Similarity of Matrices

Defn:

Let A and B be two square matrices of order n. Then B is said to be similar to A iff $\exists$ $n\times n$ invertible matrix C such that $AC = CB$. i.e, $B = C^{-1}AC$ (or) $A = CBC^{-1}$

Ex:- Let $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

Soln

$$|C| = \begin{vmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{vmatrix} = 1 \neq 0$$

$\therefore$ C is non-singular.

$\therefore$ C^{-1} exists.

$\therefore$ C is an invertible matrix.

Now $C^{-1} = \dfrac{\text{adj}\, C}{|C|}$

$$= \frac{1}{1}\begin{bmatrix} 16-9 & -12+9 & 9-12 \\ -4+3 & 4-3 & 3-3 \\ 3-4 & -3+3 & 4-3 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

Now $B = C^{-1}AC$

$$= \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 5 & 17 & 16 \\ 5 & 18 & 16 \\ 5 & 17 & 17 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 14 & 13 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ B is similar to A.

Theorem "Similarity of matrices" is an equivalence relation in the set of all $n \times n$ matrices.

Proof: Let A and B be two square matrices of order n. Then B is similar to A if $\exists$ an $n\times n$ invertible matrix C such that

$B = C^{-1}AC$ or $A = CBC^{-1}$.

$\equiv$ similarity

(i) Reflexivity:

Let A be an $n\times n$ matrix then $\exists$ an $n\times n$ invertible matrix I such that $A = I^{-1}AI$

$\Rightarrow$ A is similar to itself.

$\therefore$ Similarity of matrices is reflexive.

(ii) Symmetry:

A is similar to B

$\Rightarrow A = CBC^{-1}$

$\Rightarrow AC = CB$

$\Rightarrow C^{-1}AC = B$.

$\Rightarrow (C^{-1})A(C^{-1})^{-1} = B$

$\Rightarrow$ B is similar to A. ($\because C^{-1}$ is invertible)

$\therefore$ Similarity of matrices is symmetric.

(iii) Transitivity:

Let A, B and C be three square matrices of order $n\times n$ such that A is similar to B and B is similar to C.

$\Rightarrow \exists$ invertible $n\times n$ matrices P, Q such that

$A = PBP^{-1}$ and $B = QCQ^{-1}$

$\Rightarrow A = P(QCQ^{-1})P^{-1}$

$\Rightarrow A = (PQ)C(Q^{-1}P^{-1})$

$= (PQ)C(PQ)^{-1}$ ($\because$ P, Q are invertible $\Rightarrow$ PQ is also invertible)

$\Rightarrow$ A is similar to C

$\therefore$ Similarity of matrices is transitive.

$\therefore$ "Similarity of matrices" is an equivalence relation.

Note: $\boxed{1}$. If A is similar to B then B is similar to A and we say that A and B are similar.

$\boxed{2}$. If A and B are similar and B and C are similar then A and C are similar.

Theorem $\rightarrow$ Similar matrices have the same determinant.

proof: Let A and B be similar matrices.

Then $\exists$ an invertible matrix P such that

$B = P^{-1}AP$.

$\Rightarrow |B| = |P^{-1}AP|$

$= |P^{-1}||A||P|$

$= |P^{-1}|\ |P|\ |A|$

$= |P^{-1}P|\ |A|$

$= |I||A|$

$= |A|$

$\therefore |B| = |A|.$

Theorem 2000 ISM: Similar matrices have the same characteristic polynomial and hence the same characteristic roots.

proof: Let A and B be similar matrices.

Then $\exists$ an invertible matrix P such that

$$B = P^{-1}AP.$$

$$\begin{aligned}
\therefore B-\lambda I &= P^{-1}AP - \lambda I \\
&= P^{-1}AP - \lambda P^{-1}P \\
&= P^{-1}AP - P^{-1}\lambda P \\
&= P^{-1}AP - P^{-1}\lambda(IP) \\
&= P^{-1}AP - P^{-1}(\lambda I)P \\
&= P^{-1}(A-\lambda I)P
\end{aligned}$$

$$\begin{aligned}
\therefore |B-\lambda I| &= |P^{-1}(A-\lambda I)P| \\
&= |P^{-1}|\ |A-\lambda I|\ |P| \\
&= |A-\lambda I|\ |P^{-1}P| \\
&= |A-\lambda I|
\end{aligned}$$

$\therefore |B-\lambda I| = |A-\lambda I|$

$\therefore$ A and B have the same characteristic polynomial and hence same characteristic roots.

Ex: The similar matrices $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 14 & 13 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ have the same characteristic roots.

Solⁿ: The characteristic eqⁿ of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 2 & 1 \\ 1 & 3-\lambda & 1 \\ 1 & 2 & 2-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda = 1, 1, 5.$$

Now the characteristic equation of B is

$$|B - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 5-\lambda & 14 & 13 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda = 1, 1, 5.$$

$\therefore$ The similar matrices A & B have the same characteristic roots.

Note: If two matrices (of same order) have same characteristic roots then it is not necessary that they are similar.

Ex: $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & -1 \\ -3 & -2 & 3 \end{bmatrix}$

have the same characteristic roots but are not similar.

Theorem → If x is a characteristic vector of A corresponding to the characteristic root λ then $P^{-1}x$ is a characteristic vector of B corresponding to characteristic root λ where $B = P^{-1}AP$.

proof: If λ is a characteristic root of A and x is corresponding characteristic vector then

$$Ax = \lambda x \quad \text{——(1)}$$

Since $B = P^{-1}AP$

$$\Rightarrow B(P^{-1}x) = (P^{-1}AP)(P^{-1}x)$$
$$= P^{-1}A(PP^{-1})x$$
$$= P^{-1}AIx$$
$$= P^{-1}Ax$$
$$= P^{-1}\lambda x \quad (\because Ax = \lambda x)$$
$$= P^{-1}(x\lambda)$$
$$= (P^{-1}x)\lambda$$
$$= \lambda(P^{-1}x)$$

$$\therefore B(P^{-1}x) = \lambda(P^{-1}x)$$

$\therefore P^{-1}x$ is a characteristic vector of B corresponding to the characteristic root λ.

Theorem → If the matrix A is similar to a diagonal matrix D, then the diagonal elements of D are the characteristic roots of A.

proof: Since A and D are similar.

$\therefore$ They have the same characteristic roots.

But the characteristic roots of diagonal matrix D are its diagonal elements.

Hence the characteristic roots of A are the diagonal elements of D.

Diagonalizable matrix:

If a matrix A is similar to a diagonal matrix then A is said to be diagonalizable.

i.e., a matrix A is diagonalizable if there exists an invertible matrix P such that

$P^{-1}AP = D$ where D is diagonal matrix.

Also P is said to diagonalize A or transform A to a diagonal form.

Theorem An n-rowed square matrix is diagonalizable iff the matrix possesses 'n' linearly independent characteristic vectors.

proof: Suppose an n-rowed square matrix A is diagonalizable.

Then A is similar to the diagonal matrix

$$D = \text{dia}[\lambda_1, \lambda_2 \cdots\cdots \lambda_n]$$

$\therefore \exists$ an $n\times n$ invertible matrix $P = [x_1, x_2 \cdots\cdots x_n]$

Such that $P^{-1}AP = D$.

$\Rightarrow \quad AP = PD$

$\Rightarrow A[x_1, x_2 \cdots x_n] = [x_1, x_2 \cdots x_n] \text{ dia } [\lambda_1, \lambda_2 \cdots \lambda_n]$

$\Rightarrow [Ax_1, Ax_2 \cdots Ax_n] = [\lambda_1 x_1 \; \lambda_2 x_2 \cdots \lambda_n x_n]$

$\Rightarrow Ax_1 = \lambda_1 x_1 ; \; Ax_2 = \lambda_2 x_2, \cdots\cdots Ax_n = \lambda x_n$

$\Rightarrow x_1, x_2, \ldots x_n$ are characteristic vectors of A corresponding to $\lambda_1, \lambda_2, \ldots \lambda_n$ respectively.

Since the matrix P is non-singular matrix

$\therefore$ Its coloumn vectors $x_1, x_2, \cdots\cdots x_n$ are LI.

$\therefore$ 'A' possesses 'n' LI characteristic vectors

Conversely, suppose that A possesses 'n' LI characteristic vectors $x_1, x_2, \cdots x_n$ and let $\lambda_1, \lambda_2, \cdots \lambda_n$ be the corresponding characteristic roots then $Ax_1 = \lambda_1 x_1$, $Ax_2 = \lambda_2 x_2, \cdots\cdots Ax_n = \lambda x_n$

Let $P = [x_1, x_2, \cdots x_n]$ and $D = \text{dia} [\lambda_1, \lambda_2, \ldots \lambda_n]$

then $AP = A[x_1, x_2, \cdots x_n]$

$= [Ax_1, Ax_2 \cdots Ax_n]$

$= [\lambda_1 x_1, \lambda_2 x_2, \cdots \lambda_n x_n]$

$= [x_1, x_2, \cdots x_n] \text{ dia} [\lambda_1, \lambda_2, \cdots \lambda_n]$

$= PD$

$\therefore AP = PD$

$\Rightarrow P^{-1}AP = P^{-1}PD$

$\Rightarrow P^{-1}AP = D$

$\Rightarrow$ A is similar to a diagonal matrix D.

$\Rightarrow$ A is diagonalizable.

Note: In the proof of the above theorem we shown that if A is diagonalizable and P diagonalizes A, then $P^{-1}AP = D$.

$$= \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

iff the j^{th} coloumn of P is an eigen vector of A corresponding to the eigen value λ_j of A. $(j=1,2,\ldots n)$. The diagonal elements of D are the eigen values of A and they occur in the same order as is the order of their corresponding eigen vectors in the coloumn vectors of P.

Ex: The matrix $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}$ has characteristic roots 5, 1, 1 with corresponding characteristic vectors (1,1,1), (2,−1,0), (1,0,−1) respectively.

Sol^n: Taking $P = \begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$

we have $P^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}$

and $P^{-1}AP = \frac{1}{4}\begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}$

$$= \frac{1}{4}\begin{bmatrix} 1 & 2 & 1 \\ 1 & -2 & 1 \\ 1 & 2 & -3 \end{bmatrix}\begin{bmatrix} 5 & 2 & 1 \\ 5 & -1 & 0 \\ 5 & 0 & -1 \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} 20 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = D$$

$\therefore$ A is similar to the diagonal matrix

$D = \text{dia}(5, 1, 1)$.

Note: Every square matrix is not similar to a diagonal matrix.

Ex: $A = \begin{bmatrix} 2 & -1 & 1 \\ 2 & 2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$

Sol: The characteristic equation of A is

$|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & -1 & 1 \\ 2 & 2-\lambda & -1 \\ 1 & 2 & -1-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda = 1, 1, 1.$

$\therefore$ The characteristic roots of A are 1, 1, 1.

The characteristic vector X of A corresponding to characteristic root $\lambda = 1$:-

$(A - \lambda I)X = 0$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & 2 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - R_1$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -3 \\ 0 & 3 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\Rightarrow \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Clearly the coefficient matrix is in echelon form.

$\therefore \rho(A) = 2$

$\therefore$ there is $3-2=1$ LI eigen vectors.

and $x_1 - x_2 + x_3 = 0$

$3x_2 - 3x_3 = 0$

$\Rightarrow x_1 - x_2 + x_3 = 0$

$x_2 - x_3 = 0$

$\Rightarrow x_1 = 0$

let $x_3 = k$, k is arbitrary constant

then $x_2 = k$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ k \\ k \end{bmatrix}$$

$$= k\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = kX_1$$

Here $X_1 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is L.I. vector corresponding to characteristic root 1.

$\therefore$ The matrix A has only 1 LI characteristic vector and is consequently not diagonal matrix

$\therefore$ The square matrix A is not similar to diagonal matrix.

Note: $\boxed{1}$. If the eigen values of an $n\times n$ matrix are all distinct then it is always similar to a diagonal matrix.

$\boxed{2}$. Two $n\times n$ matrices with the same set of 'n' distinct eigen values are similar.

→ Prove that the matrices $\begin{bmatrix} -10 & 6 & 3 \\ -26 & 16 & 8 \\ 16 & -10 & -5 \end{bmatrix}$ and $\begin{bmatrix} 0 & -6 & -16 \\ 0 & 17 & 45 \\ 0 & -6 & -16 \end{bmatrix}$ are similar.

Solⁿ: The characteristic equation of A is $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix} -10-\lambda & 6 & 3 \\ -26 & 16-\lambda & 8 \\ 16 & -10 & -5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda = 0, -1, 2$$

The characteristic roots of A are $0, -1, 2$ and they are distinct.

Now the characteristic equation of B is $|B-\lambda I|=0$

$$\Rightarrow \begin{vmatrix} 0-\lambda & -6 & -16 \\ 0 & 17-\lambda & 45 \\ 0 & -6 & -16-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda = 0, -1, 2.$$

∴ The characteristic roots of B are distinct.

∴ A and B are similar.

→ Show that the matrices $A=\begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ -1 & 2 & 1 \end{bmatrix}$, $B=\begin{bmatrix} -10 & -38 & -36 \\ 4 & 14 & 13 \\ 0 & 1 & 1 \end{bmatrix}$ are similar if $P^{-1}=\begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$ and $P=\begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$

Solⁿ: If B is similar to A then $P^{-1}AP=B$

Now $$P^{-1}AP = \begin{bmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -8 & -6 \\ 1 & 2 & 1 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} -10 & -38 & -36 \\ 4 & 14 & 13 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= B.$$

problems:

→ Show that the rank of every matrix similar to A is the same as that of A.

Sol^n: Let B be a matrix similar to A.

$\therefore \exists$ a non-singular matrix P such that

$B = P^{-1}AP$.

W.K.T the rank of a matrix doesnot change on multiplication by a non-singular matrix.

$\therefore \rho(P^{-1}AP) = \rho(A)$

i.e, $\rho(B) = \rho(A)$.

→ Let A and B be n-rowed square matrices and A be non-singular. Show that the matrices $A^{-1}B$ and BA^{-1} have the same eigen values.

Sol^n: we have $A^{-1}(BA^{-1})A = A^{-1}B$.

$\therefore BA^{-1}$ is similar to $A^{-1}B$.

But the similar matrices have the same eigen values.

$\therefore A^{-1}B$ and BA^{-1} have the same eigen values.

→ If U be a unitary matrix such that $U^{\theta}AU = \text{diag}[\lambda_1, \lambda_2, \dots \lambda_n]$. Show that $\lambda_1, \lambda_2, \dots \lambda_n$ are the eigen values of A.

Sol^n: Let $\text{diag}[\lambda_1, \lambda_2, \dots \lambda_n] = D$

Since U is unitary.

$UU^{\theta} = I$

$\therefore U^{\theta} = U^{-1}$

$\Rightarrow U^{\theta} A U = D$

$\Rightarrow U^{-1} A U = D$

$\therefore$ A is similar to the diagonal matrix D.

But the similar matrices have the same eigen values and eigen values of D are its diagonal elements.

$\therefore \lambda_1, \lambda_2, \ldots \lambda_n$ are the eigen values of A.

→ If A and B are non-singular matrices of order n, show that the matrices AB and BA are similar.

Sol^n: Since A is non-singular.

$\therefore A^{-1}$ exists.

we have $A^{-1}(AB)A = BA$

$\therefore$ AB and BA are similar matrices.

→ A and B are two $n \times n$ matrices with the same set of 'n' distinct eigen values. Show that there exist two matrices P and Q (one of them is non singular) such that $A = PQ$, $B = QP$.

Sol^n: Since A and B have the same set of 'n' distinct eigen values.

$\therefore$ They are similar.

$\therefore \exists$ a non-singular matrix P such that

$P^{-1}AP = B$ —— ①

Let $P^{-1}A = Q$ then

① $\equiv QP = B$

and $P^{-1}A = Q$

$\Rightarrow A = PQ$.

→ Prove that A is similar to a diagonal matrix, then A^T is similar to A. 8

Sol^n: Let A be similar to a diagonal matrix D. Then ∃ a non-singular matrix P such that

$$P^{-1}AP = D$$

$$\Rightarrow A = PDP^{-1}$$

$$\Rightarrow A^T = (PDP^{-1})^T$$

$$= (P^{-1})^T D^T P^T$$

$$= (P^T)^{-1} D^T P^T$$

$$= (P^T)^{-1} D P^T. \quad (\because D \text{ is diagonal} \Rightarrow D^T = D)$$

$$\therefore A^T = (P^T)^{-1} D P^T.$$

∴ A^T is similar to D.

⇒ D is similar to A^T.

Finally A is similar to D and D is similar to A^T.

⇒ A is similar to A^T.

Algebraic and Geometric multiplicity of a characteristic root:

If λ_1 is a characteristic root of order 't' of the characteristic equation $|A-\lambda I|=0$, then t is called the algebraic multiplicity of λ_1.

If S is the number of linearly independent characteristic vectors corresponding to the characteristic value λ_1, then S is called the geometric multiplicity of λ_1.

Here the number of linearly independent solutions of $(A-\lambda_1 I)x=0$ will be s and
$\rho(A-\lambda_1 I) = n-s$.

Ex: (1) For the matrix O_n, zero is the characteristic root of algebraic multiplicity 'n'.

(2) For the matrix I_n, unity is the characteristic root of algebraic multiplicity 'n'.

Note: $\boxed{1}$. The geometric multiplicity of a characteristic root cannot exceed its algebraic multiplicity. i.e, $s \le t$.

$\boxed{2}$. A square matrix is similar to a diagonal matrix iff the geometric multiplicity of each of its characteristic roots is equal to its algebraic multiplicity.

Problems:

→ Show that $A=\begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$ is similar to a diagonal matrix. Also find the transforming matrix and diagonal matrix

(OR)

Show that the characteristic vectors of $A=\begin{bmatrix} 4 & 2 & -2 \\ -5 & 3 & 2 \\ -2 & 4 & 1 \end{bmatrix}$ are linearly independent. Hence find a diagonal matrix similar to A.

Solⁿ: The characteristic equation of A is

$$|A-\lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 4-\lambda & 2 & -2 \\ -5 & 3-\lambda & 2 \\ -2 & 4 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow \lambda = 1, 5, 2$

$\therefore$ The Characteristic roots of A are 1,5,2. 9

Since the eigen values of the matrix A are all distinct.

$\therefore$ A is similar to a diagonal matrix.

Since the algebraic multiplicity of each eigen value of A is 1.

$\therefore$ there will be one and only one linearly independent eigen vector of A corresponding to each eigen value of A.

Let $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be the characteristic vector corresponding to a characteristic value

$\therefore$ Characteristic vector of A corresponding to the characteristic value '1' is given by

$(A - 1I)X = O$

$$\Rightarrow \begin{bmatrix} 3 & 2 & -2 \\ -5 & 2 & 2 \\ -2 & 4 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to 3R_2 + 5R_1$, $R_3 \to 3R_3 + 2R_1$

$$\Rightarrow \begin{bmatrix} 3 & 2 & -2 \\ 0 & 16 & -4 \\ 0 & 16 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\Rightarrow \begin{bmatrix} 3 & 2 & -2 \\ 0 & 16 & -4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Clearly the coefficient matrix is in echelon form.

$\therefore \rho(A) = 2$

$\therefore$ These equations have $3 - 2 = 1$ linearly independent solutions.

$\therefore$ we have

$$3x_1+2x_2-2x_3=0$$
$$16x_2-4x_3=0 \Rightarrow 4x_2-x_3=0$$
$$\Rightarrow x_3=4x_2.$$

Take $x_2=1$. then $x_3=4$.

and $3x_1+2(1)-2(4)=0$

$\Rightarrow 3x_1-6=0$

$\Rightarrow x_1=2$

$\therefore X_1=\begin{bmatrix}2\\1\\4\end{bmatrix}$ is a characteristic vector of A corresponding to characteristic value 1 of A.

Also the characteristic vector X of A corresponding to the characteristic value 2 is given by

$(A-2I)X=0$

$$\Rightarrow \begin{bmatrix}2 & 2 & -2\\-5 & 1 & 2\\-2 & 4 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$R_2\to 2R_2+5R_1$
$R_3\to R_3+R_1$

$$\sim \begin{bmatrix}2 & 2 & -2\\0 & 12 & -6\\0 & 6 & -3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$R_1\to \frac{1}{2}R_1$, $R_2\to \frac{1}{6}R_2$, $R_3\to \frac{1}{3}R_3$

$$\sim \begin{bmatrix}1 & 1 & -1\\0 & 2 & -1\\0 & 2 & -1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$R_3\to R_3+R_2$

$$\sim \begin{bmatrix}1 & 1 & -1\\0 & 2 & -1\\0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

Clearly the coefficient is in echelon form.

$\therefore$ The rank of coefficient matrix $=2$.

So the equations have $3-2=1$ LI solution.

we have $x_1+x_2-x_3=0$

$2x_2-x_3=0 \Rightarrow x_3=2x_2$.

Take $x_3=2$ then $x_2=1$

and $x_1=1$

$\therefore X_2=\begin{bmatrix}1\\1\\2\end{bmatrix}$ is a characteristic vector of A corresponding to the characteristic root 2 of A.

Again characteristic vector X of A corresponding to the characteristic root 5 is given by

$(A-5I)X=0$

$$\Rightarrow \begin{bmatrix}-1 & 2 & -2\\-5 & -2 & 2\\-2 & 4 & -4\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\Rightarrow X_3=\begin{bmatrix}0\\1\\1\end{bmatrix}$ is a characteristic vector of A corresponding to the characteristic root 5 of A

Let $P=[X_1\ X_2\ X_3]$

$$=\begin{bmatrix}2 & 1 & 0\\1 & 1 & 1\\4 & 2 & 1\end{bmatrix}$$

The coloumns of P are LI vectors of A corresponding to the characteristic roots 1, 2, 5. respectively.

The matrix P will transform A to diagonal form D is given by the relation $P^{-1}AP=D$.

$$=\begin{bmatrix}1 & 0 & 0\\0 & 2 & 0\\0 & 0 & 5\end{bmatrix}$$

$\therefore$ The transforming matrix $P = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}$

and

diagonal matrix $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

Note: By actual multiplication we can verify

$P^{-1}AP = D$.

H.W. $\rightarrow$ Show that the matrix $A = \begin{bmatrix} 8 & -8 & -2 \\ 4 & -3 & -2 \\ 3 & -4 & 1 \end{bmatrix}$ is diagonalizable. Also find the transforming matrix and diagonal matrix.

$\rightarrow$ Show that the matrix $A = \begin{bmatrix} -9 & 4 & 4 \\ -8 & 3 & 4 \\ -16 & 8 & 7 \end{bmatrix}$ is diagonalizable.

Also find the diagonal form and a diagonalizing matrix P.

Sol: The characteristic equation of A is $|A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -9-\lambda & 4 & 4 \\ -8 & 3-\lambda & 4 \\ -16 & 8 & 7-\lambda \end{vmatrix} = 0$$

$C_1 \to C_1 + C_2 + C_3$

$$\Rightarrow \begin{vmatrix} -1-\lambda & 4 & 4 \\ -1-\lambda & 3-\lambda & 4 \\ -1-\lambda & 8 & 7-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (-1-\lambda) \begin{vmatrix} 1 & 4 & 4 \\ 1 & 3-\lambda & 4 \\ 1 & 8 & 7-\lambda \end{vmatrix} = 0$$

$R_2 \to R_2 - R_1$
$R_3 \to R_3 - R_1$

$$\Rightarrow (1+\lambda) \begin{vmatrix} 1 & 4 & 4 \\ 0 & -1-\lambda & 0 \\ 0 & 4 & 3-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1+\lambda)(1+\lambda)(3-\lambda) = 0$$

$\Rightarrow \lambda = -1, -1, 3.$

$\therefore$ The characteristic roots of A are $-1, -1, 3$.

The eigen vectors X of A corresponding to the characteristic root -1 are given by

$$(A - (-1)I)X = 0$$

$$\Rightarrow \begin{bmatrix} -8 & 4 & 4 \\ -8 & 4 & 4 \\ -16 & 8 & 8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - R_1$

$R_2 \to R_3 - 2R_1$

$$\sim \begin{bmatrix} -8 & 4 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The rank of the coefficient matrix $= 1$

$\therefore$ the equations have $3-1=2$ LI solutions.

$\therefore$ we have

$$-8x_1 + 4x_2 + 4x_3 = 0$$

$$\Rightarrow -2x_1 + x_2 + x_3 = 0$$

Let $x_2 = k_1$ and $x_3 = k_2$; k_1, k_2 are arbitrary constants.

$$\therefore x_1 = \frac{k_1 + k_2}{2}$$

$$\therefore X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{k_1+k_2}{2} \\ k_1 \\ k_2 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{k_1}{2} + \frac{k_2}{2} \\ k_1 \\ k_2 \end{bmatrix}$$

$$= k_1\begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix} + k_2\begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}$$

$$X = k_1X_1 + k_2X_2$$

Here $X_1 = \begin{bmatrix} 1/2 \\ 1 \\ 0 \end{bmatrix}$ & $X_2 = \begin{bmatrix} 1/2 \\ 0 \\ 1 \end{bmatrix}$ are LI vectors of A corresponding to characteristic root -1.

$\therefore$ The geometric multiplicity of eigen value is equal to its algebraic multiplicity

Now the eigen vectors x of A corresponding to the eigen value 3 are given by $(A-3I)x=0$

$$\Rightarrow \begin{bmatrix} -12 & 4 & 4 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \to R_1 - R_3$

$$\sim \begin{bmatrix} 4 & -4 & 0 \\ -8 & 0 & 4 \\ -16 & 8 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 + 2R_1$

$R_3 \to R_3 + 4R_1$

$$\sim \begin{bmatrix} 4 & -4 & 0 \\ 0 & -8 & 4 \\ 0 & -8 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\sim \begin{bmatrix} 4 & -4 & 0 \\ 0 & -8 & 4 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The rank of the coefficient matrix $=2$

$\therefore$ The equations have $3-2=1$ LI solution

$\therefore$ we have $4x_1 - 4x_2 = 0$

$-8x_2 + 4x_3 = 0$

$\Rightarrow x_1 = x_2$

and $2x_2 = x_3$.

Take $x_3 = 2$

then $x_2 = 1$

and $x_1 = 1$

$\therefore X_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ is an eigen vector of A corresponding to the eigen value 3.

$\therefore$ The geometric multiplicity of eigen value 3 is 1 and its algebraic multiplicity is also 1.

Since the geometric multiplicity of each eigen-value of A is equal to its algebraic multiplicity.

$\therefore$ A is similar to diagonal matrix.

$\therefore$ A is diagonalizable matrix.

Let $P = [x_1, x_2, x_3]$

$$= \begin{bmatrix} 1/2 & 1/2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

The coloumns of P are LI eigen vectors of A corresponding to the eigen values $-1, -1, 3$ respectively.

The matrix P will transform A to diagonal form D is given by the relation $P^{-1}AP = D$.

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

The transforming matrix $P = \begin{bmatrix} 1/2 & 1/2 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}$

and diagonal matrix $D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

H.W. $\rightarrow$ Show that the matrix $A = \begin{bmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{bmatrix}$ is similar to a diagonal matrix. Also find the transforming matrix and diagonal matrix.

$\rightarrow$ Show that the following matrices are not similar to diagonal matrices:

(i) $\begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$ (ii) $A = \begin{bmatrix} 2 & -1 & 1 \\ 2 & 2 & -1 \\ 1 & 2 & -1 \end{bmatrix}$

Sol^n: (i) $A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

The characteristic equation of A is $|A-\lambda I|=0$

$$\begin{vmatrix} 2-\lambda & 3 & 4 \\ 0 & 2-\lambda & -1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$\Rightarrow (2-\lambda)(2-\lambda)(1-\lambda)=0$

$\Rightarrow \lambda = 2, 2, 1$

$\therefore$ The characteristic roots of A are 2, 2, 1.

The eigen vectors x of A corresponding to the eigen value 2 are given by $(A-2I)x=0$

$$\Rightarrow \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 3 & 4 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_3 \to R_3 - R_2$$

The rank of coefficient matrix = 2

$\therefore$ The equations have $3-2=1$ LI solution corresponding to eigen value 2.

The geometric multiplicity of 2 is 1. while its algebraic multiplicity is 2.

Since the geometric multiplicity of this eigen value is not equal to its algebraic multiplicity.

$\therefore$ A is not similar to a diagonal matrix.

Orthogonal vectors:

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Inner product space:

Let $V(F)$ be a vector space, where F is the field of real numbers or the field of complex numbers. An inner product on V is a function

$f : V \times V \rightarrow F$ such that

(i) $f(\alpha, \beta) = \overline{f(\beta, \alpha)}$ where $\overline{f(\beta,\alpha)}$ is the conjugate of the complex number $f(\beta, \alpha)$.

(ii) $f(\alpha, \alpha) > 0$ for $\alpha \neq \bar{0}$ and $f(\alpha, \alpha) = 0$ for $\alpha = \bar{0}$

(iii) $f(a\alpha + b\beta, \gamma) = a f(\alpha, \gamma) + b f(\beta, \gamma)$

where $\alpha, \beta, \gamma, \bar{0} \in V$

and $a, b, 0 \in F$.

The vector space $V(F)$, in which an inner product 'f' defined as above, is called inner product space and is denoted by (V, f).

Note: → In practice $f(\alpha, \beta)$ where $\alpha, \beta \in V$ is denoted by (α, β) or $\langle \alpha, \beta \rangle$ or (α/β)

Here after we use (α, β) for $f(\alpha, \beta)$

→ $f(\alpha, \beta) = (\alpha, \beta)$, the above three conditions of the inner product are written as follows.

(i) $(\alpha, \beta) = \overline{(\beta, \alpha)}$

(ii) $(\alpha, \alpha) > 0$ for $\alpha \neq \bar{0}$ and $(\alpha, \alpha) = 0$ for $\alpha = \bar{0}$

(iii) $(a\alpha + b\beta, \gamma) = a(\alpha, \gamma) + b(\beta, \gamma)$

→ If $V(F)$ is an inner product space and F is the field of real numbers then $V(F)$ is called Euclidean space.

→ If $V(F)$ is an inner product space and F is the field of complex numbers then $V(F)$ is called unitary space.

Some important observations :

(1) For $0 \in F$ and $\bar{0} \in V \Rightarrow (\bar{0}, \bar{0}) = 0$

(2) for $a \in F$ and $\alpha, \gamma \in V$
$$\Rightarrow (a\alpha, \gamma) = a(\alpha, \gamma)$$

(3) for $\alpha, \beta, \gamma \in V \Rightarrow (\alpha + \beta, \gamma) = (\alpha, \gamma) + (\beta, \gamma)$

(4) for $\bar{0}, \beta \in V \Rightarrow (\bar{0}, \beta) = (0\alpha, \beta) = 0(\alpha, \beta) = 0$.

problems:

→ If $\alpha = (a_1, a_2, a_3)$, $\beta = (b_1, b_2, b_3)$ are elements of a vector space $\mathbb{R}^3$. then prove that $(\alpha, \beta) = a_1b_1 + a_2b_2 + a_3b_3$ defines an inner product space.

Soln: Let $\alpha = (a_1, a_2, a_3)$, $\beta = (b_1, b_2, b_3)$ and $\gamma = (c_1, c_2, c_3) \in \mathbb{R}^3$. where $a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 \in \mathbb{R}$

(i) $(\alpha, \beta) = a_1b_1 + a_2b_2 + a_3b_3$
$$= b_1a_1 + b_2a_2 + b_3a_3$$
$$= (\beta, \alpha) = \overline{(\beta, \alpha)}$$

(ii) $(\alpha, \alpha) = a_1a_1 + a_2a_2 + a_3a_3$
$$= a_1^2 + a_2^2 + a_3^2$$

If $\alpha = (a_1, a_2, a_3) \neq (0, 0, 0)$
then atleast one of a_1, a_2, a_3 is not zero.
$$\therefore (\alpha, \alpha) = a_1^2 + a_2^2 + a_3^2 > 0$$

If $\alpha = (a_1, a_2, a_3) = (0, 0, 0)$
then $a_1 = a_2 = a_3 = 0$
$$\therefore (\alpha, \alpha) = a_1^2 + a_2^2 + a_3^2 = 0.$$

(iii) for $a, b \in \mathbb{R}$ and $\alpha, \beta, \gamma \in \mathbb{R}^3$.
$$\Rightarrow a\alpha + b\beta = a(a_1, a_2, a_3) + b(b_1, b_2, b_3)$$
$$= (aa_1, aa_2, aa_3) + (bb_1, bb_2, bb_3)$$
$$= (aa_1 + bb_1, aa_2 + bb_2, aa_3 + bb_3)$$
$$\therefore (a\alpha + b\beta, \gamma) = (aa_1 + bb_1)c_1 + (aa_2 + bb_2)c_2 + (aa_3 + bb_3)c_3.$$

$= aa_1c_1 + bb_1c_1 + aa_2c_2 + bb_2c_2 + aa_3c_3 + bb_3c_3$.

$= (aa_1c_1 + aa_2c_2 + aa_3c_3) + (bb_1c_1 + bb_2c_2 + bb_3c_3)$

$= a(a_1c_1 + a_2c_2 + a_3c_3) + b(b_1c_1 + b_2c_2 + b_3c_3)$

$= a(\alpha, r) + b(\beta, r)$

$\therefore$ The product $(\alpha, \beta) = a_1b_1 + a_2b_2 + a_3b_3$ is an inner product on $\mathbb{R}^3$.

$\therefore$ $\mathbb{R}^3$ is an innerproduct space with the above product.

Note 1. The inner product of α & β is $(\alpha, \beta) = a_1b_1 + a_2b_2 + a_3b_3$ is called the dot product of α & β and is denoted by $\alpha\cdot\beta$.
This is called the standard inner product space in $\mathbb{R}^3$.

2. If $\alpha = (a_1, a_2, \ldots a_n)$, $\beta = (b_1, b_2, \ldots b_n)$ are two vectors of the vector space $V_n(\mathbb{R})$ then $(\alpha, \beta) = a_1b_1 + a_2b_2 + \ldots\ldots + a_nb_n$. is called the standard inner product

3. If $\alpha = (a_1, a_2, \ldots a_n)$ and $\beta = (b_1, b_2 \ldots b_n)$ are the elements of the vector space $V_n(C)$ where C is the field of complex numbers then $(\alpha, \beta) = a_1\bar{b}_1 + a_2\bar{b}_2 + \ldots\ldots + a_n\bar{b}_n$ defines an inner product on $V_n(C)$.
It is called standard inner product on $V_n(C)$.

→ **Inner product of two vectors:**

Let X and Y be two complex n-vectors such that $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ & $Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$

Then the inner product of X & Y denoted by (X, Y) is defined as

$$(X, Y) = \bar{x}_1 y_1 + \bar{x}_2 y_2 + \cdots + \bar{x}_n y_n$$
$$= X^\theta Y$$
$$= \begin{bmatrix} \bar{x}_1, & \bar{x}_2, & \cdots & \bar{x}_n \end{bmatrix}_{1\times n} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}_{n\times 1}$$
$$= \bar{x}_1 y_1 + \bar{x}_2 y_2 + \cdots + \bar{x}_n y_n.$$

→ If X and Y are real n-vectors written as coloumn vectors then their inner-product is defined as

$$(X, Y) = X^T Y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n.$$

→ If X and Y are complex n-vectors written as row vectors then

$$(X, Y) = X Y^\theta.$$
$$= x_1 \bar{y}_1 + x_2 \bar{y}_2 + \cdots + x_n \bar{y}_n.$$

Norm or length of a vector:

Let X be a complex n-vector. The norm (or length) of X denoted by $\|X\|$ is defined as the +ve square root of (X, X).

i.e, Norm or length of $X = \|X\|$

$$= \sqrt{(X, X)}$$
$$= \sqrt{X^\theta X}$$
$$= \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2}$$

where $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

Unit vector:

If $\|x\|=1$ then x is called a unit vector and is said to be normalized.

- A unit vector is sometimes also called a normal vector.

Ex: Normalize $x = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$

$$\|x\| = \sqrt{(x,x)} = \sqrt{x^{\theta}x} = \sqrt{x^T x} = \sqrt{6} \qquad \left(\because [2\ 1\ -1]\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = 6\right)$$

$$\hat{x} = \frac{x}{\|x\|} = \frac{1}{\sqrt{6}}x = \frac{1}{\sqrt{6}}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}$$

which is the required unit vector.

Orthogonal vectors:

Let X and Y be two complex n-vectors then X is said to be orthogonal to Y if $(X,Y)=0$

i.e., $X^{\theta}Y=0$

Orthogonal set: A set S of complex n-vectors $x_1, x_2, \ldots x_k$ is said to be an orthogonal set if any two distinct vectors in S are orthogonal.

Orthonormal set: A set S of complex n-vectors $x_1, x_2, \ldots x_k$ is said to be an orthonormal set if (i) each vector in S is a unit vector.
(ii) any two distinct vectors in S are orthogonal.

Orthogonally Similar matrices:

Let A and B be square matrices of order 'n'. Then B is said to be orthogonally similar to A if $\exists$ an orthogonal matrix P such that

$$B = P^{-1}AP.$$

- If A and B are orthogonally similar, then they are similar also.

→ Every real symmetric matrix is orthogonally similar to a diagonal matrix with real elements.

→ A real symmetric matrix of order 'n' has n mutually orthogonal real eigen vectors.

→ Any two eigen vectors corresponding to two distinct eigen values of real symmetric matrix are orthogonal.

Sol^n: Let x_1, x_2 be two eigen vectors corresponding to two distinct eigen values λ_1, λ_2 of real symmetric matrix A.

Then $AX_1 = \lambda_1 X_1$ —(1) & $Ax_2 = \lambda_2 x_2$ —(2)

Here λ_1, λ_2 are real and x_1, x_2 are real vectors.

$$\begin{aligned}
\text{Now } \lambda_1 X_2^T X_1 &= X_2^T(\lambda_1 X_1) \\
&= X_2^T(AX_1) \quad (\text{by } 1) \\
&= (X_2^T A)X_1 \\
&= (X_2^T A^T)X_1 \quad (\because A^T = A) \\
&= (AX_2)^T X_1 \\
&= (\lambda_2 X_2)^T X_1 \quad (\text{by } 2) \\
&= \lambda_2 X_2^T X_1
\end{aligned}$$

$$(\lambda_1 - \lambda_2)x_2^T X_1 = 0.$$

$\Rightarrow x_2^T x_1 = 0$ ($\because \lambda_1$ & λ_2 are distinct $\Rightarrow \lambda_1 - \lambda_2 \neq 0$)

$\therefore x_1$ & x_2 are orthogonal

→ If λ occurs exactly 'p' times as an eigen-value of a real symmetric matrix A then A has p but not more than p mutually orthogonal real eigen vectors corresponding to λ.

working rule for orthogonal reduction of a real symmetric matrix:

- Suppose A is a real symmetric matrix.
- first we find the characteristic roots of A.

If λ is characteristic root of A having p as its algebraic multiplicity then we shall able to find an orthonormal set of p characteristic vectors of A corresponding to this characteristic root.

we should repeat this process for each characteristic root of A.

- Since the characteristic vectors corresponding to two distinct characteristic roots of a real symmetric matrix are mutually orthogonal.

Therefore, the n characteristic vectors found in this manner constitute an orthonormal set.

- The matrix P, having as its coloumns the members of the orthonormal set obtained above, is orthogonal and is such that $P^{-1}AP = D$ (Diagonal matrix)

Problems

→ Find an orthogonal matrix that will diagonalize the real symmetric matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$. Also write the resulting diagonal matrix.

Solⁿ: Given real symmetric matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

The characteristic equation of A is

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 4-\lambda & 6 \\ 3 & 6 & 9-\lambda \end{vmatrix} = 0$$

$C_1 \to C_1 - 2C_2 + C_3$

$$\Rightarrow \begin{vmatrix} -\lambda & 2 & 3 \\ 2\lambda & 4-\lambda & 6 \\ -\lambda & 6 & 9-\lambda \end{vmatrix} = 0$$

$$\Rightarrow \lambda \begin{vmatrix} -1 & 2 & 3 \\ 2 & 4-\lambda & 6 \\ -1 & 6 & 9-\lambda \end{vmatrix} = 0$$

$R_2 \to R_2 + 2R_1$

$R_3 \to R_3 - R_1$

$$\Rightarrow \lambda \begin{vmatrix} -1 & 2 & 3 \\ 0 & 8-\lambda & 12 \\ 0 & 4 & 6-\lambda \end{vmatrix} = 0$$

$$\Rightarrow -\lambda[(8-\lambda)(6-\lambda) - 48] = 0$$

$$\Rightarrow \lambda(\lambda^2 - 14\lambda) = 0$$

$$\Rightarrow \lambda^2(\lambda - 14) = 0$$

$$\Rightarrow \lambda = 0, 0, 14.$$

∴ The characteristic roots of A are 0, 0, 14.

Now the characteristic root '0' is of algebraic multiplicity 2.

∴ there will be two LI eigen vectors corresponding to characteristic root

(17)

The characteristic vectors x corresponding to this eigen value is given by

$(A - 0I)x = 0$

$\Rightarrow AX = 0$

$$\Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - 3R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x_1 + 2x_2 + 3x_3 = 0.$ —— (1)

let $x_1 = 0$, $x_2 = 3$, $x_3 = -2$ then

$X_1 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$ is the characteristic vector of A. —— (2)

let $X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ be another characteristic vector of A orthogonal to (2) then

$3x_2 - 2x_3 = 0$ —— (3)

solving (1) & (3), we get

(3) $\equiv 3x_2 = 2x_3 \Rightarrow \boxed{x_2 = \frac{2}{3}x_3}$

if $x_3 = 1$ then $\boxed{x_2 = 2/3}$ and

(1) $\equiv x_1 + 2(2/3) + 3(1) = 0$

$\Rightarrow x_1 + 4/3 + 3 = 0$

$\Rightarrow x_1 + \frac{13}{3} = 0 \Rightarrow \boxed{x_1 = -\frac{13}{3}.}$

$$\therefore X_2 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -13/3 \\ 2/3. \\ 1 \end{bmatrix} \text{—— (4)}$$

$\therefore X_1 = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$ and $X_2 = \begin{bmatrix} -13/3 \\ 2/3 \\ 1 \end{bmatrix}$ are L.I.

orthogonal eigen vectors corresponding to characteristic root '0'.

The algebraic multiplicity of characteristic root 14 is 1.

$\therefore$ There will be only one L.I. characteristic vector corr. to characteristic root.

$\therefore$ The characteristic vector X corresponding to this characteristic root is given by $(A-14I)X=0$

$$\Rightarrow \begin{bmatrix} -13 & 2 & 3 \\ 2 & -10 & 6 \\ 3 & 6 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_1 \leftrightarrow R_2.$$

$$\sim \begin{bmatrix} 2 & -10 & 6 \\ -13 & 2 & 3 \\ 3 & 6 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_1 \rightarrow \tfrac{1}{2} R_1$$

$$\sim \begin{bmatrix} 1 & -5 & 3 \\ -13 & 2 & 3 \\ 3 & 6 & -5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{array}{l} R_2 \rightarrow R_2 + 13R_1 \\ R_3 \rightarrow R_3 - 3R_1 \end{array}$$

$$\sim \begin{bmatrix} 1 & -5 & 3 \\ 0 & -63 & 42 \\ 0 & 21 & -14 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \begin{array}{l} R_2 \rightarrow \frac{1}{21} R_2 \\ R_3 \rightarrow \frac{1}{7} R_3 \end{array}$$

$$\sim \begin{bmatrix} 1 & -5 & 3 \\ 0 & -3 & 2 \\ 0 & 3 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad R_3 \rightarrow R_3 + R_2.$$

$$\sim \begin{bmatrix} 1 & -5 & 3 \\ 0 & -3 & 2 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_1 - 5x_2 + 3x_3 = 0$$
$$-3x_2 + 2x_3 = 0$$

Let $x_2 = 2$ and $x_3 = 3$ then $x_1 = 1$.

$\therefore x_3 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is LI characteristic vector corresponding to characteristic root 14.

$\therefore x_1, x_2, x_3$ are required characteristic vectors corresponding to 0, 0, 14.

Since the characteristic vector corresponding to two distinct characteristic roots of ~~the~~ real symmetric matrix are mutually orthogonal.

Now let us normalize the vectors x_1, x_2, x_3 i.e let us find the unit vectors $\hat{x}_1, \hat{x}_2, \hat{x}_3$ which are scalar multiples of x_1, x_2, x_3 respectively.

~~for this~~ $\|x_1\| = \sqrt{9+4} = \sqrt{13}$.

$\|x_2\| = \sqrt{\frac{169}{9} + \frac{4}{9} + 1} = \frac{\sqrt{182}}{3}$.

and $\|x_3\| = \sqrt{1+4+9} = \sqrt{14}$

$\therefore \hat{x}_1 = \frac{x_1}{\|x_1\|}, \quad \hat{x}_2 = \frac{x_2}{\|x_2\|}, \quad x_3 = \frac{x_3}{\|x_3\|}$

$= \frac{1}{\sqrt{13}}\begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix} \quad = \frac{3}{\sqrt{182}}\begin{bmatrix} -13/3 \\ 2/3 \\ 1 \end{bmatrix}; \quad = \frac{1}{\sqrt{14}}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

Let $P = \begin{bmatrix} \hat{x}_1 & \hat{x}_2 & \hat{x}_3 \end{bmatrix} = \begin{bmatrix} 0 & \frac{-13}{\sqrt{182}} & \frac{1}{\sqrt{14}} \\ \frac{3}{\sqrt{13}} & \frac{2}{\sqrt{182}} & \frac{2}{\sqrt{14}} \\ \frac{-2}{\sqrt{13}} & \frac{3}{\sqrt{182}} & \frac{3}{\sqrt{14}} \end{bmatrix}$

which is the required orthogonal matrix.

$\therefore P^T P = I$.

and P is non-singular.

P^{-1} exists.

$P^T = P^{-1}$.

$$\therefore P^{-1} = \begin{bmatrix} 0 & \frac{3}{\sqrt{13}} & \frac{-2}{\sqrt{13}} \\ \frac{-13}{\sqrt{182}} & \frac{2}{\sqrt{182}} & \frac{3}{\sqrt{182}} \\ \frac{1}{\sqrt{14}} & \frac{2}{\sqrt{14}} & \frac{3}{\sqrt{14}} \end{bmatrix}$$

$= P^T$

since the real symmetric matrix is orthogonally similar to a diagonal matrix with real elements.

$$\therefore P^{-1} A P = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 14 \end{bmatrix} = D.$$

→ Determine diagonal matrices orthogonally similar to the following real symmetric matrices, obtaining also the transforming matrices.

(i) $A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{bmatrix}$

2003 (ii) $A = \begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$

2001 (iii) $A = \begin{bmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ -4 & -1 & -8 \end{bmatrix}$

(iv) $A = \begin{bmatrix} 7 & 0 & -2 \\ 0 & 5 & -2 \\ -2 & -2 & 6 \end{bmatrix}$.

2007 → Let A be a real 3×3 Symmetric matrix with eigen values 0,0,5. If the corresponding eigen vectors are (2,0,1), (2,1,1) & (1,0,−2) find the matrix A. (19)

Sol Given that the matrix A is a real 3×3 symmetric matrix with eigen values 0,0,5 and the corresponding eigen vectors (2,0,1), (2,1,1), (1,0,−2).
clearly these vectors are L.I. vectors.

Let $P = \begin{bmatrix} 2 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & -2 \end{bmatrix}$

The columns of P are LI vectors of A corresponding to the characteristic roots 0,0,5 respectively.

∴ The matrix P will transform A to diagonal form D. is given by the relation $P^{-1}AP = D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.

$\Rightarrow A = PDP^{-1}$ ⟵ (1).

let us find the inverse of P:

$P^{-1} = \frac{\operatorname{adj} P}{|P|} = -\frac{1}{5}\begin{bmatrix} -2 & 5 & -1 \\ 0 & -5 & 0 \\ -1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 2/5 & 1 & 1/5 \\ 0 & 1 & 0 \\ 1/5 & 0 & -2/5 \end{bmatrix}$

∴ (1) ≡ $A = \begin{bmatrix} 2 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -2 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} 2/5 & 1 & 1/5 \\ 0 & 1 & 0 \\ 1/5 & 0 & -2/5 \end{bmatrix}$

$= \begin{bmatrix} 2 & 2 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -2 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & -2 \end{bmatrix}$

$= \begin{bmatrix} 1 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 4 \end{bmatrix}$ which is the required 3×3 symmetric matrix.

* Unitarily Similar matrices:

Defn: Let A & B be two square matrices of order 'n'. Then B is said to be unitarily similar to A if $\exists$ a unitary matrix P such that $B = P^{-1}AP$.

→ If A and B are unitarily similar, then they are similar also.

→ Every Hermitian matrix is unitarily similar to a diagonal matrix.

→ An $n \times n$ Hermitian matrix H has 'n' mutually orthogonal eigen vectors in the complex vectorspace V_n.

→ Any two eigen vectors corresponding to two distinct eigen values of a hermitian matrix are orthogonal.

→ If λ occurs exactly p times as an eigen value of Hermitian matrix A then A has p but not more than p mutually eigen vectors corresponding to λ.

Problem:

Determine the diagonal matrix unitarily similar to the hermitian matrix $A=\begin{bmatrix} 2 & 1-2i \\ 1+2i & -2 \end{bmatrix}$, obtaining also the transformation matrix.

Sol^n: Given Hermitian matrix $A=\begin{bmatrix} 2 & 1-2i \\ 1+2i & -2 \end{bmatrix}$

Now the characteristic equation of A is $|A-\lambda I|=0$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1-2i \\ 1+2i & -2-\lambda \end{vmatrix}=0$$

$$\Rightarrow \quad \lambda^2-9=0$$

$$\Rightarrow \quad \lambda=-3,3.$$

$\therefore$ The characteristic roots of A are $-3, 3$.

The algebraic multiplicity of characteristic root -3 is 1.

$\therefore$ There will be one LI eigen vector.

The characteristic vector X corresponding to this eigen value is $(A-(-3)I)X=0$

$$\Rightarrow \begin{bmatrix} 5 & 1-2i \\ 1+2i & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$\Rightarrow 5x + (1-2i)y = 0$ — ①

$(1+2i)x + y = 0$ — ②

$\Rightarrow x = 1-2i;\ y = -5$

$\therefore X_1 = \begin{bmatrix} 1-2i \\ -5 \end{bmatrix}$

Similarly corresponding to $\lambda = 3$ the eigen vector

$X_2 = \begin{bmatrix} 5 \\ 1+2i \end{bmatrix}$

$\therefore X_1$ and X_2 are characteristic vectors corresponding to characteristic values $-3, 3$.

Since the characteristic vectors corresponding to the two distinct characteristic roots of the Hermitian matrix are mutually orthogonal.

<u>Now</u> <u>let us normalize the vectors X_1 & X_2:</u>

For this.

$\|X_1\| = \sqrt{|1-2i|^2 + |-5|^2} = \sqrt{5+25} = \sqrt{30}$

and $\|X_2\| = \sqrt{|5|^2 + |1+2i|^2} = \sqrt{25+5} = \sqrt{30}$.

$\therefore \hat{X}_1 = \dfrac{X_1}{\|X_1\|} = \dfrac{1}{\sqrt{30}}\begin{bmatrix} 1-2i \\ -5 \end{bmatrix} = \begin{bmatrix} \frac{1-2i}{\sqrt{30}} \\ -5/\sqrt{30} \end{bmatrix}$

$\hat{X}_2 = \dfrac{X_2}{\|X_2\|} = \dfrac{1}{\sqrt{30}}\begin{bmatrix} 5 \\ 1+2i \end{bmatrix} = \begin{bmatrix} 5/\sqrt{30} \\ \frac{1+2i}{\sqrt{30}} \end{bmatrix}$

Let $P = \left[\hat{X}_1\ \ \hat{X}_2\right]$

$= \begin{bmatrix} \frac{1-2i}{\sqrt{30}} & 5/\sqrt{30} \\ -5/\sqrt{30} & \frac{1+2i}{\sqrt{30}} \end{bmatrix} = \dfrac{1}{\sqrt{30}}\begin{bmatrix} 1-2i & 5 \\ -5 & 1+2i \end{bmatrix}$

which is the required unitary matrix.

Since P is unitary matrix.

$\therefore P^{\theta}P = I$.

and P is non-singular.

$P^{\theta} = P^{-1}$

$P^{-1} = P^{\theta} = \dfrac{1}{\sqrt{30}}\begin{bmatrix} 1+2i & -5 \\ 5 & 1-2i \end{bmatrix}$

Since every Hermitian matrix is unitarily similar to a diagonal matrix.

$$\therefore P^{-1}AP = \begin{bmatrix} -3 & 0 \\ 0 & 3 \end{bmatrix} = \text{dia}(-3,3) = D$$

$\therefore$ D is unitarily similar to A where P is transforming matrix.

H.W. → Find a unitary matrix that will diagonalize the Hermitian matrix $\begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$

IFoS 2014 → Examine whether the matrix $A = \begin{bmatrix} -2 & 2 & -3 \\ 2 & 1 & -6 \\ -1 & -2 & 0 \end{bmatrix}$ is diagonalizable.

Find all eigen values. Then obtain a matrix P s.t $P^{-1}AP$ is diagonal.

2013 ~~IAS~~ IFoS → let $H = \begin{bmatrix} 1 & 1 & 2+i \\ -i & 2 & 1-i \\ 2-i & 1+i & 2 \end{bmatrix}$ be a Hermitian matrix.

IAS 2012 Find a non-singular matrix P ~~s.t~~ s.t $P^T H \bar{P}$ is diagonal.

2011 ~~IAS~~ IFoS → ~~Find~~ let $A = \begin{bmatrix} 1 & -3 & 3 \\ 0 & -5 & 6 \\ 0 & -3 & 4 \end{bmatrix}$. Find an invertible matrix P ~~s.t~~ s.t that $P^{-1}AP$ is a diagonal matrix

2010 ~~IAS~~ IFoS → let $A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$. Find an invertible matrix P s.t $P^{-1}AP$ is a diagonal matrix

2009 ~~IAS~~ IFoS Let $A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix}$.

Is A similar to a diagonal matrix? If so, find an invertible matrix P s.t $P^{-1}AP$ is a diagonal matrix

2007 ~~IAS~~ IFoS Is the matrix $A = \begin{bmatrix} 6 & -3 & 2 \\ 4 & -1 & -2 \\ 10 & -5 & -3 \end{bmatrix}$ similar over the field $\mathbb{R}$ to a diagonal matrix? Is A similar over the field $\mathbb{C}$ to a diagonal matrix?