⇒ Dynamics & Statics :-

1) Forces :

If the line of action of forces lie in the same plane, they are called coplanar forces.

If they do not lie in same plane, they are non-coplanar or Forces in space :

Collinear Forces
* all forces act along the same line of action
* always coplanar

Concurrent Forces
* all lines of forces Intersect at a point
* may be coplanar or may not be

Non-Concurrent Forces
* lines of actions do not meet at a point
* Both coplanar & non-coplanar

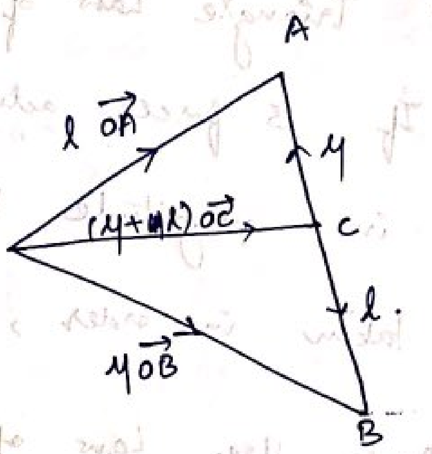
Parallel Forces
* lines of action of all the forces are parallel
* ——"——

2) $[\cos^2 A + \cos^2 B + \cos^2 C = 1 - 2\cos A \cos B \cos C]$ in ΔABC.

3) l-μ Theorem :

if 2 forces $l\,\vec{OA}$ and $\mu\,\vec{OB}$ act at a point O, then their resultant is given by $(l+\mu)\vec{OC}$, where C divides AB st $l(CA) = \mu(CB)$

$\therefore$ C divides in ratio $\mu : l$



Proof: Let C be st $l|\vec{CA}| = \mu|\vec{CB}|$

$\Rightarrow l(\vec{CA}) = \mu(\vec{BC})$

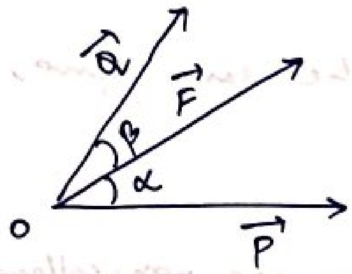
In ΔOAC: $l(\vec{OA}) = l(\vec{OC}) + l(\vec{CA})$

In ΔOBC: $\mu(\vec{OB}) = \mu(\vec{OC}) + \mu(\vec{CB})$

$\Rightarrow l(\vec{OA}) + \mu(\vec{OB}) = (l+\mu)\vec{OC} + \underbrace{l\vec{CA} + \mu\vec{CB}}_{0}$

$\Rightarrow l(\vec{OA}) + \mu(\vec{OB}) = (l+\mu)\vec{OC}$

4) Component of Force along 2 given direction :



$$P = \frac{F \sin\beta}{\sin(\alpha+\beta)} \qquad Q = \frac{F \sin\alpha}{\sin(\alpha+\beta)}$$

$$\vec{F} = \vec{P} + \vec{Q}$$

use $\vec{P}\times\vec{F} = \vec{P}\times(\vec{P}+\vec{Q}) = \vec{P}\times\vec{Q}$

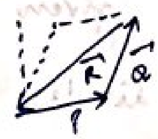
& $\vec{F}\times\vec{Q} = (\vec{P}+\vec{Q})\times\vec{Q} = \cancel{Q}\,\vec{P}\times\vec{Q}$

5) Conditions for equilibrium :

System of coplanar forces acting at a pt are in equilibrium $\iff$ Algebraic sums of the resolved parts of the forces along two mutually $\perp^r$ directions in their plane should be zero. separately.

6) Triangle Law of Forces :

If 3 forces acting at a point can be represented in magnitude & direction by the sides of a triangle, taken in order, they will be in equilibrium.

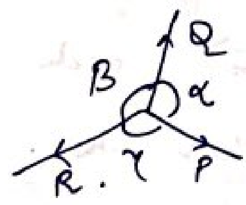


Proof: Use Law of ||gm of forces :

If 2 forces, acting at a point be represented in magnitude and direction by the two sides of a ||gm drawn from one of its angular points, their resultant is represented by the diagonal of ||gm through that point both in magn. & direction

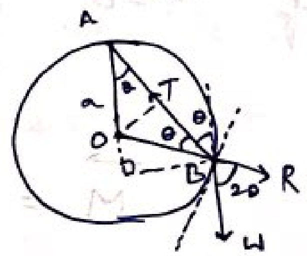
7) Lami's Theorem :

If 3 forces acting on a particle keep it in $\rightleftharpoons$, each is proportional to the sides of the angle b/w the other two, i.e., $\frac{P}{\sin\beta} = \frac{Q}{\sin\gamma} = \frac{R}{\sin\alpha}$



8) Δ law can be extended for any polygon also.

eg. One end of light inextensible string of length 'l' is fastened to the highest point of a smooth circular wire of radius 'a' which is kept fixed in a vertical plane. Other end is attached to a ring of weight 'W' which slides on wire. Find the tension of the string and the reaction of the wire.



* Draw FBD, Apply Lami's & solve

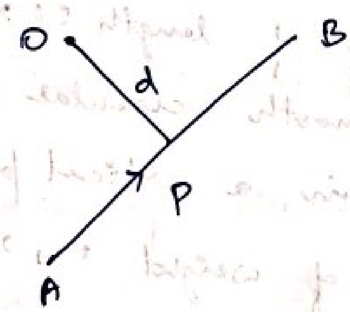
$$\frac{T}{\sin 2\theta} = \frac{W}{\sin(\pi-\theta)} = \frac{R}{\sin(\pi-\theta)}$$

& $\cancel{R^2 \le a^2}$ $\frac{\left(\frac{l}{2}\right)}{a} = \cos\theta \Rightarrow \cos\theta = \frac{l}{2a}$

⇒ Equilibrium of Rigid Body :

1) If 3 ∥ forces acting on a rigid body are in $\rightleftharpoons$, each is proportional to the distance b/w the other two.

2) Moment of a force about a point :



Force is P along $\overrightarrow{AB}$.

Moment about O = $+P.d$ or $-P.d$

where d is ⊥ dist of O from AB

+ if ACW − if CW

If $M=0$, either $F=0$ or line of action passes through the point.

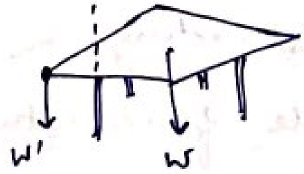
$$\vec{M} = \vec{r} \times \vec{F}$$

About a line : θ is angle b/w AB & CD.

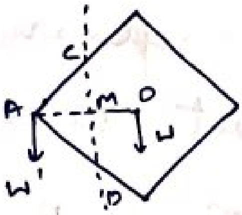
$\therefore P[AB]$ about CD = $(P \sin\theta) \cdot d$.

Resolved part of P parallel to CD has no tendency to rotate ~~AD~~. body about CD.

eg. square table stands on 4 legs placed at the middle points of its sides. Find the greatest weight which can be put on one corner w/o toppling the table (total ~~mass~~ weight being W).



$\Rightarrow$



Moment about CD

$= -W_1(AM) + W(OM)$

$= 0$

$\Rightarrow W_1 = W$.

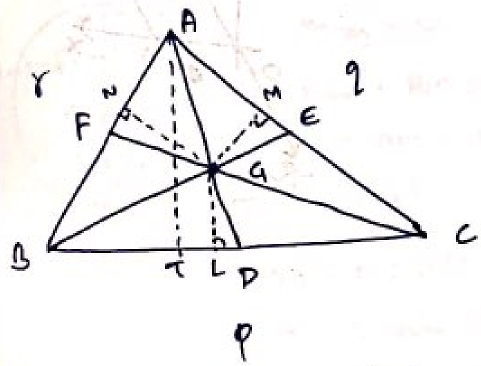
3) Varignon's Theorem :

Algebraic sum of the moments of two coplanar forces not forming a couple, about any point in their plane is equal to the moment of their resultant about that point.

It can be generalised to a system of coplanar forces which reduces to a single force

4) Remember : 3 forces P, Q, R act along the side BC, CA, AB of a ΔABC. Their resultant passes through the centre of gravity of ΔABC. Then

$$\frac{P}{\sin A} + \frac{Q}{\sin B} + \frac{R}{\sin C} = 0$$



$p \cdot GL + q\, GM + r\, GN = 0$ [$\because$ moment about G]

Now $\Delta DGL \sim \Delta DAT$

$$\frac{GL}{AT} = \frac{GD}{AD} = \frac{GD}{3GD}$$

$\Rightarrow GL = \frac{AT}{3}$ sly GM & GN can be fetched

Now Area of $\Delta ABC = S = \frac{1}{2} \times AT \times BC \Rightarrow AT = \frac{2S}{BC}$

$\therefore GL = \frac{2S}{3BC}$ $GM = \frac{2S}{3AC}$ $GN = \frac{2S}{3AB}$

$\Rightarrow \frac{p}{BC} + \frac{q}{AC} + \frac{r}{AB} = 0 \Rightarrow \frac{p}{\sin A} + \frac{q}{\sin B} + \frac{r}{\sin C} = 0$ [$\because \frac{BC}{\sin A} = \frac{AC}{\sin B} = \frac{AB}{\sin C}$]

5) Thm: Any system of coplanar forces acting on a rigid body is equivalent to a single force acting at an arbitrarily chosen point together with a single couple.

Single force $\vec{R} = \sum_{r=1}^{n} F_r$ acting at some pt O

and a couple of moment $\vec{M} = \sum_{r=1}^{n} \vec{a_r} \times \vec{F_r}$

Condition for $\rightleftharpoons$:

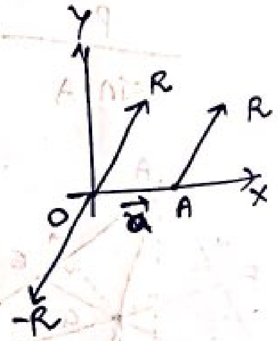
∴ A rigid body under the action of a system of coplanar forces acting as diff pts is in $\rightleftharpoons$ $\Leftrightarrow$ Sum of Resolved parts along any 2 mutually ⊥ directions vanishes separately and sum of the moments about any point in the plane of forces vanishes.

Equation of Line of Action:

Resultant $\vec{R} = R_x\hat{i} + R_y\hat{j}$ which cuts x-axis at A.

Moment about O ⇒ $M = \vec{R_x} \cdot \vec{o} + \vec{R_y} \cdot \vec{a}$

$$a = \frac{M}{R_y}$$



$$\therefore \vec{OA} = \frac{M}{R_y}\hat{i} = \vec{a}$$

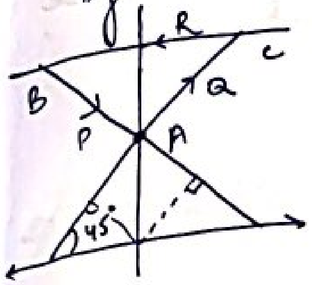
∴ Any pt on line is

$$\vec{r} = \vec{a} + t\vec{R}$$

$$x\hat{i} + y\hat{j} = \frac{M}{R_y}\hat{i} + tR_x\hat{i} + tR_y\hat{j}$$

$$\Rightarrow x = \frac{M}{R_y} + tR_x \quad y = tR_y \Rightarrow \boxed{R_y \cdot x - R_x \cdot y = M}$$

eg. 3 forces P, Q, R act along the sides of a Δ formed by $x+y=1$, $y-x=1$, $y=2$. Find equation of the resultant



$\therefore \quad R_x = \left[\frac{(Q+P)}{\sqrt{2}} - R\right] \quad R_y = \frac{(Q-P)}{\sqrt{2}}$

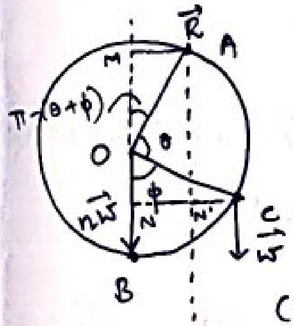
$\therefore \quad R = \left(\frac{P}{\sqrt{2}} + \frac{Q}{\sqrt{2}} - R\right)\hat{i} + \left(\frac{Q}{\sqrt{2}} - \frac{P}{\sqrt{2}}\right)\hat{j}$

$M(\text{abt } O) = +R(2) - \frac{P}{\sqrt{2}} - \frac{Q}{\sqrt{2}} = 2R - \frac{(P+Q)}{\sqrt{2}}$

$\therefore \quad eq^n \Rightarrow \quad R_y x - R_x y = M$

$\Rightarrow \quad \left(\frac{Q-P}{\sqrt{2}}\right)x + \left(R - \frac{(Q+P)}{\sqrt{2}}\right)y = 2R - \frac{(P+Q)}{\sqrt{2}}$

eg. A uniform circular disc of weight 'nW' has a heavy particle of weight 'W' attached to pt C on its rim. If disk is ~~fixe~~ suspended from pt A on its rim, B is the lowest pt & if suspended from B, A is the lowest point. St. the angle subtended by AB at the centre is $2\sec^{-1}(2(n+1))$



$m_{(A)} = 0$

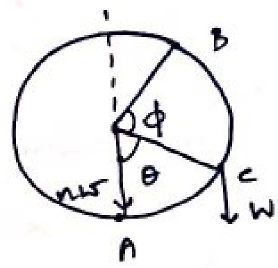
$nW \times AM = W \times CN'$

$n\, AM = (CN - AM)$

$(n+1)\, AM = CN$

$(n+1)\, r \sin(\pi - (\theta+\phi)) = r \sin\phi$

$\Rightarrow (n+1)\sin(\theta+\phi) = \sin\phi$



Here we get

$(n+1)\sin(\theta+\phi) = \sin\theta$

$\therefore$ we get $\theta = \phi$

$\Rightarrow (n+1)\sin 2\theta = \sin\theta$

$2(n+1)\cos\theta = 1$

$\therefore \quad \theta = \sec^{-1}(2(n+1))$

$\therefore \angle AOB = \theta + \phi = 2\theta = 2\sec^{-1}(2(n+1))$

6. Thm: If 3 forces acting upon a rigid body keep it in $\rightleftharpoons$ they must either meet in a point or be parallel.

⇒ Virtual Work :

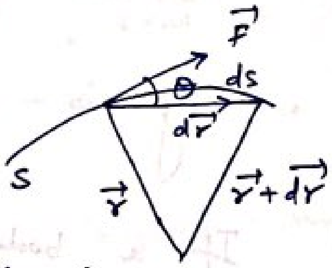
1) Work done by force $W = \vec{F}\cdot\vec{d}$

$= \vec{F}\cdot\vec{dr}$ (small displacement $\vec{dr}$)

2) Principle of Virtual Work :

Work of a force : $dW = \vec{F}\cdot\vec{dr}$

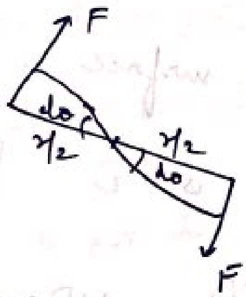
$= \vec{F}\, ds \cos\theta$



Work of a couple in rotation = $dW = \vec{F}\cdot\left(\frac{r}{2}d\theta\right) + \vec{F}\left(\frac{r}{2}d\theta\right)$

$= Frd\theta$

$= Md\theta$



⇒ If coplanar forces are acting on a particle or a rigid body in ⇌, then the algebraic sum of total virtual work done by the forces during any small displacement is zero.

$$W = \vec{U}\cdot\vec{R} + \vec{\delta\theta}\cdot\vec{M} = 0.$$

3) Advantages is that certain forces can automatically be excluded in equation of VW :

(a) WD by Tension of an inextensible string is 0 during small displacement.

(b) WD by thrust of an inextensible rod is 0 during small displacement.

Tension acts inwards & thrust acts outwards

(c) If the distance b/w 2 particles is invariable, WD by the mutual action & rxn forces b/w the 2 particles is zero.

(d) Reaction force of a smooth surface with which a body is in contact does no work.

(e) If a body rolls w/o sliding on any fixed surface, WD by the rxn force at point or axis is 0 [∵ displacement of point of contact is 0].

(f) WD by mutual rxn b/w 2 bodies of a system is zero. [∵ as they are equal & opposite]

(g) VWD by the rxn force at a point or axis around which the body is turned is 0.

4) Some other standard forces & their VW:

(a) WD by T (tension) of an extensible string $= -T\delta l$

(b) WD by T (thrust) of an extensible rod $= T\delta l$

5) Application of PoVW:

(a) Displace in a manner than forces which are not required get excluded.

(b) Note the pts & the lengths that change & do not change.

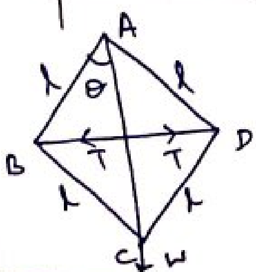
(c) To find Tension or Thrust we need to displace such that length of string or rod changes, else we cannot find T. But since we can't do so in an inextensible string, we replace the string / rod by 2 equal & opposite forces & then displace. (Remember $+T.\delta l$ & $-T.\delta l$)

(d) So essentially we try to express reqd lengths (along with the forces) in terms of some parameter θ. as $f(\theta)$. Then taking $\frac{d}{d\theta}(f(\theta))$, we get small displacement $= f'(\theta)\,d\theta$ & then we multiply by the relevant force & put in VW eqn.

Remember all these lengths must be from a fixed pt or wall which don't move on small displacement.

Basic Example :

5 weightless rods of equal length are joined together so as to form rhombus ABCD with one diagonal BD. If a weight 'W' is attached to C & system is suspended from A. Find Thrust in BD.



$AC = 2l\cos\theta$ $\therefore \delta(AC) = -2l\sin\theta\, d\theta$

$BD = 2l\sin\theta$ $\therefore \delta(BD) = 2l\cos\theta\, d\theta$

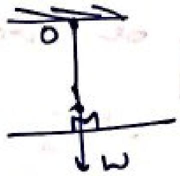
$\therefore \delta W = W(\delta AC) + T(\delta BD) \Rightarrow W(-2l\sin\theta)\,d\theta + T(2l\cos\theta)\,d\theta$

$\Rightarrow T = W\tan\theta$

But $\theta = 30^\circ$ [$\because \triangle ABD$ is equilateral] $\therefore T = W/\sqrt{3}$

6) Keep the stability of the system & presence of rod or string in mind to avoid confusion over sign.
Also vertical displacement direction in case of 'W'

7) 

$\Rightarrow WD = -W(\delta OM)$ 

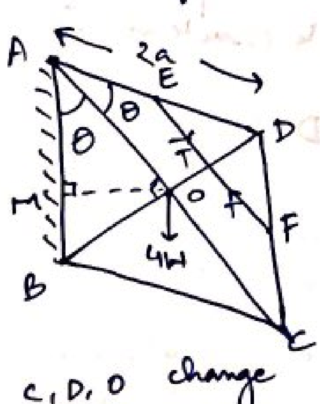
$WD = +W\,\delta(OM)$

8) Do not start with the given figure properties directly. We have to use general parameters 1st & then apply the constraints of ⇌ position

eg. 4 equal rods of weight 'W' are joined to form a square ABCD. AB is clamped in vertical position & figure is kept in shape by a string joining mid points of AD & CD. Find Tension in string.

So, here don't start with the condition that figure is a square. Start as a rhombus (as a small displacement will make the fig. look like a rhombus) & then put cond^n of square.

taking A & as fixed pt.



$4W\,\delta(AM) - T\,\delta(EF) = 0$

$EF = 2[a\cos\theta] = 2a\cos\theta$

$AM = (2a\cos\theta)\cos\theta = 2a\cos^2\theta$.

$\therefore T \times 2\cancel{a}\sin\theta\, d\theta - 4W \times \overset{2}{\cancel{4}}\cancel{a}\cos\theta\sin\theta\, d\theta = 0$

$\therefore T = 8W\cos\theta$

C, D, O change upon displacement.

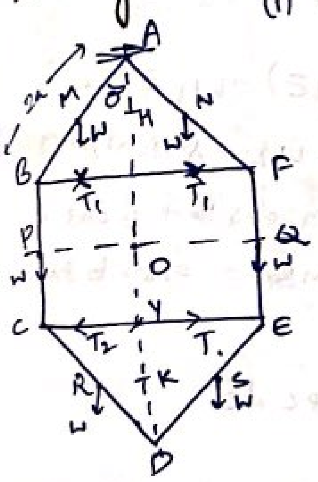
In ⇌ $\theta = 45°$

$\therefore T = 4\sqrt{2}\,W$

⇒ If 2 diff^t Ts are to be calculated, 1st find T_1 by w.r.t replacing second string | rod & then find T_2 by replacing both the strings | rods.

eg. 6 equal heavy rods hinged at the ends form a regular hexagon ABCDEF, hung by pt A & BF & CE are joined by rods. Find the thrust in both the rods.

(i) Remove T_1 rod only.



Vertical distances : $AH = a\cos\theta$

$AO = a + 2a\cos\theta$

$AK = 2a + 2a\cos\theta + YK$ ⇓ const as CE is not removed

$\therefore T_1(\delta BF) + 2W(\delta AH) + 2W(\delta AO) + 2W(\delta AK) = 0$

$T_1(\delta\, 4a\sin\theta) + 2W(\delta\, a\cos\theta) + 2W(\delta\, a + 2a\cos\theta) + 2W(\delta\, 2a + 2a\cos\theta + YK) = 0$

⇒ $4\cancel{a}\cos\theta\,\cancel{d\theta}\,T_1 - (2W + 4W + 4W) \times \cancel{a}\sin\theta\,\cancel{d\theta} = 0$. [∵ Can't take COM directly at O as YO is rigid]

⇒ $T_1 = \frac{5}{2}\frac{W\sin\theta}{\cos\theta} = \frac{5\sqrt{3}}{2}W$

(ii) Now replace both T_1 & T_2

$\therefore T_1(\delta BF) + T_2(\delta CE) + 6W(\delta AO) = 0$ [∵ we can take COM at O as all rods are free]

$T_1(\delta\, 4a\sin\theta) + T_2(\delta\, 4a\sin\theta) + 6W(\delta\, a + 2a\cos\theta) = 0$

$+ 4\cancel{a}(T_1 + T_2)\cos\theta\,\cancel{d\theta} - 12W\cancel{a}\sin\theta\,\cancel{d\theta} = 0$

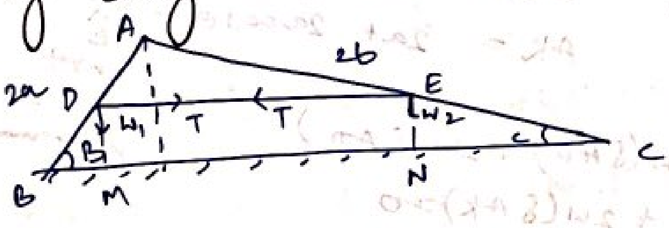
$T_1 + T_2 = 3W\tan\theta = 3\sqrt{3}\,W$

$\therefore T_2 = \left(3\sqrt{3} - \frac{5\sqrt{3}}{2}\right)W = \frac{\sqrt{3}}{2}W$

10) Some questions have multiple parameters that can change & there might be some relation b/w them or not.

(a) They are related.

eg. 2 uniform rods AB & AC smoothly joined at A are in ⇌ in a vertical plane. B & C rest on smooth horizontal plane & middle pts of AB & AC are connected by string. Find T in string.



Here: $-T(\delta DE) - W_1\,\delta(DM) - W_2\,(\delta EN) = 0$

where $DE = a\cos B + b\cos C$

$DM = a\sin B = EN = b\sin C$

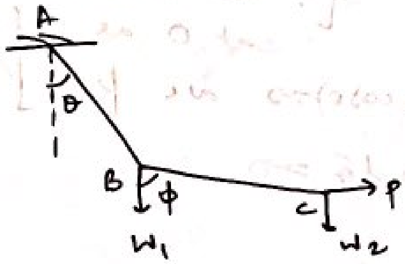
& use relation $a\sin B = b\sin C$

$\Rightarrow a\cos B\,dB = b\cos C\,dC$.

(b) They are not related.

Here proceed as usual & in the end make $d\alpha = 0$ for each parameter separately & get the cond^n.

eg.



ST: $P = (W_1 + W_2)\tan\theta = W_2\tan\phi$ for system to be in ⇌.

Here we have $P(\delta(l_1\sin\theta + l_2\sin\phi)) + W_1(\delta(l_1\cos\theta)) + W_2(\delta(l_1\cos\theta + l_2\cos\phi)) = 0$

$\Rightarrow l_1[P\cos\theta - (W_1+W_2)\sin\theta]\,\delta\theta = l_2[W_2\sin\phi - P\cos\phi]\,\delta\phi$

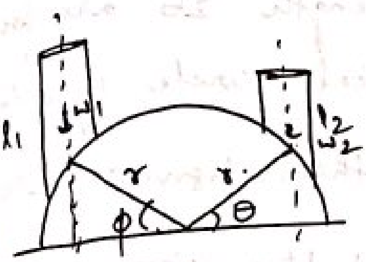
Put $\delta\phi = 0 \Rightarrow P = (W_1+W_2)\tan\theta$

Put $\delta\theta = 0 \Rightarrow P = W_2\tan\phi$

11) Some times there is an implicit relation

eg.

No. 40 in Pg 47 in Notes



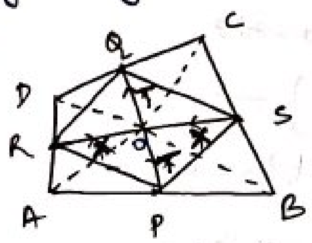
Here if the system is in $\rightleftharpoons$, horizontal distance b/w beams is constant.

$\therefore$ $r\cos\phi + r\cos\theta = $ constant

$\Rightarrow$ $\sin\phi \, d\phi + \sin\theta \, d\theta = 0$ $\Rightarrow$ use this constraint

12) Remember: Given a quadrilateral ABCD, the figure formed by joining the mid pts of the sides is a ||gm.

Example:



RS connected by rod with thrust X

& PQ connected by string with tension T.

ST: $T/PQ = X/RS$

$-T\,\delta(PQ) + X(\delta RS) = 0 \Rightarrow \dfrac{X}{T} = \dfrac{\delta(PQ)}{\delta(RS)}$

Remember

$\because$ OP is median of $\Delta OAB \Rightarrow$ $\boxed{OA^2 + OB^2 = 2OP^2 + 2AP^2}$

$= 2\left(\frac{PQ}{2}\right)^2 + 2\left(\frac{AB}{2}\right)^2$

$= \dfrac{PQ^2}{2} + \dfrac{AB^2}{2}$

Also $OC^2 + OD^2 = \dfrac{PQ^2}{2} + \dfrac{CD^2}{2}$

$\Rightarrow \Sigma OA^2 = PQ^2 + \dfrac{AB^2 + CD^2}{2}$ Sly $\Sigma OA^2 = RS^2 + \dfrac{AD^2 + BC^2}{2}$

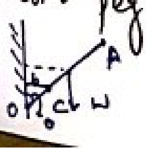
$\Rightarrow PQ^2 - RS^2 = $ constt

$\therefore 2PQ\, d(PQ) - 2RS\, d(RS) = 0$

$\Rightarrow \dfrac{d(PQ)}{d(RS)} = \dfrac{RS}{PQ}$.

13) Always take distance from fixed pt.

eg. Beam of length $2a$ rests against a smooth wall & upon a smooth peg at distance 'b' from wall. ST at $\rightleftharpoons$ $\angle$ of inclination of $\sin^{-1}\left(\frac{b}{a}\right)^{1/3}$



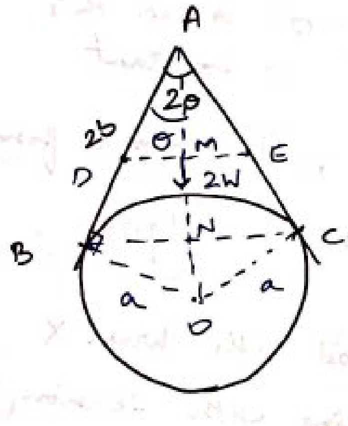
$W[\delta(a\cos\theta - b\cot\theta)] = 0 \Rightarrow a\sin\theta = b\,\mathrm{cosec}^2\theta$

we tok distance from 'C'

$\Rightarrow \sin^3\theta = \left(\frac{b}{a}\right)$

14) Choose the fixed pt carefully

Example : 2 roads AB and AC, each of length 2b are freely joined at A & rest on a smooth vertical circle of radius a. ST if the angle b/w them is 2θ, then $b\sin^3\theta = a\cos\theta$



Here 'O' is the fixed pt. upon small displacement θ changes to $\theta + d\theta$ and M also changes.

$\therefore \; W\,\delta(OM) = 0$

$OM = OA - MA$

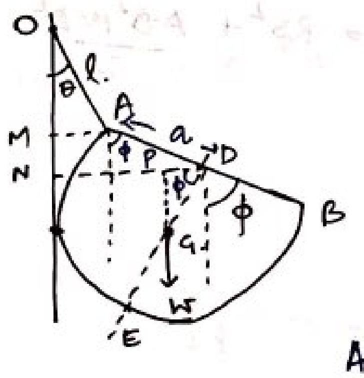
$= a\,\mathrm{cosec}\,\theta - b\cos\theta$

$\therefore \; W \times [a\cot\theta\,\mathrm{cosec}\,\theta\, d\theta + b\sin\theta\, d\theta] = 0$

$\Rightarrow \; b\sin\theta = a\cot\theta\,\mathrm{cosec}\,\theta$

$\Rightarrow \; b\sin^3\theta = a\cos\theta.$

15) String supporting a hemisphere against a smooth wall. String makes $\angle\theta$ with vertical & base of hemisphere makes $\angle\phi$. Find relation b/w θ & ϕ



COM of HS $= \frac{3a}{8} \Rightarrow PG = \frac{3a}{8}$

O is the fixed point.

$W\,\delta(PG + OM) = 0$

$PG = \frac{3a}{8}\sin\phi \qquad OM = l\cos\theta + a\cos\phi$

Also $AM = l\sin\theta = a - a\sin\phi$

$\therefore \; l\cos\theta\, d\theta = -a\cos\phi\, d\phi$

Put in $\delta\left(\frac{3a}{8}\sin\phi + l\cos\theta + a\cos\phi\right) = 0$

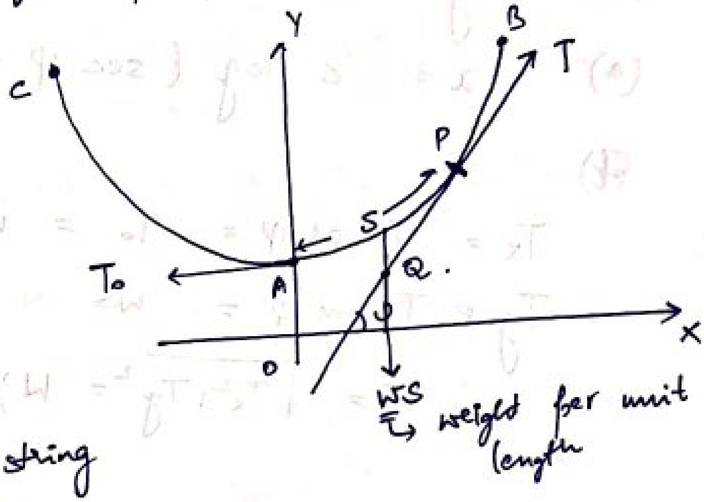
$\Rightarrow \; \tan\phi - \tan\theta = \frac{3}{8}$

⇒ Catenary :

1) A uniform string or chain hanging freely under gravity b/w 2 points not in the same vertical line.

We assume generally it to be a uniform catenary, i.e. weight per unit length of suspended chain is constant.

2) Intrinsic Equation :



AP is in ⇌

$\therefore$ $T_0 = T\cos\psi$

$ws = T\sin\psi$

Let $T_0 = wc$ = weight of the length 'c' of string

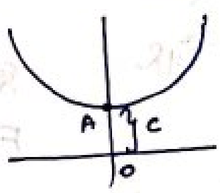
$\Rightarrow$ $\boxed{s = c\tan\psi}$

$T_0 = T\cos\psi$ = horizontal component of the tension at every pt of catenary is same & is equal to tension at the lowest point.

3) Cartesian form : put $\tan\psi = \frac{dy}{dx}$ & $\frac{ds}{dx} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$

& then sub $\frac{dy}{dx} = p$

$\Rightarrow$ $\boxed{y = c\cosh\left(\frac{x}{c}\right)}$



Axis : vertical through lowest pt of catenary

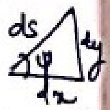
Vertex : lowest pt at which $\frac{dy}{dx} = 0$

Parameter = 'c' Directrix : Horizontal line at depth 'c' below the lowest point, i.e. x-axis.

4) Various Relations :

(a) $y = c \cosh\left(\frac{x}{c}\right)$

(b) $s = c \tan\psi = c\frac{dy}{dx} = c \sinh\left(\frac{x}{c}\right)$



(c) $y^2 = c^2 + s^2$

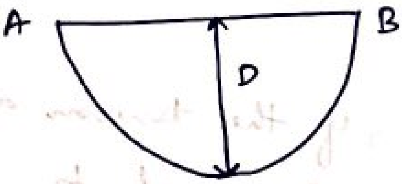
(d) $y = c \sec\psi$ $\left[\because \frac{dy}{d\psi} = \frac{dy}{ds}\cdot\frac{ds}{d\psi} = c\sin\psi\sec^2\psi \ldots\right]$

(e) $x = c \log(\sec\psi + \tan\psi)$

5) $T_x = T\cos\psi = T_0 = wc$

$T_y = T\sin\psi = ws$

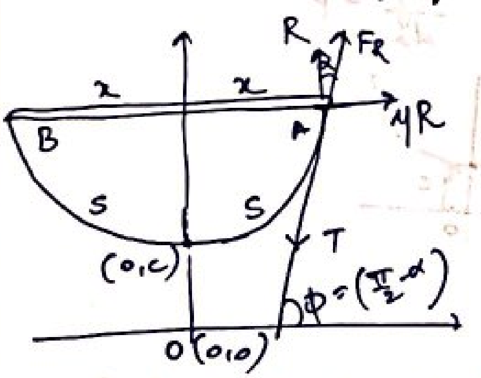
$\therefore T = \sqrt{T_x^2 + T_y^2} = wy$

6) 

AB = Span (distance b/w hanging points)

D = Sag (depth below AB)

Example : End links of a uniform chain slide along a fixed rough rod (μ). PT Ratio of max span to length of chain is $\mu \log\left\{\frac{1+\sqrt{1+\mu^2}}{\mu}\right\}$.



AB are in limiting eq.

R = Normal rxn of rod at A

F = Resultant of (F=μR) & R

$\boxed{\mu = \tan\alpha}$ ⇒ condition of limiting equilibrium

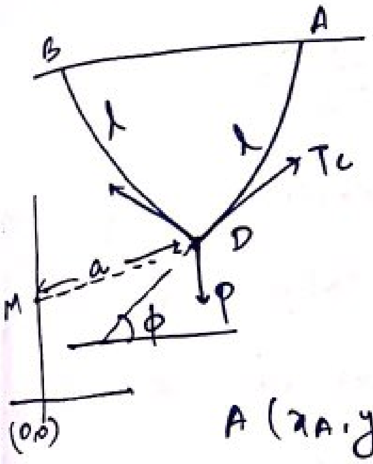
⇒ Remember

$2S = 2c\tan\phi \Rightarrow 2c\cot\alpha = \frac{2c}{\mu}$

$AB = 2x = 2c\log(\sec\phi + \tan\phi) = 2c\log(\operatorname{cosec}\alpha + \cot\alpha) = 2c\log\left(\frac{\sqrt{1+\mu^2}}{\mu} + \frac{1}{\mu}\right)$

$\therefore \frac{2x}{2s} = \mu\log\{(1+\sqrt{1+\mu^2})/\mu\}$

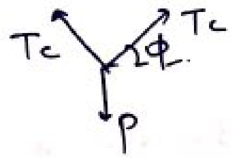
Example : Uniform chain of length $2l$ & weight W is suspended from A & B. Load 'p' is suspended from its middle pt., the resulting depth being 'h' below AB. ST terminal tension is $\frac{1}{2}\left\{ p\frac{l}{h} + W \cdot \frac{h^2+l^2}{2hl} \right\}$.



$w = \frac{W}{2l}$.

Remember

At 'D'



$2T_c \sin\phi = p$

But $T_c \sin\phi = wa$

$\therefore \quad 2wa = p \Rightarrow a = \frac{p}{2w} = \frac{pl}{W}$

$A(x_A, y_A)$, $D(x_D, y_D)$

$S_A = a + l \qquad S_D = a$

$\therefore \quad y_A^2 = (a+l)^2 + c^2$

$y_D^2 = a^2 + c^2 \Rightarrow y_A^2 - (y_A - h)^2 = l^2 + 2al$

$\Rightarrow (2y_A - h)(h) = 2al + l^2$

$\Rightarrow \quad y_a = \frac{1}{2}\left\{ h + \frac{2al + l^2}{h} \right\}$

$T_A = w y_A = \frac{W}{2l} \cdot \frac{1}{2}\left\{ h + \frac{\frac{2pl^2}{W} + l^2}{h} \right\}$

$= \frac{W}{4l}\left\{ h + \frac{2pl^2}{Wh} + \frac{l^2}{h} \right\} = \frac{1}{2}\left\{ \frac{pl}{h} + W\frac{h^2+l^2}{2hl} \right\}$.

⇒ Rectilinear Motion (SHM) :

1) Particle moves in a straight line st its acc^n is always directed towards a fixed point. on a the line & varies as the distance of particle from the fixed point

$$\frac{d^2x}{dt^2} = -\mu x$$

2) Important Relations :

(a) $v^2 = \mu(a^2 - x^2)$ $\therefore$ v is max at $x=0$, ie, centre of force

(b) $x = a\cos(\sqrt{\mu}\,t)$ [t is measured from extreme pt]

(c) Time of descent ~~from~~ to centre is independent of the initial displacement $= \frac{\pi}{2\sqrt{\mu}}$

Time Period $= 4 \times \frac{\pi}{2\sqrt{\mu}} = \frac{2\pi}{\sqrt{\mu}}$. [independent of 'a']

Frequency $= \frac{1}{T} = \frac{\sqrt{\mu}}{2\pi}$ Max acc^n $= \mu a$ max velocity $= \sqrt{\mu}\,a$ $= (\sqrt{\mu}) \times$ amplitude.

3) Particle revolving in a ~~cir~~ circle with constant angular velocity ω. its projection on x-axis follows SHM.

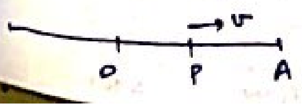
$x = a\cos\omega t$ where $T = \frac{2\pi}{\omega}$

4) Best is to go by standard relations like $\frac{d^2x}{dt^2} = -\mu x$ or $v^2 = \mu(a^2 - x^2)$ & then use conditions.

Example. Particle performing SHM of period T about O. If passes through ~~O~~P with velocity v and $OP = b$. Find the total time taken to come back to 'P'.

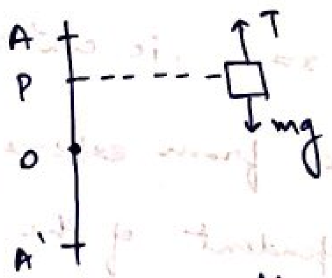
$T = 2 \times t_{AP}$ & $\frac{2\pi}{\sqrt{\mu}} = T$ use this to proceed

$\frac{dx}{dt} = \sqrt{\mu}\sqrt{a^2 - x^2} \Rightarrow \int_a^b \frac{dx}{\sqrt{a^2 - x^2}} = \int_{t=0}^{t_{AP}} \sqrt{\mu}\,dt$



Example :- Body is attached to one end of an inelastic string & other end makes SHM in vertical line of amplitude 'a' & freq 'n'. ST, string will not remain tight during the motion unless $n^2 < \dfrac{g}{4\pi^2 a}$

Suppose string remains tight which means body also undergoes SHM.



Let body move in SHM b/w A & A' with centre O. OA = a.

Also $\dfrac{2\pi}{\sqrt{\mu}} = 1/n \Rightarrow \mu = 4\pi^2 n^2$

At time 't' body is at P, OP = x, then

force acting on ~~p~~ body = T − mg

$$m\frac{d^2x}{dt^2} = T - mg \Rightarrow T = mg + \frac{md^2x}{dt^2}$$

T is least when $\dfrac{d^2x}{dt^2}$ is least whose least value is $-\mu a$

$\therefore$ Least $T = mg - m\mu a > 0 \Rightarrow mg > m \times 4\pi^2 n^2 \times a$

$$\Rightarrow n^2 < \frac{g}{4\pi^2 a}.$$

⇒ Hooke's Law :

Tension of an elastic string $\propto$ extension of string beyond its natural length.

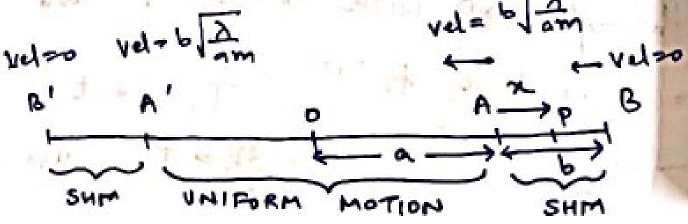
$$\therefore T = \lambda\frac{(x-l)}{l}.$$

λ = modulus of elasticity

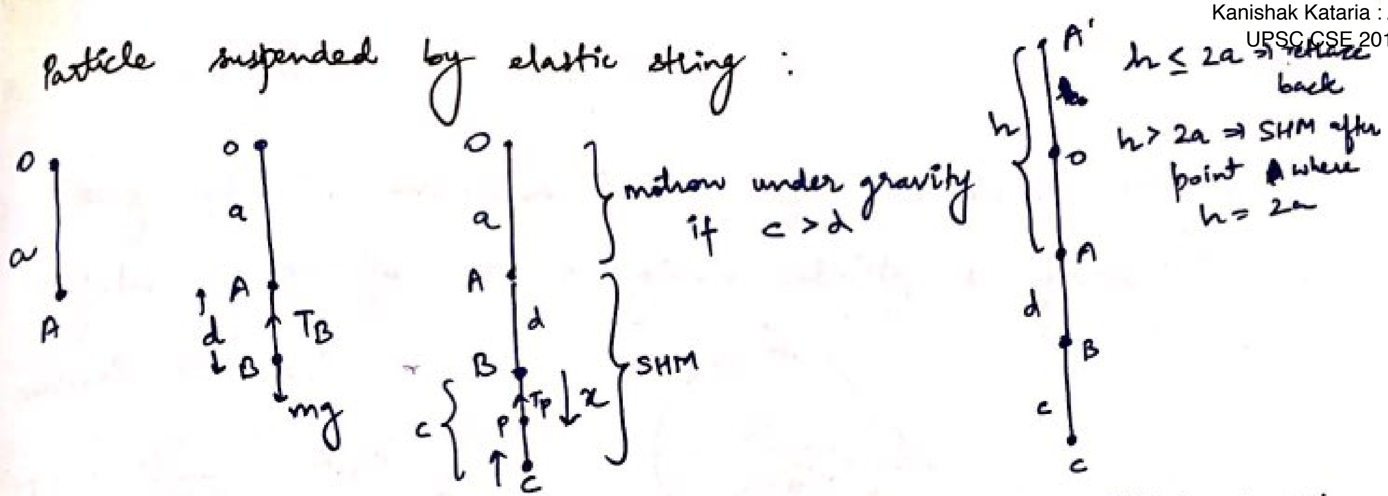
& direction of T is opposite to the extension

6> DIFFERENT MOTIONS :

(i) Particle attached to one end of horizontal elastic string



(ii) Particle suspended by elastic string :



Remember : while answering, use $\frac{d^2x}{dt^2} = -\mu x \Rightarrow$ multiplying by $2\frac{dx}{dt}$ & integrating

$$\Rightarrow \left(\frac{dx}{dt}\right)^2 = -\mu x^2 + C.$$

Then satisfy conditions to get C. suppose we get

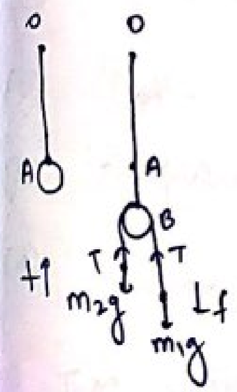
$$\left(\frac{dx}{dt}\right)^2 = \mu(a^2 - x^2) \Rightarrow \int \frac{dx}{\sqrt{a^2-x^2}} = \int \sqrt{\mu}\, dt$$

While taking sqrt, choose +/- sign as per movement in direction of increasing or decreasing 'x' & do mention it

$$\sin^{-1}\left(\frac{x}{a}\right)\Big|_{x_1}^{x_2} = \sqrt{\mu}\, t \,\Big|_{t_1}^{t_2}$$ to calculate time elapsed.

Use this standard method always!! Don't think too much ☺

Example: Smooth light pulley, suspended by spring of length 'l' & $\lambda = mg$. Masses m_1 & m_2 hang around the pulley. ST pulley executes SHM about a centre whose depth below point of suspension is $l\left\{1 + \frac{2M}{m}\right\}$ where $\frac{2}{M} = \frac{1}{m_1} + \frac{1}{m_2}$



Let f be common acceleration of masses

$$m_1 g - T = f m_1, \quad T - m_2 g = m_2 f$$

$$\Rightarrow T = \frac{2m_1m_2 g}{(m_1+m_2)} = Mg$$

$\therefore$ Tension in string $= 2T = 2Mg$ = Pressure on pulley

or pulley acts like a particle of mass $2M$.

For $\rightleftharpoons$ of pulley $\frac{\lambda AB}{l} = 2Mg \Rightarrow AB = \frac{2Mgl}{\lambda} = \frac{2Mgl}{mg} = \frac{2Ml}{m}$

Now if it is slightly pulled down.

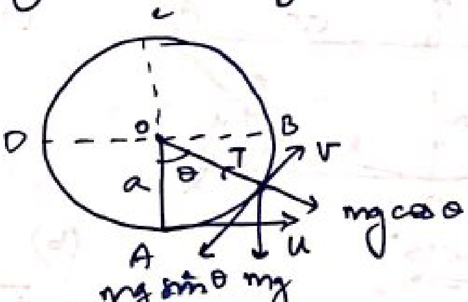
$$2Mg - T_p = 2Mg - \frac{\lambda(AB+x)}{l} = -\frac{\lambda x}{l} \Rightarrow 2Ma = -\frac{mgx}{l} \Rightarrow a = -\frac{mg}{2Ml}x$$

which is SHM

$$\therefore \text{Centre} = OB = OA + AB = l + \frac{2Ml}{m} = l\left\{1 + \frac{2M}{m}\right\}$$

⇛ Constrained Motion :

1) Heavy particle attached to a light inextensible string, projected horizontally with a given velocity 'u' from its vertical eqm pos^n



$a_T = \frac{d^2s}{dt^2}$ $a_N = \frac{v^2}{\rho}$

$m\frac{d^2s}{dt^2} = -mg\sin\theta$ — (I)

$T - mg\cos\theta = \frac{mv^2}{(\rho = a)}$ (II) $s = a\theta$

we get ⇒ $v^2 = u^2 - 2ag + 2ag\cos\theta$ $T = \frac{m}{a}(u^2 - 2ag + 3ag\cos\theta)$

If velocity vanishes at $h = h_1$ ⇒ $h_1 = OA - a\cos\theta$
$= a - a \cdot \frac{2ag - u^2}{2ag} = \frac{u^2}{2g}$

easy to remember using PE+KE = const.

If $T = 0$ at θ_2 ⇒ $\frac{2ag - u^2}{3ag} = \cos\theta_2$

at h_2 ⇒ $h_2 = a - a \cdot \frac{2ag - u^2}{3ag} = \frac{ag + u^2}{3g}$

(a) v vanishes before T ⇒ $h_1 < h_2$ ⇒ $u^2 < 2ag$ [it won't cross B]

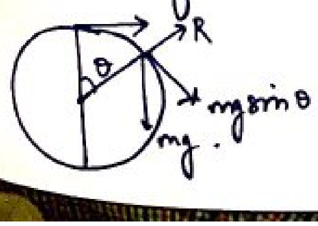
(b) Both vanish simultaneously ⇒ $h_1 = h_2$ ⇒ $u^2 = 2ag$

(c) Completes full circle ⇒ lowest value of $T > 0$ ⇒ $u^2 \geq 5ag$.

(d) Tension vanishes before ⇒ $h_1 > h_2$ ⇒ $2ag < u^2 < 5ag$ ⇒ parabolic path once string slacks
⇓
use formulas of parabola properly

Just write (I) & (II) & start solving

2) Motion of on outside of a smooth circle.



$m\frac{d^2s}{dt^2} = mg\sin\theta$

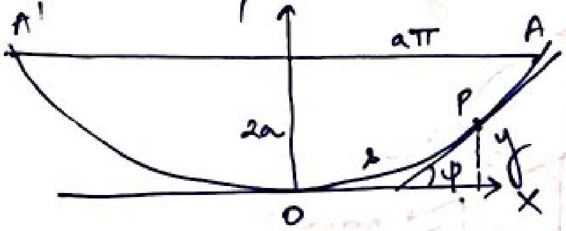
$\frac{mv^2}{a} = mg\cos\theta - R$

Now proceed

It leaves surface after covering a/3 distance vertically

3) Cycloidal Motion :

Cycloid is a curve traced by a point ~~at~~ on the circumference of a circle as the circle rolls . .



(a) $x = a(\theta + \sin\theta)$
$y = a(1 - \cos\theta)$
where $\theta = 2\psi$

A, A' are 'cusps'

(b) Intrinsic equation : $s = 4a \sin\psi$

(c) $\boxed{s^2 = 8ay}$

Particle slides down the arc of a smooth cycloid whose axis is vertical & vertex downwards :

$$m\frac{d^2s}{dt^2} = -mg\sin\psi \qquad \frac{mv^2}{\rho} = R - mg\cos\psi$$

$$\boxed{\rho = \frac{ds}{d\psi}}$$

Also $s = 4a\sin\psi$

$$\Rightarrow \frac{d^2s}{dt^2} = \frac{-g}{4a}s \Rightarrow \text{SHM} \qquad \therefore T = 4\pi\sqrt{\frac{a}{g}}$$

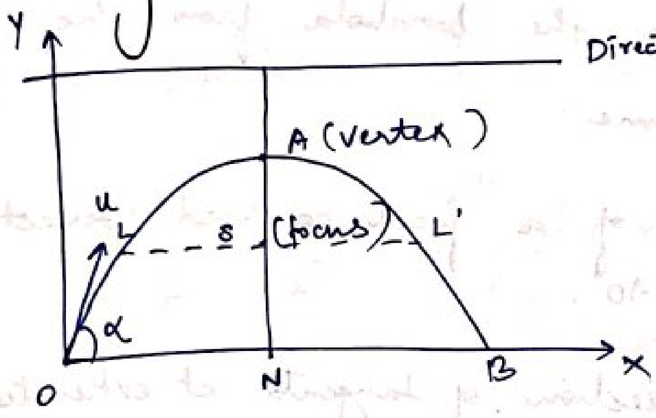
max $y = 2a$ for which $s = 4a$.

$\Rightarrow$ $\boxed{\text{APSE}}$ DE : $\boxed{h^2\left[u + \frac{d^2u}{d\theta^2}\right] = \frac{P}{u^2}}$ $\quad u = \frac{1}{r}$ $\quad P =$ central acc^n

$$v^2 = h^2\left[u^2 + \left(\frac{du}{d\theta}\right)^2\right]$$

↳ gives both h & constt of integration for initial velocity

⇒ Projectiles :

1) 

Directrix $y = u^2/2g$

$x = (u\cos\alpha)t$

$y = (u\sin\alpha)t - \frac{1}{2}gt^2$

$$\boxed{y = x\tan\alpha - \frac{1}{2}g\frac{x^2}{u^2\cos^2\alpha}} \quad ①$$

$V_x = u\cos\alpha \qquad V_y = u\sin\alpha - gt$

2) From ① : $\left(x - \frac{u^2\cos\alpha\sin\alpha}{g}\right)^2 = -\frac{2u^2\cos^2\alpha}{g}\left(y - \frac{u^2\sin^2\alpha}{2g}\right)$

$x = X + \frac{u^2\cos\alpha\sin\alpha}{g} \qquad y = Y + \frac{u^2\sin^2\alpha}{2g}$

$\Rightarrow \quad X^2 = -\frac{2u^2\cos^2\alpha}{g}Y \quad \Leftrightarrow \quad X^2 = -4aY.$

Latus Rectum : $= \frac{2}{g}u^2\cos^2\alpha = \frac{2}{g}(\text{horizontal velocity})^2$

Vertex : $\left(\frac{u^2\sin\alpha\cos\alpha}{g}, \frac{u^2\sin^2\alpha}{2g}\right)$ = Top most point

Focus : F lies on axis of parabola

$S_x = A_x = \frac{u^2\sin\alpha\cos\alpha}{g}$

$S_y = A_y - \frac{1}{4}\text{latus rectum} = \frac{-u^2}{2g}\cos 2\alpha.$

Directrix : Height of D above O = height of A above O $+ \frac{1}{4}$ Latus rectum

$y = \frac{u^2}{2g}$

3) $T = \frac{2u\sin\alpha}{g} \qquad R = \frac{u^2\sin 2\alpha}{g} \qquad H = \frac{u^2\sin^2\alpha}{2g}$

4) Some properties of Parabola :

(a) Distance of any pt on the parabola from the focus & directrix is same

(b) Tangents at extremities of a focal co chord intersect on the directrix at $90°$.

(c) Line joining pt of intersection of tangents at extremities of any chord to the middle pt of the chord is // to axis

(d) Tangent at any point bisects the angle b/w the focal distance of the pt & the ⊥ from point to directrix.

Examples from notes : 21, 25, 26, 41, 68*

5) Max^m range up the inclined plane $= \dfrac{u^2}{g(1+\sin\beta)}$

Max^m range down the inclined plane $= \dfrac{u^2}{g(1-\sin\beta)}$

Work, Energy & Impulse

1) $W = \int_A^B \vec{F} \cdot d\vec{r}$ Also $\frac{d\vec{r}}{ds} = \hat{t}$ where $\hat{t}$ is the unit vector along the tangent at P in sense of increasing s.

$$\therefore W = \int_A^B \left(\vec{F} \cdot \frac{d\vec{r}}{ds}\right) ds = \int_A^B (\vec{F} \cdot \hat{t}) ds$$

$$= \int_A^B F\cos\theta \, ds$$

2) $KE = \frac{1}{2}mu^2$ $\Delta KE =$ work done by forces. ← use this!

3) Conservative force : Work done depends only on initial & final posn & not the path.

$\vec{F} = X\hat{i} + Y\hat{j} + Z\hat{k}$ is conservative $\Leftrightarrow \exists f$ st. $\frac{\partial f}{\partial x} = X$, $\frac{\partial f}{\partial y} = Y$, $\frac{\partial f}{\partial z} = Z$

then $\vec{F} = \nabla f$

4) Conservation of Linear momentum :

If net force along a direction is 0, then momentum is ~~f~~ conserved along that line.

5) Impulse : $I = \int_{t_0}^{t_1} F(t)\,dt$

Change of momentum in a time interval = net impulse

$$I = \int m\frac{dv}{dt}dt = \int m\,dv \cdot mv_2 - mv_1 = \Delta p.$$

Impulsive force = very large force in a very small time interval

6) Apply principle of energy conservation to get work done directly.

Example : 2 men each of mass M, stand on 2 inelastic platforms each of mass m hanging over a smooth pulley. One of the men leaping from ground can raise his COG by 'h'. If the leaps with same energy from the the platform, then his COG will raise by ?

Let 'u' be the velocity when he jumps & the rest of the system has common velocity 'v'

$\therefore I =$ Impulsive force $= Mu$ [on M_1]

Net Impulsive force of $P_1 = mv$

~~Impulsive~~ $I -$ Impulsive tension $(I') = mV$

$\Rightarrow I' = Mu - mv$

Also $I' = (m+M)v$ using M_2 & P_2

$\therefore (m+M)v = Mu - mv$

$\Rightarrow (2m+M)v = Mu$

Mgh (energy imparted) $= \frac{1}{2}Mu^2 + \frac{1}{2}(2m+M)v^2$

$$Mgh = \frac{1}{2}Mu^2 + \frac{1}{2}\frac{\cancel{(2m+M)}\,M^2u^2}{(2m+M)^{\cancel{2}}}$$

$$\cancel{2}\cancel{M}gh = \cancel{M}u^2\left[\frac{\cancel{2}m+\cancel{2}M}{(2m+M)}\right]$$

$$u^2 = \frac{(2m+M)gh}{(m+M)}$$

$\therefore h'$ (new ht jumped) $= \frac{u^2}{2g} = \frac{(2m+M)h}{2(m+M)}$.

→ Stable & Unstable Equilibrium :-

1) 

Stable $\rightleftharpoons$ Unstable $\rightleftharpoons$ Neutral $\rightleftharpoons$

O → Geometric Centre
G → Centre of Gravity

2) <u>Work function</u> is stationary (maximum/ minimum) in the position of equilibrium. [∵ Principle of virtual work $dW = 0$]

3) $W' - W < 0 \Rightarrow dW < 0 \Rightarrow$ from A to B work is done against the force, which tries to restore the original pos^n^

∴ It is stable $\rightleftharpoons$.

Work Function test :

∴ W is minimum ⇒ unstable $\rightleftharpoons$

W is maximum ⇒ stable $\rightleftharpoons$

4) Potential Energy test :

V = potential energy in a position = − W (work function at that pos^n^)

[Remember $V = mgh$ but $W = -mgh$]

WD in moving from A to B : $V_A - V_B$

Let $V_A = V$ $V_B = V + dV \Rightarrow$ $WD = -dV = 0$ [∵ POVW]

∴ <u>V is stationary</u> in position of $\rightleftharpoons$.

V is maximum ⇒ unstable $\rightleftharpoons$

V is minimum ⇒ stable $\rightleftharpoons$.

5) Z-test : (If only gravity is acting on the body).

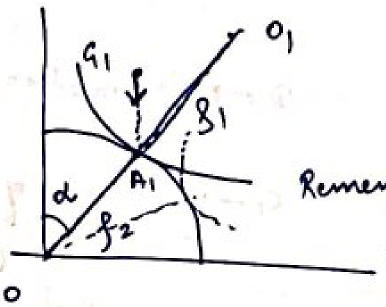
Let z = height of COG above a fixed horizontal plane & express it as $z = f(\theta)$

By POVW ⇒ $W(-\delta z) = 0 \Rightarrow \delta z = 0 \Rightarrow \frac{dz}{d\theta}\delta\theta = 0 \Rightarrow \left(\frac{dz}{d\theta}\right) = 0$

$\frac{dz}{d\theta} = 0$ gives $\rightleftharpoons$ positions & $\frac{d^2z}{d\theta^2} > 0 \Rightarrow z$ is minimum ⇒ stable $\rightleftharpoons$

$\frac{d^2z}{d\theta^2} < 0 \Rightarrow z$ is maximum ⇒ unstable $\rightleftharpoons$

6) Sphere on top of sphere :



Equilibrium is stable or unstable as

$$h < \text{ or } > \frac{\rho_1 \rho_2}{\rho_1 + \rho_2} \cos\alpha$$

Remember for $\gtreqless$, $A_1 G_1$ has to be a vertical line.

Detailed derivation in notes

Cases :

(i) $\alpha = 0$ Bodies in contact have radii of curvature ρ_1 & ρ_2

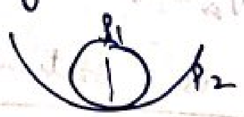
& CoG of 1^{st} body lies 'h' height above point of contact

Then : $\frac{1}{h} > \frac{1}{\rho_1} + \frac{1}{\rho_2} \Rightarrow$ stable $\gtreqless$

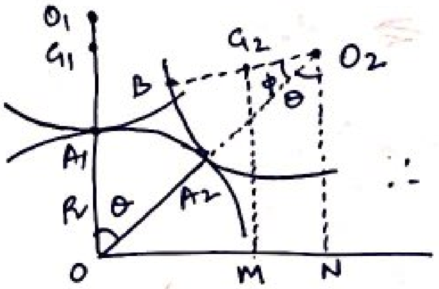
$\frac{1}{h} \leq \frac{1}{\rho_1} + \frac{1}{\rho_2} \Rightarrow$ unstable $\lesseqgtr$

If the surfaces are plane put corresponding $\rho = \infty$

If lower body is concave, take $\rho_2 < 0$ (with $-$ve sign)



Derivation :



$\angle A_1 O A_2 = \theta$
$\angle B O_2 A_2 = \phi$
$\therefore \angle G_2 O_2 N = \theta + \phi$
$O_1 A_1 = r \quad OA_1 = R \quad A_1 G = h.$

& $\overset{\frown}{A_1 A_2} = \overset{\frown}{BA_2} \Rightarrow R\theta = r\phi \Rightarrow \frac{d\phi}{d\theta} = \frac{R}{r}$

Now $z = G_2 M = OO_2 \cos\theta - (O_2 B - G_2 B) \cos (\theta + \phi)$

$= (R+r)\cos\theta - (r-h)\cos(\theta + \phi)$

$= (R+r)\cos\theta - (r-h)\cos\left(\theta + \frac{R\theta}{r}\right) = (R+r)\cos\theta - (r-h)\cos\left(\frac{r+R}{r}\theta\right)$

$\therefore \frac{dz}{d\theta} = -(R+r)\sin\theta + (r-h)\sin\left(\frac{r+R}{r}\theta\right) \times \frac{r+R}{r} = 0$

$\frac{d^2z}{d\theta^2} = -(R+r)\cos\theta + (r-h)\cos\left(\frac{r+R}{r}\theta\right) \times \left(\frac{r+R}{r}\right)^2$

for $\theta = 0$ $\quad \dfrac{d^2z}{d\theta^2} = -(R+r) + (r-h)\left(\dfrac{R+r}{r}\right)^2$

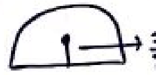
$$= (R+r)\left[\frac{(R+r)(r-h)}{r^2} - 1\right]$$

$$= \frac{(R+r)^2}{r^2}\left[r-h - \frac{r^2}{R+r}\right] = \left(\frac{r+R}{r}\right)^2\left[\frac{rR}{R+r} - h\right]$$

stable $\Rightarrow \dfrac{d^2z}{d\theta^2} > 0 \Rightarrow h < \dfrac{rR}{R+r} \Rightarrow \dfrac{1}{h} > \dfrac{1}{R} + \dfrac{1}{r}$

unstable $\Rightarrow \dfrac{d^2z}{d\theta^2} < 0 \Rightarrow h > \dfrac{rR}{R+r} \Rightarrow \dfrac{1}{h} < \dfrac{1}{R} + \dfrac{1}{r}$

If $h = \dfrac{rR}{R+r}$, then we show that $\dfrac{d^3z}{d\theta^3} = 0$ & $\dfrac{d^4z}{d\theta^4} < 0$

NOTE: For hemisphere centre of mass is at $\dfrac{3R}{8}$  $\frac{3r}{8}$