Appendix I

Beta and Gamma Functions

We have already discussed the convergence of the improper integrals

$$\int_0^1 x^{m-1}(1-x)^{n-1}\,dx, \text{ and } \int_0^\infty x^{m-1}e^{-x}\,dx,$$

for $m-1<0$, and $n-1<0$ in chapter 11 Sections 3.4 and 4.4 respectively.

We have seen that the first integral converges if $m>0$, $n>0$ and the second converges for $m>0$.

These integrals are named as Beta and Gamma functions, respectively and denoted by $\beta(m, n)$ and $\Gamma(m)$, respectively.

i.e.,
$$\beta(m, n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx = 2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta, \ (x=\sin^2\theta)$$

and

$$\Gamma(m)=\int_0^\infty x^{m-1}e^{-x}\,dx = 2\int_0^\infty r^{2m-1}e^{-r^2}\,dr \quad (x=r^2)$$

Also in chapter 17, example 25, we have established a relation between them, viz.,

$$\beta(m, n)=\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

We shall now give *Legendre's Duplication Formula*

$$\sqrt{\pi}\,\Gamma(2m)=2^{2m-1}\,\Gamma(m)\,\Gamma\left(m+\frac{1}{2}\right)$$

we have

$$\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}=\beta(m, n)=2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta \qquad \ldots(1)$$

Taking $n=m$, we have

$$\frac{(\Gamma(m))^2}{\Gamma(2m)}=\beta(m, m)=2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2m-1}\theta\,d\theta$$

$$=\frac{1}{2^{2m-2}}\int_0^{\pi/2}\sin^{2m-1}2\theta\,d\theta \qquad \ldots(2)$$

$$=\frac{1}{2^{2m-1}}\int_0^{\pi}\sin^{2m-1}\phi\,d\phi \quad (2\theta=\phi)$$

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In (1) taking $n = \frac{1}{2}$, we get

$$\frac{\Gamma(m)\,\Gamma(1/2)}{\Gamma(m+\frac{1}{2})} = 2\int_0^{\pi/2} \sin^{2m-1}\theta\, d\theta \qquad \text{...(3)}$$

From (2) and (3), we obtain

$$\frac{(\Gamma(m))^2}{\Gamma(2m)} = \frac{1}{2^{2m-1}}\,\frac{\Gamma(m)\,\Gamma(\frac{1}{2})}{\Gamma(m+\frac{1}{2})}$$

or

$$2^{2m-1}\Gamma(m)\Gamma\left(m+\frac{1}{2}\right) = \sqrt{\pi}\,\Gamma(2m), \text{ since } \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}.$$

This proves the required duplication formula.

Ex. Prove that $\Gamma(1/4)\,\Gamma(3/4) = \sqrt{2}\pi$.

Next, employing integrating by parts to the Gamma function

$$\Gamma(m) = \int_0^{\infty} x^{m-1} e^{-x}\, dx, \quad m > 0$$

we obtain

$$\Gamma(m+1) = \lim_{\substack{b\to\infty \\ a\to 0+}} \int_a^b x^m e^{-x}\, dx$$

$$= \lim_{\substack{b\to\infty \\ a\to 0+}} \left\{-b^m e^{-b} + a^m e^{-a} + \int_a^b m x^{m-1} e^{-x}\, dx\right\}$$

$$= m\,\Gamma(m), \text{ since } b^m e^{-b} \to 0, \text{ as } b \to \infty, \text{ and}$$

$$a^m e^{-a} \to 0, \text{ as } a \to 0+ \qquad (\because\ m > 0)$$

$$\therefore \qquad \Gamma(m+1) = m\Gamma(m), \qquad \forall\, m > 0.$$

Further, since $\Gamma(1) = 1$, so it can be easily shown that

$$\Gamma(n+1) = n!, \quad \forall\, n \in \mathbf{N}.$$

Ex. 1. Show that

$$\left\{\int_0^{\pi/2} \sin^p x\, dx\right\}\left\{\int_0^{\pi/2} \sin^{p+1} x\, dx\right\} = \frac{\pi}{2(p+1)}$$

Ex. 2. Show that

$$\Gamma(m)\,\Gamma(1-m) = \pi/\sin m\pi, \quad 0 < m < 1$$

[*Hint:* $\beta(m, 1-m) = \dfrac{\Gamma(m)\,\Gamma(1-m)}{\Gamma(1)} = \Gamma(m)\,\Gamma(1-m)$,

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and

$$\beta(m, 1-m) = \int_0^1 x^{m-1}(1-x)^{-m}\, dx$$

$$= \int_0^\infty \frac{y^{m-1}}{1+y}\, dy, \text{ taking } x = y/(1+y).$$

Evaluate this improper integral, and use exercise 8, chapter 14.]

Example 1. Show that

$$\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx = \beta(m, n), \text{ for } m, n > 0$$

■ Put

$$x = \frac{t}{1-t}$$

$dx = \dfrac{dt}{(1-t)^2}$, when x varies from 0 to ∞, t varies from 0 to 1.

$$\therefore \quad \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\, dx = \int_0^1 \left(\frac{t}{1-t}\right)^{m-1} \frac{(1-t)^{m+n}}{(1-t)^2}\, dt$$

$$= \int_0^1 t^{m-1}(1-t)^{n-1}\, dt = \beta(m, n)$$

Example 2. Show that for $l > 0$, $m > 0$

$$\int_a^b (x-a)^{l-1}(b-x)^{m-1}\, dx = (b-a)^{l+m-1}\,\beta(l, m)$$

■ Put $x = py + q$, where p and q are such that when $x = a$, $y = 0$ and when $x = b$, $y = 1$. This gives $p = b - a$, and $a = a$ and the integral becomes

$$= \int_0^1 [(b-a)\,y + a - a]^{l-1}\,(b - (b-a)\,y - a)^{m-1}\,(b-a)\, dy$$

$$= \int_0^1 (b-a)^{l-1+1+m-1}\, y^{l-1}(1-y)^{m-1}\, dy = (b-a)^{l+m-1}\ \beta(l, m)$$

Example 3. Show that

$$\int_0^1 x^{-1/3}(1-x)^{-2/3}(1+2x)^{-1}\, dx = \frac{1}{9^{1/3}}\ \beta(2/3, 1/3)$$

■ Put $\dfrac{x}{1-x} = \dfrac{at}{1-t}$, where a is a constant to be chosen so that the given integral becomes Beta function

$$x = \frac{at}{1-(1-a)t}.$$

$$dx = \frac{2dt}{[1-(1-a)t]^2} \text{ when } x = 0,\ t = 0$$

$$\int_0^1 \left[\frac{at}{1-(1-a)t}\right]^{-1/3} \left[\frac{1-t}{1-(1-a)t}\right]^{-2/3} \left[\frac{1-t+3at}{1-(1-a)t}\right]^{-1} \frac{adt}{[1-(1-a)t]^2}$$

$$= \int_0^1 \frac{a^{2/3}\, t^{-1/3}\, [1-t(1-3a)]^{-1}\, dt}{(1-t)^{2/3}}.$$

If we choose $a = \frac{1}{3}$ then the integral becomes a Beta function and therefore taking $a = \frac{1}{3}$, we have

$$= \int_0^1 \left(\frac{1}{3}\right)^{2/3} t^{(2/3)-1}\,(1-t)^{(1/3)-1}\, dt = \frac{1}{9^{1/3}}\, \beta(2/3,\ 1/3)$$

Example 4. If n is a positive integer, prove that the ratio of the areas enclosed by the curves

$$x^{2n} + y^2 = 1,\ x^{2n} + y^{2n} = 1 \text{ is } n2^{1/n}/(n+1)$$

■ For area under the 1st curve

Put $x^{2n} = \cos^2\theta,\ y^2 = \sin^2\theta$

then the area is

$$A_1 = 4\int_0^{\pi/2} \sin\theta \frac{1}{n} \cos^{(1/n)-1}\theta\,(-\sin\theta)\, d\theta$$

$$= -\frac{4}{n}\int_0^{\pi/2} \sin^2\theta \cos^{(1/n)-1}\theta\, d\theta$$

$$= -\frac{2}{n}\beta\left(\frac{3}{2}, \frac{1}{2n}\right) = -\frac{2}{n}\frac{\Gamma(3/2)\,\Gamma(1/2n)}{\Gamma(1/2n + 3/2)}$$

Similarly putting $x^{2n} = \cos^2\theta,\ y^{2n} = \sin^2\theta,$ the area under the 2nd curve is

$$A_2 = -4\int_0^{\pi/2} \sin^{1/n}\theta\ \frac{1}{n}\cos^{(1/n)-1}\theta \sin\theta\, d\theta$$

$$= -\frac{4}{n}\int_0^{\pi/2} \sin^{(1/n)+1}\theta \cos^{(1/n)-1}\theta d\theta$$

$$= -\frac{2}{n}\beta\left(\frac{1}{2n}+1, \frac{1}{2n}\right)$$

$$\therefore \quad \frac{A_1}{A_2} = \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2n}\right)\Gamma\left(\frac{1}{2n}+1+\frac{1}{2n}\right)}{\Gamma\left(\frac{1}{2n}+\frac{1}{2}+1\right)\Gamma\left(\frac{1}{2n}+1\right)\Gamma\left(\frac{1}{2n}\right)}$$

$$= \frac{1}{2} \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{2n}\right)\frac{1}{n}\Gamma\left(\frac{1}{n}\right)}{\left(\frac{1}{2n}+\frac{1}{2}\right)\Gamma\left(\frac{1}{2n}+\frac{1}{2}\right)\frac{1}{2n}\Gamma\left(\frac{1}{2n}\right)\Gamma\left(\frac{1}{2n}\right)}.$$

$$= \frac{2n}{n+1} \frac{\Gamma\left(\frac{1}{n}\right)\sqrt{\pi}}{\Gamma\left(\frac{1}{2n}\right)\Gamma\left(\frac{1}{2n}+\frac{1}{2}\right)}$$

Using duplication formula,

$$2^{1/n-1}\,\Gamma\left(\frac{1}{2n}\right)\Gamma\left(\frac{1}{2n}+\frac{1}{2}\right) = \sqrt{\pi}\,\Gamma\left(\frac{1}{n}\right)$$

we get

$$\frac{A_1}{A_2} = \frac{2n\sqrt{\pi}\,\Gamma\left(\frac{1}{n}\right)}{(n+1)\,2^{-1/n+1}\sqrt{\pi}\,\Gamma\left(\frac{1}{n}\right)} = 2^{1/n}\frac{n}{n+1}.$$

Example 5. Evaluate the integrals

$$\int_0^\infty e^{-ax}\,x^{m-1}\cos bx\,dx, \text{ and } \int_0^\infty e^{-ax}\,x^{m-1}\sin bx\,dx, \quad m > 0.$$

Hence or otherwise show that

$$\int_0^\infty x^{m-1}\cos bx\,dx = \frac{\Gamma(m)}{b^m}\cos\left(\frac{m\pi}{2}\right) \text{ and } \int_0^\infty x^{m-1}\sin bx\,dx = \frac{\Gamma(m)}{b^m}\sin(m\pi/2).$$

■ Now $$\int_0^\infty e^{-kx}\,x^{m-1}\,dx = \frac{\Gamma(m)}{k^m}$$

Taking $k = a - ib,\ |k| > 0$

$$\int_0^\infty e^{-(a-ib)x}\,x^{m-1}\,dx = \frac{\Gamma(m)}{(a-ib)^m}$$

$$\int_0^\infty e^{-ax}\,e^{ibx}\,x^{m-1}\,dx = \frac{\Gamma(m)\,(a+ib)^m}{(a-ib)^m(a+ib)^m}$$

$$\int_0^\infty e^{-ax}\,(\cos bx + i\sin bx)x^{m-1}\,dx = \frac{\Gamma(m)\,(a+ib)^m}{(a^2+b^2)^m}$$

Writing $a + ib = r(\cos\theta + i\sin\theta)$, and separating the real and imaginary parts, we get

$$\int_0^\infty e^{-ax}\cos bx\, x^{m-1}\,dx = \frac{\Gamma(m)\cos m\theta}{(a^2+b^2)^{m/2}},\ \text{where}\ \theta = \tan^{-1}\frac{b}{a}$$

and

$$\int_0^\infty e^{-ax}\sin bx\, x^{m-1}\,dx = \frac{\Gamma(m)\sin m\theta}{(a^2+b^2)^{m/2}},\ \text{where}\ \theta = \tan^{-1}\frac{b}{a}$$

Taking $a = 0, \theta = \pi/2$

$$\int_0^\infty \cos bx\, x^{m-1}\,dx = \frac{\Gamma(m)\cos(m\pi/2)}{b^m}$$

and

$$\int_0^\infty \sin bx\, x^{m-1}\,dx = \frac{\Gamma(m)\sin(m\pi/2)}{b^m}.$$

EXERCISE

1. Show that

$$\int_0^p x^m(p^q - x^q)^n\,dx = \frac{p^{qn+m+1}}{q}\,\beta\left(n+1, \frac{m+1}{q}\right)$$

if $p > 0, q > 0, m+1 > 0, n+1 > 0$.

2. Prove that

(i) $$\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx = \beta(m,n).$$

$$\left[\text{Hint: } ?(m,n) = \int_0^\infty \frac{x^{m-1}\,dx}{(1+x)^{m+n}} = \int_0^1 \frac{x^{m-1}}{(1+x)^{m+n}}\,dx + \int_1^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx\right.$$

$$\left.\text{Put } x = \frac{1}{t} \text{ in the second integral.}\right]$$

(ii) $$\int_0^\infty \frac{(x^{m-1}+x^{n-1})}{(1+x)^{m+n}}\,dx = 2\beta(m,n).$$

3. Show that for $m, n > 0$,

$$\int_0^1 \frac{x^{m-1}(1-x)^{n-1})}{(b+cx)^{m+n}}\,dx = \frac{\beta(m,n)}{(b+c)^m\, b^n}.$$

$$\left[\text{Hint: Put } y = \frac{(b+c)x}{b+cx}\right]$$

4. Show that

$$\int_0^n \left(1-\frac{t}{n}\right)^n t^{x-1}\,dt = n^x\,\beta(x, n+1), \text{ where } x>0.$$

5. Show that for $m>0$,

(i) $\beta(m,m) = 2^{1-2m}\,\beta(m, \frac{1}{2})$,

(ii) $\beta(m,m)\,\beta(m+\frac{1}{2}, m+\frac{1}{2}) = \pi m^{-1} 2^{1-4m}$.

6. Prove that

(i) $\displaystyle\int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} \times \int_0^{\pi/2} \sqrt{\sin\theta}\,d\theta = \pi.$

(ii) $\displaystyle\left\{\int_0^1 \frac{x^2 dx}{\sqrt{1-x^4}}\right\}\left\{\int_0^1 \frac{dx}{\sqrt{1+x^4}}\right\} = \frac{\pi}{4\sqrt{2}}.$

7. Show that

(i) $\displaystyle\int_0^1 \sqrt{1-x^4}\,dx = \frac{1}{12}\sqrt{\frac{2}{\pi}}\,[\Gamma(\tfrac{1}{4})]^2,$

(ii) $\displaystyle\int_0^1 (1-x^n)^{-1/2}\,dx = 2^{(2/n)-1}\,[\Gamma(1/n)]^2/n\Gamma(2/n).$

8. Show that the perimeter of the lemniscate $r^2 = 2a^2\cos 2\theta$ is

$$\frac{a}{\sqrt{\pi}}[\Gamma(\tfrac{1}{4})]^2.$$

9. Show that the perimeter of a loop of the curve

$$r^n = a^n \cos n\theta$$

is

$$\frac{a}{n}\cdot 2^{\left(\frac{1}{n}\right)-1}\,\frac{\left[\Gamma\left(\frac{1}{2n}\right)\right]^2}{\Gamma\left(\frac{1}{n}\right)}.$$

10. Show that the area bounded by the curve $x^n + y^n = a^n$, and the co-ordinate axes in the first quadrant is $[\Gamma(1/n)]^2/2n\Gamma(2/n)$.