IMS™
(INSTITUTE OF MATHEMATICAL SCIENCES)

INSTITUTE FOR IAS/IFOS EXAMINATIONS

MATHEMATICS OPTIONAL

By

K. VENKANNA

WORKSHEET - 1
LINEAR ALGEBRA

QLP

(QUICK LEARNING PROGRAMME FOR MAINS-2016)

Scoring Maximum Marks in Main-2016

FEATURES OF MATHEMATICS PROGRAMME

- Class Timings – 9:30 AM – 5:30 PM.
- Atleast one day one module to be discussed (Expected)
- A question based approach to be followed.
- Pin-pointed formulas to be provided.
- 15-20 problems to be provided on each formula set.
- Time-bound problem solving sessions.
- On board discussion sessions to be followed there after.
- Around 50-100 problems expected to be solved per session.
- Only problems to be discussed, no concept explanation.
- Sessions will be from Monday to Friday.

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Room No. 202, R.L'S-Kancham's Blue Supphire, Ashok Nagar, Hyderabad-20.

WORKSHEET - 1

VECTOR SPACES

Topics to be covered:

Vector Space, Subspace, Algebra of Subspaces, Linear Combination of Vectors, Linear Span of Vectors, Smallest Subspace containing any Subset of a vector Space. Linear dependence of Vectors & Linear Independence of Vectors and their properties.

1. The vectors $V_1 = (1,1,2,4)$, $V_2 = (2,-1,-5,2)$, $V_3 = (1,-1,-4,0)$ and $V_4 = (2,1,1,6)$ are linearly independent. Is it true? Justify your answer.

2. Find one vector in R^3 which generates the intersection of V and W, where V is the xy plane and W is the space generated by the vectors (1, 2, 3) and (1, –1, 1).

3. (i) Let V be the vector space of all 2×2 matrices over the field of real numbers. Let W be the set consisting of all matrices with zero determinant. Is W a subspace of V? Justify your answer.

   (ii) Find the dimension and a basis for the space W of all solutions of the following homogeneous system using matrix notation:

$$x_1 + 2x_2 + 3x_3 - 2x_4 + 4x_5 = 0$$
$$2x_1 + 4x_2 + 8x_3 + x_4 + 9x_5 = 0$$
$$3x_1 + 6x_2 + 13x_3 + 4x_4 + 14x_5 = 0$$

4. (i) In the n-space IR^n, determine whether or not the set $\{e_1 - e_2, e_2 - e_3 \ldots, e_{n-1} - e_n, e_n - e_1\}$ is linearly independent.

   (ii) Let T be a linear transformation from a vector space V over reals into V such that $T - T^2 = I$. Show that T is invertible.

5. Let S be a non empty set and let V denote the set of all functions from S into R. show that V is a vector space with respect to the vector addition $(f+g)(x) = f(x) + g(x)$ and scalar multiplication $(cf)(x) = cf(x)$

6. Let S be any non-empty subset of a vector space V over the field F. Show that the set $\{a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n : a_1, a_2, \ldots, a_n \in F, \alpha_1, \alpha_2, \ldots \alpha_n \in S, n \in N\}$ is the subspace generated by S.

7. Show that the vectors (1, 0, -1), (0, -3, 2) and (1, 2, 1) form a basis for the vector space $R^3(R)$.

8. Let V be the vector space of functions from R to IR (the real numbers). Show that f, g, h in V are linearly independent where $f(t) = e^{2t}$, $g(t) = t^2$ and $h(t) = t$

9. If w is a subspace of a finite dimensional space V, then prove that
dim v/w = dimv -dimw

10. Find a maximal linearly independent subsystem in the system of vectors:
$V_1 = (2, -2, -4)$; $V_2 = (1, 9, 3)$; $V_3 = (-2, -4, 1)$ and $V_4 = (3, 7, -1)$

11. Let V be the vector space of polynomials of degree ≤ 3. Determine whether the following vectors of V are linearly dependent or independent:

$u = t^3 - 3t^2 + 5t + 1$,

$v = t^3 - t^2 + 8t + 2$,

$w = 2t^3 - 4t^2 + 9t + 5$.

12. Show that $u_1=(1,-1,0)$, $u_2=(1,1,0)$ and $u_3=(0,1,1)$ from a basis for $\mathbb{R}^3$. Express (5, 3, 4) in terms of u_1, u_2 and u_3.

13. Let $V = \mathbb{R}^3$ and $\alpha_1 = (1,1,2)$, $\alpha_2 = (0,1,3)$, $\alpha_3 = (2,4,5)$ and $\alpha_4 = (-1,0,-1)$ be the elements of V. Find a basis for the intersection of the subspace spanned by $\{\alpha_1, \alpha_2\}$ and $\{\alpha_3, \alpha_4\}$.

14. Show that the set of all functions which satisfy the differential equation $\dfrac{d^2 f}{dx^2} + 3\dfrac{df}{dx} = 0$ is a vector space.

15. Show that vectors $(0,2,-4),(1,-2,-1),(1,-4,3)$ are linearly dependent. Also express $(0,2,-4)$ as a linear combination of $(1,-2,-1)$ and $(1,-4,3)$.

16. Show that the vectors $v_1 = (1,1,1)$, $v_2 = (0,1,1)$, $v_3 = (0,0,1)$ form a basis for IR^3. Express $v = (3,1,-4)$ as a linear combination of v_1, v_2 and v_3. Is the set $S = \{v, v_1, v_2, v_3\}$ linearly independent ?

17. Let C(IR) be the vector space of complex numbers over the field of real numbers. Under what conditions of the real numbers $\alpha, \beta, \gamma, \delta$ in the set

$S = \{\alpha + i\beta, \gamma + i\delta\}$, $i = \sqrt{(-1)}$

do we have $L(s) = C(IR)$

where L(s) denotes the linear spans of S? Justify your answer

18. Find the values of k for which the vector (1,1,1,1), (1, 3, –2, k).
(2, 2k–2, –k–2, 3k–1) and (3, k+2, –3, 2k+1) are linearly independent in $\mathbb{R}^4$.

19. Let $\alpha_1 = (1, 1, -2, 1)$, $\alpha_2 = (3, 0, 4, -1)$ $\alpha_3 = (-1, 2, 5, 2)$
Let $\alpha = (4, -5, 9, -7)$, $\beta = (3, 1, -4, 4)$ $\gamma = (-1, 1, 0, 1)$
(i) Which of the vectors α, β, γ are in the subspaces of R^4 spanned by the α_i ?
(ii) Which of the vectors α, β, γ are in the suspaces of C^4 spanned by the α_i ?
(iii) Does this suggest a theorem?

20. Let W be the set of all $(x_1, x_2, x_3, x_4, x_5)$ in R^5 which satisfy

$$2x_1 - x_2 + \frac{4}{3}x_3 - x_4 = 0$$

$$x_1 + \frac{2}{3}x_3 - x_5 = 0$$

$9x_1 - 3x_2 + 6x_3 - 3x_4 - 3x_5 = 0$, find a finite set of verctors which spans W.

21. Are the vectors
$\alpha_1 = (1, -1, -2, 4)$, $\alpha_2 = (2, -1, -5, 2)$,
$\alpha_3 = (1, -1, -4, 0)$, $\alpha_4 = (2, 1, 1, 6)$
lineraly independent in $\mathbb{R}^4$, Also find a basis for the subspace of $\mathbb{R}^4$ spanned by the four vectors.

22. Show that the vectors $X_1 = (1. \ 1 + i, \ i)$, $=$ $(i, - i \ 1- i)$ and $X_3 = (0, 1–2i, 2 – i)$ in C^3 are linearly independent over the field of real numbers but are linearly dependent over the field of complex numbers.

23. Find two subspaces A and B of $V = \mathbb{R}^4(\mathbb{R})$ such that dim A=2, dim B=3 and dim $(A \cap B) = 1$.

24. Show that the sets $w_1 = \left\{ \begin{pmatrix} a & 0 \\ a & 0 \end{pmatrix} \middle| a \in \mathbb{R} \right\}$.

$w_2 = \left\{ \begin{pmatrix} 0 & a \\ 0 & a \end{pmatrix} \middle| a \in \mathbb{R} \right\}$ are subspaces of the vector space M_2 of 2x2 matrices over $\mathbb{R}$.

25. Let R^+ be the set of all positive real numbers. Define the operations of addition and scalar multiplication as follows:

$u + v = u.v$ *for all* $u, v \in R^+$

$\alpha u = u^{\alpha}$ *for all* $u \in R^+$

and real scalar α. prove that R^+ is a real vector space.

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WORKSHEET - 2
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WORKSHEET - 2

BASIS AND DIMENSION

Topics to be covered:

Basis, Finite dimensional Vector Space (FDVS), Infinite Dimensional Vector Space (IDVS) Dimension of a vector Space & its related theorems.
Like: Basis Existence Theorem, Basis Extension etc.
Echelon form of a Matrix, row reduced echelon form of a Matrix etc.

1. Suppose U and W are distinct four-dimensional subspaces of a vector space V, where dim V = 6. Find the possible dimensions of $U \cap W$. (10)

2. Find the dimension and a basis of the solution space W of the system

$$x+2y+2z-s+3t=0,$$
$$x+2y+3z+s+t=0,$$
$$3x+6y+8z+s+5t=0$$

3. Let V be the vector space of 2×2 matrices over the field of real numbers $\mathbb{R}$. Let $W=\{A\in V \mid Trace\ \ A=0\}$. Show that W is a subspace of V. Find a basis of W and dimension of W.

4. Show that the vectors $\alpha_1=(1,0,-1)$, $\alpha_2=(1,2,1)$, $\alpha_3(0,-3,2)$ form a basis for $\mathbb{R}^3$. Find the components of $(1,0,0)$ w.r.t the basis $\{\alpha_1,\alpha_2,\alpha_3\}$.

5. Let V be the vector space of polynomials over R. Let U and W be the subspaces generated by $\{t^3+4t^2-t+3, t^3+5t^2+5, 3t^3+10t^2-5t+5\}$ and $\{t^3+4t^2+6, 2t^2-t+5, 2t^3+2t^2-3t+9)$ recpectively. Find

(i) $\dim(U+W)$

(ii) $\dim(U\cap W)$.

6. If S and T are subspace of IR^4 given by

$$S=\left\{(x,y,z,w)\in IR^4 : 2x+y+3z+w=0\right\}$$

and

$$T=\left\{(x,y,z,w)\in IR^4 : x+2y+z+3w=0\right\},$$

find $\dim(S\cap T)$.

7. Suppose U and W are subspaces of the vector space $IR^4\ (IR)$ generated by the sets

$$B_1=\{(1,1,0,-1),(1,2,3,0),(2,3,3,-1)\}$$
$$B_2=\{(1,2,2,-2),(2,3,2,-3),(1,3,4,-3)\}$$

recpectively. Determine $\dim(U+W)$.

8. Show that the solutions of the differential equation $2\frac{d^2y}{dx^2}-9\frac{dy}{dx}+2y=0$ is a subspace

of the vectorspace of all real valued continuous functions.

9. Let V be the real vector space of all polynomials in one variable with real coefficients and of degree less than, or equal to, 5. Let W be the subspace defined by
$W = \{p \in V \mid p(1) = p'(2) = 0\}$.
What is the dimension of W?

10. Let U and W be subspaces of IR^3 for which dim U = , dim w = 2. and $U \not\subset W$. Show that IR^3 = U + w and $U \cap W = \{0\}$.

11. Find a basis for $\mathbb{R}^4$ that contains vectors $v = [1,2,0,0]$ and $u = [0,0,1,2]$.

12. Suppose U and W are distinct four-dimensional subspaces of a vector space V, where dim V = 6. Find the possible dimensions of $U \cap W$.

13. Find a basis for a subspace U of V in the folliwng

(i) $U = \left\{(x_1, x_2, x_3, x_4, x_5) \in V_5 / \begin{matrix} x_1 + x_2 + x_3 = 0 \\ 3x_1 - x_4 + 7x_5 = 0 \end{matrix}\right\}, V = V_5$

(ii) U = {p∈ ρ_4/ p(x_0) = 0}, V = ρ_4 , where ρ_4 is the set of all polynomials of degree ≤ 4.

14. Show that the subspaces of IR^3 spanned by two sets of vectors {(1, 1, –1), (1, 0,1) and {1, 2, –3}, (5, 2, 1)} are identical. Also find the dimension of this subspace.

15. Show that the set
P[t] = {at² + bt + c/a, b, c $\in \mathbb{R}$ }
forms a vector space over the field $\mathbb{R}$. Find a basis for this vector space. What is the dimension of this vector space?

16. Let v = {(a, b, c, d)/ b + c + d = 0 } and w = {(a, b, c, d)/ a + b = 0, c = 2d} are the subspaces of R^4.
Find the dimension and basis of
(i) v
(ii) w
(iii) $v \cap w$

17. Let W be the subspace of R^4 generated by vectors (1, –2, 5, –3), (2, 3, 1, –4) and (3, 8, – 3, –5). Find a basis and dimension of W. Extend this basis of W to a basis of R^4.

18. (i) Let V be the set of real numbers. Regard V as a vector space over the field of rational numbers, with the usual operations. Prove that this vector space is not finite-dimensional.
(ii) Is F[x] finite dimensional ? Justify.

19. Take V = P_3, the space oif all real polynomials of degree at most 3. Let U be the subspace of P_3 consisting of those polynomials of P_3 consisting of those polynomials of P_3 whose first derivatives vanish at x = 1. Determine the subspaces U, W, U ∩W, U + W, in terms of dimension, bvasis, and the extensionof each of these bases to a basis for V = P_3.

20. (i) Let {a_1, a_2, a_3, a_4) : $a_1 \in \Re$, i = 1, 2, 3, 4 and $a_1 + a_2 + a_3 + a_4 = 0$}
and $\Gamma = \{a_1, a_2, a_3, a_4\} : a_i \in \Re$,
i = 1, 2, 3, 4, and $a_1 - a_2 + a_3 - a_4 = 0$
Find a bvasis for $S \cap \Gamma$.
(ii) Determine diagonal matrix orthogonally similar to the following real symmetric

matrix $A = \begin{bmatrix} 7 & 0 & -2 \\ 0 & 5 & -1 \\ -2 & -2 & 6 \end{bmatrix}$ obtaining

also the transforming matrix.

21. (i) Let k be a real number, and let A be the matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & k \\ 1 & 4 & k^2 \end{pmatrix}$ Determine all values of k for which the linear system $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & k \\ 1 & 4 & k^2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 10 \\ 21 \end{pmatrix}$ has exactly one solution.

(ii) Let $W \subset \mathbb{R}^4$ be the subspace defined by $W = \{x \in \mathbb{R}^4 : Ax = 0\}$ where $A \begin{bmatrix} 2 & 1 & 2 & 3 \\ 1 & 1 & 3 & 0 \end{bmatrix}$ Write down a basis for W.

22. (i) Given that the matrix $\begin{pmatrix} \alpha & 1 \\ 2 & 3 \end{pmatrix}$ has 1 as an eigenvalue, compute its trace and its determinant.

(ii) Find a basis for $\mathbb{R}^4$ that contains vectors v = [1, 2, 0, 0] and u = [0, 0, 1, 2].

23. (a) Let A and B be two subspaces of $\mathbb{R}^3(\mathbb{R})$ generated by the sets

$S_1 = \{(1, 2, 1), (3, 1, 5), (3, -4, 7)\}$

and

$S_2 = \{(1, 0, -1), (1, 2, 1), (0, -3, 2)\}$

respectively.

Determine (i) dim (A+B) (ii) dim $(A \cap B)$

(b) Find an orthogonal matrix that will diagonalize the real symmetric matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$. Also write the resulting diagonal matrix.

24. Let V and W be subspaces of R^3 defined as follows :

V = {(a, b, c) / b + 2c = 0}, W = {(a, b, c) / a + b + c = 0)

Find bases and dimentions of V, W, $V \cap W$.

Hence prove that $V + W = R^3$

25. (i) Find the largest possible number of independent vectors among

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$$

$$v_4 = \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix} \quad v_5 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} \quad v_6 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}.$$

(ii) Do the vectors (1, 1, 3), (2, 3, 6) and (1, 4, 3) form a basis for R^3?

26. Let V and W be the following subspaces of R^4: $V = \{(a,b,c,d) / b - 2c + d = 0\}$, .

$W = \{(a,b,c,d) / a = d, b = 2c\}$

Find bases and dimensions of V , W and $V \cap W$. Hence prove that $\boldsymbol{R}^4 = V + W$.

27. Describel the four subspaces of R^3 associated with

$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and

$$I + A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

28. Find a basis for the following subspaces of R^{41}
(i) The vectors for which $x_1 + x_2 + x_3 = 0$ and $x_3 + x_4 = 0$.
(ii) The subspace spanned by (1, 1, 1, 1), (1, 2, 3, 4) and (2, 3, 4, 5).

29. Find a basis for a subspace U of V in the following.

(i) $U = \left\{(x_1, x_2, x_3, x_4, x_5) \in V_5 \Big/ \begin{matrix} x_1 + x_2 + x_3 = 0 \\ 3x_1 - x_4 + 7x_5 = 0 \end{matrix}\right\}, V = V_5$

(ii) $U = \left\{ p \in P / p(x_0) = 0 \right\}, V = P_4.$

Where P_4 is the set of all polynomials of degree ≤ 4.

30. Let V be the vector space of 2×2 matrices over the field of real numbers $\mathbb{R}$.

Let $W = \{ A = V / \text{Trace } A = 0 \}$. Show that w is a subspace of V. Find a basis for w and dimension.

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WORKSHEET - 3

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WORKSHEET - 3

LINEAR TRANSFORMATIONS

Topics to be covered:

Linear Transformations and their elementary properties.
Range and Null Space of Linear Transformation, Dimension of range space and Kernal (Null PSace), Singular and Non-Singular Linear Transformation and Inverse of Linear Transformation Matrix of Linear Transformation etc.

1. (i) Find an upper triangular matrix A such that

$$A^3 = \begin{bmatrix} 8 & -57 \\ 0 & 27 \end{bmatrix}$$

(ii) Let G be the linear operator on $\mathbb{R}^3$ defined by

$$G(x, y, z) = (2y + z, x - 4y, 3x)$$

Find the matrix representation of G relative to the basis

$$S = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$$

2. Let $R_3[x] = \{a_0 + a_1 x + a_2 x^2 / a_0, a_1, a_2 \in R\}$. Define $T : R_3[x] \to R_3[x]$ by $T(f(x)) = \frac{d}{dx}(f(x))$ for all $f(x) \in R_3[x]$. Show that **T** is a linear transformation. Also find the matrix representation of T with reference to basis sets $\{1, x, x^2\}$ and $\{1, 1 + x, 1 + x + x^2\}$.

3. Show that the vectors (1, 1, 1), (2, 1, 2) and (1, 2, 3) are linearly independent in $R^{(3)}$. Let $T : IR^{(3)} \to IR^{(3)}$ be a linear transformation defined by $T(x, y, z) = (x + 2y + 3z, x + 2y + 5z, 2x + 4y + 6z)$.
Show that the images of above vectors under T are linearly dependent. Give the reason for the same

4. Prove that $g : P_2 \to P_3$ defined as follows is linear. Find the kernel and range of g. Give bases for these subspaces
$g(a_2 x^2 + a_1 x + a_0) = (a_2 - a_1)x^3 + a_1 x + 2a_0$.

5. Consider the linear transformation $T : R^3 \to R^2$ defined by $T(x, y, z) = (x - y, x + z)$. Find the matrix of T with respect to the bases (u_1, u_2, u_3) and $\{u'_1, u'_2\}$ of R^3 and R^2, where $u_1 = (1, -1, 0)$, $u_2 = (2, 0, 1)$, $u_3 = (1, 2, 1)$ and $u_1' = (-1, 0)$, $u_2' = (0, 1)$ Use this matrix to find the image of the vector $u = (3, -4, 0)$.

6. Show that the transformation $T : P_2 \to P_2$ defined as follows is linear. $T(ax^2 + bx + c) = cx^2 + a$. Find the image of $3x^2 - x + 2$. Determine another element of P_2 that has the same image.

7. Let P_n denote the vector space of all real polynomials of degree atmost n and $T : P_2 \to P_3$ be a linear transformation given by $T(p(x)) = \int_0^x p(t)dt$, $p(x) \in P_2$. Find the matrix of T with respect to the bases $\{1, x, x^2\}$ and $\{1, x, 1 + x^2, 1 + x^3\}$ of P_2 and P_3 respectively. Also, find the null space of T.

8. Let D be the operation of taking the derivative. Interpret D as an operator on P_n.

Prove that $D^2 - 2D + 1$ is a linear operator. Find the set of polynomials that are mapped by $D^2 - 2D + 1$ into $12x - 4$.

9. Consider the linear mapping $F : \mathbb{R}^2 \to \mathbb{R}^2$ given as $F(x,y) = (3x+4y, 2x-5y)$ with usual basis.
Find the matix associated with the linear transformation relative to the basis $S=\{u_1, u_2\}$ where $u_1=(1,2)$, $u_2=(2,3)$.

10. Let $T : R^4 \to R^3$ be the linear transformation defined by
$T(x_1, x_2, x_3, x_4) = (x_1 - x_2 + x_3 + x_4, x_1 + 2x_3 - x_4, x_1 + x_2 + 3x_3 - 3x_4)$.
Find the range, rank, kernel and nullity of T.

11. Describe explicitly a linear transformation from R^3 to R^3 which has its range the subspace spanned by the vectors $(1, 0, -1)$ and $(1, 2, 2)$.

12. Let $f : \mathbb{R} \to \mathbb{R}^3$ be a linear transformation defined by $f(a,b,c) = (a, a+b, 0)$.
Find the matrices A and B respectively of the linear transformation f with respect to the standard basis (e_1, e_2, e_3) and the basis (e_1', e_2', e_3') where
$e_1' = (1,1,0)$, $e_2' = (0,1,1)$, $e_3' = (1,1,1)$.
Also, show that there exists an invertible matrix P such that $B = P^{-1}AP$

13. Is the vector $(3, -1, 0, -1)$ in the subspace of R^4 spanned by the vectors $(2, -1, 3, 2)$, $(-1, 1, 1, -3)$ and $(1, 1, 9, -5)$?

14. Let $R_3(x) = \{a_0 + a_1x + a_2x^2 / a_0, a_1, a_2 \in \mathbb{R}\}$
Define $T : \mathbb{R}_3(x) \to \mathbb{R}_3(x)$ by

$$T(f(x)) = \frac{d}{dx} f(x) \text{ for all } f(x) \in \mathbb{R}_3(x).$$

Show that T is a linear transformation. Also find the matrix representation of T with reference to basis sets $\{1, x, x^2\}$ and $\{1 + x, 1 + x + x^2\}$.

15. Let F be the field of complex numbers and $T : F^3 \to F^3$ be given as
$T(x_1, x_2, x_3)$
$= (x_1 - x_2 + 2x_3, 2x_1 + x_2, -x_1 - 2x_2 + 2x_3)$.
(i) Show that T is a linear transformation.
(ii) Find its rank and nullity.
(iii) Find conditions on a, b, c such that (a, b, c) $\in$ Range T and (a, b, c) $\in$ null space of T.

16. Let T: $\mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by
$T(x_1, x_2, x_3) = (x_1 + 3x_2 + 2x_3,$
$3x_1 + 4x_2 + x_3, 2x_1 + x_2 - x_3)$.
Then find the dimension of the range space of T^2. Also find the

17. M is any 2 by 2 matrix and $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$.
The linear transformation T is defined by $T(M) = AM$. What rules of matrix multiplication show that T is linear?

18. Suppose that the matrix of a linear transformation of R^3 relative to the basis $\{e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)\}$ is

$$\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}$$

Find the matrix of T relative to the basis $\{f_1 = (1, 1, -1), f_2 = (-1, 0, 1), f_3 = (1, 2, 1)\}$

19. (i) Let T be a linear operator on C^3 defined by $T(1, 0, 0) = 1, 0, i)$, $T(0, 1, 0) = (0, 1, 1)$, $T(0, 0, 1) = (i, 1, 0)$. Is T invertible ? Justify your answer.

(ii) If T is a linear transformation on a vector space V such that $T^2 - T + I = 0$, then show that T is invertible.

20. Let F be a subfield of complex numbers and T a function from $F^3 \to F^3$ defined by
$T(x_1, x_2, x_3) =$
$(x_1 + x_2 + 3x_3, 2x_1 - x_2, -3x_1 + x_2 - x_3)$
What are the conditions on (a, b, c) such that (a, b, c) be in the null space of T ? Find the nullity of T.

21. Let T be the linear operator defined by $T(x, y, z) = (2x, 4x - y, 2x + 3y - z)$
(i) Show that T is invertible.
(ii) Find a formula for T^{-1}.

22. Let $T : P_3 \to P_3$ be the map given by $T(p(x)) = \int_1^x p'(t)dt$. If the matrix of T relative to the standard bases $B_1 = B_2 = \{1, x, x^2, x^3\}$ is M and M' denotes the transpose of the matrix M then find M + M'.

23. Suppose $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear transformation whose matrix, with respect to the standard basis is $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Find

$T^{-1}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}\right)$.

24. Consider the linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T(x, y) = (x, x + y, 2y)$. Find the matrix of **T** with respect to the bases $\{u_1, u_2\}$ and $\{u'_1, u'_2, u'_3\}$ of $\mathbb{R}^2$ and $\mathbb{R}^3$, where $u_1 = (1, 3)$, $u_2 = (4, -1)$, and $u'_1 = (1, 0, 0)$, $u'_2 = (0, 2, 0)$, $u'_3 = (0, 0, -1)$
Use this matrix to find the image of the vector u = (9, 1).

25. Find the matrix of the following linear transformation with respecto to the basis of $\{x, 1\}$ of P_1 and $\{x^2, x, 1\}$ of P_2.
$T(ax^2 + bx + c) = (b + c)x^2 + (b - c)x$ of P_2 into itself.

26. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be defined by $T(x, y, z) = (y + z, z, 0)$. Show that T is a linear transformation. If $V \in \mathbb{R}^3$ is such that $T^2(V) \neq 0$, then show that
$B = \{V, T(V), T^2(V)\}$ forms a basis of $\mathbb{R}^3$.
Compute the matrix of T with respect to B.
Also find a $V \in \mathbb{R}^3$ such that $V^2(V) \neq 0$.

27. Let V be the vector space polynomials of degree ≤ 3 over F. Let T be a linear transformation defined on V by
$T(a_0 + a_1x + a_2x^2 + a_3x^3)$
$= (a_0 + a_1(x+1) + a_2(x+1)^2 + a_3(x+1)^3$
Compute the matrix of T relative to the bases
(i) $\{1, x, x^2, x^3\}$
(ii) $\{1, 1 + x, 1 + x^2, 1 + x^3\}$.

28. Let $\{e_1, e_2, e_3\}$ be the standard basis of IR^3 and **T** be a linear transformation from R^3 into IR^2 defined by

$$T(e_1) = (2, 3)^T, T(e_2) = (1, 2)^T$$

and $T(e_3) = (-1, -4)^T$

(i) What is T $(1, -2, -1)$
(ii) What is the matrix of T with respect to the standard bases of IR^3 and IR^2 ?

29. Let $D:V(IR)\to V(IR): f(x)\to \frac{df(x)}{dx}$

$T:V(IR)\to V(IR): f(x)\to xf(x)$

be linear transformations on V(IR), the vector space of all polynomials in an indeterminate X with real coefficients. Show that

(i) DT – TD = I the identity operator

(ii) $(TD)^2 = TD^2 + TD$

find A^{-2} when $A=\begin{bmatrix}1 & 2 & 0\\ 2 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}$

30. Show that the transformation $T:P_2\to P_2$ defined as follows is linear.

$T(ax^2+bx+c)=cx^2+a.$ Find the image of $3x^2 - x + 2$. Determine another element of P_2 that has the same image.

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MATHEMATICS OPTIONAL

By

K. VENKANNA

WORKSHEET - 4
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QLP

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WORKSHEET - 4

MATRICES

1. Find the characteristic equation of the matrix $A=\begin{bmatrix}2&1&1\\0&1&0\\1&1&2\end{bmatrix}$ and, hence, find the matrix represented by $A^8-5A^7+7A^6-3A^5+A^4-5A^3+8A^2-2A+I$

2. Find the condition on a, b, and c so that the following system in unknowns x, y and z has a solution.
$x+2y-3z=a,\ 2x+6y-11z=b,\ x-2y+7z=c$

3. If H is a Hermitian matrix, show that $(I-iH)(I+iH)^{-1}=(I+iH)^{-1}(I-iH)=U$ where U is a unitary matrix and that if λ is an eigenvalue of H, then $(1-i\lambda)/(1+i\lambda)$ is an eigenvalue of U.
Find U when $H=\begin{bmatrix}1&e^{i\alpha}\\e^{-i\alpha}&-1\end{bmatrix}$

4. (*i*) If α is a characteristic root of a non-singular matrix A, then prove that $\dfrac{|A|}{\alpha}$ is a characteristic root of Adj. A.
(*ii*) Show that the matrix
$$A=\begin{bmatrix}-9&4&4\\-8&3&4\\-16&8&7\end{bmatrix}$$
is diagonalizable, Also find the diagonal form and a diagonalizing matrix P.

5. (i) Verify that the matrix $A=\begin{bmatrix}1&2&0\\2&-1&0\\0&0&-1\end{bmatrix}$ satisfies its own characteristic equation. Is it true of every square matrix ? State the theorem that applies here.
(ii) Let $V=R^4$ (R) and $W=\{(a,b,c,d)\in R^4:\ a=b+c,\ c=b+d\}$. Find a basis and the dimension of W.

6. (i) Let $M=\begin{bmatrix}1+i&2i&i+3\\0&1-i&3i\\0&0&i\end{bmatrix}$. Determine the eigen values of the matrix $B=M^2-2M+I$.
(ii) Find a 3×3 orthogonal matrix whose first two rows are $\left[\frac{1}{3},\frac{2}{3},\frac{2}{3}\right]$ and $\left[0,\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right]$.

7. Let $A=\begin{pmatrix}1&0&0\\1&0&1\\0&1&0\end{pmatrix}$
Show that for every interger $n\geq3,\ A^n=A^{n+2}+A^2-I$ and hence find the matrix A^8.

8. Find the characteristic and minimal polynomials of $A=\begin{pmatrix}2&0&0\\1&2&0\\0&0&-1\end{pmatrix}$ and determine whether A is diagonalisable

9. Let $A=\begin{pmatrix}1&1&2\\-1&2&1\\0&1&3\end{pmatrix}$. Find an invertible 3 × 3 matrix P such that P′Ap = D, D is diagonal matrix. Find D also.

10. Find an orthogonal matrix that will diagonalize the real symmetric matrix

$$A=\begin{bmatrix}1&2&3\\2&4&6\\3&6&9\end{bmatrix}$$

Also write the resulting diagonal matrix.

11. Show that if λ is a non-zero eigen value of the non singular n-square matrix A, then $\frac{|A|}{\lambda}$ is an eigen value of adjA.
Also give an example to prove that the eigen values of AB are not necessarily the product of eigen values of A and that of B.

12. Given the linear transformation $Y=\begin{bmatrix}1&1&0\\2&1&3\\-2&3&5\end{bmatrix}X$, Show that it is singular and the images of the linearly independent vectors,

$X_1=[1,1,1]^T$, $X_2=[2,1,2]^T$, $X_3=[1,2,3]^T$

are linearly dependent.

13. Calculate $f(A)=e^A-e^{-A}$ for

$$A=\begin{bmatrix}2&4&0\\6&0&8\\0&3&-2\end{bmatrix}$$

14. If $A=\begin{bmatrix}1&0&0\\1&0&1\\0&1&0\end{bmatrix}$

Show that for every integer $n \geq 3$, $\mathbf{A}^n = \mathbf{A}^{n-2} + \mathbf{A}^2 - \mathbf{I}$. Hence determine $\mathbf{A}^{50}$

15. Reduce $A=\begin{bmatrix}1&3&6&-1\\1&4&5&1\\1&5&4&3\end{bmatrix}$ to normal form N and compute the matices P & Q such that PAQ=N.

16. Let $V=P_3(IR)$ be the vector space of polynomial fucntions on reals of degree at most 3. Let $D:V\to V$ be the differentiation operator defined by

$$D\left(a_0+a_1x+a_2x^2+a_3x^3\right)=a_1+2a_2x+3a_3x^2.$$

(i) Show that D is a linear transformation.
(ii) Find kernel and image of D.
(iii) What are dimensions of V, $\ker D$ and image D ?
(iv) Give relation among them of (iii)

17. Find the eigen values and the corresponding eigen, vectors of $A=\begin{bmatrix}1&2\\2&-2\end{bmatrix}$.

18. Show that the vectors $(1,2,1),(1,0,-1)$ *and* $(0,-3,2)$ form a basis for IR^3.

19. Determine non-singular matrices P and Q such that the matrix PAQ is in canonical

form, Where $A=\begin{bmatrix}5 & 3 & 14 & 4\\ 0 & 1 & 2 & 1\\ 1 & -1 & 2 & 0\end{bmatrix}$. Hence find the rank of A.

20. Find the minimum polynomial of the matirx $A=\begin{pmatrix}1 & 2 & 2\\ 2 & 1 & 2\\ 2 & 2 & 1\end{pmatrix}$, and use it to determine whether A is similar to a diagonal matrix.

21. Show that the quadratic form $2x^2-4xy+3zx+6y^2+6yz+8z^2$ in three variables is positive definite.

22. Find the eigen values and their corresponding eigen vectors of the matrix $\begin{pmatrix}2 & 0 & 3\\ 2 & 1 & 2\\ 0 & 0 & 1\end{pmatrix}$. Is the matrix diagonalisable?

23. For what values of α has the systme of equations
$x+2y+z=1,$
$\alpha x+4y+2z=2,$
$4x-2y+2\alpha z=-1$
(i) a unique solution
(ii) infinitely many solutions
(iii) no solution.

24. Determine an orthogonal matrix which reduces the quadratic form $Q(x_1,x_2,x_3)=2x_1^2+x_2^2-4x_2x_3+x_3^2$ to a canonical form. Also , indentify the surface respresented by $Q(x_1,x_2,x_3)=7$.

25. Find the eigen values and the corresponding eigen vectors of $A=\begin{pmatrix}2 & 1\\ 2 & 3\end{pmatrix}$.

26. Determine a non-singular matirx P such that $P'AP$ is a diagonal matrix, where P' denotes the transpose of P, and $A=\begin{bmatrix}0 & 1 & 2\\ 1 & 0 & 3\\ 2 & 3 & 0\end{bmatrix}$

27. If A be an orthogonal matrix with the properly that -1 is not a characteristic root of A, then show that A is expressible as $(I-S)(I+S)^{-1}$ for some suitable skew-symmetric matrix S.

28. Show that every unitary matrix A can, by a suitable choice of skew Hermitian matrix S, be expressed as
$\mathbf{A=(I+S)(I-S)^{-1}}$, Provided that -1, is not a characteristic root of A.

29. (*i*) For what values of η the equations
$x+y+z=1$
$x+2y+4z=\eta$
$x+4y+10z=\eta^2$
have a solution and solve them completely in each case.
(*ii*) Determine whether the linear transformation defined by the following matrix is one-to-one.

$$A=\begin{bmatrix}-1 & 2 & 0 & 5\\ 3 & 7 & 2 & 8\\ -4 & 2 & 0 & 0\\ 1 & 3 & 0 & 6\end{bmatrix}$$

30. Let $A=\begin{pmatrix}1 & 4\\ 2 & 3\end{pmatrix}$. Find all eigen values of **A** and the corresponding eigen vectors.

31. Let $H=\begin{bmatrix}1 & 1+i & 2i\\ 1-i & 4 & 2-3i\\ -2i & 2+3i & 7\end{bmatrix}$ be a Hemitian matrix. Find a non-singular matrix P such that $P'A\overline{P}$ is diagonal.

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WORKSHEET - 5
FULL CHAPTER

1. (i) Show that the diagonal entries of a skew symmetric matrix are all zero, but the converse is not true.

   (ii) Let $Q' = \{ax^2 + bx + c / a \neq a, b, c \in C)$ Is Q′ a complex vector apace ? Justify your answer.

2. Find the characteristic values and bases of the corresponding characteristic spaces of the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix}.$$

   Is A similar to a diagonal matrix ? Give reasons.

3. Find a non zero vector common to the space spanned by (1, 2, 3), (3, 2, 1) and the space spanned by (1, 0, 1) and (3, 4, 3).

4. Let $V = \mathbb{R}^3$ and $\alpha_1 = (1,1,2)$, $\alpha_2 = (0,1,3)$, $\alpha_3 = (2,4,5)$ and $\alpha_4 = (-1,0,-1)$ be the elements of V. Find a basis for the intersection of the subspace spanned by $\{\alpha_1, \alpha_2\}$ and $\{\alpha_3, \alpha_4\}$.

5. Is the vector (3, –1, 0, –1) in the subspace of R^5 spanned by the vectors (2, –1, 3, 2), (–1, 1, 1, –3), and (1, 1, 9, –5) ?

6. Let

$$V = \{(x, y, z, u) \in \mathbb{R}^4 : y + z + u = 0\},$$
$$W = \{(x, y, z, u) \in \mathbb{R}^4 : x + y = 0, z = 2u\}$$

   be two subspaces of $\mathbb{R}^4$. Find bases for V, W, V+W and $V \cap N$.

7. Show that the sets

$$W_1 = \left\{ \begin{bmatrix} a & 0 \\ a & 0 \end{bmatrix} : a \in \mathbb{R} \right\},$$

$$W_2 = \left\{ \begin{bmatrix} 0 & a \\ 0 & a \end{bmatrix} : a \in \mathbb{R} \right\}$$ are subspaces of the vectorspace M_2 of 2×2 matrices over R. Further show that $W_1 \cup W_2$ is not a subspace of M_2.

8. Let W be the set of all $(x_1, x_2, x_3, x_4, x_5,)$ in $\mathbb{R}^5$ which satisfy

$$2x_1 - x_2 + \frac{4}{3}x_3 - x_4 = 0$$

$$x_1 + \frac{2}{3}x_3 - x_5 = 0$$

$$9x_1 - 3x_2 + 6x_3 - 3x_4 - 3x_5 = 0$$

Find a finite set of vectors which spans W.

9. Show that the set

$$P(t) = \left[at^2 + bt + c / a, b, c \in \mathbb{R}\right]$$

forms a vector space over the field $\mathbb{R}$. Find a basis for this vector space. What is the dimension of this vector space?

10. (i) Let V be the vectorspace of all 2×2 matrices over the field $\mathbb{R}$ of real numbers.

Let (i) $W_1 = \{A \in V : A^2 = A\}$

(ii) $W_2 = \{A \in V : \det A = 0\}$

Show that W_1 and W_2 are not subspaces of V.

(ii) Find two subspaces A and B of $V = \mathbb{R}^4(\mathbb{R})$ such that dim A = 2, B = 3 and $\dim(A \cap B) = 1$.

11. Let V the vector space of all 2×2 matrices over C.

Let $W_1 = \left\{\begin{bmatrix} x & -x \\ y & z \end{bmatrix} : x, y, z \in C\right\}$,

$W_2 = \left\{\begin{bmatrix} a & b \\ -a & c \end{bmatrix} : a, b, c \in C\right\}$.

Verify that w_1 and w_2 are subspaces of V.

12. Let V be the vectorspace of all functions from $\mathbb{R}$ into $\mathbb{R}$; let v_e be the subset of even functions $f(-x) = f(x)$; let v_0 be the subset of odd functions $f(-x) = -f(x)$.

(i) Prove that v_e and v_0 are subspaces of v.

(ii) Prove that $v_e + v_0 = v$

(iii) Prove that $v_e \cap v_0 = \{0\}$.

13. Let U = Span (u_1, u_2, u_3) and W = Span (v_1, v_2) be two subspaces of $\mathbb{R}^4$ where u_1 = (1,2,-1,3), u_2=(2,4,1,-2), u_3=(3,6,3,-7) v_1 = (1, 2, -4, 11), v_2 = (2, 4, -5, 14); show that U=W.

14. Let W_1 and W_2 be two subspaces of V(F), then the linear sum W_1+W_2 is a subsapce of V(F) and $W_1+W_2 = L(W_1 \cup W_2)$.

15. Show that the functions $f(t) = \sin t, g(t) = \cos t, h(t) = t$ are linearly independent.

16. (a) (i) If $P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, then find P^{50}.

(ii) Find the dimension of the subspace

$$W = \{(x, y, z, w) \in \mathbb{R}^4 \mid x + y + z + w = 0, \; x + y + 2z = 0, x + 3y = 0\}$$

17. Let V be the vector space of polynomials over R. Let U and W be the subspaces generated by

$$\{t^3 + 4t^2 - t + 3, t^3 + 5t^2 + 5, 3t^3 + 10t^2 - 5t + 5\}$$

and

$$\{t^3 + 4t^2 + 6, \; t^3 + 2t^2 - t + 5, \; 2t^3 + 2t^2 - 3t + 9\}$$

respectively. Find

(i) dim (U + W)

(ii) $\dim(U \cap W)$.

18. Show that the vectors $\alpha_1 = (1, 0, -1), \alpha_2 = (1, 2, 1), \alpha_3 = (0, -3, 2)$ from a basis for R^3. Find the components of (1, 0, 0) w.r.t the basis $\{\alpha_1, \alpha_2, \alpha_3\}$.

19. Find bases for both the row and column spaces of the following matrix A. Show that the dimensions of both the row space and the column space are the same.

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 1 & 4 & 1 \\ 2 & 5 & 5 \end{bmatrix}$$

20. Let V be the vector space of all 2×2 matrices over the field F. Let W_1 be the set of matrices of the form $\begin{bmatrix} x & -x \\ y & z \end{bmatrix}$ and let $\mathbf{W}_2$ be the set of matrices of the form $\begin{bmatrix} a & b \\ -a & c \end{bmatrix}$.

21. Take V = $\;_3$, the space of all real polynomials of degree at most 3. Let U be the subspace of $\;_3$ consisting of those polynomials of $\;_3$ that vanish at x = 1. Let W be the subspace of $\;_3$ consisting of those polynomials of $\;_3$ whose first derivatives vanish at x = 1. Determine the subspaces $U, W, U \cap W, U + W$ in terms of dimension, basis, and the extension of each of these bases to a basis for V = $\;_3$.

22. (i) Let A be a 3×3 upper triangular matrix with real entries. If $a_{11} = 1, a_{22} = 2$ and $a_{33} = 3$, determine α, β and γ such that $A^{-1} = \alpha A^2 + \beta A + \gamma I$.

(ii) Let $W \subset \mathbb{R}^4$ be the subspace defined by

$$W = \left\{ x \in \mathbb{R}^4 : Ax = 0 \right\}$$

where

$$A = \begin{bmatrix} 2 & 1 & 2 & 3 \\ 1 & 1 & 3 & 0 \end{bmatrix}$$

Write down a basis for W.

23. Find three vectors in R^3 which are linearly dependent, and are such that any two of them are linearly independent.

24. Show that the set of all real-valued continuous functions y = f(x) satisfying the differential equation:

$$\frac{d^3y}{dx^3} + 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} + 6y = 0$$ is a vector space over R. Give a basis for the vector space.

25. Given $S_1 = \{(1,2,3),(0,1,2),(3,2,1)\}$ and $S_2 = \{(1,-2,3),\ (-1,1,-2),\ (1,-3,4)\}$, determine the dimension and a basis for

(i) $[S_1] \cap [S_2]$ (ii) $[S_1] + [S_2]$

26. Let $\{(a_1, a_2, a_3, a_4) : a_i \in \Re,\ i = 1,2,3,4$ and $a_1 + a_2 + a_3 + a_4 = 0\}$ and $\Gamma = \{(a_1, a_2, a_3, a_4) : a_i \in \Re, i = 1,2,3,4$ and $a_1 - a_2 + a_3 - a_4 = 0\}$. Find a basis for $S \cap \Gamma$.

27. Let

$V = \mathbb{R}^4(\mathbb{R})$

$W = \left\{ (a,b,c,d) \in \mathbb{R}^4 ; a = b + c, c = b + d \right\}$,

Find a basis and the dimension of W.

28. Let V be the (real) vector space of all polynomial functions from R into R of degree 2 nor less, i.e, the space of all functions f of the form $f(x) = c_0 + c_1x + c_2x^2$.

Let t be a fixed real number and define $g_1(x) = 1,\ g_2(x) = x + t,\ g_3(x) = (x + t)^2$.

Prove that B={g_1, g_2, g_3} is a basis for V.

If $f(x) = c_0 + c_1x + c_2x^2$.

What are the coordinates of f is this ordered basis B.

29. Let V be the vector space of polynomials over $\mathbb{R}$. Let U and W be the subspaces generated by

$$\left\{ t^3 + 4t^2 - t + 3, t^3 + 5t^2 + 5, 3t^3 + 10t^2 - 5t + 5 \right\}$$

and $\left\{ t^3 + 4t^2 + 6, t^3 + 2t^2 - t + 5, 2t^3 + 2t^2 - 3t + 9 \right\}$

respectively. Find

(i) $\dim(U+W)$

(ii) $\dim(U\cap W)$

30. Are the vectors
$\alpha_1=(1,1,2,4),\ \alpha_2=(2,-1,-5,2),$
$\alpha_3=(1,-1,-4,0),\ \alpha_4=(2,1,1,6)$
linearly independent in $\mathbb{R}^4$? Also find a basis for the subspace of $\mathbb{R}^4$ spanned by the four vectors.

31. The vectorspace V of 2×2 matrices over R and the following usual basis E of V;

$$E=\left\{E_1=\begin{bmatrix}1&0\\0&0\end{bmatrix},E_2=\begin{bmatrix}0&1\\0&0\end{bmatrix},E_3=\begin{bmatrix}0&0\\1&0\end{bmatrix},E_4=\begin{bmatrix}0&0\\0&1\end{bmatrix}\right\}$$

32. Let A and B be twos subspaces of $\mathbb{R}^3$ generated by the sets
$S_1=\{(1,2,1),(3,1,5),(3,-4,7)\}$
and $S_2=\{(1,0,-1),(1,2,1),(0,-3,2)\}$
respectively. Determine
(i) $\dim(A+B)$
(ii) $\dim(A\cap B)$

33. Let W be the space generated by the polynomials
$V_1=t^3-2t^2+4t+1,\ V_2=2t^3-3t^2+9t-1,$
$V_3=t^3+6t-5$ and $V_4=2t^3-5t^2+7t+5.$
Find a basis and dimension of W.

34. (i) Show that the dimension of the vectorspace $Q\left(\sqrt{2},\sqrt{3}\right)$ over Q is 4.
(ii) Determine whether or not the vectors (6,2,3,4), (0,5,-3,1), (0,0,7,-2), (0,0,0,4) from a basis of $V_4(\mathbb{R})$.

35. (i) Prove that the union of two subspaces is a subspace iff one is contained in the other.
(ii) Let V be the vector space of 3X3 synmetric matrices over F; then show that dim V=6 by exhibiting a basis of V.

36. Let A and B be two subspaces of $\mathbb{R}^3(\mathbb{R})$ generated by the sets
S_1 = {(1,2,1), (3,1,5), (3,-4,7)} and
S_2 = {(1,0,-1), (1, 2, 1), (0, -3, 2)}, respectively.
Determine (i) dim (A+B), (ii) dim $(A\cap B)$.

37. Find the two subspaces A and B of $v=\mathbb{R}^4(\mathbb{R})$ such that dim A=2, dim B=3 and dim $(A\cap B)$=1.

38. Let W be the space generated by the polynomials
$V_1=t^3-2t^2+4t+1,\ V_2=2t^3-3t^2+9t-1,$
$V_3=t^3+6t-5$ and $V_4=2t^3-5t^2+7t+5.$
Find a basis and dimension of W.

39. Find a basis and dimension of the subspace W of generated by
$f(t)=\sin t, g(t)=\cos t, h(t)=t$ are linearly independent.

40. Find a basis and dimension fo the subspace W of generated by (1, -4, -2,1), (1,-3,-1,2), (3,-8,-2,7). All extend the basis of W to a basis of the whole space $\mathbb{R}^4$.

41. (i) If $A=\begin{bmatrix}1&0&0\\1&0&1\\0&1&0\end{bmatrix}$, then find $\mathbf{A}^{50}$.

(ii) Let T: $P_3\to P_3$ be the map given by
$T(P(x))=\int_1^x P'(t)\,dt$. If matrix of **T** relative to the standard basis

$B_1 = B_2 = \{1, x, x^2, x^3\}$ is M and M^T denotes the transpose of the matrix M, then find $M+M^T$.

42. On the Space $\mathbf{P}_3$ of cubic polynomials, what matrix represents d^2/dt^2? Construct the 4 by 4 matrix from the standard basis 1, t, t^2, t^3. Find its nullspace and column space. What do they mean in terms of polynomials?

43. If A be an orthogonal matrix with the property that -1 is not a characteristic root, then A is expressible as $(I+S)(I-S)^{-1}$ for some suitable real skew symmertic matrix S.

44. (i) Show that the transformation T : $P_2 \rightarrow P_2$ defined as folows it linear. $T(ax^2 + bx + c) = cx^2 + a$. Find the image of $3x^2 - x + 2$.
Determine another alement of P_2 that has the same image.
(ii) let A be a 5 × 5 matrix whose characteristic polynominal is given by $(\lambda - 2)^3 (\lambda + 2)^2$ If A is diagonalizable, find α and β such that $A^{-1} = \alpha A + \beta I$.

45. If H is a hermitian matrix, shown that $(I - iH)(I + iH)^{-1} (I - iH) = U$ where U is a unitary matrix and that if λ is an eigenvalue of H, then $(1 - i\lambda)/(1 + i\lambda)$ is an eigenvalue of U. Find U when $H = \begin{bmatrix} 1 & e^{ia} \\ e^{-ia} & -1 \end{bmatrix}$

46. Discuss for all values of k the system of equations
$2x + 3ky + (3k + 4) z = 0$
$x + (k + 4)y + (4k + 2) z = 0$
$x + 2(k + l) y + (3k + 4) z = 0$

47. (i) Construct a 2 × 2 matrix Awith real entries such that $A^3 = I$.
(ii) Find the inverse in $\mathbb{Z}_3$ of the following matrix :
$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$

48. Prove that the following two matrices are not row-equivalent :
$$\begin{bmatrix} 2 & 0 & 0 \\ a & -1 & 0 \\ b & c & 3 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 2 \\ -2 & 0 & -1 \\ 1 & 3 & 5 \end{bmatrix}$$

49. (i) Find a matrix P which transfomrs the matrix $A = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 5 & 1 \\ 3 & 1 & 1 \end{bmatrix}$ to diagonal form.
Hence calculate A^4.
(ii) If λ be an eigen value of a non-singular matrix A, show that $|A|/\lambda$ is an eigen value of the matrix adj A.

50. Consider the system of equations
$x_1 - x_2 + 2x_3 = 1$
$2x_1 + 2x_3 = 1$
$x_1 - 3x_2 + 4x_3 = 2$
Does this system have a solution? If so, describe explicitly all solutions.

51. Let $A = \begin{bmatrix} 3 & -6 & 2 & -1 \\ -2 & 4 & 1 & 3 \\ 0 & 0 & 1 & 1 \\ 1 & -2 & 1 & 0 \end{bmatrix}$.
For which (y_1, y_2, y_3, y_4) does the system of equations AX = Y have a solution ?

INSTITUTE FOR IAS/IFOS EXAMINATIONS

MATHEMATICS OPTIONAL

By

K. VENKANNA

WORKSHEET - 6

LINEAR ALGEBRA

QLP

(QUICK LEARNING PROGRAMME FOR MAINS-2016)

Scoring Maximum Marks in Main-2016

FEATURES OF MATHEMATICS PROGRAMME

- Class Timings – 9:30 AM – 5:30 PM.
- Atleast one day one module to be discussed (Expected)
- A question based approach to be followed.
- Pin-pointed formulas to be provided.
- 15-20 problems to be provided on each formula set.
- Time-bound problem solving sessions.
- On board discussion sessions to be followed there after.
- Around 50-100 problems expected to be solved per session.
- Only problems to be discussed, no concept explanation.
- Sessions will be from Monday to Friday.

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WORKSHEET - 6
FULL CHAPTER

1. Show that the transformation $T(ax^2 + bx + c) = 2ax + b$ of $P_2 \to P_1$ is linear. Find the image of $3x^2 - 2x + 1$. Determine another element of P_2 that has the same image.

2. Find the values of λ for which the equations
$(\lambda - 1)x + (3\lambda + 1)y + 2\lambda z = 0$
$(\lambda - 1)x + (4\lambda - 2)y + (\lambda + 3)z = 0$
$2x + (3\lambda + 1)y + 3(\lambda - 1)z = 0$
are consistent, and find the ratios of $x : y : z$ when λ has the smallest of these values. What happens when λ has the greater of these values.

3. Using Cayley-Hamilton theorem, find A^8, if

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$$

4. Let $A = \begin{bmatrix} 3 & -1 & 2 \\ 2 & 1 & 1 \\ 1 & -3 & 0 \end{bmatrix}$.

For which triples (y_1, y_2, y_3) does the system $AX = Y$ have a solution?

5. Reduce the quadratic form $x^2 + 4y^2 + 9z^2 + t^2 - 12yz + 6zx - 4xy - 2xt - 6zt$ to canonical form and find its rank and signature.

6. Let $A = \begin{bmatrix} 5 & 0 & 0 \\ 1 & 5 & 0 \\ 0 & 1 & 5 \end{bmatrix}$

For which X does there exist a scalar c such that $AX = cX$?

7. Let W be the subspace of $\mathbb{R}^5$ spanned by
$u_1 = (1, 2, -1, 3, 4)$
$u_2 = (2, 4, -2, 6, 8)$, $u_3 = (1, 3, 2, 2, 6)$
$u_4 = (1, 4, 5, 1, 8)$ and $u_5 = (2, 7, 3, 3, 9)$.
Find a subset of the vectors which form a basis of W.

8. (i) Show that the dimension of the vectorspace $Q(\sqrt{2}, \sqrt{3})$ over Q is 4.

(ii) Determine whether or not the vectors (6,2,3,4), (0,5,-3,1), (0,0,7,-2), (0,0,0,4) form a basis of $V_4(\mathbb{R})$.

9. If W_1, W_2 are two subspaces of a finite dimensional vectospace V(F), then

$$\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$

10. Let W be the space generated by the polynomials $v_1 = t^3 - 2t^2 + 4t + 1$, $v_2 = 2t^3 - 3t^2 + 9t - 1$, $v_3 = t^3 + 6t - 5$ and $v_4 = 2t^3 - 5t^2 + 7t + 5$.
Find a basis and dimension of w.

11. Let V be the set of all real valued functions $y = f(x)$ satisfying $\frac{d^2y}{dx^2} + 4y = 0$. Prove that V is a 2- dimensional real vectorspace.

12. Let V be the vectorspace of 3×3 anit symmetric matrices over F, then show that dim V = 3 by exhibiting a basis of V.

13. Find the dimension of the subspace of R^4, spanned by the set
$\{(1,0,0,0), (0,1,0,0), (1,2,0,1), (0,0,0,1)\}$
Hence find its basis.

14. (i) Show that the subspaces of IR^3 spanned by two sets of vectors $\{(1,1,-1),(1,0,1)\}$ and $\{(1,2,-3),(5,2,1)\}$ are identical. Also find the dimension of this subspace.
(ii) Find the nullity and a basis of the null space of the linear transformation $A : R^{(4)} \to R^{(4)}$ given by the matrix

$$A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}.$$

15. Prove that the set V of the vectors (x_1, x_2, x_3, x_4) in IR^4 which satisfy the equations $x_1 + x_2 + 2x_3 + x_4 = 0$ and $2x_1 + 3x_2 - x_3 + x_4 = 0$, is a subspace of IR^4. What is the dimension of this subspace? Find one of its bases.

16. Prove that the set V of all 3 × 3 real symmetric matrices forms a linear subspace of the space of all 3 × 3 real matrices. What is the dimension of this subspace? Find at least one of the bases for V.

17. Find the dimension of the subspace of R^4 spanned by the set {(1, 0, 0, 0), (0, 1, 0, 0), (1, 2, 0, 1), (0, 0, 0, 1)}. Hence find a basis for the subspace.

18. Let W be the set of all 3×3 symmetric matrices over IR. Does it form a subspace of the vector space of the 3×3 matrices over IR? In case it does, construct a basis for this space and determine its dimension.

19. Let S be the vector space of all polynomials P(x), with real coefficients, of degree less than or equal to two considered over the real field IR, such that P(0)=0 and P(1)=0. Determine a basis for S and hence its dimension.

20. Let V be the vector space of polynomials in x of degree $\le n$ over R. Prove that the set $\{1, x, x^2, ..., x^n\}$ is a basis for V. Extend this basis so that it becomes a basis for the set of all polynomials in x.

21. Let S be space generated by the vectors $\{(0,2,6),(3,1,6),(4,-2,-2)\}$. What is the dimension of the space S? find a basis for S.

22. Given two linearly independent vectors (1, 0, 1, 0) and (0, -1, 1, 0) of IR^4, find a basis of IR^4 which includes these two vectors:

23. Let V be the vector space of polynomials over IR. Find a basis and dimension of the subspace w of v spanned by the polynomials

$v_1 = t^3 - 2t^2 + 4t + 1$,

$v_2 = 2t^3 - 3t^2 + 9t - 1$,

$v_3 = t^2 + 6t - 5$,

$v_4 = 2t^3 - 5t^2 + 7t + 5,$

24. In IR^4, let w_1 be the space generated by (1, 1, 0, -1), (2, 4, 6, 0) and (-2, -3, -3, 1) and let w_2 be the space generated by (-1, -2, -2, 2), (4, 6, 4, -6) and (1, 3, 4, -3). Find a basis for the space w_1+w_2.

25. Let V and U be vector spaces over the field K and let V be of finite dimension. Let $T:V \rightarrow U$ be a linear map.
Prove that dim V = dim R (T) + dim N (T).

26. Let S= {(x,y,z)/ x+y+z=0}, x,y,z being real, prove that S is a subspace of IR^3. Find a basis of 'S'.

27. Let V(IR) be the real vector space of all 2×3 matrices with real entries.Find a basis for V(IR). what is the dimension of V(IR)?

28. Show that the rank of the product of two square matrices A, B each of order n, satisfies the inequality

$$r_A + r_B - n \le r_{AB} \le \min\left(r_A, r_B\right)$$

where r_0 stands for the rank of a square matrix C.

29. Determine a basis of the subspace spanned by the vectors
$v_1 = (1, 2, 3)$; $v_2 = (2, 1, -1)$
$v_3 = (1, -1, -4)$; $v_4 = (4, 2, -2)$
(a) show that it is impossible for any 2×2 real symmetric matrix of the form $\begin{bmatrix} a_1 & b \\ b & a_2 \end{bmatrix}$ $(b \neq 0)$ to have identical eigen values.
(b) Prove that the eigen values of a Hermitian matrix are all real and that an eigen value of a skew Hermitian matrix is either zero or a pure imaginary number.

30. Let $M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $M_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$,

$$M_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, M_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

prove that the set $\{M_1, M_2, M_3, M_4\}$ form the basis of the vector space of 2 × 2 matrices.

31. Consider the basis $S = \{V_1, V_2, V_3\}$ of IR^3 where
$V_1 = (1,1,1)$, $V_2 = (1, 1, 0)$,
$V_3 = (1,0,0)$. Express (2,–3, 5) interms of the basis V_1, V_2, V_3.
Let $T:\mathbb{R}^3 \to \mathbb{R}^2$ be defined as $T(V_1) = (1, 0), T(V_2) = (2,-1), T(V_3) = (4,3)$.
Find T (2, –3, 5)

32. If the matrix of a linear operator T on $V_3(\mathbb{R})$ with respect to the standard basis is

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}.$$

Describe explicitly $T:V_3(\mathbb{R}) \to V_3(\mathbb{R})$. What is the martix of T with respect to the basis $\{(0,1,-1),(1,-1,1),(-1,1,0)\}$.

33. Let $R_3(x) = \{a_0 + a_1x + a_2x^2 / a_0, a_1, a_2 \in \mathbb{R}\}$
Define $T:\mathbb{R}_3(x) \to \mathbb{R}_3(x)$ by
$T(f(x)) = \frac{d}{dx}f(x)$ for all $f(x) \in \mathbb{R}_3(x)$. Show that T is a linear transformation. Also find the matrix representation of T with reference to basis sets $\{1, x, x^2\}$ and $\{1, 1+x, 1+x+x^2\}$.

34. Let V be the real vector space of all polynomials in one variable with real coefficients and of degree less than, or equal to 3, provided with the standard basis $\{1, x, x^2, x^3\}$.
If $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3$, define
$T(p)(x) = a_0 + a_1(x+1) + a_2(x+1)^2 + a_3(x+1)^3$.

Write down the matrix representing the linear transformation T with respect to this basis.

35. Let ${}_3$ denote the (real) vector space of all polynomials (in one variable), with real coefficients and of degree less than, or equal to, 3, equipped with the standard basis $\{1, x, x^2, x^3\}$. Write down the matrix (with respect to this basis) of the linear transformation $L(p) = p'' - 2p' + p,\ p \in {}_3$

36. Show that the map $T : P_2 \to P_2$ defined by $T(\alpha_0 + \alpha_1 x + \alpha_2 x^2) = (\alpha_0 + \alpha_1) + (\alpha_1 + 2\alpha_2)x + (\alpha_0 + \alpha_1 + 3\alpha_2)x^2$ is non singular and find its inverse.

37. (i) Show that there does not exist any non-singular mapping of $\mathbb{R}^3$ into $\mathbb{R}^2$. Give an example of a non-singular mapping of $\mathbb{R}^2$ into $\mathbb{R}^3$.

(ii) Show that $T : \mathbb{R}^3 \to \mathbb{R}^2$ defined by T(x, y, z) = (y, x)is a linear transformation for the basis sets $S_1 = \{e_1, e_2, e_3\}$ of $\mathbb{R}^3$ and $S_2 = \{(1,0),(2,3)\}$ of $\mathbb{R}^2$, find the matrix representation of T. Here e_1, e_2, e_3 are standard unit vectors.

38. Let F be a subfield of the complex numbers and let T be the function from F^3 into F^3 defined by

$T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3,\ 2x_1 + x_2,\ -x_1 - 2x_2 + 2x_3)$

(i) Verify that T is a linear transformation.

(ii) If (a, b, c) is a vector in F^3, what are the conditions on a, b and c that the vector be in the range of T ? What is the rank of T?

(iii) What are the conditions on a, b and c that (a, b, c) be in the null space of T? What is the nullity of T?

39. Let $R_3[x] = \left\{ a_0 + a_1 x + a_2 x^2 / a_0, a_1, a_2 \in R \right\}$.

Define $T : \mathbb{R}_3[x] \to \mathbb{R}_3[x]$ by

$$T(f(x)) = \frac{d}{dx}(f(x)) \text{ for all } f(x) \in R_3[x].$$

Show that T is a linear transformation. Also find the matrix representation of T with reference to basis sets $\{1, x, x^2\}$ and $\{1, 1+x, 1+x+x^2\}$.

40. Let V be the vectorspace polynomials of degree ≤ 3 over F. Let T be a linear transformation defined on V by

$T(a_0 + a_1 x + a_2 x^2 + a_3 x^3) = (a_0 + a_1(x+1) + a_2(x+1)^2 + a3(x+1)^3).$

Compute the matrix of T relative to the bases

(i) $[1, x, x^2, x^3]$

(ii) $\left[1, 1+x, 1+x^2, 1+x^3\right]$.

41. Let T be the linear operator on $\mathbb{R}^3$ defined by $T(x_1, x_2, x_3) = (3x_1, x_1 - x_2, 2x_1 + x_2 + x_3)$. Is T invertible ? If so, find a rule for T^{-1} like the one which defines T.

42. Let V = {P(x)/P(x) is a polynomial of degree $\le n$ with real coefficients} and $T : V \to \mathbb{R}^m$ be defined as T(p(x) = (p(1), p(2),...., p(m)). Then show that T is linear and determine the Nullity of T.

43. Let T be a linear operator on C^3 defined by
T(1,0,0)=(1,0,i),
T(0,1,0)=(0,1,1),
T(0,0,1)=(i,1,0).
Is T invertible? Justify your answer.

44. Let T be a linear operator on C^2 defined by $T(x_1, x_2) = (x_1, 0)$. Let B be the, standard ordered basis for C_2 and

$\beta' = \{\alpha_1 = (1,i), \alpha_2 = (-i,2)\}$ be an ordered basis for C^2. Find a matrix P such that $[T]_{\beta'} = p^{-1}[T]_{\beta} P$

45. The matrix map $B:\mathbb{R}^3 \to \mathbb{R}^3$ defined by the matrix $B = \begin{bmatrix} 1 & 2 & 5 \\ 3 & 5 & 13 \\ -2 & -1 & -4 \end{bmatrix}$, Find the dimension and a basis of the kernal of the matrix map B.

46. Let V be the vector space polynomials of degree ≤ 3 over F. Let T be a linear transformation defined on V by
$T(a_0+a_1x+a_2x^2+a_3x^3) = (a_0+a_1(x+1)+a_2(x+1)^2+a_3(x+1)^3)$.
Compute the matrix of T relative to the bases
(i) $\{1, x, x^2, x^3\}$
(ii) $\{1, 1+x, 1+x^2, 1+x^3\}$

47. Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be defined as
$T(x,y,z) = (3x, x-y, 2x+y+z), \forall (x,y,z) \in \mathbb{R}^3$
(i) Show that T is a linear operator on $\mathbb{R}^3$.
(ii) Is T invertible ? If yes, find T^{-1}.

48. Let $\mathbb{R}^3 \to \mathbb{R}^3$ be defined by
$T(x_1, x_2, x_3) = (x_1+x_2, x_2+x_3, x_3+x_3+x_1)$.
Is T invertible ? If yes, find $T^{-1}(x_1,x_2,x_3)$.

49. If the matrix of a linear transformation $T:\mathbb{R}^3 \to \mathbb{R}^3$ relative to the basis $\{e_1=(1,0,0),\ e_2=(0,1,0),\ e_3=(0,0,1)\}$ is $\begin{pmatrix} 1 & 2 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$. Find the matrix of T relative to the basis
$\{f_1=(1,1,1),\ f_2=(0,1,1),\ f_3=(0,0,1)\}$.

50. Let $R_3(x) = \{a_0 + a_1x + a_2x^2 / a_0, a_1, a_2 \in \mathbb{R}\}$
Define $T: \mathbb{R}_3(x) \to \mathbb{R}_3(x)$
by $T(f(x)) = \frac{d}{dx} f(x)$ for all $f(x) \in \mathbb{R}_3(x)$.
Show that T is a linear transformation. Also find the matrix representation of T with reference to basis sets $\{1, x, x^2\}$ and $\{1, 1+x, 1+x+x^2\}$.

51. Let T be a linear operator on $\mathbb{R}^3$ defined by
$T(x,y,z) = (x+y+2z, -x+2y+z, y+3z)$
Let $\beta = \{\rho_1 = (1,0,0), \rho_2 = (0,1,0), \rho_3 = (0,0,1)\}$
and $\beta' = \{\alpha_1 = (1,1,1), \alpha_2 = (0,1,1), \alpha_3 = (0,0,1)\}$
be two ordred bases for $\mathbb{R}^3$. Find a matrix P such that $[T]_{\beta'} = P^{-1}[T]_{\beta} P$.

52. Let P_n denote the vector space of all real polynomials of degree atmost n and $T: P_2 \to P_3$ be a linear transformation given by
$T(p(x)) = \int_0^x p(t)\,dt,\ p(x) \in P_2$.
Find the matrix of T with respect to the bases $\{1, x, x^2\}$ and $\{1, x, 1+x^2, 1+x^3\}$ of P_2 and P_3 respectively. Also, find the null space of T.

53. Let V be an n-dimensional vector space and $T:V \to V$ be an invertible linear operator. If $\beta = \{X_1, X_2, ..., X_n\}$ is a basis of V, show that $\beta' = \{TX_1, TX_2, ...TX_n\}$ is also a basis of V.

54. Let $T:\mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by
$T(\alpha, \beta, \gamma) = (\alpha, 2\beta - 3\gamma, 2\alpha + 5\beta - 4\gamma, \alpha + 4\beta + \gamma)$
Find a basis and the dimension of the image of T and the kernel of T.

55. (i) Consider the linear mapping $f:\mathbb{R}^2 \to \mathbb{R}^2$ by

$f(x,y) = (3x+4y, 2x-5y)$

Find the matrix A relative to the basis $\{(1, 0), (0, 1)\}$ and the matrix B relative to the basis $\{(1, 2), (2, 3)\}$.

(ii) If λ is a characteristic root of a non-singular matrix A, then prove that $\frac{|A|}{\lambda}$ is a characteristic root of Adj A.

56. Show that the vectors (1, 1, 1), (2, 1, 2) and (1, 2, 3) are linearly independent in $R^{(3)}$. Let $T: IR^{(3)} \to IR^{(3)}$ be a linear transformation defined by

$$T(x,y,z) = (x+2y+3z, x+2y+5z, 2x+4y+6z)$$

Show that the images of above vectors under T are linearly dependent. Give the reason for the same.

57. What is the null space of the differentiation transformation $\frac{d}{dx}: P_n \to P_n$

Where P_n is the space of all polynomials of degree $\leq$ n over the real numbers? What is the null space of the second derivative as a transformation of P_n? What is the null space of the kth derivative?

58. Let $M = \begin{pmatrix} 4 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$. Find the unique linear transformation $T: IR^3 \to IR^2$ so that M is the matrix of T with respect to the basis.

$\beta = \{v_1 = (1,0,0), v_2 = (1,1,0), v_3 = (1,1,1)\}$ of IR^3 and $\beta = \{w_1 = (1,0), w_2 = (1,1)\}$ of IR^2. Also find T (x, y, z)

59. Let $\beta = \{(1, 1, 0), (1, 0, 1), (0, 1,1)\}$ and $\beta' = \{(2, 1, 1), (1, 2, 1), (-1, 1, 1)\}$ be the two ordered bases of IR^3. Then find a matrix representing the linear transformation $T: IR^3 \to IR^3$ which transforms β into β'. Use this matrix representation to find $T(\bar{x})$, where $\bar{x} = (2,3,1)$.

60. Let $L: IR^4 \to IR^3$ be a linear transformation defined by

$$L(x_1, x_2, x_3, x_4) = (x_3 + x_4 - x_1 - x_2, x_3 - x_2, x_4 - x_1)$$

Then find the rank and nullity of L. Also, determine null space and range space of L.

61. Show that $B = \{(1,0,0), (1,1,0), (1,1,1)\}$ is a basis of R^3. Let $T: R^3 \to R^3$ be a linear transformation such that T(1, 0, 0) = (1, 0, 0), T (1, 1, 0) = (1, 1, 1) and T (1, 1, 1) = (1, 1, 0). Find T (x, y, z).

62. Let T be the linear transformation from IR^3 to IR^4 defined by

$$(2x_1 + x_2 + x_3, x_1 + x_2, x_1 + x_3, 3x_1 + x_2 + 2x_3)$$

for each $(x_1, x_2, x_3) \in IR^3$

Determine a basis for the Null space of T. What is the dimension of the Range space of T.

63. Let T be a linear transformation on R^3, whose matrix relative to the standard basis of R^3 is

$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 2 \\ 3 & 3 & 4 \end{bmatrix}$ find the matrix of T relative to the basis

$\beta = \{(1,1,1), (1,1,0), (0,1,1)\}$.

64. Show that $f: IR^3 \to IR$ is a linear transformation, where $f(x, y, z) = 3x + y - z$. What is the dimension of the kernel? find a basis for the kernel.

65. Show that the linear transformation from IR^3 to IR^4 which is represented by the matrix

$$\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & -2 \\ 2 & 1 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$ is one to one. Find a basis for its image

66. Show that the mapping $T: R^3 \to R^3$ where $T(a,b,c) = (a-b, b-c, a+c)$ is linear and non-singular.

67. Let $T: R^5 \to R^5$ be a linear mapping given by
$T(a,b,c,d,e) = (b-d, d+e, b, 2d+e, b+e)$.
Obtain bases for its null space and range space.

68. Let V be a vector space over R and
Let $T = \{(x, y) / x, y \in V\}$
Define addition in T componentwise and scalar multiplication by a complex number $\alpha + i\beta$ by $(\alpha + i\beta)(x, y) = (\alpha x - \beta y, \beta x + \alpha y)$ $\forall \alpha, \beta \in R.$ show that T is a vector space over C.

69. Let $T = IR^3 \to IR^3$ defined by
$T(x_1, x_2, x_3) = (x_2, x_3 - cx_1, -bx_2 - ax_3)$, where a, b, c are fixed real numbers. Show that T is a linear transformation of IR^3 and that $A^3 + aA^2 + bA + cI = 0,$ where A is the matrix of T with respect to standard basis of IR^3.

70. Let V= IR^3 and v_1, v_2, v_3 be a basis of IR^3. Let $T: V \to V$ be linear transformation such that $T(v_i) = v_1 + v_2 + v_3,\ 1 \le i \le 3$. By writing the matrix of T w.r.t another basis, show that the matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ is similar to $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

71. Let $V = IR^3$ and $T: V \to V$ be the linear map defined by
$T(x, y, z) = (x+z, -2x+y, -x+2y+z)$.
What is the matrix of T w.r.t the basis (1, 0, 1), (-1, 1, 1) and (0, 1, 1)? Using this matrix, write down the matrix of T w.r.t the basis (0, 1, 2), (-1, 1, 1) and (0, 1, 1).

72. Let T be the linear operator in R^3 defined by $T(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3)$.
What is the matrix of T in the standard ordered basis for R^3? what is a basis of range space of T and a basis of null space of T?

73. Show that $f_1(t) = 1$, $f_2(t) = t-2$, $f_3(t) = (t-2)^3$ form a basis of P_3, the space of polynomials with degree $\le 2.$ Express $3t^2-5t+4$ as a linear combination of f_1, f_2, f_3.

74. Define rank and nullity of a linear transformation T. If V be a finite dimensional vector space and T a linear operator on V such that rank T^2 = rank T, then prove that the null space of T = the null space of T^2 and the intersection of the range space and null space of T is the zero subspace of V.

75. If the matrix of a linear operator T on IR^2 relative to the standard basis {(1, 0), (0, 1)} is $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, what is the matrix of T relative to the basis B = {(1, 1), (1, –1)}?

76. Verify which of the following are linear transformations.
(i) $T:IR \to IR^2$ defined by $T(x) = (2x, -x)$
(ii) $T:IR^2 \to IR^3$ defined by $T(x, y) = (xy, y, x)$
(iii) $T:IR \to IR^2$ defined by $T(x) = (1, -1)$
(iv) $T:IR^2 \to IR^3$ defined by $T(x,y) = (x+y, y, x)$

77. Let $T:M_{2\times1} \to M_{2\times3}$ be a linear transformation defined by

$$T\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}2&1&3\\4&1&5\end{pmatrix},\ T\begin{pmatrix}1\\1\end{pmatrix}=\begin{pmatrix}6&1&0\\0&0&2\end{pmatrix}.$$

Find $T\begin{pmatrix}x\\y\end{pmatrix}$

78. Let C be the field of complex numbers and let T be the function from $\mathbb{C}^3$ into $\mathbb{C}^3$ defined by
$T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3, 2x_1 + x_2, -x_1 - 2x_2 + 2x_3)$
(i) verify that T is a linear transformation.
(ii) If (a, b, c) is a vector in $\mathbb{C}^3$, what are the conditions on a, b and c so that the vector be in the range of T? what is the rank of T?
(iii) What are the conditions on a, b and c so that (a, b, c) is in the null space of T? what is the nullity of T?

79. Let $L: IR^3 \to IR^2$ be linear transformation for which we know that $L(1,0,0) = (2,-1), L(0,1,0) = (3,1)$ and $L(0,0,1) = (-1,2)$. Find $L(-3,4,2)$.

80. Let $L: IR^3 \to IR^2$ be defined by

$$L\left[\begin{pmatrix}x\\y\\z\end{pmatrix}\right]=\begin{pmatrix}x+y\\x-z\end{pmatrix}$$

Let $S=\{v_1,v_2,v_3\}$ and $T=\{w_1,w_2\}$ be bases for IR^3 and IR^2 respectively *where*

$$v_1=\begin{bmatrix}1\\0\\0\end{bmatrix},\ v_2=\begin{bmatrix}0\\1\\0\end{bmatrix},\ v_3=\begin{bmatrix}0\\0\\1\end{bmatrix},$$

$$w_1=\begin{bmatrix}1\\0\end{bmatrix}\ \&\ w_2=\begin{bmatrix}0\\1\end{bmatrix}$$

Find the matrix of L w.r.t S and T.

81. Find the characteristic equation of the matrix $A=\begin{bmatrix}2&-1&1\\-1&2&-1\\1&-1&2\end{bmatrix}$ and verify that it satisfies by A.

82. Is the matrix $A=\begin{bmatrix}6&-3&2\\4&-1&-2\\10&-5&-3\end{bmatrix}$ Similar over the field IR to a diagonal matrix ? Is **A** similarl over the field C to a diagonal matrix ?

83. Determine the difiniteness of the following quadratic form

$$q(x_1,x_2,x_3)=[x_1\ x_2\ x_3]\begin{bmatrix}2&0&-1\\1&5&2\\-2&1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}$$

84. Determine a,b and c so that the matrix $A=\begin{pmatrix}0&2b&c\\a&b&-c\\a&-b&c\end{pmatrix}$ is orthogonal.

85. Obtain the characteristic equation of the matrix $A=\begin{pmatrix}1&2\\2&-1\end{pmatrix}$ and show that **A** satisfies its characteristic equation. Hence determine the inverse of A.

86. If S be a real skew-symmetric matrix of order prove that the matrix $P=(I_n+S)^{-1}(I_n-S)$ is orthogonal where I_n stands for the identity matrix of order $'n'$

87. Find the row rank and column rank of the matrix $A=\begin{pmatrix}2&1&4&3\\3&2&6&9\\1&1&2&6\end{pmatrix}$. Hence determine the rank of A .

88. Reduce the equation
$2y^2-2xy-2yz+2zx-x-2y+3z-2=0$
into canonical form and determine the nature of the quadric.

89. Find a linear map $T:R^3\to R^4$ Whose image is generated by $(1,2,0,-4)$ *and* $(2,0,-1,-3)$

90. Let $A=\begin{pmatrix}1&-3&3\\3&-5&3\\6&-6&4\end{pmatrix}$. Is A similar to a diagonal matrix ? If so, find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

91. Show that the set $P[t]=\left\{at^2+bt+c/a,b,c\in\mathbb{R}\right\}$ forms a vector sapce over the field $\mathbb{R}$. Find a basis for this vector space. What is the dimension of this vector space ?

92. Find the characteristic polynomial of $\begin{pmatrix}0&0&1\\1&0&2\\0&1&3\end{pmatrix}$. Verify Cayley-Hamilton theorem for this matrix and hence find its inverse.

93. Let $A=\begin{pmatrix}5&-6&-6\\-1&4&2\\3&-6&-4\end{pmatrix}$ Find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

94. Find the rank of the matrix

$$\begin{pmatrix}1&2&1&1&2\\2&4&3&4&7\\-1&-2&2&5&3\\3&6&2&1&3\\4&8&6&8&9\end{pmatrix}$$

95. Let $V=\left\{(x,y,z,u)\in\mathbb{R}^4 : y+z+u=0\right\}$,

$W=\left\{(x,y,z,u)\in\mathbb{R}^4 : x+y=0\ ,\ z=2u\right\}$

be two subspaces of $\mathbb{R}^4$. Find bases for $V,W\ \ V+W$ and $V\cap W$.

96. Find the characteristic polynomial of the matrix $A=\begin{pmatrix}3&1&1\\2&4&2\\-1&-1&1\end{pmatrix}$ and hence compute A^{10}.

97. Let $A=\begin{pmatrix}1&-3&3\\0&-5&6\\0&-3&4\end{pmatrix}$ Find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

98. Find an orthogonal transformation to reduce the quadratic form $5x^2+2y^2+4xy$ to a canonical form.

99. Verify Cayley–Hamilton theorem for the matrix $A=\begin{pmatrix}1 & 4\\ 2 & 3\end{pmatrix}$ and find its inverse. Also express $A^5-4A^4-7A^3+11A^2-A-10I$ as a linear polynomial in A.

100. Find the characteristic equation of the matrix $A=\begin{bmatrix}2 & 1 & 1\\ 0 & 1 & 0\\ 1 & 1 & 2\end{bmatrix}$ and hence find the matrix represented by

$$A^8-5A^7+7A^6-3A^5$$
$$+A^4-5A^3+8A^2-2A+I.$$

101. Let V be the vecot space of 2×2 matrices over $\mathbb{R}$ and let $M=\begin{bmatrix}1 & -1\\ -2 & 2\end{bmatrix}$. Let $F:V\to V$ be the linear map defined by $F(A) = MA$. Find a basis and the dimension of
(i) The kernel of W of F
(ii) The image U of F.

102. Let $H=\begin{bmatrix}1 & i & 2+i\\ -i & 2 & 1-i\\ 2-i & 1+i & 2\end{bmatrix}$ be a Hermitian matrix. Find a non-singular matrix P such that $P^t H\overline{P}$ is diagonal and also find its signature.

103. Find an orthogonal transformation of co-ordinates which diagonalizes the quadratic form

$q(x,y) = 2x^2 - 4xy + 5y^2$.

104. Discuss the consistency and the solutions of the equations

$x + ay + az = 1$, $ax + y + 2az = -4$, $ax - ay + 4z = 2$.

for different values of a.

105. For the matrix $A=\begin{bmatrix}1 & 0 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{bmatrix}$. Prove that $\mathbf{A}^n = A^{n-2} + A^2 - I, n \geq 3$.

106. Let $B=\begin{bmatrix}1 & -1\\ 2 & -1\end{bmatrix}$. Find all eigen values and corresponding eigen vectors of B viewed as a matrix over:
(i) The real field R.
(ii) The complex field C.

107. Examine whether the matrix $A=\begin{bmatrix}-2 & 2 & -3\\ 2 & 1 & -6\\ -1 & -2 & 0\end{bmatrix}$ is diagonalizable. Find all eigen values. Then obtain a matrix p such that $P^{-1}AP$ is a diagonal matrix.

108. Find an upper triangular matrix A such that

$$A^3 = \begin{bmatrix}8 & -57\\ 0 & 27\end{bmatrix}$$

109. Find the condition on a, b, and c so that the following system in unknowns x, y and z has a solution.

$x + 2y - 3z = a$

$2x + 6y - 11z = b$

$x - 2y + 7z = c$

110. Find the minimal polynomial of the matrix

$$A = \begin{bmatrix} 4 & -2 & 2 \\ 6 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$$

111. Find a 3×3 orthogonal matrix whose first two rows are $\left[\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right]$ and $\left[0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right]$.

112. Determine diagonal matrices orthogonally similar to the following real symmetric matrices, obtaining also the transforming matrices:

$$A = \begin{bmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ -4 & -1 & -8 \end{bmatrix}$$

113. Show that every square matrix can be expressed uniquely as A + iB where A, B are Hemitian.

114. Reduce the quadratic form $x^2 + 4y^2 + 9z^2 + t^2 - 12yz + 6zx - 4xy - 2xt - 6zt$ to canonical form and find its rank and signature.

115. (*i*) If **B** is an idempotent matrix, show that **A = I – B** is also idempotent and that **AB = BA = O.**
(*ii*) If A is a non-singular matrix, then show that $adj\ adj\ A = |A|^{n-2} A$.

116. Find the characteristic roots of the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and verify Cayley-Hamilton theorem for this matrix. Find the inverse of the matrix A and also express.
$A^5 - 4A^4 - 7A^3 + 11A^2 - A - 10I$ as a linear polynomial in A.

117. Let T be the linear operator on R^4 which is represented in the standard ordered basis by the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{bmatrix}$$

Under what conditions on a, b, and c is T diagonalizable?

118. Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

Show that for every interger $n \geq 3$, $A^n = A^{n-2} + A^2 - I$ and hence find the matrix A^8.

119. Find the characteristic and minimal polynomials of $A = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ and determine whether A is diagonalisable.

120. Let $A = \begin{pmatrix} 1 & 1 & 2 \\ -1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$. Find an invertible 3×3 matrix P such that $P'Ap = D$, D is diagonal matrix. Find D also.

121. If the matrix of a linear operator T on $V_3(\mathbb{R})$ with respect to the standard basis is $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$. Describe explicitly $T : V_3(\mathbb{R}) \to V_3(\mathbb{R})$. What is the matrix of T with respect to the basis {(0, 1, –1), (1, –1,1), (–1, 1,0)}

122. Show that if λ is a non-zero eigen value of the non singular n-square matrix A, then $\frac{|A|}{\lambda}$ is an eigen value of adjA.
Also give an example to prove that the eigen values of AB are not necessarily the product of eigen values of A and that of B.

123. Given the linear transformation $Y = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 3 \\ -2 & 3 & 5 \end{bmatrix} X$, Show that it is singular and the images of the linearly independent vectors,
$X_1 = [1,1,1]^T$, $X_2 = [2,1,2]^T$, $X_3 = [1,2,3]^T$
are linearly dependent.

124. Find the inverse of the matrix : A = $\begin{bmatrix} 1 & 3 & 1 \\ 2 & -1 & 7 \\ 3 & 2 & -1 \end{bmatrix}$
by using elementary row operations. Hence solve the system of linear equations
$x + 3y + z = 10$
$2x - y + 7z = 21$
$3x + 2y - z = 4$

125. The $n \times n$ A matrix A satisfies $A^4 = -1.6A^2 - 0.64I$ show that $\underset{n\to\infty}{Lt} A^n$ exists and determine this limit.

126. Reduce $A = \begin{bmatrix} 1 & 3 & 6 & -1 \\ 1 & 4 & 5 & 1 \\ 1 & 5 & 4 & 3 \end{bmatrix}$ to normal form N and compute the matices P & Q such that PAQ=N.

127. Let $V = P_3(IR)$ be the vector space of polynomial fucntions on reals of degree at most 3. Let $D : V \to V$ be the differentiation operator defined by
$$D\left(a_0 + a_1x + a_2x^2 + a_3x^3\right) = a_1 + 2a_2x + 3a_3x^2.$$
(i) Show that D is a linear transformation.
(ii) Find kernel and image of D.
(iii) What are dimensions of V , $\ker D$ and image D ?
(iv) Give relation among them of (iii)

128. Find the eigen values and the corresponding eigen, vectors of $A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$.

129. Show that the vectors $(1,2,1),(1,0,-1)$ *and* $(0,-3,2)$ form a basis for IR^3.

130. Determine non-singular matrices P and Q such that the matrix PAQ is in canonical form, Where $A = \begin{bmatrix} 5 & 3 & 14 & 4 \\ 0 & 1 & 2 & 1 \\ 1 & -1 & 2 & 0 \end{bmatrix}$. Hence find the rank of A.

131. Find the minimum polynomial of the matirx $A = \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}$, and use it to determine whether A is similar to a diagonal matrix.

132. Show that the quadratic form $2x^2 - 4xy + 3zx + 6y^2 + 6yz + 8z^2$ in three variables is positive definite.

133. Find the eigen values and their corresponding eigen vectors of the matrix $\begin{pmatrix} 2 & 0 & 3 \\ 2 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}$. Is the matrix diagonalisable?

134. For what values of α has the systme of equations

$x+2y+z=1$,
$\alpha x+4y+2z=2$,
$4x-2y+2\alpha z=-1$
(i) a unique solution
(ii) infinitely many solutions
(iii) no solution.

135. Determine an orthogonal matrix which reduces the quadratic form $Q(x_1,x_2,x_3)=2x_1^2+x_2^2-4x_2x_3+x_3^2$ to a canonical form. Also, indentify the surface respresented by $Q(x_1,x_2,x_3)=7$.

136. Find the eigen values and the corresponding eigen vectors of $A=\begin{pmatrix}2 & 1\\ 2 & 3\end{pmatrix}$.

137. Determine a non-singular matirx P such that P^tAP is a diagonal matrix, where P^t denotes the transpose of P, and $A=\begin{bmatrix}0 & 1 & 2\\ 1 & 0 & 3\\ 2 & 3 & 0\end{bmatrix}$.

138. If A be an orthogonal matrix with the properly that -1 is not a characteristic root of A, then show that A is expressible as $(I-S)(I+S)^{-1}$ for some suitable skew-symmetric matrix S.

139. Show that the real quadratic form $\phi=n\left(x_1^2+x_2^2+......+x_n^2\right)-\left(x_1+x_2+.......+x_n\right)^2$. in n variables is positive semi-definite.

140. Find the symmetric matrix which corresponds to the following quadratic polynomial:

$q(x,y,z)=x^2-2yz+xz.$

141. Let $A=\begin{pmatrix}1 & 4\\ 2 & 3\end{pmatrix}$. Find all eigen values of A and the corresponding eigen vectors.

142. Let $H=\begin{bmatrix}1 & 1+i & 2i\\ 1-i & 4 & 2-3i\\ -2i & 2+3i & 7\end{bmatrix}$ be a Hemitian matrix. Find a non-singular matrix P such that $P^tA\overline{P}$ is diagonal

143. Let $L:IR^3\to IR^2$ be linear transformation for which we know that $L(1,0,0)=(2,-1), L(0,1,0)=(3,1)$ and $L(0,0,1)=(-1,2)$. Find $L(-3,4,2)$.

144. Let $L:IR^3\to IR^2$ be defined by

$$L\left[\begin{pmatrix}x\\ y\\ z\end{pmatrix}\right]=\begin{pmatrix}x+y\\ x-z\end{pmatrix}$$

Let $S=\{v_1,v_2,v_3\}$ and $T=\{w_1,w_2\}$ be bases for IR^3 and IR^2 respectively *where*

$$v_1=\begin{bmatrix}1\\0\\0\end{bmatrix}, v_2=\begin{bmatrix}0\\1\\0\end{bmatrix},$$

$$v_3=\begin{bmatrix}0\\0\\1\end{bmatrix}, w_1=\begin{bmatrix}1\\0\end{bmatrix} \ \& \ w_2=\begin{bmatrix}0\\1\end{bmatrix}$$

Find the matrix of L w.r.t S and T.

145. Find the characteristic equation of the matrix $A=\begin{bmatrix}2 & -1 & 1\\ -1 & 2 & -1\\ 1 & -1 & 2\end{bmatrix}$ and verify that it satisfies by A.

146. Is the matrix $A=\begin{bmatrix}6 & -3 & 2\\ 4 & -1 & -2\\ 10 & -5 & -3\end{bmatrix}$ Similar

over the field IR to a diagonal matrix ? Is **A** similarl over the field C to a diagonal matrix ?

147. Determine the difiniteness of the following quadratic form

$$q(x_1, x_2, x_3) = [x_1\ x_2\ x_3]\begin{bmatrix} 2 & 0 & -1 \\ 1 & 5 & 2 \\ -2 & 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

148. Determine a,b and c so that the matrix

$A = \begin{pmatrix} 0 & 2b & c \\ a & b & -c \\ a & -b & c \end{pmatrix}$ is orthogonal.

149. Obtain the characteristic equation of the matrix $A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$ and show that **A** satisfies its characteristic equation. Hence determine the inverse of A.

150. If S be a real skew-symmetric matrix of order prove that the matrix $P = (I_n + S)^{-1}(I_n - S)$ is orthogonal where I_n stands for the identity matrix of order $'n'$

151. Find the row rank and column rank of the matrix $A = \begin{pmatrix} 2 & 1 & 4 & 3 \\ 3 & 2 & 6 & 9 \\ 1 & 1 & 2 & 6 \end{pmatrix}$. Hence determine the rank of A.

152. Reduce the equation

$2y^2 - 2xy - 2yz + 2zx - x - 2y + 3z - 2 = 0$

into canonical form and determine the nature of the quadric.

153. Find a linear map $T : R^3 \to R^4$ Whose image is generated by $(1,2,0,-4)$ *and* $(2,0,-1,-3)$

154. Let $A = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$. Is A similar to a diagonal matrix ? If so, find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

155. Show that the set $P[t] = \{at^2 + bt + c / a,b,c \in \mathbb{R}\}$ forms a vector sapce over the field $\mathbb{R}$. Find a basis for this vector space. What is the dimension of this vector space ?

156. Find the characteristic polynomial of $\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{pmatrix}$.

Verify Cayley-Hamilton theorem for this matrix and hence find its inverse.

157. Let $A = \begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix}$. Find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

158. Find the rank of the matrix

$$\begin{pmatrix} 1 & 2 & 1 & 1 & 2 \\ 2 & 4 & 3 & 4 & 7 \\ -1 & -2 & 2 & 5 & 3 \\ 3 & 6 & 2 & 1 & 3 \\ 4 & 8 & 6 & 8 & 9 \end{pmatrix}$$

159. Let

$V = \{(x,y,z,u) \in \mathbb{R}^4 : y + z + u = 0\}$,

$W = \{(x,y,z,u) \in \mathbb{R}^4 : x + y = 0,\ z = 2u\}$

be two subspaces of $\mathbb{R}^4$. Find bases for $V, W \ \ V+W$ and $V \cap W$.

160. Find the characteristic polynomial of the matrix $A=\begin{pmatrix} 3 & 1 & 1 \\ 2 & 4 & 2 \\ -1 & -1 & 1 \end{pmatrix}$ and hence compute A^{10}.

161. Let $A=\begin{pmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{pmatrix}$ Find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

162. Find an orthogonal transformation to reduce the quadratic form $5x^2+2y^2+4xy$ to a canonical form.

163. Verify Cayley–Hamilton theorem for the matrix $A=\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ and find its inverse. Also express $A^5-4A^4-7A^3+11A^2-A-10I$ as a linear polynomial in A.

164. Find the characteristic equation of the matrix $A=\begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and hence find the matrix represented by
$$A^8-5A^7+7A^6-3A^5$$
$$+A^4-5A^3+8A^2-2A+I$$

165. Let V be the vecot space of 2×2 matrices over $\mathbb{R}$ and let $M=\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}$. Let $F: V \to V$ be the linear map defined by $F(A) = MA$. Find a basis and the dimension of
(i) The kernel of W of F
(ii) The image U of F.

166. Let $H=\begin{bmatrix} 1 & i & 2+i \\ -i & 2 & 1-i \\ 2-i & 1+i & 2 \end{bmatrix}$ be a Hermitian matrix. Find a non-singular matrix P such that $P^t H\overline{P}$ is diagonal and also find its signature.

167. Find an orthogonal transformation of co-ordinates which diagonalizes the quadratic form
$q(x,y) = 2x^2 - 4xy + 5y^2$.

168. Discuss the consistency and the solutions of the equations
$x + ay + az = 1,\ ax + y + 2az = -4,\ ax - ay + 4z = 2$.
for different values of a.

169. For the matrix $A=\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$. Prove that $A^n = A^{n-2} + A^2 - I,\ n \geq 3$.

170. Let $B=\begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}$. Find all eigen values and corresponding eigen vectors of B viewed as a matrix over:
(i) The real field R.
(ii) The complex field C.

171. Examine whether the matrix $A=\begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ is diagonalizable. Find all eigen values. Then obtain a matrix p such that $P^{-1}AP$ is a diagonal matrix.

172. Find an upper triangular matrix A such that
$A^3 = \begin{bmatrix} 8 & -57 \\ 0 & 27 \end{bmatrix}$

173. Find the condition on a, b, and c so that the following system in unknowns x, y and z has a solution.
$x + 2y - 3z = a$
$2x + 6y - 11z = b$
$x - 2y + 7z = c$

174. Find the minimal polynomial of the matrix
$$A = \begin{bmatrix} 4 & -2 & 2 \\ 6 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$$

175. Find a 3×3 orthogonal matrix whose first two rows are $\left[\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right]$ and $\left[0, \frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right]$.

176. Reduce the matrix
$$A = \begin{bmatrix} 1 & 1 & 0 \\ 4 & 3 & -5 \\ 1 & 1 & 5 \end{bmatrix}$$
to I_3 by a finite sequence of E-row transformation and express A as a product of elementary matrices. Reduce A^{-1}.

177. If H is a Hermitian matrix, show that $A = (I + iH)^{-1}$
(I-iH) is a unitary matrix. Also show that $A = (I-iH)(I+iH)^{-1}$

178. Show that the two matrices A, C^{-1} AC have the same characteristic roots.

179. Find an orthogonal matix that will diagonalise the real symmetrix matrix
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$$
Also write the resulting diagonal matix.

180. Show that the characteristic roots of any diagonal matrix are same as its elements in the diagonal.

181. Find a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ whose range is spanned by (1, 2, 0, -4), (2, 0, -1, -3).

182. Find a basis and the dimension oft he solution space 's' of the linear equations
$$x + 2y - 2z + 2s - t = 0$$
$$x + 2y - z + 3s - 2t = 0$$
$$2x + 4y - 7z + s + t = 0.$$

183. Find the characteristic roots of the matrix $A = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and verify cayley - Hamilton theorem for this matrix. Find the inverse of the matrix A.

184. Determine the diagonal matrix unitarily similar to the Hermitian matrix $A = \begin{bmatrix} 2 & 1-2i \\ 1+2i & -2 \end{bmatrix}$ obtaining also the transformation matrix.

185. Use the cayley - Hamilton theorem, to find the inverse of the following matrix
$$\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}.$$

186. Find a basis and the dimension fo the solution space 'S' of the linear equations
$$x + 2y - 2z + 2s - t = 0$$
$$x + 2y - z + 3s - 2t = 0$$
$$2x + 4y - 7z + s + t = 0.$$

187. Find the value of α for which the matrix $\begin{bmatrix} 0 & 1 & \alpha \\ 1 & \alpha & 0 \\ \alpha & 0 & 1 \end{bmatrix}$ is invertible and find its inverse.

188. Determine diagonal matrix orthogonally similar to the following real symmetric matrix

$A=\begin{bmatrix} 7 & 0 & -2 \\ 0 & 5 & -1 \\ -2 & -2 & 6 \end{bmatrix}$ obtaining also the transforming matrix.

189. Solve the equations
$\lambda x+2y-2z=1$
$4x+2\lambda y-z=2$
$6x+2y+\lambda z=3$
Considering specially the case when $\lambda=2$.

190. Find $F:\mathbb{R}^3\to\mathbb{R}^4$ is a linear transformation whose range is spanned by (1,-1,2,3) and (2,3,-1,0).

191. If S is a real skew symmetric matrix then show that (I+S) is non-singular and (I-S) $(I+S)^{-1}$ is orthogonal.

192. If the matrix A is non-singular, then show that the eigen values of A^{-1} are the reciprocals fo the eigen values of A.

193. Show that $A=\begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$ is similar to a diagonal matix. Also find the transforming matrix and diagonal matrix.

194. Show that the only real value of λ for which the following system of equations has a non-zero solution is 6.

195. If $A=\begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ then show that for every integer $n\geq 3, A^n=A^{n-2}+A^2-1$. Hence determine A^{50}.

196. Reduce the Matrix $\begin{bmatrix} 0 & 1 & 2 & 1 \\ 1 & 2 & 3 & 2 \\ 3 & 1 & 1 & 3 \end{bmatrix}$ to its normal form and hence or otherwise determine its rank.

197. Let T be the linear operator on $\mathbb{R}^3$ which is represented in the standard ordered basis by the matrix $A=\begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$ Show that **T** is diagonalisable and find the non-singular matrix P so that $P^{-1}AP$ is a diagonal matrix.

198. If S is a real skew symmetric matrixt hen show that (I+S) is non-singular and (I-S) $(I+S)^{-1}$ is orthogonal.

199. Find the characteristic roots of the matrix $A=\begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix}$ and verify cayley - Hamiltan theorem for this matrix. Find the inverse of the matrix A and also express $A^5-4A^4-7A^3+11A^2-A-10I$ as a linear polyonomial in A.

200. (i) Show that 0 is a characteristic root of a matrix if and only if the matrix is singular.
(ii) If α is a characteristic root of a non-singular matrix A, then prove that $\frac{|A|}{\alpha}$ is a characteristic root of Adj A.

201. Let $M=\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and T be the linear operation V defined by T(A) = MA. Find the matrix representation of T relative to the above usual basis of V.

202. Let A be a real 3×3 symmetric matrix with eigen values 0, 0, 5. If the corresponding eigen vectors are (2,0,1), (2,1,1) and (1,0,–2) then find the matrix A.

203. If A be an orthogonal matrix with the property that –1 is not a characteristic root, then A is expressible as $(I+S)(I-S)^{-1}$ for some suitable real skew–symmetric matrix S.

204. If H is a Hermitian matrix, show that $A=(I+iH)^{-1}(I-iH)$ is a unitary matrix.
Also show that $A=(I-iH)(I+iH)^{-1}$.
Further show that if λ is an eigen value of H, then $\frac{1-i\lambda}{1+i\lambda}$ is an eigen value of A.

205. Show that the matrix
$$A=\begin{bmatrix}-9 & 4 & 4\\ -8 & 3 & 4\\ -16 & 8 & 7\end{bmatrix}$$ is diagonalizable.
Also find the diagonal form and diagonalizing matrix P.

206. A square matrix M of order n with complex entries is called skew Hermitian if $M+\bar{M}^T=0$, where 0 is the zero matrix of order n.
Determine whether V = {M/M is a 2×2 skew Hermitian matrix} is a vector space over (i) the field $\mathbb{R}$ and (ii) the field C with the usual operations of addition and scalar multiplication for matrices ?

207. Shwo that the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^4$ which is represented by the matrix
$$\begin{bmatrix}1 & 3 & 0\\ 0 & 1 & -2\\ 2 & 1 & 1\\ -1 & 1 & 2\end{bmatrix}$$ is one to one. Find a basis for its image.

208. Let $M=\begin{bmatrix}1+i & 2i & i+3\\ 0 & 1-i & 3i\\ 0 & 0 & i\end{bmatrix}$. Determien the eigen values of the matrix $B = M2\text{-}2M\text{+}I$.

209. Let $T:\mathbb{R}^3\to\mathbb{R}^3$ be defined by $T(x,y,z)=(y+z,z,0)$.
Show that T is a linear transformation. If $V\in\mathbb{R}^3$ is such that $T^2(v)\neq 0$, then show that $B=\{v_1T(v),T_2(v)\}$ forms a basis of $\mathbb{R}^3$.
Compute the matrix of T with respect to B.
Also find a $V\in\mathbb{R}^3$ such that $T^2(v)\neq 0$.

210. Find a linear transformation $T:\mathbb{R}^3\Rightarrow\mathbb{R}^4$ whose range is spanned by (1,2,0,–4), (2,0,–1,–3).

211. Reduce the matrix
$$A=\begin{bmatrix}1 & 1 & 0\\ 4 & 3 & -5\\ 1 & 1 & 5\end{bmatrix}$$ to I_3 by a finite sequence of E-row transformation and express A as a product of elementary matrices, Reduce A^{-1}.

212. Find a basis and the dimension of the solution space 'S' of the linear equations
$$x+2y-2z+2s-t=0$$
$$x+2y-z+3s-2t=0$$
$$2x+4y-72+s+t=0$$

213. Verify that the matrix $A=\begin{bmatrix}1 & 2 & 0\\ 2 & -1 & 0\\ 0 & 0 & -1\end{bmatrix}$ satisfies its own characteristic equation. Is it true of every square matrix? State the theorem that applies here.

214. Find an orthogonal matrix that will diagonalize the real symmetric matrix

$$A=\begin{bmatrix}1&2&3\\2&4&6\\3&6&9\end{bmatrix}.$$

Also write the resulting diagonal matrix.

215. If U is a unitary matrix and $|I-U|\neq 0$, Prove that the matrix H defined by setting $iH=(I+U)(I+U)^{-1}$ is Hermitian. If $e^{i\theta},...e^{i\theta_n}$ be the eigen values of U, Show that eigen values of H are $\cot\left(\theta/2\right),....,\cot\left(\theta/2\right)$.

216. If A is a non-singular matrix, then show that adj adj $A=|A|^{n-2}A$.

217. If A be an orthogonal matrix with the property that –1 is not a characteristic root, then A is expressible as $(I+S)(I-S)^{-1}$ for some suitable real skew – symmetic matrix S.

218. Let $A=\begin{bmatrix}2&-2&2\\1&1&1\\1&3&-1\end{bmatrix}$ and C be a non-singular matrix of order 3×3. Find the eigen valuesof the matrix B^3 where $B=C^{-1}AC$

219. Let F be a field and let n be a positive integer $(n\geq 2)$. Let V be the vector space of all $n\times n$ matrices over F. Which of the following sets of matrices A in V are subspaces of V ?
(i) all invertible A
(ii) all non-invertible A
(iii) All A such that AB = BA, where B is some fixed matrix in V.

220. (i) Let $F:\mathbb{R}^3\to\mathbb{R}^3$ be the linear mapping defined by $F(x,y,s,t)=(x-y+s+t,\ x+2s-t,\ x+y+3s=3t)$.
Find a basis and the dimension of the image U of F.
(ii) Let $H:\mathbb{R}^3-\mathbb{R}^3$ be defined by $H(x,y,z)=(x+y-2z,\ x+2y+z,\ 2x+2y-3z)$. Is H nonsingular? If not, find $v\neq 0$ such that $H(v)=0$.

221. Determine diagonal matrices orthogonally similar to the real symmetric matrix.

$A=\begin{bmatrix}6&-2&2\\-2&3&-1\\2&-1&3\end{bmatrix}$, obtaining also the transforming matrix.

222. Let T be the linear operator on R^3 which is represented in the standard ordered basis by the matrix $A=\begin{bmatrix}5&-6&-6\\-1&4&2\\3&-6&-4\end{bmatrix}$. Show that T is diagonalisable and find the non-singular matrix P so that $P^{-1}AP$ is diagonal matrix.

223. Let $T:\mathbb{R}^4\to\mathbb{R}^3$ be the linear transformation defined by $T(x_1,x_2,x_3,x_4,)=(x_1-x_2+x_3+x_4,\ x_1+2x_3-x_4,\ x_1+x_2+3x_3-3x_4)$. Find the range, rank, kernel and nullity of T.

224. If A is an odd order orthogonal matrix, show that either A–I or A+I is necessarily singular.

225. (i) Prove that the eigen values of A^θ are the conjugates of the eigen values of A. If K_1 and K_2 are distinct eigen values of A, prove that any eigen vector of A corresponding to K_1 is orthogonal to any eigen vector of A^θ corresponding to K_2.
(ii) Show that the matrix

$$A=\begin{bmatrix}a+ic&-b+id\\b+id&a-ic\end{bmatrix}$$

is unitary iff $a^2+b^2+c^2+d^2=1$.

226. (i) Are the followng statements true? Give resons in favour of your answers.
(A) A, B are n-rowed square matrices such that AB=0 and B is non-singular. Then A=0.
(B) Only a square, non-singular matrix posses inverse which is unique.
(ii) Find the value of adj(P^{-1}) in terms of P where P is anon-singular matrix and hence show that adj $(Q^{-1}BP^{-1})$=PAQ, given that adj B = A and |P| = |Q| = 1.

227. If H is a Hermitian matrix, show that $A=(I+iH)^{-1}(I-iH)$ is a unitary matrix. Also show that $A=(I-iH)(I+iH)^{-1}$. Further show that if λ is an eigen value of H, then $(1-i\lambda)/(1+i\lambda)$ is an eigen value of A.

228. Define $T: M_{2,2} \to M_{2,3}$ such that

$$T\begin{bmatrix}\alpha_{11} & \alpha_{12}\\ \alpha_{21} & \alpha_{22}\end{bmatrix} = \begin{bmatrix}\alpha_{11}+\alpha_{12} & 0 & \alpha_{12}+\alpha_{22}\\ \alpha_{12} & \alpha_{21}+\alpha_{22} & 0\end{bmatrix}$$

Prove that T is linear and determine its matrix relative to the standard bases for $M_{2,2}$ and $M_{2,3}$.

229. If the product of two non-zero square matrices is a zero matrix, show that both of them must be singular matrices.

230. Find the values of k for which the equations
$x+3y=0, x-z=0, 2x+11ky+k^2z=0$
have non-zero solutions.
For the larger of these values of k check that the equations
$x+3y=1,\ x-z=2,\ 2x+11ky+k^2z=3$
are consistent.

231. Consider the matrix.

$$A=\begin{bmatrix}3 & 0 & 0\\ 2 & 2 & 0\\ 1 & 0 & 2\end{bmatrix}$$

Find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix. Write down that diagonal matrix also.

232. Determine whether matrix A is diagonalizable. If so, find a matrix P that diagonalizes A and compute A^{2004}.

$$A=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 3 & 0 & 1\end{bmatrix}$$

233. Let k be a real number, and let A be the matrix

$$A=\begin{pmatrix}1 & 1 & 1\\ 1 & 2 & k\\ 1 & 4 & k^2\end{pmatrix}$$

Determine all values of k for which the linear system

$$\begin{pmatrix}1 & 1 & 1\\ 1 & 2 & k\\ 1 & 4 & k^2\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\begin{pmatrix}5\\ 10\\ 21\end{pmatrix}$$

has exactly one solution.

234. (i) Let T be a linear operator on the n-dimensional vector space V, and suppose that T has n distinct characteristic values. Prove that T is diagonalizable.
(ii) Find a basis for the kernel of T if T

$$\left(\begin{bmatrix}x\\ y\\ z\end{bmatrix}\right)=\begin{bmatrix}x+y\\ 2x+y+z\\ x+z\end{bmatrix}.$$

235. Let T be the linear operator on R^4 which is represented in the standard ordered basis by the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \end{bmatrix}.$$

Under what conditions on a, b, and c is T diagonalizable?

236. Discuss for all values of k the system of equations

$$2x + 3ky + (3k+4)z = 0$$
$$x + (k+4)y + (4k+2)z = 0$$
$$x + 2(k+1)y + (3k+4)z = 0$$

237. (i) Construct a 2×2 matrix A(¹ I) with real entries such that $A^3 = I$.

(ii) Find the inverse in $\mathbb{Z}_5$ of the following matrix:

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & 2 & 4 \\ 0 & 0 & 3 \end{bmatrix}$$

238. If H is a Hermitian matrix, show that

$(I - iH)(I + iH)^{-1} = (I + iH)^{-1}(I - iH) = U$

where U is a unitary matrix and that if λ is an eigenvalue of H, then $(1 - i\lambda)/(1 + i\lambda)$ is an eigenvalue of U.

Find U when $H = \begin{bmatrix} 1 & e^{i\alpha} \\ e^{-i\alpha} & -1 \end{bmatrix}$

239. Let A= $\begin{pmatrix} 0 & 1 & 1 \\ 1 & -2 & 2 \\ 1 & 2 & -1 \end{pmatrix}$, a symmetric matrix.

Find a nonsingular matrix P such that $P^T AP$ is diagonal and find the diagonal matrix $P^T AP$.

240. Determine diagonal matrix orthogonally similar to the following real symmetric matrix

$$A = \begin{bmatrix} 7 & 0 & -2 \\ 0 & 5 & -1 \\ -2 & -2 & 6 \end{bmatrix}$$

obtaining also the transforming matrix.

241. Prove that similar matrices have the same characteristic polynomial and hence the same eigenvalues. If **X** is an eigenvector of **A** corresponding to the eigenvalue λ, then $\mathbf{P}^{-1}\mathbf{X}$ is an eigenvector of **B** corresponding to the eigenvalue λ where $\mathbf{B} = \mathbf{P}^{-1}\mathbf{AP}$.

242. If **A**, **B** are square matrices each of order n and **I** is the corresponding unit matrix, show that the equation **AB – BA = I** can never hold.

243. Determine whether the linear transformation defined by the following matrix is one-to-one.

$$A = \begin{bmatrix} -1 & 2 & 0 & 5 \\ 3 & 7 & 2 & 8 \\ -4 & 2 & 0 & 0 \\ 1 & 3 & 0 & 6 \end{bmatrix}$$

244. Orthogonally diagonalize the following symmetric matrix. Give the similarity transformation

$$\begin{bmatrix} 9 & -3 & 3 \\ -3 & 6 & -6 \\ 3 & -6 & 6 \end{bmatrix}.$$

245. Let $M = \begin{bmatrix} 1+i & 2i & i+3 \\ 0 & 1-i & 3i \\ 0 & 0 & i \end{bmatrix}$. Determine the eigen values of the matrix $B = M^2 - 2M + I$.

246. If A is non-singular, prove that the eigen values of A^{-1} are the reciprocals of the eigen values of A.

247. If the product of two non-zero square matrices is a zero matrix, show that both of them must be singular matrices.

248. Show that the matrix $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ is diagonalizable. Also find the diagonal form and diagonalizing matrix P.

249. Let A be a 3×3 upper triangular matrix with real entries. If $a_{11} = 1, a_{22} = 2$ and $a_{33} = 3,$ determine α, β and γ such that $A^{-1} = \alpha A^2 + \beta A + \gamma I.$

250. Find bases for both the row and column spaces of the following matrix A. Show that the dimensions of both the row space and the column space are the same.

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 3 \\ 1 & 4 & 6 \end{bmatrix}$$

251. Determine diagonal matrices orthgonally similar to the real symmetric matrix.

$A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$, obtaining also the transforming matrix.

252. (i) Let A be a 3×3 matrix with trace (A) = 3 and det(A) = 2. If 1 is an eigenvalue of A then what are the eigenvalues of the matrix $A^2 - 2I$?

(ii) What is the dimension of the solution space of linear equations $x + y + z + t = 0,$ $x - y - z - 3t = 0, -x + y - 2z = 0$?

253. Find the characteristic equation of the matrix $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}$ and hence find the matrix represented by

$$A^8 - 5A^7 + 7A^6 - 3A^5 + A^4 - 5A^3 + 8A^2 - 2A + I.$$

254. Discuss the consistency and the solutions of the equations

$$x + ay + az = 1, ax + y + 2az = -4, ax - ay + 4z = 2$$

for different values of *a*.

255. (i) Show that the subspaces of IR^3 spanned by two sets of vectors $\{(1,1,-1),(1,0,1)\}$ and $\{(1,2,-3),(5,2,1)\}$ are identical. Also find the dimension of this subspace.

(ii) Let $A = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 1 & 1 \\ 1 & 3 & -1 \end{bmatrix}$ and C be a non-singular matrix of order 3×3. Find the eigen values of the matrix B^3 where $B = C^{-1}AC$.

256. Let V be the set of real numbers. Regard V as a vector space over the field of rational numbers, with the usual operations. Prove or disprove this vector space is not finite - dimensional.

257. For what values of η the equations

$$x + y + z = 1$$
$$x + 2y + 4z = \eta$$
$$x + 4y + 10z = \eta^2$$

have a solution and solve them completely in each case.

258. Find the characteristic values and bases of the corresponding characteristic spaces of the matrix

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix}.$$

Is A similar to a diagonal matrix ? Give reasons.

259. Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear operator, where

$T(e_1) = 5e_1 - 6e_2 - 6e_3$

$T(e_2) = -e_1 + 4e_2 + 2e_3$

$T(e_3) = 3e_1 - 6e_2 - 4e_3.$

Find the characteristic values of T and compute the corresponding characteristic vectors.

260. If H is a Hermitian matrix, show that

$(I - iH)(I + iH)^{-1} = (I + iH)^{-1}(I - iH) = U$

where U is a unitary matrix and that if λ is an eigenvalue of H, then $(1 - i\lambda)/(1 + i\lambda)$ is an eigenvalue of U.

Find U when $H = \begin{bmatrix} 1 & e^{i\alpha} \\ e^{-i\alpha} & -1 \end{bmatrix}$

261. Given that the matrix $\begin{pmatrix} \alpha & 1 \\ 2 & 3 \end{pmatrix}$ has 1 as an eigenvalue, compute its trace and its determinant.

262. (i) Let A be a 5×5 matrix whose characteristic polynomial is given by

$(\lambda - 2)^3 (\lambda + 2)^2$

If A is diagonalizable, find α and β such that $A^{-1} = \alpha A + \beta I.$

(ii) Find a basis for $\mathbb{R}^4$ that contains vectors $v = [1, 2, 0, 0]$ and $u = [0, 0, 1, 2].$

263. Determine diagonal matrix orthogonally similar to the following real symmetric matrix $A = \begin{bmatrix} 7 & 0 & -2 \\ 0 & 5 & -1 \\ -2 & -2 & 6 \end{bmatrix}$ obtaining also the transforming matrix.

264. Find the eigenvalues and eigenvectors of

$$A = \begin{vmatrix} 3 & 4 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{vmatrix}$$

Check that $\lambda_1 + \lambda_2 + \lambda_3$ equals the trace and $\lambda_1 \lambda_2 \lambda_3$ equals the determinant.

265. Let $H = \begin{pmatrix} 1 & 1+i & 2i \\ 1-i & 4 & 2-3i \\ -2i & 2+3i & 7 \end{pmatrix}$, a Hermitian matrix. Find a nonsingular matrix P such that $P^T H \overline{P}$ is diagonal.

266. Find the dim ension and a basis of the solution space w of the system

x + 2y –4z + 3r – x = 0

x + 2y –2z + 2r + s = 0

2x + 4y –2z + 3r + 4s = 0

267. If A is non-singular, Prove that the eigen values of A^{-1} are the reciprocals of the eigen values of A.

268. Let $A = \begin{pmatrix} 1 & 2 & -3 \\ 2 & 5 & -4 \\ -3 & -4 & 8 \end{pmatrix}$ a symmetric matrix

Find the nonsingular matrix P such that $P^T AP$ is diagonal and find $P^T AP$.

269. (i) Ax = b has a solution under what condition on b, for the following A and B ?

$$A=\begin{bmatrix}1&2&0&3\\0&0&0&0\\2&4&0&1\end{bmatrix} \text{ and } b=\begin{bmatrix}b_1\\b_2\\b_3\end{bmatrix}$$

(ii) Find a basis for the nullspace of A.
(iii) Find the general solution to Ax = b, when a solution exists.
(iv) Find a basis for the column space of A.
(v) What is the rank of A^T ?

270. Show that the determinant equals the product of the eigenvalues by imagining that the characteristic polynomial is factored into: $det(A-\lambda I)=(\lambda_1-\lambda)(\lambda_2-\lambda)......(\lambda_n-\lambda)$ and making a clever choice of λ.

271. If $A=\begin{bmatrix}a&b\\0&d\end{bmatrix}$, then det $(A-\lambda I)$ is $(\lambda-a)(\lambda-d)$. Check the Cayley-Hamilton statement that $(A-aI)(A-dI)=zero\,matrix$.

272. Find the rank and the nullspace of
$$A=\begin{bmatrix}0&0&1\\0&0&1\\1&1&1\end{bmatrix} \text{ and } B=\begin{bmatrix}0&0&1&2\\0&0&1&2\\1&1&1&0\end{bmatrix}$$

273. (i) If $A^2 = I$, what are the possible eigen values of A ?
(ii) If this A is 2 by 2, and not I or –I, find its trace and determinant.
(iii) If the first row is (3, –1), what is the second row.

274. Obtain eigen values and eigen vectors of the differential operator $D : P_2 \to P_2$. $D(a_0 + a_1 x + a_2x^2) = a_1 + 2a_2x$, for $a_0, a_1, a_2 \in R$.

275. (i) If $A=\begin{bmatrix}4&3\\1&2\end{bmatrix}$, find $\mathbf{A}^{100}$ by diagonalizing A.
(ii) Show by direct calculation that AB and BA have the same trace when $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$ and $B=\begin{bmatrix}q&r\\s&t\end{bmatrix}$.
Deduce that AB – BA = I is impossible (except in infinite dimensions).

276. Let $T : R^3 \to R^3$ be defined by $T(x_1, x_2, x_3) = (x_1 - x_2 + 2x_3, 2x_1 + 2x_1 + x_2 - x_1 - 2x_2 + 2x_3)$ Find R(T) and Ker T

277. If we shift to A-7*I*, what are the eigenvalues and eigenvectors and how are they related to those of A?
$$B=A-7I=\begin{bmatrix}-6&-1\\2&-3\end{bmatrix}$$

278. (i) If a 3 by 3 upper triangular matrix has diagonal entries 1,2,7 how do you know it can be diagonalized? What is Λ ? When Λ an eigen value matrix.
(ii) Let V be the vector space of all real polynomials. Consider the subspace W spanned by t^2+t+2, t^2+2t+5, $5t^2+3t+4$ and $2t^2+2t+4$. Then, find the dimension of W.

279. (i) Show that $A^2 = 0$ is possible but $A^TA = 0$ is not possible.
(ii) Verify that $(AB)^T$ equals $B^T A^T$ but those are different from $A^T B^T$:
$$A=\begin{bmatrix}1&0\\2&1\end{bmatrix} \quad B=\begin{bmatrix}1&3\\0&1\end{bmatrix}$$
$$AB=\begin{bmatrix}1&3\\2&7\end{bmatrix}.$$
In case AB=BA, how do you prove that $B^T A^T = A^T B^T$?

280. If the matrix of a linear operator T on $V_3(\mathbb{R})$ with respect to the standard basis is
$$\begin{bmatrix}0&1&1\\1&0&-1\\-1&-1&0\end{bmatrix}$$

Describe explicitel $T : V_3(\mathbb{R}) \to V_3(\mathbb{R})$. What is the matrix of T with respect to the basis $\{(0,1,-1),(1,-1,1),(-1,1,0)\}$.

281. If the matrix of a linear operator T on $V_3(\mathbb{R})$ with respect to the standard basis is

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$$

Describe explicitly $T : V_3(\mathbb{R}) \to V_3(\mathbb{R})$. What is the matrix of T with respect to the basis {(0,1,-1), (1,-1,1), (-1,1,0)}.

281. If the matrix of a linear operator T on $V_3(\mathbb{R})$ with respect to the standard basis is

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}.$$

Describe explicitely $T : V_3(\mathbb{R}) \to V_3(\mathbb{R})$.

283. If $T : V_4(\mathbb{R}) \to V_3(\mathbb{R})$ is a linear transforma-tion defined by T(a, b, c, d) = (a - b + c + d, a + 2c - d, a+b + 3c-3d) for a, b, c, d, $\in \mathbb{R}$, then verify $e(T) + v(T) = \dim V_4(\mathbb{R})$.

284. Let $\mathbb{R}_3[x] = \{a_0 + a_1x + a_2x^2 / a_0, a_1, a_2 \in \mathbb{R}\}$. Define $T : \mathbb{R}_3[x] \to \mathbb{R}_3[x]$ by

$$T(f(x)) = \frac{d}{dx} f(c) \text{ for all } f(x) \in \mathbb{R}_3[x].$$

Show that T is a linear transformation. Also find the matrix representation of T with reference to basis set $\{1, x, x^2\}$ adn $\{1, 1+x, 1+x+x^2\}$.

285. Let G be the linear operator on $\mathbb{R}^3$ defined by G(x, y, z) = (2y + z, x – 4y, 3x). Find the matrix representation of G relative to the basis S = {(1, 1, 1), (1, 1, 0), (1, 0 ,0)}.

286. Let V=R^3 and T$\in$ A(V), for all $a_i \in A(V)$, be defined by
$T(a_1,a_2, a_3)=(2a_1+5a_2+a_3, -3a_1+a_2-a_3, -a_1+2a_2+3a_3)$
What is the matrix T relative to the basis $V_1 = (1,0,1)$ $V_2=(-1, 2, 1)$ $V_3=(3, -1, 1)$?

287. Show that the mapping $T : V_2(\bar{R}) \to V_3(\bar{R})$ defined as $T(a,b) = (a+b, a-b, b)$ is a linear transformation. Find the range, rank and nullity of T.

288. Let F be a subfield of complex numbers and T a function from $F^3 \to F^3$ defined by
$T(x_1, x_2, x_3) = (x_1 + x_2 + 3x_3, 2x_1 - x_2, -3x_1 + x_2 - x_3)$.
what are the conditions on (a,b,c) such that (a,b,c) be in the null space of T? Find the nullity of T.

289. Find the matrix representation of linear transformation T on $V_3(\mathbb{R})$ defined as $T(a,b,c) = (2b + c, a - 4b, 3a)$ corresponding to the basis $B = \{(1,1,1),(1,1,0),(1,0,0)\}$

290. Find a linear map $T : IR^3 \to IR^4$, Whose range is generated by $(1,2,0,-4)$ *and* $(2,0,-1,-3)$. Also find a basis and the dimension of the
(i) range U of T.
(ii) kernel W of T.

291. Find the linear transformation from $\mathbb{R}^3$ into $\mathbb{R}^3$ which has its range the subspace spanned by $(1,0,-1)$, $(1,2,2)$

292. Find the rank of the matrix:

$$A = \begin{pmatrix} 1 & 3 & 1 & -2 & -3 \\ 1 & 4 & 3 & -1 & -4 \\ 2 & 3 & -4 & -7 & -3 \\ 3 & 8 & 1 & -7 & -8 \end{pmatrix}$$

293. Let $V = P_3(IR)$ be the vector space of polynomial functions over reals of degree atmost 3. Let $D: V \to V$ be the differentiation operator defined by $D(a_0 + a_1x + a_2x^2 + a_3x^3) = a_1 + 2a_2x + 3a_3x^2, x \in IR.$

(i) Show that $B = \{1, x, x^2, x^3\}$ is a basis for V.

(ii) Find the matrix $[D]_B$ with respect to B of D.

(iii) Show that $B' = \{1, (x+1), (x+1)^2, (x+1)^3\}$ is a basis for V.

(iv) Find the matrix $[D]_{B'}$ with respect B' of D.

(v) Find the matrix $[D]_{B'B}$ of D relative to B' and B.