rods, we must have $dz/d\theta = 0$
i.e., $(r \text{ cosec } \frac{1}{2}\alpha - a \cos\frac{1}{2}\alpha) \cos\theta = 0$ *i.e.*, $\cos\theta = 0$ *i.e.*, $\theta = \frac{1}{2}\pi$.

Thus in the position of equilibrium of rods, the line AO must be vertical.

Also $d^2z/d\theta^2 = -(r \text{ cosec } \frac{1}{2}\alpha - a \cos\frac{1}{2}\alpha) \sin\theta$
$= -r \text{ cosec } \frac{1}{2}\alpha + a \cos\frac{1}{2}\alpha$, for $\theta = \frac{1}{2}\pi$.

The equilibrium will be stable or unstable according as the height z of the C.G. of the system is minimum or maximum in the position of equilibrium,

i.e., according as $d^2z/d\theta^2$ is positive or negative at $\theta = \frac{1}{2}\pi$

i.e., according as $a \cos\frac{1}{2}\alpha >$ or $< r$ cosec $\frac{1}{2}\alpha$

i.e., according as $2a >$ or $< \dfrac{2r}{\cos\frac{1}{2}\alpha \sin\frac{1}{2}\alpha}$

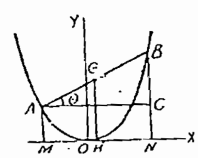
i.e., according as $2a >$ or $< \dfrac{4r}{\sin\alpha}$

i.e., according as $2a >$ or $< 4r$ cosec α.

Ex. 23. *A uniform rod, of length $2l$, is attached by smooth rings at both ends of a parabolic wire, fixed with its axis vertical and vertex downwards, and of latus rectum $4a$. Show that the angle θ which the rod makes with the horizontal in a slanting position of equilibrium is given by $\cos^2\theta = 2a/l$, and that, if these positions exist they are stable.*

Show also that the positions in which the rod is horizontal are stable or unstable according as the rod is below or above the focus.

Sol. Let AB be the rod of length $2l$. Take OX and OY as coordinate axes, so that the equation of the parabola be written as



$$x^2 = 4ay.$$

Let the coordinates of the point A be $(2at, at^2)$ and let the rod AB make an angle θ with the horizontal AC. Then the coordinates of B are $(2at + 2l \cos\theta,\ at^2 + 2l \sin\theta)$. Since B lies on the parabola $x^2 = 4ay$, therefore

$(2at + 2l \cos\theta)^2 = 4a\,(at^2 + 2l \sin\theta)$

or $\quad 8atl \cos\theta + 4l^2 \cos^2\theta = 8al \sin\theta$

or $\quad (2al \cos\theta)\, t = 2al \sin\theta - l^2 \cos^2\theta$

or $\quad t = \tan\theta - (l/2a) \cos\theta. \quad \ldots(1)$

The centre of gravity of the rod AB is at its middle point G. If z be the height of G above the fixed horizontal line OX, then

$z = GH = \frac{1}{2}(AM + BN)$
$= \frac{1}{2}[at^2 + (at^2 + 2l \sin\theta)] = at^2 + l \sin\theta$
$= a[\tan\theta - (l/2a) \cos\theta]^2 + l \sin\theta \quad$ [from (1)]
$= (l^2/4a) \cos^2\theta + a \tan^2\theta = (1/4a)\,[l^2 \cos^2\theta + 4a^2 \tan^2\theta]$.

$\therefore \quad dz/d\theta = (1/4a)\,[-2l^2 \cos\theta \sin\theta + 8a^2 \tan\theta \sec^2\theta]$
$= (1/2a) \sin\theta\,[-l^2 \cos\theta + 4a^2 \sec^3\theta]$.

For the equilibrium of the rod, we must have $dz/d\theta = 0$

i.e. $\quad (1/2a) \sin\theta\,(-l^2 \cos\theta + 4a^2 \sec^3\theta) = 0$.

$\therefore$ either $\sin\theta = 0$ *i.e.*, $\theta = 0$, which gives the horizontal position of rest of the rod

or $\quad -l^2 \cos\theta + 4a^2 \sec^3\theta = 0$ *i.e.* $l^2 \cos\theta = 4a^2/\cos^3\theta$

i.e., $\quad \cos^4\theta = 4a^2/l^2$ *i.e.*, $\cos^2\theta = 2a/l$, which gives the inclined position of rest of the rod.

Now, $d^2z/d\theta^2 = (1/2a) \cos\theta\,[-l^2 \cos\theta + 4a^2 \sec^3\theta]$
$+ (1/2a) \sin\theta\,[l^2 \sin\theta + 12a^2 \sec^3\theta \tan\theta]. \quad \ldots(2)$

When $\cos^2\theta = 2a/l$ *i.e.*, when $-l^2 \cos\theta + 4a^2 \sec^3\theta = 0$ we have

$d^2z/d\theta^2 = (1/2a) \sin\theta\,[l^2 \sin\theta + 12a^2 \sec^3\theta \tan\theta]$
$= (1/2a) \sin^2\theta\,[l^2 + 12a^2 \sec^4\theta]$, which is > 0.

Hence in the inclined position of rest of the rod, z is minimum and so the equilibrium is stable.

Again when the rod is horizontal *i.e.*, $\theta=0$, we have, from (2)

$$\frac{d^2z}{d\theta^2}=\frac{8a^2-2l^2}{4a}=\frac{4a^2-l^2}{2a}.$$

The equilibrium in this case is stable or unstable according as $d^2z/d\theta^2$ is positive or negative

i.e., according as $4a^2-l^2>$ or <0

i.e., according as $2a>$ or $<l$

i.e., according as $2l<$ or $>4a$

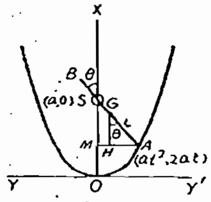
i.e., according as the rod is below or above the focus.

Ex. 24. *A uniform smooth rod passes through a ring at the focus of a fixed parabola whose axis is vertical and vertex below the focus, and rests with one end on the parabola. Prove that the rod will be in equilibrium if it makes with the vertical an angle θ given by the equation*

$$\cos^4 \tfrac{1}{2}\theta=a/2c$$

where $4a$ is the latus rectum and $2c$ the length of the rod. Investigate also the stability of equilibrium in this position. [Lucknow 81]

Sol. Let the equation of the parabola be $y^2=4ax$.



Let AB be the rod of length $2c$ with its end A on the parabola and passing through a ring at the focus S. Let the coordinates of A be $(at^2, 2at)$; the coordinates of the focus S are $(a, 0)$. If the rod AB makes an angle θ with the vertical OX, then

$\tan\theta=$ the gradient of the line AB

$$=\frac{2at-0}{at^2-a}=\frac{2t}{t^2-1}=\frac{-2t}{1-t^2}$$

$$\therefore \quad \frac{2\tan\frac{1}{2}\theta}{1-\tan^2\frac{1}{2}\theta}=\frac{2(-t)}{1-(-t)^2}, \quad \text{or} \quad \tan\tfrac{1}{2}\theta=-t.$$

Let z be the height of the centre of gravity G of the rod AB above the fixed horizontal line YOY'. Then

$$z=OM+HG=OM+AG\cos\theta$$

$$=at^2+c\cos\theta$$

[$\because$ $OM=x$-coordinate of A and $AG=\frac{1}{2}AB$]

$$=a\tan^2\tfrac{1}{2}\theta+c\cos\theta.$$

$$\therefore \quad dz/d\theta=2a(\tan\tfrac{1}{2}\theta\sec^2\tfrac{1}{2}\theta).\tfrac{1}{2}-c\sin\theta$$

$$=a\tan\tfrac{1}{2}\theta\sec^2\tfrac{1}{2}\theta-c.2\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta$$

$$=\sin\tfrac{1}{2}\theta\,[a\sec^3\tfrac{1}{2}\theta-2c\cos\tfrac{1}{2}\theta].$$

For the equilibrium of the rod, we must have $dz/d\theta=0$

i.e., $\sin\frac{1}{2}\theta\,(a\sec^3\frac{1}{2}\theta-2c\cos\frac{1}{2}\theta)=0.$

$\therefore$ either $\sin\frac{1}{2}\theta=0$ *i.e.*, $\theta=0$,

which gives the vertical position of equilibrium,

or $a\sec^3\frac{1}{2}\theta-2c\cos\frac{1}{2}\theta=0$ *i.e.*, $a\sec^3\frac{1}{2}\theta=2c\cos\frac{1}{2}\theta$

i.e., $\cos^4\frac{1}{2}\theta=a/2c$, which gives the inclined position of rest of the rod.

Now

$$\frac{d^2z}{d\theta^2}=\tfrac{1}{2}\cos\tfrac{1}{2}\theta\,[a\sec^3\tfrac{1}{2}\theta-2c\cos\tfrac{1}{2}\theta]$$

$$+\sin\tfrac{1}{2}\theta\left[\frac{3a}{2}\sec^3\tfrac{1}{2}\theta\tan\tfrac{1}{2}\theta+c\sin\tfrac{1}{2}\theta\right]$$

$$=\tfrac{1}{2}\cos\tfrac{1}{2}\theta\,[a\sec^3\tfrac{1}{2}\theta-2c\cos\tfrac{1}{2}\theta]+\sin^2\tfrac{1}{2}\theta\,[\tfrac{3}{2}a\sec^4\tfrac{1}{2}\theta+c],$$

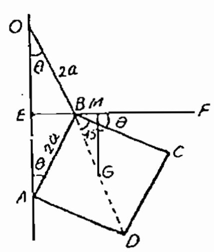
which is >0 when $\cos^4\frac{1}{2}\theta=a/2c$

i.e., when $a\sec^3\frac{1}{2}\theta-2c\cos\frac{1}{2}\theta=0$.

Thus in the inclined position of equilibrium of the rod, $d^2z/d\theta^2$ is positive *i.e.*, z is minimum. Hence the equilibrium is stable in the inclined position of rest of the rod.

Ex. 25. *A square lamina rests with its plane perpendicular to a smooth wall one corner being attached to a point in the wall by a fine string of length equal to the side of the square. Find the position of equilibrium and show that it is stable.*

Sol. $ABCD$ is a square lamina of side $2a$. It is suspended from the point O in the wall by a fine string OB of length $2a$. The corner A of the lamina touches the wall and the plane of the lamina is perpendicular to the wall.



Let $\angle BAO=\theta$.

Then $\angle AOB=\angle BAO=\theta$.

[$\because$ $AB=OB$]

Since BC is perpendicular to AB and the horizontal line EF is perpendicular to AO, therefore $\angle FBC=\theta$.

The centre of gravity of the lamina is the middle point G of the diagonal BD. We have

$$BG=\tfrac{1}{2}BD=\tfrac{1}{2}.2a\sqrt{2}=a\sqrt{2}.$$

$\therefore$ $\angle CBD=45°$ and $\angle FBG=45°+\theta$.

If z be the depth of G below the fixed point O, then

$$z=OE+MG=2a\cos\theta+BG\sin(45°+\theta)$$

$$=2a\cos\theta+a\sqrt{2}\left(\frac{1}{\sqrt{2}}\cos\theta+\frac{1}{\sqrt{2}}\sin\theta\right)$$

$$=3a\cos\theta+a\sin\theta.$$

$\therefore$ $dz/d\theta=-3a\sin\theta+a\cos\theta.$

For equilibrium, $dz/d\theta=0$

i.e., $-3a\sin\theta+a\cos\theta=0$ *i.e.*, $\tan\theta=\frac{1}{3}$.

This gives the position of equilibrium *i.e.*, in equilibrium the side AB of the lamina makes an angle $\tan^{-1}\frac{1}{3}$ with the wall.

Now $d^2z/d\theta^2=-3a\cos\theta-a\sin\theta$

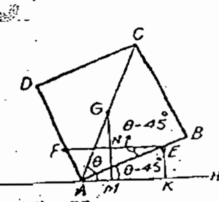
$$=-a\left(3\times\frac{3}{\sqrt{10}}+\frac{1}{\sqrt{10}}\right), \text{ when } \tan\theta=\tfrac{1}{3}$$

$=$ a negative number.

Thus in the position of equilibrium the depth z of the C.G. of the lamina below the fixed point O is maximum. Hence the equilibrium is stable.

Ex. 26. *A square lamina rests in a vertical plane on two smooth pegs which are in the same horizontal line. Show that there is only one position of equilibrium unless the distance between the pegs is greater than one-quarter of the diagonal of the square, but that if this condition is satisfied, there may be three positions of equilibrium and that the symmetrical position will be stable, but the other two positions of equilibrium will be unstable.*

Sol. $ABCD$ is a square lamina resting on the pegs E and F which are in the same horizontal line. Let $EF=c$ and $AC=2d$. Suppose the diagonal AC makes an angle θ with the horizontal AH. Then



$\angle EAK=\theta-\angle CAB=\theta-45°.$

The C.G. of the lamina is the middle point G of the diagonal AC.

Let z be the height of G above the fixed line EF.

Then $z=GN=GM-NM=GM-EK$

$=AG\sin\theta-AE\sin(\theta-45°)$

$=d\sin\theta-EF\cos(\theta-45°)\sin(\theta-45°)$

$=d\sin\theta-\frac{1}{2}c\sin 2(\theta-45°)$

$=d\sin\theta-\frac{1}{2}c\sin(2\theta-90°)$

$=d\sin\theta+\frac{1}{2}c\sin(90°-2\theta)=d\sin\theta+\frac{1}{2}c\cos 2\theta.$

$\therefore$ $dz/d\theta=d\cos\theta-c\sin 2\theta.$

For equilibrium,

$dz/d\theta=0$ *i.e.*, $d\cos\theta-c\sin 2\theta=0$

i.e., $d\cos\theta-2c\sin\theta\cos\theta=0$ *i.e.*, $\cos\theta\,(d-2c\sin\theta)=0.$

$\therefore$ $\cos\theta=0$ *i.e.*, $\theta=\frac{1}{2}\pi$,

or $d-2c\sin\theta=0$ *i.e.*, $\sin\theta=d/2c$ *i.e.*, $\theta=\sin^{-1}(d/2c)$.

In the position of equilibrium given by $\theta=\frac{1}{2}\pi$, the diagonal AC is vertical and the square rests symmetrically on the pegs.

In the position of equilibrium given by $\theta=\sin^{-1}(d/2c)$, if $d/2c<1$, the diagonal AC is not vertical but is inclined at some angle to the vertical. So it gives inclined position of equilibrium.

But we know that $\sin\theta=\sin(\pi-\theta)$.

Hence we shall have two inclined positions of equilibrium given by

$$\theta=\sin^{-1}(d/2c) \text{ and } \theta=\pi-\sin^{-1}(d/2c).$$

The inclined position of equilibrium is possible only when

$d/2c<1$. [$\because$ $\sin\theta<1$ for inclined position]

i.e., when $d<2c$ *i.e.*, when $c>\frac{1}{2}d$ *i.e.*, when $c>\frac{1}{4}.(2d)$

i.e., when the distance between the pegs $>\frac{1}{4}$ (length of the diagonal).

Thus there is only one position of equilibrium (*i.e.*, the symmetrical position) unless the distance between the pegs is greater than one-quarter of the diagonal of the square. Also if $2c>d$, there are three positions of equilibrium.

To determine the nature of equilibrium when $2c>d$.

We have,

$$d^2z/d\theta^2=-d\sin\theta-2c\cos 2\theta$$

$$=-d\sin\theta-2c\,(1-2\sin^2\theta)=-d\sin\theta-2c+4c\sin^2\theta.$$

For the symmetrical position of equilibrium $\theta=\frac{1}{2}\pi$,

$$d^2z/d\theta^2=-d-2c+4c=2c-d>0, \text{ because } 2c>d.$$

$\therefore$ $d^2z/d\theta^2$ is positive when $\theta=\frac{1}{2}\pi$ and so z is minimum for $\theta=\frac{1}{2}\pi$. Hence the symmetrical position of equilibrium given by $\theta=\frac{1}{2}\pi$ is stable.

For the inclined position of equilibrium given by $\sin\theta=d/2c$, we have

$$\frac{d^2z}{d\theta^2}=-d.\frac{d}{2c}-2c+4c.\frac{d^2}{4c^2}=-\frac{d^2}{2c}+\frac{d^2}{c}-2c$$

$$=\frac{d^2-4c^2}{2c}<0, \text{ because } 2c>d.$$

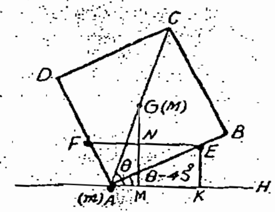
$\therefore$ $d^2z/d\theta^2$ is negative when $\sin\theta = d/2c$ and so z is maximum for the inclined positions of equilibrium. Hence the inclined positions of equilibrium are unstable.

Remark. When $2c < d$, there is only one position of equilibrium *i.e.*, the symmetrical position of equilibrium. For this position of equilibrium, $d^2z/d\theta^2 = 2c - d$ which is < 0, because $2c < d$. Hence z is maximum and the equilibrium is unstable.

Ex. 27. *A uniform square board of mass M is supported in a vertical plane on two smooth pegs on the same horizontal level. The distance between the pegs is a and the diagonal of the square is D, where D > 4a. If one diagonal is vertical and a mass m is attached to its lower end, prove that the equilibrium is stable, if*

$$4am > M(D-4a).$$

Sol. $ABCD$ is a square board resting on the pegs E and F which are in the same horizontal line.



We have
$EF = a$ and $AC = D$.

The mass M of the lamina acts at the middle point G of AC and there is a mass m attached at A. Suppose the diagonal AC makes an angle θ with the horizontal AH. Then

$$\angle EAK = \theta - 45° = \angle FEA.$$

The height of G (*i.e.*, the point where M acts) above EF
$= GN = GM - NM = GM - EK = AG\sin\theta - AE\sin(\theta - 45°)$
$= \frac{1}{2}D\sin\theta - EF\cos(\theta-45°)\sin(\theta-45°)$
$= \frac{1}{2}D\sin\theta - \frac{1}{2}a\sin 2(\theta-45°) = \frac{1}{2}D\sin\theta - \frac{1}{2}a\sin(2\theta-90°)$
$= \frac{1}{2}D\sin\theta + \frac{1}{2}a\sin(90°-2\theta) = \frac{1}{2}D\sin\theta + \frac{1}{2}a\cos 2\theta.$

Also the depth of A (*i.e.*, the point where m acts) below EF
$= EK = AE\sin(\theta-45°) = EF\cos(\theta-45°)\sin(\theta-45°)$
$= \frac{1}{2}a\sin(2\theta-90°) = -\frac{1}{2}a\cos 2\theta.$

Let z be the height of C.G. of the system consisting of the masses M and m above the fixed line EF. Then

$$z = \frac{M(\frac{1}{2}D\sin\theta + \frac{1}{2}a\cos 2\theta) + m[-(-\frac{1}{2}a\cos 2\theta)]}{M+m}$$

$$= \frac{\frac{1}{2}MD\sin\theta + (M+m)\cdot\frac{1}{2}a\cos 2\theta}{M+m}.$$

$$\therefore \quad \frac{dz}{d\theta} = \frac{1}{M+m}[\tfrac{1}{2}MD\cos\theta - a(M+m)\sin 2\theta].$$

For equilibrium, $dz/d\theta = 0$,

i.e., $\frac{1}{2}MD\cos\theta - 2a(M+m)\sin\theta\cos\theta = 0$

i.e., $\cos\theta[\frac{1}{2}MD - 2a(M+m)\sin\theta] = 0.$

$\therefore$ either $\cos\theta = 0$ *i.e.*, $\theta = \frac{1}{2}\pi$,

or $\frac{1}{2}MD - 2a(M+m)\sin\theta = 0$

i.e. $\sin\theta = MD/(4a(M+m))$.

Now $\theta = \frac{1}{2}\pi$ means the diagonal AC is vertical.

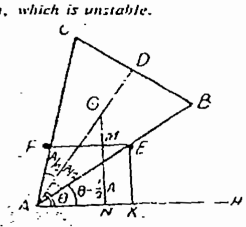
We have $$\frac{d^2z}{d\theta^2} = \frac{1}{M+m}[-\tfrac{1}{2}MD\sin\theta - 2a(M+m)\cos 2\theta]$$

$$= \frac{1}{M+m}[-\tfrac{1}{2}MD + 2a(M+m)], \text{ for } \theta = \tfrac{1}{2}\pi.$$

The equilibrium is stable at $\theta = \frac{1}{2}\pi$ if z is minimum at $\theta = \frac{1}{2}\pi$ *i.e.*, if $d^2z/d\theta^2$ is positive at $\theta = \frac{1}{2}\pi$ *i.e.*, if $-\frac{1}{2}MD - 2a(M+m) > 0$ or, $4am > MD - 4aM$ or, $4am > M(D-4a)$.

Ex. 28 (a). *A uniform isosceles triangular lamina ABC rests in equilibrium with its equal sides AB and AC in contact with two smooth pegs in the same horizontal line at a distance c apart. If the perpendicular AD upon BC is h, show that there are three positions of equilibrium, of which the one with AD vertical is stable and the other two are unstable, if h < 3c cosec A; whilst if h ≥ 3c cosec A, there is only one position of equilibrium, which is unstable.*

Sol. ABC is an isosceles triangular lamina resting on two smooth pegs E and F which are in the same horizontal line and $EF = c$. The perpendicular AD from A upon BC is of length h. We have



$$\angle BAD = \angle CAD = \tfrac{1}{2}A.$$

The weight of the lamina acts at its centre of gravity G, where

$$AD = \tfrac{2}{3}AD = \tfrac{2}{3}h.$$

Suppose AD makes an angle θ with the horizontal AH, so that

$$\angle BAH = \theta - \tfrac{1}{2}A.$$

Let z be the height of G above the fixed horizontal line EF. Then

$z = GM = GN - MN = GN - EK = AG\sin\theta - AE\sin(\theta - \frac{1}{2}A)$

$= \frac{2}{3}h\sin\theta - AE\sin(\theta - \frac{1}{2}A).$ ...(1)

Since EF is parallel to AK, therefore

$$\angle FEA = \angle EAK = \theta - \tfrac{1}{2}A.$$

Now in the $\triangle AEF$, we have

$$\angle EFA = \pi - \{A + (\theta - \tfrac{1}{2}A)\} = \pi - (\theta + \tfrac{1}{2}A).$$

Applying the sine theorem of trigonometry for the $\triangle AEF$, we have

$$\frac{AE}{\sin\angle EFA} = \frac{EF}{\sin\angle FAE}$$

i.e., $$\frac{AE}{\sin\{\pi - (\theta + \frac{1}{2}A)\}} = \frac{c}{\sin A}.$$

$$\therefore \quad AE = \frac{c}{\sin A}\sin(\theta + \tfrac{1}{2}A).$$

Substituting this value of AE in (1), we have

$$z = \tfrac{2}{3}h\sin\theta - \frac{c}{\sin A}\sin(\theta + \tfrac{1}{2}A)\sin(\theta - \tfrac{1}{2}A)$$

$$= \tfrac{2}{3}h\sin\theta - \frac{c}{2\sin A}[\cos A - \cos 2\theta]$$

$$= \tfrac{2}{3}h\sin\theta - \frac{c}{2}\cot A + \frac{c}{2\sin A}\cos 2\theta.$$

$$\therefore \quad \frac{dz}{d\theta} = \tfrac{2}{3}h\cos\theta - \frac{c}{\sin A}\sin 2\theta. \quad \text{...(2)}$$

For equilibrium, $dz/d\theta = 0$

i.e., $\frac{2}{3}h\cos\theta - \frac{2c}{\sin A}\sin\theta\cos\theta = 0$

i.e., $2\cos\theta\left[\frac{1}{3}h - \frac{c\sin\theta}{\sin A}\right] = 0.$

$\therefore$ either $\cos\theta = 0$ *i.e.* $\theta = \frac{1}{2}\pi$

or $\frac{1}{3}h - \frac{c\sin\theta}{\sin A} = 0$ *i.e.* $\sin\theta = \frac{h\sin A}{3c} = \frac{h}{3c\,\text{cosec}\,A}.$

Now $\theta = \frac{1}{2}\pi$ gives the position of equilibrium in which AD is vertical and the triangle rests symmetrically on the pegs. The values of θ given by $\sin\theta = h/(3c\,\text{cosec}\,A)$ are real and not equal to $\frac{1}{2}\pi$ if $h < 3c\,\text{cosec}\,A$. Since

$$\sin(\pi - \theta) = \sin\theta,$$

therefore if $h < 3c\,\text{cosec}\,A$, the equation $\sin\theta = h/(3c\,\text{cosec}\,A)$ gives two inclined positions of equilibrium, one θ and the other $\pi - \theta$. Thus if $h < 3c\,\text{cosec}\,A$, there are three positions of equilibrium, one symmetrical and the other two inclined.

If $h \geq 3c\,\text{cosec}\,A$, then the equation $\sin\theta = h/(3c\,\text{cosec}\,A)$ either gives no real value of θ or the value of θ given by it is also equal to $\frac{1}{2}\pi$. Thus in this case the symmetrical position of equilibrium, $\theta = \frac{1}{2}\pi$, is the only position of equilibrium.

Nature of equilibrium.

From (2), $$\frac{d^2z}{d\theta^2} = -\tfrac{2}{3}h\sin\theta - \frac{2c}{\sin A}\cos 2\theta. \quad \text{...(3)}$$

For $\theta = \frac{1}{2}\pi$, $$\frac{d^2z}{d\theta^2} = -\tfrac{2}{3}h + \frac{2c}{\sin A} = \tfrac{2}{3}(-h + 3c\,\text{cosec}\,A),$$

which is positive or negative according as

$$h < \text{ or } > 3c\,\text{cosec}\,A.$$

Thus for $\theta = \frac{1}{2}\pi$, z is minimum or maximum according as

$$h < \text{ or } > 3c\,\text{cosec}\,A.$$

Hence for $\theta = \frac{1}{2}\pi$, the equilibrium is stable or unstable according as

$$h < \text{ or } > 3c\,\text{cosec}\,A.$$

For $\theta = \frac{1}{2}\pi$, $d^2z/d\theta^2 = 0$ when $h = 3c\,\text{cosec}\,A$. In this case we can see that $d^3z/d\theta^3 = 0$ and $d^4z/d\theta^4 = -6c\,\text{cosec}\,A$, which is negative. So in this case z is maximum and the equilibrium is unstable. Thus the symmetrical position of equilibrium is stable or unstable according as

$$h < \text{ or } \geq 3c\,\text{cosec}\,A.$$

Now we consider the inclined positions of equilibrium. From (3), we can write

$$\frac{d^2z}{d\theta^2} = -\tfrac{2}{3}h\sin\theta - \frac{2c}{\sin A}(1 - 2\sin^2\theta). \quad \text{...(4)}$$

For the inclined positions of equilibrium, $\sin\theta = (h\sin A)/3c$. Putting $\sin\theta = (h\sin A)/3c$ in (4), we get

$$\frac{d^2z}{d\theta^2} = -\frac{2h}{3}\cdot\frac{h\sin A}{3c} - \frac{2c}{\sin A} + \frac{4c}{\sin A}\cdot\frac{h^2\sin^2 A}{9c^2}$$

$$= \frac{2h^2}{9c}\sin A - \frac{2c}{\sin A} = \frac{2}{9c}\sin A\,(h^2 - 9c^2\,\text{cosec}^2 A),$$

which is negative since for inclined positions of equilibrium

$$h < 3c\,\text{cosec}\,A.$$

Thus for the inclined positions of equilibrium, z is maximum and so they are positions of unstable equilibrium.

Remark. For inclined positions of equilibrium to exist, we must have $h < 3c\,\text{cosec}\,A$. For these positions of equilibrium, θ is given by $\sin\theta = (h\sin A)/3c$.

Now $\frac{1}{2}A < \theta \Rightarrow \sin\frac{1}{2}A < \sin\theta \Rightarrow \sin\frac{1}{2}A < (h\sin A)/3c$

$$\Rightarrow \sin\tfrac{1}{2}A < \frac{2h\sin\frac{1}{2}A\cos\frac{1}{2}A}{3c} \Rightarrow h > \tfrac{3}{2}c\sec\tfrac{1}{2}A.$$

Thus for inclined positions of equilibrium, we must have

$$\tfrac{3}{2}c\sec\tfrac{1}{2}A < h < 3c\operatorname{cosec} A.$$

Ex. 28. (b) *An isosceles triangular lamina of an angle 2α and height h rests between two smooth pegs at the same level, distant 2c apart; prove that if*

$$3c\sec\alpha < h < 6c\operatorname{cosec}2\alpha,$$

the oblique positions of equilibrium exist, which are unstable. Discuss the stability of the vertical position.

Sol. Proceed as in Ex. 28 (a). The complete question has been solved there.

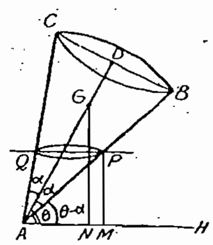
Ex. 29 (a). *A smooth solid right circular cone, of height h and vertical angle 2α, is at rest with its axis vertical in a horizontal circular hole of radius a. Show that if $16a > 3h\sin 2\alpha$, the equilibrium is stable, and there are two other positions of unstable equilibrium; and that if $16a < 3h\sin 2\alpha$, the equilibrium is unstable, and the position in which the axis is vertical is the only position of equilibrium.*

Sol. ABC is a solid right circular cone whose height AD is h and vertical angle BAC is 2α. It rests in a horizontal circular hole PQ of radius a, so that $PQ=2a$. We have

$$\angle BAD = \angle CAD = \alpha.$$

The weight of the cone acts at its centre of gravity G, where

$$AG = \tfrac{3}{4}AD = \tfrac{3}{4}h.$$



Suppose AD makes an angle θ with the horizontal AH, so that

$$\angle BAH = \theta - \alpha.$$

Let z be the height of G above the fixed horizontal line PQ. Then

$$z = GN - PM$$
$$= AG\sin\theta - AP\sin(\theta-\alpha)$$
$$= \tfrac{3}{4}h\sin\theta - AP\sin(\theta-\alpha). \quad ...(1)$$

Since PQ is parallel to AM, therefore

$$\angle QPA = \angle PAM = \theta - \alpha.$$

Now in the $\triangle APQ$, we have

$$\angle PQA = \pi - \{2\alpha + (\theta-\alpha)\} = \pi - (\theta+\alpha).$$

Applying the sine theorem of trigonometry for the $\triangle APQ$, we have

$$\frac{AP}{\sin\{\pi-(\theta+\alpha)\}} = \frac{PQ}{\sin 2\alpha}$$

$$\therefore \quad AP = \frac{2a}{\sin 2\alpha}\sin(\theta+\alpha), \text{ because } PQ = 2a.$$

Putting the value of AP in (1), we have

$$z = \tfrac{3}{4}h\sin\theta - \frac{2a\sin(\theta+\alpha)}{\sin 2\alpha}\sin(\theta-\alpha)$$

$$= \tfrac{3}{4}h\sin\theta - \frac{a}{\sin 2\alpha}[\cos 2\alpha - \cos 2\theta]$$

$$= \tfrac{3}{4}h\sin\theta - a\cot 2\alpha + \frac{a}{\sin 2\alpha}\cos 2\theta.$$

$$\therefore \quad \frac{dz}{d\theta} = \tfrac{3}{4}h\cos\theta - \frac{2a}{\sin 2\alpha}\sin 2\theta. \quad ...(2)$$

For equilibrium, $dz/d\theta = 0$

i.e., $\quad \tfrac{3}{4}h\cos\theta - \dfrac{4a}{\sin 2\alpha}\sin\theta\cos\theta = 0$

i.e., $\quad \cos\theta\left[\tfrac{3}{4}h - \dfrac{4a\sin\theta}{\sin 2\alpha}\right] = 0.$

$\therefore$ either $\cos\theta = 0$ *i.e.*, $\theta = \tfrac{1}{2}\pi$,

or $\quad \tfrac{3}{4}h - \dfrac{4a\sin\theta}{\sin 2\alpha} = 0$ *i.e.*, $\sin\theta = \dfrac{3h\sin 2\alpha}{16a}.$

Now $\theta = \tfrac{1}{2}\pi$ gives the position of equilibrium in which the axis AD of the cone is vertical. The values of θ given by

$$\sin\theta = (3h\sin 2\alpha)/16a$$

are real and not equal to $\tfrac{1}{2}\pi$ if $\sin\theta < 1$ *i.e.*, if $16a > 3h\sin 2\alpha$. Since $\sin(\pi-\theta) = \sin\theta$, therefore if $16a > 3h\sin 2\alpha$, the equation

$$\sin\theta = (3h\sin 2\alpha)/16a$$

gives two oblique positions of equilibrium one θ and the other $\pi-\theta$. Thus if $16a > 3h\sin 2\alpha$, there are three positions of equilibrium, one in which the axis AD is vertical and the other two inclined.

If $16a < 3h\sin 2\alpha$, the equation

$$\sin\theta = (3h\sin 2\alpha)/16a$$

gives no real value of θ. Thus in this case the only position of equilibrium is that in which the axis of the cone is vertical.

Nature of equilibrium

From (2), $\quad \dfrac{d^2z}{d\theta^2} = -\tfrac{3}{4}h\sin\theta - \dfrac{4a}{\sin 2\alpha}\cos 2\theta. \quad ...(3)$

For $\theta = \tfrac{1}{2}\pi$,

$$\frac{d^2z}{d\theta^2} = -\tfrac{3}{4}h + \frac{4a}{\sin 2\alpha} = \frac{1}{4\sin 2\alpha}[-3h\sin 2\alpha + 16a],$$

which is positive or negative according as

$$16a > \text{ or } < 3h\sin 2\alpha.$$

Thus for $\theta = \tfrac{1}{2}\pi$, z is minimum or maximum according as

$$16a > \text{ or } < 3h\sin 2\alpha.$$

Hence the vertical position of equilibrium is stable or unstable according as $16a >$ or $< 3h\sin 2\alpha$.

Now we consider the inclined positions of equilibrium given by $\quad \sin\theta = (3h\sin 2\alpha)/16a.$

These exist only if $16a > 3h\sin 2\alpha$. From (3), we can write

$$\frac{d^2z}{d\theta^2} = -\tfrac{3}{4}h\sin\theta - \frac{4a}{\sin 2\alpha}(1-2\sin^2\theta).$$

Putting $\sin\theta = (3h\sin 2\alpha)/16a$ in it, we get

$$\frac{d^2z}{d\theta^2} = -\tfrac{3}{4}h\cdot\frac{3h\sin 2\alpha}{16a} - \frac{4a}{\sin 2\alpha} + \frac{8a}{\sin 2\alpha}\cdot\frac{9h^2\sin^2 2\alpha}{256a^2}$$

$$= \frac{9h^2}{64a}\sin 2\alpha - \frac{4a}{\sin 2\alpha} = \frac{9h^2\sin^2 2\alpha - 256a^2}{64a\sin 2\alpha} = \frac{(3h\sin 2\alpha)^2 - (16a)^2}{64a\sin 2\alpha}$$

which is negative since for inclined positions of equilibrium

$$16a > 3h\sin 2\alpha.$$

Thus for the inclined positions of equilibrium, z is maximum and so they are positions of unstable equilibrium.

Ex. 29. (b) *A smooth cone is placed with vertex downwards in a circular horizontal hole. Prove that the position of equilibrium with the axis vertical is unstable or stable according as it is, or, is not, the only possible position of equilibrium.*

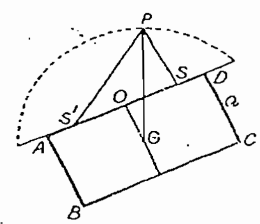
Sol. Proceed as in Ex. 29 (a). Also take help from Ex. 28.

Ex. 30. (a) *A rectangular picture hangs in a vertical position by means of a string, of length l, which after passing over a smooth nail has its ends attached to two points symmetrically situated in the upper edge of the picture at a distance c apart. If the height of the picture is a, show that there is no position of equilibrium in which a side of the picture is inclined to the horizon if $la > c\sqrt{(c^2+a^2)}$, whilst if $la < c\sqrt{(c^2+a^2)}$, there are two such positions which are both stable.*

Show also that in the latter case the position in which the side is vertical is stable for some and unstable for other displacements.

Sol. $ABCD$ is a rectangular picture which hangs by means of a string of length l passing over the peg P, the ends of the string being attached to two points S and S' symmetrically situated in the upper edge AD of the picture such that $SS' = c$. If O is the middle point of AD, then O is also the middle point of SS' because S and S' are symmetrically situated in AD. Therefore

$$OS = OS' = \tfrac{1}{2}c.$$



If G be the centre of gravity of the picture, then $OG = \tfrac{1}{2}a$, as height CD of the picture is given to be a.

We have $\quad SP + S'P = l. \quad ...(1)$

From the relation (1), it is obvious that P lies on an ellipse whose foci are S and S' and the length say 2α, of whose major axis is l, so that $\alpha = \tfrac{1}{2}l$.

We have $OS = \alpha e$, where e is the eccentricity of the ellipse.

$$\therefore \quad \alpha e = \tfrac{1}{2}c.$$

If β be the semi major axis of the ellipse, then

$$\beta^2 = \alpha^2 - \alpha^2e^2 = \tfrac{1}{4}l^2 - \tfrac{1}{4}c^2 = \tfrac{1}{4}(l^2-c^2), \text{ so that } \beta = \tfrac{1}{2}\sqrt{(l^2-c^2)}.$$

The centre of the ellipse is the middle point O of SS'. Take O as origin, OS as x-axis and a line perpendicular to OS through O as y-axis. Then the coordinates of G are $(0, -\tfrac{1}{2}a)$. Let the coordinates of P be $(\alpha\cos\theta, \beta\sin\theta)$.

Since the line PG is vertical, therefore if z be the depth of G below the fixed point P, then $z = PG$.

Now z is maximum or minimum according as z^2 or PG^2 is maximum or minimum.

Let $\quad u = PG^2 = (\alpha\cos\theta - 0)^2 + (\beta\sin\theta + \tfrac{1}{2}a)^2$

$$= \alpha^2\cos^2\theta + \beta^2\sin^2\theta + a\beta\sin\theta + \tfrac{1}{4}a^2.$$

$$\therefore \quad \frac{du}{d\theta} = 2(\beta^2-\alpha^2)\sin\theta\cos\theta + a\beta\cos\theta.$$

For equilibrium,

$$dz/d\theta = 0 \text{ i.e., } du/d\theta = 0,$$

i.e., $\quad \cos\theta\,[2(\beta^2-\alpha^2)\sin\theta + a\beta] = 0.$

$\therefore$ either $\cos\theta=0$ i.e., $\theta=\frac{1}{2}\pi$,

or $$\sin\theta=\frac{a\beta}{2(\alpha^2-\beta^2)}=\frac{a.\frac{1}{2}\sqrt{(l^2-c^2)}}{2\left[\frac{1}{4}l^2-\frac{1}{4}(l^2-c^2)\right]}=\frac{a\sqrt{(l^2-c^2)}}{c^2}, \quad ...(2)$$

after substituting the values of α and β.

Here, $\theta=\frac{1}{2}\pi$ gives the position of equilibrium, symmetrical about the peg P, in which the sides AB and CD of the picture hang vertically.

There is no inclined position of equilibrium if the value of $\sin\theta$ given by (2) is >1,

i.e., if $a\sqrt{(l^2-c^2)}>c^2$, i.e., if $a^2l^2-a^2c^2>c^4$

i.e., if $a^2l^2>c^2(c^2+a^2)$ i.e., if $al>c\sqrt{(a^2+c^2)}$.

Thus if $al>c\sqrt{(a^2+c^2)}$, there is no position of equilibrium in which a side of the picture is inclined to the horizon. In this case the symmetrical position $\theta=\frac{1}{2}\pi$ is the only position of equilibrium.

But if the value of $\sin\theta$ given by (2) is <1,

i.e., $a\sqrt{(l^2-c^2)}<c^2$, or $al<c\sqrt{(a^2+c^2)}$,

then (2) gives real values of θ. Since $\sin\theta=\sin(\pi-\theta)$, therefore when $al<c\sqrt{(a^2+c^2)}$, we have two inclined positions of equilibrium given by (2). In these positions the side CD may be inclined towards either side of the vertical. In this case there are in all three positions of equilibrium, one symmetrical, given by $\theta=\frac{1}{2}\pi$, and the other two, which are inclined, given by (2).

Nature of the positions of equilibrium.

We have,

$$d^2u/d\theta^2=2(\beta^2-\alpha^2)(\cos^2\theta-\sin^2\theta)-a\beta\sin\theta$$
$$=2(\beta^2-\alpha^2)(1-2\sin^2\theta)-a\beta\sin\theta. \quad ...(3)$$

For the symmetrical position of equilibrium given by $\theta=\frac{1}{2}\pi$,

$$\frac{d^2u}{d\theta^2}=-2(\beta^2-\alpha^2)-a\beta$$
$$=-2\left[\frac{1}{4}(l^2-c^2)-\frac{1}{4}l^2\right]-a.\frac{1}{2}\sqrt{(l^2-c^2)}$$
$$=\frac{1}{2}c^2-\frac{1}{2}a\sqrt{(l^2-c^2)}=\frac{1}{2}\left[c^2-a\sqrt{(l^2-c^2)}\right],$$

which is positive or negative according as $a\sqrt{(l^2-c^2)}<$ or $>c^2$ i.e., according as $al<$ or $>c\sqrt{(a^2+c^2)}$.

Thus if $al<c\sqrt{(c^2+a^2)}$, then u and so also z is minimum. Since z is the depth of G below the fixed point P, therefore the equilibrium is unstable in this case. Again if $al>c\sqrt{(c^2+a^2)}$, then u and so also z is maximum, and the equilibrium is stable. Hence the symmetrical equilibrium position $\theta=\frac{1}{2}\pi$ is unstable if

$$al<c\sqrt{(c^2+a^2)}$$

and stable if $al>c\sqrt{(c^2+a^2)}$.

Now consider the inclined positions of equilibrium given by

$$\sin\theta=\{a\sqrt{(l^2-c^2)}\}/c^2,$$

which give real values of θ only if

$$a\sqrt{(l^2-c^2)}<c^2, \text{ or } al<c\sqrt{(c^2+a^2)}.$$

In this case putting $\sin\theta=\{a\sqrt{(l^2-c^2)}\}/c^2$ in (3), we get

$$\frac{d^2u}{d\theta^2}=2\left[\frac{1}{4}(l^2-c^2)-\frac{1}{4}l^2\right]\left[1-2.\frac{a^2(l^2-c^2)}{c^4}\right]$$
$$-a.\frac{1}{2}\sqrt{(l^2-c^2)}.\frac{a}{c^2}\sqrt{(l^2-c^2)}$$
$$=-\frac{c^2}{2}+\frac{a^2(l^2-c^2)}{c^2}-\frac{a^2(l^2-c^2)}{2c^2}$$
$$=-\frac{c^2}{2}+\frac{a^2(l^2-c^2)}{2c^2}=\frac{1}{2c^2}\left[a^2(l^2-c^2)-c^4\right],$$

which is negative because $a\sqrt{(l^2-c^2)}<c^2$.

Thus in this case u and so also z is maximum and the equilibrium is stable. Hence if $al<c\sqrt{(c^2+a^2)}$, there are two inclined positions of equilibrium and they are both stable.

Ex. 30 (b). *A rectangular picture-frame hangs from a small perfectly smooth pulley by a string of length 2a attached symmetrically to two points on the upper edge at a distance 2c apart. Prove that if the depth of the picture is less than*

$$2c^2/\sqrt{(a^2-c^2)},$$

there are three positions of equilibrium of which the symmetrical one is unstable. If the depth exceeds the above value the symmetrical position of equilibrium is the only one and is stable.

Sol. Proceed as in Ex. 30 (a).