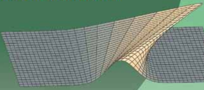
18TH EDITION

ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS

Dr. M. D. RAISINGHANIA



S. CHAND

ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS

[For BA, B.Sc. and Honours (Mathematics and Physics), M.A., M.Sc. (Mathematics and Physics), B.E. Students of Various Universities and for I.A.S., P.C.S., A.M.I.E. GATE, C.S.I.R. U.G.C. NET and Various Competitive Examinations]

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PREFACE TO THE FIFTEENTH EDITION

Questions asked in recent papers of GATE and various university examinations have been inserted at appropriate places. This enriched inclusion of solved examples and variety of new exercises at the end of each article and chapter makes this book more useful to the reader. While revising this book I have been guided by following simple teaching philosophy : "*An ideal text book should teach the students to solve all types of problems*".

Any suggestion, remarks and constructive comments for the improvement of this book are always welcome.

AUTHOR

PREFACE TO THE SIXTH EDITION

It gives me great pleasure to inform the reader that the present edition of the book has been improved, well-organised, enlarged and made up-to-date in the light of latest syllabi. The following major changes have been made in the present edition:

- Almost all the chapters have been rewritten so that in the present form, the reader will not find any difficulty in understanding the subject matter.
- The matter of the previous edition has been re-organised so that now each topic gets its proper place in the book.
- More solved examples have been added so that the reader may gain confidence in the techniques of solving problems.
- References to the latest papers of various universities and I.A.S. examination have been made at proper places.
- Errors and omissions of the previous edition have been corrected.

In view of the above mentioned features it is expected that this new edition will prove more useful to the reader.

I am extremely thankful to the Managing Director, Shri Rajendra Kumar Gupta and the Director, Shri Ravindra Kumar Gupta for showing keen interest throughout the publication of the book.

Suggestions for further improvement of the book will be gratefully received.

AUTHOR

PREFACE TO THE FIRST EDITION

This book has been designed for the use of honours and postgraduate students of various Indian universities. It will also be found useful by the students preparing for various competitive examinations. During my long teaching experience I have fully understood the need of the students and hence I have taken great care to present the subject matter in the most clear, interesting and complete form from the student's point of view.

Do not start this book with an unreasonable fear. There are no mysteries in Mathematics. It is all simple and honest reasoning explained step by step which anybody can follow with a little effort and concentration. Often a student has difficulty in following a mathematical explanation only because the author skips steps which he assumes the students to be familiar with. If the student fails to recount the missing steps, he may be faced with a gap in the reasoning and the author's conclusion may become mysterious to him. I have avoided such gaps by giving necessary references throughout the book. I have been influenced by the following wise-saying.

"My passion is for lucidity. I don't mean simple mindedness. If people can't understand it, why write it."

AUTHOR

Dedicated to the
momory of my Parents

PART-I

ELEMENTARY DIFFERENTIAL EQUATIONS

PART-II

ADVANCED ORDINARY DIFFERNTIAL EQUATIONS AND SPECIAL FUNCTIONS

PART-III

PARTIAL DIFFERENTIAL EQUATIONS

CHAPTERS

LIST OF SOME USEFUL RESULTS FOR DIRECT APPLICATIONS

I Table of elementary integrals

$\int \sin x\, dx = -\cos x$

$\int \sinh x\, dx = \cosh x$

$\int \cos x\, dx = \sin x$

$\int \cosh x\, dx = \sinh x$

$\int \sec^2 x\, dx = \tan x$

$\int \operatorname{sech}^2 x\, dx = \tanh x$

$\int \operatorname{cosec}^2 x\, dx = -\cot x$

$\int \operatorname{cosech}^2 x\, dx = -\coth x$

$\int \sec x \tan x\, dx = \sec x$

$\int \operatorname{sech} x \tanh x\, dx = -\operatorname{sech} x$

$\int \operatorname{cosec} x \cot x\, dx = -\operatorname{cosec} x$

$\int \operatorname{cosech} x \coth x\, dx = -\operatorname{cosech} x$

$\int e^x\, dx = e^x$

$\int a^x\, dx = (a^x)/\log_e a$

$\int x^n\, dx = \dfrac{x^{n+1}}{n+1},\ n \neq -1$

$\int \{f(x)\}^n f'(x) dx = \dfrac{\{f(x)\}^{n+1}}{n+1},\ n \neq -1$

$\int \dfrac{1}{x}\, dx = \log x$

$\int \dfrac{f'(x)}{f(x)}\, dx = \log x$

$\int \tan x\, dx = \log \sec x = -\log \cos x$

$\int \cot x\, dx = \log \sin x$

$\int \operatorname{cosec} x\, dx = \log \tan(x/2) = \log(\operatorname{cosec} x - \cot x)$

$\int \sec x\, dx = \log \tan(\pi/4 + x/2) = \log(\sec x + \tan x)$

$\int \dfrac{dx}{x^2 + a^2} = \dfrac{1}{a}\tan^{-1}\dfrac{x}{a};\quad \int \dfrac{dx}{x^2 - a^2} = \dfrac{1}{2a}\log\dfrac{x-a}{x+a},\ x > a;\quad \int \dfrac{dx}{a^2 - x^2} = \dfrac{1}{2a}\log\dfrac{a+x}{a-x},\ x < a$

$\int \dfrac{dx}{x(x^2 - a^2)^{1/2}} = \dfrac{1}{a}\sec^{-1}\dfrac{x}{a}$ or $-\dfrac{1}{a}\operatorname{cosec}^{-1}\dfrac{x}{a}$

$\int \dfrac{dx}{(a^2 - x^2)^{1/2}} = \sin^{-1}\dfrac{x}{a}$ or $-\cos^{-1}\dfrac{x}{a}$

$\int \dfrac{dx}{(a^2 + x^2)^{1/2}} = \sinh^{-1}\dfrac{x}{a}$ or $\log\{x + (x^2 + a^2)^{1/2}\}$

$\int \dfrac{dx}{(x^2 - a^2)^{1/2}} = \cosh^{-1}\dfrac{x}{a}$ or $\log\{x + (x^2 - a^2)^{1/2}\}$

$$\int (a^2 - x^2)^{1/2} dx = (x/2) \times (a^2 - x^2)^{1/2} + (a^2/2) \times \sin^{-1}(x/a)$$

$$\int (a^2 + x^2)^{1/2} dx = (x/2) \times (a^2 + x^2)^{1/2} + (a^2/2) \times \log\{x + (a^2 + x^2)^{1/2}\}$$

$$\int (x^2 - a^2)^{1/2} dx = (x/2) \times (x^2 - a^2)^{1/2} - (a^2/2) \times \log\{x + (x^2 - a^2)^{1/2}\}$$

Note. If x is replaced by $ax + b$ (a and b being constants) on both sides of any formula of the above table, then the standard form remains true, provided the result on R.H.S. is divided by a, the coefficient of x. For examples,

$$\int \cos(ax+b)dx = \frac{\sin(ax+b)}{a}; \qquad \int e^{ax+b} dx = \frac{e^{ax+b}}{a}$$

II. $\displaystyle\int e^{ax} \sin bx\, dx = \frac{e^{ax}(a \sin bx - b \cos bx)}{a^2 + b^2}$; $\qquad \displaystyle\int_0^\infty e^{-ax} \sin bx\, dx = \frac{b}{a^2 + b^2}$

$$\int e^{ax} \cos bx\, dx = \frac{e^{ax}(a \cos bx + b \sin bx)}{a^2 + b^2}; \qquad \int_0^\infty e^{-ax} \cos bx\, dx = \frac{a}{a^2 + b^2}$$

$$\int e^{ax} \sin(bx + c)dx = \frac{e^{ax}}{a^2 + b^2}\{a \sin(bx + c) - b \cos(bx + c)\} = \frac{e^{ax}}{\sqrt{a^2 + b^2}} \sin\left(bx + c - \tan^{-1}\frac{b}{a}\right)$$

$$\int e^{ax} \cos(bx + c)dx = \frac{e^{ax}}{a^2 + b^2}\{a \cos(bx + c) + b \cos(bx + c)\} = \frac{e^{ax}}{\sqrt{a^2 + b^2}} \cos\left(bx + c - \tan^{-1}\frac{b}{a}\right)$$

III. Integration by parts

$$\int f_1(x) f_2(x) dx = f_1(x)\left\{\int f_2(x)\, dx\right\} - \int\left[\left\{\frac{d}{dx} f_1(x)\right\} \times \left\{\int f_2(x)\, dx\right\}\right] dx$$

In words, this formula states

The integral of the product of two functions

= Ist function × integral of 2nd – integral of (diff. coeff. of 1st × integral of 2nd)

The success of this method depends upon the choosing the first and second functions in such a way that the second term on the R.H.S. is easily integrable.

Note. While choosing the first and second function, note carefully the following facts:

(*i*) The second function must be chosen in such a way that its integral is known

(*ii*) If the integrals of both the functions in the product to be integrated are known, then the second function must be chosen in much a way that the new integral on the R.H.S. should be integrable directly or it should the simpler than the original integral.

(*iii*) If the integrals of both the functions are known and if one them be of the form x^n or $a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n$ (where n is a positive integers and $a_0, a_1, ..., a_n$ are constants), then that function must be chosen as the first function. For example, in $\int x^3 e^x dx$, x^3 must be chosen as the first function.

(*iv*) If in the product of two functions the integral of one of the functions is not known, then that function must be taken as the first function. For example, in $\int x \tan^{-1} x\, dx$ and $\int x \log x\, dx$ etc we do not know the integrals of $\tan^{-1} x$ and $\log x$ and hence we must choose $\tan^{-1}x$ and $\log x$ etc as first function.

(*v*) Sometimes we are to evaluate the integral of a single function by the method of integration by parts. In such cases, unity (*i.e.*, 1) must be taken as the second function. For example, to find

$\int \tan^{-1} x\, dx$, $\int \log x\, dx$ etc, we always take 1 as the function. Thus, we write

$$\int \tan^{-1} x\, dx = \int (\tan^{-1} x)\cdot 1\, dx, \qquad \int \log x\, dx = \int (\log x)\cdot 1\, dx \text{ etc.}$$

(*vi*) The formula of integration by parts can be applied more than once, if necessary.

BERNOULLI'S FORMULA OR GENERALISED RULE OF INTEGRATION BY PARTS OR CHAIN RULE OF INTEGRATION BY PARTS.

Let u and v be two functions of x. Let dashes denote differentiation and suffixes integration with respect to x. Thus, we have

$$u' = \frac{du}{dx}, \quad u'' = \frac{du'}{dx^2} = \frac{d^2u}{dx^2}, \ldots, \quad v_1 = \int v\, dx, \quad v_2 = \int v_1\, dx = \iint v (dx)^2, \quad \text{and so on.}$$

Then
$$\int u v\, dx = u v_1 - u' v_2 + u'' v_3 - u''' v_4 + \ldots$$

The above rule is applied when u is of the form x^n or $a_0 x^n + a_1 x^{n-1} + \ldots + a_{n-1} x + a_n$ (where n is a positive integer) and v is a function of the forms e^{ax}, a^x, sin ax or cos ax.

While applying the above rule, simplification should be done only when the whole process of integration is over. Study the solutions of the following problems carefully.

Example 1. $\int x^4 e^x dx = (x^4)(e^x) - (4x^3)(e^x) + (12x^2)(e^x) - (24x)(e^x) + (24)(e^x)$

$= e^x (x^4 - 4x^3 + 12x^2 - 24x + 24)$

Example 2. $\int_0^{\pi} x^5 \sin x\, dx = \Big[(x^5)(-\cos x) - (5x^4)(-\sin x) + (20x^3)(\cos x)$

$-(60x^2)(\sin x) + (120x)(-\cos x) - (120)(-\sin x)\Big]_0^{\pi}$

$= \Big[(-x^5 + 20x^3 - 120x)\cos x + (5x^4 - 60x^2 + 120)\sin x\Big]_0^{\pi}$

$= (-\pi^5 + 20\pi^3 - 120\pi)\cos\pi + (-\pi^5 + 20\pi^3 - 120\pi)\times(-1) = \pi^5 - 20\pi^3 + 120\pi$

Some useful direct results based on integration by parts

$\int e^{ax}\{a f(x) + f'(x)\}dx = e^{ax} f(x)$. Its particulars case are

$\int e^x \{f(x) + f'(x)\}dx = e^x f(x)$; $\qquad \int e^{-x}\{-f(x) + f'(x)\}dx = e^{-x} f(x)$

IV Properties of definite integrals

(*i*) $\int_a^b f(x)dx = \int_a^b f(t)\, dt$ $\qquad$ (*ii*) $\int_a^b f(x)\, dx = -\int_b^a f(x)\, dx$

(*iii*) $\int_a^b f(x)\, dx = \int_a^b f(x) dx + \int_c^b f(x)\, dx$, where $a < c < b$

(*iv*) $\int_a^a f(x) dx = 2\int_0^a f(x)\, dx$, if $f(-x) = f(x)$; $\qquad \int_{-a}^a f(x)dx = 0$, if $f(-x) = -f(x)$

(*v*) $\int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx$, if $f(2a - x) = f(x)$; $\int_0^{2a} f(x)\,dx = 0$, if $f(2a - x) = -f(x)$

V Walli's formulas

(*i*) If n is an even positive integer, then

$$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

For example
$$\int_0^{\pi/2} \sin^{10} x\,dx = \frac{9}{10}\cdot\frac{7}{8}\cdot\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2}$$

Note carefully that answer is written down very easily by beginning with the denominator. We then have the ordinary sequence of natural numbers written down backwards. Thus, in the above example, we write (10 under 9) × (8 under 7) × (6 under 5) .... etc. stopping at (2 under 1), and writing a factor $\pi/2$ in the end.

(*ii*) If n is an odd positive integer, then

$$\int_0^{\pi/2} \sin^n x\,dx = \int_0^{\pi/2} \cos^n x\,dx = \frac{n-1}{n}\cdot\frac{n-3}{n-2}\cdots\frac{4}{5}\cdot\frac{2}{3}$$

For example.
$$\int_0^{\pi/2} \sin^{9} x\,dx = \frac{8}{9}\cdot\frac{6}{7}\cdot\frac{4}{5}\cdot\frac{2}{3}$$

Thus, as above, we begin with the denominator. We then have the ordinary sequence of natural numbers written down backwards. Thus, in the above example, we write (9 under 8) × (7 under 6) × .... etc stopping at (3 under 2) and additional factor $\pi/2$ is not written in the end.

(*iii*) If m and n are positive integers, then

$$\int_0^{\pi/2} \sin^m x \cos^n x\,dx = \frac{(m-1)\,(m-3)\,(m-5)...(n-1)\,(n-3)(n-5)...}{(m+n)\,(m+n-2)\,(m+n-4)...}\times k,$$

where k is $\pi/2$ if m and n are both both positive even integers otherwise $k = 1$. The last factor in each of the three products (namely, $(m-1)\,(m-3)\,(m-5)$ ..., $(n-1)\,(n-3)\,(n-5)$ .... and $(m+n)\,(m+n-2)\,(m+n-1)$ ...) is either 1 or 2. In case any of m or n is 1, we simply write 1 as the only factor to replace its product.

Example 1 $\int_0^{\pi/2} \sin^4 x \cos^2 x\,dx = \frac{3\cdot1\cdot1}{6\cdot4\cdot2}\times\frac{\pi}{2} = \frac{\pi}{32}$

Example 2 $\int_0^{\pi/2} \sin^4 x \cos^3 x\,dx = \frac{3\cdot1\cdot2}{7\cdot5\cdot3\cdot1}\times 1 = \frac{2}{35}$

Example 3 $\int_0^{\pi/2} \sin^4 x \cos x\,dx = \frac{3\cdot1\cdot1}{5\cdot3\cdot1} = \frac{1}{5}$

ELEMENTARY DIFFERENTIAL EQUATIONS

Where is What

1

Differential Equations Their Formation And Solutions

1.1 Differential equation

Definition. An equation involving derivatives or differentials of one or more dependent variables with respect to one or more independent variables is called a *differential equation*.

For examples of differential equations we list the following:

$$dy = (x + \sin x)\, dx, \qquad \text{... (1)}$$

$$\frac{d^4x}{dt^4} + \frac{d^2x}{dt^2} + \left(\frac{dx}{dt}\right)^5 = e^t, \qquad \text{... (2)}$$

$$y = \sqrt{x}\,\frac{dy}{dx} + \frac{k}{dy/dx}, \qquad \text{... (3)}$$

$$k\,(d^2y/dx^2) = \{1 + (dy/dx)^2\}^{3/2} \qquad \text{... (4)}$$

$$\partial^2 v/\partial t^2 = k\,(\partial^3 v/\partial x^3)^2 \qquad \text{... (5)}$$

and

$$\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + \partial^2 u/\partial z^2 = 0 \qquad \text{... (6)}$$

Note. Unless otherwise stated, y' (or y_1), y'' (or y_2), ..., $y^{(n)}$ (or y_n) will denote $\dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}, \ldots, \dfrac{d^ny}{dx^n}$ respectively. Thus, for example equation (3) may be re-written as

$$y = \sqrt{x}\, y' + k/y' \qquad \text{or} \qquad y = \sqrt{x}\, y_1 + k/y_1.$$

1.2 Ordinary differential equation

Definition. A differential equation involving derivatives with respect to a single independent variable is called an *ordinary differential equation*.

In Art. 1.1 equations (1), (2), (3) and (4) are all ordinary differential equations.

1.3 Partial differential equation

Definition. A differential equation involving partial derivatives with respect to more than one independent variables is called a *partial differential equation*.

In Art. 1.1 equations (5) and (6) are both partial differential equations.

1.4 Order of a differential equation

Definition. The order of the highest order derivative involved in a differential equation is called the *order of the differential equation*.

In Art. 1.1 equation (2) is of the fourth order, equations (1) and (3) are of the first order, equations (4) and (6) are of the second order and equation (5) is of the third order.

1.5 Degree of a differential equation

Definition. The *degree of a differential equation* is the degree of the highest derivative which occurs in it, after the differential equation has been made free from radicals and fractions as far as the derivatives are concerned. Note that the above definition of degree does not require variables x, t, u etc. to be free from radicals and fractions.

In Art 1.1 equations (1), (2) and (6) are of first degree. Making equation (3) free from fractions, we obtain $\quad y\,(dy/dx) = \sqrt{x}\,(dy/dx)^2 + k,$

which is of second degree. Again we must square both sides of (4) to make it free from radicals. Then by definition, the degree of (4) and (5) is two.

1.6 Linear and non-linear differential equations

Definition. A differential equation is called *linear* if (*i*) every dependent variable and every derivative involved occurs in the first degree only, and (*ii*) no products of dependent variables and/or derivatives occur. A differential equation which is not linear is called a *non-linear differential equation.*

In Art 1.1 equations (1) and (6) are linear and equations (2), (3), (4) and (5) are all non-linear.

SOLVED EXAMPLES

Ex. 1. *Find the order and degree of the following differential equations. Also classify them as linear and non-linear.*

(*a*) $y = \sqrt{x}\,(dy/dx) + k/(dy/dx)$ (*b*) $y = x\,(dy/dx) + a\,\{1 + (dy/dx)^2\}^{1/2}$

(*c*) $dy = (y + \sin x)\,dx$ (*d*) $(d^2y/dx^2)^3 + x\,(dy/dx)^5 + y = x^2$

(*e*) $\{y + x\,(dy/dx)^2\}^{4/3} = x(d^2y/dx^2)$ **[Rajsthan 2010]**

(*f*) $(d^2y/dx^2)^{1/3} = (y + dy/dx)^{1/2}$ **[Pune 2010]**

Sol. (*a*) Multiplying both sides of the given equation by dy/dx, we get

$$y\,(dy/dx) = \sqrt{x}\,(dy/dx)^2 + k, \qquad \dots (1)$$

which is of the first order and second degree because the order of the highest differential coefficient dy/dx is one and the highest degree of dy/dx is 2. Here (1) is non-linear differential equation because degree of dy/dx is 2 and product $y\,(dy/dx)$ of dependent variable y and its derivative (dy/dx) occurs.

(*b*) Re-writing the given equation, $\quad y - x\,(dy/dx) = a\,\{1 + (dy/dx)^2\}^{1/2}$

To get rid of radicals, square both sides to obtain

$$y^2 + x^2\,(dy/dx)^2 - 2xy\,(dy/dx) = a^2\,\{1 + (dy/dx)^2\},$$

which is of the first order and second degree because the order of the highest differential coefficient dy/dx is one and the highest degree of dy/dx is 2. Since degree of dy/dx is 2, the given equation is non-linear.

(*c*) **Ans.** It is of first order, first degree and linear.

(*d*) **Ans.** It is of second order, third degree and non-linear.

(*e*) Order 2, degree 3, non-linear (*f*) order 2, degree 2, non-linear

1.7 Solution of a differential equation

Definition. Any relation between the dependent and independent variables, when substituted in the differential equation, reduces it to an identity is called a *solution* or *integral of the differential equation*. It should be noted that a solution of a differential equation does not involve the derivatives of the dependent variable with respect to the independent variable or variables.

For example, $y = ce^{2x}$ is a solution of $dy/dx = 2y$ because by putting $y = ce^{2x}$ and $dy/dx = 2ce^{2x}$, the given differential equation reduces to the identity $2ce^{2x} = 2ce^{2x}$. Observe that $y = ce^{2x}$ is a solution of the given differential equation for any real constant c which is called an *arbitrary constant*.

SOLVED EXAMPLES

Ex. 1. *Show that* $y = (A/x) + B$ *is a solution of* $(d^2y/dx^2) + (2/x) \times (dy/dx) = 0$.

Sol. Given that $$(d^2y/dx^2) + (2/x) \times (dy/dx) = 0. \qquad \text{... (1)}$$

Also given that $$y = (A/x) + B. \qquad \text{... (2)}$$

Differentiating (2) w.r.t. 'x', $$dy/dx = -(A/x^2) \qquad \text{... (3)}$$

Differentiating (3) w.r.t. 'x', $$d^2y/dx^2 = 2A/x^3 \qquad \text{... (4)}$$

Substituting for dy/dx and d^2y/dx^2 from (3) and (4) in (1), we get

$$(2A/x^3) + (2/x) \times (-A/x^2) = 0 \qquad \text{or} \qquad 0 = 0,$$

which is true. Hence (2) is a solution of (1).

Ex. 2. *Show that* $y = a \cos (mx + b)$ *is a solution of the differential equation* $d^2y/dx^2 + m^2y = 0$.

Sol. Try yourself.

1.8 Family of curves

Definition. An n-parameter family of curves is a set of relations of the form

$$\{(x, y) : f(x, y, c_1, c_2, ..., c_n) = 0\},$$

where 'f' is a real valued function of $x, y, c_1, c_2, ..., c_n$ and each c_i $(i = 1, 2, ..., n)$ ranges over an interval of real values.

For example, the set of concentric circles defined by $x^2 + y^2 = c$ is one parameter family if c takes all non-negative real values.

Again, the set of circles, defined by $(x - c_1)^2 + (y - c_2)^2 = c_3$ is a three-parameter family if c_1, c_2 take all real values and c_3 takes all non-negative real values.

1.9 Complete primitive (or general solution). Particular solution and singular solution. Definitions

Let $$F(x, y, y_1, y_2, ..., y_n) = 0 \qquad \text{... (1)}$$

be an nth order ordinary differential equation.

(*i*) A solution of (1) containing n independent arbitrary constants is called a *general solution*.

(*ii*) A solution of (1) obtained from a general solution of (1) by giving particular values to one or more of the n independent arbitrary constants is called a *particular solution* of (1).

(*iii*) A solution of (1) which cannot be obtained from any general solution of (1) by any choice of the n independent arbitrary constants is called a *singular solution* of (1).

The student can easily verify that $$y = c_1e^x + c_2e^{2x} \qquad \text{... (2)}$$

is the general solution of $$y'' - 3y' + 2y = 0. \qquad \text{... (3)}$$

Since c_1 and c_2 are independent arbitrary constants and the order of (3) is two, (2) is a general solution of (3). Some particular solutions of (3) are given by $y = e^x + e^{2x}$, $y = e^x - 2e^{2x}$ etc.

Again, the reader can verify that $$y = (x + c)^2 \qquad \text{... (4)}$$

is the general solution of $$(dy/dx)^2 - 4y = 0. \qquad \text{... (5)}$$

The reader can also verify that $y = 0$ is also solution of (5). Moreover, $y = 0$ cannot be obtained by any choice of c in (4). Hence, $y = 0$ is a singular solution of (5).

1.10 Formation of differential equations

Suppose we are given a family of curves containing n arbitrary constants. Then we can obtain an nth order differential equation whose solution is the given family as follows.

Working rule to form the differential equation from the given equation in x and y, containing n arbitrary constants.

Step I. Write the equation of the given family of curves.

Step II. Differentiate the equation of step I, n times so as to get n additional equations containing the n arbitrary constants and derivatives.

Step III. Eliminate n arbitrary constants from the $(n + 1)$ equations obtained in steps I and II. Thus, we obtain the required differential equation involving a derivative of nth order.

1.11 Solved examples based on Art. 1.10

Ex. 1. *Find the differential equation of the family of curves $y = e^{mx}$, where m is an arbitrary constant.*

Sol. Given that $\qquad y = e^{mx}. \qquad$ ... (1)

Differentiating (1) w.r.t. 'x', we get $\qquad dy/dx = me^{mx}. \qquad$ ... (2)

Now, (1) and (2) $\Rightarrow \quad dy/dx = my \quad \Rightarrow \quad m = (1/y) \times (dy/dx). \qquad$ ... (3)

Again, from (1), $\quad mx = \log_e y \quad$ so that $\quad m = (\log_e y)/x. \qquad$ ... (4)

Eliminating m from (3) and (4), we get $\qquad (1/y) \times (dy/dx) = (1/x) \times \log_e y$.

Ex. 2. (*a*) *Find the differential equation of all straight lines passing through the origin.*

(*b*) *Find the differential equation of all the straight lines in the xy-plane.*

Sol. (*a*) Equation of any straight line passing through the origin is

$y = mx$, m being arbitrary constant. ... (1)

Differentiating (1) w.r.t. 'x', $\qquad dy/dx = m. \qquad$ ... (2)

Eliminating m from (1) and (2), we get $\qquad y = x\,(dy/dx)$.

(*b*) We know that equation of any straight line in the xy-plane is given by

$y = mx + c$, m and c being arbitrary constants. ... (1)

Differentiating (1) w.r.t. 'x', we get $\qquad dy/dx = m. \qquad$ ... (2)

Differentiating (2) w.r.t. 'x', we get $\qquad d^2y/dx^2 = 0, \qquad$ ... (3)

which is the required differential equation.

Note. Equation (3) is free from m and c and so it is not necessary to eliminate m and c from (1), (2) and (3) as usual.

Ex. 3. (*a*) *Obtain a differential equation satisfied by family of circles $x^2 + y^2 = a^2$, a being an arbitrary constant.*

(*b*) *Obtain a differential equation satisfied by the family of concentric circles.*

Sol. (*a*) Given $\qquad x^2 + y^2 = a^2. \qquad$ ... (1)

Differentiating (1) w.r.t. 'x', we get $\qquad 2x + 2y\,(dy/dx) = 0 \quad$ or $\quad x + y\,(dy/dx) = 0,$

which is the required differential equation.

(*b*) Let the centre of the given family of concentric circles be (0, 0). Then we know that the equation of the family of concentric circles is given by $x^2 + y^2 = a^2$, a being arbitrary constant.

Now proceed as in part (*a*). **Ans.** $x + y\,(dy/dx) = 0$.

Ex. 4. (*a*) *Find the differential equation of all circles which pass through the origin and whose centres are on the x-axis.* **[I.A.S. (Prel.) 2002]**

(*b*) *Find the differential equation of the system of circles touching the y-axis at the origin.* **[I.A.S. (Prel.) 1997]**

(*c*) *Find the differential equation of all circles touching a given straight line at a given point.*

Sol. (*a*) We know that the equation of any circle passing through the origin and whose centre is on the x-axis is given by

$$x^2 + y^2 + 2gx = 0, \ g \text{ being an arbitrary constant.} \qquad \dots (1)$$

Differentiating (1) w.r.t. 'x', we get $\quad 2x + 2y\,(dy/dx) + 2g = 0. \qquad \dots (2)$

From (1), $\quad 2gx = -(x^2 + y^2) \quad$ so that $\quad 2g = -(x^2 + y^2)/x \qquad \dots (3)$

Substituting for $2g$ from (3) in (2), we have

$$2x + 2y\left(\frac{dy}{dx}\right) - \frac{x^2 + y^2}{x} = 0 \qquad \text{or} \qquad 2xy\,\frac{dy}{dx} + x^2 - y^2 = 0.$$

(*b*) We note that any circle which touches the y-axis at the origin must have its centre on the x-axis and so equation of any such circle is given by

$$x^2 + y^2 + 2gx = 0, \ g \text{ being an arbitrary constant.} \qquad \dots (1)$$

Now proceed as in part (*a*) and obtain same answer.

(*c*) For the sake of simplification, without loss of any generality, take the given point as the origin and the given straight line as the y-axis.

Now proceed as in part (*b*) and get the same answer.

Ex. 5. (*a*) *Find the differential equation of all circles which pass through the origin and whose centres are on the y-axis.*

(*b*) *Find the differential equation of the system of circles touching the x-axis at the origin.* **[I.A.S. (Prel.) 1999]**

Sol. Parts (*a*) and (*b*). Here in both parts the equation of any circle is

$$x^2 + y^2 + 2fy = 0, \ f \text{ being an arbitrary constant.} \qquad \dots (1)$$

Proceed as in Ex. 4(*a*). **Ans.** $(x^2 - y^2)\,(dy/dx) - 2xy = 0.$

Ex. 6. *Find the differential equation which has $y = a \cos (mx + b)$ for its integral, a and b being arbitrary constants and m being a fixed constant.*

Sol. Given that $\quad y = a \cos (mx + b). \qquad \dots (1)$

Differentiating (1) w.r.t. 'x', we get $\quad dy/dx = -\,am \sin (mx + b). \qquad \dots (2)$

Differentiating (2) w.r.t. 'x', we get $\quad d^2y/dx^2 = -\,am^2 \cos (mx + b). \qquad \dots (3)$

or $\quad d^2y/dx^2 = -\,m^2 y$, using (1)

Thus, the required differential equation is $\quad d^2y/dx^2 + m^2 y = 0.$

Ex. 7. *Find the differential equation from the relation $y = a \sin x + b \cos x + x \sin x$, where a and b are arbitrary constants.*

Sol. Given $\quad y = a \sin x + b \cos x + x \sin x. \qquad \dots (1)$

Differentiating (1) w.r.t. 'x', $\quad dy/dx = a \cos x - b \sin x + \sin x + x \cos x. \qquad \dots (2)$

Differentiating (2) w.r.t. 'x', $\quad d^2y/dx^2 = -\,a \sin x - b \cos x + 2 \cos x - x \sin x$

or $\quad d^2y/dx^2 = 2 \cos x - (a \sin x + b \cos x + x \sin x) = 2 \cos x - y$, by (1).

$\therefore \ (d^2y/dx^2) + y = 2 \cos x$, which is the required differential equation.

Ex. 8. (*a*) *Find the differential equation of the family of curves $y = e^x (A \cos x + B \sin x)$, where A and B are arbitrary constants.* **[G.N.D.U. Amritsar 2010]**

(*b*) *Form a differential equation of which $y = e^x (A \cos 2x + B \sin 2x)$ is a solution, A and B being arbitrary constants.*

Sol. (*a*) Given that $\quad y = e^x (A \cos x + B \sin x). \qquad \dots (1)$

Differentiating (1), $y' = e^x(-A\sin x + B\cos x) + e^x(A\cos x + B\sin x)$

or $y' = e^x(-A\sin x + B\cos x) + y$, using (1). ... (2)

Differentiating (2) again with respect to x, we get

$$y'' = -e^x(A\cos x + B\sin x) + e^x(-A\sin x + B\cos x) + y'. \quad ...(3)$$

Now from (2), we get $e^x(-A\sin x + B\sin x) = y' - y$. ... (4)

Hence, eliminating A and B from (1), (3) and (4), we get

$$y'' = -y + y' - y + y' \quad \text{or} \quad y'' - 2y' + 2y = 0.$$

(*b*) Proceed as in Ex. 8(*a*). **Ans.** $y'' - 2y' + 5y = 0$

Ex. 9. *By eliminating the constants a and b obtain the differential equation for which* $xy = ae^x + be^{-x} + x^2$ *is a solution.* **[I.A.S. 1992]**

Sol. Given that $xy = ae^x + be^{-x} + x^2$. ... (1)

Diff. (1) w.r.t. 'x', we get $xy' + y = ae^x - be^{-x} + 2x$. ... (2)

Diff. (2) w.r.t. 'x', we get $xy'' + y' + y' = ae^x + be^{-x} + 2$

or $xy'' + 2y' = (xy - x^2) + 2$, using (1)

or $xy'' + 2y' - xy + x^2 - 2 = 0$.

Ex. 10. *Find the differential equation corresponding to the family of curves* $y = c(x - c)^2$, *where c is an arbitrary constant.* **[I.A.S. (Prel.) 2009; Karnatak 1995]**

Sol. Given that $y = c(x - c)^2$. ... (1)

Diff. (1) w.r.t. 'x', we get $y' = 2c(x - c)$. ... (2)

From (1) and (2), $y'/y = 2/(x - c)$ so that $c = x - (2y/y')$. ... (3)

Putting this value of c in (2), the required equation is

$$y' = 2\{x - (2y/y')\} \times (2y/y') \quad \text{or} \quad (y')^3 = 4y(xy' - 2y).$$

Ex. 11. *Find the differential equation of all circles of radius a.* **[Nagarjuna 2003]**

Sol. The equation of all circles of radius a is given by

$$(x - h)^2 + (y - k)^2 = a^2, \quad ...(1)$$

where h and k, are to be taken as arbitrary constants.

Diff. (1) w.r.t. 'x', we get $(x - h) + (y - k)y' = 0$. ... (2)

Diff. (2), $1 + (y')^2 + (y - k)y'' = 0$ or $y - k = -\{1 + (y')^2\}/y''$. ... (3)

Putting this value of $y - k$ in (2), we get

$$x - h = -(y - k)y' = \{1 + (y')^2\} \times (y'/y''). \quad ...(4)$$

Using (3) and (4), (1) gives the required equation as

$$\frac{\{1 + (y')^2\}^2 (y')^2}{(y'')^2} + \frac{\{1 + (y')^2\}^2}{(y'')^2} = a^2 \quad \text{or} \quad \{1 + (y')^2\}^3 = a^2(y'')^2.$$

Ex. 12. *Show that* $Ax^2 + By^2 = 1$ *is the solution of* $x[y(d^2y/dx^2) + (dy/dx)^2] = y(dy/dx)$. **[Gauhati 1996, Indore 1997]**

Sol. Given that $Ax^2 + By^2 = 1$. ... (1)

Diff. (1), $2Ax + 2By(dy/dx) = 0$ or $Ax + By(dy/dx) = 0$. ... (2)

Diff. (2), $A + B\{y(d^2y/dx^2) + (dy/dx) \times (dy/dx)\} = 0$. ... (3)

Multiplying (3) by x, we get $Ax + Bx\{y(d^2y/dx^2) + (dy/dx)^2\} = 0$. ... (4)

Subtracting (2) from (4), we get $Bx\{y(d^2y/dx^2) + (dy/dx)^2\} - By(dy/dx) = 0$

or $x[y(d^2y/dx^2) + (dy/dx)^2] = y(dy/dx)$, as required.

Ex. 13. *Find the third order differential equation whose solution is the 3-parameter family of curves defined by $x^2 + y^2 + 2ax + 2by + c = 0$, where a, b, c are parameters.*

Sol. Given $$x^2 + y^2 + 2ax + 2by + c = 0. \qquad \text{... (1)}$$

Differentiating (1) w.r.t. 'x' three times in succession, we have

$$x + y y^{(1)} + a + b y^{(1)} = 0, \text{ where } y^{(1)} = dy/dx \qquad \text{... (2)}$$

$$1 + [y^{(1)}]^2 + y y^{(2)} + b y^{(2)} = 0, \text{ where } y^{(2)} = d^2y/dx^2 \qquad \text{... (3)}$$

and $$3y^{(1)} y^{(2)} + y y^{(3)} + b y^{(3)} = 0., \text{ where } y^{(3)} = d^3y/dx^3 \qquad \text{... (4)}$$

We now eliminate a, b, c from (1), (2), (3) and (4). To do so, we simply eliminate b from (3) and (4). Multiplying both sides of (3) by $y^{(3)}$ and (4) by $y^{(2)}$ and subtracting, we have

$$y^{(3)} + y^{(3)} [y^{(1)}]^2 + y y^{(2)} y^{(3)} - 3y^{(1)} [y^{(2)}]^2 - y y^{(2)} y^{(3)} = 0$$

or $$[1 + (y^{(1)})^2]\, y^{(3)} - 3y^{(1)} (y^{(2)})^2 = 0.$$

EXERCISE 1(A)

1. Form the differential equations for the following:

(a) $y = Ae^{2x} + Be^{-2x}$, A, B being arbitrary constants. **Ans.** $y'' = 4y$

(b) $y = k \sin^{-1} x$, k parameter **Ans.** $y = y' \sqrt{(1 - x^2)}\, \sin^{-1}x$

(c) $y = \alpha x + \beta x^2$, α, β parameters **[Rajasthan 2010]** **Ans.** $x^2y'' - 2xy' + 2y = 0$

(d) $y = A \cos nt + B \sin nt$, ($A$, B parameters) **Ans.** $(d^2x/dt^2) + n^2x = 0$

(e) $xy = ae^x + be^{-x}$, (a, b parameters) **[Behrampur 2010]** **Ans.** $xy'' + 2y' - xy = 0$

2. Find the differential equation of the family of curves $y = Ae^{3x} + Be^{5x}$; for different values of A and B. **Ans.** $y'' - 8y' + 15y = 0$

3. Find the differential equation of all circles passing through origin and having their centres on the x-axis. **Ans.** $2xy' = y^2 - x^2$

4. Show that $v = B + A/r$ is a solution of $(d^2v/dr^2) + (2/r) \times (dv/dr) = 0$.

5. Find a differential equation with the following solution: $y = ae^x + be^{-x} + c \cos x + d \sin x$, where a, b, c and d are parameters. **Ans.** $d^4y/dx^4 - y = 0$

6. Classify the following equations as linear and non-linear equations and write down their orders

(a) $\dfrac{a^3y}{dx^3} + \dfrac{d^2y}{dx^2} \cdot \dfrac{dy}{dx} + y = x.$ (b) $x\dfrac{d^4y}{dx^4} + \dfrac{d^2y}{dx^2} = e^x.$ (c) $\dfrac{dy}{dx} + y^2 = x^2.$

Ans. (a) Non-linear; 3 (b) Linear, 4 (c) Non-linear, 1

7. Write down the order and degree of $x^2 (d^2y/dx^2)^3 + y\,(dy/dx)^4 + y^4 = 0$. How many constants does the general solution of the differential equation must contain. **Ans.** 2, 3, 2

8. Find the differential equation of the family of parabolas $y^2 = 4ax$. **Ans.** $y = 2x\,(dy/dx)$

9. Show that the differential equation of the family of circles of fixed radius r with centre on y-axis is $(x^2 - r^2)\,(dy/dx)^2 + x^2 = 0$.

10. Find the differential equation of all

(a) parabolas of latusrectum $4a$ and axis parallel to y-axis.

(b) tangent lines to the parabola $y = x^2$.

(c) ellipses centered at the origin.

(d) circles through the origin.

(e) circles tangent to y-axis.

(f) parabolas with axis parallel to the axis of y.

(g) parabolas with foci at the origin and axis along x-axis.

(h) all conics whose axes coincide with axes of co-ordinates.

Ans. (a) $2ay_2 - 1 = 0$ (b) $4\,(y - xy_1) + (y_1)^2 = 0$

(c) $xyy_2 + x\,(y_1)^2 - yy_1 = 0$ (d) $(x^2 + y^2)\, y_2 = 2\,(xy_1 - y)\,(1 + y_1{}^2)$

(e) $x^2 y_1^2 - 2xy_1y_2(1 + y_1^2) - (1 + y_1^2) = 0$ (f) $y_3 = 0$

(g) $yy_1^2 + 2xy_1 - y = 0$ (h) $xyy_2 + xy_1^2 = xy_1$.

11. Form the differential equation of family of curves $y = cx + c^2$, c being a comitent **[Pune 2010]**

Ans. $y = xy' = y''^2$

1.12 The Wronskian **[Delhi Maths (Hons) 2000]**

Definition. The Wronskian of n functions $y_1(x), y_2(x), ..., y_n(x)$ is denoted by $W(x)$ or $W(y_1, y_2, ..., y_n)(x)$ and is defined to be the determinant

$$W(y_1, y_2, ..., y_n)(x) = W(x) = \begin{vmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \cdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{vmatrix}.$$

1.13 Linearly dependent and independent set of functions

Definitions. n functions $y_1(x), y_2(x), ..., y_n(x)$ are *linearly dependent* if there exist constants $c_1, c_2, ..., c_n$ (not all zero), such that

$$c_1y_1 + c_2y_2 + ... + c_ny_n = 0. \qquad ... (1)$$

If, however, identity (1) implies that $c_1 = c_2 = ... = c_n = 0$,

then $y_1, y_2, ..., y_n$ are said to be *linearly independent*.

SOLVED EXAMPLES

Ex. 1. *Consider the two functions* $f_1(x) = x^3$ *and* $f_2(x) = x^2|x|$ *on the interval* $[-1, 1]$.

(i) *Show that their Wronskian* $W(f_1, f_2)$ *vanishes identically.*

(ii) *Show that* f_1 *and* f_2 *are not linearly dependent.*

(iii) *Do* (i) *and* (ii) *contradict theorem III, Art. 1.16 If not, why not.*

Sol. Left as an exercise.

Ex. 2. *Define the concept of linear dependence and independence. Hence, show that*

(i) *sin x and cos x*, $-\infty < x < \infty$ *are linearly independent.*

(ii) $e^{i\theta x}$, $\sin\theta x$, $\cos\theta x$, $-\infty < x < \infty$, θ *being a real number, are linearly dependent.*

(iii) $1, x, x^2, ..., x^n$, $-\infty < x < \infty$ *are linearly independent.*

(iv) x^2 *and* $x|x|$ *are linearly independent on* $-\infty < x < \infty$.

(v) $\sin x, \sin 2x, \sin 3x$ *are linearly independent on* $[0, 2\pi]$.

(vi) e^x, $\cos x$, $\sin x$ *are linearly independent on a real line.*

(vii) $\sin x$, $\sin(x + \pi/8)$, $\sin(x - \pi/8)$ *are linearly dependent on* $]-\infty, \infty[$.

($viii$) x^4 *and* $x^3|x|$ *are linearly independent on* $[-1, 1]$ *but are linearly dependent on* $[-1, 0]$ *and* $[0, 1]$

(ix) $f_1 + f_2$ *and* $f_1 - f_2$ *are linearly independent on an interval I whenever* f_1 *and* f_2 *are linearly independent on the interval I.*

(x) *f and g are linearly independent on* $[-1, 1]$, *if functions f and g are defined on* $[-1, 1]$ *as follows:*

$$\left.\begin{aligned} f(x) &= 0 \\ g(x) &= 1 \end{aligned}\right\} \text{ if } x \in [-1, 0]; \qquad \left.\begin{aligned} f(x) &= \sin x \\ g(x) &= 1 - x \end{aligned}\right\} \text{ if } x \in [0, 1]$$

Sol. A set of n real or complex valued functions $y_1(x), y_2(x), \ldots\ldots, y_n(x)$, $n \geq 2$ defined on an interval I are linearly dependent on I if there exist n constants, real or complex, $c_1, c_2, \ldots\ldots, c_n$, not all of them simultaneously zero such that

$$c_1 y_1(x) + c_2 y_2(x) + \ldots\ldots + c_n y_n(x) = 0 \text{ for each } x \text{ in I.}$$

The functions $y_1(x), y_2(x), \ldots\ldots, y_n(x)$ are linearly independent on I if they are not linearly dependent on I.

(*i*) Consider $$c_1 \sin x + c_2 \cos x = 0, -\infty < x < \infty \quad \ldots (1)$$

Differentiating (1) w.r.t. 'x', we get $$c_1 \cos x - c_2 \sin x = 0 \quad \ldots (2)$$

Solving (1) and (2), $c_1 = c_2 = 0$. Hence, $\sin x$ and $\cos x$ are linearly independent on $-\infty < x < \infty$.

(*ii*) We have, $$e^{i\theta x} = \cos \theta x + i \sin \theta x, \text{ by Euler's theorem}$$

Hence, $$e^{i\theta x} - \cos \theta x - i \sin \theta x = 0,$$

showing that $e^{i\theta x}$, $\cos \theta x$, $\sin \theta x$ are linearly dependent.

(*iii*) Consider, $$c_0 + c_1 x + c_2 x^2 + \ldots\ldots + c_n x^n = 0, -\infty < x < \infty \quad \ldots (A_1)$$

Differentiating (A_1) w.r.t. 'x' successively on $-\infty < x < \infty$ yields

$$c_1 + 2c_2 x + 3c_3 x^2 + \ldots\ldots + n\, c_n x^{n-1} = 0 \quad \ldots (A_2)$$

$$2c_2 + 6c_3 x + \ldots\ldots + n(n-1)\, c_n x^{n-2} = 0 \quad \ldots (A_3)$$

....

$$n!\, c_n = 0 \quad \ldots (A_4)$$

Solving $(A_1), (A_2), \ldots\ldots, (A_n)$ yields $c_1 = c_2 = c_3 = \ldots\ldots = c_{n-1} = c_n = 0$ and so the given functions $1, x, x^2, \ldots\ldots, x^n$ are linearly independent

(*iv*) Here, $$c_1 x^2 + c_2 x\,|\,x\,| = 0 \quad \ldots (1)$$

$\Rightarrow$ $$c_1 x^2 + c_2 x^2 = 0 \text{ for } x \geq 0 \quad \ldots (2)$$

and $$c_1 x^2 - c_2 x^2 = 0 \text{ for } x < 0 \quad \ldots (3)$$

In order that (1) may hold, (2) and (3) should hold simultaneously. This is possible only when $c_1 + c_2 = 0$ and $c_1 - c_2 = 0$, *i.e.*, $c_1 = c_2 = 0$. Hence, x^2 and $x\,|\,x\,|$ are linearly independent on $-\infty < x < \infty$.

1.14 Existence and uniqueness theorem **[Delhi B.Sc. (Hons) II 2011]**

Consider a second order linear differential equation of the form

$$a_0(x)\, y'' + a_1(x)\, y' + a_2(x)\, y = r(x), \quad \ldots (1)$$

where $a_0(x)$, $a_1(x)$, $a_2(x)$ and $r(x)$ are continuous functions on an interval (a, b) and $a_0(x) \neq 0$ for each $x \in (a, b)$. Let c_1 and c_2 be arbitrary real numbers and $x_0 \in (a, b)$. Then there exists a unique solution $y(x)$ of (1) satisfying $y(x_0) = c_1$ and $y'(x_0) = c_2$. Moreover, this solution $y(x)$ is defined over the interval (a, b).

Note 1. The above theorem is an existence theorem because it says that the initial value problem does have a solution. It is also a uniqueness theorem, because it says that there is only one solution. Clearly, this theorem also applies to an associated homogenous equation

$$a_0(x)\, y'' + a_1(x)\, y' + a_2(x)\, y = 0. \quad \ldots (2)$$

Note 2. In this chapter, we shall assume without proof, the above basic theorem for initial value problems associated with linear differential equations.

Note 3. The conditions of existence and uniqueness theorem cannot be further relaxed. For example, if $a_0(x) = 0$ for some $x \in (a, b)$, then the solution of (1) may not be unique or may not exist at all. For an example, refer solved example 1 of Art. 1.15.

Note 4. Existence and uniqueness theorem can be extended to an nth order linear differential equation.

Corollary. *If $y(x)$ be a solution of $a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$ satisfying $y(x_0) = 0$ and $y'(x_0) = 0$ for some $x_0 \in (a, b)$, then $y(x)$ is identically zero on (a, b).*

Proof. By definition, here $y(x)$ is a solution of the given equation which satisfies $y(x_0) = 0$ and $y'(x_0) = 0$. Again, by existence and uniqueness theorem, $y(x)$ is the unique solution satisfying $y(x_0) = 0$ and $y'(x_0) = 0$. It follows that $y(x) \equiv 0$ on (a, b), *i.e.*, $y(x)$ is identically zero on (a, b).

Note 1. A real valued function $y(x)$ is said to be identically zero on an interval (a, b) written as $y(x) \equiv 0$, if $y(x) = 0$ for each $x \in (a, b)$.

Note 2. A function $u(x)$ is called a solution of the equation

$$a_0(x)\,y''(x) + a_1(x)\,y'(x) + a_2(x)\,y(x) = 0$$

if $$a_0(x)\,u''(x) + a_1(x)\,u'(x) + a_2(x)\,u(x) = 0, \text{ for each } x \in (a, b).$$

1.14A (*a*) *State the existence and uniqueness theorem for the nth order differential equation*

$$L(y)(x) = y^{(n)}(x) + p_1(x)\,y^{(n-1)}(x) + \ldots\ldots + p_n(x)\,y(x) = 0,\ x \in I,$$

which is a linear homogeneous equation.

(*b*) *Show that there are three linearly independent solutions of the third order equation $y''' + p_1(x)y'' + p_2(x)y' + p_3(x)y = 0$, $x \in I$ where p_1, p_2 and p_3 are functions, defined and continuous on an interval I.*

(*c*) *Let ϕ be any solution of $y''' + p_1(x)y'' + p_2(x)y' + p_3(x)y = 0$, $x \in I$. Here p_1, p_2 and p_3 are functions defined and continuous on an interval I. Further, let ϕ_1, ϕ_2 and ϕ_3 be three linearly independent solutions of the given equation. Prove that constants c_1, c_2 and c_3 exist such that*

$$\phi = c_1\,\phi_1 + c_2\,\phi_2 + c_3\,\phi_3,\ x \in I.$$

Sol. (*a*) **Statement of the existence and uniqueness theorem for**

$$\boldsymbol{L(y)(x) = y^{(n)}(x) + p_1(x)\,y^{(n-1)}(x) + \ldots\ldots + p_n(x)\,y(x) = 0,\ x \in I} \quad \ldots (1)$$

Let $p_1, p_2, \ldots\ldots, p_n$ be defined and continous on an interval I which contains a point x_0. Let $a_0, a_1, \ldots\ldots, a_{n-1}$ be n constants. Then, there exists a unique solution ϕ on I of the nth order equation (1) satisfying the initial conditions.

$$\phi(x_0) = a_0, \qquad \phi'(x_0) = a_1, \ldots\ldots \qquad , \phi^{(n-1)}(x_0) = a_{n-1}$$

Note. Suppose that $\phi_1(x), \ldots\ldots, \phi_n(x)$ are n solutions of $L(y)(x) = 0$ given in (1) and suppose that $c_1, c_2, \ldots\ldots, c_n$ are n arbitrary constants. Since $L(\phi_1) = L(\phi_2) = \ldots\ldots = L(\phi_n) = 0$, and L is a linear operator, hence we have

$$L(c_1\,\phi_1 + c_2\,\phi_2 + \ldots\ldots + c_n\,\phi_n) = c_1\,L(\phi_1) + \ldots\ldots + c_n\,L(\phi_n) = 0$$

In case n solutions $\phi_1, \ldots\ldots, \phi_n$ are linerly independent and $c_1, c_2, \ldots, c_n$ are constants, then

$$c_1\phi_1 + c_2\phi_2 + \ldots.. + c_n\phi_n = 0,\ x \in \mathrm{I} \Rightarrow c_1 = c_2 = \ldots.\ c_n = 0$$

(*b*) Given $$y''' + p_1(x)\,y'' + p_2(x)\,y' + p_3(x)\,y = 0,\ x \in I \quad \ldots (1)$$

Using the existence and uniqueness theorem stated in part (*a*), we conclude that there exist solutions $\phi_1(x)$, $\phi_2(x)$ and $\phi_3(x)$ of (1) such that for $x_0 \in I$.

$$\left.\begin{array}{lll} \phi_1(x_0) = 0, & \phi_1'(x_0) = 0, & \phi_1''(x_0) = 0 \\ \phi_2(x_0) = 0, & \phi_2'(x_0) = 1, & \phi_2''(x_0) = 0 \\ \text{and} \quad \phi_3(x_0) = 0, & \phi_3'(x_0) = 0, & \phi_3''(x_0) = 1 \end{array}\right\} \quad \ldots (2)$$

We now proceed to prove that ϕ_1, ϕ_2 and ϕ_3 are linearly independent. Assume that

$$c_1\,\phi_1(x) + c_2\,\phi_2(x) + c_3\,\phi_3(x) = 0,\ x \in I \quad \ldots (3)$$

for some constants c_1, c_2 and c_3. At $x = x_0$, from (3), we obtain

$$c_1\,\phi_1\,(x_0) + c_2\,\phi_2\,(x_0) + c_3\,\phi_3\,(x_0) = 0 \qquad \dots (4)$$

Differentiating (3) w.r.t. 'x' and then replacing x by x_0 yields

$$c_1\,\phi_1'\,(x_0) + c_2\,\phi_2'\,(x_0) + c_3\,\phi_3'\,(x_0) = 0 \qquad \dots (5)$$

Differentiating (3) twice w.r.t. 'x' and then replacing x by x_0 yields

$$c_1\,\phi_1''\,(x_0) + c_2\,\phi_2''\,(x_0) + c_3\,\phi_3''\,(x_0) = 0 \qquad \dots (6)$$

Using (2) in (4), (5) and (6), we have $\qquad c_1 = c_2 = c_3 = 0.$

Hence, ϕ_1, ϕ_2 and ϕ_3 are linearly independent.

(*c*) Given $\qquad y'' + p_1\,(x)\,y'' + p_2\,(x)\,y' + p_3\,(x)\,y = 0,\ x \in I \qquad \dots (1)$

Given that ϕ is a solution of (1). Using the existence and uniqueness theorem stated in part (*a*), at $x = x_0 \in I$, there exist constants a_1, a_2 and a_3 such that

$$\phi(x_0) = a_1, \qquad \phi'(x_0) = a_2 \qquad \text{and} \qquad \phi''(x_0) = a_3$$

The solutions ϕ_1, ϕ_2 and ϕ_3 are as given by part (*b*). We now define a function ψ on I such that $\psi(x) = a_1\phi_1(x) + a_2\,\phi_2(x) + a_3\phi_3(x)$, $x \in$ I. Clearly ψ satisfies (1) and

$$\psi(x_0) = a_1, \qquad \psi'(x_0) = a_2 \qquad \text{and} \qquad \psi''(x_0) = a_3$$

Observe that two solutions ϕ and ψ of (1) have the same initial conditions. Hence by the existence and uniqueness theorem, it follows that $\phi(x) = \psi(x)$ for $x \in$ I.

Remark. From the parts (*b*) and (*c*), it follows that for a third order equation (1) of part (*a*) and (*b*), there are three linearly independent solutions and that any other solution of that equation is a linear combination of these solutions.

1.15 Solved examples based on Art 1.14 and 1.14A

Ex. 1. *Show that the function $y = cx^2 + x + 3$ is a solution, though not unique, of the initial value problem $x^2y'' - 2xy' + 2y = 6$ with $y\,(0) = 3$, $y'\,(0) = 1$* on $(-\infty, \infty)$.

[Delhi Maths (Hons.) 1994, 2007]

Sol. Given equation is $\qquad x^2y'' - 2xy' + 2y = 6 \qquad \dots (1)$

and given function is $\qquad y\,(x) = cx^2 + x + 3. \qquad \dots (2)$

Differentiating (2), we get $\qquad y' = 2cx + 1 \qquad$ and $\qquad y'' = 2c. \qquad \dots (3)$

$\therefore$ L.H.S. of (1) $= x^2\,(2c) - 2x\,(2cx + 1) + 2\,(cx^2 + x + 3)$, by (2) and (3)

$= 6 =$ R.H.S. of (1),

showing that (2) is a solution of (1). Again, from (2) and (3), we get

$$y\,(0) = (c \times 0) + 0 + 3 = 3 \quad \text{and} \quad y'\,(0) = (2c) \times (0) + 1 = 1.$$

Comparing (1) with $\qquad a_0\,(x)\,y'' + a_1\,(x)\,y' + a_2\,(x)\,y = r\,(x),$

here $\quad a_0\,(x) = x^2, \quad a_1\,(x) = -2x, \quad a_2\,(x) = 2 \quad$ and $\quad r\,(x) = 6,$

which are continuous functions on $(-\infty, \infty)$.

Since $a_0\,(x) = x^2 = 0$ for $x = 0 \in (-\infty, \infty)$, therefore, the solution $y = cx^2 + x + 3$ is not unique (refer note 3 of Art 1.14). We see that $y = cx^2 + x + 3$ is solution for any real value of c. For example, $y = 2x^2 + x + 3$ and $y = 3x^2 + x + 3$ are both solutions of (1) with $y\,(0) = 3$ and $y'\,(0) = 1$.

Ex. 2. *Show that $y = 3e^{2x} + e^{-2x} - 3x$ is the unique solution of the initial value problem $y'' - 4y = 12x$, where $y\,(0) = 4$. $y'\,(0) = 1$.* **[Delhi Maths (Hons.) 1996]**

Sol. Given equation is $\qquad y'' - 4y = 12x \qquad \dots (1)$

and the given function is $\qquad y = 3e^{2x} + e^{-2x} - 3x. \qquad \dots (2)$

Differentiating (2), we get $y' = 6e^{2x} - 2e^{-2x} - 3$ and $y'' = 12e^{2x} + 4e^{-2x}$. ... (3)

$\therefore$ L.H.S. of (1) $= 12e^{2x} + 4e^{-2x} - 4\,(3e^{2x} + e^{-2x} - 3x)$, using (2) and (3)

$= 12x =$ R.H.S. of (1),

showing that (2) is a solution of (1). Again, from (2) and (3), we get

$y\,(0) = 3 + 1 - (3 \times 0) = 4$ and $y'\,(0) = 6 - 2 - 3 = 1$.

Comparing (1) with $a_0\,(x)\,y'' + a_1\,(x)\,y' + a_2\,(x)\,y = r\,(x)$, here

$$a_0\,(x) = 1, \quad a_1\,(x) = 0, \quad a_2\,(x) = -4 \quad \text{and} \quad r\,(x) = 12x,$$

which are all continuous functions in $(-\infty, \infty)$ and $a_0\,(x) = 1 \neq 0$ for each $x \in (-\infty, \infty)$. Therefore, by existence and uniqueness theorem, it follows that (2) is the unique solution of (1), satisfying $y\,(0) = 4$ and $y'\,(0) = 1$.

Ex. 3. *Find the unique solution of $y'' = 1$ satisfying $y\,(0) = 1$ and $y'\,(0) = 2$.*

Sol. Given equation is $y'' = d^2y/dx^2 = 1$ ... (1)

Integrating (1), $y' = dy/dx = x + c_1$ ... (2)

Integrating (2), $y = x^2/2 + c_1 x + c_2$ ... (3)

Putting $x = 0$ in (2) and (3) and using $y\,(0) = 1$ and $y'\,(0) = 2$, we get $c_1 = 2$ and $c_2 = 1$.

Hence (3) becomes $y = x^2/2 + 2x + 1$ ... (4)

Now, comparing (1) with $a_0\,(x)\,y'' + a_1\,(x)\,y' + a_2\,(x)\,y = r\,(x)$, we have $a_0\,(x) = 1$, $a_1\,(x) = 0$, $a_2\,(x) = 0$ and $r\,(x) = 1$. These are all continuous in $(-\infty, \infty)$ and $a_0\,(x) \neq 0$ for each $x \in (-\infty, \infty)$. Hence, by existence and uniqueness theorem, the solution (4) is unique.

Exercise 1(B)

1. Show that $y = x + x \log x - 1$ is the unique solution of $xy'' - 1 = 0$ satisfying $y\,(1) = 0$ and $y'\,(1) = 2$.
2. Show that $y = (1/4) \times \sin 4x$ is a unique solution of the initial value problem $y'' + 16y = 0$ with $y\,(0) = 0$ and $y'\,(0) = 1$.
3. Show that $y_1 \equiv x$ and $y_2 \equiv x^3$ are two different solutions of $x^2\,y'' - 3xy' + 3y = 0$ satisfying the initial conditions y (0) = y' (0) = 0. Explain why these facts do not contradict the existence and uniquenen theorem. **[Delhi B.Sc (Hons) II 2011]**
4. Given that $y = c_1 + c_2 x^2$ is a two parameter family of solutions of $xy'' - y' = 0$ on the interval $-\infty < x < \infty$. Show that constants c_1 and c_2 cannot be found so that a member of the family satisfies the initial conditions $y\,(0) = 1$, $y'\,(0) = 0$. Explain why this does not violate existence and uniqueness theorem.

1.16 Some important theorems

Theorem 1. *If $y_1\,(x)$ and $y_2\,(x)$ are any two solutions of*

$$a_0\,(x)\,y''\,(x) + a_1\,(x)\,y'\,(x) + a_2\,(x)\,y\,(x) = 0,$$

then the linear combination $c_1 y_1\,(x) + c_2 y_2\,(x)$, where c_1 and c_2 are constants, is also a solution of the given equation. **[Delhi Maths (G) 2001, 02]**

Proof. Given $a_0\,(x)\,y''\,(x) + a_1\,(x)\,y'\,(x) + a_2\,(x)\,y\,(x) = 0$. ... (1)

Since $y_1\,(x)$ and $y_2\,(x)$ are solutions of (1), we have

$$a_0\,(x)\,y_1''\,(x) + a_1\,(x)\,y_1'\,(x) + a_2\,(x)\,y_1\,(x) = 0 \quad \text{... (2)}$$

and

$$a_0\,(x)\,y_2''\,(x) + a_1\,(x)\,y_2'\,(x) + a_2\,(x)\,y_2\,(x) = 0. \quad \text{... (3)}$$

Let $u\,(x) = c_1 y_1\,(x) + c_2 y_2\,(x)$. ... (4)

Differentiating (4) twice w.r.t. 'x', we have

$u'\,(x) = c_1 y_1'\,(x) + c_2 y_2'\,(x)$ and $u''\,(x) = c_1 y_1''\,(x) + c_2 y_2''\,(x)$. ... (5)

Then, $a_0(x)\, u''(x) + a_1(x)\, u'(x) + a_2(x)\, u(x)$

$$= a_0(x)\,[c_1 y_1''(x) + c_2 y_2''(x)] + a_1(x)\,[c_1 y_1'(x) + c_2 y_2'(x)] + a_2(x)\,[c_1 y_1(x) + c_2 y_2(x)], \text{ using (4) and (5)}$$

$$= c_1\,[a_0(x)\, y_1''(x) + a_1(x) y_1'(x) + a_2(x)\, y_1(x)] + c_2\,[a_0(x)\, y_2''(x) + a_1(x) y_2'(x) + a_2(x)\, y_2(x)]$$

$= c_1 \cdot 0 + c_2 \cdot 0$ using (2) and (3)

Thus, $$a_0(x)\, u''(x) + a_1(x)\, u'(x) + a_2(x)\, u(x) = 0,$$

showing that $u(x)$, *i.e.*, $c_1 y_1(x) + c_2 y_2(x)$ is also solution of (1).

Note. The result of the above theorem 1 can be generalised as follows: If $y_1(x), y_2(x), \ldots, y_n(x)$ be n solutions of $a_0(x)\, y''(x) + a_1(x)\, y'(x) + a_2(x)\, y(x) = 0$, then their linear combination $c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x)$ is also solution of the given equation, $c_1, c_2, \ldots, c_n$ being constants.

Theorem II. *There exist two linearly independent solutions $y_1(x)$ and $y_2(x)$ of the equation*

$$a_0(x)\, y''(x) + a_1(x)\, y'(x) + a_2(x)\, y(x) = 0$$

such that its every solution $y(x)$ may be written as

$$y(x) = c_1 y_1(x) + c_2 y_2(x),\ x \in (a, b)$$

where c_1 and c_2 are suitable chosen constants. **[Delhi Maths (Hons) 1996]**

Proof. Given $$a_0(x)\, y''(x) + a_1(x)\, y'(x) + a_2(x)\, y(x) = 0. \qquad \ldots (1)$$

Let $x_0 \in (a, b)$ and $y_1(x)$ and $y_2(x)$ be two solutions of (1) satisfying

$$y_1(x_0) = 1 \quad \text{and} \quad y_1'(x_0) = 0 \qquad \ldots (2)$$

and $$y_2(x_0) = 0 \quad \text{and} \quad y_2'(x_0) = 1. \qquad \ldots (3)$$

To prove that $y_1(x)$ and $y_2(x)$ are linearly independent. Let, if possible $y_1(x)$ and $y_2(x)$ be linearly dependent. Then, by definition, there must exist constants c_1 and c_2, not both zero, such that

$$c_1 y_1(x) + c_2 y_2(x) = 0 \text{ for each } x \in (a, b). \qquad \ldots (4)$$

Now (4) $\Rightarrow$ $$c_1 y_1'(x) + c_2 y_2'(x) = 0 \text{ for each } x \in (a, b). \qquad \ldots (5)$$

By assumption, $x_0 \in (a, b)$. Hence (4) and (5) give

$$c_1 y_1(x_0) + c_2 y_2(x_0) = 0 \qquad \ldots (6)$$

and $$c_1 y_1'(x_0) + c_2 y_2'(x_0) = 0. \qquad \ldots (7)$$

Using (2) and (3), (6) $\Rightarrow$ $c_1 = 0$ and (7) $\Rightarrow$ $c_2 = 0$. This is a contradiction of the fact that c_1 and c_2 are not both zero. Hence, our assumption that $y_1(x)$ and $y_2(x)$ are linearly dependent is not possible and so by definition, $y_1(x)$ and $y_2(x)$ must be linearly independent.

We now prove the last part of the theorem. Let $y(x)$ be an any solution of (1) satisfying

$$y(x_0) = c_1 \quad \text{and} \quad y'(x_0) = c_2. \qquad \ldots (8)$$

Let $$u(x) = y(x) - c_1 y_1(x) - c_2 y_2(x). \qquad \ldots (9)$$

(9) shows that $u(x)$ is a linear combination of solutions $y(x)$, $y_1(x)$ and $y_2(x)$ of (1) and hence $u(x)$ is also a solution of (1). [Refer note for generalisation of theorem I]

From (9), $$u'(x) = y'(x) - c_1 y_1'(x) - c_2 y_2'(x). \qquad \ldots (10)$$

Now, (9) $\Rightarrow$ $u(x_0) = y(x_0) - c_1 y_1(x_0) - c_2 y_2(x_0) = 0$, by (2), (3) and (8).

and (10) $\Rightarrow$ $u'(x_0) = y'(x_0) - c_1 y_1'(x_0) - c_2 y_2'(x_0) = 0$, by (2), (3) and (8).

Thus, we find that $u(x)$ is a solution of (1) satisfying $u(x_0) = 0$ and $u'(x_0) = 0$. Hence, $u(x) \equiv 0$ on (a, b) [Refer corollary of Art. 1.14] and so by (9), we have

$$y(x) - c_1 y_1(x) - c_2 y_2(x) = 0 \quad \text{or} \quad y(x) = c_1 y_1(x) + c_2 y_2(x),$$

where c_1 and c_2 are suitable chosen constants and are given by (8).

Theorem III. *Two solutions $y_1(x)$ and $y_2(x)$ of the equation,*

$$a_0(x)\, y'' + a_1(x)\, y' + a_2(x)\, y = 0,\ a_0(x) \neq 0,\ x \in (a, b)$$

are linearly dependent if and only if their Wronskian is identically zero.

[Mumbai 2010; Delhi B.Sc. (Prog.) II 2007, 09; Delhi Maths Prog II 2007, 08]
[Delhi Maths (G) 2000; Delhi Maths (H) 2000, 01, 02, 06, 08; Lucknow 1995]

Proof. *Condition is necessary.* Let $y_1(x)$ and $y_2(x)$ be linearly dependent. Then, there must exist two constants c_1 and c_2, not both zero, such that

$$c_1 y_1(x) + c_2 y_2(x) = 0 \text{ for each } x \in (a, b). \qquad \text{... (1)}$$

From (1),
$$c_1 y_1'(x) + c_2 y_2'(x) = 0 \text{ for each } x \in (a, b). \qquad \text{... (2)}$$

Since c_1 and c_2 cannot be zero simultaneously, the system of simultaneous equation (1) and (2) possess non-zero solutions for which the condition is

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = 0 \text{ for each } x \in (a, b)$$

$\Rightarrow$ $W(x) \equiv 0$ on (a, b), *i.e.*, Wronskian is identically zero.

Condition is sufficient. Suppose that Wronskian of $y_1(x)$ and $y_2(x)$ is identically zero on (a, b), *i.e.*, let

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} \equiv 0 \text{ on } (a, b). \qquad \text{... (3)}$$

Let $x = x_0 \in (a, b)$. Then from (3), we have

$$\begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = 0. \qquad \text{... (4)}$$

Now, (4) is the condition for existence of two constants k_1 and k_2, both not zero, such that

$$k_1 y_1(x_0) + k_2 y_2(x_0) = 0 \qquad \text{... (5)}$$

and
$$k_1 y_1'(x_0) + k_2 y_2'(x_0) = 0. \qquad \text{... (6)}$$

Let
$$y(x) = k_1 y_1(x) + k_2 y_2(x). \qquad \text{... (7)}$$

Then $y(x)$ being a linear combination of solutions $y_1(x)$ and $y_2(x)$ is also a solution of the given equation. [Refer theorem 1 of Art. 1.16]

Again, from (8)
$$y'(x) = k_1 y_1'(x) + k_2 y_2'(x). \qquad \text{... (8)}$$

Now, (7) $\Rightarrow$
$$y(x_0) = k_1 y_1(x_0) + k_2 y_2(x_0) = 0, \text{ using (5)}$$

and (8) $\Rightarrow$
$$y'(x_0) = k_1 y_1'(x_0) + k_2 y_2'(x_0) = 0, \text{ using (6)}.$$

Thus, we find that $y(x)$ is a solution of the given equation such that $y(x_0) = 0$ and $y'(x_0) = 0$. Hence, $y(x) \equiv 0$ on (a, b). [Refer corollary of Art. 1.14] and so by (7), we have

$$k_1 y_1(x) + k_2 y_2(x) = 0 \text{ for each } x \in (a, b),$$

where k_1 and k_2 are constants, both not zero.

Hence, by definition, $y_1(x)$ and $y_2(x)$ are linearly dependent.

Corollary to theorem III. *Two solutions $y_1(x)$ and $y_2(x)$ of the equation $a_0(x)\, y'' + a_1(x)\, y' + a_2(x)\, y = 0$, $a_0(x) \neq 0$, $x \in (a, b)$ are linearly independent if and only if their Wronskian is not zero at some point $x_0 \in (a, b)$.* **[Delhi B.A. (Prog) II 2010; Delhi B.Sc. (Hons) II 2011]**

Proof. *Condition is necessary.* Let $y_1(x)$ and $y_2(x)$ be linearly independent. Then, by definition, $y_1(x)$ and $y_2(x)$ are not linearly dependent. Hence, by theorem III, we cannot have $W(x) \equiv 0$ on

(a, b), for otherwise $y_1(x)$ and $y_2(x)$ would become linearly dependent. It follows that there must exist some $x_0 \in (a, b)$, such that $W(x_0) \neq 0$. Hence the result.

Condition is sufficient. Suppose there exist some $x_0 \in (a, b)$, such that $W(x_0) \neq 0$. Then, it follows that $W(x) \neq 0$ on (a, b) and hence $y_1(x)$ and $y_2(x)$ cannot be linearly dependent by theorem III. So by definition, $y_1(x)$ and $y_2(x)$ must be linearly independent.

Theorem IV. *The Wronskian of two solutions of the equation,*

$$a_0(x)\, y'' + a_1(x)\, y' + a_2(x)\, y = 0, \qquad a_0(x) \neq 0, \qquad x \in (a, b)$$

is either identically zero or never zero on (a, b). **[Delhi B.Sc. (Prog.) II 2009; 2010 Delhi Maths (Hons.) 2005, 07; Lucknow 2001;Nagpur 1997; Delhi Maths (G) 2005, 05]**

Proof. Given differential equation is

$$a_0(x)\, y'' + a_1(x)\, y' + a_2(x)\, y = 0,\ a_0(x) \neq 0,\ x \in (a, b). \qquad \text{... (1)}$$

Let $y_1(x)$ and $y_2(x)$ be two solutions of (1). Then their Wronskian $W(x)$ is given by

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = y_1(x)\, y_2'(x) - y_2(x)\, y_1'(x). \qquad \text{... (2)}$$

Differentiating both sides of (2) with respect to 'x', we get

$$W'(x) = \frac{d}{dx}[y_1(x)\, y_2'(x)] - \frac{d}{dx}[y_2(x)\, y_1'(x)]$$

or $$W'(x) = [y_1'(x)\, y_2'(x) + y_1(x)\, y_2''(x)] - [y_2'(x)\, y_1'(x) + y_2(x)\, y_1''(x)]$$

or $$W'(x) = y_1(x)\, y_2''(x) - y_2(x)\, y_1''(x). \qquad \text{... (3)}$$

Since $a_0(x) \neq 0$, dividing by $a_0(x)$ and re-writing, (1) becomes

$$y''(x) = -(a_1/a_0)\, y'(x) - (a_2/a_0)\, y(x). \qquad \text{... (4)}$$

Since $y_1(x)$ and $y_2(x)$ are solutions of (4), we have

$$y_1''(x) = -(a_1/a_0)\, y_1'(x) - (a_2/a_0)\, y_1(x) \qquad \text{... (5)}$$

and $$y_2''(x) = -(a_1/a_0)\, y_2'(x) - (a_2/a_0)\, y_2(x). \qquad \text{... (6)}$$

Substituting the values of $y_1''(x)$ and $y_2''(x)$ given by (5) and (6) in (3), we get

$$W'(x) = y_1(x)[-(a_1/a_0)\, y_2'(x) - (a_2/a_0)\, y_2(x)] - y_2(x)[-(a_1/a_0)\, y_1'(x) - (a_2/a_0)\, y_1(x)]$$

or $$W'(x) = -(a_1/a_0)[y_1(x)\, y_2'(x) - y_2(x)\, y_1'(x)]$$

or $$W'(x) = -(a_1/a_0)\, W(x), \text{ using (2)} \qquad \text{... (7)}$$

or $$a_0(x)\, W'(x) + a_1(x)\, W(x) = 0. \qquad \text{... (8)}$$

From (8), it follows that $W(x)$ is a solution (8). Now, the following two cases arise:

Case I. Let $W(x) \neq 0$ on (a, b). Then the second part of the theorem is proved.

Case II. If possible, let $W(x_0) = 0$ for some $x_0 \in (a, b)$.

Then, $\quad (7) \Rightarrow W'(x_0) = -(a_1/a_0)\, W(x_0) = 0.$

Thus, $W(x)$ is a solution of (8), such that $W(x_0) = 0$ and $W'(x_0) = 0$. Hence, $W(x) \equiv 0$ on (a, b), *i.e.*, Wronskian is identically zero on (a, b). [Refer corollary of Art. 1.14] This proves the first part of the theorem.

Theorem V. *The nth order homogeneous linear equation,*

$$a_0(x)\,(d^n y/dx^n) + a_1(x)\,(d^{n-1}y/dx^{n-1}) + \ldots + a_{n-1}(x)\,(dy/dx) + a_n(x)\, y = 0$$

always possesses n independent solutions $y_1(x), y_2(x), \ldots, y_n(x)$ and its general solution is given by $\quad y = c_1 y_1(x) + c_2 y_2(x) + \ldots + c_n y_n(x)$, *where $c_1, c_2, \ldots, c_n$ are n arbitrary constants.*

Proof. Left for the reader.

Theorem VI. *Let $p_1(x), p_2(x), \ldots\ldots, p_n(x)$ be real or complex valued functions defined and continuous on an internal I and $\phi_1, \phi_2, \ldots\ldots, \phi_n$ are n solutions of the equation*

$$L(y)(x) = y^{(n)}(x) + p_1(x)\, y^{(n-1)}(x) + \ldots\ldots + p_n(x)\, y(x) = 0,\ x \in I$$

existing on I. Then n solutions are linearly independent on I if and only if $W(x) \neq 0$ for every $x \in I$.

[Himachal 2008]

Sol. The condition is necessary. Let there be a point $x_0 \in I$ such that

$$W(\phi_1, \phi_2, \ldots\ldots, \phi_n)(x_0) = W(x_0) = \begin{vmatrix} \phi_1(x_0) & \phi_2(x_0) \ldots\ldots & \phi_n(x_0) \\ \phi_1'(x_0) & \phi_2'(x_0) \ldots\ldots & \phi_n'(x_0) \\ \ldots & \ldots & \ldots \\ \phi_1^{(n-1)}(x_0) & \phi_2^{n-1}(x_0) \ldots\ldots & \phi_n^{(n-1)}(x_0) \end{vmatrix} \neq 0 \quad \ldots (1)$$

Let there exist constants $c_1, c_2, \ldots\ldots, c_n$ such that

$$c_1 \phi_1(x) + c_2 \phi_2(x) + \ldots\ldots + c_n \phi_n(x) = 0; \text{ for each } x \in I \quad \ldots (2)$$

In order to prove that $\phi_1, \phi_2, \ldots\ldots, \phi_n$ are linearly independent, we must show that

$$c_1 = c_2 = \ldots\ldots = c_n = 0. \quad \ldots (3)$$

Differentiating (2) successively w.r.t. 'x', we have

$$\left.\begin{aligned} c_1 \phi_1'(x) + c_2 \phi_2'(x) + \ldots\ldots + c_n \phi_n'(x) &= 0; \text{ for each } x \in I \\ c_1 \phi_1''(x) + c_2 \phi_2''(x) + \ldots\ldots + c_n \phi_n''(x) &= 0; \text{ for each } x \in I \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ c_1 \phi_1^{(n-1)}(x) + c_2 \phi_2^{(n-1)}(x) + \ldots\ldots + c_n \phi_n^{(n-1)}(x) &= 0; \text{ for each } x \in I \end{aligned}\right\} \quad \ldots (4)$$

At $x = x_0 \in I$, (2) and (4) yield

$$\left.\begin{aligned} c_1 \phi_1(x_0) + c_2 \phi_2(x_0) + \ldots\ldots + c_n \phi_n(x_0) &= 0 \\ c_1 \phi_1'(x_0) + c_2 \phi_2'(x_0) + \ldots\ldots + c_n \phi_n'(x_0) &= 0 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ c_1 \phi_1^{(n-1)}(x_0) + c_2 \phi_2^{(n-1)}(x_0) + \ldots\ldots + c_n \phi_n^{(n-1)}(x_0) &= 0 \end{aligned}\right\} \quad \ldots (5)$$

(5) represents a system of n simultaneous homogeneous equations in $c_1, c_2, \ldots\ldots, c_n$ as n unknown constants. The determinant of the coefficients of the above n equations (5) is clearly $W(\phi_1, \phi_2, \ldots\ldots, \phi_n)(x_0)$ or $W(x_0)$, which is non-zero by (1). Hence, there is only one solution of the system (5), namely $c_1 = c_2 = \ldots\ldots = c_n = 0$. Hence, $\phi_1, \phi_2, \ldots\ldots, \phi_n$ are linearly independent on I.

The condition is sufficient. Suppose, that solutions $\phi_1, \phi_2, \ldots\ldots, \phi_n$ of

$$L(y)(x) = y^{(n)}(x) + p_1(x)\, y^{(n-1)}(x) + \ldots\ldots + p_n(x)\, y(x) = 0,\ x \in I \quad \ldots (6)$$

are linearly independent. Suppose, if possible, there is an $x_0 \in I$ such that

$$W(\phi_1, \phi_2, \ldots\ldots, \phi_n)(x_0) = W(x_0) = 0 \quad \ldots (7)$$

Then (7) implies that the system (5) has a solution $c_1, c_2, \ldots\ldots, c_n$ where not all the constants $c_1, \ldots\ldots, c_n$ are zero. Let $c_1, c_2, \ldots\ldots, c_n$ be such a non-trivial solution of (5) and consider the function $\psi(x)$ such that

$$\psi(x) = c_1 \phi_1(x) + c_2 \phi_2(x) + \ldots\ldots + c_n \phi_n(x), \text{ for each } x \in I \quad \ldots (8)$$

Since, $\phi_1, \phi_2, \ldots\ldots, \phi_n$ are solutions of (6), we have

$$L(\phi_1) = L(\phi_2) = \ldots\ldots = L(\phi_n) = 0. \quad \ldots (9)$$

Then, from (8) and (9), $\qquad L(\psi) = 0, \qquad \ldots (10)$

showing that ψ is a solution (6). From (5) and (8), we have

$$\psi(x_0) = 0, \qquad \psi'(x_0) = 0, \ldots\ldots, \qquad \psi^{(n-1)}(x_0) = 0 \qquad \ldots (11)$$

Thus, $\psi(x)$ is a solution of (6) satisfying the initial conditions (11). From the existence and uniqueness theorem (refer part (a) of Art. 1.14A), it follows that $\psi(x) = 0$ for all x in I, and hence from (8), $c_1 \phi_1(x) + \ldots\ldots + c_n \phi_n(x) = 0$ for all $x \in I$.

where all $c_1, c_2, \ldots\ldots, c_n$ are not simultaneously zero, leading to conclusion that $\phi_1, \phi_2, \ldots\ldots, \phi_n$ are linearly dependent which contradicts the fact that $\phi_1, \ldots\ldots, \phi_n$ are linearly independent on I. Thus, the supposition that there was a point $x_0 \in I$ such that (1) holds must be false. Consequently, we must have $W(\phi_1, \phi_2, \ldots\ldots, \phi_n)(x) \neq 0$ for all x in I

Theorem VII. ***Abel's formula***

Let functions $p_1(x)$ and $p_2(x)$ in $L(y)(x) = y''(x) + p_1(x)y'(x) + p_2(x)y(x) = 0, x \in I$ ... (i) be defined and continuous on an interval I. Let ϕ_1, any ϕ_2 be two linearly independent solutions of (1) existing on I containing a point x_0. Then,

$$W(\phi_1, \phi_2)(x) = \exp\left(-\int_{x_0}^{x} p_1(x)\,dx\right) W(\phi_1, \phi_2)(x_0) \qquad \ldots (ii)$$

[Note that here exp A stands for e^A] **[Delhi B.Sc. (Hons) II 2011]**

Proof. Given $y''(x) + p_1(x)y'(x) + p_2(x)y(x) = 0, x \in I$

Here,
$$W(\phi_1, \phi_2) = \begin{vmatrix} \phi_1 & \phi_2 \\ \phi_1' & \phi_2' \end{vmatrix} = \phi_1 \phi_2' - \phi_1' \phi_2 \qquad \ldots (iii)$$

From (*iii*),
$$W'(\phi_1, \phi_2) = \phi_1' \phi_2' + \phi_1 \phi_2'' - (\phi_1'' \phi_2 + \phi_1' \phi_2')$$

or
$$W'(\phi_1, \phi_2) = \phi_1 \phi_2'' - \phi_1'' \phi_2 \qquad \ldots (iv)$$

Since ϕ_1 and ϕ_2 satisfy (*i*), we have

$$\phi_1'' + p_1 \phi_1' + p_2 \phi_1 = 0 \Rightarrow \phi_1'' = -p_1 \phi_1' - p_2 \phi_1$$

and
$$\phi_2'' + p_1 \phi_2' + p_2 \phi_2 = 0 \Rightarrow \phi_2'' = -p_1 \phi_2' - p_2 \phi_2$$

Substituting the above values of ϕ_1'' and ϕ_2'' in (*iv*), we get

$$W'(\phi_1, \phi_2) = \phi_1(-p_1 \phi_1' - p_2 \phi_1) - \phi_2(-p_1 \phi_2' - p_2 \phi_2)$$

or
$$W'(\phi_1, \phi_2) = -p_1(\phi_1 \phi_2' - \phi_2 \phi_1') = -p_1 W(\phi_1, \phi_2), \text{ by } (iii)$$

Thus, $W(\phi_1, \phi_2)$ satisfies a first order linear homogeneous equation $W' + p_1 W = 0, x \in I$

or
$$\frac{dW}{dx} = -p_1 W \quad \text{or} \quad \frac{dW}{W} = -p_1\,dx \quad \text{or} \quad \log W - \log c = -\int_{x_0}^{x} p_1\,dx$$

so that
$$W(\phi_1, \phi_2)(x) = c \exp\left(-\int_{x_0}^{x} p_1(x)\,dx\right) \qquad \ldots (v)$$

where c is a constant. By putting $x = x_0$ in (*v*), we get $c = W(\phi_1, \phi_2)(x_0)$. Substituting this value of c in (*v*), we get the required result.

We now state and prove Abel's formula for general case:

Statement. *Let the functions $p_1(x), p_2(x), \ldots\ldots, p_n(x)$ in*

$$L(y)(x) = y^{(n)}(x) + p_1(x)y^{(n-1)}(x) + p_2(x)y^{(n-2)}(x) + \ldots\ldots + p_n(x)y(x) = 0, x \in I \ldots (1)$$

be defined and continous on an interval I. Let $\phi_1, \phi_2, \ldots\ldots, \phi_n$ be n linearly independent solutions of (1) existing on I containing a point x_0. Then, we have

$$W(\phi_1, \ldots\ldots, \phi_n)(x) = \exp\left(-\int_{x_0}^{x} p_1(x)\,dx\right) W(\phi_1, \ldots\ldots, \phi_n)(x_0) \qquad \ldots (2)$$

Proof. We let $W(x) = W(\phi_1, \phi_2, ..., \phi_n)(x)$ for brevity. We have,

$$W(x) = W(\phi_1 ...., \phi_n)(x) = \begin{vmatrix} \phi_1 & \phi_2 & ...... & \phi_n \\ \phi_1' & \phi_2' & ...... & \phi_n' \\ \phi_1'' & \phi_2'' & ...... & \phi_n'' \\ ...... & ...... & ...... & ...... \\ \phi_1^{(n-1)} & \phi_2^{(n-1)} & ...... & \phi_n^{(n-1)} \end{vmatrix} \quad ... (3)$$

Differentiating $W(x)$ w.r.t. 'x', we see that W' is a sum of n determinants.

$$W'(x) = V_1 + V_2 + ... + V_i + ... + V_n, \quad ... (4)$$

where V_i differs from $W(x)$ only in the i th row, and the i th row of V_i is obtained by differentiating the i th row of $W(x)$. Thus, we arrive at

$$W'(x) = \begin{vmatrix} \phi_1' & \phi_2' & ...... & \phi_n' \\ \phi_1' & \phi_2' & ...... & \phi_n' \\ \phi_1'' & \phi_2'' & ...... & \phi_n'' \\ ...... & ...... & ...... & ...... \\ \phi_1^{(n-1)} & \phi_2^{(n-1)} & ...... & \phi_n^{(n-1)} \end{vmatrix} + \begin{vmatrix} \phi_1 & \phi_2 & ...... & \phi_n \\ \phi_1'' & \phi_2'' & ...... & \phi_n'' \\ \phi_1'' & \phi_2'' & ...... & \phi_n'' \\ ...... & ...... & ...... & ...... \\ \phi_1^{(n-1)} & \phi_2^{(n-1)} & ...... & \phi_n^{(n-1)} \end{vmatrix} + . + \begin{vmatrix} \phi_1 & \phi_2 & ...... & \phi_n \\ \phi_1' & \phi_2' & ...... & \phi_n' \\ \phi_1'' & \phi_2'' & ...... & \phi_n'' \\ ...... & ...... & ...... & ...... \\ \phi_1^{(n)} & \phi_2^{(n)} & ...... & \phi_n^{(n)} \end{vmatrix}$$

The first $n - 1$ determinants $V_1, ..., V_{n-1}$ are all zero, since they each have two identical rows. Thus, we obtain

$$W'(x) = \begin{vmatrix} \phi_1 & \phi_2 & ...... & \phi_n \\ \phi_1' & \phi_2' & ...... & \phi_n' \\ ...... & ...... & ...... & ...... \\ \phi_1^{(n-2)} & \phi_2^{(n-2)} & ...... & \phi_n^{(n-2)} \\ \phi_1^{(n)} & \phi_2^{(n)} & ...... & \phi_n^{(n)} \end{vmatrix} \quad ... (5)$$

Now, since $\phi_1, \phi_2, ..., \phi_n$ are solutions of (1), we have

$$\phi_i^{(n)} + p_1 \phi_i^{(n-1)} + p_2 \phi_i^{(n-2)} + ... + p_n \phi_i = 0 \quad \text{for} \quad i = 1, 2, ..., n$$

$$\Rightarrow \quad \phi_i^{(n)} = -p_1 \phi_i^{(n-1)} - p_2 \phi_i^{(n-2)} - ...... - p_n \phi_i \quad \text{for} \quad i = 1, 2, ..., n \quad ... (6)$$

Putting $i = 1, 2, ..., n$ in (6), get values of $\phi_1^{(n)}, \phi_2^{(n)}, ..., \phi_n^{(n)}$ and substitute these values in (5) and obtain

$$W' = \begin{vmatrix} \phi_1 & \phi_2 & \phi_n \\ \phi_1' & \phi_2' & \phi_n' \\ \phi_1'' & \phi_2'' & \phi_n'' \\ ...... & ...... & ...... \\ \phi_1^{(n-2)} & \phi_2^{(n-2)} & \phi_n^{(n-2)} \\ -p_1\phi_1^{(n-1)} - p_2\phi_1^{(n-2)} ...... - p_n\phi_1 & -p_1\phi_2^{(n-1)} - p_2\phi_2^{(n-2)} ...... - p_n\phi_2 & -p_1\phi_n^{(n-1)} - p_2\phi_n^{(n-2)} ...... - p_n\phi_n \end{vmatrix}$$

From the properties of a determinant, we know that the value of the above determinant is unchanged, if we multiply any row by a number and add to the last row. Multiplying the first row by p_n, the second row by p_{n-1}, ..., the $(n - 1)$ – st row by p_2 and adding these to the last row, we obtain

$$W'(x) = \begin{vmatrix} \phi_1 & \phi_2 & \ldots\ldots & \phi_n \\ \phi_1' & \phi_2' & \ldots\ldots & \phi_n' \\ \vdots & \vdots & \ldots\ldots & \vdots \\ \phi_1^{(n-2)} & \phi_2^{(n-2)} & \ldots\ldots & \phi_n^{(n-2)} \\ -p_1\phi_1^{(n-1)} & -p_1\phi_2^{(n-1)} & \ldots\ldots & -p_1\phi_n^{(n-1)} \end{vmatrix} = -p_1 W(x), \text{ by (3)}$$

Thus, W satisfies a first order linear differential equation

$$dW/dx = -pW \quad \text{or} \quad (1/W)\, dW = -p_1\, dx$$

Integrating, $\log W - \log c = -\int_{x_0}^{x} p_1\, dx$

so that $$W(x) = c \exp\left(-\int_{x_0}^{x} p_1(x)\, dx\right), \qquad \ldots (7)$$

where c is an arbitrary constant. By putting $x = x_0$ in (4), we get $c = W(x_0)$. Substituting this value of c in (4), we get

$$W(x) = \exp\left(-\int_{x_0}^{x} p_1(x)\, dx\right) W(x_0)$$

i.e., $$W(\phi_1, \phi_2, \ldots\ldots, \phi_n)(x) = \exp\left(-\int_{x_0}^{x} p_1(x)\, dx\right) W(\phi_1, \phi_2, \ldots\ldots, \phi_n)(x_0) \qquad \ldots (8)$$

Corollary. *If the coefficients $p_1(x), p_2(x), \ldots\ldots, p_n(x)$ of equation* (1) *are constants, then*

$$W(\phi_1, \phi_2, \ldots\ldots, \phi_n)(x) = e^{-p_1(x-x_0)}\; W(\phi_1, \phi_2, \ldots\ldots, \phi_n)(x_0)$$

Proof. If p_1 is constant, then we have $\int_{x_0}^{x} p_1\, dx = p_1 \int_{x_0}^{x} dx = p_1(x - x_0)$

$\therefore$ (8) yields, $W(\phi_1, \phi_2, \ldots\ldots, \phi_n)(x) = \exp[-p_1(x - x_0)]\; W(\phi_1, \phi_2, \ldots\ldots, \phi_n)(x_0)$

i.e., $W(\phi_1, \phi_2, \ldots\ldots, \phi_n)(x) = e^{-p_1(x-x_0)}\; W(\phi_1, \phi_2, \ldots\ldots, \phi_n)(x_0)$

An important application of Abel's formula. *A consequence of Abel's formula is that n solutions $\phi_1, \phi_2, \ldots\ldots, \phi_n$ of* equation (*1*) *on an interval I are linearly independent there if and only if $W(\phi_1, \phi_2, \ldots\ldots, \phi_n)(x_0) \neq 0$ for any particular x_0 in interval I.*

From (5), it is clear that if $W(\phi_1, \phi_2, \ldots\ldots, \phi_n)(x_0) \neq 0$, then $W(\phi_1, \phi_2, \ldots\ldots, \phi_n)(x) \neq 0$ for $x \in I$. Hence, it is enough to show that $W(\phi_1, \phi_2, \ldots\ldots, \phi_n)(x) \neq 0$ only at just one point of I. This criterion yields the linear independence of n solutions of (1).

We now given a simple illustration of the use of Abel's formula.

Example. *To show that solutions $\phi_1(x) = e^{2x}$, $\phi_2(x) = xe^{2x}$ and $\phi_3(x) = x^2 e^{2x}$ are linearly independent solutions of $y''' - 6y'' + 12y' - 8y = 0$ on an interval $0 \leq x \leq 1$.*

Solution. We have

$$W(\phi_1, \phi_2, \phi_3)(x) = \begin{vmatrix} e^{2x} & xe^{2x} & x^2 e^{2x} \\ d(e^{2x})/dx & d(x e^{2x})/dx & d(x^2 e^{2x})/dx \\ d^2(e^{2x})/dx^2 & d^2(x e^{2x})/dx^2 & d^2(x^2 e^{2x})/dx^2 \end{vmatrix}$$

or $$W(\phi_1, \phi_2, \phi_3)(x) = \begin{vmatrix} e^{2x} & xe^{2x} & x^2 e^{2x} \\ 2e^{2x} & (1+2x)\, e^{2x} & (2x+2x^2)\, e^{2x} \\ 4e^{2x} & (4+4x)\, e^{2x} & (2+8x+4x^2)\, e^{2x} \end{vmatrix} \qquad \ldots (1)$$

Clearly, it is not very easy to evaluate R.H.S. of (1). We chose $0 \in [0, 1]$. Then, from (1),

$$W(\phi_1, \phi_2, \phi_3)(0) = \begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 4 & 2 \end{vmatrix} = 2 \qquad \text{... (2)}$$

By corollary of Abel's formula, we have

$$W(\phi_1, \phi_2, \phi_3)(x) = e^{-p_1(x-x_0)}\, W(\phi_1, \phi_2, \phi_3)(x_0) \qquad \text{... (3)}$$

Here $p_1 = -6$ and $x_0 = 0$. Hence, (3) reduces to $W(\phi_1, \phi_2, \phi_3)(x) = 2e^{6x}$, using (2).

1.17 Solved examples based on Art. 1.16

Ex. 1. *If $y_1(x) = \sin 3x$ and $y_2(x) = \cos 3x$ are two solutions of $y'' + 9y = 0$, show that $y_1(x)$ and $y_2(x)$ are linearly independent solutions.* **[Delhi Maths (Hons) 1996]**

Sol. The Wronskian of $y_1(x)$ and $y_2(x)$ is given by

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} \sin 3x & \cos 3x \\ 3\cos 3x & -3\sin 3x \end{vmatrix}$$

$$= -3\sin^2 3x - 3\cos^2 3x = -3(\sin^2 3x + \cos^2 3x) = -3 \neq 0.$$

Since $W(x) \neq 0$, $y_1(x)$ and $y_2(x)$ are linearly independent solutions of $y'' + 9y = 0$.

Ex. 2. *Prove that $\sin 2x$ and $\cos 2x$ are solutions of $y'' + 4y = 0$ and these solutions are linearly independent.* **[Delhi Maths (G) 1998]**

Sol. Given equation is $\qquad y'' + 4y = 0. \qquad$ ... (1)

Let $\qquad y_1(x) = \sin 2x \qquad$ and $\qquad y_2(x) = \cos 2x. \qquad$ ... (2)

Now, $\qquad y_1'(x) = 2\cos 2x \qquad$ and $\qquad y_1''(x) = -4\sin 2x. \qquad$ ... (3)

$\therefore \qquad y_1''(x) + 4y_1(x) = -4\sin 2x + 4\sin 2x = 0$, by (2) and (3)

Hence, $y_1(x) = \sin 2x$ is a solution of (1). Similarly, we can prove that $y_2(x)$ is a solution of (1). Now, the Wronskian $W(x)$ of $y_1(x)$ and $y_2(x)$ is given by

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} \sin 2x & \cos 2x \\ 2\cos 2x & -2\sin 2x \end{vmatrix}$$

$$= -2\sin^2 2x - 2\cos^2 2x = -2(\sin^2 2x + \cos^2 2x) = -2 \neq 0.$$

Since $W(x) \neq 0$, $\sin 2x$ and $\cos 2x$ are linearly independent solutions of (1).

Ex. 3. *Show that linearly independent solutions of $y'' - 2y' + 2y = 0$ are $e^x \sin x$ and $e^x \cos x$. What is the general solution? Find the solution $y(x)$ with the property $y(0) = 2$, $y'(0) = 3$.*

[Delhi B.A. (Prog.) 2009; Delhi Maths (Hons) 2002; Delhi Maths (G) 2006]

Sol. Given equation is $\qquad y'' - 2y' + 2y = 0. \qquad$ ... (1)

Let $\qquad y_1(x) = e^x \sin x \qquad$ and $\qquad y_2(x) = e^x \cos x. \qquad$ ... (2)

From (2), $\qquad y_1'(x) = e^x \sin x + e^x \cos x = e^x(\sin x + \cos x) \qquad$ ... (3)

From (3), $\qquad y_1''(x) = e^x(\sin x + \cos x) + e^x(\cos x - \sin x) = 2e^x \cos x. \qquad$ ... (4)

$\therefore \qquad y_1''(x) - 2y_1'(x) + 2y_1(x) = 2e^x \cos x - 2e^x(\sin x + \cos x) + 2e^x \sin x = 0,$

showing that $y_1(x) = e^x \sin x$ is a solution of (1).

Similarly, we can show that $y_2(x) = e^x \cos x$ is a solution of (1).

Now, the Wronskian $W(x)$ of $y_1(x)$ and $y_2(x)$ is given by

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} e^x \sin x & e^x \cos x \\ e^x (\sin x + \cos x) & e^x (\cos x - \sin x) \end{vmatrix}$$

$$= e^{2x} (\sin x \cos x - \sin^2 x) - e^{2x} (\sin x \cos x + \cos^2 x) = - e^{2x} \neq 0,$$

showing that $W(x) \neq 0$, and hence $y_1(x)$ and $y_2(x)$ are linearly independent solutions of (1). The general solution of (1) is [Refer theorem V, Art. 1.16]

$$y(x) = c_1 y_1(x) + c_2 y_2(x) = e^x (c_1 \sin x + c_2 \cos x), \qquad \text{... (5)}$$

where c_1 and c_2 are arbitrary constants.

From (5), $\quad y'(x) = e^x (c_1 \sin x + c_2 \cos x) + e^x (c_1 \cos x - c_2 \sin x)$. ... (6)

Putting $x = 0$ in (5) and using the given result $y(0) = 2$, we get $\quad y(0) = c_2 \quad$ or $\quad c_2 = 2$

Putting $x = 0$ in (6) and using the given result $y'(0) = -3$, we get

$$y'(0) = c_2 + c_1 \qquad \text{or} \qquad -3 = 2 + c_2 \qquad \text{or} \qquad c_1 = -5, \text{ as } c_2 = 2$$

$\therefore$ From (5), solution of (1)satisfying the given properties is $y = e^x (2 \cos x - 5 \sin x)$.

Ex. 4. *Show that e^{2x} and e^{3x} are linearly independent solutions of $y'' - 5y' + 6y = 0$. Find the solution $y(x)$ with the property that $y(0) = 0$ and $y'(0) = 1$.* **[Delhi Maths (G) 98, 2006]**

Sol. Given equation is $\quad y'' - 5y' + 6y = 0$. ... (1)

Let $\quad y_1(x) = e^{2x} \quad$ and $\quad y_2(x) = e^{3x}$. ... (2)

From (2), $\quad y_1'(x) = 2e^{2x} \quad$ and $\quad y_1''(x) = 4e^{2x}$. ... (3)

$\therefore \quad y_1''(x) - 5y_1'(x) + 6y_1(x) = 4e^{2x} - 5(2e^{2x}) + 6e^{2x} = 0,$

showing that $y_1(x)$ is a solution of (1). Similarly, $y_2(x) = e^{3x}$ is a solution of (1).

Now, the Wronskian $W(x)$ of $y_1(x)$ and $y_2(x)$ is given by

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = 3e^{5x} - 2e^{5x} = e^{5x} \neq 0,$$

showing that e^{2x} and e^{3x} are linearly independent solutions of (1).

The general solution of (1) is given by

$$y(x) = c_1 e^{2x} + c_2 e^{3x}, \; c_1 \text{ and } c_2 \text{ being arbitrary constants.} \qquad \text{... (4)}$$

From (4), $\quad y'(x) = 2c_1 e^{2x} + 3c_2 e^{3x}$. ... (5)

Putting $x = 0$ in (4) and using $y(0) = 0$, $\quad c_1 + c_2 = 0$. ... (6)

Putting $x = 0$ in (5) and using $y'(0) = 1$, $\quad 2c_1 + 3c_2 = 1$. ... (7)

Solving (6) and (7), $c_1 = -1$ and $c_2 = 1$ and so from (4), we have

$$y(x) = e^{3x} - e^{2x} \text{ as the required solution.}$$

Ex. 5. (*a*) *Show that $y_1(x) = \sin x$ and $y_2(x) = \sin x - \cos x$ are linearly independent solutions of $y'' + y = 0$. Determine the constants c_1 and c_2, so that the solution $\sin x + 3 \cos x \equiv c_1 y_1(x) + c_2 y_2(x)$.* **[Delhi Maths (P) 2002]**

Sol. Given equation is $\quad y'' + y = 0$. ... (1)

Here $\quad y_1(x) = \sin x, \quad$ so that $\quad y_1'(x) = \cos x \quad$ and $\quad y_1''(x) = -\sin x$. ... (2)

Hence, $y_1''(x) + y_1(x) = -\sin x + \sin x = 0$, showing that $y_1(x)$ is a solution of (1). Similarly, we can show that $y_2(x)$ is also a solution of (1).

Now, the Wronskian of $y_1(x)$ and $y_2(x)$ is given by

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} \sin x & \sin x - \cos x \\ \cos x & \cos x + \sin x \end{vmatrix}$$

$$= \sin x (\cos x + \sin x) - \cos x (\sin x - \cos x) = 1 \neq 0,$$

showing that $y_1(x)$ and $y_2(x)$ are linearly independent solutions of (1).

Given that $$\sin x + 3\cos x \equiv c_1 y_1(x) + c_2 y_2(x)$$

or $$\sin x + 3\cos x \equiv c_1 \sin x + c_2(\sin x - \cos x). \qquad \text{... (3)}$$

Comparing the coefficients of $\sin x$ and $\cos x$ on both sides of (3), we have

$$c_1 + c_2 = 1 \quad \text{and} \quad -c_2 = 3 \quad \text{so that} \quad c_1 = 4 \quad \text{and} \quad c_2 = -3.$$

Ex. 5. (*b*) *Define Wronskian. Evaluate Wronskian of the functions $y_1(x) = \sin x$ and $y_2(x) = \sin x - \cos x$ and hence conclude whether or not they are linearly independent. Also, form the differential equation.* **[Meerut 2003]**

Sol. For definition of Wronskian, refer Art. 1.12

For second part, refer Ex. 5(*a*)

For the last part, proceed as follows: Since $\sin x$ and $\sin x - \cos x$ are linearly independent functions, these will form solution of a differential equation of the form

$$y = A\sin x + B(\sin x - \cos x),\ A, B \text{ being parameters.} \qquad \text{...(1)}$$

Differentiating (1) w.r.t. 'x', we have

$$y' = A\cos x + B(\cos x + \sin x) \qquad \text{...(2)}$$

From (2), $$y'' = -A\sin x + B(-\sin x + \cos x) \qquad \text{...(3)}$$

Adding (1) and (3), $y + y'' = 0$, which in the desired differential equation.

Ex. 6. *Show that x and xe^x are linearly independent on the x-axis.*

Sol. The Wronskian $W(x)$ of x and xe^x is given by

$$W(x) = \begin{vmatrix} x & xe^x \\ dx/dx & d(xe^x)/dx \end{vmatrix} = \begin{vmatrix} x & xe^x \\ 1 & e^x + xe^x \end{vmatrix} = x(e^x + xe^x) - xe^x = x^2 e^x.$$

Thus, $W(x) \neq 0$ for $x \neq 0$ on the x-axis. Hence, x and xe^x are linearly independent on the x-axis [Refer corollary to theorem III of Art. 1.16]

Ex. 7. *Show that the Wronskian of the functions x^2 and $x^2 \log x$ is non-zero. Can these functions be independent solutions of an ordinary differential equation. If so, determine this differential equation.* **[Meerut 1998]**

Sol. Let $y_1(x) = x^2$ and $y_2(x) = x^2 \log x$.

The Wronskian $W(x)$ of $y_1(x)$ and $y_2(x)$ is given by,

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x^2 & x^2\log x \\ 2x & 2x\log x + x \end{vmatrix} = x^2(2x\log x + x) - 2x^3 \log x.$$

$\therefore$ $W(x) = x^3$, which is not identically equal to zero on $(-\infty, \infty)$. Hence, functions $y_1(x)$ and $y_2(x)$, *i.e.*, x^2 and $x\log x$ can be linearly independent solutions of an ordinary differential equation.

To form the required differential equation. The general solution of the required differential equation may be written as, $$y = A\,y_1(x) + B\,y_2(x) = Ax^2 + Bx^2\log x, \qquad \text{... (1)}$$
where A and B are arbitrary constants.

Differentiating (1), we get $$y' = 2Ax + B(2x\log x + x). \qquad \text{... (2)}$$

Differentiating (2), we get $$y'' = 2A + B(2\log x + 2 + 1). \qquad \text{... (3)}$$

We now eliminate A and B from (1), (2) and (3). To this end, we first solve (2) and (3) for A and B. Multiplying both sides of (3) by x, we get

$$xy'' = 2Ax + B(3x + 2x\log x). \qquad \text{... (4)}$$

Subtracting (2) from (4), $xy'' - y' = 2Bx$ or $B = (xy'' - y')/2x$.

Substituting this value of B in (3), we have

$$2A = y'' - (1/2x) \times (xy'' - y')(3 + 2\log x) \quad \text{or} \quad A = (1/4x) \times [2xy'' - (xy'' - y')(3 + 2\log x)].$$

Substituting the above values A and B in (1), we have

$$y = (x/4) \times [2xy'' - 3xy'' + 3y' - 2xy'' \log x + 2y' \log x] + (x/2) \times (xy'' - y') \log x$$

or $$4y = x\,(-xy'' + 3y' - 2xy'' \log x + 2y' \log x) + 2x\,(xy'' - y') \log x$$

or $$x^2y'' - 3xy' + 4y = 0, \text{ which is the required equation.}$$

Ex. 8. *Find the Wronskian of x and xe^x. Hence, conclude whether or not these are linearly independent. If they are independent, set up the differential equation having them as its independent solutions.* **[Meerut 1997]**

Sol. Let $y_1 = x$ and $y_2 = xe^x$. Then their Wronskian $W(x)$ is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & xe^x \\ 1 & e^x + xe^x \end{vmatrix} = xe^x + x^2e^x - xe^x = x^2e^x,$$

which is not identically equal to zero on $(-\infty, \infty)$. Hence, y_1 and y_2 are linearly independent.

To form the required differential equation. The general solution of the required differential equation may be written as $$y = Ay_1 + By_2 = Ax + Bx\,e^x, \qquad \text{... (1)}$$
where A and B are arbitrary constants.

Differentiating (1), we get $$y' = A + B\,(e^x + xe^x) = A + B\,(1 + x)\,e^x. \qquad \text{... (2)}$$

Differentiating (2), we get $$y'' = B\,[e^x + (1 + x)\,e^x] = Be^x\,(2 + x). \qquad \text{... (3)}$$

We now eliminate A and B from (1), (2) and (3). From (3), $B = y''/[e^x\,(2 + x)]$.

Substituting this value of B in (2), we have

$$A = y' - B\,(1 + x)\,e^x = y' - \frac{1+x}{2+x}\,y'' = \frac{(2+x)\,y' - (1+x)\,y''}{2+x}.$$

Substituting the above values of A and B in (1), we get

$$y = \left[\frac{(2+x)\,y' - (1+x)\,y''}{2+x}\right] x + \left[\frac{y''}{e^x\,(2+x)}\right] xe^x$$

or $$(2 + x)\,y = x\,(2 + x)\,y' - x\,(1 + x)\,y'' + xy''$$

or $$x^2y'' - x\,(2 + x)\,y' + (2 + x)\,y = 0, \text{ which is required equation.}$$

Ex. 9. (*a*) *Show that the solutions e^x, e^{-x}, e^{2x} of $(d^3y/dx^3) - 2\,(d^2y/dx^2) - (dy/dx) + 2y = 0$ are linearly independent and hence or otherwise solve the given equation.*

[Delhi Maths (G) 1993, 98; Meerut 1998]

Sol. Given equation is $$y''' - 2y'' - y' + 2y = 0. \qquad \text{... (1)}$$

Let $\quad y_1 = e^x, \quad y_2 = e^{-x} \quad$ and $\quad y_3 = e^{2x} \qquad$ ... (2)

Here $\quad y_1' = e^x, \quad y_1'' = e^x \quad$ and $\quad y_1''' = e^x. \qquad$ ... (3)

$\therefore \quad y_1''' - 2y_1'' - y_1' + 2y_1 = e^x - 2e^x - e^x + 2e^x = 0$, by (2) and (3)

Hence, $y_1 = e^x$ is a solution of (1). Similarly, show that e^{-x} and e^{2x} are also solutions of (1).

Now, the Wronskian $W(x)$ of y_1, y_2, y_3 is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} & e^{2x} \\ e^x & -e^{-x} & 2e^{2x} \\ e^x & e^{-x} & 4e^{2x} \end{vmatrix}$$

$$= (e^x \cdot e^{-x} \cdot e^{2x}) \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 1 & 1 & 4 \end{vmatrix} = e^{2x} \begin{vmatrix} 1 & 0 & 0 \\ 1 & -2 & 1 \\ 1 & 0 & 3 \end{vmatrix} \quad \begin{bmatrix} \text{using operations} \\ C_2 \to C_2 - C_1 \\ C_3 \to C_3 - C_1 \end{bmatrix}$$

$= -6e^{2x}$, which is not identically zero on $(-\infty, \infty)$

Hence, y_1, y_2, y_3 are linearly independent solutions of (1) [Refer corollary of theorem III of Art. 1.16]. Since the order of the given equation (1) is three, it follows that the general solution of (1) will contain three arbitrary constants c_1, c_2, c_3 and is given by [Refer Theorem V of Art. 1.16]

$$y = c_1 y_1 + c_2 y_2 + c_3 y_3, \qquad i.e., \qquad y = c_1 e^x + c_2 e^{-x} + c_3 e^{2x}.$$

Ex. 9. (*b*) *Show that the* e^{-x}, e^{3x}, e^{4x} *are linearly independent solutions of* $d^3y/dx^3 - 6(d^2y/dx^2) + 5\,(dy/dx) + 12y = 0$ *on the interval* $-\infty < x < \infty$ *are* write the general solution. **[Delhi B.A (Prog) II 2011]**

Hint. Try yourself as in Ex. 9. (*a*) **Ans.** $y = c_1 e^{-x} + c_2 e^{3x} + c_4 e^{4x}$.

Ex. 10. *Prove that the functions 1, x,* x^2 *are linearly independent. Hence, form the differential equation whose solutions are 1, x,* x^2. **[Meerut 1997]**

Sol. Let $\quad y_1(x) = 1, \quad y_2(x) = x \quad$ and $\quad y_3(x) = x^2. \quad$ ... (1)

Then the Wronskian $W(x)$ of y_1, y_2, y_3 is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} 1 & x & x^2 \\ 0 & 1 & 2x \\ 0 & 0 & 2 \end{vmatrix}, \text{using (1)}$$

or $$W(x) = 2 \neq 0 \text{ for any } x \in (-\infty, \infty).$$

Hence, y_1, y_2 and y_3 are linearly independent.

To form the required differential equation. The general solution of the required differential equation may be written as $\quad y = Ay_1 + By_2 + Cy_3 = A + Bx + Cx^2, \quad$ ... (1)

where A, B, C are arbitrary constants.

Differentiating (1) w.r.t. 'x', we get $\quad y' = B + 2Cx. \quad$ ... (2)

Differentiating (2) w.r.t. 'x', we get $\quad y'' = 2C. \quad$ ... (3)

Differentiating (3) w.r.t. 'x', we get $\quad y''' = 0, \quad i.e., \quad d^3y/dx^3 = 0. \quad$ ... (4)

Since (4) is free from arbitrary constants hence (4) is the required differential equation.

Ex. 11. *Use Wronskian to show that x,* x^2, x^3 *are independent. Determine the differential equation with these as independent solutions.* **[Meerut 1995, 2001, 2002]**

Sol. Let $\quad y_1(x) = x, \quad y_2(x) = x^2 \quad$ and $\quad y_3 = x^3. \quad$ ... (1)

The Wronskian $W(x)$ of y_1, y_2 and y_3 is given by

$$W(x) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} x & x^2 & x^3 \\ 1 & 2x & 2x^2 \\ 0 & 2 & 6x \end{vmatrix}, \text{using (1)}$$

or $$W(x) = x\,(12x^2 - 6x^2) - (1) \times (6x^3 - 2x^3) = 2x^3,$$

which is not identically equal to zero. Hence, the functions y_1, y_2 and y_3 are linearly independent.

To form the differential equation. The general solution of the required differential equation may be written as $\quad y = Ay_1 + By_2 + Cy_3 = Ax + Bx^2 + Cx^3. \quad$ ... (2)

Differentiating (2) w.r.t. 'x', we get $\quad y' = A + 2Bx + 3Cx^2. \quad$ ... (3)

Differentiating (3) w.r.t. 'x', we get $\quad y'' = 2B + 6Cx. \quad$ ... (4)

Differentiating (4) w.r.t. 'x', we get $\quad y''' = 6C. \quad$ ... (5)

From (5), $C = y'''/6$. Then, from (4), $\quad B = (y'' - xy''')/2. \quad$ ... (6)

Multiplying both sides of (3) by x, $\quad xy' = Ax + 2Bx^2 + 3Cx^3. \quad$ ... (7)

Subtracting (7) from (2), we get $\quad y - xy' = -Bx^2 - 2Cx^3$

or $$y - xy' = -(1/2) \times x^2\,(y'' - xy''') - (2x^3) \times (y'''/6), \text{ using (5) and (6)}$$

or $\quad 6y - 6xy' = -3x^2y'' + 3x^3y''' - 2x^3y''' \quad$ or $\quad x^3y''' - 3x^2y'' + 6xy' - 6y = 0,$
which is the required differential equation.

Ex. 12. *If $y_1(x)$ and $y_2(x)$ are linearly independent solutions of the differential equation $a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$, then show that any other solution of the equation can be written in the form $y(x) = C_1y_1(x) + C_2y(x)$, where C_1 and C_2 are suitably chosen constants.*

Sol. Refer theorem I of Art. 1.16. **[Delhi Maths (G) 2004]**

Ex. 13. *If $y_1(x)$ and $y_2(x)$ are linearly independent solutions of*

$$a_0(x)y'' + a_1(x)\cdot y'(x) + a_2(x)y = 0,$$

then prove that every other solution of the equation is a linear combination of two solutions $y_1(x)$ and $y_2(x)$. Hence, show that every solution of $d^2y/dx^2 + y = 0$ is a linear combination of $\cos x + \sin x$ and $\cos x - \sin x$. **[Delhi Maths (H) 2004]**

Sol. Ist part. See theorem I of Art. 1.16.

Second part. Given $\quad d^2y/dx^2 + y = 0 \quad$ or $\quad (D^2+1)y = 0,\ D \equiv d/dx \quad \ldots(1)$

$\therefore$ Solution of (1) is $\quad y = A\cos x + B\sin x$, A, B being constants. $\quad \ldots(2)$

Let $\quad y_1(x) = \cos x + \sin x \quad$ and $\quad y_2(x) = \cos x - \sin x \quad \ldots(3)$

From (3), $\quad dy_1/dx = -\sin x + \cos x$

and $\quad d^2y_1/dx^2 = -\cos x - \sin x \quad \ldots(4)$

From (3) and (4), $\quad d^2y_1/dx^2 + y_1(x) = -\cos x - \sin x + \cos x + \sin x = 0,$
showing that $y_1(x)$ is a solution of (1). Similarly, $y_2(x)$ is also a solution of (1).

Now, the Wronskian $W(x)$ of $y_1(x)$ and $y_2(x)$ is given

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} \cos x + \sin x & \cos x - \sin x \\ -\sin x + \cos x & -\sin x - \cos x \end{vmatrix}$$

$$= -(\cos x + \sin x)^2 - (\cos x - \sin x)^2 = -2 \neq 0.$$

Since the Wronskian of $y_1(x)$ and $y_2(x)$ is non-zero, it follows that $y_1(x)$ and $y_2(x)$ are linearly independent solutions of (1). Hence by the first part, every solution of (1) will be of the form $y = C_1y_1(x) + C_2y_2(x)$, *i.e.*,

$$y = C_1(\cos x + \sin x) + C_2(\cos x - \sin x) = (C_1 + C_2)\cos x + (C_1 - C_2)\sin x$$

or $\quad y = A\cos x + B\sin x, \quad$ taking $\quad A = C_1 + C_2 \quad$ and $\quad B = C_1 - C_2$
which is the same as (2). Hence, the required result follows.

Ex. 14. *Show that $\sin x$, $\cos x$ and $\sin x - \cos x$ are solutions of the differential equation $y'' + y = 0$, where $y' = dy/dx$. Prove that these solutions are linearly dependent.* (*Use the idea of Wronskian*) **[Delhi Maths (Prog) 2007]**

Sol. Given differential equation is $\quad y'' + y = 0 \quad \ldots(1)$

Let $\quad y_1 = \sin x, \quad y_2 = \cos x \quad$ and $\quad y_3 = \sin x - \cos x \quad \ldots(2)$

Then, $\quad y_1' = \cos x \quad$ and $\quad y_1'' = -\sin x \quad$ and so $\quad y_1'' + y_1 = 0,$
showing that y_1 is a solution of (1). Similary, we find that y_2 and y_3 are also solutions of (1).

Now, $$W(y_1, y_2, y_3) = \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix} = \begin{vmatrix} \sin x & \cos x & \sin x - \cos x \\ \cos x & -\sin x & \cos x + \sin x \\ -\sin x & -\cos x & -\sin x + \cos x \end{vmatrix}$$

$$= \begin{vmatrix} \sin x & \cos x & \sin x - \cos x \\ \cos x & -\sin x & \cos x + \sin x \\ 0 & 0 & 0 \end{vmatrix}, \text{ by operating } R_3 \to R_3 + R_1$$

Hence, $\quad W(y_1, y_2, y_3) = 0$ and therefore y_1, y_2, y_3 are linearly dependent as desired.

Exercise 1(C)

1. Prove that the Wronskian of the functions $e^{m_1 x}, e^{m_2 x}, e^{m_3 x}$, is equal to $(m_1 - m_2)(m_2 - m_3)(m_3 - m_1)$ $e^{(m_1 + m_2 + m_3)x}$. Are these functions linearly independent.

Ans. Given functions are linearly independent if $m_1 \neq m_2 \neq m_3$.

2. Test the linear independence of the following sets of functions:

(*i*) $\sin x, \cos x$. **Ans.** Linearly independent

(*ii*) $1 + x, 1 + 2x, x^2$. **Ans.** Linearly independent

(*iii*) $x^2 - 1, x^2 - x + 1, 3x^2 - x - 1$. **Ans.** Linearly dependent

(*iv*) $\sin x, \cos x, \sin 2x$. **[Meerut 2010]** **Ans.** Linearly independent

(*v*) $e^x, e^{-x}, \sin ax$. **Ans.** Linearly independent

(*vi*) $e^x, xe^x, \sinh x$. **Ans.** Linearly independent

(*vii*) $\sin 3x, \sin x, \sin^3 x$. **Ans.** Linearly dependent

3. Show that the functions $e^x \cos x$ and $e^x \sin x$ are linearly independent. Form the differential equation of second order having these two functions as independent solutions. **Ans.** $y'' - 2y' + 2y = 0$

4. Evaluate the Wronskian of the functions e^x and xe^x. Hence, conclude whether or not they are linearly independent. If they are independent set up the differential equation having them as its independent solutions. **Ans.** $y'' - 2y' + y = 0$

5. Show that linearly independent solutions of $y'' - 3y' + 2y = 0$ are e^x and e^{2x}. Find the solution $y(x)$ with the property that $y(0) = 0$, $y'(0) = 1$. **[Delhi Maths (G) 2000]** **Ans.** $y(x) = e^{2x} - e^x$

6. Show that the $y_1(x) = x$ and $y_2(x) = |x|$ are linearly independent on the real line, even though the Wronskian cannot be computed.

7. Show graphically that $y_1(x) = x^2$ and $y_2(x) = x|x|$ are linearly independent on $-\infty < x < \infty$, however, Wronskian vanishes for every real value of x.

8. Show that e^x and e^{-x} are linearly independent solutions of $y'' - y = 0$ on any interval.

[Lucknow 2001; Nagpur 1996]

9. Show that $y_1(x) = e^{-x/2} \sin(x\sqrt{3}/2)$ and $y_2(x) = e^{-x/2} \cos(x\sqrt{3}/2)$ are linearly independent solutions of the differential equation $y'' + y' + y = 0$. **[Delhi Maths (G) 1999, 2000]**

10. Using the idea of Wronskian, show that $e^x \cos x$ and $e^x \sin x$ are linearly independent solution of $y'' - 2y' + 2y = 0$. Find the solution with the property that $y(0) = 1$ and $y'(0) = 2$.

[Delhi Maths (H) 2004]

Hint. Proceed like Ex. 3 of Art. 1.17. **Ans.** $y = e^x(\cos x + \sin x)$

11. Define the Wronskian of two solutions $y_1(x)$ and $y_2(x)$ of the equation $a_0(x)y'' + a_1(x)y' + a_2(x)y = 0$.
[Delhi Maths (G) 2006]

Ans. Refer Art. 1.12. Accordingly, the Wronskian $W(y_1, y_2)$ of y_1 and y_2 is given by

$$W(y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

1.18 Linear differential equation and its general solution

Linear differential equation contains dependent variable and its derivative in their first degree. The most general form of linear differential equation of order n is

$$y^{(n)} + P_1 y^{(n-1)} + \dots + P_n y = Q, \qquad \dots (1)$$

where $P_1, P_2, \dots, P_n, Q$ are functions of x and are assumed to be continuous on interval I.

The differential equation $$y^{(n)} + P_1 y^{(n-1)} + \dots + P_n y = 0 \qquad \dots (2)$$

is said to be associated homogeneous equation of (1).

We now state an important theorem without proof. (Proof follows from the well known existence uniqueness theorem of differential equation of nth order, refer Art. 1.14A) In what follows we shall use the following notations $y^{(1)} = dy/dx$, $y^{(2)} = d^2y/dx^2$, $y_1^{(1)} = dy_1/dx$,

$y_1^{(2)} = d^2 y_1 / dx^2$, $y_2^{(1)} = dy_2 / dx$ and $y_2^{(2)} = d^2 y_2 / dx^2$, $W^{(1)} = dW / dx$ and so on

Theorem I. *A solution $y(x)$ of (2), satisfying the initial conditions $y(x_0) = y^{(1)}(x_0) = \ldots\ldots = y^{(n-1)}(x_0) = 0$, is identically zero.*

As a particular case ($n = 1$), we have

Theorem II. *If a solution of a first-order equation* $y^{(1)} + Py = 0$ ... (3) *vanishes at a single point x_0, the solution is identically zero.*

Theorem III. *The Wronskian of two solutions of differential equation*

$$y^{(2)} + Py^{(1)} + Qy = 0, \qquad \text{... (4)}$$

where P, Q are either constants or functions of x alone, is either identically zero or never zero.

[Delhi Maths (Hons.) 1997, 2000]

Proof. Let $y_1(x)$ and $y_2(x)$ be two solutions of (4). Then, we have

$$y_1^{(2)} + Py_1^{(1)} + Qy_1 = 0 \quad \text{and} \quad y_2^{(2)} + Py_2^{(1)} + Qy_2 = 0$$

$$\Rightarrow \quad y_1^{(2)} = -(Py_1^{(1)} + Qy_1) \quad \text{and} \quad y_2^{(2)} = -(Py_2^{(1)} + Qy_2). \qquad \text{... (5)}$$

Now, the Wronskian W of y_1 and y_2 is given by

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1^{(1)} & y_2^{(1)} \end{vmatrix} = y_1 y_2^{(1)} - y_2 y_1^{(1)}. \qquad \text{... (6)}$$

$$(6) \Rightarrow \qquad W^{(1)} = y_1^{(1)} y_2^{(1)} + y_1 y_2^{(2)} - \left[y_2^{(1)} y_1^{(1)} + y_2 y_1^{(2)} \right] = y_1 y_2^{(2)} - y_1^{(2)} y_2$$

or $$W^{(1)} = -y_1 \left\{ Py_2^{(1)} + Qy_2 \right\} + y_2 \left\{ Py_1^{(1)} + Qy_1 \right\}, \text{ using (5)}$$

or $$W^{(1)} = -P \left\{ y_1 y_2^{(1)} - y_2 y_1^{(1)} \right\} = -PW, \text{ using (6)}$$

$$\Rightarrow \qquad W^{(1)} + PW = 0,$$

showing that W is identically zero or never zero (refer theorem II).

Theorem IV. *Consider the linear differential equation* $y^{(2)} + Py^{(1)} + Qy = 0$, ... (7) *where P, Q are either constants or functions of x alone. Then two solutions of* (7) *are linearly dependent if and only if their Wronskian vanishes identically.*

Proof. Let $y_1(x)$ and $y_2(x)$ be solutions of (7). Let W be the Wronskian of y_1, y_2, so that

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1^{(1)}(x) & y_2^{(1)}(x) \end{vmatrix}$$

Assume that $W(x) \equiv 0$. If x_0 be any point, then we have

$$W(x_0) \equiv 0, \qquad \text{so that} \qquad \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1^{(1)}(x_0) & y_2^{(1)}(x_0) \end{vmatrix} = 0$$

$\Rightarrow$ There exist constants c_1 and c_2, not both zero, such that

$$c_1 y_1(x_0) + c_2 y_2(x_0) = 0 \quad \text{and} \quad c_1 y_1^{(1)}(x_0) + c_2 y_2^{(1)}(x_0) = 0.$$

Let $$y(x) = c_1 y_1(x) + c_2 y_2(x). \qquad \text{... (8)}$$

Then (8) shows that $y(x)$ is a solution of (7) satisfying the conditions $y(x_0) = 0$, $y^{(1)}(x_0) = 0$.

$\therefore$ $y(x) = 0$ for all x (using Theorem I)

$\Rightarrow$ There exist constants c_1 and c_2 not both zero, such that $c_1 y_1(x) + c_2 y_2(x) = 0$, for all x.

$\Rightarrow$ y_1 and y_2 are linearly dependent, by definition.

Conversely, let y_1, y_2 be linearly dependent. Then, there exist constants c_1, c_2 not both zero, such that $c_1 y_1(x) + c_2 y_2(x) = 0$, for all x. ... (9A)

Differentiating (9A) w.r.t. 'x', we get $c_1 y_1^{(1)}(x) + c_2 y_2^{(1)}(x) = 0$, for all x. ... (9B)

Eliminating c_1, c_2 from (9A) and (9B), we have

$$\begin{vmatrix} y_1(x) & y_2(x) \\ y_1^{(1)}(x) & y_2^{(1)}(x) \end{vmatrix} = 0, \text{ for all } x.$$

$\Rightarrow$ $W(x) = 0$, for all x $\Rightarrow$ Wronskian of y_1, y_2 vanishes identically.

Corollary. *Two solutions of* (7) *are linearly independent if their Wronskian does not vanish identically.*

Proof. Left as an exercise for the reader.

Theorem V. *The general solution of differential equation* $y^{(2)} + Py^{(1)} + Qy = 0$, ... (10) *where P, Q are either constants or functions of x alone, can be put in the form*

$$c_1 y_1(x) + c_2 y_2(x), \qquad \text{... (11)}$$

where c_1, c_2 *are constants and* y_1, y_2 *are any pair of linearly independent solutions of* (10).

Proof. Clearly $c_1 y_1 + c_2 y_2$ is a solution of (10). To prove the required result, it is sufficient to prove that every solution of (10) can be put in the form (11). To this end, we assume that

$$y = c_1 y_1 + c_2 y_2, \qquad \text{... (12)}$$

where y is any solution of (1) and c_1, c_2 are constants.

(12) $\Rightarrow$ $$y^{(1)} = c_1 y_1^{(1)} + c_2 y_2^{(1)}. \qquad \text{... (13)}$$

Solving (12) and (13) for c_1, c_2, we have

$$c_1 = (1/W) \times \left\{ y y_2^{(1)} - y^{(1)} y_2 \right\}, \qquad c_2 = (1/W) \times \left\{ y^{(1)} y_1 - y y_1^{(1)} \right\}, \qquad \text{... (14)}$$

where $W =$ Wronskian of y_1 and $y_2 = \begin{vmatrix} y_1 & y_2 \\ y_1^{(1)} & y_2^{(1)} \end{vmatrix}$

$= \{ y_1 y_2^{(1)} - y_2 y_1^{(1)} \} \neq 0$, for all x $\qquad$ ($\because$ y_1, y_2 are linearly independent)

For c_1, c_2 given by (14), y and $c_1 y_1 + c_2 y_2$ have the same value at a point x and the same result applies to their derivatives. The required result now follows by the existence theorem of solution of differential equation.

Theorem VI. *If* $y_1(x), y_2(x)$ *be any two linearly independent solutions of the homogeneous differential equation* $y^{(2)} + Py^{(1)} + Qy = 0$ ... (15)

and y_0 *is any particular solution of the non-homogeneous differential equation*

$$y^{(2)} + Py^{(1)} + Qy = R, \qquad \text{... (16)}$$

the general solution of (16) *is* $y_0 + c_1 y_1 + c_2 y_2$, ... (17)

where c_1, c_2 *are arbitrary constants.*

Proof. Since y_0 is a solution of (16), we have $y_0^{(2)} + Py_0^{(1)} + Qy_0 = R$. ... (18)

Let y be any arbitrary solution of (16). Then, we have

$$y^{(2)} + Py^{(1)} + Qy = R. \qquad \text{... (19)}$$

Let $u = y - y_0$. ... (20)

(20) $\Rightarrow$ $u^{(1)} = y^{(1)} - y_0^{(1)}$ and $u^{(2)} = y^{(2)} - y_0^{(2)}$. ... (21)

Subtracting (18) from (19), we have $y^{(2)} - y_0^{(2)} + P\{y^{(1)} - y_0^{(1)}\} + Q(y - y_0) = 0$

or $$u^{(2)} + P u^{(1)} + Q u = 0, \text{ using (20) and (21)}$$
showing that u is a solution of (15) and so we have $u = c_1 y_1 + c_2 y_2$, (c_1, c_2 are some constants)
or $$y - y_0 = c_1 y_1 + c_2 y_2, \text{ using (20)}$$
so that $$y = y_0 + c_1 y_1 + c_2 y_2,$$
showing that (17) is the general solution of (16).

All theorems of the present article have natural generalization to equation of higher order. We close this article by giving extension of Theorem VI without proof.

Theorem VII. *If $y_1, y_2, ..., y_n$ be any n linearly independent solutions of the homogeneous linear differential equation of the nth order* $$y^{(n)} + P_1 y^{(n-1)} + ... + P_n y = 0 \qquad ... (22)$$
and y_0 is any particular solution of the non-homogeneous differential equation
$$y^{(n)} + P_1 y^{(n-1)} + ... + P_n (y) = Q, \qquad ... (23)$$
the general solution of (23) is $$y_0 + c_1 y_1 + c_2 y_2 + ... + c_n y_n, \qquad ... (24)$$
where c's are arbitrary constants.

Remark. The general solution $c_1 y_1 + c_2 y_2 + ... + c_n y_n$ of (22) is called complementary function (C.F.) and the particular solution y_0 of (23) is called particular integral (P.I.).

Example. *Show that $y = C_1 e^{2x} + C_2 xe^{2x}$ is the general solution of $y'' - 4y' + 4y = 0$ on any interval.* **[Nagpur 2002]**

Sol. Given $$y'' - 4y' + 4y = 0 \qquad ... (1)$$
Let $y = e^{2x}$ so that $y' = 2e^{2x}$ and $y'' = 4e^{2x}$

Then, $$\text{L.H.S. of (1)} = 4e^{2x} - 8e^{2x} + 4e^{2x} = 0 \qquad ... (2)$$
Again, let $y = xe^{2x}$, so that $y' = e^{2x} + 2xe^{2x}$, $y'' = 2e^{2x} + 2e^{2x} + 4xe^{2x}$

Then, $$\text{L.H.S. of (1)} = 4e^{2x} + 4xe^{2x} - 4\,(e^{2x} + 2xe^{2x}) + 4xe^{2x} = 0 \qquad ... (3)$$
From (2) and (3), it follows that e^{2x} and xe^{2x} are solutions of (1).

Again, Wronskian of $e^{2x}, xe^{2x} = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & e^{2x} + 2xe^{2x} \end{vmatrix} = e^{4x} \neq 0$ for all x.

Hence by theorem V, Art. 1.18, it follows that $y = C_1 e^{2x} + C_2 xe^{2x}$ is the general solution of (1) in any interval.

OBJECTIVE PROBLEMS ON CHAPTER 1

Ex. 1. *The differential equation of the family of circles of radius 'r' whose centre lie on the x-axis, is* (*a*) $y\,(dy/dx) + y^2 = r^2$ (*b*) $y\,\{(dy/dx) + 1\} = r^2$

(*c*) $y^2\,\{(dy/dx) + 1\} = r^2$ (*d*) $y^2\,\{(dy/dx)^2 + 1\} = r^2$ **[I.A.S. (Prel.) 1993]**

Sol. Ans. (*d*). Equation of a family of circles of radius r whose centre lie on the x-axis is given by $$(x - \lambda)^2 + y^2 = r^2, \text{ where } \lambda \text{ is a parameter} \qquad ... (1)$$
Differentiating w.r.t. 'x', (1) gives
$$2\,(x - \lambda) + 2yy' = 0, \quad \text{so that} \quad x - \lambda = -\,yy' \qquad ... (2)$$
Then, (1) and (2) $\Rightarrow$ $y^2 y'^2 + y^2 = r^2$ or $y^2\,\{(dy/dx)^2 + 1\} = r^2$.

Ex. 2. *The equation of the curve, for which the angle between the tangent and the radius vector is twice the vectorial angle is $r^2 = A \sin 2\theta$. This satisfies the differential equation*

(*a*) $r\,(dr/d\theta) = \tan 2\theta$ (*b*) $r\,(d\theta/dr) = \tan 2\theta$

(*c*) $r\,(dr/d\theta) = \cos 2\theta$ (*d*) $r\,(d\theta/dr) = \cos 2\theta$. **[I.A.S. (Prel.) 1993]**

Sol. Ans. (*b*). Given that $$r^2 = A \sin 2\theta \qquad ... (1)$$
Differentiating w.r.t. 'θ', (1) gives

$2r\,(dr/d\theta) = 2A \cos 2\theta$ or $r\,(dr/d\theta) = A \cos 2\theta$ ... (2)

Dividing (1) by (2), $r\,(d\theta/dr) = \tan 2\theta$.

Ex. 3. *The maximum number of linearly independent solutions of the differential equation $d^4y/dx^4 = 0$ with the condition $y\,(0) = 1$ is*

(*a*) 4 (*b*) 3 (*c*) 2 (*d*) 1 **[GATE 2010]** **Ans. (*a*)**

Ex. 4. *The order and degree of differential equation $\{1 + (dy/dx)^2\}^{3/2} = k\,(d^2y/dx^2)$ is*

(*a*) 3_c1 (*b*) 3_c2

(*c*) 3, 3 (*d*) 2, 2 **[Garhwal 2010]**

Sol. Ans. (*d*). See Art. 1.4, 1.5 and 1.6

Ex. 5. *Consider the following differential equations:*

$$1.\ x^2\left(\frac{d^2y}{dx^2}\right)^6 + y^{-2/3}\left\{1+\left(\frac{d^3y}{dx^3}\right)^5\right\}^{1/2} + \frac{d^2}{dx^2}\left\{\left(\frac{d^2y}{dx^2}\right)^{-2/3}\right\} = 0$$

2. $dy/dx - 6x = \{ay + bx\,(dy/dx)\}^{-3/2},\ b \neq 0.$

The sum of the order of the first differential equation and degree of the second differential equation is (*a*) 6 (*b*) 7 (*c*) 8 (*d*) 9. **[I.A.S. (Prel.) 2002]**

Sol. Ans. (*d*). Re-writing the last term of the first equation

$$\frac{d^2}{dx^2}\left\{\left(\frac{d^2y}{dx^2}\right)^{-2/3}\right\} = \frac{d}{dx}\left\{\left(-\frac{2}{3}\right)\left(\frac{d^2y}{dx^2}\right)^{-5/3}\frac{d^3y}{dx^3}\right\} = -\frac{2}{3}\frac{d}{dx}\left\{\left(\frac{d^2y}{dx^2}\right)^{-5/3}\frac{d^3y}{dx^3}\right\}$$

$$= -\frac{2}{3}\left\{-\left(\frac{5}{3}\right)\left(\frac{d^3y}{dx^2}\right)^{-8/3}\left(\frac{d^3y}{dx^3}\right)^2 + \left(\frac{d^2y}{dx^2}\right)^{-5/3}\left(\frac{d^4y}{dx^4}\right)\right\},$$

which involves fourth order derivative d^4y/dx^4. So, by definition of order of a differential equation (see Art 1.4), the order of the first equation is 4.

Next, re-writing the second differential equation, we get

$$(dy/dx - 6x)\ \{ay + bx\,(dy/dx)\}^{3/2} = 1$$

Squaring both sides, we get $(dy/dx - 6x)^2\ \{ay + bx\,(dy/dx)\}^3 = 1$

or $\{(dy/dx)^2 - 12x\,(dy/dx) + 36x^2\} \times \{a^3y^3 + 3a^2y^2bx\,(dy/dx)$

$+ 3ayb^2x^2\,(dy/dx)^2 + b^3x^3\,(dy/dx)^3\} = 1$ ... (1)

On multiplying the two factors on the L.H.S. of (1), we find that $(dy/dx)^5$ occurs in the resulting equation. Hence by definition of degree of a differential equation (see Art. 1.5), the degree of the given second equation is 5. Hence, the sum of the order of the first equation and the degree of the second equation is (4 + 5), *i.e.*, 9.

Ex. 6. *The degree of the equation* $(d^3y/dx^3)^{2/3} + (d^3y/dx^3)^{3/2} = 0$ *is*

(*a*) 3 (*b*) 5 (*c*) 4 (*d*) 9. **[I.A.S. (Prel.) 2004]**

Sol. Ans. (*d*). Re-writing the given equation, we have

$(d^3y/dx^3)^{2/3} = -\,(d^3y/dx^3)^{3/2}$ or $(d^3y/dx^3)^4 = (d^3y/dx^3)^9$

So, by the definition, the degree of the given equation is 9.

Ex. 7. *Linear combinations of solutions of an ordinary differential equation are solutions if the differential equation is* (*a*) *Linear non-homogeneous* (*b*) *Linear homogeneous*

(*c*) *Non-linear homogeneous* (*d*) *Non-linear non-homogeneous* **[GATE 2002]**

Sol. Ans. (*b*). Refer theorem V of Art. 1.18.

Ex. 8. *Which of the following pair of functions is not a linearly independent solutions of* $y'' + 9y = 0$? (a) $\sin 3x, \sin 3x - \cos 3x$ (b) $\sin 3x + \cos 3x, 3 \sin x - 4 \sin^3 x$

(c) $\sin 3x, \sin 3x \cos 3x$ (d) $\sin 3x + \cos 3x, 4 \cos^3 x - 3 \cos x$. **[GATE 2001]**

Sol. Ans. (c). Use theorem IV of Art. 1.18.

Ex. 9. *Let* $y = \phi(x)$ *and* $y = \psi(x)$ *be solutions of* $y'' - 2xy' + (\sin x^2)\, y = 0$, *such that* $\phi(0) = 1$, $\phi'(0) = 1$ *and* $\psi(0) = 1$, $\psi'(0) = 2$. *The value of Wromhian* $W(\phi, \psi)$ *at* $x = 0$ *is*

(a) 0 (b) 1 (c) e (d) e^2 **[GATE 2004]**

Sol. Ans. (b). We know that

$$W(\phi, \psi) = \begin{vmatrix} \phi(x) & \phi'(x) \\ \psi(x) & \psi'(x) \end{vmatrix} \text{ and hence its value at } x = 0 \text{ is given by}$$

$$\begin{vmatrix} \phi(0) & \phi'(0) \\ \psi(0) & \psi'(0) \end{vmatrix}, \text{ i.e., } \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}, \text{ i.e., } 2 - 1, \text{ i.e., } 1.$$

Ex. 10. *What are the order and degree respectively of the differential equation*

$\dfrac{d^2}{dx^2}\left(\dfrac{d^2 y}{dx^2}\right)^{-3/2} = 0$ (a) 1, 4 (b) 4, 1 (c) 4, 4 (d) 1, 1. **[I.A.S. (Prel.) 2006]**

Sol. Ans. (b) Re-writing the given differential equation, we have

$$\frac{d}{dx}\left\{\frac{d}{dx}\left(\frac{d^2 y}{dx^2}\right)^{-3/2}\right\} = 0 \quad \text{or} \quad -\frac{3}{2}\frac{d}{dx}\left\{\left(\frac{d^2 y}{dx^2}\right)^{-5/2}\frac{d^3 y}{dx^3}\right\} = 0$$

or $$-\frac{3}{2}\left\{-\frac{5}{2}\left(\frac{d^2 y}{dx^2}\right)^{-7/2}\left(\frac{d^3 y}{dx^3}\right)^2 + \left(\frac{d^2 y}{dx^2}\right)^{-5/2}\frac{d^4 y}{dx^4}\right\} = 0$$

or $$\frac{5}{2}\left(\frac{d^2 y}{dx^2}\right)^{-7/2}\left(\frac{d^3 y}{dx^3}\right)^2 = \left(\frac{d^2 y}{dx^2}\right)^{-5/2}\frac{d^4 y}{dx^4} \quad \text{or} \quad 5\left(\frac{d^3 y}{dx^3}\right)^2 = 2\frac{d^2 y}{dx^2}\frac{d^4 y}{dx^4}$$

By definitions, its order is 4 and degree is 1.

Ex. 11. *What is the degree of the differential equation for a given curve in which* (*subtangent*)^m = (*subnormal*)^n *in cartesian form, where* $0 < n < m$, $m, n, m/n$ *are integers?*

(a) $m + n$ (b) $m - n$ (c) mn (d) m/n. **[I.A.S. (Prel.) 2006]**

Sol. Ans. (a) From calculus, we know that

subtangent $= y/(dy/dx) = y\,(dy/dx)^{-1}$, and subnormal $= y\,(dy/dx)$

Hence, the relation (subtangent)^m = (subnormal)^n

$$\Rightarrow \left\{y\left(\frac{dy}{dx}\right)^{-1}\right\}^m = \left(y\,\frac{dy}{dx}\right)^n \quad \text{or} \quad y^{m-n} = \left(\frac{dy}{dx}\right)^{m+n},$$

which is a differential equation of order $m + n$.

Ex. 12. *What are the order and degree respectivels of the differental equation of the family of curves* $y^2 = 2c\,(x + \sqrt{c})$,

(a) *1, 1* (b) *1, 2* (c) *1, 3* (d) *2, 1* **[I.A.S. Prel. 2007]**

Sol. Ans. (c) Given $\qquad y^2 = 2c\,(x + \sqrt{c}) \qquad$ ... (1)

Differentiating (1) w.r.t. 'x' $\qquad 2yy' = 2c \qquad$ so that $\qquad c = yy' \qquad$ ... (2)

Substituting the value of c given by (2) in (1), we have

$$y^2 = 2yy'\{x+(yy')^{1/2}\} \quad \text{or} \quad y^2 - 2xyy' = 2yy'(yy')^{1/2} \quad ...(3)$$

Squaring both sies of (3), we get $\qquad y^4 - 4xy^3y' + 4x^2y^2y'^2 = 4y^3y'^3$

which is a differential equation whose order is one and degree is three.

Ex. 13. *Which one of the following statement is correct ? The differential equation* $(dx/dy)^2 + 5y^{1/3} = x$ *is (a) linear equation of order 2 and degree 1 (b) nonlinear equation of order 1 and degree 2 (c) non-linear equation of order 1 and degree 6 (d) linear equation of order 1 and degree 6.* **[I.A.S. (Prel.) 2008]**

Sol. Ans. (b). Refer Art. 1.4 and Art. 1.5

Miscellaneous problems on chapter 1

Ex.1. *Show that sin 3x, cos 3x and sin 3x + cos 3x are solutions of differential equation* $y'' + 9y = 0$. Are these solutions linearly dependent ? Use the idea of Wronskian. **[Delhi 2008]**

Hint. Proceed like solved Ex. 14, page 1.27.

Ex. 2. *Show that* e^{2x} *and* e^{3x} *are linearly independent solutions of the equation* $y'' - 5y + 6 = 0$ on $-\infty < x < \infty$. *What is the general solution ? Find the solution y(x) that satisfies the conditions :* $y(0) = 2, y'(0) = 3$. **[Delhi B.A. (Prog) II 2010]**

Hint: Proceed as in Ex. 4, page 1.23. General solution is $y(x) = c_1e^{2x} + c_2 e^{3x}$. The solution satisfying the given initial conditions is $y(x) = 3e^{2x} - e^{3x}$.

Ex. 3. If $y_1(x) = 1 + x$ and $y_2(x) = e^x$ be two solution of $y''(x) + P(x)y'(x) + Q(x)y(x) = 0$, then $P(x) =$ *(a)* $1 + x$ *(b)* $-1 - x$ *(c)* $(1 + x)/x$ *(d)* $(-1 - x)/x$ **[GATE 2009]**

Ex. 4. *Consider the differential equation* $y''(x) + P(x)y'(x) + Q(x)y(x) = 0$. *The set of inital conditions for which the above differential equation has no solution is*

(*a*) $y(0) = 2;\ y'(0) = 1$ (*c*) $y(1) = 0,\ y'(1) = 1$

(*c*) $y(1) = 1,\ y'(1) = 0$ (*d*) $y(2) = 1,\ y'(2) = 2$ **[GATE 2009]**

Ex. 5. *If* $y_1(x)$ *and* $y_2(x)$ *are linearly independent solutions of the homogeneous differential equations* $y'' + P(x)y' + Q(x)y = 0$, then show that $P(x) = \{(y_1y_2'' - y_2y_1'')/W(y_1, y_2)\}$ and $Q(x) = (y_1'y_2'' - y_2'y_1'')/W(y_1, y_2)$. Hence construct the differential equation having two linearly independent solution e^{2x} and xe^{2x}. **[Mumbai 2010] Ans.** $y'' - 4y' + 4 = 0$

Ex. 6. *Let* y_1 *and* y_2 *be two solution of the differential equation* $y'' + p(x)y' + Q(x)y = 0$. *on* $[a, b]$. *If* y_1 *and* y_2 *have a maxima at* $x_0 \in (a, b)$, then show that y_2 is a constant multiple of y_1 or y_1 is a constant multiple of y_2 on $[a, b]$. **[Mumbai 2010]**

Ex. 7. *Let* y_1 *and* y_2 *be two linearly independent solutions of the second order differential equation* $y'' + p(x)y' + q(x)y = 0$. *and* $W[y_1, y_2]$ *be their Wronkian. Show that* $dW/dx = -p(x)W$. *Hence deduce that* $W = k \exp\left(-\int p(x)dx\right)$, *where k in constant.* **[Delhi B.Sc. (Hons) II 2011]**

Hint. Proceed as in theorem vii, page 1.19. Here note that, we have $\phi_1 = y_1$, $\phi_2 = y_2$, $p_1(x) = p(x)$, $p_2(x) = q(x)$, $W(\phi_1, \phi_2) = W(y_1, y_2)$ *and* $c = k$.

Thus, we get $dW/dx = -p(x)W$ *so that* $(1/W)\,dW = -p(x)\,dx$. *Integrating,*

$$\log W - \log K = -\int p(x)\,dx \qquad W = K\,esp\left(-\int p(x)\,dx\right).$$

Ex. 8. *Which one of the following equations has the same order and degree?*

(a) $d^4y/dx^4 + 8\ (dy/dx)^4 + 5y = e^x$ (b) $5\ (d^3y/dx^3)^4 + 8\ (dy/dx + 1)^2 + 5y = x^3$

(c) $\{1 + (dy/dx)^3\}^{2/3} = 4\ (d^3y/dx^3)$ (d) $y = x^2\ (dy/dx) + \{(dy/dx)^2 + 1\}^{1/2}$.**[I.A.S. (Prel.) 2005]**

Sol. Ans. (*c*). Refer Art. 1.4 and Art. 1.5.

Ex. 9. *Let y_1 and y_2 be any two solutions of a second order linear non-homogeneous ordinary differential equation and c be any arbitrary constant. Then, in general*

(a) *$y_1 + y_2$ is its solution, but cy_1 is not* (b) *cy_1 is its solution, but $y_1 + y_2$ is not*

(c) *both $y_1 + y_2$ and cy_1 are its solutions* (d) *neither $y_1 + y_2$ nor cy_1 is its solution.*

[I.A.S. (Prel.) 2005]

Sol. Ans. (*d*). Refer Art. 1.18.

Ex. 10. *Consider the following statements regarding the differential equation* $|\,dy/dx\,| + |\,y\,| = 0$, $0 < x < 1$ *satisfying* $y\ (0) = 1$: 1. *It is a linear differential equation* 2. *It has a unique solution. Which of the statements given above is/are correct?*

(a) 1 *only* (b) 2 *only* (c) *both* 1 *and* 2 (d) *neither* 1 *nor* 2. **[I.A.S. (Prel.) 2005]**

2

Equations of First Order And First Degree

2.1 Introduction

There are two standard forms of differential equations of first order and first degree, namely,

(*i*) $dy/dx = f(x, y)$ (*ii*) $M(x, y)\, dx + N(x, y)\, dy = 0$.

In what follows we shall see that an equation in one of these forms may readily be written in the other form. It will be assumed that the necessary conditions for the existence of solutions are satisfied. We now discuss various methods to solve such equations.

2.2 Separation of variables

If in an equation, it is possible to get all the functions of x and dx to one side and all the functions of y and dy to the other, the variables are said to be separable.

Working rule to solve an equation in which variables are separable.

Step 1: Let $$dy/dx = f_1(x) f_2(y), \qquad \text{... (1)}$$

be given equation. $f_1(x)$ is a function of x alone and $f_2(y)$ is a function of y alone.

Step 2: From (1), separating variables, $$[1/f_2(y)]\, dy = f_1(x)\, dx. \qquad \text{... (2)}$$

Step 3: Integrating both sides of (2), we have

$$\int [1/f_2(y)]\, dy = \int f_1(x)\, dx + c, \qquad \text{... (3)}$$

where c is constant of integration, is the required solution.

Note 1. In all solutions (3), an arbitrary constant c must be added in any one side only. If c is not added, then the solution obtained will not be a general solution of (1).

Note 2. To simplify the solution (3), the constant of integration can be chosen in any suitable form so as to get the final solution in a form as simple as possible. Accordingly, we write $\log c$, $\tan^{-1} c$, $\sin c$, e^c, $(1/2) \times c$, $(-1/3) \times c$ etc. in place of c in some solutions.

Note 3. The students are advised to remember by heart the following formulas. These will help them to write solution (3) in compact form

(*i*) $\log x + \log y = \log xy$. (*ii*) $\log x - \log y = \log (x/y)$.

(*iii*) $n \log x = \log x^n$. (*iv*) $\tan^{-1} x + \tan^{-1} y = \tan^{-1} [(x + y)/(1 - xy)]$

(*v*) $\tan^{-1} x - \tan^{-1} y = \tan^{-1} [(x - y)/(1 + xy)]$ (*vi*) $e^{\log f(x)} = f(x)$.

2.3 Examples of Type 1 based on Art 2.2

Ex. 1. (*a*) *Solve* $dy/dx = e^{x-y} + x^2 e^{-y}$

[Agra 1995; Lucknow 1998; Mysore 2004;Punjab 1994; Meerut 2009; Agra 2005]

(*b*) *Solve* $dy/dx = e^{x+y} = x^2 e^y$

Sol. (*a*) For separating variables, we re-write the given equation as

$dy/dx = e^{-y}(e^x + x^2)$ or $e^y dy = (x^2 + e^x)\, dx$.

Integrating, $e^y = x^3/3 + e^x + c$, c being an arbitrary constant.

(b) Do like part (a). **Ans.** $-e^{-y} = x^3/3 + e^x + c$.

Ex. 2. *Find the curves passing through* (0, 1) *and satisfying* $\sin (dy/dx) = c$.

[I.A.S. (Prel.) 2005]

Sol. Re-writing the given equation, we have

$dy/dx = \sin^{-1} c$ or $dy = (\sin^{-1} c)\, dx$.

Integrating, $y = x \sin^{-1} c + c'$, c' being arbitrary constant. ... (1)

Since, (1) must pass through (0, 1), we put $x = 0$ and $y = 1$ in (1) and obtain $c' = 1$. Hence, (1) reduces to $y = x \sin^{-1} c + 1$ or $(y - 1)/x = \sin^{-1} c$

or $\sin \{(y - 1)/x\} = c$, which gives the desired curves.

Ex. 3. *Solve* $(dy/dx) \tan y = \sin (x + y) + \sin (x - y)$.

Sol. Using formula $\sin C + \sin D = 2 \sin \{(C + D)/2\} \cos \{(C - D)/2\}$, the given equation can be rewritten as

$$(\tan y)\, (dy/dx) = 2 \sin x \cos y \quad \text{or} \quad \sec y \tan y\, dy = 2 \sin x\, dx.$$

Integrating, $\sec y = -2 \cos x + c$, c being an arbitrary constant.

Ex. 4. *Solve the following differential equations:*

(i) $\dfrac{dy}{dx} = \dfrac{\sin x + x \cos x}{y\,(2 \log y + 1)}$ (ii) $\dfrac{dy}{dx} = \dfrac{x\,(2 \log x + 1)}{\sin y + y \cos y}$. **[I.A.S. (Prel.) 2009]**

Sol. (i) Re-writing the given equation, $(\sin x + x \cos x)\, dx = (2y \log y + y)\, dy$.

Integrating, $$-\cos x + \int x \cos x\, dx = 2 \int y \log y\, dy + (y^2/2) + c. \quad \text{... (1)}$$

Now, $$\int x \cos x\, dx = x \sin x - \int \sin x\, dx, \text{ integrating by parts}$$

or $$\int x \cos x\, dx = x \sin x + \cos x. \quad \text{... (2)}$$

Also, $$\int y \log y\, dy = (\log y) \times (y^2/2) - \int \{(1/y) \times (y^2/2)\}\, dy, \text{ integrating by parts}$$

or $$\int y \log y\, dy = (y^2/2) \times \log y - y^2/4 \quad \text{... (3)}$$

Using (2) and (3), (1) reduces to

$$-\cos x + x \sin x + \cos x = 2\ \{(y^2/2) \times \log y - y^2/4\} + y^2/2 + c.$$

or $x \sin x = y^2 \log y + c$, c being an arbitrary constant.

(ii) Proceed exactly as in part (i). **Ans.** $x^2 \log x = y \sin y + c$.

Ex. 5. *Solve* $\log (dy/dx) = ax + by$.

Sol. Re-writing the given equation, we get $dy/dx = e^{ax+by} = e^{ax} e^{by}$ or $e^{-by}\, dy = e^{ax}\, dx$.

Integrating, $-(1/b)\, e^{-by} = (1/a)\, e^{ax} + c$, c being an arbitrary constant.

Ex. 6. *Solve* $y - x\, (dy/dx) = a\, (y^2 + dy/dx)$. **[Meerut 1993; Delhi Maths (G) 1994; Purvanchal 2006, Rajasthan 1995; Agra 1993; Indore 1993]**

Sol. The given equation can be re-written as

$$(a + x) \frac{dy}{dx} = y - ay \quad \text{or} \quad \frac{dx}{x + a} = \frac{dy}{y\,(1 - ay)}$$

or $$\frac{dx}{x + a} = \left[\frac{a}{1 - ay} + \frac{1}{y}\right] dy, \text{ on resolving into partial fractions.}$$

Integrating, $\log(x+a) = -\log(1-ay) + \log y + \log c,$

or $\log(x+a) = \log\left[\dfrac{cy}{1-ay}\right]$ or $x + a = \dfrac{cy}{1-ay}$

or $(x+a)(1-ay) = cy$, which is the required solution.

Ex. 7. *Solve* $3e^x \tan y\, dx + (1 - e^x)\sec^2 y\, dy = 0.$ **[Meerut 2008; Kanpur 1997]**

Sol. Separating the variables, we get $\dfrac{3e^x}{1-e^x}dx + \dfrac{\sec^2 y}{\tan y}dy = 0.$

Integrating, $-3\log(1-e^x) + \log(\tan y) = \log c$, c being an arbitrary constant.

or $\log(\tan y) = \log(1-e^x)^3 + \log c$ or $\tan y = c(1-e^x)^3.$

Ex. 8. *Solve* $\sqrt{(1 + x^2 + y^2 + x^2y^2)} + xy\,(dy/dx) = 0.$

Sol. Re-writing the given differential equation, we have

$\sqrt{[(1+x^2)(1+y^2)]} + xy\,(dy/dx) = 0$ or $\sqrt{(1+x^2)}\sqrt{(1+y^2)} + xy\,(dy/dx) = 0$

or $\dfrac{\sqrt{(1+x^2)}\,dx}{x} + \dfrac{y\,dy}{\sqrt{(1+y^2)}} = 0$ or $\dfrac{(1+x^2)\,dx}{x\sqrt{(1+x^2)}} + \dfrac{y\,dy}{\sqrt{(1+y^2)}} = 0.$

Integrating, $$\int \frac{dx}{x(1+x^2)^{1/2}} + \int \frac{x\,dx}{(1+x^2)^{1/2}} + \int \frac{y\,dy}{(1+y^2)^{1\backslash 2}} = \mathrm{C}. \qquad \ldots(1)$$

Now, $$\int \frac{dx}{x(1+x^2)^{1/2}} = \int \frac{(-1/t^2)\,dt}{(1/t)\sqrt{1+(1/t)^2}}, \text{ putting } x = \frac{1}{t}$$

$$= -\int \frac{dt}{\sqrt{(t^2+1)}} = -\log\{t + \sqrt{t^2+1}\}$$

$$= -\log\left\{\frac{1}{x} + \sqrt{\left(\frac{1}{x^2}+1\right)}\right\} = -\log\left\{\frac{1+\sqrt{(1+x^2)}}{x}\right\}$$

$$= \log x - \log\{1 + (1+x^2)^{1/2}\} \qquad \ldots(2)$$

Again, $$\int \frac{x\,dx}{(1+x^2)^{1/2}} = \int \frac{t\,dt}{2\sqrt{t}}, \text{ putting } 1 + x^2 = t$$

$$= \frac{1}{2}\int t^{-1/2}\,dt = t^{1/2} = (1+x^2)^{1/2}. \qquad \ldots(3)$$

Similarly, $$\int \frac{y\,dy}{(1+y^2)^{1/2}} = (1+y^2)^{1/2}. \qquad \ldots(4)$$

Using (2), (3) and (4), (1) gives the required solution as

$$\log x - \log\{1 + (1+x^2)^{1/2}\} + (1+x^2)^{1/2} + (1+y^2)^{1/2} = \mathrm{C}.$$

Ex. 9. *Solve* $dy/dx = e^{x+y} + x^2e^{x^3+y}.$

Sol. From given equation, we get $dy/dx = e^y\,(e^x + x^2e^{x^3}).$

or $$e^{-y}\,dy = (e^x + x^2e^{x^3})\,dx. \qquad \ldots(1)$$

Integrating (1), $\int e^{-y}\, dy = \int e^x\, dx + \int x^2 e^{x^3} dx$

or $-e^{-y} = e^x + (1/3)\int e^t\, dt + c,$ putting $x^3 = t$

or $-e^{-y} = e^x + (1/3)\, e^t + c = e^x + (1/3)\, e^{x^3} + c.$

Ex. 10. *If $dy/dx = e^{x+y}$ and it is given that for $x = 1$, $y = 1$; find y when $x = -1$.*

Sol. Rewriting the given equation, we get $e^{-y}\, dy = e^x\, dx.$

Integrating it, $-e^{-y} = e^x + c.$ ... (1)

Putting $x = 1, y = 1$ in (1), $-e^{-1} = e + c$ so that $c = -e^{-1} - e.$

Hence (1) becomes $-e^{-y} = e^x - e^{-1} - e.$ ... (2)

Putting $x = -1$ in (2), we obtain $-e^{-y} = e^{-1} - e^{-1} - e$ so that $y = -1.$

Exercise 2(A)

1. $(e^x + 1)\, y\, dy = (y + 1)\, e^x\, dx.$ **[Agra 1996]** **Ans.** $(e^x + 1)(y + 1) = c\, e^y$

2. $(dy/dx) - y \tan x = -y \sec^2 x.$ **Ans.** $y \cos x = c\, e^{-\tan x}$

3. $x\sqrt{(1+y^2)}\, dx + y\sqrt{(1+x^2)}\, dy = 0.$ **[Bangalore 1996]** **Ans.** $\sqrt{(1+x^2)} + \sqrt{(1+y^2)} = C$

4. $(2ax + x^2)(dy/dx) = a^2 + 2ax.$ **[Kanpur 1996]** **Ans.** $x\,(x + 2a)^3 = Ce^{(2y/a)}$

5. $dr = a\,(r \sin\theta\, d\theta - \cos\theta\, dr).$ **Ans.** $r\,(1 + a\cos\theta) = c$

6. $(e^y + 1)\cos x\, dx + e^y \sin x\, dy = 0.$ **[Lucknow 1992]** **Ans.** $(\sin x)(e^y + 1) = c$

7. (a) $\sqrt{(a+x)}\,(dy/dx) + x = 0.$ **[Rohilkhand 1995; Bundelkhand 1998]**

Ans. $y + (2/3)(x - 2a)(a + x)^{1/2} = c$

(b) $dy/dx + \sqrt{(1-y^2)/(1-x^2)} = 0$ **[Pune 2010; Bangalore 1996]** **Ans.** $\sin^{-1} x + \sin^{-1} y = c$

8. $(x^2 - yx^2)\, dy + (y^2 + xy^2)\, dx = 0.$ **Ans.** $\log(x/y) - (x + y)/(xy) = c$

9. $(xy^2 + x)\, dx + (yx^2 + y)\, dy = 0.$ **[Agra 2005, Rajsthan 2010]** **Ans.** $(x^2 + 1)(y^2 + 1) = c$

10. $\sec^2 x \tan y\, dx + \sec^2 y \tan x\, dy = 0.$ **[Agra 2006]** **Ans.** $\tan x \tan y = c$

11. $(1 + x)\, y\, dx + (1 + y)\, x\, dy = 0.$ **Ans.** $x + y + \log(xy) = c$

12. Find the function 'f' which satisfies the equation $df/dx = 2f$, given that $f(0) = e^3$. **Ans.** $f = e^{2x+3}$

13. $(1 - x^2)(1 - y)\, dx = xy\,(1 + y)\, dx.$ **Ans.** $\log[x(1 - y)^2] = \frac{1}{2}(x^2 - y^2) - 2y + c$

[Jabalpur 1993; Guwahati 1996; Vikram 1992; Nagpur 2005; Agra 1992; Meerut 1995]

14. $x^2(y + 1)\, dx + y^2(x - 1)\, dy = 0.$ **Ans.** $x^2 + y^2 + 2(x - y) + 2\log\{(x - 1)(y + 1)\} = c$

15. $(dy/dx)\tan y = \sin(x + y) + \sin(x - y).$ **Ans.** $2\cos x + \sec y = c$

16. $x\, dy - y\, dx = (a^2 + y^2)^{1/2}\, dx.$ **Ans.** $2a^2 \log(xc) = y\,(a^2 + y^2)^{1/2} + a^2 \log\{y + (a^2 + y^2)^{1/2}\} - y^2$

17. $y - x\dfrac{dy}{dx} = 3\left(1 + x^2\dfrac{dy}{dx}\right).$ **Ans.** $(y - 3)(1 + 3x) = cx$

18. $\cos y \log(\sec x + \tan x)\, dx = \cos x \log(\sec y + \tan y)\, dy.$ **[Kanpur 1994]**

Ans. $\log\dfrac{\sec x + \tan x}{\sec y + \tan y}\ \log\{(\sec x + \tan x)(\sec y + \tan y)\} = c$

2.4 Transformation of some equations in the form in which variables are separable

Equations of the form **[Nagpur 2003]**

$dy/dx = f(ax + by + c)$ or $dy/dx = f(ax + by)$

can be reduced to an equation in which variables can be separated. For this purpose, we use the substitution $ax + by + c = v$ or $ax + by = v.$

2.5 Examples of Type 2 based on Art 2.4

Ex. 1. (*a*) *Solve* $dy/dx = (4x + y + 1)^2$. **[I.A.S. (Prel.) 2006]**

(*b*) $dy/dx = (4x + y + 1)^2$ *if* $y(0) = 1$. **[Delhi Maths (G) 2006]**

Sol. Let $4x + y + 1 = v$. ... (1)

Differentiating (1) with respect to x, we get

$4 + (dy/dx) = dv/dx$ or $dy/dx = (dv/dx) - 4$... (2)

Using (1) and (2), the given equation becomes

$(dv/dx) - 4 = v^2$ or $dv/dx = 4 + v^2$

Now, separating variables x and v, $dx = (dv) / (4 + v^2)$

Integrating, $x + c' = (1/2) \times \tan^{-1}(v/2)$, where c' is an arbitrary constant.

or $2x + c = \tan^{-1}(v/2)$ or $v = 2\tan(2x + c)$, where $c = 2c'$

or $4x + y + 1 = 2\tan(2x + c)$, using (1) ... (2)

(*b*) Putting $x = 0$, $y = 1$ in (2), we get $\tan c = 1$, so that $c = \pi/4$.

$\therefore$ Required solution is $4x + y + 1 = 2\tan(2x + \pi/4)$.

Ex. 2. *Solve* $(x + y)^2 (dy/dx) = a^2$. **[Meerut 1997; Indore 1998; I.A.S. (Prel.) 1994; Delhi Maths (G) 1997; Ravishankar 1992]**

Sol. Let $x + y = v$. ... (1)

Differentiating, $1 + (dy/dx) = dv/dx$ or $dy/dx = dv/dx - 1$... (2)

Using (1) and (2), the given equation becomes

$$v^2\left(\frac{dv}{dx} - 1\right) = a^2 \quad \text{or} \quad v^2\,\frac{dv}{dx} = a^2 + v^2$$

$$\text{or} \quad dx = \frac{v^2}{v^2 + a^2}\,dv \quad \text{or} \quad dx = \left[1 - \frac{a^2}{a^2 + v^2}\right]dv.$$

Integrating, $x + c = v - a^2 \times (1/a) \times \tan^{-1}(v/a)$, where c is arbitrary constant

$$\text{or} \quad x + c = x + y - a\tan^{-1}\left(\frac{x + y}{a}\right) \quad \text{or} \quad y - a\tan^{-1}\left(\frac{x+y}{a}\right) = c.$$

Ex. 3. *Solve* $dy/dx = \sec(x + y)$ **[Delhi Maths (P) 2005]**

or $\cos(x + y)\,dy = dx$. **[Kanpur 1992]**

Sol. Let $x + y = v$ so that $dy/dx = (dv/dx) - 1$.... (1)

Using (1), the given equation becomes

$$\frac{dv}{dx} - 1 = \sec v \quad \text{or} \quad \frac{dv}{dx} = 1 + \frac{1}{\cos v}$$

$$\text{or} \quad dx = \frac{\cos v}{1 + \cos v}\,dv = \frac{2\cos^2 \frac{1}{2}v - 1}{1 + 2\cos^2\frac{1}{2}v - 1}\,dv \quad \text{or} \quad dx = (1 - \tfrac{1}{2}\sec^2\tfrac{1}{2}v)\,dv.$$

Integrating, $x + c = v - \tan\frac{1}{2}v$ or $y - \tan\frac{1}{2}(x + y) = c$, by (1).

Ex. 4. *Solve* $dy/dx = \sin(x + y) + \cos(x + y)$. **[Guwahati 2007; Garhwal 1994]**

Sol. Let $x + y = v$ ... (1)

Differentiating (1) w.r.t 'x', $1 + \dfrac{dy}{dx} = \dfrac{dv}{dx}$ or $\dfrac{dy}{dx} = \dfrac{dv}{dx} - 1$... (2)

Using (1) and (2), the given equation becomes

$$\frac{dv}{dx} - 1 = \sin v + \cos v \quad \text{or} \quad \frac{dv}{dx} = 1 + \sin v + \cos v. \quad \ldots (3)$$

But $1 + \sin v + \cos v = 1 + 2 \sin (v/2) \cos (v/2) + 2 \cos^2 (v/2) - 1 = 2 \cos^2 (v/2) [1 + \tan v/2]$.

$\therefore$ (3) reduces to $$dx = \frac{dv}{2 \cos^2 (v/2) [1 + \tan (v/2)]} = \frac{\frac{1}{2} \sec^2 (v/2)\, dv}{1 + \tan (v/2)}.$$

Integrating, $x + c = \log [1 + \tan (v/2)]$, c being an arbitrary constant

or $x + c = \log [1 + \tan \{(x + y)/2\}]$, on using (1).

Ex. 5. *Solve* $(x + y)(dx - dy) = dx + dy$. **[Calcutta 1995]**

Sol. Re-writing the given equation, we get

$$(x + y - 1)\, dx = (x + y + 1)\, dy \quad \text{or} \quad \frac{dy}{dx} = \frac{x + y - 1}{x + y + 1}. \quad \ldots (1)$$

Let $x + y = v$. ... (2)

(2) $\Rightarrow$ $1 + dy/dx = dv/dx$ so that $dy/dx = (dv/dx) - 1$. ... (3)

Using (2) and (3), (1) becomes

$$\frac{dv}{dx} - 1 = \frac{v - 1}{v + 1} \quad \text{or} \quad \frac{dv}{dx} = \frac{2v}{v + 1} \quad \text{or} \quad 2dx = \left(1 + \frac{1}{v}\right) dv.$$

$\therefore$ Integrating, $2x + c = v + \log v$ or $x - y + c = \log (x + y)$, by (2)

Ex. 6. *Solve* $dy/dx = (4x + 6y + 5) / (3y + 2x + 4)$ **[Delhi Maths (G) 2005; Calcutta 1995; Delhi Maths (H) 2002; Karnataka 1995 Rajasthan 2010]**

Sol. The given equation may be re-written as $$\frac{dy}{dx} = \frac{2(2x + 3y) + 5}{(2x + 3y) + 4}. \quad \ldots (1)$$

$\therefore$ We take $2x + 3y = v$. ... (2)

Differentiating, (2) w.r.t. 'x' $$2 + 3 \frac{dy}{dx} = \frac{dv}{dx} \quad \text{or} \quad \frac{dy}{dx} = \frac{1}{3}\left(\frac{dv}{dx} - 2\right). \quad \ldots (3)$$

Using (2) and (3), (1) gives

$$\frac{1}{3}\left(\frac{dv}{dx} - 2\right) = \frac{2v + 5}{v + 4} \quad \text{or} \quad \frac{dv}{dx} = \frac{3(2v + 5)}{v + 4} + 2 = \frac{8v + 23}{v + 4}$$

or $$\frac{dx}{dv} = \frac{v + 4}{8v + 23} = \frac{(1/8) \times (8v + 23) + 4 - (23/8)}{8v + 23} = \left[\frac{1}{8} + \frac{9}{8(8v + 23)}\right]$$

Separating variables, $$dx = \left[\frac{1}{8} + \frac{9}{8(8v + 23)}\right] dv.$$

Integrating, $x + c = (v/8) + (9/64) \log (8v + 23)$, c being an arbitrary constant

or $8x + 8c = 2x + 3y + (9/8) \log (16x + 24y + 23)$, using (2) and multiplying by 8

or $3y - 6x + (9/8) \log (16x + 24y + 23) = 8c$ or $y - 2x + (3/8) \log (16x + 24y + 23) = c'$,

where c' $(= 8c/3)$ is an arbitrary constant.

Ex.7 Solve $(x + 2y - 1)\, dx = (x + 2y + 1)\, dy$ **[Delhi Maths (H) (2007)**

Sol. Rewritting the given equation, $dy/dx = (x + 2y - 1) / (x + 2y + 1)$ ...(1)

Let $x + 2y = v$ so that $1 + 2\, (dy/dx) = dv/dx$ or $dy/dx = (dv/dx - 1)/2$

$\therefore$ (1) reduces to $\quad \dfrac{1}{2}\left(\dfrac{dv}{dx}-1\right)=\dfrac{v-1}{v+1} \quad$ or $\quad \dfrac{dv}{dx}=2\left(\dfrac{v-1}{v+1}\right)+1$

or $\quad \dfrac{dv}{dx}=\dfrac{3v-1}{v+1} \quad$ or $\quad dx=\dfrac{v+1}{3v-1}\,dx \;=\; \dfrac{1}{3}\dfrac{3v+3}{3v-1}\,dv$

or $\quad dx = \dfrac{1}{3}\,\dfrac{(3v-1)+4}{3v-1}\,dv \quad$ or $\quad 3dx=\left(1+\dfrac{4}{3v-1}\right)dv$

Intergrating $3x=v+(4/3)\times\log(3v-1)-(4/3)\times\log c$, c being an arbitrary constant.

or $\quad \dfrac{4}{3}\log\dfrac{3v-1}{c}=3x-v \quad$ or $\quad \dfrac{4}{3}\log\dfrac{3(x+2y)-1}{c}=3x-(x+2y)$

or $\quad \log\dfrac{3x+6y-1}{c}=\dfrac{3}{4}\times(2x-2y) \quad$ or $\quad 3x+6y-1=c\,e^{3(x-y)/2}$

Exercise 2(B)

Solve the following differential equations:

1. $dy/dx=(x+y)^2$. **[Nagpur 2002]** **Ans.** $x+c=\tan^{-1}(x+y)$
2. $dy/dx+1=e^{x+y}$. **[Calcutta 1996]** **Ans.** $x+e^{-(x+y)}=c$
3. $(2x+y+1)\,dx+(4x+2y-1)\,dy=0$. **Ans.** $2y+x+\log(2x+y-1)=c$
4. $(x-y-2)\,dx-(2x-2y-3)\,dy=0$. **Ans.** $x-2y-\log(x-y-1)=c$
5. $(x+y+1)\,(dy/dx)=1$. **[Meerut 1995; Delhi Maths (G) 1991; Dibrugarh 1995]** **Ans.** $x+y+2=c\,e^y$
6. $\sin^{-1}(dy/dx)=x+y$. **[Mysore 2004]** **Ans.** $-2/(x+c)=1+\tan\frac{1}{2}(x+y)$
7. $(2x+4y+3)\,(dy/dx)=2y+x+1$. **Ans.** $4x+8y+5=ce^{4(x-2y)}$
8. $\dfrac{4x+6y+5}{3y+2x+4}\cdot\dfrac{dy}{dx}=1$. **Ans.** $(2/7)(2x+3y)-(9/49)\log(14x+21y+22)=x+c$
9. $dy/dx=(x-y+3)/(2x-2y+5)$. **Ans.** $x-2y+\log(x-y+2)=c$
10. $(2x+2y+3)\,dy-(x+y+1)\,dx=0$ or $dy/dx=(x+y+1)/(2x+2y+3)$. **[Lucknow 1998; Agra 1995; Meerut 1994]** **Ans.** $x+y+(4/3)=ce^{3(x-2y)}$
11. $(x-y)^2\,(dy/dx)=a^2$. **[Delhi Maths (G) 1999]** **Ans.** $y+c=(a/2)\log\{(x-y-a)/(x-y+a)\}$
12. $\dfrac{x+y-a}{x+y-b}\cdot\dfrac{dy}{dx}=\dfrac{x+y+a}{x+y+b}$. **Ans.** $(b-a)^2\log\{(x+y)^2-ab\}=2(x-y)+c$
13. $dy/dx=\cos(x+y)$. **Ans.** $x+c=\tan\{(x+y)/2\}$
14. If $dy/dx=e^{x+y}$ and it is given that for $x=1$, $y=1$, prove $y(-1)=-1$.
15. $dy/dx=(x+y+1)/(x+y-1)$ when $y=(1/3)$ at $x=(2/3)$. **Ans.** $\log(x+y)=y-x-(1/3)$
16. $(x+y-1)\,dy=(x+y)\,dx$. **Ans.** $2(y-x)-\log(2x+2y-1)=c$
17. $dy/dx=(x-y+3)/(2x-2y+5)$. **[Garhwal 2010]** **Ans.** $x-y+2=ce^{2y-x}$

2.6 Homogeneous equation Definition. A differential equation of first order and first degree is said to be homogeneous if it can be put in the form $\qquad dy/dx=f(y/x)$

2.7 Working rule for solving homogeneous equations

Let the given equation be homogeneous. Then, by definition, the given equation can be put in the form $\qquad dy/dx=f(y/x). \qquad$ ... (1)

To solve (1), let $\qquad y/x=v, \qquad$ *i.e.*, $\qquad y=vx. \qquad$ ... (2)

Differentiating with respect to x, (2) gives $dy/dx = v + x\,(dv/dx)$. ... (3)

Using (2) and (3), (1) becomes

$$v + x\frac{dv}{dx} = f(v) \quad \text{or} \quad x\frac{dv}{dx} = f(v) - v$$

Separating the variables x and v, we have

$$\frac{dx}{x} = \frac{dv}{f(v) - v} \quad \text{so that} \quad \log x + c = \int \frac{dv}{f(v) - v}$$

where c is an arbitrary constant. After integration, replace v by y/x.

2.8 Examples of Type 3 based on Art. 2.7

Ex. 1. *Solve* $(x^3 + 3xy^2)\,dx + (y^3 + 3x^2y)\,dy = 0$. **[I.A.S. (Prel.) 2004]**

Sol. Given $$\frac{dy}{dx} = -\frac{x^3 + 3xy^2}{y^3 + 3x^2y} = -\frac{1 + 3\,(y/x)^2}{(y/x)^3 + 3\,(y/x)}. \qquad \text{... (1)}$$

Take $y/x = v$, *i.e.*, $y = vx$. so that $dy/dx = v + x\,(dv/dx)$. ... (2)

From (1) and (2), $$v + x\frac{dv}{dx} = -\frac{1 + 3v^2}{v^3 + 3v}$$

or $$x\frac{dv}{dx} = -\frac{1 + 3v^2}{v^3 + 3v} - v = -\frac{v^4 + 6v^2 + 1}{v^3 + 3v} \quad \text{or} \quad 4\frac{dx}{x} = -\frac{4v^3 + 12v}{v^4 + 6v^2 + 1}\,dv.$$

Integrating, $4 \log x = -\log (v^4 + 6v^2 + 1) + \log c$, c being an arbitrary constant.

or $\log x^4 = \log [c/(v^4 + 6v^2 + 1)]$, *i.e.*, $x^4\,(v^4 + 6v^2 + 1) = c$

or $y^4 + 6x^2y^2 + x^4 = c$ or $(x^2 + y^2)^2 + 4x^2y^2 = c$, as $y/x = v$.

Ex. 2. Solve: $x\,dy - y\,dx = (x^2 + y^2)^{1/2}\,dx$ **[Meerut 2008; Delhi Maths (G) 1999]**

Sol. Here, $$\frac{dy}{dx} = \frac{y + (x^2 + y^2)^{1/2}}{x} = \frac{y}{x} + \left\{1 + \left(y/x\right)^2\right\}^{1/2}. \qquad \text{... (1)}$$

Take $y/x = v$, *i.e.*, $y = vx$. so that $dy/dx = v + x\,(dv/dx)$.... (2)

From (1) and (2), $$v + x\frac{dv}{dx} = v + \sqrt{(1 + v^2)} \quad \text{or} \quad \frac{dx}{x} = \frac{dv}{\sqrt{(1 + v^2)}}.$$

Integrating, $$\log x + \log c = \log\,[v + \sqrt{(v^2 + 1)}] \quad \text{or} \quad xc = v + \sqrt{(v^2 + 1)}$$

or $$x^2c = y + \sqrt{(y^2 + x^2)}, \text{as } v = y/x$$

Ex. 3. *Solve* $\dfrac{dy}{dx} = \dfrac{y}{x} + \tan\dfrac{y}{x}$. **[Patna 2003, I.A.S. 2001]**

Sol. Since the R.H.S. of the given equation is function of y/x alone, we conclude that it must be a homogeneous equation.

Take $y/x = v$, *i.e.*, $y = vx$, so that $dy/dx = v + x\,(dv/dx)$. ... (1)

Using (1), the given equation becomes $$v + x\frac{dv}{dx} = v + \tan v \quad \text{or} \quad \frac{dx}{x} = \frac{\cos v}{\sin v}\,dv.$$

Integrating, $\log x + \log c = \log \sin v$, c being an arbitrary constant.

or $cx = \sin v$, *i.e.*, $cx = \sin (y/x)$, as $v = y/x$

Ex. 4. *Solve:* $x \cos (y/x)\,(y\,dx + x\,dy) = y \sin (y/x)\,(x\,dy - y\,dx)$ ... (1)

or $$\left(x \cos \frac{y}{x} + y \sin \frac{y}{x}\right) y - \left(y \sin \frac{y}{x} - x \cos \frac{y}{x}\right) x \frac{dy}{dx} = 0. \quad \text{... (2)}$$

[Mysore 2004; Kanpur 1996; Lucknow 1997]

Sol. Rewriting (1), we get (2). So (1) and (2) are the same equations.

From (2), $$\frac{dy}{dx} = \frac{\{x \cos (y/x) + y \sin (y/x)\}\, y}{\{y \sin (y/x) - x \cos (y/x)\}\, x}$$

or $$\frac{dy}{dx} = \frac{[\cos (y/x) + (y/x) \sin (y/x)]\,(y/x)}{[(y/x) \sin (y/x) - \cos (y/x)]} \quad \text{... (3)}$$

Take $y/x = v$, *i.e.*, $y = vx$, so that $dy/dx = v + x\,(dv/dx)$.... (4)

Using (4), (3) becomes $$v + x\frac{dv}{dx} = \frac{v\,(\cos v + v \sin v)}{v \sin v - \cos v}$$

or $$x\frac{dv}{dx} = \frac{v\,(\cos v + v \sin v)}{v \sin v - \cos v} - v = \frac{2v \cos v}{v \sin v - \cos v} \text{ or } 2\frac{dx}{x} = \frac{v \sin v - \cos v}{v \cos v}\,dv = \left[\frac{\sin v}{\cos v} - \frac{1}{v}\right] dv.$$

Integrating, $2 \log x = - \log \cos v - \log v + \log c$, c being an arbitrary constant.

or $\log x^2 = \log (c/v \cos v)$ or $x^2 v \cos v = c$ or $xy \cos (y/x) = c$. $[\because\ v = y/x]$

Ex. 5. *Solve* $(4y + 3x)\,dy + (y - 2x)\,dx = 0$. **[Delhi Maths (H) 1994]**

Sol. Re-writing the given equation, $$\frac{dy}{dx} = -\frac{y - 2x}{4y + 3x} = \frac{2 - (y/x)}{3 + 4\,(y/x)}. \quad \text{... (1)}$$

Let $y/x = v$ so that $y = xv$. ... (2)

From (2), $dy/dx = v + x\,(dv/dx)$. ... (3)

Using (2) and (3), (1) reduces to

$$v + x\frac{dv}{dx} = \frac{2 - v}{3 + 3v} \quad \text{or} \quad x\frac{dv}{dx} = \frac{2 - v}{3 + 4v} - v$$

or $$x\frac{dv}{dx} = \frac{2 - 4v - 4v^2}{3 + 4v} \quad \text{or} \quad \frac{2dx}{x} = \frac{3 + 4v}{1 - 2v - 2v^2}.$$

Integrating, $$2 \log x = \int \frac{(3 + 4v)\,dv}{1 - 2v - 2v^2} = \int \frac{-(-2 - 4v) + 1}{1 - 2v - 2v^2}\,dv$$

or $$\log x^2 + \log c = - \log (1 - 2v - 2v^2) + (1/2) \times \int \frac{dv}{(1/2) - v - v^2}$$

or $$\log \{cx^2 (1 - 2v - 2v^2)\} = \frac{1}{2}\int \frac{dv}{(3/4) - (v^2 + v + 1/4)} = \frac{1}{2}\int \frac{dv}{(\sqrt{3}/2)^2 - (v + 1/2)^2}$$

$$= \frac{1}{2} \times \frac{1}{2\,(\sqrt{3}/2)} \log \frac{(\sqrt{3}/2) + (v + 1/2)}{(\sqrt{3}/2) - (v + 1/2)} = \frac{1}{2\sqrt{3}} \log \frac{\sqrt{3} + 2v + 1}{\sqrt{3} - v - 1}$$

or $$\log \left[cx^2 \left(1 - \frac{2y}{x} - \frac{2y^2}{x^2}\right)\right] = \frac{1}{2\sqrt{3}} \log \frac{(\sqrt{3} + 1) + 2\,(y/x)}{(\sqrt{3} - 1) - 2\,(y/x)}, \text{ as } v = \frac{y}{x}$$

or $$c\,(x^2 - 2xy - 2y^2) = \left\{\frac{(\sqrt{3} + 1)\,x + 2y}{(\sqrt{3} - 1)\,x - 2y}\right\}^{1/2\sqrt{3}}, \ c \text{ being an arbitrary constant.}$$

Ex. 6. *Solve* (*a*) $x\,(dy/dx) = y\,\{\log y - \log x + 1\}$ **[I.A.S. (Prel.) 2005]**

(*b*) $(2\sqrt{xy} - x)\,dy + y\,dx = 0$

Sol. (*a*) Given $$\frac{dy}{dx} = \frac{y}{x}\left(\log\frac{y}{x} + 1\right). \qquad \text{... (1)}$$

Putting $y/x = v$ or $y = xv$, we have $dy/dx = v + x\,(dv/dx)$. ... (2)

From (1) and (2), $v + x\dfrac{dv}{dx} = v\,(\log v + 1)$ or $\dfrac{dx}{x} = \dfrac{dv}{v\log v}$.

Integrating, $\log x + \log c = \log\log v$ or $xc = \log v$.

$\therefore$ $v = e^{xc}$ or $(y/x) = e^{xc}$ or $y = xe^{xc}$.

Part (*b*). Given $$\frac{dy}{dx} = -\frac{y}{2\sqrt{xy} - x} = \frac{y/x}{1 - 2\,(\sqrt{y/x})}. \qquad \text{... (1)}$$

Putting $y/x = v$ or $y = xv$, we have $dy/dx = v + x\,(dv/dx)$. ... (2)

From (1) and (2), $v + x\dfrac{dv}{dx} = \dfrac{v}{1 - 2\sqrt{v}}$ or $x\dfrac{dv}{dx} = \dfrac{2v\sqrt{v}}{1 - 2\sqrt{v}}$

or $\dfrac{dx}{x} = \dfrac{1 - 2\sqrt{v}}{2v\sqrt{v}}\,dv$ or $\dfrac{dx}{x} = \left(\dfrac{1}{2}v^{-3/2} - \dfrac{1}{v}\right)dv$.

Integrating, $\log x = -v^{-1/2} - \log v + \log c$ or $\log\left(\dfrac{xv}{c}\right) = -\dfrac{1}{\sqrt{v}}$.

or $\log\left(\dfrac{xy}{cx}\right) = \dfrac{-1}{\sqrt{(y/x)}}$ or $\log\,(y/c) = -\sqrt{(x/y)}$

or $y/c = e^{-\sqrt{(x/y)}}$ so that $y = ce^{-\sqrt{(x/y)}}$

Ex. 7. *Solve* $(x^3 + y^3)\,dx = (x^2y + xy^2)\,dy$ **[Delhi Maths (H) 2002]**

Sol. Re-writing the given equation, $dy/dx = (x^3 + y^3)/(x^2y + xy^2)$ ... (1)

Putting $y = xv$ and $dy/dx = v + x\,(dv/dx)$, (1) becomes

$v + x\dfrac{dv}{dx} = \dfrac{1 + v^3}{v + v^2}$ or $x\dfrac{dv}{dx} = \dfrac{1 + v^3}{v + v^2} - v = \dfrac{(1 - v)(1 + v)}{v\,(1 + v)}$

or $\dfrac{dx}{x} + \dfrac{v\,dv}{v - 1} = 0$ or $\dfrac{dx}{x} + \left(1 + \dfrac{1}{v - 1}\right)dv = 0$

Integrating, $\log x + v + \log\,(v - 1) - \log c = 0$, c being an arbitrary constant.

or $\log\,\{x\,(v - 1)/c\} = -v$ or $x\,(v - 1) = ce^{-v}$ or $x\,(y/x) - x = ce^{-y/x}$

or $y - x = ce^{-y/x}$, c being an arbitrary constant.

Ex. 8. *Solve* $(x^2 - 4xy - 2y^2)\,dx + (y^2 - 4xy - 2x^2)\,dy = 0$. **[Delhi Maths (G) 2005, 06]**

Sol. Re-writing the given differential equation, we have

$$dy/dx = (x^2 - 4xy - 2y^2)/(2x^2 + 4xy - y^2) \qquad \text{... (1)}$$

Putting $y = xv$ and $dy/dx = v + x\,(dv/dx)$, (1) reduces to

$v + x\dfrac{dv}{dx} = \dfrac{1 - 4v - 2v^2}{2 + 4v - v^2}$ or $x\dfrac{dv}{dx} = \dfrac{1 - 4v - 2v^2}{2 + 4v - v^2} - v$

or $\quad x\dfrac{dv}{dx} = \dfrac{1-6v-6v^2+v^3}{2+4v-v^2}$ $\quad$ or $\quad 3\dfrac{dx}{x} + \dfrac{3(v^2-4v-2)}{v^3-6v^2-6v+1}dv = 0$

Integrating, $3 \log x + \log (v^3 - 6v^2 - 6v + 1) = \log c$, *being* an arbitrary constant

or $\quad x^3 (v^3 - 6v^2 - 6v + 1) = c$ $\quad$ or $\quad x^3 \{(y/x)^3 - 6 (y/x)^2 - 6 (y/x) + 1\} = c$

or $\quad y^3 - 6xy^2 - 6x^2y + x^3 = c$, c being an arbitrary constant.

Exercise 2(C)

Solve the following differential equations:

1. $(x^2 + y^2)\, dx - 2x\, dy = 0$. **[Delhi Maths (H) 1992]** **Ans.** $x^2 - y^2 = c\,x$
2. $y^2 + x^2 (dy/dx) = xy\, (dy/dx)$. **Ans.** $y = c\, e^{y/x}$
3. $(x^2 + xy)\, dy = (x^2 + y^2)\, dx$. **Ans.** $(x - y)^2 = c\,x\,e^{-y/x}$
4. $dy/dx = y/x + \sin (y/x)$. **Ans.** $\tan (y/2x) = c\,x$
5. $(x^2 + y^2)\,(dy/dx) = xy$. **[Kerala 2001]** **Ans.** $y = ce^{x^2/2y^2}$
6. $(x^2 - y^2)\, dy = 2xy\, dx$. **Ans.** $y = c\,(x^2 + y^2)$
7. $(x^3 - y^3)\, dx + xy^2\, dy = 0$. **[Kanpur 2005]** **Ans.** $x = ce^{-y^3/3x^3}$
8. $y^2\, dx + (xy + x^2)\, dy = 0$. **Ans.** $2y + x = cxy^2$
9. $x\,(dy/dx) + (y^2/x) = y$. **[Delhi Maths 1997; Dibrugarh 1996]** **Ans.** $x = ce^{x/y}$
10. $x^2y\, dx - (x^3 + y^3)\, dy = 0$. **[Andhra 2003; Bangalore 1995]** **Ans.** $y^3 = ce^{x^3/y^3}$
11. $(x + y)\, dy + (x - y)\, dx = 0$ or $y - x\,(dy/dx) = x + y\,(dy/dx)$ or $y - xp = x + yp$, $p = dy/dx$. **[Delhi Maths (H) 1994; Rajpur 1995]** **Ans.** $\tan^{-1} (y/x) + (1/2) \times \log (x^2 + y^2) = c$
12. $x\,(x - y)\, dy + y^2\, dx = 0$. **Ans.** $y = ce^{y/x}$
13. $x\,(x - y)\, dy = y\,(x + y)\, dx$. **[Dibrugarh 1995]** **Ans.** $xy = ce^{-x/y}$
14. $x \sin (y/x)\,(dy/dx) = y \sin (y/x) - x$. **[Nagpur 2002]** **Ans.** $x = ce^{\cos (y/x)}$
15. $x^2\, dy + y\,(x + y)\, dx = 0$. **Ans.** $y + 2x = cx^2y$
16. $(x^3 - 3xy^2)\, dx = (y^3 - 3x^2y)\, dy$. **(Delhi Maths (Prog) 2007)** **Ans.** $x^2 - y^2 = c\,(x^2 + y^2)^2$
17. $2\,(dy/dx) = [y\,(x + y)/x^2]$ or $2\,(dy/dx) - (y/x) = y^2/x^2$. **Ans.** $(y - x)^2 = cxy^2$
18. $(x^3 - 2y^3)\, dx + 3xy^2\, dy = 0$. **Ans.** $x^3 + y^3 = cx^2$
19. $dy/dx = (xy^2 - x^2y)/x^3$. **Ans.** $x^2y = c\,(y - 2x)$
20. $(x^2 + y^2)\, dx + 2xy\, dy = 0$ **(Guwahati 2007)** **Ans.** $x(x^2 + 3y^2) = c$
21. $(x^2y - 2xy^2)dx - (x^3 - 2x^2y)dy = 0$ **[Delhi B.Sc. (Prog) II 2010; Pune]** **Ans.** $y^3 = Cx^2e^{-x/y}$

2.9 Equations reducible to homogeneous form

Equations of the form $\quad \dfrac{dy}{dx} = \dfrac{ax + by + c}{a'x + b'y + c'}$, $\quad$ where $\quad \dfrac{a}{a'} \neq \dfrac{b}{b'}$, $\quad$ ... (1)

can be reduced to homogeneous form as explained below.

Take $x = X + h$ $\quad$ and $\quad$ $y = Y + k$,... (2)

where X and Y are new variables and h and k are constants to be so chosen that the resulting equation in terms of X and Y may become homogeneous.

From (2), $\quad dx = dX$ $\quad$ and $\quad dy = dY$, $\quad$ so that $\quad dy/dx = dY/dX$.... (3)

Using (2) and (3), (1) becomes

$$\frac{dY}{dX} = \frac{a\,(X+h) + b\,(Y+k) + c}{a'\,(X+h) + b'\,(Y+k) + c'} = \frac{aX + bY + (ah + bk + c)}{a'X + b'Y + (a'h + b'k + c')}. \quad \text{... (4)}$$

In order to make (4) homogeneous, choose h and k so as to satisfy the following two equations $\quad ah + bk + c = 0$ $\quad$ and $\quad a'h + b'k + c' = 0$. $\quad$ ... (5)

Solving (5), $h = \dfrac{bc' - b'c}{ab' - a'b}$ and $k = \dfrac{ca' - c'a}{ab' - a'b}$. ... (6)

Given that $a/a' \neq b/b'$. Therefore, $(ab' - a'b) \neq 0$. Hence, h and k given by (6) are meaningful, *i.e.*, h and k will exist. Now, h and k are known. So from (2), we get

$$X = x - h \quad \text{and} \quad Y = y - k. \qquad \ldots (7)$$

In view of (5), (4) reduces to $\dfrac{dY}{dX} = \dfrac{aX + bY}{a'X + b'Y} = \dfrac{a + b\,(Y/X)}{a' + b'\,(Y/X)}$,

which is surely homogeneous equation in X and Y and can be solved by putting $Y/X = v$ as usual. After getting solution in terms of X and Y, we remove X and Y by using (7) and obtain solution in terms of the original variables x and y.

2.10 Examples of Type 4 based on Art. 2.9

Ex. 1. *Solve* $dy/dx = (x + 2y - 3)/(2x + y - 3)$. **[Agra 1996; Bangalore 2005; Delhi Maths (G) 1993; Mysore 2004]**

Sol. Take $x = X + h$, $y = Y + k$, so that $dy/dx = dY/dX$.... (1)

$\therefore$ Given equation becomes $\dfrac{dY}{dX} = \dfrac{X + 2Y + (h + 2k - 3)}{2X + Y + (2h + k - 3)}$. ... (2)

Choose h, k so that $h + 2k - 3 = 0$ and $2h + k - 3 = 0$. ... (3)

Solving (3), we get $h = 1$, $k = 1$ so that from (1), we have

$$X = x - 1, \quad \text{and} \quad Y = y - 1. \qquad \ldots (4)$$

Using (3) in (2), we get $\dfrac{dY}{dX} = \dfrac{X + 2Y}{2X + Y} = \dfrac{1 + (2Y/X)}{2 + (Y/X)}$. ... (5)

Take $Y/X = v$, *i.e.*, $Y = vX$. Therefore, $dY/dX = v + X\,(dv/dX)$. ... (6)

From (5) and (6), we have

$$v + X\frac{dv}{dX} = \frac{1 + 2v}{2 + v} \quad \text{or} \quad X\frac{dv}{dX} = \frac{1 + 2v}{2 + v} - v = \frac{1 - v^2}{2 + v}$$

or $\dfrac{dX}{X} = \dfrac{(2 + v)\,dv}{(1 - v)\,(1 + v)} = \left[\dfrac{1}{2}\left(\dfrac{1}{1 + v}\right) + \dfrac{3}{2}\left(\dfrac{1}{1 - v}\right)\right] dv$, resolving into partial fractions

Integrating, $\log X + \log c = (1/2)\,[\log (1 + v) - 3 \log (1 - v)]$

or $2 \log (cX) = \log \dfrac{1 + v}{(1 - v)^3}$ or $X^2 c^2 = \dfrac{1 + v}{(1 - v)^3}$

or $X^2 c^2 \left(1 - Y/X\right)^3 = 1 + Y/X$, as $v = Y/X$

or $c^2 (X - Y)^3 = X + Y$ or $c^2 \{x - 1 - (y - 1)\}^2 = x - 1 + y - 1$, by (4)

or $c'(x - y)^2 = x + y - 2$, taking $c' = c^2$. c' being an arbitrary constant

Ex. 2. *Solve* $dy/dx + (x - y - 2)/(x - 2y - 3) = 0$. **[Ravishankar 1993]**

Sol. Given equation is $dy/dx = -(x - y - 2)/(x - 2y - 3)$.

Take $x = X + h$, $y = Y + k$ so that $dy/dx = dY/dX$,... (1)

The given equation becomes $\dfrac{dY}{dX} = -\dfrac{X - Y + h - k - 2}{X - 2Y + h - 2k - 3}$. ... (2)

Choose h, k so that $h - k - 2 = 0$ and $h - 2k - 3 = 0$.... (3)

Solving (3), we get $h = k$, $k = -1$ so that from (1), we have

$$X = x - 1 \qquad \text{and} \qquad Y = y + 1. \qquad \text{... (4)}$$

and (2) becomes
$$\frac{dY}{dX} = -\frac{X - Y}{X - 2Y} = -\frac{1-(Y/X)}{1-2(Y/X)}. \qquad \text{... (5)}$$

Take Y/X = v, *i.e.*, $Y = vX$. so that $dY/dX = v + X = dv/dX$. ... (6)

From (5) and (6),
$$v + X\frac{dv}{dX} = -\frac{1-v}{1-2v} \qquad \text{or} \qquad X\frac{dv}{dX} = \frac{1-2v^2}{2v-1}dv$$

or
$$\frac{dX}{X} = \frac{2v-1}{1-2v^2}dv \qquad \text{or} \qquad \frac{dX}{X} = \left[-\frac{1}{2}\frac{(-4v)}{1-2v^2} - \frac{1}{1-(v\sqrt{2})^2}\right]dv.$$

Integrating,
$$\log X = -\frac{1}{2}\log(1-2v^2) - \frac{1}{2\sqrt{2}}\log\frac{1+v\sqrt{2}}{1-v\sqrt{2}} - \frac{1}{2}\log c$$

or
$$2\log X + \log(1-2v^2) + \log c = -\frac{1}{\sqrt{2}}\log\left(\frac{1+v\sqrt{2}}{1-v\sqrt{2}}\right) \text{or} \log\{cX^2(1-2v^2)\} = \log\left(\frac{1-v\sqrt{2}}{1+v\sqrt{2}}\right)^{1/\sqrt{2}}$$

or
$$cX^2\left(1-2\frac{Y^2}{X^2}\right) = \left\{\frac{1-(Y/X)\sqrt{2}}{1+(Y/X)\sqrt{2}}\right\}^{1/\sqrt{2}} \qquad \text{or} \qquad c(X^2-2Y^2) = \left(\frac{X-Y\sqrt{2}}{X+Y\sqrt{2}}\right)^{1/\sqrt{2}}$$

or
$$c\{(x-1)^2 - 2(y+1)^2\} = \left\{\frac{x-1-(y+1)\sqrt{2}}{x-1+(y+1)\sqrt{2}}\right\}^{1/\sqrt{2}}$$

or
$$c(x^2 - 2y^2 - 2x - 4y - 1) = \left(\frac{x - y\sqrt{2} - \sqrt{2} - 1}{x + y\sqrt{2} - 1 + \sqrt{2}}\right)^{1/\sqrt{2}}$$, c being an arbitrary constant.

Ex. 3. *Solve* $dy/dx = (x + y + 4)/(x - y - 6)$. **[I.A.S. 2002]**

Sol. Given $dy/dx = (x + y + 4)/(x - y - 6)$ ... (1)

Let $x = X + h$, $Y = y + k$ so that $dy/dx = dY/dX$ ... (2)

Using (2), (1) reduces to
$$\frac{dy}{dx} = \frac{(X+Y)+(h+k+4)}{(X-Y)+(h-k-6)} \qquad \text{... (3)}$$

We choose h and k, such that $h + k + 4 = 0$, and $h - k - 6 = 0$ ... (4)

Solving (4), $h = 1$, $k = -5$ and so by (2), $X = x - 1$, $Y = y + 5$. ... (5)

Using (4), (3) reduces to
$$\frac{dY}{dX} = \frac{X+Y}{X-Y} = \frac{1+(Y/X)}{1-(Y/X)} \qquad \text{... (6)}$$

Putting $Y = xV$ and $dY/dX = v + X(dv/dX)$, (6) becomes

$$v + X\frac{dv}{dX} = \frac{1+v}{1-v} \qquad \text{or} \qquad \frac{dX}{X} = \frac{1-v}{1+v^2}dv = \frac{dv}{1+v^2} - \frac{v\,dv}{1+v^2}$$

Integrating, $\log X = \tan^{-1} v - (1/2)\log(1 + v^2) + (1/2)\log c$

or $2\log X + \log(1 + Y^2/X^2) - \log c = 2\tan^{-1}(Y/X)$, as $v = Y/X$

or $\log\{(X^2 + Y^2)/c\} = 2\tan^{-1}(Y/X)$ or $X^2 + Y^2 = ce^{2\tan^{-1}(Y/X)}$

or $(x - 1)^2 + (y + 5)^2 = ce^{2\tan^{-1}\{(y+5)/(x-1)\}}$, c being an arbitrary constant.

Ex. 4. *Solve* $dy/dx = (x - 2y + 5)/(2x + y - 1)$. **[Delhi Maths (H) 2002]**

Sol. Let $x = X + h$, $y = Y + k$ so that $dy/dx = dY/dX$ ... (1)

Then given equation becomes $$\frac{dY}{dX} = \frac{X - 2Y + h - 2k + 5}{2X + Y + 2h + k - 1} \quad \text{... (2)}$$

Choose h and k so that $h - 2k + 5 = 0$ and $2h + k - 1 = 0$ ... (3)

(3) $\Rightarrow$ $h = -3/5$, $k = 11/5$ so by (1) $X = x + 3/5$ and $Y = y - 11/5$... (4)

Using (3), (2) becomes $$\frac{dY}{dX} = \frac{X - 2Y}{2X + Y} = \frac{1 - 2(Y/X)}{2 + (Y/X)} \quad \text{... (5)}$$

Putting $Y = Xv$ and $dY/dX = v + X(dv/dX)$, (5) gives

$$v + X\frac{dv}{dX} = \frac{1 - 2v}{2 + v} \quad \text{or} \quad \frac{dX}{X} + \frac{1}{2}\frac{2v + 4}{v^2 + 4v - 1}dv = 0$$

Integrating, $\log X = (1/2)\log(v^2 + 4v - 1) = (1/2)\log C$

or $X^2(v^2 + 4v - 1) = C$ or $X^2(Y^2/X^2 + 4Y/X - 1) = C$, as $v = Y/X$

or $Y^2 + 4XY - X^2 = C$ or $(y - 11/5)^2 + 4(x + 3/5)(y - 11/5) - (x + 3/5)^2 = C$

or $x^2 - y^2 - 4xy + 10x + 2y = C_1$, where C_1 is another arbitrary constant.

Ex. 5. *Solve* $dy/dx = (x + y - 2)/(y - x - 4)$ **[Delhi Maths (G) 2004]**

Sol. Let $x = X + h$ and $y = Y + k$ so that $dy/dx = dY/dX$... (1)

Then given equation gives $$\frac{dY}{dX} = \frac{X + Y + (h + k - 2)}{Y - X + (k - h - 4)} \quad \text{... (2)}$$

Choose h, k such that $h + k - 2 = 0$ and $k - h - 4 = 0$. .. (3)

Solving (3), $h = -1$, $k = 3$. Then (1) gives $X = x + 1$ and $Y = y - 3$... (4)

Using (3), (2) becomes $$\frac{dY}{dX} = \frac{X + Y}{Y - X} = \frac{1 + (Y/X)}{(Y/X) - 1} \quad \text{... (5)}$$

Let $Y/X = v$, *i.e.*, $Y = vX$ so that $dY/dX = v + X(dv/dX)$... (6)

From (5) and (6), $$v + X\frac{dv}{dX} = \frac{1 + v}{v - 1} \quad \text{or} \quad X\frac{dv}{dX} = \frac{1 + 2v - v^2}{v - 1}$$

or $$\frac{(v - 1)\,dv}{1 + 2v - v^2} = \frac{dX}{X} \quad \text{or} \quad \frac{(2 - 2v)\,dv}{1 + 2v - v^2} = -2\frac{dX}{X}$$

Integrating, $\log(1 + 2v - v^2) + 2\log X = \log C$ or $X^2(1 + 2v - v^2) = C$

or $X^2\{1 + 2(Y/X) - (Y/X)^2\} = C$ or $X^2 + 2XY - Y^2 = C$

or $(x + 1)^2 + 2(x + 1)(y - 3) - (y - 3)^2 = C$, using (3)

Ex. 6. *Solve* $(2x^2 + 3y^2 - 7)\,x\,dx - (3x^2 + 2y^2 - 8)\,y\,dy = 0$ **[I.A.S. 1995]**

Sol. Given $(2x^2 + 3y^2 - 7)\,x\,dx - (3x^2 + 2y^2 - 8)\,y\,dy = 0$. ... (1)

Let $x^2 = u$ and $y^2 = v$ so that $2x\,dx = du$ and $2y\,dy = dv$.... (2)

From (1) and (2), $(2u + 3v - 7)\,du - (3u + 2v - 8)\,dv = 0$

or $dv/du = (2u + 3v - 7)/(3u + 2v - 8)$. ... (3)

Taking $u = U + h$, $v = V + k$ so that $dv/du = dV/dU$, ... (4)

the given equation becomes $$\frac{dV}{dU} = \frac{2U + 3V + (2h + 2k - 7)}{3U + 2V + (3h + 2k - 8)}. \quad \text{... (5)}$$

Choose h, k so that $2h + 3k - 7 = 0$ and $3h + 2k - 8 = 0$.... (6)

Solving (3), we get $h = 2$, $k = 1$ so that from (4), we have

$U = u - 2$ and $V = v - 1$.

or $\qquad U = x^2 - 2 \qquad$ and $\qquad V = y^2 - 1$, by (2)... (7)

Then (5) becomes
$$\frac{dV}{dU} = \frac{2U + 3V}{3U + 2V} = \frac{2 + 3\,(V/U)}{3 + 2\,(V/U)}. \qquad \text{... (8)}$$

Take $\quad V/U = w, \quad$ *i.e.*, $\quad V = wU \quad$ so that $\quad dV/dU = w + U\,(dw/dU)$... (9)

From (8) and (9), $\qquad w + U\dfrac{dw}{dU} = \dfrac{2+3w}{3+2w} \qquad$ or $\qquad U\dfrac{dw}{dU} = \dfrac{2\,(1-w^2)}{3+2w}$

or
$$\frac{2dU}{U} = \frac{3+2w}{1-w^2}\,dw = \left[\frac{3}{1-w^2} - \frac{-2w}{1-w^2}\right]dw.$$

Integrating, $2\log U = \dfrac{3}{2}\log\dfrac{1+w}{1-w} - \log(1-w^2) + \dfrac{1}{2}\log c$, c being an arbitrary constant

or
$$4\log U = 3\log\left(\frac{1+w}{1-w}\right) - 2\log(1-w^2) + \log c$$

or $\qquad \log\dfrac{U^4}{c} = \log\left(\dfrac{1+w}{1-w}\right)^3 - \log(1-w^2)^2 \qquad$ or $\qquad \log\dfrac{U^4}{c} = \log\left[\left(\dfrac{1+w}{1-w}\right)^3 \cdot \dfrac{1}{(1-w^2)^2}\right]$

or $\qquad \dfrac{U^4}{c} = \dfrac{(1+w)^3}{(1-w)^5\,(1+w)^2} \qquad$ or $\qquad (1-w)^5\,U^4 = c\,(1+w)$

or $\qquad \left(1 - \dfrac{V}{U}\right)^5 U^4 = c\left(1 + \dfrac{V}{U}\right) \qquad$ or $\qquad (U - V)^5 = C\,(U + V)$

or
$$(x^2 - y^2 - 1)^5 = c\,(x^2 + y^2 - 3), \text{ by (7).}$$

Exercise 2(D)

Solve the following differential equations:

1. $dy/dx = (x + 2y + 3)/(2x + 3y + 4)$.

Ans. $c\,(y\sqrt{3} + x + 2\sqrt{3} - 1)^{1/\sqrt{3}} = (y\sqrt{3} - x + 2\sqrt{3} + 1)^{1/\sqrt{3}}\,(3y^2 - x^2 - 12y + 12x + 11)^{1/2}$

2. $dy/dx = (y - x - 1)/(y + x + 5)$. **[Delhi Maths (H) 1995]**

Ans. $\log(x^2 + y^2 + 4x + 6y + 13) + 2\tan^{-1}\{(y + 3)/(x + 2)\} = c$

3. $dy/dx = (2x + 2y - 2)/(3x + y - 5)$. **Ans.** $(y - x + 3)^4 = c\,(2x + y - 3)$

4. $dy/dx = (2x - y + 1)/(x + 2y - 3)$. **Ans.** $(5y - 7)^2 + (5x - 1)\,(5y - 1) - (5x - 1)^2 = c$

5. $(x + 2y - 2)\,dx + (2x - y + 3)\,dy = 0$. **[Calicut 2004]** **Ans.** $x^2 + 4xy - y^2 - 4x + 6y = c$

6. $(2x + 3y - 5)\,(dy/dx) + (3x + 2y - 5) = 0$. **Ans.** $3x^2 + 4xy + 3y^2 - 10x - 10y = c$

7. $(x - y)\,dy = (x + y + 1)\,dx$. **Ans.** $\log\{c\,(x^2 + y^2 + x + y + 1/2)\} = 2\tan^{-1}\{(2y + 1)/(2x + 1)\}$

8. $(6x + 2y - 10)\,(dy/dx) - 2x - 9y + 20 = 0$. **Ans.** $(y - 2x)^2 = \text{c}\,(x + 2y - 5)$

9. $(6x - 2y - 7)\,dx = (2x + 3y - 6)\,dy$. **Ans.** $3y^2 + 4xy - 6x^2 + 14x - 12y - (9/2) = 0$

10. $(3y - 7x + 7)\,dx + (7y - 3x + 3)\,dy = 0$.**[Delhi Maths (H) 1995, 2008]**

Ans. $(y - x + 1)^2\,(y + x - 1)^5 = c$

11. $(x - y - 1)\,dx + (4y + x - 1)\,dy = 0$. **[I.A.S. (Prel.) 2005]**

Ans. $\log\{4y^2 + (x - 1)^2\} + \tan^{-1}\{2y/(x - 1)\} = c$

12. $(2x + 3y + 4)\,dy = (x + 2y + 3)\,dx$. **Ans.** $\{(x - 1) + (y + 2)\sqrt{3}\}^{2-\sqrt{3}} = c\,\{(x - 1) - (y + 2)\sqrt{3}\}^{2+\sqrt{3}}$

13. $(x + 2y - 3)\,dx - (2x - y + 1)\,dy = 0$ **[G.N.D.U., Amritsar 2010]**

Ans. $5(x^2 + y^2 + xy) - 9x - 15y = c$

2.11 Pfaffian differential equation

Definition. Pfaffian differential form is an expression of the form

$$\sum_{i=1}^{n} f_i\,(x_1, x_2, ..., x_n)\,dx_i,$$

where f_i are functions of some or all of the n variables $x_1, x_2, ..., x_n$.

Again, the equation $$\sum_{i=1}^{n} f_i\,(x_1, x_2, ..., x_n)\,dx_i = 0$$

is known as *Pfaffian differential equation.*

$M\,(x, y)\,dx + N\,(x, y)\,dy = 0$ and $P\,(x, y, z)\,dx + Q\,(x, y, z)\,dy + R\,(x, y, z)\,dz = 0$ are examples of Pfaffian equations in two and three variables.

2.12 Exact differential equation **[Dibrugarh 1996]**

Definition. If M and N are functions of x and y, the equation $M\,dx + N\,dy = 0$... (1)

is called exact when there exists a function $f\,(x, y)$ of x and y, such that

$$d\,[f\,(x, y)] = M\,dx + N\,dy, \qquad ...(2)$$

i.e., $$(\partial f/\partial x)\,dx + (\partial f/\partial y)\,dy = M\,dx + N\,dy. \qquad ...(3)$$

Remarks. The differential equation $$y^2\,dx + 2xy\,dy = 0, \qquad ...(4)$$

is an exact differential equation, for there exists a function xy^2, such that

$$d\,(xy^2) = \frac{\partial}{\partial x}(xy^2)\,dx + \frac{\partial}{\partial y}(xy^2)\,dy \quad \text{or} \quad d\,(xy^2) = y^2\,dx + 2xy\,dy. \qquad ...(5)$$

So, (4) may be rewritten as $d\,(xy^2) = 0$. This on integration yields $xy^2 = c$, where c is an arbitrary constant. Thus general solution of (4) is $xy^2 = c$.

In practice, however, we shall not be able to determine $f\,(x, y)$ so easily. But the method outlined here will be often useful. Note that if $xy^2 = c$ is merely differentiated, then it gives rise to (4). Thus exact equations have the following important property : *An exact differential equation can always be derived from its general solution directly by differentiating without any subsequent multiplication, elimination*, etc.

2.13 Theorem. To determine the necessary and sufficient condition for a differential equation of first order and first degree to be exact

[Guwahati 2007; Pune 2010, Agra 1995]

Statement. *The necessary and sufficient condition for the differential equation*

$$M\,dx + N\,dy = 0 \qquad ...(1)$$

to be exact is $$\partial M/\partial y = \partial N/\partial x. \qquad ...(2)$$

Proof. The condition (2) is necessary. Let (1) be exact. Hence by definition, there must exist a function $f\,(x, y)$ of x and y, such that

$$d\,[f\,(x, y)] = (\partial f/\partial x)\,dx + (\partial f/\partial y)\,dy = M\,dy + N\,dy. \qquad ...(3)$$

Equating coefficients of dx and dy in (3), we get

$$M = \partial f/\partial x \qquad ...(4)$$

and $$N = \partial f/\partial y. \qquad ...(5)$$

To remove the unknown function $f\,(x, y)$, we differentiate partially (4) and (5) with respect to y and x respectively giving

$$\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\,\partial x} \qquad ...(6)$$

and $$\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y\, \partial x} \qquad \text{... (7)}$$

Since $\partial^2 f/\partial y\, \partial x = \partial^2 f/\partial x\, \partial y$, (6) and (7) give $\partial M/\partial y = \partial N/\partial x$.

Thus, if (1) is exact, M and N satisfy condition (2).

The condition is sufficient. We assume that (2) holds and show that (1) is an exact equation. For this we must find a function $f(x, y)$, such that $d\,[f(x, y)] = M\,dx + N\,dy$.

Let $$g(x, y) = \int M\,dx \qquad \text{... (8)}$$

be the partial integral of M, that is, the integral obtained by keeping y fixed. We first prove that $(N - \partial g/\partial y)$ is a function of y only. This is clear because

$$\frac{\partial}{\partial x}\left(N - \frac{\partial g}{\partial y}\right) = \frac{\partial N}{\partial x} - \frac{\partial^2 g}{\partial x\, \partial y} = \frac{\partial N}{\partial x} - \frac{\partial^2 g}{\partial y\, \partial x} \quad \text{as} \quad \frac{\partial^2 g}{\partial x\, \partial y} = \frac{\partial^2 g}{\partial y\, \partial x}$$

$$= \frac{\partial N}{\partial x} - \frac{\partial}{\partial y}\left(\frac{\partial g}{\partial x}\right) = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}, \text{ using (8)}$$

$$= 0, \text{ using (2)}$$

Take, $$f(x, y) = g(x, y) + \int \{N - (\partial g/\partial y)\}\,dy. \qquad \text{... (9)}$$

Hence on total differentiation of (9), we get

$$df = dg + \left(N - \frac{\partial g}{\partial y}\right)dy = \left(\frac{\partial g}{\partial x}dx + \frac{\partial g}{\partial y}dy\right) + N\,dy - \frac{\partial g}{\partial y}dy$$

$$= (\partial g/\partial x)\,dx + N\,dy = M\,dx + N\,dy, \text{ using (8)}$$

Thus, if (2) is satisfied, (1) is surely an exact equation.

2.14 Working rule for solving an exact differential equation [Meerut 2008]

Compare the given equation with $M\,dx + N\,dy = 0$ and find out M and N. Then find out $\partial M/\partial y$ and $\partial N/\partial x$. If $\partial M/\partial y = \partial N/\partial x$, we conclude that the given equation is exact. If the equation is exact, then

Step 1. Integrate M with respect to x treating y as a constant.

Step 2. Integrate with respect to y only those terms of N which do not contain x.

Step 3. Equate the sum of these two integrals [found in steps 1 and 2] to an arbitrary constant and thus we obtain the required solution. In short the solution of exact equation $M\,dx + N\,dy = 0$ is

$$\underset{[\text{Treating } y \text{ as constant}]}{\int M\,dx} + \int (\text{terms in } N \text{ not containing } x)\,dy = c,$$

where c is an arbitrary constant.

2.15 Solved Examples of type 5 based on Art. 2.14

Ex. 1. *Solve* $(x^2 - 4xy - 2y^2)\,dx + (y^2 - 4xy - 2x^2)\,dy = 0.$ **[Delhi Maths (H) 1995, 2005]**

Sol. Comparing the given equation with $M\,dx + N\,dy = 0$, we have

$M = x^2 - 4xy - 2y^2$ and $N = y^2 - 4xy - 2x^2$.

$\therefore$ $\partial M/\partial y = -4x - 4y$ and $\partial N/\partial x = -4y - 4x$ so that $\partial M/\partial y = \partial N/\partial x$.

Hence, the given equation is exact and hence its solution is

$$\underset{[\text{Treating } y \text{ as constant}]}{\int M\,dx} + \int (\text{terms in } N \text{ not containing } x)\,dy = c'$$

or $$\int (x^2 - 4xy - 2y^2)\, dx + \int y^2 dy = c'$$
[Treating y as constant]

or $$x^3/3 - 4y \times (x^2/2) - 2y^2x + y^3/3 = c/3, \text{ taking } c' = c/3$$

or $$x^3 + y^3 - 6xy\,(x + y) = c, \; c \text{ being an arbitrary constant.}$$

Ex. 2. *Test whether the equation* $(x + y)^2\, dx - (y^2 - 2xy - x^2)\, dy = 0$ *is exact and hence solve it.* **[I.A.S. 1995]**

Sol. The given equation can be re-written as $(x^2 + 2xy + y^2)\, dx + (x^2 + 2xy - y^2)\, dy = 0$ ..(1)

Comparing (1) with $M\,dx + N\,dy = 0$, *here* $M = x^2 + 2xy + y^2$, $N = x^2 + 2xy - y^2$.

$\therefore$ $\partial M/\partial y = 2x + 2y$ and $\partial N/\partial x = 2x + 2y$ so that $\partial M/\partial y = \partial N/\partial x$.

Hence (1) is exact and hence its solution is

$$\underset{\text{[Treating } y \text{ as constant]}}{\int M\, dx} + \int (\text{terms in } N \text{ not containing } x)\, dy = c'$$

or $$\int (x^2 + 2xy + y^2)\, dx + \int (-y^2)\, dy = c'$$
[Treating y as constant]

or $$x^3/3 + 2y \times (x^2/2) + y^2x - y^3/3 = c/3, \text{ taking } c' = c/3$$

or $$x^3 + y^3 + 3xy\,(x + y) = c, \; c \text{ being an arbitrary constant.}$$

Ex. 3. *Solve* (*a*) $(2x - y + 1)\, dx + (2y - x - 1)\, dy = 0$.

[Delhi Maths. (G) 1996, Delhi Maths. (H) 1996, 1998]

(*b*) $(4x + 3y + 1)\, dx + (3x + 2y + 1)\, dy = 0$.

(*c*) $dy/dx = (2x - y + 1)/(x + 2y - 3)$.

Sol. (*a*) Given equation is $(2x - y + 1)\, dx + (2y - x - 1)\, dy = 0$. ... (1)

Comparing (1) with $M\,dx + N\,dy = 0$, here $M = 2x - y + 1$, $N = 2y - x - 1$.

$\therefore$ $\partial M/\partial y = -1 = \partial N/\partial x$ and hence (1) is exact and its solution is given by

$$\underset{\text{[Treating } y \text{ as constant]}}{\int M\, dx} + \int (\text{terms in } N \text{ not containing } x)\, dy = c$$

or $\displaystyle\int (2x - y + 1)\, dx + \int (2y - 1)\, dt = 0$ [Treating y as constant] or $x^2 - xy + x - y^2 - y = c$

(*b*) Do as in part (*a*). **Ans.** $2x^2 + 3xy + y^2 + x + y = c$.

(*c*) Do yourself. **Ans.** $x^2 - xy + x + 3y - y^2 = c$.

Ex. 4. *Solve* $(1 + e^{x/y})\, dx + e^{x/y}\,\{1 - (x/y)\}\, dy = 0$. **[I.A.S. Prel. 2007;Osmania 2005]**

Sol. Comparing the given equation with $M\,dx + N\,dy = 0$, $M = 1 + e^{x/y}$, $N = e^{x/y}\,\{1 - (x/y)\}$.

$\therefore$ $\partial M/\partial y = e^{x/y}\,(-x/y^2)$, $\partial N/\partial x = e^{x/y}\,(-1/y) + (1 - x/y)\, e^{x/y}\,(1/y) = (-x/y^2)\, e^{x/y}$

Thus, $\partial M/\partial y = \partial N/\partial x$ and so the given equation is exact.

Its solution is $$\underset{\text{[Treating } y \text{ as constant]}}{\int M\, dx} + \int (\text{terms in } N \text{ not containing } x)\, dy = c$$

or $\displaystyle\int (1 + e^{x/y})\, dx = c$ [Treating y as constant] or $x + ye^{x/y} = c$.

Ex. 5. *Solve* $(y^2 e^{xy^2} + 4x^3)\, dx + (2xy\, e^{xy^2} - 3y^2)\, dy = 0$.

Sol. Comparing the given equation with $M\,dx + N\,dy = 0$, here

$M = y^2 e^{xy^2} + 4x^3$ and $N = 2xye^{xy^2} - 3y^2$.

$$\partial M/\partial y = 2ye^{xy^2} + y^2 \cdot 2xye^{xy^2} = \partial N/\partial x,$$

Hence, the given equation is exact and so its solution is

$$\underset{\text{[Treating } y \text{ as constant]}}{\int M\,dx} + \int (\text{terms in N not containing } x)\,dy = c$$

or
$$\underset{\text{[Treating } y \text{ as constant]}}{\int y^2 e^{xy^2} + 4x^3)dx} + \int \left(-3y^2\right) dy = c$$

or $y^2 \times (1/y^2) \times e^{xy^2} + 4 \times (1/4) \times x^4 - 3 \times (y^3/3) = c$ or $e^{xy^2} + x^4 - y^3 = c.$

Ex. 6. *Solve* $(ax + by + g)\,dx + (hx + by + f)\,dy = 0$

or $\dfrac{dy}{dx} + \dfrac{ax + hy + g}{hx + by + f} = 0$ **[Delhi Maths. 1994, 1997]**

Sol. Comparing the given equation with $M\,dx + N\,dy = 0$, here

$M = ax + hy = g$ and $N = hx + by + f$

$\therefore$ $\partial M/\partial y = h = \partial N/\partial x$ and hence the given equation is exact and so its solution is given by

$$\underset{\text{[Treating } y \text{ as constant]}}{\int M\,dx} + \int (\text{terms in } N \text{ not containing } x)\,dy = c$$

or
$$\underset{\text{[Treating } y \text{ as constant]}}{\int (ax + hy + g)\,dx} + \int (by + f)\,dy = c$$

or
$$(1/2) \times ax^2 + hxy + gx + (1/2) \times by^2 + fy = c$$

or
$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c' = 0, \text{ where } c' = -2c.$$

Ex. 7. *Solve* $\{y\,(1 + 1/x) + \cos y\}\,dx + (x + \log x - x \sin y)\,dy = 0$

[Delhi Maths. (G) 1993; I.A.S. 1993; Osmania 2005]

Sol. Comparing the given equation with $M\,dx + N\,dy = 0$, here

$M = y\,(1 + 1/x) + \cos y$ and $N = x + \log x - x \sin y$

$\therefore$ $\partial M/\partial y = 1 + (1/x) - \sin y = \partial N/\partial x$

Hence, the given equation is exact and so its solution is

$$\underset{\text{[Treating } y \text{ as constant]}}{\int M\,dx} + \int (\text{terms in } N \text{ not containing } x)\,dy = c$$

or $\underset{\text{[Treating } y \text{ as constant]}}{\int (y + y/x + \cos y)\,dx} + 0 = c$ or $yx + y \log x + x \cos y = c$

Ex. 8(*a*) *Solve* $x\,dx + y\,dy + \dfrac{x\,dy - y\,dx}{x^2 + y^2} = 0.$

[Agra 2006; Bangalore 1995; Kanpur 1998; Lucknow 1995]

Sol. Re-writing the given equation, $\{x - y/(x^2 + y^2)\}\,dx + \{y + x/(x^2 + y^2)\}\,dy = 0$ ... (1)

Comparing (1) with $M\,dx + N\,dx = 0$, $M = x - y/(x^2 + y^2)$, and $N = y + x/(x^2 + y^2)$

$$\therefore \quad \frac{\partial M}{\partial y} = 0 - \frac{1 \cdot (x^2 + y^2) - y \cdot 2y}{(x^2 + y^2)^2} = -\frac{x^2 - y^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

and
$$\frac{\partial N}{\partial x} = 0 + \frac{1 \cdot (x^2 + y^2) - x \cdot 2x}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}. \qquad \text{Thus, } \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}.$$

Hence (1) is exact and therefore its solution is

$$\underset{\text{[Treating } y \text{ as constant]}}{\int M\,dx} + \int (\text{terms in } N \text{ not containing } x)\,dy = \frac{1}{2}c$$

or
$$\underset{\text{[Treating } y \text{ as constant]}}{\int \{x - y/(x^2+y^2)\}\,dx} + \int y\,dy = \frac{1}{2}c$$

or $\quad x^2/2 - y \times (1/y) \times \tan^{-1}(x/y) + y^2/2 = c/2 \quad$ or $\quad x^2 + y^2 - 2\tan^{-1}(x/y) = c.$

Ex. 8(*b*). *Solve* $x\,dx + y\,dy = a^2\,\dfrac{x\,dy - y\,dx}{x^2+y^2}$

[Meerut 2007; Kanpur 1998; Lucknow 1995; Purvanchal 1995]

Ans. $x^2 + y^2 + 2a^2\tan^{-1}(x/y) = c$, c being an arbitrary constant.

Ex. 8(*c*). *Solve* $(x\,dy + y\,dy)(x^2+y^2) = a^2(x\,dy - y\,dx)$.

Sol. Dividing both sides of given equation by (x^2+y^2), we get

$$x\,dx + y\,dy = a^2(x\,dy - y\,dx)/(x^2+y^2)$$

which is same as in Ex. 8(*b*). So proceed yourself as before.

Ex. 8(*d*). *Solve* $(x^3 + xy^2 + a^2y)\,dx + (y^3 + yx^2 - a^2x)\,dy = 0$ **[Guwahati 2007]**

Sol. The given equation can be re-written as

$x(x^2+y^2)\,dx + a^2y\,dx + y(y^2+x^2)\,dy - a^2x\,dy = 0 \quad$ or $(x\,dx + y\,dy)(x^2+y^2) = a^2(x\,dy - y\,dx)$, which is same as in Ex. 8(*c*). So proceed yourself as before.

Ex. 8(*e*). *Solve* $(x^2+y^2)(x\,dx + y\,dy) = x\,dy - y\,dx$. **[Delhi Maths. (H) 1996]**

Hint. This question is same as Ex. 8(*c*) with $a = 1$. Proceed as before taking $a = 1$ in whole solution.

Ans. $x^2 + y^2 + 2\tan^{-1}(x/y) = c$.

Ex. 9. *Solve* $(r + \sin\theta - \cos\theta)\,dr + r(\sin\theta + \cos\theta)\,d\theta = 0$. **[Allahabad 1996]**

Sol. Here we have r and θ in place of usual variables x and y.

Comparing the given equation with $M\,dr + N\,d\theta = 0$, $M = r + \sin\theta - \cos\theta$, $N = r(\sin\theta + \cos\theta)$.

$\therefore\ \partial M/\partial\theta = \cos\theta + \sin\theta = \partial N/\partial r$. So equation is exact with solution

$$\underset{\text{[Treating } \theta \text{ as constant]}}{\int M\,dx} + \int (\text{terms in } N \text{ not containing } r) = c$$

or $\quad \underset{\text{[Treating } \theta \text{ as constant]}}{\int (r + \sin\theta - \cos\theta)\,dr} = c \quad$ or $\quad r^2/2 + r(\sin\theta - \theta) = c.$

Ex. 10(*a*). *Solve* $y\sin 2x\,dx - (1 + y^2 + \cos^2 x)\,dy = 0$. **[I.A.S. 1996; Lucknow 1994]**

Sol. Re-writing the given equation, $\quad y\sin 2x\,dx - \{1 + y^2 + \tfrac{1}{2}(1 + \cos 2x)\}\,dy = 0. \quad$ ... (1)

Comparing (1) with $\quad M\,dx + N\,dy = 0, \quad M = y\sin 2x, \quad N = -(3/2) - y^2 - (1/2)\cos 2x,$

$\therefore\ \partial M/\partial y = 2\cos 2x = \partial N/\partial x$. Hence (1) is exact and its solution is

$$\underset{\text{[Treating } y \text{ as constant]}}{\int M\,dx} + \int (\text{terms in } N \text{ not containing } x)\,dy = c'$$

or
$$\underset{\text{[Treating } y \text{ as constant]}}{\int y\sin 2x\,dx} + \int \{(-3/2) - y^2\}\,dy = c'$$

or $\quad y \times (-1/2) \times \cos 2x - (3/2) \times y - y^3/3 = -c/6$, taking $c' = -c/6$

$\therefore$ Required solution is $\quad 3y\cos 2x + 9y + 2y^3 = c$, c being an arbitrary constant.

Ex. 10(*b*). *Solve y sin 2x dx* – (y^2 + $\cos^2 x$) *dy* = 0.

Sol. Proceed as in Ex. 10(*a*). **Ans.** $3y \cos 2x + 3y + 2y^3 = c$

Ex. 11. *Solve* $(x^2 + y^2 + x)\, dx - (2x^2 + 2y^2 - y)\, dy = 0$. **[Lucknow 1997]**

Sol. Re-writing the given equation, $\{(x^2 + y^2) + x\}\, dx + \{y - 2\,(x^2 + y^2)\}\, dy = 0$

or $$[1 + \{x/(x^2 + y^2)\}]\, dx + [\{y/(x^2 + y^2)\} - 2]\, dy = 0 \qquad \text{... (1)}$$

Comparing (1) with $M\,dx + N\,dy = 0$, we get

$$M = 1 + \frac{y}{x^2 + y^2}, \qquad N = \frac{y}{x^2 + y^2} - 2 \qquad \text{Therefore,} \quad \frac{\partial M}{\partial y} = \frac{2xy}{x^2 + y^2} = \frac{\partial N}{\partial x}.$$

Hence (1) is exact and so its solution is

$$\underset{[\text{Treating } y \text{ as constant}]}{\int M\, dx} + \int (\text{terms in } N \text{ not containing } x)\, dy = c$$

or $$\underset{[\text{Treating } y \text{ as constant}]}{\int \{1 + x/(x^2 + y^2)\}\, dx} + \int (-2)\, dy = c \qquad \text{or} \qquad x + (1/2) \times \log (x^2 + y^2) - 2y = c.$$

Ex. 12. *Show* $(4x + 3y + 1)\, dx + (3x + 2y + 1)\, dy = 0$ *is a family of hyperbolas with a common axis and tangent at the vertex.* **[I.A.S. 2000]**

Sol. Given $$(4x + 3y + 1)\, dx + (3x + 2y + 1)\, dy = 0 \qquad \text{... (1)}$$

Comparing (1) with $M\,dx + N\,dy = 0$ here, $M = 4x + 3y + 1$, $N = 3x + 2y + 1$.

Here $\partial M/\partial y = 3 = \partial N/\partial x$ and so (1) is exact. Its solution is

$$\underset{[\text{Treating } y \text{ as constant}]}{\int (4x + 3y + 1)\, dx} + \underset{[\text{Integrating terms free from } x]}{\int (3x + 2y + 1)\, dy} = 0$$

or $2x^2 + 3xy + x + y^2 + y + k = 0$, where k is an arbitrary constant. ... (2)

Comparing (2) with standard form of conic section $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, here

$a = 2$, $b = 1$, $h = 3/2$, $g = 1/2$, $f = 1/2$, $c = k$ ... (3)

Then $$h^2 - ab = (9/4) - 2 = \text{positive quantity,}$$

showing that (2) represents *a* family of hyperbolas, *k* being the parameter, with common axis and tangent at vertex.

Ex. 13. *Find the values of constant* λ *such that* $(2xe^y + 3y^2)\,(dy/dx) + (3x^2 + \lambda e^y) = 0$ *is exact. Further, for this value of* λ, *solve the equation.* **[I.A.S. 2002]**

Sol. Re-writing the given equation, $$(3x^2 + \lambda e^y)\, dx + (2xe^y + 3y^2)\, dy = 0 \quad . \quad .. (1)$$

Comparing (1) with $M\,dx + N\,dy = 0$, here $M = 3x^2 + \lambda e^y$ and $N = 2xe^y + 3y^2$.

Now, for (1) to be exact we must have

$\partial M/\partial y = \partial N/\partial x$ so that $\lambda e^y = 2e^y$ giving $\lambda = 2$.

$\therefore$ (1) becomes $$(3x^2 + 2e^y)\, dx + (2xe^y + 3y^2)\, dy = 0 \qquad \text{... (3)}$$

Equation (3) in exact and hence its solution is its solution is

$$\underset{[\text{Treating } y \text{ as constant}]}{\int M\, dx} + \int (\text{terms in N not containing } x)\, dy = c$$

or $$\int (3x^2 + 2e^x)\, dx + \int (3y^2)\, dy = c \qquad \text{or} \qquad x^3 + 2e^x + y^3 = c$$

Exercise 2(E)

Solution the following differential equations:

1. $(x + 2y - 2)\, dx + (2x - y + 3)\, dy = 0$. **Ans.** $x^2 + 4xy - 4x - y^2 + 6y = c$

2. $(2ax + by)\, y\, dx + (ax + 2by)\, x\, dy = 0$. **Ans.** $ayx^2 + by^2x = c$

3. $(x^2 - ay)\, dx = (ax - y^2)\, dy$. **[Delhi Maths. (G) 1996]** **Ans.** $x^3 - 3axy + y^3 = c$

4. $dy/dx = (2x - y)/(x + 2y - 5)$. **Ans.** $x^2 - xy + y^2 + 5y = c$

5. $(x^2 + y^2 + a^2)\, y\, dy + (x^2 + y^2 - a^2)\, x\, dx = 0$. **[S.V. University (A.P.) 1997]**

Ans. $x^4 + y^4 + 2x^2y^2 + 2a^2 (y^2 - x^2) = c$

6. $(e^y + 1) \cos x\, dx + e^y \sin x\, dy = 0$. **[Agra 2006]** **Ans.** $(e^y + 1) \sin x = c$

7. $x (x^2 + 3y^2)\, dx + y (y^2 + 3x^2)\, dy = 0$. **Ans.** $x^4 + 6x^2y^2 + y^4 = c$

8. $(a^2 - 2xy - y^2)\, dx - (x + y)^2\, dy = 0$. **[Delhi B.A. (Prog) II 2011]**

Ans. $a^2x - x^2y - xy^2 - (1/3)\, y^3 = c$

9. $(3x^2 + 6xy^2)\, dx + (6x^2y^2 + 4y^3)\, dy = 0$. **[Delhi Maths (G) 2006]** **Ans.** $x^3 + 3x^2y^2 + y^4 = c$

10. Verify that the equation $(x^4 - 2xy^2 + y^4)\, dx - (2x^2y - 4xy^2 + \sin y)\, dy = 0$ is exact and solve it

Ans. $x^5/5 - x^2y^2 + xy^4 + \cos y = c$

11. $(3x^2 + 4xy)dx + (2x^2 + 2y)dy = 0$ **[Delhi Maths (Prog) 2007]** **Ans.** $x^3 + 2x^2y + y^2 = c.$

2.16 Integrating factor. [Osmania 2005]

Definition. If an equation of the form $M\, dx + N\, dy = 0$ is not exact, it can always be made exact by multiplying by some function of x and y. Such a multiplier is called an integrating factor. We shall write I.F. for integrating factor.

Although an equation of the form $M\, dx + N\, dy = 0$ always has integrating factors, there is no general method of finding them. It should be remembered that there are an infinite number of integrating factors for an equation of the form $M\, dx + N\, dy = 0$ as established in the following theorem.

Theorem. *The differential equation $M\, dx + N\, dy = 0$ possess an infinite number of integrating factors.*

Proof. Given
$$M\, dx + N\, dy = 0. \qquad \text{... (1)}$$

Let $\mu\,(x, y)$ be an I.F. of (1). Then, by definition
$$\mu\,(M\, dx + N\, dy) = 0$$
must be an exact differential equation and so there must exist a function $V(x, y)$, such that
$$dV = \mu\,(M\, dx + N\, dy) \qquad \text{... (2)}$$
$$\therefore \qquad V = \text{constant is a solution of (1).}$$

Assume that $f(V)$ be any function of V. So, by (2), we have
$$f(V)\, dV = \mu f(V)\, (M\, dx + N\, dy). \qquad \text{... (3)}$$

Since the expression on L.H.S. of (3) is an exact differential, it follows that the expression on R.H.S. of (3) must also be an exact differential. Hence, by definition, it follows that $\mu f(V)$ is an I.F. of (1). Since $f(V)$ is an arbitrary function of V, it follows that (1) has an infinite number of integrating factors.

Remark. Although an equation of the form $M\, dx + N\, dy = 0$ always has integrating factors, there is no general method of finding them. We now explain rules for finding integrating factors.

Rule I. By inspection. Often an I.F. of given equation $M\, dx + N\, dy = 0$ can be found out by inspection as explained below.

By rearranging the terms of the given equation and/or by dividing by a suitable function of x and y, the equation thus obtained will contain several parts integrable easily. In this connection, the following list of exact differentiable should be noted carefully.

(*i*) $d\left(\dfrac{y}{x}\right) = \dfrac{x\, dy - y\, dx}{x^2}$ (*ii*) $d\left(\dfrac{x}{y}\right) = \dfrac{y\, dx - x\, dy}{y^2}$

(*iii*) $d\left(\dfrac{y^2}{x}\right) = \dfrac{2xy\,dy - y^2 dx}{x^2}$ (*iv*) $d\left(\dfrac{x^2}{y}\right) = \dfrac{2yx\,dx - x^2 dy}{y^2}$

(*v*) $d\left(\dfrac{y^2}{x^2}\right) = \dfrac{2x^2 y\,dy - 2xy^2\,dx}{x^4}$ (*vi*) $d\left(\dfrac{x^2}{y^2}\right) = \dfrac{2y^2 x\,dx - 2yx^2\,dy}{y^4}$

(*vii*) $d\,[\log(xy)] = \dfrac{x\,dy + y\,dx}{xy}$ (*viii*) $d\,(xy) = x\,dy + y\,dx$

(*ix*) $d\left(\tan^{-1}\dfrac{y}{x}\right) = \dfrac{x\,dy - y\,dx}{x^2 + y^2}$ (*x*) $d\left(\tan^{-1}\dfrac{x}{y}\right) = \dfrac{y\,dx - x\,dy}{x^2 + y^2}$

(*xi*) $d\left[\log\left(\dfrac{y}{x}\right)\right] = \dfrac{x\,dy - y\,dx}{xy}$ (*xii*) $d\left[\log\left(\dfrac{x}{y}\right)\right] = \dfrac{y\,dx - x\,dy}{xy}$

(*xiii*) $d\left[\frac{1}{2}\log(x^2 + y^2)\right] = \dfrac{x\,dx + y\,dy}{x^2 + y^2}$ (*xiv*) $d\left(-\dfrac{1}{xy}\right) = \dfrac{x\,dy + y\,dx}{x^2 y^2}$

(*xv*) $d\left(\dfrac{e^x}{y}\right) = \dfrac{y\,e^x\,dx - e^x\,dy}{y^2}$ (*xvi*) $d\,(\sin^{-1} xy) = \dfrac{x\,dy + y\,dx}{(1 - x^2 y^2)^{1/2}}$

2.17 Solved examples of Type 6 based on Rule 1 of Art. 2.16

Ex. 1. *Solve* $y\,dx - x\,dy + (1 + x^2)\,dx + x^2 \sin y\,dy = 0$. **[Allahabad 1996]**

Sol. Dividing each term of the given equation by x^2, we get

$$\frac{y\,dx - x\,dy}{x^2} + \frac{1 + x^2}{x^2}\,dx + \sin y\,dy = 0 \quad \text{or} \quad -\frac{x\,dy - y\,dx}{x^2} + \left(\frac{1}{x^2} + 1\right)dx + \sin y\,dy = 0$$

or $$-\,d\,(y/x) + (1 + 1/x^2)\,dx + \sin y\,dy = 0.$$

Integrating, $$-\,(y/x) + x - (1/x) - \cos y = c$$

or $$-\,y + x^2 - 1 - x\cos y = cx, \text{ where } c \text{ is an arbitrary constant.}$$

Ex. 2. *Solve* $y\,(2xy + e^x)\,dx = e^x\,dy$. **[Agra 1995; Lucknow 1998]**

Sol. Re-writing, $2xy^2\,dx + y\,e^x\,dx - e^x\,dy = 0$

or $$2x\,dx + \frac{y\,e^x\,dx - e^x\,dy}{y^2} = 0 \quad \text{or} \quad 2x\,dx + d\left(\frac{e^x}{y}\right) = 0.$$

Integrating, $$x^2 + e^x/y = c \quad \text{or} \quad yx^2 + e^x = cy.$$

Ex. 3. *Solve* $y \sin 2x\,dx = (1 + y^2 + \cos^2 x)\,dy$.

Sol. Re-writing, $-\,2y \sin x \cos x\,dx + \cos^2 x\,dy + (1 + y^2)\,dy = 0$

or $$d\,(y\cos^2 x) + (1 + y^2)\,dy = 0. \qquad \textbf{[Note Carefully]}$$

Integrating, $y\cos^2 x + y + y^3/3 = c$, c being an arbitrary constant.

Ex. 4. *Solve* $(x^3 + xy^2 + a^2 y)\,dx + (y^3 + yx^2 - a^2 x)\,dy = 0$.

Sol. Re-writing, the given equation, $x\,(x^2 + y^2)\,dx + y\,(x^2 + y^2)\,dy + a^2\,(y\,dx - x\,dy) = 0$

or $$x\,dx + y\,dy + a^2\,\frac{y\,dx - x\,dy}{x^2 + y^2} = 0 \quad \text{or} \quad x\,dx + y\,dy + a^2 d\left(\tan^{-1}\frac{x}{y}\right) = 0.$$

Integrating, $$x^2/2 + y^2/2 + a^2 \tan^{-1}(x/y) = c/2 \quad \text{or} \quad x^2 + y^2 + 2a^2 \tan^{-1}(x/y) = c.$$

Ex. 5. *Solve* (*a*) $x^2\,(dy/dx) + xy = \sqrt{1 - x^2 y^2}$. **[Delhi Maths (H) 1993]**

(*b*) $x^2\,(dy/dx) + xy + \sqrt{1 - x^2 y^2} = 0$. **[Delhi Maths (H) 2006]**

Sol. (*a*) Re-writing given equation, we have $\dfrac{x\,dy + y\,dx}{\sqrt{1-x^2y^2}} - \dfrac{dx}{x} = 0.$

Integrating, $\sin^{-1}(xy) - \log x = c$, c being an arbitrary constant.

(*b*) Proceed as in Part (*a*). **Ans.** $\sin^{-1}(xy) + \log x = c$

Exercise 2(F)

Solve the following differential equations:

1. $e^y\,dx + (xe^y + 2y)\,dy = 0.$ **Ans.** $xe^y + y^2 = c$

2. $x\,dx + y\,dy + (x^2+y^2)\,dy = 0.$ **Ans.** $x^2 + y^2 = ce^{-2y}$

3. $x\,dy - y\,dx = (x^2+y^2)\,dx.$ **Ans.** $\tan^{-1}(y/x) = x + c$

4. $y\,dx - x\,dy + \log x\,dx = 0.$ **Ans.** $cx + y + \log x + 1 = 0$

5. $e^{2y}\,dx + 2\,(xe^{2y} - y)\,dy = 0.$ **Ans.** $xe^{2y} - y^2 = c$

6. $y\,(2x^2y + e^x)\,dx - (e^x + y^3)\,dy = 0.$ **Ans.** $4x^3y - 3y^3 + 6e^x = 6cy$

7. $(x^3e^x - my^2)\,dx + m\,xy\,dy = 0.$ **Ans.** $e^x + (my^2)/(2x^2) = c$

8. $x\,dy - y\,dx = xy^2\,dx.$ **Ans.** $yx^2 + 2x = 2cy$

9. $y\,(axy + e^x)\,dx - e^x\,dy = 0.$ **Ans.** $ax^2y + 2e^x = cy$

10. $\{y + \cos y + 1/(2\sqrt{x})\}\,dx + (x - x\sin y - 1)\,dy = 0.$ **Ans.** $xy + x\cos y + x^{1/2} - y = c$

11. $(x^2 + y^2 - a^2)\,x\,dx + (x^2 - y^2 - b^2)\,y\,dy = 0.$ **Ans.** $(x^2 - a^2)^2 - (y^2 + b^2)^2 + 2x^2y^2 = c$

12. $a\,(x\,dy + 2y\,dx) = xy\,dy.$ **Ans.** $a\log(yx^2) - y = c$

[**Hint.** Divide by xy, *i.e.*, take $1/(xy)$ as an I.F.]

13. $dx + y\,dy = m\,(x\,dy - y\,dx).$ **[Delhi Maths(H) 2000]**

[**Hint.** Re-writing, $d\,(x^2+y^2) = 2mx^2 d\,(y/x)$

or $\dfrac{d\,(x^2+y^2)}{x^2+y^2} = 2m\,\dfrac{d\,(y/x)}{1+(y/x)^2}$ or $d\log(x^2+y^2) = 2m\,d\left(\tan^{-1}\dfrac{y}{x}\right).$

Integrating, $\log(x^2+y^2) - 2m\tan^{-1}(y/x) = c$, c being an arbitrary constant.]

14. $(x^4e^x - 2m\,xy^2)\,dx + 2mx^2y\,dy = 0.$

[**Hint.** Re-writing, the given equation is $x^4e^x\,dx + 2m\,(x\,dy - y\,dx)\,xy\,dy = 0$

or $x^4e^x\,dx + 2mx^3y\,d\left(\dfrac{y}{x}\right) = 0$ or $e^x\,dx + 2m\left(\dfrac{y}{x}\right)d\left(\dfrac{y}{x}\right) = 0$

or $d\,\{e^x + m\,(y/x)^2\} = 0$ or $e^x + m\,(y/x)^2 = c$]

15. $(1 + xy)\,y\,dx + x\,(1 - xy)\,dy = 0.$

[Calcutta 1995; I.A.S. 1994; Meerut 1993; Kanpur 1994; Ravishankar 1996; G.N.D.U. Amritsar 2010]

[**Hint.** Re-writing, the given equation is $y\,dx + x\,dy + xy\,(y\,dx - x\,dy) = 0$

or $d\,(xy) + x^2y^2\left(\dfrac{dx}{x} - \dfrac{dy}{y}\right) = 0$ or $d\,(xy) + x^2y^2\,d\left(\log\dfrac{x}{y}\right) = 0$

or $\dfrac{1}{x^2y^2}\,d\,(xy) + d\left(\log\dfrac{x}{y}\right) = 0$ or $d\left(\log\dfrac{x}{y} - \dfrac{1}{xy}\right) = 0,$

Integrating, $\log(x/y) - 1/(xy) = c$, where c is an arbitrary constant.

Rule II. *If the given equation $M\,dx + N\,dy = 0$ is homogeneous and $(Mx + Ny) \neq 0$, then $1/(Mx + Ny)$ is an integrating factor.*

Proof. Re-writing $M\,dx + N\,dy$, we have

$$M\,dx + N\,dy = \frac{1}{2}\left\{(Mx+Ny)\left(\frac{dx}{x}+\frac{dy}{y}\right)+(Mx-Ny)\left(\frac{dx}{x}-\frac{dy}{y}\right)\right\}$$

$$\Rightarrow \qquad \frac{M\,dx+N\,dy}{Mx+Ny} = \frac{1}{2}\left\{\left(\frac{dx}{x}+\frac{dy}{y}\right)+\frac{Mx-Ny}{Mx+Ny}\left(\frac{dx}{x}-\frac{dy}{y}\right)\right\} \qquad \text{... (1)}$$

Since $M\,dx + N\,dy = 0$ is a homogeneous equation, M and N must be of the same degree in variables x and y and hence we may write

$$\frac{Mx-Ny}{Mx+Ny} = \text{some function of } \frac{x}{y} = f\left(\frac{x}{y}\right), \text{ say} \qquad \text{... (2)}$$

Using (2), (1) reduces to

$$\frac{M\,dx+N\,dy}{Mx+Ny} = \frac{1}{2}\left\{\left(\frac{dx}{x}+\frac{dy}{y}\right)+f\left(\frac{x}{y}\right)\left(\frac{dx}{x}-\frac{dy}{y}\right)\right\}$$

$$=\frac{1}{2}\left\{d(\log xy)+f(e^{\log(x/y)})\,d\left(\log\frac{x}{y}\right)\right\}=\frac{1}{2}\left\{d(\log xy)+g\left(\log\frac{x}{y}\right)d\left(\log\frac{x}{y}\right)\right\}$$

[on assuming $f(e^{\log(x/y)}) = g\{\log(x/y)\}$]

$$= d\left[(1/2)\times\log xy+(1/2)\times\int g\{\log(x/y)\}\,d\{\log(x/y)\}\right]$$

showing that $1/(Mx + Ny)$ is an I.F. for the given equation $M\,dx + N\,dy = 0$.

2.18 Solved example of type 7 based on rule II of Art. 2.16

Ex. 1. *Solve* $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0.$ **[Delhi Maths (G) 1994; Garhwal 2010]**

Sol. Given $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0.$ ... (1)

Clearly (1) is a homogeneous differential equation. Comparing (1) with $M\,dx + N\,dy = 0$,

$M = x^2y - 2xy^2$ and $N = -(x^3 - 3x^2y)$ ....(2)

$\therefore \quad Mx + Ny = x(x^2y - 2xy^2) - y(x^3 - 3x^2y) = x^2y^2 \neq 0,$

showing that I.F. of (1) $= 1/(Mx + Ny) = 1/(x^2y^2)$. On multiplying (1) by $1/(x^2y^2)$,

$(1/y - 2/x)\,dx - (x/y^2 - 3/y)\,dy = 0$, which is exact where solution is

$$\underset{\text{[Treating y as constant]}}{\int \{(1/y)-(2/x)\}\,dx} + \int (3/y)\,dy = 0 \quad \text{or} \quad (x/y) - 2\log x + 3\log y = \log c$$

or $\quad \log y^2 - \log x^2 - \log c = -x/y \quad$ or $\quad \log(y^2/cx^2) = -x/y$

or $\quad y^2 = cx^2 e^{-x/y}$, where c is an arbitrary constant.

Note. All questions based on rule II can also be solved as explained in Art 2.7. Refer solved examples of type 3 in Art. 2.8.

Ex. 2. *Solve* $x^2y\,dx - (x^3 - y^3)\,dy = 0.$ **[Calicut 1993]** **Ans.** $y = ce^{x^3/3y^3}$

Rule III. *If the equation* $M\,dx + N\,dy = 0$ *is of the form* $f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0$, *then* $1/(Mx - Ny)$ *is an integrating factor of* $M\,dx + N\,dy = 0$ *provided* $(Mx - Ny) \neq 0$. **[I.A.S. 1991]**

Proof. Suppose that $M\,dx + N\,dy = 0$ ... (1)

is of the form $f_1(xy)\,y\,dx + f_2(xy)\,x\,dy = 0.$ ... (2)

Comparing (1) and (2), we have

$$\frac{M}{y f_1(xy))} = \frac{N}{x f_2(xy)} = \mu \text{ (say)}$$

$\Rightarrow \qquad M = \mu\, y\, f_1\,(xy) \qquad$ and $\qquad N = \mu\, x\, f_2\,(xy). \qquad$ ... (3)

Re-writing $M\,dx + N\,dy$, we have

$$M\,dx + N\,dy = \frac{1}{2}\left\{(Mx+Ny)\left(\frac{dx}{x}+\frac{dy}{y}\right)+(Mx-Ny)\left(\frac{dx}{x}-\frac{dy}{y}\right)\right\}$$

$$\Rightarrow \frac{M\,dx + N\,dy}{Mx - Ny} = \frac{1}{2}\left\{\frac{Mx+Ny}{Mx-Ny}\left(\frac{dx}{x}+\frac{dy}{y}\right)+\left(\frac{dx}{x}-\frac{dy}{y}\right)\right\}$$

$$=\frac{1}{2}\left\{\frac{f_1(xy)+f_2(xy)}{f_1(xy)-f_2(xy)}\,d\,(\log xy)+d\left(\log\frac{x}{y}\right)\right\}, \text{using (3)}$$

$$= \frac{1}{2}\left\{f\,(xy)\,d(\log xy)+d\left(\log\frac{x}{y}\right)\right\}, \quad \text{where } \frac{f_1(xy)+f_2(xy)}{f_1(xy)-f_2(xy)} = f\,(xy)$$

$$= \frac{1}{2}\left\{f\,(e^{\log xy})\,d(\log xy)+d\left(\log\frac{x}{y}\right)\right\} = \frac{1}{2}\left\{g\,(\log xy)\,d(\log xy)+d\,\log\left(\frac{x}{y}\right)\right\}$$

[on assuming that $f\,(e^{\log xy}) = g\,(\log xy)$]

$$= d\;\{(1/2)\times\log\,(x/y)+(1/2)\times\int g\,(\log xy)\;d\;(\log xy)\},$$

showing that $Mx - Ny$ is an I.F. of $M\,dx + N\,dy = 0$.

2.19 Solved examples of type 8 based on Rule III of Art. 2.16

Ex. 1. *Solve* $(xy \sin xy + \cos xy)\,y\,dx + (xy \sin xy - \cos xy)\,x\,dy = 0$.

[Purvanchal 1996; Kanpur 1993; Lucknow 1993, 1997

Sol. *Given* $\qquad (xy \sin xy + \cos xy)\,y\,dx + (xy \sin xy - \cos xy)\,x\,dy = 0 \qquad$ ... (1)

Comparing (1) with $M\,dx + N\,dy = 0$, we have

$M = y\,(xy \sin xy + \cos xy) \qquad$ and $\qquad N = x\,(xy \sin xy - \cos xy)$,

showing that (1) is of the form $\qquad f_1\,(xy)\,y\,dx + f_2\,(xy)\,x\,dy = 0$.

Again, $\quad Mx - Ny = xy\,(xy \sin xy + \cos xy) - xy\,(xy \sin xy - \cos xy)$

$\therefore \;\; Mx - Ny = 2xy \cos xy \neq 0$. Hence I.F. of (1) $= 1/(Mx - Ny) = 1/(2xy \cos xy)$.

On multiplying (1) by $1/(2xy \cos xy)$, we have

$$(1/2)\times(y \tan xy + 1/x)\,dx + (1/2)\times(x \tan xy - 1/y)\,dy = 0 \qquad \text{... (2)}$$

which must be exact and so by the usual rule, solution of (2) is

$$\int \{(1/2)\times(y \tan xy + 1/x)\}\,dx + \int (-1/2y)\,dy = (1/2)\times\log c$$

[Treating y as constant]

or $\qquad (1/2)\times(\log \sec xy + \log x) - (1/2)\times\log y = (1/2)\log c$

or $\qquad \log \sec xy + \log\,(x/y) = \log c \qquad$ or $\qquad (x/y) \sec xy = c$.

Ex. 2. *Solve* $y\,(1 + xy)\,dx + x\,(1 - xy)\,dy = 0$. **[I.A.S. (Prel.) 2006; Meerut 1993; G.N.D.U. Amritsar 2010]**

Sol. Given $\qquad (1 + xy)\,y\,dx + (1 - xy)\,x\,dy = 0. \qquad$ ... (1)

Comparing (1) with $M\,dx + N\,dy = 0$, $M = (1 + xy)\,y$ and $N = (1 - xy)\,x$, showing that (1) is of the form $f_1\,(xy)\,y\,dx + f_2\,(xy)\,x\,dy = 0$.

Again, $\qquad Mx - Ny = xy\,(1 + xy) - xy\,(1 - xy) = 2x^2y^2 \neq 0$,

showing that I.F. of (1) $= 1/(Mx - Ny) = 1/(2x^2y^2)$.

On multiplying (1) by $1/(2x^2y^2)$, we have $\qquad \dfrac{1}{2}\left(\dfrac{1}{x^2y}+\dfrac{1}{x}\right)dx + \dfrac{1}{2}\left(\dfrac{1}{xy^2}-\dfrac{1}{y}\right)dy = 0,$

which must be exact and so by the usual rule, solution of (2) is

$$\int\left(\frac{1}{2x^2y}+\frac{1}{2x}\right)dx+\int\left(-\frac{1}{2y}\right)dy=\frac{1}{2}\log c \qquad \text{or} \qquad \frac{1}{-2xy}+\frac{1}{2}\log x-\frac{1}{2}\log y=\frac{1}{2}\log c$$

[Treating y as constant]

or $\log (x/y) - \log c = 1/(xy)$ or $\log (x/cy) = 1/(xy)$ or $x = cy\, e^{1/(xy)}$.

Ex. 3. *Solve* $(x^3y^3 + x^2y^2 + xy + 1)\, y\, dx + (x^3y^3 - x^2y^2 - xy + 1)\, x\, dy = 0$.

Sol. Comparing the given equation with $M\,dx + N\,dy = 0$, we get

$M = y\,(x^3y^3 + x^2y^2 + xy + 1)$ and $N = x\,(x^3y^3 - x^2y^2 - xy + 1)$.

$\therefore\ Mx - Ny = xy\,(x^3y^3 + x^2y^2 + xy + 1) - xy\,(x^3y^3 - x^2y^2 - xy + 1)$.

$= 2xy\,(x^2y^2 + xy) = 2x^2y^2\,(xy + 1) \neq 0$,

showing that I.F. of the given equation $= 1/(Mx - Ny) = 1/\{2x^2y^2\,(xy + 1)\}$.

On multiplying the given equation by its I.F., we have

$$\frac{x^2y^2\,(xy+1)+(xy+1)}{2x^2y^2\,(xy+1)}\,y\,dx+\frac{(xy+1)\,(x^2y^2-xy+1)-xy\,(xy+1)}{2x^2y^2\,(xy+1)}\,x\,dy=0$$

$$\frac{x^2y^2+1}{x^2y^2}\,y\,dx+\frac{(x^2y^2-xy+1)-xy}{x^2y^2}\,x\,dy=0$$

or $$(y\,dx+x\,dy)+\frac{y\,dx+x\,dy}{x^2y^2}-\frac{2x^2y}{x^2y^2}\,dy=0 \qquad \text{or} \qquad d\,(xy)+\frac{d\,(xy)}{(xy)^2}-\frac{2}{y}\,dy=0$$

or $d\,(xy) + (1/z^2)\,dz - (2/y)\,dy = 0$, putting $xy = z$.

Integrating, $xy - (1/z) - 2\log y = c$ or $xy - (1/xy) - 2\log y = c$

Ex. 4. *Solve* $y\,(x^2y^2 + 2)\,dx + x\,(2 - 2x^2y^2)\,dy = 0$ **[Delhi Maths (H) 2001]**

Sol. Comparing the given equation with $M\,dx + N\,dy = 0$, here $M = y\,(x^2y^2 + 2)$ and $N = x\,(2 - 2x^2y^2)$, showing that the given equation is of the form $f_1\,(xy)\,y\,dx + f_2\,(x, y)\,x\,dy = 0$.

Again, $Mx - Ny = xy\,(x^2y^2 + 2) - xy\,(2 - x^2y^2) = 3x^3y^3 \neq 0$,

showing that I.F. of given equation $= 1/(Mx - Ny) = 1/(3x^3y^3)$.

Multiplying the given equation by $1/(3x^3y^3)$, we get

$$\left(\frac{1}{3x}+\frac{2}{3x^3y^3}\right)dx+\left(\frac{2}{3x^2y^3}-\frac{2}{3y}\right)dy=0, \text{ which is exact.}$$

As usual, its solution is $(1/3) \times \log x - (1/3x^2y^2) - (2/3) \times \log y = (1/3) \times \log c$

or $\log (x/cy^2) = 1/x^2y^2$ or $x = cy^2\,e^{1/x^2y^2}$, c being an arbitrary constants.

Exercise 2(G)

Solve the following differential equations:

1. $(x^2y^2 + xy + 1)\,y\,dx + (x^2y^2 - xy + 1)\,x\,dy = 0$. **Ans.** $xy - (1/xy) + \log (x/y) = c$
2. $(x^4y^4 + x^2y^2 + xy)\,y\,dx + (x^4y^4 - x^2y^2 + xy)\,x\,dy = 0$. **Ans.** $(1/2) \times x^2y^2 - (1/xy) + \log (x/y) = c$
3. $y\,(1 - xy)\,dx - x\,(1 + xy)\,dy = 0$. **[Agra 1994; I.A.S. 1969]** **Ans.** $\log (x/y) - xy = c$
4. $(xy^2 + 2x^2y^3)\,dx + (x^2y - x^3y^2)\,dy = 0$. **[I.A.S. 2004]** **Ans.** $\log (x^2/y) - (1/xy) = c$

Rule IV. *If* $\dfrac{1}{N}\left(\dfrac{\partial M}{\partial y}-\dfrac{\partial N}{\partial x}\right)$ *is a function x alone say* $f(x)$, *then* $e^{\int f(x)\,dx}$ *is an integrating factor of* $M\,dx + N\,dy = 0$. **[I.A.S. 1977, 94]**

Proof. Given equation is $$M\,dx + N\,dy = 0 \qquad \ldots (1)$$

and $$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = f(x) \qquad \text{so that} \qquad N f(x) = \frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} \ldots (2)$$

Multiplying both sides of (1) by $e^{\int f(x)dx}$, we have $$M_1\,dx + N_1\,dy = 0, \qquad \ldots (3)$$

where $$M_1 = M e^{\int f(x)dx} \qquad \text{and} \qquad N_1 = N e^{\int f(x)dx} \ldots (4)$$

From (4), $$\frac{\partial M_1}{\partial y} = \frac{\partial M}{\partial y} e^{\int f(x)dx} \qquad \ldots (5)$$

and $$\frac{\partial N_1}{\partial x} = \frac{\partial N}{\partial x} e^{\int f(x)dx} + N e^{\int f(x)dx} f(x) = e^{\int f(x)dx}\left\{\frac{\partial N}{\partial x} + N f(x)\right\}$$

$$= e^{\int f(x)dx}\,(\partial N/\partial x + \partial M/\partial y - \partial N/\partial x)\text{, by (2)}$$

so that $$\frac{\partial N_1}{\partial x} = e^{\int f(x)dx}\frac{\partial M}{\partial y}. \qquad \ldots (7)$$

$\therefore$ From (6) and (7), $\partial M_1/\partial dy = \partial N_1/\partial x$,

showing the $M_1 dx + N_1 dy = 0$ must be exact and hence $e^{\int f(x)dx}$ is its I.F.

2.20 Solved examples of type 9 based on Rule IV of Art. 2.16

Ex. 1. *Solve* $(x^2 + y^2 + x)\,dx + xy\,dy = 0.$ **[Delhi B.Sc. (Prog) II, 2009]**

Sol. Given $$(x^2 + y^2 + x)\,dx + xy\,dy = 0. \qquad \ldots (1)$$

Comparing (1) with $M\,dx + N\,dy = 0$, here $M = x^2 + y^2 + x$ and $N = xy$.

Here $\partial M/\partial y = 2y$ and $\partial N/\partial x = y$. So $\partial M/\partial y \neq \partial N/\partial x$. We have

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{xy}(2y-1) = \frac{1}{x},$$ which is a function of x alone.

$$\therefore \qquad \text{I.F. of (1)} = e^{\int (1/x)dx} = e^{\log x} = x.$$

Multiplying (1) by x, we have $(x^3 + xy^2 + x^2)\,dx + x^2 y\,dy = 0,$

which must be exact equation and so its solution as usual is

$$\int (x^3 + xy^2 + x^2)\,dx = (1/6)\times c \qquad \text{or} \qquad (1/4)\times x^4 + (1/2)\times x^2y^2 + (1/3)\times x^3 = c/6$$

[Treating y as constant]

or $$3x^4 + 6x^2y^2 + 4x^3 = c,$$ where c is an arbitrary constant.

Ex. 2. *Solve* $(y + y^3/3 + x^2/2)\,dx + (1/4) \times (x + xy^2)\,dy = 0.$ **[Allahabad 1994]**

Sol. Given $$(y + y^3/3 + x^2/2)\,dx + (1/4) \times (x + xy^2)\,dy = 0. \qquad \ldots (1)$$

Comparing (1) with $M\,dx + N\,dy = 0$, $M = y + y^3/3 + x^2/2$ and $N = (1/4) \times (x + xy^2)$.

Here $\partial M/\partial y = 1 + y^2$ and $\partial N/\partial x = (1/4) \times (1 + y^2)$.

$$\therefore \frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{4}{x(1+y^2)}\left\{(1+y^2) - \frac{1}{4}(1+y^2)\right\} = \frac{4}{x}\left(1 - \frac{1}{4}\right) = \frac{3}{x},$$

which is a function of x alone. So I.F. $= e^{\int (3/x)dx} = e^{3\log x} = e^{\log x^3} = x^3.$

Multiplying (1) with x^3, we have

$\{x^3y + (1/3)\,x^3y^3 + (1/2)\,x^5\}\,dx + (1/4)\,(x^4 + x^4y^2)\,dy = 0$ whose solution as usual is

$\int \{x^3 y + (1/3)\, x^3 y^3 + (1/2)\, x^5\}\, dx = c/12$ or $(1/4)\times x^4 y + (1/12)\times x^4 y^3 + (1/12)\times x^6 = c/6$

[Treating y as constant]

or $3x^4 y + x^4 y^3 + x^6 = c$, where c is an arbitrary constant.

Exercise 2(H)

Solve the following differential equations:

1. $(x^2 + y^2 + 2x)\, dx + 2y\, dy = 0$. **Ans.** $e^x (x^2 + y^2) = c$
2. $(x^3 - 2y^2)\, dx + 2xy\, dy = 0$. **Ans.** $x + (y^2/x^2) = c$
3. $(x^2 + y^2)\, dx - 2xy\, dy = 0$. **(Pune 2010)** **Ans.** $x^2 - y^2 = cx$
4. $(x^2 + y^2 + 1)\, dx - 2xy\, dy = 0$. **(Delhi B.Sc. (Prog.) II 2008)** **Ans.** $x^2 - 1 - y^2 = cx$
5. $(x^2 + y^2 + 1)\, dx + x\,(x - 2y)\, dy = 0$. **Ans.** $x + y - (y^2 + 1)/x = c$
6. $(5xy + 4y^2 + 1)dx + (x^2 + 2xy)dy = 0$ **[Delhi B.A. (Prog.) II 2010]** **Ans.** $x^5 y + x^4 y^2 + x^4/4 = c$

Rule V. *If* $\dfrac{1}{M}\left(\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}\right)$ *is function of y alone, say f (y), then* $e^{\int f(y)\,dy}$ *is an integrating factor of* $M\,dx + N\,dy = 0$.

Proof. Proceed exactly as for rule IV.

2.21 Solved example of type 10 based on Rule V of Art. 2.16

Ex. 1. *Solve* $(2xy^4 e^y + 2xy^3 + y)\, dx + (x^2 y^4\, e^y - x^2 y^2 - 3x)\, dy = 0.$ ... (1)

Sol. Comparing (1) with $M\,dx + N\,dy = 0$, we get

$M = 2xy^4 e^y + 2xy^3 + y$ and $N = x^2 y^4 e^y - x^2 y^2 - 3x$. ... (2)

Here $\partial M/\partial y = 8xy^3 e^y + 2xy^4 e^y + 6xy^2 + 1$ and $\partial N/\partial x = 2xy^4 e^y - 2xy^2 - 3$.

$$\therefore \quad \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = -4\,(2xy^3 e^y + 2xy^2 + 1) = -\frac{4}{y}(2xy^4 e^y + 2xy^3 + y) = -\frac{4M}{y}$$

$$\Rightarrow \quad \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = -\frac{4}{y}, \text{ which is a function of } y \text{ alone.}$$

$$\Rightarrow \quad \text{I.F. of (1)} = e^{\int(-4/y)\,dy} = e^{-4\log y} = (1/y^4).$$

Multiplying (1) by $1/y^4$, we have

$\{2xe^y + (2x/y) + (1/y^3)\}\, dx + \{x^2 e^y - (x^2/y^2) - 3\,(x/y^4)\}\, dy = 0$ whose solution as usual is

$\int \{2xe^y + (2x/y) + (1/y^3)\}\, dx = c$ or $x^2 e^y + (x^2/y) + (x/y^3) = c$.

[Treating y as constant]

Ex. 2. *Solve* $(xy^2 - x^2)\, dx + (3x^2 y^2 + x^2 y - 2x^3 + y^2)\, dy = 0.$

Sol. Given $(xy^2 - x^2)\, dx + (3x^2 y^2 + x^2 y - 2x^3 + y^2)\, dy = 0.$ ... (1)

Comparing (1) with $M\,dx + N\,dy = 0$, $M = xy^2 - x^2$, $N = 3x^2 y^2 + x^2 y - 2x^3 + y^2$.

$\therefore$ $\partial M/\partial y = 2xy$ and $\partial N/\partial x = 6xy^2 + 2xy - 6x^2$.

$$\therefore \quad \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{xy^2 - x^2}\{(6xy^2 + 2xy - 6x^2) - 2xy\} = \frac{6x\,(y^2 - x)}{x\,(y^2 - x)} = 6,$$

which being a constant can be treated as a function of y alone.

$\therefore$ I.F. of (1) $= e^{\int 6\,dy} = e^{6y}$. Multiplying (1) by e^{6y}, we have

$e^{6y}(xy^2 - x^2)\, dx + e^{6y}(3x^2 y^2 + x^2 y - 2x^3 + y^2)\, dy = 0$ whose solution is

$$\int e^{6y}\,(xy^2 - x^2)\, dx + \int e^{6y} y^2\, dy = c$$

[Treating y as constant]

or $$e^{6y}\,[(1/2)x^2y^2-(1/3)x^3]+y^2(1/6)e^{6y}-\int(2y)(1/6)e^{6y}\,dy=c$$

or $$e^{6y}\left(\tfrac{1}{2}x^2y^2-\tfrac{1}{3}x^3\right)+\tfrac{1}{6}y^2e^{6y}-\tfrac{1}{3}\left[y\times\frac{1}{6}e^{6y}-\int\left(1\times\frac{1}{6}e^{6y}\right)dy\right]=c$$

or $$e^{6y}\left(\tfrac{1}{2}x^2y^2-\tfrac{1}{3}x^3\right)+\tfrac{1}{6}y^2e^{6y}+\tfrac{1}{3}\left[\tfrac{1}{6}ye^{6y}-\tfrac{1}{36}e^{6y}\right]=c$$

or $$e^{6y}\left(\tfrac{1}{2}x^2y^2-\tfrac{1}{3}x^3+\tfrac{1}{6}y^2-\tfrac{1}{18}y+\tfrac{1}{108}\right)=c.$$ c being an arbitrary constant.

Exercise 2(I)

Solve the following differential equations:

1. $(xy^3+y)\,dx+2\,(x^2y^2+x+y^4)\,dy=0.$ **(Delhi 2009)** **Ans.** $3x^2y^4+6xy^2+2y=c$

2. $(2xy^2-2y)\,dx+(3x^2y-4x)\,dy=0.$ **[Delhi B.A (Prog) II 2011]** **Ans.** $x^2y^3+2xy^2=c$

3. $(y^4+2y)\,dx+(xy^3+2y^4-4x)\,dy=0$ **[Delhi Maths (H) 2007, 08; Delhi B.Sc. (Prog) II 2011]**

Ans. $x\,\{y+(2/y^2)\}+y^2=c$

Rule VI. *If the given equation $M\,dx+N\,dy=0$, is of the form $x^\alpha y^\beta\,(my\,dx+nx\,dy)=0$, then its integrating factor is $x^{km-1-\alpha}\,y^{kn-1-\beta}$, where k can have any value.*

Proof. By assumption, the given equation can be written as

$$x^\alpha y^\beta\,(my\,dx+nx\,dy)=0. \qquad ...(1)$$

Multiplying (1) by $x^{km-1-\alpha}\,y^{kn-1-\beta}$, we have $\quad x^{km-1}\,y^{kn-1}\,(my\,dx+nx\,dy)=0$

or $\quad km\,x^{km-1}\,y^{kn}\,dx+kn\,y^{kn-1}\,x^{km}\,dy=0 \quad$ or $\quad d\,(x^{km}\,y^{kn})=0,$

showing that $x^{km-1-\alpha}\,y^{kn-1-\beta}$ is an I.F. of the given equation (1).

Remark 1. Using rule VI, we now find the rule for finding an I.F. of the equation of the form

$$x^\alpha y^\beta\,(my\,dx+nx\,dy)+x^{\alpha'}y^{\beta'}\,(m'\,ydx+n'\,xdy)=0 \qquad ...(2)$$

By virtue of rule VI, we see that the factor that makes the first term of (2) exact differential is $x^{km-1-\alpha}\,y^{kn-1-\beta}$ and that for the second term of (2) is $x^{k'm'-1-\alpha'}\,y^{k'n'-1-\beta'}$ where k and k' can have any value.

The above mentioned two factors will be identical if we choose k and k', such that

$$km-1-\alpha=k'm'-1-\alpha' \qquad ...(3)$$

and $$kn-1-\beta=k'n'-1-\beta'. \qquad ...(4)$$

Solving (3) and (4), we evaluate the values of k and k'. Substituting these values in the factor $x^{km-1-\alpha}\,y^{kn-1-\beta}$ or $x^{k'm'-1-\alpha'}\,y^{k'n'-1-\beta'}$, we obtain the required I.F. of (2).

2.22 Solved examples of Type 11 based on rule VI of Art. 2.16

Example: *Solve* $(y^2+2x^2y)\,dx+(2x^3-xy)\,dy=0.$ ...(1)

Sol. Re-writing (1) in the standard form

$$x^\alpha y^\beta\,(my\,dx+nx\,dy)\,x^{\alpha'}y^{\beta'}\,(m'y\,dx+n'x\,dy)=0, \qquad ...(2)$$

we have $$y\,(y\,dx-x\,dy)+x^2\,(2y\,dx+2x\,dy)=0. \qquad ...(3)$$

Comparing (2) and (3), we have

$\alpha=0,\quad \beta=1,\quad m=1,\quad n=-1;\quad \alpha'=2,\quad \beta'=0,\quad m'=2,\quad n'=2.$

Hence, the I.F. for the first term on L.H.S. of (3) is $x^{k-1}\,y^{-k-1-1}$, *i.e.*, $x^{k-1}\,y^{-k-2}$... (4)

and the I.F. for the second term on L.H.S. of (3) is $2^{2k'-1-2}\,y^{2k'-1}$, *i.e.*, $x^{2k'-3}\,y^{2k'-1}$ ...(5)

For the integrating factors (4) and (5) to be identical, we have

$$k-1=2k'-3 \quad \text{and} \quad -k-2=2k'-1$$

$$\Rightarrow\quad k-2k'=-2 \quad\text{and}\quad k+2k'=-1 \quad\Rightarrow\quad k=-3/2 \quad\text{and}\quad k'=1/4 \qquad ...(6)$$

Substituting the value of k in (4) or k' in (5), the integrating factor of (3) or (1) is $x^{-5/2}\, y^{-1/2}$.

Multiplying (1) by $x^{-5/2}\, y^{-1/2}$, we have

$$(x^{-5/2}\, y^{3/2} + 2x^{-1/2}\, y^{1/2})\, dx + (2x^{1/2}\, y^{-1/2} - x^{-3/2}\, y^{1/2})\, dy = 0,$$

which must be exact and so by the usual rule its solution is given by

$$\frac{x^{-3/2}\, y^{3/2}}{(-3/2)} + \frac{2x^{1/2}\, y^{1/2}}{(1/2)} = \frac{2C}{3} \qquad \text{or} \qquad 6x^{1/2}\, y^{1/2} - x^{-3/2}\, y^{3/2} = C.$$

Remark 2. Sometimes the Rule VI of Art. 2.16 for finding I.F. is modified as given below:

If the given equation $M\, dx + N\, dy = 0$ can be put in the form

$$x^{\alpha}\, y^{\beta}\, (my\, dx + nx\, dy) + x^{\alpha'}\, y^{\beta'}\, (m'y\, dx + n'x\, dy) = 0,$$

where α, β, m, n, α', β', m', n' are constants, then the given equation has an I.F. $x^h\, y^k$, where h and k are obtained by applying the condition that the given equation must become exact after multiplying by $x^h\, y^k$.

Illustrative solved examples based on the above remark 2

Ex. 1. *Solve* $(y^2 + 2x^2y)\, dx + (2x^3 - xy)\, dy = 0.$ **[Allahabad 1993; Lucknow 1993]**

Sol. Given $\qquad (y^2 + 2x^2y)\, dx + (2x^3 - xy)\, dy = 0. \qquad \ldots (1)$

Re-writing (1), $\qquad (y^2dx - xy\, dy) + (2x^2y\, dx + 2x^3\, dy) = 0$

or $\qquad y\, (y\, dx - x\, dy) + x^2\, (2y\, dx + 2x\, dy) = 0.$ **[Delhi 2009]**

which is of the form $\qquad x^{\alpha}y^{\beta}\, (my\, dx + nx\, dy) + x^{\alpha'}y^{\beta'}\, (m'y\, dx + n'x\, dy) = 0.$

So, let $x^h\, y^k$ be an I.F. (1). Multiplying (1) by $x^h\, y^k$, we have

$$(x^h\, y^{k+2} + 2x^{h+2}\, y^{k+1})\, dx + (2x^{h+3}\, y^k - x^{h+1}\, y^{k+1})\, dy = 0, \qquad \ldots (2)$$

which must be exact. Comparing (2) with $M\, dx + N\, dy = 0$, we get

$$M = x^h\, y^{k+2} + 2x^{h+2}\, y^{k+1} \qquad \text{and} \qquad N = 2x^{h+3}\, y^k - x^{h+1}\, y^{k+1}.$$

Since (2) is exact, we must have $\qquad \partial M/\partial y = \partial N/\partial x,$

i.e., $\qquad (k + 2)\, x^h\, y^{k+1} + 2\, (k + 1)\, x^{h+2}\, y^k = 2\, (h + 3)\, x^{h+2}\, y^k - (h + 1)\, x^h\, y^{k+1}.$

Now equating the coefficients of $x^h\, y^{k+1}$ and $x^{h+2}\, y^k$, we get

$$k + 2 = -\, (h + 1) \qquad \text{and} \qquad 2\, (k + 1) = 2\, (h + 3),$$

i.e., $\qquad h + k = -\, 3 \qquad$ and $\qquad h - k = -\, 2 \qquad$ giving $\qquad h = -\, (5/2), \qquad k = -\, (1/2)$

$\therefore$ I.F. $= x^{-5/2}\, y^{-1/2}$. Multiplying (1) by I.F. $x^{-5/2}\, y^{-1/2}$, we get

$$(x^{-5/2}\, y^{3/2} + 2x^{-1/2}\, y^{1/2})\, dx + (2x^{1/2}\, y^{-1/2} + x^{-3/2}\, y^{1/2})\, dy = 0$$

which must be exact. For this new equation, as usual its solution is

$$\int (x^{-5/2}y^{3/2} + 2x^{-1/2}y^{1/2})\, dx = c \qquad \text{or} \qquad -\tfrac{2}{3}x^{-3/2}y^{3/2} + 4x^{1/2}y^{1/2} = c.$$

[Treading y as constant]

Ex. 2. *Solve* $(2y\, dx + 3x\, dy) + 2xy\, (3y\, dx + 4x\, dy) = 0.$ **[Kanpur 1998]**

Sol. Given $\qquad x^0\, y^0\, (2y\, dx + 3x\, dy) + xy\, (6y\, dx + 8x\, dy) = 0. \qquad \ldots (1)$

Since (1) is of the form $x^{\alpha}\, y^{\beta}\, (my\, dx + nx\, dy) + x^{\alpha'}\, y^{\beta'}\, (m'y\, dx + n'x\, dy) = 0$, where α, β, m, n, α', β', m', n' are constants. So $x^h\, y^k$ can be taken as I.F. of (1).

Re-writing (1), $\qquad (2y + 6xy^2)\, dx + (3x + 8x^2y)\, dy = 0. \qquad \ldots (2)$

Multiplying (2) by I.F. $x^h\, y^k$, we have

$$(2x^h\, y^{k+1} + 6x^{h+1}\, y^{k+2})\, dx + (3x^{h+1}\, y^k + 8x^{h+2}\, y^{k+1})\, dy = 0, \qquad \ldots (3)$$

which must be exact. Comparing (3) with $M\, dx + N\, dy = 0$, we get

$$M = 2x^h\, y^{k+1} + 6x^{h+1}\, y^{k+2} \qquad \text{and} \qquad N = 3x^{h+1}\, y^k + 8x^{h+2}\, y^{k+1}$$

$\therefore \qquad \partial M/\partial y = 2(k+1)\,x^h y^k + 6(k+2)\,x^{h+1} y^{k+1}$

and $\qquad \partial N/\partial x = 3(h+1)\,x^h y^k + 8(h+2)\,x^{h+1} y^{k+1}$.

For (3) to be exact, $\qquad \partial M/dy = \partial N/dx$.

$\therefore \qquad 2(k+1)\,x^h y^k + 6(k+2)\,x^{h+1} y^{k+1} = 3(h+1)\,x^h y^h + 8(h+2)\,x^{h+1} y^{k+1}$

Equating the coefficients of $x^h y^k$ and $x^{h+1} y^{k+1}$ on both sides, we get

$2(k+1) = 3(h+1) \qquad$ and $\qquad 6(k+2) = 8(h+2)$

i.e., $\qquad 3h - 2k = -1 \qquad$ and $\qquad 4h - 3k = -2$,

Solving these, $h = 1$, $k = 2$ and so I.F. $= x^h y^k = xy^2$.

Multiplying (2) by xy^2 or putting $h = 1$ and $k = 2$ in (3), we get

$$(2xy^3 + 6x^2y^4)\,dx + (3x^2y^2 + 8x^3y^3)\,dy = 0,$$

which must be exact. Hence, as usual, the required solution is

$$\int (2xy^3 + 6x^2y^4)\,dx = c \qquad \text{or} \qquad x^2y^3 + 2x^3y^4 = c$$

[Treating y as constant]

Ex. 3. *Given that the differential equation $(2x^2y^2 + y)\,dx - (x^3y - 3x)\,dy = 0$ has an I.F. of the form $x^h y^k$, find its general solution.* **[Kakitiya 1997; G.N.D.U. Amritsar 2010]**

Sol. Given $\qquad (2x^2y^2 + y)\,dx + (3x - x^3y)\,dy = 0. \qquad$ ... (1)

Multiplying both sides of (1) by I.F. $x^h y^k$, we get

$$(2x^{h+2} y^{k+2} + x^h y^{k+1})\,dx + (3x^{h+1} y^k - x^{h+3} y^{k+1})\,dy = 0, \qquad \text{... (2)}$$

which must be exact. Comparing (2) with $M\,dx + N\,dy = 0$, we have

$M = 2x^{h+2} y^{k+2} + x^h y^{k+1} \qquad$ and $\qquad N = 3x^{h+1} y^k - x^{h+3} y^{k+1}$. $\qquad$ ... (3)

For (2) to be exact, $\qquad \partial M/\partial y = \partial N/\partial x$

$\Rightarrow \quad 2(k+2)\,x^{h+2} y^{k+1} + (k+1)\,x^h y^k = 3(h+1)\,x^h y^k - (h+3)\,x^{h+2} y^{k+1}$

$\Rightarrow \quad 2(k+2) = -(h+3) \quad$ and $\quad k+1 = 3(h+1)$

$\Rightarrow \quad h + 2k = -7$ and $3h - k = -2 \Rightarrow h = -11/7$ and $k = -19/7$.

Hence an I.F. of (1) is $x^{-11/7} y^{-19/7}$. Multiplying (1) by $x^{-11/7} y^{-19/7}$, we have

$$(2x^{3/7} y^{-5/7} + x^{-11/7} y^{-12/7})\,dx + (3x^{-4/7} y^{-19/7} - x^{10/7} y^{-12/7})\,dy = 0,$$

which must be exact. Hence, as usual, the required solution is

$$\frac{2x^{10/7} y^{-5/7}}{(10/7)} + \frac{x^{-4/7} y^{-12/7}}{(-4/7)} = \frac{7c}{20} \qquad \text{or} \qquad 4x^{10/7} y^{-5/7} - 5x^{-4/7} y^{-12/7} = c.$$

Exercise 2(J)

Solve the following differential equations:

1. $(2x^2y - 3y^4)\,dx + (3x^3 + 2xy^3)\,dy = 0.$ **Ans.** $12x^{-10/13} y^{15/13} + 5x^{-36/13} y^{24/13} = c$

2. $(3x + 2y^2)\,y\,dx + 2x\,(2x + 3y^2)\,dy = 0.$ **Ans.** $x^3y^4 + x^2y^6 = c$

3. $x\,(3y\,dx + 2x\,dy) + 8y^4\,(y\,dx + 3x\,dy) = 0.$ **Ans.** $x^2y^3\,(x + 4y^4) = c$

4. $x\,(4y\,dx + 2x\,dy) + y^3\,(3y\,dx + 5x\,dy) = 0.$ **[Delhi Maths (G) 1999]** **Ans.** $x^4y^3 + x^3y^5 = c$

5. $xy^3\,(y\,dx + 2x\,dy) + (3y\,dx + 5x\,dy) = 0.$ **Ans.** $x^3y^5\,(xy^3 + 4) = c$

6. $(8x^2y^3 - 2y^4)\,dx + (5x^3y^2 - 8xy^3)\,dy = 0$ *by first finding integrating factor of the form $x^h y^k$.* **[Delhi, B.Sc. (Prog) II 2011]** **Ans.** $x^{8/3} y^{5/3} - x^{2/3} y^{8/3} = c$

2.23 Linear differential equation

Definition. A first order differential equation is called linear if it can be written in the form

$$(dx/dy) + Py = Q, \qquad \text{... (1)}$$

where P and Q are constants or functions of x alone (*i.e.*, not of y).

A method of solving (1). Suppose R (which is taken as function of x alone) is an integrating factor of (1). Multiplying (1) by R, we get

$$R(dy/dx) + RPy = RQ, \qquad \text{... (2)}$$

which must be exact. Suppose, we wish that the L.H.S. of (2) is the differential coefficient of some product. But the term R (dy/dx) can only be obtained by differentiating the product Ry. Accordingly, we take

$$R\frac{dy}{dx} + RPy = \frac{d}{dx}(Ry) \qquad \text{... (3)}$$

or $\quad R\dfrac{dy}{dx} + RPy = R\dfrac{dy}{dx} + y\dfrac{dR}{dx} \quad$ or $\quad \dfrac{dR}{R} = P dx.$

Integrating, $\log R = \int P\,dx$,taking constant of integration equal to the zero for sake of simplicity.

Thus, an integrating factor of (1) is $R = e^{\int P\,dx}$ and (2) reduces to

$\dfrac{d}{dx}(Ry) = RQ$, using (3) $\quad$ or $\quad d\,(Ry) = RQ\,dx.$

Integrating, $\quad Ry = \int RQ\,dx + c \quad$ or $\quad ye^{\int P\,dx} = \int \{Qe^{\int P\,dx}\}\,dx + c,$

which is the required solution of given linear differential equation (1).

Working rule for solving linear equations. First put the given equation in the standard form (1). Next find an integrating factor (I.F.) by using formula

$$\text{I.F.} = e^{\int P\,dx} \qquad \text{... (5)}$$

Two formulas $e^{m \log A} = A^m$ and $e^{-m \log A} = 1/A^m$ will be often used in simplifying I.F.

Lastly, the required solution is obtained by using the result

$$y \times (\text{I.F.}) = \int [Q \times (\text{I.F.})]\,dx + c, \text{ where } c \text{ is an arbitrary constant.} \qquad \text{... (6)}$$

Remarks. Sometimes a differential equation cannot be put in the form (1) of a linear equation. Then, we regard y as the independent variable and x as the dependent variable and obtain a differential equation of the form

$$\text{dx/dy} + P_1 x = Q_1, \qquad \text{... (7)}$$

where P_1 and Q_1 are constants or functions of y alone. In this, we modify the above working rule as follows.

$$\text{I.F.} = e^{\int P_1\,dy} \qquad \text{... (8)}$$

and the required solution is $\quad x \times (\text{I.F.}) = \int [Q_1 \times (I.F.)]\,dy + c.$

2.24 Examples of Type 12 based on Art. 2.23

Ex. 1. *Solve $x \cos x\,(dy/dx) + y\,(x \sin x + \cos x) = 1$.* **[Agra 1994]**

Sol. Re-writing given equation, we have $\quad \dfrac{dy}{dx} + \left(\tan x + \dfrac{1}{x}\right)y = \dfrac{\sec x}{x} \qquad$ ... (1)

$$\text{I.F. of (1)} = e^{\int (\tan x + 1/x)\,dx} = e^{\log \sec x + \log x} = e^{\log x \sec x} = x \sec x.$$

Hence the required solution is $\quad yx \sec x = \int \sec^2 x\,dx + c,$

or $\quad yx \sec x = \tan x + c$, c being arbitoary constants.

Ex. 2. (a) *Solve* $(1-x^2)(dy/dx)+2xy = x\sqrt{(1-x^2)}$. **[Kerala 2001]**

(b) solve $(1-x^2)(dy/dx)+2xy = x\sqrt{1-x^2}$, $y(0)=1$ **[Delhi Maths (Prog) 2007]**

Sol. The given equation is $$\frac{dy}{dx}+\frac{2x}{1-x^2}\,y = \frac{x}{(1-x^2)^{1/2}}. \quad \text{... (1)}$$

Comparing (1) with $dy/dx + Py = Q$, here $P = 2x/(1-x^2)$

Here $\int P\,dx = \int \frac{2x}{1-x^2}\,dx = -\log(1-x^2)$ hence $\text{I.F. of (1)} = e^{\int P\,dx} = \frac{1}{1-x^2}$

So the required solution is

$$\frac{y}{1-x^2} = \int \frac{x}{\sqrt{(1-x^2)}}\times\frac{1}{1-x^2}\,dx = -\frac{1}{2}\int t^{-3/2}\,dt + c, \text{ putting } 1-x^2 = t \text{ and } -2x\,dx = dt$$

or $\frac{y}{1-x^2} = t^{-1/2} + c = c + \frac{1}{\sqrt{t}}$ or $\frac{y}{1-x^2} = \frac{1}{(1-x^2)^{1/2}} + c$, as $t = 1-x^2$ ... (2)

(b) First do upto equation (2) as in Ex. 2(a). Putting $x = 0$ and $y = 1$ in (2), we have $1 = 1 + c$ so that $c = 0$. Hence (2) becomes

$y/(1-x^2) = 1/(1-x^2)^{1/2}$ or $y = (1-x^2)^{1/2}$

Ex. 3. *Solve sin x (dy/dx) + 3y = cos x.* **[Rohilkhand 1993]**

Sol. Re-writing, we have, $dy/dx + (3 \operatorname{cosec} x)\, y = \cot x$. ... (1)

Comparing (1) with $dy/dx + Py = Q$, here $P = 3 \operatorname{cosec} x$

Here $\int P\,dx = 3\int \operatorname{cosec} x\,dx = 3\log\tan(x/2)$ so I.F. *of* (1) $= e^{\int P\,dx} = \tan^3 x/2$

Hence, the required solution is given by

$$y\tan^3(x/2) = \int \cot x \tan^3(x/2)\,dx + c = \int \frac{1-\tan^2(x/2)}{2\tan(x/2)}\tan^3\frac{x}{2}\,dx + c$$

or $$y\tan^3(x/2) = \frac{1}{2}\int \{1-\tan^2(x/2)\}\tan^2(x/2)\,dx + c$$

or $$y\tan^3\frac{x}{2} = \frac{1}{2}\int (1-t^2)\,t^2\times\frac{2\,dt}{1+t^2} + c, \text{ } c \text{ being an arbitrary constant}$$

$$\left[\text{Put } \tan\frac{x}{2} = t \quad \text{so that} \quad \frac{1}{2}\sec^2\frac{x}{2}\,dx = dt \Rightarrow dx = \frac{2\,dt}{\sec^2(x/2)} = \frac{2\,dt}{1+\tan^2(x/2)} = \frac{2\,dt}{1+t^2}\right]$$

or $y\tan^3\frac{x}{2} = \int \frac{t^2-t^4}{1+t^2}\,dt + c$ or $y\tan^3\frac{x}{2} = \int\left[-t^2+2-\frac{2}{t^2+1}\right]dt$

or $$y\tan^3(x/2) = -(1/3)\,t^3 + 2t - 2\tan^{-1} t + c$$

or $$y\tan^3\frac{x}{2} = -\frac{1}{3}\tan^3\frac{x}{2} + 2\tan\frac{x}{2} - 2\tan^{-1}\left(\tan\frac{x}{2}\right) + c$$

or $(y + 1/3)\tan^3(x/2) = 2\tan(x/2) - x + c$. c being an arbitrary constant.

Ex. 4. *Integrate $(1 + x^2)(dy/dx) + 2xy - 4x^2 = 0$. Obtain equation of the curve satisfying this equation and passing through the origin.* **[Agra 1993]**

Sol. Re-writing the given equation, $$\frac{dy}{dx}+\frac{2x}{1+x^2}\,y=\frac{4x^2}{1+x^2}. \qquad \text{... (1)}$$

Comparing (1) with $dy/dx + Py = Q$, here $P = (2x)/(1+x^2)$

Here $\int P\,dx=\int \frac{2x}{1+x^2}dx=\log(1+x^2)$ so I.F. of (1) $= e^{\int P\,dx}=(1+x^2)$.

Hence the required solution is $$y\,(1+x^2)=\int \frac{4x^2}{1+x^2}\cdot(1+x^2)\,dx+c$$

or $$y\,(1+x^2) = (4/3)\,x^3 + c, \ c \text{ being an arbitrary constant.} \qquad \text{... (1)}$$

Since the required curve passes through origin, (1) must satisfy the condition $x = 0$, $y = 0$. Putting these in (1), we get $c = 0$. Hence the required curve is $4x^3 = 3y\,(1+x^2)$.

Ex. 5. *Solve* $(x + 2y^3)\,(dy/dx) = y$. **[Rohilkhand 1993; Agra 1995; Delhi Maths. (G) 1995, 2002; Lucknow 1995; Rajasthan 2010]**

Sol. Here it is possible to put the equation in form $dx/dy + P_1x = Q_1$.
where P_1 and Q_1 are function of y or constants

Thus, we have $$\frac{dx}{dy}=\frac{x+2y^3}{y}, \qquad \text{or} \qquad \frac{dx}{dy}-\frac{1}{y}\,x=2y^2. \qquad \text{... (1)}$$

For (1), $\int P_1\,dy=-\int(1/y)\,dy=-\log y$ so I.F. of (1) $= e^{-\log y}=1/y$.

Hence, the required solution is $$x/y=\int 2y^2\cdot(1/y)\,dx+c.$$

or $$x/y = y^2 + c, \text{ where } c \text{ is an arbitrary constant.}$$

Ex. 6. (*a*) *Solve* $(1 + y^2)\,dx = (\tan^{-1} y - x)\,dy$. **[Delhi Maths 2007]**
[Agra 2005; Delhi Maths(G) 2004; Lucknow 1996; Calicut 2004; Utkal 2003]

Sol. Re-writing the given equaion, $$\frac{dx}{dy}+\frac{x}{1+y^2}=\frac{\tan^{-1}y}{1+y^2}. \qquad \text{... (1)}$$

which is of the form dx/dy + $P_1x = Q_1$. Comparing it with (1) here $P_1 = 1/(1+x^2)$

$$\therefore \quad \int P_1\,dy=\int\frac{1}{1+x^2}\,dy=\tan^{-1}y \quad \text{and hence I.F. of (1)} = e^{\int P_1\,dy}=e^{\tan^{-1}y}.$$

Hence the required solution is $$xe^{\tan^{-1}y}=\int e^{\tan^{-1}y}\cdot\frac{\tan^{-1}y}{1+y^2}\,dy=c.$$

or $$xe^{\tan^{-1}y}=\int e^t\cdot t\,dt+c, \text{ putting } \tan^{-1}y=t \text{ and } (dy)/(1+y)=dt$$

or $$xe^{\tan^{-1}y}=te^t-e^t+c \qquad \text{or} \qquad xe^{\tan^{-1}y}=e^{\tan^{-1}y}(\tan^{-1}y-1)+c$$

or $$x=\tan^{-1}y-1+ce^{\tan^{-1}y}, \ c \text{ being an arbitrary constant.}$$

Ex. 6. (*b*) Solve $(1+y^2)+(x-e^{-\tan^{-1}y})\,(dy/dx)=0$. **[I.A.S. 2006]**

Sol. Re-writing the given equation, we have

$$\frac{dx}{dy}+\frac{x-e^{-\tan^{-1}y}}{1+y^2}=0 \qquad \text{or} \qquad \frac{dx}{dy}+\frac{1}{1+y^2}\,x=\frac{e^{-\tan^{-1}y}}{1+y^2} \qquad \text{... (1)}$$

Its I.F. $= e^{\int \{1/(1+y^2)\}\, dy} = e^{\tan^{-1} y}$ and so its solution is

$$xe^{\tan^{-1} y} = \int \left(e^{\tan^{-1} y} \times \frac{e^{-\tan^{-1} y}}{1+y^2} \right) dy + c \qquad \text{or} \qquad xe^{\tan^{-1} y} = \tan^{-1} y + c \qquad \text{... (2)}$$

Ex. 6(c). *Solve* $(1 + y^2)\, dx + (x - e^{-\tan^{-1} y})\, dy = 0,\ y(1) = 0$ **[Dehil Maths (Prog) 2007]**

Sol. First do as in Ex. 6(*b*) upto equation (2). Putting x = 1, $y = 0$, in (2), we get $c = 1$. Hence the required solution is $$xe^{\tan^{-1} y} = \tan^{-1} y + 1$$

Ex. 7. *Solve* $\dfrac{dy}{dx} + \dfrac{y}{(1-x^2)^{3/2}} = \dfrac{x + (1-x^2)^{1/2}}{(1-x^2)^2}$. **[I.A.S. (Prel.) 2005]**

Sol. Comparing the given equation with $(dy/dx) + Py = Q$, here

$$P = \frac{1}{(1-x^2)^{3/2}} \qquad \text{and} \qquad Q = \frac{x + (1-x^2)^{1/2}}{(1-x^2)^2} \qquad \text{... (1)}$$

Hence, $$\int P\, dx = \int \frac{1}{(1-x^2)^{3/2}}\, dx = \int \frac{\cos\theta\, d\theta}{\cos^3\theta}, \text{ putting } x = \sin\theta$$

$$= \int \sec^2\theta\, d\theta = \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{x}{(1-x^2)^{1/2}}.$$

Hence, $$\text{I.F. of (1)} = e^{\int P\, dx} = e^{x/(1-x^2)^{1/2}} \qquad \text{... (2)}$$

Solution of the given differential equation is $$y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\, dx + c. \qquad \text{... (3)}$$

Now, $$\int Q\,(\text{I.F.})\, dx = \int \frac{x + (1-x^2)^{1/2}}{(1-x^2)^2}\, e^{x/(1-x^2)^{1/2}}\, dx \qquad \text{... (4)}$$

Put $$x/(1 - x^2)^{1/2} = t. \qquad \text{... (5)}$$

From (5), $$\frac{(1-x^2)^{1/2} \cdot 1 - x\,(1/2)\,(1-x^2)^{-1/2} \cdot (-2x)}{1-x^2}\, dx = dt$$

or $$\frac{(1-x^2)^{1/2} + [x^2/(1-x^2)^{1/2}]}{1-x^2}\, dx = dt \qquad \text{or} \qquad \frac{1}{(1-x^2)^{3/2}}\, dx = dt. \qquad \text{... (6)}$$

Re-writing (4), we have

$$\int Q\,(\text{I.F.})\, dx = \int \frac{[x/(1-x^2)^{1/2}] + 1}{(1-x^2)^{3/2}}\, e^{x/(1-x^2)^{1/2}}\, dx = \int (t+1)\, e^t\, dt, \text{ using (5) and (6)}$$

$$\therefore \quad \int Q\,(\text{I.F.})\, dx = (t+1)\, e^t - \int e^t\, dt = (t+1)\, e^t - e^t = te^t = \left\{x/(1-x^2)^{1/2}\right\} \times e^{x/(1-x^2)^{1/2}} \qquad \text{...(7)}$$

Using (2) and (7) in (3), the required solution is

$$ye^{x/(1-x^2)^{1/2}} = \frac{x}{(1-x^2)^{1/2}}\, e^{x/(1-x^2)^{1/2}} + c \qquad \text{or} \qquad y = \frac{x}{(1-x^2)^{1/2}} + c\, e^{-x/(1-x^2)^{1/2}}$$

Ex. 8. *Solve* $x(1-x^2)\,dy + (2x^2y - y - ax^3)\,dx = 0.$

Sol. Re-writing the given equation, we have

$$x(1-x^2)\frac{dy}{dx} + y(2x^2-1) = ax^3 \qquad \text{or} \qquad \frac{dy}{dx} + \frac{2x^2-1}{x(1-x^2)}\,y = \frac{ax^2}{1-x^2}. \qquad \ldots(1)$$

Comparing (1) with $(dy/dx) + Py = Q$, we have

$$P = \frac{2x^2-1}{x(1-x^2)} = -\frac{1}{x} - \frac{1}{2(x+1)} - \frac{1}{2(x-1)} \qquad \text{and} \qquad Q = \frac{ax^2}{1-x^2} \qquad \ldots(2)$$

$$\int P\,dx = -\int\left[\frac{1}{x} + \frac{1}{2(x+1)} + \frac{1}{2(x-1)}\right]dx = -\left[\log x + \frac{1}{2}\times\log(x+1) + \frac{1}{2}\times\log(x-1)\right]$$

$$= -\left[\log x + (1/2)\times\log(x^2-1)\right] = -\log[x(x^2-1)^{1/2}] = \log[x(x^2-1)^{1/2}]^{-1}$$

$\therefore$ Integrating factor $= e^{\int P dx} = e^{\log\{x(x^2-1)^{1/2}\}^{-1}} = \{x(x^2-1)^{1/2}\}^{-1} = 1/\{x(x^2-1)^{1/2}\}$

Solution of (1) is $\qquad y\,(\text{I.F.}) = \int Q\,(\text{I.F.})\,dx + c$, c being an arbitrary constant

or
$$\frac{y}{x(x^2-1)^{1/2}} = \int \frac{ax^2}{1-x^2}\times\frac{1}{x(x^2-1)^{1/2}}\,dx + c = c - a\int\frac{x\,dx}{(x^2-1)^{3/2}}$$

$$\frac{y}{x(x^2-1)^{1/2}} = c - \frac{a}{2}\int\frac{dt}{t^{3/2}}, \text{ putting } x^2-1 = t \text{ and } 2x\,dx = dt$$

or
$$\frac{y}{x(x^2-1)^{1/2}} = c - \frac{a}{2}\left[\frac{t^{-1/2}}{-(1/2)}\right] = c + \frac{a}{\sqrt{t}} = c + \frac{a}{(x^2-1)^{1/2}} \qquad \text{or} \qquad y = ax + cx(x^2-1)^{1/2}.$$

Ex. 9. *Solve* $(x+1)(dy/dx) - ny = e^x(x+1)^{n+1}$. **[Delhi Maths. (H) 2002]**

Sol. Re-writing the given equation, $\qquad \dfrac{dy}{dx} - \dfrac{n}{x+1}\,y = e^x(x+1)^n \qquad \ldots(1)$

which is linear equation whose I.F. $= e^{\int\{-n/(x+1)\}dx} = e^{-n\log(x+1)} = (x+1)^{-n}$ and solution is

$$y(x+1)^{-n} = \int e^x(x+1)^n(x+1)^{-n} + c = e^x + c., \text{ c being an arbitrary constant.}$$

Ex. 10. *Solve* $(1 + x + xy^2)\,dy + (y + y^3)\,dy = 0.$ **[Delhi Maths. (G) 2001]**

Sol. Re-writing, $\qquad \dfrac{dx}{dy} + \dfrac{1 + x(1+y^2)}{y(1+y^2)} = 0 \qquad \text{or} \qquad \dfrac{dx}{dy} + \dfrac{1}{y}\,x = -\dfrac{1}{y(1+y^2)},$

whose $\qquad$ I.F.$= e^{\int(1/y)dy} = e^{\log y} = y$ and solution is

$$xy = -\int\frac{1}{y(1+y^2)}\cdot y\,dy + c \qquad \text{or} \qquad xy = -\tan^{-1}y + c., \text{ c being an arbitrary constant.}$$

Ex. 11. *Solve* $dy/dx + y\cos x = (1/2)\times\sin 2x$ **[I.A.S. 2004]**

Sol. Integrating factor of the given equation $= e^{\int\cos x\,dx} = e^{\sin x}$ and solution is

$$ye^{\sin x} = c + \int(1/2)\times\left(\sin 2x\,e^{\sin x}\right)dx = c + \int\sin x\,e^{\sin x}\cos x\,dx$$

$$= c + \int t\,e^t\,dt, \text{ on putting } \sin x = t \text{ and } \cos x\,dx = dt,$$

$$= c + te^t - \int e^t\,dt = c + e^t(t-1)$$

or
$$ye^{\sin x} = c + e^{\sin x}(\sin x - 1) \qquad \text{or} \qquad y = ce^{-\sin x} + \sin x - 1.$$

Exercise 2(K)

Solve the following differential equations:

1. $(1+x^2)(dy/dx)+y=e^{\tan^{-1}x}$. **Ans.** $ye^{\tan^{-1}x}=(1/2)\times e^{2\tan^{-1}x}+c$

2. $(dy/dx)+y\cot x=2\cos x$. **[Bangalore 1994]** **Ans.** $y\sin x=-(1/2)\times\cos 2x+c$

3. $x\log x\,(dy/dx)+y=2\log x$. **[Delhi (Maths (G) 2005]** **Ans.** $y\log x=c+(\log x)^2$

4. $dy/dx=y\tan x-2\sin x$. **Ans.** $y=\cos x+c\sec x$

5. $(dy/dx)+2y\tan x=\sin x$, given that $y=0$ when $x=\pi/3$. **Ans.** $y=\cos x-2\cos^2 x$

6. $\cos^2 x\,(dy/dx)+y=\tan x$. **Ans.** $y=\tan x-1+ce^{\tan^{-1}x}$

7. $dy/dx+2(y/x)=\sin x$. **Ans.** $yx^2=c-x^2\cos x+2x\sin x+2\cos x$

8. $(2x-10y^3)(dy/dx)+y=0$. **Ans.** $xy^2=c+2y^5$

9. $(x\log x)(dy/dx)+y=2\log x$. **[Delhi Maths. (G) 1996]** **Ans.** $y\log x=c+(\log x)^2$

10. $\cos x\,(dy/dx)+y=\sin x$ or $(dy/dx)+y\sec x=\tan x$.

Ans. $y(\sec x+\tan x)=\sec x+\tan x-x+c$

11. $(1+y^2)+(x-e^{\tan^{-1}y})(dy/dx)=0$. **Ans.** $xe^{\tan^{-1}y}=(1/2)\times e^{2\tan^{-1}y}+c$

12. $x(dy/dx)-y=2x^2\operatorname{cosec} x$. **[Kanpur 1996]** **Ans.** $y=cx+x\log(\tan x)$

13. $x^2(x^2-1)(dy/dx)+x(x^2+1)y=x^2-1$. **Ans.** $\{y(x^2-1)\}/x=\log x+(1/2)\times x^{-2}+c$

14. $x(x-1)(dy/dx)-(x-2)y=x^3(2x-1)$. **Ans.** $\{y(x-1)\}/x^2=x^2-x+c$

15. $(1+x^2)(dy/dx)+2xy=\cos x$. **[Meerut 2009]** **Ans.** $y(1+x^2)=c+\sin x$

16. $(dy/dx)-y\tan x=e^x\sec x$. **Ans.** $y\cos x=c+e^x$

17. $\sec x\,(dy/dx)=y+\sin x$. **Ans.** $y=ce^{\sin x}-(1+\sin x)$

18. $y\log y\,dx+(x-\log y)\,dy=0$. **Ans.** $x\log y=(1/2)\times(\log y)^2+c$

19. $\dfrac{dy}{dx}+\dfrac{4x}{x^2+1}y=\dfrac{1}{(x^2+1)^2}$. **Ans.** $y(x^2+1)^2=x+c$

20. $\sin 2x\,(dy/dx)=y+\tan x$. **Ans.** $y=\tan x+c\sqrt{(\tan x)}$

21. $(x+3y+2)(dy/dx)=1$. **Ans.** $x+3y+5=ce^y$

22. $(1-x^2)(dy/dx)-xy=1$. **Ans.** $y(x^2-1)^{1/2}=c-\log[x+(x^2-1)^{1/2}]$

23. $(dy/dx)+(y/x)=x^2$, if $y=1$ when $x=1$. **Ans.** $4xy=x^4+3$

24. $\dfrac{dy}{dx}+\dfrac{2x}{1+x^2}y=\dfrac{1}{(1+x^2)^2}$ if $y=0$, when $x=1$. **Ans.** $y(1+x^2)=\tan^{-1}x-(\pi/4)$

25. Solve $\dfrac{dy}{dx}+y\dfrac{d\phi}{dx}=\phi(x)\dfrac{d\phi}{dx}$, where ϕ is some function of x. **Ans.** $ye^{\phi}=\int\phi e^{\phi}\,dx+c$

26. $x(dy/dx)+2y=x^2\log x$ **[Guwahati 2007]**. **Ans.** $x^2y=(x^4/16)\times(4\log x-1)+c$

27. $(x^2-1)(dy/dx)+2xy=1$ **[Meerut 2010]**. **Ans.** $y(x^2-1)=x+c$

2.25 Equations reducible to linear form

An equation of the form

$$f'(y)\frac{dy}{dx}+P\,f(y)=Q, \qquad \text{... (1)}$$

where P and Q are constants or functions of x alone (and not of y) can be reduced to linear form as follows. Putting $f(y)=v$ so that $f'(y)\,(dy/dx)=dv/dx$, (1) becomes

$$dv/dx+Pv=Q, \qquad \text{... (2)}$$

which is linear in v and x and its solution can be obtained by using working rule of Art. 2.23. Thus, we have I.F. $= e^{\int P\,dx}$ and solution is $\quad v \cdot e^{\int P\,dx} = \int Q e^{\int P\,dx}\,dx + c.$

Finally, replace v by $f(y)$ to get solution in terms of x and y alone.

Another form. An equation of the form
$$f'(x)\,\frac{dx}{dy} + P_1 f(x) = Q_1, \qquad \text{... (1)'}$$
where P_1 and Q_1 are constants or functions of y alone can be reduced to linear form again as follows: Putting $f(x) = v$ so that $f'(x)\,(dx/dy) = dv/dy$, (1)′ gives $\quad dv/dy + P_1\, v = Q_1, \qquad$ ... (2)'

which is linear in variables v and y.

Integrating factor of (2)′ is $e^{\int P_1 dy}$ and hence solution of (2)′ is $v\,e^{\int P_1 dy} = \int Q_1\, e^{\int P_1 dy}\,dx + c.$

Replacing v by $f(x)$, we obtain the required solution of (1).

2.25A Bernoulli's equation A particular case of Art. 2.25.

An equation of the form
$$(dy/dx) + Py = Qy^n \qquad \text{... (1A)}$$
where P and Q are constants or functions of x alone (and not of y) and n is constant except 0 and 1, is called a *Bernoulli's differential equation*.

We first multiply by y^{-n}, thereby expressing it in the form (1) of Art. 2.25
$$y^{-n}\,(dy/dx) + Py^{1-n} = Q. \qquad \text{... (2 A)}$$

Let
$$y^{1-n} = v \qquad \text{... (3 A)}$$

Differentiating w.r.t. x, (3 A) gives $(1-n)\,y^{-n}\,\dfrac{dy}{dx} = \dfrac{dv}{dx}$, or $\quad y^{-n}\,\dfrac{dy}{dx} = \dfrac{1}{1-n}\,\dfrac{dv}{dx} \qquad$ ... (4 A)

Using (3 A) and (4 A), (2 A) reduces to
$$\frac{1}{1-n}\,\frac{dv}{dx} + Pv = Q \qquad \text{or} \qquad \frac{dv}{dx} + P\,(1-n)\,v = Q\,(1-n),$$
which is linear in v and x. Its I.F. $= e^{\int P(1-n)\,dx} = e^{(1-n)\int P\,dx}$ and hence the required solution is
$$v \cdot e^{(1-n)\int P\,dx} = \int Q \cdot e^{(1-n)\int P\,dx}\,dx + c, \ c \text{ being an arbitrary constant}$$
$$y^{1-n}\, e^{(1-n)\int P\,dx} = \int Q \cdot e^{(1-n)\int P\,dx}\,dx + c, \text{ using (3A)}$$

Remark. Equation $dx/dy + P_1\,x = Q_1\,x^n$ is also in the Bernoulli's form. Here P_1 and Q_1 are functions of y alone. Method of solution is similar to that of form (1A) above.

2.26 Example of Type 13 based on Art. 2.25

Ex. 1. *Solve* $(dy/dx) + x \sin 2y = x^3 \cos^2 y$.

[I.A.S. (Prel.) 2005; I.A.S. 1994; Calcutta 1995; Kanpur 1997; Lucknow 1996]

Sol. Dividing by $\cos^2 y$, $\qquad \sec^2 y\,(dy/dx) + 2x\,(\tan y) = x^3. \qquad$ ... (1)

Put $\tan y = v$ so that $\sec^2 y\,(dy/dx) = dv/dx$ Hence the above eqn. becomes $dv/dx + 2xv = x^3$, which is linear in v and x. Hence its I.F. $= e^{\int 2x\,dx} = e^{x^2}$ and its solution is given by
$$v \cdot e^{x^2} = \int x^3 e^{x^2}\,dx + c, \ c \text{ being an arbitrary constant}$$
$$ve^{x^2} = (1/2) \times \int t\,e^t\,dt + c, \text{ putting } x^2 = t \text{ and } 2x\,dx = dt$$
$$= (1/2) \times [t \times e^t - \int (1 \times e^t)\,dt] + c = (1/2) \times (t\,e^t - e^t) + c$$

or $\quad \tan y \cdot e^{x^2} = (1/2) \times e^{x^2}\,(x^2 - 1) + c$, as $\quad v = \tan y \quad$ and $\quad t = x^2$

or $\quad \tan y = (1/2) \times (x^2 - 1) + ce^{-x^2}$, dividing by e^{x^2}

Ex. 2. *Solve* $(dy/dx) = e^{x-y}(e^x - e^y)$. **[Agra 1995; Delhi Maths (G) 1997; Kanpur 1997; Rohilkhand 1997]**

Sol. Re-writing, $\quad dy/dx = e^{2x} \cdot e^{-y} - e^x \quad$ or $\quad dy/dx + e^x = e^{2x} \cdot e^{-y}$.

Now dividing by e^{-y}, we get $\quad e^y(dy/dx) + e^x \cdot e^y = e^{2x}$.

Putting $e^y = v$ so that $e^y(dy/dx) = dv/dx$ we get $\quad dv/dx + e^x v = e^{2x}$.

Its I.F. $= e^{\int P\,dx} = e^{\int e^x\,dx} = e^{e^x}$ and the solution is

$$v \cdot e^{e^x} = \int e^{2x} \cdot e^{e^x}\,dx + c = \int e^x e^{e^x} \cdot e^x\,dx + c = \int t e^t\,dt + c, \text{ putting } e^x = t \text{ so that } e^x\,dx = dt$$

$$= \int t \cdot e^t - \int 1 \cdot e^t\,dt + c = t \cdot e^t - e^t + c = e^t(t-1) + c$$

i.e., $\quad e^y e^{e^x} = e^{e^x}(e^x - 1) + c \quad$ or $\quad e^{e^x}(e^y - e^x + 1) = c$, as $v = e^y$ and $t = e^x$

Ex. 3. *Solve* $\dfrac{dz}{dx} + \dfrac{z}{x} \log z = \dfrac{z}{x^2} \cdot (\log z)^2$. **[I.A.S. 2001; Calcutta 1994]**

Sol. Here we have z in place of y and so the method of solution will remain similar. Dividing by $z(\log z)^2$, we get

$$\frac{1}{z(\log z)^2}\frac{dz}{dx} + \frac{1}{x}\frac{1}{(\log z)} = \frac{1}{x^2}. \qquad \text{... (1)}$$

Putting $\quad \dfrac{1}{\log z} = v \quad$ so that $\quad \dfrac{(-1)}{(\log z)^2 z}\dfrac{dz}{dx} = \dfrac{dv}{dx}, \quad$ (1) becomes

$$-\frac{dv}{dx} + \frac{1}{x} v = \frac{1}{x^2} \quad \text{or} \quad \frac{dv}{dx} - \frac{1}{x} v = -\frac{1}{x^2}, \qquad \text{... (2)}$$

whose I.F. $= e^{-\int (1/x)\,dx} = e^{-\log x} = 1/x$ and so solution is

$$\frac{v}{x} = \int\left(-\frac{1}{x^3}\right)dx + c = \frac{1}{2x^2} + c \quad \text{or} \quad \frac{1}{x(\log z)} = \frac{1}{2x^2} + c.$$

Ex. 4. $x(dy/dx) + y \log y = xy\,e^x$. **[Agra 1994]**

Sol. Dividing by xy, the given equation reduces to $\quad \dfrac{1}{y}\dfrac{dy}{dx} + \dfrac{1}{x}\log y = e^x. \quad$ ... (1)

Let $\quad \log y = v \quad$ so that $\quad (1/y) \times (dy/dx) = dv/dx \quad$ ... (2)

Using (2), (1) gives $\quad (dv/dx) + (1/x)\,v = e^x. \quad$ ... (3)

Comparing (3) with $\quad dv/dx + Pv = Q$, we have $\quad P = 1/x \quad$ and $\quad Q = e^x. \quad$ ... (4)

Since $\int P\,dx = \int (1/x)\,dx = \log x$; I.F. of (3) $= e^{\int P\,dx} = e^{\log x} = x$. Hence solution of (3) is

$$v.(\text{I.F.}) = \int Q.(\text{I.F.})\,dx + c \quad \text{or} \quad vx = \int x e^x\,dx + c \quad \text{or} \quad vx = xe^x - \int e^x\,dx + c = xe^x - e^x + c$$

or $\quad x \log y = e^x(x - 1) + c$, by (2); c being an arbitrary constant.

Ex. 5. *Solve* $(x^2 - 2x + 2y^2)\,dx + 2xy\,dy = 0$. **[I.A.S. 1991]**

Sol. Re-writing the given equation, we have

$$2xy\frac{dy}{dx} + x^2 - 2x + 2y^2 = 0 \quad \text{or} \quad 2y\frac{dy}{dx} + \frac{x^2 - 2x}{x} + \frac{2y^2}{x} = 0$$

or $$2y\frac{dy}{dx}+\frac{2}{x}y^2=\frac{2x-x^2}{x}. \qquad \text{... (1)}$$

Putting $y^2=v$ so that $2y\,(dy/dx)=dv/dx$ ... (2)

Using (2), (1) gives $$\frac{dv}{dx}+\frac{2}{x}v=\frac{2x-x^2}{x} \qquad \text{... (3)}$$

Comparing (3) with $(dv/dx)+Pv=Q$, we have $P=2/x$ and $Q=(2x-x^2)/x$ ...(4)

$\therefore$ Since $\int P\,dx=\int (2/x)\,dx=2\log x=\log x^2$, hence I.F. of (3) $=e^{\int P\,dx}=e^{\log x^2}=x^2$.

and solution of (3) is $y\,(\text{I.F.})=\int Q\cdot(\text{I.F.})\,dx+c$, c being an arbitrary constant.

or $$y^2x^2=\int\left(\frac{2x-x^2}{x}\right)x^2\,dx=\int(2x^2-x^3)\,dx+c \quad \text{or} \quad y^2x^2=\frac{2x^3}{3}-\frac{x^4}{4}+c$$

Ex. 6. (*a*) *Solve* $2xy\,dy-(x^2+y^2+1)\,dx=0$. **[Delhi Maths 2008]**

(*b*) *Solve* $\dfrac{dy}{dx}=\dfrac{x^2+y^2+1}{2xy}$, *given* $y=1$ *when* $x=1$.

Sol. (*a*) Re-writing the given equation, $2xy\dfrac{dy}{dx}=x^2+y^2+1$ or $2xy\dfrac{dy}{dx}-y^2=1+x^2$.

Dividing by x, $$2y\frac{dy}{dx}-\frac{1}{x}y^2=\left(\frac{1}{x}+x\right). \qquad \text{... (1)}$$

Putting $y^2=v$ so that $2y\,(dy/dx)=dv/dx$, ... (2)

Using (2), (1) gives $$\frac{dv}{dx}-\frac{1}{x}v=\frac{1}{x}+x. \qquad \text{... (3)}$$

Comparing (3) with $(dv/dx)+Pv=Q$, here $P=-(1/x)$ and $Q=x+(1/x)$. ... (4)

$\therefore$ Since $\int P\,dx=-\int(1/x)\,dx=-\log x$ so I.F. of (3) $=e^{\int P\,dx}=e^{\log x^{-1}}=x^{-1}$.

and solution of (3) is $v\cdot(\text{I.F.})=\int\{Q\times(\text{I.F.})\}\,dx+c$

or $y^2x^{-1}=\int(x+1/x)\,x^{-1}\,dx$ or $y^2x^{-1}=x-x^{-1}+c$

or $y^2=(c+x)\,x-1$, c being an arbitrary constant. ... (5)

(*b*) Proceed as in part (*a*) and obtain (5). Given that $y=1$ when $x=1$. Hence (5) yields $1=c+1-1$ or $c=1$. Therefore, from (5), the required solution is $y^2=x\,(x+1)-1$.

Ex. 7. *Solve* $dy/dx+(1/x)\sin 2y=x^2\cos^2 y$. **[Delhi Maths (H) 2001]**

Sol. On dividing by $\cos^2 x$, the given equation reduces to
$$\sec^2 y\,(dy/dx)+(2/x)\tan y=x^2 \qquad \text{... (1)}$$

Putting $\tan y=v$ and $\sec^2 y\,(dy/dx)=dv/dx$, (1) reduces to
$dv/dx+(2/x)\,v=x^2$, which is linear equation

Its I.F. $=e^{\int(2/x)dx}=e^{2\log x}=x^2$ and solution is

$$v\,x^2=\int(x^2\times x^2)\,dx+c \quad \text{or} \quad x^2\tan y=c+(x^5/5)$$

Ex. 8. *Solve* $(\sec x\tan x\tan y-e^x)\,dx+\sec x\sec^2 y\,dy=0$ **[Delhi Maths (H) 2004]**

Sol. Re-writing the given equation, we get $\sec x\sec^2 y\,(dy/dx)+\sec x\tan y=e^x$

or $$\sec^2 y\,(dy/dx)+\tan x\tan y=e^x\cos x. \qquad \text{...(1)}$$

Putting $\tan y = v$ and $\sec^2 y\,(dy/dx) = dv/dx$, (1) reduces to

$$dv/dx + (\tan x)\cdot v = e^x \cos x \text{ which is linear equation} \qquad ...(2)$$

I.F. of (2) $= e^{\int \tan x\, dx} = e^{\log \sec x} = \sec x$ and so its solution is

$$v \sec x = \int e^x \cos x \sec x\, dx + C \qquad \text{or} \qquad \tan y \sec x = e^x + C.$$

Ex. 9. Solve $(xy^2 + e^{-1/x^3})\, dx - x^2 y\, dy = 0.$ **[I.A.S. 2006]**

Sol. Re-writing the given equation, we have

$$x^2 y \frac{dy}{dx} = xy^2 + e^{-1/x^3} \qquad \text{or} \qquad 2y\frac{dy}{dx} - \frac{2}{x} y^2 = \frac{2}{x^2} e^{-1/x^3} \qquad ...(1)$$

Putting $y^2 = v$ and $2y\,(dy/dx) = dv/dx$, (1) reduces to

$$dv/dx - (2/x)\, v = (2/x^2)\, e^{-1/x^3}, \text{ which is linear equation} \qquad ...(2)$$

It I.F. $= e^{\int (-2/x)\,dx} = e^{-2\log x} = e^{\log x^{-2}} = x^{-2}$ and solution is

$$vx^{-2} = \int (x^{-2}) \times (2x^{-2} e^{-x^{-3}})\, dx + c \qquad \text{or} \qquad vx^{-2} = 2\int x^{-4} e^{-x^{-3}}\, dx + c \qquad ...(3)$$

Putting $-x^{-3} = u$ so that $3x^{-4}\, dx = du$ or $x^{-4}\, dx = (1/3) \times du$

$\therefore$ (3) reduces to $vx^{-2} = (2/3) \times \int e^u\, du + c$ or $vx^{-2} = (2/3) \times e^u + c$

or $\qquad y^2 x^{-2} = (2/3) \times (-x^{-3}) + c$, as $v = y^2$ and $u = -x^{-3}$

or $\qquad y^2/x^2 = (2/3) \times e^{-1/x^3} + c$, c being an arbitrary constant.

Ex. 10. *Solve* $(x^2 + y^2 + 2y)\, dy + 2x\, dx = 0$

Sol. Given $x^2 + y^2 + 2y + 2x\,(dx/dy) = 0$ or $2x\,(dx/dy) + x^2 = -(y^2 + 2y)$ ... (1)

Putting $x^2 = v$ and $2x\,(dx/dy) = dv/dy$, (1) reduces to

$$dv/dy + v = -(y^2 + 2y), \text{ which is linear} \qquad ...(2)$$

I.F. of (2) $= e^{\int dy} = e^y$ and hence solution of (2) is

$$ve^y = -\int e^y (y^2 + 2y)\, dy + c, \; c \text{ being an arbitrary constant}$$

or $$ve^y = -[e^y (y^2 + 2y) - \int e^y (2y + 2)\, dy] + c, \text{ integrating by parts}$$

or $$ve^y = -e^y (y^2 + 2y) + \int e^y (2y + 2)\, dy + c$$

or $$ve^y = -e^y (y^2 + 2y) + e^y (2y + 2) - \int (e^y \times 2)\, dy + c, \text{ integrating by parts}$$

or $$x^2 e^y = -y^2 e^y - 2y e^y + 2y e^y + 2e^y - 2e^y + c, \text{ as } v = x^2$$

or $$(x^2 + y^2)\, e^y = c$$

Exercise 2(L)

1. $\sin y\,(dy/dx) = \cos y\,(1 - x\cos y)$. **Ans.** $\sec y = x + 1 + ce^x$
2. $(dy/dx) + (1/x)\tan y = (1/x^2)\tan y \sin y$. **Ans.** $x \operatorname{cosec} y = c + \log x$
3. $(dy/dx) + 1 = e^{x-y}$. **Ans.** $e^y = ce^{-x} + (1/2) \times e^x$
4. $(dy/dx) - (\tan y)/(1 + x) = (1 + x)\, e^x \sec y$. **[Kanpur 1998; Lucknow 1996]** **Ans.** $\sin y = (1 + x)(c + e^x)$
5. $(dy/dx) + (1/x) = e^y/x^2$. **[Kerala 2001]** **Ans.** $2x\, e^{-y} = 1 + 2cx^2$
6. $(x^2 + y^2 + 2x)\, dx + 2y\, dy = 0$. **Ans.** $e^x (x^2 + y^2) = c$

2.27 Examples of Type 14 based on Art. 2.25A

Ex. 1. *Solve* $x\,(dy/dx) + y = y^2 \log x$. **[Delhi Maths (H) 2009; Kanpur 2006]**

Sol. Re-writing the given equation $y^{-2}\,(dy/dx) + (1/x) \times y^{-1} = (1/x) \times \log x$. ... (1)

Putting $y^{-1} = v$ so that $-y^{-2}\,(dy/dx) = dv/dx$. Then (1) gives

$$-\frac{dv}{dx} + \frac{1}{x}v = \frac{1}{x}\log x \qquad \text{or} \qquad \frac{dv}{dx} - \frac{1}{x}v = -\frac{1}{x}\log x \qquad \text{... (2)}$$

I.F. of (2) $= e^{-\int (1/x)\,dx} = e^{-\log x} = x^{-1} = 1/x$. and hence solution of (2) is

$$vx^{-1} = -\int x^{-2} \log x\, dx + c, \text{ c being an arbitrary constant}$$

or $$y^{-1}x^{-1} = -\left[\log x \times \frac{x^{-1}}{(-1)} - \int \frac{1}{x} \times \frac{x^{-1}}{(-1)}\,dx\right] + c \qquad \text{or} \qquad \frac{1}{y} = \log x + 1 + cx.$$

Ex. 2. *Solve* $(dy/dx) - y \tan x = -y^2 \sec x$ *or* $\cos x\, dy = (\sin x - y)\, y\, dx$. **[Kanpur 1995]**

Sol. Dividing by y^2, the given equation gives $y^{-2}\,(dy/dx) - \tan x \cdot y^{-1} = -\sec x$ ... (1)

Putting $y^{-1} = v$ so that $-y^{-2}\,(dy/dx) = dv/dx$, (1) becomes

$$-\frac{dv}{dx} - \tan x \cdot v = -\sec x \qquad \text{or} \qquad \frac{dv}{dx} + \tan x \cdot v = \sec x \text{ ... (2)}$$

which is linear whose I.F. $= e^{\int \tan x\, dx} = e^{\log \sec x} = \sec x$.

Hence solution of (2) is $v \cdot \sec x = \int \sec x \cdot \sec x\, dx + c$, c being an arbtritray constant.

or $v \sec x = \tan x + c$ or $y^{-1} \sec x = \tan x + c$, as $v = y^{-1}$

Ex. 3. *Solve the following differential equations:*

(*a*) $(x^3y^2 + xy)\, dx = dy$. **[Guwahati 2007; Delhi Maths (G) 1996; Delhi Maths (H) 1988]**

(*b*) $x^3\,(dy/dx) - x^2y + y^4 \cos x = 0$. **[Delhi Maths (H) 1993]**

(*c*) $2x^2\,(dy/dx) = xy + y^2$. **[Delhi Maths (H) 1992**

(*d*) $x\,(dy/dx) + y^2x = y$. **[Delhi Maths (H) 1994]**

Sol. (*a*) Re-writing the given equation, we have

$$\frac{dy}{dx} = x^3y^2 + xy \qquad \text{or} \qquad y^{-2}\frac{dy}{dx} - xy^{-1} = x^3 \text{ ... (1)}$$

Putting $y^{-1} = v$ so that $-y^{-2}\,(dy/dx) = dv/dx$, Hence (1) reduces to

$-(dv/dx) - xv = x^3$ or $(dv/dx) + xv = -x^3$,

which is linear whose I.F. $= e^{\int x\,dx} = e^{x^2/2}$. and hence its solution is

$$v e^{x^2/2} = -\int x^3 e^{x^2/2}\, dx + c = -\int x^2 \cdot e^{x^2/2}\, x\, dx . + c, \ c \text{ being an arbitrary constant} \qquad \text{... (2)}$$

Putting $x^2/2 = t$ so that $x\, dx = dt$, (2) gives

$$v e^{x^2/2} = -2\int t e^t\, dx = -2\,[t e^t - \int e^t\, dt] + c = -2\,(t\,e^t - e^t) + c = -2e^t\,(t-1) + c$$

$$y^{-1} e^{x^2/2} = -2e^{x^2/2}\,\{(x^2/2) - 1\} + c \qquad \text{or} \qquad y^{-1} = (2 - x^2) + ce^{-x^2/2}.$$

(*b*) Dividing by x^3y^4, the given equation becomes $y^{-4}\dfrac{dy}{dx} - \dfrac{1}{x}y^{-3} = -\dfrac{\cos x}{x^3}$. ... (1)

Putting $y^{-3} = v$ so that $-3y^{-4}\,(dy/dx) = (dv/dx)$ or $y^{-4}\,(dy/dx) = -\{(1/3) \times (dv/dx)\}$

(1) gives $$-\frac{1}{3}\frac{dv}{dx}-\frac{1}{x}v=-\frac{\cos x}{x^3} \quad \text{or} \quad \frac{dv}{dx}+\frac{3}{x}v=\frac{3\cos x}{x^3}$$

which is linear whose I.F. $= e^{\int (3/x)dx} = e^{3\log x} = x^3$, and hence its solution is

$$v x^3 = \int \left(\frac{3\cos x}{x^3}\right) x^3\, dx \quad \text{or} \quad \frac{x^3}{y^3} = 3\sin x + c.$$

(*c*) Do yourself. **Ans.** $x/y = 1 + c\sqrt{x}$

(*d*) Do yourself. **Ans.** $y^{-1}x = (x^2/2) + c$

Ex. 4. *Solve* $x\,(dx/dy) + 3y = x^3y^2$. **[I.A.S. 2002]**

Sol. Dividing by xy^2, the given equation reduces to

$$(1/y^2)\,(dy/dx) + (3/x)\,(1/y) = x^2 \qquad \text{... (1)}$$

Putting $1/y = v$ and $(-1/y^2)\,(dy/dx) = dv/dx$, (1) reduces to

$$-\frac{dv}{dx}+\frac{3}{x}v = x^2 \quad \text{or} \quad \frac{dv}{dx}-\frac{3}{x}v = -x^2,$$ which is linear equation

Its I.F. $= e^{\int (-3/x)dx} = e^{-3\log x} = x^{-3}$ and its solution is

$$x^{-3}v = \int (-x^2)(x^{-3})\,dx + c \quad \text{or} \quad x^{-3}y^{-1} = -\log x + c.$$

Ex. 5. *Solve* $dy/dx + y\cos x = y^4\sin 2x$. **[Delhi Maths (P) 2001]**

Sol. Re-writing, $$y^{-4}\,(dy/dx) + (\cos x)\,y^{-3} = \sin 2x \qquad \text{... (1)}$$

Putting $y^{-3} = v$ and $-3y^{-4}\,(dy/dx) = dv/dx$, (1) gives

$$-\{(1/3)\times(dv/dx)\} + (\cos x)\,v = \sin 2x \quad \text{or} \quad dv/dx - (3\cos x)\,v = -3\sin 2x$$

Its I.F. $= e^{\int(-3\cos x)dx} = e^{-3\sin x}$ and solution is $v\,e^{-3\sin x} = \int e^{-3\sin x}(-6\sin x\cos x)\,dx + c$

or $v\,e^{-3\sin x} = -6\int e^t\,(-1/3)\,t\,(-1/3)\,dt = -(2/3)\times\int t\,e^t\,dt$, put $-3\sin x = t$ and $-3\cos x dx =$ dt

or $$v\,e^{-3\sin x} = -(2/3)\times[t\,e^t - e^t] = (-2/3)\times e^t\,(t+1) + c$$

or $$y^{-3}e^{-3\sin x} = -(2/3)\times e^{-3\sin x}(1 - 3\sin x) + c, \text{ as } v = y^{-3},\ t = -3\sin x$$

or $y^{-3} = 2\sin x - (2/3) + ce^{3\sin x}$, c being an arbitrary constant.

Ex. 6. *Solve* $dy/dx + y\sin x = y^3\cos 2x$. **[Delhi Maths (P) 2004]**

Sol. Re-writing, the given equation, $$y^{-3}\,(dy/dx) + (\sin x)\,y^{-2} = \cos 2x \qquad \text{... (1)}$$

Let $y^{-2} = v$ so that $-2y^{-3}\,(dy/dx) = (dv/dx)$ or $y^{-3}(dy/dx) = -\{(1/2)\times(dv/dx)\}$

(1) gives, $$-\{(1/2)\times(dv/dx)\} + (\sin x)\,v = \cos 2x$$

or $dv/dx - (2\sin x)\,v = -2\cos 2x$, which is linear equation

whose I.F. $= e^{\int(-2\sin x)dx} = e^{2\cos x}$ and its solution is

$$v\,e^{2\cos x} = C + \int e^{2\cos x}(-2\cos 2x)\,dx \quad \text{or} \quad (1/y^2)\,e^{2\cos x} = C - 2\int \cos 2x\,e^{2\cos x}\,dx$$

[**Note:** Since the integral on R.H.S. cannot be evaluated, the required solution is given in the above form involving an integral]

Ex. 7. *Solve* $dy/dx + y\cos x = y^n\sin 2x$. **[Bangalore 2004]**

Sol. Re-writing the given equation, we have $$y^{-n}\,(dy/dx) + y^{1-n}\cos x = \sin 2x \qquad \text{... (1)}$$

Putting $y^{1-n} = v$ so that $$(1-n)\,y^{-n}\frac{dy}{dx} = \frac{dv}{dx} \quad \text{or} \quad y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dv}{dx} \qquad \text{... (2)}$$

Using (2), (1) reduces to

$$\frac{1}{1-n}\frac{dv}{dx} + v\cos x = \sin 2x \qquad \text{or} \qquad \frac{dv}{dx} + \{(1-n)\cos x\}\, v = (1-n)\sin 2x \quad \ldots (3)$$

I.F. of (3) $= e^{\int (1-n)\cos x\, dx} = e^{(1-n)\sin x}$ and hence its solution is

$$v\, e^{(1-n)\sin x} = \int (1-n)\sin 2x\, e^{(1-n)\sin x}\, dx + c = 2\int (1-n)\sin x\, e^{(1-n)\sin x}\cos x\, dx + c$$

$$= \frac{2}{1-n}\int t e^t\, dt + c = \frac{2}{1-n}(t e^t - e^t) + c, \text{ putting } (1-n)\sin x = t \text{ and } (1-n)\cos x\, dx = dt$$

Thus, $\quad y^{1-n}\, e^{(1-n)\sin x} = \{2/(1-n)\} \times e^{(1-n)\sin x}\{(1-n)\sin x - 1\} + c$

or $\quad y^{1-n} = 2\sin x - \{2/(1-n)\} + ce^{-(1-n)\sin x,}$ c being an arbitrary constant

Ex. 8. *Solve* $(x^2y^3 + xy)(dy/dx) = 1$. **[Calcutta 1995]**

Sol. Re-writing, $\quad (dx/dy) = x^2y^3 + xy \quad$ or $\quad (dx/dy) - yx = y^3x^2$

or $\quad x^{-2}(dx/dy) - x^{-1}y = y^3. \quad \ldots (1)$

Putting, $x^{-1} = v$, and $-x^{-2}(dx/dy) = dv/dy$, (1) becomes,

$$-(dv/dy) - yv = y^3, \qquad \text{or} \qquad (dv/dy) + yv = -y^3,$$

which is linear in v and y and I.F. $= e^{\int y\,dy} = e^{y^2/2}$. and its solution is

$$v \cdot e^{y^2/2} = -\int y^3 e^{y^2/2}\, dy + c = -2\int t e^t\, dt + c, \text{ putting } y^2/2 = t \text{ and } y\, dy = dt$$

$$= -2\left[t\cdot e^t - \int 1\cdot e^t\, dt + c\right] = -2(t e^t - e^t) + c.$$

or $\quad x^{-1}e^{y^2/2} = -2e^{y^2/2}(y^2/2 - 1) + c$, as $v = x^{-1}$

$\therefore\quad$ Required soution is $\quad 1/x = 2 - y^2 + ce^{-y^2/2}$., c being an arbitrary constant.

Exercise 2(M)

1. $(dy/dx) - 2y\tan x = y^2\tan^2 x$. **Ans.** $(-1/y)\sec^2 x = c + (1/2)\times\tan^3 x$

2. $2(dy/dx) - y\sec x = y^3\tan x$. **Ans.** $-(\sec x + \tan x)/y^2 = c + \sec x + \tan x - x$

3. $(dy/dx)(x^3y^3 + xy) = 1$ **Ans.** $x^2(1 - y^2 - ce^{-y^2}) = 1$

4. $xy - (dy/dx) = y^3 e^{-x^2}$. **[I.A.S. 1998; Nagpur 2005]** **Ans.** $y^{-2}e^{x^2} = 2x + c$

5. $(1 - x^2)(dy/dx) + xy = xy^2$. **[Purvanchal 2007]** **Ans.** $cy = (1-y)\sqrt{(1-x^2)}$

6. $dy/dx = x^3y^3 - xy$. **[Rohilkhand 1995]** **Ans.** $y^{-2} = x^2 + 1 + ce^{x^2}$

7. $(dy/dx) - y\sec x = y^2\sin x\cos x$. **Ans.** $y^{-1}(\sec x + \tan x) = \cos x - (x/2) + (1/4)\sin 2x + c$

8. $dy/dx + y = xy^3$. **[Delhi B.Sc. (Hons) II 2011]** **Ans.** $y = x + 1/2 + c\, e^{2x}$

9. $(x - y^2)\, dx + 2xy\, dy = 0$. **Ans.** $y^2 + x\log(cx) = 0$

10. $y(2xy + e^x)\, dx - e^x\, dy = 0$. **[Delhi Maths (G) 1995]** **Ans.** $e^{x/y} + x^2 + c = 0$

11. $\dfrac{dy}{dx} + \dfrac{xy}{1-x^2} = x\sqrt{y}$. **[Delhi Maths (H) 2004]** **Ans.** $\sqrt{y} = -\frac{1}{3}(1-x^2) - c(1-x^2)^{1/4}$

12. $3(dy/dx) + 2y/(x+1) = x^3/y^2$. **Ans.** $y^3(x+1)^2 = c + (1/6)\times x^6 + (2/5)\times x^5 + (1/4)\times x^4$

13. $x(dy/dx) + y = x^3y^6$. **[Delhi Maths (G) 2006]** **Ans.** $y^{-5}x^{-5} = (5/2)\times x^{-2} + c$

14. $(dy/dx) + xy = y^2 e^{x^2/2}\sin x$. **[Agra 2006]** **Ans.** $1/y = (x + c)e^{x^2/2}$

2.28 Geometrical meaning of a differential equation of the first order and first degree

Let $$dy/dx = f(x, y), \qquad \text{... (1)}$$

be a differential equation of the first order and first degree. Let P_1 (x_1, y_1) be any point on the xy-plane. Substituting the co-ordinates of P_1 in (1), we obtain a corresponding particular value of dy/dx, say m_1, which is slope (or direction) of the tangent at P_1. Suppose it moves from P_1 in the direction m_1 (*i.e.*, along the tangent at P_1) for an infinitesimal distance, to a point P_2 (x_2, y_2). Let m_2 be the slope of the tangent at P_2, determined by equation (1). Suppose it moves from P_2 in the direction of m_2 for an infinitesimal distance, to a point P_3 (x_3, y_3). Let m_3 be the slope of the tangent at P_3, determined by equation (1). Let the point move from P_3 in the direction of m_3 for an infinitesimal distance, to a point P_4 (x_4, y_4) and so on through successive points.

Proceeding likewise, the point will describe a curve, the co-ordinates of every point of which, and the direction of the tangent there at, will satisfy the differential equation (1). If the moving point starts at any other point, not on the curve already described, and proceeds as before, it will describe another curve, the co-ordinates of whose points and the direction of the tangents thereat satisfy the equation.

Thus, through every point on the xy-plane, there will pass a particular curve, for every point of which x, y, dy/dx, will satisfy (1). The equation of each curve is thus a particular solution of (1); the equation of the system of such curves is the general solution; and all curves represented by the general solution, taken together, make the locus of the differential equation. Since there is one arbitrary constant in the general solution of an equation of the first order, it follows that the latter is made up of a single infinity of curves.

2.29 Applications of equation of first order and first degree **(Meerut 2008)**

We now discuss some problems which give rise to differential equations of first order and first degree. By using given data of the problem, we shall first prepare a differential equation which will then be solved by a suitable method. In forming differential equation, we shall use the following results which must be remembered by heart.

2.30 List of important results for direct applications:

List A. Facts for cartesian curve $y = f(x)$ as studied in chapter on tangent and normal in differential calculus.

Let P (x, y) be any point on the curve UPV whose equation is $y = f(x)$. Let the tangent PT and the normal PG at P meet the x-axis in T and G respec-tively. Let PN be the ordinate of P. Let ψ be the angle which the tangent at P makes with x-axis. Then, from figure $\angle GPN = \psi = \angle PTG$. Now, we have

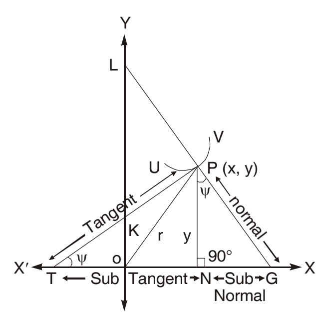


Fig. 2.1

A:1. $dy/dx = \tan \psi$ = gradient (or slope) of tangent PT.

A:2. NG = subnormal $= y\ (dx/dy)$.

A:3. NT = subtangent $= y\ (dx/dy)$.

A:4. PG = The length of normal at $(x, y) = y\ \{1 + (dy/dx^2)\}^{1/2}$.

A:5. PT = The length of tangent at $(x, y) = y\ \{1 + (dx/dy)^2\}^{1/2}$.

A:6. Equation of tangent at (x, y) is $\quad Y - y = (dy/dx)\ (X - x)$.

A:7. Equation of normal at (x, y) is $\quad (dy/dx)\ (Y - y) + X - x = 0$.

A:8. $$\begin{cases} x - \text{intercept } OT \text{ of the tangent at } (x, y) = x - y\,(dx/dy). \\ y - \text{intercept } OK \text{ of the tangent at } (x, y) = y - x\,(dy/dx). \end{cases}$$

A:9. $\begin{cases} x - \text{intercept } OG \text{ of the normal at } (x, y) = x + y\,(dy/dx). \\ y - \text{intercept } OL \text{ of the normal at } (x, y) = y + x\,(dx/dy). \end{cases}$

A:10. If 's' denotes the length of the arc of a curve from a fixed point on the curve.

Then, $ds/dx = \sqrt{1+(dy/dx)^2}$ and $ds/dy = \sqrt{1+(dx/dy)^2}$

A11. ρ = radius of curvature $= [1 + (dy/dx)^2]^{3/2}/(d^2y/dx^2)$.

Note. In results A_6 and A_7, (X, Y) denote the current coordinates of any point on tangent or normal as the case may be.

List B. Facts for polar curve $r = f(\theta)$ as studied in chapter on tangent and normal in differential calculus.

Let $P(r, \theta)$ be any point on the curve UPV whose equation is $r = f(\theta)$. Through the pole O, draw GOT perpendicular to the radius vector OP meeting the tangent PT in T and the normal PG in G. Let OM be perpendicular to the tangent PT. Here $\angle POX = \theta$, $\angle OPT = \phi$ and $\angle PKX = \psi$. Then, we have

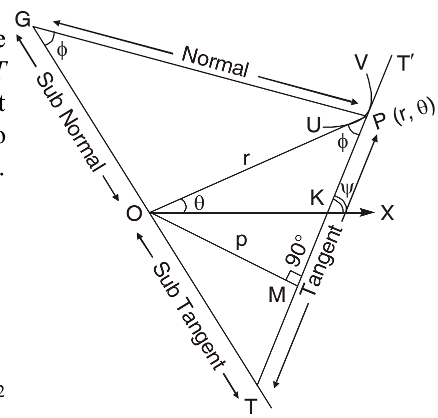


Fig. 2.2

B1. $\psi = \theta + \phi$.

B2. $\tan\phi = r\,(d\theta/dr)$.

B3. Polar subtangent $= OT = r\tan\phi = r^2\,(d\theta/dr)$.

B4. Polar subnormal $= OG = r\cot\phi = dr/d\theta$.

B5. Length of polar tangent $= PT = r\,\{1 + r^2\,(d\theta/dr)^2\}^{1/2}$

B6. Length of polar normal $= PG = \{r^2 + (dr/d\theta)^2\}^{1/2}$.

B7. If 'p' is the perpendicular PM from the pole O on the tangent PT at the point $P(r, \theta)$, then $p = r\sin\phi$ and $1/p^2 = (1/r^2) + (1/r^4) \times (dr/d\theta)^2$.

B8. If 's' denotes the length of the arc of a curve from a fixed point the curve

Then $ds/dr = \{1 + r^2\,(d\theta/dr)^2\}^{1/2}$ and $ds/d\theta = \{r^2 + (dr/d\theta)^2\}^{1/2}$.

B9. If cartesian coordinates of P be (x, y), then we have

$r^2 = x^2 + y^2$, $\tan\theta = y/x$, $x = r\cos\theta$ and $y = r\sin\theta$.

List C. Some important results of dynamics of a particle

C1. If a particle is moving in a straight line OX where O is a fixed point on the line, then velocity v of the particle at any time t is given by dx/dt. Remember that the velocity dx/dt is along the line OX itself and is taken with positive or negative sign according as the particle is moving in the direction of x increasing or x decreasing.

C2. If a particle is moving in a straight line OX where O is a fixed point on the line, then acceleration of the particle at any time t is given by d^2x/dt^2.Remember that that acceleration d^2x/dt^2 is along OX itself and is taken with positive or negative sign according as the particle is moving in the direction of x increasing or x decreasing.

Other expressions for acceleration f are given by $f = d^2x/dt^2 = dv/dt = v\,(dv/dx)$.

Note. Some authors use s for x and a for f.

List D. Population growth problems. Rate of growth of population is proportional to the population. For example, the bacteria population grows at a rate proportional to the population, *i.e.*, the growth rate dx/dt is proportional to x, where $x = x(t)$ denotes the number of bacteria present at time t. Mathematically, the above fact can be expressed as $dx/dt = kx$, k being a positive constant of proportionality.

List E. Radioactive decay problems. Rate of decay of a certain radioactive material is proportional to the amount present. If $x = x(t)$ represents the amount of radioactive material present at time t, then $dx/dt = kx$, k being a negative constant of proportionality.

List F. Newton's law of cooling problems. According to this law, the rate at which the temperature $T = T(t)$ changes in a cooling body is proportional to the difference between the temperature T of the body and the constant temperature. T_0 of the surrounding medium. Thus, $dT/dt = -k(T - T_0)$, $T > T_0$ and k being a positive constant of proportionality. Note that negative sign is taken due to the reduction in the temperatuer of the hot body, when it cools.

2.31 Solved examples of Type 15 based on Art. 2.30

Ex. 1. *Show that the curve in which the slope of the tangent any point equals the ratio of the abscissa to the ordinate of the point is a rectangular hyperbola.*

Sol. Given, $\quad dy/dx$ = slope of tangent at $(x, y) = x/y$.

$\therefore\ 2x\,dx - 2y\,dy = 0 \quad$ so that $\quad x^2 - y^2 = c^2$,

which is a rectangular hyperbola, c being an arbitrary constant.

Ex. 2(*a*). *Show that the parabola is the only curve in which the subnormal is constant.*

[I.A.S (Prel.) 2009]

Sol. Given that, $\quad$ subnormal $= y\,(dy/dx)$ = constant $= k$, (say).

$\therefore\ 2y\,dy = 2k\,dx \quad$ so that $\quad y^2 = 2kx + c$, ... (1)

where c is an arbitrary constant. (1) is the equation of a parabola, since second degree terms of this quadratic equation form a perfect square.

Ex. 2(*b*). *Find the equation to the curve for which cartesian subtangent is constant. What would be its equation if it passes through (0, 1)?*

Sol. Given that, $\quad$ the subtangent $= y\,(dx/dy)$ = constant $= k$, (say)

$\therefore\ dx = (k/y)\,dy \quad$ so that $\quad x = k \log y - k \log c$

or $\quad k \log(y/c) = x \quad$ or $\quad \log(y/c) = x/k \quad$ or $\quad y = ce^{x/k}$ ... (1)

If (1) passes through (0, 1), we have $\quad 1 = ce^0 \quad$ or $\quad c = 1$.

Then from (1), the required curve is $\quad y = e^{x/k}$.

Ex. 2(*c*). *Find curve in which the cartesian subnormal is equal to abscissa.*

Sol. Do yourself. $\quad$ **Ans.** $y^2 = x^2 + c$.

Ex. 2(*d*). *Find the curve for which the cartesian subnormal varies as the square of its radius vector.*

or $\quad$ *The normal at any point P of a curve cuts OX in G and N is the foot of the ordinate of P. If NG varies as the square of the radius vector from O, find the curve.*

Sol. Given that subnormal $NG = k\,r^2$, where r is radius vector and $r^2 = x^2 + y^2$. Refer figure 2.1 of Art. 2.30.

$\therefore\ y\,(dy/dx) = k\,(x^2 + y^2) \quad$ or $\quad 2y\,(dy/dx) - 2k\,y^2 = 2k\,x^2$.... (1)

Putting $y^2 = v$ so that $2y\,(dy/dx) = dv/dx$, (1) becomes

$$(dv/dx) - 2k\,v = 2k\,x^2, \text{ which is linear.} \qquad ... (2)$$

Its integrating factor $= e^{\int(-2kx)\,dx} = e^{-2kx}$ and hence its solution is

$$v\,e^{-2kx} = \int (2kx^2)\,e^{-2kx}\,dx + c = c + 2k\int x^2\,e^{-2kx}\,dx$$

$= c + 2k\{x^2(-1/2k)e^{-2kx} - \int (2x)(-1/2k)e^{-2kx}\,dx\}$, integrating by parts

$= c - x^2 e^{-2kx} + \int x e^{-2kx}\,dx = c - x^2 e^{-2kx} + 2\{x(-1/2k)e^{-2kx} - \int 1\cdot(-1/2k)e^{-2kx}\,dx\}$,

or $$v\, e^{-2kx} = c - x^2\, e^{-2kx} - (x/k)\, e^{-2kx} - (1/2k^2)\, e^{-2kx}$$

or $y^2 = ce^{2kx} - x^2 - (x/k) - (1/2k^2)$, as $v = y^2$., c being an arbitrary constant.

Ex. 2(*e*). *Determine the curve in which the subtangent is n times the subnormal.*

Sol. Try yourself. **Ans.** $y\, n^{1/2} = x + c$.

Ex. 2(*f*). *Find the curve for which the product of the subtangent at any point and the abscissa of that point is constant.* **Ans.** $y = ce^{x^2/2k}$.

Ex. 2(*g*). *Determine the curve whose subtangent is n times the abscissa of the point of contact and find the particular curve which passes through the point (2, 3). What is the curve when (a) n = 1* (b) $n = 2$.

Sol. Do yourself. **Ans.** $2y^n = 3^n x$. (*a*) $2y = 3x$ (*b*) $2y^2 = 9x$.

Ex. 2(*h*). *Find the curve in which the length of the subnormal is proportional to the square of the abscissa.* **Ans.** $3y^2 = 2kx^2 + c$.

Ex. 2(*i*). *Find the curve in which the length of the subnormal is proportional to the square of the ordinate.* **Ans.** $y = ce^{kx}$.

Ex. 3(*a*). *Show that the curve for which the normal at every point passes through a fixed point is a circle.*

(*b*) *Find the curve for which the normal at any point passes through origin.*

Sol. (*a*) The equation of the normal at any point (x, y) of the curve is

$$(dy/dx)\,(Y - y) + X - x = 0. \qquad \ldots (1)$$

Let (h, k) be the coordinates of the fixed point. Since the required curve (1) passes through (h, k), we have

$(dy/dx)\,(k - y) + (h - x) = 0$ or $2\,(k - y)\,dy + 2\,(h - x)\,dx = 0$.

Integrating, $2ky - y^2 + 2hx - x^2 + c = 0$, where c is an arbitrary constant

or $x^2 + y^2 - 2hx - 2ky - c = 0$, which represents a circle. ...(2)

(*b*) Proceed as above. Here $h = k = 0$. So (2) gives $x^2 + y^2 = c$, c which is a circle.

Ex. 4. *Find the curve in which the subtangent is always bisected at origin.*

Sol. Refere Figure 2.1 of Art. 2.30. PT is the tangent to the curve at $P\,(x, y)$, meeting x-axis in T and PN is the ordinate of P. Then TN is the subtangent. Now, the equation of PT is

$$Y - y = (dy/dx)\,(X - x). \qquad \ldots (1)$$

Let (1) meet x-axis in T whose coordinates are $(h, 0)$.

$\therefore\ 0 - y = (dy/dx)\,(h - x)$ so that $h = x - y\,(dx/dy)$.

Then, the coordinates of T are $[x - y\,(dx/dy), 0]$. Also, the coordinates of N are $(x, 0)$.

$\therefore$ x-coordinate of middle point of $TN = [x - y\,(dx/dy) + x]/2$.

But according to the given problem origin (0, 0) is middle point of TN.

$\therefore$ $[2x - y\,(dx/dy)]/2 = 0$ or $2x = y\,(dx/dy)$ or $(2/y)\,dy = (1/x)\,dx$.

Integrating, $2 \log y = \log x + \log c$ or $y^2 = xc$, c being an arbitrary constant.

Ex. 5. *The normal PG to a curve meets x-axis in G. If distance of G from the origin is twice the abscissa of P, prove that curve is a rectangular hyperbola.* **[I.A.S. (Prel.) 2009]**

Sol. Refer Figure 2.1 of Art. 2.30. Let coordinates of P be (x, y) and let PN be perpendicular from P to x-axis.

Given that $OG = 2x$ or $ON + NG = 2x$ or $x + NG = 2x$ or $NG = x$. ... (1)

But NG = the length of the subnormal = $y\,(dy/dx)$.

$\therefore$ (1) $\Rightarrow$ $y\,(dy/dx) = x$ or $2x\,dx - 2y\,dy = 0$.

Integrating, $x^2 - y^2 = c^2$ which is a rectangular hyperbola.

Ex. 6. *The normal PG to a curve meets the x-axis in G. If OP = PG, find the equation to the curve.*

or *Find the equation of the family of curves for which the length of the normal is equal to the radius vector.*

or *A curve is such that any point P on it is as far from the origin as from the point in which the normal at P meets the axis. Show that it must be an equilateral hyperbola (rectangular hyperbola) or a circle.*

Sol. Refer Figure 2.1 of Art. 2.30. Then, if the coordination of P be (x, y), then $OP = (x^2 + y^2)^{1/2}$ and PG = length of normal $= y\,\{1 + (dy/dx)^2\}^{1/2}$.

Given that $OP = PG$ or $OP^2 = PG^2$.

$\therefore$ $x^2 + y^2 = y^2\,\{1 + (dy/dx)^2\}$ or $2x\,dx = \pm\, 2y\,dx$.

Integrating, $x^2 = c^2 \pm y^2$ where c is an arbitrary constant. It represents a circle or rectangular hyperbola according as we take the – ve or + ve sign.

Ex. 7. *The normal at each point of the curve and the line from that point to the origin form an isosceles triangle with the base on the x-axis. Find the equation of the curve.*

Sol. Refer Figure 2.1 of Art 2.30. Let the coordinate of P be (x, y) and PG be the normal at P and OP be the line joining O and P. Then, according to given problem $\Delta\, OPG$ is an isosceles triangle so that,

$ON = NG$, where NG is subnormal $\Rightarrow$ $x = y\,(dy/dx)$ or $2x\,dx - 2y\,dy = 0$.

Integrating, $x^2 - y^2 = c$, where c is an arbitrary constant.

Ex. 8. *Find the curve which is such that portion of x-axis cut off between origin and the tangent at any point is proportional to the ordinate of the point.*

Sol. Refer Figure 2.1 of Art 2.30. Here OT is the portion of the x-axis cut off between the origin O and the tangent PT at any point $P\,(x, y)$. Clearly OT is x-intercept of the tangent at P. According to the problem, we have

$OT = ky$ or $x - y\,(dx/dy) = ky$, where k is a constant

or $(dx/dy) - (1/y)\,x = -\,k$, which is linear equation. ... (1)

Its integrating factor of (1) $= e^{-\int (1/y)\,dy} = e^{-\log y} = e^{\log y^{-1}} = y^{-1} = 1/y$ and solution is

$x/y = \int (-k)(1/y)\,dy + k\log c$ or $x/y = -k\log y + k\log c$,

or $\log\,(y/c) = -\,x/(ky)$ or $y = ce^{-x/(ky)}$

Ex. 9. *Find the curve for which the portion of y-axis cut off between the origin and the tangent varies as the cube of the abscissa of the point of contact.* **[I.A.S. 1992]**

Sol. Refer Fig. 2.1 of Art 2.30. Here OK is the portion of the y-axis cut off between the origin O and the tangent PK at any point $P\,(x, y)$. Clearly OK is y-intercept of the tangent at P. According to the problem, we have

$OK = kx^3$ or $y - x\,(dy/dx) = kx^3$, where k is a constant

or $(y/x) - (dy/dx) = kx^2$ or $(dy/dx) - (1/x)\,y = -\,kx^2$. ... (1)

(1) is linear equation whose integrating factor $= e^{-\int (1/x)\,dx} = e^{-\log x} = x^{-1} = 1/x.$

$\therefore$ Solution is $\quad y/x = \int (-kx^2)(1/x)\,dx + c/2 \quad$ or $\quad y/x = -(kx^2)/2 + c/2$

or $\quad 2y = -kx^3 + cx$, where c is an arbitrary constant.

Ex. 10. *Find the equation to the family of curves in which the length of the tangent between the point of contact and x-axis is of constant length equal to k.*

Sol. Refer Fig. 2.1 of Art 2.30. Let the point of contact be $P(x, y)$. Given that

$PT = k \quad$ or $\quad \{1 + (dx/dy)^2\}^{1/2} = k \quad$ or $\quad (dx/dy)^2 = (k^2 - y^2)/y^2$

or $\quad dx = \{(k^2 - y^2)^{1/2}/y\}\,dy$ Integrate it and get the final solution yourself.

Ans. $(k^2 - y^2)^{1/2} - k \log [\{k + (k^2 - y^2)^{1/2}\}/y] = c + x.$

Ex. 11. *Find the curve for which the intercept cut off by a tangent on the x-axis is equal to 4 times the ordinate of the point of contact.*

Sol. Refer Fig. 2.1 of Art. 2.30. Here, given that

$OT = 4\,PN \quad$ or $\quad x - y\,(dx/dy) = 4y \quad$ or $\quad dy/dx = y\,(x - 4y),$

which is homogeneous. Get its solution as $\quad cy^4 = e^{-x/y}.$

Ex. 12. *Find the equation of the curve in which the perpendicular from the origin on any tangent is equal to the abscissa of the point of contact.*

Sol. The equation of the tangent to the curve at any point (x, y) is

$Y - y = (dy/dx)(X - x) \quad$ or $\quad (dy/dx)\,X - Y + \{y - x\,(dy/dx)\} = 0.... \; (1)$

Given $\quad x =$ the length of the perpendicular from (0, 0) on (1)

or $$x = \frac{y - x\,(dy/dx)}{\sqrt{(dy/dx)^2 + (-1)^2}} \quad \text{or} \quad x^2\,\{(dy/dx)^2 + 1\} = \{y - x\,(dy/dx)\}^2$$

or $\quad x^2\,(dy/dx)^2 + x^2 = y^2 - 2xy\,(dy/dx) + x^2\,(dy/dx)^2 \quad$ or $\quad 2xy\,(dy/dx) = y^2 - x^2$

Solve this homogeneous equation to get the required curve $\quad x^2 + y^2 = cx.$

Ex. 13. *Find the cartesian equation of the curve in which the perpendicular from the foot of the ordinate on the tangent is of constant length.*

Sol. Let $P(x, y)$ be any point on the required curve UPV, PN is the ordinate and NM is the perpendicular from the foot N of the ordinate on the tangent at PT at P.

From Fig. 2.3, $\quad PN = y \quad$ and $\quad MN = y \cos \psi.$

Given that $MN =$ constant $= k$, say $\Rightarrow y \cos \psi = k \quad$ or $\quad y = k \sec \psi$

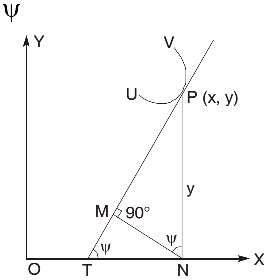


Fig. 2.3

or $\quad y = k\sqrt{1 + \tan^2 \psi} = k\sqrt{1 + (dy/dx)^2}$, as $\tan \psi = dy/dx$

or $\quad y^2 = k^2\,[1 + (dy/dx)^2] \quad$ or $\quad k^2\,(dy/dx)^2 = y^2 - k^2$

or $$k\,\frac{dy}{dx} = \sqrt{y^2 - k^2} \quad \text{or} \quad \frac{dy}{\sqrt{y^2 - k^2}} = \frac{1}{k}\,dx.$$

Integrating, $\cosh^{-1}(y/k) = x/k + c/k$, c being an arbitrary constant

$\therefore \quad y = k \cosh \{(x + c)/k\}$ is the required curve.

Ex. 14. *A curve passes through (2, 1) and is such that the square of the ordinate is twice the rectangle contained by the abscissa and x-intercept of the normal. Find its equation.*

Sol. Refer Fig. 2.1 of Art. 2.30. By problem, we have

$$y^2 = 2x \cdot OG \quad \text{or} \quad y^2 = 2x\,[x + y\,(dy/dx)]$$

or $\quad 2xy\dfrac{dy}{dx} = y^2 - 2x^2 \quad$ or $\quad \dfrac{dy}{dx} = \dfrac{y^2 - 2x^2}{2xy} = \dfrac{(y/x)^2 - 2}{2(y/x)}$ ...(i)

Now solve this homogeneous equation. **Ans.** $2y^2 + 4x^2 - 9x = 0$.

Ex. 15. *The tangent at a point P of a curve meets the axis of y in K and a line through P parallel to the axis of y meets the axis of x at N, O is the origin. If the area of triangle KON is constant, show that the curve is a hyperbola.*

Sol. Refer Figure 2.1 of Art. 2.30. Given that, area of triangle KON = constant = a, (say)

$\therefore \quad (1/2) \times OK \times ON = a \quad$ or $\quad (1/2) \times \{y - x\,(dy/dx)\} \times x = a.$

or $\quad xy - x^2\,(dy/dx) = 2a \quad$ or $\quad (dy/dx) - (1/x)\,y = -(2a/x^2)$,... (1)

which is linear. Its integrating factor = $e^{\int(-1/x)\,dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = 1/x$ and solution is

$$y(1/x) = c + \int(-2a/x^2)(1/x)\,dx = c - 2a\,\{x^{-2}/(-2)\}$$

or $\quad xy = cx^2 + a \quad$ or $\quad cx^2 - xy + a = 0.$ ... (2)

Comparing (2) with $Ax^2 + 2Hxy + By^2 + 2Gx + 2Fy + C = 0$, here, we have

$A = c, \quad H = -1/2, \quad B = 0,$ and therefore $\quad H^2 - AB = (1/4) - 0 = (1/4) > 0,$

and hence (2) must represent a hyperbola.

Ex. 16. *Find the curve in which the length of the arc measured from a fixed point A to any point P is proportional to the square root of the abscissa of P.*

Sol. Let $P\,(x, y)$ be any point on the curve UPV as show in Fig. 2.1 of Art. 2.30. Then, if arc $UP = s$, we have $s \propto x^{1/2}$ so that $\quad s = kx^{1/2}$, where k is a constant. ... (1)

Differentiating (1), $\quad ds/dx = k/2x^{1/2} \quad$ or $\quad \{1 + (dy/dx)^2\}^{1/2} = k/2x^{1/2}$

or $\quad 1 + (dy/dx)^2 = k^2/4x = a/x$, where $a = k^2/4$

or $\quad (dy/dx)^2 = (a - x)/x \quad$ or $\quad dy = \{(a - x)/x\}^{1/2}\,dx.$ ... (2)

Putting $x = a\sin^2\theta$ so that $dx = 2a\sin\theta\cos\theta\,d\theta$, (1) gives

$$dy = \{a\,(1 - \sin^2\theta)/a\sin^2\theta\}^{1/2}\,(2a\sin\theta\cos\theta)\,d\theta = a\,(2\cos^2\theta)\,d\theta$$

$\therefore \quad \int dy = a\int(1 + \cos 2\theta)\,d\theta \quad$ or $\quad y = c + a\,\{\theta + (1/2)\sin 2\theta\}$

or $\quad y = c + a\,(\theta + \sin\theta\cos\theta) = c + a\,[\theta + \sin\theta\,(1 - \sin^2\theta)^{1/2}]$

or $\quad y = c + a\,[\sin^{-1}(x/a)^{1/2} + (x/a)^{1/2}\,(1 - x/a)^{1/2}]$, as $\sin\theta = (x/a)^{1/2}$

or $\quad y = c + a\sin^{-1}(x/a)^{1/2} + (ax - x^2)^{1/2}$., c being an arbitrary constant.

Ex. 17. *Find the equation of curve for which the cartesian subtangent varies as the reciprocal of the square of the abscissa.*

Sol. Here the sub-tangent varies as $1/x^2$. So (Sub-tangent) $x^2 = k$, where k is a constant

or $\quad y\,(dx/dy)\,x^2 = k \quad$ or $\quad x^2dx = (k/y)dy$

Integrating, $x^3/3 = k\log y + c$, where c is an arbitrary constant.This gives the desired curve.

Ex. 18. *Find the curve in which the length of the portion of the normal intercepted between the curve and the x-axis varies as the square of the ordinate.*

Sol. We know that the length of the portion of the normal intercepted between the curve and the x-axis is also called the length of normal. Thus, the length of normal varies as y^2, i.e.,

$y\{1 + (dy/dx)^2\}^{1/2} = ky^2 \quad$ or $\quad 1 + (dy/dx)^2 = k^2y^2$

$\dfrac{dy}{dx} = \pm(k^2y^2 - 1)^{1/2} \quad$ or $\quad dx = \pm\dfrac{dy}{(k^2y^2 - 1)^{1/2}}$

Integrating, $\quad x = c \pm (1/k)\cosh^{-1}(ky)$, c being an arbitrary constants.

Ex. 19. *Find the family of curves whose tangent form an angle $\pi/4$ with the hyperbolas $xy = c$.* **[I.A.S. 1994, 2006]**

Sol. Here the required angle is given by

$$\tan^{-1}\left(\frac{m_1 - m_2}{1 + m_1 m_2}\right) = \frac{\pi}{4} \quad \text{or} \quad \tan\left(\frac{\pi}{4}\right) = \frac{m_1 - m_2}{1 + m_1 m_2} \quad \ldots (1)$$

where $m_1 = dy/dx$ for the required family at (x, y)

and m_2= value of the dy/dx for the second curve $(xy = c) = -c/x^2$, as $y = c/x \Rightarrow dy/dx = -(c/x^2)$

Putting values of m_1 and m_2 in (1), we get

$$1 = \frac{\dfrac{dy}{dx} + \dfrac{c}{x^2}}{1 - \dfrac{c}{x^2} \times \dfrac{dy}{dx}} \quad \text{or} \quad 1 - \left(\frac{c}{x^2} \times \frac{dy}{dx}\right) = \frac{dy}{dx} + \frac{c}{x^2} \quad \text{or} \quad \left(1 + \frac{c}{x^2}\right)\frac{dy}{dx} = 1 - \frac{c}{x^2}$$

or $$dy = \frac{x^2 - c}{x^2 + c}\,dx \quad \text{or} \quad dy = \left[\frac{x^2 + c - 2c}{x^2 + c}\right]dx = \left[1 - \frac{2c}{x^2 + c}\right]dx.$$

Integrating, $$y = x - 2c\,(1/\sqrt{c})\tan^{-1}(x/\sqrt{c}) + c'$$

or $y = x - 2\sqrt{c}\tan^{-1}(x/\sqrt{c}) + c'$, where c' is an arbitrary constant.

Ex. 20. *Show that the curve in which the angle between the tangent and the radius vector at any point is half of the vectorial angle is a cardioid.* **[Kurukshetra 1993; Magadh 1993]**

Sol. Here $$\phi = \frac{\theta}{2} \Rightarrow \tan\phi = \tan\frac{\theta}{2} \Rightarrow r\frac{d\theta}{dr} = \tan\frac{\theta}{2},$$

Separating variables, $(1/r)\,dr = \cot(\theta/2)\,d\theta$

Integrating, $\log r = 2\log\sin(\theta/2) + \log c$ or $r = c\sin^2(\theta/2)$, *i.e.*, $r = (c/2) \times (1 - \cos\theta)$

i.e., $r = c'(1 - \cos\theta)$, where $c' = c/2$ and it represents a cardioid.

Ex. 21. *Find the curve for which the sum of the radius vector and the subnormal varies as the square of the radius vector.*

Sol. Here given that (radius vector + subnormal) $= kr^2$, where k is a constant

or $r + (dr/d\theta) = kr^2$ or $(dr/d\theta) = r(kr - 1)$.

or $$d\theta = \frac{dr}{r(kr-1)} = \left(\frac{k}{kr-1} - \frac{1}{r}\right)dr,$$ resolving into partial fractions

Integrating, $\theta = \log(kr - 1) - \log r - \log c$, c being an arbitrary constant

or $\log\{(kr - 1)/rc\} = \theta$ or $kr - 1 = rc\,ce^{\theta}$.

Ex. 22(*a*). *Find the curve in which the angle between the radius vector and the tangent is n times the vectorial angle. What is the curve when n = 1, n = 1/2?*

Sol. Here given that $\phi = n\theta$ so that $\tan\phi = \tan n\theta$

or $r(d\theta/dr) = \tan n\theta$ or $(1/r)\,dr = \cot n\theta\,d\theta$.

Integrating, $\log r = (1/n)\log\sin n\theta + (1/n)\log c$, being an arbitrary constant

or $n\log r - \log c = \log\sin n\theta$ or $r^n = c\sin n\theta$.... (1)

When $n = 1$, (1) becomes $r = c\sin\theta$, which is a circle.

When $n = 1/2$, (1) becomes $r^{1/2} = c\sin(\theta/2)$ or $r = c^2\sin^2(\theta/2)$ or $r = c^2(1 - \cos\theta)/2$ which is equation of a cardioid.

Ex. 22(*b*). *Find the equation of the curve in which the angle between the radius vector and tangent is supplementary of half the vectorial angle.*

Sol. Given that $\phi = \pi - (\theta/2)$ so that $\tan\phi = \tan(\pi - \theta/2)$

or $r\,(d\theta/dr) = -\tan\theta/2$ or $(1/r)\,dr + \cot(\theta/2)\,d\theta = 0.$

Integrating, $\log r + 2\log\sin(\theta/2) = \log c$ or $\log r + \log\sin^2(\theta/2) = \log c$

or $r\sin^2(\theta/2) = c$ or $r(1 - \cos\theta)/2 = c$

or $(2c)/r = 1 - \cos\theta$, which is polar equation of parabola.

Ex. 23. *Find the curve for which the length of the perpendicular from the pole to the tangent varies as the radius vector.*

Sol. If 'p' is length of the perpendicular from the pole on the tangent at any point $P(r, \theta)$ on the curve, then $p = r\sin\phi,$ ... (1)

where ϕ is the angle between the tangent and the radius vector at P.

According to the given condition, $p = kr$, where k is constant. ... (2)

(1) and (2) $\Rightarrow$ $kr = r\sin\phi$ $\Rightarrow$ $\sin\phi = k.$ $\Rightarrow$ $\tan\phi = \text{constant} = a$, (say)

or $r\,(d\theta/dr) = a,$ as $\tan\phi = r\,(d\theta/dr)$

or $(1/r)\,dr = (1/a)\,d\theta$ so that $\log r = \log c + (1/a)\,\theta$ or $r = ce^{\theta/a}.$

Ex. 24. *The tangent at any point P of a curve meets the x-axis in Q. If Q is on the positive side of the origin O and OP = OQ, show that the family of curves having this property are parabolas whose common axis is the x-axis.*

Sol. Here TQ is the tangent at $P(r, \theta)$.

Also $\angle POQ = \theta$ and $\angle OPQ = \phi$.

Given that $OP = OQ$ so that $\angle OQP = \angle OPQ = \phi$.

Now, in $\Delta\, OPQ$, $\theta + \phi + \phi = 180°$ or $2\phi = 180° - \theta$

Thus, $\phi = 90° - (\theta/2).$

$\therefore$ $\tan\phi = \tan\{90° - (\theta/2)\} = \cot(\theta/2).$

or $r\,(d\theta/dr) = \cot(\theta/2)$ or $(1/r)\,dr = \tan(\theta/2)\,d\theta.$

Integrating, $\log r = \log c - 2\log\cos(\theta/2),$

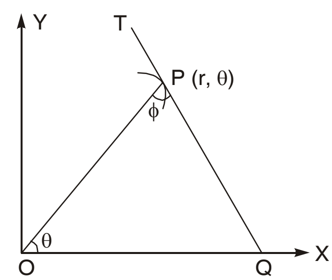


Fig. 2.4

where c in an arbitrary constant.

or $\log r + \log\cos^2(\theta/2) = \log c$ or $\log\{r\cos^2(\theta/2)\} = \log c$

or $r\cos^2(\theta/2) = c$ or $r(1 + \cos\theta)/2 = c$ or $2c/r = 1 + \cos\theta,$

which is the standard polar equation of a family of parabolas with initial line (x-axis) as the common axis, c being the parameter.

Ex. 25. *A point moves in a fixed straight path so that $s = t^{1/2}$, show that the acceleration is negative and proportional to the cube of the velocity.*

Sol. Given that $s = t^{1/2}.$ ... (1)

(1) $\Rightarrow$ $ds/dt = (1/2) \times t^{-1/2}$ or $v = (1/2) \times t^{-1/2}$.... (2)

$\therefore$ (2) $\Rightarrow$ $dv/dt = -(1/4) \times t^{-3/2}$ $\Rightarrow$ $f = -(1/4) \times (t^{-1/2})^3$

$\therefore$ $f = -(1/4) \times (2v)^3$, as by (2), $t^{-1/2} = 2v$

showing that acceleration f is – ve and proportional to cube of velocity v.

Ex. 26(*a*). *Particle moving in a straight line is subject to a resistance which produces retardation kv^3, where v is the velocity. Show that v and t are given in terms of s by the equations $v = u/(1 + ksu)$ and $t = (1/2)\,ks^2 + (s/u)$, where u is the initial velocity.*

Sol. Given that retardation $= kv^3$ so that $v\,(dv/ds) = -kv^3$ or $v^{-2}\,dv = -k\,ds.$

Integrating, $-(1/v) = -ks + A$, where A is an arbitrary constant ... (1)

But initially when $s = 0$, $v = u$, so from (1), $A = -(1/u.)$

Hence (1) becomes $$-\frac{1}{v} = -ks - \frac{1}{u} = -\frac{kus+1}{u} \quad \text{or} \quad v = \frac{u}{1+kus} \qquad \text{... (2)}$$

Since $v = (ds/dt)$, so (2) becomes $ds/dt = u/(1 + kus)$

or $dt = (1/u)(1 + kus)\, ds = \{(1/u) + ks\}\, ds.$

Integrating, $t = (s/u) + (1/2)\, ks^2 + B$, where B is an arbitrary constant. .. (3)

But initially, when $t = 0$, $s = 0$, so from (3), $B = 0$. Then from (3), $t = (1/2)\, ks^2 + (s/u)$

Ex. 26(*b*). *A particle is projected with velocity u along a smooth horizontal plane in a medium whose resistance per unit mass is k (velocity), show that the velocity after a time t and the distance s in that time are given by* $v = ue^{-kt}$ *and* $s = u\,(1 - e^{-kt})/k$.

Hints: Here $f = -kv$. Now proceed as in Ex. 26(*a*).

Ex. 27(*a*). *If the population of a country doubles in 50 years, in how many years will it treble under the assumption that the rate of increase is proportional to the number of inhabitants.*

[Delhi B.Sc. I (Hons) 2010; Delhi Maths (H) 1995, 1998, 2000, 08]

Sol. Let the population be x at time t (in years) and x_0 be the population when $t = 0$. Then, given that dx/dt is proportional to x, *i.e.*,

$dx/dt = kx$, k being the constant of proportionality, ... (1)

$$(1) \Rightarrow (1/x)\, dx = k\, dt \Rightarrow \int (1/x)\, dx = \int k\, dt \Rightarrow \log x - \log c = kt.$$

$\therefore \log (x/c) = kt$ so that $x = ce^{kt}$.... (2)

By our assumption, when $t = 0$, $x = x_0$ so that

(2) $\Rightarrow x_0 = c$ and then (2) $\Rightarrow x = x_0\, e^{kt}$.... (3)

Given $x = 2x_0$ when $t = 50$, so (3) yields

$$2x_0 = x_0\, e^{50k} \Rightarrow 50k = \log 2 \Rightarrow k = (\log 2)/50. \qquad \text{... (4)}$$

Next, let the population treble in t' years.

$\therefore$ From (3), $3x_0 = x_0\, e^{kt'} \Rightarrow kt' = \log 3 \Rightarrow t' = (\log 3)/k = (50 \log 3)/\log 2$, by (4)

or $t' = (50 \times .47712)/.30103 = 78.25$ years.

Ex. 27(*b*). *The number of bacteria in a yeast culture grows at a rate which is proportional to the number present. If the population of a colony of yeast bacteria triples in 1 hour, find the number of bacteria which will be present at the end of 5 hours.* **[Delhi Maths (H) 1993]**

Sol. Suppose that the number of bacteria is x_0 when $t = 0$, and it is x at time t (in hours). Then given that dx/dt is proportional to x, *i.e.*,

$(dx/dt) = kx$, k being the constant of proportionality, ... (1)

$$(1) \Rightarrow (1/x)\, dx = k\, dt \Rightarrow \int (1/x)\, dx = k \int dt \Rightarrow \log x - \log c = kt.$$

$\therefore$ $\log (x/c) = kt$ so that $x = ce^{kt}$.... (2)

By our assumption, when $t = 0$, $x = x_0$ Therefore,

(2) $\Rightarrow x_0 = c$ and then (2) $\Rightarrow$ $x = x_0\, e^{kt}$. ... (3)

Given $x = 3x_0$ when $t = 1$, so (3) yields $3x_0 = x_0\, e^k \Rightarrow e^k = 3.$ ... (4)

Next, let $x = x'$ when $t = 5$. Then (3) yields $x' = x_0\, e^{5k} = x_0\, (e^k)^5 = x_0 \cdot 3^5$, by (4)

Hence, the bacteria is expected to grow 3^5 times at the end of 5 hours.

Ex. 28. *Radium is known to decay at a rate proportional to the amount present. If the half life of radium is 1600 years, what percentage of radium will remain in a given sample after 800* *years? Also determine the number of years, after which only one-tenth of the original amount of radium would remain?* **[Delhi Maths (H) 1998]**

[**Note.** The time required to reduce a decaying material to one-half the original mass is called the half-life of the material]

Sol. Let $x(t)$ denote the amount of material present at time t. Then, according to given condition, $dx/dt = kx$, where k is negative constant

$\therefore \quad (1/x)\, dx = kt \qquad$ so that $\qquad \log x = kt + c. \qquad ...(1)$

Let x_0 be the amount of radium at $t = 0$ so that $x = x_0$, $t = 0$. Then (1) gives $c = \log x_0$ and hence (1) becomes

$\log x = kt + \log x_0 \qquad$ or $\qquad x = x_0\, e^{kt}. \qquad ...(2)$

Since half life of radium is 1600 years, so $x = x_0/2$ when $t = 1600$.

Then, $\qquad (2) \Rightarrow x_0/2 = x_0\, e^{1600\,k} \Rightarrow k = -(1/1600)\log 2. \qquad ...(3)$

Let $x = x'$ when $t = 800$ years. Then, from (2), $\qquad x' = x_0\, e^{800\,k} = x_0\, e^{-(1/2)\log 2}$, using (3)

$$\therefore \qquad x' = x_0\, e^{\log(2)^{-1/2}} = 2^{-1/2}\, x_0 = x_0/\sqrt{2} = 0.707\, x_0.$$

Hence the given percentage of radium that remains in given sample after 800 years

$$= (x'/x_0) \times 100 = (0.707) \times 100 = 70.7\%.$$

Second part. Proceed upto relation (3) as before. Let $t = t'$ when $x = x_0/10$. Then (2) gives

or $\quad \dfrac{x_0}{10} = x_0\, e^{kt'} \quad$ or $\quad kt' = \log\dfrac{1}{10} \quad$ or $\quad -\dfrac{t'}{1600}\log 2 = \log\dfrac{1}{10}$, by (3)

or $\qquad t' = [1600 \log 10]/\log 2 = 5317$ years, approximately.

Ex. 29(*a*). *According to Newton's law of cooling, the rate at which a substance cools in moving air is proportional to the difference between the temperature of the substance and that of the air. If the temperature of the air is 290 K and the substance cools from 370 K to 330 K in 10 minutes, find when the temperature will be 295 K.* **[Delhi Maths (H) 1994, 2005]**

Sol. Let T be the temperature of the substance at the time t (in minutes). Then, by hypothesis,

we have $\qquad \dfrac{dT}{dt} = -\lambda(T - 290) \qquad$ or $\qquad \dfrac{dT}{T-290} = -\lambda\, dt, \qquad ...(1)$

where λ is a positive constant of proportionality.

Integrating (1) between the limits $t = 0$, $T = 370\,K$ and $t = 10$ minutes, $T = 330$ K, we have

$$\int_{370}^{330} \frac{dT}{T-290} = -\lambda \int_0^{10} dt \qquad \text{or} \qquad \Big[\log(T-290)\Big]_{370}^{330} = -10\lambda$$

$$\Rightarrow -10\lambda = \log 40 - \log 80 \quad \Rightarrow \quad \lambda = (1/10)\log 2.$$

Again, assuming that $t = t'$ minutes when $T = 295\,K$ and so integrating (1) between the limits $t = 0$, $T = 370$ K and $t = t'$ (minutes), $T = 295\,K$, we have

$$\int_{370}^{295} \frac{dT}{T-290} = -\lambda \int_0^{t'} dt \qquad \text{or} \qquad \Big[\log(T-290)\Big]_{370}^{295} = -\lambda t'$$

$\Rightarrow -\lambda t' = \log 5 - \log 80 \quad \Rightarrow \quad \lambda t' = \log 16 \quad \Rightarrow \quad \lambda t' = 4\log 2$

$\Rightarrow [(1/10)\log 2]\, t' = 4\log 2$, using (2). So $t' = 40$ minutes.

Ex. 29(b). *A metal bar at a temperature of 100° F is placed in a room at a constant temperature of 0° F. If after 20 minutes the temperature of the bar is half, find an experssion for the temperature of the bar at any time.* **[Delhi Maths (H) 1996, 2006]**

Sol. Let T be the temperature of the substance at any time t (in minutes). Then, by Newton's cooling law, we have $(dT/dt) = -1\,(T - 0)$ or $(1/T)\,dT = -\lambda\,dt$.

Integrating, $\log T = -\lambda t + c$, c being an arbitrary constant. ... (1)

Initially, when $t = 0$, $T = 100$. So (1) gives $c = \log 100$.

Then, (1) becomes $\log T = -\lambda t + \log 100$ or $T = 100\, e^{-\lambda t}$. ... (2)

When $t = 20$ minutes, we are given that $T = 50$.

$\therefore$ From (2), $50 = 100\, e^{-20\lambda}$ so that $e^{-20\lambda} = (1/2)$

or $-20\lambda = \log (1/2) = -\log 2$ or $\lambda = (\log 2)/20 = 0.035$.

Then, from (2), $T = 100\, e^{(-0.035)\,t}$, as required.

Ex. 29(*c*). *A body whose temperature is initially 100°C is allowed to cool in air, whose temperature remains at a constant temperature 20°C. It is given that after 10 minutes, the body has cooled to 40°C. Find the temperature of the body after half an hour.* **[Delhi Maths (H) 2000]**

Sol. Let T be the temperature of the body in degree celsius and t be time in minutes. Then, by Newton's law of cooling, we get

$$\frac{dT}{dt} = -\lambda\,(T - 20) \qquad \text{or} \qquad \frac{dT}{T - 20} = -\lambda\, dt \quad ...(1)$$

where λ is a positive constant of proportionality.

Integrating (1), $\log (T - 20) - \log C = -\lambda t$ or $T = 20 + Ce^{-\lambda t}$ ... (2)

Initially, when $t = 0$, $T = 100$. So (2) gives $C = 80$

Then (2) reduces to $T = 20 + 80\, e^{-\lambda t}$ ... (3)

Given that $T = 40$ when $t = 10$. (3) gives $40 = 20 + 80\, e^{-10\lambda}$

or $80\, e^{-10\lambda} = 20$ or $e^{-10\lambda} = 4^{-1}$ or $e^{-\lambda} = (4^{-1})^{1/10}$

$\therefore$ (3) reduces to $T = 20 + 80\,(e^{-\lambda})^t = 20 + 80\,(4^{-1})^{t/10}$ ... (4)

Let $T = T'$ when t = half an hour = 30 minutes. Then (4) gives

$T = 20 + 80\,(4^{-1})^3 = 20 + 8\,(1/4^3) = 20 + 1.25 = 21.25$°C.

Ex. 30(*a*). *A certain radioactive material is known to decay at rate proportional to the amount present. If initially 500 mg of the material is present and after 3 years 20 per cent of the original mass has decayed, find an expression for the mass at any time.* **[Delhi Maths (H) 2001]**

Sol. Let $x\,(t)$ denote the amount of material present at any time t. Then according to the given condition, we have

$dx/dt = kx$ where k is a negative constant.

or $(1/x)\, dx = k\, dt$ so that $\log x = kt + c$... (1)

Let x_0 be the amount of the material at $t = 0$ so that $x = x_0$ when $t = 0$.

Then (1) gives $c = \log x_0$ and so (1) reduces to

$\log x = kt + \log x_0$ or $\log (x/x_0) = k$ or $x = x_0\, e^{kt}$... (2)

Also, given that when $t = 3$ years, $x = x_0 - (20/100)\, x_0 = (4/5)\, x_0$

Hence, (2) $\Rightarrow$ $(4/5) \times x_0 = x_0\, e^{3k}$ $\Rightarrow$ $e^{3k} = 0.8 \Rightarrow k = (1/3) \times \log (0.8) = -0.07438$

Putting $x_0 = 500$ and $k = -0.07438$ in (2), the required expression for mass at any time t is given by $x = 500\, e^{-0.07438\, t}$ mg

Ex. 30(b). *Assume that the rate at which radio active nuclei decay is proportional to the number of nuclei in the sample. In a certain sample 10% of the original number of radio active nuclei have undergone disintegration in a period of 200 years. (i) What percentage of the original radio active nuclei will remain after 1000 years (ii) In how many years will only one – fourth of the original number remain .* **[Delhi Maths (Prog) 2007]**

Sol. Let $x(t)$ denote the number of nuclei present in a sample at any time t. Then, according to the given condtion of the problem, we have

$$dx/dt = kx, \text{ where } k \text{ is a negative constant} \qquad \text{... (1)}$$

Rewriting (1), $\quad (1/x)\, dx = k\, dt \quad$ so that $\quad \log x = kt + c, \quad$ ... (2)

where c is an arbitrary constant. Let x_0 denote the number of nuclei present in the given sampel at time $t = 0$. Thus, we have $x = x_0$ when $t = 0$. Putting $t = 0$ and $x = x_0$ in (2), we have $c = \log x_0$. Hence, (2) reduces to

$$\log x = kt + \log x_0 \quad \text{or} \quad \log(x/x_0) = kt \quad \text{or} \quad x = x_o e^{kt} \qquad \text{... (3)}$$

Also, given that when $t = 200$ years, $x = x_0 - (10/100) \times x_0 = (9/10) \times x_0$.

Using this fact, $\quad (3) \Rightarrow \quad (9/10) \times x_0 = x_0\, e^{200k} \quad \Rightarrow \quad e^{200k} = 9/10 \quad$ ... (4)

Part (i): From (3), x' = the number of nuclei in the sampe after 1000 years $= x_0 e^{1000k}$

$\therefore$ Required percentage of nuclei after 1000 years.

$$= \frac{x'}{x_0} \times 100 = \frac{x_0 e^{1000k}}{x_0} \times 100 = \left(e^{200k}\right)^5 \times 100 = (9/10)^5 \times 100, \text{ using (4)}$$

$$= 9^5/10^3 = 59045/1000 = 59\% \text{ (approximately)}$$

Part (ii): Suppose that after t' years, there will remain only $x_0/4$ nuclei in the smaple, Then, from (3) we have

$$x_0/4 = x_0^{kt'} \Rightarrow kt' = \log(1/4) \Rightarrow kt' = -\log 4 \Rightarrow t' = -(\log 4/k) \qquad \text{...(5)}$$

Again, $\quad (4) \Rightarrow \quad 200k = \log(9/10) \quad \Rightarrow \quad k = (\log 0.9)/200 \quad$ ...(6)

From (5) and (6), $t' = -\dfrac{200 \log 4}{\log 0.9} = 2631$ years, on simplification and using log tables.

Ex. 30(*c*). *In a certain city the population gets doubled in 2 years and after 3 years the population is 20,000. Find the number of people initially being living in the city.*

[Delhi Maths (H) 2004]

Sol. Let the number of people initially living in the city be x_0 and let the population of the city at time t (in years) be x. Assuming that the rate of increase of population is proportional to x, we have $\quad dx/dt \propto x \quad$ or $\quad dx/dt = kx$, k being a constant.

or $\quad (1/x)\, dx = k\, dt \quad$ so that $\quad \log x - \log C = kt \quad$ or $\quad x = Ce^{kt} \quad$ ... (1)

By our assumption, when $t = 0$, $x = x_0$, so (1) gives $C = x_0$

Then (1) reduces to $\quad x = x_0\, e^{kt} \quad$ ... (2)

Given that $x = 2x_0$ when $t = 2$. So (2) gives

$$2x_0 = x_0\, e^{2k} \quad \text{or} \quad e^{2k} = 2 \quad \text{or} \quad (e^k)^2 = 2 \quad \text{or} \quad e^k = \sqrt{2} \ \text{... (3)}$$

Also, given that, $x = 20{,}000$, when $t = 3$. Hence (2) gives

$$20{,}000 = x_0\, e^{3k} \quad \text{or} \quad x_0\,(e^k)^3 = 20{,}000 \quad \text{or} \quad x_0(\sqrt{2})^3 = 20{,}000, \text{ by (3)}$$

$$\therefore \quad x_0 = \frac{20{,}000}{2 \times \sqrt{2}} = \frac{10{,}000}{1.414} = 7072.14 = 7072, \text{ nearly.}$$

Hence, initially 7072 people were living in the city.

Ex. 31. *Assume that a spherical rain drop evaporates at a rate proportional to its surface area. If its radius originally is 3 mm, and one hour later has been reduced to 2 mm, find an expression for the radius of the rain drop at any time.* **[I.A.S. 1997]**

Sol. Let r mm be the radius of the rain drop at time t hours from start. If V and S be volume and surface area of the rain drop, then we have

$V = (4/3)\,\pi r^3$ cubic mm and $S = 4\pi r^2$ sq. mm.... (1)

Given $dV/dt = -kS$, where k (> 0) is the constant of proportionality.

Using (1), this $\Rightarrow$ $4\pi r^2\,(dr/dt) = -k\,(4\pi r^2)$ or $dr = -k\,dt$

Integrating $r = -kt + c$, where c is an arbitrary constant. ... (2)

Now, initially when $t = 0$, $r = 3$ mm. Then (2) $\Rightarrow$ $c = 3$

Hence (2) reduces to $r = 3 - kt$ ... (3)

Again, given that $r = 2$ mm when $t = 1$ hour. Hence (3) reduces to $2 = 3 - k$ so that $k = 1$.

With $k = 1$, (3) reduces to $r = 3 - t$, which is the required expression for radius r at any time t.

Ex. 32. *A particle begins to move from a distance 'a' towards a fixed centre, which repels it with retardation μx. If its initial velocity is $a\sqrt{\mu}$, show that it will continually approach the fixed centre, but will never reach it.*

Sol. Since the particle is repelled from the centre O in the direction OX in which x increases and so by hypothesis $v(dv/dx) = \mu x$ or $2v/dv = 2\mu x\,dx$, ... (1)

where the distance x is measured from O.

Integrating (1) between the limits $v = -a\sqrt{\mu}$, $x = a$ and $v = v$, $x = x$, we have

$$\int_{-a\sqrt{\mu}}^{v} 2v\,dv = \int_a^x 2\mu x\,dx \qquad \text{or} \qquad \left[v^2\right]_{-a\sqrt{\mu}}^{v} = \mu\left[x^2\right]_a^x$$

or $v^2 - \mu a^2 = \mu x^2 - \mu a^2$ or $v = \pm\sqrt{\mu}\,x$. ... (2)

Since the particle is moving in the direction of x decreasing, v is negative.

$\therefore$ $v = -\sqrt{\mu}\,x$ or $dx/dt = -\sqrt{\mu}\,x$ or $(1/x)\,dx = -\sqrt{\mu}\,dt$.

Integrating, $\log x = -\sqrt{\mu}\,t + c$, c being an arbitrary constant. ... (3)

Initially, when $x = a$, $t = 0$, so from (3), $c = \log a$.

Hence (3) gives $\log x = -\sqrt{\mu}\,t + \log a$ or $\log(x/a) = -\sqrt{\mu}\,t$

or $t = -(1/\sqrt{\mu})\log(x/a)$. ... (4)

When the particle reaches at centre O, $x = 0$, then from (4), $t \to \infty$.($\because$ $\log 0 \to -\infty$ as $x \to 0$). Thus, we find that the particle does not reach the centre for any finite value of t.

Ex. 33. *Determine a family of curves for which the ratio of the y-intercept of the tangent to the radius vector is constant.* **[I.A.S. 1995]**

Sol. The equation of the tangent PT to the curve at $P\,(x, y)$ is

$$Y - y = (dy/dx)\,(X - x). \qquad \text{... (1)}$$

It meets y-axis at A (say), where $X = 0$ and $OA = Y$.

$\therefore$ from (1), $OA - y = (dy/dx)\,(0 - x)$ $\Rightarrow$ $OA = y - x\,(dy/dx)$.

By hypothesis, OA/OP = constant = k, say

$\therefore$ $y - x\,(dy/dx) = k\,(x^2 + y^2)^{1/2}$ $\Rightarrow$ $dy/dx = (y/x) - k\,(1 + y^2/x^2)^{1/2}$ ... (2)

Putting $y/x = v$ so that $y = xv$ and $dy/dx = v + x\,(dv/dx)$, ... (3)

(2) yields $v + x\dfrac{dv}{dx} = v - k(1+v^2)^{1/2}$ or $\dfrac{dv}{(1+v^2)^{1/2}} + k\dfrac{dx}{x} = 0$

Integrating, $\log\{v+(1+v^2)^{1/2}\}+k\log x=\log c$, c being an arbitrary constant.

or $\quad x^k\{v+(v^2+1)^{1/2}\}=c$ $\quad$ or $\quad x^k\{y/x+(y^2/x^2+1)^{1/2}\}=c$

Hence $\quad x^{k-1}\{y+(x^2+y^2)^{1/2}\}=c$.

Ex. 34. Show that the only curves having constant curvature are circles and straight lines.

Sol. By hypothesis, $\quad$ curvature $=\dfrac{(d^2y/dx^2)}{[1+(dy/dx)^2]^{3/2}}=k$, where k is constant. $\quad$ ... (1)

Putting $dy/dx=p$ so that $d^2y/dx^2=dp/dx$, (1) yields

$$\frac{(dp/dx)}{(1+p^2)^{3/2}}=k \quad \text{or} \quad k\,dx=\frac{dp}{(1+p^2)^{3/2}}$$

or $\quad k\,dx=\dfrac{\sec^2\theta\,d\theta}{(1+\tan^2\theta)^{3/2}}$ $\quad$ | on putting $p=\tan\theta$ and $dp=\sec^2\theta\,d\theta$

or $\quad k\,dx=\cos\theta\,d\theta$ $\quad$ so that $\quad kx=\sin\theta+c$

or $\quad kx-c=p/(1+p^2)^{1/2}$, as $p=\tan\theta \Rightarrow \sin\theta=p/(1+p^2)^{1/2}$

or $\quad 1+p^2=p^2/(px-c)^2$ $\quad$ or $\quad p^2[1-(kx-c)^2]=(kx-c)^2$

or $\quad \dfrac{dy}{dx}=\pm\dfrac{kx-c}{\sqrt{[1-(kx-c)^2]}}$ $\quad$ or $\quad dy=\pm\dfrac{(kx-c)\,dx}{\sqrt{[1-(kx-c)^2]}}$

or $\quad k\,dy=\pm\dfrac{z\,dz}{(1-z^2)^{1/2}}$, putting $kx-c=z$

Integrating, $\quad ky=\pm(1-z^2)^{1/2}$ $\quad$ or $\quad k^2y^2=1-z^2=1-(kx-c)^2$

or $\quad k^2x^2+k^2y^2-2kcx+c^2-1=0$, which represents a circle for all values of k except $k=0$.

If $k=0$, then from (1), we have $\quad \dfrac{d^2y}{dx^2}=0$ $\quad$ or $\quad \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right)=0$

Integrating it, $\quad (dy/dx)=a$, a being constant of integration $\quad$ ... (2)

Integrating (2), $\quad y=ax+b$, b being constant of integration

$y=ax+b$ is general equation of a straight line. Thus, we see that the only curves having constant curvature are circles or straight lines.

Ex. 35. *Find the curve for which sum of the reciprocals of the radius vector and the polar subtangent is constant.* **[I.A.S. 1996]**

Sol. Reciprocal of polar substangent $=\left(r^2\dfrac{d\theta}{dr}\right)^{-1}=\dfrac{1}{r^2}\dfrac{dr}{d\theta}$.

Given that $\dfrac{1}{r}+\dfrac{1}{r^2}\dfrac{dr}{d\theta}=k$, where k is constant $\quad$ or $\quad \dfrac{1}{r^2}\dfrac{dr}{d\theta}=k-\dfrac{1}{r}=\dfrac{kr-1}{r}$

or $\quad d\theta=\dfrac{dr}{r(kr-1)}$ $\quad$ or $\quad d\theta=\left(\dfrac{k}{kr-1}-\dfrac{1}{r}\right)dr$.

Integrating, $\theta+c=\log\{(kr-1)/r\}$ $\quad$ or $\quad kr-1=re^{\theta+c}$ $\quad$ or $\quad kr-1=c'r\,e^{\theta}$, where $c'\,(=e^c)$ is an arbitrary constant.

Exercise 2(N)

1. Show that the equation of the curve whose slope at any point is equal to $y+2x$ and which passes through the origin in $y=2(e^x-x-1)$. **(Guwahati 2007)**

2. Find the cartesian equation of the curve whose gradient at (x, y) is $y/2x$ and passes through the point $(a, 2a)$. **Ans.** $y^2 = 4ax$

3. Find the equations to the curves for which

(i) Cartesian sub-tangent is constant. **Ans.** $y = ke^{x/c}$

(ii) Cartesian sub-normal is constant. **Ans.** $y^2 = 2cx + k$

(iii) Polar sub-tangent is constant. **Ans.** $r(k - \theta) = c$

(iv) Polar sub-normal is constant. **Ans.** $r = c\theta + k$

4. Find the curve for which the angle between the tangent and the radius vector at anypoint is

(a) Constant ($= \alpha$). **Ans.** $r = ce^{\theta \cos \alpha}$

(b) Twice the vectorial angle. **Ans.** $r^2 = c \sin 2\theta$

5. Find the curves in which the cartesian subtangent varies as the abscissa. **Ans.** $x = cy^k$

6. Find the polar equation of the family of curves for which the sum of the radius vector and the polar subnormal varies as the kth power of the radius vector. **Ans.** $1/r^{k-1} = ce^{(k-1)\theta} + c'$

7. Find the curve for which the sum of the reciprocals of polar radius and polar subtangent at any point on it is of 5 units and the curve passes through a point whose polar coordinates are (1/2, 0). **Ans.** $5r - 1 = 3re^{\theta}$

8. Prove that all curves for which the square of the normal is equal to the square of the radius vector are either circles or rectangular hyperbolas.

9. Find the curve in which the length of the portion of the normal intercepted between the curve and the x-axis varies as the square of the ordinate. **Ans.** $ky = \cosh(kx + c)$

10. The population of a country doubles in 40 years. Assuming that the rate of increase is proportional to the number of inhabitants, find the number of years, in which it would triple (becomes three times) itself. **[Delhi B.A. (Prog) II 2010]** **Ans.** (40 log 3)/(log 2)

11. Air at temperature 200 K is passed over a substance at 300 K. The temperature of the substance cools down to 260 K in 30 minutes. Assuming that the rate at which a substance cools in moving air is proportional to the difference between the temperature of the substance and that of the air, find after what time the temperature of the substance would be 240 K. (K stands for Kelvin, a unit of measurement of temperature) **Ans.** 53.8 minutes

12. A steam boat is moving at velocity V when steam is shut off. Given that the retardation at any subsequent time is equal to the magnitude of the velocity at that time. Find the velocity and distance travelled in time t after the steam is shut off. **Ans.** $v = Ve^{-t}, x = V(1 - e^{-t})$

13. The rate of increase of bacteria in a culture is proportional to the number of bacteria present and it is found that the number doubles in 5 hours. Express this mathematically, using rate of increase of bacteria with respect to time. Hence, calculate how many times the bacteria may be expected to grow at the end of 15 hours. **Ans.** 8 times

14. Determine the curve for which the radius of curvature is proportional to the slope of the tangent. **[I.A.S. 1993]**

15. A thermometer reading 18°F is brought into a room, the temperature of which is 70°F. One minute later, the thermometer reading is 31°F. Determine the thermometer reading as a function of time. **[Kuvempa 2005]**

16. Find the family of curves that intersect the family of spirals $r = a\theta$ at a constant angle α.

Ans. $\log (r/c) - \log (\theta + \tan \alpha)^{\sec \alpha} + \theta \tan \alpha = 0$

2.32 Some typical examples on Chapter 2

Ex. 1. *Solve* $(dy/dx) - x \tan (y - x) = 1$.

Sol. Given $(dy/dx) - x \tan (y - x) = 1$. ... (1)

Putting $y - x = v$ so that $(dy/dx) - 1 = dv/dx$ or $dy/dx = 1 + (dv/dx)$, (1) becomes

$1 + (dv/dx) - x \tan v = 1$ or $dv/dx = x \tan v$ or $\cot v \, dv = x \, dx$.

Integrating, $\log \sin v = x^2/2 + \log c$ or $\log \sin (y - x) - \log c = x^2/2$

or $\log \{\sin (y - x)/c\} = x^2/2$ or $\sin (y - x) = ce^{x^2/2}$, c being an arbitrary constant.

Ex. 2. *Solve* $x (dy/dx) - y = x (x^2 + y^2)^{1/2}$ **[Kolkata 2003]**

Sol. Given $dy/dx = \{y + (x^2 + y^2)^{1/2}\}/x$ or $dy/dx = (y/x) + (x^2 + y^2)^{1/2}$. ... (1)

Putting $y = vx$ so that $dy/dx = v + x (dv/dx)$, (1) becomes

$$v + x\frac{dv}{dx} = v + (x^2 + x^2 y^2)^{1/2} \quad \text{or} \quad x\frac{dv}{dx} = (1 + v^2)^{1/2} \quad \text{or} \quad \frac{dv}{(1+v^2)^{1/2}} = dx.$$

Integrating, $\sinh^{-1} v = x + c$ or $v = \sinh (x + c)$ or $y/x = \sinh (x + c)$.

Ex. 3. *If* $2\int v\,dx = v - \log_e(1 + v) + A$, *where v is a function of x which has value 0 when x = 0, prove that* $v = 2e^x \sinh x$.

Sol. Differentiating both sides of given equation, w.r.t. 'x', we get

$$2v = \frac{dv}{dx} - \frac{1}{1+v}\frac{dv}{dx} \quad \text{or} \quad 2v = \left(1 - \frac{1}{1+v}\right)\frac{dv}{dx} \quad \text{or} \quad \frac{1}{1+v}\,dv = 2dx.$$

Integrating, $\log (1 + v) - \log c = 2x$ or $1 + v = ce^{2x}$. ... (1)

we are given that when $x = 0$, $v = 0$. Hence (1) gives $1 = c$ and so (2) reduces to

$v = e^{2x} - 1 = e^x (e^x - e^{-x})$ or $v = 2e^x \sinh x$, as $\sinh x = (e^x - e^{-x})/2$

Ex. 4. *By the substitution* $y^2 = v - x$ *reduce the equation* $y^3 (dy/dx) + x + y^2 = 0$ *to the homogeneous form and hence solve the equation.*

Sol. Given $y^2 \cdot y\,(dy/dx) + x + y^2 = 0$. ... (1)

Putting $y^2 = v - x$ so that $2y\,(dy/dx) = (dv/dx) - 1$ or $y\,(dy/dx) = \{(dy/dx) - 1\}/2$,

(1) becomes $$(v - x)\cdot\frac{1}{2}\left(\frac{dv}{dx} - 1\right) + x + (v - x) = 0 \quad \text{or} \quad \frac{dv}{dx} = \frac{v + x}{x - v}$$

which is homogeneous. To solve (1) we proceed as usual.

Putting $v = ux$ so that $dv/dx = u + x (du/dx)$, (1) becomes

$$u + x\frac{du}{dx} = \frac{ux + x}{x - ux} \quad \text{or} \quad x\frac{du}{dx} = \frac{u+1}{1-u} - u = \frac{1+u^2}{1-u}$$

or $$\frac{1-u}{1+u^2}\,du = \frac{dx}{x} \quad \text{or} \quad \left[\frac{1}{1+u^2} - \frac{1}{2}\frac{2u}{1+u^2}\right]du = \frac{dx}{x}$$

Integrating, $\tan^{-1} u - (1/2) \times \log (1 + u^2) = \log x - \log c$

$\log x + \log (1 + u^2)^{1/2} - \log c = \tan^{-1} u$ or $x\,(1 + u^2)^{1/2} = ce^{\tan^{-1} u}$

or $x\,\{1 + (v/x)^2\}^{1/2} = ce^{\tan^{-1}(v/x)}$ or $\{x^2 + (y^2 + x)^2\}^{1/2} = ce^{\tan^{-1}(y^2+x)/x}$

Ex. 5(*a*). *Solve* $dy/dx = (x + y - 1)^2/4(x - 2)^2$. **[Srivenkateshwar 2003]**

Sol. Given $dy/dx = (x + y - 1)^2/4(x - 2)^2$. ... (1)

Put $x = X + h$ and $y = Y + k$ so that $dx = dX$ and $dy = dY$.... (2)

Then, from (1), $$\frac{dY}{dX} = \frac{(X + h + Y + k - 1)^2}{4(X + h - 2)^2} = \frac{(X + Y + h - k - 1)^2}{4(X + h - 2)^2} \quad \text{... (3)}$$

Choose h and k such that $h - k - 1 = 0$ and $h - 2 = 0$ so that $h = 2$ and $k = -1$.

Then, (2) $\Rightarrow$ $X = x - h = x - 2$ and $Y = y - k = y + 1$. ... (4)

Also, from (3), $dY/dX = (X + Y)^2/4X^2$ which is homogeneous ... (5)

Putting $Y = vX$ so that $dY/dX = v + X\,(dv/dX)$, (5) becomes

$$v + X\frac{dv}{dX} = \frac{(X + vX)^2}{4X^2} \quad \text{or} \quad X\frac{dv}{dX} = \frac{(v+1)^2}{4} - v = \frac{(1-v)^2}{4}.$$

Separating the variables, $4\,(1 - v)^{-2}\,dv = (1/X)\,dX$.

Integrating, $4\,(1 - v)^{-1} = \log X + c$ or $4\,(1 - Y/X)^{-1} = \log X + c$.

or $$\frac{4X}{X - Y} = \log X + c \quad \text{or} \quad \frac{4\,(x-2)}{x - y - 3} = \log(x - 2) + c, \text{ by (4)}$$

Ex. 5(*b*). *Solve* $dy/dx = (x + y - 1)/5\,(x - 2)^2$.

Ans. $Ye^{1/5X} = \int (1/5X)\,e^{1/5X}\,dX + c$ where $X = x - 2$, $Y = y + 1$.

Ex. 6. *The integration factor of the following equation is of the form* y^n. *Find* n *and hence solve the equation.* $y \sec^2 x\,dx + [3 \tan x - \{(\sec y)/y\}^2]\,dy = 0$. **[Kanpur 1992, 1993]**

Sol. Multiplying the given equation by y^n, we have

$$y^{n+1} \sec^2 x\,dx + \{3y^n \tan x - \sec^2 y \cdot y^{n-2}\}\,dy = 0. \qquad \text{... (1)}$$

Comparing (1) with $M\,dx + N\,dy = 0$, we have

$M = y^{n+1} \sec^2 x$ and $N = 3y^n \tan x - y^{n-2} \sec^2 y$... (2)

Since y^n is the I.F. of the given equation, so (1) must be exact and hence

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{or} \quad \frac{\partial}{\partial y}(y^{n+1} \sec^2 x) = \frac{\partial}{\partial x}[3y^n \tan x - y^{n-2} \sec^2 y]$$

or $(n + 1)\,y^n \sec^2 x = 3y^n \sec^2 x$ or $n + 1 = 3$ so that $n = 2$.

With $n = 2$, (1) gives, $y^3 \sec^2 x\,dx + (3y^2 \tan x - \sec^2 y)\,dy = 0$. ... (3)

Solution of exact equation (3) is given by

$$\underset{\text{[Treating } y \text{ as costant]}}{\int y^3 \sec^2 x\,dx} + \underset{\text{[Integrate terms free from } x\text{]}}{\int (-\sec^2 y)\,dy = c}$$

or $$y^3 \tan x - \int \sec^2 y\,dy = c \quad \text{or} \quad y^3 \tan x - \tan y = c.$$

Ex. 7. *Prove that* $1/(x + y + 1)^4$ *is an integrating factor of* $(2xy - y^2 - y)\,dx + (2xy - x^2 - x)\,dy = 0$, *and find the solution of this equation.*

Sol. Multiplying the given equation by $1/(x + y + 1)^4$, we have

$$(2xy - y^2 - y)\,(x + y + 1)^{-4}\,dx + (2xy - x^2 - x)\,(x + y + 1)^{-4}\,dy = 0. \qquad \text{... (1)}$$

Comparing (1) with $M\,dx + N\,dy = 0$, we have

$M = (2xy - y^2 - y)\,(x + y + 1)^{-4}$ and $N = (2xy - x^2 - x)\,(x + y + 1)^{-4}$

Now, $\partial M/\partial y = (2x - 2y - 1)\,(x + y + 1)^{-4} - 4\,(2xy - y^2 - y)\,(x + y + 1)^{-5}$

$= (x + y + 1)^{-5}\,\{(2x - 2y - 1)\,(x + y + 1) - 4\,(2xy - y^2 - y)\}$

$\Rightarrow$ $\partial M/\partial y = (x + y + 1)^{-5}\,(2x^2 + 2y^2 - 8xy + x + y + 1)$ ... (2)

and $\partial N/\partial x = (2y - 2x - 1)\,(x + y + 1)^{-4} - 4\,(2xy - x^2 - x)\,(x + y + 1)^{-5}$

$= (x + y + 1)^{-5}\,\{(2y - 2x - 1)\,(x + y + 1) - 4\,(2xy - x^2 - x)\}$

$\Rightarrow$ $\partial N/\partial x = (x + y + 1)^{-5}\,(2x^2 + 2y^2 - 8xy + x + y - 1)$. ... (3)

From (2) and (3), $\partial M/\partial y = \partial N/\partial x$ and so (1) is exact. Solution of (1) is

$$\underset{\text{[Treating } y \text{ as constant]}}{\int M\,dx} + \int (\text{terms free from } x \text{ in } N)\,dy = c$$

or $\displaystyle\int \frac{2xy - y^2 - y}{(x+y+1)^4}\,dx = c$ or $\displaystyle\int \frac{2y\,(x+y+1-y-1)-(y^2+y)}{(x+y+1)^4}\,dx = c$

or $$\int (x+y+1)^{-4}\{2y\,(x+y+1) - 2y\,(y+1) - (y^2+y)\}\,dx = c$$

or $$2y\int (x+y+1)^{-3}\,dx - [2y\,(y+1) + y\,(y+1)]\int (x+y+1)^{-4}\,dx = c \qquad \text{... (3)}$$

Intergrating (3) w.r.t. x while treating y as $c$onstant, we get.

or $\displaystyle\frac{y\,(y+1)}{(x+y+1)^3} - \frac{y}{(x+y+1)^2} = c$ or $\displaystyle\frac{y\,(y+1) - y\,(x+y+1)}{(x+y+1)^3} = c$

or $y^2 + y - xy - y^2 - y = c\,(x+y+1)^3$ or $c\,(x+y+1)^3 + xy = 0.$

Ex. 8(*a*). *Solve* $\displaystyle\frac{x\,dx + y\,dy}{x\,dy - y\,dx} = \sqrt{\left(\frac{a^2 - x^2 - y^2}{x^2 + y^2}\right)}$ **[Delhi Maths (H) 2009; I.A.S. 1999; Kumaun 1998; Garhwal 2010]**

Sol. We transform the given equation to polars, by taking $x = r\cos\theta$, $y = r\sin\theta$, so that $x^2 + y^2 = r^2$ ... (1)

and $y/x = \tan\theta.$ ... (2)

From (1), $2x\,dx + 2y\,dy = 2r\,dr$ or $x\,dx + y\,dy = r\,dr$ ... (3)

(2) $\Rightarrow$ $(x\,dy - y\,dx)\,x^2 = \sec^2\theta\,d\theta$ $\Rightarrow$ $x\,dy - y\,dx = r^2\,d\theta.$ ... (4)

Using (1), (3) and (4) the given equation reduces to

$(r\,dr)/(r^2\,d\theta) = \{(a^2 - r^2)/r^2\}^{1/2}$ or $d\theta = \{1/(a^2 - r^2)^{1/2}\}\,dr.$

Integrating, $\theta + c = \sin^{-1}(r/a)$ or $\tan^{-1}(y/x) + c = \sin^{-1}\{(x^2 + y^2)^{1/2}/a\}.$

Ex. 8(*b*). *Solve* $x\,dx + y\,dy = a^2\,(x\,dy - y\,dx)/(x^2 + y^2)$.

Sol. Let $x = r\cos\theta$ and $y = r\sin\theta$.... (1)

(1) $\Rightarrow$ $x^2 + y^2 = r^2$ and $y/x = \tan\theta$.... (2)

Now, (2) $\Rightarrow$ $2x\,dx + 2y\,dy = 2r\,dr$ and $(x\,dy - y\,dx)/x^2 = \sec^2\theta\,d\theta$

Thus, $x\,dx + y\,dy = r\,dr$ and $x\,dy - y\,dx = r^2\,d\theta$, as $x\sec\theta = r$

Then, the given equation becomes $r\,dr = (a^2r^2\,d\theta)/r^2$ or $2r\,dr = 2a^2\,d\theta.$

Integrating, $r^2 = 2a^2\,\theta + c$ or $x^2 + y^2 = 2a^2\tan^{-1}(y/x) + c$, by (2)

Ex. 9(*a*). *Show that the equation* $(4x + 3y + 1)\,dx + (3x + 2y + 1)\,dy = 0$ *represents a family of hyperbolas having as asymptotes the lines* $x + y = 0$ *and* $2x + y + 1 = 0$. **[I.A.S. 1998]**

Sol. Given $(4x + 3y + 1)\,dx + (3x + 2y + 1)\,dy = 0.$ ... (1)

Comparing (1) with $M\,dx + N\,dy = 0$, $M = 4x + 3y + 1$ and $N = 3x + 2y + 1$.

Here $\partial M/\partial y = 3 = \partial N/\partial x$ and so (1) is exact and as usual, its solution is given by

$$\int (4x + 3y + 1)\,dx + \int (2y + 1)\,dy = c$$

[Treating y as constant]

$2x^2 + 3xy + x + y^2 + y + c = 0$, c being arbitrary is constant ... (2)

Comparing (2) with standard form of conic section

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we have, $a = 2$, $h = 3/2$, $b = 1$.

Here $h^2 - ab = (9/4) - 2 =$ positive quantity $\Rightarrow$ (2) is hyperbola.

Since the equation of the hyperbola and asymptotes differ by *a* constant, so the combined equations of two asymptotes of the hyperbola (2) may be taken as

$$2x^2 + 3xy + y^2 + x + y + k = 0, \text{ where } k \text{ is some constant.} \quad \ldots (3)$$

Comparing (3) with standard equation of pair of lines $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$, we have $a = 2$, $h = 3/2$, $b = 1$, $g = 1/2$, $f = 1/2$, $c = k$.

Condition for (3) to represent two lines is $abc + 2fgh - af^2 - bg^2 - ch^2 = 0$

or $\quad 2k + 2 \cdot (1/2) \cdot (1/2) \cdot (3/2) - 2 \cdot (1/4) - 1 \cdot (1/4) - k \cdot (9/4) = 0 \Rightarrow k = 0.$

Hence the required equation of the two asymptotes of (2) is

$2x^2 + 3xy + y^2 + x + y = 0 \quad$ or $\quad (x + y)(2x + y + 1) = 0,$

showing that $x + y = 0$ and $2x + y + 1 = 0$ are the required asymptotes.

Ex. 9(*b*). *Show that the equation $(12x + 7y + 1)\,dx + (7x + 4y + 1)\,dy = 0$ represents a family of curves having as asymptotes the lines $3x + 2y - 1 = 0$ and $2x + y + 1 = 0$.*

Sol. Proceed as in Ex. 10.

Ex. 10. *Solve $dy/dx = (y - x)^{1/2}$.*

Sol. Given $\quad dy/dx = (y - x)^{1/2} \quad \ldots (1)$

Let $\quad y - x = v^2 \quad$ so that $(dy/dx) - 1 = 2v\,(dv/dx)$ or $(dy/dx) = 1 + 2v\,(dv/dx) \ldots (2)$

From (1) and (2), $\quad 1 + 2v\,\dfrac{dv}{dx} = v \quad$ or $\quad dx = \dfrac{2v\,dv}{v-1}$

or $\quad dx = \dfrac{2(v-1)+2}{v-1}\,dx \quad$ or $\quad dx = \left(2 + \dfrac{2}{v-1}\right)dv.$

Integrating, $\quad x + c = 2v + 2\log(v - 1)$, c being an arbitrary constant.

or $\quad x + c = 2(y - x)^{1/2} + 2\log\{(y - x)^{1/2} - 1\}$, using (2)

Ex. 11(*a*). *Solve $(y^2 + x^2 - a^2x)\,x\,dx + (y^2 + x^2 - b^2y)\,y\,dy = 0$*

Sol. Re-writing the given equation, $\quad (x^2 + y^2)(x\,dx + y\,dy) - a^2x^2\,dx - b^2y^2\,dy = 0$

or $\quad (1/2) \times (x^2 + y^2)(2x\,dx + 2y\,dy) - a^2x^2\,dx - b^2y^2\,dy = 0 \quad \ldots (1)$

Let $\quad x^2 + y^2 = z \quad$ so that $\quad 2x\,dx + 2y\,dy = dz \ldots (2)$

From (1) and (2), $\quad (1/2) \times z\,dz - a^2x^2\,dx - b^2y^2\,dy = 0$

Integrating, $\quad (z^2/4) - (1/3) \times (a^2x^3 + b^2y^3) = c/12$, being an arbitrary constant.

or $\quad 3z^2 - 4(a^2x^3 + b^2y^3) = c \quad$ or $\quad 3(x^2 + y^2)^2 - 4(a^2x^3 + b^2y^3) = c$, by (2)

Ex. 11(*b*). Solve $(a^2 - 2xy - y^2)\,dx = (x + y)^2\,dy$

Sol. Re-writing the given equation, $\quad \{(a^2 + x^2) - (x^2 + 2xy + y^2)\}\,dx = (x + y)^2\,dy$

or $\quad (a^2 + x^2)\,dx = (x + y)^2(dx + dy) \quad \ldots (1)$

Let $\quad x + y = z \quad$ so that $\quad dx + dy = dz \ldots (2)$

From (1) and (2), $\quad (a^2 + x^2)\,dx = z^2\,dz.$

Integrating, $\quad a^2x + (1/3) \times x^3 = (1/3) \times z^3 + c/3 \quad$ or $\quad 3a^2x + x^3 = z^3 + c$

or $\quad 3a^2x + x^3 = (x + y)^3 + c \quad$ or $\quad 3a^2x - 3x^2y - 3xy^2 - y^3 = c.$

MISCELLANEOUS PROBLEMS ON CHAPTER 2

1. Solve $(1 - x^2y^2)\,dx = y\,dx + x\,dy$. **[Rohilkhand 1994]**

   Hint: Put $xy = v$. **Ans.** $(1 + xy)/(1 - xy) = ce^{2x}$

2. Show that if y_1 and y_2 be solutions of the equation $dy/dx + Py = Q$, where P and Q are functions of x alone, and $y_2 = y_1 z$, then $z = 1 + ae^{(-Q/y_1)}\,dx$, *a* being an arbitrary constant.

OBJECTIVE PROBLEMS ON CHAPTER 2

Ex. 1. *The solution of* $(x - y^2)\,dx + 2xy\,dy = 0$ *is*
(a) $ye^{y^2/x} = A$ (b) $xe^{y^2/x} = A$ (c) $ye^{x/y^2} = A$ (d) $xe^{x/y^2} = A$. **[I.A.S. (Prel.) 1993]**

Sol. Ans. (*b*). Rewriting given equation, we have

$x - y^2 + 2xy\,(dy/dx) = 0$ or $2y\,(dy/dx) - (1/x)\,y^2 = -1$ ... (1)

Putting $y^2 = v$ so that $2y\,(dy/dx) = dv/dx$. ... (2)

(1) and (2) $\Rightarrow$ $(dv/dx) - (1/x)\,v = -1$, which is linear. ... (3)

Its I.F. $= e^{-\int(1/x)\,dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = 1/x$. and hence its solution is

$$v\,(1/x) = \int (-1)\,(1/x)\,dx = -\log x + \log A, \text{ A being an arbitrary constant.}$$

or $v/x = \log(A/x)$ or $A/x = e^{v/x}$ or $A = xe^{y^2/x}$, by (2).

Ex. 2. *The solution of* $(dy/dx) + y\,(d\phi/dx) = \phi\,(x)\,(d\phi/dx)$ *is* (a) $y = \phi\,(x) - 1 + ce^{-\phi}$
(b) $y = ce^{\phi}$ (c) $y = x\phi\,(x) - ce^{-\phi}$ (d) $y = [\phi\,(x) - 1]\,e^{-\phi} + c$. **[I.A.S. (Prel.) 1993]**

Sol. Ans. (*a*). Given equation is linear whose I.F. $= e^{\int(d\phi/dx)dx} = e^{\phi}$.

$\therefore$ Solution is $$ye^{\phi} = \int e^{\phi}\,\phi(x)\,\frac{d\phi}{dx}\,dx = \int e^{\phi}\,\phi\,d\phi = \phi e^{\phi} - \int e^{\phi}\,d\phi$$

or $e^{\phi}\,y = \phi e^{\phi} - e^{\phi} + c$ or $y = \phi\,(x) - 1 + ce^{-\phi}$.

Ex. 3. *The solution of* $dy/dx + 2xy = e^{-x^2}$ is
(a) $ye^{-x^2} = x + c$ (b) $ye^{x^2} = x + c$ (c) $x\,e^{x^2} = y + c$ (d) $xe^{y^2} = x + c$ **[Garhwal 2010]**

Hint. Ans. (*b*). Proceed as in Art. 2.24, page 2.33

Ex. 4. *Differential equation* $x\,dy - y\,dx - 2x^3\,dx = 0$ *has the solution*
(a) $y + x^3 = c_1 x$ (b) $-y + x^3 = c_2 x$, (c) $y - x^2 = c_3 x$, (d) $y^3 - x^3 = c_4 x$ **[I.A.S. (Prel.) 1993]**

Sol. Ans. (*b*). Re-write given equation as $(dy/dx) - (1/x)\,y = 2x^2$,
which is linear whose I.F. $= e^{-\int(1/x)dx} = e^{-\log x} = 1/x$ and solution is

$y/x = \int (1/x)(2x^2)\,dx + c = x^2 + c$ or $-y + x^3 = c_2 x$, where $c_2 = -c$.

Ex. 5. *Primitive* $(2xy^4 e^y + 2xy^3 + y)\,dx + (x^2y^4 e^y - x^2y^2 - 3x)\,dy = 0$, is
(a) $x^2e^y + (x^2/y) + (x/y^3) = c$ (b) $x^2e^v - (x^2/y) + (x/y^3) = c$
(c) $x^2e^v + (x^2/y) - (x/y^3) = c$ (d) $x^2e^v - (x^2/y) - (x/y^3) = c$. **[I.A.S. (Prel.) 1994]**

Sol. Ans. (*a*). Dividing throughout by y^4, we have

$$\{2xe^y + (2x/y) + (1/y^3)\}\,dx + \{x^2e^y - (x^2/y^2) - (3x/y^4)\}\,dy = 0.$$

$$(2xe^y\,dx + x^2e^y\,dy) + \left(\frac{2x\,dx}{y} - \frac{x^2\,dy}{y^2}\right) + \left(\frac{dx}{y^3} - \frac{3x\,dy}{y^4}\right) = 0$$

or $$d\,(x^2e^x) + d\,(x^2/y) + d\,(x/y^3) = 0.$$

Integrating, $x^2e^x + (x^2/y) + (x/y^3) = c$, c being an arbitrary constants.

Ex. 6. *The solution of* $(dy/dx) + 2xy = 2xy^2$, *is* (a) $y = (cx)/(1 + e^{-x^2})$ (b) $y = 1/(1 - ce^x)$
(c) $y = 1/(1 + ce^{x^2})$ (d) $y = (cx)/(1 + e^{x^2})$. **[I.A.S. (Prel.) 1994]**

Sol. Ans. (*c*). Dividing by y^2, $y^{-2}\,(dy/dx) + 2x\,y^{-1} = 2x$. ... (1)

Put $y^{-1} = v$ so that $-y^{-2}\,(dy/dx) = dv/dx$, ... (2)

$\therefore$ (1) $\Rightarrow$ $-(dv/dx) + 2xv = 2x$ or $(dv/dx) - 2xv = -2x$, ... (3)

which is linear whose I.F. $= e^{-2\int x\,dx} = e^{-x^2}$. and hence required solution is

$$ve^{-x^2} = \int (-2x)\, e^{-x^2} dx = \int e^t\, dt, \text{ putting } (-x^2) = t$$

or $\quad y^{-1} e^{-x^2} = e^t + c = e^{-x^2} + c,\quad$ or $\quad y^{-1} = 1 + ce^{x^2}\quad$ or $\quad y = 1/(1 + ce^{x^2})$.

Ex. 7. *The solution of* $(x + y)^2\, (dy/dx) = a^2$ *is given by*

(a) $y + x = a \tan \{(y - c)/a\}$ (b) $y - x = a \tan (y - c)$

(c) $y - x = \tan \{(y - c)/a\}$ (d) $a\, (y - x) = \tan \{(y - c)/a\}$. **[I.A.S. (Prel.) 1994]**

Sol. Ans. (*a*). Refer solved example 2, Art. 2.5.

Ex. 8. *A solution curve of the equation* $xy' = 2y$, *passing through* $(1, 2)$, *also passes through*

(a) $(2, 1)$ (b) $(0, 0)$ (c) $(4, 24)$ (d) $(24, 4)$. **[I.A.S. (Prel.) 1995]**

Sol. Ans. (*b*). Given $x\, (dy/dx) = 2y$ or $(1/y)\, dy = (2/x)\, dx$.

Integrating, $\log y = 2 \log x + \log c$ or $y = cx^2$.... (1)

Since (1) passes through (1, 2), so putting $x = 1, y = 2$ in (1), we have $2 = c$ and so (1) becomes $y = 2x^2$ which clearly passes through (0, 0).

Ex. 9. *An integral curve of* $x\, (4y\, dx + 2x\, dy) + y^3\, (3y\, dx + 5x\, dy) = 0$ *is* (a) $x^4y^2 + x^3y^5 = 1$

(b) $x^4y^2 + x^3y^4 = 1$ (c) $x^3y^3 + x^4y^3 = 1$ (d) $x^2y^4 + x^3y^4 = 1$ **[I.A.S. (Prel.) 1995]**

Sol. Ans. (*a*). Do as explained in Art. 2.22.

Ex. 10. *The integrating factor of* $y^2 dx + (1 + xy)\, dy = 0$ *is*

(a) e^y (b) e^x (c) e^{xy} (d) e^{-xy}. **[I.A.S. (Prel.) 1995]**

Sol. Ans. (*c*). Multiplying given equation by e^{xy}, $\quad y^2 e^{xy}\, dx + e^{xy}\, (1 + xy)\, dy = 0$. ... (1)

Comparing (1) with $M\, dx + N\, dy = 0$, $\quad M = y^2 e^{xy}$, $\quad N = e^{xy}\, (1 + xy)$.

$\therefore\quad \partial M/\partial y = e^{xy}\, xy^2 + e^{xy} \cdot 2y = e^{xy}\, (xy^2 + 2y)$, $\quad \partial N/\partial x = e^{xy}\, y\, (1 + xy)\, e^{xy}\, y = e^{xy}\, (xy^2 + 2y)$.

Since $\partial M/\partial y = \partial N/\partial x$, so (1) is exact and hence e^{xy} is I.F.

Ex. 11. *The homogeneous differential equation* $M\, (x, y)\, dx + N\, (x, y)\, dy = 0$ *can reduced to a differential equation, in which the variables are separated, by the substitution.*

(a) $y = vx$ (b) $xy = v$ (c) $x + y = v$ (d) $x - y = v$. **[I.A.S. (Prel.) 1996]**

Sol. Ans. (*a*). Refer Art. 2.7.

Ex. 12. *The solution of the differential equation* $(dy/dx) + (y/x) = x^2$ *under the condition that* $y = 1$ *when* $x = 1$ *is* (a) $4xy = x^3 + 3$ (b) $4xy = y^4 + 3$ (c) $4xy = x^2 + 3$ (d) $4xy = y^3 + 3$.

[I.A.S. (Prel.) 1996]

Sol. Ans. (*a*). The given equation is linear equation. Try yourself.

Ex. 13. *The differential equation* $M\, (x, y)\, dx + N\, (x, y)\, dy = 0$ *is an exact equation if*

(a) $\dfrac{\partial M}{\partial y} + \dfrac{\partial N}{\partial x} = 0$ (b) $\dfrac{\partial M}{\partial y} - \dfrac{\partial N}{\partial x} = 0$ (c) $\dfrac{\partial N}{\partial y} + \dfrac{\partial M}{\partial x} = 0$ (d) $\dfrac{\partial N}{\partial y} - \dfrac{\partial M}{\partial x} = 0$ **[I.A.S. (Prel.) 1997]**

Sol. Ans. (*b*). Refer Art 2.13.

Ex. 14. *The differential equation* $(4x + 3y + 1)\, dx + (3x + 2y + 1)\, dy = 0$ *represents a family of*

(a) *circles* (b) *parabolas* (c) *ellipses* (d) *hyperbolas*. **[I.A.S. (Prel.) 1997]**

Sol. Ans. (*d*). Refer Ex. 9(a), Art. 2.32.

Ex. 15. *The general solution of the differential equation* $(x^2 + y^2)\, dx - 2x\, dy = 0$ *is*

(a) $x^2 - cx - y^2 = 0$ (b) $(x - y)^2 = cx$ (c) $x + y + 2xy = c$ (d) $y = x^2 - 2x + c$. **[I.A.S. (Prel.) 1997]**

Sol. Ans. (*a*). Use Art. 2.7.

Ex. 16. *The differential equation* $y\, dx - 2x\, dy = 0$ *represents a family of*

(a) *straight lines* (b) *parabolas* (c) *circles* (d) *catenaries*. **[I.A.S. (Prel.) 1998]**

Sol. Ans. (*b*). The given equation is $(2/y)\,dy = (1/x)\,dx$. Integrating it, $2 \log y = \log x + \log c$ or $y^2 = cx$ which is a family of parabolas.

Ex. 17. *If c be an arbitrary constant, the general solution of the equation* $(x + 2y^3)(dx/dy) = y$ *is* (*a*) $x = cy - y^2$ (*b*) $x = cy + y^2$ (*c*) $x = cy + y^3$ (*d*) $x = cy - y^3$. **[I.A.S. (Prel.) 1998]**

Sol. Ans. (*b*). Re-writing the given equation, we have

$x + 2y^3 = y\,(dx/dy)$ or $dx/dy - (1/y)\,x = 2y^2$, which is linear.

Its IF $= e^{-\int (1/y)dy} = e^{-\log y} = e^{y^{-1}} = y^{-1} = 1/y$ and hence solution is

$$x \times (1/y) = \int (2y^2)(1/y)\,dy + c = y^2 + c \qquad \text{or} \qquad x = cy + y^2.$$

Ex. 18. *Consider the following statements: The equation* $(2x/y^3)\,dx + [(y^2 - 3x^2)/y^4]\,dy = 0$ *is 1. Exact 2. Homogeneous 3. Linear. Of these statements* (*a*) *1 and 2 are correct* (*b*) *1 and 3 are correct* (*c*) *2 and 3 are correct* (*d*) *1, 2 and 3 are correct.* **[I.A.S. (Prel.) 1998]**

Sol. Ans. (*c*). Apply definitions of exact, homogeneous and linear equation as given in articles 2.12, 2.8, 2.22 respectively.

Ex. 19. *The rate at which bacteria multiply is proportional to the instantaneous number present. If the original number doubles in 2 hours, then it will triple in*
(*a*) *2 log 3/log 2* (*b*) *2 log 2/log 3* (*c*) *log 3/log 2* (*d*) *log 2/log 3.* **[I.A.S. (Prel.) 1998]**

Sol. Ans. (*a*). Suppose that the number of bacteria is x_0 when $t = 0$ and it is x at time t (in hours). Then given that $dx/dt \propto x$ so that $dx/dt = kx$, ... (1)
where k is the constant of proportionality. Now, (1) gives

$(1/x)\,dx = k\,dt$ so that $\log x - \log c = kt$ or $x = ce^{kt}$ ... (2)

But $x = x_0$ when $t = 0$. So (2) gives $c = x_0$ and so $x = x_0\,e^{kt}$. ... (3)

Given that $x = 2x_0$ when $t = 2$. Also, let $x = 3x_0$, when $t = t'$.

Then (3) gives $2x_0 = x_0\,e^{2k}$ and $3x_0 = x_0\,e^{t'k}$.

$\Rightarrow$ $2k = \log 2$ and $t'k = \log 3$ $\Rightarrow$ $t' = 2 \log 3/\log 2$.

Ex. 20. *Which of the following is not an integrating factor of* $x\,dy - y\,dx = 0$?

(*a*) $1/x^2$ (*b*) $1/(x^2 + y^2)$ (*c*) $1/xy$ (*d*) x/y **[GATE 2001]**

Hint: Ans. (*d*). Refer results (*i*), (*ix*) and (*xi*) of Rule I, Art. 2.16.

Ex. 21. *The general solution of* $dy/dx + \tan y \tan x = \cos x \sec y$ *is*

(*a*) $2 \sin y = (x + c - \sin x \cos x) \sec x$ (*b*) $\sin y = (x + c) \cos x$

(*c*) $\cos y = (x + c) \sin x$ (*d*) $\sec y = (x + c) \cos x$ **[GATE 2001]**

Sol. Ans. (*b*). Re-writing $\cos y\,(dy/dx) + \tan x \sin x = \cos x$.

Put $\sin y = v$ to reduce to linear equation and proceed as usual.

Ex. 22. *If the integrating factor of* $(x^7y^2 + 3y)\,dx + (3x^8y - x)\,dy = 0$ *is* $x^m y^n$ *then*
(*a*) $m = -7, n = 1$ (*b*) $m = 1, n = -7$ (*c*) $m = n = 0$ (*d*) $m = n = 1$. **[GATE 2002]**

Hint: Ans. (*a*). Use rule VI, Art. 2.16.

Ex. 23. *A curve* γ *in the xy-plane is such that the line joining the origin to any point P* (x, y) *on the curve and the line parallel to the y-axis through P are equally inclined to the tangent to the curve at P. Differential equation curve* γ *is*

(*a*) $x\,(dy/dx)^2 - 2y\,(dy/dx) = x$ (*b*) $x\,(dy/dx)^2 + 2y\,(dy/dx) = 0$

(*c*) $x\,(dy/dx)^2 + 2y\,(dx/dy) = 0$ (*d*) $x\,(dy/dx)^2 + 2y\,(dx/dy) = x$ **[GATE 2005]**

Sol. Ans. (*a*). We use (X, Y) as current coordinates. Here $P(x, y)$ is a given point on the given curve $Y = f(X)$, say. Now, we have

$$\tan \psi_1 = \text{gradient of OP} = (y - 0)/(x - 0) = y/x \quad \dots (1)$$

ψ_2 = slope of line through P parallel to y-axis = 90°

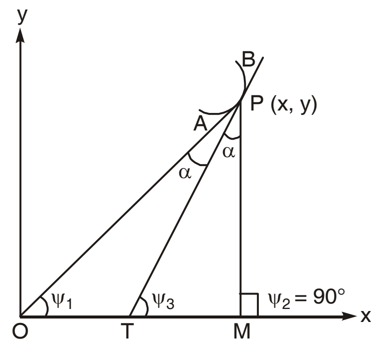


Fig. 2.5

Let PT be the tangent drawn to the given curve at $P(x, y)$. Then, by problem we have $\angle OPT = \angle MPT = \alpha$, say. Now, from ΔOPM, we find that

$\psi_1 + 2\alpha = \pi/2$ so that $\alpha = \pi/4 - \psi_1/2$ ... (2)

Again, from ΔTPM, ψ_3 = slope of $PT = \pi/2 - \alpha$

$\therefore \quad \psi_3 = \pi/2 - (\pi/4 - \psi_1/2)$ or $2\psi_3 = \pi/2 + \psi_1$, by (2)

$\Rightarrow \tan(2\psi_3) = -\cot \psi_1$

$\Rightarrow (2 \tan \psi_3)/(1 - \tan^2 \psi_3) = -(1/\tan \psi_1)$

$\Rightarrow (2y')/(1 - y'^2) = -x/y$, since $\tan \psi_3 = y' = dy/dx$ and $\tan \psi_1 = y/x$

$\Rightarrow 2yy' = -x(1 - y'^2) \Rightarrow x(dy/dx)^2 - 2y(dy/dx) = x$

Ex. 24. *An integrating factor for* $(\cos y \sin 2x)\, dx + (\cos^2 y - \cos^2 x)\, dy = 0$ *is*

(*a*) $\sec^2 y + \sec y \tan y$ (*b*) $\tan^2 y + \sec y \tan y$

(*c*) $1/(\sec^2 y + \sec y + \tan y)$ (*d*) $1/(\tan^2 y + \sec y \tan y)$ **[GATE 2006]**

Sol. Ans. (*a*). Comparing the given equation with $M\, dx + N\, dy = 0$, here

$M = \cos y \sin 2x$ and $N = \cos^2 y - \cos^2 x$.

$$\therefore \quad \frac{1}{M}\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) = \frac{1}{\cos y \sin 2x}\{2 \cos x \sin x - (-\sin y \sin 2x)\} = \frac{\sin 2x + \sin y \sin 2x}{\cos y \sin 2x}$$

$= \sec y + \tan y$, which is function of y alone. So by rule V, Art. 2.16,

$$\therefore \quad \text{I.F.} = e^{\int (\sec y + \tan y)\, dy} = e^{\log(\sec y + \tan y) + \log \sec y} = e^{\log\{(\sec y + \tan y)\sec y\}} = (\sec y + \tan y)\sec y$$

Ex. 25. *Which equation represents the set of all curves in the xy-plane which have slope at each P equal to the reciprocal of the slope of the straight lines through P and the origin?*

(*a*) $y^2 = x + c$ (*b*) $x^2 + y^2 = c^2$ (*c*) $x^2 - y^2 = c^2$ (*d*) $xy = c^2$. **[I.A.S. (Prel.) 2006]**

Sol. Ans. (*c*). Let the coordinates of P be (x, y). Then, slope of straight line through $P(x, y)$ and origin $O(0, 0)$ is $(y - 0)/(x - c)$, *i.e.*, $y/x = m$, say. By problem, if dy/dx be the slope of the required curve, then $d\, y/dx = 1/m$ or $dy/dx = x/y$ so that $2x\, dx - 2y\, dy = 0$.

Integrating, $x^2 - y^2 = c^2$, c being an arbitrary constant.

Ex. 26. *The population P of a city increases at a rate which is jointly proportional to the current population and the difference between 200,000 and the current population. The differential eauation for this* is

(*a*) $dP/dt = P(P - 200{,}000)$ (*b*) $dP/dt = kP(200{,}000 - P)$

(*c*) $dP/dt = k(P - 200{,}000)$ (*d*) $dP/dt = 200{,}000\, P$ **[M.S. Univ. T.N. 2007]**

Sol. Here $dP/dt \propto P$ and $dP/dt \propto (200{,}00 - P)$.

Hence $dP/dt = kP(200{,}000)$, k being the constant of proportionality

Ex. 27. *The cooling law "The rate at which a hot body cools is proportional to the difference in temperature between the body and the surrounding medium",*

(*a*) *Ohm's law* (*b*) *Kepler's law*

(*c*) *Newton's law* (*d*) *Kelvin's law* **[M.S.Univ. T.N. 2007]**

Sol. Ans. (c). Refer list F of Art. 2.30.

Ex. 28. *If in a culture of yeast the active ferment doubles itself in three hours. By what ratio will it increase in 15 hours on the assumption that the quantity increases at rate proportional to itself* (*a*) *22 times* (*b*) *12 times* (*c*) *32 times* (*d*) *42 times* **[M.S. Univ. T.N. 2007]**

Sol. Ans. (c). Let the quantity of yeast at any time t be x. Then, according to the given problem $dx/dt \propto x$

Thus, $dx/dt = kx$, where k is constant of proportionality

or $(1/x)dx = kdt$ so that $\log x - \log c = kt$

or $\log (x/c) = kt$ or $x = ce^{kt}$, c being an arbitrary constant

Let $x = x_0$ when $t = 0$. Then the above equation givens $c = x_0$

Then, we have $$x = x_0 e^{kt} \qquad \text{... (1)}$$

Given that $x = 2x_0$ when $t = 3$ hours. Hence (1) $\Rightarrow$ $2x_0 = x_0\, e^{3k}$

Then, we have $$e^{3k} = 2 \qquad \text{... (2)}$$

Let $x = x'$ when $t = 15$ hours. Then, (1) $\Rightarrow$ $x' = x_0\, e^{15k}$

or $x' = x_0\, (e^{3k})^5 = x_0 \times 2^5$, using (2)

Then, $x' = 32\, x_0$ so that $x'/x_0 = 32$, which is required result

MISCELLANEOUS PROBLEMS ON CHAPTER 2

Ex. 1. *Solve the ordinary differential equation (cos 3x) × (dy/dx) – 3y sin 3x = (1/2) × sin 6x + sin² 3x, 0 < x < π/2* **[I.A.S. 2007]**

Sol. Re-writing the given equation, we have

$$(dy/dx) - (3 \tan 3x)y = \sec 3x\ \{(1/2) \times \sin 6x + \sin^2 3x\} \qquad \text{... (1)}$$

which is linear whose I.F. $= e^{\int (-3 \tan 3x)dx} = e^{\log \cos 3x} = \cos 3x$ and hence its solution is

$$y \cos 3x = \int \cos 3x \sec 3x\ \{(1/2) \times \sin 6x + \sin^2 3x\}\ dx + c$$

or $$y \cos 3x = \int \{(1/2) \times \sin 6x + (1/2) \times (1 - \cos 6x)\}\ dx + c$$

or $$y \cos 3x = -(1/12) \times \cos 6x + x/2 - (1/12) \times \sin 3x + c$$

or $y \cos 3x = (1/12) \times (6x - \cos 6x - \sin 6x) + c$, c being an arbitrary constant

Ex. 2. *Find the solution of the equation* $(1/y)dy + xy^2 dx = -\,4x\, dx$ **[I.A.S. 2007]**

Sol. Re-writing the given equation, we have

$$\frac{dy}{y} + x(4 + y^2)\, dx = 0 \qquad \text{or} \qquad \frac{dy}{y(y^2+4)} + xdx = 0$$

or $$\frac{1}{4}\left(\frac{1}{y} - \frac{y}{y^2+4}\right) dy + x\, dx = 0 \qquad \text{or} \qquad \left(\frac{2y}{y^2+4} - \frac{2}{y}\right) dy = 8x\, dx$$

Integrating, $\log (y^2 + 4) - 2 \log y - \log c = 4x^2$, c being an arbitrary constant

or $\log \{(y^2 + 4)/cy^2\} = 4x^2$ or $y^2 + 4 = cy^2 e^{4x^2}$

Ex. 3 (a). *A particle falls from rest in a medium whose resistance varies as the velocity. Find the relation between velocity (v) and the distance (x).* **[M.S. Univ. T.N. 2007]**

(b) *A particle falls from rest in a medium whose resistance varies as the velocity of the particle. Find the distance fallen by the particle and its velocity at time t.* **[I.A.S. 2007]**

Sol. Let a particle of mass m fall from rest under gravity from a fixed point O. Let P be the position of the particle at any time t such that $OP = x$. Let v be its velocity at P. Let kv be the force of resistance per unit mass so that mkv is resistance of the medium on the particle acting in vertical upward direction.Then, the equation of the particle at any time t is given by

$$m\ddot{x} = mg - mkv \qquad \text{or} \qquad \ddot{x} = g(1 - kv/g) \qquad \text{... (1)}$$

Let V be the terminal velocity of the particle so that $v = V$ when $\ddot{x} = 0$. Then, (1) yields $0 = g(1 - kV/g)$ giving $k = g/V$. Hence, (1) reduces to

$$\ddot{x} = g(1 - v/V) \qquad \text{or} \qquad \ddot{x} = (g/V) \times (V - v) \qquad \text{... (2)}$$

Part (a). Since $\ddot{x} = v(dv/dx)$, (2) $\Rightarrow$ $\qquad v(dv/dx) = (g/V) \times (V - v)$

$$\text{or} \qquad \frac{v}{V - v}\, du = \frac{g}{V}\, dx \qquad \text{or} \qquad \left(\frac{V}{V - v} - 1\right) dv = \frac{g}{V}\, dx$$

Integrating, $\qquad - V \log (V - v) - v = gx/V + A$, A being an arbitrary constant ... (3)

Initially at O, when $x = 0$, $v = 0$. Hence, (3) givens $\qquad A = - V \log V$.

Thus, (3) becomes $\qquad - V \log (V - v) - v = gx/V - V \log V$

$$\text{or} \qquad \frac{gx}{V} = V \log \frac{V}{V - v} - v \qquad \text{or} \qquad x = \frac{V^2}{g} \log \frac{V}{V - v} - \frac{Vv}{g}$$

Part (b). To find the velocity of the particle at any time t

Since $\ddot{x} = dv/dt$, (2) reduces to $\qquad dv/dt = (g/V) \times (V - v)$

or $\qquad \{V/(V - v)\}dv = g\, dt$

Integrating, $\qquad - V \log (V - v) = gt + B$, B being an arbitrary constant ... (4)

Initially at O, when $t = 0$, $v = 0$. Hence, (4) givens $\qquad B = - V \log V$

Hence (4) reduces to $\qquad - V \log (V - v) = gt - V \log V$

$\log \{(V - v)/V\} = - gt/V \qquad$ so that $\qquad v = V(1 - e^{-gt/V})$ ... (5)

To find the distance fallen by the particle at any time *t*

Since $v = dx/dt$, (5) reduces to $\qquad dx/dt = V(1 - e^{-gt/V})$

so that $\qquad dx = V(1 - e^{-gt/V})dt$

Integrating, $\quad x = Vt + (V^2/g) \times e^{-gt/V} + C$, C being an arbitrary constant ... (6)

Initially at O, when $t = 0$, $x = 0$. Hence (6) yields $\qquad C = - (V^2/g)$

Hence (6) reduces to $\qquad x = Vt - (V^2/g) \times (1 - e^{-gt/V})$.

Ex. 4. *A solid sphere of salt dissolves in running water at rate proportional to the surface area of the sphere. If half the solid dissolves in 15 minutes in what time it all will be dissolved.* **[M.S. Univ. T.N. 2007]**

Sol. Let original volume and radius of the given solid sphere of salt be V_0 and r_0 respectively:

Let its volume, surface and radius at any time be V, S and r respectively. According to the given problem we have

$$\frac{dV}{dt} = kS \qquad \text{or} \qquad \frac{d}{dt}\left(\frac{4}{3}\pi r^3\right) = k \times (4\pi r^2), \qquad \text{... (1)}$$

where k is constant of proportionality.

Re-writing (1), $\frac{4}{3}\times(3\pi r^2)\times\frac{dr}{dt}=k\times(4\pi r^2)$ or $dr = k\,dt$.

Integrating, $r = kt + c$, c being an arbitrary constant ... (2)

Initially, when $t = 0$, $r = r_0$. So (2) yields $c = r_0$.

Then, (2) reduces to $r = r_0 + kt$ ... (3)

Given that when $t = 15$ minutes, $V = V_0/2$. Then, let the radius of the solid sphere of solid be r' when its volume is $V_0/2$. Thus, we have

$$\frac{4}{3}\pi r'^3=\frac{1}{2}V_0=\frac{1}{2}\times\frac{4}{3}\pi r_0^3 \quad\Rightarrow\quad r'=\frac{1}{2^{1/3}}r_0 \qquad \text{... (4)}$$

Now, given that $r = r'$ when $t = 15$ minutes, so (3) gives

$r' = r_0 + 15k$ or $k = (r' - r_0) / 15$

$\therefore$ (3) becomes $r = r_0 + \{(r' - r_0) / 15\}t$ ... (5)

Let the whole solid sphere dissolves in time T minutes. Then, $r = 0$ when $t = T$. Hence (5) gives $0 = r_0 + \{(r' - r_0) / 15\}T$ or $\{(r_0 - r') / 15\}T = r_0$

giving $T=\dfrac{15\,r_0}{r_0-r'}$ or $T=\dfrac{15}{1-(r'/r_0)}=\dfrac{15}{1-(1/2^{1/3})}$, using (4)

Thus, required time = $\{(15 \times 2^{1/3}) / (2^{1/3} - 1)\}$ hours

Ex. 5. The equation $(\alpha xy^3+y\cos x)dx+(x^2y^2+\beta\sin x)dy=0$ is exact if

(a) $\alpha = 3/2,\ \beta = 1$ (b) $\alpha = 1,\ \beta = 3/2$

(c) $\alpha = 2/3,\ \beta = 1$ (c) $\alpha = 1,\ \beta = 2/3$ **(GATE 2009)**

Sol. Ans.(c). Comparing the given equation with $Mdx + Ndy = 0$, we have $M=\alpha xy^3+y\cos x$ and $N=x^2y^2+\beta\sin x$. Since the given equation in exact, hence by definition $\partial.M/\partial y=\partial N/\partial x$

Hence $3\alpha xy^2+\cos x=2xy^2+\beta\cos x\Rightarrow 3\alpha=2,\ \beta=1\Rightarrow\alpha=2/3,\ \beta=1$

Ex. 6. The rate at which a substance cools in air is proportional to the difference between the temperatures of the substance and air. If the temperature of the air is 30° and the substance cools from 100°C to 70°C in 15 minutes, find when the temperature will be 40°C.

[Delhi Maths (H) 2009]

Hint: Do like Ex. 29 (a), page 2.56. **Ans.** 52 minutes (approx.)

Ex. 7. Show that the solution curves of the differential equation $dy/dx = -\{y\,(2x^3 - y^3)/x\,(2y^3 - x^3)\}$ are of the form $x^3 + y^3 = 3cxy$ **[Delhi B.Sc. (Hons) III 2011]**

Sol. Given $$\frac{dy}{dx}=-\left(\frac{y}{x}\right)\times\frac{2x^3-y^3}{2y^3-x^3}=-\left(\frac{y}{x}\right)\times\frac{2-(y/x)^3}{2(y/x)^3-1} \qquad \text{.... (1)}$$

Take $\dfrac{y}{x}=v$ i.e., $y = vx$ *so that* $dy/dx = v + x\,(dv/dx)$ .... (2)

From (1) and (2), $v+x\dfrac{dv}{dx}=-v\dfrac{2-v^3}{2v^3-1}$ or $x\dfrac{dv}{dx}=\dfrac{v^4-2v}{2v^3-1}-v$

or $\quad x\dfrac{Dv}{dx} = -\dfrac{v^4+v}{2v^3-1}$ $\quad$ or $\quad \dfrac{dx}{x} = -\dfrac{2v^3-1}{v\,(v^3+1)}\,v^2\,dv$ ... (3)

Putting $v^3 = u$ and $3v^2\,dv = du$, (3) reduces to

$\dfrac{dx}{x} = -\dfrac{1}{3}\displaystyle\int \dfrac{2u-1}{u(u+1)}du$ $\quad$ or $\quad -3\dfrac{dx}{x} = \displaystyle\int \left(\dfrac{3}{u+1} - \dfrac{1}{u}\right) du$

[On resolving into partial fractions]

Integrating, $-3 \log x = 3 \log (u + 1) - \log u - \log c'^3$ c' being an arbitrary constant

or $\quad \log (u+1)^3 + \log x^3 = \log c'^3 + \log u$ $\quad$ or $\quad x^3 (u+1)^3 = u\,c'^3$

or $\quad x^3 (v^3 + 1)^3 = v^3 c'^3$ $\quad$ or $\quad x (v^3 + 1) = c'v$, as $u = v^3$

or $\quad x (y^3/x^3 + 1) = c' \times (y/x)$

or $\quad x^3 + y^3 = 3cxy$, $\quad$ where $\quad c' = 3c$

Ex. 8. *Solve* $6 \cos^2 x\,(dy/dx) - y \sin x + 2y^4 \sin^3 x = 0$ **[Pune 2010]**

Sol. Re-writing the given equation, $-y^{-4}(dy/dx) + (\sin x / 6\cos^{2x})y^{-3} = (1/3) \times \sin x \tan^2 x$

Putting $y^{-3} = v$ and $-3y^{-4}(dy/dx) = dv/dx$, the above equation yields

$$\frac{1}{3}\frac{dv}{dx} + \frac{\sin x}{6\cos^2 x}v = \frac{1}{3}\sin x \tan^2 x$$

or
$$\frac{dv}{dx} + \frac{1}{2}(\tan x \sec x)v = \frac{\sin x \tan^2 x}{3}$$

which in liner differential equation whose integrating factor is

$$e^{\int (1/2) \times \sec x \tan x\,dx} \qquad i.e \qquad e^{(1/2) \times \sec x}$$

and hence its solution is given by

$$v \times e^{(1/2)\times\sec x} = \frac{1}{3}\int e^{(1/2)\times \sec x} \sin x \tan^2 x\,dx + c,\ c \text{ being an arbitrary constant.}$$

or
$$y^{-3} \times e^{(1/2)\times\sec x} = \frac{1}{3}\int e^{(1/2)\times\sec x} \sin x \tan^2 x\,dx + c$$

Ex. 9. *Find r, if* x^r *is an integrating factor of the equation* $(x + y^3)\,dx + 6xy^2\,dy = 0$ and hence solve it. **[Pune 2010]**

Sol. Multiplying the given differential equation by x^r, we have

$$(x^{r+1} + x^r y^3)dx + 6x^{r+1}y^2 dy = 0 \qquad \text{... (1)}$$

Comparing (1) with $Mdx + Ndy = 0$, $\quad$ here $\quad M = x^{r+1} + x^r y^3$ and $N = 6x^{r+1} y^2$ ... (2)

Since x^r is an integrating factor of the given equation, so (1) must be exact and hence $\partial M/\partial y = \partial N/\partial x$. Hence, using (2), we obtain

$3x^r y^2 = 6\,(r+1)\,x^r y^2 \quad \Rightarrow \quad 3 = 6\,(r+1) \quad \Rightarrow \quad r + 1 = 1/2 \quad \Rightarrow \quad r = -(1/2)$

Putting $r = -(1/2)$ in (1) yields $\quad \{x^{1/2} + x^{-(1/2)}y^3\}\,dx + 6\,x^{1/2}y^2 dy = 0$ ... (3)

Solution of exact equation (3), as usual, is given by

$$\underset{\text{(Treating y as constant)}}{\int \{x^{1/2} + x^{-(1/2)}y^3\}\,dx} \quad + \quad \underset{\text{[Integrating terms free from } x]}{\int \{6x^{1/2} + y^2\}\,dy} = c/3$$

giving $\quad (2/3) \times x^{3/2} + 2\,x^{1/2}\,y^3 = c/3$ $\quad$ or $\quad x^{3/2} + 3\,x^{1/2}\,y^3 = c.$

Ex.10. *Explai Terricell's law*

Sol. Suppose that a water tank has a hole with area A at its bottom, from which water is leaking. Denote by $y(t)$ the depth of water in the tank at time t and by $V(t)$ the volume of water in the tank at that instant. Then, under suitable condition, the velocity v of the water exiting through the hole is given by

$$v = \sqrt{2gy} \qquad \text{... (1)}$$

using (1), we have

$$\frac{dV}{dt} = -Av = -A\sqrt{2gy} \qquad \text{... (2)}$$

or
$$\frac{dV}{dt} = -k\sqrt{y}, \qquad \text{where} \qquad k = A\sqrt{2g} \qquad \text{... (3)}$$

This is a statement of Torricelli's law for a draining tank. If S(y) denotes the horizontal cross-sectional area of the tank at height y above the hole, the method of volume by crosssection given by

$$V = \int_0^y S(y)\, dy \qquad \text{... (4)}$$

and hence using the fundamental theorem of calculus, (4) implies that

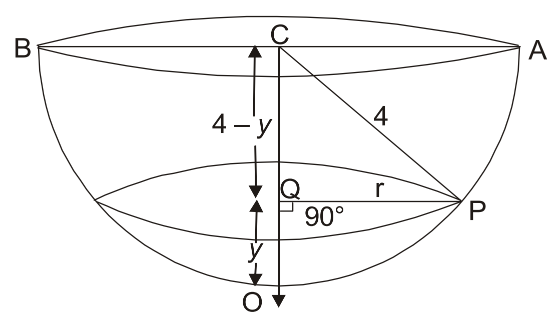
$$\frac{dV}{dt} = \frac{dV}{dy}\frac{dy}{dt} = S(y)\,\frac{dy}{dt} \qquad \text{... (5)}$$

From equations (2), (3) and (5) we obtain

$$S(y)\frac{dy}{dt} = -\sqrt{2gy} = -k\sqrt{y}, \qquad \text{... (6)}$$

which is an alternative form of Torricelli's law

Ex.11(b). *A hemispherical bowl has top radius 4ft and at time t = 0 is full of water. A that moment a circular hole with diameter 1in. is opened in the bottom of the tank. How long will it take for the water to drain from the tank ?*



Sol. Let AOB be a hemispherical lowl of radius CA = CB = CO = 4 ft. Let y(t) be the depth of water in the tank at time t. Let S (y) denote the horizontal cross sectional area of the tank at height by above the hole suppose that the bowl has a hole with area A at its bottom 0, from which water is leaking. Them, by Torricelli's law,

$$S(y)\frac{dy}{dt} = -A\sqrt{2gy} \qquad \text{... (1)}$$

Here, $A = \pi \times (1/24)^2 \;\; g = 32 ft\ /\sec^2$. Also from right angled triangle PCQ, we have

$$S(y) = \pi r^2 = \pi\{4^2 - (4-y)^2\} = \pi(8y - y^2)$$

Hence, (1) reduces to $\quad \pi(8y - y^2)\,\dfrac{dy}{dt} = -\pi\left(\dfrac{1}{24}\right)^2 \sqrt{2 \times 32y}$

so that $$(8y^{1/2} - y^{3/2})dy = -(1/72)\ dt$$

Integrating, $$(16/3)\times y^{3/2} - (2/5)\times y^{5/2} = -(t/72) + C,\ C \text{ being a constant} \qquad ...(2)$$

Initially when the bowl is full of water, y = 4 and t = 0. Hence (2) yields

$C = (16/3)\times 4^{3/2} - (2/5)\times 4^{5/2} = 448/15$ Therefore, (2) reduces to

$$(16/3)\times y^{3/2} - (2/5)\times y^{5/2} = -(t/72) + (448/15) \qquad ...(3)$$

The tank is empty when y = 0, thus when

$$0 = -(t/72) + (448/15), \text{ i.e., } t = 72 \times (448/15) \approx 2150 \text{ sec.}$$

that is about 35 min 50 sec. so it takes slightly less than 36 min. for the tank to drain.

Ex. 11.A. A spherical tank of radius 4ft is full of gasoline, when a circular bottom hole with radius 1in. is opened. How long will be required for all the gasoline to drain from the tank ?

[**Ans.** 14 min. 29 sec.]

Ex. 11B. A spherical tank of radius 2ft in full of gasoline, when a circular bottom hole with radius 1 in. is opened. How long will be required for the liquid 1in. is opened. How long will be required for the liquid to drain completely. **[Delhi B.Sc. (Hons.) 2011]**

3

Trajectories

3.1 Trajectories

Definition. A curve which cuts every member of a given family of curves in accordance with some given law is called *trajectory* of the given family of curves. In the present chapter we shall study only the case when the given law is that the angle at which the curve cuts every member is constant.

If a curve cuts every member of given family of curves at right angles, it is called an *orthogonal trajectory*. Again, if a curve cuts every member of a given family of curves at an angle α ($\neq 90°$), it is called an *oblique trajectory*.

As an example, consider two family of curves $y = mx$ and $x^2 + y^2 = a^2$, where m and a are parameters. Then, we find that every line (given by $y = mx$) through the origin of coordinates is an orthogonal trajectory of the family of concentric circles (given by $x^2 + y^2 = a^2$) with centre at origin. Thus, $y = mx$ is the orthogonal trajectory of the family of circles $x^2 + y^2 = a^2$. Since m is parameter, it follows that the orthogonal trajectories of a given family of curves themselves form a family of curves.

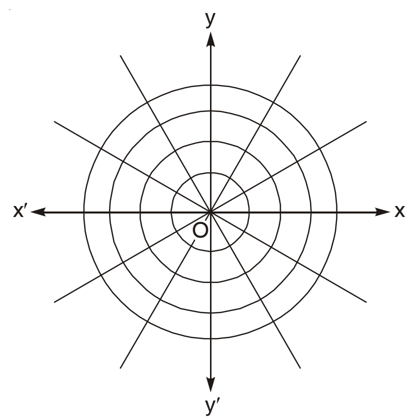


Fig. 3.1

3.2 Determination of orthogonal trajectories in cartesian coordinates

Let the equation of the given family of curves be $$f(x, y, c) = 0, \qquad \text{... (1)}$$

where c is a parameter. Differentiating (1) with respect to x and eliminating c, between (1) and the derived result, we shall arrive at the differential equation of the given family of curves (1). Let it be

$$F(x, y, dy/dx) = 0. \qquad \text{... (2)}$$

Let ψ be the angle between the tangent PT to a member PQ of the family of curves and x-axis at any point $P(x, y)$. Then,

$$\tan \psi = dy/dx \qquad \text{.... (3)}$$

Let (X, Y) be the current coordinates of any point of a trajectory. At point of intersection P of any member of (2) with the trajectory PQ', let ψ' be the angle which the tangent PT' to the trajectory makes with x-axis.

$$\therefore \qquad \tan \psi' = dY/dX. \qquad \text{... (4)}$$

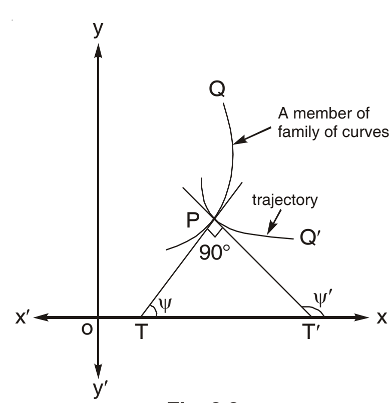


Fig. 3.2

Let PT and PT' intersect at 90°. Hence $\tan\psi \tan\psi' = -1$ or $(dy/dx) \times (dY/dX) = -1$

$$\therefore \qquad \frac{dy}{dx} = -\frac{1}{dY/dX} = -\frac{dX}{dY} \qquad \ldots (5)$$

At a point of intersection of any member of (2) with the trajectory, we have

$$x = X, \qquad \text{and} \qquad y = Y. \qquad \ldots (6)$$

Eliminating x, y and dy/dx from (2), (5) and (6), we get $F(X, Y, -dX/dY) = 0$, ... (7)
which is the differential equation of the required family of trajectories. In the usual notation, we observe that the differential equation of the family of trajectories of the family of curves given by

$$F(x, y, dy/dx) = 0, \qquad \ldots (8)$$

is

$$F(x, y, -dx/dy) = 0, \qquad \ldots (9)$$

showing that it can be obtained on replacing, dy/dx, by $-(dx/dy)$.

3.3 Self-orthogonal family of curves.

Definition. If each member of *a* given family of curves intersects all other members orthogonally, then the given family of curves is said to be self orthogonal.

Remark. From the above definition, it follows that if the differential equation of the family of curves is identical with the differential equation of its orthogonal trajectories, then such a family of curves must be self orthogonal.

3.4 Working rule for finding the orthogonal trajectories of the given family of curves in cartesian coordinates

Step I. Differentiate the given equation of the family of curves. Eliminate the parameter between this derived equation and the given equation of the family of curves to obtain the differential equation of the given family of curves.

Step II. In the differential equation found in step I, replace dy/dx by $-dx/dy$ and thus obtain the differential equation of the required orthogonal trajectories.

Step III. Obtain the general solution of the differential equation of the orthogonal trajectories found in step II. The general solution so obtained will give us the desired orthogonal trajectories.

3.5 Solved examples of Type I based on Art. 3.4

Ex. 1. *Find the orthogonal trajectories of family of curves $y = ax^2$, a being a parameter.*

[Patna 2003, Kanpur 2011]

Sol. The given family of curves is $y = ax^2$, with a as parameter. ... (1)

Differentiating (1) with respect to x, we get $dx/dy = 2ax$. ... (2)

From (1), $a = y/x^2$. ... (3)

Eliminating a from (2) and (3), we have

$$dy/dx = 2\,(y/x^2)\,x \qquad \text{or} \qquad dy/dx = 2y/x, \qquad \ldots (4)$$

which is the differential equation of the given family of curves (1). Replacing dy/dx by $-dx/dy$, the differential equation of the required orthogonal trajectories is given by

$$-dx/dy = 2y/x \qquad \text{or} \qquad x\,dx + 2y\,dy = 0.$$

Integrating, $x^2/2 + y^2 = b^2$ or $x^2/2b^2 + y^2/b^2 = 1$, ... (5)

which is the required orthogonal trajectories, b being parameter.

Remark. Here given family (1) represents parabolas. Their orthogonal trajectories is family of ellipses given by (5). Each member of (1) cuts each member of (5) orthogonally.

Ex. 2. *Find the orthogonal trajectories of the family of curves $3xy = x^3 - a^3$, a being parameter of the family.* **[Kanpur 1998; Agra 1995]**

Sol. The given family of curves is $3xy = x^3 - a^3$, with a as parameter. ... (1)

Differentiating (1) with respect to x, we get

$3y + 3x\,(dy/dx) = 3x^2$ or $y + x\,(dy/dx) = x^2$. ... (2)

Since (2) does not contain parameter a, so (2) is the differential equation of the given family of curves (1). Replacing dy/dx by $-\,dx/dy$, the differential equation of the required orthogonal trajectories is given by

$y - x\,(dx/dy) = x^2$ or $x\,(dx/dy) + x^2 = y$.... (3)

To solve (3), we reduce it to linear differential equation as follows:

Putting $x^2 = v$ so that $2x\,(dx/dy) = dv/dy$, (3) gives

$(1/2) \times (dv/dy) + v = y$ or $(dv/dy) + 2v = 2y$, ...(4)

which is linear differential equation whose I.F. $= e^{\int 2\,dy} = e^{2y}$. Hence solution of (4) is

$v\,(\text{I.F.}) = \int (2y)\,(\text{I.F.})\,dy + c$ or $ve^{2y} = \int 2y\,e^{2y}\,dy + c$

or $$ve^{2y} = (2y)\left(\frac{1}{2}e^{2y}\right) - \int (2)\left(\frac{1}{2}e^{2y}\right)dy + c, \text{ integrating by parts}$$

or $ve^{2y} = ye^{2y} - (1/2) \times e^{2y} + c$ or $v = y - (1/2) \times e^{-2y} + c$

or $x^2 = y - (1/2) + ce^{-2y}$, which is the required orthogonal trajectories, c being parameter.

Ex. 3. *Find the orthogonal trajectories of the system of curves* $(dy/dx)^2 = a/x$.

[I.A.S. Prel. 2007; Meerut 1995]

Sol. The differential equation of the given family of curves is $(dy/dx)^2 = a/x$, ... (1)

where a is a given constant. Replacing dy/dx by $-\,dx/dy$, the differential equation of the required orthogonal trajectories is given by

$(-\,dx/dy)^2 = a/x$ or $dy = \pm\,(x^{1/2}/a^{1/2})\,dx$. ... (2)

Integrating (2), $y + c = (1/a^{1/2}) \times (2/3) \times x^{3/2}$ or $3\sqrt{a}\,(y + c) = \pm\, 2x^{3/2}$

or $9a\,(y + c)^2 = 4x^3$, which is the required orthogonal trajectories, c being parameter.

Ex. 4. *Find the orthogonal trajectories of the system of circles touching a given straight line at a given point.* **[Purvanchal 1993]**

OR

Find the orthogonal trajectories of $x^2 + y^2 = 2ax$. **[GATE 2003]**

Sol. Let the given point be $O\,(0, 0)$ and the given straight line be y-axis. Now, if a be the radius, then equation family of given circles is

$(x - a)^2 + (y - 0)^2 = a^2$ or $x^2 + y^2 = 2ax$, where a is a parameter. ... (1)

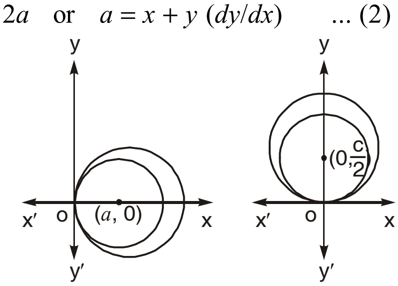
Differentiating (1) with respect to x, $2x + 2y\,(dy/dx) = 2a$ or $a = x + y\,(dy/dx)$ ... (2)

Eliminating a from (1) and (2), we get

$$x^2 + y^2 = 2x\left(x + y\,\frac{dy}{dx}\right)$$

or $2xy\,(dy/dx) = y^2 - x^2$, ... (3)

which is the differential equation of the given family of circles (1). Replacing dy/dx by $-\,dx/dy$, the differential equation of the required orthogonal trajectories is



Given system of circles Orthogonal trajectories

Fig. 3.3

$$-2xy\frac{dx}{dy} = y^2 - x^2 \qquad \text{or} \qquad \frac{dy}{dx} = \frac{2xy}{x^2-y^2} = \frac{2(y/x)}{1-(y/x)^2}, \qquad \ldots (3)$$

which is a homogeneous differential equation.

Put $y/x = v$ or $y = xv$ so that $dy/dx = v + x\,(dv/dx)$

$\therefore$ (3) gives $\quad v + x\dfrac{dv}{dx} = \dfrac{2v}{1-v^2} \quad$ or $\quad x\dfrac{dv}{dx} = \dfrac{2v}{1-v^2} - v$

or $\quad x\dfrac{dv}{dx} = \dfrac{v+v^3}{1-v^2} \quad$ or $\quad \dfrac{dx}{x} = \dfrac{1-v^2}{v\,(1+v^2)}\,dv$

or $\quad \dfrac{dx}{x} = \left(\dfrac{1}{v} - \dfrac{2v}{1+v^2}\right) dv$, on resolving into partial fractions

Integrating, $\quad \log x = \log v - \log(1+v^2) + \log c \quad$ or $\quad x = (cv)/(1+v^2)$

or $\quad x\,(1+v^2) = cv \quad$ or $\quad x\,(1+y^2/x^2) = c\,(y/x)$, as $v = y/x$

$\therefore \quad x^2 + y^2 = cy$, c being parameter. ... (4)

Note: Here the orthogonal trajectories (4) again represents a family of circles touching x-axis at O (0, 0) and having variable radius ($c/2$).

Ex. 5. *Find the orthogonal trajectories of the parabola* $6ay^2 = (x-3)$, *where a is a variable parameter.* **[Kanpur 2006]**

Sol. Given $\quad 6ay^2 = x - 3$ ... (1)

Differentiating (1) w.r.t. 'x', $\quad 12ay\,(dy/dx) = 1$ ... (2)

Dividing (2) by (1), $\quad (2/y) \times (dy/dx) = 1/(x-3)$, ... (3)

which is the differential equation of the given family of curves (1). Replacing dy/dx by $-(dx/dy)$, the differential equation of the required orthogonal trajectories is given by

$-(2/y) \times (dx/dy) = 1/(x-3) \quad$ or $\quad 2\,(x-3)\,dx + y\,dy = 0$

Integrating, $\quad (x-3)^2 + y^2/2 = c^2$, c being an arbitrary constant.

Ex. 6. *Find the orthogonal trajectories of the family of co-axial circles* $x^2 + y^2 + 2gx + c = 0$, *where g is the parameter.* **[Kanpur 2005; I.A.S. 2005; Guwahati 1996; Lucknow 1998; Meerut 1998; Osmania 1997]**

Sol. The given family of curves is $\quad x^2 + y^2 + 2gx + c = 0$, with g as parameter. ... (1)

Differentiating (1) with respect to x, $\quad 2x + 2y\dfrac{dy}{dx} + 2g = 0 \quad$ or $\quad g = -\left(x + y\dfrac{dy}{dx}\right)$. ... (2)

Eliminating g from (1) and (2), we get

$$x^2 + y^2 + 2x\left(-x - y\frac{dy}{dx}\right) + c = 0 \qquad \text{or} \qquad y^2 - x^2 - 2xy\frac{dy}{dx} + c = 0, \qquad \ldots (3)$$

which is the differential equation of the given family of circles (1). Replacing dy/dx by $-dx/dy$ in (3), the differential equation of the required orthogonal trajectories is

$$y^2 - x^2 + 2xy\frac{dx}{dy} + c = 0 \qquad \text{or} \qquad 2x\frac{dx}{dy} - \frac{1}{y}x^2 = -\frac{c}{y} - y, \qquad \ldots (4)$$

which can be reduced to linear differential equation as follows:

Putting $x^2 = v$ so that $2x\,(dx/dy) = dv/dy$, (4) gives $\quad \dfrac{dv}{dy} - \dfrac{1}{y}v = -\dfrac{c}{y} - y$, ... (5)

whose I.F. $= e^{\int (-1/y)\,dy} = e^{-\log y} = (1/y)$. Hence solution of (5) is

$$v\,(\text{I.F.}) = \int\left(-\frac{c}{y}-y\right)\cdot(\text{I.F.})\,dy + d \;=\; -\int\left(\frac{c}{y}+y\right)\frac{1}{y}\,dy + d = -\int (cy^{-2}+1)\,dy + d$$

or $\qquad \dfrac{v}{y} = cy^{-1} - y + d \qquad$ or $\qquad \dfrac{x^2}{y} = \dfrac{c}{y} - y + d,\quad$ as $v = x^2$

or $\qquad x^2 + y^2 - dy - c = 0$, d being parameter.

Ex. 7. *Find the orthogonal trajectories of the family of curves:*

(*a*) $(x^2/a^2) + \{y^2/(b^2 + \lambda)\} = 1$, λ *being the parameter.* **[Guwahati 2007]**

(*b*) $(x^2/a^2) + \{y^2/(a^2 + \lambda)\} = 1$, λ *being the parameter.* **[Calicut 2004; Calcutta 1995]**

Sol. (*a*) Given $\qquad (x^2/a^2) + \{y^2/(b^2 + \lambda)\} = 1$, with λ as parameter. ... (1)

Differentiating (1), $\qquad \dfrac{2x}{a^2} + \dfrac{2y}{b^2+\lambda}\dfrac{dy}{dx} = 0 \qquad$ or $\qquad \dfrac{1}{b^2+\lambda} = -\dfrac{x}{a^2 y}\dfrac{dx}{dy}$ ... (2)

Eliminating λ from (1) and (2), we get

$$\frac{x^2}{a^2} + y^2\left(-\frac{x}{a^2 y}\frac{dx}{dy}\right) = 1 \qquad \text{or} \qquad \frac{x^2}{a^2} - \frac{xy}{a^2}\frac{1}{dy/dx} = 1, \qquad \text{... (3)}$$

which is the differential equation of the given family of curves (1). Replacing dy/dx by $-\,dx/dy$ in (3), the differential equation of the required orthogonal trajectories is

$$\frac{x^2}{a^2} - \frac{xy}{a^2}\left(-\frac{1}{dx/dy}\right) = 1 \qquad \text{or} \qquad \frac{x^2}{a^2} - \frac{xy}{a^2}\frac{dy}{dx} = 1$$

or $\qquad (xy/a^2)\,(dy/dx) = (a^2 - x^2)/a^2 \qquad$ or $\qquad y\,dy = \{(a^2/x) - x\}\,dx$

Integrating, $\qquad y^2/2 = a^2 \log x - x^2/2 + c/2 \qquad$ or $\qquad x^2 + y^2 - 2a^2 \log x = c$,

which is the required equation of the orthogonal trajectories.

(*b*) Proceed as in part (*a*). **Ans.** $x^2 + y^2 - 2a^2 \log x = c$.

Ex. 8(*a*). *Find the orthogonal trajectories of the family of circles* $x^2 + y^2 + 2fy + 1 = 0$, *where* f *is parameter.* **[Meerut 1999]**

(*b*) *Find the orthogonal trajectories of the family of curves* $x^2 + y^2 + 2fy - 1 = 0$, f *being a parameter.*

Sol. (*a*) Given $\qquad x^2 + y^2 + 2fy + 1 = 0$, where f is parameter. ... (1)

Differentiating (1) w.r.t. 'x', $\qquad 2x + 2y\,(dy/dx) + 2f\,(dy/dx) = 0$. ... (2)

From (1) and (2), $\qquad 2fy = -\,(1 + x^2 + y^2) \qquad$ and $\qquad 2f\,(dy/dx) = -\,[2x + 2y\,(dy/dx)]$

On dividing, these give $\qquad \dfrac{2f\,(dy/dx)}{2fy} = \dfrac{2x + 2y\,(dy/dx)}{1 + x^2 + y^2}$

or $\qquad (1 + x^2 + y^2)\,(dy/dx) = 2y\,[x + y\,(dy/dx)]$, ... (3)

which is the differential equation of (1). Replacing dy/dx by $-\,dx/dy$, the differential equation of the required orthogonal trajectories is

$$(1 + x^2 + y^2)\,(-\,dx/dy) = 2y\,[x + y\,(-\,dx/dy)] \qquad \text{or} \qquad (dx/dy)\,(y^2 - x^2 - 1) = 2xy$$

or $\qquad 2xy\dfrac{dy}{dx} = y^2 - x^2 - 1 \qquad$ or $\qquad 2y\dfrac{dy}{dx} - \dfrac{1}{x}y^2 = -\dfrac{x^2+1}{x}$. ... (4)

Putting $y^2 = v$ so that $2y\,(dy/dx) = dv/dx$, (4) reduces to

$(dv/dx) - (1/x)\,v = -\,(x^2 + 1)/x$, which is linear equation, ... (5)

Integrating factor of (5) $= e^{\int (-1/x)\,dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = 1/x$.

and solution is $\qquad v/x = -\displaystyle\int \{(x^2+1)/x\}\cdot(1/x)\,dx + c = -\int (1 + x^{-2})\,dx + c$

or $\quad y^2/x = -x + (1/x) + c \quad$ or $\quad x^2 + y^2 - cx - 1 = 0$

or $\quad x^2 + y^2 + 2gx - 1 = 0$, where $2g = -c$, g being parameter.

(b) Proceed as in part (a). **Ans.** $x^2 + y^2 + 2gx + 1 = 0$.

Ex. 9. *Find the differential equation of the family of curves given by the equation* $x^2 - y^2 + 2\lambda xy = 1$, *where* λ *is a parameter. Obtain the differential equation of its orthogonal trajectories and solve it.*

Sol. The given family of curves is $\quad x^2 - y^2 + 2\lambda xy = 1$, where λ is parameter ... (1)

Differentiating (1), $\quad x - y\,(dy/dx) + \lambda\,[y + x\,(dy/dx)] = 0.$ ... (2)

From (1), $\quad \lambda = (1 + y^2 - x^2)/(2xy).$ ... (3)

Eliminating λ between (2) and (3), $\quad x - y\dfrac{dy}{dx} + \dfrac{1 + y^2 - x^2}{2xy}\left[y + x\dfrac{dy}{dx}\right] = 0,$ ... (4)

which is the differential equation of the family of curves given by (1).

Replacing dy/dx by $(-dx/dy)$ in (4), the differential equation of the required orthogonal trajectories is

$$x + y\frac{dx}{dy} + \frac{1 + y^2 - x^2}{2xy}\left(y - x\frac{dx}{dy}\right) = 0$$

or $\quad x^2y + y^3 + y + (xy^2 - x + x^3)\,(dx/dy) = 0 \quad$ or $\quad y\,(x^2 + y^2 + 1)\,dy + x\,(y^2 + x^2 - 1)\,dx = 0$

or $\quad (x^2 + y^2)\,(2x\,dx + 2y\,dy) + 2y\,dy - 2x\,dx = 0.$ ... (5)

Putting $\quad x^2 + y^2 = z \quad$ so that $\quad 2x\,dx + 2y\,dy = dz$, (5) becomes

$z\,dz + 2y\,dy - 2x\,dx = 0 \quad$ so $\quad z^2/2 + y^2 - x^2 = -c/2$, by integration

or $\quad (x^2 + y^2)^2 + 2y^2 - 2x^2 = c$, as $z = x^2 + y^2$.

Ex. 10. *A system of rectangular hyperbolas pass through the fixed points* $(\pm a, 0)$ *and have the origin as centre; show that the orthogonal trajectories is given by* $(x^2 + y^2)^2 = 2a^2\,(x^2 - y^2) + c$.

Sol. Equation of any rectangular hyperbola with origin as centre is

$Ax^2 + 2Hxy - Ay^2 = 1$, where A and H are parameters. ... (1)

(1) passes through $(\pm a, 0) \;\Rightarrow\; Aa^2 = 1 \;\Rightarrow\; A = 1/a^2$.

$\therefore$ (1) gives $\quad (x^2 + y^2)/a^2 + 2Hxy = 1 \quad$ or $\quad x^2 - y^2 + 2Ha^2xy = a^2$

or $\quad x^2 - y^2 + 2\lambda xy = a^2$, where $\lambda\ (= Ha^2)$ is a parameter. ... (1)

Note the equation (1) is similar to equation (1) of Ex. 9 except that now we have a^2 on R.H.S. in place of 1. Now proceed exactly as in Ex. 9 to find the required family of orthogonal trajectories.

Ex. 11. *Find the orthogonal trajectories of family of parabolas* $y^2 = 4a\,(x + a)$, *where a is parameter.* **[Meerut 2000; Purvanchal 2007]**

or *Show that the system of confocal and co-axial parabolas* $y^2 = 4a\,(x + a)$ *is self orthogonal, a being parameter.* **[Purvanchal 1999; I.A.S. (Prel.) 1997, 2001; Pune 2010]**

or *Show that the orthogonal trajectories of the system of parabolas* $y^2 = 4a\,(x + y)$ *belongs to the system itself, a being parameter.* **[Bangalore 2005; Kerala 2001]**

Sol. Given $\quad y^2 = 4a\,(x + a)$, with a as parameter. ... (1)

Differentiating (1), $\quad 2y\,(dy/dx) = 4a \quad$ so that $\quad a = (y/2)\,(dy/dx).$ ... (2)

Eliminating a from (1) and (2), we have

$y^2 = 2y\,(dy/dx)\,\{x + (y/2)\,(dy/dx)\} \quad$ or $\quad y = 2x\,(dy/dx) + y\,(dy/dx)^2,$ ... (3)

which is differential equation of (1). Replacing dy/dx by $-dx/dy$ in (3), the differential equation of the required orthogonal trajectories is

$$y = -2x\frac{dx}{dy} + y\left(-\frac{dx}{dy}\right)^2 \qquad \text{or} \qquad y = -\frac{2x}{(dy/dx)} + \frac{y}{(dy/dx)^2}$$

or $\quad y\,(dy/dx)^2 = -\,2x\,(dy/dx) + y \qquad$ or $\qquad y = 2x\,(dy/dx) + y\,(dy/dx)^2, \qquad \ldots (4)$

which is the same as the differential equation (3) of the given system (1). Hence, the system of parabolas (1) is self orthogonal, *i.e.*, each member of the given system of parabolas intersects its own members orthogonally.

Ex. 12. *Find the orthogonal trajectories of the family of curves* $x^2/(a^2+\lambda) + y^2/(b^2+\lambda) = 1$, *where* λ *is a parameter.* **[Gorakhpur 1996; Kumaun 1995]**

or *Show that the system of confocal conics* $\{x^2/(a^2+\lambda)\} + \{y^2/(b^2+\lambda)\} = 1$ *is self orthogonal.*

[I.A.S. 1993; Bilaspur 1995; Kumaun 1997; Purvanchal 1998, 2007; Meerut 1998]

Sol. Given $\qquad x^2/(a^2+\lambda) + y^2(b^2+\lambda) = 1. \qquad \ldots (1)$

Differentiating (1), $\qquad \dfrac{2x}{a^2+\lambda} + \dfrac{2y}{b^2+\lambda}\dfrac{dy}{dx} = 0 \qquad$ or $\qquad \dfrac{x}{a^2+\lambda} + \dfrac{y}{b^2+\lambda}\dfrac{dy}{dx} = 0$

or $\qquad x(b^2+\lambda) + y(a^2+\lambda)\dfrac{dy}{dx} = 0 \qquad$ or $\qquad \lambda\left(x + y\dfrac{dy}{dx}\right) = -\left(b^2x + a^2y\dfrac{dy}{dx}\right).$

$$\therefore \qquad \lambda = -\ \{b^2x + a^2y\,(dy/dx)\}/\{x + y\,(dy/dx)\}$$

$$\therefore \qquad a^2 + \lambda = a^2 - \frac{b^2x + a^2y\,(dy/dx)}{x + y\,(dy/dx)} = \frac{(a^2 - b^2)\,x}{x + y\,(dy/dx)}$$

and $$b^2 + \lambda = b^2 - \frac{b^2x + a^2y\,(dy/dx)}{x + y\,(dy/dx)} = \frac{-(a^2 - b^2)\,y\,(dy/dx)}{x + y\,(dy/dx)}.$$

Putting the above values of $(a^2+\lambda)$ and $(b^2+\lambda)$ in (1), we have

$$\frac{x^2\,\{x + y\,(dy/dx)\}}{(a^2 - b^2)\,x} - \frac{y^2\,\{x + y\,(dy/dx)\}}{(a^2 - b^2)\,y\,(dy/dx)} = 1$$

or $$\{x + y\,(dy/dx)\}\ \{x - y\,(dx/dy)\} = a^2 - b^2, \qquad \ldots (2)$$

which is the differential equation of the family (1). Replacing dy/dx by $(-\,dx/dy)$ in (2), the differential equation of the required orthogonal trajectories is

$\{x + y\,(-\,dx/dy)\}\ \{x - y\,(-\,dy/dx)\} = a^2 - b^2$ or $\{x + y\,(dy/dx)\}\ \{x - y\,(dx/dy)\} = a^2 - b^2, \ \ldots (3)$

which is the same as the differential equation (2) of the given family of curves (1). Hence, the system of given curves (1) is self orthogonal, *i.e.*, each member of the given family of curves intersects its own members orthogonally.

Ex. 13. *Prove that the orthogonal trajectories of the family of conics* $y^2 - x^2 + 4xy - 2cx = 0$ *consists of a family of cubics with the common asymptote* $x + y = 0$. **[Meerut 2009]**

Sol. Given $\qquad y^2 - x^2 + 4xy - 2cx = 0$, c being a parameter $\qquad \ldots (1)$

Differentating (1), $\qquad 2y\,(dy/dx) - 2x + 4\,[x\,(dy/dx) + y] = 2c. \qquad \ldots (2)$

From (1), $\qquad 2c = (y^2 - x^2 + 4xy)/x. \qquad \ldots (3)$

Eliminating c between (2) and (3), $2y\,(dy/dx) - 2x + 4\,\{x\,(dy/dx) + y\} = (y^2 - x^2 + 4xy)/x$

or $$2xy\,(dy/dx) - 2x^2 + 4x^2\,(dy/dx) + 4xy = y^2 - x^2 + 4xy$$

or $$2x\,(y + 2x)\,(dy/dx) = x^2 + y^2, \qquad \ldots (4)$$

which is the differential equation of (1). Replacing dy/dx by $(-\,dx/dy)$ in (4), the differential equation of the required orthogonal trajectories is

$$2x\,(y + 2x)\,(-\,dx/dy) = x^2 + y^2 \qquad \text{or} \qquad \frac{dy}{dx} = -\frac{2x\,(y + 2x)}{x^2 + y^2} = -\frac{2\,(y/x) + 4}{1 + (y/x)^2} \qquad \ldots (5)$$

Putting $y/x = v$ or $y = xv$ so that $dy/dx = v + x\,(dv/dx)$, (5) gives

$$v + x\frac{dv}{dx} = -\frac{2v+4}{1+v^2} \qquad \text{or} \qquad x\frac{dv}{dx} = -v - \frac{2v+4}{1+v^2} = -\frac{4+3v+v^3}{1+v^2}$$

or $$\frac{1+v^2}{4+3v+v^3}\,dv = -\frac{dx}{x} \qquad \text{or} \qquad 3\frac{dx}{x} + \frac{3v^2+3}{4+3v+v^3}\,dv = 0.$$

Integrating, $3\log x + \log(4 + 3v + v^3) = \log c$, c being an arbitrary constant

or $\log x^3 + \log(4 + 3v + v^3) = \log c$ or $x^3(4 + 3v + v^3) = c$

or $x^3\{4 + 3(y/x) + (y/x)^3\} = c$ or $y^3 + 3x^2y + 4x^3 = \text{c}.$ ... (6)

Re-writing (6), $(y + x)(y^2 - xy + 4x^2) = c.$ ... (7)

The asymptote of (7) corresponding to the factor $(x + y)$ is given by

$$x + y = \lim_{x\to\infty,\, y/x\to -1}\left(\frac{c}{y^2 - xy + 4x^2}\right)$$

$$= \lim_{x\to\infty,\, y/x\to -1}\left[\frac{(c/x^2)}{(y/x)^2 - (y/x) + y}\right] = \lim_{x\to\infty}\frac{(c/x^2)}{(-1)^2 - (-1) + 4} = 0.$$

Hence $x + y = 0$ is the asymptote common to the family of cubics (6).

Exercise 3(A)

1. Find the orthogonal trajectories of the following family of curves:
   (*i*) $y = ax^n$ **[Agra 1996]** **Ans.** $x^2 + ny^2 = c$
   (*ii*) $y = ax^3$ **[Garhwal 2005, 2010]** **Ans.** $x^2 + 3y^2 = c$
   (*iii*) $cx^2 + y^2 = 1$ **[Delhi B.A. (Prog) II 2011]** **Ans.** $y = ce^{(x^2+y^2)/2}$
   (*iv*) $y^2 = 4ax$ **[Kanpur 2009]** **Ans.** $2x^2 + y^2 = c^2$
   (*v*) $x^2 + y^2 = a^2$ **[I.A.S. 2008]** **Ans.** $y = cx$
   (*vi*) $x^{2/3} + y^{2/3} = a^{2/3}$ **[Kakatiya 2003]** **Ans.** $y^{4/3} - x^{4/3} = c^{4/3}$
   (*vii*) $x^2 + y^2 = cx^3$ **[Delhi B.A. (Prog) II 2009, 10]** **Ans.** $yx^2 + y^3 = c'$
2. Find the orthogonal trajectories of the family of semicubical parabolas $ay^2 = x^3$ where a is a prameter. **[Garhwal 1993; I.A.S. 1992; Nagpur 2005]** **Ans.** $2x^2 + 3y^2 = c^2$
3. Find the orthogonal trajectories of the family of the curves $xy = k^2$. **[Kanpur 2007; Mysore 2004]** **Ans.** $x^2 - y^2 = a^2$
4. Find the orthogonal trajectories of $x^2 + y^2 + 2gx + c = 0$, where c is the parameter. **Ans.** $x + g = ay$
5. Show that the orthogonal trajectories of $x^2 + y^2 + 2gx + 1 = 0$ is $x^2 + y^2 + 2fy - 1 = 0$.
6. Find the orthogonal trajectories of the curve $y^2 = x^3/(a - x)$, a being parameter. **Ans.** $(x^2 + y^2)^2 = c\,(y^2 + 2x^2)$
7. Find the orthogonal trajectories of the family of circles $(x - 1)^2 + y^2 + 2ax = 0$, a being the parameter. **Ans.** $x^2 - y^2 - cy + 1 = 0$

3.6 Determination of orthogonal trajectories in polar coordinates

Let the equation of the given family of curves be $f(r, \theta, c) = 0,$ ... (1)

where c is a parameter. Differentiating (1) with respect to θ and eliminating c, between (1) and the derived result, we shall arrive at the differentiated equation of the given family of curves (1). Let it be

$$F(r, \theta, dr/d\theta) = 0. \qquad \ldots (2)$$

Let ϕ be the angle between the tangent PT to a member PQ of the family of given curves and radius vector OP at any point $P(r, \theta)$.

$$\therefore \qquad \tan\phi = r\,(d\theta/dr). \qquad \ldots (3)$$

Let (R, Θ) be the current coordinates of any point of a trajectory. At point of intersection P of any member of (2) with the trajectory PQ', let ϕ' be angle which the tangent PT' to the trajectory makes with the common radius vector OP.

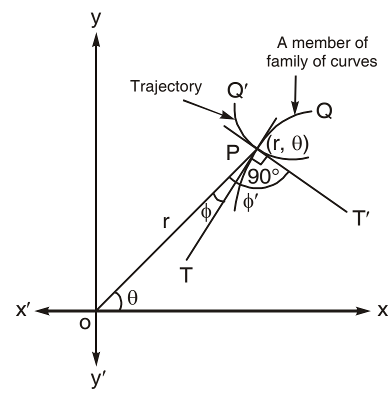


Fig. 3.4

$$\therefore \qquad \tan\phi' = R\,(d\Theta/dR). \qquad \ldots (4)$$

Let PT and PT' intersect at $90°$. Then, from figure, $\phi' - \phi = 90°$ so that $\phi' = 90° + \phi$.

$$\therefore \quad \tan\phi' = \tan(90° + \phi) = -\cot\phi \qquad \text{or} \qquad \tan\phi\tan\phi' = -1. \quad \ldots (5)$$

Using (3) and (4), (5) gives $\left(r\dfrac{d\theta}{dr}\right)\left(R\dfrac{d\Theta}{dR}\right) = 1$ or $\dfrac{dr}{d\theta} = -rR\dfrac{d\Theta}{dR}$ .... (6)

At a point of intersection of any member of (2) with the trajectory, $r = R, \quad \theta = \Theta.$ ... (7)

Eliminating r, θ and $dr/d\theta$ from (2), (6) and (7), we have

$$F\left(R, \Theta, -R^2\frac{d\Theta}{dR}\right) = 0, \qquad \ldots (8)$$

which is the differential equation of the required family of trajectories. In the usual notation, we observe that the differential equation of the family of orthogonal trajectories of the family of curves given by

$$F(r, \theta, dr/d\theta) = 0 \qquad \ldots (9)$$

is

$$F\left(r, \theta, -r^2\frac{d\theta}{dr}\right) = 0, \qquad \ldots (10)$$

showing that it can be obtained by replacing $dr/d\theta$ by $-r^2(d\theta/dr)$.

3.7 Working rule for getting orthogonal trajectories in polar coordinates

Step I. Differentiate the given equation of the family of curves, w.r.t. 'θ'. Eliminate the parameter between this derived equation and the given equation of the family of curves to obtain the differential equation of the given family of curves. Note that while differentiating the given equation, we shall generally take logarithm and then differentiate.

Step II. In the differential equation found in step I, replace $(dr/d\theta)$ by $-r^2(d\theta/dr)$ and thus obtain the differential equation of the required orthogonal trajectories.

Step III. Obtain the general solution of the differential equation obtained in step II. The solution so obtained will give us the desired orthogonal trajectories.

3.8 Solved examples of Type 2 based on Art. 3.7

Ex. 1(*a*). *Find the orthogonal trajectories of cardioids $r = a(1 - \cos\theta)$, a being parameter.*

[Meerut 1993; Gorakhpur 1996; Kumaun 1996; I.A.S. (Prel.) 2001; Nagarjuna 2003]

Sol. The given family of cardioids is $r = a(1 - \cos\theta)$. ... (1)

Taking logarithm of both sides of (2), we get $\log r = \log a + \log(1 - \cos\theta)$. ... (2)

Differentiating (2) with respect to 'θ', we get $\quad \dfrac{1}{r}\dfrac{dr}{d\theta} = \dfrac{\sin\theta}{1-\cos\theta}$. ... (3)

Since (3) is free from parameter 'a', hence (3) is the differential equation of the given family (1).

Replacing $dr/d\theta$ by $-r^2\,(d\theta/dr)$ in (3), the differential equation of the required orthogonal trajectories is

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \frac{\sin\theta}{1-\cos\theta} = \frac{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{2\sin^2\frac{1}{2}\theta} = \cot\frac{1}{2}\theta$$

or $\quad (1/r)\,dr = -\tan(\theta/2)\,d\theta$, on separating variables

Integrating, $\quad \log r = 2\log\cos\frac{1}{2}\theta + \log c \quad$ or $\quad \log r = \log\left(c\cos^2\frac{1}{2}\theta\right)$

or $\quad r = (c/2)(1+\cos\theta) \quad$ or $\quad r = b\,(1+\cos\theta)$, ... (4)

where $b\ (= c/2)$ is arbitrary constant. (4) gives another family of cardioids.

Note: We know that (r, θ) and $(-r, \pi+\theta)$ represent the same point in polar coordinates. Accordingly, replacing r and θ by $-r$ and $\pi+\theta$ respectively in (4), we obtain

$-r = b\,[1+\cos(\pi+\theta)] \quad$ or $\quad r = (-b)(1-\cos\theta)$. ... (5)

If we set $(-b) = a$ in (5), (5) reduces to (1). Hence, the given family of cardioids (1) is self orthogonal.

Ex. 1(*b*). *Find the orthogonal trajectories of the family of curves $r = a\,(1+\cos\theta)$, where a is the parameter.* **[Agra 1994, 1995; Garhwal 1994; Gorakhpur 1993; Lucknow 1992, 1995; Meerut 1993; Purvanchal 1992; G.N.D.U Amritsar 2010]**

Sol. The given family of curves is $\quad r = a\,(1+\cos\theta)$. ... (1)

Taking logarithm of both sides $\quad \log r = \log a + \log(1+\cos\theta)$. ... (2)

Differentiating (2) w.r.t. 'θ', $\quad (1/r)(dr/d\theta) = (-\sin\theta)/(1+\cos\theta)$, ... (3)

which is the differential equation (1). Replacing $dr/d\theta$ by $-r^2\,(d\theta/dr)$ in (3), the differential equation of the required orthogonal trajectories is

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = \frac{-\sin\theta}{1+\cos\theta} = -\frac{2\sin(\theta/2)\cos(\theta/2)}{1+2\cos^2(\theta/2)-1} = -\tan(\theta/2)$$

or $\quad r\,(d\theta/dr) = \tan(\theta/2) \quad$ or $\quad (1/r)\,dr = \cot(\theta/2)\,d\theta$

Integrating, $\quad \log r = 2\log\sin(\theta/2) + \log c \quad$ or $\quad r = c\sin^2(\theta/2)$

or $\quad r = c\,\{(1-\cos\theta)/2\} \quad$ or $\quad r = b\,(1-\cos\theta)$, taking $b = c/2$

which is the equation of the required trajectories with b as parameter.

Ex. 1(*c*). *Show that the families of curves given by the equation $r = a\,(1+\cos\theta)$ and $r = b\,(1-\cos\theta)$ intersect orthogonally.* **[Purvanchal 1996]**

Sol. Here we are to show that the family of orthogonal trajectories of the family of curves $r = a\,(1+\cos\theta)$ is $r = b\,(1-\cos\theta)$, where a and b are parameters of the respective families. This is same as proved in Ex. 1(*a*).

Ex. 1(*d*). *Find the orthogonal trajectories of $r = a\,(1+\cos n\theta)$.*

Sol. Given family is $\quad r = a\,(1+\cos n\theta)$, where a is parameter. ... (1)

Taking logarithm of both sides, $\quad \log r = \log a + \log(1+\cos n\theta)$. ... (2)

Differentiating (2) w.r.t. θ $\quad (1/r)(dr/d\theta) = -(n\sin n\theta)/(1+\cos n\theta)$ ... (3)

which is differential equation of the family of curves (1). Replacing $dr/d\theta$ by $-r^2\,(d\theta/dr)$ in (3), the differential equation of the required trajectories is

$$\frac{1}{r}\left(-r^2\frac{d\theta}{dr}\right) = -\frac{n\sin n\theta}{1+\cos n\theta} \quad \text{or} \quad \frac{n\,dr}{r} = \frac{1+\cos n\theta}{\sin n\theta}\,d\theta$$

or $$\frac{n\,dr}{r} = \frac{2\cos^2(n\theta/2)\,d\theta}{2\sin(n\theta/2)\cos(n\theta/2)} \qquad \text{or} \qquad n\frac{dr}{r} = \cot(n\theta/2)\,d\theta.$$

Integrating, $n \log r = (2/n) \times \log \sin (n\theta/2) + (1/n) \times \log c$, c being arbitrary constant

or $n^2 \log r = \log \sin^2 (n\theta/2) + \log c$ or $r^{n^2} = c \sin^2 (n\theta/2)$

or $r^{n^2} = (c/2)(1 - \cos n\theta)$ or $r^{n^2} = b(1 - \cos n\theta)$, taking $b = c/2$

which is the equation of required orthogonal trajectories with b as parameter.

Ex. 2. *Find the equation of the system of orthogonal trajectories of the parabolas* $r = 2a/(1 + \cos\theta)$, *where a is the parameter.* **[Purvanchal 1995; Ravishankar 1995]**

Sol. From the given equation, we get $\log r = \log 2a - \log (1 + \cos\theta)$. ... (1)

Differentiating (1) w.r.t. θ, we get $(1/r)(dr/d\theta) = (\sin\theta)/(1 + \cos\theta)$ ... (2)

(2) is the differential equation of the given system of parabolas. Replacing $dr/d\theta$ by $-r^2(d\theta/dr)$ in (2), the differential equation of the required orthogonal trajectories is

$$\frac{1}{r}(-r^2)\frac{d\theta}{dr} = \frac{2\sin(\theta/2)\cos(\theta/2)}{2\cos^2(\theta/2)} \qquad \text{or} \qquad \frac{dr}{r} = -\cot\frac{\theta}{2}\,d\theta.$$

Integrating, $\log r = -2 \log \sin (\theta/2) + \log c$, or $r = c/\sin^2(\theta/2)$

or $r = 2c/(1 - \cos\theta)$, as $\sin^2(\theta/2) = (1 - \cos\theta)/2$

Ex. 3. *Find the orthogonal trajectories of the curves* $A = r^2 \cos\theta$, *where A is the parameter.*

or Prove that the orthogonal trajectories of the curves $A = r^2\cos\theta$ *are the curves* $B = r\sin^2\theta$.

[I.A.S. (Prel.) 2005; Purvanchal 1994]

Sol. Given family of curves is $r^2 \cos\theta = A$. ... (1)

Taking logarithm of both sides, $2 \log r + \log \cos\theta = \log A$. ... (2)

Differentiating (2) w.r.t. 'θ', we get $(2/r)(dr/d\theta) - \cot\theta = 0$, ... (3)

which is differential equation of the family (1). Replacing $dr/d\theta$ by $-r^2(d\theta/dr)$ in (3), the differential equation of the required orthogonal trajectories is

$(2/r)(-r^2)(d\theta/dr) - \tan\theta = 0$ or $(1/r)\,dr + 2\cot\theta\,d\theta = 0$.

Integrating, $\log r + 2 \log \sin\theta = \log B$ or $r\sin^2\theta = B$.

Ex. 4(*a*). *Prove that the orthogonal trajectories of* $r^n \cos n\theta = a^n$ *is* $r^n \sin n\theta = c^n$.

Sol. Given $r^n \cos n\theta = a^n$, where a is a parameter. ... (1)

From (1), $n \log r + \log \cos n\theta = \log a^n$. ... (2)

Differentiating (2), $(n/r)(dr/d\theta) - n\tan n\theta = 0$ or $(1/r)(dr/d\theta) - \tan n\theta = 0$, ... (3)

which is the differential equation of family (1). Replacing $dr/d\theta$ by $-r^2(d\theta/dr)$ in (3), the differential equation of the required orthogonal trajectories is

$(1/r)(-r^2)(d\theta/dr) - \tan n\theta = 0$ or $(1/r)\,dr + \cot n\theta\,d\theta = 0$.

Integrating, $\log r + (1/n) \log \sin n\theta = \log c$, c being an arbitrary constant.

or $n \log r + \log \sin n\theta = n \log c$ or $r^n \sin n\theta = c^n$,

which is the required equation of orthogonal trajectories.

Ex. 4(*b*). *Find the orthogonal trajectories of* $r^n \sin n\theta = a^n$.

[I.A.S. (Prel.) 1999; Agra 1996; Kanpur 1995; Meerut 1996]

Sol. Proceed as in Ex. 4(*a*). **Ans.** $r^n \cos n\theta = c^n$.

Exercise 3(B)

1. (*a*) Find the orthogonal trajectories of the system of curves $r^n = a^n \cos n\theta$, where a is parameter. **[Kanpur 2008; Meerut 2007; Kumaun 1998]**

(*b*) Find the orthogonal trajectories of the system of curves $r^n = a^n \sin n\theta$, where a is parameter. **[Lucknow 1993]**

(*c*) Determine the orthogonal trajectories of the system of curves $r^n = a^n \cos n\theta$ and hence find the orthogonal trajectories of the series of lemniscates $r^2 = a^2 \cos 2\theta$.

(*d*) Find the orthogonal trajectories of the family of curves given by $r^2 = a^2 \cos 2\theta$, a being the parameter.

Ans. (*a*) $r^n = c^n \sin n\theta$, (*b*) $r^n = c^n \cos n\theta$, (*c*) $r^n = c^n \sin n\theta$, $r^2 = c^2 \sin 2\theta$, (*d*) $r^2 = c^2 \sin 2\theta$

2. Find the orthogonal trajectories of the family of curves $r = c(\cos\theta + \sin\theta)$, where c is the parameter.

Ans. $r = d(\sin\theta - \cos\theta)$, d being parameter

3. Find the orthogonal trajectories of the family of curves $r = a + \sin 5\theta$, a being the parameter.

Ans. $\sec 5\theta + \tan 5\theta = ce^{25/r}$, c being parameter

4. Find the orthogonal trajectories of the following family of curves: (*i*) $r\theta = a$ (*ii*) $r = a\theta$; a being the parameter.

[Sol (*i*). Given family is $r\theta = a$, a being a parameter ... (1)

Differentiating (1) w.r.t. 'θ', we get $(dr/d\theta)\,\theta + r = 0$, ... (2)

which in the differential equation of (1). Replacing $dr/d\theta$ by $-r^2(d\theta/dr)$ in (2), the differential equation of the required orthogonal trahectories is

$-r^2(d\theta/dr) + r = 0$ or $(1/r)\,dr = \theta d\theta$ so that $\log r - \log c = \theta^2/2$ or $r = c\,e^{\theta^2/2}$

(*ii*) Try yourself. **Ans.** $r = c\,e^{-\theta^2/2}$ **]**

5. Find the orthogonal trajectories of $r = e^{a\theta}$ with a is parameter.

Ans. $\theta^2 + (\log r)^2 = c^2$, c being parameter

6. Find the orthogonal trajectories of $\{r + (k^2/r)\}\cos\theta = \alpha$, α being the parameter.

Ans. $r^2 - k^2 = rc \operatorname{cosec}\theta$

3.9 Determination of oblique trajectories in cartesian coordinates

Let the equation of the given family of curves be $f(x, y, c) = 0$, ... (1)

where c is a parameter. Differentiating (1) w.r.t. 'x' and eliminating parameter c between (1) and the derived result, we shall arrive at the differential equation the given family of curves (1). Let it be

$$F(x, y, dy/dx) = 0. \qquad \text{... (2)}$$

Let ψ be the angle between the tangent PT to a member PQ of the family of curves and x-axis at any point $P(x, y)$. Hence, $\tan\psi = dy/dx$. ... (3)

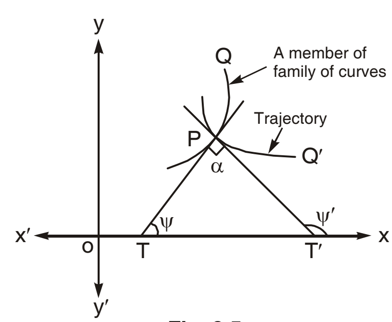


Fig. 3.5

Let (X, Y) be the current coordinates of any point of trajectory. At point of intersection P of any member of (2) with the trajectory PQ', let ψ' be the angle which the tangent PT' to the trajectory makes with x-axis.

$$\therefore \qquad \tan\psi' = dY/dX. \qquad \text{... (4)}$$

Let PT and PT' intersect at angle α. Then, we have

$\therefore \qquad \tan\alpha = \dfrac{(dy/dx) - (dY/dX)}{1 + (dy/dx)(dY/dX)}$ so that $\dfrac{dy}{dx} = \dfrac{(dY/dX) + \tan\alpha}{1 - (dY/dX)\tan\alpha}$. ... (5)

At a point of intersection of any member of (2) with the trajectory, $x = X, \quad y = Y$. ... (6)

Eliminating x, y and dy/dx from (2), (5) and (6), $F\left(X, Y, \dfrac{(dY/dX) + \tan\alpha}{1 - (dY/dX)\tan\alpha}\right) = 0$, ... (7)

which is the differential equation of the required family of trajectories. In the usual notation, we observe that the differential equation of the family of trajectories of the family of curves given by

$$F(x, y, dy/dx) = 0, \qquad \text{... (8)}$$

is

$$F\left(x, y, \frac{(dy/dx) + \tan\alpha}{1 - (dy/dx)\tan\alpha}\right) = 0, \qquad \text{... (9)}$$

showing that it can be obtained on replacing dy/dx by $[(dy/dx) + \tan\alpha]/[1 - (dy/dx)\tan\alpha]$, *i.e.*, $(p + \tan\alpha)/(1 - p\tan\alpha)$, where $p = dy/dx$

3.10 Working rule for finding the oblique trajectories which cut every member of the given family of curves at a constant angle α

All the steps are similar to working rule of Art. 3.4. For oblique trajectories we replace dy/dx (*i.e.*, p) by $(p + \tan\alpha)/(1 - p\tan\alpha)$.

3.11 Solved examples of type 3 based on Art. 3.10

Ex. 1. *Find the family of curves whose tangents form the angle of $\pi/4$ with the hyperbola $xy = c$.* **[I.A.S. 1994, 2006]**

Sol. The given family of curves is $\quad xy = c$, where c is a parameter ... (1)

Differentiating (1), $\quad y + x(dy/dx) = 0 \quad$ or $\quad y + xp = 0$, where $p = dy/dx$. ... (2)

(2) is the differential equation of given family (1).

Replacing p by $\dfrac{p + \tan(\pi/4)}{1 - p\tan(\pi/4)}$, *i.e.*, $\dfrac{p+1}{1-p}$ in (2) the differential equation of the desired family of curves is

$$y + \frac{p+1}{1-p}\,x = 0 \quad \text{or} \quad p = \frac{y+x}{y-x} \quad \text{or} \quad \frac{dy}{dx} = \frac{(y/x)+1}{(y/x)-1}. \qquad \text{... (3)}$$

Let $\quad y/x = v, \quad$ *i.e.*, $\quad y = xv \quad$ so that $\quad dy/dx = v + x(dv/dx)$. ... (4)

$\therefore$ From (3), $\quad v + x\dfrac{dv}{dx} = \dfrac{v+1}{v-1} \quad$ or $\quad x\dfrac{dv}{dx} = -\dfrac{v^2 - 2v - 1}{v - 1}$

or $\qquad (2/x)\,dx = -\{2(v-1)/(v^2 - 2v - 1)\}\,dv.$

Integrating, $\quad 2\log x = -\log(v^2 - 2v - 1) + \log c$, c being an arbitrary constant

or $\quad \log x^2 + \log(v^2 - 2v - 1) = \log c \quad$ or $\quad x^2(v^2 - 2v - 1) = c$

or $\quad x^2(y^2/x^2 - 2y/x - 1) = c \quad$ or $\quad y^2 - 2xy - x^2 = c.$

Ex. 2. *Find the equation of the family of oblique trajectories which cut the family of concentric circles at 30°.* **[Gorakhpur 1997]**

Sol. Let the equation of the given family of concentric circles, having (0, 0) as common centre be $\quad x^2 + y^2 = a^2$, where a is the parameter. ... (1)

Differentiating (1), $\quad 2x + 2y(dy/dx) = 0 \quad$ or $\quad x + yp = 0$, where $p = dy/dx$. ... (2)

Replacing p by $\dfrac{p+\tan 30°}{1-p\tan 30°}$, *i.e.*, $\dfrac{p+(1/\sqrt{3})}{1-p(1/\sqrt{3})}$, *i.e.*, $\dfrac{\sqrt{3}p+1}{\sqrt{3}-p}$ in (2),

the differential equation of the desired family of curves is

$$x+y\{(\sqrt{3}p+1)/(\sqrt{3}-p)\}=0 \qquad \text{or} \qquad x(\sqrt{3}-p)+y(\sqrt{3}p+1)=0$$

or $$p=\frac{x\sqrt{3}+y}{x-y\sqrt{3}} \qquad \text{or} \qquad \frac{dy}{dx}=\frac{\sqrt{3}+(y/x)}{1-\sqrt{3}(y/x)}. \qquad \text{... (3)}$$

Putting $y/x = v$ or $y = xv$ so that $dy/dx = v + x\,(dv/dx)$, (3) gives

$$v+x\frac{dv}{dx}=\frac{\sqrt{3}+v}{1-v\sqrt{3}} \qquad \text{or} \qquad x\frac{dv}{dx}=\frac{\sqrt{3}+v}{1-v\sqrt{3}}-v=\frac{\sqrt{3}(v^2+1)}{1-v\sqrt{3}}$$

or $$\sqrt{3}\frac{dx}{x}=\frac{1-v\sqrt{3}}{v^2+1} \qquad \text{or} \qquad \sqrt{3}\frac{dx}{x}-\frac{dv}{v^2+1}+\frac{\sqrt{3}}{2}\frac{2v\,dv}{v^2+1}=0.$$

Integrating, $$\sqrt{3}\log x-\tan^{-1}v+(\sqrt{3}/2)\log(v^2+1)=(\sqrt{3}/2)\log c$$

or $$2\log x+\log(v^2+1)-\log c=(2/\sqrt{3})\tan^{-1}v$$

or $$\log\{x^2(v^2+1)/c\}=(2/\sqrt{3})\tan^{-1}v \qquad \text{or} \qquad x^2(v^2+1)/c=e^{(2/\sqrt{3})\tan^{-1}v}$$

or $$x^2[(y/x)^2+1]=ce^{(2/\sqrt{3})\tan^{-1}(y/x)} \qquad \text{or} \qquad y^2+x^2=ce^{(2/\sqrt{3})\tan^{-1}(y/x)},$$

which is the required family of curves, c being a parameter.

Exercise 3(C)

1. Determine the 45° trajectories of the family of concentric circles $x^2 + y^2 = a^2$, a being the parameters.
[Delhi Math (H) 1996] **Ans.** $x^2+y^2=ce^{-2\tan^{-1}(y/x)}$, where c is an arbitrary constant.

2. Given the set of lines $y = ax$, a being arbitrary, find all the curves that cut these lines at a constant angle α. **Ans.** $(x^2+y^2)^{\tan\alpha}=ce^{-2\tan^{-1}(y/x)}$

3. Find the family of curves cutting the family of parabolas $y^2 = 4ax$ at 45°.

Ans. $x^2+y^2=ce^{-2\tan^{-1}(y/x)}$, c being an arbitrary constant

Objective Problem on Chapter 3

Ex. 1. *The differential equation of the orthogonal trajectories of the system of parabolas* $y = ax^2$ is (*a*) $y' = x^2 + y$ (*b*) $y' = x - y^2$ (*c*) $y' = -(x/2y)$ (*d*) $y' = x/(2y)$ **[I.A.S. (Prel.) 1993]**

Sol. Ans. (*c*). Sol. Given $y = ax^2$, where a is a parameter ... (1)

From (1), $dy/dx = 2ax = 2x\,(y/x^2)$, using (1) ... (2)

Replacing dy/dx by $-dx/dy$ in (2), the required differential equation of the orthogonal trajectory is $-dx/dy = 2y/x$ or $dy/dx = -(x/2y)$ or $y' = -(x/2y)$.

Ex. 2. *Consider the Assertion* (*A*) *and Reason* (*R*) *given below:*

Assertion (*A*): *The curves* $y = ax^3$ *and* $x^2 + 3y^2 = c^2$ *form orthogonal trajectories.*

Reason (*R*): *The differential equation of the second curve is obtained from the differential equation of the first by replacement of* dy/dx *by* $-(dx/dy)$.

The correct answer is

(*a*) *Both A and R are true and R is the correct explanation of A.*

(*b*) *Both A and R are true but R is not a correct explanation of A.*

(*c*) *A is true but R is false.* (*d*) *A is false but R is true.* **[I.A.S. (Prel.) 1993]**

Sol. Ans. (*a*). Refer Art. 3.4.

Ex. 3. *The orthogonal trajectories of the hyperbolas $xy = c$ is*
(*a*) $x^2 - y^2 = c$ (*b*) $x^2 = cy^2$ (*c*) $x^2 + y^2 = c$ (*d*) $x = cy^2$. **[I.A.S. (Prel.) 1994]**

Sol. Ans. (*a*). Use Art. 3.4.

Ex. 4. *The equation $y - 2x = c$ represents the orthogonal trajectories of the family*
(*a*) $y = Ce^{-2x}$ (*b*) $x^2 + 2y^2 = C$ (*c*) $xy = C$ (*d*) $x + 2y = C$. **[I.A.S. (Prel.) 1995]**

Sol. Ans. (*d*). Use Art. 3.4.

Ex. 5. *The orthogonal trajectories of the parabola $y^2 = 4a\,(x + a)$, a being the parameter, are given by* (*a*) $y^2 = 4b\,(x + b)$ (*b*) $y^2 = 4b\,(x - b)$
(*c*) $y^2 = 4bx$ (*d*) $x^2 = 4by$.

Sol. Ans. (*a*). Refer solved Ex. 11, Art. 3.5. **[I.A.S. (Prel.) 1997]**

Ex. 6. *The orthogonal trajectories of the system of curves $r^n \sin n\theta = k^n$, k being arbitrary constant, are* (*a*) $r^n \cos n\theta = a$ (*b*) $r^n \cos \theta = a$
(*c*) $r^2 \cos n\theta = a$ (*d*) $r^n \tan n\theta = a$. **[I.A.S. (Prel.) 1999]**

Sol. Ans. (*a*). Sol. Given $r^n \sin n\theta = k^n$, k being parameter ... (1)

From (1), $n \log r + \log \sin n\theta = \log k^n$ ... (2)

Differentiating (2), $(n/r)\,(dr/d\theta) + n \cot n\theta = 0$ or $(1/r)\,(dr/d\theta) + \cot n\theta = 0$ ... (3)

which is the differential equation of family (1). Replacing $dr/d\theta$ by $-r^2\,(d\theta/dr)$ in (3), the differential equation of the required orthogonal trajectories is

$$(1/r)\,(-r^2)\,(d\theta/dr) + \cot n\theta = 0 \quad \text{or} \quad (1/r)\,dr - \tan n\theta\,d\theta = 0$$

Integrating, $\log r + (1/n) \log \cos n\theta = (1/n) \log a$ or $r^n \cos n\theta = a$

Ex. 7. *The equation whose solution is self orthogonal is* (taking p = dy/dx)
(*a*) $p - (1/p) = p^2$ (*b*) $(px + y)\,(x + yp) - \lambda p = 0$
(*c*) $(px - y)\,(x + yp) - \lambda p = 0$ (*d*) $(px + y)\,(x - yp) - \lambda p = 0$ **[I.A.S. (Prel.) 1999]**

Sol. Ans. (*d*). Replace dy/dx (*i.e.*, p) by $(-\,dx/dy)$, *i.e.*, $(-\,1/p)$ in all given equations one by one. Then, we find that equation given in part (*d*) remains unchanged even after replacing p by $-\,1/p$. Hence, equation whose solution family is self orthogonal is given by (*d*). Refer Art. 3.3 and Art.3.4.

Ex. 8. *If k is a parameter, then the orthogonal trajectories of the cardioia $r = k\,(1 - \cos \theta)$ is* (*a*) $r = c\,(1 + \cos \theta)$ (*b*) $r = c\,(1 - \sin \theta)$ (*c*) $r\,(1 + \cos \theta) = c$ (*d*) $r\,(1 - \sin \theta) = c$.

Sol. Ans. (*a*). Refer solved Ex. 1, Art. 3.8. **[I.A.S. (Prel.) 2000]**

Ex. 9. *The orthogonal trajectories of the system of parabolas $y^2 = 4a\,(x + a)$, a being parameter, is given by the system of curves*
(*a*) $y^2 = 4a\,(x + a)$ (*b*) $y^2 = 4a\,(x - a)$ (*c*) $y^2 = 4ax$ (*d*) $x^2 = 4ay$. **[I.A.S. (Prel.) 2001]**

Sol. Ans. (*a*). Refer Ex. 11, Art. 3.5.

Ex. 10. *The orthogonal trajectories of family $r^n \sin n\theta = a^n$, a being parameter, is*
(*a*) $r^n \sin n\theta = c$ (*b*) $r^n \cos n\theta = c$ (*c*) $r^n \sin^n \theta = c$ (*d*) $r^n \cos^n \theta = c$. **[I.A.S. (Prel.) 2002]**

Sol. Ans. (*b*). Refer Ex. 4(*b*) of Art. 3.8.

Ex. 11. *The orthogonal trajectories of the family $y^2 = 4ax + 4a^2$ is the family*
(*a*) $x^2 = 4ay + 4a^2$ (*b*) $y^2 = 4ay + 4a^2x$ (*c*) $y^2 = 4ax + 4a^2$ (*d*) $x^2 = 4ax + 4a^2y$

Sol. Ans. (*c*). Refer solved Ex. 11, Art. 3.5. **[I.A.S. (Prel.) 2003]**

Ex. 12. *If $2x\,(1 - y) = K$ and $g\,(x, y) = L$ are orthogonal families of curves where K and L are constants, then $g\,(x, y)$ is* (*a*) $x^2 + 2y - y^2$ (*b*) $2y\,(1 - x)$ (*c*) $x^2 + 2x - y^2$ (*d*) $x^2 + 2y + y^2$.

Sol. Ans. (*a*). Proceed as in Art. 3.4. **[GATE 1999]**

Ex. 13. *The orthogonal trajectories to family of straight lines $y = k\,(x - 1)$, $k \in R$, are given by* (*a*) $(x - 1)^2 + (y - 1)^2 = C^2$ (*b*) $x^2 + y^2 = C^2$ (*c*) $x^2 + (y - 1)^2 = C^2$ (*d*) $(x - 1)^2 + y^2 = C^2$

Sol. Ans. (*d*). Proceed as in Art. 3.4. **[GATE 2004]**

Ex. 14. *The orthogonal trajectory of the cardioid $r = a\ (1 + \cos\theta)$, a being the parameter, is* (*a*) $r = a\ (1 - \cos\theta)$ (*b*) $r = a \cos\theta$ *c*) $r = a\ (1 + \cos\theta)$ (*d*) $r = a\ (1 + \sin\theta)$.

Sol. Ans. (*a*). Refer solved Ex. 1(*b*), Art. 3.8. **[I.A.S. (Prel.) 2004]**

Ex. 15. *Which one of the following curves is the orthogonal trajectory of straight lines passing through a fixed point* (a, b)? (*a*) $x - a = k\ (y - b)$ (*b*) $(x - a)\ (y - b) = k$ (*c*) $(x - a)^2 = k\ (y - b)$ (*d*) $(x - a)^2 + (y - b)^2 = k$. **[I.A.S. (Prel.) 2006]**

Sol. Ans. (*d*). Equation of family of straight lines passing through (a, b) is

$$y - b = m\ (x - a),\ m \text{ being the parameter.} \qquad \dots (1)$$

Differentiating w.r.t. 'x', (1) gives $dy/dx = m$ ... (2)

Eliminating m from (1) and (2), $y - b = (x - a)\ (dy/dx)$, ... (3)

which is the differential equation of the given family of straight lines (1). Replacing dy/dx by $-(dx/dy)$, the differential equation of the required orthogonal trajectories is given by

$$y - b = -(x - a)\ (dy/dx) \qquad \text{or} \qquad 2\ (x - a)\ dx + 2\ (y - b)\ dy = 0$$

Integrating, $(x - a)^2 + (y - b)^2 = k$, k being arbitrary constant.

Ex. 16. *The equation of the family of curves orthogonal to the family $y = ax^3$ is :*

(*a*) $x^2 + 3y^2 = 0$ (*b*) $x^2 - 3y^2 = 0$ (*c*) $y^2 + 3x^2 = c^2$ (*d*) $x^2 + 3y^2 = c^2$ **[Garhwal 2010]**

Sol. Ans. Given $y = ax^3$, a being parameter ... (1)

Differentiating (1) w.r.t 'x', $dy/dx = 3\ ax^2$ ... (2)

Eliminating a between (1) and (2), $dy/dx = 3 \times (y/x^3) \times x^2$ or $dy/dx = 3y/x$, ... (3)

which is the differential equation of (1). Replacing dy/dx by $-dx/dy$, the differential equation of the required family of orthogonal trojectories is given by

$$-(dx/dy) = 3y/x \qquad \text{so that} \qquad 2xdx + 6y\ dy = 0 \qquad \dots(4)$$

Integrating (4), $x^2 + 3y^2 = c^2$, c being an arbitrary constant

4

Equations of the First Order But Not of The First Degree, Singular Solutions And Extraneous Loci

PART 1: DIFFERENT METHODS OF FINDING GENERAL SOLUTIONS

4.1 Equations of the first order but not of the first degree

Defintion. The general first order differential equation of degree $n > 1$ is

$$P_0 (dy/dx)^n + P_1 (dy/dx)^{n-1} + P_2 (dy/dx)^{n-2} + \dots + P_{n-1} (dy/dx) + P_n = 0, \qquad \dots (1)$$

where $P_0, P_1, P_2, \dots, P_{n-1}, P_n$ are functions of x and y.

It is a convention to denote dy/dx by p and so (1) becomes

$$P_0 p^n + P_1 p^{n-1} + P_2 p^{n-2} + \dots + P_{n-1} p + P_n = 0.$$

Such equations can be solved by one (or more) of some methods given in this chapter. In each of these methods, the given problem is reduced to that of solving one or more equations of the first order and the first degree (already discussed in chapter 2).

Remarks. In this Chapter, we shall present solutions in the following two forms:

(*i*) **The cartesian form** will consist of x, y and an arbitrary constant c, say. For example, $y^2 = 2cx + c^2$ is general solution in the cartesian form.

(*ii*) **The parametric form** will consist of two equations of form $x = f_1 (p, c)$ and $y = f_2 (p, c)$, where c is an arbitrary constant. These two equations together will give the parametric equations of general solution. This situation arises only when the elimination of p is not possible during the process of solving certain equations. Note that p is treated as parameter for this purpose. Sometimes even the above form is not possible. Then we regard the following forms of solutions to be in parametric form: (*i*) $y = f(x, p)$, $F(x, p, c) = 0$ (*ii*) $x = g(y, p)$, $G(y, p, c) = 0$.

4.2 Method I. Equations solvable for p

Let

$$P_0 p^n + P_1 p^{n-1} + P_2 p^{n-2} + \dots + P_{n-1} p + P_n = 0 \qquad \dots (1)$$

be the given differential equation of the first order and degree $n > 1$.

Since (1) is solvable for p, it can be put in the form

$$[p - f_1 (x, y)]\, [p - f_2 (x, y)] \dots [p - f_n (x, y)] = 0. \qquad \dots (2)$$

Equating each factor of (2) to zero, we obtain n equations of the first order and the first degree, namely,

$$p = dy/dx = f_1 (x, y), \qquad p = dy/dx = f_2 (x, y), \qquad \dots., \qquad p = dy/dx = f_n (x, y). \qquad \dots (3)$$

Let the solutions of these n component equations be respectively

$$F_1 (x, y, c_1) = 0. \qquad F_2 (x, y, c_2) = 0, \dots, \qquad F_n (x, y, c_n) = 0,$$

which $c_1, c_2, \dots, c_n$ are the arbitrary constants of integration.

Since all the c's can have any one of an infinite number of values, the above solutions (4) will remain general if we replace $c_1, c_2, ..., c_n$ by a single arbitrary constant c. Then the n solutions (4) can be re-written as $F_1 (x, y, c) = 0, \quad F_2 (x, y, c) = 0, ..., \quad F_n (x, y, c) = 0.$

These can be combined into one equation, namely,

$$F_1 (x, y, c)\, F_2 (x, y, c), \ldots\ldots\, F_n (x, y, c) = 0.$$

Remark. Since the given equation (1) is of the first order, its general solution cannot have more than one arbitrary constant.

4.3 Solved Examples based on method I of Art. 4.2

Ex. 1. *Solve the following differential equations*:

(a) $(dy/dx)^2 - ax^3 = 0.$ (b) $p^2 - x^5 = 0.$ (c) $p^3 = ax^4$

(d) $4p^2x (x - a) (x - b) = [3x^2 - 2 (a + b) x + ab]^2$

(e) $4p^2x (x - 1) (x - 2) = (3x^2 - 6x + 2)^2$

Sol. (*a*) Given $(dy/dx)^2 = ax^3$ so that $dy/dx = \pm a^{1/2} x^{3/2}$.

Separating variables, $dy = \pm a^{1/2} x^{3/2} dx$.

Integrating, $y + c = \pm (2/5) a^{1/2} x^{5/2}$ or $5 (y + c) = \pm 2a^{1/2} x^{5/2}$.

Squaring both sides, $25 (y + c)^2 = 4ax^5$ which is the required solution.

(b) Proceed as in part (a). **Ans.** $49 (y + c)^2 = 4x^7$.

(c) Given $(dy/dx)^3 = ax^4$ or $dy/dx = a^{1/3} x^{4/3}$ or $dy = a^{1/3} x^{4/3} dx$.

Integrating, $y + c = (3/7) a^{1/3} x^{7/3}$, c being an arbitrary constant.

Cubing, $343 (y + c)^3 = 27ax^7$, which is the required solution.

(d) Here $$p = \frac{dy}{dx} = \pm \frac{3x^2 - 2(a+b)x + ab}{2[x^3 - x^2(a+b) + abx]^{1/2}}$$

or $$dy = \pm (1/2) [x^3 - x^2 (a + b) + abx]^{-1/2} [3x^2 - 2 (a + b) x + ab]\, dx. \qquad \ldots (1)$$

Putting $x^3 - x^2 (a + b) + abx = v$ so that $[3x^2 - 2 (a + b) x + ab]\, dx = dv$,

(1) becomes $dy = \pm (1/2) v^{-1/2} dv$ so that $y + c = \pm v^{1/2}$.

Squaring and putting the value of v, we have

$$(y + c)^2 = x^3 - x^2 (a + b) + abx \quad \text{or} \quad (y + c)^2 = x (x - a) (x - b), \qquad \ldots (2)$$

which is the required solution, c being an arbitrary constant.

(e) This part is a particular case of part (d). Here $a = 1$, $b = 2$, proceed now as in part (d).

Ans. $(y + c)^2 = x (x - 1) (x - 2)$.

Ex. 2. *Solve* (i) $4xp^2 = (3x - a)^2$

(ii) $xp^2 = (x - a)^2$.

Sol. (i) Solving for p, $p = dy/dx = \pm (3x - a)/2\sqrt{x}$

i.e., $$dy = \pm\{(3/2) \times x^{1/2} - (1/2) \times ax^{-1/2}\}dx$$

Integrating, $y + c = \pm (x^{3/2} - ax^{1/2})$ or $y + c = \pm\sqrt{x}\, (x - a)$

On squaring, the general solution is $(y + c)^2 = x (x - a)^2$.

(ii) Do like part (i). **Ans.** $(y + c)^2 = (4x/9) \times (x - 3a)^2$.

Ex. 3. *Solve the following differential equations*:

(a) $p^2 - 7p + 12 = 0$

(b) $p^2 - 2p \cosh x + 1 = 0$ **[Kolkata 1993]**

Sol. (*a*) Resolving into linear factors, $p^2 - 7p + 12 = 0$ becomes $(p - 3) (p - 4) = 0$.

Its component equations are $p - 3 = 0$ and $p - 4 = 0$

or $(dy/dx) - 3 = 0$ and $(dy/dx) - 4 = 0$ or $dy - 3dx = 0$ and $dy - 4dx = 0$.

Integrating, $y - 3x - c = 0$ and $y - 4x - c = 0$.

Hence, the required combined solution is

$(y - 3x - c)(y - 4x - c) = 0$, c being an arbitrary constant.

(b) The given equation is $p^2 - p(e^x + e^{-x}) + 1 = 0$, as $\cosh x = (e^x + e^{-x})/2$

or $p^2 - p\,e^x - pe^{-x} + e^x \cdot e^{-x} = 0$, as $e^x \cdot e^{-x} = 1$

or $p(p - e^x) - e^{-x}(p - e^x) = 0$ or $(p - e^x)(p - e^{-x}) = 0$.

Its component equations are $p - e^x = 0$ and $p - e^{-x} = 0$

or $(dy/dx) - e^x = 0$ and $(dy/dx) - e^{-x} = 0$ or $dy - e^x dx = 0$ and $dy - e^{-x}\,dx = 0$.

Integrating, $y - e^x - c = 0$ and $y + e^{-x} - c = 0$.

Hence, the required combined solution is $(y - e^x - c)(y + e^{-x} - c) = 0$.

Ex. 4. *Solve the following differential equations:*

(a) $x^2p^2 + xyp - 6y^2 = 0$. **[Delhi Maths (G) 1993]**

(b) $p^2 + (x + y - 2y/x)\,p + xy + (y^2/x^2) - y - (y^2/x) = 0$

Sol. (a) Given $x^2p^2 + xyp - 6y^2 = 0$ or $x^2p^2 + 3xyp - 2xyp - 6y^2 = 0$

or $xp(xp + 3y) - 2y(xp + 3y) = 0$ or $(xp + 3y)(xp - 2y) = 0$.

Its component equations are $x(dy/dx) + 3y = 0$ and $x(dy/dx) - 2y = 0$

or $(1/y)\,dy + 3(1/x)\,dx = 0$ and $(1/y)\,dy - 2(1/x)\,dx = 0$.

Integrating, $\log y + 3\log x = \log c$, *i.e.*, $yx^3 = c$ and $\log y - 2\log x = \log c$, *i.e.*, $y/x^2 = c$

Hence, the general solution is $(yx^3 - c)(y/x^2 - c) = 0.$, c being an arbitrary constant

(b) Re-writing, $p^2 + [\{x - (y/x)\} + \{y - (y/x)\}]\,p + \{x - (y/x)\}\{y - (y/x)\} = 0$

or $[p + \{x - (y/x)\}][p + \{y - (y/x)\}] = 0$.

Its component equations are $(dy/dx) + x - (y/x) = 0$ ... (1)

and $(dy/dx) + y - (y/x) = 0.$ ... (2)

Re-writing (1), $(dy/dx) - (1/x)\,y = -x$, which is linear equation.

Its I.F. $= e^{\int(-1/x)\,dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = 1/x$ and solution is

$y \times (1/x) = \int(-x)(1/x)\,dx + c$ or $y/x = -x + c$. ... (3)

Re-writing (2), $dy/dx = y\,\{(1/x) - 1\}$ or $(1/y)\,dy = \{(1/x) - 1\}\,dx$

Integrating, $\log y = \log x - x + \log c$ or $y = cx\,e^{-x}$.

Hence, the combined solution of the given equation is

$(y + x^2 - cx)(y - cx\,e^{-x}) = 0$, c being an arbitrary constant.

Ex. 5. *Solve differential equation* $p^3 + 2xp^2 - y^2p^2 - 2xy^2p = 0$.

Sol. The given equation can be re-written as

$p(p^2 + 2xp - y^2p - 2xy^2) = 0$ or $p[p(p + 2x) - y^2(p + 2x)] = 0$

or $p(p + 2x)(p - y^2) = 0$.

Its component equations are $dy/dx = 0$, $(dy/dx) + 2x = 0$ and $(dy/dx) - y^2 = 0$

or $dy = 0$, $dy + 2x\,dx = 0$ and $dx - y^{-2}\,dy = 0$.

Integrating, $y = c$, $y + x^2 = c$ and $x + (1/y) = c$.

Hence, the combined solution is $(y - c)(y + x^2 - c)[x + (1/y) - c] = 0$.

Ex. 6. *Solve the following differential equations:*

(a) $p(p + x) = y(x + y)$ or $p^2 - xy = y^2 - px$

(b) $p(p + y) = x(x + y)$ **[Delhi Maths (G) 2006]**

(c) $p(p - y) = x(x + y)$ **[Meerut 1997]**

Sol. (*a*) Given $p\,(p + x) = y\,(x + y)$ or $(p^2 - y^2) + x\,(p - y) = 0$.

or $(p - y)\,(p + y) + x\,(p - y) = 0$ or $(p - y)\,(p + y + x) = 0$.

Its component equation are $(dy/dx) - y = 0$ ... (1)

and $(dy/dx) + y + x = 0$ or $(dy/dx) + y = -x$, which is linear ... (2)

Now, from (1), $(1/y)\,dy = dx$ so that $\log y = x + \log c$ or $y = ce^x$. ... (3)

For (2), I.F. $= e^{\int dx} = e^x$ and hence its solution is

$$ye^x = \int (-x)\,e^x\,dx + c = -[xe^x - \int (1 \cdot e^x)\,dx] + c \qquad \text{or} \qquad ye^x = -xe^x + e^x + c$$

or $y + x - 1 - ce^{-x} = 0$, c being an arbitrary constant ... (4)

From (3) and (4), the required combined solution is $(y - ce^x)\,(y + x - 1 - ce^{-x}) = 0$.

(*b*) Try yourself. **Ans.** $(x^2 + y^2 - c)\,(2xy + x^2 - c) = 0$.

(*c*) Proceed as in part (*a*). **Ans.** $(x^2 + 2y - c)\,(x + y + 1 - ce^x) = 0$.

Ex. 7. *Solve the following differential equations:*

(*a*) $x^2p^2 - xyp - y^2 = 0$.

(*b*) $p^2 - px + 1 = 0$. **[Delhi Maths (G) 1996]**

(*c*) $x^2p^2 - 2xyp + 2y^2 - x^2 = 0$. **[Osmania 2003; Kanpur 2005; Rajsthan 2010]**

Sol. (*a*) Solving for p, $p = \{xy \pm (x^2y^2 + 4x^2y^2)^{1/2}\}/(2x^2) = (1 \pm 5^{1/2})\,(y/2x)$

or $dy/dx = (1 \pm 5^{1/2})\,(y/2x)$ or $(2/y)\,dy = (1 \pm 5^{1/2})\,(1/x)\,dx$.

Integrating, $2 \log y = (1 \pm 5^{1/2}) \log x + \log c$ or $y^2 = cx^{(1 \pm 5^{1/2})}$.

(*b*) Solving for p, $p = dy/dx = \{x \pm (x^2 - 4)^{1/2}\}/2$ or $dy = (1/2)\,[x \pm \sqrt{x^2 - 2^2}\,]\,dx$.

Integrating, $$y = \frac{1}{2}\left[\frac{1}{2}x^2 \pm \left\{\left(\frac{x}{2}\right)\sqrt{(x^2 - 4)} - \left(\frac{2^2}{2}\right)\log\left(x + \sqrt{(x^2 - 4)}\right)\right\}\right] + \frac{c}{4}$$

or $4y = x^2 \pm [x\,(x^2 - 4)^{1/2} - 4 \log\{x + (x^2 - 4)^{1/2}\}] + c$, c being an arbitrary constant

(*c*) Solving for p, $p = [2xy + \{4x^2y^2 - 4x^2\,(2y^2 - x^2)\}^{1/2}]/2x^2$

or $dy/dx = (1/2x^2)\,\{2xy \pm 2x\,(x^2 - y^2)^{1/2}\} = \{y \pm (x^2 - y^2)^{1/2}\}/x$ ... (1)

Putting $y/x = v$ or $y = xv$ so that $dy/dx = v + x\,(dv/dx)$, (1) gives

$$v + x\,(dv/dx) = \{xv \pm (x^2 - x^2v^2)^{1/2}\}/x = v \pm (1 - v^2)^{1/2}$$

$\therefore$ $(1/x)\,dx = \pm\,(1/\sqrt{1 - v^2})\,dv$ and hence $\log x - \log c = \pm \sin^{-1} v$

or $\log (x/c) = \pm \sin^{-1}(y/x)$ or $x = ce^{\pm \sin^{-1}(y/x)}$

Ex. 8. *Solve* $(1 - y^2 + y^4/x^2)\,p^2 - 2\,(yp/x) + (y^2/x^2) = 0$.

Sol. Re-writing the given equation, we have

$p^2 - (2y/x)\,p + (y/x)^2 = p^2y^2 - (p^2y^4/x^2)$ or $(p - y/x)^2 = p^2y^2\,(1 - y^2/x^2)$

or $px - y = \pm\,py\,(x^2 - y^2)^{1/2}$

or $$\{x \mp y\,(x^2 - y^2)^{1/2}\}\frac{dy}{dx} = y \qquad \text{or} \qquad \frac{dx}{dy} = \frac{x \mp (x^2 - y^2)^{1/2}}{y} \qquad \text{... (1)}$$

To solve it, put $x = vy$ so that $dx/dv = v + y$ (dv/dy). Then, (1) gives

$$v + y\,\frac{dv}{dy} = \frac{vy \mp \sqrt{(v^2y^2 - y^2)}}{y} = v \mp y\sqrt{(v^2 - 1)}$$

or $$\frac{dv}{dy} = \mp\sqrt{(v^2 - 1)} \qquad \text{or} \qquad \frac{dv}{\sqrt{(v^2 - 1)}} = \mp\,dy$$

Integrating, $\cosh^{-1} v = c \mp y$ or $\cosh^{-1}(x/y) = c \mp y$.

Ex. 9. *Solve* $p^2y^2 \cos^2 \alpha - 2pxy \sin^2 \alpha + y^2 - x^2 \sin^2 \alpha = 0$.

Sol. Given $(py)^2 - 2\,(py)\,x \tan^2 \alpha + (y^2 \sec^2 \alpha - x^2 \tan^2 \alpha) = 0$.

$$\therefore \qquad py = \frac{2x \tan^2 \alpha \pm \sqrt{\{4x^2 \tan^4 \alpha - 4\,(y^2 \sec^2 \alpha - x^2 \tan^2 \alpha)\}}}{2}$$

or $$py = x \tan^2 \alpha \pm \sqrt{\{x^2 \tan^2 \alpha\,(\tan^2 \alpha + 1) - y^2 \sec^2 \alpha\}}$$

or $$y\,(dy/dx) = x \tan^2 \alpha \pm \sec \alpha \sqrt{(x^2 \tan^2 \alpha - y^2)}$$

or $$y\,dy - x \tan^2 \alpha\,dx = \pm \sec \alpha \sqrt{(x^2 \tan^2 \alpha - y^2)}\,dx$$

or $$\pm \frac{x \tan^2 dx - y\,dy}{\sqrt{(x^2 \tan^2 \alpha - y^2)}} = -\sec \alpha\,dx.$$

Integrating, $\pm \sqrt{(x^2 \tan^2 \alpha - y^2)} = c - x \sec \alpha.$, c being an arbitrary constant

Squaring, $x^2 \tan^2 \alpha - y^2 = c^2 - 2cx \sec \alpha + x^2 \sec^2 \alpha$ or $x^2 + y^2 - 2c\,x \sec \alpha + c^2 = 0$.

Ex. 10. *Solve* $p^2 + 2py \cot x = y^2$. **[Andhra 2003; Kanpur 1997 Srivenkateshwara 2003; Kanpur 2008; Gulbarga 2005; Delhi Math (G) 1994]**

(*b*) *If the curve whose differential equation is* $p^2 + 2py \cot x = y^2$ *passes through* $(\pi/2, 1)$, *show that the equation of the curve is given by* $(2y - \sec^2 x/2)\,(2y - \operatorname{cosec}^2 x/2) = 0$.

Sol. (*a*) Given $p^2 + (2y \cot x)\,p - y^2 = 0$. Solving it for p, we get

$$p = [-2y \cot x \pm \sqrt{(4y^2 \cot^2 x + 4y^2)}]/2$$

or $$p = -y \cot x \pm y\,(\cot^2 x + 1)^{1/2} = -y\,(\cot x \pm \operatorname{cosec} x).$$

Its component equations are $dy/dx = -y\,(\cot x + \operatorname{cosec} x)$ ... (1)

and $dy/dx = -y\,(\cot x - \operatorname{cosec} x)$. ... (2)

By (1), $$\frac{dy}{dx} = -y\left(\frac{\cos x}{\sin x} + \frac{1}{\sin x}\right) = -\frac{1+\cos x}{\sin x}\,y = -\frac{2y \cos^2 (x/2)}{2 \sin (x/2) \cos (x/2)}$$

or $$(1/y)\,dy = -\cot (x/2)\,dx.$$

Integrating, $\log y = \log c - 2 \log \sin (x/2)$ or $y = c \operatorname{cosec}^2 (x/2)$. ... (3)

By (2), $$\frac{dy}{dx} = -y\left(\frac{\cos x}{\sin x} - \frac{1}{\sin x}\right) = \frac{1-\cos x}{\sin x}\,y = \frac{2y \sin^2 (x/2)}{2 \sin (x/2) \cos (x/2)}$$

or $$(1/y)\,dy = \tan (x/2)\,dx.$$

Integrating, $\log y = \log c - 2 \log \cos (x/2)$ or $y = c \sec^2 (x/2)$. ... (4)

$\therefore$ From (3) and (4), the combined solution is $(y - c \sec^2 x/2)\,(y - c \operatorname{cosec}^2 x/2) = 0$.

(*b*) As in part (*a*), the general equation of the curve is

$$(y - c \sec^2 x/2)\,(y - c \operatorname{cosec}^2 x/2) = 0. \qquad \text{... (5)}$$

Since (1) is to pass through $(\pi/2, 1)$, (5) $\Rightarrow$ $(1 - 2c)^2 = 0$ $\Rightarrow$ $c = 1/2$.

Putting $c = 1/2$ in (5), the equation of the required curve is

$$(2y - \sec^2 x/2)\,(2y - \operatorname{cosec}^2 x/2) = 0.$$

Ex. 11. *Solve* $x^2p^2 - 2xyp + y^2 - x^2y^2 - x^4 = 0$. **[Delhi Maths (H) 2004]**

Sol. Given $(xp - y)^2 = x^2\,(x^2 + y^2)$ or $xp - y = \pm x\,(x^2 + y^2)^{1/2}$

or $\quad x\dfrac{dy}{dx} - y = \pm x\,(x^2+y^2)^{1/2}$ $\quad$ or $\quad \dfrac{x\,dy - y\,dx}{x^2} = \pm\left(1+\dfrac{y^2}{x^2}\right)^{1/2}$

or $\quad \dfrac{d\,(y/x)}{\{1+(y/x)^2\}^{1/2}} = \pm\,dx$ $\quad$ or $\quad \dfrac{dt}{(1+t^2)^{1/2}} = \pm\,dx$, where $\dfrac{y}{x} = t$

Integrating, $\quad \sinh^{-1} t = C \pm x$ $\quad$ or $\quad \sinh^{-1}(y/x) = C \pm x$, as $t = y/x$

Thus, $\quad y/x = \sinh(C \pm x)$ $\quad$ or $\quad y = x\sinh(C \pm x)$

The required solution is $\quad \{y - x\sinh(C+x)\}\ \{y - x\sinh(C-x)\} = 0$

Exercise 4(A)

Solve the following differential equations:

1. $p^2 - 5p + 6 = 0$ **[Delhi Maths (G) 2005]** **Ans.** $(y-2x-c)(cy-3x-c)=0$
2. $p^2 + p = 6$ **Ans.** $(y+3x-c)(y-2x-c)=0$
3. $p^2 - p(e^2+e^{-2}) + 1 = 0$ **Ans.** $(y - x\,e^2 - c)(y + x\,e^2 - c) = 0$
4. $yp^2 + (x-y)\,p - x = 0$ **[Delhi B.A. (Prog.) II 2011]** **Ans.** $(y-x-c)(y^2+x^2-c)=0$
5. $x^2p^2 + 3\,xyp + 2\,y^2 = 0$ **[Madras 2005, Delhi Maths (G) 1993]** **Ans.** $(xy-c)(x^2y-c)=0$
6. $xyp^2 + (3\,x^2 - 2\,y^2)\,p - 6xy = 0$ **Ans.** $(y^2+3x^2-c)(y-cx^2)=0$
7. $p^3 + 3\,xp^2 - y^3p^2 - 3\,xy^3p = 0$ **Ans.** $(y-c)(2y+3x^2-c)(y^{-2}+2x-c)=0$
8. $x^2p^3 + y\,(1+x^2y)\,p^2 + y^3p = 0$ **Ans.** $(y-c)(x-y^{-1}-c)(ye^{-1/x}-c)=0$
9. $xyp^2 + (x^2+xy+y^2)\,p + x^2 + xy = 0$ **Ans.** $(x^2+y^2-c)(2xy+x^2-c)=0$
10. $x^2p^2 - 2xyp + y^2 = x^2y^2 + x^4$ **Ans.** $x + c = \pm\log\,[(y/x) + \{1+(y/x)^2\}^{1/2}]$
11. $3\,p^2y^2 - 2\,xyp + 4\,y^2 - x^2 = 0$ **Ans.** $2x \pm (x^2 - 3\,y^2)^{1/2} = c$
12. $y = x\,\{p + (1+p^2)^{1/2}\}$ **[Kanpur 1996]** **Ans.** $x^2 + y^2 = cx$
13. $xyp^2 - (x^2 - y^2)\,p - xy = 0$ **[Kolkata 1994]** **Ans.** $(y^2 - x^2 - c)(xy - c) = 0$
14. $4\,y^2p^2 + 2\,pxy\,(3x+1) + 3x^3 = 0$ **[Delhi 2008]** **Ans.** $(x^2+2y^2-c)(x^2+y^2-c)=0$
15. $p^3\,(x+2\,y) + 3\,p^2\,(x+y) + p\,(y+2x) = 0$ **[Delhi B.Sc. (Prog) II 2010]** **Ans.** $(y-c)(x+y-c)(xy+x^2+y^2-c)=0$

4.4 Method II. Equations solvable for x

If the given equation $f(x, y, p) = 0$ is solvable for x, we can express x explicitly in terms of y and p. Thus, an equation solvable for x can be put in the form. $\quad x = F(y, p) \quad$ ... (1)

Differentiating (1) w.r.t. 'y' and writing $1/p$ for dx/dy, we get

$$1/p = \phi\,(y, p, dp/dy), \qquad \text{... (2)}$$

which is a equation involving two variables y and p. Let its solution be

$$\psi\,(y, p, c) = 0,\ c \text{ being an arbitrary constant.} \qquad \text{... (3)}$$

Eliminating p between (1) and (3), we get the solution of (1) in the form $\quad g(x, y, c) = 0.$

If the elimination of p between (1) and (3) is not possible, then we solve (1) and (3) to express x and y in terms of p and c in the form

$$x = f_1\,(p, c), \qquad y = f_2\,(p, c). \qquad \text{... (4)}$$

These two equations together form the general solution of (1) in the parametric form, the parameter being p.

Sometimes even the form (4) of the desired solution is not possible. In that case (1) and (3) may be regarded as giving x and y in terms of p, that is, (1) and (3) together are said to form the solution in parametric form.

Remark 1. In some problems (2) can be expressed as $\phi_1\,(y, p)\,\phi_2\,(y, p, dp/dy) = 0.$ ... (5)

In such cases we ignore the first factor $\phi_1\,(y, p)$ which does not involve dp/dy and proceed with $\phi_2\,(y, p, dp/dy) = 0$ as already discussed in Art. 4.3.

Remark 2. If instead of ignoring the factor $\phi_1(y, p)$, we eliminate p between (1) and $\phi_1(y, p) = 0$, we obtain an equation involving no constant c. This is known as *singular solution* of (1) and we shall discuss it later on in this chapter. Singular solution will be obtained only if asked to do so in a given question. Refer part II of this chapter for more details.

Special case of Method II. Equations that do not contain *y*.

If the given equation does not contain y, then it can be put in the form $f(x, p) = 0$. ... (6)

If (6) is solvable for p, it can be put in the form $p = F(x)$ or $dy/dx = F(x)$, which can be easily integrated to give the required solution.

If (6) is solvable for x, it can be put in the form $x = G(p)$, which is of the form (1) considered in Art 4.3 and so it can be solved as before.

4.5 Solved examples based on method II of Art. 4.4

Ex. 1. *Solve the following differential equations* :

(a) $y = 2px + y^2p^3$. **[Delhi B.Sc. (Prog) II 2008; Purvanchal 2007]**

(b) $p^3 - 4xyp + 8y^2 = 0$. **[Delhi B.Sc. Maths (Prog) II 2007]**

(c) $(2x - b)\, p = y - ayp^2,\ a > 0$ **[I.A.S. 2001]**

Sol. (*a*) Given $y = 2px + y^2p^3$ where $p = dy/dx$. ... (1)

Solving (1) for x, $x = y\,(1/2p) - y^2\,(p^2/2)$. ... (2)

Differentiating (2) w.r.t. 'y' and writing $1/p$ for dx/dy, we get

$$\frac{1}{p} = \frac{1}{2p} - \frac{y}{2p^2}\frac{dp}{dy} - \frac{2y\,p^2}{2} - \frac{y^2}{2}\cdot 2p\frac{dp}{dy}$$

or $$\frac{1}{2p} + yp^2 + \frac{dp}{dy}\left(\frac{y}{2p^2} + yp^2\right) = 0 \qquad \text{or} \qquad p\left(py + \frac{1}{2p^2}\right) + y\frac{dp}{dy}\left(py + \frac{1}{2p^2}\right) = 0$$

or $$[py + (1/2\,p^2)]\ [p + y\,(dp/dy)] = 0.$$

*Neglecting the first factor which does not involve dp/dy, (3) reduces to

$$p + y\,(dp/dy) = 0 \qquad \text{or} \qquad (1/p)\,dp + (1/y)\,dy = 0.$$

Integrating, $\log p + \log y = \log c$ or $py = c$ or $p = c/y$.

Substituting this value of p in (1), we get the required solution

$$y = (2x)\,(c/y) + y^2\,(c/y)^3 \qquad \text{or} \qquad y^2 = 2\,cx + c^3.$$

(*b*) Given $p^3 - 4xyp + 8\,y^2 = 0$, where $p = dy/dx$. ... (1)

Solving (1) for x, $x = (1/4y)\,p^2 + (1/p)\,(2y)$. ... (2)

Differentiating (2) w.r.t. 'y and writing $1/p$ for dx/dy, we get

$$\frac{1}{p} = -\frac{p^2}{4y^2} + \frac{2p}{4y}\frac{dp}{dy} + \frac{2}{p} - \frac{2y}{p^2}\frac{dp}{dy} \qquad \text{or} \qquad \frac{p^2}{4y} - \frac{1}{p} - \frac{dp}{dy}\left(\frac{p}{2y} - \frac{2y}{p^2}\right) = 0$$

or $$\left(\frac{p^2}{4y} - \frac{1}{p}\right) - \frac{2y}{p}\frac{dp}{dy}\left(\frac{p^2}{4y} - \frac{1}{p}\right) = 0 \qquad \text{or} \qquad \left(\frac{p^2}{4y} - \frac{1}{p}\right)\left(1 - \frac{2y}{p}\frac{dp}{dy}\right) = 0 \qquad \text{... (3)}$$

Neglecting the first factor which does not involve dp/dy, (3) reduces to

$$1 - (2y/p)\,(dp/dy) = 0 \qquad \text{or} \qquad (2/p)\,dp = (1/y)\,dy.$$

Integrating, $2 \log p = \log y + \log c'$ so that $p^2 = c'\,y$. ... (4)

We now proceed to eliminate p between (1) and (4). Re-writing (1), $p\,(p^2 - 4xy) = -\,8\,y^2$

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* If we take $py + (1/2p^2) = 0$, then we shall get singular solution which is not to be discussed unless asked. Thus, the factor which does not involve a derivative of p w.r.t. 'x' or 'y' will be omitted. For more details refer part II of this chapter.

Squaring, $p^2(p^2 - 4xy)^2 = 64\,y^4$ or $c'y\,(c'y - 4xy)^2 = 64\,y^4$, using (4) ... (5)

Let $c' = 4c$, where c in new arbitrary constant. Then (5) gives

$4cy\,(4cy - 4xy)^2 = 64y^4$ or $c\,(c - x)^2 = y$. ... (6)

(c) Solving the given equation for x, we have $2x = b + (y/p) - ayp$. ... (1)

Differentiating (1) w.r.t. 'y', we have

$$\frac{2}{p} = \frac{1}{p} - \frac{y}{p^2}\frac{dp}{dy} - ap - ay\frac{dp}{dy} \quad \text{or} \quad \left(\frac{1}{p} + ap\right)\left(1 + \frac{y}{p}\frac{dp}{dy}\right) = 0$$

Neglecting the first factor which does not involve dp/dy, we get

$1 + (y/p)\,(dp/dy) = 0$ or $(1/p)\,dp + (1/y)\,dy = 0$ so that $py = c$.

Putting $p = c/y$ in the given equation, the required solution is

$(a\,c^2/y) + (2x - b)\,(c/y) - y = 0$ or $ac^2 + (2x - b)\,c - y^2 = 0$.

Ex. 2. *Solve* $y = 2px + p^2y$. **[Kakatiya 2003, Purvanchal 2007 ; Delhi (Prog.) II 2011]**

Sol. Solving for x, we get $2x = -\,py + y/p$. ... (1)

Differentiating (1) w.r.t. y, we get $\dfrac{2}{p} = -\,p - y\dfrac{dp}{dy} + \dfrac{1}{p} - \dfrac{y}{p^2}\dfrac{dp}{dy}$, as $\dfrac{dx}{dy} = \dfrac{1}{p}$

or $p + 1/p = -\,y\,(dp/dy)\,(1 + 1/p^2)$ or $p\,(1 + 1/p^2) + y\,(dp/dy)\,(1 + 1/p^2) = 0$

or $(1 + 1/p^2)\,[p + y\,(dp/dy)] = 0$.

Omitting the first factor which leads to a singular solution, we get

$p + y\,(dp/dy) = 0$ or $(1/p)\,dp + (1/y)\,dy = 0$

Integrating $\log p + \log y = \log c$ or $py = c$. ... (2)

To eliminate p between (1) and (2), first solve (2) for p. So $p = c/y$. Putting this value of p in (1), we get $2x = -\,c + y^2/c$ or $2xc - y^2 + c^2 = 0$.

Ex. 3. *Solve* $y^2 \log y = xpy + p^2$. **[G.N.D.U. Amritsar 2010]**

Sol. Solving for x, $x = \dfrac{y \log y}{p} - \dfrac{p}{y}$. ... (1)

Differentiating (1) w.r.t. y and remembering that $dx/dy = 1/p$, we get

$$\frac{1}{p} = \left(\log y + y\cdot\frac{1}{y}\right)\frac{1}{p} - y\log y\cdot\frac{1}{p^3}\frac{dp}{dy} - \left[-\frac{1}{y^2}\cdot p + \frac{1}{y}\frac{dp}{dy}\right]$$

or $$\frac{1}{p} = \frac{\log y}{p} + \frac{1}{p} + \frac{p}{y^2} - \frac{dp}{dy}\left(\frac{y\log y}{p^2} + \frac{1}{y}\right) \quad \text{or} \quad 0 = \frac{p}{y}\left(\frac{y\log y}{p^2} + \frac{1}{y}\right) - \frac{dp}{dy}\left(\frac{y\log y}{p^2} + \frac{1}{y}\right)$$

or $$\left(\frac{y\log y}{p^2} + \frac{1}{y}\right)\left(\frac{p}{y} - \frac{dp}{dy}\right) = 0.$$

Omitting the first factor, we have $\dfrac{p}{y} - \dfrac{dp}{dy} = 0$ or $\dfrac{dp}{p} = \dfrac{dy}{y}$.

Integrating, $\log p = \log y + \log c$ so that $p = cy$. ... (2)

Putting the value of p given by (2) in (1), we get $x = (\log y)/c - c$ or $\log y = cx + c^2$.

Ex. 4. *Solving the following differential equations :*

(a) $x = y + a\,\log p$. **[Delhi Maths. (G) 2001, Indore 1995]**

(b) $x = y + p^2$. **[Delhi Maths. (Pass) 2004]**

Sol. (*a*) Given $x = y + a \log p$, where $p = dy/dx$. ... (1)

Differentiating (1) w.r.t. 'y' and writing $1/p$ for dx/dy, we get

$$\frac{1}{p} = 1 + \frac{a}{p}\frac{dp}{dy} \quad \text{or} \quad \frac{1-p}{p} = \frac{a}{p}\frac{dp}{dy} \quad \text{or} \quad dy = \frac{a}{1-p}\,dp.$$

Integrating, $y = c - a \log(1 - p)$, c being an arbitrary constant ... (2)

Substituting this value of y in (1), we get $x = c - a \log(1 - p) + a \log p$. ... (3)

(2) and (3) together form the required general solution of (1) in parametric form, p being regarded as parameter.

(*b*) Differentiating the given equation w.r.t. 'x', we have

$$1/p = 1 + 2p\,(dp/dy) \quad \text{or} \quad dp/dy = (1-p)/2p^2$$

or
$$dy = -\frac{2p^2}{p-1}\,dp \quad \text{or} \quad dy = -2\left(p + 1 + \frac{1}{p-1}\right)dp$$

Integrating, $y = c - \{p^2/2 + p + \log(p - 1)\}$, c being an arbitrary constant ... (1)

Substituting this value of y in the given equation, we get $x = c - 2\,\{p + \log(p - 1)\}$. ... (2)

(1) and (2) together constitute solution in parametric form, p being regarded as parameter.

Ex. 5. *Solve the following differential equations* :

(*a*) $p^3 - p\,(y + 3) + x = 0$ **[Lucknow 1997]**

(*b*) $x = py + p^2$ **[Delhi Maths. (G) 2004]**

(*c*) $x = py - p^2$

(*d*) $x = py + ap^2$

Sol. (*a*) Given $p^3 - p\,(y + 3) + x = 0$, where $p = dy/dx$. ... (1)

Solving for x, $x = p\,(y + 3) - p^3$. ... (2)

Differentiating (2) w.r.t. 'y' and writing $1/p$ for dx/dy, we get

$$\frac{1}{p} = p + (y+3)\frac{dp}{dy} - 3p^2\frac{dp}{dy} \quad \text{or} \quad \frac{1}{p} - p = \frac{dp}{dy}(y + 3 - 3p^2)$$

or
$$\frac{1-p^2}{p}\frac{dy}{dp} = y + 3 - 3p^2 \quad \text{or} \quad \frac{dy}{dp} = \frac{p}{1-p^2}[y + 3(1-p^2)]$$

or $(dy/dp) - \{p/(1 - p^2)\}\,y = 3p$, which is linear equation. ... (3)

Its I.F. $= e^{-\int [p/(1-p^2)]dp} = e^{(1/2)\times \log(1-p^2)} = (1-p^2)^{1/2}$.

$\therefore$ Solution of (3) is $y\,(1 - p^2)^{1/2} = c + \int 3p\,(1-p^2)^{1/2}dp$. ... (4)

Putting $1 - p^2 = v$ so that $-2p\,dp = dv$ or $pdp = -(1/2)\,dv$, (4) gives

$$y\,(1 - p^2)^{1/2} = c - (3/2)\times\int v^{1/2}dv = c - v^{3/2} = c - (1-p^2)^{3/2} \quad \text{... (5)}$$

$$y = c\,(1 - p^2)^{-1/2} - (1 - p^2), \text{ where } |\,p\,| < 1, c \text{ being an arbitrary constant}$$

Putting this value of y in (2), we get

$$x = p\,[c\,(1 - p^2)^{-1/2} - 1 + p^2 + 3] - p^3 \quad \text{or} \quad x = cp\,(1 - p^2)^{-1/2} + 2p. \quad \text{... (6)}$$

(5) and (6) together form the solution of (1) in parametric form, p being treated as parameter.

(*b*) Given $x = py + p^2$ ... (1)

Differentiating (1) w.r.t. 'y', $1/p = p + y\,(dp/dy) + 2p\,(dp/dy)$

or $(1 - p^2)/p = (y + 2p)\,(dp/dy)$ or $dy/dp = p\,(y + 2p)/(1 - p^2)$

or $$dy/dp - \{p/(1-p^2)\}\, y = 2p^2/(1-p^2), \qquad \text{... (2)}$$

which is linear with I.E. $e^{\int P\,dx}$, where $P = -p/(1-p^2)$.

Now, $$\int P\,dx = -\int \frac{p}{1-p^2}dp = \frac{1}{2}\int \frac{(-2p)dp}{1-p^2} = \frac{1}{2}\log(1-p^2) = \log(1-p^2)^{1/2}$$

$\therefore$ I.F. of (2) $= e^{\log(1-p^2)^{1/2}} = (1-p^2)^{1/2}$ and so solution of (2) is

$$y\,(1-p^2)^{1/2} = \int \frac{2p^2}{1-p^2}(1-p^2)^{1/2}dp + C = C + 2\int \frac{1-(1-p^2)}{(1-p^2)^{1/2}}dp$$

$$= C + 2\int \frac{dp}{(1-p^2)^{1/2}} - 2\int (1-p^2)^{1/2}dp$$

$$= C + 2\sin^{-1}p - 2\,\{(p/2)\times(1-p^2)^{1/2} + (1/2)\times \sin^{-1}p\}$$

or $$y = C\,(1-p^2)^{-1/2} + (1-p^2)^{-1/2}\sin^{-1}p - p \quad \text{or} \quad y = (C + \sin^{-1}p)\,(1-p^2)^{-1/2} - p \; \text{... (3)}$$

Substituting the above value of y in (1), we have

$$x = p\,\{C\,(1-p^2)^{-1/2} + (1-p^2)^{-1/2}\sin^{-1}p - p\} + p^2$$

or $$x = Cp\,(1-p^2)^{-1/2} + p\,(1-p^2)^{-1/2}\sin^{-1}p$$

or $$x = p\,(C + \sin^{-1}p)\,(1-p^2)^{-1/2}. \qquad \text{... (4)}$$

(3) and (4) together give the solution is parametric form, p being the parameter.

(*c*) Do as in part (*b*). Solution in parametric form is

$$x = p(C - \sin^{-1}p)\times(1-p^2)^{-1/2}, \qquad y = (C - \sin^{-1}p)\times(1-p^2)^{-1/2} + p$$

(*d*) Do as in part (*b*). Solution in parametric form is

$$x = p\,(C + a\sin^{-1}p)\,(1-p^2)^{-1/2}, \qquad y = (C + a\sin^{-1}p)\,(1-p^2)^{-1/2} - ap$$

Ex. 6. *Solve the following differential equations*: *(a)* $x = 4\,(p + p^3)$

(b) $x\,(1+p^2) = 1$

(c) $x + \{p/(1+p^2)^{1/2}\} = a$

Sol. (*a*) Given $$x = 4\,(p + p^3), \text{ where } p = dy/dx. \qquad \text{... (1)}$$

Differentiating (1) w.r.t. 'y' and writing $1/p$ for dx/dy, we get

$$1/p = 4\,(1 + 3p^2)\,(dp/dy) \qquad \text{or} \qquad dy = 4\,(p + 3p^3)\,dp.$$

Integrating, $$y = 4\,[(p^2/2) + 3\,(p^4/4)] + c = 2p^2 + 3p^4 + c. \qquad \text{... (2)}$$

(1) and (2) together form the solution in parametric form, p being treated as parameter.

(*b*) Given $$x = (1+p^2)^{-1}, \text{ where } p = dy/dx. \qquad \text{... (1)}$$

Differentiating (1) w.r.t. 'y' and writing $1/p$ for dx/dy, we get

$$\frac{1}{p} = -(1+p^2)^{-2}\cdot 2p\frac{dp}{dy} \qquad \text{or} \qquad dy = -\frac{2p^2}{(1+p^2)^2}dp.$$

Integrating, $y = c - 2\int \frac{p^2 dp}{(1+p^2)^2} = c - 2\int \frac{\tan^2\theta\cdot\sec^2\theta\,d\theta}{(1+\tan^2\theta)^2}$, putting $p = \tan\theta$, $dp = \sec^2\theta\,d\theta$

or $$y = c - 2\int \sin^2\theta\,d\theta = c - \int (1-\cos 2\theta)\,d\theta = c - \theta + (1/2)\sin 2\theta$$

or $$y = c - \theta + \frac{1}{2}\frac{2\tan\theta}{1+\tan^2\theta} = c - \tan^{-1}p + \frac{p}{1+p^2}. \qquad \text{... (2)}$$

(1) and (2) together form the solution in parametric form, p being parameter

(*c*) Given $$x = a - p\,(1+p^2)^{-1/2}. \qquad \text{... (1)}$$

Differentiating (1) w.r.t. 'y' and writing $1/p$ for dx/dy, we get

$$\frac{1}{p} = 0 - \left[(1+p^2)^{-1/2} + p\cdot\left(-\frac{1}{2}\right)(1+p^2)^{-3/2}\cdot 2p\right]\frac{dp}{dy}$$

or
$$\frac{1}{p} = \frac{dp}{dy}\left[\frac{p^2}{(1+p^2)^{3/2}} - \frac{1}{(1+p^2)^{3/2}}\right] = \frac{dp}{dy}\frac{p^2-(1+p^2)}{(1+p^2)^{3/2}}$$

or
$$dy = -(1+p^2)^{-3/2}\, p\, dp.$$

Integrating, $y = c - \frac{1}{2}\int (1+p^2)^{-3/2}(2p\,dp) = c - \frac{1}{2}\int v^{-3/2}\,dv$, putting $1 + p^2 = v$, $2\,p\,dp = \mathrm{dv}$

or $\quad y = c + v^{-1/2} \quad$ or $\quad y = c + (1+p^2)^{-1/2}$. ... (2)

We now try to eliminate p from (1) and (2) as follows.

Here, (1) and (2) $\Rightarrow$ $x - a = p/(1+p^2)^{1/2}$ and $y - c = 1/(1+p^2)^{1/2}$.

Squaring and adding these, $\quad (x-a)^2 + (y-c)^2 = 1$,

which is the required solution, c being an arbitrary constant.

Exercise 4(B)

Solve the following differential equations

1. $p^3y^2 - 2px + y = 0$ **[Delhi B.Sc. (Prog) II 2011]** **Ans.** $y^2 = 2\,cx - c^3$

2. $yp^2 - 2xp + y = 0$ **Ans.** $y^2 = 2\,cx - c^2$

3. (*i*) $y - 2xp + ayp^2 = 0$ (*ii*) $y - 2xp + yp^2$ (I.A.S. 2008) **Ans.** (*i*) $y^2 = 2cx - ac^2$ (*ii*) $y = 2cx - c^2$

4. $4\,(xp^2 + yp) = y^4$ **Ans.** $y = 4c\,(xyc + 1)$

5. $y = 3\,px + 6\,p^2y^2$ **[Delhi Maths (H) 2001; Delhi Maths (G) 2000]** **Ans.** $y^3 = 3\,cx + 6\,c^2$

6. $p^2 - 2xp + 1 = 0$ **Ans.** $x = (p/2) + (1/2p)$, $y = (p^2/4) - (1/2) \times \log p + c$

7. $xp^3 = a + bp$ **Ans.** $x = a/p^3 + b/p^2$, $y = (3\,a/2p^2) + (2\,b/p) + c$

8. $xp^3 - p^2 - 1 = 0$ **Ans.** $x = p^{-1} + p^{-3}$, $y = (3/2) \times p^{-2} - \log p + c$

9. $p = \tan\{x - p/(1+p^2)\}$ or $p \cot\{x - p/(1+p^2)\} = 1$

Ans. $x = \{p/(1+p^2)\} + \tan^{-1} p$, $y = c - (p^2+1)^{-1}$

4.6 Method III. Equations solvable for *y*

If the given equation $f(x, y, p) = 0$ is solvable for y, we can express y explicitly in terms of x and p. Thus, an equation solvable for y can be put in the form $\quad y = F(x, p)$. ... (1)

Differentiating (1) w.r.t. x and writing p for dy/dx, we get $\quad p = \phi(x, p, dp/dx)$, ... (2)

which is an equation involving two variables x and p. Let its solution be

$\psi(x, p, c) = 0$, c being an arbitrary constant. ... (3)

Eliminating p between (1) and (3), the solution of (1) is in the form $\quad g(x, y, c) = 0$.

If the elimination of p between (1) and (3) is not possible, then we solve (1) and (3) to express x and y in terms of p and c in the form

$x = f_1(p, c)$, $\quad y = f_2(p, c)$. ... (4)

These two equations together form the general solution of (1) in the parametric form, the parameter being p.

Sometimes even the form (4) of the desired solution is not possible. In that case (1) and (3) may be regarded as giving x and y in terms of p, that is, (1) and (3) together are said to form the solution in parametric form.

Remark 1. In some problems (2) can be expressed as $\phi_1(x, p)\,\phi_2(x, p, dp/dx) = 0$. ... (5)

In such cases we ignore the first factor $\phi_1(x, p)$ which does not involve dp/dx and proceed with $\phi_2(x, p, dp/dx) = 0$ as discussed in Art. 4.6.

Remark 2. If instead of ignoring the factor $\phi_1\,(x, p)$, we eliminate p between (1) and $\phi_1\,(x, p) = 0$, we obtain an equation involving no constant c. This is known as *singular solution* of (1) and we shall discuss it later on in this chapter. Singular solution will be obtained only if asked to do so in a given question. For details refer part II of this chapter.

First special case of Method III. Lagrange's equation

The differential equation of the form $\qquad y = x\,F(p) + f(p) \qquad$ ... (6)

is known as *Lagrange's equation.* To solve it we proceed as in Art. 4.6.

Differentiating (6) w.r.t. 'x', we have $\qquad p = F(p) + x\,F'(p)\,(dp/dx) + f'(p)\,(dp/dx)$

or $\quad p - F(p) = \dfrac{dp}{dx}[x F'(p) + f'(p)] \qquad$ or $\qquad \dfrac{dx}{dp} = \dfrac{x F'(p) + f'(p)}{p - F(p)}$

or
$$\frac{dx}{dp} - \frac{F'(p)}{p - F(p)}\,x = \frac{f'(p)}{p - F(p)},$$

which is linear equation in x and p and can be solved by usual method to give a relation of the form
$$x = \phi\,(p, c) \qquad \text{... (7)}$$

We now eliminate p between (6) and (7) to get the required solution

It p cannot be eliminated, then putting the value of x in (6), we get
$$y = \phi\,(p, \text{c})\;F(p) + f(p). \qquad \text{... (8)}$$

Then (7) and (8) together form the required solution in parametric form, p being regarded as parameter.

Second special case of method III. Equation that do not contain *x*.

If given equation does not contain x, then it can be put in the form $\quad f(y, p) = 0. \quad$ ... (9)

If (9) is solvable for p, it can be put in the form $\qquad p = F(y) \qquad$ or $\qquad dy/dx = F(y)$,

which can be easily integrated to give the required solution.

If (9) is solvable for y, it can be put in the form $\qquad y = G(p)$,

which is of the form (1) of Art. 4.6 and so it can be solved as before.

4.7 Solved examples based on method III of Art. 4.6

Ex. 1. Solve (*a*) $y = 3x + \log p$.

(*b*) $y = x\,\{p + (1 + p^2)^{1/2}\}$. **[Kanpur 2009]**

Sol. (*a*) Given $\qquad y = 3x + \log p$, where $p = dy/dx$. $\qquad$ ... (1)

Differentiating (1) w.r.t. 'x' and writing p for dy/dx, we get

$p = 3 + (1/p)\,(dp/dx) \qquad$ or $\qquad p\,(p - 3) = (dp/dx)$

or
$$dx = \frac{dp}{p(p-3)} = \frac{1}{3}\left[\frac{1}{p-3} - \frac{1}{p}\right]dp, \text{ resolving into partial fractions.}$$

Integrating, $\quad x = (1/3)\,[\log(p - 3) - \log p] - (1/3)\log c$, c being an arbitrary constant

or $\quad \log\left[\dfrac{p-3}{pc}\right] = 3x \qquad$ or $\qquad \dfrac{p-3}{pc} = e^{3x} \qquad$ or $\qquad p = \dfrac{3}{1 - ce^{3x}}$

Putting this value of p in (1), the required solution is
$$y = 3x + \log\{3/(1 - ce^{3x})\}, \; c \text{ being an arbitrary constant}$$

(*b*) Given $\qquad y = x\,\{p + (1 + p^2)^{1/2}\}, \qquad$ where $\qquad p = dy/dx. \qquad$ ... (1)

Differentiating (1) w.r.t. 'x' and writing p for dy/dx, we get

$$p = p + (1+p^2)^{1/2} + x\left[1 + \frac{1}{2}(1+p^2)^{-1/2} \times 2p\right]\frac{dp}{dx}$$

or $\dfrac{x[p+(1+p^2)^{1/2}]}{(1+p^2)^{1/2}}\dfrac{dp}{dx} = -(1+p^2)^{1/2}$ or $\left[\dfrac{1}{2}\cdot\dfrac{2p}{1+p^2} + \dfrac{1}{(1+p^2)^{1/2}}\right]dp = -\dfrac{dx}{x}.$

Integrating, $(1/2) \times \log(1+p^2) + \log[p + (1+p^2)^{1/2}] = \log c - \log x$

$\log(c/x) = \log(1+p^2)^{1/2} + \log[p + (1+p^2)^{1/2}]$

or $c/x = (1+p^2)^{1/2}[p + (1+p^2)^{1/2}]$ or $c/x = (1+p^2)^{1/2}(y/x)$, by (1)

or $c = y(1+p^2)^{1/2}$ or $c^2 = y^2(1+p^2)$ or $c^2 - y^2 = y^2p^2$. ... (2)

Multiplying both sides of (1) by y, $y^2 = xyp + xy(1+p^2)^{1/2}$

or $y^2 = xyp + cx$, since from (2) $y(1+p^2)^{1/2} = c$

or $y^2 - cx = xyp$ or $(y^2 - cx)^2 = x^2y^2p^2$

or $y^4 - 2cxy^2 + c^2x^2 = x^2y^2p^2$ or $y^4 - 2cxy^2 + c^2x^2 = x^2(c^2 - y^2)$, by (2)

or $y^4 - 2cxy^2 + x^2y^2 = 0$ or $x^2 + y^2 - 2xc = 0$

Ex. 2. *Solve the following differential equations:*

(a) $y + px = x^4p^2$. **[M.S. Univ. T.N. 2007; Agra 1996, Delhi Maths (Hons.) 1998]**

(b) $y = yp^2 + 2px$ **[Allahabad 1994, Kanpur, 1993]**

(c) $y = 2px + f(xp^2)$ **Meerut 2007, Delhi Maths (G) 1998]**

(d) $y = 2px + \tan^{-1}(xp^2)$ **[Nagarjuna 2003]**

Sol. (*a*) Given $y + px = x^4p^2$, where $p = dy/dx$. ... (1)

Solving (1) for y, $y = x^4p^2 - px$. ... (2)

Differentiating (2) w.r.t. 'x' and writing p for dy/dx, we get

$$p = 4x^3p^2 + 2x^4p\,(dp/dx) - [p + x\,(dp/dx)]$$

or $2p - 4x^3p^2 + (dp/dx)(x - 2x^4p) = 0$ or $2p(1 - 2x^3p) + x(dp/dx)(1 - 2x^3p) = 0$

or $(1 - 2x^3p)\,[2p + x\,(dp/dx)] = 0.$ ... (3)

Neglecting the first factor which does not involve dp/dx, (3) reduces to

$2p + x\,(dp/dx) = 0$ or $(1/p)\,dp + 2\,(1/x)\,dx = 0.$

Integrating, $\log p + 2\log x = \log c$ or $px^2 = c$ or $p = c/x^2$.

Putting this value of p in (1), the required solution is

$y + x\,(c/x^2) = x^4\,(c^2/x^4)$ or $xy + c = c^2x$.

(*b*) Given $y = yp^2 + 2px$, where $p = dy/dx$. ... (1)

Solving (1) for y, $y(1 - p^2) = 2px$ or $y = 2px/(1 - p^2)$. ... (2)

Differentiating (2) w.r.t. 'x' and writing p for dy/dx, we get

$$p = [(1 - p^2)\,\{2p + 2x\,(dp/dx)\} - 2px\,(-2p)\,(dp/dx)]/(1 - p^2)^2$$

or $p(1 - p^2)^2 = 2p(1 - p^2) + 2x(1 - p^2)(dp/dx) + 4p^2x\,(dp/dx)$

or $p(1 - p^2)[(1 - p^2) - 2] - 2x\,(dp/dx)\,(1 - p^2 + 2p^2) = 0$

or $p(p^2 - 1)(1 + p^2) - 2x\,(dp/dx)\,(1 + p^2) = 0$

or $(1 + p^2)\,[p(p^2 - 1) - 2x\,(dp/dx)] = 0.$ ... (3)

Neglecting the first factor which does not involve dp/dx, (3) reduces to

$p(p^2 - 1) - 2x\dfrac{dp}{dx} = 0$ or $\left(\dfrac{1}{p-1} + \dfrac{1}{p+1} - \dfrac{2}{p}\right)dp = \dfrac{dx}{x}.$

Integration, $\log(p - 1) + \log(p + 1) - 2\log p = \log x + \log c$

or $[(p-1)(p+1)/p^2] = cx$ or $(p^2-1)/p^2 = cx$

or $p^2 - 1 = cxp^2$ or $p = [1/(1+cx)]^{1/2}$.

Putting this value of p in (1), the required solution is

$$y = \frac{y}{1+cx} + \frac{2x}{(1+cx)^{1/2}} \quad \text{or} \quad y\left[1 - \frac{1}{1+cx}\right] = \frac{2x}{(1+cx)^{1/2}} \quad \text{or} \quad -\frac{cy}{(1+cx)^{1/2}} = 2$$

Squaring both sides of above equation, $c^2y^2 = 4(1+cx)$, c being an arbitrary constant

(*c*) Given $y = 2px + f(xp^2)$, where $p = dy/dx$. ... (1)

Differentiating (1) w.r.t. 'x' and writing p for dy/dx, we get

$$p = 2p + 2x\,(dp/dx) + f'(xp^2) \times \{p^2 + 2xp\,(dp/dx)\}$$

or $p\{1 + pf'(xp^2)\} + 2x\,(dp/dx)\{1 + pf'(xp^2)\} = 0$

or $\{1 + pf'(xp^2)\}\{p + 2x\,(dp/dx)\} = 0$. ... (2)

Neglecting the first factor which does not involve dp/dx, (2) reduces to

$p + 2x\,(dp/dx) = 0$ or $2(1/p)\,dp + (1/x)\,dx = 0$.

Integrating, $2\log p + \log x = 2\log c$ or $p^2x = c^2$ or $p = c/x^{1/2}$.

Putting this value of p in (1), the required solution is

$y = 2x\,(c/x^{1/2}) + f(c^2)$ or $y = 2cx^{1/2} + f(c^2)$.

(*d*) Do as in part (*c*). **Ans.** $y = 2(cx)^{1/2} + \tan^{-1} c$.

Ex. 3. *Solve* : $yp^2 - 2xp + y = 0$

Sol. Solving for y, $y = (2px)/(1+p^2)$. ... (1)

Differentiating (1) w.r.t. 'x' and writing p for dy/dx, we get

$$p = \frac{(1+p^2)[2p + 2x(dp/dx)] - 2px \cdot (2p)(dp/dx)}{(1+p^2)^2}$$

or $p(1+p^2)^2 = 2p(1+p^2) + 2x\,(dp/dx)\{1 + p^2 - 2p^2\}$

or $p(1+p^2)(1+p^2-2) = -2x(p^2-1)(dp/dx)$ ... (2)

or $(p^2-1)[p(1+p^2) + 2x\,(dp/dx)] = 0$.

Neglecting the first factor which does not involve dp/dx, (2) reduces to

$$p(1+p^2) + 2x\frac{dp}{dx} = 0 \quad \text{or} \quad \frac{dx}{x} + \left[\frac{2}{p} - \frac{2p}{1+p^2}\right]dp = 0,$$

$\therefore$ $\log x + 2\log p - \log(1+p^2) = \log c$ or $x = c(1+p^2)/p^2$ ... (3)

Putting this value of x in (1), we get $y = 2c/p$. ... (4)

(2) and (4) together form the solution in parametric form, p being treated as parameter.

Ex. 4. *Solve the following differential equations* :

(a) $y = x + a\tan^{-1} p$ **[Delhi Maths (G) 1999, 2000]**

(b) $4y = x^2 + p^2$ **[Delhi Maths (H) 2009; Delhi Maths (G) 1998]**

Sol. (*a*) Given $y = x + a\tan^{-1} p$, where $p = dy/dx$. ... (1)

Differentiating (1) w.r.t. 'x' and writing p for dy/dx, we have

$$p = 1 + \frac{a}{1+p^2}\frac{dp}{dx} \quad \text{or} \quad p - 1 = \frac{a}{1+p^2}\frac{dp}{dx}$$

or $$dx = \frac{a}{(p-1)(1+p^2)}dp = \frac{a}{2}\left[\frac{1}{p-1} - \frac{1+p}{1+p^2}\right], \text{ by partial fractions}$$

Integrating, $x = (a/2) \times [\log (p-1) - \tan^{-1} p - (1/2) \times \log (1 + p^2) + \log c]$

or $$x = (a/2) \times [\log \{c\,(p-1)/(1+p^2)^{1/2}\} - \tan^{-1} p] \qquad \text{... (2)}$$

Substituting the value of x given by (2) in (1), we have

$$y = (a/2) \times [\log \{c\,(p-1)/(1+p^2)^{1/2}\} + \tan^{-1} p]. \qquad \text{... (3)}$$

(2) and (3) together form the required solution in parametric form, p being parameter

(*b*) Given $$4\,y = x^2 + p^2, \text{ where } p = dy/dx. \qquad \text{... (1)}$$

Differentiating (1) w.r.t. x, $$4\,p = 2x + 2\,p\,(dp/dx)$$

so that $$\frac{dp}{dx} = \frac{2p - x}{p} = \frac{2(p/x) - 1}{(p/x)}, \text{ which is } a \text{ homogeneous equation} \qquad \text{... (2)}$$

Putting $p/x = v$ or $p = xv$ so that $dp/dx = v + x\,(dv/dx)$, (2) gives

$$v + x\frac{dv}{dx} = \frac{2v-1}{v} \qquad \text{or} \qquad x\frac{dv}{dx} = \frac{2v-1}{v} - v = \frac{2v - 1 - v^2}{v} = -\frac{(v-1)^2}{v}$$

or $$\frac{dx}{x} = -\frac{v\,dv}{(v-1)^2} = -\frac{(v-1)+1}{(v-1)^2}\,dv = \left[-\frac{1}{v-1} - \frac{1}{(v-1)^2}\right]dv.$$

Integrating, $\log x = -\log (v-1) + 1/(v-1) + c$, c being an arbitrary constant

or $$\log \{x\,(v-1)\} = \frac{1}{v-1} + c \qquad \text{or} \qquad \log (p - x) = \frac{x}{p-x} + c, \text{ as } v = \frac{p}{x} \qquad \text{... (3)}$$

(1) and (3) together give solution, p being the parameter.

Ex. 5. *Solve the following the so called Lagrange's equations.*

(*a*) $y = 2\,px - p^2$. **[Kanpur 2007, S.V. University (A.P.) 1997]**

(*b*) $x = yp + ap^2$.

(*c*) $9\,(y + x\,p\,\log p) = (2 + 3\,\log p)\,p^3$

(*d*) $y = abx + bp^3$

(*e*) $y = 3px + 4p^2$ **[I.A.S. 1998]**

(*f*) $y = 2\,px + p^2$ **[Delhi Maths (G) 2005, 06]**

Sol. (*a*) Given $$y = 2\,px - p^2, \text{ where } p = dy/dx. \qquad \text{... (1)}$$

Differeintiating (1) w.r.t. 'x', $$p = 2p + 2x\,(dp/dx) - 2p\,(dp/dx)$$

or $$p + 2\,(x - p)\,(dp/dx) = 0 \qquad \text{or} \qquad p\,(dx/dp) + 2\,(x - p) = 0$$

or $$(dx/dp) + (2/p)\,x = 2, \text{ which is linear equation.}$$

Its I.F. $= e^{\int (2/p)dp} = e^{2\log p} = e^{\log p^2} = p^2$ and solution is

$$xp^2 = \int 2p^2\,dp + c = (2/3)\,p^3 + c \qquad \text{or} \qquad x = (2/3)\,p + cp^{-2}. \qquad \text{... (2)}$$

Putting this value of x in (1), we get

$$y = 2p\,[(2/3)\,p + cp^{-2}] - p^2 \qquad \text{or} \qquad y = (1/3)\,p^2 + 2\,cp^{-1}. \qquad \text{... (3)}$$

(2) and (3) together form the required solution, p being the parameter.

(*b*) Solving for y, we have $$y = (x/p) - ap. \qquad \text{... (1)}$$

Differentiating (1) w.r.t., 'x', $$p = (1/p) - (x/p^2)\,(dp/dx) - a\,(dp/dx)$$

or $$(1/p) - p = (dp/dx)\,(x/p^2 + a) \qquad \text{or} \qquad (1 - p^2)\,p = (x + ap^2)\,(dp/dx)$$

or $$\frac{dx}{dp} = \frac{x + ap^2}{(1 - p^2)\,p} \qquad \text{or} \qquad \frac{dx}{dp} - \frac{1}{p(1-p^2)}x = \frac{ap}{1-p^2}, \qquad \text{... (2)}$$

which is a linear equation. Comparing (2) with $(dx/dp) + Px = Q$, here, we have

$$P = -\,1/\{p\,(1 - p^2)\} \qquad \text{and} \qquad Q = (ap)/(1 - p^2). \qquad \text{... (3)}$$

$$\therefore \quad \int P dx = -\int \frac{dp}{p(1-p^2)} = -\int \left(\frac{1}{p} + \frac{p}{1-p^2}\right), \text{ on resolving into partial fractions}$$

$$= \int [(1/p) - (1/2)\{-2p/(1-p^2)\}]\,dp = -\log p + (1/2) \times \log (1 - p^2)$$

$$= \log\,(1 - p^2)^{1/2} - \log p = \log\,\{(1 + p^2)^{1/2}/p\}.$$

$\therefore$ I.F. of (2) $= e^{\int P dp} = e^{\log\{(1-p^2)^{1/2}/p\}} = (1 - p^2)^{1/2}/p$ and solution is

or $$\frac{x(1+p^2)^{1/2}}{p} = c + \int \frac{ap}{1-p^2} \cdot \frac{(1-p^2)^{1/2}}{p} dp = c + a\int \frac{dp}{(1-p^2)^{1/2}} = c + a\sin^{-1} p$$

$$\therefore \qquad x = \{p/(1 + p^2)^{1/2}\}\ (c + a\ \sin^{-1} p). \qquad \text{... (3)}$$

Using (3), $\qquad (1) \Rightarrow y = \{1/(1 + p^2)^{1/2}\}\ (c + a\ \sin^{-1}\ p) - ap.$ ... (4)

(3) and (4) together form the required solution, p being the parameter.

(*c*) Solving the given equation for y, $\qquad y = -\,xp \log p + (1/9)\ (2 + 3 \log p)\ p^3.$ ... (1)

Differentiating (1) w.r.t. 'x' and writing p for dy/dx, we have

$$p = -\,p \log p - x\ (\log p + 1)\ (dp/dx) + (1/9)\ [3p^2\ (2 + 3 \log p) + 3\ p^2]\ (dp/dx)$$

or $$p\ (1 + \log p) = -\ x\ (1 + \log p)\ (dp/dx) + (1/9)\ (9p^2 + 9p^2 \log p)\ (dp/dx)$$

or $$p\ (1 + \log p) = -\ x\ (1 + \log p)\ (dp/dx) + p^2\ (1 + \log p)\ (dp/dx)$$

or $$(1 + \log p)\ \{p + x\ (dp/dx) - p^2\ (dp/dx)\} = 0. \qquad \text{... (2)}$$

Rejecting the first factor since it does not involve dp/dx, (2) gives

$$p + (x - p^2)\ (dp/dx) = 0 \qquad \text{or} \qquad p\ (dx/dp) + x - p^2 = 0$$

or $$(dx/dp) + (1/p)\ x - p = 0 \qquad \text{or} \qquad (dx/dp) + (1/p)\ x = p. \qquad \text{... (3)}$$

(3) is linear equation. Its I.F. $= e^{\int (1/p)\,dp} = e^{\log p} = p$ and solution is

$$xp = \int (p \cdot p)\,dp + c \qquad \text{or} \qquad xp = p^3/3 + c$$

$$\therefore \qquad x = p^2/3 + cp^{-1},\ c \text{ being an arbitrary constant} \qquad \text{... (4)}$$

Then, $\qquad (1) \Rightarrow y = -\ p \log p\ [(1/3)\ p^2 + cp^{-1}] + (1/9)\ (2 + 3 \log p)\ p^3$

or $$y = (2/9) \times p^3 - c \log p. \qquad \text{... (5)}$$

(1) and (5) together form the required solution, p being the parameter

(*d*) Differentiating w.r.t. x the given equation $\quad y = apx + bp^3,$ ... (1)

we get $\quad p = ap + ax\ (dp/dx) + 3\ bp^2\ (dp/dx) \qquad$ or $\qquad p\ (1 - a) = (ax + 3bp^3)\ (dp/dx)$

or $$\frac{dx}{dp} = \frac{ax + 3bp^2}{(1-a)\,p} \qquad \text{or} \qquad \frac{dx}{dp} + \frac{a}{(a-1)\,p}x = \frac{3bp}{1-a}, \qquad \text{... (2)}$$

I.F. $= e^{\int [(a \cdot dp)/(a-1)p]} = e^{\{a/(a-1)\}\log p} = p^{[a/a-1)]}$ and solution is

$$xp^{[a/(a-1)]} = \frac{3b}{1-a}\int p^{[a/(a-1)]} p dp + c = \frac{3b}{1-a}\int p^{[(2a-1)/(a-1)]} dp + c$$

or $$xp^{[a/(a-1)]} = \frac{3b}{1-a}\,\frac{p^{\{(3a-2)/(a-1)\}}}{\{(3a-2)/(a-1)\}} + c = \frac{3b}{2-3a}\,p^{[(3a-2)/(a-1)]} + c$$

or $$x = (3\,bp^2)/(2 - 3a) + cp^{[a/(1-a)]}. \qquad \text{... (3)}$$

The required solution is given by (1) and (3) in parametric form, p being the parameter.

(*e*) Given that $$y = 3\,px + 4\,p^2. \qquad \text{... (1)}$$

Differentiating (1) w.r.t. 'x', $\quad p = 3\,p + 3\,x\,(dp/dx) + 8\,p\,(dp/dx)$

or $(3x+8p)\dfrac{dp}{dx} = -2p$ $\quad$ or $\quad \dfrac{dx}{dp} = \dfrac{3x+8p}{-2p} = -\dfrac{3}{2p}x - 4$

or $$(dx/dp) + (3/2\,p)\,x = -4, \text{ which is linear equation} \qquad \text{... (2)}$$

Its I.F. $= e^{\int (3/2p)dp} = e^{(3/2)\log p} = e^{\log p^{3/2}} = p^{3/2}$ and solution is

$$xp^{3/2} = -4\int p^{3/2}\,dp + c = -4\frac{p^{5/2}}{(5/2)} + C = -\frac{8}{5}p^{5/2} + c$$

or $$x = -(8/5)\,p + c\,p^{-3/2},\ c \text{ being an arbitrary constant} \qquad \text{... (3)}$$

Substituting the above value of x in (1), we get

$$y = 3\,p\,\{(-8/5)\,p + cp^{-3/2}\} + 4\,p^2 = 3\,cp^{-1/2} - (4/5)\,p^2 \qquad \text{... (4)}$$

The required solution is given by (3) and (4) in parametric form, p being the parameter.

(*f*) Given that $$y = 2\,px + p^2 \qquad \text{... (1)}$$

Differentiating (1) w.r.t. 'x' and writing p for dy/dx, we get

$p = 2\,p + 2x\,(dp/dx) + 2\,p\,(dp/dx)$ $\quad$ or $\quad 2\,(x+p)\,(dp/dx) = -p$

or $\dfrac{dx}{dp} = -\dfrac{2(x+p)}{p}$ $\quad$ or $\quad \dfrac{dx}{dp} + \dfrac{2}{p}x = -2, \qquad \text{... (2)}$

Integrating factor of (2) $= e^{\int (2/p)dp} = e^{2\log p} = e^{\log p^2} = p^2$ and solution of (2) is

$xp^2 = \int (-2)\,p^2\,dp + c$ $\quad$ or $\quad xp^2 = -(2/3)\,p^3 + c$

Thus, $$x = -(2/3)\,p + c/p^2,\ c \text{ being an arbitrary constant} \qquad \text{... (3)}$$

Substituting the value of x given by (3) in (1), we get

$y = 2\,p\{(-2/3)\,p + c/p^2\} + p^2$ $\quad$ or $\quad y = -(1/3)\,p^2 + (2c/p) \qquad \text{... (4)}$

The required solution is given by (3) and (4) in parametric form, p being the parameter.

Ex. 6. *Solve the following differential equations* :

(a) $y = p \tan p + \log \cos p$.

(b) $p^3 + p = e^y$

Sol. (*a*) Given $$y = p\tan p + \log\cos p. \qquad \text{... (1)}$$

Differentiating (1) w.r.t. 'x' and writing p for dy/dx, we get

$$p = [\tan p + p\sec^2 p + \{1/\cos p\}\,(-\sin p)]\,(dp/dx)$$

or $p = p\sec^2 p\,(dp/dx)$ $\quad$ or $\quad dx = \sec^2 p\,dp. \qquad \text{... (2)}$

Integrating, (2), $\quad x = \tan p + c$, c being an arbitrary constant $\qquad$ ... (3)

(1) and (3) form the solution in parametric form, p being the parameter

(*b*) Taking logarithm of both sides of the given equation we have

$\log e^y = \log\{p\,(1+p^2)\}$ $\quad$ or $\quad y = \log p + \log(1+p^2). \qquad \text{... (1)}$

Differentiating (1) w.r.t. 'x', we have

$p = \left(\dfrac{1}{p} + \dfrac{2p}{1+p^2}\right)\dfrac{dp}{dx}$ $\quad$ or $\quad dx = \left(\dfrac{1}{p^2} + \dfrac{2}{1+p^2}\right)dp$

Integrating, $\quad x = -(1/p) + 2\tan^{-1} p + c$, c being an arbitrary constant. $\qquad$ ... (2)

(1) and (2) together form the required solution in parametric form, p being the parameter.

Exercise 4(C)

Solve following differential equations :

1. $y = 2px + p^4x^2$ **[Delhi Maths (G) 1998]** **Ans.** $(y - c^2)^2 = 4cx$
2. $xp^2 - 2yp + ax = 0$ **[Delhi Maths (H) 2006; G.N.D.U. Amritsar 2010]** **Ans.** $c^2x^2 - 2yc + a = 0$
3. $y = 2px - xp^2$ **[Delhi B.A. (Prog) II 2007, 10]** **Ans.** $(y + c)^2 = 4cx$
4. $xp^2 - 3yp + 9x^2 = 0$ **Ans.** $3cy = c^2x^3 + 9$
5. $p^2 + p = y/x$ or $x(p^2 + p) = y$. **Ans.** $x = (c/p^2)\, e^{1/p}$, $y = c(1 + 1/p)\, e^{1/p}$
6. $y = p^2 + x$. **Ans.** $x = 2p + 2\log(p-1) + c$, $y = p^2 + 2p + 2\log(p-1) + c$
7. $y = p^3 + x$. **Ans.** $x = 3p^2/2 + 3p - 3\log(p-1) + c$,
   $y = p^3 + 3p^2/2 + 3p + 3\log(p-1) + c$
8. $y - px + x - (y/b) = a$. **Ans.** $x = c(p-b)^{-b}$, $y = \{b/(b-1)\} \times \{c(p-1)(p-b)^{-b} + a\}$
9. $y = p + xp^2$ **[Agra 1995]** **Ans.** $x = (\log p - p + c)/(p-1)^2$, $y = p\{1 - p(2 - \log p)\}/(p-1)^2$
10. $x^2 + p^2x = yp$. **Ans.** $x = cp^{1/2} - (p^2/3)$, $y = \{c - (1/3)p^{3/2}\} \times \{c + (2/3)p^{3/2}\}$
11. $x = yp - p^2$ **Ans.** $x = \{p/(1+p^2)^{1/2}\}(c - \sin^{-1} p)$, $y = \{1/(1+p^2)^{1/2}\}(c - \sin^{-1} + p) + p$
12. $x + py = p^3$ **[Delhi B.Sc. (Prog) II 2009; Delhi Maths (G) 2000, Delhi Maths (H) 1999]**
    Ans. $x = 2p + cp(p^2+1)^{-1/2}$, $y = p^2 - c(p^2+1)^{-1/2} - 2$
13. (*a*) $y = (1+p)x + ap^2$. **Ans.** $x = ce^{-p} - 2a(p-1)$, $y = (1+p)\{ce^{-p} - 2a(p-1)\} + ap^2$
    (*b*) $y = (1+p)x + p^2$ **Ans.** $x = ce^{-p} - 2(p-1)$, $y = (1+p)\{ce^{-p} - 2(p-1)\} + p^2$
14. $4p^3 + 3xp = y$. **Ans.** $x = -(12p^2/7) + cp^{-3/2}$, $y = -(8p^3/7) + 3cp^{1/2}$
15. $y = 2px + p^n$. **Ans.** $x = (c/p^2) - (np^{n-1})/(n+1)$, $y = (2c/p) - \{(n-1)p^n/(n+1)\}$
16. $p^3 + xp^2 = y$. **Ans.** $x = (2x - 2p^3 + 3p^2)/2(p-1)^2$, $y = (2cp^2 + 2p^3 - p^4)/2(p-1)^2$
17. $y = p^2x + p^4$ **Ans.** $x = \{3c + p^3(4 - 3p)\}/(1-p)^2$,
    $y = \{3cp^2 + p^5(4 - 3p) + 3p^4(1-p)^2\}/3(1-p)^2$.
18. $y = \sin p - p\cos p$ **[Kanpur 1995]** **Ans.** $x = c - \cos p$, $y = \sin p - p\cos p$
19. $y = p\sin p + \cos p$. **Ans.** $x = c + \sin p$, $y = p\sin p + \cos p$
20. $e^{p-y} = p^2 - 1$. **Ans.** $x = \log p - \log\{(p-1)/(p+1)\} + c$, $y = p - \log(p^2 - 1)$
21. $y = 2p + 3p^2$. **Ans.** $x = c + 6p + 2\log p$, $y = 2p + 3p^2$
22. $y = a(1+p^2)^{1/2}$ **[Kanpur 1994, 96]** **Ans.** $x = c + a\sinh^{-1} p$, $y = a(1+p^2)^{1/2}$
23. $y - \{1/(1-p^2)^{1/2}\} = a$ **Ans.** $(x+c)^2 + (y-a)^2 = 1$
24. $y = 2p + (1+p^2)^{1/2}$ **Ans.** $y = 2\log p + \sinh^{-1} p + c$, $y = 2p + (1+p^2)^{1/2}$
25. $p - y = \log(p^2 - 1)$. **Ans.** $x = \log\{p(p+1)/(p-1)\} + c$, $y = p - \log(p^2 - 1)$
26. $y = a + bp + dp^2$. **Ans.** $x = b\log p + 2dp + c$, $y = a + bp + dp^2$

4.8 Method IV : Equations in Clairaut's form

Clairaut's equation. Definition. **[Andra 2003, Garhwal 2005]**

An equation of the form $y = px + f(p)$ is known as *Clairaut's equation.*

General solution of Clairaut's equation. *To show that the general solution of Clairaut's equation $y = px + f(p)$ is $y = cx + f(c)$ which is obtained by replacing p by c, where c is an arbitrary constant*

Proof. Given Clairaut's equation is $y = px + f(p)$. ... (1)

Differentiating (1) w.r.t. 'x' and writing p for dy/dx, we have

$p = p + x(dp/dx) + f'(p)(dp/dx)$ or $[x + f'(p)](dp/dx) = 0$ ... (2)

Omitting the factor $x + f'(p)$ which does not involve dp/dx, (2) gives,

$dp/dx = 0$ so that $p = c$, c being on arbitrary constant ... (3)

Putting the value of p given by (3) in (1), the required solution is

$$y = cx + f(c). \qquad \text{... (4)}$$

Working rule for solving Clairaut's equation *i.e.* (1):

Replace p, by c in (1) to obtain the general solution of (1) c being an arbitrary constant.

Remark. It we eliminate p between $x + f'(p) = 0$ and (1), we shall arrive at a solution which is free from an arbitrary constant and is not a particular case of (4). Such a solution is known as a *singular solution* of (1) and will be discussed in part II of this chapter.

4.9 Solved examples based on Art. 4.8

Ex. 1. *Solve the following differential equations :*

(a) $y = px + (1 + p^2)^{1/2}$ **[Kanpur 1995]**

(b) $y = px + ap\ (1 - p)$ **[Agra 2006]**

(c) $y = px + (a/p)$ or $y = x\ (dy/dx) + a\ (dx/dp)$ **[Agra 2005]**

(d) $y = x\ (dy/dx) + e^{dy/dx}$ or $y = xp + e^p$

(e) $y = x\ (dy/dx) + (dy/dx)^2$ or $y = px + p^2$ **[I.A.S. (Prel.) 1997]**

Sol. (*a*) The given equation is in Clairaut's form $y = px + f(p)$. So replacing p by c its general solution is $y = cx + (1 + c^2)^{1/2}$, c being an arbitrary constant.

(*b*) Proceed as in part (*a*). **Ans.** $y = cx + ac\ (1 - c)$

(*c*) Proceed as in part (*a*). **Ans.** $y = cx + (a/c)$

(*d*) Writing p for dy/dx, we get $y = xp + e^p$ which is in Clairaut's form $y = px + f(p)$. So replacing p by c its general solution is $y = cx + e^c$.

(*e*) Writing p for dy/dx, the given equation reduces to $y = px + p^2$, which is in Clairaut's form $y = px + f(p)$. Replacing p by c, the required solution is $y = cx + c^2$, c being an arbitrary constant

Ex. 2. *Solve the following differential equations :*

(a) $p = \log\ (px - y)$. **I.A.S. (Prel.) 2005]**

(b) $p = \tan\ (px - y)$. **[Jabalpur 1993, Poona 1991]**

(c) $\sin px \cos y = \cos px \sin y + p$ **[I.A.S. (Prel.) 2005]**

(d) $(y - px)^2 / (1 + p^2) = a^2$

(e) $p^2\ (x^2 - a^2) - 2\ pxy + y^2 - b^2 = 0$ **[G.N.D.U. Amritsar 2010]**

(f) $y^2 + x^2\ (dy/dx)^2 - 2xy\ (dy/dx) = 4\ (dx/dy)^2$

Sol. (*a*) Given $p = \log\ (px - y)$ so that $e^p = px - y$ or $y = px - e^p$, which is in Clairaut's form. So replacing p by c, the required general solution is

$$y = cx - e^c, \ c \text{ being an arbitrary constant.}$$

(*b*) Given $p = \tan\ (px - y)$ so that $\tan^{-1} p = px - y$ or $y = px - \tan^{-1} p$, which is in Clairaut's form. So replacing p by c, the required solution is

$$y = cx - \tan^{-1} c, \ c \text{ being an arbitrary constant}$$

(*c*) Given $\sin px \cos y = \cos px \sin y + p$ or $\sin px \cos y - \cos px \sin y = p$ or $\sin\ (px - y) = p$ or $px - y = \sin^{-1} p$ or $y = px - \sin^{-1} p$, which is in Clairaut's form. So replacing p by c, the required general solution is

$$y = cx - \sin^{-1} c, \ c \text{ being an arbitrary constant.}$$

(*d*) Given $(y - px)^2 / (1 + p^2) = a^2$ or $y - px = \pm\ a\ (1 + p^2)^{1/2}$

or $y = cx \pm a\ (1 + p^2)^{1/2}$, which is in Clairaut's form.

Replacing p by c, its general solution is $y = cx \pm a\ (1 + c^2)^{1/2}$.

or $y - cx = \pm\ a\ (1 + c^2)^{1/2}$ or $(y - cx)^2 = a^2\ (1 + c^2)$, on squaring.

(*e*) Given $p^2(x^2-a^2)-2pxy+y^2-b^2=0$ or $p^2x^2-2pxy+y^2=p^2a^2+b^2$.

or $(y-px)^2=p^2a^2+b^2$ or $y=px\pm(p^2a^2+b^2)^{1/2}$.

Replacing p by c, the required general solution is

$y=cx\pm(c^2a^2+b^2)^{1/2}$ or $y-cx=\pm(c^2a^2+b^2)^{1/2}$.

or $(y-cx)^2=c^2a^2+b^2$ or $c^2(x^2-a^2)-2c\,xy+y^2-b^2=0$.

(*f*) Writing p for dy/dx and $1/p$ or dx/dy, we get

$y^2+x^2p^2-2\,xyp=4/p^2$ or $(y-px)^2=4/p^2$ or $y=px\pm(2/p)$,

which is of Clairaut's form. So replacing p by c, the required solution is

$y=cx\pm(2/c)$ or $y-cx=\pm(2/c)$ or $y^2+x^2c^2-2\,xyc=4/c^2$.

Ex. 3. *Solve* $p^2x\,(x-2)+p\,(2y-2xy-x+2)+y^2+y=0$.

Sol. Given $p^2x^2-2\,p^2x+2py-2pxy-px+2p+y^2+y=0$

or $(y^2-2pxy+p^2x^2)+2p\,(y-px)+(y-px)+2p=0$

or $(y-px)^2+2p\,(y-px)+(y-px)+2p=0$

or $(y-px)\,[(y-px)+2p]+\{(y-px)+2p\}=0$ or $(y-px+2p)\,(y-px+1)=0$

$\Rightarrow$ $y-px+2p=0$ or $y-px+1=0$.

$\Rightarrow$ $y=px-2p$ or $y=px-1$. ... (1)

Both the component equations of (1) are in Clairaut's form and hence replacing p by c, their general solution are $y=cx-2c$ and $y=cx-1$.

$\therefore$ general solution is $(y-cx+2c)\,(y-cx+1)=0$

or $c^2x\,(x-2)+c\,(2y-2xy-x+2)+y^2+y=0$, c being an arbitrary constant

Exercise 4(*d*)

Solve the following differential equations, taking $p=dy/dx$

1. $y=px+p^n$ **Ans.** $y=cx+c^n$
2. $y=px+p-p^2$ **[Kanpur 2011]** **Ans.** $y=cx+c-c^2$
3. $y=px+\log p$ **Ans.** $y=cx+\log c$
4. $y=px+\sin^{-1}p$ **Ans.** $y=cx+\sin^{-1}c$
5. $y=x\,(dy/dx)+(dy/dx)^3$ **Ans.** $y=cx+c^3$
6. $y=px+a\tan^{-1}p$ **Ans.** $y=cx+a\tan^{-1}c$
7. $px-y+p^3=m^3/p^3$ **Ans.** $y=cx+c^3-(m^3/c^3)$
8. $y=px+(a^2p^2+b^2)^{1/2}$ **Ans.** $y=cx+(a^2c^2+b^2)^{1/2}$
9. $xp^2-yp+a=0$ or $y=px+a/p$ **[Purvanchal 2007]** **Ans.** $y=cx+(a/c)$
10. $\sin(y-px)=p$ **[Delhi Maths (Hons.) 2005]** **Ans.** $y=cx+\sin^{-1}c$
11. $\cos px\cos y+\sin px\sin y=p$ **Ans.** $y=cx-\cos^{-1}c$
12. $(y-px)\,(p-1)=p$ **Ans.** $y=cx+\{c/(c-1)\}$
13. $(xp-y)^2=p^2+1$ **[Guwahati 2007]** **Ans.** $(cx-y)^2=c^2+1$
14. $(xp-y)^2=p^2-1$ **Ans.** $(cx-y)^2=c^2-1$
15. $(y-px)^2(1+p^2)=a^2p^2$ **Ans.** $(y-cx)^2(1+c^2)=a^2c^2$
16. $p^2(x^2-a^2)-2\,pxy+y^2+a^4=0$ **Ans.** $c^2(x^2-a^2)-2cxy+y^2+a^4=0$
17. $y^2-2pxy+p^2(x^2-1)=m^2$ **Ans.** $c^2-2cx+c^2(x^2-1)=m^2$
18. $(x^2-1)\,p^2-2xyp+y^2=1$ **Ans.** $(x^2-1)\,c^2-2xyc+y^2=1$

4.10 Method V : Equations reducible to Clairaut's form

By using suitable substitutions, some equations can be reduced to Clairaut's form. There is no general method of deciding about proper substitution in *a* certain problem. These can be learned only by practice. However, the students are adviced to remember the three important substitutions given in following solved examples 1, 2, 3 and indicated as forms I, II and III respectively.

4.11 Solved examples based on Art. 4.10

Form I : To solve $y^2 = (py/x)\, x^2 + f\,(py/x)$, put $x^2 = u$ and $y^2 = v$.

Now, $x^2 = u$ and $y^2 = v$ $\Rightarrow$ $2x\,dx = du$ and $2y\,dy = dv$.

$\therefore$

$$\frac{2y\,dy}{2x\,dx} = \frac{dv}{du} \quad \text{or} \quad \frac{py}{x} = P, \quad \text{where} \quad P = \frac{dv}{du}.$$

Hence the given equation becomes $v = Pu + f\,(P)$,

which is in Clairaut's form and so its solution is

$v = cu + f\,(c)$ or $y^2 = cx^2 + f\,(c)$, c being an arbitrary constant.

We now illustrate form I in the following example 1

Ex. 1. (*a*) *Solve $x^2\,(y - px) = yp^2$ or $yp^2 + x^3p - x^2y = 0$.* **[Allahabad 1994, Delhi Maths (G) 1994, Kumaun 1998, Agra 1995, I.A.S. 1996, Lucknow 1995, S.V. Univ. (A.P.) 1997]**

(*b*) *Solve $(px - y)\,(py + x) = h^2p$, using the transformations $x^2 = u$, $y^2 = v$.* **[Delhi Maths (H) 2007, Agra 2006, Delhi Maths (G) 1995, 96, Osmania 2004 Kumaun 1996, Meerut 1997]**

(*c*) *Reduce the differential equation $(px - y)\,(x - yp) = 2p$ to Clairaut's form by the substitution $x^2 = y$, $y^2 = v$ and find its complex primitive.* **[Allahabad 1993, Kanpur 1992]**

(*d*) *Solve $axy\,p^2 + (x^2 - ay^2 - b)\,p - xy = 0$ by putting $u = x^2$, $v = y^2$.*

(*e*) *Solve $xyp^2 - (x^2 + y^2 - 1)\,p + xy = 0$.* **[G.N.D.U. Amritsar 2010]**

(*f*) *Solve $xy\,(y - px) = x + yp$.*

Sol. (*a*) Given $x^2\,(y - px) = yp^2$ or $y - px = (yp^2)/x^2$

or $y^2 = pxy + (py/x)^2$ or $y^2 = (py/x)\,x^2 + (py/x)^2$. ... (1)

Putting $x^2 = u$ and $y^2 = v$ so that $2x\,dx = du$ and $2y\,dy = dv$, ... (2)

we get $$\frac{2y\,dy}{2x\,dx} = \frac{dv}{du} \quad \text{or} \quad \frac{Py}{x} = P, \quad \text{where} \quad P = \frac{dv}{du}. \qquad \text{... (3)}$$

Using (2) and (3), (1) becomes $v = Pu + P^2$, which is in Clairaut's form. So replacing P by arbitrary constant c, the required general solution is

$v = cu + c^2$ or $y^2 = cx^2 + c^2$, c being an arbitrary constant

(*b*) Given $(px - y)\,(py + x) = h^2p$ or $(pxy - y^2)\,x\,\{1 + (py/x)\} = h^2py$

or $$(pxy - y^2) = \frac{h\,(py/x)}{1 + (py/x)} \quad \text{or} \quad y^2 = \left(\frac{py}{x}\right)x^2 - \frac{h\,(py/x)}{1 + (py/x)}. \qquad \text{... (1)}$$

Putting $x^2 = u$ and $y^2 = v$ so that $2x\,dx = du$ and $2y\,dy = dv$, ... (2)

we get $$\frac{2y\,dy}{2x\,dx} = \frac{dv}{du} \quad \text{or} \quad \frac{py}{x} = P, \quad \text{where} \quad P = \frac{dv}{du}. \qquad \text{... (3)}$$

Using (2) and (3), (1) becomes $v = Pu - (hP)/(1 + P)$, which is in Clairaut's form. So replacing P by arbitrary constant c, the required solution is

$v = cu - (hc)/(1 + c)$ or $y^2 = cx^2 - (hc)/(1 + c)$, c being an arbitrary constant

(*c*) Proceed as in part (*b*). **Ans.** $y^2 = cx^2 - (2c)/(1 - c)$.

(*d*) Given $axy\,p^2 + (x^2 - ay^2 - b)\,p - xy = 0$. ... (1)

Putting $x^2 = u$ and $y^2 = v$ so that $2x\,dx = du$ and $2y\,dy = dv$, ... (2)

we get $\dfrac{2y\,dy}{2x\,dx} = \dfrac{dv}{du}$ or $\dfrac{py}{x} = P$ or $p = \dfrac{xP}{y}$, where $P = \dfrac{dv}{du}$. ... (3)

Replacing p by (xP/y) in (1), we have

$axy(x^2P^2/y^2) + (x^2 - ay^2 - b)(xP/y) - xy = 0$ or $ax^2P^2 + (x^2 - ay^2 - b)P - y^2 = 0$

or $auP^2 + (u - av - b)\,P - v = 0$ or $v\,(1 + aP) = uP\,(1 + aP) - bP$, as $x^2 = u, y^2 = v$

or $v = uP - (bP)/(1 + aP)$, which is in Clairaut's form.

So replacing P by arbitrary constant c, the required general solution is

$v = uc - (bc)/(1 + ac)$ or $y^2 = cx^2 - (bc)/(1 + ac)$, c being an arbitrary constant

(*e*) Do like part (*d*). **Ans.** $y^2 = cx^2 - c/(c - 1)$

(*f*) Given $xy\,(y - px) = x + yp$ or $xy^2 = px^2y + x + yp$

or $y^2 = pxy + 1 + (py/x)$ or $y^2 = (py/x)\,x^2 + 1 + (py/x)$. ... (1)

Putting $x^2 = u$ and $y^2 = v$ so that $2x\,dx = du$ and $2y\,dy = dv$, ... (2)

we get $\dfrac{2y\,dy}{2x\,dx} = \dfrac{dv}{du}$ or $\dfrac{py}{x} = P$, where $P = \dfrac{dv}{du}$. ... (3)

Using (2) and (3), (1) becomes $v = Pu + (1 + P)$, which is in Clairaut's form. So replacing P by arbitrary constant c, the required general solution is

$v = cu + (1 + c)$ or $y^2 = cx^2 + (1 + c)$, c being an arbitrary constant

Form II : To solve equations of the form $e^{by}\,(a - bp) = f\,(pe^{by - ax})$, we use the transformation $e^{ax} = u$ and $e^{by} = v$

We now illustrate form II in the following example 2.

Ex. 2. *Solve the following differential equations :*

(a) $e^{3x}\,(p - 1) + p^3\,e^{2y} = 0$. **[Kanpur 1993]**

(b) $e^{4x}\,(p - 1) + e^{2y}\,p^2 = 0$. **[Delhi 2008; Kakitiya 1997]**

Sol. (*a*) Given $e^{3x}\,(p - 1) + p^3\,e^{2y} = 0$. ... (1)

Re-writing (1), $1 - p = p^3\,e^{2y - 3x}$ or $e^y\,(1 - p) = (p\,e^{y - x})^3$,

which is of the form II. Note that here $a = 1$, $b = 1$.

Putting $e^x = u$ and $e^y = v$ so that $e^x\,dx = du$ and $e^y\,dy = dv$, we get

$\dfrac{e^y\,dy}{e^x dx} = \dfrac{dv}{du}$ or $\dfrac{v}{u}p = P$ or $p = \dfrac{uP}{v}$, where $P = \dfrac{dv}{du}$.

Putting $e^x = u$, $e^y = v$ and $p = uP/v$ in (1), we have

$u^3\left(\dfrac{uP}{v} - 1\right) + \dfrac{u^3P^3}{v^3} \times v^2 = 0$ or $uP - v + P^3 = 0$ or $v = uP + P^3$,

which is in Clairaut's form. So replacing P by c, the required solution is

$v = uc + c^3$ or $e^y = c\,e^x + c^3$, c being an arbitrary constant

(*b*) Given $e^{4x}\,(p - 1) + e^{2y}\,p^2 = 0$. ... (1)

Re-writing (1), $1 - p = p^2\,e^{2y - 4x}$ or $e^{2y}\,(1 - p) = (p\,e^{2y - 2x})^2$,

which is of the form II. Note that have here $a = 2$, $b = 2$.

Putting $e^{2x} = u$ and $e^{2y} = v$ so that $2\,e^{2x}\,dx = du$ and $2\,e^{2y}\,dy = dv$, we get

$\dfrac{2e^{2y}\,dy}{2e^{2x}\,dx} = \dfrac{dv}{du}$ or $\dfrac{v}{u}p = P$ or $p = \dfrac{uP}{v}$, where $P = \dfrac{dv}{du}$, Putting $e^{2x} = u$, $e^{2y} = v$ and $p = uP/v$ in (1), we get

$$u^2\left(\frac{uP}{v}-1\right)+v\left(\frac{u^2P^2}{v^2}\right)=0 \qquad \text{or} \qquad uP - v + P^2 = 0 \qquad \text{or} \qquad v = uP + P^2,$$

which is in Clairaut's form. So replacing P by c, the required solution is

$v = uc + c^2$ or $e^{2y} = c\, e^{2x} + c^2$, c being an arbitrary constant.

Form III. Sometimes the substitution $y^2 = v$ will transform the given equation to Clairaut's form as shown in the following example 3

Ex. 3. *Solve the following differential equations :*

(a) $y = 2px + y^2 p^3$ **[Delhi 2008]**

(b) $y = 2px + ay\, p^2$

(c) $y = 2px + 4yp^2 = 0$

(d) $y = 2px + yp^2$.

(e) $yp^2 - 2xp + y = 0$.

Sol. (*a*) Given $$y = 2px + y^2p^3 \qquad \text{... (1)}$$

Multiplying both sides of (1) by y, we get

$$y^2 = 2\,pxy + y^3 p^3 \qquad \text{or} \qquad y^2 = x\,(2yp) + (1/8) \times (2y\,p)^3. \qquad \text{... (2)}$$

Put $y^2 = v$ so that $2y\,(dy/dx) = dv/dx$ or $2yp = P$, where $P = dv/dx$.

Then (2) becomes $v = xP + P^3/8$, which is in Clairaut's form. ... (3)

So replacing P by arbitrary constant c in (3), the required solution is

$$v = xc + (c^3/8) \qquad \text{or} \qquad y^2 = cx + (c^3/8).$$

(*b*) Given $$y = 2px - ayp^2. \qquad \text{... (1)}$$

Multiplying both sides of (1) by y, we get

$$y^2 = 2pxy - ay^2p^2 \qquad \text{or} \qquad y^2 = x\,(2yp) - (a/4) \times (2yp)^2.$$

Put $y^2 = v$ so that $2y\,(dy/dx) = dv/dx$ or $2yp = P$, where $P = dv/dx$.

Then (2) becomes $v = xp - (a/4) \times P^2$, which is in Clairaut's form. So replacing P by arbitrary constant c, the required solution is

$$v = xc - (ac^2/4) \qquad \text{or} \qquad y^2 = xc - (ac^2/4).$$

(*c*) Try yourself. **Ans.** $y^2 = cx + 4c^2$.

(*d*) Try yourself. **Ans.** $y^2 = cx + (c^2/4)$.

(e) Try yourself. **Ans.** $y = cx - (c^2/4)$.

Note. In the following solved examples, we now present a variety of transformations employed to reduce a given equation to Clairaut's form.

Ex. 4. *Reduce the equation $y^2\,(y - xp) = x^4p^2$ to Clairaut's form by the substitution $x = 1/u$, $y = 1/v$ and hence solve the equation.* **[Gulberga 2005]**

Sol. Give equation is $$y^2\,(y - xp) = x^4p^2. \qquad \text{... (1)}$$

Putting $x = 1/u$, $y = 1/v$ so that $dx = -(1/u^2)\,du$, $dy = -(1/v^2)\,dv$, we get

$$\frac{dy}{dx}=\frac{u^2}{v^2}\frac{dv}{du} \qquad \text{or} \qquad p=\frac{u^2}{v^2}P, \qquad \text{where} \qquad P=\frac{dv}{du} \qquad \text{and} \qquad p=\frac{dy}{dx}.$$

$\therefore$ Putting $x = 1/u$, $y = 1/v$, $p = (u^2P)/v^2$ in (1), we have

$$(1/v^2)\,\{(1/v) - (1/u)\,(u^2P/v^2)\} = (1/u^4)\,(u^4P^2/v^4) \qquad \text{or} \qquad v = uP + P^2,$$

which is in Clairaut's form. Replacing P by c, the required general solution is

$v = uc + c^2$ or $1/y = (c/x) + c^2$ or $x = cy + c^2xy$.

Ex. 5. *Reduce the equation $xp^2 - 2yp + x + 2y = 0$ to Clairaut's form by using the substitution $y - x = v$ and $x^2 = u$.*

Sol. Given equation is $xp^2 - 2yp + x + 2y = 0$. ... (1)

Given $y - x = v$ and $x^2 = u$. ... (2)

Differentiating (2), $dy - dx = dv$ and $2x\, dx = du$

$$\therefore \quad \frac{dy - dx}{2x\,dx} = \frac{dv}{du} \quad \text{or} \quad \frac{dy}{dx} - 1 = 2x\frac{dv}{du} \quad \text{or} \quad p = 1 + 2xP,$$

where $p = dy/dx$ and $P = dv/du$. Putting $p = 1 + 2xP$ in (1), we get

$$x(1 + 2xP)^2 - 2y(1 + 2xP) + x + 2y = 0$$

or $2x^2p^2 - 2(y - x)P + 1 = 0$ or $2uP^2 - 2vP + 1 = 0$, using (2)

or $v = uP + 1/(2P)$, which is of Clairaut's form. Its solution is

$v = uc + 1/(2c)$ or $y - x = cx^2 + 1/(2c)$, c being an arbitrary constant.

Ex. 6. *Solve $x^2p^2 + yp(2x + y) + y^2 = 0$ by using the substitution $y = u$. $xy = v$.*

[Kumaun 1997, Gorakhpur 1993]

Sol. Given equation is $x^2p^2 + yp(2x + y) + y^2 = 0$. ... (1)

Given $y = u$ and $xy = v$. ... (2)

Differentiation (2), $dy = du$ and $x\,dy + y\,dx = dv$.

$$\therefore \quad \frac{x\,dy + y\,dx}{dy} = \frac{dv}{du} \quad \text{or} \quad x + y\frac{dx}{dy} = \frac{dv}{du} \quad \text{or} \quad x + \frac{y}{p} = P$$

or $y/p = P - x$ or $p = y/(P - x)$, where $p = dy/dx$, $P = dv/du$.

Putting $p = y/(P - x)$ in (1), we have

$$\frac{x^2y^2}{(P-x)^2} + \frac{y^2}{P-x}(2x + y) + y^2 = 0 \quad \text{or} \quad x^2 + (P - x)(2x + y) + (P - x)^2 = 0$$

or $Py - xy + P^2 = 0$ or $v = uP + P^2$, using (2). ... (3)

(3) is in Clairaut's form. So replacing P by c its general solution is

$v = uc + c^2$ or $xy = yc + c^2$, c being an arbitrary constant.

Ex. 7. *Solve (a) $y = 2px + f(xp^2)$.* **[Meerut 2007; Nagrajuna 2003, Delhi 1998]**

(b) $y = 2px + \tan^{-1}(xp^2)$.

Sol. (*a*) Given equation is $y = 2px + f(xp^2)$. ... (1)

Let $x = u^2$ so that $dx/dy = 2u\,(du/dy)$.

or $dy/dx = (1/2u)(dy/du)$ or $p = (1/2u)P$, where $p = dy/dx$, $P = dy/du$.

Putting $x = u^2$ and $p = P/(2u)$ in (1), we have

$$y = 2u^2\left(\frac{P}{2u}\right) + f\left(u^2\frac{P^2}{4u^2}\right) \quad \text{or} \quad y = uP + f(P^2/4),$$

which is in Clairaut's form. So replacing P by c, its solution is $y = cu + f(c^2/4)$

or $y = c\,x^{1/2} + f(c^2/4)$, c being an arbitrary constant

(*b*) Proceed as in part (*a*). **Ans.** $y = cx^{1/2} + \tan^{-1}(c^2/4)$.

Ex. 8. *Solve* $(y + xp)^2 = x^2p$.

Sol. Given equation is $(y + xp)^2 = x^2p$. ... (1)

Let $xy = v$ so that $y + x(dy/dx) = dv/dx$. ... (2)

or $y + xp = P$ or $p = (P - y)/x$, where $p = dy/dx$, $P = dv/dx$.

Putting $p = (P - y)/x$ in (1), we get $(y + P - y)^2 = x\,(P - y)$

or $P^2 = xP - xy$ or $v = xP - P^2$, by (2).

It is in Clairaut's form. Hence its solution is $v = xc - c^2$ or $xy = xc - c^2$.

Ex. 9. *Solve* $y^2 \log y = pxy + p^2$.

Sol. Given equation is $y^2 \log y = pxy + p^2$. ... (1)

Let $\log y = v$ so that $(1/y)\,(dy/dx) = dv/dx$ or $(1/y)\,p = P$

so that $p = Py$, where $p = dy/dx$ and $P = dv/dx$.

Putting $p = Py$ in (1), we get $y^2 \log y = (Py)\,(xy) + P^2y^2$

or $\log y = Px + P^2$ or $v = Px + P^2$, as $v = \log y$. ... (2)

(2) is in Clairaut's form. Replacing P by c, its general solution is

$v = cx + c^2$ or $\log y = cx + c^2$, c being an arbitrary constant.

Ex. 10. *Solve* $(px^2 + y^2)\,(px + y) = (p + 1)^2$.

Sol. Let $x + y = u$ and $xy = v$. [Note] ... (1)

Differentiating (1), $dx + dy = du$ and $x\,dy + y\,dx = dv$.

$$\therefore \quad \frac{dv}{du} = \frac{x\,dy + y\,dx}{dx + dy} = \frac{x(dy/dx) + y}{1 + (dy/dx)} \quad \text{or} \quad P = \frac{xp + y}{1 + p}, \qquad \text{... (2)}$$

where $P = dv/du$ and $p = dy/dx$. Now, $(px^2 + y^2)$ can be re-written as

$$px^2 + y^2 = (px + y)\,(x + y) - xy\,(p + 1).$$

Hence the given equation can be re-written as

$$\{(px + y)\,(x + y) - xy\,(p + 1)\}\,(px + y) = (p + 1)^2$$

$$\text{or} \quad \left(\frac{px + y}{p + 1}\right)^2 (x + y) - xy\left(\frac{px + y}{p + 1}\right) = 1 \quad \text{or} \quad P^2u - vP = 1, \text{ by (1) and (2)}$$

or $v = uP - (1/P)$, which is in Clairaut's form.

Its general solution is $v = uc - (1/c)$ or $xy = c\,(x + y) - (1/c)$

or $c^2\,(x + y) - cxy - 1 = c$, c being an arbitrary constant.

Ex. 11. *Solve* $(x^2 + y^2)\,(1 + p)^2 - 2\,(x + y)\,(1 + p)\,(x + yp) + (x + yp)^2 = 0$ **[I.A.S. 2005]**

Sol. Let $x^2 + y^2 = v$ and $x + y = u$ ... (1)

Differentiating (1), $2\,(x\,dx + y\,dy) = dv$ and $dx + dy = du$

$$\therefore \quad \frac{dv}{du} = \frac{2(x\,dx + y\,dy)}{dx + dy} = \frac{2\,\{x + y\,(dy/dx)\}}{1 + (dy/dx)} \quad \text{or} \quad P = \frac{2(x + y\,p)}{1 + p}, \qquad \text{... (2)}$$

where $p = dv/du$ and $p = dy/dx$. Re-writing the given equation, we get

$$(x^2 + y^2) - 2(x + y)\left(\frac{x + yp}{1 + p}\right) + \left(\frac{x + yp}{1 + p}\right)^2 = 0 \quad \text{or} \quad v - 2u \times \frac{P}{2} + \left(\frac{P}{2}\right)^2 = 0, \text{ using (1) and (2)}$$

or $v = uP - (P^2/4)$, which is in Clairaut's, form. Hence its general solution is

$v = uc - (c^2/4)$ or $x^2 + y^2 = C\,(x + y) - (c^2/4)$, by (1)

Exercise 4(E)

Solve the following differential equations:

1. $p^2 \cos^2 y + p \sin x \cos x \cos y - \sin y \cos^2 x = 0$

Hint : Put $\sin x = u$, $\sin y = v$. **Ans.** $\sin y = c \sin x + c^2$.

2. $y = 2\,px + y^{n-1}\,p^n$.

Hint : Put $y = v^{1/2}$. **Ans.** $v = 2cx + c^n$.

3. Solve $y = 2\,px + 6\,y^2p^2$.

Hint : Put $y^3 = v$, **Ans.** $y^2 = 3\,cx + 6\,c^2$.

4. $x^2 - (xy/p) = f\,(y^2 - xyp)$.

Hint : Put $y^2/x^2 = v$, $1/x^2 = u$. **Ans.** $cx^2 = (c - y^2) + f\,(c)$

5. $(2\,y/x) - p = f\,(px - y/x^2)$

Hint: Put $y/x = v$. **Ans.** $y = cx^2 + c\,f\,(c)$.

Part II : SINGULAR SOLUTIONS

4.12 Introduction

Let the given differential equation be $\quad y = px + (a/p), \quad$ where $p = dy/dx$. ... (1)

(1) is in Clairaut's form. So its solution is

$$y = mx + (a/m), \text{ where } m \text{ is an arbitrary constant.} \quad \text{... (2)}$$

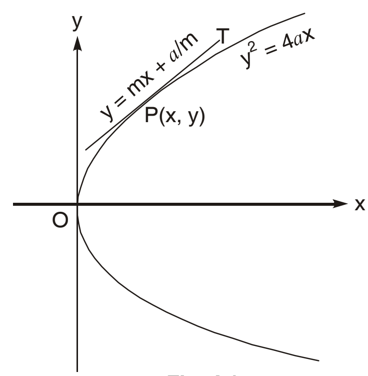

Fig. 4.1

(2) represents a family of curves whose each member corresponds to a definite value of m. From coordinate geometry, we know that the straight line (2) touches the parabola

$$y^2 = 4\,ax. \quad \text{... (3)}$$

Thus assigning different values of m we obtain different solutions (2), all of which satisfy (1) and they all touch parabola (3) as shown in fig. 4.1.

Consider the point of contact $P\,(x, y)$ of any particular tangent. At P the tangent and parabola have the same direction, so they have a common values of dy/dx, as well as of x and y. As P may be any point on the parabola, the equation of the parabola $y^2 = 4ax$ must be a solution of (1) as shown below :

Differentiating $y^2 = 4\,ax$, $p = dy/dx = 2\,a/y$. Again from $y^2 = 4ax$, we have $x = y^2/4a$. Putting these values of p and x, $\quad$ the R.H.S. of (1) $= \dfrac{2a}{y} \cdot \dfrac{y^2}{4a} + a \cdot \dfrac{y}{2a} = \dfrac{y}{2} + \dfrac{y}{2} = y =$ R.H.S. of (1)

$y^2 = 4ax$ is known as a *singular solution* of (1). Such a solution does not contain any arbitrary constant and is not a particular case of the general solution. It is sometimes possible to get this solution from the general solution by assigning a particular value to the arbitrary constant. In such a case the singular solution is also a particular solution.

Again, we see that the singular solution $y^2 = 4ax$ is the *envelope of the general solution (2) of (1).

4.13 Relation between the singular solution of a differential equation and the envelope of family of curves represented by that differential equation

Theorem. Whenever the family of curves $\quad \phi\,(x, y, c) = 0 \quad$ ... (1)

represented by the differential equation $\quad f\,(x, y, p) = 0, \quad$ where $\quad p = dy/dx \quad$ ... (2)

possesses *an *envelope, the equation of the envelope is the singular solution of the differential equation* (2).

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* **Envelope. Definition.** A curve which touches each member of a one-parameter family of curves and at each point is touched by some member of the family, is called the *envelope* of that one-parameter family of curves.

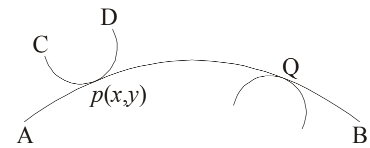


Fig. 4.2

Proof. Suppose that the family of curves (1) possesses an envelope AB. Let $P(x, y)$ be any point on the envelope. Then there exists a curve CPD of the family (1), say, $\phi(x, y, c_1) = 0$, which touches the envelope at $P(x, y)$. The values of x, y, dy/dx for the curve at P satisfy the given differential equation (2). Again the values of x, y, dy/dx at P for the envelope are the same as those for the curve. Hence it follows that the values of x, y, dy/dx at every point of the envelope satisfy the given differential equation (2). Therefore the equation of the envelope is a solution of the differential equation.

4.14 *c*-discriminant and *p*-discriminant relations.

Definitions. The discriminant of an equation involving a single parameter is the simplest function of the coefficients in the rational integral form, whose vanishing is the condition that the equation have two equal roots. When the equation is quadratic in parameter p or c, the corresponding p and c-discriminant relations can be obtained by using the following simple formula of theory of equations.

If $AP^2 + Bp + C = 0$ (or $Ac^2 + Bc + C = 0$) be quadratic in p (or c) where A, B, C are functions of x and y, then p-discriminant (or c-discriminant) relation is given by $B^2 - 4AC = 0$.

However, if the equation in p (or c) is of higher degree (than 2), then p (or c) discriminant relation is found in the following way.

Method of getting *p*-discriminant relation. Let the given equation be $f(x, y, p) = 0$

Let p $(= dy/dx)$ be regarded as parameter. Then the p discriminant relation is obtained by eliminating p between the equations, $f(x, y, p) = 0$ and $\partial f / \partial p = 0$.

Thus the p discriminant relation represents the locus, for each point of which $f(x, y, p) = 0$ has equal values of p.

Method of getting c-discriminant relation. Let $\phi(x, y, c) = 0$ be the general solution of the given differential equation $f(x, y, p) = 0$. Let the arbitrary constant c be regarded as parameter. Then the discriminant relation is obtained by eliminating c between the equations,

$$\phi(x, y, c) = 0 \qquad \text{and} \qquad \partial\phi / \partial c = 0.$$

Thus the c discriminant relation represents the locus, for each point of which $\phi(x, y, c) = 0$ has equal values of c.

4.15 Determination of singular solutions

There are two methods of determining singular solution of *a* given differential equation. We may find it either from the general solution of the given differential equation or directly from the given differential equation.

First method. Determination of singular solution from the general solution $\phi(x, y, c) = 0$ of the given differential equation $f(x, y, p) = 0$.

We know that the envelope of the family of curves $\phi(x, y, c) = 0$ is contained in c-discriminant relation, say, $\psi(x, y) = 0$.

As the c-discriminant $\psi(x, y) = 0$ may represent loci other than the envelope, it follows that only that part of locus $\psi(x, y) = 0$ is the singular solution which also satisfies the given differential equation $f(x, y, p) = 0$. Accordingly, if the equation $\psi(x, y) = 0$ fails to satisfy $f(x, y, p) = 0$, then $\psi(x, y) = 0$ should be resolved into more simpler factors. Now, try to verify if any part so obtained is or is not a solution of $f(x, y, p) = 0$. Only those parts will constitute the required envelope which satisfy $f(x, y, p) = 0$.

Second Method. Determination of singular solution directly from the given differential equation $f(x, y, p) = 0$.

Let $$\phi(x, y, c) = 0 \qquad \text{... (1)}$$
be the general solution of $$f(x, y, p) = 0 \qquad \text{... (2)}$$
Let $$\psi(x, y) = 0 \qquad \text{... (3)}$$
be the c discriminant relation. From the theory of equations, it is known that the equation (3) represents the locus of points (x, y) such that at least two of the corresponding values of c, are equal *i.e.* such that at least two of the curves of the family through (x, y) coincide. As the equation (2) determines the slopes of the tangents to the curves of the family through (x, y), it follows that for a point (x, y) satisfying (3) at least two of the corresponding values of p must coincide. Hence we see that the envelope and hence the singular solution is also contained in p-discriminant relation.

4.16 Working rule for finding the singular solution

Method I. Let $$f(x, y, p) = 0 \qquad \text{... (1)}$$
be the given differential equation.

Differentiating (1) partially w.r.t. 'p', $$\partial f/\partial p = 0. \qquad \text{... (2)}$$

Eliminate p between (1) and (2) to get p-discriminant relation $$F(x, y) = 0. \qquad \text{... (3)}$$

It (3) satisfies (1), then (3) is the required singular solution. If (3) does not satisfy (1), then resolve $F(x, y)$ into simpler factors. Now it is necessary to verify if any part of (3) is or is not a solution of (1). Only those parts will constitute the singular solution which satisfy (1).

From the above discussion, it follows that if p occurs only in the first degree in the given differential equation, there will be no singular solution. Similarly, if the differential equation can be resolved into a number of factors, each linear in p, there will be no singular solution.

Method II. Let $$\phi(x, y, c) = 0 \qquad \text{... (4)}$$
be the general solution of given differential equation (1).

Differentiating (4) partially w.r.t. 'c', $$\partial\phi/\partial c = 0. \qquad \text{... (5)}$$

Eliminate c between (4) and (5) to get c discriminant relation $$\psi(x, y) = 0. \qquad \text{... (6)}$$

It (6) satisfies (1), then (6) is the required singular solution. On the other hand, if (6) does not satisfy (1), then resolve $\psi(x, y)$ into simpler factors. Now it is necessary to verify if any part of (6) is or is not a solution of (1). Only those parts will be the singular solutions which satisfy (1).

Particular case. Singular solution of Clairaut's equation.

We know that the general solution of the Clairaut's equation
$$y = px + f(p) \qquad \text{... (1)}$$
is
$$y = cx + f(c). \qquad \text{... (2)}$$

Differentiating (2) partially w.r.t. 'c', $$0 = x + f'(c). \qquad \text{... (3)}$$

The singular solution, which is the envelope of (2), is obtained by elimination c between (2) and (3).

Now, differentiating (1) partially w.r.t. 'p', $$x + f'(p) = 0. \qquad \text{... (4)}$$

Note that the equations (1) and (4) differ from the equations (2) and (3) only in having p in place of c. Accordingly, the c discriminant relation from (2) and (3) and p discriminant relation from (1) and (4) are identical and both of them gives us the required singular solution.

The students are, therefore, advised to make use of this fact in doing problems. Thus there is no need to get general solution (2) and subsequently c-discriminant relation while dealing with Clairaut's form of differential equation. Accordingly, the singular solution of the Clairaut's equation (1) should be always obtained from p-discriminant relation.

Note. In part I of this chapter we have explained five different methods of solving $f(x, y, p) = 0$. We now discuss the determination of singular solution in the following five types of problems corresponding to five methods of part I. In some solutions we have given reference to solved examples of part I. Students are advised to give full solution in examination for complete answer to the problem.

4.17 Solved examples based on singular solutions (See Art. 4.16)

Type I : Equations solvable for p

Ex. 1. *Find general and singular solutions of $9p^2 (2 - y)^2 = 4 (3 - y)$.*

Sol. Solving for p, $\quad p = (dy/dx) = \pm (2/3) \times \{(3 - y)^{1/2} / (2 - y)\}$

Separating variables, $\quad dx = \pm \dfrac{3}{2} \dfrac{2-y}{(3-y)^{1/2}} dy = \pm \dfrac{3}{2} \dfrac{(3-y)-1}{(3-y)^{1/2}} dy.$

Integrating, $\quad x + c = \pm (3/2) \int [(3-y)^{1/2} - (3-y)^{-1/2}] dy$

or $\quad x + c = \pm (3/2) [(-2/3) (3 - y)^{3/2} + 2(3 - y)^{1/2}] = \pm (3 - y)^{1/2} [-(3 - y) + 3]$

or $\quad x + c = \pm y (3 - y)^{1/2}$ $\quad$ or $\quad (x + c)^2 = y^2 (3 - y)$, on squaring.

This is the general solution. The singular solution can be found by any of the two methods given below :

First Method. The general solution is $\quad (x + c)^2 = y^2 (3 - y)$

or $\quad c^2 + 2xc + \{x^2 - y^2 (3 - y)\} = 0.$

This is a quadratic equation in the parameter c. So the c discriminant relation is

$B^2 - 4AC = 0$, $\quad$ *i.e.*, $\quad 4x^2 - 4 \times 1 \times \{x^2 - y^2 (3 - y)\} = 0$ $\quad$ or $\quad y^2 (3 - y) = 0.$

Now $y = 0$ gives $dy/dx = p = 0$. Substitution of $y = 0$ and $p = 0$ in the given differential equation does not satisfy it. Hence $y = 0$ is not a singular solution.

Again $3 - y = 0$ gives $dy/dx = p = 0$. Substitution of $y = 3$ and $p = 0$ in the given differential equation satisfies it. Hence $y = 3$ the required singular solution.

Second Method. Given equation is $\quad 9p^2 (2 - y)^2 + 0 \cdot p - 4 (3 - y) = 0.$

This is a quadratic in the parameter p. So p-discriminant relation is given by

$B^2 - 4AC = 0$, $\quad$ *i.e.*, $\quad 0^2 - 4 \cdot 9 (2 - y)^2 \cdot \{-4 (3 - y)\} = 0$ $\quad$ or $\quad (2 - y)^2 (3 - y) = 0.$

Now $2 - y = 0$ gives $dy/dx = p = 0$. Substitution of $y = 2$ and $p = 0$ in the given equation does not satisfy it. Hence $y = 2$ is not a singular solution.

Again $3 - y = 0$ gives $dy/dx = p = 0$. Substitution of $y = 3$ and $p = 0$ in the given equation satisfies it. Hence $y = 3$ is the required singular solution.

Ex. 2. *Find the general solution and singular solution of*

(a) $4p^2 = 9x$

(b) $xp^2 = (x - a)^2$

Sol. (*a*) Solving for p, $\quad p = dy/dx = \pm (3/2) x^{1/2}$ $\quad$ or $\quad dy = \pm (3/2) x^{1/2}$

Integrating, $\quad y + c = \pm x^{3/2}$ $\quad$ or $\quad (y + c)^2 = x^3$ $\quad$ or $\quad c^2 + 2yc + (y^2 - x^3) = 0,$

which is is a quadratic equation in the parameter c. Hence c–discriminant relation is

$B^2 - 4AC = 0$, $\quad$ *i.e.*, $\quad 4y^2 - 4 \times 1 \times (y^2 - x^3) = 0$ $\quad$ or $\quad x^3 = 0$ $\quad$ or $\quad x = 0$

Re-writing the given equation, $4 = 9x\,(1/p)^2$ ...(1)

Now $x = 0$ gives $dx/dy = 1/p = 0$. Substitution of $x = 0$ and $1/p = 0$ in the given equation (1) does not satisfy it. Hence $x = 0$ is not a singular solution.

Thus, the given equation has no singular solution.

(*b*) Solving for p, $p = dy/dx = \pm\,(x - a)/x^{1/2} = \pm\,(x^{1/2} - a\,x^{-1/2})$.

or $dy = \pm\,(x^{1/2} - ax^{-1/2})\,dx$ so that $y + c = \pm\,\{(2/3)\,x^{3/2} - 2a\,x^{1/2}\}$

or $y + c = \pm\,(2/3)\,x^{1/2}\,(x - 3a)$ or $9\,(y + c)^2 = 4x\,(x - a)^2$

or $9c^2 + 18\,yc + \{9y^2 - 4x\,(x - a)^2\} = 0,$

which is a quadratic in the parameter c. So the c discriminant relation is $B^2 - 4\,AC = 0$, *i.e.*, $(18)^2 \times y^2 - 4 \times 9 \times \{9y^2 - 4x\,(x - a)^2\} = 0$ or $4x\,(x - a)^2 = 0$.

Given equation is $xp^2 = (x - a)^2$ or $x = (x - a)^2\,(1/p)^2$. ... (1)

Now $x = 0$ gives $dx/dy = 1/p = 0$. Substitution of $x = 0$ and $1/p = 0$ in the given equation (1) satisfies it. Hence $x = 0$ in the required singular solution.

Again, $x = a$ gives $dx/dy = 1/p = 0$. Substitution of $x = a$ and $1/p = 0$ in the given equation (1). does not satisfy it. Hence $x = a$ is not singular solution.

Ex. 3. *Find general and singular solutions of* $8ap^3 = 27y$ **[I.A.S. 1993]**

Sol. Solving for p, $p = dy/dx = (3/2)\,(1/a^{1/3})\,y^{1/3}$

or $dx = (2/3)\,a^{1/3}\,y^{-1/3}\,dy$ so that $x + c = a^{1/3}\,y^{2/3}$

or $(x + c)^3 = ay^2$, on cubing both sides. ... (1)

Now, differenting (2) partially w.r.t. 'c', we get

$3\,(x + c)^2 = 0$ or $x + c = 0$ or $c = -\,x$. ... (2)

Eliminating c between (1) and (2), the c-discriminant relation is

$0 = ay^2$ or $y = 0$.

Now $y = 0$ gives $p = dy/dx = 0$. Substitution of $y = 0$ and $p = 0$ in the given differential equation satisfy it. Hence $y = 0$ is the singular solution.

Ex. 4. *Find the singular solution of* $xp^2 - (y - x)\,p - y = 1$.

Sol. Since the given equation is the quadratic in the parameter p, hence the p-discriminant relation is $B^2 - 4\,AC = 0$, *i.e.*, $(y - x)^2 - 4x \cdot (-\,y - 1) = 0$ or $(y + x)^2 = -\,4x$. ... (1)

Differentiating (1) w.r.t. 'x', $2\,(y + x)\,(p + 1) = -\,4$ or $p = -\,(x + y + 2)/(x + y)$.

Now, substituting this value of p in the given differential equation and using (1), we verify that it is satisfied. Hence $(x + y)^2 + 4x = 0$ is the required singular solution.

Ex. 5. *Find the general and singular solution of* $p^2 + y^2 = 1$.

Sol. Given equation is $p^2 + \text{o}\cdot\text{p} + (y^2 - 1) = 0$ ... (1)

Solving for p, $p = dy/dx = \pm\,(y^2 - 1)^{1/2}$ or $dx = \pm\,[1/(1 - y^2)^{1/2}]\,dy$.

Integrating, $x + c = \pm\,\cos^{-1} y$ or $\cos^{-1} y = \pm\,(x + c)$ or $y = \cos\,(x + c)$.

From (1), the p-discriminant relation is $B^2 - 4\,AC = 0$, *i.e.*,

$0 - 4 \cdot 1 \cdot (y^2 - 1) = 0$ or $y^2 - 1 = 0$ or $(y - 1)\,(y + 1) = 0$.

Now, $y - 1 = 0$ gives $p = dy/dx = 0$. Substitution of $y = 1$ and $p = 0$ in (1) satisfies it. Hence $y = 1$ is a singular solution. Similarly we see that $y = -\,1$ is also a singular solution.

Hence $y = \cos\,(x + c)$ is general solution and $y = \pm\,1$ are singular solutions.

Ex. 6. *Find the general and singular solution of* $y^2\,(1 + p^2) = r^2$ *or* $y^2\,\{1 + (dy/dx^2\} = r^2$.

[I.A.S. (Prel.) 2000, 01, 02, 06]

Sol. Re-writing the given equation, we have

$p = dy/dx = \pm (r^2 - y^2)^{1/2}/y$ or $dx = \pm (1/2) \times (r^2 - y^2)^{-1/2} (-2y)\, dy$

Integrating, $x + c = \pm (r^2 - y^2)^{1/2}$ or $(x + c)^2 + y^2 = r^2$, ... (1)

which is the general solution of the given differential equation.

Now differentiating (1) partially w.r.t. 'c', we get

$2(x + c) + 0 = 0$ so that $c = -x$. ... (2)

Eliminating c between (1) and (2), we get $y^2 = r^2$ or $(y - r)(y + r) = 0$, which is the c-discriminant relation. Since $y = r$ and $y = -r$ both satisfy the given differential equation and hence form the singular solutions.

Ex. 7. *Find the general and singular solutions of*

(a) $4p^2x(x - a)(x - b) = \{3x^2 - 2x(a + b) + ab\}^2$.

(b) $4p^2x(x - 1)(x - 2) = (3x^2 - 6x + 2)^2$.

Sol. (*a*) Given $4p^2 x(x - a)(x - b) - \{3x^2 - 2x(a + b) + ab\}^2 = 0$. ... (1)

Proceed now as in Ex. 1(*d*) of Art. 4.3 and get the solution as

$(y + c)^2 = x(x - a)(x - b)$ or $c^2 + 2cy + \{y^2 - x(x - a)(x - b)\} = 0$.

This is a quadratic equation in the parameter c. Hence the c-discriminant is given

$B^2 - 4AC = 0$, *i.e.*, $4y^2 - 4\cdot 1\cdot \{y^2 - x(x - a)(x - b)\} = 0$ or $x(x - a)(x - b) = 0$.

Re-writing (1), $4x(x - a)(x - b) = \{3x^2 - 2x(a + b) + ab\}^2(1/p)^2$ ... (2)

Now, $x = 0$ gives dx/dy = 0 or $1/p = 0$. Substitution of $x = 0$ and $1/p = 0$ in (2) satisfies it. Hence $x = 0$ is a singular solution Similarly, $x = a$ and $x = b$ are also singular solutions

(*b*) Proceed as in part (*a*).**Ans.** $(y + c)^2 = x(x - 1)(x - 2)$ is general solution and $x = 0$, $x = 1$, $x = 2$ are singular solutions.

Ex. 8. *Find the general and singular solution of* $p^2y^2 \cos^2 \alpha - 2pxy \sin^2 \alpha + y^2 - x^2 \sin^2 \alpha = 0$

Sol. For general solution refer Ex. 9 of Art. 4.3. Then we have

$$c^2 + 2(x \sec \alpha)x + x^2 + y^2 = 0, \ c \text{ being an arbitrary constant}$$

This is quadratic in c. So here c-discriminant relation is

$4x^2 \sec^2 \alpha - 4\cdot 1\cdot (x^2 + y^2) = 0$ or $x^2(\sec^2 \alpha - 1) - y^2 = 0$

or $y^2 - x^2 \tan^2 \alpha = 0$ or $(y - x \tan \alpha)(y + x \tan \alpha) = 0$.

Now, $y = x \tan \alpha$ gives $p = dy/dx = \tan \alpha$. Substitution of $p = \tan \alpha$ and $y = x \tan \alpha$ in the given equation satisfies it. Hence $y = x \tan \alpha$ is a singular solution. Similarly, we easily verify that $y = -x \tan \alpha$ is also a singular solution.

Ex. 9. *Find the differential equation of the family of circles* $x^2 + y^2 + 2cx + 2c^2 - 1 = 0$ *(c arbitrary constant). Determine singular solution of the differential equation.*

Sol. To get differential equation proceed as in chapter 1 and get

$$2y^2p^2 + 2xyp + x^2 + y^2 - 1 = 0, \text{ where } p = dy/dx \qquad \text{... (1)}$$

Here p-discrimation relation is

$x^2y^2 - 2y^2(x^2 + y^2 - 1) = 0$ *i.e.* $y^2(x^2 + 2y^2 - 2) = 0$. ... (2)

From the given equation, c-discriminant relation is

$x^2 - 2(x^2 + y^2 - 1) = 0$ or $x^2 + 2y^2 - 2 = 0$. ... (3)

From (2) and (3), we conclude that $x^2 + 2y^2 - 2 = 0$ is the only singular solution since it occurs once both in p and c-discriminant relations and also satisfies (1).

Type 2 : Equations solvable for x

Ex. 10. *Find the general and singular solution of* $p^3 - 4xyp + 8y^2 = 0.$

Sol. Given equation is $$p^3 - 4xyp + 8y^2 = 0. \quad ... (1)$$

As in solved example 1(*b*) of Art. 4.5, the required general solution is

$$c\,(c - x)^2 = y, \text{ c being an arbitrary constant} \quad ... (2)$$

Differentiating (2) partially w.r.t. 'c', we have

$(c - x)^2 + 2c\,(c - x) = 0$ or $(c - x)\,(3c - x) = 0$ so that $c = x$ or $c = x/3$

When $c = x$, (4) gives $y = 0$; when $c = x/3$, (4) gives $y = 4x^3/27$. .. (3)

Now, $y = 0$ gives $p = dy/dx = 0$. Substitution of $y = 0$ and $p = 0$ in (1) satisfy (1). Hence $y = 0$ is a singular solution of (1). Again $y = 4x^3/27$ gives $p = dy/dx = 4x^2/9$. These satisfy (1). Hence $y = 4x^3/27$ is also singular solution.

Ex. 11. *Find the solution of the differential equation* $y = 2xp - yp^2$ *where* $p = dy/dx$. *Also find the singular solution.* **[Guwahati 1996]**

Sol. Given $$y = 2xp - yp^2 \quad ... (1)$$

Solving (1) for x, $$x = y/2p + yp/2 \quad ... (2)$$

Diff. (2) w.r.t. y and noting that $dx/dy = 1/p$, we get $$\frac{1}{p} = \frac{1}{2p} - \frac{y}{2p^2}\frac{dp}{dy} + \frac{p}{2} + \frac{y}{2}\frac{dp}{dy}$$

or $$\frac{y}{2}\frac{dp}{dy}\left(1 - \frac{1}{p^2}\right) + \frac{p}{2}\left(1 - \frac{1}{p^2}\right) = 0 \quad \text{or} \quad \frac{1}{2}\left(1 - \frac{1}{p^2}\right)\left(y\frac{dp}{dy} + p\right) = 0.$$

Omitting the first factor, for general solution we have

$y\,(dp/dy) + p = 0$ or $(1/p)\,dp + (1/y)\,dy = 0$

Integrating, $\log p + \log y = \log c$ or $py = c$ or $p = c/y$. ... (3)

Eliminating p from (1) and (3), the general solution is

$y = (2xc)/y - (yxc^2)/y^2$ or $y^2 = 2cx - c^2$ .... (4)

The p-disc. relation from (1) *i.e.* $yp^2 - 2xp + y = 0$ is given by

$4x^2 - 4y^2 = 0$ or $x^2 - y^2 = 0$ or $(x - y)\,(x + y) = 0.$

The c-disc. relation from (4) *i.e.* $c^2 - 2cx + y^2 = 0$ is given by

$4x^2 - 4y^2 = 0$ or $x^2 - y^2 = 0$ or $(x - y)\,(x + y) = 0.$

Hence $x - y = 0$ and $x + y = 0$ are singular solutions because these appear once in both the descriminants and also satisfy (1).

Type 3 : Equations solvable for y

Ex. 12. *Solve the general and singular solutions of* $x^3p^2 + x^2yp + a^3 = 0.$ **[Kanpur 1994]**

Sol. The given equation is $$x^3p^2 + x^2yp + a^3 = 0. \quad ... (1)$$

Solving for y, $$y = -\,xp - a^3/(x^2p). \quad ... (2)$$

Differentiating (2) w.r.t. 'x' and writing p for dy/dx, we have

$$p = -\,p - x\frac{dp}{dx} - a^3\left(-\frac{2}{x^3p} - \frac{1}{x^2p^2}\frac{dp}{dx}\right) \quad \text{or} \quad 2p + x\frac{dp}{dx} - \frac{2a^3}{x^3p} - \frac{a^3}{x^2p^2}\frac{dp}{dx} = 0$$

or $$2p\left(1 - \frac{a^3}{x^3p^2}\right) + x\frac{dp}{dx}\left(1 - \frac{a^3}{x^3p^2}\right) = 0 \quad \text{or} \quad \left(1 - \frac{a^3}{x^3p^2}\right)\left(2p + x\frac{dp}{dx}\right) = 0.$$

Omitting the first factor since it does not involve dp/dx, we get

$2p + x\,(dp/dx) = 0$ or $(1/p)\,dx + (2/x)\,dx = 0.$

Integrating, $\log p + 2\log x = \log c$ or $px^2 = c$ or $p = c/x^2.$ ... (3)

Eliminating p between (1) and (3), the required general solution is

$x^3\,(c^2/x^4) + x^2y\,(c/x^2) + a^3 = 0$ or $c^2 + xyc + a^3x = 0.$ ... (4)

By (4), c-discriminant relation is $(xy)^2 - 4 \times 1 \times (a^3x) = 0$ or $x\,(xy^2 - 4a^3) = 0.$

Now $x = 0$ and $xy^2 - 4a^3 = 0$ both satisfy (1) and the hence required singular solutions are $x = 0$ and $xy^2 - 4a^3 = 0$.

Ex. 13. *Find general and singular solutions of* $3xy = 2px^2 - 2p^2$ *or* $y = (2x/3)\,p - (2/3x)\,p^2$ **[I.A.S. Prel. 1995, 2001]**

Sol. Given equation is $3xy = 2px^2 - 2p^2.$ ... (1)

Solving (1) for y, $y = (2/3)\,px - (2/3)\,p^2\,x^{-1}.$ ... (2)

Differentiating (2) w.r.t. 'x' and writing p for dy/dx, we get

$$p = \frac{2p}{3} + \frac{2x}{3}\frac{dp}{dx} - \frac{2}{3}\left[2p\frac{dp}{dx}x^{-1} - x^{-2}p^2\right] \quad \text{or} \quad 3p - 2p - \frac{2p^2}{x^2} - 2x\frac{dp}{dx} + \frac{4p}{x}\frac{dp}{dx} = 0$$

or $$p - \frac{2p^2}{x^2} - 2\frac{dp}{dx}\left(x - \frac{2p}{x}\right) = 0 \quad \text{or} \quad p\left(1 - \frac{2p}{x^2}\right) - 2x\frac{dp}{dx}\left(1 - \frac{2p}{x^2}\right) = 0$$

or $\{1 - (2\,p/x^2)\}\ \{p - 2x\,(dp/dx)\} = 0.$ ... (3)

Omitting the first factor which does not involve dp/dx, we get

$p - 2x\,(dp/dx) = 0$ or $(2/p)\,dp = (1/x)\,dx$

Integrating, $2\log p = \log x + \log c$ or $p^2 = xc$ or $p = \pm\,(xc)^{1/2}.$

Putting this value of p in (2), the required general solution is

$3y = \pm\,2x\,(xc)^{1/2} - 2c$ or $3y + 2c = \pm\,2x\,(xc)^{1/2}$

or $(3y + 2c)^2 = 4cx^3$ or $4c^2 + 4c\,(3y - x^3) + 9y^2 = 0.$ ... (4)

From (4), the c-discriminant relation is $B^2 - 4\,AC = 0$, *i.e.*,

$16\,(3y - x^3)^2 - 4 \times 4 \times 9y^2 = 0$ or $x^3(x^3 - 6y) = 0.$

Now $x^3 = 0$ gives $x = 0$ and $dx/dy = 1/p = 0$ and these values do not satisfy (1). So $x = 0$ is not a singular solution.

Again $x^3 - 6y = 0$ gives $y = x^3/6$ and $p = dy/dx = x^2/2$. These values satisfy (1). Hence $x^3 - 6y = 0$ is singular solution.

Ex. 14. *Solve the differential equation* $(8p^3 - 27)\,x = 12p^2y$ *and investigate whether a singular solution exists.*

Sol. The given equation is $(8p^3 - 27)\,x = 12\,p^2y.$ ... (1)

Solving (1) for y, $y = (2/3)\,px - (9/4)\,(x/p^2).$ ... (2)

Differentiating (2) partially w.r.t. 'x' and writing p for dy/dx, we get

$$p = \frac{2}{3}\left(p + \frac{dp}{dx}\right) - \frac{9}{4}\left(\frac{1}{p^2} - \frac{2x}{p^3}\frac{dp}{dx}\right) \quad \text{or} \quad \frac{1}{3}p + \frac{9}{4p^2} - x\frac{dp}{dx}\left(\frac{2}{3} + \frac{9}{2p^3}\right) = 0$$

or $$\frac{1}{3}p\left(1 + \frac{27}{4p^3}\right) - \frac{2x}{3}\frac{dp}{dx}\left(1 + \frac{27}{4p^3}\right) = 0 \quad \text{or} \quad \frac{1}{3}\left(1 + \frac{27}{4p^3}\right)\left(p - 2x\frac{dp}{dx}\right) = 0.$$

Omitting the first factor which does not involve dp/dx, we get

$p - 2x\,(dp/dx) = 0$ or $(2/p)\,dp = (1/x)\,dx.$

Integrating, $2\log p = \log x + \log(9/4c)$ or $p^2 = x\,(9/4c)$ or $p = \pm\,(3/2)\,(x/c)^{1/2}$

Putting this value of p in (2), the required general solution is

$y = \pm (x^{3/2} / c^{1/2}) - c$ or $c^{1/2}(y + c) = \pm x^{3/2}$ or $c(y + c)^2 = x^3$. ... (3)

Differentiating (3) w.r.t. c, $(y + c)^2 + 2c(y + c) = 0$ or $(y + c)(y + 3c) = 0$

$\therefore$ $y + c = 0$ or $y + 3c = 0$ so that $c = -y$ or $c = -y/3$.

When $c = -y$, (3) gives $x^3 = 0$ or $x = 0$; when $c = -y/3$, (3) gives $4y^3+27x^3 = 0$.

We easily verity that $x = 0$ and $4y^3 + 27x^3 = 0$ both satisfy (1). Hence these are required singular solutions.

Ex. 15. *Solve the differential equation $y = x - 2ap + ap^2$. Find the singular solution and interpret it geometrically.* **[I.A.S. 2000]**

Sol. Given that $y = x - 2ap + ap^2$, where $p = dy/dx$ ... (1)

Differentiating (1) w.r.t. 'x', $p = 1 - 2a(dp/dx) + 2ap(dp/dx)$

or $p - 1 = 2a(p - 1)(dp/dx)$ or $(p - 1)\{2a(dp/dx) - 1\} = 0$

Omitting the first factor since it does not involve dp/dx, we get

$2a(dp/dx) - 1 = 0$ or $dx = 2a\,dp$.

Integrating, $x = 2ap + c$ so that $p = (x - c)/2a$ ... (2)

Substituting the value of p from (2) in (1), general solution of (1) is

$y = x - (x - c) + (1/4a)(x - c)^2$ or $4a(y - c) = x^2 + c^2 - 2xc$

or $c^2 - 2xc + 4ac + x^2 - 4ay = 0$ or $c^2 - 2c(x - 2a) + (x^2 - 4ay) = 0$, ... (3)

which is a quadratic equation in parameter c. So the c-discriminant relation is

$4(x - 2a)^2 - 4(x^2 - 4ay) = 0$ or $y - x + a = 0$ ... (4)

Again, re-writing (1), $ap^2 - 2ap + (x - y) = 0$, ... (5)

which is a quadratic in parameter p. Hence the p-discriminant relation is

$4a^2 - 4a(x - y) = 0$ or $y - x + a = 0$ ... (6)

From (4) and (6), we find that $y - x + a = 0$ is present in both p and c discriminant relations. Further $y - x + a = 0$ gives $y = x - a$ and $p = dy/dx = 1$. These satisfy (1). Hence $y - x + a = 0$ is singular solution of (1).

Geometrical interpretation of singular solution $y - x + a = 0$.

Re-writing (3), $(x - c)^2 = 4a(y - c)$, ... (7)

which represents a family of parabolas all of which touch the line $y - x + a = 0$, which is the envelope of this family of parabolas.

Type 4 : Equations in Clairaut's form

Ex. 16. *Find the general and singular solutions of $y = px + (a/p)$.*

Sol. The given equation is $y = px + (a/p)$, ... (1)

which is in Clairaut's form. So replacing p by c in (1) the solution is

$y = cx + (a/c)$ or $c^2x - yc + a = 0$. ... (2)

Now, c – discriminant relation of (2) is $B^2 - 4AC = 0$,. *i.e.*,

$(-y)^2 - 4xa = 0$ or $y^2 = 4ax$. ... (3)

Now, $y^2 = 4ax$ gives $2y(dy/dx) = 4a$ or $p = 2a/y$. Putting this value of p in (1), we get $y = (2ax)/y + (y/2)$ or $y^2 = 4ax$ which is true by (3). Hence (3) satisfies (1) and so $y^2 = 4ax$ is the required singular solution.

Ex. 17. *Find the complete primitive (general solution) and singular solution of the following equations. Interpret your results geometrically.*

(a) $y = px + (b^2 + a^2p^2)^{1/2}$

(b) $y = x\,(dy/dx) + a\,\{1 + (dy/dx)^2\}^{1/2}$ *or* $y = px + a\,(1 + p^2)^{1/2}$

(c) $(xp - y)^2 = p^2 - 1.$ **[I.A.S. Prel 1997, 2000, 01, 02, 08]**

Sol. (*a*) The given equation is $y = px + (b^2 + a^2p^2)^{1/2}$, ... (1)

which is in Clairaut's form. So replacing p by the arbitrary constant c in (1), the required complete primitive is $y = cx + (b^2 + a^2c^2)^{1/2}$ ... (2)

or $(y - cx)^2 = b^2 + a^2c^2$ or $c^2\,(x^2 - a^2) - 2xyc + (y^2 - b^2) = 0$ ... (3)

From (3), the c-discriminant relation is $B^2 - 4AC = 0$, *i.e.*,

$4x^2y^2 - 4(x^2 - a^2)\,(y^2 - b^2) = 0$ or $b^2x^2 + a^2y^2 = a^2b^2$

or $x^2/a^2 + y^2/b^2 = 1.$ ... (3)

This relation satisfies (1), and hence it is the required singular solution.

Geometrical interpretation. The complete primitive (2) represents a family of straight lines all of which touch the ellipse (3) which is the envelope of this family of straight lines.

(*b*) Proceed as in part (*a*). Note that here $b = a$.

The general solution is $y = cx + a\,(1 + c^2)^{1/2}$ and the singular solution is $x^2 + y^2 = a^2$.

Geometrical interpretation. The complete primitive $y = cx + a\,(1 + c^2)^{1/2}$ represents a family of straight lines all of which touch the circle $x^2 + y^2 = a^2$ which is the envelope of this family of straight lines.

(*c*) The given equation is $(xp - y)^2 = p^2 - 1.$ ... (1)

Re-writing (1), $xp - y = \pm\,(p^2 - 1)^{1/2}$ or $y = px \pm (p^2 - 1)^{1/2}$ ... (2)

either of which is in Clairaut's form. So replacing p by the arbitrary constant c,

the required complete primitive is $y = cx \pm (c^2 - 1)^{1/2}$ ... (3)

$(y - cx)^2 = c^2 - 1$ or $c^2\,(x^2 - 1) - 2xyc + (y^2 + 1) = 0.$ ... (4)

From (4), the c-discriminant relation is $B^2 - 4AC = 0$, *i.e.*,

$(-\,2xy)^2 - 4\,(x^2 - 1)\,(y^2 + 1) = 0$ or $x^2 - y^2 = 1.$ ... (5)

This relation satisfies (1) and hence it is the singular solution.

Geometrical interpresation. The complete primitive gives by (3) represents a family of straight lines, each member of which touches the rectangular hyperbola $x^2 - y^2 = 1$, which is the envelope of this family of straight lines.

Ex. 18. *Find the general and singular solutions of* $(px - y)^2 = p^2 + m^2$

or $y^2 - 2pxy + p^2\,(x^2 - 1) = m^2.$

Sol. The given equation is $y^2 - 2pxy + p^2x^2 = p^2 + m^2$ ... (1)

Re-writing (1), $(y - px)^2 = m^2 + p^2$ or $y = px \pm (m^2 + p^2)^{1/2}.$

either of which is in Clairaut's form. So replacing p by the arbitrary constant c, the required general solution is $y = cx \pm (m^2 + c^2)^{1/2}$ ... (2)

or $(y - cx)^2 = m^2 + c^2$ or $c^2\,(x^2 - 1) - 2cxy + (y^2 - m^2) = 0.$ ... (3)

From (3), the c-discriminant relation is $B^2 - 4AC = 0$, *i.e.*,

$(-\,2xy)^2 - 4\,(x^2 - 1)\,(y^2 - m^2) = 0$ or $y^2 + m^2x^2 = m^2.$ ... (4)

This relation satisfies (1), and hence it is the singular solution

Ex. 19. *Find the general and singular solution of equation* $p = \log\,(px - y)$

[I.A.S. (Prel.) 1994, Purvanchal 1993]

Sol. The given equation is $p = \log\,(px - y).$ ... (1)

Re-writing (1), $e^p = px - y$ or $y = px - e^p$, ... (2)

which is in Clairaut's form and hence its general solution is $y = cx - e^c$. ... (3)

Differentiating (3) partially w.r.t. 'c', $0 = x - e^c$ or $e^c = x$ or $c = \log x$. ... (4)

Eliminating c between (2) and (3), the c-discriminant is given by

$y = x \log x - e^{\log x}$ or $y = x \log x - x$. ... (5)

Now, $y = x \log x - x$ gives $p = dy/dx = \log x + 1 - 1 = \log x$. These values of y and p satisfy (1). Hence the required singular solution is $y = x \log x - x$.

Ex. 20. *Find the general and singular solutions of sin px cos y = cos px sin y + p.*

Sol. Given equation is $\sin px \cos y = \cos px \sin y + p$ ... (1)

(1) $\Rightarrow \sin (px - y) = p$ or $px - y = \sin^{-1} p$ *or* $y = px - \sin^{-1} p$. ...(2)

which is of Clairaut's form and hence its general solution is $y = cx - \sin^{-1} c$. ... (3)

Differentiating (3) partially w.r.t. 'c', we get

$0 = x - 1/(1 - c^2)^{1/2}$ or $1 - c^2 = (1/x^2)$ so that $c = (x^2 - 1)^{1/2}/x$. ... (4)

Eliminating c between (3) and (4), the c discriminant relation is

$$y = (x^2 - 1)^{1/2} - \sin^{-1} \{(x^2 - 1)^{1/2}/x\} \qquad \text{... (5)}$$

Since (5) satisfies (2), so (5) is also the required singular solution

Ex. 21. *Find the general and singular solution of y = px + p – p², where p = dy/dx. Discuss the relation between the two solutions.* **[Calcutta 2003]**

Sol. Given equation is in Clairaut's form and so its general solution is

$$y = cx + c - c^2, \; c \text{ being the parameter.} \qquad \text{... (1)}$$

Re-writing (1), $c^2 - c\,(x + 1) + y = 0$, which is quadratic in c. Hence, its c-discriminant relation is given by $(x + 1)^2 - 4y = 0$ or $(x + 1)^2 = 4y$ ... (2)

Since (2) satisfies (1), so (2) is singular solution. Here (2) represents a parabola which is envelope of family of straight lines given by (1).

Type 5 : Equations reducible to Clairaut's form

Ex. 22. *Solve and examine for singular solution of x² (y – xp) = yp².*

Sol. The given equation is $x^2 (y - xp) = yp^2$. ... (1)

The general solution of (1) is [Refer Ex. 1(a) of Art. 4.10]

$y^2 = cx^2 + c^2$ or $c^2 + cx^2 - y^2 = 0$, ... (2)

which is a quadratic equation in c. Its c-discriminant relation is

$(x^2)^2 - 4\cdot1\cdot(-y^2) = 0$ or $x^4 + 4y^2 = 0$. ... (3)

Since (3) satisfies (1), so $x^4 + 4y^2 = 0$ is the required singular solution.

Ex. 23. *Solve and find the singular solution of axy p² + p (x² – ay² – b) = xy.*

Sol. The given equation is $axy\, p^2 + p\,(x^2 - ay^2 - b) - xy = 0$. ... (1)

The general solution of (1) is [Refer Ex. 1 (*d*) of Art. 4.10]

$y^2 = cx^2 - (bc)/(1 + ac)$ or $ax^2c^2 + c(x^2 - ay^2 - b) - y^2 = 0$ ... (2)

which is a quadratic equation in c and so its c-discriminant relation is

$$(x^2 - ay^2 - b)^2 + 4ax^2 y^2 = 0. \qquad \text{... (3)}$$

This relation satisfies (1), and hence it is the singular solution.

Ex. 24. *Reduce the equation xyp² – p(x² + y² – 1) + xy = 0 to Clairaut's form by the substitutions x² = u and y² = v. Hence show that the equation represents a family of conics touching the four sides of a square.* **[I.A.S. 2004]**

Sol. The given equation is $xyp^2 - p\,(x^2 + y^2 - 1) + xy = 0$. ... (1)

The general solution of (1) is [Refer Ex. 1(*e*) of Art. 4.10]

$$y^2 = cx^2 - c/(c-1) \quad \text{or} \quad c^2x^2 - c\,(x^2 + y^2 - 1) + y^2 = 0, \quad \ldots (2)$$

which represents a family of conics. Since (2) is a quadratic equation is c so c-discriminant relation is given by $(x^2 + y^2 - 1)^2 - 4x^2y^2 = 0$ or $(x^2 + y^2 - 1)^2 - (2xy)^2 = 0$

or $(x^2 + y^2 - 1 + 2xy)(x^2 + y^2 - 1 - 2\,xy) = 0$ or $\{(x + y)^2 - 1^2\}\{(x - y)^2 - 1^2\} = 0$

or $$(x + y + 1)\,(x + y - 1)\,(x - y + 1)\,(x - y - 1) = 0 \quad \ldots (3)$$

Now $x + y + 1 = 0$ gives $y = -x - 1$ and $p = dy/dx = -1$. These values satisfy (1). Hence $x + y + 1 = 0$ is a singular solution. Similarly $x + y - 1 = 0$, $x - y + 1 = 0$ and $x - y - 1 = 0$ are singular solutions. Clearly $x + y + 1 = 0$, $x + y - 1 = 0$, $x - y + 1 = 0$ and $x - y - 1 = 0$ form a square.

Geometrical interpretation. General solution (2) represents a family of conics all of which touch the straight lines $x + y + 1 = 0$, $x + y - 1 = 0$, $x - y + 1 = 0$ and $x - y - 1 = 0$ (forming a square) which are the envelopes of family of conics.

Ex. 25. *Reduce the differential equation $(px - y)\,(x - py) = 2p$ to Clairaut's form by the substitution $x^2 = u$ and $y^2 = v$ and find its complete primitive and its singular solution, if any.*

Sol. The given equation is $$(px - y)\,(x - py) = 2p. \quad \ldots (1)$$

The general solution of (1) is [Refer Ex. 1(*c*) of Art. 4.10]

$$y^2 = cx^2 - (cx)/(1 - c) \quad \text{or} \quad x^2c^2 - c\,(x^2 + y^2 - 2) + y^2 = 0, \quad \ldots (2)$$

which is a quadratic in and c so its c-discriminant relation is

$$(x^2 + y^2 - 2)^2 - (2xy)^2 = 0 \quad \text{or} \quad (x^2 + y^2 - 2 - 2xy)\,(x^2 + y^2 - 2 + 2xy) = 0$$

or $$\{(x - y)^2 - (\sqrt{2})^2\}\ \{(x + y)^2 - (\sqrt{2})^2\} = 0$$

or $$(x - y + \sqrt{2})\,(x - y - \sqrt{2})\,(x + y + \sqrt{2})\,(x + y - \sqrt{2}) = 0.$$

Now, $x - y + \sqrt{2} = 0$ gives $y = x + \sqrt{2}$ and $p = dy/dx = 1$. These values satisfy (1). So $x - y + \sqrt{2} = 0$ is a singular solution. Similarly $x - y - \sqrt{2} = 0$, $x + y + \sqrt{2} = 0$ and $x + y - \sqrt{2} = 0$ are also singular solutions.

Ex. 26. *Find the general and singular solution of $xy\,(y - px) = x + py$.*

Sol. The given equation is $$xy\,(y - px) = x + py. \quad \ldots (1)$$

The general solution of (1) is [Refer Ex 1(*f*) of Art. 4.10] $$y^2 = cx^2 + 1 + c, \quad \ldots (2)$$

which is linear in the parameter c. Hence c-discriminant does not exist and hence there is no singular solution.

Ex. 27. *Reduce $y = 2px + y^2p^3$ to Clairaut's form by putting $y^2 = v$ and hence find its general and singular solutions.*

Sol. The given equation is $$y = 2px + y^2p^3. \quad \ldots (1)$$

The general solution of (1) is [Refer Ex. 3(*a*) of Art. 4.10

$$y^2 = cx + (c^3/8). \quad \ldots (2)$$

Differentiating (2) w.r.t. 'c', $0 = x + 3c^2/8$ or $c^2 = (-8/3)\,x. \quad \ldots (3)$

From (2), $y^2 = c\,(x + c^2/8)$ or $y^4 = c^2\,(x^2 + c^2/8)^2$

or $$y^4 = (-8/3)\,x\,(x^2 - x/3), \quad \text{using (3)}$$

or $y^4 = -(32/27)\,x^3$ or $27\,y^4 + 32\,x^3 = 0, \quad \ldots (4)$

which is the c-discriminant relation (since it has been obtained by eliminating c between (2) and (3)). Since (4) satisfies (1), hence (4) is also singular solution.

Ex. 28. *Find the general and singular solution of $y^2\,(y - xp) = x^4p^2$.*

Sol. The given equation is $$y^2\,(y - xp) = x^4p^2. \quad \ldots (1)$$

The general solution of (1) is [Refer Ex. 4 of Art. 4.10]

$$x = cy + c^2xy \qquad \text{or} \qquad xyc^2 + yc - x = 0, \qquad \dots (2)$$

which is a quadratic equation in c and so its c-discriminant relation is

$$y^2 - 4\,(xy)\,(-x) = 0 \qquad \text{or} \qquad y\,(y + 4x^2) = 0.$$

Now, $y = 0$ gives $p = dy/dx = 0$. These values satisfy (1). So $y = 0$ is a singular solution. Again $y = -4x^2$ gives $p = dy/dx = -8x$. These values satisfy (1). Hence $y + 4x^2 = 0$ is also singular solution.

Ex. 29. *Reduce the equation $xp^2 - 2yp + x + 2y = 0$ to Clairaut's form by putting $y - x = v$ and $x^2 = u$. Hence obtain and interpret the primitive and singular solution of the equation. Show that the given equation represents a family of parabolas touching a pair of straight lines.*

Sol. The given equation is $\qquad xp^2 - 2yp + x + 2y = 0. \qquad \dots (1)$

The general solution of (1) is [Refer Ex. 5 of Art. 4.10]

$$2\,c^2x^2 - 2c\,(y - x) + 1 = 0 \qquad \dots (2)$$

which represents a family of parabolas since the second degree terms form a perfect square. Since (2) is a quadratic equation so c-discriminant relation is

$$4\,(y - x)^2 - 4\cdot 2\,x^2 = 0 \qquad \text{or} \qquad y - x = \pm x\sqrt{2}. \qquad \dots (3)$$

Now $y - x = x\sqrt{2}$ gives $y = x(\sqrt{2} + 1)$ and $p = dy/dx = \sqrt{2} + 1$. These values satisfy (1). So $y - x = x\sqrt{2}$ is a singular solution. Similarly, we can show that $y - x = -x\sqrt{2}$ is also singular solution. Thus the given equation (1) represents a family of parabolas given by (2) and this family is being touched by a pair of straight lines $y - x = \pm x\sqrt{2}$.

Ex. 30. *Reduce the equation $x^2p^2 + py\,(2x + y) + y^2 = 0$ where $p = dy/dx$ to Clairaut's form by putting $u = y$ and $v = xy$ and find its complete primitive and its singular solution.*

[I.A.S. 2006, Kumaun 1995]

Sol. The given equation is $\qquad x^2p^2 + py\,(2x + y) + y^2 = 0 \qquad \dots (1)$

The complete primitive of (1) is [Refer Ex 6 of Art. 4.10]

$$xy = cy + c^2 \qquad \text{or} \qquad c^2 + cy - xy = 0$$

which is a quadratic equation in c and hence c-discriminant relation is

$$y^2 - 4\cdot 1\cdot (-xy) = 0 \qquad \text{or} \qquad y\,(y + 4x) = 0.$$

Since $y = 0$ and $y + 4x = 0$ both satisfy (1), so these are both singular solutions.

Ex. 31. *Solve $(px^2 + y^2)\,(px + y) = (p + 1)^2$ by reducing it to Clairaut's form and find its singular solution.*

Sol. Given $\qquad (px^2 + y^2)\,(px + y) = (p + 1)^2. \qquad \dots (1)$

The general solution of (1) is [Refer Ex 10 of Art. 4.10] $\quad c^2\,(x + y) - xyc - 1 = 0.$

Its c-discriminant relation is $\quad B^2 - 4AC = 0,\quad$ *i.e.*,

$$(xy)^2 - 4\,(x + y) \times (-1) = 0 \qquad \text{or} \qquad x^2y^2 + 4\,(x + y) = 0.$$

This relation satisfies (1), and hence it is the singular solution.

Note. In what follows, G.S. and S.S. will stand for general solution and singular solution respectively.

Exercise 4(F)

1. **(Type I) : Equations solvable for p.** *Find the general and singular solution of the following equations* :

   (*a*) $p^2\,(2 - 3y)^2 = 4\,(1 - y)$. **Ans.** G.S. $(x + c)^2 = y^2\,(1 - y)$; S.S. $y = 1$

   (*b*) $y = p^2$ **Ans.** G.S. $4y = (x + c)^2$; S.S. $y = 0$

   (*c*) $4\,xp^2 = (3x - a)^2$ **Ans.** G.S. $(y + c)^2 = x\,(x - a)^2$; S.S. $x = 0$

(*d*) $8p^3 = 27y$ **Ans.** G.S. $(x+c)^3 = ay^2$; S.S. $y = 0$

(*e*) $p^2(1-x^2) = 1-y^2$ **Ans.** G.S. $c^2 - 2cxy + x^2 + y^2 - 1 = 0$; S.S. $x = \pm 1, y = \pm 1$

2. **Type 2, Equations solvable for *x*.** *Find the general and singular solutions of the following equations :*

(*a*) $3p^2 e^{4y} - px + 1 = 0$ **Ans.** G.S. $3c^2 e^{3y} - xce^y + 1 = 0$; S.S. $x^2 - 12e^y = 0$

(*b*) $8p^3 x = y(12p^2 - 9)$ **Ans.** G.S. $(x+c)^3 = 3cy^2$; S.S. $3y + 2x = 0$

(*c*) $xp^2 - yp - y = 0$ **Ans.** S.S. $y^2 + 4xy = 0$

(*d*) $y^2(1+4p^2) - 2pxy - 1 = 0$ **Ans.** S.S. $x^2 - 4y^2 + 4 = 0$

3. **Type 3, Equations solvable for *y*.** *Find the general and singular solutions of the following equations:*

(*a*) $y - px + x - (y/p) = a$ **Ans.** G.S. $xc^2 + c(a-x-y) + y = 0$; S.S. $x^2 + y^2 - 2xy - 2ax - 2ay + a^2 = 0$

(*b*) $y + px = x^4p^2$ **Ans.** G.S. $xc^2 - c - xy = 0$, S.S. $4x^2y + 1 = 0$

(*c*) $xp^2 - 2yp + 4x = 0$ **(Rajasthan 2010)** **Ans.** G.S. $c^2x^2 - 2cy + 4 = 0$; S.S. $y = 2x, y = -2x$

(*d*) $xp^2 - 2yp + ax = 0$ **Ans.** G.S. $x^2c^2 - 2cy + a = 0$; S.S. $y^2 - ax^2 = 0$

4. **Type 4 : Equations in Clairaut's form.** *Find the general and singular solutions of the following differential equations (here $p \equiv dy/dx$)*

(*a*) $p^2x - py + a = 0$ **Ans.** G.S. $c^2x - yc + a = 0$; S.S. $y^2 = 4ax$

(*b*) $y = x(dy/dx) + (1/2) \times (dx/dy)$ **Ans.** G.S. $y = cx + (1/2c)$; S.S. $y^2 = 2x$

(*c*) $y = px + p/2$ **Ans.** G.S. $y = cx + c/2$; S.S. $y = 0$

(*d*) $p^2 + px - y = 0$ **Ans.** G.S. $y = cx + c^2$; S.S. $x^2 + 4y = 0$

(*e*) $y = px - p^2$ **Ans.** G.S. $y = cx - c^2$; S.S. $x^2 = 4y$

(*f*) $y = px + ap(1-p)$ **Ans.** G.S. $y = cx - ac(1-c)$; S.S. $(x+a)^2 = 4ay$

(*g*) $(a^2 - x^2)p^2 + 2xyp + b^2 - y^2 = 0$ **Ans.** G.S. $c^2(a^2-x^2) + 2xyc + b^2 - y^2 = 0$; S.S. $x^2/a^2 + y^2/b^2 = 1$

(*h*) $(x^2 - a^2)p^2 - 2xyp + y^2 + a^2 = 0$ **Ans.** G.S. $c^2(x^2-a^2) - 2xyc + y^2 + a^2 = 0$; S.S. $x^2 - y^2 = a^2$

(*i*) $(y - px)^2 + a^2p = 0$ **Ans.** G.S. $x^2c^2 + c(a^2 - 2xy) + y^2 = 0$; S.S. $4xy = a^2$

(*j*) $p^3 + px - y = 0$ **Ans.** G.S. $y = cx + c^3$; S.S. $27y^2 + 4x^3 = 0$

(*k*) $(y-px)^2(1+p^2) = a^2p^2$ **[Purvanchal 1996]** **Ans.** G.S. $(y-cx)^2(1+c^2) = a^2c^2$; S.S. $x^{2/3} + y^{2/3} = a^{2/3}$

(*l*) $y = px + \cos p$. **Ans.** G.S. $y = cx + \cos c$; S.S. $(y - x\sin^{-1} x)^2 = 1 - x^2$

(*m*) $y = px + (1+p^2)^{1/2} - p\cos^{-1} p$ **Ans.** G.S. $y = cx + (1+c^2)^{1/2} - c\cos^{-1}c$

PART III. Extraneous Loci

4.18 Extraneous Loci *i.e.* Relations, not solutions; that may appear in *p* and *c*-discriminant relations

We have seen that if $\psi(x, y) = 0$ be a singular solution, then it must be contained in both p and c-discriminants. These discriminants, however, may contain other factors which give rise to other loci associated with the general solution of the given differential equation. Since the equation of these loci generally do not satisfy the differential equations, they are known as *extraneous loci.* We now discuss tac-locus, node-lucus and cusp-locus which are all extraneous loci.

4.19 Tac-locus

Let P be a point on the p-discriminant relation. Then by definition there are two equal values of p at P. These equal p's, however, may belong to two curves of the family of curves that are not consecutive, but which happen to touch at P. Then P lies on a point of contact of two non-consecutive curves and the locus of P is called the *tac-locus* of the family of curves. If $T(x, y) = 0$ is the equation of the tac-locus, then $T(x, y)$ must be a factor of the p-discriminant and will not be contained in the c-discriminant, since the touching curves being non-consecutive will have different c's. As an example, consider a system of circles, all of equal radii, whose centres lie on straight line TT' as shown in Fig. 4.3. Here TT', which is the locus of points of contact of non-consecutive circles of the system, represents the tac-locus.

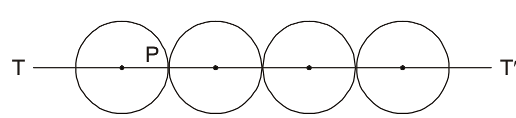


Fig. 4.3

4.20 Node-locus

The c-discriminant relation simply represents the locus of the points for which two values of c are equal. These equal c's may belong to the nodes which are also the ultimate points of intersection of the consecutive curves. Locus of such points is known as the *node-locus*. If $N(x, y) = 0$ is the equation of the node-locus, then $N(x, y)$ is not a factor of p-discriminant and is always contained in c-discriminant. In general, it does not satisfy the differential equation [See Fig. 4.4(a)]. In exceptional case [See Fig. 4.4(b)] however, $N(x, y) = 0$ may satisfy the differential equation. In that case the node-locus would also be an envelope. In both figures S_1, S_2, S_3 are curves and NN' represents the node-locus passing through the nodes N_1, N_2, N_3.

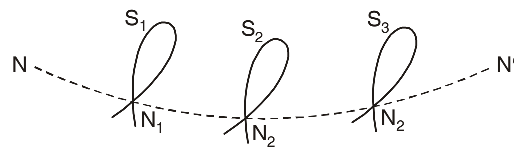


Fig. 4.4(a)

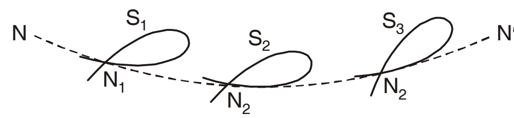


Fig. 4.4(b)

4.21 Cusp-locus

The c-discriminant relation simply represents the locus of points for which two values of c are equal. These equal c's may belong to the cusps which are also the ultimate points of intersection of the consecutive curves. Locus of such points is called the *cusp-locus*. If $C(x, y) = 0$ is the equation of the cusp-locus, then $C(x, y)$ is a factor of both the p- and c-discriminants. In general, it does not satisfy the differential equation [See Fig. 4.5 (a)]. In exceptional case [See Fig. 4.5(b)]. however, $C(x, y) = 0$ may satisfy the differential equation. In that case the cusp-locus would also be an envelope. In both the figures, S_1, S_2, S_3 are curves and CC' represents the cusp-locus passing through the cusps C_1, C_2, C_3.

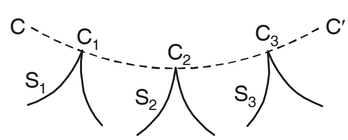


Fig. 4.5(a)

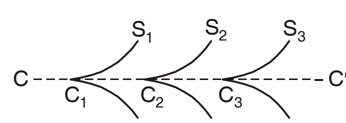


Fig. 4.5(b)

4.22 Working rule for finding singular solution (envelope) and extraneous loci

Step 1. First find the p-discriminant relation.

Step 2. By using any method determine the complete primitive or general solution of the given differential equation. Then find out the c-discriminant. In case the given equation is of Clairaut's form or is reducible to this form, then the p-discriminant relation will also be the c-discriminant relation. Thus in such a case step 2 is not necessary.

	Singular Solution (Envelope)	*Tac-Locus*	*Node-Locus*	*Cusp-Locus*
p-discrt. Relation	Once	Squard	Absent	Once
c-disct. Relation	Once	Absent	Squared	Cubed
Whether or not satisfy diff. eqn.	Yes always	Not, in general	Not, in general	Not, in general

Step 3. From the appearance of factors in p and c-discriminants, we shall decide about a particular solution or loci by remembering the table given in step 2.

When any locus falls in two of the categories, the multiplicity of its equation in a discriminant relation is the sum of the multiplicities for each category. As an example, a cusp-locus which is also an envelope must appear in degree two in the p-discriminant relation and degree four in the c-discriminant relation.

Aid to memory. The results of the above table are symbolically written as

p-discriminant $= ET^2 C$; $\qquad$ c = discriminant. $= EN^2 C^3$

when E stands for envelope (singular solution), T for tac-locus, N for node-locus and C for cusp-locus.

4.23 Solved Examples based on Art. 4.22

Ex. 1. *Obtain the complete primitive and singular solution of the following equations, explaining the geometrical significance of the irrelevent factors that present themselves.*

(i) $4xp^2 = (3x - a)^2$.

(ii) $xp^2 = (x - a)^2$. **[Ravishankar 1996; Vikram 1993]**

Sol. (*i*) The given differential equation is $\quad 4xp^2 - (3x - a)^2 = 0.$ ... (1)

The general solution of (1) is [Refer Ex. 2 (*i*) of Art. 4.3] $\quad (y + c)^2 = x(x - a)^2.$ ... (2)

Rewriting (2) as quadratic in c, we have $\quad c^2 + 2cy + y^2 - x(x - a)^2 = 0.$ ... (3)

Now from (1), the p-discriminant relation is

$0 - 4 \cdot 4x \{-(3x - a)^2\} = 0$ $\quad$ or $\quad$ $x(3x - a)^2 = 0.$ ... (4)

Similarly from (3), the c-discriminant relation is given by

$4y^2 - 4[y^2 - x(x - a)^2] = 0$ $\quad$ or $\quad$ $x(x - a)^2 = 0.$ ... (5)

Here $x = 0$ appears once in both the discriminants. Again (1) may be re-written as $4x - (3x - a)^2/p^2 = 0$ which is satisfied because $x = 0$ gives $dx/dy = 0$ *i.e.* $1/p = 0$. Thus, by definition $x = 0$ is a singular solution.

$3x - a = 0$ is a tac-locus since it appears squared in the p-discriminant relation (4), does not occur in the c-discriminant relation (5), and does not satisfy the differential equation (1).

$x - a = 0$ is a node-locus since it appears squared in the c-discriminant relation (5), does not occur in the p-discriminant relation (4), and does not satisfy the differential equation (1).

(*ii*) Proceed as above. **Ans.** $x = 0$ is singular solution, $x - a = 0$ is tac locus, $x - 3a = 0$ in node-locus.

Ex. 2. *(i) Obtain the primitive and singular solution of the following equation* $4p^2x(x - a)(x - b) = \{3x^2 - 2x(a + b) + ab\}^2$. *Specify the nature of the loci which are not solutions but which are obtained with the singular solution.*

(ii) Investigate fully for singular solution, explaining the geometrical significance of irrelevent factors that present themselves in $4x(x - 1)(x - 2)p^2 = (3x^2 - 6x + 2)^2$.

Sol. (*i*) The given differential equation is

$$4x(x - a)(x - b)p^2 - \{3x^2 - 2x(a + b) + ab\}^2 = 0. \qquad ... (1)$$

Refer Ex.1 (*d*) of Art. 4.3. The general solution of (1) is

$(y + c)^2 = x(x - a)(x - b)$ $\quad$ or $\quad$ $c^2 + 2cy + y^2 - x(x - a)(x - b) = 0.$ .... (2)

From (1), the p-discriminant relation is

$$0 - 4x(x - a)(x - b)[-\{3x^2 - 2x(a + b) + ab\}^2] = 0$$

i.e. $$x(x-a)(x-b)\{3x^2-2x(a+b)+ab\}^2=0. \quad \text{... (3)}$$

From (2), the c-discriminant relation is

$4y^2 - 4\cdot 1\cdot [y^2 - x(x-a)(x-b)] = 0$ or $x(x-a)(x-b)=0$ ... (4)

Here $x = 0$ appears once in both the discriminants. Again $x = 0$ with corresponding value $dx/dy = 0$ *i.e.* $1/p = 0$ satisfies (1) [after dividing (1) by p^2]. Hence $x = 0$ is a singular solution. For similar reasons $x - a = 0$ and $x - b = 0$ are also singular solutions.

$3x^2 - 2x(a+b) + ab = 0$ is a tac-locus since it appears squared in the p-discriminant relation only. Again solving it for x, we get

$x = [2(a+b) \pm \{4(a+b)^2 - 12\,ab\}^{1/2}]/6$ or $3x = a + b \pm (a^2 - ab + b^2)^{1/2}$ ... (5)

Thus the components of the above-mentioned tac-locus are given by (5). Thus there are two tac loci given by (5).

(*ii*) It is a particular case of part (*i*). Here $a = 1$, $b = 2$. **Ans.** $x = 0$, $x - 1 = 0$, $x - 2 = 0$ are singular solutions and $x = 1 \pm (1/\sqrt{3})$ are tac loci.

Ex. 3. *Obtain the primitive and the singular solution of the equation* $p^2(1-x^2) = 1-y^2$. *Specify the nature of the geometrical loci which are not singular solutions, but which may be obtained with the singular solution.*

Sol. The given equation is $p^2(1-x^2) - (1-y^2) = 0$. ... (1)

$\therefore$ p-disc. relation is $0 + 4(1-x^2)(1-y^2) = 0$

i.e. $$(1-x)(1+x)(1-y)(1+y) = 0 \quad \text{... (2)}$$

Solving (1) for p, $p = \dfrac{dy}{dx} = \dfrac{\sqrt{(1-y^2)}}{\sqrt{(1-x^2)}}$ so that

Integrating, $\sin^{-1} y - \sin^{-1} x = c'$, where c' is an arbitrary constant.

or $(\pi/2) - \cos^{-1} y - [(\pi/2) - \cos^{-1} x] = c'$

or $\cos^{-1}[xy + \sqrt{(1-x^2)}\sqrt{(1-y^2)}] = c'$ or $\sqrt{(1-x^2)}\sqrt{(1-y^2)} + xy = \cos c' = c$, say

or $(1-y)^2(1-x^2) = (c-yx)^2$ or $c^2 - 2xyc + x^2 + y^2 - 1 = 0$. ... (3)

$\therefore$ The c-disc. relation is $4x^2y^2 - 4(x^2 + y^2 - 1) = 0$

or $(1-x^2)(1-y^2) = 0$ or $(1-x)(1+x)(1-y)(1+y) = 0$. ... (4)

$1 - x = 0$ occurs once in p-disc. and c-disc. relation, satisfies (1), and hence it is a singular solution. Similarly $1 + x = 0$, $1 - y = 0$ and $1 + y = 0$ are also singular solutions. Furthermore the general solution (3) represents a family of conics which are here touched by the lines $x = \pm 1$ and $y = \pm 1$. In our present problem there exist no geometrical loci which are not singular solutions, that is, there does not exist any extraneous locus.

Ex. 4. *Examine* $p^2(2-3y)^2 = 4(1-y)$ *for singular solution and extraneous loci.*

Sol. The given equation is $p^2(2-3y)^2 - 4(1-y) = 0$. ... (1)

$\therefore$ p-disc. relation is $0 + 4\cdot(2-3y)^2 \cdot 4(1-y) = 0$ or $(2-3y)^2(1-y) = 0$. ... (2)

Solving (1) for p, $p = dy/dx =$

or $$dx = \frac{2-3y}{2\sqrt{(1-y)}}\,dy = \frac{3-3y-1}{2\sqrt{(1-y)}}\,dy = \left[\frac{3}{2}\sqrt{(1-y)} - \frac{1}{2}\frac{1}{\sqrt{(1-y)}}\right]dy.$$

Integrating, $x + c = -(1-y)^{3/2} + (1-y)^{1/2}$, c being an arbitrary constant.

or $x + c = (1-y)^{1/2}\{1 - (1-y)\} = (1-y)^{1/2}\,y$.

Squaring, $$(x + c)^2 = y^2 (1 - y)$$

or $$c^2 + 2xc + x^2 - y^2 (1 - y) = 0. \qquad \text{... (3)}$$

$\therefore$ c-disc. relation is $4x^2 - 4 \cdot 1 \, [x^2 - y^2 (1 - y)] = 0$ or $y^2 (1 - y) = 0$ ... (4)

$1 - y = 0$, which occurs once in both the discriminants, gives singular solution. $y = 0$, which occurs squared in the c-disc. and not at all in the p-disc., gives a node-locus. $2 - 3y = 0$, which occurs squared in the p-disc. and not at all in the c-disc., gives a tac locus.

Ex. 5. *Examine the following equations for singular solution and extraneous loci, if any.*

(i) $x^3p^2 + x^2yp + a^3 = 0.$

(ii) $y + px = x^4p^2.$

(iii) $3y = 2px - 2p^2/x.$ *or* $2p^2 - 2px^2 + 3xy = 0.$

Sol. (*i*) The given equation is $$x^3p^2 + x^2yp + a^3 = 0. \qquad \text{... (1)}$$

Solving for y, $$y = -xp - (a^3/x^2p)$$

Differentiating w.r.t. x, $$p = -p - x\frac{dp}{dx} - a^3\left(-\frac{2}{x^3 p} - \frac{1}{x^2 p^2}\frac{dp}{dx}\right)$$

or $$2p + x\frac{dp}{dx} = \frac{a^3}{x^3 p^2}\left(2p + x\frac{dy}{dx}\right)$$ or $$\left(1 - \frac{a^3}{x^3 p^2}\right)\left(2p + x\frac{dp}{dx}\right) = 0$$

or $$2p + x\,(dp/dx) = 0 \qquad \text{[For general solution omit first factor]}$$

or $$(1/p)\, dp + 2\,(1/x)\, dx = 0.$$

$\therefore$ Integrating, $\log p + 2 \log x = \log c$ or $p = c/x^2$

Putting this value of p in (1) and simplifying, we get

$$c^2 + xyc + a^3x = 0 \text{ as the general solution.} \qquad \text{... (2)}$$

From (1), the p-disc. relation is $x^4y^2 - 4x^3a^3 = 0$ *i.e.* $x^2x \cdot (xy^2 - 4a^3) = 0.$ ... (3)

From (2), the c-disc. relation is, $x^2y^2 - 4a^3x = 0$ *i.e.* $x\,(xy^2 - 4a^3) = 0.$ ... (4)

$\therefore$ As usual $x = 0$ and $xy^2 - 4a^3 = 0$ are singular solutions. $x = 0$ is tac locus since it appears as squared in p-disc. relation only.

(*ii*) The given equation is $$x^4p^2 - xp - y = 0. \qquad \text{... (1)}$$

Its general solution [Refer Ex. 2(*a*) is of Art. 4.7]

$$xc^2 - c - xy = 0. \qquad \text{... (2)}$$

As usual, the p-disc. relation is $x^2\,(4x^2y + 1) = 0$ and the c-disc. relation is $4x^2y + 1 = 0.$

$\therefore$ $4x^2y + 1 = 0$ is singular solution and $x = 0$ is a tac-locus.

(*iii*) Proceed as in part (*i*), the general solution is $(3y + 2c)^2 = 4cx^3.$

$\therefore$ As usual the p-disc. relation is $x\,(x^3 - 6y) = 0$ and the c-disc. relatioin is $x^3\,(x^3 - 6y) = 0.$

$\therefore$ $x^3 - 6y = 0$ is the singular solution and $x = 0$ is the cusp-locus.

Ex. 6. *Solve and find cusp locus of* $p^2 + 2px - y = 0$

Ans. G.S. $(2x^3 + 3xy + c)^2 = 4\,(x^2 + y^2)^3$; cusp locus $x^2 + y = 0$

Ex. 7. *Examine* $y^2\,(1 + p^2) = r^2$ *for singular solution and extraneous loci.*

Ans. $y = r$, $y = -r$ are singular solutions; $y = 0$ is tac locus.

Ex. 8. *Examine* $8ap^3 = 27y$ *for singular solutions and extraneous loci.*

Ans. $y = 0$ in singular solution as well as cusp locus.

Ex. 9. *Examine* $y^2\,(1 + 4p^2) - 2pxy - 1 = 0$ *for singular and extraneous loci.*

Ans. $x^2 - 4y^2 + 4 = 0$ is singular solution and $y = 0$ is tac locus.

Objective Problem on Chapter 4

Ex. 1. *The differential equation $x\,(dy/dx)^2 - (x-3)^2 = 0$ has p-discriminant relation as $x\,(x-3)^2 = 0$ and c-discriminant relation as $x\,(x-9)^2 = 0$. The singular solution is*

(a) $x - 3 = 0$ *(b)* $x - 9 = 0$ *(c)* $x = 0$ *(d)* $x\,(x-3)\,(x-9) = 0$. **[I.A.S. Prel. 1993]**

Sol. Ans. (*c*). Given $x\,(dy/dx)^2 - (x-3)^2 = 0$ or $x - (x-3)^2\,(dx/dy)^2 = 0$. ... (1)

Now $x = 0 \Rightarrow dx/dy = 0$ and so $x = 0$ satisfies (1). Thus $x = 0$ is the only singular solution since it occurs once in both the p and c discriminant relations and satisfies the given differential equation (1).

Ex. 2. *The equation $8ap^3 = 27y$, where $p = dy/dx$, has singular solution*

(a) $y = 0$ *(b)* $y = c$ *(c)* $y^2 = (x-c)^2/a$ *(d)* $y = (x-c)^2/a$. **[I.A.S. Prel. 1993]**

Sol. Ans. (*a*). Refer Ex. 3 of Art. 4.17

Ex. 3. *The general and singular solutions of $(dy/dx)^2 + x\,(dy/dx) - y = 0$ are*

(a) $(y - c_1 x)\,(y - x^2/4 - c_2) = 0,\ x^2 + 4y = 0.$

(b) $y = cx + c^2\ ;\ x^2 + 4y = 0.$ *(c)* $(y - 2x)^2 = cx\ ;\ x^2 + y^2 - xy = 0$

(d) $x^2 + y^2 = cxy + c^2\ ;\ (xy)^2 - 4\,(x^2 + y^2) = 0.$ **[I.A.S. Prel. 1994]**

Sol. Ans. (*b*). Let $p = dy/dx$. Then, we get $y = px + p^2$. This is in Clairaut's form. Hence putting c for p, the general solution is $y = cx + c^2$.

Since $p^2 + px - y = 0$ is quadratic in p, so the p-discriminant relation is given by $x^2 - 4 \cdot 1 \cdot (-y) = 0$ or $x^2 + 4y = 0$. Similarly, the c-discriminant is also $x^2 + 4y = 0$.

Now, $x^2 + 4y = 0 \Rightarrow y = -(1/4)\,x^2 \Rightarrow p = dy/dx = -(x/2)$.

Hence $y = px + p^2$ reduces to $y = (-x/2)\,x + (-x/2)^2$ or $-x^2/4 = -x^2/4$ which is an identity, showing that $x^2 + 4y = 0$ satisfies the given equation.

Now, $x^2 + 4y = 0$ appears in both the p and c discriminant relations and satisfies the given differential equation. Hence it forms the singular solution.

Ex. 4. *The singular solution/solutions of $x\,(dy/dx)^2 - 2y\,(dy/dx) + 4x = 0,\ (x > 0)$ is/are*

(a) $y = \pm x^2$ *(b)* $y = 2x + 3$ *(c)* $y = x^2 - 2x$ *(d)* $y = \pm 2x$. **[I.A.S. Prel. 1994]**

Sol. Ans. (*d*). Given $xp^2 - 2\,yp + 4x = 0$ ... (1)

Solving for y, (1) $\Rightarrow$ $y = (1/2)\,xp - (2x)/p$. ... (2)

Diff. (2) w.r.t. 'x', or $\left(x\dfrac{dp}{dx} - p\right)\left(\dfrac{1}{2} + \dfrac{2}{p^2}\right) = 0,$

For general solution, we take only first factor.

$\therefore\ x\dfrac{dp}{dx} - p$ or $\dfrac{dx}{x} = \dfrac{dp}{p}$ so that $\log p = \log c + \log x$ or $p = cx$.

Putting this value of p in (1), the general solution of (1) is

$$c^2x^3 - 2y\,cx + 4x = 0 \quad \text{or} \quad c^2x^2 - 2cy + 4 = 0. \qquad ... (3)$$

Both the p-discriminant and c-discriminant relations are same and are given by

$4y^2 - 16x^2 = 0$ or $y^2 - 4x^2 = 0$. ... (4)

From (4), $y^2 = 4x^2$ so that $2\,yp = 8x$ or $p = (4x)/y$.

Putting this value of p in (1), we have

$x\,(16x^2/y^2) - 2y\,(4x/y) + 4x = 0$ or $(4x^3)/y^2 - x = 0$

or $4x^3 - xy^2 = 0$ or $4x^3 - x\,(4x^2) = 0$, by (4), showing that (4) satisfies the given diff. equation. Now, $y^2 - 4x^2 = 0$ appears both in p and c discriminant relations and satisfy the given differential equation. So $y^2 - 4x^2 = 0$ or $y = \pm 2x$ are two singular solutions.

Ex. 5. *The singular solution of* $p = \log (px - y)$, *is*

(a) $y = x (\log - 1)$ (b) $y = x \log x - 1$ (c) $y = \log x - 1$ (d) $y = x \log x$

Sol. Ans. (*a*). Refer Ex. 19, Art. 4.17. **[I.A.S. Prel. 1994]**

Ex. 6. *The singular solution of* $y = (2x/3)(dy/dx) - (2/3x)(dy/dx)^2$, $x > 0$ *is*

(a) $y = \pm x^2$ (b) $y = x^3/6$ (c) $y = x$ (d) $x = y^2/6$. **[I.A.S. P. 1995]**

Sol. Ans. (*b*). Refer Ex. 13 of Art. 4.17

Ex. 7. *The singular solution of* $(xp - y)^2 = p^2 - 1$, *where p has the usual meaning, is*

(a) $x^2 + y^2 = 1$ (b) $x^2 - y^2 = 1$ (c) $x^2 + y^2 = 2$ (d) $x^2 - y^2 = 2$. **[I.A.S. Prel. 1996]**

Sol. Ans. (*b*). Refer Ex. 17 of Art. 4.17

Ex. 8. *The general solution of* $y = x (dy/dx) + (dy/dx)^2$ *is*

(a) $y = cx - c^2$ (b) $y = cx + c$ (c) $y = cx - c$ (d) $y = cx + c^2$. **[I.A.S. Prel. 1997]**

Sol. Ans. (*d*). Refer Ex. 1(*e*) of Art. 4.9.

Ex. 9. *The singular solution of the differential equation* $(xp - y)^2 = p^2 - 1$ *is*

(a) $x^2 + y^2 = 1$ (b) $x^2 - y^2 = 1$ (c) $x^2 + 2y^2 = 1$ (d) $2x^2 + y^2 = 1$ **[I.A.S. Prel. 1997]**

Sol. Ans. (*b*). Refer Ex. 17, Art. 4.17.

Ex. 10. The singular solution of the equation $y = px + f(p)$, $p = dy/dx$ is obtained on eliminating p between original equation and the equation

(*a*) $x - f'(p) = 0$ (*b*) $x + f'(p) = 0$ (*c*) $y - f'(p) = 0$ (*d*) $y + f'(p) = 0$

Sol. Ans. (*b*). Refer Art. 4.8 **[I.A.S. Prel. 1998]**

Ex. 11. *The singular solution of* $y = px + p^3$, $(p = dy/dx)$ *is* (a) $4y^3 + 27x^2 = 0$

(b) $4x^2 + 27y^3 = 0$ (c) $4y^2 - 27x^3 = 0$ (d) $4x^3 + 27y^2 = 0$ **[I.A.S. Prel. 1998]**

Sol. Ans. (*d*). Given equation is $y = p + px^3$... (1)

is in Clairaut's form. So its p- and c- relations are same.

Diff. (1) partially w.r.t. 'p', we get $0 = x + 3p^2$ or $p^2 = - x/3$. ... (2)

To find singular solution we eliminate p from (1) and (2). Re-writing (1), we have

$y = p (x + p^2)$ or $y^2 = p^2 (x + p^2)^2$ or $y^2 = (- x/3)(x - x/3)^2$, using (2)

or $y^2 = - 4x^3/27$ or $4x^3 + 27y^2 = 0$, which is singular solution.

Ex. 12. *Which of the following statements associated with a first order non-linerar differential equation* $f(x, y, dy/dx) = 0$ *are correct.*

(1) *Its general solution must contain only one arbitrary constant*

(2) *Its singular solution can be obtained by substituting particular value of the arbitrary constant in its general solution*

(3) *Its singular solution is an envelope of its general solution which also satisfies the equation.*

Select the correct answer using the codes given below :

(a) 1, 2 and 3 (b) 1 and 2 (c) 1 and 3 (d) 2 and 3. **[I.A.S. Prel. 1998]**

Sol. (*c*). Refer Art 4.2, Art 4.12 and Art 4.13

Ex. 13. *The singular solution of* $p^3 - 4xyp + 8y^2 = 0$, $(p = dy/dx)$, *is*

(a) $27y = 4x$ (b) $27y = 4x^2$ (c) $27 y = 4x^3$ (d) $27y^3 = 4x$ **[I.A.S. Prel. 1999]**

Sol. Ans. (*c*). Refer Ex. 10 of Art 4.17

Ex. 14. *The singular solution of the differential equation* $(px - y)^2 = p^2 - 1$ *is*

(a) $x^2 + y^2 = 1$ (b) $x^2 - y^2 = 1$ (c) $x^2 + 2y^2 = 1$ (d) $x^2 - 2y^2 = 1$

Sol. Ans. (*b*). Refer Ex. 17 of Art 4.17 **[I.A.S. Prel. 2000]**

Ex. 15. *The singular solution of diff. eqn.* $(xp - y)^2 = p^2 - 1$ *is*

(a) $x^2 + y^2 = 1$ (b) $x^2 - y^2 = 1$ (c) $x^2 + 2y^2 = 1$ (d) $2x^2 + y^2 = 1$. **[I.A.S. Prel. 2001]**

Sol. Ans. (*b*). Refer Ex. 17 of Art 4.17

Ex. 16. *The singular solution of* $y = (2x/3) \times (dy/dx) - (2/3x) \times (dy/dx)^2$, $x > 0$, *is*

(a) $y = x^3$ (b) $y = x$ (c) $y = x^3/6$ (d) $y = x^3/2$. **[I.A.S. Prel. 2001]**

Sol. Ans. (*c*). Refer Ex. 13 of Art 4.17.

Ex. 17. *The singular solution of diff. eqn.* $(xp - y)^2 = p^2 - 1$ *is*

(a) $x^2 + y^2 = 1$ (b) $y^2 - x^2 = 1$ (c) $x^2 + 2y^2 = 1$ (d) $x^2 - y^2 = 1$ **[I.A.S. Prel. 2002]**

Sol. Ans. (*d*). Refer Ex. 17 of Art. 4.17

Ex. 18. *Consider the Assertion (A) and Reason (R) given below* :

Assertion (*A*): $y = 0$ *is the singular solution of the differential equation* $9yp^2 + 4 = 0$, *where* $p = dy/dx$.

Reason (*R*): $y = 0$ *occurs both in p-discriminant and c-discriminant obtained from its general solution* $y^3 + (x + c)^2 = 0$ *of* $9yp^2 + 4 = 0$.

The correct answer is

(a) Both A and R are true and R is correct explanation of A

(b) Both A and R are true and R is not correct explanation of A

(c) A is true but R is false *(d) A is false but R is true* **[I.A.S. Prel. 2002]**

Sol. Ans. (*d*). Given equation is $9yp^2 + o.p + 4 = 0$ which is quadratic eqn is p. So its p-discriminant is $(0)^2 - 4\,(9y)\,(4) = 0$ *i.e.* $y = 0$.

Again the general solution is given by $y^3 + (x + c)^2 = 0$... (1)

Differentiating (1) partially w.r.t. 'c', $2\,(x + c) = 0$ giving $c = -x$.

Putting $c = -x$ in (1), c-discriminant is $y^3 = 0$.

Thus $y = 0$ is present in both p-discriminant and c-discriminant. But $y = 0 \Rightarrow dy/dx = 0 \Rightarrow p = 0$. Putting $y = 0$ and $p = 0$ in the given diff. eqn. $9\,yp^2 + 4 = 0$ we get $0 = 4$, which is absurd. Thus, $y = 0$ does not satisfy the given diff. eqn. and so $y = 0$ is not singular solution. Hence A is false and R is true.

Ex. 19. *The singular solution of the differentiate equation* $y^2\{1 + (dy/dx)^2\} = R^2$ *is*

(a) $y = R/2$ (b) $y = R$ (c) $y = 3R/2$ (d) $y = 2R$ **[I.A.S. P. 2003]**

Sol. Ans. (*b*). Refer Ex. 6 of Art. 4.17

Ex. 20. *The singular solution of* $y = px + a\,(1 + p^2)^{1/2}$ *is*

(a) parabola *(b) hyperbola* *(c) circle* *(d) straight line* **[I.A.S. Prel. 2003]**

Sol. Ans. (*c*). Re-writing given equation, $y - px = a\,(1 + p^2)^{1/2}$

or $(y - px)^2 = a^2\,(1 + p^2)$ or $p^2\,(x^2 - a^2) - 2pxy + y^2 - a^2 = 0$

Its p-discriminant relation is $4x^2y^2 - 4\,(x^2 - a^2)\,(y^2 - a^2) = 0$

or $x^2y^2 - x^2y^2 + x^2a^2 + y^2a^2 - a^4 = 0$ or $x^2 + y^2 = a^2$.

Since the given equation is in Clairaut's form so its c-discriminant relation is also $x^2 + y^2 = a^2$. Hence $x^2 + y^2 = a^2$ is singular solution which is a circle.

Ex. 21. *The singular solution of* $xyp^2 - (x^2 + y^2 - 1)\, p + xy = 0,$ *where* $p = dy/dx$

(a) is $y = 0$ *(b) is* $y^2 = (x - 1)^3$ *(c) does not exist* *(d) is none of the above.*

Sol. Ans. (*d*). Refer Ex. 24 of Art. 4.17 **[I.A.S. (Prel.) 2004]**

Ex. 22. *What is the singular solution of the differential equation* $p = \ln (px - y)$ *?*

(a) $y = cx - e^x$ *(b)* $y = x + x \ln x$ *(c)* $y = \ln x - x$ *(d)* $y = x \ln c - c,$

where ln x stands for $\log_e x$. **[I.A.S. Prel. 2005]**

Sol. Ans. (*c*). Refer Ex. 19 of Art. 4.17

Ex. 23. *What is the solution of the differential equation* $\sin px \cos y = \cos px \sin y + p$*?*

(a) $y = cx^2 - \sin^{-1} c$ *(b)* $y = cx - \sin^{-1} c$ *(c)* $y = cy - \sin^{-1} c$ *(d)* $y = cy^2 - \sin^{-1} c$.

Sol. Ans. (*b*). Refer Ex. 2 (*c*) of Art. 4.9. **[I.A.S. Prel. 2005]**

Ex. 24. *What is the singular solution of* $y^2 (1 + y'^2) = r^2$ *where r is a constant ?*

(a) $y^2 = 4x$ *(b)* $y^2 = 4r$ *(c)* $y^2 = r^2$ *(d)* $y^2 = r^3$ **[I.A.S. Prel. 2006]**

Sol. Ans. (*c*). Refer Ex. 6 of Art. 4.17. Here y' stands for dy/dx.

Ex. 25. *The singular solution of the differential equation* $y = px + f(p)$ *will be obtained by eliminating p between the equation* $y = px + f(p)$ *and which of the following equation ?*

(a) $x + df/dp = 0$ *(b)* $dy/dp = x + (df/dp)$ *(c)* $dy/dx = p$ *(d)* $dy/dx = p + (df/dp)$

Sol. Ans. (a). Refer remark of Art. 4.8 **[I.A.S. Prel. 2007]**

Ex. 26. *Consider the following statements:*

Assertion (A): The function $y = x^2/y$ *is a singular solution of* $(dy/dx)^2 - x(dy/dx) + y = 0$

Reason (R): The general solution of the given equation is $y = cx - c^2$ *and the given solution cannot be obtained by assigning a definite value to c in the general solution.*

Of these statements

(a) Both A and R are individually true and R is the correct explanation of A

(b) Both A and R are individually true and R is not the correct explanation of A

(c) A is true but R in false *(d) A is false but R is true* **[I.A.S. Prel. 2007]**

Sol. Ans. (c) Re-writing the given equation, $y = px - p^2$, where $p = dy/dx$ ... (1)

Since (1) in is Clairaut's form, its solution is $y = cx - c^2$, c being an arbitrary constant... (2)

Differentiating (2) w.r.t. 'c' $0 = x - 2c$ giving $c = x/2$ ... (3)

Substituting the value of c given to (3) in (2), we get

$y = (x^2/2) - (x^2/4)$ or $y = x^2/4$, which is singular solution of (1)

Assertion (A) is true. But the Reason (R) is always true in view of discussion in Art. 4.12.

MISCELLANEOUS EXAMPLES ON CHAPTER 4

Ex.1 What are the general and singular solutions respectively of the differential equation $(xp - y)^2 = p^2 - 1$? (*a*) $(cx + y)^2 = c^2 + 1$, $x = \pm 1$ (*b*) $(cx - y)^2 = c^2 - 1$, $x^2 + y^2 = !$

(*c*) $(cx - y)^2 = c^2 - 1$, $x^2 - y^2 = 1$ (*d*) $(x - cy)^2 = 1 - c^2$, $x^2 \pm y^2 = 1$ **[I.A.S. 2008]**

Sol. Ans (c). Refer part (c) of Ex. 17, page 4.34

Ex.2. The general solution of $(x - p)\, p = p^2 + y$ is (*a*) $y = cx - 2c^2$

(*b*) $y = cx + c^2$ (c) $y = cx$ (*d*) $y = c^2 - cx$ **[Madurai Kamraj 2008]**

Sol. Ans. (a). Use Art. 4.8.

Ex.3. Solution of $(x - a) p^2 + (x - y) p - y = 0$ is

(*a*) $y = cx - (ac^2)/(c+1)$ (*b*) $y = cy + (ac^2)/(x+1)$

(c) $y = cx - (ac^2)/(y+1)$ (*d*) $y = cy - (ac^2)/(c+1)$ **[Garhwal 2010]**

Sol. Ans. (*a*) Re-writing given equation, $y(p+1) = xp(p+1) - ap^2$ or $y = xp - (ap^2)/(p+1)$ which is in Clariaut's form. Hence the required solution is $y = cx - (ac^2)/(c+1)$

5

Linear Differential Equations With Constant Coefficients

PART 1: USUAL METHODS OF SOLVING LINEAR DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS

5.1 Some useful results

Let D stand for d/dx ; D^2 for d^2/dx^2; and so on. The symbols D, D^2, etc., are called operators. The index of D indicates the number of times the operation of differentiation must be carried out. For example, D^3x^4 shows that we must differentiate x^4 three times. Thus, $D^3x^4 = 24x$. The following results are valid for such operators.

1. $D^m + D^n = D^n + D^m$
2. $D^mD^n = D^n D^m = D^{m+n}$
3. $D(u + v) = Du + Dv$, where u and v are functions of x.
4. $(D - \alpha)(D - \beta) = (D - \beta)(D - \alpha)$, where α and β are constants.

Negative index of *D*. D^{-1} is equivalennt to an integration. For example, $D^{-1}x = \int x\, dx = x^2/2$. But it is important to note that the main object of D^{-1} is to find an integral but not the complete integral. Consequently the arbitrary constant which arises in integration must be omitted. The index of D^{-1}, say $(D^{-1})^5$ is denoted by D^{-5} The negative index of D indicates the number of times the operation of integration is to be carried out. For example,

$$D^{-2}\ x = \int \left[\int x dx \right] dx = \int (x^2/2)\, dx = x^3/6$$

It is usual to write $1/D^m$ for D^{-m}. It is to be remembered that $DD^{-1} = 1$ and the symbol D with negative indices also satisfy the above four results. Furthermore we write $(d^2y/dx^2) + a_1(dy/dx) + a_2y = (D^2 + a_1D + a_2)y = f(D)y$, where $f(D)$ is the operator now. If $f_1(D)$ and $f_2(D)$ be two operators, then $f_1(D) f_2(D)$ is also an operator such that $f_1(D) f_2(D) = f_2(D) f_1(D)$.

If u be a function of x and k be a constant then $f(D)(k u) = k f(D)u$.

From the above discussion we note that the symbol D obviously satisfies the fundamental laws of algebra and hence it can be regarded as an algebraic quantity in several respects.

5.2 Linear differential equations with constant coefficients

A linear differential equation with constant coefficients is that in which the dependent variable and its differential coefficients occur only in the first degree and are not multiplied together, and the coefficients are all constants.

The general from of the equation is $$\frac{d^n y}{dx^n} + a_1 \frac{d^{n-1}y}{dx^{n-1}} + a_2 \frac{d^{n-2}y}{dx^{n-2}} + ... + a_n y = X, \qquad ... (1)$$

where X is a function of x only and $a_1, a_2 ... a_n$ are constants.

Using the symbols D, D^2, ... D^n of Art. 5.1, (1) becomes

$$D^n y + a_1 D^{n-1} y + a_2 D^{n-2} y + ... + a_n y = X \quad \text{or} \quad (D^n + a_1 D^{n-1} + a_2 D^{n-2} + ... + a_n) y = X \quad ... (2)$$

or $$f(D) y = X \quad ... (3)$$

where $$f(D) = D^n + a_1 D^{n-1} + a_2 D^{n-2} + ... + a_n \quad ... (4)$$

and $f(D)$ now acts as operator and operates on y to yield X. The forms (2) and (3) are called the symbolic forms of the given equation (1).

Consider the differential equation $$f(D)\, y = 0, \quad ...(5)$$

obtained on replacing the right hand member of (3) by zero. We will, now, show that if $y_1, y_2 ..., y_n$ are n linearly independent solutions of (5) then, $c_1 y_1 + c_2 y_2 + ... + c_n y_n$ is also a solution of (5); $c_1, c_2 ..., c_n$ being arbitrary constants.

Since $y_1, y_2 ..., y_n$ are solutions of (5), $f(D)\, y_1 = 0$, $f(D)\, y_2 = 0$, $... f(D) y_n = 0$ ... (6)

If $c_1, c_2 ..., c_n$ are an arbitrary constants, we get

$$f(D)\,(c_1 y_1 + c_2 y_2 + ...\, c_n\, y_n) = f(D)\,(c_1 y_1) + f(D)\,(c_2 y_2) + ... + f(D)\,(c_n y_n)$$
$$= c_1 f(D)\, y_1 + c_2 f(D)\, y_2 + ... + c_n f(D)\, y_n = c_1.0 + c_2.0 + ... + c_n.0 = 0, \text{ using (6)}$$

This proves the statement made above.

Since the general solution of a differential equaiton of the n^{th} order contains n arbitrary constants, we conclude that $$c_1 y_1 + c_2 y_2 + ... + c_n\, y_n = u, \text{ say}$$
is the general solution of (5).

Thus, $$f(D)\, u = 0. \quad ... (7)$$

Again, let v be any particular solution of (3) and hence $f(D)\, v = X$. ... (8)

Now, we have $f(D)\,(u + v) = f(D)\, u + f(D)\, v = 0 + X$, using (7) and (8)

This shows that $(u + v)$, *i.e.*, $c_1 y_1 + c_2 y_2 + ... + c_n y_n + v$ is the general solution of (3), *i.e.*, (1), containing n arbitrary constants $c_1.\ c_2, ..., c_n$. The part $c_1 y_1 + c_2 y_2 + ... + c_n y_n$, is known as the *ComplementaryFunction* (C.F.) and v, not involving any arbitrary constant, is called the *Particular Integral (P.I.). or particular solution (P.S.)*

Thus, the general solution of (1) is y = C.F. + P.I., where C.F. involves n arbitrary constants and P.I. does not involve any arbitrary constant.

Remarks. It should be remembered that P.I. appears due to X in (1). Hence if a linear differential equation with constant coefficients is given with $X = 0$, then its general solution will not involve P.I and so for such differential equations the general solution will be given by y = C.F.

5.3 To find complementary function (C.F.) of the given equation [Mumbai 2010]

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + ... + a_n) y = X \quad \text{or} \quad f(D) y = X. \quad ... (1)$$

By definition, C.F. of (1) is the general solution of

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + ... + a_n) y = 0 \quad \text{or} \quad f(D) y = 0. \quad ... (2)$$

Let (2) be equivalent to $$(D - m_1)\,(D - m_2)\, ...(D - m_n)\, y = 0 \quad ... (3)$$

Then solution of any one of the equations

$$(D - m_1)\, y = 0, \quad (D - m_2)\, y = 0, \quad ....., \quad (D - m_n)\, y = 0 \quad ... (4)$$

is also a solution of (3) because if $\phi_r(x)$, $1 \le r \le n$ be a solution of $(D - m_r)\, y = 0$ ... (5)

then, $$(D - m_1)(D - m_2) ... (D - m_n)\, \phi_r\,(x)$$
$$= (D - m_1) .... (D - m_{r-1})(D - m_r)(D - m_{r+1}) .... (D - m_n)\, \phi_r\,(x)$$

$= (D - m_1) \ldots. (D - m_{r-1})(D - m_{r+1}) \ldots. (D - m_n)(D - m_r)\,\phi_r(x)$

$= 0$, since $\phi_r(x)$ is a solution of (5) $\Rightarrow (D - m_r)\,\phi_r(x) = 0$

We now proceed to find the general solution of (5), *i.e.*,

$(dy/dx) - m_r y = 0$ or $(1/y)\,dy = m_r\,dx$.

Integrating, $\log y - \log c_r = m_r x$ or $y = c_r e^{m_r x}$, ... (6)

where c_r is the constant of integration.

Since the general solution of (5) is given by (6), we can assume that a solution of the equation (2) is of the form $y = e^{mx}$. Then since $y = e^{mx}$, $Dy = m\,e^{mx}$, $D^2 y = m^2\,e^{mx}$, ..., $D^n y = m^n\,e^{mx}$, so (2) becomes.

$$(m^n + a_1 m^{n-1} + a_2 m^{n-2} + \ldots. + a_n)\,e^{mx} = 0$$

Cancelling e^{mx} as $e^{mx} \neq 0$ for any m, we obtain

$$m^n + a_1 m^{n-1} + a_2 m^{n-2} + \ldots. + a_n = 0 \qquad \ldots (7)$$

Equation (7) is called the *auxiliary equation* (A. E.). Replacing m by D in (7), we have

$$D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots. + a_n = 0 \qquad \ldots(8)$$

Clearly, equaiton (7) gives the same values of m as equation (8) gives of D. In practice, we may also take equation (8) as the auxiliary equation which is obtained by equating to zero the symbolic coefficient of y in (1) or (2). Here D is considered as an algebraic quantity.

(7) or (8) will give, in genral, n roots say, $m_1, m_2, \ldots, m_n$.

Case I. When all the roots of the A.E. (7) or (8) are real and different.

Let $m_1, m_2, m_3 \ldots, m_n$ be the n real and different roots of (8). Then $y = e^{m_1 x}, y = e^{m_2 x}, \ldots, y = e^{m_n x}$ are n independent solutions of (2). So the solution of (2) is

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x} + \ldots.. + c_n e^{m_n x},\ c_1, c_2, \ldots.., c_n \text{ being arbitrary constants} \qquad \ldots (9)$$

Case II. When the auxilary equaiton has equal roots

Let the roots m_1 and m_2 of the A.E. (7) or (8) be equal. Then the general solution (9) becomes

$$y = (c_1 + c_2)\,e^{m_1 x} + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \ldots. + c_n e^{m_n x}$$

or $y = Ae^{m_1 x} + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \ldots + c_n e^{m_n x}$ where $A = c_1 + c_2$. ... (10)

This solution contains $(n - 1)$ arbitrary constants. $A, c_3, c_4, \ldots., c_n$ and not n. But we know that C.F. of (1), *i.e.*, solution of (1) must contain as many arbitrary constants as is the order of the given differential equation. Hence (10) is not the general solution of (2). To obtain the general solution of (2), consider the differential equation

$(D - m_1)^2\,y = 0$ or $(D - m_1)(D - m_1)\,y = 0$, ... (11)

in which the two roots are equal.

Let $(D - m_1)\,y = v$. ... (12)

Then, (11) $\Rightarrow (D - m_1)v = 0$ or $dv/dx = m_1 v$ or $(1/v)\,dv = m_1 dx$.

Integrating, $\log v - \log c_1 = m_1 x$ or $v = c_1 e^{m_1 x}$ ...(13)

Using (13), (12), becomes $(D - m_1)\,y = c_1 e^{m_1 x}$ or $Dy - m_1 y = c_1 e^{m_1 x}$

or $(dy/dx) - m_1 y = c_1 e^{m_1 x}$, which is linear equation ...(14)

Its I.F. $= e^{\int (-m_1)dx} = e^{-m_1 x}$ and so its solution is

$$ye^{-m_1 x} = \int \left\{ (c_1 e^{m_1 x}).e^{-m_1 x}) \right\} dx + c_2 = c_1 \int dx + c_2 = c_1 x + c_2$$

or $y = (c_1 x + c_2)e^{m_1 x}$ or $y = (c_2 + c_1 x)\,e^{m_1 x}$

Hence the general solution of (2) in this case is of the form

$$y = (c_2 + c_1 x)\, e^{m_1 x} + c_3 e^{m_3 x} + c_4 e^{m_4 x} + \ldots + c_n e^{m_1 x}. \qquad \ldots(15)$$

Similarly, it three roots of the A.E. are equal, say $m_1 = m_2 = m_3$, then the general solution of (2) is of the form $y = (c_3 + c_2 x + c_1 x^2)\; e^{m_1 x} + c_4 e^{m_4 x} + \ldots + c_n e^{m_n x}$

Case III. When the A.E. has complex roots

Let the two roots of the A.E. be complex, say $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$. Then the corresponding part of the C.F.

$$= c_1' \, e^{(\alpha + i\beta)x} + c_2' \, e^{(\alpha - i\beta)x}, \; c_1', \; c_2' \text{ being arbitrary constants}$$

$$= e^{\alpha x}\, (c_1' \, e^{i\beta x} + c_2' \, e^{-i\beta x}) = e^{\alpha x}\, [c_1' \,(\cos \beta x + i \sin \beta x) + c_2' \,(\cos \beta x - i \sin \beta x)]$$

[As by Euler's theorem, $e^{i\theta} = \cos\theta + i \sin\theta$, $e^{-i\theta} = \cos\theta - i\sin\theta$]

$$= e^{\alpha x}\, [(c_1' + c_2') \cos \beta x + i\,(c_1' - c_2') \sin \beta x] = e^{\alpha x}\, (c_1 \cos \beta x + c_2 \sin \beta x),$$

[Taking $c_1 = c_1' + c_1'$ and $c_2 = i\,(c_1' - c_2')$]

where c_1 and c_2 are arbitrary constants.

Similarly, if the complex roots are repeated, say $\alpha + i\beta$ and $\alpha - i\beta$ occur twice, then the corresponding part of C.F. is of the form $e^{\alpha x}\, [(c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x]$ and so on.

Remark. After suitably adjusting the constants, $e^{\alpha x}\,(c_1 \cos \beta x + c_2 \sin \beta x)$ may also be written as $c_1 e^{\alpha x} \cos (\beta x + c_2)$ or $c_1 e^{\alpha x} \sin (\beta x + c_2)$.

Case IV. When the A.E. has surd roots.

Let the two roots of the A.E. be surds, say $m_1 = \alpha + \sqrt{\beta}, m_2 = \alpha - \sqrt{\beta}$. Then the corresponding part of the C.F. of (1)

$$= c_1' \, e^{(\alpha + \sqrt{\beta})x} + c_2' \, e^{(\alpha - \sqrt{\beta})x} = e^{\alpha x} [c_1' \, e^{\sqrt{\beta} x} + c_2' \, e^{-\sqrt{\beta} x}]$$

$$= e^{\alpha x}\, [c_1' \,(\cosh x\sqrt{\beta} + \sinh x\sqrt{\beta}) + c_2' \,(\cosh x\sqrt{\beta} - \sinh x\sqrt{\beta})]$$

$$[\because e^{\theta} = \cosh\theta + \sinh\theta \quad \text{and} \quad e^{-\theta} = \cosh\theta - \sinh\theta]$$

$$= e^{\alpha x}\, [(c_1' + c_2') \cosh x\sqrt{\beta} + (c_1' - c_2') \sinh x\sqrt{\beta})] = e^{\alpha x}\, [c_1 \cosh x\sqrt{\beta} + c_2 \sinh x\sqrt{\beta})],$$

where c_1 and c_2 are arbitrary constants given by $c_1 = c_1' + c_2'$ and $c_2 = c_1' - c_2'$.

Similarly, if the surd roots are repeated, say $\alpha + \sqrt{\beta}$ and $\alpha - \sqrt{\beta}$ occur twice, then the corresponding part of C.F. is of the form

$$e^{\alpha x}\, [(c_1 + c_2 x) \cosh x\sqrt{\beta} + (c_3 + c_4 x) \sin x\sqrt{\beta}] \text{ and so on.}$$

Remark. After suitably adjusting the constants, $e^{\alpha x}\,(c_1 \cosh\, x\sqrt{\beta} + c_2 \sinh x\sqrt{\beta}\,)$ may be written as $c_1 e^{\alpha x} \cosh\,(x\sqrt{\beta} + c_2)$ or $c_1 e^{\alpha x} \sinh (x\sqrt{\beta} + c_2)$.

5.4 Working rule for finding C.F. of the given equation

$$(d^n y / dx^n) + a_1\,(d^{\,n-1} y/dx^{n-1}) + a_2\,(d^{\,n-2} y/dx^{n-2}) + \ldots + a_n y = X \qquad \ldots (1)$$

Step I. Re-write the equation (1) in the symbolic form

$$(D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n)\, y = X. \qquad \ldots (2)$$

Step II. The auxiliary equation is $m^n + a_1 m^{n-1} + a_2 m^{n-2} + \ldots + a_n = 0$. ... (3)

or $D^n + a_1 D^{n-1} + a_2 D^{n-2} + \ldots + a_n = 0$. ... (3)′

Step III. From the roots of A.E. (3) or (3)′, write down the corresponding part of the C.F. as given in the following table

S. No.	*Corresponding part of C.F.*	*Nature of roots of auxiliary equation (A.E)*
1.	(*i*) One real root m_1	$c_1 e^{m_1 x}$
	(*ii*) Two real and different roots m_1, m_2	$c_1 e^{m_1 x} + c_2 e^{m_2 x}$
	(*iii*) Three real and different roots m_1, m_2, m_3	$c_1 e^{m_1 x} + c_2 e^{m_2 x} + c_3 e^{m_3 x}$
2.	(*i*) Two real and equal roots m_1, m_1	$(c_1 + c_2 x)\, e^{m_1 x}$
	(*ii*) Three real and equal roots m_1, m_1, m_1	$(c_1 + c_2 x + c_3 x^2)\, e^{m_1 x}$
3.	(*i*) One pair of complex roots $\alpha \pm i\beta$	$e^{\alpha x}\,(c_1 \cos \beta x + c_2 \sin \beta x)$ or $c_1 e^{\alpha x} \cos(\beta x + c_2)$ or $c_1\, e^{\alpha x} \sin(\beta x + c_2)$
	(*ii*) Two pairs of complex and equal roots $\alpha \pm i\beta$, $\alpha \pm i\beta$	$e^{\alpha x}\,[(c_1 + c_2 x) \cos \beta x + (c_3 + c_4 x) \sin \beta x]$
4.	(*i*) One pair of surd roots $\alpha \pm \sqrt{\beta}$	$e^{\alpha x}\,(c_1 \cosh x\sqrt{\beta} + c_2 \sinh x\sqrt{\beta})$ or $c_1\, e^{\alpha x} \cosh(x\sqrt{\beta} + c_2)$ or $c_1 e^{\alpha x} \sinh(x\sqrt{\beta} + c_2)$
	(*ii*) Two pairs of surd and equal roots $\alpha \pm \sqrt{\beta}, \alpha \pm \sqrt{\beta}$	$e^{\alpha x}\,[(c_1 + c_2 x) \cosh x\sqrt{\beta} + (c_3 + c_4 x) \sinh x\sqrt{\beta}\,]$

5.5 Solved examples based on Art 5.4

Ex. 1. *Solve* $(d^3y/dx^3) + 6\,(d^2y/dx^3) + 11\,(dy/dx) + 6y = 0.$

Sol. The given equation can be re-written as

$$(D^3 + 6\,D^2 + 11\,D + 6)\,y = 0, \text{ where } D \equiv d/dx \qquad \text{... (1)}$$

The auxiliary equation of (1) is $\quad D^3 + 6\,D^2 + 11\,D + 6 = 0$

or $\quad (D + 1)\,(D + 2)\,(D + 3) = 0 \quad$ so that $\quad D = -1, -2, -3.$

$\therefore$ The required general solution is $y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x}$, c_1, c_2, c_3 being arbitrary constants.

Ex. 2. *Solve* $(D^3 + 3D^2 + 3D + 1)\,y = 0$ **[Delhi Maths. (G) 1994]**

Sol. The auxiliary equation is $D^3 + 3D^2 + 3D + 1 = 0$ or $(D + 1)^3 = 0 \Rightarrow -1, -1, -1.$

$\therefore$ The required solution is $\quad y = (c_1 + c_2 x + c_3 x^2)\, e^{-x}$, c_1, c_2, c_3 being arbitrary constants.

Ex. 3 *Solve* $(d^4y/dx^4) - (d^3y/dx^3) - 9(d^2y/dx^2) - 11(dy/dx) - 4y = 0.$ **[Delhi Maths. (G) 1997]**

Sol. Let $D = d/dx$. Then the given equation can be written as

$(D^4 - D^3 - 9\,D^2 - 11\,D - 4)\,y = 0 \quad$ or $\quad (D + 1)^3\,(D - 4) = 0 \quad$ so that $\quad D = 4, -1, -1, -1.$

$\therefore$ The required solution is $y = c_1 e^{4x} + (c_2 + c_3 x + c_4 x^2)\, e^{-x}$, c_1, c_2, c_3, c_4 being arbitrary constants.

Ex. 4. *Solve* ***(a)*** $(D^4 - 5D^2 + 4)\,y = 0$

(b) $(D^4 + 2D^3 - 3D^2 - 4D + 4)\,y = 0$

(c) $(D^3 - 3D^2 + 2D)\,y = 0$

Sol. (*a*) Here auxiliary equation is $\quad D^4 - 5D^2 + 4 = 0$

or $\quad (D^2 - 4)\,(D^2 - 1) = 0 \quad$ or $\quad D^2 = 4 \quad$ or $\quad 1 \quad$ so that $\quad D = 2, -2, 1, -1.$

$\therefore$ The required general solution is $\quad y = c_1 e^{2x} + c_2 e^{-2x} + c_3 e^{x} + c_4 e^{-x}$,

c_1, c_2, c_3, c_4 being arbitrary constants

(*b*) Here auxiliary equation is $\quad D^4 + 2D^3 - 3D^2 - 4D + 4 = 0 \quad$ or $\quad (D - 1)\,(D^3 + 3D^2 - 4) = 0$

or $\quad (D - 1)\,\{(D - 1)\,(D^2 + 4D + 4)\} \quad$ or $\quad (D - 1)^2\,(D + 2)^2 = 0 \quad$ so that $\quad D = 1, 1, -2, -2.$

$\therefore$ The required solution is $y = (c_1 + c_2 x)\, e^{x} + (c_3 + c_4 x)^{-2x}$, c_1, c_2, c_3 being arbitrary constants.

(*c*) Here the auxiliary equation is $D^3 - 3D^2 + 2D = 0$ or $D(D^2 - 3D + 2) = 0$

or $D(D-1)(D-2) = 0$ so that $D = 0, 1, 2.$

Hence the required solution is $y = c_1 e^{0x} + c_2 e^{x} + c_3 e^{2x}$

or $y = c_1 + c_2 e^{x} + c_3 e^{2x}$, c_1, c_2, c_3 being arbitrary constants

Ex. 5. *Solve* $(D^3 - 8)\,y = 0.$

Sol. (*a*) Here auxiliary equation is $D^3 - 8 = 0$ or $(D-2)(D^2 + 2D + 4) = 0$ so that

$D = 2$ or $D = \{-2 \pm (4-16)^{1/2}\}/2$ or $D = 2, -1 \pm i\sqrt{3}$.

$\therefore$ The required solution is $y = c_1 e^{2x} + e^{-x}\,\{c_2 \cos(x\sqrt{3}) + c_3 \sin(x\sqrt{3})\}$,

c_1, c_2, c_3 being arbitrary constants

Ex. 6. *Solve* (*i*) $d^4 y/dx^4 + m^4 y = 0$

(*ii*) $d^4 y/dx^4 + y = 0$ **[(I.A.S.Prel 2001; Agra 2006]**

Sol. (*i*) Let $D \equiv d/dx$. Then, the given equation can be rewritten as $(D^4 + m^4)\,y = 0$

Its auxiliary equation is $D^4 + m^4 = 0$ or $(D^2 + m^2)^2 - (\sqrt{2}D\,m)^2 = 0$

or $(D^2 + m^2 + \sqrt{2}Dm)(D^2 + m^2 - \sqrt{2}Dm) = 0 \Rightarrow D^2 + m^2 + \sqrt{2}\,Dm = 0$ or $D^2 + m^2 - \sqrt{2}\,Dm = 0$

$\therefore$ $D = \{-\sqrt{2}m \pm (2m^2 - 4m^2)^{1/2}\}/2 = -(m/\sqrt{2}) \pm i\,(m/\sqrt{2})$,

and $D = \{\sqrt{2}m \pm (2m^2 - 4m^2)^{1/2}\}/2 = m/\sqrt{2} \pm i\,(m/\sqrt{2})$

Hence the required general solution is $y = e^{-(mx/\sqrt{2})}\ \{c_1 \cos(mx/\sqrt{2}) + c_2 \sin(mx/\sqrt{2})\}$

$$+ e^{mx/\sqrt{2}}\left\{c_3 \cos(mx/\sqrt{2}) + c_4 \sin(mx/\sqrt{2})\right\},$$ c_1, c_2, c_3, c_4 being arbitrary constants.

(*ii*) This is a particular case of part (*i*). Here $m = 1$. Solution is

$$y = e^{-(x/\sqrt{2})}\ \{c_1 \cos(x/\sqrt{2}) + c_2 \sin(x/\sqrt{2})\} + e^{x/\sqrt{2}}\ \{c_3 \cos(x/\sqrt{2}) + c_4 \sin(x/\sqrt{2})\}.$$

Ex. 7. *Solve* (*i*) $(D^4 - m^4)\,y = 0$

(*ii*) $(D^4 - 81)\,y = 0$

Sol. (*i*) Here auxiliary equation is $D^4 - m^4 = 0$ or $(D^2 - m^2)(D^2 + m^2) = 0$

Hence $D = m, -m, \pm im$. Now the part of C.F. corresponding to roots $m, -m$ is $c_1 e^{mx} + c_2 e^{-mx}$ and the part of the C.F. corresponding to roots $0 \pm mi$ is given by (noting that $\alpha = 0$ and $\beta = m$ in S.No. 3 (*i*) of table of Art. 5.4) $e^{0 \cdot x}(c_3 \cos mx + c_4 \sin mx)$, *i.e.*, $c_3 \cos mx + c_4 \sin mx$.

Solution is $y = c_1 e^{mx} + c_2 e^{-mx} + c_3 \cos mx + c_4 \sin mx$, c_1, c_2, c_3, c_4 being arbitrary constants.

(*ii*) Take $m = 3$ in part, (*i*). **Ans.** $y = c_1 e^{3x} + c_2 e^{-3x} + c_3 \cos 3x + c_4 \sin 3x$

Ex. 8. (*a*) *Solve* $(D^2 + 1)^2 y = 0$, *where* $D \equiv d/dx$. **[I.A.S. Prel. 1993]**

Sol. Here auxiliary equation is $(D^2 + 1)^2 = 0$ so that $D^2 + 1 = 0$ (twice)

Hence $D = 0 \pm i$ (twice). Therefore, required solution is

$$y = e^{0 \cdot x}\,\{(A_1 + A_2 x)\cos x + (A_3 + A_4 x)\sin x\}$$

or $y = (A_1 + A_2 x)\cos x + (A_3 + A_4 x)\sin x$, A_1, A_2, A_3, A_4 being arbitrary constants

Ex. 8. (*b*) *Find the primtive of* $(D^2 - 2D + 5)^2\,y = 0.$ **[I.A.S. Prel. 1995]**

Sol. Here auxiliary equation is $(D^2 - 2D + 5)^2 = 0$ so that

$D^2 - 2D + 5 = 0$ (twice) and hence $D = (2 \pm \sqrt{-16})/2 = 1 \pm 2i$ (twice)

$\therefore$ Required solution is $y = e^{x}\{(c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x\}$,

c_1, c_2, c_3 and c_4 being arbitrary constants.

Ex. 9. *Solve* $(D^4 - 6D^3 + 12D^2 - 8D)\,y = 0$ **[I.A.S. Prel. 1996]**

Sol. Here A.E. is $D(D^3 - 6D^2 + 12D - 8) = 0$ or $D\{D^2(D-2) - 4D(D-2) + 4(D-2)\} = 0$ or $D(D-2)(D^2 - 4D + 4) = 0$ or $D(D-2)^3 = 0$ so that $D = 0, 2, 2, 2$

$\therefore$ Required solution is $y = c_1 e^{0\cdot x} + (c_2 + c_3 x + c_4 x^2)\,e^{2x}$ or $y = c_1 + (c_2 + c_3 x + c_4 x^2)\,e^{2x}$, c_1, c_2, c_3 and c_4 being arbitrary constants

Ex. 10. *Solve* $(D^6 - 1)\,y = 0.$

Sol. Here the auxiliary equation is $D^6 - 1 = 0$ or $(D^2)^3 - (1)^3 = 0$

or $(D^2 - 1)(D^4 + D^2 + 1) = 0$ or $(D^2 - 1)\{(D^4 + 2D^2 + 1) - D^2\} = 0$

or $(D^2 - 1)\{(D^2 + 1)^2 - D^2\} = 0$

or $(D - 1)(D + 1)(D^2 + 1 + D)(D^2 + 1 - D) = 0$

$$\therefore \quad D = 1, -1, \frac{-1 \pm (1-4)^{1/2}}{2}, \frac{1 \pm (1-4)^{1/2}}{2} = 1, -1, -\frac{1}{2} \pm \frac{i\sqrt{3}}{2}, \frac{1}{2} \pm \frac{i\sqrt{3}}{2}$$

Hence the solution is $y = c_1 e^x + c_2 e^{-x} + e^{-x/2}\,[c_5 \cos(x\sqrt{3}/2) + c_4 \sin(x\sqrt{3}/2)]$ $+ e^{x/2}\,[c_5 \cos(x\sqrt{3}/2) + c_6 \sin x\sqrt{3}/2)]$, $c_1, c_2, c_3, c_4, c_5, c_6$ being arbitrary constants

Ex. 11. *Solve* (a) $(D^4 + 8D^2 + 16)\,y = 0$ **[I.A.S. (Prel.) 1994]**

(b) $(D^2 + D + 1)^2 = 0.$

Sol. (*a*) Here the auxiliary equation is $D^4 + 8D^2 + 16 = 0$ or $(D^2 + 4)^2 = 0$ or $D^2 + 4 = 0$ (twice) so that $D = 0 \pm 2i$ (twice). Here complex roots of A.E. are repeated twice. Hence, the required general solution is $y = e^{0x}\,[(c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x]$

or $y = (c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x$, c_1, c_2, c_3 and c_4 being arbitrary constants

(*b*) Ans. $y = e^{-x/2}\,[(c_1 + c_2 x)\cos(x\sqrt{3}/2) + (c_3 + c_4 x)\sin(x\sqrt{3}/2)]$.

Ex. 12. *Solve (a)* $(D^2 + D + 1)^2 (D - 2)\,y = 0.$

(b) $(D^2 + 1)^2 (D^2 + D + 1)\,y = 0$

(c) $(D^2 + 1)^3 (D^2 + D + 1)^2\,y = 0$

Sol. (*a*) Here the auxiliary equation is $(D^2 + D + 1)^2 (D - 2)\,y = 0$

$\Rightarrow D^2 + D + 1 = 0$ (twice) or $D - 2 = 0$

$\Rightarrow D = [-1 \pm \sqrt{(1-4)}\,]/2$ (twice) or $D = 2$

$\Rightarrow D = (-1/2) \pm i\,(\sqrt{3}/2)$ (twice) or $D = 2.$

So required general solution is $y = c_1 e^{2x} + e^{-x/2}\,[(c_2 + c_3 x)\cos(x\sqrt{3}/2)$ $+ (c_4 + c_5 x)\sin(x\sqrt{3}/2)]$, c_1, c_2, c_3, c_4 and c_5 being arbitrary constants.

(*b*) Here the auxiliary equation is $(D^2 + 1)^2 (D^2 + D + 1) = 0$

$\therefore$ $D = 0 \pm i$ (twice), $\{-1 \pm (1-4)^{1/2}\}/2$ *i.e.*, $0 \pm i$ (twice), $-(1/2) \pm i(\sqrt{3}/2)$.

Hence the required general solution is $y = (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x$ $+ e^{-x/2}\,[c_5 \cos(x\sqrt{3}/2) + c_6 \sin(x\sqrt{3}/2)]$, $c_1, c_2, c_3, c_4, c_5, c_6$ being arbitrary constants.

(*c*) Here the auxiliary solution is $(D^2 + 1)^3 (D^2 + D + 1)^2 = 0 \Rightarrow D^2 + 1 = 0$ (thrice), $D^2 + D + 1 = 0$ (twice). Hence $D = 0 \pm i$ (thrice), $-(1/2) \pm i\,(\sqrt{3}/2)$ (twice).

Solution is $y = (c_1 + c_2 x + c_3 x^2)\cos x + (c_4 + c_5 x + c_6 x^2)\sin x + e^{-x/2}\,[(c_7 + c_8 x)\cos(\sqrt{3}x/2)$ $+ (c_9 + c_{10} x)\sin(\sqrt{3}x/2)]$, where $c_1, c_2, \ldots, c_{10}$ are arbitrary constants.

Ex. 13. (*a*) *Solve* $(d^4y/dx^4) - 4(d^3y/dx^3) + 8(d^2y/dx^2) - 8(dy/dx) + 4y = 0$ **[Rohilkhand 1995]**

Sol. Let $D \equiv d/dx$. Then the given equation, becomes $(D^4 - 4D^3 + 8D^2 - 8D + 4)\, y = 0$

Here the auxiliary equation is $D^4 - 4D^3 + 8D^2 - 8D + 4 = 0$

or $(D^2 + 4/D^2) - 4\,(D + 2/D) + 8 = 0$, on dividing both sides by D^2 ... (1)

Let $D + 2/D = z$ so that $D^2 + 4/D^2 = z^2 - 4$. ... (2)

Using (2), (1) becomes $(z^2 - 4) - 4z + 8 = 0$ or $z^2 - 4z + 4 = 0$

or $(z - 2)^2 = 0$ or $[D + (2/D) - 2]^2 = 0$, using (2)

or $(D^2 - 2D + 2)^2 = 0$ $\Rightarrow$ $D = [2 \pm (4 - 8)]/2$ (twice), *i.e.*, $D = 1 \pm i$ (twice).

Hence the required general solution is $y = e^x\, [(c_1 + c_2x) \cos x + (c_3 + c_4x) \sin x)]$, c_1, c_2, c_3 and c_4 being arbitrary constants.

Ex. 13. (*b*) *Solve* $(D^4 + 2D^3 + 3D^2 + 2D + 1)\, y = 0.$

Sol. A.E. is $D^4 + 2D^3 + 3D^2 + 2D + 1 = 0$ or $(D^4 + 2D^3 + D^2) + (2D^2 + 2D + 1) = 0$

or $(D^2 + D)^2 + 2\,(D^2 + D) + 1 = 0$ or $(D^2 + D + 1)^2 = 0$ or $D^2 + D + 1 = 0$ (twice)

$\therefore$ $D = \{-1 \pm (1 - 4)^{1/2}\}/2 = -(1/2) \pm i\left(\sqrt{3}/2\right)$ (twice). The required general solution is

$$y = e^{-x/2}\, [(c_1 + c_2x) \cos(\sqrt{3}x/2) + (c_3 + c_4x) \sin(\sqrt{3}x/2)]$$

c_1, c_2, c_3 and c_4 being arbitrary constants.

Ex. 14 (*a*). *Solve* $(d^2y/dx^2) + 4y = 0$, *given that* $y = 2$ *and* $dy/dx = 0$ *when* $x = 0$.

Sol. Let $D \equiv d/dx$. Then the equation is $(D^2 + 4)\, y = 0$. ... (1)

Its auxiliary equation is $D^2 + 4 = 1$ so that $D = 0 \pm 2i$.

Hence the general solution of (1) is $y = c_1 \cos 2x + c_2 \sin 2x$, ... (2)

where c_1 and c_2 are arbitrary constants. These constants will be determined by using the given conditioins of the problem, namely,

$y = 2$ when $x = 0$ ... (3)

and $dy/dx = 0$ when $x = 0$ ... (4)

Now, from (2), $dy/dx = -2c_1 \sin 2x + 2c_2 \cos 2x$. ... (5)

Using condition (3), (2) gives $2 = c_1$ so that $c_1 = 2$

Using condition (4), (5) gives $0 = 2c_2$ so that $c_2 = 0$.

Putting $c_1 = 2$, $c_2 = 0$ in (2), the required solution is $y = 2 \cos 2x$.

Ex. 14 (*b*). *Solve* $(d^2y/dx^2) + y = 0$ *given* $y = 2$ *for* $x = 0$ *and* $y = -2$ *for* $x = \pi/2$.

Sol. Proceed as in part (*a*). **Ans.** $y = 2(\cos x - \sin x)$.

Ex. 15 (*a*). *Solve* $l(d^2\theta/dt^2) + g\theta = 0$ *given that* $\theta = \theta_0$ *and* $d\theta/dt = 0$ *when* $t = 0$.

Sol. Let $D \equiv d/dt$. Then the given equation can be written as $[D^2 + (g/l)]\, \theta = 0$. ... (1)

Its auxiliary equation is $D^2 + (g/l) = 0$ so that $D = 0 \pm i(g/l)^{1/2}$

The general solution of (1) is $\theta = c_1 \cos\left\{t\sqrt{(g/l)}\right\} + c_2 \sin\left\{t\sqrt{(g/l)}\right\}$, ... (2)

where c_1 and c_2 are arbitrary constants.

From (2), $d\theta/dt = -c_1\sqrt{(g/l)} \sin\{t\sqrt{g/l)}\} + c_2\sqrt{(g/l)} \cos\{t\sqrt{g/l)}\}$ ... (3)

Given that $\theta = \theta_0$ when $t = 0$ ... (4)

and $d\theta/dt = 0$ when $t = 0$. ... (5)

Using the condition (4), (2) $\Rightarrow \theta_0 = c_1$ so that $c_1 = \theta_0$.

Using the condition (5), (3) $\Rightarrow 0 = c_2\sqrt{(g/l)}$ so that $c_2 = 0$.

Putting $c_1 = \theta_0$, $c_2 = 0$ in (2), the required solution is $\theta = \theta_0 \cos\{t\sqrt{(g/l)}\}$.

Ex. 15 (*b*). *Find the solution of* $(d^2i/dt^2) + (R/L)(di/dt) + (1/LC)\, i = 0$, *where* $R^2C = 4L$ *and* R, C, L *are constants.*

Sol. Let $D \equiv d/dt$. Then the given equation can be written as $[(D^2 + (R/L)D + (1/LC)]\, i = 0$.

Here the auxiliary equation is $$D^2 + (R/L)D + (1/LC) = 0$$

so that $$D = [-(R/L) \pm \{(R^2/L^2) - (4/LC)\}^{1/2}]/2 = -(R/2L), \text{ as } R^2C = 4L$$

Thus, $D = -(R/2L)$ (twice). Hence the required general solution is

$y = (c_1 + c_2 t)\, e^{-t\,(R/2L)}$, c_1, c_2 being arbitrary constants.

Exercise 5(A)

Solve the following differential equations :

1. (*a*) $(D^3 + 6D^2 + 11D + 6)y = 0$ **(Meerut 2010)** **Ans.** $y = c_1 e^{-x} + c_2 e^{-2x} + c_3 e^{-3x}$

(*b*) $d^2y/dx^2 + 2(dy/dx) + 5y = 0$ **(Guwahati 2007)** **Ans.** $y = e^{-x}(c_1 \cos 4x + c_2 \sin 4x)$

(*c*) $d^3y/dx^3 - 6(d^2y/dx^2) + 9\,(dy/dx) = 0$ **(Pune 2010)** **Ans.** $y = c_1 + (c_2 + xc_3)\, e^{3x}$

2. $(D^3 + 6D^2 + 12D + 8)\, y = 0$. **Ans.** $y = (c_1 + c_2 x + c_3 x^2)\, e^{-2x}$

3. $(d^2y/dx^2) + 2p\,(dy/dx) + (p^2 + q^2)\, y = 0$. **Ans.** $y = e^{-px}(c_1 \cos qx + c_2 \sin qx)$

4. $(D^4 - 2D^3 + 5D^2 - 8D + 4)\, y = 0$. **Ans.** $y = (c_1 + c_2 x)\, e^x + c_3 \cos 2x + c_4 \sin 2x$

5. $(D^4 + D^2 + 1)\, y = 0$.

Ans. $y = e^{x/2}[c_1 \cos(x\sqrt{3}/2) + c_2 \sin(x\sqrt{3}/2)] + e^{-x/2}[c_3 \cos(x\sqrt{3}/2) + c_4 \sin(x\sqrt{3}/2)]$

6. $(D^4 + 4D^3 - 5D^2 - 36D - 36)\, y = 0$. **Ans.** $y = c_1 e^{-3x} + c_2 e^{3x} + (c_3 + c_4 x)\, e^{-2x}$

7. $(D^4 - 7D^3 + 18D^2 - 20D + 8)\, y = 0$. **Ans.** $y = c_1 e^x + (c_2 + c_3 x + c_4 x^2)\, e^{2x}$

8. $(D^2 \pm w^2)\, y = 0$, $w \neq 0$. **Ans.** $y = c_1 \cos wx + c_2 \sin wx + c_3 e^{wx} + c_4 e^{-wx}$

9. $\{D^3 + D^2(2\sqrt{3} - 1) + D(3 - 2\sqrt{3}) - 3\}\, y = 0$ **Ans.** $y = c_1 e^x + (c_2 + c_3 x)\, e^{-x\sqrt{3}}$

10. (*a*) $(D^5 - 13D^3 + 26D^2 + 82D + 104)\, y = 0$

Ans. (*a*) $y = c_1 e^{-4x} + e^{-x}(c_2 \cos x + c_3 \sin x) + e^{3x}(c_4 \cos 2x + c_5 \sin 2x)$

(*b*) $(D^6 + 9D^4 + 24D^2 + 16)\, y = 0$ (*b*) $y = c_1 \cos x + c_2 \sin x + (c_3 + c_4 x) \cos 2x + (c_5 + c_6 x) \sin 2x$

11. $d^2x/dt^2 - 3\,(dx/dt) + 2x = 0$ given when $t = 0$, $x = 0$ and $dx/dt = 0$ **Ans.** $x = 0$

5.6 The symbolic function 1/*f*(*D*).

Definition. The expression $\dfrac{1}{f(D)} X$ is defined to be that function of x which when operated upon by $f(D)$ gives X.

For example, $$\frac{1}{D^2 + 3D}(2 + 6x) = x^2 \qquad [\because\ (D^2 + 3D)\, x^2 = 2 + 6x]$$

The operator $1/f(D)$, according to this definition, is the inverse of the operator $f(D)$.

Thus, $$\frac{1}{D} X = \int X\, dx. \qquad \textbf{[Remember]}$$

5.7 Determination of the particular integral (P.I.) of

$$f(D)\, y = X. \qquad \ldots (1)$$

In view of the definition 5.6, it follows that, Particular integral of (1) $= \dfrac{1}{f(D)} X.$ ... (2)

5.8 General method of getting particular integral

Theorem. If X is a function of x, then $$\frac{1}{D - \alpha} X = e^{\alpha x} \int X e^{-\alpha x}\, dx.$$

Proof. Let $$y = \frac{1}{D-\alpha} X$$

On operating by $(D - \alpha)$, we get $$(D-\alpha)\, y = X$$

or $$\left(\frac{d}{dx} - \alpha\right) y = X \quad \text{or} \quad \frac{dy}{dx} - \alpha y = X,$$

which is a linear differential equation whose I.F. $= e^{-\int \alpha dx} = e^{-\alpha x}$ and hence its solution is given by

$ye^{-\alpha x} = \int Xe^{-\alpha x}\, dx$, after omitting constant of integration, since P.I. is required.

$$\therefore \qquad y = e^{\alpha x} \int Xe^{-\alpha x} dx$$

Thus, $$\frac{1}{D-\alpha} X = e^{\alpha x} \int Xe^{-\alpha x}\, dx \qquad \text{... (1)}$$

Similarly $$\frac{1}{D+\alpha} X = e^{-\alpha x} \int Xe^{\alpha x}\, dx . \qquad \text{... (2)}$$

Remark 1. Since we require only a particular integral, we shall never add a constant of integration after integration is performed in connection with any method of finding P.I. Hence P.I. will never contain any arbitrary constant.

Remark 2. The above method can be used to evaluate P.I. in any problem. Since shorter methods depending upon the special form of function X are available (to be discussed later on), the above general method, however, must be used for problems in which X is of the forms sec ax, cosec ax, sec^2 ax, cosec^2 ax, tan ax, cot ax or any other form not covered by shorter methods (employed for special forms).

5.9 Corollary. If n is a positive integer, then $$\frac{1}{(D-\alpha)^n} e^{\alpha x} = \frac{x^n}{n!} e^{\alpha x}$$

Proof L.H.S.

$$= \frac{1}{(D-\alpha)^n} e^{\alpha x} = \frac{1}{(D-\alpha)^{n-1}} \frac{1}{D-\alpha} e^{\alpha x} = \frac{1}{(D-\alpha)^{n-1}} e^{\alpha x} \int e^{\alpha x} e^{-\alpha x} dx$$

[Using the theorem of Art. 5.8 with $X = e^{\alpha x}$]

$$= \frac{1}{(D-\alpha)^{n-1}} e^{\alpha x} x = \frac{1}{(D-\alpha)^{n-2}} \frac{1}{D-\alpha} xe^{\alpha x} = \frac{1}{(D-\alpha)^{n-2}} e^{\alpha x} \int xe^{\alpha x} e^{-\alpha x} dx$$

[Using the theorem of Art. 5.8 with $X = xe^{\alpha x}$]

$$= \frac{1}{(D-\alpha)^{n-2}} e^{\alpha x} \int x dx = \frac{1}{(D-\alpha)^{n-2}} e^{\alpha x} \cdot \frac{x^2}{2!} \qquad \text{... } (i)$$

$$= \frac{1}{(D-\alpha)^{n-3}} \frac{1}{D-\alpha} e^{\alpha x} \frac{x^2}{2!} = \frac{1}{(D-\alpha)^{n-3}} e^{\alpha x} \int \{e^{\alpha x} (x^2/2!)\, e^{-\alpha x}\}\, dx$$

[Using the theorem of Art. 5.8 with $X = e^{\alpha x}\, (x^2/2!)$]

$$= \frac{1}{(D-\alpha)^{n-3}} \frac{e^{\alpha x}}{2!} \int x^2 dx = \frac{1}{(D-\alpha)^{n-3}} \frac{x^3}{3!} e^{\alpha x} \qquad \text{... } (ii)$$

Continuing as before and noting (*i*) and (*ii*), we finally obtain

$$\frac{1}{(D-\alpha)^n}e^{\alpha x}=\frac{1}{(D-\alpha)^{n-n}}\frac{x^n}{n!}e^{\alpha x}=\frac{x^n}{n!}e^{\alpha x} \qquad \text{... }(iii)$$

5.10 Working rule of finding the particular integral (P.I.) *i.e.* $\dfrac{1}{f(D)}X$.

There are two following ways to obtain P.I.

Method I. The oeprator $1/f(D)$ may be factored into linear factors;

Then, $$\text{P.I.}=\frac{1}{D-\alpha_1}\cdot\frac{1}{D-\alpha_2}\cdot\frac{1}{D-\alpha_3}\cdots\frac{1}{D-\alpha_n}X$$

On operating with the first symbolic factor, beginning at the right, there is obtained (keeping the result (1) of theorem 5.8)

$$\text{P.I.}=\frac{1}{D-\alpha_1}\frac{1}{D-\alpha_2}\cdots\frac{1}{D-\alpha_{n-1}}e^{\alpha_n x}\int Xe^{-\alpha_n x}\,dx;$$

then, on operating with the second and remaining factors in succession, taking them from right to left, required P.I. can be obtained.

Method II. The operator $1/f(D)$ may be decomposed into its partial fractions, then

$$\text{P.I.}=\left[\frac{A_1}{D-\alpha_1}+\frac{A}{D-\alpha_2}+\ldots+\frac{A_n}{D-\alpha_n}\right]X = A_1e^{\alpha_1 x}\int Xe^{-\alpha_1 x}dx+\ldots+A_n e^{\alpha_n x}\int Xe^{-\alpha_n x}dx.$$

Of these two methods, the latter is generally used in practice.

5.11 Solved examples based on Art. 5.10

Ex. 1. *Solve $(D^2+a^2)\,y=\cot ax$.* **[Delhi Maths. (G) 2005]**

Sol. Here the auxiliary equation is $D^2+a^2=0$ so that $D=0\pm ia$.

$\therefore$ C.F. $= e^{ax}(c_1\cos ax+c_2\sin ax)=c_1\cos ax+c_2\sin ax$, c_1, c_2 being arbitrary constants

Now, $$\text{P.I.}=\frac{1}{D^2+a^2}\cot ax=\frac{1}{(D+ai)(D-ia)}\cot ax$$

$[\because D^2+a^2=D^2-(ia)^2=(D+ai)(D-ai)]$

$$=\frac{1}{2ia}\left[\frac{1}{D-ia}+\frac{1}{D+ia}\right]\cot ax, \text{ on resolving into partial fractions}$$

Now, $$\frac{1}{D-ia}\cot ax = e^{iax}\int e^{-iax}\cot ax\,dx=e^{iax}\int(\cos ax-i\sin ax)\frac{\cos ax}{\sin ax}dx$$

[$\because$ by Euler's theorem, $e^{-iax}=\cos ax-i\sin ax$]

$$=e^{iax}\int\left(\frac{\cos^2 ax}{\sin ax}-i\cos ax\right)dx=e^{iax}\int\left(\frac{1-\sin^2 ax}{\sin ax}-i\cos ax\right)dx$$

$$=e^{iax}\int(\operatorname{cosec} ax-\sin ax-i\cos ax)\,dx = e^{iax}[(1/a)\log\tan(ax/2)+(1/a)\cos ax-(i/a)\sin ax]$$

$= e^{iax}[(1/a)\log\tan(ax/2)+(1/a)(\cos ax-i\sin ax)]$

$= e^{iax}[(1/a)\log\tan(ax/2)+(1/a)e^{-iax}]$, by Euler's theorem

$$\therefore \qquad \frac{1}{D-ia}\cot ax=\frac{1}{a}\left[e^{iax}\log\tan\frac{ax}{2}+1\right]. \qquad \text{... }(2)$$

Replacing i by $-i$ in (2), $$\frac{1}{D-ia}\cot ax=\frac{1}{a}\left[e^{-iax}\log\tan\frac{ax}{2}+1\right] \qquad ...(3)$$

Using (2) and (3), (1) reduces to

$$\text{P.I.} = \frac{1}{2ia}\left[\frac{1}{a}\{e^{iax}\log\tan\frac{ax}{2}+1\}-\frac{1}{a}\{e^{-iax}\log\tan\frac{ax}{2}+1\}\right]$$

$$= \frac{1}{a^2}\cdot\frac{e^{iax}-e^{-iax}}{2i}\log\tan\frac{ax}{2}=\frac{1}{a^2}\sin ax\log\tan\frac{ax}{2}.$$

Hence the required general solution is $y=$ C.F. + P.I., i.e.,

$y=c_1\cos ax+c_2\sin ax+(1/a^2)\sin ax\log\tan(ax/2)$, where c_1 and c_2 are arbitrary constants.

Ex. 2. *Solve* $(D^2+a^2)y=\tan ax$. **(*Meerut 1996*)**

Sol. Here the auxiliary equation $D^2+a^2=0$ gives $D=\pm ia$.

$\therefore$ C.F. $= c_1\cos ax+c_2\sin ax$, c_1, c_2 is being arbitrary constants.

Now, $$\text{P.I.}=\frac{1}{D^2+a^2}\tan ax=\frac{1}{(D+ai)(D-ai)}\tan ax=\frac{1}{2ia}\left[\frac{1}{D-ia}-\frac{1}{D+ia}\right]\tan ax. \qquad ...(1)$$

Now, $$\frac{1}{D-ia}\tan ax=e^{iax}\int e^{-iax}\tan ax\,dx = e^{iax}\int(\cos ax-i\sin ax)\frac{\sin ax}{\cos ax}dx,$$ by Euler's theorem

$$= e^{iax}\int\left[\sin ax-i\,\frac{1-\cos^2 ax}{\cos ax}\right]dx=e^{iax}\int[\sin ax-i(\sec ax-\cos ax)]dx$$

$$= e^{iax}\left[-\frac{\cos ax}{a}-\frac{i}{a}\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)+i.\frac{\sin ax}{a}\right]=-\frac{e^{iax}}{a}\left[(\cos ax-i\sin ax)+i\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)\right]$$

$$= -\frac{e^{iax}}{a}\left[e^{-iax}+i\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)\right],$$ by Euler's theorem

Thus, $$\frac{1}{D-ia}\tan ax=-\frac{1}{a}\left[1+i\,e^{iax}\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)\right] \qquad ...(2)$$

Replacing i by $-i$ in (2), $$\frac{1}{D+ia}\tan ax=-\frac{1}{a}\left[1-ie^{-iax}\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)\right] \qquad ...(3)$$

$\therefore$ By (1), (2) and (3), we have

$$\text{P.I.}=\frac{1}{2ia}\left\{-\frac{i}{a}(e^{iax}+e^{-iax})\log\tan\left(\frac{\pi}{4}+\frac{ax}{2}\right)\right\} = -(1/a)^2\cos ax\log\tan(\pi/4+ax/2).$$

$\therefore$ The required solution is $y=$ C.F. + P.I., that is,

$$y=c_1\cos ax+c_2\sin ax-(1/a^2)\cos ax\log\tan(\pi/4+ax/2).$$

Ex. 3. *Solve* $(D^2+a^2)y=\sec ax$. **[Rohilkhand 1995, Purvanchal 2007, Agra 2006, Kanpur 1997, Delhi, Maths (G) 2006, Lucknow 1995]**

Sol. Here the auxiliary equation is $D^2+a^2=0$. so that $D=\pm ia$.

and hence C.F. $= c_1\cos ax+c_2\sin ax$, c_1, c_2 being arbitrary constants.

$$\text{P.I.}=\frac{1}{D^2+a^2}\sec ax=\frac{1}{2ia}\left[\frac{1}{D-ia}-\frac{1}{D+ia}\right]\sec ax. \qquad ...(1)$$

Now, $$\frac{1}{D-ia}\sec ax = e^{iax}\int e^{-iax}\sec ax\,dx = e^{iax}\int \frac{\cos ax - i\sin ax}{\cos ax}dx$$

$$= e^{iax}\int (1-i\tan ax)\,dx = e^{iax}\{x+(i/a)\log\cos ax\} \quad \text{... (2)}$$

Replacing i by $-i$ in (2), $$\frac{1}{D+ia}\sec ax = e^{-iax}[x-(i/a)\log\cos ax] \quad \text{... (3)}$$

$\therefore$ From (1), (2) and (3), we get

P.I. $= (1/2ia)\,[e^{iax}\{x+(1/a)\log\cos ax\} - e^{-iax}\{x-(i/a)\log\cos ax\}]$

$$= x\frac{(e^{iax}-e^{-iax})}{2ia} + \frac{1}{a^2}(\log\cos ax).\frac{(e^{iax}+e^{-iax})}{2}$$

$= (x/a)\sin ax + (1/a^2)\cos ax \log\cos ax$

$\therefore$ Hence the required solution is $y =$ C.F. + P.I., that is,

$y = c_1 \cos ax + c_2 \sin ax + (x/a)\sin ax + (1/a^2)\cos ax \log\cos ax.$

Ex. 4. *Solve (a)* $(d^2y/dx^2) + y = \sec^2 x$. **[Delhi Maths. (Prog) 2009]**

(b) $(d^2y/dx^2) + y = \operatorname{cosec}^2 x$.

Sol. (*a*) Let $D \equiv d/dx$. Then the given equation is $(D^2+1)\,y = \sec^2 x$.

Its auxiliary equation is $D^2 + 1 = 0$ so that $D = \pm i$ and C.F. $= c_1 \cos x + c_2 \sin x$.

and $$\text{P.I.} = \frac{1}{D^2+1}\sec^2 x = \frac{1}{(D-i)(D+i)}\sec^2 x = \frac{1}{2i}\left[\frac{1}{D-i} - \frac{1}{D+i}\right]\sec^2 x$$

Now, $$\frac{1}{D-i}\sec^2 x = e^{ix}\int e^{-ix}\sec^2 x\,dx = e^{ix}\int\frac{\cos x - i\sin x}{\cos^2 x}dx$$

$$= e^{ix}\int(\sec x - i\sec x\tan x)\,dx = e^{ix}[\log(\sec x+\tan x) - i\sec x] \quad \text{... (2)}$$

Replacing i by $-i$ in (2), $$\frac{1}{D+i}\sec^2 x = e^{-ix}[\log(\sec x+\tan x) + i\sec x] \quad \text{... (3)}$$

From (1) (2), and (3), we have

P.I. $= (1/2i)\,[e^{ix}\log(\sec x+\tan x) - ie^{ix}\sec x - e^{-ix}\log(\sec x+\tan x) - e^{-ix}\sec x]$

$$= \frac{e^{ix}-e^{-ix}}{2i}\log(\sec x+\tan x) - \frac{e^{ix}+e^{-ix}}{2}\sec x$$

$= \sin x\log(\sec x+\tan x) - \cos x\sec x = \sin x\log(\sec x+\tan x) - 1.$

Hence the required solution. $y = c_1\cos x + c_2\sin x + \sin x\log(\sec x+\tan x) - 1.$

(*b*) Do yourself **Ans.** $y = c_1\cos x + c_2\sin x - \cos x\log(\operatorname{cosec} x - \cot x) - 1.$

Exercise 5 (B)

Solve the following differential equations :

1. $(D^2+4)\,y = \tan 2x$ **Ans.** $y = c_1\cos 2x + c_2\sin x - (1/4)\times\cos 2x\log\tan(\pi/4 + x)$

2. $(D^2+1)\,y = \operatorname{cosec} x$. **Ans.** $y = c_1\cos x + c_2\sin x - x\cos x + \sin x\log\sin x$

3. (a) $(D^2+1)\,y = \sec x$. **Ans.** $y = c_1\cos x + c_2\sin x + x\sin x + \cos x\log\cos x$

3. (b) $(D^2+9)\,y = \sec 3x$. **[Meerut 2007; Rajasthan 2010]**

Ans. $y = c_1\cos 3x + c_2\sin 3x + (x/3)\times\sin 3x + (1/9)\times\cos 3x\log\cos 3x$

4. $(\mathrm{D}^2+a^2)\,y = \operatorname{cosec} ax$. **Ans.** $y = c_1\cos ax + c_2\sin ax + (1/a^2)\sin ax\log\sin ax - (x/a)\cos ax$

5. $d^2x/dy^2 + 4x = \tan 2y$. **Ans.** $x = c_1 \cos 2y + c_2 \sin 2y - (1/4) \times \cos 2y \tan (y + \pi/4)$

Hint: It is same as Ex. 1 by interchanging x and y.

6. $(D^2 - 3D + 2)\, y = \sin e^{-x}$ **Ans.** $y = c_1 e^x + c_2 e^{-x} - e^{2x} \sin e^{-x}$

7. $(D^2 - 9D + 18)y = e^{e^{-3x}}$ **Ans.** $y = c_1 e^{2x} + c_2 e^{6x} + (1/9) \times e^{6x}\, e^{e^{-3x}}$

5.12 Short methods for finding the particular integral of given equation $f(D)\, y = X$, when X is of certain special form.

The general method of finding P.I. given in Art. 5.8 leads to cumbersome calculations in most of the problems. However, the P.I. can be obtained by methods that are shorter than general methods provided X is one of the following special forms :

Form I. When $X = e^{ax}$, where a is any constant.

Form II. When $X = \sin ax$ or $\cos ax$

Form III. When $X = x^n$ or *a polynomial* $a_0 x^n + a_1 x^{n-1} + \ldots\ldots\ldots\ldots + a_{n-1} x + a_n$ where n is any positive integer

Form IV. When $X = x^n V$, where V is a function of x and n is positive integer.

5.13 Short method of finding P.I. when $X = e^{ax}$, where 'a' is constant.

Formula IA. $$\text{P.I.} = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}, \text{ when } f(a) \neq 0.$$

Formula IIA. $$\text{P.I.} = \frac{1}{(D-a)^n} e^{ax} = \frac{x^n}{n!} e^{ax}, \; n = 1, 2, 3. \ldots$$

Proof of formula IA. Let $f(D) = D^n + c_1 D^{n-1} + c_2 D^{n-2} + \ldots + c_{n-1} D + c_n$.

But $D e^{ax} = a\, e^{ax}$, $D^2 e^{ax} = a^2 e^{ax}$, ..., $D^{n-1} e^{ax} = a^{n-1} e^{ax}$, $D^n e^{ax} = a^n e^{ax}$.

$$\therefore \quad f(D)\, e^{ax} = (D^n + c_1 D^{n-1} + \ldots c_{n-1} D + c_n)\, e^{ax}$$
$$= (a^n + c_1 a^{n-1} + \ldots + c_{n-1} a + c_n)\, e^{ax} = f(a)\, e^{ax}$$

Thus, $$f(D)\, e^{ax} = f(a)\, e^{ax}.$$

Operating upon both sides by $1/f(D)$, we get

$$\frac{1}{f(D)} f(D) \cdot e^{ax} = \frac{1}{f(D)} f(a) e^{ax} \quad \text{or} \quad e^{ax} = f(a) \frac{1}{f(D)} e^{ax}$$

or $$\frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}, \text{ provide } f(a) \neq 0.$$

Proof of formula IIA. Refer corollary in Art. 5.8.

5.14 Working rule for finding P.I. of $f(D)\, y = X$ when $X = e^{ax}$

Step 1. If $f(a) \neq 0$, then use formula IA of Art. 5.13. Note that we put a for D in $f(D)$ provided $f(a) \neq 0$.

Step II. If $f(a) = 0$, then the following two cases arise.

Case (*i*) If $f(D) = (D - a)^n$, where $n = 1, 2, 3, \ldots$ Then we shall use formula IIA of Art. 5.13.

Case (*ii*) If $f(D) = (D - a)^r \phi(D)$, where $\phi(a) \neq 0$ and $r = 1, 2, 3, \ldots$ Then we use formulas IA and IIA of Art. 5.13 in succession as below :

$$\text{P.I.} = \frac{1}{(D-a)^r\, \phi(D)} e^{ax} = \frac{1}{(D-a)^r} \frac{1}{\phi(a)} e^{x}, \text{ using formula IA of Art. 5.13}$$

$$= \frac{1}{\phi(a)} \frac{1}{(D-a)^r} e^{ax} = \frac{1}{\phi(a)} \frac{x^r}{r!} e^{ax}, \text{ using formula IIA of Art. 5.13} \qquad \text{... (1)}$$

Alternative form of above result (1). Since $f(D) = (D - a)^r \phi(D)$, $f^{(r)}(D) = r!\, \phi(D)$ + terms containing $(D - a)$ and its higher powers. So $f^{(r)}(a) = r!\, \phi(a)$. Hence, (1) takes new form

$$P.I. = (x^r e^{ax}) / f^{(r)}(a) = x^r \left[f^{(r)}(a) \right]^{-1} e^{ax} \qquad \text{... (2)}$$

Note. As a particular case, if $a = 0$ so that $X = e^{0.x} = 1$, then formulae I and II of Art. 5.13 take the following forms :

Formula IB. $\dfrac{1}{f(D)} 1 = \dfrac{1}{f(D)} e^{0.x} = \dfrac{1}{f(0)} e^{0.x} = \dfrac{1}{f(0)}$, if $f(0) \neq 0$

Formula IIB. $\dfrac{1}{D^n} 1 = \dfrac{1}{(D-0)^n} e^{0.x} = \dfrac{x^n}{n!} e^{0.x} = \dfrac{x^n}{n!}$, $n = 1, 2, 3 \ldots$

5.15 Solved examples based on working rule 5.14

Ex. 1. *Solve the following differential equations :*

(*a*) $(D^2 - 3D + 2)\, y = e^{3x}$. **[I.A.S. (Preliminary) 1993, Meerut 1994]**

(*b*) $(4D^2 + 12D + 9)\, y = 144\, e^{-3x}$. **[Rohilkhand 1992, 93]**

(*c*) $[D^2 + 2pD + (p^2 + q^2)]\, y = e^{ax}$.

(*d*) $D^2 (D + 1)^2 (D^2 + D + 1)^2 y = e^x$

Sol. (*a*) Here the auxiliary equation is $D^2 - 3D + 2 = 0$ so that $D = 1, 2$

$\therefore$ C.F. $= c_1 e^x + c_2 e^{2x}$, c_1, c_2 being arbitrary constants.

and P.I. $= \dfrac{1}{D^2 - 3D + 2} e^{3x} = \dfrac{1}{3^3 - (3 \times 3) + 2} e^{3x} = \dfrac{1}{2} e^{3x}$

$\therefore$ The required general solution is $y = c_1 e^x + c_2 e^{2x} + (1/2)\, e^{3x}$.

(*b*) Here the A.E. is $(2D + 3)^2 = 0$ so that $D = -3/2, -3/2$

$\therefore$ C.F. $= (c_1 + c_2 x)\, e^{-3x/2}$, c_1, c_2 being arbitrary constants.

and P.I. $= \dfrac{1}{4D^2 + 12D + 9} 144 e^{-3x} = 144 \dfrac{1}{(2D+3)^2} e^{-3x} = \dfrac{144}{(-6+3)^2} e^{-3x} = 16 e^{-3x}$

Hence the required solution is $y = (c_1 + c_2 x)\, e^{-3x/2} + 16\, e^{-3x}$.

(*c*) Here the auxiliary equation is $D^2 + 2pD + (p^2 + q^2) = 0$

Solving $D = \dfrac{-2p \pm \sqrt{4p^2 - 4(p^2 + q^2)}}{2} = -p \neq iq$ (complex roots)

$\therefore$ C.F. $= e^{-px} (c_1 \cos qx + c_2 \sin qx)$, c_1, c_2 being arbitrary constants

and P.I. $= \dfrac{1}{D^2 + 2pD + (p^2 + q^2)} e^{ax} = \dfrac{1}{a^2 + 2pa + p^2 + q^2} e^{ax} = \dfrac{e^{ax}}{(p+a)^2 + q^2}$

$\therefore$ Required solution is $y = e^{-p} (c_1 \cos qx + c_2 \sin qx) + e^{ax}/\{(p + a)^2 + q^2\}$

(*d*) Here the auxiliary equation is $D^2 (D + 1)^2 (D^2 + D + 1)^2 = 0$

Solving, we get $D = 0, \; 0, \; -1, \; -1, \; -(1/2) \pm i\, (\sqrt{3}/2), \; -(1/2) \pm i\, (\sqrt{3}/2)$.

$\therefore$ C.F. $= (c_1 + c_2x)\, e^{0x} + (c_3 + c_4x)\, e^{-x} + e^{-x/2}\, [(c_5 + c_6x) \cos(\sqrt{3}x/2)]$

$+ [(c_7 + c_8x) \sin(\sqrt{3}x/2)]$, $c_1, c_2, \ldots, c_7, c_8$ being arbitrary constants.

and
$$\text{P.I.} = \frac{1}{D^2 (D+1)^2 (D^2 + D + 1)^2} e^x = \frac{1}{1^2 (1+1)^2 (1^2 + 1 + 1)^2} e^x = \frac{e^x}{36}$$

Hence the required solution is $y = c_1 + c_2x + (c_3 + c_4x)\, e^{-x}$

$+ e^{-x/2}\, [(c_5 + c_6x) \cos(\sqrt{3}x/2)] + [(c_7 + c_8x) \sin(\sqrt{3}x/2)] + (1/36)\, e^x$.

Ex. 2. *Solve the following differential equations* :

(a) $(4D^2 - 12D + 9)\, y = 144\, e^{3x/2}$.

(b) $(D^2 + 4D + 4)\, y = e^{2x} - e^{-2x}$ or $(D^2 + 4D + 4)\, y = 2 \sinh 2x$.

Sol. (*a*) Here the auxiliary equation is $4D^2 - 12D + 9 = 0$.

or $(2D - 3)^2 = 0$ so that $D = 3/2, 3/2$.

$\therefore$ C.F. $= (c_1 + c_2x)\, e^{3x/2}$, c_1, c_2 being arbitrary constants.

and
$$\text{P.I.} = \frac{1}{(4D^2 - 12D + 9)} 144\, e^{3x/2} = 144 \frac{1}{(2D-3)} e^{3x/2} = \frac{144}{4} \frac{1}{\{D - (3/2)\}^2} e^{3x/2}$$

$$= 36 \cdot \frac{x^2}{2!} e^{3x/2}, \text{ as } \frac{1}{(D-a)^n} e^{ax} = \frac{x^n}{n!} e^{ax}$$

$\therefore$ Solution is $y = (c_1 + c_2x)\, e^{3x/2} + 18x^2\, e^{3x/2}$. c_1, c_2, being arbitrary constants.

(*b*) Given equation is $(D^2 + 4D + 4)\, y = 2 \sinh 2x$.

or $(D + 2)^2\, y = e^{2x} - e^{-2x}$, as $\sinh 2x = (e^{2x} - e^{-2x})/2$.

Here the auxiliary equation is $(D + 2)^2 = 0$ so that $D = -2, -2$

$\therefore$ C.F. $= (c_1 + c_2x)\, e^{-2x}$, c_1, c_2 being arbitrary constants.

and
$$\text{P.I.} = \frac{1}{(D+2)^2} (e^{2x} - e^{-2x}) = \frac{1}{(D+2)^2} e^{2x} - \frac{1}{(D+2)^2} e^{-2x}$$

$$= \frac{1}{(2+2)^2} e^{2x} - \frac{x^2}{2!} e^{-2x} = \frac{1}{16} e^{2x} - \frac{x^2}{2} e^{-2x}.$$

Hence the required solution is $y = (c_1 + c_2x)\, e^{-2x} + (1/16) \times e^{2x} - (x^2/2) \times e^{-2x}$

Ex. 3. *Solve the following differential equations* :

(a) $(D + 2)\, (D - 1)^3\, y = e^x$.

(b) $(D - 1)^2\, (D^2 + 1)^2\, y = e^x$

Sol. (*a*) Here auxiliary equation is $(D + 2)\, (D - 1)^3 = 0$ so that $D = -2, 1, 1, 1$.

$\therefore$ C.F. $= c_1e^{-2x} + (c_2 + c_3x + c_4x^2)\, e^x$, c_1, c_2, c_3 being arbitrary constants

$$\text{P.I.} = \frac{1}{(D-1)^3} \frac{1}{D+2} e^x = \frac{1}{(D-1)^3} \cdot \frac{1}{1+2} e^x$$

$$= \frac{1}{3} \frac{1}{(D-1)^3} e^x = \frac{1}{3} \cdot \frac{x^3}{3!} e^x, \quad \text{as} \quad \frac{1}{(D-a)^n} e^{ax} = \frac{x^n}{n!} e^{ax}$$

$\therefore$ The required solution is $y = c_1e^{-2x} + (c_2 + c_3x + c_4x^2)\, e^x + (1/18)\, x^3\, e^x$.

(*b*) **Ans.** $y = (c_1 + c_2x)\, e^x + (c_3 + c_4x) \cos x + (c_5 + c_6x) \sin x + (x^2/8)\, e^x$.

Ex. 4. *Solve* : (*a*) $(D^2 + D - 2)\, y = e^x$ **[I.A.S. (Prel.) 1997]**

(*b*) $(D - 1)(D^2 - 2D + 2)\, y = e^x$ **[I.A.S. 2002]**

(*c*) $(D^3 - D)\, y = e^x + e^{-x}$ or $(D^3 - D)\, y = 2 \cosh x$ **[I.A.S. Prel. 1993]**

Sol. (*a*) Here the auxiliary equation is $D^2 + D - 2 = 0$ so that $D = -2, 1$

Hence C.F. $= c_1e^{-2x} + c_2e^x$, c_1, c_2 being arbitrary constants.

$$\text{P.I.} = \frac{1}{D^2 + D - 2} e^x = \frac{1}{(D-1)(D+2)} e^x = \frac{1}{D-1} \cdot \frac{1}{1+2} e^x = \frac{1}{3} \frac{1}{D-1} e^x = \frac{1}{3} \times \frac{x}{1!} e^x = \frac{x}{3} e^x$$

So required solution is $y = c_1e^{-2x} + c_2e^x + (x/3) \times e^x$

(*b*) Here the auxiliary equation is $(D - 1)(D^2 - 2D + 2) = 0$

Hence $D = 1, \{2 \pm (4 - 8)^{1/2}\}/2$ *i.e.* $D = 1, 1 \pm i$

Hence C.F. $= c_1e^x + e^x (c_2 \cos x + c_3 \sin x)$, c_1, c_2, c_3 being arbitrary constants

and
$$\text{P.I.} = \frac{1}{D-1} \cdot \frac{1}{D^2 - 2D + 2} e^x = \frac{1}{D-1} \frac{1}{1-2+2} e^x = \frac{1}{D-1} e^x = \frac{x}{1!} e^x$$

So the required solution is $y = c_1e^x + e^x (c_2 \cos x + c_3 \sin x) + x\, e^x$

i.e. $y = e^x (c_1 + c_2 \cos x + c_3 \sin x + x)$.

(*c*) Hence the auxiliary equation is $D^3 - D = 0$ so that $D = 0, 1, -1$

Hence C.F. $= c_1\, e^{0.x} + c_2\, e^x + c_2\, e^{-x} = c_1 + c_2\, e^x + c_3\, e^{-x}$

$$\text{P.I.} = \frac{1}{D^3 - D}(e^x + e^{-x}) = \frac{1}{D\,(D-1)\,(D+1)}(e^x + e^{-x})$$

$$= \frac{1}{D-1} \frac{1}{D(D+1)} e^x + \frac{1}{D+1} \frac{1}{D(D-1)} e^{-x} = \frac{1}{D-1} \frac{1}{1 \times 2} e^x + \frac{1}{D+1} \frac{1}{(-1) \times (-2)} e^{-x}$$

$$= \frac{1}{2} \frac{1}{D-1} e^x + \frac{1}{2} \frac{1}{D+1} e^{-x} = \frac{1}{2} \frac{x}{1!} e^x + \frac{1}{2} \frac{x}{1!} e^{-x} = \frac{x(e^x + e^{-x})}{2}$$

$\therefore$ Required solution is $y = c_1 + c_2\, e^x + c_3e^{-x} + (x/2) \times (e^x + e^{-x})$

Ex. 5. *Solve (a)* $(D^4 - 2D^3 + 5D^2 - 8D + 4)\, y = e^x$ **[I.A.S. 1991]**

(*b*) $(D^4 + D^3 + D^2 - D - 2)\, y = e^x$ **[Bhopal 1993; Lucknow 1994]**

Sol. (*a*) A.E. is $D^4 - 2D^3 + 5D^2 - 8D + 4 = 0$ or $(D - 1)(D^3 - D^2 + 4D - 4) = 0$

or $(D - 1)[D^2 (D - 1) + 4 (D - 1)] = 0$ or $(D - 1)^2 (D^2 + 4) = 0$ so that $D = 1, 1, \pm 2i$.

$\therefore$ C.F. $= (c_1 + c_2x)\, e^x + c_3 \cos 2x + c_4 \sin 2x$, c_1, c_2, c_3, c_4 being arbitrary constants.

and
$$\text{P.I.} = \frac{1}{(D-1)^2(D^2+4)} e^x = \frac{1}{(D-1)^2} \frac{1}{5} e^x = \frac{1}{5} \frac{x^2}{2!} e^x$$

The solution is $y = (c_1 + c_2x)\, e^x + c_3 \cos 2x + c_4 \sin 2x + (x^2\, e^x)/10$.

(*b*) The given equation is $(D^4 + D^3 + D^2 + D - 2)\, y = e^x$.

or $\{D^3 (D - 1) + 2D^2 (D - 1) + 3D (D - 1) + 2 (D - 1)\}\, y = e^x$

or $(D - 1)(D^3 + 2D^2 + 3D + 2)\, y = e^x$

or $(D - 1)\, \{(D^2 (D + 1) + D (D + 1) + 2 (D + 1)\}\, y = e^x$

or $(D - 1)(D + 1)(D^2 + D + 2)\, y = e^x$

Here auxiliary equation is $(D - 1)(D + 2)(D^2 + D + 2) = 0$

$\therefore$ $D = 1, -1, \{-1 \pm (1-8)^{1/2}\}/2$ or $D = 1, -1, -(1/2) \pm i(\sqrt{7}/2)$

and C.F. $= c_1 e^x + c_2 e^{-x} + e^{-x/2}\{c_3 \cos(\sqrt{7}x/2) + c_4 \sin(\sqrt{7}x/2)\}$,
c_1, c_2, c_3 and c_4 being arbitrary constants.

Also, $$\text{P.I.} = \frac{1}{(D-1)(D+1)(D^2+D+2)}e^x = \frac{1}{(D-1)}\frac{1}{(1+1)(1+1+2)}e^x$$

$$= \frac{1}{8}\frac{1}{D-1}e^x = \frac{1}{8}\frac{x}{1!}e^x = \frac{x}{8}e^x, \quad \text{as} \quad \frac{1}{(D-a)^n}e^{ax} = \frac{x^n e^{ax}}{n!}$$

Required solution is $y = c_1 e^x + c_2 e^{-x} + e^{-x/2}\{c_3 \cos(\sqrt{7}x/2) + c_4 \sin(\sqrt{7}x/2)\} + (x/8)e^x$.

Ex. 6. *Solve : (a) $D^2 - 3D + 2)\ y = e^x + e^{2x}$.* **[Delhi Maths (G) 1996]**

(b) $(D^2 - 3D + 2)\ y = \cosh x$. **[I.A.S. Prel. 2005]**

(c) $(D^3 - 5D^2 + 7D - 3)\ y = e^{2x} \cosh x$

(d) $(d^3y/dx^3) - y = (e^x + 1)^2$. **[Delhi Maths (H) 1993, 1996]**

Sol. (*a*) Here the auxiliary equation is $D^3 - 3D + 2 = 0$ so that $D = 1, 2.$

$\therefore$ C.F. $= c_1 e^x + c_2 e^{2x}$, c_1, c_2 being arbitrary constants.

and $$\text{P.I.} = \frac{1}{D^2 - 3D + 2}(e^x + e^{2x}) = \frac{1}{(D-1)(D-2)}e^x + \frac{1}{(D-2)(D-1)}e^{2x}$$

$$= \frac{1}{D-1}\ \frac{1}{1-2}e^x + \frac{1}{D-2}\ \frac{1}{2-1}e^{2x} = -\frac{x}{1!}e^x + \frac{x}{1!}e^{2x}$$

Hence the general solution is $y = c_1 e^x + c_2 e^{2x} - xe^x + xe^{2x}$.

(*b*) Here auxiliary equation is $D^2 - 3D + 2 = 0$ so that $D = 1, 2$

$\therefore$ C.F. $= c_1 e^x + c_2 e^{2x}$, c_1, c_2 being arbitrary constants

and $$\text{P.I} = \frac{1}{D^2 - 3D + 2}\cosh x = \frac{1}{(D-1)(D-2)}\frac{(e^x + e^{-x})}{2}$$

$$= \frac{1}{2}\frac{1}{(D-1)(D-2)}e^x + \frac{1}{2}\frac{1}{(D-1)(D-2)}e^{-x}$$

$$= \frac{1}{2}\frac{1}{D-1}\frac{1}{1-2}e^x + \frac{1}{2}\frac{1}{(-2)\times(-3)}e^{-x} = -\frac{1}{2}\frac{1}{D-1}e^x + \frac{1}{2}\times\frac{1}{6}e^{-x}$$

$$= -\frac{1}{2}\times\frac{x}{1!}e^x + \frac{1}{12}e^{-x} = -\frac{x}{2}e^x + \frac{1}{12}e^{-x}$$

$\therefore$ the required solution is $y = c_1 e^x + c_2 e^{2x} - (x/2) \times e^x + (1/12) \times e^{-x}$

(*c*) Re-writing, the given equation becomes

$\{D^2(D-1) - 4D(D-1) + 3(D-1)\}y = e^{2x} \cosh x$ or $(D-1)^2(D-3)\ y = e^{2x} \cosh x$.

Here auxiliary equation is $(D-1)^2(D-3) = 0$ so that $D = 1, 1, 3.$

$\therefore$ C.F. $= (c_1 + c_2 x)\ e^x + c_3 e^{3x}$, c_1, c_2, c_3 being arbitrary constants

$$\text{P.I.} = \frac{1}{(D-1)^2(D-3)}e^{2x}\cosh x = \frac{1}{(D-1)^2(D-3)}\frac{e^{2x}(e^x + e^{-x})}{2}$$

$$= \frac{1}{2}\ \frac{1}{D-3}\ \frac{1}{(D-1)^2}e^{3x} + \frac{1}{2}\ \frac{1}{(D-1)^2}\ \frac{1}{D-3}e^x$$

$$= \frac{1}{8}\ \frac{1}{D-3}e^{3x} - \frac{1}{4}\ \frac{1}{(D-1)^2}e^x = \frac{1}{8}\ \frac{x}{1!}e^{3x} - \frac{1}{4}\ \frac{x}{2!}e^x$$

So the required solution is $y = (c_1 + c_2 x)\, e^x + c_3 e^{3x} + (x/8)\, e^{3x} - (x^2/8)\, e^x$.

(*d*) Let $D \equiv d/dx$. Then the given equation reduces to

$(D^3 - 1)\, y = (e^x + 1)^2$ or $(D - 1)(D^2 + D + 1)\, y = e^{2x} + 2e^x + 1$.

Here auxiliary equation is $(D - 1)(D^2 + D + 1) = 0$ so that $D = 1, -(1/2) \pm i\ (\sqrt{3}/2)$.

$\therefore$ C.F. $= c_1 e^x + e^{-(x/2)} \left[c_2 \cos(\sqrt{3}x/2) + c_3 \sin(\sqrt{3}x/2)\right]$

$$\text{P.I.} = \frac{1}{D^3 - 1}(e^{2x} + 2e^x + 1) = \frac{1}{D^3 - 1} e^{2x} + 2\frac{1}{D^3 - 1} e^x + \frac{1}{D^3 - 1} e^{0.x}$$

$$= \frac{1}{2^3 - 1} e^{2x} + 2\frac{1}{(D-1)(D^2 + D + 1)} e^x + \frac{1}{(0)^3 - 1} e^{0.x}$$

$$= \frac{1}{7} e^{2x} + 2\frac{1}{D-1}\ \frac{1}{1^2 + 1 + 1} e^x - 1 = \frac{1}{7} e^{2x} + \frac{2}{3} \cdot \frac{x}{1!} e^x - 1$$

So solution is $y = c_1 e^x + e^{-(x/2)}\, [c_2 \cos(\sqrt{3}x/2) + c_3 \sin(\sqrt{3}x/2)] + (1/7) \times e^{2x} + (2x/3) \times e^x - 1$, $c_1\ c_2, c_3$ being arbitrary constants.

Ex. 7. *If* $(d^2x/dt^2) + (g/b)\,(x - a) = 0$, *(a, b and g being constants) and* $x = a'$ *and* $dx/dt = 0$ *when* $t = 0$, *show that* $x = a + (a' - a) \cos t\sqrt{(g/b)}$. **[I.A.S. 1994, Kurushetra 1994]**

Sol. With $D \equiv d/dt$, given equation is $\{D^2 + (g/b)\}\, x = ga/b$. ... (1)

Here auxiliary equation $D^2 + g/b = 0$ gives $D = 0 \pm i\sqrt{(g/b)}$.

$\therefore$ C.F. $= c_1 \cos t\sqrt{(g/b)} + c_2 \sin t\sqrt{(g/b)}$, c_1, c_2, being arbitrary constants

$$\text{P.I.} = \frac{1}{D^2 + (g/b)} \frac{ga}{b} = \frac{ga}{b} \frac{1}{D^2 + (g/b)} e^{0.t} = \frac{ga}{b} \frac{1}{0 + (g/b)} e^{0.t} = a.$$

So general solution is $x = c_1 \cos t\sqrt{(g/b)} + c_2 \sin t\sqrt{(g/b)} + a$. ...(2)

From (2), $dx/dt = -\ c_1\sqrt{(g/b)} \sin t\sqrt{(g/b)} + c_2\sqrt{(g/b)} \cos t\sqrt{(g/b)}$ ...(3)

Given $x = a'$ when $t = 0$. So (2) $\Rightarrow a' = c_1 + a$ or $c_1 = a' - a$. ...(4)

Given $dx/dt = 0$ when $t = 0$. So (3) $\Rightarrow 0 = c_2\sqrt{g/b}$ or $c_2 = 0$. ...(5)

Substituting the values of c_1 and c_2 in (2), the required solution is

$x = (a' - a) \cos t\sqrt{g/b} + a$, as required.

Ex. 8. *Solve* $(D^2 - 6D + 8)\, y = (e^{2x} + 1)^2$. **[Delhi Maths (G) 2006]**

Sol. Here auxiliary equation is $D^2 - 6D + 8 = 0$ giving $D = 2, 4$.

Hence C.F. $= c_1 e^{2x} + c_2 e^{4x}$, c_1, c_2 being arbitrary constants.

$$\text{P.I.} = \frac{1}{D^2 - 6D + 8}(e^{2x} + 1)^2 = \frac{1}{(D-2)(D-4)}(e^{4x} + 2e^{2x} + 1)$$

$$= \frac{1}{(D-4)(D-2)} e^{4x} + 2\frac{1}{(D-2)(D-4)} e^{2x} + \frac{1}{(D-2)(D-4)} e^{0.x}$$

$$= \frac{1}{D-4}\ \frac{1}{4-2} e^{4x} + 2\frac{1}{D-2}\ \frac{1}{2-4} e^{2x} + \frac{1}{(0-2)(0-4)} e^{0.x}$$

$$= \frac{1}{2}\frac{x}{1!}e^{4x} - \frac{x}{1!}e^{2x} + \frac{1}{8} = \frac{1}{8}(4xe^{4x} - 8xe^{2x} + 1).$$

$\therefore$ Required solution is $y = c_1 e^{2x} + c_2 e^{4x} + (1/8) \times (4\, xe^{4x} - 8xe^{2x} + 1)$.

Ex. 9. *Find the solution of the equation $(D^2 - 1) = 1$, which vanishes when $x = 0$ and tends to a finite limit as $x \to -\infty$. D stands for d/dx.*

Sol. Here auxiliary equation. is $D^2 - 1 = 0$ so that $D = 1, -1$.

$\therefore$ C.F. $= c_1 e^x + c_2 e^{-x}$, c_1, c_2 being arbitrary constants.

and P.I. $= \dfrac{1}{D^2 - 1} 1 = \dfrac{1}{D^2 - 1} e^{0.x} = \dfrac{1}{0^2 - 1} e^{0.x} = -1.$

So the general solution is $y = c_1 e^x + c_2 e^{-x} - 1$. ... (1)

Given $y = 0$ when $x = 0$. So (1) $\Rightarrow 0 = c_1 + c_2 - 1$ or $c_1 + c_2 = 1$. ... (2)

Multiplying both sies of (1) by e^x, $ye^x = c_1 e^{2x} + c_2 - e^x$. ... (3)

We know that $e^x = 0$ as $x \to -\infty$ ... (4)

Taking limit of both sides of (3) as $x \to -\infty$ and using (4) and the given fact that y in finite, we get (finite) $\times\, 0 = c_1 \times 0 + c_2 - 0$ so that $c_2 = 0$. ... (5)

Solving (2) and (5), $c_1 = 1$, $c_2 = 0$. Hence, from (1), $y = e^x - 1$, which is the required solution.

Exercise 5(C)

Solve the following differential equations.

1. $(D - 3)^2 y = 2e^{4x}$ **[Guwahati 2007]** **Ans.** $y = (c_1 + c_2 x)\, e^{3x} + 2e^{4x}$
2. $d^2y/dx^2 - 3\,(dy/dx) + 2y = e^{5x}$. **[Merrut 1994]** **Ans.** $y = c_1 e^x + c_2 e^{2x} + (1/12)\, e^{5x}$
3. $(D^2 + D + 1)\, y = e^{-x}$ **Ans.** $y = e^{-x/2}\, [c_1 \cos (x\sqrt{3}/2) + c_2 \sin (x\sqrt{3}/2)\,] + e^{-x}$
4. $(D^2 + 5D + 6)\, y = e^{2x}$. **Ans.** $y = c_1 e^{-2x} + c_2 e^{-3x} + (1/20) e^{2x}$
5. $(D^2 - 1)\, y = \cosh x$. **[Utkal 2003; I.A.S. 2008]** **Ans.** $y = c_1 e^x + c_2 e^{-x} + (x/2) \times \sinh x$
6. $(D^3 + 3D^2 + 3D + 1)\, y = e^{-x}$. **[Pune 2010]** **Ans.** $y = (c_1 + c_2 x + c_3 x^2)\, e^{-x} + (x^3/6)\, e^{-x}$
7. $(D^3 - D^2 - 4D + 4)\, y = e^{3x}$. **Ans.** $y = c_1 e^x + c_2 e^{2x} + c_3 e^{-2x} + (1/10)\, e^{3x}$
8. $(D^3 + 1)\, y = (e^x + 1)^2$. **Ans.** $y = c_1\, e^{-x} + e^{x/2}\, [c_2 \cos\, (x\sqrt{3}/2) + c_3 \sin\, (x\sqrt{3}/2)\,] + 1 + e^x + (1/9) e^{2x}$
9. $(D^2 - 2k\, D + k^2)\, y = e^{kx}$. **Ans.** $y = (c_1 + c_2 x)\, e^{kx} + (x^2/2)\, e^{kx}$
10. $(D^2 - 3D + 2)\, y = e^x$, given $y = 3$ and $dy/dx = 3$ when $x = 0$. **Ans.** $y = 2e^x + e^{2x} - xe^x$
11. $(D^2 - a^2)\, y = \cosh ax$. **Ans.** $y = c_1 e^{ax} + c_2 e^{-ax} + (x/2a) \sinh ax$
12. $(D^2 + 4D + 4)\, y = 2 \sinh 2x$. **[Garhwal 2010]** **Ans.** $y = (c_1 + c_2 x)\, e^{-2x} + (116) \times e^{2x} - (x^2/2) \times e^{-2x}$

5.16 Short method of finding P.I. when $X = \sin ax$ or $X = \cos ax$.

Case I. When $f(D)$ can be expressed as $\phi\,(D^2)$ and $\phi\,(-a^2) \neq 0$, we shall use the following formulas
$$\frac{1}{\phi(D^2)} \sin ax = \frac{1}{\phi(-a^2)} \sin ax \quad \text{and} \quad \frac{1}{\phi(D^2)} \cos ax = \frac{1}{\phi(-a^2)} \cos ax$$

Thus, the rule is to replace D^2 by $-a^2$.

Proof of the above formula. By successive differentiation, we have

$D \sin ax = a \cos ax$

$D^2 \sin ax = -a^2 \sin ax \Rightarrow (D^2)^1 \sin ax = (-a^2)^1 \sin ax$ ... (A_1)

$D^3 \sin ax = -a^3 \cos ax$

$D^4 \sin ax = a^4 \sin ax \Rightarrow (D^2)^2 \sin ax = (-a^2)^2 \sin ax$ ... (A_2)

....

$D^{2n} \sin ax = -a^{2n} \sin ax \Rightarrow (D^2)^n \sin ax = (-a^2)^n \sin ax$. ... ($A_n$)

Let $\qquad \phi(D^2) = (D^2)^n + a_1 (D^2)^{n-1} + \ldots + a_{n-1}(D^2)^1 + a_n.$ ... (1)

Then, from (A_1), (A_2), ..., (A_n) and (1), it follows that $\phi(D)^2 \sin ax = \phi(-a^2) \sin ax$. ... (2)

Operating upon both sides of (2) by $1/\phi(D^2)$, we have

$$\frac{1}{\phi(D^2)}\phi(D^2)\sin ax = \frac{1}{\phi(D^2)}\phi(-a^2)\ \sin ax \qquad \text{or} \qquad \sin ax = \phi(-a^2)\ \frac{1}{\phi(D^2)}\sin ax$$

Dividing both sides by $\phi(-a^2)$ which is not zero, we get

$$\frac{1}{\phi(D^2)}\sin ax = \frac{1}{\phi(-a^2)}\sin ax, \text{ provided } \phi(-a^2) \neq 0.$$

Similarly, we have $\qquad \dfrac{1}{\phi(D^2)}\cos ax = \dfrac{1}{\phi(-a^2)}\cos ax$, provided $\phi(-a^2) \neq 0$.

An important sub case. If $f(D)$ contains odd powers also, it can be put in the form $f(D) = f_1(D^2) + D f_2(D^2)$, where $f_1(-a^2) \neq 0$ and $f_2(-a^2) \neq 0$. Then

$$\text{P.I.} = \frac{1}{f_1(D^2) + D f_2(D^2)}\sin ax = \frac{1}{f_1(-a^2) + D\, f_2(-a^2)}\sin ax$$

[Use case I so that replace D^2 by $-a^2$]

$$= \frac{1}{p+qD}\sin ax \text{, where } p = f_1(-a^2) \text{ and } q = f_2(-a^2)$$

$$= (p-qD)\cdot\frac{1}{(p-qD)(p+qD)}\sin ax = (p-qD)\frac{1}{p^2 - q^2D^2}\sin ax$$

$$= (p-qD)\frac{1}{p^2 - q^2(-a^2)}\sin ax = \frac{1}{p^2+q^2a^2}(p\sin ax - qD\sin ax)$$

$$= \frac{1}{p^2+q^2a^2}\ (p \sin ax - qa \cos ax), \text{ as } D \sin ax = \frac{d}{dx}\ \sin ax.$$

Similarly, $\qquad \text{P.I.} = \dfrac{1}{f_1(D^2) + D\, f_2(D^2)}\cos ax \qquad$ can be evaluated.

Case II. When $f(D)$ can be expressed as $\phi(D^2)$ where $\phi(-a^2) = 0$.

Then we shall use the following formula (for proof refer Art. 5.20).

$$\frac{1}{f(D)}e^{ax}V = e^{ax}\ \frac{1}{f(D+a)}V \text{, where } V \text{ is a function of } x \qquad \ldots (i)$$

The above result says that e^{ax} which is on the right of $1/f(D)$ may be taken out to the left provided D is replaced by $D + a$.

We shall now evaluate $\dfrac{1}{D^2+a^2}\sin ax$ and $\dfrac{1}{D^2+a^2}\cos ax$. **[Pune 2010]**

Note that here $\qquad f(D^2) = D^2 + a^2 \qquad$ and $\qquad f(-a^2) = -a^2 + a^2 = 0.$

But $\qquad \dfrac{1}{D^2+a^2}\sin ax$ = Imaginary part of $\dfrac{1}{D^2+a^2}$ $(\cos ax + i \sin ax)$

Thus, $\qquad \dfrac{1}{D^2+a^2}\sin ax$ = Imaginary part of $\dfrac{1}{D^2+a^2}e^{aix}$. ... (1)

Now, $\dfrac{1}{D^2+a^2}e^{aix} = \dfrac{1}{D^2+a^2}e^{aix}\cdot 1$, taking $V = 1$

$= e^{aix}\dfrac{1}{(D+ai)^2+a^2}1$, by formula (*i*) of case II.

$= e^{aix}\dfrac{1}{D(D+2ai)^2}e^{0.x} = \dfrac{1}{2ia}e^{aix}\dfrac{1}{D}1$ by formula IA of Art 5.13.

$$= \frac{1}{2ia}e^{aix}\,x = \frac{x}{2ia}(\cos ax + i\sin ax) = \frac{x}{2a}\sin ax + \frac{1}{i}\frac{x}{2a}\cos ax = \frac{x}{2a}\sin ax - i\frac{x}{2a}\cos ax. \quad \text{... (2)}$$

Using (2), (1) reduces to

$$\frac{1}{D^2+a^2}\sin ax = \text{imaginary part of}\left[\frac{x}{2a}\sin ax - i\frac{x}{2a}\cos ax\right] = -\frac{x}{2a}\cos ax.$$

Similarly, $$\frac{1}{D^2+a^2}\cos ax = \text{Real part of } \frac{1}{D^2+a^2}e^{aix} \quad \text{... (3)}$$

$\therefore$ (2) and (3) $\Rightarrow$ $\dfrac{1}{D^2+a^2}\cos ax = \text{Real part of}\left[\dfrac{x}{2a}\sin ax - i\dfrac{x}{2a}\cos ax\right] = \dfrac{x}{2a}\sin ax$

Remark. Note carefully and remember the following formulas :

$$\frac{1}{D^2+a^2}\sin ax = -\frac{x}{2a}\cos ax = \frac{x}{2}\int \sin ax\, dx. \quad \text{... (4)}$$

$$\frac{1}{D^2+a^2}\cos ax = \frac{x}{2a}\sin ax = \frac{x}{2}\int \cos ax\, dx. \quad \text{... (5)}$$

5.17 Solved examples based on Art. 5.16

Ex. 1. *Solve the following differential equations*

(a) $(D^2 + 1)\, y = \cos 2x$

(b) $(D^2 + 9)\, y = \cos 4x$.

Sol. (*a*) Here the auxiliary eqution is $D^2 + 1 = 0$ so that $D = \pm i$,

$\therefore$ C.F. $= c_1 \cos x + c_2 \sin x$, c_1, c_2 being arbitrary constants.

Now, P.I. $= \dfrac{1}{D^2+1}\cos 2x = \dfrac{1}{-2^2+1}\cos 2x = -\dfrac{1}{3}\cos 2x$.

The required general solution is $y = c_1 \cos x + c_2 \sin x - (1/3)\cos 2x$.

(*b*) Try yourself. **Ans.** $y = c_1 \cos 3x + c_2 \sin 3x - (1/7)\cos 4x$.

Ex. 2. (*a*) *Solve* $(D^2 - 3D + 2)\, y = \sin 3x$. **[Delhi Maths (G) 1996]**

(*b*) $(D^2 - 4D + 4)\, y = \sin 2x$. **[G.N.D.U. (Amritsar) 2010]**

Sol. (*a*) Here auxiliary equation $D^2 - 3D + 2 = 0$ gives $D = 1, 2$.

C.F. $= c_1e^x + c_2e^{2x}$, c_1, c_2 being arbitrary constants.

and P.I. $= \dfrac{1}{D^2-3D+2}\sin 3x = \dfrac{1}{-3^2-3D+2}\sin 3x$

$$= -\frac{1}{3D+7}\sin 3x = -(3D-7)\frac{1}{(3D-7)(3D+7)}\sin 3x$$

$$= -(3D-7)\frac{1}{9D^2-49}\sin 3x = -(3D-7)\frac{1}{9(-3^2)-49}\sin 3x$$

$$= (1/130)\times(3D-7)\sin 3x = (1/130)\times(9\cos 3x - 7\sin 3x).$$

$\therefore$ Solution is $\quad y = c_1e^x + c_2e^{2x} + (1/130)\times(9\cos 3x - 7\sin 3x).$

(*b*) **Ans.** $\quad y = (c_1 + c_2x)\,e^{2x} + (3\sin 2x + 8\cos 2x)/25$

Ex. 3. *Solve* $(D^2 + a^2)\,y = \sin ax.$ **[Lucknow 1997, Delhi 1997, Meerut 1995]**

(*b*) $(D^2 + a^2)\,y = \cos ax$

Sol. (*a*) Here the auxiliary equation is $\quad D^2 + a^2 = 0 \quad$ so that $\quad D = \pm\, ia.$

$\therefore$ C.F. $= c_1\cos ax + c_2\sin ax$, c_1, c_2 being arbitrary constants.

and $$\text{P.I.} = \frac{1}{D^2+a^2}\sin ax = \text{Imaginary part of } \frac{1}{D^2+a^2}(\cos ax + i\sin ax)$$

or $$\text{P.I.} = \text{Imaginary part of } \frac{1}{D^2+a^2}e^{iax}\text{, by Euler's theorem.} \qquad \text{... (1)}$$

Now, $$\frac{1}{D^2+a^2}e^{iax} = \frac{1}{D^2+a^2}e^{iax}\cdot 1 = e^{iax}\frac{1}{(D+ia)^2+a^2}\cdot 1$$

$$\left[\because \text{ From Art. 5.20, } \frac{1}{f(D)}e^{ax}V = e^{ax}\frac{1}{f(D+a)}V\right]$$

$$= e^{iax}\frac{1}{D^2+2iaD}e^{iax} = \frac{1}{D}\frac{1}{D+2ia}e^{0.x} = e^{iax}\frac{1}{D}\frac{1}{0+2ia}e^{0.x}$$

$$= \frac{1}{2ia}e^{iax}\frac{1}{D}1 = \frac{1}{2ia}e^{iax}x = \frac{x}{2ia}(\cos ax + i\sin ax)$$

$= (x/2a)\sin ax - i\,(x/2a)\cos ax$, as $(1/i) = -\,i.$

$\therefore$ From (1), $$\text{P.I.} = \text{Imaginary part of } \left(\frac{x}{2a}\sin ax - \frac{ix}{2a}\cos ax\right) = -\frac{x}{2a}\cos ax$$

Hence the required general solution of the given equation is

$y = \text{C.F.} + \text{P.I.} \qquad$ or $\qquad y = c_1\cos ax + c_2\sin ax - (x/2a)\cos ax.$

Note: You can also use remark of Art. 4.15 to write $\dfrac{1}{D^2+a^2}\sin ax = -\dfrac{x}{2a}\cos ax$

(*b*) Proceed as in part (*a*). C.F. is same as in part (*a*).

$$\text{P.I.} = \frac{1}{D^2+a^2}\cos ax = \text{Real part of } \frac{1}{D^2+a^2}(\cos ax + i\sin ax)$$

or $$\text{P.I.} = \text{Real part of } \frac{1}{D^2+a^2}e^{iax} \qquad \text{...(1)}'$$

Now , we have $$\frac{1}{D^2+a^2}e^{iax} = \left(\frac{x}{2a}\right)\sin ax - \left(i\frac{x}{2a}\right)\cos ax\text{, do as in part } (a).$$

Hence from (1)′, $$\text{P.I.} = \text{real part of } \left(\frac{x}{2a}\sin ax - \frac{ix}{2a}\cos ax\right) = \frac{x}{2a}\sin ax$$

$\therefore$ The required solution is $\qquad y = c_1\cos ax + c_2\sin ax + (x/2a)\sin ax$

Note: You can also use the result $\qquad \dfrac{1}{D^2+a^2}\cos ax = \dfrac{x}{2a}\sin ax$

Ex. 4. *Solve the following differential equations:*

(a) $(D^3 + a^2 D)\, y = \sin ax$ **[I.A.S. Prel. 2006, Rajsthan 2010, Purvanchal 1999]**

(b) $(D^3 + 9D)\, y = \sin 3x.$

(c) $(d^3x/dy^3) + b^2(dx/dy) = \sin by.$

Sol. (*a*) Here auxiliary equation is $D^3 + a^2 D = 0$ so that $D = 0, 0 \pm ia.$

$\therefore$ C.F. $= c_1 e^{0.x} + e^{0.x}(c_2 \cos ax + c_3 \sin ax) = c_1 + c_2 \cos ax + c_3 \sin ax,$

where c_1, c_2 and c_3 arbitrary constants.

$$\text{P.I.} = \frac{1}{D^3 + a^2 D}\sin ax = \frac{1}{D^2 + a^2}\frac{1}{D}\sin ax = \frac{1}{D^2 + a^2}\left(-\frac{1}{a}\cos ax\right)$$

$$= -\frac{1}{a}\left[\text{Real part of } \frac{1}{D^2 + a^2}(\cos ax + i \sin ax)\right]$$

$$= -\frac{1}{a}\left[\text{Real part of } \frac{1}{D^2 + a^2} e^{iax}\right], \text{ by Euler's theorem}$$

$$= -\frac{1}{a}\left[\text{Real part of } \left(\frac{x}{2a}\sin ax - \frac{ix}{2a}\text{ as } ax\right)\right]$$

$$\left[\text{As in Ex. 3. } (a)\text{, prove that } \frac{1}{D^2 + a^2} e^{iax} = \frac{x}{2a}\sin ax - \frac{ix}{2a}\cos ax\right]$$

$= -(1/a)(x/2a)\sin ax = -(x/2a^2)\sin ax.$

Hence the general solution is $y = c_1 + c_2 \cos ax + c_3 \sin ax - (x/2a^2)\sin ax.$

(*b*) Do as in part (*a*) **Ans.** $y = c_1 + c_2 \cos 3x + c_3 \sin 3x - (x/18)\sin 3x.$

(*c*) Proceed as in part (*a*). Here y is independent and x is dependent variable

Ans. $x = c_1 + c_2 \cos by + c_3 \sin by - (y/2b^2)\sin by$

Ex. 5. *Solve* $(D - 1)^2 (D^2 + 1)^2 y = \sin x.$ **[Gorakhpur 1994]**

Sol. Here the auxiliary equation is $(D - 1)^2 (D^2 + 1)^2 = 0$ so that $D = 1, 1, 0 \pm i, 0 \pm i.$

$\therefore$ C.F. $= (c_1 + c_2 x)e^x + (c_3 + c_4 x)\cos x + (c_5 + c_6 x)\sin x,$

where c_1, c_2, c_3, c_4, c_5 and c_6 arbitrary constants.

$$\text{P.I.} = \frac{1}{(D-1)^2 (D^2+1)^2}\sin x = \frac{1}{(D^2+1)^2}(D+1)^2 \frac{1}{(D+1)^2 (D-1)^2}\sin x$$

$$= \frac{1}{(D^2+1)^2}(D^2 + 2D + 1)\frac{1}{(D^2-1)^2}\sin x = \frac{1}{(D^2+1)^2}\frac{(D^2 + 2D + 1)}{(-1^2 - 1)^2}\sin x$$

$$= \frac{1}{4}\frac{1}{(D^2+1)^2}(D^2 \sin x + 2D \sin x + \sin x)$$

$$= \frac{1}{4}\frac{1}{(D^2+1)^2}(-\sin x + 2\cos x + \sin x) = \frac{1}{2}\frac{1}{(D^2+1)^2}\cos x$$

$$= \frac{1}{2}\text{Real part of } \frac{1}{(D^2+1)^2} e^{ix}\cdot 1 = \frac{1}{2}\text{Real part of } e^{ix}\frac{1}{[(D+i)^2+1]}\cdot 1$$

$$\left[\because \text{ From Art. 5.20, } \frac{1}{f(D)} e^{ax} V = e^{ax}\frac{1}{f(D+a)} V\right]$$

$$= \frac{1}{2} \text{Real part of } e^{ix} \frac{1}{D^2 + 2iD} \cdot 1 = \frac{1}{2} \text{Real part of } e^{ix} \frac{1}{D(D+2i)} e^{0.x}$$

$$= \frac{1}{2} \text{Real part of } e^{ix} \frac{1}{D} \frac{1}{0+2i} e^{0.x} = \frac{1}{2} \text{Real part of } \frac{e^{ix}}{2i} \frac{1}{D} \cdot 1$$

$= -(i/4) \times$ Real part of x ($\cos x + i \sin x$) $= (x/4) \times \sin x$

$\therefore$ Solution is $y = (c_1 + c_2 x) e^x + (c_3 + c_4 x) \cos x + (c_5 + c_6 x) \sin x + (x/4) \sin x$.

Ex. 6. *Solve $(d^4y/dx^4) - m^4 y = \sin mx$.* **[I.A.S. 1991]**

Sol. Given $(D^4 - m^4)\, y = \sin mx$, where $D \equiv d/dx$ ... (1)

whose auxililary equation is $D^4 - m^4 = 0$ giving $D = \pm\, m,\ 0 \pm im$.

C.F. $= c_1 e^{mx} + c_2 e^{-mx} + c_3 \cos mx + c_4 \sin mx$, c_1, c_2, c_3, c_4 being arbitrary constans

$$\text{P.I.} = \frac{1}{(D^2 + m^2)} \frac{1}{(D^2 - m^2)} \sin mx = \frac{1}{D^2 + m^2} \frac{1}{-m^2 - m^2} \sin mx$$

$$= -\frac{1}{2m^2} \frac{1}{D^2 + m^2} \sin mx = -\frac{1}{2m^2} \times \left(-\frac{x}{2m} \cos mx\right) = \frac{x}{4m^3} \cos mx.$$

So solution is $y = c_1 e^{mx} + c_2 e^{-mx} + c_3 \cos mx + c_4 \sin mx + (x/4m^3) \cos mx$.

Ex. 7. (a) *Find the solution of $(d^2y/dx^2) + 4y = 8 \cos 2x$, given that $y = 0$ and $dy/dx = 0$, when $x = 0$.* **[I.A.S. 1995]**

(b) *Solve $(D^2 + 4)\, y = \sin 2x$, given that when $x = 0$, $y = 0$ and $dy/dx = 2$.* **[I.A.S. 1992]**

Sol. (a) Re-writing the given equation, we get $(D^2 + 4)\, y = 8 \cos 2x$. ... (1)

Also given that when $x = 0$, $y = 0$ ... (2)

and when $x = 0$, $dy/dx = 0$. ... (3)

The auxiliary equation of (1) is $D^2 + 4 = 0$ so that $D = \pm\, 2i$

C.F. $= e^{0.x} (c_1 \cos 2x + c_2 \sin 2x) = c_1 \cos 2x + c_2 \sin 2x$, c_1, c_2 being arbitrary constants

Also, $$\text{P.I.} = 8 \frac{1}{D^2 + 2^2} \cos 2x = 8 \cdot \frac{x}{(2 \times 2)} \sin 2x = 2x \sin 2x.$$

Solution of (1) is $y = c_1 \cos 2x + c_2 \sin 2x + 2x \sin 2x$. ... (4)

Putting $x = 0$ and $y = 0$ (due to (2)), (4) yields $c_1 = 0$. Then (4) gives

$y = c_2 \sin 2x + 2x \sin 2x = (c_2 + 2x) \sin 2x$. ... (5)

From (5) $dy/dx = 2 \sin 2x + 2(c_2 + 2x) \cos 2x$ ... (6)

Putting $x = 0$ and $dy/dx = 0$, (6) yields $0 = 2c_2 \Rightarrow c_2 = 0$.

Hence from (5), the required solution is $y = 2x \sin 2x$.

(b) Proceed as in part (a). Solution is $y = (1/8)\,(9 \sin 2x - 2x \cos 2x)$

Ex. 8. *Solve $(d^2y/dx^2) - 8\,(dy/dx) + 9y = 40 \sin 5x$.*

Sol. Let $D \equiv d/dx$. Then given equation becomes $(D^2 - 8D + 9)\, y = 40 \sin 5x$. ... (1)

Here auxiliary equation $D^2 - 8D + 9 = 0 \Rightarrow D = \dfrac{8 \pm \sqrt{(64 - 36)}}{2} = \dfrac{8 \pm 2\sqrt{7}}{2} = 4 \pm \sqrt{7}$.

$\therefore$ C.F. $= e^{4x} (c_1 \cosh x\sqrt{7} + c_2 \sinh x\sqrt{7})$, c_1, c_2 being arbitrary constants

and $$\text{P.I.} = \frac{1}{D^2 - 8D + 9} 40 \sin 5x = 40 \frac{1}{-5^2 - 8D + 9} \sin 5x = 40 \frac{1}{-8(D+2)} \sin 5x$$

$$= -5(D-2)\frac{1}{(D-2)(D+2)}\sin 5x = -5(D-2)\frac{1}{D^2-4}\sin 5x = -5(D-2)\frac{1}{-5^2-4}\sin 5x$$

$= (5/29) \times (D \sin 5x - 2 \sin 5x) = (5/29) \times (5 \cos 5x - 2 \sin 5x)$.

Hence the required general solution is $y = \text{C.F.} + \text{P.I.}$

or $$y = e^{4x}(c_1 \cosh x\sqrt{7} + c_2 \sin x\sqrt{7}) + (5/29) \times (5 \cos 5x - 2 \sin 5x).$$

Ex. 9. *Solve $(d^2y/dx^2) + 2(dy/dx) + 10y + 37 \sin 3x = 0$, and find the value of y when $x = \pi/2$ if it is given that $y = 3$ and $dy/dx = 0$ when $x = 0$.* **[I.A.S. 1997, Lucknow 1996]**

Sol. Re-writing, the given equation is $(D^2 + 2D + 10)\, y = -37 \sin 3x.$, $D \equiv d/dx$

Its auxiliary equation $D^2 + 2D + 10 = 0 \Rightarrow D = \dfrac{-2 \pm \sqrt{(4-40)}}{2} = \dfrac{-2 \pm 6i}{2} = -1 \pm 3i.$

$\therefore$ C.F. $= e^{-x}(c_1 \cos 3x + c_2 \sin 3x)$, c_1, c_2 being arbitrary constants.

$$\text{P.I.} = \frac{1}{D^2+2D+10}(-37\sin 3x) = -37\frac{1}{-3^2+2D+10}\sin 3x$$

$$= -3\frac{1}{2D+1}\sin 3x = -37(2D-1)\frac{1}{(2D-1)(2D+1)}\sin 3x$$

$$= -37(2D-1)\frac{1}{4D^2-1}\sin 3x = -37(2D-1)\frac{1}{4(-3^2)-1}\sin 3x$$

$= (2D - 1) \sin 3x = 6 \cos 3x - \sin 3x.$

Hence the general solution of the given equation is

$$y = e^{-x}(c_1 \cos 3x + c_2 \sin 3x) + 6 \cos 3x - \sin 3x. \qquad \text{... (1)}$$

Differentiating both sides of (1) w.r.t. 'x', we have

$dy/dx = e^{-x}(-3c_1 \sin 3x + 3c_2 \cos 3x) - e^{-x}(c_1 \cos 3x + c_2 \sin 3x)$
$\qquad - 18 \sin 3x - 3 \cos 3x.$ ... (2)

It is given that $y = 3$, $dy/dx = 0$ when $x = 0$. So (1) and (2) give

$3 = c_1 + 6$ and $0 = 3c_2 - c_1 - 3$ so that $c_1 = -3$, $c_2 = 0$.

$\therefore$ From (1), $y = -3e^{-x} \cos 3x + 6 \cos 3x - \sin 3x.$ ... (3)

Putting $x = \pi/2$ in (3), the corresponding value of y is given by

$y = -3e^{-\pi/2} \cos(3\pi/2) + 6 \cos(3\pi/2) - \sin(3\pi/2) = 1$, as $\cos(3\pi/2) = 0$.

Ex. 10. *Find the integral of the equation $(d^2x/dt^2) + 2n \cos\alpha\, (dx/dt) + n^2x = a \cos nt$, which is such that when $t = 0$, $x = 0$ and $dx/dt = 0$.*

Sol. Let $D \equiv d/dt$. Then the given equation can be written as

$$[D^2 + (2n \cos\alpha)\, D + n^2]\, x = a \cos nt.$$

Its auxiliary equation is $D^2 + (2n \cos\alpha)\, D + n^2 = 0.$

$$\therefore \quad D = \frac{-2n\cos\alpha \pm \sqrt{4n^2\cos^2\alpha - 4n^2}}{2} = -n\cos\alpha \pm \sqrt{(-n^2\sin^2\alpha)} = -n\cos\alpha \pm i\,n\sin a.$$

$\therefore$ C.F. $= e^{-nt\cos\alpha}[c_1 \cos(nt \sin\alpha) + c_2 \sin(nt \sin\alpha)]$, c_1, c_2 being arbitrary constants

and $$\text{P.I.} = \frac{1}{D^2+(2n\cos\alpha)D+n^2}a\cos nt = a\frac{1}{-n^2+(2n\cos\alpha)D+n^2}\cos nt$$

$$= \frac{a}{2n\cos\alpha}\frac{1}{D}\cos nt = \frac{a}{2n\cos\alpha}\int\cos nt\,dt = \frac{a\sin nt}{2n^2\cos\alpha}.$$

[Note that here $(1/D)$ stands for integration w.r.t. 't']

Hence the general solution of the given equation is $x = \text{C.F.} + \text{P.I.}$

or $x = e^{-nt\cos\alpha}\,[c_1 \cos(nt\sin\alpha) + c_2 \sin(nt\sin\alpha)] + (a\sin nt)/(2n^2\cos\alpha)$ ....(1)

Differentiating both sides of (1) w.r.t. 't', we have

$$dx/dt = e^{-nt\cos\alpha}\,n\sin\alpha\,[-c_1\sin(nt\sin\alpha) + c_2\cos(nt\sin\alpha)]$$

$-n\cos\alpha\, e^{-nt\cos\alpha}\,[c_1\cos(nt\sin\alpha) + c_2\sin(nt\sin\alpha)] + (a\cos nt)/(2n\cos\alpha)$ ... (2)

Since, it is given that $x = 0$, $dx/dt = 0$ when $t = 0$, so from (1) and (2)

$0 = c_1$ and $0 = c_2 n\sin\alpha - n\,c_1\cos\alpha + a/(2n\cos\alpha)$

$\Rightarrow$ $c_1 = 0$ and $c_2 = -a/(2n^2\sin\alpha\cos\alpha) = -a/(n^2\sin 2\alpha)$.

Putting the above values of c_1 and c_2 in (1), the required solution is

$$x = [-a/(n^2\sin 2\alpha)]\,e^{-nt\cos\alpha}\sin(nt\sin\alpha) + (a\sin nt)/(2n^2\cos\alpha).$$

Ex. 11. *Solve* $(D^2 + 4)\,y = \sin^2 x$. **[I.A.S. Prel. 1998; Merrut 1996]**

Sol. Here auxiliary equation is $D^2 + 4 = 0$ so that $D = \pm 2i$.

Hence C.F. $= \cos 2x + c_2\sin 2x$, c_1, c_2 being arbitrary constants

Also $$\text{P.I.} = \frac{1}{D^2+4}\sin^2 x = \frac{1}{2}\frac{1}{D^2+4}(1-\cos 2x) = \frac{1}{2}\left[\frac{1}{D^2+4}e^{0.x} - \frac{1}{D^2+4}\cos 2x\right]$$

$$= \frac{1}{2}\left[\frac{1}{0+4}e^{0.x} - \left(\frac{x}{2\times 2}\right)\sin 2x\right] = \frac{1}{8} - \frac{1}{8}x\sin 2x.$$

$\therefore$ Solution is $y = c_1\cos 2x + c_2\sin 2x + (1/8) - (x/8)\sin 2x$.

Exercise 5(D)

Solve the following differential equations :

1. $y'' + y = \sin x$ **[Delhi B.Sc. (Hons) II 2011]** **Ans.** $y = c_1\cos x + c_2\sin x - (x/2)\times\cos x$
2. $(D^3 + D^2 - D - 1)\,y = \cos 2x$. **[Kanpur 2006]** **Ans.** $y = c_1 e^x + (c_2 + c_3 x)\,e^{-x} - (1/25)\,(2\sin 2x + \cos 2x)$
3. $(D^2 - 5D + 6)\,y = \sin 3x$. **[Gorakhpur 1995]** **Ans.** $y = c_1 e^{2x} + c_2 e^{3x} + (1/78)\,(5\cos 3x - \sin 3x)$
4. $(D^2 + D + 1)\,y = \sin 2x$. **Ans.** $y = e^{-x/2}\left\{c_1\cos(x\sqrt{3}/2) + c_2\sin(x\sqrt{3}/2)\right\} - (1/13)(2\cos 2x + 3\sin 2x)$
5. $(D^2 - 4)\,y = \cos^2 x$. **Ans.** $y = c_1 e^{2x} + c_2 e^{-2x} - (1/16)\,(2 + \cos 2x)$
6. $(D^2 - a^2)\,y = \cos mx$. **Ans.** $y = c_1 e^{ax} + c_2 e^{-ax} - [1/(m^2 + a^2)]\cos mx$
7. $(D^2 + 1)\,y = \cos 2x$. **Ans.** $y = c_1 e^{-x} + e^{x/2}\{c_2\cos(x\sqrt{3}/2) + c_3\sin(x\sqrt{3}/2)\} + (\cos 2x - 8\sin 2x)/65$
8. $(D^2 + 9)\,y = \cos 2x + \sin 2x$. **Ans.** $y = c_1\cos 3x + c_2\sin 3x + (1/5)\,(\cos 2x + \sin 2x)$
9. (*a*) $(D^2 + 4)\,y = \sin 2x$. **[Pune 2011]** **Ans.** $y = c_1\cos 2x + c_2\sin 2x - (x/4)\times\cos 2x$

   (*b*) $(D^2 + 4)\,y = 4 + \sin^2 x$. **[Guwahati 2007]** **Ans.** $y = c_1\cos 2x + c_2\sin 2x + 9/8 - (x/8)\times\sin 2x$
10. $d^2x/dt^2 + 4x = a\sin t\cos t$. **Ans.** $x = c_1\cos 2t + c_2\sin 2t - (at/8)\times\cos 2t$
11. $(D^2 + 1)\,y = \sin x\sin 2x$ **Ans.** $y = c_1\cos x + c_2\sin x + (x/4)\times\sin x + (1/16)\times\cos 3x$
12. $(D^2 + 1)\,y = \cos x\sin 3x$ **Ans.** $y = c_1\cos x + c_2\sin x - (1/30)\times\sin 4x - (1/6)\times\sin 2x$
13. $(D^4 - 1)\,y = \sin 2x$. **[Delhi Maths.(G) 2000]** **Ans.** $y = c_1 e^x + c_2 e^{-x} + c_3\cos x + c_4\sin x + (1/15)\times\sin 2x$
14. $(D^3 + 1)\,y = \cos 2x$ **Ans.** $y = c_1 e^{-x} + e^{x/2}\{c_2\cos(x\sqrt{3}/2) + c_3\sin(x\sqrt{3}/2)\} + (1/65)\times(\cos 2x - 8\sin 2x)$
15. $d^2y/dx^2 - (dy/dx) - 2y = \sin 2x$. **[Meerut 2008, 10]**

    Ans. $y = c_1 e^{-x} + c_2 e^{2x} + (\cos 2x - 3\sin 2x)/20$

5.18 Short method of finding P.I. when $X = x^m$, m being a positive integer.

Working rule for evaluating $\{1/f(D)\}\, x^m$.

Step I. Bring out the lowest degree term from $f(D)$ so that the remaining factor in the denominator is of the form $[1 + \phi(D)]^n$ or $[1 - \phi(D)]^n$, n being a positive integer.

Step II. We take $[1 + \phi(D)]^n$ or $[1 - \phi(D)]^n$ in the numerator so that it takes the form $[1 + \phi(D)]^{-n}$ or $[1 - \phi(D)]^{-n}$.

Step III. We expand $[1 \pm \phi(D)]^{-n}$ by the binomial theorem, namely

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + ...$$

In particular the following binomial expansions should be remembered.

$(1-x)^{-1} = 1 + x + x^2 + x^3 + \dots$; $\qquad (1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$

$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$; $\qquad (1-x)^{-2} = 1 - 2x + 3x^2 - 4x^3 + \dots$

In any case, the expansion is to be carried upto D^m, since $D^{m+1} x^m = 0$, $D^{m+2} x^m = 0$, and all the higher differential coefficients of x^m vanish.

Remark. If we are given a polynomial $x^m + a_1 x^{m-1} + \dots + a_{m-1} x + a_m$ of degree m in place of x^m, then also we proceed to evaluate $[1/f(D)]\,(x^m + a_1 x^{m-1} + \dots + a_m)$ in the same manner as we did for $\{1/f(D)\}\, x^m$. Also, if X is a constant, the above method can be used.

5.19 Solved examples based on Art. 5.18

Ex. 1. *Solve $(D^4 - D^2)\, y = 2$.* **[Agra 2005]**

Sol. Here auxiliary equation is $D^4 - D^2 = 0$ or $D^2(D^2 - 1) = 0 \Rightarrow D = 0, 0, 1, -1$

$\therefore$ C.F. $= (c_1 + c_2 x)\, e^{0.x} + c_3 e^x + c_4 e^{-x} = c_1 + c_2 x + c_3 e^x + c_4 e^{-x}$,

where c_1, c_2, c_3 and c_4 are arbitrary constants.

$$\text{P.I.} = \frac{1}{D^4 - D^2}\,2 = -\frac{2}{D^2}\frac{1}{(1-D^2)}\,1 = -\frac{2}{D^2}(1-D^2)^{-1}\cdot 1$$

$$= -\frac{2}{D^2}(1 + D^2 + D^4 + \dots)\cdot 1 = -\frac{2}{D^2}\,1 = -\frac{2}{D}\,x = (-2)\times\frac{x^2}{2} = -x^2$$

$\therefore$ Solution is $\qquad y = c_1 + c_2 x + c_3 e^x + c_4 e^{-x} - x^2$

Ex. 2. *Find the particular integral of $(D^2 + D)\, y = x^2 + 2x + 4$.* **[I.A.S. Prel. 1994]**

Sol. The required particular integral

$$= \frac{1}{D^2 + D}\,(x^2 + 2x + 4) = \frac{1}{D(1+D)}\,(x^2 + 2x + 4) = \frac{1}{\mathrm{D}}\,(1 + \mathrm{D})^{-1}\,(x^2 + 2x + 4)$$

$= (1/D)\,(1 - D + D^2 - D^3 + \dots)\,(x^2 + 2x + 4)$

$= (1/D)\,[(x^2 + 2x + 4) - D\,(x^2 + 2x + 4) + D^2\,(x^2 + 2x + 4)]$

$= (1/D)\,[x^2 + 2x + 4 - (2x + 2) + 2] = (1/D)\,(x^2 + 4) = x^3/3 + 4x$.

Ex. 3. *Solve $(D^4 - a^4)\, y = x^4$.* **[Delhi Maths Hons. 1994]**

Sol. Here the auxiliary equation is $D^4 - a^4 = 0$ or $(D^2 - a^2)(D^2 + a^2) = 0$

so that $D^2 - a^2 = 0$ or $D^2 + a^2 = 0$ and hence $D = a, -a, 0 \pm ia$.

$\therefore$ C.F. $= c_1 e^{ax} + c_2 e^{-ax} + c_3 \cos ax + c_4 \sin ax$, c_1, c_2 being arbitrary constants

and $$\text{P.I.} = \frac{1}{D^4 - a^4}\,x^4 = \frac{1}{-a^4(1 - D^4/a^4)}\,x^4 = -\frac{1}{a^4}\left(1 - \frac{D^4}{a^4}\right)^{-1}$$

$= -(1/a^4)\,\{1 + (D^4/a^4) + \dots\}\, x^4 = -(1/a^4)\,[x^4 + (1/a^4) \times D^4 x^4]$

$= -(1/a^4)\ [x^4 + (1/a^4) \times 24]$, as $D^4 x^4 = 4 \times 3 \times 2 \times 1 = 24$

$\therefore$ Solution is $\quad y = c_1 e^{ax} + c_2 e^{-ax} + c_3 \cos ax + c_4 \sin ax - (1/a^4)\ (x^4 + 24/a^4)$.

Ex. 4. *Solve* $(D^3 + 8)\ y = x^4 + 2x + 1$. **[Delhi Maths Hons. 1998]**

Sol. Here the auxiliary equation is $\quad D^3 + 2^3 = 0 \quad$ or $\quad (D + 2)\ (D^2 - 2D + 4) = 0$

so that $\quad D = -2,\ \{2 \pm (4 - 16)^{1/2}\}/2 = -2,\ 1 \pm i\sqrt{3}$.

$\therefore$ C.F. $= c_1 e^{-2x} + e^x\ (c_2 \cos x\sqrt{3} + c_3 \sin x\sqrt{3})$, c_1, c_2, c_3 being arbitrary constans

$$\therefore \quad \text{P.I.} = \frac{1}{D^3 + 8}(x^4 + 2x + 1) = \frac{1}{8(1 + D^3/8)}(x^4 + 2x + 1)$$

$$= \frac{1}{8}\left(1 + \frac{D^3}{8}\right)^{-1}(x^4 + 2x + 1) = \frac{1}{8}\left(1 - \frac{D^3}{8} + \dots\right)(x^4 + 2x + 1)$$

$= (1/8) \times [\ (x^4 + 2x + 1) - (1/8) \times D^3(x^4 + 2x + 1)]$

$= (1/8) \times [(x^4 + 2x + 1) - (1/8)\ (24x)] = (1/8) \times (x^4 + 2x + 1 - 3x) = (x^4 - x + 1)/8$.

$\therefore$ Required solution is $\quad y = c_1 e^{-2x} + e^x\ (c_2 \cos x\sqrt{3} + c_3 \sin x\sqrt{3}) + (x^4 - x + 1)/8$.

Ex. 5. *Solve (a)* $(D^2 + 2D + 2)\ y = x^2$.

(b) $(D^2 - 4D + 4)\ y = x^2$.

Sol. (*a*) Here the auxiliary equation is $\quad D^2 + 2D + 2 = 0 \quad$ so that $\quad D = -1 \pm i$.

$\therefore$ C.F. $= e^{-x}\ (c_1 \cos x + c_2 \sin x)$, c_1, c_2 being arbitrary constants

$$\text{and} \quad \text{P.I.} = \frac{1}{D^2 + 2D + 2}x^2 = \frac{1}{2[1 + (D + D^2/2)]}x^2 = \frac{1}{2}\{1 + (D + D^2/2)\}^{-1}$$

$= (1/2)\ \{1 - (D + D^2/2) + (D + D^2/2)^2 - \dots\}x^2 = (1/2)\ \{1 - (D + D^2/2) + D^2 + \dots\}x^2$

$= (1/2)\ \{1 - D + D^2/2 + \dots\}x^2 = (1/2)\ (x^2 - 2x + 1)$

$\therefore$ The required solution is $\quad y = e^{-x}\ (c_1 \cos x + c_2 \sin x) + (x^2 - 2x + 1)/2$.

(*b*) Try yourself. **Ans.** $y = (c_1 + c_2 x)\ e^{2x} + (2x^2 + 4x + 3)/4$.

Ex. 6. *Solve (a)* $(D^3 + 3D^2 + 2D)\ y = x^2$. **[Delhi Maths(G) 2006]**

(*b*) $(D^3 + 3D^2 + 2D)\ y = x$.

Sol. (*a*) Here the auxiliary equation is $\quad D^3 + 3D^2 + 2D = 0 \quad$ or $\quad D\ (D^2 + 3D + 2) = 0$

or $\quad D\ (D + 1)\ (D + 2) = 0 \quad$ so that $\quad D = 0, -1, -2$.

$\therefore$ C.F. $= c_1 + c_2 e^{-x} + c_2 e^{-2x}$, where c_1, c_2, c_3 are arbitrary constants.

$$\text{and} \quad \text{P.I.} = \frac{1}{D^3 + 3D^2 + 2D}x^2 = \frac{1}{2D[1 + (3/2)\,D + (1/2)\,D^2]}x^2$$

$$= \frac{1}{2D}\left[1 + \left(\frac{3D}{2} + \frac{D^2}{2}\right)\right]^{-1} x^2 = \frac{1}{2D}\left[1 - \left(\frac{3D}{2} + \frac{D^2}{2}\right) + \left(\frac{3D}{2} + \frac{D^2}{2}\right)^2 - \dots\right]x^2$$

$$= \frac{1}{2D}\left[1 - \left(\frac{3D}{2} + \frac{D^2}{2}\right) + \left(\frac{9D^2}{4} + \dots\right) + \dots\right]x^2 = \frac{1}{2D}\left(1 - \frac{3D}{2} + \frac{7D^2}{4} + \dots\right)x^2$$

$$= \frac{1}{2D}\left[x^2 - \frac{3}{2}(2x) + \frac{7}{4}(2)\right] = \frac{1}{2}\int\left(x^2 - 3x + \frac{7}{2}\right)dx = \frac{x^3}{6} - \frac{3x^2}{4} + \frac{7x}{4}.$$

$\therefore$ The required solution $\quad y = c_1 + c_2 e^{-x} + c_2 e^{-2x} + (x^3/6) - (3x^2/4) + (7x/4)$.

(*b*) Try yourself. **Ans.** $y = c_1 + c_2 e^{-x} + c^2 e^{-2x} + (x^2/4) - (3x/4)$.

Ex. 7. *Solve* $(D^4 - 2D^3 + D^2)\, y = x$.

Sol. Here the auxiliary equation is $D^4 - 2D^3 + D^2 = 0$ giving $D = 0, 0, 1, 1$.

$\therefore$ C.F. $= (c_1 + c_2 x)\, e^{0.x} + (c_3 + c_4 x)\, e^x = c_1 + c_2 x + (c_3 + c_4 x)\, e^x$,

where c_1, c_2, c_3 and c_4 are arbitrary constants.

$$\text{P.I.} = \frac{1}{D^4 - 2D^3 + D^2}\, x = \frac{1}{D^2 (D^2 - 2D + 1)}\, x = \frac{1}{D^2 (1 - D)^2}\, x$$

$$= \frac{1}{D^2}(1 - D)^{-2}\, x = \frac{1}{D^2}(1 + 2D + \ldots)\, x = \frac{1}{D^2}(x + 2)$$

$$= \frac{1}{D}\int (x + 2)\, dx = \frac{1}{D}\left(\frac{x^2}{2} + 2x\right) = \int\left(\frac{x^2}{2} + 2x\right) dx = \frac{x^3}{6} + x^2.$$

$\therefore$ The required solution is $y = c_1 + c_2 x + (c_3 + c_4 x)\, e^x + (x^3/6) + x^2$.

Ex. 8. *Solve* $(D^4 + D^2 + 16)\, y = 16x^2 + 256$.

Sol. Here the auxiliary equation is $D^4 + D^2 + 16 = 0$ or $(D^2 + 4)^2 - (D\sqrt{7}\,)^2 = 0$

or $(D^2 + D\sqrt{7} + 4)\,(D^2 - D\sqrt{7} + 4) = 0 \;\Rightarrow\; D^2 + D\sqrt{7} + 4 = 0$ or $D^2 - D\sqrt{7} + 4 = 0$

$$\therefore\; D = \frac{-\sqrt{7} \pm \sqrt{7 - 16}}{2}, \frac{\sqrt{7} \pm \sqrt{7 - 16}}{2} = -\frac{\sqrt{7}}{2} \pm \frac{3i}{2}, \frac{\sqrt{7}}{2} \pm \frac{3i}{2}$$

C.F. $= e^{-x\sqrt{7}/2}\,(c_1 \cos 3x/2 + c_2 \sin 3x/2) + e^{x\sqrt{7}/2}\,(c_3 \cos 3x/2 + c_4 \sin 3x/2)$,

c_1, c_2, c_3 and c_4 being arbitrary constants.

$$\text{and P.I.} = \frac{1}{D^4 + D^2 + 16}(16x^2 + 256) = \frac{1}{16\{1 + (D^2 + D^4)/16\}}\,(16x^2 + 256)$$

$$= \frac{1}{16}\left[1 + \left(\frac{D^2}{16} + \frac{D^4}{16}\right)\right]^{-1}(16x^2 + 256) = \frac{1}{16}\left[1 - \left(\frac{D^2}{16} + \frac{D^4}{16}\right) + \ldots\right](16x^2 + 256)$$

$$= (1/16) \times [(16x^2 + 256) - (1/16) \times (32)] = x^2 + (127/8)$$

Hence the required solution is

$$y = e^{-x\sqrt{7}/2}\,(c_1 \cos 3x/2 + c_2 \sin 3x/2) + e^{x\sqrt{7}/2}\,(c_3 \cos 3x/2 + c_4 \sin 3x/2) + x^2 + (127/8).$$

Ex. 9. *Solve the equation* $(d^2y/dx^2) = a + bx + cx^2$, *given that* $dy/dx = 0$ *when* $x = 0$ *and* $y = d$, *when* $x = 0$.

Sol. Let $D \equiv d/dx$. Then, we have $D^2 y = a + bx + cx^2$. The A.E. is $D^2 = 0$ so that $D = 0, 0$. Hence C.F. $= (c_1 + c_2 x)\, e^{0.x} = c_1 + c_2 x$, c_1 and c_2 being arbitrary constants.

$$\text{P.I.} = \frac{1}{D^2}(a + bx + cx^2) = \frac{1}{D}\int (a + bx + cx^2)\, dx = \frac{1}{D}\left(ax + \frac{1}{2}bx^2 + \frac{1}{3}cx^3\right)$$

$$= \int (ax + \frac{1}{2}bx^2 + \frac{1}{3}cx^3)\, dx = \frac{1}{2}ax^2 + \frac{1}{6}bx^3 + \frac{1}{12}cx^4.$$

$\therefore$ The general solution is $y = c_1 + c_2 x + (1/2)\, ax^2 + (1/6)\, bx^3 + (1/12)\, cx^4$. ... (1)

From (1), $(dy/dx) = c_2 + ax + 2\, bx^2 + (1/3)\, cx^3$. ... (2)

Putting $x = 0$ and $y = d$ in (1), we get $c_1 = d$. Next, putting $x = 0$ and $dy/dx = 0$ in (2), we get $c_2 = 0$. Putting values of c_1 and c_2 in (1), the desired solution is

$$y = d + (1/2) \times ax^2 + (1/6) \times bx^3 + (1/12) \times cx^4.$$

Ex. 10. *Find solution of* $(D^3 - D^2 - D - 2)\, y = x$. **[Agra 2005]**

Sol. Auxiliary equation of the given equation is given by

$D^3 - D^2 - D - 2 = 0,$ or $(D-2)(D^2+D+1)=0$

$\therefore$ $D = 2, (-1 \pm \sqrt{-3})/2$ *i.e.,* $D = 2, (-1/2) \pm i(\sqrt{3}/2)$.

$\therefore$ C.F. $= c_1 e^{2x} + e^{-x/2}\{c_3 \cos(x\sqrt{3}/2) + c_4 \sin(x\sqrt{3}/2)\}$, c_1, c_2 being arbitrary constants

$$\text{P.I.} = \frac{1}{D^3-D^2-D-2}x = \frac{1}{-2\{1+(D+D^2-D^3)/2\}}x$$

$$= -\frac{1}{2}\{1+\frac{1}{2}(D+D^2-D^3)\}^{-1}x = -\frac{1}{2}\{1-\frac{1}{2}(D+D^2-D^3)+\ldots\}x$$

$$= -\frac{1}{2}\left(x-\frac{1}{2}Dx\right) = -\frac{1}{2}\left(x-\frac{1}{2}\times\frac{x^2}{2}\right) = -\frac{1}{8}(4x-x^2)$$

$\therefore$ The required solution is $y = c_1 e^{2x} + e^{-x/2}\{c_3\cos(x\sqrt{3}/2) + c_4 \sin(x\sqrt{3}/2)\} - (1/8)\times(4x-x^2)$.

Ex. 11. *Solve* $(D^3 - D^2 - 6D)\, y = x^2 + 1$. **[Agra 1995, Garhwal 1996; Bangalore 1995, Lucknow 1992, Allahabad 1994]**

Sol. The auxiliary equation is $D^3 - D^2 - 6D = 0$ giving

$D(D^2 - D - 6) = 0$ or $D(D-3)(D+2) = 0$ so that $D = 0, 3, -2$

$\therefore$ C.F. $= c_1 e^{0.x} + c_2 e^{3x} + c_3 e^{-2x} = c_1 + c_2 e^{3x} + c_3 e^{-2x}$.

$$\text{P.I.} = \frac{1}{D^3-D^2-6D}(x^2+1) = \frac{1}{-6D(1+D/6-D^2/6)}(x^2+1) = -\frac{1}{6D}\left[1+\left(\frac{D}{6}-\frac{D^2}{6}\right)\right]^{-1}(x^2+1)$$

$$= -\frac{1}{6D}\left[1-\left(\frac{D}{6}-\frac{D^2}{6}\right)+\left(\frac{D}{6}-\frac{D^2}{6}\right)^2 \ldots\right](x^2+1) = -\frac{1}{6D}\left[1-\frac{D}{6}+\frac{7}{36}D^2+\ldots\right](x^2+1)$$

$= -(1/6)\times(1/D)\,\{x^2+1-(1/6)\times D(x^2+1)+(7/36)\times D^2(x^2+1)\}$

$$= -\frac{1}{6D}\left[x^2+1-\frac{1}{6}(2x+0)+\frac{7}{36}(2+0)+\ldots\right] = -\frac{1}{6D}\left[x^2-\frac{1}{3}x+\frac{25}{18}\right] = -\frac{1}{6}\left[\frac{x^3}{3}-\frac{x^2}{6}+\frac{25}{18}x\right].$$

Hence the required solution is $y =$ C.F. + P.I., that is,

$$y = c_1 + c_2 e^{3x} + c_3 e^{-2x} - (1/18)\times(x^3 - x^2/2 + 25x/6)$$

Exercise 5(E)

Solve the following differential equations:

1. $(D^2 + D - 6)\, y = x$. **Ans.** $y = c_1 e^{2x} + c_2 e^{-3x} - (1/36)(6x+1)$
2. $(D^2 - 2D + 1)\, y = x - 1$. **Ans.** $y = (c_1 + c_2 x)\, e^x + x + 1$
3. $(D^2 - 4D + 4)\, y = x^2$. **Ans.** $y = (c_1 + c_2 x)\, e^{2x} + (1/4)(x^2 + 2x + 3/2)$
4. $(D^2 - 4)\, y = x^2$. **Ans.** $y = c_1 e^{2x} + c_2 e^{-2x} - (1/8)(2x^2+1)$
5. $(D^3 - 8)\, y = x^3$. **Ans.** $y = c_1 e^{2x} + e^{-x}(c_2 \cos x\sqrt{3} + c_3 \sin x\sqrt{3}) - (1/32)(3 + 4x^3)$
6. $(D^2 + 2D + 1)\, y = 2x + x^2$. **Ans.** $y = (c_1 + c_2 x)\, e^{-x} + x^2 - 2x + 2$
7. $(x^3 + 8)\, y = x^4 + 2x + 1$. **Ans.** $y = c_1 e^{-2x} + e^x(c_2 \cos x\sqrt{3} + c_3 \sin x\sqrt{3}) + (1/8)(x^4 - x + 1)$
8. $(D^4 + 8D^2 + 16)\, y = 16x + 256$. **Ans.** $y = (c_1 + c_2 x)\cos 2x + (c_3 + c_4 x)\sin 2x + x + 16$
9. $(D^4 - 2D^3 + 5D^2 - 8D + 4)\, y = x^2$.
   Ans. $y = (c_1 + c_2 x)\, e^x + c_3 \cos 2x + c_4 \sin 2x + (1/4)(x_2 + 4x + 11/2)$
10. $(D^2 - 1)\, y = 1$ **Ans.** $y = c_1 e^x + c_2 e^{-x} - 1$

11. $(D^2 + 2D + 1)\, y = 2x + x^2$. **Ans.** $y = (c_1 + c_2 x)\, e^{-x} + x^2 - 2x + 2$

12. $(D^2 + 2D - 20)\, y = (x + 1)^2$

Ans. $y = e^{-x}\, \{c_1 \cosh(x\sqrt{21}) + c_2 \sinh(x\sqrt{21})\} - (1/20) \times \{x^2 + (11/5)\, x + 33/25\}$

13. $(D^3 - 3D^2 - 6D + 8)\, y = x$. **Ans.** $y = c_1 e^x + c_2 e^{4x} + c_3 e^{-2x} + (4x + 3)/32$

14. $(D^4 + D^3 + D^2)\, y = x^2 (a + bx)$. **Ans.** $y = e^{-x/2}\, \{c_1 \cos(x\sqrt{3}/2) + c_2 \sin(x\sqrt{3}/2)\} + c_3 + c_4 x$
[G.N.D.U. Amritsar 2010] $+ (1/20) \times bx^5 + (1/12) \times (a - 3b)\, x^4 - (1/3) \times ax^3 - 3bx^2$.

5.20 Short method of finding P.I. when $X = e^{ax} V$, where V is any function of x.

Theorem. $\dfrac{1}{f(D)} e^{ax} V = e^{ax} \dfrac{1}{f(D+a)} V$, V **being a function of** x.

Proof. By successive differentiation, we have

$$D\,(e^{ax}V) = e^{ax}\, DV + a\, e^{ax}\, V = e^{ax}\, (D + a)\, V,$$

$$D^2\,(e^{ax}V) = e^{ax}\, D^2V + a\, e^{ax}\, DV + a\, e^{ax}\, DV + a^2\, e^{ax}V$$

$$= e^{ax}\, (D^2 + 2aD + a^2)\, V = e^{ax}\, (D + a)^2\, V.$$

Similarly, $$D^3\,(e^{ax}V) = e^{ax}\, (D + a)^3\, V,$$

....

$$D^n\,(e^{ax}V) = e^{ax}\, (D + a)^n\, V.$$

$\therefore$ $$f(D)\, e^{ax}V = e^{ax} f(D + a)\, V. \qquad \text{... (1)}$$

The above result (1) is true for any function of x. Taking $\{1/f(D + a)\}\ V$ in place of V in (1), we have

$$f(D)\left\{e^{ax}\, \frac{1}{f(D+a)} V\right\} = e^{ax} f(D+a) \left\{\frac{1}{f(D+a)} V\right\}$$

or $$e^{ax}V = f(D) \left\{e^{ax}\, \frac{1}{f(D+a)} V\right\}. \qquad \text{... (2)}$$

Operating by $1/f(D)$ on both sides of (2), we have

$$\frac{1}{f(D)}\, e^{ax}V = e^{ax}\, \frac{1}{f(D+a)} V. \qquad \text{... (3)}$$

Working rule. Read carefully the above formula, (3). Accordingly, e^{ax} which is on the right of $1/f(D)$ may be taken to the left provided D is replaced by $D + a$. After applying the above formula, $\{1/f(D + a)\}\ V$ is evaluated by short methods of Art 5.16 or 5.18 as the case may be.

5.21 Solved examples based on Art. 5.20

Ex. 1. *Solve* $(D^2 - 2D + 1)\, y = x^2\, e^{3x}$.

[Purvanchal 2007, Delhi 1993, 2005, 08; Agra 2003]

Sol. The auxiliary equation of the given equation $D^2 - 2D + 1 = 0 \Rightarrow D = 1, 1$.

$\therefore$ C.F. $= (c_1 + c_2 x)\, e^x$, c_1, c_2 being arbitrary constants.

$$\text{P.I.} = \frac{1}{D^2 - 2D + 1} x^2\, e^{3x} = \frac{1}{(D-1)^2} x^2\, e^{3x} = e^{3x}\, \frac{1}{(D+3-1)^2} x^2$$

$$= e^{3x}\, \frac{1}{(D+2)^2} x^2 = e^{3x}\, \frac{1}{4(1+D/2)^2} x^2 = \frac{1}{4} e^{3x} \left(1 + \frac{D}{2}\right)^{-2} x^2$$

$$= \frac{1}{4} e^{3x} \left[1 - \frac{D}{2} + \frac{(-2)(-3)}{2!} \frac{D^2}{4} + \ldots\right] x^2 = \frac{1}{4} e^{3x} \left(1 - \frac{D}{2} + \frac{3}{4} D^2 + \ldots\right) x^2$$

$= (1/4) \times e^{3x} \{x^2 - (1/2) \times (2x) + (3/4) \times 2\} = (1/8) \times e^{3x} (2x - 4x + 3).$

$\therefore$ Required solution is $\quad y = (c_1 + c_2 x) e^x + (1/8) \times e^{3x} (2x^2 - 4x + 3).$

Ex. 2. *Solve (a)* $(D^2 - 2D + 1) y = x^2 e^x$

(b) $(D^2 - 6D + 9) y = x^2 e^{3x}$ **[G.N.D.U. Amritsar 2010]**

Sol. (*a*) Here the auxiliary equation $\quad D^2 - 2D + 1 = 0 \Rightarrow D = 1, 1.$

$\therefore$ C.F. $= (c_1 + c_2 x) e^x$, c_1, c_2 being arbitrary constants.

and

$$\text{P.I.} = \frac{1}{D^2 - 2D + 1} e^x x^2 = \frac{1}{(D-1)^2} e^x x^2 = e^x \frac{1}{(D+1-1)^2} x^2$$

$$= e^x \frac{1}{D^2} x^2 = e^x \frac{1}{D} \int x^2 dx = e^x \frac{1}{D} \frac{x^3}{3} = e^x \int \frac{x^3}{3} dx = \frac{e^x}{3} \cdot \frac{x^4}{4}.$$

Hence the required solution is $\quad y = (c_1 + c_2 x) e^x + (1/12) \times x^4 e^x.$

(*b*) Try yourself. **Ans.** $y = (c_1 + c_2 x) e^{3x} + (x^4/12) \times e^{3x}$

Ex. 3. *Find the particular solution of* $(D - 1)^2 y = e^x \sec^2 x \tan x.$ **[Kuvempa 2005]**

Sol.

$$\text{P.I.} = \frac{1}{(D-1)^2} e^x \sec^2 x \tan x = e^x \frac{1}{(D+1-1)^2} \tan x \sec^2 x$$

$$= e^x \frac{1}{D} \int \tan x \sec^2 x \, dx = e^x \frac{1}{D} \left(\frac{\tan^2 x}{2} \right) = \frac{e^x}{2} \int \tan^2 x \, dx$$

$$= \frac{e^x}{2} \int (\sec^2 x - 1) dx = \frac{e^x}{2} (\tan x - x)$$

Ex. 4. *Solve* $(D - a)^2 y = e^{ax} f'(x)$ **[Agra 2006]**

Sol. Here auxiliary equation of the given equation is $\quad (D - a)^2 = 0 \quad$ so that $\quad D = a, a.$

$\therefore$ C.F. $= (c_1 + c_2 x) e^{ax}$, c_1 and c_2 being arbitrary constants

$$\text{P.I.} = \frac{1}{(D-a)^2} e^{ax} f'(x) = e^{ax} \frac{1}{(D+a-a)^2} f'(x) = e^{ax} \frac{1}{D} \frac{1}{D} f'(x) = e^{ax} \frac{1}{D} f(x) = e^{ax} \int f(x) dx$$

$\therefore$ The required solution is $\quad y = (c_1 + c_2 x) e^{ax} + e^{ax} \int f(x) dx$

Ex. 5. *Solve* $(D^3 - 3D - 2) y = 540 x^3 e^{-x}$ **[Lucknow 1994]**

Sol. Here the auxiliary equation of the given equation is $\quad D^3 - 3D - 2 = 0$

or $\quad (D - 2)(D^2 + 2D + 1) = 0 \quad$ so that $\quad D = 2, -1, -1.$

$\therefore$ C.F. $= c_1 e^{2x} + (c_2 + c_3 x) e^{-x}$, c_1, c_2, c_3 being arbitrary constants and

$$\text{P.I.} = \frac{1}{D^3 - 3D - 2} 540 x^3 e^{-x} = 540 e^{-x} \frac{1}{(D-1)^3 - 3(D-1) - 2} x^3 = 540 e^{-x} \frac{1}{D^3 - 3D^2} x^3$$

$$= 540 e^{-x} \frac{1}{-3D^2 (1 - D/3)} x^3 = -180 e^{-x} \frac{1}{D^2} \left(1 - \frac{D}{3}\right)^{-1} x^3 = -180 e^{-x} \frac{1}{D^2} \left(1 + \frac{D}{3} + \frac{D^2}{9} + \frac{D^3}{27} + \ldots\right) x^3$$

$$= -180 e^{-x} \frac{1}{D^2} \left(x^3 + x^2 + \frac{2}{3} x + \frac{2}{9} \right) = -180 e^{-x} \frac{1}{D} \left(\frac{x^4}{4} + \frac{x^3}{3} + \frac{x^2}{3} + \frac{2x}{9} \right)$$

$$= -180 e^{-x} \left(\frac{x^5}{20} + \frac{x^4}{12} + \frac{x^3}{9} + \frac{x^2}{9} \right) = -e^{-x} (9x^5 + 15x^4 + 20x^3 + 20x^2).$$

$\therefore$ Solution is $\quad y = c_1 e^{2x} + (c_2 + c_3 x) e^{-x} - e^{-x} (9x^5 + 15x^4 + 20x^3 + 20x^2).$

Ex. 6. *Solve* $(D^2 + 3D + 2)\, y = e^{2x} \sin x$. **[Delhi Maths (Prog.) 2007; Delhi Maths(H) 1996]**

Sol. Here the auxiliary equation $\quad D^2 + 3D + 2 = 0 \Rightarrow D = -2, -1.$

$\therefore$ C.F. $= c_1 e^{-2x} + c_2 e^{-x}$, where c_1 and c_2 are arbitrary constants

$$\text{P.I.} = \frac{1}{D^2+3D+2} e^{2x} \sin x = e^{2x} \frac{1}{(D+2)^2+3(D+2)+2} \sin x = e^{2x} \frac{1}{D^2+7D+12} \sin x$$

$$= e^{2x} \frac{1}{-1^2+7D+12} \sin x = e^{2x} \frac{1}{11+7D} \sin x = e^{2x}(11-7D) \frac{1}{(11-7D)(11+7D)} \sin x$$

$$= e^{2x}(11-7D) \frac{1}{121-49D^2} \sin x = e^{2x}(11-7D) \frac{1}{121-49(-1^2)} \sin x = \frac{e^{2x}}{170}(11-7D) \sin x$$

$= (1/170) \times e^{2x}\, (11 \sin x - 7D \sin x) = (1/170) \times e^{2x}\, (11 \sin x - 7 \cos x).$

$\therefore$ Required solution is $\quad y = c_1 e^{-2x} + c_2 e^{-x} + (1/170)\, e^{2x}\, (11 \sin x - 7 \cos x).$

Ex. 7. *Solve* $(D^4 - 1)\, y = e^x \cos x$.

Sol. Here the auxiliary equation is $\quad D^4 - 1 = 0 \quad$ or $\quad (D^2 - 1)(D^2 + 1) = 0$

so that $\quad D^2 - 1 = 0 \quad$ or $\quad D^2 + 1 = 0 \quad$ giving $\quad D = 1, -1, \pm i.$

$\therefore$ C.F. $= c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$, c_1, c_2, c_3, c_4 being arbitrary constants.

and
$$\text{P.I.} = \frac{1}{D^4-1} e^x \cos x = e^x \frac{1}{(D+1)^4-1} \cos x = e^x \frac{1}{(D^4+4D^3+6D^2+4D+1)-1} \cos x$$

$$= e^x \frac{1}{(D^2)^2+4D\cdot D^2+6D^2+4D} \cos x = e^x \frac{1}{(-1^2)^2+4D(-1^2)+6(-1^2)+4D} e^x = -\frac{1}{5} e^x \cos x$$

$\therefore$ Required solution is $\quad y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x - (1/5) \times e^x \cos x.$

Ex. 8. *Solve* $(D^3 - D^2 + 3D + 5)\, y = e^x \cos x$. **[Delhi Maths (G) 1995]**

Sol. Here the auxiliary equation is $\quad D^3 - D^2 + 3D + 5 = 0$

or $\quad (D + 1)(D^2 - 2D + 5) = 0 \quad$ so that $\quad D = -1, 1 \pm 2i.$

$\therefore$ C.F. $= c_1 e^{-x} + e^x (c_2 \cos 2x + c_3 \sin 2x)$, c_1, c_2, c_3 being arbitrary constants

$$\text{P.I.} = \frac{1}{D^3-D^2+3D+5} e^x \cos x = e^x \frac{1}{(D+1)^3-(D+1)^2+3(D+1)+5} \cos x$$

$$= e^x \frac{1}{D^3-D^2+4D+7} \cos x = e^x \frac{1}{D(-1^2)-(-1^2)+4D+7} \cos x$$

$$= e^x \frac{1}{3D+5} \cos x = e^x (3D-5) \frac{1}{(3D-5)(3D+5)} \cos x$$

$$= e^x (3D-5) \frac{1}{9D^2-25} \cos x = e^x (3D-5) \frac{1}{9(-1^2)-25} \cos x$$

$= -(1/34) \times e^x\, (3D \cos x - 5 \cos x) = (1/34) \times e^x\, (3 \sin x + 5 \cos x).$

$\therefore$ Required solution is $\; y = c_1 e^{-x} + e^x (c_2 \cos 2x + c_3 \sin 2x) + (1/34) \times e^x\, (3 \sin x + 5 \cos x).$

Ex. 9 (*a*). *Solve* $(D^2 - 1)\, y = \cosh x \cos x$. **[Rohilkhand 1994]**

Sol. Given $\quad (D^2 - 1)\, y = \cosh x \cos x = (1/2) \times (e^x + e^{-x}) \cos x.$

or $\quad (D^2 - 1)\, y = (1/2) \times e^x \cos x + (1/2) \times e^{-x} \cos x \quad \ldots (1)$

Here auxiliary equation is $D^2 - 1 = 0$, giving $D = 1, -1$. So $\quad$ C.F. $= c_1 e^x + c_2 e^{-x}$.

P.I. corresponding to $(1/2) \times e^x \cos x$

$$= \frac{1}{2}\frac{1}{D^2-1}e^x\cos x = \frac{e^x}{2}\frac{1}{(D+1)^2-1}\cos x = \frac{e^x}{2}\frac{1}{D^2+2D}\cos x$$

$$= \frac{e^x}{2}\frac{1}{-1^2+2D}\cos x = \frac{e^x}{2}(2D+1)\cdot\frac{1}{(2D+1)\,(2D-1)}\cos x$$

$$= \frac{e^x}{2}(2D+1)\cdot\frac{1}{4D^2-1}\cos x = \frac{e^x}{2}(2D+1)\frac{1}{-4-1}\cos x = -\frac{e^x}{10}(2D+1)\cos x$$

$= -\,(1/10) \times e^x\,(2D \cos x + \cos x) = -\,(1/10) \times e^x\,(-\,2 \sin x + \cos x)$

P.I. corresponding to $(1/2) \times e^{-x} \cos x$.

$$= \frac{1}{2}\frac{1}{D^2-1}e^{-x}\cos x = \frac{1}{2}e^{-x}\frac{1}{(D-1)^2-1}\cos x = \frac{e^{-x}}{2}\frac{1}{D^2-2D}\cos x$$

$$= \frac{e^x}{2}\frac{1}{(-1^2-2D)}\cos x = -\frac{e^{-x}}{2}(2D-1)\frac{1}{(2D-1)(2D+1)}\cos x$$

$$= -\frac{e^{-x}}{2}(2D-1)\frac{1}{4D^2-1}\cos x = -\frac{e^{-x}}{2}(2D-1)\frac{1}{4(-1^2)-1}\cos x$$

$= (1/10) \times e^{-x}\,(2D - 1) \cos x = (1/10) \times e^{-x}\,(-\,2 \sin x - \cos x).$

$\therefore$ Solution is $y = c_1e^x + c_2e^{-x} - (1/10) \times e^x\,(-2 \sin x + \cos x) + (1/10) \times e^{-x}\,(-2\sin x - \cos x)$

or $\quad y = c_1e^x + c_2e^{-x} + (2/5) \times \sin x\ \{(e^x - e^{-x})/2\} - (1/5) \times \cos x\ \{(e^x + e^{-x})/2\}$

or $\quad y = c_1e^x + c_2e^{-x} + (2/5) \times \sin x \sinh x - (1/5) \times \cos x \cosh x.$

Ex. 9 (*b*). *Solve* $(D^2 - 1)\,y = \cosh x \cos x + a^x$.

Sol. As in Ex. 19 (*a*), find C.F. and P.I. corresponding to $\cosh x \cos x$.

Now, P.I. corresponding to a^x.

$$= \frac{1}{D^2-1}a^x = \frac{1}{D^2-1}e^{x\log a} = \frac{1}{(\log a)^2-1}e^{x\log a} = \frac{1}{(\log a)^2-1}a^x$$

$\therefore$ solution is $y = c_1e^x + c_2e^{-x} + (2/5) \sinh x \sin x - (1/5) \cosh x \cos x + a^x/\{(\log a)^2 - 1\}$

Ex. 10. *Solve* $(D^4 + D^2 + 1)\,y = e^{-x/2} \cos\,(x\sqrt{3}/2)$ **[I.A.S. 1993]**

Sol. Given $\quad (D^4 + D^2 + 1)\,y = e^{-x/2} \cos\,(x\sqrt{3}/2).$ ... (1)

Here auxiliary equation is $\quad D^4 + D^2 + 1 = 0 \quad$ or $\quad (D^2 + 1)^2 - D^2 = 0$

or $\quad (D^2 + 1 + D)\,(D^2 + 1 - D) = 0 \Rightarrow D^2 + D + 1 = 0 \quad$ or $\quad D^2 - D + 1 = 0,$

Solving these, $\quad D = -\,(1/2) \pm i\,(\sqrt{3}/2), \quad (1/2) \pm i\,(\sqrt{3}/2).$

$\therefore$C.F. $= e^{-x/2}\,[c_1 \cos\,(x\sqrt{3}/2) + c_2 \sin\,(x\sqrt{3}/2)] + e^{x/2}\,[c_3 \cos\,(x\sqrt{3}/2) + c_4 \sin\,(x\sqrt{3}/2)]$, c_1, c_2, c_3 and c_4 being arbitrary constants

$$\text{P.I.} = \frac{1}{D^4+D^2+1}e^{-x/2}\cos(x\sqrt{3}/2) = e^{-x/2}\frac{1}{(D-1/2)^4+(D-1/2)^2+1}\cos(x\sqrt{3}/2)$$

$$= e^{-x/2}\frac{1}{D^4-4D^3\cdot(1/2)+6D^2(1/2)^2-4D(1/2)^3+(1/2)^4+D^2-D+(1/4)+1}\cos\frac{x\sqrt{3}}{2}$$

$$= e^{-x/2}\frac{1}{D^4-2D^3+(5/2)D^2-(3/2)D+(21/16)}\cos\frac{x\sqrt{3}}{2}$$

$= e^{-x/2}\dfrac{1}{[(D^2+(3/4)][D^2-2D+(7/4)]}\cos\dfrac{x\sqrt{3}}{2}$ [Since denominator is zero when $D^2 = -3/4$ so $D^2 + (3/4)$ must be a factor of the denominator]

$$= e^{-x/2}\frac{1}{D^2+(3/4)}\cdot\frac{1}{(-3/4)-2D+(7/4)}\cos\frac{x\sqrt{3}}{2} = e^{-x/2}\frac{1}{D^2+(3/4)}\frac{1}{1-2D}\cos\frac{x\sqrt{3}}{2}$$

$$= e^{-x/2}\frac{1}{D^2+(3/4)}\cdot(1+2D)\cdot\frac{1}{(1+2D)(1-2D)}\cos\frac{x\sqrt{3}}{2}$$

$$= e^{-x/2}\frac{1}{D^2+(3/4)}(1+2D)\frac{1}{1-4D^2}\cos\frac{x\sqrt{3}}{2} = e^{-x/2}\frac{1}{D^2+(3/4)}(1+2D)\frac{1}{1-4\times(-3/4)}\cos\frac{x\sqrt{3}}{2}$$

$$= \frac{1}{4}e^{-x/2}\frac{1}{D^2+(3/4)}(1+2D)\cos\frac{x\sqrt{3}}{2} = \frac{1}{4}e^{-x/2}\frac{1}{D^2+(\sqrt{3}/2)^2}\left[\cos\frac{x\sqrt{3}}{2}-\sqrt{3}\sin\frac{x\sqrt{3}}{2}\right]$$

$$= \frac{1}{4}e^{-x/2}\left[\frac{1}{D^2+(\sqrt{3}/2)^2}\cos\frac{x\sqrt{3}}{2}-\sqrt{3}\frac{1}{D^2+(\sqrt{3}/2)^2}\sin\frac{x\sqrt{3}}{2}\right]$$

$= \dfrac{1}{4}e^{-x/2}\left[\dfrac{x}{2\times(\sqrt{3}/2)}\sin\dfrac{x\sqrt{3}}{2}+\sqrt{3}\cdot\dfrac{x}{2\times(\sqrt{3}/2)}\cos\dfrac{x\sqrt{3}}{2}\right]$, using results (4) and (5) of Art. 5.16

$$= (x/4\sqrt{3})\times e^{-x/2}\ \{\sin(x\sqrt{3}/2)+\sqrt{3}\cos(x\sqrt{3}/2)\}.$$

$\therefore$ The required solution is $y = e^{-x/2}\,[c_1\cos(x\sqrt{3}/2) + c_2\sin(x\sqrt{3}/2)] + e^{x/2}\,[c_3\cos(x\sqrt{3}/2)$
$+ c_4\sin(x\sqrt{3}/2)] + (x/4\sqrt{3}) \times e^{-x/2}\,[\sin(x\sqrt{3}/2) + \sqrt{3}\cos(x\sqrt{3}/2)]$

Ex. 11. *Solve $(d^4y/dx^4) + 6(d^3y/dx^3) + 11\,(d^2y/dx^2) + 6\,(dy/dx) = 20\,e^{-2x}\sin x$.* **[I.A.S. 1997]**

Sol. Re - writing the given equation, $(D^4 + 6D^3 + 11D^2 + 6D)\,y = 20\,e^{-2x}\sin x$. ...(1)

Its auxiliary equation is $D^4 + 6D^3 + 11D^2 + 6D = 0$ or $D\,(D + 1)\,(D + 2)\,(D + 3) = 0$

Solving it, we get $D = 0, -1, -2, -3$.

$\therefore$ C.F. $= c_1 e^{0.x} + c_2 e^{-x} + c_3 e^{-2x} + c_4 e^{-3x}$, c_1, c_2, c_3 and c_4 being arbitrary constants.

$$\text{P.I.} = \frac{1}{D^4+6D^3+11D^2+6D}20e^{-2x}\sin x$$

$$= 20e^{-2x}\frac{1}{(D-2)^4+6(D-2)^3+11(D-2)^2+6(D-2)}\sin x$$

$$= 20\,e^{-2x}\frac{1}{D^4-8D^3+24D^2+32D+16+6(D^3-6D^2+12D-8)+11(D^2-4D+4)+6(D-2)}\sin x$$

$$= 20\,e^{-2x}\frac{1}{D^4-2D^3-D^2+2D}\sin x = 20\,e^{-2x}\frac{1}{(D^2)^2-2D(D^2)-D^2+2D}\sin x$$

$$= 20\,e^{-2x}\frac{1}{(-1^2)^2-2(-1^2)D-(-1^2)+2D}\sin x = 20e^{-2x}\frac{1}{2(1+2D)}\sin x$$

$$= 10\,e^{-2x}(1-2D)\frac{1}{1-4D^2}\sin x = 10e^{-2x}(1-2D)\frac{1}{1-4(-1^2)}\sin x$$

$= 2\, e^{-2x}\,(1 - 2D) \sin x = 2\, e^{-2x}\,(\sin x - 2\cos x)$

$\therefore$ The required solution is $y = c_1 + c_2 e^{-x} + c_3 e^{-2x} + c_4 e^{-3x} - 2\, e^{-2x}\,(\sin x - 2\cos x)$

Ex. 12. *Solve* $(D^2 + 2D + 1)\, y = xe^x \sin x$. **[Agra 1996, Lucknow 1996, Purvanchal 1998]**

Sol. Here the auxiliary equation is $D^2 - 2D + 1 = 0$ so that $D = 1, 1$.

$\therefore$ C.F. $= (c_1 + c_2 x)\, e^x$, c_1, c_2 being arbitrary constants.

$$\text{P.I.} = \frac{1}{(D-1)^2} e^x (x \sin x) = e^x \frac{1}{(D+1-1)^2} x \sin x = e^x \frac{1}{D^2} x \sin x$$

$$= e^x \frac{1}{D}\int x \sin x\, dx = e^x \frac{1}{D}\left[-x\cos x - \int 1\cdot(-\cos x)\, dx\right], \text{ integrating by parts}$$

$$= e^x \frac{1}{D}[-x\cos x + \sin x] = e^x \int (\sin x - x\cos x)\, dx$$

$$= e^x \left[\int \sin x\, dx - \int x \cos dx\right] = e^x\left[-\cos x - \{x \sin x - \int 1\cdot \sin x\, dx\}\right]$$

$$= e^x\,(-\cos x - x\sin x - \cos x) = -\,e^x\,(x \sin x + 2\cos x).$$

$\therefore$ The required solution is $y = (c_1 + c_2 x)\, e^x - e^x\,(x\sin x + 2\cos x)$.

Ex. 13(*a*) Solve $(D^2 - 4D + 4)\, y = e^{2x} \sin 2x$ **[Purvanchal 2007]**

Sol. Given $(D^2 - 4D + 4)\, y = e^{2x} \sin 2x,\; D \equiv d/dx$ ... (1)

Its auxiliary equation is $D^2 - 4D + 4 = 0$ or $(D-2)^2 = 0 \Rightarrow 0 = 2,2$.

$\therefore$ C.F. $= (c_1 + c_2 x)\, e^{2x}$, c_1, c_2 being arbitrary constants

$$\text{P.I.} = \frac{1}{D^2 - 4D + 4} e^{2x} \sin x = \frac{1}{(D-2)^2} e^{2x} \sin x = e^{2x} \frac{1}{(D+2-2)^2} \sin x$$

$$= e^{2x}\frac{1}{D^2}\sin x = e^{2x}\frac{1}{D}\int \sin x\, dx = e^{2x}\frac{1}{D}(-\cos x) = -e^{2x}\int \cos x\, dx = -\,e^{2x}\sin x$$

$\therefore$ Required solution is $y = (c_1 + c_2 x)\, e^{2x} - e^{2x} \sin x$

Ex. 13(*b*) *Solve* $(D^2 - 4D + 4)\, y = 8x^2 e^{2x} \sin 2x$. **[Kanpur 1997, Lucknow 1993, 97]**

Sol. Here the auxiliary equation is $D^2 - 4D + 4 = 0$ giving $D = 2, 2$.

$\therefore$ C.F. $= (c_1 + c_2 x)\, e^{2x}$, c_1, c_2 being arbitrary constants.

$$\text{P.I.} = \frac{1}{(D-2)^2} 8x^2 e^{2x} \sin 2x = 8e^{2x} \frac{1}{(D+2-2)^2} x^2 \sin 2x = 8e^{2x} \frac{1}{D^2} x^2 \sin 2x$$

$$= 8e^{2x}\frac{1}{D}\int x^2 \sin 2x\, dx = 8e^{2x}\frac{1}{D}\left[x^2\left(-\frac{\cos 2x}{2}\right) - \int (2x)\left(-\frac{\cos 2x}{2}\right)dx\right], \text{ integrating by parts}$$

$$= 8e^{2x}\frac{1}{D}\left[-\frac{1}{2}x^2\cos 2x + \int x\cos 2x\, dx\right] = 8e^{2x}\frac{1}{D}\left[-\frac{1}{2}x^2\cos 2x + x\left(\frac{\sin 2x}{2}\right) - \int 1\cdot\frac{\sin 2x}{2}dx\right]$$

$$= 8e^{2x}\frac{1}{D}\left[-\frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x\right] = 8e^{2x}\int\left(-\frac{1}{2}x^2\cos 2x + \frac{1}{2}x\sin 2x + \frac{1}{4}\cos 2x\right)dx$$

$$= 8e^{2x}\left[-\frac{1}{2}\int x^2\cos 2x\;\; dx + \frac{1}{2}\int x\sin 2x\;\; dx + \frac{1}{4}\int \cos 2x\;\; dx\right]$$

$$= 8e^{2x}\left[-\frac{1}{2}\left\{x^2\left(\frac{1}{2}\sin 2x\right) - \int 2x\left(\frac{1}{2}\sin 2x\right)dx\right\} + \frac{1}{2}\int x\sin 2x\, dx + \frac{1}{8}\sin 2x\right]$$

$$= 8e^{2x}\left[-\frac{1}{4}x^2\sin 2x\;\; dx + \frac{1}{2}\int x\sin 2x\, dx + \frac{1}{2}\int x\sin 2x\, dx + (1/8)\times\sin 2x\right]$$

$$= 8e^{2x}[-\frac{1}{4}x^2 \sin 2x + \int x \sin 2x\, dx + \frac{1}{8}\sin 2x]$$

$$= 8e^{2x}\left[-\frac{1}{4}x^2 \sin 2x + x(-\frac{1}{2}\cos 2x) - \int 1\cdot(-\frac{1}{2}\cos 2x)\, dx + \frac{1}{8}\sin 2x\right]$$

$$= 8e^{2x}[-\frac{1}{4}x^2 \sin 2x - \frac{1}{2}x\cos 2x + \frac{1}{4}\sin 2x + \frac{1}{8}\sin 2x]$$

$$= 8e^{2x}[-\frac{1}{4}x^2 \sin 2x - \frac{1}{2}x\cos 2x + \frac{3}{8}\sin 2x]$$

$\therefore$ The required solution is $y = (c_1 + c_2x)\, e^{2x} + e^{2x}\,(3 \sin 2x - 4x \cos 2x - 2x^2 \sin 2x)$.

Ex. 14. *Solve* $(D^3 + 1)\, y = e^{2x} \sin x + e^{x/2} \sin (\sqrt{3}x/2)$.

Sol. Given $(D^3 + 1)\, y = e^{2x} \sin x + e^{x/2} \sin (\sqrt{3}x/2)$. ... (1)

The auxiliary equation is $D^3 + 1 = 0$, or $(D + 1)(D^2 - D + 1) = 0$,

giving $D = -1,\ \{1 \pm (1-4)^{1/2}\}/2$ or $D = -1,\ 1/2 \pm i(\sqrt{3}/2)$

$\therefore$ C.F. $= c_1e^{-x} + e^{x/2}\,[c_2 \cos (x\sqrt{3}/2) + c_3 \sin (x\sqrt{3}/2)]$,

c_1, c_2 and c_3 being arbitrary constants.

P.I. corresponding to $e^{2x} \sin x$.

$$= \frac{1}{D^3+1}e^{2x}\sin x = e^{2x}\frac{1}{(D+2)^3+1}\sin x = e^{2x}\frac{1}{D^3+3D^2\cdot 2+3D\cdot 2^2+2^3+1}\sin x$$

$$= e^{2x}\frac{1}{D^3+6D^2+12D+9}\sin x = e^{2x}\frac{1}{-D-6+12D+9}\sin x$$

$$= e^{2x}\frac{1}{11D+3}\sin x = e^{2x}(11D-3)\frac{1}{121D^2-9}\sin x = e^{2x}\frac{(11D-3)}{-121-9}\sin x$$

$= -(1/130) \times e^{2x}\,(11 \cos x - 3 \sin x)$.

P.I. corresponding to $e^{x/2} \sin (x\sqrt{3}/2)$

$$= \frac{1}{D^3+1}e^{x/2}\sin\left(\frac{x\sqrt{3}}{2}\right) = e^{x/2}\frac{1}{(D+1/2)^3+1}\sin\left(\frac{x\sqrt{3}}{2}\right)$$

$$= e^{x/2}\frac{1}{D^3+3D^2\cdot(1/2)+3D\cdot(1/2)^2+(1/2)^3+1}\sin\left(\frac{x\sqrt{3}}{2}\right)$$

$$= e^{x/2}\frac{1}{D^3+(3/2)D^2+(3/4)D+(9/8)}\sin\left(\frac{x\sqrt{3}}{2}\right) = e^{x/2}\frac{1}{[D^2+(3/4)][D+(3/2)]}\sin\left(\frac{x\sqrt{3}}{2}\right)$$

[As denominator is zero when $D^2 = -3/4$, factorize denominator]

$$= e^{x/2}\frac{1}{D^2+(3/4)}\left(D-\frac{3}{2}\right)\cdot\frac{1}{D^2-(9/4)}\sin\left(\frac{x\sqrt{3}}{2}\right)$$

$$= e^{x/2}\frac{1}{D^2+(3/4)}\left(D-\frac{3}{2}\right)\frac{1}{(-3/4)-(9/4)}\sin\left(\frac{x\sqrt{3}}{2}\right)$$

$$= -\frac{e^{x/2}}{3}\frac{1}{D^2+(3/4)}\left(D-\frac{3}{2}\right)\sin\left(\frac{x\sqrt{3}}{2}\right) = -\frac{e^{x/2}}{3}\frac{1}{D^2+(3/4)}\left[\frac{\sqrt{3}}{2}\cos\left(\frac{x\sqrt{3}}{2}\right)-\frac{3}{2}\sin\left(\frac{x\sqrt{3}}{2}\right)\right]$$

$$= -\frac{e^{x/2}}{6}\left[\sqrt{3}\,\frac{1}{D^2+(\sqrt{3}/2)^2}\cos\left(\frac{x\sqrt{3}}{2}\right)-3\frac{1}{D^2+(\sqrt{3}/2)^2}\sin\left(\frac{x\sqrt{3}}{2}\right)\right]$$

$$= -\frac{e^{x/2}}{6}\left[\sqrt{3}\,\frac{x}{2(\sqrt{3}/2)}\sin\left(\frac{x\sqrt{3}}{2}\right)+3\cdot\frac{1}{2(\sqrt{3}/2)}\cos\left(\frac{x\sqrt{3}}{2}\right)\right]$$

$$= -\frac{xe^{x/2}}{6}\left[\sin\left(\frac{x\sqrt{3}}{2}\right)+\sqrt{3}\cos\left(\frac{x\sqrt{3}}{2}\right)\right].$$

$\therefore$ Required solution is $y = c_1e^{-x} + e^{x/2}\,[c_2\cos(x\sqrt{3}/2) + c_2\sin(x\sqrt{3}/2)]$
$- (x/6) \times e^{x/2}\,[\sin(x\sqrt{3}/2) + \sqrt{3}\cos(x\sqrt{3}/2)]$.

Ex. 15. *Solve* $(d^2y/dx^2) - 5\,(dy/dx) + 6y = e^{4x}\,(x^2 + 9)$. **[I.A.S. 1998]**

Sol. Given $(D^2 - 5D + 6)\,y = e^{4x}\,(x^2 + 9)$, *where* $D \equiv d/dx$ ... (1)

Its auxiliary equation is $D^2 - 5D + 6 = 0$ so that $D = 2, 3$.

Hence, here C.F. $= c_1 e^{2x} + c_2 e^{3x}$, c_1 and c_2 being arbitrary constants.

$$\text{P.I.} = \frac{1}{D^2-5D+6}e^{4x}(x^2+9) = e^{4x}\frac{1}{(D+4)^2-(D+4)+6}(x^2+9) = e^{4x}\frac{1}{D^2+3D+2}(x^2+9)$$

$$= \frac{e^{4x}}{2}\,\frac{1}{1+D^2/2+3D/2}\,(x^2+9) = \frac{1}{2}e^{4x}\{1+(D^2/2+3D/2)\}^{-1}$$

$$= (1/2)\times e^{4x}\left\{1-(D^2/2+3D/2)+(D^2/2+3D/2)^2\ldots\ldots\right\}(x^2+9)$$

$= (1/2) \times e^{4x}\,[1 - (D^2/2) - (3D/2) + 9D^2/4 + \ldots]\,(x^2 + 9)$

$= (1/2) \times e^{4x}\,\{(x^2 + 9) - (3/2)\,D\,(x^2 + 9) + (7/4)\,D^2\,(x^2 + 9)]$

$= (1/2) \times e^{4x}\,\{x^2 + 9 - (3/2)\,(2x) + (7/4)\,(2)\}$

$= (1/2) \times e^{4x}\,(x^2 - 3x + 25/2) = (1/4)\,e^{4x}\,(2x^2 - 6x + 25)$

Hence the required solution is $y = c_1 e^{2x} + c_2 e^{3x} + (1/4)\,e^{4x}\,(2x^2 - 6x + 25)$.

Ex. 16. *Solve* $d^2y/dx^2 - 4\,(dy/dx) - 5y = xe^{-x}$, *given that* $y = 0$ *and* $dy/dx = 0$ *when* $x = 0$.

Sol. Re-writting the given equation, $(D^2 - 4D - 5)\,y = xe^{-x}$ where $D = d/dx$ ... (1)

Also, given that $y = 0$, when $x = 0$ ... (2)

and $dy/dx = 0$, when $x = 0$ ... (3)

The auxiliary equation of (1) is $D^2 - 4D - 5 = 0$ or $(D - 5)\,(D + 1) = 0 \Rightarrow D = 5, -1$.

So C.F. $= c_1e^{5x} + c_2e^{-x}$, c_1, c_2 being arbitrary constants

and $$\text{P.I.} = \frac{1}{D^2-4D-5}xe^{-x} = e^{-x}\frac{1}{(D-1)^2-4(D-1)-5}x$$

$$= e^{-x}\frac{1}{D^2-6D}x = -\frac{1}{6}e^{-x}\frac{1}{D(1-D/6)}x = -\frac{1}{6}e^{-x}\frac{1}{D}\left(1-\frac{D}{6}\right)^{-1}x = -\frac{1}{6}e^{-x}\frac{1}{D}\left(1+\frac{D}{6}+\ldots\right)x$$

$$= -\frac{1}{6}e^{-x}\frac{1}{D}\left(x+\frac{1}{6}\right) = -\frac{1}{6}e^{-x}\left(\frac{x^2}{2}+\frac{x}{6}\right).$$

$\therefore$ The solution of (1) is $y = c_1 e^{5x} + c_2 e^{-x} - (1/12)\,e^{-x}(x^2 + x/3)$ ... (4)

Putting $x = 0$, $y = 0$ (due to (2)), in (4), we get $\qquad c_1 + c_2 = 0.$ ... (5)

From (4), $$\frac{dy}{dx} = 5c_1e^{5x} - c_2e^{-x} + \frac{1}{12}e^{-x}\left(x^2 + \frac{x}{3}\right) - \frac{1}{12}e^{-x}\left(2x + \frac{1}{3}\right) \qquad ...(6)$$

Putting $x = 0$, $dy/dx = 0$ (due to (3)) in (6), we get $\qquad 5c_1 - c_2 - (1/36) = 0.$ ... (7)

Solving (5) and (7), $c_1 = 1/216$ and $c_2 = -(1/216)$. With these values, (4) reduces to

$$y = (1/216) \times (e^{5x} - e^{-x}) - (1/12) \times e^{-x} (x^2 + x/3).$$

Exercise 5(F)

Solve the following differential equations:

1. $(D - 2)^3\, y = xe^{2x}$. **Ans.** $y = (c_1 + c_2x + c_3x^2)\, e^{2x} + (x^4/24) \times e^{2x}$

2. $(D + 1)^3\, y = x^2 e^{-x}$ **Ans.** $y = (c_1 + c_2x + c_3x^2)\, e^{-x} + (x^5/60) \times e^{-x}$

3. $(D^2 - 3D + 2)\, y = xe^x$ **[Agra 1994]** **Ans.** $y = c_1e^x + c_2\, e^{2x} + (1/2)e^x(x^2 + 2x)$

4. $(D^2 - 1)\, y = e^x \cos x$. **Ans.** $y = c_1\, e^x + c_2\, e^{-x} + (1/5)\, e^x\, (2 \sin x - \cos x)$

5. $(D^2 - 4D + 1)\, y = e^{2x} \sin x$. **Ans.** $y = c_1\, e^{2x} \cosh (x\sqrt{3} + c_2) - (1/7)\, e^{2x} \sin 3x$

6. $(D^2 - 2D + 5)\, y = e^{2x} \sin x$ **[Agra 1996, Rohilkhand 1996]**

Ans. $y = e^x\, (c_1 \cos 2x + c_2 \sin 2x) - (1/10)\, e^{2x}\, (\cos x - 2 \sin x)$

7. $(D^2 - 2D + 4)\, y = e^x \cos x$ **[Agra 1995; Calcutta 1996, Delhi Math(H) 1995, Kanpur 1996, S.V. University (A.P.) 1997]** **Ans.** $y = e^x\, [c_1 \cos (x\sqrt{3}) + c_2 \sin (x\sqrt{3})] + (1/2)\, e^x \cos x$

8. $D^2y = e^x \cos x$. **Ans.** $y = c_1x + c_2 + (1/2)\, e^x \sin x$

9. $(D^2 + 4D - 12)\, y = (x - 1)\, e^{2x}$ **[Bangalore 1996; Lucknow 1997, 98]**

Ans. $y = c_1e^{2x} + c_2\, e^{-6x} + (1/64)\, e^{2x}\, (4x^2 - 9x)$

10. $(D^3 - 7D - 6)\, y = (x + 1)\, e^{2x}$ **[Delhi Maths (G) 2002, Kanpur 1996]**

Ans. $y = c_1\, e^x + c_2\, e^{-2x} + c_3\, e^{3x} - (1/144)\, e^{2x}\, (12x + 17)$

11. $(D^3 - D - 6)\, y = (x + 1)\, e^{2x}$ **Ans.** $y = c_1\, e^{2x} + c_2\, e^{-x} \cos (x\sqrt{2} + c_3) + (1/242)\, (10x + 11x^2)$

12. $(D^3 - 3D^2 + 3D - 1)\, y = (x + 1)\, e^x$ **Ans.** $y = (c_1 + c_2x + c_3x^2)\, e^x + (1/24)\, (x + 1)^4\, e^x$

13. $(D^2 - 2D + 6)\, y = e^x \cos x$ **[Agra 1996; Meerut 1999]**

Ans. $y = e^x\, (c_1 \cos x\sqrt{5} + c_2 \sin x\sqrt{5}) + (1/4) \times e^x \cos x$

14. $(D^2 - 2D + 4)\, y = e^x \sin x$ **Ans.** $y = e^x\, \{c_1 \cos (x\sqrt{3}) + c_2 \sin (x\sqrt{3})\} + (1/2)\, e^x \sin x$

15. $(D^2 - 4D + 1)\, y = e^{2x} \sin 2x$. **Ans.** $y = e^{2x}\{c_1 \cosh (x\sqrt{3}) + c_2 \sinh (x\sqrt{3})\} + (1/7)\, e^{2x} \sin 2x$

16. $(D^2 - 2D + 4)\, y = e^{2x} \cos x$.

Ans. $y = e^x\{c_1 \cos (x\sqrt{3}) + c_2 \sin (x\sqrt{3})\} + (1/13)\, e^{2x}\, (2 \sin x + 3 \cos x)$

17. $(D^2 + 1)\, y = xe^{2x}$. **Ans.** $y = c_1 \cos x + c_2 \sin x + (1/25)\, (5x - 4)\, e^{2x}$

18. $(D^2 + 2)\, y = x^2 e^{3x}$. **[Delhi Maths (G) 1996]**

Ans. $y = c_1 \cos (x\sqrt{2}) + c_2 \sin (x\sqrt{2}) - (1/11)\, \{x^2 - (12/11)\, x - (50/121)\}e^{3x}$

19. $(D^2 - 1)\, y = e^x\, (1 + x^2)$. **Ans.** $y = c_1e^x + c_2e^{-x} + (1/12)\, e^x\, (2x^3 - 3x^2 + 9x)$

20. $(D^2 - 1)\, y = \sinh x \cosh x \cos x$ **Ans.** $y = c_1\, e^x + c_2\, e^{-x} + (1/20) \times (\cos x \sinh 2x - 2 \sin x \cosh 2x)$

5.22 Short method of finding P.I. when $X = xV$, where V is any function of x.

For this purpose, we have the following theorem

Theorem. To prove that $$\frac{1}{f(D)}xV = x\frac{1}{f(D)}V + \left(\frac{d}{dD}f(D)\right)V.$$

Proof. Let U be a function x. If u and v be functions of x, then by Leibnitz's theorem,

$$D^n (uv) = D^n u\cdot v + {}^nc_1\, D^{n-1}\, u\cdot Dv + {}^nc_2\, D^{n-2}\, u\cdot D^2 v + \dots u.\, D^n\, v. \qquad \dots (1)$$

Using Leibnitz's theorem (1), we have

$$D^n (xU) = D^n (Ux) = D^n U\cdot x + {}^nc_1\, D^{n-1}U\cdot 1 = x\, D^n U + n\, D^{n-1}\, U$$

or $$D^n (xU) = x\, D^n\, U + \frac{d}{dD}\,(D^n)\, U, \qquad \text{as} \qquad \frac{d}{dD}\, D^n = n\, D^{n-1} \qquad \dots (2)$$

Since $f(D)$ is a polynomial in D, from (2) we have

$$f(D)\,(xU) = x f(D)\, U + \left(\frac{d}{dD} f(D)\right)U. \qquad \dots (3)$$

Putting $f(D)\, U = V$, we have

$$\frac{1}{f(D)}(f(D)U) = \frac{1}{f(D)}V \qquad \text{so that} \qquad U = \frac{1}{f(D)}V.$$

Moreover, since U is function of x, so is V.

$\therefore$ From (3), $$f(D)\left(x\frac{1}{f(D)}V\right) = xV + \left(\frac{d}{dD}f(D)\right)\frac{1}{f(D)}V. \qquad \dots (4)$$

Operating on both sides of (4) by $1/f(D)$, we have

$$x\frac{1}{f(D)}V = \frac{1}{f(D)}(xV) + \frac{f'(D)}{[f(D)]^2}V, \qquad \text{where} \qquad f'(D) = \frac{df(D)}{dD}$$

or $$\frac{1}{f(D)}(xV) = x\frac{1}{f(D)}V - \frac{f'(D)}{[f(D)]^2}V \qquad \dots (5A)$$

or $$\frac{1}{f(D)}(xV) = x\frac{1}{f(D)}V + \left(\frac{d}{dD}\frac{1}{f(D)}\right)V \qquad \dots (5B)$$

Working rule of finding P.I. when $X = xV$. We shall apply the formula (5B) to compute $\{1/f(D)\}\,(xV)$. Proceeding by repated application of the above formula, $\{1/f(D)\}\,(x^m V)$ can be evaluated, if m is a positive integer. We shall apply the above formula (5A) when V is of the form $\sin ax$ or $\cos ax$. In practice the above formula should not be used when $V = \sin ax$ or $\cos ax$ and $f(-a^2) = 0$ i.e., $f(D^2)$ vanishes by putting $-a^2$ for D^2. In such situations we shall apply the following alternative method. This alternative method can also be applied even when $f(-a^2) \neq 0$.

Alternative Working rule for finding P.I. when $X = x^m \sin ax$ or $x^m \cos ax$.

(*i*) P.I $= \dfrac{1}{f(D)}x^m \cos ax =$ Real part of $\dfrac{1}{f(D)}x^m (\cos ax + i\sin ax)$

$= $ R.P. of $\dfrac{1}{f(D)}x^m e^{aix}$, by Euler's theorem, where R.P. stands for real part

(*ii*) P.I $= \dfrac{1}{f(D)}x^m \sin ax =$ Imaginary part of $\dfrac{1}{f(D)}x^m (\cos ax + i\sin ax)$

$= $ I.P. of $\dfrac{1}{f(D)}x^m e^{aix}$, by Euler's theorem, where I.P. stands for imaginary part.

5.23 Solved examples based an Art. 5.22

Ex. 1. *Solve* $(D^2 + 9) = x \sin x$.

Sol. Method I. Here auxilary equation $\qquad D^2 + 9 = 0 \Rightarrow D = \pm 3i$.

$\therefore$ C.F. $= c_1 \cos 3x + c_2 \sin 3x$, c_1, c_2 being arbitrary constants.

$$\text{P.I} = \frac{1}{D^2+9} x \sin x = \text{I.P. of } \frac{1}{D^2+9} x e^{ix}\text{, where I.P. stands for imaginary part}$$

$$= \text{I.P. of } e^{ix} \frac{1}{(D+i)^2+9} x, \qquad \text{as} \qquad \frac{1}{f(D)} V e^{ax} = e^{ax} \frac{1}{f(D+a)} V$$

$$= \text{I.P. of } e^{ix} \frac{1}{D^2+2iD+8} x = \text{I.P. of } e^{ix} \frac{1}{8[1+(1/8)(D^3+2iD)]} x$$

$$= \text{I.P. of } \frac{e^{ix}}{8}\left[1+\left(\frac{iD}{4}+\frac{D^2}{8}\right)\right]^{-1} x = \text{I.P. of } \frac{e^{ix}}{8}\left[1-\frac{iD}{4}+\ldots\right] x$$

$= \text{I.P. of } (1/8) \times (\cos x + i \sin x)\,(x - i/4) = (1/8) \times \{x \sin x - (1/4) \cos x\}$.

$\therefore$ Required solution is $\qquad y = c_1 \cos 3x + c_2 \sin 3x + (1/8)\, x \sin x - (1/32) \cos x$.

Method II. As in Method I, $\qquad$ C.F. $= c_1 \cos 3x + c_2 \sin 3x$.

$$\text{P.I.} = \frac{1}{D^2+9} x \sin x = x \frac{1}{D^2+9} \sin x + \left[\frac{d}{dD}\frac{1}{D^2+9}\right] \sin x$$

$$\left[\because \frac{1}{f(D)} xV = x \frac{1}{f(D)} V + \left\{\frac{d}{dD}\frac{1}{f(D)}\right\} V\right]$$

$$= x \frac{1}{D^2+9} \sin x - \frac{2D}{(D^2+9)^2} \sin x = x \frac{1}{-1^2+9} \sin x - 2D \frac{1}{(-1^2+9)^2} \sin x$$

$= (x/8) \times \sin x - (1/32) \times D \sin x = (x/8) \times \sin x - (1/32) \times \cos x$

$\therefore$ Required solution is $\qquad y = c_1 \cos 3x + c_2 \sin 3x + (1/8)\, x \sin x - (1/32) \cos x$.

Ex. 2. *Solve* ***(a)*** $(D^2 + 2D + 1)\, y = x \cos x$ **[Agra 1995]**

(b) $(D^2 + 2D + 1)\, y = x \sin x$ **[IAS 1998, Delhi Maths (G) 1995]**

Sol. (*a*) Here the auxiliary equation is $D^2 + 2D + 1 = 0$ so that $D = -1, -1$.

$\therefore$ C.F. $= (c_1 + c_2 x)\, e^{-x}$, c_1, c_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{(D+1)^2} x \cos x = \text{R.P. of } \frac{1}{(D+1)^2} x e^{ix}\text{, where R.P. stands for real part}$$

$$= \text{R.P. of } e^{ix} \frac{1}{[(D+i)+1)]^2} x = \text{R.P. of } \frac{e^{ix}}{(1+i)^2}\left(1+\frac{D}{1+i}\right)^{-2}$$

$$= \text{R.P. of } \frac{e^{ix}}{2i}\left(1-\frac{2D}{1+i}+\ldots\right) x = \text{R.P. of } \frac{e^{ix}}{2i}\left(x-\frac{2}{1+i}\right)$$

$$= \text{R.P. of } i\frac{e^{ix}}{2i^2}\left[x-\frac{2(1-i)}{(1-i)(1+i)}\right] = \text{R.P. of } \frac{ie^{ix}}{(-2)}\left[x-\frac{2(1-i)}{1+1}\right]$$

$= \text{R.P. of } (-i/2) \times (\cos x + i \sin x)\ \{(x-1)+i\}$

$= \text{R.P. of } (-1/2) \times (i \cos x - \sin x)\ \{(x-1)+i\} \qquad \ldots (1)$

$\therefore$ P.I.$= (-1/2) \times [-\sin x \cdot (x-1) - \cos x] = (1/2) \times [(x-1)\sin x + \cos x]$

Solution is $y = (c_1 + c_2 x)\, e^{-x} + (1/2) \times [(x-1)\sin x + \cos x]$.

(b) Proceed as in part (*a*). As before C.F. $= (c_1 + c_2 x)\, e^{-x}$

$$\text{P.I.} = \frac{1}{(D+1)^2} x \sin x = \text{I.P. of } \frac{1}{(D+1)^2} x e^{ix}, \text{ where I.P. stands for imaginary part}$$

$= \text{I.P. of } (-1/2) \times (i \cos x - \sin x)\,[(x-1) + i],$

[Proceeding as in part (*a*) upto equation (1) replacing R.P. by I.P.]

$= (-1/2) \times \{(x-1)\cos x - \sin x\} = (1/2) \times \{\sin x - (x-1)\cos x\}$.

$\therefore$ The required solution is $y = (c_1 + c_2 x)\, e^{-x} + (1/2) \times \{\sin x - (x-1)\cos x\}$.

Ex. 3. *Solve* $(D^2 - 2D + 1)\, y = x \sin x$.

[Agra 1996, Kanpur 1994, Allahabad 1998, Meerut 1993, Ravishankar 1994]

Sol. Here the auxiliary equation $D^2 - 2D + 1 = 0$ so that $D = 1, 1$.

$\therefore$ C.F. $= (c_1 + c_2 x)\, e^x$, c_1, c_2 being arbitrary constants.

$$\text{P.I.} = \frac{1}{D^2 - 2D + 1} x \sin x = x \frac{1}{D^2 - 2D + 1} \sin x + \left(\frac{d}{dD} \frac{1}{D^2 - 2D + 1}\right) \sin x$$

$$= x \frac{1}{-1^2 - 2D + 1} \sin x - \frac{(2D-2)}{(D^2 - 2D + 1)^2} \sin x = -\frac{x}{2}\frac{1}{D} \sin x - 2(D-1) \frac{1}{(-1^2 - 2D + 1)^2} \sin x$$

$$= \frac{x}{2} \cos x - \frac{1}{2}(D-1) \frac{1}{4D^2} \sin x = \frac{x}{2} \cos x - \frac{1}{2}(D-1) \frac{1}{D} \int \sin x \, dx$$

$$= \frac{x}{2} \cos x - \frac{1}{2}(D-1) \frac{1}{D}(-\cos x) = \frac{x}{2} \cos x - \frac{1}{2}(D-1) \int (-\cos x)\, dx$$

$= (x/2) \cos x + (1/2)\,(D-1) \sin x = (x/2) \cos x + (1/2)\,(\cos x - \sin x)$

$\therefore$ Required solution is $y = (c_1 + c_2 x)\, e^x + (1/2)\,(x \cos x + \cos x - \sin x)$.

Ex. 4. *Solve* $(D^2 + 1)\, y = x^2 \sin 2x$. **[Kanpur 1995 Delhi Maths (H) 2000]**

Sol. Here the auxiliary equation is $D^2 + 1 = 0$ so that $D = \pm\, i$.

$\therefore$ C.F. $= c_1 \cos x + c_2 \sin x$, c_1, c_2 being arbitrary constants.

$$\text{P.I.} = \frac{1}{D^2 + 1} x^2 \sin 2x = \text{I.P. of } \frac{1}{D^2 + 1} x^2 e^{2ix}, \text{ where I.P. stands for imaginary part}$$

$$= \text{I.P. of } e^{2ix} \frac{1}{(D + 2i)^2 + 1} x^2 = \text{I.P. of } e^{2ix} \frac{1}{D^2 + 4iD - 3} x^2, \text{ as } i^2 = -1$$

$$= \text{I.P. of } \frac{e^{2ix}}{-3} \frac{1}{\{1 - (4iD + D^2)/3\}} x^2 = \text{I.P. of } \frac{e^{2ix}}{-3} \left[1 - \left(\frac{4iD}{3} + \frac{D^2}{3}\right)\right]^{-1} x^2$$

$$= \text{I.P. of } \frac{e^{2ix}}{-3} \left[1 + \left(\frac{4iD}{3} + \frac{D^2}{3}\right) + \left(\frac{4iD}{3} + \frac{D^2}{3}\right)^2 + \ldots\right] x^2$$

$$= \text{I.P. of } \frac{e^{2ix}}{-3} \left[1 + \frac{4iD}{3} + \frac{D^2}{3} - \frac{16D^2}{9} + \ldots\right] x^2$$

$$= \text{I.P. of } \frac{e^{2ix}}{-3} \left[1 + \frac{4iD}{3} - \frac{13D^2}{9} + \ldots\right] x^2 = \text{I.P. of } \frac{e^{2ix}}{-3} \left[x^2 + \left(\frac{4i}{3} \times 2x\right) - \left(\frac{13}{9} \times 2\right)\right]$$

= I.P. of $(-1/3)(\cos 2x + i \sin 2x)\{x^2 + (8/3)ix - (26/9)\}$

$= -(1/3)[(x^2 - 26/9)\sin 2x + (8/3)x\cos 2x]$

$\therefore$ Solution is $\quad y = c_1 \cos x + c_2 \sin x - (1/3)[(x^2 - 26/9)\sin 2x + (8/3)x\cos 2x)]$.

Ex. 5. *Solve* $(D^4 + 2D^2 + 1)y = x^2 \cos x$. **[Purvanchal 2007; Agra 1995, Meeruth 1997, Kumaun 1995, Kanpur 1998, Lucknow 1998, Delhi Maths (H) 2004]**

Sol. Here the auxiliary equation is $D^4 + 2D^2 + 1 = 0$ or $(D^2+1)^2 = 0$ so that $D = \pm i, \pm i$.

$\therefore$ C.F. $= (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x$, c_1, c_2, c_3, c_4 being arbitrary constants.

$$\text{P.I.} = \frac{1}{(D^2+1)^2} x^2 \cos x = \text{R.P. of } \frac{1}{(D^2+1)^2} x^2 e^{ix}, \text{ where R.P. stands for real part}$$

$$= \text{R.P. of } e^{ix} \frac{1}{\{(D+i)^2+1\}^2} x^2 = \text{R.P. of } e^{ix} \frac{1}{(D^2+2Di)^2} x^2$$

$$= \text{R.P. of } e^{ix} \frac{1}{(2Di)^2(1+D/2i)^2} x^2 = \text{R.P. of } \frac{e^{ix}}{-4D^2}\left(1+\frac{D}{2i}\right)^{-2} x^2$$

$$= \text{R.P. of } \frac{e^{ix}}{-4D^2}\left(1-\frac{iD}{2}\right)^{-2} x^2 = \text{R.P. of } \frac{e^{ix}}{-4D^2}\left(1+iD+\frac{3i^2D^2}{4}+\dots\right)x^2$$

$$= \text{R.P. of } \frac{e^{ix}}{-4D^2}\left(1+iD-\frac{3D^2}{4}\right)x^2 = \text{R.P. of } \frac{e^{ix}}{-4D^2}\left[x^2+2ix-\frac{3}{4}\cdot 2\right]$$

$$= \text{R.P. of } \frac{e^{ix}}{-4}\frac{1}{D}\left[\frac{x^3}{3}+ix^2-\frac{3}{2}x\right] = \text{R.P. of } \frac{e^{ix}}{-4}\left[\frac{x^4}{12}+\frac{ix^3}{3}-\frac{3x^2}{4}\right]$$

= R.P. of $(-1/4) \times (\cos x + i \sin x)\{(x^4/12) - (3x^2/4) + i(x^3/3)\}$

$= -(1/4) \times [(x^4/12 - 3x^2/4)\cos x - (x^3/3)\sin x]$

$\therefore$ Solution is $y = (c_1 + c_2 x)\cos x + (c_3 + c_4 x)\sin x - (1/4) \times [(x^4/12 - 3x^2/4)\cos x - (x^3/3) \times \sin x]$.

Ex. 6. *Solve* $(D^4 - 1)y = x \sin x$. **[Meerut 2000, Lucknow 1993, 98]**

Sol. Here the auxiliary equation is $\quad D^4 - 1 = \Rightarrow D = 1, -1 \pm i$.

$\therefore$ C.F. $= c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x$, c's being arbitrary constants.

$$\text{P.I.} = \frac{1}{D^4-1} x \sin x = I.P.\ of\ \frac{1}{D^4-1} xe^{ix}, \text{ where I.P. stand for imaginary part}$$

$$= \text{I.P. of } e^{ix}\frac{1}{(D+i)^4-1}x = \text{I.P. of } e^{ix}\frac{1}{D^4+4D^3i+6D^2i^2+4Di^3+i^4-1}x$$

$$= \text{I.P of } e^{ix}\frac{1}{D^4+4iD^3-6D^2-4iD}x = \text{I.P. of } \frac{e^{ix}}{-4iD}\frac{1}{[1+3D/2i-D^2-D^3/4i]}x$$

$$= \text{I.P. of } \frac{ie^{ix}}{4D}\left[1+\left(-\frac{3Di}{2}-D^2+\frac{iD^3}{4}\right)\right]^{-1} x$$

$$= \text{I.P. of } \frac{ie^{ix}}{4D}\left[1+\left(\frac{3Di}{2}+D^2-\frac{iD^3}{4}\right)+\dots\right]x = \text{I.P. of } \frac{ie^{ix}}{4}\frac{1}{D}\left(x+\frac{3i}{2}\right)$$

= I.P. of $(1/4) \times ie^{ix}\{(x^2/2) + (3/2)ix\}$ = I.P. of $(1/8) \times (\cos x + i \sin x)(ix^2 - 3x)$

Thus, $\quad$ P.I. $= (1/8) \times (x^2 \cos x - 3x \sin x)$

$\therefore$ Solution is $\quad y = c_1 e^x + c_2 e^{-x} + c_3 \cos x + c_4 \sin x + (1/8) \times (x^2 \cos x - 3x \sin x)$.

Ex. 7. *Solve $(D^2 + 1)^2 = 24x \cos x$ given that $y = Dy = D^2y = 0$ and $D^3y = 12$ when $x = 0$.* **[I.A.S. 2001]**

Sol. Given $$(D^2 + 1)^2 y = 24 x \cos x \qquad \text{... (1)}$$

with $y = 0, \quad y' = 0, \quad y'' = 0$ and $y''' = 12$ when $x = 0$. ... (2)

The auxiliary equation is $(D^2 + 1)^2 = 0$ giving $D = \pm i$ (twice)

$\therefore$ C.F. $= (c_1 + c_2x) \cos x + (c_3 + c_4 x) \sin x$, c_1, c_2, c_3, c_4 being arbitrary constants ... (3)

$$\text{P.I.} = \frac{1}{(D^2+1)^2} 24 x \cos x = \text{Real part of } \frac{1}{(D^2+1)^2} 24 x e^{ix} \qquad \text{... (4)}$$

$$\text{Now, } \frac{1}{(D^2+1)^2} 24x\, e^{ix} = 24e^{ix} \frac{1}{\{(D+i)^2+1\}^2} x = 24e^{ix} \frac{1}{(D^2+2Di)^2} x$$

$$= 24\, e^{ix} \frac{1}{(2Di)^2} \left(1 + \frac{D}{2i}\right)^{-2} x = \frac{24\, e^{ix}}{(-4D^2)} \left(1 - \frac{D}{i} + ...\right) x$$

$$= -6\, e^{ix} \frac{1}{D^2} (x + i) = -6e^{ix} \left(\frac{x^3}{6} + \frac{ix^2}{2}\right) \qquad \text{... (5)}$$

Using (5), from (4) we, have

P.I. = Real part of $(\cos x + \sin x)(-x^3 + 3i\, x^2) = 3x^2 \sin x - x^3 \cos x$

Hence the general solution of (1) is $y =$ C.F. + P.I. *i.e.*

$$y = c_1 \cos x + c_3 \sin x + x\,(c_2 \cos x + c_4 \sin x) + 3x^2 \sin x - x^3 \cos x \qquad \text{... (6)}$$

Putting $x = 0$, in (6) and noting that y = 0 when x = 0, we get $c_1 = 0$. Putting $c_1 = 0$ in (6),

$$y = c_3 \sin x + x\,(c_2 \cos x + c_4 \sin x) + 3x^2 \sin x - x^3 \cos x \qquad \text{... (7)}$$

Differentiating (7) w.r.t. 'x', we get $y' = c_3 \cos x + c_2 \cos x + c_4 \sin x$

$$+ x\,(-c_2 \sin x + c_4 \cos x) + 6x \sin x + 3x^2 \cos x - (3x^2 \cos x - x^3 \sin x) \qquad \text{... (8)}$$

Putting $x = 0$ in (8) and noting that $y' = 0$ when $x = 0$, we get

$$0 = c_3 + c_2 \qquad \text{so that} \qquad c_3 = -c_2 \qquad \text{... (9)}$$

$\therefore$ (8) gives $$y' = c_4 \sin x + x\,(-c_2 \sin x + c_4 \cos x) + 6\,x \sin x + x^3 \sin x \qquad \text{... (10)}$$

Differentiating (10) w.r.t. 'x', we get $y'' = c_4 \cos x - c_2 \sin x + c_4 \cos x$

$$+ x\,(-c_2 \cos x - c_4 \sin x) + 6 \sin x + 6x \cos x + 3x^2 \sin x + x^3 \cos x \qquad \text{... (11)}$$

Putting $x = 0$ in (11) and noting that $y'' = 0$ when $x = 0$ (11) gives $c_4 = 0$. Putting $c_4 = 0$ in (11),

$$y'' = -c_2 \sin x - c_2\, x \cos x + 6 \sin x + 6x \cos x + 3x^2 \sin x + x^3 \cos x \qquad \text{... (12)}$$

Differentiating (12) w.r.t. 'x', we get $y''' = -c_2 \cos x - c_2 (\cos x - x \sin x) + 6 \cos x$

$$+ 6\,(\cos x - x \sin x) + 3\,(2x \sin x + x^2 \cos x) + 3x^2 \cos x - x^3 \sin x \qquad \text{... (13)}$$

Putting $x = 0$ in (13) and noting that $y''' = 12$ when $x = 0$, (13) reduces to

$12 = -2\, c_2 + 12$ so that $c_2 = 0$. So by (9), $c_3 = 0$.

Thus, finally, $c_1 = c_2 = c_3 = c_4 = 0$ and so (6) reduces to

$y = 3x^2 \sin x - x^3 \cos x$. which is the required solution.

Exercise 5(G)

Solve the following differential equations :

1. (*a*) $(D^2 + 4)\, y = x \sin x$ **[Delhi Maths (G) 1994)**

Ans. $y = c_1 \cos 2x + c_2 \sin 2x + (x/3) \cdot \sin x - (2/9) \cos x$

(*b*) $(D^2 + m^2)\, y = x \cos mx$ **Ans.** $y = c_1 \cos mx + c_2 \sin mx + (x/4m^2) \cos mx + (x^2/4m) \sin mx$

2. $(D^2 - 5D + 6)\, y = x \sin 3x$ **[Delhi Maths (G) 1996]**

Ans. $y = c_1 e^{2x} + c_2 e^{3x} - \{(78\, x + 40) \sin 3x - (123 - 390x) \cos 3x\}/6084$

3. $(D^2 + D)\, y = x \cos x.$ **Ans.** $y = c_1 + c_2 e^{-x} + (1/2) \times (\sin x - \cos x) + (1/2) \times \sin x$

4. $(D^2 - 1)\, y = x^2 \cos x$ **[Delhi:Maths(H) 2007]** **Ans.** $y = c_1 e^x + c_2 e^{-x} + (1/2) \times (1 - x^2) \cos x + x \sin x$

5. $(D^2 - 1)\, y = x^2 \sin x$ **Ans.** $y = c_1 e^x + c_2 e^{-x} - x \cos x - (1/2) \times (x^2 - 1) \sin x$

6. $(D^2 + 4)\, y = x \sin^2 x.$ **Ans.** $y = c_1 \cos 2x + c_2 \sin 2x + (x/8) \times (1 + 2 \sin 2x)$

5.24 More about particular integral of $f(D)\, y = X$

We now consider examples in which X is sum of two or more special functions of x considered sperarately. In such cases we obtain P.I. corresponding each function separately and then add these to get the required P.I. of a differential equation.

5.25 Solved Examples based on Art. 5.24 and Miscellaneous examples

Ex. 1. *Solve* $(D^2 - 4D + 4)\, y = x^2 + e^x + \sin 2x.$ **[Delhi B.A. (Prog) II 2010]**

Sol. Given $(D^2 - 4D + 4)\, y = x^2 + e^x + \sin 2x.$ ... (1)

Here the auxiliary equation is $D^2 - 4D + 4 = 0$ so that $D = 2, 2.$

$\therefore$ C.F. $= (c_1 + c_2 x)\, e^{2x}$, c_1, c_2 being arbitrary constants.

P.I. corresponding to x^2

$$= \frac{1}{D^2 - 4D + 4} x^2 = \frac{1}{(2 - D)^2} x^2 = \frac{1}{4[1 - (D/2)]^2} x^2 = \frac{1}{4}\left(1 - \frac{D}{2}\right)^{-2} x^2$$

$$= \frac{1}{4}\left[1 + D + \frac{(-2)(-3)}{1 \cdot 2} \cdot \frac{D^2}{4} + \dots\right] x^2 = \frac{1}{4}(1 + D + \frac{3}{4} D^2 + \dots)\, x^2 = \frac{1}{4}\left(x^2 + 2x + \frac{3}{2}\right)$$

P.I. corresponding to $e^x = \dfrac{1}{D^2 - 4D + 4} e^x = \dfrac{1}{1 - 4 + 4} e^x = e^x$

and P.I. Corresponding to $\sin 2x$

$$= \frac{1}{D^2 - 4D + 4} \sin 2x = \frac{1}{-2^2 - 4D + 4} \sin 2x = -\frac{1}{4} \cdot \frac{1}{D} \sin 2x = \frac{1}{8} \cos 2x$$

$\therefore$ Required solution is $y = (c_1 + c_2 x)\, e^{2x} + (1/8) \times (2x^2 + 4x + 3) + (1/8) \times \cos 2x.$

Ex. 2. *Solve* $(D^2 - 1)\, y = x\, e^x + \cos^2 x.$ **[Delhi Maths 2006, I.A.S. 1992]**

Sol. Given: $(D^2 - 1)\, y = x\, e^x + (1/2) \times (1 + \cos 2x).$ ... (1)

The auxiliary equation is $D^2 - 1 = 0$ so that $D = \pm 1.$

So C.F. $= c_1 e^x + c_2 e^{-x}$, c_1, c_2 being arbitrary constants.

P.I. corresponding to xe^x

$$= \frac{1}{D^2 - 1} x e^x = e^x \frac{1}{(D + 1)^2 - 1} x = e^x \frac{1}{D^2 + 2D} x = \frac{e^x}{2} \frac{1}{D(1 + D/2)} x = \frac{e^x}{2} \cdot \frac{1}{D}\left(1 + \frac{D}{2}\right)^{-1} x$$

$$= \frac{e^x}{2} \cdot \frac{1}{D}\left(1 - \frac{D}{2} + \dots\right) x = \frac{e^x}{2} \cdot \frac{1}{D}\left(x - \frac{1}{2}\right) = \frac{1}{2} e^x \left(\frac{x^2}{2} - \frac{x}{2}\right) = \frac{1}{4} e^x\, (x^2 - x)$$

P.I. corresponding to $\dfrac{1}{2} = \dfrac{1}{2} \dfrac{1}{D^2 - 1} e^{0.x} = \dfrac{1}{2} \dfrac{1}{O^2 - 1} e^{O.x} = -\dfrac{1}{2}.$

P.I. corresponding to $\dfrac{1}{2}$ cos $2x$ = $\dfrac{1}{2}\dfrac{1}{D^2-1}\cos 2x = \dfrac{1}{2}\;\dfrac{1}{-2^2-1}\;\cos 2x = -\dfrac{1}{10}\cos 2x.$

$\therefore$ The required solution $y = c_1 e^x + c_2 e^{-x} + (1/4) \times e^x (x^2 - x) - (1/2) - (1/10) \times \cos 2x.$

Ex. 3. *Solve* $(D^4 - 4D^2 - 5)\, y = e^x (x + \cos x)$ **[I.A.S. 2004]**

Sol. Here auxiliary equation is $D^4 - 4D^2 - 5 = 0$ or $(D^2 - 5)(D^2 + 1) = 0$

giving $D^2 = 5$ or -1 so that $D = \pm\sqrt{5}, \quad \pm i$

$\therefore$ C.F. $= c_1 \cosh\ x\sqrt{5} + c_2\ \sinh\ x\sqrt{5} + c_3\ \cos x + c_4 \sin x$

P.I. corresponding to $x\, e^x$

$$= \frac{1}{D^4 - 4D^2 - 5} xe^x = e^x \frac{1}{(D+1)^4 - 4(D+1)^2 - 5} x \;=\; e^x \frac{1}{D^4 + 4D^3 + 2D^2 - 4D - 8} x$$

$$= -\frac{e^x}{8}\;\frac{1}{1 + D/2 - D^2/4 - D^3/2 - D^4/8} x \;=\; -\frac{e^x}{8}\left(1 + \frac{D}{2} - \frac{D^2}{4} - \frac{D^3}{2} - \frac{D^4}{8}\right)^{-1} x$$

$= -(e^x/8) \times \{1 - (D/2 - D^2/4 - D^3/2 - D^4/8) + \ldots\}\, x = -(e^x/8) \times (x - 1/2) = -(e^x/16) \times (2x - 1)$

P.I. corresponding to $e^x \cos x = \dfrac{1}{D^4 - 4D^2 - 5} e^x \cos x = e^x \dfrac{1}{(D+1)^4 - 4(D+!)^2 - 5}\cos x$

$$= e^x \frac{1}{D^4 + 4D^3 + 2D^2 - 4D - 8}\cos x = e^x \frac{1}{(D^2)^2 + 4D^2 D + 2D^2 - 4D - 8}\cos x$$

$$= e^x \frac{1}{(-1^2)^2 + 4(-1^2)\, D + 2(-1^2) - 4D - 8}\cos x$$

$$= -e^x \frac{1}{9 + 8D}\cos x = -e^x \frac{9 - 8D}{(9+8D)(9-8D)}\cos x$$

$$= -e^x \frac{9 - 8D}{81 - 64D^2}\cos x = -e^x \frac{9 - 8D}{81 - 64 \times (-1^2)}\cos x \;=\; -\frac{1}{145} e^x\ (9 \cos x + 8 \sin x)$$

$\therefore$ Required solution is $y = c_1\ \cosh\ x\sqrt{5} + c_2 \sinh x\sqrt{5} + c_3\ \cos x + c_4 \sin x - (e^x/16) \times (2x - 1)$
$- (1/145)\, e^x (9 \cos x + 8 \sin x)$, c_1, c_2, c_3, c_4 being arbitrary constants.

Ex. 4. *Solve* $(d^3y/dx^3) - 3\,(d^2y/dx^2) + 4\,(dy/dx) - 2y = e^x + \cos x.$ **[I.A.S. 1999]**

Sol. Given $(D^3 - 3D^2 + 4D - 2)\, y = e^x + \cos x$, where $D \equiv d/dx$ ... (1)

Its auxiliary equation is $D^3 - 3D^2 + 4D - 2 = 0$ or $D^2(D - 1) - 2D\,(D - 1) + 2(D - 1) = 0$

or $(D - 1)(D^2 - 2D + 2) = 0$ giving $D = 1$, $(2 \pm \sqrt{4-8})/2$, *i.e.*, $D = 1, 1 \pm i$.

$\therefore$ C.F. $= c_1\, e^x + e^x (c_2 \cos x + c_3 \sin x)$, c_1, c_2, c_3 being arbitrary constants

P.I. corresponding to $e^x = \dfrac{1}{D^3 - 3D^2 + 4D - 2} e^x = \dfrac{1}{(D-1)(D^2 - 2D + 2)} e^x$

$$= \frac{1}{D-1}\;\frac{1}{1 - 2 + 2} e^x \;=\; \frac{1}{D-1} e^x \cdot 1 = e^x \frac{1}{(D+1) - 1}\cdot 1 = e^x \frac{1}{D}\cdot 1 = xe^x$$

P.I. corresponding to $\cos x = \dfrac{1}{D^3 - 3D^2 + 4D - 2}\cos x = \dfrac{1}{D^2 \cdot D - 3D^2 + 4D - 1}\cos x$

$$= \frac{1}{(-1^2)D - 3(-1^2) + 4D - 2} \cos x = \frac{1}{3D+1} \cos x = (3D-1) \frac{1}{9D^2 - 1} \cos x$$

$$= (3D-1) \frac{1}{9(-1^2) - 1} \cos x = -\frac{1}{10}(3D \cos x - \cos x) = -\frac{1}{10}(-3 \sin x - \cos x)$$

$\therefore$ Required solution is $\quad y = e^x (c_1 + c_2 \cos x + c_3 \sin x) + x e^x + (3 \sin x + \cos x)/10$

Ex. 5. *Solve* $(D^2 + a^2) y = \sin ax + x e^{2x}$ **[Delhi Maths (G) 1999]**

Sol. Auxiliary equation $\quad D^2 + a^2 = 0 \quad$ gives $\quad D^2 = - a^2 \quad$ or $\quad D = \pm ia.$

$\therefore$ C.F. $= c_1 \cos ax + c_2 \sin ax$, c_1, c_2 being arbitrary constants

P.I. corresponding to $\sin ax = \dfrac{1}{D^2 + a^2} \sin ax = -\dfrac{x}{2a} \cos ax$

P.I. corresponding to $e^{2x} x$

$$= \frac{1}{D^2 + a^2} e^{2x} x = e^{2x} \frac{1}{(D+2a)^2 + a^2} x = e^{2x} \frac{1}{D^2 + 4Da + 4 + a^2} x$$

$$= \frac{e^{2x}}{4+a^2} \left[1 + \frac{D^2 + 4D}{4+a^2}\right]^{-1} x = \frac{e^{2x}}{4+a^2} \left[1 - \frac{D^2 + 4D}{4+a^2} + \ldots\right] x$$

$$= \frac{e^{2x}}{4+a^2} \left(x - \frac{4}{4+a^2}\right) = \frac{e^{2x}}{(4+a^2)^2} \{x(4+a^2) - 4\}$$

$\therefore$ Solution is $\quad y = c_1 \cos x + c_2 \sin ax - (x/2a) \cos ax + e^{2x} (4 + a^2)^{-2} (4x + x a^2 - 4)$

Ex. 6. *Solve* $(D^2 - 6D + 8) y = (e^{2x} - 1)^2 + \sin 3x.$ **[Delhi Maths (G) 2000]**

Sol. The auxiliary equation is $\quad D^2 - 6D + 8 = 0 \quad$ giving $\quad D = 4, 2.$

$\therefore$ C.F. $= c_1 e^{2x} + c_2 e^{4x}$, $\quad c_1, c_2$ being arbitrary constants

P.I. corresponding to $(e^{2x} - 1)^2 = \dfrac{1}{(D-4)(D-2)} (e^{4x} - 2e^{2x} + 1)$

$$= \frac{1}{(D-4)(D-2)} e^{4x} - 2 \frac{1}{(D-2)(D-4)} e^{2x} + \frac{1}{(D-4)(D-2)} e^{0.x}$$

$$= \frac{1}{D-4} \frac{1}{(4-2)} e^{4x} - 2 \frac{1}{(D-2)(2-4)} e^{2x} + \frac{1}{(0-4)(0-2)} e^{0.x} = \frac{1}{2} x e^{2x} + x e^{2x} + \frac{1}{8}$$

P.I. corresponding to $\sin 3x = \dfrac{1}{D^2 - 6D + 8} \sin 3x = \dfrac{1}{-3^2 - 6D + 8} \sin 3x$

$$= -\frac{1}{6D+1} \sin 3x = -\frac{6D-1}{36D^2 - 1} \sin 3x = -\frac{(6D-1) \sin 3x}{36(-3^2) - 1} = \frac{1}{325} (18 \cos 3x - \sin 3x)$$

$\therefore$ Solution is $\quad y = c_1 e^{2x} + c_2 e^{4x} + (x/2) e^{4x} + x e^{2x} + 1/8 + (1/325) (18 \cos 3x - \sin 3x)$

Ex. 7(*a*). *Solve* $(D^4 + 4) y = e^x + x^2$, where $D \equiv d/dx$. **[Delhi Maths (Prog) 2007]**

Sol. Auxiliary equation of the given equation is $\quad D^4 + 4 = 0$

or $\quad (D^2 + 2)^2 - (2D)^2 = 0 \quad$ or $\quad (D^2 + 2D + 2)(D^2 - 2D + 2) = 0 \quad \Rightarrow \quad D = 1 \pm i, -1 \pm i$

$\therefore$ C.F. $= e^x (c_1 \cos x + c_2 \sin x) + e^{-x} (c_3 \cos x + c_4 \sin x)$, c_1, c_2, c_3, c_4 being arbitrary constants

P.I. corresponding to $\quad e^x = \dfrac{1}{D^4 + 4} e^x = \dfrac{1}{1^4 + 4} e^x = \dfrac{1}{5} e^x$

and P.I. corresponding to $x^2 = \dfrac{1}{D^4+4}x^2 = \dfrac{1}{4(1+D^4/4)}x^2 = \dfrac{1}{4}\left(1+\dfrac{D^4}{4}\right)^{-1}x^2$

$= (1/4) \times (1 - D^4/4 + ...)\, x^2 = x^2/4$

$\therefore$ Required solution is $y = e^x (c_1 \cos x + c_2 \sin x) + e^{-x} (c_3 \cos x + c_4 \sin x) + (1/5) \times e^x + x^2/4$

Ex. 7(*b*). *Solve* $(D^2 + 2)\, y = x^2 e^{3x} + e^x \cos 2x$ **[Delhi Maths (H) 2006, 08]**

Sol. Here auxiliary equation is $D^2 + 2 = 0$ giving $D = \pm i\sqrt{2}$

$\therefore$ C.F. $= c_1 \cos\,(x\sqrt{2}) + c_2 \sin\,(x\sqrt{2})$, c_1, c_2 being arbitrary constants

P.I. corresponding to $x^2 e^{3x}$

$$= \frac{1}{D^2+2}x^2e^{3x} = e^{3x}\frac{1}{(D+3)^2+2}x^2 = e^{3x}\frac{1}{D^2+6D+11}x^2 = e^{3x}\frac{1}{11(1+6D/11+D^2/11)}x^2$$

$$= \frac{e^{3x}}{11}\left\{1+\left(\frac{(6D+D^2)}{11}\right)\right\}^{-1}x^2 = \frac{e^{3x}}{11}\left\{1-\frac{6D+D^2}{11}+\left(\frac{6D+D^2}{11}\right)^2 - .....\right\}x^2$$

$$= \frac{e^{3x}}{11}\left(1-\frac{6D}{11}-\frac{D^2}{11}+\frac{36D^2}{121}+...\right)x^2 = \frac{e^{3x}}{11}\left(1-\frac{6D}{11}+\frac{25D^2}{121}+...\right)x^2 = \frac{e^{3x}}{11}\left(x^2-\frac{12x}{11}+\frac{50}{121}\right)$$

$$= e^{3x}(121x^2-132x+50)/(11)^3$$

P.I. corresponding to $e^x \cos 2x$

$$= \frac{1}{D^2+2}e^x\cos 2x = e^x\frac{1}{(D+1)^2+2}\cos 2x = e^x\frac{1}{D^2+2D+3}\cos 2x$$

$$= e^x\frac{1}{-2^2+2D+3}\cos 2x = e^x\frac{1}{2D-1}\cos 2x = e^x\frac{2D+1}{4D^2-1}\cos 2x$$

$$= e^x\frac{2D+1}{4\times(-2^2)-1}\cos 2x = -\frac{e^x}{17}(2D+1)\cos 2x = -\frac{e^x}{17}(-4\sin 2x+\cos 2x)$$

$\therefore$ The required solution is $y = c_1 \cos\,(x\sqrt{2}) + c_2 \sin\,(x\sqrt{2})$

$+ \{1/(11)^3\} \times e^{3x} (121x^2 - 132x + 50) - (1/17) \times e^x (\cos 2x - 4 \sin 2x)$.

Ex. 8. *Solve* $(D^4 + D^2 + 1)\, y = ax^2 + be^{-x} \sin 2x$. **[Ravishankar 1994]**

Sol. Given $(D^4 + D^2 + 1)\, y = ax^2 + be^{-x} \sin 2x$. ... (1)

A.E. is $D^4 + D^2 + 1 = 0$ or $(D^2 + 1)^2 - D^2 = 0$ or $(D^2 + D + 1)(D^2 - D + 1) = 0$

so that $D^2 + D + 1 = 0$ or $D^2 - D + 1 = 0$, giving $D = (-1 \pm i\sqrt{3})/2,\ (1 \pm i\sqrt{3}/2)$.

$\therefore$ C.F. $= e^{-x/2}[c_1 \cos(x\sqrt{3}/2) + c_2 \sin(x\sqrt{3}/2)] + e^{x/2}[c_3 \cos(x\sqrt{3}/2) + c_4 \sin(x\sqrt{3}/2)]$,

c_1, c_2, c_3 and c_4 being arbitrary constants

Now, P.I. corresponding to ax^2

$$= a\frac{1}{D^4+D^2+1}x^2 = a[1+(D^4+D^2)]^{-1}x^2 = a\,[1 - D^2 + ...]\, x^2 = a\,(x^2 - 2).$$

Next, P.I. corresponding to $be^{-x} \sin 2x$

$$= b\frac{1}{D^4+D^2+1}e^{-x}\sin 2x = be^{-x}\frac{1}{(D-1)^4+(D-1)^2+1}\sin 2x$$

$$= be^{-x}\frac{1}{D^4-4D^3+6D^2-4D+1+D^2-2D+1+1}\sin 2x$$

$$= be^{-x}\frac{1}{(D^2)^2-4D(D^2)+7D^2-6D+3}\sin 2x$$

$$= be^{-x}\frac{1}{(-2^2)^2-4D(-2^2)+7(-2^2)-6D+3}\sin 2x$$

$$= be^{-x}\frac{1}{10D-9}\sin 2x = be^{-x}(10D+9)\cdot\frac{1}{100D^2-81}\sin 2x$$

$$= be^{-x}(10D+9)\frac{1}{(100)\times(-2^2)-81}\sin 2x$$

$$= -be^{-x}(10D+9)\frac{1}{481}\sin 2x = -\frac{be^{-x}}{481}(20\cos 2x+9\sin 2x).$$

$\therefore$ Required solution is $y = e^{-x/2}\,[c_1 \cos(x\sqrt{3}/2) + c_2 \sin(x\sqrt{3}/2)] + e^{x/2}\,[c_3 \cos(x\sqrt{3}/2)$
$+ c_4 \sin(x\sqrt{3}/2)] + a\,(x_2 - 2) - (b/481)\,e^{-x}\,(20\cos 2x + 9\sin 2x).$

Ex. 9. *Solve* $(D-1)^2(D^2+1)^2 y = \sin^2(x/2) + e^x + x.$

Sol. Re-writing, $(D-1)^2(D^2+1)^2 y = (1-\cos x)/2 + e^x + x.$

Here auxiliary equation $(D-1)^2(D^2+1)^2 = 0$ gives $D = 1, 1, \pm i$ (twice).

So C.F. $= (c_1 + c_2x)\,e^x + (c_3 + c_4x)\cos x + (c_5 + c_6x)\sin x,$
where c_1, c_2, c_3, c_4, c_5 and c_6 are arbitrary constants.

$\therefore$ P.I. corresponding to $\frac{1}{2} = \frac{1}{2}\frac{1}{(D-1)^2(D^2+1)^2}e^{0.x} = \frac{1}{2}\frac{1}{(0-1)^2(0+1)^2}e^{0.x} = \frac{1}{2},$

P.I. corresponding to $(-1/2)\times\cos x$

$$= -\frac{1}{2}\frac{1}{(D^2+1)^2(D^2-2D+1)}\cos x = -\frac{1}{2}\frac{1}{(D^2+1)^2(-1^2-2D+1)}\cos x$$

$$= \frac{1}{4}\frac{1}{(D^2+1)^2}\frac{1}{D}\cos x = \frac{1}{4}\frac{1}{(D^2+1)^2}\sin x = \text{Imaginary part of } \frac{1}{4}\frac{1}{(D^2+1)^2}e^{ix} \qquad \text{... (1)}$$

Now, $\frac{1}{(D^2+1)^2}e^{ix}\cdot 1 = e^{ix}\frac{1}{[(D+i)^2+1]^2}\cdot 1 = e^{ix}\frac{1}{(D^2+2iD)^2}\cdot 1 = e^{ix}\frac{1}{(2iD)^2(1+D/2i)^2}e^{0.x}$

$$= -\frac{1}{4}e^{ix}\frac{1}{D^2}\frac{1}{(1+0)^2}e^{0.x} = -\frac{1}{4}e^{ix}\frac{1}{D^2}\cdot 1 = -\frac{1}{4}e^{ix}\left(\frac{x^2}{2}\right) = -\frac{1}{8}x^2(\cos x + i\sin x). \qquad \text{... (2)}$$

Using (1) and (2), P.I. corresponding to $(-1/2)\times\cos x = -(x^2/32)\times\sin x$

Again, P.I. corresponding to e^x

$$= \frac{1}{(D-1)^2\,(D^2+1)^2}e^x = \frac{1}{(D-1)^2\,(1+1)^2}e^x = \frac{1}{4}\frac{1}{(D-1)^2}e^x = \frac{1}{4}\frac{x^2}{2!}e^x = \left(\frac{x^2}{8}\right)e^x.$$

Finally, P.I. corresponding to x.

$$= \frac{1}{(D-1)^2(D^2+1)^2}x = (1-D)^{-2}\,(1+D^2)^{-2}\,x = (1+2D+\ldots)\,(1+\ldots)\,x = (1+2D+\ldots)\,x = x+2.$$

$\therefore$ Required solution is $y = (c_1 + c_2 x)\, e^x + (c_3 + c_4 x) \cos x + (c_5 + c_6 x) \sin x$
$+ (1/2) - (x^2/32) \sin x + (x^2/8)\, e^x + x + 2.$

Ex. 10. *solve* $(D^5 - D)\, y = 12\, e^x + 8 \sin x - 2x.$

Sol. A.E. $D(D^4 - 1) = 0 \Rightarrow D(D^2 - 1)(D^2 + 1) = 0 \Rightarrow D = 0, 1, -1, \pm i.$

$\therefore$ C.F. $= c_1 + c_2 e^x + c_3 e^{-x} + c_4 \cos x + c_5 \sin x$, $c_1, \ldots, c_5$ being arbitrary constants

Now P.I. corresponding to $12e^x$

$$= 12 \frac{1}{(D-1)D(D+1)(D^2+1)} e^x = 12 \frac{1}{(D-1)\,1\cdot(1+1)(1+1)} e^x = 3 \frac{1}{D-1} e^x = 3 \cdot \frac{x}{1!} e^x = 3xe^x.$$

P.I. corresponding to $8 \sin x$ is

$$= 8 \frac{1}{(D^2+1)\, D\, (D^2-1)} \sin x = 8 \frac{1}{(D^2+1)\, D\, (-1^2-1)} \sin x = -4 \frac{1}{(D^2+1)} \left[\frac{1}{D} \sin x \right]$$

$$= 4 \frac{1}{D^2+1} \cos x = 4 \left(\frac{x}{2 \times 1} \sin x \right) = 2x \sin x \qquad \left[\because \frac{1}{D^2+a^2} \cos ax = \frac{x}{2a} \sin x \right]$$

P.I. corresponding to $(-2x)$ is

$$= -2 \frac{1}{D(D^2-1)(D^2+1)} x = 2 \frac{1}{D(1-D^2)(1+D^2)} x = 2 \frac{1}{D} (1-D^2)^{-1} (1+D^2)^{-1} x$$

$$= 2 \frac{1}{D} (1 + D^2 + \ldots)(1 - D^2 + \ldots)\, x = 2 \frac{1}{D} (1 + D^2 - D^2 + \ldots)\, x = 2 \frac{1}{D} x = 2 \left(\frac{x^2}{2} \right) = x^2.$$

$\therefore$ Solution is $y = c_1 + c_2 e^x + c_3 e^{-x} + c_4 \cos x + c_5 \sin x + 3\, xe^x + 2x \sin x + x^2.$

Exercise 5(H)

1. $d^3y/dx^3 + d^2y/dx^2 + dy/dx = e^{2x} + x^2 + x.$ **[Meerut 2009; Lucknow 1994, 98]**

Ans. $y = c_1 + (c_2 + c_3 x)\, e^{-x} + (1/18) \times e^{2x} + (1/3) \times x^3 - (3/2) \times x^2 + 4x$

2. $(D^2 - 4D + 4)\, y = \sin 2x + x^2.$ **[G.N.D.U. Amritsar 2011]**

Ans. $y = (c_1 + c_2 x)\, e^{2x} + (3 \sin 2x + 8 \cos 2x)/25 + (2x^2 + 4x + 3)/8$

3. $(D^2 + 4)\, y = e^x + \sin 2x.$ **[Allahabad 1994 ; Agra 2005 ; Rohilkhand 1994]**

Ans. $y = c_1 \cos 2x + c_2 \sin 2x + (1/5) \times e^x - (1/4) \times x \cos 2x$

4. $(D^4 + 2D^3 - 3D^2)\, y = 3e^{2x} + 4 \sin x.$ **[Kanpur 1993]**

Ans. $y = c_1 + c_2 x + c_3 e^x + c_4 e^{-3x} + (3/20) \times e^{2x} + (2/5) \times (2 \sin x + \cos x)$

5. $(D^2 + D - 2)\, y = x + \sin x.$ **[Guwahati 1998; Meerut 1998 ; Delhi 2007, 09 ; Utkal 2003]**

Ans. $y = c_1 e^x + c_2 e^{-2x} - (x/2) - (1/4) - (1/10) \times (\cos x + 3 \sin x)$

6. (*i*) $D^3 - 3D^2 + 3D - 1)\, y = x\, e^{-x} + e^x$.**Ans.** $y = (c_1 + c_2 x + c_3 x^2)\, e^x - (1/16) \times (2x + 3)\, e^{-x} + (x^3/6)\, e^x$

(*ii*) $(D^3 - 3D^2 + 3D - 1)\, y = x\, e^x + e^x.$ **Ans.** $y = e^x\, [c_1 + c_2 x + c_3 x^2 + (1/6) \times x^3 + (1/24) \times x^4]$

7. $(D^2 + 5D + 6)\, y = e^{-2x} + 5 \sin 4x.$ **Ans.** $y = c_1 e^{-3x} + c_2 e^{-2x} + xe^{-2x} - (1/10) \times (\sin 4x + 2 \cos 4x)$

8. $(D^2 + 1)\, y = e^{-x} + \cos x.$ **Ans.** $y = c_1 \cos x + c_2 \sin x + (1/2) \times e^{-x} + (1/2) \times x \sin x$

9. $(D^2 + 4)\, y = \sin^2 x.$ **Ans.** $y = c_1 \cos 2x + c_2 \sin 2x + (1/8) \times (9 - x \sin 2x)$

10. $(D^2 + 1)\, y = e^{-x} + \cos x + x^3 + e^x \sin x.$

Ans. $y = c_1 \cos x + c_2 \sin x + (1/2) \times e^{-x} + (1/2) \times x \sin x + x^3 - 6x - (1/5) \times e^x\, (2 \cos x - \sin x)$

11. $(D^2 + 1)\, y = \cos^2 (x/2).$ **Ans.** $y = c_1 \cos x + c_2 \sin x + (1/2) + (1/4) \times x \sin x$

12. $(D^2 + 4)\, y = x^2 + 3 \sin x$ **Ans.** $y = c_1 \cos 2x + c_2 \sin 2x + (2x - 1)/8 + \sin x.$

13. $(D^2 + 4)\, y = \sin 2x + x^2.$ **Ans.** $y = c_1 \cos 2x + c_2 \sin 2x - (1/4) \times x \cos 2x + (1/8) \times (2x^2 - 1)$

14. $(2D^2 - D - 6)\, y = e^{-(3x/2)} + \sin x$ **[Pune 2010]**

Ans. $y = c_1 e^{2x} + c_2 e^{-(3x/2)} + (\cos x - 8 \sin x) / 65 - (x/7) \times e^{-(3x/2)}$

15. $(D^2 + 1)\, y = x^3 + e^x \sin x$. **Ans.** $y = c_1 \cos x + c_2 \sin x + x^3 - 6x - (1/5) \times e^x (2 \cos x - \sin x)$

16. $(D^2 - 1)\, y = x \sin x + e^x (1 + x^2)$.

Ans. $y = c_1 e^x + c_2 e^{-x} + (1/12) \times e^x (9x - 3x^2 + 2x^3) - (1/2) \times (x \sin x + \cos x)$

17. $(D^3 - 3D^2 + 4D - 2)\, y = e^x + \cos x$. **[I.A.S. 1999 ; Kanpur 1994]**

Ans. $y = e^x (c_1 + c_2 \cos x + c_3 \sin x + x) + (1/10) \times (3 \sin x + \cos x)$

18. $(D^2 - 4D + 4)\, y = e^{2x} + \sin 2x$. **Ans.** $y = (c_1 + c_2 x)\, e^{2x} + (1/2) \times x^2 e^{2x} + (1/8) \times \cos 2x$

19. $(D^2 - 5D + 6)\, y = x + \sin 3x$ **Ans.** $y = c_1 e^{3x} + c_2 e^{2x} + (1/36) \times (6x + 5) + (1/78) \times (5 \cos 3x - \sin 3x)$

20. $(D^2 - 3D + 2)\, y = 6\, e^{3x} + \sin 2x$. **[Kanpur 1994]**

Ans. $y = c_1 e^x + c_1 e^{2x} + 3e^{3x} - (1/20) \times (3 \cos 2x - \sin 2x)$

21. (*a*) $(D^2 - 4D + 4)\, y = x^2 + e^{2x}$ **[Delhi Maths (G) 1998]**

(*b*) $(D^2 + 10\, D + 29)\, y = xe^{5x} + \sin 2x$ **[Delhi Maths (G) 1998]**

Ans. (*a*) $y = (c_1 + c_2\, x)\, e^{2x} + (1/8) \times (2x^2 + 4x + 3) + (x^2/2) \times e^{2x}$

(*b*) $y = e^{-5x} (c_1 \cos 2x + c_2 \sin 2x) + (1/104) \times e^{5x} (x - 5/26) + (1/205) \times (5 \sin 2x - 4 \cos 2x)$

22. $(D^2 + 2)\, y = x^2\, e^{2x} + x^2 \cos 2x$ **[Delhi Maths (G) 1997]**

Ans. $y = c_1 \cos (x\sqrt{2}\,) + c_2 \sin (x\sqrt{2}\,) + (1/6) \times e^{2x} \{x^2 - (4/3)\, x + (5/9)\} - (1/2) \times \{(x^2 - 7) \cos 2x - 4x \sin 2x\}$

23. $(d^2y/dx^2) - 5\, (dy/dx) + 6y = x + e^{mx}$ **[Kanpur 2007]**

Ans. $y = c_1\, e^{2x} + c_2\, e^{3x} + (6x + 5)/36 + e^{mx}/(m^2 - 5m + 6)$, where $m \neq 2$ and $m \neq 3$.

PART II: METHOD OF UNDETERMINED COEFFICIENTS

5.26 Method of undetermined coefficients for solving linear differential equation, with constant coefficients $f(D)\, y = X$. [Pune 2010]

As explained in Art. 5.4, we first evaluate C.F. of the given equation $f(D)\, y = X$. The method of undetermined coefficients is yet another method of finding a particular integral of $f(D)\, y = X$. Now, we shall not make use of various short methods discussed is articles 5.14, 5.16, 5.18, 5.20 and 5.22 for finding P.I. In place of using these methods we shall use the method of undetermined coefficients. This method is useful only when X contains terms in some special forms given in the following table. The method of undetermined coefficients consists in making a guess of the **trial solution** y^* from the form of X. Then we substitute the trial solution y^* in $f(D)\, y = X$ and determine constants by comparing like terms on both sides of the equation $f(D)\, y^* = X$. Finally the required general solution is given by $y =$ C.F. $+ y^*$.

The following table suggests the form of the trial solution y^* (for particular integral) to be used corresponding to a speical form of X.

Table

S.No.	*Special form of X*	*Trial solution y* for P.I.*
1.	x^n or $a^n x^n$ or $a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$	$A_0 + A_1 x + \ldots + A_n\, x^n$
2.	e^{ax} or pe^{ax}	Ae^{ax}
3.	$a_n x^n\, e^{ax}$ or $e^{ax} (a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n)$	$e^{ax} (A_0 + A_1 x + A_2 x^2 + \ldots + A_n x^n)$
4.	$p \sin ax$ or $q \cos ax$ or $p \sin ax + q \cos ax$	$A \sin ax + B \cos ax$
5.	$p\, e^{bx} \sin ax$ or $q\, e^{bx} \cos ax$ or $e^{bx} (p \sin ax + q \cos ax)$	$e^{bx} (A \sin ax + B \cos ax)$
6.	$x^n \sin ax$ or $a_n\, x^n \sin ax$ or $(a_0 + a_1 x + \ldots + a_n x^n) \sin ax$ or $x^n \cos ax$ or $a_n x^n \cos ax$ or $(a_0 + a_1 x + \ldots + a_n x^n) \cos ax$	$(A_0 + A_1 x + \ldots + A_n x^n) \sin ax + (A_0' + A_1' x + \ldots + A_n' x^n) \cos ax$

Remark 1. In the above table, n is a positive integer and $a_0, a_1 \ldots, a_n, p, q, a, b, A_0, A_1, \ldots A_n, A_0', A_1', \ldots, A_n'$ are constants. The constants occuring in second column are known and the constants occuring in third column are determined by substituting the trial solution in given equation *i.e.* they are found from the resulting identity $f(D)\, y^* = X$.

Remark 2. If R.H.S. X of given equation $f(D)\, y = X$ is a linear combination of more than one special forms of the above table, then the trial solution must be taken as the sum of the corresponding trial solutions with appropriate constant coefficients to be evaluated later on.

Remark 3. The trial solution indicated in the above table is modified in the following situations (say) :

Situation (*i*) A term of X is also a term of C.F. of the given equation.

If a term of X, say u is also a term of the C.F. corresponding to an s-fold root m, then in the trial solution y^* of the table we introduce a term $x^s u$ plus terms arising from it by differentiation.

For example, consider $$(D-2)^2 (D+3)\, y = e^{2x} + x^2. \qquad \dots (1)$$

As usual, $$\text{C.F. of (1)} = (c_1 + c_2 x)\, e^{2x} + c_3 e^{-3x}. \qquad \dots (2)$$

Note x^2 is not occurring in C.F., so the corresponding contribution to trial solution y^* is $A_1 + A_2 x + A_3 x^2$. Next, note carefully that e^{2x} occurs in X and e^{2x} is also present in C.F. corresponding to a double root $m = 2$; hence we employ $x^2 e^{2x}$ and all terms arising from it by differentiation. Thus contribution to trial solution y^* corresponding to e^{2x} of X is taken as $A_4 e^{2x} + A_5 x\, e^{2x} + A_6 x^2 e^{2x}$. Thus, the total P.I. y^* is

$$y^* = A_1 + A_2 x + A_3 x^2 + A_4 e^{2x} + A_5 x\, e^{2x} + A_6 x^2 e^{2x}.$$

Situation (*ii*) *A* term of X is $x^r u$ and u is a term of the C.F.

If u correspond to an s-fold root m, the trial solution y^* must contain the term $x^{r+s} u$ plus terms arising from it by differentiation.

For example, consider $$(D-2)^3 (D+3)\, y = x^2 e^{2x} + x^2. \qquad \dots (1)$$

AS usual, $$\text{C.F. of (1)} = (c_1 + c_2 x + c_3 x^2)\, e^{2x} + c_4 e^{-3x}. \qquad \dots (2)$$

Note carefully that $x^2 e^{2x}$ occurs in X and e^{2x} is also present in C.F. corresponding to triple root $m = 3$; hence we employ $x^{2+3} e^{2x}$, *i.e.* $x^5 e^{2x}$ and all terms arising from it by differentiation. Thus the contribution to trial solution y^* corresponding to $x^2 e^{2x}$ of X is taken as $A_1 e^{2x} + A_2 x\, e^{2x} + A_3 x^2 e^{2x} + A_4 x^3 e^{2x} + A_5 x^4 e^{2x} + A_6 x^5 e^{2x}$. Note that x^2 is not occuring in C.F., so the corresponding contribution to trial solution y^* is $A_7 + A_8 x + A_9 x^2$. Hence the net trial solution y^* for P.I. is given by

$$y^* = A_1 e^{2x} + A_2 x\, e^{2x} + A_3 x^2 e^{2x} + A_4 x^3 e^{2x} + A_5 x^4 e^{2x} + A_6 x^5 e^{2x} + A_7 + A_8 x + A_9 x^2.$$

5.27 Solved examples based on Art. 5.26

Ex. 1. *(a) By the method of undetermined coefficients, solve $(D^2 + 4)\, y = x^2$.*

Sol. Here given that $$(D^2 + 4)\, y = x^2. \qquad \dots (1)$$

Its auxiliary equation is $D^2 + 4 = 0$ so that $D = \pm\, i$.

$\therefore$ C.F. $= c_1 \cos 2x + c_2 \sin 2x$, c_1, c_2 being arbitrary constants. ... (2)

Let the trial solution be $y^* = A_0 + A_1 x + A_2 x^2$. [Refer result 1 in table of Art. 5.26] ... (3)

Since y^* must satisfy (1), we have $(D^2 + 4)\, y^* = x^2$ or $D^2 y^* + 4y^* = x^2$. ... (4)

Now, (3) $\Rightarrow D\, y^* = A_1 + 2A_2 x$ and $D^2 y^* = 2A_2$. ... (5)

Using (3) and (5), (4) reduces to $$2A_2 + 4\, (A_0 + A_1 x + A_2 x^2) = x^2$$

or $$2A_2 + 4A_0 + 4A_1 x + 4A_2\, x^2 = x^2. \qquad \dots (6)$$

(6) is an identify. Comparing the coefficients of like terms, we get

$2A_2 + 4A_0 = 0$, $\quad 4A_1 = 0$, $\quad 4A_2 = 1$. ... (7)

Solving (7), $A_1 = 0$, $A_2 = 1/4$. $A_0 = -\,1/8$. Then, from (3), we have

$$y^* = -\,(1/8) + x^2/4 = (1/8)\,(2x^2 - 1).$$

Hence the required general solution is y = C.F. + P.I. = C.F. + y^*

or $$y = c_1 \cos 2x + c_2 \sin 2x + (1/8)(2x^2 - 1).$$

Ex. 1(*b*). *Using the method of undetermined coefficients, solve* $y_2 - 2y_1 + y = x^2$.

[Delhi Maths (G) 1995]

Sol. Let $D \equiv d/dx$. Then, we have $$(D^2 - 2D + 1)\, y = x^2. \qquad \text{... (1)}$$

Its auxiliary equation is $D^2 - 2D + 1 = 0$ so that $D = 1, 1$.

$\therefore$ C.F. $= (c_1 + c_2 x)\, e^x$, c_1, c_2 being arbitrary constants. ... (2)

Let the trial solution be $y^* = A_0 + A_1 x + A_2 x^2$. [Refer result 1 in table of Art. 5.26] ... (3)

Since y^* must satisfy (1), we have $D^2 y^* - 2D\, y^* + y^* = x^2$. ... (4)

Now, (3) $\Rightarrow D\, y^* = A_1 + 2A_2 x$ and $D^2 y^* = 2A_2$. ... (5)

Using (3) and (5), (4) reduces to $$2A_2 - 2(A_1 + 2A_2 x) + A_0 + A_1 x + A_2 x^2 = x^2$$

or $$(A_0 - 2A_1 + 2A_2) + x\,(A_1 - 4A_2) + A_2 x^2 = x^2.$$

Comparing the coefficients of like terms in above identity, we have

$A_0 - 2A_1 + 2A_2 = 0$, $A_1 - 4A_2 = 0$ and $A_2 = 1$ so that $A_2 = 1$, $A_1 = 4$, $A_0 = 6$.

From (3), $y^* = 6 + 4x + x^2$ and so solution is y = C.F. + $y^* = (c_1 + c_2 x)\, e^x + 6 + 4x + x^2$.

Ex. 2. *Using the method of undetermined coefficients, solve* $y_2 + 2y_1 + y = x - e^x$.

[Delhi Maths (G) 1996]

Sol. Re-writing the given equation, $$(D^2 + 2D + 1)\, y = x - e^x \qquad \text{... (1)}$$

Its auxiliary equation $D^2 + 2D + 1 = 0$ so that $D = -1, -1$.

$\therefore$ C.F. $= (c_1 + c_2 x)\, e^{-x}$, c_1, c_2 being arbitrary constants. ... (2)

Let the trial solution be $y^* = Ax + B + C\, e^x$. [Refer results 1 and 2 in table of Art. 5.26] ... (3)

Since y^* must satisfy (1), $(D^2 + 2D + 1)\, y^* = x - e^x$ or $D^2 y^* + 2D\, y^* + y^* = x - e^x$. ... (4)

From (3), $D\, y^* = A + C\, e^x$ and $D^2 y^* = Ce^x$. ... (5)

Using (3) and (5), (4) gives $$C\, e^x + 2(A + C\, e^x) + Ax + B + C\, e^x = x - e^x$$

or $$(2A + B) + Ax + 4C\, e^x = x - e^x.$$

Equating the coefficients of like terms in the above identity, we get

$2A + B = 0$, $A = 1$, $4C = -1$ so that $A = 1$, $B = -2$, $C = -1/4$.

$\therefore$ from (3), $y^* = x - 2 - (1/4)\, e^x$ and so the general solution is

y = C.F. + y^* or $y = (c_1 + c_2 x)\, e^{-x} + x - 2 - (1/4)\, e^x$.

Ex. 3(a). *Solve* $(D^2 + 3D + 2)\, y = x + \cos x$ *by using the method of undetermined coefficients.* **[Delhi Maths (Prog) 2007]**

Sol. Given $$(D^2 + 3D + 2)\, y = x + \cos x, \text{ where } D \equiv d/dx \qquad \text{... (1)}$$

Its auxiliary equation is $D^2 + 3D + 2 = 0$ giving $D = -1, -2$

$\therefore$ C.F. $= c_1 e^{-x} + c_2 e^{-2x}$, c_1, c_2 being arbitrary constants ... (2)

Corresponding to special form x of R.H.S. of (1), we choose trial solution for P.I. as $A_0 + A_1 x$ and corresponding to special form $\cos x$ of R.H.S. of (1), we choose trial solution for P.I. as $A_2 \cos x + A_3 \sin x$. Combining these, we attempt a trial solution for P.I. as

$$y^* = A_0 + A_1 x + A_2 \cos x + A_3 \sin x, \qquad \text{... (3)}$$

where A_0, A_1, A_2 and A_3 are constants to be determined. Since y^* must satisfy (1), we get

$(D^2 + 3D + 2) y^* = x + \cos x$ or $D^2 y^* + 3D\, y^* + 2y^* = x + \cos x$ ... (4)

From (3), $$Dy^* = A_1 - A_2 \sin x + A_3 \cos x \qquad \text{... (5)}$$

From (5), $$D^2 y^* = -A_2 \cos x - A_3 \sin x \qquad \text{... (6)}$$

Using (3), (5) and (6), (4) reduces to

$-A_2 \cos x - A_3 \sin x + 3\,(A_1 - A_2 \sin x + A_3 \cos x) + 2\,(A_0 + A_1 x + A_2 \cos x + A_3 \sin x) = x + \cos x$

or $\qquad 3A_1 + 2A_0 + 2A_1 x + (A_2 + 3A_3) \cos x + (A_3 - 3A_2) \sin x = x + \cos x,$

which is an identity and so equating the coefficients of like terms, we get

$3A_1 + 2A_0 = 0, \qquad 2A_1 = 1, \qquad A_2 + 3A_3 = 1 \qquad$ and $\qquad A_3 - 3A_2 = 0.$

Solving these, $A_0 = -(3/4)$, $A_1 = 1/2$, $A_2 = 1/10$, $A_3 = 3/10$. Then, from (3), we get

$$y^* = -(3/4) + x/2 + (1/10) \times (\cos x + 3 \sin x)$$

Hence the required general solution is $\qquad y = \text{C.F.} + \text{P.I.} \qquad$ or $\qquad y = C.F. + y^*$

or $\qquad y = c_1 e^{-x} + c_2 e^{-2x} - (3/4) + x/2 + (1/10) \times (\cos x + 3 \sin x)$

Ex. 3(*b*). *Using the method of undetermined coefficients, solve $y_2 - 2y_1 + 3y = \cos x + x^2$.* **[Delhi Maths (G) 1996]**

Sol. (*a*) Let $D \equiv d/dx$. Then given $\qquad (D^2 - 2D + 3)\, y = \cos x + x^2. \qquad$ ... (1)

Its auxiliary equation is $\qquad D^2 - 2D + 3 = 0 \qquad$ so that $\qquad D = 1 \pm i\sqrt{2}.$

$\therefore \qquad$ C.F. $= e^x (c_1 \cos x\sqrt{2} + c_2 \sin x\sqrt{2})$, c_1, c_2 being arbitrary constants ... (2)

Let the trial solution be $\qquad y^* = A_0 + A_1 x + A_2 x^2 + A_3 \cos x + A_4 \sin x. \qquad$ ... (3)

[Refer results 1 and 4 in table of Art. 5.26]

Since y^* must satisfy (1), $\qquad D^2 y^* - 2D\, y^* + 3y^* = \cos x + x^2. \qquad$ ... (4)

From (3), $\qquad D\, y^* = A_1 + 2A_2 x - A_3 \sin x + A_4 \cos x \qquad$ ... (5)

and from (6), $\qquad D^2 y^* = 2A_2 - A_3 \cos x - A_4 \sin x. \qquad$ ... (6)

Using (3), (5) and (6), (4) reduces to

$2A_2 - A_3 \cos x - A_4 \sin x - 2\,(A_1 + 2A_2 x - A_3 \sin x + A_4 \cos x)$
$\qquad + 3\,(A_0 + A_1 x + A_2 x^2 + A_3 \cos x + A_4 \sin x) = \cos x + x^2.$

or $\qquad (2A_2 + 2A_1 + 3A_0) + x\,(3A_1 - 4A_2) + 3A_2 x^2 + (2A_3 - 2A_4) \cos x$
$\qquad + (2A_4 - 2A_3) \sin x = \cos x + x^2.$

Equating the coefficients of like terms on both sides, we get

$2A_2 + 2A_1 - 3A_0 = 0, \qquad 3A_1 - 4A_2 = 0, \qquad 3A_2 = 1, \qquad 2(A_3 - A_4) = 1, \qquad 2(A_4 - A_3) = 0$

Solving these, $\qquad A_0 = 2/27, \qquad A_1 = 4/9, \qquad A_2 = 1/3, \qquad A_3 = 1/4, \qquad A_4 = -1/4.$

$\therefore$ From (3), $\qquad y^* = (2/27) + (4/9)\, x + (1/3)\, x^2 + (1/4)\,(\cos x - \sin x).$

Hence the required general solution is $\qquad y = \text{C.F.} + y^*$, *i.e.*,

$y = e^x (c_1 \cos x\sqrt{2} + c_2 \sin x\sqrt{2}) + (1/27)(2 + 12x + 9x^2) + (1/4)(\cos x - \sin x)$

Ex. 3(*c*). *Solve $(D^2 - 2D + 3)\, y = x^3 + \sin x$* **[Delhi Maths (Hons) 2001]**

Sol. Given $\qquad (D^2 - 2D + 3)\, y = x^3 + \sin x \qquad$ ... (1)

Its auxiliary equation is $\qquad D^2 - 2D + 3 = 0, \qquad$ giving $\qquad D = 1 \pm i\sqrt{2}$

$\therefore$ C.F. $= e^x \{c_1 \cos(x\sqrt{2}) + c_2 \sin(x\sqrt{2})\}$, c_1, c_2 being arbitrary constatns ... (2)

Corresponding to special form x^3 of R.H.S. of (1), we choose trial solution for P.I. as $A_0 + A_1 x + A_2 x^2 + A_3 x^3$ and corresponding to special form $\sin x$ of R.H.S. of (1), we choose trial solution for P.I. as $A_4 \cos x + A_5 \sin x$. Combining these, we attempt a trial solution for P.I. of the form

$$y^* = A_0 + A_1 x + A_2 x^2 + A_3 x^3 + A_4 \cos x + A_5 \sin x. \qquad \text{... (3)}$$

Since y^* must satisfy (1), we have

$$(D^2 - 2D + 3)\, y^* = x^3 + \sin x \Rightarrow D^2 y^* - 2Dy^* + 3\, y^* = x^3 + \sin x. \qquad \text{... (4)}$$

From (3), $\quad D y^* = A_1 + 2A_2 x + 3A_3 x^2 - A_4 \sin x + A_5 \cos x.$ ... (5)

From (5), $\quad D^2 y^* = 2A_2 + 6A_3 x - A_4 \cos x - A_5 \sin x.$ ... (6)

Using (3), (5) and (6), (4) reduces to

$$2A_2 + 6 A_3 x - A_4 \cos x - A_5 \sin x - 2 (A_1 + 2A_2 x + 3A_3 x^2 - A_4 \sin x + A_5 \cos x) + 3 (A_0 + A_1 x + A_2 x^2 + A_3 x^3 + A_4 \cos x + A_5 \sin x) = x^3 + \sin x$$

or $$3A_0 - 2A_1 + 2A_2 + x (6A_3 - 4A_2 + 3A_1) + (3A_2 - 6A_3) x^2 + 3A_3 x^3 + 2 (A_4 - A_5) \cos x + 2 (A_4 + A_5) \sin x = x^3 + \sin x.$$

which is an identity and so equating coefficients of like terms, we have

$3A_0 - 2A_1 + 2A_2 = 0,\ 6A_3 - 4A_2 + 3A_1 = 0,\ 3A_2 - 6A_3 = 0,\ 3A_3 = 1,\ 2(A_4 - A_5) = 0,\ 2(A_4 + A_5) = 1$

$\Rightarrow A_0 = -(8/27), \quad A_1 = 2/9, \quad A_2 = 2/3, \quad A_3 = 1/3, \quad A_4 = A_5 = 1/4.$

$\therefore$ (3) $\Rightarrow$ P.I. $= -(8/27) + (2/9)\, x + (2/3)\, x^2 + (1/3)\, x^3 + (1/4)\, \sin x + \cos x)$

and hence the required general solution of (1) is $\quad y = \text{C.F.} + \text{P.I.}, \quad$ *i.e.*,

$$y = e^x (c_1 \cos \sqrt{2}x + c_2 \sin \sqrt{2}x) + (1/27)(9x^3 + 18x^2 + 6x - 8) + (1/4)(\sin x + \cos x).$$

Ex. 4. *Using the method of undetermined coefficients to solve $(d^2y/dx^2) - 2(dy/dx) - 3y = 2e^x - 10 \sin x$.* **[Delhi Maths Hons. 1997]**

Sol. Given $\quad (D^2 - 2D - 3)\, y = 2\, e^x - 10 \sin x$, where $D \equiv d/dx$. ... (1)

Its auxiliary equation is $\quad D^2 - 2D - 3 = 0 \quad$ so that $\quad D = -1, 3.$

$\therefore$ C.F. $= c_1 e^{-x} + c_2 e^{3x}$, c_1, c_2 being arbitrary constants. ... (2)

Let the trial solution be $\quad y^* = A\, e^x + B \sin x + C \cos x.$ ... (3)

Since y^* must satisfy (1), $\quad D^2 y^* - 2D\, y^* - 3\, y^* = 2\, e^x - 10 \sin x.$ ... (4)

From (3), $\quad D\, y^* = Ae^x + \text{B} \cos x - \text{C} \sin x,\ D^2 y^* = Ae^x - B \sin x - C \cos x$ ... (5)

Using (3) and (5), (4) reduces to

$$(Ae^x - B \sin x - C \cos x) - 2\, (Ae^x + B \cos x - C \sin x) - 3\, (Ae^x + B \sin x + C \cos x) = 2e^x - 10 \sin x$$

or $$-4\, Ae^x - (4B - 2C) \sin x - (4C + 2B) \cos x = 2e^x - 10 \sin x.$$

Equating the coefficients of like terms in above identity, we have

$-4A = 2, \quad -(4B - 2C) = -10 \quad$ and $\quad -(4C + 2B) = 0 \quad \Rightarrow \quad A = -1/2, \quad B = 2, \quad C = -1$

$\therefore$ From (3), $\quad y^* = (-1/2)\, e^x + 2 \sin x - \cos x.$

and general solution is $y = \text{C.F.} + y^*$ *i.e.*, $\quad y = c_1\, e^{-x} + c_2\, e^{3x} - (1/2)\, e^x + 2 \sin x - \cos x.$

Ex. 5(*a*). *Solve $(d^2y/dx^2) - 9y = x + e^{2x} - \sin 2x$, using method of undetermined coefficients for finding particular integral.* **[Delhi Maths Hons. 1995, Delhi Maths (G) 2002, 06]**

Sol. Given $\quad (D^2 - 9)\, y = x + e^{2x} - \sin 2x$, where $D \equiv d/dx$. ... (1)

The auxiliary equation is $\quad D^2 - 9 = 0 \quad$ so that $\quad D = \pm 3.$

$\therefore$ C.F. $= c_1\, e^{3x} + c_2\, e^{-3x}$, c_1, c_2 being arbitrary constants ... (2)

Corresponding to special form x of R.H.S. of (1), we choose trial solution for P.I. as $A_0 + A_1 x$. Next, corresponding to special form e^{2x} of R.H.S. of (1), we choose trial solution for P.I. as $A_2\, e^{2x}$. Finally corresponding to the special form $\sin 2x$ of R.H.S. of (1), we choose trial solution for P.I. as $A_3 \cos 2x + A_4 \sin 2x$. Combining these, we attempt a trial solution y^* for P.I. of the form

$$y^* = A_0 + A_1 x + A_2\, e^{2x} + A_3 \cos 2x + A_4 \sin 2x. \quad ... (3)$$

Since y^* must satisfy (1), $\quad D^2 y^* - 9\, y^* = x + e^{2x} - \sin 2x.$ ... (4)

From (3), $\quad D\, y^* = A_1 + 2A_2\, e^{2x} - 2A_3 \sin 2x + 2A_4 \cos 2x$ ... (5)

and so $\quad D^2 y^* = 4A_2\, e^{2x} - 4A_3 \cos 2x - 4A_4 \sin 2x.$ ... (5)

Using (5) and (6), (4) reduces to $\quad 4A_2\, e^{2x} - 4A_3 \cos 2x - 4A_4 \sin 2x$

$$-9\,(A_0 + A_1 x + A_2 e^{2x} + A_3 \cos 2x + A_4 \sin 2x) = x + e^{2x} - \sin 2x$$

or $$-9\,A_0 - 9\,A_1 x - 5\,A_2 e^{2x} - 13A_3 \cos 2x - 13A_4 \sin 2x = x + e^{2x} - \sin 2x,$$

which is an identity and so equating coefficients of like terms, we have

$-9\,A_0 = 0,\quad 9A_1 = 1,\quad -5A_2 = 1,\quad -13A_3 = 0,\quad$ and $\quad -13A_4 = -1$

$\Rightarrow\quad A_0 = 0,\quad A_1 = -(1/9),\quad A_2 = -(1/5),\quad A_3 = 0\quad$ and $\quad A_4 = 1/13.$

(3) $\Rightarrow y^* = -(1/9)\,x - (1/5)\,e^{2x} - (1/13) \sin x$ and solution of (1) is

$y = \text{C.F.} + y^* = c_1\, e^{3x} + c_2\, e^{-3x} - (1/9)\,x - (1/5)\,e^{2x} - (1/13) \sin x.$

Ex. 5(*b*). *Use the method of undetermined coefficients to solve $(d^2y/dx^2) + 9y = e^{3x} + e^{-3x} + e^{3x} \sin 3x$.* **[Delhi Maths (G) 1993]**

Sol. Given $\quad (D^2 + 9)\,y = e^{3x} + e^{-3x} + e^{3x} \sin 3x,\quad$ where $\quad D = d/dx.$ ... (1)

Its auxiliary equation is $\quad D^2 + 9 = 0\quad$ so that $\quad D = \pm\, 3i.$

$\therefore\;$ C.F. $= c_1 \cos 3x + c_2 \sin 3x$, c_1, c_2 being arbitrary constants. ... (2)

Let the trial solution be $\quad y^* = A\,e^{3x} + \mathrm{B}\,e^{-3x} + C\,e^{3x} \cos 3x + D\,e^{3x} \sin 3x$ ... (3)

Since y^* must satisfy (1), $\quad D^2 y^* + 9y^* = e^{3x} + e^{-3x} + e^{3x} \sin x.$ ... (4)

From (3), $D\,y^* = 3A\,e^{3x} - 3Be^{-3x} + C\,(3e^{3x} \cos 3x - 3\,e^{3x} \sin 3x) + D\,(3\,e^{3x} \sin 3x + 3\,e^{3x} \cos 3x)$

or $$D\,y^* = 3\,[A\,e^{3x} - B\,e^{-3x} + (C + D)\,e^{3x} \cos 3x + (D - C)\,e^{3x} \sin 3x]$$

and $$D^2 y^* = 3\,[3\,A\,e^{3x} + 3\,\mathrm{B}e^{-3x} + (C + D)\,(3\,e^{3x} \cos 3x - 3\,e^{3x} \sin 3x) + (D - C)\,(3\,e^{3x} \sin 3x + 3\,e^{3x} \cos 3x)]$$

or $$D^2 y^* = 9\,(A\,e^{3x} - B\,e^{-3x} + 2D\,e^{3x} \cos 3x - 2C\,e^{3x} \sin 3x)$$

Putting the above values of $D^2 y^*$ and y^* in (4), we have

$$9\,(A\,e^{3x} + B\,e^{-3x} + 2D\,e^{3x} \cos 3x - 2C\,e^{3x} \sin 3x) + 9\,(A\,e^{3x} + B\,e^{-3x} + C\,e^{3x} \cos 3x + D\,e^{3x} \sin 3x) = e^{3x} + e^{-3x} + e^{3x} \sin 3x.$$

or $$18\,A\,e^{3x} + 18\,B\,e^{-3x} + (18D + 9C)\,e^{3x} \cos 3x + (9D - 18C)\,e^{3x} \sin 3x = e^{3x} + e^{-3x} + e^{3x} \sin 3x. \quad ...\,(5)$$

Equating the coefficients of like terms on both sides of (5), we get

$18A = 1,\quad 18\,\mathrm{B} = 1,\quad 18D + 9C = 0,\quad 9D - 18C = 1.\quad$ Solving these,

$A = 1/18,\quad B = 1/18,\quad C = -\,2/45,\quad D = 1/45\quad$ and so from (3), we have

$$y^* = (1/18)\,(e^{3x} + e^{-3x}) + (1/45)\,e^{3x}\,(\sin 3x - 2 \cos 3x).$$

Hence the required general solution is $\quad y = \text{C.F.} + y^*$

or $$y = c_1 \cos 3x + c_2 \sin 3x + (1/18)\,(e^{3x} + e^{-3x}) + (1/45)\,e^{3x}\,(\sin 3x - 2 \cos 3x)$$

Ex. 5. (*c*) *Solve by using the method of undetermined coefficients $(D^2 + 1)\,y = 12 \cos^2 x$.* **[Kuvempa 2005]**

Sol. Given $\quad (D^2 + 1)\,y = 6\,(1 + \cos 2x)\quad$ or $\quad (D^2 + 1)\,y = 6 + 6 \cos 2x$ ... (1)

Here auxiliary equation is $\quad D^2 + 1 = 0\quad$ so that $\quad D = \pm\, i.$

$\therefore\;$ C.F. $= c_1 \cos x + c_2 \sin x$, c_1, c_2 being arbitrary constants. ... (2)

Let the trial solution be $\quad y^* = A_0 + A_1 \cos 2x + A_2 \sin 2x$ ... (3)

[Using results 1 and 4 of table of Art. 5.26]

Since y^* must satisfy (1), $\quad D^2 y^* + y^* = 6 + 6 \cos 2x$ ... (4)

From (3), $\quad D\,y^* = -\,2A_1 \sin 2x + 2A_2 \cos 2x,$ ... (5A)

and $\quad D^2 y^* = -\,4A_1 \cos 2x - 4A_2 \sin 2x$ ... (5B)

Using (3) (5A) and (5B), (4) reduces to

$$-\,4A_1 \cos 2x - 4\,A_2 \sin 2x + A_0 + A_1 \cos 2x + A_2 \sin 2x = 6 + 6 \cos 2x$$

or $$A_0 - 3A_1 \cos 2x - 3A_2 \sin 2x = 6 + 6 \cos 2x \qquad \text{... (6)}$$

Equating the coefficients of like terms on both sides of (6), we get

$A_0 = 6, \quad -3A_1 = 6 \quad$ and $\quad -3A_2 = 0 \quad \Rightarrow \quad A_0 = 6, \quad A_1 = -2, \quad A_2 = 0$

$\therefore$ From (3), $\quad y^* = 6 - 2\cos 2x$.

$\therefore$ the required solution is $\quad y = \text{C.F.} + y^*$, *i.e.*, $\quad y = c_1 \cos x + c_2 \sin x + 6 - 2\cos 2x$

Ex. 6(*a*). *Using the method of undetermined coefficients, solve* $\quad (D^2 - 2D)\, y = e^x \sin x$.

[Delhi Maths (G) 1997]

Sol. Given that $$(D^2 - 2D)\, y = e^x \sin x. \qquad \text{... (1)}$$

Its auxiliary equation is $\quad D^2 - 2D = 0 \quad$ so that $\quad D = 0, 2$.

$\therefore$ C.F. $= c_1 e^{0.x} + c_2 e^{2x} = c_1 + c_2 e^{2x}$, c_1, c_2 are arbitrary constants. ... (2)

Let the trial solution be $$y^* = e^x (A_1 \cos x + A_2 \sin x). \qquad \text{... (3)}$$

[Refer result 5 in table of Art. 5.26]

Since y^* must satisfy (1), $\quad (D^2 - 2D)\, y^* = e^x \sin x \quad$ or $\quad D^2 y^* - 2D\, y^* = e^x \sin x$. ... (4)

Now, $$(3) \Rightarrow D\, y^* = e^x (A_1 \cos x + A_2 \sin x) + e^x (-A_1 \sin x + A_2 \cos x). \qquad \text{... (5)}$$

and $$(5) \Rightarrow D^2 y^* = e^x (A_1 \cos x + A_2 \sin x) + 2e^x (-A_1 \sin x + A_2 \cos x)$$
$$+ e^x (-A_1 \cos x - A_2 \sin x). \qquad \text{... (6)}$$

Using (5) and (6), (4) reduces to

$$e^x (A_1 \cos x + A_2 \sin x) + 2e^x (-A_1 \sin x + A_2 \cos x) + e^x (-A_1 \cos x$$
$$- A_2 \sin x) - 2[e^x (A_1 \cos x + A_2 \sin x) + e^x (-A_1 \sin x + A_2 \cos x)] = e^x \sin x$$

or $$-2A_1 e^x \cos x - 2A_2 e^x \sin x = e^x \sin x \qquad \text{... (7)}$$

Comparing the coefficients of $e^x \cos x$ and $e^x \sin x$ in identity (7), we get

$-2A_1 = 0 \quad$ and $\quad -2A_2 = 1 \quad$ so that $\quad A_1 = 0 \quad$ and $\quad A_2 = -1/2 \quad$ So $\quad (3) \Rightarrow y^* = (-1/2)\, e^x \sin x$

$\therefore$ The general solution is $$y = \text{C.F.} + y^* = c_1 + c_2 e^{2x} - (1/2)\, e^x \sin x$$

Ex. 6(*b*). *Solve* $(d^2y/dx^2) + 2\,(dy/dx) + 4y = 111\, e^{2x} \cos 3x$, *using the method of undetermined coefficients.* **[Delhi Maths Hons. 1993]**

Sol. Given, $$(D^2 + 2D + 4)\, y = 111\, e^{2x} \cos 3x. \qquad \text{... (1)}$$

The auxiliary equation $\quad D^2 + 2D + 4 = 0 \quad \Rightarrow \quad D = (-2 \pm \sqrt{-12})/2 = -1 \pm i\sqrt{3}$

$\therefore$ C.F. $= e^{-x} (c_1 \cos \sqrt{3}x + c_2 \sin \sqrt{3}x)$, c_1, c_2 being arbitrary constants ... (2)

Since R.H.S. contains $111\, e^{2x} \cos 3x$, as a P.I. we take trial solution

$$y^* = e^{2x} (A_1 \cos 3x + A_2 \sin 3x). \qquad \text{... (3)}$$

Since y^* must satisfy (1), we have $\quad (D^2 + 2D + 4)\, y^* = 111\, e^{2x} \cos 3x$

or $$D^2 y^* + 2D\, y^* + 4\, y^* = 111\, e^{2x} \cos 3x. \qquad \text{... (4)}$$

$$(3) \Rightarrow D\, y^* = A_1 (2e^{2x} \cos 3x - 3e^{2x} \sin 3x) + A_2 (2e^{2x} \sin 3x + 3e^{2x} \cos 3x)$$

or $$Dy^* = (2A_1 + 3A_2)\, e^{2x} \cos 3x + (2A_2 - 3A_1)\, e^{2x} \sin 3x. \qquad \text{... (5)}$$

$$(5) \Rightarrow D^2 y^* = (2A_1 + 3A_2)(2e^{2x} \cos 3x - 3e^{2x} \sin 3x) + (2A_2 - 3A_1)(2e^{2x} \sin 3x + 3e^{2x} \cos 3x)$$

or $$D^2 y^* = (12A_2 - 5A_1)\, e^{2x} \cos 3x - (12A_1 + 5A_2)\, e^{2x} \sin 3x. \qquad \text{... (6)}$$

Using (3), (5) and (6), (4) reduces to

$$(12A_2 - 5A_1)\, e^{2x} \cos 3x - (12A_1 + 5A_2)\, e^{2x} \sin 3x + 2\,\{(2A_1 + 3A_2)\, e^{2x} \cos 3x$$
$$+ (2A_2 - 3A_1)\, e^{2x} \sin 3x\} + 4\, e^{2x} (A_1 \cos 3x + A_2 \sin 3x) = 111\, e^{2x} \cos 3x$$

or $$(3A_1 + 18A_2)\, e^{2x} \cos 3x + (3A_2 - 18A_1)\, e^{2x} \sin 3x = 111\, e^{2x} \cos 3x,$$

which is an identity and hence equating coefficients of like terms, we have

$3A_1 + 18A_2 = 111$ and $3A_2 - 18A_1 = 0.$ ... (7)

Solving (7), $A_1 = 1$ and $A_2 = 6$ and so form (3), we have

P.I. $= e^{2x}$ (cos $3x$ + 6 sin $3x$) and hence the required solution is $y = C.F. + P.I.,$

or $y = e^{-x}(c_1 \cos\sqrt{3}x + c_2 \sin\sqrt{3}x) + e^{2x}(\cos 3x + 6 \sin 3x).$

Ex. 7(*a*). *Using the method of undetermined coefficients, solve* $(D^3 + 3D^2 + 2D)\, y = x^2 + 4x + 8.$

[Delhi Maths (G) 1997]

Sol. Given that $(D^3 + 3D^2 + 2D)\, y = x^2 + 4x + 8.$ ... (1)

Its auxiliary equation is $D^3 + 3D^2 + 2D = 0$ so that $D = 0, -1, -2.$

$\therefore$ C.F. $= c_1 + c_2 e^{-x} + c_3 e^{-2x}$, c_1, c_2 and c_3 being arbitrary constants. ... (2)

Since constant 8 occurs in R.H.S. of (1) and it also occurs as constant c_1 in the C.F. (2) corresponding to root of multiplicity one, so we choose trial solution y^* as $x\,(A_0 + A_1x + A_2x^2)$ [Refer result 1 in table of Art. 5.26 and situation (*i*) of remark 3 of Art. 5.26].

Thus, here $y^* = A_0 x + A_1 x^2 + A_2 x^3.$ ... (3)

Since y^* must satisfy (1), $D^3y^* + 3D^2y^* + 2Dy^* = x^2 + 4x + 8.$ ... (4)

(3) $\Rightarrow D\,y^* = A_0 + 2A_1x + 3A_2x^2,$ $D^2y^* = 2A_1 + 6A_2 x,$ $D^3 y^* = 6A_2.$ ... (5)

Using (5), (4) gives $6A_2 + 3(2A_1 + 6A_2x) + 2\,(A_0 + 2A_1x + 3A_2x^2) = x^2 + 4x + 8$

or $2A_0 + 6A_1 + 6A_2 + x\,(4A_1 + 18A_2) + 6A_2\, x^2 = 8 + 4x + x^2.$

Comparing the coefficients of like terms in above identity, we have

$2A_0 + 6A_1 + 6A_2 = 8,\quad 4A_1 + 18A_2 = 4,\quad 6A_2 = 1 \Rightarrow A_0 = 11/4,\quad A_1 = 1/4,\quad A_2 = 1/6$

$\therefore$ (3) $\Rightarrow y^* = (11/4)\,x + (1/4)\,x^2 + (1/6)\,x^3 = (x/12)\,(33 + 3x + 2x^2).$

Hence solution is $y = C.F. + y^*$ or $y = c_1 + c_2 e^{-x} + c_2 e^{-2x} + (x/12)\,(33 + 3x + 2x^2)$

Ex. 7(*b*). *Using the method of undetermined coefficients, solve* $(D^2 + 5D + 6)\, y = 2e^{-3x} + 4e^{-2x}.$

[Delhi Maths (G) 1994]

Sol. Given $(D^2 + 5D + 6)\, y = 2e^{-3x} + 4e^{-2x}.$ ... (1)

Its auxiliary equation is $D^2 + 5D + 6 = 0$ so that $D = -3, -2.$

$\therefore$ C.F. $= c_1 e^{-3x} + c_2 e^{-2x}$, c_1, c_2 being arbitrary constants. ... (2)

Since the terms e^{-3x} and e^{-2x} in the R.H.S. of (1) appear in C.F. (2) corresponding to the roots -3 and -2 respectively, each occuring once, hence we take the trial solution y^* as

$$y^* = A\,x\,e^{-3x} + Bx\,e^{-2x}. \qquad ... (3)$$

Since y^* must satisfy (1), $D^2 y^* + 5D\,y^* + 6\,y^* = 2\,e^{-3x} + 4\,e^{-2x}$ ... (4)

Now, (3) $\Rightarrow D\,y^* = -\,3Ax\,e^{-3x} + 3A\,e^{-3x} - 2Bx\,e^{-2x} + B\,e^{-2x}$ ... (5)

and $D^2 y^* = -\,3A(e^{-3x} - 3x\,e^{-3x}) - 3A\,e^{-3x} - 2B\,(e^{-2x} - 2x\,e^{-2x}) - 2B\,e^{-2x}$

or $D^2 y^* = -\,6A\,e^{-3x} + 9x\,A\,e^{-3x} - 4B\,e^{-2x} + 4Bx\,e^{-2x}.$ ... (6)

Using (3), (5) and (6), (4) reduces to

$$-\,6A\,e^{-3x} + 9x\,A\,e^{-3x} - 4B\,e^{-2x} + 4Bx\,e^{-2x} + 5\,(-\,3Ax\,e^{-3x} + A\,e^{-3x} - 2Bx\,e^{-2x} + B\,e^{-2x}) + 6\,(Ax\,e^{-3x} + Bx\,e^{-2x}) = 2e^{-3x} + 4e^{-2x}$$

or $-\,A\,e^{-3x} + B\,e^{-2x} = 2\,e^{-3x} + 4\,e^{-3x} \Rightarrow A = -\,2,\ B = 4$

$\therefore$ (3) $\Rightarrow$ $y^* = -\,2x\,e^{-3x} + 4x\,e^{-2x}$

and solution is $y = C.F. + y^*$ or $y = c_1\,e^{-3x} + c_2\,e^{-2x} - 2x\,e^{-3x} + 4x\,e^{-3x}.$

Ex. 7(*c*). *Using the method of undetermined coefficients, solve* $y'' + y = 2 \cos x.$

[Lucknow 1996]

Sol. Let $D \equiv d/dx$. Then given equation reduces to $(D^2 + 1)\, y = 2 \cos x.$ ... (1)

Its auxiliary equation is $D^2 + 1 = 0$ so that $D = \pm i$.

$\therefore$ C.F. $= c_1 \cos x + c_2 \sin x$, c_1, c_2 being arbitrary constants. ... (2)

Since $\cos x$ occurs on R.H.S. of (1) and $\cos x$ is a part of C.F. (2) corresponding to root of multiplicity one, choose trial solution y^* of the form

$$y^* = x\,(A \cos x + B \sin x) = Ax \cos x + Bx \sin x. \quad ... (3)$$

Since y^* must satisfy (1) $\quad D^2 y^* + y^* = 2 \cos x$. ... (4)

From (3), $\quad Dy^* = A \cos x - A\,x \sin x + B \sin x + Bx \cos x$. ... (5)

and $\quad D^2 y^* = -A \sin x - A\,(\sin x + x \cos x) + B \cos x + B\,(\cos x - \sin x)$. ... (6)

Using (3) and (6), (4) reduces to

$$-2A \sin x - Ax \cos x + 2B \cos x - Bx \sin x + Ax \cos x + Bx \sin x = 2 \cos x$$

or $\quad -2A \sin x + 2\text{B} \cos x = 2 \cos x \Rightarrow -2A = 0,\ 2B = 2 \Rightarrow A = 0,\ B = 1$.

From (3), $y^* = x \sin x$ and solution is $\quad y = \text{C.F.} + y^* = c_1 \cos x + c_2 \sin x + x \sin x$.

Ex. 7(*d*). *Using the method of undetermined coefficients, solve* $\quad y_3 + y_1 = 2x^2 + 4 \sin x$.

[Delhi Maths (G) 1998]

Sol. Let $D \equiv d/dx$. Then, given equation reduced to $\quad (D^3 + D)\,y = 2x^2 + 4 \sin x$. ... (1)

Its auxiliary equation is $D^3 + D = 0$ so that $D = 0, \pm i$

$\therefore \quad$ C.F. $= c_1 e^{0.x} + e^{0.x}(c_2 \cos x + c_3 \sin x) = c_1 + c_2 \cos x + c_3 \sin x$. ... (2)

Since the constant term c_1 appears in C.F. (2), corresponding to the term x^2 on R.H.S. of (1), contribution to trial solution y^* is taken $x\,(A_0 + A_1 x + A_2 x^2)$. Again, since $c_2 \sin x$ appears is C.F. (2), corresponding to the term $4 \sin x$ on R.H.S. of (1), contribution to trial solution y^* is taken as $x\,(A_3 \cos x + A_4 \sin x)$. So, suppose that

$$y^* = A_0 x + A_1 x^2 + A_2 x^3 + A_3 x \cos x + A_4 x \sin x. \quad ... (3)$$

Since y* satisfies (1), $\quad D^3 y^* + D\,y^* = 2x^2 + 4 \sin x$. ... (4)

Differentiate (3) w.r.t. 'x', we have

$$D\,y^* = A_0 + 2A_1 x + 3A_2 x^2 + A_3 \cos x - A_3 x \sin x + A_4 \sin x + A_4 x \cos x,$$

$$D^2 y^* = 2A_1 + 6A_2 x - A_3 \sin x - A_3 (\sin x + x \cos x) + A_4 \cos x + A_4 (\cos x - x \sin x)$$

or $\quad D^2 y^* = 2A_1 + 6A_2 x - 2A_3 \sin x - A_3 x \cos x + 2A_4 \cos x - A_4 x \sin x$

and $\quad D^3 y^* = 6A_2 - 2A_3 \cos x - A_3 (\cos x - x \sin x) - 2A_4 \sin x - A_4 (\sin x + x \cos x)$

or $\quad D^3 y^* = 6A_2 - 3A_3 \cos x + A_3 x \sin x - 3A_4 \sin x - A_4 x \cos x$.

Putting the above values of $D\,y^*$ and $D^3 y^*$ in (4), we have

$$6A_2 - 3A_3 \cos x + A_3 x \sin x - 3A_4 \sin x - A_4 x \cos x + A_0 + 2A_1 x$$
$$+ 3A_2 x^2 + A_3 \cos x - A_3 x \sin x + A_4 \sin x + A_4 x \cos x = 2x^2 + 4 \sin x$$

or $\quad (6A_2 + A_0) + 2A_1 x + 3A_2 x^2 - 2A_3 \cos x - 2A_4 \sin x = 2x^2 + 4 \sin x$.

Equating the coefficients of like terms, we have

$6A_2 + A_0 = 0, \quad 2A_1 = 0, \quad 3A_2 = 2, \quad -2A_3 = 0 \quad$ and $\quad -2A_4 = 4$. These give $A_0 = -4, \quad A_1 = 0, \quad A_2 = 2/3, \quad A_3 = 0, \quad A_4 = -2$. So (3) $\Rightarrow y^* = -4x + (2/3)\,x^3 - 2x \sin x$.

$\therefore$ Required solution is $\quad y = C.F. + y^* = c_1 + c_2 \cos x + c_3 \sin x - 4x + (2/3)\,x^3 - 2x \sin x$.

Ex. 7(*e*). *Solve* $(D^3 + 2D^2 - D - 2)\,y = e^x + x^2$.

Sol. Given $\quad (D^3 + 2D^2 - D - 2)\,y = e^x + x^2$ ... (1)

The auxiliary equation is $\quad D^3 + 2D^2 - D - 2 = 0 \quad$ or $\quad (D^2 - 1)(D + 2) = 0 \Rightarrow D = 1, -1, -2$.

$\therefore$ C.F. $= c_1 e^x + c_2 e^{-x} + c_2 e^{-2x}$, c_1, c_2, c_3 being arbitrary constants. ... (2)

Corresponding to special form x^2 of R.H.S. of (1), we choose trail solution for P.I. as $A_0 + A_1x + A_2x^2$. Since e^x occurs in R.H.S. of (1) and it also occurs in the C.F. (2) corresponding to a root of multiplicity one, so we choose trial solution for P.I. as $A_3x\, e^x$ (note that term e^x is not included, since it already appears in C.F. (2) with arbitrary coefficient c_1). Combining the above two trial solutions, we attempt a trial solution for P.I. of the form

$$y^* = A_0 + A_1x + A_2x^2 + A_3x\, e^x. \qquad \text{... (3)}$$

Since y^* must satisfy (1), we have $\quad (D^3 + 2D^2 - D - 2)\, y^* = e^x + x^2$

or $$D^3 y^* + 2D^2 y^* - Dy^* - 2y^* = e^x + x^2. \qquad \text{... (4)}$$

Now, $$(3) \Rightarrow Dy^* = A_1 + 2A_2x + A_3e^x\, (x + 1). \qquad \text{... (5)}$$

$$(5) \Rightarrow D^2y^* = 2A_2 + A_3\, e^x\, (x + 2). \qquad \text{... (6)}$$

and $$(6) \Rightarrow D^3y^* = A_3\, e^x\, (x + 3). \qquad \text{... (7)}$$

Using (3), (5), (6) and (7), (4) reduces to

$$A_3\, e^x\, (x + 3) + 4A_2 + 2A_3e^x\, (x + 2) - A_1 - A_2x - A_3\, e^x\, (x + 1) - 2A_0 - 2A_1x - 2A_2\, x^2 - 2A_3x\, e^x = e^x + x^2$$

or $$-2A_2\, x^2 - 2(A_1 + A_2)\, x + 4A_2 - A_1 - 2A_0 + 6A_3\, e^x = e^x + x^2,$$

which is an identity and so equating coefficients of like terms,

$-2A_2 = 1,\quad -2\,(A_1 + A_2) = 0,\quad 4A_2 - A_1 - 2A_0 = 0,\quad 6A_3 = 1$; hence

$A_2 = -(1/2),\quad A_1 = 1/2,\quad A_0 = -(5/4),\quad A_3 = 1/6$. So from (3),

P.I. $= -(5/4) + (1/2)\, x - (1/2)\, x^2 + (1/6)\, x\, e^x$ and general solution is $y = C.F. + P.I.$, *i.e.*,

$$y = c_1\, e^x + c_2\, e^{-x} + c_3\, e^{-2x} - (5/4) + (1/2)\, x - (1/2)\, x^2 + (1/6)\, x\, e^x.$$

Ex. 8(*a*). *Solve* $(D^2 - 4D + 4)\, y = x^3\, e^{2x} + x\, e^{2x}$.

Sol. Given $$(D^2 - 4D + 4)\, y = x^3\, e^{2x} + x\, e^{2x} \qquad \text{... (1)}$$

The auxiliary equation is $\quad D^2 - 4D + 4 = 0 \quad$ so that $\quad D = 2, 2.$

C.F. $= (c_1 + c_2x)\, e^{2x}$, c_1 and c_3 being arbitrary constants. ... (2)

Here e^{2x} occurs in R.H.S. of (1) and it also occurs in the C.F. corresponding to a root of multiplicity two. So here, as a trial solution for P.I., we choose

$$y^* = A_1x^5\, e^{2x} + A_2\, x^4\, e^{2x} + A_3\, x^3\, e^{2x} + A_4\, x^2\, e^{2x}, \qquad \text{... (3)}$$

wherein the terms involving e^{2x} and $x\, e^{2x}$ are not included, since they already appear in the C.F. (2) with arbitrary constants c_1 and c_2 respectively.

Since y^* must satisfy (1), so $\quad (D^2 - 4D + 4)\, y^* = x^3\, e^{2x} + x\, e^{2x}.$

or $$D^2\, y^* - 4Dy^* + 4y^* = x^3\, e^{2x} = x\, e^{2x}. \qquad \text{... (4)}$$

$$(3) \Rightarrow \quad Dy^* = 2A_1x^5\, e^{2x} + (5A_1 + 2A_2)\, x^4\, e^{2x} + (4A_2 + 2A_3)\, x^3\, e^{2x} + (3A_3 + 2A_4)\, x^2\, e^{2x} + 2A_4\, x\, e^{2x} \qquad \text{... (5)}$$

$$(5) \Rightarrow \quad D^2y^* = 4A_1x^5\, e^{2x} + (20A_1 + 4A_2)\, x^4\, e^{2x} + (20A_1 + 16A_2 + 4A_3)\, x^3\, e^{2x} + (12A_2 + 12A_3 + 4A_4)\, x^2\, e^{2x} + (6A_3 + 8A_4)\, x\, e^{2x} + 2A_4\, e^{2x}. \qquad \text{... (6)}$$

Using (3), (5) and (6), (4) reduces, after simplification, to

$$20\, A_1x^3\, e^{2x} + 12A_2\, x^2\, e^{2x} + 6A_3x\, e^{2x} + 2A_4e^{2x} = x^3\, e^{2x} + xe^{2x},$$

which is an identity and so equating coefficients of like terms, we have

$20A_1 = 1,\quad 12A_2 = 0,\quad 6A_3 = 1,\quad 2A_4 = 0,\quad$ so that $\quad A_1 = 1/20,\quad A_2 = 0,\quad A_3 = 1/6,\quad A_4 = 0.$

$\therefore\quad (3) \Rightarrow y^* = (1/20)\, x^5\, e^{2x} + (1/6)\, x^3\, e^{2x}$ and general solution of (1) is

$$y = \text{C.F.} + y^* = (c_1 + c_2x)\, e^{2x} + (1/20)\, x^5\, e^{2x} + (1/6)\, x^3\, e^{2x}.$$

Ex. 8(*b*). *Solve* $(D^2 + 4)\, y = x^2 \sin 2x$.

Sol. Given $(D^2 + 4)\, y = x^2 \sin 2x.$ ... (1)

Auxiliary equation $D^2 + 4 = 0 \Rightarrow D = \pm\, 2i.$ Hence C.F. $= c_1 \cos 2x + c_2 \sin 2x$.... (2)

Since $x^2 \sin 2x$ occurs on R.H.S. of (1) and $\sin 2x$ is a part of the C.F. (2) corresponding to a root of multiplicity one, we choose trial solution y^* as

$y^* = A_1 x^3 \cos 2x + A_2 x^3 \sin 2x + A_3 x^2 \cos 2x + A_4 x^2 \sin 2x + A_5 x \cos 2x + A_6\, x \sin 2x,$ ... (3)

wherein $A_7 \cos x + A_6 \sin x$ is not included, since these terms are already occuring in the C.F. with arbitrary coefficients c_1 and c_2.

Since y^* must satisfy (1), so $D^2 y^* + 4y^* = x^2 \sin 2x.$ ... (4)

(3) $\Rightarrow Dy^* = 2A_2 x^3 \cos 2x - 2A_1\, x^3 \sin 2x + (3A_1 + 2A_4)\, x^2 \cos 2x + (2A_2 - 3A_3)\, x^2 \sin 2x$
$+ (2A_3 + 2A_6)\, x \cos 2x + (2A_4 - 2A_5)\, x \sin 2x + A_5 \cos 2x + A_6 \sin 2x,$... (5)

(5) $\Rightarrow D^2 y^* = -\, 4A_1\, x^3 \cos 2x - 4A_2\, x^3 \sin 2x + (12A_2 - 4A_3)\, x^2 \cos 2x$
$-\, (12A_1 + 4A_4)\, x^2 \sin 2x + (6A_1 + 8A_4 - 4A_5)\, x \cos 2x + (6A_2 - 8A_3 - 4A_6)\, x \sin 2x$
$+ (2A_3 + 4A_6) \cos 2x + (2A_4 - 4A_5) \sin 2x$ (on simplification) ... (6)

Using (3) and (6), (4) reduces to

$12A_2 x^2 \cos 2x - 12A_1 x^2 \sin 2x + (6A_1 + 8A_4)\, x \cos 2x + (6A_2 - 8A_3)\, x \sin 2x$
$+ (2A_3 + 4A_6) \cos 2x + (2A_4 - 4A_5) \sin 2x = x^2 \sin 2x,$

which is an identity and so equating coefficients of like terms, we have

$-12A_1 = 1, \quad 12A_2 = 0, \quad 6A_1 + 8A_4 = 0, \quad 6A_2 - 8A_3 = 0, \quad 2A_3 + 4A_6 = 0, \quad 2A_4 - 4A_5 = 0;$

hence, $A_1 = -\, 1/12, \quad A_2 = 0, \quad A_3 = 0, \quad A_4 = 1/16, \quad A_5 = 1/32, \quad A_6 = 0$ and so from (3),

P.I. $= y^* = -\, (1/12)\, x^3 \cos 2x + (1/16)\, x^2 \sin 2x + (1/32)\, x \cos 2x.$

Solution is $y =$ C.F. $+ y^*$, *i.e.*, $y = c_1 \cos 2x + c_2 \sin 2x - (1/12) \times x^3 \cos 2x$
$+ (1/16) \times x^2 \sin 2x + (1/32) \times x \cos 2x$, where c_1 and c_2 are arbitrary constants.

Ex. 9. *Using the method of undetermined coefficients, solve*

(a) $(d^2y/dx^2) - 6\, (dy/dx) + 9y = x^2 e^{3x}.$

(b) $(D^2 - 2D + 1)\, y = x^2 e^x$ **[Delhi Maths (Hons.) 2000, 06]**

(c) $(d^3y/dx^3) - 3\, (d^2y/dx^2) + 2\, (dy/dx) = x^2 e^x.$

Sol. (*a*) Given $(D^2 - 6D + 9)\, y = x^2 e^{3x}$, where, $D \equiv d/dx.$ ... (1)

Its auxiliary equation is $D^2 - 6D + 9 = 0$ so that $D = 3, 3$

$\therefore$ C.F. $= (c_1 + c_2 x)\, e^{3x}$, c_1, c_2 being arbitrary constants ... (2)

Here $x^2 e^{3x}$ appears on the R.H.S. of (1) and e^{3x} is a term of C.F. (2) corresponding to the root 3 occuring twice. So the trial solution y^* will contain $x^{2+2} e^{3x}$ *i.e.* $x^4 e^{3x}$ plus terms arising from it by differentiation. Therefore, suppose that

$$y^* = Ax^4 e^{3x} + B\, x^3 e^{3x} + C\, x^2 e^{3x}, \qquad \text{... (3)}$$

in which the terms involving $Dx\, e^{3x}$ and $E\, e^{3x}$ are not included, since they already appear in the C.F. (2) with arbitrary constants c_1 and c_2.

Since y^* must satisfy (1), $D^2 y^* - 6D^2 y^* + 9y^* = x^2 e^{3x}.$ ... (4)

(3) $\Rightarrow Dy^* = 4Ax^3 e^{3x} + 3Ax^4 e^{3x} + 3Bx^2 e^{3x} + 3Bx^3 e^{3x} + 2Cx\, e^{3x} + 3C\, x^2 e^{3x}$
$= 3Ax^4 e^{3x} + (4A + 3B)\, x^3 e^{3x} + (3B + 3C)\, x^2 e^{3x} + 2Cx\, e^{3x}$ ... (5)

and $D^2 y^* = 12Ax^3 e^{3x} + 9A\, x^4 e^{3x} + 3(4A + 3B)\, x^2 e^{3x} + 3\, (4A + 3B)\, x^3 e^{3x}$
$+ 2(3B + 3C)\, x\, e^{3x} + 3(3B + 3C)\, x^2 e^{3x} + 2C\, e^{3x} + 6Cx\, e^{3x}$
$= 9A\, x^4 e^{3x} + (24A + 9B)\, x^3 e^{3x} + (12A + 18B + 9C)\, x^2 e^{3x} + (6B + 12C)\, x\, e^{3x} + 2C\, e^{3x}.$ ... (6)

Using (3), (5) and (6), (4) reduces to

$9Ax^4 e^{3x} + (24A + 9B) x^3 e^{3x} + (12A + 18B + 9C) x^2 e^{3x} + (6B + 12C) x e^{3x} + 2C e^{3x} - 6 [3A x^4 e^{3x} + (4A + 3B) x^3 e^{3x} + (3B + 3C) x^2 e^{3x} + 2C x e^{3x}] + 9 (Ax^4 e^{3x} + Bx^3 e^{3x} + Cx^2 e^{3x}) = x^2 e^{3x}.$

or $$12Ax^2 e^{3x} + 6Bx e^{3x} + 2C e^{3x} = x^2 e^{3x}. \qquad ... (7)$$

Equating the coefficients of like powers on both sides of (7), we get

$12A = 1,\quad B = 0,\quad C = 0$ so that $A = 1/12,\quad B = 0,\quad C = 0$. So $y^* = (x^4/12) \times e^{3x}$

$\therefore$ required solution is $\quad y = \text{C.F.} + y^* = (c_1 + c_2x) e^{3x} + (x^4/12) \times e^{3x}$

(*b*) Proceed by taking $y^* = Ax^4 e^x + Bx^3 e^x + Cx^2 e^x$. **Ans.** $y = (c_1 + c_2x) e^x + (x^4/12) \times e^x$

(*c*) Proceed as is part (*a*) by taking $y^* = Ax^3 e^x + Bx^2 e^x + C x e^x$.

Ans. $y = c_1 + c_2 e^x + c_3 e^{2x} - 2x e^x - (x^3/3) \times e^x$

Ex. 10. *Using the method of undetermined coefficients, solve* $(d^3y/dx^3) - 3 (d^2y/dx^2) + 2(dy/dx) = 3x e^{2x} + 5x^2$.

Sol. Given $\quad (D^3 - 3D^2 + 2D) y = 3x e^{2x} + 5x^2,\quad$ where $\quad D \equiv d/dx. \qquad ... (1)$

Its auxiliary equation is $\quad D^3 - 3D^2 + 2D = 0 \quad$ so that $\quad D = 0, 1, 2.$

$\therefore$ C.F. $= c_1 + c_2 e^x + c_3 e^{2x}$, c_1, c_2, c_3 being arbitrary constants. ... (2)

Here $x e^{2x}$ appears on the R.H.S. of (1), where e^{2x} is a term of the C.F. (2) corresponding to the root 2 occuring once. So the trial solution y^* will contain the term $x^{1+1} e^{2x}$ *i.e.* $x^2 e^{2x}$ plus terms arising from it by differentiation. Thus, contribution of $3x e^{2x}$ to y* will be $Dx^2 e^{2x} + Ex e^{2x}$, which does not involve the term $F e^{2x}$ because it already appears in C.F. (2) with arbitrary constants c_2. Since constant c_1 appears in C.F. (2), the contribution of $5x^2$ on R.H.S. of (1) to trial solution y^* is taken as $x (Ax^2 + Bx + C)$. Hence the total trial solution y^* is taken as

$$y^* = Ax^3 + Bx^2 + Cx + Dx^2 e^{2x} + Ex e^{2x}. \qquad ... (3)$$

Since y^* must satisfy (1), $\quad D^3y^* - 3D^2y^* + 2Dy^* = 3x e^{2x} + 5x^2. \qquad ... (4)$

$$(3) \Rightarrow Dy^* = 3Ax^2 + 2Bx + C + 2Dx^2 e^{2x} + (2D + 2E) x e^{2x} + E e^{2x} \qquad ... (5)$$

$$(5) \Rightarrow D^2y^* = 6Ax + 2B + 4Dx^2 e^{2x} + (8D + 4E) x e^{2x} + (2D + 4E) e^{2x} \qquad ... (6)$$

$$(6) \Rightarrow D^3y^* = 6A + 8Dx^2 e^{2x} + (24D + 8E) x e^{2x} + (12D + 12E) e^{2x}. \qquad ... (7)$$

Using (3), (5), (6) and (7), (4) reduces to

$$4Dx e^{2x} + (6D + 2E) e^{2x} + 6Ax^2 + (4B - 18A) x + (6A - 6B + 2C) = 3x e^{2x} + 5x^2.$$

Equating the coefficients of like terms in above identity, we have

$4D = 3,\quad 6D + 2E = 0,\quad 6A = 5,\quad 4B - 18A = 0\quad$ and $\quad 6A - 6B + 2C = 0.$

Solving these, $\quad A = 5/6,\quad B = 15/4,\quad C = 35/4,\quad D = 3/4,\quad E = -9/4.$

$\therefore$ from (3), $\quad y^* = (5/6) x^3 + (15/4) x^2 + (35/4) x + (3/4) e^{2x} (x^2 - 3x).$

Hence the required general solution is $\quad y = \text{C.F.} + y^*,\quad$ *i.e.*

$$y = c_1 + c_2 e^x + c_3 e^{2x} + (5/6) x^3 + (15/4) x^2 + (35/4) x + (3/4) e^{2x} (x^2 - 3x).$$

Ex. 11. *Find the complementary solution of the differential equation* $y^{(4)} + 9y'' = (x^2 + 1) \sin 3x$. Set up the appropriate form of a particular solution, but do not determine the values of the coefficients. **[Delhi B.Sc. (Hons) II 2011]**

Sol. The Given equation may be written

$$(D^4 - 9D^2) y = (x^2 + 1) \sin 3x,\quad \text{where}\quad D \equiv d/dx. \qquad ... (1)$$

Auxiliary equation of (1) is $\quad D^4 + 9D^2 = 0$ giving D $= 0, 0, \pm 3i$.

Hence the complementary solution (i.e., complementary function $y_c (x)$ is given by

$y_c (x) = c_1 + c_2x + c_3 \cos 3x + c_4\, sin\, 3x$, c_1, c_2, c_3 and c_4 being arbitrary constants. ... (2)

As a first step toward the form of a particular solution, let us examine the following sum

$$(A + Bx + Cx^2)\cos 3x + (D + Ex + Fx^2)\sin 3x$$

In order to eliminate duplication of terms of $y_c(x)$, both parts given in (3) must be multiplied by x. Hence, the desired appropriate particular solution $y^*(x)$ is given by

$$y^*(x) = x(A + Bx + Cx^2)\cos 3x + x(D + Ex + Fx^2)\sin 3x,$$

where A, B, C, D, E and F are six unknown coefficients.

Ex. 12. (*a*) *The method of undetermined coefficients to solve* $x^2 y'' - xy' - 3y = x^2 \log x$ **[Mumbai 2010]**

Sol. Re-writing, the given equation can be written as

$$(x^2 D^2 - xD - 3)\, y = x^2 \log x, \quad \text{where} \quad D \equiv d/dx. \qquad \dots (1)$$

Let $x = e^z$, $\log x = z$ and $D_1 \equiv d/dz$ then, $xD = D_1$, $x^2 D^2 = D_1(D_1 - 1)$ and so (1) yields

$$\{D_1(D_1 - 1) - D_1 - 3\}\, y = ze^{2z} \quad \text{or} \quad (D_1^2 - 2D_1 - 3)\, y = z\, e^{2z} \qquad \dots (2)$$

which is linear differential equation with consten coefficients. We shall now apply the usual method of undetermined coefficients in order to solve (2). In what follows note carefully that we have new independent variable z in place of old variable x. Also, note that we have operator D_1 in place of D.

Auxiliary equation of (2) is $\quad D_1^2 - 2D_1 - 3 = 0$ giving $D_1 = 3, -1$.

Hence, C.F. of (2) is $\quad c_1 e^{3z} + c_2 e^{-z}$, c_1 and c_2 being arbitrary constants

Let the trial solution y^* be given by (using result 3 of the table on page 5.52)

$$y^* = (A_0 + A_1 z)\, e^{2z}, \text{ where } A_0 \text{ and } A_1 \text{ are unknown coefficients} \qquad \dots (3)$$

Since y^* must satisfy (2), we have $\quad (D_1^2 - 2D_1 - 3)\, y^* = z\, e^{2z} \qquad \dots (4)$

or

$$D_1^2 y^* - 2D_1 y^* - 3y^* = z\, e^{2z}$$

Now, from (3), $\quad D_1 y^* = A_1 e^{2z} + (A_0 + A_1 z) \times (2e^{2z}) = (2A_0 + A_1 + 2A_1 z)\, e^{2z} \qquad \dots (5)$

Then, from (5), $D_1^2 y^* = 2A_1 e^{2z} + (2A_0 + A_1 + 2A_1 z) \times (2e^{2z}) = (4A_0 + 4A_1 + 4A_1 z)\, e^{2z} \qquad \dots (6)$

Using (3), (5) and (6), (4) reducer to

$$(4A_0 + 4A_1 + 4A_1 z)\, e^{2z} - (4A_0 + 2A_1 + 4A_1 z)\, e^{2z} - 3(A_0 + A_1 z)\, e^{2z} = ze^{2z} \qquad \dots (7)$$

Equating the coefficients of e^{2z} and ze^{2z} in the above identity, we get

$$-3A_0 + 2A_1 = 0 \quad \text{and} \quad -3A_0 = 1 \text{ so that } A_1 = -(1/3) \text{ and } A_0 = -(2/9)$$

Hence, (3) yields $\quad y^* = -(2/9 + z/3)\, e^{2z} = -(2 + 3z) \times (e^{2z}/9)$

Therefore, the general solution of (2) is $y = \text{C.F.} + y^*$, i.e.,

$$y = c_1 e^{2z} + c_2 e^{-z} - (2 + 3z) \times (e^{2z}/9) \qquad \dots (8)$$

Since $e^z = x$ and $\log x = z$, from (8) the required solution is given by

$$y = c_1 (e^z)^3 + c_2 (e^z)^{-1} - (2 + 3z) \times \{(e^z)^2/9\} = c_1 x^3 + c_2 x^{-1} - (x^2/9) \times (2 + 3\log x)$$

Ex. 12(b). *Use the method of undetermined coefficients to solve* $x^2 y'' + xy' + 4y = 2x \log x$ **[Mumbai 2010]**

Hint. Try like Ex. 12(*a*). **Ans.** $y = c_1 x^2 + c_2 x^{-2} - (2x/9) \times (2 + 3 \log x)$

Exercise 5(J)

Solve, using the method of undetermined coefficients :

1. $y'' + y = \sin x$ **[Delhi B.Sc. (Hons) II 2011]** **Ans.** $y = c_1 \cos x + c_2 \sin x - (x/2)\cos x$

1. $y''' - 7y' - 6y = x^{-2x}$ **[Mumbai 2010]** **Ans.** $y = c_1 e^{-x} + c_2 e^{3x} + c_3 e^{-2x} + (x/5) \times e^{-2x}$

3. $(D^2 - D - 2)\, y = 4x^2$ **[Nagpur 2002]** **Ans.** $y = c_1 e^{2x} + c_2 e^{-x} - 3 + 2x - 2x^2$

4. $(D^2 - 1)\, y = e^x \sin 2x$. **Ans.** $y = c_1 e^x + c_2 e^{-x} - e^x (\sin 2x + \cos 2x)/8$

5. $(D^2 - 4D + 4)\,y = e^x \sin x$ **[Delhi Maths (Hons.) 2004]**

Ans. $y = e^x \{c_1 \cos (x\sqrt{2}) + c_2 \sin (x\sqrt{2})\} + x/3 + 2/9 + (\sin x + \cos x)/4$

6. $(D^2 - 3D + 2)\,y = e^{2x} \sin x$ **[Delhi Mahts (Hons.) 1996]**

Ans. $y = c_1\, e^{-x} + c_2\, e^{-2x} - (1/170)\, e^{2x} (11 \sin x - 7 \cos x)$

7. $(D^2 + 2D + 1)\,y = x^2 - \cos x$ **[Delhi 1998]** **Ans.** $y = (c_1 + c_2 x)\, e^{-x} + x^2 - 4x + 6 - (1/6) \sin x$

8. $(D^2 + 2D + 2)\,y = \sin x + x^2$ **[Delhi Maths (G) 1998]**

Ans. $y = e^{-x} (c_1 \cos x + c_2 \sin x) + (1/2)\, x^2 - x + (1/2) + (\sin x - 2 \cos x)/5$

9. $(D^2 - 2D + 3)\,y = x + \sin x$ **[Delhi Maths (G) 2004, 06]**

Ans. $y = c_1 \cos (x\sqrt{2}) + c_2 \sin (x\sqrt{2}) + (3x + 2)/9 + (\sin x + \cos x)/4.$

10. $y'' + y' - 2y = -2\, e^{-x} - 5 \cos x$ **[Delhi Maths (H) 2009; Delhi Maths (G) 1995, 2005]**

Ans. $y = c_1\, e^x + c_2\, e^{-2x} + e^{-x} + (3 \cos x + \sin x)/2$

11. $y'' - 2y' + 3y = x^2 + \cos x$ **[Delhi Maths (Hons.) 2005]**

Ans. $y = e^x (c_1 \cos x\sqrt{2} + c_2 \sin x\sqrt{2}) + (2 + 12x + 9x^2)/27 + (\cos x - \sin x)/4$

12. $(4D^2 + 4D + 1)\,y = e^x + 2 \cos 2x$ **[Delhi Maths (G) 1999)**

Ans. $y = (c_1 + c_2 x)\, e^{-x/2} + (1/9)\, e^x + (16 \sin 2x - 30 \cos 2x)/289$

13. $y_3 - 3y_2 + 2y_1 = 3x\, e^{2x} + 5x^3$ **[Delhi Maths (Hons) 1999]**

Ans. $y = c_1 + c_2 e^x + c_3 e^{2x} - (9/4)\, x\, e^{2x} + (5x^4 + 30\, x^3 + 105\, x^2 + 225\, x)/8$

14. $(D^2 + 1)\,y = 4x \cos x - 2 \sin x.$ **Ans.** $y = c_1 \cos x + c_2 \sin x + 2x \cos x + x^2 \sin x$

15. $(D^2 - 3D)\,y = 8\, e^{3x} + 4 \sin x$ **Ans.** $y = c_1 + c_2\, e^{3x} + (8/3)\, x\, e^{3x} + (6 \cos x - 2 \sin x)/5$

16. $(D^2 + 4)\,y = x^2 \sin 2x$ **Ans.** $y = c_1 \cos 2x + c_2 \sin 2x - (1/12)\, x^3 \cos 2x + (1/16)\, x^2 \sin 2x + (1/32)\, x \cos 2x$

17. $(D^3 - D^2 - 4D + 4)\,y = 2x^2 - 4x - 1 + 2x^2\, e^{2x} + 5x\, e^{2x} + e^{2x}$

Ans. $y = c_1\, e^x + c_2\, e^{2x} + c_3\, e^{-2x} + (1/2)\, x^2 + (1/6)\, x^3\, e^{2x}.$

18. $(D^2 + 3D + 2)\,y = x + \cos x$ **[Delhi BA / B.Sc. Maths (Prog) 2007]**

Ans. $y = c_1\, e^{-x} + c_2 e^{-2x} + (x/2) - (3/4) + (1/10) \times (3 \sin x + \cos x)$

19. $(D^2 + 4)y = \sin 3x + e^x + x^2$ **[Delhi B.A. (Prog) II 2010]**

Ans. $y = c_1 \cos 2x + c_2 \sin 2x - (1/5) \times \sin 3x + (1/5) \times e^x + (2x^2 - 1)/8$

Obejective Problems on Chapter 5

Ex. 1. *The solution of $(D^2 + 1)^2 y = 0$, is : (a) $A \cos x + B \sin x$ (b) $e^x (A \cos x + B \sin x)$ (c) $(A_1 + A_2) \cos x + (A_3 + A_4) \sin x$ (d) $(A_1 + A_2 x) \cos x + (A_3 + A_4 x) \sin x$*

Sol. Ans. (*d*). See Ex. 8(*a*), Art. 5.5 **[I.A.S. Prel. 1993]**

Ex. 2. *A particular integral of $(d^2y/dx^2) - (dy/dx) - 2y = \cos x + 3 \sin x$ is (a) $\sin x$ (b) $\cos x$ (c) $-\sin x$ (d) $-\cos x$* **[I.A.S. Prel. 1995]**

Sol. Ans. (*c*). Given equation is $(D^2 - D - 2)\, y = \cos x + 3 \sin x$

$$\text{P.I.} = \frac{1}{D^2 - D - 2}\cos x + 3\frac{1}{D^2 - D - 2}\sin x = \frac{1}{-1^2 - D - 2}\cos x + 3\frac{1}{-1^2 - D - 2}\sin x$$

$$= -\frac{1}{D+3}\cos x - 3\frac{1}{D+3}\sin x = -\frac{D-3}{D^2 - 9}\cos x - 3\frac{D-3}{D^2 - 9}\sin x = -\frac{D-3}{(-10)}\cos x - 3\frac{D-3}{(-10)}\sin x$$

$= (1/10)\, [(D - 3) \cos x + 3\, (D - 3) \sin x] = (1/10)\, [-\sin x - 3 \cos x + 3 \cos x - 9 \sin x] = -\sin x.$

Ex. 3. *A particular integral of $d^2y/dx^2 - (a + b)\, (dy/dx) + aby = Q(x)$ is*

(a) $e^{ax}\int\{e^{(a-b)x}(Qe^{bx}dx)\}dx$ (b) $e^{ax}\int\{e^{(b-a)x}(Qe^{-bx}dx)\}dx$

(c) $e^{-ax}\int\{e^{(b-a)x}(Qe^{bx}dx)\}dx$ (d) $e^{-ax}\int\{e^{(b-a)x}(Qe^{-bx}dx)\}dx$ **[I.A.S. Prel. 1999]**

Sol. Ans. (*b*). Given $\{D^2-(a+b)D+ab\}\,y=Q(x)$ or $(D-a)(D-b)\,y=Q(x)$, $D\equiv d/dx$

$$\text{P.I.} = \frac{1}{(D-a)(D-b)}Q(x) = \frac{1}{D-a}e^{bx}\int Qe^{-bx}dx, \text{ using result of Art. 5.8}$$

$$= e^{ax}\int\{e^{-ax}(e^{bx}\int Qe^{-bx}dx)\}dx = e^{ax}\int\{e^{(b-a)x}(Qe^{-bx}dx)\}dx.$$

Ex. 4. *The solution of $d^2s/dt^2 = g$. (g is a constant, s = 0 and ds/dt = u when t = 0), is*

(a) $s = gt$ (b) $s = ut + (1/2)\times gt^2$ (c) $s = (1/2)\times gt^2$ (d) none of these **[I.A.S. Prel. 2000]**

Sol. Ans. (*b*). Integrating $d^2s/dt^2 = g$, we get $ds/dt = gt + c$. ... (1)

Given that $ds/dt = u$ when $t = 0$. So (1) gives $c = u$.

Then (1) becomes $ds/dt = gt + u$ or $ds = (gt + u)\,dt$

Integrating it, $s = (1/2)\times gt^2 + ut + c'$, c' is another arbitrary constant ... (2)

Given that $s = 0$ when $t = 0$. So (2) gives $c' = 0$. So (2) $\Rightarrow$ $s = (1/2)\times gt^2 + ut$.

Ex. 5. *The solution of $(d^2y/dx^2) - y = k$, (k = a non-zero constant) which vanishes when x = 0 and which tends to finite limit as x tends to infinity is*

(a) $y = k\,(1 + e^{-x})$ *(b) $y = k\,(e^{-x} - 1)$*

(c) $y = k\,(e^{x} + e^{-x} - 2)$ *(d) $y = k\,(e^{x} - 1)$* **[I.A.S. Prel. 2001]**

Sol. Ans. (*b*). Re-writing given equation, $(D^2 - 1)\,y = k$ ... (1)

Its auxiliary equation is $D^2 - 1 = 0$ so that $D = 1, -1$.

Hence its C.F. $= C_1 e^x + C_2 e^{-x}$, c_1, c_2 being arbitrary constants

and its $$\text{P.I.} = \frac{1}{D^2-1}k = k\frac{1}{D^2-1}e^{0.x} = k\frac{1}{0^2-1}e^{0.x} = -k.$$

So solution of (1) is $y = C_1 e^x + C_2 e^{-x} - k$ ... (2)

Given that $y = 0$ when $x = 0$. Hence (2) gives $0 = C_1 + C_2 - k$ or $C_1 + C_2 = k$. ... (3)

Multiplying both sides of (2), by e^{-x}, we get $y\,e^{-x} = C_1 + C_2\,(e^{-x})^2 - k\,e^{-x}$ ... (4)

Given that $y \to m$ when $x \to \infty$, m being a finite quantity

So (4) $\Rightarrow$ $m \times 0 = C_1 + (C_2 \times 0) - (k \times 0)$ or $C_1 = 0$ ... (5)

Solving (4) and (5), $C_1 = 0$ and $C_2 = k$ So from (2), the required solution is $y = k\,(e^{-x} - 1)$.

Ex. 6. *If $\phi_1(x)$ is a particular integral of $Ly = d^2y/dx^2 - a\,(dy/dx) + by = e^{ax} + f(x)$ and $\phi_2(x)$ is a particular integral of $Ly = e^{ax} - f(x)$, a, b being constants, then the particular integral of $Ly = 2b\,e^{ax}$ is* *(a) $b\,\phi_1(x) + \phi_2(x)$* *(b) $\phi_1(x) - b\phi_2(x)$*

(c) $a\,\phi_1(x) + b\,\phi_2(x)$ *(d) $b\,\{\phi_1(x) + \phi_2(x)\}$* **[I.A.S. Prel. 2002]**

Sol. Ans. (*d*). Note that a particular solution of a given differential equation will satisfy the entire given equation. Therefore,

$\phi_1(x)$ satisfies $d^2y/dx^2 - a\,(dy/dx) + by = e^{ax} + f(x)$

$$\Rightarrow d^2\phi_1/dx^2 - a\,(d\phi_1/dx) + b\,\phi_1 = e^{ax} + f(x) \quad \text{... (1)}$$

$\phi_2(x)$ satisfies $d^2y/dx^2 - a\,(dy/dx) + by = e^{ax} - f(x)$

$$\Rightarrow d^2\phi_2/dx^2 - a\,(d\phi_2/dx) + b\phi_2 = e^{ax} - f(x) \quad \text{... (2)}$$

Adding (1) and (2), $$\frac{d^2(\phi_1+\phi_2)}{dx^2} - a\,\frac{d(\phi_1+\phi_2)}{dx} + b\,(\phi_1+\phi_2) = 2e^{ax}.$$

Multiplying both sides by b and rewriting, we get

$$\frac{d^2\{b(\phi_1+\phi_2)\}}{dx^2} - a\,\frac{d\{b(\phi_1+\phi_2)\}}{dx} + b\{b(\phi_1+\phi_2)\} = 2be^{ax},$$

showing that $b\,(\phi_1+\phi_2)$ is a particular integral of $d^2y/dx^2 - a\,(dy/dx) - by = 2b\,e^{ax}$ i.e. $Ly = 2b\,e^{ax}$

Ex. 7. *If $e^{ax}\,u(x)$ is particular integral of $d^2y/dx^2 - 2a(dy/dx) + a^2y = f(x)$, where a is a constant, then d^2u/dx^2 is equal to*

(a) $f(x)$ (b) $f(x)\,e^{ax}$ (c) $f(x)\,e^{-ax}$ (d) $f(x)\,(e^{ax}+e^{-ax})$ **[I.A.S. Prel. 2002]**

Sol. Ans. (*c*). Recall the following formula for finding a particular integral (Refer Art. 5.8)

$$\frac{1}{D-\alpha}X = e^{\alpha x}\int e^{-\alpha x}X\,dx \quad \text{where } D \equiv d/dx \text{ and } X \text{ is a function of } x. \qquad \dots (1)$$

Given equation is $(D^2 - 2aD + a^2)\,y = f(x)$ or $(D-a)^2\,y = f(x)$

$$\therefore \quad \text{P.I.} = \frac{1}{(D-a)^2}f(x) \quad \text{or} \quad e^{ax}\,u(x) = \frac{1}{D-a}\frac{1}{D-a}f(x) = \frac{1}{D-a}e^{ax}\int e^{-ax}f(x)\,dx$$

[Using (1) and the given value of P.I. namely, $e^{ax}\,u\,(x)$]

or $$e^{ax}\,u\,(x) = e^{ax}\int e^{-ax}\{e^{ax}\int e^{-ax}f(x)\,dx\}\,dx, \text{ using result (1) again}$$

$$\Rightarrow \quad u(x) = \int\!\!\int e^{-ax}f(x)\,dx \quad \Rightarrow \quad \frac{du}{dx} = \int e^{-ax}f(x)\,dx \quad \Rightarrow \quad d^2u/dx^2 = e^{-ax}f(x).$$

Ex. 8. *If $y = \phi(x)$ is a particular solution of $y'' + (\sin x)\,y' + 2y = e^x$ and $y = \psi(x)$ is a particular solution of $y'' + (\sin x)\,y' + 2y = \cos 2x$, then a particular solution of $y'' + (\sin x)\,y' + 2y = e^x + 2\sin^2 x$, is given by*

(a) $\phi(x) - \psi(x) + 1/2$ *(b) $\psi(x) - \phi(x) + 1/2$*

(c) $\phi(x) - \psi(x) + 1$ *(d) $\psi(x) - \phi(x) + 1$* **[GATE 2004]**

Sol. Ans. (*a*). By definition of particular solution (*i.e.* particular integral), taking $D \equiv d/dx$

$$\phi(x) = \frac{1}{D^2+(\sin x)D+2}e^x \quad \text{and} \quad \psi(x) = \frac{1}{D^2+(\sin x)D+2}\cos 2x \qquad \dots (1)$$

$\therefore$ Particular solution (P.S.) of $[D^2 + (\sin x)\,D + 2]\,y = e^x + 2\sin^2 x$ is

$$= \frac{1}{D^2+(\sin x)D+2}e^x + \frac{1}{D^2+(\sin x)D+2}(2\sin^2 x) = \phi(x) + \frac{1}{D^2+(\sin x)D+2}(1-\cos 2x),$$

$$= \phi(x) + \frac{1}{D^2+(\sin x)D+2}e^{0.x} - \frac{1}{D^2+(\sin x)D+2}(\cos 2x) = \phi(x) + 1/2 - \psi(x), \text{ using (1).}$$

Ex. 9. *The set of linearly independent solutions of the differential equation $(dy^4/dx^4) - (dy^2/dx^2) = 0$ is*

(a) $\{1, x, e^x, e^{-x}\}$ (b) $\{1, x, e^{-x}, x\,e^{-x}\}$ (b) $\{1, x, e^x, x\,e^x\}$ (d) $\{1, x, e^x, x\,e^{-x}\}$ **[GATE 2005]**

Sol. Ans. (*a*). Here auxiliary equation is $D^4 - D^2 = 0$, *i.e.* $D^2\,(D^2 - 1) = 0$ so that $D = 0, 0, 1, -1$. Hence C.F. $= C_1 + C_2x + C_3e^x + C_4e^{-x}$ and so $\{1, x, e^x, e^{-x}\}$ forms a set of linearly independent solutions.

Ex. 10. *All real solutions of the differential equation $y'' + 2ay' + by = \cos x$ (where a and b are real constants) are periodic if (a) $a = 1$ and $b = 0$ (b) $a = 0$ and $b = 1$*
(c) $a = 1$ and $b \neq 0$ (d) $a = 0$ and $b \neq 1$ **[GATE 2003]**

Ex. 11. Let $y\ (x)$ be the solution of the initial value problem $y''' - y'' + 4y' - 4y = 0,$ $y(0) = y'(0) = 2$ and $y''(0) = 0.$ Then, the value of $y(\pi/2)$ is

(*a*) $(4e^{\pi/2} - 6)/5$ (*b*) $(6e^{\pi/2} - 4)/5$ (*c*) $(8e^{\pi/2} - 2)/5$ (*d*) $(8e^{\pi/2} + 2)/5$ **[GATE 2010]**

Sol. Ans. (*c*). Re-writing the given equation, $(D^3 - D^2 + 4D - 4)y = 0$, where $D = d/dx$. Its auxiliary equation is $D^3 - D^2 + 4D - 4 = 0$ or $(D-1)(D^2+4) = 0$ giving $D = 1,\ \pm 2i$. Hence, the general solution of the given differential equation is given by

$y(x) = c_1 e^x + c_2 \sin 2x + c_3 \cos 2x, c_1, c_2$ and c_3 bening arbitrary constants ... (1)

From (1), $y'(x) = c_1 e^x + 2c_2 \cos 2x - 2c_3 \sin 2x$ ... (2)

From (2), $y''(x) = c_1 e^x - 4c_2 \sin 2x - 4c_3 \cos 2x$ ... (3)

Putting $x = 0$ in (1), (2), (3) and using the given initial values $y(0) = 2$, $y'(0) = 2$ and $y''(0) = 0$, we obtain

$c_1 + c_3 = 2, \quad c_1 + 2c_2 = 2, \ c_1 - 4c_3 = 0$ giving $c_1 = 8/5,\ c_2 = -(4/5), \quad c_3 = 2/5$

Hence, from (1), $y(x) = (8/5) \times e^x - (4/5) \times \sin 2x + (2/5) \times \cos 2x$ ... (4)

Now from (4), $y(\pi/2) = (8/5) \times e^{\pi/2} - (2/5) = (8e^{\pi/2} - 2)/5$

MISCELLANEOUS PROBLEMS ON CHAPTER 5

Ex. 1. Describe the general solution of the differential equation $d^n y/dx^n + a_{n-1}\ (d^{n-1}y/dx^{n-1}) + \ldots + a_1\ (dy/dx) + a_0\ y = 0$ where $a_0, a_1, a_2, \ldots, a_{n-11}$ are constants depending on the roots of multiplicity of auxiliary equation. **[Mumbai 2010]**

Hints. Refer Art. 5.3, page 5.2

Ex.2 A particular solution of $(D^2 + 4)y = x,\ D \equiv d/dx$ is

(*a*) $x\ e^{2x}$ (*b*) $x \cos 2x$ (*c*) $x \sin 2x$ (*d*) $d/4$

Sol. Ans. (*d*) As usual particular integral

$$= \frac{1}{D^2+4} x = \frac{1}{4(1 + D^2/4)} x = \frac{1}{4}\left(1 + \frac{D^2}{4}\right)^{-1} x$$

$$= \frac{1}{4}\left(1 - \frac{D^2}{4}\right) x = \frac{x}{4}.$$

Ex. 3. The number of lineraly independent solutions of

$$d^4 y/dx^4 - (d^3 y/dx^3) - 3\ (dy/dx)\ + 5(dy/dx) - 2y = 0$$

of the form e^{ax} (a being a real number) is

(a) 1 (b) 2 (c) 3 (d) 4

Sol. Ans. (b). Re-writing, the given equation

$$(D^4 - D^3 - 3D^2 + 5D - 2)y = 0,\ D \equiv d/x$$

Itsw auxiliary equation is

$$D^4 - D^3 - 3D^2 + 5D - 2 = 0$$

or $(D-1)^3(D+2) = 0$ so that D = 1, 1, 1, –2

Hence, the general solution of the given equation is given by

$$y = (c_1 + c_2 x + c_3 x^2)\ e^x + c_4\ e^{-2x},$$

Which contain two solutions e^x and e^{-2x} of the form e^{ax}. Now, we here

$$W(e^x, e^{-2x})$$

$$= \text{Wronskian of } e^x \text{ and } e^{-2x}$$

$$= \begin{vmatrix} e^x & e^{-2x} \\ e^x & -2e^{-2x} \end{vmatrix}$$

$$= -2e^{-x} = e^{-x} = -3e^{-x} \neq 0,$$

Showing that two linearly independent solutions of the given equation are of the form e^{ax}.

6

Homogeneous Linear Equations Or Cauchy-Euler Equations

6.1 Homogeneous linear equations (or Cauchy-Euler-Equations)

A linear differential equation of the form

$$a_0 x^n (d^n y/dx^n) + a_1 x^{n-1} (d^{n-1} y/dx^{n-1}) + ... + a_{n-1} x (dy/dx) + a_n \, y = X \qquad ... (1)$$

i.e., $$(a_0 x^n D^n + a_1 x^{n-1} D^{n-1} + ... + a_{n-1} x D + a_n) y = X, \quad D \equiv d/dx \qquad ... (2)$$

where $a_0, a_1, a_2, ... a_n$ are constants, and X is either a constant or a function of x only is called a *homogeneous linear differential equation.* Note that the index of x and the order of derivative is same in each term of such equations. These are also known as *Cauchy-Euler equations.*

6.2 Method of solution of homogeneous linear differential equation [Mumbai 2010]

$$(a_0 x^n D^n + a_1 x^{n-1} D^{n-1} + ... + a_{n-1} x D + a_n) y = X. \qquad ... (1)$$

In order to solve (1) introduce a new independent variable z such that

$x = e^z$ or $\log x = z$ so that $1/x = dz/dx$. ... (2)

Now, $$\frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{1}{x}\frac{dy}{dz}, \text{ using (2)} \qquad ... (3)$$

or $$x\frac{dy}{dx} = \frac{dy}{dz} \quad \text{or} \quad xD = x\frac{d}{dx} \equiv \frac{d}{dz} = D_1, \text{ say} \qquad ... (4)$$

Again, $$\frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{x}\frac{dy}{dz}\right) = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$= -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dz}\left(\frac{dy}{dz}\right) \cdot \frac{dz}{dx} = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2 y}{dz^2}, \text{ by (2)}$$

or $$x^2 D^2 = x^2 \frac{d^2 y}{dx^2} = \frac{d^2 y}{dz^2} - \frac{dy}{dz} = (D_1^2 - D_1)\, y = D_1 (D_1 - 1)\, y. \qquad ... (5)$$

and so on. Proceeding likewise, we can show that

$$x^n D^n = x^n \frac{d^n y}{dx^n} = D_1 (D_1 - 1)(D_1 - 2) ... (D_1 - n + 1)\, y. \qquad ... (6)$$

Substituting the above values of x, xD, $x^2 D^2$, $x^3 D^3$, ..., $x^n D^n$ in (1) and thus changing the independent variable from x to z, we have

$$[a_0 D_1 (D_1 - 1) ... (D_1 - n + 1) + ... + a_{n-2} D_1 (D_1 - 1) + a_{n-1} D_1 + a_n] y = Z \quad \text{or} \quad f(D_1)\, y = Z, \qquad ... (7)$$

where Z is now a function of z only. The method of solving (7) is same as done in chapter 5.

6.3 Working rule for solving linear homogeneous differential equation

$$(a_0x^nD^n + a_1x^{n-1}D^{n-1} + a_2x^{n-2}D^{n-2} + ... + a_{n-2}x^2D^2 + a_{n-1}xD + a_n)\, y = X, \qquad ... (1)$$

Step I. Put $x = e^z$ or $z = \log x$, where $x > 0$

Step II. Assume that $D_1 = d/dz$ and $D \equiv d/dx$. Then, we have

$xD = D_1$, $\quad x^2D^2 = D_1(D_1 - 1)$, $\quad x^3D^3 = D_1(D_1 - 1)(D_1 - 2)$ and so on.

Then (1) reduces to $f(D_1)\, y = Z$, where Z is now function of z only ... (2)

Step III. Using methods of chapter 5, (2) gives the general solution $y = \phi(z)$. ... (3)

Since $z = \log x$, the desired solution is $y = \phi(\log x)$, $x > 0$. ... (4)

Note. While solving any equation, remember that $e^{m \log n} = n^m$.

6.4 Solved examples based on Art. 6.3

Ex. 1. *(a) Solve* $x^2y_2 + xy_1 - 4y = 0$ **[Delhi Maths (G) 1993]**

Sol. Given $(x^2D^2 + xD - 4)\, y = 0$, where $D \equiv d/dx$. ... (1)

Let $x = e^z$ (or $z = \log x$) and $D_1 \equiv d/dz$ so that $xD = D_1$ and $x^2D^2 = D_1(D_1 - 1)$.

Then (1) reduces to $[D_1(D_1 - 1) + D_1 - 4]\, y = 0$ or $(D_1^2 - 4)\, y = 0$. ... (2)

Its auxiliary equation is $D_1^2 - 4 = 0$ so that $D_1 = 2, -2$. Hence the general solution of (2) is

$$y = c_1 c^{2z} + c_2 e^{-2z} = c_1 e^{2 \log x} + c_2 e^{-2 \log x} = c_1x^2 + c_2x^{-2}, \text{ as } z = \log x,$$

where c_1 and c_2 are arbitrary constants.

Ex. 1. *(b) Solve* $x^2(d^2y/dx^2) - 3x\,(dy/dx) + 4y = 0$ **[I.A.S. Prel. 1994]**

Sol. Let $d/dx \equiv D$. Then the given equation reduces to $(x^2D^2 - 3xD + 4)\, y = 0$. ... (1)

Let $x = e^z$, *i.e.*, $z = \log x$ and $D_1 = d/dz$ ... (2)

Then, $xD = D_1$ and $x^2D^2 = D_1(D_1 - 1)$. Hence (1) reduces to

$$\{D_1(D_1 - 1) - 3D_1 + 4\}\, y = 0 \quad \text{or} \quad (D_1 - 2)^2\, y = 0$$

Its auxiliary equation is $(D_1 - 2)^2 = 0$ giving $D_1 = 2, 2$.

The general solution is $y = (c_1 + c_2z)\, e^{2z} = (c_1 + c_2 z)\,(e^z)^2 = (c_1 + c_2 \log x)\, x^2$, using (2). where c_1 and c_2 are arbitrary constants.

Ex. 1. *(c) Solve* $x^3(d^3y/dx^3) + 2x^2\,(d^2y/d^2x) + 3x\,(dy/dx) - 3y = 0$ **[Meerut 2007]**

Sol. Rewriting the given equation, $(x^3D^3 + 2x^2D^2 + 3xD - 3)y = 0$, $D \equiv d/dx$ ....(1)

Let $x = e^z$, *i.e.*, $z = \log x$ and $D_1 \equiv d/dz$ ...(2)

Then, $xD = D_1$, $\quad x^2D^2 = D_1(D_1 - 1)$ and $x^3D^3 = D_1(D_1 - 1)(D_1 - 2)$

Using (2) and (3), (1) becomes $\{D_1(D_1 - 1)(D_1 - 2) + 2D_1(D_1 - 1) + 3D_1 - 3\}\, y = 0$ ...(4)

Auxiliary equation of (4) is $D_1(D_1 - 1)(D_1 - 2) + 2D_1(D_1 - 1) + 3D_1 - 3 = 0$

or $(D_1 - 1)(D_1^2 + 3) = 0$ giving $D_1 = 1, \pm i\sqrt{3}$

$\therefore$ Solution of (4) is $y = c_1 e^z + c_2 \sin(z\sqrt{3}) + c_3 \cos(z\sqrt{3})$

or $y = c_1x + c_2 \sin(\sqrt{3} \log x) + c_3 \cos(\sqrt{3} \log x)$, using (2); c_1, c_2, c_3 being arbitrary constants.

Ex. 2. *Solve* $(x^3D^3 + 3x^2D^2 - 2xD + 2)\, y = 0$, *where* $D \equiv d/dx$.

Sol. Let $x = e^z$ (or $z = \log x$) and $D_1 \equiv d/dz$, so that $xD = D_1$, $x^2D^2 = D_1(D_1 - 1)$ and $x^3D^3 = D_1(D_1 - 1)(D_1 - 2)$. Then the given equation reduces to

$$[D_1(D_1 - 1)(D_1 - 2) + 3D_1(D_1 - 1) - 2D_1 + 2] = 0$$

or

$$[D_1(D_1 - 1)(D_1 - 2) + 3D_1(D_1 - 1) - 2(D_1 - 1)]\, y = 0$$

or $(D_1 - 1)[D_1(D_1 - 2) + 3D_1 - 2]\, y = 0$ or $(D_1 - 1)(D_1^2 + D_1 - 2)\, y = 0$

or $(D_1 - 1)(D_1 - 1)(D_1 + 2)\, y = 0$ or $(D_1 - 1)^2 (D_1 + 2)\, y = 0$

Its auxiliary equation is $(D_1 - 1)^2 (D_1 + 2) = 0$ so that $D_1 = 1, 1, -2.$

$\therefore$ The general solution is $y = \text{C.F.}$ *i.e.*, $y = (c_1 + c_2 z)\, e^z + c_3 e^{-2z}$

or $y = (c_1 + c_2 z)\, e^z + c_3 (e^z)^{-2}$ or $y = (c_1 + c_2 \log x)\, x + c_3 x^{-2}$, as $x = e^z, z = \log x$

Ex. 3. *Solve the following differential equations :*

(*i*) $x^2 y_2 + y = 3x^2$ **[Delhi Maths (G) 1993]**

(*ii*) $xy_3 + y_2 = 1/x$. **[Delhi Maths (G) 1995, 96]**

(*iii*) $(x^2D^2 - 3xD + 4)\, y = 2x^2$. **[Agra 2005, Lucknow 1992]**

(*iv*) $x^2D^2 - 2y = x^2 + (1/x)$ **[Rohilkhand 1993]**

Sol. (*i*) Given $x^2 y_2 + y = 3x^2$ or $(x^2D^2 + 1)\, y = 3x^2$, where $D \equiv d/dx$. ... (1)

Let $x = e^z$ (or $z = \log x$) and $D_1 \equiv d/dz$ so that $x^2D^2 = D_1(D_1 - 1)$.

$\therefore$ (1) $\Rightarrow [D_1(D_1 - 1) + 1]\, y = 3\, e^{2z}$ or $(D_1^2 - D_1 + 1) = 3e^{2z}$.

Its auxiliary equation is $D_1^2 - D_1 + 1 = 0$ so that $D_1 = (1 \pm i\sqrt{3})/2.$

$\therefore$ C.F.$= e^{z/2}\, [c_1 \cos\, (z\sqrt{3}/2) + c_2 \sin\, (z\sqrt{3}/2)] = (e^z)^{1/2}\, [c_1 \cos\, (z\sqrt{3}/2) + c_2 \sin\, (z\sqrt{3}/2)]$

$= x^{1/2}\, [c_1 \cos\, \{(\sqrt{3}/2) \log x\} + c_2 \sin\, \{(\sqrt{3}/2)) \log x\}]$, as $x = e^z$;

c_1 and c_2 being arbitrary constants.

and $$\text{P.I.} = \frac{1}{D_1^2 - D_1 + 1} 3e^{2z} = 3\frac{1}{2^2 - 2 + 1} e^{2z} = (e^z)^2 = x^2.$$

Hence the required general solution is $y = \text{C.F.} + \text{P.I.}$, *i.e.*,

$$y = x^{1/2} [c_1 \cos \{(\sqrt{3}/2) \log x\} + c_2 \sin \{(\sqrt{3}/2) \log x\} + x^2.$$

(*ii*) Given $x^3\, (d^3y/dx^3) + x^2\, (d^2y/dx^2) = x$ or $(x^3D^3 + x^2D^2)\, y = x$, $D \equiv d/dx$ ... (1)

Let $x = e^z$ (or $z = \log x$) and $D_1 \equiv d/dz$ ... (2)

so that $x^2D^2 = D_1(D_1 - 1)$, $x^3D^3 = D_1(D_1 - 1)(D_1 - 2)$. Then (1) transforms to

$[D_1(D_1 - 1)(D_1 - 2) + D_1 (D_1 - 1)]\, y = e^z$ or $(D_1^3 - 2D_1^2 + D_1)\, y = e^z$.

Here the auxiliary equation is $D_1^3 - 2D_1^2 + D_1 = 0$ so that $D_1 = 0, 1, 1.$

$\therefore$ C.F. $= c_1 e^{0.z} + (c_2 + c_3 z)\, e^z = c_1 + (c_2 + c_3 \log x)\, x$, as $e^z = x$ and $z = \log x$.

$$\text{P.I.} = \frac{1}{D_1^3 - 2D_1^2 + D_1} e^z = \frac{1}{(D_1 - 1)^2} \frac{1}{D_1} e^z = \frac{1}{(D_1 - 1)^2} e^z, \qquad \text{as} \qquad \frac{1}{D_1} e^z = \int e^z dz = e^z$$

$$= \frac{z^2}{2!} e^z, \qquad \text{since} \qquad \frac{1}{(D_1 - \alpha)^m}\; e^{\alpha z} = \frac{z^m}{m!}\; e^{\alpha z}$$

$$= (x/2) \times (\log x)^2, \quad \text{since} \quad x = e^z \quad \text{and} \quad z = \log x$$

$\therefore$ The required solution is $y = c_1 + (c_2 + c_3 \log x)\, x + (x/2) \times (\log x)^2$,

c_1, c_2 and c_3 being arbitrary constants.

(*iii*) Given that $(x^2D^2 - 3xD + 4)\, y = 2x^2$. ... (1)

Let $x = e^z$ (or $z = \log x$) and $D_1 \equiv d/dz$. Then (1) becomes

$\{D_1(D_1 - 1) - 3D_1 + 4\}\, y = 2\, e^{2z}$ or $(D_1 - 2)^2\, y = 2\, e^{2z}$

Its auxiliary equation is $(D_1 - 2)^2 = 0$ so that $D_1 = 2, 2.$

$\therefore$ C.F. $= (c_1 + c_2 z)\, e^{2z} = (c_1 + c_2 z)\, (e^z)^2 = (c_1 + c_2 \log x)\, x^2$, since $x = e^z$ and $z = \log x$

P.I. $= \dfrac{1}{(D_1-2)^2} 2e^{2z} = 2\dfrac{z^2}{2!}e^{2z} = z^2(e^z)^2 = (\log x)^2 x^2,$ as $\dfrac{1}{(D_1-\alpha)^m}e^{\alpha z} = \dfrac{z^m e^{\alpha z}}{m!}$

Hence the required solution is $y = C.F. + P.1,$

i.e., $y = (c_1 + c_2 \log x)\ x^2 + (\log x)^2 x^2$ or $y = x^2[c_1 + c_2 \log x + (\log x)^2]$

(*iv*) Given $(x^2D^2 - 2)\ y = x^2 + x^{-1},$ where $D \equiv d/dx$ ...(1)

Let $x = e^z$ (or $z = \log x$) and $D_1 \equiv d/dz$ Then (1) becomes

$[D_1(D_1-1)-2]\ y = e^{2z} + e^{-z}$ or $(D_1^2 - D_1 - 2)\ y = e^{2z} + e^{-z}.$

Its auxilary equation is $D_1^2 - D_1 - 2 = 0$ so that $D_1 = 2, -1.$

$\therefore$ $C.F. = c_1 e^{2z} + c_2 e^{-z} = c_1(e^z)^2 + c_2(e^z)^{-1} = c_1x^2 + c_2x^{-1}$, as $x = e^z$,

$$\text{P.I.} = \frac{1}{(D^2 - D_1 - 2)}(e^{2z} + e^{-z}) = \frac{1}{(D_1-2)}\frac{1}{(D_1+1)}e^{2z} + \frac{1}{(D_1+1)}\frac{1}{(D_1-2)}e^{-z}$$

$$= \frac{1}{D_1-2}\frac{1}{2+1}e^{2z} + \frac{1}{D_1+1}\frac{1}{-1-2}e^{-z} = \frac{1}{3}\frac{z}{1!}e^{2z} - \frac{1}{3}\frac{z}{1!}e^{-z} = \frac{1}{3}\log x\left(x^2 + x^{-1}\right), \text{ as } x = e^z$$

$\therefore$ Solution is $y = c_1x^2 + c_2x^{-1} + (1/3) \times (x^2 + x^{-1}) \log x$, c_1 and c_2 being arbitrary constants .

Ex. 4. *Solve the differential equations*

(*i*) $x^2(d^2y/dx^2) + 2x(dy/dx) = \log x.$ **[Agra 1994]**

(*ii*) $(x^2D^2 + 7xD + 13)\ y = \log x.$ **[Meerut 1997, 99]**

Sol. (*i*) given $(x^2D^2 + 2xD)y = \log x$, where $D = d/dx$ ... (1)

Let $x = e^z$ (or $z = \log x$ and $D_1 \equiv d/dz$. Then (1) becomes

$[D_1(D_1 - 1) + 2D_1]\ y = z$ or $(D_1^2 + D_1)\ y = z.$

Its auxiliary equation is $D_1^2 + D_1 = 0$ so that $D_1 = 0, -1.$

$\therefore$ C.F. $= c_1 e^{0.z} + c_2 e^{-z} = c_1 + c_2(e^z)^{-1} = c_1 + c_2x^{-1}.$, c_1 and c_2 being arbitrary constants.

$$\text{P.I.} = \frac{1}{D_1^2 + D_1} = \frac{1}{D_1(1+D_1)}z = \frac{1}{D_1}(1+D_1)^{-1}z = \frac{1}{D_1}(1 - D_1 + \ldots)\,z = \frac{1}{D_1}(z-1)$$

$$= (1/2)\times z^2 - z = (1/2)\times(\log x)^2 - \log x, \text{ as } x = e^z \text{ and } z = \log x.$$

$\therefore$ The required solution is $y = c_1 + c_2x^{-1} + (1/2) \times (\log x)^2 - \log x,$

(*ii*) Given that $(x^2D^2 + 7xD + 13)\ y = \log x,$ $D \equiv d/dx$ ...(1)

Let $x = e^z$ (or $z = \log x$) and $D_1 \equiv d/dz$. Then, (1) becomes

$[D_1(D_1 - 1) + 7D_1 + 13]\ y = z$ or $(D_1^2 + 6D_1 + 13)\ y = z.$

Its auxiliary equation is $D_1^2 + 6D_1 + 13 = 0$ so that $D_1 = -3 \pm 2i.$

$\therefore$ C.F. $= e^{-3z}(c_1 \cos 2z + c_2 \sin 2z) = x^{-3}[c_1 \cos(2\log x) + c_2 \sin(2\log x)]$,

where c_1 and c_2 being arbitrary constants.

$$\text{P.I.} = \frac{1}{D_1^2 + 6D_1 + 13}z = \frac{1}{13[1 + (6/13)D_1 + (1/13)D_1^2]}z = \frac{1}{13}\left[1 + \left(\frac{6}{13}D_1 + \frac{1}{13}D_1^2\right)\right]^{-1} z$$

$$= \frac{1}{13}\left[1 - \left(\frac{6}{13}D_1 + \frac{1}{13}D_1^2\right) + \ldots\right]z = \frac{1}{13}\left(z - \frac{6}{13}\right) = \frac{1}{13}\left(\log x - \frac{6}{13}\right) = \frac{1}{169}(13\log x - 6)$$

$\therefore$ Required solution is $y = x^{-3}[c_1 \cos(2\log x) + c_2 \sin(2\log x)] + (1/169) \times (13\log x - 6)$

Ex. 5. *Solve* $x^3 (d^3y/dx^3) + 3x^2 (d^2y/dx^2) + x (dy/dx) + y = \log x + x.$

[Agra 1995, Lucknow 1996, Meerut 1995, Rohilkhand 1997]

Sol. Given $(x^3D^3 + 3x^2D^2 + xD + 1)\, y = \log x + x$, where $D \equiv d/dx$. ... (1)

Let $x = e^z$ (or $z = \log x$) and $D_1 \equiv d/dz$. Then (1) becomes

$[D_1(D_1 - 1)(D_1 - 2) + 3D_1(D_1 - 1) + D_1 + 1]\, y = z + e^z$ or $(D_1^3 + 1)\, y = e^z + z.$

Its auxiliary equation is $D_1^3 + 1 = 0$ or $(D_1 + 1)(D_1^2 - D_1 + 1) = 0$

so that $D_1 = -1,\ (1 \pm i\sqrt{3})/2$ *i.e.*, $D_1 = -1,\ (1/2) \pm i(\sqrt{3}/2)$.

$\therefore$ C.F. $= c_1 e^{-z} + e^{z/2} [c_2 \cos \{(\sqrt{3}/2)z\} + c_3 \sin \{(\sqrt{3}/2)z\}]$

$= c_1x^{-1} + x^{1/2} [c_2 \cos \{(\sqrt{3}/2) \log x\} + c_3 \sin \{(\sqrt{3}/2)\log x\}]$, as $x = e^z$

where c_1 and c_2 being arbitrary constatns

$$\text{P.I.} = \frac{1}{D_1^3+1}(e^z + z) = \frac{1}{D_1^3+1}e^z + \frac{1}{D_1^3+1}z = \frac{1}{1^3+1}e^z + (1+D_1^3)^{-1}z$$

$= (1/2) \times e^z + (1 - D_1^3 + \ldots)\, z = (1/2) \times e^z + z = x/2 + \log x$

Hence the required general solution is $y =$ C.F. + P.I. *i.e.*,

$$y = c_1x^{-1} + x^{1/2}[c_2 \cos\{(\sqrt{3}/2)\log x\} + c_3 \sin\{(\sqrt{3}/2)\log x\}] + x/2 + \log x$$

Ex. 6. *Solve the following differential equations :*

(*i*) $(x^2D^2 - 3xD + 5)\, y = \sin (\log x)$.

(*ii*) $3x^2y_2 - 5xy_1 + 5y = \sin (\log x)$. **[S.V. (Univ.) A.P. (1997)]**

(*iii*) $x^3(d^3y/dx^3) + 3x^2 (d^2y/dx^2) + x (dy/dx) + 8y = 65 \cos (\log x)$.

(*iv*) $x^4(d^4y/dx^4) + 6x^3(d^3y/dx^3) + 4x^2(d^2y/dx^2) - 2x (dy/dx) - 4y = 2 \cos (\log x)$.

Sol. (*i*) Given $(x^2D^2 - 3xD + 5) = \sin (\log x)$, where D = d/dx ... (1)

Let $x = e^z$ (or $z = \log x$) and $D_1 \equiv d/dz$. Then (1) becomes

$[D_1(D_1 - 1) - 3D_1 + 5]\, y = \sin z$ or $(D_1^2 - 4D_1 + 5)\, y = \sin z.$

Its auxiliary equation is $D_1^2 - 4D_1 + 5 = 0$ so that $D_1 = 2 \pm i$.

$\therefore$ C.F. $= e^{2z} (c_1\cos z + c_2 \sin z) = x^2 [c_1 \cos (\log x) + c_2 \sin (\log x)]$,

where c_1 and c_2 are arbitrary constants

$$\text{P.I.} = \frac{1}{D_1^2 - 4D_1 + 5}\sin z = \frac{1}{-1^2 - 4D_1 + 5}\sin z = \frac{1}{4}\,\frac{1}{1-D_1}\sin z$$

$$= \frac{1}{4}(1+D_1)\frac{1}{1-D_1^2}\sin z = \frac{1}{4}(1+D_1)\frac{1}{1-(-1^2)}\sin z = \frac{1}{8}(\sin z + D_1 \sin z)$$

$= (1/8) \times [\sin z + \cos z] = (1/8) \times [\sin (\log x) + \cos (\log x)]$, as $z = \log x$

$\therefore$ Solution is $y = x^2[c_1 \cos (\log x) + c_2 \sin (\log x)] + (1/8) \times [\sin (\log x) + \cos (\log x)]$.

(*ii*) **Ans.** $y = c_1x + c_2x^{5/3} + (1/16) \times [\sin (\log x) + \cos (\log x)]$.

(*iii*) **Ans.** $y = c_1x^{-2} + x\, [c_2 \cos(\sqrt{3}\log x) + c_3 \sin(\sqrt{3}\log x)] + 8 \cos (\log x) - \sin (\log x)$.

(*iv*) Given $(x^4D^4 + 6x^3D^3 + 4x^2D^2 - 2xD - 4)\, y = 2 \cos (\log x)$, $D \equiv d/dx$ ... (1)

Let $x = e^z$ (or $z = \log x$) and $D_1 \equiv d/dz$. Then (1) becomes

$[D_1(D_1 - 1)(D_1 - 2)(D_1 - 3) + 6D_1(D_1 - 1)(D_1 - 2) + 4D_1(D_1 - 1) - 2D_1 - 4]\, y = 2 \cos z$

or $\{(D_1^3 - 3D_1^2 + 2D_1)(D_1 - 3) + 6(D_1^3 - 3D_1^2 + 2D_1) + 4D_1^2 - 6D_1 - 4\}\, y = 2 \cos z$

or $[D_1^4 + 11D_1^2 - 6D_1 + 6(D_1^3 - 3D_1^2 + 2D_1) + 4D_1^2 - 6D_1 - 4]\, y = 2 \cos z$

or $(D_1^4 - 3D_1^2 - 4)\, y = 2 \cos z$

Its auxiliary equation is $D_1^4 - 3D_1^2 - 4 = 0$ so that $D_1 = 2, -2, 0 \pm i$.

$\therefore$ C.F. $\equiv c_1 e^{2z} + c_2 e^{-2z} + e^{0.z}(c_3 \cos z + c_4 \sin z)$

$\equiv c_1 x^2 + c_2 x^{-2} + c_3 \cos(\log x) + c_4 \sin(\log x)$, as $x = e^z$, $z = \log x$,

where c_1, c_2, c_3 and c_4 are arbitrary constants.

$$\text{P.I.} = \frac{1}{D_1^4 - 3D_1^2 - 4} 2\cos z = 2\frac{1}{(D_1^2+1)(D_1^2-4)}\cos z = 2\frac{1}{D_1^2+1}\frac{1}{-1^2-4}\cos z$$

$$= -\frac{2}{5}\frac{1}{D_1^2+1}\cos z = -\frac{2}{5}\frac{z}{(2\times 1)}\sin z = -\frac{\log x \ \ \sin(\log x)}{5}, as \ \frac{1}{D_1^2+a^2}\cos az = \frac{z}{2a}\sin az$$

$\therefore$ Solution is $y = c_1 x^2 + c_2 x^{-2} + c_3 \cos(\log x) + c_4 \sin(\log x) - (1/5) \times \log x \sin(\log x)$.

Ex. 7. *Solve the following differential equations :*

(i) $x^2(d^2y/dx^2) + 5x(dy/dx) + 4y = x \log x$. **[Allahabad 1994]**

(ii) $\{x^2D^2 - (2m-1)xD + (m^2+n^2)\} y = n^2 x^m \log x$, where $D \equiv d/dx$

Sol. (*i*) Given $(x^2D^2 + 5xD + 4)y = x \log x$, where $D \equiv d/dx$ ... (1)

Let $x = e^z$ (or $z = \log x$) and $D_1 \equiv d/dz$. Then (1) becomes

$[D_1(D_1-1) + 5D_1 + 4]y = ze^z$ or $(D_1+2)^2 y = ze^z$.

Its auxiliary equation is $(D_1+2)^2 = 0$ so that $D_1 = -2, -2$.

$\therefore$ C.F. $= (c_1 + c_2 z)e^{-2z} = (c_1 + c_2 z)(e^z)^{-2} = (c_1 + c_2 \log x)x^{-2}$,

where c_1 and c_2 are arbitrary constants.

$$\text{P.I.} = \frac{1}{(D_1+2)^2} ze^z = e^z\frac{1}{[(D_1+1)+2]^2} z = e^z\frac{1}{(3+D_1)^2} z = \frac{e^z}{9}\frac{1}{(1+D_1/3)^2} z = \frac{e^z}{9}\left(1+\frac{D_1}{3}\right)^{-2} z$$

$$= \frac{e^z}{9}\left(1 - \frac{2D_1}{3} + \ldots\right)z = \frac{e^z}{9}\left(z - \frac{2}{3}D_1 z\right) = \frac{e^z}{9}\left(z - \frac{2}{3}\right) = \frac{e^z}{27}(3z-2) = \frac{x}{27}(3\log x - 2).$$

Hence the solution is $y = (c_1 + c_2 \log x)x^{-2} + (x/27) \times (3\log x - 2)$

(*ii*) Let $x = e^z$ or $z = \log x$ and $D_1 \equiv d/dz$. So the given equation becomes

$[D_1(D_1-1) - (2m-1)D_1 + (m^2+n^2)]y = n^2 e^{mz} z$ or $[D_1^2 - 2mD_1 + (m^2+n^2)]y = n^2 e^{mz} z$.

Its auxilary equations is $D_1^2 - 2mD_1 + (m^2+n^2) = 0$ so that $D_1 = m \pm in$.

$\therefore$ C.F. $= e^{mz}[c_1 \cos nz + c_2 \sin nz] = x^m[c_1 \cos(n \log x) + c_2 \sin(n \log x)]$, as $x = e^z$

where c_1 and c_2 are arbitrary constants

$$\text{P.I.} = \frac{1}{D_1^2 - 2mD_1 + (m^2+n^2)} n^2 e^{mz} z = n^2 e^{mz}\frac{1}{(D_1+m)^2 - 2m(D_1+m) + m^2 + n^2} z$$

$$= n^2 e^{mz}\frac{1}{D_1^2+n^2} z = n^2 e^{mz}\frac{1}{n^2(1+D_1^2/n^2)} z = e^{mz}\{1 + (D_1^2/n^2)\}^{-1} z$$

$$= e^{mz}\{1 - (D_1^2/n^2) + \ldots\} z = e^{mz}\ z = (e^z)^m\ z = x^m \log x, \text{as } x = e^z$$

$\therefore$ Solution is $y = C.F. + P.I. = x^m[c_1 \cos(n \log x) + c_2 \sin(n \log x)] + x^m \log x$.

Ex. 8. *Solve* $(x^2D^2 - xD + 4)y = \cos(\log x) + x \sin(\log x)$. **[Delhi Maths (H) 2009]**

Sol. Let $x = e^z$ or $z = \log x$ and $D_1 \equiv d/dz$. So given equation gives

$[D_1(D_1-1) - D_1 + 4]y = \cos z + e^z \sin z$ or $(D_1^2 - 2D_1 + 4)y = \cos z + e^z \sin z$.

Its auxiliary equation is $D_1^2 - 2D_1 + 4 = 0$ so that $D_1 = 1 \pm i\sqrt{3}$.

$\therefore$ C.F. $= e^z[c_1 \cos(z\sqrt{3}) + c_2 \sin(z\sqrt{3}) = x[c_1 \cos(\sqrt{3}\log x) + c_2 \sin(\sqrt{3}\log x)]$, as $x = e^z$

P.I. corresponding to $\cos z$

$$= \frac{1}{D_1^2 - 2D_1 + 4}\cos z = \frac{1}{-1^2 - 2D_1 + 4}\cos z = \frac{1}{3 - 2D_1}\cos z$$

$$= (3 + 2D_1)\frac{1}{(3 + 2D_1)(3 - 2D_1)}\cos z = (3 + 2D_1)\frac{1}{9 - 4D_1^2}\cos z$$

$$= (3 + 2D_1)\frac{1}{9 - 4(-1^2)}\cos z = \frac{1}{13}(3\cos z + 2D_1 \cos z)$$

$= (1/13) \times (3 \cos z - 2 \sin z) = (1/13) \times [3 \cos (\log x) - 2 \sin (\log x)]$, as $x = e^z$

P.I. corresponding to $e^z \sin z$

$$= \frac{1}{D_1^2 - 2D_1 + 4} e^z \sin z = e^z \frac{1}{(D_1 + 1)^2 - 2(D_1 + 1) + 4}\sin z$$

$$= e^z \frac{1}{D_1^2 + 3}\sin z = e^z \frac{1}{-1^2 + 3}\sin z = \frac{1}{2}x \sin(\log x), \text{ as } x = e^z \text{ and } z = \log x$$

$\therefore$ Solution is $y = x[c_1 \cos(\sqrt{3}\log x) + c_2 \sin(\sqrt{3}\log x)]$

$+ (1/13) \times [3 \cos (\log x) - 2 \sin (\log x)] + (x/2) \times \sin (\log x)$.

Ex. 9. *Solve* $x^2 (d^2y/dx^2) - 2x (dy/dx) + 2y = x + x^2 \log x + x^3$.

Sol. Given $(x^2D^2 - 2xD + 2)\, y = x + x^2 \log x + x^3$, where $D \equiv d/dx$ ... (1)

Let $x = e^z$ or $z = \log x$ and $D_1 \equiv d/dz$. Then (1) becomes

$[D_1(D_1 - 1) - 2D_1 + 2]\, y = e^z + ze^{2z} + e^{3z}$ or $(D_1^2 - 3D_1 + 2)\, y = e^z + ze^{2z} + e^{3z}$.

Here auxiliary equation is $D_1^2 - 3D_1 + 2 = 0$ so that $D_1 = 1, 2$.

$\therefore$ C.F. $= c_1 e^z + c_2 e^{2z} = c_1 e^z + c_2 (e^z)^2 = c_1x + c_2x^2$, c_1, c_2 being arbitrary constants

P.I. corresponding to $(e^z + e^{3z})$

$$= \frac{1}{D_1^2 - 3D_1 + 2}(e^z + e^{3z}) = \frac{1}{(D_1 - 1)(D_1 - 2)}e^z + \frac{1}{(D_1 - 1)(D_1 - 2)}e^{3z}$$

$$= \frac{1}{D_1 - 1}\frac{1}{1 - 2}e^z + \frac{1}{(3-1)(3-2)}e^{3z} = -\frac{1}{(D_1 - 1)}e^z + \frac{1}{2}e^{3z} = -\frac{z}{1!}e^z + \frac{1}{2}(e^z)^3$$

$= - z\, e^z - (1/2) \times (e^z)^3 = -x \log x + (x^3/2)$, as $x = e^z$ and $z = \log x$

P.I. corresponding to ze^{2z}

$$= \frac{1}{D_1^2 - 3D_1 + 2}ze^{2z} = e^{2z}\frac{1}{(D_1 + 2)^2 - 3((D_1 + 2) + 2}z = e^{2z}\frac{1}{D_1^2 + D_1}z$$

$$= e^{2z}\frac{1}{D_1}(1 + D_1)^{-1}z = e^{2z}\frac{1}{D_1}(1 - D_1 + \dots\dots)z = e^{2z}\frac{1}{D_1}(z - 1) = (e^z)^2\{(z^2/2) - z\}$$

$= x^2\, [(1/2) \times (\log x)^2 - \log x] = (x^2/2) \times [(\log x)^2 - 2 \log x]$

$\therefore$ Solution is $y = c_1x + c_2x^2 - x \log x + x^3/2 + (x^2 / 2) \times [(\log x)^2 - 2 \log x]$.

Ex. 10. *Solve* $(x^4D^3 + 2x^3D^2 - x^2D + x)\, y = 1$. **[Purvanchal 1996, Agra 1994]**

Sol. Re-writing $(x^3D^3 + 2x^2D^2 - xD + 1)\, y = 1/x.$, where $D = d/dx$ ... (1)

Let $x = e^z$ so that $z = \log x$ and let $D_1 = d/dz$. Then (1) becomes

$[D_1(D_1 - 1)(D_1 - 2) + 2D_1(D_1 - 1) - 2D_1 + 1]\, y = e^{-z}$ or $(D_1^3 - D_1^2 - D_1 + 1)\, y = e^{-z}$ ...(2)

Here auxiliary equation is $D_1^3 - D_1^2 - D_1 + 1 = 0$ gives $D_1 = 1, 1, -1$.

$\therefore$ C.F. $= (c_1 + c_2 z)\, e^z + c_3 e^{-z} = (c_1 + c_2 \log x)\, x + c_3 x^{-1}$, c_1, c_2 being arbitrary constants

$$\text{P.I.} = \frac{1}{(D_1+1)}\frac{1}{(D_1-1)^2} e^{-z} = \frac{1}{(D_1+1)(-1-1)^2} e^{-z} = \frac{1}{4}\frac{1}{D_1+1} e^{-z}\cdot 1$$

$$= \frac{1}{4} e^{-z} \frac{1}{D_1 - 1 + 1}\cdot 1 = \frac{1}{4} e^{-z}\cdot z = \frac{1}{4} x^{-1}\log x,$$

$\therefore$ Solution is $y = (c_1 + c_2 \log x)\, x + c_3 x^{-1} + (1/4) \times x^{-1} \log x$.

Ex. 11. *Solve* $(x^2D^2 - xD + 2)\, y = x \log x$. **[Delhi Maths (G) 2002; Bangalore 1993; Kanpur 1997, 98; Lucknow 1997 ; Utkal 2003]**

Sol. Let $x = e^z$ so that $z = \log x$ and let $D_1 \equiv d/dz$.... (1)

Then the given equation becomes $[D_1(D_1 - 1) - D_1 + 2]\, y = ze^z$ or $(D_1^2 - 2D_1 + 2)\, y = ze^z$,

Its auxiliary equation is $D_1^2 - 2D_1 + 2 = 0$. giving $D_1 = 1 \pm i$.

$\therefore$ C.F. $= e^z (c_1 \cos z + c_2 \sin z) = x\, [c_1 \cos(\log x) + c_2 \sin(\log x)]$, using (1)

where c_1 and c_2 are arbitrary constants.

$$\text{P.I.} = \frac{1}{D_1^2 - 2D + 2} ze^z = e^z \frac{1}{(D_1+1)^2 - 2(D_1+1) + 2} z = e^z \frac{1}{D_1^2 + 1}\cdot z$$

$$= e^z (1 + D_1^2)^{-1}\cdot z = e^z (1 - ...)\, z = e^z \cdot z = x \log x, \text{ using (1)}$$

$\therefore$ Required solution is $y = x\, [c_1 \cos(\log x) + c_2 \sin(\log x)] + x \log x$.

Ex. 12. *Solve* $\dfrac{d^2y}{dx^2} + \dfrac{1}{x}\cdot\dfrac{dy}{dx} = \dfrac{12 \log x}{x^2}$. **[Delhi Maths (G) 1997]**

Sol. Given $(x^2D^2 + xD)\, y = 12 \log x$, where $D \equiv d/dx$. ... (1)

Let $x = e^z$ *i.e.* $z = \log x$ and $D_1 \equiv d/dz$. Then (1) becomes

$[D_1(D_1 - 1) + D_1]\, y = 12\, z$ or $D_1^2\, y = 12\, z$. A.E. $D_1^2 = 0$ gives $D_1 = 0, 0$

C.F. $= c_1 + c_2 z = c_1 + c_2 \log x$, c_1 and c_2 being arbitrary constants.

$$\text{and P.I.} = \frac{1}{D_1^2} 12z = 12\frac{1}{D_1^2} z = 12\frac{1}{D_1}\cdot\frac{z^2}{2} = 12 \times \frac{z^3}{6} = 2(\log x)^3.$$

$\therefore$ Required Solution is $y = c_1 + c_2 \log x + 2 (\log x)^3$, c_1, c_2 being arbitrary constants.

Ex. 13. *Solve* $x^2D^2y - 3x\, Dy + 5y = x^2 \sin \log x$.

[Delhi Maths (G) 2001; Delhi Maths (Hons) 2007]

Sol. Given $(x^2D^2 - 3xD + 5)\, y = x^2 \sin \log x$, where $D \equiv d/dx$ ... (1)

Let $x = e^z$ so that $z = \log x$ and let $D_1 \equiv d/dz$ ... (2)

Then $x D = D_1$ and $x^2D^2 = D_1 (D_1 - 1)$ ... (3)

Using (2) and (3), (1) reduces to

$[D_1(D_1 - 1) - 3D_1 + 5]\, y = e^{2z} \sin z$ or $(D_1^2 - 4D_1 + 5)\, y = e^{2z} \sin z$.... (4)

Auxiliary equation for (4) is $D_1^2 - 4D_1 + 5 = 0$, giving $D_1 = (4 \pm \sqrt{16 - 20})/2 = 2 \pm i$.

$\therefore$ C.F. $= e^{2z} (c_1 \cos z + c_2 \sin z) = x^2 (c_1 \cos \log x + c_2 \sin \log x)$, by (2)

where c_1 and c_2 are arbitrary constants.

$$\text{P.I.} = \frac{1}{D_1^2 - 4D_1 + 5} e^{2z} \sin z = e^{2z} \frac{1}{(D_1+2)^2 - 4(D_1+2)+5} \sin z = e^{2z} \frac{1}{D_1^2+1} \sin z$$

$$= e^{2z}\left(-\frac{z}{2}\cos z\right) = -\frac{x^2}{2}\log x \cos \log x, \text{ by (2);} \qquad \text{as } \frac{1}{D_1^2+a^2}\sin az = -\frac{z}{2a}\cos az$$

$\therefore$ Required solution is $\quad y = x^2 (c_1 \cos \log x + c_2 \sin \log x) - (x^2/2) \times \log x \cos \log x.$

Ex. 14(*a*). *Solve* $x^3 (d^3y/dx^3) + 2x^2 (d^2y/dx^2) + 2y = 10(x + 1/x)$. **[Agra 2006, I.A.S. 1999, Delhi Maths (G) 1996, Delhi Maths (H) 1997, Rohilkhand 1997, Kanpur 1995]**

Sol. Given $\quad (x^3D^3 + 2\, x^2D^2 + 2)\, y = 10\, (x + x^{-1})$, $\quad$ where $\quad D \equiv d/dx$. ... (1)

Let $x = e^z$ so that $x = \log x$ and let $D_1 \equiv d/dx$. Then (1) becomes

$$[D_1(D_1 - 1)(D_1 - 2) + 2D_1(D_1 - 1) + 2]\, y = 10\, (e^z + e^{-z})$$

or
$$(D_1^3 - D_1^2 + 2)\, y = 10\, e^z + 10\, e^{-z}. \qquad \text{... (2)}$$

A.E. of (2) is $\; D_1^3 - D_1^2 + 2 = 0 \;$ or $\; (D_1 + 1)(D_1^2 - 2D_1 + 2) = 0 \;$ giving $\; D_1 = -1, 1 \pm i$.

C.F. = $\; c_1e^{-z} + e^z (c_1 \cos z + c_2 \sin z) = c_1x^{-1} + x\, (c_2 \cos \log x + c_3 \sin \log x)$

$$\text{P.I. corresponding to } 10\, e^z = 10 \frac{1}{(D_1+1)(D_1^2 - 2D_1 + 2)} e^z = 10 \frac{1}{2(1-2+2)} e^z = 5x$$

$$\text{and P.I. corresponding to } 10\, e^{-z} = 10 \frac{1}{(D_1+1)(D_1^2 - 2D_1 + 2)} e^{-z} = 10 \frac{1}{D_1+1} \cdot \frac{1}{1+2+2} e^{-z}$$

$$= 2\frac{1}{D_1+1} e^{-z} \cdot 1 = 2e^{-z} \frac{1}{D_1 - 1 + 1} \cdot 1 = 2e^{-z} \frac{1}{D_1} \cdot 1 = 2e^{-z} z = 2x^{-1} \log x.$$

$\therefore$ Required solution is $\quad y = c_1x^{-1} + x\, (c_2 \cos \log x + c_3 \sin \log x) + 5x + 2x^{-1} \log x.$

Ex. 14(*b*). *Solve* $x^2(d^3y/dx^3) + 2x(d^2y/dx^2) + 2(y/x) = 10(1 + 1/x^2)$. **[I.A.S. 2006]**

Sol. Multiplying both sides by x, the given equation becomes

$$x^3(d^3y/dx^3) + 2x^2(d^2y/dx^2) + 2y = 10(x + 1/x)$$

which is same as given in Ex. 14(*a*). Now, proceed as in Ex. 14(*a*).

Ex. 15. $(x^4D^4 + 6x^3D^3 + 9x^2D^2 + 3xD + 1)\, y = (1 + \log x)^2$.

Sol. Given $\quad (x^4D^4 + 6x^3D^3 + 9x^2D^2 + 3xD + 1)\, y = (1 + \log x)^2$, $\quad$ where $\quad D \equiv d/dx$ ... (1)

Let $\quad x = e^z \quad$ so that $\quad z = \log x \quad$ and $\quad$ let $D_1 \equiv d/dz$. ...(2)

Then (1) becomes

$$[D_1(D_1 - 1)(D_1 - 2)(D_1 - 3) + 6D_1(D_1 - 1)(D_1 - 2) + 9D_1(D_1 - 1) + 3D_1 + 1]\, y = (1 + z)^2$$

or
$$(D_1^4 + 2D_1^2 + 1)\, y = (1 + z)^2, \text{ on simplification.} \qquad \text{.. (3)}$$

$\therefore$ Auxiliary equation for (3) is $\; D_1^4 + 2D_1^2 + 1 = 0 \;$ or $\; (D_1^2 + 1)^2 = 0 \;$ so $D_1 = 0 \pm i$ (twice).

$\therefore$ C.F. $= e^{0.z}\, [(c_1 + c_1z) \cos z + (c_3 + c_4z) \sin z]$

$= (c_1 + c_2 \log x) \cos \log x + (c_3 + c_4 \log x) \sin \log x$, using (2)

where c_1, c_2, c_3 and c_4 are arbitrary constant.

$$\text{P.I.} = \frac{1}{(D_1^2+1)^2}(1+z)^2 = (1 + D_1^2)^{-2}(1+z)^2 = (1 - 2D_1^2 + \ldots)(1 + 2z + z^2)$$

$= (1 + 2z + z^2) - 2D_1^2\, (1 + 2z + z^2) = 1 + 2z + z^2 - 4 = z^2 + 2z - 3 = (\log x)^2 + 2 \log x - 3,$

$\therefore$ Solution is $\quad y = (c_1 + c_2\log x) \cos \log x + (c_3 + c_4 \log x)\sin \log x + (\log x)^2 + 2 \log x - 3.$

Ex. 16. *Solve* $(x^2D^2 + xD + 1)\, y = \log x . \sin \log x$.

Sol. Let $x = e^z$ or $z = \log x$ and $D_1 \equiv d/dz$. Given equation becomes

$[D_1(D_1 - 1) + D_1 + 1]\, y = z \sin z$ or $(D_1^2 + 1)\, y = z \sin z$.

Its auxiliary equation is $D_1^2 + 1 = 0$ so that $D_1 = \pm i$.

$$\text{P.I.} = \frac{1}{D_1^2+1} z \sin z = \text{I.P. of } \frac{1}{D_1^2+1} ze^{iz}, \text{ by Euler's theorem.}$$

[Here I.P. stands for imaginary part.]

$$= \text{I.P. of } e^{iz} \frac{1}{(D_1+i)^2+1} z = \text{I.P. of } e^{iz} \frac{1}{D_1^2+2D_1 i} z = \text{I.P. of } e^{iz} \frac{1}{2iD_1(1+D_1/2i)} z$$

$$= \text{I.P. of } \frac{e^{iz}}{2i} \frac{1}{D_1} \left(1+\frac{D_1}{2i}\right)^{-1} z = \text{I.P. of } \frac{e^{iz}}{2i} \frac{1}{D_1} \left(1-\frac{D_1}{2i}+\ldots\right) z = \text{I.P. of } \frac{e^{iz}}{2i} \frac{1}{D_1} \left(z-\frac{1}{2i}\right)$$

$= \text{I.P. of } (-i/2)\, e^{iz} \{(z^2/2) + (z/2)\, i\} = \text{I.P. of } (1/4) \times (\cos z + i \sin z)(-iz^2 + z)$

$= (1/4) \times (z \sin z - z^2 \cos z) = (1/4) \times \log x \sin (\log x) - (1/4) \times (\log x)^2 \cos (\log x)$.

$\therefore$ Solution is $y = c_1 \cos \log x + c_2 \sin \log x + (1/4) \times \log x \sin(\log x) - (1/4) \times (\log x)^2 \cos (\log x)$

Ex. 17. *Solve* $x^2 \dfrac{d^2y}{dx^2} - 3x \dfrac{dy}{dx} + y = \dfrac{\log x \sin \log x + 1}{x}$. **[Meerut 1996, Agra 1993]**

Sol. Given $(x^2D^2 - 3xD + 1)\, y = x^{-1}[1 + \log x \sin \log x]$, where $D = d/dx$ ... (1)

Let $x = e^z$ so that $z = \log x$ and let $D_1 \equiv d/dz$. Then (1) becomes

$[D_1(D_1 - 1) - 3D_1 + 1]\, y = e^{-z}(1 + z \sin z)$ or $(D_1^2 - 4D_1 + 1)\, y = e^{-z} + e^{-z} z \sin z$.

Here auxiliary equation for (2) is $D_1^2 - 4D_1 + 1 = 0$ so that $D_1 = 2 \pm \sqrt{3}$.

C.F. $= e^{2z} [c_1 \cosh (\sqrt{3}z) + c_2 \sinh (\sqrt{3}z)] = x^2[c_1 \cosh(\sqrt{3} \log x) + c_2 \sinh (\sqrt{3} \log x)]$,

where c_1 and c_2 are arbitrary constants.

$$\text{P.I. corresponding to } e^{-z} = \frac{1}{D_1^2-4D_1+1} e^{-z} = \frac{1}{1+4+1} e^{-z} = \frac{1}{6} x^{-1}.$$

and P.I. corresponding to $e^{-z} z \sin z$

$$= \frac{1}{D_1^2-4D_1+1} e^{-z}(z \sin z) = e^{-z} \frac{1}{(D_1-1)^2-4(D_1-1)+1} z \sin z$$

$$= e^{-z} \frac{1}{D_1^2-6D_1+6} z \sin z = e^{-z} \left[z \frac{1}{D_1^2-6D_1+6} \sin z - (2D_1-6) \frac{1}{(D_1^2-6D_1+6)^2} \sin z \right]$$

[Using result of theorem of Art. 5.22, Chapter 5]

$$= e^{-z} \left[z \frac{1}{-1-6D_1+6} \sin z - (2D_1-6) \frac{1}{(-1-6D_1+6)^2} \sin z \right]$$

$$= e^{-z} \left[z \frac{1}{5-6D_1} \sin z - (2D_1-6) \frac{1}{(5-6D_1)^2} \sin z \right]$$

$$= e^{-z} \left[z(5+6D_1) \frac{1}{25-36D_1^2} \sin z - (2D_1-6) \frac{1}{25-60D_1+36D_1^2} \sin z \right]$$

$$= e^{-z} \left[z\,(5+6D_1) \frac{1}{25+36} \sin z - (2D_1-6) \frac{1}{25-60D_1-36} \sin z \right]$$

$$= e^{-z}\left[\frac{z}{61}(5+6D_1)\sin z+(2D_1-6)\frac{1}{11+60D_1}\sin z\right]$$

$$= e^{-z}\left[\frac{z}{61}(5\sin z+6\cos z)+(2D_1-6)(60D_1-11)\frac{1}{3600D_1^2-121}\sin z\right]$$

$$= e^{-z}\left[\frac{z}{61}(5\sin z+6\cos z)+\frac{120D_1^2-382D_1+66}{-3600-121}\sin z\right]$$

$$= e^{-z}\left[\frac{z}{61}(5\sin z+6\cos z)+\frac{120(-\sin z)-382\cos z+66\sin z}{-3721}\right]$$

$$= \frac{1}{x}\left[\frac{\log x}{61}(5\sin\log x+6\cos\log x)+\frac{54\sin\log x+382\cos\log x}{3721}\right]$$

Solution is $\quad y=x^2[c_1\cosh(\sqrt{3}\log x)+c_2\sinh(\sqrt{3}\log x)]+1/(6x)$

$$+\frac{1}{x}\left[\frac{\log x}{61}(5\sin\log x+6\cos\log x)+\frac{54\sin\log x+382\cos\log x}{3721}\right]$$

Ex. 18. *Reduce* $2x^2y\,(d^2y/dx^2)+4y^2=x^2\,(dy/dx)^2+2xy\,(dy/dx)$ *to homogeneous form by making the substitution* $y=z^2$ *and hence solve it.*

Sol. Given $\quad 2x^2y(d^2y/dx^2)+4y^2=x^2(dy/dx)^2+2xy(dy/dx)$ ... (1)

and $\quad y=z^2.$ ... (2)

From (2), $\quad \dfrac{dy}{dx}=2z\dfrac{dz}{dx}\quad$ and $\quad \dfrac{d^2y}{dx^2}=2\left(\dfrac{dz}{dx}\right)+2z\dfrac{d^2z}{dx^2}.$ ... (3)

Using (2) and (3), (1) reduces to

$$2x^2z^2\left\{2\left(\frac{dz}{dx}\right)^2+2z\frac{d^2z}{dx^2}\right\}+4z^4=x^2\cdot 4z^2\left(\frac{dz}{dx}\right)^2+2xz^2\cdot 2z\frac{dz}{dx}$$

or $\quad x^2(d^2z/dx^2)-x\,(dz/dx)+z=0 \quad$ or $\quad (x^2D^2-xD+1)\,z=0$... (4)

Let $\quad x=e^t\quad$ so that $\quad t=\log x\quad$ and $\quad$ let $D_1=d/dt.\quad$ Also, here $\quad D\equiv d/dx.$ ... (5)

Then $\quad xD=D_1 \quad$ and $\quad x^2D^2=D_1(D_1-1)$.... (6)

Using (5) and (6), (4) reduces to $[D_1(D_1-1)-D_1+1]\,z=0$ or $(D_1^2-2D_1+1)\,z=0$... (7)

The auxiliary equation of (7) is $\quad (D_1-1)^2=0,\quad$ giving $\quad D_1=1,1.$

$\therefore$ The solution of (4) is $\quad z=(c_1+c_2t)\,e^t=(c_1+c_2\log x)\,x$, by (5)

From (2), $\;y=z^2=(c_1+c_2\log x)^2x^2$, giving the required solution, where c_1 and c_2 are arbitrary constants.

Ex. 19. *Solve* $(d^3y/dx^3)-(4/x)\times(d^2y/dx^2)+(5/x^2)\times(dy/dx)-(2y/x^3)=1$

[Delhi Maths (Prog) 2007), Delhi Maths 2000, Delhi Maths (G) 2000]

Sol. Re-writing, the given equation is $\quad x^3\,(d^3y/dx^3)-4x^2\,(d^2y/dx^2)+5x\,(dy/dx)-2y=x^3$

or $\quad (x^3D^3-4x^2D^2+5xD-2)\,y=x^3,\quad$ where $\quad D\equiv d/dx$ ... (1)

Let $\quad x=e^z,\quad \log x=z\quad$ and $\quad D_1=d/dz$ ... (2)

Then $\quad xD=D_1,\quad x^2D^2=D_1(D_1-1)\quad$ and $\quad x^3D^3=D_1(D_1-1)(D_1-2).$

$\therefore$ (1) gives $\quad \{D_1(D_1-1)(D_1-2)-4D_1(D_1-1)+5D_1-2\}y=e^{3z}$

or $\quad (D_1^3-7D_1^2+11D_1-2)\,y=e^{3z}$ ... (3)

The auxiliary equation for (3) is $D_1^3 - 7D_1^2 + 11D_1 - 2 = 0$

or $(D_1 - 2)(D_1^2 - 5D_1 + 1) = 0$ giving $D_1 = 2,\ (5 \pm \sqrt{21})/2$

C.F. $= C_1 e^{2z} + C_2 e^{(5+\sqrt{21})z/2} + C_3 e^{(5-\sqrt{21})z/2}$, c_1 and c_2 being arbitrary constants.

or C.F. $= C_1 x^2 + C_2 x^{(5+\sqrt{21})/2} + C_3 x^{(5-\sqrt{21})/2}$, using (2)

$$\text{P.I.} = \frac{1}{D_1^3 - 7D_1^2 + 11D_1 - 2} e^{3z} = \frac{1}{3^3 - (7\times 3^2) + (11\times 3) - 2} e^{3z} = -\frac{1}{5}x^3$$

The required solution is $y = C_1 x^2 + C_2 x^{(5+\sqrt{21})/2} + x^{(5-\sqrt{21})/2} - x^3/5.$

Ex. 20. *Solve* $x^3 (d^3y/dx^3) + 2x (dy/dx) - 2y = x^2 \log x + 3x$.

[Delhi Maths (H) 2001; Delhi Maths (G) 2005; Delhi B.Sc. (Prog) II 2011]

Sol. Re-writing, the given equation is $(x^3D^3 + 2xD - 2)y = x^2 \log x + 3x$, $D \equiv d/dx$. ... (1)

Let $x = e^z$, $\log x = z$ and $D_1 \equiv d/dz$ ... (2)

Then $xD = D_1$ and $x^3D^3 = D_1(D_1 - 1)(D_1 - 2)$ and so (1) becomes

$\{D_1(D_1 - 1)(D_1 - 2) + 2D_1 - 2\}\, y = ze^{2z} + 3e^z$ or $(D_1^3 - 3D_1^2 + 4D_1 - 2)\, y = ze^{2z} + 3e^z$

Its auxiliary equation is $D_1^3 - 3D_1^2 + 4D_1 - 2 = 0,$ giving

$(D_1 - 1)(D_1^2 - 2D_1 + 2) = 0$ so that $D_1 = 1,\ 1 \pm i$

$\therefore$ C.F. $= C_1 e^z + e^z (C_2 \cos z + C_3 \sin z) = x(C_1 + C_2 \cos \log x + C_3 \sin \log x)$,

where C_1, C_2 and C_3 are arbitrary constants.

P.I. corresponding to ze^{2z}

$$= \frac{1}{D_1^3 - 3D_1^2 + 4D_1 - 2} ze^{2z} = e^{2z} \frac{1}{(D_1+2)^3 - 3(D_1+2)^2 + 4(D_1+2) - 2} z$$

$$= e^{2z} \frac{1}{D_1^3 + 3D_1^2 + 4D + 2} z = \frac{e^{2z}}{2}\left[1 + \frac{D_1^3 + 3D_1^2 + 4D_1}{2}\right]^{-1} z$$

$= (e^{2z}/2)\{1 - (1/2) \times (D_1^3 + 3D_1^2 + 4D_1) + \ldots\} z = (e^{2z}/2)\{z - (1/2) \times 4\} = (x^2/2)(\log x - 2)$

P.I. corresponding to $3e^z$

$$= 3\frac{1}{(D_1 - 1)(D_1^2 - 2D_1 + 2)} e^z = 3\frac{1}{D_1 - 1}\ \frac{1}{1^2 - 2\cdot 1 + 2} e^z = 3\frac{z}{1!} e^z = 3ze^z = 3x \log x$$

$\therefore$ Solution is $y = x(C_1 + C_2 \cos \log x + C_2 \sin \log x) + (x^2/2)(\log x - 2) + 3x \log x.$

Ex. 21. *Find the values of* λ *for which all solutions of* $x^2 (d^2y/dx^2) - 3x (dy/dx) - \lambda y = 0$ *tend to zero* $x \to \infty$. **[I.A.S. 2002]**

Sol. Given $(x^2D^2 - 3xD + \lambda)y = 0,$ where $D \equiv d/dx.$ ... (1)

Let $x = e^z$ so that $z = \log x.$ Also let $D_1 \equiv d/dz.$ ... (2)

Then $xD = D_1$ and $x^2D^2 = D_1(D_1 - 1)$ and so (1) reduces to

$\{D_1(D_1 - 1) + 3D_1 - 1\}\, y = 0$ or $(D_1^2 + 2D_1 - \lambda)\, y = 0$ ... (3)

Its auxiliary equations is $D_1^2 + 2D_1 - \lambda = 0,$ giving

$D_1 = \{-2 \pm (4 + 4\lambda)^{1/2}\}/2 = -1 \pm (1 + \lambda)^{1/2},$ where $\lambda \geq -1.$ ... (4)

Hence the required general solution is given by

$y = C_1 e^{-[1-(1+\lambda)^{1/2}]z} + C_2 e^{-\{1+(1+\lambda)^{1/2}\}z} \equiv C_1 x^{-[1-(1+\lambda)^{1/2}]} + C_2 x^{-\{1+(1+\lambda)^{1/2}\}}$, using (2) ... (5)

Since all solutions (4) must tend to zero as $x \to \infty$, λ must be chosen to satisfy the following condition $\quad 1 - (1+\lambda)^{1/2} > 0 \quad$ or $\quad (1+\lambda)^{1/2} < 1 \quad$ so that $\quad \lambda < 0 \quad$ ... (6)

(4) and (6) $\Rightarrow\ -1 \le \lambda < 0$, which are the required values of λ.

Exercise 6(A)

Solve the following differential equations, taking $D \equiv d/dx$

1. $(x^2D^2 - 4xD + 6)\,y = x^4$. **[Meerut 1998]** **Ans.** $y = c_1x^2 + c_2x^3 + x^4/2$

2. $(x^2D^2 + xD - 1)\,y = x^m$, **[G.N.D.U. Amritser 2010]** **Ans.** $y = c_1x + c_2x^{-1} + \{1/(m^2-1)\}\,x^m$

3. $x^2\,(d^2y/dx^2) + 2x\,(dy/dx) - 20\,y = (x+1)^2$. **[Delhi Maths (Prog) 2007; [Delhi Maths (G) 2000]**

Ans. $y = c_1x^4 + c_2x^{-5} - (x^2/14) - (x/9) - (1/20)$

4. $(x^3D^3 - x^2D^2 + 2xD - 2)\,y = x^3 + 3x$ **Ans.** $y = x\,(c_1 + c_2\log x) + c_3x^2 + (x^3/4) - (3x/2)\times(\log x)^2$

5. $x^2\,(d^2y/dx^2) + 2\,(dy/dx) = 6\,x$ **[Kanpur 1997]** **Ans.** $y = c_1 + c_2x^{-1} + x^2$

6. $(x^2D^2 + xD - 4)\,y = x^2$ **[I.A.S. 1992 ; Kanpur 1995]** **Ans.** $y = c_1x^2 + c_2x^{-2} + (x^2/4)\times\log x$

7. (*a*) $(x^2D^2 + 7xD + 5)\,y = x^5$. **Ans.** $y = c_1x^{-1} + c_2x^{-5} + (x^5/60)$

(*b*) $2x^2\,(d^2y/dx^2) + 3x\,(dy/dx) - 3y = x^3$ **[I.A.S. 2007]** **Ans.** $y = c_1x + c_2x^{-3/2} + (x^3/8)$

8. $(x^2D^2 - 2xD - 4)\,y = x^4$ **[Guwahati 2007]** **Ans.** $y = c_1x^4 + c_2x^{-1} + (x^4/5)\times\log x$

9. $x^3\,(d^3y/dx^3) + 2x^2\,(d^2y/dx^2) + 3x\,(dy/dx) - 3y = x^2 + x$ **[Agra 1995]**

Ans. $y = c_1x + c_2\cos(\sqrt{3}\log x) + c_3\sin(\sqrt{3}\log x) + (x^2/7) + (x/4)\times\log x$

10. $x^2\,(d^2y/dx^2) - x\,(dy/dx) + y = 2\log x$ **[Agra 1995, Osmania 2005]**

Ans. $y = x\,(c_1 + c_2\log x) + 2\log x + 4$

11. $(x^2D^2 - xD + 3)\,y = x^2\log x$ **[Gulberga 2005; Guwahati 1996; Kanpur 2006, 2007 Guwahati 2007; Garhwal 2010; Lucknow 1998; Madras 2005; Meerut 1994; Rohilkhand 1996;]**

Ans. $y = x\{c_1\cos(\sqrt{2}\log x) + c_2\sin(\sqrt{2}\log x\} + (x^2/9)(3\log x - 2)$

12. $d^2y/dx^2 - (6/x^2)\,y = x\log x$. **[Delhi 2008]** **Ans.** $y = c_1x^{-2} + c_2x^{-3} + (x^2/50)\,\{5\,(\log x)^2 - 2\log x\}$

13. (***a***) $(x^3D^3 + 3x^2D^2 + xD + 1)\,y = \log x$

Ans. $y = c_1x^{-1} + x^{1/2}[c_2\cos\{(\sqrt{3}/2)\log x\} + c_3\sin\{(\sqrt{3}/2)\log x\}] + \log x$

(***b***) $(x^3D^3 + 3x^2D^2 + xD + 1)\,y = x\log x$ **[I.A.S. 1996]**

Ans. $y = c_1x^{-1} + x^{1/2}[c_2\cos\{(\sqrt{3}/2)\log x)\} + c_3\sin\{(\sqrt{3}/2)\log x\}] + (x/4)\times(2\log x - 3)$

14. $(x^3D^3 - 3x^2D^2 + 6\,xD - 6)\,y = (\log x)^2$

Ans. $y = c_1x + c_2x^2 + c_3x^3 - (1/6)\,\{(\log x)^2 - (11/3)\log x + (85/18)\}$

15. $(x^2D^2 - xD - 3)\,y = x^2\,(\log x)^2$. **Ans.** $y = c_1x^3 + c_2x^{-1} - (x^3/3)\,\{(\log x)^2 + (4/3)\log x + (14/9)\}$

16. $x^2\,(d^2y/dx^2) + x\,(dy/dx) - y = x^8$. **Ans.** $y = c_1x + c_2x^{-1} + (x^8/63)$

17. $(x^2D^2 + xD - 1)\,y = 4$ **Ans.** $y = c_1x + c_2x^{-1} - 4$

18. $(x^2D^2 - 2xD + 2)\,y = 1/x$ **Ans.** $y = c_1x + c_2x^2 + (1/6x)$

19. $x^2y_2 - 2xy_1 + 2\,y = (\log x)^2 - \log x^2$ **[Delhi Maths (G) 2004]**

Ans. $y = c_1x + c_2x^2 + \{2\,(\log x)^2 + 2\log x + 1\}/4$

20. $x^2y'' + xy' - 16\,y = 0$ **[Nagpur 2002]** **Ans.** $y = c_1x^4 + c^2x^{-4}$

6.5 Definition of $\{1/f(D_1)\}\,X$, where $D_1 \equiv d/dz$, $x = e^z$ and X is a function of x

Definition. The function $[1/f(D_1)]\,X$ is defined to be that function which when operated upon by $f(D_1)$ gives X.

Note that the operator $1/f(D_1)$ can be resolved into factors which can occur in any order and can be resolved into partial fractions.

To find the value of $\dfrac{1}{D_1 - a}X$ **where** $D_1 \equiv xD \equiv x\dfrac{d}{dx}$.

Let $\dfrac{1}{D_1 - a}X = u$ or $(D_1 - a)u = X$ or $x\dfrac{du}{dx} = au + X$ or $\dfrac{du}{dx} - \dfrac{a}{x}u = \dfrac{X}{x}$

which is linear differential equation in variables u and x.

Its integrating factor $= e^{\int(-a/x)dx} = e^{-a\log x} = x^{-a}$ and solution is

$ux^{-a} = \int (X/x)\, x^{-a}\, dx$ or $u = x^{a}\int x^{-a-1} X dx.$

Thus,
$$\frac{1}{D_1 - a}X = x^{a}\int x^{-a-1}X\,dx. \qquad \textbf{(Remember)} \qquad \ldots (1)$$

Replacing a by $-a$ in the above result, we have
$$\frac{1}{D_1 + a}X = x^{-a}\int x^{a-1}X\,dx. \qquad \textbf{(Remember)} \qquad \ldots (2)$$

6.6A An alternative method of getting P.I. of homogeneous equation $f(D_1)\, y = X$, where $x = e^z$, $D_1 \equiv d/dz$ and X is any function of x

P.I. $= \dfrac{1}{f(D_1)}X$ can be obtained in either of the following two ways:

(i) The operator $1/f(D_1)$ may be resolved into fators. Then
$$\text{P.I.} = \frac{1}{f(D_1)}X = \frac{1}{D_1 - a_1}\frac{1}{D_1 - a_2}\cdots\frac{1}{D_1 - a_n}X,$$
wherein the operations indicated by factors are to be taken in succession beginning with the first on the right (making use of results of result Art 6.5).

(ii) The operator $1/f(D_1)$ may be resolved into partial fractions. Then
$$\text{P.I.} = \frac{1}{f(D_1)}X = \left[\frac{A_1}{D_1 - a_1} + \frac{A_2}{D_1 - a_2} + \ldots + \frac{A_n}{D_1 - a_n}\right]X$$
$= A_1 x^{-a_1}\int x^{-a_1-1}X\,dx + A_2 x^{-a_2}\int x^{-a_2-1}X\,dx + \ldots + A_n x^{-a_n}\int x^{-a_n-1}X\,dx$, using result (1) of Art 6.5

6.6B Particular cases

Case I. To find $\dfrac{1}{f(D_1)}x^m$, **where** $f(m) \neq 0$.

We have $D_1 x^m = x\dfrac{d}{dx}(x^m) = mx^m.$

$\therefore\; D_1^2 x^m = D_1(D_1 x^m) = D_1(mx^m) = x\dfrac{d}{dx}(mx^m) = mx\dfrac{d}{dx}x^m = m^2 x^m.$

$\therefore$ In general, $D_1^n x^m = m^n x^m.$ Hence $f(D_1)\, x^m = f(m)\, x^m.$

Operating both sides of the above result by $1/f(D_1)$, we have

$\dfrac{1}{f(D_1)}f(D_1)\,x^m = \dfrac{1}{f(D_1)}\{f(m)\,x^m\}$ or $x^m = f(m)\dfrac{1}{f(D_1)}x^m$

Dividing both sides by $f(m)$ we have

$$\text{P.I.} = \frac{1}{f(D_1)} x^m = \frac{1}{f(m)} x^m, \text{ where } f(m) \neq 0 \qquad \textbf{(Remember)} \qquad \ldots (1)$$

Thus, D_1 is replaced by m provided $f(m) \neq 0$.

Case II. If $f(m) = 0$, the above formula (1) fails. Then we shall use the following formula

$$\frac{1}{(D_1 - m)^n} x^m = \frac{(\log x)^n}{n!} x^m. \qquad \textbf{(Remember)} \qquad \ldots (2)$$

Proof. We have $$\frac{1}{(D_1 - m)} X = x^m \int x^{-m-1} X\, dx \qquad \ldots (3)$$

$$\text{L.H.S. of (2)} = \frac{1}{(D_1 - m)^{n-1}} \frac{1}{D_1 - m} x^m = \frac{1}{(D_1 - m)^{n-1}} x^m \int x^{-m-1} \cdot x^m\, dx \text{, by formula (3)}$$

$$= \frac{1}{(D_1 - m)^{n-1}} x^m \int \frac{1}{x} dx = \frac{1}{(D_1 - m)^{n-1}} (x^m \log x)$$

$$= \frac{1}{(D_1 - m)^{n-2}} \frac{1}{D_1 - m} (x^m \log x) = \frac{1}{(D_1 - m)^{n-2}} x^m \int x^{-m-1} \cdot (x^m \log x) dx, \text{ by formula (3) again}$$

$$= \frac{1}{(D_1 - m)^{n-2}} x^m \int \log x \cdot \frac{1}{x} dx = \frac{1}{(D_1 - m)^{n-2}} x^m \frac{(\log x)^2}{2!}$$

$$= \frac{1}{(D_1 - m)^{n-3}} \frac{1}{D_1 - m} \frac{x^m (\log x)^2}{2} = \frac{1}{(D_1 - m)^{n-3}} x^m \int x^{-m-1} \cdot \frac{x^m (\log x)^2}{2} dx, \text{ by formula (3) again}$$

$$= \frac{1}{(D_1 - m)^{n-3}} \frac{x^m}{2} \int (\log x)^2 \frac{1}{x} dx = \frac{1}{(D_1 - m)^{n-3}} x^m \frac{(\log x)^3}{3!}$$

Proceeding likewise and using formula (3), we finally obtain

$$\frac{1}{(D_1 - m)^n} x^m = \frac{1}{(D_1 - m)^{n-n}} x^m \frac{(\log x)^n}{n!} = \frac{(\log x)^n}{n!} x^m.$$

6.7 Solved examples based an Art. 6.5 and Art. 6.6A

Ex. 1. *Solve* $(x^2D^2 + 3xD + 1)\, y = 1/(1 - x)^2$.

[S.V. (Univ.) A.P. 1997, Purvanchal 1998, Garhwal 1996]

Sol. Put $x = e^z$ *i.e.* $z = \log x$ and take $d/dz \equiv D_1$.

Then without changing R.H.S., the given equation becomes

$[D_1(D_1 - 1) + 3D_1 + 1]\, y = 1/(1 - x)^2$ or $(D_1 + 1)^2 y = 1/(1 - x)^2$. ... (1)

The auxiliary equation for (2) is $(D_1 + 1)^2 = 0$ giving $D_1 = -1, -1$.

$\therefore$ C.F. $= (c_1 + c_2 z)\, e^{-z} = (c_1 + c_2 \log x)\, x^{-1}$, c_1, c_2 being arbitrary constants

$$\text{P.I.} = \frac{1}{D_1 + 1} \cdot \frac{1}{D_1 + 1} (1 - x)^{-2} = \frac{1}{D_1 + 1} x^{-1} \int x^{1-1} (1 - x)^{-2}\, dx$$

$$= \frac{1}{D_1 + 1} x^{-1} (1 - x)^{-1} = x^{-1} \int x^{1-1} x^{-1} (1 - x)^{-1}\, dx$$

$$= x^{-1}\int \frac{dx}{x(1-x)} = x^{-1}\int\left(\frac{1}{x}+\frac{1}{1-x}\right)dx$$

$$= x^{-1}\,[\log x - \log(1-x)] = x^{-1}\log\{x/(1-x)\}.$$

$\therefore$ The solution is $\quad y = (c_1 + c_2 \log x)\, x^{-1} + x^{-1}\log\{x/(1-x)\}$.

Ex. 2. *Solve* $x^2\,(d^2y/dx^2) + 4x\,(dy/dx) + 2y = e^x$. **[Madurai Kamraj 2008; Rajsthan 2010]**

Sol. Let $\quad x = e^z \quad$ *i.e.*, $\quad z = \log x \quad$ and $\quad d/dz \equiv D_1$. ...(1)

Then the given equation becomes

$[D_1(D_1 - 1) + 4D_1 + 2]\, y = e^x \qquad$ or $\qquad (D_1{}^2 + 3D_1 + 2)\, y = e^x$. ...(2)

$\therefore$ The auxiliary equation is $\quad D_1{}^2 + 3D_l + 2 = 0 \quad$ so that $\quad D_1 = -2, -1$. ...(3)

$\therefore$ C.F. $= c_1 e^{-2z} + c_3 e^{-z} = c_1 (e^z)^{-2} + c_2 (e^z)^{-1} = c_1 x^{-2} + c_2 x^{-1}$, as $x = e^z$

$$\text{P.I.} = \frac{1}{(D_1+2)(D_1+1)}e^x = \left[\frac{1}{D_1+1} - \frac{1}{D_1+2}\right]e^x = \frac{1}{D_1+1}e^x - \frac{1}{D_1+2}e^x$$

$$= x^{-1}\int x^{1-1}e^x\,dx - x^{-2}\int x^{2-1}e^x\,dx = x^{-1}\int e^x\,dx - x^{-2}\int xe^x\,dx$$

$$= x^{-1}e^x - x^{-2}[xe^x - \int 1\cdot e^x\,dx] = x^{-1}e^x - x^{-2}[xe^x - e^x] = x^{-2}e^x.$$

$\therefore$ Solution is $\quad y = c_1 x^{-2} + c_2 x^{-1} + x^{-2} e^x$, c_1, c_2, c_3 being arbitrary constants.

Ex. 3. *Solve* $x^2\,(d^2y/dx^2) + 4x\,(dy/dx) + 2y = x + \sin x$.

Sol. Given $\qquad (x^2D^2 + 4xD + 2)\, y = x + \sin x$. ...(1)

Let $x = e^z$ so that $z = \log x$ and let $D_1 \equiv d/dz$. Then (1) becomes ....(2)

$[D_1(D_1 - 1) + 4D_1 + 2]\, y = e^z + \sin e^z \qquad$ or $\qquad (D_1{}^2 + 3D_1 + 2)\, y = e^z + \sin e^z$ ...(3)

Auxiliary equation for (3) is $\quad D_1{}^2 + 3D_1 + 2 = 0 \quad$ gives $\quad D_1 = -2, -1$.

$\therefore$ C.F. $= c_1 e^{-2z} + c_2 e^{-z} = c_1(e^z)^{-2} + c_2(e^z)^{-1} = c_1x^{-2} + c_2x^{-1}$, using (2)

P.I. corresponding to $e^z = \dfrac{1}{D_1^2 + 3D_1 + 2}e^z = \dfrac{1}{1+3+2}e^z = \dfrac{1}{6}x.$

P.I. corresponding to $\sin e^z$ or $\sin x$ (as $e^z = x$)

$$= \frac{1}{(D_1+2)(D_1+1)}\sin x = \frac{1}{D_1+2}x^{-1}\int x^{1-1}\sin x\,dx = \frac{1}{D_1+2}x^{-1}(-\cos x)$$

$$= x^{-2}\int x^{2-1}(-x^{-1}\cos x)\,dx = -\frac{1}{x^2}\int \cos x\,dx = -\frac{\sin x}{x^2}.$$

$\therefore$ Solution is $\quad y = c_1x^{-2} + c_2x^{-1} + x/6 - (1/x^2) \times \sin x$, c_1, c_2 being arbitrary constants

Exercise 6(B)

Solve the following differential equations, taking $D \equiv d/dx$

1. $(x^2D^2 + xD - 1)\, y = x^2 e^x$ **Ans.** $y = c_1x + c_2x^{-1} + e^x(1 - x)$
2. $(x^2D^2 + xD - 1)\, y = x^2 e^{2x}$ **Ans.** $y = c_1x + c_2x^{-1} + (1/8) \times e^{2x}(2 - x^{-1})$

6.8 Solved examples based on Art. 6.5 and Art. 6.6B

Ex. 1. *Solve* $x^2(d^2y/dx^2) - 4x\,(dy/dx) + 6y = x$.

Sol. Given $\qquad (x^2D^2 - 4xD + 6)\, y = x, \qquad$ where $\qquad D \equiv d/dx$. ... (1)

Let $x = e^z$ or $z = \log x$ and $D_1 \equiv d/dz$. ... (2)

Then (1) gives, $[D_1(D_1 - 1) - 4D_1 + 6]\, y = x$ or $(D_1^2 - 5D_1 + 6)\, y = x$. ... (3)

Auxiliary equation for (3) is $D_1^2 - 5D_1 + 6 = 0$ so that $D_1 = 2, 3$.

$\therefore$ C.F. $= c_1 e^{2z} + c_2 e^{3z} = c_1 (e^z)^2 + c_2 (e^z)^3 = c_1x^2 + c_2x^3$., using (2)

$$\text{P.I.} = \frac{1}{D_1^2 - 5D_1 + 6} x = \frac{1}{1^2 - (5 \cdot 1) + 6} x = \frac{x}{2}, \qquad \text{as} \qquad \frac{1}{f(D_1)} x^m = \frac{1}{f(m)} x^m$$

$\therefore$ Solution is $y =$ C.F. + P.I. $= c_1x^2 + c_2x^3 + x/2$, c_1, c_2 being arbitrary constants

Ex. 2. *Solve* $(x^2D^2 - 3xD + 4)\, y = 2x^2$.

Sol. Let $x = e^z$ or $z = \log x$ and $D_1 \equiv d/dz$. ...(1)

Then, the given equation becomes $[D_1(D_1 - 1) - 3D_1 + 4]\, y = 2x^2$ or $(D_1 - 2)^2\, y = 2x^2$...(2)

Its auxiliary equation for (2) is $(D_1 - 2)^2 = 0$ so that $D_1 = 2, 2$

$\therefore$ C.F. $= (c_1 + c_2z)\, e^{2z} = (c_1 + c_2z)\, (e^z)^2 = (c_1 + c_2 \log x)\, x^2$, using (1)

$$\text{P.I.} = \frac{1}{(D_1 - 2)^2} 2x^2 = 2\frac{(\log x)^2}{2!} x^2, \qquad \text{as} \quad \frac{1}{(D_1 - m)^n} x^m = \frac{(\log x)^n}{n!} x^m$$

$\therefore$ Solution is $y =$ C.F. + P.I. $= (c_1 + c_1 \log x)\, x^2 + x^2 (\log x)^2$.

Ex. 3. *Solve* $(x^3D^3 + 2x^2D^2 + 2)\, y = 10\, (x + 1/x)$.

[Agra 1996, Delhi Maths (G) 1996, Delhi (Hons) 1994, 97, Rohilkhand 1997]

Sol. Given $(x^3D^3 + 2x^2D^2 + 2)\, y = 10\, (x + x^{-1})$ ...(1)

Let $x = e^z$ or $z = \log x$ and $D_1 \equiv d/dz$. ...(2)

Then (1) becomes $[D_1(D_1 - 1)(D_1 - 2) + 2D_1(D_1 - 1) + 2]\, y = 10\, (x + x^{-1})$

or $(D_1^3 - D_1^2 + 2)\, y = 10x + 10x^{-1}$ ...(3)

Auxiliary Equation *for (3) is* $D_1^3 - D_1^2 + 2 = 0$ giving $D_1 = -1, 1 \pm i$.

$\therefore$ C.F. $= c_1e^{-z} + e^z\, [c_2\cos z + c_3\sin z] = c_1x^{-1} + x\, [c_2\cos \log x + c_3\sin \log x]$, as $x = e^z$

where c_1, c_2 and c_3 are arbitrary constants.

P.I. corresponding to $10x$

$$= \frac{1}{D_1^3 - D_1^2 + 2} 10x = 10\frac{1}{1^3 - 1^2 + 2} x = 5x, \qquad \text{as} \quad \frac{1}{f(D_1)} x^m = \frac{1}{f(m)} x^m, \text{if } f(m) \neq 0$$

P.I. corresponding to $10\, x^{-1}$

$$= \frac{1}{D_1^3 - D_1^2 + 2} 10x^{-1} = 10\frac{1}{(D_1 + 1)} \; \frac{1}{(D_1^2 - 2D_1 + 2)} x^{-1} = 10\frac{1}{D_1 + 1}\,\frac{1}{5} x^{-1}$$

$$= 2\,\frac{1}{(D_1 + 1)} x^{-1} = 2\,\frac{(\log x)^1}{1!}\; x^{-1} = 2x^{-1} \log x, \qquad \text{as} \quad \frac{1}{(D_1 - m)^n} x^m = \frac{(\log x)^n}{n!} x^m.$$

$\therefore$ Require *soluion is* $y = c_1x^{-1} + x[c_2\cos \log x + c_3\sin \log x] + 5x + 2x^{-1}\log x$.

Ex. 4. *Solve* $x^2(d^2y/dx^2) + x\,(dy/dx) - 4y = x^2$.

Sol. Given $(x^2D^2 + xD - 4)\, y = x^2$, where $D \equiv d/dx$ ... (1)

Let $x = e^z$ or $z = \log x$ and $D_1 \equiv d/dz$. Then (1) becomes

$[D_1(D_1 - 1) + D_1 - 4]\, y = x^2$ or $(D_1^2 - 4)\, y = x^2$. ... (2)

Its auxiliary equation is $D_1^2 - 4 = 0$ so that $D_1 = 2, -2$.

$\therefore$ C.F. $= c_1 e^{2z} + c_2 e^{-2z} = c_1 (e^z)^2 + c_2 (e^z)^{-2} = c_1x^2 + c_2x^{-2}$, as $x = e^z$

$$\text{P.I.} = \frac{1}{D_1^2 - 4} x^2 = \frac{1}{D_1 - 2} \frac{1}{D_1 + 2} x^2 = \frac{1}{D_1 - 2} \frac{x^2}{2+2} \qquad \text{as} \qquad \frac{1}{f(D_1)} x^m = \frac{1}{f(m)} x^m$$

$$= \frac{1}{4} \frac{1}{D_1 - 2} x^2 = \frac{1}{4} \frac{(\log x)^1}{1!} x^2, \qquad \text{as} \qquad \frac{1}{(D_1 - m)^n} x^m = \frac{(\log x)^n}{n!} x^m$$

$\therefore$ the required solution is $\qquad y = \text{C.F.} + \text{P.I.} = c_1 x^2 + c_2 x^{-2} + (1/4)\, x^2 \log x.$

Exercise 6(C)

Solve the following differential equations:

1. $3x^2 (d^2y/dx^2) + x\,(dy/dx) + y = x$ **Ans.** $y = x^{1/3}[c_1 \cos\{(\sqrt{2}/3)\log x\} + c_2 \sin\{(\sqrt{2}/3)\log x\}] + (x/2)$

2. $x^4 (d^3y/dx^3) + 2x^3 (d^2y/dx^2) - x^2 (dy/dx) + xy = 1$ **Ans.** $y = (c_1 + c_2 \log x)\, x + c_3 x^{-1} + (1/4x) \times \log x$

6.9 Equations reducible to homogeneous linear form. Legendre's linear equation

A linear differential equation of the form

$$[a_0(a + bx)^n D^n + a_1 (a + bx)^{n-1} D^{n-1} + \dots + a_{n-1}(a + bx) D + a_n] y = X, \qquad \dots (1)$$

where $a, b, a_0, a_1, a_2, \dots, a_n$ are constants and X is either a constant or a function of x only, is called *Legendre's linear equation.* Note that the index of $(a + bx)$ and the order of derivative is same in each term of such equations.

Method of solution. To solve (1), introduce a new variable z such that

$$a + bx = e^z \qquad \text{or} \qquad \log(a + bx) = z \qquad \dots (2)$$

Let $\qquad D_1 \equiv d/dz \qquad$ and $\qquad D \equiv d/dx \qquad$ .. (3)

From (2), we have $\qquad dz/dx = b/(a + bx). \qquad$ ... (4)

$$\therefore \qquad \frac{dy}{dx} = \frac{dy}{dz} \cdot \frac{dz}{dx} = \frac{b}{a + bx} \frac{dy}{dz}, \text{ using (4)} \qquad \dots (5)$$

$$\Rightarrow \quad (a + bx)(dy/dx) = b\,(dy/dz) \qquad \Rightarrow \qquad (a + bx) D = bD_1 \quad \dots (6)$$

$$\text{Again, } \frac{d^2 y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{b}{a + bx} \frac{dy}{dz}\right), \text{ using (5)}$$

$$= -\frac{b^2}{(a + bx)^2} \frac{dy}{dz} + \frac{b}{a + bx} \frac{d}{dx}\left(\frac{dy}{dz}\right) = -\frac{b^2}{(a + bx)^2} \frac{dy}{dz} + \frac{b}{a + bx} \frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx}$$

$$= -\frac{b^2}{(a + bx)^2} \frac{dy}{dz} + \frac{b}{a + bx} \frac{d^2 y}{dz^2} \frac{b}{a + bx}, \text{ using (4)}$$

$$\Rightarrow \qquad (a + bx)^2 \frac{d^2 y}{dx^2} = b^2 \left(\frac{d^2 y}{dz^2} - \frac{dy}{dz}\right) \quad \Rightarrow \quad (a + bx)^2 D^2 y = b^2 (D_1^2 - D_1)\, y.$$

$$\therefore \qquad (a + bx)^2 D^2 = b^2 D_1 (D_1 - 1). \qquad \dots (7)$$

Similarly, $\qquad (a + bx)^3 D^3 = b^3 D_1 (D_1 - 1)(D_1 - 2). \qquad$ ... (8)

and so on. Proceeding likewise, we finally have

$$(a + bx)^n D^n = b^n D_1 (D_1 - 1)(D_1 - 2) \dots (D_1 - n + 1).$$

Substituting the above values of $(a + bx)^n D^n, \dots, (a + bx)^2 D^2, (a + bx) D$ etc in (1), we have

$$[a_0 b^n D_1(D_1 - 1)(D_1 - 2)..(D_1 - n + 1) + ... + a_{n-1} bD + a_n]\, y = Z, \qquad ...(9)$$

which is a linear differential equation with constant coefficients in variables y and z; Z is now function of z only and is obtained by using transformation (2) by replacing x by $(e^z - a)/b$. Let a solution of (1) be $y = F(z)$. Then, the required solution is given by

$$y = F[\log (a + bx)], \qquad \text{as} \qquad z = \log (a + bx)$$

6.10 Working rule for solving Legendre's linear equation, *i.e.*

$$\{a_0(a + bx)^n D^n + a_1(a + bx)^{n-1} D^{n-1} + ... + a_{n-1}(a + bx) D + a_n\}\, y = X, \qquad ...(1)$$

where $D \equiv d/dx$, a, b, a_0, a_1, ..., a_n are constants and X is either a constant or a function of x only.

Step I. Introduce a new variable z such that $\quad a + bx = e^z \quad$ or $\quad \log (a + bx) = z \quad ...(2)$

Step II. Assume that $D_1 \equiv d/dz$. Then, we have

$(a + bx) D = bD_1$, $(a + bx)^2 D^2 = b^2 D_1(D_1 - 1)$, $\quad (a + bx)^3 D^3 = b^3 D_1(D_1 - 1)(D_1 - 2)$ and so on.

As a particular case, when $b = 1$, we have

$(a + x) D = D_1$, $\quad (a + x)^2 D^2 = D_1(D_1 - 1)$, $\quad (a + x)^3 D^3 = D_1(D_1 - 1)(D_1 - 2)$ and so on.

Then (1) reduces to $\quad f(D_1)\, y = Z$, where Z is now function of z only. $\qquad ...(3)$

Step III. We now use the methods of Chapter 5 to solve (3) and get a solution of the form

$$y = F(z) \qquad ...(4)$$

Using (2), the required solution is given by $\quad y = F\{\log (a + bx)\} \qquad ...(5)$

6.11 Solved examples based on Art. 6.10

Ex. 1(*a*). *Solve* $(1 + x)^2 (d^2y/dx^2) + (1 + x)(dy/dx) + y = 4 \cos \log (1 + x)$.

[Andhra 1997, Delhi Maths (H) 1993, Delhi Maths (G) 2005, Meerut 1997, Purvanchal 1999]

Sol. Given $\quad [(1 + x)^2 D^2 + (1 + x) D + 1]\, y = 4 \cos \log (1 + x), \quad D \equiv d/dx. \quad ...(1)$

Let $\quad 1 + x = e^z \quad$ or $\quad \log (1 + x) = z. \quad$ Also, let $\quad D_1 \equiv d/dz. \quad ...(2)$

Then, we have $\quad (1 + x) D = D_1$, $(1 + x)^2 D^2 = D_1(D_1 - 1)$ and hence (1) gives

$[D_1(D_1 - 1) + D_1 + 1] = 4 \cos z \quad$ or $\quad (D_1^2 + 1)\, y = 4 \cos z. \quad ...(3)$

Its auxiliary equation is $\quad D_1^2 + 1 = 0 \quad$ so that $\quad D_1 = 0 \pm i$.

$\therefore\;$ C.F. $= e^{0.z}(c_1 \cos z + c_2 \sin z) = c_1 \cos \log(1 + x) + c_2 \sin \log(1 + x)$, using (2)

where, c_1 and c_2 are arbitrary constants.

P.I. $= \dfrac{1}{D_1^2 + 1} 4 \cos z = \text{R.P. of } \dfrac{1}{D_1^2 + 1} 4e^{iz}$, where R.P. stands for real part

$$= \text{R.P. of } \frac{1}{D_1^2 + 1} e^{iz} \cdot 4 = \text{R.P. of } e^{iz} \frac{1}{(D_1 + i)^2 + 1} \cdot 4$$

$$= \text{R.P. of } e^{iz} \frac{1}{D_1^2 + 2Di} 4 = \text{R.P. of } e^{iz} \frac{1}{2D_1 i(1 + D_1/2i)} \cdot 4$$

$$= \text{R.P. of } \frac{e^{iz}}{2i} \frac{1}{D_1}\left(1 + \frac{D_1}{2i}\right)^{-1} 4 = \text{R.P. of } \frac{e^{iz}}{2i} \frac{1}{D_1}\left(1 - \frac{D_1}{2i} + ...\right) 4$$

$= \text{R.P. of } e^{iz}(1/2i) \times (4z) = \text{R.P. of } (-2iz) \times (\cos z + i \sin z)$, as $1/i = -i$

$= 2z \sin z = 2 \log (1 + x) \sin \log (1 + x)$ as $z = \log (1 + x)$

$\therefore\;$ Solution is $y = c_1 \cos \log (1 + x) + c_2 \sin \log (1 + x) + 2 \log (1 + x) \sin \log (1 + x)$.

Ex. 1(*b*) Solve $\{(x + 1)^4 D^3 + 2(x + 1)^3 D^2 - (x + 1)^2 D + (x + 1)\}y = 1/(x + 1)$, $D \equiv d/dx$. **[I.A.S. 2005]**

Sol. Dividing both sides by $(x + 1)$, the given equation reduces to

$$\{(x+1)^3 D^3 + 2(x+1)^2 D^2 - (x+1)\, D + 1\}\, y = (1+x)^{-2} \qquad \text{...(1)}$$

Let $1 + x = e^z$ or $\log (1 + x) = z$, Also, let $D_1 \equiv d/dz$ ...(2)

Then, we have $xD = D_1$, $x^2 D^2 = D_1(D_1 - 1)$ and $x^3 D^3 = D_1 (D_1 - 1)(D_1 - 2)$ ...(3)

Using (2) and (3), (1) reduces to

$$\{D_1(D_1-1)(D_1-2) + 2D_1\,(D_1-1) - D_1 + 1\}\; y = e^{-2z}$$

or $(D_1^3 - D_1^2 - D_1 + 1)y = e^{-2z}$ or $(D_1 - 1)^2 (D_1 + 1)y = e^{-2z}$ ...(4)

Here auxiliary equation for (4) is $(D_1 - 1)^2 (D_1 + 1) = 0$ giving $D_1 = 1, 1, -1$

$\therefore$ C.F. $= (c_1 + c_2 z)\, e^z + c_3 e^{-z}$, c_1, c_2 and c_3 being arbitrary constants

and P.I. $= \dfrac{1}{(D_1-1)^2 (D_1+1)} e^{-2z} = \dfrac{1}{(-2-1)^2(-2+1)} e^{-2z} = -\dfrac{1}{9} e^{-2z}$

$\therefore$ The required solution is $y = (c_1 + c_2 z)\, e^z + c_3 (e^z)^{-1} - (1/9) \times (e^z)^{-2}$

or $y = \{c_1 + c_2 \log (1+x)\}(1+x) + c_3 (1+x)^{-1} - (1/9) \times (1+x)^{-2}$, using (2)

Ex. 2(*a*). *Solve* $(x + a)^2 (d^2y/dx^2) - 4(x + a)(dy/dx) + 6y = x$. **[Kanpur 2011]**

Sol. Given $[(x + a)^2 D^2 - 4(x + a)\, D + 6]\, y = x$. ... (1)

Let $x + a = e^z$ or $\log (x + a) = z$. Also, let $D_1 \equiv d/dz$. ... (2)

Then, $(a + x)\, D = D_1$, $(a + x)^2 D^2 = D_1(D_1 - 1)$ and (1) hence gives

$[D_1(D_1 - 1) - 4D_1 + 6]\, y = e^z - a$ or $(D_1^2 - 5D_1 + 6)\, y = e^z - a$. ... (3)

Its auxiliary equation is $D_1^2 - 5D_1 + 6 = 0$ so that $D_1 = 2, 3$.

$\therefore$ C.F. $= c_1 e^{2z} + c_2 e^{3z} = c_1 (e^z)^2 + c_2 (e^z)^3 = c_1 (x + a)^2 + c_2 (x + a)^3$.

$$\text{P.I.} = \frac{1}{D_1^2 - 5D_1 + 6}(e^z - a e^{0.z}) = \frac{1}{D_1^2 - 5D_1 + 6} e^z - a \frac{1}{D_1^2 - 5D_1 + 6} e^{0.z}$$

$$= \frac{1}{1^2 - (5 \cdot 1) + 6} e^z - a \frac{1}{0^2 - (5 \cdot 0) + 6} e^{0.z} = \frac{x+a}{2} - \frac{a}{6} = \frac{3x + 2a}{6}$$

$\therefore$ Solution is $y = c_1 (x + a)^2 + c_2 (x + a)^3 + (3x + 2a)/6$.

Ex. 2(*b*). *Solve* $(x + 3)^2 y^2 - 4(x + 3)\, y_1 + 6y = x$. **[Delhi Maths (G) 1998]**

Sol. Given $[(x + 3)^2 D^2 - 4(x + 3)\, D + 6]\, y = x$, $D \equiv d/dx$ ...(1)

which is the same as equation (1) of Ex. 2(*a*). Here $a = 3$. Proceeding as before, the solution is

$$y = c_1 (x + 3)^2 + c_2 (x + 3)^3 + (x + 2)/2, \; c_1, c_2 \text{ being arbitrary constants.}$$

Ex. 3. *Solve* $(x + 1)^2 (d^2y/dx^2) - 4(x + 1)(dy/dx) + 6y = 6(x + 1)$, **[Delhi Maths (G) 2006]**

Sol. Given $[(x + 1)^2 D^2 - 4\,(x + 1)\, D + 6]\, y = 6\,(x + 1)$, where $D \equiv d/dx$ ... (1)

Let $x + 1 = e^z$ or $z = \log (x + 1)$. Also, $D_1 \equiv d/dz$... (2)

Then, $(x + 1)\, D = D_1$ and $(x + 1)^2 D^2 = D_1(D_1 - 1)$. So (1) gives

$[D_1(D_1 - 1) - 4D_1 + 6\}\; y = 6\, e^z$ or $(D_1^2 - 5D_1 + 6)\, y = 6\, e^z$ ... (3)

Auxiliary equation of (3) is $D_1^2 - 5D_1 + 6 = 0$ giving $D_1 = 2, 3$.

$\therefore$ C.F. $= c_1 e^{2z} + c_2 e^{3z} = c_1 (e^z)^2 + c_2 (e^z)^3 = c_1(x+1)^2 + c_2(x+1)^3$

$$\text{P.I.} = \frac{1}{D_1^2 - 5D_1 + 6} 6e^z = 6\frac{1}{1^2 - (5\times 1) + 6} e^z = 3e^z = 3(x+1), \text{ as } x + 1 = e^z$$

$\therefore$ Solution is $y = c_1(x+1)^2 + c_2(x+1)^3 + 3(x+1)$, c_1, c_2 being arbitrary constants.

Ex. 4. *Solve* $(x+1)^2 (d^2y/dx^2) + (x+1)(dy/dx) = (2x+3)(2x+4)$

[Delhi Maths (G) 2006; Nagpur 1993; Rajasthan 1995]

Sol. Let $D \equiv d/dx$. Given equation reduces to $\{(x+1)^2 D^2 + (x+1) D\} y = 4x^2 + 14x + 12$ .. (1)

Let $x + 1 = e^z$ or $z = \log(x+1)$ Also let $D_1 \equiv d/dz$ ... (2)

Then, $(x+1) D = D_1$ and $(x+1)^2 D^2 = D_1(D_1 - 1)$. Hence, (1) gives

$\{D_1(D_1 - 1) + D_1\} y = 4(e^z - 1)^2 + 14(e^z - 1) + 12$ or $D_1^2 y = 4e^{2z} + 6e^z + 2$ ... (3)

Auxiliary equation of (5) is $D_1^2 = 0$ giving $D_1 = 0, 0$.

$\therefore$ C.F. $= (c_1 + c_2 z) e^{0.z} = c_1 + c_2 z = c_1 + c_2 \log(x+1)$, using (2)

$$\text{P.I.} = \frac{1}{D_1^2}(4e^{2z} + 6e^z + 2) = \frac{1}{D_1}(2e^{2z} + 6e^z + 2z) = e^{2z} + 6e^z + z^2 = (e^z)^2 + 6e^z + z^2$$

$= (1+x)^2 + 6(1+x) + [\log(1+x)]^2$, using (2)

Thus, P.I. $= x^2 + 8x + 7 + [\log(1+x)]^2$ and solution of (1) is

$y = c_1 + c_2 \log(1+x) + x^2 + 8x + 7 + [\log(1+x)]^2 = c_1' + c_2 \log(1+x) + x^2 + 8x + [\log(1+x)]^2$,

where c_1' $(= c_1 + 7)$ and c_2 are arbitrary constants.

Ex. 5. *Solve* $16(x+1)^4 (d^4y/dx^4) + 96(x+1)^3 (d^3y/dx^3) + 104(x+1)^2 (d^2y/dx^2) + 8(x+1)(dy/dx) + y = x^2 + 4x + 3$.

Sol. Let $D \equiv d/dx$. Then the given equation reduces to

$$\{16(x+1)^4 D^4 + 96(x+1)^3 D^3 + 104(x+1)^2 D^2 + 8(x+1)D + 1\} y = x^2 + 4x + 3 \quad \ldots (1)$$

Let $x + 1 = e^z$ or $z = \log(x+1)$. Also, let $D_1 \equiv d/dz$ ... (2)

Then, $(x+1) D = D_1$, $(x+1)^2 D^2 = D_1(D_1 - 1)$, $(x+1)^3 D^3 = D_1(D_1 - 1)(D_1 - 2)$ and $(x+1)^4 D^4 = D_1(D_1 - 1)(D_1 - 2)(D_1 - 3)$. Hence, (1) reduces to

$$\{16D_1(D_1 - 1)(D_1 - 2)(D_1 - 3) + 96D_1(D_1 - 1)(D_1 - 2) + 104D_1(D_1 - 1) + 8D_1 + 1\} y = (e^z - 1)^2 + 4(e^z - 1) + 3$$

or $(16D_1^4 - 8D_1^2 + 1) y = e^{2z} + 2e^z$, on simplifying ...(3)

Auxiliary equation of (3) is $16D_1^4 - 8D_1^2 - 1 = 0$ or $(4D_1^2 - 1)^2 = 0$,

giving $D_1 = 1/2, 1/2, -1/2, -1/2$.

$\therefore$ C.F. $= (c_1 + c_2 z)e^{z/2} + (c_3 + c_4 z)e^{-z/2} = (c_1 + c_2 z)(e^z)^{1/2} + (c_3 + c_4 z)(e^z)^{-1/2}$

$= [c_1 + c_2 \log(1+x)](1+x)^{1/2} + [c_3 + c_4 \log(1+x)](1+x)^{-1/2}$, using (2)

$$\text{P.I. corresponding to } e^{2z} = \frac{1}{(4D_1^2 - 1)^2} e^{2z} = \frac{1}{(16-1)^2} e^{2z} = \frac{1}{225}(e^z)^2 = \frac{(1+x)^2}{225}$$

$$\text{P.I. corresponding to } 2e^z = \frac{1}{(4D_1^2 - 1)^2} 2e^z = 2\frac{1}{\{(4\times 1^2) - 1\}^2} e^z = \frac{2}{9}(1+x), \text{ using (2)}$$

Hence the required solution is $y =$ C.F. + P.I., *i.e.*

$y = \{c_1 + c_2 \log(1+x)\}(1+x)^{1/2} + \{c_3 + c_4 \log(1+x)\}(1+x)^{-1/2} + (1/225) \times (1+x)^2 + (2/9) \times (1+x)$, c_1, c_2, c_3 and c_4 being arbitrary constants.

Ex. 6. *Solve* $[(3x+2)^2 D^2 + 3(3x+2) D - 36]\, y = 3x^2 + 4x + 1,\ D \equiv d/dx.$

[Agra 1994, Allahabad 1996, Delhi Maths (H.) 1997, Indore 1993]

Sol. Given $\qquad [(3x+2)^2 D^2 + 3(3x+2) D - 36]\, y = 3x^2 + 4x + 1. \qquad \text{... (1)}$

Let $\quad 3x + 2 = e^z \quad$ or $\quad \log(3x+2) = z. \quad$ Also, let $\quad D_1 \equiv d/dz.$ .. (2)

$\therefore\quad (2+3x) D = 3D_1,\ (2+3x)^2 D^2 = 3^2 D_1(D_1 - 1)$. Then (1) gives

$$[3^2 D_1(D_1-1) + 3\cdot 3D_1 - 36]\, y = 3\ \{(e^z-2)/3\}^2 + 4\{(e^z-2)/3\} + 1$$

$[\because (2) \Rightarrow 3x = e^z - 2 \Rightarrow x = (e^z-2)/3]$

or $\qquad 9[D_1(D_1-1) + D_1 - 4] = (1/3) \times (e^{2z} - 4e^z + 4) + (4/3) \times (e^z - 2) + 1$

or $\quad 9(D_1^{\,2} - 4) = (1/3) \times e^{2z} - (1/3) \qquad$ or $\qquad (D_1^{\,2} - 4) = (1/27) \times e^{2z} - (1/27)$

Here auxiliary equation is $\qquad D_1^{\,2} - 4 = 0 \qquad$ so that $\qquad D_1 = 2, -2.$

$\therefore\quad$ C.F. $= c_1 e^{2z} + c_2 e^{-2z} = c_1(e^z)^2 + c_2(e^z)^{-2} = c_1(3x+2)^2 + c_2(3x+2)^{-2}.$

P.I. corresponding to $\dfrac{1}{27}\, e^{2z} = \dfrac{1}{27}\dfrac{1}{D_1^2 - 4} e^{2z} = \dfrac{1}{27}\dfrac{1}{D_1 - 2}\dfrac{1}{D_1 + 2} e^{2z} = \dfrac{1}{27}\dfrac{1}{D_1 - 2}\dfrac{1}{2+2} e^{2z}$

$$= \frac{1}{108}\frac{1}{(D_1 - 2)} e^{2z} = \frac{1}{108}\frac{z}{1!} e^{2z}, \quad \text{as} \quad \frac{1}{(D_1 - a)^n} e^{az} = \frac{z^n}{n!} e^{az}$$

$= (1/108) \times z(e^z)^2 = (1/108) \times (3x+2)^2 \log(3x+2)$, using (2)

P.I. corresponding to $-\dfrac{1}{27} = -\dfrac{1}{27}\dfrac{1}{D_1^2 - 4}\cdot 1 = -\dfrac{1}{27}\dfrac{1}{D_1^2 - 4} e^{0z} = -\dfrac{1}{27}\dfrac{1}{0^2 - 4} e^{0z} = \dfrac{1}{108}$

$\therefore$ Solution is $\qquad y = c_1(3x+2)^2 + c_2(3x+2)^{-2} + (1/108)\, [(3x+2)^2 \log(3x+2) + 1]$

where c_1 and c_2 are arbitrary constants.

Ex. 7(*a*). *Solve* $[(1+2x)^2 D^2 - 6(1+2x) D + 16]\, y = 8(1+2x)^2.$

(*b*) *Solve* $[(1+2x)^2 (d^2y/dx^2) - 6(1+2x)(dy/dx) + 16y = 8(1+2x)^2$ *given that* $y(0) = 0,$ $y'(0) = 2.$ **[I.A.S. 1997]**

Sol. (*a*) Given $\qquad [(1+2x)^2 D^2 - 6(1+2x) D + 16]\, y = 8(1+2x)^2. \qquad \text{... (1)}$

Let $\quad 1 + 2x = e^z \quad$ or $\quad \log(1+2x) = z. \quad$ Also, let $D_1 \equiv d/dz.$ ... (2)

Then $(1+2x) D = 2D_1,\ (1+2x)^2 D^2 = 2^2 D_1(D_1 - 1)$ and so (1) becomes

$[2^2 D_1(D_1-1) - 6\cdot 2\, D_1 + 16]\, y = 8\, e^{2z} \qquad$ or $\qquad (D_1 - 2)^2 y = 2e^{2z}.$ ... (3)

Its auxiliary equation is $\qquad (D_1 - 2)^2 = 0 \qquad$ so that $\qquad D_1 = 2, 2.$

$\therefore\quad$ C.F. $= (c_1 + c_2 z)\, e^{2z} = (c_1 + c_2 z)(e^z)^2 = [c_1 + c_2 \log(1+2x)](1+2x)^2,$

where c_1 and c_2 are arbitrary constants.

$$\text{P.I.} = \frac{1}{(D_1 - 2)^2} 2e^{2z} = 2\frac{z^2}{2!} e^{2z}, \qquad \text{as } \frac{1}{(D_1 - a)^n} e^{az} = \frac{z^n}{n!} e^{az}$$

$= z^2 (e^z)^2 = [\log(1+2x)]^2 (1+2x)^2$, using (2)

$\therefore\quad$ Solution is $\qquad y = [c_1 + c_2 \log(1+2x)]\,(1+2x)^2 + [\log(1+2x)]^2 (1+2x)^2$

or $\qquad y = (1+2x)^2\, [c_1 + c_2 \log(1+x) + \{\log(1+2x)\}^2]. \qquad \text{... (4)}$

(*b*) Proceed as in part (*a*) upto equation (4) to get

$$y(x) = (1+2x)^2\, [c_1 + c_2 \log(1+2x) + \{\log(1+2x)\}^2]. \qquad \text{... (5)}$$

Differentiating both sides of (5) w.r.t. 'x' we have

$$y'(x) = (2 \times 2)(1+2x)^2 [c_1 + c_2 \log(1+2x) + \{\log(1+2x)\}^2]$$
$$+ (1+2x)^2 \left[\frac{2c_2}{1+2x} + \frac{2\log(1+2x)}{1+2x} \times 2\right] \qquad \dots (6)$$

Putting $x = 0$ in (5) and noting that $y(0) = 0$ (given), we get

$0 = 1 \times [c_1 + (c_2 \times 0) + 0^2]$ so that $c_1 = 0$.

Putting $x = 0$ in (6) and noting that $y'(0) = 2$ (given), we get

$2 = 4[c_1 + (c_2 \times 0) + 0^2] + 1 \times [2c_2 + (2 \times 0 \times 2)]$ so that $c_2 = 1$, as $c_1 = 0$

Putting the above values of c_1 and c_2 in (5), the required solution is

$$y(x) = (1+2x)^2 \log(1+2x)[1 + \log(1+2x)].$$

Exercise 6(D)

Solve the following differential equations, taking $D \equiv d/dx$

1. $(x+1)^2 (d^2y/dx^2) - 3(x+1)(dy/dx) + 4y = x$. **[Delhi Maths. 1999, 2004]**

Ans. $y = (x+1)^2 \{c_1 + c_2 \log(x+1)\} + x + (3/4)$

2. $(x+1)^2 y_2 - 3(x+1) y_1 + 4y = x^2$ **[Kanpur 2010; Delhi Maths (G) 2005]**

Ans. $y = (x+1)^2 \{c_1 + c_2 \log(x+1)\} + (1/2) \times (x+1)^2 \{\log(x+1)\}^2 - 2x - (7/4)$

3. $\{(5+2x)^2 D^2 - 6(5+2x) D + 8\}y = 0$ **[Delhi Maths 2001]**

Ans. $y = (5+2x)^2 \{c_1 (5+2x)^{\sqrt{2}} + c_2 (5+2x)^{-\sqrt{2}}\}$

4. $\{(2x-1)^3 D^3 + (2x-1) D - 2\}\, y = 0$ **[Delhi 2008]**

Ans. $y = c_1(2x-1) + (2x-1)[c_2 \cosh\{(\sqrt{3}/2)\log(2x-1)\} + c_3 \sinh\{(\sqrt{3}/2)\log(2x-1)\}]$

5. $(1+x)^2 (d^2y/dx^2) + (1+x)(dy/dx) + y = 2 \sin \log(1+x)$

Ans. $y = c_1 \cos \log(1+x) + c_2 \sin \log(1+x) - \log(1+x) \cos \log(1+x)$

6. $(3x+2)^2 y_2 + 5(3x+2) y_1 - 3y = x^2 + x + 1$ **[G.N.D.U. Amritsar 2010]**

Ans. $y = c_1 (3x+2)^{1/3} + c_2 (3x+2)^{-1} + (36x^2 + 3x - 434)/620$

MISCELLANEOUS PROBLEMS ON CHAPTER 6

Ex. 1. *Let $D \equiv d/dx$. Then, the value of $\{1/(xD+1)\}x^{-1}$ is*

(*a*) $\log x$ (*b*) $(\log x)/x$ (*c*) $(\log x)/x^2$ (*d*) $(\log x)/x^3$

[GATE 2009]

Sol. Ans. (*b*). Let $x = e^z$ or $z = \log x$. Then $xD = D_1$, where $D_1 \equiv d/dz$. Then,

$$\frac{1}{xD+1}x^{-1} = \frac{1}{D_1+1}e^{-z} = e^{-z}\frac{1}{(D_1-1)+1}1 = e^{-z}\frac{1}{D_1}1 = e^{-z}z = \frac{\log x}{x}$$

Ex.2. *Let* $y(x)$ be the solution of the initial value problem $x^2y'' + xy' + y = x$, $y(1) = y'(1) = 1$. Then, the value of $y(e^{\pi/2})$ is

(*a*) $(1-e^{\pi/2})/2$ (*b*) $(1+e^{\pi/2})/2$ (*c*) $1/2+\pi/4$ (*d*) $1/2-\pi/4$ **[GATE 2010]**

Sol. Ans. (*b*). Re-writing the given equation, $(x^2D^2 + xD + 1)y = x$, where $D \equiv d/dx$ ... (1)

Let $x = e^z$ or $z = \log x$ and let $D_1 \equiv d/dz$. ... (2)

Then, $xD = D_1$ and $x^2D^2 = D_1(D_1 - 1)$. Hence, (1) may be re-written as

$\{D_1(D_1-1) + D_1 + 1\}y = e^z$ or $(D_1^2 + 1)\, y = e^z$ ... (3)

whose auxiliary equation is $D_1^2 + 1 = 0$ giving $D_1 = \pm i.$

Hence, C.F. $= c_1 \cos z + c_2 \sin z = c_1 \cos \log x + c_2 \sin \log x$

and P.I. $= \dfrac{1}{D_1^2+1} e^z = \dfrac{1}{1^2+1} e^z = \dfrac{1}{2} x$, *as* $e^z = x$

Hence the general solution of the given differential equation is given by

$y(x) = c_1 \cos \log x + c_2 \sin \log x + x/2$, c_1 and c_2 being arbitrary constants ... (4)

From (4), $y'(x) = -(c_1/x) \times \sin \log x + (c_2/2) \times \cos \log x + 1/2$ ... (5)

Putting x = 1 in (4) and (5) and using the given intial conditions, we get

$c_1 + 1/2 = 1$ and $c_2 + 1/2 = 1$ so that $c_1 = c_2 = 1/2.$

Hence, from (4) $y(x) = (\cos \log x + \sin \log z + x)/2$ ... (6)

Now from (6), $y(e^{\pi/2}) = \{\cos(\pi/2) + \sin(\pi/2) + e^{\pi/2}\}/2 = (1 + e^{\pi/2})/2$

Ex. 3. *A particular solution of* $4x^2 (d^2y/dx^2) + 8x (dy/dx) + y = 4/\sqrt{x}$ *is*

(a) $1/2\sqrt{x}$ *(b)* $(\log x)/2\sqrt{x}$

(c) $(\log x)^2/2\sqrt{x}$ *(d)* $\{(\log x)\sqrt{x}\}/2$ **[GATE 2006]**

Sol. Ans. (*c*). Let $d/dx \equiv D$. Then, the given equation becomes

$$(4x^2D^2 + 8xD + 1)\, y = 4/\sqrt{x} \qquad \dots (1)$$

Let $x = e^z$ or $z = \log x$. Also, let $D_1 \equiv d/dz$. Then, $xD = D_1$ and $x^2D^2 = D_1(D_1 - 1)$ and hence (1) reduces to $\{4D_1(D_1 - 1) + 8D_1 + 1\}\, y = 4e^{-z/2}$ or $(2D_1 + 1)^2\, y = 4\, e^{-3/2}$

$\therefore$ Required particular integral

$$= \frac{1}{(2D_1+1)^2} 4e^{-z/2} = \frac{1}{(D_1+1/2)^2} e^{-z/2} = \frac{z^2}{2!} e^{-z/2} = \frac{1}{2} z^2 (e^z)^{-1} = \frac{(\log x)^2}{2\sqrt{x}}, \text{ as } x = e^z$$

Ex. 4. Solve $(x^2D^2 - 5xD + 8)y = 2x^3$ **[Delhi Maths (Prog.) 2009]**

Sol. Do like Ex. 3, page 6.3 **Ans.** $y = c_1x^2 + c_2x^4 - 2x^3$

Ex. 5. Show that the substitution $x = e^z$ transforms the differential equation $x^n (d^ny/dx^n) + p_1 x^{n-1} (d^{n-1}y/dx^{n-1}) + \dots + p_{n-1} x (dy/dx) + p_n y = R$ into a linear differential equation with constant coefficients (where $p_0, p_1, \dots p_n$ are real constants). **[Mumbai 2010]**

Hints. Refer Art. 6.2, Page 6.1

7

Method of Variation of Parameters

7.1 Method of variation of parameters for solving $dy/dx + P(x)\, y = Q(x)$

Consider a first order linear differential equation

$$dy/dx + Py = Q, \qquad i.e., \qquad y_1 + Py = Q, \qquad \text{where} \qquad y_1 = dy/dx \qquad \dots (1)$$

and P and Q are functions of x or constants. Suppose that the general solution of

$$y_1 + Py = 0, \qquad \dots (2)$$

be given by $$y = au, \qquad \dots (3)$$

where a is an arbitrary constant and u is a function of x. Since u must be a solution of (2), we have

$$u_1 + Pu = 0 \qquad \text{where} \qquad u_1 = du/dx. \qquad \dots (4)$$

When $Q \neq 0$, (3) cannot be the general solution of (1).

Now assume that $$y = Au \qquad \dots (5)$$

is the general solution of (1), where A is no longer constant but function of x to be so chosen that (1) is satisfied.

From (3) and (5), we note that the form of y is the same for two equations (1) and (2), but the constant which occurs in the former case is changed in the latter into a function of the independent variable x. For this reason, the present method is known as *variation of parameters*.

Differentiating (5) w.r.t. 'x', we have

$$y_1 = A_1 u + A u_1, \qquad \text{where} \qquad A_1 = dA/dx \qquad \dots (6)$$

Putting the values of y and y_1 given by (5) and (6) in (1), we get

$$A_1 u + A u_1 + PAu = Q \qquad \text{or} \qquad A_1 u + A(u_1 + Pu) = Q$$

or $$A_1 u = Q, \text{ using } (4) \qquad \dots (7)$$

From (7), $\quad A_1 = Q/u \quad$ or $\quad dA/dx = Q/u \quad$ or $\quad dA = (Q/u)\, dx$

Integrating, $\quad A = \int (Q/u)\, dx + c,\;$ where c is an arbitrary constant $\qquad \dots (8)$

Using (8) in (5), the general solution of (1) is given by

$$y = u(x)\{c + \int (Q/u) dx\} \qquad \text{or} \qquad y = c\, u(x) + u(x) \int (Q/u) dx \qquad \dots (9)$$

7.2 Working rule for solving $y_1 + Py = Q$ by variation of parameters, where P and Q are functions of x or constants. [Meerut 2009]

Step 1. Re-write the given equation in the standard form

$$(dy/dx) + Py = Q, \qquad i.e., \qquad y_1 + Py = Q, \qquad \dots (1)$$

in which the coefficient of y_1 must be unity.

Step 2. Consider $$y_1 + Py = 0, \qquad \dots (2)$$

which is obtained by taking $Q = 0$ in (1). Solve (2) by methods of chapter 2. Let the general solution of (2) be $\quad y = au$, a being an arbitrary constant $\qquad \dots (3)$

Step 3. General solution of (1) is given by

$$y = c\,u\,(x) + u(x)\int (Q/u)\,dx, \text{ where } c \text{ is an arbitrary constant.} \qquad \dots (4)$$

An illustrative solved example: *Solve (x + 4) (dy/dx) + 3y = 3 by the method of variation of parameters.* **[Calicut 2004]**

Sol. Re-writing the given equation in standard form $dy/dx + \{3/(x + 4)\}\, y = 3/(x + 4)$... (1)

Compare (1) with $dy/dx + Py = Q$, here $Q = 3/(x + 4) = 3(x + 4)^{-1}$ ... (2)

Now with $Q = 0$, we consider the following equation

$$\frac{dy}{dx} + \frac{3}{x+4}\,y = 0 \qquad \text{or} \qquad \frac{dy}{y} + \frac{3}{x+y}\,dx = 0$$

Integrating, $\log y + 3\,\log\,(x + 4) = \log a$ or $y = a\,(x + 4)^{-3}$,
where a in an arbitrary constant.

Hence, we choose $u = (x + 4)^{-3}$ ... (3)

Then the required general solution is given by

$$y = c\,u(x) + u(x)\int \frac{Q}{u}\,dx \qquad \text{or} \qquad y = c(x+4)^{-3} + (x+4)^{-3}\int \frac{3(x+4)^{-1}}{(x+4)^{-3}}\,dx$$

or $y = c\,(x + 4)^{-3} + 3(x + 4)^{-3}\int (x+4)^2 dx$ or $y = c\,(x + 4)^{-3} + 3(x + 4)^{-3} \times \{(x+4)^3/3\}$

$\therefore$ Required solution is $y = c\,(x + 4)^{-3} + 1$, c being an arbitrary constant.

7.3 Method of variation of parameters for solving $d^2y/dx^2 + P(dy/dx) + Qy = R$, where P, Q and R are functions of x or constants

Consider a second order linear differential equation

$$d^2y/dx^2 + P\,(dy/dx) + Q\,y = R, \qquad i.e., \qquad y_2 + P\,y_1 + Q\,y = R \qquad \dots(1)$$

where P, Q and R are functions of x or constants.

Suppose that C.F. of (1), *i.e.*, the general solution of $y_2 + Py_1 + Q\,y = 0$ ... (2)
be given by $y = au + bv$, ... (3)
where a and b are arbitrary constants and u and v are functions of x. Again, u and v must be solutions of (2). Hence, we have

$$u_2 + Pu_1 + Qu = 0 \qquad \text{and} \qquad v_2 + Pv_1 + Qv = 0 \qquad \dots (4)$$

When $R \neq 0$, (3) cannot be the general solution of (1).

Now assume that $y = Au + Bv$ ... (5)
is the general solution of (1), where A and B are no constants but functions of x to be so chosen that (1) is satisfied.

From (3) and (5), we note that the form of y is the same for the two equations (1) and (2), but the constants which occur in the former case are changed in the latter into functions of the independent variable x. For this reason, the present method is known as *variation of parameters.*

In order to find A and B, we take $uA_1 + vB_1 = 0$ ... (6)

Now differentiating (5) and using (6), we get $y_1 = Au_1 + Bv_1$ ... (7)

Differentiating (7), we get $y_2 = Au_2 + A_1u_1 + Bv_2 + B_1v_1$ ... (8)

Putting values of y, y_1 and y_2 given by (5), (7) and (8) in (1), we get

$$Au_2 + A_1u_1 + Bv_2 + B_1v_1 + P(Au_1 + Bv_1) + Au + Bv = R$$

or $$A(u_2 + Pu_1 + Qu) + B\,(v_2 + Pv_1 + Ov) + A_1u_1 + B_1v_1 = R$$

or $$A_1u_1 + B_1v_1 = R, \text{ using (4)} \qquad \dots (9)$$

Solving (6) and (9) for A_1 and B_1, we get

$$A_1 = \frac{dA}{dx} = \frac{-vR}{uv_1 - u_1 v} = -\frac{vR}{W} \quad \text{and} \quad \frac{dB}{dx} = \frac{uR}{uv_1 - u_1 v} = \frac{uR}{W}, \qquad \text{... (10)}$$

where $$W = \text{Wronskian of } u \text{ and } v = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = uv_1 - u_1 v \neq 0 \qquad \text{... (11)}$$

Here W is non-zero because u and v are linearly independent.

Integrating (10), $A = f(x) + C_1$ and $B = g(x) + C_2$ ... (12)

where $$f(x) = -\int \frac{vR}{W} dx \quad \text{and} \quad g(x) = \int \frac{uR}{W} dx \qquad \text{... (13)}$$

and C_1 and C_2 are arbitrary constants.

Using (12) in (5), the general solution of (1) is given by

$$y = u\{f(x) + C_1\} + v\{g(x) + C_2\} = C_1 u + C_2 v + u f(x) + v g(x)$$

7.4A. Working rule for solving $y_2 + Py_1 + Q y = R$ by variation of parameters, where P, Q and R are functions of x or constants **[Mumbai 2010]**

Step 1. Re-write the given equation as $y_2 + Py_1 + Q y = R$, ... (1)

Step 2. Consider $y_2 + Py_1 + Q y = 0$ ... (2)

which is obtained by taking $R = 0$ in (1). Solve (2) by methods of chapters 4 and 5 as the case may be. Let the general solution of (2) *i.e.*, C.F. of (1) be

$$y = C_1 u + C_2 v,\ C_1,\ C_2 \text{ being arbitrary constants} \qquad \text{... (3)}$$

Step 3. General solution of (1) is $y = \text{C.F.} + \text{P.I.}$ ... (4)

where $\text{C.F.} = C_1 u + C_2 v$, C_1, C_2 being arbitrary constants ... (5)

and $\text{P.I.} = u f(x) + v g(x)$ ... (6)

where $$f(x) = -\int \frac{vR}{W} dx \quad \text{and} \quad g(x) = \int \frac{uR}{W} dx, \qquad \text{... (7)}$$

where $$W = \text{Wronskian of } u \text{ and } v = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = uv_1 - u_1 v \qquad \text{... (8)}$$

Important Note. Variation of parameters is an elegant but somewhat artificial method for finding the complete primitive of a linear equation whose complementary function is known. This method is very effective to find particular integral and it can be applied where the earlier methods cease to be applicable.

7.5A. Solved examples based on Art. 7.4A

Ex. 1. *Apply the method of variation of parameters to solve*

(*i*) $y_2 + n^2 y = \sec nx$ **[Agra 2006, Madras 2005; Gulbarga 2005; Delhi Maths 99, 2004; Bundelhand 2001; Himachal 2004; Kanpur 2007; Meerut 2008, 09; Madurai 2001; Rohilkhand**

2004; Ravishanka 2002, 2004;Purvanchal 2007, I.A.S. 99 Venkenkateshwar 2003]

(*ii*) $y_2 + y = \sec x$ **[Mysore 2004;Delhi Maths (P) 2001; 02, Delhi Maths (G) 2002]**

(*iii*) $y_2 + 4y = \sec 2x$

(*iv*) $y_2 + 9y = \sec 3x$ **[Meerut 2007; Delhi Maths (H) 1999]**

Sol. (*i*) Given $y_2 + n^2 y = \sec nx$ ... (1)

Comparing (1) with $y_2 + Py_1 + Q y = R$, we have $R = \sec nx$

Consider $y_2 + n^2 y = 0$ or $(D^2 + n^2)\, y = 0$, where $D \equiv d/dx$ ... (2)

Auxiliary equation of (2) is $D^2 + n^2 = 0$ so that $D = \pm\, in$.

C.F. of (1) = $C_1 \cos nx + C_2 \sin nx$, C_1 and C_2 being arbitrary constants ... (3)

Let $u = \cos nx$, $v = \sin nx$ Also, here $R = \sec nx$ ... (4)

Here $$W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} \cos nx & \sin nx \\ -n \sin nx & n \cos nx \end{vmatrix} = n \neq 0 \qquad \text{... (5)}$$

Then, P.I. of (1) = $u\, f(x) + v\, g(x)$ ... (6)

where $$f(x) = -\int \frac{vR}{W}\, dx = -\int \frac{\sin nx \sec nx}{n}\, dx = \frac{1}{n^2} \log \cos nx, \text{ by (4) and (5)}$$

and $$g(x) = \int \frac{uR}{W}\, dx = \int \frac{\cos nx \;\; \sec nx}{n}\, dx = \frac{x}{n}, \textit{ by } (4) \text{ and } (5)$$

$\therefore$ P.I. of (1) = $(\cos nx) \times (1/n^2) \log \cos nx + (\sin nx) \times (x/n)$, by (6)

Hence the general solution of (1) is y = C.F. + P.I.

i.e., $y = C_1 \cos nx + C_2 \sin nx + (1/n^2) \times \cos nx \log \cos nx + (x/n) \times \sin nx$

(*ii*) Compare it with part (*i*). Here $n = 1$. Now do as in part (*i*).

The required solution is $y = C_1 \cos x + C_2 \sin x + \cos x \log \cos x + x \sin x$.

(*iii*) Proceed as in part (*i*). Note that here $n = 2$.

Ans. $y = C_1 \cos 2x + C_2 \sin 2x + (1/4) \times \cos 2x \log \cos 2x + (x/2) \times \sin 2x$

(*iv*) Proceed as in part (*i*). Note that here $n = 3$.

Ans. $y = C_1 \cos 3x + C_2 \sin 3x + (1/9) \times \cos 3x \log \cos 3x + (x/3) \times \sin 3x$

Ex. 2. *Apply the method of variation of parameters to solve*

(*i*) $y_2 + a^2 y = \textit{cosec } ax$ **[Meerut 2004, 10; Kakatiay 2003; S.V. University A.P. 199, Rajsthan 2003, 01]**

(*ii*) $y_2 + y = \textit{cosec } x$ **[Meerut 2007, 11; Bangalore 1996, Delhi Maths (G) 1998, 2003] Nagpur 2002, Delhi Maths (H) 1997; Guwahati 1996; Bilaspur 2000, 04 Indore 2001, 07]**

(*iii*) $y_2 + 9y = \textit{cosec } 3x$ **[Delhi Maths (Pass) 2004]**

Sol. (*i*) Given $y_2 + a^2 y = \text{cosec } ax$ ... (1)

Comparing (1) with $y_2 + Py_1 + Q\,y = R$, we have $R = \text{cosec } ax$

Consider $y_2 + a^2 y = 0$ or $(D^2 + a^2)\, y = 0$, $D \equiv d/dx$ ... (2)

Auxiliary equation of (2) is $D^2 + a^2 = 0$ so that $D = \pm\, ai$

$\therefore$ C.F. of (1) = $C_1 \cos ax + C_2 \sin ax$, C_1 and C_2 being arbitrary constants ... (3)

Let $u = \cos ax$, $v = \sin ax$. Also, here $R = \text{cosec } ax$ ... (4)

Here $$W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a \sin ax & a \cos ax \end{vmatrix} = a \neq 0 \qquad \text{... (5)}$$

Then, P.I. of (1) = $u\, f(x) + v\, g(x)$, ... (6)

where $$f(x) = -\int \frac{vR}{W}\, dx = -\int \frac{\sin ax \operatorname{cosec} ax}{a}\, dx = -\frac{x}{a}, \text{ by (4) and (5)}$$

and $$g(x) = \int \frac{uR}{W}\, dx = \int \frac{\cos ax \operatorname{cosec} ax}{a}\, dx = (1/a^2) \times \log \sin ax, \text{ by (4) and (5)}$$

$\therefore$ P.I. of (1) = $(\cos ax) \times (-\,x/a) + (\sin ax) \times (1/a^2) \times \log \sin ax$, by (6)

Hence the general solution of (1) is $y = \text{C.F.} + \text{P.I.}$

i.e., $y = C_1 \cos ax + C_2 \sin ax - (x/a) \times \cos ax + (1/a^2) \times \sin ax \log \sin ax$

(*ii*) Proceed as in part (*i*). Note that here $a = 1$.

Ans. $y = C_1 \cos x + C_2 \sin x - x \cos x + \sin x \log \sin x$

(*iii*) Proceed as in part (*i*). Note that have $a = 3$

Ans. $y = C_1 \cos 3x + C_2 \sin 3x - (x/3) \times \cos 3x + (1/9) \times \sin 3x \log \sin 3x$

Ex. 3. *Apply the method of variation of parameters to solve*

(*i*) $y_2 + a^2 y = \tan ax$ **[Osmania 2004]**

(*ii*) $y_2 + 4y = 4 \tan 2x$ **[Himiachal 2002, 03;Garhwal 2005, Delhi Maths (G) 1997, 2001; Rohilkhanad 2001; Delhi B.A. (Prog) II 2010; Kanpur 2002, 08; Nagpur 1996]**

(*iii*) $y_2 + y = \tan x$ **[Delhi B.A (Prog.) H 2007, 08, 11; Delhi B.A (G) 2000; Bangalore 2005; Delhi B.Sc. (Prog.) II 2008; Delhi Maths (H.) 1996, 2002]**

(*iv*) $y_2 + a^2 y = \cot ax$ **[Delhi Maths (G) 2005]**

(*v*) $y_2 + 4y = \cot 2x$

Sol. (*i*) Given $y_2 + a^2 y = \tan ax$ ... (1)

Comparing (1) with $y_2 + Py_1 + Q\,y = Q$, we have $R = \tan ax$

Consider $y_2 + a^2 y = 0$ or $(D^2 + a^2)\, y = 0$, where $D \equiv d/dx$ ... (2)

Auxiliary equation of (2) is $D^2 + a^2 = 0$ so that $D = \pm\, ia$

$\therefore$ C.F. of (1) $= c_1 \cos ax + c_2 \sin ax$, c_1 and c_2 being arbitrary constants ... (3)

Let $u = \cos ax$, $v = \sin ax$. Also, here $R = \tan ax$ ... (4)

Here $$W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a \sin ax & a \cos ax \end{vmatrix} = a \neq 0 \qquad \text{... (5)}$$

Then P.I. of (1) $= u\, f(x) + v\, g(x)$, ... (6)

where $$f(x) = -\int \frac{vR}{W}\, dx = -\int \frac{\sin ax \tan ax}{a} dx = -\frac{1}{a}\int \frac{1 - \cos^2 ax}{\cos ax} dx \text{, using (4) and (5)}$$

$$= -\frac{1}{a}\int (\sec ax - \cos ax) dx = -\frac{1}{a}\left[\frac{1}{a} \log(\sec ax + \tan ax) - \frac{\sin ax}{a}\right]$$

$= (1/a^2) \times \{\sin ax - \log(\sec ax + \tan ax)\}$

and $$g(x) = \int \frac{uR}{W}\, dx = \int \frac{\cos ax \tan ax}{a}\, dx = \frac{1}{a}\int \sin ax\, dx = -\frac{1}{a^2} \cos ax, \text{using (4) and (5)}$$

Using (6), P.I. of (1) $= \cos ax \times (1/a^2)\, \{\sin ax - \log(\sec ax + \tan ax)\} + \sin ax \times (-1/a^2) \cos ax$
$= -(1/a^2) \times \cos ax \log(\sec ax + \tan ax)$

Hence the general solution of (1) is $y = \text{C.F.} + \text{P.I.}$

i.e., $y = c_1 \cos ax + c_2 \sin ax - (1/a^2) \times \cos ax \log(\sec ax + \tan ax)$

(*ii*) Given $y_2 + 4y = 4 \tan 2x$ ... (1)

Comparing (1) with $y_2 + Py_1 + Q\,y = R$, here $R = 4 \tan 2x$

Consider $y_2 + 4y = 0$ or $(D^2 + 4)\, y = 0$, $D \equiv d/dx$ ... (2)

Auxiliary equation of (2) is $D^2 + 4 = 0$ so that $D = \pm\, 2i$.

C.F. of (1) $= C_1 \cos 2x + C_2 \sin 2x$, C_1 and C_2 being arbitrary constants ... (3)

Let $u = \cos 2x$, $v = \sin 2x$. Also, here $R = 4 \tan 2x$ ... (4)

Here $$W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2 \neq 0 \qquad \text{... (5)}$$

Then, $$\text{P.I. of (1)} = uf(x) + vg(x), \qquad \text{... (6)}$$

where $$f(x) = -\int \frac{vR}{W} dx = -4\int \frac{\sin 2x \tan 2x}{2} dx = -2\int \frac{1-\cos^2 2x}{\cos 2x} dx, \text{ using (4) and (5)}$$

$$= 2\int (\cos 2x - \sec 2x)\, dx = \sin 2x - \log(\sec 2x + \tan 2x)$$

and $$g(x) = \int \frac{uR}{W} dx = 4\int \frac{\cos 2x \tan 2x}{2} dx = -\cos 2x, \text{ by (4) and (5)}$$

$\therefore$ P.I of (1) = $(\cos 2x)\{\sin 2x - \log(\sec 2x + \tan 2x)\} + (\sin 2x)(-\cos 2x)$, by (6)

or $$\text{P.I. of (1)} = -\cos 2x \log(\sec 2x + \tan 2x)$$

Hence the general solution of (1) is $y = \text{C.F.} + \text{P.I.}$

i.e., $$y = C_1 \cos 2x + C_2 \sin 2x - \cos 2x \log(\sec 2x + \tan 2x).$$

(*iii*) Proceed as in part (*i*) by taking $a = 1$. The general solution is

$$y = C_1 \cos x + C_2 \sin x - \cos x \log(\sec x + \tan x)$$

(*iv*) Given $$y_2 + a^2 y = \cot ax \qquad \text{... (1)}$$

Comparing (1) with $y_2 + Py_1 + Qy = R$, we have $R = \cot ax$

Consider $y_2 + a^2 y = 0$ or $(D^2 + a^2)y = 0$, $D \equiv d/dx$ ...(2)

Auxiliary equation of (2) is $D^2 + a^2 = 0$ so that $D = \pm ia$.

$\therefore$ C.F. of (1) = $c_1 \cos x + c_2 \sin ax$, c_1 and c_2 being arbitrary constants ... (3)

Let $u = \cos ax$, $v = \sin ax$. Also, here $R = \cot ax$ ...(4)

Here $$W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix} = a \neq 0 \qquad \text{... (5)}$$

Then $$\text{P.I. of (1)} = u\,f(x) + v\,g(x), \qquad \text{... (6)}$$

where $$f(x) = -\int \frac{vR}{W} dx = -\int \frac{\sin ax \cot ax}{a} dx = -\frac{1}{a}\int \cos ax\, dx = -\frac{\sin ax}{a^2}, \text{ using (4) and (5)}$$

and $$g(x) = \int \frac{uR}{W} dx = \int \frac{\cos ax \cot ax}{a} dx = \frac{1}{a}\int \frac{1-\sin^2 ax}{\sin ax} dx, \text{ by (4) and (5)}$$

$$= \frac{1}{a}\int (\operatorname{cosec} ax - \sin ax)\, dx = \frac{1}{a^2}\left(\log \tan \frac{ax}{2} + \cos ax\right)$$

Using (6), P.I. of (1) $= \cos ax \times (-1/a^2) \times \sin ax + \sin ax \times (1/a^2) \times \{\log \tan (ax/2) + \cos ax\}$

$= (1/a^2) \times \log \tan (ax/2)$

Hence, the general solution of (1) is $y = \text{C.F.} + \text{P.I.}$

i.e., $$y = c_1 \cos ax + c_2 \sin ax + (1/a^2) \times \log \tan (ax/2)$$

(*v*) Proceed like part (*iv*) with $a = 2$. The solution is

$$y = c_1 \cos 2x + c_2 \sin 2x + (1/4) \times \log \tan x$$

Ex. 4. *Apply the method of variation of parameters to solve*

(*i*) $y_2 - y = 2/(1 + e^x)$ **[Delhi Maths (H) 2001; Delhi Maths (G) 1999; Rohilkhand 2002; Allahabad 2000, 05; Kanpur 2007; Nagpur 2001, 06; Bangalore 2004**

(*ii*) $y_2 - 3y_1 + 2y = e^x/(1 + e^x)$ **Delhi B.Sc. (Prog) 2009]**

(*iii*) $y_2 - 4y_1 + 3y = e^x/(1 + e^x)$.

Sol. (*i*) Given $y_2 - y = 2/(1 + e^x)$ ... (1)

Comparing (1) with $y_2 + Py_1 + 6\,y = R,$ here $R = 2/(1 + e^x)$

Consider $y_2 - y = 0$ or $(D^2 - 1)\,y = 0,$ $D \equiv d/dx$ ... (2)

Its auxiliary equation is $D^2 - 1 = 0$ so that $D = \pm\, 1.$

$\therefore$ C.F. of (1) $= C_1 e^x + C_2 e^{-x}$, C_1 and C_2 being arbitrary constants ... (3)

Let $u = e^x,$ $v = e^{-x}.$ Also, here $R = 2/(1 + e^x)$ ... (4)

Here $$W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{vmatrix} = -2 \neq 0 \qquad \text{... (5)}$$

Then, P.I. of (1) $= u\,f(x) + v\,g(x),$ ... (6)

where $$f(x) = -\int \frac{vR}{W}\,dx = -\int \frac{e^{-x}\{2/(1+e^x)\}}{(-2)}\,dx = \int \frac{dx}{e^x(e^x+1)}, \text{ by (4) and (5)} \qquad \text{... (7)}$$

Putting $e^x = z$ so that $e^x\,dx = dz$ or $dx = (1/z)\,dz$ in (7), we get

$$f(x) = \int \frac{dz}{z^2(1+z)} = \int \left(\frac{1}{z^2} - \frac{1}{z} + \frac{1}{1+z}\right) dz, \text{ on resolving into partial fractions.}$$

$$= -z^{-1} - \log z + \log(1 + z) = -e^{-x} - x + \log(1 + e^x), \quad \text{as } z = e^x \text{ and } \log z = x$$

and $$g(x) = \int \frac{uR}{W}\,dx = \int \frac{e^x\{(2/(1+e^x)\}}{(-2)}\,dx = -\log(1+e^x), \text{ using (4) and (5)}$$

$\therefore$ P.I. of (1) $= e^x\{-e^{-x} - x + \log(1 + e^x)\} + e^{-x}\{-\log(1 + e^x)\}$, by (6)

$= -1 - x\,e^x + (e^x - e^{-x})\log(1 + e^x)$

Hence the general solution of (1) is $y = \text{C.F.} + \text{P.I.}$

i.e., $$y = C_1 e^x + C_2 e^{-x} - 1 - x\,e^x + (e^x - e^{-x})\log(1 + e^x)$$

(*ii*) Given $y_2 - 3\,y_1 + 2\,y = e^x/(1 + e^x)$ ... (1)

Comparing (1) with $y_2 + Py_1 + Q\,y = R,$ here $R = e^x/(1 + e^x)$

Consider $y_2 - 3y_1 + 2y = 0$ or $(D^2 - 3D + 2)\,y = 0,$ $D \equiv d/dx$ ... (2)

Auxiliary equation of (2) is $D^2 - 3D + 2 = 0$ so that $D = 1, 2.$

$\therefore$ C.F. of (1) $= C_1 e^x + C_2 e^{2x}$, C_1 and C_2 being arbitrary constants ... (3)

Let $u = e^x,$ $v = e^{2x}.$ Also, here $R = e^x/(1 + e^x)$ ... (4)

Here $$W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = e^{3x} \neq 0 \qquad \text{... (5)}$$

Then, P.I. of (1) $= u\,f(x) + v\,g(x),$ ... (6)

where $$f(x) = -\int \frac{vR}{W}\,dx = -\int \frac{e^{2x}\{e^x/(1+e^x)\}}{e^{3x}}\,dx = -\int \frac{dx}{1+e^x}, \text{ by (4) and (5)}$$

$$= \int \frac{(-e^{-x})\,dx}{e^{-x}+1} = \log(e^{-x}+1)$$

and $$g(x) = \int \frac{uR}{W}\,dx = \int \frac{e^x\{e^x/(1+e^x)\}}{e^{3x}}\,dx = \int \frac{dx}{e^x(1+e^x)}, \text{ by (4) and (5)}$$

$$= \int \frac{dz}{z^2(1+z)}, \text{ putting } e^x = z \text{ and } dx = \frac{1}{z}\,dz$$

$= \int \left(\frac{1}{z^2} - \frac{1}{z} + \frac{1}{1+z} \right) dx$, resolving into partial fractions

$= -(1/z) - \log z + \log(1+z) = -e^{-x} - x + \log(1+e^x)$

$\therefore$ P.I. of (1) $= e^x \log(e^{-x}+1) + e^{2x}\{-e^{-x} - x + \log(1+e^x)\}$, by (6)

Hence the required general solution is $y = \text{C.F.} + \text{P.I.}$

i.e., $y = C_1 e^x + C_2 e^{2x} - e^x - x\, e^{2x} + e^x \log(e^{-x}+1) + e^{2x} \log(1+e^x)$.

(*iii*) Try Yourself. **Ans.** $y = C_1 e^x + C_2 e^{3x} + (1/2) \times (e^x - e^{3x}) \log(1+e^{-x}) + (1/2) \times e^{2x}$

Ex. 5.(a) *Solve by using the method of variation of parameters.* $d^2y/dx^2 - 2(dy/dx) = e^x \sin x$.

[Delhi B.Sc. (Prog) II 2010; Delhi Maths (G) 1998; Delhi Maths(H) 2008]

Sol. Given $(D^2 - 2D)\, y = e^x \sin x$, where $D \equiv d/dx$ ... (1)

Comparing (1) with $y_2 + Py_1 + Q\,y = R$, here $R = e^x \sin x$

Consider $(D^2 - 2D)\, y = 0$ ... (2)

Auxiliary equation of (2) is $D^2 - 2D = 0$ so that $D = 0, 2$.

C.F. of (1) $= C_1 + C_2 e^{2x}$, C_1 and C_2 being arbitrary constants. ... (3)

Let $u = 1$ and $v = e^{2x}$. Also, here $R = e^x \sin x$ ... (4)

Here $$W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} 1 & e^{2x} \\ 0 & 2e^{2x} \end{vmatrix} = 2e^{2x} \neq 0$$ ... (5)

Then, P.I. of (1) $= u\, f(x) + v\, g(x)$, ... (6)

where $$f(x) = -\int \frac{vR}{W} dx = -\int \frac{e^{2x} e^x \sin x}{2e^{2x}} = -\frac{1}{2} \int e^x \sin x\, dx, \text{ by (4) and (5)}$$

$$= -\frac{1}{2} \frac{e^x}{1^2+1^2} (\sin x - \cos x), \text{as} \int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx)$$

$= -(1/4) \times e^x (\sin x - \cos x)$

and $$g(x) = \int \frac{uR}{W} dx = \int \frac{e^x \sin x}{2e^{2x}} dx = \frac{1}{2} \int e^{-x} \sin dx, \text{by (4) and (5)}$$

$$= \frac{1}{2} \frac{e^{-x}}{(-1)^2+1^2} \{(-1) \sin x - \cos x\} = -\frac{e^{-x}}{4} (\sin x + \cos x)$$

$\therefore$ P.I. of (1) $= -(1/4) \times e^x(\sin x - \cos x) + e^{2x} \times (-1/4) \times e^{-x} (\sin x + \cos x)$, by (6)

$= -(1/4) \times e^x \{(\sin x - \cos x) + (\sin x + \cos x)\} = -(1/2) \times e^x \sin x$

Hence the required general solution is $y = \text{C.F} + \text{P.I.}$

i.e., $y = C_1 + C_2 e^{2x} - (1/2) \times e^x \sin x$, C_1, C_2 being arbitrary constants.

Ex. 5(*b*) Solve $(d^2y / dx^2) - 2\,(dy/dx) = e^x \cos x$ **Ans.** $y = c_1 + c_2 e^{2x} - (1/2) \times e^x \cos x$

Ex. 6. *Using method of variation of parameters, solve* $d^2y/dx^2 - 2(dy/dx) + y = x\, e^x \sin x$ *with* $y(0) = 0$ *and* $(dy/dx)_{x=0} = 0$. **[I.A.S. 2002]**

Sol. Given $(D^2 - 2D + 1)\, y = x\, e^x \sin x$, where $D \equiv d/dx$ ... (1)

Comparing (1) with $y_2 + \text{P}y_1 + Q\,y = R$, here $R = x\, e^x \sin x$

Consider $(D^2 - 2D + 1)\, y = 0$ or $(D-1)^2\, y = 0$, $D \equiv d/dx$ ... (2)

Auxiliary equation of (2) is $(D-1)^2 = 0$ so that $D = 1, 1$.

$\therefore$ C.F. of (1) $= (C_1 + C_2 x)\, e^x = C_1 e^x + C_2\, x\, e^x$, C_1 and C_2 being arbitrary constants. ... (3)

Let $u = e^x$, $v = x\,e^x$. Also, here $R = x\,e^x \sin x$ ...(4)

Here $$W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x} \neq 0 \qquad ...(5)$$

Then, P.I. of (1) $= u\,f(x) + v\,g(x)$, ... (6)

where $$f(x) = -\int \frac{vR}{W} dx = -\int \frac{xe^x(xe^x \sin x)}{e^{2x}} dx = -\int x^2 \sin x\, dx, \text{by (4) and (5)}$$

$= -\{x^2(-\cos x) - (2x)(-\sin x) + (2)(\cos x)\}$, using chain rule of integration by parts

and $$g(x) = \int \frac{uR}{W} dx = \int \frac{e^x(x\,e^x \sin x)}{e^{2x}} dx = \int x \sin x\, dx, \text{by (4) and (5)}$$

$= (x)(-\cos x) - (1)(-\sin x) = \sin x - x\cos x$

$\therefore$ P.I. of (1) $= e^x(x^2\cos x - 2x\sin x - 2\cos x) + x\,e^x(\sin x - x\cos x)$, by (6)

$= -x\,e^x \sin x - 2\,e^x \cos x$.

Hence the general solution of (1) is $y = \text{C.F.} + \text{P.I.}$

i.e., $y = C_1e^x + C_2xe^x - xe^x \sin x - 2e^x\cos x = e^x(C_1 + C_2x - x\sin x - 2\cos x)$ ... (7)

Given that $y = 0$ when $x = 0$. Hence (7) gives $0 = C_1 - 2$ or $C_1 = 2$. Putting $C_1 = 2$ in (7),

$$y = e^x(2 + C_2x - x\sin x - 2\cos x) \qquad ... (8)$$

(8) $\Rightarrow dy/dx = e^x(2 + C_2x - x\sin x - 2\cos x) + e^x\{C_2 - (\sin x + x\cos x) + 2\sin x\}$

Given that $dy/dx = 0$ when $x = 0$. So the above equation gives $0 = C_2$. Putting $C_1 = 2$ and $C_2 = 0$ in (8), the required solution is $y = e^x(2 - x\sin x - 2\cos x)$.

Ex. 7. *Apply the method of variation oof parameters to solve.*

(i) $x^2y_2 + xy_1 - y = x^2e^x$ **[Purvanchal 2007; Agra 2000, 02; Delhi Maths (Prog.) 2009; Indore 2000; Delhi Maths (H.) 2004, 06; Kanpur 2006; Calcutta 2003; Garhwal 2011; Meerut 2011; Rajsthan 2010]**

(ii) $x^2y_2 + xy_1 - y = x$, *given that the C.F. is* $C_1x + C_2x^{-1}$. **[Meerut 2008]**

(iii) $x^2y_2 - xy_1 = x^3\,e^x$.

Sol. (*i*) Re-writing the given equation, $y_2 + (1/x)\,y_1 - (1/x^2)\,y = e^x$ ... (1)

Comparing (1) with $y_2 + Py_1 + Q\,y = R$, here $R = e^x$

Consider $y_2 + (1/x)\,y_1 - (1/x^2)\,y = 0$ or $(x^2D^2 + xD - 1)\,y = 0$, $D \equiv d/dx$ ... (2)

Let $x = e^z$, $\log x = z$ and $D_1 \equiv d/dz$ ... (3)

Then $xD = D_1$ and $x^2D^2 = D_1(D_1 - 1)$. Now, (2) becomes

$\{D_1(D_1 - 1) + D_1 - 1\}\,y = 0$ or $(D_1^{\,2} - 1)\,y = 0$ ... (4)

Auxiliary equation of (4) is $D_1^{\,2} - 1 = 0$ so that $D_1 = 1, -1$

$\therefore$ C.F. of (1) $= C_1\,e^z + C_2\,e^{-z} = C_1x + C_2x^{-1}$, as $x = e^z$ ... (5)

Let $u = x$ and $v = x^{-1}$. Also, here $R = e^x$ ... (6)

Here $$W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} x & x^{-1} \\ 1 & -x^{-2} \end{vmatrix} = -2x^{-1} = -\frac{2}{x} \neq 0 \qquad ... (7)$$

Then P.I. of (1) $= u\,f(x) + v\,g(x)$,

where $$f(x) = -\int \frac{vR}{W} dx = -\int \frac{x^{-1}e^x}{(-2x^{-1})} dx = \frac{1}{2} e^x, \text{ by (6) and (7)}$$

and $$g(x) = \int \frac{uR}{W} dx = \int \frac{x\,e^x dx}{(-2x^{-1})} = -\frac{1}{2}\int x^2e^x\, dx, \text{ by (6) and (7)}$$

$= -(1/2) \times [(x^2) \times (e^x) - (2x) \times (e^x) + (2) \times (e^x)]$, using chain rule of integration by parts

$\therefore$ P.I. of (1) $= x \times (1/2) \times e^x - x^{-1} \times (1/2) \times e^x (x^2 - 2x + 2) = e^x - x^{-1} e^x$, using (6)

Hence the general solution of (1) is $y = \text{C.F.} + \text{P.I.}$

i.e., $y = C_1 x + C_2 x^{-1} + e^x (1 - x^{-1})$, C_1, C_2 being arbitrary constants

(*ii*) We have the same problem. Start with given C.F.

(*iii*) Try yourself. **Ans.** $y = C_1 + C_2 x^2 + (x - 1) e^x$.

Ex. 8. *Apply the method of variation of parameters to solve*

(i) $x^2 y_2 + 3xy_1 + y = 1/(1 - x)^2$

(ii) $x^2 y_2 + xy_1 - y = x^2 \log x,\ x > 0$

(iii) $x^2 y'' - 2xy' + 2y = x \log x,\ x > 0$ **[I.A.S. 2005]**

Sol. (*i*) Re-writing, $y_2 + (3/x) y_1 + (1/x^2) y = x^{-2}(1 - x)^{-2}$ ... (1)

Comparing (1) with $y_2 + Py_1 + Q y = R$, here $R = x^{-2} (1 - x)^{-2}$

Consider $y_2 + (3/x) y_1 + (1/x^2) y = 0$ or $x^2D^2 + 3xD + 1) y = 0$, $D \equiv d/dx$ ... (2)

Let $x = e^z$, $\log x = z$ and $D_1 \equiv d/dz$. ... (3)

Then $xD = D_1$, and $x^2D^2 = D_1(D_1 - 1)$ and so (2) becomes

$\{D_1(D_1 - 1) + 3D_1 + 1\} y = 0$ or $(D_1 + 1)^2 y = 0$

whose auxiliary equation is $(D_1 + 1)^2 = 0$ giving $D_1 = -1, -1$

$\therefore$ C.F. of (1) $= (C_1 + C_2 z) e^{-z} = (C_1 + C_2 \log x) x^{-1}$, by (3)

$\therefore$ C.F. $= C_1 x^{-1} + C_2 x^{-1} \log x$, C_1 and C_2 being arbitrary constants ... (4)

Let $u = x^{-1}$, $v = x^{-1} \log x$. Also, here $R = x^{-2} (1 - x)^{-2}$ ... (5)

Here $$W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} x^{-1} & x^{-1}\log x \\ -x^{-2} & x^{-2} - x^{-2}\log x \end{vmatrix} = x^{-3} \neq 0$$ ... (6)

$\therefore$ P.I. of (1) $= u f(x) + v g(x)$, where ... (7)

$$f(x) = -\int \frac{vR}{W} dx = -\int \frac{x^{-1}\log x . x^{-2}(1-x)^{-2}}{x^{-3}} dx = -\int (1-x)^{-2} \log x \, dx$$, using (5) and (6)

$$= -\left[\frac{1}{1-x}\log x - \int \frac{1}{x(1-x)} dx\right]$$, integrating by parts

$$= -\frac{\log x}{1-x} + \int\left(\frac{1}{x} + \frac{1}{1-x}\right) dx = -(1-x)^{-1}\log x + \log x - \log(1-x)$$

and $$g(x) = \int \frac{uR}{W} dx = \int \frac{x^{-1}\cdot x^{-2}(1-x)^{-2}}{x^{-3}} dx = (1-x)^{-1}$$, by (5) and (6)

$\therefore$ Using (7), P.I. of (1) $= x^{-1}\{-(1 - x)^{-1}\log x + \log x - \log (1 - x)\} + x^{-1}\log x\cdot(1 - x)^{-1}$

$= x^{-1} \{\log x - \log (1 - x)\} = x^{-1} \log \{x/(1 - x)\}$

Hence the general solution of (1) is $y = C_1x^{-1} + C_2x^{-1} \log x + x^{-1} \log\{x/(1 - x)\}$

(*ii*) Try yourself **Ans.** $y = C_1x + C_2x^{-1} + (x^3/3) \times \log x - (4x^2/9)$

(*iii*) Re-writing, given equation is $y'' - (2/x)) y' + (2/x^2) y = (1/x)\log x$ ... (1)

or $\{D^2 - (2/x)D + (2/x^2)\} y = (1/x)\log x$, where $D \equiv d/dx$

Comparing (1) with $y'' + Py' + Q y = R$, here $R = (1/x) \log x$

Consider $\{D^2 - (2/x) D + (2/x^2)\} y = 0$ or $\{x^2D^2 - 2xD + 2\} y = 0$ ...(2)

Let $x = e^z$ or $\log x = z$. Also let $D_1 \equiv d/dz$ ...(3)

Then $xD = D_1$ and $x^2D^2 = D_1(D_1 - 1)$ and so (2) reduces to

$\{D_1(D_1 - 1) - 2D_1 + 2\}\, y = 0$ or $(D_1^{\,2} - 3D_1 + 2)\, y = 0$

whose auxiliary equation is $D_1 - 3D_1 + 2 = 0$ giving $D_1 = 1, 2$

$\therefore$ C.F. of (1) $= C_1 e^z + C_2 e^{2z} = C_1 e^z + C_2 (e^z)^2 = C_1 x + C_2 x^2$, ... (4)

C_1 and C_2 being arbitrary constants.

Let $u = x$ and $v = x^2$. Also, here $R = (1/x)\log x$ ... (5)

Here $$W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} x & x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - x^2 = x^2 \neq 0, \qquad \text{... (6)}$$

$\therefore$ P.I. of (1) $= u\, f(x) + v\, g(x)$, ... (7)

where $$f(x) = -\int \frac{vR}{W}\,dx = -\int \frac{x^2 \log x}{x^2 \times x}\,dx = -\int \log x . \frac{1}{x}\,dx = -\frac{(\log x)^2}{2}, \text{ using (5) and (6)}$$

and $$g(x) = \int \frac{uR}{W}\,dx = \int \frac{\log x}{x^2}\,dx = \int \log x \cdot x^{-2}\,dx, \text{ by (5) and (6)}$$

$$= \log x \times \frac{x^{-1}}{(-1)} - \int \frac{1}{x} \times \frac{x^{-1}}{(-1)}\,dx, \text{ on integrating by parts}$$

$$= -\frac{\log x}{x} + \int x^{-2}\,dx = -\frac{\log x}{x} + \frac{x^{-1}}{(-1)} = -\frac{1}{x}(1 + \log x)$$

$\therefore$ Using (7), P.I. of (1) $= x \times (-1/2) \times (\log x)^2 + x^2 \times (-1/x) \times (1 + \log x)$

$= -(x/2) \times (\log x)^2 - x(1 + \log x)$

$\therefore$ The solution of (1) is $y = C_1 x + C_2 x^2 - (x/2) \times (\log x)^2 - x(1 + \log x)$

Ex. 9. *Solve $y_2 - 2y_1 + y = x e^x \log x,\ x > 0$ by the method of variation of parameters.*

[Delhi Maths (Prog) 2007, Delhi Maths (H) 2005, 09; Delhi Maths (G) 2005]

Sol. Given $(D^2 - 2D + 1)\, y = x e^x \log x$, $x > 0$, $D \equiv d/dx$ ... (1)

Comparing (1) with $y_2 + Py_1 + Q\, y = R$, here $R = xe^x \log x$

Consider $(D^2 - 2D + 1)\, y = 0$ or $(D - 1)^2\, y = 0$ ... (2)

Auxiliary equation of (2) is $(D - 1)^2 = 0$ so that $D = 1, 1$.

$\therefore$ C.F. of (1) $= (C_1 + C_2 x)\, e^x = C_1 e^x + C_2\, x\, e^x$, C_1 and C_2 being arbitrary constants ... (3)

Let $u = e^x$, $v = x\, e^x$. Also, here $R = x\, e^x \log x$. ... (4)

Here $$W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} e^x & x\,e^x \\ e^x & e^x + xe^x \end{vmatrix} = e^{2x} \neq 0 \qquad \text{... (5)}$$

Then, P.I. of (1) $= u\, f(x) + v\, g(x)$, ...(6)

where $$f(x) = -\int \frac{vR}{W}\,dx = -\int \frac{x\,e^x \cdot x\,e^x \log x}{e^{2x}}\,dx = -\int x^2 \log x\,dx, \textit{ by } \text{(4) and (5)}$$

$$= -\left[\log x \cdot \frac{x^3}{3} - \int \frac{1}{x} \cdot \frac{x^3}{3}\,dx\right] = -\left[\frac{1}{3}x^3 \log x - \frac{1}{9}x^3\right]$$

and $$g(x) = \int \frac{uR}{W}\,dx = \int \frac{e^x \cdot x\,e^x \log x}{e^{2x}}\,dx = \int x \log x\,dx, \text{ by (4) and (5)}$$

$$= \frac{x^2}{2}\log x - \int \frac{x^2}{2} \cdot \frac{1}{x}\,dx = \frac{x^2}{2}\log x - \frac{x^2}{4}$$

$\therefore$ (6) $\Rightarrow$ P.I. $= -e^x \{(x^3/3)\log x - (x^3/9)\} + x\,e^x \{(x^2/2)\log x - (x^2/4)\}$

or P.I $= x^3 e^x \log x\,(1/2 - 1/3) - x^3 e^x (1/4 - 1/9) = (1/6) \times x^3 e^x \log x - (5/36) \times x^2 e^x$

Hence the general solution of (1) is $y = C_1 e^x + C_2 x\,e^x + (1/6) \times x^3 e^x \log x - (5/36) \times x^3 e^x$

Ex. 10. *Solve the following equations by the method of variations :*

(i) $y'' + y = \sec^2 x$ **[Delhi Maths (H) 2004]**

(ii) $y'' + 4y = 4\sec^2 2x$ **[Delhi Maths(G) 2006]**

(iii) $y'' + 4y = 4\,\mathrm{cosec}^2\, 2x$

(iv) $y'' + y = \mathrm{cosec}^2 x$ **[Delhi B.Sc. (Hons) II 2011]**

Sol. (*i*) Given $y'' + y = \sec^2 x$ ... (1)

Comparing (1) with $y'' + Py' + Q\,y = R$, here $R = \sec^2 x$

Consider $y'' + y = 0$ or $(D^2 + 1)\,y = 0$, $D \equiv d/dx$ ... (2)

Auxiliary equation of (1) is $D^2 + 1 = 0$ so that $D = \pm\, i$.

$\therefore$ C.F. of (1) $= C_1 \cos x + C_2 \sin x$, C_1 and C_2 being arbitrary constants ... (3)

Let $u = \cos x$, $v = \sin x$. Also, here $R = \sec^2 x$ ... (4)

Here $$W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = 1 \neq 0$$ ... (5)

Then, P.I. of (1) $= u\,f(x) + v\,g(x)$, ... (6)

where $$f(x) = -\int \frac{vR}{W}\,dx = -\int \sin x \sec^2 x\,dx = -\int \sec x \tan x\,dx = -\sec x, \text{ by (4) and (5)}$$

$$g(x) = \int \frac{uR}{W}\,dx = \int \cos x \sec^2 x\,dx = \int \sec x\,dx = \log(\sec x + \tan x), \text{ by (4) and (5)}$$

$\therefore$ P.I. $= \cos x\,(-\sec x) + \sin x \log(\sec x + \tan x)$, using (6)

Hence the general solution of (1) is $y = C_1 \cos x + C_2 \sin x - 1 + \sin x \log(\sec x + \tan x)$.

(*ii*) Given $y'' + 4y = 4\sec^2 2x$ ... (1)

Comparing (1) with $y'' + Py' + Qy = R$, here $R = 4\sec^2 2x$

Consider $y'' + 4y = 0$ or $(D^2 + 4)\,y = 0$, $D \equiv d/dx$ ... (2)

Auxiliary equation of (2) is $D^2 + 4 = 0$ so that $D = \pm\, 2i$

$\therefore$ C.F. of (1) $= C_1 \cos 2x + C_2 \sin 2x$, C_1 and C_2 being arbitrary constants ... (3)

Let $u = \cos 2x$, $v = \sin 2x$. Also, here $R = 4\sec^2 2x$ ... (4)

Here $$W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2 \neq 0.$$ ... (5)

Then, P.I. of (1) $= u\,f(x) + v\,g(x)$, ... (6)

where $$f(x) = -\int \frac{vR}{W}\,dx = -\int \frac{(\sin 2x) \times (4\sec^2 2x)}{2}\,dx, \text{ by (4) and (5)}$$

$$= -2\int \sec 2x \tan 2x\,dx = -\sec 2x$$

and $$g(x) = \int \frac{uR}{W}\,dx = \int \frac{(\cos 2x) \times (4\sec^2 2x)}{2}\,dx, \text{ by (4) and (5)}$$

$$= 2\int \sec 2x\,dx = \log(\sec 2x + \tan 2x)$$

$\therefore$ P.I. of (1) $= \cos 2x \times (-\sec 2x) + \sin 2x \log (\sec 2x + \tan 2x)$, by (6)

$= \sin 2x \log (\sec 2x + \tan 2x) - 1$

Hence the required general solution is $y = \text{C.F.} + \text{P.I.}$

i.e., $y = C_1 \cos 2x + C_2 \sin 2x + \sin 2x \log (\sec 2x + \tan 2x) - 1$

(*iii*) Given $y'' + 4y = 4 \operatorname{cosec}^2 2x$ ... (1)

Comparing (1) with $y'' + Py' + Qy = R$, here $R = 4 \operatorname{cosec}^2 2x$

Consider $y'' + 4y = 0$ or $(D^2 + 4)y = 0$, $D \equiv d/dx$ ... (2)

Auxiliary equation of (2) is $D^2 + 4 = 0$ so that $D = \pm 2i$

$\therefore$ C.F. of (1) $= C_1 \cos 2x + C_2 \sin 2x$, C_1 and C_2 being arbitrary constants ... (3)

Let $u = \cos 2x$, $v = \sin 2x$. Also, here $R = 4 \operatorname{cosec}^2 2x$ ... (4)

Here $$W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2 \neq 0 \qquad \text{... (5)}$$

Then, P.I. of (1) $= u f(x) + v g(x)$, where ... (6)

$$f(x) = -\int \frac{vR}{W} dx - \int \frac{(\sin 2x) \times (4 \operatorname{cosec}^2 2x)}{2} dx = -2 \int \operatorname{cosec} 2x \, dx = \log \tan x, \text{ by (4) and (5)}$$

and $$g(x) = \int \frac{uR}{W} dx = \int \frac{(\cos 2x) \times (4 \operatorname{cosec}^2 2x)}{2} dx = 2\int \operatorname{cosec} 2x \cot 2x \, dx = -\operatorname{cosec} 2x, \text{ by (4) and (5)}$$

$\therefore$ Using (6) P.I. of (1) $= \cos 2x \times (-\log \tan x) + \sin 2x (-\operatorname{cosec} 2x) = -1 - \cos 2x \log \tan x$

$\therefore$ The solution is $y = C_1 \cos 2x + C_2 \sin 2x - 1 - \cos 2x \log \tan x$

(*iv*) Do as it part (*iii*). **Ans.** $y = C_1 \cos x + C_2 \sin x - 1 - \cos x \log \tan(x/2)$

Ex. 11(a). *Solve the differential equaion* $(D^2 - 2D + 2)\, y = e^x \tan x$, $D \equiv d/dx$ *by method of variation of parameters.* **[I.A.S. 2006]**

Sol. Given $(D^2 - 2D + 2)\, y = e^x \tan x$ or $y_2 - 2y_1 + 2y = e^x \tan x$ ...(1)

Comparing (1) with $y_2 + Py_1 + Qy = R$, here $R = e^x \tan x$

Consider $y_2 - 2y_1 + 2y = 0$ or $(D^2 - 2D + 2)y = 0$. ...(2)

Auxiliary equation for (2) in $D^2 - 2D + 2 = 0$ giving $D = 1 \pm i$

$\therefore$ C.F. of (1) $= e^x (C_1 \cos x + C_2 \sin x)$, C_1 and C_2 being arbitrary constants ...(3)

Let $u = e^x \cos x$ and $v = e^x \sin x$. Also, here $R = e^x \tan x$ ...(4)

Here $$W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} e^x \cos x & e^x \sin x \\ e^x (\cos x - \sin x) & e^x (\cos x + \sin x) \end{vmatrix}$$

or $$W = e^{2x} \{\cos x (\cos x + \sin x) - \sin x (\cos x - \sin x)\} = e^{2x} \neq 0 \qquad \text{...(5)}$$

Then P.I. of (1) $= u f(x) + v g(x)$, ...(6)

where $$f(x) = -\int \frac{vR}{W} dx = -\int \frac{(e^x \sin x) \times (e^x \tan x)}{e^{2x}} dx, \text{ using (4) and (5)}$$

$$= -\int \frac{1 - \cos^2 x}{\cos x} dx = \int (\cos x - \sec x)\, dx = \sin x - \log (\sec x + \tan x) \qquad \text{...(7)}$$

and $$g(x) = \int \frac{uR}{W} dx = \int \frac{(e^x \cos x) \times (e^x \tan x)}{e^{2x}} dx, \text{ using (4) and (5)}$$

$$= \int \sin x \, dx = -\cos x \qquad \text{...(8)}$$

Using (6), (7) and (8), we have

$$\text{P.I. of (1)} = e^x \cos x\,\{\sin x - \log(\sec x + \tan x)\} + (e^x \sin x) \times (-\cos x)$$

$$= -e^x \cos x \log(\sec x + \tan x)$$

$\therefore$ The required solution of (1) $y = e^x (C_1 \cos x + C_2 \sin x) - e^x \cos x \log(\sin x + \tan x)$

Ex. 11(b). *Let $y = C_1 u(x) + C_2 v(x)$ be the general solution of $y'' + P(x) y' + Q(x) y = 0$. Show that $y = f(x) u(x) + g(x) v(x)$ is a solution of $y'' + P(x) y' + Q(x) y = R(x)$, where*

$$f(x) = -\int \frac{vR}{W} dx \text{ and } g(x) = \int \frac{uR}{W} dx, \; W \text{ being the Wronskian of } u \text{ and } v.$$

Hence find particular solution of $y'' + 2y' + 5y = e^{-x} \sec 2x$. **[Nagpur 1996]**

Sol. For the first part refer Art. 7.3

Second part. given $\qquad y'' + 2y' + 5y = e^{-x} \sec 2x \qquad$ ... (1)

Comparing (1) with $\quad y'' + Py' + Q\,y = R, \quad$ here $\quad R = e^{-x} \sec 2x$

Consider $\quad y'' + 2y + 5y = 0 \quad$ or $\quad (D^2 + 2D + 5)\,y = 0, \quad D \equiv d/dx$ ... (2)

Auxiliary equation of (2) is $\quad D^2 + 2D + 5 = 0 \quad$ giving $\quad D = -1 \pm 2i$

$\therefore$ C.F. of (1) $= e^{-x} (C_1 \cos 2x + C_2 \sin 2x)$, C_1 and C_2 being arbitrary constants ... (3)

Let $\quad u = e^{-x} \cos 2x, \quad v = e^{-x} \sin 2x. \quad$ Also, here $\quad R = e^{-x} \sec 2x$ ... (4)

Here $$W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} e^{-x} \cos 2x & e^{-x} \sin 2x \\ -e^{-x}(\cos 2x + 2\sin 2x) & e^{-x}(2\cos 2x - \sin 2x) \end{vmatrix}$$

$$= e^{-2x} \cos 2x\,(2 \cos 2x - \sin 2x) + e^{-2x} \sin 2x\,(\cos 2x + 2 \sin 2x)$$

Thus, $\qquad W = 2e^{-2x} (\cos^2 2x + \sin^2 2x) = 2\,e^{-2x} \neq 0 \qquad$ ... (5)

Then, $\qquad$ P.I. of (1) $= u\,f(x) + v\,g(x)$, where $\qquad$ ... (6)

$$f(x) = -\int \frac{(e^{-x} \sin 2x) \times (e^{-x} \sec 2x)}{2e^{-2x}} dx = -\frac{1}{2} \times \int \tan 2x\, dx = \frac{1}{4} \times \log \cos 2x, \text{ by (4) and (5)}$$

and $$g(x) = \int \frac{(e^{-x} \cos 2x) \times (e^{-x} \sec 2x)}{2\,e^{-2x}} dx = \frac{1}{2} \int dx = \frac{x}{2}$$

(6) $\Rightarrow$ P.I. of (1) $= (e^{-x} \cos 2x) \times (1/4) \log \cos 2x + (e^{-x} \sin 2x) \times (1/2)$.

Ex. 12. *Use the method of variation of parameters to solve $y'' + y = 1 / (1+ \sin x)$*

[Delhi Maths(H) 2007]

Sol. Given $\qquad y'' + y = 1/ (1 + \sin x) \qquad$ ... (1)

Comparing (1) with $\quad y'' + Py' + Q\,y = R, \quad$ here $\quad R = 1/ (1 + \sin x)$... (2)

Consider $\quad y'' + y = 0 \quad$ or $\quad (D^2 + 1)\,y = 0, \quad$ where $\quad D \equiv d/dx$... (3)

The auxilary equation is $\quad D^2 + 1 = 0 \quad$ giving $\quad D = \pm\, i.$

$\therefore$ C.F. of (1) $= C_1 \cos x + C_2 \sin x$, C_1 and C_2 being arbitrary constants.

Let $\qquad u = \cos x \quad$ and $\quad v = \sin x \qquad$ ... (4)

Here $$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 + \sin^2 x = 1 \neq 0 \qquad \text{... (5)}$$

Then, $\qquad$ P.I. of (1) $= u\,f(x) + v\,g(x)$, where

$$f(x) = -\int \frac{vR}{W}dx = -\int \frac{\sin x}{1+\sin x}dx = -\int \frac{\sin x(1-\sin x)}{1-\sin^2 x}dx, by\ (2), (4) \text{ and } (5)$$

$$= \int \frac{\sin x - \sin^2 x}{\cos^2 x}dx = -\int (\sec x \tan x - \tan^2 x)dx$$

$$= -\int \{\sec x \tan x - (\sec^2 x - 1)\}\, dx = -(\sec x - \tan x + x)$$

and $$g(x) = \int \frac{uR}{W}dx = \int \frac{\cos x}{1+\sin x}dx = \log(1+\sin x) \text{ by (2), (4) and (5)}$$

$\therefore$ From (6), P.I. of (1) $= -\cos x(\sec x - \tan x + x) + \sin x \log(1+\sin x)$

$$= -1 + \sin x - x\cos x + \sin x \log(1+\sin x)$$

$\therefore$ Required solution is $y = C_1 \cos x + C_2 \sin x - 1 + \sin x - x \cos x + \sin x \log(1 + \sin x)$

Ex. 13. Solve $y'' + 3y' + 2y = x + \cos x$ by the method of variation of parameters.

[Delhi Maths (Prog) 2007]

Sol. Given $y'' + 3y' + 2y = x + \cos x$ ... (1)

Compains (1) with $y'' + Py' + Qy = R$, here $R = x + \cos x$ ...(2)

Consider $y'' + 3y' + 2y = 0$ or $(D^2 + 3D + 2)\, y = 0$, $D \equiv d/dx$ ...(3)

Its auxilary equation is $D^2 + 3D + 2 = 0$ giving $D = -1, -2$

$\therefore$ C.F. of (1) $= C_1 e^{-x} + C_2 e^{-2x}$, C_1 and C_2 being arbitrary constants.

Let $u = e^{-x}$ and $v = e^{-2x}$ ...(4)

Here $$W = \begin{vmatrix} u & v \\ u' & v' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -2e^{-3x} + e^{-3x} = -e^{-3x} \neq 0$$ ...(5)

Then, P. I. of (1) $= u\, f(x) + v\, g(x)$, where ...(6)

$$f(x) = -\int \frac{vR}{W}dx = -\int \frac{e^{-2x}(x+\cos x)}{(-e^{-3x})}dx = \int e^x\,(x+\cos x), \text{ by (2), (4) and (5)}$$

$$= \int xe^x dx + \int e^x \cos x\, dx = xe^x - \int (1\times e^x)\, dx + (1/2)\times e^x(\cos x + \sin x)$$

$$\left[\because \int e^{ax}\cos bx\, dx = \{1/(a^2+b^2)\}\times e^{ax}(a\cos bx + b\sin bx)\right]$$

Thus $$f(x) = xe^x - e^x + (1/2)\times e^x\,(\cos x + \sin x)$$ ...(7)

and $$g(x) = \int \frac{uR}{W}dx = \int \frac{e^{-x}(x+\cos x)}{(-e^{-3x})}dx = -\int e^{2x}(x+\cos x)\, dx$$

$$= -\int xe^{2x}dx - \int e^{2x}\cos x\, dx = -\left[x\times(1/2)\times e^{2x} - \int \{1\times(1/2)\times e^{2x}\}\, dx\right] - \frac{1}{5}e^{2x}(2\cos x + \sin x)$$

Thus, $$g(x) = -(x/2)\times e^{2x} + (1/4)\times e^{2x} - (1/5)\times e^{2x}(2\cos x + \sin x)$$ ...(8)

From (4), (6), (7) and (8), we have

P.I. of (1) $= e^{-x}\{xe^x - e^x + (1/2)\times e^x(\cos x + \sin x)\} + e^{-2x}\{-(x/2)\times e^{2x} + (1/4)\times e^{2x}$

$-(1/5)\times e^{2x}(2\cos x + \sin x)\} = x - 1 + (\cos x + \sin x)/2 - (x/2) + 1/4 - (2\cos x + \sin x)/5$

$= x/2 - (3/4) + (1/10)\times(3\sin x + \cos x)$

$\therefore$ Required solution is $y = C_1 e^{-x} + C_2 e^{-2x} + x/2 - (3/4) + (1/10)\times(3\sin x + \cos x)$

Ex. 14. *Use the variation of parameters method to show that the solution of equation $d^2y/dx^2 + k^2y = \phi(x)$ satisfying the initial conditions* $y(0) = 0, y'(0) = 0$ *is* $y(x) = \dfrac{1}{k}\displaystyle\int_0^x \phi(t)\sin k(x-t)\, dt$.

[Himachal 2002, 05, 06, Kolkata 2003, 04, 06]

Sol. Given $y'' + k^2y = \phi(x)$, *i.e.*, $(D^2 + k^2)y = \phi(x)$, $D \equiv d/dx$ ...(1)

Comparing (1) with $y'' + Py' + Qy = R$ here $R = \phi(x)$ ...(2)

Consider $(D^2 + k^2)y = 0$ whose auxiliary equation is

$D^2 + k^2 = 0$ so that $D = \pm ik$

$\therefore$ C.F. of (1) $= c_1 \cos kx + c_2 \sin kx$, c_1 and c_2 being arbitrary constants. ...(3)

Let $u = \cos kx$ and $v = \sin kx$ ...(4)

Here W = Wranskian of u and v $= \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} \cos kx & \sin kx \\ -k\sin kx & k\cos kx \end{vmatrix} = k \neq 0$ ...(5)

$\therefore$ P.I. of (1) $= u\,f(x) + v\,g(x)$, where ...(6)

$$\therefore \quad f(x) = -\int \frac{vR}{W}dx = -\int_0^x \frac{\sin kx\ \phi(x)}{k}dx = -\frac{1}{k}\int_0^x \phi(t)\sin kt\,dt \qquad ...(7)$$

and
$$g(x) = \int \frac{u\,R}{W}dx = \int_0^x \frac{\cos kx\ \phi(x)}{k}dx = \frac{1}{k}\int_0^x \phi(t)\cos kt\,dt \qquad ...(8)$$

Using (6), (7) and (8), we have

$$\text{P.I. of (1)} = -\frac{1}{k}\cos kx\int_0^x \phi(t)\sin kt\,dt + \frac{1}{k}\sin kx\int_0^x \phi(t)\cos kt\,dt$$

$$= \frac{1}{k}\int_0^x \phi(t)\left(\sin kx\ \cos kt - \cos kx\ \sin kt\right)dt = \frac{1}{k}\int_0^x \phi(t)\sin(k\,x - kt)dt$$

Hence the general solution of (1) is y = C.F. + P.I.,

i.e.,
$$y = c_1\ \cos kx + c_2\ \sin kx + \frac{1}{k}\int_0^x \phi(t)\sin k\left(x-t\right)dt \qquad ...(9)$$

Putting $x = 0$ in (9) and using the given condition $y(0) = 0$, we get $c_1 = 0$

$$\therefore \quad (9) \Rightarrow \quad y = c_2\ \sin kx + \frac{1}{k}\int_0^x \phi(t)\sin k\left(x-t\right)dt \qquad ...(10)$$

Differentiating both sides of (10) w.r.t. 'x' and using * Leibnitz's rule of differentiation under integral sign, we have

$$y'(x) = c_2 k\cos kx + \frac{1}{k}\left[\int_0^x \frac{\partial}{\partial x}\{\phi(t)\sin k(x-t)\}dt + \phi(x)\sin k(x-x)\frac{dx}{dx} - \phi(0)\sin kx\frac{do}{dx}\right]$$

or
$$y'(x) = c_2\ k\cos kx + \int_0^x \phi(t)\cos k(x-t)\,dt \qquad ...(11)$$

***Leibnitz's rule of differentiation under integral sign**

Let $F(x, t)$ and $\partial F / \partial x$ be continuous functions of both x and t and let the first deviatives of $G(x)$ and $H(x)$ be continous. Then $\displaystyle\frac{d}{dx}\int_{G(x)}^{H(x)} F(x,t)\,dt = \int_{G(x)}^{H(x)}\frac{\partial F}{\partial x}dt + F(x, H(x))\frac{dH}{dx} - F(x, G(x))\frac{dG}{dx}$

Putting $x = 0$ in (11) and using the boundary condition $y'(0) = 0$, we get

$$0 = c_2 k + 0 \qquad \text{so that} \qquad c_2 = 0, \qquad \text{as} \qquad k \neq 0$$

Putting $c_1 = 0$ and $c_2 = 0$ (10), the required solution is $\quad y = \dfrac{1}{k}\displaystyle\int_0^x \phi(t)\, \sin k\,(x-t)\, dt$

7.4B Alternative working rule for solving $y_2 + P y_1 + Qy = R$, where P, Q and R are functions of x or contants by variation of parameters, where $y_1 = dy/dx$ and $y_2 = d^2y/dx^2$

Step 1. In order to make coefficient of y_2 unity, divide the given equation by the coefficient of y_2 throughout and obtain it in the standard form $\quad y_2 + P\, y_1 + Q\, y = R \quad$ ...(1)

Step 2. Consider $\quad y_2 + P\, y_1 + Q\, y = 0 \quad$ ...(2)

which is obtained from (1) by taking R = 0. Solve (2) completely by any method of Chapter 5 or 6. Let $y = au + bv$ be solution of (2), where a and b are arbitarary constants and u and v are known functions of x. Then $au + bv$ is complementary function of (1).

Step 3. Let $\quad y = Au + Bv \quad$ ...(3)

be the general solution of (1). Then A and B are functions of x to be determined.

Step 4. Differentiating (3) w.r.t. 'x', we get $\quad y_1 = Au_1 + A_1 u + Bv_1 + B_1 v \quad$ ...(4)

where $u_1 = du/dx$, $v_1 = dv/dx$, $A_1 = dA/dx$ and $B_1 = dB/dx$

Step 5. Choose A and B such that $\quad A_1 u + B_1 v = 0 \quad$ ...(5)

Then (4) reduces to $\quad y_1 = Au_1 + Bv_1 \quad$ ...(6)

Step 6. Differentiating (6) w.r.t. 'x', we get $\quad y_2 = A_1\, u_1 + Au_2 + B_1 v_1 + Bv_2 \quad$ ...(7)

Put these values of y, y_1 and y_2 from equations (3), (6) and (7) repectively in (1). We observe that the terms containing A and B disappear, giving finally $\quad A_1\, u_1 + B_1\, v_1 = R \quad$ ...(8)

Note that L.H.S. of (8) is free from A and B and R.H.S. of (8) is the same as R.H.S. of (1). This fact will be used in all problems.

Step 7. Solve (5) and (8) and get A_1 and B_1 i.e., dA/dx and dB/dx. Integrate these to get A and B. Putting the values of A and B so obtained in (3), we get the desired general solution.

7.5B Solved examples based on working rule 7.4B

Apply the method of variation of parameters to solve the following equations:

(a) $y_2 + n^2 y = \sec nx$ **[Agra 2005; I.A.S. 1999 Delhi Maths (G) 2004]**

(b) $y_2 + a^2 y = \operatorname{cosec} ax$ **[Kakatiya 2003, S.V. Univ. A.P. 1997]**

(c) $y_2 + y = x$ **[Delhi Maths (G)1993, Meerut 2005; Nagpur 2000; Ravishankar 2007]**

(d) $y_2 + 4y = 4 \tan 2x$ **[Gorhwal 2005; Delhi Maths (G) 2004]**

(e) $y_2 - y = 2/(1 + e^x)$ **[Delhi Maths (H) 2001, Rohilkhand 2000]**

(f) $y_2 - 3y_1 + 2y = e^x/(1 + e^x)$ **[Delhi Maths (G) 1993]**

(g) $(d^2y/dx^2) - 2(dy/dx) = e^x \sin x$ **[Delhi Maths (G) 1998]**

(h) $x^2 y_2 + xy_1 - y = x^2 e^x$ **[Delhi Maths (H) 2004, 06, Kanpur 2006]**

(i) $x^2 y_2 + 3xy_1 + y = 1/(1 - x)^2$

Sol (a) Given $\quad y_2 + n^2 y = \text{sex } nx \quad$ ...(1)

Consider $\quad y_2 + n^2 y = 0 \quad$ or $\quad (D^2 + n^2)y = 0, \quad$ where $D \equiv d/dx$. ...(2)

Its auxiliary equation is $D^2 + n^2 = 0$ so that $D = \pm\, i\, n$ and hence solution of (2) is

$$y = a \cos nx + b \sin nx, \; a \text{ and } b \text{ being arbitrary constants.}$$

Let $\quad y = A \cos nx + B \sin nx \quad$ ...(3)

be the complete solution of (1). Then A and B are functions of x which are so chosen that (1) will be satisfied. Differentiating (3) w.r.t. 'x', we have

$$y_1 = A_1 \cos nx - A\, n \sin nx + B_1 \sin nx + B\, n \cos nx. \qquad ...(4)$$

Choose A and B such that $A_1 \cos nx + B_1 \sin nx = 0$ ...(5)

Then (4) becoms $y_1 = -A\, n \sin nx + B\, n \cos nx.$ ...(6)

Differentiating both sides of (6) with respect to x, we get

$$y_2 = -(A_1 n \sin nx + A\, n^2 \cos nx) + (B_1 n \cos nx - B\, n^2 \sin nx). \qquad ...(7)$$

Using (3) and (7), (1) reduces to $-A_1 n \sin nx + B_1 n \cos nx = \sec nx$ ...(8)

We now solve (5) and (8). Multiplying (5) by $n \sin nx$ and (8) by $\cos nx$ and adding the resulting equations, we have

$nB_1 (\cos^2 nx + \sin^2 nx) = \sec nx \cos nx$ or $B_1 = dB/dx = 1/n$ ...(9)

Integrating it, $B = (x/n) + c_1$, c_1 being an arbitrary constant. ...(10)

Using (9), (5) gives $A_1 = dA/dx = -(1/n) \tan nx.$ ...(11)

Integrating it, $A = (1/n^2) \log \cos nx + c_2$, c_2 being an arbitrary constant ...(12)

Using (10) and (12) in (3), the required general solution of (1) is

$$y = [(1/n^2) \log \text{con } nx + c_2] \cos nx + [(x/n) + c_1] \sin nx$$

or $$y = c_1 \sin nx + c_2 \cos nx + (x/n) \sin nx + (1/n^2) \cos nx \log \cos nx.$$

(*b*) Given $y_2 + a^2 y = \text{cosec } ax.$ ...(1)

Consider $y_2 + a^2 y = 0$ or $(D^2 + a^2)y = 0,$ where $D \equiv d/dx$ ...(2)

Its auxiliary equation is $D^2 + a^2 = 0$ so that $D = \pm ia$ and hence solution of (2) is

$$y = a' \cos ax + b' \sin ax, \; a' \text{ and } b' \text{ being arbitrary constants.}$$

Let $y = A \cos ax + B \sin ax.$ ...(3)

be the complete solution of (1). Then A and B are functions of x which are so chosen that (1) will be satisfied. Differentiating (3) with respect to x, we have

$$y_1 = A_1 \cos ax - Aa \sin ax + B_1 \sin ax + Ba \cos ax. \qquad ...(4)$$

Choose A and B such that $A_1 \cos ax + B_1 \sin ax = 0$ ...(5)

Then (4) becomes $y_1 = -A\, a \sin ax + B\, a \cos ax.$ ...(6)

Differentiating both sides of (6) with respect to x, we get

$$y_2 = -(A_1\, a \sin ax + Aa^2 \cos ax) + (B_1 a \cos ax - Ba^2 \sin ax). \qquad ...(7)$$

Using (3) and (7), (1) reduces to $-A_1\, a \sin ax + B_1 a \cos ax = \text{cosec } ax$ ...(8)

Solving (5) and (8), $A_1 = dA/dx = -1/a$ and $B_1 = dB/dx = (1/a) \cot ax.$

Integrating these, $A = (-x/a) + c_1$ and $B = (1/a^2) \log \sin ax + c_2$

where c_1 and c_2 are arbitrary constants.

Putting these values of A and B in (3), the required solution is

$$y = [-(x/a) + c_1] \cos ax + [(1/a^2) \log \sin ax + c_2] \sin ax.$$

or $$y = c_1 \cos ax + c_2 \sin ax - (x/a) \cos ax + (1/a^2) \sin ax \log \sin ax.$$

(*c*) Given equation is $y_2 + y = x.$ ...(1)

Consider $y_2 + y = 0$ or $(D^2 + 1)y = 0,$ where $D \equiv d/dx.$ ...(2)

Its auxiliary equation is $D^2 + 1 = 0$ so that $D = \pm i$ and hence solution of (2) is

$$y = a \cos x + b \sin x, \; a, b \text{ being arbitrary constants.}$$

Let $y = A \cos x + B \sin x$ ...(3)

be the complete solution of (1). Then A and B are functions of x which are so chosen that (1) will be satisfied. Differentiating (3) with respect to x, we have

$$y_1 = A_1 \cos x - A \sin x + B_1 \sin x + B \cos x. \qquad ...(4)$$

Choose A and B such that $A_1 \cos x + B_1 \sin x = 0$ ...(5)

Then (4) becomes $y_1 = -A \sin x + B \cos x$ ...(6)

Differentiating (6), $y_2 = -(A_1 \sin x + A \cos x) + (B_1 \cos x - B \sin x)$ ...(7)

Using (3) and (7), (1) reduces to $-A_1 \sin x + B_1 \cos x = x.$ ...(8)

Solving (5) and (8), $A_1 = dA/dx = -x \sin x,$ $B_1 = dB/dx = x \cos x.$

Integrating these, $A = -\int x \sin x\, dx + c_1 = -[x(-\cos x) - \int 1.(-\cos x)\, dx + c_1$

or $A = x \cos x - \sin x + c_1$, c_1 being an arbitrary constant. ...(9)

and $B = \int x \cos x + c_2 = x \sin x - \int (1 \times \sin x)\, dx + c_2$

or $B = x \sin x + \cos x + c_2$, c_2 being an arbitrary constant. ...(10)

Using (9) and (10) in (3), the required general solution is

$y = (x \cos x - \sin x + c_1) \cos x + (x \sin x + \cos x + c_2) \sin x$ or $y = c_1 \cos x + c_2 \sin x + x.$

(*d*) Given that $y_2 + 4y = 4 \tan 2x.$ ...(1)

Consider $y_2 + 4y = 0$ or $(D^2 + 4)y = 0,$ where $D \equiv d//dx$ ...(2)

Its auxiliary equation is $D^2 + 4 = 0$ so that $D = \pm 2i$ and hence solution of (2) is $y = a \cos 2x + b \sin 2x$, a and b being arbitrary constants.

Let $y = A \cos 2x + B \sin 2x$ ...(3)

be the complete solution of (1). Then A and B are functions of x which are so chosen that (1) will be satisfied. Differentiating (3) w.r.t. x, we have

$$y_1 = A_1 \cos 2x - 2A \sin 2x + B_1 \sin 2x + 2B \cos 2x. \quad ...(4)$$

Choose A and B such that $A_1 \cos 2x + B_1 \sin 2x = 0$ ...(5)

Then (4) becomes $y_1 = -2A \sin 2x + 2B \cos 2x.$ ...(6)

Differentiating both sides of (6) with respect to x, we have

$$y_2 = -(2A_1 \sin 2x + 4A \cos 2x) + 2B_1 \cos 2x - 4B \sin 2x. \quad ...(7)$$

Using (3) and (7), (1) reduces to

$-2A_1 \sin 2x + 2B_1 \cos 2x = 4 \tan 2x$ or $-A_1 \sin 2x + B_1 \cos 2x = 2 \tan 2x$ ...(8)

Solving (5) and (8), $A_1 = dA/dx = -(2 \sin^2 2x)/\cos 2x,$ $B_1 = dB/dx = 2 \sin 2x.$

Integrating these, $A = -2\int \frac{1-\cos^2 2x}{\cos 2x} dx + c_1 = -2\int (\sec 2x - \cos 2x) dx + c_1$

or $A = -\log(\sec 2x + \tan 2x) + \sin 2x + c_1$, c_1 being an arbitrary constant. ...(9)

and $B = 2\int \sin 2x\, dx = -\cos 2x + c_2$, c_2 being an arbitrary constant. ...(10)

Using (9) and (10) in (3), the required general solution is

$$y = [-\log(\sec 2x + \tan 2x) + \sin 2x + c_1] \cos 2x + [-\cos 2x + c_2] \sin 2x$$

or $y = c_1 \cos 2x + c_2 \sin 2x - \cos 2x \log(\sec 2x + \tan 2x).$

(*e*) Given $y_2 - y = 2/(1 + e^x)$ ...(1)

Consider $y_2 - y = 0$ or $(D^2 - 1)y = 0$ where $D \equiv d/dx.$ ...(2)

Its auxiliary equation is $D^2 - 1 = 0$ so that $D = \pm 1$ and hence solution of (2) is $y = ae^x + be^{-x}$, a and b being arbitrary constants.

Let $y = Ae^x + Be^{-x}$

be the complete solution of (1). Then A and B are functions of x which are so chosen that (1) will be satisfied. Differentiating (3) with respect to x, we have

$$y_1 = A_1 e^x + Ae^x + B_1 e^{-x} - Be^{-x} \qquad ...(4)$$

Choose A and B such that $\quad A_1e^x + B_1e^{-x} = 0 \qquad ...(5)$

Then (4) becomes $\quad y_1 = Ae^x - Be^{-x}. \qquad ...(6)$

Differentiating (6), we get $\quad y_2 = A_1e^x + Ae^x - (B_1e^{-x} - Be^{-x}) \qquad ...(7)$

Using (3) and (7), (1) reduces to $\quad A_1e^x - B_1e^{-x} = 2/(1 + e^x) \qquad ...(8)$

Solving (5) and (8), $\quad A_1 = \dfrac{dA}{dx} = \dfrac{e^{-x}}{1+e^x} \quad$ and $\quad B_1 = \dfrac{dB}{dx} = -\dfrac{e^x}{1+e^x}$

Integrating these in succession, we have

$$A = \int \frac{e^{-x}}{1+e^x}\,dx + c_1 = \int \frac{dx}{e^x(e^x+1)} + c_1 = \int \frac{dz}{z^2(1+z)} + c_1$$

[Putting $e^x = z$ so that $e^x\,dx = dz$ or $z\,dx = dz$ or $dx = (1/z)dz$]

or $\quad A = \displaystyle\int \left(\frac{1}{z^2} - \frac{1}{z} + \frac{1}{1+z}\right) dz + c_1$, on breaking into partial fractions.

or $\quad A = -(1/z) - \log z + \log(1+z) + c_1 = -(1/z) + \log[(1+z)/z] + c_1$

or $\quad A = -e^{-x} + \log[(1+e^x)/e^x] + c_1$, as $z = e^x$, c_1 being an arbitrary constant $\qquad ...(9)$

and $\quad B = -\displaystyle\int \frac{e^x}{1+e^x}\,dx + c_2 = -\log(1+e^x) + c_2$, c_2 being an arbitrary constant $\qquad ...(10)$

Using (9) and (10) in (3), the required general solution is

$$y = [-e^{-x} + \log\{(1+e^x)/e^x\} + c_1]e^x + [-\log(1+e^x) + c_2]\,e^{-x}$$

or $\quad y = c_1e^x + c_2e^{-x} - 1 + e^x \log\{(1+e^x)/e^x\} - e^{-x}\log(1+e^x).$

(*f*) Given $\quad y_2 - 3y_1 + 2y = e^x/(1+e^x) \qquad ...(1)$

Consider $\quad y_2 - 3y_1 + 2y = 0 \quad$ or $\quad (D^2 - 3D + 2)y = 0,\quad$ where $D \equiv d/dx \qquad ...(2)$

Its auxiliary equation is $D^2 - 3D + 2 = 0$ so that $D = 1, 2$ and hence solution of (2) is

$y = ae^x + be^{2x}$, a and b being arbitrary constants.

Let $\quad y = Ae^x + Be^{2x} \qquad ...(3)$

be the complete solution of (1). Then A and B are functions of x which are so chosen that (1) will be satisfied. Differentiating (3) with respect to x, we have

$$y_1 = A_1e^x + Ae^x + B_1e^{2x} + 2Be^{2x}. \qquad ...(4)$$

Choose A and B such that $\quad A_1e^x + B_1e^{2x} = 0 \qquad ...(5)$

Then (4) becomes $\quad y_1 = Ae^x + 2Be^{2x}. \qquad ...(6)$

Differentiating (6), we get $\quad y_2 = A_1e^x + Ae^x + 2B_1e^{2x} + 4Be^{2x}. \qquad ...(7)$

Using (3), (6) and (7), (1) reduces to $\quad A_1e^x + 2B_1e^{2x} = e^x/(1+e^x). \qquad ...(8)$

Solving (5) and (8), $\quad A_1 = \dfrac{dA}{dx} = \dfrac{1}{e^x(1+e^x)} \quad$ and $\quad B_1 = \dfrac{dB}{dx} = -\dfrac{1}{1+e^x}.$

Integrating these in succession, we get

$$A = \int \frac{dx}{e^x(1+e^x)} + c_1 = \int \frac{dz}{z^2(1+z)} + c_1 = \int \left(\frac{1}{z^2} - \frac{1}{z} + \frac{1}{1+z}\right) + c_1$$

[Putting $e^x = z$ so that $e^x\,dx = dz$ or $z\,dx = dz$ or $dx = (1/z)dz$]

or $\quad A = (1/z) - \log z + \log (1 + z) + c_1 = (-1/z) + \log [(1+ z)/z] + c_1$ ...(9)

or $\quad A = -e^{-x} + \log [(1+e^x) /e^x] + c_1$, as $z = e^x$; c_1 being an arbitrary constant

and $B = -\int \frac{dx}{1+e^x} + c_2 = \int \frac{(-e^{-x})}{e^{-x}+1} dx + c_2 = \log(e^{-x}+1) + c_2$, c_2 being an arbitrary constant. ...(10)

Using (9) and (10) in (3), the required general solution is

$$y = [-e^{-x} + \log (e^{-x} + 1) + c_1]e^x + [\log (e^{-x} + 1) + c_2]e^{-x}$$

or
$$y = c_1e^x + c_2e^{-x} - 1 + (e^x + e^{-x}) \log (e^{-x} + 1).$$

(*g*) Given that $\quad y_2 - 2y_1 = e^x \sin x.$ ...(1)

Consider $\quad y_2 - 2y_1 = 0 \quad$ or $\quad (D^2 - 2D)\, y = 0, \quad$ where $D \equiv d/dx.$ ...(2)

Its auxiliary equation is $D^2 - 2D = 0$ so that $D = 0, 2$ and hence solution of (2) is $y = ae^{0x} + be^{2x} = a + be^{2x}$, a and b being arbitrary constants.

Let $\quad y = A + Be^{2x}$ ...(3)

be the complete solution of (1). Then A and B are functions of x which are so chosen that (1) will be satisfied. Differentiationg (3) with respect to x, we have

$$y_1 = A_1 + B_1e^{2x} + 2Be^{2x} \qquad ...(4)$$

Choose A and B such that $\quad A_1 + B_1e^{2x} = 0.$ ...(5)

Then (4) reduces to $\quad y_1 = 2Be^{2x}.$ ...(6)

Differentitting (6), we get $\quad y_2 = 2B_1e^{2x} + 4Be^{2x}.$ ...(7)

Using (6) and (7), (1) reduces to $\quad 2B_1e^{2x} = e^x \sin x.$ ...(8)

Solving (5) and (8), $\quad A_1 = \frac{dA}{dx} = -\frac{1}{2}e^x \sin x, \qquad B_1 = \frac{dB}{dx} = \frac{1}{2}e^{-x} \sin x.$

Integrating these, $\quad A = -\frac{1}{2}\int e^x \sin x\, dx + c_1 \quad$ and $\quad B = \frac{1}{2}\int e^{-x} \sin x\, dx + c_2,$

or $\quad A = -\frac{1}{2}\frac{e^x(\sin x - \cos x)}{1^2+1^2} + c_1 \quad$ and $\quad B = \frac{1}{2}\frac{e^{-x}(-\sin x - \cos x)}{(-1)^2+1^2} + c_2$

$$\left[\because \int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx)\right]$$

Putting these values of A and B in (3), the required solution is

$$y = -(1/4) \times e^x (\sin x - \cos x) + c_1 + e^{2x} [-(1/4) \times e^{-x} (\sin x + \cos x) + c_2]$$

or $\quad y = c_1 + c_2e^{2x} - (1/2) \times e^x \sin x$, c_1 and c_2 being arbitrary constants

(*h*) The given equation in standard form $\quad y_2 + Py_1 + Qy = R \quad$ is given by

$$y_2 + (1/x)y_1 - (1/x^2)y = e^x \qquad ...(1)$$

Consider $\quad y_2 + (1/x)y_1 - (1/x^2)y = 0 \quad$ or $\quad x^2y_2 + xy_1 - y = 0$

or $\quad (x^2D^2 + xD - 1)y = 0, \quad$ where $\quad D \equiv d/dx.$ ...(2)

which is a homogeneous equation. Putting $x = e^z$ and $D_1 \equiv d/dz$, (2) becomes

$[D_1(D_1-1) + D_1 - 1]y = 0 \quad$ or $\quad (D^2_1 - 1)y = 0$ ...(2)

Its auxiliary equation is $D_1^2 - 1 = 0$ so that $D_1 = \pm 1$ and hence solution of (2) is

$$y = ae^z + be^{-z} = ae^z + b\,(e^z)^{-1} = ax + bx^{-1}, \text{ a and b being arbitrary constants.}$$

Let $y = Ax + Bx^{-1}$ ...(3)

be the complete solution of (1). Then A and B are functions of x which are so chosen that (1) will be satisfied. Differentiating (3) with respect to x, we have

$$y_1 = A_1x + A + B_1x^{-1} - Bx^{-2} \quad ...(4)$$

Choose A and B such that $A_1x + B_1x^{-1} = 0.$ ...(5)

Then, (4) reduces to $y_1 = A - Bx^{-2}$ ...(6)

Differentiating (6), we get $y_2 = A_1 - (B_1 x^{-2} - 2Bx^{-2})$ ...(7)

Using (3), (6) and (7), (1) reduces to $A_1 - B_1 x^{-2} = e^x$ ...(8)

Solving (5) and (8), $A_1 = dA/dx = (1/2)\times e^x,$ $B_1 = dB/dx = -(1/2)\times x^2e^x.$

Integrating these, $A = (1/2)\times\int e^x dx + c_1 = (1/2)\times e^x + c_1$ ...(9)

and $B = -(1/2)\times\int x^2e^x dx + c_2 = c_2 - (1/2)\times[x^2e^x - \int(2x)e^x dx]$

or $B = c_2 - (1/2)\times x^2 e^x + [xe^x - \int (1.e^x)\, dx] = c_2 - (1/2)\times x^2e^x + xe^x - e^x$ ...(10)

Substituting these values of A and B in (3), the required solution is

$$y = [(1/2)\times e^x + c_1]x + [c_2 - (1/2)\times x^2e^x + xe^x - e^x]\, x^{-1}$$

or $y = c_1x + c_2x^{-1} + e^x - x^{-1} e^x$, c_1 and c_2 being arbitrry constants

(i) The given equation in standard form $y_2 + Py_1 + Qy = R$ is

$$y_2 + (3/x)y_1 + (1/x^2)\, y = x^{-2}\, (1-x)^{-2} \quad ...(1)$$

Consider $y_2 + (3/x)y_1 + (1/x^2)y = 0$ or $(x^2D^2 + 3xD + 1)\, y = 0,$ $D \equiv d/dx,$ ...(2)

which is a homogeneous equation. Putting $x = e^z$ (or $z = \log x$) and $D_1 = d/dz$, (2) becomes

$$[D_1(D_1 - 1) + 3\, D_1 + 1]y = 0 \qquad \text{or} \qquad (D_1 + 1)^2y = 0 \quad ...(2)$$

Its auxiliary equation is $(D_1 + 1)^2 = 0$ so that $D_1 = -1, -1$ and hence solution of (2) is $y = (a + bz)e^{-z} = (a + bz)(e^z)^{-1} = (a + b \log x)\, x^{-1}$, a and b being arbitrary constants.

Let $y = Ax^{-1} + Bx^{-1} \log x$ ...(3)

be the complete solution of (1). Then A and B are functions of x which are so chosen that (1) will be satisfied. Differentiating (3) with respect to x, we have

$$y_1 = A_1x^{-1} - Ax^{-2} + B_1x^{-1} \log x - Bx^{-2} \log x + Bx^{-2} \quad ...(4)$$

Choose A and B such that $A_1x^{-1} + B_1x^{-1} \log x = 0.$ ...(5)

Then (4) reduces to $y_1 = -Ax^{-2} - Bx^{-2} \log x + Bx^{-2}$ ...(6)

Diff. (6), $y_2 = -(A_1x^{-2} - 2Ax^{-3}) - (B_1x^{-2} \log x - 2Bx^{-3} \log x + Bx^{-3}) + B_1x^{-2} - 2Bx^{-3}$ ...(7)

Using (3), (6) and (7), (1) reduces to $-A_1x^{-2} - B_1x^{-2} \log x + B_1x^{-2} = x^{-2}\, (1 - x)^{-2}$

or $A_1 + B_1\, (\log x - 1) = -(1 - x)^{-2}$ ...(8)

Solving (5) and (8), $A_1 = \dfrac{dA}{dx} = -\dfrac{\log x}{(1-x)^2},$ $B_1 = \dfrac{dB}{dx} = \dfrac{1}{(1-x)^2}$ ...(9)

Integrating these, B = $\int (1-x)^{-2} dx + c_2 = (1-x)^{-1} + c_2, c_2$ being an arbitrary constant.

and $A = \int (\log x).(1-x)^{-2}\, dx + c_1 = c_1 - \left[\dfrac{(\log x)}{(1-x)} - \int \dfrac{dx}{x(1-x)} dx\right]$, intergrating the parts

$$= c_1 - \frac{\log x}{1-x} + \int \frac{dx}{x(1-x)} = c_1 - \frac{\log x}{1-x} + \int\left(\frac{1}{x} + \frac{1}{1-x}\right)dx$$

[On resolving into partial fractions]

or $\quad A = c_1 - (\log x) / (1- x) + \log x - \log (1- x)$, c_1 being an arbitrary constant ...(10)

Using (9) and (10) in (3), the required general solution is

$$y = [\, c_1 - (\log x)/(1 - x) + \log x \ \{x/(1- x)\}]\, x^{-1} + [c_2 + (1 - x)^{-1}]\, x^{-1} \log x$$

or $$y = c_1x^{-1} + c_2x^{-1} \log x + x^{-1} \log \{x/(1- x)\} = x^{-1}[c_1+c_2 \log x + \log \{x/(1- x)\}]$$

7.6 Working rule for solving a third order differential equation $y_3 + Py_2 + Q\,y_1 + R\,y = S$, where P, Q, R and S are functions of x or constants by variation of parameters

The method explained in Art 7.3 for a second order equation can be extended to a third order equation.

Step 1. Re-write the given equation as $\quad y_3 + Py_2 + Qy_1 + Ry = S \quad$ ... (1)

in which the coefficient of y_3 must be unity.

Step 2. Consider $\quad y_3 + Py_2 + Q\,y_1 + R\,y = 0 \quad$ ...(2)

which is obtained by taking $S = 0$ in (1). Solve (2) by methods of chapters 5 or 6 as the case may be. Let the general solution of (2), *i.e.*, C.F. of (1), be

$$y = C_1u + C_2v + C_3w,\ C_1,\ C_2 \text{ and } C_3 \text{ being arbitrary constants} \quad \text{... (3)}$$

Step 3. General solution of (1) is $\quad y = \text{C.F.} + \text{P.I.} \quad$ ... (4)

where $\quad \text{C.F.} = C_1u + C_2v + C_3w$, C_1, C_2 and C_3 being arbitrary constants ... (5)

and $\quad \text{P.I.} = u\,f(x) + v\,g(x) + w\,h(x)$, ... (6)

where $f(x)$, g (x) and h (x) are obtained by solving the following differential equations:

$$\frac{du}{dx} = \frac{S}{W}\begin{vmatrix} v & w \\ v_1 & w_1 \end{vmatrix}, \qquad \frac{dv}{dx} = -\frac{S}{W}\begin{vmatrix} u & w \\ u_1 & w_1 \end{vmatrix} \qquad \text{and} \qquad \frac{dw}{dx} = \frac{S}{W}\begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} \quad \text{... (7)}$$

and $$W = \text{Wroskian of } u, v \text{ and } w = \begin{vmatrix} u & v & w \\ u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \end{vmatrix} \quad \text{... (8)}$$

7.7 Examples based on Art 7.6

Ex. 1. *Apply the method of variation of parameters to solve*

(i) $y_3 + y_1 = \sec x$

(ii) $y_3 + y_1 = \operatorname{cosec} x$

Sol. (*i*) Given $\quad y_3 + y_1 = \operatorname{cosec} x \quad$ ... (1)

Comparing (1) with $\quad y_3 + Py_2 + Qy_1 + Ry = S$, $\quad$ here $\quad S = \sec x$

Consider $\quad y_3 + y_1 = 0 \quad$ or $\quad (D^3 + D)\,y = 0$, $\quad$ where $\quad D \equiv d/dx \quad$ ...(2)

Auxiliary equation of (2) is $\quad D^3 + D = 0 \quad$ giving $\quad D = 0, \pm i$

$\therefore$ C.F. of (1) $= C_1 + C_2 \cos x + C_3 \sin x$, C_1, C_2 and C_3 *being arbitrary constants.* ... (3)

Let $\quad u = 1, \quad v = \cos x, \quad w = \sin x. \quad$ Also, here $\quad S = \sec x.$ ... (4)

Here $$W = \begin{vmatrix} u & v & w \\ u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \end{vmatrix} = \begin{vmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{vmatrix} = \sin^2 x + \cos^2 x = 1 \neq 0 \quad \text{... (5)}$$

Then $\quad \text{P.I. of (1)} = u\,f(x) + v\,g(x) + w\,h(x)$, ... (6)

where $$\frac{df(x)}{dx} = \frac{S}{W}\begin{vmatrix} v & w \\ v_1 & w_1 \end{vmatrix} = \sec x \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \sec x$$

$\Rightarrow \quad d\,f(x) = \sec x\, dx \qquad \Rightarrow \qquad f(x) = \log(\sec x + \tan x)$

$$\frac{dg(x)}{dx} = -\frac{S}{W}\begin{vmatrix} u & w \\ u_1 & w_1 \end{vmatrix} = -\sec x \begin{vmatrix} 1 & \sin x \\ 0 & \cos x \end{vmatrix} = -1 \qquad \Rightarrow \qquad g(x) = -x$$

and
$$\frac{dh(x)}{dx} = \frac{S}{W}\begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \sec x \begin{vmatrix} 1 & \cos x \\ 0 & -\sin x \end{vmatrix} = -\tan x \qquad \Rightarrow \qquad h(x) = \log \cos x$$

$\therefore$ P.I. of (1) $= 1 \cdot \log(\sec x + \tan x) + \cos x\,(-x) + \sin x \log \cos x$, by (6)

Hence the required general solution is $\qquad y = \text{C.F.} + \text{P.I.},$

i.e., $\quad y = C_1 + C_2 \cos x + C_3 \sin x + \log(\sec x + \tan x) - x \cos x + \sin x \log \cos x.$

(*ii*) **Ans.** $y = C_1 + C_2 \cos x + C_3 \sin x - \log(\operatorname{cosec} x + \cot x) - \cos x \log \sin x - x \sin x$

Ex. 2. *Solve $y_3 - 6y_2 + 11y_1 - 6y = e^{2x}$ by variaton of parameters.* **[Delhi Maths (G) 2003]**

Sol. Given $\qquad y_3 - 6\,y_2 + 11\,y - 6\,y = e^{2x} \qquad$ ... (1)

Comparing (1) with $\quad y_3 + Py_2 + Qy_1 + Ry = S, \qquad$ here $\quad S = e^{2x}$

Consider $\quad y_3 - 6y_2 + 11y - 6y = 0 \quad$ or $\quad (D^3 - 6D^2 + 11D - 6)\,y = 0, \quad D \equiv d/dx \qquad$ ... (2)

Auxiliary equation of (2) is $\quad D^3 - 6D^2 + 11D - 6 = 0 \quad$ giving $\quad D = 1, 2, 3.$

$\therefore \qquad$ C.F. of (1) $= c_1 e^x + c_2 e^{2x} + c_3 e^{3x}, \qquad$ ...(3)

Let $\quad u = e^x, \quad v = e^{2x}, \quad w = e^{3x}. \quad$ Also, here $\quad S = e^{2x} \qquad$ ...(4)

Here
$$W = \begin{vmatrix} u & v & w \\ u_1 & v_1 & w_1 \\ u_2 & v_2 & w_2 \end{vmatrix} = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix}$$

$$= e^{6x}\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{vmatrix} = e^{6x}\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 5 \end{vmatrix}, \text{using } C_2 \to C_2 - C_1 \text{ and } C_3 \to C_3 - C_2$$

Thus, $\qquad W = e^{6x}(5 - 3) = 2\,e^{6x} \neq 0. \qquad$ ... (5)

Then, $\qquad$ P.I. of (1) $= u\,f(x) + v\,g(x) + w\,h(x), \qquad$ ... (6)

where
$$\frac{df}{dx} = \frac{S}{W}\begin{vmatrix} v & w \\ v_1 & w_1 \end{vmatrix} = \frac{e^{2x}}{2e^{6x}}\begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = \frac{e^x}{2} \Rightarrow df = \frac{e^x}{2}dx \Rightarrow f(x) = \frac{e^x}{2}$$

$$\frac{dg}{dx} = -\frac{S}{W}\begin{vmatrix} u & w \\ u_1 & w_1 \end{vmatrix} = -\frac{e^{2x}}{2e^{6x}}\begin{vmatrix} e^x & e^{3x} \\ e^x & 3e^{3x} \end{vmatrix} = -1 \Rightarrow dg = -dx \Rightarrow g(x) = -x$$

and
$$\frac{dh}{dx} = \frac{S}{W}\begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \frac{e^{2x}}{2e^{6x}}\begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = \frac{e^{-x}}{2} \Rightarrow dh = \frac{e^{-x}}{2}dx \Rightarrow h(x) = -\frac{e^{-x}}{2}$$

Hence, (6) $\Rightarrow$ P.I. of (1) $= e^x \times (1/2) \times e^x + e^{2x} \times (-x) + e^{3x} \times (-1/2) \times e^{-x} = -x\,e^{2x}$

Hence the general solution of (1) is $\quad y = \text{C.F.} + \text{P.I.}$, *i.e.*,

$$y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x} - x\,e^{2x},\ c_1, c_2, c_3 \text{ being arbitrary constants.}$$

Exercise

1. *Apply the method of variation of parameters to solve the equations:*

(*i*) $y_2 - 2y_1 + y = e^x$ **Ans.** $y = (C_1 + C_2 x)\,e^x + (x^2/2) \times e^x$

(*ii*) $y_2 - 6y_1 + 9y = x^{-2}\,e^{2x}$ **Ans.** $y = (C_1 + C_2 x)\,e^{2x} - (1 + \log x)\,e^{2x}$

(*iii*) $y_2 + a^2y = \cos ax$ **Ans.** $y = C_1 \cos ax + C_2 \sin ax + (x/2a) \times \sin ax$

(*iv*) $y_2 - 3y_1 + 2y = 2$ **Ans.** $y = C_1e^x + C_2e^{2x} + 1$

(*v*) $y_2 + 4y = e^x$ **Ans.** $y = C_1 \cos 2x + C_2 \sin 2x + (e^x/5)$

(*vi*) $y_2 + 4y = \sin x$ **Ans.** $y = C_1 \cos 2x + C_2 \sin 2x + (1/3) \times \sin x$

(*vii*) $y_2 - 2y_1 + y = (1/x)\, e^x$ **Ans.** $y = (c_1 + c_2x)\, e^x + xe^x (\log x - 1)$

(*viii*) $y_2 - 2y_1 + y = (1/x^3)\, e^x$ **Ans.** $y = (c_1 + c_2x)\, e^x - (1/2x) \times e^x$

(*ix*) $y_2 - y = e^{2x}$ **Ans.** $y = c_1\, e^x + c_2\, e^{-x} + (1/3) \times e^{2x}$

(*x*) $y_2 - 3y_1 + 2y + \{e^{2x}/(e^x + 1)\} = 0$ **Ans.** $y = c_1\, e^x + c_2\, e^{2x} + e^x \log(1 + e^x) + e^{2x} \log(1 + e^{-x})$

2. Find the particular integral of $(d^2y/dx^2) - 2(dy/dx) + y = 2x$ by the method of variation of parameters. **[Nagpur 1997, 2005]**

Hint: Use Art. 7.3 to find particular integral. **Ans.** P.I. $= 2x + 4$

3. Solve $x^2y_2 + xy_1 - y = x^2 e^x$ by the method of variation of parameters when the complementary function of the equation is given by $ax + bx^{-1}$. **[Delhi Maths (H) 2004]**

Hint: Proceed as in solved Ex. 7(*i*) of Art. 7.5. Note that here we have C.F. $= ax + bx^{-1}$, which is the same as relation of Ex. 7 (*i*) with $C_1 = a$ and $C_2 = b$. Now proceed after relation (5) exactly as in Ex. 7 (*i*).

4. Solve the following equations by the method of variation of parameters :

(*i*) $d^2y/dx^2 + y = x$ **[Meerut 2010]** **Ans.** $y = c_1 \cos + c_2 \sin x + x$

(*ii*) $d^2y/dx^2 + y = 2 - x$ **[Delhi Maths (G) 1999]** **Ans.** $y = c_1 \cos x + c_2 \sin x + 2 - x$

(*iii*) $d^2y/dx^2 - 3\,(dy/dx) + 2y = -1$ **[Delhi Maths (P) 2005]** **Ans.** $y = c_1e^x + c_2\, e^{2x} - (1/2)$

(*iv*) $d^2y/dx^2 - dy/dx = \sec^2 x - \tan x$, $|\, x \,| < \pi/2$. **Ans.** $y = c_1 + c_2\, e^x + \log \sec x$

(*v*) $d^2y/dx^2 + y = \operatorname{cosec} x \cot x$ **[Agra 2005]**

Ans. $y = c_1 \cos x + c_2 \sin x + \cos x \log \operatorname{cosec} x - \cos x - x \sin x$

(*vi*) $d^2y/dx^2 - 2\,(dy/dx) + y = e^x/x^2$ **[Delhi Maths (H) 2003]**

Ans. $y = (c_1 + c_2x)\, e^x - e^x (1 + \log x)$

(*vii*) $d^2y/dx^2 + y = \sin^2 x$ **[Delhi Maths. (G) 2003]**

Ans. $y = c_1 \cos x + c_2 \sin x + (1/3) \times \sin^4 x + (1/12) \times \cos x\,(9 \cos x - \cos 3x)$

(*viii*) $(d^2y/dx^2) - 2(dy/dx) + 3y = x + \sin x$ **[Delhi Maths (G) 2006]**

Ans. $y = C_1 \cos(x\sqrt{2}) + C_2 \sin(x\sqrt{2}) + (3x + 2)/9 + (\sin x + \cos x)/4$

(*ix*) $d^2y/dx^2 + 3(dy/dx) + 2y = 2e^x$ **(I.A.S. 2007)**

Ans. $y = c_1x^{-1} + c_2x^{-2} + (1/3) \times e^x$

5. Solve the following equations by variation of parameters :

(*i*) $y_3 + 4y_1 = 4 \cot 2x$

Ans. $y = C_1 + C_2 \cos 2x + C_3 \sin 2x + (1/2) \times \log \sin 2x - (1/2) + (1/2) \times \cos 2x \log \tan x$

(*ii*) $x^3y_3 + x^2y_2 - 2xy_1 + 2y = x \log x$, $x > 0$

Ans. $y = C_1x + C_2x^{-1} + C_3x^2 - (x/4) \times \{(\log x)^2 + \log x\} - (3x/8)$

(*iii*) $(D^2 - 9)\, y = e^{2x} + x$ **[Delhi Maths (Prog.) 2008]**

[Ans. $y = c_1e^{3x} + c_2\, e^{-3x} - (1/5) \times e^{2x} - (x/9)$ **]**

(*iv*) $(D^2 + 1)y = \sec 3x$, $D \equiv d/dx$ **[Delhi B.A. (Prog.) II 2009]**

[Ans. $y = c_1 \cos x + c_2 \sin x + (1/6) \times (\log_e |\, 4\cos^2 x - 3\,|$

$-2 \log_e |\cos x|) \cos x + (1/2\sqrt{3}) \times \log_e \{(1 + \sqrt{3} \tan x)/(1 - \sqrt{3} \tan x)\} \times \sin x$

(*v*) $y'' + 4y = \sin^2 2x$ **[Delhi B.A. (Prog) II 2010]**

[Ans. $y = c_1 \cos 2x + c_2 \sin 2x + (3 - \cos 4x)/24$]

(*vi*) $d^2y/dx^2 + y = \sec^3 x$, **[Delhi B.Sc. (Prog) II 2011]**

[Ans. $y = c_1 \cos x + c_2 \sin x + (1/2) \times \sin x \tan x$]

(*vii*) $y'' + 2y' + 2y = 4e^{-x} \sec^3 x$ **[Mumbai 2011]**

[Ans. $y = e^{-x}(c_1 \cos x + c_2 \sin x + 2 \sin x \tan x)$]

(*viii*) $x^2 y'' + xy' - 9y = 48x^5$ **[Mumbai 2010]** **[Ans.** $y = c_1 x^3 + c_2 x^{-3} + 3x^5$]

6. Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of the differential equation $y'' + P(x)y' + Q(x)y = 0$ on $[a, b]$ where $P(x)$ and $Q(x)$ are continuous on $[a, b]$. If $R(x)$ is a continuous function on $[a, b]$, then show that $v_p(x) = v_1(x)y_1(x) + v_2(x)y_2(x)$ is a particular solution of differential equation $y'' + P(x)y' + Q(x)y = R(x)$, where

$$v_1 = -\int \frac{y_2(x)R(x)}{W(y_1,(x), y_2(x))} dx, \qquad v_2 = \int \frac{y_1(x)R(x)}{W(y_1(x), y_2(x)} dx$$

Hints : Refer Art. 7.4 A, page 7.3. **[Mumbai 2010]**

8

Ordinary Simultaneous Differential Equations

8.1 Introduction

In this chapter, we shall discuss differential equations is which there is one independent variable and two or more than two dependent variables. To solve such equations completely, there must be as many equations as there are dependent variables. Such equations are called its *ordinary simultaneous differential equations*.

8.2 Methods for solving ordinary simultaneous differential equations with constant coefficients

Let x and y be the dependent variables and t be the independent variable. Thus, in such equations there occur differential coefficients of x, y with respect to t. Let $D \equiv d/dt$. Then such equations can be put in the form

$$f_1(D)\, x + f_2(D)\, y = T_1 \qquad \text{... (1)}$$

and

$$g_1(D)\, x + g_2(D)\, y = T_2, \qquad \text{... (2)}$$

where T_1 and T_2 are functions of the independent variable t and $f_1(D)$, $f_2(D)$, $g_1(D)$ and $g_2(D)$ are all rational integral functions of D with constant coefficients. Such equations can be solved by the following two methods.

First method. Method of elimination (use of operator *D*).

In order to eliminate y between (1) and (2), operating on both sides of (1) by $g_2(D)$ and on both sides of (2) by $f_2(D)$ and subtracting, we have

$$\{f_1(D)\, g_2(D) - g_1(D)\, f_2(D)\}x = g_2(D)\, T_1 - f_2(D)\, T_2, \qquad \text{... (3)}$$

which is a linear differential equation with constant coefficients in x and t and can be solved to give the value of x in terms of t. Substituting this value of x in either (1) or (2), we get the value of y in terms of t. Equation (3) is solved by using methods of chapter 5.

Note 1. The above equations (1) and (2) can be also solved by first eliminating x between them and solving the resulting equation to get y in terms of t. Substituting this value of y in either (1) or (2), we get the value of x in terms of t.

Note 2. Since $f_2(D)$ and $g_2(D)$ are functions of D with constant coefficients, so

$$f_2(D)\, g_2(D) = g_2(D)\, f_2(D).$$

Note 3. In the general solutions of (1) and (2) the number of arbitrary constants is equal to the degree of D in the determinant $\Delta = \begin{vmatrix} f_1(D) & f_2(D) \\ g_1(D) & g_2(D) \end{vmatrix}$, provided $\Delta \neq 0$.

If $\Delta = 0$, then the system of equations (1) and (2) is dependent and such cases will not be considered.

Second method. Method of differentiation.

Sometimes, x or y can be eliminated easily if we differentiate (1) or (2). For example, assume that the given equations (1) and (2) connect four quantities x, y, dx/dt and dy/dt. Differentiating (1) and (2) with respect to t, we obtain four equations containing x, dx/dt, d^2x/dt^2, y, dy/dt and d^2y/dt^2. Eliminating three quantities y, dy/dt, d^2y/dt^2 from these four equations, y is eliminated and we get an equation of the second order with x as the dependent and t as the independent variable. Solving this equation we get value of x in terms of t. Substituting this value of x in either (1) or (2), we get value of y in terms of t.

In what follows we present solution of an ordinary simultaneous differential equations by above two methods. In future, we shall use first method or second method as per requirement of the problem.

AN ILLUSTRATIVE SOLVED EXAMPLE

Solve the simultaneous equations $(dx/dt) - 7x + y = 0$ *and* $(dy/dt) - 2x - 5y = 0$. **[Delhi Maths (Prog) 2007-09, 11; Lucknow 2001, 2000, Sagar 2000; Vikram 2003; Meerut 2007, 10]**

Sol. We shall solve the given system by two methods given in Art. 8.2.

First method. Method of elimination (use of operator *D*)

Step 1. Writing D for d/dt, the given equations can be rewritten in the symbolic form as follows:

$$(D-7)\,x + y = 0 \qquad \text{...(1)}$$

and

$$-2x + (D-5)\,y = 0. \qquad \text{...(2)}$$

Step 2. We now eliminate x (say) as follows. Multiplying (1) by 2 and operating (2) by $(D-7)$, we get

$$2(D-7)\,x + 2y = 0 \qquad \text{...(3)}$$

$$-2\,(D-7)\,x + (D-7)\,(D-5)\,y = 0 \qquad \text{...(4)}$$

Adding (3) and (4), $\quad [(D-7)\,(D-5)+2]\,y = 0 \quad$ or $\quad (D^2 - 12D + 37)\,y = 0$, which is linear equation with constants coefficients.

Its. auxiliary equation is $\quad D^2 - 12D + 37 = 0 \quad$ so that $\quad D = 6 \pm i$

$\therefore \quad y = e^{6t}\,(c_1 \cos t + c_2 \sin t)$, c_1 and c_2 being arbitrary constants. ...(5)

Step 3. We now try to get x by using (5). In this connection remember that we must avoid integration to get x. Thus if we use (1) to get x, then after putting value of y we have to integrate for getting x. Hence we must use (2) because this will not involve any subsequent integration to obtain x. Now from (5), differentiating w.r.t. 't', we get

$$Dy = 6e^{6t}\,[(c_1 \cos t + c_2 \sin t) + e^{6t}(-\,c_1 \sin t + c_2 \cos t)$$

or

$$Dy = e^{6t}\,\{(6c_1 + c_2)\,\text{cost}\,t + (6c_2 - c_1)\sin t] \qquad \text{...(6)}$$

Substituting the values of y and Dy given by (5) and (6) in (2), we have

$$2x = Dy - 5y = e^{6t}\,[6c_1 + c_2)\cos t + (6c_2 - c_1)\sin t - 5(c_1 \cos \mathrm{t} + c_2 \sin t)]$$

or

$$x = (1/2) \times e^{6t}\,[(c_1 + c_2)\cos t + (c_2 - c_1)\sin t] \qquad \text{...(7)}$$

Thus (5) and (7) together give the required solution.

Remark. We can also eliminate y first (as we did to eliminate x) and then obtain x. This value of x can be put in (1) to get the desired value of y.

Second method. Method of differentiation. Given that

$$(dx/dt) - 7x + y = 0 \qquad \text{...(1)}$$

and

$$(dy/dt) - 2x - 5y = 0. \qquad \text{...(2)}$$

To eliminate x, we differentiate (2) w.r.t. 't' and obtain

$$(d^2y/dt^2) - 2(dx/dt) - 5(dy/dt) = 0 \qquad \text{...(3)}$$

Now, from (2), we have

$$x = \frac{1}{2}\left(\frac{dy}{dt} - 5y\right). \qquad \text{...(4)}$$

Then, from (1), we get $$\frac{dx}{dt} = 7x - y = \frac{7}{2}\left(\frac{dy}{dt} - 5y\right) - y, \text{ using (4)}$$

$\therefore$ $$dx/dt = (7/2) \times (dy/dt) - (37y/2)$$

Substituting this value of dx/dt in (3), we have

$$(d^2y/dt^2) - 7\,(dy/dt) + 37y - 5\,(dy/dt) = 0 \qquad \text{or} \qquad (D^2 - 12D + 37)\,y = 0.$$

Now get y as done in first method. In fact repeat the whole method after this step. Thus we get the same values of x and y as in first method.

Note 1. Second method will be used when found very necessary. In almost all problems we shall use the first method.

Note 2. Generally t will be the independent variable and x and y will be dependent variables. In some problems any other variable, x say, will be given as the independent variable and y and z as the dependent variables. This point should be noted carefully while doing any problem.

8.3 Solved examples based on Art 8.2

Ex. 1. *Solve $dx/dt - y = t$, $dy/dt + x = 1$.* **[Agra 2000, Delhi Maths (G) 1998]**

Sol. Writing D for d/dt, the given equations become

$$Dx - y = t \qquad \text{... (1)}$$

and $$x + Dy = 1 \qquad \text{... (2)}$$

Differentiating (1) w.r.t. 't', $$D^2x - Dy = 1 \qquad \text{...(3)}$$

To eliminate y between (2) and (3), we add them and get

$$D^2x + x = 2 \qquad \text{or} \qquad (D^2 + 1)\,x = 2. \qquad \text{... (4)}$$

Now the auxiliary equation of (4) is $D^2 + 1 = 0$ so that $D = \pm\, i$.

$\therefore$ C.F. $= c_1 \cos t + c_2 \sin t$, c_1 and c_2 being arbitrary constants.

and $$P.I. = \frac{1}{1+D^2}\,2 = (1+D^2)^{-1}\,2 = (1 - D^2 + \ldots)\,2 = 2$$

Hence the general solution of (4) is $$x = c_1 \cos t + c_2 \sin t + 2 \qquad \text{... (5)}$$

From (5), $$Dx = dx/dt = -\,c_1 \sin t + c_2 \cos t \qquad \text{... (6)}$$

$\therefore$ From (1), $$y = Dx - t = -\,c_1 \sin t + c_2 \cos t - t. \qquad \text{... (7)}$$

The required solution is given by (5) and (7).

Ex. 2. *Solve the simultaneous differential equations $dx/dt = 3x + 2y$, $dy/dt = 5x + 3y$.*

[Kanpur 2004, Lucknow 2001, 03]

Sol. Writing D for d/dt, the given equations become

$$(D - 3)\,x - 2y = 0 \qquad \text{... (1)}$$

and $$-\,5x + (D - 3)\,y = 0 \qquad \text{... (2)}$$

Operating on both sides of (1) by $(D - 3)$ and multiplying both sides of (2) by 2 and then adding, we have

$$\{(D - 3)^2 - 10\}\;x = 0 \qquad \text{or} \qquad (D^2 - 6D - 1)\,x = 0. \qquad \text{... (3)}$$

Now, auxiliary equation of (3) is $D^2 - 6D - 1 = 0$ so that $D = 3 \pm \sqrt{10}$.

$\therefore$ $$x = \text{C.F.} = e^{3t}[c_1 \cosh(t\sqrt{10}) + c_2 \sinh(t\sqrt{10})\}. \qquad \text{... (4)}$$

From (4), $$Dx = dx/dt = 3\,e^{3t}\,\{c_1 \cosh\,(t\sqrt{10}) + c_2 \sinh\,(t\sqrt{10})\}$$
$$+\, e^{3t}\{c_1\sqrt{10}\sinh(t\sqrt{10}) + c_2\sqrt{10}\cosh(t\sqrt{10})\}$$

or $$Dx = e^{3t}\{(3c_1 + c_2\sqrt{10})\cosh(t\sqrt{10}) + (3c_2 + c_1\sqrt{10})\sinh(t\sqrt{10})\} \qquad \ldots (5)$$

Then, from (1), we have $y = (1/2) \times (D-3)\,x = (1/2) \times (Dx - 3x)$

i.e., $$y = (1/2)\times[e^{3t}\{(3c_1 + c_2\sqrt{10})\cosh(t\sqrt{10}) + (3c_2 + c_1\sqrt{10})\sinh(t\sqrt{10})\}$$
$$-3e^{3t}\{c_1\cosh(t\sqrt{10}) + c_2\sinh(t\sqrt{10})\}], \text{ using (4) and (5)}$$

$\therefore$ $$y = (\sqrt{10}/2)\times e^{3t}[c_2\cosh(t\sqrt{10}) + c_1\sinh(t\sqrt{10})] \qquad \ldots (6)$$

The general solution is given by (4) and (6).

Ex. 3. *Solve the simultaneous differential equations* $(D-17)\,y + (2D-8)\,z = 0$, $(13D-53)\,y - 2z = 0$, *where* $D \equiv d/dt$.

Sol. Given $$(D-17)\,y + 2\,(D-4)\,z = 0 \qquad \ldots (1)$$

and $$(13D-53)\,y - 2z = 0 \qquad \ldots (2)$$

Operating on both sides of (2) by $(D-4)$ and then adding to (1), we have
$$\{(D-17) + (D-4)(13D-53)\}\,y = 0 \quad \text{or} \quad (D^2 - 8D - 15)\,y = 0 \qquad \ldots (3)$$

Here auxiliary equation is $D^2 - 8D - 15 = 0$ so that $D = 3, 5$.

$\therefore$ $y = \text{C.F.} = c_1 e^{3x} + c_2 e^{5x}$, c_1 and c_2 being arbitrary constants ... (4)

From (4), $$Dy = dy/dx = 3c_1 e^{3x} + 5c_2 e^{5x} \qquad \ldots (5)$$

From (2), $$2z = 13Dy - 53y$$

or $$2z = 13\,(3c_1 e^{3x} + 5c_2 e^{5x}) - 53\,(c_1 e^{3x} + c_2 e^{5x}), \text{ by (4) and (5)}$$

$\therefore$ $$z = 6C_2 e^{5x} - 7C_1 e^{3x} \qquad \ldots (6)$$

The required general solution is given by (4) and (6).

Ex. 4(*a*). *Solve* $(dx/dt) + 5x + y = e^t$, $(dy/dt) - x + 3y = e^{2t}$. **[Kanpur 2005, Garhwal 2005, Delhi Maths (Hons.) 2000, 02, Delhi Maths (G) 2000]**

Sol. Given $$(D+5)\,x + y = e^t \qquad \ldots (1)$$

and $$-x + (D+3)\,y = e^{2t} \qquad \ldots (2)$$

Operating on both sides of (2) by $(D+5)$, we get
$$-(D+5)\,x + (D+5)(D+3)\,y = (D+5)\,e^{2t} = 2e^{2t} + 5e^{2t}, \qquad \ldots (3)$$

Adding (1) and (3), $$\{1 + (D+5)(D+3)\}\,y = e^t + 7e^{2t}$$

or $$(D+4)^2\,y = e^t + 7e^{2t} \qquad \ldots (4)$$

Its auxiliary equation is $(D+4)^2 = 0$ so that $D = -4, -4$.

$\therefore$ C.F. $= (c_1 + c_2 t)\,e^{-4t}$ c_1 and c_2, being arbitrary constants.

$$P.I. = \frac{1}{(D+4)^2}(e^t + 7e^{2t}) = \frac{1}{(D+4)^2}e^t + 7\frac{1}{(D+4)^2}e^{2t} = \frac{1}{(1+4)^2}e^t + 7\frac{1}{(2+4)^2}e^{2t} = \frac{1}{25}e^t + \frac{7}{36}e^{2t}.$$

$\therefore$ Solution of (4) is $$y = \text{C.F.} + \text{P.I.} = (c_1 + c_2 t)\,e^{-4t} + (1/25)\,e^t + (7/36)\,e^{2t} \qquad \ldots (5)$$

From (5), $$Dy = dy/dt = -4(c_1 + c_2 t)e^{-4t} + c_2 e^{-4t} + (1/25)\,e^t + (7/18)\,e^{2t} \qquad \ldots (6)$$

$\therefore$ From (2), $x = Dy + 3y - e^{2t}$, Using (5) and (6), this gives
$$x = -4\,(c_1 + c_2 t)\,e^{-4t} + c_2 e^{-4t} + (1/25)\,e^t + (7/18)\,e^{2t} + 3\,[(c_1 + c_2 t)\,e^{-4t} + (1/25)\,e^t + (7/36)\,e^{2t}] - e^{2t}$$

or $$x = -(c_1 + c_2 t)\,e^{-4t} + c_2\,e^{-4t} + (4/25)\,e^t - (1/36)\,e^{2t}. \qquad \ldots (7)$$

The required general solution is given by (5) and (7).

Ex. 4(*b*). *Solve* $dx/dt + 2y + x = e^t$, $dy/dt + 2x + y = 3e^t$. **[Delhi Maths (H) 2009]**

Sol. Writing D for d/dt, the given equations become
$$(D+1)\,x + 2y = e^t \qquad \ldots (1)$$

and $$2x + (D+1)\,y = 3e^t \qquad \ldots (2)$$

Operating on both sides of (1) by $(D + 1)$ and multiplying both sides of (2) by 2 and then subtracting, we get

$$[(D + 1)^2 - 4]\, x = (D + 1)\, e^t - 6e^t \quad \text{or} \quad (D^2 + 2D - 3)x = - 4e^t \qquad \ldots (3)$$

The auxiliary equation is $D^2 + 2D - 3 = 0$ so that $D = 1, - 3$.

$\therefore$ C.F. $= c_1 e^t + c_2 e^{-3t}$, c_1 and c_2, being arbitrary constants.

and
$$\text{P.I.} = \frac{1}{D^2 + 2D - 3}(-4e^t) = -4\frac{1}{(D-1)}\frac{1}{(D+3)}e^t$$

$$= -4\frac{1}{D-1}\frac{1}{1+3}e^t = -\frac{1}{(D-1)}e^t = -\frac{t}{1!}e^t, \text{ as } \frac{1}{(D-a)^n}e^{at} = \frac{t^n}{n!}e^{at}$$

$\therefore$ Solution of (3) is
$$x = c_1 e^t + c_2 e^{-3t} - t\, e^t \qquad \ldots (4)$$

From (4),
$$Dx = dx/dt = c_1 e^t - 3c_2 e^{-3t} - (e^t + t\, e^t) \qquad \ldots (5)$$

Now, $2y = e^t - Dx - x$, using (1)

or
$$2y = e^t - (c_1 e^t - 3c_2 e^{-3t} - e^t - t\, e^t) - (c_1 e^t + c_2 e^{-3t} - t\, e^t), \text{ using (4) and (5)}$$

or
$$y = e^t - c_1 e^t + c_2 e^{-3t} + t\, e^t \qquad \ldots (6)$$

The required general solution is given by (4) and (6).

Ex. 4(*c*). *Solve $(dx/dt) + 2\,(dy/dt) - x + y = 0$ and $2\,(dx/dt) + (dy/dt) + 2x + y = 3e^{-t}$.*

[Delhi Maths (Hons.) 1998]

Sol. Writing D for d/dt, the given equations become

$$(D - 1)\, x + (2D + 1)\, y = 0 \qquad \ldots (1)$$

and
$$2(D + 1)\, x + (D + 1)\, y = 3e^{-t} \qquad \ldots (2)$$

Operating on both sides of (1) by $(D + 1)$ and (2) by $(2D + 1)$ and then subtracting, we have

$$[(D + 1)(D - 1) - 2(2D + 1)(D + 1)]\, x = 0 - (2D + 1)(3e^{-t})$$

or
$$[D^2 - 1 - 2(2D^2 + 3D + 1)]\, x = - 6D\, e^{-t} - 3e^{-t} = 6e^{-t} - 3e^{-t} = 3e^{-t}$$

or
$$(- 3D^2 - 6D - 3)\, x = 3e^{-t} \quad \text{or} \quad (D + 1)^2\, x = - e^{-t} \qquad \ldots (3)$$

Its auxiliary equation is $(D + 1)^2 = 0$ so that $D = - 1, - 1$.

$\therefore$ C.F. $= (c_1 + c_2 t)\, e^{-t}$, c_1 and c_2 being arbitrary constants.

and
$$\text{P.I.} = \frac{1}{(D+1)^2}(-e^{-t}) = -\frac{t^2}{2!}e^{-t}, \quad \text{as } \frac{1}{(D-a)^n}e^{at} = \frac{t^n}{n!}e^{at}$$

$\therefore$ Solution of (3) is
$$x = (c_1 + c_2 t) - (1/2) \times t^2 e^{-t}. \qquad \ldots (4)$$

From (4),
$$Dx = - (c_1 + c_2 t)\, e^{-t} + c_2 e^{-t} - (1/2) \times (2t\, e^{-t} - t^2 e^{-t}) \qquad \ldots (5)$$

Multiplying both sides of (2) by 2, we have

$$(4D + 4)\, x + (2D + 2)\, y = 6e^{-t} \qquad \ldots (6)$$

Subtracting (1) from (6), we have

$$(3D + 5)\, x + y = 6\, e^{-t} \quad \text{or} \quad y = 6\, e^{-t} - 3Dx - 5x$$

or
$$y = 6\, e^{-t} - 3\,[- (c_1 + c_2 t)\, e^{-t} + c_2 e^{-t} - (1/2) \times (2t\, e^{-t} - t^2 e^{-t})] - 5[(c_1 + c_2 t)\, e^{-t} - (1/2) \times t^2 e^{-t}]$$

or
$$y = 6\, e^{-t} - 2\,(c_1 + c_2 t)\, e^{-t} - 3c_2 e^{-t} + 3t\, e^{-t} + t^2 e^{-t}$$

or
$$y = - 2\,(c_1 + c_2 t)\, e^{-t} - 3c_2 e^{-t} + (t^2 + 3t + 6)\, e^{-t}. \qquad \ldots (7)$$

The required general solution is given by (4) and (7).

Ex. 5. *Solve $(dx/dt) - y = t^2$, $(dy/dt) + 4x = t$, given $x(0) = 0$ and $y(0) = 3/4$.*

Sol. Writing D for d/dt, the given equations become

$$Dx - y = t^2 \qquad \ldots (1)$$

and
$$4x + Dy = t \qquad \ldots (2)$$

Operating on both sides of (1) by D and adding to (2), we get

$$D^2x + 4x = Dt^2 + t \quad \text{or} \quad (D^2 + 4)\, x = 2t + t = 3t \qquad \text{.... (3)}$$

Its auxiliary equation is $D^2 + 4 = 0$ so that $D = \pm\, 2i$

$\therefore$ C.F. $= c_1 \cos 2t + c_2 \sin 2t$, c_1 and c_2 being arbitrary constants.

and
$$\text{P.I.} = \frac{1}{D^2+4}\,3t = 3\,\frac{1}{4(1+D^2/4)}\,t = \frac{3}{4}\left(1+\frac{D^2}{4}\right)^{-1} = \frac{3}{4}\left(1-\frac{D^2}{4}+\ldots\right)t = \frac{3t}{4}$$

$\therefore$ Solution of (3) is $\quad x = c_1 \cos 2t + c_2 \sin 2t + (3t/4)$. ... (4)

From (4), $\quad Dx = dx/dt = -\,2c_1 \sin 2t + 2c_2 \cos 2t + (3/4)$. ... (5)

From (1) and (5), $\quad y = Dx - t^2 = -\,2c_1 \sin 2t + 2c_2 \cos 2t + (3/4) - t^2$ ... (6)

Putting $t = 0$ in (4) and using the fact that $x(0) = 0$, we get $c_1 = 0$. Again, putting $t = 0$ in (6) and using the fact that $y(0) = 3/4$, we get $3/4 = 2c_2 + 3/4$ so that $c_2 = 0$.

Hence, from (4) and (6), the required solution is $\quad x = (3t/4), \quad y = (3/4) - t^2$.

Ex. 6. *Solve dydt = y, dx/dt = 2y + x.* **[Delhi Maths (G) 2000]**

Sol. Given that $\quad dy/dt = y$ ... (1)

and $\quad dx/dt = 2y + x$ ... (2)

From (1), $\quad (1/y)\, dy = dt$.

Integrating, $\quad \log y - \log c_1 = t \quad$ or $\quad y = c_1 e^t$ ... (3)

Substituting this value of y in (2), we have $\quad (dx/dt) = 2c_1e^t + x \quad$ or $\quad (dx/dt) - x = 2c_1e^t$,

which is a linear equation. Its I.F. $= e^{\int(-1)\,dt} = e^{-t}$ and solution is

$$x \cdot e^{-t} = \int (2c_1e^t)\cdot e^{-t}\, dt + c_2 = 2c_1t + c_2, \qquad \text{or} \qquad x = (2c_1t + c_2)\,e^t$$

where c_1 and c_2 are arbitrary constants.

Hence the required solution is given by $\quad x = (2\,c_1t + c_2)\, e^t, \quad y = c_1e^t$.

Ex. 7(*a*). *Solve* $(dx/dt) + 4x + 3y = t$, $(dy/dt) + 2x + 5y = e^t$. **[Garhwal 2003; Lucknow 2003; Kerala 2001; Karnataka 2002; Vikram 2000; Osmania 2004, Meerut 2011; Delhi Maths (G) 1994, Delhi Maths (Hons.) 1999]**

Sol. Writing D for d/dt, the given equations become

$$(D + 4)\, x + 3y = t \qquad \text{... (1)}$$

and
$$2x + (D + 5)\, y = e^t \qquad \text{... (2)}$$

Operating on both sides of (1) by $(D + 5)$ and multiplying both sides of (2) by 3 and then subtracting, we get

$$\{(D + 5)(D + 4) - 6\}\, x = (D + 5)\, t - 3e^t \quad \text{or} \quad (D^2 + 9D + 14)\, x = 1 + 5t - 3e^t \quad \text{...(3)}$$

Its auxiliary equation is $D^2 + 9D + 14 = 0$ so that $D = -\,2, -\,7$.

$\therefore$ C.F. $= c_1 e^{-2t} + c_2 e^{-7t}$, c_1 and c_2 being arbitrary constants.

P.I. corresponding to $(1 + 5t)$

$$= \frac{1}{14+9D+D^2}(1+5t) = \frac{1}{14[1+(9/14)\,D+(1/14)D^2]}(1+5t) = \frac{1}{14}\left[1+\left(\frac{9}{14}D+\frac{1}{14}D^2\right)\right]^{-1}(1+5t)$$

$$= \frac{1}{14}\left[1-\left(\frac{9}{14}D+\frac{1}{14}D^2\right)+\ldots\right](1+5t) = \frac{1}{14}\left[1+5t-\frac{9}{14}D(1+5t)\right] = \frac{1}{14}\left[1+5t-\frac{9}{14}\times 5\right] = \frac{5t}{14}-\frac{31}{196}.$$

P.I. corresponding to $(-3e^t) = \dfrac{1}{14+9D+D^2}(-3e^t) = -3\dfrac{1}{14+9\cdot1+1^2}e^t = -\dfrac{3}{24}e^t = -\dfrac{e^t}{8}.$

$\therefore$ Solution of (3) is $x = \text{C.F.} + \text{P.I.} = c_1e^{-2t} + c_2e^{-7t} + (5/14)\,t - (31/196) - (1/8)\,e^t$ ...(4)

$\therefore$ $Dx = dx/dt = -2c_1e^{-2t} - 7c_2e^{-7t} + (5/14) - (1/8)\,e^t.$ ...(5)

From (1), $3y = t - Dx - 4x.$ Using (4) and (5), this gives

$3y = t - [-2c_1e^{-2t} - 7c_2e^{-7t} + (5/14) - (1/8)\,e^t] - 4\,[c_1e^{-2t} + c_2e^{-7t} + (5/14)\,t - (31/196) - (1/8)\,e^t]$

or $y = (1/3)\,[-2c_1\,e^{-2t} + 3c_2\,e^{-7t} + (5/8)\,e^t + (27/98) - (3/7)\,t]$ ... (5)

The required general solution is given by (3) and (5).

Ex. 7(*b*). *Solve $dx/dt + 2x - 3y = t,\ dy/dt - 3x + 2y = e^{2t}$.*

[Ujjain 2003, Delhi Maths 2001; Delhi B.A. (Prog) II 2010]

Sol. Let $D \equiv d/dt$. Then the given equations become

$$(D+2)\,x - 3y = t \qquad \text{... (1)}$$

and
$$-3x + (D+2)\,y = e^{2t} \qquad \text{... (2)}$$

Eliminating y from (1) and (2), we have $(D+2)^2\,x - 9x = (D+2)\,t + 3\,e^{2t}$

or
$$(D^2 + 4D - 5)\,x = 2t + 1 + 3\,e^{2t} \qquad \text{... (3)}$$

Auxiliary equation for (3) is $D^2 + 4D - 5 = 0.$ Hence $D = 1, -5.$

$\therefore$ C.F. of (3) $= c_1\,e^t + c_2\,e^{-5t}$, c_1 and c_2 being arbitrary constants

P.I. corresponding to $(2t+1)$

$$= \frac{1}{D^2+4D-5}(2t+1) = -\frac{1}{5}\left[1 - \left(\frac{4D}{5} + \frac{D^2}{5}\right)\right]^{-1}(2t+1) = -\frac{1}{5}\left(1 + \frac{4D}{5} + \dots\right)(2t+1)$$

$= -(1/5)\times(2t+1+8/5) = (10t+13)/25$

P.I. corresponding to $3\,e^{2t} = 3\dfrac{1}{D^3+4D-5}e^{2t} = 3\dfrac{1}{4+8-5}e^{2t} = \dfrac{3}{7}e^{2t}.$

Hence the general solution of (3) is

$$x = c_1e^t + c_2\,e^{-5t} + (3/7)\,e^{2t} - (1/25)\,(10\,t + 13). \qquad \text{... (4)}$$

$\therefore$
$$Dx = c_1\,e^t - 5c_2\,e^{-5t} + (6/7)\,e^{2t} - (2/5). \qquad \text{... (5)}$$

From (1), $3y = Dx + 2x - t$. Using (4) and (5), it gives

$$3y = 3c_1\,e^t - 3c_2\,e^{-5t} + (12/7)\,e^{2t} - (9/5)\,t - (36/25)$$

$\therefore$
$$y = c_1\,e^t - c_2\,e^{-5t} + (4/7)\,e^{2t} - (3/5)\,t - (12/25) \qquad \text{... (6)}$$

The required solution is given by (4) and (6).

Ex. 7(c). *Solve $dx/dt + dy/dt - 2y = 2\cos t - 7\sin t,\ dx/dt - dy/dt + 2x = 4\cos t - 3\sin t$.*

[Lucknow 2005; Pune 2000; Delhi Maths (G) 2005; Agra 2002; Kanpur 1998]

Sol. Let $D \equiv d/dt$. Then the given equations become

$$Dx + (D-2)\,y = 2\cos t - 7\sin t \qquad \text{... (1)}$$

and
$$(D+2)\,x - Dy = 4\cos t - 3\sin t \qquad \text{... (2)}$$

Eliminating y from (1) and (2), we get

$$[D^2 + (D-2)\,(D+2)]x = D(2\cos t - \sin t) + (D-2)(4\cos t - 3\sin t)$$

or
$$(D^2 - 2)\,x = -9\cos t, \text{ on simplification} \qquad \text{... (3)}$$

Auxiliary equation is $D^2 - 2 = 0$ giving $D = \pm\sqrt{2}$

$\therefore$ C.F. $= c_1 e^{\sqrt{2}\,t} + c_2 e^{-\sqrt{2}\,t}$, c_1 and c_2 being arbitrary constants.

Also, $$\text{P.I.} = -\frac{9}{D^2-2}\cos t = \frac{-9}{-1^2-2}\cos t = 3\cos t.$$

$\therefore$ Solution of (3) is $$x = c_1 e^{\sqrt{2}\,t} + c_2 e^{\sqrt{2}t} + 3\cos t. \qquad \text{... (4)}$$

From (4), $$Dx = c_1\sqrt{2}\, e^{\sqrt{2}\,t} - c_2\sqrt{2}\, e^{-\sqrt{2}\,t} - 3\sin t \qquad \text{... (5)}$$

Adding (1) and (2), $$2Dx + 2x - 2y = 6\cos t - 10\sin t$$

$\therefore$ $$y = Dx + x - 3\cos t + 5\sin t$$

or $y = c_1\sqrt{2}\, e^{\sqrt{2}\,t} - c_2\sqrt{2}\, e^{-\sqrt{2}\,t} - 3\sin t + c_1 e^{\sqrt{2}\,t} + c_2 e^{-\sqrt{2}\,t} + 3\cos t - 3\cos t + 5\sin t$, by (4) and (5)

Thus, $$y = (1+\sqrt{2})\,c_1 e^{\sqrt{2}\,t} + (1-\sqrt{2})\,c_2 e^{-\sqrt{2}\,t} + 2\sin t \qquad \text{... (6)}$$

The required solution is given by (4) and (6).

Ex. 7(*d*). *Solve the equations $4(dx/dt) + 9\,(dy/dt) + 11x + 31y = e^t$, $3\,(dx/dt) + 7\,(dy/dt) + 8x + 24y = e^{2t}$* **[Lucknow 1998, Meerut 1996]**

Sol. Writing D for d/dt, the given equations become

$$(4D + 11)\,x + (9D + 31)\,y = e^t \qquad \text{... (1)}$$

and $$(3D + 8)\,x + (7D + 24)\,y = e^{2t} \qquad \text{... (2)}$$

Operating on both sides of (1) by $(7D + 24)$ and (2) by $(9D + 31)$ and then subtracting, we have $$\{(7D + 24)\,(4D + 11) - (9D + 31)\,(3D + 8)\}\,x = (7D + 24)\,e^t - (9D + 31)\,e^{2t}$$

or $$(D + 4)^2\,x = 31\,e^t - 49\,e^{2t} \qquad \text{... (3)}$$

Its auxiliary equation is $(D + 4)^2 = 0$ so that $D = -4, -4$

$\therefore$ C.F. $= (c_1 + c_2\,t)\,e^{-4t}$, c_1 and c_2 being arbitrary constants

and $$\text{P.I.} = \frac{1}{(D+4)^4}(31e^t - 49e^{2t}) = 31\frac{1}{(D+4)^2}e^t - 49\frac{1}{(D+4)^4}e^{2t}$$

$$= 31\frac{1}{(1+4)^2}e^t - 49\frac{1}{(2+4)^2}e^{2t} = \frac{31}{25}e^t - \frac{49}{36}e^{2t}.$$

$\therefore$ Solution of (3) is $$x = \text{C.F.} + \text{P.I.} = (c_1 + c_2 t)\,e^{-4t} + (31/25)\,e^t - (49/36)\,e^{2t} \qquad \text{... (4)}$$

From (4), $$Dx = dx/dt = c_2\,e^{-4t} - 4\,(c_1 + c_2 t)\,e^{-4t} + (31/25)\,e^t - (49/18)e^{2t} \qquad \text{... (5)}$$

Now, multiplying both sides of (1) by 7 and (2) by 9, we get

$$(28D + 77)\,x + (63D + 217)\,y = 7\,e^t \qquad \text{... (6)}$$

and $$(27D + 72)\,x + (63D + 216)\,y = 9\,e^{2t}. \qquad \text{... (7)}$$

Subtracting (7) from (6), $Dx + 5x + y = 7\,e^t - 9\,e^{2t}$ or $y = -Dx - 5x + 7e^t - 9e^{2t}$

or $$y = -[c_2 e^{-4t} - 4\,(c_1 + c_2\,t)\,e^{-4t} + (31/25)\,e^t - (49/18)\,e^{2t}]$$
$$- 5\,[(c_1 + c_2\,t)\,e^{-4t} + (31/25)\,e^t - (49/36)\,e^{2t}] + 7\,e^t - 9\,e^{2t}, \text{ by (4) and (5)}$$

or $$y = -(c_2 + c_1 + c_2 t)\,e^{-4t} + (19/36)\,e^{2t} - (11/25)\,e^t \qquad \text{.... (8)}$$

The required general solution is given by (4) and (8).

Ex. 8. *Solve the following simultaneous equations :*

(i) $dy/dx + y = z + e^x$, $dz/dx + z = y + e^x$. **[Delhi Maths (P) 2005]**

(ii) $dx/dt + x = y + e^t$, $dy/dt + y = x + e^t$. **[Delhi Maths Hons. 2005]**

Sol. (*i*) Writing D for d/dx, the given equations become

$$(D + 1)\,y - z = e^x \qquad \text{... (1)}$$

and $$-y + (D + 1)\,z = e^x \qquad \text{... (2)}$$

Operating (1) by $(D + 1)$, we get $(D + 1)^2 y - (D + 1) z = (D + 1) e^x$ ... (3)

Adding (2) and (3), we get $[(D + 1)^2 - 1] y = e^x + (e^x + e^x)$

or $(D^2 + 2D) y = 3e^x$ or $D (D + 2) y = 3e^x$ ...(4)

Auxiliary equation of (4) is $D (D + 2) = 0$ giving $D = 0, - 2$.

$\therefore$ C.F. $= c_1 + c_2 e^{-2x}$ and P.I. $= 3\dfrac{1}{D(D+2)}e^x = 3\dfrac{1}{1\times(1+2)}e^x = e^x$.

$\therefore$ Solution of (4) is $y = c_1 + c_2 e^{-2x} + e^x$, c_1, c_2 being arbitrary constants. ... (4)

From (4), $Dy = dy/dx = - 2c_2 e^{-2x} + e^x$ ... (5)

$\therefore$ From (1), $z = Dy + y - e^x = - 2c_2 e^{-2x} + e^x + c_1 + c_2 e^{-2x} + e^x - e^x$, by (4) and (5)

or $z = c_1 - c_2 e^{-2x} + e^x$ ... (6)

The required solution is given by (4) and (6).

(*ii*) This is just the same as (*i*). Here we have t in place of x and x and y in place of y and z. You have to denote d/dt by D. **Ans.** $x = c_1 + c_2 e^{-2t} + e^t$, $y = c_1 - c_2 e^{-2t} + e^t$.

Ex. 9(*a*). *Solve* $dx/dt = ax + by$, $dy/dt = b x + ay$

[Punjab 2005; G.N.D.U. Amritsar 2000; Garhwal 1998, Lucknow 1999]

Sol. Writing D of d/dt, the given equations become

$$(D - a) x - by = 0 \qquad ... (1)$$

and
$$- bx + (D - a) y = 0 \qquad ... (2)$$

Operating both sides of (1) by $(D - a)$ and multiplying (2) by b, we get

$$(D - a)^2 x - b (D - a) y = 0$$

and
$$- b^2x + b (D - a) y = 0$$

Adding these, $[(D - a)^2 - b^2] x = 0$ or $(D - a - b) (D - a + b) x = 0$ ... (3)

Its auxiliary equation $(D - a - b) (D - a + b) = 0$ yields $D = a + b$ and $D = a - b$.

Hence, solution of (3) is $x = c_1e^{(a + b) t} + c_2 e^{(a - b) t}$, c_1, c_2 being arbitrary constants ... (4)

From (4), $dx/dt = c_1 (a + b) e^{(a + b)t} + c_2(a - b) e^{(a - b)t}$ ... (5)

From the first given differential equatioin, we have

$y = (1/b) \times \{dx/dt - ax\}$

$= (1/b) \times \{c_1(a + b) e^{(a + b)t} + c_2 (a - b) e^{(a - b)t} - ac_1e^{(a + b)t} - ac_2e^{(a - b)t}]$, using (4) and (5)

or $y = c_1 e^{(a + b)t} - c_2 e^{(a - b)t}$ on simplification ... (6)

(4) and (6) together give the required solution.

Ex. 9(*b*). *Solve* $dx/dt = ax + by$, $dy/dt = a' x + b' y$.

[Garhwal 1999, G.N.D.U. Amritsar 2000, Lucknow 1999]

Sol. Writing D for d/dt, the given equations become

$$(D - a) x - by = 0 \qquad ... (1)$$

and
$$- a'x + (D - b') y = 0 \qquad ... (2)$$

Operating both sides of (1) by $(D - b')$ and multiplying (2) by b, we get

$$(D - b') (D - a) x - b (D - b') y = 0 \qquad ... (3)$$

and
$$- a'bx + b (D - b) y = 0 \qquad ... (4)$$

Adding (3) and (4), $[(D - b') (D - a) - a'b] x = 0$

or $[D^2 - D (a + b') + (ab' - a'b)] x = 0$, ... (5)

Its auxiliary equation is $D^2 - D (a + b') + (ab' - a'b) = 0$,

giving
$$D = \frac{a+b' \pm \sqrt{\{(a+b')^2 - 4(ab'-a'b)\}}}{2} = \frac{a+b' \pm \sqrt{\{(a-b')^2 + 4a'b\}}}{2}$$

$\therefore$ $D = (1/2) \times [a + b' + \{(a - b')^2 + 4a'b\}^{1/2}] = \alpha_1$, say

and $D = (1/2) \times [a + b' - \{(a - b')^2 + 4a'b\}^{1/2}] = \alpha_2$, say

$\therefore$ Solution of (5) is $x = c_1 e^{\alpha_1 t} + c_2 e^{\alpha_2 t}$, c_1 and c_2 being arbitrary constants ... (6)

(1) $\Rightarrow by = (D - a)\, x$ or $y = (1/b) \times \{(dx/dt) - ax\}$

$\therefore$ $y = (1/b) \times [c_1 \alpha_1 e^{\alpha_1 t} + c_2 \alpha_2 e^{\alpha_2 t} - a\,(c_1 \alpha^{\alpha_1 t} + c_2 e^{\alpha_2 t})]$, by (6)

or $y = (1/b) \times [c_1 (\alpha_1 - a)\, e^{\alpha_1 t} + c_2\, (\alpha_2 - a)\, e^{\alpha_2 t}]$. ... (8)

(6) and (8) together give the required solution.

Ex. 9(*c*). *Solve dx/dt = – wy and dy/dt = wx. Also show that the point (x, y) lies on a circle.* **[I.A.S. 2002, Meerut 2006; Nagpur 2007; Sagar 2001, 04]**

Sol. Writing D for d/dt, the given equations become $Dx + wy = 0$ ... (1)

and $wx - Dy = 0$ ... (2)

Operating (1) by D and multiplying (2) by w, we get

$D^2 x + w\,Dy = 0$ and $w^2 x - w\,Dy = 0$.

Adding the above two equations, we get $(D^2 + w^2)\, x = 0$ ....(3)

Auxiliary equation for (3) is $D^2 + w^2 = 0$ giving $D = \pm\, iw$

Solution of (3) is $x = c_1 \cos wt + c_2 \sin wt$, c_1, c_2 being arbitrary constants ... (3)

(3) $\Rightarrow$ $dx/dt = Dx = -\, c_1 w \sin wt + c_2 w \cos wt$. ... (4)

$\therefore$ From (1), $y = -\,(1/w) \times Dx = -\,(1/w) \times (-\, c_1 w \sin wt + c_2 w \cos wt)$, by (4)

Thus, $y = c_1 \sin wt - c_2 \cos wt$ ... (5)

Thus (3) and (5) together give the required solution.

Squaring and adding (3) and (5), $x^2 + y^2 = (c_1 \cos wt + c_2 \sin wt)^2 + (c_1 \sin wt - c_2 \cos wt)^2$

Thus, $x^2 + y^2 = c_1^{\,2} + c_2^{\,2} = \{(c_1^{\,2} + c_2^{\,2})^{1/2}\}^2$, which is a circle.

Hence the point (x, y) lies on a circle.

Ex. 10(*a*). *Solve for x and y :(dx/dt) + 2 (dy/dt) – 2x + 2y = 3e^t and 3 (dx/dt) + (dy/dt) + 2x + y = 4e^{2t}.* **[Delhi B.Sc. (Prog) II 2010; Kanpur 2002, 07; Meerut 2007]**

Sol. Given $(dx/dt) + 2(dy/dt) - 2x + 2y = 3e^t$ ... (1)

and $3\,(dx/dt) + (dy/dt) + 2x + y = 4e^{2t}$ ... (2)

Multiplying both sides of (2) by 2, we have $6\,(dx/dt) + 2\,(dy/dt) + 4x + 2y = 8\,e^{2t}$ ... (3)

Subtracting (1) from (3), we have

$5\dfrac{dx}{dt} + 6x = 8\,e^{2t} - 3\,e^t$ or $\dfrac{dx}{dt} + \dfrac{6}{5}\,x = \dfrac{8}{5}\,e^{2t} - \dfrac{3}{5}\,e^t$, ... (4)

which is a linear differential equation of order one.

I.F. of (4) $= e^{\int (6/5)\,dt} = e^{(6/5)\,t}$ and its solution is

$$x\,e^{(6/5)\,t} = \int \left(\frac{8}{5}\,e^{2t} - \frac{3}{5}\,e^t\right) e^{(6/5)\,t} dt + c_1 = \int \left[\frac{8}{5}\,e^{(16/5)\,t} - \frac{3}{5}\,e^{(11/5)\,t}\right] dt + c_1$$

or $$x\,e^{(6/5)\,t} = (8/5)\cdot(5/16)\,e^{(16/5)\,t} - (3/5)\cdot(5/11)\,e^{(11/5)\,t} + c_1$$

$x = (1/2)\,e^{2t} - (3/11)\,e^t + c_1\,e^{-(6/5)\,t}$, c_1 being an arbitrary constant ... (5)

Multiplying both sides of (1) by 3, $3\,(dx/dt) + 6\,(dy/dt) + 6x + 6y = 9e^t$ ... (6)

Subtracting (2) from (6), we have

$5\,(dy/dt) - 8x + 5y = 9e^t - 4e^{2t}$ or $5\,(dy/dt) + 5y = 8x + 9e^t - 4e^{2t}$

or $$5\frac{dy}{dt}+5y=8\left[\frac{1}{2}e^{2t}-\frac{3}{11}e^{t}+c_1 e^{-(6/5)t}\right]+9e^{t}-4e^{2t}, \text{by (5)}$$

or $$5\frac{dy}{dt}+5y=\frac{75}{11}e^{t}+8c_1 e^{-(6/5)t} \quad \text{or} \quad \frac{dy}{dt}+y=\frac{15}{11}e^{t}+\frac{8c_1}{5}e^{-(6/5)t}$$

which is again a linear differential equation of order one.

Its integrating factor = $e^{\int dt}=e^{t}$ and its solution is

$$y\,e^{t}=\int\left[\frac{15}{11}e^{t}+\frac{8c_1}{5}e^{(-6/5)t}\right]e^{t}\,dt+c_2 \quad \text{or} \quad y\,e^{t}=\int\left[\frac{15}{11}e^{2t}+\frac{8c_1}{5}e^{-(1/5)t}\right]dt+c_2$$

or $$y\,e^{t}=(15/11)\cdot(1/2)e^{2t}+(8c_1/5)\cdot(-5)\cdot e^{-(1/5)t}+c_2$$

or $$y=c_2\,e^{-t}-8c_1\,e^{(-6/5)t}+(15/22)e^{t}., \; c_2 \text{ being an arbitrary constant.} \qquad \text{... (7)}$$

(5) and (7) together give the required solution.

Ex. 10(*b*). *Solve $dx/dt + 2x + 3y = 0$, $dy/dt + 3x + 2y = 2e^{2t}$.* **[Delhi Maths 2002, 04]**

Sol. Writing D for d/dt, the given equations become

$$dx/dt + 2x + 3y = 0 \quad \text{or} \quad (D+2)\,x + 3y = 0 \qquad \text{... (1)}$$

and $$dy/dt + 3x + 2y = 2e^{2t} \quad \text{or} \quad 3x + (D+2)\,y = 2e^{2t} \qquad \text{... (2)}$$

Operating (2) by $(D + 2)$ and multiplying (1) by 3 and then subtracting, we have

$$[(D+2)^2-9]\,y=(D+2)\,2e^{2t} \quad \text{or} \quad (D^2+4D-5)\,y=8e^{2t} \qquad \text{... (3)}$$

Auxiliary equation of (3) is $D^2 + 4D - 5 = 0$ so that $D = 1, -5$

$\therefore$ C.F. of (3) = $C_1e^{t} + C_2\,e^{-5t}$, c_1 and c_2 being arbitrary constants

$$\text{P.I. of (3)} = \frac{1}{D^2+4D-5}8\,e^{2t}=8\,\frac{1}{2^2+4\cdot 2-5}e^{2t}=\frac{8}{7}e^{2t}$$

$\therefore$ solution of (3) is $$y = C_1e^{t} + C_2e^{-5t} + (8/7)\,e^{2t} \qquad \text{... (4)}$$

From (4), $$dy/dt = C_1e^{t} - 5C_2\,e^{-5t} + (16/7)\,e^{2t} \qquad \text{... (5)}$$

From (2), $$3x = 2e^{2t} - 2y - dy/dt$$

or $$3x=2e^{2t}-2\{C_1e^{t}+C_2e^{-5t}+(8/7)e^{2t}\}-\{C_1\,e^{t}-5C_2\,e^{-5t}+(16/7)e^{2t}\}$$

[On putting values of y and dy/dt from (4) and (5)]

or $$3x = -3C_1e^{t} + 3C_2e^{-5t} - (18/7)\,e^{2t} \quad \text{or} \quad x = -\,C_1e^{t} + C_2e^{-5t} - (6/7)\,e^{2t} \qquad \text{... (6)}$$

The required solution is given by (4) and (6).

Ex. 10(*c*). *Solve $(dx/dt) - (dy/dt) + 3x = \sin t$, $dx/dt + y = \cos t$, given that $x = 1$, $y = 0$ for $t = 0$.* **[Delhi Maths (H) 2001]**

Sol. Writing D for d/dt, the given equations become

$$(dx/dt) - (dy/dt) + 3x = \sin t \quad \text{or} \quad (D+3)\,x - Dy = \sin t \qquad \text{... (1)}$$

and $$dx/dt + y = \cos t \quad \text{or} \quad Dx + y = \cos t \qquad \text{... (2)}$$

Operating (2) by D and adding it to (1), we get

$$[(D+3)+D^2]\,x=\sin t + D\cos t \quad \text{or} \quad (D+D+3)\,x=0 \qquad \text{... (3)}$$

Auxiliary equation of (3) is $D^2 + D + 3 = 0$, giving

$$D=\{-1\pm(1-12)^{1/2}\}/2=(-1/2)\pm i(\sqrt{11}/2)$$

So solution of (3) is $$x=e^{-t/2}\{C_1\cos(t\sqrt{11}/2)+C_2\sin(t\sqrt{11}/2)\} \qquad \text{... (4)}$$

Diff. (4) w.r.t 't', $dx/dt = -(1/2)\,e^{-t/2}\{C_1 \cos(t\sqrt{11}/2) + C_2 \sin(t\sqrt{11}/2]$

$$+\, e^{-t/2}\{-(C_1\sqrt{11}/2)\sin(t\sqrt{11}/2) + (C_2\sqrt{11}/2)\cos(t\sqrt{11}/2)\} \quad \text{... (5)}$$

From (2), $y = \cos t - dx/dt$

or $y = \cos t + (1/2)\; e^{-t/2}\; \{C_1 \cos\; (t\sqrt{11}/2)\; +\; C_2 \sin\; (t\sqrt{11}/2)\}$

$$-\, e^{-t/2}\{-(C_1\sqrt{11}/2)\sin(t\sqrt{11}/2) + (C_2\sqrt{11}/2)\cos(t\sqrt{11}/2)\}\text{, using (5)} \quad \text{... (6)}$$

Given that $y = 0$ for $t = 0$. So the above equation gives

$$0 = 1 + (1/2)\,C_1 - (C_2\sqrt{11}/2) \quad \text{... (7)}$$

Again, given that $x = 1$ for $t = 0$. So (4) gives $C_1 = 1$. With this value of C_1, (7) gives $C_2 = 3/\sqrt{11}$. Therefore, (4) and (6) give

$$x = e^{-t/2}\{\cos(t\sqrt{11}/2) + (3/\sqrt{11})\sin(t\sqrt{11}/2)\} \quad \text{... (7)}$$

and $y = \cos t + (1/2)\,e^{-t/2}[\cos(t\sqrt{11}/2) + (3/\sqrt{11})\sin(t\sqrt{11}/2)]$

$$-\, e^{-t/2}\{-(\sqrt{11}/2)\sin(t\sqrt{11}/2) + (3/2)\cos(t\sqrt{11}/2)\}$$

or $$y = \cos t - e^{-t/2}\cos(t\sqrt{11}/2) + e^{-t/2}(3/2\sqrt{11} + \sqrt{11}/2)\sin(t\sqrt{11}/2) \quad \text{... (8)}$$

The required solution is given by (7) and (8).

Ex. 10(*d*). *Solve $dx/dt - 3x + 4y = e^{-2t}$, $dy/dt - x + 2y = 3e^{-2t}$.* **[Delhi Maths (H) 2004, 06]**

Find also the particular solution, if $x = 12$, $y = 7$ when $t = 0$

Sol. Let $D \equiv d/dt$. Then, the given equation reduce to

$$(D - 3)\,x + 4y = e^{-2t} \quad \text{... (1)}$$

and $$-\,x + (D + 2)\,y = 3e^{-2t}$$

Eliminating y from (1) and (2), $(D + 2)(D - 3)x + 4x = (D + 2)e^{-2t} - 12e^{-2t}$

or $$(D^2 - D - 2)\,x = -\,12e^{-2t} \quad \text{...(3)}$$

Its auxiliary equation is $D^2 - D - 2 = 0$, giving $D = 2, -1$.

Its C.F. $= C_1e^{2t} + C_2e^{-t}$, C_1 and C_2 being arbitrary constants.

Its $$\text{P.I.} = \frac{1}{D^2 - D - 2}(-12e^{-2t}) = -12\frac{1}{(-2)^2 + 2 - 2}e^{-2t} = -3\,e^{-2t}$$

So solution of (3) is $x = C_1e^{2t} + C_2e^{-t} - 3e^{-2t}$ ... (4)

From (1), $4y = e^{-2t} + 3x - (dx/dt)$

$$= e^{-2t} + 3\,(C_1e^{2t} + C_2e^{-t} - 3e^{-2t}) - (2C_1e^{2t} - C_2e^{-t} + 6e^{-2t})\text{, using (4)}$$

or $$y = (1/4) \times (C_1e^{2t} + 4C_2e^{-t} - 14e^{-2t}) \quad \text{... (5)}$$

(4) and (5) together give the required solution.

Second part Given that $x = 12$ and $y = 7$ when $t = 0$. So (4) and (5) reduce to

$C_1 + C_2 - 3 = 12$ giving $C_1 + C_2 = 15$ ...(6)

and $(1/4) \times (C_1 + 4C_2 - 14) = 7$ giving $C_1 + 4C_2 = 42$ ... (7)

Solving (6) and (7), $C_1 = 6$ and $C_2 = 9$. Hence, the required solution is given by

$$x = 6\,e^{2t} + 9\,e^{-t} - 3e^{-2t}, \qquad y = (3/2)\,e^{2t} + 9e^{-t} - (7/2)\,e^{-2t}$$

Ex. 10(*e*). *Solve $dx/dt + dy/dt + 2x + y = e^t$, $dy/dt + 5x + 3y = t$.* **[Delhi Maths (G) 2004]**

Sol. Let $D \equiv d/dt$. Then, the given equations reduce to

$$(D + 2)\, x + (D + 1)\, y = e^t \qquad \text{... (1)}$$

$$5x + (D + 3)\, y = t \qquad \text{... (2)}$$

Eliminating x from (1) and (2), $\{5(D + 1) - (D + 2)(D + 3)\}y = 5e^t - (D + 2)t$

or $(-D^2 - 1)\, y = 5e^t - 1 - 2t$ or $(D^2 + 1)\, y = 1 + 2t - 5e^t$ ... (3)

Its auxiliary equation is $D^2 + 1 = 0$, giving $D = \pm i$

$\therefore$ C.F. of (3) $= C_1 \cos t + C_2 \sin t$, C_1 and C_2 being arbitrary constants.

P.I. of (3) corresponding to $(1 + 2t)$ $= \dfrac{1}{1+D^2}(1+2t)$

$$= (1 + D^2)^{-1}(1 + 2t) = (1 - D^2 + \ldots)(1 + 2t) = 1 + 2t$$

and P.I. of (3) corresponding to $(-5e^t) = \dfrac{1}{D^2+1}(-5e^t) = -\dfrac{5}{2}e^t$

$\therefore$ Solution of (3) is $y = C_1 \cos t + C_2 \sin t + 1 + 2t - (5/2)\, e^t$ ... (4)

From (2), $5x = t - 3y - (dy/dt) = t - 3\,\{C_1 \cos t + C_2 \sin t + 1 + 2t - (5/2)\, e^t\}$

$$- (- C_1 \sin t + C_2 \cos t + 2 - (5/2)\, e^t\}, \text{ by (4)}$$

or $x = \{(C_1 - 3C_2)/5\} \sin t - \{(3C_1 + C_2)/5\} \cos t - t - 1 + 2e^t$ ... (5)

(4) and (5) together give the required solution.

Ex. 10(*f*). *Solve $dx/dt + dy/dt + 2x - y = 3\,(t^2 - e^{-t})$, $2\,(dx/dt) - (dy/dt) - x - y = 3\,(2t - e^{-t})$* **[I.A.S. 2003; Rajasthan 2007]**

Sol. Let $x_1 = dx/dt$, $x_2 = d^2x/dt^2$, $y_1 = dy/dt$ and $y_2 = d^2y/dt^2$

Then, re-writing the given equation, we have

$$x_1 + y_1 + 2x - y = 3\,(t^2 - e^{-t}) \qquad \text{... (1)}$$

and $$2x_1 - y_1 - x - y = 3\,(2t - e^{-t}) \qquad \text{... (2)}$$

Differentiating (1) and (2) w.r.t. 't', we have $x_2 + y_2 + 2x_1 - y_1 = 3\,(2t + e^{-t})$ ... (3)

and $2x_2 - y_2 - x_1 - y_1 = 3\,(2 + e^{-t})$ ... (4)

Adding (3) and (4), $3x_2 + x_1 - 2y_1 = 6\,(t + 1 + e^{-t})$ ... (5)

Subtracting (2) from (1), $x_1 - 2y_1 - 3x = 3\,(2t - t^2)$ ... (6)

Subtracting (6) from (5), $3x_2 + 3x = 6 + 3t^2 + 6\,e^{-t}$

or $(D^2 + 1)\, x = 2 + t^2 + 2e^{-t}$, where $D \equiv d/dt$ ... (7)

Auxiliary equation of (7) is $D^2 + 1 = 0$ so that $D = \pm\, i$.

$\therefore$ C.F. of (7) $= c_1 \cos t + c_2 \sin t$, c_1 and c_2 being arbitrary constants

P.I. corresponding to $(2 + t^2)$

$$= \frac{1}{D^2+1}(2+t^2) = (1+D^2)^{-1}(2+t^2) = (1-D^2+\ldots)(2+t^2) = 2 + t^2 - 2 = t^2.$$

P.I. corresponding to $(2e^{-t}) = \dfrac{1}{D^2+1}2e^{-t} = 2\dfrac{1}{1+1}e^{-t} = e^{-t}$.

$\therefore$ Solution of (7) is $x = c_1\cos t + c_2 \sin t + e^{-t} + t^2$ ... (8)

From (8), on differentiating, $x_1 = - c_1 \sin t + c_2 \cos t - e^{-t} + 2t$ ... (9)

From (6), $2y_1 = x_1 - 3x - 6t + 3t^2 = - c_1 \sin t + c_2 \cos t - e^{-t} + 2t$

$$- 3\,(c_1 \cos t + c_2 \sin t + e^{-t} + t^2) - 6t + 3t^2, \text{ by (8) and (9)}$$

$\therefore$ $y_1 = [(c_2 - 3c_1) \cos t - (c_1 + 3c_2) \sin t - 4t - 4e^{-t}]/2$ ... (10)

$\therefore$ From (2), $\qquad y = 2x_1 - y_1 - x - 6t + 3e^{-t}$

or $\qquad y = 2\,(-c_1 \sin t + c_2 \cos t + 2t - e^{-t}) - (1/2)\,[(c_2 - 3c_1) \cos t$

$- (c_1 + 3c_2) \sin t - 4t - 4e^{-t}] - (c_1 \cos t + c_2 \sin t + e^{-t} + t^2) - 6t + 3e^{-t}$, by (8) (9) and (10)

or $$y = (1/2) \times (3c_2 + c_1) \cos t + (1/2) \times (c_2 - 3c_1) \sin t + 2e^{-t} - t^2 \qquad \text{... (11)}$$

(8) and (11) together give the desired solution.

Ex. 10(*g*). *Solve* $4x_1 + 9y_1 + 44x + 49y = t$, $3x_1 + 7y_1 + 34x + 38y = e^t$ *where* $x_1 = dx/dt$ *and* $y_1 = dy/dt$. **[Kanpur 2005; Meerut 1997; Delhi Maths (Prog) 2007]**

Sol. Let $D \equiv d/dt$. Then the given equations can be re-written as

$$(4D + 44)\, x + (9D + 49)\, y = t \qquad \text{... (1)}$$

and $$(3D + 34)\, x + (7D + 38)\, y = e^t \qquad \text{... (2)}$$

Eliminating y from the above equations, we have

$$[(7D + 38)\,(4D + 44) - (9D + 49)\,(3D + 34)]\, x = (7D + 38)\, t - (9D + 49)\, e^t$$

or $$(D^2 + 7D + 6)\, x = 7 + 38t - 58t^2 \qquad \text{... (3)}$$

$\therefore$ C.F. of (3) $= c_1e^{-t} + c_2e^{-6t}$, c_1 and c_2 being arbitrary constants

P.I. corresponding to $(7 + 38\,t)$ is

$$= \frac{1}{D^2 + 7D + 6}(7 + 38t) = \frac{1}{6[1 + (D^2 + 7D)/6]}(7 + 38t) = \frac{1}{6}\left[1 + \frac{D^2 + 7D}{6}\right]^{-1}(7 + 38t)$$

$$= \frac{1}{6}\left(1 - \frac{D^2 + 7D}{6} + \dots\right)(7 + 38t) = \frac{1}{6}\left[7 + 38t - \frac{7}{6} \times (38)\right] = \frac{19}{3}t - \frac{56}{9}.$$

P.I. corresponding to $(-58\, e^t) = -58\dfrac{1}{D^2 + 7D + 6}e^t = -\dfrac{29}{7}e^t$.

Hence, the solution of (3) is $\quad x = c_1e^{-t} + c_2e^{-6t} + (19/3)t - (29/77)\, e^t - (56/9) \qquad$ ... (4)

Now, $\qquad (4) \Rightarrow x_1 = dx/dt = -c_1e^{-t} - 6c_2\, e^{-6t} + (19/3) - (29/7)\, e^t \qquad$ ... (5)

Eliminating y_1 from given equations, we have

$x_1 + 2x + y = 7t - 9e^t$ so that $y = 7t - 9e^t - x_1 - 2x$

or $\quad y = 7t - 9e^t - \{-c_1e^{-t} - 6\, c_2e^{-6t} + (19/3) - (29/7)\, e^t\}$

$- 2\{c_1e^{-t} + c_2e^{-6t} + (19/3)\, t - (29/7)\, e^t - (56/9)\}$, using (4) and (5)

or $$y = -c_1e^{-t} + 4c_2\, e^{-6t} - (17/3)\, t + (24/7)\, e^t + (55/9) \qquad \text{... (6)}$$

(4) and (6) together give the required solution.

Ex. 10(*h*). *Solve :* $dx/dt = ax + by + c$, $dy/dt = a'x + b'y + c'$. **[Rajasthan 2004, 05]**

Sol. Given $$dx/dt - ax - by = c \qquad \text{... (1)}$$

and $$dy/dt - a'x - b'y = c' \qquad \text{... (2)}$$

Let $d/dt \equiv D$. Then (1) and (2) can be written as

$$(D - a)\, x - by = c \qquad \text{... (3)}$$

and $$-a'x + (D - b')\, y = c' \qquad \text{... (4)}$$

Eliminating y from (3) and (4), we have

$$[(D - b')\,(D - a) - a'b]\, x = (D - b')\, c + bc'$$

or $$[D^2 - (a + b')\, D + ab' - a'b]\, x = c'b - cb' \qquad \text{... (5)}$$

Here auxiliary equation of (5) is $\quad D^2 - (a + b')\, D + ab' - a'b = 0$

$$\Rightarrow D = \frac{a + b' \pm \sqrt{(a + b')^2 - 4(ab' - a'b)}}{2} = \frac{(a + b') \pm \sqrt{(a - b')^2 + 4a'b}}{2} \Rightarrow D = m_1, m_2 \text{ (say)}$$

$\therefore$ C.F. $= c_1 e^{m_1 t} + c_2 e^{m_2 t}$, c_1 and c_2 being arbitrary constants

$$\text{P.I.} = (c'b - cb')\frac{1}{D^2 - (a+b)D + ab' - a'b} e^{0.t} = \frac{c'b - cb'}{ab' - a'b}, \text{ provided } (ab' - a'b) \neq 0.$$

Hence, the general solution of (5) is $\quad x = c_1 e^{m_1 t} + c_2 e^{m_2 t} + \{(c'b - cb')/(ab' - a'b)\}$. ... (6)

Now, $\quad (6) \Rightarrow \quad dx/dt = c_1 m_1 e^{m_1 t} + c_2 m_2 e^{m_2 t}$. ... (7)

From (1), we have $\quad by = (dx/dt) - ax - c$

$\therefore \quad by = c_1 m_1 e^{m_1 t} + c_2 m_2 e^{m_2 t} - a[c_1 e^{m_1 t} + c_2 e^{m_2 t} + \{(c'b - cb')/(ab' - a'b)\}] - c$, by (6) and (7)

$$= (m_1 - a) c_1 e^{m_1 t} + (m_2 - a) c_2 e^{m_2 t} - \frac{a(c'b - cb') + c(ab' - a'b)}{ab' - a'b}$$

$$= (m_1 - a) c_1 e^{m_1 t} + (m_2 - a) c_2 e^{m_2 t} - \{b(ac' - ca')\}/(ab' - a'b)$$

$$\therefore \quad y = \frac{c_1}{b}(m_1 - a) e^{m_1 t} + \frac{c_2}{b}(m_2 - a) e^{m_2 t} - \frac{ac' - ca'}{ab' - a'b}. \quad ... (8)$$

(6) and (8) together give the required solution.

Ex. 11(*a*). *Solve $d^2x/dt^2 - 3x - 4y = 0$, $d^2y/dt^2 + x + y = 0$.* **[Agra 2001, 04; Kanpur 2003; [Garhwal 2005, Gorakhpur 1999, Delhi Maths Hons. 1992, Meerut 2009]**

Sol. Writing D for d/dt, the given equations become

$$(D^2 - 3)x - 4y = 0 \quad ... (1)$$

and

$$x + (D^2 + 1)y = 0 \quad ... (2)$$

Eliminating y from (1) and (2), $\quad [(D^2 + 1)(D^2 - 3) + 4]x = 0 \quad$ or $\quad (D^2 - 1)^2 y = 0$ ... (3)

Auxiliary equation for (3) is, $\quad (D^2 - 1)^2 = 0 \quad$ so that $\quad D = 1, 1, -1, -1$.

Hence solution of (3) is $\quad x = (c_1 + c_2 t) e^t + (c_3 + c_4 t) e^{-t}$, ... (4)

where c_1, c_2, c_3 and c_4 are arbitrary constants.

(4) $\Rightarrow \quad Dx = c_2 e^t + (c_1 + c_2 t) e^t + c_4 e^{-t} - (c_3 + c_4 t) e^{-t} = (c_1 + c_2 + c_2 t) e^t + (c_4 - c_3 - c_4 t) e^{-t}$.

$$\therefore \quad D^2 x = c_2 e^t + (c_1 + c_2 + c_2 t) e^t - c_4 e^{-t} - (c_4 - c_3 - c_4 t) e^{-t}$$

or

$$D^2 x = (c_1 + 2c_2 + c_2 t) e^t - (2c_4 - c_3 - c_4 t) e^{-t} \quad ... (5)$$

But from (1), $\quad 4y = D^2 x - 3x$ ... (6)

Hence using (4) and (5), (6) becomes

$$4y = (c_1 + c_2 + 2c_2 t) e^t - (2c_4 - c_3 - c_4 t) e^{-t} - 3[(c_1 + c_2 t) e^t + (c_3 + c_4 t) e^{-t}]$$

or

$$4y = 2(c_2 - c_1 - c_2 t) e^t - 2(c_4 - c_3 + c_4 t) e^{-t}$$

or

$$y = (1/2) \times (c_2 - c_1 - c_2 t) e^t - (1/2) \times (c_4 - c_3 + c_4 t) e^{-t} \quad ... (7)$$

The required solution is given by (4) and (7).

Ex. 11(*b*). *Solve $D^2x + m^2y = 0$, $D^2y - m^2x = 0$, where $D \equiv d/dt$.*

[Gwaliar 2004; Rajasthan 1997; Rohilkhand 1995, Agra 1998, Poona 1994]

Sol. Given

$$D^2 x + m^2 y = 0 \quad ... (1)$$

and

$$-m^2 x + D^2 y = 0. \quad ... (2)$$

Eliminating y from (1) and (2), $\quad (D^4 + m^4)x = 0 \quad$ whose auxiliary equation is $\quad D^4 + m^4 \equiv 0$.

or $\quad (D^4 + 2m^2D^2 + m^4) - 2m^2D^2 = 0 \quad$ or $\quad (D^2 + m^2)^2 - (m\sqrt{2}D)^2 = 0$

or

$$(D^2 + \sqrt{2}mD + m^2)(D^2 - \sqrt{2}mD + m^2) = 0.$$

$\therefore \quad D = \{-m\sqrt{2} \pm (2m^2 - 4m^2)^{1/2}\}/2, \qquad \{m\sqrt{2} \pm (2m^2 - 4m^2)^{1/2}\}/2$

or $\quad D = -(m/\sqrt{2}) \pm i\,(m/\sqrt{2}), \qquad (m/\sqrt{2}) \pm i(m/\sqrt{2})$

$$\therefore \; x = e^{-mt/\sqrt{2}}[c_1 \cos(mt/\sqrt{2}) + c_2 \sin(mt/\sqrt{2})] + e^{mt/\sqrt{2}}[c_3 \cos(mt/\sqrt{2}) + c_4 \sin(mt/\sqrt{2})]. \quad \text{... (3)}$$

$$Dx = e^{-mt/\sqrt{2}}\left[-\frac{c_1 m}{\sqrt{2}}\sin(mt/\sqrt{2}) + \frac{c_2 m}{\sqrt{2}}\cos(mt/\sqrt{2})\right] - \frac{m}{\sqrt{2}}e^{-mt/\sqrt{2}}[c_1\cos(mt/\sqrt{2}) + c_2\sin(mt/\sqrt{2})]$$

$$+ e^{mt/\sqrt{2}}\left[-(1/\sqrt{2})c_3 m \sin(mt/\sqrt{2}) + (1/\sqrt{2})c_4 m\cos(mt/\sqrt{2})\right]$$

$$+(m/\sqrt{2})e^{mt/\sqrt{2}}\left[c_3\cos(mt/\sqrt{2}) + c_4\sin(mt/\sqrt{2})\right]$$

$$= -(m/\sqrt{2})e^{-mt/\sqrt{2}}[(c_1 + c_2)\sin(mt/\sqrt{2}) + (c_1 - c_2)\cos(mt/\sqrt{2})]$$

$$+(m/\sqrt{2})e^{mt/\sqrt{2}}[(c_3 + c_4)\cos(mt/\sqrt{2}) + (c_4 - c_3)\sin(mt/\sqrt{2})]$$

$$\therefore \quad D^2 x = -\frac{m}{\sqrt{2}}e^{-mt/\sqrt{2}}\left[\frac{m(c_1 + c_2)}{\sqrt{2}}\cos\left(\frac{mt}{\sqrt{2}}\right) - \frac{m(c_1 - c_2)}{\sqrt{2}}\sin\left(\frac{mt}{\sqrt{2}}\right)\right]$$

$$+\frac{m^2}{2}e^{-mt/\sqrt{2}}\left[(c_1 + c_2)\sin\left(\frac{mt}{\sqrt{2}}\right) + (c_1 - c_2)\cos\left(\frac{mt}{\sqrt{2}}\right)\right]$$

$$+\frac{m}{\sqrt{2}}e^{mt/\sqrt{2}}\left[-\frac{m(c_3 + c_4)}{\sqrt{2}}\sin\left(\frac{mt}{\sqrt{2}}\right) + \frac{m(c_4 - c_3)}{\sqrt{2}}\cos\left(\frac{mt}{\sqrt{2}}\right)\right]$$

$$+\frac{m^2}{2}e^{mt/\sqrt{2}}\left[(c_3 + c_4)\cos\left(\frac{mt}{\sqrt{2}}\right) + (c_4 - c_3)\sin\left(\frac{mt}{\sqrt{2}}\right)\right]$$

$$= m^2 e^{-mt/\sqrt{2}}[c_1\sin(mt/\sqrt{2}) - c_2\cos(mt/\sqrt{2})] + m^2 e^{mt/\sqrt{2}}[c_4\cos(mt/\sqrt{2}) - c_3\sin(mt/\sqrt{2})]. \quad \text{... (4)}$$

Now (1), $\qquad y = -(1/m^2) \times D^2 x.$

or $\quad y = e^{-mt/\sqrt{2}}[c_2\cos(mt/\sqrt{2}) - c_1\sin(mt/\sqrt{2})] + e^{mt/\sqrt{2}}[c_3\sin(mt/\sqrt{2}) - c_4\cos(mt/\sqrt{2})]$, by (4) ... (5)

The required solution is given by (3) and (6).

Ex. 11(*c*). *Solve* $d^2x/dt^2 - 3x - 4y + 3 = 0, \quad d^2y/dt^2 + y + x + 5 = 0.$

[Delhi Maths (G) 1999]

Sol. Let $D \equiv d/dt$. Then the given equations become

$d^2x/dt^2 - 3x - 4y + 3 = 0 \qquad$ or $\qquad (D^2 - 3)\,x - 4y = -3 \qquad$ ... (1)

and $\quad d^2y/dt^2 + y + x + 5 = 0 \qquad$ or $\qquad x + (D^2 + 1)\,y = -5 \qquad$ ... (2)

Operate (1) by $(D^2 + 1)$ and multiply (2) by 4 and then add. Thus, we get

$\{(D^2 + 1)(D^2 - 3) + 4]\,x = -(D^2 + 1)3 - 20 \qquad$ or $\qquad (D^4 - 2D^2 + 1)\,x = -23 \qquad$ ... (3)

The auxiliary equation of (3) is $\quad (D^2 - 1)^2 = 0 \quad$ gives $\quad D = 1, 1, -1, -1$

$\therefore$ C.F. $= (C_1 + C_2 t)\,e^t + (C_3 + C_4 t)\,e^{-t}$, C_1, C_2, C_3 and C_4 being arbitrary constants.

$$\text{P.I.} = \frac{1}{D^4 - 2D^2 + 1}(-23)\,e^{0.t} = \frac{1}{0^2 - (2 \times 0^2) + 1}(-23)\,e^{0.t} = -23$$

$\therefore$ Solution of (3) is $\qquad x = (C_1 + C_2 t)\,e^t + (C_3 + C_4 t)\,e^{-t} - 23 \qquad$ ... (4)

From (4), $dx/dt = (C_1 + C_2 t)\, e^t + C_2 e^t - (C_3 + C_4 t)\, e^{-t} + C_4 e^{-t}$ ... (5)

From (5), $d^2x/dt^2 = (C_1 + C_2 t)\, e^t + 2C_2 e^t + (C_3 + C_4 t)\, e^{-t} - 2C_4 e^{-t}$ ... (6)

From (1), $4y = d^2x/dt^2 - 3x + 3$

or $4y = (C_1 + 2C_2 + C_2 t)\, e^t + (C_3 - 2C_4 + C_4 t)\, e^{-t} - 3(C_1 + C_2 t)\, e^t$
$+ (C_3 + C_4 t)\, e^{-t} - 23\} + 3$, using (4) and (6)

or $4y = (2C_2 - 2C_1 - 2C_2 t)\, e^t - (2C_4 + 2C_3 + 2C_4 t)\, e^{-t} + 72$

or $y = (1/2) \times (C_2 - C_1 - C_2 t)\, e^t - (1/2) \times (C_4 + C_3 + C_4 t)\, e^{-t} + 18$ ... (7)

The required solution is given by (4) and (7).

Ex. 11(*d*). *Solve the simultaneous equations $(d^2x/dt^2) + 4x + y = t\, e^{3t}$ and $(d^2y/dt^2) + y - 2x = \cos^2 t$.* **[Meerut 1997]**

Sol. Writing D for d/dt, the given equations become $(D^2 + 4)\, x + y = t\, e^{3t}$ ... (1)

and $-2x + (D^2 + 1)\, y = \cos^2 t.$ ... (2)

Operating both sides of (1) by $(D^2 + 1)$, we get

$$(D^2 + 1)\,(D^2 + 4)\, x + (D^2 + 1)\, y = (D^2 + 1)\,(t\, e^{3t})$$

or $(D^4 + 5D^2 + 4)\, x + (D^2 + 1)\, y = D\, \{D\, (te^{3t})\} + t\, e^{3t}$

or $(D^4 + 5D^2 + 4)\, x + (D^2 + 1)\, y = D\, (e^{3t} + 3t\, e^{3t}) + t\, e^{3t}$

or $(D^4 + 5D^2 + 4)\, x + (D^2 + 1)\, y = 3\, e^{3t} + 3\, (e^{3t} + 3t\, e^{3t}) + t\, e^{3t}.$

or $(D^4 + 5D^2 + 4)\, x + (D^2 + 1)\, y = 6\, e^{3t} + 10\, t\, e^{3t}.$ ... (3)

Subtracting (2) from (3), $(D^4 + 5D^2 + 6)\, x = 6e^{3t} + 10t\, e^{3t} - \cos^2 t$

or $(D^4 + 5D^2 + 6)\, x = 6\, e^{3t} + 10\, t\, e^{3t} - (1/2) \times (1 + \cos 2t),$ ... (4)

whose auxiliary equation is $D^2 + 5D^2 + 6 = 0$ so that $D = \pm i\sqrt{3}, \pm i\sqrt{2}$

$\therefore$ C.F. $= c_1 \cos\sqrt{3}t + c_2 \sin\sqrt{3}t + c_3 \cos\sqrt{2}t + c_4 \sin\sqrt{2}t$, c_1, c_2, c_3 and c_4 being arbitrary constants.

P.I. corresponding to $6e^{3t} = 6\dfrac{1}{D^4 + 5D^2 + 6} e^{3t} = 6\dfrac{1}{3^4 + (5 \times 3^2) + 6} e^{3t} = \dfrac{e^{3t}}{22}.$

P.I. corresponding to $10\, te^{3t} = 10\dfrac{1}{D^4 + 5D^2 + 6} t\, e^{3t} = 10\, e^{3t} \dfrac{1}{(D+3)^4 + 5(D+3)^2 + 6} t$

$$= 10\, e^{3t} \frac{1}{132 + 138D + \ldots} t = 10\, e^{3t} \frac{1}{132\{1 + (23/22)\, D + \ldots\}} t$$

$$= \frac{5\, e^{3t}}{66}\left(1 + \frac{23}{22} D + \ldots\right)^{-1} t = \frac{5\, e^{3t}}{66}\left(1 - \frac{23}{22} D + \ldots\right) t = \frac{5\, e^{3t}}{66}\left(t - \frac{23}{22}\right).$$

P.I. corresponding to $\left(-\dfrac{1}{2}\right) = \dfrac{1}{D^4 + 5D^2 + 6}\left(-\dfrac{1}{2}\right) = -\dfrac{1}{2}\dfrac{1}{D^4 + 5D^2 + 6} e^{0.t} = -\dfrac{1}{2}\cdot\dfrac{1}{6} = -\dfrac{1}{12}.$

P.I. corresponding to $\left(-\dfrac{1}{2}\cos 2t\right) = -\dfrac{1}{2}\dfrac{1}{D^4 + 5D^2 + 6}\cos 2t = -\dfrac{1}{2}\dfrac{1}{(D^2)^2 + 5D^2 + 6}\cos 2t$

$$= -\frac{1}{2}\frac{1}{\times (-2^2)^2 + 5 \times (-2^2) + 6}\cos 2t = -\frac{1}{4}\cos 2t.$$

$\therefore$ Solution of (4) is $x = c_1 \cos\sqrt{3}\, t + c_2 \sin\sqrt{3}\, t + c_3 \cos\sqrt{2}\, t + c_4 \sin\sqrt{2}\, t + (1/22)\, e^{3t}$
$+ (5/66)\, e^{3t}\, (t - 23/22) - (1/12) - (1/4) \cos 2t.$ ... (5)

Differentiating both sides of (5) w.r.t. 't' twice, we get

$$dx/dt = -c_1\sqrt{3}\sin\sqrt{3}\,t + c_2\sqrt{3}\cos\sqrt{3} - c_3\sqrt{2}\,\sin\sqrt{2}\,t + c_4\sqrt{2}\,\cos\sqrt{2}\,t$$
$$+ (3/22)\,e^{3t} + (5/22)\,e^{3t}\,(t - 23/22) + (5/66)\,e^{3t} + (1/2)\sin 2t$$

and $$d^2x/dt^2 = -3c_1\cos\sqrt{3}\,t - 3c_2\sin\sqrt{3}\,t - 2\,c_3\cos\sqrt{2}\,t - 2\,c_4\sin\sqrt{2}\,t$$
$$+ (9/22)\,e^{3t} + (15/22)\,e^{3t}\,(t - 23/22) + (5/22)\,e^{3t} + (5/22)\,e^{3t} + \cos 2t. \quad \text{... (6)}$$

Now, (1) $\Rightarrow$ $y = -(d^2x/dt^2) - 4x + t\,e^{3t}$

$\therefore$ $y = 3c_1\cos\sqrt{3}\,t + 3c_2\sin\sqrt{3}\,t + 2c_3\cos\sqrt{2}t + 2c_4\sin\sqrt{2}\,t - (9/22)e^{3t} - (15/22)\,e^{3t}\,(t - 23/22)$
$+ (5/11)\,e^{3t} - \cos 2t - 4[c_1\cos\sqrt{3}\,t + c_2\sin\sqrt{3}\,t + c_3\cos\sqrt{2}\,t + c_4\sin\sqrt{2}\,t + (1/22)\,e^{3t}$
$+ (5/66)\,e^{3t}\,(t - 23/22) - (1/12) - (1/4)\cos 2t] + t\,e^{3t}$, using (5) and (6)

$\therefore$ $y = -c_1\cos\sqrt{3}t - c_2\sin\sqrt{3}t - 2c_3\cos\sqrt{2}t - 2c_4\sin\sqrt{2}t + (1/66)te^{3t} - (23/2452)e^{3t} + (1/3).$ ... (7)

(5) and (7) together give the required solution.

Ex. 11(*e*). *Solve $(d^2x/dt^2) - (dy/dt) = 2x + 2t$, $(dx/dt) + 4(dy/dt) = 3y$.* **[G.N.D.U. 1997]**

Sol. Given $(d^2x/dt^2) - (dy/dt) - 2x = 2t$ ... (1)

and $(dx/dt) + 4\,(dy/dt) - 3y = 0.$ ... (2)

Writing D for d/dt, the given equations (1) and (2) become $(D^2 - 2)\,x - Dy = 2t$ ... (3)

and $Dx + (4D - 3)\,y = 0.$ ... (4)

Operating both sides of (3) and (4) by $(4D - 3)$ and D respectively, we get

$(4D - 3)\,(D^2 - 2)\,x - (4D - 3)\,Dy = 2\,(4D - 3)\,t$ ... (5)

and $D^2x + (4D - 3)\,Dy = 0.$ ... (6)

Adding (5) and (6), $\{(4D - 3)\,(D^2 - 2) + D^2\}\,x = 8\,Dt - 6t$

or $(2D^3 - D^2 - 4D + 3)\,x = 4 - 3t.$ ... (7)

Its auxiliary equation is $2D^3 - D^2 - 4D + 3 = 0$, giving $D = 1, 1, -3/2.$

$\therefore$ C.F. of (7) $= (c_1 + c_2t)\,e^t + c_3\,e^{-3t/2}$, c_1 and c_2 being arbitrary constants.

$$\text{P.I. of (7)} = \frac{1}{2D^3 - D^2 - 4D + 3}(4 - 3t) = \frac{1}{3[1 - (4D/3 + D^2/3 - 2D^3/3)]}(4 - 3t)$$
$$= (1/3) \times [1 - (4D/3 + D^2/3 - 2D^3/3)]^{-1}\,(4 - 3t)$$
$$= (1/3) \times \{1 + 4D/3 + \ldots\}\,(4 - 3t) = (1/3)\,\{4 - 3t + (4/3)\cdot(-3)\} = -t$$

$\therefore$ Solution of (7) is, $x = (c_1 + c_2t)\,e^t + c_3\,e^{-3t/2} - t.$ ... (8)

where c_1, c_2 and c_3 are arbitrary constants.

From (8), $dx/dt = (c_1 + c_2t)\,e^t + c_2\,e^t - (3/2)\,c_3\,e^{-3t/2} - 1$

or $dx/dt = (c_1 + c_2 + c_2t)\,e^t - (3/2)\,c_3\,e^{-3t/2} - 1.$ ... (9)

From (9), $d^2x/dt^2 = (c_1 + c_2 + c_2t)\,e^t + c_2\,e^t + (9/4)\,c_3\,e^{-3t/2}$

or $d^2x/dt^2 = (c_1 + 2c_2 + c_2t)\,e^t + (9/4)\,c_3\,e^{-3t/2}.$ ... (10)

(1) $\Rightarrow$ $dy/dt = (d^2x/dt^2) - 2x - 2t$

$= (c_1 + 2c_2 + c_2t)\,e^t + (9/4)\,c_3\,e^{-3t/2} - 2\,[(c_1 + c_2t)\,e^t + c_3\,e^{-3t/2} - t] - 2t$, by (8) and (10)

$\therefore$ $dy/dt = (2c_2 - c_1 - c_2t) + (1/4)\,c_3\,e^{-3t/2}$ ... (11)

(2) $\Rightarrow$ $3y = (dx/dt) + 4\,(dy/dt)$

$= (c_1 + c_2 + c_2t)\,e^t - (3/2)\,c_3\,e^{-3t/2} - 1 + 4\,[(2c_2 - c_1 - c_2t)\,e^t + (1/4)\,c_3\,e^{-3t/2}]$, by (9) and (11)

Thus, $3y = (9c_2 - 3c_1 - 3c_2\,t)\,e^t - (1/2)\,c_3\,e^{-3t/2} - 1.$

$\therefore$ $y = (3c_2 - c_1 - c_2t)\,e^t - (1/6)\,c_3\,e^{-3t/2} - (1/3).$ ... (12)

(8) and (12) together give the required solution.

Ex. 11(*f*). *Solve* $d^2x/dt^2 - 2(dy/dt) - x = e^t \cos t,\ d^2y/dt^2 + 2(dx/dt) - y = e^t \sin t.$

Sol. Given $$d^2x/dt^2 - x - 2(dy/dt) = e^t \cos t \qquad \text{... (1)}$$
and $$2\,(dx/dt) + d^2y/dt^2 - y = e^t \sin t. \qquad \text{... (2)}$$

Writing D for d/dt, (1) and (2) become
$$(D^2 - 1)\,x - 2\,Dy = e^t \cos t \qquad \text{... (3)}$$
and $$2Dx + (D^2 - 1)\,y = e^t \sin t. \qquad \text{... (4)}$$

Operating both sides of (3) and (4) by $(D^2 - 1)$ and $2D$ respectively, we get
$$(D^2 - 1)^2 x - 2D\,(D^2 - 1)\,y = (D^2 - 1)\,(e^t \cos t) \qquad \text{... (5)}$$
and $$4D^2x + 2D\,(D^2 - 1)\,y = 2D\,(e^t \sin t). \qquad \text{... (6)}$$

Adding (5) and (6), we have
$$[(D^2 - 1)^2 + 4D^2]\,x = D^2\,(e^t \cos t) - e^t \cos t + 2D\,(e^t \sin t)$$
or $$(D^2 + 1)^2\,x = D\,[e^t \cos t - e^t \sin t] - e^t \cos t + 2\,[e^t \sin t + e^t \cos t]$$
or $$(D^2 + 1)^2\,x = e^t \cos t - e^t \sin t - (e^t \sin t + e^t \cos t) - e^t \cos t + 2(e^t \sin t + e^t \cos t)$$
or $$(D^2 + 1)^2\,x = e^t \cos t. \qquad \text{... (7)}$$

whose auxiliary equation is $(D^2 + 1)^2 = 0$ so that $D = \pm\, i, \pm\, i.$

$\therefore$ C.F. of (7) $= (c_1 + c_2t) \cos t + (c_3 + c_4t) \sin t$, c_1, c_2, c_3, c_4 being arbitrary constants.

$$\text{P.I.} = \frac{1}{(D^2+1)^2} e^t \cos t = e^t \frac{1}{[(D+1)^2+1]^2} \cos t$$
$$= e^t \frac{1}{(D^2+2D+2)^2} \cos t = e^t \frac{1}{D^4+4D^2+4+4D^3+8D+4D^2} \cos t$$
$$= e^t \frac{1}{(-1)^2+4(-1)+4+4D(-1)+8D+4(-1)} \cos t$$
$$= e^t \frac{1}{4D-3} \cos t = e^t \frac{4D+3}{(4D-3)\,(4D+3)} \cos t$$
$$= e^t\,(4D+3) \frac{1}{16D^2-9} \cos t = e^t\,(4D+3) \frac{1}{-16-9} \cos t$$
$$= -(1/25) \times e^t (4\,D \cos t + 3 \cos t) = -(1/25) \times e^t (-4 \sin t + 3 \cos t).$$

$\therefore$ Solution of (7) is
$$x = (c_1 + c_2\,t) \cos t + (c_3 + c_4t) \sin t + (1/25)\, e^t\,(4 \sin t - 3 \cos t) \qquad \text{... (8)}$$

(8) $\Rightarrow$ $dx/dt = c_2 \cos t - (c_1 + c_2t) \sin t + c_4 \sin t + (c_3 + c_4\,t) \cos t$
$$+ (1/25)\,\{e^t\,(4 \sin t - 3 \cos t) + e^t\,(4 \cos t + 3 \sin t)\}$$
or $$dx/dt = (c_2 + c_3 + c_4t) \cos t + (c_4 - c_1 - c_2t) \sin t + (1/25) \times e^t\,(7 \sin t + \cos t) \qquad \text{... (9)}$$

(9) $\Rightarrow$ $d^2x/dt^2 = c_4 \cos t - (c_2 + c_3 + c_4t) \sin t - c_2 \sin t + (c_4 - c_1 - c_2t) \cos t$
$$+ (1/25) \times \{e^t\,(7 \sin t + \cos t) + e^t\,(7 \cos t - \sin t)\}$$
or $$d^2x/dt^2 = (2c_4 - c_1 - c_2t) \cos t - (2c_2 + c_3 + c_4t) \sin t + (1/25) \times e^t\,(6 \sin t + 8 \cos t) \qquad \text{... (10)}$$

From (1), we have $2\,(dy/dt) = (d^2x/dt^2) - x - e^t \cos t$

$\therefore$ $2(dydt) = (2c_4 - c_1 - c_2t) \cos t - (2\,c_2 + c_3 + c_4t) \sin t + (1/25)\, e^t\,(6 \sin t + 8 \cos t) - (c_1 + c_2\,t) \cos t$
$$- (c_3 + c_4t) \sin t - (1/25) \times e^t\,(4 \sin t - 3 \cos t) - e^t \cos t, \text{ by (8) and (10)}$$
$$= (2c_4 - 2c_1 - 2c_2t) \cos t - (2c_2 + 2c_3 + 2c_4t) \sin t + (1/25) \times e^t\,(2 \sin t - 14 \cos t).$$
or $$dy/dt = (c_4 - c_1 - c_2t) \cos t - (c_2 + c_3 + c_4t) \sin t + (1/25) \times e^t\,(\sin t - 7 \cos t). \qquad \text{... (11)}$$

(11) $\Rightarrow d^2y/dt^2 = -c_2 \cos t - (c_4 - c_1 - c_2 t) \sin t - c_4 \sin t - (c_2 + c_3 + c_4) \cos t$
$+(1/25) \times [e^t (\sin t - 7 \cos t) + e^t (\cos t + 7 \sin t)]$

or $d^2y/dt^2 = -(2c_2 + c_3 + c_4 t) \cos t - (2c_4 - c_1 - c_2 t) \sin t + (1/25) \times e^t (8 \sin t - 6 \cos t)$ ... (12)

Now, from (2), we have $y = 2\,(dx/dt) + (d^2y/dt^2) - e^t \sin t$

or $y = 2\,(c_2 + c_3 + c_4 t) \cos t + 2\,(c_4 - c_1 - c_2 t) \sin t + (2/25) \times e^t (7 \sin t + \cos t) - (2c_2 + c_3 + c_4 t) \cos t$

$-(2c_4 - c_1 - c_2 t) \sin t + (1/25) \times e^t (8 \sin t - 6 \cos t) - e^t \sin t$, using (9) and (12)

or $y = (c_3 + c_4 t) \cos t - (c_1 + c_2 t) \sin t + (1/25) \times e^t (4 \cos 3t - 3 \sin t)$ ... (13)

(8) and (13) give the required solution.

Exercise 8

Solve the following simultaneous differential equations :

1. (a) $dx/dt = x - 2y,\ dy/dt = 5x + 3y$ **[Andhra 2003]**

Ans. $x = e^{2t} (c_1 \cos 3t + c_2 \sin 3t),\ y = \{(3c_1 - c_2) \sin 3t - (c_1 + 3c_2) \cos 3t\}/2$

(b) $dx/dt = 3x + 2y,\ dy/dt = -5x - 3y$ **[Lucknow 2002; Meerut 2000]**

Ans. $x = c_1 \cos t + c_2 \sin t,\ y = (1/2) \times (c_2 - 3c_1) \cos t - (1/2) \times (c_1 + 3c_2) \sin t$

2. $dz/dx = x + y,\ dy/dx = x + z$ **Ans.** $y = \{c_1 e^{-x} + c_2 e^x - 2(x+1)\}/2,\ z = \{c_2 e^x - c_1 e^{-x} - 2(x+1)\}/2$

3. $dx/dt + 7x - y = 0,\ dy/dt + 2x + 5y = 0$ **[Agra 2006; Kanpur 2003; Lucknow 2005]**

Ans. $x = e^{-6t} (c_1 \cos t + c_2 \sin t),\ y = e^{-6t} \{(c_1 + c_2) \cos t - (c_1 - c_2) \sin t\}$

4. (a) $dx/dt = 3x + 2y,\ dy/dt + 5x + 3y = 0$ **Ans.** $x = c_1 \cos t + c_2 \sin t,\ y = \{(c_2 - 3c_1) \cos t - (c_1 + 3c_2) \sin t\}/2$

(b) $dx/dt + dy/dt + 2x + y = 0,\ dy/dt + 5x + 3y = 0$

[Gujrat 2005, Indore 2003; Karnatak 2000; Meerut 1998; Vikram 1998)

Ans. $x = c_1 \cos t + c_2 \sin t,\ y = (1/2) \times (c_2 - 3c_1) \cos t - (1/2) \times (c_1 + 3c_2) \sin t$

5. $dx/dt + x - y = e^t,\ dy/dt + y - x = 0$ **Ans.** $x = c_1 + c_2 e^{-2t} + (2t/3),\ y = c_1 - c_2 e^{-2t} + (1/3)\, e^t$

6. $dx/dt - y = e^{-t},\ dy/dt + x = e^t$ **Ans.** $x = c_1 \cos t + c_2 \sin t + (e^t - e^{-t})/2,\ y = c_2 \cos t - c_1 \sin t + (e^t - e^{-t})/2$

7. $(dx/dt) - y = t^2,\ (dy/dt) + 4x = t$, given $x(0) = 0,\ y(0) = 3/4$. **Ans.** $x = 3t/4,\ y = (3/4) - t^2$

8. $(5D + 4)\, y - (2D + 1)\, z = e^{-x},\ (D + 8)\, y - 3z = 5e^{-x}$, where $D \equiv d/dx$ **[Pune 2000; Kolkata 2003]**

Ans. $y = c_1 e^{-2x} + c_2 e^x + 2e^{-x},\ z = 2c_1 e^{-2x} + 3c_2 e^x + 3e^{-x}$

9. $\left(\dfrac{d}{dt} + 2\right) x + 3y = 0,\ 3x + \left(\dfrac{d}{dt} + 2\right) y = 2e^{3t}$ **Ans.** $x = c_1 e^t - c_2 e^{-5t} - (3/8)\, e^{3t},\ y = c_2 e^{-5t} - c_1 e^t + (5/8)\, e^{3t}$

10. $dx/dt + 2x + 4y = 1 + 4t,\ dy/dt + x - y = 3t^2/2$ **[I.A.S. 1987]**

Ans. $x = c_1 e^{2t} + c_2 e^{-3t} + t^2 + (t/3) - (5/6),\ y = -4c_1 e^{2t} + c_2 e^{-3t} - 2t^2 + (4t/3) + (7/3)$

11. $dx/dt - dy/dt - y = e^t,\ dy/dt + x - y = e^{2t}$ **[Agra 1994]**

Ans. $x = c_1 + \cos t\, c_2 \sin t + (3/5)\, e^{2t},\ y = (1/2) \times (c_1 + c_2) \cos t - (1/2)(c_1 - c_2) \sin t + (1/2)\, e^t + (2/5)\, e^{2t}$

12. $4(dx/dt) - (dy/dt) + 3x = \sin t,\ (dx/dt) + y = \cos t$, given that $x = 1,\ y = 1$ for $t = 0$. **Ans.** $x = 2e^{-t} - e^{-3t},\ y = 2e^{-t} - 3e^{-3t} + \cos t$

13. $(dx/dt) + 2\,(dy/dt) + x + 7y = e^t - 3,\ (dy/dt) - 2x = 3y = 12 - 3e^t$

Ans. $x = (1/2) \times e^{-4t} \{(c_2 - c_1) \cos t - (c_1 + c_2) \sin t\} + (31/26)\, e^t - (3/17)$
$y = e^{-4t} (c_1 \operatorname{cost} t + c_2 \sin t) + (6/17) - (2/13)\, e^t$

14. $3\,(dx/dt) + 2\,(dy/dt) - 4x + 3y = 8e^{-3t},\ 4\,(dx/dt) + (dy/dt) + 3x + 4y = 8e^{-3t}$, given that $x = 1/5,\ y = 0$ when $t = 0$.

Ans. $x = e^{-t} \{\cos 2t - (1/18) \times \sin 2t\} - (4/5)\, e^{-3t},\ y = e^{-t} \{(21/10) \sin 2t - (4/5) \cos 2t\} + (4/5)\, e^{-3t}$

15. Solve $4x_1 + 9y_1 + 2x + 31y = e^t,\ 3x_1 + 7y_1 + x + 24y = 3$, where, $x_1 = dx/dt$ and $y_1 = dy/dt$.

Ans. $x = e^{-4t} (c_1 \cos t + c_2 \sin t) + (31/26)\, e^t - (93/17)$.
$y = e^{-4t} [(c_2 - c_1) \sin t - (c_1 + c_2) \cos t] - (2/13)\, e^t + (6/17)$

16. $d^2x/dt^2 + 16x - 6(dy/dt) = 0,\ 6(dx/dt) + d^2y/dt^2 + 164 = 0$. **[Agra 1994]**

Ans. $x = c_1 \cos 2t + c_2 \sin 2t + c_3 \cos 8t + c_4 \sin 8t,\ y = c_1 \sin 2t + c_2 \cos 2t + c_3 \sin 8t + c_4 \cos 8t$

17. $d^2x/dt^2 - 4(dx/dt) + 4x = y,\ d^2y/dt^2 + 4(dy/dt) + 4y = 25 + 16\, e^t$.

Ans. $x = c_1 e^{3t} + c_2 e^{-3t} + c_3 \cos t + c_4 \sin t - e^t$, $y = c_1 e^{3t} + 25c_2 e^{-3t} + 7c_3 \cos t - c_4 \sin t - e^t$

18. $d^2x/dt^2 + 4x + y = t\,e^t,\ d^2y/dt^2 + y - 2x = \sin^2 t$.

Ans. $x = c_1 \cos \sqrt{2}\,t + c_2 \sin \sqrt{2}\,t + c_3 \cos \sqrt{3}\,t + c_4 \sin \sqrt{3}\,t + (1/6)\, e^t (6t-1) - (1/12) + (1/4) \cos 2t$;

$y = -2\, c_1 \cos \sqrt{2}\,t - 2\, c_2 \sin \sqrt{2}\,t - c_3 \cos \sqrt{3}\,t - c_4 \sin \sqrt{3}\,t + (1/36)\, e^t (6t-7) + (1/3)$

19. Solve $d^2x/dt^2 - 2(dy/dt) - x = e^t \cos t,\ d^2y/dt^2 + 2(dx/dt) - y = e^t \sin t$.

Ans. $x = (c_1 + c_2 t) \cos t + (c_3 + c_4 t) \sin t - (1/25)\, e^t (3 \cos t - 4 \sin t)$;

$y = -(c_1 + c_2 t) \sin t + (c_3 + c_4 t) \cos t - (1/25)\, e^t (3 \sin t + 4 \cos t)$

8.4 Solution of simultaneous differential equations involving operators $x\,(d/dx)$ or $t\,(d/dt)$

In such problems we begin with use of methods of chapter 6. We transform the given equations into ordinary simultaneous differential equations and then proceed as explained in Art. 8.3.

8.5 Solved Examples based on Art. 8.4

Ex. 1. *Solve* $t\,(dx/dt) + y = 0,\ t\,(dy/dt) + x = 0.$

[G.N.D.U. Amritsar 2004; Rajasthan 2010, Meerut 2001, 08, Lucknow 2001, 06]

Sol. Let $t = e^z$. Let $D_1 \equiv d/dz \equiv t(d/dt)$. Then given equations become

$$D_1 x + y = 0 \qquad \text{... (1)}$$

and

$$x + D_1 y = 0. \qquad \text{... (2)}$$

Eliminating y from (1) and (2), $\quad D_1^2 x - x = 0 \quad$ or $\quad (D_1^2 - 1)\, x = 0.$ ... (3)

Its auxiliary equation is $\quad D_1^2 - 1 = 0 \quad$ so that $\quad D_1 = 1, -1.$

$\therefore$ Solution of (3) is $\quad x = c_1 e^z + c_2 e^{-z} \quad$ and so $\quad D_1 x = c_1 e^z - c_2 e^{-z}$

$\therefore$ From (1), $\quad y = -D_1 x = c_2 e^{-z} - c_1 e^z$

Since $t = e^z$, the required solution is $\quad x = c_1 t + c_2 t^{-1}, \quad y = c_2 t^{-1} - c_1 t.$

Ex. 2. *Solve :* $t^2\,(d^2x/dt^2) + t\,(dx/dt) + 2y = 0,\ t^2\,(d^2y/dt^2) + t\,(dy/dt) - 2x = 0.$

Sol. Let $t = e^z$ so that $z = \log t$. Let $D_1 \equiv d/dz = t\,(d/dt)$. Then $t^2\,(d^2/dt^2) = D_1\,(D_1 - 1)$.

Using the above values, given equations become

$$[D_1(D_1 - 1) + D_1]\, x + 2y = 0 \quad \text{and} \quad [D_1(D_1 - 1) + D_1]\, y - 2x = 0.$$

i.e.,

$$D_1^2 x + 2y = 0 \qquad \text{... (1)}$$

and

$$-2x + D_1^2 y = 0 \qquad \text{... (2)}$$

Eliminating y from (1) and (2), $\quad (D_1^4 + 4)\, x = 0$ ... (3)

Its auxiliary equation is $\quad D_1^4 + 4 = 0 \quad$ or $\quad (D_1^2 + 2)^2 - (2D_1)^2 = 0.$

$(D_1^2 - 2D_1 + 2)\,(D_1^2 + 2D_1 + 2) \quad$ so that $\quad D = 1 \pm i, -1, \pm i.$

$\therefore$ Solution of (3) is $\quad x = e^z\,(c_1 \cos z + c_2 \sin z) + e^{-z}\,(c_3 \cos z + c_4 \sin z).$ ... (4)

where c_1, c_2, c_3 and c_4 are arbitrary constants.

$$\therefore\ D_1 x = e^z\,(c_1 \cos z + c_2 \sin z) - e^{-z}\,(c_3 \cos z + c_4 \sin z) + e^z\,(-c_1 \sin z + c_2 \cos z) + e^{-z}\,(-c_3 \sin z + c_4 \cos z)$$

$$= e^z\,[(c_1 + c_2) \cos z + (c_2 - c_1) \sin z] + e^{-z}\,[(c_4 - c_3) \cos z - (c_3 + c_4) \sin z].$$

$$\therefore\ D_1^2 x = e^z\,[(c_1 + c_2) \cos z + (c_2 - c_1) \sin z] + e^z\,[-(c_1 + c_2) \sin z + (c_2 - c_1) \cos z] - e^{-z}\,[(c_4 - c_3) \cos z - (c_3 + c_4) \sin z] + e^{-z}\,[-(c_4 - c_3) \sin z - (c_3 + c_4) \cos z]$$

or

$$D_1^2 x = 2e^z\,(c_2 \cos z - c_1 \sin z) + 2e^{-z}\,(c_3 \sin z - c_4 \cos z).$$

Using this value of $D_1^2 x$ in (1), we get

$$y = e^z\,(c_1 \sin z - c_2 \cos z) + e^{-z}\,(c_4 \cos z - c_3 \sin z). \qquad \text{... (5)}$$

Since $t = e^z$ and $\log t = z$, from (4) and (5) the required solution is given by

$$x = t\,(c_1 \cos \log t + c_2 \sin \log t) + t^{-1}\,(c_3 \cos \log t + c_4 \sin \log t)$$

and

$$y = t\,(c_1 \sin \log t - c_2 \cos \log t) + t^{-1}\,(c_4 \cos \log t - c_3 \sin \log t).$$

Ex. 3. $t\,Dx + 2\,(x - y) = t,\ t\,Dy + x + 5y = t^2$, *where* $D \equiv d/dt$. **[Lucknow 2002]**

Sol. Let $t = e^z$ and $D_1 \equiv d/dz \equiv t\,(d/dt)$. Given equations become

$$(D_1 + 2)\, x - 2y = e^z \qquad \text{... (1)}$$

and

$$x + (D_1 + 5)\, y = e^{2z}. \qquad \text{... (2)}$$

Eliminating y from (1) and (2), $\quad (D_1 + 5)(D_1 + 2)\, x + 2x = (D_1 + 5)e^z + 2e^{2z}$

or

$$(D_1^2 + 7D_1 + 12)\, x = 6\, e^z + 2\, e^{2z}. \qquad \text{... (3)}$$

Its auxiliary equation is $\quad D_1^2 + 7D_1 + 12 = 0 \quad$ giving $\quad D_1 = -3, -4,$

C.F. of (3) $= c_1e^{-3z} + c_2e^{-4z}$, where c_1 and c_2, are arbitrary constants.

$$\text{P.I. corresponding to } 6e^z = 6\frac{1}{D_1^2+7D_1+12}e^z = \frac{3}{10}e^z.$$

$$\text{P.I. corresponding to } 2e^{2z} = 2\frac{1}{D_1^2+7D_1+12}e^{2z} = \frac{1}{15}e^{2z}.$$

$\therefore$ Solution of (3) is $\quad x = c_1e^{-3z} + c_2e^{-4z} + (3/10)\times e^z + (1/15)\times e^{2z}$ ... (4)

$\therefore \quad D_1x = -3c_1e^{-3z} - 4c_2e^{-4z} + (3/10)\,e^z + (2/15)\times e^{2z}$ ... (5)

From (1) and (5), $\quad y = -(1/2)\,c_1e^{-3z} - c_2e^{-4z} - (1/20)\,e^z + (2/15)\,e^{2z}$ ... (6)

Putting $t = e^z$ in (4) and (6), the required general solution is

$$x = c_1t^{-3} + c_2t^{-4} + 3t/10 + t^2/15, \qquad y = -(1/2)\,c_1t^{-3} - c_2t^{-4} + 2t^2/15 - t/20$$

8.6 Miscellaneous examples on chapter 8

Ex. 1. *Solve* $t\,dx = (t-2x)\,dt$ *and* $t\,dy = (tx+ty+2x-t)\,dt$.

Sol. Given $\quad t\,dx = (t-2x)\,dt.$ ... (1)

and $\quad t\,dy = (tx+ty+2x-t)\,dt.$ ... (2)

From (1), $\quad \dfrac{dx}{dt} = 1 - \dfrac{2x}{t} \quad$ or $\quad \dfrac{dx}{dt} + \dfrac{2}{t}x = 1$, which is a linear equation.

Its $\quad$ I.F. $= e^{\int(2/t)dt} = e^{2\log t} = t^2 \quad$ and so its solution is

$$xt^2 = c_1 + \int t^2\,dt = c_1 + (t^2/3) \qquad \text{or} \qquad x = c_1t^{-2} + (t/3) \qquad \text{... (3)}$$

Now from (2), $\quad t\,dy = t\,(x+y)\,dt - (t-2x)\,dt \quad$ or $\quad t\,dy = t\,(x+y)\,dt - t\,dx$, using (1)

or $\quad dx + dy = (x+y)\,dt \quad$ or $\quad (dx+dy)/(x+y) = dt.$

Integrating, $\quad \log(x+y) - \log c_2 = t \quad$ or $\quad x + y = c_2e^t \quad$ or $\quad y = c_2e^t - x.$

or $\quad y = c_2e^t - c_1t^{-2} - (1/3)\,t$, using (3) ... (4)

The required solution is given by (3) and (4).

Ex. 2. *Solve* $\quad dx/dt = ny - mz, \quad dy/dt = lz - nx, \quad dz/dt = mx - ly.$ **[Meertut 1996]**

Sol. Given $\quad dx/dt = ny - mz.$ ... (1)

$dy/dt = lz - nx$ ... (2)

and $\quad dz/dt = mx - ly.$ ... (3)

Multiplying (1), (2) and (3) by $2x$, $2y$ and $2z$ respectively and adding,

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} + 2z\frac{dz}{dt} = 0 \qquad \text{or} \qquad \frac{d}{dt}(x^2+y^2+z^2) = 0.$$

Integrating, $\quad x^2 + y^2 + z^2 = c_1$, c_1 being an arbitrary constant ... (4)

Again multiplying (1), (2), (3) by $2lx$, $2my$, $2nz$ respectively and then adding, we have

$$2lx\frac{dx}{dt} + 2my\frac{dy}{dt} + 2nz\frac{dz}{dt} = 0 \qquad \text{or} \qquad \frac{d}{dt}(lx^2+my^2+nz^2) = 0.$$

Integrating, $\quad lx^2 + my^2 + nz^2 = c_2$, c_2 being an arbitrary constant ... (5)

Now multiplying (1), (2) and (3) by l, m and n respectively and adding,

$$l\frac{dx}{dt} + m\frac{dy}{dt} + n\frac{dz}{dt} = 0 \qquad \text{or} \qquad \frac{d}{dt}(lx+my+nz) = 0.$$

Integrating, $\quad lx + my + nz = c_3$, c_3 being an arbitrary constant ... (6)

The required solution is given by (4), (5) and (6).

Ex. 3. *Solve :* $\quad lt\dfrac{dx}{dt} = mn(y-z), \quad mt\dfrac{dy}{dt} = nl(z-x), \quad nt\dfrac{dz}{dt} = lm(x-y).$

Sol. Re-writing the given equations, we have

$$\frac{ldx}{(1/t)\,dt} = mn(y-z), \qquad \frac{mdy}{(1/t)\,dt} = nl(z-x), \qquad \frac{ndz}{(1/t)\,dt} = lm(x-y). \qquad \text{... (1)}$$

Putting $\quad ldx = dX, \quad mdy = dY, \quad ndz = dZ \quad$ and $\quad (1/t)\,dt = dT,$ ... (2)

(1) $\Rightarrow dX/dT = nY - mZ, \qquad dY/dT = lZ - nX, \qquad dZ/dT = mX - lY.$

Now proceed as in Ex. 3 and finally replace X, Y, Z and T by their values given by (2), namely $X = lx$, $Y = my$, $Z == nz$ and $T = \log t$.

Ex. 4. *$dx/dt = 2y$, $dy/dt = 2z$ and $dz/dt = 2x$.*

Sol. Let as an exercise. **Ans.** $x = c_1 e^{2t} + c_2 e^{-t} \cos\,(\sqrt{3}\,t + c_3)$,

$$y = c_1 e^{2t} + c_2 e^{-t} \cos(\sqrt{3}\,t + c_3 + 2\pi/3), \qquad z = c_1 e^{2t} + c_2 e^{-t} \cos(\sqrt{3}\,t + c_3 + 4\pi/3)$$

Ex. 5. *Solve* $\quad (D + 1)\,x + (D - 1)\,y = e^t.$ ... (1)

$$(D^2 + D + 1)\,x + (D^2 - D + 1)\,y = t^2 \qquad \text{... (2)}$$

where $D \equiv d/dt$. **[Kurukshetra 2005; 07; Mysore 2001, 03]**

Sol. Here the determinant Δ formed by operator 'coefficients' is given by

$$\Delta = \begin{vmatrix} D+1 & D-1 \\ D^2+D+1 & D^2-D+1 \end{vmatrix} = (D+1)(D^2-D+1) - (D-1)(D^2+D+1)$$

or $\qquad \Delta = (D^3 + 1) - (D^3 - 1) = 2.$

Since the degree of D in Δ is zero, the general solution of the given system should not contain any arbitrary constant (refer note of Art 8.2 for understanding).

Now operating (1) by $(D^2 - D + 1)$, (2) by $(D - 1)$ and then subtracting the equations thus obtained, we get

$$[(D^2 - D + 1)(D + 1) - (D - 1)(D^2 + D + 1)]\,x = (D^2 - D + 1)\,e^t - (D - 1)\,t^2$$

or $\qquad [(D^3 + 1) - (D^3 - 1)]\,x = e^t - e^t + e^t - 2t + t^2$

or $\qquad 2x = e^t - 2t + t^2 \qquad$ or $\qquad x = (1/2)\,(e^t - 2t + t^2).$ ... (3)

Similarly, on eliminating x, (1) and (2) give

$$[(D^2 + D + 1)(D - 1) - (D + 1)(D^2 - D + 1)]\,y = (D^2 + D + 1)\,e^t - (D + 1)\,t^2$$

or $\qquad [D^3 - 1 - (D^3 + 1)]\,y = e^t + e^t + e^t - 2t - t^2$

or $\qquad -2y = 3e^t - 2t - t^2 \qquad$ or $\qquad y = (2t + t^2 - 3e^t)/2$ ... (4)

The required solution is given by (3) and (4).

Ex. 6. Solve $dx/dt = x^2 + xy$, $dy/dt = y^2 + xy$, satisfying the initial condition $x = 1$, $y = 2$ when $t = 0$.

Sol. Given $\qquad dx/dt = x\,(x + y)$ ...(1)

$\qquad dy/dt = y\,(x + y)$ ...(2)

Given initial condition are $\qquad x = 1, \quad y = 2 \quad$ when $\quad t = 0$ ...(3)

Dividing (2) by (1), we get $\qquad dy/dx = y/x \qquad$ or $\qquad (1/y)dy = (1/x)\,dx$

Integrating it, we get $\qquad \log y = \log x + \log c \qquad$ or $\qquad y = cx$, ...(4)

where c in an arbitrary constant.

Putting $x = 1$ and $y = 2$ in (4), we get $c = 2$. Hence (4) reduce to

$$y = 2x \qquad \text{...(5)}$$

Using (5), (1) gives $\qquad dx/dt = x\,(x + 2x) \qquad$ or $\qquad (1/x^2)dx = 3dt$

Integrating it, $\qquad -(1/x) = 3t + c_1$, c_1 being an arbitrary constant ...(6)

Putting $x = 1$ and $t = 0$ in (6) we get $\qquad c_1 = -1$

$\therefore$ (6) reduces to $\qquad -(1/x) = 3t - 1 \qquad$ or $\qquad x = 1/(1-3t)$ ...(7)

From (5) and (7), we get $\qquad y = 2/(1-3t)$ ...(8)

Hence the required solution is given by (7) and (8).

Objective problems on chapter 8

Ex. 1. *The general solution of the system of equations $y + (dz/dx) = 0$, $dy/dx - z = 0$ is given by*

(a) $y = \alpha e^x + \beta e^{-x}$, $z = \alpha e^x - \beta e^{-x}$ *(b)* $y = \alpha \cos x + \beta \sin x$, $z = \alpha \sin x - \beta \cos x$

(c) $y = \alpha \sin x - \beta \cos x$, $z = \alpha \cos x + \beta \sin x$ *(d)* $y = \alpha e^x - \beta e^{-x}$, $z = \alpha e^x + \beta e^{-x}$

[GATE 2005]

Sol. Ans. (*c*). Use Art 8.2 and Art. 8.3.

Ex. 2. The general solution $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ *of the system $x' = -x + 2y$, $y' = 4x + y$ is given by*

(a) $\begin{pmatrix} c_1e^{3t} - c_2e^{-3t} \\ 2c_1e^{3t} + c_2e^{-3t} \end{pmatrix}$ (b) $\begin{pmatrix} c_1e^{3t} \\ c_2e^{-3t} \end{pmatrix}$ (c) $\begin{pmatrix} c_1e^{3t} + c_2e^{-3t} \\ 2c_1e^{3t} + c_2e^{-3t} \end{pmatrix}$ (d) $\begin{pmatrix} c_1e^{3t} - c_2e^{-3t} \\ -2c_1e^{3t} + c_2e^{-3t} \end{pmatrix}$

[GATE 2004]

Sol. Ans. (*a*) Writing D for d/dt, the given equations become

$$(D + 1)\, x - 2y = 0 \qquad \text{... (1)}$$

and
$$-4x + (D - 1)\, y = 0 \qquad \text{... (2)}$$

Operating (1) by $(D - 1)$, multiplying (2) by 2 and then adding the resulting equations, we get

$$(D - 1)(D + 1)\, x - 8x = 0 \qquad \text{or} \qquad (D^2 - 9)x = 0$$

whose solution is
$$x = c_1'e^{3t} + c_2'e^{-3t} \qquad \text{... (3)}$$

From (1), $\quad 2y = dx/dt + x = 3c_1'e^{3t} - 3c_2'e^{-3t} + c_1'e^{3t} + c_2'e^{-3t}$, using (3)

Thus,
$$y = 2c_1'e^{3t} - c_2'e^{-3t} \qquad \text{...(4)}$$

Setting $c_1' = c_1$ and $c_2' = -c_2$, (3) and (4) yield, $x = c_1e^{3t} - c_2e^{-3t}$, $y = 2c_1e^{3t} + c_2e^{-3t}$ ...(5)

Putting equations of (5) in matrix form, we get $\begin{pmatrix} x(t) \\ 3(t) \end{pmatrix} = \begin{pmatrix} c_1e^{3t} - c_2e^{-3t} \\ 2c_1e^{3t} + c_2e^{-3t} \end{pmatrix}$

Miscellaneous Examples on Chapter 8

Ex. 1. *Solve the simultaneous differential equations $2(D - 2)x + (D + 1)y = e^{2t}$, $(D + 2)x + (D - 3)y = 0$, $D = d/dt$* **[Guwahati 2007]**

Sol. Given
$$2(D - 2)x + (D + 1)y = e^{2t} \qquad \text{... (1)}$$
$$(D+2)x + (D-3)y = 0 \qquad \text{... (2)}$$

Operating (1) by $(D - 3)$ and (2) by $(D + 1)$ and then subtracting, we have

$$\{2(D-3)(D-2) - (D+1)(D+2)\}x = (D-3)e^{2t} \qquad \text{or} \qquad (D^2 - 13D + 10)x = -e^{2t} \quad \text{... (3)}$$

Auxiliary equation of (3) is $\quad D^2 - 13D + 10 = 0 \quad$ giving $\quad D = (13 \pm \sqrt{129})/2$

$\therefore$ C.F. of (3) = $c_1\, e^{x(13+\sqrt{129})/2} + c_2\, e^{x(13-\sqrt{129})/2}$, c_1, c_2 being arbitrary constants

$$\text{P.I of (3)} = \frac{1}{D^2 - 13D + 10}(-e^{2t}) = -\frac{1}{2^2 + (13 \times 2) + 10}e^{2t} = -\frac{1}{40}e^{2t}$$

$\therefore$ General solution of (3) is given by

$$x = c_1\, e^{x(13+\sqrt{129})/2} + c_2\, e^{x(13-\sqrt{129})/2} - (1/40) \times e^{2t} \qquad \text{... (4)}$$

From (2),
$$Dy = 3y - Dx - 2x \qquad \text{... (5)}$$

From (1),
$$y = e^{2t} - Dy - 2Dx + 4x$$

or
$$y = e^{2t} - (3y - Dx - 2x) - 2Dx + 4x, \text{ using (5)}$$

or
$$4y = e^{2t} - Dx + 6x \qquad \text{... (6)}$$

From (4), we have

$$Dx = (c_1/2) \times (13 + \sqrt{129})e^{x(13+\sqrt{129})/2} + (c_2/2) \times (13 - \sqrt{129})\, e^{x(13-\sqrt{129}/2} - (1/20) \times e^{2t} \quad \text{..... (7)}$$

Substituting the valvue of x and Dx given by (4) and (7) respectively in (6), we have

$$4y = e^{2t} - (c_1/2) \times (13 + \sqrt{129})\, e^{x(13+\sqrt{129})/2} - (c_2/2) \times (13 - \sqrt{129})\, e^{x(12-\sqrt{129})/2}$$
$$+ (1/20) \times e^{2t} + 6c_1e^{x(13+\sqrt{129})/2} + 6c_2e^{x(13-\sqrt{129})/2} - (3/20)\, e^{2t}$$

or
$$4y = (9/10) \times e^{2t} - (c_1/2) \times (1 + \sqrt{129})\, e^{x(13+\sqrt{129})/2} - (c_2/2) \times (1 - \sqrt{129})\, e^{x(12-\sqrt{129})/2}$$

or
$$y = (9/40) \times e^{2t} - (c_1/8) \times (1 + \sqrt{129})\, e^{x(13+\sqrt{129})/2} - (c_2/8) \times (1 - \sqrt{129})\, e^{x(12-\sqrt{129})/2} \quad \text{... (8)}$$

The required solution is given by (4) and (8).

9

Exact Differential Equations And Equations of Special Forms

9.1 Exact Differential Equation. Definition.

A differential equation, $$f(d^n y/dx^n, d^{n-1}y/dx^{n-1}, ..., dy/dx, y) = \phi(x), \qquad ...(1)$$
is said to be *exact* when it can be derived by differntiation only, and without any further process, from an equation of the next lower order of the form

$$f(d^{n-1}y/dx^{n-1}, d^{n-2}y/dx^{n-2}, ..., dy/dx, y) = \int \phi(x)dx + C. \qquad ...(2)$$

Remark 1. Equation (2) is said to be a *first integral* of (1). If (2) is also exact, then it can be obtained from an equation of the next lower order of the form

$$f(d^{n-2}y/dx^{n-2}, d^{n-3}y/dx^{n-3}, ..., dy/dx, y) = \iint \phi(x)(dx)^2 + C. \qquad ...(3)$$

as before. Equation (3) is said to be a *second integral* of (1). In general, there will be n integrals for a differential equation of the nth order.

9.2 Condition of exactness of a linear differential equation of order *n* [Garhwal 2005]

Let the linear differential equation of order n be

$$P_0 \ (dy^n/dx^n) + P_1 \ (dy^{n-1}/dx^{n-1}) + ... + P_n y = \phi(x), \qquad ...(1)$$

where P_0, P_1, ..., P_n and ϕ are functions of x alone. Let (1) be exact *i.e.*, it can be obtained from an equation of next lower order simply by differentition. In what follows the successive derivatives will be denoted by dashes. Since $P_0(d^ny/dx^n)$ can be obtained by simply differentiating once $P_0(d^{n-1}y/dx^{n-1})$, we assume that (1) can be obtained by differentiating once the equation

$$P_0 \frac{d^{n-1}y}{dx^{n-1}} + Q_1 \frac{d^{n-2}y}{dx^{n-2}} + ... + Q_{n-1}y = \int \phi(x)dx = C, \qquad ...(2)$$

where Q_1, Q_2, ..., Q_{n-1} are some functions of x alone. Differentiating (2) with respect to x, we get

$$\left(P_0 \frac{d^n y}{dx^n} + P_0' \frac{d^{n-1}y}{dx^{n-1}}\right) + \left(Q_1 \frac{d^{n-1}y}{dx^{n-1}} + Q_1' \frac{d^{n-2}y}{dx^{n-2}}\right) + ... + Q_{n-1}\frac{dy}{dx} + Q_{n-1}' y = \phi(x)$$

or $$P_0 \frac{d^n y}{dx^n} + (P_0' + Q_1)\frac{d^{n-1}y}{dx^{n-1}} + (Q_1' + Q_2)\frac{d^{n-2}y}{dx^{n-2}} + ... + (Q_{n-2}' + Q_{n-1})\frac{dy}{dx} + Q_{n-1}' y = \phi(x) \qquad ...(3)$$

Now (1) and (3) must be the same equations; so equating coefficients of $d^{n-1}y/dx^{n-1}$, $d^{n-2}y/dx^{n-2}$, ..., dy/dx, y, we have

$$P_1 = P_0' + Q_1, \quad P_2 = Q_1' + Q_2, \quad P_3 = Q_2' + Q_3, \quad P_{n-1} = Q_{n-2}' + Q_{n-1} \quad \text{and} \quad P_n = Q_{n-1}' \qquad ...(4)$$

To get the desired condition, we must eliminate all Q's and get a certain relation between P's alone. Furthermore, we shall calculate Q's in terms of P's so that their substitution in (2) may give the *first integral* of the given equation.

Now, the relations (4) give

$$Q_1 = P_1 - P_0', \qquad Q_2 = P_2 - Q_1' = P_2 - \frac{d}{dx}(P_1 - P_0') = P_2 - P_1' - P_0'',$$

$$Q_3 = P_3 - Q_2' = P_3 - \frac{d}{dx}(P_2 - P_1' + P_0'') = P_3 - P_2' + P_1'' - P_0''',$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

$$Q_{n-1} = P_{n-1} - P'_{n-2} + P''_{n-3} - \ldots + (-1)^{n-1} P_0^{(n-1)}$$

$\therefore$ $$P_n = Q'_{n-1} = \frac{d}{dx}[P_{n-1} - P'_{n-2} + P''_{n-3} - \ldots + (-1)^{n-1} P_0^{(n-1)}]$$

or $$P_n = P'_{n-1} - P''_{n-2} + P'''_{n-3} - \ldots + (-1)^{n-1} P_0^{(n)}$$

or $$P_n - P'_{n-1} + P''_{n-2} - P'''_{n-3} + \ldots - (-1)^{n-1} P_o^{(n)} = 0$$

or $$P_n - P'_{n-1} + P''_{n-2} - P'''_{n-3} - \ldots + (-1)^n P_0^{(n)} = 0. \qquad \ldots (6)$$

Again putting the above values of $Q_1, Q_2, \ldots, Q_{n-1}$ in (2), we get

$$P_0(d^{n-1}y/dx^{n-1}) + (P_1 - P_0')(d^{n-2}y/dx^{n-2}) + (P_2 - P_1' + P_0'')(d^{n-3}y/dx^{n-3}) + \ldots$$

$$\ldots + \{P_{n-1} - P'_{n-2} + P''_{n-3} - \ldots + (-1)^{n-1} P_0^{(n-1)}\}y = \int \phi(x)dx + C. \qquad \ldots (7)$$

Thus (6) is the desired condition of exactness of (1). When this condition is satisfied, the first integral of (1) is given by (7).

Note. Remember (6) and (7) for direct application in problems.

9.3 Working rule for solving exact equations

First of all write the given equation in full (by writing zero for the missing coefficents, if necessary). Then write values of $P_0, P_1, \ldots$ by comparing the given equation with standard equation (1) of Art 9.2 Now write $P_n, P_{n-1}, P_{n-2}, \ldots$ starting from the highest and put +ve and –ve sign before them. Finally put no dash, one dash, two dashes...on them and then find the value of this expression. If it is zero, the given equation is exact and its first integral is given by (7) of Art. 9.2

After getting the first integral, we test it for exactness in a similar manner. If the first integral is also exact, then we get its first integral (*i.e.*, second integral of the given equation) by using formula (7). Proceeding in this way we generally get an equation of the form $dy/dy + Py = Q$ which is solved as above (if it exact) or by using the standard methods of Chapter 2.

9.4 Examples (Type 1) based on working rule of Art 9.3

Ex. 1. *Solve* $(1 + x + x^2)(d^3y/dx^3) + (3 + 6x)(d^2y/dx^2) + 6(dy/dx) = 0.$ **(Garhwal 2005)**

Sol. Supplying the missing coefficient of y, the given equation in the standard form is

$$(1 + x + x^2)(d^3y/dx^3) + (3 + 6x)(d^2y/dx^2) + 6(dy/dx) + 0 \cdot y = 0. \qquad \ldots (1)$$

Comparing (1) with $P_0y''' + P_1y'' + P_2y' + P_3y = \phi(x)$, here

$$P_0 = 1 + x + x^2, \quad P_1 = 3 + 6x, \quad P_2 = 6, \quad P_3 = 0 \quad \text{and} \quad \phi(x) = 0. \qquad \ldots (2)$$

The given equation will be exact if $\quad P_3 - P_2' + P_1'' - P_0''' = 0. \qquad \ldots (3)$

Now using (2), $\quad$ L.H.S. of (3) $= 0 - 0 + 0 - 0 = 0 =$ R.H.S. of (3).

$\therefore$ The given equation is exact and its first integral is

$$P_0(d^2y/dx^2) + (P_1 - P_0')(dy/dx) + (P_2 - P_1' + P_0'')y = c_1. \qquad \ldots (4)$$

Using (2), (4) reduces to

or $$(1 + x + x^2)(d^2y/dx^2) + \{3 + 6x - (1 + 2x)\}(dy/dx) + (6 - 6 + 2)y = c_1.$$

or $$(1 + x + x^2)(d^2y/dx^2) + (2 + 4x)(dy/dx) + 2y = c_1 \qquad \ldots (5)$$

Now let us examine (5) for exactness. We shall repeat the whole process for (5) as we did for (1). On comparing (5) with $P_0 y'' + P_1 y' + P_2 y = \phi(x)$, we have

$$P_0 = 1 + x + x^2, \qquad P_1 = 2 + 4x, \qquad P_2 = 2 \qquad \text{and} \qquad \phi(x) = c_1. \quad \ldots (6)$$

Here $P_2 - P_1' + P_0'' = 2 - 4 + 2 = 0$, which shows that (5) is exact and hence its integral is

$$P_0 \frac{dy}{dx} + (P_1 - P_0')y = \int c_1 dx + c_2, \; c_1 \text{ and } c_2 \text{ being arbitrary constants.}$$

or $$(1 + x + x^2)(dy/dx) + [2 + 4x - (1 + 2x)] = c_1 x + c_2, \text{ using (6)} \quad \ldots (7)$$

Let us again examine (7) for exactness. For (7), we have on comparing with $P_0 y' + P_1 y = \phi(x)$,

$P_0 = 1 + x + x^2$ and $P_1 = 1 + 2x$, $\phi(x) = c_1 x + c_2$. Hence $P_1 - P_0' = 1 + 2x - (1 + 2x) = 0$.

Hence (7) is exact and its first integral (which will be required solution because it will be free from derivatives) is $$P_0 y = \int (c_1 x + c_2) dx + c_3$$

i.e., $(1 + x + x^2)y = (c_1/2) \times x^2 + c_2 x + c_2 x + c_3$, which is the desired solution, containg c_1, c_2 and c_3 as arbitrary constants.

Ex. 2. *Solve* $x\,(d^3y/dx^3) + (x^2 + x + 3)\,(d^2y/dx^2) + (4x + 2)\,(dy/dx) + 2y = 0$.

Sol. Comparing the given equation with $P_0 y''' + P_1 y'' + P_2 y' + P_3 y = \phi(x)$, here

$$P_0 = x, \qquad P_1 = x^2 + x + 3, \qquad P_2 = 4x + 2, \qquad P_3 = 2 \quad \ldots (1)$$

and $$P_3 - P_2' + P_1'' - P_0''' = 2 - 4 + 2 - 0 = 0,$$

which shows that the given equation is exact and hence its first integral is

$$P_0 (d^2y/dx^2) + (P_1 - P_0')(dy/dx) + (P_2 - P_1' + P_0'')y = c_1$$

or $$x\,(d^2y/dx^2) + (x^2 + x + 2)\,(dy/dx) + (2x + 1)y = c_1, \text{ by (1)} \quad \ldots (2)$$

On comparing (2) with $P_0 y'' + P_1 y' + P_2 y = \phi(x)$, here

$$P_0 = x, \qquad P_1 = x^2 + x + 2, \qquad P_2 = 2x + 1 \qquad \text{and} \qquad \phi(x) = c_1 \quad \ldots (3)$$

and $$P_2 - P_1' + P_0'' = 2x + 1 - (2x + 1) + 0 = 0,$$

which shows that (2) is exact and hence its first integral is

$$P_0 \frac{dy}{dx} + (P_1 - P_0')y = \int c_1 dx + c_2 \qquad \text{or} \qquad x\frac{dy}{dx} + (x^2 + x + 1)y = c_1 x + c_2, \quad \ldots (4)$$

which is not exact (verify orally) and can be put in the linear form by dividing by x.

$$\therefore \qquad (dy/dx) + (1 + x + 1/x)y = c_1 + c_2/x \quad \ldots (5)$$

$$\text{I.F. of (5)} = e^{\int [(1/x) + 1 + x] dx} = e^{\log x + x + x^2/2} = e^{\log x}\, e^{(x + x^2/2)} = x\, e^{x + x^2/2}$$

$\therefore$ Solution of (5) is $$y\, x\, e^{x + x^2/2} = \int (c_1 + c_2/x)\, x\, e^{x + x^2/2}\, dx + c_3$$

or $$x\, y\, e^{x + x^2/2} = \int (c_1 x + c_2)\, e^{x + x^2/2}\, dx + c_3, \; c_1, c_2 \text{ and } c_3 \text{ being arbitrary constants.}$$

Remark. Sometimes when the integral cannot be evaluated by known standard methods, the final answer is given in terms of an integral as shown above.

Ex. 3. *Show that* $\cos x\, y'' + 2 \sin x\, y' + 3 \cos x\, y = \tan^2 x$ *is exact.* **[Bangalore 1997]**

Sol. Comparing the given equation with $P_0 y'' + P_1 y' + P_2 y = \phi(x)$, here

$$P_0 = \cos x, \qquad P_1 = 2 \sin x, \qquad P_2 = 3 \text{ cox } x \quad \ldots (1)$$

The given equation will exact if $\qquad P_2 - P_1' + P_0'' = 0.$

Now, here $\qquad P_2 - P_1' + P_0'' = 3 \cos x - 2 \cos x - \cos x = 0$, using (1)

Hence the given equation is exact.

Ex. 4. (a) *Show that the equation* $(1 + x^2)y'' + 3xy' + y = 1 + 3x^2$ is exact and hence solve the equation. **[Bangalore 1995]**

(b) Test for the exacteness of $(1 + x^2)\, y'' + 3xy' + y = 0$ and write its first integral.

(G.N.D.U. Amritsar 1997)

Sol. (*a*) Given $$(1 + x^2)y'' + 3xy' + y = 1 + 3x^2. \qquad \text{... (1)}$$

Comparing (1) with $P_0 y'' + P_1 y' + P_2 y = \phi(x)$, here $P_0 = 1 + x^2$, $P_1 = 3x$, $P_2 = 1$ and $\phi(x) = 1 + 3x^2$ Now, we have $P_2 - P_1' + P_0'' = 1 - 3 + 2 = 0$,

showing that (1) is exact and its first integral is

$$P_0 \frac{dy}{dx} + (P_1 - P_0')y = \int (1+3x^2)\, dx + c_1 \quad \text{or} \quad (1 + x^2)\,(dy/dx) + (3x - 2x)y = x + x^3 + c_1$$

or $$\frac{dy}{dx} + \frac{x}{1+x^2} y = x + \frac{c_1}{1+x^2}, \text{ which is a linear equation} \qquad \text{... (2)}$$

I.F. of (2) = $e^{\int [x/(1+x^2)]dx} = e^{(1/2)\times \log(1+x^2)} = e^{\log(1+x^2)^{1/2}=(1+x^2)^{1/2}}$

and solution of (2) is $$y(1 + x^2)^{1/2} = \int (1+x^2)^{1/2} \left\{x + c_1/(1+x^2)\right\} dx + c_2$$

or $$y(1 + x^2)^{1/2} = \int (1+x^2)^{1/2} x\, dx + c_1 \int \frac{dx}{(1+x^2)^{1/2}} + c_2$$

or $$y(1 + x^2)^{1/2} = (1/3) \times (1 + x^2)^{3/2} + c_1 \log [x + (1 + x^2)^{1/2}] + c_2,$$

where c_1 and c_2 are arbitrary constants.

(*b*) Try yourself **Ans.** $(1 + x^2)\, y' + xy = x + x^3 + c$

Ex. 5. *Test for exactness and solve* $(1 + x^2)y'' + 4xy' + 2y = \sec^2 x$ *given that* $y = 0$, $y' = 1$ *when* $x = 0$. **[Bangalore 1992]**

Sol. Given $$(1 + x^2)y'' + 4xy' + 2y = \sec^2 x \qquad \text{... (1)}$$

Comparing (1) with $P_0 y'' + P_1 y' + P_2 y = \phi(x)$, here $P_0 = 2 + x^2$, $P_1 = 4x$, $P_2 = 2$, $\phi(x) = \sec^2 x$

Also, $P_2 - P_1' + P_0'' = 2 - 4 + 2 = 0$, so (1) is exact. Then, the first integral of (1) is

$$P_0 \frac{dy}{dx} + (P_1 - P_0')y = \int \sec^2 x\, dx + c_1 \quad \text{or} \quad (1+x^2)\frac{dy}{dx} + (4x - 2x)y = \tan x + c_1$$

or $$\frac{dy}{dx} + \frac{2x}{1+x^2} y = \frac{\tan x}{1+x^2} + \frac{c_1}{1+x^2}, \text{ which is a linear equation}$$

Its I.F. = $e^{\int \{2x/(1+x^2)\}dx} = e^{\log(1+x^2)} = 1 + x^2$ and solution is

$$y(1+x^2) = \int (1+x^2)\{(\tan x)/(1+x^2) + c_1/(1+x^2)\}\, dx + c_2$$

or $$y(1+x^2) = \log \sec x + c_1 \tan^{-1} x + c_2,\ c_1,\ c_2 \text{ being arbitrary constants.} \qquad \text{... (2)}$$

Putting $x = 0$ and $y = 0$ in (2), we get $C_2 = 0$ Now, differentiating (2) w.r.t. 'x', we get

$$y'(1+x^2) + 2xy = \frac{1}{\sec x}(\sec x \tan x) + \frac{c_1}{1+x^2}. \qquad \text{... (3)}$$

Putting $x = 0$, $y = 0$ and $y' = 1$ in (2), we get $1 = c_1$. Putting $c_1 = 1$ and $c_2 = 0$ in (2), we get

$$y(1+x^2) = \log \sec x + \tan^{-1} x, \text{ which is the required solution.}$$

Ex. 6. (*a*) $(d^3y/dx^3) + \cos x\,(d^2y/dx^2) - 2 \sin x\,(dy/dx) - y \text{ cox } x = \sin 2x$.

[Rajasthan 2010]

(*b*) Prove that $y''' + \cos x\, y'' - 2 \sin x\, y' - y \cos x = 0$ is exact and write its first integral.

[G.N.D.U. Amritsar 1996]

Sol. (*a*) Comparing the given equation with $P_0 y''' + P_1 y'' + P_2 y' + P_3 y = \phi(x)$, we have

$P_0 = 1$, $P_1 = \cos x$, $P_2 = -2 \sin x$, $P_3 = -\cos x$ and $\phi(x) = \sin 2x$ ... (1)

We have, $$P_3 - P_2' + P_1'' - P_0''' = -\cos x + 2 \cos x - \cos x + 0 = 0.$$

$\therefore$ The given equation is exact and its first integral is

$$P_0 \frac{d^2 y}{dx^2} + (P_1 - P_0')\frac{dy}{dx} + (P_2 - P_1' + P_0'')y = \int \sin 2x\, dx + c_1$$

or $$d^2 y/dx^2 + \cos x (dy/dx) - \sin x \cdot y = -(1/2) \times \cos 2x + c_1 \text{ using (1)} \qquad \text{... (2)}$$

For (2), on comparing with $P_0 y'' + P_1 y' + P_2 y = \phi(x)$, $P_0 = 1$, $P_1 = \cos x$, $P_2 = -\sin x$ ... (3)

$\therefore$ $P_2 - P_1' + P_0'' = -\sin x + \sin x + 0 = 0$, using (3)

Hence (2) is exact and its first integral is

$$P_0 \frac{dy}{dx} + (P_1 - P_0')y = \int\left(-\frac{1}{2}\cos 2x + c_1\right)dx + c_2 \quad \text{or} \quad \frac{dy}{dx} + \cos x . y = -\frac{1}{4}\sin 2x + c_1 x + c_2, \text{ by (3)}$$

which is linear. Its I.F. = $e^{\int \cos x\, dx} = e^{\sin x}$ and so solution is

$$ye^{\sin x} = -(1/4) \times \int \sin 2x\, e^{\sin x} dx + \int (c_1 x + c_2)e^{\sin x} dx + c_2 . \qquad \text{... (4)}$$

But $\int \sin 2x\, e^{\sin x} dx = 2\int \sin x e^{\sin x} \cos x\, dx = 2\int te^t dt$, putting $\sin x = t$ and $\cos x\, dx = dt$

$$= 2\left[te^t - \int (1 \times e^t)\, dt\right] = 2(te^t - e^t) = 2(t-1)e^t = 2(\sin x - 1)e^{\sin x}$$

$\therefore$ solution of (4) is $\quad ye^{\sin x} = -(1/2) \times (\sin x - 1)e^{\sin x} + \int (c_1 x + c_2)e^{\sin x} dx + c_3$.

(b) Try yourself. **Ans.** $y'' + \cos x\, y' - \sin x = c_1 - (1/2) \times \cos 2x$.

Ex. 7. *Solve* $\sin x\, (d^2y/dx^2) - \cos x\, (dy/dx) + 2y \sin x = 0$. **(Mysore 2005)**

Sol. Comparing (1) with $P_0 y'' + P_1 y' + P_2 y = \phi(x)$, $P_0 = \sin x$, $P_0 = -\cos x$, $P_2 = 2 \sin x$.

and hence $\quad P_2 - P_1' + P_0'' = 2 \sin x - \sin x - \sin x = 0$.

Hence the given equation is exact and its first integral is

$P_0(dy/dx) + (P_1 - P_0')y = c_1$. $\quad$ or $\quad \sin x\, (dy/dx) - 2 \cos x \cdot y = c_1$,

which is not exact. Dividing by $\sin x$, we get

$$(dy/dx) - 2 \cot x \cdot y = c_1 \operatorname{cosec} x, \qquad \text{... (1)}$$

which is linear and its integrating factor (I.F.) is given by

$$\text{I.F.} = e^{\int (-2\cot x)dx} = e^{-2 \log \sin x} = (\sin x)^{-2} = \operatorname{cosec}^2 x$$

$\therefore$ The solution of (1) is $\quad y \operatorname{cosec}^2 x = c_1 \int \operatorname{cosec} x \cdot \operatorname{cosec}^2 x dx + c_2 \qquad$ ... (2)

Now $$\int \operatorname{cosec} x \cdot \operatorname{cosec}^2 x\, dx = -\operatorname{cosec} x \cdot \cot x - \int (-\operatorname{cosec} x \cot x)(-\cot x) dx$$

[on integrating by parts]

$$= -\operatorname{cosec} x \cdot \cot x - \int -\operatorname{cosec} x\, (\operatorname{cosec}^2 x - 1)\, dx$$

$$= -\operatorname{cosec} x \cdot \cot x - \int -\operatorname{cosec} x \operatorname{cosec}^2 x + \int \operatorname{cosec} x\, dx$$

or $$2\int \operatorname{cosec} x \operatorname{cosec}^2 x\, dx = -\operatorname{cosec} x + \log \tan(x/2)$$

or $$\int \operatorname{cosec} x \cdot \operatorname{cosec}^2 x\, dx = -(1/2) \times \operatorname{cosec} x \cot x + (1/2) \times \log \tan(x/2).$$

Using this value in (2), the required solution is

$y \operatorname{cosec}^2 x = (1/2) \times c_1[\log \tan (x/2) - \operatorname{cosec} x \cot x] + c_2$, c_1, c_2 being arbitrary constants

Ex. 8. *Solve* $\sin^2 x(d^2y/dx^2) = 2y$ or $d^2y/dx^2 = 2y \operatorname{cosec}^2 x$. **(Kanpur 2008)**

Sol. Re-writing the given equation, we get $\quad (d^2y/dx^2) - \operatorname{cosec}^2 x \cdot y = 0$.

Multiplying by $\cot x$, $\quad \cot x\, (d^2y/dx^2) + 0 \cdot (dy/dx) - 2 \cot x \operatorname{cosec}^2 x \cdot y = 0$. ... (1)

Comparing (1) with $P_0 y'' + P_1 y' + P_2 y = \phi(x)$, we get

$P_0 = \cot x, \quad P_1 = 0, \quad P_2 = -2 \cot x \operatorname{cosec}^2 x \quad$ and $\quad \phi(x) = 0$ ... (2)

and $\quad P_2 - P_1' + P_0'' = -2 \cot x \operatorname{cosec}^2 x - 0 + 2 \cot x \operatorname{cosec}^2 x = 0.$

Hence (1) is exact and its first intergrat is given by

$$P_o(dy/dx) + (P_1 - P_o')y = c_1 \quad \text{or} \quad \cot x\,(dy/dx) + \operatorname{cosec}^2 x . y = c_1, \text{ by (2)} \quad ...(3)$$

which is not exact. Dividing by cot x, we get $\dfrac{dy}{dx} + \dfrac{\operatorname{cosec}^2 x}{\cot x} y = c_1 \tan x$.

$$\text{Its I.F.} = e^{\int (\operatorname{cosec}^2 x/\cot x)dx} = e^{-\log \cot x} = (\cot x)^{-1} = \tan x.$$

$\therefore$ The required solution of (3) is

$$y \cdot \tan x = \int c_1 \tan^2 x\,dx + c_2 = \int c_1(\sec^2 x - 1)dx + c_2 \quad \text{or} \quad y \tan x = c_1 \tan x - c_1 x + c_2.$$

Ex. 9. *Prove that the equation* $x^3\,(d^3y/dx^3) + 9x^2(d^2y/dx^2) + 18x\,(dy/dx) + 6y = \cos x$ *is exact and find its first integral.*

Sol. Comparing the given equation with $\quad P_0 y''' + P_1 y'' + P_2 y' + P_3 y = \phi(x)$,

$P_0 = x^3, \quad P_1 = 9x^2, \quad P_2 = 18x, \quad P_3 = 6. \quad$ and $\quad \phi(x) = \cos x$ ... (1)

and $\quad P_3 - P_2' + P_1'' - P_0''' = 6 - 18 + 18 - 6 = 0$, using (1)

Hence the given equation is exact and its first integral is

$$P_0(d^2y/dx^2) + (P_1 - P_0')(dy/dx) + (P_2 - P_1' - P_0'')y = \int \cos x\,dx + c_1 .$$

or $\quad x^3\,(d^2y/dx^2) + 6x^2\,(dy/dx) + 6xy = \sin x + c_1$. using (1)

Ex. 10. *Find the integral of the exact equation* $x^2\,(1 + x)\,y'' + 2x\,(2 + 3x)y' + 2\,(1 + 3x)\,y = 0.$

[Gullbarga 2005]

Sol. Comparing the given equation with $P_0 y'' + P_1 y' + P_2 y = 0$, here

$P_0 = x^2 + x^3, \quad P_1 = 4x + 6x^2, \quad$ and $\quad P_2 = 2 + 6x$

$\therefore \quad P_2 - P_1' + P_0'' = 2 + 6x - (4 + 12x) + (2 + 6x) = 0,$

showing that the given equation is exact and its first integral is given by

$$P_0 \frac{dy}{dx} + (P_1 - P_0')\,y = C_1 \quad \text{or} \quad x^2(1+x)\frac{dy}{dx} + \{4x + 6x^2 - (2x + 3x^2)\}\,y = C_1$$

or
$$\frac{dy}{dx} + \frac{2x + 3x^2}{x^2(x+1)} y = \frac{C_1}{x^2(1+x)} \quad ...(1)$$

which is a linear differential equation whose

$$\text{I.F.} = e^{\int \frac{2x+3x^2}{x^2(x+1)}dx} = e^{\int\left(\frac{2}{x} + \frac{1}{x+1}\right)dx} = e^{2\log x + \log(x+1)} = x^2(x+1) \text{ and solution is}$$

$$yx^2\,(x+1) = \int \frac{C_1}{x^2(1+x)} \times x^2(x+1)dx + C_2 \quad \text{or} \quad yx^2\,(x+1) = C_1 x + C_2,$$

where C_1 and C_2 are arbitrary constants.

EXERCISE 9 (A)

Solve the following differential equations:

1. $(x^3 - 4x)\,y''' + (9x^2 - 12)\,y'' + 18xy' + 6y = 0.$ **Ans.** $(x^3 - 4x)\,y = (1/2) \times c_1 x^2 + c_2 x + c_3$

2. $xy''' + (x^2 - 3)\,y'' + 4xy' + 2y = 0.$ **Ans.** $\dfrac{ye^{x^2/2}}{x^5} = c_1 \displaystyle\int \frac{e^{x^2/2}}{x^5} dx + c_2 \int \frac{e^{x^2/2}}{x^6} dx + c_2$

3. $xy'' + (1 - x)\,y' - y = e^x.$ **Ans.** $y = e^x \log x + c_1\,e^x \displaystyle\int \frac{1}{x} e^{-x} dx + c_2 e^x$

4. $(ax - bx^2)\, y'' + 2ay' + 2by = x$ **Ans.** $xy = x^3 / 6a + c_1 / 3b + c_1 (a - bx)^3$.

5. $x^2y'' + 3xy' + y = 1 / (1 - x)^2$. **[Mysore 2004]** **Ans.** $xy = \log [x / (1 - x)] + c_1 \log x + c_2$

6. $(x^3 - x)\, y''' + (8x^2 - 3)y'' + 14xy' + 4y = 2 / x^3$.

Ans. $xy\,(x^2-1)^{1/2} = \sec^{-1} x + c_1\,(x^2-1)^{1/2} + c_2 \log [x + (x^2-1)^{1/2}] + c_3$.

7. Find a first integral of $x^3 y'' + 4x^2 y'' + x\,(x^2 + 2)\, y' + 3x^2y = 2x$. **Ans.** $x^3 y'' + x^2y' + x^3y = x^2 + c$.

8. $y'' + 2 \tan x \cdot y' + 3y = \tan^2 x \sec x$

Hint. The given equation is not exact. Multiplying both sides by cos x the resulting equation becomes exact. Later on do as usual. **Ans.** $y \sec^3 x = (2/3) \times \tan^2 x + (1/4) \times \tan^4 x - x \tan x$ $+ (2/3) \times \log \sec x - (x/3) \times \tan^3 x + c_1 \tan x + (c_1/3) \times \tan^3 x + c_2$.

9. Show that $\{(x^3 - 2x)\, D^3 + 3\,(3x^2 - 2)\, D^2 + 18x\, D + 6\}\; y = 24\, x$ is exact. Hence solve it completely.

10. $(2x^2 + 3x)\, y'' + (6x + 3)\, y' + 2y = (x + 1)\, e^x$. **[Bangalore 1996]** **Ans.** $y\,(3 + 2x) = e^x + c_1 \log x + c_2$.

9.5 Integrating Factor

Suppose that the equation considered in Art. 9.2 is not exact, and let the coefficients P_0, P_1, ... etc. be of the type $(a\, x^q + b\, x^q + ...)$ etc. Then in such cases x^m can be taken as an integrating factor. Multiply the given equation by x^m and apply the condition of exactness which will give a particular value of m. Thus the exact value of the desired integrating factor will be known to us. The rest of the method is same as discussed in working Rule 9.3.

9.6 Examples (Type 2) based on Art 9.5

Ex. 1. *Solve* $x^{1/2}\,(d^2y/dx^2) + 2x\,(dy/dx) + 3y = x.$ **(G.N.D.U. Amritsar 1998)**

Sol. We see that the given equation is not exact (verity yourself as usual). Let its integrating factor be x^m. Multiplying the given equation by x^m, we get

$$x^{m+1/2}(d^2y/dx^2) + 2x^{m+1}\,(dy/dx) + 3x^m y = x^{m+1}. \qquad \text{... (1)}$$

which must be exact. Comparing (1) with $P_0 y'' + P_1 y' + P_2 y = \phi(x)$, here

$$P_0 = x^{m+1/2}, \qquad P_1 = 2\, x^{m+1}, \qquad P_2 = 3x^m, \qquad \phi(x) = x^{m+1}. \qquad \text{... (2)}$$

Now, (1) is exact if $\quad P_2 - P_1' + P_0'' = 0$

or $\quad 3x^m - 2\,(m + 1)\, x^m + (m + 1/2)\,(m - 1/2)\, x^{m-3/2} = 0,$

or $\quad (1 - 2m)\, x^m + (1/4) \times (2m + 1)\,(2m - 1) \times x^{m-3/2} = 0$

or $\quad (1 - 2m)\,[x^m - (1/4) \times (2m + 1)\, x^{m-3/2}] = 0 \quad$ or $\quad 1 - 2m = 0$ for all x. Hence $m = 1/2$.

Putting this value of m in (1) we get $\quad x\,(d^2y/dx^2) + 2x^{3/2}\,(dy/dx) + 3x^{1/2}\, y = x^{3/2} \quad$ ... (3)

which must be exact. For this equation, on comparing with $P_o y'' + P_1 y' + P_2 y = \phi(x)$, we get

$P_0 = x, \qquad P_1 = 2x^{3/2}, \qquad P_2 = 3x^{1/2}, \qquad \phi(x) = x^{3/2}$ and hence first integral of (3) is

$$P_0 \frac{dy}{dx} + (P_1 - P_0')\, y = \int \phi(x)\, dx + c_1 \qquad \text{or} \qquad x\frac{dy}{dx} + (2x^{3/2} - 1)\, y = \frac{2}{5}x^{5/2} + c_1,$$

which is not exact. Dividing both sides of it by x, we get

$$\frac{dy}{dx} + \left(2x^{1/2} - \frac{1}{x}\right) = \frac{2}{5}x^{5/2} + \frac{c_1}{x}, \text{ which is a linear equation}$$

its I.F. $= e^{\int (2x^{1/2} - 1/x)} = e^{(4/3)x^{3/2} - \log x} = (1/x)e^{(4/3)x^{3/2}}$ and solution is

$$y \cdot \frac{1}{x} e^{(4/3)x^{3/2}} = \int \frac{1}{x} e^{(4/3)x^{3/2}} \left\{\frac{2}{5}x^{5/2} + \frac{c_1}{x}\right\} dx + c_2$$

or

$$\frac{y}{x} e^{(4/3)x^{3/2}} = \frac{2}{5}\int e^{(4/3)x^{3/2}} \cdot x^{1/2} dx + c_1 \int \frac{1}{x^2} e^{(4/3)x^{3/2}}\, dx + c_2\,. \qquad \text{... (4)}$$

Now, $$\int e^{(4/3)x^{3/2}} \cdot x^{1/2} dx = \frac{2}{3}\int e^{4t/3} dt = \frac{2}{3} \times \frac{3}{4} e^{4t/3} = \frac{1}{2} e^{(4/3)x^{3/2}},$$

[putting $x^{3/2} = t$ so that $x^{1/2}\ dx = (2/3) \times dt$]

Putting this in (3), the required solution is

$$(y/x) e^{(4/3)x^{3/2}} = (1/5) \times e^{(4/3)x^{3/2}} + c_1 \int (1/x^2)\, e^{(4/3)x^{3/2}} dx + c_2.$$

Ex. 2. *Solve* $x^5 (d^2y / dx^2) + 3x^3 (dy / dx) + (3 - 6x)\, x^2 y = x^4 + 2x - 5$.

Sol. Verify that the given equation is not exact. Let the integrating factor be x^m. Multiplying the given equation by x^m, we get

$$x^{m+5} \frac{d^2 y}{dx^2} + 3x^{m+3} \frac{dy}{dx} + (3 - 6x)\, x^{m+2} y = x^{m+4} + 2x^{m+3} - 5x^m, \qquad \text{... (1)}$$

which must be exact. Comparing (1) with $P_0'' + P_1 y' + P_2 y = \phi(x)$, here

$P_0 = x^{m+5}$, $P_1 = 3x^{m+3}$, $P_2 = (3 - 6x)\, x^{m+2}$, $\phi(x) = x^{m+4} + 2x^{m+3} - 5x^m$.

Since (1) is exact, we must have $P_2 - P_1' + P_0'' = 0$

i.e., $(3 - 6x)\, x^{m+2} - 3(m+3)\, x^{m+2} \quad (m+5)(m+4)\, x^{m+3} = 0$

or $[3 - 3(m+3)]\, x^{m+2} + [(m+5)(m+4) - 6]\, x^{m+3} = 0$

or $-3(m+2)\, x^{m+2} + (m^2 + 9m + 14)\, x^{m+3} = 0$

or $-3(m+2)\, x^{m+2} + (m+2)(m+7)\, x^{m+3} = 0$

or $(m+2)\,[-3x^{m+2} + (m+7)\, x^{m+3}] = 0$ giving $m + 2 = 0$ *i.e.,* $m = -2$ for all x.

Putting this value of m in (1), $$x^3 \frac{d^2 y}{dx^2} + 3x \frac{dy}{dx} + (3 - 6x)\, y = x^2 + \frac{2}{x} - \frac{5}{x^2}, \qquad \text{... (2)}$$

which is exact and for which $P_0 = x^3$, $P_1 = 3x$, $P_2 = 3 - 6x$, $\phi(x) = x^2 + 2/x - 5/x^2$.

Hence its first integral is $$P_0 \frac{dy}{dx} + (P_1 - P_0')\, y = \int (x^2 + \frac{2}{x} - \frac{5}{x^2})\, dx + c_1$$

or $$x^3 \left(dy/dx\right) + 3x(1-x)\, y = x^3/3 + 2 \log x + 5/x + c_1,$$

which is not exact. Dividing both sides by x^3, we get

$$\frac{dy}{dx} + \frac{3(1-x)}{x^2}\, y = \frac{1}{3} + \frac{2}{x^3} \log x + \frac{5}{x^4} + \frac{c_1}{x^3}, \text{ which is linear.}$$

Its $\text{I.F.} = e^{3\int \{(1-x)/x^2\} dx} = e^{3\int (1/x^2 - 1/x) dx} = e^{3(-1/x - \log x)} = e^{-3/x}\, e^{-3 \log x}$

Thus, $\text{I.F.} = e^{\log x^{-3}}\, e^{-3/x} = x^{-3}\, e^{-3/x}$

$\therefore$ The required solution is $$\frac{y}{x^3} e^{-3/x} = \int \left(\frac{1}{3} + \frac{2}{x^3} \log x + \frac{5}{x^4} + \frac{c_1}{x^3} \right) \frac{1}{x^3} e^{-3/x}\, dx + c_2.$$

EXERCISE 9 (B)

Solve the following differential equations:

1. $2x^2(x + 1)\, (d^2y / dx^2) + x(7x + 3)\, (dy/dx) - 3y = x^2$. **Ans.** $5(x + 1) = 5x^2/7 + c_1 x - c_2 x^{-3/2}$

2. $x^4 (d^2y / dx^2) + x^2 (x - 1)\, (dy/dx) + xy = x^3 - 4$ **Ans.** $y e^{1/x} = \int (1 + c_1/x)\, e^{1/x} dx - 2e^{1/x}(1/x - 1) + c_2$.

3. Show that $\{x^2 D^3 + 4xD^2 + (x^2 + 2)D + 3x\} y = 2$ becomes integrable on being multiplied by some power of x. Obtain its first integral. Here $D \equiv d/dx$

Ans. $(x^3 D^2 + x^2 D + x^3) y = x^2 + c$, c being an arbitrary constant

9.7 Exactness of Non-Linear Equations-Solution by trial

There is no simple test for exactness of non-linear equations. So we solve these by trial. The methods of solving such equations can also be used to solve linear exact equations.

9.8 Examples (Type 3) based on Art. 9.7

Ex. 1. Solve $x^2 y \dfrac{d^2 y}{dx^2} + \left(x\dfrac{dy}{dx} - y \right)^2 - 3y^2 = 0.$

Sol. If possible, let us write

$$\frac{du}{dx} = x^2 y \frac{d^2 y}{dx^2} + x^2 \left(\frac{dy}{dx} \right)^2 - 2xy\frac{dy}{dx} - 2y^2 = 0. \qquad \text{... (1)}$$

We note that $x^2y\ (d^2y/dx^2)$ can be obtained by differentiating $x^2y\ (dy/dx)$ [Remember that we shall first consider the term conatining the highest order derivative].

Let $\qquad u_1 = x^2y\ (dy/dx) \qquad$ ... (2)

Differentiating both sides w.r.t, 'x', (2) gives

$$\frac{du_1}{dx} = x^2 y \frac{d^2 y}{dx^2} + x^2 \left(\frac{dy}{dx} \right)^2 + 2xy\frac{dy}{dx} \qquad \text{... (3)}$$

Substracting (3) from (1), we get $\qquad \dfrac{du}{dx} - \dfrac{du_1}{dx} = -4xy\dfrac{dy}{dx} - 2y^2 \qquad$ ... (4)

As before, $-4xy\ (dy/dx)$ can be obtained by differentiating $-2xy^2$

$\therefore$ Let $\qquad u_2 = -2xy^2. \qquad$ ... (5)

Differentiating, (5) w.r.t, 'x', we get $\qquad \dfrac{du_2}{dx} = -4xy\dfrac{dy}{dx} - 2y^2$

Putting this in (4), we get $\quad \dfrac{du}{dx} - \dfrac{du_1}{dx} = \dfrac{du_2}{dx} \qquad$ or $\qquad \dfrac{du}{dx} = \dfrac{d}{dx}(u_1 + u_2) \quad$ ... (6)

By virtue of (6), we see that (1) is exact. Thus (1) becomes $\qquad \dfrac{d}{dx}(u_1 + u_2) = 0.$

Integrating, $\quad u_1 + u_2 = c. \qquad$ so that $\qquad x^2y\ (dy/dx) - 2xy^2 = c$, using (2) and (5).

Dividing by x^2, $\qquad y\ (dy/dx) - (2/x)\ y^2 = c\ /\ x^2. \qquad$ ... (7)

Take $y^2 = v$ so that $2y\ (dy/dx) = (dv/dx)$ and so (7) gives

$$\frac{1}{2}\frac{dv}{dx} - \frac{2}{x}v = \frac{c_1}{x^2} \qquad \text{or} \qquad \frac{dv}{dx} - \frac{4}{x}v = \frac{2c_1}{x^2}$$

Its I.F. $= e^{\int (-4/x)dx} = e^{-4\log x} = e^{\log x^{-4}} = x^{-4} = 1/x^4$. So the required solution is

$$v/x^4 = 2c_1 \int (1/x^6)dx + c_2 \qquad \text{or} \qquad v/x^4 = -(2c_1/5x^5) + c_2$$

or $\qquad y^2/x^4 = -2c_1 \times (1/5x^5) + c_2 \qquad$ or $\qquad y^2x = c_2x^5 - 2c_1/5.$

Ex. 2. *Show that the following equation is exact and find its first integral*

$$y + 3x\frac{dy}{dx} + 2y\left(\frac{dy}{dx} \right)^3 + \left(x^2 + 2y^2 \frac{dy}{dx} \right)\frac{d^2 y}{dx^2} = 0$$

Sol. If possible, let us write

$$\frac{du}{dx} = x^2 \frac{d^2y}{dx^2} + 2y^2 \frac{dy}{dx}\frac{d^2y}{dx^2} + 2y\left(\frac{dy}{dx}\right)^3 + 3x\frac{dy}{dx} + y = 0. \qquad \text{... (1)}$$

Take $$u_1 = x^2(dy/dx) \qquad \text{... (2)}$$

$$\therefore \qquad \frac{du_1}{dx} = x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx}. \qquad \text{... (3)}$$

Substracting (3) from (1), we get

$$\frac{du}{dx} - \frac{du_1}{dx} = 2y^2\frac{dy}{dx}\frac{d^2y}{dx^2} + 2y\left(\frac{dy}{dx}\right)^3 + x\frac{dy}{dx} + y. \qquad \text{... (4)}$$

Now, take $$u_2 = y^2 \,(dy/dx)^2. \qquad \text{... (5)}$$

$$\therefore \qquad \frac{du_2}{dx} = 2y^2\frac{dy}{dx}\frac{d^2y}{dx^2} + 2y\left(\frac{dy}{dx}\right)^3 \qquad \text{... (6)}$$

Substracting (6) from (4), we get $$\frac{du}{dx} - \frac{du_1}{dx} - \frac{du_2}{dx} = x\frac{dy}{dx} + y. \qquad \text{... (7)}$$

Next, take $$u_3 = xy. \qquad \text{... (8)}$$

$$\therefore \qquad \frac{du_3}{dx} = x\frac{dy}{dx} + y$$

Putting this in (7), we get $\dfrac{du}{dx} - \dfrac{du_1}{dx} - \dfrac{du_2}{dx} = \dfrac{du_3}{dx}$ or $\dfrac{du}{dx} = \dfrac{d}{dx}(u_1 + u_2 + u_3)$.

Hence (1) is exact and it can be written as $$\frac{d}{dx}(u_1 + u_2 + u_3) = 0.$$

Integrating, $u_1 + u_2 + u_3 = c$ or $x^2(dy/dx) + y^2(dy/dx)^2 + xy = c$, using (2), (5), (8) which is the required first integral.

Second Method [Sometimes we group the terms of the given equation in such a manner so that they become perfect differentials and their integrals may be found directly.]

We rewrite the equation by grouping the terms as follows:

$$\left(x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx}\right) + \left[2y^2\frac{dy}{dx}\frac{d^2y}{dx^2} + 2y\left(\frac{dy}{dx}\right)^3\right] + \left(x\frac{dy}{dx} + y\right) = 0$$

or $$\frac{d}{dx}\left(x^2\frac{dy}{dx}\right) + \frac{d}{dx}\left[y^2\left(\frac{dy}{dx}\right)^2\right] + \frac{\mathrm{d(xy)}}{\mathrm{dx}} = 0.$$

Integrating, $x^2(dy/dx) + y^2(dy/dx)^2 + xy = c$, c being an arbitrary constant.

Ex. 3. *Show that the equation* $\left(y^2 + 2x^2\dfrac{dy}{dx}\right)\dfrac{d^2y}{dx^2} + 2(y+x)\left(\dfrac{dy}{dx}\right)^2 + x\dfrac{dy}{dx} + y = 0$ *is exact and find its first intergal.*

Sol. Rewriting the given equation by grouping the terms, we get

$$\left[y^2\frac{d^2y}{dx^2} + 2y\left(\frac{dy}{dx}\right)^2\right] + \left[2x^2\frac{dy}{dx}\frac{d^2y}{dx^2} + 2x\left(\frac{dy}{dx}\right)^2\right] + \left(x\frac{dy}{dx} + y\right) = 0$$

or $$\frac{d}{dx}\left(y^2\frac{dy}{dx}\right) + \frac{d}{dx}\left[x^2\left(\frac{dy}{dx}\right)^2\right] + \frac{d}{dx}(xy) = 0.$$

Integrating, $y^2(dy/dx)+x^2(dy/dx)^2+(xy)=c$, c being an arbitrary constant.

Ex. 4. *Solve* $x^2y(d^2y/dx^2)+\{x(dy/dx)-y\}^2=0$

Sol. The given equation can be re-written as

$$x^2\left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right]-\left[2xy\frac{dy}{dx}-y^2\right]=0 \quad \text{or} \quad \left[y\frac{d^2y}{dx^2}+\left(\frac{dy}{dx}\right)^2\right]-\frac{2xy(dy/dx)-y^2}{x^2}=0$$

or
$$\frac{d}{dx}\left(y\frac{dy}{dx}\right)-\frac{d}{dx}\left(\frac{y^2}{x}\right)=0$$

Integrating, $y(dy/dx)-(1/x)y^2=c_1$. c_1 being an arbitray constant

Putting $y^2 = v$ and $2y\,(dy/dx) = (dv/dx)$, the above equation reduces to

$$\frac{1}{2}\frac{dv}{dx}-\frac{v}{x}=c_1 \quad \text{or} \quad \frac{dv}{dx}-\frac{2}{x}v=2c_1,$$

which is linear. Its I.F. = $e^{-\int(2/x)dx}=e^{-2\log x}=1/x^2$. and solution is $\dfrac{y}{x^2}=2c_1\int\dfrac{1}{x^2}dx+c_2$

or $y^2/x^2 = -2c_1/x + c_2$ or $y^2 = x\,(c_2x - 2c_1)$.

Ex. 5. *Find a first integral of* $\dfrac{dy}{dx}\dfrac{d^2y}{dx^2}-x^2y\dfrac{dy}{dx}=xy^2$.

Sol. Rewriting the given equation, we have

$$2\frac{dy}{dx}\frac{d^2y}{dx^2}-\left[2x^2y\frac{dy}{dx}+2xy^2\right]=0 \quad \text{or} \quad \frac{d}{dx}\left(\frac{dy}{dx}\right)^2-\frac{d}{dx}(x^2y^2)=0$$

Integrating, $(dy/dx)^2 - x^2y^2 = c$ or $(dy/dx) = (c+x^2y^2)^{1/2}$

EXERCISE 9 (C)

Solve the following differential equations:

1. $y'' + y' = e^x$ **Ans.** $y = (1/2)\,e^x + c_1 + c_2e^{-x}$.
2. $2x^2\cos y\cdot y'' - 2x^2\sin y\cdot(y')^2 + x\cos y\cdot y' - \sin y = \log x$ **Ans.** $\sin y = 1-\log x + c_1\,(x/3) + c_2/\sqrt{x}$.
3. $2\,y\,y''' + 2(y+3y')y''' + 2(y')^2 = 2$ **Ans.** $y^2 = x^2 + c_1 + c_2x + c_3e^{-x}$.
4. $\cos y\cdot y'' - \sin y\cdot(y')^2 + \cos y\cdot y' = x + 1$ **Ans.** $2\sin y = x^2 + c_1 + c_2e^{-x}$
5. $2\sin x\cdot y'' + 2\cos x\cdot y' + 2\sin x\,y' + 2y\cos x = \cos x$ **Ans.** $ye^x = c_2 + \int(c_1\operatorname{cosec}x+1/2)e^x\,dx$

9.9 Equations of form $(d^ny/dx^n) = f(x)$

This is an exact differential equation. Integrating directly, it gives

$$d^{n-1}y/dx^{n-1} = \int f(x)\,dx + c_1,$$

which is also exact. Again integrating it, we get

$$d^{n-2}y/dx^{n-2} = \iint f(x)(dx)^2 + c_1x + c_2,$$

which is again exact. We integrate again and again till we get the complete solution.

9.10 Examples (Type 4) based on Art 9.9

Ex. 1. *Solve* $d^3y/dx^3 = xe^x$.

Sol. Integrating, the given equation gives

$$d^2y / dx^2 = \int xe^x dx + c_1 = xe^x - e^x + c_1. \qquad \text{... (1)}$$

Again integrating (1), $dy/dx = \int xe^x dx - \int e^x dx + \int c_1 dx + c_2$

i.e., $dy/dx = xe^x - e^x - e^x + c_1 x + c_2$. ... (2)

Again integrating (2), $y = \int xe^x dx - 2\int e^x dx + \int (c_1 x + c_2) dx + c_3$

or $y = xe^x - e^x - 2e^x + (1/2) \times c_1 x^2 - c_2 x + c_3$. or $y = xe^x - 3e^x + (1/2) \times c_1 x^2 - c_2 x + c_3$.

EXERCISE 9 (D)

Solve the following differential equations.

1. $d^2y / dx^2 = x^2 \sin^2 x$. **Ans.** $y = -x^2 \sin x - 4x \cos x + \cos x + 6 \sin x + c_1 x + c_2$.
2. $x^2 (d^4 y / dx^4) + 1 = 0$ **Ans.** $y = (1/2) \times x^2 \log x + c_1 x^3 + c_2 x^2 + c_3 x + c_4$.
3. $d^2 y / d x^2 = \log x$. **Ans.** $36y = 6x^3 \log x - 11x^3 + c_1 x^2 + c_2 x + c_3$.

9.11 Equations of the form $(d^2y / dx^2) = f(y)$

To solve it, multiply both sides by $2(dy / dx)$. Then we get

$$2\frac{dy}{dx}\frac{d^2y}{dx^2} = 2f(y)\frac{dy}{dx} \quad \text{or} \quad \frac{d}{dx}\left(\frac{dy}{dx}\right)^2 = 2f(y)\frac{dy}{dx}.$$

Integrating, $\left(\frac{dy}{dx}\right)^2 = 2\int f(y)dy + c_1$ or $\frac{dy}{dx} = \{2\int f(y)dy + c_1\}^{1/2}$

Separating variables, $$dx = \frac{dy}{\left\{2\int f(y)dy + c_1\right\}^{1/2}}$$

Integrating, $x + c_2 = \int \frac{dy}{\left\{2\int f(y)dy + c_1\right\}^{1/2}}$, c_1 and c_2 being arbitrary constants.

9.12 Examples (Type 5) based on art. 9.11

Ex. 1. *Solve* $d^2y / dx^2 = \sec^2 y \tan y$.

Sol. Multiplying both sides by $2(dy/dx)$, we get

$$2\frac{dy}{dx}\frac{d^2y}{dx^2} = 2\sec^2 y \tan y \frac{dy}{dx} \quad \text{or} \quad \frac{d}{dx}\left(\frac{dy}{dx}\right)^2 = 2\sec^2 y \tan y \frac{dy}{dx}.$$

Integrating, $(dy / dx)^2 = 2\int \tan y \sec^2 y \, dy + c_1$, c_1 being an arbitrary constant

or $(dy / dx)^2 = \tan^2 y + c_1$ or $(dy / dx) = (\tan^2 y + c_1)^{1/2}$. ... (1)

Separating variables, $dx = dy / (\tan^2 y + c_1)^{1/2}$

or $$dx = \frac{dy}{\left(\sin^2 y / \cos^2 y + c_1\right)^{1/2}} = \frac{\cos y \, dy}{\sqrt{\sin^2 y + c_1 \cos^2 y}} \quad \text{or} \quad dx = \frac{\cos y \, dy}{\sqrt{[\sin^2 y + c_1(1 - \sin^2 y]}}$$

Put $\sin y = v$ so that $\cos y \, dy = dv$. ... (2)

$\therefore \quad dx = \dfrac{dv}{\sqrt{[c_1 - (c_1 - 1)v^2]}}$ or $dx = \dfrac{1}{\sqrt{(c_1 - 1)}} \dfrac{dy}{\sqrt{[\{c_1 / (c_1 - 1)\} - v^2]}}$

Integrating, $x + c_2 = \dfrac{1}{\sqrt{(c_1 - 1)}} \sin^{-1}\left[v\sqrt{\{(c_1 - 1)/c_1\}}\right]$, c_2 being an arbitrary constant.

Using (2), we have $\quad x + c_2 = \dfrac{1}{\sqrt{(c_1 - 1)}} \sin^{-1}\left[\sin y\sqrt{\{(c_1 - 1)/c_1\}}\right]$ ... (3)

Ex. 2. *Solve* $y'' = \sec^2 y \tan y$ *and modify the solution under the condition that when* $x = 0$, *then* $y = 0$ *and* $dy/dx = 1$.

Sol. For solution see Ex. 1. Putting $y = 0$, $dy/dx = 1$ in (1) of Ex.1, we get $c_1 = 1$. Again putting $x = 0$, $y = 0$ in (3) of Ex.1, we cannot get c_2, So we put $c_1 = 1$ in (1) and get

$dy/dx = \sqrt{(\tan^2 y + 1)} = \sec y$ or $dx = \cos y\, dy$ so that $x + c_2 = \sin y$

Now put given data $x = 0$, $y = 0$. Then we get $c_2 = 0$. Hence we get $x = \sin y$ or $y = \sin^{-1} x$.

EXERCISE 9 (E)

Solve the following differential equations:

1. $y'' = 1/\sqrt{(ay)}$. **Ans.** $3x + c_2 = 2a^{1/4}\left(\sqrt{y} - 2c_1\right)\sqrt{\left(\sqrt{y} + c_1\right)}$.
2. $y^3 y'' = a$ **Ans.** $(c_1 x + c_2)^2 = c_1 y^2 - a$.
3. $y'' + a^2/y^2 = 0$. **Ans.** $\sqrt{c_1 y^2 + y} - \sqrt{2}ac_1 + c_2 = (1/\sqrt{c_1})\log\left[\sqrt{(c_1 y + 1)} + \sqrt{(c_1 y)}\right]$

9.13 Reduction of order. Equations that do not contains *y* directly *i.e.*, equations of the form

$$f(d^n y/dx^n, d^{n-1} y/dx^{n-1} \,\ldots\ldots\ldots\, d^m y/dx^m, x) = o \qquad \ldots (1)$$

which involves lowest derivative m. In such cases we put $d^m y/dx^m = q$. Differentiate this and solve the modified form of (1). Remember that if $m = 1$, we put $dy/dx = p$, where p is standard symbol for dy/dx. In case x and y both are absent, the same procedure may be used.

9.14 Examples (Type 6) based on Art. 9.13

Ex. 1. *Solve* $(1 - x^2)(d^2y/dx^2) - x(dy/dx) = 2$.

Sol. The given equation does not contain y directly and dy/dx is the lowest derivative.

Here we put $dy/dx = p$ so that $d^2y/dx^2 = dp/dx$ Then given equation gives

$(1 - x^2)\dfrac{dp}{dx} - xp = 2$ or $\dfrac{dp}{dx} - \dfrac{x}{1 - x}p = \dfrac{2}{1 - x^2}$, ... (1)

which is linear in p and x. Its I.F. $= e^{-\int [x/(1-x^2)]\,dx} = e^{(1/2)\times\log(1-x^2)} = e^{\log(1-x^2)^{1/2}} = \sqrt{1 - x^2}$.

$\therefore$ The solution of (1) is

$p \times \sqrt{1 - x^2} = \displaystyle\int \frac{2}{1 - x^2}\sqrt{1 - x^2}\, dx + c_1$ or $\dfrac{dy}{dx}\sqrt{1 - x^2} = 2\displaystyle\int \frac{dx}{\sqrt{1 - x^2}} + c_1 = 2\sin^{-1} x + c_1$.

Separating variables, $\quad dy = \left[\dfrac{2\sin^{-1} x}{\sqrt{1 - x^2}} + \dfrac{c_1}{\sqrt{1 - x^2}}\right] dx$

Integrating, $\quad y = (\sin^{-1} x)^2 + c_1 \sin^{-1} x + c_2$, c_1 and c_2 being arbitrary constants.

Ex. 2. *Solve* $(d^2y / dx^2) + (dy/dx) + (dy/dx)^3 = 0$

Sol. The given equation does not contain y directly and dy/dx is the lowest derivative. Hence put $dy/dx = p$ so that $d^2y / dx^2 = dp/dx$. Then the given equation reduces to

$$\frac{dp}{dx} + p + p^3 = 0 \quad \text{or} \quad \frac{dp}{p(1+p^2)} = -\,dx$$

or $$\left[\frac{1}{p} - \frac{p}{1+p^2}\right] dp = -\,dx, \text{ on resolving into partial fractions.}$$

Integrating, $$\log p - \log\sqrt{1+p^2} - \log c_1 = -\,x.$$

or $$\log\frac{p}{c_1\sqrt{1+p^2}} = -\,x \quad \text{or} \quad \frac{p}{c_1\sqrt{1+p^2}} = e^{-x} \quad \text{or} \quad \frac{p^2}{1+p^2} = c_1^2 e^{-2x}$$

or $$p^2 = \frac{c_1^2 e^{-2x}}{1 - c_1^2 e^{-2x}} \quad \text{or} \quad p = \frac{dy}{dx} = \frac{c_1 e^{-x}}{\sqrt{(1 - c_1^2 e^{-2x})}} \quad \text{or} \quad dy = \frac{c_1 e^{-x}\,dx}{\sqrt{(1 - c_1^2 e^{-2x})}}.$$

Integrating, $y = -\sin^{-1}(c_1 e^{-x}) + c_2$, c_1 and c_2 being arbitrary constants.

Ex. 3. *Solve* $x\,(d^2 y / dx^2) + (dy/dx) = 4x$, by reducing its order. **(Nagpur 2007)**

Sol. Putting $dy/dx = p$ and $d^2y / dx^2 = dp / \mathrm{dx}$, given equation becomes

$$x(dp/dx) + p = 4x \quad \text{or} \quad (dp/dx) + (1/x)p = 4, \quad \ldots (1)$$

which is linear in variables p and x. Its I.F. $= e^{\int (1/x)dx} = e^{\log x} = x$

$\therefore$ Solution of (1) is $\quad px = \int 4x\,dx + c_1 \quad$ or $\quad x\,(dy/dx) = 2x^2 + c_1$

or $\quad dy = (2x + c_1/x)\,dx \quad$ so that $\quad y = x^2 + c_1 \log x + c_2$,

where c_1 and c_2 are arbitrary constants.

Ex. 4. *Solve* $\dfrac{d^2 y}{dx^2} - \dfrac{a^2}{x(a^2 - x^2)}\dfrac{dy}{dx} = \dfrac{x^2}{a(a^2 - x^2)}$

Sol. Putting $dy/dx = p$ and $d^2y/dx^2 = dp/dx$, the given equation reduces to

$$\frac{dp}{dx} - \frac{a^2}{x(a^2 - x^2)}\,p = \frac{x^2}{a(a^2 - x^2)}, \text{ which is linear in } p \text{ and } x. \quad \ldots (1)$$

Its I.F. $= e^{-\int \{a^2 / x(a^2 - x^2)\}dx} = e^{-\int \{(1/x + 1/2(a-x) - 1/2(a+x)\}dx}$, by partial fractions

$$= e^{-\log x + (1/2)\times \log (a-x) + (1/2)\times \log (a+x)} = e^{-\log x + (1/2)\times \log (a^2 - x^2)} = (a^2 - x^2)^{1/2} / x$$

$\therefore$ The solution of (1) is $$p\frac{\sqrt{(a^2 - x^2)}}{x} = \int \frac{x^2}{a(a^2 - x^2)}\frac{\sqrt{(a^2 - x^2)}}{x}\,dx + c_1$$

or $$\frac{dy}{dx}\frac{\sqrt{(a^2 - x^2)}}{x} = \frac{1}{a}\int \frac{x\,dx}{\sqrt{(a^2 - x^2)}} + c_1 \quad \text{or} \quad \frac{dy}{dx}\frac{\sqrt{(a^2 - x^2)}}{x} = -\frac{\sqrt{(a^2 - x^2)}}{a} + c_1.$$

Separating variables, $$dy = \left[-\frac{x}{a} + \frac{c_1 x}{\sqrt{(a^2 - x^2)}}\right] dx.$$

Integrating, $y = -(x^2 / 2a) - c_1\sqrt{(a^2 - x^2)} + c_2$, c_1 and c_2 being arbitrary constants.

EXERCISE 9 (F)

Solve the following differential equations:

1. $(1+x^2)(d^2y/dx^2)+1+(dy/dx)^2=0.$ **Ans.** $y=\{(1+c_1^2)/c_1^2\}\log(1+c_1x)-(x/c_1)+c_2$.

2. $dy/dx-x(d^2y/dx^2)=f(d^2y/dx^2).$ **Ans.** $y=(1/2)\times cx^2+xf(c)+c'.$

3. $2x(d^3y/dx^3)(d^2y/dx^2)=(d^2y/dx^2)^2-a^2.$ **Ans.** $y=4(a^2+c_1x)^{5/2}/15c_1^2+c_2x+c_3$

4. $d^4y/dx^4-a^2(d^2y/dx^2)=0.$ **Ans.** $y=(c_1/a^2)e^{ax}+(c_2/a^2)e^{-ax}+c_3x+c_4$

5. $d^4y/dx^4-\cot x\,(d^3y/dx^3)=0.$ **Ans.** $y=c_1\cos x+(c_2/2)\times x^2+c_3x+c_4$

6. $x^2(d^3y/dx^3)-4x(d^2y/dx^2)+6(dy/dx)=4.$ **Ans.** $y=c_1x^4+c_2x^3+(2x/3)+c_3$

7. $d^2y/dx^2=\{1+(dy/dx)^2\}^{1/2}$ **Ans.** $y=c_2+\cosh(x+c_1).$

8. $(d^2y/dx^2)=a^2+k^2(d^2y/dx^2)^2.$ **Ans.** $y=(1/k^2)\log\sec\{ak(x+c_1)\}+c_2.$

9. $x(d^2y/dx^2)+x(dy/dx)^2-dy/dx=0.$ **Ans.** $y=\log(x^2+2c_1)+c_2$

10. $(d^3y/dx^3)^2+x(d^2y/dx^2)-d^2y/dx^2=0.$ **Ans.** $y=(1/6)\times c_1x^3+(1/2)\times c_1^2x^2+c_2x+c_3$

9.15 Equations that do not contain *x* directly *i.e.*, equations of the form

$$\int\left(d^ny/dx^n,d^{n-1}y/dx^{n-1},\ldots\ldots\ldots\ldots.dy/dx.y'=0\right) \quad \ldots(1)$$

In such equations we put $dy/dx=p$ ... (2)

and
$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{dp}{dx}=\frac{dp}{dy}\cdot\frac{dy}{dx}=p\frac{dp}{dy}. \quad \ldots(3)$$

$$\frac{d^3y}{dx^3}=\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)=\frac{d}{dx}\left(p\frac{dp}{dy}\right)=\frac{d}{dy}\left(p\frac{dp}{dy}\right)\frac{dy}{dx}=\left[p\frac{d^2p}{dy^2}+\left(\frac{dp}{dy}\right)^2\right]p$$

or
$$\frac{d^3y}{dx^3}=p^2\frac{d^2p}{dy^2}+p\left(\frac{dp}{dy}\right)^2 \quad \ldots(4)$$

and so on. We put these in the given equation and get an equation in two variables p and y.

Important note : We shall use formulae (3) and (4) directly in the solutions. The students are advised to prove these before using in examination. When x and y are both absent, you can apply both Art. 9.13 and 9.15.

9.16 Examples (Type 7) based on Art 9.15

Ex. 1. *Solve* $y(d^2y/dx^2)-(dy/dx)^2=y^2\log y$

Sol. The given equation does not contain x directly.

Let $dy/dx=p$ so that $d^2y/dx^2=p(dp/dy)$

Then the given equation becomes

Putting $yp\dfrac{dp}{dy}-p^2=y^2\log y$ or $p\dfrac{dp}{dy}-\dfrac{1}{y}p^2=y\log y.$ ... (1)

Putting $p^2=v$ so that $2p(dp/dy)=dv/dy$, (1) gives

$$\frac{1}{2}\frac{dv}{dy}-\frac{1}{y}v=y\log y \quad \text{or} \quad \frac{dv}{dy}-\frac{2}{y}v=2y\log y, \ldots(2)$$

which is linear in v and y. Its integration factor (I.F) is given by

$$\text{I.F.} = e^{-\int (2/y)dy} = e^{-2\log y} = e^{\log y^{-2}} = (1/y^2) \text{ and so solution of (2) is}$$

$$v \cdot \frac{1}{y^2} = \int 2y \log y \cdot \frac{1}{y^2} dy + c_1 \quad \text{or} \quad \frac{p^2}{y^2} = 2\int \log y \cdot \frac{dy}{y} + c_1$$

or $$\left(\frac{dy}{dx}\right)^2 \frac{1}{y^2} = 2\frac{(\log y)^2}{2} + c_1 \quad \text{or} \quad dx = \frac{dy}{y\sqrt{[(\log y^2) + c_1]}} \quad \text{.... (3)}$$

To integrate, put $\log y = t$ so that $(1/y)dy = dt$. Then, (3) gives $dt/\sqrt{(t_1^2 + c_1)} = dx$.

Integrating, $\sinh^{-1}(t/\sqrt{c_1}) = x + c_2$ or $t = \sqrt{c_1} \sinh (x + c_2)$

or $\log y = \sqrt{c_1} \sinh (x + c_2)$, as $t = \log y$, c_1 and c_2 being arbitrary constants.

Ex. 2. *Solve* $y(1 - \log y)(d^2 y/dx^2) + (1 + \log y)(dy/dx)^2 = 0$.

Sol. The given equation does not contain x directly.

Hence put $(dy/dx) = p$ so that $(d^2y / dx^2) = p\ (dp/dy)$. Then the given equation gives

$$y(1 - \log y)p\frac{dp}{dy} + (1 + \log y)p^2 = 0. \quad \text{or} \quad \frac{dp}{dy} + \frac{1 + \log y}{y(1 - \log y)} dy = 0. \quad \text{... (1)}$$

Put $\log y = t$ so that $(1/y)dy = dt$. ... (2)

$\therefore$ From (1), $\dfrac{dp}{p} - \dfrac{t+1}{t-1} dt = 0$ or $\dfrac{dp}{p} - \left[1 + \dfrac{2}{t-1}\right] dt = 0$

Integrating, $\log p - 2 \log (t - 1) - \log c_1 = t$ or $\log \{p/c_1(t - 1)^2\} = t$

or $p/c_1\ (t - 1)^2 = e^t$ or $p = c_1(t - 1)^2 e^t$

$$\frac{dy}{dx} = c_1(\log y - 1)^2 e^{\log y}, \text{ as } t = \log y \quad \text{or} \quad c_1 dx = \frac{dy}{y(\log y - 1)^2}.$$

Using (2) again, $c_1 dx = \dfrac{dt}{(t-1)^2}$ so that $c_1 x + c_2 = -\dfrac{1}{t-1}$

or $c_1 x + c_2 = -1/(\log y - 1)$, c_1 and c_2 being arbitrary constants.

Ex. 3. *Solve* $(dy/dx)^2 - y(d^2 y/dx^2) = n\{(dy/dx)^2 + a^2(d^2 y/dx^2)\}^{1/2}$

Sol. Put $(dy/dx) = p$ so that $(d^2y / dx^2) = p(dp/dy)$. Then, the given equation gives

$$\therefore \quad p^2 - yp\left(dp/dy\right) = n\left\{p^2 + a^2 p^2 \left(dp/dy\right)^2\right\}^{1/2}$$

or $p = yP + n(1 + a^2P^2)^{1/2}$, (taking $dp/dy = P$)

which is of Clairaut's form. So its solution is

$p = yc + n(1 + a^2c^2)^{1/2}$ or $dy/dx = yc + n(1 + a^2c^2)^{1/2}$

or $$\frac{c\,dy}{yc + n(1 + a^2c^2)^{1/2}} = c\,dx, \text{ on separating the variables.}$$

Integrating, $\log\{yc + n(1 + a^2c^2)^{1/2}\} - \log c' = cx, c, c'$, being arbitrary constants.

or $\quad [yc + n\sqrt{(1 + a^2c^2)}]/c' = e^{cx} \quad$ or $\quad yc + n\sqrt{(1 + a^2c^2)} = c'e^{cx}$.

Ex. 4. *Solve:* $y\,(d^2y/dx^2) - 2(dy/dx)^2 = y^2$.

Sol. Putting $dy/dx = p$ and $d^2y/dx^2 = p\,(dp/dy)$, the given equation reduces to

$$\therefore \quad yp\frac{dp}{dy} - 2p^2 = y^2 \quad \text{or} \quad 2p\frac{dp}{dy} - \frac{4}{y}p^2 = 2y. \quad \text{... (1)}$$

Putting $p^2 = v$ and $2p(dp/dy) = dv/dy$, (1) gives $\quad (dv/dy) - (4/y)v = 2y$

whose $\quad$ I.F. $= e^{-4\int (1/y)dy} = e^{-4\log y} = y^{-4}$ and so its solution is

$$vy^{-4} = 2\int y^{-3}dy + c_1 \quad \text{or} \quad p^2y^{-4} = -y^{-2} + c_1$$

or $$\left(\frac{dy}{dx}\right)^2 = c_1y^4 - y^2 \quad \text{or} \quad dx = \frac{dy}{y\sqrt{c_1y^2 - 1}}.$$

Putting $y = t^{-1}$ so that $dy = -t^{-2}\,dt$, we get

$$dx = -\frac{t^{-2}dt}{t^{-1}\sqrt{(c_1/t^2) - 1}} \quad \text{or} \quad dx = -\frac{dt}{\sqrt{(c_1 - t^2)}}.$$

Integrating, $x + c_2 = \cos^{-1}(t/\sqrt{c_1}) \quad$ or $\quad \cos(x + c_2) = 1/(y\sqrt{c_2})$.

or $\quad y\sqrt{c_1}\cos(x + c_2) = 1$, where c_1 and c_2 are arbitrary constants.

EXERCISE 9 (G)

Solve the following differential equations:

1. $y(d^2y/dx^2) + (dy/dx)^2 = 1$. **Ans.** $y^2 = x^2 + 2c_2x + c_1$
2. $y^2(d^2y/dx^2) = a$. **Ans.** $(xc_1 + c_2)^2 = c_1y^2 - a$
3. $y(d^2y/dx^2) + (dy/dx)^2 = y^2$. **Ans.** $y^2 = c_1\sinh(x\sqrt{2} + c_2)$
4. $y(d^2y/dx^2) + (dy/dx)^2 = dy/dx$. **Ans.** $y + (1/c_1)\log(c_1y - 1) = x + c_2$
5. $(1 - y^2)(d^2y/dx^2) = 2(dy/dx)\{1 + y\,(dy/dx)\}$. **Ans.** $8x + c_2 = -2y^2 + 2c_1y + (4 - c_1^2)\log(2y + c_1)$
6. $yy'' + 1 = (y')^2$. **Ans.** $c_1y = \sinh(c_1x + c_2)$
7. $yy'' + (y')^2 = yy'$. **[Delhi B.Sc. (Hons.) II 2011]** **Ans.** $y = \pm\,(c_1 + c_2e^x)^{1/2}$

OBJECTIVE PROBLEMS ON CHAPTER 9

Select (a), (b), (c) or (d) whichever is correct.

Ex. 1 *The differential equation* $P_0(d^3y/dx^3) + P_1(d^2y/dx^2) + P_2(dy/dx) + P_3y = 0$ *is exact if*

(*a*) $P_3 - P_2 + P_1 - P_0 = 0$ $\quad$ (*b*) $P_3 + P_2 + P_1 + P_0 = 0$

(*c*) $P_3 + P'_2 + P'_1 + P''_0 = 0$ $\quad$ (*d*) $P_3 - P'_2 + P''_1 - P'''_0 = 0$

Sol. Ans. (d). Refer Art. 9.2 **[Garhwal 2005]**

Ex. 2. *The solution of the differential equation* $x\{y(d^2y/dx^2) + (dy/dx)^2\} = y(dy/dx)$ is

(*a*) $ax + by = x$ $\quad$ (*b*) $ax^2 + by = 0$

(*c*) $ax^2 + by^2 = 1$ $\quad$ (*d*) $ax^2 + by^2 = 0$ **[I.A.S. (Prel) 1998]**

Sol. Ans. (*c*) Given $\quad \{xy(d^2y/dx^2) + x(dy/dx)^2 + y(dy/dx)\} - 2y(dy/dx) = 0$

or $\quad d\{xy(dy/dx)\} - d(y^2) = 0 \quad$ so that $\quad xy(dy/dx) - y^2 = c$.

or $\quad 2y(dy/dx) - (2/x)y^2 = 2c/x, c$ being an arbitrary constant. $\quad$ ... (1)

Put $y^2 = v$ and $2y(dy/dx) = dv/dx$. Then $(1) \Rightarrow (dv/dx) - (2/x)v = 2c/x$. ... (2)

I.F. of (2) = $e^{\int(-2/x)dx} = e^{-2\log x} = e^{\log x^{-2}} = x^{-2}$ and solution of (2) is

$$vx^{-2} = \int (2c/x)(x^{-2})\,dx + c', c' \text{ being another arbitrary constant.}$$

or $\quad y^2/x^2 = (2c)\{x^{-2}/(-2)\} + c' \quad$ or $\quad y^2 = -c + c'x^2 \quad$ or $\quad c'x^2 - y^2 = c$

or $\quad (c'/c)x^2 - (1/c)y^2 = 1 \quad$ or $\quad ax^2 + by^2 = 1$, taking $a = c'/c$ and $b = -(1/c)$

Ex. 3. *What is the solution of the differential equation* $xy(d^2y/dx^2) + x(dy/dx)^2 + y(dy/dx) = 0$?

(*a*) $y^2 = A \ln x + B$ (*b*) $y^2 = A \ln^2 x$

(*c*) $y = A \ln x + B$ (*d*) $y = A \ln^2 x + B$ **[I.A.S. (Prel) 2006]**

Sol. Ans. *(a)* We shall proceed as explained in Art. 9.8. If possible, Let us write

$$du/dx = xy\,(d^2y/dx^2) + x(dy/dx)^2 + y(dy/dx) = 0 \qquad \text{... (1)}$$

Take $$u_1 = xy(dy/dx) \qquad \text{... (2)}$$

Then $$du_1/dx = xy(d^2/dx^2) + y(dy/dx) + x(dy/dx)^2 \qquad \text{... (3)}$$

From (1) and (3), $$\frac{du}{dx} = \frac{du_1}{dx} = \frac{d}{dx}\left(xy\frac{dy}{dx}\right), \quad \text{by (2)}$$

Then, $\quad (1) \Rightarrow \dfrac{d}{dx}\left(xy\dfrac{dy}{dx}\right) = 0 \quad$ so that $\quad xy\dfrac{dy}{dx} = c_1$

or $$2ydy = 2c_1(1/x)dx \qquad \text{... (4)}$$

Integrating, $$y^2 = A \ln x + B$$

where A (= $2c_1$) and B are aribitrary constants and $\ln x = \log_e x$.

10

Linear Equations of Second Order

10.1 The general (standard) form of the linear equations of the second order

Such equations are of the form $$d^2y/dx^2 + P(dy/dx) + Qy = R, \qquad \text{... (1)}$$
where P, Q, and R are functions of x or constants. We have already read some methods of solving (1), for example, see chapters on linear equations with constant coefficients (Chapter 5), homogeneous equations (Chapter 6) and exact equations (Chapter 9). When (1) cannot be solved by these mehtods, we shall try the methods of this chapter.

10.2 Complete solution of $y'' + Py' + Qy = R$ in terms of one known integral belonging to the complementary function. Solution of $y'' + Py' + Qy = R$ by reduction of its order

(Agra 1995, Gujrat 2001, 05; Himanchal 2001, Delhi 2001, 03, 07, Vikram 2002)

Given $$(d^2y/dx^2) + P(dy/dx) + Qy = R. \qquad \text{... (1)}$$

Let $y = u$ be a known integral of the complementary function. So u is a solution of (1) when its right hand side is taken to be zero. Thus $y = u$ is a solution of

$$(d^2y/dx^2) + P(dy/dx) + Qy = 0$$

so that $$d^2u/dx + P(du/dx) + Qu = 0. \qquad \text{... (2)}$$

Now let the complete solution of (1) be $$y = uv \qquad \text{... (3)}$$
where v is a function of x. v will now be determined.

From (3),
$$\left.\begin{aligned} \frac{dy}{dx} &= v\frac{du}{dx} + u\frac{dv}{dx} \\ \text{and} \quad \frac{d^2y}{dx^2} &= v\frac{d^2u}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + u\frac{d^2v}{dx^2} \end{aligned}\right\} \qquad \text{... (4)}$$

Using (3) and (4), (1) becomes

$$\left(v\frac{d^2u}{dx^2} + 2\frac{du}{dx}\frac{dv}{dx} + u\frac{d^2v}{dx}\right) + P\left(v\frac{du}{dx} + u\frac{dv}{dx}\right) + Quv = R$$

or $$v\left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right) + u\left(\frac{d^2v}{dx^2} + P\frac{dv}{dx}\right) + 2\frac{du}{dx}\frac{dv}{dx} = R$$

or $$u\left(\frac{d^2v}{dx^2} + P\frac{dv}{dx}\right) + 2\frac{du}{dx}\frac{dv}{dx} = R, \text{ using (2)}$$

or $$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}, \text{ dividing by } u. \qquad \text{... (5)}$$

Now put $dv/dx = q$ so that $d^2v / dx^2 = dq/dx$. Hence (5) gives

$$\frac{dq}{dx} + \left(P + \frac{2}{u}\frac{du}{dx}\right)q = \frac{R}{u}, \qquad \text{... (6)}$$

which is linear in q and x. The original equation has therefore had its oredr depressed by unity.

$$\text{I.F. of (6)} = e^{\int [P+(2/u)(du/dx)]dx} = e^{\int Pdx + 2\log u} = e^{\log u^2} \cdot e^{\int Pdx} = u^2 e^{\int Pdx}$$

and solution of (6) is $$qu^2 e^{\int Pdx} = \int \frac{R}{u} u^2 e^{\int Pdx} dx + c_1$$

or $$q = \frac{dv}{dx} = \frac{e^{-\int P\,dx}}{u^2} \int Ru\, e^{\int P\,dx} dx + \frac{c_1 e^{-\int P\,dx}}{u^2}.$$

Integrating, $$v = \int \left\{ \frac{1}{u^2} e^{-\int Pdx} \int Ru\, e^{\int P\,dx} dx \right\} dx + c_1 \int \frac{1}{u^2} e^{-\int Pdx}\, dx + c_2.$$

Putting this value of v in (3), we get

$$y = c_2 u + c_1 u \int \frac{1}{u^2} e^{-\int Pdx}\, dx + u \int \left\{ \frac{1}{u^2} e^{-\int Pdx} \int Ru\, e^{\int P\,dx} dx \right\} dx \qquad \ldots (7)$$

which includes the given solution $y = u$; and since it contains two arbitrary constants c_1 and c_2, hence it is the required complete solution.

10.3 Rules for getting an integral belonging to complementary function (C.F.) i.e. solution of

$$y'' + Py' + Qy = 0. \qquad \ldots (1)$$

Rules 1. $y = e^{ax}$ is a solution of (1) if $\quad$ **$a^2 + Pa + Q = 0$**

Proof. If $y = e^{ax}$, then $dy/dx = ae^{ax}$ and $d^2y/dx^2 = a^2e^{ax}$. Putting these in (1), we get

$(a^2 + Pa + Q)\, e^{ax} = 0$ $\quad$ or $\quad$ $a^2 + Pa + Q = 0.$

Particular case (i). Take $a = 1$. Then $y = e^x$ is a solution of (1) if $\quad$ **$1 + P + Q = 0.$**

Particular case (ii). Take $a = -1$. Then $y = e^{-x}$ is a solution of (1) if $\quad$ **$1 - P + Q = 0.$**

Rule II. $y = x^m$ is *a* solution of (1) if $\quad$ **$m\,(m - 1) + Pmx + Qx^2 = 0.$**

Proof. If $y = x^m$, then $\quad$ $dy/dx = mx^{m-1}$ $\quad$ and $\quad$ $d^2y/dx^2 = m(m-1)x^{m-2}$.

Putting these in (1), we get

$[m(m-1) + Pmx + Qx^2]\, x^{m-2} = 0$ $\quad$ so that $\quad$ $m(m-1) + Pmx + Qx^2 = 0.$

Particular case (i). Take $m = 1$, then $y = x$ is solution of (1) if $\quad$ **$P + Qx = 0.$**

Particular case (ii). Take $m = 2$, then $y = x^2$ is a solution of (1) if $\quad$ **$2 + 2Px + Qx^2 = 0.$**

[Garhwal 2010]

10.4 Working rule for finding complete primitive solution when an integral of C.F. is known or can be obtained by rules of Art. 10.3

Step 1. Put the given equation in standard form $y'' + Py' + Qy = R$, in which the coeffiecient of $d^2y\,/\,dx^2$ is unity.

Step 2. Find an integral u of C.F. by using the following table :

	Condition satisfied	An integral of C.F. is
(*i*)	$1 + P + Q = 0$	$u = e^x$
(*ii*)	$1 - P + Q = 0$	$u = e^{-x}$
(*iii*)	$a^2 + aP + Q = 0$	$u = e^{ax}$
(*iv*)	$P + Qx = 0$	$u = x$
(*v*)	$2 + 2Px + Ox^2 = 0$	$u = x^2$
(*vi*)	$m(m-1) + Pmx + Qx^2 = 0$	$u = x^m$

If a solution (or integral) u is given in a problem, then this step is omitted.

Step 3. Assume that the complete solution of given equation is $y = uv$, where u has been obtained in step 2. Then as explained in Art. 10.2, given equaiton reduces to

$$\frac{d^2 v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u} \qquad \text{... (1)}$$

Step 4. Take $dv/dx = q$ so that $d^2v/dx^2 = dq/dx$. Put in (1). Then (1) will come out to be a linear equation in x and q if $R \neq 0$. Solve it as usual. If $R = 0$, then variables q and x will be separable.

Step 5. Now replace q by dv/dx and separate the variables v and x. Integrate and determine v. Put this value of v in the assumed solution $y = uv$. This will lead us to the desired complete solution of the given equaiton.

10.4.A. Theorem. *If $y = f(x)$ is a solution of the equation $p(x)\, y'' + q(x)\, y' + r(x)\, y = 0$, then $y = c_1 f(x) + c_2 v f(x)$ is the general solution of the given equaiton, where*

$$v = \int \frac{\exp\left[-\int \{q(x)/p(x)\}\right]}{\{f(x)\}^2}, dx, \; c_1, c_2 \text{ being arbitrary constants}$$

Here exp (a) stands for exponential of a, *i.e.*, e^a.

[Himachal 2003, 04, 06, Kolkata 2002, 04, 06, Kurukshetra 2003, 06]

Proof Given $$p(x)\, y'' + q(x)\, y' + r(x)\, y = 0 \qquad \text{...(1)}$$

Given that f(x) is a solution of (1), hence

$$p(x) f''(x) + q(x) f''(x) + r(x) f(x) = 0 \qquad \text{...(2)}$$

Let $$g(x) = v(x) f(x) \qquad \text{...(3)}$$

be the second solution of (1). Hence, we have

$$p(x)\, g''(x) + q(x)\, g'(x) + r(x)\, g(x) = 0 \qquad \text{...(4)}$$

From (3), $g'(x) = v'(x) f(x) + v(x) f'(x)$, $\qquad g''(x) = v''(x) f(x) + 2v'(x) f'(x) + v(x) f''(x)$

$\therefore$ (4) reduces to $$p(x)\{v''(x) f(x) + 2v'(x) f'(x) + v(x) f''(x)\}$$
$$+ q(x)\{v'(x) f(x) + v(x) f'(x)\} + r(x) v(x) f(x) = 0$$

or $$p(x) v''(x) f(x) + v'(x)\{2p(x) f'(x) + q(x) f(x)\}$$
$$+ v(x)\{p(x) f''(x) + q(x) f'(x) + r(x) f(x)\} = 0$$

or $$p(x) v''(x) f(x) + v'(x)\{2p(x) f'(x) + q(x) f(x)\} = 0, \text{using (2)}$$

Let $v'(x) = w(x)$ so that $v''(x) = w'(x)$ Then, the above equation becomes

$$p(x)\, f(x)\, (dw/dx) + w\{2\, p(x) f'(x) + q(x) f(x)\} = 0$$

or $$\frac{dw}{w} = -\left\{2\frac{f'(x)}{f(x)} + \frac{q(x)}{p(x)}\right\} dx, \text{ by separating the variables}$$

Integrating it, $$\log w = -2 \log f(x) - \int \{q(x)/f(x)\} dx, \qquad \text{...(5)}$$

where we have omitted the usual constant of integration because we wish to find second particular solution of (1).

$$(5) \Rightarrow \log\left[w\{f(x)\}^2\right] = -\int \{q(x)/f(x)\}\,dx \Rightarrow w = \frac{1}{\{f(x)\}^2}\exp[-\int \{q(x)/p(x)\}\,dx]$$

Thus, $$\frac{d\nu}{dx} = \frac{\exp\left[-\int \{q(x)/p(x)\}\,dx\right]}{\{f(x)\}^2} \Rightarrow \nu = \int \frac{\exp\left[-\int \{q(x)/p(x)\}dx\right]}{\{f(x)\}^2}\,dx \qquad \text{...(6)}$$

which is the required form of the function ν (*x*).

We shall now show that *f* (*x*) and *g*(*x*) are linearly independent solutions of (1). We have,

$W(f, g)$ = Wronskian of $f(x)$ and $g(x)$

$$= \begin{vmatrix} f(x) & g(x) \\ f'(x) & g'(x) \end{vmatrix} = \begin{vmatrix} f(x) & \nu f(x) \\ f'(x) & \nu' f(x) + \nu f'(x) \end{vmatrix}, \text{ using (3)}$$

$$= \begin{vmatrix} f(x) & 0 \\ f'(x) & \nu' f(x) \end{vmatrix}, \text{ on applying the operation } C_2 \to C_2 - \nu C_1$$

$$= \{f(x)\}^2 \nu' = \exp\left[-\int \{q(x)/p(x)\}dx\right], \text{ using (6)}$$

which is non-zero, being an exponential function. Since $W(f, g) \neq 0$, it follows that $f(x)$ and $g(x)$ are linearly independent solutions of (1).

Then, the general solution of (1) is $y = c_1 f(x) + c_2\, g(x)$, i.e., $y = c_1 f(x) + c_2 \nu f(x)$

10.4B. Solved examples based on Art. 10.4A

Ex. 1. *Find the second linearly independent solution of* $xy'' - (x + 1)y' + y = 0$ $(x > 0)$ *while one solution is* e^x. **[Madurai 2001, 03, 05, 07]**

Sol. Comparing the given equation with $p(x)y'' + q(x)\,y' + r(x)\,y = 0$, we have $p(x) = x$ and $q(x) = -(x + 1)$. Also here $f(x) = e^x$. Hence the second solution = νe^x, where

$$\nu = \int \frac{\exp\left[-\int \{q(x)/p(x)\}dx\right]}{\{f(x)^2}\,dx, \text{ using formula (6) of Art. 10.4.A.} \qquad \text{...}(i)$$

Now, $$\int \frac{q(x)}{p(x)}\,dx = -\int \frac{x+1}{x}dx = \int (1 + 1/x)\,dx = -(x + \log x)$$

$\therefore$ $$\exp[-\int \{q(x)/p(x)\}dx] = \exp\,(x + \log x) = e^{x + \log x} = e^x e^{\log x} = xe^x$$

$\therefore$ from (i), $$\nu = \int \frac{xe^x}{(e^x)^2}\,dx = \int xe^{-x}dx = x(-e^{-x}) - \int \{1 \times (-e^{-x})\}\,dx = -xe^{-x} - e^{-x}$$

$\therefore$ Required second solution = $\nu e^x = e^x(-xe^{-x} - e^{-x}) = -(x+1)$

Ex. 2. *Verify that* $\phi_1(x) = x^2$ *in a solution of the differential equation* $(d^2y/dx^2) - (2/x^2)y = 0$, $0 < x < \infty$ *and find a second independent solution. Also obtain the solution of the given equation.* **[Himachal 2000, 2001, I.A.S. 1995, 97]**

Sol. Given $$x^2 y'' + 0 \times y' - 2y = 0 \qquad ...(1)$$

and $$\phi_1(x) = x^2 \qquad ...(2)$$

(2) $\Rightarrow$ $\phi_1'(x) = 2x$ and $\phi_1''(x) = 2$ ...(3)

Now, $x^2\phi_1''(x) - 2\phi_1(x) = 2x^2 - 2x^2 = 0$, using (2) and (3),

showing that $\phi_1(x)$ solution of (1).

Companing (1) with $p(x)\, y'' + q(x)\, y' + r(x)\, y = 0$, we have $p(x) = x^2$ and $q(x) = 0$, Also, here usual $f(x) = \phi_1(x) = x^2$. Hence the second solution $= \nu x^2$, where

$$\nu = \int \frac{\exp[-\int \{q(x)/p(x)\}dx}{\{f(x)\}^2} dx \text{, using formula (6) of Art. 10.4A} \qquad ...(i)$$

Now, $\displaystyle\int \frac{q(x)}{p(x)} dx = \int \frac{0}{x^2} dx = 0$ and hence $\exp[-\int \{q(x)/p(x)\}dx] = \exp 0 = e^0 = 1$

$\therefore$ From (*i*), we have $$\nu = \int (1/x^4)dx = \int x^{-4}dx = -(1/3x^3)$$

Hence, required second solution $= \nu x^2 = -x^2 \times (1/3x^3) = -(1/3x)$.

Also, the general solution is $$y = c_1 x^2 + c_2 \times (-1/3) \times (1/x).$$

i.e., $$y = c_1 x^2 + c_2'/x \quad \text{where} \quad c_2' = -(c_2/3)$$

Hence the second linearly independent solution can be taken as $1/x$.

Ex. 3. *Given that the equation* $x(1 - x)y'' + (3/2 - 2x)y' - y/4 = 0$ *has a particular integral of the form* x^n, *prove that* $n = -(1/2)$ *and that the primitive of the equation is* $y = x^{12}(A + B \sin^{-1} x^{1/2})$ *where A and B are arbitrary constants.* **[Gwaliar 2004, Pune 2001]**

Sol. Given $$x(1-x)y'' + (3/2 - 2x)y' - y/4 = 0 \qquad ...(1)$$

Let $$f(x) = x^n \qquad ...(2)$$

From (2), $f'(x) = nx^{n-1}$ and $f''(x) = n(n-1)x^{n-2}$ ...(3)

Since $f(x)$ is a particular integral of (1), we must have

$$x(1-x)f''(x) + (3/2 - 2x)f'(x) - (1/4) \times f(x) = 0$$

or $$x(1-x)n(n-1)x^{n-2} + (3/2 - 2x)nx^{n-1} - (1/4) \times x^n = 0 \text{ using (2) and (3)}$$

or $$x^n \{-n(n-1) - 2n - 1/4\} + x^{n-1}\{n(n-1) - (3/2) \times n\} = 0,$$

which must be an identity in x. Hence, we must have

$$-\{n(n-1) + 2n + 1/4\} = 0, \qquad \textit{i.e.,} \qquad (2n+1)^2 = 0, \qquad \text{giving } n = -(1/2)$$

and $$n(n-1) - (3/2) \times n = 0, \text{ which is also satisfied by } n = -(1/2)$$

Hence, we have $$f(x) = x^n = x^{-1/2} \qquad ...(4)$$

Comparing (1) with $p(x)y'' + q(x)y' + r(x)y = 0$, here $p(x) = x(1-x)$ and $q(x) = 3/2 - 2x$

Also, here $f(x) = x^{-1/2}$ and hence the second solution of (1) is $\nu f(x)$, *i.e.*, $\nu x^{-1/2}$, where

$$\nu = \int \frac{\exp\left[-\int \{q(x)/p(x)\}\,dx\right]}{\{f(x)\}^2}\,dx, \text{ using formula (6) of Art.10.4.A} \qquad ...(i)$$

Now, $\displaystyle\int\left\{-\frac{q(x)}{p(x)}\right\}dx = -\int \frac{3/2-2x}{x(1-x)}\,dx = -\int\left\{\frac{3}{2x} - \frac{1}{2(1-x)}\right\}dx$, on resolving into partial fractions.

$$= -(3/2)\times\log x + (1/2)\times\log(1-x) = -(1/2)\times\log\{x^3(1-x)\} = \log\{x^3(1-x)\}^{-1/2}$$

$$\therefore\ \exp\left[-\int\{q(x)/p(x)\}\,dx\right] = e^{\log[x^3(1-x)]^{-1/2}} = \{x^3(1-x)\}^{-1/2} = x^{-1}\{x(1-x)\}^{-1/2}$$

$$\therefore\ \text{From } (i),\ \nu = \int \frac{x^{-1}\{x(1-x)\}^{-1/2}}{(x^{-1/2})^2}\,dx = 2\int \frac{1}{2(1-x)^{1/2}x^{1/2}}\,dx = 2\sin^{-1} x^{1/2}$$

Hence, the second solution of (1) $= \nu x^{-1/2} = 2x^{-1/2}\sin^{-1} x^{1/2}$

$\therefore$ General solution of (1) is $$y = c_1 f(x) + c_2\,\nu f(x) = c_1x^{-1/2} + 2c_2x^{-1/2}\sin^{-1} x^{1/2}$$

or $$y = x^{-1/2}(A + B\sin^{-1} x^{1/2}), \qquad \text{by taking} \qquad c_1 = A \quad \text{and} \quad c_2 = B.$$

10.5 Solved examples based on Art. 10.4

Ex. 1. *Prove that* $y = \sin x$ *is a part of C.F. of the equation* $(\sin x - x\cos x)y'' - x\sin x\, y' + y\sin x = 0$. **[I.A.S. 2005; Bangalore 1994]**

Sol. Given $$(\sin x - x\cos x)y'' - x\sin x\, y' + y\sin x = 0. \qquad ...(1)$$

Given that $y = \sin x$ so that $y' = \cos x$ and $y'' = -\sin x$.

With these values of y, y' and y'', we have

$$\text{L.H.S. of (1)} = (\sin x - x\cos x)(-\sin x) - x\sin x\cos x + \sin^2 x$$
$$= -\sin^2 x + x\sin x\cos x - x\sin x\cos x + \sin^2 x = 0,$$

showing that $y = \sin x$ is a part of C.F. of (1).

Ex. 2(a). *Solve* $xy'' - (2x-1)y' + (x-1)y = 0$. **[Patna 2003; Delhi Maths (G) 2006; Bangalore 2002, 05. Osmania 2001, 04, 07 Kanpur 1994; Meerut 2010]**

or

Solve by reducing the order. $x\,y^{(2)} - (2x-1)\,y^{(1)} + (x - 11y = 0$, *given that* e^x *is one integral part. where* $y^{(1)} = dy/dx$ *and* $y^{(2)} = d^2y/dx^2$. **[Delhi B.Sc/ B.A. Maths (Prog) 2007]**

Sol. Putting the given equation in standard form, we get

$$\frac{d^2y}{dx^2} - \left(2 - \frac{1}{x}\right)\frac{dy}{dx} + \left(1 - \frac{1}{x}\right) = 0. \qquad ...(1)$$

Comparing (1) with $y'' + Py' + Qy = R$, we have

$$P = -(2 - 1/x), \qquad Q = 1 - (1/x), \qquad R = 0. \qquad ...(2)$$

Here, $1 + P + Q = 1 - 2 + (1/x) + 1 - (1/x) = 0$,

showing that $u = e^x$ ... (3)

is a part of C.F. of the solution of (1).

Let the complete solution (1) be $y = uv$ ... (4)

Then v is given by $$\frac{d^2 v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$$

or $$\frac{d^2 v}{dx^2} + \left(-2 + \frac{1}{x} + \frac{2}{e^x}\frac{de^x}{dx}\right)\frac{dv}{dx} = 0, \quad \text{using (2) and (3)}$$

or $$\frac{d^2 v}{dx^2} + \left(-2 + \frac{1}{x} + 2\right)\frac{dv}{dx} = 0 \qquad \text{or} \qquad \frac{d^2 v}{dx^2} + \frac{1}{x}\frac{dv}{dx} = 0. \quad ...(5)$$

Let $dv/dx = q$ so that $d^2v/dx^2 = dq/dx$... (6)

Then (5) becomes $\dfrac{dq}{dx} + \dfrac{q}{x} = 0$ or $\dfrac{dq}{q} = -\dfrac{dx}{x}$.

Integrating, $\log q = \log c_1 - \log x$ or $q = c_1/x$

or $\dfrac{dv}{dx} = \dfrac{c_1}{x}$ or $dv = \dfrac{c_1 dx}{x}$ $\left[\because q = \dfrac{dv}{dx}\right]$

Integrating, $v = c_1 \log x + c_2$, c_1, c_2 being arbitrary constants ... (7)

From (3), (4) and (7), the required complete solution is

$y = e^x v$ or $y = c_1 e^x \log x + c_2 e^x$.

Ex. 2(b). *Solve* $(3 - x)y'' - (9 - 4x)y' + (6 - 3x)y = 0$. **[Delhi Maths (G) 1998; Allahabad 2003; Garhwal 1997; Kurukshetra 2000, 05]]**

Sol. Re-writing the given equation in standard form, we get

$$\frac{d^2 y}{dx^2} - \frac{9-4x}{3-x}\frac{dy}{dx} + \frac{6-3x}{3-x} = 0. \quad ... (1)$$

Comparing (1) with $y'' + Py' + Qy = R$, we have

$P = -(9 - 4x) / (3 - x)$, $Q = (6 - 3x) / (3 - x)$, $R = 0$.... (2)

Here $$1 + P + Q = 1 - \frac{9-4x}{3-x} + \frac{6-3x}{3-x} = \frac{3 - x - (9-4x) + 6 - 3x}{3-x} = 0,$$

showing that $u = e^x$ ... (3)

is a part of C.F. of the solution of (1).

Let the complete solution of (1) be $y = uv$ ... (4)

Then v is given by $$\frac{d^2 v}{dx^2}\left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$$

or $$\frac{d^2 v}{dx^2} + \left(-\frac{9-4x}{3-x} + \frac{2}{e^x}\frac{de^x}{dx}\right)\frac{dv}{dx} = 0, \quad \text{using (2) and (3)}$$

or $$\frac{d^2 v}{dx^2} + \frac{2(3-x)-(9-4x)}{3-x}\frac{dv}{dx} = 0 \qquad \text{or} \qquad \frac{d^2 v}{dx^2} + \frac{2x-3}{3-x}\frac{dv}{dx} = 0 \quad ... (5)$$

Let $dv/dx = q$ so that $d^2v / dx^2 = dq/dx$. ... (6)

Then (5) becomes $\dfrac{dq}{dx}+\dfrac{2x-3}{3-x}q=0$ or $\dfrac{dq}{dx}=-\dfrac{2x-3}{3-x}q$

or $\dfrac{dq}{q}=\dfrac{2x-3}{x-3}dx$ or $\dfrac{dq}{q}=\left(2+\dfrac{3}{x-3}\right)dx.$

Integrating, $\log q = 2x + 3\ \log (x-3) + \log c_1$

or $\log q - \log(x-3)^3 - \log c_1 = 2x$ or $q/[c_1(x-3)^3] = e^{2x}$

or $q = c_1 e^{2x}(x-3)^3$ or $dv/dx = c_1 e^{2x}(x-3)^3$, by (6)

or $dv = c_1 e^{2x}(x-3)^3\ dx.$

Integrating, $v=c_1\int (x-3)^3 e^{2x}dx + c_2$, c_1, c_2 being arbitrary constants.

or $v=c_2+c_1\left[(x-3)^3\dfrac{e^{2x}}{2}-\int 3(x-3)^2\cdot\dfrac{e^{2x}}{2}dx\right]$, integrating by parts

$=c_2+\dfrac{c_1}{2}(x-3)^3e^{2x}-\dfrac{3}{2}c_1\left[(x-3)^2\cdot\dfrac{e^{2x}}{2}-\int 2(x-3)\cdot\dfrac{e^{2x}}{2}dx\right]$, integrating by parts again

$=c_2+\dfrac{c_1}{2}(x-3)^3e^{3x}-\dfrac{3}{4}c_1(x-3)^2\,e^{2x}+\dfrac{3}{2}c_1\left[(x-3)\cdot\dfrac{e^{2x}}{2}-\int 1\cdot\dfrac{e^{2x}}{2}dx\right]$

(Integrating by parts again)

$= c_2+(1/2)\times c_1(x-3)^3e^{2x}-(3/4)\times c_1(x-3)^2e^{2x}+(3/4)\times c_1(x-3)e^{2x}-(3/8)\times c_1e^{2x}$

$= c_2+(1/8)\times c_1e^{2x}[4(x-3)^3-6(x-3)^2+6(x-3)-3]$

or $$v=c_2+(1/8)\times c_1e^{2x}(4x^3-42x^2+150x-183). \qquad \text{... (7)}$$

From (3), (4) and (7), the required complete solution is

$y = e^x v$ or $y = c_2e^x + (1/8)\times c_1e^{3x}(4x^3-42x^2+150x-183).$

Ex. 3. *Solve* $(x+2)y''-(4x+9)y'+(3x+7)y=0.$ **[Delhi Maths (G) 1994]**

Hint. Do as in Ex. 2(*b*). **Ans.** $y = c_1(2x+3)e^{3x}+c_2e^x$

Ex.4(*a*) *Find general solution of* $(1-x^2)y''-2xy'+2y=0$, *if* $y = x$ *is a solution of it.*

(*b*) *If* $y = x$ *is a solution of* $x^2y''+xy'-y=0$, *find the solution.* **[Mumbai 2010]**

Sol. (*a*) Re-writing the given equation in standard form, we get

$$\frac{d^2y}{dx^2}-\frac{2x}{1-x^2}\frac{dy}{dx}+\frac{2}{1-x^2}y=0. \qquad \text{... (1)}$$

Comparing (1) with $y''+Py'+Qy=R$, we get

$P=-(2x)/(1-x^2)$, $Q=2/(1-x^2)$, $R=0.$ ... (2)

Here $u = x$ ... (3)

is given to be part a of C.F. of the solution of (1).

Let the complete solution of (1) be $y = uv.$ ... (4)

Then v is given by $$\frac{d^2y}{dx^2}+\left(P+\frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx}=\frac{R}{u}$$

or $$\frac{d^2 y}{dx^2}+\left(-\frac{2x}{1-x^2}+\frac{2}{x}\frac{dx}{dx}\right)\frac{dv}{dx}=0, \quad \text{using (2) and (3)}$$

or $$\frac{d^2 v}{dx^2}+\left(\frac{2}{x}-\frac{2x}{1-x^2}\right)\frac{dv}{dx}=0. \qquad \text{... (5)}$$

Let $dv/dx = q$ so that $d^2v/dx^2 = dq/dx$.... (6)

Then (5) becomes $\dfrac{dq}{dx}+\left(\dfrac{2}{x}-\dfrac{2x}{1-x^2}\right)q=0$ or $\dfrac{dq}{q}+\left(\dfrac{2}{x}-\dfrac{2x}{1-x^2}\right)dx=0$

Integrating, $\log q + 2\log x + \log(1-x^2) = \log c_1$

or $qx^2(1-x^2)c_1$ or $dv/dx = c_1/x^2(1-x^2)$ by (6)

or $$dv=c_1\frac{1}{x^2(1-x^2)}dx=c_1\left(\frac{1}{x^2}+\frac{1}{1-x^2}\right)dx, \text{ on resolving into partial fractions}$$

Integrating, $$v=c_1\left[\frac{x^{-1}}{-1}+\frac{1}{2}\log\frac{1+x}{1-x}\right]+c_2,\; c_1, c_2 \text{ beng arbitrary constant} \qquad \text{... (7)}$$

From (3), (4) and (7), the required general solution is

$y = uv = xv$ or $y=c_2 x+c_1\left(-1+\dfrac{x}{2}\log\dfrac{1+x}{1-x}\right).$

(*b*) **Hint :** Proceed as in part (*a*). **Ans.** $y = x + x^{-1}$.

Ex. 5. *Solve* $x^2 y'' + xy' - y = 0$, *given that* $x + (1/x)$ *is one integral by using the method of reduction of order.* **[Delhi Maths (G) 2001]**

Sol. Re-writing the given equation in standard form, we get

$$y'' + (1/x)y' - (1/x^2)y = 0; \qquad \text{... (1)}$$

Comparing (1) with $y'' + Py' + Q\,y = R$, we get $P = 1/x$, $Q = -(1/x^2)$. $R = 0$ ... (2)

Here given that $u = x + 1/x$ ... (3)

is part of C.F. of the solution of (1).

Let the complete solution of (1) be $y = uv$. ... (4)

Then v is given by $$\frac{d^2 v}{dx^2}+\left(P+\frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx}=\frac{R}{u}$$

or $$\frac{d^2 v}{dx^2}+\left[\frac{1}{x}+\frac{2}{x+(1/x)}\frac{d}{dx}\left(x+\frac{1}{x}\right)\right]\frac{dv}{dx}=0$$

or $\dfrac{d^2 v}{dx^2}+\left[\dfrac{1}{x}+\dfrac{2x}{x^2+1}\left(1-\dfrac{1}{x^2}\right)\right]\dfrac{dv}{dx}=0$ or $\dfrac{d^2 v}{dx^2}+\dfrac{3x^2-1}{x(x^2+1)}\dfrac{dv}{dx}=0$ ... (5)

Let $dv/dx = q$ so that $d^2v/dx^2 = dq/dx$. ... (6)

Then (5) becomes $\dfrac{dq}{dx}+\dfrac{3x^2-1}{x(x^2+1)}q=0$ or $\dfrac{dq}{q}+\dfrac{3x^2-1}{x(x^2+1)}dx=0$

or $$\frac{dq}{q}+\left(\frac{4x}{x^2+1}-\frac{1}{x}\right)dx=0, \quad \text{on resolving into partial fractions.}$$

Integrating, $\log q + 2\log(x^2+1) - \log x = \log C_1$, C_1 being an arbitrary constant

or $$\frac{q(x^2+1)^2}{x} = C_1 \quad \text{or} \quad \frac{dv}{dx} = \frac{C_1 x}{(x^2+1)^2}, \quad \text{as} \quad q = \frac{dv}{dx}$$

or $$dv = \frac{C_1 x}{(x^2+1)^2}\,dx = \frac{C_1}{2t^2}\,dt, \quad \text{putting } x^2+1=t \quad \text{so that} \quad 2x\,dx = dt$$

Integrating, $v = C_2 - \dfrac{C_1}{2t} = C_2 - \dfrac{C_1}{2(x^2+1)}$, C_2 being an arbitrary constant. ... (7)

From (3), (4) and (7), the required general solution is

$$y = uv = \left(x + \frac{1}{x}\right)v = \frac{x^2+1}{x}\left[C_2 - \frac{C_1}{2(x^2+1)}\right] \quad \text{or} \quad y = C_2\frac{x^2+1}{x} - \frac{C_1}{2}\cdot\frac{1}{x}$$

or $$y = C_2(x + 1/x) + C_1'(1/x), \quad \text{where} \quad C_1' = -C_1/2.$$

Ex. 6. *Solve* $xy'' - (x+2)y' + 2y = 0.$ **[Delhi Maths (G) 2000, 2002]**

Hint. Do as in Ex.1. **Ans.** $y = C_1e^x + C_2(x^2+2x+2)$

Ex. 7. *Solve* $x^2y'' + xy' - 9y = 0$, *given that* $y = x^3$ *is a solution.*

Hint. Here $u = x^3$. **Ans.** $y = C_1x^{-3} + C_2x^3$.

Ex. 8. *Solve the following differential equations :*

(*i*) $(x\sin x + \cos x)\,y'' - x\cos x \cdot y' + y\cos x = 0.$

[Nagpur 2005; Delhi Maths (Hons.) 1992; Rohilkhand 1997]

(*ii*) $x(x\cos x - 2\sin x)y'' + (x^2+2)\sin x \cdot y' - 2(x\sin x + \cos x)y = 0.$

(*iii*) $(\sin x - x\cos x)y'' - x\sin x\, y' + y\sin x = 0$, given that $y = \sin x$ is a solution.

(*iv*) $\sin^2 x\,(d^2y/dx^2) = 2y$, given that $y = \cot x$ is a solution. **[Bangalore 2005; Meerut 2000; Kanpur 2001, 07, 08; Kurukshetra 2001; Nagpur 2003; Rajsthan 2010)**

(*v*) $(1+x^2)y'' - xy' - a^2y = 0$, given that $y = e^{a\sin^{-1}x}$ is an integral.

Sol. (*i*) Re-writing the given equation in standard form, we get

$$\frac{d^2y}{dx^2} - \frac{x\cos x}{x\sin x + \cos x}\frac{dy}{dx} + \frac{\cos x}{x\sin x + \cos x}\,y = 0. \qquad \text{... (1)}$$

Comparing (1) with $y'' + Py' + Qy = R$, we have

$$P = -(x\cos x)/(x\sin x + \cos x), \qquad Q = \cos x/(x\sin x + \cos x), \qquad R = 0 \qquad \text{... (2)}$$

Here $P + Qx = 0$, showing that $u = x$, ... (3)

is a part of the C.F. of the solution of (1).

Let the complete solution of (1) be $y = uv$. ... (4)

Then v is given by $$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$$

or $$\frac{d^2v}{dx^2} + \left[-\frac{x\cos x}{x\sin x + \cos x} + \frac{2}{x}\frac{dx}{dx}\right]\frac{dv}{dx} = 0. \qquad \text{... (5)}$$

Let $dv/dx = q$ so that $d^2v/dx^2 = dq/dx$. ... (6)

$$\therefore \quad (5) \Rightarrow \frac{dq}{dx} + \left(-\frac{x\cos x}{x\sin x + \cos x} + \frac{2}{x}\right)q = 0 \quad \text{or} \quad \frac{dq}{q} = \left(\frac{x\cos x}{x\sin x + \cos x} - \frac{2}{x}\right)dx.$$

Integrating, $\log q = \log(x\sin x + \cos x) - 2\log x + \log C_1$

or $\quad q = C_1 \dfrac{x\sin x + \cos x}{x^2}$ $\quad$ or $\quad \dfrac{dv}{dx} = C_1\left(\dfrac{\sin x}{x} + \dfrac{\cos x}{x^2}\right)$, by (6)

$\therefore \quad \displaystyle\int dv = C_1 \int \frac{1}{x}\sin x\, dx + C_1 \int \frac{1}{x^2}\cos x\, dx$

or $\quad v = C_1\left[\dfrac{1}{x}(-\cos x) - \displaystyle\int\left(-\frac{1}{x^2}\right)(-\cos x)dx\right] + C_1 \int \frac{1}{x^2}\cos x\, dx + C_2$

[Integrating by parts only the first integral]

or $\quad v = -\dfrac{C_1}{x}\cos x - C_1 \displaystyle\int \frac{1}{x^2}\cos x\, dx + C_1 \int \frac{1}{x^2}\cos x\, dx + C_2 = -\frac{C_1}{x}\cos x + C_2$ ... (7)

From (3), (4) and (7), the required general solution is

$y = uv = xv$ $\quad$ or $\quad$ $y = C_2 x - C_1 \cos x.$

(ii) Do as in part (i). **Ans.** $y = C_1 \sin x + C_2 x^2$

(iii) Re-writing the given equation, $\dfrac{d^2y}{dx^2} - \dfrac{x\sin x}{\sin x - x\cos x}\dfrac{dy}{dx} + \dfrac{\sin x}{\sin x - x\cos x}\, y = 0.$ ... (1)

Comparing (1) with $y'' + Py' + Qy = R$, we have

$P = -(x\sin x)/(\sin x - x\cos x)$, $\quad Q = (\sin x)/(\sin x - x\cos x)$, $\quad R = 0$... (2)

Here given that $\quad u = \sin x$, ... (3)

is a part of C.F. of (1).

Let the general solution of (1) be $\quad y = uv.$ ... (4)

Then v is given by $\quad \dfrac{d^2v}{dx^2} + \left(P + \dfrac{2}{u}\cdot\dfrac{du}{dx}\right)\dfrac{dv}{dx} = \dfrac{R}{u}$

or $\quad \dfrac{d^2v}{dx^2} + \left[-\dfrac{x\sin x}{\sin x - x\cos x} + \dfrac{2}{\sin x}\dfrac{d\sin x}{dx}\right]\dfrac{dv}{dx} = 0.$ ... (5)

Let $\quad dv/dx = q \quad$ *so that* $\quad d^2v/dx^2 = dq/dx$ ...(6)

Then (5) becomes $\dfrac{dq}{dx} + \left(-\dfrac{x\sin x}{\sin x - x\cos x} + 2\cot x\right)q = 0$ or $\dfrac{dq}{q} = \left(\dfrac{x\sin x}{\sin x - x\cos x} - 2\cot x\right)dx$.

Integrating, $\quad \log q = \log(\sin x - x\cos x) - 2\log\sin x + \log C_1$

or $\quad q = \dfrac{C_1(\sin x - x\cos x)}{\sin^2 x}$ $\quad$ or $\quad \dfrac{dv}{dx} = C_1\dfrac{\sin x - x\cos x}{\sin^2 x}$

or $\quad dv = C_1\dfrac{\sin x - x\cos x}{\sin^2 x}dx$ $\quad$ or $\quad dv = C_1 d\left(\dfrac{x}{\sin x}\right)$

Integrating, $\quad v = C_1(x/\sin x) + C_2.$ ... (7)

$\therefore$ The required solution is $\quad y = uv = (\sin x)[C_1(x/\sin x) + C_2].$

or $\quad y = C_1 x + C_2 \sin x$, where C_1 and C_2 are arbitrary constants.

(iv) Rewriting the given equation in standard form, we have

$y'' + 0\cdot y' - 2\,\mathrm{cosec}^2 x\cdot y = 0.$ ... (1)

Comparing (1) with $y'' + Py' + Qy = R$, $\quad P = 0, \quad Q = -2\,\mathrm{cosec}^2 x, \quad R = 0$.... (2)

Given that $\quad u = \cot x$, ... (3)

is a part of C.F. of (1).

Let the general solution of (1) be $\quad y = uv.$ ... (4)

Then v is given by $$\frac{d^2v}{dx^2}+\left(P+\frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx}=\frac{R}{u}$$

or $$\frac{d^2v}{dx^2}+\left(0+\frac{2}{\cot x}\frac{d\cot x}{dx}\right)\frac{dv}{dx}=0 \qquad \text{or} \qquad \frac{d^2v}{dx^2}-\frac{2\operatorname{cosec}^2 x}{\cot x}\frac{dv}{dx}=0. \quad \ldots (5)$$

Let $dv/dx = q$ so that $d^2v/dx^2 = dq/dx$. ... (6)

Then (5) becomes $\dfrac{dq}{dx}-\dfrac{4}{2\sin x\cos x}q=0$ or $\dfrac{dq}{dx}=4\operatorname{cosec} 2x\, dx.$

Integrating, $\log q = 4 \times (1/2) \times \log \tan x + \log C_1$ or $q = C_1 \tan^2 x$

or $dv/dx = C_1 \tan^2 x$ or $dv = C_1 (\sec^2 x - 1)\, dx.$

Integrating, $v = C_1 (\tan x - x) + C_2$, C_1, C_2 being arbitrary constants. ... (7)

From (3), (4) and (7), the required general solution is given by

$y = uv$ or $y = \cot x\, [C_1(\tan x - x) + C_2] = C_1 (1 - x \cot x) + C_2 \cot x.$

(v) Rewriting the given equation in standard form, we have

$$\frac{d^2y}{dx^2}-\frac{x}{1-x^2}\frac{dy}{dx}-\frac{a^2}{1-x^2}y=0. \quad \ldots (1)$$

Comparing (1) with $y'' + Py' + Q\,y = R$, we have

$P = -x/(1-x^2)$, $Q = -a^2/(1-x^2)$, $R = 0$.... (2)

Since $e^{a\sin^{-1}x}$ is a solution of (1), therefore $u=e^{a\sin^{-1}x}$ ... (3)

is a part of C.F. of (1)

Let the general solution of (1) be $y = uv$ ... (4)

Then v is given by $$\frac{d^2v}{dx^2}+\left(P+\frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx}=\frac{R}{u}$$

or $$\frac{d^2v}{dx^2}+\left(-\frac{x}{1-x^2}+\frac{2}{e^{a\sin^{-1}x}}\frac{de^{a\sin^{-1}x}}{dx}\right)\frac{dv}{dx}=0. \quad \ldots (5)$$

Let $dv/dx = q$ so that $d^2v/dx^2 = dq/dx$. ... (6)

Then (5) becomes $$\frac{dq}{dx}+\left(-\frac{x}{1-x^2}+\frac{2}{e^{a\sin^{-1}x}}e^{a\sin^{-1}x}\cdot\frac{a}{\sqrt{(1-x^2)}}\right)q=0$$

or $$\frac{dq}{dx}=\left[\frac{x}{1-x^2}-\frac{2a}{\sqrt{(1-x^2)}}\right]q \qquad \text{or} \qquad \frac{dq}{q}=\left[-\frac{1}{2}\cdot\frac{-2x}{1-x^2}-\frac{2a}{\sqrt{(1-x^2)}}\right]dx.$$

Integrating, $\log q = -(1/2) \times \log(1-x^2) - 2a\sin^{-1}x + \log C_1$

or $\log[(q/C_1)\sqrt{(1-x^2)}]=-2a\sin^{-1}x$ or $q=(C_1/\sqrt{(1-x^2)})\,e^{-2a\sin^{-1}x}$

or $$\frac{dv}{dx}=\frac{C_1}{\sqrt{(1-x^2)}}e^{-2a\sin^{-1}x} \qquad \text{or} \qquad dv=\frac{C_1}{\sqrt{(1-x^2)}}\left(e^{-a\sin^{-1}x}\right)^2 dx. \quad \ldots (7)$$

Putting $-a\sin^{-1}x=t$ and $-[a/\sqrt{(1-x^2)}]\,dx=dt$, (7) gives

$dv=-(C_1/a)(e^t)^2\,dt$ or $dv=-(C_1/a)e^{2t}dt.$

Integrating, $v = -(C_1/2a)e^{2t} + C_2 = -(C_1/2a)e^{-2a\sin^{-1}x} + C_2.$ ... (8)

From (3), (4) and (8), the required general solution is given by

$$y = uv = e^{a\sin^{-1}x}[-(C_1/2a)e^{-2a\sin^{-1}x} + C_2] = -(C_1/2a)e^{-a\sin^{-1}x} + C_2e^{a\sin^{-1}x}$$

or $$y = C_1' e^{-a\sin^{-1}x} + C_2e^{a\sin^{-1}x}, \quad \text{talking} \quad C_1' = -(C_1/2a).$$

Ex. 9. *Solve* $x^2 y_2 - 2x(1+x)y_1 + 2(1+x)y = x^3$.

(Delhi Maths (G) 2001, 02; Bangalogre 2003, 05; Garhwal 1994; Merrut 2004)
(Agra 1999; Rohilkhond 2001; Mumbai 2000; Indore 2000, 02)

Sol. Dividing by x^2, the given equaiton in standard form is

$$\frac{d^2y}{dx^2} - \frac{2(1+x)}{x}\frac{dy}{dx} + \frac{2(1+x)}{x^2}y = x. \quad \text{... (1)}$$

Comparing (1) with $y'' + Py' + Qy = R$, we have

$P = -2(1+x)/x$, $\quad Q = 2(1+x)/x^2 \quad$ and $\quad R = x$. ... (2)

Here $P + Qx = 0$, showing that $u = x$ ... (3)

is a part of C.F. of (1)

Let the general solution of (1) be $y = uv$ ... (4)

Then v is given by $$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$$

or $$\frac{d^2v}{dx^2} + \left[-\frac{2(1+x)}{x} + \frac{2}{x}\frac{dx}{dx}\right]\frac{dv}{dx} = \frac{x}{x} \quad \text{or} \quad \frac{d^2v}{dx^2} - 2\frac{dv}{dx} = 1. \quad \text{... (5)}$$

Let $dv/dx = q$ so that $d^2v/dx^2 = dq/dx$. ... (6)

Then (5) becomes $(dq/dx) - 2q = 1$, which is linear in q and x.

Its integrating factor = I.F. = $e^{\int(-2)dx} = e^{-2x}$ and solution is

$$q\,e^{-2x} = \int 1\cdot e^{-2x}dx + C_1 = -(1/2)\times e^{-2x} + C_1$$

$\therefore \quad q = -(1/2) + C_1e^{2x} \quad$ or $\quad dv/dx = -(1/2) + C_1e^{2x}$

or $$dv = [-(1/2) + C_1e^{2x}]\,dx.$$

Integrating, $v = -(x/2) + (C_1/2)e^{2x} + C_2$. ... (7)

From (3), (4) and (7), the required general solution is $y = uv = x[-(x/2) + (C_1/2)e^{2x} + C_2]$

or $y = C_1' xe^{2x} + C_2x - (x^2/2)$, where $C_1' = C_1/2$; C_1' and C_2 being arbitrary constants.

Ex. 10. *Solve* $x^2y'' - (x^2 + 2x)y' + (x+2)y = x^3e^x$.

[Himanchal 2008; Delhi Maths (G) 1997; Meerut 2005, 09]

Sol. Dividing by x^2, the given equation in standard form is

$$\frac{d^2y}{dx^2} - \left(1 + \frac{2}{x}\right)\frac{dy}{dx} + \left(\frac{1}{x} + \frac{2}{x^2}\right)y = x\,e^x. \quad \text{... (1)}$$

Comparing (1) with $y'' + Py' + Qy = R$, we have

$P = -(1 + 2/x)$, $\quad Q = 1/x + 2/x^2$, $\quad R = x\,e^x$. ... (2)

Here $P + Qx = 0$, showing that $u = x$ ... (3)

is a part of C.F. of (1).

Let the general solution of (1) be $y = uv$. ... (4)

Then v is given by $$\frac{d^2v}{dx^2}+\left(P+\frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx}=\frac{R}{u}$$

or $$\frac{d^2v}{dx^2}+\left(-1-\frac{2}{x}+\frac{2}{x}\frac{dx}{dx}\right)\frac{dv}{dx}=\frac{xe^x}{x}$$ or $$\frac{d^2v}{dx^2}-\frac{dv}{dx}=e^x. \quad ... (5)$$

Let $dv/dx = q$ so that $d^2v / dx^2 = dq/dx$. ... (6)

Then (5) becomes $(dq/dx) - q = e^x$, which is linear in q and x.

Its integraging factor I.F. $= e^{\int(-1)dx} = e^{-x}$ and so solution is

$$qe^{-x} = \int (e^x \cdot e^{-x})\, dx + C_1 = x + C_1$$ or $$q = (x + C_1)\, e^x$$

or $dv/dx = (x + C_1)e^x$ or $dv = (x + C_1)e^x dx$

Integrating, $v = (x + C_1)e^x - \int (1 \cdot e^x)\, dx + C_2$, C_1, C_2 being arbitrary constants.

or $$v = (x + C_1)\, e^x - e^x + C_2 = (x + C_1 - 1)\, e^x + C_2. \quad ... (7)$$

From (3), (4) and (7), the required general solution is

$y = uv = x\, [(x + C_1 - 1)\, e^x + C_2]$ or $y = C_1xe^x + C_2e^x + (x - 1)\, xe^x$.

Ex. 11.(*a*) *Solve* $(x + 1)\, (d^2y / dx^2) - 2(x + 3)\, (dy/dx) + (x + 5)y = e^x$. **[Garhwal 1993]**

Sol. Dividing by $(x + 1)$, the given equation in standard form is

$$\frac{d^2y}{dx^2}-\frac{2(x+3)}{x+1}\frac{dy}{dx}+\frac{x+5}{x+1}y=\frac{e^x}{x+1}. \quad ... (1)$$

Comparing (1) with $y'' + Py' + Q\,y = R$, we get

$P = -\, 2(x + 3)/(x + 1)$, $Q = (x + 5)/(x + 1)$, $R = e^x/(x + 1)$. ... (2)

Here $$1 + P + Q = 1-\frac{2x+6}{x+1}+\frac{x+5}{x+1}=\frac{x+1-(2x+6)+x+5}{x+1}=0,$$

showing that $u = e^x$ ... (3)

is a part of C.F. of (1).

Let the general solution of (1) be $y = uv$. ... (4)

Then v is given by $$\frac{d^2v}{dx^2}+\left(P+\frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx}=\frac{R}{u}$$

or $$\frac{d^2v}{dx^2}+\left(-\frac{2x+6}{x+1}+\frac{2}{e^x}\frac{de^x}{dx}\right)\frac{dv}{dx}=\frac{e^x}{e^x(x+1)}$$

or $$\frac{d^2v}{dx^2}+\left(2-\frac{2x+6}{x+1}\right)\frac{dv}{dx}=\frac{1}{x+1}$$ or $$\frac{d^2v}{dx^2}-\frac{4}{x+1}\frac{dv}{dx}=\frac{1}{x+1}. \quad ... (5)$$

Let $dv/dx = q$ so that $d^2v / dx^2 = dq/dx$. ... (6)

Then (5) becomes $\dfrac{dq}{dx}-\dfrac{4}{x+1}q=\dfrac{1}{x+1}$, which is linear in q and x.

Its integrating factor I.F. $= e^{-\int[4/(x+1)]dx} = e^{-4\log(x+1)} = (x+1)^{-4}$. and solution is

$$q(x+1)^{-4} = \int \frac{1}{x+1}\cdot(x+1)^{-4}\,dx + C_1 = \int (x+1)^{-5}\,dx + C_1$$

or $\quad dv/dx = -(1/4) + C_1\ (x+1)^4 \quad$ or $\quad dv = [-(1/4) + C_1(x+1)^4]dx.$

Integrating, $\quad v = -(1/4)x + (C_1/5)\ (x+1)^5 + C_2.$ ... (7)

From (3), (4) and (7), the required general solution is

$$y = uv = e^x[-(1/4)x + (C_1/5)\ (x+1)^5 + C_2]$$

or $\quad y = C_1'\ e^x\ (x+1)^5 + C_2\ e^x - (1/4)x\ e^x, \quad$ where $C_1' = C_1/5.$

Ex. 11(*b*) *Solve* $xy'' - 2(x+1)y' + (x+2)y = (x-2)e^x.$ **[Bangalore 2001, 04]**

Sol. Do as in Ex. 11(*a*). **Ans.** $y = (1/3) \times C_1x^3e^x + C_2e^x + (x - x^2/2)e^x.$

Ex. 11(*c*) *Solve* $d^2y\ /\ dx^2 - \cot x\ (dy/dx) - (1 - \cot x)y = e^x \sin x.$

[Delhi Maths (G) 2005; Meerut 1996; S.V. University (A.P.) 1997, Kanpur 2006]

Sol. Comparing the given equation with $y'' + Py' + Q\,y = R,$ we get

$P = -\cot x, \qquad Q = -1 + \cot x, \qquad R = e^x \sin x.$ ... (1)

Here $1 + P + Q = 0,$ showing that $\quad u = e^x$ ... (2)

is a part of C.F. of the given equation.

Let the required general solution be $\quad y = uv$ ... (3)

Then v is given by $\quad \dfrac{d^2v}{dx^2} + \left(P + \dfrac{2}{u}\dfrac{du}{dx}\right)\dfrac{dv}{dx} = \dfrac{R}{u}$

or $\quad \dfrac{d^2v}{dx^2} + \left(-\cot x + \dfrac{2}{e^x}\dfrac{de^x}{dx}\right)\dfrac{dv}{dx} = \dfrac{e^x \sin x}{e^x} \quad$ or $\quad \dfrac{d^2v}{dx^2} + (2 - \cot x)\dfrac{dv}{dx} = \sin x$

Let $\quad dv/dx = q \quad$ so that $\quad d^2v/dx^2 = dq/dx$ ... (4)

Hence we get $\quad dq/dx + (2 - \cot x)q = \sin x,$ which linear in q and x

Its I.F. $= e^{\int (2-\cot x)dx} = e^{2x - \log \sin x} = e^{2x} \cdot e^{\log (\sin x)^{-1}} = e^{2x}(\sin x)^{-1}$ and solution is

$q\dfrac{e^{2x}}{\sin x} = \displaystyle\int \left(\sin x \cdot \dfrac{e^{2x}}{\sin x}\right) dx + c_1 = \dfrac{1}{2}e^{2x} + c_1$, c_1 being an arbitrary constant

or $\quad q = (1/2) \times \sin x + c_1e^{-2x} \sin x \quad$ or $\quad dv/dx = (1/2) \times \sin x + c_1e^{-2x} \sin x.$

or $\quad dv = [(1/2) \times \sin x + c_1e^{-2x} \sin x]\ dx.$

Integrating, $\quad v = (-1/2) \times \cos x + c_1 \int e^{-2x} \sin x\ dx + c_2$, c_2 being an arbitrary constant

or $\quad v = -\dfrac{\cos x}{2} + \dfrac{c_1}{1^2 + (-2)^2}\ (-2 \sin x - \cos x) + c_2$ or $v = -\dfrac{\cos x}{2} - \dfrac{c_1}{5}\ (2 \sin x + \cos x) + c_2$ ....(5)

$$\left[\because \int e^{ax} \sin bx\ dx = \frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx)\right]$$

Hence form (2), (3) and (5), the required general solution is

$$y = uv = e^x\ [-(1/2) \times \cos x - (c_1/5) \times e^{-2x}\ (2 \sin x + \cos x) + c_2]$$

or $\quad y = c_1'\ e^{-2x}\ (2 \sin x + \cos x) + c_2e^x - (1/2) \times e^x \cos x,$ where $c_1' = -(c_1/5).$

Ex. 11(*d*) *Solve* $y'' + (1 - \cot x)y' - y \cot x = \sin^2 x.$

[Meerut 1999,Delhi Maths (G) 2000; Rohilkhand 1997]

Hint. Do as in Ex.11(*c*). General solution is given by

$$y = c_1\ (\sin x - \cos x) + c_2e^{-x} - (1/10) \times (\sin 2x - 2 \cos 2x).$$

Ex. 11(*e*) *For the equation* $y_2 + (1 - \cot x)y_1 - y \cot x = \sin^2 x$, *find an integral of C.F.* **[Bangalore 1995]**

Sol. Given $$y_2 + (1 - \cot x)y_1 - y \cot x = \sin^2 x. \qquad ... (1)$$

Comparing (1) with $y_2 + Py_1 + Q\,y = R$, here $P = 1 - \cot x$, $Q = -\cot x$.

Hence $1 - P + Q = 1 - (1 - \cot x) - \cot x = 0$ and so e^{-x} is an integral of C.F.

Ex. 12. *Solve* $x(d^2y/dx^2) - (dy/dx) + (1 - x)y = x^2e^{-x}$. **[Delhi Maths (G) 1996]**

Sol. Re-writing, $$(d^2y/dx^2) - (1/x)(dy/dx) + (1/x - 1)y = xe^{-x}. \qquad ... (1)$$

Comparing (1) with $y'' + Py' + Q\,y = R$, here $P = -1/x$, $Q = (1/x) - 1$, $R = xe^{-x}$ ... (2)

Hence $1 + P + Q = 0$, showing that $$u = e^x \qquad ... (3)$$
is a part of C.F. of the given equation.

Let the required general solution be $$y = uv. \qquad ... (4)$$

Then v is given by $$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$$

or $$\frac{d^2v}{dx^2} + \left(-\frac{1}{x} + \frac{2}{e^x}\frac{de^x}{dx}\right)\frac{dv}{dx} = \frac{xe^{-x}}{e^x} \quad \text{or} \quad \frac{d^2v}{dx^2} + \left(2 - \frac{1}{x}\right)\frac{dv}{dx} = xe^{-2x} \qquad .... (5)$$

Let $dv/dx = q$ so that $d^2v/dx^2 = dq/dx$.... (6)

Then (5) becomes $$\frac{dq}{dx} + \left(2 - \frac{1}{x}\right)q = xe^{-2x},$$ which is linear in q and x.

Its I.F. $= e^{\int (2-1/x)dx} = e^{2x - \log x} = e^{2x} \cdot e^{-\log x} = e^{2x} \cdot x^{-1}$ and solution is

$$\therefore \quad q \cdot e^{2x}x^{-1} = \int (xe^{-2x}) \cdot (e^{2x}x^{-1})\,dx + c_1 = x + c_1$$

or $q = xe^{-2x}(x + c_1)$ or $dv/dx = (x^2 + c_1x)e^{-2x}$, by (6)

$$\therefore \quad \int dv = \int (x^2 + c_1x)e^{-2x}dx + c_2$$

$$\therefore \quad v = c_2 + (x^2 + c_1x) \times (-1/2) \times e^{-2x} - \int \{(2x + c_1) \times (-1/2) \times e^{-2x}\}dx,$$ integrating by parts

$$= c_2 - (1/2) \times (x^2 + c_1x)e^{-2x} + (1/2) \times \int (2x + c_1)e^{-2x}dx$$

$$= c_2 - \frac{1}{2}(x^2 + c_1x)e^{-2x} + \frac{1}{2}\left[(2x + c_1)\left(\frac{e^{-2x}}{-2}\right) - \int (2)\left(\frac{e^{-2x}}{-2}\right)dx\right]$$

$$= c_2 - (1/2) \times (x^2 + c_1x)e^{-2x} - (1/4) \times (2x + c_1)e^{-2x} - (1/4) \times e^{-2x}.$$

$$= c_2 - (1/4) \times e^{-2x}(2x^2 + 2c_1x + 2x + c_1 + 1).$$

or $$v = c_2 - (1/4) \times e^{-2x}(2x^2 + 2x + 1) - (1/4) \times e^{-2x}c_1(2x + 1). \qquad ... (7)$$

From (3), (4) and (7), the required general solution is

$$y = uv = e^x[c_2 - (1/4) \times e^{-2x}(2x^2 + 2x + 1) - (1/4) \times e^{-2x}c_1(2x + 1)]$$

or $$y = c_1'(2x + 1)e^{-x} + c_2e^x - (1/4) \times (2x^2 + 2x + 1)e^{-x}, \quad \text{where } c_1' = -(1/4) \times c_1.$$

Ex. 13. *Solve the following differential equations :*

(*a*) $(d^2y/dx^2) - (1 + x)(dy/dx) + xy = x$.

[Delhi Maths (G) 1993, 94, Gujrat 2003, 05; Kurukshetra 2004]

(*b*) $x(d^2y/dx^2) - (dy/dx) + (1 - x)y = xe^{-x}$. **[Delhi Maths (G) 1995]**

(*c*) $(d^2y/dx^2) - x^2(dy/dx) + xy = x$. **[Delhi Maths (G) 1994, 99]** (*d*) $x(d^2y / dx^2) + (1 - x)(dy/dx) - y = e^x$. **[Lucknow 2006; Srivenkateshwara 2003]**

(*e*) $x(d^2y / dx^2) - (x - 2)(dy/dx) - 2y = x^3$. **[Luknow 2002, Delhi Maths (G) 1996;Kanpur 2006]**

(*f*) $(2x - 1)(d^2y / dx^2) - 2(dy/dx) + (3 - 2x)y = 2e^x$.

(*g*) $(d^2y / dx^2) + [1 + (2/x)\cot x - (2/x^2)]y = x \cos x$, *given that* $(\sin x)/x$ *is an integral included in C.F.* **[Bhopal 2002, 05, Indore 2001, 02, Purvanchal 2000, 04, 07]**

(*h*) $x^2y_2 + xy_1 - y = x^2e^x$. **[Rohilkhand 1994]**

Sol. (*a*) Given $$y'' - (1 + x)y' + xy = x. \qquad \text{... (1)}$$

Comparing (1) with $y'' + Py' + Qy = R$, *here* $P = -1 - x$, $Q = x$, $R = x$. ... (2)

Here $1 + P + Q = 0$, showing that $u = e^x$, ... (3)

is a part of C.F. of (1).

Let the required general solution of (1) be $y = uv$. ... (4)

Then v is given by $$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$$

or $$\frac{d^2v}{dx^2} + \left(-1 - x + \frac{2}{e^x}\frac{de^x}{dx}\right)\frac{dv}{dx} = \frac{x}{e^x} \quad \text{or} \quad \frac{d^2v}{dx^2} + (1 - x)\frac{dv}{dx} = xe^{-x}. \qquad \text{... (5)}$$

Let $dv/dx = q$ so that $d^2v / dx^2 = dq/dx$. ... (6)

Then (5) becomes $(dq/dx) + (1 - x)q = xe^{-x}$.

Its I.F. $= e^{\int (1-x)dx} = e^{x - x^2/2}$ and so its solution is

$$qe^{x-x^2/2} = \int (xe^{-x} \cdot e^{x-x^2/2})\, dx + c_1 = \int xe^{-(x^2/2)} dx + c_1.$$

Putting $-x^2/2 = t$ so that $-xdx = dt$, we have

$$\therefore \quad qe^{x-x^2/2} = \int e^t(-dt) + c_1 = -e^t + c_1 = -e^{-(x^2/2)} + c_1.$$

$$\therefore \quad q = e^{-(x-x^2/2)}\left[c_1 - e^{-x^2/2}\right] = c_1 e^{-x+x^2/2} - e^{-x}$$

or $$dv/dx = c_1 e^{-x+x^2/2} - e^{-x} \quad \text{or} \quad dv = [c_1 e^{-x+x^2/2} - e^{-x}]\, dx.$$

Integrating, $v = c_1 \int e^{-x+x^2/2} dx + e^{-x} + c_2$, c_1, c_2 being arbitrary constants ... (7)

From (3), (4) and (7), the required general solution is given by

$$y = uv = e^x\left[c_1 \int e^{-x+x^2/2} dx + e^{-x} + c_2\right] = c_1 e^x \int e^{-x-x^2/2} dx + c_2 e^x + 1.$$

Remark: Some solutions are left without evaluating the integral which are not integrable by well known standard methods.

(*b*) Do as in part (*a*). **Ans.** $y = c_1(2x + 1)e^{-x} + c_2 e^x + e^x \int x \log x\, e^{-2x} dx.$

(*c*) Given $$y'' - x^2 y' + xy = x \qquad \text{... (1)}$$

Comparing (1) with $y'' + Py' + Qy = R$, $P = -x^2$, $Q = x$, $R = x$. ... (2)

Here $P + Qx = 0$, showing that a part of C.F. of (1) is given by $u = x$. ... (3)

Let the required general solution of (1) be $y = uv$. ... (4)

Then v is given by $$\frac{d^2v}{dx^2}+\left(P+\frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx}=\frac{R}{u}$$

or $$\frac{d^2v}{dx^2}+\left(-x^2+\frac{2}{x}\frac{dx}{dx}\right)\frac{dv}{dx}=\frac{x}{x} \quad \text{or} \quad \frac{d^2v}{dx^2}+\left(\frac{2}{x}-x^2\right)\frac{dv}{dx}=1. \quad \dots (5)$$

Let $dv/dx = q$ so that $d^2v / dx^2 = dq/dx$. ... (6)

Then (5) becomes $\dfrac{dq}{dx}+\left(\dfrac{2}{x}-x^2\right)q=1$, which is linear in q and x.

Its I.F. $= e^{\int[(2/x)-x^2]dx} = e^{2\log x-(x^3/3)}=e^{\log x^2}\cdot e^{-x^3/3}=x^2e^{-x^3/3}$ and solution is

$$\therefore \quad qx^2e^{-x^3/3} = \int(1\cdot x^2e^{-x^3/3})\,dx+c_1=c_1-\int e^t\,dt=c_1-e^t=c_1-e^{-x^3/3}$$

[Putting $-x^3/3 = t$ so that $-x^2\,dx = dt$]

or $q = (1/x^2)e^{x^3/3}(c_1-e^{-x^3/3})$ or $dv/dx=(c_1/x^2)e^{x^3/3}-x^{-2}$

or $$dv = [(c_1/x^2)e^{x^3/3}-x^{-2}]\,dx.$$

Integrating, $v = c_1\int(1/x^2)e^{x^3/3}dx+x^{-1}+c_2$, c_1, c_2 being arbitrary constants ... (7)

From (3), (4) and (7), the required general solution is

$$y=uv= x\left[c_1\int(1/x^2)e^{x^3/3}dx+x^{-1}+c_2\right] = c_1x\int(1/x^2)e^{x^3/3}dx+c_2x+1$$

(*d*) Try yourself. **Ans.** $y = c_1e^x\int x^{-1}e^x\,dx+c_2e^x+e^x\log x.$

(*e*) Dividing by x, the given equation in standard form is

$$y''-(1-2/x)y'-(2/x)y=x^2. \quad \dots (1)$$

Comparing (1) with $y''+Py'+Q\,y=R$, here $P=-1+2/x$, $Q=-2/x$, $R=x^2$.. (2)

Here $1+P+Q=0$, showing that a part of C.F. of (1) is given by $u=e^x$. ... (3)

Let the reqired general solution of (1) be $y=uv$. ... (4)

Then v is given by $$\frac{d^2v}{dx^2}+\left(P+\frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx}=\frac{R}{u}$$

or $$\frac{d^2v}{dx^2}+\left(-1+\frac{2}{x}+\frac{2}{e^x}\frac{de^x}{dx}\right)\frac{dv}{dx} = \frac{x^2}{e^x} \quad \text{or} \quad \frac{d^2v}{dx^2}+\left(1+\frac{2}{x}\right)\frac{dv}{dx}=x^2e^{-x}. \quad \dots (5)$$

Let $dv/dx = q$ so that $d^2v / dx^2 = dq/dx$. ... (6)

Then (5) becomes $(dq/dx)+(1+2/x)q=x^2e^{-x}$, which is a linear equation.

Its I.F. $= e^{\int(1+2/x)dx}=e^{x+2\log x}=e^x\cdot e^{\log x^2}=x^2e^x$ and solution is

$$q(x^2e^x) = \int[(x^2e^{-x})\cdot(x^2e^x)]\,dx+c_1=x^2/5+c_1$$

or $q=dv/dx=x^{-2}e^{-x}[x^2/5+c_1]=(x^3/5)\times e^{-x}+c_1x^{-2}e^{-x}$ or $dv=\{(x^3/5)\times e^{-x}+c_1x^{-2}e^{-x}\}dx$

Integrating, $v = (1/5)\times\int x^3e^{-x}dx+c_1\int x^{-2}e^{-x}dx+c_2$, c_1, c_2 being arbitrary constants

or $$v = (1/5)\times\left[x^3(-e^{-x})-\int(3x^2)(-e^{-x})\,dx\right]+c_1\int x^{-2}e^{-x}dx+c_2$$

[Integrating by parts only the first integral]

$$= -(1/5)\times x^3e^{-x} + (3/5)\times\int x^2e^{-x}dx + c_1\int x^{-2}e^{-x}dx + c_2$$

$$= -(1/5)\times x^3e^{-x} + (3/5)\times\left[x^2(-e^{-x}) - \int(2x)(-e^{-x})\,dx\right] + c_1\int x^{-2}e^{-x}dx + c_2$$

$$= -(1/5)\times x^3e^{-x} - (3/5)\times x^2e^{-x} + (6/5)\times\int xe^{-x}\,dx + c_1\int x^{-2}e^{-x}dx + c_2$$

$$= -(1/5)\times x^3e^{-x} - (3/5)\times x^2e^{-x} + (6/5)\times\left[x(-e^{-x}) - \int\{1\cdot(-e^{-x})\}dx\right] + c_1\int x^{-2}e^{-x}dx + c_2$$

$$= -(1/5)\times x^3e^{-x} - (3/5)\times x^2e^{-x} - (6/5)\times xe^{-x} - (6/5)\times e^{-x} + c_1\int x^{-2}e^{-x}dx + c_2$$

or $$v = -(1/5)\times e^{-x}(x^3 + 3x^2 + 6x + 6) + c_1\int x^{-2}e^{-x}dx + c_2. \qquad \text{... (7)}$$

Hence from (3), (4) and (7), the required solution is

$$y = uv = e^x\left[-(1/5)\times e^{-x}(x^3+3x^2+6x+6) + c_1\int x^{-2}e^{-x}dx + c_2\right]$$

or $$y = c_1e^x\int x^{-2}e^{-x}dx + c_2e^x - (1/5)\times(x^3+3x^2+6x+6).$$

(*f*) Try yourself. **Ans.** $y = -c_1\,x\,e^{-x} + c_2\,e^x - x\,e^x$

$$+ e^x\int(2x-1)\log(2x-1)dx - 2e^x\int\left[e^{-2x}(2x-1)\int e^{2x}\log(2x-1)\,dx\right]dx.$$

(*g*) Comparing the given equation with $y'' + Py' + Q\,y = R$, we have

$P = 0$, $Q = 1 + (2/x)\cot x - (2/x^2)$, $R = x\cos x$.... (1)

Here a part of C.F. of given equation is given by $u = (\sin x)/x$. ... (2)

Let the required general solution be $y = uv$. ... (3)

Then v is given by $$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$$

or $$\frac{d^2v}{dx^2} + \left[0 + \frac{2x}{\sin x}\frac{d}{dx}\left(\frac{\sin x}{x}\right)\right]\frac{dv}{dx} = \frac{x\cos x}{(\sin x)/x}, \text{ using (2)}$$

or $$\frac{d^2v}{dx^2} + \left(\frac{2x}{\sin x}\right)\left(\frac{x\cos x - \sin x}{x^2}\right)\frac{dv}{dx} = x^2\cot x. \qquad \text{... (4)}$$

Let $dv/dx = q$ so that $d^2v/dx^2 = dq/dx$.

Then (4) becomes $\dfrac{dq}{dx} + 2\left(\cot x - \dfrac{1}{x}\right)q = x^2\cot x$, which is a linear equation

Its I.F. $= e^{2\int[\cot x - (1/x)]dx} = e^{2(\log\sin x - \log x)} = (\sin x/x)^2$ and its solution is

$$\therefore \quad q\left(\frac{\sin x}{x}\right)^2 = \int\left[x^2\cot x\cdot\left(\frac{\sin x}{x}\right)^2\right]dx + c_1 = \int\sin x\cos x\,dx + c_1 = \frac{1}{2}\sin^2 x + c_1$$

or $$q = \frac{dv}{dx} = \left(\frac{x}{\sin x}\right)^2\left[\frac{1}{2}\sin^2 x + c_1\right] = \frac{1}{2}x^2 + c_1x^2\operatorname{cosec}^2 x$$

or $$\int dv = \int[x^2/2 + c_1x^2\operatorname{cosec}^2 x]\,dx + c_2, \; c_1, c_2 \text{ being arbitrary constants}$$

or $$v = x^3/6 + c_1\left[x^2(-\cot x) - \int 2x(-\cot x)dx\right] + c_2, \text{ integrating by parts}$$

or $$v = x^3/6 - c_1 x^2 \cot x + 2c_1 \left[x \log \sin x - \int \log \sin x \, dx \right] + c_2 \qquad \text{... (5)}$$

[Integrating by parts again]

From (2), (3) and (5), the required general solution is

$$y = uv = \frac{\sin x}{x}\left[\frac{x^3}{6} - c_1 x^2 \cot x + 2c_1 x \log \sin x - 2c_1 \int \log \sin x \, dx + c_2\right]$$

or $$y = c_1\left[-x\cos x + 2\sin x \log \sin x - 2\frac{\sin x}{x}\int \log \sin x \, dx\right] + c_2\frac{\sin x}{x} + \frac{1}{6}x^2 \sin x.$$

(*h*) Try yourself. **Ans.** $y = c_1 x + c_2/x + x\int e^x x^{-3}(x^2 - 2x + 2)dx$

Ex. 14(*a*). *Solve* $xy_1 - y = (x-1)(y_2 - x + 1)$. **[Agra 2002; Delhi Maths (G) 2004]**

Sol. Dividing by $(x-1)$, the given equation in standard form is

$$\left(\frac{x}{x-1}\right)y_1 - \frac{1}{x-1}y = y_2 - (x-1) \qquad \text{or} \qquad \frac{d^2y}{dx^2} - \frac{x}{x-1}\frac{dy}{dx} + \frac{1}{x-1}y = x - 1 \quad \text{... (1)}$$

Comparing (1) with $y'' + Py' + Qy = R$, we have

$P = -x/(x-1)$, $Q = 1/(x-1)$, and $R = x - 1$. ... (2)

Here $P + Qx = 0$, showing that a part of C.F. of (1) is $u = x$. ... (3)

Let the required general solution be $y = uv$. ... (4)

Then v is given by $$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$$

or $$\frac{d^2v}{dx^2}\left[-\frac{x}{x-1} + \frac{2}{x}\frac{dx}{dx}\right]\frac{dv}{dx} = \frac{x-1}{x}. \qquad \text{... (5)}$$

Let $dv/dx = q$ so that $d^2v/dx^2 = dq/dx$. ... (6)

Then (5) becomes $$\frac{dq}{dx} + \left(\frac{2}{x} - \frac{x}{x-1}\right)q = \frac{x-1}{x}. \qquad \text{... (7)}$$

Now, $$E = \int\left(\frac{2}{x} - \frac{x}{x-1}\right)dx = \int\left(\frac{2}{x} - \frac{x-1+1}{x-1}\right)dx = \int\left(\frac{2}{x} - 1 - \frac{1}{x-1}\right)dx$$

$$= 2\log x - x - \log(x-1) = \log x^2 - \log(x-1) - x$$

$\therefore$ I.F. of (7) $= e^E = e^{\log x^2 - \log(x-1) - x} = e^{\log[x^2/(x-1)]} \cdot e^{-x} = [x^2/(x-1)]\, e^{-x}$

and its solution is $$q \cdot \frac{x^2}{x-1}e^{-x} = \int\left(\frac{x-1}{x} \times \frac{x^2}{x-1}\right)e^{-x}dx + c_1$$

or $$\frac{qx^2 e^{-x}}{x-1} = \int xe^{-x}dx + c_1 = x(-e^{-x}) - \int\{1\cdot(-e^{-x})\}dx + c_1 = -xe^{-x} - e^{-x} + c_1 = c_1 - e^{-x}(x+1)$$

$\therefore$ $$q = \frac{dv}{dx} = \frac{x-1}{x^2}e^x[c_1 - e^{-x}(x+1)] = c_1\frac{x-1}{x^2}e^x - \frac{x^2-1}{x^2}$$

or $$\int dv = c_1\int\frac{1}{x}e^x dx - c_1\int\frac{1}{x^2}e^x dx - \int(1 - x^{-2})dx + c_2$$

or $$v = c_1\left[\frac{1}{x}e^x - \int\left(-\frac{1}{x^2}\right)e^x dx\right] - c_1\int\frac{e^x}{x^2}dx - (x + x^{-1}) + c_2$$

or $\qquad v = (c_1/x)e^x - x - (1/x) + c_2$, c_1 and c_2 being arbitrary constants ... (8)

From (3), (4) and (8), the required general solution is

$$y = uv = x[(c_1/x)e^x - x - (1/x) + c_2] = c_1 e^x + c_2 x - (x^2 + 1).$$

Ex. 14(*b*). *Solve* $xy_2 - (2x+1)y_1 + (x+1)y = (x^2 + x - 1)e^{2x}$. **[Meerut 1994,95]**

Hint: Try yourself as in Ex.14(*a*). **Ans.** $y = c_1 x^2 e^x + c_2 e^x + xe^{2x}$.

Ex. 14(*c*). *Solve* $xy'' + 2(x+1)y' + (x+2)y = (x-2)e^{2x}$.

Hint: Do as in Ex.14(*a*). **Ans.** $y = (1/3) \times x^3 e^x + c_2 e^x + e^{2x}$

Ex. 15. *Solve* $(x+2)y'' - (2x+5)y' + 2y = (x+1)e^x$. **[Rajasthan 2004, 06, I.A.S. 2004]**
[Kanpur 2000, 07; Meerut 1994; Rohilkhand 1998; Vikram 2000]

Sol. Dividing by $(x+2)$, the given equation in standard form is

$$\frac{d^2y}{dx^2} - \frac{2x+5}{x+2}\frac{dy}{dx} + \frac{2}{x+2}y = \frac{x+1}{x+2}e^x. \quad \text{... (1)}$$

Comparing (1) with $\quad y'' + Py' + Qy = R$, $\quad$ we have

$P = -(2x+5)/(x+2)$, $\qquad Q = 2/(x+2)$, $\qquad R = [(x+1)/(x+2)]e^x$.... (2)

Here $\quad 2^2 + 2P + Q = 4 + 2\left[-\dfrac{2x+5}{x+2}\right] + \dfrac{2}{x+2} = \dfrac{4(x+2) - 2(2x+5) + 2}{x+2} = 0,$

showing that a part of C.F. of (1) is $\qquad u = e^{2x}$ ... (3)

Let the general solution be $\qquad y = uv.$ ... (4)

Then v is given by $\qquad \dfrac{d^2v}{dx^2} + \left(P + \dfrac{2}{u}\dfrac{du}{dx}\right)\dfrac{dv}{dx} = \dfrac{R}{u}$

or $$\frac{d^2v}{dx^2} + \left(-\frac{2x+5}{x+2} + \frac{2}{e^{2x}}\frac{de^{2x}}{dx}\right)\frac{dv}{dx} = \frac{x+1}{x+2}\frac{e^x}{e^{2x}}$$

or $\quad \dfrac{d^2v}{dx^2} + \left(4 - \dfrac{2x+5}{x+2}\right)\dfrac{dv}{dx} = \dfrac{x+1}{x+2}e^{-x} \qquad$ or $\qquad \dfrac{d^2v}{dx^2} + \dfrac{2x+3}{x+2}\dfrac{dv}{dx} = \dfrac{x+1}{x+2}e^{-x}$. ... (5)

Let $\qquad dv/dx = q \qquad$ so that $\qquad d^2v/dx^2 = dq/dx$. ... (6)

Then (5) becomes $\qquad \dfrac{dq}{dx} + \dfrac{2x+3}{x+2}q = \dfrac{x+1}{x+2}e^{-x}$. ... (7)

Now, $\qquad E = \displaystyle\int \frac{2x+3}{x+2}dx = \int\left(2 - \frac{1}{x+2}\right)dx = 2x - \log(x+2).$

$\therefore \quad$ I.F. of (7) $= e^E = e^{2x - \log(x+2)} = e^{2x}e^{-\log(x+2)} = e^{2x}e^{\log(x+2)^{-1}} = e^{2x}(x+2)^{-1}$ and solution is

$$\therefore \quad q \cdot e^{2x}(x+2)^{-1} = c_1 + \int \frac{x+1}{x+2}e^{-x} \cdot e^{2x}(x+2)^{-1}dx = \int \frac{x+1}{(x+2)^2}e^x dx + c_1$$

$$= \int \frac{(x+2)-1}{(x+2)^2}e^x dx + c_1 = \int \frac{1}{x+2}e^x dx - \int \frac{1}{(x+2)^2}e^x dx + c_1$$

$$= \frac{1}{x+2}e^x - \int\left\{-\frac{1}{(x+2)^2}\right\}e^x dx - \int \frac{1}{(x+2)^2}e^x dx + c_1$$

[Integrating by parts only the first integral] or $\quad (qe^{2x})/(x+2) = (x+2)^{-1}e^x + c_1 \quad$ or $\quad q = dv/dx = e^{-x} + c_1e^{-2x}(x+2)$, by (6)

$\therefore \quad dv = [e^{-x} + c_1e^{-2x}(x+2)]dx.$

Integrating, $\quad v = -e^{-x} + c_1\int e^{-2x}(x+2)\,dx + c_2$, c_1, c_2 being arbitrary constants.

or
$$v = -e^{-x} + c_1\left[(x+2)\left(-\frac{1}{2}\right)e^{-2x} - \int 1\cdot\left(-\frac{1}{2}\right)e^{-2x}dx\right] + c_2$$

or $\quad v = -e^{-x} + c_1[(x+2)\times(-1/2)e^{-2x} - (1/4)\times e^{-2x}] + c_2 = -e^{-x} + (c_1/4)\times(2x+5) + c_2. \quad \text{... (8)}$

From (3), (4) and (8), the required general solution is

$y = uv = e^{2x}[-e^{-x} + (c_1/4)\times(2x+5) + c_2] = c'_1 e^{2x}(2x+5) + c_2e^{2x} - e^x$, where $c'_1 = c_1/4$

Ex. 16. *Solve* $(1-x^2)y_2 + xy_1 - y = x(1-x^2)^{3/2}$. **[Allahabad 2001, Kurukshetra 2002]**

Sol. Dividing by $(1-x^2)$, the given equation is standard form is

$$\frac{d^2y}{dx^2} + \frac{x}{1-x^2}\frac{dy}{dx} - \frac{1}{1-x^2}y = x(1-x^2)^{1/2}. \quad \text{... (1)}$$

Comparing (1) with $\quad y'' + Py' + Qy = R, \quad$ we have

$P = x/(1-x^2), \qquad Q = -1/(1-x^2), \qquad R = x(1-x^2)^{1/2} \quad$ ... (2)

Here $\quad P + Qx = 0$, showing that a part of C.F. of (1) is $\qquad u = x. \quad$ ... (3)

Let the required general solution be $\qquad y = uv. \quad$ ... (4)

Then v is given by
$$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$$

or
$$\frac{d^2v}{dx^2} + \left(\frac{x}{1-x^2} + \frac{2}{x}\frac{dx}{dx}\right)\frac{dv}{dx} = \frac{x(1-x^2)^{1/2}}{x}. \quad \text{... (5)}$$

Let $\qquad dv/dx = q \qquad$ so that $\qquad d^2v/dx^2 = dq/dx. \quad$ ... (6)

Then (5) reduces to
$$\frac{dq}{dx} + \left(\frac{2}{x} + \frac{x}{1-x^2}\right)q = (1-x^2)^{1/2}. \quad \text{... (7)}$$

Here
$$E = \int\left(\frac{2}{x} + \frac{x}{1-x^2}\right)dx = \int\frac{2}{x}dx - \frac{1}{2}\int\frac{(-2x)dx}{1-x^2} = 2\log x - \frac{1}{2}\log(1-x^2)$$
$$= \log x^2 - \log(1-x^2)^{1/2} = \log\{x^2/(1-x^2)^{1/2}\}$$

$\therefore\quad$ I.F. of (7) $= e^E = e^{\log\left[x^2/(1-x^2)^{1/2}\right]} = x^2/(1-x^2)^{1/2}$ and solution is

$$q\cdot\frac{x^2}{(1-x^2)^{1/2}} = \int(1-x^2)^{1/2}\times\frac{x^2}{(1-x^2)^{1/2}}dx + c_1 = \frac{1}{3}x^3 + c_1$$

or
$$q = dv/dx = (x/3)(1-x^2)^{1/2} + (c_1/x^2)(1-x^2)^{1/2}$$

Integrating, $\quad \int dv = \int(1/3)(-1/2)(1-x^2)^{1/2}(-2x)dx + c_1\int x^{-2}(1-x^2)^{1/2}dx + c_2$

or
$$v = -\left(\frac{1}{6}\right)\frac{(1-x^2)^{3/2}}{(3/2)} + c_1\left[(1-x^2)^{1/2}(-x^{-1}) - \int\frac{1}{2}(1-x^2)^{-1/2}(-2x)(-x^{-1})dx\right] + c_2$$

[Integrating by parts second integral]

or $$v = -\frac{1}{9}(1-x^2)^{3/2} - \frac{c_1}{x}(1-x^2)^{1/2} - c_1\int \frac{dx}{(1-x^2)^{1/2}} + c_2$$

or $$v = -(1/9)(1-x^2)^{3/2} - (c_1/x)(1-x^2)^{1/2} - c_1 \sin^{-1} x + c_2. \qquad \text{... (8)}$$

From (3), (4) and (8), the required general solution is

$$y = uv = x[-(1/9)(1-x^2)^{3/2} - (c_1/x)(1-x^2)^{1/2} - c_1 \sin^{-1} x + c_2]$$

or $$y = -c_1[(1-x^2)^{1/2} + x\sin^{-1} x] + c_2 x - (x/9)(1-x^2)^{3/2}.$$

Ex. 17 *Find the complementary function of the equation* $xy'' - 2(x+1)y' + (x+2)y = (x-2)e^{2x}$, $x > 0$. **[Gulbarga 2005]**

Sol. Rewriting given equation, $$\frac{d^2y}{dx} - \frac{2(x+1)}{x}\frac{dy}{dx} + \frac{x+2}{x}y = \frac{x-2}{x}e^{2x}$$

Comparing it with $y'' + Py' + Qy = R$, here

$P = -\dfrac{2x+2}{x}$ and $Q = \dfrac{x+2}{x}$ so that $1 + P + Q = 1 - \dfrac{2x+2}{x} + \dfrac{x+2}{x} = 0$,

showing that e^x is a part of C.F. (see step 2 of Art 10.4)

Ex. 18.(*a*) *Solve* $(D^2+1)y = \text{cosec}^3 x$ *by reduction of order.* **[Kuvempa 2005]**

(*b*) *Use the method of reduction of order to solve* $(D^2+1)y = \sec^3 x$.

[G.N.D.U. Amritsar 2010]

Sol. (a.) Given $$y'' + y = \text{cosec}^3 x \qquad \text{... (1)}$$

Comparing (1) with $y'' + Py' + Qy = R$, we have $P = 0$, $Q = 1$, $R = \text{cosec}^3 x$ ... (2)

By inspection $y = \sin x$ is a part of C.F., *i.e.*, $y = \sin x$ is a solution of $y'' + y = 0$.

Thus, we take $$u = \sin x \qquad \text{... (3)}$$

Let the complete solution of (1) be $$y = uv \qquad \text{... (4)}$$

Then v is given by $$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$$

or $$\frac{d^2v}{dx^2} + \left(0 + \frac{2}{\sin x} \times \cos x\right)\frac{dv}{dx} = \frac{\text{cosec}^3 x}{\sin x}, \text{ by (2) and (3)}$$

or $$d^2v/dx^2 + (2\cot x) \times (dv/dx) = \text{cosec}^4 x \qquad \text{... (5)}$$

Let $dv/dx = q$ so that $d^2v/dx^2 = dq/dx$

Then (5) yields $$(dq/dx) + (2\cot x)q = \text{cosec}^4 x \qquad \text{... (6)}$$

which is linear equation in variables q and x.

Its integrating factor $= e^{\int (2\cot x)dx} = e^{2\log \sin x} = \sin^2 x$ and solution is

$$q\sin^2 x = C_1 + \int \{(\text{cosec}^4 x) \times \sin^2 x\}\, dx = C_1 - \cot x$$

or $(dv/dx)\sin^2 x = C_1 - \cot x$ or $dv = (C_1 \text{cosec}^2 x - \cot x \text{ cosec}^2 x)\, dx$

Integrating, $$v = -C_1 \cot x + (\cot^2 x)/2 + C_2 \qquad \text{... (7)}$$

From (3), (4) and (7), the required general solution is $y = uv$

or $$y = (\sin x)\,\{C_2 - C_1 \cot x - (\cot^2 x)/2\}$$

or $y = C_2 \sin x - C_1 \cos x - (1/2) \times \cos^2 x \text{ cosec } x$, C_1, C_2 being arbitrary constants.

Part (*b*). *Do* as in Ex. 12(a). **Ans.** $y = c_1 \sin x + c_2 \cos x + (1/2) \times \tan^2 x \sec x$

Ex. 21. *Solve* $y_2 + xy_1 - y = f(x)$.

Sol. Given $$y_2 + xy_1 - y = f(x) \qquad \text{... (1)}$$

Comparing (1) with $y_2 + Py_1 + Qy = R,$ here $P = x,$ $Q = -1,$ $R = f(x)$ ...(2)

Here $P + Qx = x - x = 0$, showing that $u = x$ ...(3)

is a par tof C.F. Let the complete solution of (1) be $y = uv$ ...(4)

Then ν is given by $$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$$

or $$\frac{d^2v}{dx^2} + \left(x + \frac{2}{x}\frac{dx}{dx}\right)\frac{dv}{dx} = \frac{f(x)}{x} \quad \text{or} \quad \frac{d^2v}{dx^2} + \left(x + \frac{2}{x}\right)\frac{dv}{dx} = \frac{f(x)}{x} \qquad ...(5)$$

Let $dv/dx = q$ so that $d^2v/dx^2 = dq/dx$

Hence (5) reduces to $\dfrac{dq}{dx} + \left(x + \dfrac{2}{x}\right)q = \dfrac{f(x)}{x}$, which is a linear equation.

Its I.F. = $e^{\int (x+2/x)dx} = e^{x^2/2 + 2\log x} = e^{x^2/2}\ e^{\log x^2} = x^2\ e^{x^2/2}$ and solution is

$$qx^2e^{x^2/2} = \int \frac{f(x)}{x} x^2 e^{x^2/2}\, dx + c_1 \quad \text{or} \quad \frac{dv}{dx} x^2\ e^{x^2/2} = \int x\, f(x) e^{x^2/2}\, dx + c_1$$

or $$\frac{dv}{dx} = x^{-2}e^{-x^2/2} \int x\, f(x) e^{x^2/2} dx + c_1\ x^{-2}e^{-x^2/2}$$

Integrating, $$\text{ν} = \int x^{-2}\ e^{-x^2/2} \left\{x\, f(x) e^{x^2/2}\ dx\right\} dx + c_1 \int x^{-2}e^{-x^2/2}\ dx + c_2 \qquad ...(6)$$

where c_1 and c_2 are arbitrary constants.

From (3) (4) and (6), the required general solution is

$$y = u\text{ν} \quad \text{or} \quad y = x\int x^{-2}\ e^{-x^2/2} \int x\, f(x)\, e^{x^2/2}(dx)^2 + c_1\, x\int x^{-2}e^{-x^2/2} dx + c_2 x$$

10.5 A. Some typical solved examples. Important note.

Sometimes the method discussed in articles 10.2, 10.3 and 10.4 can be used to solve a third order differential equation $y_3 + p(x)y_2 + q(x)y_1 + r(x)y = s(x)$, *provided a part of C.F. is either given or can be obtained by inspection (similar to rules discussed in Art. 10.3) We now explain the whole procedure with help of the following two examples Ex.1 and Ex. 2.*

Ex. 1. *Solve* $(x^2 + x)y_3 - (x^2 + 3x + 1)y_2 + (x + 4 + 2/x)y_1 - (1 + 4/x + 2/x^2)y = 3x^2(x+1)^2$ *of which* $y = x$ *is a particular interal.*

[Agra 1997; Delhi Maths (H) 1994; Gwaliar 2005; Gujrat 2007; Nagpur 1994]

Sol. $(x^2 + x)y_3 - (x^2 + 3x + 1)y_2 + (x + 4 + 2/x)y_1 - (1 + 4/x + 2/x^2)y = 3x^2(x+1)^2$ ... (1)

Since x is a particular integral of (1), let complete solution of (1) be $y = xv$ ...(2)

$(2) \Rightarrow$ $y_1 = v + xv_1$, $y_2 = 2v_1 + xv_2$, $y_3 = 3v_2 + xv_3$ ...(3)

Using (2) and (3), (1) reduces to

$$(x^2+x)(3v_2+xv_3)-(x^2+3x+1)(2v_1+xv_2)+(x+4+2/x))(v+xv_1)$$
$$-(1+4/x+2/x^2)xv=3x^2(x+1)^2$$

or $$x(x^2+x)v_3-\left\{3(x^2+x)-x(x^2+3x+1)\right\}v_2-\{2(x^2+3x+1)-x\left(x+4+2/x\right)\}v_1$$
$$+\left\{x+4+2/x-x(1+4/x+2/x^2)\right\}v=3x^2\left(x+1\right)^2$$

or $$x(x^2+x)v_3-x(x^2-2)v_2-x(x+2)v_1=3x^2(x+1)^2$$

or $$(x^2+x)v_3-(x^2-2)v_2-(x+2)v_1=3x(x+1)^2 \qquad ...(4)$$

Let $v_1 = q$ so that $v_2 = q_1$ and $v_3 = q_2$. Then, (4) reduces to

$$(x^2+x)q_2-(x^2-2)q_1-(x+2)q=3x(x+1)^2 \qquad ...(5)$$

or $$q_2-\frac{x^2-2}{x^2+x}q_1-\frac{x+2}{x^2+x}q=3\left(x+1\right) \qquad ...(5)$$

which is standard equation of linear equation of second order $q_2 + Pq_1 + Qq = R$. Here we have q in place of y of Art. 10.4 On comparing (5) with $q_2 + Pq_1 + Qq = R$, we have

$$P=-\{(x^2-2)/(x^2+x)\}, \qquad Q=-\{(x+2)/(x^2+x)\} \qquad \text{and} \qquad R=3(x+1) \qquad ...(6)$$

Here $$1+P+Q=1-\frac{x^2-2}{x^2+x}-\frac{x+2}{x^2+x}=0,$$

showing that a part of C.F. of the solution (5) is $u=e^x$ ...(7)

Let the complete solution of (5) be $q = u\,V = e^xV$ ...(8)

Then V is given by (See Art 10.4) $$\frac{d^2V}{dx^2}+\left(P+\frac{2}{u}\frac{du}{dx}\right)\frac{dV}{dx}=\frac{R}{u}$$

or $$\frac{d^2V}{dx^2}+\left(-\frac{x^2-2}{x^2-x}+\frac{2}{e^x}\frac{de^x}{dx}\right)\frac{dV}{dx}=3e^{-x}(x+1) \quad \text{or} \quad \frac{d^2V}{dx^2}+\left(2-\frac{x^2-2}{x^2+x}\right)\frac{dV}{dx}=3e^{-x}\left(x+1\right)$$

or $$\frac{d^2V}{dx^2}+\frac{x^2+2x+2}{x^2+x}\frac{dV}{dx}=3e^{-x}(x+1) \qquad ...(9)$$

Let $dV/dx = P$ so that $d^2V/dx^2=dp/dx$ ...(10)

Then (9) becomes $$\frac{dp}{dx}+\frac{x^2+2x+2}{x^2+x}p=3e^{-x}(x+1) \qquad ...(11)$$

I.F. of (11) = e^E, say, where

$$E=\int\frac{x^2+2x+2}{x(x+1)}dx=\int\left(1+\frac{2}{x}-\frac{1}{1+x}\right)dx \text{, on resolving into partial fractions.}$$

$$= x+2\ \log x-\log(1+x)=x+\log\{x^2/(1+x)\}$$

I.F. of (11) = $e^{x+\log\{x^2/(1+x)\}} = e^x\, e^{\log\{x^2/(1+x)\}} = e^x \times \{x^2/(1+x)\}$ and solution is

$$p\times\frac{e^x\, x^2}{x+1} = \int \frac{e^x\, x^2}{x+1}\,.\,3e^{-x}(x+1)dx + c_1 = x^3 + c_1 \quad \text{or} \quad p = \frac{dV}{dx} = x(x+1)e^{-x} + c_1\,\frac{x+1}{x^2}e^{-x}$$

Integrating, $$V = \int (x^2+x)e^{-x}dx + c_1\int (1/x + 1/x^2)e^{-x}dx + c_2$$

or $$V = (x^2+x)(-e^{-x}) - \int (2x+1)(-e^{-x})dx + c_1\int \frac{1}{x}e^{-x}dx + c_1\int \frac{1}{x^2}e^{-x}dx + c_2$$

or $$V = -e^x(x^2+x) + \int (2x+1)e^{-x}\,dx + c_1\{(1/x)(-e^{-x})\}$$

$$-\int\left(-\frac{1}{x^2}\right)(-e^{-x})dx\} + c_1\int \frac{1}{x^2}e^{-x}dx + c_2,\ c_1 \text{ and } c_2 \text{ being arbitrary constants.}$$

or $$V = -e^{-x}(x^2+x) + \left\{(2x+1)(-e^{-x}) - \int 2(-e^{-x})dx\right\} + (c_1/x)\times e^{-x} + c_2$$

or $$V = -e^{-x}(x^2+x) - e^{-x}(2x+1) - 2e^{-x} + (c_1/x)\times e^{-x} + c_2$$

or $$V = -e^{-x}(x^2+3x+3) + (c_1/x)\times e^{-x} + c_2, c_1, c_2 \text{ being arbitrary constants} \qquad ...(12)$$

From (7), (8) and (12) the solution of (5) in given by

$$q = uV = e^x\left\{-e^{-x}(x^2+3x+3) + (c_1/x)e^{-x} + c_2\right\}$$

or $$dv/dx = e^x\left\{-e^{-x}(x^2+3x+3) + (c_1/x)e^{-x} + c_2\right\}, \qquad \text{as} \qquad q = v_1 = dv/dx$$

Integrating $$v = \int\left\{-(x^2+3x+3) + c_1/x + c_2e^x\right\}dx + c_3$$

or $$v = -(x^3/3) - (3x^2/2) - 3x + c_1\log x + c_2e^x + c_3$$

Using (2), the required general solution of (1) given by

$$y = xv = -(x^4/3) - (3x^3/2) - 3x^2 + c_1 x\log x + c_2xe^x + c_3x$$

Ex. 2. *Solve $y_3 - xy_2 - y_1 + xy = 0$. Though it is a third order linear differential equation show that even then it can be solved by the usual method of of this chapter.*

Sol. Given $$y_3 - xy_2 - y_1 + xy = 0 \qquad ...(1)$$

Since the sum of the coeffcients of (1) is zero, by inspection e^x is a part of C.F of (1).

Let the complete solution of (1) be $$y = e^x\, v. \qquad ...(2)$$

$$(2) \Rightarrow y_1 = e^x v + e^x v_1, \quad y_2 = e^x v + 2e^x v_1 + e^x v_2, \quad y_3 = e^x v + 3e^x v_1 + 3e^x v_2 + e^x v_3$$

Substituting the above values in (1), we have

$$e^x v + 3e^x v_1 + 3e^x v_2 + e^x v_3 - x(e^x v + 2e^x v_1 + e^x v_2) - (e^x v + e^x v_1) + xe^x v = 0$$

$$v_3 + (3-x)v_2 + (2-2x)v_1 = 0 \qquad ...(3)$$

Let $v_1 = q$ so that $v_2 = q_1$ and $v_3 = q_2$. Then (3) becomes

$$q_2 + (3 - x)\, q_1 + (2 - 2x)\, q = 0 \qquad ...(4)$$

Comparing (4) with $q_2 + Pq_1 + Qq = R$, here $P = 3 - x$, $Q = 2 - 2x$ and $R = 0$ Also, we have

$$(-2)^2 + (-2) \times P + Q = 4 - 2\,(3 - x) + 2 - 2x = 0, \qquad \text{[Refer rule 1 of Art. 10.3]}$$

showing that e^{-2x} is a part of C.F. of (1)

Thus, we take $$u = e^{-2x} \qquad ...(5)$$

Let the complete solution of (4) be $$q = uV = e^{-2x}\, V \qquad ...(6)$$

Then V is given by (refer Art 10.4) $$\frac{d^2V}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dV}{dx} = \frac{R}{u}$$

or $$\frac{d^2V}{dx^2} + \left(3 - x + \frac{2}{e^{-2x}}\,\frac{de^{-2x}}{dx}\right)\frac{dV}{dx} - 0 \qquad \text{or} \qquad \frac{d^2V}{dx^2} - (1 + x)\,V = 0 \qquad ...(7)$$

Let $dV/dx = p$ so that $d^2V/dx^2 = dp/dx$ ...(8)

Then (7) becomes $$\frac{dp}{dx} - (1 + x)\,p - 0 \qquad \text{or} \qquad \frac{dp}{p} = (1 + x)\,dx$$

Intergrating, $$\log p - \log c_1 = x + x^2/2 \quad \Rightarrow \quad p = c_1\, e^{x + x^2/2}$$

or $$dV/dx = c_1\, e^{x + e2/2} \qquad \text{or} \qquad dV = c_1 e^{x + x^2/2}\, dx$$

Integrating, $$V = c_1 \int e^{x + x^2/2}\, dx + c_2,\; c_1,\; c_2 \text{ being arbitrary constants.} \qquad ....(9)$$

From (6) and (9), $$q = e^{-2x}\left\{c_2 + c_1 \int e^{x + x^2 2} dx\right\} \qquad \text{or} \qquad q = c_2 e^{-2x} + c_1\, e^{-2x} \int e^{x + x^2/2} dx$$

or $$dv/dx = c_2 e^{-2x} + c_1\, e^{-2x} \int e^{x + x^2/2} dx, \qquad \text{as} \qquad q = v_1 = dv/dx$$

Integrating, $$v = (-1/2) \times c_2\, e^{-2x} + c_1 \int e^{-2x} \left\{\int e^{x + x^2/2} dx\right\} dx + c_3$$

Using (2), the required general solution is given by

$$y = v e^x = e^x [(-1/2) \times c_2\, e^{-2x} + c_1 \int e^{-2x} \int e^{x + x^2/2} (dx)^2 + c_3]$$

or $$y = c_2'\; e^{-x} + c_1 e^x \int e^{-2x} \int e^{x + x^2/2} (dx)^2 + c_3\, e^x,$$

where c_1, $c_2'\ (= -c_2/_2)$ and c_3 arbitrary constants.

EXERCISE 10 (A)

1. Solve $xy'' - (2x + 1)y' + (x + 1)y = x^3 e^x$. **Ans.** $y = (c_1/2)x^2 e^x + c_2 e^x + (1/3)x^3 e^x$.

2. Solve $xy'' + (x - 1)y' - y = x^2$. **Ans.** $y = c_1(x - 1) + c_2 e^{-x} + x^2 - 2x + 2$.

3. Solve $xy'' - (x + 2)y' + 2y = x^3$. **Ans.** $y = c_1(x^2 + 2x + 2) + c_2 e^x - x^3$.

4. Solve $xy'' + (x - 2)y' - 2y = x^3$. **Ans.** $y = c_1(x^2 - 2x + 2) + c_2 e^{-x} + x^3$.

5. Solve $(x - x^2)y_2 - (1 - 2x)y_1 + (1 - 3x + x^2)y = (1 - x)^3$. **Ans.** $y = (1/2)c_1 x^2 e^{-x} + c_2 e^x - x$

6. Solve $(x \sin x + \cos x)y_2 - x \cos x\, y_1 + y \cos x = \sin x\,(x \sin x + \cos x)^2$. **[Mumbai 2001,05]**

Ans. $y = -c_1 \cos x + c_2 x + (1/4)x \cos 2x - (1/2) \sin 2x$.

7. (*a*) Solve $xy_2 - y_1 - 4x^3 y = -4x^5$, given that $y = e^{x^2}$ is a solution if the left hand side is equated to zero.

Ans. $y = c_1 e^{-x^2} + c_2 e^{x^2} + x^2$

(*b*) Solve $y'' + y = \sec x$ given that $\cos x$ is a part of C.F.

Ans. $y = c_1 \sin x + c_2 \cos x + x \sin x - \cos x \log \sec x$

8. Solve $(\sin x - x \cos x)y_2 - (x \sin x)y_1 + (\sin x)y = 0$, given that $y = \sin x$ is a solution.

Ans. $y = c_1 x + c_2 \sin x$

9. Verify that the left hand side of the equation $(\sin x - x \cos x)y'' - x \sin x\, y' + y \sin x = x$ vanishes when $y = \sin x$ and hence obtain the general solution of the whole equation. **Ans.** $y = c_1 x + c_2 \sin x + \cos x$

10. Solve $(x + 1)y'' + (x - 1)y' - 2y = 0$, given that $y = e^{-x}$ is a solution. **Ans.** $y = c_1(1 + x^2) + c_2 e^{-x}$.

11. Find a particular solution of $(x^2 + x)y'' + (2 - x^2)y' - (2 + x)y = x(x + 1)^2$ if $y = e^x$ is a solution of the corresponding hemogeneous equation. **[Lucknow 1995, Nagpur 1997]**

12. If $y_1(x)$ is a known (non-zero)solution of $y'' + P(x)y' + Q(x) = 0$ then determine the other solution of the differential equation. **[Allahabad 2003, 07, Lucknow 2004]**

Hint: Proceed as in Art.10.2 with $R = 0$ and $u = y_1(x)$. Then the complete solution will be

$$y = c_2 y_1 + c_1 y_1 \int \frac{1}{y_1^2} e^{-\int P dx} \quad \text{and hence the other solution} = y_1 \int \frac{1}{y_1^2} e^{-\int P dx} dx$$

13. $(x + 1)\, y_2 - (2x + 3)\, y_1 + (x + 2)\, y = x^2 + 2x - 1$ **[Pune 2003, 05]**

Ans. $y = -(1/9) \times (3x^2 + x + 3) + (1/16) \times c_1 e^{3x}(4x + 3) + c_2\, e^{-x}$

14. $(x - x^2)y_2 - (1 - 2x)y_1 + (1 - 3x + x^2)y = (1 - x)^2$ **Ans.** $y = -x + (1/2) \times c_1 x^2\, e^{-x} + c_2 e^x$

15. $(1 - x^2)y'' - xy' = 2$ **Ans.** $y = c_1 \sin^{-1} x + c_2 + (\sin^{-1} x)^2$

16. $y_2 - (3/x)y_1 + (3/x^2)y = 2x - 1$ **Ans.** $y = x(c_1 x^2 + c_2 + x^2 \log x + x)$

10.6. Removal of the first derivative. Reduction to Normal Form.

Transformation of the equation $y'' + Py' + Q\,y = R$ by changing the dependent variable

Statement. *Obtain a suitable substitution for the dependent variable which transforms the equation, $y'' + Py' + Q\,y = R$ into normal form i.e. form where the first derivative is absent.*

or

Reduce the differential equation $y'' + Py' + Q\,y = R$, where P, Q and R are functions of x, to the form $d^2v / dx^2 + Iv = S$ which is known as the normal form of the given equation.

[I.A.S. 2000; Guwahati 1996]

Sol. The given equation is $$d^2y / dx^2 + P(dy/dx) + Qy = R \qquad \text{... (1)}$$

Let the complete solution of (1) be $y = uv$, where u and v are functions of x.

Differentiating twice, $y = uv$ gives

$$\frac{dy}{dx} = \frac{du}{dx}v + u\frac{dv}{dx}, \qquad \text{and} \qquad \frac{d^2y}{dx^2} = \frac{d^2u}{dx^2}v + 2\frac{du}{dx}\frac{dv}{dx} + u\frac{d^2v}{dx^2}.$$

$\therefore$ by (1), $$\frac{d^2u}{dx^2}v + 2\frac{du}{dx}\frac{dv}{dx} + u\frac{d^2v}{dx^2} + P\left(\frac{du}{dx}v + u\frac{dv}{dx}\right) + Quv = R$$

or $$u\frac{d^2v}{dx^2} + \left(Pu + 2\frac{du}{dx}\right)\frac{dv}{dx} + v\left(\frac{d^2u}{dx^2} + P\frac{du}{dx} + Qu\right) = R$$

or $$\frac{d^2v}{dx^2}+\left(P+\frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx}+\frac{1}{u}\left(\frac{d^2u}{dx^2}+P\frac{du}{dx}+Qu\right)v = \frac{R}{u}. \qquad \text{... (2)}$$

In order to remove the first derivative dv/dx from (2), we take

$$P+\frac{2}{u}\frac{du}{dx}=0 \qquad \text{or} \qquad \frac{du}{u}=-\frac{1}{2}Pdx. \qquad \text{... (3)}$$

Integrating, $\log u = -(1/2)\times\int Pdx$ or $u = e^{-\frac{1}{2}\int Pdx}$ ... (4)

Thus, the required suitable subsitiution for the dependent variable is $y = uv$ where u is given by (4)., Now, from (3), we have

$$\frac{du}{dx} = -\frac{1}{2}Pu \qquad \text{so that} \qquad \frac{d^2u}{dx^2}=-\frac{1}{2}P\frac{du}{dx}-\frac{1}{2}\frac{dP}{dx}u$$

or $$\frac{d^2u}{dx^2} = -\frac{1}{2}P\left(-\frac{1}{2}Pu\right)-\frac{1}{2}\frac{dP}{dx}u, \quad \text{putting value of } \frac{du}{dx} \qquad \text{... (5)}$$

$$\therefore \qquad \frac{1}{u}\left(\frac{d^2u}{dx^2}+P\frac{du}{dx}+Qu\right) = \frac{1}{u}\left(\frac{1}{4}P^2u-\frac{1}{2}\frac{dP}{dx}u-\frac{1}{2}P^2u+Qu\right), \text{ by (5)}$$

Thus, $$\frac{1}{u}\left(\frac{d^2u}{dx^2}+P\frac{du}{dx}+Qu\right)= Q-\frac{1}{4}P^2-\frac{1}{2}\frac{dP}{dx}=I, \text{ say} \qquad \text{... (6)}$$

Also take $S = R/u$ ... (7)

Using (3), (6) and (7), (2) becomes $d^2v / dx^2 + Iv = S$,

where I and S are given by (6) and (7). (7) is known as *normal form* of (1).

10.7 Working rule for solving problems by using normal form

Step 1. Put the equation in the standard form $y'' + Py' + Qy = R$,
in which the coefficient of d^2y / dx^2 must be unity.

Step 2. To remove the first derivative, we choose $u=e^{-\frac{1}{2}\int P\,dx}$

Step 3. We now asume that the complete solution of given equation is $y = uv$.

Then the given equation reduces to normal form

$$\frac{d^2v}{dx^2}+Iv = S, \qquad \text{where} \qquad I=Q-\frac{1}{4}P^2-\frac{1}{2}\frac{dP}{dx} \qquad \text{and} \qquad S=\frac{R}{u}.$$

Important Note. The success in solving the given equation depends on the success in solving $d^2v / dx^2 + Iv = S$. Now this latter equation can be solved easily if I takes two special forms (*i*) when I = constant, then resulting equation being with constant coefficients can be solved by usual methods of chapter 5

(*ii*) when I = (constant) / x^2, then the resulting equation reduces to homogeneous form and hence it can be solved by using usual methods of chapter 6.

Step 4. After getting v, the complete solution is given by $y = uv$.

10.8 Solved examples based on working rule 10.7

Ex. 1. *Solve the following differential equations :*

(*i*) $y'' - 2\tan x \cdot y' + 5y = 0$. **[Agra 2006, 07; Delhi Maths (G) 1993]**

(*ii*) $y'' - 2\tan x \cdot y' + y = 0$. **[Delhi Maths (G) 1996, 98]**

(*iii*) $y'' - 2\tan x \cdot y' - 5y = 0$. **[Delhi Maths (G) 1995]**

Sol. (*i*) Given $y'' - 2\tan x \cdot y' + 5y = 0.$ ... (1)

Comparing (1) with $y'' + Py' + Qy = R,$ $P = -2\tan x,$ $Q = 5,$ $R = 0.$ ... (2)

To remove the first derivative from (1), we choose

$$u = e^{-\frac{1}{2}\int P\,dx} = e^{-\frac{1}{2}\int(-2\tan x)dx} = e^{\log\sec x} = \sec x. \quad \text{... (3)}$$

Let the required general solution be $y = uv.$ ... (4)

Then v is given by normal form $(d^2v/dx^2) + Iv = S,$ ... (5)

where, $I = Q - (P^2/4) - (1/2)\times(dP/dx) = 5 - (1/4)\times(4\tan^2 x) - (1/2)\times(-2\sec^2 x)$, by (2)

$$= 5 - \tan^2 x + \sec^2 x = 5 - \tan^2 x + (\tan^2 x + 1) = 6$$

and $S = R/u = 0$, since $R = 0$ by (2).

Then (5) becomes $(d^2v/dx^2) + 6v = 0$ or $(D^2 + 6)v = 0$, where $D \equiv d/dx.$ ...(6)

Here the auxiliary equation of (6) is $D^2 + 6 = 0$ so that $D = \pm i\sqrt{6}$.

So solution of (6) is $v = \text{C.F.} = c_1\cos(x\sqrt{6}) + c_2\sin(x\sqrt{6}).$ ... (7)

From (3), (4) and (7), the required general solution is

$y = uv$ or $y = \sec x[c_1\cos(x\sqrt{6}) + c_2\sin(x\sqrt{6})]$

(*ii*) Proceed as in part (*i*). **Ans.** $y = \sec x[c_1\cos(x\sqrt{2}) + c_2\sin(x\sqrt{2})]$

(*iii*) Proceed as in part (*i*). **Ans.** $y = \sec x[c_1 e^{2x} + c_2 e^{-2x}]$

Ex. 2. Make use of the transformation $y(x) = v(x)\sec x$ to obtain the solution of $y'' - 2y'\tan x + 5y = 0$, $y(0) = 0$, $y'(0) = \sqrt{6}$. **[I.A.S. 1997]**

Sol. Given, $y'' - 2y'\tan x + 5y = 0.$ ... (1)

Also given $y = v(x)\sec x.$ ... (2)

From (2), $y' = v'\sec x + v\sec x\tan x.$ ... (3)

From (3), $y'' = v''\sec x + 2v'\sec x\tan x + v[\sec x\tan^2 x + \sec^3 x].$

Substituting the above values of y, y' and y'' in (1), we get

$$v''\sec x + 2v'\sec x\tan x + v(\sec x\tan^2 x + \sec^3 x)$$
$$- 2\tan x\,(v'\sec x + v\sec x\tan x) + 5v\sec x = 0$$

or $v''\sec x + v(\sec^3 x - \sec x\tan^2 x + 5\sec x) = 0$

or $v'' + v(\sec^2 x - \tan^2 x + 5) = 0$ or $(D^2 + 6)v = 0$, $D \equiv d/dx.$

Its auxiliary equation is $D^2 + 6 = 0$ so that $D = \pm i\sqrt{6}$.

$\therefore$ $v = \text{C.F.} = c_1\cos(x\sqrt{6}) + c_2\sin(x\sqrt{6})$, c_1 and c_2 being arbitrary constants.

Hence from (2), the general solution of (1) is

$$y(x) = \sec x[c_1\cos(x\sqrt{6}) + c_2\sin(x\sqrt{6})] \quad \text{.. (4)}$$

Putting $x = 0$ in (4) and using the given fact $y(0) = 0$, we get $0 = c_1$. Hence (4) reduces to

$$y(x) = c_2\sec x\sin(x\sqrt{6}). \quad \text{... (5)}$$

From (5), $y'(x) = c_2\sin x\tan x\sin(x\sqrt{6}) + c_2\sqrt{6}\sec x\cos(x\sqrt{6}).$

Putting $x = 0$ and using the given fact $y'(0) = \sqrt{6}$, we get $\sqrt{6} = c_2\sqrt{6}$ so that $c_2 = 1$. Then, from (5), the required solution is $y = \sec x\sin(x\sqrt{6}).$

Ex. 3(*a*). *Solve* $y'' - 2\tan x \cdot y' + 5y = \sec x \cdot e^x$. **[Agra 2006; Garhwal 2010; Kanpur 2009; Meerut 1998; Rhilkhand 2001; S.V.University (A.P.) 1997;**

(*b*) *Solve* $v'' - 2\tan x \cdot y' - (a^2 + 1)y = e^x \cdot \sec x$. **Gulbarga 2005]**

Sol. (*a*). Given $$y'' - 2\tan x \cdot y' + 5y = \sec x \cdot e^x. \qquad \text{... (1)}$$

Comparing (1) with $y'' + Py' + Qy = R$, $P = -2\tan x$, $Q = 5$, $R = \sec x \cdot e^x$. ... (2)

To remove the first derivative from (1), we choose

$$u = e^{-\frac{1}{2}\int P dx} = e^{-\frac{1}{2}\int(-2\tan x)dx} = e^{\log\sec x} = \sec x \qquad \text{... (3)}$$

Let the required general solution be $y = uv$. ... (4)

Then v is given by normal form $(d^2v / dx^2) + Iv = S$, ... (5)

where, $I = Q - P^2/4 - (1/2) \times (dP/dx) = 5 - (1/4) \times (4\tan^2 x) - (1/2) \times (-2\sec^2 x)$, by (2)

$= 5 - \tan^2 x + \sec^2 x = 5 - \tan^2 x + (\tan^2 x + 1) = 6$

and $S = R/u = (\sec x \cdot e^x) / \sec x = e^x$, using (2)

Then (5) becomes $(d^2v / dx^2) + 6v = e^x$ or $(D^2 + 6)v = e^x$, ... (6)

Its auxiliary equation is $D^2 + 6 = 0$ so that $D = \pm i\sqrt{6}$.

$\therefore$ C.F. of (6) $= c_1 \cos(x\sqrt{6}) + c_2 \sin(x\sqrt{6})$, c_1 and c_2 being arbitrary constants

and $$\text{P.I.} = \frac{1}{D^2+6}e^x = \frac{1}{1^2+6}e^x = \frac{1}{7}e^x.$$

Hence solution of (6) is $v = c_1\cos(x\sqrt{6}) + c_2\sin(x\sqrt{6}) + (1/7) \times e^x$ ... (7)

From (3), (4) and (7), the required general solution is

$y = uv$ or $y = \sec x[c_1\cos(x\sqrt{6}) + c_2\sin(x\sqrt{6})] + (1/7) \times e^x]$.

(*b*) Do as in part (*a*). **Ans.** $y = \sec x\,[c_1 e^{ax} + c_2 e^{-ax} + e^x / (1 - a^2)$

Ex. 4(a). *Solve* $(d^2y / dx^2) - (2/x) \times dy/dx) + (n^2 + 2/x^2)y = 0$. **[Delhi Maths (G) 1997]**

(*b*) *Solve* $x\dfrac{d}{dx}\left(x\dfrac{dy}{dx} - y\right) - 2x\dfrac{dy}{dx} + 2y + x^2 y = 0$. **[Agra 2000, 03; Rohilkhand 2002, 04]**

Sol. (*a*) Comparing the given equation with $y'' + Py' + Qy = R$, we get

$P = -(2/x)$, $Q = (n^2 + 2/x^2)$, and $R = 0$. ... (1)

To reduce the given equation into normal form, we choose

$$u = e^{-\frac{1}{2}\int P\,dx} = e^{-\frac{1}{2}\int(-2/x)dx} = e^{\log x} = x. \qquad \text{... (2)}$$

Let the required general solution be $y = uv$. ... (3)

Then v is given by normal form $(d^2v / dx^2) + Iv = S$, ... (4)

where $$I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx} = n^2 + \frac{2}{x^2} - \frac{1}{4}\left(\frac{4}{x^2}\right) - \frac{1}{2}\left(\frac{2}{x^2}\right) = n^2, \text{ by (1).}$$

and $S = R/u = 0$, as $R = 0$.

Then (4) becomes $(d^2v / dx^2) + n^2 v = 0$ or $(D^2 + n^2)v = 0$. ... (5)

Its auxiliary equation is $D^2 + n^2 = 0$ so that $D = \pm in$.

$\therefore$ Solution of (5) is $v = \text{C.F.} = c_1\cos nx + c_2\sin nx$. ... (6)

From (2), (3) and (6), the required solution is $y = uv$ or $y = x(c_1\cos nx + c_2\sin nx)$.

(*b*) Re-writing the given equation, $$x\left(x\frac{d^2y}{dx^2}+\frac{dy}{dx}-\frac{dy}{dx}\right)-2x\frac{dy}{dx}+2y+x^2y=0$$

or $$(d^2y/dx^2)-(2/x)(dy/dx)+(1+2/x^2)y=0,$$

which is just the same as part (*a*) taking $n = 1$. **Ans.** $y = x(c_1 \cos x + c_2 \sin x)$

Ex. 5. *Solve* $\dfrac{d}{dx}\left(\cos^2 x\dfrac{dy}{dx}\right)+y\cos^2 x=0.$ **[Agra 2004, Rajasthan 2003, 06]**

Sol. Re-writing the given equation, we have

$$\cos^2 x\frac{d^2y}{dx^2}-2\cos x\sin x\frac{dy}{dx}+y\cos^2 x=0 \qquad \text{or} \qquad \frac{d^2y}{dx^2}-2\tan x\frac{dy}{dx}+y=0.$$

Now proceed as in Ex. 1(*a*). **Ans.** $y=\sec x\,[c_1\cos(x\sqrt{2})+c_2\sin(x\sqrt{2})]$

Ex. 6(*a*). *Solve* $(y''+y)\cot x+2(y'+y\tan x)=\sec x$.

(*b*) Solve $(y''+y)\cot x+2(y'+y\tan x)=0$. **[Delhi Maths (H) 1999]**

Sol. (*a*) Given $$\cot x\cdot y''+2y'+(\cot x+2\tan x)y=\sec x.$$

or $$y''+2\tan x+(1+2\tan^2 x)y=\sec x\tan x. \qquad \ldots(1)$$

Comparing (1) with $y''+Py'+Qy=R$, we have

$P=2\tan x$, $\quad Q=1+2\tan^2 x\quad$ and $\quad R=\sec x\tan x$, ... (2)

In order to remove the first derivative from (1), we choose

$$u=e^{-\frac{1}{2}\int P\,dx}=e^{-\frac{1}{2}\int(2\tan x)dx}=e^{\log\cos x}=\cos x. \qquad \ldots(3)$$

Let the required general solution be $y=uv$. ... (4)

Then v is given by normal form $(d^2v/dx^2)+Iv=S$, ... (5)

where $$I=Q-\frac{1}{4}P^2-\frac{1}{2}\frac{dP}{dx}=1+2\tan^2 x-\frac{1}{4}(4\tan^2 x)-\frac{1}{2}(2\sec^2 x)$$
$$=1+\tan^2 x-\sec^2 x=\sec^2 x-\sec^2 x=0$$

and $$S=R/u=(\sec x\tan x)/\cos x=\sec^2 x\tan x.$$

Then (5) becomes $(d^2v/dx^2)=\sec^2 x\tan x=(\sec x)(\sec x\tan x)$

Integrating it, $dv/dx=(1/2)\times\sec^2 x+c_1$. ... (6)

Integrating (6), $v=(1/2)\times\tan x+c_1x+c_2$, c_1, c_2 being arbitrary constants. ... (7)

From (3), (4) and (7), the required general solution is

$y=uv \quad$ or $\quad y=\cos x\,[(1/2)\times\tan x+c_1x+c_2]$.

(*b*) **Hint :** Proceed as in part (*a*). Note that $R=0$ in this case and so $S=R/u=0$. Hence (5) reduces to $d^2v/dx^2=0$.

Integrating, $dv/dx=c_1$ so that $v=c_1x+c_2$.

Hence the required solution is $y=uv=\cos x\,(c_1x+c_2)$.

Ex. 7. *Solve* $y''-(2/x)y'+(1+2/x^2)y=xe^x$ *by changing the dependent variable.*

[Kanpur 2009; Patna 2003; Bangalore 2005]

Sol. Comparing the given equation with $y''+Py'+Qy=R$, we get

$P=-2/x$, $\quad Q=1+(2/x^2)\quad$ and $\quad R=xe^x$. ... (1)

We choose $$u=e^{-\frac{1}{2}\int P\,dx}=e^{-\frac{1}{2}\int(-2/x)\,dx}=e^{\int(1/x)\,dx}=e^{\log x}=x. \qquad \ldots(2)$$

Let the required general solution be $y = uv$. ... (3)

Then v is given by the normal form $(d^2v / dx^2) + Iv = S,$ ... (4)

where $$I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx} = 1 + \frac{2}{x^2} - \frac{1}{4}\left(\frac{4}{x^2}\right) - \frac{1}{2}\left(\frac{2}{x^2}\right) = 1$$

and $S = R/u = (xe^x)/x = e^x$.

Then (4) becomes $(d^2v / dx^2) + v = e^x$ or $(D^2 + 1)v = e^x$. ... (5)

Its auxiliary equation is $D^2 + 1 = 0$ so that $D = \pm i$

$\therefore$ C.F. of (5) $= c_1 \cos x + c_2 \sin x$, c_1 and c_2 being arbitrary constants.

and $$\text{P.I.} = \frac{1}{D^2+1}e^x = \frac{1}{1^2+1}e^x = \frac{1}{2}e^x$$

Hence the solution of (5) is $v = \text{C.F.} + \text{P.I.} = c_1 \cos x + c_2 \sin x + (1/2) \times e^x$

and so the required solution is $y = uv = x[c_1 \cos x + c_2 \sin x + (1/2) \times e^x]$.

Ex. 8(*a*). *Solve* $y'' - 4xy' + (4x^2 - 1)y = -3e^{x^2} \sin 2x$.

[Guwahati 2007; Meerut 2004; Delhi Maths (G) 2004, 05; I.A.S. 2000]

Sol. Comparing the given equation with $y'' + Py' + Q\,y = R$, we get

$P = -4x$, $Q = 4x^2 - 1$ and $R = -3e^{x^2} \sin 2x$. ... (1)

We choose $$u = e^{-\frac{1}{2}\int P\,dx} = e^{-\frac{1}{2}\int(-4x)\,dx} = e^{x^2}.$$ ... (2)

Let the required general solution be $y = uv$. ... (3)

Then v is given by the normal form $(d^2v / dx^2) + Iv = S,$ ... (4)

where $$I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx} = 4x^2 - 1 - \frac{1}{4}(16x^2) - \frac{1}{2}(-4) = 1$$

and $S = R/u = (-3e^{x^2} \sin 2x)/e^{x^2} = -3 \sin 2x$.

Then (4) becomes $(d^2v / dx^2) + v = -3 \sin 2x$ or $(D^2 + 1)v = -3 \sin 2x$.

Its auxiliary equation is $D^2 + 1 = 0$ so that $D = \pm i$

$\therefore$ Its C.F. $= c_1 \cos x + c_2 \sin x$, c_1 and c_2 being arbitrary constants

and $$\text{P.I.} = \frac{1}{D^2+1}(-3\sin 2x) = -3\frac{1}{-2^2+1}\sin 2x = \sin 2x.$$

$\therefore$ $v = \text{C.F.} + \text{P.I.} = c_1 \cos x + c_2 \sin x + \sin 2x$. ... (5)

From (2), (3) and (5), the required general solution is

$y = uv$ or $y = e^{x^2}(c_1 \cos x + c_2 \sin x + \sin 2x)$.

Ex. 8(*b*). *Solve* $y'' - 4xy' + (4x^2 - 3)y = e^{x^2}$ **[Delhi Maths (G) 2006, Bangalore 2005]**

Hint : Do as in part (*a*) **Ans.** $y = e^{x^2}(c_1 e^x + c_2 e^{-x} - 1)$

Ex. 8(*c*). *Solve* $y'' - 4xy' + (4x^2 - 1)y = -3e^{x^2}(\sin 2x + 5e^{-2x} + 6)$.

Hint : As in part (*a*), here $P = -4x$, $Q = 4x^2 - 1$, $R = -3e^{x^2}(\sin 2x + 5e^{-2x} + 6)$.

As before, $u = e^{x^2}$ and $I = 1$. Also $S = R/u = -3 \sin 2x - 15e^{-2x} - 18$.

Hence normal form is $(d^2v / dx^2) + v = -3 \sin 2x - 15e^{-2x} - 18$

or $(D^2 + 1)v = -3 \sin 2x - 15e^{-2x} - 18.$

As before, we get C.F. $= c_1 \cos x + c_2 \sin x$.

$$\text{P.I.} = \frac{1}{D^2+1}(-3\sin 2x - 15e^{-2x} - 18) = -3\frac{1}{D^2+1}\sin 2x - 15\frac{1}{D^2+1}e^{-2x} - 18\frac{1}{D^2+1}e^{0.x}$$

$$= -3\frac{1}{-2^2+1}\sin 2x - 15\frac{1}{(-2)^2+1}e^{-2x} - 18\frac{1}{0^2+1}e^{0.x} = \sin 2x - 3e^{-2x} - 18.$$

$\therefore$ $v = \text{C.F.} + \text{P.I} = c_1 \cos x + c_2 \sin x + \sin 2x - 3e^{-2x} - 18.$

and required solution is $y = uv = e^{x^2}(c_1 \cos x + c_2 \sin x + \sin 2x - 3e^{-2x} - 18).$

Ex. 9(*a*). *Solve* $y'' - 2bxy' + b^2x^2y = x.$ **[Sagar 2002]**

(*b*) *Solve* $y'' - 2bxy' + b^2x^2y = 0.$

Sol. (*a*) Comparing the given equation with $y'' + Py' + Q\,y = R$, we get

$P = -2bx,$ $Q = b^2x^2$ and $R = x.$ ... (1)

We choose $u = e^{-\frac{1}{2}\int P\,dx} = e^{-\frac{1}{2}\int(-2bx)dx} = e^{(bx^2/2)}$ ... (2)

Let the required general solution be $y = uv.$ ... (3)

Then v is given by the normal form $(d^2v / dx^2) + Iv = S,$ ... (4)

where $$I = Q - \frac{P^2}{4} - \frac{1}{2}\frac{dP}{dx} = b^2x^2 - b^2x^2 - \frac{1}{2}(-2b) = b$$

and $S = R/u = xe^{-(bx^2/2)}$, using (1) and (2)

Then (4) becomes $(d^2v / dx^2) + bv = xe^{-(bx^2/2)}$ or $(D^2 + b)v = xe^{-(bx^2/2)}$ ... (5)

Now the auxiliary equation of (5) is $D^2 + b = 0$ so that $D = \pm i\sqrt{b}$.

$\therefore$ C.F. of (5) $= c_1 \cos(x\sqrt{b}) + c_2 \sin(x\sqrt{b})$, c_1, c_2 being arbitrary constants

and P.I. $= \dfrac{1}{D^2+b}xe^{-(bx^2/2)}$, which cannot be evaluated by well known methods.

$\therefore$ $$v = \text{C.F.} + \text{P.I.} = c_1 \cos(x\sqrt{b}) + c_2 \sin(x\sqrt{b}) + \frac{1}{D^2+b}xe^{-(bx^2/2)} \quad ...\ (6)$$

From (2), (3) and (6), the required general solution is

$y = uv$ or $$y = e^{(bx^2/2)}\left[c_1 \cos(x\sqrt{b}) + c_2 \sin(x\sqrt{b}) + \frac{1}{D^2+b}xe^{-(bx^2/2)}\right]$$

(*b*) Do as in part (*a*). Then (5) takes the form $(D^2 + b)v = 0.$

Hence $v = c_1 \cos(x\sqrt{b})\, c_2 \sin(x\sqrt{b})$ and $u = e^{(bx^2/2)}$ as before

and the required solution is $y = uv = e^{(bx^2/2)}[c_1 \cos(x\sqrt{b}) + c_2 \sin(x\sqrt{b})].$

Ex. 10(*a*). *Solve* $y'' + 2xy' + (x^2 + 1)y = x^3 + 3x.$ **[Merrut 1997]**

Sol. Comparing the given equation with $y'' + Py' + Q\,y = R$, we get

$P = 2x,$ $Q = x^2 + 1$ and $R = x^3 + 3x.$ ... (1)

we choose $u = e^{-\frac{1}{2}\int P\,dx} = e^{-\frac{1}{2}\int(2x)dx} = e^{-x^2/2}.$ ... (2)

Let the required general solution be $y = uv$. ... (3)

Then v is given by the normal term $(d^2v / dx^2) + Iv = S$, ... (4)

where $I = Q - \dfrac{P^2}{4} - \dfrac{1}{2}\dfrac{dP}{dx} = x^2 + 1 - x^2 - \dfrac{1}{2} \times 2 = 0$ and $S = \dfrac{R}{u} = (x^3 + 3x)e^{x^2/2}$

Then (4) reduces to $d^2 v / dx^2 = (x^3 + 3x)e^{x^2/2}$ ... (5)

Integrating (5), $\dfrac{dv}{dx} = \int x^3 e^{x^2/2} dx + 3\int x\, e^{x^2/2} dx + c_1.$ ... (6)

Putting $x^2/2 = t$ so that $xdx = dt$, (6) becomes

$dv / dx = \int (2t)e^t dt + 3\int e^t dt + c_1$ or $dv/dx = 2(t\, e^t - \int e^t dt) + 3e^t + c_1$, integrating by parts.

or $dv/dx = 2t\, e^t - 2\, e^t + 3\, e^t + c_1 = 2t\, e^t + e^t + c_1$

or $dv/dx = x^2\, e^{x^2/2} + e^{x^2/2} + c_1$, as $t = x^2/2$. ... (7)

Integrating (7), $v = \int x^2\, e^{x^2/2} dx + \int e^{x^2/2} dx + c_1 x + c_2.$ ... (8)

Now, $\int x\, e^{x^2/2} dx = \int e^t dt$, putting $x^2/2 = t$ and $x\, dx = dt$

Thus, we have $\int x\, e^{x^2/2} dx = e^t = e^{x^2/2}.$ ... (9)

Then $\int x^2 e^{x^2/2} dx = \int x\,(xe^{x^2/2})\, dx = xe^{x^2/2} - \int (1 \cdot e^{x^2/2})\, dx$, integrating by parts by taking x as first function and $xe^{x^2/2}$ as the second function and using result (9) directly

$\therefore$ (8) reduces to $v = xe^{x^2/2} - \int e^{x^2/2} dx + \int e^{x^2/2} dx + c_1 x + c_2 = xe^{x^2/2} + c_1 x + c_2$

and so the required general solution is given by

$$y = uv = e^{-x^2/2}[xe^{x^2/2} + c_1 x + c_2] = x + e^{-x^2/2}(c_1 x + c_2)$$

Ex. 10(*b*). *Solve the equation* $xy'' - 2(x + 1)y' + (x + 2)y = (x - 2)e^x$, $(x > 0)$ by changing into normal form. **[Bangalore 1995]**

Sol. Try yourself. **Ans.** $y = -(1/2) \times x^2 e^x + xe^x + (1/3) \times c_1 x^3 e^x + c_2 e^x$

Ex. 11. *Solve the following differential equations :*

(*a*) $\dfrac{d^2 y}{dx^2} + \dfrac{1}{x^{1/3}}\dfrac{dy}{dx} + \left(\dfrac{1}{4x^{2/3}} - \dfrac{1}{6x^{4/3}} - \dfrac{6}{x^2}\right) = 0.$ **[Punjab 2003; Vikram 2001, 03]**

(*b*) $\dfrac{d^2 y}{dx^2} - \dfrac{1}{x^{1/2}}\dfrac{dy}{dx} + \dfrac{1}{4x^2}(x + x^{1/2} - 8)y = 0.$ **[Agra 2005; Pune 2006, Vikram 2001, 03**

(*c*) $4x^2(d^2y / dx^2) + 4x^5(dy/dx) + (x^8 + 6x^4 + 4)y = 0.$ **[Vikram 2001,03]**

(*d*) $(x^3 - 2x^2)\,(d^2y / dx^2) + 2x^2(dy/dx) + 12(x - 2)y = 0.$

(*e*) $x^2\,(\log x)^2\,(d^2y / dx^2) - 2x \log x\,(dy/dx) + [2 + \log x - 2\,(\log x)^2]y = x^2(\log x)^3$ **[Vikram 2001]**

(*f*) $x^2\,(d^2y / dx^2) - 2x(3x - 2)(dy/dx) + 3x(3x - 4)y = e^{2x}.$ **[Agra 1997]**

Sol. (*a*) Comparing the given equation with $y'' + Py' + Q\,y = R$, we get

$P = x^{-x/3}$, $Q = (1/4x^{2/3}) - (1/6x^{4/3}) - (6/x^2)$ and $R = 0$ ... (1)

We choose $u = e^{-\frac{1}{2}\int P\,dx} = e^{-\frac{1}{2}\int x^{-1/3} dx} = e^{-(3/4)x^{2/3}}.$ ... (2)

Let the required general solution be $y = uv$. ... (3)

Then v is given by the normal form $(d^2v / dx^2) + Iv = S$, ... (4)

where $I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx} = \frac{1}{4x^{2/3}} - \frac{1}{6x^{4/3}} - \frac{6}{x^2} - \frac{1}{4x^{2/3}} - \frac{1}{2}\left(-\frac{1}{3x^{4/3}}\right) = -\frac{6}{x^2}$

and $S = R/u = 0$, as $R = 0$.

Then (4) becomes $(d^2v / dx^2) - (6/x^2)v = 0$ or $(x^2D^2 - 6)v = 0$... (5)

which is a homogeneous linear equation. Here $D \equiv d/dx$.

To solve it, let $x = e^x$ (or $z = \log x$) and $D_1 \equiv d/dz$.... (6)

We have $x^2D^2 = D_1(D_1 - 1)$. Then (6) reduces to

$[D_1(D_1 - 1) - 6]v = 0$ or $(D^2{}_1 - D_1 - 6)v = 0$.... (7)

Here the auxiliary equation of (7), is $D_1{}^2 - D_1 - 6 = 0$. so that $D_1 = 3, -2$.

$\therefore$ C.F. $= c_1e^{3z} + c_2e^{-2z} = c_1(e^z)^3 + c_2(e^z)^{-2} = c_1x^3 + c_2x^{-2}$

Hence the solution of (7) is $v =$ C.F. $= c_1x^3 + c_1x^{-2}$. ... (8)

From (2), (3) and (8), the required solution is $y = uv$ or $y = e^{-(3/4)x^{2/3}}(c_1\, x^3 + c_2x^{-2})$

(*b*) Try as in part (*a*). **Ans.** $y = e^{x^{1/2}}(c_1x^2 + c_2x^{-1})$

(*c*) Dividing by $4x^2$, the given equation in standard form is

$$y'' + x^3y' + [(x^8 + 6x^4 + 4)/4x^2]y = 0. \quad ... (1)$$

Comparing (1) with $y'' + Py' + Q\,y = R$, we have

$P = x^3$, $Q = x^6/4 + 3x^2/2 + 1/x^2$ and $R = 0$. ... (2)

We choose $u = e^{-\frac{1}{2}\int P\,dx} = e^{-\frac{1}{2}\int x^3\,dx} = e^{-x^4/8}$. ... (3)

Let the required general solution be $y = uv$. ... (4)

Then v is given by the normal form $(d^2v / dx^2) + Iv = S$, ... (5)

where $I = Q - \frac{1}{4}P^2 - \frac{1}{2}\frac{dP}{dx} = \frac{x^6}{4} + \frac{3x^2}{2} + \frac{1}{x^2} - \frac{x^6}{4} - \frac{1}{2}\times(3x^2) = \frac{1}{x^2}, S = \frac{R}{u} = 0.$

Then (5) becomes $(d^2v / dx^2) + (1/x^2)v = 0$ or $(x^2D^2 + 1)v = 0$.... (6)

Let $x = e^z$ (or $z = \log x$), $D \equiv d/dx$ and $D_1 \equiv d/dz$

so that $x^2D^2 = D_1(D_1 - 1)$. Then (6) reduces to

$\{D_1(D_1 - 1) + 1\}v = 0$ or $(D^2{}_1 - D_1 + 1)v = 0$. ... (7)

Its auxiliary equation is $D_1{}^2 - D_1 + 1 = 0$ so that $D_1 = (1 \pm \sqrt{1-4})/2 = (1/2) \pm i\,(\sqrt{3}/2)$

$$\therefore \quad \text{C.F. of (7)} = e^{z/2}[c_1 \cos\{(\sqrt{3}/2)z\} + c_2 \sin\{(\sqrt{3}/2)z\}]$$
$$= (e^z)^{(1/2)}[c_1 \cos\{(\sqrt{3}/2)z\} + c_2 \sin\{(\sqrt{3}/2)z\}]$$
$$= x^{1/2}[c_1 \cos\{(\sqrt{3}/2)\log x\} + c_2 \sin\{(\sqrt{3}/2)\log x\}], \text{ as } x = e^z$$

Hence the solution of (7) is given by $v =$ C.F.

or $$v = x^{1/2}[c_1 \cos\{(\sqrt{3}/2)\log x\} + c_2 \sin\{(\sqrt{3}/2)\log x\}] \quad ...(8)$$

From (3), (4) and (8), the required general solution is

$$y = u\,v \quad \text{or} \quad y = e^{-x^4/8}\, x^{1/2}\left[c_1 \cos\{(\sqrt{3}/2)\log x\} + c_2 \sin\{(\sqrt{3}/2)\log x\}\right]$$

(*d*) Try yourself. **Ans.** $y = (x-2)^{-1}(c_1x^4 + c_2x^{-3})$

(*e*) Dividing by x^2 $(\log x)^2$, the given equation in standard form is

$$\frac{d^2y}{dx^2} - \frac{2}{x\log x}\frac{dy}{dx} + \frac{2+\log x - 2(\log x)^2}{x^2(\log x)^2}\,y = \log x. \qquad \text{... (1)}$$

Comparing (1) with $y'' + Py' + Q\,y = R$, we have

$$P = -\frac{2}{x\log x}, \qquad Q = \frac{2}{x^2(\log x)^2} + \frac{1}{x^2\log x} - \frac{2}{x^2} \qquad \text{and} \qquad R = \log x. \qquad \text{... (2)}$$

We choose $u = e^{-\frac{1}{2}\int P\,dx} = e^{\int \frac{(1/x)}{\log x}dx} = e^{\log(\log x)} = \log x.$ ... (3)

Let the required general solution be $y = uv.$ ... (4)

Then v is given by the normal form $(d^2v/dx^2) + Iv = S,$ ... (5)

where $I = Q - (1/4)\times P^2 - (1/2)\times(dP/dx)$

$$= \frac{2}{x^2(\cos x)^2} + \frac{1}{x^2\cos x} - \frac{2}{x^2} - \frac{1}{x^2(\cos x)^2} + \left[-\frac{1}{x^2\cos x} - \frac{1}{x^2(\cos x)^2}\right]$$

or $I = -2/x^2$ and $S = R/u = (\log x)/(\log x) = 1.$

Then (5) reduce to $(d^2v/dx^2) - (2/x^2)v = 1$ or $(x^2D^2 - 2)v = x^2.$ ... (6)

Let $x = e^x$ (or $z = \log x$), $D \equiv d/dx$ and $D_1 = d/dz$ so that $x^2D^2 = D_1(D_1 - 1).$

Then (6) reduces to $[D_1(D_1 - 1) - 2]v = e^{2z}$ or $(D^2_1 - D_1 - 2)v = e^{2z}.$ ... (7)

Its auxiliary equation is $D_1^2 - D_1 - 2 = 0$ so that $D_1 = 2, -1$

$\therefore$ C.F. $= c_1e^{2z} + c_2e^{-z} = c_1(e^z)^2 + c_2(e^z)^{-1} = c_1x^2 + c_2x^{-1}$

and P.I. $= \dfrac{1}{D_1^2 - D_1 - 2}e^{2z} = \dfrac{1}{(D_1-2)}\dfrac{1}{(D_1+1)}e^{2z} = \dfrac{1}{D_1-2}\dfrac{1}{2+1}e^{2z}$

$= \dfrac{1}{3}\dfrac{1}{(D_1-2)^1}e^{2z} = \dfrac{1}{3}\dfrac{z}{1!}e^{2z}$ $\left[\because \dfrac{1}{(D_1-a)^n}e^{az} = \dfrac{z^n}{n!}e^{az}\right]$

$= (1/3)\times z\,(e^z)^2 = (1/3)\times x^2\log x,$ as $z = \log x,$ and $e^z = x.$

$\therefore$ Solution of (7) is $v = \text{C.F.} + \text{P.I.} = c_1x^2 + c_2x^{-1} + (1/3)\times x^2\log x$ ... (8)

From (3), (4) and (8), the required general solution is

$y = uv$ or $y = \log x\,[c_1x^2 + c_2x^{-1} + (1/3)x^2\log x]$

(*f*) Try yourself. **Ans.** $y = x^{-2}e^{3x}[c_1x^2 + c_2x^{-1} + (1/3)\times x^2\log x$

Ex. 12(*a*). *Reduce the equation* $x^2y'' - 2x(1+x)y' + 2(1+x)y = x^3$, $(x > 0)$ *into the normal form and hence solve it.* **[Bangalore 1995]** **Ans.** $y = c_1xe^{2x} + c_2x - (x^2/2).$

Ex. 12(*b*). *Solve* $y'' + (4\text{ cosec } 2x)y' + (2\tan^2 x)y = e^x \cot x$ *by changing the dependent variable.* **[Bangalore 2005]**

Sol. Comparing the given equation with $y'' + Py' + Q\,y = R$, here

$P = 4\text{ cosec } 2x,$ $Q = 2\tan^2 x$ and $R = e^x\cot x$... (1)

Hence $I = Q - \dfrac{1}{4}P^2 - \dfrac{1}{2}\dfrac{dP}{dx} = 2\tan^2 x - \dfrac{1}{4}\times(16\text{ cosec}^2 2x) - \dfrac{1}{2}\times(-8\text{ cosec } 2x\cot 2x)$

$$= 2\tan^2 x - 4\operatorname{cosec} 2x\left(\frac{1}{\sin 2x} - \frac{\cos 2x}{\sin 2x}\right)$$

$$= 2\tan^2 x - 4\operatorname{cosec} 2x\left(\frac{1-\cos 2x}{\sin 2x}\right) = 2\tan^2 x - \frac{8\operatorname{cosec} 2x \sin^2 x}{\sin 2x}$$

Then, $$\mathrm{I} = \frac{2\sin^2 x}{\cos^2 x} - \frac{8\sin^2 x}{(2\sin x\cos x)^2} = \frac{-2(1-\sin^2 x)}{\cos^2 x} = -2 \quad \text{... (2)}$$

which is a constant. Hence to solve the given equation, we choose

$$u = e^{\int(-P/2)\,dx} = e^{\int(-2\operatorname{cosec} 2x)\,dx} = e^{-\log\tan x} = \cot x \quad \text{... (3)}$$

Let the required complete solution be $y = uv$ ... (4)

Then v is given by the normal form $d^2v/dx^2 + Iv = S$ ... (5)

where $S = R/4 = (e^x \cot x)/\cot x = e^x$ ... (6)

Then (2), (6) and (5) yield $d^2v/dx^2 - 2v = e^x$ or $(D^2 - 2)v = e^x$ ... (7)

Here auxiliary equation is $D^2 - 2 = 0$ giving $D = \pm\sqrt{2}$

$\therefore$ C.F. of (7) = $C_1 e^{x\sqrt{2}} + C_2 e^{-x\sqrt{2}}$, C_1, C_2 being arbitrary constants

and P.I. of (7) = $\dfrac{1}{D^2 - 2}e^x = \dfrac{1}{(1-2)}e^x = -e^x$

$\therefore$ v = C.F. + P.I or $v = C_1 e^{x\sqrt{2}} + C_2 e^{-x\sqrt{2}} - e^x$ ... (8)

From (3), (4) and (8), the required solution is

$y = uv$ or $y = \cot x\,(C_1 e^{x\sqrt{2}} + C_2 e^{-x\sqrt{2}} - e^x)$

EXERCISE 10(B)

Solve the following differential equations by reducing to normal form:

1. $x^2 y_2 - 2(x^2 + x)y_1 + (x^2 + 2x + 2)y = 0$. **[Rohilkhand 2001; Mumbai 1997, Pune 1998; Nagpur 2000; Delhi Maths (G) 2001; Madurai Kamraj 2008; Kanpur 2008]** **Ans.** $y = xe^x(c_1 x + c_2)$
2. $y_2 + 4xy_1 + 4x^2 y = 0$ **[Karnataka 2001, Vikram 1999]** **Ans.** $y = e^{-x^2}(c_1 e^{x\sqrt{2}} + c_2 e^{-x\sqrt{2}})$
3. (*a*) $y_2 + (2/x)y_1 - n^2 y = 0$. **Ans.** $y = x^{-1}(c_1 e^{nx} + c_2 e^{-nx})$

   (*b*) $y_2 + (2/x)y_1 + n^2 y = 0$. **[Delhi Maths (G) 2000]** **Ans.** $y = x^{-1}(c_1 \cos nx + \sin nx)$
4. $y_2 + 2xy_1 + (x^2 + 5)y = xe^{-x^2/2}$ **Ans.** $y = e^{-x^2/2}(c_1 \cos x + c_2 \sin x + x/4)$
5. $(1 - x^2)y_2 - 4xy_1 - (1 + x^2)y = x$. **[I.A.S. 2004]** **Ans.** $y = (1 - x^2)^{-1}(c_1 \sin x + c_2 \cos x + x)$
6. $y_2 - 4xy_1 + (4x^2 - 1)y = e^{x^2}(5 - 3\cos 2x)$. **Ans.** $y = e^{x^2}(c_1 \cos x + c_2 \sin x + 5 + \cos 2x)$
7. $y'' - 2\cot x \cdot y' + (1 + 2\cot^2 x)y = 0$. **Ans.** $y = \sin x.\ (c_1 + c_2 x)$
8. $y'' - 4xy' + 4x^2 y = e^{x^2}$. **Ans.** $y = e^{x^2}[c_1 \cos(x\sqrt{2}) + c_2 \sin(x\sqrt{2})]$
9. $y'' + 2xy' + (x^2 - 8)y = xe^{-x^2/2}$. **Ans.** $y = e^{-x^2/2}(c_1 e^{3x} + c_2 e^{-2x} - x^2/9 - 2/81)$
10. $y'' + (2/x)y' + y = (\sin 2x)/x$. **Ans.** $y = x^{-1}[c_1 \cos x + c_2 \sin x - (1/3) \times \sin 2x]$

11. $y'' - 2xy' + (x^2 + 2)y = e^{(x^2+x)/2}$. **Ans.** $y = e^{x^2/2}[c_1 \cos x\sqrt{3} + c_2 \sin x\sqrt{3} + (1/4) \times e^{(x^2+2x)/2}]$

12. $x^2 y'' - 2xy' + (x^2 + 2)y = x^3 e^x$. **Ans.** $y = x(c_1 \cos x + c_2 \sin x + e^x / 2$

13. $y'' + (2/x)y' - y = 0$. **[Nagpur 1996]** **Ans.** $y = x^{-1}(c_1 e^x + c_2 e^{-x})$

10.9. Transformation of the equation by changing the independent variable.

Consider $$d^2y / dx^2 + P(dy/dx) + Q\, y = R, \qquad \text{... (1)}$$

where P, Q and R are functions of x and let the independent variable be changed from x to z, where $z = f(x)$, say

Using the formula, $\dfrac{df}{dx} = \dfrac{df}{dz} \cdot \dfrac{dz}{dx}$, we have $\dfrac{dy}{dx} = \dfrac{dy}{dz} \cdot \dfrac{dz}{dx}$

and $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dz}\frac{dz}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dz}\right)\frac{dz}{dx} + \frac{dy}{dz}\frac{d^2z}{dx^2}$$

or $$\frac{d^2y}{dx^2} = \frac{d}{dz}\left(\frac{dy}{dz}\right) \times \frac{dz}{dx} \times \frac{dz}{dx} + \frac{dy}{dz}\frac{d^2z}{dx^2} = \frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\frac{d^2z}{dx^2}$$

Putting the above values of dy/dx and d^2y / dx^2 in (1), we obtain

$$\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\frac{d^2z}{dx^2} + P\frac{dy}{dz}\frac{dz}{dx} + Qy = R \qquad \text{or} \qquad \frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \left(\frac{d^2z}{dx^2} + P\frac{dz}{dx}\right)\frac{dy}{dz} + Qy = R$$

Dividing by $\left(\dfrac{dz}{dx}\right)^2$, we have $$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz}\; Q_1 y = R_1, \qquad \text{... (2)}$$

where $$P_1 = \frac{(d^2z/dx^2) + P(dz/dx)}{(dz/dx)^2}, \qquad Q_1 = \frac{Q}{(dz/dx)^2} \qquad \text{and} \qquad R_1 = \frac{R}{(dz/dx)^2} \qquad (3)$$

Here P_1, Q_1 and R_1 are functions of x but these can be converted to functions of z by using the relation $z = f(x)$. If by equating Q_1 to a constant quantity we see that P_1 also becomes constant then (2) can be solved (since it will be linear equation with constant co-efficient) to obtain the required solution.

10.10. Working Rule for solving equation by changing the independent variable:

Step 1. Put the given equation in standard form by keeping the coefficient of d^2y/dx^2 as unity, *i.e.* $$y'' + Py' + Q\, y = R. \quad \text{... (1)}$$

Step 2. Suppose $Q = \pm\, k\, f(x)$, then we assume a relation between the new independent variable z and the old independent variable x given by $(dz/dx)^2 = k\, f(x)$. **Note carefully that we omit –ve sign of Q while writing this step. This is extremely important to find real values.** Sometimes we assume that $(dz/dx)^2 = f(x)$ whenever we anticipate complicated relation between z and x.

Step 3. We now solve $(dz/dx)^2 = k\, f(x)$. Rejecting negative sign, we get

$$dz/dx = \sqrt{[k\, f(x)]}. \qquad \text{... (2)}$$

Now separating variables, (2) gives $dz = \sqrt{[k\, f(x)]}\, dx$ so that $z = \int \sqrt{[kf(x)]}\, dx$, ... (3)

where we have omitted constant of integration since we are interested in finding just a relation between z and x.

Step 4. With the relationship (3) between z and x, we transform (1) to get an equation of the form $$(d^2y/dz^2) + P_1(dy/dz) + Q_1 y = R_1, \qquad \text{... (4)}$$

where $P_1 = \dfrac{(d^2z/dx^2) + P(dz/dx)}{(dz/dx)^2}$, $\quad Q_1 = \dfrac{Q}{(dz/dx)^2}$, $\quad$ and , $\quad R_1 = \dfrac{R}{(dz/dx)^2}$ ... (5)

Now by virtue of (2), $Q_1 = \dfrac{\pm k\, f(x)}{d\, f(x)} = \pm k$, a constant. Then we calculate P_1. **If P_1 is also constant, then (4) can be solved because it will be a linear equation with constant coefficients. If, however, P_1 does not become constant, then this rule will not be useful. The students must, therefore, be sure that P_1 comes out to be constant before proceeding further. The value of P_1, Q_1 and R_1 must be remembered for direct use in problems. R_1 can be converted to a function of z by using (3).**

Step 5. After solving equation (4) by usual methods the variable z is replaced by x by using (3).

10.11. Solved examples based on Art. 10.10.

Ex. 1. *Solve* $\sin^2 x\, y'' + \sin x \cos x \cdot y' + 4y = 0.$ or $y'' + \cot x \cdot y' + 4 \operatorname{cosec}^2 x \cdot y = 0.$

[Agra 2006; Kanput 2006; Delhi Maths (G) 1997; Meerut 2001; Rohilknand 2001]

Sol. Dividing by $\sin^2 x$, the given equation becomes

$$y'' + \cot x \cdot y + 4 \operatorname{cosec}^2 x \cdot y = 0. \qquad \text{... (1)}$$

Comparing (1) with $\quad y'' + Py' + Q\,y = R,\quad$ we have

$P = \cot x, \qquad Q = 4 \operatorname{cosec}^2 x \qquad$ and $\qquad R = 0.$ ... (2)

We choose z such that $\qquad (dz/dx)^2 = 4 \operatorname{cosec}^2 x$ ... (3)

*so that $\qquad dz/dx = 2 \operatorname{cosec} x \qquad$ giving $\qquad z = 2 \log \tan (x/2)$ ... (4)

Now changing the independent variable from x to z by using relation (4), (1) becomes

$$(d^2y/dz^2) + P_1(dy/dz) + Q_1 y = R_1, \qquad \text{... (5)}$$

where $$P_1 = \frac{(d^2z/dx^2) + P(dz/dx)}{(dz/dx)^2} = \frac{-2\operatorname{cosec} x \cot x + \cot x \cdot (2 \operatorname{cosec} x)}{4 \operatorname{cosec}^2 x} = 0,$$

[Using relations (2), (3) and (4)]

$Q_1 = \dfrac{Q}{(dz/dx)^2} = \dfrac{4\operatorname{cosec}^2 x}{4\operatorname{cosec}^2 x} = 1 \qquad$ and $\qquad R_1 = \dfrac{R}{(dz/dx)^2} = 0$, by (2) and (3).

$\therefore$ From (5), $\quad (d^2y/dz^2) + y = 0 \quad$ or $\quad (D_1^2 + 1)y = 0$, where $D_1 \equiv d/dz$

Its auxiliary equation is $\quad D_1^2 + 1 = 0 \quad$ so that $\quad D_1 = \pm i.$

Hence the required solution is $\quad y = \text{C.F.} = c_1 \cos z + c_2 \sin z$

or $$y = c_1 \cos \{2 \log \tan (x/2)\} + c_2 \sin \{2 \log \tan (x/2)\}.$$

Ex. 2. *Solve* (*a*) $y'' + (2/x)y' + (a^2/x^4)\, y = 0.$ **[Agra 2000;Kanpur 2005; Sagar 2004;]**

(*b*) $x^4 y'' + 2x^3 y + n^2 y = 0.$ **[Kurukshetra 2000]**

Sol. (*a*) Comparing the given equation with $y'' + Py' + Q\,y = R$, we get

$P = 2/x, \qquad Q = a^2/x^4 \qquad$ and $\qquad R = 0.$ ... (1)

We choose z such that $\qquad (dz/dx)^2 = a^2/x^4$ ... (2)

*so that $\qquad dz/dx = a/x^2 \qquad$ giving $\qquad z = -a/x.$ ... (3)

* In practice, while extracting square root on both sides of (3), we shall take positive sign.

Now changing the independent variable from x to z by using relation (3), the given equation is transformed into $(d^2y/dz^2) + P_1(dy/dz) + Q_1 y = R_1$, ... (4)

where $P_1 = \dfrac{(d^2z/dx^2) + P(dz/dx)}{(dz/dx)^2} = \dfrac{-2a/x^3 + (2/x)\times(a/x^2)}{(a^2/x^4)} = 0$, using (1), (2) and (3)

$Q_1 = \dfrac{Q}{(dz/dx)^2} = \dfrac{a^2/x^4}{a^2/x^4} = 1$ and $R_1 = \dfrac{R}{(dz/dx)^2} = 0$, by (1) and (2)

$\therefore$ From (4), $(d^2y/dz^2) + y = 0$ or $(D_1^2 + 1)y = 0$, where $D_1 \equiv d/dz$

Its auxiliary equation is $D_1^2 + 1 = 0$ so that $D_1 = \pm i$.

Hence the required solution is $y = c_1 \cos z + c_2 \sin z$

or $y = c_1 \cos(-a/x) + c_2 \sin(-a/x) = c_1 \cos(a/x) - c_2 \sin(a/x)$, by (3)

(*b*) **Hint.** Divide by x^4, we have $y'' + (2/x)y' + (n^2/x^4)y = 0$, which is the same as in part (*a*) with $a = n$. **Ans.** $y = c_1 \cos(n/x) - c_2 \sin(n/x)$.

Ex. 3. *Solve* $(1 + x^2)^2 y'' + 2x(1 + x^2)y' + 4y = 0$. **[Meerut 2004; Vikram 2005]**

Sol. Dividing by $(1 + x^2)^2$ the given equation in standard form is

$$\frac{d^2y}{dx^2} + \frac{2x}{1+x^2}\frac{dy}{dx} + \frac{4}{(1+x^2)^2}y = 0. \quad \text{... (1)}$$

Comparing (1) with $y'' + Py' + Qy = R$, we have

$P = (2x)/(1 + x^2)$, $Q = 4/(1 + x^2)^2$ and $R = 0$ ... (2)

Choose z such that $(dz/dx)^2 = 4/(1 + x^2)^2$ so that $dz/dx = 2/(1 + x^2)$. ... (3)

Integrating, $z = 2\displaystyle\int \frac{dx}{1+x^2}$ or $z = 2\tan^{-1}x$... (4)

Now changing the independent variable from x to z by using relation (4), (1) becomes

$$(d^2y/dz^2) + P_1(dy/dz) + Q_1 y = R_1, \quad \text{... (5)}$$

where $P_1 = \dfrac{\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx}}{(dz/dx)^2} = \dfrac{-\dfrac{4x}{(1+x^2)^2} + \dfrac{2x}{1+x^2}\times\dfrac{2}{1+x^2}}{4/(1+x^2)^2} = 0$, by (2) and (3)

$Q_1 = \dfrac{Q}{(dz/dx)^2} = \dfrac{4/(1+x^2)^2}{4/(1+x^2)^2} = 1$ and $R_1 = \dfrac{R}{(dz/dx)^2} = 0$, by (2) and (3)

$\therefore$ From (5), $(d^2y/dz^2) + y = 0$ or $(D_1^2 + 1)y = 0$, where $D_1 \equiv d/dz$.

Its auxiliary equation is $D_1^2 + 1 = 0$ so that $D_1 = \pm i$.

Hence the required solution is $y = c_1 \cos z + c_2 \sin z$, c_1, c_2 being arbitrary constants.

or $y = c_1 \cos(2\tan^{-1}x) + c_2 \sin(2\tan^{-1}x)$. ... (6)

Let $\tan^{-1}x = \theta$ so that $x = \tan\theta$. Then, we have

$\cos(2\tan^{-1}x) = \cos 2\theta = \dfrac{1-\tan^2\theta}{1+\tan^2\theta} = \dfrac{1-x^2}{1+x^2}$, $\sin(2\tan^{-1}x) = \sin 2\theta = \dfrac{2\tan\theta}{1+\tan^2\theta} = \dfrac{2x}{1+x^2}$.

$\therefore$ (6) becomes $y = c_1[(1 - x^2)/(1 + x^2)] + c_2[(2x)/(1 + x^2)]$ or $(1 + x^2)y = c_1(1 - x^2) + 2c_2 x$.

Ex. 4. *Solve* $x^6 y'' + 3x^5 y' + a^2 y = 1/x^2$. **[Delhi Maths (G) 2006; Rajasthan 2010]**

Sol. Dividing by x^6, given equation becomes $\dfrac{d^2y}{dx^2} + \dfrac{3}{x}\dfrac{dy}{dx} + \dfrac{a^2}{x^6}y = \dfrac{1}{x^8}$ ... (1)

Comparing (1) with $y'' + Py' + Q\,y = R$, $P = 3/x$, $Q = a^2/x^6$, $R = 1/x^8$... (2)

Choose z such that $\left(\dfrac{dz}{dx}\right)^2 = \dfrac{a^2}{x^6}$ or $\dfrac{dz}{dx} = \dfrac{a}{x^3}$ so that $z = -\dfrac{a}{2x^2}$. ... (3)

With this change, (1) becomes $(d^2y/dz^2) + P_1(dy/dz) + Q_1 y = R_1$. ... (4)

where $$P_1 = \frac{\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx}}{(dz/dx)^2} = \frac{-\dfrac{3a}{x^4} + \dfrac{3}{x} \times \dfrac{a}{x^3}}{(dz/dx)^2} = 0, \qquad Q_1 = \frac{Q}{(dz/dx)^2} = 1$$

and $$R_1 = \frac{R}{(dz/dx)^2} = \frac{1/x^8}{a^2/x^6} = \frac{1}{a^2x^2} = -\frac{2z}{a^3}, \text{ by (2) and (3).}$$

$\therefore$ (4) gives $(D_1^2 + 1)y = -2z/a^3$. Its auxiliary equation is $D_1^2 + 1 = 0$, where $D_1 \equiv d/dz$

$\therefore$ $D_1 = \pm i$ and hence C.F. $= c_1 \cos z + c_2 \sin z$.

Now, $$\text{P.I.} = \frac{1}{D_1^2+1}\left(-\frac{2z}{a^3}\right) = -\frac{2}{a^3}(1+D_1^2)^{-1} z = -\frac{2}{a^3}(1 - ...)\cdot z = -\frac{2z}{a^3}$$

$\therefore$ Required solution is $y = \text{CF.} + \text{P.I.} = c_1 \cos z + c_2 \sin z - 2z/a^3$

or $$y = c_1 \cos\left(-\frac{a}{2x^2}\right) + c_2 \sin\left(-\frac{a}{2x^2}\right) + \frac{1}{a^2x^2} = c_1 \cos\left(\frac{a}{2x^2}\right) - c_2 \sin\left(-\frac{a}{2x^2}\right) + \frac{1}{a^2x^2}.$$

Ex. 5. *Solve* $xy'' - y' + 4x^3y = x^5$ or $y'' - (1/x)\cdot y' + 4x^2y = x^4$.

[Pune 2001, 05; Andhra 2003; Osmania 2003; Nagpur 1996; Garhwal 2005]

Sol. Given $y'' - (1/x)\cdot y' + 4x^2y = x^4$ ... (1)

Comparing (1) with $y'' + Py' + Q\,y = R$, $P = -1/x$, $Q = 4x^2$, $R = x^4$. ... (2)

Choose z such that $\left(\dfrac{dz}{dx}\right)^2 = 4x^2$ or $\dfrac{dz}{dx} = 2x$ so that $z = x^2$ ... (3)

Then, (1) reduces to $d^2y/dz^2 + P_1\,(dy/dz) + Q_1\,y = R_1$, ... (4)

where $$P_1 = \frac{d^2z/dx^2 + P(dz/dx)}{(dz/dx)^2} = \frac{2 + (-1/x)\times 2x}{4x^2} = 0, \qquad Q_1 = \frac{Q}{(dz/dx)^2} = \frac{4x^2}{4x^2} = 1$$

and $$R_1 = \frac{R}{(dz/dx)^2} = \frac{x^4}{4x^2} = \frac{x^2}{4} = \frac{z}{4}, \text{ by (2) and (3).}$$

$\therefore$ (4) gives $(D_1^2 + 1)y = z/4$, where $D_1 \equiv d/dz$ ... (5)

Auxiliary equation is $D_1^2 + 1 = 0$ giving $D_1 = \pm i$

$\therefore$ C.F. of (5) $= c_1 \cos z + c_2 \sin z$, c_1 and c_2 being arbitrary constants.)

and $$\text{P.I.} = \frac{1}{D_1^2+4}\cdot\frac{z}{4} = \frac{1}{4}(1+D_1^2)^{-1} z = \frac{1}{4}(1 - D_1^2 + ...)z = \frac{z}{4}.$$

$\therefore$ So the required solution is $y = c_1 \cos z + c_2 \sin z + z/4 = c_1 \cos x^2 + c_2 \sin x^2 + x^2/4$

Ex. 6(*a*) *Solve* $\cos x\, y'' + y' \sin x - 2y \cos^3 x = 2 \cos^5 x$.

[Guwahati 2007; Bangalore 1995; Meerut 1997; Purvanchal 2007]

Sol. Dividing by $\cos x$, given equation in standard form is

$$y'' + \tan x \cdot y' - (2\cos^2 x)y = 2\cos^4 x. \quad ...(1)$$

Comparing (1) with $y'' + Py' + Qy = R$, we have

$P = \tan x$, $Q = -2\cos^2 x$ and $R = 2\cos^4 x$. ... (2)

Choose z such that $(dz/dx)^2 = 2\cos^2 x$ or $(dz/dx) = \sqrt{2}\cos x$

$\therefore$ $dz = \sqrt{2}\cos x\, dx$ so that $z = \sqrt{2}\sin x$. ... (3)

With this z, (1) transforms to $(d^2y/dz^2) + P_1\,(dy/dz) + Q_1 y = R_1$, ... (4)

where $$P_1 = \frac{(d^2z/dx^2) + P(dz/dx)}{(dz/dx)^2} = \frac{-\sqrt{2}\sin x + \tan x\cdot\sqrt{2}\cos x}{2\cos^2 x} = 0,$$

$$Q_1 = \frac{Q}{(dz/dx)^2} = -1 \quad \text{and} \quad R_1 = \frac{R}{(dz/dx)^2} = \frac{2\cos^4 x}{2\cos^2 x} = \cos^2 x = 1 - \sin^2 x = 1 - z^2/2, \text{by (2) and (3).}$$

$\therefore$ (4) gives $(D_1^2 - 1)y = 1 - z^2/2$, where $D_1 = d/dz$

Auxiliary equation of (5) is $D_1^2 - 1 = 0$ giving $D_1 = \pm 1$.

$\therefore$ C.F. of (5) $= c_1 e^z + c_2 e^{-z}$, where c_1 and c_2 are arbitrary constants

and $$\text{P.I.} = \frac{1}{D_1^2 - 1}\left[1 - \frac{1}{2}z^2\right] = \frac{1}{D_1^2 - 1}e^{0.z} + \frac{1}{2}\frac{1}{(1 - D_1^2)}z^2 = \frac{1}{0^2 - 1}e^{0\cdot z} + \frac{1}{2}(1 - D_1^2)^{-1}z^2$$

$$= -1 + (1/2)\times(1 + D_1^2 + \ldots)z^2 = -1 + (1/2)\times(z^2 + 2) = z^2/2.$$

Hence the required solution is $y = \text{C.F.} + \text{P.I.}$, *i.e.*, $y = c_1 e^z + c_2 e^{-z} + z^2/2$

or $$y = c_1 e^{\sqrt{2}\sin x} + c_2 e^{-\sqrt{2}\sin x} + \sin^2 x, \text{ as } z = \sqrt{2}\sin x.$$

Remarks. Sometimes a relation between new independent variable z and given independent variable is given in some problems and we are required to transform the given differential equation and hence solve it. We adopt the method explained in the following examples 6(*b*) and 6(d).

Ex. 6(*b*). *Transform the differential equation* $\cos x \cdot y'' + \sin x \cdot y' - 2y\cos^3 x = 2\cos^5 x$ into *the one having z as indpendent variable, where z = sin x and solve it.* **[Himachal 2003]**

Sol. Given that $z = \sin x$ so that $dz/dx = \cos x$. ... (1)

Now $$\frac{dy}{dx} = \frac{dy}{dx}\cdot\frac{dz}{dx} = \cos x\frac{dy}{dz}, \text{ by (1)} \qquad \text{... (2)}$$

and $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dz}\right) = \frac{d}{dx}\left(\cos x\frac{dy}{dz}\right) = -\sin x\frac{dy}{dz} + \cos x\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

$$= -\sin x\frac{dy}{dz} + \cos x\frac{d}{dz}\left(\frac{dy}{dz}\right).\frac{dz}{dx} = -\sin x\frac{dy}{dz} + \cos^2 x\frac{d^2y}{dz^2}, \text{ by (1)} \qquad \text{... (3)}$$

Using (2) and (3), the given equation becomes

$$\cos x\left(-\sin x\frac{dy}{dz} + \cos^2 x\frac{d^2y}{d^2z}\right) + \sin x\cos x\frac{dy}{dz} - 2\cos^3 x\cdot y = 2\cos^5 x$$

or $d^2y/dz^2 - 2y = 2\cos^2 x$ or $d^2y/dz^2 - 2y = 2(1 - \sin^2 x)$

or $(D_1^2 - 2)y = 2(1 - z^2)$, *where* $D_1 \equiv d/dz$ ... (4)

The auxiliary equation of (4) is $D_1^2 - 2 = 0$ so that $D_1 = \pm\sqrt{2}$.

Here, C.F. of (4) $= c_1 e^{z\sqrt{2}} + c_2 e^{-z\sqrt{2}}$, c_1 and c_2 being arbitrary constants

Also P.I. $= 2\dfrac{1}{D_1^2 - 2}(1-z^2) = 2\dfrac{1}{-2(1-D_1^2/2)}(1-z^2)$

$= -(1 - D_1^2/2)^{-1}(1 - z^2) = -(1 + D_1^2/2 + \ldots)(1 - z^2) = -(1 - z^2 - 1) = z^2.$

Hence complete solution of (4) is $\qquad y = c_1 e^{z\sqrt{2}} + c_2 e^{-z\sqrt{2}} + z^2.$

or $\qquad y = c_1 e^{\sqrt{2}\sin x} + c_2 e^{-\sqrt{2}\sin x} + \sin^2 x$, $\quad$ as $z = \sin x$

Ex. 6(*c*). *Solve* $xy'' + (2x^2 - 1)y' - 24x^3 y = 4x^3 \sin x^2$ *using the transformation* $z = x^2$, $x > 0$.

Sol. Try yourself as in Ex. 6(*b*). **[Bangalore 1996]**

Ex. 6(*d*). *Solve* $(1+x^2)^2 y'' + 2x(1+x^2)y' + y = 0$ *using the transformation* $z = \tan^{-1}x$.

[Bangalore 2005]

Sol. Given $\qquad z = \tan^{-1}x \qquad \ldots(1)$

From (1) $\qquad dz/dx = 1/(1 + x^2) \qquad \ldots(2)$

Now, $\qquad \dfrac{dy}{dx} = \dfrac{dy}{dz}\cdot\dfrac{dz}{dx} = \dfrac{1}{1+x^2}\dfrac{dy}{dz} \qquad \ldots(3)$

and $\qquad \dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dx}\left(\dfrac{1}{1+x^2}\dfrac{dy}{dz}\right) = -\dfrac{2x}{(1+x^2)^2}\dfrac{dy}{dz} + \dfrac{1}{1+x^2}\dfrac{d}{dx}\left(\dfrac{dy}{dz}\right)$

$$\frac{d^2y}{dx^2} = -\frac{2x}{(1+x^2)^2}\frac{dy}{dz} + \frac{1}{1+x^2}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx} = -\frac{2x}{(1+x^2)^2}\frac{dy}{dz} + \frac{1}{(1+x^2)^2}\frac{d^2y}{dz^2}, \text{ by (2)}$$

Substituting values of dy/dx and d^2y/dx^2 as given by (3) and (4) in the given equation, we get

$$(1+x^2)^2\left\{\frac{1}{(1+x^2)^2}\frac{d^2y}{dz^2} - \frac{2x}{(1+x^2)^2}\frac{dy}{dz}\right\} + 2x(1+x^2) \times \frac{1}{1+x^2}\frac{dy}{dz} + y = 0$$

or $\qquad d^2y/dz^2 + y = 0 \qquad$ or $\qquad (D_1^2 + 1)y = 0$, *where* $D_1 \equiv d/dz$

whose general solution is $\qquad y = C_1 \cos z + C_2 \sin z, C_1, C_2$ being arbitrary constants.

or $\qquad y = C_1 \cos(\tan^{-1} x) + C_2 \sin(\tan^{-1} x)$, $\quad$ by (1) $\qquad \ldots(5)$

From Trigonomery, $\qquad \tan^{-1} x = \cos^{-1}\left\{1/(1+x^2)^{1/2}\right\} = \sin^{-1}\left\{x/(1+x^2)^{1/2}\right\}$

Hence, from (5), the required solution takes the form

$$y = C_1 \cos\cos^{-1}\left\{1/(1+x^2)^{1/2}\right\} + C_2 \sin\sin^{-1}\left\{x/(1+x^2)^{1/2}\right\}$$

or $\qquad y = C_1/(1 + x^2)^{1/2} + (C_2 x)/(1 + x^2)^{1/2} \qquad$ or $\qquad y(1+x^2)^{1/2} = C_1 + C_2 x$

Ex. 7(*a*) *Solve the equation* $d^2y/dx^2 + (2\cos x + \tan x) \times (dy/dx) + y\cos^2 x = \cos^4 x$ *by changing the independent variable.* **[Gulbarga 2005]**

Sol. Comparing the given equation with $\quad y'' + Py' + Qy = R, \quad$ we have

$P = 2\cos x + \tan x, \qquad Q = \cos^2 x \qquad$ and $\qquad R = \cos^4 x \ldots (1)$

Choose z such that $\qquad (dz/dx)^2 = \cos^2 x \qquad$ or $\qquad dz/dx = \cos x \ldots (2)$

From (2), $\qquad dz = \cos x\, dx \qquad$ so that $\qquad z = \sin x \ldots (3)$

With this value of z, the given equation transforms to

$$d^2y/dz^2 + P_1(dy/dz) + Q_1 y = R_1 \qquad \ldots(4)$$

where $\qquad P_1 = \dfrac{d^2z/dx^2 + P(dz/dx)}{(dz/dx)^2} = \dfrac{-\sin x + (2\cos x + \tan x)\times\cos x}{\cos^2 x} = 2$

$$Q_1 = \frac{Q}{(dz/dx)^2} = 1 \qquad \text{and} \qquad R_1 = \frac{R}{(dz/dx)^2} = \frac{\cos^4 x}{\cos^2 x} = \cos^2 x = 1 - \sin^2 x = 1 - z^2$$

Hence (4) yields $\quad d^2y/dz^2 + 2(dy/dz) + y = 1 - z^2 \quad$ or $\quad (D^2_1 + 2D_1 + 1)y = 1 - z^2$

or $\quad (D_1 + 1)^2 y = 1 - z^2, \quad$ where $\quad D_1 \equiv d/dz. \quad$ ... (5)

The auxiliary equation of (5) is $\quad (D_1 + 1)^2 = 0 \quad$ giving $\quad D_1 = -1, -1$

$\therefore$ C.F. $= (C_1 + C_2 z)e^{-z}$, C_1 and C_2 being arbitrary constants

and $$\text{P.I.} = \frac{1}{(D_1+1)^2}(1-z^2) = (1+D_1)^{-2}(1-z^2) = (1 - 2D_1 + 3D_1^2 + ...)(1-z^2)$$

$= 1 - z^2 - 2D_1(1 - z^2) + 3D_1^2(1 - z^2) + ... = 1 - z^2 - 2 \times (-2z) + 3 \times (-2) = -z^2 + 4z - 5$

Hence the required solution is $\quad y =$ C.F. + P.I $\quad$ or $\quad y = (C_1 + C_2 z)e^{-z} - z^2 + 4z - 5$

or $\quad y = (C_1 + C_2 \sin x)e^{-\sin x} - \sin^2 x + 4\sin x - 5, \quad$ as $z = \sin x$

Ex. 7(*b*) *Solve* $x(d^2y/dx^2) - (dy/dx) - 4x^2 y = 8x^3 \sin x^2$.

[Kanpur 2002, Rohilkhand 2001; Bangalore 2005; Garhwal 1994; Vikram 2001, 05]
[Gurukul Kangri U. 2004; Agra 2005]

Sol. Dividing by x, $\quad (d^2y/dx^2) - (1/x)(dy/dx) - 4x^2 y = 8x^2 \sin x^2. \quad$ ... (1)

Comparing (1) with $\quad y'' + Py' + Qy = R, \quad$ we have

$P = -1/x, \quad Q = -4x^2, \quad$ and $\quad R = 8x^2 \sin x^2$. ... (2)

Choose z such that $\quad (dz/dx)^2 = 4x^2 \quad$ or $\quad (dz/dx) = 2x \quad$ so that $\quad z = x^2$. ... (3)

As usual, $\quad P_1 = 0, \quad Q_1 = -1, \quad R_1 = (8x^2 \sin x^2)/4x^2 = 2\sin x^2 = 2\sin z.$

$\therefore$ You will get $\quad (D_1^2 - 1)y = 2 \sin z. \quad$ whose $\quad$ C.F. $= c_1 e^z + c_2 e^{-z}$

and $$\text{P.I.} = \frac{1}{D_1^2 - 1} 2\sin z = 2\frac{1}{-1^2 - 1}\sin z = -\sin z.$$

$\therefore$ The required solution is $\quad y = c_1 e^z + c_2^{-z} - \sin z = c_1 e^{x^2} + c_2 e^{-x^2} - \sin x^2.$

Ex. 8(*a*). *Solve* $\quad y'' - y' \cot x - y \sin^2 x = \cos x - \cos^3 x.$

(*b*) *Solve* $\quad y'' - y' \cot x - y \sin^2 x = 0.$ **[Rohilkhand 1996]**

Sol. (*a*) Comparing the given equation with $\quad y'' + Py' + Qy = R, \quad$ we have

$P = -\cot x, \quad Q = -\sin^2 x \quad$ and $\quad R = \cos x - \cos^3 x = \cos x \sin^2 x. \quad$ ... (1)

Choose z such that $\quad (dz/dx)^2 = \sin^2 x \quad$ and $\quad dz/dx = \sin x$... (2)

Integrating, $\quad z = \int \sin x \, dx \quad$ or $\quad z = -\cos x$.... (3)

Now changing the independent variable form x to z by using relation (3), the given equaiton is transformed into $\quad (d^2y/dz^2) + P_1(dy/dz) + Q_1 y = R_1, \quad$ ... (4)

where $$P_1 = \frac{d^2z/dx^2 + P(dz/dx)}{(dz/dx)^2} = \frac{\cos x + (-\cot x)(\sin x)}{\sin^2 x} = 0, \quad \text{by (1) and (2)}$$

$$Q_1 = \frac{Q}{(dz/dx)^2} = \frac{(-\sin^2 x)}{\sin^2 x} = -1, \qquad R_1 = \frac{R}{(dz/dx)^2} = \frac{\cos x \cdot \sin^2 x}{\sin^2 x} = \cos x = -z.$$

$\therefore$ From (5), $\quad (d^2y/dz^2) - y = -z \quad$ or $\quad (D^2_1 - 1)y = -z, \quad$ where $D_1 \equiv d/dz$, ... (5)

Its auxiliary equation is $\quad D_1^2 - 1 = 0 \quad$ so that $\quad D_I = \pm 1$

$\therefore \quad$ C.F. of (5) is $= c_1 e^z + c_2 e^{-z} = c_1 e^{-\cos x} + c_2 e^{\cos x}, \quad$ by (3)

$$\text{P.I.} = \frac{1}{D_1^2 - 1}(-z) = \frac{1}{1 - D_1^2} z = (1 - D_1^2)^{-1} z = (1 + D_1^2 + ...)z = z = -\cos x.$$

Hence the required solution is $\quad y = c_1 e^{-\cos x} + c_2 e^{\cos x} - \cos x.$

(*b*) Try yourself as in part (*a*). **Ans.** $y = c_1 e^{-\cos x} + c_2 e^{\cos x}$

Ex. 9. *Solve* (*a*) $(1+x)^2 (d^2y/dx^2) + (1+x)(dy/dx) + y = 4 \cos \log (1+x)$.

[Delhi Maths (Hons.) 1993]

(*b*) $(1+x)^2(d^2y/dx^2)+(1+x)(dy/dx)+y=4\sin\log(1+x)$.

Sol. (*a*) Dividing by $(1+x)^2$, the given equation in standard form is

$$\frac{d^2y}{dx^2}+\frac{1}{1+x}\frac{dy}{dx}+\frac{1}{(1+x)^2}y=\frac{4\cos\log(1+x)}{(1+x)^2}. \quad \text{... (1)}$$

Comparing (1) with $y'' + Py' + Qy = R$, we have

$P = (1+x)^{-1}$, $Q = (1+x)^{-2}$ and $R = 4(1+x)^{-2} \cos \log(1+x)$. ... (2)

Choose z such that $(dz/dx)^2 = 1/(1+x)^2$ so that $dz/dx = 1/(1+x)$ ... (3)

Integrating it $z = \int \frac{1}{1+x}dx$ or $z = \log(1+x)$. ... (4)

With this z, (1) reduces to $(d^2y/dz^2) + P_1(dy/dz) + Q_1 y = R_1$, ... (5)

where $$P_1 = \frac{(d^2z/dx^2)+P(dz/dx)}{(dz/dx)^2}=\frac{-(1+x)^{-2}+(1+x)^{-1}(1+x)^{-1}}{(1+x)^{-2}}=0,$$

$$Q_1 = \frac{Q}{(dz/dx)^2}=\frac{(1+x)^{-2}}{(1+x)^{-2}}=1, \qquad R_1 = \frac{R}{(dz/dx)^2}=\frac{4(1+x)^{-2}\cos\log(1+x)}{(1+x)^{-2}}=4\cos z$$

∴ From (5), $(d^2y/dz^2) + y = 4\cos z$ or $(D_1^2+1)y = 4\cos z$, where $D_1 \equiv d/dz$. ... (6)

Its auxiliary equation is $D_1^2 + 1 = 0$ so that $D_1 = \pm i$

∴ C.F. of (6) $= c_1 \cos z + c_2 \sin z = c_1 \cos \log(1+x) + c_2 \sin \log(1+x)$, by (4)

$$\text{P.I.} = \frac{1}{D_1^2+1}4\cos z = 4\frac{1}{D_1^2+1^2}\cos z = 4\frac{z}{(2\times 1)}\sin z, \quad \text{as} \quad \frac{1}{D_1^2+a^2}\cos az = \frac{z}{2a}\sin az$$

$= 2\log(1+x)\sin\log(1+x)$, using (4)

Hence the required general solution is $y = \text{C.F.} + \text{P.I.}$, *i.e.*

$y = c_1 \cos \log(1+x) + c_2 \sin \log(1+x) + 2\log(1+x)\sin\log(1+x)$.

(*b*) Try yourself as in part (a) Use the result $\frac{1}{D_1^2+a^2}\sin az = -\frac{z}{2a}\cos az$, $D_1 \equiv \frac{d}{dz}$

Ans. $y = c_1 \cos\log(1+x) + c_2 \sin\log(1+x) - 2\log(1+x)\cos\log(1+x)$

Ex. 10. *Solve* $(d^2y/dx^2) + (\tan x - 1)^2 (dy/dx) - n(n-1)y\sec^4 x = 0$.

Sol. Comparing the given equation with $y'' + Py' + Qy = R$, we get

$P = (\tan x - 1)^2$, $Q = -n(n-1)\sec^4 x$ and $R = 0$. ... (1)

Choose z such that $(dz/dx)^2 = \sec^4 x$ ... (2)

or $dz/dx = \sec^2 x$ so that $z = \tan x$ ... (3)

With this z, the given equation becomes $(d^2y/dz^2) + P_1(dy/dz) + Q_1 y = R_1$, ... (4)

where $$P_1 = \frac{(d^2z/dx^2)+P(dz/dx)}{(dz/dx)^2}=\frac{2\sec^2 x\tan x+(\tan x-1)^2\sec^2 x}{\sec^4 x}=1,$$

$$Q_1 = \frac{Q}{(dz/dx)^2}=\frac{-n(n-1)\sec^4 x}{\sec^4 x}=-n(n-1) \quad \text{and} \quad R_1 = \frac{R}{(dz/dx)^2}=0.$$

∴ From (4), $(d^2y/dz^2) + (dy/dz) - n(n-1)y = 0$ or $[D_1^2 + D_1 - n(n+1)]y = 0$, $D_1 \equiv d/dz$

Its auxiliary eqaution is $D_1^2 + D_1 - n(n-1) = 0$ or $(D_1^2 - n^2) + (D_1 + n) = 0$

or $(D_1 + n)(D_1 - n) + (D_1 + n) = 0$ or $(D_1 + n)(D_1 - n + 1) = 0$ so that $D_1 = -n, n-1$

$\therefore$ The required solution is $y = c_1 e^{-nz} + c_2 e^{(n-1)z} = c_1 e^{-n\tan x} + c_2 e^{(n-1)\tan x}$, as $z = \tan x$.

EXERCISE 10(C)

Solve the following differential equations :

1. $y'' + y' \tan x + y \cos^2 x = 0$. **[Delhi Maths (G) 2004; Meerut 1996; Mysore 2004]**

Ans. $y = c_1 \cos(\sin x) + c_2 \sin(\sin x)$

2. $xy'' + (4x^2 - 1)y' + 4x^3 y = 2x^3$. **Ans.** $y = e^{-x^2}(c_1 + c_2 x^2) + (1/2)$

3. $(x^3 - x)y'' + y' + n^2 x^3 y = 0$. **Ans.** $y = c_1 \cos\{n(x^2-1)^{1/2}\} + c_2 \sin\{n(x^2-1)^{1/2}\}$.

4. $y'' + (\tan x - 3\cos x)y' + 2y\cos^2 x = \cos^4 x$.

Ans. $y = c_1 e^{\sin x} + c_2 e^{-\sin x} - (5/4) - (3/2)\times\sin x - (1/2)\times\sin^2 x$.

5. (*a*) $(a^2 - x^2)y'' - (a^2/x)y' + (x^2/a)y = 0$. **[BundelKhand 2001]**

Ans. $y = c_1 \cos\{(a^2 - x^2)/a\}^{1/2} + c_2 \sin\{(a^2 - x^2)/a\}^{1/2}$

(*b*) $(81 - x^2)y'' - (81/x)y' + (x^2/9)y = 0$. **Ans.** $y = c_1 \cos\{(81 - x^2)^{1/2}/9\} + c_2 \sin\{(81 - x^2)^{1/2}/9\}$

6. $y'' + (3\sin x - \cot x)y' + 2y\sin^2 x = e^{-\cos x}\sin^2 x$. **Ans.** $y = c_1 e^{\cos x} + c_2 e^{2\cos x} + (1/6)\times e^{-\cos x}$

7. $y'' - (1 + 4e^x)y' + 3e^{2x}y = e^{2(x+e^x)}$. **[Agra 2006;Kanpur 1997]** **Ans.** $y = c_1 e^{3e^x} + c_2 e^{e^x} - e^{2e^x}$

8. $y'' + (1 - 1/x)y' + 4x^2 e^{-2x} y = 4(x^2 + x^3)e^{-3x}$.

Ans. $y = c_1 \cos\{2e^{-x}(1+x)\} + c_2 \sin\{2e^{-x}(1+x)\} + e^x(1+x)$.

9. $y'' - (8e^{2x} + 2)y' + 4e^{4x}y = e^{6x}$. **Ans.** $y = e^{2x}[c_1\cos(x\sqrt{3}) + c_2\sin(x\sqrt{3})] + (1/4)\times e^{2x} + 1$

10. 12 An important theorem.

If $y = y_1(x)$ and $y = y_2(x)$ are two solutions of the equation $(d^2y/dx^2) + P(x)(dy/dx) + Q(x)y = 0$, where $P(x)$, $Q(x)$ are continuous function of x, prove that

$$y_1 \frac{dy_2}{dx} - y_2 \frac{dy_1}{dx} = c\, e^{-\int P\,dx}, \text{ c being an arbitrary constant.}$$

[Himachal 2000; Kalkata 2001, 03 05, 07; Kurukshetra 200, 03; Allahabad 2002, 04, 07; Lucknow 2001, 04]

Proof. Since y_1 and y_3 are solutions of the given equation, we have

$$\frac{d^2y_1}{dx^2} + P(x)\frac{dy_1}{dx} + Q(x)y_1 = 0. \qquad \text{... (1)}$$

and
$$\frac{d^2y_2}{dx^2} + P(x)\frac{dy_2}{dx} + Q(x)y_2 = 0. \qquad \text{... (2)}$$

Multiplying (1) by y_2 and (2) by y_1 and then substracting, we get

$$y_1 \frac{d^2y_2}{dx^2} - y_2 \frac{d^2y_1}{dx^2} + P\left(y_1 \frac{dy_2}{dx} - y_2 \frac{dy_1}{dx}\right) = 0. \qquad \text{... (3)}$$

Let
$$y_1 \frac{dy_2}{dx} - y_2 \frac{dy_1}{dx} = v. \qquad \text{... (4)}$$

Differentiating both sides of (4) w.r.t. x, we get

$$\left(y_1\frac{d^2y_2}{dx^2}+\frac{dy_1}{dx}\frac{dy_2}{dx}\right)-\left(y_2\frac{d^2y_1}{dx}+\frac{dy_2}{dx}\frac{dy_1}{dx}\right)=\frac{dv}{dx} \qquad \text{or} \qquad y_1\frac{d^2y_2}{dx^2}-y_2\frac{d^2y_1}{dx^2}=\frac{dv}{dx}.$$

$\therefore$ (3) becomes $\qquad \dfrac{dv}{dx}+Pv=0 \qquad$ or $\qquad \dfrac{dv}{v}=-Pdx.$

Integrating, $\qquad \log v - \log c = -\int P\,dx \qquad$ or $\qquad v = ce^{-\int P\,dx}$

or $$y_1\frac{dy_2}{dx}-y_2\frac{dy_1}{dx}=ce^{-\int P\,dx}, \text{ using (4)}$$

10.13 Method of variation of parameters

We have already explained the method of variation of parameters for solving $dy^2/dx^2 + P(dy/dx) + Qy = R$, where P,Q and R are functions of x in Art 7.3, Art. 7.4*A* and Art. 7.4*B* in chapter 7. So far we have used the method of variation of parameters to solve linear differential equations with constant coefficients or Cauchy-Euler equations (refer chapter 7). In this article, we purpose to solve differential equations whose complementary function can be obtained by methods of the present chapter However, it should be carefully noted that the method of variation of parameters is used when

(*i*) The solution of $y_2 + Py_1 + Qy = R$ coannot be obtained by mehtods explained in Art. 10.3, Art. 10.7 and Art. 10.10.

(*ii*) You are asked in solve a given equation by using variation of parameters.

10.14 Solved examples based on Art. 10.13

Ex. 1. *Verify that e^x and x are solutions of the homogeneous equation corresponding to* $(1 - x)y_2 + xy_1 - y = 2(x - 1)^2e^{-x}$, $0 < x < 1$. *Thus find its general solution.*

Sol. The given equation in standard form is

$$y_2 + [x/(1 - x)]y_1 - y/(1 - x) = 2(1 - x)e^{-x}. \qquad \text{... (1)}$$

Consider $$y_2 + [x/(1 - x)]y_1 - y/(1 - x) = 0 \qquad \text{... (2)}$$

which is said to be the homogeneous equaiton corresponding to (1). Take $y = e^x$ so that $y_1 = e^x$, $y_2 = e^x$. With these values,

$$\text{the L.H.S. of (2)} = e^x+\frac{x}{1-x}e^x-\frac{e^x}{1-x}=e^x\,\frac{1-x+x-1}{1-x}=0.$$

So e^x is a solution of (2). Next, take $y = x$ so that $y_1 = 1$, $y_2 = 0$. with these values,

$$\text{the L.H.S. of (2)} = 0+\frac{1}{1+x}x-\frac{x}{1-x}=0.$$

So x is also a solution of (2). Now, the Wronskian $W(e^x, x)$ of e^x and x is given by

$$W(e^x, x) = \begin{vmatrix} e^x & x \\ e^x & 1 \end{vmatrix} = e^x - xe^x \neq 0$$

Hence e^x and x are linearly independent solutions of (2) [refer chapter 1]. Hence the general solution of (2) is $y = ae^x + bx$. So the C.F. of (1) is $ae^x + bx$, a and b being arbitrary constants. We shall now use the method discussed in Art. 7.4B of chapter 7

Let $$y = Ae^x + Bx \qquad \text{... (3)}$$

be the complete solution of (1). Then A and B are functions of x which are so chosen that (1) will be satisfied. Differentiating (3), we get

$$y_1 = A_1 e^x + Ae^x + B_1 x + \mathrm{B}. \qquad \ldots (4)$$

Choose A and B such that $\quad A_1 e^x + B_1 x = 0. \qquad \ldots (5)$

Then (4) reduces to $\quad y_1 = Ae^x + B. \qquad \ldots (6)$

Differentiating (6), $\quad y_2 = A_1 e^x + Ae^x + B_1$, where $\mathrm{A}_1 = \mathrm{dA/dx}$, $\mathrm{B}_1 = \mathrm{dB/dx} \qquad \ldots (7)$

Using (3), (6) and (7), (1) reduces to $\quad A_1 e^x + B_1 = 2(1-x)e^{-x}. \qquad \ldots (8)$

Subtracting (5) from (8), we have

$(1-x)B_1 = 2(1-x)e^{-x}$ or $B_1 = dB/dx = 2e^{-x}$ so that $B = c_1 - 2e^{-x}$.

Then (5) gives $\quad A_1 = dA/dx = (-2\mathrm{x}e^{-x})/\mathrm{e}^x = -2xe^{-2x}$.

Integrating and using the chain rule of integration by parts, we have

$$A = -2\int xe^{-2x}dx + c_2 = c_2 - 2\left[(x)\left(-\frac{1}{2}e^{-2x}\right) - (1)\left(\frac{1}{4}e^{-2x}\right)\right] = c_2 + e^{-2x}\left(x + \frac{1}{2}\right)$$

Putting the value of A and B in (3), the required solution is

$$y = [c_2 + e^{-2x}(x + 1/2)]e^x + (c_1 - 2e^{-x})x = c_1 x + c_2 e^x + e^{-x}[(1/2) - x]$$

Ex. 2. *(i) Using the method variation of parameters, solve the differential equation $(x-1)D^2y - xDy + y = (x-1)^2$, where $D \equiv d/dx$.* **[Guwahati 2007]**

(ii) Apply the method of variation of parameters to solve $(x-1)y_2 - xy_1 + y = (x-1)^2$, given that the integrals in the complementary function are x and e^x

Sol. (i) The given equation in the standard form $y_2 + Py_1 + Qy = R$ *is*

$$y_2 - \frac{x}{x-1}y_1 + \frac{1}{x-1}y = x-1. \qquad \ldots (1)$$

Consider
$$y_2 - \frac{x}{x-1}y_1 + \frac{1}{x-1}y = 0. \qquad \ldots (2)$$

Comparing (2) with $\quad y_2 + Py_1 + Qy = R$, we have

$P = x/(x-1)$, $\quad Q = 1/(x-1)$. We easily verify that

$$P + Qx = -\frac{x}{x-1} + \frac{x}{x-1} = 0 \quad \text{and} \quad 1 + P + Q = 1 - \frac{x}{x-1} + \frac{1}{x-1} - 0.$$

So x and e^x are integrals of C.F. (1) or solutions of (2). [See Art. 10.3].Again, we have

$$\text{Wronskian of } e^x \text{ and } x = W(e^x, x) = \begin{vmatrix} e^x & x \\ e^x & 1 \end{vmatrix} = e^x - xe^x \neq 0.$$

Hence, e^x and x are linearly independent solutions of (2) [See chapter 1].

Hence, the general solution of (2) is $y = ae^x + bx$. So the C.F. of (1) is $ae^x + bx$, a and b being arbitrary constants. We now use Art. 7.4B of chapter 7.

Let $\quad y = Ae^x + Bx \qquad \ldots (3)$

be the complete solution of (1). Then A and B are functions of x which are so chosen that (1) will be satisfied. Differentiatin (3), we get

$$y_1 = A_1 e^x + Ae^x + B_1 x + B. \qquad \ldots (4)$$

Choose A and B such that $\quad A_1 e^x + B_1 x = 0. \qquad \ldots (5)$

Then (4) reduces to $\quad y_1 = Ae^x + B. \qquad \ldots (6)$

Differentiating (6), $\quad y_2 = A_1 e^x + Ae^x + B_1$, where $A_1 = dA/dx$, $B_1 = dB/dx \qquad \ldots (7)$

Using (3), (6) and (7), (1) reduces to $\quad A_1 e^x + B_1 = x - 1. \qquad \ldots (8)$

Subtracting (5) from (8), we have $(1 - x)B_1 = x - 1$

Hence $B_1 = dB/dx = -1$ so that $B = c_1 - x$.

Then (5) gives $A_1 = dA/dx = x/e^x = xe^{-x}$.

Integrating, $A = c_2 + \int xe^{-x}dx = c_2 + (x)(-e^{-x}) - \int \{1\cdot(-e^{-x})\}\, dx = c_2 - e^{-x}(x+1)$

Putting the above values of A and B in (3), the required solution is

$$y = [c_2 - e^{-x}(x + 1)]e^x + (c_1 - x)x = c_1x + c_2e^x - (x^2 + x + 1).$$

Part (*ii*). Since x and e^x are integrals in C.F. So $ae^x + bx$ is C.F. of (1). So now start from equation (3) onwards as in part (*i*).

Ex. 3. *Apply the method of variation of parameters to solve the equation*

$$(x + 2)y_2 - (2x + 5)y_1 + 2y = (x + 1)e^x.$$ **[Kanpur 2009]**

Or *Solve* $(x + 2)y_2 - (2x + 5)y_1 + 2y = (x + 1)e^x$ by method of variation of parameters when C.F. is $a\,(2x + 5) + be^{2x}$. **[Kakitya 1997]**

Sol. Putting the given equation in standard form $y_2 + Py_1 + Q\,y = R$, we get

$$y_2 - \frac{2x+5}{x+2}y_1 + \frac{2}{x+2}y = \frac{x+1}{x+2}e^x. \qquad \text{... (1)}$$

Consider $$y_2 - \frac{2x+5}{x+2}y_1 + \frac{2}{x+2}y = 0. \qquad \text{... (2)}$$

Comparing (2) with $y_2 + Py_1 + Q\,y = R$, we have

$P = -(2x + 5)/(x + 2)$, $Q = 2/(x + 2)$ and $R = 0$.

Here $$2^2 + 2P + Q = 4 - \frac{2(2x+5)}{x+2} + \frac{2}{x+2} = 0.$$

Hence $u = e^{2x}$ [See Art. 10.3] is an integral of (2). We now use method of Art. 10.4 to find solution of (2). Let the complete solution of (2) be $y = uv$. Then (2) reduces to

$$\frac{d^2v}{dx^2} + \left[P + \frac{2}{u}\frac{du}{dx}\right]\frac{dv}{dx} = \frac{R}{u} \quad \text{or} \quad \frac{d^2v}{dx^2} + \left[-\frac{2x+5}{x+2} + \frac{1}{e^{2x}} \times 2e^{2x}\right]\frac{dv}{dx} = 0$$

or $$\frac{d^2v}{dx^2} + \frac{2x+3}{2x+2}\frac{dv}{dx} = 0. \qquad \text{... (2)}'$$

Putting $dv/dx = q$ so that $d^2v / dx^2 = dq/dx$, (2)′ becomes

$$\frac{dq}{dx} + \left[2 - \frac{1}{x+2}\right]q = 0 \quad \text{or} \quad \frac{dq}{q} + \left[2 - \frac{1}{x+2}\right]dx = 0$$

Integrating, $\log q - \log a' - \log (x + 2) = a'$ being an arbitrary constant

or $q = a'\,(x + 2)e^{-2x}$ or $dv/dx = a'\,(x + 2)e^{-2x}$

Integrating by chain rule of integration by parts, we have

or $$v = a'\left[(x+2)\left(-\frac{1}{2}e^{-2x}\right) - (1)\left(\frac{1}{4}e^{-2x}\right)\right] + b$$

or $$v = -(a'/4) \times e^{-2x}\,(2x+4+1) + b = a\,(2x+5) + b, \textit{where}\; a = -(a'/4)$$

Hence solution of (2) is $$y = uv = e^{2x}\left\{a\,(2x+5)\,e^{-2x} + b\right\} = a\,(2x+5) + be^{2x}$$

Thus, $a(2x + 5) + be^{2x}$ is C.F. of (1), a and b being arbitrary constants.

Let $$y = A(2x + 5) + Be^{2x}. \qquad \text{... (3)}$$

be the completet solution of (1). Then A and B are functions of x which are so chosen that (1) will be satisfied. Differentiating (3), we get

$$y_1 = A_1(2x + 5) + 2A + B_1e^{2x} + 2Be^{2x}. \quad ...(4)$$

Choose A and B such that $\quad A_1(2x + 5) + B_1e^{2x} = 0. \quad ...(5)$

Then (4) reduces to $\quad y_1 = 2A + 2Be^{2x} \quad ...(6)$

Differentiating (6), $\quad y_2 = 2A_1 + 2B_1e^{2x} + 4Be^{2x}. \quad ...(7)$

Using (3), (6) and (7), (1) reduces to $\quad 2A_1 + 2B_1e^{2x} = [(x + 1)/(x + 2)]e^x. \quad ...(8)$

Multiplying (5) by 2 and subtracting it from (8), we get

$$A_1(-4x - 8) = \frac{x+1}{x+2}e^x \quad \text{or} \quad A_1 = \frac{dA}{dx} = -\frac{(x+1)}{4(x+2)^2}e^x. \quad ...(9)$$

Integrating, $\quad A = -\frac{1}{4}\int \frac{x+1}{(x+2)^2}e^x\,dx + c_1$, c_1 being an arbitrary constant

or $$A = c_1 - \frac{1}{4}\int \frac{(x+2)-1}{(x+2)^2}e^x\,dx = c_1 - \frac{1}{4}\int e^x[(x+2)^{-1} - (x+2)^{-2}]dx$$

or $$A = c_1 - (1/4)\times e^x(x+2)^{-1}, \qquad \text{as} \qquad \int e^x[f(x) + f'(x)]\,dx = e^x f(x) \quad ...(10)$$

From (5) and (9), $$B_1 = \frac{dB}{dx} = \frac{(2x+5)(x+1)e^{-x}}{4(x+2)^2} = \frac{(2x^2+7x+5)e^{-x}}{4(x+2)^2} = \frac{2(x+2)^2 - (x+3)}{4(x+2)^2}e^{-x}$$

or $$\frac{dB}{dx} = \frac{1}{2}e^{-x} - \frac{(x+2)+1}{4(x+2)^2}e^{-x} = \frac{1}{2}e^{-x} + \frac{1}{4}e^{-x}[-(x+2)^{-1} - (x+2)^{-2}]$$

Integrating, $B = c_2 - (1/2)\times e^{-x} + (1/4)\times e^{-x}(x+2)^{-1}$, c_2 being an arbitrary constant ... (11)

[Using formula $\int e^{ax}[af(x) + f'(x)]dx = e^{ax}f(x)$ for $a = -1$]

Using (10) and (11) in (3), the required solution is

$$y = \left[c_1 - (1/4)\times e^x(x+2)^{-1}\right](2x+5) + \left[c_2 - (1/2)\times e^{-x} + (1/4)\times e^{-x}(x+2)^{-1}\right]e^{2x}$$

or $$y = c_1(2x+5) + c_2e^{2x} + \frac{1}{4}e^x\left[\frac{1}{x+2} - 2 - \frac{2x+5}{x+2}\right] = c_1(2x + 5) + c_2e^{2x} - e^x$$

Ex. 4. *Apply the method of variation of parameters to solve* $(1 - x)y_2 + xy_1 - y = (1 - x)^2$

[Delhi Maths (G) 2006; Kanpur 2005]

Sol. Dividing by $(1 - x)$ and re-writing, the given equation becomes

$$y_2 - \{x/(x - 1)\}y_1 + \{1/(x - 1)\}\,y = -(x - 1) \quad ...(1)$$

Consider $\quad y_2 - \{x/(x - 1)\}y_1 + \{1/(x - 1)\}y = 0 \quad ...(2)$

Comparing (2) with $y_2 + Py_1 + Q\,y = 0$, $\quad P = (-x)/(x - 1) \quad$ and $\quad Q = 1/(x - 1)$.

$\therefore\ P + Qx = (-x)/(x-1) + x/(x-1) = 0 \quad$ and $\quad 1 + P + Q = 1 + (-x)/(x - 1) + 1/(x - 1) = 0$.

Hence by working rule 10.4, we see that x and e^x are integrals of C.F. of (1) or solutions of (2). Again the Wronkian W of x and e^x is given by

$$W = \begin{vmatrix} x & e^x \\ dx/dx & d(e^x)/dx \end{vmatrix} = \begin{vmatrix} x & e^x \\ 1 & e^x \end{vmatrix} = e^x(x-1) \neq 0, \quad ...(3)$$

showing that x and e^x are linearly independent solutions of (2). Hence the general solution of (2) is $y = ax + be^x$ and therefore C.F. of (1) is $ax + be^x$, a and b being arbitrary constants.

We now use working rule 7.4 A of chapter 7.

Comparing (1) with $y_2 + Py_1 + Q\,y = R,$ here $R = -(x-1)$ ... (4)

Let $u = x$ and $v = e^x$ ... (5)

Then, P.I. of (1) $= u\,f(x) + vg(x),$ ... (6)

where $f(x) = -\int \frac{vR}{W}dx = \int \frac{e^x(x-1)}{e^x(x-1)}dx = \int dx = x$ by (3), (4) and (5)

and $g(x) = \int \frac{uR}{W}dx = -\int \frac{x(x-1)}{e^x(x-1)}dx = -\int xe^{-x}dx$

$$= -\left\{x(-e^{-x}) - \int(-e^{-x})dx\right\} = -(-xe^{-x} - e^{-x}) = e^{-x}(x+1)$$

Substituting the above values of $u, v, f(x)$ and $g(x)$ in (6), we have

P.I. of (7) $= x \cdot x + e^x \cdot e^{-x}(x+1) = x^2 + x + 1$. Hence the general solution of (1) is

$y =$ C.F + P.I., *i.e.* $y = ax + be^x + x^2 + x + 1$, a and b being arbitrary constants.

Ex. 5. *Solve by the method of variation of parameters* $x(dy/dx) - y = (x-1)(d^2y/dx^2 - x + 1)$ **[I.A.S. 2000]**

Sol. Re-writing the given equation, we have

$xy_1 - y = (x-1)y_2 - (x-1)^2$ or $y_2 - \{x/(x-1)\}y_1 + \{1/(x-1)\}y = x - 1$ ... (1)

Consider $y_2 - \{x/(x-1)\}y_1 + \{1/(x-1)\}y = 0$ ... (2)

Comparing (2) with $y_2 + Py_1 + Q\,y = 0,$ here $P = (-x)/(x-1)$ and $Q = 1/(x-1)$.

Then, $P + Qx = (-x)/(x-1) + x/(x-1) = 0,\ 1 + P + Q = 1 + (-x)/(1-x) + 1/(x-1) = 0.$

Hence by working rule 10.4, we see that x and e^x are integrals of C.F. of (1) or solutions of (2). Again the Wronikian W of x and e^x is given by

$$W = \begin{vmatrix} x & e^x \\ dx/dx & d(e^x)/dx \end{vmatrix} = \begin{vmatrix} x & e^x \\ 1 & e^x \end{vmatrix} = e^x(x-1) \neq 0, \qquad \text{... (3)}$$

showing that x and e^x are linearly independent solutions of (2).

Hence, the general solution of (2) is $y = ax + b\,e^x$ and therefore C.F. of (1) is $ax + b\,e^x$, a and b being arbitrary constants. We now use working rule 7.4B of chapter 7.

Comparing (1) with $y_2 + Py_1 + Q\,y = R,$ here $R = x - 1.$ ... (4)

Let $u = x$ and $v = e^x.$ ... (5)

Then, P. I. of (1) $= u\,f(x) + v\,g(x),$ ... (6)

where $f(x) = -\int \frac{vR}{W}dx = -\int \frac{e^x(x-1)}{e^x(x-1)}dx = -\int dx = -x$, using (2), (4) and (5)

and $g(x) = \int \frac{uR}{W}dx = \int \frac{x(x-1)}{e^x(x-1)}dx = \int xe^{-x}dx$, by (2), (4) and (5)

$$= x(-e^{-x}) - \int 1\cdot(-e^{-x})dx = -xe^{-x} - e^{-x} = -e^{-x}(x+1)$$

Substituting the above values of $u, v, f(x)$ and $g(x)$ in (6), we have

$$\text{P.I. of (1)} = x \times (-x) + e^x\{-e^{-x}(x+1)\} = -(x^2 + x + 1)$$

Hence the general solution of (1) is $y =$ C.F. + P.I., $y = ax + b\,e^x - (x^2 + x + 1).$

Ex. 6 *Solve by the method of variation of parameters* $d^2y/dx^2 + (1 - \cot x)(dy/dx) - y \cot x = \sin^2 x$
[Agra 1995, 99; Garhwal 1997; Meerut 1999; Rohilkhanad 1997]

Sol. Given $$y'' + (1 - \cot x)y' - y\cot x = \sin^2 x \qquad \text{... (1)}$$

First we shall find C.F. of (1), *i.e.*, solution of $$y'' + (1 - \cot x)y' - y\cot x = 0 \qquad \text{... (2)}$$

Comparing (2) with $y'' + Py' + Qy = 0$, here $P = 1 - \cot x$, $Q = -\cot x$, $R = o$ ...(3)

$\therefore\ 1 - P + Q = 1 - (1 - \cot x) - \cot x = 0$, showing that $u = e^{-x}$ ... (4)
is a part of C.F. of (2)

Let the complete solution of (2) be $$y = uv \qquad \text{... (5)}$$

Then v is given by $$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$$

or $$\frac{d^2v}{dx^2} + \{1 - \cot x + \frac{2}{x^{-x}} \times (-e^{-x})\}\frac{dv}{dx} = 0, \text{ using (3) and (4)}$$

or $$(d^2v/dx^2) - (1 + \cot x)(dv/dx) = 0 \qquad \text{... (6)}$$

Let $dv/dx = q$ so that $d^2v/dx^2 = dq/dx$ ... (7)

Then (6) becomes $dq/dx - (1 + \cot x)q = 0$ or $(1/q)dq = (1 + \cot x)dx$

Intergrating it, $\log q - \log c_1 = x + \log \sin x$ or $q/(c_1 \sin x) = e^x$

or $q = c_1 e^x \sin x$ or $dv/dx = c_1 e^x \sin x$

or $dv = c_1 e^x \sin x\, dx$ so that $v = c_1 \int e^x \sin x\, dx + c_2$

or $v = \frac{1}{2} c_1 e^x (\sin x - \cos x) + c_2$, as $\int e^{ax} \sin bx\, dx = \frac{e^{ax}}{a^2 + b^2}(a \sin bx - b \cos bx)$

or $v = c_1' e^x (\sin x - \cos x) + c_2$, where $c_1' = c_1/2$ ... (8)

From (4) (5) (8), the complete solution of (2), *i.e.*, C.F. of (1) is given by

$$v = e^{-x}\{c_1' e^x (\sin x - \cos x) + c_2\} = c_1'(\sin x - \cos x) + c_2 e^{-x}$$

Let $$y = A(\sin x - \cos x) + Be^{-x} \qquad \text{... (9)}$$

be the complete solution of (1). Then, A and B are functions of x which are so chosen that (1) will be satisfied. Differentiating (9) w.r.t 'x', we have

$$y' = A(\cos x + \sin x) + A_1(\sin x - \cos x) - Be^{-x} + B_1 e^{-x}, \qquad \text{... (10)}$$

where $A_1 = dA/dx$ $B_1 = dB/dx$. Choose A and B such that

$$A_1(\sin x - \cos x) + B_1 e^{-x} = 0 \qquad \text{... (11)}$$

Then (10) reduces to $$y' = A(\cos x + \sin x) - Be^{-x} \qquad \text{... (12)}$$

Differentiating (12), $$y'' = A_1(\cos x + \sin x) + A(-\sin x + \cos x) - B_1 e^{-x} + Be^{-x} \qquad \text{... (13)}$$

Substituting the values of y, y', and y'' given by (9), (12) and (13) in (1), we get

$$A_1(\cos x + \sin x) - B_1 e^{-x} = \sin^2 x \qquad \text{... (14)}$$

Solving (11) and (14) for A_1 and B_1, we have

$A_1 = dA/dx = (1/2)\times \sin x$ and $B_1 = dB/dx = (1/2)\times e^x(\sin x\cos x - \sin^2 x)$

$\Rightarrow \quad dA = (1/2)\times \sin x\, dx$ so that $A = -(1/2)\times \cos x + c_1$

$$dB = (1/4)\times e^x(2\sin x\cos x - 2\sin^2 x)dx = (1/4)\times e^x(\sin 2x + \cos 2x - 1)dx$$

so that $\quad B = (1/4)\times \int e^x \sin 2x\, dx + (1/4)\times \int e^x \cos 2x\, dx - (1/4)\times e^x + c_2$

or $\quad B = (1/4)\times(1/5)\times e^x(\sin 2x - 2\cos 2x) + (1/4)\times(1/5)\times e^x(\cos 2x + 2\sin 2x) - (1/4)\times e^x + c_2$

$$\left[\because \int e^{ax}\sin bx\, dx = \frac{e^{ax}(a\sin bx - b\cos bx)}{a^2+b^2} \text{ and } \int e^{ax}\cos bx\, dx = \frac{e^{ax}(a\cos bx + b\sin bx)}{a^2+b^2}\right]$$

Thus, $\quad B = (1/20)\times e^x(3\sin 2x - \cos 2x) - (1/4)\times e^x + c_2$

Substituting the above values of A and B is (9), the required solution of (1) is

$$y = \{-(1/2)\times \cos x + c_1\}\times(\sin x - \cos x) + e^{-x}[(1/20)\times e^x(3\sin 2x - \cos 2x) - (1/4)\times e^x + c_2]$$

or $y = c_1(\sin x - \cos x) + c_2 e^{-x} - (1/4)\times(2\sin x\cos x - 2\cos^2 x) + (1/20)\times(3\sin 2x - \cos 2x) - (1/4)$

or $\quad y = c_1(\sin x - \cos x) + c_2 e^{-x} - (1/4)\times(\sin 2x - 1 - \cos 2x) + (1/20)\times(3\sin 2x - \cos 2x) - (1/4)$

or $$y = c_1(\sin x - \cos x) + c_2 e^{-x} - (1/10)\times(\sin 2x - 2\cos 2x)$$

Ex. 7. *Solve by the method of variation of parameters* $x^2 y'' - 2x(1+x)y' + 2(x+1)y = x^3$.

[Rajasthan 1994; Rohilkhanad 1994]

Sol. Re-writing the given equation in sandard form, we get

$$\frac{d^2y}{dx^2} - \frac{2(1+x)}{x}\frac{dy}{dx} + \frac{2(x+1)}{x^2}y = x \qquad \text{...(1)}$$

Fisrst we shall find the C.F. of (1), that is, solution of

$$\frac{d^2y}{dx^2} - \frac{2(1+x)}{x}\frac{dy}{dx} + \frac{2(x+1)}{x^2}y = 0 \qquad \text{...(2)}$$

Comparing (2) with $y'' + Py' + Qy = R$, $\quad P = -\dfrac{2(1+x)}{x}, Q = \dfrac{2(x+1)}{x^2}$ and $R = 0$ ...(3)

Here, $\quad P + xQ = 0,$ showing that $\quad u = x$ ...(4)

is a part of C.F. of (2).

Let the complete solution of (1) be $\quad y = uv$ ...(5)

Then v is given by $$\frac{d^2v}{dx^2} + \left(P + \frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx} = \frac{R}{u}$$

or $\quad \dfrac{d^2v}{dx} + \left[-\dfrac{2(1+x)}{x} + \dfrac{2}{x}\dfrac{dx}{dx}\right]\dfrac{dv}{dx} = 0$ or $\quad \dfrac{d^2v}{dx^2} - 2\dfrac{dv}{dx} = 0,$ using (3) and (4)

or $\quad (D^2 - 2D)v = 0,$ where $\quad D \equiv d/dx$ ...(6)

Auxiliary equation of (6) is $D^2 - 2D = 0$ giving $D = 0, 2$

$\therefore$ Solution of (6) is $y = c_1 e^{o.x} + c_2 e^{2x} = c_1 + c_2 e^{2x}$, c_1 and c_2 being arbitrary constants ...(7)

From (4), (5) and (7), the complete solution of (1), *i.e.*, C.F. of (1) is given by

$$y = x(c_1 + c_2 e^{2x}) \quad \text{or} \quad y = c_1 x + c_2 x e^{2x} \qquad ...(8)$$

Let $$y = Ax + Bx e^{2x} \qquad ...(9)$$

be the complete solution of (1). Then, A and B are functions of x which are so choosen that (1) will be satisfied. Differentiating (9), w.r.t. 'x', we get

$$y' = A + A_1 x + B(e^{2x} + 2x e^{2x}) + B_1 x e^{2x}, \qquad ...(10)$$

where $A_1 = dA/dx$ and $B_1 = dB/dx$. Choose A and B such that

$$A_1 x + B_1 x e^{2x} = 0 \qquad ...(11)$$

Then, (10) reduces to $$y' = A + B e^{2x}(1+2x) \qquad ...(12)$$

Differentiating (12), $$y'' = A_1 + B_1 e^{2x}(1+2x) + B\{2x^{2x}(1+2x) + 2e^{2x}\} \qquad ...(13)$$

Substituting the values of y, y' and y'' given by (9), (12) and (13) is (1), we have

$$x^2\{A_1 + B_1 e^{2x}(1+2x) + 4B e^{2x}(1+x)\} - 2x(1+x)\{A + Be^{2x}(1+2x)\} + 2(x+1)(Ax + Bx e^{2x}) = x^3$$

or $$A_1 x^2 + x^2 B_1 e^{2x}(1+2x) = x^3 \quad \text{or} \quad A_1 + B_1(1+2x)e^{2x} = x \qquad ...(14)$$

Solving (11) and (14) for A_1 and B_1, we have

$$A_1 = dA/dx = -(1/2) \quad \text{and} \quad B_1 = dB/dx = (1/2) \times e^{-2x}$$

Integrating these, $A = -(x/2) + c_1$, and $B = -(1/4) \times e^{-2x} + c_2$

Substituting the above values of A and B is (9), the required solution is

$$y = \{-(x/2) + c_1\}x + \{-(1/4) \times e^{-2x} + c_2\}x e^{2x} \quad \text{or} \quad y = c_1 x + c_2 x e^{2x} - (x^2/2) - (x/4)$$

EXERCISE 11(D)

Apply the method of variation of parameters to solve the following equations

1. $(x^2 + 1)y_2 - 2xy_1 + 2y = 6(x^2 + 1)^2$ **[Bangalore 1992]**

2. $(x^2 - 1)y_2 - 2xy_1 + 2y = (x^2 - 1)^2$, given that x and (x^2+1) are solutions of the reduced equation **[Kanpur 1996]**

3. Solve $(1 - x^2)y_2 - 4xy_1 - (1 + x^2)y = x$ when $y_1 = (\cos x)/(1 - x^2)$, $y_2 = (\sin x)/(1 - x^2)$ are its two complementary solutions. **[Ravishankar 1995]**

4. $(1-x^2)y'' + xy' - y = x(1-x^2)^{3/2}$

Ans. $y = c_1\{(1-x^2)^{1/2} + x\sin^{-1} x\} + c_2 x - (1/9) \times x(1-x^2)^{3/2}$

10.15 Solutions by Operators

Let the given equation be $$S(d^2y/dx^2) + P(dy/dx) + Qy = R, \qquad ...(1)$$

where P, Q, R and S are functions of x.

Writing D for d/dx, (1) gives $$[SD^2 + PD + Q]y = R. \qquad ... (2)$$

Sometimes it will be possible to factorise the left-hand side into two linear operators acting on y. In such a case the equation is integrated in two stages. We illustrate the method by the following solved examples.

Important Remarks. Remember that the factors are not commutative since these will involve functions of x directly. Hence care should be taken while using the factorised operators in the correct order. So test the correctness of the order before using the operators.

10.16 Solved Examples based on Art. 10.15

Ex. 1. *Solve* $y'' + (1 - x)y' - y = e^x$. **[Kurukshetra 2000, Rohilkhand 1995]**

Sol. Writing D for d/dx, the given equation becomes $[xD^2 + (1 - x)D - 1]y = e^x$. ... (1)

Now, $xD^2 + (1 - x)D - 1 = xD^2 - xD + D - 1 = xD(D - 1) + D - 1$

$\therefore$ $$xD^2 + (1 - x)D - 1 = (xD + 1)(D - 1). \qquad ... (2)$$

If we take the other order *i.e.* $(D - 1)(xD + 1)$, then we get

$$(D - 1)(xD + 1) = D(xD + 1) - (xD + 1) = 1\cdot D + x \cdot D^2 + D - xD - 1,$$

which is different from L.H.S. of (2). So the order in (2) is correct.

Using (2), (1) gives $$(xD + 1)(D - 1)y = e^x. \qquad ... (3)$$

Let $$(D - 1)y = v. \qquad ... (4)$$

Then (3) gives $$(Dx + 1)v = e^x. \qquad ... (5)$$

We first slove (5), *i.e.* $x\dfrac{dv}{dx} + v = e^x$ or $\dfrac{dv}{dx} + \dfrac{1}{x}v = \dfrac{1}{x}e^x$,

which is linear. Its I.F. = $e^{\int (1/x)\cdot dx} = e^{\log x} = x$ and solution is

$$vx = \int x \cdot (1/x)e^x dx + c_1 = e^x + c_1; \; c_1 \text{ being on arbitary constant.}$$

$\therefore$ $v = (1/x)e^x + (1/x)c_1$. Putting this value of v in (4), we get

$(D - 1)y = (1/x)e^x + (1/x)c_1$ or $(dy/dx) - y = (1/x)e^x + (1/x)c_1$,

which is again a linear equation of the first order.

Its I.F. = $e^{-\int dx} = e^{-x}$. Hence its solution is given by

$$ye^{-x} = c_2 + \int \left(\frac{1}{x}e^x + \frac{1}{x}c_1\right)e^{-x}dx = c_2 + \log x + c_1 \int \frac{e^{-x}}{x}dx.$$

$\therefore$ The solution of the given equation is $y = c_1 e^x \int (e^{-x}/x)\,dx + c_2 e^x + e^x \log x.$

Ex. 2. *Factorise the operator on the L.H.S of* $[(x + 2)D^2 - (2x + 5)D + 2]y = (x + 1)e^x$ *and hence solve it.* **[Guwahat 1997, Kanpur 1998]**

Sol. L.H.S. of the given equation

$$= (x + 2)D^2 - [2(x + 2) + 1] + 2 = (x + 2)D^2 - 2(x + 2)D - (D - 2)$$
$$= (x + 2)D\,(D - 2) - (D - 2) = [(x + 2)D - 1](D - 2). \qquad ... (1)$$

We cannot reverse the order, for then

$(D - 2)[(x + 2)D - 1] = D + (x + 2)D^2 - D - 2(x + 2)D + 2 = (x + 2)D^2 - (2x + 4)D + 2.$

which is clearly different from L.H.S. of the given equation.

Thus the order (1) is correct. Hence the given equation gives

$$[(x + 2)D - 1](D - 2)y = (x + 1)e^x. \qquad ... (2)$$

Put $$(D - 2)y = v. \qquad ... (3)$$

Then, (2) gives $$[(x + 2)D - 1]v = (x + 1)e^x. \qquad ... (4)$$

We first solve (4), *i.e.*, $(x+2)\dfrac{dv}{dx} - v = (x+1)e^x$ or $\dfrac{dv}{dx} - \dfrac{1}{x+1}v = \dfrac{x+1}{x+2}e^x$,

Its I.F. = $e^{-\int dx/(x+2)} = e^{-\log(x+2)} = 1/(x+2)$ and its solution is given by

$$v \cdot \frac{1}{x+2} = \int \frac{x+1}{(x+2)^2} e^x dx + c_1 = \int \frac{(x+2)-1}{(x+2)^2} e^x dx + c_1 = \int \frac{e^x dx}{x+2} - \int \frac{e^x dx}{(x+2)^2} + c_1$$

$$= \int \frac{e^x dx}{x+2} - \left[\frac{(x+2)^{-1}}{(-1)} e^x - \int \frac{(x+2)^{-1}}{(-1)} e^x \right] + c_1$$, integrating by parts only the second integral

or $\quad v/(x+2) = c_1 + e^x/(x+2)$ so that $v = c_1(x+2) + e^x$.

Putting this in (3), $(D-2)y = c_1(x+2) + e^x$ or $dy/dx - 2y = c_1(x+2) + e^x$

Its I.F. $= e^{-2\int dx} = e^{-2x}$ and its solution is

$$ye^{-2x} = c_2 + \int [c_1(x+2) + e^x] e^{-2x} dx = c_2 + c_1 \int (x+2) e^{-2x} dx + \int e^{-x} dx$$

$$= c_2 + c_1 \left[(x+2)\left(-\frac{1}{2} e^{-2x}\right) + \int \left(\frac{1}{4} e^{-2x}\right) dx \right] - e^{-x} = c_2 - \frac{1}{2} c_1 (x+2) e^{-2x} - \frac{1}{4} c_1 e^{-2x} - e^{-x}$$

or $\quad y = c_2 e^{2x} - (c_1/4) \times (2x+5) - e^x$, c_1 and c_2 being arbitrary constants

Ex. 3. *Solve* $xy'' + (x-2)y' - 2y = x^3$.

Sol. Writing $D \equiv d/dx$, the given equation may be written as

$$[xD^2 + (x-2)D - 2]y = x^2. \qquad \text{... (1)}$$

But $xD^2 + (x-2)D - 2 = xD^2 + xD - 2D - 2 = xD(D+1) - 2(D+1) = (xD-2)(D+1)$

Hence (1) may be re-written as $(xD-2)(D+1)y = x^3$. ... (2)

Let $(D+1)y = v$. ... (3)

Then, (2) gives $(xD-2)v = x^3$.

or $\quad x\dfrac{dv}{dx} - 2v = x^3$ or $\dfrac{dv}{dx} - \dfrac{2}{x} v = x^2$. ... (4)

which is linear. Its. I.F. $= e^{-\int (2/x) dx} = e^{-2 \log x} = x^{-2}$ and so solution of (4) is

$vx^{-2} = \int x^2 \cdot x^{-2} dx + c_1$ or $vx^{-2} = x + c_1$ or $v = x^3 + c_1 x^2$ ... (5)

Using (5), (3) reduces to $dy/dx + y = x^3 + c_1 x^2$ ... (6)

which is linear. Its I.F. $= e^{\int dx} = e^x$. So solution of (6) is

$$y \cdot e^x = \int e^x (x^3 + c_1 x^2) dx + c_2$$, c_1 and c_2 being a arbitrary constants

or $\quad ye^x = (x^3 + c_1 x^2)(e^x) - (3x^2 + 2c_1 x)(e^x) + (6x + 2c_1)e^x - 6e^x + c_2$

[By chain rule of integration by parts]

or $\quad y = x^3 + c_1 x^2 - 3x^2 - 2c_1 x + 6x + 2c_1 - 6 + c_2 e^{-x} = x^3 + (c_1 - 3)x^2 + (6 - 2c_1)x + 2(c_1 - 3) + c_2 e^{-x}$.

EXERCISE 10(E)

Solve the following differential equaitons:

1. $3x^2 y'' + (2 - 6x^2)y - 4y = 0$. **Ans.** $y = c_2 e^{2x} \int e^{-2x + (2x/3)} dx + c_1 e^{2x}$
2. $3x^2 y'' + (2 + 6x - 6x^2)y' - 4y = 0$. **[Rajasthan 2010]** **Ans.** $y = c_2 e^{2/3x} + c_1 e^{2/3x} \int (1/x^2) e^{2x - (2/3x)} dx$
3. $(x+1)y'' + (x-1)y' - 2y = 0$. **Ans.** $y = c_1(x^2+1) + c_2 e^{-x}$
4. $xy'' + (x-1)y'\, y = 0$. **Ans.** $y = c_1(x-1) + c_2 e^{-x}$

5. $xy'' + (x^2 + 1)y' + 2xy = x^2$ given that $y = 2$, $y' = 0$, when $x = 0$. **Ans.** $y = 1 + e^{-x^2/2}$

6. $xy'' + (x - 1)y' - y = x^2$. **Hint.** $(xD - 1)(D + 1)y = x^2$. **Ans.** $y = c_1(x-1) + c_2 e^{-x} + x^2$

7. $xy'' + (x - 1)y' - y = x^4$.

Hint. $(xD - 1)(D + 1)y = x^4$. **Ans.** $y = c_1 e^{-x} + c_2(x-1) + (1/3)x^4 - (4/3)x^3 + 4x^2$

8. $[(x + 3)D^2 - (2x + 7)D + 2]y = (x + 3)^2 e^x$.

Hint. $\{(x+3)D - 1\}(D-2)y = (x+3)^2 e^x$. **Ans.** $y = c_2 e^{2x} + c_1(2x+7) - e^x(x+4)$

9. $xy'' - (x + 2)y' + 2y = x^3$. **Hint.** $(xD - 2)(D - 1)y = x^3$ **Ans.** $y = -x^3 - (c_1+3)(x^2+2x+2) + c_2 e^x$

10. $x^2y'' + y' - (1 + x^2)y = e^{-x}$. **Ans.** $y = c_1 e^x \int e^{-2x+(1/x)} dx + c_2 e^x - (1/2) \times e^{-x}$.

11. $xy_2 + (1 + x)y_1 + y = e^x$. **[Rohilkhand 1995]** **Ans.** $y = c_1 e^{-x} \int e^x x^{-1} dx + c_2 e^{-x} + e^{-x} \int e^{2x} x^{-1} dx$

Miscellaneous problem in Chapter 10

Ex. 1. If $y = x$ is a solution of the differential equation $y'' - (2/x^2 + 1/x)(xy' - y) = 0,\ 0 < x < \infty$, then its general solution is (*a*) $(\alpha + \beta e^{-2x})x$ (*b*) $(\alpha + \beta e^{2x})x$ (*c*) $\alpha x + \beta e^x$ (*d*) $(\alpha e^x + \beta)\, x$ **[GATE 2009]**

Sol. Ans. (d). Proceed as in Ex. 4(a), page 10.8

Ex. 2. Show that the change of independent variable from x to z by $z = \int \sqrt{q(x)}\, dx$ transforms the differential equations $y'' + p(x)y' + q(x)y = 0$ into a differential with constant coefficients if $\{q'(x) + 2p(x)q(x)\} / \{q(x)\}^{3/2}$ is constant **[Mumbai 2010]**

Hints. Proceed as in Art. 10.9

11

Applications of Differential Equations

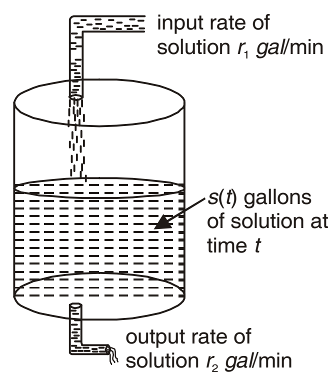
PART I: APPLICATIONS OF FIRST ORDER DIFFERENTIAL EQUATIONS

11.1 Introduction

Differential equations originate from the mathematical formulation of a number of problems in science and engineering. We have already discussed many applications in chapter 2 (refer Art.2.29 to Art. 2.31). In this part, we propose to discuss some variety of problems. For various methods of solving first order differential equations refer chapter 2.

11.2 Mixture problems

Let us suppose that a large mixing tank initialy holds s_o gallons of a solution in which x_0 pounds of a substance S is dissolved. Let another solution, containing x_1 *lb*/gal of S, flows into the tank at a given rate r_1 gal/min. When the solution in the tank is well stirred, it in pumped out at a given rate r_2 gal/min.



Let $x(t)$ denote the amount of substance S (measured in pounds) in the tank at time t. Then the rate at which x changes with time t is given by

dx/dt = (input rate of S) – (output rate of S) $= R_1 - R_2$, say ...(1)

Now, the input rate R_1 at which S enters the tank is the product of the inflow concentration x_1 *lb/sec* of S and the inflow rate r_1 gal/min of the fluid. Note that R_1 is measured in *lb*/min. Thus, we have

$$R_1 = (x_1\ lb/gal) \times (r_1\ gal/\text{min}) = x_1 r_1\ lb/\text{min} \qquad ...(2)$$

Let $s(t)$ denote the number of gallons of solution in the tank at time t. Then the concentration of S in the tank, as well as in the outflow, is (x/s) *lb*/*gal* at any time t. Hence the output rate R_2 of S in given by

$$R_2 = \left(\frac{x}{s}\ lb/gal\right) \times (r_2\ gal/\text{min}) = \frac{xr_2}{s}\ lb/\text{min} \qquad ...(3)$$

From (1), (2) and (3), $\qquad dx/dt = x_1 r_1 - (xr_2)/s \qquad$ ...(4)

which is a first order differential equation. On solving (4), we obtain the amount of substance S in the tank at any time t.

Remarks: 1. If $r_1 = r_2$, then clearly $s(t) = s_0$ = constant value.

2. When $r_1 > r_2$ or $r_1 < r_2$, then the number of gallons of soluion in he tank is either increasing $(r_1 > r_2)$ or decreasing $(r_1 < r_2)$ at the net rate $r_1 - r_2$.

3. In some problems, x_0 or x_1 may be zero.
4. In some problems r_1 or r_2 may be zero.

11.3. Solved examples based on Art 11.2

Ex. 1. *A tank initially contains 50 gallons of pure water. Starting at t = 0 a brine containing 2 lb of dissolved salt per gallon flows into the tank at the rate of 3 gal/min. The mixture is kept uniform by stirring and the well-stirred mixture simultaneously flows out of the tank at the same rate. Then (i) How much salt is in the tank at any time t > 0? (ii) How much salt is pressent at the end of 25 minutes? (ii) How much salt is present after a long time?*

Sol. *(i)* Let x denote the amount of salt (measured in pounds) in the tank at any time t. Then the rate at which x changes with time t is given by

$$dx/dt = \text{(input rate of salt)} - \text{(output rate of salt)} = R_1 - R_1 \text{ say} \qquad ...(1)$$

Now, the input rate R_1 at which salt enters the tank is the product of the inflow concentration 2 *lb*/sec of salt and the inflow rate 3 *gal*/min of brine.

$$\therefore \qquad R_1 = (2\ lb/gal) \times (3\ gal/\text{min}) = 6\ lb/\text{min} \qquad ...(2)$$

Since the rate of outflow equals the rate of inflow, the tank conains 50 gallons of brine at any time t. This 50 gallons contains x *lb* of salt at time t. Hence the concentration of salt in the tank, as well as in the outflow, is $(x/50)$ *lb*/*gal* at any time t. Hence, as before, the output rate R_2 of salt is given by

$$R_2 = (\frac{x}{50}\ lb/gal) \times (3\ gal/\text{min}) = \frac{3x}{50} \text{ lb/min} \qquad ...(3)$$

Using (2) and (3), (1) takes the form $dx/dt = 6 - (3x/50)$

or $$(dx)/(100 - x) = (3/50)\ dt$$

Integrating, $-\log(100 - x) + \log C = 3t/50$ or $\log(100 - x) - \log C = -3t/50$

or $(100 - x)/C = e^{-3t/50}$ or $x = 100 - C\ e^{-3t/50}$ ...(4)

Since initially there was no salt in the tank, so initial condition is

$$x = 0 \quad \text{when} \quad t = 0 \qquad ...(5)$$

Applying the initial condition (5), (4) gives $0 = 100 - C$ or $C = 100.$

Hence (4) reduces to $x = 100\ (1 - e^{-3t/50})$ ...(6)

which gives the amount of salt in the tank at any time t.

(ii) Let x_1 be the amount of the salt present in the tank at the end of 25 minutes.

Thus, $x = x_1$ when $t = 25$. Then (6) yields.

$$x_1 = 100\ (1 - e^{-3/2}) = 100\ (1 - e^{-1.5}) = 78\ lb \text{ (approximitey)}$$

(iii) Here we require to find out the amount of salt present in the tank as $t \to \infty$ To find this value, we let $t \to \infty$ in (6) and note that $x \to 100$.

Ex. 2. *Initially 50 pounds of salt is dissolved in a large tank having 300 gallons of water. A brine solution is pumped into the tank at a rate of 3 gal/min and well-stirred solution is then pumped out at the same rate. If the concentration of the solution entering is 2lb/gal, find the amount of salt in the tank at any time. How much salt is present after 50 min and after a long time.*

Sol. Let x denote the amount of salt (measured in pounds) in the tank at any time t. Then the rate at which x changes with time t is given by

$$dx/dt = \text{(input rate of salt)} - \text{(output rate of salt)} = R_1 - R_2 \text{ say} \qquad ...(1)$$

Now, the input rate R_1 at which salt enters the tank is the product of the inflow concentration 2 *lb*/*gal* of salt and the inflow rate 3 *gal*/min of solution.

$$\therefore \qquad R_1 = (2\ lb/gal) \times (3\ gal/.\text{min}) = 6\ lb/\text{min} \qquad ...(2)$$

Since the rate of outflow equals the rate of inflow, the tank contain 300 gallons of brine at any time t. This 300 gallons contain x *lb* of salt at time t. Hence the concentration of salt in the tank, as well as in the outflow, is $(x/300)$ *lb/gal* at any time t. Hence as before, the output rate R_2 of salt is given by

$$R_2 = \left(\frac{x}{300} lb/gal\right) \times (3\ gal/.\text{min}) = \frac{x}{100} lb/\text{min} \qquad ...(3)$$

Using (2) and (3), (1) takes the form $\qquad dx/dt = 6 - x/100$

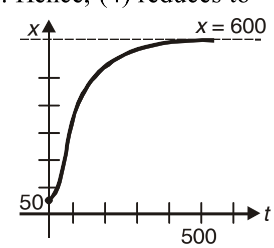
or $$(dx)/(600 - x) = (1/100)\ dt$$

Integrating, $\quad -\log(600 - x) + \log C = t/100 \quad$ or $\quad \log(600 - x) - \log C = -t/100$

or $\quad (600 - x)/C = e^{-t/100} \quad$ or $\quad x = 600 - Ce^{-t/100} \qquad ...(4)$

Since initially 50 *lb* salt was present in the tank, hence the initial condition is : $x = 50$ when $t = 0$. Using this condition, (4) reduces to $50 = 600 - C$ so that $C = 550$. Hence, (4) reduces to

$$x = 600 - 550\ e^{-t/100} \qquad ...(5)$$



which gives the desired amount of salt at any time t.

Second part : Let $x = x_1$, when $t = 50$. Then (5) reduces to $x_1 = 600 - 550\ e^{-1/2} = 600 - 550\ e^{-0.5} = 266.41$ *lb* (approx)

Third part : From (5), we see that as $t \to \infty$, $x \to 600$ (see figure) which is what we would expect; over a long period of time the number of pounds of salt in the solution must be (300 *gal*) × (2*lb*/*gal*) = 600 *lb*.

Ex. 3. *A large tank initially contains 50 gallons of brine in which there is dissolved 10 lb of salt. Brine containing 2lb of dissolved salt per gallon flows into the tank at the rate of 5 gal/min. The mixture is kept uniform by stirring, and stirred mixture simultaneously flows out at the slower rate 3 gal/min. How much salt is in the tank at time t > 0?*

Sol. Let x denote the amount of salt (measured in pounds) in the tank at any time t. Then the rate at which x changes with time t is given by

$$dx/dt = (\text{input rate of salt}) - (\text{output rate of salt}) = R_1 - R_2, \text{ say} \qquad ...(1)$$

Now, the input rate R_1 at which salt enters the tank is the product of the inflow concentration 2 *lb*/*gal* of salt and the inflow rate 5 *gal*/min of brine.

$\therefore$ $$R_1 = (2\ lb/gal) \times (5\ gal/\text{min}) = 10\ lb/\text{min} \qquad ...(2)$$

At $t = 0$, the tank contains 50 gallons of brine. Since brine flows in at the rate of 5 *gal*/min whereas flows out at the slower rate 3 *gal*/min, there is a net gain of 5 – 3 = 2 *gal*/min of brine in the tank.

Therefore, $s(t)$ = the amount of brine in the tank at the end of time $t = (50 + 2t)$ gallons. Hence the concentration of salt in the tank, as well as in the outflow, is $x(t)/s(t)$, *i.e.*, $x/(50 + 2t)$ *lb*/*gal* at any time t. Hence, as before, the output rate R_2 of salt is given by

$$R_2 = \left(\frac{x}{50 + 2t}\ lb/gal\right) \times (3\ gal/\text{min}) = \frac{3x}{50 + 2t}\ lb/\text{min} \qquad ...(3)$$

Using (2) and (3), (1) takes the form $\qquad dx/dt = 10 - (3x)/(50 + 2t)$

or $$\frac{dx}{dt} + \frac{3}{50 + 2t}x = 10, \text{ which is a linear differential equation.} \qquad ...(4)$$

Its integrating factor $= e^{\int \{3/(50+2t)\} dt} = e^{(3/2) \times \log(50 + 2t)} = (50 + 2t)^{3/2}$

and hence solution of (4) is given by $\qquad x(50 + 2t)^{3/2} = \int 10 (50 + 2t)^{3/2} dt + C$

or $\quad x(50 + 2t)^{3/2} = 2(50 + 2t)^{5/2} + C \quad$ or $\quad x = 4\,(t + 25) + C/(2t\,/\,50)^{3/2} \quad$ ...(5)

Since there was initially 10 *lb* of salt in the tank, we have the initial condition: $x = 10$ when $t = 0$. Using this condition, (5) reduces to $10 = 100 + C/(50)^{3/2}$ so that $C = -(90) \times (50)^{3/2} = -22.500\sqrt{2}$. Substituting this value of C in (5), the amount of salt at any time t is given by

$$x = 4t + 100 - (22.500\sqrt{2})/(50 + 2t)^{3/2}$$

Ex. 4. *A tank contains 100 gallons brine in which 10 lb of salt are dissolved. Brine containing 2 lb sat per gallon flows into the tank at 5 gal/min. If the well-stirred mixture is drawn off at 4 gal/min, find (a) the amount of salt in the tank at time t, and (b) the amount of the salt at t = 10 minutes.*

Sol. Proceed as in **Ex 3.**

Ans *(a)* $x(\mathrm{t}) = 2(100 + t) - 190\,(100)^4\,(100 + t)^{-4}$ *(b)* $x(10) = 90.2$ *lb* (approx).

EXERCISE 11(A)

1. A tank initially contains 100 gallons of brine in which there is dissolved 20 *lb* of salt. Starting at $t = 0$, brine containing 3 *lb* of dissolved salt per gallon flows into the tank at the rate of 4 *gal*/min. The mixture is kept uniform by stirring and the well-stirred mixture simultaneously flows out of the tank at the same rate. Find *(a)* How much salt is in the tank at the end of 10 minutes and *(b)* When is there 160 *lb* of salt in the tank. **Ans.** *(a)* 112.31 lb *(b)* 17.33 minutes.

2. A tank initially contains 100 gallons of pure water. Starting at $t = 0$, a brine containing 4 lb of salt per gallon flows into the tank at the rate of 5 gal/min. The mixure is kept uniform by stirring and the well stirred mixture flows out at the slower rate of 3 *gal*/min. Find *(a)* How much salt is in the tank at the end of 20 mintes? *(b)* When is there 50 *lb* of salt in the tank?

Ans. *(a)* 318.53 lb *(b)* 2.74 minutes.

3. A large tank initially contains 100 gallons of brine in which 10 lb of salt is dissolved. Starting at $t = 0$, pure water flows into the tank at the rate of 5 *gal*/min. The mixture is kept uniform by stirring and the well stirred mixtune simultaneously flow out at the slower rate of 2 *gal*/min. Find *(a)* How much salt is in tank at the end of 15 minutes and what is the concentration at that time? *(b)* If the capacity of the tank is 250 gallons, what is the concentration at the instant the tank overflows.

4. A large tank initially contains 200 gallons of brine in which 15 *lb* of salt is dissolved. Starting at $t = 0$, brine containing 4 *lb* of salt per gallons flows into the tank at the rate of 3.5 *gal*/min. The mixture is kept uniform by stirring and the well-stirred mixture leaves the tank at the rate of 4 *gal*/min. Find *(a)* How much salt is in the tank at the end of one hour? *(b)* How much salt in the tank when the tank contains only 50 *gallons* of brine?

PART II: APPLICATIONS OF SECOND ORDER LINEAR DIFFERENTIAL EQUATIONS

11.4 Introduction

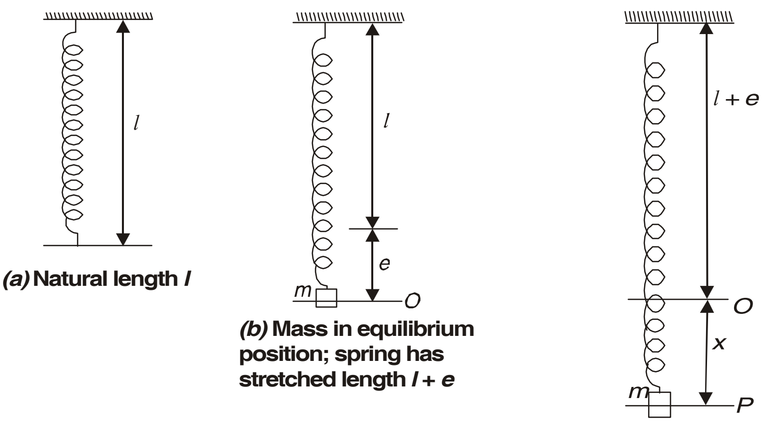
Second order linear differential equations with constant coefficients have a number of applications in physics, electrical and mechanical engineering, medical science, economics and other linear systems. In this part, we propose to study the applications of second order linear differential equations to some of these disciplines. For various methods of solving second order linear equations with constant coefficient, refer chapter 5.

11.5 Newton's second law and Hooke's law

Newton's second law. Suppose m be the mass of a body, F the resultant force acting upon it and a be the acceleration produced in the body. Then, by Newton's second low, we have $F = ma$. Note that this is a vector equation.

Hooke's law. According to Hooke's law, the magnitude of the force needed to produce a certain elongation of a spring is directly proportional to the amount of this elongation. The positive constant of proporionality k is called the *spring constant.* Thus, 1F1 = ks, were F is the magnitude of the force and s is the amount of elongation.

11.6 The differential equation of the vibrations of a mass on a spring



(a) **Natural length *l***

(b) **Mass in equilibrium position; spring has stretched length *l* + *e***

(c) **Mass distance x below equilibrium position; spring stretched to length *l* + *e* + *x***

As shown in figure *(a),* let the coil spring have natural (unstretched) length l. The mass m is attached to its lower end and comes to rest in its equilibrium position O, threreby stretching the spring by an amount e so that its stretched length is $l + e$. In the position of equilibrium O, the mass m in acted upon by two forces: *(i)* weight mg acting vertically downwards *(ii)* The spring force ke acting vertically upwards (see figure b).

Thus, we have $$mg = ke \qquad ...(1)$$

We choose the axis along the line of the spring, with the origin at equilibrium position O and the positive direction downward. Let P be the position of the mass at any time t such that $OP = x$. Then x is positive, zero, or negative according to whether the mass is below, at, or above its equilibrium position (see figure c).

When the mass is situated at P, it is acted upon by the following forces. The forces tending to pull the mass downward are positive, while those pulling it vertically upward are negative.

(i) F_1 = mg, acting in the vertically downward direction

(ii) Let F_2 be he restoring force of the spring. When the mass is at P, F_2 is acting is the upward direction and so it is negative. By Hooke's law, we have

$$F_2 = -k(x + e) = -kx - ke \qquad \text{or} \qquad F_2 = -kx - mg, \text{ using (1)}$$

(iii) Let F_3 be the resisting force of the medium, called the *damping force.* It is known that for small velocities F_3 is approximately proportional to the magnitude of the velocity. When the mass is moving downward (at P, say), F_3 acts in the upward direction (opposite to that of the motion) and so F_3 is negative and is given by

$F_3 = -a\,(dx/dt)$, where a (> 0) is called the *damping constant.*

(iv) External impressed force F (t) acting in downward direction.

By Newton's second law $F = ma$ where $F = F_1 + F_2 + F_3 + F_4$ and $a = d^2x/dt^2$

Thus, we obtain $$m\,(d^2x/dt^2) = mg - kx - mg - a\,(dx/dt) + F(t)$$

or $$m\,(d^2x/dt^2) + a\,(dx/dt) + kx = F(t) \qquad ...(2)$$

which is the differentical equation for the motion of the mass on the spring. If $a = 0$ the motion is called *undamped* otherwise it is called *damped.* If there are no external impressed forces, $F(t) = 0$ for all t and the motion is called *free*; otherwise it is called *forced.* In the following articles we propose to discuss the solution of (2) in each of these cases.

11.7 Free, undamped moion

Refer Art 11.6 Setting $a = 0$ and $F(t) = 0$ in equation (2) of Art 11.6, the differential equation for free, undamped motion is given by

$$m\,(d^2x/dt^2) + kx = 0 \qquad \text{or} \qquad d^2x/dt^2 + \mu^2 x = 0 \qquad ...(1)$$

where $\mu^2 = \text{k/m}$ and $D \equiv d/dt$ ...(2)

Auxiliary equation of (1) is $D^2 + \mu^2 = 0$ so that $D = \pm i\mu$

$\therefore$ Solution of (1) is $x = C_1 \sin \mu t + C_2 \cos \mu t$, C_1, C_2 being arbitrary contants ...(3)

Suppose that the mass was initially displaced a distance x_0 from its equilibrium position O and released from that point with initial velocity v_0. Then, we have the initial conditions:

$$x(0) = x_0 \qquad \text{and} \qquad x'(0) = v_0 \qquad ...(4)$$

Differentiating (3) w.r.t. 't' gives $dx/dt = C_1\, \mu \cos \mu t - C_2\, \mu \sin \mu t$ ...(5)

Applying conditions (4) to equation (3) and (5), we have

$$x_o = C_2 \;\text{ and }\; v_o = C_1 \qquad \Rightarrow \qquad C_2 = x_0 \;\text{ and }\; C_1 = v_0/\mu$$

$\therefore$ (3) yields $x = (v_0/\mu) \sin \mu t + x_0 \cos \mu t$ ...(6)

Equation (6) describes the free vibrations or free motion of the mechanical system, since it is free of external influencing forces other than those imposed by gravity and the spring itself.

We would like to re-write (6) in the form $x = \text{C} \cos(\mu t + \phi)$ so that we can graph (and understand) the superposition of the sine and cosine functions in (6). Re-writing (6), we have

$$x = C\left\{\frac{(v_0/\mu)}{C}\sin \mu t + \frac{x_0}{C}\cos \mu t\right\} \qquad ...(7)$$

where $$C = \left\{(v_0/\mu)^2 + x_0^2\right\}^{1/2} > 0 \qquad ...(8)$$

Assume that $$\frac{(v_0/\mu)}{C} = -\sin\phi \qquad \text{and} \qquad \frac{x_0}{C} = \cos\phi \qquad ...(9)$$

Then, (7) yields $x = C(\cos \mu t \cos \phi - \sin \mu t \sin \phi)$ or $x = C\cos(\mu t + \phi)$ ...(10)

where C is given by (8) and ϕ is determined by (9). We have to be careful to determine which quadrant ϕ is in. Since $\mu = \sqrt{k/m}$, (10) may be re-written as

$$x = C\cos(t\sqrt{k/m} + \phi), \qquad ...(11)$$

giving the displacement x of the mass from the equilibrium position O as a function of t ($t > 0$). Clearly free, undamped motion of the mass is a simple harmonic motion. The constant C is called the amplitude of the motion and gives the maximum (positive) displacement of the mass from the O. The motion is a periodic motion, and the mass oscillates back and forth between $x = C$ and $x = -C$. We have $x = C$ if and only if

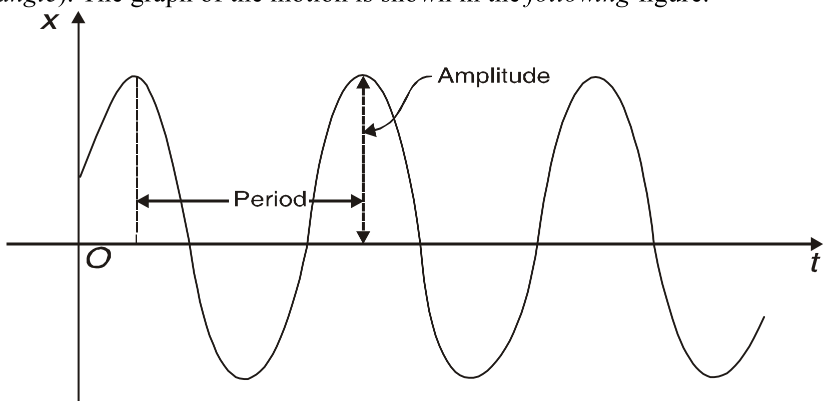
$$\cos(t\sqrt{k/m} + \phi) = 0 \qquad \text{or} \qquad t\sqrt{k/m} + \phi = \pm 2n\pi, \text{ where } n = 0, 1, 2, 3,$$

Thus, the maximum (positive) displacement occurs if and only if

$$t = (m/k)^{1/2}(\pm 2x\pi - \phi) > 0, \qquad \text{where} \qquad n = 0, 1, 2, 3, ... \qquad ...(12)$$

The time interval between two successive maxima is called the *period of the motion.* Using (12), periodic time T is given by $$T = (2\pi)/\sqrt{k/m} = (2\pi)/\mu \qquad ...(13)$$

The reciprocal of the period, which gives the number of oscillations per second is called the *natural frequency* (or simply *frequency*) of the motion. The number ϕ is called the *phase* constant (or *phase angle*). The graph of the motion is shown in the *following* figure.



Illustrative example: *An 8-lb weight is placed upon the lower end of a coil spring suspended from the ceilling. The weight comes to rest in its equilibrium position, thereby stretching the spring 6 in. The weight is then pulled down 3 in below its equilibrium position and released at t = 0 with an initial velocity of 1ft/sec directed downwards. Neglecting the resistance of the medium and assuming that no external forces are present, determine the amplitude, period and frequency of the resulting motion.*

Sol. Refer figure of Art. 11.6 As shown in figure, the natural length of spring = l in. The mass m (= w/g = 8/32 = ¼(slugs) is attached to its lower end and comes to rest in its equilibrium position O, thereby stretching the spring by an amount e (= 6 in = ½ft). In the position of equilibrium, the mass m is acted upon by two forces: *(i)* weight 8 - *lb* acting in the vertically downward direction *(ii)* the spring force ke *i.e.*, $(1/2) \times k$ acting in the vertically upwords. Thus,

$$8 = (1/2) \times k \qquad \text{so that} \qquad k = 16\ lb/\text{ft}.$$

We choose the axis along the line of the spring, with the origin at equilibrium position O and the positive direction downwad. Let P be the position of the mass at any time t.such that $OP = x$. Then x is positive, zero or negative according to whether the mass is below, at, or above O.

When the mass is situated at P, it is acted upon by the following forces: *(i)* $F_1 = mg$, acting in the vertically downward direction *(ii)* Restoring force F_2 of the spring acting in the vertically upward divectium.

Then, $\quad F_2 = -k\,(x + e) = -kx - ke \qquad \text{or} \qquad F_2 = -kx - mg, \qquad \text{as} \qquad ke = mg$

By Newton's second law $F = ma$, we have

$$F_1 - F_2 = m\,(d^2x/dt^2) \qquad \text{or} \qquad mg - kx + mg = m\,(d^2x/dt^2)$$

$$\text{or} \qquad d^2x/dt + (k/m)\,x = 0 \qquad \text{or} \qquad d^2x/dt^2 + 64\,x = 0 \qquad ...(1)$$

[$\because\ k = 16\ lb/\text{ft}$ and $m = (1/2)$ slugs]

Since the weight was released with a downward initial velocity of 1 ft/sec from a point 3 in (= 1/4 ft) below its equilibrium position O, we have the initial conditions:

$$x\,(0) = 1/4 \qquad \text{and} \qquad x'(0) = 1 \qquad ...(2)$$

Re-writting (1), $\quad (D^2 + 64)\,x = 0 \qquad \text{where} \qquad D \equiv d/dt \qquad ...(3)$

whose auxiliary equation is $\quad D^2 + 64 = 0 \qquad \text{so that} \qquad D = \pm 8i$

$\therefore$ Solution of (3) is $\quad x = C_1 \sin 8t + C_2 \cos 8t, C_1, C_2$ being arbitrary constants. $\qquad ...(4)$

Applying the condition (2) to equations (4) and (6), we get C_2 = 1/4 and C_1 = 1/8. Substituting these in (5), we get

$$x = (1/8) \times \sin 8t + (1/4) \times \cos 8t \quad ...(6)$$

We have, $\{(1/8)^2 + (1/4)^2\}^{1/2} = \sqrt{5}/8$.

Rewriting (6), we have
$$x = \frac{\sqrt{5}}{8}\left\{\frac{(1/8)}{(\sqrt{5}/8)}\sin 8t + \frac{(1/4)}{(\sqrt{5}/8)}\cos 8t\right\} \quad ...(7)$$

Let $\cos\phi = \frac{(1/4)}{(\sqrt{5}/8)} = \frac{2\sqrt{5}}{5}$ and $\sin\phi = -\frac{(1/8)}{(\sqrt{5}/8)} = -\frac{\sqrt{5}}{5}$ ...(8)

Then (7) yields, $$x = (\sqrt{5}/8)\ \cos(8t + \phi) \quad ...(9)$$

where ϕ is determined by equations (8). Since $\cos\phi$ is positive and $\sin\phi$ is negative, it follows that the phase angle ϕ is located in fourth quadrant. To compute ϕ, we have $\tan\phi = -(1/2) = -0.5$, using (9). So $\phi = \tan^{-1}(-0.5) = -0.46$ radians (approximately). Taking $\sqrt{5} = 2.236$ (apparox.), (9) reduces to

$$x = 0.280 \cos(8t - 0.46) \quad ...(10)$$

The amplitude of the motion is 0.280 ft. The period = $T = 2\pi/\sqrt{64}$ (= $\pi/4$) sec, and the frequency is t/T *i.e.,* $4/\pi$ oscillations/sec.

11.8 Free, damped motion

We now wish to examine the effect of the resistance of the medium upon the mass on the spring. We assume that no external force acts on the mass. Thus, we have the so called *free-damped motion.* Hence setting F*(t)* = 0, the basic differential equation of the vibrations of a mass on the spring for free damped motion is (refer equation (2) of Art 11.6)

$$m\,(d^2x/dt^2) + a\,(dx/dt) + kx = 0 \quad a > 0,\ k > 0$$

or $$(D^2 + 2b\,D + \mu^2)\,x = 0 \quad ...(1)$$

where $D \equiv d/dt$, $2b = a/m$ and $\mu^2 = k/m$ ...(2)

Observe that since a is positive, b is also positive Auxillary equation of (1) is

$D^2 + 2bD + \mu^2 = 0$, giving $D = \{-2b \pm (4b^2 - 4\mu^2)^{1/2}\}/2 = -b \pm (b^2 - \mu^2)^{1/2}$ ...(3)

Three different cases arise, depending upon the nature of these roots, which in turn depends upon the sign of $b^2 - \mu^2$.

Since each solution of (1) contains the damping factor e^{-bt}, $b > o$, the displacements of the mass become negligible over a long period of time.

Case I: Motion of an over damped system: Here we consider the case in which $b^2 - \mu^2 > 0$. In this situation the system is said to be *over damped* because the damping coefficent a is large when compared to the spring constant k. The corresponding solution of (1) is

$$x(t) = e^{-\lambda t}\{C_1\, e^{t(b^2-\mu^2)^{1/2}} + C_2\, e^{-t(b^2-\mu^2)^{1/2}}\}, \quad ...(4)$$

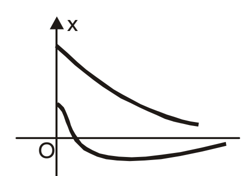


Fig (i). Motion of an over damped system

which represents a smooth and nonoscillatory motion. Figure *(i)* shows two graphs of $x(t)$. Here C_1 and C_2 are arbitrary constants.

Case II: Motion of a critically damped system: Here we consider the case in which $b^2 - \mu^2 = 0$. In this situation the system is said to be *critically damped system* because slight decrease in the damping force would result in oscillatory motion. When $b^2 - \mu^2 = 0$, (3) $\Rightarrow D = -b, -b$. The corresponding solution of (1) is

$x(t) = e^{-bt}(C_1 + C_2 t)$, C_1, C_2 being arbitrary constants ...(5)

Some graphs of typical motion are given in figure *(ii)*. Observe that the motion is quite similar to that of an overdamped system. It is also apparent from (5) that the mass can pass through the equilibrium position at most one time.

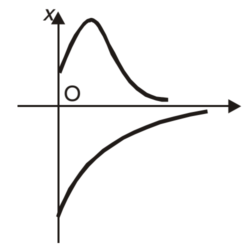


Fig. (ii). Motion of a critically damped system

Case III. Motion of an underdamped system: Here we consider the case in which $b^2 - \mu^2 < 0$ so that $\mu^2 - b^2 > 0$. Re-writting (3) we have $D = -b \pm \{-(\mu^2 - b^2)\}^{1/2}$

or $\quad D = -b \pm i(\mu^2 - b^2)^{1/2}$, $\quad$ where $\quad i = \sqrt{-1}$.

In this situation the system is said to be *underdamped* because the damping coefficient a is small compared to the spring constant k. The corresponding solution of (1) is

$x(t) = e^{-bt}\{C_1 \cos t(\mu^2 - b^2)^{1/2} + C_2 \sin t(\mu^2 - b^2)^{1/2}\}$ ...(6)

As shown in figure *(iii)*, the motion described by (6) is oscillatory, but because of the coefficient e^{-bt}, the amplitudes of vibration $\to 0$ as $t \to \infty$.

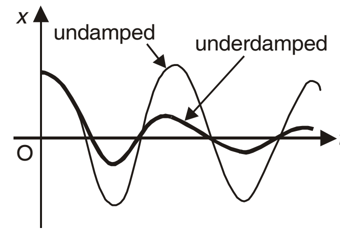


Fig. (iii). Motion of an under damped system

11.9 Solved examples based on Art. 11.8

Example *1. An 8-pound weight stretches a spring 2 feet. Assuming that a damping force numerically equal to 2 times the instantaneous velocity acts on the system, determine the equation of motion if the weight is released from the equilibrium position with an upward velocity of 3 ft/sec.*

Sol. Using Hooke's law we have $8 = k \times 2$ so that $k = 4$ *lb*/ft. Again $W = mg \Rightarrow 8 = m \times 32$ so that $m = 1/4$ slug. Also, here damping factor = 2. Using the above facts, the basic differential equation of the vibrations of the given mass on the spring for free damped motion (refer Art 11.8) namely $m(d^2x/dt^2) + a(dx/dt) + kx = 0$ reduces to $\quad (1/4) \times (d^2x/dt^2) + 2x(dx/dt) + 4x = 0$

or $\quad (D^2 + 8D + 16)x = 0 \quad$ where $\quad D \equiv d/dt$ ...(1)

The initial condition are $\quad x(0) = 0 \quad$ and $\quad x'(0) = -3$ ...(2)

Its auxiliary equation is $D^2 + 8D + 16 = 0$ or $(D + 4)^2 = 0$ so that $D = -4, -4$ (equal roots). Hence the system is critically damped (refer case II of Art. 11.8) and hence, we have

$x(t) = (C_1 + C_2 t)e^{-4t}$, C_1, C_2 being arbitrary constants....(3)

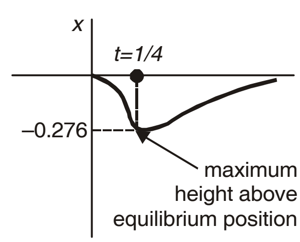
Differentiating (3) w.r.. 't', we have

$$x'(t) = C_2 e^{-4t} - 4(C_1 + C_2 t)e^{-4t} \qquad ...(4)$$

Applying the initial conditions (2) in (3) and (4), we find, in turn, that $C_1 = 0$ and $C_2 = -3$. Hence from (3), we get

$$x(t) = -3t\, e^{-4t} \qquad ...(5)$$

From (5), $x'(t) = -3e^{-4t}(1 - 4t)$, showing that $x'(t) = 0$ when $t = 1/4$. The corresponding extreme displacement is given by



Critically damped system

$x(1/4) = -3 \times (1/4) \times e^{-1} = -0.276$. As shown in the adjoining figure, we interpret this value to mean that the weight reaches the maximum height of 0.276 foot above the equilibrium position.

Example 2. *A 32–lb weight is attached to the lower end of a coil spring suspended from the ceiling. The weight comes to rest in its equilibrium position, there by stretching the spring 2ft. The weight in then pulled down 6 inches below its equilibrium position and released at t = 0. No external forces are present; but the resistance of the medium in pounds is numerically equal to*

8 *(dx/dt), where dx/dt is the instantaneous velocity in feet per second. Determine the resulting motion of the weight on the spring.*

Sol. Here e = the elongation of the spring after the weight is attached = 2 feet. Using Hooke's law, we have $32 = k \times 2$ so that $k = 16$ *lb*/ft. Again, $W = mg \Rightarrow 32 = m \times 32 \Rightarrow m = 1$ slug. Here damping factor = $a = 8$.

Using these facts, the basic differential equation of the vibrations of the given mass on the spring for free damped motion (refer Art 11.8) namely $m\,(d^2x/dt^2) + a\,(dx/dt) + k\,x = 0$ reduces to

$$d^2x/dt + 8\,(dx/dt) + 16x = 0 \qquad \text{or} \qquad (D^2 + 8D + 16)x = 0, \qquad D \equiv d/dt \quad ...(1)$$

The initial conditions are : $x(0) = 6/12 = 1/2$ and $x'(0) = 0$ ...(2)

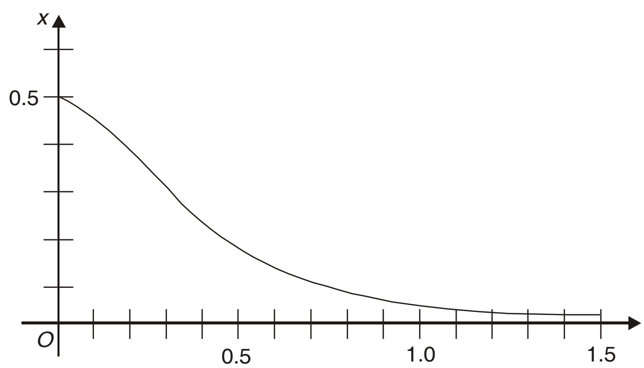
The auxiliary equation for (1) is $D^2 + 8D + 16 = 0$ or $(D + 4)^2 = 0$ giving $D = -4, -4$: Hence the system is critically damped (refer case II or Art 11.8) and

$$x(t) = (C_1 + C_2 t)e^{-4t},\ C_1 \text{ and } C_2 \text{ being arbitrary constants} \quad ...(3)$$

From (3), $x'(t) = C_2\, e^{-4t} - 4(C_1 + C_2\, t)\, e^{-4t} = (C_2 - 4C_1 - 4C_2\, t)e^{-4t}$ ...(4)

Applying the initial conditions (2) in (3) and (4), we find, in turn, $C_1 = 1/2$ and $C_2 - 4\,C_1 = 0$ so that $C_1 = 1/4$ and $C_2 = 2$. Hence, from (3), we get $x(t)=(1/2) \times (1 + 4t)e^{-4t}$ ...(5)

Interpretation: The motion is critically damped. Using (5), we have $x = 0 \Leftrightarrow t = -(1/4)$. It follows that $x \neq 0$ for $t > 0$ and the weight does not pass through its equilibrium position. Also, from (5), $x'(t) = (2–2–8t)\, e^{-4t} = -8te^{-4t} < 0$ for all t > 0. Thus, the displacement of the weight from its equilibrum position is decreasing function of t for all $t > 0$. In other words the weight starts to move back towards its equilibrium position at once and $x \to 0$ monotonically as $t \to \infty$. The graph of the solution (5) is shown in the following figure.



Example 3. *A 16-pound weight is attached to a 5-foot long spring. At equilibrium the spring measures 8.2 feet. If the weight is pushed up and released from rest at a point 2 feet above the equilibrium position, find the displacement x(t) if it is further known that surrounding medium offers a resistance numerically equal to the instantaneous velocity.*

Sol. Here e = the elongation of the spring after the weight is attached = 8.2 – 5 = 3.2 ft. Using Hooke's law, we have $16 = k \times 3.2$ so that $k = 5$ *lb*/ft. Again, $W = mg \Rightarrow 16 = m \times 32$ so that $m = 1/2$ slug. Also here damping factor = $a = 2$. Using the above facts the basic differential equation of the vibrations of the given mass on the spring for free damped motion (refer Art 11.8), namely

$m\,(d^2x/dt) + a\,(dx/dt) + kx = 0$ reduces to $(1/2) \times (d^2x/dt) + dx/dt + 5x = 0$

or $$(D^2 + 2D + 10)x = 0 \quad ...(1)$$

The initial conditions are: $x(0) = -2$ and $x'(0) = 0$ ...(2)

The auxiliary equation for (1) is $D^2 + 2D + 10 = 0$ so that $D = -1 \pm 3i$, which then implies that the system is underdamped and

$$x(t) = e^{-t}\,(C_1 \cos 3t + C_2 \sin 3t),\ C_1 \text{ and } C_2 \text{ being arbitrary constants} \quad ...(3)$$

From (3), $x'(t) = -e^{-t}(C_1 \cos 3t + C_2 \sin 3t) + 3e^{-t}(-C_1 \sin 3t + C_2 \cos 3t)$ ...(4)

Applying the initial conditions $x(0) = -2$ and $x'(0) = 0$ in (3) and (4), yield, in turn, $C_1 = -2$ and $C_2 = -(2/3)$. Hence from (3), we obtain $x(t) = -(2/3) \times e^{-t}(3 \cos 3t + \sin 3t)$

Example 4. *A 32–lb weight is attached to the lower end of a coil spring suspended from the ceiling. The weight comes to rest in its equilibrium position, thereby stretching the spring 2 ft. The weight is then pulled down 6 inches below its equilibrium position and released at t = 0. No external forces are present; but the resistance of the medium is numerically equal to 4(dx/dt), where dx/dt is the instantaneous velocity in feet per second. Determine the resulting motion of the weight on the spring.*

Sol. Here e = the elongation of the spring after the weight is attached = 2 feet. Using Hooke's law,we have $32 = k \times 2$ so that k = 16 *lb*/ft. Again, $W = mg \Rightarrow 32 = m \times 32$ so that $m = 1$ slug. Here damping factor = $a = 4$.

Using these facts, the basic differential equation of the vibrations of the given mass on the spring for free damped motion (refer Art 11.8), namely, $m(d^2x/dt^2) + a\,(dx/dt) + kx = 0$ reduces to

$$d^2x/dt + 4\,(dx/dt) + 16 = 0 \qquad \text{or} \qquad (D^2 + 4D + 16)\,x = 0, \qquad D \equiv d/dt \quad ...(1)$$

The initial conditions are: $x(0) = 6/12 = 1/2$, and $x'(0) = 0$ ...(2)

The auxiliary equation for (1) is $D + 4D + 6 = 0$ giving $D = -2 \pm 2\sqrt{3}$

which then implies that the system is underdamped and

$$x(t) = e^{-2t}(C_1 \sin 2\sqrt{3}\,t + C_2 \cos 2\sqrt{3}\,t),\ C_1 \text{ and } C_2 \text{ being arbitrary constants} \quad ...(3)$$

From (3), $x'(t) = -2e^{-2t}(C_1 \sin 2\sqrt{3}\,t + C_2 \cos 2\sqrt{3}\,t) + 2\sqrt{3}\,e^{-2t}(C_1 \cos 2\sqrt{3}\,t - C_2 \sin 2\sqrt{3}\,t)$

or $$x'(t) = e^{-2t}\{(-2\,C_1 - 2\sqrt{3}\,C_2) \sin 2\sqrt{3}\,t + (2\sqrt{3}\,C_1 - C_2) \cos 2\sqrt{3}\,t\} \quad ...(4)$$

Applying the initial conditions (2) to equation (3) and (4), we have $C_2 = 1/2$ and $2\sqrt{3}\,C_1 - 2C_2 = 0$ so that $C_1 = \sqrt{3}/6$ and $C_2 = 1/2$.

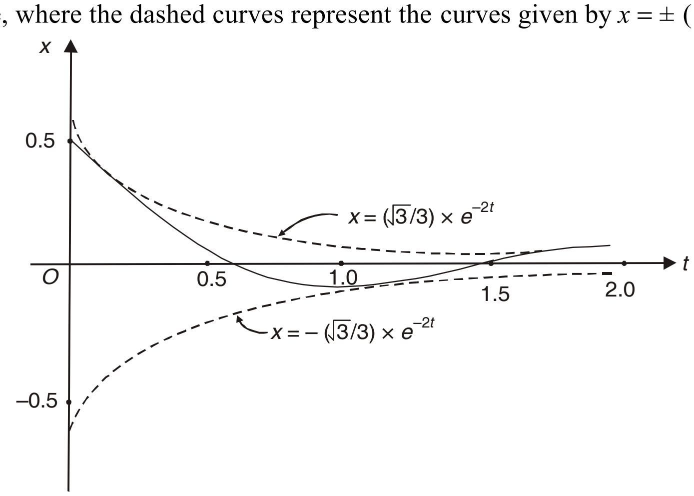
Hence from (3), the solution of (1) is given by

$$x = e^{-2t}\{(\sqrt{3}/6) \times \sin 2\sqrt{3}t + (1/2) \times \cos 2\sqrt{3}t\} \quad ...(5)$$

But $\dfrac{\sqrt{3}}{6} \sin 2\sqrt{3}t + \dfrac{1}{2} \cos 2\sqrt{3}t = \dfrac{\sqrt{3}}{3}\left(\dfrac{1}{2} \sin 2\sqrt{3}t + \dfrac{\sqrt{3}}{2} \cos 2\sqrt{3}t\right) = \dfrac{\sqrt{3}}{3} \cos\left(2\sqrt{3}t - \dfrac{\pi}{6}\right)$

$\therefore$ (5) takes the form $x = \left(\sqrt{3}/3\right) \times e^{-2t} \cos(2\sqrt{3}t - \pi/6)$ ...(6)

Interpretation: (6) represents a damped oscillatory motion. The damping factor is $(\sqrt{3}/3)\,e^{-2t}$, the perod is $(2\pi)/(2\sqrt{3}) = (\sqrt{3}\,\pi)/3$. The graph of the solution (6) is shown in the following figure, where the dashed curves represent the curves given by $x = \pm(\sqrt{3}/3) \times e^{-2t}$.



Example 5. *A 32-pound weight is attached to the lower end of a coil spring suspended from the ceiling. The weight comes to rest in its equilibrium position, thereby stretching the spring 2 feet. The weight is then pulled down 6 inches below its equilibrium position and released at t = 0. No external forces are present; but the resistance of the medium is numerically equal to 10 (dx/dt), where dx/dt is the instantaneous velocity in feet per second. Determine the resulting motion of the weight on the spring.*

Sol. Here e = the elongation of the spring after the weight is attached = 2 feet. Using Hooke's law, we have $32 = k \times 2$ so that $k = 16$ *lb*/ft. Again, $W = mg \Rightarrow 32 = m \times 32 \Rightarrow m = 1$ slug. Here damping factor = $a = 10$.

Using these facts, the basic differential equation of the vibrations of the given mass on the spring for free damped motion (refer Art 11.8) namely, $m(d^2x/dt^2) + a\,(dx/dt) + k\,x = 0$ reduces to

$d^2x/dt + 10(dx/dt) + 16x = 0$ or $(D^2 + 10\,D + 16)\,x = 0,\ D \equiv d/dt$...(1)

The initial conditions are: $x(0) = 6/12 = 1/2$ and $x'\,(0) = 0$...(2)

The auxiliary equaion for (1) as $D^2 + 10D + 16 = 0$ giving $D = -\,2, -\,8$.
Hence the system is over-damped (refer case I of Art 11.8). The general solution of (1) is

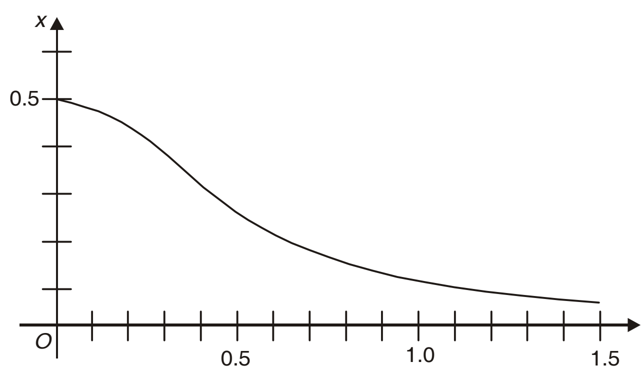
$$x(t) = C_1\, e^{-\,2t} + C_2\, e^{-\,8t},\ C_1 \text{ and } C_2 \text{ being arbitrary constants.}$$

From (3), $x'\,(t) = -\,2\, C_1\, e^{-\,2t} - 8\, C_2\, e^{-\,8t}$ ...(4)

Applying the initial conditions (2) to (3) and (4), we get $C_1 + C_2 = 1/2$ and $-\,2C_1 - 8C_2 = 0$ so that $C_1 = 2/3$ and $C_2 = -\,(1/6)$ Hence, from (3), the solution of the given problem is

$$x = (2/3) \times e^{-\,2t} - (1/6) \times e^{-\,8t} \qquad ...(5)$$

Interpretation: Qualitatively the motion is the same as that of the solution (5) of Ex. 2. Here, however, due to the increased damping, the weight returns to its equilibrium position at a slower rate. The graph of (5) is shown in the following figure.



11.10 Forced Motion

In the present article, we propose to discuss an important special case of forced motion. That is, we not only consider the effect of damping upon the mass on the spring but also the effect upon it of a periodic external impressed force F defined by $F(t) = p \cos \omega t$ for all $t \geq 0$, where p and ω are constants. Then the basic differential equation of forced motion is given by (refer equation (2) of Art 11.6).

$$m(d^2x/dt^2) + a\,(dx/dt) + kx = p \cos \omega t \qquad ...(1)$$

or $d^2x/dt^2 + 2b(dx/dt) + \lambda^2\, x = E \cos \omega t.$ or $(D^2 + 2bD + \lambda^2)x = E \cos \omega t.$ ...(2)

where, $2b = a/m,$ $k/m = \lambda^2,$ $p/m = E$ and $D \equiv d/dt$ ...(3)

We shall assume that the positive damping constant a is small enough so that the damping is less than critical. In other words we assume that $b < \lambda$. Now, the auxiliary equation for (2) is

$$D^2 + 2b\,D + \lambda^2 = O \qquad \text{so that} \qquad D = -\,b \pm i\,(\lambda^2 - b^2)^{1/2}$$

$\therefore$ Complementary function of (2) $= C\,e^{-bt} \cos\,((\lambda^2 - b^2)^{1/2}\,t + \phi)$, where C and ϕ, are arbitrary constants. Again, as usual, P.I., (*i.e.*,) particular integral of (2) is given by

$$\text{P.I.} = \frac{1}{D^2 + 2bD + \lambda^2}\,E\cos\omega t = E\frac{1}{-\omega^2 + 2bD + \lambda^2}\cos\omega t$$

$$= E\left\{(\lambda^2 - \omega^2) - 2bD\right\} \times \frac{1}{\left\{(\lambda^2 - \omega^2) - 2bD\right\}\left\{(\lambda^2 - \omega^2) + 2bD\right\}}\cos\omega t$$

$$= E\left\{(\lambda^2 - \omega^2) - 2bD\right\} \times \frac{1}{(\lambda^2 - \omega^2)^2 - 4b^2D^2}\cos\omega t$$

$$= E\left\{(\lambda^2 - \omega^2) - 2bD\right\}\frac{1}{(\lambda^2 - \omega^2)^2 + 4b^2\omega^2}\cos\omega t$$

$$= \frac{E}{(\lambda^2 - \omega^2)^2 + 4b^2\omega^2}\left\{(\lambda^2 - \omega^2)\cos\omega t + 2b\omega\sin\omega t\right\}$$

$$= \frac{E}{\{(\lambda^2 - \omega^2)^2 + (2b\omega)^2\}^{1/2}}\left[\frac{\lambda^2 - \omega^2}{\{(\lambda^2 - \omega^2)^2 + (2b\omega)^2\}^{1/2}}\cos\omega t + \frac{2b\omega}{\{(\lambda^2 - \omega^2)^2 + (2b\omega)^2\}^{1/2}}\sin\omega t\right]$$

$$= \frac{E(\cos\theta\,\cos\omega t + \sin\theta\,\sin\omega t)}{\left\{(\lambda^2 - \omega^2)^2 + (2b\omega)^2\right\}^{1/2}} = \frac{E\cos(\omega t - \theta)}{\left\{(\lambda^2 - \omega^2)^2 + (2b\omega)^2\right\}^{1/2}}$$

where $$\frac{\lambda^2 - \omega^2}{\left\{(\lambda^2 - \omega^2)^2 + (2b\omega)^2\right\}^{1/2}} = \cos\theta, \qquad \frac{2b\omega}{\left\{(\lambda^2 - \omega^2)^2 + (2b\omega)^2\right\}^{1/2}} = \sin\theta \qquad \text{...(4)}$$

The general soluton of (2) is given by

$$x(t) = C\,e^{-bt}\cos\,((\lambda^2 - b^2)^{1/2}\,t + \phi) + E\{(\lambda^2 - b^2)^2 + 4b^2\omega^2\}^{-1/2}\cos\,(\omega t - \theta) \qquad \text{...(5)}$$

Observe that this solution is sum of two terms. The first term, $C\,e^{-bt}\cos\,((\lambda^2 - b^2)^{1/2}\,t + \phi)$ represents the damped oscillation which would be the entire motion of the system if the external force $p\cos\omega t$ were not present. The second term $E\{(\lambda^2 - \omega^2)^2 + 4b^2\omega^2\}^{-1/2}\cos\,(\omega t - \theta)$, which results from the presence of the external force, represents a simple harmonic motion of period $2\pi/\omega$. Because of the damping factor Ce^{-bt} the contribution of the first term will decrease as time increases and will eventually become negligible. The first term is thus known as *transient term*. The second term however, being cosine term of constant amplitude, continues, to contribute to the motion in a periodic, oscillatory manner. Eventually, the transient term having become relatively small, the entire motion will consist essentially of that given by this second term. This second term in thus known as the *steady state term.*

An illustrative example. *A 16-lb weight is attached to the lower end of a coil spring suspended from the ceiling, the spring constant of the spring being 10 lb/ft. The weigth comes to rest in its equilibrium position. Beginning at t = 0 an external force given by F (t) = 5 cos 2t is applied to the system. Determine. the resulting motion if the damping force is numerically equal to 2 (dx/dt), where dx/dt is the instantaneous velocity in feet per second.* **[Delhi Maths 2007]**

Sol. For the present problem, k = the spring constant = 10 lb/ft. Again, $W = mg \Rightarrow 16 = 32$ m $\Rightarrow m = 1/2$ (slug) and the damping factor $= a = 2$.Also, external force $= F(t) = 5\cos 2t$. Using the above facts the basic differential equation of the vibrations of the given mass on the spring for forced motion (refer Art 11.6), namely, $m\,(d^2x/dt^2) + a(dx/dt) + kx = F(t)$ reduces to

$$(1/2) \times (d^2x/dt^2) + 2(dx/dt) + 10x = 5 \cos 2t \quad \text{or} \quad (D^2 + 4D + 20)\,x = 10 \cos 2t \quad ...(1)$$

The initial conditions are: $x(0) = 0$ and $x'(0) = 0$ ...(2)

The auxiliary equation for (1) is $D^2 + 4D + 20 = 0$ giving $D = -2 \pm 4i$. Hence, for (1),

Complementary function $= e^{-2t}(C_1 \sin 4t + C_2 \cos 4t)$, C_1, C_2 being arbitrary constants

$$\text{P.I.} = \frac{1}{D^2+4D+20} 10\cos 2t = 10\,\frac{1}{-2^2+4D+20}\cos 2t = \frac{5}{2}\,\frac{1}{D+4}\cos 2t$$

$$= \frac{5}{2}(D-4)\frac{1}{(D-4)(D+4)}\cos 2t = \frac{5}{2}(D-4)\frac{1}{D^2-16}\cos 2t$$

$$= \frac{5}{2}(D-4)\frac{1}{-2^2-16}\cos 2t = -\frac{1}{8}(D\cos 2t - 4\cos 2t)$$

$$= -(1/8) \times (-2 \sin 2t - 4 \cos 2t) = (1/2) \times \cos 2t + (1/4) \times \sin 2t$$

Hence the general solution of (1) is given by

$$x(t) = e^{-2t}(C_1 \sin 4t + C_2 \cos 4t) + (1/2) \times \cos 2t + (1/4) \times \sin 2t \quad ...(3)$$

Differentiating (3) w.r.t. 't', we obtain

$$x'(t) = -e^{-2t}(C_1 \sin 4t + C_2 \cos 4t) + 4e^{-2t}(C_1 \cos 4t - C_2 \sin 4t) - \sin 2t + (1/2) \times \cos 2t$$

or $$x'(t) = e^{-2t}\{(-2C_1 - 4C_2)\sin 4t + (-2C_2 + 4C_1)\cos 4t\} - \sin 2t + (1/2) \times \cos 2t \quad ...(4)$$

Putting $t = 0$ in (3) and (4) and using the condition (2), we get

$C_2 + 1/2 = 0$ and $4C_1 - 2C_2 + 1/2 = 0$ so that $C_1 = -(3/8)$ and $C_2 = -(1/2)$

$\therefore$ (3) reduces to $x(t) = -(1/8) \times e^{-2t}(3 \sin 4t + 4 \cos 4t) + (1/4) \times (2 \cos 2t + \sin 2t)$ ...(5)

We now re-write (5) in the "phase angle" form as follows:

We have, $3 \sin 4t + 4 \cos 4t = 5\{(3/5) \times \sin 4t + (4/5) \times \cos 4t\} = 5 \cos(4t - \phi)$

where, $\cos\theta = 4/5$ and $\sin\phi = 3/5$ ...(6)

Again, $2 \cos 2t + \sin 2t = \sqrt{5}\,\{(2/\sqrt{5}) \times \cos 2t + (1/\sqrt{5}) \times \sin 2t\} = \sqrt{5}\cos(2t - \theta)$

where, $\cos\theta = 2/\sqrt{5}$ and $\sin\theta = 1/\sqrt{5}$ ...(7)

Using above results, (5) may be re-written as

$$x = -(5/8) \times e^{-2t}\cos(4t - \phi) + (\sqrt{5}/4) \times \cos(2t - \theta), \quad ...(8)$$

where ϕ and θ are given by (7) and (8) respectively. Thus, we obtain $\phi \approx 0.64$ (rad) and $\theta \approx 0.46$ (rad). Thus the solution (8) is given approximately by

$$x = -0.63\, e^{-2t}\cos(4t - 0.64) + 0.56 \cos(2t - 0.46) \quad ...(9)$$

Explanation: The first term on R.H.S. of (9) is the transient term, representing a damped oscillatory motion. It becomes negligble in a short time, for example, for $t > 3$, its numerical value is less han 0.002. Its graph is shown in figure *(i).* The second term of the R.H.S. of *(9)* is the steady term, representing a simple harmoic motion of amlitude 0.56 and period π. Its graph is shown in figure *(ii).* The graph in figure *(iii)* is that of the complete solution. From this figure we see that the effect of the transient term soon becomes negligible, and after a short time the contribution of the steady state term is essentially all that remains.

Fig (i)

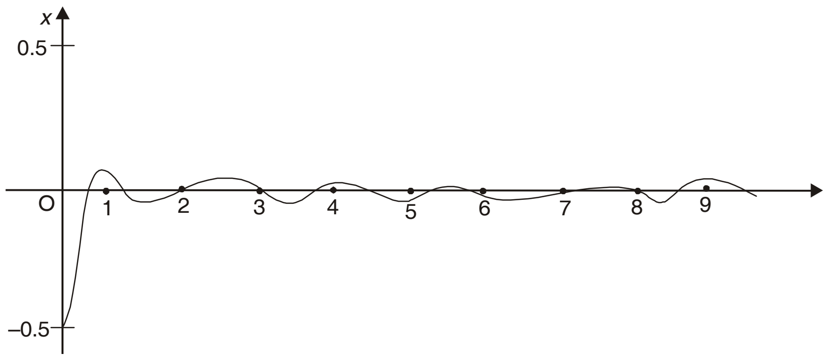


Fig (ii)

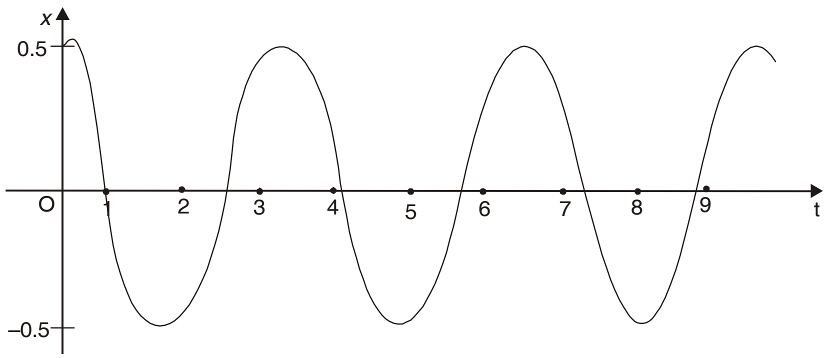
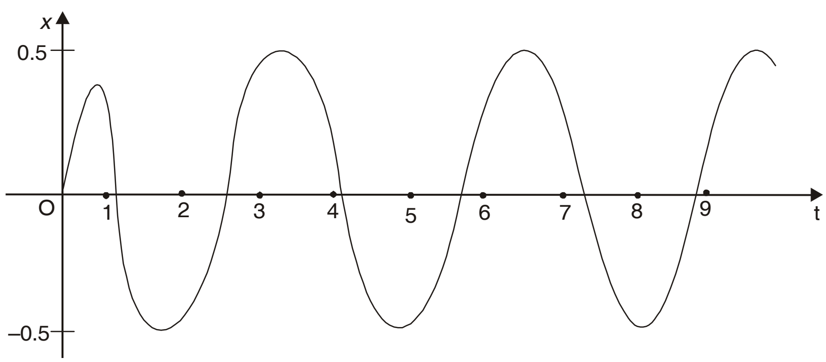


Fig (iii)



11.11 Resonance Phenomena

Refer Art 11.10 Let us examine the amplitude of steady-state vibration which results from the periodic external force defined for all t by $F(t) = p \cos \omega t$, where we assume that $p >$o. For fixed b, λ and E, we obtain from equation (5) of Art. 11.10 that this is the function f of ω defined by

$$f(\omega) = E/\{(\lambda^2 - \omega^2)^2 + 4b^2\omega^2\}^{1/2} \qquad \text{...(i)}$$

If $\omega = 0$, then $F(t)$ is the constant p and the amplitude $f(\omega)$ has the value $E/\lambda^2 > 0$. Also, from (i), as $\omega \to \infty$, $f(\omega) \to 0$. Let us examine the function f for $0 < \omega < \infty$.

From *(i)*, $\quad f'(\omega) = -(1/2) \times E\{(\lambda^2 - \omega^2)^2 + 4b^2\omega^2\}^{-3/2} \times \{2(\lambda^2 - \omega^2) \times (-2\omega) + 8b^2\omega\}$

or $\quad f'(\omega) = -2\omega E\{2b^2 - (\lambda^2 - \omega^2)\} \times \{(\lambda^2 - \omega^2)^2 + 4b^2\omega^2\}^{-3/2}$

$\therefore \quad f'(\omega) = 0 \Rightarrow \omega\{2\lambda^2 - (\lambda^2 - \omega^2)\} = 0$, as $E > 0$

Thus, $f'(\omega) = 0$ only if $\omega = 0$ or $\omega = (\lambda^2 - 2b^2)^{1/2}$. Three cases arise:

Case 1. If $\lambda^2 < 2b^2$, $(\lambda^2 - 2b^2)^{1/2}$ is a complex number. Hence in this case f has no extremum for $0 < \omega < \infty$, but rather f decreases monotonically for $0 < \omega < \infty$ from the value E/λ^2 at $\omega = 0$ and approaches zero as $\omega \to \infty$.

Case 2. If $\lambda^2 > 2b^2$, then f has a relative maximum at $\omega_1 = (\lambda^2 - 2b^2)^{1/2}$ and this maximum value is given by

$$f(\omega_1) = \frac{E}{\left\{(2b^2)^2 + 4b^2(\lambda^2 - 2b^2)\right\}^{1/2}} = \frac{E}{2b(\lambda^2 - b^2)^{1/2}} \quad ...(ii)$$

When the frequency of the forcing function $p \cos \omega t$ is such that $\omega = \omega_1$, then the forcing function is said to be in *resonance* with the system. In other words, the forcing function defined by $p \cos \omega t$ is in resonance with system when ω assumes the value ω_1 at which $f(\omega)$ is a maximum. The value $\omega_1/2\pi$ is known as the *resonance frequency* of the system. It is to be noted carefully that resonance occurs only when $\lambda^2 > 2b^2$. Since $\lambda^2 > b^2$, the damping force must be less than critical in such a case.

Refer equation (1) of Art. 11.10. Using relations of equation (3) of Art. 11.10, we express $f(\omega)$ in terms of m, a, k and p and obtain.

$$f(\omega) = \frac{(p/m)}{\left\{(k/m - \omega^2)^2 + (a/m)^2\omega^2\right\}^{1/2}} \quad ...(iii)$$

Also, the resonance frequency $= \dfrac{\omega_1}{2\pi} = \dfrac{(\lambda^2 - 2b^2)^{1/2}}{2\pi} = \dfrac{1}{2\pi}\left(\dfrac{k}{m} - \dfrac{a^2}{2m^2}\right)^{1/2}$ ...(iv)

Since the frequency of the corresponding free, damped oscillation is given by

$$(1/2\pi) \times (k/m - a^2/4m^2)^{1/2}, \quad ...(v)$$

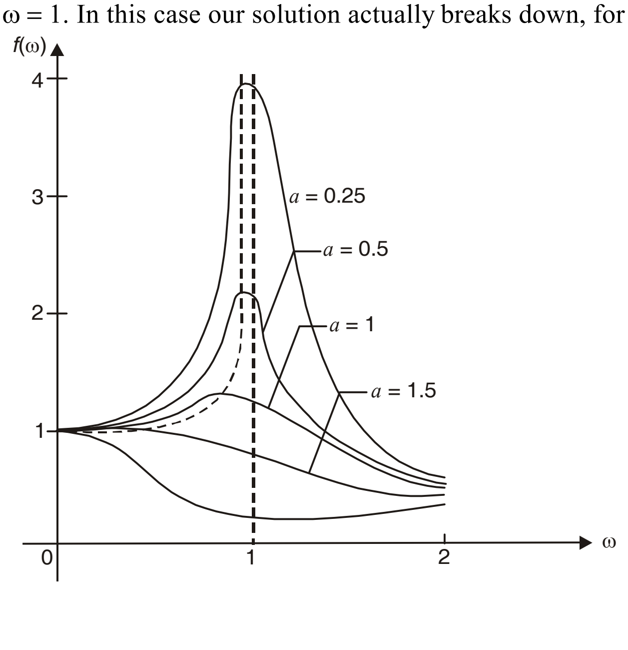
we find that the resonance frequency is less than that of the corresponding free, damped oscillation.

The graph of $f(\omega)$ is known as the *resonance curve* of the system. For a given system with m, k and p, there is a resonance curve corresponding to each value of the damping coefficient $a \geq 0$. Taking $m = k = p = 1$, we now, graph the resonance curves corresponding to certain selected values of a. For this particular case, we have

$$f(\omega) = \{(1 - \omega^2)^2 + a^2\ \omega)^2\}^{-1/2}$$

and the resonance frequency is given by $(1/2\pi) \times (1 - a^2/2)^{½}$. See figure ($i$) For the present case note that resonance occurs only if $a < \sqrt{2}$. As a decrease from $\sqrt{2}$ to 0, the value ω_1 at which resonance occurs increases from 0 to 1 and the corresponding maximum value of $f(\omega)$ becomes larger and larger. In the limiting case $a = 0$, the maximum has disappeared and an infinite discontinuity occurs at $\omega = 1$. In this case our solution actually breaks down, for then

Fig. (i)



$$f(\omega) = 1/\{(1 - \omega^2)^2\}^{1/2} = 1/(1 - \omega^2), \qquad ...(vi)$$

showing that $f(1)$ is undefined. This limiting case is an example of undamped resonance, a phenomenon which we shall now discuss.

Undamped resonance occurs when there is no damping and the frequency of the impressed force in equal to the natural frequency of the system. Since in this case $a = 0$ and the frequency $\omega/2\pi$ equals the natural frequency $(1/2\pi) \times (k/m)^{1/2}$ (use equation (v) with $a = 0$), the differential equaion (1) of Art. 11.10 reduces to

$$m(d^2x/dt^2) + kx = p \cos \{t(k/m)^{1/2}\} \quad \text{or} \quad d^2x/dt^2 + (k/m)x = E \cos \{t(k/m)^{1/2}\}$$

or $$\{D^2 + (k/m)\}x = E \cos \{t(k/m)^{1/2}\}, \quad \text{where} \quad E = p/m \qquad ...(vii)$$

The auxiliary equation for (vii) is $D^2 + k/m = 0$ giving $D = \pm\, i(k/m)^{1/2}$.

$\therefore$ $C.F. = C \cos (t\sqrt{k/m} + \phi)$, C, and ϕ being arbitrary constants

and $$\text{P.I.} = x_p = \frac{1}{D^2\{(k/m)^{1/2}\}^2} E \cos\{t(k/m)^{1/2}\}$$

$$= E\frac{t}{2\times(k/m)^{1/2}}\sin\{t(k/m)^{1/2}\}, \quad \text{as} \quad \frac{1}{D^2+a^2}\cos ax = \frac{x}{2a}\sin ax$$

Hence the general soluion of (vii) is given by

$$x = C \cos (t\sqrt{k/m} + \phi) + (E\,t/2) \times (m/k)^{1/2} \sin \{t(k/m)^{1/2}\} \quad ...(viii)$$

The motion defined by ($viii$) is thus the sum of a periodic term and an oscillatory term whose magnitude $(E/2) \times (m/k)^{1/2}t$ increases with t. The graph of the function defined by the second term on R.H.S. of ($viii$) is shown in the adjoining figure. As t increases, this term clearly dominates the entire motion. One might argue that ($viii$) informs us that as $t \to \infty$ the oscillations will become infinite. However, common sense intervenes and convinces us that before this exciting phenomenon can occur the system will break down and then ($viii$) will no longer apply.

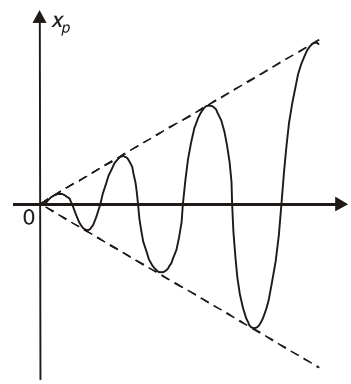


Figure (ii)

An Illustrative Solved Example

A 64–lb weight is attached to the lower end of a coil spring suspended from the ceiling, the spring constant being 18lb/ft. The weight comes to rest in its equlitium position. It in then pulled down 6 inches below its equilibrium position and released at t = 0. At this instant an external force given by F(t) = 3 cos ωt is applied to the system. (i) Assuming the damping force in pounds is numerically equal to 4(dx/dt), where dx/dt is the instantaneouse velocity in feet per second, determine the resonance frequency of resulting motion. (ii) Assuming there is no damping, determine the value of ω which gives rise to undamped resonance.

Sol. For the present problem, k = the spring constant = 18lb/ft, $W = mg \Rightarrow 64 = 32m$ $\Rightarrow m = 2$ (slugs), damping factor = a and external force = $F(t) = 3 \cos \omega t$. Using these facts, the basic differential equation of the vibrations of the given mass on the spring for forced motion (refer Art. 11.6), namely, $m\,(d^2x/dt) + a\,(dx/dt) + kx = F(t)$ reduces to

$$2\,(d^2x/dt^2) + a\,(dx/dt) + 18x = 3\cos \omega t \qquad ...(1)$$

Part (i). For this case $a = 4$. Hence (1) reduces to

$$2\,(d^2x/dt^2) + 4\,(dx/dt) + 18x = 3 \cos \omega t \qquad ...(2)$$

In part (*i*), we are not required to solve (2). Using formula (4) of Art. 11.11 we have

$$\text{Resonance frequency} = \frac{1}{2\pi}\left(\frac{k}{m} - \frac{a^2}{2m^2}\right)^{1/2} = \frac{1}{2\pi}\left(\frac{18}{2} - \frac{16}{2\times 4}\right)^{1/2} = \frac{\sqrt{7}}{2\pi} \approx 0.42 \text{ (cycles/sec)}$$

Therefore resonance occurs when $\omega = \sqrt{7} \approx 2.65$

Part (*ii*). In this case $a = 0$. Hence (1) reduces to

$$d^2x/dt^2 + 9x = (3/2) \times \cos \omega t \quad \text{or} \quad (D^2 + 9)x = (3/2) \times \cos \omega t, \text{ where } \quad D \equiv d/dt \quad ...(3)$$

Undamped resonance occurs when the frequency $\omega/2\pi$ of the impressed force is equal to the natural frequency. The *C.F.* of (3) is $C_1 \sin 3t + C_2 \cos 3t$, where C_1 and C_2 are arbitrary constants. From this we find that the natural frequency is $3/2\pi$. Thus, $\omega = 3$ gives rise to undamped resonance and equation (3) in this case reduces to

$$d^2x/dt^2 + 9x = (3/2) \times \cos 3t \qquad \text{or} \qquad (D^2 + 9)x = (3/2) \times \cos 3t \qquad ...(4)$$

The initial conditions are $\qquad x(0) = 1/2 \qquad$ and $\qquad x'(0) = 0 \qquad$ ...(5)

C.F. of (4) $= C_1 \cos 3t + C_2 \sin 3t$, C_1 and C_2 being arbitrary constants

and $\quad$ $$\text{P.I.} = \frac{1}{D^2+9}\frac{3}{2}\cos 3t = \frac{3}{2}\frac{1}{D^2+3^2}\cos 3t = \frac{3}{2}\frac{t}{2\times 3}\sin 3t = \frac{1}{4}t\sin 3t,$$

$$\left[\because \frac{1}{D^2+a^2}\cos at = \frac{t}{2a}\sin at\right]$$

Hence the general solution of (4) satisfying (5) is $x = C_1 \cos 3t + C_2 \sin 3t + (t/4) \times \sin 3t$

EXERCISE 11(B)

1. A 12–*lb* weight is placed upon the lower end of a coil spring suspended from the ceiling. The weight comes to rest in its equilibrium position, thereby stretching the spring 1.5 inches. The weight is then pulled 2 inches below its equilibrium position and released from rest at $t = 0$. Find the displacement of the weight as a function of the time, determine the amplitude, period, and frequency of the resulting motion.

Ans. $x(t) = (1/6) \times \cos 16t$, $\quad$ 1/6ft, $\quad$ $\pi/8$ sec, $\quad$ $8/\pi$ oscillations/sec.

2. A 4–*lb* weight is attached to the lower end of a coil spring suspended from the ceiling. The weight comes to rest in its equilibrium position, thereby stretching the spring 6 inches. At $t = 0$ the weight is then struck so as to set it into motion with an initial velocity 2ft/sec, directed downward. (*i*) Deermine the resulting displacement and velocity of the weight as function of time (*ii*) Find the amplitude, period, and frequency of the motion (*iii*) Determine the times at which the weight is 1.5 inches below its equilibrium position and moving downward (*iv*) Determine the times at which it is 1.5 inches below its equilibrium position and moving upward.

Ans. (*i*) $x = (1/4) \times \sin 8t$ $\qquad$ (*ii*) 1/4 ft, $\pi/4$ sec, $4/\pi$ oscillations/sec.

(*iii*) $t = \pi/8 + (n\pi)/4$ $(n = 0, 1, 2, ...)$ $\quad$ (*iv*) $t = (5\pi)/48 + (n\pi)/4$ $(n = 0, 1, 2, ...)$

3. An 8–*lb* weight is attached to the lower end of a coil spring suspended from the ceiling and comes to rest in its equilibrium position, thereby stretching the spring 0.4 ft. The weight is then pulled down 6 inches below its equilibrium position and released at $t = 0$. The resistance of the medium in pounds is numerically equal to 2 (dx/dt), where dx/dt is the instantaneous velocity in feet per second. (*i*) Set up the differential equation for the motion and list the initial conditions (*ii*) Solve the initial value problem set up in part (*i*) to determine the displacement of the weight as a function of the time. (*iii*) Express the solution found in part (*ii*) in an alternative form also (*iv*) what is the so called "period" of the motion. (v) Graph the displacement as a function of the time.

Ans. (*i*) $(1/4) \times (d^2x/dt^2) + 2\,(dx/dt) + 20\,x = 0$, $x(0) = 1/2$, $x'(0) = 0$

(*ii*) $x = e^{-4t}\,\{(1/4) \times \sin 8t + (1/2) \times \cos 8t\}$

(*iii*) $x = (\sqrt{5}/4)\ e^{-4t} \cos(8t - \phi)$, where $\phi \approx 0.46$ (*iv*) $\pi/4$ sec.

4. An 8–*lb* weight is attached to the lower end of a coil spring suspended from a fixed support. The weight comes to rest in its equilibrium position, thereby stretching the spring 6 inches. The weight is then pulled down 9 inches below its equilibrium position and released at $t = 0$. The medium offers a resistance in pounds equal to $4(dx/dt)$, where dx/dt is the instantaneous velocity in feet per second. Determine the displacement of the weight as a function of the time.

[Delhi B.Sc. II (Prog) 2009, 10] **Ans.** $x = (6t + 3/4)e^{-8t}$

5. A 6–*lb* weight is attached to the lower end of a coil spring suspended from the ceiling, the spring constant being 27 *lb*/ft. The weight comes to rest in its equilibrium position, and beginning at $t = 0$ an external force given by $F(t) = 12 \cos 20\,t$ is applied to the system. Determine the resulting displacement as a function of the time, assuming damping is negligible.

[Delhi B.Sc. II (Prog) 2011] **Ans.** $x = (\cos 12t - \cos 20t)/4$

6. A 10–*lb* weight is hung on the lower end of a coil spring suspended from the ceiling, the spring constant of the spring being 20 *lb/ft*. The weight comes to rest in its equlibrium position, and beginning at $t = 0$ an external force given by $F(t) = 10 \cos 8t$ is applied to the system. The medium offers a resistance in pounds numerically equal to $5(dx/dt)$, where dx/dt is the instantaneous velocity in feet per second. Find the displacement of the weight of the weight as a function of the time. **Ans.** $x(t) = -\,2t\,e^{-8t} + (1/4) \times \sin 8t$

7. A 6–*lb* weight is hung on the lower end of a coil spring suspended from the ceiling. The weight comes to rest in its equilibrium position, thereby stretching the spring 4 inches. Then beginning at $t = 0$ an external force given by $F(t) = 27 \sin 4t - 3\cos 4t$ is applied to the system. If the medium offers a resistance in pounds numerically equal to three times the instantaneous velocity, measured in feet per second, find the displacement as a function of the time.

Ans. $x(t) = (1/2) \times e^{-8t}\,(\sqrt{2}\ \sin 4\sqrt{2}\,t + 2 \cos 4\sqrt{2}\,t) + \sin 4t - \cos 4t$

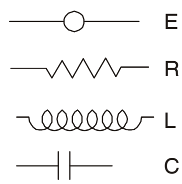
8. A 12–*lb* weight is attached to the lower end of a coil spring suspended from the ceiling. The weight comes to rest in its equilibrium position thereby stretching the spring 6 inches. Begninning at $t = 0$ an external force gives by $F(t) = 2\cos \omega t$ is applied to the system. *(i)* If the damping force in pounds is numerically equal to $3(dx/dt)$, where dx/dt is the instantaneous velocity in feet per second, determine the resonance frequency of the resulting motion and find the displacement as a function of the time when the forcing function is in resonance with the system.*(ii)* Assuming there is no damping, determine the value of ω which gives rise to undamped resonance and find the displacement as a function of the time in this case.

Ans. (*i*) $(2\sqrt{2})/\pi$; $x(t) = -\,(1/18) \times e^{-4t}\,(\sqrt{3} \sin 4\sqrt{3}\,t + \cos 4\sqrt{3}\,t) + (1/18) \times (\sqrt{2}\ \sin 4\sqrt{2}\,t$ $+ \cos 4\sqrt{2}\,t)$ *(ii)* 8; $x(t) = (t/3) \times \sin 8t$

9. The differential equation for the motion of a unit mass on a certain coil spring under the action of an external force of the form $F(t) = 30 \cos \omega t$ is $(d^2x/dt^2) + a\,(dx/dt) + 24x = 30 \cos \omega t$, where $a \geq 0$ is the damping coefficient *(i)* If $a = 4$, find the resonance frequency and determine the amplitude of the steady–state vibration when the forcing function is in resonance with the system. *(b)* Proceed as in part *(i)* if $a = 2$. **Ans.** (*i*) $2/\pi$; $3\sqrt{5}/4$ (*ii*) $\sqrt{22}/2\pi$; $15\sqrt{23}/23$

11.12 Electric Circuit Problems

In this article we propose to study the application of differential equations to series circuits containing (1) an electromotive force, and (2) resistors,inductors, and capacitors. In what follows, the following conventional symbols will be used.



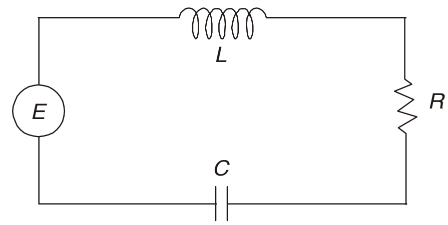


Figure showing LRC-series circuit

Some useful results related to series circuits:

Electromotive force (for example, a battery or generator) produces a flow of current in a closed circuit and that this current produces a so called voltage drop across each resistor, inductor and capacitor. See the following table for symbols and unis.

Quantity and symbol	*Unit*
emf or voltage E	volt (V)
current i	ampere
charge q	coulomb
resistance R	ohm (Ω)
inductance L	henry (H)
capacitance C	farad

Recall the following three laws concerning the voltage drops across resistor, inductor and capacitor:

Law I: The voltage drop E_R acorss a resisor is given by $E_R = R\ i,$ ...(1)
where R is a constant of proportionally called the *resistance*, and i the *current.*

Law II: The voltage drop E_L across an inductor is given by $E_L = L(di/dt),$ ...(2)
where L is a constant of proportionality called the *inductane.*

Law III: The voltage drop E_C across a capacitor is given $E_C = q/C,$ ...(3)
where C is a constant of proportionality called the *capacitance* and q is instantaneous *charge* on the capacitor.

The fundamental law in the study of electric circuits is the following:

Kirchhoff' Voltage Law: *The sum of the voltage drops across resistor, inductors, and capacitors is equal to the total electromotive force in a closed circuit.*

Let us apply Kirchhoff's law to the circuit of figure. Let E denote the electromotive force. Then, using the above mentioned laws 1, 2 and 3 for voltage drops, we obtain

$$L\ (di/dt) + R\ i + q/C = E \qquad ...(4)$$

containing two dependent variables i and q. But, we also have

$$i = dq/dt \qquad \text{so that} \qquad di/dt = d^2q/dt^2 \qquad ...(5)$$

Using (5), (4) takes the form $L\ (d^2q/dt^2) + R\ (dq/dt) + q/C = E,$ ...(6)
which is a second – order linear differential equation in the single dependent variable q. So we can obtain q from (6).

Now, differentiating (4) w.r.t. 't', gives $L\,(d^2i/dt^2) + R\,(di/dt) + (1/C) \times (dq/dt) = dE/dt$

or $$L\,(d^2i/dt^2) + R\,(di/dt) + (1/C) \times i = dE/dt, \text{ using (5)} \qquad ...(7)$$

which is a second order linear differential equation in the single dependent variable i. So we can obtain i from (7).

Particular cases: We now consider two very simple cases in which the problem reduces to a first order linear differential equation.

Case I: If the circuit contains no capacitor (so that $C = 0$), then (4) reduces to

$$L\,(di/dt) + R\,i = E \qquad ...(8)$$

Case II: If the circuit contains no inductor (so that $L = 0$), then (6) reduces to

$$R\,(dq/dt) + q/C = \mathrm{E} \qquad ...(9)$$

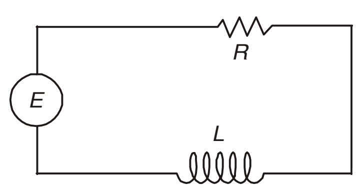
Electro-mechanical analogy: Observe that the differential equation (6) for the charge is exactly the same as the differential equation (2) of Art. 11.6 for the vibrations of a mass on a coil spring, except for the notations used. That is, the electrical system described by (6) is analogous to the mechanical system described by equation (2) of Art. 11.6.

Since electrical circuits are easy to assemble and the currents and voltages are accurately measured very easily, this affords a practical method of studying the oscillations of complicated mechanical systems which are expensive to make and unwieldy to handle by cosidering an equivalent electric circuit. While making an electric equivalent of a mechanical system, the correspondence between the elements shown in the following table should be kept in mind.

Mechanical system	*Electric system*
mass m	inductance L
damping constant a	Resistance R
spring constant k	reciprocal of capacitance = $1/C$
impressed force $F(t)$	impressed voltage or emf E
displacement x	charge q
velocity = $v = dx/dt$	current = $i = dq/dt$

11.13 Solved examples based on Art. 11.2

Ex.1. *A circuit has in series an electromotive force given by $E = 100 \sin 40t$ V, a resistor of 10 Ω and an inductor of 0.5 H. If the initial current in 0, find the current at time $t > 0$.*



Sol. For the given problem the circuit diagram is shown in the adjoining figure. Let i denote the current in amperes at time t. The total electromotive force is $100 \sin 40\,t$. Then, as usual (refer laws 1 and 2 of Art 11.12), we have

The voltage drop across the resistor $= E_R = Ri = 10i$

and the voltage drop across the inductor $= E_L = L\,(di/dt) = (1/2) \times (di/dt)$

Applying Kirchhoff's law, we have $(1/2) \times (di/dt) + 10i = 100 \sin 40t$

or $di/dt + 20i = 200 \sin 40t$, which is first order linear equation ...(1)

Since the initial current is 0, the initial condition is $i\,(0) = 0$ ...(2)

Integrating factor of (1) = $e^{\int 20 dt} = e^{20t}$ and hence its solution is

$$i\, e^{20t} = \int \{(200 \sin 40\, t) \times e^{20t}\}\, dt + C = 200 \int e^{20t} \sin 40t\, dt + C \quad ...(3)$$

$\therefore$ Since, from Integral Calculus, $\int e^{ax} \sin bx\, dx = \{e^{ax}(a \sin bx - b \cos bx)\}/(a^2 + b^2)$,

hence, $$\int e^{20t} \sin 40t\, dt = \frac{e^{20t}(20 \sin 40t - 40 \cos 40t)}{(20)^2 + (40)^2} = \frac{e^{20t}(\sin 40t - 2\cos 40t)}{100}$$

$\therefore$ (3) reduces to $$i\, e^{20t} = 2\, e^{20t} (\sin 40t - 2\cos 40t) + C$$

or $$i = 2 (\sin 40t - 2\cos 40t) + C\, e^{-20t} \quad ...(4)$$

Applying the condition (2), $i = 0$ when $t = 0$, (4) gives $C = 4$. Hence (4) becames

$$i = 2 (\sin 40 - 2 \cos 40t) + 4\, e^{-20t} \quad ...(5)$$

We transform (5) in a "phase – angle" form as follows:

$$\sin 40t - 2 \cos 40t = \sqrt{5}\left\{(1/\sqrt{5})\sin 40t - (2/\sqrt{5})\cos 40t\right\} = \sqrt{5} \sin (40t + \phi) \quad ...(6)$$

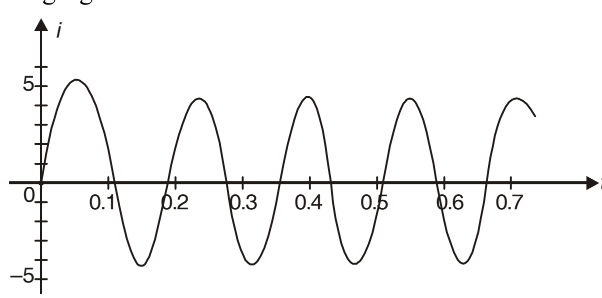
where $\cos \phi = 1/\sqrt{5}$ and $\sin \phi = 2/\sqrt{5}$ ...(7)

From (7), $\phi \approx -1.11$ radians Hence, $\sin 40t - 2 \cos 40t = \sqrt{5} \sin (40t - 1.11)$

$\therefore$ (5) transforms to $$i = 2\sqrt{5} \sin (40t - 1.11) + 4e^{-20t}$$

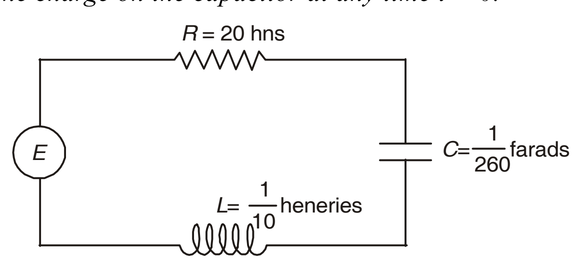
or $$i = 4.47 \sin (40t - 1.11) + 4e^{-20t} \quad ...(8)$$

Interpretation. The current is presented as the sum of a sinusodial term and an exponential. The exponential becomes so very small in a short time that its effect is soon practically negligible; it is the *transient term*. Thus, after a short time, essentially all that remains is the sinusodial term; it is the *steady current*. Observe that its period $\pi/20$ is the same as that of the electromotive force. However, the phase angle $\phi \approx -1.11$ radians indicates that the electromotive force leads the steady–state current by approximately (1/46) × 1.11. The grah of the current as a function of time is shown in the following figure.



Ex. 2. *A circuit has in series an electromotive force given by E = 100 sin 60 t V, a resistor of 2Ω, an inductor of 0.1H, and a capacior of 1/260 farads. If the initial current and the initial charge on the capacitor are both zero, find the charge on the capacitor at any time t > 0.*

Sol. The circuit diagram is shown in the adjoining figure. We have $L = 1/10$ heneries, $C = 1/260$ farads, $R = 2$ ohms and $E = 100 \sin 60t\ V$. Let q denote the instantaneous charge on the capacitor. Then q is given in terms of L, C, R and E by the following second–order linear differential equation:



$L\,(d^2q/dt^2) + R\,(dq/dt) + q/C = E$ or $(1/10) \times (d^2q/dt^2) + 2\,(dq/dt) + 260\,q = 100 \sin 60\,t$

or $(D^2 + 20\,D + 2600)q = 1000 \sin 60t$, where $D \equiv d/dt$ ...(1)

Since the charge q is initially zero, we have first initial condition: $q(0) = 0$ ...(2)

Since the current i is also initially zero and $i = dq/dt = q'(t)$, we have

Second initial condition: $q'(0) = 0$ ...(3)

The auxiliary equation for (1) is $D^2 + 20\,D + 2600 = 0$ giving $D = -10 \pm 50\,i$.

$\therefore$ q_c = C.F. of (1) $= e^{-10t}(C_1 \sin 50\,t + C_2 \cos 50\,t)$, C_1, C_2 being arbitrary constants

and $$q_p = \text{P.I.} = \frac{1}{D^2 + 20D + 2600} 1000 \sin 60t = 1000 \frac{1}{-(60)^2 + 20D + 2600} \sin 60t$$

$$= 50 \frac{1}{D - 50} \sin 60t = 50\,(D + 50)\, \frac{1}{D^2 - (50)^2} \sin 60t$$

$$= 50\,(D + 50)\, \frac{1}{-(60)^2 - (50)^2} \sin 60t = -\frac{1}{(2 \times 61)}(60 \cos 60t + 50 \sin 60t)$$

$$= -(25/61) \times \sin 60t - (30/61) \times \cos 60t$$

Hence the general solution of (1) is $q = q_c + q_p$, that is,

$$q(t) = e^{-10t}(C_1 \sin 50t + C_2 \cos 50t) - (25/61) \times \sin 60t - (30/61) \times \cos 60t \quad ...(4)$$

Differentiating (4) w.r.t. 't' and simplifying, we obtain

$$q'(t) = e^{-10t}\,\{(-10C_1 - 50C_2\}\sin 50t + (50C_1 - 10C_2)\cos 50t\}$$
$$- (500/61) \times \sin 60t + (1800/61) \times \sin 60t \quad ...(5)$$

Applying condition (2) to equation (4) and condition (3) to equation (5), we get

$C_2 - (30/61) = 0$ and $50\,C_1 - 10\,C_2 - (1500/61) = 0$ giving $C_1 = 36/61$, $C_2 = 30/61$

Substituting these values in (4), the required solution is

$$q = (6/61) \times e^{-10t}(6 \sin 50t + 5 \cos 50t) - (5/61) \times (5 \sin 60t + 6 \cos 60t) \quad ...(6)$$

We shall now re-write (6) in a "phase-angle" form. We have

$$6 \sin 50t + 5 \cos 50t = \sqrt{61}\left\{(6/\sqrt{61} \sin 50t + (5/\sqrt{61}) \cos 50t\right\} = \sqrt{61} \cos(50t - \phi), \quad ...(7)$$

where $\cos \phi = 5/\sqrt{61}$ and $\sin \phi = 6/\sqrt{61}$ ...(8)

and $5 \sin 60\text{t} + 6 \cos 60\text{t} = \sqrt{61}\left\{(5/\sqrt{61}) \sin 60t + (6/\sqrt{61}) \cos 60t\right\} = \sqrt{61} \cos(60t - \theta)$, ...(9)

where $\cos \theta = 6/\sqrt{61}$ and $\sin \theta = 5/\sqrt{61}$ ...(10)

From (8) and (9), we get $\phi \approx 0.88$ radians and $\theta \approx 0.69$ radians

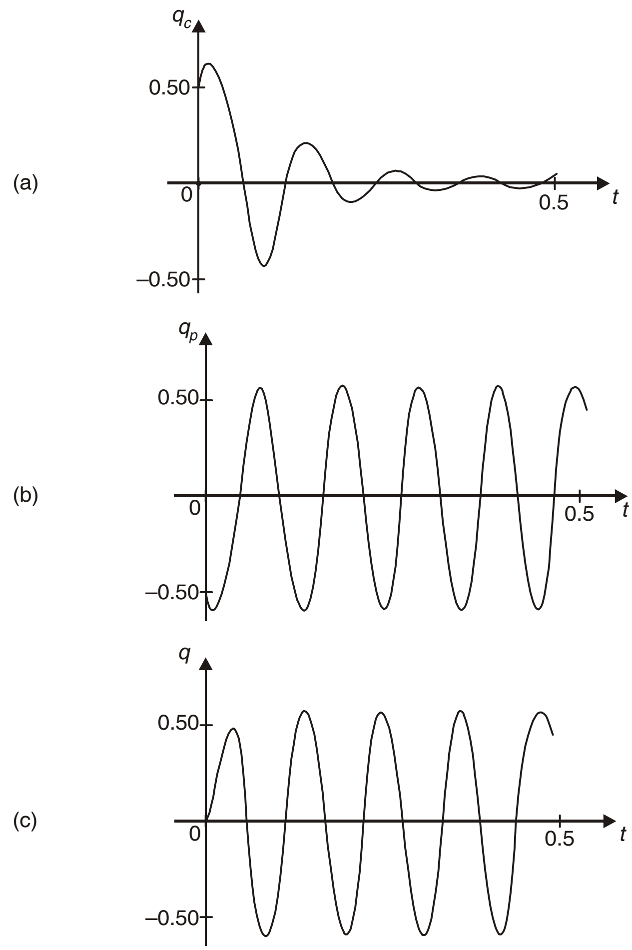
$\therefore$ (7) reduces to $6 \sin 50t + 5 \cos 50t = \sqrt{61} \cos(50t - 0.88)$

and (9) reduces to $5 \sin 60t + 6 \cos 60t = \sqrt{61} \cos(60t - 0.69)$

Using the above results (6) takes the form

$$q = 0.77\, e^{-10t} \cos(50t - 0.88) - 0.64 \cos(60t - 0.69) \quad ...(11)$$

Interpretation: Clearly the first term on R.H.S. of (11) becomes negligible after a relatively short time, it is the *transient term*. After a sufficient time essentially all that remains is the periodic second term on the R.H.S. of (11); this is the *steady–state term.* The graph of these two components and that of their sum (the complete solution) are shown in the following diagrams.



EXERCISE 11(C)

1. A circuit has in series a constant electromotive force of 40 V, a resistor of 10Ω and an inductor of 0.2H. If the initial current is 0, find the current at time $t > 0$. **Ans.** $i = 4(1 - e^{-50t})$

2. A circuit has in series a constant electromotive force of 100 V, a resistor of 10Ω and a capacitor of 2×10^{-4} farads. The switch is closed at time $t = 0$, and the charge on the capacitor at this instant is zero. Find the charge and current at time $t > 0$. **Ans.** $q = (1 - e^{-500t})/50$; $i = 10\, e^{-500t}$

3. A circuit has in series an electromotive force given by $E(t) = 100 \sin 200t\ V$, a resistor of 40Ω, an inductor of 0.25 H, and *a* capacitor of 4×10^{-4} farads. If the inital current is zero, and he initial charge on the capacitor is 0.01 coulombs, find the current at any time $t > 0$.

Ans. $i = e^{-80t}(-4.588 \sin 60t + 1.247 \cos 60t) - 1.247 \cos 200t + 1.331 \sin 200t$

4. A circuit has in series a resistor R Ω, an inductor L H, and a capacitor of C farads. The initial current is zero and the initial charge on the capacitor in Q_0 coulombs.

(a) Show that the charge and the current are damped oscillatory functions of time if and only if $R < 2\,(L/C)^{1/2}$, and find the expressions for the charge and the current in this case.

(b) If $R \geq 2\,(L/C)^{1/2}$, discuss the nature of the charge and current as functions of time.

Ans. (*a*) $i = -\dfrac{2Q_0}{(4LC - R^2C^2)^{1/2}} e^{-Rt/2L} \sin\left(\dfrac{(4L - R^2C)^{1/2}}{2L\sqrt{C}} t\right)$;

$$q = e^{-Rt/2L}\left[\frac{Q_0 R\sqrt{C}}{(4L - R^2C)^{1/2}} \sin\left(\frac{(4L-R^2C)^{1/2}}{2L\sqrt{C}} t\right) + Q_0 \cos\left(\frac{(4L-R^2C)^{1/2}}{2L\sqrt{C}} t\right)\right]$$

Part III. Applications to Simultaneous Differential Equations

11.14 Applications to Mechanics

System of linear differential equations originate in the mathematical formulation of various problems in mechanics. In the next article we shall discuss such problems in details. For details of method of solution refer Chapter 8.

11.15 Solved example based on Art. 11.14

Ex.1. *On a smooth horizontal plane BC an object A_1 is connected to a fixed point P by a massless spring S_1 of natural length L_1. An object A_2 is then connected to A_1 by a massless spring S_2 of natural length L_2 in such a way that the fixed point P and the centres of gravity A_1 and A_2 all lie in a straight line (refer fig. (i))*

The object A_1 is then displaced a distance a_1 to the right or left of its equilibrium position O_1, the object A_2 is displaced a distance a_2 to the right or left of its equilibrium position O_2 and at time $t = 0$ the two objects are released (see fig. (ii)). What are the positions of the two objects at any time $t > 0$.

Sol. Let m_1 and m_2 be masses of objects A_1 and A_2 respectively Also, assume that spring constants of springs S_1 and S_2 be k_1 and k_2 respectively.

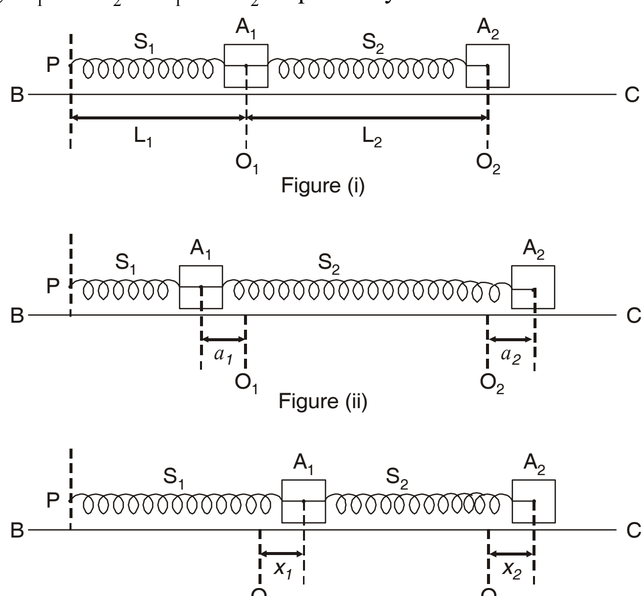


Figure (i)

Figure (ii)

Figure (iii)

Let x_1 denote the displacement of A_1 from its equilibrium position at time $t \geq 0$ and assume that x_1 is positive when A_1 is to the right of O_1. Similarly, let x_2 denote the displacement of A_2 from its equilibrium position O_2 at time $t \geq 0$ and assume that x_2 is positive when A_2 is to the right of O_2 (see figure (*iii*)).

The forces acting on A_1 at time $t > 0$ are: *(i)* Force F_1 exerted by the spring S_1 *(ii)* Force F_2 exerted by spring S_2. By Hooke's law, the force F_1 is of magnitude $k_1|x_1|$. Since this force is exterted toward the left when A_1 is to the right of O_1 and toward the right when A_1 is to the left of O_1, we have $F_1 = - k_1 x_1$. Again using Hooke's law, the force F_2 is of magnitude $k_2 e$, where e is the elongation of S_2 at time t. But $e = |x_2 - x_1|$ and hence magnitude of F_2 is $k_2\ |x_2 - x_1|$. Again, since this force is exerted toward the left when $x_2 - x_1 < 0$ and toward the right when $x_2 - x_1 > 0$, we have $F_2 = k_2\ (x_2 - x_1)$.

Applying Newton's second law to the object A_1, we have

$$m_1(d^2x/dt^2) = - kx_1 + k_2\ (x_2 - x_1) \qquad \text{or} \qquad (m_1D^2 + k_1 + k_2)\ x_1 - k_2\ x_2 = 0, \quad ...(1)$$

where $D \equiv d/dt$. The object A_2 is acted upon by only one force F_3 which is exerted by spring S_2. By Hooke's law, the magnitude of F_3 is $k_2|x_2 - x_1|$. Since F_3 is exerted toward the left when $x_2 - x_1 > 0$ and toward the right when $x_2 - x_1 < 0$, we have $F_3 = - k_2(x_2 - x_1)$. Applying Newton's second law to the object A_2, we have

$$m_2\ (d^2x_2/dt^2) = - k_2(x_2 - x_1) \qquad \text{or} \qquad (m_2D^2 + k_2)\ x_2 - k_2\ x_1 = 0 \qquad ...(2)$$

From the statement of the problem, the inditial conditions are:

$$x_1(0) = a_1, \qquad x'(0) = 0, \qquad x_2(0) = a_2, \qquad \text{and} \qquad x'_2\ (0) = 0 \qquad ...(3)$$

Solution of a specific case: Suppose the two objects A_1 and A_2 are each of unit mass, so that $m_1 = m_2 = 1$. Also, suppose that the springs S_1 and S_2 have spring constants $k_1 = 3$ an $k_2 = 2$, repectively. Further, we take $a_1 = -1$ and $a_2 = 2$. Then, (1), (2) and (3) take the following forms:

$$(D^2 + 5)x_1 - 2x_2 = 0 \qquad ...(4)$$

$$- 2x_1 + (D^2 + 2)\ x_2 = 0 \qquad ...(5)$$

$$x_1(0) = - 1, \qquad x'_1(0) = 0, \qquad x_2(0) = 2 \qquad \text{and} \qquad x'_2(0) = 0 \qquad ...(6)$$

Operating both sides of (4) by $(D^2 + 2)$ and multiplying (5) by 2 and then adding the resulting equations, we find

$$\{(D^2 + 2)\ (D^2 + 5) - 4\}x_1 = 0 \qquad \text{or} \qquad (D^4 + 7D^2 + 6)x_1 = 0 \qquad ...(7)$$

The auxiliary equation for (7) is $\quad D^4 + 7D^2 + 6 = 0 \quad$ or $\quad (D^2 + 6)\ (D^2 + 1) = 0$

so that $D = \pm\ i\sqrt{6}$, $\pm\ i$ and hence the general solution of (7) is

$$x_1 = C_1 \sin t + C_2 \cos t + C_3 \sin t\sqrt{6} + C_4 \cos t\sqrt{6}\ , \ C_1, C_2, C_3, C_4 \text{ being arbitrary contants} \quad ...(8)$$

$$(8) \Rightarrow Dx_1 = C_1 \cos t - C_2 \sin t + C_3\sqrt{6} \cos t\sqrt{6} - C_4\sqrt{6} \sin t\sqrt{6} \qquad ...(9)$$

$$(9) \Rightarrow D^2x_1 = - C_1 \sin t - C_2 \cos t - 6\ C_3 \sin t\sqrt{6} - 6C_4 \cos t\sqrt{6} \qquad ...(10)$$

Now, $(4) \Rightarrow 2x_2 = D^2x_1 + 5x_1 = - C_1 \sin t - C_2 \cos t - 6\ C_3 \sin t\sqrt{6} - 6\ C_4 \cos t\sqrt{6}$

$$+ 5(C_1 \sin t + C_2 \cos t + C_3 \sin t\sqrt{6} + C_4 \cos t\sqrt{6}\ ), \text{ using (8) and (10)}$$

or

$$x_2 = 2C_1 \sin t + 2C_2 \cos t - (1/2) \times C_3 \sin t\sqrt{6} - (1/2) \times C_4 \cos t\sqrt{6} \qquad ...(11)$$

From (6), we have $x_1 = -1$ and $x'_1 = dx_1/dt = Dx_1 = 0$ when $t = 0$. Hence (8) and (9) give

$$- 1 = C_2 + C_4 \qquad \text{and} \qquad 0 = C_1 + C_3\sqrt{6} \qquad ...(12)$$

From (11), $x'_2 = dx_2/dt = 2C_1 \cos t - 2C_2 \sin t - \left(\sqrt{6}/2\right) \times C_3 \cos t\sqrt{6} + \left(\sqrt{6}/2\right) \times C_4 \sin t\sqrt{6}$ ...(13)

From (6), we have $x_2 = 2$ and $x'_2 = dx_2/dt = 0$ when $t = 0$. Hence (11) and (13) give

$$2 = 2C_2 - (1/2) + C_4 \qquad \text{and} \qquad 0 = 2C_1 - \left(\sqrt{6}/2\right) \times C_3 \qquad ...(14)$$

Solving (12) and (14), $C_1 = 0$, $C_2 = 3/5$, $C_3 = 0$ and $C_4 = -(8/5)$.

Substituting the above values in (8) and (11), the particular solution of the specific problem consisting of the system of equations (4) and (5) and initial conditions (6) is given by

$$x_1(t) = (3/5) \times \cos t - (8/5) \times \cos t\sqrt{6} \quad \text{and} \quad x_2(t) = (6/5) \times \cos t + (4/5) \times \cos t\sqrt{6}$$

Ex 2. *Solve the problem of Ex. 1. for the case in which the objects A_1 has mass $m_1 = 2$, the object A_2 has mass $m_2 = 1$, the spring S_1, has spring constant $k_1 = 4$, the spring S_2 has* s_2 has spring *constant $k_2 = 2$, and the initial conditions are $x_1(0) = 1$, $x_1'(0) = 0$, $x_2(0) = 5$ and $x_2'(0) = 0$.* **Ans.** $x_1 = 2\cos t - \cos 2t$, $x_2 = 4\cos t + \cos 2t$

Ex. 3. A projectile of mass m is fired into the air from a gun which is inclimed at an angle θ with the horizontal, and suppose the initial velocity of the projectile is v_0 feet per second. Neglect all forces except that of gravity and the air resistance, and assume that this latter force (in pounds) is numerically equal to k times the velocity (in feet/second). (*i*) Taking the origin at the position of the gun, with x-axis horizontal and the y-axis vertical, show that the differential equations of the resulting motion are $m\,(d^2x/dt^2) + k\,(dx/dt) = 0$ and $m(d^2y/dt^2) + k\,(dx/dt) + mg = 0$. (*ii*) Find the solution of the system of differential equation of part (*i*).

MISCELLANEOUS EXAMPLES ON CHATPER 11

Ex. 1. *When a switch is closed in circuit containing a battery E, a resistor R and an inductance L, the current i builds up at a rate given by $L\,(di/dt) + Ri = E$.*

Find i as a function of t. **[M.S. Univ. T.N. 2007]**

Sol. Re-writing the given equation, $\quad di/dt + (R/L) \times i = E/L \qquad \ldots (1)$

Integrating factor of linear equation (1) $= e^{\int (R/L)\,dt} = e^{Rt/L}$

$\therefore$ Solution of (1) is $\quad ie^{Rt/L} = \int (E/L)\, e^{Rt/L} dt + c$, c being an arbitrary constant

or $\quad i\,e^{Rt/L} = (E/L) \times (L/R)\, e^{Rt/L} + c \quad$ or $\quad i = E/R + ce^{-(Rt/L)} \qquad \ldots (2)$

Initially, at $t = 0$, $i = 0$. So (2) givens $\quad 0 = E/R + c \quad$ so that $\quad c = -E/R$

Hence, (2) reduces to $\quad i = (E/R) \times (1 - e^{-Rt/L}) \qquad \ldots (3)$

Ex. 2. *A 12 volt battery is connected to a simple series circuit in which the inductance is (1/2) H and the resistance is 10 Ω. Determine the current i if $i(0) = 0$.* **[M.S.Univ. T.N. 2007]**

Sol. If a circuit has in series an electromotive force E volt, a resistor R ohm and an inductor L heneries, then current i in amperes at time t is given by

$$L\,(di/dt) + Ri = E \qquad \ldots (1)$$

Here $L = (1/2)$ H, $R = 10\Omega$ and $E = 12$ volt, So (1) reduces to

$(1/2) \times (di/dt) + 10i = 12 \quad$ or $\quad di/dt + 20i = 24 \qquad \ldots (2)$

Its I.F. $= e^{\int 20\,dt} = e^{20t}$ and solution is given by

$i\,e^{20i} = \int (24\,e^{20t})\,dt + c \quad$ or $\quad i\,e^{20t} = (6/5) \times e^{20t} + c$

or $\quad i = (6/5) + c\,e^{-20t}$, c being an arbitrary constant $\qquad \ldots (3)$

Since $i\,(0) = 0$, putting $i = 0$ and $t = 0$ in (3), we get $\quad c = -6/5$

Hence, (3) reduces to $\quad i = (6/5) \times (1 - e^{-20t})$

Ex. 3. *16 lb weight is placed upon the lower end of a coil spring suspended from the ceiling and comes to rest in the equilibrium position, thereby stretching the spring 8 in At time $t = 0$ the weight is then struck so as to set it into motion with initial velocity of 2ft/sec directed downward. The mediam ofters a resistance in pounds numerically equal to 6 (dx/dt),* where dx/dt is the instantaneous velocity in feet per second. Determine the resulting displacement of the weight as a function of time. **[Delhi B.Sc. II (Prog.) 2008]**

4. For an electric circuit with circuit constants *L, R, C* the charge *q* on a plate of condenser is given by $L(d^2q/dt^2)+R(dq/dt)+q/c=100$. Given $L = 1$, $R = 1200$, $C = 10^{-6}$, $q = dq/dt = 0$ for $t > 0$, find the charge *q*. **[Madurai Kamraj 2008]**

5. A current of electricity on a circuit of resistance *R* ohms commences at time $t = 0$. The self induction of the circuit is L and when t = 0, the electromotive force is E. The current satisfies the equation $L(di/dt)+Ri=E$. Solve the equation and show that $i=(E/R)\times(1-e^{-Rt/2})$

[Madurai Kamraj 2008]

MISCELLANEOUS PROBLEMS BASED ON THIS PART OF THE BOOK

Ex. 1. *What is the general solution of $2x\ (dy/dx) = 10x^3y^5 + y$?*

(a) $y^4 = cx^2 - 4x^3$ (b) $y^{-4} = c/x^2 + 4x^3$

(c) $y^4 = c/x^2 + 4x^3$ (d) $y^{-4} = c/x^2 - 4x^3$ **[I.A.S. Prel. 2008]**

Sol. Ans. (*d*). Re-writing, the given equation reduces to

$$y^{-5}\frac{dy}{dx} = 5x^2 + \frac{y^{-4}}{2x} \quad \text{or} \quad y^{-5}\frac{dy}{dx} - \frac{1}{2x}y^{-4} = 5x^2 \quad \ldots(1)$$

Let $y^{-4} = v$ so that $-(4y^{-5}) \times (dy/dx) = dv/dx$. Then, (1) reduces to

$-(1/4) \times (dv/dx) - (1/2x) \times v = 5x^2$ or $dv/dx + (2/x) \times v = -20x^2$, …(2)

whose I.F. $= e^{\int (2/x)dx} = e^{2\log x} = x^2$ and hence its solution is

$$vx^2 = \int \left\{(-20x^2) \times x^2\right\} dx + c \quad \text{or} \quad y^{-4}x^2 = -4x^5 + c$$

or $y^{-4} = c/x^2 - 4x^3$, c being an arbitrary constant.

Ex. 2. *Solve the differential equation $ydx + (x + x^3y^2)dy = 0$* **[I.A.S. 2008]**

Sol. Re-writing the given equation, $y\ (dx/dy) + x + x^3 y^2 = 0$

or $dx/dy + (1/y) \times x = -x^3y$ or $x^{-3}\ (dx/dy) + (1/y) \times x^{-2} = -y$ ...(1)

Let $x^{-2} = v$ so that $-(2x^{-3}) \times (dx/dy) = dv/dy$. Then, (1) reduces to

$-(1/2) \times (dv/dy) + v/y = -y$ or $dv/dy - (2/y) \times v = 2y$, …(2)

whose I.F. $= e^{-\int (2/y)dy} = e^{-2\log y} = y^{-2}$ and hence its solution is

$$v\,y^{-2} = \int (2y \times y^{-2})dy + c \quad \text{or} \quad x^{-2}y^{-2} = 2\log y - 2\log c$$

or $\log (y/c) = 1/(2x^2y^2)$ or $y = c\,e^{(1/2x^2y^2)}$

Ex. 3. *Solve $(1 + x + xy^2)dy + (y + y^3)dx = 0$* **[Delhi 2008]**

Sol. Re-writing the given equation, $y(1 + y^2)dx + \{1 + x\ (1 + y^2)\}dy = 0$

or $(1 + y^2)\ (ydx + xdy) + dy = 0$ or $d(xy) + \{1/(1+y^2)\}dy = 0$

Integrating, $xy + \log (1 + y^2) - \log c = 0$ or $\log \{(1+y^2)/c\} = -xy$

or $1 + y^2 = c\ e^{-xy}$, c being an arbitrary constant.

Ex. 4. *What is the solution of the equation $x(dy/dx) + y^2/x = y$?* Here $\ln x = \log_e x$

(a) $\ln(y/x) - (1/x) = C$ (b) $\ln x - (x/y) = C$

(c) $\ln(x/y) - (1/x) = C$ (d) $\ln x + (x/y) = C$ **[I.A.S. (Prel.) 2009]**

Sol. Ans. (*b*). Given $x(dy/dx) + y^2/x = y$ or $dy/dx - (1/x)y = -(y^2/x^2)$

or $y^{-2}\ (dy/dx) - (1/x)y^{-1} = -(1/x^2)$ …(1)

Putting $y^{-1} = v$ so that $-y^{-2}\ (dy/dx) = dv/dx$, (1) reduces to

$-(dv/dx) - (1/x)\,v = -(1/x^2)$ or $dv/dx + (1/x)\,v = 1/x^2$, …(2)

which is a linear differential equation whose I.F. $= e^{\int(1/x)dx} = e^{logx} = x$ and hence solution of (2) is

$$v\,x = \int \left\{(1/x^2) \times x\right\} dx + c' = \log_e x + c' = \ln x + c', \ c' \text{ being an arbitrary constant.}$$

or $x/y = \ln x + c'$ or $\ln x \sim (x/y) = c$,

where $c = (= -c')$ is an arbitrary constant.

Ex. 5. *Which one of the following differential equations represents the orthogonal trajectories of the family of curves $xy = k^2$?*

Sol. Ans. (*c*). Given $xy = k^2$, where k is a parameter ... (1)

Differentiating (1) w.r.t. 'x', $y + x\,(dy/dx) = 0$, ... (2)

which is the differential equation of the given family of curves (1). Replacing dy/dx by $-(dx/dy)$, the differential of the required orthogonal trajectories is given by

$y - x\,(dx/dy) = 0$ or $xdx - ydy = 0$

Ex. 6. *Solve* $(D^3 + D)y = 2x^2 + 4 \sin x$ **[Delhi 2008]**

Sol. The auxiliary equation of the given equation is given by

$D^3 + D = 0$, *i.e.,* $D(D^2 + 1) = 0$ giving $D = 0, \pm i$

Hence C.F. $= C_1 e^{o.x} + C_2 \cos x + C_3 \sin x$, C_1, C_2 and C_3 being arbitrary constants

Here, P.I. corresponding to $2x^2$

$$= \frac{1}{D(D^2+1)} 2x^2 = \frac{2}{D}(1+D^2)^{-1} x^2 = \frac{2}{D}(1 - D^2 + D^4 \ldots)x^2 = \frac{2}{D}(x^2 - 2) = \frac{2x^3}{3} - 4x$$

and P.I. corresponding to 4 sin x

$$= \frac{1}{D^3 + D} 4 \sin x = 4 \frac{1}{D^2+1} \frac{1}{D} \sin x = 4 \frac{1}{D^2+1}(-\cos x) = -4 \frac{1}{D^2+1} \cos x$$

$$= -4 \times \frac{x}{2 \times 1} \sin x = -2x \sin x, \qquad \text{as} \qquad \frac{1}{D^2 + a^2} \cos ax = \frac{x}{2a} \sin ax$$

Hence the required solution is y = C.F. + total P.I., *i.e.,*

$$y = C_1 + C_2 \cos x + C_3 \sin x + (2x^3/3) - 4x - 2x \sin x$$

Ex. 7. *Solve the differential equation* $(d^2 y/dx^2) - 9y = e^{2x} + x$ *by the method of undetermined coefficients.* **[Delhi 2008]**

Hint. Do like Ex 2, page.5.54. **Ans.** $y = c_1 e^{3x} + c_2\, e^{-3x} - (e^{2x}/5) - (x/9)$

Ex. 8. *One particular solution of* $y^{111} - y^{11} - y^{1} + y = - e^x$ *is a constant multiple of*

(*a*) xe^{-x} (*b*) xe^x (*c*) x^2e^{-x} (*d*) x^2e^x **[GATE 2008]**

Sol. Ans. (*d*). Let $D = d/dx$. Then, the given equation reduces to

$(D^3 - D^2 - D + 1)y = - e^x$ or $(D - 1)^2 (D + 1)\, y = - e^x$

$$\text{P.I.} = \frac{1}{(D-1)^2} \frac{1}{D+1}(-e^x) = \frac{1}{(D-1)^2}\left(-\frac{1}{2} e^x\right) = -\frac{1}{2} \frac{1}{(D-1)^2} e^x = -\frac{1}{2} \times \frac{x^2}{2!} \times e^x = -\frac{x^2 e^x}{4}$$

which is a constant multiple of $x^2 e^{x}$.

Ex. 9. *Solve* $(x^2D^2 - xD + 1)y = (\log x \sin \log x + 1)/x$ **[Madurai Kamraj 2008]**

Sol. Try yourself as in Ex. 17, page 6.10 **Ans.** $y = x(c_1 + c_2 \log x) + (1/4x)$

$+ \{(\log x) \times (3 \log \sin x + 4 \log \cos x)\}/25x + (110 \cos \log x + 20 \sin \log x)/625x$

Ex. 10. *Solve the differential equation* $(D^2 - 2D + 1)y = x \log x$, $(x > 0)$ *by using the method of variation of parameters.* **[Delhi 2008]**

Sol. Re-writing the given equation, $y_2 - 2y_1 + y = x \log x$ ... (1)

Comparing (1) with $y_2 + Py_1 + Qy = R$, here $R = x \log x$

Consider $y_2 - 2y_1 + y = 0$ or $(D^2 - 2D + 1)y = 0$, where $D \equiv d/dx$ ... (2)

The auxiliary equation of (2) is $D^2 - 2D + 1 = 0$ giving $D = 1, 1$.

Hence, C.F. of (1) $= (c_1 + c_2 x)e^x$, c_1 and c_2 being arbitrary constants ... (3)

Let $u = e^x$ and $v = x\, e^x$. Also, here $R = x \log x$ ... (4)

Now,
$$W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & e^x + xe^x \end{vmatrix} = e^x(e^x + xe^x) - xe^{2x} = e^{2x} \neq 0$$

Then, P.I of (1) $= u\, f(x) + v\; g(x)$, where ... (5)

$$f(x) = -\int \frac{vR}{W} dx = -\int \frac{(xe^x)\,(x\log x)}{e^{2x}} dx = -\int e^{-x}(x^2 \log x) dx$$

$$= -\left\{(x^2 \log x)\,(-e^{-x}) - \int (2x\log x + x)\,(-e^{-x}) dx\right\}, \text{ integrating by parts}$$

$$= e^{-x}x^2 \log x - \int e^{-x}(2x\log x + x) dx$$

$$= e^{-x}x^2 \log x - \left\{(2x\log x + x)\,(-e^{-x}) - \int (2\log x + 2 + 1)\,(-e^{-x})\, dx\right\}$$

$$= e^{-x}x^2 \log x + e^{-x}(2x\log x + x) - \int e^{-x}(2\log x + 3) dx$$

$$= e^{-x}(x^2 \log x + 2x\log x + x) - \left\{(2\log x + 3)(-e^{-x}) - \int (2/x)\,(-e^{-x}) dx\right\}$$

$$= e^{-x}(x^2 \log x + 2x\log x + x + 2\log x + 3) - 2\int (e^{-x}/x) dx \qquad \text{... (6)}$$

$$g(x) = \int \frac{uR}{W} dx = \int \frac{e^x \times (x\log x)}{e^{2x}} dx = \int e^{-x}(x\log x) dx$$

$$= (x\log x)\,(-e^{-x}) - \int (\log x + 1)\,(-e^{-x}) dx = -xe^{-x}\log x + \int e^{-x}(\log x + 1) dx$$

$$= -xe^{-x}\log x + (\log x + 1)\,(-e^{-x}) - \int \{(1/x) \times (-e^{-x})\} dx, \text{ integrating by parts}$$

$$= -e^{-x}(x\log x + \log x + 1) + \int (e^{-x}/x) dx \qquad \text{... (7)}$$

Using (4), (6) and (7), (5) yields

$$\text{P.I. of (1)} = e^x \left\{e^{-x}(x^2 \log x + 2x\log x + x + 2\log x + 3) - 2\int (e^{-x}/x) dx\right\}$$

$$+\; xe^x \left\{-e^{-x}(x\log x + \log x + 1) + \int (e^{-x}/x) dx\right\}$$

$$= x^2 \log x + 2x\log x + x + 2\log x + 3 - 2e^x \int (e^{-x}/x) dx$$

$$-x^2 \log x - x\log x - x + xe^x \int (e^{-x}/x) dx$$

$$= x\log x + 2\log x + 3 - (2 - x)e^x \int (e^{-x}/x) dx$$

Hence the required solution is y = C.F. + P.I., i.e.,

$$y = (c_1 + c_2 x)e^x + x\log x + 2\log x + 3 - (2 - x)e^x \int (e^{-x}/x) dx$$

Ex. 11. *Use the method of variation of parameters to find the general solution of* $x^2y'' - 4xy' + 6y = -x^4 \sin x$. **[I.A.S. 2008]**

Sol. Re-writing the given equation, $y_2 - (4/x) \times y_1 + (6/x^2) \times y = -x^2 \sin x$. ...(1)

Comparing (1) with $y_2 + Py_1 + Qy = R$, here $R = -x^2 \sin x$

Consider $y_2 - (4/x) \times y_1 + (6/x^2) \times y = 0$ or $(x^2D^2 - 4xD + 6)y = 0$, $D \equiv d/dx$ ...(2)

In order to apply the method of variation of parameters, we shall reduce (2) into linear differential equation with constant coefficients.

Let $x = e^z$, *i.e.,* $\log x = z$ and let $D_1 \equiv d/dz$ ...(3)

Then, $xD = D_1$, $x^2D^2 = D_1(D_1 - 1)$ and so (2) reduces to

$$\{D_1(D_1 - 1) - 4D_1 + 6\}y = 0 \qquad \text{or} \qquad (D_1^2 - 5D_1 + 6)y = 0,$$

whose auxiliary equation is $D_1^2 - 5D_1 + 6 = 0$ giving $D_1 = 2, 3$.

$\therefore$ C.F. of (1) $= c_1 e^{2z} + c_2 e^{3z} = c_1 (e^z)^2 + c_2 (e^z)^3 = c_1x^2 + c_2x^3$ ... (4)

Let $u = x^2$ and $v = x^3$. Also, here $R = -x^2 \sin x$ ...(5)

Here $$W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = 3x^4 - 2x^4 = x^4 \neq 0$$

Hence, P.I. of (1) $= u\,f(x) + v\,g(x)$, where ...(6)

$$f(x) = -\int \frac{vR}{W} dx = -\int \frac{x^3 \times (-x^2 \sin x)}{x^4} dx = \int x \sin x\, dx$$

$$= x(-\cos x) - \int \{1 \times (-\cos x)\} dx = -x\cos x + \sin x \qquad \text{...(7)}$$

and $$g(x) = \int \frac{uR}{W} dx = \int \frac{x^2 \times (-x^2 \sin x)}{x^4} dx = -\int \sin x\, dx = \cos x \qquad \text{...(8)}$$

Using (5), (7) and (8), (6) reduces to

P.I. of (1) $= x^2(-x\cos x + \sin x) + x^3 \cos x = x^2 \sin x$

Hence the required general solution is y = C.F. + P.I., i.e.,

$y = c_1x^2 + c_2x^3 + x^2 \sin x$, c_1 and c_2 being arbitrary constants.

Ex. 12. Solve $2(dx/dt) + dy/dt - x - y = 1$ and $dx/dt + dy/dt + 2x - y = t$.

[Delhi Maths (Prog.) 2008]

Sol. Given $2(dx/dt) + dy/dt - x - y = 1$ ... (1)

and $dx/dt + dy/dt + 2x - y = t$ ... (2)

Subtracting (2) from (1) $dx/dt - 3x = 1 - t$, ... (3)

which is linear differential equation whose I.F. $= e^{\int(-3)dt} = e^{-3t}$ and solution is

$$xe^{-3t} = \int (1-t)e^{-3t} dt + c_1 = (1-t)(-e^{-3t}/3) - (-1) \times (e^{-3t}/9) + c_1$$

[Using the chain rule of integration by parts]

or $xe^{-3t} = \{(3t-2)/9\}e^{-3t} + c_1$ or $x = c_1e^{3t} + (3t-2)/9$ ... (4)

From (4), $$dx/dt = 3c_1e^{3t} + 1/3 \qquad ...(5)$$

From (2), $$dy/dt - y = t - (dx/dt) - 2x$$

or $$dy/dt - y = t - (3c_1e^{3t} + 1/3) - 2\left\{c_1e^{3t} + (3t-2)/9\right\}$$

or $$dy/dt - y = -5c_1e^{3t} + t/3 + 1/9 \qquad ...(6)$$

which is linear differentail equation whose I.F. $= e^{\int(-1)dt} = e^{-t}$ and solution is

$$ye^{-t} = \int(-5c_1e^{3t} + t/3 + 1/9)e^{-t}dt + c_2 = -5c_1\int e^{2t}dt + \int(t/3+1/9)e^{-t}dt + c_2$$

or $$ye^{-t} = -(5c_1/2)\times e^{2t} + (t/3+1/9)\times(-e^{-t}) - (1/3)\times(e^{-t}) + c_2$$

[Using the chain rule of integration by parts]

or $$ye^{-t} = -(5c_1/2)\times e^{2t} - (t/3+4/9)\times(e^{-t}) + c_2$$

$$y = -(5c_1/2)\times e^{3t} - t/3 - 4/9 + c_2e^t \qquad ...(7)$$

The required solution is given by (4) and (7), where c_1 and c_2 are arbitrary constants.

Ex. 13. Solve the following simultaneous differential equations.

(a) $2(dx/dt) + (dy/dt) - x - y = 1,\ (dx/dt) + (dy/dt) + 2x - y = t$

(Delhi B.Sc. II (Prog) 2008)

(b) $dx/dt + 4y = \sec^2 2t,\ dy/dt = x.$ **[Delhi B.A. II (Prog) 2009]**

(c) $dx/dt + 9(dy/dt) + 2x + 31y = e^t,\ 3(dx/dt) + 7(dy/dt) + x + 24y = 3$

[Delhi B.Sc. II (Prog) 2009]

Ans. (a) $x = c_1e^{3t} + (3t-2)/9;\ y = c_2e^t - (5c_1/2)\times e^{3t} - (t/3) - (4/9)$

(b) $x = -2c_1\sin 2t + 2c_2\cos 2t + (1/2)\times\{\cos 2t\log(\sec 2t + \tan 2t) + \tan 2t\}$

$y = c_1\cos 2t + c_2\sin 2t + (1/4)\times\{\sin 2t\log(\sec 2t + \tan 2t) - 1\}$

Ex. 14. *If $f(D) = xD + 2$ and $g(D) = D + 5$, then find $[f(D)\,g(D)]y$, where $D = d/dx$.*

[Pune 2010]

Sol. $[f(D)\,g(D)]\,y = [(xD + 2)(D + 5)]y = [xD(D + 5) + 2(D + 5)]y$
$= (xD^2 + 5xD + 2D + 10)y$

Ex. 15. (a) *Find the particular solution of the equation $(D^2 - 3D + 2)\,y = \cos(e^{-x})$ by using the method of variation of parameters.*

(b) *Solve by variation of parameters $(D^2 - 3D + 2)\,y = \cos(e^{-x})$* **[G.N.D.U. Amritsar 2011]**

Sol. (*a*). Given $y_2 - 3y_1 + 2y = \cos(e^{-x})$ ... (1)

Comparing (1) with $y_2 + Py_1 + Qy = R$, we have $R = \cos(e^{-x})$ ... (2)

Auxiliary equation of (2) is $D^2 - 3D + 2 = 0$ giving $D = 1, 2$

Hence, C.F. of (1) $= c_1e^x + c_2e^{2x}$, c_1 and c_2 being arbitrary constants ... (3)

Let $u = e^x$ and $v = e^{2x}$. also, here $R = \cos(e^{-x})$ ... (4)

Now, $$W = \begin{vmatrix} u & v \\ u_1 & v \end{vmatrix} = \begin{vmatrix} e^x & e^{2x} \\ e^x & 2e^{2x} \end{vmatrix} = 2e^{3x} - e^{3x} = e^{3x} \neq 0 \qquad ...(5)$$

Then, particular solution (or integral) of (1) $= uf(x) + vg(x)$, ... (6)

where $$f(x) = -\int \frac{vR}{W}dx = -\int \frac{e^{2x}\cos(e^{-x})}{e^{3x}}dx = -\int e^{-x}\cos(e^{-x})dx = \int \cos t\, dt = \sin t = \sin e^{-x}$$

[Putting $e^{-x} = t$ so that $e^{-x}dx = -\,dt$]

and $$g(x) = \int \frac{uR}{W}dx = \int \frac{e^{x}\cos(e^{-x})}{e^{3x}}dx = \int e^{-2x}\cos(e^{-x})dx = -\int t\cos dt, \text{ putting } e^{-x} = t$$

$$= -[t\sin t - \int \sin t\, dt] = -t\sin t - \cos t = -e^{-x}\sin(e^{-x}) - \cos(e^{-x})$$

Substituting the above values of $f(x)$ and $g(x)$ in (6), we have

P.S. = Particular solution = $e^{x}\sin(e^{-x}) + e^{2x}\{-e^{-x}\sin(e^{-x}) - \cos(e^{-x})\} = -e^{2x}\cos(e^{-x})$

Part (*b*). Proceed as in part (*a*) to get C.F. and P.S. Then, the required solution is

$$y = C.F. + P.I. = c_1 e^{x} + c_2 e^{2x} - e^{2x}\cos(e^{-x})$$

Ex. 16. *Given that the complementary solution of the differential equation* $(x^2 - 1)y'' - 2x y' + 2y = x^2 - 1$ is $y_c = c_1 x + c_2(1 + x^2)$, *find the particular solution using the method of variation of parameters.* **[Delhi B.Sc. (Hons) II 2011]**

Sol. Re-writing the given equation, $y_2 - \{2x/(x^2 - 1)\}y_1 + \{2/(x^2 - 1)\}\, y = 1$ ... (1)

Comparing (1) with $y_2 + Py_1 + Qy = R$, we have $R = 1$ ... (2)

Given that C.F. of (11 $= y_c = c_1 x + c_2(1 + x_2)$, c_1 and c_2 being arbitrary constants ... (3)

Let $u = x$, $v = 1 + x^2$. Also here $R = 1$ ... (4)

Now, $$W = \begin{vmatrix} u & v \\ u_1 & v_1 \end{vmatrix} = \begin{vmatrix} x & 1+x^2 \\ 1 & 2x \end{vmatrix} = 2x^2 - (1 + x^2) = x^2 - 1 \neq 0 \quad \text{... (5)}$$

Then, the required particular solution $= y_p = uf(x) + vg(x)$, ... (6)

where $$f(x) = -\int \frac{vR}{W}dx = -\int \frac{1+x^2}{x^2-1}dx = \int \frac{2-(1-x^2)}{1-x^2}dx = \log_e\left|\frac{1+x}{1-x}\right| - x$$

and $$g(x) = \int \frac{uR}{W}dx = \int \frac{x}{x^2-1}dx = \frac{1}{2}\int \frac{(-2x)}{1-x^2}dx = \frac{1}{2}\log_e(1-x^2)$$

Using the above values of $f(x)$ and $g(x)$ in (6), the required particular solution is

$$y_p = x\left\{\log_e\left|\frac{1+x}{1-x}\right| - x\right\} + (1+x^2)\times\frac{1}{2}\log_e(1-x^2) = -x^2 + x\log_e\left|\frac{1+x}{1-x}\right| + \frac{1}{2}(1+x^2)\log_e(1-x^2)$$

Ex. 17.(*a*) *Given that $y = x$ is a solution of $(x^2 + 1)(d^2y/dx^2) - 2x(dy/dx) + 2y = 0$. Find a linearly independent solution by reducing the order. Write the general solution.*

[Delhi B.Sc. (Hons) II 2011]

(*b*) *Verify that $y_1 = x/(x-1)^2$ is a solution of the differential equation $x(x-1)y'' + 3xy' + y = 0$. Find the other linearly independent solution of the equation and hence its general solution.*

[Mumbai 2010]

Sol. (a) Comparing the given equation with $p(x)\,y'' + q(x)\,y' + r(x)\,y = 0$, we have $p(x) = x^2 + 1$ and $q(x) = -2x$. Also here $f(x) = x$. Hence the required second linearly independent solution is $vf(x)$, i.e., xv, where v is given by (refer formula (6) off Art. 10.4A of chapter 10).

$$v = \int \frac{\exp[-\int\{q(x)/p(x)\}dx]}{\{f(x)\}^2}dx \quad \text{... (1)}$$

Here, $$-\int \frac{q(x)}{p(x)}dx = \int \frac{2x}{1+x^2}dx \;=\; \log_e(x^2+1) \quad \text{and hence we have}$$

$$\exp\left[-\int \{q(x)/p(x)\}\,dx\right] \;=\; \exp\left[\log_e(x^2+1)\right] = e^{\log_e(x^2+1)} = x^2+1$$

Hence, from (1), $$v = \int \frac{x^2+1}{(x^2+1)^2}dx \;=\; \int \frac{dx}{x^2+1} = \tan^{-1} x$$

and hence the required second linearly independent solution of the given equation is $vf(x)$, i.e., $x \tan^{-1} x$. Hence, the required solution is given by

$y = c_1 x + c_2 x \tan^{-1} x$, c_1 and c_2 being arbitrary constants

Part (*b*). Hint. Proceed as in part (*a*). Here $f(x) = y_1 = x/(x-1)^2$. Then, as before, show that $v = x^{-1} + \log x$ and hence the second linearly solution of the given equation in $vf(x)$, i.e., $\{x/(x-1)^2\} \times (x^{-1} + \log x)$. Required general solution is given by

$y = (c_1 x)/(x-1)^2 + c_2 \times \{x/(x-1)^2\} \times (x^{-1} + \log x)$, c_1 and c_2 being arbitrary constants.

Ex. 18. *Let f(x) and x f(x) be the particular solutions of the differential equation* $y'' + R(x)y' + S(x)y = 0$. *Then the solution of the differential equation* $y'' + R(x)y' + S(x)y = f(x)$ *is*

(*a*) $y = (-x^2/2 + \alpha x + \beta) f(x)$ (*b*) $y = (x^2/2 + \alpha x + \beta) f(x)$

(*c*) $y = (-x^2 + \alpha x + \beta) f(x)$ (*d*) $y = (x^2 + \alpha x + \beta) f(x)$ **[GATE 2012]**

Sol. Ans. (b). For the given objective type problem, use working rule of Art. 7.4B. Accordingly, here $u = xf(x)$, $v = f(x)$ and $R = f(x)$ $u^{-2} e^{-\frac{1}{2}\int p dx} = e^{-\frac{1}{2}\int\left\{\frac{-4f'(x)}{f(x)}\right\}dx} = e^{2\int \frac{f'(x)}{f(x)}dx} = e^{2\log f(x)} = [f(x)]^2$ ... (1)

Let $$y = Au + Bv \qquad ...(2)$$

be the general solution of $y'' + R(x)y' + S(x)y = f(x)$ ... (3)

Then A and B are given by [Refer equations (5) and (8) of Art 7.4B]

$A_1 u + B_1 v = 0$ i.e., $A_1 x f(x) + B_1 f(x) = 0$ or $A_1 x + B_1 = 0$ ...(4)

and $A_1 u_1 + B_1 v_1 = R$ i.e., $A_1 \dfrac{d}{dx}(xf(x)) + B \dfrac{d}{dx}(f(x)) = f(x)$

or $$A_1\{f(x) + xf'(x)\} + B_1 f'(x) = f(x) \qquad (5)$$

Eliminating B, between (4) and (5), $A_1 f(x) + A_1 x f'(x) - A_1 x f'(x) = f(x)$, giving

$A_1 = 1$ or $dA/dx = 1$ giving $A = x + \alpha$, α being a constant ...(6)

Since $A_1 = 1$, (5) yields $B_1 = -x$ or $dB/dx = -x$ or $dB = -xdx$

Integrating, $B = -(x^2/2) + \beta$, β being a constant ... (7)

using (6) and (7), (2) yields $y = (x+\alpha)xf(x) + (\beta - x^2/2)f(x) = (x^2/2 + \alpha x + \beta)f(x)$

Ex. 19. If a transformation $y = uv$ transforms the given differential equation $f(x)y'' - 4f'(x)y' + g(x)y = 0$ into the equation of the form $v'' + 4h(x)v = 0$, then u must be

(*a*) $1/f^2$ (*b*) xf (*c*) $1/2f$ (*d*) f^2 **[GATE 2012]**

Sol. Ans. (*d*). Re-writing the given equation, we have

$$y'' = \{4f'(x)/f(x)y' + \{g(x)/f(x)\}y = 0 \qquad \text{... (1)}$$

Comparing (1) with $y'' + Py' + Qy = 0$, here $P = \{4f'(x)/f(x)$

Using result of working rule of Art. 10.7, the required value of u is given by

$$u = e^{-\frac{1}{2}\int p dx} = e^{-\frac{1}{2}\int\left\{\frac{4f'(x)}{f(x)}\right\}dx} = e^{2\int\frac{f'(x)}{f(x)}dx} = e^{2\log f(x)} = [f(x)]^2$$

ADVANCED ORDINARY DIFFERENTIAL EQUATIONS AND SPECIAL FUNCTIONS

WHERE IS WHAT

1

Picard's Iterative Method Uniqueness And Existence Theorems

1.1. Introduction. In many practical problems we come across with a differential equation which cannot be solved by one of the standard methods known so far. Various methods have been formulated for getting to any desired degree of accuracy the numerical solution of the above mentioned type of differential equation with numerical coefficients and given conditions. In this chapter we propose to discuss Picard's iteration method for finding an approximate solution of the initial value problem of the form

$$dy/dx = f(x, y), \qquad y(x_0) = y_0.$$

The condition $y(x_0) = y_0$ is called the *initial condition*. Here $y(x_0)$ denotes the value of y at $x = x_0$. Sometimes $y(x_0) = y_0$ is also expressed by saying that $y = y_0$ when $x = x_0$.

An *iteration method* is a method which consists of a repeatred application of exactly the same type of steps where in each step we use the result of the previous step (or steps).

1.2A. Picard's method of successive approximations (or Picard's iteration method)
[Himanchal 2004, Bangalore 2002, 06; Allahabad 2001; Meerut 2000, 10; Ujjan 2003]

Consider an initial value problem of the form

$$dy/dx = f(x, y), \qquad y(x_0) = y_0. \qquad ...(1)$$

By integrating over the interval (x_0, x), (1) gives

$$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x,y)\, dx \qquad \text{or} \qquad y(x) - y_0 = \int_{x_0}^{x} f(x,y)\, dx$$

or

$$y(x) = y_0 + \int_{x_0}^{x} f(x,y)\, dx. \qquad ...(2)$$

Thus, the solving of initial value problem (1) is equivalent to finding a function $y(x)$ which satisfies the equation (2), since by differentiating (2) we get $dy/dx = f(x, y)$ and putting $x = x_0$ in (2) yields $y(x_0) = y_0 + 0$ *i.e.*, $y(x_0) = y_0$. Conversely, (2) has been obtained from (1) by integration over the interval (x_0, x) and employing the initial condition $y(x_0) = y_0$.

Since the information concerning the expression of y in terms of x is absent, the integral on the R.H.S. of (2) cannot be evaluated. Hence the exact value of y cannot be obtained. Therefore we determine a sequence of approximations to the solution (2) as follows. As a crude approximation, we put $y = y_0$ in the integral on the right of (2) and obtain

$$y_1(x) = y_0 + \int_{x_0}^{x} f(x, y_0)\, dx, \qquad ...(3)$$

where $y_1(x)$ is the corresponding value of $y(x)$ and is called *first approximation* and is better approximation of $y(x)$ at any x. To determine still better approximation we replace y by y_1 in the integral on R.H.S. in (2) and obtain the *second approximation* y_2 as

$$y_2(x) = y_0 + \int_{x_0}^{x} f(x, y_1)\, dx. \qquad ...(4)$$

Proceeding in this way, the *nth approximation* y_n is given by

$$y_n(x) = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\,dx. \qquad ...(5)$$

Thus, we arrive at a sequence of approximate solutions $y_1(x), y_2(x), y_3(x), \ldots\ldots\ldots\ldots y_n(x), \ldots\ldots$

1.2B. Solved examples based on Art. 1.2A

Ex. 1. *Apply Picard's method to solve the following initial value problem upto third approximation :* $dy/dx = 2y - 2x^2 - 3$ *given that* $y = 2$ *when* $x = 0$.

[Agra 2005; Gwalior 2003; Delhi Maths (Hons.) 1998, 2005; Meerut 2000, 04, 05, 11]

Sol. Given problem is $dy/dx = 2y - 2x^2 - 3$, where $y = 2$, $x = 0$. ... (1)

We know that the nth approximation y_n of the initial value problem

$$dy/dx = f(x, y), \quad \text{where } y = y_0 \quad \text{when } x = x_0 \qquad ...(2)$$

is given by

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\,dx. \qquad ...(3)$$

Comparing (1) and (2), $f(x, y) = 2y - 2x^2 - 3$, $x_0 = 0$ and $y_0 = 2$. ...(4)

$\therefore$ from (3),

$$y_n = 2 + \int_0^x (2y_{n-1} - 2x^2 - 3)\,dx. \qquad ...(5)$$

First approximation. Putting $n = 1$ in (5), we have

$$y_1 = 2 + \int_0^x (2y_0 - 2x^2 - 3)\,dx = 2 + \int_0^x (4 - 2x^2 - 3)\,dx, \text{ using (4)}$$

or

$$y_1 = 2 + \int_0^x (1 - 2x^2)\,dx = 2 + \left[x - 2\left(\frac{x^3}{3}\right)\right]_0^x = 2 + x - \frac{2x^3}{3}. \qquad ...(6)$$

Second approximation. Putting $n = 2$ in (5), we have

$$y_2 = 2 + \int_0^x (2y_1 - 2x^2 - 3)\,dx = 2 + \int_0^x \left[2\left(2 + x - \frac{2x^3}{3}\right) - 2x^2 - 3\right]dx, \text{ using (6)}$$

or

$$y = 2 + \int_0^x \left(1 + 2x - 2x^2 - \frac{4x^3}{3}\right)dx = 2 + x + x^2 - \frac{2x^3}{3} - \frac{x^4}{3}. \qquad ...(7)$$

Third approximation : Putting $n = 3$ in (5), we have

$$y_3 = 2 + \int_0^x \left(2y_2 - 2x^2 - 3\right)dx = 2 + \int_0^x \left[2\left(2 + x + x^2 - \frac{2x^3}{3} - \frac{x^4}{3}\right) - 2x^2 - 3\right]dx, \text{ using (7)}$$

or

$$y = 2 + \int_0^x \left[1 + 2x - \frac{4x^3}{3} - \frac{2}{3}x^4\right]dx = 2 + x + x^2 - \frac{x^4}{3} - \frac{2x^5}{15}.$$

Ex. 2. *Using the Picard's method of successive approximations, find the third approximation of the solution of the equation :* $dy/dx = x + y^2$, *where* $y = 0$ *when* $x = 0$.

[Delhi Maths (Hons) 1996; Meerut 2007; Ravishankar 2001 Indore 2001; Jabalpur 2000, 02, 05; Gwalior 2006; Rohilkhand 2004]

Sol. Given problem is $dy/dx = x + y^2$, where $y = 0$ when $x = 0$. ...(1)

We know that the nth approximation y_n of the initial problem

$$dy/dx = f(x, y) \quad \text{where} \quad y = y_0 \quad \text{when} \quad x = x_0. \qquad ...(2)$$

is given by

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\,dx. \qquad ...(3)$$

Comparing (1) and (2), $f(x, y) = x + y^2$, $x_0 = 0$ and $y_0 = 0$. ...(4)

$\therefore$ from (3) , $$y_n = \int_0^x \left[x + y_{n-1}^2\right] dx. \qquad ...(5)$$

First approximation. Putting $n = 1$ in (5) and using (4) we have

$$y_1 = \int_0^x \left(x + y_0^2\right) dx = \int_0^x x\, dx = \frac{1}{2} x^2. \qquad ...(6)$$

Second approximation. Putting $x = 2$ in (5) and using (6), we have

$$y_2 = \int_0^x \left(x + y_1^2\right) dx = \int_0^x \left[x + \frac{x^4}{4}\right] dx = \frac{1}{2} x^2 + \frac{1}{20} x^5. \qquad ...(7)$$

Third approximation. Putting $n = 3$ in (5), we get

$$y_3 = \int_0^x \left(x + y_2^2\right) dx = \int_0^x \left[x + \left(\frac{1}{2} x^2 + \frac{1}{20} x^5\right)^2\right] dx, \text{ using (7)}$$

$$= \int_0^x \left[x + \frac{1}{4} x^4 + \frac{1}{400} x^{10} + \frac{1}{20} x^7\right] dx = \frac{1}{2} x^2 + \frac{1}{20} x^5 + \frac{1}{4400} x^{11} + \frac{1}{160} x^8.$$

Ex. 3. *Find the third approximation of the solution of the equation $dy/dx = 2 - (y/x)$ by Picard's method, where $y = 2$ when $x = 1$.*

[Delhi Maths (Hons.) 1997, 99, 2008, 2009; Gwalior 2004; Meerut 2002, 11; Rohilkhand 2000]

Sol. Given problem is $dy/dx = 2 - (y/x)$, where $y = 2$ when $x = 1$. ...(1)

We know that the nth approximation y_n of the initial value problem

$$dy/dx = f(x, y), \quad \text{where} \quad y = y_0 \quad \text{when} \quad x = x_0. \qquad ...(2)$$

is given by $$y_n = y_0 + \int_{x_0}^x f(x, y_{n-1})\, dx. \qquad ...(3)$$

Comparing (1) and (2), $f(x, y) = 2 - (y/x)$, $x_0 = 1$ and $y_0 = 2$. ...(4)

$\therefore$ from (3), $$y_n = 2 + \int_1^x [2 - (1/x) y_{n-1}]\, dx. \qquad ...(5)$$

First Approximation. Putting $n = 1$ in (5), we get

$$y_1 = 2 + \int_1^x \left[2 - (1/x) y_0\right] dx = 2 + \int_1^x \left[2 - (2/x)\right] dx, \text{ using (4)}$$

$$= 2 + \left[2x - 2\log x\right]_1^x = 2 + 2x - 2\log x - 2 = 2x - 2\log x. \qquad ...(6)$$

Second Approximation. Putting $n = 2$ in (5), we get

$$y_2 = 2 + \int_1^x \left(2 - \frac{y_1}{x}\right) dx = 2 + \int_1^x [2 - \frac{1}{x}(2x - 2\log x)]\, dx, \text{ by (6)}$$

$$= 2 + 2\int_1^x \log x . \frac{1}{x} dx = 2 + \left[(\log x)^2\right]_1^x = 2 + (\log x)^2. \qquad ...(7)$$

Third Approximation. Putting $n = 3$ in (5), we get

$$y_3 = 2 + \int_1^x \left[2 - \left(\frac{1}{x}\right) y_2\right] dx = 2 + \int_1^x \left[2 - \left(\frac{1}{x}\right)\left\{2 + (\log x)^2\right\}\right] dx, \text{ by (7)}$$

$$= 2 + \int_1^x \left[2 - \frac{2}{x} - (\log x)^2 \frac{1}{x}\right] dx = 2 + \left[2x - 2\log x - \frac{(\log x)^3}{3}\right]_1^x$$

$$= 2 + 2x - 2\log x - (1/3) \times (\log x)^3 - 2 = 2x - 2\log x - (1/3) \times (\log x)^3.$$

Ex. 4. *(a) Using Picard's method of successive approximation, find a sequence of two functions which approach solution of the initial value problem $dy/dx = e^x + y^2$, $y(0) = 1$.*

[Delhi Maths (Hons.) 1994, 2002]

Sol. Given problem is $dy/dx = e^x + y^2$, where $y = 1$ when $x = 0$. ...(1)

We know that the nth approximation y_n of the initial value problem

$dy/dx = f(x, y)$, where $y = y_0$ when $x = x_0$ ...(2)

is given by
$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx. \qquad ...(3)$$

Comparing (1) and (2), $f(x, y) = e^x + y^2$, $x_0 = 0$ and $y_0 = 1$. ...(4)

$\therefore$ from (3),
$$y_n = 1 + \int_0^x (e^x + y_{n-1}^2)\, dx. \qquad ...(5)$$

First Approximation. Putting $n = 1$ in (5), and using (4), we get

$$y_1 = 1 + \int_0^x (e^x + y_0^2)\, dx = 1 + \int_0^x (e^x + 1)\, dx = 1 + \left[e^x + x\right]_0^x = 1 + e^x + x - 1 = e^x + x. \qquad ...(6)$$

Second Approximation. Putting $n = 2$ in (5), we have

$$y_2 = 1 + \int_0^x (e^x + y_1^2)\, dx = 1 + \int_0^x \left[e^x + (e^x + x)^2\right] dx, \text{ by (6)}$$

$$= 1 + \int_0^x (e^x + e^{2x} + x^2 + 2x\, e^x)\, dx = 1 + \left[e^x + \frac{1}{2}e^{2x} + \frac{x^3}{3}\right]_0^x + 2\int_0^x x\, e^x\, dx$$

$$= 1 + e^x + \frac{1}{2}e^{2x} + \frac{x^3}{3} - 1 - \frac{1}{2} + 2\left[\left[x\, e^x\right]_0^x - \int_0^x (1 . e^x)\, dx\right] = e^x + \frac{1}{2}e^{2x} + \frac{1}{3}x^3 - \frac{1}{2} + 2\left[xe^x - [e^x]_0^x\right]$$

$$= e^x + (1/2) \times e^{2x} + x^3/3 - (1/2) + 2x\, e^x - 2\,(e^x - 1) = (1/2) \times e^{2x} + x^3/3 + 3/2 + (2x - 1)\, e^x.$$

Ex. 4(b). *Find three successive approximations of the solution of $dy/dx = e^x + y^2$, $y(0) = 0$.*

[Delhi Maths (Hons.) 2007]

Sol. Given problem in $dy/dx = e^x + y^2$, $y(0) = 0$ ... (1)

We know that the nth approximation y_n of the initial value problem.

$dy/dx = f(x, y)$, where $y = y_0$ when $x = x_0$ ... (2)

in given by
$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx \qquad ... (3)$$

Comparing (1) and (2), here $f(x, y) = e^x + y^2$, $x_0 = 0$, $y_0 = 0$ ... (4)

$\therefore$ (3) reduces to
$$y_n = \int_0^x (e^x + y_{n-1}^2)\, dx \qquad ... (5)$$

First approximation : Putting $n = 1$ in (5) and using (4), we get

$$y_1 = \int_0^x (e^x + y_0^2) dx = \int_0^x e^x dx = [e^x]_0^x = e^x - 1 \qquad ... (6)$$

Second approximation : Putting $n = 2$ in (5) and using (4) and (6), we get

$$y_2 = \int_0^x (e^x + y_1^2) dx = \int_0^x \{e^x + (e^x - 1)^2\} dx = \int_0^x (e^{2x} - e^x + 1)\, dx$$

$$= [(1/2) \times e^{2x} - e^x + x]_0^x = (1/2) \times (e^{2x} - 2e^x + 2x + 1) \qquad ... (7)$$

Third approximation : Putting n = 3 in (5) and using (4) and (7), we get

$$y_3 = \int_0^x (e^x + y_2^2)dx = \int_0^x \{e^x + (1/4)\times(e^{2x} - 2e^x + 2x + 1)^2\}dx$$

$$= \frac{1}{4}\int_0^x (4e^x + e^{4x} + 4e^{2x} + 4x^2 + 1 - 4e^{3x} + 4xe^{2x} + 2e^{2x} - 8xe^x - 4e^x + 4x)\, dx$$

$$= \frac{1}{4}\int_0^x (e^{4x} - 4e^{3x} + 2e^{2x} + 4x^2 + 4x + 1 + 4xe^{2x} - 8xe^x)\, dx$$

$$= \frac{1}{16}\Big[e^{4x}\Big]_0^x - \frac{1}{3}\Big[e^{3x}\Big]_0^x + \frac{1}{4}\Big[e^{2x}\Big]_0^x + \frac{1}{4}\left[\frac{4x^3}{3} + 2x^2 + x\right]_0^x + \left[x\times\left(\frac{1}{2}e^{2x}\right) - (1)\times\left(\frac{1}{4}e^{2x}\right)\right]_0^x$$

$$-2\Big[(x)\,(e^x) - (1)\times(e^x)\Big]_0^x, \text{ on integrating by parts the last two terms}$$

$$= (1/16)\times(e^{4x} - 1) - (1/3)\times(e^{3x} - 1) + (1/4)\times(e^{2x} - 1) + (1/12)\times(4x^3 - 6x^2 + 3x)$$

$$+ (x/2)\times e^{2x} - (1/4)\times e^{2x} + 1/4 - 2(xe^x - e^x + 1)$$

$$= (1/16)\times e^{4x} - (1/3)\times e^{3x} + (1/4)\times e^{2x} + (1/12)\times(4x^3 - 6x^2 + 3x) + (1/4)\times(2x-1)e^{2x} - 2\,(x-1)e^x - (83/48).$$

Ex. 5. *Use Picard's method to obtain a solution of the differential equation:* $dy/dx = x^2 - y$, $y(0) = 0$. *Find at least the fourth approximation to each solution.* **[Meerut 1996]**

Sol. Given $dy/dx = x^2 - y$, where $y = 0$ when $x = 0$. ...(1)

We know that the nth approximation y_n of the initial value problem

$dy/dx = f(x, y)$, when $y = y_0$ when $x = x_0$ ...(2)

is given by $$y_n = y_0 + \int_{x_0}^x f(x, y_{n-1})\, dx. \qquad ...(3)$$

Comparing (1) and (2), $f(x, y) = x^2 - y$, $x_0 = 0$ and $y_0 = 0$. ...(4)

$\therefore$ from (3), $$y_n = \int_0^x (x^2 - y_{n-1})\, dx. \qquad ...(5)$$

First approximation. Putting n = 1 in (5) and using (4), we get

$$y_1 = \int_0^x (x^2 - y_0)\, dx = \int_0^x x^2\, dx = \left[\frac{1}{3}x^3\right]_0^x = \frac{1}{3}x^3. \qquad ...(6)$$

Second approximation. Putting n = 2 in (5) and using (6), we get

$$y_2 = \int_0^x (x^2 - y_1)\, dx = \int_0^x \left(x^2 - \frac{1}{3}x^3\right)dx = \left[\frac{1}{3}x^3 - \frac{1}{12}x^4\right]_0^x = \frac{1}{3}x^3 - \frac{1}{12}x^4. \qquad ...(7)$$

Third approximation. Putting n = 3 in (5) and using (7), we get

$$y_3 = \int_0^x \left(x^2 - y_2\right)dx = \int_0^x \left[x^2 - \left(\frac{1}{3}x^3 - \frac{1}{12}x^4\right)\right]dx = \left[\frac{1}{3}x^3 - \frac{1}{12}x^4 + \frac{1}{60}x^5\right]_0^x$$

or $$y_3 = x^3/3 - x^4/12 + x^5/60. \qquad ...(8)$$

Fourth approximation. Putting n = 4 in (5) and using (7), we get

$$y_4 = \int_0^x (x^2 - y_3)\,dx = \int_0^x \left[x^2 - \left(\frac{1}{3}x^3 - \frac{1}{12}x^4 + \frac{1}{60}x^5\right)\right] dx = \left[\frac{1}{3}x^3 - \frac{1}{12}x^4 + \frac{1}{60}x^5 - \frac{1}{360}x^6\right]_0^x$$

or $$y = x^3/3 - x^4/12 + x^5/60 - x^6/360.$$

Ex. 6. (a) *Apply Picard's method to find the solution of the problem* $dy/dx = y - x$, $y(0) = 2$. *Show that the iterative solution approaches the exact solution.* **[Meerut 1995]**

Sol. Given $dy/dx = y - x$, where $y = 2$ when $x = 0$. ...(1)

We know that the nth approximation of the initial value problem

$$dy/dx = f(x, y) \quad \text{where} \quad y = y_0 \quad \text{when} \quad x = x_0 \qquad ...(2)$$

is given by $$y_n = y_0 + \int_{x_0}^x f(x, y_{n-1})\,dx. \qquad ...(3)$$

Comparing (1) and (2), $f(x, y) = y - x$, $x_0 = 1$ and $y_0 = 2$. ...(4)

$\therefore$ from (3), $$y_n = 2 + \int_0^x (y_{n-1} - x)\,dx. \qquad ...(5)$$

First approximation. Putting $n = 1$ in (5) and using (4), we get

$$y_1 = 2 + \int_0^x (y_0 - x)\,dx = 2 + \int_0^x (2 - x)\,dx = 2 + 2x - \frac{1}{2}x^2. \qquad ...(6)$$

Second approximation. Putting $n = 2$ in (5) and using (6) we get

$$y_2 = 2 + \int_0^x (y_1 - x)\,dx = 2 + \int_0^x \left(2 + 2x - \frac{1}{2}x^2 - x\right) dx = 2 + 2x + \frac{1}{2}x^2 - \frac{1}{6}x^3 \qquad ...(7)$$

Third approximation. Putting n = 3 in (5) and using (7), we get

$$y_3 = 2 + \int_0^x (y_2 - x)\,dx = 2 + \int_0^x [2 + 2x + x^2/2 - x^3/6 - x]\,dx$$

$$= 2 + 2x + \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{24} = 1 + x + 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} - \frac{x^4}{4!} \qquad ...(8)$$

To find the exact solution of (1). Rewriting (1), we have

$(dy/dx) - y = -x$, which is a linear differential equation ...(9)

Its I.F. $= e^{\int(-1)\,dx} = e^{-x}$ and hence its solution is

$$ye^{-x} = \int (-x)(e^{-x})\,dx + c = -\left[x(-e^{-x}) - \int 1.(-e^{-x})\,dx\right] + c,$$ c being an arbitrary constant

or $ye^{-x} = xe^{-x} + e^{-x} + c$ or $y = x + 1 + c\,e^x$. ...(10)

Given that $y = 2$, when $x = 0$, so (10) gives $2 = 1 + c$ or $c = 1$.

Hence, from (10), the exact solution is $y = x + 1 + e^x$. ...(11)

We know that $e^x = 1 + x + (x^2/2!) + (x^3/3!) + ...$ ad inf. ...(12)

Keeping (12) and (8) in view, we find that the approximate solution tends to

$y = 1 + x + 1 + x/1! + x^2/2! + x^3/3! + ... = 1 + x + e^x$ *i.e.*, which is exact solution of (4).

Ex. 6 (b) *Apply Picard's iteration method to the initial value problem* $dy/dx = y$, $y(0) = 1$ *and show that the successive approximations tend to the limit* $y = e^x$, *the exact solution.*

[Allahabad 2001, 05; Gwalior 2005; Indore 2002, 03; Kurukshetra 2003; Pune 2002]

Sol. Proceeding as in Ext. 6 (a), we have

$y_1 = 1 + x,\qquad y_2 = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!},\qquad y_3 = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!}.$

In general, $y_n = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots\ldots + \dfrac{x^n}{n!}.$ ...(1)

Also, solving $dy/dx = y$, we have $(1/y)dy = dx$ so that $y = c\, e^x$. ... (2)

Given that $y = 1$ when $x = 0$, so (2) given $1 = c$ and hence the exact solution (2) becomes $y = e^x$. ...(3)

From (1), we see that the successive approximations tend to the limit $y = e^x$ as $n \to \infty$, which is the exact solution.

Ex. 6. (*c*). *Show that successive approximation* $\phi_n(x)$ *for the equation* $y'(x) = y(x)$ $(-\infty < x < \infty)$ *with initial condition* $y(0) = 1$ *are given by* $\phi_n(x) = 1 + x + x^2/2! + x^3/3! + \ldots + x^n/n!$
[Kolkata 2002, 04, 07]

EXERCISE 1 (A)

1. *Apply Picard's method to the following initial value problems and find the first three successive approximations.*

(*i*) $dy/dx = 2xy$, $y(0) = 1$. **[Allahabad 1994, Delhi Maths (Hons.) 95, 2001, 04, 06]**

Ans. $y_1 = 1 + x^2$, $y_2 = 1 + x^2 + (x^4/2\ )$, $y_3 = 1 + x^2 + (x^4/2) + (x^6/6)$

(*ii*) $dy/dx = 3e^x + 2y$, $y(0) = 0$.

[Himanchal 2002; Meerut 1993, 94, M.K.U. (Tamil Nadu), 2002, 02]

Ans. $y_1 = 3(e^x - 1)$, $y_2 = 9e^x - 6x - 9$, $y_3 = 21\ e^x - 6x^2 - 18x - 21$.

(*iii*) $dy/dx = x + y$, $y(0) = 1$. **[Allahabad 1999; Calicut 2004; Meerut 1993, 95]**

Ans. $y_1 = 1 + x + (x^2/2)$, $y_2 = 1 + x + x^2 + (x^3/6)$, $y_3 = 1 + x + x^2 + (x^3/3) + (x^4/24)$

(*iv*) $dy/dx = 1 + xy$, $y(0) = 2$.

[Agra 2001, 04, 05 Himanchal 2004, 05; Lucknow 2003, Meerut 2001, 06]

Ans. $y_1 = 2 + x + x^2$, $y_2 = 2 + x + x^2 + (x^3/3) + (x^4/4)$,
$y_3 = 2 + x + x^2 + (x^3/3) + (x^4/4) + (x^5/15) + (x^6/24)$

(*v*) $dy/dx = 2x - y^2$, where $y = 0$ at $x = 0$.

Ans. $y_1 = x^2$, $y_2 = x^2 - (x^5/5)$, $y_3 = x^2 - (x^5/5) + (x^8/120) - (x^{11}/275)$.

(*vi*) $dy/dx = e^x + y^2$, $y(0) = 0$. **Ans.** $y_1 = e^x - 1$, $y_2 = (1/2)e^{2x} - e^x + x + (1/2)$, $y_3 = (1/16)e^{4x}$ $- (1/3)e^{3x} + (1/2)xe^{2x} + (1/2)e^{2x} - 2xe^x + 2e^x + (1/3)x^3 + (1/2)x^2 + (1/4)x - (107/48)$.

2. *Solve the differential equation* $dy/dx = x - y$ *with the condition* $y = 1$ *when* $x = 0$ *and show that the sequence of approximations given by Picard's method tend to the exact solution as a limit.*
[Agra 2001; Meerut 2003]

Ans. $y_1 = 1 - x + \dfrac{x^2}{2!}$, $y_2 = 1 - x + \dfrac{2x^2}{2!} - \dfrac{x^3}{3!}$, $y_3 = 1 - x + \dfrac{2x^2}{2!} - \dfrac{2x^3}{3!} + \dfrac{x^4}{4!}$;

$y_4 = 1 - x + \dfrac{2x^2}{2!} - \dfrac{2x^3}{3!} + \dfrac{2x^4}{4!} - \dfrac{x^5}{5!}$ tending to $y = -1 + x + 2e^{-x}$, which is exact solution.

3. *Use Picard's method to approximate the solution of the equation* $dy/dx + 2xy^2 = 0$ *with* $y = 1$ *when* $x = 0$ *and hence show that* $y = 1/(1 + x^2)$. **[Rohtak 2001, Meerut 1995]**

Ans. $y_1 = 1 - x^2, y_2 = 1 - x^2 + x^4 - (1/3)x^6, y_3 = 1 - x^2 + x^4 - x^6 + (2/3)x^8 - (1/3)x^{10} + (1/9)x^{12} - (1/63)x^{14}$.

4. *State the conditions under which the initial value problem* $dy/dx = x^2$, $x_0 = 2$, $y_0 = 1$ *has unique solution by Picard's method of successive approximation. Obtain the solution of the initial value problem* $dy/dx = x^2 + y^2$, $y(0) = 1$, *by Picards method as far as the term, involving* x^4.

[Himanchal 2003, 05; Jabalpur 2004, 06; Kurukshetra 2006]

Ans. $y_1(x) = 1 + x + x^3/3, y_2(x) = 1 + x + x^2 + (2x^3/3) + (2x^4)/12 + (2x^5)/15 + x^6/63$

5. *Using Picard's method find the third approximation of the solution of the initial value problem* $dy/dx = 1 + y^2$, $y(0) = 0$. **[Kolkata 2004, 07, Pune 2002]**

Ans. $y_1(x) = x,\ y_2(x) = x + x^3/3, y_3(x) = x + x^3/3 + (2x^5)/15 + x^7/63$

6. *Find the exact solution of initial value problem* $dy/dx = x + y$, $y(0) = 0$. *Next apply Picard's iterative method to obtain three successive approximate solutions* $y_1(x)$, $y_2(x)$, $y_3(x)$.

[Allahabad 2003, 07]

7. *Solve the initial value problem* $dy/dx = y^2$, $y(0) = 1$ *by method of successive approximation.*

[Jabalpur 2004; Osmania 2006]

Ans. $y_1(x) = 1 + x;\ y_2(x) = 1 + x + x^2 + x^3/3;\ y_3(x) = 1 + x + x^2 + x^3 + (2x^4)/3 + x^5/3 + x^6/9 + x^7/63$

8. (*a*) *Under what condition does the initial value problem of the form* $y' = f(x, y)$, $y(x_0) = y_0$ *has unique solution ? Give statement only.*

(b) Consider the initial value problem $y' = x - y^2$, $y(0) = 1/2$. (*i*) *Does this have a unique solution ? Justify your answer* (*ii*) *Applying Picard method find an approximate solution of the above initial value problem containing at least four no-zero terms (iii) Give your comments regarding the utility of Picard's method in view of the above initial value problem.*

[Lucknow 2004, 05]

9. *Consider the initial value problem* $dy/dx = f(x, y)$, $y(a) = b$ *and discuss Picard's method of successive approximation to solve it.* **[Bangalore 2002, 06; Himanchal 2004]**

10. *Let* $\{y_n\}$ *be a sequence of successive approximation to the solution of* $dy/dx = f(x, y)$, $y(x_0) = y_0$ *such that* $y_{10} = y_9$. *Show* y_{10} *is exact solution.* **[Himanchal 2002, 03, 05, 07]**

11. *Explain initial value problem and equivalent integral equation.*

[Kanpur 2002, 07; Rohilkhand 2001, Ujjain 2000, 01, 06]

1.3 A. Working rule for Picard's method of solving simultaneous differntial equations with initial conditions, namely

$dy/dx = f(x, y, z)$, $dz/dx = g(x, y, z)$ where $y = y_0$, $z = z_0$ when $x = x_0$. ...(1)

The nth approximation (y_n, z_n) to the initial value problem (1) is given by

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1}, z_{n-1})\, dx \qquad ...(2)$$

and

$$z_n = z_0 + \int_{x_0}^{x} g(x, y_{n-1}, z_{n-1})\, dx . \qquad ...(3)$$

1.3B. Solved examples based on Art 1.3A.

Ex. 1(a) *Find the third approximation of the solution of the equation* $dy/dx = z$, $dz/dx = x^3(y + z)$ *by Picard's method where* $y = 1$, $z = 1/2$ *where* $x = 0$.

[Meerut 2006,07; Rohilkhand 2002; Gwalior 2003]

Sol. Given $dy/dx = z$, $dz/dx = x^3(y + z)$, $y = 1$, $z = 1/2$ when $x = 0$. ...(1)

We know that the nth approximation (y_n, z_n) to the initial value problem

$dy/dx = f(x, y, z)$, $dz/dx = g(x, y, z)$, when $y = y_0$, $z = z_0$ when $x = x_0$ ...(2)

is given by

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1}, z_{n-1})\, dx \qquad ...(3)$$

and $$z_n = z_0 + \int_{x_0}^{x} g(x, y_{n-1}, z_{n-1})\, dx. \qquad ...(4)$$

Comparing (1) and (2), we have

$$f(x, y, z) = z, \qquad g(x, y, z) = x^3 (y + z), \qquad y_0 = 1, \qquad z_0 = 1/2, \qquad x_0 = 0. \qquad ...(5)$$

$\therefore$ From (3), $$y_n = 1 + \int_0^x z_{n-1}\, dx \qquad ...(6)$$

and from (4), $$z_n = \frac{1}{2} + \int_0^x x^3 \left(y_{n-1} + z_{n-1}\right) dx. \qquad ...(7)$$

First approximation. Putting $n = 1$ in (6) and using (5), we get

$$y_1 = 1 + \int_0^x z_0\, dx = 1 + \int_0^x \frac{1}{2} dx = 1 + \frac{1}{2} x. \qquad ...(8)$$

Next, putting $n = 1$ in (7) and using (5), we gets

$$z_1 = \frac{1}{2} + \int_0^x x^3 (y_0 + z_0)\, dx = \frac{1}{2} + \int_0^x x^3 \left(1 + \frac{1}{2}\right) dx = \frac{1}{2} + \frac{3}{8} x^4. \qquad ...(9)$$

Second approximation. Putting $n = 2$ in (6) and using (9), we get

$$y_2 = 1 + \int_0^x z_1\, dx = 1 + \int_0^x \left(\frac{1}{2} + \frac{3}{8} x^4\right) dx = 1 + \frac{1}{2} x + \frac{3x^5}{40}. \qquad ...(10)$$

Next, putting $n = 1$ in (7) and using (8) and (9), we get

$$z_2 = \frac{1}{2} + \int_0^x x^3 (y_1 + z_1)\, dx = \frac{1}{2} + \int_0^x x^3 \left(1 + \frac{1}{2} x + \frac{1}{2} + \frac{3}{8} x^4\right) dx$$

$$= \frac{1}{2} + \int_0^x \left(\frac{3}{2} x^3 + \frac{1}{2} x^4 + \frac{3}{8} x^7\right) dx = \frac{1}{2} + \frac{3}{8} x^4 + \frac{1}{10} x^5 + \frac{3}{64} x^8. \qquad ...(11)$$

Third approximation. Putting $n = 3$ in (6) and (11), we get

$$y_3 = 1 + \int_0^x z_2\, dx = 1 + \int_0^x \left(\frac{1}{2} + \frac{3}{8} x^4 + \frac{1}{10} x^5 + \frac{3}{64} x^8\right) dx = 1 + \frac{1}{2} x + \frac{3}{40} x^5 + \frac{1}{60} x^6 + \frac{1}{192} x^9.$$

Next, putting $n = 3$ in (7) and using (10) and (11), we get

$$z_3 = \frac{1}{2} + \int_0^x x^3 (y_2 + z_2)\, dx = \frac{1}{2} + \int_0^x x^3 \left[1 + \frac{x}{2} + \frac{3x^5}{40} + \frac{1}{2} + \frac{3x^4}{8} + \frac{x^5}{10} + \frac{3x^8}{64}\right] dx$$

$$= \frac{1}{2} + \int_0^x \left(\frac{3}{2} x^3 + \frac{1}{2} x^4 + \frac{3}{8} x^7 + \frac{7}{40} x^8 + \frac{3}{64} x^{11}\right) dx = \frac{1}{2} + \frac{3x^4}{8} + \frac{x^5}{10} + \frac{3x^8}{64} + \frac{7x^9}{360} + \frac{x^{12}}{256}$$

Ex. 1 (b) *Find the third approximation of the solution of the equation* $d^2y/dx^2 = x^3(y + dy/dx)$, *where* $y = 1$ *and* $dy/dx = 1/2$ *when* $x = 0$

(*Agra 2000, 02; Himanchal 2004, 05; Meerut 2000; Rohilkhand 2002*)

Sol. Given that $d^2y/dx^2 = x^3(y + dy/dx)$, where $y = 1$ and $dy/dx = 1/2$ when $x = 0$. ...(1)

Let $dy/dx = z$ so that $d^2y/dx^2 = dz/dx$. Then, we have

$dy/dx = z$, $\quad dz/dx = x^3(y + z)$, where $y = 1$ and $z = 1/2$ when $x = 0$. ...(2)

which, is the same as given in solved Ex. 1 (a) Proceed now as before and obtain the required third approximation: $$y_3 = 1 + x/2 + 3x^5/40 + x^6/60 + x^9/192.$$

Note : For complete solution of Ex. 1(b), you need not find z_3.

Ex. 2 (a) *Find the third approximation of the solution of the equation* $dy/dx = z$, $dz/dx = x^2 z + x^4 y$ *by Picard's method,* $y = 5$ *and* $z = 1$ *when* $x = 0$. **[Bangalore 2001, Meerut 2006, 07]**

Sol. Given $dy/dx = z$, $dz/dx = x^2 z + x^4 y$, $y = 5$, $z = 1$ when $x = 0$. ...(1)

We know that the nth approximation (y_n, z_n) to initial value problem

$dy/dx = f(x, y, z)$, $dz/dx = g(x, y, z)$ where $y = y_0$, $z = z_0$, when $x = x_0$ ...(2)

is given by
$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1}, z_{n-1})\, dx \qquad ...(3)$$

and
$$z_n = z_0 + \int_{x_0}^{x} g(x, y_{n-1}, z_{n-1})\, dx. \qquad ...(4)$$

Comparing (1) and (2), we have

$f(x, y, z) = z$, $g(x, y, z) = x^2 z + x^4 y$, $y_0 = 5$, $z_0 = 1$, $x_0 = 0$. ...(5)

$\therefore$ from (3),
$$y_n = 5 + \int_0^x z_{n-1}\, dx \qquad ...(6)$$

and from (4),
$$z_n = 1 + \int_0^x (x^2 z_{n-1} + x^4 y_{n-1})\, dx. \qquad ...(7)$$

First approximation. Putting $n = 1$ in (6) and using (5), we get
$$y_1 = 5 + \int_0^x z_0\, dx = 5 + \int_0^x dx = 5 + x. \qquad ...(8)$$

Next putting $n = 1$ in (7) and using (5), we get
$$z_1 = 1 + \int_0^x (x^2 z_0 + x^4 y_0)\, dx = 1 + \int_0^x (x^2 + 5x^4)\, dx = 1 + \frac{x^3}{3} + x^5. \qquad ...(9)$$

Second approximation. Putting $n = 2$ in (6) and using (9), we get
$$y_2 = 5 + \int_0^x z_1\, dx = 5 + \int_0^x \left(1 + \frac{x^3}{3} + x^5\right) dx = 5 + x + \frac{x^4}{12} + \frac{x^6}{6}. \qquad ...(10)$$

Next, putting $n = 2$ in (7) and using (8) and (9), we get
$$z_2 = 1 + \int_0^x \left(x^2 z_1 + x^4 y_1\right) dx = 1 + \int_0^x \left[x^2\left(1 + \frac{x^3}{3} + x^5\right) + x^4(5 + x)\right] dx$$
$$= 1 + \int_0^x \left(x^2 + 5x^4 + \frac{4}{3}x^5 + x^7\right) dx = 1 + \frac{1}{3}x^3 + x^5 + \frac{2}{9}x^6 + \frac{1}{8}x^8. \qquad ...(11)$$

Third approximation. Putting $n = 3$ in (6) and using (11), we get
$$y_3 = 5 + \int_0^x z_2\, dx = 5 + \int_0^x \left(1 + \frac{1}{3}x^3 + x^5 + \frac{2}{9}x^6 + \frac{1}{8}x^8\right) dx = 5 + x + \frac{x^4}{12} + \frac{x^6}{6} + \frac{2x^7}{63} + \frac{x^9}{72}$$

Next, putting $n = 3$ in (7) and using (10) and (11), we have
$$z_3 = 1 + \int_0^x \left(x^2 z_2 + x^4 y_2\right) dx = 1 + \int_0^x \left[x^4\left(5 + x + \frac{x^4}{12} + \frac{x^6}{6}\right) + x^2\left(1 + \frac{x^3}{3} + x^5 + \frac{2x^6}{9} + \frac{x^8}{8}\right)\right] dx$$
$$= 1 + (1/3)x^3 + x^5 + (2/9)x^6 + (1/8)x^8 + (11/224)x^9 + (7/264)x^{11}.$$

Ex. 2 (b) *Find the third approximation of the solution of the equation* $d^2y/dx^2 = x^2(dy/dx) + x^4 y$ *where* $y = 5$ and $dy/dx = 1$ *when* $x = 0$.

Sol. Let $dy/dx = z$ so that $dz/dx = d^2y/dx^2 = x^2 z + x^4 y$, where $y = 5$ and $z = 1$ when $x = 0$. $(\because dy/dx = 1 \Rightarrow z = 1)$

This is exactly the same problem as given in Ex. 2.(a) Proceed as above and obtain the required value of y_3.

In the solution of Ex. 2 (b), you need not compute z_3 because we want third approximation of the solution of the original equation in which y is dependent variable.

$\therefore$ The required third approximation $= y_3(x) = 5 + x + (1/12)x^4 + (1/6)x^6 + (2/63)x^7 + (1/72)x^9$.

Ex. 3. *Find the third approximation of the initial value problem*
$d^2y/dx^2 = xy + 1$, *when* $(y)_0 = 1$ and $(dy/dx)_0 = 0$. **[Meerut 1994]**

Sol. Let $dy/dx = z$ so that $dz/dx = d^2y/dx^2 = xy + 1$.

Again, re–writing the initial conditions, $y_0 = 1$, $z_0 = 0$ when $x = 0$.

In order to solve the given problem, we shall solve the following initial value problem involving differential equations : $dy/dx = z$, $dz/dx = 1 + xy$, $y = 1$, $z = 0$ when $x = 0$ ...(1)

We know that the nth approximation (y_n, z_n) to the initial value problem

$$dy/dx = f(x, y, z), \quad dz/dx = g(x, y, z), \quad \text{when} \quad y = y_0, \quad z = z_0 \quad \text{when} \quad x = x_0 \qquad ...(2)$$

is given by
$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1}, z_{n-1})\, dx \qquad ...(3)$$

and
$$z_n = z_0 + \int_{x_0}^{x} g(x, y_{n-1}, z_{n-1})\, dx. \qquad ...(4)$$

Comparing (1) and (2), we have

$$f(x, y, z) = z, \quad g(x, y, z) = 1 + xy, \quad y_0 = 1, \quad z_0 = 0 \quad \text{and} \quad x_0 = 0. \qquad ...(5)$$

$\therefore$ From (3),
$$y_n = 1 + \int_0^x z_{n-1}\, dx \qquad ...(6)$$

and from (4),
$$z_n = \int_0^x (1 + xy_{n-1})\, dx. \qquad ...(7)$$

First approximation. Putting $n = 1$ in (6) and using (5), we get

$$y_1 = 1 + \int_0^x z_0\, dx = 1 + \int_0^x (0)\, dx = 1. \qquad ...(8)$$

Next, putting $n = 1$ in (7) and using (5), we get

$$z_1 = \int_0^x (1 + xy_0)\, dx = \int_0^x (1 + x)\, dx = x + \frac{x^2}{2}. \qquad ...(9)$$

Second approximation. Putting $n = 2$ in (6) and using (9), we get

$$y_2 = 1 + \int_0^x z_1\, dx = 1 + \int_0^x \left(x + \frac{x^2}{2}\right) dx = 1 + \frac{1}{2}x^2 + \frac{1}{6}x^3. \qquad ...(10)$$

Next, putting $n = 2$ in (7) and using (8), we get

$$z_2 = \int_0^x (1 + xy_1)\, dx = 1 + \int_0^x (1 + x)\, dx = x + \frac{1}{2}x^2. \qquad ...(11)$$

Third approximation. Putting $n = 3$ in (6) and using (11), we get

$$y_3 = 1 + \int_0^x z_2\, dx = 1 + \int_0^x \left(x + \frac{1}{2}x^2\right) dx = 1 + \frac{1}{2}x^2 + \frac{1}{6}x^3.$$

Next, putting $n = 3$ in (7) and using (10), we get

$$z_3 = \int_0^x (1 + xy_2)\, dx = \int_0^x \left[1 + x\left(1 + \frac{1}{2}x^2 + \frac{1}{6}x^3\right)\right] dx = x + \frac{x^2}{2} + \frac{x^4}{8} + \frac{x^5}{30}$$

EXERCISE 1 (B)

1. Apply Picard's method upto third approximation to solve the equations:
$dy/dx = x + z$, $dz/dx = x - y^2$, given that $y = 2$, $z = 1$ when $x = 0$

[Bhopal, 2002; Meerut 2000; Agra 2003, 04; Gwalior 2004]

[**Ans.** $y_1 = 2 + x + (1/2)x^2$, $z_1 = 1 - 4x + (1/2)x^2$; $y_2 = 2 + x - (3/2)x^2 + (1/6)x^3$; $z_2 = 1 - 4x - (3/2)x^2 - x^3 - (1/4)x^4 - (1/20)x^5$; $y_3 = 2 + x - (3/2)x^2 - (1/2)x^3 - (1/4)x^4 - (1/20)x^5 - (1/120)x^6$; $z_3 = 1 - 4x - (3/2)x^2 + (5/3)x^3 + (7/12)x^4 - (31/60)x^5 + (1/12)x^6 - (1/252)x^8$.]

2. Find the third approximation of the solution of the following equations:
$dy/dx = 2x + z$, $dz/dx = 3xy + x^2 z$ where $y = 2$ and $z = 0$ when $x = 0$.

[Meerut 2000, 05; I.A.S. 2000]

[**Ans.** $y_3 = 2 + x^2 + x^3 + (3/20)x^5 + (1/10)x^6$, $z_3 = 3x^2 + (3/4)x^4 + (6/5)x^5 + (3/28)x^7 + (3/40)x^8$.]

1.4. Problems of existence and uniqueness : An Introduction.

Consider the initial value problem

$$| dy/dx | + | y | = 0, \qquad y(0) = 1. \qquad ...(1)$$

If possible, let $y \neq 0$. Then division by $| y |$ and integration leads to an absurd result. Hence $y = 0$ is the only solution of the differential equation. Clearly this solution does not satisfy the initial condition $y(0) = 1$. Thus, we see that the initial value problem (1) has no solution at all.

Now let us consider the initial value problem

$$dy/dx = x, \qquad y(0) = 1. \qquad ...(2)$$

Separating the variables, we get $\qquad dy = x\,dx.$

Integrating, $\qquad y = (x^2/2) + c$, where c is an arbitrary constant.

Using the initial condition $y(0) = 1$ *i.e.*, $x = 0$, $y = 1$, we get $c = 1$. Hence, the initial value problem (2) has only one solution, namely, $\qquad y = (x^2/2) + 1$.

Finally consider the following initial value problem

$$dy/dx = (y - 1)/x, \qquad y(0) = 1. \qquad ...(3)$$

Separating the variables, $\qquad (dy)/(y - 1) = (dx)/x.$

Integrating, $\qquad \log (y - 1) = \log x + \log c \qquad$ or $\qquad y - 1 = xc.$

Using the given initial condition, *i.e.*, $x = 0$, $y = 1$, we see that c cannot be dertermined. Thus, the given initial value problem (3) has infinite solutions given by $y - 1 = xc$, where c is an arbitrary constant.

From the above examples, we conclude that an initial value problem

$$dy/dx = f(x, y), \qquad y(x_0) = y_0 \qquad ...(4)$$

may have none, exactly one, or more than one solution. This leads us to the following two fundamental questions.

Problem of existence. *Under what conditions does an initial value problem of the form (4) has at least one solution ?*

Problem of uniqueness. *Under what conditions does that problem has a unique solution, that is, only one solution ?*

Theorems which state such conditions are called *existence theorem* and *uniqueness theorem*, respectively.

It may be noted that the above three examples are very simple and investigation about their existence and uniqueness is evident by mere inspection (or by actually solving), without using any theorem. However, when the equation cannot be solved by standard methods, existence and uniqueness theorem will play an important role.

1.5. Lipschitz condition. A function $f(x, y)$ is said to satisfy a *Lipschitz condition* in a region D in xy–plane if there exists a positive constant k such that

$$| f(x, y_2) - f(x, y_1) | \le K | y_2 - y_1 |$$

whenever the points (x, y_1) and (x, y_2) both lie in D. The constant K is called a *Lipschitz constant* for the function $f(x, y)$.

1.6. Picard's Theorem. Existence and uniqueness theorem.

[Himanchal 2008, 09; Agra 2003, Calicut 2003, I.A.S. 1985 ; Meerut 2001, 02, 07, 11; Gwalior 2004, 05; G.N.D.U. Amritsar 2000; Jiwaji 2002 Ravishankar 2002, Rajasthan 2004, Rohilkhand 2007; Kolkata 2003; Ujjain 2003, 06]

Statement. Let $f(x, y)$ *be continuous in a domain D of the* (x, y) *plane and let M be a constant such that* $\quad | f(x, y) | \le M \quad$ in D. ...(1)

Let $f(x, y)$ *satisfy in D the Lipschitz condition in y namely*

$$| f(x, y_1) - f(x, y_2) | \le K | y_1 - y_2 |, \qquad \text{...(2)}$$

where the constant K is independent of x, y_1, y_2.

Let the rectangle R, defined by $\quad | x - x_0 | \le h, \quad | y - y_0 | \le k,$ ...(3)

lie in D, where $Mh < k$. *Then, for* $| x - x_0 | \le h$, *the differential equation* $dy/dx = f(x, y)$ *has a unique solution* $y = y(x)$ *for which* $y(x_0) = y_0$.

Proof. [Read article 1.2A also for this proof carefully.]

We shall prove this theorem by the method of successive approximations. Let x be such that $| x - x_0 | \le h$. We now define a sequence of functions $y_1(x), y_2(x), \dots y_n(x) \dots$, called the successive approximations (or Picard Iterants) as follows :

$$\left.\begin{aligned} y_1(x) &= y_0 + \int_{x_0}^{x} f(x, y_0)\, dx \\ y_2(x) &= y_0 + \int_{x_0}^{x} f(x, y_1)\, dx \\ &\dots \quad \dots \quad \dots \quad \dots \quad \dots \\ y_{n-1}(x) &= y_0 + \int_{x_0}^{x} f(x, y_{n-2})\, dx \\ y_n(x) &= y_0 + \int_{x_0}^{x} f(x, y_{n-1})\, dx \end{aligned}\right\} \qquad \text{...(4)}$$

We shall divide the proof into five main steps.

First Step. *We prove that, for* $x_0 - h \le x \le x_0 + h$ *the curve* $y = y_n(x)$ *lies in the rectangle R, that is to say* $y_0 - k < y < y_0 + k$.

Now, $\quad \left| y_1 - y_0 \right| = \left| \int_{x_0}^{x} f(x, y_0)\, dx \right| \le \int_{x_0}^{x} | f(x, y_0) | . | dx |$, by (4)

or $\quad \left| y_1 - y_0 \right| \le M | x - x_0 | \le Mh < k$, using (1), (3) and the given result viz $Mh < k$.

This proves the desired result for $n = 1$. Assume that $\quad y = y_{n-1}(x)$ lies in R and so $f(x, y_{n-1})$ is defined and continuous and satisfies

$$\left| f(x, y_{n-1}) \right| \le M \qquad \text{on} \qquad [x_0 - h, x_0 + h].$$

From (4), we have $\quad \left| y_n - y_0 \right| = \left| \int_{x_0}^{x} f(x, y_{n-1})\, dx \right| \le \int_{x_0}^{x} \left| f(x, y_{n-1}) \right| . | dx | \le M \left| x - x_0 \right| \le Mh < k,$

as before which shows that $y_n(x)$ lies in R and hence $f(x, y_n)$ is defined and continuous on $[x_0 - h, x_0 + h]$. The above arguments show that the desired result holds for all n by induction.

Second Step. *We prove again by induction, that* $\quad \left| y_n - y_{n-1} \right| \le \dfrac{MK^{n-1}}{n!} \left| x - x_0 \right|^n.$ ...(5)

We have already verified (5) for $n = 1$ in first step where we have shown that $| y_1 - y_0 | \le M \; | x - x_0 |$. Assume that this inequality (5) holds for $n - 1$ in place of n, that is, let

$$\left|y_{n-1} - y_{n-2}\right| \le \frac{MK^{n-2}}{(n-1)!} |x - x_0|^{n-1}. \qquad ...(6)$$

Then, we have
$$\left|y_n - y_{n-1}\right| = \left|\int_{x_0}^{x} \left\{f(x, y_{n-1}) - f(x, y_{n-2})\right\} dx\right|, \text{ by (4)}$$

or
$$\left|y_n - y_{n-1}\right| \le \int_{x_0}^{x} \left|f(x, y_{n-1}) - f(x, y_{n-2})\right| . \left|dx\right|. \qquad ...(7)$$

Lipschitz condition (2) gives
$$\left|f(x, y_{n-1}) - f(x, y_{n-2})\right| \le K \left|y_{n-1} - y_{n-2}\right| \qquad ...(8)$$

From (7) and (8), we get

$$\left|y_n - y_{n-1}\right| \le \int_{x_0}^{x} K\left|y_{n-1} - y_{n-2}\right| . \left|dx\right| \le \text{K} \; . \; \frac{MK^{n-1}}{(n-1)!} . \frac{|x - x_0|^n}{n}, \text{ by (6)}$$

Hence by mathematical induction, we conclude that (5) is true for each natural number n.

Third Step. We shall now prove that the sequence y_n converges uniformly to a limit for $x_0 - h \le x \le x_0 + h$. For the interval under consideration, $\left|x - x_0\right| \le h$.

Hence from second step, we get
$$\left|y_n - y_{n-1}\right| \le \frac{MK^{n-1} h^n}{n\,!} \text{ is true for all } n.$$

Using this, the infinite series

$$y_0 + (y_1 - y_0) + (y_2 - y_1) - ... + (y_n - y_{n-1}) + ... \qquad ...(9)$$
$$\le \; y_0 + Mh + \frac{1}{2!} MKh^2 + \ldots + \frac{1}{n!} MK^{n-1} h^n + \ldots \le y_0 + \frac{M}{K}\left[e^{Kh} - 1\right],$$

which is known to be convergent for all values of K, h and M. Consequently, the series (9) is surely convergent. Thus, by the Weirstrass M–test, the series (9) converges uniformly on $[x_0 - h, x_0 + h]$. Now since the terms of (9) are continuous functions of x, therefore,

$$\text{its sum} = \lim_{n\to\infty} y_n(x) = y(x), \text{ say,} \qquad \text{as} \qquad y_n = y_0 + \sum_{n=1}^{n} (y_n - y_{n-1}) \qquad ...(10)$$

must be continuous.

Fourth Step. *We now show that $y = y(x)$ satisfies the differential equation $dy/dx = f(x, y)$.*

Since $y_n(x)$ tends uniformly to $y(x)$ in $[x_0 - h, x_0 + h]$ and by Lipschitz condition,

$$\left|f(x, y) - f(x, y_n)\right| \le K\left|y - y_n\right|,$$

it follows that $f[x, y_n(x)]$ tends uniformly to $f[x, y(x)]$. Again from (4) we have

$$y_n(x) = y_0 + \int_{x_0}^{x} f[x, y_{n-1}(x)]\, dx$$

or
$$\lim_{n\to\infty} y_n(x) = y_0 + \lim_{n\to\infty} \int_{x_0}^{x} f[x, y_{n-1}(x)]\, dx, \text{ letting } n \to \infty.$$

Since the sequence $f[x, y_n(x)]$, consisting of continuous functions on the given interval, converge uniformly to $f[x, y(x)]$ on the same interval, the interchanges of limiting operations given below are valid. Thus using (10), we have

$$y(x) = y_0 + \int_{x_0}^{x} \lim_{n\to\infty} f[x, y_{n-1}(x)]\, dx \qquad \text{or} \qquad y(x) = y_0 + \int_{x_0}^{x} f[x, y(x)]\, dx \quad ....(11)$$

The integrand on the right–hand side of (11) being a continuous function of x, we conclude that the integral has the derivative. Thus, the limit function $y(x)$ satisfies the differential equation $dy/dx = f(x, y)$ on $[x_0 - h, x_0 + h]$ and is such that $y(x_0) = y_0$.

[**In the above four steps we have thus proved the existence of a solution of the given initial value problem. The next step will show that the solution $y(x)$ is unique.**]

Fifth Step. Uniqueness of the solution : We now prove that the solution $y = y(x)$ just found is the only solution for which $y(x_0) = y_0$.

Assume if possible $y = Y(x)$, say, is another solution of the given initial value problem.

Let $\quad \left|Y(x) - y(x)\right| \le B, \quad$ where $\quad x_0 - h \le x \le x_0 + h. \quad$ ...(12)

It may be noted here that we can surely take $B = 2K$. From (11), we get

$$\left|Y(x) - y(x)\right| = \left|\int_{x_0}^{x} \left[f\{x, Y(x)\} - f\{x, y(x)\}\right]\right| \text{ or } \left|Y(x) - y(x)\right| \le \int_{x_0}^{x} \left|f\{x, Y(x)\} - f\{x, y(x)\}\right|.\left|dx\right|$$

or
$$\left|Y(x) - y(x)\right| \le K \int_{x_0}^{x} \left|Y(x) - y(x)\right|.\left|dx\right|. \qquad \text{...(13)}$$

$\left[\because \; \left|f\{x, Y(x)\} - f\{x, y(x)\}\right| \le K\left|Y(x) - y(x)\right)\right|$, by Lipschitz condition

or
$$\left|Y(x) - y(x)\right| \le K.B\left|x - x_0\right|, \text{ using (12)} \qquad \text{...(14)}$$

Now substituting (14) for integrand in (13), we get

$$\left|Y(x) - y(x)\right| \le K^2 B \int_{x_0}^{x} \left|x - x_0\right| dx \le \frac{K^2 B\left|x - x_0\right|^2}{2\,!}. \qquad \text{...(15)}$$

Again, substituting (15) for the integrand in (13), we get

$$\left|Y(x) - y(x)\right| \le \frac{K^3 B}{2!} \int_{x_0}^{x} \left|x - x_0\right|^2 dx \le \frac{K^3 B\left|x - x_0\right|^3}{3\,!}$$

Continuing in this way, we shall surely get

$$\left|Y(x) - y(x)\right| \le \frac{K^n B\left|x - x_0\right|^n}{n\,!} \le B\frac{(Kh)^n}{n!}, \qquad \text{as} \qquad \left|x - x_0\right| \le h \qquad \text{...(16)}$$

Now the series $\displaystyle\sum_{n=0}^{n} B\frac{(Kh)^n}{n!}$ converges, and so $\displaystyle\lim_{n\to\infty} B\frac{(Kh)^n}{n!} = 0$.

Thus $\left|Y(x) - y(x)\right|$ can be made less than any number however small and consequently we conclude that $\quad Y(x) - y(x) = 0 \quad$ *i.e.*, $\quad Y(x) = y(x)$

This shows that the solution $y = y(x)$ is always unique, and the proof of the theorem is complete.

Important note : The above theorem is an existence theorem because it says that the initial value problem does have a solution. It is also a uniqueness theorem, because it says that there is only one solution.

Remark 1. If the existence theorem is asked then you need not mention uniqueness in the statement of theorem and finish up the proof just after the fourth step. Again if you are asked to state and prove the uniqueness theorem, then give the complete proof of first four steps and fifth step.

Remark 2. If $f(x, y)$ satisfies the condition $\quad \left|\partial f/\partial y\right| \le K \quad$ ...(*i*)

for all values of x, y in the given range then for the same constant K the Lipschitz's condition is also satisfied.

By the mean value theorem of differential calculus, we get

$$f(x, y_2) - f(x, y_1) = (y_2 - y_1)\left(\frac{\partial f}{\partial y}\right)_{y=\bar{y}}, \qquad \text{where } y_1 < \bar{y} < y_2, \qquad ...(ii)$$

where (x, y_1) and (x, y_2) are assumed in the given range.

Now (i) and (ii) $\Rightarrow$ $\qquad |f(x,y_2) - f(x,y_1)| \le K|y_2 - y_1|,$ $\qquad ...(iii)$

which is Lipschitz condition. It follows that the Lipschitz condition (iii) can be replaced by the stronger condition (i).

1.7. An Important Theorem. *If S in either a rectangle* $|x - x_0| \le h$, $|y - y_0| \le k$, $(h, k > 0)$ *or a strip* $|x - x_0| \le h$, $|y| < \infty$ $(h > 0)$, *and if* $f(x, y)$ *is a real valued function defined on S such that* $\partial f/\partial y$ *exists, is continuous on S, and* $\left|\frac{\partial}{\partial y} f(x,y)\right| \le K$, $(x, y) \in S$ *for a positive constant K, then* $f(x, y)$ *satisfies a Lipschitz condition on S with Lipschitz constant K.* **[Meerut 1994]**

Proof. Now, $\quad |f(x,y_1) - f(x,y_2)| = \left|\int_{y_1}^{y_2} \frac{\partial}{\partial y} f(x,y)\,dy\right| = \int_{y_1}^{y_2} \left|\frac{\partial}{\partial y} f(x,y)\right| |dx| \le K \int_{y_1}^{y_2} |dy|.$

Thus, $\quad |f(x,y_1) - f(x,y_2)| = K|y_1 - y_2| \quad$ for $\quad (x, y_1), (x, y_2) \in S,$

showing that $f(x, y)$ satisfies Lipschitz condition on S with Lipschitz constant K.

1.8. Solved examples based on Articles 1.4 to 1.7

Ex. 1. *Show that* $f(x, y) = xy^2$ *satisfies the Lipschitz condition on the rectangle R:* $|x| \le 1$, $|y| \le 1$ *but does not satisfy a Lipschitz condition on the strip* $S\, |x| \le 1$, $|y| < \infty$.

[Meerut 2005, 07, Kanpur 2002, Bilaspur 2004]

Sol. Method I. We have $\quad |f(x,y_2) - f(x,y_1)| = |xy_2^2 - xy_1^2|, \quad$ as $\quad f(x, y) = xy^2$

or $$|f(x,y_2) - f(x,y_1)| = |x|\,|y_2 + y_1|\,|y_2 - y_1|. \qquad ...(1)$$

Hence in the rectangle $|x| \le 1$, $|y| \le 1$, (1) $\Rightarrow |f(x,y_2) - f(x,y_1)| \le (1) \times (2) \times |y_2 - y_1|$, showing that Lipschitz condition is satisfied.

Next, $\quad \left|\frac{f(x,y_2) - f(x,0)}{y_2 - 0}\right| = |x||y_2| \to \infty \quad$ when $\quad |y_2| \to \infty \quad$ if $\quad |x| \ne 0,$

showing that the Lipschitz condition is not satisfied on the strip $|x| \le 1, |y| < \infty$.

Method II. We have $$|\partial f/\partial y| = 2|xy| = 2|x||y| \qquad ...(1)$$

$\therefore$ In the rectangle $|x| \le 1, |y| \le 1, \; |\partial f/\partial y| \le 2,$ for eah $(x, y) \in R,$

showing that $f(x, y)$ satisfies Lipschitz condition in R, with Lipschitz constant 2.

On the other hand, on the strip $S : |x| \le 1, \; |y| < \infty,$ (1) shows that $|\partial f/\partial y|$ is unbounded on the strip S as $|y| \to \infty$. Hence Lipschitz condition is not satisfied on the strip S.

Ex. 2. *If S is defined by the rectangle* $|x| \le a$, $|y| \le b$, *show that the* $f(x, y) = x^2 + y^2$, *satisfies the Lipschitz condition. Find the Lipschitz constant.*

[Meerut 2000, 05, 07; Himanchal 2003; Rohilkhand 2007]

Sol. Let (x, y_1) and (x, y_2) be two arbitrary points in the rectangle S. Then, we have

$$|f(x,y_2) - f(x,y_1)| = \left|(x^2 + y_2^2) - (x^2 + y_1^2)\right|, \quad \text{as } f(x, y) = x^2 + y^2$$

$$= \left|y_2^2 - y_1^2\right| = \left|y_2 + y_1\right|\left|y_2 - y_1\right|.$$

Thus, $\quad \left|f(x, y_2) - f(x, y_1)\right| \le 2b\left|y_2 - y_1\right|, \quad$ since $\quad |y| < b$ in S.

showing that the Lipschitz condition is satisfied. Here the Lipschitz constant $K = 2b$.

Ex. 3. *Prove that the continuity of $f(x, y)$ is not enough to guarantee the uniqueness of the solution of the initial value problem* : $dy/dx = f(x, y) = \sqrt{|y|}$, $y(0) = 0$.

[Himanchal 2003; Lucknow 2006; Meerut 1998]

OR Show that the solution of the initial value problem $dy/dx = f(x, y)$, $y(x_0) = y_0$ may not be unique although $f(x, y)$ is continuous.

Sol. Consider the initial value problem $\quad dy/dx = \sqrt{|y|}, \quad y(0) = 0. \quad$ ...(1)

Clearly $f(x, y) = \sqrt{|y|}$ is continuous for all y and (1) has the following two solutions :

$$y \equiv 0 \qquad \text{and} \qquad y = \begin{cases} x^2/4, & \text{when } x \ge 0 \\ -x^2/4, & \text{when } x < 0. \end{cases}$$

We now show that Lipschitz condition $\quad \dfrac{\left|f(x, y_2) - f(x, y_1)\right|}{\left|y_2 - y_1\right|} \le K \quad$ ...(2)

does not hold good in any region which includes the line $y = 0$. For example, when $y_1 = 0$ and y_2 is positive, then $\quad \dfrac{\left|f(x, y_2) - f(x, y_1)\right|}{\left|y_2 - y_1\right|} = \dfrac{\sqrt{y_2}}{y_2} = \dfrac{1}{\sqrt{y_2}}, \quad$ as $\quad \sqrt{y_2} > 0$

showing that quantity on L.H.S. of (2) can be made as large as we please by choosing y_2 sufficiently small. This violates (2) because the quantity on L.H.S. of (2) should not exceed a fixed constant K.

Ex. 4. *Illustrate by an example that a continuous function may not satisfy a Lipschitz condition on a rectangle.* **[Meerut 1995]**

Sol. As an example, consider the function $f(x, y) = y^{2/3}$ on the rectangle $S : |x| \le 1, |y| \le 1$. ..(1)

Clearly, $f(x, y)$ is continuous in the rectangle S.

Here $\quad \left|\dfrac{\partial}{\partial y} f(x, y)\right| = \left|\dfrac{2}{3y^{1/3}}\right| \to \infty, \quad$ when, $\quad y \to 0. \quad$ ...(2)

Since $y = 0$ is a point of the rectangle S, (2) shows that the Lipschitz condition is not satisfied by the function $f(x, y) = y^{2/3}$ on the rectangle S.

Ex. 5. *(a) For the initial value problem $dy/dx = e^y$, $y(0) = 0$. Find the largest interval $|x| \le a$ in which the Picard's theorem guarantees existence of a unique solution.*

[Meerut 2007, 11 Gwalior 2000, 02, 05]

(*b*) *For the initial value problem $dy/dx = e^y$, $y(0) = 0$; find the largest interval $|x| \le a$ in which the Picard's theorem holds.* **[Meerut 2007]**

Sol. Here the condition of boundedness of $f(x, y)$, namely, $|f(x, y)| \le M$ for $|y - y_0| \le Ma$,

reduces to $\quad e^y \le M \quad$ for $\quad |y - 0| \le Ma, \quad$ as $\quad f(x, y) = e^y$

Let y_1, y_2 lie in the range $|y| \le Ma$ and $y_1 < y_2$. Then, using the mean value theorem, we have

$$e^{y_2} - e^{y_1} = (y_2 - y_1)\left(\frac{\partial e^y}{\partial y}\right)_{y = \bar{y}}, \qquad \text{where} \qquad y_1 < \bar{y} < y_2$$

or $$e^{y_2} - e^{y_1} \le (y_2 - y_1)\, M, \quad \text{since} \quad e^{\bar{y}} \le M,$$
showing that the function e^y satisfies the Lipschitz condition.

Again, the inequality $e^y \le M$ will be satisfied for all values of y such that $|y| \le Ma$ provided it is satisfied for $y = Ma$. Hence $e^{Ma} \le M$ or $a \le (\log M)/M$.

Using the well known methods of finding maximum and minimum values of a function, we can easily show that the function $(\log M)/M$ is maximum when $M = e = 2.718$. Hence uniqueness theorem of the given initial value problem gives $|x| \le a$ where $a = 1/e = 0.308$. Thus, the required largest interval is $|x| \le 0.308$.

Ex. 6. *Show that for the problem $dy/dx = y$, $y(0) = 1$, the constant a in Picard's theorem must be smaller than unity.* **[Agra 1992; Meerut 1993]**

Sol. The condition of bounded of $f(x, y)$, namely $|f(x,y)| \le M$ for $|y - y_0| \le Ma$

reduces to $$|y| \le M \quad \text{for} \quad |y-1| \le Ma.$$

Choose $M \ge 1$. Then the Lipschitz condition is also satisfied because in this situation, the Lipschitz condition takes the form

$$|f(x,y_2) - f(x,y_1)| = |y_2 - y_1| \le M\,|y_2 - y_1|, \quad \text{as} \quad M \ge 1.$$

Again, $$|y - 1| \le Ma \quad \Rightarrow \quad |y| - 1 \le Ma.$$

It follows that the condition $|y| \le M$ will be satisfied for all values of $|y-1| \le Ma$ provided it is satisfied for $|y| = 1 + Ma$.

$\therefore \quad 1 + Ma \le M \quad$ or $\quad a < (\mathrm{M} - 1)/\mathrm{M} = 1 - (1/\mathrm{M}) < 1, \quad$ as $\quad 1 \le M < \infty$.

Ex. 7. *For the initial value problem statement (I.V.P.) $y' = f(x, y)$ with $y(0) = 0$ which of the following statement is true:*

(*a*) *$f(x, y) = (xy)^{1/2}$ satisfies Lipschitz's condition and so I.V.P. has unique solution.*

(*b*) *$f(x, y) = (xy)^{1/2}$ does not satisfy Lipschitz's condition and so, I.V.P. has no solution.*

(*c*) *$f(x, y) = |y|$ satisfies Lipschitz's condition and so I.V.P. has unique solution.*

(*d*) *$f(x, y) = |y|$ does not satisfy Lipschitz's condition still I.V.P. has unique solution.*

Sol. Ans. (a) Proceed as in Ex. 3 of Art. 1.8. **(GATE 2003)**

Ex. 8. (*i*) *Define Lipschitz conditions and Lipschitz constant.*

(*ii*) *Give an example to show that there exist functions which do not satisfy the Lipschitz condition.*

(*iii*) *Give an example to show that the existence of partial derivative of $f(x, y)$ is not necessary for $f(x, y)$ to be a Lipschitz function.*

Sol. (*i*) **Lipschitz condition and Lipschitz constant**

Definitions. Let $f(x, y)$ be a function defined for all (x, y) in a domain D. We say, (x, y) satisfies a *Lipschitz condition* on D, if there exists a constant $K > 0$, such that

$$|f(x, y_1) - f(x, y_2)| \le K |y_1 - y_2|$$

for all (x, y_1), (x, y_2) in D. The constant K is known as *Lipschitz constant.*

As a consequence of the definition, a function $f(x, y)$ satisfies Lispschitz condition if and only if there exists a constant $K > 0$ such that

$$\frac{|f(x, y_1) - f(x, y_2)|}{|y_1 - y_2|} \le K, \quad y_1 \ne y_2, \text{ whenever } (x, y_1), (x, y_2) \text{ belongs to } D.$$

We now state a general criterion which would ensure the Lipschitz condition :

Let $f(x, y)$ be a continuous function defined over a rectangle $D = \{(x, y); |x - x_0| \le a, |y - y_0| \le b\}$.

Here a, b are some positive real numbers. Let $\partial f / \partial y$ be defined and continuous on D and

$$|\partial f / \partial y| \le K, \text{ for each } (x, y) \in D$$

for some $K > 0$. Then f satisfies a Lipschitz conditon on D with Lipschitz constant K.

(*ii*) Let $f(x, y) = y^{1/2}$ be defined on the rectangle $D = \{(x, y) : |x| \le 2, |y| \le 2\}$, Let $y_1 > 0$.

Then, $$\frac{|f(x, y_1) - f(x, 0)|}{|y_1 - 0|} = \frac{y_1^{1/2}}{y_1} = \frac{1}{y_1^{1/2}},$$ which is unbounded as $y_1 \to 0$.

Hence $f(x, y)$ does not satisfy the Lipschitz condition in D.

(*iii*) Let $f(x, y) = |y|$ be defined on the square $D = \{(x, y) : \; |x| \le 1, \; |y| \le 1\}$.

Recall that the partial derivatives of $f(x, y)$ w.r.t. 'y' at the point (x', y') is defined as

$$\left(\frac{\partial f}{\partial y}\right)_{(x', y')} = \lim_{k \to 0} \frac{f(x', y' + k) - f(x', y')}{k}$$

$\therefore$ Using the above definition, we get $$\left(\frac{\partial f}{\partial y}\right)_{(x, 0)} = \lim_{k \to 0} \frac{f(x, 0 + k) - f(x, 0)}{k} = \lim_{k \to 0} \frac{|k|}{k},$$

which does not exist. Thus $\partial f / \partial y$ fails to exist at $(x, 0)$.

Again, $$\frac{|f(x, y_1) - f(x, y_2)|}{|y_1 - y_2|} = \frac{||y_1| - |y_2||}{|y_1 - y_2|} \le 1, \qquad \text{as} \qquad |y_1 - y_2| \ge ||y_1| - |y_2||$$

showing that $f(x, y)$ satisfies Lipschitz condition in x on D with Lipschitz constant $K = 1$.

Ex.9. *If S is defined by the rectangle* $|x| \le a, \; |y| \le b$, *show that the function* $f(x, y) = x \sin y + y \cos x$, *satisfy the Lipschitz condition. Find the Lipschitz constant.*

[Agra 2006; Jabalpur 2005; Kanpur 2002; Meerut 2003]

Sol. From the given function, $$\partial f / \partial y = x \cos y + \cos x \qquad \ldots (1)$$

Since x, $\cos y$ and $\cos x$ are continuous functions, $\partial f / \partial y$ is also continuous on S. Also,

we have $$|\partial f / \partial y| = |x \cos y + \cos x| \le |x \cos y| \le |\cos x|$$

or $$|\partial f / \partial y| = |x||\cos y| + |\cos x| \le |x| + 1 \le a + 1 \text{ for each } (x, y) \in S, \qquad [\because |x| < a]$$

showing that $f(x, y)$ satisfies Lipschitz condition and Lipschitz constant is $a + 1$.

Ex. 10. *Show that the function* $f(x, y) = y^{2/3}$ *does not satisfy the Lipschitz condition on the rectangle* $R : |x| \le 1, |y| \le 1$.

Sol. We have $|\partial f / \partial y| = |2/3y^{1/3}|$, which is unbounded in every neighbourhood of the origin.

Hence $f(x, y)$ does not satisfy the Lipschitz condition

Second method. We have $$\frac{|f(0, y) - f(0, 0)|}{|y - 0|} = \frac{|y^{2/3} - 0|}{y} = \frac{1}{|y|^{1/3}},$$

which is unbounded in every neighbourhood of the origin and so $f(x, y)$ does not satisfy the Lipschitz condition.

Ex. 11. *Give an example to prove that we cannot drop the Lipschitz condition in the statement of Picard's theorem.*

Sol. Consider the problem $dy/dx = 3\, y^{2/3}$, $y(0) = 0$ ... (1)

and let R be the rectangle $|x| \le 1,\ |y \le 1|$. Clearly $f(x, y) = 3\, y^{2/3}$, which is continuous on R.

We easily verify that $y_1(x) = x^3$ and $y_2(x) = 0$ are two distinct solutions of (1) valid for all x. Thus (1) has solution that is not unique. The reason for this non uniqueness lies in the fact that $f(x, y)$ does not satisfy a Lipschitz condition on the rectangle R, since

$$\frac{|f(0,y)-f(0,0)|}{|y-0|} = \left|\frac{3y^{2/3}}{y}\right| = \frac{3}{|y|^{1/3}}$$

is unbounded in every neighbourhood of the origin.

Ex. 12. *Show that the* $f(x, y) = 4x^2 + y^2$ *on* $R: |x| \le 1, |y| \le 1$ *satisfies Lipschitz condition.*

Sol. Since $|\partial f/\partial y| = |2y| \le 2$, for each $(x, y) \in R$.

Hence $f(x, y)$ satisfies the Lipschitz condition with Lipschitz constant 2.

Ex. 13. *Consider* $f(x, y) = x^3 |y|$. *Prove that* f *satisfies a Lipschitz condition on* $R: |x| \le 2, |y| \le 2$ *even though* $\partial f/\partial y$ *does not exist at* $(x, 0)$ *if* $x \ne 0$.

Sol. By definition of partial derivative, we have

$$\left(\frac{\partial f}{\partial y}\right)_{(x,0)} = \lim_{k\to 0} \frac{f(x,k)-f(x,0)}{k} = \lim_{k \to 0} \frac{x^3|k|}{k},$$

which does not exist. However, we have

$$|f(x,y_1)-f(x,y_2)| = \left|x^3|y_1| - x^3|y_2|\right| = |x^3| \times \left||y_1|-|y_2|\right| \quad \text{... (1)}$$

Also, we have $\left||y_1|-|y_2|\right| \le |y_1-y_2|$ ... (2)

(1) and (2) $\Rightarrow$ $|f(x,y_1)-f(x,y_2)| \le |x|^3 |y_1-y_2|$

$\therefore$ $|f(x,y_1)-f(x,y_2)| \le 8|y_1-y_2|$, which is true for all (x, y_1), (x, y_2) in R. ... (3)

(3) shows that $f(x, y)$ satisfies Lipschitz condition with Lipschitz constant 8.

Ex. 14. *For the initial value problem* $dy/dx = y^2 + \cos^2 x$, $y(0) = 0$, *determine the interval of existence of its solution given that R is the rectangle containing origin,*

$$R: \{(x,y): \quad 0 \le x \le a, \quad |y| \le b, \quad a > 1/2, \quad b > 0\}$$

Sol. Let $f(x, y) = y^2 + \cos^2 x$ Also, $|f(x,y)| = |y^2+\cos^2 x| \le |y|^2 + |\cos x|^2 \le b^2+1$

Let $b^2 + 1 = M$ so that $|f(x, y)| \le M$

Again, since $|\partial f/\partial y| = |2y| = 2b = K$ (say), we see that $f(x, y)$ satisfies Lipschitz condition

We find that $y(x)$ exists for $0 \le x \le h = \min(a, b/M) = \min(a, b/(1+b^2))$ ... (1)

Now, $\dfrac{b}{1+b^2} = \dfrac{1}{1/b+b} = \dfrac{1}{\left(1/\sqrt{b}-\sqrt{b}\right)^2+2}$, $\Rightarrow$ The maximum value of $\dfrac{b}{1+b^2}$ is $\dfrac{1}{2}$

Hence (1) $\Rightarrow$ $h = 1/2$ and so $y(x)$ exists on the interval $0 \le x \le 1/2$.

Ex. 15. *Consider the initial value problem $dy/dx = y^2$, $y(0) = 2$. Let R be the rectangle $R:\{(x, y):|x|\le a, |y-2|\le b, a>0, b>0\}$. Find the largest interval of existence of its solution. By explicitly solving the given initial value problem, show that there exists an interval of existence which is larger than that obtained by the application of Picards' method.*

Sol. Given $dy/dx = y^2$, $y(0) = 2$ ... (1)

To find the interval of existence by application of Picard's theorem: Let $f(x, y) = y^2$. Then, in R, we have $|f(x,y)|=|y^2|\le(b+2)^2 = M$, say and hence the interval of existence of solution is given by $|x|\le h$, where $h=\min\left(a, b/(b+2)^2\right)=1/8.$

$\Rightarrow$ The interval of existence is $-(1/8)\le x\le(1/8)$.

***To find the interval of existence by solving* (1).** We have $dy/dx = y^2$ so that $y^{-2}dy = dx$.

Integrating, $-y^{-1} = x + c$, c being an arbitrary constant. ... (2)

Putting $x = 0$ and $y = 2$, (2) gives $-2^{-1} = c$. Then (2) gives $-\dfrac{1}{y}=x-\dfrac{1}{2}$ or $y(x)=\dfrac{2}{1-2x}$,

showing that $y(x)$ exists on $-\infty < x < 1/2$. This interval of existence is much larger than that obtained by the application of Picard's method.

Ex. 16. *The Picard's theorem assumes Lipschitz condition. Can we drop this condition. If answer is no, give an example to illustrate this point.*

Sol. In the following two examples, we shall show that if Lipschitz condition is not satisfied then we do not arrive at unique solution as stated in Picard's theorem.

Consider $dy/dx = 4y^{3/4}$, $y(0) = 0$, $x\ge 0$ ... (1)

Let $f(x, y) = 4y^{3/4}$. Then, we get $\dfrac{f(x,y)-f(x,0)}{y-0}=\dfrac{4y^{3/4}}{y}=\dfrac{4}{y^{1/4}},\ y\ne 0$

which is unfounded for $x\ge 0$, since it can be made as large as possible by choosing y close to zero. Thus $f(x, y)$ fails to satisfy Lipschitz condition.

Using Picard's method of successive approximation (refer Art. 1.2), we easily see that $y_n(x) = 0$ for $n = 0, 1, 2, 3$..... Hence $y(x)=\lim_{n\to\infty} y_n(x)=0$ on $[0,\infty[$. We also not that $y(x) = x^4$ is also solution of (1). Hence (1) does not possess unique solution.

EXERCISE 1(C)

Ex. 1 (*a*) Show that the function f given by $f(x, y) = y^{1/2}$ does not satisfy a Lipschitz condition on $R:|x|\le 1, 0\le y\le 1$. Show that f satisfies a Lipschitz condition on any rectangle R of the form $R:|x|\le a, b\le y\le c, (a, b, c>0)$

(*b*) Show that a function f given by $f(x, y) = x^2 \mid y \mid$ satisfies a Lipschitz condition on $R:|x|\le 1, |y|\le 1$

(*c*) Show that $f(x, y) = xy^2$ (*i*) satisfies a Lipschitz condition on any rectangle $a\le x\le b$, $c\le y\le d$. (*ii*) does not satisfy a Lipschitz condition on any strip $a\le x\le b$, $-\infty< y\le\infty$

(*d*) Show that $f(x, y) = xy$ (*i*) satisfies Lipschitz condition on any rectangle $a\le x\le b$, $c\le y\le d$ (*ii*) satisfies a Lilpschitz condition on any strip $a\le x\le b$, $-\infty<y<\infty$ (*iii*) does not satisfy a Lipschitz condition on the entire plane.

(e) For what points (x_0, y_0) does Picard's theorem imply that the initial value problem $dy/dx = y\,|\,y\,|$, $y(x_0) = y_0$ has a unique solution on some interval $|\,x - x_0\,| \le h$? **[Ans.** All point (x_0, y_0)]

(f) Consider the initial value problem : $dy/dx = 2y/x$, if $x > 0$ and $dy/dx = 0$, if $x = 0$; $y(0) = 0$. Show that $2y/x$ does not satisfy Lipschitz condition in any closed rectangle containing (0, 0) and the given initial value problem has solution which is not unique.

Sol. Left as an exercise for the reader.

Ex. 2. By computing appropriate Lipschitz constants, show that the following functions satisfy Lipschitz conditions on the domain D of xy–plane indicated.

(i) $f(x, y) = 4x^2 + y^2$, on $D : |\,x\,| \le 1, \quad |\,y\,| \le 1$

(ii) $f(x, y) = x^2 \cos^2 y + y \sin^2 x$, on $D : |\,x\,| \le 1, \quad |\,y\,| < \infty$

(iii) $f(x, y) = x^3 e^{-xy^2}$, on $D : 0 \le x \le a, \quad |\,y\,| < \infty, \quad (a > 0)$

(iv) $f(x, y) = a(x)\, y^2 + b(x)\, y + c(x)$, on $D : |\,x\,| \le 1, \quad |\,y\,| \le 2$,

where $a(x)$, $b(x)$ and $c(x)$ are continuous functions on $|\,x\,| \le 1$.

(v) $f(x, y) = a(x)\, y + b(x)$, on $D : |\,x\,| \le 1, |\,y\,| < \infty$, where $a(x)$ and $b(x)$ are continuous on $|\,x\,| < 1$

Ans. (i) $K = 2$ (ii) $K = 3$ (iii) $K = \max\,\{2a^3, 2a^4\}$

(iv) $K = 4\,M_a + M_b$; where $M_a = \max\limits_{|x| \le 1} |\,a(x)\,|, M_b = \max\limits_{|x| \le 1} |\,b(x)\,|$ (v) $K = \max\limits_{|x| \le 1} |\,a(x)\,|$

3. Let (x_0, y_0) be an interior point of a closed rectangle R, $a \le x \le b$, $c \le y \le d$ in which $f(x, y)$ is continuous. Let $f(x, y)$ satisfy the Lipschitz condition. $|\,f(x, y_1) - f(x, y_2)| \le K\,(y_1 - y_2)$, for all possible (x, y_1) and (x, y_2) in R and some fixed constant K. Prove that $dy/dx = f(x, y)$, $y(x_0) = y_0$ has a unique solution.

4. Write a note on Picard's existence theorem regarding the existence and uniquencess of the solution of the equation $dy/dx = f(x, y)$, where f satisfies Lipschitz condition.

5. (a) Show that (i) $|\,y'\,| + |\,y\,| = 0$, $y(0) = 1$ has no solution.

(ii) $y' = x$, $y(0) = 1$ has one solution (iii) $y' = (y - 1)/x$, $y(0) = 1$ has an infinity of solutions.

Explain what you mean by existence and uniqueness of a differential equation.

(b) State carefully any existence and uniqueness theorem for differential equation you know and apply it to the above three examples.

6. Show that the conditions for the exitence and uniqueness of a solution of the following initial value problem are not satisfied by the function $f(x, y) = (y - 1)/x$ in any rectangle R of xy–plane with (0, 1) as its centre : $y' = (y - 1)/x$, $y(0) = 1$; but a solution does exist of above problem. Give reason for your answer. Draw some possible solution curves.

7. Apply Picard's iteration process to get the solution of $y' = (y - 1)/x$, $y(1) = 1$. Give your arguments, why Picard's iteration process is applicable to the above initial value problem.

8. If $f(x, y) = y^{2/3}$, show that Lipschitz condition in not satisfied in any region containing the origin and that the solution of the differential equation $dy/dx = f(x, y)$ satisfying the initial condition $y = 0$ when $x = 0$ is not unique. **[Meerut 1999]**

9. By giving suitable example prove that a function does not satisfy the Lipschitz condition on the prescribed domain. **[Madurai 2001, 05]**

10. Examine existence and uniqueness of the solution of the initial value problem $dy/dx = y^2$, $y(1) = -1$. **[Meerut 2011]**

11. Show that the Picard's theorem ensures a unique solution in the interval $|\,x\,| \le 1/2$ for the initial value problem $dy/dx = x + y^2$, $y(0) = 0$.

12. State Picard's theorem on existence of solutions of differential equations. Show that $f(x, y) = y^{1/2}$ does not satisfy Lipschtiz's condition on $|x| \le 1$, $0 \le y \le 1$ while $f(x, y) = x^2 \cos^2 y + y \sin^2 x$ satisfies Lipschitz's condition on $|x| \le 1$, $|y| < \infty$. Find the Lipschitz's constant. **[I.A.S. 2000]**

13. Examine the exitence and uniqueness of solution of the initial value problem $dy/dx = y^{1/3}$, $y(0) = 0$ **[Himanchal 2003; Kanpur 2000; Kolkata 2000, Osmania 2005, 07; G.N.D.U. Amritsar 2003, 05; Meerut 2000]**

14. Show that for the problem $dy/dx = y$, $y(0) = 1$, the constant h in Picard's theorem must be smaller than unity. **[Rohilkhand 2007; Meerut 1995]**

15. Discuss the existence and uniqueness of a solution of the initial value problem $dy/dx = y^{4/3}$, $y(x_0) = y_0$. **[Bhopal 2010; Rajasthan 2010; Himachal 2010]**

16. Examine existence and uniqueness of the solution of the initial value problem $dy/dx = y^2$, $y(1) = -1$. **[Gwalior 2000, 01, 03; Ujjain 2000, 02, 04, Rajasthan 2000; Agra 1997; Kanpur 2000; Meerut 1994, 96]**

17. State and prove the uniqueness theorem for the initial value problem $dy/dx = f(x, y)$, $y(x_0) = y_0$. **[Meerut 1998]**

18. Examine whether the following differential equation possesses unique solution. Justify your answer. $\dfrac{dy}{dx} = \begin{cases} y(1-2x), x > 0 \\ y(2x-1), x < 0 \end{cases}$ subject to the condition: $y = 1$ at $x = 1$ **[Jabalpur 2003]**

19. Verify that the initial value problem $xy' - y = 0, y(0) = 0$ has two solutions $y_1(x) = 0$ and $y_2(x) = x$. Does it contradict the Picard's theorem. **[Himanchal 2002, 03, 05]**

20. Let $f(x, y) = (\cos y)/(1 - x^2)$, $(|x| < 1)$. Show that f satisfied a Lipschitz condition on every strip $S_a : |x| \le a,$, where $0 < a < 1$. **[Himanchal 2009]**

21. Define Lipschitz condition (with respect to y) of the function $f(x, y)$ and investigate its geometrical significance. Show that the function $f(x, y) = x^2 e^{x+y}$ satisfies Lipschitz condition in the rectangle defined by $|x| \le a, |y| \le b$. **[Himanchal 2009]**

22. State and prove Picard's existence theorem for the solution of differential equation $dy/dx = f(x, y)$, $y(x_0) = y_0$. Show that $dy/dx = x^2 + y^2$, $y(0) = 0$ has a unique solution. **[Himanchal 2009]**

2

Simultaneous Equations of the Form $(dx)/P = (dy)/Q = (dz)/R$

2.1. Introduction.

In this chapter we shall study simultaneous equations of the first order and of the first degree in the derivatives. Equations containing only three variables will be studied. It will be noted that the method of solution presented here can be applied to equations involving any number of variables.

The general type of a set of simultaneous equations of the first order having three variables is

$$P_1dx + Q_1dy + R_1dz = 0 \qquad \text{and} \qquad P_2dx + Q_2dy + R_2dz = 0, \qquad \ldots(1)$$

where the coefficients are functions of x, y, z. Solving these equations simultaneously, we have

$$\frac{dx}{Q_1R_2 - Q_2R_1} = \frac{dy}{R_1P_2 - R_2P_1} = \frac{dz}{P_1Q_2 - P_2Q_1},$$

which is of the form $\qquad (dx)/P = (dy)/Q = (dz)/R, \qquad \ldots(2)$

where P, Q, and R are functions of x, y, z. Thus we note that the simultaneous equations (1) can always be put in the form (2).

2.2. The nature of solution of (dx)/P = (dy)/Q = (dz)/R.

The given equations are said to be completely solved when we get a solution of the form $u_1(x, y, z) = c_1$ and $u_2(x, y, z) = c_2$, where u_1 and u_2 are two independent integrals (solutions) of the given equations. u_1 and u_2 are said to be independent integrals if u_1/u_2 is not merely a constant. For example, $u_1 = x^2 + y^2 + z^2$ and $u_2 = x + y + z$ are independent integrals whereas $u_1 = 2x + 2y + 2z$ and $u_2 = 2(x + y + z)$ are not independent.

2.3. Geometrical Interpretation of (dx)/P = (dy)/Q = (dz)/R.

From three dimensional coordinate geometry, it is known that the direction cosines of the tangent to a curve are proportional to dx: dy: dz. The given differential equations, therefore, express the fact that the direction cosines of the tangent to the curve at that point are proportional to $P : Q : R$. Suppose that the solution of the given equations is given by $u_1(x, y, z) = c_1$ and $u_2(x, y, z) = c_2$. Then we observe that the solution represents the curves of intersection of the surfaces $u_1(x, y, z) = c_1$ and $u_2(x, y, z) = c_2$. Since c_1 and c_2 can take any values in infinite number of ways, we get a doubly infinite number of such curves.

2.4. Rule I for solving (dx)/P = (dy)/Q = (dz)/R. ...(1)

By equating two of the three fractions of (1), we may be able to get an equation in only two variables. Sometimes such an equation is obtained after cancellation of some factor from the chosen two fractions of (1). On integrating the differential equation in only two variables by well known methods, we shall obtain one of the relations in the general solution of (1). This method may be repeated to give another relation with help of two other fractions of (1).

2.5. Solved examples based on Art. 2.4.

Ex. 1. *Solve* (*a*) $\dfrac{xdx}{y^2z} = \dfrac{dy}{xz} = \dfrac{dz}{y^2}$. **[Nagpur 1996; Bangalore 2005]**

(b) $\dfrac{xdx}{y^2z} = \dfrac{dy}{xz} = \dfrac{dz}{y^2z}$ **[Poona 2006; Vikram 1996]**

Sol. (a) Taking the first two fractions, we get $x^2dx = y^2dy$

or $3x^2dx - 3y^2dy = 0$ and so $x^3 - y^3 = c_1$. ...(1)

Next, taking the first and the third fractions, we get $xdx = zdz$

or $2xdx - 2zdz = 0$ and so $x^2 - z^2 = c_2$. ...(2)

Since $x^3 - y^3$ and $x^2 - z^2$ are independent, the required general solution is given by the relations (1) and (2), c_1 and c_2 being arbitrary constants.

(b) Proceed as in part (a). **Ans.** $x^3 - y^3 = c_1$, $x^2 - 2z = c_2$

Ex. 2. *Solve* $\dfrac{dx}{yz} = \dfrac{dy}{zx} = \dfrac{dz}{xy}$. **[Agra 1996; Delhi Maths (G) 1998]**

Sol. Taking the first two fractions, we have

$xdx = ydy$ or $2xdx - 2ydy = 0$ so that $x^2 - y^2 = c_1$. ...(1)

Again, taking the first and the third fractions, we have

$xdx = zdz$ or $2xdx - 2zdz = 0$ so that $x^2 - z^2 = c_2$. ...(2)

The required general solution is given by the relations (1) and (2).

Ex. 3. $(dx)/x = (dy)/0 = (dz)/(-x)$

Sol. From the second fraction, we have $dy = 0$ so that $y = c_1$. ...(1)

Taking the first and the third fractions, $xdx + zdz = 0$ or $2xdx + 2zdz = 0$.

Integrating, $x^2 + z^2 = c_2$, c_2 being an arbitrary to constant. ...(2)

The required solution is given by the relations (1) and (2).

Ex. 4. *Solve* $\dfrac{dx}{y^2} = \dfrac{dy}{x^2} = \dfrac{dz}{x^2y^2z^2}$. **[Delhi Maths (G) 2000]**

Sol. Taking the first two fractions, $3x^2dx - 3y^2dy = 0$.

Integrating, $x^3 - y^3 = c_1$, being an arbitrary constant ...(1)

Taking the first and third fractions, $3x^2dx - 3z^{-2}dz = 0$.

Integrating, $x^3 + 3z^{-1} = c_2$, c_2 being an arbitrary constant ...(2)

The required general solution is given by the relations (1) and (2).

Ex. 5. *Solve* $\dfrac{dx}{x^2 + 2y^2} = \dfrac{dy}{-xy} = \dfrac{dz}{xz}$. **[Delhi Maths (Hons.) 1994]**

Sol. Taking the last two fractions, $(1/y)dy + (1/z)dz = 0$.

Integrating, $\log y + \log z = \log c_1$ or $yz = c_1$. ...(1)

Again, taking the first two fractions, we have

$$\frac{dx}{dy} = \frac{x^2 + 2y^2}{-xy} \quad \text{or} \quad x\frac{dx}{dy} = -\frac{x^2}{y} - 2y \quad \text{or} \quad 2x\frac{dx}{dy} + \frac{2}{y}x^2 = -4y \qquad \text{... (2)}$$

Putting $x^2 = v$ so that $2x(dx/dy) = dv/dx$, (2) reduces to

$(dv/dx) + (2/y)v = -4y$, which is linear differential equation ...(3)

Integrating factor of (3) $= e^{\int (2/y)dy} = e^{2\log y} = y^2$ and so its solution is

$vy^2 = \int \{(-4y) \times y^2\} \times dy + c_2$ or $x^2y^2 + y^4 = c_2$. ...(4)

The required general solution is given by the relations (1) and (4).

EXERCISE 2 (A)

Solve the following simultaneous differential equations :

1. $dx = dy = dz$ **Ans.** $x - y = c_1, x - z = c_2$
2. $(dx)/a = (dy)/a = dz$ **Ans.** $x - y = c_1, y - az = c_2$
3. $(dx)/x = (dy)/y = (dz)/z$ **Ans.** $x/y = c_1, x/z = c_2$
4. $(dx)/\tan x = (dy)/\tan y = (dz)/\tan z$ **Ans.** $(\sin x)/(\sin y) = c_1, (\sin x)/(\sin z) = c_2$
5. $dx = dy = (dz)/\sin x$ **Ans.** $x - y = c_1, z + \cos x = c_2$

2.6. Rule II for solving (dx)/P = (dy)/Q = (dz)/R. ...(1)

Suppose only one relation $u_1(x, y, z) = c_1$ can be found by using rule I of Art 2.4. Then, sometimes we try to use this relation in expressing one variable in terms of the others. This may help us to obtain an equation in two variables. The solution of this equation will give us second relation for the general solution of (1). Note that the second relation will involve the arbitrary constant c_1. To find the final form the second relation, the arbitrary constant c_1 must be removed with help of the first relation $u_1(x, y, z) = c_1$.

2.7. Solved examples based on Art. 2.6

Ex. 1. *Solve* $\dfrac{dx}{xz(z^2 + xy)} = \dfrac{dy}{-yz(z^2 + xy)} = \dfrac{dz}{x^4}$. **[Delhi Maths (H) 2001]**

Sol. Cancelling $z(z^2 + xy)$, the first two fractions give

$(1/x)dx = -(1/y)dy$ or $(1/x)dx + (1/y)dy = 0,$

Integrating, $\log x + \log y = \log c_1$ or $xy = c_1$. ...(1)

Using (1), the first and third fractions give

$x^4 dx = xz(z^2 + c_1)dz$ or $x^3 dx - (z^3 + c_1 z)dz = 0.$

Integrating, $\frac{1}{4}x^4 - \left(\frac{1}{4}z^4 + \frac{1}{2}c_1 z^2\right) = \frac{1}{4}c_2$ or $x^4 - z^4 - 2c_1 z^2 = c_2$.

Using (1) to remove c_1, we get $x^4 + z^4 - 2xyz^2 = c_2$. ...(2)

The complete solution is given by the relations (1) and (2).

Ex. 2. *Solve* $\dfrac{dx}{xy} = \dfrac{dy}{y^2} = \dfrac{dz}{zxy - 2x^2}$. **[Rohilkhand 1993]**

Sol. Taking the first two fractions, $(1/x)dx - (1/y)dy = 0$ so that $x/y = c_1$. ...(1)

From (1), $x = c_1 y$. So the second and third fractions give

$\dfrac{dy}{y^2} = \dfrac{dz}{c_1 zy^2 - 2c_1^2 y^2}$ or $c_1 dy = \dfrac{dz}{z - 2c_1^2}$.

Integrating, $c_1 y - \log (z - 2c^2_1) = c_2$, c_1 and c_2 being arbitrary constants. ...(2)

Using (1) to remove c_1, (2) gives $x - \log (z - 2x^2/y^2) = c_2$. ...(3)

The complete solution is given by the relations (1) and (3).

Ex. 3. *Solve* $\dfrac{dx}{1} = \dfrac{dy}{2} = \dfrac{dz}{5z + \tan(y - 2x)}$. **(Delhi Maths (H) 2005; Mumbai 2007)**

Sol. Taking the first two fractions, $dy - 2dx = 0$.

Integrating, $y - 2x = c_1$, c_1 being an arbitrary constant. ...(1)

Using (1), the first and the third fractions give

$dx = dz/(5z + \tan c_1)$ so that $x - (1/5) \times \log (5z + \tan c_1) = c_2/5$.

Using (1) to remove c_1, this gives on simplification

$$5x - \log [5z + \tan (y - 2x)] = c_2, \ c_2 \text{ being an arbitrary constant.} \qquad ...(2)$$

The complete solution is given by the relations (1) and (2).

Ex. 4. *Solve* $\dfrac{dx}{y} = \dfrac{dy}{x} = \dfrac{dz}{xyz^2(x^2 - y^2)}$.

Sol. Taking the first two fractions, $\quad 2xdx - 2ydy = 0.$

Integrating, $\quad x^2 - y^2 = c_1, \ c_1$ being an arbitrary constant $\qquad ...(1)$

Using relation (1), the first and third fractions give

$$\frac{dx}{y} = \frac{dz}{xyz^2 c_1} \qquad \text{or} \qquad 2c_1 x dx - 2z^{-2} dz = 0.$$

Integrating, $\quad c_1 x^2 + 2z^{-1} = c_2 \quad$ or $\quad (x^2 - y^2)x^2 + 2z^{-1} = c_2$, by (1) $\quad ...(2)$

The required general solution is given by the relations (1) and (2).

Ex. 5. *Solve* $(dx)/(xz) = (dy)/(yz) = (dz)/(xy)$.

Sol. Taking the first two fractions, $\quad (1/x)dx - (1/y)dy = 0.$

Integrating, $\quad \log x - \log y = \log c_1 \quad$ or $\quad x/y = c_1. \qquad ...(1)$

From the second and third fractions, $\quad xdy = zdz.$

or $\quad 2c_1 ydy = 2zdz \qquad [\because \text{ from (1), } x = c_1 y]$

Integrating, $\quad c_1 y^2 - z^2 = c_2 \quad$ or $\quad (x/y) \times y^2 - z^2 = c_2$, using (1).

Thus, $\quad xy - z^2 = c_2, \ c_2$ being an arbitrary constant $\qquad ...(2)$

The required general solution is given by the relations (1) and (2).

Ex. 6. *Solve* $\dfrac{dx}{-xy^2} = \dfrac{dy}{y^3} = \dfrac{dz}{axz}$.

Sol. Taking the first two fractions, $\quad (1/x)dx + (1/y)dy = 0.$

Integrating, $\quad \log x + \log y = \log c_1 \quad$ or $\quad xy = c_1. \qquad ...(1)$

From (1), $x = c_1/y$. Hence the last two fractions give

$$\frac{dy}{y^3} = \frac{dz}{az \times (c_1 / y)} \qquad \text{or} \qquad \frac{dz}{z} - ac_1 \ y^{-4} dy = 0.$$

Integrating, $\quad \log z - (ac_1) \times [(y^{-3}/(-3)] = c_2, \ c_2$ being an arbitrary constant.

Using (1), we get $\quad \log z + (axy)/3y^3 = c_2 \quad$ or $\quad \log z + (ax/3y^2) = c_2. \quad ...(2)$

The required general solution is given by the relations (1) and (2).

EXERCISE 2 (B)

Solve the following simultaneous differential equations :

1. *Solve* $\dfrac{dx}{1} = \dfrac{dt}{-2} = \dfrac{dz}{3x^2 \sin(y + 2x)}$ **(Gulbarga 2005) Ans.** $y + 2x = c_1, x^3 \sin (y + 2x) - z = c_2$

2. *Solve* $\dfrac{dx}{z} = \dfrac{dy}{-z} = \dfrac{dz}{z^2 + (x + y)^2}$. **Ans.** $x + y = c_1$ and $z^2 + (x + y)^2 = c_1 e^{2x}$

3. *Solve* $\dfrac{dx}{1} = \dfrac{dy}{-1} = \dfrac{dz}{z/(x + y)}$ **Ans.** $x + y = c_1$ and $x - (x + y) \log z = c_2$

4. $(dx)/(zx) = (dy)/(-zy) = (dz)/(z + xy)$ **[Delhi Maths (G) 2006]**

Ans. $xy = c_1, \log x + xy \log (z + xy) - z = c_2$

2.8. Rule III for solving (dx)/P = (dy)/Q = (dz)/R ...(1)

Let P_1, Q_1, R_1 be functions of x, y, z. Then, by a well–unknown principle of algebra, each fraction in (1) will be equal to

$$(P_1dx + Q_1dy + R_1dz) \,/\, (P_1P + Q_1Q + R_1R). \qquad ...(2)$$

If $P_1P + Q_1Q + R_1R = 0$ in (2), then we know that numerator of (2) is also zero. This gives $P_1dx + Q_1dy + R_1dz = 0$ which can be integrated to give $u_1(x, y, z) = c_1$. This method may be repeated to get another integral $u_2(x, y, z) = c_2$. P_1, Q_1, R_1 are called multipliers. As a special case, these can be constants also. Sometimes only one integral is possible with help of multipliers. In such cases second integral should be obtained by using Rule I of Art. 2.4 or Rule II of Art. 2.6 as the case may be.

2.9. Solved examples based on Art. 2.8

Ex. 1. *Solve the simultaneous equations* $\dfrac{adx}{(b-c)yz} = \dfrac{bdy}{(c-a)zx} = \dfrac{cdz}{(a-b)xy}$.

[Kolkata 2001; Kumaoun 2002; Guwahati 2001, 02; Nagpur 2003, 04; Delhi Maths 1995, 1996; Bangalore 2005; Lucknow 2006]

Sol. Choosing x, y, z as multipliers, each fraction of given equations

$$= \frac{axdx + bydy + czdz}{xyz[(b-c)+(c-a)+(a-b)]} = \frac{axdx + bydy + czdz}{0}.$$

$\therefore \quad ax\,dx + by\,dy + cz\,dz = 0 \quad$ or $\quad 2ax\,dx + 2by\,dy + 2cz\,dz = 0.$

Integrating, $\quad ax^2 + by^2 + cz^2 = c_1$, c_1 being an arbitrary contant.

Again choosing ax, by, cz as multipliers, each fraction of the given equations

$$= \frac{a^2xdx + b^2ydy + c^2zdz}{xyz[a(b-c)+b(c-a)+c(a-b)]} = \frac{a^2xdx + b^2ydy + c^2zdz}{0}.$$

$\therefore \quad a^2x\,dx + b^2y\,dy + c^2z\,dz = 0 \quad$ or $\quad 2a^2x\,dx + 2b^2y\,dy + 2c^2z\,dz = 0.$

Integrating, $\quad a^2x^2 + b^2y^2 + c^2z^2 = c_2$, c_2 being an arbitrary constant ...(2)

The complete solution is given by relations (1) and (2).

Ex. 2. *Solve :* $\dfrac{dx}{z(x+y)} = \dfrac{dy}{z(x-y)} = \dfrac{dz}{x^2+y^2}$. **[Delhi Maths (H) 2009]**

[Agra 2005; Gauhati 1996, Meerut 2006, 11; Kanpur 2002; Rajasthan 2003]

Sol. Choosing, x, $-y$, $-z$ as multipliers, each fraction of the given equations

$$= \frac{xdx - ydy - zdz}{xz(x+y) - yz(x-y) - z(x^2+y^2)} = \frac{xdx - ydy - zdz}{0}.$$

$\therefore \quad x\,dx - y\,dy - z\,dz = 0 \quad$ or $\quad 2x\,dx - 2y\,dy - 2z\,dz = 0.$

Integrating, $\quad x^2 - y^2 - z^2 = c_1$, c_1 being an arbitrary constants

Now choosing, y, x, $-z$ as multipliers, each fraction of the given equations

$$= \frac{ydx + xdy - zdz}{yz(x+y) + xz(x-y) - z(x^2+y^2)} = \frac{ydx + xdy - zdz}{0}.$$

$\therefore \quad 2y\,dx + 2x\,dy - 2z\,dz = 0 \quad$ or $\quad 2d(xy) - d(z^2) = 0.$

Integrating, $\quad 2xy - z^2 = c_2$, c_2 being an arbitrary constant ...(2)

The complete solution is given by the relations (1) and (2).

Ex. 3. *Solve* $\dfrac{dx}{mz - ny} = \dfrac{dy}{nx - lz} = \dfrac{dz}{ly - mx}$. **[Bangalore 1997, Delhi Maths (H) 2001; Lucknow 2002; Karnataka 2004; Mysore 2005; Rajasthan 2006]**

Sol. Choosing l, m, n as multipliers, each fraction of given equations

$$= \frac{ldx + mdy + ndz}{l(mz - ny) + m(nx - lz) + n(ly - mx)} = \frac{ldx + my + ndz}{0}.$$

Thus, $\quad l\,dx + m\,dy + n\,dz = 0 \quad$ so that $\quad lx + my + nz = c_1$. ...(1)

Similarly, choosing x, y, z as multipliers, each fraction of the given equations

$$= \frac{xdx + ydy + zdz}{x(mz - ny) + y(nx - ly) + z(ly - mx)} = \frac{xdx + ydy - zdz}{0}.$$

$\therefore \quad x\,dx + y\,dy + z\,dz = 0 \quad$ or $\quad 2x\,dx + 2y\,dy + 2z\,dz = 0.$

Integrating, $\quad x^2 + y^2 + z^2 = c_2$, c_2 being an arbitrary constant. ...(2)

The complete solution consists of (1) and (2).

Ex.4. *Solve* $\dfrac{dx}{x(y^2 - z^2)} = \dfrac{dy}{y(z^2 - x^2)} = \dfrac{dz}{z(x^2 - y^2)}$. **[Bangalore1993,DelhiMaths(Hons.)2006]**

Sol. Choosing x, y, z as multipliers, each fraction of the given equations

$$= \frac{xdx + ydy + zdz}{x^2(y^2 - z^2) + y^2(z^2 - x^2) + z^2(x^2 - y^2)} = \frac{xdx + ydy + zdz}{0}.$$

$\therefore \quad x\,dx + y\,dy + z\,dz = 0 \quad$ or $\quad 2x\,dx + 2y\,dy + 2z\,dz = 0.$

Integrating, $\quad x^2 + y^2 + z^2 = c_1$, c_1 being an arbitrary constant. ...(1)

Again choosing $1/x$, $1/y$, $1/z$ as multipliers, each fraction of the given equations

$$= \frac{dx/x + dy/y + dz/z}{(y^2 - z^2) + (z^2 - x^2) + (x^2 - y^2)} = \frac{dx/x + dy/y + dz/z}{0}.$$

$\therefore \quad dx/x + dy/y + dz/z = 0. \quad$ so that $\quad \log x + \log y + \log z = \log c_2$

or $\quad \log xyz = \log c_2 \quad$ or $\quad xyz = c_2$, c_2 being an arbitrary constant ...(2)

The complete solution is given by the relations (1) and (2).

Ex. 5. *Solve* $(dx)/y = (dy)/(-x) = (dz)/(bx - ay)$.

Sol. Taking a, b, 1 as multipliers, each fraction of the given equations.

$= (a\,dx + b\,dy + dz)/0 \quad$ so that $\quad a\,dx + b\,dy + dz = 0$.

Integrating, $\quad ax + by + z = c_1$, c_1 being an arbitrary constant. ... (1)

From first two fractions, $\quad x\,dx + ydy = 0 \quad$ so that $\quad x^2 + y^2 = c_2$ ... (2)

The required solution is given by (1) and (2).

Ex. 6. *Solve* $\dfrac{xdx}{z^2 - 2yz - y^2} = \dfrac{dy}{y + z} = \dfrac{dz}{y - z}$. **[Bangalore 2007, Rohilkhand 1997]**

Sol. Choose 1, y, z as multipliers, each fraction of the given equations

$= (xdx + ydy + zdz)/0 \quad$ so that $\quad 2x\,dx + 2y\,dy + 2z\,dz = 0.$

Integrating, $\quad x^2 + y^2 + z^2 = c_1$, c_1 being an arbitrary constant. ...(1)

From the last two fractions, $\quad (y - z)dy = (y + z)dz$

or $\quad 2(ydz + zdy) - 2ydy + 2zdz = 0 \quad$ or $\quad 2d(yz) - d(y^2) + d(z^2) = 0.$

Integrating, $\quad 2yz - y^2 + z^2 = c_2$, c_2 being an arbitrary constant. ...(2)

The required complete solution is given by (1) and (2).

Ex. 7. *Solve* $\dfrac{dx}{y^3x - 2x^4} = \dfrac{dy}{2y^4 - x^3y} = \dfrac{dz}{9z(x^3 - y^3)}$.

Sol. Choosing $1/x$, $1/y$, $1/3z$ as multipliers, each fraction of the given equations

$$= \frac{dx/x + dy/y + dz/3z}{(y^3 - 2x^3) + (2y^3 - x^3) + 3(x^3 - y^3)} = \frac{dx/x + dy/y + dz/3z}{0}.$$

$\therefore \quad (1/x)dx + (1/y)dy + (1/3z)\,dz = 0$ so that $\log x + \log y + (1/3) \times \log z = \log c_1$

or $\quad xyz^{1/3} = c_1$, c_1 being an arbitrary constant. ...(1)

Now the first two fractions give $(2y^4 - x^3y)dx = (y^3x - 2x^4)dy.$

Dividing by x^3y^3, $\left(\frac{2y}{x^3} - \frac{1}{y^2}\right)dx = \left(\frac{1}{x^2} - \frac{2x}{y^3}\right)dy$ or $\left(\frac{1}{x^2}dy - \frac{2y}{x^3}dx\right) + \left(\frac{1}{y^2}dx - \frac{2x}{y^3}dy\right) = 0.$

or $\quad d(y/x^2) + d(x/y^2) = 0.$ so that $y/x^2 + x/y^2 = c_2.$ ...(2)

The required solution is given by the relations (1) and (2).

Ex. 8. *Solve* $(dx)/y^2 = (dy)/x^2 = (dz)/x^2y^2z^2.$ **[Mysore 2004]**

Sol. First two fractions give $3x^2dx - 3y^2dy = 0$ so that $x^3 - y^3 = c_1.$ ...(1)

Choosing x^2, y^2, $-2/z^2$ as multipliers, each fraction of the given equations

$$= \frac{x^2dx + y^2dy - (2/z^2)dz}{x^2y^2 + x^2y^2 - 2x^2y^2} = \frac{x^2dx + y^2dy - (2/z^2)dz}{0}.$$

$\therefore\ x^2dx + y^2dy - (2/z^2)dz = 0$ or $3x^2dx + 3y^2dy - (6/z^2)dz = 0.$

Integrating, $x^3 + y^3 + 6/z = c_2$, c_2 being an arbitrary constant. ...(2)

(1) and (2) together give the complete solution.

Ex. 9. $\dfrac{dx}{x(y^2 + z)} = \dfrac{dy}{-y(x^2 + z)} = \dfrac{dz}{z(x^2 - y^2)}.$ **[Meerut 2007]**

Sol. Choosing $1/x$, $1/y$, $1/z$ as multipliers, each fraction of the given equations

$$= \frac{(1/x)dx + (1/y)dy + (1/z)dz}{y^2 + z - (x^2 + z) + x^2 - y^2} = \frac{(1/x)dx + (1/y)dy + (1/z)dz}{0}.$$

$\therefore\ (1/x)dx + (1/y)dy + (1/z)dz = 0$ so that $xyz = c_1.$ ...(1)

Next choosing x, y, -1 as multipliers, each fraction of the given equations

$$= \frac{xdx + ydy - dz}{x^2(y^2 + z) - y^2(x^2 + z) - z(x^2 - y^2)} = \frac{xdx + ydy - dz}{0}$$

$\therefore\ 2xdx + 2ydy - 2dz = 0$ so that $x^2 + y^2 - 2z = c_2.$ ...(2)

(1) and (2) together give the complete solution.

Ex. 10. (*a*) *Solve* $\dfrac{dx}{z - y} = \dfrac{dy}{x - z} = \dfrac{dz}{y - x}.$ **[Delhi Maths (G) 1994]**

(*b*) *Show that* $u = x + y + z$, $v = x^2 + y^2 + z^2$ *are integrals of the linear system* $dx/dt = y - z$, $dy/dt = z - x$, $dz/dt\ \ x - y$. **[Amravati 2003]**

Sol. (*a*) Choosing 1, 1, 1 as multipliers, each fraction of the given equations

$$= \frac{dx + dy + dz}{z - y + x - z + y - x} = \frac{dx + dy + dz}{0}.$$

$\therefore\ dx + dy + dz = 0$ so that $x + y + z = c_1.$ ...(1)

Again, choosing x, y, z as multipliers, each fraction

$$= \frac{xdx + ydy + zdz}{x(z - y) + y(x - z) + z(y - x)} = \frac{xdx + ydy + zdz}{0}.$$

$\therefore\ xdx + ydy + zdz = 0$ or $2xdx + 2ydy + 2zdz = 0.$

Integrating, $x^2 + y^2 + z^2 = c_2$, c_2 being arbitrary constant. ...(2)

The required solution is given by the relations (1) and (2).

(*b*) Given system yields $(dx)/(y - z) = (dy)/(z - x) = (dz)/(x - y)$. Now proceed as in part to show that $x + y + z = c_1$ and $x^2 + y^2 + z^2 = c_2$. Hence $u = x + y + z$ and $v = x^2 + y^2 + z^2$, as required.

Ex. 11. *Solve* $\dfrac{yzdx}{y-z} = \dfrac{zxdy}{z-x} = \dfrac{xydz}{x-y}$. **[Delhi Maths (G) 1996]**

Sol. Choosing 1, 1, 1 as multipliers, each fraction of the given equations

$$= \frac{yzdx + zxdy + xydz}{y - z + z - x + x - y} = \frac{d(xyz)}{0}.$$

$\therefore \quad d(xyz) = 0 \quad$ so that $\quad xyz = c_1$. ...(1)

Again choosing $1/yz$, $1/zx$, $1/xy$ as multipliers, each fraction of the given equations

$$= \frac{dx + dy + dz}{(y-z)/yz + (z-x)/zx + (x-y)/xy} = \frac{dx + dy + dz}{1/z - 1/y + 1/x - 1/z + 1/y - 1/x} = \frac{dx + dy + dz}{0}$$

$\therefore \quad dx + dy + dz = 0 \quad$ so that $\quad x + y + z = c_2$. ...(2)

The required general solution is given by the relations (1) and (2).

Ex. 12. *Solve* $\dfrac{dx}{x(y^2 - z^2)} = \dfrac{dy}{-y(z^2 + x^2)} = \dfrac{dz}{z(x^2 + y^2)}$. **[Delhi Maths (H) 1993]**

Sol. Choosing x, y, z as multipliers, each fraction of the given equations

$$= \frac{xdx + ydy + zdz}{x^2(y^2 - z^2) - y^2(z^2 + x^2) + z^2(x^2 + y^2)} = \frac{xdx + ydy + zdz}{0}$$

$\therefore \quad xdx + ydy + zdz = 0 \quad$ so that $\quad x^2 + y^2 + z^2 = c_1$. ...(1)

Again, choosing $1/x$, $-1/y$, $-1/z$ as multipliners, each fraction

$$= \frac{(1/x)dx - (1/y)dy - (1/z)dz}{(y^2 - z^2) + (z^2 + x^2) - (x^2 + y^2)} = \frac{(1/x)dx - (1/y)dy - (1/z)dz}{0}.$$

$\therefore \quad (1/x)dx - (1/y)dy - (1/z)dz = 0 \quad$ so that $\quad \log x - \log y - \log z = \log c_2$

or $\quad \log (x/yz) = \log c_2 \quad$ or $\quad x/yz = c_2$. ...(2)

The required general solution is given by the relations (1) and (2).

Ex. 13. *Solve* $\dfrac{dx}{y - zx} = \dfrac{dy}{x + yz} = \dfrac{dz}{x^2 + y^2}$. **[Nagpur 1996, Delhi Maths (H) 1998]**

Sol. Choosing x, $-y$, z as multipliers, each fraction of the given equations

$$= \frac{xdx - ydy + zdz}{x(y - zx) - y(x + yz) + z(x^2 + y^2)} = \frac{xdx - ydy + zdz}{0}$$

$\therefore \quad xdx - ydy + zdz = 0 \quad$ so that $\quad 2xdx - 2ydy + 2zdz = 0$.

Integrating it, $x^2 - y^2 + z^2 = c_1$, c_1 being an arbitrary constants. ...(1)

Again, choosing y, x, -1 as multipliers, each fraction of the given equation

$$= \frac{y\,dx + x\,dy - dz}{y(y - zx) + x(x + yz) - (x^2 + y^2)} = \frac{y\,dx + x\,dy - dz}{0}.$$

$\therefore \quad y\,dx + x\,dy - dz = 0 \quad$ or $\quad d(xy) - dz = 0$.

Integrating it, $xy - z = c_2$, c_2 being an arbitrary constant. ...(2)

The required general solution is given by the relations (1) and (2).

Ex. 14. *Solve* $\dfrac{dx}{x^2} = \dfrac{dy}{y^2} = \dfrac{dz}{nxy}$. **[Delhi Maths (G) 1995]**

Sol. Taking the first two fractions, $x^{-2}dx = y^{-2}dy$.

Integrating $\quad -\dfrac{1}{x} = -\dfrac{1}{y} - c_1 \quad$ or $\quad \dfrac{1}{x} - \dfrac{1}{y} = c_1$

or $\quad y - x = c_1 xy \quad$ or $\quad x - y + c_1 xy = 0. \quad$ ...(1)

Choosing $1/x$, $-1/y$, c_1/n as multipliers, each fraction of the given equations

$$= \frac{(1/x)dx - (1/y)dy + (c_1/n)dz}{x - y + c_1 xy} = \frac{(1/x)dx - (1/y)dy + (c_1/n)dz}{0}, \text{ using (1)}$$

$\therefore \quad (1/x)dx - (1/y)dy + (c_1/n)dz = 0.$

Integrating, $\quad \log x - \log y + (c_1/n)z = (c_1/n)c_2$, c_2 being arbitrary constant

or $\quad \dfrac{c_1}{n}z = \dfrac{c_1}{n}c_2 + \log\dfrac{y}{x} \quad$ or $\quad z = c_2 + \dfrac{n}{c_1}\log\dfrac{y}{x} = c_2 + \dfrac{nxy}{y-x}\log\dfrac{y}{x}$, using (1)

The required general solution is given by the relations (1) and (2).

Ex. 15. *Solve* $\dfrac{dx}{y^2 + z^2} = \dfrac{dy}{-xy} = \dfrac{dz}{-xz}$. **[Delhi Maths. (H) 1995]**

Sol. Taking the last two fractions, $(1/y)dy - (1/z)dz = 0$.

Integrating, $\quad \log y - \log z = \log c, \quad$ or $\quad y/z = c_1. \quad$ ...(1)

Choosing x, y, z or multipliers, each given fraction $= \dfrac{xdx + ydy + zdz}{x(y^2 + z^2) - xy^2 - xz^2} = \dfrac{xdx + ydy + zdy}{0}$.

$\therefore \quad xdx + ydy + zdz = 0 \quad$ so that $\quad x^2 + y^2 + z^2 = c_2$.

The required general solution is given by the relations (1) and (2).

Ex. 16. *Solve* $\dfrac{dx}{x(2y^4 - z^4)} = \dfrac{dy}{y(z^4 - 2x^4)} = \dfrac{dz}{z(x^4 - y^4)}$.

Sol. Choosing $1/x$, $1/y$, $2/z$ as multipliers, each fraction

$$= \frac{(1/x)dx + (1/y)dy + (2/z)dz}{(2y^4 - z^4) + (z^4 - 2x^4) + 2(x^4 - y^4)} = \frac{(1/x)dx + (1/y)dy + 2(1/z)dz}{0}$$

$\therefore (1/x)dx + (1/y)dy + 2(1/z)dz = 0 \quad$ so that $\quad \log x + \log y + 2\log z = \log c_2$

or $\quad \log x + \log y + \log z^2 = \log c_2 \quad$ or $\quad xyz^2 = c_2. \quad$ ...(1)

Again, choosing x^3, y^3, z^3 as multipliers, each fraction

$$= \frac{x^3dx + y^3dy + z^3dz}{x^4(2y^4 - z^4) + y^4(z^4 - 2x^4) + z^4(x^4 - y^4)} = \frac{x^3dx + y^3dy + z^3dz}{0}$$

$\therefore \quad x^3dx + y^3dy + z^3dz = 0 \quad$ so that $\quad x^4 + y^4 + z^4 = c_2. \quad$ ...(2)

The required general solution is given by the relations (1) and (2).

Important Note: *Sometimes multipliers are chosen by using a trial method. The whole procedure is explained in the next solved Ex. 17.*

Ex. 17. Solve $\dfrac{dx}{4y - 3z} = \dfrac{dy}{4x - 2z} = \dfrac{dz}{2y - 3x}$.

Sol. Choosing l, m, n as multipliers, each fraction of the given equations

$$= \frac{ldx + mdy + ndz}{l(4y - 3z) + m(4x - 2z) + n(2y - 3x)} \quad ...(1)$$

Now, choose l, m, n such that $\quad l(4y - 3z) + m(4x - 2z) + n(2y - 3x) = 0$

or $\quad (4m - 3n)x + (4l + 2n)y + (-3l - 2m)z = 0,$

which is satisfied if $\quad 4m - 3n = 0, \quad 4l + 2n = 0, \quad -3l - 2m = 0 \quad$ or $\quad l : m : n = 2 : -3 : -4.$

$\therefore$ from (1), each given ratio = $(2dx - 3dy - 4dz)/0$.

$\therefore \quad 2dx - 3dy - 4dz = 0 \quad$ so that $\quad 2x - 3y - 4z = c_1$.

Again, choose l, m, n such that $\quad l(4y - 3z) + m(4x - 2z) + n(2y - 3x) = 0$

or $\quad 4(ly + mx) + 3(-lz - nx) + 2(ny - mz) = 0,$

which is satisfied if $\quad ly + mx = 0, \quad -lz - nx = 0, \quad ny - mz = 0 \quad$ or $\quad l : m : n = x : -y : -z.$

$\therefore$ from (1), each given ratio = $(xdx - ydy - zdz)/0$.

$\therefore \quad xdx - ydy - zdz = 0 \quad$ so that $\quad x^2 - y^2 - z^2 = c_2. \quad$ ...(3)

The required general solution is given by the relations (1) and (2).

Ex. 18. *Solve* $\dfrac{dx}{z^2 - 2yz - y^2} = \dfrac{dy}{y + z} = \dfrac{dz}{y - z}$.

Sol. From the last two fractions, $\quad (y - z)dy = (y + z)dy$

or $\quad ydy - zdz - (zdy + ydz) = 0 \quad$ or $\quad 2ydy - 2zdz - 2d(yz) = 0.$

Integrating, $\quad y^2 - z^2 - 2yz = c_1$, c_1 being an arbitrary constant $\quad$ ...(1)

Taking, 1, y, z as multipliers, each fraction

$$= \frac{dx + ydy + zdz}{z^2 - 2yz - y^2 + y(y + z) + z(y - z)} = \frac{dx + ydy + zdz}{0}.$$

$\therefore \quad dx + ydy + zdz = 0 \quad$ or $\quad 2dx + 2ydy + 2zdz = 0.$

Integrating, $\quad 2x + y^2 + z^2 = c_2$, c_2 being an arbitrary constant $\quad$ ...(2)

The required general solution is given by the relations (1) and (2).

Ex. 19. *Solve* $\dfrac{dx}{y - xz} = \dfrac{dy}{yz + x} = \dfrac{dz}{x^2 + y^2}$.

Sol. Choosing y, x, -1 and x, $-y$, z as multipliers by turn each given fraction,

$$= \frac{ydx + xdy - dz}{0} = \frac{xdx - ydy + zdz}{0}$$

$\therefore \quad ydx + xdy - dz = 0 \quad$ and $\quad xdx - ydy + zdz = 0.$

Integrating, $\quad xy - z = c_1 \quad$ and $\quad x^2 - y^2 + z^2 = c_2. \quad$ ... (3)

The required general solution is given by (3).

Ex. 20. *Solve* $\dfrac{dx}{x^2(y - z)} = \dfrac{dy}{y^2(z - x)} = \dfrac{dz}{z^2(x - y)}$.

Sol. Choosing $1/x$, $1/y$, $1/z$ and $1/x^2$, $1/y^2$, $1/z^2$ as multipliers by turn, each fraction

$$= \frac{(1/x)dx + (1/y)dy + (1/z)dz}{0} = \frac{(1/x^2)dx + (1/y^2)dy + (1/z^2)dz}{0}$$

$\therefore \quad (1/x)dx + (1/y)dy + (1/z)dz = 0 \quad$ and $\quad x^{-2}dx + y^{-2}dy + z^{-2}dz = 0.$

Integrating, $\quad \log x + \log y + \log z = \log c_1 \quad$ and $\quad -x^{-1} - y^{-1} - z^{-1} = -c_2$

or $\quad xyz = c_1 \quad$ and $\quad x^{-1} + y^{-1} + z^{-1} = c_2$, which give the desired solution.

Ex. 21. $\dfrac{dx}{x + 2z} = \dfrac{dy}{4zx - y} = \dfrac{dy}{2x^2 + y}$.

Sol. Choosing y, x, $-2z$ as multipliers, each fraction

$$= \frac{ydx + xdy - 2zdz}{y(x + 2z) + x(4zx - y) - 2z(2x^2 + y)} = \frac{d(xy) - 2zdz}{0}$$

$\therefore \quad d(xy) - 2zdz = 0$ so that $xy - z^2 = c_1$. ...(1)

Choosing $2x, -1, -1$ as multipliers, each fraction

$$= \frac{2xdx - dy - dz}{2x(x+2z) - (4zx - y) - (2x^2 + y)} = \frac{2xdx - dy - dz}{0}.$$

$\therefore \; 2xdx - dy - dz = 0$ so that $x^2 - y - z = c_2$. ...(2)

The required general solution is given by relations (1) and (2).

Ex. 22. *Solve* $\dfrac{dx}{x-y} = \dfrac{dy}{x+y} = \dfrac{dz}{2xz}$.

Sol. Choosing $1, 1, -1/z$ as multipliers, each given fraction

$$= \frac{dx + dy - (1/z)dz}{(x-y) + (x+y) - (1/z)\times(2xz)} = \frac{dx + dy - (1/z)dz}{0}.$$

$\therefore \quad dx + dy - (1/z)dz = 0$ so that $x + y - \log z = c_1$. ...(1)

Taking the first two fractions $\dfrac{dy}{dx} = \dfrac{x+y}{x-y} = \dfrac{1+(y/x)}{1-(y/x)}$. ...(2)

Let $y/x = v$ so that $y = xv$. ...(3)

From (3), $dy/dx = v + x(dv/dx)$. ...(4)

Using (3) and (4), (2) reduces to

$$v + x\frac{dv}{dx} = \frac{1+v}{1-v} \quad \text{or} \quad x\frac{dv}{dx} = \frac{1+v}{1-v} - v = \frac{1+v-v(1-v)}{1-v} = \frac{1+v^2}{1-v}$$

or $\quad \dfrac{1-v}{1+v^2}dv = \dfrac{dx}{x} \quad$ or $\quad \left(\dfrac{2}{1+v^2} - \dfrac{2v}{1+v^2}\right)dv = \dfrac{2dx}{x}$

Integrating, $\quad 2\tan^{-1} v - \log(1+v^2) = 2\log x - \log c_2 \quad$ or $\quad \log x^2 - \log(1+v^2) - \log c_2 = 2\tan^{-1} v$

or $\quad \log\{x^2(1+v^2)/c_2\} = 2\tan^{-1} v \quad$ or $\quad x^2(1+v^2) = c_2 e^{2\tan^{-1} v}$

or $\quad x^2\{1 + (y^2/x^2)\} = c_2\, e^{2\tan^{-1}(y/x)}$, as $v = y/x$ by (3)

or $\quad (x^2 + y^2)e^{-2\tan^{-1}(y/x)} = c_2$, c_2 being an arbitrary constant ...(5)

The required general solution is given by the relations (1) and (5).

Ex. 23. *Solve* $\dfrac{dx}{x(x^2+3y^2)} = \dfrac{dy}{-y(3x^2+y^2)} = \dfrac{dz}{2z(y^2-x^2)}$.

Sol. Choosing $1/x, 1/y, -1/z$ as multipliers, each fraction of the given equations.

$= \dfrac{(1/x)dx + (1/y)dy - (1/z)dz}{0}$ so that $\dfrac{dx}{x} + \dfrac{dy}{y} - \dfrac{dz}{z} = 0.$

Integrating, $\quad \log x + \log y - \log z = \log c_1$ so that $(xy)/z = c_1$. ...(1)

Taking the first two fractions, $\dfrac{dy}{dx} = -\dfrac{y(3x^2+y^2)}{x(x^2+3y^2)} = -\dfrac{y}{x}\times\dfrac{3+(y/x)^2}{1+3(y/x)^2}$

Putting $\quad y/x = v \quad$ or $\quad y = xv \quad$ so that $\quad dy/dx = v + x\,(dv/dx)$, we get

$$v + x\frac{dv}{dx} = -v\,\frac{3+v^2}{1+3v^2} \quad \text{or} \quad x\frac{dv}{dx} = -v\left[\frac{3+v^2}{1+3v^2} + 1\right]$$

or $\quad x\dfrac{dv}{dx} = -\dfrac{4(1+v^2)v}{1+3v^2}$ $\quad$ or $\quad 4\dfrac{dx}{x} + \dfrac{1+3v^2}{v(1+v^2)}dv = 0$

or $\quad 4\dfrac{dx}{x} + \left(\dfrac{1}{v} + \dfrac{2v}{1+v^2}\right)dv = 0$, on resolving into partial fractions

Integrating, $4 \log x + \log v + \log (1 + v^2) = \log c_2'$ $\quad$ or $\quad x^4 v(1 + v^2) = c_2'$

or $\quad x^4(y/x)[1 + (y/x)^2)] = c_2'$ $\quad$ or $\quad xy(x^2 + y^2) = c_2'$ $\quad$ or $\quad c_1 z(x^2 + y^2) = c_2'$, using (1)

or $\quad z(x^2 + y^2) = c_2$, where $c_2 = c_2'/c_1$, c_2 being an arbitrary constant ...(2)

The required general solution is given by the relations (1) and (2).

Ex. 24. *Solve* $\dfrac{dx}{y+z} = \dfrac{dy}{-(x+z)} = \dfrac{dz}{x-z}$. **[Mysore 2004]**

Sol. Choosing 1, 1, 1 as multipliers each fraction $= \dfrac{dx+dy+dz}{y+z-(x+z)+x-y} = \dfrac{dx+dy+dz}{0}$

so that $\quad dx + dy + dz = 0$ $\quad$ and so $\quad x + y + z = C_1$ ... (1)

From (1), $\quad y + z = C_1 - x$ $\quad$ and $\quad x + z = C_1 - y$

Hence the first two fractions of the given problem may be re-written as

$$\frac{dx}{C_1 - x} = \frac{dy}{-(C_1 - y)} \quad \text{or} \quad \frac{dx}{C_1 - x} + \frac{dy}{C_1 - y} = 0$$

Integrating, $\quad -\log (C_1 - x) - \log (C_1 - y) = -\log C_2$, C_2 being an arbitrary constant

or $\quad (C_1 - x)(C_1 - y) = C_2$ $\quad$ or $\quad (y + z)(x + z) = C_2$ ... (2)

The required solution is given by the relations (1) and (2).

EXERCISE 2 (C)

Solve the following simultaneous differential equations :

1. $\dfrac{dx}{x^2(y^3-z^3)} = \dfrac{dy}{y^2(z^3-x^3)} = \dfrac{dz}{z^2(x^3-y^3)}$ **[Delhi Maths (G) 2005]**

Hint. Do like Ex. 4. of Art. 2.9, **Ans.** $x^2 + y^2 + z^2 = c_1$, $1/x + 1/y + 1/z = c_2$

2. $\dfrac{dx}{x(y-z)} = \dfrac{dy}{y(z-x)} = \dfrac{dz}{z(x-y)}$. **[Pune 2010; Nagpur 1996; Bangalore 1993]**

Sol. Try yourself **Ans.** $x + y + z = c_1$ and $xyz = c_2$

3. $\dfrac{l dx}{mn(y-z)} = \dfrac{m dy}{nl(z-x)} = \dfrac{n dz}{lm(x-y)}$. **[Delhi Maths (G) 2001]**

Sol. Try yourself as in Ex. 1 of Art. 2.9 **Ans.** $l^2 x + m^2 y + n^2 z = c_1$, $l^2 x^2 + m^2 y^2 + n^2 z^2 = c_2$

4. $(dx)/y = (dy)/(-x) = (dz)/(2x - 3y)$ **[Osmania 2003]** **Ans.** $x^2 + y^2 = C_1$, $3x + 2y + z = C_2$

5. $(dx)/(zx) = (dy)/(-zy) = (dz)/(y^2 - x^2)$ **Ans.** $xy = C_1$, $x^2 + y^2 + z^2 = C_2$

2.10. Rule IV for solving (dx)/P = (dy)/Q = (dz)/R. ...(1)

Let P_1, Q_1, R_1 be functions of x, y, z. Then, by a well known principle of algebra, each fraction in (1) will be equal to

$$(P_1 dx + Q_1 dy + R_1 dz)/(P_1 P + Q_1 Q + R_1 R). \qquad ...(2)$$

Suppose the numerator of (2) is exact differential of the denominator of (2). Then (2) can be combined with a suitable fraction in (1) to give an integral. However, in some problems, another set of multipliers P_2, Q_2 and R_2 are so chosen that the fraction

$$(P_2dx + Q_2dy + R_2dz)/(P_2P + Q_2Q + R_2R) \qquad ...(3)$$

is such that its numerator is exact differential of denominator. Fractions (2) and (3) are then combined to give an integral. This method may also be repeated in some problems to get another integral.

Sometimes only one integral is possible with help of multipliers. In such cases second integral should be obtained by using rule I of Art. 2.4 or rule II of Art. 2.6 or rule III of Art 2.8 as the case may be.

2.11. Solved examples based on Art. 2.10

Ex. 1. *Solve* $\dfrac{dx}{y^2(x-y)} = \dfrac{dy}{-x^2(x-y)} = \dfrac{dz}{z(x^2+y^2)}$

Sol. Given $$\frac{dx}{y^2(x-y)} = \frac{dy}{-x^2(x-y)} = \frac{dx}{z(x^2+y^2)}. \qquad ... (1)$$

Taking the first two fractions in (1), we get

$x^2dx = -y^2dy$ or $3x^2dx + 3y^2dy = 0.$

Integrating, $x^3 + y^3 = c_1$, c_1 being an arbitrary constant ...(2)

Choosing 1, –1, 0 as multipliers, each fraction of (1)

$$= \frac{dx - dy}{y^2(x-y) + x^2(x-y)} = \frac{dx - dy}{(x-y)(x^2+y^2)}. \qquad ...(3)$$

Combining the third fraction in (1) with fraction (3), we get

$$\frac{dz}{z(x^2+y^2)} = \frac{dx - dy}{(x-y)(x^2+y^2)} \qquad \text{or} \qquad \frac{dz}{z} = \frac{dx - dy}{x - y}$$

Integrating $\log (x - y) - \log z = \log c_2$ or $(x - y)/z = c_2.$ ...(4)

The required solution is given by (2) and (4).

Ex. 2. *Solve* $\dfrac{dx}{x^2 - y^2 - z^2} = \dfrac{dy}{2xy} = \dfrac{dz}{2xz}.$ **[Guwahati 2007; Delhi Maths (H) 2000, Delhi Maths (Prog) 2009]**

or $\dfrac{dx}{y^2 + z^2 - x^2} = \dfrac{dy}{-2xy} = \dfrac{dz}{-2xz}.$ **[Delhi Maths (G) 2000; Nagpur 1995]**

Sol. Given $$\frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2xy} = \frac{dz}{2xz} \qquad ... (1)$$

Taking the last two fractions of (1), $(1/y)dy - (1/z)dz = 0$

Integrating, $\log y - \log z = \log c_1$ or $y/z = c_1.$ ...(2)

Choosing x, y, z as multipliers, each fraction in (1)

$$= \frac{xdx + ydy + zdz}{x^3 - xy^2 - xz^2 + 2xy^2 + 2xz^2} = \frac{xdx + ydy + zdz}{x(x^2+y^2+z^2)}. \qquad ...(3)$$

Combining the third fraction in (1) with fraction (3), we get

$$\frac{xdx + ydy + zdz}{x(x^2+y^2+z^2)} = \frac{dz}{2xz} \qquad \text{or} \qquad \frac{2(xdx + ydy + zdz)}{x^2+y^2+z^2} = \frac{dz}{z}.$$

Integrating, $\log (x^2 + y^2 + z^2) - \log z = \log c_2$ or $(x^2 + y^2 + z^2)/z = c_2.$ ...(4)

The required solution is given by (2) and (4).

Ex. 3. *Solve* $\dfrac{dx}{x^2 - yz} = \dfrac{dy}{y^2 - zx} = \dfrac{dz}{z^2 - xy}$ **[Bangalore 2005, Delhi Maths (G) 1998, 2008]**

Sol. Given $$\frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy} \qquad \text{... (1)}$$

Choosing 1, –1, 0 and 0, 1, –1 as multipliers by turn, each fraction of (1)

$$= \frac{dx - dy}{x^2 - y^2 + z(x-y)} = \frac{dy - dz}{y^2 - z^2 + x(y-z)}$$

$$\therefore \quad \frac{dx - dy}{(x-y)(x+y+z)} = \frac{dy - dz}{(y-z)(y+z+x)} \quad \text{or} \quad \frac{dx - dy}{x-y} = \frac{dy - dz}{y-z}.$$

Integrating, $\log(x-y) - \log(y-z) = \log c_1$ so that $(x-y)/(y-z) = c_1$. ...(2)

Choosing x, y, z as multipliers, each fraction in (1)

$$= \frac{xdx + ydy + zdz}{x^3 + y^3 + z^3 - 3xyz} = \frac{xdx + ydy + zdz}{(x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx)} \qquad \text{...(3)}$$

Again, choosing 1, 1, 1 as multipliers, each fraction in (1)

$$= \frac{dx + dy + dz}{x^2 + y^2 + z^2 - xy - yz - zx} \qquad \text{...(4)}$$

From (3) and (4), we get $(xdx + ydy + zdz)/(x + y + z) = dx + dy + dz$

or $$2(x + y + z)(dx + dy + dz) - 2(xdx + ydy + zdz) = 0$$

Integrating, $$(x+y+z)^2 - (x^2 + y^2 + z^2) = 2c_2$$

or $$(x^2 + y^2 + z^2 + 2xy + 2yz + 2zx) - (x^2 + y^2 + z^2) = 2c_2$$

or $xy + yz + zx = c_2$, c_2 being an arbitrary constant. ...(5)

The required solution is given by (2) and (5).

Ex. 4. *Solve* $\dfrac{dx}{y+z} = \dfrac{dy}{z+x} = \dfrac{dz}{x+y}$. **[Delhi Maths 1999; 2002]**

Sol. Given $$\frac{dx}{y+z} = \frac{dy}{z+x} = \frac{dz}{x+y} \qquad \text{... (1)}$$

Choosing, 1, –1, 0 and 0, 1, –1 as multipliers each fraction of (1)

$$= \frac{dx - dy}{(y+z) - (z+x)} = \frac{dy - dz}{(z+x) - (x+y)}. \qquad \text{...(2)}$$

So $$\frac{dx - dy}{-(x-y)} = \frac{dy - dz}{-(y-z)} \quad \text{or} \quad \frac{dx - dy}{x-y} = \frac{dy - dz}{y-z}.$$

Integrating, $\log(x-y) - \log(y-z) = \log c_1$ so that $(x-y)/(y-z) = c_1$. ...(3)

Choosing 1, 1, 1 as multiplers, each given fraction of (1) $= \dfrac{dx + dy + dz}{2(x+y+z)}$. ...(4)

Combining the first fraction in (2) which fraction (4), we have

$$\frac{dx - dy}{-(x-y)} = \frac{dx + dy + dz}{2(x+y+z)} \quad \text{or} \quad \frac{dx - dy}{x-y} + \frac{dx + dy + dz}{2(x+y+z)} = 0.$$

Integrating, $\log(x-y) + (1/2) \times \log(x+y+z) = \log c_2$ or $(x-y)(x+y+z)^{1/2} = c_2$. ...(5)

The required solution is given by (3) and (5).

Ex. 5. *Solve* $\dfrac{dx}{1+y} = \dfrac{dy}{1+x} = \dfrac{dz}{z}$. **[Delhi Maths (Hons) 2007, 08; Meerut 2005]**

Sol. Given
$$\frac{dx}{1+y} = \frac{dy}{1+x} = \frac{dz}{z} \qquad \text{...(1)}$$

Taking the first two fractions in (1), we have

$2(1 + x)dx - 2(1 + y)dy = 0$ so that $(1 + x)^2 - (1 + y)^2 = c_1$. ...(2)

Taking 1, 1, 0 as multipliers, each fraction in (1) = $(dx + dy)/(2 + x + y)$. ...(3)

Combining the last fraction in (1) with (3), we get

$$\frac{dx+dy}{2+x+y} = \frac{dz}{z} \quad \text{so that} \quad \log(2 + x + y) - \log z = \log c_2$$

or $(2 + x + y)/z = c_2$, c_2 being an arbitrary constant. ...(4)

The required solution is given by (2) and (4).

Ex. 6. *Solve* $\dfrac{dx}{x} = \dfrac{dy}{y} = \dfrac{dz}{z - a(x^2+y^2+z^2)^{1/2}}$.

Sol. Given
$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z - a(x^2+y^2+z^2)^{1/2}} \qquad \text{... (1)}$$

Taking the first two fractions in (1), $(1/x)dx - (1/y)dy = 0$

Integrating, $\log x - \log y = \log c_1$ or $x/y = c_1$. ...(2)

Choosing x, y, z as multipliers, each fraction in (1)

$$= \frac{xdx+ydy+zdz}{x^2+y^2+z^2-az(x^2+y^2+z^2)^{1/2}} = \frac{tdt}{t^2-azt} = \frac{dt}{t-az}. \qquad \text{...(3)}$$

[Put $x^2 + y^2 + z^2 = t^2$ so that $xdx + ydy + zdz = tdt$]

Putting $x^2 + y^2 + z^2 = t^2$ in (1), we get
$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z-at} \qquad \text{... (4)}$$

Then, (3) and (4) $\Rightarrow$
$$\frac{dz}{z-at} = \frac{dt}{t-az} = \frac{dx}{x}. \qquad \text{...(5)}$$

Choosing 1, 1, 0 as multipliers, each fraction in (5)

$$= \frac{dz+dt}{z+t-a(t+z)} = \frac{dz+dt}{(z+t)(1-a)}. \qquad \text{...(6)}$$

Combining the last fraction in (5) with (6), we get

$$\frac{dz+dt}{(z+t)(1-a)} = \frac{dx}{x} \quad \text{or} \quad (1-a)\frac{dx}{x} - \frac{dz+dt}{z+t} = 0.$$

Integrating, $(1 - a)\log x - \log(z + t) = \log c_2$

or
$$\frac{x^{a-1}}{z+t} = c_2 \quad \text{or} \quad \frac{x^{a-1}}{z+(x^2+y^2+z^2)^{1/2}} = c_2. \qquad \text{...(7)}$$

The complete solution is given by (2) and (7).

Ex. 7. *Solve* $(dx)/\cos(x + y) = (dy)/\sin(x + y) = (dz)/z$.

Sol. Given $(dx)/\cos(x + y) = (dy)/\sin(x + y) = (dz)/z$. ... (1)

From (1), using 1, 1, 0 and 1, –1, 0 as multipliers by turn, we have

$$\frac{dz}{z} = \frac{dx + dy}{\cos(x+y) + \sin(x+y)} = \frac{dx - dy}{\cos(x+y) - \sin(x+y)}. \quad \text{...(2)}$$

Putting $x + y = t$ so that $dx + dy = dt$, the first two fractions give

$$\frac{dz}{z} = \frac{dt}{\cos t + \sin t} \quad \text{or} \quad \sqrt{2}\,\frac{dz}{z} = \operatorname{cosec}\left(t + \frac{\pi}{4}\right)dt$$

$$\left[\because \cos t + \sin t = \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos t + \frac{1}{\sqrt{2}}\sin t\right) = \sqrt{2}\sin\left(t + \frac{\pi}{4}\right)\right]$$

Integrating, $\sqrt{2}\ \log z = \log\tan\frac{1}{2}\left(t + \frac{\pi}{4}\right) + \log c_1$ or $z^{\sqrt{2}} = c_1 \tan\frac{1}{2}\left(t + \frac{\pi}{4}\right)$

or $$z^{\sqrt{2}}\cot\frac{1}{2}\left(x + y + \frac{\pi}{4}\right) = c_1, \quad \text{as} \quad t = x + y. \quad \text{...(3)}$$

Now from the last two fractions in (2), we get

$$\frac{\cos(x+y) - \sin(x+y)}{\cos(x+y) + \sin(x+y)}(dx + dy) = dx - dy \quad \text{or} \quad \frac{\cos t - \sin t}{\cos t + \sin t}dt = dx - dy.$$

Integrating, $\log(\cos t + \sin t) - \log c_2 = x - y$ or $(\cos t + \sin t)/c_2 = e^{x-y}$.

or $[\cos(x+y) + \sin(x+y)]e^{y-x} = c_2$, as $t = x + y$. ...(4)

The complete solution is given by (3) and (4).

Ex. 8. *Solve* $\dfrac{dx}{x^2 - y^2 - yz} = \dfrac{dy}{x^2 - y^2 - zx} = \dfrac{dz}{z(x-y)}$. **[Garhwal 2010]**

Sol. Choosing 1, –1, 0 and x, $-y$, 0 as multipliers by turn, each fraction of the given equations.

$$= \frac{dx - dy}{z(x-y)} = \frac{xdx - ydy}{(x^2 - y^2)(x-y)}$$

In view of the last fraction of the given equations and the above fractions, we have

$$\frac{dz}{z(x-y)} = \frac{dx - dy}{z(x-y)} = \frac{xdx - ydy}{(x^2 - y^2)(x-y)} \quad \text{... (1)}$$

From the first two fractions of (1), $dz = dx - dy$.

Integrating, $z = x - y + c_1$ so that $z - x + y = c_1$.

...(2)

Now taking the first and the last fraction in (1), $\dfrac{2dz}{z} = \dfrac{2(xdx - ydy)}{x^2 - y^2}$.

Integrating, $2\log z = \log(x^2 - y^2) - \log c_2$. or $(x^2 - y^2)/z^2 = c_2$. ...(3)

The required general solution is given by the relations (2) and (3).

Ex. 9. *Solve* $(dx)/xz = (dy)/yz = (dz)/xy$.

Sol. Given $(dx)/xy = (dy)/yz = (dz)/xy$ ... (1)

Taking the first two fraction, $(1/x)dx - (1/y)dy = 0$.

Integrating, $\log x - \log y = \log c_1$ or $x/y = c_1$. ...(2)

Choosing $1/x$, $1/y$, 0 as multipliers, each fraction of (1)

$$= \frac{(1/x)dx + (1/y)dy}{(1/x)\times xz + (1/y)\times yz} = \frac{ydx + xdy}{2xyz} \quad \text{...(3)}$$

Combining the last fraction of (1) with fraction (3), we have

$$\frac{ydx + xdy}{2xyz} = \frac{dz}{xy} \qquad \text{or} \qquad ydx + xdy = 2zdz$$

or $\quad d(xy) = 2zdz \quad$ so that $\quad xy - z^2 = c_2$. ...(4)

The required general solution is given by the relations (2) and (4).

Ex. 10. *Solve* $\dfrac{dx}{x^2 + y^2 + yz} = \dfrac{dy}{x^2 + y^2 - xz} = \dfrac{dz}{z(x+y)}$. **[Kanpur 2009]**

Sol. Given
$$\frac{dx}{x^2 + y^2 + yz} = \frac{dy}{x^2 + y^2 - xz} = \frac{dz}{z(x+y)} \qquad \text{... (1)}$$

Choosing 1, –1, 0 as multipliers, each fraction of (1)

$$= \frac{dx - dy}{(x^2 + y^2 + yz) - (x^2 + y^2 - xz)} = \frac{dx - dy}{z(x+y)}. \qquad \text{...(2)}$$

Choosing x, y, 0 as multipliers, each fraction of (1)

$$= \frac{xdx + ydy}{x(x^2 + y^2 + yz) + y(x^2 + y^2 - xz)} = \frac{xdx + ydy}{(x+y)(x^2 + y^2)} \qquad \text{...(3)}$$

From (1), (2) and (3),
$$\frac{dz}{z(x+y)} = \frac{dx - dy}{z(x+y)} = \frac{xdx + ydy}{(x+y)(x^2 + y^2)} \qquad \text{...(4)}$$

Taking the first two fractions of (4), $\quad dz - dx + dy = 0$.

Integrating, $\quad z - x + y = c_1$, c_1 being an arbitrary constant ...(5)

Taking the first and third fractions of (4),
$$\frac{d(x^2 + y^2)}{x^2 + y^2} - 2\frac{dz}{z} = 0.$$

Integrating, $\quad \log (x^2 + y^2) - 2 \log z = \log c_2 \quad$ or $\quad (x^2 + y^2)/z^2 = c_2$. ...(6)

The required general solution is given by relations (1) and (2)

Ex. 11. *Solve* $\dfrac{dx}{x^3 + 3xy^2} = \dfrac{dy}{y^3 + 3x^2y} = \dfrac{dz}{2z(x^2 + y^2)}$.

Sol. Given
$$\frac{dx}{x^3 + 3xy^2} = \frac{dy}{y^3 + 3x^2y} = \frac{dz}{2z(x^2 + y^2)} \qquad \text{... (1)}$$

Choosing 1, 1, 0 as multipliers, each fraction of (1) $= \dfrac{dx + dy}{x^3 + 3xy^2 + 3x^2y + y^3} = \dfrac{d(x+y)}{(x+y)^3}$. ... (2)

Choosing 1, –1, 0 as multipliers, each fraction of (1) $= \dfrac{dx - dy}{x^3 + 3xy^2 - y^3 - 3x^2y} = \dfrac{d(x-y)}{(x-y)^3}$. ...(3)

From (2) and (3), $\quad (x + y)^{-3}d(x + y) = (x - y)^{-3}d(x - y)$.

or $\quad u^{-3}du - v^{-3}dv = 0, \quad$ [putting $\quad u = x + y \quad$ and $\quad v = x - y$]

Integrating, $\quad u^{-2}/(-2) - v^{-2}/(-2) = c_1/2 \quad$ or $\quad v^{-2} - u^{-2} = c_1$

or $\quad (x - y)^{-2} - (x + y)^{-2} = c_1, \quad$ as $\quad u = x + y, \quad v = x - y$. ...(4)

Choosing $1/x$, $1/y$, 0 as multipliers, each fraction of (1)

$$= \frac{(1/x)dx + (1/y)dy}{(1/x)(x^3 + 3xy^2) + (1/y)(y^3 + 3x^2y)} = \frac{(1/x)dx + (1/y)dy}{4(x^2 + y^2)}. \qquad \text{...(5)}$$

Combining the last fraction of (1) with fraction (5), we have

$$\frac{dz}{2z(x^2+y^2)} = \frac{(1/x)dx+(1/y)dy}{4(x^2+y^2)} \qquad \text{or} \qquad \frac{dx}{x}+\frac{dy}{y}-2\frac{dz}{z}=0.$$

Integrating, $\log x + \log y - 2 \log z = \log c_2$ or $(xy)/z^2 = c_2$. ...(6)

The required general solution is given by the relations (4) and (6).

Ex. 12. *Solve* $dx = dy = (dz)/(x + y + z)$.

Sol. Given $$\frac{dx}{1}=\frac{dy}{1}=\frac{dz}{x+y+z} \qquad \text{... (1)}$$

Taking the first two fractions, $dx - dy = 0$ so that $x - y = c_1$. ...(2)

Choosing 1, 1, 1 as multipliers, each fraction of (1)

$$=\frac{dx+dy+dz}{1+1+(x+y+z)}=\frac{d(2+x+y+z)}{2+x+y+z} \qquad \text{...(3)}$$

Combining the first fraction of (1) with fraction (3), we get

$$\frac{d(2+x+y+z)}{2+x+y+z}=dx \qquad \text{so that} \qquad \log(2+x+y+z)-\log c_2 = x.$$

or $[(2 + x + y + z)/c_2] = e^x$ or $e^{-x}(2 + x + y + z) = c_2$. ...(4)

The required general solution is given by (2) and (4).

Ex. 13. *Solve* $\dfrac{dx}{y^2+yz+z^2}=\dfrac{dy}{z^2+zx+x^2}=\dfrac{dz}{x^2+xy+y^2}$. **[Delhi Maths (H) 2004; Meerut 1996]**

Sol. Given $$\frac{dx}{y^2+yz+z^2}=\frac{dy}{z^2+zx+x^2}=\frac{dz}{x^2+xy+y^2} \qquad \text{... (1)}$$

Choosing 1, –1, 0 as multipliers, each ratio of (1)

$$=\frac{dx-dy}{(y^2+yz+z^2)-(z^2+zx+x^2)}=\frac{dx-dy}{(y^2-x^2)+z(y-x)}=-\frac{dy-dx}{(y-x)(y+x+z)}. \qquad \text{...(2)}$$

Again, choosing 0, 1, –1 as multipliers, each ratio of (1)

$$=\frac{dy-dz}{(z^2+zx+x^2)-(x^2+xy+y^2)}=\frac{dy-dz}{(z^2-y^2)+x(z-y)}=-\frac{dz-dy}{(z-y)(z+y+x)}. \qquad \text{...(3)}$$

From (2) and (3), $$\frac{d(y-x)}{y-x}-\frac{d(z-y)}{z-y}=0.$$

Integrating, $\log(y - x) - \log(z - y) = \log c_1$ or $(y - x)/(z - y) = c_1$. ...(4)

Choosing x, y, z as multipliers, each ratio of (1)

$$=\frac{xdx+ydy+zdz}{x(y^2+yz+z^2)+y(z^2+zx+x^2)+z(x^2+xy+y^2)}=\frac{xdx+ydy+zdz}{(x+y+z)(xy+yz+zx)} \qquad \text{...(5)}$$

Again, choosing $y + z$, $z + x$, $x + y$ as multipliers, each ratio of (1)

$$=\frac{(y+z)dx+(z+x)dy+(x+y)dz}{(y+z)(y^2+yz+z^2)+(z+x)(z^2+zx+x^2)+(x+y)(x^2+xy+y^2)}$$

$$=\frac{(ydx+xdy)+(ydz+zdy)+(zdx+xdz)}{2(x^3+xy^2+xz^2+y^3+yx^2+yz^2+z^3+zx^2+zy^2)}$$

$$=\frac{d(xy)+d(yz)+d(zx)}{2(x+y+z)(x^2+y^2+z^2)}=\frac{d(xy+yz+zx)}{2(x+y+z)(x^2+y^2+z^2)}. \qquad \text{...(6)}$$

From fractions (5) and (6), we have

$$\frac{xdx + ydy + zdz}{(x+y+z)(xy+yz+zx)} = \frac{d(xy+yz+zx)}{2(x+y+z)(x^2+y^2+z^2)}.$$

or $\quad (xy + yz + zx)\, d(xy + yz + zx) - (x^2 + y^2 + z^2)\,(2xdx + 2ydy + 2zdz) = 0.$

Integrating, $\quad (xy + yz + zx)^2 - (x^2 + y^2 + z^2)^2 = c_2.$ ...(7)

The required general solution is given by the relations (4) and (7).

Ex. 14. *Solve* $\dfrac{dx}{x(x+y)} = \dfrac{dy}{-y(x+y)} = \dfrac{dz}{-(x-y)(2x+2y+z)}.$

Sol. Given $\quad \dfrac{dx}{x(x+y)} = \dfrac{dy}{-y(x+y)} = \dfrac{dz}{-(x-y)(2x+2y+z)}$ ... (1)

Taking the first two fractions in (1), we get $\quad (1/x)dx + (1/y)dy = 0.$

Integrating, $\quad \log x + \log y = \log c_1 \quad$ or $\quad xy = c_1.$ ...(2)

Choosing, 1, 1, 0 as multipliers, each fraction of (1)

$$= \frac{dx+dy}{x(x+y)-y(x+y)} = \frac{dx+dy}{(x+y)(x-y)}. \qquad ...(3)$$

Again, choosing 1, 1, 1 as multipliers, each fraction of (1)

$$= \frac{dx+dy+dz}{x(x+y)-y(x+y)-(x-y)(2x+2y+z)} = \frac{dx+dy+dz}{(x-y)(x+y)-(x-y)(2x+2y+z)}$$

$$= \frac{dx+dy+dz}{(x-y)\{x+y-(2x+2y+z)\}} = -\frac{dx+dy+dz}{(x-y)(x+y+z)}. \qquad ...(4)$$

From fractions (3) and (4), $\quad \dfrac{dx+dy}{(x+y)(x-y)} = -\dfrac{dx+dy+dz}{(x-y)(x+y+z)}$

or $\quad \dfrac{dx+dy}{x+y} + \dfrac{dx+dy+dz}{x+y+z} = 0.$

Integrating, $\log (x + y) + \log (x + y + z) = \log c_2$, so that $(x + y)(x + y + z) = c_2.$ ...(5)

The required general solution is given by the relations (2) and (5).

Ex. 15. *Solve* $(dx)/x^2 = (dy)/y^2 = (dz)/z(x + y)$

Sol. Given $\quad \dfrac{dx}{x^2} = \dfrac{dy}{y^2} = \dfrac{dz}{z(x+y)}$ ... (1)

From the first two fractions in (1), $\quad x^{-2}dx - y^{-2}dy = 0.$

Integrating, $\quad -x^{-1} + y^{-1} = c_1 \quad$ or $\quad (1/y) - (1/x) = c_1.$ ...(2)

Choosing 1, –1, 0 as multipliers, each fraction of (1) $\quad = \dfrac{dx-dy}{x^2-y^2} = \dfrac{dx-dy}{(x-y)(x+y)}.$

...(3)

Taking the last fraction of (1) and fraction (3), we have

$$\frac{dz}{z(x+y)} = \frac{dx-dy}{(x-y)(x+y)} \quad \text{or} \quad \frac{d(x-y)}{x-y} - \frac{dz}{z} = 0.$$

Integrating, $\quad \log (x - y) - \log z = \log c_2 \quad$ or $\quad (x - y)/z = c_2.$ ...(4)

The required general solution is given by the relations (2) and (4).

Ex. 16. *Solve* $(dx)/(x^2 + y^2) = (dy)/(2xy) = (dz)/z(x + y).$

Sol. Given $\quad \dfrac{dx}{x^2+y^2} = \dfrac{dy}{2xy} = \dfrac{dz}{(x+y)z}$ ... (1)

Choosing 1, 1, 0 as multipliers, each fraction of (1) $= \dfrac{dx+dy}{x^2+y^2+2xy} = \dfrac{dx+dy}{(x+y)^2}$. ...(2)

Choosing 1, –1, 0 as multipliers, each fraction of (1) $= \dfrac{dx-dy}{x^2+y^2-2xy} = \dfrac{dx-dy}{(x-y)^2}$. ...(3)

From fractions (2) and (3), $(x+y)^{-2}(dx+dy) = (x-y)^{-2}(dx-dy)$.

Integrating, $-(x+y)^{-1} = -(x-y)^{-1} + c_1$ or $(x-y)^{-1} - (x+y)^{-1} = c_1$....(4)

From the last fraction of (1) and fraction (2), we have

$$\frac{dx+dy}{(x+y)^2} = \frac{dz}{(x+y)z} \quad \text{or} \quad \frac{d(x+y)}{x+y} - \frac{dz}{z} = 0.$$

Integrating, $\log(x+y) - \log z = \log c_2$ or $(x+y)/z = c_2$. ...(5)

The required general solution is given by relations (4) and (5).

Ex. 17. *Solve* $(dx)/y = (dy)/x = (dz)/z$.

Sol. Given $(dx)/y = (dy)/x = (dx)/z$ ... (1)

From the first two fractions, $2xdx - 2ydy = 0$ so that $x^2 - y^2 = c_1$. ...(2)

Choosing 1, 1, 0 as multipliers, each fraction of (1) = $(dx+dy)/(y+x)$.

Combining this fraction which the last fraction (1), we get

$$\frac{dx+dy}{x+y} = \frac{dz}{z} \quad \text{so that} \quad \log(x+y) - \log z = \log c_2.$$

or $\log[(x+y)/z] = \log c_2$ or $(x+y)/z = c_2$. ...(3)

The required general solution is given by the relations (2) and (3).

Ex. 18. *Solve* $(dx)/x = (dy)/z = (dz)/(-y)$.

Sol. Given $(dx)/x = (dy)/z = (dz)(-y)$ ... (1)

From the last two fractions, $2ydy + 2zdz = 0$. so that $y^2 + z^2 = c_1$. ...(2)

Choosing, 0, z, $-y$ as multipliers each fraction of (1)

$$= \frac{zdy - ydz}{z.z - y(-y)} = \frac{zdy - ydz}{z^2+y^2} = \frac{(1/z)dy - (y/z^2)dz}{1+(y/z)^2} = \frac{d(y/z)}{1+(y/z)^2} = \frac{dt}{1+t^2}, \text{ where } t = \frac{y}{z} \quad ...(3)$$

Combining the above fraction with the first fraction of (1), we get

$$\frac{dx}{x} = \frac{dt}{1+t^2} \quad \text{so that} \quad \log x - \log c_2 = \tan^{-1} t = \tan^{-1}(y/z).$$

or $\log(x/c_2) = \tan^{-1}(y/z)$ or $x = c_2 e^{\tan^{-1}(y/z)}$ ...(4)

The required general solution is given by the relations (2) and (4).

Ex. 19. *Solve* $(dx)/(x^2+a^2) = (dy)/(xy-az) = (dz)/(xz+ay)$.

Sol. Given $$\frac{dx}{x^2+a^2} = \frac{dy}{xy-az} = \frac{dz}{xz+ay} \quad ... (1)$$

Taking 0, z, $-y$ as multipliers, each fraction of (1)

$$= \frac{zdy - ydz}{z(xy-az) - y(xz+ay)} = \frac{zdy - ydz}{-a(y^2+z^2)}. \quad ...(2)$$

Taking 0, y, z as multipliers, each fraction of (1)

$$= \frac{ydy + zdz}{y(xy-az) + z(xz+ay)} = \frac{ydy+zdz}{x(y^2+z^2)}. \quad ...(3)$$

Taking the first fraction of (1) and fraction (3), we have

$$\frac{dx}{x^2+a^2} = \frac{ydy+zdz}{x(y^2+z^2)} \qquad \text{or} \qquad \frac{2xdx}{x^2+a^2} - \frac{2ydy+2zdz}{y^2+z^2} = 0.$$

Integrating, $\log(x^2+a^2) - \log(y^2+z^2) = \log c_2$.

or $(x^2+a^2)/(y^2+z^2) = c_2$, c_2 being an arbitrary constant ...(4)

Taking the first fraction of (1) and fraction (2), we have

$$\frac{dx}{x^2+a^2} = \frac{zdy-ydz}{-a(y^2+z^2)} \qquad \text{or} \qquad \frac{adx}{x^2+a^2} + \frac{zdy-ydz}{z^2+y^2} = 0$$

or
$$\frac{adx}{x^2+a^2} + \frac{(1/z)dy-(y/z^2)dz}{1+(y/z)^2} = 0 \qquad \text{or} \qquad \frac{adx}{x^2+a^2} + \frac{d(y/z)}{1+(y/z)^2} = 0$$

or
$$\frac{adx}{x^2+a^2} + \frac{dt}{1+t^2} = 0, \qquad \text{where} \qquad t = \frac{y}{z}.$$

Integrating, $a\tan^{-1}(x/a) + \tan^{-1} t = c_2$ or $a\tan^{-1}(x/a) + \tan^{-1}(y/z) = c_2$. ...(5)

The required general solution is given by the relations (4) and (5).

Ex. 20. *Solve* $\dfrac{dx}{y(x+y)+az} = \dfrac{dy}{x(x+y)-az} = \dfrac{dz}{z(x+y)}$.

[Delhi Maths (Hons) 2005, Nagpur 2005, 10]

Sol. Given
$$\frac{dx}{y(x+y)+az} = \frac{dy}{x(x+y)-az} = \frac{dz}{z(x+y)} \qquad \text{... (1)}$$

Choosing 1, 1, 0 as multipliers, each fraction of (1)

$$= \frac{dx+dy}{y(x+y)+az+x(x+y)-az} = \frac{dx+dy}{(x+y)^2}. \qquad \text{...(2)}$$

From the last fraction of (1) and the fraction (2), we have

$$\frac{dx+dy}{(x+y)^2} = \frac{dz}{z(x+y)} \qquad \text{or} \qquad \frac{d(x+y)}{x+y} - \frac{dz}{z} = 0.$$

Integrating, $\log(x+y) - \log z = \log c_1$ or $(x+y)/z = c_1$. ...(3)

Choosing x, $-y$, 0 as multipliers, each fraction of (1)

$$= \frac{xdx-ydy}{x[y(x+y)+az]-y[x(x+y)-az]} = \frac{xdx-ydy}{az(x+y)}. \qquad \text{...(4)}$$

From the first fraction of (1) and fraction (4), we have

$$\frac{xdx-ydy}{az(x+y)} = \frac{dz}{z(x+y)} \qquad \text{or} \qquad 2xdx - 2ydy - 2adz = 0.$$

Integrating, $x^2 - y^2 - 2az = c_2$, c_2 being an arbitrary constant ...(5)

The required general solution is given by relations (3) and (5).

Ex. 21. *Solve* $\dfrac{dx}{x+y-xy^2} = \dfrac{dy}{x^2y-x-y} = \dfrac{dz}{z(y^2-x^2)}$ **[Delhi Maths (H) 1997, 2002, 07]**

Sol. Given
$$\frac{dx}{x+y-xy^2} = \frac{dy}{x^2y-x-y} = \frac{dz}{z(y^2-x^2)} \qquad \text{... (1)}$$

Each fraction of (1) $= \dfrac{y\,dx + x\,dy}{y(x+y-xy^2)+x(x^2y-x-y)} = \dfrac{ydx+xdy}{(y^2-x^2)(1-xy)}$

Combining the above fraction with last fraction of (1), we get

$$\frac{y\,dx+x\,dy}{(y^2-x^2)(1-xy)} = \frac{dz}{z(y^2-x^2)} \qquad \text{or} \qquad \frac{-y\,dx-x\,dy}{1-xy}+\frac{dz}{z}=0.$$

Integrating, $\log(1-xy)+\log z = \log c_1$ or $z(1-xy)=c_1$ ... (2)

Again, each fraction of (1) = $[xdx + ydy + (1/z)dz]/0$

Hence $2x\,dx+2y\,dy+2(1/z)\,dz=0$ so that $x^2+y^2+2\log z = c_2'$

or $x^2+y^2+2\log\{c_1/(1-xy)\}=c_2'$, using (2)

or $x^2+y^2-2\log(1-xy)=c_2$, where $c_2=c_2'-2\log c_1$ ... (3)

The required general solution is given by (2) and (3)

EXERCISE 2 (D)

Solve the following simultaneous differential equations :

1. $(dx)/x = (dy)/z = (dz)/y$ **Ans.** $y^2 - z^2 = C_1$, $(y+z)/x = C_2$
2. $(dx)/\cos(x+y) = (dy)/\sin(x+y) = (dz)/(z+1/z)$

   Ans. $e^{y-x}\{\cos(x+y)+\sin(x+y)\}=C_1$, $(z^2+1)^{1/2}\tan\{3\pi/8-(x+y)/2\}=C_2$
3. $(dx)/x = (dy)/(-y) = (dz)/(y^2-x^2)$ **(Bangalore 2005)** **Ans.** $xy = C_1$, $x^2+y^2-2z = C_2$
4. $(dx)/y^2 = (dy)/x^2 = (dz)/z^2(x^2-y^2)$ **(Bangalore 2005)** **Ans.** $x^3-y^3 = c_1$, $x+y+(1/z) = C_2$
5. Solve $(dx)/(xz-y) = (dy)/(yz-x) = (dz)/(1-z^2)$ **(Pune 2010)**

   Ans. $(x-y)(1-z)=c_1$, $(x+y)(1+z)=c_2$

2.12. Orthogonal trajectories of a system of curves on a surface

Let the given surface be $f(x, y, z) = 0$ ... (1)

and let the given system of surface be $\phi(x,y,z)=c$, c being a parameter. ... (2)

Then, the given system of curves lying on the surface (1) are the curves of intersection of (1) and (2).

Clearly the direction ratios dx, dy, dz of the tangent at any point (x, y, z) on the given curve lying on the surfaces (1) and (2) are given by

$$(\partial f/\partial x)dx+(\partial f/\partial y)dy+(\partial f/\partial z)dz=0 \qquad \text{and} \qquad (\partial\phi/\partial x)dx+(\partial\phi/\partial y)dy+(\partial\phi/\partial z)dz=0$$

Solving these for dx, dy, dz, we have

$$(dx)/P = (dy)/Q = (dz)/R \qquad \text{... (3)}$$

where $P=\dfrac{\partial f}{\partial y}\dfrac{\partial\phi}{\partial z}-\dfrac{\partial f}{\partial z}\dfrac{\partial\phi}{\partial y}$, $Q=\dfrac{\partial f}{\partial z}\dfrac{\partial\phi}{\partial x}-\dfrac{\partial f}{\partial x}\dfrac{\partial\phi}{\partial z}$, $R=\dfrac{\partial f}{\partial x}\dfrac{\partial\phi}{\partial y}-\dfrac{\partial f}{\partial y}\dfrac{\partial\phi}{\partial x}$ ... (4)

Thus, P, Q, R are the direction ratio of tangents to the given curves of intersection of (1) and (2). By, definition, P, Q, R will be direction ratios of normal of the required orthogonal trajectories. Therefore, if dx, dy, dz be the direction ratios of the required orthogonal trajectorie, then the differential equations of the orthogonal trajectories are given by

$$(\partial f/\partial x)dx+(\partial f/\partial y)dy+(\partial f/\partial z)\,dz=0 \qquad \text{and} \qquad Pdx + Qdy + Rdz = 0$$

Solving these for dx, dy, dz, we obtain

$$(dx)/P' = (dy)/Q' = (dz)/R', \qquad \text{... (5)}$$

where $P' = R(\partial f/\partial y) - Q(\partial f/\partial z),\ \ Q' = P(\partial f/\partial z) - R(\partial f/\partial x),\ \ R' = Q(\partial f/\partial x) - P(\partial f/\partial y)$ ... (6)

The solution of simultaneous differential equations (5) together with the given surface (1) gives the system of required orthogonal trajectories.

2.12A. Solved Example based on Art. 2.12.

Ex. 1. *Find the orthogonal trajectories on the cone $x^2 + y^2 = z^2 \tan^2\alpha$ of its intersection with the family of planes parallel to $z = 0$.* **[Madurai 2005; Pune 2010]**

Sol. Given surface is $f(x, y, z) = x^2 + y^2 - z^2\tan^2\alpha = 0$ ... (1)

and the family of planes parallel to $z = 0$ is $z = k$. ... (2)

where k is parameter. Then, the system of differential equations of the given curves of intersection of (1) and (2) is given by

$$2x\,dx + 2y\,dy - 2z\tan^2\alpha\,dz = 0; \qquad dz = 0$$

Solving these equation for dx, dy, dz, we get

$$(dx)/y = (dy)/(-x) = (dz)/0$$

Hence the system of differential equations of the required orthogonal trajectories of the given curves is $x\,dx + y\,dy - 2z\tan^2\alpha\,dz = 0$ and $y\,dx - x\,dy + 0 \cdot dz = 0$ ... (3)

Solving these for dx, dy, dz, we have

$$\frac{dx}{xz\tan^2\alpha} = \frac{dy}{yz\tan^2\alpha} = \frac{dz}{x^2 + y^2} \qquad \text{... (4)}$$

Taking x, y, 0 as multipliers, each fraction of (4) $= \dfrac{x\,dx + y\,dy}{(x^2 + y^2)z\tan^2\alpha}$

Combining this fraction with last fraction in (4), we have

$$\frac{x\,dx + y\,dy}{(x^2 + y^2)z\tan^2\alpha} = \frac{dz}{x^2 + y^2} \qquad \text{so that} \qquad 2x\,dx + 2y\,dy - 2z\tan^2\alpha\,dz = 0$$

Integrating, $x^2 + y^2 - z^2\tan^2\alpha = c'$, c' being an arbitrary constant.

Choosing $c' = 0$, we obtain the given surface (1).

Taking the first and second fractions of (4), $(1/x)\,dx - (1/y)dy = 0$.

Integrating, $\log x - \log y = \log c$ or $x/y = c$, c being an arbitrary constant.

Hence the required family of the orthogonal trajectories is given by $x^2 + y^2 = z^2\tan^2\alpha$ and $x = cy$.

Ex. 2. *Find the orthogonal trajectories on the conicoid* $(x + y)z = 1$ of the conics in which it is cut by the system of planes $x - y + z = k$, where k is parameter. **[Kolkata 2001]**

Sol. Given surface is $f(x, y, z) = xz + yz - 1 = 0$ ... (1)

and the given system of planes is $x - y + z = k$ ... (2)

Then the system of differential equations of the given curves of intersection of (1) and (2) is given by

$$z\,dx + z\,dy + (x + y)dz = 0 \qquad \text{and} \qquad dx - dy + dz = 0$$

Solving these for dx, dy, dz we see $\dfrac{dx}{z + x + y} = \dfrac{dy}{x + y - z} = \dfrac{dz}{-2z}$

Hence the system of differential equations of the required orthogonal trajectories of the given curves is

$$\left.\begin{aligned} z\,dx + z\,dy + (x+y)\,dz &= 0 \\ \text{and} \qquad (z+x+y)\,dx + (x+y-z)\,dy - 2z\,dz &= 0 \end{aligned}\right\} \qquad \ldots(3)$$

Solving the above simultaneous system of equations for dx, dy, dz, we get

$$\frac{dx}{-2z^2-(x+y)^2+z(x+y)} = \frac{dy}{(x+y)^2+z(x+y)+2z^2} = \frac{dz}{-2z^2} \qquad \ldots(4)$$

$\therefore$ (4) $\Rightarrow$ $\dfrac{dx+dy}{2z(x+y)} = \dfrac{dz}{-2z^2}$ or $\dfrac{dx+dy}{x+y} + \dfrac{dz}{z} = 0$

Integrating, $\log(x+y)+\log z = \log c$ or $(x+y)z = c$, ... (5)
where c is an arbitrary constant. Choosing $c = 1$, we get the given surface (1), namely, $(x+y)\,z = 1$ ... (6)

Now, choosing the first and the last fractions in (4) and using (6), we have

$\dfrac{dx}{-2z^2-1/z^2+1} = \dfrac{dz}{-2z^2}$ or $dx = \{1+(1/2)\times z^{-4}-(1/2)\times z^{-2}\}dz$

Integrating, $x + c' = z - (1/6z^3) + (1/2z)$, c' being an arbitrary constant.

Hence the required family of the orthogonal trajectories is given by

$(x+y)z = c$ and $x + c' = z - (1/6z^3) + (1/2z)$.

EXERCISE 2(E)

1. Find the orthogonal trajectories on the surface $x^2+y^2+2fyz+d=0$ of its curves of intersection with planes parallel to xy-plane. **[Nagarajuna 1993]**

Ans. $x^2+y^2+2fyz+d=0$, $fyz+d=c'x$, c' being an arbitrary constant.

2. Find the equations of the system of curves on the cylinder $2y = x^2$ orthogonal to intersection with the hyperboids $xy = z + c$. **Ans.** $2y = x^2$, $3z + 2(x - 1/x) = c'$.

3. Find the orthogonal trajectories on the sphere $x^2 + y^2 + z^2 = a^2$ of its intersections with the paraboloids $xy = cz$, c being parameter.

4. Show that the orthogonal trajectories on the hyperboloid $x^2+y^2-z^2 = 1$ of the conics in which it is cut by the system of planes $x + y = c$ are its curves of intersection with the surfaces $(x-y)\,z = k$, where k is parameter.

MISCELLANEOUS PROBLEMS ON CHAPTER 2

1. If $u\,(x, y, z) = c_1$ and $v\,(x, y, z) = c_2$ are integral curves of $dx/P = dy/Q = dz/R$, then what is the geometrical meanings of P, Q, R. **[Pune 2010]**

3

Total (or Pfaffian) Differential Equations

3.1. Introduction. In this chapter, we propose to discuss differential equations with one independent variable and more than one dependent variables.

Pfaffian differential form. Pfaffian differential equation. Definitions

Let u_i, $i = 1, 2, .., n$ be n functions of some or all of n independent variables $x_1, x_2, .., x_n$. Then, $\sum_{i=1}^{n} u_i\, dx_i$ is called a Pfaffian differential form in n variables and $\sum_{i=1}^{n} u_i\, dx_i = 0$ is called a Pfaffian differential equation in n variables $x_1, x_2, .., x_n$.

3.2. Total (or single) differential equation (or Pfaffian differential equation).

An equation of the form $$Pdx + Qdy + Rdz = 0, \qquad ...(1)$$
where P, Q, R are functions of x, y, z is called the *single or total differential equation* in three variables x, y, z.

Equation (1) can be directly integrated if there exists a function $u(x, y, z)$ whose total differential du is equal to the left hand number of (1). In other cases (1) may or may not be integrable. We now proceed to find the condition which P, Q, R must satisfy, so that (1) may be integrable. This will be called the *condition or criterian of integrability* of the single differential equation (1).

3.3. Necessary and sufficient conditions for integrability of total (or single) differential equation Pdx + Qdy + Rdz = 0. **[Bangalore 2005; Patna 2003; Delhi Maths Hons. 1994;**
Agra 2002, 05, 07; Indore 2001, 02; Kanpur 1999; Lucknow 2003, 05; Meerut 2006, 07; Gwalior 2006; Karnataka 2003; Nagarajuna 2004; Sagar 2001, 04, Ujjain 2001, 02]

Necessary condition : Consider the total (or single) differential equation
$$Pdx + Qdy + Rdz = 0, \text{ where } P, Q, R \text{ are functions of } x, y, z. \qquad ...(1)$$
Let (1) have an integral $$u(x, y, z) = c. \qquad ...(2)$$
Then total differential du must be equal to $Pdx + Qdy + Rdz$, or to it multiplied by a factor.
But, we know that $$du = (\partial u/\partial x)dx + (\partial u/\partial y)dy + (\partial u/\partial z)dz. \qquad ...(3)$$
Since (2) is an integral of (1), P, Q, R must be proportional to $\partial u/\partial x$, $\partial u/\partial y$, $\partial u/\partial z$

Therefore $$\frac{\partial u / dx}{P} = \frac{\partial u / dy}{Q} = \frac{\partial u / dz}{R} = \lambda(x, y, z), \text{ say.}$$

$$\therefore \lambda P = \partial u / \partial x, \qquad \lambda Q = \partial u / \partial y \qquad \text{and} \qquad \lambda R = \partial u / \partial z. \qquad ...(4)$$

From the first two equations of (4), we get
$$\frac{\partial}{\partial y}(\lambda P) = \frac{\partial^2 u}{\partial y\, \partial x} = \frac{\partial^2 u}{\partial x\, \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial x}(\lambda Q)$$

or $$\lambda \frac{\partial P}{\partial y} + P\frac{\partial \lambda}{\partial y} = \lambda \frac{\partial Q}{\partial x} + Q\frac{\partial \lambda}{\partial x}$$

i.e. $$\lambda\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = Q\frac{\partial \lambda}{\partial x} - P\frac{\partial \lambda}{\partial y}. \qquad ...(5)$$

Similarly, $$\lambda\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) = R\frac{\partial \lambda}{\partial y} - Q\frac{\partial \lambda}{\partial z} \qquad ...(6)$$

and $$\lambda\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) = P\frac{\partial \lambda}{\partial z} - R\frac{\partial \lambda}{\partial x}. \qquad ...(7)$$

Multiplying (5), (6) and (7) by R, P and Q respectively and adding, we get

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0. \qquad ...(8)$$

This is, therefore, the necessary condition for the integrability of the equation (1).

Sufficient condition : Suppose that the coefficients P, Q, R, of (1) satisfy the relation (8). It will now be proved that this relation gives the required sufficient condition for the existence of an integral of (1). For this we show that an integral of (1) can be found when relation (8) holds.

We first prove that if we take $P_1 = \mu P$, $Q_1 = \mu Q$, $R_1 = \mu R$, where μ is any function of x, y and z, the same condition is satisfied by P_1, Q_1, R_1 as by P, Q, R. We have

$$\frac{\partial Q_1}{\partial z} - \frac{\partial R_1}{\partial y} = \mu\frac{\partial Q}{\partial z} + Q\frac{\partial \mu}{\partial z} - \left(\mu\frac{\partial R}{\partial y} + R\frac{\partial \mu}{\partial y}\right), \quad \text{as} \quad Q_1 = \mu Q, \quad \text{and} \quad R_1 = \mu R$$

or $$\frac{\partial Q_1}{\partial z} - \frac{\partial R_1}{\partial y} = \mu\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\frac{\partial \mu}{\partial z} - R\frac{\partial \mu}{\partial y}. \qquad ...(9)$$

Similarly, $$\frac{\partial R_1}{\partial x} - \frac{\partial P_1}{\partial z} = \mu\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\frac{\partial \mu}{\partial x} - P\frac{\partial \mu}{\partial z} \qquad ...(10)$$

and $$\frac{\partial P_1}{\partial y} - \frac{\partial Q_1}{\partial x} = \mu\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) + P\frac{\partial \mu}{\partial y} - Q\frac{\partial \mu}{\partial x}. \qquad ...(11)$$

Multiplying (9), (10) and (11) by P_1, Q_1, R_1 respectively, adding and replacing P_1, Q_1, R_1 by μP, μQ, μR respectively in resulting R.H.S., we obtain

$$P_1\left(\frac{\partial Q_1}{\partial z} - \frac{\partial R_1}{\partial y}\right) + Q_1\left(\frac{\partial R_1}{\partial x} - \frac{\partial P_1}{\partial z}\right) + R_1\left(\frac{\partial P_1}{\partial y} - \frac{\partial Q_1}{\partial x}\right)$$

$$= \mu\left\{P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\right\} = 0, \text{ by (8)} \qquad ...(12)$$

Now $Pdx + Qdy$ may be regarded as an exact differential. For if it is not so, then multiplying the equation (1) by the integrating factor $\mu(x, y, z)$, we can make it so. *Thus there is no loss of generality in regarding $Pdx + Qdy$ as an exact differential. For this the condition is

$$\partial P/\partial y = \partial Q/\partial x \qquad ...(13)$$

Let $$V = \int (Pdx + Qdy) \qquad ...(14)$$

then it follows that $P = \partial V/\partial x$ and $Q = \partial V/\partial y$. ...(15)

*By this we mean here that if $Pdx + Qdy$ is not exact differential, then as explained $\mu Pdx + \mu Qdy$, *i.e.*, $P_1dx + Q_1dy$ will be exact differential and in the whole discussion (12) may be used in place of (8).

From (15), $\qquad \dfrac{\partial P}{\partial z} = \dfrac{\partial^2 V}{\partial z\,\partial x} \qquad$ and $\qquad \dfrac{\partial Q}{\partial z} = \dfrac{\partial^2 V}{\partial z\,\partial y}$.

Using the above relations, (13) and (15), (8) gives

$$\frac{\partial V}{\partial x}\left(\frac{\partial^2 V}{\partial z\,\partial x} - \frac{\partial R}{\partial y}\right) + \frac{\partial V}{\partial y}\left(\frac{\partial R}{\partial x} - \frac{\partial^2 V}{\partial z\,\partial x}\right) = 0. \quad \text{or} \quad \frac{\partial V}{\partial x}\frac{\partial}{\partial y}\left(\frac{\partial V}{\partial z} - R\right) - \frac{\partial V}{\partial y}\frac{\partial}{\partial x}\left(\frac{\partial V}{\partial z} - R\right) = 0.$$

or
$$\begin{vmatrix} \dfrac{\partial V}{\partial x} & \dfrac{\partial}{\partial x}\left(\dfrac{\partial V}{\partial z} - R\right) \\ \dfrac{\partial V}{\partial y} & \dfrac{\partial}{\partial y}\left(\dfrac{\partial V}{\partial z} - R\right) \end{vmatrix} = 0.$$

*This shows that a relation independent of x and y exists between V and $(\partial V/\partial z) - R$. Consequently $(\partial V/\partial z) - R$ can be expressed as a function of z and V alone. That is, we can take

$$(\partial V/\partial z) - R = \phi(z, V). \qquad \text{... (16)}$$

Now, $\quad Pdx + Qdy + Rdz = \dfrac{\partial V}{\partial x}\,dx + \dfrac{\partial V}{\partial y}\,dy + \left(\dfrac{\partial V}{\partial z} - \phi\right)dz$, using (14) and (16)

$$= \left(\frac{\partial V}{\partial x}\,dx + \frac{\partial V}{\partial y}\,dy + \frac{\partial V}{\partial z}\,dz\right) - \phi\,dz = dV - \phi\,dz.$$

Thus (1) may be written as $dV - \phi\,dz = 0$ which is an equation in two variables. Hence its integration will give an integral of the form $\qquad F(V, z) = 0.$

Hence the condition (8) is sufficient.

Thus (8) is both the necessary and sufficient condition that (1) has an integral.

Theorem. *Prove that the necessary condition for integrability of the total differential equation.* $\mathbf{A}\cdot d\mathbf{r} = Pdx + Qdy + Rdz = 0$ *is* $\mathbf{A}\cdot curl\ \mathbf{A} = 0$.

[Himanchal 2003, 04, 05, 06; Indore 2004; Lucknow 2001, 04]

Proof. Given $\qquad \mathbf{A}\cdot d\mathbf{r} = Pdx + Qdy + Rdz = 0. \qquad$ ...(1)

Let $\quad \mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \quad$ so that $\quad d\mathbf{r} = dx\,\mathbf{i} + dy\,\mathbf{j} + dz\,\mathbf{k} \qquad$ ...(2)

and $\qquad \mathbf{A} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}. \qquad$ ...(3)

Then we see that (1) is satisfied by usual rule of dot product of two vectors **A** and $d\mathbf{r}$.

Now show (as explained in Art 3.2) that the necessary condition for integrability of (1) is

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0. \qquad \text{...(4)}$$

From vector calculus, we know that

$$\text{Curl } \mathbf{A} = \left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right)\mathbf{i} + \left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right)\mathbf{j} + \left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)\mathbf{k}. \qquad \text{...(5)}$$

Hence, using (3) and (5) and applying the usual rule of dot product of two vectors, the necessary condition (4) may be rewritten as $\mathbf{A}\cdot$ Curl $\mathbf{A} = 0$ as desired.

3.4. The conditions for exactness of Pdx + Qdy + Rdz = 0.

The given total differential equation is said to be exact if the following three conditions are

*Refer a chapter on Jacobians in any text book on advanced Differential Calculus.

satisfied

$\partial P/\partial y = \partial Q/\partial x, \qquad \partial Q/\partial z = \partial R/\partial y \qquad$ and $\qquad \partial R/\partial x = \partial P/\partial z.$ ...(1)

Note that when conditions (1) are satisfied, the condition for integrability of $Pdx + Qdy + Rdz = 0$, namely,

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0 \qquad \text{... (2)}$$

is also satisfied, for each term of (2), vanishes identically.

3.5. Methods of solving Pdx + Qdy + Rdz = 0. ...(1)

There are several methods of solving (1). We know that (1) is integrable when the following condition is satisfied

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0. \qquad \text{...(2)}$$

Before applying any method of solving (1), the students are strongly advised not to forget to verify the condition (2) for integrability of (1). Since this verification is quite simple, we have verified the same in some problems only and have left the same for the students to do themselves.

3.6. Special Method I. Solution by inspection. Sometimes by the rearranging the terms of the given equation and/or by dividing by a suitable function of x, y, z, the equation thus obtained will contain several parts which are exact differentials. The following list will help to re–write the given equation. Students should note that each formula of the list is quite general, that is, we can replace x by y (or z) and so on as may be necessary in a particular problem:

(*i*) $\dfrac{xdy - ydx}{x^2} = d\left(\dfrac{y}{x}\right)$ (*ii*) $\dfrac{xdy - ydx}{xy} = d\left(\log\dfrac{y}{x}\right)$

(*iii*) $\dfrac{xdy - ydx}{x^2+y^2} = d\left(\tan^{-1}\dfrac{y}{x}\right)$ (*iv*) $\dfrac{xdy + ydx}{xy} = d[\log(xy)]$.

(*v*) $\dfrac{xdy + ydx}{x^2+y^2} = d\left[\frac{1}{2}\log(x^2+y^2)\right]$. (*vi*) $\dfrac{2xy\,dy - y^2\,dx}{x^2} = d\left(\dfrac{y^2}{x}\right)$.

(*vii*) $x\,dy + y\,dx = d\,(xy)$. (*viii*) $d(xyz) = xy\,dz + xz\,dy + yz\,dx$.

(*ix*) $\displaystyle\int [f(x, y, z)]^n\, df(x, y, z) = \frac{[f(x, y, z)]^{n+1}}{n+1}$, $(n \neq -1)$. [Power formula]

(*x*) $\displaystyle\int \frac{df(x, y, z)}{f(x, y, z)} = \log f(x, y, z)$. [Log formula] (*xi*) $\dfrac{2x^2y\,dy - 2xy^2\,dx}{x^4} = d\left(\dfrac{y^2}{x^2}\right)$

(*xii*) $-\dfrac{xdy + ydx}{x^2y^2} = d\left(\dfrac{1}{xy}\right)$ (*xiii*) $\dfrac{ye^x dx - e^x dy}{y^2} = d\left(\dfrac{e^x}{y}\right)$

(*xiv*) $y^2dx + 2xy\,dy = d(y^2x)$ (*xv*) $2(xdx + ydy) = d(x^2 + y^2)$.

(*xvi*) $2(xdx + ydy + zdz) = d(x^2 + y^2 + z^2)$. (*xvii*) $3x^2ydx + x^3dy = d(x^3y)$.

3.7. Solved examples based on Special Method I of Art. 3.6.

Ex. 1(a). *Verify the condition of integrability for* $z\,dx + z\,dy + 2(x + y + \sin z)dz = 0$.

[Bangalore 2004, 07]

Sol. Comparing the given equation with $Pdx + Qdy + Rdz = 0$, here

$P = z, \qquad Q = z \qquad$ and $\qquad R = 2(x + y + \sin z).$ ...(1)

Now, $P(\partial Q/\partial z - \partial R/\partial y) + Q(\partial R/\partial x - \partial P/\partial z) + R(\partial P/\partial y - \partial Q/\partial x)$

$= z(1 - 2) + z(2 - 1) + 2(x + y + \sin z)\,(0 - 0) = 0$, using (1)

showing that the condition of integrability is satisfied.

Ex. 1(b). *Show that* $(2x + y^2 + 2xz)dx + 2xydy + x^2dz = 0$ *is integrable.* **[Bangalore 1995]**

Sol. Comparing the given equation with $Pdx + Qdy + Rdz = 0$, here

$$P = 2x + y^2 + 2xz, \qquad Q = 2xy \qquad \text{and} \qquad R = x^2. \qquad \ldots (1)$$

Now, $P(\partial Q/\partial z - \partial R/\partial y) + Q(\partial R/\partial x - \partial P/\partial z) + R(\partial P/\partial y - \partial Q/\partial x)$

$= (2x + y^2 + 2xz)\,(0 - 0) + 2xy\,(2x - 2x) + x^2\,(2y - 2y) = 0$, using (1)

showing that the condition of integrability is satisfied and hence the given equation is integrable.

Ex. 1(c). *Test for the integrability of the equation* $(y^2 + yz)\,dx + (xz + z^2)\,dy + (y^2 - xy)\,dz = 0$ **[Gulberga 2005]**

Sol. Comparing the given equation with $Pdx + Qdy + Rdq = 0$, we get

$$P = y^2 + yz, \qquad Q = xz + z^2 \qquad \text{and} \qquad R = y^2 - xy \qquad \ldots (1)$$

Here, $P(\partial Q/\partial z - \partial R/\partial y) + Q(\partial R/\partial x - \partial P/\partial z) + R(\partial P/\partial y - \partial Q/\partial x)$

$= (y^2 + yz)\,\{(x + 2z) - (2y - x)\} + (xz + z^2)\,(-y - y) + (y^2 - xy)\,\{(2y + z) - z\}$

$= 2(y^2 + yz)\,(x - y + z) - 2y(xz + z^2) + 2y(y^2 - xy)$

$= 2(xy^2 + xyz - y^3 - y^2z + y^2z + yz^2 - xyz - yz^2 + y^3 - xy^2) = 0,$

showing that the condition of integrability is satisfied and hence the given equation is integrable.

Ex. 2(a). *Solve* $(yz + xyz)dx + (zx + xyz)dy + (xy + xyz)dz = 0$.

[Delhi Maths (G) 2002; Agra 2002; Gujrat 2001, 05; Indore 1997; Karnataka 2001; Meerut 1998; Rajasthan 2004; Vikram 1999]

Sol. Comparing the given equation with $Pdx + Qdy + Rdz = 0$, we get

$$P = yz + xyz, \qquad Q = zx + xyz \qquad \text{and} \qquad R = xy + xyz.$$

$\therefore\ P(\partial Q/\partial z - \partial R/\partial y) + Q(\partial R/\partial x - \partial P/\partial z) + R(\partial P/\partial y - \partial Q/\partial x)$

$= yz(1 + x)\,\{(x + xy) - (x + xz)\} + zx(1 + y)\,\{(y + yz) - (y + xy) + xy(1 + z)\,\{(z + xz) - (z + yz)\}$

$= yz(1 + x)\,x(y - z) + zx(1 + y)\,y(z - x) + xy(1 + z)\,z(x - y)$

$= xyz\,\{(1 + x)\,(y - z) + (1 + y)\,(z - x) + (1 + z)\,(x - y)\}$

$= xyz\,[\{(y - z) + (z - x) + (x - y)\} + \{x(y - z) + y(z - x) + z(x - y)\}] = xyz\,[0 + 0] = 0,$

showing that the given total differential equation is integrable.

Dividing each term by xyz, the given equation becomes

$$(1/x + 1)dx + (1/y + 1)dy + (1/z + 1)dz = 0.$$

Integrating it, $\log x + x + \log y + y + \log z + z = c$ or $\log (xyz) + x + y + z = c$,

which is the required general solution, c being an arbitrary constant.

Ex. 2(b). *Solve* $(yz + 2x)dx + (zx - 2z)dy + (xy - 2y)dz = 0$.

[Delhi Maths (G) 1996; Garhwal 1993; Meerut 1997, 2007]

Sol. Comparing the given equation with $Pdx + Qdy + Rdz = 0$, we get

$$P = yz + 2x, \qquad Q = zx - 2z \qquad \text{and} \qquad R = xy - 2y. \qquad \ldots (1)$$

$\therefore\quad P(\partial Q/\partial z - \partial R/\partial y) + Q(\partial R/\partial x - \partial P/\partial z) + R(\partial P/\partial y - \partial Q/\partial x)$

$= (yz + 2x)\,\{(x - 2) - (x - 2)\} + (zx - 2z)\,(y - y) + (xy - 2y)\,(z - z) = 0$, by (1)

showing that the given total differential equation is integrable.

On rearranging, the given equation can be rewritten as

$(yz\,dx + zx\,dy + xy\,dz) + 2xdx - 2(zdy + ydz) = 0.$ or $d(xyz) + d(x^2) - 2d(yz) = 0$

Integrating, $xyz + x^2 - 2yz = c,$

which is the required general solution, c being an arbitrary constant.

Ex. 3. *Solve* $(y^2 + z^2 - x^2)dx - 2xy\,dy - 2xz\,dz = 0$.

[Delhi B.A. (Prog) II 2010; Kumaun 2005; Meerut 2007]

Sol. As usual, verify that the given equation is integrable.

Adding and subtracting x^2dx, the given equation becomes

$$(x^2 + y^2 + z^2)dx - x(2x\,dx + 2y\,dy + 2z\,dz) = 0 \qquad \text{or} \qquad \frac{dx}{x} = \frac{2x\,dx + 2y\,dy + 2z\,dz}{x^2 + y^2 + z^2}$$

Integrating, $\log x + \log c = \log (x^2 + y^2 + z^2)$ or $xc = x^2 + y^2 + z^2$, which is the required solution, c being an arbitrary constant.

Ex. 4. *Solve* $2yz\,dx + zx\,dy - xy(1 + z)dz = 0$. **[Meerut 1998; Garhwal 2010]**

Sol. As usual, verify that the given equation is integrable.

Dividing each term by xyz, the given equation becomes

$$(2/x)dx + (1/y)dy - (1 + 1/z)dz = 0.$$

Integrating, $2 \log x + \log y - z - \log z - \log c = 0$ or $\log [(x^2y)/(cz)] = z$

or $x^2y = cze^z$, which is the required general solution, c being an arbitrary constant.

Ex. 5(*a*). *Solve* $x\,dy - y\,dx - 2x^2z\,dz = 0$. **[Rajasthan 2001; Banglore 1997]**

Sol. As usual, verify that the given equation is integrable.

Dividing the given equation by x^2, we have

$$\frac{xdy - ydx}{x^2} - 2z\,dz = 0 \qquad \text{or} \qquad d\left(\frac{y}{x}\right) - 2z\,dz = 0.$$

Integrating, $(y/x) - z^2 = c$, c being an arbitrary constant.

Ex. 5(*b*). *Solve* $zy\,dx = zx\,dy + y^2\,dz$.

[Mysore 2004; Delhi B.Sc. (H) 2000, 04, 06, 08; Delhi BA/B.Sc (Prog.) Maths 2007]

Sol. As usual, verify that the given equation is integrable.

Dividing by zy^2, the given equation can be rewritten as

$$\frac{y\,dx - x\,dy}{y^2} = \frac{dz}{z} \qquad \text{or} \qquad d\left(\frac{x}{y}\right) = \frac{dz}{z}.$$

Integrating, $(x/y) = \log z - \log c$ or $\log (z/c) = x/y$ or $z = c\,e^{x/y}$.

Ex. 6. *Solve* $yz \log z\,dx - zx \log z\,dy + xy\,dz = 0$. **[Delhi Maths (G) 1997, 1999; Agra 1998; Kanpur 1999; Garhwal 1997; Sgar 1996; Delhi Math (Prog.) 2007]**

Sol. As usual, verify that the given equation is integrable.

Dividing by $xy \log z$, the given equation becomes $\dfrac{1}{x}\,dx - \dfrac{1}{y}\,dy + \dfrac{(1/z)dz}{\log z} = 0.$

Integrating, $\log x - \log y + \log \log z = \log c$ or $x \log z = cy$.

Ex. 7. *Solve* $(a - z)\,(ydx + xdy) + xy\,dz = 0$. **[Delhi Maths (G) 2002; Meerut 1999]**

Sol. As usual, verify that the given equation is integrable.

Dividing by $xy(a - z)$, the given equation becomes

$$\frac{dx}{x} + \frac{dy}{y} + \frac{dz}{a - z} = 0 \qquad \text{so that} \qquad \log x + \log y - \log (a - z) = \log c$$

or $$xy = c(a - z), \; c \text{ being an arbitrary constant.}$$

Ex. 8(a). *Solve* $yz^2(x^2 - yz)dx + zx^2(y^2 - xz)dy + xy^2(z^2 - xy)dz = 0$. **[Meerut 1997]**

(b) $(x^2y - y^3 - y^2z)dx + (xy^2 - x^2z - x^3)dy + (xy^2 + x^2y)dz = 0$.

[Lucknow 2006; Sagar 2003, 05; Rajasthan 2002, 07; Agra 1998; Gorakhpur 2003,05; [Delhi Maths (G) 2001; 2004; Meerut 1996; Rohilkhand 1997]

Sol. (*a*) As usual, verify that the given equation is integrable.

Dividing each term by $x^2y^2z^2$, the given equation becomes

$$(1/y)dx - (z/x^2)dx + (1/z)dy - (x/y^2)dy + (1/x)dz - (y/z^2)dz = 0$$

or $$\left(\frac{dx}{y} - \frac{x}{y^2}\,dy\right) + \left(\frac{dy}{z} - \frac{y}{z^2}\,dz\right) + \left(\frac{dz}{x} - \frac{z}{x^2}\,dx\right) = 0, \text{ on rearranging}$$

or $$\frac{ydx - xdy}{y^2} + \frac{zdy - ydz}{z^2} + \frac{xdz - zdx}{x^2} = 0 \qquad \text{or} \qquad d\left(\frac{x}{y}\right) + d\left(\frac{y}{z}\right) + d\left(\frac{z}{x}\right) = 0.$$

Integrating, $(x/y) + (y/z) + (z/x) = c$ or $x^2z + y^2x + z^2y = cxyz.$

(*b*) As usual, verify that the given equation is integrable.

Dividing each term by x^2y^2, the given equation becomes

$$\frac{dx}{y} - \frac{ydx}{x^2} - \frac{zdx}{x^2} + \frac{dy}{x} - \frac{zdy}{y^2} - \frac{xdy}{y^2} + \frac{dz}{x} + \frac{dz}{y} = 0$$

or $$\left(\frac{dx}{y} - \frac{xdy}{y^2}\right) + \left(\frac{dy}{x} - \frac{ydx}{x^2}\right) + \left(\frac{dz}{x} - \frac{zdx}{x^2}\right) + \left(\frac{dz}{y} - \frac{zdy}{y^2}\right) = 0, \text{ on re-arranging}$$

or $$\frac{ydx - xdy}{y^2} + \frac{xdy - ydx}{x^2} + \frac{xdz - zdx}{x^2} + \frac{ydz - zdy}{y^2} = 0$$

or $$d(x/y) + d(y/x) + d(z/x) + d(z/y) = 0.$$

Integrating, $(x/y) + (y/x) + (z/x) + (z/y) = c$ or $x^2 + y^2 + z(x + y) = cxy.$

Ex. 9. *Solve* $(2x^2 + 2xy + 2xz^2 + 1)dx + dy + 2zdz = 0.$ **[Delhi Maths (Hons) 1993, 2001]**

Sol. As usual verify that the given equation is integrable. On rearranging,

$$2x(x + y + z^2)dx + (dx + dy + 2zdz) = 0 \qquad \text{or} \qquad \frac{dx + dy + 2zdz}{x + y + z^2} = -2xdx.$$

Integrating, $\log(x + y + z^2) - \log c = -x^2$ or $x + y + z^2 = ce^{-x^2}.$

Ex. 10. *Solve* (*a*) $xz^3dx - zdy + 2ydz = 0.$ **[Delhi Maths (G) 1996]**

(*b*) $(x^2z - y^3)dx + 3xy^2dy + x^3dz = 0.$ **[Himanchal 2004; Delhi Maths (G) 1995, 96]**

Sol. (*a*) As usual verify that the given equation is integrable.

Dividing each term of the given equation by z^3, we get

$$xdx - \left(\frac{dy}{z^2} - \frac{2ydz}{z^3}\right) = 0 \qquad \text{or} \qquad 2xdx - 2d\left(\frac{1}{z^2}\cdot y\right) = 0.$$

Integrating, $x^2 - 2\,(y/z^2) = c$ or $x^2z^2 - 2y = cz^2.$

(*b*) Dividing each term by x^2, the given equation becomes

$$zdx - \frac{y^3}{x^2}dx + \frac{3y^2}{x}dy + xdz = 0 \qquad \text{or} \qquad (zdx + xdz) + \left(\frac{3y^2\,dy}{x} - \frac{y^3\,dx}{x^2}\right) = 0.$$

Integrating, $xz + (y^3/x) = c$ or $x^2z + y^3 = cx.$

Ex. 11. *Solve* $2xz\,dx + zdy - dz = 0.$ **[Delhi Maths (Hons) 1998]**

Sol. Dividing by z, $2xdx + dy - (1/z)dz = 0.$

Integrating, $x^2 + y - \log z = c$, c being an arbitrary constant.

Ex. 12. (*a*) *Solve* $2yzdx - 3zxdy - 4xydz = 0.$ **[Delhi Maths 2003]**

(*b*) *Solve* $yzdx + 2xzdy - 3xydz = 0.$ **[Delhi Maths (Hons) 1993]**

Sol. (*a*) Dividing each term by xyz, the given equation becomes

$(2/x)dx - (3/y)dy - (4/z)dz = 0$ so that $2\log x - 3\log y - 4\log z = \log c$

or $\log x^2 = \log y^3 + \log z^4 + \log c$ or $x^2 = cy^3z^4.$

(*b*) Proceed as in part (*a*). **Ans.** $x^2y = cz.$

Ex. 13. *Solve* $(x - 3y - z)dx + (2y - 3x)dy + (z - x)dz = 0.$ **[Delhi BA (Prog) II 2011]**

Sol. As usual verify that the given equation is integrable.

One rearranging, the given equation can be written as

$$x\,dx + 2y\,dy + z\,dz - 3(y\,dx + x\,dy) - (z\,dx + x\,dz) = 0.$$

Integrating, $\quad x^2/2 + y^2 + z^2/2 - 3xy - xz = c/2$

or $\quad x^2 + 2y^2 + z^2 - 6xy - 2xz = c$, c being an arbitrary constant.

Ex. 14. *Very that* $x(y^2 - a^2)dx + y(x^2 - z^2)sdy - z(y^2 - a^2)dz = 0$ *is integrable and solve it.*

[G.N.D.U. (Amritar) 2004, 07; Himanchal 2003, 04, 05]

Sol. As usual verify that the given equation is integrable.

Dividing each term by $(y^2 - a^2)\,(x^2 - z^2)$, the given equation becomes

$$\frac{2(xdx - zdz)}{x^2 - z^2} + \frac{2ydy}{y^2 - a^2} = 0 \quad \text{or} \quad d\{\log\,(x^2 - z^2)\} + d\{\log\,(y^2 - a^2)\} = 0.$$

Integrating, $\quad \log\,(x^2 - z^2) + \log\,(y^2 - a^2) = \log c \quad$ or $\quad (x^2 - z^2)\,(y^2 - a^2) = c.$

Ex. 15. *Solve* $\dfrac{yz\,dx}{x^2 + y^2} - \dfrac{xz\,dy}{x^2 + y^2} - \tan^{-1}\dfrac{y}{x}\,dz = 0.$

Sol. As usual, verify that the given equation is integrable. On rearranging it, we get

$$\frac{z(ydx - xdy)}{x^2 + y^2} - \tan^{-1}\frac{y}{x}\,dz = 0 \quad \text{or} \quad \frac{1}{\tan^{-1}(y/x)}\cdot\frac{1}{1 + (y^2/x^2)}\cdot\frac{xdy - ydx}{x^2} + \frac{dz}{z} = 0.$$

Integrating, $\quad \log\tan^{-1}(y/x) + \log z = \log c \quad$ or $\quad \tan^{-1}(y/x) = c/z$

or $\quad y/x = \tan\,(c/z) \quad$ or $\quad y = x\tan\,(c/z)$, c being an arbitrary constant.

Ex. 16. *Solve* (*a*) $\quad xdx + ydy - \sqrt{(a^2 - x^2 - y^2)}\,dz = 0.$ **[Gujrat 1997]**

(*b*) $zdz + (x - a)dx = \sqrt{[h^2 - z^2 - (x - a)^2]}\,dy.$ **[Agra 1994; Rajasthan 2000, 07]**

Sol. (*a*) On rearranging, $\quad [a^2 - (x^2 + y^2)]^{-1/2}\,(2xdx + 2ydy) - 2dz = 0 \quad$ ...(1)

Putting $x^2 + y^2 = t$ so that $2x\,dx + 2y\,dy = dt$. Then (1) gives $\quad (a^2 - t)^{-1/2}\,dt - 2dz = 0.$

Integrating, $\quad -2(a^2 - t)^{1/2} - 2z = 2c \quad$ or $\quad -(a^2 - t^2)^{1/2} = z + c.$

Squaring, $a^2 - t^2 = (z + c)^2 \quad$ or $\quad t^2 + (z + c)^2 = a^2 \quad$ or $\quad x^2 + y^2 + (z + c)^2 = a^2.$

(*b*)On rearranging, $\quad [h^2 - z^2 - (x - a)^2]^{-1/2}\,[2zdz + 2(x - a)dx] = 2dy.$

Putting $\quad z^2 + (x - a)^2 = t \quad$ so that $\quad 2z\,dz + 2(x - a)dx = dt$

$\therefore \quad (h^2 - t)^{-1/2}dt = 2dy \quad$ so that $\quad -2(h^2 - t)^{1/2} = 2y - 2c$

or $\quad (h^2 - t)^{1/2} = c - y \quad$ or $\quad h^2 - t = (c - y)^2 \quad$ or $\quad t + (c - y)^2 = h^2$

or $\quad z^2 + (x - a)^2 + (c - y)^2 = h^2$, when c is an arbitrary constant.

Ex. 17. *Solve* $\dfrac{xdx + ydy + zdz}{(x^2 + y^2 + z^2)^{1/2}} + \dfrac{z\,dx - x\,dz}{x^2 + z^2} + 3a\,x^2dx + 2b\,y\,dy + c\,dz = 0.$

Sol. We notice by inspection that all the parts are exact differentials. Hence the given equation must be exact and so integrable. Integrating, we obtain

$(x^2 + y^2 + z^2)^{1/2} + \tan^{-1}(x/z) + ax^3 + by^2 + cz = k$, where k is an arbitary constant.

Ex. 18. *Solve* $(x^2 - y^2 - z^2 + 2xy + 2xz)dx + (y^2 - z^2 - x^2 + 2yz + 2yx)dy + (z^2 - x^2 - y^2 + 2zx + 2zy)dz = 0.$

Sol. Adding and subtracting x^2dx, y^2dy, z^2dz in first, second and third term respectively and simplifying, we get

$$[-(x^2 + y^2 + z^2) + 2x(x + y + z)]dx + [-(x^2 + y^2 + z^2) + 2y(x + y + z)]dy + [-(x^2 + y^2 + z^2) + 2z(x + y + z)]dz = 0$$

or $\quad -(x^2 + y^2 + z^2)\,(dx + dy + dz) + 2(x + y + z)\,(xdx + ydy + zdz) = 0$

or $$\frac{dx+dy+dz}{x+y+z} = \frac{2xdx+2ydy+2zdz}{x^2+y^2+z^2}.$$

Integrating, $\log(x+y+z) = \log(x^2+y^2+z^2) + \log c$

or $x+y+z = c(x^2+y^2+z^2)$, c being arbitrary constant.

Ex. 19. *Solve* $\dfrac{y+z-2x}{(y-x)(z-x)}dx + \dfrac{z+x-2y}{(z-y)(x-y)}dy + \dfrac{x+y-2z}{(x-z)(y-z)} = 0$

[Delhi Maths (G) 2004]

Sol. Re-writing, the given euqation yields

$$\frac{(y-x)+(z-x)}{(y-x)(z-x)}dx + \frac{(z-y)+(x-y)}{(z-y)(x-y)}dy + \frac{(x-z)+(y-z)}{(x-z)(y-z)}dz = 0$$

or $$\left(\frac{1}{z-x}+\frac{1}{y-x}\right)dx + \left(\frac{1}{x-y}+\frac{1}{z-y}\right)dy + \left(\frac{1}{y-z}+\frac{1}{x-z}\right)dz = 0$$

or $$-\frac{dx-dy}{x-y} - \frac{dy-dz}{y-z} - \frac{dz-dx}{z-x} = 0$$

Integrating, $-[\log(x-y) + \log(y-z) + \log(z-x)] = -\log c$

or $(x-y)(y-z)(z-x) = c$, c being an arbitrary constant.

Ex. 20. *Find $f(y)$ such that the total differential equation $\{(yz+z)/x\}dx - zdy + f(y)dz = 0$ is integrable. Hence solve it.* **[Indore 2000; Meerut 1994; Pune 2000]**

Sol. Multiplying throughout by x, the given equation reduces to

$$(yz+z)dx - xzdy + xf(y)dz = 0. \qquad ...(1)$$

Comparing (1) with $Pdx + Qdy + Rdz = 0$, we have

$P = yz + z$, $\quad Q = -xz \quad$ and $\quad R = xf(y)$. ...(2)

Suppose that (1) is integrable so that the following condition of integrability is satisfied by it.

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0. \qquad ...(3)$$

Using (2) and denoting df/dy by f', (3) gives

$$(yz+z)(-x-xf') - xz[f-(y+1)] + xf[z-(-z)] = 0.$$

or $xz(1+y)f' = xzf$ $\quad$ or $\quad (df)/f = (dy)/(y+1)$.

Integrating, $\log f = \log(y+1) + \log k$ or $f = k(y+1)$, where k is arbitrary constant.

Thus the required value of $f(y)$ is $k(y+1)$ Putting this value $f(y)$ in (1), we get

$$z(y+1)dx - xzdy + xk(y+1)dz = 0. \qquad ...(4)$$

Dividing by $xz(1+y)$, (4) becomes $\quad (1/x)dx - (dy)/(y+1) + (k/z)dz = 0$.

Integrating, $\log x - \log(y+1) + k\log z = \log c$. Hence $xz^k = c(y+1)$ is required solution.

Ex. 21. *Find $f(z)$ such that $[(y^2+z^2-x^2)/2x]dx - y\,dy + f(z)dz = 0$ is integrable. Hence solve it.*

[Himanchal 2001; Indore 2002; Karnataka 2002; P.C.S. (U.P) 2002; Rajasthan 2003]

Sol. Multiplying throughout by $2x$, the given equation reduces to

$$(y^2+z^2-x^2)dx - 2xydy + 2xf(z)dz = 0. \qquad ...(1)$$

Comparing (1) with $Pdx + Qdy + Rdz = 0$, we have

$P = y^2 + z^2 - x^2$, $\quad Q = -2xy$, $\quad R = 2xf(z)$. ...(2)

Suppose that (1) is integrable so that the following condition of integrability is satisfied by it.

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0. \qquad ...(3)$$

Using (2) and denoting $f(z)$ by f, (3) gives

$$(y^2+z^2-x^2)(0-0)-2xy(2f-2z)+2xf[2y-(-2y)]=0$$

or $\quad -4xy(f-z)+8xfy=0 \quad$ or $\quad f=-z.$

Putting $f(y)=-z$ in (1) and re–writing, $\quad (y^2+z^2-x^2)dx-2xydy-2xzdz=0. \qquad ...(4)$

Adding and subtracting x^2dx, (4) reduces to

$$(x^2+y^2+z^2)dx-x(2xdx+2ydy+2zdz)=0 \quad \text{or} \quad \frac{dx}{x}=\frac{2xdx+2ydy+2zdz}{x^2+y^2+z^2}$$

Integrating, $\quad \log x+\log c=\log(x^2+y^2+z^2) \quad$ or $\quad xc=x^2+y^2+z^2.$

Ex. 22. (*a*) *Solve* $x^2dx^2+y^2dy^2-z^2dz^2+2xydxdy=0.$ **[Meerut 2002, 04]**

(*b*) $dx\,dy\,dz=0$ **[Meerut 2004, 05, 06, 07]**

Sol. Re–writing, $\quad (xdx+ydy)^2-(zdz)^2=0 \quad$ or $\quad (xdx+ydy+zdz)(xdx+ydy-zdz)=0. \qquad ...(1)$

Equation (1) can be resolved into two component equations

$$xdx+ydy+zdz=0 \qquad ...(2)$$

and

$$xdx+ydy-zdz=0. \qquad ...(3)$$

Integrating (2) and (3), $\quad x^2+y^2+z^2=c_1 \quad$ and $\quad x^2+y^2-z^2=c_2.$

Hence the complete solution of the given differential equation is

$$(x^2+y^2+z^2-c_1)(x^2+y^2-z^2-c_2)=0.$$

(*b*) Given equation can be resolved in the following three component equations :

$$dx=0, \qquad dy=0 \qquad \text{and} \qquad dz=0$$

Integrating these, $\quad x-c_1=0, \quad y-c_2=0 \quad$ and $\quad z-c_3=0,$

where c_1, c_2 and c_3 are arbitrary constants. Hence the complete solution of the given equation is given by

$$(x-c_1)(y-c_2)(z-c_3)=0.$$

Ex. 23. (*i*) *Solve* $(z+z^2)\cos x\,(dx/dt)-(z+z^2)(dy/dt)+(1-z^2)(y-\sin x)(dz/dt)=0.$

(*ii*) *Solve* $(z+z^2)\cos x\,dx-(z+z^2)dy+(1-z^2)(y-\sin x)dz=0.$

Sol. (*i*) The given differential equation can be re–written as

$$(z+z^2)\cos x\,dx-(z+z^2)dy+(1-z^2)(y-\sin x)dz=0$$

or

$$(z+z^2)(\cos x\,dx-dy)+(1-z^2)(y-\sin x)dz=0$$

or

$$\frac{\cos x\,dx-dy}{y-\sin x}+\frac{1-z^2}{z+z^2}\,dz=0 \quad \text{or} \quad \frac{dy-\cos x\,dx}{y-\sin x}-\frac{1-z}{z}\,dz=0$$

Integrating, $$\log(y-\sin x)-\int\left(\frac{1}{z}-1\right)dz=\log c$$

or $\quad \log(y-\sin x)-\log z+z=\log c \quad$ or $\quad (y-\sin x)/(cz)=-z$

or $\quad y-\sin x=cz\,e^{-z}$, c being an arbitrary constant.

(*ii*) This equation is same as given in part (*i*).

Ex. 24. *Solve* $(e^xy+\cos x)dx+(e^x+e^yz)dy+e^zdz=0.$

Sol. On rearranging, $\quad (ye^xdx+e^xdy)+\cos x\,dx+(ze^ydy+e^ydz)=0.$

Integrating, $\quad ye^x+\sin x+ze^y=c$, c being an arbitrary constant.

Ex. 25. *Solve* $(2x^2y+2xy^2+2xyz+1)dx+(x^3+x^2y+x^2z+2xyz+2y^2z+2yz^2+1)dy+(xy^2+y^3+y^2z+1)dz=0.$

Sol. On rearranging the given reduces to

$$[2xy(x+y+z)+1]dx+[x^2(x+y+z)+2yz(x+y+z)+1]dy+[y^2(x+y+z)+1]dz=0$$

or $$(x + y + z)\,(2xydx + x^2dy + 2yzdy + y^2dz) + (dx + dy + dz) = 0$$

or $$(2xydx + x^2dx) + (2yzdy + y^2dz) + \frac{dx + dy + dz}{x + y + z} = 0.$$

Integrating, $x^2y + y^2z + \log\,(x + y + z) = c$, c being an arbitrary constant.

Ex. 26. (*a*) *Solve* $(2x + y^2 + 2xz)(dx/dt) + 2xy(dy/dt) + x^2(dz/dt) = 1$.

[Kolkata 2002, 05, 06; Meerut 2003, 04, Vikram 2002]

(*b*) $y(y + z)dx + z(u - x)dy + y(x - u)dz + y(y + z)du = 0$.

Sol. (*a*) Re–writing, $(2x + y^2 + 2xz)dx + 2xydy + x^2dz = dt$

or $$2xdx + (y^2dx + 2xydy) + (2xzdx + x^2dz) - dt = 0.$$

Integrating, $x^2 + xy^2 + zx^2 - t = c$, c being an arbitrary constant

(*b*) Re–writing, $(y + z)\,(zdx + ydu) + z(u - x)dy + y(x - u)dz = 0$.

Adding and subtracting $(y + z)\,(xdz + udy)$ and simplifying, we get

$$(y + z)\,(zdx + ydu + xdz + udy) - (xz + uy)\,(dy + dz) = 0$$

or $$\frac{z\,dx + y\,du + x\,dz + u\,dy}{xz + uy} = \frac{dy + dz}{y + z}.$$

Integrating, $\log\,(xz + uy) = \log\,(y + z) + \log c$ or $xz + uy = c(y + z)$.

Ex. 27. *Solve* $(y^2 + yz + z^2)dx + (z^2 + zx + x^2)dy + (x^2 + xy + y^2)dz = 0$.

[Meerut 2005, 07, 11; Garhwal 1996; Karnataka 2001, 06: Rajasthan 2005; Vikram 2001]

Sol. We have, $y^2 + yz + z^2 = (y + z)^2 - yz = (y + z)^2 + x(y + z) - (xy + yz + zx)$

$$= (y + z)\,(x + y + z) - (xy + yz + zx).$$

Similarly, $z^2 + zx + x^2 = (z + x)\,(x + y + z) - (xy + yz + zx)$

and $x^2 + xy + y^2 = (x + y)\,(x + y + z) - (xy + yz + zx)$.

Using these new forms of $y^2 + yz + z^2$ etc, the given equation becomes

$$(x + y + z)\,\{(y + z)dx + (z + x)dy + (x + y)dz\} - (xy + yz + zx)\,(dx + dy + dz) = 0$$

or $$(x + y + z)\,\{(ydx + xdy) + (ydz + zdy) + (zdx + xdz)\} - (xy + yz + zx)\,(dx + dy + dz) = 0$$

or $$(x + y + z)\,d(xy + yz + zx) = (xy + yz + zx)\,d(x + y + z)$$

or $$\frac{d(xy + yz + zx)}{xy + yz + zx} = \frac{d(x + y + z)}{x + y + z}.$$

Integrating, $\log\,(xy + yz + zx) = \log\,(x + y + z) + \log c$

or $xy + yz + xz = c(x + y + z)$, c being an arbitrary constant.

Ex. 28. *Solve* $y(x + 4)\,(y + z)\,dx - x(y + 3z)dy + 2xy\,dz = 0$. **[Delhi Maths (Hons) 2007]**

Sol. Dividing by $xy(y + z)$, the given equation reduces to

$$\frac{x+4}{x}dx + \frac{2ydz - y(y+3z)dy}{y(y+z)} = 0 \qquad \text{or} \qquad \frac{x+4}{x}dx + \frac{(2y\,dz + 2y\,dy) - 3(y+z)dy}{y(y+z)} = 0$$

or $$\left(1 + \frac{4}{x}\right)dx + 2\frac{dz + dy}{y + z} - 3\frac{dy}{y} = 0$$

Integrating, $x + 4\log x + 2\log\,(y + z) - 3\log y = \log c$

or $$\log\frac{x^4(y+z)^2}{y^3c} = -x \qquad \text{or} \qquad \frac{x^4(y+z)^2}{y^3c} = e^{-x}$$

or $e^x x^4 (y+z)^2 = cy^3$, c being an arbitrary constant.

EXERCISE 3 (A)

Solve the following differential equations :

1. **(*a*)** $(y + z)dx + (z + x)dy + (x + y)dz = 0$. **Ans.** $xy + yz + zx = c$.

   (*b*) $y\,dx + x\,dy + 2z\,dz = 0$. **[Pune 2010]** **Ans.** $xy + z^2 = c$.

2. $(y + z)dx + dy + dz = 0$. **[Agra 2001, 02]** **Ans.** $y + z = c\,e^{-x}$.
3. $(yz + 2x)dx + (zx + 2y)dy + (xy + 2z)dz = 0$. **Ans.** $x^2 + y^2 + z^2 + xyz = c$.
4. $(2x + y^2 + 2xz)dx + 2xydy + x^2dz = du$. **Ans.** $x^2 + y^2x + x^2z = u + c$.
5. $(x - y)dx - x\,dy + z\,dz = 0$. **[Delhi Maths (G) 2006]** **Ans.** $x^2 - 2xy + z^2 = c$.
6. $(2x^3y + 1)dx + x^4dy + x^2 \tan z\,dz = 0$. **Ans.** $x^2y - (1/x) + \log \sec z = c$.
7. $(y + b)(z + c)\,dx + (x + a)(z + c)\,dy + (x + a)(y + b)\,dz = 0$.

   Ans. $(x + a)(y + b)(z + c) = c$, c being an arbitrary constant

8. $(x^2+z^2)(xdx+ydy+zdz)+\sqrt{(x^2+y^2+z^2)}\,(zdx-xdz)=0$. **Ans.** $\sqrt{(x^2+y^2+z^2)}+\tan^{-1}(x/z)=c$.
9. $(2xy + z^2)dx + (x^2 + 2yz)dy + (y^2 + 2xz)dz = 0$. **Ans.** $x^2y + y^2z + z^2x = c$
10. $dx + dy + (x + y + z + 1)dz = 0$. **[Meerut 2005]** **Ans.** $(x + y + z)e^z = c$.
11. $(y - z)(y + z - 2x)dx + (z - x)(z + x - 2y)dy + (x - y)(x + y - 2z)dz = 0$.

    Ans. $y^2x - z^2x + z^2y - x^2y + x^2z - y^2z = c$.

12. $(2x^3 - z)zdx + 2x^2yzdy + x(z + x)dz = 0$. **Ans.** $x^2 + y^2 + \log z + (z/x) = c$.
13. $(y + a)^2dx + zdy - (y + a)dz = 0$. **Ans.** $(x + c)(y + a) = z$.
14. $(a + z)ydx + (a + z)xdy - xydz = 0$. **Ans.** $xy = c(a + z)$
15. $(2xyz + y^2z + z^2y)\,dx + (x^2z + 2xyz + xz^2)dy + (x^2y + xy^2 + 2xyz)dz = 0$. **Ans.** $xyz\,(x + y + z) = c$
16. $a^2y^2z^2dx + b^2z^2x^2dy + c^2x^2y^2dz = 0$ **[Himanchal 2001]** **Ans.** $a^2/x + b^2/y + c^2/z = k$
17. $y(1 + z^2)dx - x(1 + z^2)\,dy + (x^2 + y^2)\,dz = 0$ **[Kurukshetra 1998]**

    Ans. $x^2 + 4y^2 - 12\,xy - 4\,xy + 2z^2 = c$

18. $2y\,dx = 2\,xdy + y^2dz$ **[Meerut 2006]** **Ans.** $x^2 + 2yz + 2z^2 = c$
19. $yz\,dx + zx\,dy + xy\,dz = 0$ **[U.P. (P.C.S.) 2005]** **Ans.** $xyz = c$
20. $(2yz + y^2z + yz^2)dx + (x^2z + 2xy\,z + zx^2)dy + (x^2y + xy^2 + 2xyz)dz = 0$ **[Indore 2001, 02]**

    Ans. $x^2yz + xy^2z + xyz^2 = c$

21. If $\boldsymbol{X}$ is vector such that $\boldsymbol{X} \cdot \text{curl}\,\boldsymbol{X} = 0$ and μ is an arbitrary function of x, y, z then prove that $(\mu\boldsymbol{X}) \cdot \text{curl}\,(\mu\boldsymbol{X}) = 0$. **[Amravati 2005; Kerala 2002, 05, 06]**

3.8. Special Method II. Solution of homogeneous equation.

The equation $Pdx + Qdy + Rdz = 0$ is called a homogeneous equation if P, Q, R are homogeneous functions of x, y, z of the same degree.

There are two following working rules to solve such equations.

Working rule I. Solution by use of an integrating factor (I.F.)

Step 1. As usual, verify that the given equation is integrable.

Step 2. Calculate $Px + Qy + Rz$. If it is not equal to zero, then $1/(Px + Qy + Rz)$ is taken as I.F. (Integrating factor) of the given equation. Thus, I.F. = $1/D$, where $D = Px + Qy + Rz$.

Step 3. Multiply the given equation by I.F. (say $1/D$) where D denotes the denominator of I.F. Find $d(D)$ *i.e.* total differential of D. Now add and subtract $d(D)$ from the numerator. Write the given equation in the form $\dfrac{d(D) \pm \dots}{D} = 0$ or $\dfrac{d(D)}{D} \pm \dots = 0$

and then integrate. Remember that the several terms in the resulting equations will be exact differential and hence rules of Art. 3.6 have to be used while regrouping the remaining terms.

Working rule II. The first method fails when $Px + Qy + Rz = 0$. In such cases we apply the following method which is applicable to all homogeneous equations.

Step 1. Do same as done in step 1 of working rule I.

Step 2. Put $x = zu$, $y = zv$ so that $dx = udz + zdu$ and $dy = zdv + vdz$.

Substituting these in the given equation two cases may arise.

Case I. If the coefficient of dz is zero, we shall have an equation in only two variables u and v. By regrouping properly, it can be easily integrated.

Case II. If the coefficient of dz is not zero, then we shall be able to separate z from u and v. Thus the resulting equation will be of the form

$$\frac{f_1(u, v)\, du + f_2(u, v)\, dv}{f(u, v)} + \frac{dz}{z} = 0. \qquad \text{...(A)}$$

We now denote $f(u, v)$ by D and find $d(D)$. Add and subtract $d(D)$ as done in step 3 of method I. Remember all this is done only in first term. Finally we integrate. After integration u and v are replaced by x/z and y/z respectively so as to get the desired solution in x, y and z.

Note. Sometimes integration of (A) is possible without assuming D etc. Hence we use D only when it helps to integrate equation (A).

3.9. Solved examples based on homogeneous equations (see Art. 3.8)

Ex. 1. *Solve* $(yz + z^2)dx - xzdy + xydz = 0$. **[Delhi Maths (Hons) 2007; Meerut 1997 Guwahati 2007; Gorakhpur 2003, 07; Gwalior 2003; Rohilkhand 2000; Mysore 2004]**

Sol. Given $$(yz + z^2)dx - xzdy + xydz = 0. \qquad \text{...(1)}$$

Comparing the given equation with $Pdx + Qdy + Rdz = 0$, we have

here $P = yz + z^2$, $Q = -xz$, $R = xy$ and let $D = Px + Qy + Rz$.

The condition of integrability is satisfied. (Do yourself)

Now, $$D = x(yz + z^2) - xyz + xyz = xz(y + z) \neq 0. \qquad \text{...(2)}$$

Multiplying (1) by integrating factor $1/D$, $$\frac{(yz + z^2)dx - xzdy + xydz}{D} = 0. \qquad \text{...(3)}$$

Now, $$d(D) = d[xz(y + z)] = (zdx + xdz)\,(y + z) + xz(dy + dz)$$

or $$d(D) = z(y + z)dx + x(y + 2z)dz + xz\,dy. \qquad \text{...(4)}$$

Re–writting, the numerator of (3)

$$= d(D) - d(D) + (yz + z^2)dx - xzdy + xydz = d(D) - 2xz(dy + dz), \text{ by (4).}$$

$\therefore$ (3) becomes $$\frac{d(D)}{D} - \frac{2xz(dy + dz)}{D} = 0$$

or $$\frac{d(D)}{D} - \frac{2xz(dy + dz)}{xy(y + z)} = 0 \quad \text{so that} \quad \frac{d(D)}{D} - \frac{2(dy + dz)}{y + z} = 0.$$

Integrating, $\log D - 2\log(y + z) = \log c$ or $D = c(y + z)^2$

or $xz(y + z) = c(y + z)^2$ or $xz = c(y + z)$,

which is the required solution, c being an arbitrary constant.

Alternative Method. Verify the usual condition of integrability.

Let $x = uz$ and $y = \ z$ so that $dx = u\,dz + z\,du$ and $dy = \ dz + z\,d\ $.

Putting these values of x, y, dx, dy in (1), we get

$$(\ z^2 + z^2)\,(u\,dz + z\,du) - uz^2(\ dz + z\,d\ ) + \ uz^2\,dz = 0$$

$$(\ + 1)z^3\,du - uz^3 d\ + (\ + 1)uz^2 dz = 0.$$

Dividing by $u\ (\ \ + 1)z^3$, we get $$\frac{du}{u} - \frac{d}{v+1} + \frac{dz}{z} = 0.$$

Integrating, $\log u - \log(\ \ + 1) + \log z = \log c$ or $uz = c(\ \ + 1)$

or $(x/z) \times z = c(1 + y/z)$ or $xz = c(y + z)$, as $u = x/z$ and $\ = y/z$.

Ex. 2. *Solve* $z^2dx + (z^2 - 2yz)dy + (2y^2 - yz - xz)dz = 0$. **[Meerut 1993]**

Sol. Verify yourself the condition of integrability. Since the given equation is homogeneous, we put

$x = uz$ and $y = \ z$, so that $dx = z\,du + u\,dz$ and $dy = z\,d\ \ + \ \ dz$. ...(1)

Putting these in the given equation, we get

$$z^2(z\,du + u\,dz) + z^2(1 - 2\ \ )(z\,d\ \ + \ \ dz) + (2\ \ ^2 - \ \ - u)z^2\,dz = 0$$

or $$z^3\,du + z^3(1 - 2\ \ )d\ \ + [u + \ \ (1 - 2\ \ ) + 2\ \ ^2 - \ \ - u]z^2\,dz = 0$$

or $$z^3\,du + z^3(1 - 2\ \ )d\ \ + (0 \times z^2)dz = 0 \quad \text{or} \quad du + (1 - 2\ \ )d\ \ = 0.$$

Integrating, $u + \ \ - \ \ ^2 = c$ or $(x/z) + (y/z) - (y^2/z^2) = c$, by (1)

or $(x + y)z - y^2 = cz^2$, c being an arbitrary constant.

Ex. 3. $2(y + z)dx - (x + y)dy + (2y - x + z)dz = 0$. **[Bangalore 2003, 07]**

Sol. The condition of integrability is satisfied. (Do yourself).

Since the given equation is homogeneous, we put

$x = uz$ and $y = \ z$ so that $dx = zdu + udz$ and $dy = zd\ \ + \ \ dz$. ...(1)

Putting these in the given equation, we get

$$2(\ \ + 1)z(zdu + udz) - (u + 1)\,z\,(zd\ \ + \ \ dz) + (2\ \ - u + 1)\,zdz = 0$$

or $$2z^2(\ \ + 1)du - z^2(u + 1)d\ \ + [2u(\ \ + 1) - \ \ (u + 1) + (2\ \ - u + 1)]zdz = 0$$

or $$z^2[2(\ \ + 1)du - (u + 1)d\ \ ] + (u\ \ + u + \ \ + 1)zdz = 0$$

or $$z^2[2(\ \ + 1)du - (u + 1)d\ \ ] + (u + 1)\,(\ \ + 1)\,dz = 0.$$

Dividing by $z^2(u + 1)(v + 1)$, $$2\frac{du}{u+1} - \frac{d}{\ \ +1} + \frac{dz}{z} = 0.$$

Integrating, $2\log(u + 1) - \log(\ \ + 1) + \log z = \log c$

or $(u + 1)^2z = c(\ \ + 1)$ or $(x/z + 1)^2\,z = c(y/z + 1)$, by (1)

i.e. $(x + z)^2 = c(x + z)$, which is the required solution, c being an arbitrary constant.

Ex. 4. $yz(y + z)dx + zx(x + z)dy + xy(x + y)dz = 0$. **[Delhi Maths Hons. 1990; Himanchal 2002; Kurukshetra 2000, 01; Delhi B.Sc. (Prog) II 2011]**

Sol. As usual verify the condition of integrability. Since the given equation is homogeneous,

Put $x = uz$ and $y = \ z$ so that $dx = zdu + udz$ and $dy = zd\ \ + \ \ dz$. ...(1)

Putting these in the given equation, we get

$$(\ \ + 1)z^3\,(zdu + udz) + u(u + 1)z^3\,(zd\ \ + \ \ dz) + u\ \ (u + \ \ )z^3dz = 0$$

or $$[\ \ (\ \ + 1)du + u(u + 1)d\ \ ]z^4 + [u\ \ (\ \ + 1) + u\ \ (u + 1) + u\ \ (u + \ \ )]z^3dz = 0$$

or $$[\ \ (\ \ + 1)du + u(u + 1)d\ \ ]z^4 + 2u\ \ (u + \ \ + 1)z^3\,dz = 0$$

Dividing by $u\ \ (u + \ \ + 1)z^4$, we get $$\frac{(\ \ +1)du}{u(u+\ \ +1)} + \frac{(u+1)d}{(u+\ \ +1)} + 2\frac{dz}{z} = 0.$$

or $$\left(\frac{1}{u} - \frac{1}{u+\ \ +1}\right)du + \left(\frac{1}{\ } - \frac{1}{u+\ \ +1}\right)d\ \ + 2\frac{dz}{z} = 0 \quad \text{or} \quad \frac{du}{u} + \frac{d}{\ } - \frac{du + d}{u+\ \ +1} + 2\frac{dz}{z} = 0.$$

Integrating, $\log u + \log\ \ - \log(u + \ \ + 1) + 2\log z = \log c$

or $u\ \ z^2 = c(u + \ \ + 1)$ or $(x/z) \times (y/z) \times z^2 = c(x/z + y/z + 1)$, by (1)

i.e. $xyz = c(x + y + z)$, which is the required solution, c being an arbitrary constant.

Ex. 5. *Solve* $(y^2 + yz)dx + (xz + z^2)dy + (y^2 - xy)dz = 0$. **[Delhi B.Sc. (Prog) II 2009]**
[I.A.S. 1999; Delhi Maths (G) 1993; Garhwal 1994; Meerut 2002; Rohilkhand 1995; Calicut 2000; G.N.D.U. Amritsar 2002; Gwalior 2004; Lucknow 2001, 04; Mumbai 2004; Nagarjuna 2000, 04, 07; Sagar 2001, 04; Vikram 2000]

Sol. As usual, verify that the given equation is integrable.

Since the given equation is homogeneous, we put

$x = uz$ and $y = \;\; z$ so that $dx = zdu + udz$ and $dy = zd \;\; + \;\; dz$. ...(1)

Putting these in the given equation, we get

$$(\;^2 + \;\;)z^2 (zdu + udz) + (u + 1)z^2 (zd \;\; + \;\; dz) + (\;^2 - u \;\;)z^2 \, dz = 0$$

or $$(\;^2 + \;\;)z^3 du + (u + 1)z^3 \, d \;\; + [u(\;^2 + \;\;) + \;\; (u + 1) + \;^2 - u \;\;]z^2 \, dz = 0$$

or $$z^3[(\;^2 + \;\;)du + (u + 1)d \;\;] + (u \;^2 + u \;\; + \;\; + \;^2)z^2 \, dz = 0$$

or $$z^3[(\;^2 + \;\;)du + (u + 1)d \;\;] + (u + 1)(\;^2 + \;\;)z^2 \, dz = 0.$$

Dividing by $z^3(u + 1)(\;^2 + \;\;)$, we get

$$\frac{(\;^2 + \;\;)du + (u+1)d \;\;}{(u+1)(\;^2 + \;\;)} + \frac{dz}{z} = 0 \quad \text{or} \quad \frac{du}{u+1} + \frac{d \;\;}{\;^2 + \;\;} + \frac{dz}{z} = 0$$

or $$\frac{du}{u+1} + \left[\frac{1}{\;\;} - \frac{1}{\;\;+1}\right] d \;\; + \frac{dz}{z} = 0, \text{ by resolving into partial fractions}$$

Integrating, $\log (u + 1) + \log \;\; - \log (\;\; + 1) + \log z = \log c$

or $(u + 1) \;\; z = c(\;\; + 1)$ or $(x/z + 1) \times (y/z) \times z = c(y/z + 1)$, by (1)

or $(x + z)y = c(y + z)$, which is the required solution, c being an arbitrary constant.

Ex. 6. *Solve* $(2xz - yz)dx + (2yz - xz)dy - (x^2 - xy + y^2)dz = 0$. **[Agra 2002, 03; Garhwal 2005; Delhi Maths (H) 2005; Delhi Maths (G) 1998; Kolkata 2002]**

Sol. As usual, verify that the given equation is integrable.

Since the given equation is homogeneous, we put

$x = uz$ and $y = \;\; z$ so that $dx = zdu + udz$ and $dy = zd \;\; + \;\; dz$ ...(1)

Putting these in the given equation, we get

$$(2uz^2 - \;\; z^2)(udz + zdu) + (2 \;\; z^2 - uz^2)(\;\; dz + zd \;\;) - (u^2z^2 - u \;\; z^2 + \;^2 z^2)dz = 0$$

or $$(2u - \;\;)(udz + zdu) + (2 \;\; - u)(\;\; dz + zd \;\;) - (u^2 - u \;\; + \;^2)dz = 0$$

or $$z[(2u - \;\;)du + (2 \;\; - u)d \;\;] + [u(2u - \;\;) + \;\; (2 \;\; - u) - (u^2 - u \;\; + \;^2)]dz = 0$$

or $z[2udu - (ud \;\; + \;\; du) + 2 \;\; d \;\;] + (u^2 - u \;\; + \;^2)dz = 0$ or $z[du^2 - d(u \;\;) + d \;^2] + (u^2 - u \;\; + \;^2)dz = 0$

or $$\frac{d(u^2 - u \;\; + \;^2)}{u^2 - u \;\; + \;^2} + \frac{dz}{z} = 0.$$

Integrating, $\log (u^2 - u \;\; + \;^2) + \log z = \log c$ or $z(u^2 - u \;\; + \;^2) = c$

or $$z\left(\frac{x^2}{z^2} - \frac{x}{z} \times \frac{y}{z} + \frac{y^2}{z^2}\right) = c \quad \text{or} \quad x^2 - xy + y^2 = cz.$$

Ex. 7. *Solve* (*a*) $z(z - y)dx + z(z + x)dy + x(x + y)dz = 0$. **[Meerut 1996]**

(*b*) $y(y + z)dx + x(x - z)dy + x(x + y)dz = 0$.

Sol. (*a*) As usual, verify that the given equation is integrable.

Comparing the given equation with $Pdx + Qdy + Rdz = 0$, we get

$P = z(z - y)$, $Q = z(z + x)$ and $R = x(x + y)$. ...(1)

The given equation is homogeneous and we have

$Px + Qy + Rz = zx(z - y) + zy(z + x) + zx(x + y)$, using (1)

$$= z(xz - xy + yz + xy + x^2 + xy) = z(xz + yz + x^2 + xy) = z(z + x)\,(y + x) \neq 0,$$

showing that I.F. of the given equation $= \dfrac{1}{Px + Qy + Rz} = \dfrac{1}{z(z + x)(x + y)}$.

Multiplying the given equation by the above I.F., we have

$$\frac{(z - y)dx}{(x + y)\,(z + x)} + \frac{dy}{x + y} + \frac{xdz}{z(z + x)} = 0$$

or $$\left(\frac{1}{x + y} - \frac{1}{x + z}\right) dx + \frac{dy}{x + y} + \left(\frac{1}{z} - \frac{1}{z + x}\right) dz = 0 \quad \text{or} \quad \frac{dx + dy}{x + y} - \frac{dx + dz}{x + z} + \frac{dz}{z} = 0.$$

Integrating, $\qquad \log\,(x + y) - \log\,(x + z) + \log z = \log c$

or $\qquad z(x + y) = c(x + z)$, c being an arbitrary constant.

(*b*) Proceed as in part (*a*). **Ans.** $x(y + z) = c(x + y)$

Ex. 8. *Solve* $(y^2z - y^3 + x^2y)dx - (x^2z + x^3 - xy^2)dy + (x^2y - xy^2)dz = 0.$

Sol. As usual, verify that the given equation is integrable.

Since the given equation is homogeneous, we put

$x = uz \quad$ and $\quad y = \ \ z \quad$ so that $\quad dx = zdu + udz \quad$ and $\quad dy = zd \ \ + \ \ dz.$ ...(1)

Putting these in the given equation, we get

$$(\ \ ^2z^3 - \ \ ^3z^3 + u^2\ \ z^3)\,(udz + zdu) - (u^2z^3 + u^3z^3 - u\ \ ^2z^3)(\ \ dz + zd\ \ ) + (u^2vz^3 - u\ \ ^2z^3)dz = 0$$

or $$(\ \ ^2 - \ \ ^3 + u^2\ \ )\,(udz + zdu) - (u^2 + u^3 - u\ \ ^2)\,(\ \ dz + zd\ \ ) + (u^2\ \ - u\ \ ^2)dz = 0$$

or $$z[(\ \ ^2 - \ \ ^3 + u^2\ \ )du - (u^2 + u^3 - u\ \ ^2)d\ \ ] + [u(\ \ ^2 - \ \ ^3 + u^2\ \ ) - \ (u^2 + u^3 - u\ \ ^2) + (u^2\ \ - u\ \ ^2)]dz = 0$$

or $$z[(\ \ ^2 - \ \ ^3 + u^2\ \ )du - (u^2 + u^3 - u\ \ ^2)d\ \ ] + 0 \times dz = 0 \text{ or } (\ \ - \ \ ^2 + u^2)\ du - (u + u^2 - \ \ ^2)ud\ \ = 0$$

or $$\ \ ^2du + (u^2 - \ \ ^2)\ du - u^2d\ \ - (u^2 - \ \ ^2)ud\ \ = 0 \quad \text{or} \quad \ \ ^2du - u^2d\ \ + (u^2 - \ \ ^2)\,(\ \ du - ud\ \ ) = 0$$

or $$\frac{du}{u^2} - \frac{d\ \ }{\ \ ^2} + \left(\frac{u^2 - \ \ ^2}{u^2}\right)\left(\frac{\ \ du - ud\ \ }{\ \ ^2}\right) = 0, \text{ on dividing by } u^2\ \ ^2$$

or $$u^{-2}du - \ \ ^{-2}d\ \ + \{1 - (u/\ \ )^{-2}\}\,d\{u/\ \ \} = 0 \quad \text{or} \quad u^{-2}du - \ \ ^{-2}d\ \ + \{1 - t^{-2}\}dt = 0, \text{ taking } t = u/\ \ .$$

Integrating, $\ -u^{-1} + \ \ ^{-1} + t + t^{-1} = c \qquad$ or $\qquad -(1/u) + (1/\ \ ) + (u/\ \ ) + (u/\ \ )^{-1} = c$

or $\qquad -(z/x) + (z/y) + (x/y) + (y/x) = c$, using (1)

or $\qquad x^2 + y^2 - zy + xz = cxy$, c being an arbitrary constant.

EXERCISE 3 (B)

Solve the following differential equations :

1. $(x^2y - y^3 - y^2z)dx + (xy^2 - x^2z - x^3)dy + (xy^2 + x^2y)dz = 0.$ **[Rohilkhand 1997]**

[**Hint :** Do as in Ex. 8 of Art. 3.9. For solution by inspection see Ex. 8(*b*) of Art. 3.7]

Ans. $x^2 + y^2 + z(x + y) = cxy$

2. $(x - y)dx - xdy + zdz = 0.$ **[Meerut 2011]** **Ans.** $x^2 - 2xy + z^2 = c$

[**Hint :** Do as in Ex. 3 of Art 3.9. We now solve it by inspection also.

Re–writing, $\qquad xdx - (ydx + xdx) + zdz = 0 \qquad$ or $\qquad 2xdx - 2d(xy) + 2zdz = 0.$

Integrating, $x^2 - 2xy + z^2 = c$, c being an arbitrary constant.]

3. $yz^2(x^2 - yz)dx + x^2z(y^2 - xz)dy + xy^2(z^2 - xy)dz = 0.$ **[Meerut 1997]**

[**Hint :** Do as in Ex. 8 of Art. 3.9. For solution be inspection see Ex. 8(a) of Art 3.7.]

Ans. $x^2z + y^2x + z^2y = cxyz$

4. $(x^2 - y^2 - z^2 + 2xy + 2xz)dx + (y^2 - z^2 - x^2 + 2yz + 2yx)dy + (z^2 - x^2 - y^2 + 2xz + 2zy)dz = 0.$

[**Hint :** Put $x = uz$ and $y = vz$. Alternatively, solution can be obtained by inspection. For solution refer Ex. 18 of Art. 3.7] **Ans.** $x^2 + y^2 + z^2 = c(x + y + z)$

5. $(2z^2 - xy + y^2)zdx + (2z^2 + x^2 - xy)zdy - (x + y)(xy + z^2)dz = 0.$ **Ans.** $(x + y)^2 z = c(z^2 - xy)$

6. $2(2y^2 + yz - z^2)dx + x(4y + z)dy + x(y - 2z)dz = 0.$ **Ans.** $x^2(2y^2 + yz - z^2) = c$

3.10. Special Method III. Use of auxiliary equations.

Let
$$Pdx + Qdy + Rdz = 0 \qquad ...(1)$$
be the given equation. Its condition of integrability is
$$P(\partial Q/\partial z - \partial R/\partial y) + Q(\partial R/\partial x - \partial P/\partial z) + R(\partial P/\partial y - \partial Q/\partial x) = 0. \qquad ...(2)$$
Comparing (1) and (2), we obtain simultaneous equations, known as auxiliary equations
$$\frac{dx}{\partial Q/\partial z - \partial R/\partial y} = \frac{dy}{\partial R/\partial x - \partial P/\partial z} = \frac{dz}{\partial P/\partial y - \partial Q/\partial x}. \qquad ...(3)$$
Equations (3) are solved by methods of chapter 2. Let $u = c_1$ and $\quad = c_2$ be two integrals so obtained. With these formulate the following equation
$$A\,du + B\,d\quad = 0. \qquad ...(4)$$
Compare (1) and (4) and thus get values of A and B. Put these values of A and B in (4) and then integrate the resulting equation. Now substitute the values of u and in the relation after integration. We thus obtain the required general solution.

Note 1. Method III discussed in Art 3.10 will fail in case the given equation (1) in exact, *i.e.*, when $\partial Q/\partial z = \partial R/\partial y$, $\partial R/\partial x = \partial P/\partial z$ and $\partial P/\partial y = \partial Q/\partial x$.

Note 2. This method is generally applied when solution by method I of Art. 3.6 and method II of Art. 3.8 are not convenient.

3.11. Solved examples based on special method III of Art. 3.10.

Ex. 1. *Solve* $xz^3dx - zdy + 2ydz = 0$.

Sol. Given
$$xz^3dx - zdy + 2ydz = 0. \qquad ...(1)$$
Comparing (1) with $Pdx + Qdy + Rdz = 0$, here $P = xz^3$, $Q = -z$, $R = 2y$. ...(2)

The auxiliary equations of the given equation are
$$\frac{dx}{\partial Q/\partial z - \partial R/\partial y} = \frac{dy}{\partial R/\partial x - \partial P/\partial z} = \frac{dz}{\partial P/\partial y - \partial Q/\partial x}.$$
or $\dfrac{dx}{-1-2} = \dfrac{dy}{0 - 3xz^2} = \dfrac{dz}{0}$, using (2) or $\dfrac{dx}{1} = \dfrac{dy}{xz^2} = \dfrac{dz}{0}$. ...(3)

Taking the third member of (3), we have

$dz = 0$ so that $z = c_1 = u$, say ...(4)

Taking the first and 2nd members of (3), we have

$xz^2dx - dy = 0$ or $2xu^2dx - 2dy = 0$, using (4)

Integrating, $x^2u^2 - 2y = c_2 = \quad$, say or $x^2z^2 - 2y = \quad$, using (4) ...(5)

Substituting the value of u and from (4) and (5) in $Adu + Bd\quad = 0$, we get

$Adz + Bd(x^2z^2 - 2y) = 0$ or $Adz + B(2xz^2dx + 2x^2zdz - 2dy) = 0$

or
$$2Bxz^2dx - 2Bdy + (A + 2Bx^2z)dz = 0. \qquad ...(6)$$
Comparing (1) and (6), $xz^3 = 2Bxz^2$, $-z = -2B$, $2y = A + 2Bx^2z$.

$\therefore$ $B = z/2$ and $A = 2y - 2Bx^2z = 2y - x^2z^2$

or $B = u/2$ and $A = -\quad$, using (4) and (5).

Substituting these values of A and B in $Adu + Bd\quad = 0$, we get

$- \quad du + (1/2) \times u d \quad = 0$ or $(1/\ )d \quad = (2/u)du.$

Integrating, $\log \quad = 2 \log u + \log c$ or $\quad = cu^2.$ ...(7)

Putting the values of u and from (4) and (5) in (7), we get

$x^2z^2 - 2y = cz^2$, which is the required general solution.

Ex. 2. *Solve* $(2xz - yz)dx + (2yz - zx)dy - (x^2 - xy + y^2)dz = 0.$ **[Delhi Maths (G) 1998]**

Sol. Given $(2xz - yz)dx + (2yz - zx)dy - (x^2 - xy + y^2)dz = 0.$ ...(1)

Comparing (1) with $Pdx + Qdy + Rdz = 0$, we have

$P = 2xz - yz, \qquad Q = 2yz - zx, \qquad R = -(x^2 - xy + y^2)$ ...(2)

The auxiliary equations of the given equation are

$$\frac{dx}{\partial Q/\partial z - \partial R/\partial y} = \frac{dy}{\partial R/\partial x - \partial P/\partial z} = \frac{dz}{\partial P/\partial y - \partial Q/\partial x}$$

or $$\frac{dx}{(2y-x)-(x-2y)} = \frac{dy}{(-2x+y)-(2x-y)} = \frac{dz}{-z-(-z)}, \text{ using (2)}$$

or $$\frac{dx}{2(2y-x)} = \frac{dy}{2(y-2x)} = \frac{dz}{0}. \qquad ...(3)$$

Taking the third member of (3), we have

$dz = 0$ so that $z = c_1 = u$, say ...(4)

Taking the first and second members of (3), we have

$(y - 2x)dx = (2y - x)dy$ or $ydx + xdy - 2xdx - 2ydy = 0.$

Integrating, $xy - x^2 - y^2 = c_2 = \quad$, say ...(5)

We now proceed to determine two functions A and B in such a manner so that the given equation (1) becomes idential with $A\,du + B\,d \quad = 0.$ ...(6)

Using (4) and (5), (6) reduces to

$Adz + Bd(xy - x^2 - y^2) = 0$ or $Adz + B(xdy + ydx - 2xdx - 2ydy) = 0$

or $(By - 2xB)dx + (xB - 2yB)dy + Adz = 0.$ ...(7)

Comparing (7) with (1), we have

$B(y - 2x) = z(2x - y), \quad B(x - 2y) = z(2y - x)$ and $A = -(x^2 - xy + y^2),$

giving $B = -z = -u$ and $A = \quad$, using (4) and (5)

Putting these values of A and B in (6), we have

$du - ud \quad = 0$ or $(1/\ )d \quad - (1/u)du = 0.$

Integrating, $\log \quad - \log u = \log c$ or $\quad = cu$

or $xy - x^2 - y^2 = cz$, by (4) and (5), c being an arbitrary constant.

Ex. 3. *Solve* $(y^2 + yz + z^2)dx + (z^2 + zx + x^2)dy + (x^2 + xy + y^2)dz = 0.$ **[Garhwal 1996]**

Sol. Given $(y^2 + yz + z^2)dx + (z^2 + zx + x^2)dy + (x^2 + xy + y^2)dz = 0.$ ...(1)

Comparing (1) with $Pdx + Qdy + Rdz = 0$, we have

$P = y^2 + yz + z^2, \qquad Q = z^2 + zx + x^2$ and $R = x^2 + xy + y^2.$ ...(2)

The auxiliary equations of the given equation are

$$\frac{dx}{\partial Q/\partial z - \partial R/\partial y} = \frac{dy}{\partial R/\partial x - \partial P/\partial z} = \frac{dz}{\partial P/\partial y - \partial Q/\partial x}$$

$$\frac{dx}{(x+2z)-(x+2y)} = \frac{dy}{(2x+y)-(y+2z)} = \frac{dz}{(2y+z)-(2x+z)}.$$

or $$\frac{dx}{z-y} = \frac{dy}{x-z} = \frac{dz}{y-x}. \qquad ...(3)$$

Each member of (3) $= \dfrac{dx + dy + dz}{z - y + x - z + y - x} = \dfrac{dx + dy + dz}{0}$

so that $$dx + dy + dz = 0.$$

Integrating, $$x + y + z = c_1 = u, \text{ say} \qquad ...(4)$$

Again, using multipliers $z + y$, $x + z$, $y + x$, each member of (3)

$$= \frac{(z+y)dx + (x+z)dy + (y+x)dz}{(z+y)(z-y) + (x+z)(x-z) + (y+x)(y-x)}$$

$$= \frac{(xdy + ydx) + (ydz + zdy) + (zdx + xdz)}{0} = \frac{d(xy) + d(yz) + d(zx)}{0}$$

so that $$d(xy) + d(yz) + d(zx) = 0.$$

Integrating, $$xy + yz + zx = c_2 = \ \ , \text{ say} \qquad ...(5)$$

We now proceed to determine two functions A and B in such a manner so that the given equation (1) becomes identical with $$A\,du + B\,d\ \ = 0. \qquad ...(6)$$

Using (4) and (5), (6) reduces to

$$Ad(x + y + z) + Bd(xy + yz + zx) = 0$$

or $$A(dx + dy + dz) + B(ydx + xdy + zdy + ydz + xdz + zdx) = 0$$

or $$\{A + B(y + z)\}dx + \{A + B(z + x)\}dy + \{A + B(x + y)\}dz = 0. \qquad ...(7)$$

Comparing (7) with (1), we have

$$A + B(y + z) = y^2 + yz + z^2 \qquad ...(8)$$

$$A + B(z + x) = z^2 + xz + x^2 \qquad ...(9)$$

$$A + B\,(x + y) = x^2 + xy + y^2 \qquad ...(10)$$

Subtracting (9) from (8), we get $B(y - x) = y^2 - x^2 + z(y - x) = (y - x)\,(x + y + z)$

$\therefore$ $$B = x + y + z = u, \text{ by } (4) \qquad ...(11)$$

From (8), $A = y^2 + yz + z^2 - B(y + z) = y^2 + yz + z^2 - (x + y + z)\,(y + z)$, by (1)

$= y^2 + yz + z^2 - (y^2 + z^2 + 2yz + xy + zx)$

Thus, $A = -\,(xy + yz + zx)$ or $A = -\ \ $, by (5) ...(12)

Using (11) and (12), (6) becomes

$-\ \ du + ud\ \ = 0$ or $(1/\ \ )d\ \ = (1/u)du.$

Integrating, $\log\ \ = \log u + \log c$ or $\ \ = cu$

or $$xy + yz + zx = c(x + y + z), \text{ by (4) and (5).}$$

Ex. 4. (*a*) *Solve* $(y^2 + yz)dx + (xz + z^2)dy + (y^2 - xy)dz = 0$.

[Delhi Maths (G) 1993; Garhwal 1994; Meerut 1997, 1998; Rohilkhand 1995]

(*b*) $z(z - y)dx + (z + x)zdy + x(x + y)dz = 0$. **[Meerut 1996]**

Sol. Try yourself. **Ans.** (*a*) $(x + z)y = c(y + z)$; (*b*) $z(x + y) = c(x + y)$

Ex. 5. *Solve* $3x^2dx + 3y^2dy - (x^3 + y^3 + e^{2z})dz = 0$. **Ans.** $x^3 + y^3 = e^{2z} + ce^z$

3.12. General method of solving Pdx + Qdy + Rdz = 0 by taking one variable as constant.

Step 1. First verify the condition of integrability.

Step 2. We now treat one of the variables, say z, as a constant *i.e.* $dz = 0$, then the resulting equations is reduced to $$Pdx + Qdy = 0. \qquad ...(1)$$

We should select a proper variable to be constant so that the resulting equation in the remaining variables is easily integrable. Thus this selection will vary from problem to problem. The present discussion is for the choice z = constant. For other cases the necessary changes have to be made in the entire procedure.

Step 3. Let the solution of (1) by $u(x, y) = f(z)$, where $f(z)$ is an arbitrary function of z. Note that in place of taking merely an absolute constant, we have taken $f(z)$. This is possible because the arbitrary function $f(z)$ is constant with respect to x and y. This is in keeping with out starting assumption, namely z = constant. Thus the solution of (1) is of the from

$$u(x, y) = f(z). \qquad ...(2)$$

Step 4. We now differentiate (2) totally with respect to x, y, z and then compare the result with the given equation $Pdx + Qdy + Rdz = 0$.

After comparing we shall get an equation in two variables f and z. If the coefficient of df or dz involve functions of x and y, it will always be possible to remove them by using (2).

Step 5. Solve the equation got in step 4 and obtain f. Putting this value of f in (2), we shall get the requried solution of the given equation.

Remarks. Many questions can be solved by this method, but the method may become tedius in some problems. Number of exercises based on methods I, II and III can be done by this method. We shall apply this method whenever there is no difficulty in solving the equation obtained by treating one variable as constant.

3.13. Solved examples based on Art. 3.12.

Ex. 1 (*a*). *Solve* $3x^2dx + 3y^2dy - (x^3 + y^3 + e^{2z})dz = 0$. **[Delhi Maths (G) 1993; Meerut 2007]**

Sol. Given $$3x^2dx + 3y^2dy - (x^3 + y^3 + e^{2z})dz = 0. \qquad ...(1)$$

As usual, verify that (1) satisfies the condition of integrability.

Let z be treated as constant, so that $dz = 0$. Then (1) becomes

$$3x^2dx + 3y^2dy = 0. \qquad ...(2)$$

Integrating (2) $$x^3 + y^3 = f(z), \text{ say}, \qquad ...(3)$$

where the constant of integration has been taken as a function $f(z)$ of z because we have treated z as constant.

Differentiating (3), $$3x^2dx + 3y^2dy - f'(z)dz = 0. \qquad ...(4)$$

Comparing (4) with (1), we have

$$f'(z) = x^3 + y^3 + e^{2z} \qquad \text{or} \qquad f'(z) = f(z) + e^{2z}, \text{ by (2)}$$

or $(df/dz) - f = e^{2z}$, which is a linear equation. Its I.F. $= e^{\int(-1)dz} = e^{-z}$ and hence its solution is

$$fe^{-z} = \int [e^{2z} . e^{-z}]\, dz + c = \int e^z + c = e^z + c$$

or $f(z) = e^{2z} + ce^z$ or $x^3 + y^3 = e^{2z} + ce^z$, using (3),

which is the required solution of (1), c being an arbitrary constant.

Ex. 1 (*b*). *Verify that the condition of integrability for the differential equation* $(2x + y + 2xz)dx + 2xy\, dy + x^2dz = 0$. *Also solve it.* **[Delhi Maths (Prog.) 2007]**

Sol. Given $$(2x + y^2 + 2xz)dx + 2xydy + x^2dz = 0 \qquad ... (1)$$

For the first part, refer Ex. 1 (*b*) of Art. 3.7

Second Part: Let x be treated as constant so that $dx = 0$.

Then, (1) reduces to $2xy\, dy + x^2dz = 0$ or $2y\, dy + x\, dz = 0$

Integrating it, $$y^2 + xz = f(x) \qquad ... (2)$$

where $f(x)$ is taken as constant of integration as x is treated as constant. Differentiating (2), we have

$$2y\, dy + x\, dz + \text{z}dx = f'dx \qquad \text{or} \qquad x(\text{z} - f')\, dx + 2xy\, dy + x^2dz = 0 \qquad ... (3)$$

Comparing (3) with (1), we get $2x + y^2 + 2xz = xz - xf'$

or $2x + y^2 + xz = -xf'$ or $2x + f = -xf'$, using (2)

or $x(df/dx) + f = -2x$ or $(df/dx) + (1/x) \times f = -2 \qquad ... (4)$

which is a linear equation whose $I.F. = e^{\int (1/x)dx} = e^{\log x} = x.$

$\therefore$ Solution of (4) is $\quad xf = \int(-2x)\,dx + c \quad$ or $\quad xf = -x^2 + c$

or $\quad x(y^2 + xz) = -x^2 + c \quad$ or $\quad x^2 + xy^2 + x^2z = c$, using (2)

which is the equired solution, c being an orbitrary constant.

Ex. 2. *Solve* $(2x^2 + 2xy + 2xz^2 + 1)dx + dy + 2zdz = 0$.

[Gwalior 2005; Indorer 1999; Jiwaji 2004; Punjab 2001, 05; Rajastan 2006; Sager 2002; Bangalore 1995; Delhi Maths (Hons) 2008; Meerut 1997]

Sol. Given $\quad (2x^2 + 2xy + 2xz^2 + 1)dx + dy + 2zdx = 0. \quad$ ...(1)

As usual, verify that (1) is integrable. Let x be treated as constant, so that $dx = 0$. Then (1) becomes $\quad dy + 2zdz = 0. \quad$ ...(2)

Integrating (1), $\quad y + z^2 = f(x)$, (say) $\quad$ ...(3)

where the constant of integration has been taken as a function $f(x)$ of x because we have treated x as constant

Differentiating (3), $\quad dy + 2zdz - f'(x)dx = 0. \quad$ ...(4)

Comparing (4) with (1), $\quad f'(x) = -(2x^2 + 2xy + 2xz^2 + 1)$

or $\quad f'(x) = -(2x^2 + 1) - 2x(y + z^2) = -(2x^2 + 1) - 2xf(x)$, using (3).

or $\quad (df/dx) + 2xf = -(2x^2 + 1)$, which is a linear equation. Its I.F. $= e^{\int(2x)dx} = e^{x^2}$ and hence its solution is

$$f\,e^{x^2} = \int e^{x^2}\{-(2x^2 + 1)\}dx + c = -\int x(2xe^{x^2})\,dx - \int e^{x^2}\,dx + c. \qquad ...(5)$$

Now, $\quad \int 2x\,e^{x^2}\,dx = \int e^t\,dt = e^t = e^{x^2}$, putting $x^2 = t$ and $2xdx = dt \quad$ ...(6)

Applying the formula of integration by parts in first term on R.H.S. of (5) and noting that $\int 2x\,e^{x^2}\,dx = e^{x^2}$ as proved in (6), we have

$$f\,e^{x^2} = -\left[xe^{x^2} - \int(1.e^{x^2})dx\right] - \int e^{x^2}\,dx + c = -\,xe^{x^2} + c$$

or $\quad f(x) = -x + c\,e^{-x^2} \quad$ or $\quad y + z^2 = -x + c\,e^{-x^2}$, by (3).

Ex. 3. *Solve* $(x^2 + y^2 + z^2)dx - 2xy\,dy - 2xz\,dz = 0$.

[Delhi Maths (H) 1999; Delhi Maths (G) 2005]

Sol. Given $\quad (x^2 + y^2 + z^2)dx - 2xy\,dy - 2xz\,dz = 0 \quad$ ...(1)

As usual, verify that (1) is integrable. Let x be treated as constant, so that $dx = 0$. Then (1) becomes $\quad -2xydy - 2xzdz = 0 \quad$ or $\quad 2ydy + 2zdz = 0. \quad$ ...(2)

Integrating (2), $\quad y^2 + z^2 = f(x)$, (say) $\quad$ ...(3)

where $f(x)$ is taken as constant of integration as x is treated as constant.

Differentiating (3), $\quad 2ydy + 2zdz = f'(x)dx \quad$ or $\quad xf'(x)dx - 2xydy - 2xzdz = 0,$...(4)

Comparing (4) with (1), $\quad xf'(x) = x^2 + y^2 + z^2 \quad$ or $\quad xf'(x) = x^2 + f(x)$, by (3)

or $\quad (df/dx) - (1/x)f = x$, which is a differential linear equation

Its I.F. $= e^{\int(-1/x)dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = 1/x$ and its solution is

$$(1/x) \times f = \int\{x \times (1/x)\}dx + c = x + c \qquad \text{or} \qquad f(x) = x^2 + cx$$

or $\quad y^2 + z^2 = x^2 + cx$, using (3); c being an arbitrary constant.

Ex. 4. *Solve* $xz^3dx - z\,dy + 2y\,dz = 0$. **[Delhi Maths (Hons) 1996]**

Sol. Given $\quad xz^3dx - z\,dy + 2y\,dz = 0. \quad$ ...(1)

Let x be treated as constant, so that $dx = 0$. Then (1) becomes

$-zdy + 2ydz = 0 \quad$ or $\quad -(1/y)dy + (2/z)dz = 0. \quad$ ...(2)

Integrating (2), $\quad -\log y + 2\log z = \log f(x) \quad$ or $\quad z^2/y = f(x), \quad$ ...(3)
when $f(x)$ is taken as constant of integration as x is treated as constant.

Differentiating (3), $\quad \dfrac{y\times 2z\,dz - z^2 dy}{y^2} = f'(x) \quad$ or $\quad -\left(\dfrac{y^2}{z}\right)\times f'(x) - z\,dy + 2y\,dz = 0. \quad$ ...(4)

Comparing (4) with (1), $\quad -(y^2/z)f'(x) = xz^3$

or $\quad f'(x) = -x(z^2/y)^2 \quad$ or $\quad f'(x) = -x[f(x)]^2$, using (3)

or $\quad df/dx = -xf^2 \quad$ or $\quad -f^{-2}\,df = x\,dx.$

Integrating, $\quad (1/f) = (x^2/2) + c \quad$ or $\quad y/z^2 = (x^2/2) + c$, using (3)

or $\quad 2y = x^2z^2 + 2cz^2$, c being an arbitrary constant

Ex. 5. *Solve* $y^2z(x\cos x - \sin x)dx + x^2z(y\cos y - \sin y)dy + xy(y\sin x + x\sin y + xy\cos z)dz = 0.$ **[Meerut 1998]**

Sol. As usual verify that the given equation is integrable. Let z be treated as constant, so that $dz = 0$. Then the given equation becomes

$$y^2z(x\cos x - \sin x)dx + x^2z(y\cos y - \sin y)dy = 0$$

or $\quad \dfrac{x\cos x - \sin x}{x^2}dx + \dfrac{y\cos y - \sin y}{y^2}dy = 0 \quad$ or $\quad d\left(\dfrac{\sin x}{x}\right) + d\left(\dfrac{\sin y}{y}\right) = 0.$

Integrating it, $\quad \dfrac{\sin x}{x} + \dfrac{\sin y}{y} = f(z)$, say $\quad$ ...(1)

where $f(z)$ is taken as constant of integration as z is treated as constant.

Differentiating (1), $\quad \dfrac{x\cos x - \sin x}{x^2}dx + \dfrac{y\cos y - \sin y}{y^2}dy = f'(z)\,dz$

or $\quad zy^2(x\cos x - \sin x)dx + zx^2(y\cos y - \sin y)dy - x^2y^2zf'(z)dz = 0. \quad$ ...(2)

Comparing (2) with the given equation, we have

$$-x^2y^2zf'(z) = xy(y\sin x + x\sin y + xy\cos z)$$

or $\quad -zf'(z) = \dfrac{\sin x}{x} + \dfrac{\sin y}{y} + \cos z = f(z) + \cos z$, using (1)

or $\quad \dfrac{df}{dz} + \dfrac{1}{z}f = -\dfrac{\cos z}{z}$, which is a linear differential equation.

Its I.F. $= e^{\int (1/z)dz} = e^{\log z} = z$ and its solution is

$z\,f(z) = \displaystyle\int z\left(-\frac{\cos z}{z}\right)dz + c = -\sin z + c \quad$ or $\quad z\left(\dfrac{\sin x}{x} + \dfrac{\sin y}{y}\right) = c - \sin z$, using (1).

Ex. 6. *Solve* $(e^xy + e^z)dx + (e^yz + e^x)dy + (e^y - e^xy - e^yz)dz = 0.$

Sol. As usual, verify that the given equation is integrable.

Let z be treated as constant, so that $dz = 0$. Then the given equation becomes

$(e^xy + e^z)dx + (e^yz + e^x)dy = 0 \quad$ or $\quad (e^xy\,dx + e^x dy) + e^z dx + ze^y dy = 0$

or $\quad d(e^xy) + e^z dx + ze^y dy = 0. \quad$ ...(1)

Integrating (1), $\quad e^xy + e^zx + e^yz = f(z)$, say $\quad$ ...(2)

where $f(z)$ is taken as constant of integration as z is treated as constant.

Differentiating (2), $\quad ye^x dx + e^x dy + xe^z dz + e^z dx + ze^y dy + e^y dz = f'(z)dz$

or $\quad (e^xy + e^z)dx + (e^yz + e^x)dy + [xe^z + e^y - f'(z)]dz = 0. \quad$ ...(3)

Comparing (3) with the given equation, we have

$e^y - e^x y - e^y z = xe^z + e^y - f'(z)$ or $f'(z) = [e^x y + e^z x + e^y z] = f(z)$, by (2)

or $df/dz = f(z)$ or $(1/f)df = dz$.

Integrating, $\log f - \log c = z$ or $f(z) = ce^z$ or $e^x y + e^z x + e^y z = ce^z$, using (2).

Ex. 7. *Find $f(y)$ if $f(y)dx - zx\, dy - xy \log y\, dz = 0$ is integrable. Find the corresponding integral.* **[Meerut 2011]**

Sol. Comparing the given equation with $Pdx + Qdy + Rdz = 0$, have $P = f(y)$, $Q = -zx$, $R = -xy \log y$. If the given equation is integrable, the following condition of integrability must be satisfied by P, Q, R

$$P(\partial Q/\partial z - \partial R/\partial y) + Q(\partial R/\partial x - \partial P/\partial z) + R(\partial P/\partial y - \partial Q/\partial x) = 0.$$

i.e. $f(y)[-x - (-x \log y - x)] - xz(-y \log y - 0) - xy \log y[f'(y) - (-z)] = 0$

or $x \log y\, f(y) - xy \log y\, f'(y) = 0$ or $f(y) - yf'(y) = 0$

or $f'(y)/f(y) = 1/y$ whose integration gives, $\log f(y) = \log y + \log c$

i.e. $f(y) = cy$, which is the required value of $f(y)$.

Then the given equation becomes $cy\, dx - zx\, dy - xy \log y\, dz = 0$. ...(1)

Treating z as constant *i.e.* $dz = 0$, (1) becomes

$$cy\, dx - zx\, dy = 0 \quad \text{or} \quad (c/x)dx - (z/y)dy = 0.$$

Integrating and remembering that z is to be regarded as constant, we get

$c \log x - z \log y = F(z)$, where $F(z)$ is an arbitrary function of z. ...(2)

Differentiating (2), $(c/x)\, dx - (z/y)\, dy - \log y\, dz = F'(z)\, dz$

or $cy\, dx - zx\, dy - [xy \log y + xy\, F'(z)]\, dz = 0$. ...(3)

Comparing (1) and (3), we get $xy \log y + xy\, F'(z) = xy \log y$ or $F'(z) = 0$.

Integrating, $F(z) = c'$, where c' is an arbitrary constant.

Putting $F(z) = c'$ in (2) the required solution is

$c \log x - z \log y = c'$ or $x^c = ky^z$, taking $c' = \log k$.

Ex. 8. *Solve* $(z + z^3) \cos x\, dx - (z + z^3)dy + (1 - z^2)(y - \sin x)dz = 0$

or $(z + z^3) \cos x\, (dx/dt) - (z + z^3)(dy/dt) + (1 - z^2)(y - \sin x)(dz/dt) = 0$.

[Allahabad 2005; Kolkata 2007; Nagpur 2004, 06]

Sol. Treating z as constant so that $dz = 0$, the given equation becomes

$(z + z^3) \cos x\, dx - (z + z^3)dy = 0$ or $\cos x\, dx - dy = 0$.

Integrating, $\sin x - y = f(z)$, where f is an arbitrary function. ...(1)

Differentiating (1), $\cos x\, dx - dy - (df/dz)\, dz = 0$

or $(z + z^3) \cos x\, dx - (z + z^3)\, dy - (z + z^3) f'(z)\, dz = 0$ ...(2)

Comparing (2) and the given equation, we have

$(1 - z^2)(y - \sin x) = -(z + z^3) f'(z)$ or $(1 - z^2)(-f) = -(z + z^3)(df/dz)$, using (1)

or $$\frac{df}{f} = \frac{1 - z^2}{z(1 + z^2)}dz \quad \text{or} \quad \frac{df}{f} = \left(\frac{1}{z} - \frac{2z}{z^2 + 1}\right)dz.$$

Integrating, $\log f = \log z - \log(z^2 + 1) + \log c$ or $(z^2 + 1)f = cz$

or $(\sin x - y)(z^2 + 1) = cz$, by (1), c being on arbitrary constant.

Ex. 9. *Verify that the following equation is integrable and find its primitive :*

$$zy\, dx + (x^2y - zx)\, dy + (x^2z - xy)\, dz = 0.$$

Sol. Given $zy\, dx + (x^2y - zx)\, dy + (x^2z - xy)\, dz = 0$. ...(1)

Treating x as constant so that $dx = 0$, (1) reduces to

$(x^2y - zx)dy + (x^2z - xy)\, dz = 0$ or $x^2(ydy + zdz) - x(zdy + ydz) = 0$. ...(2)

Integrating (2) and remembering that x is being regarded as constant, we get

$$(x^2/2)\times(y^2+z^2)-xyz = f(x), f \text{ being an arbitrary function.} \qquad ...(3)$$

Differentiating (3), we have

$$x\,dx(y^2+z^2) + (x^2/2)\times(2ydy+2zdz) - xy\,dz - yzdx - zx\,dy = f'(x)dx$$

or $$[x(y^2+z^2) - yz - f'(x)]dx + (x^2y - xy)dy + (x^2z - xy)dz = 0.$$

Comparing the above equation with (1), we have

$x(y^2+z^2) - yz - f'(x) = yz$ or $x(y^2+z^2) - 2yz = f'(x)$

or $(x^2/2)\times(y^2+z^2) - xyz = (x/2)\times f'(x)$ or $f = (x/2)\times f'(x)$ by (3)

or $f = (x/2)\times(df/dx)$ or $(1/f)df = (2/x)dx$ so that $f(x) = cx^2$.

Putting this value of $f(x)$ in (3), the required primitive is

$(x^2/2)\times(y^2+z^2) - xyz = cx^2$ or $x^2(y^2+z^2-2c) = 2xyz.$

EXERCISE 3 (C)

Solve the following differential equations :

1. $z^2dx + (z^2 - 2yz)dy + (2y^2 - yz - zx)dz = 0.$ **Ans.** $xz + yz - y^2 = cz^2$
2. $(y^2 + yz)dx + (z^2 + zx)dy + (y^2 - xy)dz = 0.$ **Ans.** $y(x+z) = c(y+z)$
3. $2yzdx + zxdy - xy(1+z)dz = 0.$ **Ans.** $x^2y = cze^z$
4. $(mz - ny)dx + (nx - lz)dy + (ly - mx)dz = 0.$ **[Rajasthan 2001, 07] Ans.** $nx - lz = c(mz - ny)$
5. $z(z+y^2)dx + z(z+x^2)dy - xy(x+y)dz = 0.$ **[Pune 2010]** **Ans.** $x(y^2+z) = z(x+y)(1-cy)$
6. $z(x^2-yz-z^2)dx+(x+z)xzdy+x(z^2-x^2-xy)dz=0.$ **Ans.** $z^{-1}(x+y)+x^{-1}(y+z)=c$
7. $xdy - ydx - 2x^2zdz = 0.$ **Ans.** $y = x(c - z^2)$
8. $3ydx = 3xdy + y^2dz.$ **Ans.** $x = y\log(cz)$
9. $zy(1+4xz)dx - xz(1+2xz)dy - xydz = 0.$ **[Delhi 2008, 09, 10]** **Ans.** $x(1+2xz) = cyz$
10. $yz\log x\,dx - zx\log z\,dy + xy\,dz = 0.$ **Ans.** $x\log z = cy$
11. $y^2dx - 2x^2dy + (xy - zy^3)dz = 0.$ **[Bangalore 1994]** **Ans.** $2y^{-1} - x^{-1} = ce^z$
12. $yz\,dx + (xz - yz^3)dy - 2xy\,dz.$ **[Bangalore 2005]**

3.14. Solution of Pdx + Qdy + Rdz = 0, if it is exact and homogeneous of degree n ≠ 1.

Theorem. $xP + yQ + zR = c$ *is the solution of* $Pdx + Qdy + Rdz = 0$, *when it is exact and homogeneous of degree* $n \neq 1$. **[Bilaspur 2001; Meerut 1998; Guwahati 1996]**

Proof : Give solution is $xP + yQ + zR = c.$ ...(1)

Differentiating (1), we obtain

$$\left(P + x\frac{\partial P}{\partial x} + y\frac{\partial Q}{\partial x} + z\frac{\partial R}{\partial x}\right)dx + \left(x\frac{\partial P}{\partial y} + Q + y\frac{\partial Q}{\partial y} + z\frac{\partial R}{\partial y}\right)dy + \left(x\frac{\partial P}{\partial z} + y\frac{\partial Q}{\partial z} + R + z\frac{\partial R}{\partial z}\right)dz = 0 \quad ....(2)$$

Since $Pdx + Qdy + Rdz = 0$ is exact, we have

$\partial P/\partial y = \partial Q/\partial x,$ $\partial Q/\partial z = \partial R/\partial y$ and $\partial R/\partial x = \partial P/\partial z.$ ...(3)

Using relation (3), (2) may be re–written as

$$\left(P + x\frac{\partial P}{\partial x} + y\frac{\partial P}{\partial y} + z\frac{\partial P}{\partial z}\right)dx + \left(Q + x\frac{\partial Q}{\partial x} + y\frac{\partial Q}{\partial y} + z\frac{\partial Q}{\partial z}\right)dy + \left(R + x\frac{\partial R}{\partial x} + y\frac{\partial R}{\partial y} + z\frac{\partial R}{\partial z}\right)dz = 0 \quad ...(4)$$

Since $Pdx + Qdy + Rdz = 0$ is homogeneous of degree n, it follows that P, Q and R are all homogeneous functions of degree n. Using Euler's theorem on homogeneous functions $P(x, y, z)$, $Q(x, y, z)$ and $R(x, y, z)$ of degree n, we get

$$\left.\begin{aligned} x(\partial P/\partial x) + y(\partial P/\partial y) + z(\partial P/\partial z) &= nP, \\ x(\partial Q/\partial x) + y(\partial Q/\partial y) + z(\partial Q/\partial z) &= nQ \\ \text{and} \qquad x(\partial R/\partial x) + y(\partial R/\partial y) + z(\partial R/\partial z) &= nR. \end{aligned}\right\} \qquad \text{...(5)}$$

Using (5), (4) reduces to

$$(P + nP)dx + (Q + nQ)dy + (R + nR)dz = 0 \quad \text{or} \quad (n + 1)(Pdx + Qdy + Rdz) = 0$$

or $\quad Pdx + Qdy + Rdz = 0, \quad$ as $\quad n \neq -1 \quad$ so that $\quad (n + 1) \neq 0 \qquad$ ... (6)

which is given differential equation and hence (1) is solution of (6), as required.

ILLUSTRATIVE SOLVED EXAMPLE

Solve $(x - 3y - z)dx + (2y - 3x)dy + (z - x)dz = 0$. **[Guwahati 1996]**

Sol. Given $\quad (x - 3y - z)dx + (2y - 3x)dy + (z - x)dz = 0. \qquad$ ...(1)

Comparing (1) with $Pdx + Qdy + Rdz = 0$, we have

$P = x - 3y - z, \qquad Q = 2y - 3x \qquad$ and $\qquad R = z - x. \qquad$ ...(2)

(1) is homogeneous equation of degree $n = 1 \neq -1$. Also from (2), we get

$(\partial P/\partial y) = -3 = (\partial Q/\partial x), \qquad (\partial Q/\partial z) = 0 = (\partial R/\partial y), \qquad (\partial R/\partial x) = -1 = (\partial P/\partial z).$ ...(3)

(3) shows that (1) is exact. Thus, (1) is exact and homogeneous of degree $n = 1 \neq -1$. Hence, solution of (1) is given by

$$xP + yQ + zR = c \qquad \text{or} \qquad x(x - 3y - z) + y(2y - 3x) + z(z - x) = c$$

or $\qquad x^2 + 2y^2 + z^2 - 6xy - 2xz = c$, c being an arbitrary constant.

3.15. The non–integrable single equation

Suppose that the condition of integrability is not satisfied by the equation

$$Pdx + Qdy + Rdz = 0. \qquad \text{...(1)}$$

Then (1) represents a family of curves orthogonal to the family represented by the equations

$$dx/P = dy/Q = dz/R. \qquad \text{...(2)}$$

However, in the present case there exists no family of surfaces orthogonal to the second family of curves. In such cases we can find an infinite number of curves that lie on any given surface and satisfy (1). However, note that the method of finding the above mentioned infinite number of curves is equally applicable to integrable equation (1).

3.16. Working Rule for finding the curves represented by the solution of non–integrable total differential equation

$$Pdx + Qdy + Rdz = 0 \qquad \text{...(1)}$$

which lie on given surface $\qquad f(x, y, z) = c. \qquad$ ....(2)

Step 1. Verify that (1) is not integrable.

Step 2. Differentiate (2) and then eliminate z and dz from the equation so obtained with help of (1) and (2).

Step 3. Integrate the equation obtained in step 2. This equation will involve x and y only. The resulting solution and (2) together represent the desired curves.

3.17. Solved Examples based on working rule 3.16.

Ex. 1. *Find the curves represented by the solution of* $y\,dx + (z - y)\,dy + x\,dz = 0$, *which lie in the plane* $2x - y - z = 1$. **[Allhabad 2006; Bangalore 2003, 07]**

Sol. Given $\qquad y\,dx + (z - y)\,dy + x\,dz = 0 \qquad$ ...(1)

and $\qquad 2x - y - z = 1. \qquad$ ...(2)

Comparing (1) with $Pdx + Qdy + Rdz = 0$, here $\quad P = y, \quad Q = z - y \quad$ and $\quad R = x \qquad$ ...(3)

Using (3), $\qquad P(\partial Q/\partial z - \partial R/\partial y) + Q(\partial R/\partial x - \partial P/\partial z) + R(\partial P/\partial y - \partial Q/\partial x)$

$$= y(1 - 0) + (z - y)(1 - 0) + x(1 - 0) = z + x \neq 0.$$

This shows that the condition of integrability is not satisfied by (1).

Differentiating (2), $\qquad 2dx - dy - dz = 0 \qquad$ or $\qquad dz = 2dx - dy. \qquad$ ...(4)

Using (4) to eliminate dz, (1) gives $\quad y\,dx + (z - y)\,dy + x(2dx - dy) = 0$

or $\quad ydx + (2x - y - 1 - y)dy + x(2dx - dy) = 0$, using (2) to eliminate z

or $\quad (y + 2x)dx + (x - 2y - 1)dy = 0 \quad$ or $\quad (ydx + xdy) + 2xdx - 2ydy - dy = 0.$

Integrating, $\quad xy + x^2 - y^2 - y = c$, c being an arbitrary constant. ...(5)

The required curves are given by the intersection of plane (2) and rectangular hyperbolic cylinders (5).

Ex. 2. *Show that there is no single integral of* $dz = 2ydx + xdy$. *Prove that the curves of this equation that lie in the plane* $z = x + y$ *lie also on surfaces of the family* $(x - 1)^2(2y - 1) = c$.

[Agra 2001, 03, 07; Meerut 2006]

Sol. Given $\quad 2ydx + xdy - dz = 0$ ...(1)

and $\quad z = x + y.$ ...(2)

As in Ex. 1, show that the condition of integrability is not satisfied

Differentiating (2), $\quad dz = dx + dy.$ ...(3)

Using (3), (1) gives $\quad 2ydx + xdy - (dx + dy) = 0$

or $\quad (2y - 1)dx + (x - 1)dy = 0$ ...(3)

or $\quad \{2/(x - 1)\}dx + \{2/(2y - 1)\}\, dy = 0$

Integrating, $\; 2 \log (x - 1) + \log (2y - 1) = \log c \quad$ or $\quad (x - 1)^2(2y - 1) = c.$

Ex. 3. *Show that the curves of* $\quad xdx + ydy + c\sqrt{(1 - x^2/a^2 - y^2/b^2)}\,dz = 0$ ...(1)

that lie on the ellipsoid $\quad x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$ ...(2)

lie also on the family of concentric spheres $\quad x^2 + y^2 + z^2 = a^2.$

Sol. Verify that (1) does not satisfy the condition of integrability. Form (2), $1 - x^2/b^2 - y^2/b^2 = z^2/c^2$. Using this, (1) may be re–written as

$$2xdx + 2ydy + 2zdz = 0$$

Integrating, $\quad x^2 + y^2 + z^2 = a^2$, a being an arbitrary constant. ...(3)

Hence the curves of (1) that lie on (2) also lie on the family of concentric spheres (3).

Ex. 4. *Find the orthogonal projection on the plane of xz of curve which lie on the paraboloid* $3z = x^2 + y^2$ *and satisfy the equation* $2dz = (x + z)dx + y\,dy$. **[Kurukshetra 1997]**

Sol. Given $\quad (x + z)dx + y\,dy - 2dz = 0$ ...(1)

and $\quad x^2 + y^2 - 3z = 0.$ ...(2)

Verify yourself that (1) does not satisfy the usual condition of integrability. Differentiating (2), we get $\quad 2x\,dx + 2y\,dy - 3dz = 0.$ ...(3)

Multiplying (1) by 3 and (3) by 2 and then subtracting the resulting equations, we get

$$3(x + z)dx + 3y\,dy - 3(2x\,dx + 2y\,dy) = 0$$

or $\quad -x\,dx - y\,dy + 3z\,dx = 0 \quad$ or $\quad x\,dx + y\,dy = (x^2 + y^2)dx$, using (2)

or $\quad \dfrac{2xdx + 2ydy}{x^2 + y^2} = 2dx \quad$ so that $\quad \log (x^2 + y^2) - \log c = 2x$

or $\quad x^2 + y^2 = ce^{2x} \quad$ or $\quad 3z = ce^{2x}$, by (2) ...(4)

which gives the required projection on xy plane.

EXERCISE 3 (D)

1. Obtain the system of the curves lying on the system of surfaces $zx = c$ and satisfying the differential equation $yz\,dx + z^2dy + y(z + x)dz = 0.$ **Ans.** $x = c/y$, $zx = c$

2. Show that the differential equation $3ydx + (z - 3y)dy + xdz = 0$ is not integrable.

Prove that the projection on the plane of xy of the curves that satisfy the equation and lie on the plane $2x + y - z = a$ are the rectangular hyperbolas $x^2 + 3xy - y^2 - ay = b$.

[Meerut 2003, 05, 06; Allahabad 2001, 07; Lucknow 2003, 05, 06]

3. Find the equation of the cylinder, with generators parallel to the axis of y, passing through the point (2, 1, –1), and also through a curve that lies on the sphere $x^2 + y^2 + z^2 = 4$ and satisfies the equation $(xy + 2xz)dz + y^2dy + (x^2 + yz)dz = 0$.

3.18. Geometrical Interpretation of Pdx + Qdy + Rdz = 0

[G.N.D.U. (Amritsar) 1997; Lucknow 2003]

The given differential equation expresses that the tangent to a curve is perpendicular to a certain line, the direction cosines of this tangent line and another line being proportional to dx, dy, dz and P, Q, R respectively. Suppose that the equation

$$Pdx + Qdy + Rdz = 0. \qquad ...(1)$$

satisfies the condition of integrability and that its solution is

$$F(x, y, z) = c. \qquad ...(2)$$

Since (2) has one arbitrary constant, it represents a single infinity of surfaces. Choosing this constant in an appropriate manner, (2) can be made to pass through any given point of space. If a point is moving upon this surface in any direction, its co–ordinates and direction cosines of its path at any moment (which are proportional to dx, dy, dz since the point moves along tangent) must satisfy (1), since (2) is the integral of (1). Again for each point (x, y, z) there will be an infinite number of values of dx, dy, dz which will satisfy (1). Thus it follows that a point which is moving in such a manner that its co–ordinates and the direction cosines of its path always satisfy (1) can pass through any point in an infinity of directions. However, while passing through any point, it must remain on the particular surface given by (2) which passes through the point. Thus, infinite number of such possible curves which it can describe through that point must lie on that surface.

3.19. To show that the locus of Pdx + Qdy + Rdz = 0 is orthogonal to the locus of (dx)/P = (dy)/Q = (dz)/R.

The equation
$$Pdx + Qdy + Rdz = 0 \qquad ...(1)$$
means, geometrically, that a straight line whose direction cosines are proportional to dx, dy, dz, is perpendicular to a line whose direction cosines are proportional to P, Q, R. As a consequence a point which satisfies (1) must move in a direction at right angles to a line whose direction cosines are proportional to P, Q, R. On the other hand, the equations

$$(dx)/P = (dy)/Q = (dz)/R. \qquad ...(2)$$

mean, geometrically, that a straight line whose direction cosines are proportional to dx, dy, dz is parallel to a line whose direction cosines are proportional to P, Q, R. As a consequence, a point which satisfies (2) must move in a direction parallel to line whose direction cosines are proportional to P, Q, R.

From the above discussion it follows that the curves traced out by the points that are moving according to the condition (1) are orthogonal to the curves traced out by the points that are moving according to the conditions (2). The former curves are any of the curves upon the surfaces given by (1). Thus, geometrically, the curves represented by (2) are normal to the surfaces represented by (1). In case (1) is not integrable, there cannot exist a family of surfaces which is orthogonal to all lines that form the locus of (2).

3.20. Total differential equations containing more that three variables.

The total differential equation in four variables is

$$Pdx + Qdy + Rdz + Tdt = 0, \qquad ...(1)$$

where P, Q, R and T are the functions of x, y, z and t.

Now (1) is integrable if it is integrable in any three variables also. Accordingly, the condition of integrability of (1) can be obtained by treating t, x, y, z as constants by turn.

Let t be regarded as constant so that $dt = 0$. Then (1) gives

$$Pdx + Qdy + Rdz = 0. \qquad ...(2)$$

The condition of integrability of (2) is

$$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0. \qquad \text{...(3)}$$

Similarly, taking x, y and z as constants successively, the conditions of integrability are

$$Q\left(\frac{\partial R}{\partial t}-\frac{\partial T}{\partial z}\right)+R\left(\frac{\partial T}{\partial y}-\frac{\partial Q}{\partial t}\right)+T\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)=0, \qquad \text{...(4)}$$

$$R\left(\frac{\partial P}{\partial t}-\frac{\partial T}{\partial x}\right)+P\left(\frac{\partial T}{\partial z}-\frac{\partial R}{\partial t}\right)+T\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)=0 \qquad \text{...(5)}$$

and

$$P\left(\frac{\partial Q}{\partial t}-\frac{\partial T}{\partial y}\right)+Q\left(\frac{\partial T}{\partial x}-\frac{\partial P}{\partial t}\right)+T\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0. \qquad \text{...(6)}$$

Hence (1) is integrable if conditions (3), (4), (5) and (6) are satisfied.

Remark 1. Proceeding as above the conditions of integrability can be derived for the total differential equations containing more than four variables.

Remark 2. Multiplying equation (3) by T, (4) by $(-P)$, (5) by Q and (6) by $(-R)$ and adding, we have $T \times (3) - P \times (4) + Q \times (5) - R \times (6) = 0$,
showing that the conditions (3) to (6) are not independent. Thus any three of the conditions (3) to (6) give the conditions of integrability of (1).

Remark 3. If given total differential equation contains n indepdent variables, there will be $(1/2) \times (n-1)(n-2)$ independent conditions for its integrability.

Methods of solving (1) : First of all satisfy any of three conditions (3) to (6) and hence declare that (1) is integrable.

Now proceed by any of the following three methods to solve (1) :

Method I. Solution by inspection. This method is the same as discussed in Art. 3.6 for three variables. Read that article carefully.

Method II. Homogeneous equation. Let P, Q, R and T be functions of x, y, z and t of the same degree.

Put $x = ut$, $y = vt$, $z = wt$ in (1) and then integrate the transformed equation as in the case of three variables (Refer Art. 3.8.)

Alternative method for solving homogeneous equation

If $(Px + Qy + Rz + Tt) \neq 0$, then we may make use of the integrating factor (I.F) given by I.F. $= 1/(Px + Qy + Rz + Tt)$. Procedure is same as explained in Art. 3.8.

Method III. Treating two variables as constants.

This method is merely an extension of the method adopted in Art. 3.12 for three variables. Take two suitable variables, z and t (say) as constants. Then $dz = 0$ and $dt = 0$. Now integrate the reduced equation and take the constant of integration as the function of the variables which were kept constant. Thus, for the above assumption we take $f(z, t)$ as constant of integration. Now proceed as discussed in Art. 3.12.

To understand the above methods, read carefully the following examples.

3.21. Solved examples based on Art. 3.20.

Ex. 1. *Solve* : $z(y + z)dx + z(u - x)dy + y(x - u)dy + y(y + z)du = 0$.

Sol. Re–writing the given equation, $\quad (y + z)(zdx + ydu) + z(u - x)dy + y(x - u)dz = 0$

or $\quad (y + z)(zdx + ydu) + (y + z)(xdz + udy) - (y + z)(xdz + udy) + z(u - x)dy + y(x - u)dz = 0$

or $\quad (y + z)(zdx + ydu + xdz + udy) - (xz + uy)(dy + dz) = 0$, by simplification

or
$$\frac{zdx + ydu + xdz + udy}{xz + uy} = \frac{dy + dz}{y + z}. \qquad \text{...(1)}$$

Integrating (1), $\quad \log(xz + uy) = \log(y + z) + \log c \quad$ or $\quad xz + uy = c(y + z)$.

Ex. 2. *Solve* : $(2x + y^2 + 2xz)(dx/dt) + 2xy(dy/dt) + x^2(dz/dt) = 1$.

Sol. Re–writing the given equation, $(2x + y^2 + 2xz)dx + 2xydy + x^2dz - dt = 0$

or $2xdx + (y^2dx + 2xydy) + (2xzdx + x^2dz) - dt = 0$ or $d(x^2) + d(xy^2) + d(x^2z) - dt = 0$.

Integrating, $x^2 + xy^2 + x^2z - t = c$, c being an arbitrary constant.

Ex. 3. Solve $t(y + z)dx + t(y + z + 1)dy + tdz - (y + z)dt = 0$.

Sol. Given $t(y + z)dx + t(y + z + 1)dy + tdz - (y + z)dt = 0$. ...(1)

Comparing (1) with $Pdx + Qdy + Rdz + Tdt = 0$, we have

$P = t(y + z)$, $Q = t(y + z + 1)$, $R = t$, $T = -(y + z)$. ...(2)

Now, $$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right)$$

$$= t(y + z)(t - 0) + t(y + z + 1)(0 - t) + t(t - 0) = 0. \qquad ...(3)$$

Similarly, we can prove that

$$P\left(\frac{\partial Q}{\partial t} - \frac{\partial T}{\partial y}\right) + Q\left(\frac{\partial T}{\partial x} - \frac{\partial P}{\partial t}\right) + T\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0 \qquad ...(4)$$

$$R\left(\frac{\partial P}{\partial t} - \frac{\partial T}{\partial x}\right) + P\left(\frac{\partial T}{\partial z} - \frac{\partial R}{\partial t}\right) + T\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) = 0 \qquad ...(5)$$

and $$Q\left(\frac{\partial R}{\partial t} - \frac{\partial T}{\partial z}\right) + R\left(\frac{\partial T}{\partial y} - \frac{\partial Q}{\partial t}\right) + T\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) = 0. \qquad ...(6)$$

Hence (1) satisfies the conditions of integrability (3), (4), (5) and (6). Therefore, (1) must be integrable. We now solve (1) by the following two methods :

Method I. (By inspection). Re–writing (1), we have

$(y + z)dx + (y + z + 1)dy + dz - (y + z)(1/t)dt = 0$ or $(y + z)(dx + dy) + (dy + dz) - (y + z)(1/t)dt = 0$

or $$dx + dy + \frac{dy + dz}{y + z} - \frac{dt}{t} = 0.$$

Integrating, $x + y + \log(y + z) - \log t = \log c$, c being arbitrary constant.

or $\log\left(\dfrac{y + z}{ct}\right) = -(x + y)$ or $y + z = cte^{-(x + y)}$ or $(y + z)e^{x + y} = ct$.

Method II. (Regarding two variables as constants)

Treat z and t as constants so that $dz = 0$ and $dt = 0$. Then (1) reduces to

$$t(y + z)dx + t(y + z + 1)dy = 0 \quad \text{or} \quad dx + \left(1 + \frac{1}{y + z}\right)dy = 0. \qquad ...(7)$$

Integrating (7), $x + y + \log(y + z) = f(z, t)$, say ...(8)

Differentiating (8), we have $$dx + dy + \frac{dy + dz}{y + z} = df$$

or $(y + z)dx + (y + z)dy + dy + dz - (y + z)df = 0$

or $t(y + z)dx + t(y + z + 1)dy + tdz - (y + z)tdf = 0$. ...(9)

Comparing (9) with (1), $-(y + z)t\,df = -(y + z)dt$ or $df = (1/t)dt$. ...(10)

Integraing (10), $f = \log t + \log c$ or $f = tc$.

or $x + y + \log(y + z) = \log(tc)$, using (8)

or $\log\left(\dfrac{y + z}{tc}\right) = -(x + y)$ or $\dfrac{y + z}{tc} = e^{-(x + y)}$ or $(y + z)e^{x + y} = tc$.

Ex. 4. *Solve :* $z(y + z)dx + z(t - x)dy + y(x - t)dz + y(y + z)dt = 0$.

Sol. Given $$z(y + z)dx + z(t - x)dy + y(x - t)dz + y(y + z)dt = 0. \qquad ...(1)$$

Comparing (1) with $Pdx + Qdy + Rdz + Tt = 0$, we have

$$P = z(y + z), \qquad Q = z(t - x), \qquad R = y(x - t), \qquad T = y(y + z). \qquad ...(2)$$

As in Ex. 3, we can show that the conditions of integrability are satisfied.

We now solve (1) by the following two methods.

Method I. (Regardings two variables as constants)

Treat y and z as constants so that $dy = 0$ and $dz = 0$. Then (1) reduces to

$$z(y + z)dx + y(y + z)dt = 0 \qquad \text{or} \qquad zdx + ydt = 0. \qquad ...(3)$$

Integrating (3), $$zx + yt = f(y, z), \text{ (say)}. \qquad ...(4)$$

Differentiating (1), $zdx + tdy + xdz + ydt = df$ or $(y + z)(zdx + tdy + xdz + ydt) = (y + z)df$

or $$z(y + z)dx + t(y + z)dy + x(y + z)dz + y(y + z)dt = (y + z)df. \qquad ...(5)$$

Comparing (5) with (1), we have

$$t(y + z)dy + x(y + z)dz - (y + z)df = z(t - x)dy + y(x - t)dz.$$

or $$(ty + xz)dy + (ty + xz)dz = (y + z)df$$

or $(ty + xz)(dy + dz) = (y + z)df$ or $f \times (dy + dz) = (y + z)df$, using (4)

or $$(1/f)df = (dy + dz)/(y + z). \qquad ...(6)$$

Integrating (6), $\log f = \log (y + z) + \log c$ or $f = c(y + z)$

or $zx + yt = c(y + z)$, using (4), c being an arbitrary constant.

Method II. (Homogeneous equation. Use of integrating factor)

Here P, Q, R and T are functions of x, y, z and t of the same degree 2.

Also, $$D = Px + Qy + Rz + Tt = xz(y + z) + yz(t - x) + yz(x - t) + ty(y + z)$$
$$= xz(y + z) + ty(y + z) + yz(t - x + x - t) = (y + z)(xz + ty) \neq 0.$$

$\therefore$ $$\text{Integrating factor} = \text{I.F.} = \frac{1}{D} = \frac{1}{(y + z)(xz + ty)}. \qquad ...(3)'$$

Multiplying the given equation (1) by I.F., we have

$$\frac{y(y + z)dx + x(t - x)dy + y(x - t)dz + y(y + z)dt}{(y + z)(xz + ty)} = 0. \qquad ...(4)'$$

Now, the numerator of (4)′

$= [(y + z)(zdx + xdz + ydt + tdy) + (xz + ty)(dy + dz)] + z(t - x)dy + y(x - t)dz$
$\qquad - (y + z)(xdz + tdy) - (xz + ty)(dy + dz)$

$= d\{(y + z)(xz + ty)\} - 2(xz + ty)(dy + dz)$.

Using the above value of numerator of (4)′, (4)′ becomes

$$\frac{d\{(y + z)(xz + ty)\} - 2(xz + ty)(dy + dz)}{(y + z)(xz + ty)} = 0 \quad \text{or} \quad \frac{d[(y + z)(xz + ty)]}{(y + z)(xz + ty)} - 2\frac{dy + dz}{y + z} = 0 \quad ....(5)'$$

Integrating (5)′, $$\log \{(y + z)(xz + ty)\} - 2 \log (y + z) = \log c$$

or $$\log \frac{(y + z)(xz + ty)}{(y + z)^2} = \log c \qquad \text{or} \qquad xz + ty = c(y + z).$$

Ex. 5. *Show that the equation* $y \sin w\, dx + x \sin w\, dy - xy \sin w\, dz - xy \cos w\, dw = 0$ *satisfies the conditions of integrability and obtain its integral.* **[Meerut 1999]**

Sol. Given $$y \sin w\, dx + x \sin w\, dy - xy \sin w\, dz - xy \cos w\, dw = 0. \qquad ...(1)$$

Compare (1) with $Pdx + Qdy + Rdz + Wdw = 0$, we have

$$P = y \sin w, \qquad Q = x \sin w, \qquad R = -xy \sin w \quad \text{and} \quad W = -xy \cos w. \qquad ...(2)$$

Here, $$P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)$$

$$= y \sin w\,(0 + x \sin w) + x \sin w\,(-y \sin w - 0) + (-xy \sin w)(\sin w - \sin w) = 0. \qquad ...(3)$$

Similarly, we can prove that

$$P\left(\frac{\partial Q}{\partial w}-\frac{\partial W}{\partial y}\right)+Q\left(\frac{\partial W}{\partial x}-\frac{\partial P}{\partial w}\right)+W\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right)=0, \qquad ...(4)$$

$$R\left(\frac{\partial P}{\partial w}-\frac{\partial W}{\partial x}\right)+P\left(\frac{\partial W}{\partial z}-\frac{\partial R}{\partial w}\right)+W\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)=0 \qquad ...(5)$$

and $$Q\left(\frac{\partial R}{\partial w}-\frac{\partial W}{\partial z}\right)+R\left(\frac{\partial W}{\partial y}-\frac{\partial Q}{\partial w}\right)+W\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)=0 \qquad ...(6)$$

Hence (1) satisfies the conditions of integrability (3), (4), (5) and (6). Hence (1) is integrable.

Treat z and w as constants so that $dz = 0$ and $dw = 0$. Then (1) reduces to

$y \sin w\, dx + x \sin w\, dy = 0$ or $ydx + xdy = 0$. ...(7)

Integrating (7), $xy = f(z, w)$, (say). ...(8)

Differentiating (8), we have $y\,dx + x\,dy = df$.

or $y \sin w\, dx + x \sin w\, dy - \sin w\, df = 0$. ...(9)

Comparing (9) with (1), $\sin w\, df = xy \sin w\, dz + xy \cos w\, dw$

or $df = xy(dz + \cot w\, dw)$ or $df = f \times (dz + \cot w\, dw)$, using (8)

or $(1/f)df = dz + \cot w\, dw$. ...(10)

Integrating, $\log f = z + \log \sin w + \log c$, c being an arbitrary constant.

or $f = c \sin w\, e^z$ or $xy = c \sin w\, e^z$, using (8).

3.22. Working rule (based on Art. 3.3) for solving *Pdx + Qdy + Rdz* = 0 ...(i)

Step 1: Verify that (*i*) satisfies the condition of integrability (8) of Art. 3.3

Step 2: Let z = constant so that $dz = 0$. Then, (*i*) yields $Pdx + Qdy = 0$ ...(*ii*)

Step 3: Let solution of (*ii*) be $f(x, y, z) = V$ = constant ...(*iii*)

Step 4: From (*ii*) and (*iii*), $\lambda P = \partial V/\partial x$ and $\lambda Q = \partial V/\partial y$. Determine λ.

Step 5: $\lambda P\,dx + \lambda Q\,dy + \lambda R\,dz = (\partial V/\partial x)\,dx + (\partial V/\partial y)\,dy + (\partial V/\partial z)\,dz + (\lambda R - \partial V/\partial z)\,dz$

$\Rightarrow \quad \lambda P\,dx + \lambda Q\,dy + \lambda R\,dz = dV + S\,dz$, where $S = \lambda R - (\partial V/\partial z)$ ...(*iv*)

Compute S in terms of z and V.

Step 6: Given equation (*i*) reduces to $dV + S\,dz = 0$ ...(*v*)

Substitute the value of S (obtained in step 5) in (*v*) and then integrate it. Finally, using (*iii*), we obtain the required solution.

Remark. Select a proper variable to be constant so that the resulting equation is easily integrable. Accordingly, necessary corresponding changes must be done in the above working rule. The following example will make this point clear.

Solved example : *Solve the following partial differential equation: zy(1 + 4xz)dx – xz(1 + 2xz) dy – xy dz = 0* **[Delhi B.A. (Prog) II 2010; Delhi Maths (H) 2008, 09]**

Solution. Comparing the given equation with $P\,dx + Q\,dy + R\,dz = 0$, we have

$P = zy\,(1 + 4xz)$, $Q = -xz\,(1 + 2xz)$ and $R = -xy$ ...(1)

Hence, $P(\partial Q/\partial z - \partial R/\partial y) + Q(\partial R/\partial x - \partial P/\partial z) + R(\partial P/\partial y - \partial Q/\partial x)$

$= zy\ (1 + 4xz)\ (-x - 4x^2z + x) - xz(1+ 2xz)\ (-y - y - 8xyz) - xy\ (z + 4xz^2 + z + 4xz^2) = 0$

Thus, the given equation is integrable. Let x = constant. Then, the given equation yields

$$-xz\ (1 + 2xz)\ dy - xydz = 0 \qquad \text{or} \qquad \frac{dy}{y} + \frac{dz}{z(1+2xz)} = 0 \qquad \text{or} \qquad \frac{dy}{y} + \left(\frac{1}{z} - \frac{2x}{1+2xz}\right)dz = 0$$

Integrating, $\log y + \log z - \log (1 + 2xz) = \log c$ or $(yz)/(1 + 2xz) = c =$ constant

Let us take $$V = (yz)/(1 + 2xz) \qquad \text{... (2)}$$

Now, $\lambda Q = \partial V/\partial y \quad \Rightarrow \quad -\lambda xz\,(1+2xz) = z\,/\,(1 + 2xz)$, using (1) and (2)

Thus, $$\lambda = -\{1/x(1+2xz)^2\} \qquad \text{... (3)}$$

Hence, $$S = \lambda P - \frac{\partial V}{\partial x} = -\frac{zy(1+4xz)}{x(1+2xz)^2} + \frac{2z^2y}{(1+2xz)^2}, \text{ using (1), (2) and (3)}$$

or $$S = -\frac{yz(1+4xz-2xz)}{x(1+2xz)^2} = -\frac{yz(1+2xz)}{x(1+2xz)^2} = -\frac{yz}{x(1+2xz)} = -\frac{V}{x}, \text{ using (2)} \quad \text{...(4)}$$

Now, the given equation reduces to $dV + S\,dx = 0$ or $dV - (V/x)dx = 0$, by (4)

or $(1/V)\,dV = (1/x)\,dx$ so that $\log V = \log x + \log A$ or $V = xA$ or $(yz)/(1+2xz) = xA$, which is the required solution, A being an arbitrary constant

Note: The reader can solve problems of Art 3.13 and Exercise 3 (*C*) by using the above working rule of Art. 3.22.

Miscellaneous Problems on Chapter 3

1. Solve $yz(1+x)dx + z\,xy\ (1+y)dy\ +\ xy\ (1+z)dz = 0$ **[Delhi B.A. (Prog). II 2010]**

Ans. $x + y + z + y^2/2 + \log(xz) = c.$

2. State true or false with justification : If curl $(P, Q, R) = 0$, then $Pdx + Qdy + Rdz = 0$ is exact. **[Pune 2010]**

Hints. Refer theorem on page 3.3.

4

Riccati's Equation

4.1. Introduction. A differential equation of the form

$$dy/dx = P + Qy + Ry^2 \qquad i.e. \qquad y_1 = P + Qy + Ry^2, \qquad ...(1)$$

where P, Q and R are functions of x, is known as the generalized Riccati's equation. Throughout this chapter suffixes will denote differentiation with respect to x. In 1841, the French mathematician Liouville proved that (1) is one of the simplest differential equations of the first order and first degree that cannot, in general, be integrated by quadratures. Due to historical and theoretical importance and occurrence of (1) in appalications, the study of certain aspect of (1) becomes quite useful.

4.2. General solution of Riccati's equation namely,

$$y_1 = P + Qy + Ry^2. \qquad ...(1)$$

We introduce another dependent variable u such that

$$y = -u_1/(Ru) = -u_1(Ru)^{-1}. \qquad ...(2)$$

Differentiating (2) w.r.t 'x', $\quad y_1 = -u_2(Ru)^{-1} + u_1(Ru)^{-2}[R_1u + Ru_1]. \qquad ...(3)$

Using (2) and (3), (1) becomes

$$-\frac{u_2}{Ru} + \frac{R_1u_1}{R^2u} + \frac{u_1^2}{Ru^2} = P + Q\left(-\frac{u_1}{Ru}\right) + R\left(\frac{u_1^2}{R^2u^2}\right)$$

or $$-Ru_2 + R_1u_1 = PR^2u - Qu_1R$$

or $$Ru_2 - (QR + R_1)u_1 + PR^2u = 0, \qquad ...(4)$$

which is linear differential equation of the second order. We know that the general solution of (4) is of the form

$$u = Af(x) + BF(x), \text{ where } A \text{ and } B \text{ are arbitrary constants.} \qquad ...(5)$$

$\therefore$ (2) gives $$y = -\frac{Af_1 + BF_1}{R(Af + BF)} = \frac{(A/B)(-f_1/R) + (-F_1/R)}{(A/B)f + F}$$

which is of the form $$y = \frac{cg(x) + G(x)}{cf(x) + F(x)} \qquad ...(6)$$

where $c = (A/B)$ is an arbitrary constant. Hence the general solution of (1) is of the form (6).

SOLVED EXAMPLES BASED ON ART. 4.2

Ex. 1. *Solve* (*i*) $y_1 = -2 - 5y - 2y^2$ $\qquad$ (*ii*) $x^2y_1 + 2 - 2xy + x^2y^2 = 0.$

Sol. (*i*) Given $$y_1 = -2 - 5y - 2y^2. \qquad ...(1)$$

Comparing (1) with $y_1 = P + Qy + Ry^2$, here $R = -2$ and so we take

$$y = -u_1/(Ru) = -u_1/(-2u) = u_1(2u)^{-1}. \qquad ...(2)$$

Differentiating (2) w.r.t. 'x', $\quad y_1 = u_2(2u)^{-1} - u_1(2u)^{-2}.(2u_1). \qquad ...(3)$

Using (2) and (3) in (1), $$\frac{u_2}{2u} - \frac{u_1^2}{2u^2} = -2 - 5\left(\frac{u_1}{2u}\right) - 2\left(\frac{u_1}{2u}\right)^2$$

or $\qquad u_2 + 5u_1 + 4u = 0 \qquad$ or $\qquad (D^2 + 5D + 4)u = 0, \quad ...(4)$

where $D \equiv d/dx$. The usual auxiliary equation is $D^2 + 5D + 4 = 0$, giving $D = -4, -1$ and hence solution of (4) is

$$u = Ae^{-x} + Be^{-4x} \qquad \text{so that} \qquad u_1 = -Ae^{-x} - 4Be^{-4x}. \quad ...(5)$$

Putting these values of u and u_1 in (2), the required general solution of (1) is

$$y = \frac{u_1}{2u} = -\frac{-Ae^{-x} - 4Be^{-4x}}{2(Ae^{-x} + Be^{-4x})} = -\frac{(A/B)e^{3x} + 4}{2[(A/B)e^{3x} + 1]}$$

or $\qquad 2y(1 + ce^{3x}) = -(4 + ce^{3x})$, where $c(= A/B)$ is an arbitrary constant.

Part (*ii*) Put the equation in standard form by dividing by x^2 and get $y_1 = -(2/x^2) + (2y/x) - y^2$. Now proceed as above, **Ans.** $y(cx + x^2) = c + 2x$.

4.3. Theorem. *The cross–ratio of any four particular integrals of a Riccati's equation is independent of x.*

Proof. We know that the general solution of Riccati's equation

$$y_1 = P + Qy + Ry^2 \qquad ...(1)$$

is of the form

$$y = (cg + G)/(cf + F). \qquad ...(2)$$

where g, G, f, F are appropriate functions of x and c is an arbitrary constant. Let $p(x)$, $q(x)$, $r(x)$, $s(x)$ be any four integrals of (1). Then these can be obtained from (2) by giving suitable values to c, say α, β, γ and δ. Thus, we have

$$p = \frac{\alpha g + G}{\alpha f + F}, \qquad q = \frac{\beta g + G}{\beta f + F}, \qquad r = \frac{\gamma g + G}{\gamma f + F}, \qquad s = \frac{\delta g + G}{\delta f + F}$$

Then, $\qquad p - q = \dfrac{(\alpha - \beta)(gF - fG)}{(\alpha f + F)(\beta f + F)}$, on simplifying.

Similarly $(p - s)$, $(r - s)$, $(r - q)$ can be obtained. Hence the cross–ratio

$$\frac{(p-q)(r-s)}{(p-s)(r-q)} = \frac{(\alpha - \beta)(\gamma - \delta)}{(\alpha - \delta)(\gamma - \beta)} = k, \text{ say,}$$

where k is indepedent of x. While forming the above cross–ratio, all the factors involving x cancel out and we are left with a mere constant. This completes the proof.

4.4. Method of solving Riccati's equation when three particular integrals are known.

Let the three particular integrals be $q(x)$, $r(x)$, $s(x)$ and let the corresponding values of c be β, γ, δ. Then as explained in Art. 4.3, we have

$$y = \frac{cg + G}{cf + F}, \qquad q = \frac{\beta g + G}{\beta f + F}, \qquad r = \frac{\gamma g + G}{\gamma f + F}, \qquad s = \frac{\delta g + G}{\delta f + F}$$

and then $\qquad \dfrac{(y-q)(r-s)}{(y-s)(r-q)} = \dfrac{(c-\beta)(\gamma-\delta)}{(c-\delta)(\gamma-\beta)} = k$, say

where k is independent of x. Thus, the general solution of Riccati's eqn. $y_1 = P + Qy + Ry^2$ is given by

$$\frac{(y-q)(r-s)}{(y-s)(r-q)} = k, \qquad ...(1)$$

where k is an arbitrary constant. It should be noted that in the present situation solution has been obtained without quadratures (*i.e.* integrations)

An illustration : *Show that 1, x, x^2 are three integrals of $x(x^2 - 1)y_1 + x^2 - (x^2 - 1)y - y^2 = 0$, and hence obtain the general solution $y(x + k) = x + kx^2$.*

Solution. Re–writing the given eqn. in standard form, we have

$$y_1 = -\frac{x}{x^2-1} + \frac{1}{x}y + \frac{1}{x(x^2-1)}y^2. \qquad \text{...(1)}'$$

Putting $y = 1$ so that $y_1 = 0$ in (1)′, we have

$$0 = -\frac{x}{x^2-1} + \frac{1}{x}\cdot 1 + \frac{1}{x(x^2-1)}\cdot 1 = 0,$$

showing that 1 is an integral of (1)′. Similarly we see that x and x^2 are also integrals of (1). Take $q(x) = 1$, $r(x) = x$ and $s(x) = x^2$. Then using formula (1) of above method of Art. 4.4, the general solution is

$$\frac{(y-1)(x-x^2)}{(y-x^2)(x-1)} = k \qquad \text{or} \qquad \frac{(y-1)(-x)}{(y-x^2)} = k$$

or $\quad k(y - x^2) + x(y - 1) = 0 \quad$ or $\quad y(k + x) = x + kx^2$,

where k is an arbitrary constant.

4.5. Method of solving Riccati's equation when two particular integrals are known.

Let $q(x)$ and $r(x)$ be two known integrals of the Riccati's equation

$$y_1 = P + Qy + Ry^2 \qquad \text{...(1)}$$

so that $\quad q_1 = P + Qq + Rq^2 \qquad$ ...(2)

and $\quad r_1 = P + Qr + Rr^2. \qquad$ ...(3)

From (1) and (2), we get

$$y_1 - q_1 = (y - q)Q + (y^2 - q^2)R \qquad \text{or} \qquad y_1 - q_1 = (y - q)\{Q + (y + q)R\}$$

or $\quad (y_1 - q_1)/(y - q) = Q + (y + q)R. \qquad$ ...(4)

Similarly, $\quad (y_1 - r_1)/(y - r) = Q + (y + r)R. \qquad$ ...(5)

From (4) and (5), we get $\quad \dfrac{y_1 - q_1}{y - q} - \dfrac{y_1 - r_1}{y - r} = (q - r)R$

Integrating this, we get $\quad \log (y - q) - \log (y - r) = c + \displaystyle\int (q - r)R \, dx$

or $\quad \log [(y - q)/(y - r)] = c + \displaystyle\int (q - r)R \, dx, \qquad$ ...(6)

which is the required general solution and it requires only one integration.

An illustration. *Show that there are two values of the constant k for which k/x is an integral of* $x^2(y_1 + y^2) = 2$, *and hence obtain the general solution.*

Solution. Re–writing the given equation in standard form

$$y_1 = P + Qy + Ry^2 \qquad \text{...(1)}$$

we get $\quad y_1 = (2/x^2) - y^2, \qquad$ ...(2)

Let $q(x)$ and $r(x)$ be two–particular integrals of (1), then as above we have (to be proved in examination for complete solution)

$$\log[(y - q)/(y - r)] = c + \int (q - r)R \, dx. \qquad \text{...(3)}$$

Let $y(x) = k/x$ so that $y_1 = -k/x^2$. Putting these in (2), we get

$-k/x^2 = 2/x^2 - k^2/x^2 \qquad$ or $\qquad k^2 - k - 2 = 0 \qquad$ so that $\qquad k = 2, -1.$

Hence $2/x$ and $-1/x$ are two particular integrals of (2). We take

$q(x) = 2/x \qquad$ and $\qquad r(x) = -1/x. \qquad$ ...(4)

Comparing (1) and (2), here $\qquad R = -1 \qquad$ ...(5)

Using (4) and (5), (3) gives the desired solution as

$$\log \frac{y-(2/x)}{y+(1/x)} = \log k + \int\left(\frac{2}{x}+\frac{1}{x}\right)(-1)\,dx, \text{ taking } c = \log k$$

or $\quad \log \dfrac{xy-2}{xy+1} = \log k - 3\log x \quad$ or $\quad \left(\dfrac{xy-2}{xy+1}\right)x^3 = k$

or $\quad x^3(xy-2) = k(xy+1)$, k being arbitrary constant.

4.6. Method of solving Riccati's equation when one particular integral is known

Let $q(x)$ be the known integral of the given Riccati's equation

$$y_1 = P + Qy + Ry^2 \qquad \text{...(1)}$$

so that $\quad q_1 = P + Qq + Rq^2. \qquad$ ...(2)

Let v be another dependent variable such that

$$y = q(x) + 1/v. \qquad \text{...(3)}$$

$\therefore \quad y_1 = q_1 - v_1/v^2 = P + Qq + Rq^2 - v_1/v^2$, by (2) $\qquad$ ...(4)

Using (3) and (4), (1) becomes

$$P + Qq + Rq^2 - \frac{v_1}{v^2} = P + Q\left(q+\frac{1}{v}\right) + R\left(q^2+\frac{2q}{v}+\frac{1}{v^2}\right)$$

or $\quad -\dfrac{v_1}{v} = \dfrac{Q}{v} + R\left(\dfrac{2q}{v}+\dfrac{1}{v^2}\right) \quad$ or $\quad \dfrac{dv}{dx} + (Q+2Rq)v = -R, \qquad$ ...(5)

which is linear differential equation of first order and first degree in v and x. Its integrating factor I.F. is given by I.F. $= e^{\int (Q+2Rq)\,dx}$ and hence the required general solution is

$$v e^{\int (Q+2Rq)\,dx} = -\int \mathrm{Re}^{\int (Q+2Rq)\,dx} + c,$$

where c is an arbitrary constant and v is to be replaced by $1/(y-q)$ [by using (3)].

Remark. The above mentioned method is very useful in practice.

4.7 Solved examples based on Art. 4.6.

Ex. 1. *Solve* $\quad y_1 = \cos x - y\sin x + y^2. \qquad$ ...(1)

Sol. Take $y = \sin x$ so that $y_1 = \cos x$. Putting these in (1), we get

$\cos x = \cos x - \sin^2 x + \sin^2 x$, showing that $\sin x$ is a particular solution of (1).

Let $\quad y = \sin x + 1/v \quad$ so that $\quad y_1 = \cos x - v_1/v^2$.

Using (2), (1) becomes

$$\cos x - v_1/v^2 = \cos x - \sin x\,(\sin x + 1/v) + (\sin x + 1/v)^2$$

or $\quad -\dfrac{v_1}{v^2} = \dfrac{\sin x}{v} + \dfrac{1}{v^2} \quad$ or $\quad \dfrac{dv}{dx} = (\sin x)\,v = -1 \qquad$ ...(3)

whose I.F. $= e^{\int \sin x\,dx} = e^{-\cos x}$ and so solution of (3) is

$$ve^{-\cos x} = c - \int e^{-\cos x}\,dx. \qquad \text{...(4)}$$

From (2), $v = 1/(y - \sin x)$ and hence (4) gives

$$\frac{e^{-\cos x}}{y-\sin x} = c - \int e^{-\cos x}\,dx$$

Ex. 2. *Solve* $y_1 = 2 + (1/2)\times(x - 1/x)y - y^2/2 \qquad$ ...(1)

Sol. Take $y = x + 1/x$ so that $y_1 = 1 - 1/x^2$. Then (1) gives

$$1 - 1/x^2 = 2 + \tfrac{1}{2}(x - 1/x)(x + 1/x) - \tfrac{1}{2}(x + 1/x)^2$$

or $$1 - 1/x^2 = 2 + \tfrac{1}{2}(x - 1/x^2) - \tfrac{1}{2}(x^2 + 2 + 1/x^2) \qquad \text{or} \qquad 1 - 1/x^2 = 1 - 1/x^2.$$

Hence $x + 1/x$ is a particular integral of (1). So we take

$y = x + 1/x + 1/v$ so that $y_1 = 1 - 1/x^2 - v_1/v^2$. ...(2)

Using (2) in (1), we get

$$1 - \frac{1}{x^2} - \frac{v_1}{v^2} = 2 + \frac{1}{2}\left(x - \frac{1}{x}\right)\left(x + \frac{1}{x} + \frac{1}{v}\right) - \frac{1}{2}\left(x + \frac{1}{x} + \frac{1}{v}\right)^2$$

or $$1 - \frac{1}{x^2} - \frac{1}{v^2} = 2 + \frac{1}{2}\left(x^2 - \frac{1}{x^2}\right) + \frac{1}{2v}\left(x - \frac{1}{x}\right) - \frac{1}{2}\left[\left(x + \frac{1}{x}\right)^2 + \frac{2}{v}\left(x + \frac{1}{x}\right) + \frac{1}{v^2}\right]$$

or $$\frac{dv}{dx} - \left(\frac{x}{2} + \frac{3}{2x}\right)v = \frac{1}{2}, \text{ on simplification.} \qquad ...(3)$$

Its I.F. $= e^{-\int(x/2 + 3/2x)dx} = e^{-x^2/4 - (3/2)\log x} = x^{-3/2} e^{-x^2/4}$. Hence solution of (3) is

$$vx^{-3/2} e^{-x^2/4} = c + \int \frac{1}{2} x^{-3/2} e^{-x^2/4} \, dx \cdot \qquad ...(4)$$

From (2), $1/v = y - x - 1/x$ so that $v = 1/(xy - x^2 - 1)$. ...(5)

Hence from (4) and (5), the required solution is

$$(x^{-3/2} e^{-x^2/4})/(xy - x^2 - 1) = c + \frac{1}{2}\int x^{-3/2} e^{-x^2/4} \, dx,$$

EXERCISE

1. $y_1 = P(1 - xy) - y^2$, P is a function of x.

Hint. $1/x$ is a particular integral. The solution is $\{1/(x^2 y - x\} e^{-\int xPdx} = c - \int x^{-2} e^{-\int xPdx} \, dx$

2. $x(1 - x^3)y_1 = x^2 + y - 2xy^2$, x^2 is an integral. **Ans.** $(x^4 - x)/(y - x^2) = c - 2x^3/3$

3. $y_1 = 1 + y^2$, $\tan x$ is an integral. **Ans.** $y(c - \tan x) = c \tan x + 1$.

4. $y_1 = 2x - (x^2 + 1)y + y^2$, $x^2 + 1$ is an integral.

5. $x^3 y_1 = x^2 y + y^2 - x^2$, x is an integral. **Ans.** $y(ce^{2/x} - 1) = x + cx\, e^{2/x}$

6. $y_1 = x + y(1 - 2x) - y^2(1 + x)$, 1 is an integral. **Ans.** $y = 1 + 1/(x + ce^x)$

7. $x(x - 1)y_1 - (2x + 1)y + y^2 + 2x = 0$, x is a solution. **Ans.** $y = (x^2 + c)/(x + c)$

8. $y_1 = x + [(1/x) - x^3]y + xy^2$, x^2 is a solution. **Ans.** $y = x^2 - (xe^{x^2/4})/(c + \int x^2 e^{x^2/4} \, dx$

9. $y_1 = e^{2x} - ye^x + e^{-x}y^2$, e^x is a solution. **Ans.** $y = e^x - (e^{2x - e^x}) \Big/ \left[c + \int e^{x - e^x} \, dx\right]$

5

Chebyshev Polynomials

5.1. Chebyshev Polynomials

The Chebyshev polynomials of first kind, $T_n(x)$, and second kind, $U_n(x)$ are defined by

$$T_n(x) = \cos(n \cos^{-1} x) \qquad \text{... (1)}$$

and
$$U_n(x) = \sin(n \cos^{-1} x) \qquad \text{... (2)}$$

where n is a non-negative integer.

Remark. Chebyshev polynomials are also known as Tchebicheff, Tchebieheff or Tschebysheff.

5.2. Theroem. *$T_n(x)$ and $U_n(x)$ are independent solutions of Chebyshev's equation*

$$(1-x^2)(d^2y/dx^2) - x(dy/dx) + n^2 y = 0. \qquad \textbf{[Kanpur, 2005; Garhwal 2005]}$$

Proof. The Chebyshev's equation is $\quad (1-x^2)(d^2y/dx^2) - x(dy/dx) + n^2 y = 0 \quad$ ... (1)

To show that $T_n(x)$ is a solution of (1) : By definition, we have

$$T_n(x) = \cos(n \cos^{-1} x) \qquad \text{... (2)}$$

$$\therefore \qquad \frac{d}{dx} T_n(x) = \frac{d}{dx} \cos(n \cos^{-1} x) = -\sin(n \cos^{-1} x) \cdot n \cdot \frac{-1}{(1-x^2)^{1/2}}$$

or
$$\frac{d}{dx} T_n(x) = \frac{n}{(1-x^2)^{1/2}} \sin(n \cos^{-1} x) \qquad \text{... (3)}$$

and
$$\frac{d^2}{dx^2} T_n(x) = \frac{d}{dx}\left(\frac{d}{dx} T_n(x)\right) = n \frac{d}{dx}[(1-x^2)^{-1/2} \sin(n \cos^{-1} x)]$$

$$= n\left[-\frac{1}{2}(1-x^2)^{-3/2}(-2x) \cdot \sin(n \cos^{-1} x) + (1-x^2)^{-1/2} \cos(n \cos^{-1} \cdot x) \cdot n \cdot \frac{1}{(1-x^2)^{1/2}}\right]$$

Thus,
$$\frac{d^2}{dx^2} T_n(x) = \frac{nx}{(1-x^2)^{3/2}} \sin(n \cos^{-1} x) - \frac{n^2}{1-x^2} \cos(n \cos^{-1} x). \qquad \text{... (4)}$$

Using (2), (3) and (4), we have

$$(1-x^2)\frac{d^2}{dx^2} T_n(x) - x \frac{d}{dx} T_n(x) + n^2 T_n(x)$$

$$= \frac{nx}{(1-x^2)^{1/2}} \sin(n \cos^{-1} x) - n^2 \cos(n \cos^{-1} x) - \frac{nx}{(1-x^2)^{1/2}} \sin(n \cos^{-1} x) + n \cos(n \cos^{-1} x) = 0,$$

showing that $T_n(x)$ is a solution of (1).

To show that $U_n(x)$ is a solution of (1) : Proceed as above. Left as an exercise for students.

To show that $T_n(x)$ and $U_n(x)$ are independent solutions of (1) : We have, by definition

$$T_n(x) = \cos(n \cos^{-1} x) \qquad \text{and} \qquad U_n(x) = \sin(n \cos^{-1} x).$$

$$\therefore \; T_n(1) = \cos(n \cos^{-1} 1) = \cos(n \times 0) = 1 \quad \text{and} \quad U_n(1) = \sin(n \cos^{-1} 1) = \sin(n \times 0) = 0.$$

Hence $U_n(x)$ cannot be expressed as a constant multiple of $T_n(x)$. This shows that $T_n(x)$ and $U_n(x)$ are independent solutions of (1).

Remark. Two solutions $u(x)$ and $v(x)$ are said to be linearly independent if $u(x)$ cannot be expressed as a constant multiple of $v(x)$ *i.e.*, $u(x) = k\, v(x)$, where k is a constant.

5.3. Orthogonal properties of Chebyshev polynomials.

To show that

$$(i)\ \int_{-1}^{1} \frac{T_m(x)T_n(x)}{\sqrt{(1-x^2)}}dx = \begin{cases} 0, & m \neq n \\ \pi/2, & m = n \neq 0 \\ \pi, & m = n = 0 \end{cases} \qquad (ii)\ \int_{-1}^{1} \frac{U_m(x)U_{(x)}}{\sqrt{(1-x^2)}}dx = \begin{cases} 0, & m \neq n \\ \pi/2, & m = n \neq 0 \\ 0, & m = n = 0 \end{cases}$$

[*Kanpur 2011*]

Proof. (i) We have, by definition

$T_m(x) = \cos(m \cos^{-1} x)$ and $T_n(x) = \cos(n \cos^{-1} x)$. ... (1)

$\therefore\ T_m(\cos\theta) = \cos(m\cos^{-1}\cos\theta) = \cos m\theta,\qquad T_n(\cos\theta) = \cos(n\cos^{-1}\cos\theta) = \cos n\theta$ ... (2)

Let $$I = \int_{-1}^{1} \frac{T_m(x)T_n(x)}{\sqrt{(1-x^2)}}dx. \qquad \text{... (3)}$$

Putting $x = \cos\theta$ so that $dx = -\sin\theta\, d\theta$ and using (2), (3), reduces to

$$I = \int_{\pi}^{0} \frac{\cos m\theta \cos n\theta}{\sin\theta}(-\sin\theta)\, d\theta \qquad \text{or} \qquad I = \int_{\pi}^{0} \cos m\theta \cos n\theta\, d\theta \qquad \text{... (4)}$$

Case 1. Let $m \neq n$ so that $(m-n) \neq 0.$ then, (4) gives

$$I = \frac{1}{2}\int_0^{\pi} 2\cos m\theta \cos n\theta d\theta = \frac{1}{2}\int_0^{\pi} [\cos(m+n)\,\theta + \cos(m-n)\,\theta]\, d\theta \; = \frac{1}{2}\left[\frac{\sin(m+n)\theta}{m+n} + \frac{\sin(m-n)\theta}{m-n}\right]_0^{\pi} = 0$$

Case 2. Let $m = n \neq 0.$ Then (4) gives

$$I = \int_0^{\pi} \cos^2 m\theta\, d\theta = \int_0^{\pi} \frac{1+\cos 2m\theta}{2} d\theta = \frac{1}{2}\left[\theta + \frac{\sin 2m\theta}{2m}\right]_0^{\pi} = \frac{\pi}{2}.$$

Case 3. Let $m = n = 0$. Then $\cos m\theta = \cos n\theta = 1.$ Then (4) gives

$$I = \int_0^{\pi} (1)\,(1) dx = [\theta]_0^{\pi} = \pi.$$

From cases 1, 2 and 3, the required result (i) follows.

Part (*ii*). We have, by definition

$U_m(x) = \sin(m \cos^{-1} x)$ and $U_n(x) = \sin(n \cos^{-1} x)$ ... (1)

$\therefore\ U_m(\cos\theta) = \sin(m\cos^{-1}\cos\theta) = \sin m\theta,\qquad U_n(\cos\theta) = \sin(n\cos^{-1}\cos\theta) = \sin n\theta.$ ... (2)

Let $$J = \int_{-1}^{1} \frac{U_m(x)\, U_n(x)}{\sqrt{1-x^2}}dx. \qquad \text{... (3)}$$

Putting $x = \cos\theta$ so that $dx = -\sin\theta d\theta$ and using (2), (3) gives

$$J = \int_{\pi}^{0} \frac{\sin m\theta \sin n\theta}{\sin\theta}(-\sin\theta d\theta) \qquad \text{or} \qquad J = \int_0^{\pi} \sin m\theta \sin n\theta\, d\theta \qquad \text{... (4)}$$

Case 1. Let $m \neq n$ so that $(m-n) \neq 0.$ Then, (4) gives

$$J = \frac{1}{2}\int_0^{\pi} 2\sin m\theta \sin n\theta d\theta = \frac{1}{2}\int_0^{\pi} [\cos(m-n)\theta - \cos(m+n)\theta]\, d\theta = \frac{1}{2}\left[\frac{\sin(m-n)\theta}{m-n} - \frac{\cos(m+n)\,\theta}{m+n}\right]_0^{\pi} = 0.$$

Case 2. Let $m = n \neq 0.$ Then (4) gives

$$J = \int_0^{\pi} \sin^2 m\theta \, d\theta = \int_0^{\pi} \frac{1-\cos 2m\theta}{2} d\theta = \frac{1}{2}\left[\theta - \frac{\sin 2m\theta}{2m}\right]_0^{\pi} = \frac{\pi}{2}.$$

Case 3. Let $m = n = 0$. Then $\sin m\theta = \sin n\theta = 0.$ Hence (4) gives

$$J = \int_0^{\pi} (0)\,(0)\, d\theta = 0.$$

From cases 1, 2 and (3), the required result (*ii*) follows

5.4. Recurrence relations (formulae) [Kanpur 2011]

I. $T_{n+1}(x) - 2xT_n(x) + T_{n-1}(x) = 0.$

II. $(1 - x^2)\, T_n'(x) = -nx\, T_n(x) + nT_{n-1}(x).$

III. $U_{n+1}(x) - 2x\, U_n(x) + U_{n-1}(x) = 0.$

IV. $(1 - x^2)\, U_n'(x) = -nx\, U_n(x) + nU_{n-1}(x).$

Proof 1. We have, by definition $\quad T_n(x) = \cos (n \cos^{-1} x) \quad$ ... (1)

$$\therefore \qquad T_n(\cos\theta) = \cos(n\cos^{-1}\cos\theta) = \cos n\theta \qquad \text{... (2)}$$

so that $\quad T_{n+1}(\cos\theta) = \cos(n+1)\theta \quad$ and $\quad T_{n-1}(\cos\theta) = \cos(n-1)\theta$ ... (3)

We are to show that $\quad T_{n+1}(x) - 2x\, T_n(x) + T_{n-1}(x) = 0.$ ... (4)

Replacing x by $\cos\theta$ in (4), we must now prove that

$$T_{n+1}(\cos\theta) - 2\cos\theta\, T_n(\cos\theta) + T_{n-1}(\cos\theta) = 0 \qquad \text{... (5)}$$

i.e. $\quad \cos(n+1)\theta - 2\cos\theta\cos n\theta + \cos(n-1)\theta = 0,$ by (2) and (3)

i.e., $\quad \cos(n+1)\theta + \cos(n-1)\theta - 2\cos\theta\cos n\theta = 0.$ ... (6)

Now, L.H.S. (6) $= 2\cos n\theta\cos\theta - 2\cos\theta\cos n\theta = 0,$

which proves (6) and hence (4) is true.

II. From (1), $T_n'(x) = -\sin(n\cos^{-1} x)\cdot\dfrac{-n}{\sqrt{(1-x^2)}}$ or $\quad T_n'(\cos\theta) = \sin(n\cos^{-1}\cos\theta)\cdot\dfrac{n}{\sqrt{(1-\cos^2\theta)}}$

Thus, $\quad T_n'(\cos\theta) = (n \sin n\,\theta)/\sin\theta$ ... (7)

We are to show that $\quad (1-x^2)T_n'(x) = -nxT_n(x) + n\,T_{n-1}(x).$ ... (8)

Putting $x = \cos\theta$ and using (3) and (7), (8) may be re-written as

$\sin^2\theta\dfrac{n\sin n\theta}{\sin\theta} = -n\cos\theta\cos n\theta + n\cos(n-1)\theta \quad$ or $\quad \sin\theta\sin n\theta = \cos(n-1)\theta - \cos\theta\cos n\theta.$ ...
(9)

R.H.S. of (9) $= \cos(n\theta - \theta) - \cos\theta\cos n\theta = \cos n\theta\cos\theta + \sin n\theta\sin\theta - \cos\theta\cos n\theta$

$= \sin n\theta\sin\theta$ = L.H.S. of (9),

which proves (9) and hence (8) is true.

III and IV. Proceed as above. Left as exercises for students.

5.5. Some theorems on Chebyshev polynomials

Theorem I. *To show that* (*i*) $T_n(x) = (1/2)\times[\{x + i(1-x^2)^{1/2}\}^n + \{x - i(1-x^2)^{1/2}\}^n].$

(*ii*) $U_n(x) = -(i/2)\times[\{x + i(1-x^2)^{1/2}\}^n - \{x - i(1-x^2)^{1/2}\}^n].$

Proof. (*i*) Putting $x = \cos\theta$, and using definition, we have

$$T_n(x) = \cos(n\cos^{-1} x), \quad = \cos(n\cos^{-1}\cos\theta) = \cos n\theta = (e^{in\theta} + e^{-in\theta})/2$$
$$= (1/2)\times\{(e^{i\theta})^n + (e^{-i\theta})^n\} = (1/2)\times\{(\cos\theta + i\sin\theta)^n + (\cos\theta - i\sin\theta)^n\}$$
$$= (1/2)\times[\{\cos\theta + i(1-\cos^2\theta)^{1/2}\}^n + \{\cos\theta - i(1-\cos^2\theta)^{1/2}\}^n]$$
$$= (1/2)\times[\{x + i(1-x^2)^{1/2}\}^n + \{x - i(1-x^2)^{1/2}\}^n], \text{ as } \quad x = \cos\theta$$

Part (*ii*) Putting $x = \cos\theta$, and using definition, we have

$$U_n(x) = \sin(n\cos^{-1} x)$$
$$= \sin n\theta = \frac{1}{2i}(e^{in\theta} - e^{in\theta}) = \frac{1}{2i}\{(e^{i\theta})^n - (e^{-i\theta})^n\} = \frac{1}{2i}\{(\cos\theta + i\sin\theta)^n - (\cos\theta - i\sin\theta)^n\}$$
$$= [\{\cos\theta + i(1-\cos^2\theta)^{1/2}\}^n - \{\cos\theta - i(1-\cos^2\theta)^{1/2}\}^n]/2i$$
$$= \frac{1}{2i}[\{x + i(1-x^2)^{1/2}\}^n - \{x - i(1-x^2)^{1/2}\}^n] = \frac{-i}{2}[\{x + i(1-x^2)^{1/2}\}^n - \{x - i(1-x^2)^{1/2}\}^n], \text{ as } x = \cos\theta$$

Theorem II. *To show that* (*i*) $T_n(x) = \sum_{r=0}^{[n/2]} (-1)^r \dfrac{n!}{(2r)!\,(n-2r)!}(1-x^2)^r x^{n-2r}$

(*ii*) $U_n(x) = \sum_{r=0}^{[(n-1)/2]} (-1)^r \dfrac{n!}{(2r+1)!\,(n-2r-1)!}(1-x^2)^{r+1/2} x^{x-2r-1}.$

Proof. (*i*) As in part (*i*) of Art. 5.5, we have

$$T_n(x) = \frac{1}{2}[\{x + i(1-x^2)^{1/2}\}^n + \{x - i(1-x^2)^{1/2}\}^n] = \frac{1}{2}\left[\sum_{s=0}^{n} {}^nC_s x^{n-s}\{i(1-x^2)^{1/2}\}^s + \sum_{s=0}^{n} {}^nC_s x^{n-s}\{-i(1-x^2)^{1/2}\}^s\right]$$

[Since by binomial theorem, we have

$$(a+b)^n = a^n + {}^nC_1 a^{n-1}b + {}^nC_2 a^{n-2}b^2 + \ldots + {}^nC_s a^{n-s}b^s + \ldots + {}^nC_n b^n = \sum_{s=0}^{n} {}^nC_s a^{n-s} b^s\ ]$$

$$\therefore \qquad T_n(x) = (1/2)\times\sum_{s=0}^{n} {}^nC_s x^{n-s}(1-x^2)^{s/2} i^s\{1 + (-1)^s\}. \qquad \ldots (1)$$

But
$$1 + (-1)^s = \begin{cases} 0, \text{ if } s \text{ is odd} \\ 2, \text{ if } s \text{ is even} \end{cases} \qquad \ldots (2)$$

Using (2), (1) reduces to
$$T_n(x) = \sum_{s\ even,\ \le n} {}^nC_s x^{n-s}(1-x^2)^{s/2} i^s \qquad \ldots (3)$$

Since s is even in (3), we take s = 2r where r is an integer. Hence $s \le n \Rightarrow 2r \le n \Rightarrow r \le n/2$. Now, if n is even, *r* goes from 0 to *n*/2, while if *n* is odd, *r* goes from 0 to (*n* – 1)/2; that is, in all cases, *r* goes from 0 to [*n*/2], where

$$[n/2] = \begin{cases} n/2, \text{ if } n \text{ is even} \\ (n-1)/2, \text{ if } n \text{ is odd.} \end{cases}$$

Hence with *s* = 2*r* and the above arguments, (3) reduces to

$$T_n(x) = \sum_{r=0}^{[n/2]} {}^nC_{2r} x^{n-2r}(1-x^2)^r i^{2r} = \sum_{r=0}^{[n/2]} \frac{n!}{(n-2r)!\,(2r)!} x^{n-2r}(1-x^2)^r(-1)^r, \text{ as } i^{2r} = (i^2)^r = (-1)^r$$

Part (*ii*). As in part (*ii*) of Art. 5.5, we have

$$U_n(x) = (-i/2)\times[\{x + i(1-x^2)^{1/2}\}^n - \{x - i(1-x^2)^{1/2}\}^n]$$

$$= (-i/2)\times\left[\sum_{s=0}^{n} {}^nC_s x^{n-s}\{i(1-x^2)^{1/2}\}^s - \sum_{s=0}^{n} {}^nC_s x^{n-s}\{-i(1-x^2)^{1/2}\}^s\right], \text{ using binomial theorem}$$

$$= (-i/2)\times \sum_{s=0}^{n} {}^nC_s x^{n-s}(1-x^2)^{s/2} i^s\{1-(-1)^s\}. \qquad \dots (4)$$

But $$1-(-1)^s = \begin{cases} 0, \text{ if } s \text{ is even} \\ 2, \text{ if } s \text{ is odd} \end{cases} \qquad \dots (5)$$

Using (5), (4) reduces to $$U_n(x) = -i \sum_{s \text{ odd}, s\le n} {}^nC_s x^{n-s}(1-x^2)^{s/2} i^s. \qquad \dots (6)$$

Since s is odd in (6), we take $s = 2r + 1$, where r is an integer. Hence $s \le n \;\Rightarrow\; 2r+1 \le n$ $\Rightarrow\; r \le (n-1)/2.$ Now, if n is odd, r goes from 0 to $(n - 1)/2$, while if n is even r goes from 0 to $(n - 2)/2$; that is in all cases, r goes from 0 to $[(n - 1)/2]$, where

$$[(n-1)/2] = \begin{cases} (n-1)/2, \text{ if } n \text{ is odd} \\ (n-2)/2, \text{ if } n \text{ is even.} \end{cases}$$

Hence with $s = 2r + 1$ and the above arguments, (6) reduces to

$$U_n(x) = -i \sum_{r=0}^{[(n-1)/2]} {}^nC_{2r+1} x^{n-2r-1}(1-x^2)^{r+1/2} i^{2r+1} = \sum_{r=0}^{[(n-1)/2]} (-1)^r \frac{n!}{(2r+1)!(n-2r-1)!}(1-x^2)^{r+1/2} x^{n-2r-1}$$

$$[\because\; i^{2r+1} = (i^2)^r \times i = (-1)^r i \text{ and } i^2 = -1]$$

5.6. First few Chebyshev polynomials.

We have $$T_n(x) = \cos(n\cos^{-1} x) \qquad \dots (1)$$

and $$U_n(x) = \sin(n\cos^{-1} x) \qquad \dots (2)$$

Putting $n = 0$ and 1 successively in (1), we get

$T_0(x) = \cos(0 \times \cos^{-1} x) = \cos 0 = 1$ and $T_1(x) = \cos(\cos^{-1} x) = x.$

Putting $n = 2$ in part (*i*) of theorem II of Art. 5.5, we have

$$T_2(x) = \sum_{r=0}^{1} (-1)^r \frac{2!}{(2r)!(2-2r)!}(1-x^2)^r x^{2-2r}, \text{ as } [n/2] = n/2 \text{ if } n \text{ is even}$$

$$= x^2 - (1 - x^2) = 2x^2 - 1.$$

Next, putting $n = 3$ in part (*i*) of theorem II of Art. 5.5, we have

$$T_3(x) = \sum_{r=0}^{1} (-1)^r \frac{3!}{(2r)!(3-2r)!}(1-x^2)^r x^{3-3r}, \quad \text{as} \quad [n/2] = (n-1)/2, \text{ if } n \text{ is odd}$$

$$= x^3 - 3(1 - x^2) \,.\, x = 4x^3 - 3x.$$

and so on. Thus, we see that $T_n(x)$ is a polynomial of degree n.

Now, putting $n = 0$ and 1 successively in (2), we get

$$U_0(x) = \sin(0\times\cos^{-1} x) = \sin 0 = 0 \quad \text{and} \quad U_1(x) = \sin\cos^{-1} x = \sin\sin^{-1}(1-x^2)^{1/2} = (1-x^2)^{1/2}.$$

Putting $n = 2$ in part (*ii*) of theorem II of Art 5.5, we have

$$U_2(x) = \sum_{r=0}^{0} (-1)^r \frac{2!}{(2r+1)!(2-2r-1)!}(1-x^2)^{r+1/2} x^{2-2r-1} = 2(1-x^2)^{1/2}\cdot x = 2x\sqrt{(1-x^2)}.$$

Putting $n = 3$ in part (*ii*) of theorem II of Art. 5.5, we have

$$T_3(x) = \sum_{r=1}^{1} (-1)^r \frac{3!}{(2r+1)!(3-2r-1)!}(1-x^2)^{r+1/2}\, x^{3+2r-1}$$

$$= 3\,(1-x^2)^{1/2} \,.\, x^2 + (-1)^1\,(1-x^2)^{3/2} = (4x^2-1)(1-x^2)^{1/2}. \quad (\because 0! = 1)$$

Remark 1. We observe that $U_n(x)$ is not a polynomial. However, if we define the Chebyshev polynomial of second kind by

$$U_u^*(x) = \sin\{(n+1)\cos^{-1} x\}(1-x^2)^{1/2} = \{1/(1-x^2)^{1/2}\}/U_{n+1}, \qquad \text{... (3)}$$

then $U_n^*(x)$ will be polynomial of degree n. This can be easily verified by using (3).

Remark 2. Putting $x = \cos\theta$ in (1) and (2), we have

$$T_n(\cos\theta) = \cos n\theta \qquad \text{and} \qquad U_n(\cos\theta) = \sin n\theta. \quad \text{... (4)}$$

Thus, we conclude that Chebashev polynomials can be used to obtain explansions of $\cos n\theta$ and $\sin n\theta/\sin\theta$ in terms of $\cos\theta$.

5.7. Generating functions for Chebyshev polynomials.

$$(i)\ \frac{1-t^2}{1-2tx-t^2} = T_0(x) + 2\sum_{n=1}^{\infty} T_n(x)\, t^n \qquad (ii)\ \frac{(1-x^2)^{1/2}}{1-2tx+t^2} = \sum_{n=0}^{\infty} U_{n+1}(x)\, t^n.$$

Proof. (*ii*) To prove the desired result we must show that in the expansion of L.H.S. in ascending powers of t the coefficient of t^n for $n \geq 1$ is $2T_n(x)$ and the coefficient of t^n is $T_0(x)$ for $n = 0$. Putting $x = \cos\theta = (e^{i\theta} + e^{-i\theta})/2$, we find that

L.H.S. of the required result

$$= \frac{1-t^2}{1-(e^{i\theta}+e^{-i\theta})t+t^2} = \frac{1-t^2}{1-e^{i\theta}t-e^{-i\theta}t+e^{i\theta}t\cdot e^{-i\theta}t} = \frac{1-t^2}{(1-e^{i\theta}t)-e^{-i\theta}t\,(1-e^{i\theta}t)} = \frac{1-t^2}{(1-e^{i\theta}t)\,(1-e^{-i\theta}t)}$$

$$= (1-t^2)\,(1-e^{i\theta}t)^{-1}(1-e^{-i\theta}t)^{-1} = (1-t^2)\sum_{r=0}^{\infty}(e^{i\theta}t)^r \sum_{s=0}^{\infty}(e^{-i\theta}t)^s$$

$$= (1-t^2)\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} e^{i(r-s)\theta} t^{r+s} = \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} e^{i(r-s)\theta} t^{r+s} + \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} e^{i(r-s)\theta} t^{r+s+2} \qquad \text{... (1)}$$

Note that we need not consider the summation for the coefficient of t^0. Taking $r = 0$, $s = 0$ in the first summation in (1), we find that

Coefficient in t^n in L.H.S. of the required result $= e^{i(0-0)\theta} = e^0 = 1 = T_0(x)$. $[\because\ T_0(x) = 1]$

For $n \geq 1$, we get the coefficient of t^n by taking $r + s = n$, (*i.e.*, $s = n - r$, so that $s \geq 0$ $\Rightarrow\ n - r \geq 0\ \Rightarrow\ r \leq n$ for the total coefficient of t^n) in the first summation in (1) and $r + s + 2 = n$, (*i.e.*, $s = n - r - 2$ so that $s \geq 0\ \Rightarrow\ n-r-2 \geq 0\ \Rightarrow\ r \leq n-2$ for the total coefficient of t^n), in the second summation in (1).

$\therefore$ the total coefficient of t^n in (1)

$$= \sum_{r=0}^{n} e^{i\{r-(n-r)\}\theta} - \sum_{r=0}^{n-2} e^{i\{r-(n-r-2)\}\theta} = e^{-in\theta}\sum_{r=0}^{n} e^{2ir\theta} - e^{-i(n-2)\theta}\sum_{r=0}^{n-2} e^{2ir\theta}$$

$$= e^{-in\theta}[1 + e^{2\theta i} + e^{4\theta i} + \ldots \text{ to } (n+1) \text{ terms}] - e^{-i(n-2)\theta}[1 + e^{2\theta i} + e^{4\theta i} + \ldots \text{ to } (n-1) \text{ terms}]$$

$$= e^{-in\theta}\,\frac{1-(e^{2i\theta})^{n+1}}{1-e^{2i\theta}} - e^{-i(n-2)\theta}\,\frac{1-(e^{2i\theta})^{n-1}}{1-e^{2i\theta}}$$

[$\because$ For geometric series, $a + ar + ar^2 + \ldots$ to n term $= \{a(1-r^n)\}/(1-r)$]

$$= \frac{e^{-in\theta} - e^{i(n+2)\theta}}{1-e^{2i\theta}} - \frac{e^{-i(n-2)\theta} - e^{in\theta}}{1-e^{2i\theta}} = \frac{e^{-in\theta} - e^{-i(n-2)\theta} + e^{in\theta} - e^{i(n+2)\theta}}{1-e^{2i\theta}}$$

$$= \frac{e^{-in\theta}(1-e^{2i\theta}) + e^{in\theta}(1-e^{2i\theta})}{(1-e^{2i\theta})} = \frac{(1-e^{2i\theta})(e^{-in\theta}+e^{in\theta})}{(1-e^{2i\theta})} = e^{in\theta} + e^{-in\theta}$$

$$= 2\cos n\,\theta, \text{ as } \cos n\,\theta = (e^{in\theta}+e^{-in\theta})/2$$

$= 2T_n(x).$ $[\because\ T_n(x) = \cos(n\cos^{-1} x) = \cos(n\cos^{-1}\cos\theta) = \cos n\theta]$

From the above arguments, the required result follows.

Part (ii). To Prove the desired result we must show that in the expansion of L.H.S. in ascending powers to t, the coefficient of t^n for $n \geq 1$ is $U_{n+1}(x)$.

Now, L.H.S. $= \dfrac{\sqrt{(1-x^2)}}{1-2tx+t^2} = \dfrac{\sqrt{(1-\cos^2\theta)}}{1-2t\cos\theta+t^2}$, putting $x = \cos\theta$

$$= \frac{\sin\theta}{1-t(e^{i\theta}+e^{-i\theta})+t^2} = \frac{\sin\theta}{1-e^{i\theta}t-e^{-i\theta}t+e^{i\theta}t\times e^{-i\theta}t} = \frac{\sin\theta}{(1-e^{i\theta}t)-e^{-i\theta}t(1-e^{i\theta}t)} - \frac{\sin\theta}{(1-e^{i\theta}t)(1-e^{-i\theta}t)}$$

$$= \sin\theta\,(1-e^{i\theta}t)^{-1}(1-e^{-i\theta}t)^{-1} = \sin\theta \sum_{r=0}^{\infty} (e^{i\theta}t)^r \sum_{s=0}^{\infty} (e^{-i\theta}t)^s = \sin\theta \sum_{r=0}^{\infty}\sum_{s=0}^{\infty} e^{i(r-s)\theta} t^{r+s} \qquad \text{... (2)}$$

We get the coefficient of t^n by taking $r + s = n$ (*i.e.*, $s = n - r$, so that $s \geq 0 \Rightarrow n - r \geq 0$ $\Rightarrow r \leq n$ for the total coefficient of t^n)

$\therefore$ The total coefficient of t^n in (2) $= \sin\theta \sum_{r=0}^{n} e^{i[r-(n-r)]\theta} = \sin\theta \sum_{r=0}^{n} e^{-in\theta}\cdot e^{2r\theta i}$

$$= \sin\theta\, e^{-in\theta} \sum_{r=0}^{n} e^{2r\theta i} = \sin\theta\, e^{-in\theta}[1+e^{2i\theta}+e^{4i\theta}+e^{6i\theta}+\ldots \text{ to } (n+1) \text{ terms}]$$

$$= \sin\theta\, e^{-in\theta}\,\frac{1-(e^{2i\theta})^{n+1}}{1-e^{2i\theta}} = \frac{e^{i\theta}-e^{-i\theta}}{2i} \times \frac{e^{-in\theta}(1-e^{2i\theta(n+1)})}{-e^{i\theta}(e^{i\theta}-e^{-i\theta})}$$

$$= \frac{e^{-i(n+1)\theta}[e^{2i\theta(n+1)}-1]}{2i} = \frac{e^{i(n+1)\theta}-e^{-i(n+1)\theta}}{2i} = \sin(n+1)\,\theta$$

$= \sin\{(n+1)\cos^{-1} x\} = U_{n+1}(x)$ by definition, as $x = \cos\theta \Rightarrow \theta = \cos^{-1} x$

Form the above argument, we get the required result.

5.8. Special values of the Chebyshev polynomials.

To show that (*i*) $T_n(1) = 1$, $T_n(-1) = (-1)^n$, $T_{2n}(0) = (-1)^n$, $T_{2n+1}(0) = 0$.

(*ii*) $U_n(1) = 0$, $U_n(-1) = 0$, $U_{2n}(0) = 0$, $U_{2n+1}(0) = (-1)^n$.

Proof (*i*) We have, by definition $T_n(x) = \cos(n\cos^{-1} x)$. ... (1)

Putting $x = 1$ in (1), we have $T_n(1) = \cos(n\cos^{-1} 1) = \cos(n \times 0) = 1$.

Putting $x = -1$ in (1), we have $T_n(-1) = \cos[n\cos^{-1}(-1)] = \cos(n\pi) = (-1)^n$.

Replacing x by 0 and n by $2n$ in (1), we have

$$T_{2n}(0) = \cos(2n\cos^{-1} 0) = \cos\{(2n)\times(\pi/2)\} = \cos n\pi = (-1)^n.$$

Replacing x by 0 and n by $2n + 1$ in (1), we have

$$T_{2n+1}(0) = \cos[(2n+1)\cos^{-1} 0] = \cos\{(2n+1)\times(\pi/2)\} = 0.$$

Part (ii) Proceed as above yourself.

5.9. Illustrative Solved Examples.

Ex. 1. *Show that* $T_m\{T_n(x)\} = T_n\{T_m(x)\} = T_{nm}(x)$.

Sol. We have, by definition

$$T_m\{T_n(x)\} = T_m[\cos(n\cos^{-1}x)]$$
$$= \cos[m\cos^{-1}\{\cos(n\cos^{-1}x)\}], \text{ by definition again}$$
$$= \cos(nm\cos^{-1}x). \quad \text{... (1)}$$

Again, $$T_n\{T_m(x)\} = T_n[\cos(m\cos^{-1}x)], \text{ by definition}$$
$$= \cos[n\cos^{-1}\{\cos(m\cos^{-1}x)\}], \text{ by definition again}$$
$$= \cos(nm\cos^{-1}x), \quad \text{... (2)}$$

Finally, $$T_{mn}(x) = \cos(mn\cos^{-1}x), \text{ by definition} \quad \text{... (3)}$$

From (1), (2) and (3), we get the required result.

Ex. 2. *Show that* $(1-x^2)^{1/2}\,T_n(x) = U_{n+1}(x) - xU_n(x)$.

Sol. By definition, $T_n(x) = \cos(n\cos^{-1}x)$ and $U_n(x) = \sin(n\cos^{-1}x)$ ... (1)

Putting $x = \cos\theta$, (1) reduces to $T_n(\cos\theta) = \cos n\theta$ and $U_n(\cos\theta) = \sin n\theta$. ... (2)

Replacing x by $\cos\theta$, the required result takes the following form

$$\sin\theta T_n(\cos\theta) = U_{n+1}(\cos\theta) - \cos\theta\, U_n(\cos\theta),\ i.e.,\ \sin\theta\cos n\theta = \sin(n+1)\theta - \cos\theta\sin n\theta, \text{ by (2)} \quad \text{...(3)}$$

Now, R.H.S. of (3) $= \sin(n\theta+\theta) - \cos\theta\sin n\theta = \sin n\theta\cos\theta + \cos n\theta\sin\theta - \cos\theta\sin n\theta$

$= \sin\theta\cos n\theta$ = L.H.S. of (3)

This proves the required result.

Ex. 3. *Show that* $\displaystyle\sum_{r=0}^{n} T_{2r}(x) = \frac{1}{2}\left(1 + \frac{1}{(1-x^2)^{1/2}}U_{2n+1}(n)\right)$. **[Kanpur 2007, 11]**

Sol. Let $x = \cos\theta$. Then by definition, we have

$$\sum_{r=0}^{n} T_{2r}(x) = \sum_{r=0}^{n}\cos(2r\cos^{-1}x) = \sum_{r=0}^{n}\cos(2r\cos^{-1}\cos\theta) = \sum_{r=0}^{n}\cos 2r\theta$$

Now, $$\sum_{r=0}^{n} T_{2r}(x) = \text{Real part of } \sum_{r=0}^{n} e^{2ri\theta}. \quad \text{... (1)}$$

But $$\sum_{r=0}^{n} e^{2ri\theta} = 1 + e^{2i\theta} + e^{4i\theta} + \ldots \text{ to } (n+1) \text{ terms}$$

$$= \frac{1-(e^{2i\theta})^{n+1}}{1-e^{i\theta}} \qquad \left[\because\ a + ar + ar^2 + \ldots \text{ to } n \text{ terms } = \frac{a(1-r^n)}{1-r}\right]$$

$$= \frac{(1-e^{i(2n+2)\theta})(1-e^{-2i\theta})}{(1-e^{2i\theta})(1-e^{-2i\theta})} = \frac{(1-e^{i(2n+2)\theta})(1-e^{-2i\theta})}{1+1-(e^{i\theta}+e^{-i\theta})}$$

$$= \frac{\{1-\cos(2n+2)\theta - i\sin(2n+2)\theta\}\{1-\cos 2\theta + i\sin 2\theta\}}{2-2\cos 2\theta}, \text{ as } e^{\pm i\theta} = \cos\theta \pm i\sin\theta$$

$\therefore$ (1) reduces to

$$\sum_{r=0}^{n} T_{2r}(x) = \frac{1-\cos(2n+2)\theta - \cos 2\theta + \cos(2n+2)\theta\cos 2\theta + \sin(2n+2)\theta\sin 2\theta}{2(1-\cos 2\theta)}$$

$$= \frac{(1-\cos 2\theta) - \cos(2n+2)\theta\,(1-\cos 2\theta) + \sin(2n+2)\theta\,\sin 2\theta}{2(1-\cos 2\theta)}$$

$$= \frac{1}{2}\left[1-\cos(2n+2)\theta+\frac{\sin(2n+2)\theta\sin 2\theta}{1-\cos 2\theta}\right] = \frac{1}{2}\left[1-\cos(2n+2)\theta+\frac{\sin(2n+2)\theta\cdot 2\sin\theta\cos\theta}{2\sin^2\theta}\right]$$

$$= \frac{1}{2}\left[1+\frac{\sin(2n+2)\theta\cos\theta-\cos(2n+2)\theta\sin\theta}{\sin\theta}\right] = \frac{1}{2}\left[1+\frac{\sin\{(2n+2)\theta-\theta\}}{\sin\theta}\right]$$

$$= \frac{1}{2}\left[1+\frac{\sin(2n+1)\theta}{(1-\cos^2\theta)^{1/2}}\right] = \frac{1}{2}\left[1+\frac{U_{2n+1}(x)}{(1-x^2)^{1/2}}\right]$$

$[\because\ x=\cos\theta$ so that $U_{2n+1}(x)=\sin[(2n+1)\cos^{-1}\cos\theta]=\sin(2n+1)\theta]$

Ex. 4. *Show that* $\{T_n(x)\}^2 - T_{n+1}(x)\ T_{n-1}(x) = 1 - x^2$.

Sol. We have, by definition $T_n(x) = \cos(n\cos^{-1}x)$.

$$\therefore \qquad T_n(\cos\theta)=\cos(n\cos^{-1}\cos\theta)=\cos n\theta. \qquad \text{... (1)}$$

With $x=\cos\theta$, the required result takes the form

$$\{T_n(\cos\theta)\}^2-T_{n+1}(\cos\theta)\,T_{n-1}(\cos\theta)=1-\cos^2\theta,\ \textit{i.e.},\ \cos^2 n\theta-\cos(n+1)\theta\cos(n-1)\theta=\sin^2\theta, \quad \text{...(2)}$$

L.H.S. of (2) $=\cos^2 n\theta-\cos(n\theta+\theta)\cos(n\theta-\theta)$

$=\cos^2 n\theta-(\cos^2 n\theta-\sin^2\theta)$, as $\cos(A+B)\cos(A+B)=\cos^2 A-\sin^2 A$

$=\sin^2\theta$ = R.H.S. of (2).

This proves the required result.

EXERCISE

1. Show that (*i*) $(1-x^2)^{1/2}U_n(x)=xT_n(x)-T_{n+1}(x)$.

(*ii*) $T_{m+n}(x)+T_{m-n}(x)=2T_m(x)\,T_n(x)$. **[Kanpur 2006]**

2. Show that (*i*) $T_m{}'(x)=\dfrac{n}{(1-x^2)^{1/2}}U_n(x)$. **[Kanpur 2009]**

(*ii*) $2\{T_n(x)\}^2 = 1 + T_{2n}(x)$.

3. Show that $\{1/(1-x^2)^{1/2}\}U_n(x)$ satisfies $(1-x^2)y''+3xy'+(n^2-1)y=0$.

[Kanpur 2006, 07]

4. Use Chebyshev's differential equation and the equation of exercise 3 above to show that

$$T_n(x)=\frac{n}{2}\sum_{r=0}^{[n/2]}(-1)^r\frac{(n-r-1)\,!}{r\,!(n-2r)\,!}(2x)^{n-2r}$$

and

$$U_n(x)=\sqrt{(1-x^2)}\sum_{r=0}^{[(n-1)/2]}(-1)^r\frac{(n-r-1)\,!}{r\,!(n-2r-1)\,!}(2x)^{n-2r-1}.$$

5. Show that the set of Chebyshev polynomials $T_n(x) = \cos(n\cos^{-1}x)$, (n = 0, 1, 2 .....) is orthogonal on the interval (–1, 1) with respect to the weight function $p(x)=1/(1-x^2)^{1/2}$.

6. Prove that (*i*) $\int_{-1}^{1}x^6(1-x^2)^{-1/2}T_8(x)dx=0$. (*ii*) $T_n(x) - 2x\,T_{n-1}(x) + T_{n-2}(x) = 0$.

7. Show that Chebyshev's polynomials $T_n(x) = \cos(n\cos^{-1}x)$ are solutions of

$$(1-x^2)(d^2y/dx^2)-x(dy/dx)+n^2y=0.$$

8. Prove that $T_n(x)=\displaystyle\sum_{m=0}^{[n/2]}\frac{(-1)^m n\,!\,(1-x)^m x^{n-2m}}{(2m)!\ \ (n-2m)!}$ **[Kanpur 2007]**

9. Prove that $T_n(x) - 2x\,T_{n-1}(x) + T_{n-2}(x) = 0$ **[Kanpur 2010]**

6

Beta and Gamma Functions

6.1. INTRODUCTION

In this chapter we propose to discuss the Gamma and Beta functions. These functions arise in the solution of physical problems and are also of great importance in various branches of mathematical analysis. The reader in strongly advised to master important results of these functions in order to understand the topics coverved by this book.

6.2. Euler's integrals. Beta and Gamma functions

Beta function. Definition

The definite integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx, \quad \text{for } m > 0, \quad n > 0$

is known as the *Beta function* and is denoted by $B(m, n)$ [read as "Beta m, n"]. Beta function is also called the *Eulerian integral of the first kind*.

Thus, $$B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\ dx,\ m > 0,\ n > 0 \qquad \text{... (1)}$$

Gamma function. Definition **(Agra 2000)**

The definite integral $\int_0^\infty e^{-x}x^{n-i}dx, \text{ for } n > 0$

is known as the *Gamma function* and is denoted by $\Gamma(n)$ [read as "Gamma n"]. Gamma function is also called the *Eulerain integral of the second kind*.

Thus, $$\Gamma(n) = \int_0^\infty e^{-x}\, x^{n-1}\ dx, \text{ for } n > 0 \qquad \text{... (2)}$$

Remark. The integral (1) is valid only for $m > 0$ and $n > 0$ and the integral (2) is valid only for $n > 0$, because it is for just these values of m and n that the above integrals are convergent.

6.3. Properties of Gamma function. (Agra 1999)

I. To show that $\Gamma(1) = 1$.

Proof. By the definition of Gamma function, $\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx, n > 0$ ... (1)

From (1), $$\Gamma(1) = \int_0^\infty e^{-x}x^{1-1}dx = \int_0^\infty e^{-x}dx = \left[-e^{-x}\right]_0^\infty = 1.$$

II. To show that $\Gamma(n+1) = n\Gamma(n), n > 0$. **(Agra 1998, Rohilkhand 1997, Delhi Physics (H) 2000)**

Proof. We have from the definition of Gamma function,

$$\Gamma(n+1) = \int_0^\infty e^{-x}x^{n+1-1}dx = \int_0^\infty x^n e^{-x}dx$$

$$= \left[x^n (-e^{-x})\right]_0^\infty - \int_0^\infty (nx^{n-1})\,(-e^{-x})\,dx, \text{ on integrating by parts}$$

$\therefore$ $$\Gamma(n+1) = -\lim_{x\to\infty} \frac{x^n}{e^x} + 0 + n\int_0^\infty e^{-x} x^{n-1}\,dx. \qquad ...(1)$$

$$\left(\because \lim_{x\to 0} x^n e^{-x} = 0 \text{ as } n > 0\right)$$

Now, we have

$$\lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{x^n}{1 + \frac{x}{1!} + \frac{x^2}{2!} + ... + \frac{x^n}{n!} + \frac{x^{n+1}}{(n+1)!} + ...} = \lim_{x\to\infty} \frac{1}{\frac{1}{x^n} + \frac{1}{1!\,x^{n-1}} + ... + \frac{1}{n!} + \frac{x}{(n+1)!} + ...} = 0$$

Also, by definition $$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1}\,dx.$$

Using the above facts (1) reduces to $$\Gamma(n+1) = n\,\Gamma(n).$$

III. If n is a non-negative integer, then $\Gamma(n+1) = n\,!$.

Proof. We known that for $n > 0$, we have (from property II)

$\Gamma(n+1) = n\Gamma(n) = n\,\Gamma(n-1+1) = n\,(n-1)\Gamma(n-1)$, by property II again

$= n\,(n-1)\,(n-2)\,\Gamma\,(n-2)$, by property II again

$= n\,(n-1)\,(n-2)...3\cdot 2\cdot 1\,\Gamma\,(1)$

(by repeated use of property II and the fact that n is positive integer)

$= n\,!$, as $\Gamma(1) = 1$

Remark. *Gauss's Pi-function* is denoted by $\Pi\,(n)$ and is defined by $\Pi\,(n) = \Gamma(n+1)$. When n is +ve integer, $\Pi\,(n) = n\,!$.

6.4. Extension of definition of Gamma function $\Gamma(n)$ for n < 0.

When $n > 0$, we known that $$\Gamma(n+1) = n\,\Gamma(n)$$

so that $$\Gamma(n) = \Gamma(n+1)/n. \qquad ...\ (1)$$

Let $-1 < n < 0$. Then $-1 < n \Rightarrow n + 1 > 0$ so that $\Gamma(n+1)$ is well defined by definition 6.2 and so R.H.S. of (1) is well defined. Thus $\Gamma(n)$ is defined for $-1 < n < 0$ by (1). Similarly, $\Gamma(n)$ is given by (1) for $-2 < n < -1$, $-3 < n < -2$ and so on. Thus (1) defines $\Gamma(n)$ for all values of n except $n = 0, -1, -2, -3, ...$

Property : To show that $\Gamma(n) = \infty$, if n is zero or a negative integer.

Proof. Putting $n = 0$ in (1), we get $\Gamma(0) = \Gamma(1)/0 \Rightarrow \Gamma(0) = \infty$ ... (2)

Again, putting $n = -1$ in (1), we get $\Gamma(-1) = \dfrac{\Gamma(0)}{-1} = \infty$, by (2) ... (3)

Next putting $n = -2$ in (1) and using (3), we get $\Gamma(-2) = \dfrac{\Gamma(-1)}{-2} = \infty$,

and so on. Thus, we find that $\Gamma(n)$ is ∞ if n is zero or negative integer.

6.5. Theorem. To show that $\Gamma(1/2) = \sqrt{\pi}$. **[Agra 2000, 05; Meerut 2007]**

Proof. From definition of gamma function, $$\Gamma(n) = \int_0^\infty e^{-t} t^{n-1} dt, \; n > 0 \qquad \ldots (1)$$

Replacing n by 1/2 in (1), we have $$\Gamma(1/2) = \int_0^\infty e^{-t} t^{-1/2} dt = 2\int_0^\infty e^{-u^2} du \qquad \ldots (2)$$

[Putting $t = u^2$ so that $dt = 2u\, du$]

$$\therefore \; \Gamma(1/2) = 2\int_0^\infty e^{-x^2} dx \quad \text{and} \quad \Gamma(1/2) = 2\int_0^\infty e^{-y^2} dy. \qquad \ldots (3)$$

[Limits remaining the same, we can write x or y as the variable in the integrand of (2)].
Multiplying the corresponding sides of two equations of (3), we get

$$[\Gamma(1/2)]^2 = \left(\int_0^\infty e^{-x^2} dx\right)\left(\int_0^\infty e^{-y^2} dy\right) = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\, dy = 4\int_{\theta=0}^{\pi/2}\int_{r=0}^\infty e^{-r^2} r\, d\theta\, dr$$

(on changing the variables to polar co-ordinates (r, θ), where $x = r\cos\theta$, $y = r\sin\theta$ so that $x^2 + y^2 = r^2$ and $dx\, dy = r\, d\theta\, dr$. The area of integration is the positive quadrant of xy– plane).

$$\therefore \; [\Gamma(1/2)]^2 = 2\int_0^{\pi/2}\left\{\int_0^\infty 2e^{-r^2} r\, dr\right\} d\theta = 2\int_0^{\pi/2}\left\{\int_0^\infty e^{-v}\, dv\right\} d\theta,$$ putting $r^2 = v$ so that $2r\, dr = dv$

Hence, $$[\Gamma(1/2)]^2 = 2\int_0^{\pi/2}\left[-e^{-v}\right]_0^\infty d\theta = 2\int_0^{\pi/2} d\theta = 2[\theta]_0^{\pi/2} = \pi$$

Thus, $[\Gamma(1/2)]^2 = \pi$ so that $\Gamma(1/2) = \sqrt{\pi}$. ... (4)

Remark. From (3) and (4), $$2\int_0^\infty e^{-x^2} dx = \sqrt{\pi} \qquad \ldots (5)$$

6.6. Transformation of Gamma function.

Form I. To show that $\Gamma(n) = \dfrac{1}{n}\displaystyle\int_0^\infty e^{-x^{1/n}} dx, \; n > 0$. **(Meerut 1996)**

Proof. By definition, $$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx, \qquad n > 0. \qquad \ldots (i)$$

Put $x^n = t$ so that $nx^{n-1}\, dx = dt$. Then (*i*) gives

$$\Gamma(n) = \frac{1}{n}\int_0^\infty e^{-t^{1/n}} dt \quad \text{or} \quad \Gamma(n) = \frac{1}{n}\int_0^\infty e^{-x^{1/n}} dx. \qquad \ldots (ii)$$

Particular Case. Put $n = 1/2$ in (*ii*). Then, $$\Gamma(1/2) = 2\int_0^\infty e^{-x^2} dx. \qquad \ldots (iii)$$

Form II. Show that $\displaystyle\int_0^\infty e^{-kx} x^{n-1} dx = \frac{\Gamma(n)}{k^n}$, $n > 0$, $k > 0$. **(Meerut 1996)**

Proof. By definition, $$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx, \qquad n > 0. \qquad \ldots (i)$$

Put $x = kt$ so that $dx = k\, dt$, Then (*i*) gives

$$\Gamma(n) = \int_0^\infty e^{-kt} k^{n-1} t^{n-1} k\, dt \quad \text{or} \quad \Gamma(n) = k^n \int_0^\infty e^{-kx} x^{n-1} dx$$

or $$\int_0^\infty e^{-kx} x^{n-1} dx = \frac{\Gamma(n)}{k^n}, \quad n > 0, \quad k > 0.$$

Form III. To show that $\Gamma(n) = \int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx, \quad n > 0$. [Meerut 1997, Kumaun 2000, Agra 2008; Garhwal 2003 Purvanchal 2005, Delhi Physics (H) 2001]

Proof. By definition, $$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx. \qquad \ldots (i)$$

Put $e^{-x} = t$ so that $-e^{-x}\,dx = dt$. Then (i) gives

$$\Gamma(n) = -\int_1^0 \left(\log \frac{1}{t}\right)^{n-1} dx = \int_0^1 \left(\log \frac{1}{t}\right)^{n-1} dt, \quad \text{as} \quad e^{-x} = t \quad \Rightarrow \quad e^x = \frac{1}{t} \quad \Rightarrow \quad x = \log \frac{1}{t}$$

$$\therefore \qquad \Gamma(n) = \int_0^1 \left(\log \frac{1}{x}\right)^{n-1} dx, \quad n > 0.$$

Form IV. To show that $\Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1} dx, \; n > 0.$

Proof. By definition, $$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} dx. \qquad \ldots (i)$$

Put $x = t^2$ so that $dx = 2t\,dt$. Then (i) gives

$$\Gamma(n) = \int_0^x e^{-t^2} (t^2)^{n-1} 2t\,dt \qquad \text{or} \qquad \Gamma(n) = 2\int_0^\infty e^{-t^2} t^{2n-1} dt \qquad \text{or} \qquad \Gamma(n) = 2\int_0^\infty e^{-x^2} x^{2n-1} dx.$$

6.7. Solved examples based on Gamma function

Ex. 1. *Evaluate* (i) $\int_0^\infty x^4 e^{-x} dx$ $\qquad (ii)$ $\int_0^\infty x^6 e^{-2x} dx.$

Sol. (i) $\int_0^\infty x^4 e^{-x} dx = \int_0^\infty e^{-x} x^{5-1} dx = \Gamma(5) = 4! = 24$, by definition of Gamma function

(ii) Let $I = \int_0^\infty x^6 e^{-2x} dx.$ Put $2x = t$ so that $dx = \frac{1}{2} dt.$ Then, we have

$$I = \int_0^\infty \left(\frac{t}{2}\right)^6 e^{-t} \cdot \frac{1}{2} dt = \frac{1}{2^7} \int_0^\infty e^{-t} t^{7-1} dt = \frac{1}{2^7} \Gamma(7), \text{ by definition of Gamma function.}$$
$$= (1/2^7) \times 6! = 45/8$$

Ex. 2. *Compute* (i) $\Gamma\left(-\frac{1}{2}\right)$ **[Delhi, Physics (H) 2001; Agra 2006]** (ii) $\Gamma\left(-\frac{3}{2}\right)$ (iii) $\Gamma\left(-\frac{5}{2}\right)$.

(iv) Prove that $\Gamma(-9/2) = -(32\sqrt{\pi})/945$ **[Kanpur 2004]**

Sol. We know that $$\Gamma(n) = \Gamma(1+n)/n \qquad \ldots (i)$$

Part (*i*). Putting $n = -\frac{1}{2}$ in (1), $\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma(1/2)}{(-1/2)} = \frac{\sqrt{\pi}}{(-1/2)} = -2\sqrt{\pi}$, as $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Part (*ii*). Putting $n = -3/2$ in (1), we have

$$\Gamma\left(-\frac{3}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}\right)}{(-3/2)} = -\frac{2}{3}\Gamma\left(-\frac{1}{2}\right) = -\frac{2}{3}\left(-2\sqrt{\pi}\right) = \frac{4\sqrt{\pi}}{3}, \text{ using part } (i)$$

Part (*iii*). Putting $n = -\dfrac{5}{2}$ in (*i*), $\Gamma\left(-\dfrac{5}{2}\right) = \dfrac{\Gamma\left(-\dfrac{3}{2}\right)}{(-5/2)} = -\dfrac{2}{5}\left(\dfrac{4\sqrt{\pi}}{3}\right)$, using part (*ii*)

Part (*iv*) Left as an exercise

Ex. 3. *If n is a positive integer, prove that* $2^n \Gamma(1+1/2) = 1\cdot 3\cdot 5 ... (2n+1)\sqrt{\pi}.$

[Delhi Physics (H) 2002]

Sol. Using the formula $\Gamma(n+1) = n\Gamma(n), \quad n > 0.$ ... (1)

$$\Gamma\left(n+\frac{1}{2}\right) = \Gamma\left(n-\frac{1}{2}+1\right) = \left(n-\frac{1}{2}\right)\Gamma\left(n-\frac{1}{2}\right) = \left(n-\frac{1}{2}\right)\Gamma\left(n-\frac{3}{2}+1\right), \text{ using (1)}$$

$$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\Gamma\left(n-\frac{3}{2}\right) = \frac{2n-1}{2}\cdot\frac{2n-3}{2}\cdot\Gamma\left(\frac{2n-3}{2}\right) = \frac{2n-1}{2}\cdot\frac{2n-3}{2}\cdots\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\ \Gamma\left(\frac{1}{2}\right)$$

[By repeated application of (1) and noting that ($2n$ – 1), ($2n$ – 3), ... are all odd].

$$\therefore \quad \Gamma\left(n+\frac{1}{2}\right) = \frac{(2n-1)(2n-3)...5\cdot 3\cdot 1}{2^n}\sqrt{\pi}, \quad \text{as} \quad \Gamma(1/2) = \sqrt{\pi}$$

$$\therefore \quad 2^n \Gamma(n+1/2) = 1\cdot 3\cdot 5 ...\ (2n-1)\sqrt{\pi}$$

Ex. 4. *If n is a positive integer and m > –1, prove that* $\displaystyle\int_0^1 x^m (\log x)^n\, dx = \frac{(-1)^n n!}{(m+1)^{n+1}}.$

Sol. Let $I = \displaystyle\int_0^1 x^m (\log x)^n\, dx$. Put $\log x = -t$ so that $x = e^{-t}$ and $dx = -e^{-t}\, dt$.

$$\therefore \quad I = \int_\infty^0 (e^{-t})^m (-t)^n (-e^{-t}\, dt) \qquad [\because \ \log 0 = -\infty \text{ and } \log 1 = 0]$$

$$= (-1)^n \int_0^\infty e^{-(m+1)t} t^{(n+1)-1} dt = (-1)^n \cdot \frac{\Gamma(n+1)}{(m+1)^{n+1}}, \text{ provided } m+1>0 \text{ i.e., } m > -1$$

[using form II of Art. 6.6]

$$= \frac{(-1)^n n!}{(m+1)^{n+1}}. \qquad [\because \ \Gamma(n+1) = n!, \ n \text{ being the integer}]$$

Ex. 5 (*a*) *With certain limitations on the values of a, b, m and n, prove that*

$$\int_0^\infty \int_0^\infty e^{-(ax^2+by^2)} x^{2m-1} y^{2n-1} dx\, dy = \frac{\Gamma(m)\Gamma(n)}{4a^m b^n}.$$

Sol. Let $$I = \int_0^\infty \int_0^\infty e^{-(ax^2+by^2)} x^{2m-1} y^{2n-1} dx\, dy \quad ... (1)$$

or $$I = \int_0^\infty e^{-ax^2} x^{2m-1} dx \times \int_0^\infty e^{-by^2} y^{2n-1} dy = I_1 \times I_2 \quad ... (2)$$

where $$I_1 = \int_0^\infty e^{-ax^2} x^{2m-1} dx \quad ... (3)$$

and $$I_2 = \int_0^\infty e^{-by^2} y^{2n-1} dy. \quad ... (4)$$

Put $ax^2 = t$, *i.e.*, $x = (t/a)^{1/2}$ so that $dx = dt/2\sqrt{at}$. Then (3) becomes

$$I_1 = \int_0^\infty e^{-t}\left[\frac{t}{a}\right]^{(2m-1)/2}\frac{dt}{2\sqrt{at}} = \frac{1}{2a^m}\int_0^\infty e^{-t}t^{m-1}dt$$

$= \Gamma(m)/2a^m$, by definition of Gamma function, taking $m > 0$, $a > 0$

Similarly, $I_2 = \Gamma(n)/2b^n$, if $n > 0$, $b > 0$

$\therefore$ From (1) and (2), we obtain $I = I_1 \times I_2 = \dfrac{\Gamma(m)\,\Gamma(n)}{4a^m b^n}$.

Ex. 5 (*b*) *Show that* $\displaystyle\int_0^\infty e^{-ax^2}x^{2n-1}dx = \frac{\Gamma(n)}{2a^n}$ **(Purvanchal 2007)**

[**Hint.** Same as I_1 of Ex. 5(*a*)]

Ex. 6. Evaluate $\displaystyle\int_0^1 \frac{dx}{\sqrt{(-\log x)}}$ **[Meerut 1996, Agra 1998]**

Sol. Put $-\log x = t$ so that $x = e^{-t}$ and $dx = -e^{-t}\,dt$.

$$\therefore \quad \int_0^1 \frac{dx}{\sqrt{(-\log x)}} = \int_\infty^0 \frac{-e^{-t}dt}{\sqrt{t}} = \int_0^\infty e^{-t}t^{-1/2}dt = \int_0^\infty e^{-t}t^{(1/2)-1}dt = \Gamma(1/2) = \sqrt{\pi}$$

Ex. 7. Evaluate $\displaystyle\int_0^\infty t^{-3/2}(1-e^{-t})\,dt$

Sol. $$\int_0^\infty t^{-3/2}(1-e^{-t})\,dt = \left[(1-e^{-t})\left(\frac{t^{-1/2}}{-1/2}\right)\right]_0^\infty - \int_0^\infty (e^{-t})\left(\frac{t^{-1/2}}{-1/2}\right)dt$$

$$= 0 + 2\int_0^\infty e^{-t}t^{(1/2)-1}dt = 2\Gamma(1/2) = 2\sqrt{\pi}$$

Ex. 8. *Evaluate* (*i*) $\displaystyle\int_0^\infty x^m e^{-a\,x^n}dx$ (*ii*) $\displaystyle\int_0^\infty x^m e^{-x^n}dx.$

Sol. Part (*i*). Put $ax^n = t$ so that $x = t^{1/n}/a^{1/n} = (t/a)^{1/n}$, $dx = (1/a^{1/n}) \times (1/n) \times t^{(1/n)-1}\,dt$

$$\therefore \quad \int_0^\infty x^m e^{-a\,x^n}dx = \int_0^\infty \left(\frac{t}{a}\right)^{m/n} e^{-t}\,\frac{1}{a^{1/n}}\cdot\frac{1}{n}t^{(1/n)-1}\,dt = \int_0^\infty \frac{e^{-t}}{a^{(m/n)+(1/n)}}t^{(m/n)+(1/n)-1}\cdot\frac{1}{n}dt$$

$$= \frac{1}{n\,a^{(m+1)/n}}\int_0^\infty e^{-t}t^{[(m+1)/n]-1}dt = \frac{1}{n\,a^{(m+1)/n}}\Gamma\left(\frac{m+1}{n}\right),$$ by definition of Gamma function

Part (*ii*). Put $a = 1$ in part (*i*) **Ans.** $(1/2)\times\Gamma((m+1)/n)$

Ex. 9. *Show that* $\displaystyle\int_0^\infty \frac{x^c}{c^x}dx = \frac{\Gamma(c+1)}{(\log c)^{c+1}}, c > 0$ **(Purvanchal 2006)**

Sol. $$\int_0^\infty \frac{x^c}{c^x}dx = \int_0^\infty x^c c^{-x}dx = \int_0^\infty x^c[e^{\log_e c}]^{-1}dx, \text{ as } c = e^{\log_e c}, \text{ if } c > 0$$

$$= \int_0^\infty x^{(c+1)-1}e^{-\log_e c}dx = \frac{\Gamma(c+1)}{(\log_e c)^{c+1}} \quad \left[\because \int_0^\infty x^{n-1}e^{-kx}dx = \frac{\Gamma(n)}{k^n}, n > 0, k > 0\right]$$

Ex. 10. *Show that* $[\Gamma(1/2)]^2 = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\, dy = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2} r\, dr\, d\theta.$

Sol. Let $$I = 4\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\, dy. \qquad ...(1)$$

$$\therefore \quad I = 4\int_0^\infty \int_0^\infty e^{-x^2} e^{-y} dx\, dy = \left(2\int_0^\infty e^{-x^2} dx\right)\left(2\int_0^x e^{-y^2} dy\right) = \Gamma\left(\frac{1}{2}\right)\cdot\Gamma\left(\frac{1}{2}\right),$$ refer Art. 6.5

$$\therefore \qquad I = [\Gamma(1/2)]^2 \qquad ...\ (2)$$

Again, put $x = r\cos\theta,\ y = r\sin\theta$ so that $x^2 + y^2 = r^2$ and $dxdy = r\, d\theta\, dr$.

Furthermore, the region of integration in integral I is the first quadrant of xy plane and so in polar coordinates the corresponding limits will be $r = 0$ to $r = \infty$ and $\theta = 0$ to $\theta = \pi/2$ for the same first quadrant. Hence $$I = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2} r\, d\theta\, dr. \qquad ...\ (3)$$

From (1), (2) and (3), the required result follows

Ex. 11. Show that $\int_0^\infty \exp\,(2ax - x^2)dx = \frac{1}{2}\sqrt{\pi}\exp a^2$, where $\exp k = e^k$.

Sol. $$\int_0^\infty \exp\,(2ax - x^2)dx = \int_0^\infty e^{2ax - x^2} dx = \int_0^\infty e^{a^2 - (x^2 - 2ax + a^2)} dx = \int_0^\infty e^{a^2 - (x-a)^2} dx$$

$$= e^{a^2}\int_0^\infty e^{-(x-a)^2} dx = e^{a^2}\int_0^\infty e^{-t^2} dt,$$ on putting $x - a = t$ and $dx = dt$

$$\therefore \qquad \int_0^\infty \exp(2ax - x^2)dx = \exp a^2 \int_0^\infty e^{-t^2} dt. \qquad ...\ (1)$$

Now, $\Gamma(n) = \int_0^\infty e^{-u} u^{n-1} du. \quad \Rightarrow \quad \Gamma(1/2) = \int_0^\infty e^{-u} u^{-1/2} du. \qquad ...\ (2)$

Putting $u = t^2$ so that $du = 2t$ in (2), we get $\Gamma(1/2) = \int_0^\infty (e^{-t^2}\cdot t^{-1}\cdot 2t)\, dt$

or $\sqrt{\pi} = 2\int_0^\infty e^{-t^2} dt$ or $\int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2} \qquad ...\ (3)$

Using (3), (1) reduces to $$\int_0^\infty \exp(2ax - x^2)dx = \frac{1}{2}\sqrt{\pi}\ \exp a^2.$$

EXERCISE 6(A)

1. Prove that (*i*) $\int_0^\infty e^{-4x} x^{3/2} dx = \frac{3\sqrt{\pi}}{128}$ (*ii*) $\int_0^\infty e^{-x^2} x^2 dx = \frac{\sqrt{\pi}}{4}$

2. Show that $\Gamma\left(-\frac{15}{2}\right) = \frac{2^8\sqrt{\pi}}{1\cdot 3\cdot 5\cdot 7\cdot 9\cdot 11\cdot 13\cdot 15}$

3. Show that if n is a positive integer, then $\Gamma\left(-n + \frac{1}{2}\right) = \frac{(-1)^n 2^n \sqrt{\pi}}{1\cdot 3\cdot 5\ ...\ (2n-1)}$

4. Prove that $\int_0^1 x^{n-1}\left(\log\frac{1}{x}\right)^{m-1} dx = \frac{\Gamma(m)}{n^m},\ m > 0,\ n > 0.$ **[Garhwal 2003]**

5. Prove that $\int_0^\infty \frac{e^{-st}}{\sqrt{t}}\,dt = \left(\frac{\pi}{s}\right)^{1/2}$, $s > 0$

6. Prove that (*i*) $\int_0^\infty \sqrt{x}\, e^{-x^3}\,dx = \frac{\sqrt{\pi}}{3}$ (*ii*) $\int_0^\infty 3^{-4x^2}\,dx = \frac{\sqrt{\pi}}{4\sqrt{(\log 3)}}$

7. Prove that $\int_0^\infty e^{-ax} x^n\,dx = \frac{\Gamma(n+1)}{a^{n+1}}$, $n > -1$, $a > 0$.

6.8. Symmetrical property of Beta function, i.e., B(m, n) = B(n, m).

Proof. By the definition of Beta function, we have

$$B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx = \int_0^1 (1-x)^{m-1}[1-(1-x)]^{n-1}\,dx, \text{ as } \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

$$= \int_0^1 (1-x)^{m-1} x^{n-1}\,dx = \int_0^1 x^{n-1}(1-x)^{m-1}\,dx = B(n, m), \text{ by the def. of Beta function.}$$

$$\therefore \qquad B(m, n) = B(n, m).$$

6.9. Evaluation of *B*(*m*, *n*) in an explicit form when *m* or *n* is a positive integer

By the definition of Beta function, $B(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx.$ ... (1)

The following three cases arise:

Case 1. When only *n* is positive integer. If $n = 1$, (1) gives

$$B(m, 1) = \int_0^1 x^{m-1}(1-x)^{1-1}\,dx = \int_0^1 x^{m-1}\,dx = \left[\frac{x^m}{m}\right]_0^1 = \frac{1}{m}, \qquad \text{... (2)}$$

showing that $B(m, n)$ can be evaluated when $n = 1$.

Now, let $n > 1$. Then from (1), we have

$$B(m, n) = \int_0^1 (1-x)^{n-1} x^{m-1}\,dx = \left[(1-x)^{n-1}\cdot\frac{x^m}{m}\right]_0^1 - \int_0^1 (n-1)\,(1-x)^{n-2}.(-1)\frac{x^m}{m}\,dx$$

$$= 0 + \frac{n-1}{m}\int_0^1 x^m (1-x)^{n-2}\,dx, \quad \text{as} \quad n > 1, \quad \text{so} \quad \lim_{x\to 0}(1-x)^{n-1}\frac{x^m}{m} = 0$$

$$= \frac{n-1}{m}\int_0^1 x^{(m+1)-1}(1-x)^{(n-1)-1}\,dx = \frac{n-1}{m}B(m+1, n-1), \text{ by def. of Beta function}$$

Thus, $B(m, n) = \frac{n-1}{m}B(m+1, n-1).$ ... (3)

Replacing m by $m + 1$ and n by $n - 1$ in (3), we get

$$B(m+1, n-1) = \frac{n-1-1}{m+1}B(m+2, n-2). \qquad \text{... (4)}$$

Using (4), (3) becomes $B(m, n) = \frac{n-1}{m}\cdot\frac{n-2}{m+1}B(m+2, n-2)$ ... (5)

Since n is a positive integer and $n > 1$, after applying the above process repeatedly, we get

$$B(m, n) = \frac{n-1}{m}\cdot\frac{n-2}{m+1}\cdot\frac{n-3}{m+1}\cdots\frac{1}{m+n-2}\,B(m+n-1, 1). \qquad \text{... (6)}$$

Replacing m by $m + n - 1$ is (2), we get $\qquad B(m+n-1, 1) = \dfrac{1}{m+n-1}$ ... (7)

Using (7), (6) becomes $\qquad B(m, n) = \dfrac{n-1}{m}\cdot\dfrac{n-2}{m+1}\cdot\dfrac{n-3}{m+2}\cdots\dfrac{1}{m+n-2}\cdot\dfrac{1}{m+n-1}$

or
$$B(m, n) = \frac{(n-1)!}{m(m+1)\,(m+2)\ldots(m+n-2)\,(m+n-1)} \quad \text{... (8)}$$

Case II. When only *m* is a positive integer.

Since the Beta function is symmetrical in m and n, *i.e.*, $B(m, n) = B(n, m)$, hence from case *I*, interchanging m and n in (8), we get

$$B(m, n) = \frac{(m-1)!}{n(n+1)\,(n+2)\ldots(n+m-2)\,(n+m-1)} \quad \text{... (9)}$$

Case III. When both *m* and *n* are positive integers.

Since n is a positive integer, so by case *I*, we have

$$B(m, n) = \frac{(n-1)!}{m(m+1)\,(m+2)\ldots(m+n-2)\,(m+n-1)}$$

$$= \frac{[1\,.\,2\,.\,3\,\ldots\,(m-1)]\,(n-1)!}{1\,.\,2\,.\,3\,\ldots\,(m-1)m(m+1)(m+2)\ldots(m+n-2)\,(m+n-1)} = \frac{(m-1)!(n-1)!}{(m+n-1)!}$$

6.10. Transformation of Beta function

From I. To show that $B(m, n) = \displaystyle\int_0^\infty \frac{x^{n-1}dx}{(1+x)^{m+n}} = \int_0^\infty \frac{x^{m-1}dx}{(1+x)^{m+n}}$, $m > 0$, $n > 0$

Proof. By definition, $\qquad B(m, n) = \displaystyle\int_0^1 x^{m-1}(1-x)^{n-1}\,dx.$ ... (1)

Put $x = 1/(1 + t)$, so that $dx = -dt/(1 + t)^2$. Then, from (1), we have

$$B(m, n) = -\int_\infty^0 \frac{1}{(t+1)^{m+1}}\left(1-\frac{1}{1+t}\right)^{n-1}\frac{dt}{(1+t)^2} = \int_0^\infty \frac{1}{(t+1)^{m+1}}\left(\frac{1}{1+t}\right)^{n-1} dt = \int_0^\infty \frac{t^{n-1}dt}{(1+t)^{m+n}}$$

or
$$B(m, n) = \int_0^\infty \frac{x^{n-1}dx}{(1+x)^{m+n}} \quad \text{... (2)}$$

Since m and n are interchangeable in Beta function, (2) gives

$$B(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx. \quad \text{... (3)}$$

Thus, (2) and (3) $\Rightarrow$ $\qquad B(m, n) = \displaystyle\int_0^\infty \frac{x^{n-1}dx}{(1+x)^{m+n}} = \int_0^\infty \frac{x^{m-1}dx}{(1+x)^{m+n}}$ ... (4)

Deduction. Show that $B(m,n) = \dfrac{1}{2}\displaystyle\int_0^\infty \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx$ **[Delhi 2008; Kanpur 2004]**

Proof. Adding (3) and (4), we have

$$2B(m,n) = \int_0^\infty \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx \quad \Rightarrow \quad B(m,n) = \frac{1}{2}\int_0^\infty \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}}\,dx$$

From II. To show that $B(m, n) = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx.$ **[Delhi Phy (H) 2002, Kumaun 2002]**

Proof. From form I, we have

$$B(m, n) = \int_0^\infty \frac{x^{m-1}dx}{(1+x)^{m+n}} = \int_0^1 \frac{x^{m-1}dx}{(1+x)^{m+n}} + \int_1^\infty \frac{x^{m-1}dx}{(1+x)^{m+n}}. \quad \text{... (1)}$$

Put $x = 1/t$, so that $dx = -1/t^2\, dt$. Then, we have

$$\int_1^\infty \frac{x^{m-1}dx}{(1+x)^{m+n}} = \int_1^0 \frac{(1/t)^{m-1}(-1/t^2)\, dt}{(1+1/t)^{m+n}} = \int_0^1 \frac{t^{n-1}dt}{(1+t)^{m+n}} = \int_0^1 \frac{x^{n-1}dx}{(1+x)^{m+n}}. \quad \text{... (2)}$$

Using (2), (1) reduces $\quad B(m, n) = \int_0^1 \frac{x^{m-1}dx}{(1+x)^{m+n}} + \int_0^1 \frac{x^{n-1}dx}{(1+x)^{m+n}} = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}} dx.$

From III. To show that $\int_0^\infty \frac{x^{m-1}dx}{(ax+b)^{m+n}} = \frac{B(m, n)}{a^m b^n}$ **[Delhi Maths (H) 2006]**

Proof. From form I, we have $\quad B(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} dx \quad$ ... (1)

Put $x = (at)/b$ so that $dx = (a\, dt)/b.$ Then, (1) reduces to

$$B(m, n) = \int_0^\infty \frac{(at/b)^{m-1} \times (a/b)}{(1+at/b)^{m+n}} dt = a^m b^n \int_0^\infty \frac{t^{m-1}dt}{(at+b)^{m+n}} = a^m b^n \int_0^\infty \frac{x^{m-1}dx}{(ax+b)^{m+n}}.$$

$$\therefore \quad \int_0^\infty \frac{x^{m-1}dx}{(ax+b)^{m+n}} = \frac{B(m, n)}{a^m b^n}.$$

From IV. To show that $\int_0^{\pi/2} \frac{\sin^{2m-1}\theta \cos^{2n-1}\theta\, d\theta}{(a\sin^2\theta + b\cos^2\theta)^{m+n}} = \frac{B(m, n)}{2a^m b^n}$

Proof. Put $x = \tan^2\theta$ so that $dx = 2\tan\theta \sec^2\theta\, d\theta.$ Then from form III above, we have

$$\frac{B(m, n)}{a^m b^n} = \int_0^{\pi/2} \frac{\tan^{2m-2}\theta \cdot 2\tan\theta \cdot \sec^2\theta\, d\theta}{(a\tan^2\theta + b)^{m+n}}$$

$$= 2\int_0^{\pi/2} \frac{\sin^{2m-2}\theta(\sin\theta/\cos\theta)\,(\cos\theta)^{2m+2n}\, d\theta}{\cos^{2m-2}\theta\cos^2\theta(a\sin^2\theta + b\cos^2\theta)^{m+n}} = 2\int_0^{\pi/2} \frac{\sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta}{(a\sin^2\theta + b\cos^2\theta)^{m+n}}$$

$$\therefore \quad \int_0^{\pi/2} \frac{\sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta}{(a\sin^2\theta + b\cos^2\theta)^{m+n}} = \frac{B(m,n)}{2a^m b^n}$$

Form V. To show that $\int_0^1 \frac{x^{m-1}(1-x)^{n-1}dx}{(x+a)^{m+n}} = \frac{B(m, n)}{a^n(1+a)^m}$

Proof. Put $\frac{x}{1+a} = \frac{t}{t+a}$ so that $dx = a(1+a)\frac{dt}{(t+a)^2}$. Then (1) reduces to

$$B(m,\,n)=\int_0^1 (1+a)^{m-1}\left(\frac{t}{t+a}\right)^{m-1} a^{n-1}\left(\frac{1-t}{1+t}\right)^{n-1}\frac{a(a+1)\,dt}{(t+a)^2}$$

$$=a^n\,(1+a)^m\int_0^1\frac{t^{m-1}(1-t)^{n-1}}{(t+a)^{m+n}}\,dt\;=a^n\,(1+a)^m\int_0^1\frac{x^{m-1}(1-x)^{n-1}}{(x+a)^{m+n}}\,dx$$

$$\therefore \qquad \int_0^1\frac{x^{m-1}(1-x)^{n-1}}{(x+a)^{m+n}}\,dx=\frac{B(m,\,n)}{a^n\,(1+a)^m}.$$

Form VI. To show that $\int_b^a (x-b)^{m-1}(a-x)^{n-1}\,dx=(a-b)^{m+n-1}B(m,\,n)$, $m>0,\ n>0$

[Delhi Maths (H) 2009]

Proof. Put $x=\dfrac{t-b}{a-b}$ so that $dx=\dfrac{dt}{a-b}$. Then (1) gives

$$B(m,\,n)=\int_b^a\left(\frac{t-b}{a-b}\right)^{m-1}\left(\frac{a-t}{a-b}\right)^{n-1}\frac{dt}{a-b}$$

$$=\frac{1}{(a-b)^{m+n-1}}\int_b^a (t-b)^{m-1}(a-t)^{n-1}\,dt\;=\frac{1}{(a-b)^{m+n-1}}\int_b^a (x-b)^{m-1}(a-x)^{n-1}\,dx$$

$$\therefore \qquad \int_b^a (x-b)^{m-1}(a-x)^{n-1}\,dx=(a-b)^{m+n-1}B(m,\,n) \qquad \text{... (1)}$$

Remark 1. By putting $a=1$, $b=-1$ in (1), we get

$$\int_{-1}^1 (x+1)^{m+1}(1-x)^{n-1}\,dx=2^{m+n-1}B(m,\,n)\;=2^{m+n-1}\frac{(m)\,\Gamma\,(n)}{\Gamma\,(m+n)}$$ **[Delhi Maths(H) 2002]**

Remark 2. Putting $b=5$, $a=7$, $m=7$, $n=4$ in (1), we get

$$\int_5^7 (x-5)^6(7-x)^3\,dx=2^{10}B(7,\,4).$$

Form VII. To show that (*i*) $\displaystyle\int_0^1\frac{x^{m-1}(1-x)^{n-1}\,dx}{\{a+(b-a)x\}^{m+n}}=\frac{1}{a^n b^m}B(m,\,n)$

(*ii*) $\displaystyle\int_0^1\frac{x^{m-1}(1-x)^{n-1}dx}{(b+cx)^{m+n}}=\frac{1}{(b+c)^m\,b^n}B(m,\,n), m>0,\ n>0.$ **[Delhi Maths (H) 2006, 09]**

Proof. By definition, $\quad B(m,n)=\int_0^1 x^{m-1}(1-x)^{n-1}\,dx \qquad$ ... (1)

Let $\dfrac{a}{y}-\dfrac{b}{x}=a-b$ so that $\quad x=\dfrac{by}{a+(b-a)y} \qquad$ ... (2)

From (2), $\quad dx=\dfrac{b[a+(b-a)y]-by(b-a)}{\{a+(b-a)y\}^2}dy \quad$ or $\quad dx=\dfrac{ab\,dy}{[a+(b-a)y]^2} \qquad$ ... (3)

Again from (2), we see that when $x=1$, $y=1$ and when $x=0$, $y=0$. Again, using (2), we have

$$1-x=1-\frac{by}{a+by-ay}=\frac{a(1-y)}{a+(b-a)} \qquad \text{... (4)}$$

Using (2), (3) and (4), (1) gives

$$B(m,n)=\int_0^1\left\{\frac{by}{a+(b-a)y}\right\}^{m-1}\left\{\frac{a(1-y)}{a+(b-a)y}\right\}^{n-1}\frac{ab\,dy}{\{a+(b-a)y\}^2}$$

$$=a^nb^m\int_0^1\frac{y^{m-1}(1-y)^{n-1}dy}{\{a+(b-a)y\}^{m+n}}=a^nb^m\int_0^1\frac{x^{m-1}(1-x)^{n-1}dx}{\{a+(b-a)x\}^{m+n}}$$

$$\therefore\qquad \int_0^1\frac{x^{m-1}(1-x)^{n-1}dx}{\{a+(b-a)x\}^{m+n}}=\frac{1}{a^nb^m}B(m,n). \qquad \text{... (5)}$$

Part (ii). Interchanging a and b in (5), $\displaystyle\int_0^1\frac{x^{m-1}(1-x)^{n-1}dx}{\{b+(a-b)x\}^{m+n}}=\frac{1}{a^mb^n}B(m,n)$ ... (6)

Putting $a-b=c$, *i.e.*, $a=b+c$ in (6), we get

$$\int_0^1\frac{x^{m-1}(1-x)^{n-1}dx}{(b+cx)^{m+n}}=\frac{1}{(b+c)^mb^n}B(m,n). \qquad \text{... (7)}$$

6.11. Relation between Beta and Gamma Functions.

$B(m,n)=\dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$, $m>0$, $n>0$. **[Agra 2000, Meerut 2004, Delhi Maths (H) 2000, 04]**

Proof. From form II of Art. 6.6, $\displaystyle\frac{\Gamma(m)}{z^m}=\int_0^\infty e^{-zx}x^{m-1}dx$ ... (1)

or $$\Gamma(m)=\int_0^\infty z^me^{-zx}x^{m-1}dx. \qquad \text{... (2)}$$

Multiplying both sides of (2) by $e^{-z}z^{n-1}$, $\displaystyle\Gamma(m)e^{-z}z^{n-1}=\int_0^\infty e^{-z(1+x)}z^{m+n-1}x^{m-1}dx.$ ... (3)

Integrating both sides of (3) w.r.t. z from 0 to ∞, we have

$$\Gamma(m)\int_0^\infty e^{-z}z^{n-1}dz=\int_0^\infty\left\{\int_0^\infty e^{-z(1+x)}z^{m+n-1}x^{m-1}dx\right\}dz \quad\text{or}\quad \Gamma(m)\Gamma(n)=\int_0^\infty\left\{\int_0^\infty e^{-z(1+x)}z^{m+n-1}dz\right\}x^{m-1}\,dx$$

[using definition of Gamma function on L.H.S. and interchanging the order of integration on R.H.S.]

or $$\Gamma(m)\Gamma(n)=\int_0^\infty\frac{\Gamma(m+n)}{(1+x)^{m+n}}x^{m-1}dx, \text{ by (1)}$$

$$=\Gamma(m+n)\int_0^\infty\frac{x^{m-1}dx}{(1+x)^{m+n}}=\Gamma(m+n)B(m,n), \text{ by form I of Art. 6.10.}$$

$$\therefore\qquad B(m,n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}.$$

Deduction IA. To show that $\Gamma(n)\,\Gamma(1-n)=\pi/\sin n\pi,\ 0<n<1$

[Agra 1999, Delhi Maths (H) 2002, 08; Delhi Physics (H) 2000; Kanpur 2006]

Proof. We know that $\displaystyle B(m,n)=\int_0^\infty\frac{x^{n-1}dx}{(1+x)^{m+n}},\ m>0, n>0.$ ... (1)

The relation between Beta and Gamma is $B(m, n) = [\Gamma(m)\,\Gamma(n)]/\Gamma(m+n)$ ... (2)

From (1) and (2), $$\int_0^\infty \frac{x^{n-1}dx}{(1+x)^{m+n}} = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}. \quad \text{... (3)}$$

Taking $m + n = 1$ so that $m = 1 - n$, (3) reduces to

$$\int_0^\infty \frac{x^{n-1}dx}{1+x} = \frac{\Gamma(1-n)\Gamma(n)}{\Gamma(1)}, 0 < n < 1.$$ as $m > 0 \Rightarrow 1 - n > 0 \Rightarrow n < 1$; Also $n > 0$

But we know that $\int_0^\infty \frac{x^{n-1}dx}{1+x} = \frac{\pi}{\sin n\pi}$ and $\Gamma(1) = 1.$

$\therefore$ $$\pi/\sin n\pi = \Gamma(1-n)\Gamma(n), \quad 0 < n < 1$$

Deduction IB. To show that $\Gamma(1+n)\,\Gamma(1-n) = (n\pi)/\sin n\pi$.

Proof. L.H.S. $= n\,\Gamma(n)\,\Gamma(1-n)$, as $\Gamma(n+1) = n\,\Gamma(n)$

$= (n\pi)/\sin n\pi$, by deduction IA.

Deduction II. To show that $\Gamma(1/2) = \sqrt{\pi}$.

Proof. We have just proved that $\Gamma(n)\Gamma(1-n) = \pi/\sin n\pi$ ... (1)

Putting $n = 1/2$ in (1), we obtain

$\Gamma(1/2)\Gamma(1-1/2) = \pi/\sin(\pi/2)$ or $[\Gamma(1/2)]^2 = \pi$ or $\Gamma(1/2) = \sqrt{\pi}$

Deduction III. To show that

$$\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}.$$ **[Meerut 1996; Rohilkhand 1999, Delhi Maths (H) 2005]**

Proof. From the definition of Gamma function, we have

$$\Gamma(n) = \int_0^\infty e^{-x}x^{n-1}dx \quad \text{and so} \quad \Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-x}x^{-1/2}dx. \quad \text{... (1)}$$

Let $x = t^2$ so that $dx = 2t\,dt$. Then (1) becomes

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-t^2}(t^2)^{-1/2}2t\,dt \quad \text{or} \quad \Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-t^2}dt. \quad \text{or} \quad \Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-x^2}dx. \quad \text{...(2)}$$

Also $$\Gamma(1/2) = \sqrt{\pi}. \quad \text{... (3)}$$

From (2) and (3), we obtain $2\int_0^\infty e^{-x^2}dx = \sqrt{\pi}$ or $\int_0^\infty e^{-x^2}dx = \frac{\sqrt{\pi}}{2}.$

Deduction IV. (*i*) $B(x+1, y) = \frac{x}{x+y}B(x, y)$ (*ii*) $B(x, y+1) = \frac{y}{x+y}B(x, y)$

[Delhi Maths (H) 2009]

Proof. (*i*) $B(x+1, y) = \frac{\Gamma(x+1)\Gamma(y)}{\Gamma(x+1+y)} = \frac{x\Gamma(x)\,\Gamma(y)}{(x+y)\,\Gamma(x+y)}$, as $\Gamma(n+1) = n\,\Gamma(n)$

$= \frac{x}{x+y}B(x, y).$

(*ii*) Proceed as in part (*i*)

Deduction V. To show that for $m > 0$, $n > 0$, $B(m, n) = B(m + 1, n) + B(m, n + 1)$.

Proof. Using results (*i*) and (*ii*) of deduction IV, we have **[Agra 2006, 07]**

$$B(m+1, n) + B(m, n+1) = \frac{m}{m+n}B(m, n) + \frac{n}{m+n}B(m, n) = \frac{m+n}{m+n}B(m, n) = B(m, n)$$

Deduction VI. To show that

(*i*) $\int_0^{\pi/2} \sin^{2m-1}\theta\ \cos^{2n-1}\theta\, d\theta = \dfrac{\Gamma(m)\,\Gamma(n)}{2\Gamma(m+n)} = \dfrac{B(m,n)}{2}$, $m > 0$, $n > 0$

[Delhi Maths (H) 2008; Purvanchal 2005; Agra 2008]

(*ii*) $\int_0^{\pi/2} \sin^p\theta\cos^q\theta\, d\theta = \dfrac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$, $p > -1, q > -1$ **[Agra 1999]**

(*iii*) $\int_0^{\pi/2} \sin^p\theta\, d\theta = \int_0^{\pi/2} \cos^p\theta\, d\theta\ = \dfrac{1\cdot3\cdot5\ldots(p-1)}{2\cdot4\cdot6\ldots p}\,\dfrac{\pi}{2}$, if p is even +ve integer

$= \dfrac{2\cdot4\cdot6\ldots(p-1)}{1\cdot3\cdot5\ldots p}$, is p is odd +ve integer

(*iv*) $\int_0^{\pi/2} \sin^{p-1}\theta\cos^{q-1}\theta\, d\theta = \dfrac{\Gamma(p/2)\Gamma(q/2)}{2\Gamma\left(\frac{p+q}{2}\right)}$

(*v*) $\int_0^{\pi/2} \sin^{p-1}\theta\, d\theta = \int_0^{\pi/2} \cos^{p-1}\theta\, d\theta\ = \dfrac{\Gamma(p/2)\Gamma(1/2)}{2\Gamma\left(\frac{p+1}{2}\right)} = \dfrac{\sqrt{\pi}}{2}\,\dfrac{\Gamma(p/2)}{\Gamma\left(\frac{p+1}{2}\right)}$, as $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Proof. (*i*) By definition of Beta function, $B(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$

Let $x = \sin^2\theta$ so that $dx = 2\sin\theta\cos\theta\, d\theta$. Then, we have

$B(m,n) = \int_0^{\pi/2} \sin^{2m-2}(1-\sin^2\theta)^{n-1}(2\sin\theta\cos\theta d\theta)$ or $\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta = \dfrac{B(m,n)}{2}$.

$\therefore \int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta = \dfrac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)}$, as $B(m,n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$ ... (1)

Part (*ii*). Let $p = 2m - 1$ and $q = 2n - 1$, so that $m = (p + 1)/2$ and $n = (q + 1)/2$.

Then (1) becomes $\int_0^{\pi/2} \sin^p\theta\cos^q\theta\, d\theta = \dfrac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$ ... (2)

Part (*iii*) Replacing q by 0 in (2), $\int_0^{\pi/2} \sin^p\theta\, d\theta = \dfrac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{p+2}{2}\right)}$ ... (3)

Next, putting $p = 0$ and $q = p$ in (2), $\int_0^{\pi/2} \cos^p\theta\, d\theta = \dfrac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{p+2}{2}\right)}$ ... (4)

Let p be even, say $p = 2r$. Then R.H.S. of (3) or (4)

$$=\frac{\Gamma\left(\frac{2r+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{2r+2}{2}\right)}=\frac{\Gamma\left(r+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma(r+1)}=\frac{\left(r-\frac{1}{2}\right)\left(r-\frac{3}{2}\right)\ldots\frac{3}{2}\cdot\frac{1}{2}\cdot\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\,.\,r\,(r-1)\,(2r-3)\ldots 3\,.\,2\,.\,1}$$

$$=\frac{(2r-1)\,(2r-3)\ldots 3\,.\,1}{(2r)\,(2r-2)\,(2r-4)\ldots 6\,.\,4\,.\,2}\,\frac{\pi}{2}=\frac{1\,.\,3\,.\,5\ldots(2r-3)\,(2r-1)}{2\,.\,4\,.\,6\ldots(2r-2)\,(2r)}\,\frac{\pi}{2},\text{ as }\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$$

$$=\frac{1\,.\,3\,.\,5\ldots(p-3)\,(p-1)}{2\,.\,4\,.\,6\ldots(p-2)\,p}\,\frac{\pi}{2},\text{ as } p=2r \qquad \ldots (5)$$

Next, let $p = 2r + 1$ *i.e.*, odd +ve integer. Then R.H.S. of (3) and (4)

$$=\frac{\Gamma(r+1)\Gamma(1/2)}{2\Gamma\left(r+\frac{3}{2}\right)}=\frac{r(r-1)\ldots 3\,.\,2\,.\,1\,\sqrt{\pi}}{2.\left(r+\frac{1}{2}\right)\left(r-\frac{1}{2}\right)\ldots\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}=\frac{2\,.\,4\,.\,6\ldots(2r-2)\,(2r)}{1\,.\,3\,.\,5\ldots(2r-1)\,(2r+1)}=\frac{2\,.\,4\,.\,6\ldots(p-1)}{1\,.\,3\,.\,5\ldots p}, \qquad \ldots (6)$$

since $2r + 1 = p$. Thus from (3), (4), (5) and (6) the required results follow.

Part (*iv*) Let $2m = p$ and $2n = q$ so that $m = p/2$ and $n = q/2$. Then (1) becomes

$$\int_0^{\pi/2}\sin^{p-1}\theta\cos^{q-1}\theta\,d\theta=\frac{\Gamma(p/2)\,\Gamma\,(q/2)}{2\Gamma\left(\frac{p+q}{2}\right)} \qquad \ldots (7)$$

Part (*v*) Replacing q by 1 in (7), $\displaystyle\int_0^{\pi/2}\sin^{p-1}\theta\,d\theta=\frac{\Gamma(p/2)\Gamma(1/2)}{2\Gamma\left(\frac{p+1}{2}\right)}$ ... (8)

Next, replacing p by 1 and q by p, in (7), $\displaystyle\int_0^{\pi/2}\cos^{p-1}\theta\,d\theta=\frac{\Gamma(1/2)\Gamma(p/2)}{2\Gamma\left(\frac{1+p}{2}\right)}$ ... (9)

From (8) and (9), the required results follow.

6.12. Solved Examples

Ex. 1. *Evaluate the following Integrals :*

(*i*) $\displaystyle\int_0^1 x^4(1-x)^2\,dx$ (*ii*) $\displaystyle\int_0^2\frac{x^2dx}{\sqrt{(2-x)}}$,

(*iii*) $\displaystyle\int_0^a y^4\sqrt{(a^2-y^2)}\,dy$ (*iv*) $\displaystyle\int_0^2 x(8-x^3)^{1/3}dx.$ **[Agra 2000, 03]**

Sol. We know that $\displaystyle\int_0^1 x^{m-1}(1-x)^{n-1}dx=B(m,n)=\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$ ... (1)

Part. (*i*). $\displaystyle\int_0^1 x^4(1-x)^2\,dx=\int_0^1 x^{5-1}(1-x)^{3-1}\,dx=\frac{\Gamma\,(5)\,\Gamma\,(3)}{\Gamma\,(5+3)}=\frac{4\,!\,2\,!}{7\,!}=\frac{4!\times 2}{7\times 5\times 4!\times 6}=\frac{1}{105}$

Part. (*ii*) Let $I=\displaystyle\int_0^2 x^2(2-x)^{-1/2}\,dx.$ Let $x = 2t$, so that $dx = 2dt$, Then

$$I=\int_0^1(2t)^2(2-2t)^{-1/2}(2dt)=4\sqrt{2}\int_0^1 t^2(1-t)^{-1/2}\,dt=4\sqrt{2}\int_0^1 t^{3-1}(1-t)^{1/2-1}\,dt$$

$$= 4\sqrt{2}\,\frac{\Gamma(3)\Gamma(1/2)}{\Gamma(3+1/2)} = 4\sqrt{2}\,\frac{2\,!\,\Gamma(1/2)}{\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma(1/2)} = \frac{64\sqrt{2}}{15}$$

Part (*iii*) Let $I = \int_0^a y^4\sqrt{(a^2-y^2)}\,dy$. Let $y^2 = a^2t$, so that $dy = \dfrac{a^2dt}{2y} = \dfrac{a\,dt}{2\sqrt{t}}$. Then

$$I = \int_0^1 (a^2t)^2\sqrt{(a^2-a^2t)}\,\frac{(a\,dt)}{2\sqrt{t}} = \frac{a^6}{2}\int_0^1 t^{3/2}(1-t)^{1/2}dt = \frac{a^6}{2}\int_0^1 t^{(5/2)-1}(1-t)^{(3/2)-1}dt$$

$$= \frac{a^6}{2}\times\frac{\Gamma(5/2)\Gamma(3/2)}{\Gamma(5/2+3/2)} = \frac{a^6}{2}\times\frac{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}\cdot\frac{1}{2}\sqrt{\pi}}{3\,!} = \frac{\pi a^6}{32}$$

Part (*iv*) Let $I = \int_0^2 x(8-x^3)^{1/3}dx$. Put $x^3 = 8t$ or $x = 2t^{1/3}$ so that $dx = (2/3)\times t^{-2/3}$ dt

$$\therefore\ I = \int_0^1 (2t^{1/3})(8-8t)^{1/3}(2/3)t^{-2/3}dt = \frac{8}{3}\int_0^1 t^{-1/3}(1-t)^{1/3}dt = \frac{8}{3}\int_0^1 t^{(2/3)-1}(1-t)^{(4/3)-1}dt$$

$$= \frac{8}{3}\times\frac{\Gamma(2/3)\Gamma(4/3)}{\Gamma(2/3+4/3)} = \frac{8}{3}\times\frac{\Gamma(1-1/3)\Gamma(1+1/3)}{\Gamma(2)} = \frac{8}{3}\Gamma\left(1-\frac{1}{3}\right)\frac{1}{3}\Gamma\left(\frac{1}{3}\right), \text{ as } \quad \Gamma(n+1) = n\,\Gamma(n)$$

$$= \frac{8}{9}\times\frac{\pi}{\sin(\pi/3)} = \frac{16\pi}{2\sqrt{3}}, \text{ as } \Gamma(1-n)\Gamma(n) = \frac{\pi}{\sin n\pi}$$

Ex. 2. *Show that* (*a*) $\displaystyle\int_0^1 \frac{dx}{\sqrt{(1-x^n)}} = \frac{\Gamma(1/n)}{\Gamma(1/2+1/n)}\cdot\frac{\sqrt{\pi}}{n}.$ **[Meerut 2004, Purvanchal 2006]**

(*b*) $\displaystyle\int_0^1 \frac{dx}{(1-x^4)^{1/2}} = \frac{\sqrt{\pi}}{4}\cdot\frac{\Gamma(1/4)}{\Gamma(1/2+1/4)}$ **[Meerut 2007]**

Sol. (*a*) Let $I = \displaystyle\int_0^1 \frac{dx}{(1-x^n)^{1/2}}$. Putting x^n = so that $x = t^{1/n}$ and $dx = (1/n)t^{(1/n)-1}\,dt$, we get

$$I = \int_0^1 \frac{1}{(1-t)^{1/2}}\cdot\frac{1}{n}t^{(1/n)-1}dt = \frac{1}{n}\int_0^1 t^{(1/n)-1}(1-t)^{(1/2)-1}dt = \frac{1}{n}B\left(\frac{1}{n},\frac{1}{2}\right)$$

$$= \frac{1}{n}\,\frac{\Gamma(1/n)\Gamma(1/2)}{\Gamma(1/n+1/2)} = \frac{\sqrt{\pi}\,\Gamma\,(1/n)}{n\,\Gamma(1/n+1/2)}.$$

(*b*) Taking $n = 4$ in part (*a*), we get the required result.

Ex. 3. *Show that* $\displaystyle\int_0^a \frac{dx}{(a^n-x^n)^{1/n}} = \frac{\pi}{n\sin(\pi/n)}.$ **[Meerut 1998]**

Sol. Let $I = \displaystyle\int_0^a \frac{dx}{(a^n-x^n)^{1/n}}$. Putting $x^n = a^n\,t$ so that $x = at^{1/n}$ and $dx = a(1/n)\,t^{(1/n)-1}\,dt$, gives

$$I = \int_0^1 \frac{1}{(a^n-a^nt)^{1/n}}\,\frac{a}{n}t^{(1/n)-1}dt = \frac{1}{n}\int_0^1 t^{(1/n)-1}(1-t)^{-1/n}\,dt = \frac{1}{n}\int_0^1 t^{(1/n)-1}(1-t)^{(1-1/n)-1}dt$$

$$=\frac{1}{n}B\left(\frac{1}{n},1-\frac{1}{n}\right),\quad =\frac{1}{n}\cdot\frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(1-\frac{1}{n}\right)}{\Gamma\left(\frac{1}{n}+1-\frac{1}{n}\right)}=\frac{\Gamma\left(\frac{1}{n}\right)\Gamma\left(1-\frac{1}{n}\right)}{n}\quad=\frac{1}{n}\cdot\frac{\pi}{\sin(\pi/n)}.$$

$[\because\ \Gamma(p)\Gamma(1-p)=\pi/\sin p\pi\,]$

Ex. 4. *Show that* $\int_0^\infty \frac{x\,dx}{1+x^6}=\frac{1}{6}B\left(\frac{1}{3},\frac{2}{3}\right)=\frac{\pi}{3\sqrt{3}}$ **[Delhi Maths (H) 2007, 08]**

Sol. Let $I=\int_0^\infty \frac{x\,dx}{1+x^6}$ Putting $x^6=t$ so that $x=t^{1/6}$ and $dx=(1/6)t^{-5/6}\,dt$, we get

$$I=\int_0^\infty \frac{t^{1/16}\cdot(1/6)t^{-5/6}}{1+t}=\frac{1}{6}\int_0^\infty \frac{t^{-2/3}dt}{1+t}\quad=\frac{1}{6}\int_0^\infty \frac{t^{(1/3)-1}}{(1+t)^{1/3+2/3}}dt$$

$$=\frac{1}{6}B\left(\frac{1}{3},\frac{2}{3}\right)\qquad\qquad \because\ \int_0^\infty \frac{x^{m-1}dx}{(1+x)^{m+n}}=B(m,n)$$

$$=\frac{1}{6}\times\frac{\Gamma(1/3)\Gamma(2/3)}{\Gamma(1/3+2/3)}=\frac{1}{6}\Gamma\left(\frac{1}{3}\right)\Gamma\left(1-\frac{1}{3}\right)$$

$$=\frac{1}{6}\times\frac{\pi}{\sin\,(\pi/3)}=\frac{\pi\sqrt{3}}{9}\qquad\qquad \because\ \Gamma(n)\Gamma(1-n)=\frac{\pi}{\sin n\pi}$$

$$=(\pi\sqrt{3})/9.$$

Ex. 5. *Evaluate* (i) $\int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}}dx$ $\qquad(ii)$ $\int_0^\infty \frac{x^{m-1}-x^{n-1}}{(1+x)^{m+1}}dx$

(iii) $\int_0^\infty \frac{x^8(1-x^6)}{(1+x)^{24}}dx.$ **[Garhwal 2000]**

Sol. Part (*i*) $\int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}}dx=\int_0^\infty \frac{x^4dx}{(1+x)^{15}}+\int_0^\infty \frac{x^9}{(1+x)^{15}}dx\ =\int_0^\infty \frac{x^{5-1}dx}{(1+x)^{5+10}}+\int_0^\infty \frac{x^{10-1}dx}{(1+x)^{10+5}}$

$= B\ (5,\ 10) + B\ (10,\ 5)$, as $\int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx=B(m,n)$

$= 2B\ (5,\ 10)$, as $B\ (5,\ 10) = B\ (10,\ 5)$

$$=2\frac{\Gamma(5)\,\Gamma\,(10)}{\Gamma\,(5+10)}=\frac{2\times 4\,!\times 9\,!}{14\,!}=\frac{1}{5005}.$$

Part (*ii*) $\int_0^\infty \frac{x^{m-1}-x^{n-1}}{(1+x)^{m+n}}dx\ =\int_0^\infty \frac{x^{m-1}dx}{(1+x)^{m+n}}-\int_0^\infty \frac{x^{n-1}dx}{(1+x)^{m+n}}$

$= B\ (m,\ n) - \mathrm{B}\ (n,\ m) = 0.$ $\qquad[\because\ B(m,\ n) = B(n,\ m)]$

Part (*iii*) $\int_0^\infty \frac{x^8(1-x^6)dx}{(1+x)^{24}}\ =\int_0^\infty \frac{x^8dx}{(1+x)^{24}}-\int_0^\infty \frac{x^{14}dx}{(1+x)^{24}}\ =\int_0^\infty \frac{x^{9-1}dx}{(1+x)^{9+15}}-\int_0^\infty \frac{x^{15-1}}{(1+x)^{15+9}}$

$= B\ (9,\ 15) - B\ (15,\ 9) = 0$ $\qquad[\because\ B(9,\ 15) = B\ (15,\ 9)]$

Ex. 6. *Show by means of Beta function, that* $\displaystyle\int_t^z \frac{dx}{(z-x)^{1-\alpha}(x-t)^{-\alpha}} = \frac{\pi}{\sin \pi\alpha},\ 0<\alpha<1$

Sol. Let $$I = \int_t^z \frac{dx}{(z-x)^{1-\alpha}(x-t)^{-\alpha}} \qquad \text{... (1)}$$

Putting $x - t = (z - t)y$ so that $x = t + (z - t)\,y$ and $dx = (z - t)dy$, (1) becomes

$$I = \int_0^1 \frac{(z-t)dy}{[z-t-(z-t)y]^{1-\alpha}[(z-t)y]^{-\alpha}} = \int_0^1 \frac{(z-t)dy}{(z-t)^{1-\alpha}(1-y)^{1-\alpha}(z-t)^{-\alpha}\,y^{-\alpha}}$$

$$= \int_0^1 (1-y)^{\alpha-1} y^{\alpha} dy = \int_0^1 y^{(1-\alpha)-1}(1-y)^{\alpha-1} dy = B(1-\alpha,\alpha), \text{ by definition of Beta function.}$$

$$= \frac{\Gamma(1-\alpha)\,\Gamma(\alpha)}{\Gamma(1-\alpha+\alpha)} = \frac{\pi}{\sin\pi\alpha}, \qquad \text{as} \qquad \Gamma(p)\Gamma(1-p) = \frac{\pi}{\sin \pi p}$$

Ex. 7. *Prove that* (*a*) $\displaystyle\int_0^{\pi/2} \sqrt{\tan\theta}\ d\theta = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(\frac{3}{4}\right) = \frac{\pi\sqrt{2}}{2}$ **[Kumaun 2000 Meerut 2004]**

(*b*) $\displaystyle\int_0^{\pi/2} \tan^n x\ dx = \frac{\pi}{2}\sec\frac{n\pi}{2},\ -1<n<1.$

Sol. Part (*a*) $\displaystyle\int_0^{\pi/2} \sqrt{\tan\theta}\ d\theta = \int_0^{\pi/2} \left(\frac{\sin\theta}{\cos\theta}\right)^{1/2} d\theta = \int_0^{\pi/2} \sin^{1/2}\theta\cos^{-1/2}\theta\ d\theta$

$$= \frac{\Gamma\left(\frac{1+1/2}{2}\right)\Gamma\left(\frac{1-1/2}{2}\right)}{2\Gamma\left(\frac{1/2-1/2+2}{2}\right)}$$ Refer deduction VI (ii) of Art. 6.11.

$$= \frac{\Gamma(3/4)\Gamma(1/4)}{2\Gamma(1)} = \frac{1}{2}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right) = \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\Gamma\left(1-\frac{1}{4}\right) = \frac{1}{2}\frac{\pi}{\sin(\pi/4)} = \frac{\pi\sqrt{2}}{2}$$

Part (*b*) $\displaystyle\int_0^{\pi/2} \tan^n x\ dx = \int_0^{\pi/2} \sin^n x\cos^{-n} x\ dx$

$$= \frac{\Gamma\left(\frac{1+n}{2}\right)\Gamma\left(\frac{1-n}{2}\right)}{2\Gamma\left(\frac{n-n+2}{2}\right)}$$ Refer deduction VI (ii) of Art. 6.11
Here $(1 + n)/2 > 0$ and $(1 - n)/2 > 0$
$\Rightarrow$ $n > -1$ and $n < 1$ $\Rightarrow$ $-1 < n < 1$.

$$= \frac{1}{2}\Gamma\left(\frac{1+n}{2}\right)\Gamma\left(1-\frac{1+n}{2}\right) = \frac{1}{2}\frac{\pi}{\sin\left(\frac{1+n}{2}\right)\pi} = \frac{\pi}{2\sin\left(\frac{\pi}{2}+\frac{n\pi}{2}\right)} = \frac{\pi}{2\cos\frac{n\pi}{2}}$$

$$= (\pi/2)\times\sec(n\pi/2), \quad \text{where } -1<n<1. \qquad [\because\ \Gamma(p)\Gamma(1-p) = \pi/\sin p\pi\,]$$

Ex. 8. *If* $p > 0,\ q > 0,\ m+1 > 0,\ n+1 > 0$, *prove* $\displaystyle\int_0^p x^m (p^q - x^q)^n\,dx = \frac{p^{nq+m+1}}{q} B\left(n+1, \frac{m+1}{q}\right)$

Sol. Let
$$I = \int_0^p x^m (p^q - x^q)^n\,dx \qquad \text{... (1)}$$

Putting $x^q = p^q t$ so that $x = pt^{1/q}$ and $dx = (p/q)\ t^{(1/q)-1}\ dt$, (1) reduces to

$$I = \int_0^1 (pt^{t/q})^m (p^q - p^q t)^n (p/q) t^{(1/q)-1} dt = \frac{p^m \cdot p^{nq} \cdot p}{q} \int_0^1 t^{(m/q)+(1/q)-1} (1-t)^{(n+1)-1} dt$$

$$= \frac{p^{nq+m+1}}{q} B\left(\frac{m+1}{q}, n+1\right) = \frac{p^{nq+m+1}}{q} B\left(n+1, \frac{m+1}{q}\right)$$

Ex. 9. *Compute* $\displaystyle I = \int_0^\infty x^2 e^{-x^4} dx \cdot \int_0^\infty e^{-x^4} dx.$

Sol. Putting $x^4 = t$ so that $x = t^{1/4}$ and $dx = (1/4)\ t^{-3/4}\ dt$, we get

$$I = \int_0^\infty (t^{1/4})^2 e^{-t} \left(\frac{1}{4} t^{-3/4}\right) dt \cdot \int_0^\infty e^{-t} \frac{1}{4} t^{-3/4} dt = \frac{1}{16} \int_0^\infty e^{-t} t^{-1/4} dt \cdot \int_0^\infty e^{-t} e^{-3/4} dt$$

$$= \frac{1}{16} \int_0^\infty e^{-t} t^{(3/4)-1} \cdot \int_0^\infty e^{-t} t^{(1/4)-1} dt = \frac{1}{16} \Gamma\left(\frac{3}{4}\right) \Gamma\left(\frac{1}{4}\right) = \frac{1}{16} \Gamma\left(\frac{1}{4}\right) \Gamma\left(1 - \frac{1}{4}\right)$$

$$= \frac{1}{16} \cdot \frac{\pi}{\sin(\pi/4)} = \frac{\pi\sqrt{2}}{16}. \qquad \Big|\ \because\ \Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$$

Ex. 10. *Show that* $\displaystyle I = \int_0^{\pi/2} \sqrt{\sin\theta}\ d\theta . \int_0^{\pi/2} \frac{d\theta}{\sqrt{\sin\theta}} = \pi.$ **[Delhi B.Sc. (Prog.) 2009]**

[Garhwal 2002, Meerut 1998, Delhi Maths (H) 2005, 08]

Sol. We know that
$$\int_0^{\pi/2} \sin^p \theta d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\sqrt{\pi}}{2\Gamma\left(\frac{p+2}{2}\right)} \qquad \text{... (1)}$$

$$I = \int_0^{\pi/2} \sin^{1/2}\theta\ d\theta \cdot \int_0^{\pi/2} \sin^{-1/2}\theta\ d\theta = \frac{\Gamma\left(\frac{1/2+1}{2}\right)\sqrt{\pi}}{2\Gamma\left(\frac{1/2+2}{2}\right)} \cdot \frac{\Gamma\left(\frac{-1/2+1}{2}\right)\sqrt{\pi}}{2\Gamma\left(\frac{-1/2+2}{2}\right)}, \text{ using (1)}$$

$$= \frac{\Gamma(3/4)\sqrt{\pi}}{2\Gamma(5/4)} \cdot \frac{\Gamma(1/4)\sqrt{\pi}}{2\Gamma(3/4)} = \frac{\pi\Gamma(1/4)}{4\Gamma(1+1/4)} = \frac{\pi\Gamma(1/4)}{4 \times (1/4) \times \Gamma(1/4)} = \pi.$$

EXERCISE 6 B

1. Prove that (a) $\displaystyle\int_0^1 \frac{dx}{(1-x^3)^{1/3}} = \frac{2\pi}{3\sqrt{3}}$ $\qquad (b)$ $\displaystyle\int_0^1 \frac{dx}{(1-x^n)^{1/n}} = \frac{\pi}{\sin(\pi/n)}$

(c) $\displaystyle\int_0^1 (1-x^n)^{1/n} dx = \frac{\{\Gamma(1/n)\}^2}{2\Gamma(2/n)}$ $\qquad (d)$ $\displaystyle\int_0^1 x^{m-1}(1-x^k)^n dx = \frac{1}{a} \frac{n!\Gamma(m/k)}{\Gamma(n+1+m/k)}$

(*e*) $\int_0^1 x^{m-1}(1-x^2)^{n-1}\,dx = \frac{1}{2}B\left(\frac{1}{2}m, n\right)$ **[Garhwal 2003]** (*f*) $\int_0^1 \frac{dx}{\sqrt{(1+x^4)}} = \frac{\Gamma(1/4)\Gamma(1/2)}{4\sqrt{2}\,\Gamma(3/4)}$

(*g*) $\int_0^1 \frac{x^{n-1}dx}{\sqrt{(1-x^2)}} = \frac{\sqrt{\pi}\,\Gamma\{(n-1)/2\}}{2\Gamma(n/2)}$ (*h*) $\int_0^1 \frac{dx}{\sqrt{(1-x^4)}} = \frac{1}{6\sqrt{2\pi}}[\Gamma(1/4)]^2$

[Delhi Math (H) 2003, Meerut 2007]

2. Prove that (*a*) $\int_0^1 \left(\frac{1}{x}-1\right)^{1/4} dx = B\left(\frac{5}{4}, \frac{3}{4}\right) = \frac{\pi}{2\sqrt{2}}$ (*b*) $\int_0^\infty \frac{dt}{\sqrt{t}(1+t)} = B\left(\frac{1}{2}, \frac{1}{2}\right) = \pi.$

3. Show that $\int_0^1 \frac{x^n dx}{\sqrt{(1-x^2)}} = \frac{1.3.5\ldots(n-1)}{2.4.6\ldots n}\times\frac{\pi}{2}$ or $\frac{2.4.6\ldots(n-1)}{1.3.5\ldots n}$

according as n is even or odd positive integer.

4. Show that if p and q are positive, then

$B(p, q) = 2\int_0^{\pi/2} \cos^{2p-1}\theta \sin^{2q-1}\theta\, d\theta = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$ and deduce that $\int_0^\pi e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$

5. Prove that (*i*) $B(l, m)\, B(l+m, n) = B(m, n)\, B(m+n, l) = B(n, l)\, B(n+l, m)$.

(*ii*) $B(l, m)\, B(l+m, n)\, B(l+m+n, p) = \frac{\Gamma(l)\Gamma(m)\Gamma(n)\Gamma(p)}{\Gamma(l+m+n+p)}$

(*iii*) $l\, B(l, m+l) = mB\,(l+l, m)$. **[Agra 2005]**

6. Prove that $\int_{-a}^{a} (a+x)^{m-1}(a-x)^{n-1}\,dx = (2a)^{m+n-1}B(m, n)$

7. Prove $\int_{-1}^{1} (1+x)^{p-1}(1-x)^{q-1}\,dx = 2^{p+q-1}B(p, q)$

8. Using the integral $\int_0^\infty \frac{x^{n-1}}{1+x}dx = \frac{\pi}{\sin n\pi}, 0 < n < 1,$ prove that $\Gamma(n)\,\Gamma(1-n) = \pi/\sin n\pi,$ $0 < n < 1$ Hence obtain value of $\Gamma(1/2)$.

9. Prove that $\int_{-1}^{1} \frac{(x+1)^{a-1}(1-x)^{b-1}}{(x+2)^{a+b}}dx = \frac{2^{a+b-1}}{3a}B(a, b), a > 0, b > 0$

10. Show that the perimeter of a loop of the curve $r^n = a^n \cos n\theta$ can be expressed as

$$(a/n)\times 2^{(1/n)-1}\{\Gamma(1/2n)/\Gamma(1/n)\}$$

11. Show that the area enclosed by the curve $x^4 + y^4 = 1$ is $[\Gamma(1/4)]^2/2\sqrt{\pi}$.

12. With the help of double integral, prove that $\int_0^\infty e^{-x^2} dx = \sqrt{\pi}/2.$

[Delhi Maths (H) 2007]

6.13. Legendre-Duplication Formula. $\Gamma(n)\Gamma\left(n+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2n-1}}\Gamma(2n),\ n>0.$

[Delhi Maths (H) 2005, 07, Delhi Phy (H) 2002 Agra 2000, 01, 02, 03, 06, 08; Meerut 1999]

Proof. We know that $$B(m, n)=\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)},\ \text{where } m>0,\ n>0. \qquad \ldots (1)$$

Putting $m = n$ in (1), we get $$B(n, n)=[\Gamma(n)]^2/\Gamma(2n) \qquad \ldots (2)$$

By the definition of the Beta function, $$B(n, n)=\int_0^1 x^{n-1}(1-x)^{n-1}\,dx. \qquad \ldots (3)$$

Putting $x=\sin^2\theta$ so that $dx=2\sin\theta\cos\theta\, d\theta$, (1) gives

$$B(n, n)=\int_0^{\pi/2}(\sin^2\theta)^{n-1}(1-\sin^2\theta)^{n-1}\cdot 2\sin\theta\cos\theta\, d\theta=2\int_0^{\pi/2}(\sin\theta\cos\theta)^{2n-1}d\theta$$

$$=2\int_0^{\pi/2}\left(\frac{\sin 2\theta}{2}\right)^{2n-1}d\theta=\frac{1}{2^{2n-2}}\int_0^{\pi/2}\sin^{2n-1}2\theta\, d\theta=\frac{1}{2^{2n-2}}\int_0^{\pi}\sin^{2n-1}\phi\frac{d\phi}{2}=\frac{1}{2^{2n-1}}\int_0^{\pi}\sin^{2n-1}\phi\, d\phi$$

[On putting $2\theta=\phi$ and $d\theta=(d\phi)/2$]

$$=\frac{1}{2^{2n-1}}\times 2\int_0^{\pi/2}\sin^{2n-1}\phi\, d\phi, \quad \text{as} \quad \int_0^{2a}f(x)dx=2\int_0^a f(x)\,dx \quad \text{when} \quad f(2a-x)=f(x)$$

$$=\frac{1}{2^{2n-2}}\int_0^{\pi/2}\sin^{2n-1}\phi\,(\cos\phi)^0\, d\phi=\frac{1}{2^{2n-2}}\frac{\Gamma\left(\frac{2n-1+1}{2}\right)\cdot\Gamma\left(\frac{0+1}{2}\right)}{2\Gamma\left(\frac{2n-1+0+2}{2}\right)}$$

$$\therefore \qquad B(n, n)=\frac{1}{2^{2n-1}}\frac{\Gamma(n)\sqrt{\pi}}{\Gamma(n+1/2)},\ \text{as } \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi} \qquad \ldots (3)$$

Equating two values of $B(n, n)$ given by (2) and (3), we obtain

$$\frac{[\Gamma(n)]^2}{\Gamma(2n)}=\frac{1}{2^{2n-1}}\frac{\Gamma(n)\sqrt{\pi}}{\Gamma(n+1/2)} \quad \text{or} \quad \Gamma(n)\Gamma(n+1/2)=\frac{\sqrt{\pi}}{2^{2n-1}}\Gamma(2n). \qquad \ldots (4)$$

Deduction 1. To show that $B(n, n)=\dfrac{\sqrt{\pi}\,\Gamma(n)}{2^{2n-1}\Gamma(n+1/2)},\ n>0$

Proof. From (3), we get the required result.

Deduction II For all positive real values of p, $2^p\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{p+2}{2}\right)=\sqrt{\pi}\,\Gamma(p+1).$

Proof. Putting $2n-1=p$ so that $n=(p+1)/2$ in (4), we get

$$\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{p+1}{2}+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^p}\Gamma(p+1) \quad \text{or} \quad 2^p\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{p+2}{2}\right)=\sqrt{\pi}\,\Gamma(p+1).$$

Deduction III. When n is positive integer, to show that

[Delhi Maths (H) 2003]

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{(2n)!}{2^{2n}n!}\sqrt{\pi}.$$

Proof. Let n be positive integer, then $\dfrac{\Gamma(2n)}{\Gamma(n)}=\dfrac{(2n-1)!}{(n-1)}=\dfrac{(2n)\,(2n-1)!}{2\cdot n\,(n-1)!}=\dfrac{(2n)!}{2\cdot n!}$

Now, from the duplication formula (4) and the above result, we have

$$\Gamma\left(n+\frac{1}{2}\right)=\frac{\sqrt{\pi}}{2^{2n-1}}\cdot\frac{\Gamma(2n)}{\Gamma(n)}=\frac{\sqrt{\pi}}{2^{2n-1}}\cdot\frac{(2n)!}{2\cdot n!}=\frac{(2n)!}{2^{2n}\,n!}\sqrt{\pi}.$$

6.14. SOLVED EXAMPLES.

Ex. 1. Express $\Gamma(1/6)$ in terms of $\Gamma(1/3)$.

Sol. From the duplication formula, $\Gamma(n)\Gamma\left(n+\dfrac{1}{2}\right)=\dfrac{\sqrt{\pi}}{2^{2n-1}}\Gamma(2n).$ ... (1)

Putting n = 1/6 in (1), we get

$$\Gamma\left(\frac{1}{6}\right)\Gamma\left(\frac{2}{3}\right)=\frac{\sqrt{\pi}\,\Gamma(1/3)}{2^{-2/3}} \quad \text{or} \quad \Gamma\left(\frac{1}{6}\right)=\frac{\sqrt{\pi}\,\Gamma(1/3)}{2^{-2/3}\,\Gamma(2/3)} \qquad \text{... (2)}$$

Now, we know that $\Gamma(n)\Gamma(1-n)=\pi/\sin n\pi.$ ... (3)

Putting n = 1/3 in (3) we get

$$\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)=\frac{\pi}{\sin(\pi/3)}=\frac{2\pi}{\sqrt{3}} \quad \text{or} \quad \Gamma\left(\frac{2}{3}\right)=\frac{2\pi}{\sqrt{3}\,\Gamma(1/3)} \qquad \text{... (4)}$$

Substituting the value of $\Gamma(2/3)$ given by (4) in (2), we get

$$\Gamma\left(\frac{1}{6}\right)=\frac{\sqrt{\pi}\,\Gamma(1/3)}{2^{-2/3}}\cdot\frac{\sqrt{3}\,\Gamma(1/3)}{2\pi} \quad \text{or} \quad \Gamma\left(\frac{1}{6}\right)=\frac{\sqrt{3}}{2^{1/3}\sqrt{\pi}}\left[\Gamma\left(\frac{1}{3}\right)\right]^2.$$

Ex. 2. Prove that $\Gamma(n)\Gamma\left(\dfrac{1-n}{2}\right)=\dfrac{\sqrt{\pi}\,\Gamma(n/2)}{2^{1-n}\cos(n\pi/2)},\ 0<n<1.$

Sol. We know that $\Gamma(m)\,\Gamma(1-m)=\pi/\sin m\pi,\ 0<m<1$ ... (1)

and $\Gamma(m)\Gamma\left(m+\dfrac{1}{2}\right)=\dfrac{\sqrt{\pi}\,\Gamma(2m)}{2^{2m-1}},\ m>0,$ ... (2)

Putting $m = (n+1)/2$ in (1), we get

$$\Gamma\left(\frac{n+1}{2}\right)\Gamma\left(\frac{1-n}{2}\right)=\frac{\pi}{\sin\{(n+1)\pi/2\}}=\frac{\pi}{\sin(\pi/2+n\pi/2)}=\frac{\pi}{\cos(n\pi/2)} \qquad \text{... (3)}$$

Putting $m = n/2$ in (2), we get $\Gamma\left(\dfrac{n}{2}\right)\Gamma\left(\dfrac{n+1}{2}\right)=\dfrac{\sqrt{\pi}\,\Gamma(n)}{2^{n-1}}$ ... (4)

Dividing the corresponding sides of (3) and (4), we get

$$\frac{\Gamma[(1-n)/2]}{\Gamma(n/2)}=\frac{\pi}{\cos(n\pi/2)}\times\frac{2^{n-1}}{\sqrt{\pi}\,\Gamma(n)} \quad \text{or} \quad \Gamma(n)\Gamma\left(\frac{1-n}{2}\right)-\frac{\sqrt{\pi}\,\Gamma(n/2)}{2^{1-n}\cos(n\pi/2)}$$

Ex. 3. Prove that $B(m,m)B(m+1/2,m+1/2)=(\pi m^{-1})/2^{4m-1}$

Ex. 4. By evaluating $I=\int_0^{\pi/2}\sin^{2p}x\,dx$ and $J=\int_0^{\pi/2}\sin^{2p}x\,dx$ derive the Legendre's duplication formula for gamma function. **[Kanpur 2006]**

7

Power Series

7.1. INTRODUCTION

In this chapter we propose to study the theory of power series which is very useful tool in the study of analyis. We shall present a summary of the pertinent results of infinite series specially power series. These will be used in solving second order differential equations in the next chapter 8, namely, 'Integration in series'.

7.2. *SUMMARY OF USEFUL RESULTS

In what follows, we shall deal with infinit series and hence we shall write simply Σu_n to denote $\sum_{n=1}^{\infty} u_n$ etc.

List A : Results related to convergence of infinite series of positive terms

A-1. Let $\sum_{n=1}^{\infty} u_n$ be an infinite series and let $S_n = \sum_{i=1}^{n} u_i$. Then $\sum_{n=1}^{\infty} u_n$ is said to be convergent or divergent according as the sequence $< S_n >$ is convergent or divergent.

A-2. Geometric series. The positive term infinite geometric series $1 + r + r^2 + .. + r^n + ...,$ $(r \geq 0)$ is convergent if and only if $r < 1$.

A-3. Harmonic series. The positive term series $\Sigma 1/n^{\lambda}$ is converted iff $\lambda > 1$.

A-4. Comparisian test. If Σu_n and Σv_n are two positive term series such that $\lim_{n\to\infty} (u_n / v_n) = l \neq 0,$ then the two series Σu_n and Σv_n have identical behaviours in relation to convergence.

A-5. D'Alembert's ratio test. Let Σu_n be a positive term series such that $\lim_{n\to\infty} (u_{n+1} / u_n) = l.$ Then the series is (*i*) convergent if $l < 1$ (*ii*) divergent if $l > 1$. (*iii*) No firm decision if $l = 1$. Also, if $l = \infty,$ then the series is divergent.

A-6 Cauchy's *n*th root test. Let Σu_n be a positive term series and let $\lim_{n\to\infty} (u_n)^{1/n} = l.$ Then the series is (*i*) convergent if $l < 1$ (*ii*) divergent if $l > 1$ (*iii*) No firm decision if $l = 1$.

A-7. Raabe's test. Let Σu_n be a positive term series and let $\lim_{n\to\infty} n\{(u_n / u_{n+1}) - 1\} = l.$ Then the series is (*i*) convergent if $l > 1$ (*ii*) divergent if $l < 1$ (*iii*) No firm decision if $l = 1$.

A-8. Logarithmic test. Let Σu_n be a positive term series and let $\lim_{n\to\infty} n \log\{u_n / u_{n+1}) = l.$ Then the series is (*i*) convergent if $l > 1$ (*ii*) divergent if $l < 1$ (*iii*) No firm decision if $l = 1$.

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*For all results of this article, refer chapters 6, 7 and 15 of Real Analysis by Shanti Narayan and M.D. Raisinghania, published by S.Chand & Co. New Delhi.

List B : Results related to convergence of infinite series with positive and negative terms

B-1. Absolutely convergent series. A series Σu_n is said to be absolutely convergent if the positive term series $\Sigma |u_n|$ formed by the moduli of the terms of the series is convergent.

B-2. Alternating series. A series whose terms are alternatively positive and negative is referred to an alternating series.

B-3. Leibnitz's test. Let $< u_n >$ be a sequence such that for each natural number n (*i*) $u_n \geq 0$ (*ii*) $u_{n+1} \leq u_n$ (*iii*) $\lim\limits_{n\to\infty} u_n = 0.$ Then the alternating series $\Sigma(-1)^{n-1} u_n$ is convergent.

B-4. Every absolutely convergent series is convergent. The converse need not be true.

List C. Results related uniform convergence of infinite series of functions.

C-1. A series $\Sigma f_n(x)$ will converge uniformly in [*a*, *b*], if there exists a convergent series ΣM_n of numbers such that $\forall\ x \in [a, b], |f_n(x)| \leq M_n$

C-2. The sum of a uniformly convergent series of continuous functions is continuous.

C-3. If $\Sigma f_n(x)$ be a uniformly convergent series of intergrable functions in [*a*, *b*], then the series is term by integrable, that is,
$$\int_a^b \sum_{n=1}^{\infty} f_n(x)\, dx = \sum_{n=1}^{\infty} \int_a^b f_n(x)\, dx$$

C-4. If $\Sigma f_n(x)$ be a uniformly convergent series of differentiable functions, then the series is term by term differentiable, *i.e.*,
$$\frac{d}{dx} \sum_{n=1}^{\infty} f_n(x) = \sum_{n=1}^{\infty} \frac{d}{dx} f_n(x)$$

7.3. POWER SERIES

A series of the form
$$\sum_{n=0}^{\infty} a_n x^n \qquad \text{... (1)}$$
is known as real infinite power series where $a_0, a_1, ..., a_n, ...$ are real coefficients free from x, and x is the real variable. More generally $\sum\limits_{n=0}^{\infty} a_n (x - x_0)^n$ is taken to represent a general power series. Since with a shift of origin to x_0 *i.e.*, with change of variable $x - x_0$ to x this precisely reduces to the form (1), hence without any loss of generality our studies shall be confined to the form (1).

For the sake of brevity we shall write $\Sigma a_n x^n$ instead of $\sum\limits_{n=0}^{\infty} a_n x^n$

7.4. SOME IMPORTANT FACTS ABOUT THE POWER SERIES $\Sigma a_n x^n$.

(*i*) For all values of the coefficients, every power series converges for $x = 0$. Hence if a power series converges for no value other than $x = 0$, we say that the given power series is *nowhere convergent.* For example, the power series $\Sigma n^n x^n$ is nowhere convergent.

(*ii*) If a given series converges for all values of x, we say that the given power series is *everywhere convergent*. For example, the power series $\Sigma (x^n / n!)$ is everywhere convergent.

(*iii*) If the given power series converges for some value of x and diverges for other values of x, then the set of all values of x for which it is convergent is known as its *region of convergence*.

7.5. RADIUS OF CONVERGENCE AND INTERVAL OF CONVERGENCE

If a given power series does not converge everywhere or nowhere, then a definite positive number R exists such that the given power series converges (indeed absolutely) for every $|x| < R$ and diverges for every $|x| > R$. Such a number R is known as the *radius of convergence* and the interval $]-R, R[$, the *interval of convergence*, of the given power series.

7.6. FORMULAS FOR DETERMINING THE RADIUS OF CONVERGENCE

Theorem I. *If the power series* $\Sigma a_n x^n$ *is such that* $a_n \neq 0$ *for all n and* $\lim_{n\to\infty} |a_{n+1}/a_n| = 1/R$, *then* $\Sigma a_n x^n$ *is convergent (indeed absolutely) for* $|x| < R$ *and divergent for* $|x| > R$..

[Delhi Maths (H) 2006]

Proof. Let $u_n = a_n x^n$ so that $u_{n+1} = a_{n+1} x^{n+1}$. Then, we have

$$\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{a_{n+1}\,x}{a_n}\right| = |x| \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x|}{R} \qquad \text{... (1)}$$

$\therefore$ By D' Alembert's ratio test, $\Sigma a_n x^n$ converges absolutely if $|x|/R < 1$, *i.e.*, $|x| < R$. Also, $\Sigma a_n x^n$ diverges if $|x| > R$.

Theorem II. *If the power series* $\Sigma a_n x^n$ *is such that* $a_n \neq 0$ *for all n and* $\lim_{n\to\infty} |a_n|^{1/n} = 1/R$, *then* $\Sigma a_n x^n$ *is convergent (indeed absolutely) for* $|x| < R$ *and divergent for* $|x| > R$.

Proof. According to Cauchy's second theorem on limits, if $<|a_n|>$ is a sequence of positive constants, then

$$\lim_{n\to\infty} |u_n|^{1/n} = \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right|, \qquad \text{... (1)}$$

provided the limit on the right side of (1) exists, whether finite or infinite. Also given that

$$\lim_{n\to\infty} |u_n|^{1/n} = 1/R \qquad \text{... (2)}$$

$\therefore$ (1) and (2) $\Rightarrow$

$$\lim_{n\to\infty} |u_{n+1}/u_n| = 1/R \qquad \text{... (3)}$$

Using (3), the result of the theorem follows from theorem I.

In view of the above discussion, the radius of convergence R of the power series $\Sigma a_n x^n$ can be determined as follows :

$$R = 1 \div \lim_{n\to\infty} \frac{a_{n+1}}{a_n} \quad \text{or} \quad R = 1 \div \lim_{n\to\infty} |a_n|^{1/n} \qquad \text{... (*)}$$

$$R = \infty \quad \text{if} \quad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0 \quad \text{or} \quad \lim_{n\to\infty} |a_n|^{1/n} = 0$$

$$R = 0, \quad \text{if} \quad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \infty \quad \text{or} \quad \lim_{n\to\infty} |a_n|^{1/n} = \infty$$

Note 1. Every power series converges absolutely within its interval of convergence.

Note 2. Observe that formula (*) is derived with the supposition of existence of the finite limit lim $|a_n / a_{n+1}|$, that is, with the supposition that the power series $\Sigma a_n x^n$ contains all powers of x. Indeed for the power series $\Sigma\{(2x)^{2n+1}/(2n+1)\}$, the coefficients of even powers of x are equal to zero, $a_{2m} = 0$ and hence $\lim_{n\to\infty}(a_{2n+1}/a_{2n}) = \infty$ and $\lim_{n\to\infty}(a_{2n+2}/a_{2n+1}) = 0$. This shows that we cannot apply the formula (*) to the given power series. However, a direct application of D' Alembert's ratio test leads to the desired result:

Here, let $u_n = (2x)^{2n+1}/(2n+1)$ so that $u_{n+1} = (2x)^{2n+3}/(2n+3)$

$$\therefore \qquad \lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{(2x)^{2n+3}}{(2x)^{2n+1}}\times\frac{2n+1}{2n+3}\right| = 4\,|\,x\,|^2 \lim_{n\to\infty}\left|\frac{2+1/n}{2+3/n}\right| = 4\,|\,x\,|^2$$

Therefore, by D' Alembert's ratio test, the given power series converges absolutely if

$$4\,|\,x\,|^2 < 1 \qquad \text{or} \qquad |\,x\,|^2 < 1/4 \qquad \text{or} \qquad |\,x\,| < 1/2.$$

Note. If the given power series is given in general form $\Sigma a_n (x - x_0)^n$, then formula (*) is used to find the radius of convergence R. In such a case, we say that the given power series converges if $|\,x - x_0\,| < R$ and diverges if $|\,x - x_0\,| > R$. The interval of convergence is given by $]x_0 - R,\, x_0 + R[$.

7.7. SOLVED EXAMPLES BASED ON ART. 7.6.

Ex. 1. *Find the radius of convergence of the following series*

(*i*) $\dfrac{x}{2} + \dfrac{1\cdot 3}{2\cdot 5}x^2 + \dfrac{1\cdot 3\cdot 5}{2\cdot 5\cdot 8}x^3 + ...$ **[Delhi Maths (H) 2006]**

(*ii*) $1 + \dfrac{a\cdot b}{1\cdot c} + \dfrac{a(a+1)\,b\,(b+1)}{1\cdot 2\,c\,(c+1)} + ...$ **[Delhi Maths (H) 2003]**

Sol. (*i*) Let the given series be denoted by $\sum\limits_{n=1}^{\infty} a_n x^n$.

Then, here $\quad a_n = \dfrac{1\cdot 3\cdot 5\,...\,(2n-1)}{2\cdot 5\cdot 8 ... (3n-1)} \quad$ and $\quad a_{n+1} = \dfrac{1\cdot 3\cdot 5\,...\,(2n-1)\,(2n+1)}{2\cdot 5\cdot 8\,...(3n-1)\,(3n+2)}$

$$\therefore \text{ Radius of convergence } = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{3n+2}{2n+1}\right| = \lim_{n\to\infty}\left|\frac{3+2/n}{2+1/n}\right| = \frac{3}{2}$$

(*ii*) Omitting the first term, let the given series be denoted by $\Sigma a_n x^n$. Then, here we have

$$a_n = \frac{a\,(a+1)...(a+n-1)\,b\,(b+1)\,...\,(b+n-1)}{1\cdot 2\,...\,n\,c\,(c+1)\,...\,(c+n-1)}$$

and $$a_{n+1} = \frac{a\,(a+1)...(a+n-1)\,(a+n)\,b\,(b+1)\,...\,(b+n-1)\,(b+n)}{1\cdot 2\,...\,n\,(n+1)\,c\,(c+1)\,...\,(c+n-1)\,(c+n)}$$

$$\text{Radius of convergence } = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)\,(c+n)}{(a+n)\,(b+n)}\right| = \lim_{n\to\infty}\left|\frac{(1+1/n)\,(1+c/n)}{(1+a/n)\,(1+b/n)}\right| = 1$$

Ex. 2. *Find the radius of convergence the exact interval of convergence of the power series*

$$\Sigma\frac{(n+1)}{(n+2)\,(n+3)}x^n.$$

Sol. Let the given series be denoted by $\Sigma a_n x^n$ or Σu_n. Then, we have

$a_n = (n+1)\,/\,\{(n+2)\,(n+3)\}$ and $a_{n+1} = (n+2)\,/\,\{(n+3)\,(n+4)\}$

$$\therefore \qquad \text{R = radius of convergence } = \lim_{n\to\infty}\left|\frac{a_n}{a_{n+1}}\right| = \lim_{n\to\infty}\frac{(n+1)\,(n+4)}{(n+2)^2} = 1$$

Hence the given series converges for $| x | <$ and diverges for $| x | > 1$. We now investigate the nature of the given power series when $| x | = 1$, *i.e.*, when $x = 1$ and $x = -1$.

For $x = 1$, $$u_n = \frac{(n+1)}{(n+2)\,(n+3)} = \frac{1}{n} \times \frac{(1+1/n)}{(1+2/n)\,(1+3/n)} \qquad \text{... (1)}$$

Let the companion series Σv_n be such that $v_n = 1/n.$

Then, $\lim_{n \to \infty} (u_n / v_n) = 1$, which is finite and non zero.

Again, $\Sigma v_n = \Sigma(1/n)$ is a divergent series. So by comparison test Σu_n diverges for $x = 1$.

Next, for $x = -1$, the given series is an alternating series for which $u_{n+1} < u_n$ for each natural number n and $\lim_{n \to \infty} u_n = 0$, by (1). Hence, by Leibnitz's test the given series converges for $x = -1$. Hence the exact interval of convergence is $[-1, 1[$.

Ex. 3. *Determine the interval of convergence of the power series* $\Sigma\{(1/n) \times (-1)^{n+1} (x-1)^n\}$.

Sol. Let the given series be denoted by $\Sigma a_n (x - x_0)^n$. Then, we have

$$a_n = (-1)^{n+1} / n \qquad \text{and} \qquad a_{n+1} = (-1)^{n+2} /(n+1).$$

$$\therefore \qquad R = \text{radius of convergence } = \lim_{n \to \infty} \left| \frac{a_n}{a_{n+1}} \right| = \lim_{n \to \infty} \left| -\frac{n+1}{n} \right| = 1$$

Since the given power series is about the point $x = x_0 = 1$, the interval of convergence is

$$x_0 - R < x < x_0 + R, \qquad i.e., \qquad -1 + 1 < x < 1 + 1, \qquad i.e., \qquad 0 < x < 2.$$

For $x = 2$, the given series reduces to the alternating series $\Sigma(-1)^{n-1} / n \ (= \Sigma(-1)^{n-1} u_n,$ say) for which $u_{n+1} < u_n$ for each natural number n and $\lim_{n \to \infty} u_n = \lim_{n \to \infty} (1/n) = 0.$ Hence by Leibnitz's test the given series is convergent when $x = 2$.

Next, for $x = 0$, clearly the given series diverges. Hence the exact interval of convergence is $]0, 2]$.

EXERCISE 7 (A)

Determine the radius of convergence and the exact interval of convergence of each of the following power series.

1. (*i*) $\Sigma \dfrac{nx^n}{(n+1)^2}$ (*ii*) $\Sigma \dfrac{2^n x^n}{n\,!}$ (*iii*) $\Sigma \dfrac{x^n}{n^3}$ (*iv*) $\Sigma \dfrac{x^n}{n^n}$

2. (*i*) $\Sigma \dfrac{(2n)!\,x^{2n}}{(n!)^2}$ (*ii*) $\Sigma \dfrac{(-1)^n x^{2n}}{(n!)^2\, 2^{2n}}$ (*iii*) $\Sigma(-1)^n \dfrac{x^{2n+1}}{(2n+1)!}$

(*iv*) $\Sigma \dfrac{(n!)^2 x^{2n}}{(2n)!}$ (*v*) $\Sigma(-1)^n \dfrac{x^{2n+1}}{(2n+1)}$

3. (*i*) $\dfrac{(x-1)^n}{2^n}$ (*ii*) $\Sigma \dfrac{(-1)^n (x-1)^n}{2^n (3n-1)}$ (*iii*) $\Sigma \dfrac{n!(x+2)^n}{n^n}$

4. If the power series $\Sigma a_n x^n$ has radius of convergence R, then prove that, for any positive integer m, $\Sigma a_n x^{mn}$ has radius of convergence $R^{1/m}$.

ANSWERS

1. (*i*) $R = 1$, $[-1, 1[$; (*ii*) $R = \infty$, R; (*iii*) $R = 1$, $[-1, 1]$ (*iv*) $R = \infty$, R

2. (*i*) R = 1/4; (*ii*) $R = \infty$, R; (*iii*) $R = 1$, $[-1, 1]$ (*iv*) $R = 4$, $]-4, 4[$ (*v*) $R = 1$, $[-1, 1]$

3. (*i*) $R = 2$, $]-1, 3[$, (*ii*) $R = 2$, $]-1, 3]$ (*iii*) $R = e$, $]-2 - e, -2 + e[$

7.8. Some theorems about power series $\Sigma a_n x^n$

Theorem I. *If a power series* $\Sigma a_n x^n$ *converges for* $x = x_0$, *then*

(*i*) *it is absolutely convergent in the interval* $|x| < |x_0|$

(*ii*) *it is uniformly convergent in the interval* $|x| \le |x_1|$, *where* $|x_1| < |x_0|$.

[Delhi Maths (H) 2004]

Solution (*i*) $\Sigma a_n x_0^n$ is convergent $\Rightarrow \lim_{n\to\infty} a_n x_0^n = 0$

$\Rightarrow$ there exists a positive integer m such that

$|a_n x_0^n - 0| < 1 \quad \forall\, n \ge m$ so that $|a_n| < 1/|x_0|^n, \quad \forall\, n \ge m$ ... (1)

Now, $|a_n x^n| = |a_n||x|^n < |x|^n / |x_0|^n, \quad \forall\, n \ge m,$ by (1)

Then, $|a_n x^n| < (|x|/|x_0|)^n, \ \forall\, n \ge m$ ... (2)

The series on the R.H.S. of (2) converges for $|x| < |x_0|$ (being a geometric series with common ratio < 1). Hence, by the comparison test $\Sigma a_n x^n$ is convergent for $|x| < |x_0|$. Therefore, $\Sigma a_n x^n$ is absolutely convergent for $|x| < |x_0|$

(*ii*) Let $M_n = |x_1|^n / |x_0|^n$. Then ΣM_n converges since $|x_1| < |x_0|$ (being a geometric series with common ratio < 1).

Now, $|a_n x_n| = |a_n||x|^n < |x|^n / |x_0|^n, \quad \forall\, n \ge m$ using (1)

$< |x_1|^n / |x_0|^n, \forall\, n \ge m$ since $|x| \le |x_1|$

Thus, $|a_n x^n| < M_n, \forall\, n \in N,$ where ΣM_n converges.

Hence, from the Weierstrass M-test, it follows that $\Sigma a_n x^n$ is uniformly convergent in the interval $|x| \le |x_1|$ where $|x_1| < |x_0|$.

Theorem II. *If a power series* $\Sigma a_n x^n$ *converges for* $|x| < R$ *and if a function f(x) is defined as* $f(x) = \Sigma a_n x^n, |x| < R,$ *then* $f(x) = \Sigma a_n x^n$ *converges uniformly on* $[-R+\varepsilon, R-\varepsilon]$ *for every* $\varepsilon > 0$.

Proof. Let $\varepsilon > 0$ be any given number. Then, we have

$$|x| \le R - \varepsilon \quad \Rightarrow \quad |a_n x^n| \le |a_n|(R-\varepsilon)^n \qquad \text{... (1)}$$

Since every power series converges absolutely within its interval of convergence, it follows that $\Sigma a_n (R-\varepsilon)^n$ converges absolutely. Hence by Weierstrass's M-test it follows that the series $\Sigma a_n x^n$ converges uniformly on $[-R+\varepsilon, R-\varepsilon]$.

Theorem III. *The series obtained by integrating and differentiating power series term by term has the same radius of convergence as the original series.*

Proof. Let R be the radius of convergence of the given power series $\sum_{n=0}^{\infty} a_n x^n$. ... (1)

On integrating (1) term by term, we get $\sum_{n=0}^{\infty} \frac{a_n x^{n+1}}{n+1}$ ... (2)

Let R' be the radius of convergence of (2). Then, we have

$$R = \lim_{n\to\infty} \frac{1}{|a_n|^{1/n}} \quad \text{and} \quad R' = \lim_{n\to\infty} \frac{(n+1)^{1/n}}{|a_n|^{1/n}} \qquad \text{... (3)}$$

Let $l = \lim_{n\to\infty} (n+1)^{1/n}$ so that $\log l = \lim_{n\to\infty} \frac{\log(1+n)}{n} = \lim_{n\to\infty} \frac{1/(n+1)}{1}$, by L' Hopital's rule

$\Rightarrow \quad \log l = 0 \quad \Rightarrow \quad l = e^0 = 1 \quad \Rightarrow \quad \lim_{n\to\infty} (n+1)^{1/n} = 1$ ... (4)

Using (4), (3) $\Rightarrow$ $R' = R$.

Next, differentiating (1) term by term, we get $\sum_{n=1}^{\infty} n a_n x^{n-1}$ ... (5)

Let R'' be the radius of convergence of (5). Then

$$R'' = \lim_{n\to\infty} \frac{1}{n^{1/n} |a_n|^{1/n}} = \lim_{n\to\infty} \frac{1}{n^{1/n}} \times \lim_{n\to\infty} \frac{1}{|a_n|^{1/n}} \qquad \text{... (6)}$$

Let $m = \lim_{n\to\infty} n^{1/n}$ so that $\log m = \lim_{n\to\infty} \frac{\log n}{n} = \lim_{n\to\infty} \frac{(1/n)}{1}$, by L' Hopital's rule

$\Rightarrow \quad \log m = 0 \quad \Rightarrow \quad m = e^0 = 1 \quad \Rightarrow \quad \lim_{n\to\infty} n^{1/n} = 1$ ... (7)

From (3), (6) and (7), we have $R'' = R$.

Exercise. Show that both the power series $\sum_{n=0}^{\infty} a_n x^n$ and corresponding series of derivatives $\sum_{n=1}^{\infty} n a_n x^{n-1}$ have the same radius of convergence. **[Delhi Maths (H) 2001]**

[**Hint.** Refer first part of the above theorem III.]

Theorem IV. *Let the given power series* $\Sigma a_n x^n$ *converges for* $|x| < R$ *and let* $f(x) = \Sigma a_n x^n$. *Then* (*i*) *f*(*x*) *is continuous in*] – *R*, *R* [. **[Delhi Maths (H) 1995]**

(*ii*) $\Sigma a_n x^n$ *can be integrated term by term in*] – *R*, *R* [.

(*iii*) $\Sigma a_n x^n$ *can be differentiated term by term in*] – *R*, *R* [.

[**Note :** If $f(x) = \Sigma a_n x^n$, then *f*(*x*) is known as the *sum function* of the series.]

Proof (*i*) Since each term of the series $\Sigma a_n x^n$ is continuous on] – *R*, *R* [and $\Sigma a_n x^n$ is uniformly convergent on $[-R+\varepsilon, R-\varepsilon]$, hence the sum function *f*(*x*) of $\Sigma a_n x^n$ is also continuous.

(*ii*) Since each term of $\Sigma a_n x^n$ is continuous on [– *R*, *R* [and $\Sigma a_n x^n$ is uniformly covergent in $[-R+\varepsilon, R-\varepsilon]$, hence $\Sigma a_n x^n$ is term by term integrable.

(*iii*) Since each term of $\Sigma a_n x^n$ in continuous, possess continuous derivatives in $]-R, R[$ and $\Sigma a_n x^n$ is uniformly convergent in $[-R+\varepsilon, R-\varepsilon]$, hence $\Sigma a_n x^n$ is term by term differentiable.

Exercise. Let the power series $\Sigma a_n x^n$ converge for | *x* | < *R*, and $f(x) = \Sigma a_n x^n, |x| < R$, prove that $\Sigma a_n x^n$ converge uniformly on $[-R+\varepsilon, R-\varepsilon]$, no matter which $\varepsilon > 0$ is chosen, and that the function *f* is continuous and differentiable on [–*R*, *R*]. **[Delhi Maths (H) 2007]**

8

Integration In Series

8.1. INTRODUCTION

In may happen that a given linear differential equation comes under none of the standard classes which are all of some particular form, and thus we may fail to express its solution in terms of elementary functions, namely, polynomials, rational functions, exponentials, trigonometric fucntions, hyperbolic functions, logarithms etc. In such situations we have to find a convergent series arranged according to powers of the independent variable, which will approximately express the value of the dependent variable. The solution in the form of an infinite series is called '*integration in series*'. In this chapter we propose to discuss some methods of getting solution in the form of infinite series for second order linear equation.

8.2. Some basic definitions. **(Jabalpur 2004)**

Power series. An infinite series of the form

$$\sum_{n=0}^{\infty} C_n (x - x_0)^n = C_0 + C_1(x - x_0) + C_2(x - x_0)^2 + \ldots \qquad \ldots(1)$$

is called a power series in $(x - x_0)$. In particular, a power series in x is an infinite series

$$\sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + \ldots \qquad \ldots(2)$$

For example, the exponential function e^x has the power series

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$$

The power series (1) converges (absolutely) for $|x| < R$, where

$$R = \lim_{n\to\infty} \left| \frac{C_n}{C_{n+1}} \right|, \text{ provided the limit exists.} \qquad \ldots(3)$$

R is said to be the *radius of convergence* of power series (1). The interval $(-R, R)$ is said to be the *interval of convergence*.

Since $R = \infty$ for the power series (2), hence the interval of convergence of the power series (2) is $(-\infty, \infty)$ *i.e.* the real line.

In what follows we shall use the following results :

(*i*) *A power series represents a continuous function within its interval of convergence.*

(*ii*) *A power series can be differentiated termwise within its interval of convergence.*

For more results about power series, refer chapter 7.

Analytic function. A function $f(x)$ defined on an interval containing the point $x = x_0$ is called *analytic at* x_0 if its Taylor series.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0) \qquad \ldots(4)$$

exists and converges to $f(x)$ for all x in the interval of convergence of (4).

Hence, we find that all polynomial functions, e^x, $\sin x$, $\cos x$, $\sinh x$ and cosh are analytic everywhere. A rational function is analytic except at those values of x at which its denominator is zero. For example, the rational function defined by $x/(x^2 - 3x + 2)$ is analytic everywhere except at $x = 1$ and $x = 2$.

8.3. Ordinary and singular points. **[Nagpur 1996, Ravishankar 1993]**

Definitions. A point $x = x_0$ is called an *ordinary point* of the equation

$$y'' + P(x)y' + Q(x)y = 0 \qquad \text{...(1)}$$

if both the functions $P(x)$ and $Q(x)$ are analytic at $x = x_0$.

If the point $x = x_0$ is not an ordinary point of the differential equation (1), then it is called a *singular point* of the differential equation of (1). There are two types of singular points :

(*i*) regular singular points and (*ii*) irregular singular points.

A singular point $x = x_0$ of the differential equation (1) is called a *regular singular point* of the differential equation (1) if both $(x - x_0)\, P(x)$ and $(x - x_0)^2 Q(x)$ are analytic at $x = x_0$.

A singular point, which is not regular is called an *irregular singular point*.

8.4. Solved examples based on Art. 8.3.

Ex. 1. *Determine whether* $x = 0$ *is an ordinary point or a regular singular point of the differential equation* $2x^2(d^2y/dx^2) + 7x(x + 1)\,(dy/dx) - 3y = 0$. **[Delhi Maths (Hons) 1993, 2000]**

Sol. Dividing by $2x^2$, the given equation becomes

$$\frac{d^2y}{dx^2} + \frac{7(x+1)}{2x}\frac{dy}{dx} - \frac{3}{2x^2}\,y = 0. \qquad \text{...(1)}$$

Comparing (1) with standard equation $y'' + P(x)\,y' + Q(x)\,y = 0$, we have

$P(x) = [7(x + 1)]/2x$ and $Q(x) = -3/(2x^2)$. ...(2)

Since both $P(x)$ and $Q(x)$ are undefined at $x = 0$, so both $P(x)$ and $Q(x)$ are not analytic at $x = 0$. Thus $x = 0$ is not an ordinary point and so $x = 0$ is a singular point.

Also, $(x - 0)P(x) = 7(x + 1)/2$ and $(x - 0)^2\, Q(x) = -3/2$, showing that both $(x - 0)P(x)$ and $(x - 0)^2Q(x)$ are analytic at $x = 0$. Therefore $x = 0$ is a regular singular point.

Ex. 2. *Show that* $x = 0$ *is an ordinary point of* $(x^2 - 1)y'' + xy' - y = 0$, *but* $x = 1$ *is a regular singular point.* **[Ranchi 2010]**

Sol. Dividing by $(x^2 - 1)$, the given equation becomes

$$\frac{d^2y}{dx^2} + \frac{x}{(x-1)(x+1)}\frac{dy}{dx} - \frac{1}{(x-1)(x+1)}\,y = 0. \qquad \text{...(1)}$$

Comparing (1) with with standard equation $y'' + P(x)y' + Q(x)y = 0$, we have

$P(x) = x/[(x - 1)(x + 1)]$ and $Q = -1/\{(x - 1)\,(x + 1)\}$.

Since both $P(x)$ and $Q(x)$ are analytic at $x = 0$, so $x = 0$ is an ordinary point of the given equation (1).

Since both $P(x)$ and $Q(x)$ are undefined at $x = 1$, so they are not analytic at $x = 0$. Thus $x = 1$ is not an ordinary point and so $x = 1$ is a singular point.

Also $(x - 1)P(x) = x/(x + 1)$ and $(x - 1)^2Q(x) = -(x - 1)/(x + 1)$, showing that both $(x - 1)P(x)$ and $(x - 1)^2Q(x)$ are analytic at $x = 1$. Therefore $x = 1$ is a regular singular point.

Ex. 3. *Show that* $x = 0$ *and* $x = -1$ *are singular points of* $x^2(x + 1)^2y'' + (x^2 - 1)y' + 2y = 0$, *where the first is irregular and the other is regular.*

Sol. Dividing by $x^2(x + 1)^2$, the given equation becomes

$$\frac{d^2y}{dx^2} + \frac{x-1}{x^2(x+1)}\frac{dy}{dx} + \frac{2}{x^2(x+1)^2}y = 0. \qquad ...(1)$$

Comparing (1) with standard equation $y'' + P(x)y' + Qy = 0$, we get

$P(x) = (x-1)/[x^2(x+1)]$ and $Q(x) = 2/[x^2(x+1)^2]$.

Since both $P(x)$ and $Q(x)$ are undefined at $x = 0$ and $x = -1$, so they are not analytic at $x = 0$ and $x = -1$. Hence $x = 0$ and $x = -1$ are both singular points.

Also $(x-0)P(x) = (x-1)/[x(x+1)]$ and $(x-0)^2Q(x) = 2/(x+1)^2$, showing that $P(x)$ is not analytic at $x = 0$ and so $x = 0$ is an irregular singular point.

Again, $(x+1)P(x) = (x-1)/x^2$ and $(x+1)^2Q(x) = 2/x^2$, showing that both $(x+1)P(x)$ and $(x+1)^2Q(x)$ are analytic at $x = -1$ and hence $x = -1$ is a regular singular point.

Ex. 4. *Discuss the singularities of the equation* $x^2y'' + xy' + (x^2 - n^2)y = 0$ *at* $x = 0$ *and* $x = \infty$. **(Delhi Physics (Hons.) 2000, 02; Bhopal 2010)**

Sol. Discussion about singularity at $x = 0$. Re-writing the given equation

$$y'' + (1/x)y' + \{(x^2 - n^2)/x^2\}y = 0 \qquad ... (1)$$

Comparing (1) with $y'' + P(x)y' + Q(x)y = 0$, $P(x) = 1/x$ and $Q(x) = (x^2 - n^2)/x^2$.

Here $(x-0)P(x) = 1$ and $(x-0)^2Q(x) = x^2 - n^2$, showing that both $(x-0)P(x)$ and $(x-0)^2 Q(x)$ are analytic at $x = 0$. Therefore, $x = 0$ is a regular singular point.

Discussion about singularity at $x = \infty$.

Let $x = 1/t$ or $t = 1/x$. Then, $dt/dx = -1/x^2$ ... (2)

Now, $$y' = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\left(-\frac{1}{x^2}\right) = -t^2\frac{dy}{dt}, \text{ by (2)} \qquad ... (3)$$

and $$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \frac{d}{dt}\left(-t^2\frac{dy}{dt}\right)\left(-\frac{1}{x^2}\right), \text{ by (2) and (3)}$$

or $$y'' = \left(-t^2\frac{d^2y}{dt^2} - 2t\frac{dy}{dx}\right)\times(-t^2) = t^4\frac{d^2y}{dt^2} + 2t^3\frac{dy}{dt} \qquad ... (4)$$

Using (2), (3) and (4), the given equation reduces to

$$\frac{1}{t^2}\left(t^4\frac{d^2y}{dt^2} + 2t^3\frac{dy}{dt}\right) + \frac{1}{t}\left(-t^2\frac{dy}{dt}\right) + \left(\frac{1}{t^2} - n^2\right)y = 0 \quad \text{or} \quad t^2\frac{d^2y}{dt^2} + t\frac{dy}{dt} + \frac{1-n^2t^2}{t^2}y = 0$$

or $$(d^2y/dt^2) + (1/t)\times(dy/dt) + \{(1-n^2t^2)/t^4\} = 0 \qquad ... (5)$$

Comparing (5) with $(d^2y/dt^2) + P(t)\,(dy/dt) + Q(t)y = 0$, here

$P(t) = 1/t$ and $Q(t) = (1 - x^2t^2)/t^4$. Then, we have

$(t-0)\,P(t) = 1$ and $(t-0)^2\,Q(t) = (1 - n^2\,t^2)/t^2$.

Since $(t-0)^2\,Q(t)$ is not analytic at $t = 0$, so $t = 0$ is irregular singular point of (5). In view of (2), $x = \infty$ is an irregular singular point of the given equation.

EXERCISE 8 (A)

1. Show that $x = 0$ is an ordinary point of $y'' - xy' + 2y = 0$.

2. Determine whether $x = 0$ is an ordinary point or regular singular point for the differential equation $2x^2y'' - xy' + (x - 5)y = 0$. **Ans.** $x = 0$ is regular singular point

3. Show that $x = 0$ is an ordinary point of $(x^2 + 1)y'' + xy' - xy = 0$.

4. Show that $x = 0$ is a regular singular point of $x^2y'' + xy' + (x^2 - 1/4)y = 0$.

5. Show that $x = 0$ is a regular singular point and $x = 1$ is an irregular singular point of $x(x - 1)^3y'' + 2(x - 1)^3y' + 3y = 0$.

6. Verify that origin is regular singular point of the equation $2x^2y'' + xy' - (x + 1)y = 0$.

7. Determine the nature of the point $x = 0$ for the equations

(*i*) $xy'' + y \sin x = 0$ **[Nagpur 1996]** (*ii*) $x^3y'' + y \sin x = 0$ **[Nagpur 2005]**

Ans. (*i*) Regular singular point (*ii*) Irregular singular point

8. Determine the singular points and their nature for the following differential equations :

(*i*) $3xy'' + 2x(x - 1)y' + 5y = 0$ (*ii*) $y'' + (1 - x)y' + (1 - x)^2 y = 0$ **[Utkal 2003]**

Ans. (*i*) $x = 0$ is regular singular point (*ii*) There is no singular point.

8.5. Power series solution in power of $(x - x_0)$ or the power series solution near the ordinary point $x = x_0$ or power series solution about the ordinary point $x = x_0$.

Let the given equation be $$y'' + P(x)y' + Q(x)y = 0. \qquad ...(1)$$

If $x = x_0$ is an ordinary point of (1), then (1) has two non–trivial linearly independent power series solutions of the form

$$\sum_{n=0}^{\infty} C_n (x - x_0)^n \qquad ...(2)$$

and these power series converge in some interval of convergence $| x - x_0 | < R$, (where R is the radius of convergence of (2)) about x_0. In order to get the coefficients C_n's is (2), we take

$$y = \sum_{n=0}^{\infty} C_n (x - x_0)^n . \qquad ...(3)$$

Differentiating twice in succession, (3) gives

$$y' = \sum_{n=1}^{\infty} n\, C_n\, (x - x_0)^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1)\, C_n\, (x - x_0)^{n-2} . \qquad ...(4)$$

Putting the above values of y, y' and y'' in (1), we get an equation of form

$$A_0 + A_1(x - x_0) + A_2(x - x_0)^2 + ... + A_n(x - x_0)^n + ... = 0, \qquad ...(5)$$

where the coefficients A_0, A_1, A_2 .... etc. are now some functions of the coefficients C_0, C_1, C_2, ... etc.

Since (5) is an identity, all the coefficients A_0, A_1, A_2, ... of (5) must be zero, *i.e.*,

$$A_0 = 0, \quad A_1 = 0, \quad A_2 = 0, \quad ...., \quad A_n = 0. \qquad ...(6)$$

Solving equation (6), we obtain the coefficients of (3) in terms of C_0 and C_1. Substituting these coefficients in (3), we obtain the required series solution of (1) in powers of $(x - x_0)$.

8.6. Solved examples based on Art. 8.5.

Ex. 1. *Find the power series solution of the equation* $(x^2 + 1)y'' + xy' - xy = 0$ in powers of x (*i.e.* about $x = 0$). **[Delhi B.Sc. (Hons.) 1993, 2000, 06, 07]**

Sol. Given that $$(x^2 + 1)y'' + xy' - xy = 0. \qquad ...(1)$$

Dividing by $(x^2 + 1)$, (1) can be written in standard form as

$$\frac{d^2y}{dx^2} + \frac{x}{x^2+1}\frac{dy}{dx} - \frac{x}{x^2+1}y = 0. \qquad ...(2)$$

Comparing (2) with $y'' + P(x)y' + Q(x)y = 0$, here we have

$P(x) = x/(x^2 + 1)$ and $Q(x) = -x/(x^2 + 1)$, showing that $P(x)$ and $Q(x)$ are analytic at $x = 0$. So $x = 0$ is an ordinary point. Therefore, to solve (1), we take power series $$y = C_0 + C_1x + C_2x^2 + C_3x^3 + ... = \sum_{n=0}^{\infty} C_n x^n \quad ...(3)$$

Differentiating (3) twice in succession w.r.t. 'x', we get

$$y' = \sum_{n=1}^{\infty} nC_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=1}^{\infty} n(n-1) C_n x^{n-2} \quad ....(4)$$

Substituting the above values of y, y' and y'' in (1), we get

$$(x^2 + 1)\sum_{n=2}^{\infty} n(n-1)C_n x^{n-2} + x\sum_{n=1}^{\infty} nC_n x^{n-1} - x\sum_{n=0}^{\infty} C_n x^n = 0$$

or $$\sum_{n=2}^{\infty} n(n-1)C_n x^n + \sum_{n=2}^{\infty} n(n-1)C_n x^{n-2} - \sum_{n=1}^{\infty} nC_n x^n - \sum_{n=0}^{\infty} C_n x^{n+1} = 0$$

or $$\sum_{n=2}^{\infty} n(n-1)C_n x^n + \sum_{n=0}^{\infty}(n+2)(n+1)C_{n+2} x^n + \sum_{n=1}^{\infty} nC_n x^n - \sum_{n=1}^{\infty} C_{n-1} x^n = 0$$

or $$2C_2 + (6C_3 + C_1 - C_0)x + \sum_{n=2}^{\infty}[n(n-1)C_n + (n+2)(n+1)C_{n+2} + nC_n - C_{n-1}]x^n = 0. \quad ...(5)$$

Since (5) is an identity, equating the constant term and the coefficients of various powers of x to zero, we get $2C_2 = 0$ so that $C_2 = 0$ ...(6)

$6C_3 + C_1 - C_0 = 0$ so that $C_3 = (C_0 - C_1)/6$ ...(7)

$n(n-1)C_n + (n+2)(n+1)C_{n+2} + nC_n - C_{n-1} = 0$ for all $n \geq 2$

or $$C_{n+2} = \frac{C_{n-1} - n^2C_n}{(n+1)(n+2)}, \quad \text{for all} \quad n \geq 2. \quad ...(8)$$

The above relation (8) in known as *recurrence relation.*

Putting $n = 2$ is (8), $C_4 = (C_1 - 4C_2)/12 = C_1/12$, as $C_2 = 0$. ...(9)

Putting $n = 3$ in (8), $$C_5 = -\frac{9C_3}{20} = -\frac{9}{20}\left(\frac{C_0 - C_1}{6}\right) = -\frac{3}{40}(C_0 - C_1). \quad ...(10)$$

Putting the above values of C_2, C_3, C_4, C_5, ... etc. in (3), we have

$$y = C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + C_5x^5 + ... \text{ ad. inf.}$$

or $$y = C_0 + C_1x + (1/6) \times (C_0 - C_1)x^3 + (1/12) \times C_1x^4 - (3/40) \times (C_0 - C_1)x^5 + ...$$

or $$y = C_0\left(1 + \frac{1}{6}x^3 - \frac{3}{40}x^5 + ...\right) + C_1\left(x - \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{3}{40}x^5 - ....\right),$$

which is the required solution near $x = 0$, where C_0 and C_1 are arbitrary constants.

Ex. 2(*a*). *Find the solution in series of* $(d^2y/dx^2) + x(dy/dx) + x^2y = 0$ about $x = 0$.

[Delhi Maths (Hons.) 2005, 08; Ranchi 2010]

(***b***) *Solve* $y'' - xy' + x^2y = 0$ *in powers of* x **[Guwahati 2007]**

Sol. (*a*) Given that $$y'' + xy' + x^2y = 0. \quad ...(1)$$

Comparing (1) with $y'' + P(x)y' + Q(x)y = 0$, here $P(x) = x$ and $Q(x) = x^2$. Since $P(x)$ and $Q(x)$ are both analytic at $x = 0$, it follows that $x = 0$ is an ordinary point. To solve (1), we take

$$y = C_0 + C_1x + C_2x^2 + C_2x^3 + ... = \sum_{n=0}^{\infty} C_nx^n. \quad ...(2)$$

Differentiating (2) twice in succession w.r.t. 'x',

$$y' = \sum_{n=1}^{\infty} C_nnx^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} C_nn(n-1)x^{n-2}. \quad ...(3)$$

Putting the above values of y, y' and y'' is (1), $\sum_{n=2}^{\infty} C_n n(n-1)x^{n-2} + x\sum_{n=1}^{\infty} C_n nx^{n-1} + x^2 \sum_{n=0}^{\infty} C_n x^n = 0$

or $$\sum_{n=2}^{\infty} C_n n(n-1)x^{n-2} + \sum_{n=1}^{\infty} C_n nx^n + \sum_{n=0}^{\infty} C_n x^{n+2} = 0$$

or $$\sum_{n=0}^{\infty} C_{n+2}(n+2)(n+1)x^n + \sum_{n=1}^{\infty} C_n nx^n + \sum_{n=2}^{\infty} C_{n-2}x^n = 0$$

or $$2C_2 + (6C_3 + C_1)x + \sum_{n=2}^{\infty}[(n+1)(n+2)C_{n+2} + nC_n + C_{n-2}]x^n = 0. \quad ...(4)$$

Since (4) is an identity, equating the constant term and the coefficients of various powers of x to zero, we get $2C_2 = 0$ so that $C_2 = 0$ ...(5)

$6C_3 + C_1 = 0$ so that $C_3 = -(C_1/6)$ ...(6)

$(n+1)(n+2)C_{n+2} + nC_n + C_{n-2} = 0$, for all $n \geq 2$

or $$C_{n+2} = -\frac{nC_n + C_{n-2}}{(n+1)(n+2)} \quad \text{for all} \quad n \geq 2. \quad ...(7)$$

Putting $n = 2$ in (7), $$C_4 = -\frac{2C_2 + C_0}{12} = -\frac{C_0}{12}, \text{ by (5).} \quad ...(8)$$

Putting $n = 3$ in (7), $$C_5 = -\frac{3C_3 + C_1}{20} = -\frac{-3 \times (C_1/6) + C_1}{20} = -\frac{C_1}{40}, \text{ by (6)}$$

Putting $n = 4$ in (7), $$C_6 = -\frac{4C_4 + C_2}{30} = -\frac{-(C_0/3)}{30} = \frac{C_0}{90}, \text{ by (8)}$$

and so as. Putting these values in (1), we get

$$y = C_0 + C_1x - (1/6) \times C_1x^3 - (1/12) \times C_0x^4 - (1/40) \times C_1x^5 + (1/90) \times C_0x^6 + ...$$

or $$y = C_0\left(1 - \frac{1}{12}x^4 + \frac{1}{90}x^6 - ....\right) + C_1\left(x - \frac{1}{6}x^3 - \frac{1}{40}x^5 - ....\right),$$

which is the required general solution about $x = 0$, where C_0 and C_1 are arbitrary constants.

(*b*) Ans. $y = C_0(1 - x^4/12 - x^6/90 + ...) + C_1(x + x^3/16 - x^5/40 + ...)$

Ex. 3. *Find the general power series solution near* $x = 0$ *of the Legendre's equation* $(1 - x^2)(d^2y/dx^2) - 2x(dy/dx) + p(p+1)y = 0$, *where p is an arbitrary constant.*

[Delhi Maths (Hons.) 2006]

Sol. Given $$(1 - x^2)y'' - 2xy' + p(p+1)y = 0. \quad ...(1)$$

or $$y'' - [(2x)/(1 - x^2)]y' + [p(p+1)/(1 - x^2)]y = 0. \quad ...(2)$$

Comparing (2) with $y'' + P(x)y' + Q(x)y = 0$, we have

$P(x) = -(2x)/(1 - x^2)$ and $Q(x) = p(p+1)/(1 - x^2)$,

showing that both $P(x)$ and $Q(x)$ are analytic at $x = 0$ and hence $x = 0$ is an ordinary point of (1).

To solve (1), let $$y = C_0 + C_1x + C_2x^2 + C_3x^3 + ... = \sum_{n=0}^{\infty} C_n x^n. \quad ...(3)$$

Differentiating (3) twice in succession w.r.t. 'x', we get

$$y' = \sum_{n=1}^{\infty} C_n nx^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} C_n n(n-1)x^{n-2}. \quad ...(4)$$

Putting the above values of y, y' and y'' in (1), we get

$$(1-x^2)\sum_{n=2}^{\infty}C_n n(n-1)x^{n-2} - 2x\sum_{n=1}^{\infty}C_n n x^{n-1} + p(p+1)\sum_{n=0}^{\infty}C_n x^n = 0$$

or $$\sum_{n=2}^{\infty}C_n n(n-1)x^{n-2} - \sum_{n=2}^{\infty}C_n n(x-1)x^n - 2\sum_{n=1}^{\infty}C_n n x^n + p(p+1)\sum_{n=0}^{\infty}C_n x^n = 0$$

or $$\sum_{n=0}^{\infty}C_{n+2}(n+2)(n+1)x^n - \sum_{n=2}^{\infty}C_n n(n-1)x^n - \sum_{n=1}^{\infty}2C_n n x^n + \sum_{n=0}^{\infty}p(p+1)C_n x^n = 0$$

or $$[2C_2+p(p+1)C_0]+[6C_3-2C_1+p(p+1)C_1]x+\sum_{n=2}^{\infty}[(n+1)(n+2)C_{n+2}-n(n-1)C_n-2nC_n-p(p+1)C_n]x^n = 0$$

or $$[2C_2+p(p+1)C_0]+[6C_3+(p^2+p-2)C_1]x+\sum_{n=2}^{\infty}[(n+1)(n+2)C_{n+2}-\{n(n-1)+2n-p(p+1)\}C_n]x^n = 0$$

or $$[2C_2+p(p+1)C_0]+[6C_3+(p-1)(p+2)C_1]x+\sum_{n=2}^{\infty}[(n+1)(n+2)C_{n+2}+\{(p^2-n^2)+(p-n)\}C_n]x^n = 0$$

or $$[2C_2+p(p+1)C_0]+[6C_3+(p-1)(p+2)C_1]x+\sum_{n=2}^{\infty}[(n+1)(n+2)C_{n+2}+(p-n)(p+n+1)C_n]x^n = 0. \quad ...(5)$$

Since (5) is an identity, we equate the coefficients of various powers of x to zero and obtain

$2C_2 + p(p+1)C_0 = 0$ so that $C_2 = -\dfrac{p(p+1)}{2!}C_0$ ...(6)

$6C_3 + (p-1)(p+2)C_1 = 0$ so that $C_3 = -\dfrac{(p-1)(p+2)}{3!}C_1$ ...(7)

$$(n+1)(n+2)C_{n+2} + (p-n)(p+n+1)C_n = 0$$

or $$C_{n+2} = -\frac{(p-n)(p+n+1)}{(n+1)(n+2)}C_n, \quad \text{for} \quad n \geq 2. \quad ...(8)$$

Putting $n = 2, 3, ...$ in (8) and using (6) and (7), we have

$$C_4 = -\frac{(p-2)(p+3)}{4\cdot 3}C_2 = \frac{p(p-2)(p+1)(p+3)}{4!}C_0$$

$$C_5 = -\frac{(p-3)(p+4)}{5\cdot 4}C_3 = \frac{(p-1)(p-3)(p+2)(p+4)}{5!}C_1$$

and so on. Putting the above values of C_2, C_3, C_4, C_5, ... in (3), we get

$$y = C_0 + C_1 x - \frac{p(p+1)}{2!}C_0 x^2 - \frac{(p-1)(p+2)}{3!}C_1 x^3 + \frac{p(p-2)(p+1)(p+3)}{4!}C_0\, x^4 + ...$$

or $$y = C_0\left[1 - \frac{p(p+1)}{2!}x^2 + \frac{p(p-2)(p+1)(p+3)}{4!}x^4 - .... \text{ad. inf}\right]$$

$$+ C_1\left[x - \frac{(p-1)(p+2)}{3!}x^3 + \frac{(p-1)(p-3)(p+2)(p+4)}{5!}x^5 - .... \text{ad. inf}\right],$$

which is the required general solution, C_0 and C_1 being arbitrary constants.

Ex. 4. *Solve $y'' - xy' - py = 0$, where p is any constant.*

Sol. Given $y'' - xy' - py = 0.$ ...(1)

Comparing (1) with $y'' + P(x)y' + Q(x)y = 0$, here $P(x) = -x$ and $Q(x) = -p$. Since $P(x)$ and $Q(x)$ are both analytic at $x = 0$, so $x = 0$ is an ordinary point. To solve (1), we take

$$y = C_0 + C_1 x + C_2 x^2 + C_2 x^3 + C_4 x^4 + ... = \sum_{n=0}^{\infty}C_n x^n. \quad ...(2)$$

Differentiating (2) twice in succession w.r.t. 'x', we get

$$y' = \sum_{n=1}^{\infty} C_n n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} C_n n(n-1) x^{n-2}. \qquad ...(3)$$

Putting the above values of y, y' and y'' in (1), we have

$$\sum_{n=2}^{\infty} C_n n(n-1)x^{n-2} - x\sum_{n=1}^{\infty} C_n n x^{n-1} - p\sum_{n=0}^{\infty} C_n x^n = 0 \text{ or } \sum_{n=2}^{\infty} n(n-1)C_n x^{n-2} - \sum_{n=1}^{\infty} nC_n x^n - \sum_{n=0}^{\infty} pC_n x^n = 0$$

or $$\sum_{n=0}^{\infty} (n+2)(n+1)C_{n+2}x^n - \sum_{n=1}^{\infty} nC_n x^n - \sum_{n=0}^{\infty} pC_n x^n = 0$$

or $$(2C_2 - pC_0) + (6C_3 - C_1 - pC_1)x + \sum_{n=2}^{\infty}[(n+2)(n+1)C_{n+2} - nC_n - pC_n] = 0$$

or $$(2C_2 - pC_0) + \{6C_3 - (p+1)C_1\}x + \sum_{n=2}^{\infty}[(n+2)(n+1)C_{n+2} - (p+n)C_n] = 0. \qquad ...(4)$$

Since (4) is an identity, we equate the coefficients of various powers of x to zero and obtain

$2C_2 - pC_0 = 0$ so that $C_2 = (p/2)C_0$ ...(5)

$6C_3 - (p+1)C_1 = 0$ so that $C_3 = [(p+1)/6]C_1$ ...(6)

$(n+2)(n+1)C_{n+2} - (p+n)C_n = 0$ so that $C_{n+2} = \dfrac{p+n}{(n+1)(n+2)}C_n$, for all $n \geq 2$ ...(7)

Putting $n = 2, 3, 4, ...$ in (7) and using (5) and (6), we get

$$C_4 = \frac{p+2}{3\cdot 4}C_2 = \frac{p+2}{3\cdot 4}\times\frac{p}{2}C_0 = \frac{p(p+2)}{4!}C_0,\ C_5 = \frac{p+3}{4\cdot 5}C_3 = \frac{p+3}{4\cdot 5}\times\frac{p+1}{2\cdot 3}C_1 = \frac{(p+1)(p+3)}{5!}C_1$$

and so on. Putting these values in (3), we obtain

$$y = C_0 + C_1 x + \frac{p}{2}C_0 x^2 + \frac{p+1}{6}C_1 x^3 + \frac{p(p+2)}{4!}C_0 x^4 + \frac{(p+1)(p+3)}{5!}C_1 x^5 + ..$$

or $$y = C_0\left[1 + \frac{p}{2!}x^2 + \frac{p(p+2)}{4!}x^4 + \dots\right] + C_1\left[x + \frac{p+1}{3!}x^3 + \frac{(p+1)(p+3)}{5!}x^5 + \dots\right]$$

which is the required solution, C_0 and C_1 being arbitrary constants.

Ex. 5(a). *Find the general solution of* $y'' + (x-3)y' + y = 0$ near $x = 2$.

(b) *Obtain power series solution of* $y'' + (x-1)y' + y = 0$ in powers of $(x-2)$.

Sol. Given $$y'' + (x-3)y' + y = 0. \qquad ...(1)$$

Comparing (1) with $y'' + P(x)y' + Q(x)y = 0$, here $P(x) = x - 3$ and $Q(x) = 1$. Since both $P(x)$ and $Q(x)$ are analytic at $x = 2$, so $x = 2$ is an ordinary point of (1). To find solution near $x = 2$, we shall find series solution in powers of $(x-2)$. We assume that

$$y = C_0 + C_1(x-2) + C_2(x-2)^2 + C_3(x-2)^3 + ... = \sum_{n=0}^{\infty} C_n (x-2)^n. \qquad ...(2)$$

Differentiating (2) twice in succession w.r.t. 'x', we get

$$y' = \sum_{n=1}^{\infty} nC_n(x-2)^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1)C_n(x-2)^{n-2}. \qquad ...(3)$$

Putting the above values of y, y' and y'' in (1), we get

$$\sum_{n=2}^{\infty} n(n-1)C_n(x-2)^{n-2} + (x-3)\sum_{n=1}^{\infty} nC_n(x-2)^{n-1} + \sum_{n=0}^{\infty} C_n(x-2)^n = 0$$

or
$$\sum_{n=2}^{\infty} n(n-1)C_n(x-2)^{n-2} + [(x-2)-1]\sum_{n=1}^{\infty} nC_n(x-2)^{n-1} + \sum_{n=0}^{\infty} C_n(x-2)^n = 0$$

or
$$\sum_{n=2}^{\infty} n(n-1)C_n(x-2)^{n-2} + \sum_{n=1}^{\infty} nC_n(x-2)^n - \sum_{n=1}^{\infty} nC_n(x-2)^{n-1} + \sum_{n=0}^{\infty} C_n(x-2)^n = 0$$

or
$$\sum_{n=0}^{\infty} (n+2)(n+1)C_{n+2}(x-2)^n + \sum_{n=1}^{\infty} nC_n(x-2)^n - \sum_{n=0}^{\infty} (n+1)C_{n+1}(x-2)^n + \sum_{n=0}^{\infty} C_n(x-2)^n = 0$$

or
$$(2C_2 - C_1 + C_0) + \sum_{n=2}^{\infty} [(n+2)(n+1)C_{n+2} + nC_n - (n+1)C_{n+1} + C_n](x-2)^n = 0. \quad ...(4)$$

which is an identity. Equating to zero the coefficients of various powers of $(x-2)$, we get

$2C_2 - C_1 + C_0 = 0$ so that $C_2 = (C_1 - C_0)/2$. ...(5)

$(n+2)(n+1)C_{n+2} + (n+1)C_n - (n+1)C_{n+1} = 0$ for all $n \geq 1$

or $C_{n+2} = (C_{n+1} - C_n)/(n+2)$, for all $n \geq 1$. ...(6)

Putting $n = 1, 2, 3, ...$ in (6) and using (5) etc., we get

$$C_3 = \frac{C_2 - C_1}{3} = \frac{1}{3}\left[\frac{C_1 - C_0}{2} - C_1\right] = -\frac{C_0 + C_1}{6} \quad ...(7)$$

$$C_4 = \frac{C_3 - C_2}{4} = \frac{1}{4}\left[-\frac{C_0 + C_1}{6} - \frac{C_1 - C_0}{2}\right] = \frac{1}{12}C_0 - \frac{1}{6}C_1. \quad ...(8)$$

and so on. Putting these values in (2), the required solution near $x = 2$ is

$$y = C_0 + C_1(x-2) + \left(\frac{C_1 - C_0}{2}\right)(x-2)^2 - \left(\frac{C_0 + C_1}{6}\right)(x-2)^3 + \left(\frac{1}{12}C_0 - \frac{1}{6}C_1\right)(x-2)^4 + ...$$

or $y = C_0[1 - (1/2) \times (x-2)^2 - (1/6) \times (x-2)^3 - (1/12) \times (x-2)^4 + ...$ ad. inf.]
$+ C_1[(x-2) + (1/2) \times (x-2)^2 - (1/6) \times (x-2)^3 - (1/6) \times (x-2)^4 + ...$ ad. inf.]

(*b*) Do as in part (*a*) yourself.

Ex. 6. *Find the power series solution in powers of* $(x - 1)$ *of the initial value problem* $xy'' + y' + 2y = 0$, $y(1) = 1$, $y'(1) = 2$. **[Purvanchal 2007; CDLU 2004]**

Sol. Given equation is $y'' + (1/x)y' + (2/x)y = 0$. ...(1)

Comparing (1) with $y'' + P(x)y' + Q(y) = 0$, here $P(x) = 1/x$ and $Q(x) = 2/x$, which are analytic at $x = 1$. Hence $x = 1$ is an ordinary point of (1). To find solution near $x = 1$, we shall find series solution in powers of $(x - 1)$.

Let $y = C_0 + C_1(x-1) + C_2(x-1)^2 + C_3(x-1)^3 + ... = \sum_{n=0}^{\infty} C_n(x-1)^n$. ...(2)

Differentiating (2) twice in succession w.r.t. 'x', we get

$y' = \sum_{n=1}^{\infty} nC_n(x-1)^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1)C_n(x-1)^{n-2}$. ...(3)

Putting these values of y, y' and y'' in given equation, we get

$$x\sum_{n=2}^{\infty} n(n-1)C_n(x-1)^{n-2} + \sum_{n=1}^{\infty} nC_n(x-1)^{n-1} + 2\sum_{n=0}^{\infty} C_n(x-1)^n = 0$$

or $$[(x-1)+1]\sum_{n=2}^{\infty} n(n-1)C_n(x-1)^{n-2} + \sum_{n=1}^{\infty} nC_n(x-1)^{n-1} + 2\sum_{n=0}^{\infty} C_n(x-1)^n = 0$$

or $$\sum_{n=2}^{\infty} n(n-1)C_n(x-1)^{n-1} + \sum_{n=2}^{\infty} n(n-1)C_n(x-1)^{n-2} + \sum_{n=1}^{\infty} nC_n(x-1)^{n-1} + 2\sum_{n=0}^{\infty} C_n(x-1)^n = 0$$

or $$\sum_{n=1}^{\infty} (n+1)nC_{n+1}(x-1)^n + \sum_{n=0}^{\infty} (n+2)(n+1)C_{n+2}(x-1)^n$$

$$+\sum_{n=0}^{\infty} (n+1)C_{n+1}(x-1)^n + 2\sum_{n=0}^{\infty} C_n(x-1)^n = 0,$$

which is an identity. Equating to zero the coefficients of various powers of $(x-1)$, we get

$2C_2 + C_1 + 2C_0 = 0$ so that $C_2 = -(C_1 + 2C_0)/2$. ...(4)

and $(n+1)nC_{n+1} + (n+2)(n+1)C_{n+2} + (n+1)C_{n+1} + 2C_n = 0$, for all $n \geq 1$

or $(n+1)(n+2)C_{n+2} + (n+1)^2C_{n+1} + 2C_n = 0$, for all $n \geq 1$

or $$C_{n+2} = -\frac{(n+1)^2C_{n+1} + 2C_n}{(n+1)(n+2)}, \text{ for all } n \geq 1. \quad ...(5)$$

Given that $y = 1$ and $y' = 2$ when $x = 1$. Hence putting $x = 1$ in (2) and (3), we have

$C_0 = 1$ and $C_1 = 2$. ...(6)

Using (6), (4) gives $C_2 = -(2+2)/2 = -2$. ...(7)

Putting $n = 1, 2, 3, \ldots$ (5) and using (6) and (7) etc., we get

$$C_3 = -\frac{2^2C_2 + 2C_1}{2\cdot 3} = -\frac{4\times(-2)+(2\times 2)}{2\cdot 3} = \frac{2}{3},\ C_4 = -\frac{3^2C_3 + 2C_2}{3\cdot 4} = -\frac{9\times(2/3)+2\times(-2)}{3\cdot 4} = -\frac{1}{6}$$

$$C_5 = -\frac{4^2C_4 + 2C_3}{4\cdot 5} = -\frac{16\times(-1/6)+2\times(2/3)}{4\cdot 5} = \frac{1}{15}.$$

and so on. Putting these values in (2), we have

$$y = 1 + 2(x-1) - 2(x-1)^2 + (2/3)\times(x-1)^3 - (1/6)\times(x-1)^4 + (1/15)\times(x-1)^5 + \ldots.$$

Note. Ex. 5 and 6 can also be solved by shifting the origin. The following example is given to provide an alternative method of solving 5 and 6.

Ex. 7. *Find the power series solution of the initial value problem* $(x^2-1)y'' + 3xy' + xy = 0$, $y(2) = 4$, $y'(2) = 6$.

Sol. Given equation is $(x^2-1)y'' + 3xy' + xy = 0$. ...(1)

Dividing by (x^2-1), (1) gives $y'' + [(3x)/(x^2-1)]y' + [x/(x^2-1)]y = 0$. ...(2)

Comparing (2) with $y'' + P(x)y' + Q(x)y = 0$, here $P(x) = (3x)/(x^2-1)$ and $Q(x) = x/(x^2-1)$. Since $P(x)$ and $Q(x)$ are both analytic at $x = 2$, so $x = 2$ is an ordinary point of (1).

Since the initial values of (1) are prescribed at $x = 2$ and $x = 2$ is an ordinary point, hence we shall find the required solution near $x = 2$, *i.e.*, in powers of $(x-2)$.

Let $$y = C_0 + C_1(x-2) + C_2(x-2)^2 + \ldots. = \sum_{n=0}^{\infty} C_n(x-2)^n. \quad ...(3)$$

We now shift the origin to $x = 2$ by writing $\quad t = x - 2 \quad$ so that $\quad x = t + 2$ ...(4)

Then $\quad \dfrac{dy}{dx} = \dfrac{dy}{dt}\dfrac{dt}{dx} = \dfrac{dy}{dt} \quad$ and $\quad \dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dt}\left(\dfrac{dy}{dt}\right) = \dfrac{d^2y}{dt^2}$. ...(5)

Using (4) and (5), (1) reduces to $\quad [(t+2)^2 - 1](d^2y/dt^2) + 3(t+2)(dy/dt) + (t+2)y = 0$

or $\quad (t^2 + 4t + 3)(d^2y/dt^2) + (3t+6)(dy/dt) + (t+2)y = 0.$ ...(6)

Also, (3) reduces to $\quad y = C_0 + C_1 t + C_2 t^2 + \dots = \sum_{n=0}^{\infty} C_n t^n$. ...(7)

Differentiating (7) twice in succession w.r.t. 't', we get

$dy/dt = \sum_{n=1}^{\infty} nC_n t^{n-1} \quad$ and $\quad d^2y/dt^2 = \sum_{n=2}^{\infty} n(n-1)C_n t^{n-2}$. ...(8)

Using (7) and (8), (6) reduces to

$$(t^2 + 4t + 3)\sum_{n=2}^{\infty} n(n-1)C_n t^{n-2} + (3t+6)\sum_{n=1}^{\infty} nC_n t^{n-1} + (t+2)\sum_{n=0}^{\infty} C_n t^n = 0$$

or $\sum_{n=2}^{\infty} n(n-1)C_n t^n + \sum_{n=2}^{\infty} 4n(n-1)C_n t^{n-1} + \sum_{n=2}^{\infty} 3n(n-1)C_n t^{n-2} + \sum_{n=1}^{\infty} 3nC_n t^n + \sum_{n=1}^{\infty} 6nC_n t^{n-1} + \sum_{n=0}^{\infty} C_n t^{n+1} + \sum_{n=0}^{\infty} 2C_n t^n = 0$

or $\quad \sum_{n=2}^{\infty} n(n-1)C_n t^n + \sum_{n=1}^{\infty} 4(n+1)nC_{n+1} t^n + \sum_{n=0}^{\infty} 3(n+2)(n+1)C_{n+2} t^n$

$$+ \sum_{n=1}^{\infty} 3nC_n t^n + \sum_{n=0}^{\infty} 6(n+1)C_{n+1} t^n + \sum_{n=1}^{\infty} C_{n-1} t^n + \sum_{n=0}^{\infty} 2C_n t^n = 0$$

or $\quad (6C_2 + 6C_1 + 2C_0) + (8C_2 + 18C_3 + 3C_1 + 12C_2 + C_0 + 2C_1)t$

$+ \sum_{n=2}^{\infty} [n(n-1)C_n + 4n(n+1)C_{n+1} + 3(n+1)(n+2)C_{n+2} + 3nC_n + 6(n+1)C_{n+1} + C_{n-1} + 2C_n]t^n = 0$

or $\quad 2(3C_2 + 3C_1 + C_0) + (18C_3 + 20C_2 + 5C_1 + C_0)t$

$+ \sum_{n=2}^{\infty} [3(n+1)(n+2)C_{n+2} + 2(2n+3)(n+1)C_{n+1} + (n^2 + 2n + 2)C_n + C_{n-1}]t^n = 0$ ...(9)

From (3), $\quad y' = C_1 + 2C_2(x-2) + 3C_3(x-2)^2 + \dots$ ...(10)

Putting $x = 2$ in (3) and (10) and using the given initial conditions, namely, $y = 4$ and $y' = 6$ when $x = 2$, we get $C_0 = 4$ and $C_1 = 6$. Hence (9) reduces to

$2(3C_2+22) + (18C_3+20C_2+34)t + \sum_{n=2}^{\infty} \{3(n+1)(n+2)C_{n+2} + 2(2n+3)(n+1)C_{n+1} + (n^2+2n+2)C_n + C_{n-1}\}t^n = 0,$

which is an identity in t. Equating to zero the coefficients of various powers of t, we have

$2(3C_2 + 22) = 0 \quad$ so that $\quad C_2 = -(22/3).$ ...(11)

$18C_3 + 20C_2 + 34 = 0 \quad$ or $\quad 18C_3 - 20 \times (22/3) + 34 = 0 \quad$ or $\quad C_3 = 169/27$ ...(12)

and $\quad 3(n+1)(n+2)C_{n+2} + 2(2n+3)(n+1)C_{n+1} + (n^2+2n+2)C_n + C_{n-1} = 0,\;$ for all $\; n \geq 2.$ ...(13)

Putting $n = 2$ in (13), we get $\quad 36C_4 + 42C_3 + 10C_2 + C_1 = 0$

or $\quad 36C_4 + 42 \times (169/27) + 18 \times (-22/3) + 6 = 0 \quad \Rightarrow \quad C_4 = 344/81$

Putting the above values in (3), the required solution is

$$y = 4 + 6(x-2) - (22/3) \times (x-2)^2 + (169/27) \times (x-2)^3 + (344/81) \times (x-2)^4 + \dots$$

Ex. 8. *Solve* $\; y'' - 2x^2y' + 4xy = x^2 + 2x + 4$ *in powers of* x.

Sol. Given equation is $\quad y'' - 2x^2y' + 4xy = x^2 + 2x + 4.$ ...(1)

Clearly $x = 0$ is an ordinary point of (1). To solve (1) it, let

$$y = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots = \sum_{n=0}^{\infty} C_n x^n . \qquad \text{...(2)}$$

Differentiating (2) twice in succession w.r.t. 'x', we have

$$y' = \sum_{n=1}^{\infty} n C_n x^{n-1} \qquad \text{and} \qquad y'' = \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} . \qquad \text{...(3)}$$

Substituting these values of y, y' and y'' in (1), we have

$$\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} - 2x^2 \sum_{n=1}^{\infty} n C_n x^{n-1} + 4x \sum_{n=0}^{\infty} C_n x^n = x^2 + 2x + 4$$

or
$$\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} - \sum_{n=1}^{\infty} 2n C_n x^{n+1} + \sum_{n=0}^{\infty} 4 C_n x^{n+1} - x^2 - 2x - 4 = 0$$

or
$$\sum_{n=0}^{\infty} (n+2)(n+1) C_{n+2} x^n - \sum_{n=2}^{\infty} 2(n-1) C_{n-1} x^n + \sum_{n=1}^{\infty} 4 C_{n-1} x^n - x^2 - 2x - 4 = 0$$

or
$$(2C_2 - 4) + (6C_3 + 4C_0 - 2)x + (12C_4 + 2C_1 - 1)x^2 + \sum_{n=3}^{\infty} [(n+2)(n+1)C_{n+2} - 2(n-1)C_{n-1} + 4C_{n-1}\} x^n = 0. \qquad \text{...(4)}$$

Equating to zero the coefficients of various powers of x in (4), we get

$2C_2 - 4 = 0$ so that $C_2 = 2$, ...(5)

$6C_3 + 4C_0 - 2 = 0$ so that $C_3 = (1/3) - (2C_0/3)$ ...(6)

$12C_4 + 2C_1 - 1 = 0$ so that $C_4 = (1/12) - (C_1/6)$ ...(7)

and $(n + 2)(n + 1)C_{n+2} - 2(n - 1)(C_{n-1} + 4C_{n-1} = 0$, for all $n \geq 3$. ...(8)

Putting $n = 3, 4, 5, \dots$ in (8) and using (5), (6), (7), etc, we get

$20C_5 - 4C_2 + 4C_2 = 0$ so that $C_5 = 0$, ...(9)

$30C_6 = 2C_3$ so that $C_6 = \dfrac{1}{15}\left(\dfrac{1}{3} - \dfrac{2}{3}C_0\right) = \dfrac{1}{45} - \dfrac{2}{45}C_0$

$42C_7 = 4C_4$ so that $C_7 = \dfrac{2}{21}\left(\dfrac{1}{12} - \dfrac{1}{6}C_1\right) = \dfrac{1}{125} - \dfrac{1}{63}C_1$

and so on. Putting these values in (2), the required solution is

$$y = C_0 + C_1 x + 2x^2 + \left(\frac{1}{3} - \frac{2C_0}{3}\right)x^3 + \left(\frac{1}{12} - \frac{C_1}{6}\right)x^4 + \left(\frac{1}{45} - \frac{2C_0}{45}\right)x^6 + \left(\frac{1}{126} - \frac{C_1}{63}\right)x^7 + \dots$$

or
$$y = C_0\left(1 - \frac{2}{3}x^3 - \frac{2}{45}x^6 \dots\right) + C_1\left(x - \frac{1}{6}x^4 - \frac{1}{63}x^7 \dots\right) + 2x^2 + \frac{1}{3}x^3 + \frac{1}{12}x^4 + \frac{1}{45}x^6 + \frac{1}{126}x^7 + \dots$$

Ex. 9. (*i*) *Explain the method of integrating in series for solving a first order differential equation.*

(*ii*) *Find a power series solution of the form* $\sum_{n=0}^{\infty} a_n x^n$ *for the differential equation* $y' = 2xy$.

[Nagpur 2005]

(*iii*) *Solve* $y' = x^2 - 4x + y + 1$ *satisfying* $y = 3$ *when* $x = 2$.

Sol. (*i*) The Picard's theorem of Art. 1.6 (refer chapter 1) for a differential equation of the form

$$dy/dx = f(x, y) \qquad \text{... (1)}$$

gives a sufficient condition for a solution. In the proof using power series, y is found in the form of a Taylor series

$$y = a_0 + a_1 (x - x_0) + a_2(x - x_0)^2 + \dots + a_n (x - x_0)^n + \dots \qquad \text{... (2)}$$

where for convenience y_0 has been replaced by a_0. This series has the following properties:

(a) It satisfies the differential equation (1) (b) It has the value y_0 when $x = x_0$.

(c) It is convergent for all values of x sufficiently near $x = x_0$.

To find the solution of (1) satisfying the condition $y = y_0$ when $x = 0$, we assume the solution to be of the form
$$y = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots = \sum_{n=0}^{\infty} a_n x^n \qquad \ldots (3)$$
in which $a_0 = y_0$ and the remaining a's are constant to be determined.

Substitute the assumed series (3) in (1) and proceed to find $a_1, a_2, a_3, \ldots a_n \ldots$ as usual.

Remark. If we are to find the solution of (1) satisfying $y = y_0$ when $x = x_0$, we modify the above procedure as follows

Make the substitution $x - x_0 = v$, that is, $x = v + x_0$, $dy/dx = dy/dv$ resulting in
$$dy/dv = F(y, v) \qquad \ldots (4)$$
Use the above procedure to obtain the solution of (4) satisfying $y = y_0$ when $v = 0$. Finally make the substitution $v = x - x_0$ in the solution.

(ii) Given
$$y' = 2xy \qquad \ldots (1)$$
Let (1) possess series solution
$$y = \sum_{n=0}^{\infty} a_n x^n \qquad \ldots (2)$$
From (2),
$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1} \qquad \ldots (3)$$
Inserting (2) and (3) into (1), we have
$$a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots + n a_n x^{n-1} + \ldots = 2x\left(a_0 + a_1 x + a_2 x^2 + \ldots + a_{n-2} x^{n-2} + \ldots\right)$$
Collecting like powers of x yields.
$$a_1 + (2a_2 - 2a_0)x + (3a_3 - 2a_1)x^2 + (4a_4 - 2a_2)x^3 + (5a_5 - 3a_3)x^4 + (6a_6 - 5a_4)x^5 + \ldots + (na_n - 2a_{n-2})x^{n-1} + \ldots = 0$$
In order that this series vanish for all values of x in some region surrounding $x = 0$, it is necessary and sufficient that coefficients of each power of x vanish. Thus, we obtain

$a_1 = 0$, $2a_2 - 2a_0 = 0$, $3a_3 - 2a_1 = 0$, $4a_4 - 2a_2 = 0$, $5a_5 - 2a_3 = 0$, $6a_6 - 2a_4 = 0$

$\Rightarrow a_1 = 0$, $a_2 = a_0$, $a_3 = (2/3)a_1 = 0$, $a_4 = (1/2)a_2 = (1/2)a_0$, $a_6 = (1/3)a_4 = (1/3) \times (a_0/2) = a_0/3!$ and so on

In general, $a_{2n+1} = 0$ and $a_{2n} = a_0/n!$, for all $n = 1, 2, 3$

Substituting these values into (2), we obtain the required power series solution.
$$y = a_0 + a_2 x^2 + a_4 x^4 + a_6 x^6 + \ldots + a_{2n} x^{2x} + \ldots \text{ or } y = a_0 + a_0 x^2 + (a_0/2!)x^4 + (a_0/3!)x^6 + \ldots + (a_0/n!)x^{2n} + \ldots$$
or $\quad y = a_0\,(1 + x^2/1! + x^4/2! + x^6/3! + x^{2n}/n! + \ldots)$, a_0 being an arbitrary constant

(iii) Given
$$dy/dx = x^2 - 4x + y + 1 \qquad \ldots (1)$$
where $\quad y = 3 \quad$ when $\quad x = 2 \qquad \ldots (2)$

Let $x = v + 2$. Then (1) and (2) reduce to
$$dy/dv = v^2 + y - 3 \qquad \ldots (3)$$
and $\quad y = 3 \quad$ when $\quad v = 0 \qquad \ldots (4)$

We now proceed with (3) and (4) as in part (ii). We assume the series solution
$$y = 3 + a_1 v + a_2 v^2 + a_3 v^3 + a_4 v^4 + \ldots + a_n v^n + \ldots \qquad \ldots (5)$$
From (5),
$$dy/dv = a_1 + 2a_2 v + 3a_3 v^2 + 4a_4 v^3 + \ldots + n a_n v^{n-1} + \ldots \qquad \ldots (6)$$
Inserting (5) and (6) into (3), we have, as before
$$a_1 + (2a_2 - a_1)v + (3a_3 - a_2 - 1)v^2 + (4a_4 - a_3)v^3 + \ldots + (na_n - a_{n-1})v^{n-1} + \ldots = 0$$

Equating each of the coefficients to zero, we obtain

$a_1 = 0, 2a_2 - a_1 = 0$ so that $a_2 = 0$; $3a_3 - a_2 - 1 = 0$ so that $a_3 = 1/3$; $4a_4 - a_3 = 0$ so that $a_4 = 1/12$, and so on.

Again, $na_n - a_{n-1} = 0 \quad \Rightarrow \quad a_n = (1/n)a_{n-1}$ for all $n \ge 1$ ... (7)

Using the recursion formula (7), we have

$$a_n = \frac{1}{n}a_{n-1} = \frac{1}{n(n-1)}a_{n-2} = \cdots = \frac{1}{n(n-1)\,(n-2)\cdots 4}a_3 = \frac{2}{n!}, \qquad \text{for all } n \ge 2$$

Substituting the above values of the coefficients into (5), we have

$$y = 3 + v^3/3 + v^4/12 + \ldots + (2/n!)v^n + \ldots \qquad \ldots (8)$$

Replacing v by $x - 2$, (8) gives the required solution

$$y = 3 + (2/3!) \times (x-2)^3 + (2/4!) \times (x-2)^4 + \ldots + (2/n!) \times (x-2)^n + \ldots$$

Ex. 10. *Solve by power series method :* $y' - y = 0$. **[Sagar 2004]**

Sol. Given $\qquad y' - y = 0 \qquad$ ... (1)

Assume that a solution of (1) is given by power series

$$y = C_0 + C_1 x + C_2 x^2 + \ldots + C_n x^n + \ldots = \sum_{n=0}^{\infty} C_n x^n \qquad \ldots (2)$$

Differentiating (2) w.r.t. 'x', $\qquad y' = \sum_{n=1}^{\infty} n\, C_n x^{n-1}$

Substituting the above values of y and y' in (1), we get $\quad \sum_{n=1}^{\infty} n\, C_n x^{n-1} - \sum_{n=0}^{\infty} C_n x^n = 0$

or $\qquad (C_1 + 2C_2 x + 3C_3 x^2 + \ldots.) - (C_0 + C_1 x + C_2 x^2 + \ldots) = 0$

or $\qquad (C_1 - C_0) + (2C_2 - C_1)x + (3C_3 - C_2)x^2 + \ldots = 0 \qquad$ ... (3)

Since (3) is an identity, we must have

$C_1 - C_0 = 0, \qquad 2C_2 - C_1 = 0, \qquad 3C_3 - C_2 = 0, \ldots. \qquad$ ... (4)

Solving (4), $\quad C_1 = C_0, \qquad C_2 = C_1/2 = C_0/2, \qquad C_3 = C_2/3 = C_0/3!, \ldots$

Substituting these values in (2), we obtain

$y = C_0\,(1 + x + x^2/2! + x^3/3! + \ldots) \qquad$ or $\qquad y = C_0 e^x,$

which is the required solution, C_0 being an arbitrary constant.

EXERCISE 8 (B)

Find the series solution of the following equations :

1. $(1 - x^2)y'' + 2xy' - y = 0$ about $x = 0$. **[Purvanchal 2007; Meerut 2000]**

Ans. $y = C_0(1 + x^2/2 - x^4/24 + \cdots) + C_1(x - x^3/6 - x^5/120 + \cdots)$

2. $(2 + x^2)y'' + xy' - (1 + x)y = 0$ near $x = 0$. **(Delhi Maths (H) 2002)**

Ans. $y = C_0(1 + x^2/4 + x^3/12 - 3x^4/96 + \cdots) + C_1(x + x^4/24 + \cdots)$

3. $(1 + x^2)y'' + xy' - y = 0$ near $x = 0$. **[Delhi Maths (Hons) 1999]**

Ans. $y = C_0(1 + x^2/2 - x^4/8 + x^6/15 + \cdots) + C_1 x.$

4. $(x^2 - 1)y'' + xy' - y = 0$ near $x = 0$. **Ans.** $y = C_0\{1 + (x^2/2) + (x^4/4) + \ldots.\} + C_1 x$

5. $(x^2 - 1)y'' + 4xy' + 2y = 0$ near $x = 0$. **Ans.** $y = C_0(1 + x^2 + x^4 + \ldots.) + C_1(x + x^3 + x^5 + \ldots)$

6. $(1 - x^2)y'' + 2xy' - y = 0$ about $x = 0$.

Ans. $y = C_0(1 + x^2/2! - x^4/4! + \ldots) + C_1\,(1 - x^3/3! - x^5/5! + \ldots)$

7. $y'' - xy' + 2y = 0$ near $x = 1$.

Ans. $y = C_0[1 - (x-1)^2 - (1/3) \times (x-1)^3 - ....] + C_1[(x-1) + (1/2) \times (x-1)^2 + ...]$

8. $(x^2 - 1)y'' + 3xy' + xy = 0, y(0) = 2, y'(0) = 3$. **Ans.** $y = 2 + 3x + (11/6) \times x^3 + (1/4) \times x^4 - ....$

9. (*i*) $(1 - x^2)y'' + 2y = 0$ near $x = 0$. **Ans.** $y = C_0(1 - x^2) + C_1(x - x^3/3 - x^5/5 - x^7/35 - ....$

(*ii*) $(1 - x^2)y'' + 2y = 0, y(0) = 4, y'(0) = 5$. **Ans.** $y = 4 + 5x - 4x^2 - (5/3) \times x^3 - (1/3) \times x^5 + ...$

10. $(x^2 + 2x)y'' + (x+1)y' - y = 0$ near $x = -1$.

Ans. $y = C_0[1 - (1/2) \times (x+1)^2 - (1/8) \times (x+1)^4 - (1/16) \times (x+1)^6 + ...] + C_1(x+1)$

11. $y'' - xy' = e^{-x}, y(0) = 2, y'(0) = -3$.

[Hint : Use the expansion $e^{-x} = 1 - (x/1!) + (x^2/2!) - (x^3/3!) + ...$

Ans. $y = 2 - 3x + (1/2) \times x^2 - (2/3) \times x^3 + (1/8) \times x^4 - ....$

Ex. 12(*a*). *Solve* $y'' + x^2y = 2 + x + x^2$ *about* $x = 0$.

Sol. Proceed as in Ex. 8. of Art. 8.6. Its general solution is given by

$$y = C_0\left(1 - \frac{x^4}{12} + \frac{x^8}{672} - ....\right) + C_1\left(x - \frac{x^5}{20} + \frac{x^9}{1440} - ....\right) + x^2 + \frac{x^3}{6} + \frac{x^4}{12} - \frac{x^6}{30} - \frac{x^7}{252} - ....$$

Ex. 12(*b*). *Apply power method to solve* $y'' - y = x$.

Sol. Proceed as in Ex. 8. of Art 8.6. Its general solution is given by

$$y = C_0\left(1 + \frac{x^2}{2!} + \frac{x^4}{4!} + ...\right) + C_1\left(x + \frac{x^3}{3!} + \frac{x^5}{5!} + ...\right) + \left(\frac{x^3}{3!} + \frac{x^5}{5!} + ...\right)$$

13. $y'' - 4y = 0$ near $x = 0$ **[Agra 2007]**

8.7. Series solution about regular singular point x = 0. Frobenius Method.

If $x = 0$ is regular point, we shall use the Frobenius method for finding series solution about $x = 0$. However if we wish to find the series solution about regular singular point $x = a$, then we first shift the origin to point $x = a$ and later on proceed as before.

If $x = 0$ is an irregular point of the given equation, then discussion of solution of the equation is beyond the scope of this book.

We now discuss Frobenius method. Consider the differential equation of the form

$$y_2 + \frac{F(x)}{x}y_1 + \frac{G(x)}{x^2}y = 0, \qquad \text{where} \qquad y_1 = \frac{dy}{dx}, \qquad y_2 = \frac{d^2y}{dx^2} \qquad ...(1)$$

and the functions $F(x)$ and $G(x)$ are analytic at $x = 0$. Then the following method for solving (1) is called Frobenius method. We assume a trial solution

$$y = x^r \sum_{m=0}^{\infty} c_m x^m = x^r(c_0 + c_1x + c_2x^2 + ...), \text{ where } c_0 \neq 0. \qquad ...(2)$$

Differentiating (2) term by term, we have

$$y_1 = rc_0x^{r-1} + (r+1)c_1x^r + ... = x^{r-1}[rc_0 + (r+1)c_1x + ...]$$
$$y_2 = r(r-1)c_0x^{r-2} + r(r+1)c_1x^{r-1} + ... = x^{r-2}[r(r-1)c_0 + (r+1)rc_1x + ...]$$

Since $F(x)$ and $G(x)$ are analytic at $x = 0$, we can write

$F(x) = a_0 + a_1x + a_2x^2 + ...$ and $G(x) = b_0 + b_1x + b_2x^2 + ...$

Putting the values of $y, y_1, y_2, F(x)$ and $G(x)$ in (1) and then multiplying both sides by x^2, gives

$$x^r[r(r-1)c_0 + ...] + (a_0 + a_1x + ...)x^r(rc_0 + ...) + (b_0 + b_1x + ...)x^r(c_0 + c_1x + c_2x^2 + ...) = 0. \quad ...(3)$$

Since (3) is an identity, we can equate to zero the coefficients of various powers of x. This will give us a system of equations involving the unknown coefficients c_m. The smallest power is x^r, and the corresponding equation is $[r(r-1) + a_0r + b_0]c_0 = 0$.

Since by assumption $c_0 \neq 0$, we obtain $r^2 + (a_0 - 1)r + b_0 = 0$. ...(4)

This important quadratic equation is known as the *indicial equation* of (1). We shall see that this method will give rise to a fundamental system of solutions ; one of these solutions will always be of the form (2), but for the form of the other solutions there will be three different possibilities corresponding to the following cases :

Case I. The roots of the indicial equation are distinct and do not differ an integer.

Let r_1 and r_2 be the roots of (4). We substitute $r = r_1$ in the above mentioned system of equations and obtain a solution $u(x) = x^{r_1}(c_0 + c_1 x + \ldots)$.

Proceeding similarly by using $r = r_2$, we obtain a second solution $v(x) = x^{r_2}(c'_0 + c'_1 x + \ldots)$ where c'_0, c_1', ... are the new values of the coefficients corresponding to $r = r_2$. Since $r_1 - r_2$ is not an integer, u/v is never constant. Thus u and v are two independent solutions of (1). Hence the general solution will be $y = au + bv$, where a and b are arbitrary constants.

Case II. The indicial equation has equal roots. The indicial equation (4) has double root r if

$$(a_0 - 1)^2 - 4b_0 = 0 \quad \text{giving} \quad r = (1 - a_0)/2. \qquad \ldots(5)$$

We first obtain the coefficients c_1, c_2.... successively from the system of equations connecting them. Thus we obtain a first solution

$$u(x) = x^r(c_0 + c_1 x + \ldots), \qquad \ldots(6)$$

wherein we write $r = (1 - a_0)/2$ afterwards.

To determine another solution we use the method of variation of parameters, that is, we replace the constant c in the solution cu by a function $w(x)$ to be determined such that

$$v(x) = u(x)\, w(x) \qquad \ldots(7)$$

is solution of (1). Differentiating (7), we get

$$v_1 = u_1 w + u w_1 \quad \text{and} \quad v_2 = u_2 w + 2u_1 w_1 + u w_2. \qquad \ldots(8)$$

Since $v(x)$ is a solution of (1) by assumption, $x^2 v_2 + F(x)\ x\, v_1 + G(x)\, v = 0. \qquad \ldots(9)$

Putting the values of v, v_1 and v_2 from (7) and (8) in (9), we get

$$x^2(u_2 w + 2u_1 w_1 + u w_2) + xF(x)[u_1 w + u w_1] + G(x)uw = 0$$

or

$$[x^2 u_2 + F(x) x u_1 + G(x)u]w + x^2(2u_1 w_1 + u w_2) + xF(x)u w_1 = 0$$

Since $u(x)$ is a solution of (1), we have $x^2 u_2 + F(x) x u_1 + G(x) u = 0.$

$\therefore$ The above equation reduces to $x^2(2u_1 w_1 + u w_2) + xF(x) u w_1 = 0.$

Dividing by $x^2 u$ and putting the value of $F(x)$ gives, $w_2 + \left(2\dfrac{u_1}{u} + \dfrac{a_0}{x} + \ldots\right) w_1 = 0. \quad \ldots(10)$

In what follows we write dots to represent terms which are constant or involve positive powers of x. Now from (6), we obtain

$$\frac{u_1}{u} = \frac{x^{r-1}[rc_0 + (r+1)c_1 x + \ldots]}{x^r[c_0 + c_1 x + \ldots]} = \frac{r}{x} + \ldots$$

$\therefore$ (10) becomes

$$w_2 + \left(\frac{2r + a_0}{x} + \ldots\right) w_1 = 0. \qquad \ldots(11)$$

But from (5), $2r + a_0 = 1$. Hence the above equation reduces to

$$w_2 + \left(\frac{1}{x} + \ldots\right) w_1 = 0 \quad \text{or} \quad \frac{w_2}{w_1} = -\frac{1}{x} + \ldots$$

Integrating, $\log w_1 = -\log x + \ldots$ or $w_1 = \dfrac{1}{x} e^{(\ldots)}$

Expanding the exponential function in powers of x and integrating once more, we obtain the expression of w always in the following form $w = \log x + k_1 x + k_2 x^2 + \ldots$

Putting this value of w in (7), we obtain the desired another independent solution $v(x)$. Then the general solution is $y = au + bv$, a and b being arbitrary constants.

Case III. The roots of the indicial equation differ by an integer.

Let the roots r_1 and r_2 of the indicial equation differ by an integer, say $r_1 = r$ and $r_2 = r - p$, where p is a positive integer, then as before one solution will be given by

$$u(x) = x^{r_1}\,(c_0 + c_1 x + c_2 x^2 + \ldots.)$$

corresponding to the root r_1. However, while, dealing with the root r_2 it may not be possible to determine another independent solution $v(x)$ as in case I. In such cases we determine $v(x)$ by using method outlined in case II. Thus as before we first obtain (11).

From (5), $\quad r_1 + r_2 = -(a_0 - 1)$, using theory of equations

Since $r_1 = r$, $r_2 = r - p$, this gives $2r + a_0 = p + 1$. Thus (11) becomes

$$\frac{w_2}{w_1} = -\left(\frac{p+1}{x} + \ldots.\right).$$

Integrating, $\quad \log w_1 = -(p+1)\log x + \ldots.\quad$ or $\quad w_1 = x^{-(p+1)}e^{(\ldots)}$

Expanding the exponential function and simplifying, we get

$$w_1 = \frac{1}{x^{p+1}} + \frac{k_1}{x^p} + \ldots. + \frac{k_p}{x} + k_{p+1} + k_{p+2}x + \ldots.$$

Integrating again, $$w = -\frac{1}{px^p} - \ldots. + k_p \log x + k_{p+1}x + \ldots.$$

Putting this value of w in (7), we obtain another solution $v(x)$ and as usual the general solution $y = au + bv$, where a and b are arbitrary constants.

8.8. Working rule for solution by Frobenius method

Consider linear differential equation of order two $\quad f(x)y'' + g(x)y' + r(x)y = 0.\quad$ ...(1)

Step 1. Suppose that a trial solution of (1) be of the form

$$y = x^k(c_0 + c_1x + c_2x^2 + \ldots. + c_mx^m + \ldots.)\qquad i.e.\qquad y = x^k \sum_{m=0}^{\infty} c_m x^m. \qquad \ldots(2)$$

Thus we take $$y = \sum_{m=0}^{\infty} c_m x^{m+k}, \qquad \text{where} \qquad c_0 \neq 0. \qquad \ldots(3)$$

Step 2. Differentiate (3) and obtain

$$y' = \sum_{m=0}^{\infty} c_m (m+k)x^{m+k-1} \qquad \text{and} \qquad y'' = \sum_{m=0}^{\infty} c_m (m+k)(m+k-1)x^{m+k-2}. \qquad \ldots(4)$$

Using (3) and (4), (1) reduces to an identity.

Step 3. Equating to zero the coefficient of the smallest power of x in the identity obtained in step 2 above, we obtain a quadratic equation is k. The quadratic equation so obtained is called the *indicial equation.*

Step 4. Solve the indicial equation. The following cases arise :

(*i*) The roots of indicial equation unequal and not differing by an integer.

(*ii*) The roots of indicial equation unequal, differing by an integer and making a coefficient of y indeterminate.

(*iii*) The roots of indicial equation unequal, differing by an integer and making a coefficient of y infinite.

(*iv*) The roots of indicial equation equal.

Step 5. We equate to zero the coefficient of general power (*e.g.* x^{k+m}, x^{k+m-1} etc. whichever may be the lowest) in the identity obtained in step 2. The equation so obtained will be called the *recurrence relation*, because it connects together the coefficients c_m, c_{m-2} or c_m, c_{m-1} etc.

Step 6. If the recurrence relation connects c_m and c_{m-2}, then we, in general, determine c_1 by equating to zero the coefficient of the next higher power (than already used for getting the indicial equation). On the other hand, if the recurrence relation connects c_m and c_{m-1}, this step may be omitted.

Step 7. After getting various coefficinets with help of steps 5 and 6 above, solution is obtained by substituting these in (2) or (3) above. Other necessary working will be shown in details and necessary modifications in method will be discussed as we proceed with different four cases outlined in step 4. The readers are advised to study carefully the first example of each case. In each problem the two series of solution should be linearly independent.

8.9. Examples of Type–1 on Frobenius method. Roots of indicial equation unequal and not differing by an integer :

In this connection the following rule should be noted carefully.

Rule. *Let k_1 and k_2 be the roots of the indicial equation. It k_1 and k_2 do not differ by an integer. Then, in general, two independent solutions u and v are obtained by putting $k = k_1$ and k_2 in the series for y. Then the general solution is $y = au + bv$, where a and b are arbitrary constants.*

Ex. 1. *Solve in series :* $9x(1-x)y'' - 12y' + 4y = 0$. **[Delhi Maths (H) 2007, 08; Meerut 1997]**

Sol. Given equation is $$9x(1-x)y'' - 12y' + 4y = 0. \qquad ...(1)$$

Dividing by $9x(1-x)$, (1) can be put in standard form

$$\frac{d^2y}{dx^2} - \frac{4}{3x(1-x)}\frac{dy}{dx} + \frac{4}{9x(1-x)}y = 0.$$

Comparing it with $y'' + P(x)y' + Q(x)y = 0$, we have $P(x) = -4/[3x(1-x)]$ and $Q(x) = 4/[9x(1-x)]$. Since $P(x)$ and $Q(x)$ are not both analytic at $x = 0$, so $x = 0$ is not ordinary point of (1). Again $xP(x) = -4/[3(1-x)]$ and $x^2Q(x) = \quad 4x/(1-x)$, showing that both $P(x)$ and $Q(x)$ are analytic at $x = 0$. So $x = 0$ is a regular singular point of (1). To find solution of (1), we take

$$y = \sum_{m=0}^{\infty} c_m x^{k+m}, \text{ where } c_0 \neq 0. \qquad ...(2)$$

$$\therefore \quad y' = \sum_{m=0}^{\infty} c_m (k+m)x^{k+m-1} \quad \text{and} \quad y'' = \sum_{m=0}^{\infty} c_m (k+m)(k+m-1)x^{k+m-2}. \qquad ...(3)$$

Substituting the series (2) and (3) in (1), we have

$$9x(1-x)\sum_{m=0}^{\infty} c_m (k+m)(k+m-1)x^{k+m-2} - 12\sum_{m=0}^{\infty} c_m (k+m)x^{k+m-1} + 4\sum_{m=0}^{\infty} c_m x^{k+m} = 0$$

or $$9x\sum_{m=0}^{\infty} c_m (k+m)(k+m-1)x^{k+m-2} - 9x^2\sum_{m=0}^{\infty} c_m (k+m)(k+m-1)x^{k+m-2} - 12\sum_{m=0}^{\infty} c_m (k+m)x^{k+m-1} + 4\sum_{m=0}^{\infty} c_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} c_m \{9(k+m)(k+m-1) - 12(k+m)\}x^{k+m-1} + \sum_{m=0}^{\infty} c_m \{4 - 9(k+m)(k+m-1)\}x^{k+m} = 0. \qquad ...(4)$$

But $$9(k+m)(k+m-1) - 12(k+m) = 3(k+m)(3k+3m-7) \qquad ...(5)$$

and $4 - 9(k+m)(k+m-1) = 4 - 9(k+m)^2 + 9(k+m)$

$= -[9(k+m)^2 - 9(k+m) - 4] = -[9(k+m)^2 - 12(k+m) + 3(k+m) - 4]$

$= -[3(k+m)\{3(k+m) - 4\} + 3(k+m) - 4] = -\{3(k+m) - 4\}\{3(k+m) + 1\}$

Thus, $$4 - 9(k+m)(k+m-1) = -(3k+3m-4)(3k+3m+1). \qquad ...(6)$$

Using (5) and (6), (4) can be re–written as

$$3\sum_{m=0}^{\infty} c_m (k+m)(3k+3m-7)x^{k+m-1} - \sum_{m=0}^{\infty} c_m (3k+m-4)(3k+3m+1)x^{k+m} = 0, \qquad ...(7)$$

which is an identity in x. Equating to zero the coefficient of the smallest power of x, namely x^{k-1}, (7) gives the indicial equation

$3c_0 k(3k-7) = 0$ or $k(3k-7) = 0$ $[\because\ c_0 \neq 0]$

Thus $k = 0$ and $7/3$, ...(8)

which are unequal and not differing by an integer. To obtain the recurrence relation, we equate to zero the coefficient of x^{k+m-1} and obtain

$$3c_m(k+m)(3k+3m-7) - c_{m-1}[3k+3(m-1)-4] \times [3k+3(m-1)+1] = 0$$

or
$$c_m = \frac{3k+3m-2}{3(k+m)} c_{m-1}. \qquad ...(9)$$

Taking $m = 1$ in (9) gives $c_1 = \dfrac{c_0}{3} \times \dfrac{3k+1}{k+1}$. ...(10)

Next, taking $m = 2$ in (9) gives $c_2 = \dfrac{3k+4}{3(k+2)} c_1 = \dfrac{c_0}{3^2} \times \dfrac{(3k+1)(3k+4)}{(k+1)(k+2)}$, by (10)

...(11)

and so on. Putting these values in (2) *i.e.*, $y = x^k(c_0 + c_1 x + c_2 x^2 + ...)$, gives

$$y = c_0 x^k \left[1 + \frac{1}{3}\frac{3k+1}{k+1}x + \frac{1}{3^2}\frac{(3k+1)(3k+4)}{(k+1)(k+2)}x^2 + ...\right] \qquad ...(12)$$

Putting $k = 0$ and replacing c_0 by a in (12), $y = a\left(1 + \dfrac{1}{3}x + \dfrac{1\cdot 4}{3\cdot 6}x^2 + ...\right) = au$, say

Next, putting $k = \dfrac{7}{2}$ and replacing C_0 by b in (12), $y = bx^{7/3}\left(1 + \dfrac{8}{10}x + \dfrac{8\cdot 11}{10\cdot 13}x^2 + ...\right) = bv$, say.

The required solution is given by $y = au + bv$, *i.e.*

$$y = a\left(1 + \frac{1}{3}x + \frac{1\cdot 4}{3\cdot 6}x^2 + ...\right) + bx^{7/3}\left(1 + \frac{8}{10}x + \frac{8\cdot 11}{10\cdot 13}x^2 + ...\right).$$

Ex. 2. *Solve the Bessel equation $x^2y'' + xy' + (x^2 - n^2)y = 0$ in series, taking $2n$ as non–integral.* **[Delhi 1997; G.N.D.U. Amritsar 2010; Ranchi 2010]**

Sol. Given $x^2y'' + xy' + (x^2 - n^2)y = 0$. ...(1)

Dividing by x^2, $y'' + (1/x)y' + \{(x^2 - n^2)/x^2\}y = 0$.

Comparing it with $y'' + P(x)y' + Q(x)y = 0$, here $P(x) = 1/x$ and $Q(x) = (x^2 - n^2)/x^2$ so that $xP(x) = 1$ and $x^2Q(x) = x^2 - n^2$. Thus both $P(x)$ and $Q(x)$ are analytic at $x = 0$ and so $x = 0$ is a regular singular point of (1). Let the series solution of (1) be of the form

$$y = \sum_{m=0}^{\infty} C_m x^{k+m}, \quad \text{where } C_0 \neq 0. \qquad ...(2)$$

$$\therefore \quad y' = \sum_{m=0}^{\infty} C_m(k+m)x^{k+m-1} \quad \text{and} \quad y'' = \sum_{m=0}^{\infty} C_m(k+m)(k+m-1)x^{k+m-2}. \qquad ...(3)$$

Substitution for y, y', y'' in (1), we have

$$x^2 \sum_{m=0}^{\infty} C_m(k+m)(k+m-1)x^{k+m-2} + x\sum_{m=0}^{\infty} C_m(k+m)x^{k+m-1} + (x^2 - n^2)\sum_{m=0}^{\infty} C_m x^{k+m} = 0$$

or
$$\sum_{m=0}^{\infty} C_m(k+m)(k+m-1)x^{k+m} + \sum_{m=0}^{\infty} C_m(k+m)x^{k+m} + \sum_{m=0}^{\infty} C_m x^{k+m+2} - n^2 \sum_{m=0}^{\infty} C_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} C_m\{(k+m)(k+m-1)+(k+m)-n^2\}x^{k+m} + \sum_{m=0}^{\infty} C_m x^{k+m+2} = 0$$

or $$\sum_{m=0}^{\infty} C_m\{(k+m)^2-n^2\}x^{k+m} + \sum_{m=0}^{\infty} C_m x^{k+m+2} = 0$$

or $$\sum_{m=0}^{\infty} C_m (k+m+n)(k+m-n)x^{k+m} + \sum_{m=0}^{\infty} C_m x^{k+m+2} = 0, \qquad ...(4)$$

which is an identity. Equating to zero the coefficient of the smallest power of x, namely x^k, (4) gives the indicial equation $C_0(k+n)(k-n)=0$ or $(k+n)(k+n)=0$ $[\because C_0 \neq 0]$

so that $k = n$ and $-n$. ...(5)

Since $2n$ is non–integral (given), the roots given by (5) are unequal and not differing by an integer. To obtain the recurrence relation, we equate to zero the coefficient of x^{k+m} and obtain

$$C_m(k+m+n)(k+m-n)+C_{m-2}=0$$

giving $$C_m = -\frac{1}{(k+m+n)(k+m-n)}C_{m-2} \qquad ...(6)$$

[Since (6) gives relationship between C_m and C_{m-2}, we proceed to find C_1 as explained in step 6 in Art. 8.8].

Equating to zero the coefficient of x^{k+1} in (4) gives $C_1(k+1+n)(k+1-n)=0$ giving $C_1 = 0$ for both $k = n$ and $k = -n$. Then from (6) and $C_1 = 0$, we have

$$C_1 = C_3 = C_5 = ... = 0. \qquad ...(7)$$

Further, taking $n = 2$ in (6) gives $$C_2 = -\frac{1}{(k+2+n)(k+2-n)}C_0. \qquad ...(8)$$

Next, taking $n = 4$ in (6) and using (8) gives

$$C_4 = -\frac{1}{(k+4+n)(k+4-n)}C_2 = \frac{1}{(k+2+n)(k+2-n)(k+4+n)(k+4-n)}C_0 \qquad ...(9)$$

and so on. Putting these values in (2), *i.e.*, $y = x^k(C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + ...)$, gives

$$y = c_0x^k\left[1-\frac{x^2}{(k+2+n)(k+2-n)}+\frac{x^4}{(k+2+n)(k+2-n)(k+4+n)(k+4-n)}-....\right]. \qquad ...(10)$$

Putting $k = n$ and replacing C_0 by a in (10), gives

$$y = ax^n\left[1-\frac{x^2}{4(n+1)}+\frac{x^4}{4\cdot 8(n+1)(n+2)}-....\right] = au \text{, say.}$$

Next, putting $k = -n$ and replacing C_0 by b in (10) gives

$$y = bx^{-n}\left[1-\frac{x^2}{4(1-n)}+\frac{x^4}{4\cdot 8(1-n)(2-n)}-....\right] = bv \text{, say.}$$

Required general series solution is given by $y = au + bv$, where a and b are arbitrary constants.

Ex. 3. *Solve the following differential equations in series :*

(*a*) $2x^2y'' - xy' + (1-x^2)y = 0$ **[Garhwal 2010]**

(*b*) $2x^2y'' - xy' + (1-x^2)y = x^2$. **[Meerut 1996, 98]**

Sol. (*a*) Given $$2x^2y'' - xy' + (1-x^2)y = 0 \qquad ...(1)$$

Dividing by x^2, (1) takes standard form $y'' - (1/2x)y' + \{(1-x^2)/2x^2\}y = 0$.

Comparing it with $y'' + P(x)y' + Q(x)y = 0$, here $P(x) = -(1/2x)$ and $Q(x) = \{(1-x^2)/2x^2\}$ so that $xP(x) = -(1/2)$ and $x^2Q(x) = (1-x^2)/2$. Thus both $P(x)$ and $Q(x)$ are analytic at $x = 0$ and so $x = 0$ is a regular singular point. Let the series solution of (1) be of the form

$$y = \sum_{m=0}^{\infty} C_m x^{k+m}, \text{ where } C_0 \neq 0. \qquad ...(2)$$

$$\therefore \quad y' = \sum_{m=0}^{\infty} C_m (k+m) x^{k+m-1} \quad \text{and} \quad y'' = \sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2}. \qquad ...(3)$$

Substitution for y, y', y'' in (1), we have

$$2x^2 \sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2} - x \sum_{m=0}^{\infty} C_m (k+m) x^{k+m-1} + (1-x^2) \sum_{m=0}^{\infty} C_m x^{k+m} = 0$$

or $$2 \sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m} - \sum_{m=0}^{\infty} C_m (k+m) x^{k+m} + \sum_{m=0}^{\infty} C_m x^{k+m} - \sum_{m=0}^{\infty} C_m x^{k+m+2} = 0$$

or $$\sum_{m=0}^{\infty} C_m \{2(k+m)(k+m-1) - (k+m) + 1\} x^{k+m} - \sum_{m=0}^{\infty} C_m x^{k+m+2} = 0$$

or $$\sum_{m=0}^{\infty} C_m \{2(k+m)^2 - 3(k+m) + 1\} x^{k+m} - \sum_{m=0}^{\infty} C_m x^{k+m+2} = 0$$

or $$\sum_{m=0}^{\infty} C_m (2k+2m-1)(k+m-1) x^{k+m} - \sum_{m=0}^{\infty} C_m x^{k+m+2} = 0, \qquad ...(4)$$

$$[\because 2(k+m)^2 - 3(k+m) + 1 = 2(k+m)^2 - 2(k+m) - (k+m) + 1 = 2(k+m)[(k+m)-1] - [(k+m)-1]$$
$$= (k+m-1)[(2(k+m)-1] = (k+m-1)[(2k+2m-1)]$$

(4) is an identity. Equating to zero the coefficient of the smallest power of x namely x^k, (4) gives the indicial equation $C_0(2k-1)(k-1) = 0$ or $(2k-1)(k-1) = 0$ $[\because C_0 \neq 0]$
so that $k = 1$, and $k = 1/2$, ...(5)
which are unequal and not differing by an integer. To obtain the recurrence relation, we equate to zero the coefficient of x^{k+m} and obtain $C_m (2k+2m-1)(k+m-1) - C_{m-2} = 0$

giving $$C_m = \frac{1}{(2k+2m-1)(k+m-1)} C_{m-2}. \qquad ...(6)$$

To obtain C_1, we now equate to zero the coefficient of x^{k+1} and get $C_1(2k+1)k = 0$ so that $C_1 = 0$ for both roots $k = 1$, and $k = 1/2$ of the indicial equation. Then from (6) and $C_1 = 0$, we have

$$C_1 = C_3 = C_5 = ... = 0. \qquad ...(7)$$

Further, taking $n = 2$ in (6) gives $$C_2 = \frac{1}{(2k+3)(k+1)} C_0. \qquad ...(8)$$

Next, taking $n = 4$ in (6) and using (8) gives

$$C_4 = \frac{1}{(2k+5)(k+3)} C_2 = \frac{1}{(k+1)(k+3)(2k+3)(2k+5)} C_0$$

and so on. Putting these values in (2), *i.e.*, $y = x^k(C_0 + C_1x + C_1x^2 + C_3x^3 + ...)$, gives

$$x = C_0 x^k \left[1 + \frac{x^2}{(k+1)(2k+3)} + \frac{x^4}{(k+1)(k+3)(2k+3)(2k+5)} + ...\right] \qquad ...(10)$$

Putting $k = 1$ and replacing C_0 by a in (10) gives

$$y = ax\left[1 + \frac{x^2}{2 \cdot 5} + \frac{x^4}{2 \cdot 4 \cdot 5 \cdot 9} + ...\right] = au \text{, say}$$

Next, putting $k = 1/2$ and replacing C_0 by b in (10) gives

$$y = bx^{1/2}\left[1 + \frac{x^2}{2\cdot 3} + \frac{x^4}{2\cdot 3\cdot 4\cdot 7} + \ldots.\right] = bv \text{, say}$$

The required series solution is given by $y = au + bv$ where a and b are arbitrary constants.

Part (b) Given $$2x^2y'' - xy' + (1 - x^2)y = x^2. \qquad \ldots(11)$$

Since R.H.S. is involved in (11), the general solution of (11) is made up of complementary function (C.F.) and particular integral (P.I.) as usual. To find C.F. of (11), we solve

$$2x^2y'' - xy' + (1 - x^2)y = 0,$$

which is the same as equation (1) of part (a). So as above, C.F. is $au + bv$. Next assume that the particular solution of (11) is of the form [similar to (2)].

$$y = x^k \sum_{m=0}^{\infty} A_m x^m, \qquad \text{where} \qquad A_0 \neq 0, \qquad \ldots(12)$$

Find y' and y'' and then substitute values of y, y' and y'' in (11). Since L.H.S. of (11) and (1) is the same, proceeding as in part (a) we shall get [compare with (4) of part (a)]

$$\sum_{m=0}^{\infty} A_m(2k + 2m - 1)(k + m - 1)x^{k+m} - \sum_{m=0}^{\infty} A_m x^{k+m+2} = x^2, \qquad \ldots(13)$$

which is an identity. Hence the leading term (the term containing the smallest power of x) of L.H.S. of (13) must be x^2 and the coefficients of each of the remaining terms of L.H.S. of (13) must vanish. These conditions are satisfied by taking

$$k = 2, \qquad A_0(k - 1)(2k - 1) = 1 \qquad \text{and} \qquad A_1 = 0 \qquad \ldots(14)$$

and $$A_m = \frac{A_{m-2}}{(2k + 2m - 1)(k + m - 1)}. \qquad \ldots(15)$$

Note that (15) is similar to the recurrence relation (6) of part (a). Since $k = 2$, (14) gives $A_0 = 1/3$. Since $A_1 = 0$, so (15) gives

$$A_1 = A_3 = A_5 = \ldots. = 0. \qquad \ldots(16)$$

Putting $m = 2$ and $k = 2$ in (15), we obtain $$A_2 = \frac{A_0}{7\cdot 3} = \frac{1}{7\cdot 3\cdot 3\cdot 1}, \text{ as } A_0 = \frac{1}{3\cdot 1}. \qquad \ldots(17)$$

Putting $m = 4$ and $k = 2$ in (15) and using (17), we obtain

$$A_4 = \frac{A_2}{11\cdot 5} = \frac{1}{11\cdot 5\cdot 7\cdot 3\cdot 3\cdot 1}, \qquad \ldots(18)$$

Putting these values in (12), *i.e.*, $y = x^k(A_0 + A_1x + A_2x^2 + A_3x^3 + A_4x^4 + \ldots.)$, we obtain

$$\text{P.I.} = \frac{x^2}{3\cdot 1} + \frac{x^4}{7\cdot 3\cdot 3\cdot 1} + \frac{x^6}{11\cdot 5\cdot 7\cdot 3\cdot 3\cdot 1} + \ldots. = f(x) \text{, say.}$$

Hence the required solution is given by $y = au + bv + f(x)$, a and b being arbitrary constants.

Ex. 4. *Solve* $xy'' + (x + n)y' + (n + 1)y = 0$, *where n is not an integer.* **[Meerut 1993]**

Sol. Given $$xy'' + (x + n)y' + (n + 1)y = 0. \qquad \ldots(1)$$

As usual verify that $x = 0$ is a regular singular point of (1). To solve (1), take

$$y = x^k(C_0 + C_1x + C_2x^2 + \ldots.) = \sum_{m=0}^{\infty} C_m x^{k+m}, \quad \text{where} \quad C_0 \neq 0. \qquad \ldots(2)$$

$$\therefore \quad y' = \sum_{m=0}^{\infty} (k + m)C_m x^{k+m-1} \quad \text{and} \quad y'' = \sum_{m=0}^{\infty} (k + m)(k + m - 1)C_m x^{k+m-2}. \quad \ldots(3)$$

Putting the above values of y, y' and y'' in (1), we have

$$x\sum_{m=0}^{\infty}(k+m)(k+m-1)C_m x^{k+m-2}+(x+n)\sum_{m=0}^{\infty}(k+m)C_m x^{k+m-1}+(n+1)\sum_{m=0}^{\infty}C_m x^{k+m}=0$$

or $$\sum_{m=0}^{\infty}(k+m)(k+m-1)C_m x^{k+m-1}+\sum_{m=0}^{\infty}(k+m)C_m x^{k+m}+\sum_{m=0}^{\infty}n(k+m)C_m x^{k+m-1}+\sum_{m=0}^{\infty}(n+1)C_m x^{k+m}=0$$

or $$\sum_{m=0}^{\infty}(k+m)(k+m-1+n)C_m x^{k+m-1}+\sum_{m=0}^{\infty}(k+m+n+1)C_m x^{k+m}=0. \qquad ...(4)$$

Equating to zero the coefficient of the smallest power of x, namely x^{k-1}, the above identity (4) gives the indicial equation

$$C_0 k(k-1+n)=0 \quad \text{so that} \quad k=0 \quad \text{and} \quad 1-n, \quad \text{as} \quad C_0\neq 0.$$

Given that n is not an integer. So the roots 0 and $1-n$ of the indicial are unequal and do not differ by an integer. Next, we equate to zero the coefficient of x^{k+m-1} in the above identity (4) and obtain the recurrence relation

$$(k+m)(k+m+n-1)C_m+(k+m+n)C_{m-1}=0$$

so that $$C_m=-\frac{k+m+n}{(k+m)(k+m+n-1)}c_{m-1}. \qquad ...(5)$$

Putting $m=1, 2, 3, ...$ in (5), we have $$C_1=-\frac{k+n+1}{(k+1)(k+n)}C_0, \qquad ...(6)$$

$$C_2=-\frac{k+n+2}{(k+2)(k+n+1)}C_1=\frac{k+n+2}{(k+2)(k+n+1)}\times\frac{k+n+1}{(k+1)(k+n)}C_0, \text{ by (6)}$$

or $$C_2=\frac{k+n+2}{(k+1)(k+2)(k+n)}C_0, \qquad ...(7)$$

$$C_3=-\frac{k+n+3}{(k+3)(k+n+2)}C_2=-\frac{(k+n+3)}{(k+3)(k+n+2)}\times\frac{(k+n+2)}{(k+1)(k+2)(k+n)}C_0, \text{ using (7)}$$

or $$C_3=-\frac{k+n+3}{(k+1)(k+2)(k+3)(k+n)}C_0 \qquad ...(8)$$

and so on. Putting these values of C_1, C_2, C_3, ... in (2), we have

$$y=C_0x^k\left[1-\frac{k+n+1}{(k+1)(k+n)}x+\frac{k+n+2}{(k+1)(k+2)(k+n)}x^2-\frac{k+n+3}{(k+1)(k+2)(k+3)(k+n)}x^3+....\right] \qquad ...(9)$$

Putting $k=0$ and replacing C_0 by a in (9), we get

$$y=a\left[1-\frac{n+1}{n}x+\frac{n+2}{n}\frac{x^2}{2!}-\frac{n+3}{n}\frac{x^3}{3!}+....\right]=au, \text{ say}$$

Putting $k=1-n$ and replacing C_0 by b in (9), we get

$$y=bx^{1-n}\left[1-\frac{2}{2-n}x+\frac{3}{(2-n)(3-n)}x^2-\frac{4}{(2-n)(3-n)(4-n)}x^3-....\right]=bv, \text{ say}$$

The required solution is $y=au+bv$, where a and b are arbitrary constants.

Ex. 5. *Verify that the origin is a regular singular point of* $2x^2y'' + xy' - (x + 1)y = 0$ *and find two independent Frobenius series solutions of it.* **[Lucknow 1995]**

Sol. Given $$2x^2y'' + xy' - (x + 1)y = 0. \qquad ...(1)$$

Dividing by $2x^2$, (1) takes the standard form $y'' + (1/2x)y' - \{(x + 1)/2x^2\}y = 0.$

Comparing it with $y'' + P(x)y' + Q(x)y = 0$, here $P(x) = 1/2x$ and $Q(x) = -(x + 1)/2x^2$ so that $xP(x) = 1/2$ and $Q(x) = -(x + 1)/2$. Since $xP(x)$ and $x^2Q(x)$ are both analytic, so $x = 0$ is a regular singular point of x. The solve (1), we take

$$y = x^k(C_0 + C_1x + C_2x^2 + ...) = \sum_{m=0}^{\infty} C_m x^{k+m}, \text{ where } C_0 \neq 0. \qquad ...(2)$$

$$\therefore \quad y' = \sum_{m=0}^{\infty}(k+m)C_m x^{k+m-1} \quad \text{and} \quad y'' = \sum_{m=0}^{\infty}(k+m)(k+m-1)C_m x^{k+m-2}. \qquad ...(3)$$

Putting the above values of y, y' and y'' in (1), we get

$$2x^2 \sum_{m=0}^{\infty}(k+m)(k+m-1)C_m x^{k+m-2} + x\sum_{m=0}^{\infty}(k+m)C_m x^{k+m-1} - (x+1)\sum_{m=0}^{\infty}C_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty}2(k+m)(k+m-1)C_m x^{k+m} + \sum_{m=0}^{\infty}(k+m)C_m x^{k+m} - \sum_{m=0}^{\infty}C_m x^{k+m+1} - \sum_{m=0}^{\infty}C_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty}\{2(k+m)(k+m-1)+(k+m)-1\}C_m x^{k+m} - \sum_{m=0}^{\infty}C_m x^{k+m+1} = 0$$

or $$\sum_{m=0}^{\infty}\{2(k+m)^2-(k+m)-1\}C_m x^{k+m} - \sum_{m=0}^{\infty}C_m x^{k+m+1} = 0$$

or $$\sum_{m=0}^{\infty}\{2(k+m)+1\}(k+m-1)C_m x^{k+m} - \sum_{m=0}^{\infty}C_m x^{k+m+1} = 0. \qquad ...(4)$$

Equating to zero the coefficient of the smallest power of x, namely, x^k, the above identity (4) in x gives the indicial equation, namely,

$C_0(2k + 1)(k - 1) = 0$ so that $k = 1$ and $-1/2$, as $C_0 \neq 0$.

Here the difference of these roots $= 1 - (- 1/2) = 3/2 \neq$ not an integer.

Next, we equate to zero the coefficient of x^{k+m} in (4) and obtain the recurrence relation

$$\{2(k + m) + 1\}(k + m - 1)C_m - C_{m-1} = 0$$

so that $$C_m = \frac{1}{(2k+2m+1)(k+m-1)}C_{m-1}. \qquad ...(5)$$

Putting $m = 1, 2, 3, ...$ in (5), we have $C_1 = \{1/k\,(2k + 3)\}C_0$, ...(6)

$$C_2 = \frac{1}{(2k+5)(k+1)}C_1 = \frac{1}{(2k+3)(2k+5)k(k+1)}C_0, \text{ by (6).} \qquad ...(7)$$

and so on. Putting these values in (2), we get

$$y = C_0x^k\left[1 + \frac{x}{(2k+3)k} + \frac{x^2}{(2k+3)(2k+5)k(k+1)} + ...\right]. \qquad ...(8)$$

Putting $k = 1$ and replacing C_0 by a in (8), we get

$$y = ax[1 + x/5 + x^2/70 + ...] = au, \text{ say}$$

Next, putting $x = -1/2$ and replacing C_0 by b in (8), we get

$$y = bx^{-1/2}[1 - x - (x^2/2) + ...] = bv, \text{ say.}$$

The required solution is $y = au + bv$, a and b being arbitrary constants.

Ex. 6. *Show that* $x = 0$ *is a regular singular point of* $(2x + x^3)y'' - y' - 6xy = 0$ *and find its solution about* $x = 0$. **[Delhi Maths (H) 1995, 96; Meerut 1996]**

Sol. Given $$(2x + x^3)y'' - y' - 6xy = 0. \qquad ...(1)$$

Dividing $(2x + x^3)$, (1) can be put in standard form $y'' - [1/(2x + x^3)\}y' - \{6/(2 + x^2)\}y = 0$.

Comparing it with $y'' + P(x)y' + Q(x)y = 0$, here $P(x) = -1/(2x + x^3)$ and $Q(x) = -6/(2 + x^2)$ so that $xP(x) = -1/(2 + x^2)$ and $x^2Q(x) = -(6x^2)/(2 + x^2)$. Since $P(x)$ and $Q(x)$ are both analytic at $x = 0$, so $x = 0$ is a regular singular point of (1). Let series solution of (1) be

$$y = x^k(C_0 + C_1x + C_2x^2 + ...) = \sum_{m=0}^{\infty} C_m x^{k+m}, \text{ where } \quad C_0 \neq 0. \qquad ...(2)$$

$$\therefore \quad y' = \sum_{m=0}^{\infty} (k+m)C_m x^{k+m-1} \quad \text{and} \quad y'' = \sum_{m=0}^{\infty} (k+m)(k+m-1)C_m x^{k+m-2}. \qquad ...(3)$$

Putting the above values of y, y' and y'' in (1), we get

$$(2x + x^3)\sum_{m=0}^{\infty}(k+m)(k+m-1)C_m x^{k+m-2} - \sum_{m=0}^{\infty}(k+m)C_m x^{k+m-1} - 6x\sum_{m=0}^{\infty}C_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty}2(k+m)(k+m-1)C_m x^{k+m-1} + \sum_{m=0}^{\infty}(k+m)(k+m-1)C_m x^{k+m+1} - \sum_{m=0}^{\infty}(k+m)C_m x^{k+m-1} - \sum_{m=0}^{\infty}6C_m x^{k+m+1} = 0$$

or $$\sum_{m=0}^{\infty}\{2(k+m)(k+m-1) - (k+m)\}C_m x^{k+m-1} + \sum_{m=0}^{\infty}\{(k+m)(k+m-1) - 6\}C_m x^{k+m+1} = 0$$

or $$\sum_{m=0}^{\infty}\{2(k+m)^2 - 3(k+m)\}C_m x^{k+m-1} + \sum_{m=0}^{\infty}\{(k+m)^2 - (k+m) - 6\}C_m x^{k+m+1} = 0$$

or $$\sum_{m=0}^{\infty}(k+m)(2k+2m-3)C_m x^{k+m-1} + \sum_{m=0}^{\infty}(k+m-3)(k+m+2)C_m x^{k+m+1} = 0. \qquad ...(4)$$

Equating to zero the coefficient of the smallest power of x, namely x^{k-1}, the above identity (4) gives the indicial equation, namely

$C_0k(2k - 3) = 0$ so that $k = 0$ and $3/2$, as $C_0 \neq 0$.

Here the difference of these roots $= (3/2) - 0 = 3/2 \neq$ not an integer.

Here the difference of the powers of x in (4) $= (k + m + 1) - (k + m - 1) = 2$. Hence we equate to zero the coefficient of x^k in the identity (4) and obtain

$C_1(k + 1)(2k - 1) = 0$ so that $C_1 = 0$ for both $k = 0$ and $k = 3/2$.

Next, equating to zero the coefficient of x^{k+m-1} in (4), we get

$$(k + m)(2k + 2m - 3)C_m + (k + m - 5)(k + m)C_{m-2} = 0$$

or $$C_m = -\frac{k+m-5}{2k+2m-3}C_{m-2}. \qquad ...(5)$$

Putting $m = 3, 5, 7, ...$ in (5) and noting that $C_1 = 0$, we get

$$C_1 = C_3 = C_5 = C_7 = ... = 0. \qquad ...(6)$$

Next, putting $m = 2, 4, 6, ...$ in (5), we have

$$C_2 = -\frac{k-3}{2k+1}C_0, \qquad C_4 = -\frac{k-1}{2k+5}C_2 = \frac{(k-1)(k-3)}{(2k+1)(2k+5)}C_0, ... \qquad ...(7)$$

Putting these values in (2), we have

$$y = C_0 x^k \left[1 - \frac{k-3}{2k+1}x^2 + \frac{(k-1)(k-3)}{(2k+1)(2k+5)}x^4 - ....\right]. \qquad ...(8)$$

Putting $k = 0$ and replacing C_0 by a in (8), $y = a[1 + 3x^2 + (3/5) \times x^4 - ....] = au$, say

Putting $k = 3/2$ and replacing C_0 by b in (8), $y = x^{3/2}[1 + (3/8) \times x^2 - (3/128) \times x^4 + ...] = bv$, say.

Hence the required solution is $y = au + bv$, a, b being arbitrary constants.

Ex. 7. *Solve the hypergeometric equation* $x(1-x)y'' + \{\gamma - (\alpha + \beta + 1)x\}y' - \alpha\beta y = 0$ *near* $x = 0$, if γ is not an integer. **[Meerut 1999, Kanpur 2006]**

Sol. Given $\quad x(1-x)y'' + \{\gamma - (\alpha + \beta + 1)x\}y' - \alpha\beta y = 0. \qquad ...(1)$

Dividing by $x(1-x)$, (1) gives $\qquad \dfrac{d^2y}{dx^2} + \dfrac{\gamma - (\alpha+\beta+1)x}{x(1-x)}\dfrac{dy}{dx} - \dfrac{\alpha\beta}{x(1-x)}y = 0.$

Comparing it with $y'' + P(x)y' + Q(x)y = 0$, we have

$$P(x) = \frac{\gamma - (\alpha + \beta + 1)x}{x(1-x)} \qquad \text{and} \qquad Q(x) = -\frac{\alpha\beta}{x(1-x)}.$$

Since $xP(x)$ and $x^2Q(x)$ are both analytic at $x = 0$, so $x = 0$ is a regular singular point of (1).

Let the series solution of (1) be $\qquad y = \sum_{m=0}^{\infty} c_m x^{k+m}, \quad \text{where} \quad c_0 \neq 0 \qquad ... (2)$

$$\therefore \quad y' = \sum_{m=0}^{\infty} c_m (k+m)x^{k+m-1} \quad \text{and} \quad y'' = \sum_{m=0}^{\infty} c_m (k+m)(k+m-1)x^{k+m-2} \qquad ...(3)$$

Putting the above values of y, y', y'' in (1) gives

$$(x - x^2)\sum_{m=0}^{\infty} c_m (k+m)(k+m-1)x^{k+m-2} + [\gamma - (\alpha+\beta+1)x]\sum_{m=0}^{\infty} c_m (k+m)x^{k+m-1} - \alpha\beta \sum_{m=0}^{\infty} c_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} c_m\{(k+m)(k+m-1) + \gamma(k+m)\}x^{k+m-1} - \sum_{m=0}^{\infty} c_m\{(k+m)(k+m-1) + (\alpha+\beta+1)(k+m) + \alpha\beta\}x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} c_m (k+m)(k+m-1+\gamma)x^{k+m-1} - \sum_{m=0}^{\infty} c_m\{(k+m)^2 + (\alpha+\beta)(k+m) + \alpha\beta\}x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} c_m (k+m)(k+m-1+\gamma)x^{k+m-1} - \sum_{m=0}^{\infty} c_m (k+m+\alpha)(k+m+\beta)x^{k+m} = 0. \qquad ...(4)$$

which is an identity. Equating to zero the coefficient of the smallest power of x, namely x^{k-1}, (4) gives the indicial equation

$$c_0 k(k - 1 + \gamma) = 0 \qquad \text{or} \qquad k(k - 1 + \gamma) = 0 \qquad [\because c_0 \neq 0]$$

so the roots of the indicial equation are $k = 0$ $1 - \gamma$, which are unequal and not differing by an integer because by assumption γ is not an integer.

To obtain the recurrence relation, we equate to zero the coefficient of x^{k+m-1}. Then we have

$$c_m(k+m)(k+m-1+\gamma) - c_{m-1}(k+m-1+\alpha)(k+m-1+\beta) = 0$$

or $$c_m = \frac{(k+m-1+\alpha)(k+m-1+\beta)}{(k+m)(k+m-1+\gamma)}c_{m-1}. \qquad ...(5)$$

Case 1. When k = 0. Putting $m = 1, 2, 3, ...$ succesively in (5), we get

$$c_1 = \frac{\alpha\beta}{1\cdot\gamma}c_0, \qquad c_2 = \frac{(\alpha+1)(\beta+1)}{2\cdot(\gamma+1)}c_1 = \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\gamma(\gamma+1)}c_0,$$

$$c_3 = \frac{(\alpha+2)(\beta+2)}{3(\gamma+2)}c_2 = \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\cdot 2\cdot 3\cdot \gamma(\gamma+1)(\gamma+2)}c_0 \text{ and so on.}$$

Putting these values and $k = 0$ and replacing c_0 by a in (2) gives

$$y = a\left[1+\frac{\alpha\beta}{1\cdot\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\gamma(\gamma+1)}x^2+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1\cdot 2\cdot 3\cdot\gamma(\gamma+1)(\gamma+2)}x^3+\ldots.\right] \qquad \ldots(6)$$

If we take $a = 1$ in (6), the series on the right hand side of (6) is called *hypergeometric series* and is represented by ${}_2F_1(\alpha, \beta ; \gamma ; x)$. Thus we see that ${}_2F_1(\alpha, \beta ; \gamma ; x)$ is a solution of (1).

Case 2. When k = 1 – γ. Then (5) reduces to

$$c_m = \frac{(1-\gamma+m-1+\alpha)(1-\gamma+m-1+\beta)}{(1-\gamma+m)(1-\gamma+m-1+\gamma)}c_{m-1}$$

or

$$c_m = \frac{(\alpha'+m-1)(\beta'+m-1)}{m(\gamma'+m-1)}c_{m-1}. \qquad \ldots(7)$$

where $\alpha' = 1-\gamma+\alpha,$ $\beta' = 1-\gamma+\beta$ and $\gamma' = 2-\gamma.$ …(8)

Replacing m by 1, 2, 3, ... successively in (7) gives as before

$$c_1 = \frac{\alpha'\beta'}{1\cdot\gamma'}c_0, \qquad c_2 = \frac{(\alpha'+1)(\beta'+1)}{2\cdot(\gamma'+1)}c_1 = \frac{\alpha'(\alpha'+1)\beta'(\beta'+1)}{1\cdot 2\cdot\gamma'(\gamma'+1)}c_0 \text{ etc.}$$

Hence putting $k = 1 - \gamma$, using the above values of c_1, c_2, ... in (2) and replacing c_0 by b gives

$$y = bx^{1-\gamma}\left[1+\frac{\alpha'\cdot\beta'}{1\cdot\gamma'}x+\frac{\alpha'(\alpha'+1)\beta'(\beta'+1)}{1\cdot 2\cdot\gamma'(1+\gamma')}x^2+\ldots.\right] \qquad \ldots(9)$$

If we take $b = 1$ in (9), the series on the R.H.S. of (9) would $x^{\gamma-1}{}_2F_1(\alpha', \beta'; \gamma' ; x)$ *i.e.* $x^{\gamma-1}{}_2F_1(1-\gamma+\alpha, 1-\gamma+\beta ; 2-\gamma ; x)$ which is another independent solution of (1). Hence the general series solution of (1) is

$$y = a\ {}_2F_1(\alpha, \beta ; \gamma ; x) + b\ x^{\gamma-1}{}_2F_1(1-\gamma+\alpha, 1-\gamma+\beta : 2-\gamma : x),$$

where a and b are arbitrary constants. ${}_2F_1(\alpha, \beta; \gamma; x)$ is called *hypergeometric function*.

Ex. 8. *Find the series solution of* $4xy''+2y'+y=0$.

[Bilaspur 2004, Purvanchal 2005, Ravishankar 1998, 2004, Vikram 2004]

Sol. Given $\quad 4xy''+2y'+y=0 \quad$ ...(1)

Re-writing (1), $\quad y''+(1/2x)y'+(1/4x)y=0 \quad$ ... (2)

Comparing (2) with $y''+P(x)y'+Q(x)y=0,$ we have

$P=1/2x$ and $Q(x)=1/4x$ ... (3)

Since $P(x)$ and $Q(x)$ are not with analytic at $x = 0$, so $x = 0$ is not ordinary point of (1).

Also, $xP(x) = 1/2$ and $x^2Q(x) = x/4$,

showing that both $P(x)$ and $Q(x)$ are analytic at $x = 0$. Hence $x = 0$ is a regular singular point of (1). We, therefore, use Frobenium method to solve (1). Let a solution of (1) be of the form

$$y = \sum_{m=0}^{\infty} C_m x^{k+m}, \qquad \text{where} \qquad C_0 \neq 0 \qquad \ldots(4)$$

$$\therefore \quad y' = \sum_{m=0}^{\infty} C_m (k+m)x^{k+m-1} \qquad \text{and} \qquad y'' = \sum_{m=0}^{\infty} C_m (k+m)(k+m-1)x^{k+m-2} \quad \ldots(5)$$

Substituting the series (4) and (5) in (1), we have

$$4x\sum_{m=0}^{\infty} C_m (k+m)(k+m-1)x^{k+m-2} + 2\sum_{m=0}^{\infty} C_m (k+m)x^{k+m-1} + \sum_{m=0}^{\infty} C_m x^{k+m} = 0$$

or
$$\sum_{m=0}^{\infty} C_m \{4(k+m)(k+m-1)+2(k+m)\} x^{k+m-1} + \sum_{m=0}^{\infty} C_m x^{k+m} = 0$$

or
$$\sum_{m=0}^{\infty} C_m \{2(k+m)(2k+2m-1)\} x^{k+m-1} + \sum_{m=0}^{\infty} C_m x^{k+m} = 0, \qquad \text{... (6)}$$

which is an identity in x. Equating to zero the coefficients of the smallest power of x, namely, x^{k-1}, (6) gives the indicial equation

$$2C_0 k(2k-1) = 0 \qquad \text{giving} \qquad k = 0,\ 1/2 \qquad [\because\ C_0 \neq 0]$$

Here roots of indicial equation are unequal and do not differ by integer. To obtain the recurrence relation, we equate to zero the coefficient of x^{k+m-1} and obtain

$$2C_m(k+m)(2k+2m-1) + C_{m-1} = 0$$

or
$$C_m = -\frac{C_{m-1}}{2(k+m)(2k+2m-1)},\ m = 1, 2, 3, \ldots \qquad \text{... (7)}$$

Taking $m = 1$ in (7),
$$C_1 = -\frac{C_0}{2(k+1)(2k+1)} \qquad \text{... (8)}$$

Next, Taking $m = 2$ in (7), we have

$$C_2 = -\frac{C_1}{2(k+2)(2k+3)} = \frac{C_0}{4(k+1)(k+2)(2k+1)(2k+3)},\ \text{by (8)} \qquad \text{... (9)}$$

Taking $m = 3$, in (7) and using (9), $C_3 = -\dfrac{C_0}{8(k+1)(k+2)(k+3)(2k+1)(2k+3)(2k+5)}$

Substituting the above values of C_1, C_2, C_3, ... in (4), we get

$$y = C_0 x^m \left\{1 - \frac{x}{2(k+1)(2k+1)} + \frac{x^2}{4(k+1)(k+2)(2k+1)(2k+3)}\right.$$
$$\left. - \frac{x^3}{8(k+1)(k+2)(k+3)(2k+1)(2k+3)(2k+5)} + \ldots\right\} \qquad \text{... (10)}$$

Taking $k = 0$ and replacing C_0 by a in (10), we have

$$y = a\left(1 - \frac{x}{1\cdot 2} + \frac{x^2}{1\cdot 2\cdot 3\cdot 4} - \frac{x^3}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} + \ldots\right) = a\left(1 - \frac{(\sqrt{x})^2}{2!} + \frac{(\sqrt{x})^4}{4!} - \frac{(\sqrt{x})^6}{6!} + \ldots\right) = a\cos x^{1/2}$$

Taking $k = 1/2$ and replacing C_0 by b in (10), we have

$$y = bx^{1/2}\left(1 - \frac{x}{2\cdot 3} + \frac{x^2}{2\cdot 3\cdot 4\cdot 5} - \frac{x^3}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7} + \ldots\right) = b\left(\sqrt{x} - \frac{(\sqrt{x})^3}{3!} + \frac{(\sqrt{x})^5}{5!} - \frac{(\sqrt{x})^7}{7!} + \ldots\right) = b\sin x^{1/2}$$

Hence the required solution is $y = a\cos x^{1/2} + b\sin x^{1/2}$, a and b being arbitrary constants.

Ex. 9. *Using Frobenius method solve the differential equation* $d^2y/dx^2 + (1/4x) \times (dy/dx) + (1/8x^2)y = 0$ **[Delhi Maths (Hons.) 1993 , 2006]**

Sol. Re-writing the given differential equation, $$8x^2 y'' + 2xy' + y = 0 \qquad \text{... (1)}$$

Let a series solution of (1) be of the form $$y = \sum_{m=0}^{\infty} C_m x^{k+m}, \; C_0 \neq 0 \qquad \text{... (2)}$$

$$\therefore \quad y' = \sum_{m=0}^{\infty} C_m (k+m) x^{k+m-1} \quad \text{and} \quad y'' = \sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2}$$

Substituting the above values of y, y' and y'' in (1), we get

$$8x^2 \sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2} + 2x \sum_{m=0}^{\infty} C_m (k+m) x^{k+m-1} + \sum_{m=0}^{\infty} C_m x^{k+m} = 0$$

or $\sum_{m=0}^{\infty} C_m \{8(k+m)(k+m-1) + 2(k+m) + 1\} x^{k+m} = 0$ or $\sum_{m=0}^{\infty} C_m \{8(k+m)^2 - 6(k+m) + 1\} x^{k+m} = 0$

or $$\sum_{m=0}^{\infty} C_m \{4(k+m) - 1\}\{2(k+m) - 1\} x^{k+m} = 0, \qquad \text{... (3)}$$

which is an identity in x. Equating to zero the coefficient of the smallest power in x, namely, x^k, (6) gives the indicial equation

$C_0(4k - 1)(2k - 1) = 0$ giving $k = 1/2, 1/4,$ as $C_0 \neq 0$

which are unequal and do not differ by an integer. To obtain the recurrence relation, we equate to zero the coefficient of x^{k+m} in (3) and obtain

$$C_m \{4(k+m) - 1\}\{2(k+m) - 1\} = 0, \quad \text{for all} \quad m \geq 1 \qquad \text{... (4)}$$

Relation (4) is satisfied by both values of $k = 1/2$ and $k = 1/4$ by choosing $C_m = 0$ for $m \geq 1$. Hence for $k = 1/2$ and $k = 1/4$, (2) reduces to

$$y = x^k (C_0 + C_1 x + C_2 x^2 + \ldots) \quad \text{or} \quad y = C_0 x^k \qquad \text{... (5)}$$

Putting $k = 1/2$ and replacing C_0 by a in (5), we have $y = ax^{1/2}$ ... (6)

Next, putting $k = 1/4$ and replacing C_0 by b in (5), we have $y = bx^{1/4}$ ... (7)

The required solution is $y = ax^{1/2} + bx^{1/4}$, where a and b are arbitrary constants.

EXERCISE 8 (C)

Find the series solution following equations near $x = 0$.

1. $2x(1 - x)y'' + (1 - x)y' + 3y = 0.$ **Ans.** $y = a\{1 - 3x + 3x^2/(1 \cdot 3) - \ldots\} + bx^{1/2}(1 - x)$

2. $2xy'' + (x + 1)y' + 3y = 0.$ **[Delhi Maths (H) 2005]**

Ans. $y = a(1 - 3x + 2x^2 - 2x^3/3 \ldots) + bx^{1/2}(1 - 7x/6 + 21x^2/40 - 11x^3/80 \ldots)$

3. $2x^2y'' - xy' + (x^2 + 1)y = 0.$

4. $3xy'' + 2y' + x^2y = 0.$

8.10. Examples of Type 2 on Frobenius method. Roots of indicial equation unequal, differing by an Integer and making a coefficient of y indeterminate :

In this connection the following rule should be noted.

Rule : *If the indicial equation has two roots k_1 and k_2 (say $k_1 < k_2$) and if the one of the coefficients of y becomes indeterminate when $k = k_2$, the complete solution is given by putting $k = k_2$ in y, which then contains two arbitrary constants. The result of putting $k = k_1$ in y merely gives a numerial multiple of one of the series contained in the first solution. Hence we reject the solution obtained by putting $k = k_1$.*

Ex. 1. *Find solution near* $x = 0$ *of* $x^2y'' + (x + x^2)y' + (x - 9)y = 0$.

Sol. Given $$x^2y'' + (x + x^2)y' + (x - 9)y = 0. \qquad ...(1)$$

Dividing by x^2, (1) can be put in standard form as $y'' + \{(1 + x)/x\}y' + \{(x - 9)/x^2\}y = 0$.

Comparing it with $y'' + P(x)y' + Q(x)y = 0$, here $P(x) = (1 + x)/x$ and $Q(x) = (x - 9)/x^2$ so that $xP(x) = 1 + x$ and $x^2Q(x) = x - 9$. Since both $xP(x)$ and $x^2Q(x)$ are analytic at $x = 0$, hence $x = 0$ is a regular singular point of (1). Let the series solution of (1) be

$$y = x^k(C_0 + C_1x + C_2x^2 + ... \text{ ad. inf.}) = \sum_{m=0}^{\infty} C_m x^{k+m}, \text{ where } C_0 \neq 0. \qquad ...(2)$$

$$\therefore\ y' = \sum_{m=0}^{\infty} (k+m)C_m x^{k+m-1} \quad \text{and} \quad y'' = \sum_{m=0}^{\infty}(k+m)(k+m-1)C_m x^{k+m-2}.$$

...(3)

Putting the above values of y, y' and y'' in (1), we have

$$x^2 \sum_{m=0}^{\infty}(k+m)(k+m-1)C_m x^{k+m-2} + (x+x^2)\sum_{m=0}^{\infty}(k+m)C_m x^{k+m-1} + (x-9)\sum_{m=0}^{\infty}C_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty}(k+m)(k+m-1)C_m x^{k+m} + \sum_{m=0}^{\infty}(k+m)C_m x^{k+m} + \sum_{m=0}^{\infty}(k+m)C_m x^{k+m+1} + \sum_{m=0}^{\infty}C_m x^{k+m+1} - 9\sum_{m=0}^{\infty}C_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty}\{(k+m)(k+m-1) + (k+m) - 9\}C_m x^{k+m} + \sum_{m=0}^{\infty}\{(k+m)+1\}C_m x^{k+m+1} = 0$$

or $$\sum_{m=0}^{\infty}\{(k+m)^2 - 3^2\}C_m x^{k+m} + \sum_{m=0}^{\infty}(k+m+1)C_m x^{k+m+1} = 0$$

or $$\sum_{m=0}^{\infty}(k+m+3)(k+m-3)C_m x^{k+m} + \sum_{m=0}^{\infty}(k+m+1)C_m x^{k+m+1} = 0. \qquad ...(4)$$

Equating to zero the coefficient of the smallest power of x, namely x^k, the above identity (4) gives the indicial equation

$$(k+3)(k-3)C_0 = 0 \quad \text{so that} \quad k = 3,\ -3 \quad \text{as} \quad C_0 \neq 0.$$

Next, equating to zero the coefficient of x^{k+m} in (4), we get

$$(k + m + 3)(k + m - 3)C_m + (k + m)C_{m-1} = 0$$

or $$C_m = -\frac{(k+m)}{(k+m+3)(k+m-3)}C_{m-1}. \qquad ...(5)$$

Putting $m = 1, 2, 3, ...$ in (5), we get $\quad C_1 = -\dfrac{k+1}{(k+4)(k-2)}C_0,$

$$C_2 = -\frac{(k+2)}{(k+5)(k-1)}C_1 = \frac{(k+1)(k+2)}{(k-1)(k-2)(k+4)(k+5)}C_0,$$

$$C_3 = -\frac{(k+3)}{(k+6)k}C_2 = -\frac{(k+1)(k+2)(k+3)}{k(k-1)(k-2)(k+4)(k+5)(k+6)}C_0$$

and so on. Putting these values in (2), we have

$$y = C_0x^k\left[1 - \frac{(k+1)}{(k-2)(k+4)}x + \frac{(k+1)(k+2)}{(k-1)(k-2)(k+4)(k+5)}x^2 - \frac{(k+1)(k+2)(k+3)}{k(k-1)(k-2)(k+4)(k+5)(k+6)}x^3 + ... \text{ad. inf.}\right] \quad ...(6)$$

Putting $k = 3$ and replacing C_0 by a in (6), we have

$$y = ax^3\left[1 - \frac{4}{2\cdot 7}x + \frac{4\cdot 5}{2\cdot 1\cdot 7\cdot 8}x^2 + \frac{4\cdot 5\cdot 6}{3\cdot 2\cdot 1\cdot 7\cdot 8\cdot 9}x^3 - ....\right] = au, \text{ say}$$

Putting $k = -3$ and replacing C_0 by b in (6), we have

$$y = bx^{-3}\{1 - (2/5)x + (1/20)x^2\} = bv, \text{ say.}$$

The required solution is $y = au + bv$, where a and b are arbitrary constants.

Important note. When $x = 0$ is an ordinary point of $y'' + P(x)y' + Q(x)y = 0$, we have already explained the method of solution in Art. 8.5. We have solved many problems based on Art. 8.5. All those problems can also be solved by Frobenius method as given in Art. 8.10 and explained in above solved Ex. 1. We now solve the same problems by method of solved Ex. 1 as follows.

Ex. 2. *Solve in series* $(1 - x^2)y'' - xy' + 4y = 0$.

Sol. Given $$(1 - x^2)y'' - xy' + 4y = 0. \qquad ...(1)$$

Clearly $x = 0$ is an ordinary point of (1). Let the series solution of (1) be of the form

$$y = \sum_{m=0}^{\infty} C_m x^{k+m}, \qquad \text{where} \qquad C_0 \neq 0. \qquad ...(2)$$

$$\therefore \qquad y' = \sum_{m=0}^{\infty} C_m (k+m) x^{k+m-1} \qquad \text{and} \qquad y'' = \sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2} \qquad ...(3)$$

Putting the above values of y, y', y'' in (1) gives

or $$(1 - x^2) \sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2} - x \sum_{m=0}^{\infty} C_m (k+m) x^{k+m-1} + 4 \sum_{m=0}^{\infty} C_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2} - \sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m} - \sum_{m=0}^{\infty} C_m (k+m) x^{k+m} + 4 \sum_{m=0}^{\infty} C_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2} - \sum_{m=0}^{\infty} C_m [(k+m)(k+m-1) + (k+m) - 4] x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2} - \sum_{m=0}^{\infty} C_m [(k+m)^2 - 4] x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2} - \sum_{m=0}^{\infty} C_m (k+m+2)(k+m-2) x^{k+m} = 0, \qquad ...(4)$$

which is an identity. Equating to zero the coefficient of the smallest power of x, namely x^{k-2}, (4) gives the indicial equation

$$C_0 k(k-1) = 0 \qquad \text{or} \qquad k(k-1) = 0 \qquad [\because C_0 \neq 0]$$

giving $k = 1$ and $k = 0$. These are unequal and differ by an integer. To get the recurrence relation, we equate to zero the coefficient of x^{k+m-2}. Thus,

$$C_m (k+m)(k+m-1) - C_{m-2}(k+m)(k+m-4) = 0$$

giving $$C_m = \frac{k+m-4}{k+m-1} C_{m-2}. \qquad ...(5)$$

Next, we equate to zero the coefficient of x^{k-1} and get $\qquad C_1(k+1)k = 0. \qquad ...(6)$

If we take $k = 0$, (6) shows that C_1 is indeterminate. With $k = 0$ and using (5), we can express C_2, C_4, C_6 .... in terms of C_0 and C_3, C_5, C_7 .... in terms of C_1 if we assume C_1 to be finite. Thus,

$$C_m = \{(m-4)/(m-1)\} \; C_{m-2}$$

so that $\quad C_2 = -2C_0, \quad C_4 = 0 \quad$ and hence $\quad C_6 = C_8 = ... = 0$

and $\quad C_3 = (-C_1)/(2) = -C_1/2, \quad C_5 = C_3/(4) = -C_1/8, \quad C_7 = (3C_5)/(6) = -C_1/16,$

and so on. Putting $k = 0$ and these values in (2), *i.e.*

$$y = x^k(C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + C_5 x^5 + ...)$$

i.e. $$y = x^0(C_0 + C_2 x^2 + C_4 x^4 + ...) + x^0(C_1 x + C_3 x^3 + C_5 x^5 + ...), \text{ we get}$$

$$y = C_0(1 - 2x^2) + C_1(x - x^3/2 - x^5/8 + x^7/16 - ....) \qquad ...(7)$$

which is the required series solution, C_0 and C_1 being two arbitrary constants.

Remarks. The reader can easily verify that the root $k = 1$ of the indicial equation gives another solution as usual (Refer examples of Type 1). But this solution, will be a constant multiple of one of the series occurring in (7). So we reject it.

Ex. 3. *Solve in series the Legendre's equation* $(1 - x^2)y'' - 2xy' + n(n+1)y = 0$

[Delhi Maths (Hons.) 1996, Nagpur 1996, Utkal 2003]

Sol. Given $$(1 - x^2)y'' - 2xy' + n(n+1)y = 0 \qquad \text{...(1)}$$

Here $x = 0$ is an ordinary point of (1). Let the series solution of (1) be of the form

$$y = \sum_{m=0}^{\infty} C_m x^{k+m}, \text{ where } C_0 \neq 0. \qquad \text{...(2)}$$

$$\therefore \; y' = \sum_{m=0}^{\infty} C_m (k+m) x^{k+m-1} \quad \text{and} \quad y'' = \sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2}. \qquad \text{...(3)}$$

Putting the above values of y, y', y'' into (1) gives

$$\sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2} - x^2 \sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2}$$

$$- 2x \sum_{m=0}^{\infty} C_m (k+m) x^{k+m-1} + n(n+1) \sum_{m=0}^{\infty} C_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2} - \sum_{m=0}^{\infty} \{(k+m)(k+m-1) + 2(k+m) - n(n+1)\} x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2} - \sum_{m=0}^{\infty} C_m \{(k+m)^2 + (k+m) - n^2 - n\} x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2} - \sum_{m=0}^{\infty} C_m \{(k+m+n)(k+m-n) + (k+m-n)\} x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2} - \sum_{m=0}^{\infty} C_m (k+m-n)(k+m+n+1) x^{k+m} = 0, \qquad \text{...(4)}$$

which is an identity. Equating to zero the coefficient of the smallest power of x, namely x^{k-2}, (4) gives the identical equation

$$c_0 k(k-1) = 0 \quad \text{or} \quad k(k-1) = 0, \qquad [\because \; c_0 \neq 0]$$

giving $k = 1$ and $k = 0$. These are unequal and differ by an integer. To get the recurrence relation, we equate to zero the coefficient of x^{k+m-2}. Thus

$$C_m(k+m)(k+m-1) - C_{m-2}(k+m-2-n)(k+m-2+n+1) = 0$$

so that $$C_m = \frac{(k+m-2-n)(k+m-1+n)}{(k+m)(k+m-1)} C_{m-2}. \qquad \text{...(5)}$$

Next, equating to zero the coefficient of x^{k-1} gives $$C_1(k+1)k = 0. \qquad \text{...(6)}$$

If we take $k = 0$, (6) shows that C_1 is indeterminate. For $k = 0$, (5) gives

$$C_m = \frac{(m-2-n)(m-1+n)}{m(m-1)} C_{m-2}. \qquad \text{...(7)}$$

We now express $C_2, C_4, C_6, ...$ in terms of C_0 and $C_3, C_5, C_7, ...$ in terms of C_1 by assuming that C_1 is finite. Putting $m = 2$ in (7) gives

$$C_2 = \frac{(-2)(n+1)}{2 \cdot 1} C_0 = -\frac{n(n+1)}{2!} C_0. \qquad ...(8)$$

Putting $m = 4$ in (7) and using (8) gives

$$C_4 = \frac{(2-n)(3+n)}{3 \cdot 2} C_2 = \frac{(n-2)n(n+1)(n+3)}{4!} C_0. \qquad ...(9)$$

Next, putting $m = 3$ in (7) gives

$$C_3 = \frac{(1-n)(2+n)}{3 \cdot 2} C_1 = -\frac{(n-1)(n+2)}{3!} C_1. \qquad ...(10)$$

Again, putting $m = 5$ in (7) and using (10) gives

$$C_5 = \frac{(3-n)(4+n)}{5 \cdot 4} C_3 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!} C_1$$

and so on. Now (2) can be re–written as

$$y = x^k(C_0 + C_1x + C_2x^2 + C_3x^3 + C_4x^4 + ...)$$

or $$y = x^0(C_0 + C_2x^2 + C_4x^4 + ...) + x^0(C_1x + C_3x^3 + C_5x^5 + ...) \qquad [\because\ k = 0]$$

Using the values of C_2, C_3, C_4, C_5, ... as given by (8), (9), (10), (11) etc. in the above equation gives

$$y = C_0\left[1 - \frac{n\,(n+1)}{2!}x^2 + \frac{(n-2)n\,(n+1)(n+3)}{4!}x^4 - ....\right] + C_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 + ...\right]$$

which is the required general series solution, C_0 and C_1 being arbitrary constants.

Ex. 4. *Find the series solution of* $x^2(d^2y/dx^2) - x\,(dy/dx) - (x^2 + 5/4)y = 0$ about $x = 0$.

[Kurukshetra 2004]

Sol. Given $$x^2y'' - xy' - (x^2 + 5/4)y = 0 \qquad ...(1)$$

Re-writing (1), $$y'' - (1/x)y' - (1 + 5/4x^2)y = 0 \qquad ... (2)$$

Comparing (2) with $y'' + P(x)y' + Q(x)y = 0$, we have

$$P(x) = -(1/x) \qquad \text{and} \qquad Q(x) - (1 + 5/4x^2) \qquad ... (3)$$

$x = 0$ is not an ordinary point of (1) because neither $P(x)$ nor $Q(x)$ is analytic at x. Again, from (3),

$$x\,P(x) = -1 \qquad \text{and} \qquad x^2Q(x) = -(x^2 + 5/4),$$

which are both analytic at $x = 0$. So $x = 0$ is a regular singular point of (1). To find solution of (1), we choose a series solution of (1) about $x = 0$ in the form

$$y = \sum_{m=0}^{\infty} C_m x^{k+m}, \text{ where } C_0 \neq 0 \qquad ... (4)$$

$$\therefore\quad y' = \sum_{m=0}^{\infty} C_m(k+m)x^{k+m-1} \qquad \text{and} \qquad y'' = \sum_{m=0}^{\infty} C_m(k+m)\,(k+m-1)x^{k+m-2} \qquad ... (5)$$

Substituting the above values of y, y' and y'' in (1), we get

$$x^2 \sum_{m=0}^{\infty} C_m(k+m)\,(k+m-1)x^{k+m-2} - x\sum_{m=0}^{\infty} C_m(k+m)x^{k+m-1} - (x^2 + 5/4)\sum_{m=0}^{\infty} C_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} C_m(k+m)\,(k+m-1)x^{k+m} - \sum_{m=0}^{\infty} C_m(k+m)x^{k+m} - \sum_{m=0}^{\infty} C_m x^{k+m+2} - \frac{5}{4}\sum_{m=0}^{\infty} C_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} C_m \{(k+m)(k+m-1)-(k+m)-5/4\} x^{k+m} - \sum_{m=0}^{\infty} C_m x^{k+m+2} = 0$$

or $$\sum_{m=0}^{\infty} C_m \{(k+m)(k+m-2)-5/4\} x^{k+m} - \sum_{m=0}^{\infty} C_m x^{k+m+2} = 0, \qquad \text{... (6)}$$

which is an identity in x. Equating to zero the coefficient of the smallest power of x, namely, x^k, (6) gives the indicial equation

$C_0\{k(k-2)-5/4\} = 0$ or $4k^2 - 8k - 5 = 0$ or $(2k-5)(2k+1) = 0$, as $C_0 \neq 0$

Hence $k = 5/2, -1/2$ are roots of the indicial equation. These are distinct roots and differ by an integer.

We begin with determination of the solution corresponding to the smallest root $k = -1/2$.

Here the exponents in x^{k+m} and x^{k+m+2} differ by 2 in identity (6). So we first equote to zero the coefficient of x^{k+1}, and obtain $9\{(k+1)(k-1)-(5/4)\} = 0 \Rightarrow C_1 = 0$, since the second factor does not vanish for both $k = -1/2$ and $k = 5/2$.

For recurrence relation, equating to zero the coefficient of x^{k+m}, (6) yields

$$C_m \{(k+m)(k+m-2)-5/4\} C_m - C_{m-2} = 0, \text{ for all } m \geq 2 \qquad \text{... (7)}$$

Putting $m = 3$ and $k = -1/2$ in (7), $O \times C_3 - C_1 = 0$ or $O \times C_3 = 0$, as $C_1 = 0$

Hence, C_3 can be chosen as any arbitrary constant. Putting $k = -1/2$ in (7), we have

$$\{(m-1/2)(m-5/2)-5/4\} C_m - C_{m-2} = 0, \text{ for all } m \geq 2$$

Thus, $$C_m = \frac{1}{m(m-3)} C_{m-2}, \qquad \text{for all } m \geq 2 \text{ and } m \neq 3 \qquad \text{... (8)}$$

Putting $m = 2$ in (8), $C_2 = -(1/2)\, C_0$

Putting $m = 4$ in (8), $C_4 = (1/4) \times C_2 = -(1/8) \times C_0$, by (9)

Similarly, $C_5 = (1/10) \times C_3$, $C_6 = -(1/144) \times C_0$, $C_7 = (1/280) \times C_3, \ldots.$

Substituting the above values in (4) and taking $k = -1/2$, we get

$$y = x^{-1/2}\left\{C_0 - (1/2)C_0x^2 + C_3x^3 - (1/8)C_0x^4 + (1/10)C_3x^5 - (1/144)C_0x^6 + (1/280)C_3x^7 - (1/5760)C_0x^8 + \ldots\right)$$

or $y = C_0x^{-1/2}(1 - x^2/2 - x^4/8 - x^6/144 - x^8/5760) + \ldots) + C_3x^{5/2}(1 + x^2/10 + x^4/280 + \ldots)$, ... (10)

which is the required solution containing a_0 and a_3 as arbitrary constants.

Remark. The reader can easily verify that the roots $k = 5/2$ of the indicial equation gives another solution $C_0x^{5/2}$ $(1 + x^2/10 + x^4/280 + \ldots)$, which already occurs in (10). So we need not consider $k = 5/2$.

EXERCISE 8 (D)

Find the series solution of the following equations near $x = 0$.

1. $(x^2 + 1)y'' + xy' - xy = 0$ about $x = 0$. **[Delhi Maths (Hons.) 1993]**

Ans. $y = C_0\{1 + (1/6)x^3 - (3/40)x^5 + \ldots.\} + C_1\{x - (1/6)x^3 + (1/12)x^4 + \ldots.\}$

2. $y'' + xy' + x^2y = 0$ near $x = 0$. **[Delhi Maths (Hons.) 1996]**

Ans. $y = C_0\{1 - (1/12)x^4 + (1/90)x^6 - \ldots.\} + C_1\{x - (1/6)x^3 - (1/40)x^5 - \ldots\}$

3. $y'' - xy' - py = 0$, where p is any constant, near $x = 0$.

Ans. $$y = C_0\left\{1 + \frac{p}{2!}x^2 + \frac{p(p+2)}{4!}x^4 + \ldots\right\} + C_1\left\{x + \frac{p+1}{3!}x^3 + \frac{(p+1)(p+3)}{5!}x^5 + \ldots\right\}$$

4. $(1-x^2)y''+2xy'-y=0$ about $x=0$.

Ans. $y=C_0\{(1+(1/2)x^2-(1/24)x^4-....\}+C_1\{x-(1/6)x^3-(1/120)x^5-....\}$

5. $(2+x^2)y''+xy'-(1+x)y=0$ near $x=0$.

Ans. $y=C_0\{1+(1/4)x^2+(1/12)x^3-....\}+C_1\{x+(1/24)x^4+....\}$

6. $(1+x^2)y''+xy'-y=0$ near $x=0$. **Ans.** $y=C_0\{1+(1/2)x^2-(1/8)x^4+....\}+C_1x$

7. $(x^2-1)y''+4xy'+2y=0$ near $x=0$. **Ans.** $y=C_0(1+x^2+x^4+....)+C_1(x+x^3+x^5+....)$

8. $y''+x^2y=0$.

$$\textbf{Ans.}\quad y=a\left(1-\frac{1}{3\cdot 4}x^4+\frac{1}{3\cdot 4\cdot 7\cdot 8}x^8-....\right)+b\left(x-\frac{1}{4\cdot 5}x^5+\frac{1}{4\cdot 5\cdot 8\cdot 9}x^9-....\right).$$

9. $(1-x^2)y''+2xy'+y=0$. **[Guwahati 2007; Meerut 1997]**

Ans. $y=a[1-(1/2)x^2+(1/8)x^4+(1/80)x^6+....]+b[x-(1/2)x^3+(1/40)x^5+....$

10. Find two independent Frobenius series solutions of the equation $xy''+2y'+xy=0$.
[Delhi B.Sc. (Hons) II 2011] **Ans.** $(\cos x)/x$ and $(\sin x)/x$

8.11. Examples of Type 3 on Frobenius method. Roots of indicial equation, unequal, differing by an integer and making a coefficient of y infinite.

In this connection the following rule should be noted carefully.

Rule. *If the indicial equation has two unequal roots k_1 and k_2 (say $k_1>k_2$) differing by an integer, and if some of the coefficients of y become infinite when $k=k_2$, we modify the form of y by replacing c_0 by do $(k-k_2)$. We then obtain two independent solutions by putting $k=k_2$ in the modified form of y and $\partial y/\partial k$. The result of putting $k=k_1$ in y gives a numerical multiple of that obtained by putting $k=k_2$ and hence we reject the solution obtained by putting $k=k_1$ in y.*

Ex. 1. *Solve in series $x(1-x)y''-3xy'-y=0$ near $x=0$.* **[Delhi Maths (H) 2009]**

Sol. Given
$$x(1-x)y''-3xy'-y=0. \qquad ...(1)$$

Dividing by $x(1-x)$, (1) yields
$$\frac{d^2y}{dx^2}-\frac{3}{1-x}\frac{dy}{dx}-\frac{1}{x(1-x)}y=0.$$

Comparing it with $y''+P(x)y'+Q(x)y=0$, hence $P(x)=-3/(1-x)$ and $Q(x)=-1/\{x(1-x)\}$ so that $xP(x)=-3x/(1-x)$ and $x^2Q(x)=-x/(1-x)$. Since $xP(x)$ and $x^2Q(x)$ are both analytic at $x=0$, so $x=0$ is a regular singular point of (1). Let the series solution of (1) be

$$y=\sum_{m=0}^{\infty}c_m x^{k+m},\quad \text{where}\quad c_0\neq 0. \qquad ...(2)$$

$$\therefore\ y'=\sum_{m=0}^{\infty}c_m(k+m)x^{k+m-1}\qquad\text{and}\qquad y''=\sum_{m=0}^{\infty}c_m(k+m)(k+m-1)x^{k+m-2}. \quad ...(3)$$

Putting the above values of y, y', y'' into (1) gives

$$(x-x^2)\sum_{m=0}^{\infty}c_m(k+m)(k+m-1)x^{k+m-2}-3x\sum_{m=0}^{\infty}c_m(k+m)x^{k+m-1}-\sum_{m=0}^{\infty}c_m x^{k+m}=0$$

or
$$\sum_{m=0}^{\infty}c_m(k+m)(k+m-1)x^{k+m-1}-\sum_{m=0}^{\infty}c_m(k+m)(k+m-1)x^{k+m}-3\sum_{m=0}^{\infty}c_m(k+m)x^{k+m}-\sum_{m=0}^{\infty}c_m x^{k+m}=0$$

or
$$\sum_{m=0}^{\infty}c_m(k+m)(k+m-1)x^{k+m-1}-\sum_{m=0}^{\infty}C_m\{(k+m)(k+m-1)+3(k+m)+1\}x^{k+m}=0$$

or
$$\sum_{m=0}^{\infty}c_m(k+m)(k+m-1)x^{k+m-1}-\sum_{m=0}^{\infty}c_m\{(k+m)^2+2(k+m)+1\}x^{k+m}=0$$

or $$\sum_{m=0}^{\infty} c_m (k+m)(k+m-1)x^{k+m-1} - \sum_{m=0}^{\infty} c_m (k+m+1)^2 x^{k+m} = 0. \qquad ...(4)$$

which is an identity. Equating to zero the coefficient of the smallest power of x, namely x^{k-1}, (4) gives the indicial equation

$$c_0 k(k-1) = 0 \qquad \text{or} \qquad k(k-1) = 0. \qquad [\because\ c_0 \neq 0]$$

which gives $k = 0$ and $k = 1$. These are unequal and differ by an integer. Next, to find the recurrence relation we equate to zero the coefficient of x^{k+m-1} and obtain

$$c_m(k+m)(k+m-1) - c_{m-1}(k+m)^2 = 0 \qquad \text{or} \qquad c_m = \frac{k+m}{k+m-1}c_{m-1}. \qquad ...(5)$$

Putting $m = 1$ in (5) gives $\qquad c_1 = \{(k+1)/k\}C_0. \qquad ...(6)$

Putting $m = 2$ in (5) and using (6) gives $\qquad c_2 = \dfrac{k+2}{k+1}c_1 = \dfrac{k+2}{k}c_0. \qquad ...(7)$

Putting $m = 3$ in (5) and using (7) gives $\qquad c_3 = \dfrac{k+3}{k+2}c_2 = \dfrac{k+3}{k}c_0. \qquad ...(8)$

Putting these values in (2), *i.e.,* $y = x^k(c_0 + c_1x + c_2x^2 + ...)$, gives

$$y = c_0 x^k\left[1 + \frac{k+1}{k}x + \frac{k+2}{k}x^2 + \frac{k+3}{k}x^3 + ...\right]. \qquad ...(9)$$

If we put $k = 0$ in (9), we find that due to presence of the factor k in their denominators, the coefficients becomes infinite. To remove this difficulty, we write $c_0 = k$ do in (9). Then (9) becomes

$$y = d_o\, x^k[k + (k+1)x + (k+2)x^2 + (k+3)k^3 + ...] \qquad ...(10)$$

Putting $k = 0$ and replacing do by a in (10) gives

$$y = a(x + 2x^2 + 3x^3 + ...) = au, \text{ say} \qquad ...(11)$$

To obtain a second solution, if we put $k = 1$ in (9) we obtain

$$y = c_0(x + 2x + 3x^2 + ...) \qquad ...(12)$$

which is not distinct (*i.e.* not linearly independent because ratio of the two series in (11) and (12) is a constant) from (11). Hence (12) will not serve the purpose of a second solution. In such a case the second independent solution is given by $(\partial y/\partial k)_{k=0}$. Differentiating (10) partially w.r.t. 'k'

$$\partial y/\partial k = do\ x^k \log x\ [k + (k+1)x + (k+2)x^2 + ...] + do\ x^k\ [1 + x + x^2 + ...]. \qquad ...(13)$$

Putting $k = 0$ and replacing *do* by b (13), gives

$$(\partial y/\partial k)_{k=0} = b \log x\ (x + 2x^2 + 3x^3 + ...) + b(1 + x + x^2 + ...)$$

or $$(\partial y/\partial k)_{k=0} = b[u \log x + (1 + x + x^2 + ...)] = bv, \text{ by (11)} \qquad ...(14)$$

The required solution is $y = au + bv$, where a and b are arbitrary constants.

Ex. 2. *Solve in series the Bessel's equation of order* 2, *near* $x = 0$, $x^2y'' + xy' + (x^2 - 4)y = 0$.

[Delhi Maths (Hons.) 1995, Meerut 1995]

Sol. Given $\qquad x^2y'' + xy' + (x^2 - 4)y = 0. \qquad ...(1)$

Dividing by x^2, $\qquad y'' + (1/x)y' + [(x^2 - 4)/x^2]y = 0.$

Comparing it with $y'' + P(x)y' + Q(x)y = 0$, here $P(x) = 1/x$ and $Q(x) = (x^2 - 4)/x^2$ so that $xP(x) = 1$ and $x^2Q(x) = x^2 - 4$. Since $xP(x)$ and $x^2Q(x)$ are both analytic at $x = 0$, hence $x = 0$ is a regular singular point of (1). Let the series solution of (1) be

$$y = \sum_{m=0}^{\infty} c_m x^{k+m}, \text{ where } c_0 \neq 0. \qquad ...(2)$$

$$\therefore \quad y' = \sum_{m=0}^{\infty} c_m (k+m)x^{k+m-1} \quad \text{and} \quad y'' = \sum_{m=0}^{\infty} c_m (k+m)(k+m-1)x^{k+m-2} \qquad ...(3)$$

Putting the above values of y, y', y'' into (1) gives

$$x^2 \sum_{m=0}^{\infty} c_m (k+m)(k+m-1)x^{k+m-2} + x \sum_{m=0}^{\infty} c_m (k+m)x^{k+m-1} + x^2 \sum_{m=0}^{\infty} c_m x^{k+m} - 4 \sum_{m=0}^{\infty} c_m x^{k+m} = 0$$

or $\sum_{m=0}^{\infty} c_m \{(k+m)(k+m-1)+(k+m)-4\}x^{k+m} + \sum_{m=0}^{\infty} c_m x^{k+m+2} = 0$ or $\sum_{m=0}^{\infty} c_m \{(k+m)^2-4\}x^{k+m} + \sum_{m=0}^{\infty} c_m x^{k+m+2} = 0$

or
$$\sum_{m=0}^{\infty} c_m (k+m+2)(k+m-2)x^{k+m} + \sum_{m=0}^{\infty} c_m x^{k+m+2} = 0, \qquad ...(4)$$

which is an identity. Equating to zero the coefficient of the smallest power of x, namely x^k, gives the indicial equation $\quad c_0(k+2)(k-2) = 0 \quad$ or $\quad (k+2)(k-2) = 0 \quad [\because c_0 \neq 0]$

This gives $k = 2$ and $k = -2$. These are unequal and differ by an integer. For the recurrence relation, we equate to zero the coefficient of x^{k+m} and get

$$c_m(k+m+2)(k+m-2) + c_{m-2} = 0 \qquad \text{so} \qquad c_m = -\frac{1}{(k+m+2)(k+m-2)} c_{m-2}. \qquad ...(5)$$

To determine c_1, we equate to zero the coefficient of x^{k+1} and get $c_1(k+3)(k-1) = 0$ giving $c_1 = 0$ for both the roots $k = 2$ and $k = -2$ of the indicial equation. Now using $c_1 = 0$ and (5), we get

$$c_1 = c_3 = c_5 = c_7 = ... = 0. \qquad ...(6)$$

Next, putting $m = 2, 4, 6, ...$ in (5) and simplifying, we have

$c_2 = -c_0/k(k+4), \qquad c_4 = -c_2/(k+2)(k+6) = c_0/k(k+2)(k+4)(k+6),$

$c_6 = -c_4/(k+4)(k+8) = -c_0/k(k+2)(k+4)^2(k+6)(k+8)$

and so on. Putting these values in (2), *i.e.*,

$y = x^k(c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + c_6x^6 + c_7x^7 + ...)$, we get

$$y = c_0 x^k \left\{1 - \frac{x^2}{k(k+4)} + \frac{x^4}{k(k+2)(k+4)(k+6)} - \frac{x^6}{k(k+2)(k+4)^2(k+6)(k+8)} + ...\right\} \qquad ...(7)$$

Putting $k = 2$ in (7) yields $y = c_0 x^2 \left\{1 - \dfrac{x^2}{2\cdot 6} + \dfrac{x^4}{2\cdot 4\cdot 8\cdot 6} - \dfrac{x^6}{2\cdot 4\cdot 6^2\cdot 8\cdot 10} + ...\right\} = c_0\,\omega$, say $\quad ...(8)$

Next, if we put $k = -2$ in (7), the coefficients of $x^4, x^6, ...$ become infinite. To get rid of this difficulty, we put $c_0 = do(k+2)$ in (7) and obtain modified solution as

$$y = do\, x^k \left\{(k+2) - \frac{(k+2)x^2}{k(k+4)} - \frac{x^4}{k(k+4)(k+6)} - \frac{x^6}{k(k+4)^2(k+6)(k+8)} + ...\right\} \qquad ...(9)$$

Putting $k = -2$ and replacing do by a in (9) gives

$$y = ax^{-2}\left\{0 - 0.x^2 + \frac{x^4}{(-2)(2)(4)} - \frac{x^6}{(-2)(2)^2(4)(6)} + ...\right\} \text{ or } y = \frac{-ax^2}{16}\left\{1 - \frac{x^2}{2\cdot 6} + \frac{x^4}{2\cdot 4\cdot 8\cdot 6} - ....\right\} = au, \text{ say} \quad ...(10)$$

Now, $\qquad$ (8) and (10) $\quad \Rightarrow \quad u = -\dfrac{x^2}{16}\left\{1 - \dfrac{x^2}{2\cdot 6} + \dfrac{x^4}{2\cdot 4\cdot 8\cdot 6} - ....\right\} = -\dfrac{\omega}{16},$

showing that w and u are dependent solutions. Hence we must find one more independent solution in order to obtain the required general solution.

To get another independent solution, substituting (9) into the L.H.S. of (1) and simplifying, we find $\qquad x^2y'' + xy' + (x^2 - 4)y = d_0(k-2)(k+2)^2x^k. \qquad ... (11)$

Differentiating both sides of (11) partially w.r.t. k, we get

$$\frac{\partial}{\partial k}\left[x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+(x^2-4)y\right] = d_0\frac{\partial}{\partial k}[(k+2)^2\cdot x^k(k-2)]$$

or $$\left[x^2\frac{d^2}{dx^2}+x\frac{d}{dx}+(x^2-4)\right]\frac{\partial y}{\partial k}=2d_0(k+2)\cdot x^k(k-2)+d_0(k+2)^2[x^k\log x\cdot(k-2)+x^k\cdot 1] \quad ...(12)$$

The presence of the factor $(k + 2)$ in each term on R.H.S. of (12) shows that a second solution is $(\partial y/\partial k)_{k=-2}$, where y is given by (9). Differentiating (9) partially w.r.t. 'k' we get

$$\frac{\partial y}{\partial k} = d_0x^k\log x\left\{(k+2)-\frac{(k+2)x^2}{k(k+4)}+\frac{x^4}{k(k+4)(k+6)}-\frac{x^6}{k(k+4)^2(k+6)(k+8)}+....\right\}$$

$$+\,d_0x^k\left\{1-\frac{(k+2)x^2}{k(k+4)}\left(\frac{1}{k+2}-\frac{1}{k}-\frac{1}{k+4}\right)+\frac{x^4}{k(k+4)(k+6)}\left(-\frac{1}{k}-\frac{1}{k+4}-\frac{1}{k+6}\right)\right.$$

$$\left.-\frac{x^6}{k(k+4)^2(k+6)(k+8)}\times\left(-\frac{1}{k}-\frac{2}{k+4}-\frac{1}{k+6}-\frac{1}{k+8}\right)+....\right\} \quad ...(13)$$

$\left[\text{To find } \frac{d}{dk}\left\{\frac{k+2}{k(k+4)}\right\}\right.$, we proceed as follows: Take $z=\frac{k+2}{k(k+4)}$ so that $\log z = \log\left[\frac{k+4}{k(k+4)}\right]$

or $$\log z = \log(k+4) - \log k - \log(k+4).$$

Differentiating it w.r.t. 'k', we get

$$\frac{1}{z}\frac{dz}{dk}=\frac{1}{k+4}-\frac{1}{k}-\frac{1}{k+4} \quad\Rightarrow\quad \frac{d}{dk}\left\{\frac{k+2}{k(k+4)}\right\}=\frac{k+2}{k(k+4)}\left(\frac{1}{k+4}-\frac{1}{k}-\frac{1}{k+4}\right) \text{ etc.}$$

Other terms can be similarly differentiated easily]

Putting $k = -2$ and replacing *do* by b in (13) gives

$$\left(\frac{\partial y}{\partial k}\right)_{k=-2} = bx^{-2}\log x\left\{0-0.x^2+\frac{x^4}{(-2)(2)(4)}-\frac{x^6}{(-2)(2)^2(4)(6)}+....\right\}+bx^{-2}\left\{1+\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot 4}\left(\frac{1}{4}\right)\right.$$

$$\left.+\frac{x^6}{2^3\cdot 4\cdot 6}\left(\frac{1}{2}-1+\frac{1}{4}-\frac{1}{6}\right)+....\right\} = b\left[u\log x+x^{-2}\left\{1+\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot 4^2}-....\right\}\right]=bv, \text{ by (10)}$$

The required solution is $y = au + bv$, where a and b are arbitrary constants.

Ex. 3. *Solve Bessel's equation of first order,* $x^2y'' + xy' + (x^2 - 1)y = 0$ in series near $x = 0$.

[Delhi Maths (H) 2007; Ravishankar 2005; Meerut 1994; Agra 2006]

Sol. Given $$x^2y'' + xy' + (x^2 - 1)y = 0. \quad ...(1)$$

As in Ex. 2, show that $x = 0$ is a regular singular point of (1). Let the series solution of (1) be of the form

$$y = x^k(C_0 + C_1x + C_2x^2 + ...) = \sum_{m=0}^{\infty} C_m x^{k+m},\ C_0 \neq 0. \quad ...(2)$$

$$\therefore\ y' = \sum_{m=0}^{\infty}(k+m)C_m x^{k+m-1} \quad\text{and}\quad y'' = \sum_{m=0}^{\infty}(k+m)(k+m-1)C_m x^{k+m-2}. \quad ...(3)$$

Putting the above values of y, y' and y'' in (1), we have

$$x^2 \sum_{m=0}^{\infty}(k+m)(k+m-1)C_m x^{k+m-2} + x\sum_{m=0}^{\infty}(k+m)C_m x^{k+m-1} + (x^2-1)\sum_{m=0}^{\infty}C_m x^{k+m} = 0$$

or
$$\sum_{m=0}^{\infty}(k+m)(k+m-1)C_m x^{k+m} + \sum_{m=0}^{\infty}(k+m)C_m x^{k+m} + \sum_{m=0}^{\infty}C_m x^{k+m+2} - \sum_{m=0}^{\infty}C_m x^{k+m} = 0$$

or
$$\sum_{m=0}^{\infty}[(k+m)(k+m-1)+(k+m)-1]C_m x^{k+m} + \sum_{m=0}^{\infty}C_m x^{k+m+2} = 0$$

or
$$\sum_{m=0}^{\infty}[(k+m)^2-1]C_m x^{k+m} + \sum_{m=0}^{\infty}C_m x^{k+m+2} = 0$$

or
$$\sum_{m=0}^{\infty}(k+m-1)(k+m+1)C_m x^{k+m} + \sum_{m=0}^{\infty}C_m x^{k+m+2} = 0. \qquad ...(4)$$

(4) is an identity. Equating to zero the coefficient of the smallest power of x, namely x^k, we obtain the indicial equation. $C_0(k-1)(k+1) = 0$ so that $k = 1$ and $k = -1$ as $C_0 \neq 0$. These are unequal and differ by an integer. Here the difference in powers of x in (4) is 2. Hence we equate to zero the coefficient of x^{k+1} in identity (4) and obtain.

$k(k+2)C_1 = 0$ giving $C_1 = 0$ for both the roots $k = 1$ and $k = -1$.

Next, equating to zero the coefficient of x^{k+m} in (4), $(k+m-1)(k+m+1)C_m + C_{m-2} = 0$

or
$$C_m = -\frac{1}{(k+m+1)(k+m+1)}C_{m-2}. \qquad ...(5)$$

Putting $m = 3, 5, 7, ...$ in (5) and noting that $C_1 = 0$, $C_1 = C_3 = C_5 = C_7 = ... = 0$. ...(6)

Next, putting $m = 2, 4, 6, ...$ in (5), we get $C_2 = -\dfrac{1}{(k+1)(k+3)}C_0$, ...(7)

$$C_4 = -\frac{1}{(k+3)(k+5)}C_2 = \frac{1}{(k+1)(k+3)^2(k+5)}C_0, \text{ by (7)}$$

and so on. Putting these values in (2), we have

$$y = C_0 x^k \left\{1 - \frac{x^2}{(k+1)(x+3)} + \frac{x^4}{(k+1)(k+3)^2(k+5)} + ...\right\} \qquad ...(8)$$

Now if we take $k = -1$ in the above series, the coefficients become infinite because of the factor $(k+1)$ in the denominator. To get rid of this difficulty, we put $C_0 = d_0(k+1)$ in (8) and get the modified solution as

$$y = d_0 x^k \left\{(k+1) - \frac{x^2}{k+3} + \frac{x^4}{(k+3)^2(x+5)} + ...\right\}. \qquad ...(9)$$

Differentiating (9) partially w.r.t. 'k', we have

$$\frac{\partial y}{\partial k} = d_0 x^k \log x\left\{(k+1) - \frac{x^2}{k+3} + \frac{x^4}{(k+3)^2(x+5)} + ...\right\} + d_0 x^k\left[1 + \frac{x^2}{(k+3)^2} - \left\{\frac{2}{(k+3)^3(k+5)} + \frac{1}{(k+3)(k+5)^2}\right\}x^4 + ...\right] \qquad ...(10)$$

Putting $k = -1$ and replacing do by a, (9) gives

$$y = ax^{-1}\left\{0 - \frac{x^2}{2} + \frac{x^4}{2^2 \cdot 4} + \frac{x^6}{2^2 \cdot 4^2 \cdot 6} + ...\right\} = au \text{ , say.} \qquad ...(11)$$

Putting $k = -1$ and replacing *do* by b, (10) gives

$$\left(\frac{\partial y}{\partial k}\right)_{k=-1} = bx^{-1}\log x\left\{0 - \frac{x^2}{2} + \frac{x^4}{2^2\cdot 4} + \frac{x^6}{2^2\cdot 4^2\cdot 6} + \ldots\right\} + bx^{-1}\left\{1 + \frac{x^2}{2^2} - \frac{1}{2^2\cdot 4}\left(\frac{2}{2}+\frac{1}{4}\right)x^4 + \ldots\right\}$$

or $$\left(\frac{\partial y}{\partial k}\right)_{k=-1} = bu\log x + bx^{-1}\left\{1 + \frac{x^2}{2^2} - \frac{1}{2^2\cdot 4}\left(\frac{2}{2}+\frac{1}{4}\right)x^4 + \ldots\right\} = bv\text{, say. [using (11)]} \quad \ldots(12)$$

Putting $k = 1$ in (9), we get

$$y = do\, x\left[2 - \frac{x^2}{4} + \frac{x^4}{4^2\cdot 6} + \ldots\right] \quad \text{or} \quad y = -2^2\, do\, x^{-1}\left[-\frac{x^2}{2} + \frac{x^4}{2^2\cdot 4} - \frac{x^6}{2^2\cdot 4^2\cdot 6} + \ldots\right]. \quad \ldots(13)$$

From (11) and (13), we find that these are linearly dependent solutions. Thus out of three solutions (11), (12) and (13) only two are linearly independent. From (11) and (12) the required general solution is $y = au + bv$, a and b being arbitrary constants.

Ex. 4. *Find the series solution near* $x = 0$ *of* $(x + x^2 + x^3)y'' + 3x^2y' - 2y = 0$.

Sol. Given $$(x + x^2 + x^3)y'' + 3x^2y' - 2y = 0. \quad \ldots(1)$$

Dividing by $(x + x^2 + x^3)$, (1) can be put in standard form

$$y'' + \{3x/(1 + x + x^2)\}y' - \{2/(x + x^2 + x^3)\}y = 0.$$

Comparing it with $y'' + P(x)y' + Q(x)y = 0$, here $P(x) = 3x/(1 + x + x^2)$ and $Q(x) = -2/(x + x^2 + x^3)$. Since $P(x)$ and $Q(x)$ are both analytic at $x = 0$, hence $x = 0$ is a regular singular point of (1). Let series solution of (1) be of the form

$$y = x^k(C_0 + C_1x + C_2x^2 + \ldots) = \sum_{m=0}^{\infty} C_m x^{k+m}, \quad \text{where} \quad C_0 \neq 0. \quad \ldots(2)$$

$$\therefore \quad y' = \sum_{m=0}^{\infty}(k+m)C_m x^{k+m-1} \quad \text{and} \quad y'' = \sum_{m=0}^{\infty}(k+m)(k+m-1)C_m x^{k+m-2}. \quad \ldots(3)$$

Putting the above values of y, y' and y'' in (1), we have

$$(x + x^2 + x^3)\sum_{m=0}^{\infty}(k+m)(k+m-1)C_m x^{k+m-2} + 3x^2\sum_{m=0}^{\infty}(k+m)C_m x^{k+m-1} - 2\sum_{m=0}^{\infty}C_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty}(k+m)(k+m-1)C_m x^{k+m-1} + \sum_{m=0}^{\infty}(k+m)(k+m-1)C_m x^{k+m}$$
$$+ \sum_{m=0}^{\infty}(k+m)(k+m+1)C_m x^{k+m+1} + \sum_{m=0}^{\infty}3(k+m)C_m x^{k+m+1} - \sum_{m=0}^{\infty}2C_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty}(k+m)(k+m-1)C_m x^{k+m-1} + \sum_{m=0}^{\infty}[(k+m)(k+m-1)-2]C_m x^{k+m} + \sum_{m=0}^{\infty}[(k+m)(k+m+1)+3(k+m)]C_m x^{k+m+1} = 0$$

or $$\sum_{m=0}^{\infty}(k+m)(k+m-1)C_m x^{k+m-1} + \sum_{m=0}^{\infty}[(k+m)^2-(k+m)-2]C_m x^{k+m} + \sum_{m=0}^{\infty}(k+m)(k+m+2)C_m x^{k+m+1} = 0$$

or $$\sum_{m=0}^{\infty}(k+m)(k+m-1)C_m x^{k+m-1} + \sum_{m=0}^{\infty}(k+m-2)(k+m+1)C_m x^{k+m} + \sum_{m=0}^{\infty}(k+m)(k+m+2)C_m x^{k+m+1} = 0. \quad \ldots(4)$$

Equating to zero the coefficient of the smallest power of x, namely x^{k-1}, in identity (4), the indicial equation is given by $C_0k(k-1) = 0$ giving $k = 0$ and $k = 1$ as $C_0 \neq 0$. These roots are unequal and differ by an integer. Next, equating to zero the coefficient of x^k in (4), we get

$(k+1)kC_1 + (k-2)(k+1)C_0 = 0$ so that $C_1 = -[(k-2)/k]C_0$. ...(5)

Finally, equating to zero the coefficient of x^{k+m-1} in (4), we get

$(k+m)(k+m-1)C_m + (k+m-3)(k+m)C_{m-1} + (k+m-2)(k+m)C_{m-2} = 0$, for all $m \geq 2$

or $(k+m-1)C_m + (k+m-3)C_{m-1} + (k+m-2)C_{m-2} = 0$ for all $m \geq 2$.

[Note that $(k+m) \neq 0$ for $k = 0$ and 1 and for all $m \geq 2$]

or $$C_m = -\frac{1}{k+m-1}[(k+m-3)C_{m-1} + (k+m-2)C_{m-2}], \quad \text{for all} \quad m \geq 2 \qquad ...(6)$$

Putting $m = 2$ in (6), we have

$$C_2 = -\frac{1}{k+1}[(k-1)C_1 + kC_0] = -\frac{1}{k+1}\left[-\frac{(k-1)(k-2)}{k}C_0 + kC_0\right], \text{ by (5)}$$

or $$C_2 = -\frac{C_0}{k(k+1)}[k^2 - (k-1)(k-2)] = -\frac{3k-2}{k(k+1)}C_0. \qquad ...(7)$$

Putting $m = 3$ in (6), we have

$$C_3 = -\frac{1}{k+2}[kC_2 + (k+1)C_1] = -\frac{1}{k+2}\left[-\frac{k(3k-2)C_0}{k(k+1)} - \frac{(k+1)(k-1)}{k}C_0\right]$$

or $$C_3 = \frac{C_0}{k+2}\left[\frac{3k-2}{k+1} + \frac{k^2-1}{k}\right] = \frac{k^3+3k^2-5k-2}{k(k+1)(k+2)}$$

and so on putting the above values in (2), we have

$$y = C_0 x^k\left[1 - \frac{k-2}{k}x - \frac{3k-2}{k(k+1)}x^2 + \frac{k^3+3k^2-5k-2}{k(k+1)(k+2)}x^3 + ..\right] \qquad ...(8)$$

Now if we put $k = 0$ in the above series, the coefficients become infinite because of the factor k in the denominator. To get rid of this difficulty, we put $C_0 = d_0 k$ in (8) and get the modified solution as

$$y = d_0 x^k\left[k - (k-2)x - \frac{3k-2}{k+1}x^2 + \frac{k^3+3k^2-5k-2}{(k+1)(k+2)}x^3 + ...\right] \qquad ...(9)$$

Differentiating (9) partially w.r.t. 'k', we get

$$\frac{\partial y}{\partial k} = d_0 x^k \log x\left[k - (k-2)x - \frac{3k-2}{k+1}x^2 + \frac{k^3+3k^2-5k-2}{(k+1)(k+2)}x^3 + ...\right] + d_0 x^k[1 - x$$

$$-\left\{\frac{3}{k+1} - \frac{(3k-2)}{(k+1)^2}\right\}x^2 + \left\{\frac{3k^2+6k-5}{(k+1)(k+2)} - \frac{k^3+3k^2-5k-2}{(k+1)^2(k+2)} - \frac{k^3+3k^2-5k-2}{(k+1)(k+2)^2}\right\}x^3 + ...\Bigg]$$

...(10)

Putting $k = 0$ and replacing *do* by a in (9), we get

$$y = a(2x + 2x^2 - x^3 + ...) = au, \text{ say.} \qquad ...(11)$$

Putting $k = 0$ and replacing *do* by b in (10), we get

$$\partial y/\partial k)_{k=0} = b \log x\,(2x + 2x^2 - x^3 + ...) + b(1 - x - 5x^2 - x^3 + ...)$$

or $$(\partial y/\partial k)_{k=0} = b[\log x\,(2x + 2x^2 - x^3 + ...) + (1 - x - 5x^2 - x^3 + ...)] = bv, \text{ say.} \qquad ...(12)$$

Putting $k = 1$ in (9), we get

$$y = d_0 x[1 + x - x^2/2 - x^3/3 + ...] = (d_0/2) \times [2x + 2x^2 - x^3 + ...] \qquad ...(13)$$

From (11) and (13), we find that these are linearly dependent solutions. Thus out of three solutions (11), (12) and (13) only two are linearly independent. From (11) and (12), the required general solution is $y = au + bv$, a and b being arbitrary constants.

Ex. 5. *Solve* (a) $(x^2 - x)y'' + 3y' - 2y = 0$ near $x = 0$.

(b) $(x^2 - x)y'' + 3y' - 2y = x + (3/x^2)$ near $x = 0$.

Sol. (a) Given $$(x^2 - x)y'' + 3y' - 2y = 0 \qquad \text{... (1)}$$

Re-writing (1), $$y'' + \{3/(x^2 - x)\}y' - \{2/(x^2 - x)\}y = 0$$

Comparing it with $y'' + P(x)y' + Q(x)y = 0$, $P(x) = 3/(x^2 - x)$ and $Q(x) = -\{2/(x^2 - x)\}$, showing that $P(x)$ and $Q(x)$ are undefined at $x = 0$ and so $P(x)$ and $Q(x)$ are not analytic at $x = 0$. Thus, $x = 0$ is not an ordinary point and therefore $x = 0$ is a singular point.

Also, $(x - 0)\, P(x) = 3/(x - 1)$ and $(x - 0)^2 Q(x) = -\{2x / (x - 1)\}$, showing that both $(x - 0)\, P(x)$ and $(x - 0)^2\, Q(x)$ are both analytic at $x = 0$. so $x = 0$ is a regular singular point.

To solve (1), we assume a series solution of the form

$$y = \sum_{m=0}^{\infty} C_m x^{k+m}, \text{ where } C_0 \neq 0 \qquad \text{... (2)}$$

$$(2) \Rightarrow \quad y' = \sum_{m=0}^{\infty} C_m (k+m) x^{k+m-1} \quad \text{and} \quad y'' = \Sigma C_m (k+m)(k+m-1) x^{k+m-2}$$

Substituting the above values of y, y' and y'' in (1), we get

$$(x^2 - x) \sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-1} + 3 \sum_{m=0}^{\infty} C_m (k+m) x^{k+m-1} - 2 \sum_{m=0}^{\infty} C_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m} - \sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-1} + 3 \sum_{m=0}^{\infty} C_m (k+m) x^{k+m-1} - 2 \sum_{m=0}^{\infty} C_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} C_m \{(k+m)(k+m-1) - 2\} x^{k+m} - \sum_{m=0}^{\infty} C_m \{(k+m)(k+m-1) - 3(k+m)\} x^{k+m-1} = 0$$

or $$\sum_{m=0}^{\infty} C_m (k+m-2)(k+m+1) x^{k+m} - \sum_{m=0}^{\infty} C_m (k+m)(k+m-4) x^{k+m-1} = 0$$

or $$-k(k-4)C_0 x^{k-1} + \{(k-2)(k+1)C_0 - (k-3)(k+1)C_1\} x^k + \{(k-1)(k+2)C_1 - (k-2)(k+2)C_2\} x^{k+1} + \ldots$$

$$\ldots\ldots + \{(k+m-3)(k+m)\, C_{m-1} - (k+m-4)(k+m)\, C_m x^{k+m-1} + \ldots = 0 \qquad \text{... (3)}$$

(3) is identity in x. Equating to zero the coefficient of the lowest power of x, namely x^{k+1}, the indicial equation is $\quad -k(k-4)\, C_0 = 0 \quad$ giving $\quad k = 0, 4 \quad$ as $\quad C_0 \neq 0$

Equating to zero the coefficient of x^{k+m-1}, the recurrence relation is given by

$$C_m = \frac{k+m-3}{k+m-4} C_{m+1}, \quad \text{for all } m \geq 1 \qquad \text{... (4)}$$

Putting m = 1, 2, 3, ... in (4), we have

$$C_1 = \frac{k-2}{k-3} C_0, \qquad C_2 = \frac{k-1}{k-2} \quad C_1 = \frac{k-1}{k-2} \times \frac{k-2}{k-3} \quad C_0 = \frac{k-1}{k-3} C_0,$$

$$C_3 = \frac{k}{k-1}C_2 = \frac{k}{k-1}\times\frac{k-1}{k-3}\ C_0 = \frac{k}{k-3}C_0, \qquad C_4 = \frac{k+1}{k}C_3 = \frac{k+1}{k}\times\frac{k}{k-3}\ C_0 = \frac{k+1}{k-3}C_0$$

and so on. Substituting these values of C_1, C_2, C_3, ... in (2), we get

$$y = C_0 x^k \left\{1 + \frac{k-2}{k-3}x + \frac{k-1}{k-3}x^2 + \frac{k}{k-3}x^3 + \frac{k+1}{k-3}x^4 + \ldots\right\}, \qquad \ldots (5)$$

The roots $k = 0, 4$ of the indicial equation differ by an integer. However, when $k = 0$ the expected vanishing of the denominator in the coefficent of x^4 does not occur since the factor k appears in both numerator and denominator (refer calculation of C_4) and thus cancels out. Observe that the coefficient C_3 of x^3 is zero when $k = 0$.

Putting $k = 0$ and replacing C_0 by a, (5) gives

$$y = a\ (1 + 2x/3 + x^2/3 - x^4/3 - 2x^5/3 - x^6 \ldots) = au, \text{ say,}$$

Putting $k = 4$ and replacing C_0 by b, (5) gives

$$y = bx^4\ (1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots) = bv, \text{ say}$$

Clearly, $$u = (1 + 2x/3 + x^2/3) - v/3 \qquad \ldots (6)$$

The required general solution is $y = au + bv$, that is,

$$y = a(1 + 2x/3 + x^2/3) + (b - a/3)v, \text{ using (6)}$$

or $$y = A\ (x^2 + 2x + 3) + Bx^4\ (1 + 2x + 3x^2 + 4x^3 + \ldots), \text{ taking } A = a/3 \text{ and } B = b - a/3$$

or $$y = A(x^2 + 2x + 3) + Bx^4\ (1 - x)^{-2}, \ A \text{ and } B \text{ being arbitrary constants.} \qquad \ldots (7)$$

(*b*) Given $$(x^2 - x)y'' + 3y' - 2y = x + 3/x^2 \qquad \ldots (i)$$

Here $x = 0$ is rgular singular point of (i). To solve (*i*), we assume a series solution of the form

$$y = \sum_{m=0}^{\infty} C_m x^{k+m}, \qquad \text{where} \qquad C_0 \neq 0 \qquad \ldots (ii)$$

$$\therefore \qquad y' = \sum_{m=0}^{\infty} C_m (k+m) x^{k+m-1} \qquad\qquad y'' = \sum_{m=0}^{\infty} C_m (k+m)\,(k+m-1) x^{k+m-2}$$

Substituting the above values of y, y' and y'' in (1) and proceeding as in part (*a*) gives

$$-k(k-4)C_0 x^{k-1} + \{(k-2)\,(k+1)C_0 - (k-3)(k+1)C_1\}x^k + \{(k-1)\,(k+2)C_1 - (k-2)\,(k+2)C_2\}x^{k+1}$$

$$+ \ldots + \{(k+m-3)\,(k+m)\,C_{m-1} - (k+m-4)\,(k+m)\,C_m\}x^{k+m-1} + \ldots = x + 3/x^2 \ \ldots (iii)$$

To determine complementary function of (i) : Setting the left member of (*iii*) equal to zero,

$$-\,k(k-4)\ C_0 x^{k-1} + \{(k-2)\ (k+1)\ C_0 - (k-3)\ (k+1)\ C_1\}\ x^k + \ldots$$
$$+ \{(k+m-3)\ (k+m)\ C_{m-1} - (k+m-4)\ (k+m)\ C_m\}\ x^{k+m-1} + \ldots = 0 \qquad \ldots (iv)$$

Starting with (*iv*), we now proceed as in part (*a*) upto equation (6) of part (*a*). Thus, as in part (*a*) we obtain $C_m = \{(k+m-3)/(k+m-4)\}\ C_{m-1}$ for all $m \geq 1$ ... (*v*)

and $$y = C_0 x^k \left(1 - \frac{k-2}{k-3}x + \frac{k-1}{k-3}x^2 + \frac{k}{k-3}x^3 + \frac{k+1}{k-3}x^4 + \ldots\right) \qquad \ldots (vi)$$

which satisfies $$(x^2 - x)y'' + 3y' - 2y = -k(k-4)C_0 x^{k-1} \qquad \ldots (vii)$$

The required complementary function (C.F.) of (*i*)

$= au + bv = a\ (1 + 2x/3 + x^3/3) + (b - a/3)v$

$= A(x^2 + 2x + 3) + Bx^4\ (1 + 2x + 3x^2 + 4x^3 + \ldots)$, setting $A = a/3$ and $B = b - a/3$

$= A\ (x^2 + 2x + 3) + Bx^4\ (1 - x)^{-2}$

To determine particular integral (P.I.). of (*i*) : To this end we shall consider each of the terms of the right member of the given equation (*i*).

Setting the right member of (*vii*) equal to x, that is, $-k(k-4)\ C_0 x^{k-1} = x$, identically,
we have $k = 2$ and $C_0 = 1/4$. Then recurrence relation (v) takes the form

$$C_m = \{(m-1)/(m-2)\}\ C_{m-1}, \quad \text{for all} \quad m \geq 1 \qquad \text{... } (viii)$$

Putting $m = 1, 2, 3, ..$ in (*viii*) yields $C_1 = C_2 = C_3 = ... = 0$.

Putting $k = 2$, $C_0 = 1/4$ and using the fact that $C_m = 0$ for all $m \geq 1$, from (*ii*), we have P.I. corresponding to the term x is $x^2/4$.

Again, setting the right member of (*vii*) equal to $3/x^2$, *i.e.*, $-k(k-4)C_0 x^{k-1} = 3/x^2$, identically, we have $k = -1$ and $C_0 = -3/5$. Then recurrence relation (v) takes the form

$$C_m = \{(m-4)/(m-5)\}\ C_{m-1}, \quad \text{for all} \quad m \geq 1 \qquad \text{... } (ix)$$

Putting $m = 1, 2, 3, ...$ in (*ix*), we have $C_1 = 3C_0/4$, $C_2 = C_0/2$, $C_3 = C_0/4$, $C_4 = C_5 = C_8 = ... = 0$.

Putting these values in (*ii*) and taking $k = -1$, $C_0 = -3/5$, we have

$\therefore$ Particular integral corresponding to the term $3/x^2$ is $-(3/5) \times x^{-1}(1 + 3x/4 + x^2/2 + x^3/4)$

Hence the complete integral of (1) is $\qquad y = \text{C.F.} + \text{total P.I.}$

or $\qquad y = A(x^2 + 2x + 3) + Bx^4(1-x)^{-2} + x^2/4 - 3/5x - 9/20 - 3x/10 - 3x^2/20$

or $\qquad y = A(x^2 + 2x + 3) + Bx^4(1-x)^{-2} + x^2/4 - 3/5x - (3/20) \times (x^2 + 2x + 3)$

or $\qquad y = C(x^2 + 2x + 3) + Bx^4(1-x)^{-2} - (3/5x) + (x^2/4)$,

where $C = (A - 3/20)$. Here C and B are arbitrary constants.

EXERCISE 8 (E)

Find the series solution of the following equations near $x = 0$.

1. $x(1-x)y'' - (1+3x)y' - y = 0$. **Ans.** $y = a(1\cdot 2x^2 + 2\cdot 3x^3 + 3\cdot 4x^4 + ...)$

$$+b\{\log x(1\cdot 2x^2 + 2\cdot 3x^3 + 3\cdot 4x^4 + ...) + (-1 + x + 3x^2 + 5x^3 + ...)\}$$

2. $x^2y'' - 3xy' + (3+4x)y = 0$. **[Meerut 1997]**

3. $x^2y'' + xy' - (1+x^2)y = 0$.

8.12. Examples of Type 4 on Frobenius method. Roots of indicial equation equal.

In this connection the following rule should be noted carefully.

Rule. *If the indicial equation has two equal roots $k = k'$, we obtain two independent solutions by substituting this value of k in y and $\partial y/\partial k$.*

Ex. 1. *Solve the Bessel's equation of order zero $xy'' + y' + xy = 0$ in series.*

[Delhi Maths (H) 2009; Meerut 1996, G.N.D.U. Amritsar 1997, Garhwal 2005]

Sol. Given $\qquad xy'' + y' + xy = 0. \qquad$ ...(1)

Dividing by x, $y'' + (1/x)y' + y = 0$. Comparing it with $y'' + P(x)y' + Q(x)y = 0$, here $P(x) = 1/x$ and $Q(x) = 1$, so that $xP(x) = 1$and $x^2Q(x) = x^2$, which are analytic at $x = 0$. So $x = 0$ is a regular singular point. Let the series solution of (1) be of the form

$$y = \sum_{m=0}^{\infty} C_m x^{k+m}, \qquad \text{where} \qquad C_0 \neq 0. \qquad \text{...(2)}$$

$$\therefore \quad y' = \sum_{m=0}^{\infty} C_m (k+m) x^{k+m-1} \quad \text{and} \quad y'' = \sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2}. \quad \text{...(3)}$$

Putting the above values of y, y', y'' into (1) gives

$$\sum_{m=0}^{\infty} C_m (k+m)(k+m-1)x^{k+m-1} + \sum_{m=0}^{\infty} C_m (k+m)x^{k+m-1} + \sum_{m=0}^{\infty} C_m x^{k+m+1} = 0$$

or $$\sum_{m=0}^{\infty} C_m \{(k+m)(k+m-1)+(k+m)\}x^{k+m-1} + \sum_{m=0}^{\infty} C_m x^{k+m+1} = 0$$

or $$\sum_{m=0}^{\infty} C_m (k+m)^2 x^{k+m-1} + \sum_{m=0}^{\infty} C_m x^{k+m+1} = 0, \qquad ...(4)$$

which is an identity. Equating to zero the coefficient of the smallest power of x, namely x^{k-1}, (4) gives the indicial equation $C_0 k^2 = 0$ or $k^2 = 0$ ($\because C_0 \neq 0$). So $k = 0, 0$ (equal roots.)

For recurrence relation, equating to zero the coefficient of x^{k+m-1} in (4) gives

$$C_m(k+m)^2 + C_{m-2} = 0 \qquad \text{or} \qquad C_m = -\frac{C_{m-2}}{(k+m)^2}. \qquad ...(5)$$

To compute C_1, equating to zero the coefficient of x^k gives $C_1(k+1)^2 = 0$ so that $C_1 = 0$ (for $k = 0$ is root of indicial equation). Using $C_1 = 0$ and (5), we get

$$C_1 = C_3 = C_5 = C_7 = ... = 0. \qquad ...(6)$$

Putting $m = 2, 4, 6, ...$ in turn in (5) and simplifying, we get

$$C_2 = -C_0/(k+2)^2, \qquad C_4 = -C_2/(k+4)^2 = C_0/(k+2)^2(k+4)^2,$$
$$C_6 = -C_4/(k+6)^2 = -C_0/(k+2)^2(k+4)^2(k+6)^2,$$

and so on. Putting these values in (2), *i.e.* $y = x^k(C_0 + C_1 x + C_2 x^2 + C_3 x^3 + ...)$,

$$y = C_0 x^k \left\{1 - \frac{x^2}{(x+2)^2} + \frac{x^4}{(k+2)^2(k+4)^2} - \frac{x^6}{(k+2)^2(k+4)^2(k+6)^2} + \dots\right\} \qquad ...(7)$$

Putting $k = 0$ and replacing C_0 by a in (7) gives

$$y = a\left(1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^4 \cdot 6^2} + ...\right) = au \text{ say} \qquad ...(8)$$

To get another independent solution, substituting, (7) into the L.H.S. of (1) and simplifying, we find $$x^2 y'' + y' + xy = C_0 k^2 . x^{k-1}. \qquad ...(9)$$

Differentiating both sides of (9) partially w.r.t. 'k' gives

$$\frac{\partial}{\partial k}\left[x^2 \frac{d^2 y}{dx^2} + \frac{dy}{dx} + xy\right] = \frac{C_0}{x}\frac{\partial}{\partial k}(k^2 x^k) \quad \text{or} \quad \left[x^2 \frac{d^2}{dx^2} + \frac{d}{dx} + x\right]\frac{\partial y}{\partial k} = \frac{C_0}{x}[2kx^k + k^2 \cdot x^k \log x]. \; ...(10)$$

The presence of the factor k in each term on R.H.S. of (10) shows that a second solution is $(\partial y/\partial k)_{k=0}$, where y is given by (7). Differentiating partially w.r.t. 'k' (7) gives

$$\partial y / \partial k = C_0 x^k \log x \{1$$
$$- \frac{x^2}{(k+2)^2} + \frac{x^4}{(k+2)^2(k+4)^2} - \frac{x^6}{(k+2)^2(k+4)^2(k+6)^2} + ...\} + c_0 x^k \left\{-\frac{x^2}{(k+2)^2} \times \frac{(-2)}{k+2}\right.$$
$$\left. + \frac{x^4}{(k+2)^2(k+4)^2}\left[-\frac{2}{k+2} - \frac{2}{k+4}\right] - \frac{x^6}{(k+2)^2(k+4)^2(k+6)^2}\left[-\frac{2}{k+2} - \frac{2}{k+4} - \frac{2}{k+6}\right] + ...\right\} ...(11)$$

[To find $\frac{d}{dx}\left[\frac{1}{(k+2)^2(k+4)^2(k+6)^2}\right]$, we have proceeded as follows. Suppose that

$z = 1/[(k+2)^2 (k+4)^2 (k+6)^2]$ so that $\log z = -2\log(k+2) - 2\log(k+4) - 2\log(k+6)$.

Differentiating w.r.t. 'k', we have

$$\frac{1}{z}\frac{dz}{dk} = -\frac{2}{k+2} - \frac{2}{k+4} - \frac{2}{k+6} \qquad \text{or} \qquad \frac{dz}{dk} = z\left(-\frac{2}{k+2} - \frac{2}{k+4} - \frac{2}{k+6}\right),$$

where $z = 1/\{(k+2)^2 (k+4)^2 (k+6)^2\}$; etc. Other terms can be similarly differentiated easily]

Putting $k = 0$ and replacing c_0 by b in (11) gives

$$(\partial y/\partial k)_{k=0} = b\log x\left(1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \ldots\right) + b\left\{\frac{x^2}{2^2} - \frac{x^2}{2^2 \cdot 4^2}\left(1+\frac{1}{2}\right) + \frac{x^2}{2^2 \cdot 4^2 \cdot 6^2}\left(1 + \frac{1}{2} + \frac{1}{3}\right) + \ldots\right\}$$

$$= b\left[u \log x + \left\{\frac{x^2}{2^2} - \frac{x^2}{2^2 \cdot 4^2}\left(1 + \frac{1}{2}\right) + \ldots\right\}\right] = bv \text{, (say)} \qquad \text{[using (8)]}$$

The required solution is $y = au + bv$, where a and b are arbitrary constants.

Ex. 2. *Solve in series* $(x - x^2)y'' + (1 - 5x)y' - 4y = 0$.

[Agra 2009; Delhi Maths (H) 2005; Nagpur 1996, Meerut 1995, 96]

Sol. Given $(x - x^2)y'' + (1 - 5x)y' - 4y = 0$. ...(1)

Dividing by $(x - x^2)$, (1) gives $y'' + \{(1 - 5x)/(x - x^2)\}y' - \{4/(x - x^2)\}y = 0$.

Comparing it with $y'' + P(x)y' + Q(x)y = 0$, here $P(x) = (1 - 5x)/(x - x^2)$ and $Q(x) = -4/(x - x^2)$ so that $xP(x) = (1 - 5x)/(1 - x)$ and $x^2Q(x) = -4x/(1 - x)$. Since $xP(x)$ and $x^2Q(x)$ are analytic at $x = 0$, hence $x = 0$ is a regular singular point of (1).

Let the series solution of (1) be of the form

$$y = \sum_{m=0}^{\infty} C_m x^{k+m}, \qquad \text{where} \qquad C_0 \neq 0. \qquad \ldots(2)$$

$$\therefore\ y' = \sum_{m=0}^{\infty} C_m (k+m)x^{k+m-1} \qquad \text{and} \qquad y'' = \sum_{m=0}^{\infty} C_m (k+m)(k+m-1)x^{k+m-2}. \qquad \ldots(3)$$

Putting the above values of y, y' y'' into (1) gives

$$(x - x^2)\sum_{m=0}^{\infty} C_m (k+m)(k+m-1)x^{k+m-2} + (1-5x)\sum_{m=0}^{\infty} C_m (k+m)x^{k+m-1} - 4\sum_{m=0}^{\infty} C_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} C_m\{(k+m)(k+m-1) + (k+m)\}x^{k+m-1} - \sum_{m=0}^{\infty} C_m\{(k+m)(k+m-1) + 5(k+m) + 4\}x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} C_m (k+m)^2 x^{k+m-1} - \sum_{m=0}^{\infty} C_m\{(k+m)^2 + 4(k+m) + 4\}x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} C_m (k+m)^2 x^{k+m-1} - \sum_{m=0}^{\infty} C_m (k+m+2)^2 x^{k+m} = 0, \qquad \ldots(4)$$

which is an identity. Equating to zero the coefficient of the smallest power of x, namely x^{k-1}, (4) gives the indicial equation

$C_0 k^2 = 0$ or $k^2 = 0$, as $C_0 \neq 0$. Thus $k = 0, 0$ (equal roots).

Finally, equating to zero the coefficient of x^{k+m-1} in (4), we get

$$C_m (k+m)^2 - C_{m-1}(k+m+1)^2 = 0 \qquad \text{or} \qquad C_m = \frac{(k+m+1)^2}{(k+m)^2} C_{m-1} \qquad \ldots(5)$$

Putting $m = 1, 2, 3, \ldots$ in turn in (5) and simplifying $C_1 = [(k+2)^2/(k+1)^2]C_0$,

$C_2 = \frac{(k+3)^2}{(k+2)^2} C_1 = \frac{(k+3)^2}{(k+2)^2} \times \frac{(k+2)^2}{(k+1)^2} C_0 = \frac{(k+3)^2}{(k+1)^2} C_0, \;\; C_3 = \frac{(k+4)^2}{(k+3)^2} C_2 = \frac{(k+4)^2}{(k+3)^2} \times \frac{(k+3)^2}{(k+1)^2} C_0 = \frac{(k+4)^2}{(k+1)^2} C_0$

and so on. Putting these in (2), *i.e.*, $y = x^k(C_0 + C_1x + C_2x^2 + C_3x^3 + ...)$, gives

$$y = x^k C_0 \left\{ 1 + \frac{(k+2)^2}{(k+1)^2} x + \frac{(k+3)^2}{(k+1)^2} x^2 + \frac{(k+4)^2}{(k+1)^2} x^3 + ... \right\}. \qquad ...(6)$$

Putting $k = 0$ and replacing C_0 by a in (6) gives

$$y = a(1 + 2^2x + 3^2x^2 + 4^2x^3 + ...) = au, \text{ say} \qquad ...(7)$$

Differentiating partially w.r.t. 'k', (6) gives

$$\frac{\partial y}{\partial k} = C_0 x^k \log x \left\{ 1 + \frac{(k+2)^2}{(k+1)^2} x + \frac{(k+3)^2}{(k+1)^2} x^2 + \frac{(k+4)^2}{(k+1)^2} x^3 + ... \right\}$$

$$+ C_0 x^k \left\{ \frac{(k+2)^2}{(k+1)^2} x \left[\frac{2}{k+2} - \frac{2}{k+1} \right] + \frac{(k+3)^2}{(k+1)^2} x^2 \left[\frac{2}{k+3} - \frac{2}{k+1} \right] + ... \right\}$$

Putting $k = 0$ and replacing C_0 by b gives

$$(\partial y/\partial k)_{k=0} = b \log x(1 + 2^2x + 3^2x^2 + 4^2x^3 + ...) + b\{2^2 x(1-2) + 3^2 \cdot x^2 \cdot 2(2/3 - 2) + ...\}$$

$$= b[u \log x - 2(1 \cdot 2x + 2 \cdot 3x^2 + ...)] = bv, \text{ say} \qquad [\text{using (7)}]$$

The required solution is $y = au + bv$, where a and b are arbitrary constants.

Ex. 3. *Find the series solution of* $(x - x^2)y'' + (1 - x)y' - y = 0$ near $x = 0$.

[Delhi Maths (Hons.) 1998, 2008; Ravishenker 2010]

Sol. Given $$(x - x^2)y'' + (1 - x)y' - y = 0. \qquad ...(1)$$

As usual, $x = 0$ is a regular singular point of (1). Let its series solution be

$$y = x^k(C_0 + C_1x + C_2x^2 + ...) = \sum_{m=0}^{\infty} C_m x^{k+m}, \text{ where } C_0 \neq 0. \qquad ...(2)$$

$$\therefore \quad y' = \sum_{m=0}^{\infty} (k+m) C_m x^{k+m-1} \quad \text{and} \quad y'' = \sum_{m=0}^{\infty} (k+m)(k+m-1) C_m x^{k+m-2}. \qquad ...(3)$$

Putting the above values of y, y' and y'' in (1), we get

$$(x - x^2) \sum_{m=0}^{\infty} (k+m)(k+m-1) C_m x^{k+m-2} + (1-x) \sum_{m=0}^{\infty} (k+m) C_m x^{k+m-1} - \sum_{m=0}^{\infty} C_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} (k+m)(k+m-1) C_m x^{k+m-1} - \sum_{m=0}^{\infty} (k+m)(k+m-1) C_m x^{k+m}$$

$$+ \sum_{m=0}^{\infty} (k+m) C_m x^{k+m-1} - \sum_{m=0}^{\infty} (k+m) C_m x^{k+m} - \sum_{m=0}^{\infty} C_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} [(k+m)(k+m-1) + (k+m)] C_m x^{k+m-1} - \sum_{m=0}^{\infty} [(k+m)(k+m-1) + (k+m) + 1] C_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} (k+m)^2 C_m x^{k+m-1} - \sum_{m=0}^{\infty} [(k+m)^2 + 1] C_m x^{k+m} = 0, \qquad ...(4)$$

which is an identity is x. Equating to zero the coefficient of the smallest power of x, namely x^{k-1} in (4), the indicial equation is

$C_0 k^2 = 0$ so that $k = 0, 0$ as $C_0 \neq 0$.

Equating to zero the coefficient of x^{k+m-1} in (4), we get

$$(k+m)^2 C_m - \{(k+m-1)^2 + 1\} C_{m-1} = 0, \quad \text{for all} \quad m \geq 1$$

or
$$C_m = \frac{(k+m-1)^2 + 1}{(k+m)^2} C_{m-1}, \qquad \text{for all} \qquad m \geq 1 \qquad \text{...(5)}$$

Putting $m = 1, 2, 3, ...$ in (5), we have $C_1 = \{(k^2+1)/(k+1)^2\} C_0$, ...(6)

$$C_2 = \frac{(k+1)^2+1}{(k+2)^2} C_1 = \frac{(k^2+1)\{(k+1)^2+1\}}{(k+1)^2(k+2)^2} C_0, \text{ by (6)} \qquad \text{...(7)}$$

$$C_3 = \frac{(k+2)^2+1}{(k+3)^2} C_2 = \frac{(k^2+1)\{(k+1)^2+1\}\{(k+2)^2+1\}}{(k+1)^2(k+2)^2(k+3)^2} C_0$$

and so on. Putting these values in (2), we have

$$y = C_0 x^k \left[1 + \frac{k^2+1}{(k+1)^2} x + \frac{(k^2+1)\{(k+1)^2+1\}}{(k+1)^2(k+2)^2} x^2 + \frac{(k^2+1)\{(k+1)^2+1\}\{(k+2)^2+1\}}{(k+1)^2(k+2)^2(k+3)^2} x^3 + ...\right] \qquad \text{...(8)}$$

Differentiating (8) partially w.r.t. 'k', we have

$$\frac{\partial y}{\partial k} = C_0 x^k \log x \left\{1 + \frac{k^2+1}{(k+1)^2} x + \frac{(k^2+1)[(k+1)^2+1]}{(k+1)^2(k+2)^2} x^2 + \frac{(k^2+1)\{(k+1)^2+1\}\{(k+2)^2+1\}}{(k+1)^2(k+2)^2(k+3)^2} x^3 + ...\right\}$$

$$+ C_0 x^k \left\{\left[\frac{2k}{(k+1)^2} - \frac{2(k^2+1)}{(k+1)^3}\right] x + \left[\frac{2k\{(k+1)^2+1\} + 2(k+1)(k^2+1)}{(k+1)^2(k+2)^2}\right.\right.$$

$$\left.\left. - \frac{2(k^2+1)\{(k+1)^2+1\}}{(k+1)^3(k+2)^2} - \frac{2(k^2+1)\{(k+1)^2+1\}}{(k+1)^2(k+2)^3}\right] x^2 + ...\right\} \qquad \text{...(9)}$$

Putting $k = 0$ and replacing C_0 by a in (8), $y = a\left(1 + x + \frac{2}{4}x^2 + \frac{2\cdot 5}{4\cdot 9}x^3 + ...\right) = au$, say.

Putting $k = 0$ and replacing C_0 by b in (9), we get

$$\left(\frac{\partial y}{\partial k}\right)_{k=0} = b \log x \left(1 + x + \frac{2}{4}x^2 + \frac{2\cdot 5}{4\cdot 9}x^3 + ...\right) + b(-2x - x^2 - ....)$$

or
$$\left(\frac{\partial y}{\partial k}\right)_{k=0} = b\left\{\log x \left(1 + x + \frac{2}{4}x^2 + \frac{2\cdot 5}{4\cdot 9}x^3 + ...\right) + (-2x - x^2 - ....)\right\} = bv, \text{ say}$$

The required solution is $y = au + bv$, a, b being arbitrary constants.

Ex. 4. *Solve in series the following differential equations* :

(a) $x(x-1)y'' + (3x-1)y' + y = 0$. **[Delhi Physics (H) 2001]**

(b) $x(x-1)y'' + (3x-1)y' + y = 4x$ **[Jabalpur 2004]**

Sol. (a) Given $(x^2 - x)y'' + (3x-1)y' + y = 0$ ... (1)

As usual show that $x = 0$ is regular singular point. To solve (1), we assume a series solution of the form

$$y = \sum_{m=0}^{\infty} C_m x^{k+m}, \qquad \text{where} \qquad C_0 \neq 0 \qquad \text{... (2)}$$

$$\therefore \quad y' = \sum_{m=0}^{\infty} C_m (k+m) x^{k+m-1} \qquad \text{and} \qquad y'' = \sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2}$$

Substituting the above values of y, y' and y'' in (1), we get

$$(x^2 - x) \sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2} + (3x-1) \sum_{m=0}^{\infty} C_m (k-m) x^{k+m-1} + \sum_{m=0}^{\infty} C_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} C_m \{(k+m)(k+m-1) + 3(k+m) + 1\} x^{k+m} - \sum_{m=0}^{\infty} C_m \{(k+m)(k+m-1) + (k+m)\} x^{k+m-1} = 0$$

or $$\sum_{m=0}^{\infty} C_m (k+m+1)^2 x^{k+m} - \sum_{m=0}^{\infty} C_m (k+m)^2 x^{k+m-1} = 0$$

or $$-C_0 k^2 x^{k-1} + (C_0 - C_1)(k+1)^2 x^k + \ldots\ldots + (C_{m-1} - C_m)(k+m)^2 x^{k+m-1} + \ldots = 0 \qquad \ldots (3)$$

(3) is identity in x. Equating to zero the coefficient of the lowest power of x, namely, x^{k-1}, the identical equation is $\quad -C_0 k^2 = 0 \quad$ giving $\quad k = 0, 0$ (equal roots), $\quad$ as $C_0 \neq 0$

Equating to zero the coefficient of x^{k+m-1}, the recurrence relation is given by

$$C_{m-1}(k+m)^2 - C_m (k+m)^2 = 0, \qquad \text{giving} \qquad C_m = C_{m-1} \text{ for all } m \geq 1 \qquad \ldots (4)$$

Putting $m = 1, 2, 3, \ldots$ in (4), we have

$C_1 = C_0$, $C_2 = C_1 = C_0$, $C_3 = C_2 = C_0$ and so on. Substituting these values in (2), we have

$y = C_0 x^k + C_1 x^{k+1} + C_2 x^{k+2} + \ldots \qquad$ or $\qquad y = C_0 x^k (1 + x + x^2 + \ldots)$

or $$y = C_0 x^k (1 - x)^{-1}. \qquad \ldots (5)$$

Putting $k = 0$ and replacing C_0 by a in (5), we get $\qquad y = a(1 - x)^{-1} \qquad \ldots (6)$

Since the indicial equation has equal roots, so the second independent solution of (1) is given by $(\partial y/\partial k)_{k=0}$ (refer Art. 8.12).

Differentiating (5) partially w.r.t. 'k', yields $\qquad \partial y/\partial k = x^k C_0 (1-x)^{-1} \log x \qquad \ldots (7)$

Putting $k = 0$ and replacing C_0 by b in (7), we get $\quad (\partial y/\partial k)_{k=0} = b(1-x)^{-1} \log x \qquad \ldots (8)$

From (6) and (8), the required solution of (1) is

$$y = a(1-x)^{-1} + b(1-x)^{-1} \log x = (a + b \log x)(1-x)^{-1}, \qquad \ldots (9)$$

where a and b are arbitrary constants.

(b) Given $$(x^2 - x) y'' + (3x-1) y' + y = 4x \qquad \ldots (i)$$

Here $x = 0$ is regular singular point of (i). To solve (1), assume a series solution of the form

$$y = \sum_{m=0}^{\infty} C_m x^{k+m}, \qquad \text{where} \qquad C_0 \neq 0 \qquad \ldots (ii)$$

$$\therefore \quad y' = \sum_{m=0}^{\infty} C_m (k+m) x^{k+m-1} \qquad \text{and} \qquad y'' = \sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-1}$$

Substituting the above values of y, y' and y'' in (i) and proceeding exactly as in part (a), we get

$$-C_0 k^2 x^{k-1} + (C_0 - C_1)(k+1)^2 x^k + \ldots + (C_{m-1} - C_m)(k+m)^2 x^{k+m-1} + \ldots = 4x \qquad \ldots (iii)$$

To determine complementary function of (*i*) Setting the left number of (iii) equal to zero,

$$-C_0 k^2 x^{k-1} + (C_0 - C_1)(k+1)^2 x^k + \ldots + (C_{m-1} - C_m)(k+m)^2 x^{k+m-1} + \ldots = 0 \qquad \ldots (iv)$$

Starting with (iv), we now proceed as in part (a) upto equation (8) of part (a). Thus, we get

$$C_m = C_{m-1}, \qquad \text{for all} \qquad m \geq 1 \qquad \text{... }(v)$$

and $$y = C_0 x^k (1-x)^{-1} \qquad \text{... }(vi)$$

which satisfies $$(x^2 - x)y'' + (3x-1)\, y' + y = -C_0 k^2 x^{k-1} \qquad \text{... }(vii)$$

The required complementary function (C.F.) is given by

$$\text{C.F. }(i) = a(1-x)^{-1} + b(1-x)^{-1} \log x = (a + b \log x)\,(1-x)^{-1}$$

To determine particular integral of (*i*) To this end we shall consider the term of the right member of the given equation (i). Thus, setting the right member of (vii) equal to $4x$, that is,

$-C_0 k^2 x^{k-1} = 4x$, identically, we have $k = 2$ and $C_0 = -1$.

The recurrence relation (v) yields $C_1 = C_2 = C_3 = \dots = C_0$, as before

Putting $k = 2$, $C_0 = -1$ and $C_1 = C_2 = C_3 = \dots = C_0$ in (ii), we have

$$\text{P. I. of }(i) = \sum_{m=0}^{\infty} C_m x^{2+m} = x^2(C_0 + C_1 x + C_2 x^2 + \dots) = x^2\, C_0\, (1 + x + x^2 + \dots) = -\,x^2\,(1-x)^{-1}$$

Hence the required complete integral of (i) is $y = \text{C.F.} + \text{P.I.}$, *i.e.*,

$y = (a + b \log x)\,(1-x)^{-1} - x^2\,(1-x)^{-1}$, a, b being arbitrary constants.

EXERCISE 8 (F)

Find the series solutions of the following equations near $x = 0$.

1. $xy'' + (1 + x)y' + 2y = 0$. **[Garhwal 2005]**

Ans. $y = a(1 - 2x + (3/2!)x^2 - (4/3!)x^3 + \dots) + b[\log x\{1 - 2x + (3/2!) \times x^2 - (4/3!) \times x^3 + \dots\}$
$+ 2(2 - 1/2)x - (3/2!) \times (-1/3 + 2 + 1/2)x^2 + \dots\}$

2. $x^2y'' - x(1 + x)y' + y = 0$. **[Delhi Maths (Hons.) 1998]**

Ans. $y = ax[1 + x + (1/2) \times x^2 + (1/2\cdot 3) \times x^3 + \dots] + b[\log x\ \{1 + x + (1/2) \times x^2$
$+ (1/2\cdot 3) \times x^3 + \dots\} + x^2\ \{-1 - (3/4) \times x + \dots\}]$

3. $4(x^4 - x^2)y'' + x^3y' - y = 0$.

Ans. $$y = ax^{1/2}\left(1 + \frac{1\cdot 3}{4^2}x^2 + \frac{1\cdot 3\cdot 5\cdot 7}{4^2\cdot 8^2}x^4 + \dots\right) + b\left[x^{1/2}\log x\left(1 + \frac{1\cdot 3}{4^2}x^2 + \frac{1\cdot 3\cdot 5\cdot 7}{4^2\cdot 8^2}x^4 + \dots\right)\right.$$
$$\left. + 2x^{1/2}\left\{\frac{1\cdot 3}{4^2}\left(1 + \frac{1}{3} - \frac{1}{2}\right)x^2 + \frac{1\cdot 3\cdot 5\cdot 7}{4^2\cdot 8^2}\left(1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4}\right)x^4 + \dots\right\}\right]$$

4. $xy'' + (p - x)y' - y = 0$ when (i) $p = 1$ (ii) p is not an integer.

Ans. (i) $$y = a\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right) + b\left[\log x\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots\right)\right.$$
$$\left. - \left\{x + \frac{x^2}{1\cdot 2}\left(1 + \frac{1}{2}\right) + \frac{x^3}{1\cdot 2\cdot 3}\left(1 + \frac{1}{2} + \frac{2}{3}\right) + \dots\right\}\right]$$

(ii) $$y = a\left(1 + \frac{x}{p} + \frac{x^2}{p(p+1)} + \dots\right) + b\left\{\log x\left(1 + \frac{x}{p} + \frac{x^2}{p(p+1)} + \dots\right) + x^{1-p}\left(1 + x/1! + x^2/2! + \dots\right)\right.$$

5. $xy'' + y' + x^2y = 0$. **[Delhi Maths (Hons.) 2001]** **Ans.** $y = au + bv$, where

$$u = 1 - \frac{x^3}{3^2} + \frac{x^6}{3^4(2!)^2} - \frac{x^9}{3^6(3!)^2} + \dots \quad \text{and} \quad v = u \log x + 2\left[\frac{x^3}{3^3} - \frac{1}{3^5(2!)^2}\left(1 + \frac{1}{2}\right)x^6 + \dots\right]$$

6. $xy'' + y' - y = 0.$ **[G.N.D.U. Amritsar 2010]**

7. $xy'' + y' + y = 0$ **[Meerut 1992]**

8. $x(1 - x^2)y'' + (1 - 3x^2)y' - xy = 0.$ **[Meerut 1994]**

8.13. Series solution about regular singular point at infinity.

Suppose the given equation is $\qquad y'' + P(x)\, y' + Q(x)\, y = 0.$ ...(1)

We propose to find the solution of (1) for large values of the independent variable, *i.e.,* about $x = \infty$. For this purpose we change the independent variable from x to t with help of the following transformation : $\qquad x = 1/t \qquad$ *i.e.* $\qquad t = 1/x.$ ...(2)

Clearly large values of x correspond to small values of t. Using (2), we re–write (1) and obtain the transformed equation near $t = 0$, say

$$d^2y/dt^2 + P_1(t)\,(dy/dt) + Q_1(t)\, y = 0. \qquad ...(3)$$

Then the given equation (1) is said to have regular singular point at $x = \infty$ if the transformed equation (3) has regular singular at $t = 0$.

Working rule for solving (1) near $x = \infty$. Assume transformation (2). With help of (2), we obtain the transformed equation (3). We verify that $t = 0$ is regular singular point of (3). Solve (3) by Frobenius method (refer Art. 8.8). Finally putting $t = 1/x$ in the solution, we obtain the required general solution of (1) near $x = \infty$.

8.14. Solved examples based on Art. 8.13

Ex. 1. *Show that $x = \infty$ is a regular singular point of $x^2y'' + 4xy' + 2y = 0$.*

Sol. Given $\qquad x^2y'' + 4xy' + 2y = 0.$ ...(1)

Let $\quad x = 1/t \quad$ or $\quad t = 1/x \quad$ so that $\quad dt/dx = -1/x^2.$ ...(2)

Now, $$y' = \frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\left(-\frac{1}{x^2}\right) = -t^2\frac{dy}{dt}, \text{ by (2)} \qquad ...(3)$$

and $$y'' = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \frac{d}{dt}\left(-t^2\frac{dy}{dt}\right)\times\left(-\frac{1}{x^2}\right), \text{ by (2) and (3)}$$

or $$y'' = \left(-t^2\frac{d^2y}{dt^2} - 2t\frac{dy}{dt}\right)(-t^2) = t^4\frac{d^2y}{dt^2} + 2t^3\frac{dy}{dt}. \qquad ...(4)$$

Using (2), (3) and (4), (1) transforms to $$\frac{1}{t^2}\left(t^4\frac{d^2y}{dt^2} + 2t^3\frac{dy}{dt}\right) + \frac{4}{t}\left(-t^2\frac{dy}{dt}\right) + 2y = 0$$

or $$(d^2y/dt^2) - (2/t)(dy/dt) + (2/t^2)y = 0. \qquad ...(5)$$

Comparing (5) with $(d^2y/dt^2) + P(t)(dy/dt) + Q(t)y = 0$, here $P(t) = -2/t$ and $Q(t) = 2/t^2$ so that $tP(t) = -2$ and $t^2Q(t) = 2$. Since $tP(t)$ and $t^2Q(t)$ are the both analytic at $t = 0$, so $t = 0$ is a regular singular point of (5). In view of (2), $x = \infty$ is a regular singular point of (1).

Ex. 2. *Find the power series solution of* $(1 - x^2)(d^2y/dt^2) - 2x(dy/dx) + 6y = 0$ about $x = \infty$. **[Delhi Maths (Hons.) 1997]**

Sol. Given $\qquad (1 - x^2)(d^2y/dx^2) - 2x(dy/dx) + 6y = 0.$ ...(1)

Let $\quad x = 1/t \quad$ or $\quad t = 1/x \quad$ so that $\quad dt/dx = -1/x^2.$ ...(2)

Now, $$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\left(-\frac{1}{x^2}\right) = -t^2\frac{dy}{dt}, \text{ by (2).} \qquad ...(3)$$

and $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \frac{d}{dt}\left(-t^2\frac{dy}{dt}\right)\times\left(-\frac{1}{x^2}\right), \text{ by (2) and (3).}$$

or $$\frac{d^2y}{dx^2} = t^2\frac{d}{dt}\left(t^2\frac{dy}{dt}\right) = t^2\left(2t\frac{dy}{dt} + t^2\frac{d^2y}{dt^2}\right). \qquad ...(4)$$

Using (2), (3) and (4), (1) is transformed to

$$\left(1-\frac{1}{t^2}\right)\left(2t^3\frac{dy}{dt} + t^4\frac{d^2y}{dt^2}\right) - \frac{2}{t}\left(-t^2\frac{dy}{dt}\right) + 6y = 0$$

or $$t^2(t^2-1)(d^2y/dt^2) + 2t^3(dy/dt) + 6y = 0. \qquad ...(5)$$

Dividing by $t^2(t^2-1)$, $\quad d^2y/dt^2 + \{2t/(t^2-1)\}(dy/dt) + [6/\{t^2(t^2-1)\}]y = 0.$

Comparing it with $(d^2y/dt^2) + P(t)\ (dy/dt) + Q(t)y = 0$, here $P(t) = 2t/(t^2-1)$ and $Q(t) = 6/[t^2(t^2-1)]$ so that $tP(t) = 2t^2/(t^2-1)$ and $t^2Q(t) = 6/(t^2-1)$. Since $tP(t)$ and $t^2Q(t)$ are both analytic at $t = 0$, so $t = 0$ is a regular singular point of (5). To solve (5), let its series solution be

$$y = t^k(C_0 + C_1t + C_2t^2 + ...) = \sum_{m=0}^{\infty} C_m t^{k+m}, \quad \text{where} \quad C_0 \neq 0. \qquad ...(6)$$

$$\therefore \quad dy/dt = \sum_{m=0}^{\infty}(k+m)C_m t^{k+m-1} \quad \text{and} \quad d^2y/dt^2 = \sum_{m=0}^{\infty}(k+m)(k+m-1)C_m t^{k+m-2}.$$

Putting these values of y, dy/dt and d^2y/dt^2 in (5), we get

$$(t^4 - t^2)\sum_{m=0}^{\infty}(k+m)(k+m-1)C_m t^{k+m-2} + 2t^3\sum_{m=0}^{\infty}(k+m)C_m t^{k+m-1} + 6\sum_{m=0}^{\infty}C_m t^{k+m} = 0$$

or $$\sum_{m=0}^{\infty}(k+m)(k+m-1)C_m t^{k+m-2} - \sum_{m=0}^{\infty}(k+m)(k+m-1)C_m t^{k+m} + \sum_{m=0}^{\infty}2(k+m)C_m t^{k+m+2} + 6\sum_{m=0}^{\infty}C_m t^{k+m} = 0$$

or $$-\sum_{m=0}^{\infty}\{(k+m)(k+m-1) - 6\}C_m t^{k+m} + \sum_{m=0}^{\infty}\{(k+m)(k+m-1) + 2(k+m)\}C_m t^{k+m+2} = 0$$

or $$\sum_{m=0}^{\infty}\{(k+m)^2 - (k+m) - 6\}C_m t^{k+m} - \sum_{m=0}^{\infty}\{(k+m)^2 + (k+m)\}C_m t^{k+m+2} = 0$$

or $$\sum_{m=0}^{\infty}(k+m-3)(k+m+2)C_m t^{k+m} - \sum_{m=0}^{\infty}(k+m)(k+m+1)C_m t^{k+m+2} = 0, \qquad ...(7)$$

which is an identity. Equating to zero the coefficient of the smallest power of t, namely t^k, (7) gives the indicial equation $C_0(k-3)(k+2) = 0$ so that $k = 3$ and $k = -2$ as $C_0 \neq 0$. The roots of indicial equation are unequal and differ by an integer.

Next equating to zero the coefficient of t^{k+1} in (7), we get $(k-2)(k+3)C_1 = 0$ giving $C_1 = 0$ for both $k = 3$ and $k = -2$. Finally, equating to zero the coefficient of t^{k+m} in (4), we get

$$(k+m-3)(k+m+2)C_m - (k+m-2)(k+m-1)C_{m-2} = 0$$

or $$C_m = \frac{(k+m-2)(k+m-1)}{(k+m-3)(k+m+2)}C_{m-2} \quad \text{for all } m \geq 2. \qquad ...(8)$$

Putting $m = 3, 5, 6, ...$ in (8) and noting that $C_1 = 0$, we get

$$C_1 = C_3 = C_5 = C_7 = ... = 0. \qquad ...(9)$$

Next, putting $m = 2, 4, 6, ...$ in (8), we have $\quad C_2 = \dfrac{k(k+1)}{(k-1)(k+4)}C_0,$

$$C_4 = \frac{(k+2)(k+3)}{(k+1)(k+6)}C_2 = \frac{k(k+1)(k+2)(k+3)}{(k-1)(k+1)(k+4)(k+6)}C_0, \text{ using (10)}$$

or $$C_4 = \frac{k(k+2)(k+3)}{(k-1)(k+4)(k+6)} C_0, \qquad \text{since } (k+1) \neq 0.$$

and so on. Putting these values in (6), we have

$$y = t^k C_0 \left[1 + \frac{k(k+1)}{(k-1)(k+4)} x^2 + \frac{k(k+2)(k+3)}{(k-1)(k+4)(k+6)} x^4 + \ldots\right]. \qquad \ldots(11)$$

Putting $k = 3$ in (11) and replacing C_0 by a, we get

$$y = at^3 \left[1 + \frac{3 \cdot 4}{2 \cdot 7} t^2 + \frac{3 \cdot 5 \cdot 6}{2 \cdot 7 \cdot 9} t^4 + \ldots\right] = au \text{, say} \qquad \ldots(12)$$

Putting $k = -2$ in (11) and replacing C_0 by b, we get $y = bt^{-2}[1 - t^2/3] = bv$, say. ...(13)

The required solution is $y = au + bv$, *i.e.*

$$y = at^3 \left(1 + \frac{3 \cdot 4}{2 \cdot 7} t^2 + \frac{3 \cdot 5 \cdot 6}{2 \cdot 7 \cdot 9} t^4 + \ldots\right) + \frac{b}{t^2}\left(1 - \frac{t^2}{3}\right)$$

i.e., $$y = \frac{a}{x^3}\left(1 + \frac{3 \cdot 4}{2 \cdot 7}\frac{1}{x^2} + \frac{3 \cdot 5 \cdot 6}{2 \cdot 7 \cdot 9}\frac{1}{x^4} + \ldots\right) + bx^2\left(1 - \frac{1}{3x^2}\right), \text{ as, } t = \frac{1}{x}.$$

Ex. 3. *Find the power series solution of* $(1 - x^2)(d^2y/dx^2) - 2x(dy/dx) + 12y = 0$ at $x = \infty$.

[Delhi Maths (Hons.) 1998]

Hint. Proceed just like Ex. 2. Its general solution is

$$y = ax^3(1 - 3/5x^2) + (b/x^4) \times (1 + 10/9x^2 + 35/33x^4 + \ldots).$$

Ex. 4. *Solve in series the equation* $x^3y'' + x(1 - x)y' + y = 0$ near $x = \infty$.

Sol. Given $\quad x^3(d^2y/dx^2) + x(1 - x)(dy/dx) + y = 0.$ ...(1)

Let $\quad x = 1/t \quad$ or $\quad t = 1/x \quad$ so that $\quad dt/dx = -1/x^2.$ ...(2)

Now, $$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\left(-\frac{1}{x^2}\right) = -t^2\frac{dy}{dt}, \text{ by (2).} \qquad \ldots(3)$$

and $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx} = \frac{d}{dt}\left(-t^2\frac{dy}{dt}\right) \times \left(-\frac{1}{x^2}\right), \text{ by (2) and (3)}$$

or $$\frac{d^2y}{dx^2} = t^2\frac{d}{dt}\left(t^2\frac{dy}{dt}\right) = t^2\left(2t\frac{dy}{dt} + t^2\frac{d^2y}{dt^2}\right). \qquad \ldots(4)$$

Using (2), (3) and (4), (1) is transformed to

$$\frac{1}{t^3}\left(2t^3\frac{dy}{dt} + t^4\frac{d^2y}{dt^2}\right) + \frac{1}{t}\left(1 - \frac{1}{t}\right)\left(-t^2\frac{dy}{dt}\right) + y = 0$$

$t(d^2y/dt^2) + 2(dy/dt) - (t - 1)(dy/dt) + y = 0 \quad$ or $\quad t(d^2y/dt^2) + (3 - t)(dy/dt) + y = 0.$ ...(5)

We now solve (5) in series about $t = 0$ which is a regular singular point of (5). (Prove yourself as in solved Ex. 2). Let a series solution of (5) be

$$y = t^k(C_0 + C_1t + C_2t^2 + \ldots) = \sum_{m=0}^{\infty} C_m t^{k+m}, \text{ where } C_0 \neq 0. \qquad \ldots(6)$$

$$\therefore \quad dy/dt = \sum_{m=0}^{\infty}(k+m)C_m t^{k+m-1} \quad \text{and} \quad d^2y/dt^2 = \sum_{m=0}^{\infty}(k+m)(k+m-1)C_m t^{k+m-2}.$$

Putting these values of y, dy/dt and d^2y/dt^2 in (5), we get

$$t\sum_{m=0}^{\infty}(k+m)(k+m-1)C_m t^{k+m-2}+(3-t)\sum_{m=0}^{\infty}(k+m)C_m t^{k+m-1}+\sum_{m=0}^{\infty}C_m t^{k+m}=0$$

or $$\sum_{m=0}^{\infty}(k+m)(k+m-1)C_m t^{k+m-1}+\sum_{m=0}^{\infty}3(k+m)C_m t^{k+m-1}-\sum_{m=0}^{\infty}(k+m)C_m t^{k+m}+\sum_{m=0}^{\infty}C_m t^{k+m}=0$$

or $$\sum_{m=0}^{\infty}\{(k+m)(k+m-1)+3(k+m)\}C_m t^{k+m-1}-\sum_{m=0}^{\infty}\{(k+m)-1\}C_m t^{k+m}=0$$

or $$\sum_{m=0}^{\infty}(k+m)(k+m+2)C_m t^{k+m-1}-\sum_{m=0}^{\infty}(k+m-1)C_m t^{k+m}=0, \qquad ...(7)$$

which is an identity. Equating to zero the coefficients of the smallest power of t, namely t^{k-1}, (7) gives the indicial equation $C_0 k(k+2)=0$ so that $k=0,-2$ as $C_0 \neq 2$.

Equating to zero the coefficients of t^{k+m-1} in (7), we get

$$(k+m)(k+m+2)C_m-(k+m-2)C_{m-1}=0$$

so that $$C_m=\frac{k+m-2}{(k+m)(k+m+2)}C_{m-1}, \quad \text{for all} \quad m \geq 1. \qquad ...(8)$$

Putting $m = 1, 2, 3, 4, ...$ in (8) we get $$C_1=\frac{k-1}{(k+1)(k+3)}C_0, \qquad ...(9)$$

$$C_2=\frac{k}{(k+2)(k+4)}C_1=\frac{k(k-1)}{(k+1)(k+2)(k+3)(k+4)}C_0, \text{ using (9)} \qquad ...(10)$$

$$C_3=\frac{k+1}{(k+3)(k+5)}C_2=\frac{k(k-1)}{(k+2)(k+3)^2(k+4)(k+5)}C_0, \text{ using (10)} \qquad ...(11)$$

$$C_4=\frac{k+2}{(k+4)(k+6)}C_3=\frac{k(k-1)}{(k+3)^2(k+4)^2(k+5)(k+6)}C_0, \text{ using (11)}$$

and so on. Putting these values in (6), we have

$$y=C_0 t^k\left[1+\frac{k-1}{(k+1)(k+3)}t+\frac{k(k-1)}{(k+1)(k+2)(k+3)(k+4)}t^2\right.$$
$$\left.+\frac{k(k-1)}{(k+2)(k+3)^2(k+4)(k+5)}t^3+\frac{k(k-1)}{(k+3)^2(k+4)^2(k+5)(k+6)}t^4+\dots\right] \qquad ...(12)$$

If we put $k=-2$ in (12), then the coefficients of t^2 and t^3 become infinite. To get rid of this difficulty, we put $C_0=d_0(k+2)$ in (12) and obtain the modified solution

$$y=d_0 t^k\left[(k+2)+\frac{(k-1)(k+2)}{(k+1)(k+3)}t+\frac{k(k-1)}{(k+1)(k+3)(k+4)}t^2\right.$$
$$\left.+\frac{k(k-1)}{(k+3)^2(k+4)(k+5)}t^3+\frac{k(k-1)(k+2)}{(k+3)^2(k+4)^2(k+5)(k+6)}t^4+\dots\right]. \qquad ...(13)$$

Differentiating (13) partially w.r.t. 'k', we get

$$\frac{\partial y}{\partial k}=d_0 t^k \log t\left[(k+2)+\frac{(k-1)(k+2)}{(k+1)(k+3)}t+\frac{k(k-1)}{(k+1)(k+3)(k+4)}t^2+\frac{k(k-1)}{(k+3)^2(k+4)(k+5)}t^3\right.$$

$$+\frac{k(k-1)(k+2)}{(k+3)^2(k+4)^2(k+5)(k+6)}t^4+\ldots\Bigg]+d_0t^k\Bigg[1+\left\{\frac{2k+1}{(k+1)(k+3)}-\frac{(k-1)(k+2)}{(k+1)^2(k+3)}-\frac{(k-1)(k+2)}{(k+1)(k+3)^2}\right\}t$$

$$+\left\{\frac{2k-1}{(k+1)(k+3)(k+4)}-\frac{k(k-1)}{(k+1)^2(k+3)(k+4)}-\frac{k(k-1)}{(k+1)(k+3)^2(k+4)}-\frac{k(k-1)}{(k+1)(k+3)(k+4)^2}\right\}t^2$$

$$+\left\{\frac{2k-1}{(k+3)^2(k+4)(k+5)}-\frac{2k(k-1)}{(k+3)^3(k+4)(k+5)}-\frac{k(k-1)}{(k+3)^2(k+4)^2(k+5)}-\frac{k(k-1)}{(k+3)^2(k+4)(k+5)^2}\right\}t^3+\ldots\Bigg] \quad \ldots(14)$$

Putting $k=-2$ in (13) and replacing do by a, $\quad y = at^{-2}(-3t^2+t^3) = au$, say $\quad \ldots(15)$

Putting $k=-2$ in (14) and replacing *do* by b, we get

$$(\partial y/\partial t)_{t=-2} = bt^{-2}(-3t^2+t^3)\log t + bt^{-2}[1+3t+4t^2-(11/3)t^3+\ldots]$$

or $\quad (\partial y/\partial t)_{t=-2} = bt^{-2}[(-3t^2+t^3)\log t+1+3t+4t^2-(11/3)t^3+\ldots) = bv$, say. $\quad \ldots(16)$

Finally putting $k=0$ in (13), $\quad y = d_0\,[2-(2/3)t] = (-2/3)d_0\,t^{-2}(-3t^2+t^3)$. $\quad \ldots(17)$

From (15) and (17), we see that these solutions are not independent. So out of three solutions (15), (16) and (17) only two are linearly independent, namely (15) and (16). So the required solution is

$$y = au+bv, \quad i.e. \quad y = at^{-2}(-3t^2+t^3)+bt^{-2}[(-3t^2+t^3)\log t+1+3t+4t^2-(11/3)t^3+\ldots]$$

or $\quad y = ax^2\left(-\dfrac{3}{x}+\dfrac{1}{x^3}\right)+bx^2\left[\left(-\dfrac{3}{x}+\dfrac{1}{x^3}\right)\log\dfrac{1}{x}+1+\dfrac{3}{x}+\dfrac{4}{x^2}-\dfrac{11}{3x^3}+\ldots\right], \quad$ as $\quad t=\dfrac{1}{x}$

Ex. 5. *Find the solution in series of the equation* $(1-x^2)(d^2y/dx^2)+2x(dy/dx)-y=0$ about $x=\infty$. **[Delhi Maths (Hons.) 1995]**

8.15. Series solution in descending powers of independent variable.

So far we obtained series solutions in ascending powers of the independent variable. However, the following cases may arise :

(*i*) There exists no solution of the form $\sum_{m=0}^{\infty} C_m x^{k+m}$

(*ii*) The usual Frobenius method may break down.

(*iii*) The series solution obtained by earlier methods does not converge within a particular range of values of independent variable.

In such cases we obtain series solution in descending powers of independent variable. Sometimes the series solutions in descending powers are desirable and are more useful in practice.

Working Rule. For details see Art. 8.8. However, the following changes should be noted carefully.

(*i*) We assume a trial solution of the form $\quad y = x^k(c_0+c_1x^{-1}+c_2x^{-2}+\ldots) = \sum_{m=0}^{\infty} c_m x^{k-m}$, $c_0 \neq 0$

(*ii*) For indicial equation the coefficient of the highest power of x in the identity is equated to zero.

(*iii*) For recurrence relation the coefficient of the higher power, in the identity is equated to zero.

Another Working Rule. In order to get a series solution in descending powers of x of the differential equation $\quad f(x)(d^2y/dx^2)+g(x)(dy/dx)+h(x)y=0, \quad \ldots(1)$

we change the variable from x to t by putting $x=1/t$ so that

$$dy/dx = (dy/dt)(dt/dx) = (dy/dt)(-1/x^2) = -t^2\,(dy/dt) \quad \ldots(2)$$

From (2), $\quad d/dx \equiv -t^2(d/dt). \quad \ldots(3)$

$$\therefore \quad \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = -t^2\frac{d}{dt}\left(-t^2\frac{dy}{dt}\right), \text{ using (2) and (3)}$$

or
$$\frac{d^2y}{dx^2} = -t^2\left[-2t\frac{dy}{dt} - t^2\frac{d^2y}{dt^2}\right] = t^4\frac{d^2y}{dt^2} + 2t^3\frac{dy}{dt}. \qquad ...(4)$$

Using $x = 1/t$, (2) and (4), (1) transforms in the form
$$F(t)(d^2y/dt^2) + G(t)(dy/dt) + H(t)y = 0. \qquad ...(5)$$

We now solve (5) by assuming series solution in the form $\quad y = \sum_{m=0}^{\infty} c_m t^{k+m}$

Proceed as explained in Art. 8.8. After getting solution replace t by $1/x$ to get the desired series in descending powers of x.

8.16. Solved examples based on Art. 8.15

Ex. 1. *Integrate in descending series the Legendre's equation.*

Sol. The differential equation of the form $\quad (1 - x^2)y'' - 2xy' + n(n + 1)y = 0 \quad ...(1)$ is called Legendre's equation, where n is a positive integer.

Let the series solution of (1) be of the form $\quad y = \sum_{m=0}^{\infty} c_m x^{k-m}, \quad$ where $\quad c_0 \neq 0. \quad ...(2)$

$$\therefore \quad y' = \sum_{m=0}^{\infty} c_m(k-m)x^{k-m-1} \quad \text{and} \quad y'' = \sum_{m=0}^{\infty} c_m(k-m)(k-m-1)x^{k-m-2}. \qquad ...(3)$$

Putting the above values of y, y', y'' in (1) gives

$$(1-x^2)\sum_{m=0}^{\infty} c_m(k-m)(k-m-1)x^{k-m-2} - 2x\sum_{m=0}^{\infty} c_m(k-m)x^{k-m-1} + n(n+1)\sum_{m=0}^{\infty} c_m x^{k-m} = 0$$

or
$$\sum_{m=0}^{\infty} c_m(k-m)(k-m-1)x^{k-m-2} - \sum_{m=0}^{\infty}\{(k-m)(k-m-1) + 2(k-m) - n(n+1)\}x^{k-m} = 0$$

or
$$\sum_{m=0}^{\infty} c_m(k-m)(k-m-1)x^{k-m-2} - \sum_{m=0}^{\infty} c_m\{(k-m)^2 - n^2 + (k-m) - n\}x^{k-m} = 0$$

or
$$\sum_{m=0}^{\infty} c_m(k-m)(k-m-1)x^{k-m} - \sum_{m=0}^{\infty} c_m(k-m-n)(k-m+n+1)x^{k-m} = 0 \qquad ...(4)$$

which is an identity. Equating to zero the coefficient of the highest power of x, namely x^k, (4) gives the indicial equation $\quad c_0(k - n)(k + n + 1) = 0 \quad$ or $\quad (k - n)(k + n + 1) = 0 \quad [\because c_0 \neq 0]$ giving $k = n, -(n + 1)$. For recurrence relation, we equate to zero the coefficient of x^{k-m} and obtain

$$c_{m-2}[k - (m - 1)][k - (m - 2) - 1] - c_m(k - m - n)(k - m + n + 1) = 0$$

or
$$c_{m-2}(k - m + 2)(k - m + 1) + c_m(n - k + m)(n + k - m + 1) = 0$$

or
$$c_m = -\frac{(k-m+2)(k-m+1)}{(n-k+m)(n+k-m+1)}c_{m-2}. \qquad ...(5)$$

Equating to zero the coefficient of x^{k-1} in (4) gives $c_1(k - 1 - n)(k + n) = 0$ so that $c_1 = 0$ for both $k = n$ and $k = -(n + 1)$. Using $c_1 = 0$ and (5), we note that

$$c_1 = c_3 = c_5 = c_7 = ... = 0 \qquad ...(6)$$

Putting $m = 2$ in (5) gives
$$c_2 = -\frac{k(k-1)}{(n-k+2)(n+k-1)}c_0 \qquad ...(7)$$

Putting $m = 4$ in (5) and using (7) gives

$$c_4 = -\frac{(k-2)(k-3)}{(n-k+4)(n+k-3)}c_2 = \frac{k(k-1)(k-2)(k-3)}{(n-k+2)(n+k-1)(n-k+4)(n+k-3)}c_0$$

and so on. Putting these values in (2), *i.e.* $y = c_0x^k + c_1x^{k-1} + c_2x^{k-2} + c_3x^{k-3} + c_4x^{k-4} + ...$, gives

$$y = c_0\left[x^k - \frac{k(k-1)x^{k-2}}{(n-k+2)(n+k-1)} + \frac{k(k-1)(k-2)(k-3)x^{k-4}}{(n-k+2)(n+k-1)(n-k+4)(n+k-3)} + \dots\right]$$

Putting $k = n$ and $-(n+1)$ in turn and replacing c_0 by a and b respectively, the above equation gives

$$y = a\left[x^n - \frac{n(n-1)}{2(2n-1)}x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2\cdot 4(2n-1)(2n-3)}x^{n-4} - \cdots\right] = au, \text{ say} \qquad ...(8)$$

$$y = b\left[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2\cdot 4(2n+3)(2n+5)}x^{-n-5} + \dots\right] = bv, \text{ say} \qquad ...(9)$$

The required solution is $y = au + bv$, a, b being arbitrary constants.

Ex. 2. *Find a series solution in descending powers of x of* $4x^3y'' + 6x^2y' + y = 0$.

Sol. Let t be another independent variable such that $x = 1/t$. Then we have (See Art. 8.15)

$$\frac{4}{t^3}\left(t^4\frac{d^2y}{dt^2} + 2t^3\frac{dy}{dt}\right) + \frac{6}{t^2}\left(-t^2\frac{dy}{dt}\right) + y = 0 \qquad \text{or} \qquad 4t\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 0.$$

Let solution of (2) be the form $\quad y = \sum_{m=0}^{\infty} c_m t^{k+m}, \quad$ where $\quad c_0 \neq 0$.

Now, proceed as explained in Art. 8.8 and obtain the required solution

$$y = a(1 - t/2! + t^2/4! + ...) + bt^{1/2}(1 - t/3! + t^2/5! - ....), \text{ where } t = 1/x.$$

8.17. Method of differentiation. In this connection, remember the following two results.

(*i*) **Maclaurin's theorem.** $\quad y = (y)_0 + \frac{x}{1!}(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \dots$

(*ii*) **Leibnitz theorem.** If u and v be two functions, then

$D^n(uv) = u_n\ v + {}^nc_1\ u_{n-1}v_1 + {}^nc_2\ u_{n-2}\ v_2 + ... + u\ v_n$, where $u_n = d^nu/dx^n$ and $v_n = d^nv/dx^n$.

Ex. 1. *Solve* $(1 - x^2)y_2 - xy_1 + m^2y = 0$ *where* $x = 0$, $y = 0$, $dy/dx = m$. **[Meerut 1995]**

Sol. Given that $\quad (1 - x^2)y_2 - xy_1 + m^2y = 0. \qquad ...(1)$

Differentiating (1) n times by Leibnitz's theorem w.r.t. 'x', we have

$$y_{n+2}(1-x^2) + {}^nC_1\ y_{n+1}(-2x) + {}^nC_2\ y_n(-2) - [y_{n+1}x + {}^nC_1\ y_n.1] + m^2y_n = 0$$

or
$$(1-x^2)y_{n+2} - 2nx\,y_{n+1} - \frac{n(n-1)}{2}y_n \times 2 - xy_{n+1} - ny_n + m^2y_n = 0$$

or
$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} + (m^2 - n^2)y_n = 0. \qquad ...(2)$$

Given that $\quad (y)_0 = 0 \quad$ and $\quad (y_1)_0 = m. \qquad ...(3)$

Putting $x = 0$ in (1), $\quad (y_2)_0 = -m^2(y)_0 = 0$, by (3). $\qquad ...(4)$

Putting $x = 0$ in (2), $\quad (y_{n+2})_0 = (n^2 - m^2)(y_n)_0. \qquad ...(5)$

Putting $n = 2, 4, 6, ...$ in (5) and noting that $(y_2)_0 = 0$, we get

$$(y_2)_0 = (y_4)_0 = (y_6)_0 = ... = 0. \qquad ...(6)$$

Putting $n = 1, 3, 5, ...$ in (5), we get

$$(y_3)_0 = (1^2 - m^2)(y_1)_0 = (1^2 - m^2)m, \text{ by (3)} \qquad ...(7)$$

$$(y_5)_0 = (3^2 - m^2)(y_3)_0 = (3^2 - m^2)(1^2 - m^2)m, \text{ by (7)}$$

and so on. Now, from Maclaurin's theorem, $y = (y)_0 + \frac{x}{1!}(y_1)_0 + \frac{x^2}{2!}(y_2)_0 + \frac{x^3}{3!}(y_3)_0 + \dots$

or $y = mx + (1^2 - m^2)m(x^3/3!) + (3^2 - m^2)(1^2 - m^2)m(x^5/5!) + \dots$

Ex. 2. *Solve* $(1 - x^2)y_2 - xy_1 - a^2y = 0$, where $x = 0$, $y = 1$ and $dy/dx = 0$.

Hint. Do as in Ex. 1. **Ans.** $y = 1 + ax + (a^2/2!)x^2 + a(a^2 + 1^2)(x^3/3!) + \dots$

Ex. 3. *Solve* $(1 + x^2)y_2 + 2xy_1 = 0$ where $x = 0$, $y_1 = 1$, $y = 0$.

Hint. Do as in Ex. 1. **Ans.** $y = x - (x^3/3) + (x^5/5) - (x^7/7) + \dots$

Ex. 4. *Solve* $xy_2 + y_1 + my = 0$ given that $y = 1$ when $x = 0$. **[Meerut 1994, 95]**

Sol. Put $x = 0$, $y = 1$ in given equation, we get $y_1 = 0$. Now do as it Ex. 1. **Ans.** $y = e^{-mx}$

MISCELLANEOUS PROBLEMS IN CHAPTER 8.

Ex. 1. *Find the series solutions of the following equation about* $x = 0$.

(*i*) $2x^2y'' + xy' + (x^2 - 1)y = 0$ **[MDU Rohtak 2004]**

Ans. $y = ax(1 - x^2/14 + x^4/616 + \dots) + bx^{-1/2}(1 - x^2/2 + x^4/40 + \dots)$

(*ii*) $x^2y'' - xy' + (x^2 + 8/9)y = 0$

Ans. $y = ax^{4/3}(1 - 3x^2/16 + 9x^4/896 + \dots) + bx^{2/3}(1 - 3x^2/8 + 9x^4/320 + \dots)$

(*iii*) $xy'' + y' - y = 0$ **[MDU Rohtak 2005]**

Ans. $y = (a + b\log x)(1 + x + x^2/(2!)^2 + \dots) - 2b\{x + (x^2/(2!)^2) \times (1 + 1/2) + \dots\}$

(*iv*) $y'' + x^2y = 0$ **Ans.** $y = a\left(1 - \frac{x^4}{4\cdot 3} + \frac{x^8}{8\cdot 7\cdot 4\cdot 3} + \dots\right) + b\left(x - \frac{x^5}{5\cdot 4} + \frac{x^9}{9\cdot 8\cdot 5\cdot 4}\dots\right)$

Ex. 2. Find the power series solution of $xy'' + y' + 2y = 0$ about $x = 1$, *i.e.*, in power of $(x - 1)$. **[KU Kurukshetra 2005]**

Ans. $y = a\left\{1 - (x-1)^2 + (2/3)\times(x-1)^3 - (1/3)\times(x-1)^4 + \dots\right\} + b\left\{(x-1) - (1/2)\times(x-1)^2 + (1/12)\times(x-1)^4 + \dots\right\}$

Ex. 3. *Find the general solution of the differential equation* $(x-1)^2y'' + (x-1)y' - 4y = 0$ *in powers of x using the Frobenius method.* **[GATE 2002]**

Ex. 4. *Find the general solution of* $\left(1 + x^2\right)y'' + 2xy' - 2y = 0$ in terms of power series in x.

[Nagpur 2005]

Ex. 5. *Show that the differential equation* $x^2y'' - 3xy' + (4x + 4)y = 0$ *has only one Frobenius series solution and find it.* **[Nagpur 2005]**

OBJECTIVE PROBLEMS ON CHAPTER 8.

Write (a), (b), (c) or (d), whichever is correct.

1. *Singular points are* (*a*) *regular* (*b*) *irregular* (*c*) regular or irregular (*d*) none of these

Sol. Ans. (*c*). Refer Art. 8.3. **[Agra 2005, 06, 08]**

2. *For the differential equatin* $(x-1)(d^2y/dx^2) + (\cot\pi x)(dy/dx) + (\operatorname{cosec}^2\pi x)y = 0$, *which of the following statement is true ?* (*a*) 0 *is regular and* 1 *is irregular* (*b*) 0 *is irregular and* 1 *is regular* (*c*) *Both* 0 *and* 1 *are regular* (*d*) *Both* 0 *and* 1 *are irregular*. **[Gate 2006]**

Sol. Ans. (a) Re-writing the given equation, $\dfrac{d^2y}{dx^2}+\dfrac{\cot\pi x}{x-1}\dfrac{dy}{dx}+\dfrac{\operatorname{cosec}^2\pi x}{x-1}y=0,$ ... (1)

Comparing (1) with standard equation $y''+P(x)y'+Q(x)y=0,$ here

$P(x)=(\cot\pi x)/(x-1)$ and $Q(x)=(\operatorname{cosec}^2\pi x)/(x-1)$ ... (2)

Since both $P(x)$ and $Q(x)$ are undefined at $x = 0$, so $x = 0$ is not an ordinary point. Also,

$$(x-0)P(x)=\frac{x(\cot\pi x)}{x-1}=\frac{\cos\pi x}{\pi(x-1)}\times\frac{\pi x}{\sin\pi x} \quad\Rightarrow\quad \lim_{x\to 0}(x-0)P(x)=-\frac{1}{\pi}$$

and $$(x-0)^2Q(x)=\frac{x^2\operatorname{cosec}^2\pi x}{x-1}=\left(\frac{\pi x}{\sin\pi x}\right)^2\times\frac{1}{\pi^2(x-1)} \quad\Rightarrow\quad \lim_{x\to 0}(x-0)^2Q(x)=-\frac{1}{\pi^2}$$

Thus, $(x-0)\,P(x)$ and $(x-0)^2\,Q(x)$ are both analytic at $x = 0$ and hence $x = 0$ is a regular point. Again, from (2), we see that both $P(x)$ and $Q(x)$ are undefined at $x = 1$, so $x = 1$ is not an ordinary point. Also, we have

$(x-1)P(x)=\cot\pi x$ and $(x-1)^2Q(x)=(x-1)\operatorname{cosec}^2\pi x$

Clearly, $(x-1)\,P(x)$ is not analytic at $x = 1$. Hence $x = 1$ is irregular point.

3. *Let* $y_1(x)$ and $y_2(x)$ *be two linearly independent solutions of* $xy''+y'+x^2y=0,$ *in the neighbourhood of* $x = 0$. *If* $y_1(x)$ *is a power series around* $x = 0$, then (a) $y_2(x)$ *is bounded around* $x = 0$ (b) $y_2(x)$ *is unbounded around* $x = 0$. (c) $y_2(x)$ has a power series solution (d) $y_2(x)$ has solution of the form $\sum_{n=0}^{\infty} b_n x^{n+r}$ *where* $r\neq 0$ and $b_0\neq 0$. **[GATE 2003]**

4. *For the differential equation* $x^2(1-x)y''+xy'+y=0$

(a) $x = 1$ is an ordinary point (b) $x = 1$ is a regular singular point

(c) $x = 0$ is an irregular singular point. (d) $x = 0$ is an ordinary point.

[GATE 2005]

Sol. Ans. (*b*) Proceed as in Ex. 1 of Art. 8.4.

5. *It is required to find the solution of* $2x(2+x)y''-2(3+x)y'+xy=0$ *around* $x = 0$*. The roots of the indicial equation are*

(a) 0, 1/2 *(b)* 0, 2 *(c)* 1/2, 1/2 *(d)* 0, –1/2 **[GATE 2005]**

Sol. Ans. (*b*) See working rule of Art. 8.8.

6. *If* $y=\sum_{m=0}^{\infty}a_m x^m$ *is a solution of* $y''+xy'+3y=0,$ *then* a_m/a_{m+2} *equals :*

(a) $\dfrac{(m+1)(m+2)}{m+3}$ (b) $-\dfrac{(m+1)(m+2)}{m+3}$ (c) $-\dfrac{m(m-1)}{m+3}$ (d) $\dfrac{m(m-1)}{m+3}$ **[GATE 2004]**

Sol. Ans. (*b*). $y = \sum_{m=0}^{\infty} a_m x^m \quad \Rightarrow \quad y' = \sum_{m=1}^{\infty} m a_m x^{m-1}, \quad y'' = \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2}$

Substituting these values in the given equation, we obtain

$$\sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + x\sum_{m=1}^{\infty} m a_m x^{m-1} + 3\sum_{m=0}^{\infty} a_m x^m = 0 \text{ or } \sum_{m=2}^{\infty} m(m-1)a_m x^{m-2} + \sum_{m=1}^{\infty} m a_m x^m + 3\sum_{m=0}^{\infty} a_m x^m = 0 \quad \dots (1)$$

Equating coefficient of x^m to zero in the above identity, we get

$$(m+2)(m+1)a_{m+2} + ma_m + 3a_m = 0 \Rightarrow \qquad \frac{a_m}{a_{m+2}} = -\frac{(m+2)(m+1)}{m+3}$$

7. *The indicial equation for* $x(1+x^2)y'' + (\cos x)y' + (1-3x+x^2)y = 0$ *is*

(*a*) $r^2 - r = 0$ (*b*) $r^2 + r = 0$ (*c*) $r^2 = 0$ (*d*) $r^2 - 1 = 0$ **[GATE 2004]**

Sol. Ans. (*c*). Use working rule of Art. 8.8.

8. For the equation $x(x-1)y'' + (\sin x)y' + 2x(x-1)y = 0$, consider the following statements. A: $x = 0$ is a regular singular point; B: $x = 1$ is a regular singular points. Then (a) both A and B are true (b) A is false but B is true (c) A is true but B is false (d) both A and B are false.**[Gate 2008]**

Sol. Ans. (*d*). Proceed as in solved Ex.1, page 8.4

9. In the equation $x(x-1)^3 y'' + 2(x-1)^3 y' + 3y = 0$ the singular point is

(*a*) $x = 1$ (*b*) $x = -1$ (*c*) $x = 0$ (*d*) None of these **[Agra 2009]**

Sol. (*c*) Proved as in Ex-1, page 8.2

10. Solve the differential equation $y'' + x^2 y' + 2xy = 0$ by the power series method. **[Lucknow 2010]**

11. Write the different forms of the series solutions of the differential equation $P_0(x) y'' + P_1(x) y' + P_2(x) y = 0$ when

(*i*) $x = a$ is an ordinary point and (*ii*) when $x = a$ in a regular sigularity. **[Ranchi 2010]**

9

Legendre Polynomials

PART I : LEGENDRE FUNCTION OF THE FIRST KIND

9.1. Legendre's equation and its solution. [Bilaspur 1996; Meerut 1994]

The differential equation of the form $(1-x^2)y'' - 2xy' + n(n+1)y = 0$ ...(1)

is called *Legendre's equation*, where n is a positive interger. We now solve (1) in series of descending powers of x. Let the series solution of (1) be

$$y = \sum_{m=0}^{\infty} c_m x^{k-m}, \qquad \text{where} \qquad c_0 \neq 0. \qquad ...(2)$$

Differnetiating (2) and then putting the values of y, dy/dx and d^2y/dx^2 into (1), we have

$$(1-x^2)\sum_{m=0}^{\infty} c_m (k-m)(k-m-1)\, x^{k-m-2} - 2x \sum_{m=0}^{\infty} c_m (k-m)\, x^{k-m-1} + n(n+1) \sum_{m=0}^{\infty} c_m x^{k-m} = 0$$

or $$\sum_{m=0}^{\infty} c_m (k-m)(k-m-1) x^{k-m-2} - \sum_{m=0}^{\infty} c_m \{(k-m)(k-m-1) + 2(k-m) - n(n+1)\} x^{k-m} = 0 \qquad ....(3)$$

Now, $(k-m)(k-m-1) + 2(k-m) - n(n+1)$

$$= (k-m)^2 - (k-m) + 2(k-m) - n(n+1) = (k-m)^2 - n^2 + (k-m) - n$$
$$= (k-m+n)(k-m-n) + (k-m-n) = (k-m-n)(k-m+n+1).$$

Hence (3) may be re–written as

$$\sum_{m=0}^{\infty} c_m (k-m)(k-m-1) x^{k-m-2} - \sum_{m=0}^{\infty} c_m (k-m-n)(k-m+n+1) x^{k-m} = 0. \qquad ...(4)$$

(4) is an identity. To get the indicial equation, we equate to zero the coefficient of the highest power of x, namely x^k in (4) and obtain

$$c_0(k-n)(k+n+1) = 0 \qquad \text{or} \qquad (k-n)(k+n+1) = 0, \quad \text{as} \quad c_0 \neq 0. \qquad ...(5)$$

So the roots of (5) are $k = n, -(n+1)$. They are unequal and differ by an integer. The next lower power of x is $k-1$. So we equate to zero the coefficient of x^{k-1} in (4) and obtain

$$c_1(k-1-n)(k+n) = 0. \qquad ...(6)$$

For $k = n$ and $-(n+1)$, neither $(k-1-n)$ nor $(k+n)$ is zero. So from (6), $c_1 = 0$. Finally, equating to zero the coefficient of x^{k-m} in (4), we have

$$c_{m-2}(k-m+2)(k-m+1) - c_m(k-m-n)(k-m+n+1) = 0$$

$$c_m = \frac{(k-m+2)(k-m+1)}{(k-m-n)(k-m+n+1)} c_{m-2}. \qquad ...(7)$$

Putting $m = 3, 5, 7, ...$ in (7) and noting that $c_1 = 0$, we have

$$c_1 = c_3 = c_5 = c_7 = ... = 0, \qquad ...(8)$$

which hold good for both $k = n$ and $k = -(n+1)$.

To obtain c_2, c_4, c_6 .... etc., we consider two cases.

Case I. When k = n. Then, (7) becomes $$c_m = -\frac{(n-m+2)(n-m+1)}{m(2n-m+1)} c_{m-2}. \quad ...(9)$$

Putting $m = 2, 4, 6$ .... in (9), we have

$$c_2 = -\frac{n(n-1)}{2(2n-1)} c_0, \qquad c_4 = -\frac{(n-2)(n-3)}{4(2n-3)} c_2 = \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 \cdot (2n-1)(2n-3)} c_0$$

and so on. Re–writting (2), we have for $k = n$

$$y = c_0 x^n + c_1 x^{n-1} + c_2 x^{n-2} + c_3 x^{n-3} + c_4 x^{n-4} + \ldots\ldots \quad ...(10)$$

Using (8) and the above values of c_2, c_4, c_6. etc., (10) becomes (after replacing c_0 by a)

$$y = a\left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 \cdot (2n-1)(2n-3)} x^{n-4} - \ldots\right]. \quad ...(11)$$

Case II. When k = –(n + 1). Then, (7) becomes $$c_m = \frac{(n+m-1)(n+m)}{m(2n+m+1)} c_{m-2}. \quad ...(12)$$

Putting $m = 2, 4, 6, \ldots$ in (12), we have

$$c_2 = \frac{(n+1)(n+2)}{2(2n+3)} c_0, \qquad c_4 = \frac{(n+3)(n+4)}{4(2n+5)} c_2 = \frac{(n+1)(n+2)(n+3)(n+4)}{2 \cdot 4 \cdot (2n+3)(2n+5)} c_0$$

and so on. For $k = -(n + 1)$, (2) gives

$$y = c_0 x^{-n-1} + c_1 x^{-n-2} + c_2 x^{-n-3} + c_3 x^{-n-4} + c_4 x^{-n-5} + \ldots \quad ...(13)$$

Using (8) and the above values of c_2, c_4, c_6....etc., (13) becomes (after replacing c_0 by b)

$$y = b\left[x^{-n-1} + \frac{(n+1)(n+2)}{2(2n+3)} x^{-n-3} + \frac{(n+1)(n+2)(n+3)(n+4)}{2 \cdot 4 \cdot (2n+3)(2n+5)} x^{-n-5} + \ldots\right]. \quad ...(14)$$

Thus, two independent solutions of (1) are given by (11) and (14). If we take $a = [1 \cdot 3 \cdot 5 \ldots (2n-1)]/n!$, the solution (11) is denoted by $P_n(x)$ and is called *Legendre's function of the first kind* or *Legendre's polynomial of degree n*. Notice that (11) is a terminating series and so it gives rise to a polynomial of degree n. Thus $P_n(x)$ is a solution of (1), Again, if we take $b = n/[1 \cdot 3 \cdot 5 \ldots (2n+1)]$ the solution (14) is denoted by $Q_n(x)$ and is called *Legendre's function of the second kind*. Since n is a positive integer, (14) is an infinite or non–terminating series and hence $Q_n(x)$ is not a polynomial. Thus $P_n(x)$ and $Q_n(x)$ are two linearly independent solutions of (1). Hence the general solution of (1) is

$$y = A\, P_n(x) + B\, Q_n(x), \text{ where } A \text{ and } B \text{ are arbitrary constants.} \quad ...(15)$$

Remark 1. When there is no confusion regarding the variable x, we shall use a shorter notation P_n for $P_n(x)$ and P_n' for $dP_n(x)/dx$, Q_n for $Q_n(x)$ and Q_n' for $dQ_n(x)/dx$ etc.

Another form of Legendre's polynomial $P_n(x)$

Legendre's polynomial of degree n is denoted and defined by

$$P_n(x) = \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{n!}\left[x^n - \frac{n(n-1)}{2 \cdot (2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4(2n-1)(2n-3)} x^{n-4} - \ldots\right] \quad ...(1)$$

We now re–write (1) in a compact form. The general term of polynomial (1) is given by

$$\frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{n} \cdot (-1)^r \frac{n(n-1)\ldots(n-2r+1)}{2 \cdot 4 \cdot 2r(2n-1)(2n-3)\ldots(2n-2r+1)} x^{n-2r}. \quad ...(2)$$

Now, $1 \cdot 3 \cdot 5 ... (2n-1)$,

$$= \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 ... (2n-1) \,.\, 2n}{2 \cdot 4 \cdot 6 ... 2n} = \frac{(2n)!}{(2 \cdot 1)(2 \cdot 2)(2 \cdot 3) ... (2 \cdot n)} = \frac{(2n)!}{2^n \cdot 1 \cdot 2 \cdot 3 ... n} = \frac{(2n)!}{2^n \cdot n!}. \quad ...(3)$$

Also, $n(n-1) ... (n-2r+1)$

$$= \frac{n(n-1)(n-2r+1)(n-2r)(n-2r-1) ... 3 \cdot 2 \cdot 1}{(n-2r)(n-2r-1) ... 3 \cdot 2 \cdot 1} = \frac{n!}{(n-2r)!} \quad ...(4)$$

and
$$2 \cdot 4 \cdot 6 ... 2r = (2 \cdot 1)(2 \cdot 2)(2 \cdot 3) ... (2 \cdot r) = 2^r \cdot r!. \quad ...(5)$$

Finally, $(2n-1)(2n-3) ... (2n-r+1)$

$$= \frac{(2n)(2n-1)(2n-2)(2n-3) \dots (2n-2r+2)(2n-2r+1)}{(2n)(2n-2)(2n-4) \dots (2n-2r+2)} \times \frac{(2n-2r)!}{(2n-2r)!}$$

$$= \frac{(2n)(2n-1)(2n-2) ... (2n-2r+1)(2n-2r)(2n-2r-1) ... 3 \cdot 2 \cdot 1}{2 \cdot n \cdot 2(n-1) \cdot 2(n-2) ... 2(n-r+1) \,.\, (2n-2r)!}$$

$$= \frac{(2n)!}{2^n \cdot n(n-1)(n-2) ... (n-r+1)(2n-2r)!}$$

$$= \frac{(2n)!}{2^n (2n-2r)!} \times \frac{(n-r)(n-r-1) ... 3 \cdot 2 \cdot 1}{n(n-1)(n-2) ... (n-r+1)(n-r)(n-r-1) ... 3 \cdot 2 \cdot 1}$$

$$= \frac{(2n)!}{2^n (2n-2r)!} \times \frac{(n-r)!}{n!}. \quad ... (6)$$

Using (3), (4), (5) and (6), the general term (2) becomes

$$\frac{(2n)!}{2^n \, n!}(-1)^r \frac{n!}{(n-2r)!} \times \frac{1}{2^r \, r!} \times \frac{2^n (2n-2r)! \, n!}{(2n)! \, (n-r)!} x^{n-2r}$$

i.e.
$$(-1)^r \frac{(2n-2r)!}{2^n \, r! \, (n-r)! \, (n-2r)!} x^{n-2r}. \quad ...(7)$$

Since (1) is polynomial of degree n, r must be chosen so that $n - 2r \geq 0$, *i.e.*, $r \leq n/2$.

Thus, if n is even, r goes from 0 to $\frac{1}{2}n$, while if n is odd, r goes from 0 to $\frac{1}{2}(n-1)$; that is, for the complete polynomial (1), r goes from 0 to $\left[\frac{1}{2}n\right]$, where

$$\left[\tfrac{1}{2}n\right] = \begin{cases} n/2, \text{ if } n \text{ is even} \\ (n-1)/2, \text{ if } n \text{ is odd} \end{cases}$$

Hence the Legendre polynomial of degree n is given by

$$P_n(x) = \sum_{r=0}^{[n/2]} (-1)^r \frac{(2n-2r)!}{2^r \, r! \, (n-r)! \, (n-2r)!} x^{n-2r}. \quad ...(8)$$

9.2. Legendre's function of the first kind or Legendre's polynomial of degree n. [Kanpur 2011, Ranchi 2010]

Definition. Legendre's polynomial of degree n is denoted and defined by

$$P_n(x) = \frac{1 \cdot 3 \cdot 5 ... (2n-1)}{n!} \left[x^n - \frac{n(n-1)}{2(2n-1)} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2 \cdot 4 \cdot (2n-1)(2n-3)} x^{n-4} - ... \right] \quad ...(1)$$

or $$P_n(x) = \sum_{r=0}^{[n/2]} (-1)^r \frac{(2n-2r)!}{2^n\, r!\,(n-r)!\,(n-2r)!}\, x^{n-2r}, \qquad \text{...(2)}$$

where $$[n/2] = \begin{cases} n/2, \text{ if } n \text{ is even} \\ (n-1)/2, \text{ if } n \text{ is odd} \end{cases} \qquad \text{...(3)}$$

Legendre's function of the second kind. Definition. This is denoted and defined by $Q_n(x) =$

$$\frac{n!}{1\cdot 3\cdot 5 \ldots (2n+1)}\left[x^{-(n+1)} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-(n+3)} + \frac{(n+1)(n+2)(n+3)(n+4)}{2\cdot 4\cdot (2n+3)(2n+5)}x^{-(n+5)} + \ldots\right]. \quad \text{...(4)}$$

Determination of first few Legendre's polynomials.

Putting $n = 0, 1, 2, 3, 4, 5, \ldots$ in result (1), we have

$$P_0(x) = \frac{1}{0!}x^0 = 1, \qquad P_1(x) = \frac{1}{1!}x^1 = x, \qquad P_2(x) = \frac{1\cdot 3}{2!}\left[x^2 - \frac{2\cdot 1}{2\cdot 3}x^0\right] = \frac{1}{2}(3x^2-1),$$

[Bhopal 2010]

$$P_3(x) = \frac{1\cdot 3\cdot 5}{3!}\left[x^3 - \frac{3\cdot 2}{2\cdot 5}x^1\right] = \frac{1}{2}(5x^3 - 3x),$$

$$P_4(x) = \frac{1\cdot 3\cdot 5\cdot 7}{4!}\left[x^4 - \frac{4\cdot 3}{2\cdot 7}x^2 + \frac{4\cdot 3\cdot 2\cdot 1}{2\cdot 4\cdot 7\cdot 5}x^0\right] = \frac{1}{8}(35x^4 - 30x^2 + 3),$$

$$P_5(x) = \frac{1\cdot 3\cdot 5\cdot 7\cdot 9}{5!}\left[x^5 - \frac{5\cdot 4}{2\cdot 9}x^3 + \frac{5\cdot 4\cdot 3\cdot 2}{2\cdot 4\cdot 9\cdot 7}x^1\right] = \frac{1}{8}(63x^5 - 70x^3 + 15x)$$ **[Kanpur 2004]**

Ex. 1. *Express* $2 - 3x + 4x^2$ *in terms of Legendre polynomials.* **[Bangalore 1995]**

Sol. We have $1 = P_0(x)$, $x = P_1(x)$, $(3x^2-1)/2 = P_2(x) \Rightarrow x^2 = [2P_2(x) + 1]/3$ ...(1)

Now, $2 - 3x + 4x^2 = 2P_0(x) - 3P_1(x) + (4/3)\times[2P_2(x)+1]$, by (1)

$= 2P_0(x) - 3P_1(x) + (8/3)\times P_2(x) + (4/3)\times P_0(x) = (10/3)\times P_0(x) - 3P_1(x) + (8/3)\times P_2(x)$.

Ex. 2. *Show that* $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = (3x^2-1)/2$, $P_3(x) = (5x^3-3)/2$ *and also expand* $x^4 + 2x^3 + 2x^2 - x - 3$ *in terms of Legendre's polynomials.* **[Bhopal 2004, Kakitiya 1997]**

Ex. 3. *Show that* $x^3 = (2/5)\times P_3(x) + (3/5)\times P_2(x)$. **[Nagpur 1996]**

9.3. Generating function for Legendre polynomials. [Nagpur 2005]

Theorem. *To show* $(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n\, P_n(x), |x| \le 1, |z| < 1$ **[M.D.U. Rohtak 2006, Delhi Physics (H) 2000; Kanpur 2005; Banaglore 1994; Nagpur 2003; Meerut 1994, 96]**

Or *To show that* $P_n(x)$ *is the coefficient of* z^n *in the exapansion of* $(1-2xz+z^2)^{-1/2}$ *in assending powers of z.* **[Garhwal 2004; Meerut 1998, Ravishankar 2010; Ranchi 2010]**

[**Note :** $(1-2xz+z^2)^{-1/2}$ is called the *generating function* for Legendre polynomial $P_n(x)$].

Proof. (*a*) since $|z| < 1$ and $|x| \le 1$, we have

$$(1-2xz+z^2)^{-1/2} = [1 - z(2x-z)]^{-1/2}$$

$$= 1 + \tfrac{1}{2}z(2x-z) + \frac{1\cdot 3}{2\cdot 4}z^2(2x-z)^2 + \ldots + \frac{1\cdot 3\ldots(2n-3)}{2\cdot 4\ldots(2n-2)}z^{n-1}(2x-z)^{n-1}$$

$$+ \frac{1\cdot 3\ldots(2n-1)}{2\cdot 4\ldots(2n)}z^n(2x-z)^n + \ldots \qquad \text{...(1)}$$

Now, the coefficient of z^n in $\dfrac{1\cdot 3\ldots(2n-1)}{2\cdot 4\ldots(2n)}z^n(2x-z)^n$

$$= \frac{1\cdot 3\ldots(2n-1)}{2\cdot 4\cdot 6\ldots(2n)}(2x)^n = \frac{1\cdot 3\cdot 5\ldots(2n-1)\cdot 2^n\cdot x^n}{(2\cdot 1)(2\cdot 2)(2\cdot 3)\ldots(2\cdot n)} = \frac{1\cdot 3\cdot 5\ldots(2n-1)}{2^n\cdot n!}2^n\cdot x^n = \frac{1\cdot 3\cdot 5\ldots(2n-1)}{n!}x^n. \quad \ldots(2)$$

Again, the coefficient of z^n in $\dfrac{1\cdot 3\ldots(2n-3)}{2\cdot 4\ldots(2n-2)}z^{n-1}(2x-z)^{n-1}$

$$= \frac{1\cdot 3\ldots(2n-3)}{(2\cdot 1)(2\cdot 2)\ldots 2(n-1)}\{-(n-1)(2x)^{n-2}\}$$

$$= -\frac{1\cdot 3\ldots(2n-3)}{2^{n-1}\,1\cdot 2\cdot 3\ldots(n-1)}\cdot\frac{2n-1}{n}\cdot\frac{n}{2n-1}[(n-1)\times 2^{n-2}\times x^{n-2}], \text{ on multiplying and dividing by } (2n-1)/n$$

$$= -\frac{1\cdot 3\ldots(2n-1)}{n!}\cdot\frac{n(n-1)}{2(2n-1)}x^{n-2} \quad \ldots(3)$$

and so on. Using (2), (3), ... we see that the coefficient of z^n in the expansion of $(1-2xz+z^2)^{-1/2}$, namely (1) is given by $\dfrac{1\cdot 3\ldots(2n-1)}{n!}\left[x^n - \dfrac{n(n-1)}{2(2n-1)}x^{n-2} + \dfrac{n(n-1)(n-2)(n-3)}{2\cdot 4\cdot(2n-1)(2n-3)}x^{n-4} - \ldots\right]$

i.e., $P_n(x)$, by definition of Legendre polynomial.

We find that $P_1(x)$, $P_2(x)$, ... will be the coefficients of z, z^2,.... in the expansion of $(1-2xz+z^2)^{-1/2}$. Thus, we may write

$$(1-2xz+z^2)^{-1/2} = 1 + zP_1(x) + z^2P_2(x) + \ldots + z^nP_n(x) + \ldots \quad \text{or} \quad (1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^nP_n(x).$$

9.4. Solved examples based on Art. 9.2 and 9.3.

Ex. 1. *Prove that* (*a*) : $1 + \frac{1}{2}P_1(\cos\theta) + \frac{1}{3}P_2(\cos\theta) + \ldots = \log[(1+\sin\frac{1}{2}\theta)/\sin\frac{1}{2}\theta]$.

(*b*) $\displaystyle\sum_{n=0}^{\infty}\frac{P_n(\cos\theta)}{n+1} = \log\{1+\operatorname{cosec}(\theta/2)\}$ **[Ravishankar 2004]**

Sol. (*a*) From the generating function, $\displaystyle\sum_{n=0}^{\infty} z^nP_n(x) = (1-2xz+z^2)^{-1/2}$. ...(1)

Integrating (1) w.r.t. z from 0 to 1, $\displaystyle\sum_{n=0}^{\infty}\int_0^1 z^nP_n(x)\,dz = \int_0^1\frac{dz}{\sqrt{(1-2xz+z^2)}}$. ...(2)

Replacing x by $\cos\theta$ on both sides, (2) gives

$$\sum_{n=0}^{\infty}P_n(\cos\theta)\int_0^1 z^n dz = \int_0^1\frac{dz}{\sqrt{(1-2z\cos\theta+z^2)}} \quad \text{or} \quad \sum_{n=0}^{\infty}P_n(\cos\theta)\left[\frac{z^{n+1}}{n+1}\right]_0^1 = \int_0^1\frac{dz}{\sqrt{[(z-\cos\theta)^2+\sin^2\theta]}}$$

or $$\sum_{n=0}^{\infty}\frac{P_n(\cos\theta)}{n+1} = \left[\log\left\{(z-\cos\theta)+\sqrt{[(z-\cos\theta)^2+\sin^2\theta]}\right\}\right]_0^1$$

$$= \log\left\{(1-\cos\theta)+\sqrt{[(1-\cos\theta)^2+\sin^2\theta]}\right\} - \log(1-\cos\theta) = \log\left\{(1-\cos\theta)+\sqrt{[2(1-\cos\theta)]}\right\} - \log(1-\cos\theta)$$

$$= \log \frac{(1-\cos\theta)+\sqrt{2}\sqrt{(1-\cos\theta)}}{1-\cos\theta} = \log \frac{\sqrt{(1-\cos\theta)}\,\sqrt{(1-\cos\theta)}+\sqrt{2}\,\sqrt{(1-\cos\theta)}}{\sqrt{(1-\cos\theta)}\,\sqrt{(1-\cos\theta)}}$$

$$= \log \frac{\sqrt{(1-\cos\theta)}+\sqrt{2}}{\sqrt{(1-\cos\theta)}} = \log \frac{\sqrt{(2\sin^2\frac{1}{2}\theta)}+\sqrt{2}}{\sqrt{(2\sin^2\frac{1}{2}\theta)}} = \log \frac{1+\sin\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}$$

$$\therefore \qquad \frac{P_0(\cos\theta)}{1} + \tfrac{1}{2}P_1(\cos\theta) + \tfrac{1}{3}P_2(\cos\theta) + \ldots = \log \frac{1+\sin\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}$$

or $\quad 1 + \tfrac{1}{2}P_1(\cos\theta) + \tfrac{1}{3}P_2(\cos\theta) + \ldots = \log \dfrac{1+\sin\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}, \quad$ as $\quad P_0(\cos\theta) = 1 \quad \ldots(3)$

(*b*) Proceed as in part (*a*) upto equation (3). Re-writing (3), we have

$$\sum_{n=0}^{\infty} \frac{P_n(\cos\theta)}{n+1} = \log\{1+\operatorname{cosec}(\theta/2)\}.$$

Ex. 2. *Prove that* (*i*) $P_n(1) = 1$ **[Agra 2010; Purvanchal 2005, Meerut 2007, 11; Bhopal 2010; Lucknow 2010; Nagpur 2010]**

(*ii*) $P_n(-1) = (-1)^n$ **[Agra 2009; Kanpur 2007]**

(*iii*) $P_n'(1) = \frac{1}{2}n(n+1)$. **[Delhi Physics (H) 2001]**

(*iv*) $P'_n(-1) = (-1)^{n-1} \times \frac{1}{2}n(n+1)$. **[Agra 2009; Gulbarge 2005]**

(*v*) $P_n(-x) = (-1)^n P_n(x)$. Deduce $P_n(-1) = (-1)^n$. **[Lucknow 2010, Nagpur 2010 Agra 2010; Kanpur 2005, 06; Kakitiya 1997; Meerut 2005; Purvanchal 2005]**

Sol. The generating function formula is $(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x), |z|<1, |x|\le 1. \quad \ldots(1)$

Part (i). Putting $x = 1$ in (1), we have

$$(1-2z+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(1) \qquad \text{or} \qquad (1-z)^{-1} = \sum_{n=0}^{\infty} z^n P_n(1).$$

Since $|z| < 1$, the binomial theorem can be used for expansion of $(1-z)^{-1}$.

$$\therefore \qquad 1 + z + z^2 + \ldots. + z^n + \ldots = \sum_{n=0}^{\infty} z^n P_n(1). \qquad \ldots(2)$$

Equating the coefficient of z^n from both sides, (2) gives $\qquad P_n(1) = 1$

Part (ii). Putting $x = -1$ in (1), we have as before

$$(1+2z+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(-1) \qquad \text{or} \qquad (1+z)^{-1} = \sum_{n=0}^{\infty} z^n P_n(-1)$$

or $$1 - z + z^2 \ldots + (-1)^n z^n + \ldots. = \sum_{n=0}^{\infty} z^n P_n(-1). \qquad \ldots(3)$$

Equation the coefficients of z^n from both sides, (3) gives $\qquad P_n(-1) = (-1)^n$.

Part (iii). Since $P_n(x)$ satisfies Legendre's equation $\quad (1-x^2)y'' - 2xy' + n(n+1)y = 0$,

we get $$(1-x^2)\,P''_n(x) - 2xP_n'(x) + n(n+1)P_n(x) = 0. \qquad \ldots(4)$$

Putting $x = 1$ in (4) and using $P_n(1) = 1$, we get

$$0 - 2P'_n(1) + n(n+1) = 0 \qquad \text{or} \qquad P'_n(1) = \tfrac{1}{2}n(n+1).$$

Part (iv). Putting $x = -1$ in (4) and using $P_n(-1) = (-1)^n$, we get

$0 + 2P'_n(-1) + n(n+1)(-1)^n = 0$ or $P'_n(-1) = -(-1)^n \times \frac{1}{2} n(n+1)$.

or $P_n'(-1) = (-1)^{n-1} \times \frac{1}{2} n(n+1)$ $[\because -(-1)^n = -(-1)^{n-1}(-1) = (-1)^{n-1}]$

Part (v). Replacing x by $-x$ in (1), $$(1 + 2xz + z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(-x). \qquad ...(5)$$

Next, replacing z by $-z$ in (1), $$(1 + 2xz + z)^{-1/2} = \sum_{n=0}^{\infty} (-z)^n P_n(x). \qquad ...(6)$$

From (5) and (6), $$\sum_{n=0}^{\infty} z^n P_n(-x) = \sum_{n=0}^{\infty} (-1)^n z^n \; P_n(x). \qquad ...(7)$$

Equating the coefficients of z^n from both sides of (8), we get
$$P_n(-x) = (-1)^n \; P_n(x). \qquad ...(8)$$

Deduction. Replacing x by 1 and noting that $P_n(1) = 1$, (8) gives $P_n(-1) = (-1)^n$.

Note. When n is odd, $(-1)^n = -1$ and so (8) becomes $P_n(-x) = -P_n(x)$. Thus, $P_n(x)$ is an odd function of x when n is odd. Similarly, $P_n(x)$ is an even function of x when n is even.

Ex. 3. *Prove that* (*i*) $P_{2m+1}(0) = 0$ (*ii*) $P_{2m}(0) = (-1)^m \dfrac{(2m)!}{2^{2m}(m!)^2}$ **[Kanpur 2008]**

i.e. $P_{2m}(0) = \{ (-1)^m 1 \cdot 3 \cdot 5 \cdot \cdots \; (2m - 1)\}/2^m \; m!$ **[Nagpur 1995, 2002]**

(*iii*) $P_n(0) = 0$, if n is odd. **[Kanpur 2007; Agra 2006; Meerut 1996]**

and (*iv*) $P_n(0) = \dfrac{(-1)^{n/2} n!}{2^n \{(n/2)!\}^2}$, *if n is even.* **[Kanpur 2007; Meeerut, 1994, 95, 96]**

Sol. We have $$\sum_{n=0}^{\infty} z^n P_n(x) = (1 - 2xz + z^2)^{-1/2}, \; |z| < 1, \; |x| \le 1. \qquad ...(1)$$

Putting $x = 0$ in (1), $$\sum_{n=0}^{\infty} z^n P_n(0) = (1 + z^2)^{-1/2}, \qquad i.e.,$$

$$\sum_{n=0}^{\infty} z^n P_n(0) = \sum_{n=0}^{\infty} \frac{\left(-\frac{1}{2}\right)\left(-\frac{1}{2}-1\right)\cdots\left(-\frac{1}{2}-n+1\right)}{n!}(z^2)^n \quad \text{or} \quad \sum_{n=0}^{\infty} z^n P_n(0) = \sum_{n=0}^{\infty} (-1)^n \frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{2^n \; n!} z^{2n}.$$
...(2)

Part (i). Note that the R.H.S. of (2) consists of even powers of z alone. So equating the coefficients of z^{2m+1} from both sides of (2), we have $P_{2m+1}(0) = 0$. ...(3)

Part (ii). Equating the coefficients of z^{2m} from both sides of (2), we get

$$P_{2m}(0) = (-1)^m \frac{1 \cdot 3 \cdot 5 \ldots (2m-1)}{2^m \, m!} = (-1)^m \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \ldots (2m-1)\,(2m)}{2^m \, m! \, 2 \cdot 4 \cdot 6 \ldots (2m)}$$

$$= (-1)^m \frac{(2m)!}{2^m \, m! \, (2 \cdot 1)\,(2 \cdot 2)\,(2 \cdot 3) \ldots (2 \cdot m)} = (-1)^m \frac{(2m)!}{2^m \, m!} \cdot \frac{1}{2^m \, m!} = (-1)^m \frac{(2m)!}{2^{2m} \, (m!)^2} \qquad ...(4)$$

Part (iii). Proceed as in part (*i*). Here $2m + 1 = n =$ odd. So from (3) $P_n(0) = 0$, if n is odd.

Part (iv). Proceed as in part (*ii*) upto (4). Here $2m = n$ = even so that $m = n/2$. Putting $m = n/2$ in (4), we have
$$P_n(0) = (-1)^{n/2} \frac{n!}{2^n \{(n/2)!\}^2}.$$

Ex. 4. *Express $x^4 + 2x^3 + 2x^2 - x - 3$ in terms of Legendre's polynomials.*

Sol. We have $P_0(x) = 1$, $P_1(x) = x$, $P_2(x) = (3x^2 - 1)/2$, $P_3(x) = (5x^3 - 3x)/2$,

and $P_4(x) = (35x^4 - 30x^2 + 3)/8$.

These $\Rightarrow$ $x^4 = (8/35)P_4(x) + (6/7)x^2 - (3/35)$, ...(1)

$x^3 = (2/5)P_3(x) + (3/5)x$, $x^2 = (2/3)P_2(x) + (1/3)$, ...(2)

$x = P_1(x)$ and $1 = P_0(x)$ ...(3)

Now, $x^4 + 2x^3 + 2x^2 - x - 3 = (8/35)P_4(x) + (6/7)x^2 - (3/35) + 2[(2/5)P_3(x) + (3/5)x] + 2x^2 - x - 3$

[Putting values of x^4 and x^3 with help of (1) and (2)]

$= (8/35)P_4(x) + (4/5)P_3(x) + (20/7)x^2 + (1/5)x - (108/35)$

$$= \frac{8}{35} P_4(x) + \frac{4}{5} P_3(x) + \frac{20}{7}\left[\frac{2}{3} P_2(x) + \frac{1}{3}\right] + \frac{1}{5} P_1(x) - \frac{108}{35}, \text{ using (2) and (3)}$$

$= (8/35)P_4(x) + (4/5)P_3(x) + (40/21)P_2(x) + (1/5)P_1(x) - (224/105)P_0(x)$, using (3)

Ex. 5. *Prove that* $P_n\left(-\tfrac{1}{2}\right) = P_0\left(-\tfrac{1}{2}\right)P_{2n}\left(\tfrac{1}{2}\right) + P_1\left(-\tfrac{1}{2}\right)P_{2n-1}\left(\tfrac{1}{2}\right) + \ldots.\ P_{2n}\left(-\tfrac{1}{2}\right)P_0\left(\tfrac{1}{2}\right)$.

Sol. We have
$$(1 - 2xz + z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x). \qquad \ldots (1)$$

Replacing x by 1/2 and –1/2 successively, (1) gives
$$(1 - z + z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n\left(\tfrac{1}{2}\right) \qquad \ldots(2)$$

and
$$(1 + z + z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n\left(-\tfrac{1}{2}\right). \qquad \ldots(3)$$

Next, replacing z by z^2 in (3),
$$(1 + z^2 + z^4)^{-1/2} = \sum_{n=0}^{\infty} z^{2n} P_n\left(-\tfrac{1}{2}\right). \qquad \ldots(4)$$

But $1 + z^2 + z^4 = (1 + z^2)^2 - z^2 = (1 + z^2 + z)(1 + z^2 - z)$

$\therefore$ $(1 + z^2 + z^4)^{-1/2} = (1 + z + z^2)^{-1/2} . (1 - z + z^2)^{-1/2}$

or
$$\sum_{n=0}^{\infty} z^{2n} P_n\left(-\tfrac{1}{2}\right) = \sum_{n=0}^{\infty} z^n P_n\left(-\tfrac{1}{2}\right) . \sum_{n=0}^{\infty} z^n P_n\left(\tfrac{1}{2}\right), \text{ by (2), (3) and (4)}$$

or
$$\sum_{n=0}^{\infty} z^{2n} P_n\left(-\tfrac{1}{2}\right) = \left[P_0\left(-\tfrac{1}{2}\right) + z P_1\left(-\tfrac{1}{2}\right) + \ldots.. + z^{2n-1} P_{2n-1}\left(-\tfrac{1}{2}\right) + z^{2n} P_{2n}\left(-\tfrac{1}{2}\right) + \ldots\right] \times \left[P_0\left(\tfrac{1}{2}\right) + z P_1\left(\tfrac{1}{2}\right) + \ldots\right.$$
$$\left. + z^{2n-1} P_{2n-1}\left(\tfrac{1}{2}\right) + z^{2n} P_{2n}\left(\tfrac{1}{2}\right) + \ldots\right]$$

Equating the coefficients of z^{2n} from both sides of the above equation, we get the desired result.

Ex. 6. *Prove that* $\dfrac{1 - z^2}{(1 - 2xz + z^2)^{3/2}} = \sum_{n=0}^{\infty} (2n + 1) z^n P_n.$

[Kanpur 2005; Purvanchal 2007; Meerut 2010]

Sol. We have $$(1 - 2xz + z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n. \qquad ...(1)$$

Differentiating both sides of (1) w.r.t. 'z', $-\frac{1}{2}(1 - 2xz + z^2)^{-3/2}(-2x + 2z) = \sum_{n=0}^{\infty} n\, z^{n-1} P_n$

or $$(x - z)(1 - 2xz + z^2)^{-3/2} = \sum_{n=0}^{\infty} n\, z^{n-1} P_n. \qquad ...(2)$$

Multiplying both sides of (2) by $2z$, $$2z(x - z)(1 - 2xz + z^2)^{-3/2} = 2\sum_{n=0}^{\infty} n\, z^n P_n. \qquad ...(3)$$

Adding (1) and (3), $$\frac{1}{(1 - 2xz + z^2)^{1/2}} + \frac{2z(x - z)}{(1 - 2xz + z^2)^{3/2}} = \sum_{n=0}^{\infty} z^n P_n + \sum_{n=0}^{\infty} 2nz^n P_n$$

or $$\frac{1 - 2xz + z^2 + 2z(x - z)}{(1 - 2xz + z^2)^{3/2}} = \sum_{n=0}^{\infty}(2n + 1)z^n P_n \quad \text{or} \quad \frac{1 - z^2}{(1 - 2xz + z^2)^{3/2}} = \sum_{n=0}^{\infty}(2n + 1)z^n P_n.$$

Ex. 7. *Prove that* $$\frac{1 + z}{z\sqrt{(1 - 2xz + z^2)}} - \frac{1}{z} = \sum_{n=0}^{\infty}(P_n + P_{n+1})z^n.$$

Sol. We have, $$(1 - 2xz + z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n. \qquad ...(1)$$

$\therefore$ L.H.S. of the required result $= (1/z) \times (1 - 2xz + z^2)^{-1/2} + (1 - 2xz + z^2)^{-1/2} - (1/z)$

$$= \frac{1}{z}\sum_{n=0}^{\infty} z^n P_n + \sum_{n=0}^{\infty} z^n P_n - \frac{1}{z}, \text{ by (1)} \qquad ...(2)$$

But $\sum_{n=0}^{\infty} z^n P_n = P_1 + zP_1 + z^2P_2 + ... + z^nP_n + z^{n+1}P_{n+1} + ... = 1 + z(P_1 + zP_2 + ... + z^nP_{n+1} + ...)$, as $P_0 = 1$

Thus, $$\sum_{n=0}^{\infty} z^n P_n = 1 + z\sum_{n=0}^{\infty} z^n P_{n+1}. \qquad ...(3)$$

Using (3) in (2), the L.H.S. of the required result

$$= \frac{1}{z}\left[1 + z\sum_{n=0}^{\infty} z^n P_{n+1}\right] + \sum_{n=0}^{\infty} z^n P_n - \frac{1}{z} = \sum_{n=0}^{\infty} z^n P_{n+1} + \sum_{n=0}^{\infty} z^n P_n = \sum_{n=0}^{\infty}(P_n + P_{n+1})z^n$$

= R.H.S. of the required result.

Ex. 8. *Prove that* $(1 - 2xz + z^2)^{-1/2}$ *is a solution of the equation* $z\dfrac{\partial^2(zv)}{\partial z^2} + \dfrac{\partial}{\partial x}\left[(1 - x^2)\dfrac{dv}{dx}\right] = 0.$

Sol. Let $$v = (1 - 2xz + z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n. \qquad ...(1)$$

$$\therefore \qquad zv = z\sum_{n=0}^{\infty} z^n P_n = \sum_{n=0}^{\infty} z^{n+1} P_n.$$

$$\therefore \qquad \frac{\partial^2 (zv)}{\partial z^2} = \frac{\partial^2}{\partial z^2}\left[\sum_{n=0}^{\infty} z^{n+1} P_n\right] = \sum_{n=0}^{\infty} n(n+1)\, z^{n-1} P_n .$$

$$\therefore \qquad z\frac{\partial^2 (zv)}{\partial z^2} = z\sum_{n=1}^{\infty} n(n+1)\, z^{n-1} P_n = \sum_{n=1}^{\infty} n(n+1)\, z^{n} P_n . \qquad ...(2)$$

From (1),
$$\frac{\partial v}{\partial x} = \sum_{n=0}^{\infty} \frac{\partial}{\partial x}(z^n\, P_n) = \sum_{n=0}^{\infty} z^n\, P_n' . \qquad ...(3)$$

$$\therefore \qquad \frac{\partial^2 v}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x}\right) = \frac{\partial}{\partial x}\sum_{n=0}^{\infty} z^n\, P_n' = \sum_{n=0}^{\infty} z^n\, P_n'' . \qquad ...(4)$$

But
$$\frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial v}{\partial x}\right] = (1-x^2)\frac{\partial^2 v}{\partial x^2} - 2x\frac{\partial v}{\partial x}$$

$$\therefore \qquad \frac{\partial}{\partial x}\left[(1-x^2)\frac{\partial v}{\partial x}\right] = (1-x^2)\sum_{n=0}^{\infty} z^n\, P_n'' - 2x\sum_{n=0}^{\infty} z^n\, P_n', \text{ using (3) and (4)} \qquad ...(5)$$

Adding the corresponding sides of (2) and (5), we have

$$z\frac{\partial^2 (zv)}{\partial v^2} + \frac{\partial}{\partial x}\left\{(1-x^2)\frac{\partial v}{\partial x}\right\} = \sum_{n=0}^{\infty} n(n+1)\, z^n\, P_n + \sum_{n=0}^{\infty}(1-x^2) z^n\, .\, P_n'' - \sum_{n=0}^{\infty} z^n (2x\, P_n')$$

$$= \sum_{n=0}^{\infty} z^n [(1-x^2)\, P_n'' - 2x\, P_n' + n(n+1)\, P_n] = 0,$$

as P_n is a solution of Legendre equation $(1-x^2)y'' - 2xy' + n(n+1)y = 0.$

9.5. Trigonometrical series for $P_n(x)$.

To show that $\quad P_n(\cos\theta)$

$$= \frac{1\cdot 3\cdot 5\ldots(2n-1)}{2^{n-1}\cdot n\,!}\left[\cos n\theta + \frac{1\cdot n}{1\cdot(2n-1)}\cos(n-2)\,\theta + \frac{1\cdot 3\cdot n(n-1)}{1\cdot 2\cdot(2n-1)\,(2n-3)}\cos(n-4)\,\theta + \ldots\right]$$

Sol. Since $\displaystyle\sum_{n=0}^{\infty} z^n P_n(x) = (1 - 2xz + z^2)^{-1/2}$, we have

$$\sum_{n=0}^{\infty} z^n P_n(\cos\theta) = (1 - 2z\cos\theta + z^2)^{-1/2} = [1 - z(e^{i\theta} + e^{-i\theta}) + z^2]^{-1/2}$$

$$= [1 - ze^{i\theta} - ze^{-i\theta} + ze^{i\theta}\, .\, ze^{-i\theta}]^{-1/2} = (1 - ze^{i\theta})^{-1/2}\,(1 - ze^{-i\theta})^{-1/2}$$

$$= \left[1 + \tfrac{1}{2} ze^{i\theta} + \frac{1\cdot 3}{2\cdot 4} z^2 e^{2i\theta} + \ldots + \frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots(2n)} z^n e^{ni\theta} + \cdots\right]$$

$$\times\left[1 + \tfrac{1}{2} ze^{-i\theta} + \frac{1\cdot 3}{2\cdot 4} z^2 e^{-2i\theta} + \ldots + \frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots(2n)} z^n e^{-ni\theta} + \cdots\right].$$

Equating the coefficients of z^n from both sides, we get

$$P_n(\cos\theta) = \frac{1\cdot 3\cdot 5\ldots(2n-1)}{2\cdot 4\cdot 6\ldots(2n)}(e^{ni\theta} + e^{-ni\theta}) + \frac{1\cdot 3\cdot 5\ldots(2n-3)\, .\, 1}{2\cdot 4\cdot 6\ldots(2n-2)\, .\, 2}[e^{(n-2)i\theta} + e^{-(n-2)i\theta}]$$

$$+\frac{1\cdot 3\cdot 5...(2n-5)\cdot 1\cdot 3}{2\cdot 4\cdot 6\,...\,(2n-2)\cdot 2\cdot 4}[e^{(n-4)i\theta}+e^{-(n-4)i\theta}]+\cdots$$

$$=\frac{1\cdot 3\cdot 5...(2n-1)}{2\cdot 4\cdot 6\,...\,(2n)}\left[2\cos n\theta+\frac{2n}{2n-1}\cdot\frac{1}{2}\cdot 2\cos(n-2)\theta\right.$$

$$\left.+\frac{2n\,.\,(2n-2)}{(2n-1)\,(2n-3)}\cdot\frac{1\cdot 3}{2\cdot 4}\cdot 2\cos(n-4)\theta+\cdots.\right] \quad ...(A)$$

$$=\frac{1\cdot 3\cdot 5...(2n-1)}{2^{n-1}\cdot n\,!}\left[\cos n\theta+\frac{1\,.\,n}{1\cdot(2n-1)}\cos(n-4)\theta+\frac{1\cdot 3\cdot n(n-1)}{1\cdot 2\cdot(2n-1)\,(2n-3)}\cos(n-4)\theta+\cdots\right]$$

SOLVED EXAMPLE BASED ON ART 9.5

Example. *Prove that* $\displaystyle\int_0^{\pi} P_n(\cos\theta)\cos n\theta\,d\theta=\frac{1\cdot 3\cdot 5\,...(2n-1)}{2\cdot 4\cdot 6\,...(2n)}\cdot\pi = B\left(n+\tfrac{1}{2},\tfrac{1}{2}\right).$

Sol. From equation (A) of Art 9.5, we have

$$P_n(\cos\theta)=\frac{1\cdot 3\cdot 5...(2n-1)}{2\cdot 4\cdot 6...(2n)}\left[2\cos n\theta+\frac{2n}{2n-1}\cdot\frac{1}{2}\cdot 2\cos(n-2)\theta+\frac{2n\cdot(2n-2)}{(2n-1)\,(2n-3)}\cdot\frac{1\cdot 3}{2\cdot 4}\cdot 2\cos(n-4)\theta+\cdots\right]$$

$$\therefore\ \int_0^{\pi}P_n(\cos\theta)\cos n\theta\,d\theta=\frac{1\cdot 3\cdot 5...(2n-1)}{2\cdot 4\cdot 6\,...\,(2n)}\left[2\int_0^{\pi}\cos^2 n\theta\,d\theta+\frac{2n}{2n-1}\cdot\frac{1}{2}\cdot 2\int_0^{\pi}\cos(n-2)\theta\cos n\theta\,d\theta+\cdots\right]$$

$$=\frac{1\cdot 3\cdot 5...(2n-1)}{2\cdot 4\cdot 6\,...\,(2n)}\int_0^{\pi}(1+\cos 2n\theta)\,d\theta\text{, since for } m\neq n,\text{ we have}$$

$$\int_0^{\pi}\cos mx\cos nx\,dx=\frac{1}{2}\int_0^{\pi}[\cos(m+n)x+\cos(m-n)x]dx=\frac{1}{2}\left[\frac{\sin(m+n)\,x}{m+n}+\frac{\sin(m-n)\,x}{m-n}\right]_0^{\pi}=0$$

$$\therefore\ \int_0^{\pi}P_n(\cos\theta)\cos n\theta\,d\theta=\frac{1\,.\,3\,.\,5\,...(2n-1)}{2\,.\,4\,.\,6\,...(2n)}\pi=\frac{\frac{2n-1}{2}\cdot\frac{2n-3}{2}\cdots\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}\,.\,\sqrt{\pi}}{1\,.\,2\,.\,3\,...\,n}$$

$$=\frac{\Gamma\left(\frac{2n+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(n+1)}=\frac{\Gamma\left(n+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(n+\frac{1}{2}+\frac{1}{2}\right)}=B\left(n+\tfrac{1}{2},\tfrac{1}{2}\right) \qquad \left[\because B(m,n)=\frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}\right]$$

EXERCISE 9 (A)

1. Show that Legendre's polynomial $P_n(x)$ satisfies the Legendre's equation

$$(1-x^2)\,(d^2y/dx^2)-2x(dy/dx)+n(n+1)y=0.$$

2. Show that the general solution of Legendre's equation $(1-x^2)y''-2xy'+n(n+1)y=0$, where n is a positive integer is $y=AP_n(x)+BQ_n(x)$, where $P_n(x)$ and $Q_n(x)$ have their usual meanings.

3. Express the following expressions in terms of Legendre's polynomial to show that

(*i*) $x^2=(1/3)\times P_0(x)+(2/3)\times P_2(x)$. (*ii*) $x^3=(2/5)\times P_3(x)+(3/5)\times P_1(x)$.

(*iii*) $x^4=(8/35)\times P_4(x)+(4/7)\times P_2(x)+(1/5)\times P_0(x)$. **(Nagpur 2003, Bilaspur 2004)**

(*iv*) $x^4-3x^2+x=-(4/5)\times P_0(x)+P_1(x)-(10/7)\times P_2(x)+(8/35)\times P_4(x)$.

9.6. Laplace's definite integrels for $P_n(x)$

(I) Laplace's first integral $P_n(x)$. When is n is a positive integer, then

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi} [x \pm (x^2-1)^{1/2}\cos\phi]^n \, d\phi . \qquad \textbf{(Agra 2010; Meerut 2005, 06)}$$

Proof. From integral calculus, $\displaystyle\int_0^{\pi} \frac{d\phi}{a \pm b\cos\phi} = \frac{\pi}{(a^2-b^2)^{1/2}}$, when $a^2 > b^2$. ...(1)

Let $a = 1 - zx$ and $b = z(x^2-1)^{1/2}$. Then $a^2 - b^2 = (1-zx)^2 - z^2(x^2-1) = 1 - 2xz + z^2$.
Using these values of a, b and $a^2 - b^2$, (1) becomes

$$\pi(1-2zx+z^2)^{-1/2} = \int_0^{\pi}\left[1 - zx \pm z(x^2-1)^{1/2}\cos\phi\right]^{-1} d\phi$$

or
$$\pi\sum_{n=0}^{\infty} z^n P_n(x) = \int_0^{\pi}(1-zt)^{-1}\,d\phi, \qquad \text{if} \qquad t = x \pm (x^2-1)^{1/2}\cos\phi$$

$$= \int_0^{\pi}(1 + zt + z^2t^2 + \dots)\,d\phi = \int_0^{\pi}\sum_{n=0}^{\infty}(zt)^n d\phi = \sum_{n=0}^{\infty} z^n \int_0^{\pi} t^n\,d\phi.$$

$$\therefore \qquad \pi\sum_{n=0}^{\infty} z^n P_n(x) = \sum_{n=0}^{\infty} z^n \int_0^{\pi}[x \pm \sqrt{(x^2-1)}\cos\phi]^n\,d\phi. \qquad ...(2)$$

Equating coefficients of z^n from both sides, (2) gives

$$\pi P_n(x) = \int_0^{\pi}[x \pm (x^2-1)^{1/2}\cos\phi]^n\,d\phi \qquad \text{or} \qquad P_n(x) = \frac{1}{\pi}\int_0^{\pi}[x \pm (x^2-1)^{1/2}\cos\phi]^n\,d\phi \quad ...(3)$$

Deductions : Show that (*i*) $P_n(\cos\theta) = \dfrac{1}{\pi}\displaystyle\int_0^{\pi}(\cos\theta + i\sin\theta\cos\phi)^n\,d\phi$

(*ii*) $P_1(x) = \dfrac{1}{\pi}\displaystyle\int_0^{\pi}\{x + (x^2-1)^{1/2}\cos\theta\}\,d\theta$

Proof. (i) Let $x = \cos\theta$. Then, we have

$$(x^2-1)^{1/2} = \sqrt{(\cos^2\theta - 1)} = \sqrt{\{(-1)(1-\cos^2\theta)\}} = \sqrt{\{(i^2\sin^2\theta)\}} = i\sin\theta$$

With these value and +ve sign, (3) gives $\quad P_n(\cos\theta) = \dfrac{1}{\pi}\displaystyle\int_0^{\pi}(\cos\theta + i\sin\theta\cos\phi)^n d\phi$.

Part (ii). Take $n = 1$ and +ve sign in (3). Thus, we obtain the required result.

(II) Laplace's second integral for $P_n(x)$. When n is a positive integer,

$$P_n(x) = \frac{1}{\pi}\int_0^{\pi}\frac{d\phi}{[x \pm (x^2-1)^{1/2}\cos\phi]^{n+1}} . \qquad \textbf{[Kanpur 2007; Meerut 2007]}$$

Proof. From integral calculus, $\displaystyle\int_0^{\pi}\frac{d\phi}{a \pm b\cos\phi} = \frac{\pi}{(a^2-b^2)^{1/2}}$, where $a^2 > b^2$. ...(1)

Let $a = zx - 1$ and $b = (x^2-1)^{1/2}$. Then, $a^2 - b^2 = (zx-1)^2 - z^2(x^2-1) = 1 - 2zx + z^2$
Using these values of a, b and $a^2 - b^2$, (1) gives

$$\pi(1-2zx+z^2)^{-1/2} = \int_0^{\pi}\left[-1 + zx \pm z(x^2-1)^{1/2}\cos\phi\right]^{-1} d\phi$$

or $$\frac{\pi}{z}\left(1-2x\,\frac{1}{z}+\frac{1}{z^2}\right)^{-1/2} = \int_0^{\pi}[-1+z\,\{x \pm (x^2-1)^{1/2}\cos\phi\}]^{-1}\,d\phi. \qquad ...(2)$$

Let $$t = x \pm (x^2-1)^{1/2}\cos\phi. \qquad ...(3)$$

We know that $$(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x). \qquad ...(4)$$

Replacing z by $\frac{1}{z}$ in (4), we have $$\left(1-2x\times\frac{1}{z}+\frac{1}{z^2}\right)^{-1/2} = \sum_{n=0}^{\infty}\frac{1}{z^n}P_n(x). \qquad ...(5)$$

Using (3) and (5), we have

$$\frac{\pi}{z}\sum_{n=0}^{\infty}\frac{1}{z^n}P_n(x) = \int_0^{\pi}(-1+zt)^{-1}d\phi = \int_0^{\pi}(zt)^{-1}\left(1-\frac{1}{zt}\right)^{-1}d\phi = \int_0^{\pi}\frac{1}{zt}\sum_{n=0}^{\infty}\left(\frac{1}{zt}\right)^n d\phi = \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\int_0^{\pi}\frac{d\phi}{t^{n+1}}.$$

$$\therefore \qquad \sum_{n=0}^{\infty}\frac{\pi}{z^{n+1}}P_n(x) = \sum_{n=0}^{\infty}\frac{1}{z^{n+1}}\int_0^{\pi}\frac{d\phi}{\{x \pm (x^2-1)^{1/2}\cos\phi\}^{n+1}}. \qquad ...(6)$$

Equating the coefficients of $1/z^{n+1}$ from sides, (6) gives

$$\pi P_n(x) = \int_0^{\pi}\frac{d\phi}{\{x \pm (x^2-1)^{1/2}\cos\phi\}^{n+1}} \qquad \text{or} \qquad P_n(x) = \frac{1}{\pi}\int_0^{\pi}\frac{d\phi}{\{x \pm (x^2-1)^{1/2}\cos\phi\}^{n+1}} \qquad ...(7)$$

Deduction. Replacing n by $-(n+1)$ in (7), we have

$$P_{-(n+1)}(x) = \frac{1}{\pi}\int_0^{\pi}\frac{d\phi}{\{x \pm (x^2-1)^{1/2}\cos\phi\}^{-n}} = \int_0^{\pi}\frac{1}{\pi}\{x \pm (x^2-1)^{1/2}\cos\phi\}^n\,d\phi$$

$= P_n(x)$, by form I of Laplace's integral

Thus, $$P_n(x) = P_{-(n+1)}(x).$$

9.7. Some bounds on $P_n(x)$. If $-1 < x < 1$ and n is any positive integer, then

(*i*) $|P_n(x)| < 1$ (*ii*) $|P_n(x)| < \{\pi/2n(1-x^2)\}^{1/2}$.

Proof. If $-1 < x < 1$, we have

$|x+(x^2-1)^{1/2}\cos\phi| = |x+i(1-x^2)^{1/2}\cos\phi| = [x^2+(1-x^2)\cos^2\phi]^{1/2}$

$= [\cos^2\phi + x^2(1-\cos^2\phi]^{1/2} = [1-\sin^2\phi + x^2\sin^2\phi]^{1/2}$.

Thus, $$|x+(x^2-1)^{1/2}\cos\phi| = [1-(1-x^2)\sin^2\phi]^{1/2}. \qquad ...(1)$$

(*i*) If $\phi \neq 0$ or $\phi \neq \pi$, $$(1) \Rightarrow |x+(x^2-1)^{1/2}\cos\phi| < 1. \qquad ...(2)$$

Now, $$|P_n(x)| = \left|\frac{1}{\pi}\int_0^{\pi}[x+(x^2-1)^{1/2}\cos\phi]^n\,d\phi\right|,$$ using Laplace's first integral for $P_n(x)$

$$\Rightarrow \qquad |P_n(x)| \le \frac{1}{\pi}\int_0^{\pi}|x+(x^2-1)^{1/2}\cos\phi|^n\,d\phi < \frac{1}{\pi}\int_0^{\pi}1\,.\,d\phi, \text{ using (2)}$$

Thus, $|P_n(x)| < 1$, if $-1 < x < 1$.

(*ii*) We have, $$|P_n(x)| = \left|\frac{1}{\pi}\int_0^{\pi}[x+(x^2-1)^{1/2}\cos\phi]^n d\phi\right| \le \frac{1}{\pi}\int_0^{\pi}|x+(x^2-1)\cos\phi|^n\,d\phi$$

or $$| P_n(x) | \le \frac{1}{\pi}\int_0^{\pi} [1-(1-x^2)\sin^2 \phi]^{n/2} d\phi, \text{ using (1)}$$

or $$| P_n(x) | \le \frac{2}{\pi}\int_0^{\pi/2} [1-(1-x^2)\sin^2 \phi]^{n/2} d\phi. \qquad ...(3)$$

But, we know that, if $0 < \phi < \pi$, then $\sin \phi > 2\phi/\pi$.

This $\Rightarrow 1 - (1 - x^2) \sin^2\phi < 1 - (1 - x^2) \times (2\phi/\pi)^2 \Rightarrow 1 - (1 - x^2) \sin^2\phi < \exp[-4\phi^2 (1 - x^2)/\pi^2]$
[Here $4\phi^2 (1 - x^2)/\pi^2 < 1$ and $\exp a = e^a$]

Then, (3) $\Rightarrow$ $$| P_n(x) | < \frac{2}{\pi}\int_0^{\pi/2} \exp[-2n\phi^2(1-x^2)/\pi^2]\, d\phi$$

Thus, $$|P_n(x)| < \frac{2}{\pi}\int_0^{\infty} \exp[-2n\, \phi^2(1-x^2)/\pi^2]\, d\phi. \qquad ...(4)$$

Putting $[2n\phi^2 \quad (1 - x^2)/\pi^2]^{1/2} = t$ so that $d\phi = [\pi/\{2n(1 - x^2)\}^{1/2}]dt$, (4) becomes

$$| P_n(x) | < \frac{2}{\pi}\cdot\frac{\pi}{\{2n(1-x^2)\}^{1/2}}\int_0^{\infty} e^{-t^2} dt = \frac{2}{\{2n(1-x^2)\}^{1/2}}\times\frac{\sqrt{\pi}}{2} \Rightarrow |P_n(x)| < \left[\frac{\pi}{2n\,(1-x^2)}\right]^{1/2}.$$

9.8. Orthogonal properties of Legendre's polynomials

Prove that (i) $\int_{-1}^{1} P_m(x)\, P_n(x)\, dx = 0$ *if* $m \ne n$.

[Agra 2006, 08, 09, 10; Kanpur 2006, 08; Meerut 2010, 11]

(*ii*) $\int_{-1}^{1} [P_n(x)]^2\, dx = \frac{2}{2n+1}$. **[Agra 2009, 10; Meerut 1992, 93]**

(*iii*) $$\int_{-1}^{1} P_m(x)\, P_n(x)\, dx = \begin{cases} 0 \text{ if, } m \ne n \\ 2/(2n+1), \text{ if } m = n. \end{cases}$$

[Agra 2005, Bilaspur 1998; Ravishankar 1996; Nagpur 2005]

or $$\int_{-1}^{1} P_m(x)\, P_n(x)\, dx = \frac{2}{2n+1}\delta_{mn}, \qquad \text{where} \qquad \delta_{mn} = \begin{cases} 0 \text{ if } m \ne n \\ 1 \text{ if } m = n. \end{cases}$$

Here δ_{mn} is called Kronecker delta. It is also denoted by δ_n^m or δ^{mn} or δ_{mn}.

[Kanpur 2006; Delhi Physics (H) 2000]

Proof (i). When m ≠ n. Since $P_m(x)$ and $P_n(x)$ satisfy Legendre's equation we have

$$(1 - x^2)P_m'' - 2xP_m' - m(m + 1)P_m = 0 \qquad ...(1)$$

and $$(1 - x^2)P_n'' - 2xP_n' + n(n + 1)P_n = 0. \qquad ...(2)$$

Multiplying (1) by P_n and (2) by P_m and then subtracting the resulting equations, we have

$$(1 - x^2)(P_n P_m' - P_m P_n') - 2x(P_n P_m' - P_n' P_m) + [m(m + 1) - n(n + 1)]P_m P_n = 0$$

or $$(1 - x^2)\frac{d}{dx}(P_n P_m' - P_n' P_m) - 2x(P_n P_m' - P_m P_n') = (n^2 - m^2 + n - m)\, P_m P_n$$

or $$\frac{d}{dx}\{(1 - x^2)(P_n P_m' - P_n' P_m)\} = (n - m)(n + m + 1)P_m P_n.$$

Integrating both sides w.r.t. 'x' from -1 to 1, we get

$$(n - m)(n + m + 1)\int_{-1}^{1} P_m(x)\, P_n(x)\, dx = \Big[(1 - x^2)(P_n P_m' - P_n' P_m)\Big]_{x=-1}^{x=1}$$

$$\therefore \qquad \int_{-1}^{1} P_m(x)\, P_n(x)\, dx = 0, \quad \text{as} \quad m \ne n. \qquad ...(3)$$

(ii). When m = n. Then the required result takes the form $\int_{-1}^{1} [P_n(x)]^2\, dx = 2/(2n+1)$.

To prove this, start with generating function $(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x)$. ...(4)

Also, (4) may be re-written as $(1-2xz+z^2)^{-1/2} = \sum_{m=0}^{\infty} z^m P_m(x)$ ...(5)

Multiplying the corresponding sides of (4) and (5), we get

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} P_m(x)\, P_n(x)\, z^{m+n} = (1-2xz+z^2)^{-1}$$

Integrating both sides of the above equation w.r.t. 'x' we get

$$\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\left\{\int_{-1}^{1} P_m(x)\, P_n(x)\, dx\right\} z^{m+n} = \int_{-1}^{1} (1-2xz+z^2)^{-1}\, dx \qquad ...(6)$$

Making use of (3), (6) reduces to

$$\sum_{n=0}^{\infty}\left[\int_{-1}^{1}\{P_n(x)\}^2\, dx\right] z^{2n} = \int_{-1}^{1}\frac{dx}{1+z^2-2xz} = \left[\frac{\log(1+z^2-2zx)}{-2z}\right]_{-1}^{1} = -\frac{1}{2z}[\log(1-z)^2 - \log(1+z)^2]$$

$= -(1/2z)\times[2\log(1-z) - 2\log(1+z)] = (1/z)\times[\log(1+z) - \log(1-z)]$

$$= \frac{1}{z}\left[\left(z-\frac{z^2}{2}+\frac{z^3}{3}-\ldots\ldots\right)-\left(-z-\frac{z^2}{2}-\frac{z^2}{3}-\ldots\ldots\right)\right] = \frac{2}{z}\left(z-\frac{z^3}{3}+\frac{z^5}{5}+\ldots\ldots\right) = \frac{2}{z}\sum_{n=0}^{\infty}\frac{z^{2n+1}}{2n+1}.$$

$$\therefore \qquad \sum_{n=0}^{\infty}\left[\int_{-1}^{1}\{P_n(x)\}^2\, dx\right] = \sum_{n=0}^{\infty}\frac{2}{2n+1}z^{2n}. \qquad ...(7)$$

Equating coefficients of z^{2n} from both sides, (7) gives $\int_{-1}^{1}[P_n(x)]^2\, dx = 2/(2n+1)$. ... (8)

(*iii*) Combining results (3) and (8), we have

$$\int_{-1}^{1} P_m(x)P_n(x)dx = \begin{cases} 0, \text{ if } m = n \\ 2/(2n+1), \text{ if } m = n \end{cases} \qquad \text{or} \qquad \int_{-1}^{1} P_m(x)P_n(x)dx = \frac{2}{2n+1}\delta_{mn},$$

where $\delta_{mn} = \text{Kronecker delta} = \begin{cases} 0, \text{ if } m \neq n \\ 1, \text{ if } m = n \end{cases}$

9.9. Recurrence relations (formulae). To show that

I. $nP_n = (2n-1)\, xP_{n-1} - (n-1)P_{n-2}$., $n \geq 2$ **[Delhi Physics (H) 2000; Agra 2005; Bilaspur 1998, Purvanchal 2005, Kanpur 2006, 09]**

or $(n+1)P_{n+1} = (2n+1)\, xP_n - nP_{n-1}$, $n \geq 1$ **[Nagpur 2005; Kakitiya 1997; Agra 2005, 06; KU Kurukshetra 2006; Meerut 2006; Ravishankar 2004; Vikram 2004]**

or $xP_n(x) = \dfrac{n+1}{2n+1}P_{n+1}(x) + \dfrac{n}{2n+1}P_{n-1}(x)$. **[Utkar 2003; Agra 1998]**

II. $nP_n = xP'_n - P'_{n-1}$. **[Purvanchal 2004; Kanpur 2011; Vikram 2000; Bangalore 1995; Meerut 1996]**

III. $(2n+1)P_n = P'_{n+1} - P'_{n-1}$. **[Bilaspur 1993; Bangalore 1992, 93]**

IV. $(n+1)P_n = P'_{n+1} - xP'_n$. or $P_n' - xP'_{n-1} = nP_{n-1}$. **[Kanpur 2007]**

V. $(1-x^2)P_n' = n(P_{n-1} - xP_n)$ or $(x^2-1)P'_n = n\,x\,P_n - nP_{n-1}$.

[Delhi Physics (H) 2001]

VI. $(1-x^2)\,P'_n = (n+1)\,(xP_n - P_{n+1})$. **[Meerut 1997]**

Proof I. From generating function, we have $$(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x). \qquad ...(1)$$

Differentiating both sides of (1) w.r.t. 'z', we get

$$-\tfrac{1}{2}(1-2xz+z^2)^{-3/2}\,(-2x+2z) = \sum_{n=0}^{\infty} n\,z^{n-1}P_n(x). \qquad ...(2)$$

Multiplying both sides by $1-2xz+z^2$, (2) gives

$$(x-z)\,(1-2xz+z^2)^{-1/2} = (1-2xz+z^2)\sum_{n=0}^{\infty} n\,z^{n-1}P_n(x)$$

or $$(x-z)\sum_{n=0}^{\infty} z^n P_n(x) = (1-2xz+z^2)\sum_{n=0}^{\infty} n\,z^{n-1}P_n(x), \text{ by (1)}$$

or $$x\sum_{n=0}^{\infty} z^n P_n - \sum_{n=0}^{\infty} z^{n+1}P_n(x) = \sum_{n=0}^{\infty} n\,z^{n-1}P_n - 2x\sum_{n=0}^{\infty} n\,z^n P_n + \sum_{n=0}^{\infty} n\,z^{n+1}P_n.$$

Equating coefficients of z^n from both sides, we get

$$xP_n - P_{n-1} = (n+1)P_{n+1} - 2xnP_n + (n-1)P_{n-1}$$

or $$(n+1)P_{n+1} = (2n+1)\,xP_n - nP_{n-1}. \qquad ...(3)$$

Replacing n by $n-1$ in (3), we get $$nP_n = (2n-1)\,x\,P_{n-1} - (n-1)P_{n-2}. \qquad ...(4)$$

Again (3) can be re–arranged to give another form $$xP_n = \frac{n+1}{2n+1}P_{n+1} + \frac{n}{2n+1}P_{n-1}.$$

II. We have $$(1-2xz+z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x). \qquad ...(1)$$

Differentiating (1) w.r.t. 'z', we get $$-\tfrac{1}{2}(1-2xz+z^2)^{-3/2}\,(-2x+2z) = \sum_{n=0}^{\infty} n\,z^{n-1}P_n$$

or $$(x-z)\,(1-2xz+z^2)^{-3/2} = \sum_{n=0}^{\infty} n\,z^{n-1}P_n. \qquad ...(2)$$

Again, differentating (1) w.r.t. 'x' and simplifying, we get

$$z(1-2xz+z^2)^{-3/2} = \sum_{n=0}^{\infty} z^n P_n'. \quad \text{or} \quad z(x-z)\,(1-2xz+z^2)^{-3/2} = (x-z)\sum_{n=0}^{\infty} z^n P_n'$$

or $$z\sum_{n=0}^{\infty} n\,z^{n-1}P_n = (x-z)\sum_{n=0}^{\infty} z^n P_n', \text{ by (2)} \quad \text{or} \quad \sum_{n=0}^{\infty} n\,z^n P_n = x\sum_{n=0}^{\infty} z^n P_n' - \sum_{n=0}^{\infty} z^{n+1}P_n'.$$

Equating coefficient of z^n on both sides we get $$nP_n = xP'_n - P'_{n-1}.$$

III. From recurrence relation I, $$(2n+1)\,xP_n = (n+1)P_{n+1} + nP_{n-1}.$$

Differentiating it w.r.t. 'x', $$(2n+1)\,xP'_n + (2n+1)P_n = (n+1)P'_{n+1} + nP'_{n-1}$$

or $$(2n+1)\,(nP_n + P'_{n-1}) + (2n+1)P_n = (n+1)P'_{n+1} - nP'_{n-1}$$

[$\because$ from recurrence II, $xP'_n = nP_n + P'_{n-1}$]

or $$(2n+1)\,(n+1)P_n = (n+1)P'_{n+1} - (n+1)P'_{n-1}$$

or $$(2n+1)P_n = P'_{n+1} - P'_{n-1}. \qquad ...(1)$$

Replacing n by $n-1$ in (1), we have $(2n-1)P_{n-1} = P'_n - P'_{n-2}$

or $$\frac{dP_n(x)}{dx} = \frac{dP_{n-2}(x)}{dx} + (2n-1)P_{n-1}(x) \qquad ...(2)$$

(1) and (2) are the required forms of the results.

IV. From recurrence relations II and III, we get

$$nP_n = xP_n' - P'_{n-1} \qquad ...(1)$$

and $$(2n+1)P_n = P'_{n+1} - P'_{n-1} \qquad ...(2)$$

Subtracting (1) from (2), $(n+1)P_n = P'_{n+1} - xP'_n$

V. From recurrence relations II and IV, we get

$$nP_n = xP'_n - P'_{n-1} \qquad ...(1)$$

and $$(n+1)P_n = P'_{n+1} - xP'_n. \qquad ...(2)$$

Replacing n by $n-1$ in (2), $nP_{n-1} = P'_n - xP'_{n-1}$. ...(3)

Multiplying both sides of (1) by x, $xnP_n = x^2P'_n - xP'_{n-1}$. ...(4)

Subtracting (4) from (3), we have

$n(P_{n-1} - xP_n) = (1-x^2)P'_n$ or $(x^2-1)P'_n = nxP_n - nP_{n-1}$.

VI. From recurrence relations I and V, we have

$$(2x+1)xP_n = (n+1)P_{n+1} + nP_{n-1} \qquad ...(1)$$

and $$(1-x^2)P'_n = n(P_{n-1} - xP_n). \qquad ...(2)$$

Re–writting (1), $[(n+1)+n]xP_n = (n+1)P_{n+1} + nP_{n-1}$

or $$(n+1)\,(xP_n - P_{n+1}) = n(P_{n-1} - xP_n). \qquad ...(3)$$

Now, from (2) and (3), $(1-x^2)P'_n = (n+1)\,(xP_n - P_{n+1})$.

9.10. Beltrami's result.

Prove that $(2n+1)\,(x^2-1)\,P_n' = n(n+1)(P_{n+1} - P_{n-1})$. **[Ravishankar 1998]**

Sol. Recurrence relations V and VI are

$$(1-x^2)P'_n = n(P_{n-1} - xP_n) \qquad ...(1)$$

and $$(1-x^2)P'_n = (n+1)\,(xP_n - P_{n+1}) \qquad ...(2)$$

Multiplying (1) by $n+1$ and (2) by n and adding, we get

$$(n+1)\,(1-x^2)P'_n + n(1-x^2)P_n' = n(n+1)P_{n-1} - n(n+1)P_{n+1}$$

or $$(2n+1)\,(1-x^2)P'_n = n(n+1)\,(P_{n-1} - P_{n+1})$$

or $$(2n+1)\,(x^2-1)P'_n = n(n+1)\,(P_{n+1} - P_{n-1}),$$

which is known as *Beltrami's result.*

9.11. Christoffel's summation formula.

Prove that $$\sum_{k=0}^{l}(2k+1)\,P_k(x)\,P_k(y) = \frac{l+1}{x-y}\{P_{l+1}(x)\,P_l(y) - P_l(x)\,P_{l+1}(y)\}.$$

[Bilaspur 1997, Jodhpur 2004; Ravishankar 2000; Garhwal 2005]

Deduce that $$\sum_{k=0}^{l}(2k+1)P_k(x) = \frac{l+1}{x-1}\{P_{l+1}(x) - P_l(x)\}.$$

Sol. From recurrence relation I, we have

$$(2k+1)\,xP_k(x) = (k+1)P_{k+1}(x) + kP_{k-1}(x) \qquad ...(1)$$

and $$(2k+1)\,yP_k(y) = (k+1)P_{k+1}(y) + kP_{k-1}(y). \qquad ...(2)$$

Multiplying (1) by $P_k(y)$ and (2) by $P_k(x)$ and then substracting, we get

$(2k + 1)\,(x - y)P_k(x)P_k(y)$

$$= (k + 1)\,[P_{k+1}(x)P_k(y) - P_{k+1}(y)\,P_k(x)] - k[P_{k-1}(x)\,P_k(y) - P_{k-1}(y)\,P_k(x)]. \qquad ...(3)$$

Replacing k by 0, 1, 2, 3, ..., $l - 1$, l successively in (3) and adding the resulting equations, we have

$$(x - y)\sum_{k=0}^{l}(2k + 1)\,P_k(x)\,P_k(y) = (l + 1)\,[P_{l+1}(x)\,P_l(y) - P_{l+1}(y)\,P_l(x)]$$

or

$$\sum_{k=0}^{l}(2k + 1)\,P_k(x)\,P_k(y) = \frac{l+1}{x-y}[P_{l+1}(x)\,P_l(y) - P_{l+1}(y)\,P_l(x)]. \qquad ...(4)$$

Deduction. Replace y by 1 in (4) and use $P_l(1) = P_{l+1}(1) = P_k(1) = 1$ to get the required result.

9.12. Christoffel's expansion. To show that $P'_n = (2n - 1)P_{n-1} - (2n - 5)P_{n-3} + (2n - 9)P_{n-5} + ...$, the last terms of the series being $3P_1$ or P_0 according as n is even or odd.

or $$P_n'(x) = \sum_{r=0}^{\left[\frac{1}{2}(n-1)\right]}(2n - 4r - 1)\,P_{n-2r-1}(x), \quad \text{where} \quad \left[\tfrac{1}{2}(n-1)\right] = \begin{cases}(n-1)/2, & \text{if } n \text{ is even}\\(n-2)/2, & \text{if } n \text{ is odd}\end{cases}$$

Sol. Replacing n by $n - 1$ in recurrence relation $(2n+1)P_n = P'_{n+1} - P'_{n-1}$, we have

$$P'_n = (2n - 1)P_{n-1} + P'_{n-2} \qquad ...(1)$$

Case I. Let n be even, Replacing n by n, $n - 2$, $n - 4$, ..., 4, 2, successively in (1) and using the fact that $P_0 = 1$ and $P'_0 = 0$, we get

$$P'_n = (2n - 1)P_{n-1} + P'_{n-2},$$
$$P'_{n-2} = (2n - 5)P_{n-3} + P'_{n-4},$$
$$P'_{n-1} = (2n - 9)\,P_{n-5} + P'_{n-6},$$
$$\text{....} \qquad \text{.... ....}$$
$$P'_4 = 7P_3 + P'_2$$

and $$P'_2 = 3P_1 + P'_0$$

Adding these and simplifying, $P'_n = (2n - 1)P_{n-1} + (2n - 5)P_{n-3} + ... + 3P_1.$ ...(2)

Case II. Let *n* is odd. Replacing n by n, $n - 2$, $n - 4$, ..., 5, 3 successively in (1) and using the fact that $P_1(x) = x$, $P_0 = 1$ so that $P'_1 = 1 = P_0$, we get

$$P'_n = (2n - 1)P_{n-1} + P'_{n-2}$$
$$P'_{n-2} = (2n - 5)P_{n-3} + P'_{n-4}$$
$$P'_{n-4} = (2n - 9)P_{n-5} + P'_{n-6}$$
$$\text{...} \qquad \text{.... ....}$$
$$\text{...} \qquad \text{.... ....}$$
$$P_5' = 9P_4 + P'_3$$

and $$P_3' = 5P_2 + P'_1 = 5P_2 + P_0.$$

Adding these and simplifying, we get $P'_n = (2n - 1)P_{n-1} + (2n - 5)P_{n-3} + ... + 5P_2 + P_0.$ ...(3)

Combining (2) and (3), we have $$P'_n(x) = \sum_{r=0}^{\left[\frac{1}{2}(n-1)\right]}(2n - 4r - 1)\,P_{n-2r-1}(x).$$

9.13. Solved examples based on Art. 9.8 and 9.9

Ex. 1. *If $P_n(x)$ is a Legendre polynomial of degree n and α is such that $P_n(\alpha) = 0$. Show that $P_{n-1}(\alpha)$ and $P_{n+1}(\alpha)$ are of opposite signs.* **[Purvanchal 2006]**

Sol. From recurrence relation I, $(2n + 1)\,xP_n(x) = (n + 1)P_{n+1}(x) + nP_{n-1}(x).$ ...(1)

Given that $\qquad P_n(\alpha) = 0.$ ... (2)

Putting $x = \alpha$ in (1) and using (2), $\quad (2n+1)\,\alpha\,.\,0 = (n+1)P_{n+1}(\alpha) + nP_{n-1}(\alpha)$

or
$$\frac{P_{n+1}(\alpha)}{P_{n-1}(\alpha)} = -\frac{n}{n+1}. \qquad ...(3)$$

Since n is a positive integer, so R.H.S. of (3) is negative. Hence (3) shows that $P_{n+1}(\alpha)$ and $P_{n-1}(\alpha)$ must be of opposite signs.

Ex. 2. *Prove that* $P'_{n+1} + P'_n = P_0 + 3P_1 + ... + (2n+1)P_n$.

or $\quad \sum_{r=0}^{n} (2r+1)\,P_r(x) = P'_{n+1}(x) + P'_n(x).$ **[Meerut 1993]**

Sol. From recurrence relation III, we have $\qquad (2n+1)P_n = P'_{n+1} - P'_{n-1}.$ ...(1)

Replacing n by 1, 2, ...$n-1$, n successively in (!), we get

$$3P_1 = P'_2 - P'_0,$$
$$5P_2 = P'_3 - P'_1,$$
$$7P_3 = P'_4 - P'_2,$$
$$\cdots \qquad \cdots\cdots$$
$$\cdots \qquad \cdots\cdots$$
$$(2n-1)P_{n-1} = P'_n - P'_{n-2}$$

and
$$(2n+1)P_n = P'_{n+1} - P'_{n-1}.$$

Adding these and noting that in the sum of right hand sides all the terms cancel except the first two of the second column and the last two of the first column, we get

$$3P_1 + 5P_2 + 7P_3 + ... + (2n+1)P_n = -P_0' - P_1' + P_n' + P'_{n+1}. \qquad ...(2)$$

Since $P_0 = 1$ and $P_1 = x$, we have $P_0' = 0$ and $P_1' = 1 = P_0$. Using these results in (2), we have

$$3P_1 + 5P_2 + 7P_3 + ... + (2n+1)P_n = 0 - P_0 + P_n' + P'_{n+1}$$

or
$$P_0 + 3P_1 + 5P_2 + ...+ (2n+1)P_n = P'_n + P'_{n+1}$$

or
$$\sum_{r=0}^{n} (2r+1)\,P_r(x) = P'_n(x) + P'_{n+1}(x).$$

Ex. 3. *Prove that* (*i*) $c + \int P_n\,dx = (P_{n+1} - P_{n-1})\,/(2n+1).$

(*ii*) $\int_x^1 P_n\,dx = (P_{n-1} - P_{n+1})\,/(2n+1).$

Sol. From recurrence relations III, we have

$(2n+1)P_n = P'_{n+1} - P'_{n-1}$ $\qquad$ or $\qquad P_n = \dfrac{1}{2n+1}\dfrac{d}{dx}(P_{n+1} - P_{n-1}).$ ...(1)

(*i*) Integrating (1), $\quad c + \int P_n\,dx = (P_{n+1} - P_{n-1})/(2n+1)$, c being an arbitrary constant

(*ii*) Integrating both sides of (1) w.r.t. 'x' between limits of x to 1, we get

$$\int_x^1 P_n(x)\,dx = \frac{1}{2n+1}\Big[P_{n+1}(x) - P_{n-1}(x)\Big]_x^1 = \frac{1}{2n+1}[P_{n+1}(1) - P_{n-1}(1) - \{P_{n+1}(x) - P_{n-1}(x)\}]$$
$$= \{P_{n-1}(x) - P_{n+1}(x)\}/(2n+1), \text{ as } P_{n+1}(1) = P_{n-1}(1) = 1.$$

Ex. 4. *Prove that* $P_0^2 + 3P_1^2 + 5P_2^2 + ... + (2n+1)P_n^2$

$= (n+1)[P_n(x)\,P'_{n+1}(x) - P_{n+1}(x)\,P'_n(x)] = (n+1)^2\,[\{P_n(x)\}^2 + (1-x^2)\,\{P'_n(x)\}^2.$

Sol. From Christoffel's summation formula (Refer Art 9.11), we have

$$(x-y)\sum_{r=0}^{n}(2r+1)\,P_r(x)\,P_r(y) = (n+1)[P_{n+1}(x)\,P_n(y) - P_{n+1}(y)\,P_n(x)]. \qquad ...(1)$$

Putting $y = x + h$, where h is a small quantity in (1), we obtain

$$-h\sum_{r=0}^{n}(2r+1)\,P_r(x)\,P_r(x+h) = (n+1)[P_{n+1}(x)\,P_n(x+h) - P_{n+1}(x+h)\,P_n(x)].$$

Expanding by Taylor's theorem on both sides, we have

$$-h\sum_{r=0}^{n}(2r+1)\,P_r(x)\,\{P_r(x) + h\,P_r'(x) + \ldots\}$$

$$=(n+1)[P_{n+1}(x)\,\{P_n(x)+hP'_n(x)+\frac{h^2}{2\,!}P''_n(x)+\ldots\}-P_n(x)\,\{P_{n+1}(x)+hP'_{n+1}(x)+\frac{h^2}{2\,!}P''_{n+1}(x)+\ldots\}]$$

$$= -h(n+1)\,[\{P_n(x)\,P'_{n+1}(x) - P'_n(x)\,P_{n+1}(x)\} + \text{terms containing } h]$$

or $$\sum_{r=0}^{n}(2r+1)\,P_r(x)\,\{P_r(x)+h\,P_r'(x)+\ldots\} = (n+1)\,[P_n(x)\,P'_{n+1}(x) - P'_n(x)\,P_{n+1}(x)\} + \text{terms containing } h].$$

Taking limit on both sides as $h \to 0$, we obtain

$$\sum_{r=0}^{\infty}(2r+1)\,[P_r(x)]^2 = (n+1)\,[P_n(x)P'_{n+1}(x) - P'_n(x)\,P_{n+1}(x)]$$

or $$P_0^2(x) + 3P_1^2(x) + 5P_2^2(x) + \ldots.. + (2n+1)\,P_n^2(x) = (n+1)\,[P_n(x)\,P'_{n+1}(x) - P'_n(x)\,P_{n+1}(x)] \qquad ...(2)$$

Now, $(n+1)^2\ \{P_n(x)\}^2 + (1-x^2)\ \{P'_n(x)\}^2$

$$= (n+1)P_n(x)\ [(n+1)P_n(x)] + P'_n(x)\ \{(1-x^2)\ P'_n(x)\}$$

$$= (n+1)P_n(x)\ [P'_{n+1}(x) - xP'_n(x)] + P'_n(x)\ [(n+1)\ \{xP_n(x) - P_{n+1}(x)\}]$$

(Using recurrence formulas IV and VI)

Thus, $$(n+1)^2\ \{P_n(x)\}^2 + (1-x^2)\ \{P'_n(x)\}^2 = (n+1)\ [P_n(x)P'_{n+1}(x) - P_{n+1}(x)P'_n(x)] \qquad ...(3)$$

From (2) and (3), we have

$$P_0^2(x) + 3P_1^2(x) + \ldots + (2n+1)\ P_n^2(x) = (n+1)^2\ \{P_n(x)\}^2 + (1-x^2)\ \{P'_n(x)\}^2.$$

Ex. 5. *Evaluate :* (*i*) $\int_0^1 P_n(x)\,dx$, when n is odd. **[Jiwaji 2004]**

(*ii*) $\int_0^1 P_n(x)\,dx$, when n is *even.* **[Gulbarga 2005]**

Sol. (*i*) From recurrence relation III, we have $(2n+1)P_n = P'_{n+1} - P'_{n-1}$

or $$P_n(x) = \frac{1}{2n+1}\frac{d}{dx}[P_{n+1}(x) - P_{n-1}(x)].$$

Integrating both sides w.r.t. 'x' from 0 to 1, we get

$$\int_0^1 P_n(x)\,dx = \frac{1}{2n+1}\Big[P_{n+1}(x) - P_{n-1}(x)\Big]_0^1 = \frac{1}{2n+1}\Big[P_{n+1}(1) - P_{n-1}(1) - P_{n+1}(0) + P_{n-1}(0)\Big] \qquad ....(1)$$

But $\qquad P_{n+1}(1) = 1, \qquad P_{n-1}(1) = 1 \qquad$ and $\qquad P_{2l}(0) = \dfrac{(-1)^l}{2^{2l}}\,\dfrac{(2l)\,!}{(l\,!)^2}. \qquad ...(2)$

Since n is odd, $n-1$ and $n+1$ are both even. Taking $2l = n-1$, *i.e.*, $l = (n-1)/2$ in (2), we have $$P_{n-1}(0) = \frac{(-1)^{(n-1)/2}}{2^{n-1}} \; \frac{(n-1)\,!}{[\{\frac{1}{2}(n-1)\}\,!]^2}. \qquad ...(3)$$

Next, taking $2l = n + 1$ *i.e.* $l = (n+1)/2$ in (2), we have

$$P_{n+1}(0) = \frac{(-1)^{(n+1)/2}}{2^{n+1}} \; \frac{(n+1)\,!}{[\{\frac{1}{2}(n+1)\}\,!]^2} = -\frac{(-1)^{(n-1)/2}(n+1)\,n\,(n-1)\,!}{2^2 \cdot 2^{n-1}\left[\dfrac{n+1}{2}\cdot\dfrac{n}{2}\cdot\left(\dfrac{n-1}{2}\right)!\right]^2}. \qquad ...(4)$$

Using (2), (3) and (4) in (1), we have

$$\int_0^1 P_n(x)\,dx = \frac{1}{2n+1} \times \frac{(-1)^{(n-1)/2}}{2^{n-1}} \times \frac{(n-1)!}{\left[\left(\dfrac{n-1}{2}\right)!\right]^2}\left[\frac{n}{n-1}+1\right]$$

$$= \frac{(-1)^{(n-1)/2}\,(n-1)!}{2^n\left(\dfrac{n}{2}+\dfrac{1}{2}\right)\left(\dfrac{n}{2}-\dfrac{1}{2}\right)!\left(\dfrac{n}{2}-\dfrac{1}{2}\right)!} = \frac{(-1)^{(n-1)/2}\,(n-1)!}{2^n\left(\dfrac{n}{2}+\dfrac{1}{2}\right)!\left(\dfrac{n}{2}-\dfrac{1}{2}\right)!} \qquad [\because\ (p+1)\,p\,! = (p+1)!]$$

Part (ii). Proceed upto equation (1) as in part (*i*). Since n is even, $n-1$ and $n+1$ are both odd and so $P_{n-1}(0) = 0$, $\quad P_{n+1}(0) = 0$, $\quad P_{n+1}(1) = 0 \quad$ and $\quad P_{n-1}(1) = 0$ ...(5)

Making use of (5), (1) reduces to $\displaystyle\int_0^1 P_n(x)\,dx = 0$, when n is even.

Ex. 6. *Prove that* $\displaystyle\int_{-1}^1 x\,P_n(x)\,P_{n-1}(x)\,dx = \frac{2n}{4n^2-1}$.

[Purvanchal 2007; Gulbarga 2005; Sagar 2004; Kanpur 2007]

Sol. From recurrence relation *I*, $$xP_n(x) = \frac{n+1}{2n+1}P_{n+1}(x) + \frac{n}{2n+1}P_{n-1}(x). \qquad ...(1)$$

Multiplying both sides of (1) by $P_{n-1}(x)$ and then integrating w.r.t. x from -1 to 1, we get

$$\int_{-1}^1 x\,P_n(x)\,P_{n-1}(x)\,dx = \frac{n+1}{2n+1}\int_{-1}^1 P_{n+1}(x)\,P_{n-1}(x)\,dx + \frac{n}{2n+1}\int_{-1}^1 [P_{n-1}(x)]^2 dx. \qquad ...(2)$$

But, $$\int_{-1}^1 P_m(x)\,P_n(x)\,dx = \begin{cases} 0, \text{ if } m \neq n \\ 2/(2n+1), \text{ if } m = n \end{cases} \qquad ...(3)$$

Making use of (3), (2) becomes

$$\int_{-1}^1 xP_n(x)\,P_{n-1}(x)\,dx = 0 + \frac{n}{2n+1} \times \frac{2}{2(n-1)+1} = \frac{2n}{(2n+1)(2n-1)} \quad \text{or} \quad \frac{2n}{4n^2-1}.$$

Ex. 7. *Prove that* $$\int_{-1}^1 (1-x^2)\,P_l'\,.\,P_m'\,dx = \begin{cases} 0, \text{ if } l \neq m \\ \dfrac{2l(l+1)}{2l+1}, \text{ if } l = m \end{cases}$$

Or *Prove that* $\displaystyle\int_{-1}^1 (1-x^2)\,P_l'\,P_m'\,dx = \frac{2l(l+1)}{2l+1}\delta_{lm}$, where $\delta_{lm} = \begin{cases} 0, \text{ if } l \neq m \\ 1, \text{ if } l = m \end{cases}$

[Indore 2004; Purvanchal 2004; Ravishakar 2005]

Sol. Case I. Let $l \neq m$. Then integrating by parts, we have

$$\int_{-1}^{1} [(1-x^2)\,P_l']\,P_m'\,dx = \left[(1-x^2)P_l'.P_m\right]_{-1}^{1} - \int_{-1}^{1} [(1-x^2)\,P_l'' - 2x\,P_l']\,P_m\,dx = -\int_{-1}^{1} [(1-x^2)\,P_l'' - 2x\,P_l']\,P_m\,dx. \quad ...(1)$$

Since P_l satisfies Legendre's equation $(1-x^2)y'' - 2xy' + l(l+1)y = 0$, hence

$$(1-x^2)\,P_l'' - 2xP_l' + l(l+1)P_l = 0 \qquad \text{or} \qquad (1-x^2)P_l'' - 2xP_l' = -l(l+1)P_l. \quad ...(2)$$

But $$\int_{-1}^{1} P_l\,P_m\,dx = 0, \text{ if } l \neq m \quad ...(3)$$

Using (2), (1) reduces to

$$\int_{-1}^{1} (1-x^2)\,P_l'\,P_m'\,dx = l(l+1)\int_{-1}^{1} P_l\,P_m\,dx = 0, \text{ using (3).} \quad ...(4)$$

Case II. Let $l = m$. Then the required result takes the form

$$\int_{-1}^{1} (1-x^2)\,(P_l')^2\,dx = \frac{2l(l+1)}{2l+1}. \quad \textbf{[Agra 2010]} \quad ...(5)$$

We have, by using integration by parts,

$$\int_{-1}^{1} (1-x^2)\,(P_l')^2\,dx = \int_{-1}^{1} [(1-x^2)\,P_l'].\,P_l'\,dx$$

$$= \left[(1-x^2)P_l'P_l\right]_{-1}^{1} - \int_{-1}^{1}\left[(1-x^2)P_l'' - 2xP_l'\right]P_l\,dx = 0 + l(l+1)\int_{-1}^{1} (P_l)^2\,dx, \text{ using (2)}$$

$$= l(l+1)\,.\frac{2}{2l+1} = \frac{2l(l+1)}{2l+1}.$$

Combining (4) and (5) and using symbol δ_{lm}, we get $\int_{-1}^{1} (1-x^2)\,P_l'\,P_m'\,dx = \frac{2l(l+1)}{2l+1}\,\delta_{lm}.$

Ex. 8. *Prove that* $(2n+1)\,(x^2-1)P'_n = n(n+1)\,(P_{n+1} - P_{n-1})$

and hence prove that $\int_{-1}^{1} (x^2-1)\,P_{n+1}\,P_n'\,dx = \frac{2n\,(n+1)}{(2n+1)\,(2n+3)}$ **[Agra 2009; Nagpur 2010]**

Sol. Refer Art. 9.10 to show that $(2n+1)\,(x^2-1) = n(n+1)\,(P_{n+1} - P_{n-1})$. ...(1)

From (1), $$(x^2-1)P'_n = \frac{n\,(n+1)}{2n+1}\,(P_{n+1} - P_{n-1}). \quad ...(2)$$

Multiplying both sides of (2) by P_{n+1} and then integrating w.r.t. x from -1 to 1, we have

$$\int_{-1}^{1} (x^2-1)\,P_{n+1}\,P_n'\,dx = \frac{n(n+1)}{2n+1}\int_{-1}^{1} P_{n+1}\,(P_{n+1} - P_{n-1})dx = \frac{n\,(n+1)}{2n+1}\left[\int_{-1}^{1} P_{n+1}^2\,dx - \int_{-1}^{1} P_{n+1}\,P_{n-1}\,dx\right] \quad ...(3)$$

But $$\int_{-1}^{1} P_m\,P_n\,dx = \begin{cases} 0, \textit{ if } m \neq n \\ n/(2n+1), \textit{ if } m = n \end{cases} \quad ...(4)$$

Using (4), (3) reduces to

$$\int_{-1}^{1} (x^2-1)\,P_{n+1}\,P_n'\,dx = \frac{n\,(n+1)}{2n+1}\left[\frac{2}{2(n+1)+1} - 0\right] = \frac{2n\,(n+1)}{(2n+1)\,(2n+3)}$$

Ex. 9. (a). *Show that* $\int_{-1}^{1} x^2\,P_{n+1}(x)\,P_{n+1}(x)\,dx = \frac{2n\,(n+1)}{(2n-1)\,(2n+1)\,(2n+3)}$

[Kanpur 2007, 08]

(b) *Deduce the value of* $\int_0^1 x^2 P_{n+1}(x) P_{n+1}(x)\,dx$. *i.e., prove that* $\int_0^1 x^2 P_{n+1}(x) P_{n+1}(x)\,dx = \frac{n(n+1)}{(4n^2-1)(2n+3)}$.

Sol. (*a*) From recurrence relation *I*, $(2n-1)xP_{n-1} = nP_n + (n-1)P_{n-2}$. ...(1)

Replacing *n* by *n* + 2 in (1), $(2n+3)xP_{n+1} = (n+2)P_{n+2} + (n+1)P_n$. ...(2)

Multiplying the corresponding sides of (1) and (2), we get

$$(2n-1)(2n+3)x^2 P_{n-1}P_{n+1} = n(n+1)P_n^{\ 2} + n(n+2)P_{n+2}P_n + (n-1)(n+2)P_{n-2}P_{n+2} + (n-1)(n+1)P_{n-2}P_n. \quad ...(3)$$

Also, $$\int_{-1}^{1} P_m P_n\,dx = \begin{cases} 0, \text{ if } m \neq n \\ 2/(2n+1), \text{ if } m = n \end{cases} \quad ...(4)$$

Integrating (3) w.r.t. 'x' from –1 to 1 and using (4), we get

$$(2n-1)(2n+3)\int_{-1}^{1} x^2 P_{n-1}(x) P_{n+1}dx = n(n+1)\,.\,\frac{2}{2n+1} + 0 + 0$$

or $$\int_{-1}^{1} x^2 P_{n-1}(x) P_{n+1}dx = \frac{2n(n+1).}{(2n+3)(2n-1)(2n+1)} \quad ...(5)$$

Part (b). Deduction. Since $P_n(x)$ is a polynomial of degree *n*, so $x^2P_{n-1}(x) P_{n+1}(x)$ is a polynomial of degree 2 + (*n* – 1) + (*n* + 1) *i.e.* 2(*n* + 1). Since 2(*n* + 1) is even, we see that $x^2P_{n-1} P_{n+1}$ is an even function of *x* and hence

$$\int_{-1}^{1} x^2 P_{n-1}(x) P_{n+1}dx = 2\int_{0}^{1} x^2 P_{n-1} P_{n+1}\,dx$$

or $$\int_{0}^{1} x^2 P_{n-1}(x) P_{n+1}dx = \tfrac{1}{2}\int_{-1}^{1} x^2 P_{n-1} P_{n+1}\,dx = \frac{n(n+1)}{(2n+3)(4n^2-1)}, \text{ by (5).}$$

Ex. 10. *Prove that* $\int_{-1}^{1} x^2 P_n^2\,dx = \frac{1}{8(2n-1)} + \frac{3}{4(2n+1)} + \frac{1}{8(2n+3)}$.

Sol. From recurence relation I, $(2n+1)xP_n = (n+1)P_{n+1} + nP_{n-1}$.

Squaring both sides, we have

$$(2n+1)^2 x^2P_n^{\ 2} = (n+1)^2P^2_{n+1} + n^2P^2_{n-1} + 2n(n+1)P_{n+1}P_{n-1}. \quad ...(1)$$

Also, $$\int_{-1}^{1} P_m P_n\,dx = \begin{cases} 0, \text{ if } m \neq n \\ 2/(2n+1), \text{ if } m = n \end{cases} \quad ...(2)$$

Integrating both sides of (1) w.r.t. 'x' between the limits –1 to 1 and using (2), we have

$$(2n+1)^2\int_{-1}^{1} x^2 P_n^2\,dx = (n+1)^2\frac{2}{2(n+1)+1} + n^2\frac{2}{2(n-1)+1} + 0$$

$$\therefore \int_{-1}^{1} x^2 P_n^2\,dx = \frac{2}{(2n+1)^2}\left[\frac{(n+1)^2}{2n+3} + \frac{n^2}{2n-1}\right] = \frac{1}{8(2n-1)} + \frac{3}{4(2n+1)} + \frac{1}{8(2n+3)}$$

(on resolving into partial fractions)

Ex. 11. *Prove that* $xP_n' = nP_n + (2n-3)P_{n-2} + (2n-7)P_{n-4} + \ldots$ and hence or otherwise show that (*a*) $\int_{-1}^{1} x P_n P_n'\,dx = (2n)/(2n+1)$.

(*b*) $\int_{-1}^{1} x P_n P_m'\,dx =$ either 0 or 2 or $(2n)/(2n+1)$.

Sol. From recurrence relation II, we have

$xP'_n = nP_n + P'_{n-1}$ or $xP'_n - P'_{n-1} = nP_n$. ...(1)

Again from recurrence relation III, we have

$P'_{n+1} = (2n+1)P_n + P'_{n-1}$ or $P'_{n+1} - P'_{n-1} = (2n+1)P_n$ ...(2)

Replacing n by $n-2$, $n-4$, $n-6$, ... successively in (2), we get

$$\begin{aligned} P'_{n-1} - P'_{n-3} &= (2n-3)P_{n-2} \\ P'_{n-3} - P'_{n-5} &= (2n-7)P_{n-4} \\ \cdots\cdot \quad & \cdots\cdot\ \cdots\cdot\ \cdots \\ \cdots \quad & \cdots\cdot\ \cdots\cdot\ \cdots \end{aligned} \qquad ...(3)$$

Adding (1) and (3) and simplifying, we get

$$xP'_n = nP_n + (2n-3)P_{n-2} + (2n-7)P_{n-4} + \cdots\cdot \qquad ...(4)$$

Part (a). Multiplying both sides of (4) by P_n, we get

$$xP_nP'_n = nP_n^{\,2} + (2n-3)P_{n-2}P_n + (2n-5)P_{n-4}P_n + \cdots \qquad ...(5)$$

Also $$\int_{-1}^{1} x\, P_m\ P_n\, dx = \begin{cases} 0, \text{ if } m \neq n \\ 2/(2n+1), \text{ if } m = n \end{cases}. \qquad ...(6)$$

Integrating both sides of (5) w.r.t. x from -1 to 1 and using (6), we have

$$\int_{-1}^{1} x\, P_n\, P_n'\, dx = n \,.\, \frac{2}{2n+1} + 0 + 0 + \ldots = \frac{2n}{2n+1}. \qquad ...(7)$$

Part (b). Replaing n by m in (4), we get

$$xP'_m = mP_m + (2m-3)P_{m-2} + (2m-7)P_{m-4} + \cdots \qquad ...(8)$$

Multiplying both sides of (8) by P_n, we get

$$xP_nP'_m = mP_mP_n + (2m-3)P_{m-2}P_n + (2m-7)P_{m-4}P_n + \cdots \qquad ...(9)$$

Integrating both sides of (9) w.r.t. 'x' from -1 to 1 and using (6), three cases arise:

Case I. When n is different from m, $m-2$, $m-4$, ... and so on. Then

$$\int_{-1}^{1} x\, P_n\, P_m'\, dx = 0 + 0 + 0 + \ldots = 0.$$

Case II. When $n = m$. Then, $$\int_{-1}^{1} x\, P_n\, P_n'\, dx = n \cdot \frac{2}{2n+1} + 0 + 0 + \ldots = \frac{2n}{2n+1}.$$

Case III. When $n = m-2$. Then $n \neq m$, $n \neq (m-4)$, $n \neq (m-6)$, ... and so on. So we obtain

$$\int_{-1}^{1} x\, P_n\, P_m'\, dx = 0 + (2m-3) \times \frac{2}{2n+1} = [2(n+2)-3] \times \frac{2}{2n+1} \qquad \left[\begin{matrix} \because\ n = m-2 \\ \Rightarrow m = n+2 \end{matrix}\right]$$

$$= (2n+1) \times \frac{2}{2n+1} = 2.$$

Similarly we can prove that $\int_{-1}^{1} x\, P_n\, P_m'\, dx = 2$, when $n = m-4$ or $n = m-6$ etc.

Thus $$\int_{-1}^{1} x\, P_n\, P_m'\, dx = 0 \quad \text{or} \quad 2 \quad \text{or} \quad \frac{2n}{2n+1}.$$

Ex. 12. *Prove that* $\int_{-1}^{1} (P_n')^2\, dx = n(n+1)$.

Sol. From Christoffel's expansion, we get

$$P'_n = (2n-1)P_{n-1} + (2n-5)P_{n-3} + (2n-9)P_{n-5} + \cdots \qquad ...(1)$$

The last term on R.H.S. of (1) is $3P_1$ or P_0 according as n is even or odd.

But $$(a + b + c + d + \ldots)^2 = (a^2 + b^2 + c^2 + \ldots) + 2\Sigma ab. \qquad \ldots(2)$$

Squaring both sides of (1) and using (2), we have

$$P'^2_n = (2n-1)^2 P^2_{n-1} + (2n-5)^2 P^2_{n-3} + (2n-9)^2 P^2_{n-5} + \ldots + 3^2 P_1^2$$
$$+ 2\Sigma(2n-1)(2n-5)P_{n-1}P_{n-3}, \text{ if } n \text{ is even} \qquad \ldots(3A)$$

or $$P'^2_n = (2n-1)^2 P^2_{n-1} + (2n-5)^2 P^2_{n-3} + \ldots + P_0^2$$
$$+ 2\Sigma(2n-1)(2n-5)P_{n-1}P_{n-3}, \text{ if } n \text{ is odd} \qquad \ldots(3B)$$

Also, $$\int_{-1}^{1} P_m P_n \, dx = \begin{cases} 0, & \text{if } m \neq n \\ 2/(2n+1), & \text{if } m = n \end{cases}. \qquad \ldots(4)$$

We consider two cases :

Case I. When n is even. Integrating both sides of (3A) w.r.t. 'x' from -1 to 1 and using (4),

$$\int_{-1}^{1} (P_n')^2 \, dx = (2n-1)^2 \cdot \frac{2}{2(n-1)+1} + (2n-5)^2 \cdot \frac{2}{2(n-3)+1} + \ldots + 3^2 \frac{2}{2\cdot 1+1} + 0 + 0 + \ldots$$

$$= 2[(2n-1) + (2n-5) + \ldots + 3]. \qquad \ldots(5)$$

Let m be the number of terms in A.P. on R.H.S. of (5). Then, we have

$$3 = (2n-1) + (m-1) \times (-4) \qquad \text{so that} \qquad m = n/2.$$

But $$\text{sum of A.P.} = \frac{\textit{number of terms}}{2} (\text{first term} + \text{last term})$$

$$\therefore \qquad (2n-1) + (2n-5) + \ldots + 3 = \frac{(n/2)}{2}(2n-1+3) = \tfrac{1}{2} n(n+1).$$

Hence (5) reduces to $$\int_{-1}^{1} P_n'^2 \, dx = 2 \times \tfrac{1}{2} n(n+1) = n(n+1).$$

Case II. When n is odd. Integrating both sides of (3B) w.r.t. 'x' from -1 to 1 ans using (4),

$$\int_{-1}^{1} P_n'^2 \, dx = (2n-1)^2 \cdot \frac{2}{2(n-1)+1} + (2n-5)^2 \cdot \frac{2}{2(n-3)+1} + \ldots + \frac{2}{2\cdot 0+1}$$

$$= 2[(2n-1) + (2n-5) + \ldots + 1]. \qquad \ldots(6)$$

Let p be the number of terms in A.P. on R.H.S. of (6). Then, we have

$$1 = (2n-1) + (p-1) \times (-4) \qquad \text{so that} \qquad p = (n+1)/2.$$

As before, $$(2n-1) + (2n-5) + \ldots + 1 = \frac{\{(n+1)/2\}}{2}(2n-1+1) = \tfrac{1}{2} n(n+1).$$

Hence (6) reduces to $$\int_{-1}^{1} P_n'^2 \, dx = 2 \times \tfrac{1}{2} n(n+1) = n(n+1).$$

Thus for all values of n, we have $$\int_{-1}^{1} P_n'^2 \, dx = n(n+1).$$

Ex. 13. Show that, when $|z| < 1$ and $|x| \leq 1$,

$$\int_{-1}^{1} P_n(x)\,(1 - 2zx + z^2)^{-1/2} \, dx = (2z^n)/(2n+1).$$ **[Meerut 2005]**

Sol. We know that $$(1 - 2zx + z^2)^{-1/2} = \sum_{n=0}^{\infty} z^n P_n(x).$$

Multiplying both sides by $P_n(x)$, we have

$$P_n(x).\ (1 - 2zx + z^2)^{-1/2} = P_n(x)\ [P_0(x) + zP_1(x) + ... + z^n P_n(x) + ...].$$

Integrating both sides w.r.t. 'x' between -1 to 1, we get

$$\int_{-1}^{1} P_n(x)\ (1 - 2zx + z^2)^{-1/2}\ dx = \int_{-1}^{1} P_0(x)\ P_n(x)\ dx + z\int_{-1}^{1} P_1(x)\ P_n(x) dx\ + ...$$

$$+ z^n \int_{-1}^{1} [P_n(x)]^2\ dx + z^{n+1}\int_{-1}^{1} P_{n+1}(x)\ P_n(x)\ dx + ... \qquad ...(1)$$

But $\int_{-1}^{1} P_m(x)\ P_n(x)\ dx = 0$, if $m \neq n$ and $\int_{-1}^{1} [P_n(x)]^2\, dx = \dfrac{2}{2n+1}$. ...(2)

Using (2), (1) reduces to $\int_{-1}^{1} P_n(x)\ (1 - 2zx + z^2)^{-1/2}\ dx = \dfrac{2z^n}{2n+1}$.

Ex. 14. Evaluate $\int_{-1}^{1} P_3^2(x)\ dx$. **[Nagpur 1995, 96]**

Sol. Since $\int_{-1}^{1} P_n^2(x)\ dx = 2/(2n + 1)$, so $\int_{-1}^{1} P_3^2(x)\ dx = 2/(6 + 1) = 2/7$.

EXERCISE 9 (B)

1. Evaluate (*i*) $\int_{-1}^{1} x\ P_n^2(x)\ dx$ (*ii*) $\int_{-1}^{1} x\ P_n(x)\ P_{n+1}(x)\ dx$ (*iii*) $\int_{-1}^{1} x^3\ P_4(x)\ dx$.

Ans. (*i*) 0 (*ii*) $2(n + 1)/[(2n + 1)\ (2n + 3)]$ (*iii*) 0.

2. Prove that $\int_{-1}^{1} (1 - x^2)\ P_m'\ P_n'\ dx = 0$, if $m \neq n$.

3. If $u_n = \int_{-1}^{1} x^{-1}\ P_n(x)\ P_{n-1}(x)\ dx$, show that $nu_n + (n - 1)u_{n-1} = 2$ and hence evaluate u_n.

Ans. $u_n = 2/n$, if n is even; $u_n = 0$ if n is odd

4. Prove that $\int_{-1}^{1} x\ P_n(x)\ P_n'(x)\ dx = \dfrac{2n}{2n+1}$.

5. Obtain the relation : $xP_n'(x) = P_{n+1}'(x) - (n+1)P_n(x)$ **[Guwahati 2007]**

9.14. Rodrigue's formula. To show that $P_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}(x^2 - 1)^n$

[Punjab 2005; Purvanchal 2006; Gulbarga 2005; Ultak 2003; Nagpur 1996; Meerut 2007, 11; Garhwal 2004; Bilaspur 1998; Bhopal 2004, 10; KU Kurukshetra 2005; Agra 2010; MDU Rohtak 2005; Kanpur 2008, 11; Ranchi 2010; Lucknow 2010]

Proof. By the definition of Legendre polynomial, we get

$$P_n(x) = \sum_{n=0}^{[n/2]} (-1)^r \frac{(2n - 2r)!\, x^{n-2r}}{2^n\ r!\,(n - r)!\,(n - 2r)!}. \qquad ...(1)$$

where

$$\left[\tfrac{1}{2}n\right] = \begin{cases} n/2, \text{ if } n \text{ is even} \\ (n-1)/2, \text{ if } n \text{ is odd} \end{cases} \qquad ...(2)$$

Now, by binomial theorem, $(x^2 - 1)^n = \sum_{r=0}^{n} {}^nC_r\ (x^2)^{n-r}(-1)^r = \sum_{r=0}^{n} {}^nC_r\ (-1)^r x^{2n-2r}$.

$$\therefore \qquad \frac{1}{2^n\, n!}\frac{d^n}{dx^n}(x^2-1)^n = \frac{1}{2^n \times n!}\sum_{r=0}^{n} {}^nC_r(-1)^r \frac{d^n}{dx^n}x^{2n-2r}. \qquad ...(3)$$

$$\text{But} \qquad \frac{d^n}{dx^n}x^m = 0 \text{ if } m < n \quad \text{and} \quad \frac{d^n}{dx^n}x^m = \frac{m!}{(m-n)!}x^{m-n}, \text{ if } m \geq n \qquad ...(4)$$

$$\therefore \qquad \frac{d^n}{dx^n}x^{2n-2r} = 0, \text{ if } 2n-2r < n, \quad i.e., \quad r > \frac{n}{2}. \qquad ...(5)$$

Making use of (5) in (3), we see that we must replace $\sum\limits_{r=0}^{n}$ by $\sum\limits_{r=0}^{n/2}$ if n is even and by $\sum\limits_{r=0}^{(n-1)/2}$ if n is odd. *i.e.* we must replace $\sum\limits_{r=0}^{n}$ by $\sum\limits_{r=0}^{[n/2]}$. Hence (3) reduces to

$$\frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1) = \frac{1}{2^n\, n!}\sum_{r=0}^{[n/2]} {}^nC_r(-1)^r \frac{d^n}{dx^n}x^{2n-2r} = \frac{1}{2^n\, n!}\sum_{r=0}^{[n/2]} {}^nC_r(-1)^r \frac{(2n-2r)!}{(2n-2r-n)!}x^{2n-2r-n}, \text{ by (4)}$$

$$= \sum_{r=0}^{[n/2]} \frac{1}{2^n\, n!}\frac{n!(-1)^r}{r!(n-r)!}\cdot\frac{(2n-2r)!}{(n-2r)!}x^{n-2r} = P_n(x), \text{ using (1).}$$

9.15. Solved examples based on Art. 9.14.

Ex. 1. *Using Rodrigue's furmula, find values of* $P_0(x)$, $P_1(x)$, $P_2(x)$ *and* $P_3(x)$.

[MDU Rohtak 2004; Bangalore 1995]

Sol. Rodrigue's formula is given by
$$P_n(x) = \frac{1}{2^n\, n!}\frac{d^n}{dx^n}(x^2-1)^n. \qquad ...\ (1)$$

Putting $n = 0$ in (1), $\qquad P_0(x) = \dfrac{1}{2^0\, 0!}(x^2-1)^0 = 1.$

Putting $n = 1$ in (1), $\qquad P_1(x) = \dfrac{1}{2^1 . 1!}\dfrac{d}{dx}(x^2-1) = \dfrac{1}{2}(2x) = x.$

Putting $n = 2$ in (1), we have

$$P_2(x) = \frac{1}{2^2\times 2!}\frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{8}\frac{d}{dx}\left[\frac{d}{dx}(x^2-1)^2\right] = \frac{1}{8}\frac{d}{dx}\left[2(x^2-1)\times 2x\right] = \frac{1}{2}\frac{d}{dx}(x^3-x) = \frac{1}{2}(3x^2-1).$$

Putting $n = 3$ in (1), we have

$$P_3(x) = \frac{1}{2^3\times 3!}\frac{d^3}{dx^3}(x^2-1)^3 = \frac{1}{48}\frac{d^2}{dx^2}\left[\frac{d}{dx}(x^2-1)^3\right] = \frac{1}{48}\frac{d^2}{dx^2}\left[3(x^2-1)^2\times 2x\right]$$

$$= \frac{1}{8}\frac{d}{dx}\left[\frac{d}{dx}x(x^2-1)^2\right] = \frac{1}{8}\frac{d}{dx}\left[(x^2-1)^2 + x\times 2(x^2-1)\times 2x\right] = \frac{1}{8}\times\frac{d}{dx}(5x^4-6x^2+1)$$

$=(1/8)\times (20x^3 - 12x) = (1/2) \times (5x^3 - 3x)$.

Ex. 2. *If* $x > 1$, *show that* $P_n(x) < P_{n+1}(x)$. **[Garhwal 2005; Ravishankar 2000]**

Sol. We are to prove $\qquad P_n(x) < P_{n+1}(x). \qquad ...(1)$

Since $x > 1$, from Rodrigues's formula, we find that $P_n(x) > 0$ for each value of n. We now use the mathematical induction to prove (1).

Now $x > 1 \Rightarrow 1 < x \Rightarrow P_0(x) < P_1(x)$ [$\because$ $P_0(x) = 1$ and $P_1(x) = x$]

This implies that (1) is true for $n = 0$. Let (1) be true for $n - 1$. Then, we have

$$P_{n-1} < P_n \quad \text{so that} \quad P_{n-1}/P_n < 1. \qquad ...(2)$$

From recurrence relation I, $(2n + 1)xP_n = (n + 1)P_{n+1} + nP_{n-1}$

or $$\frac{(2n+1)x}{n+1} = \frac{P_{n+1}}{P_n} + \frac{n}{n+1}\frac{P_{n-1}}{P_n} \quad \text{or} \quad \frac{P_{n+1}}{P_n} = \frac{(2n+1)x}{n+1} - \frac{n}{n+1}\frac{P_{n-1}}{P_n}$$

or $$\frac{P_{n+1}}{P_n} > \frac{2n+1}{n+1} - \frac{n}{n+1},$$ using (2) and noting that $x > 1$

$\therefore$ $P_{n+1}/P_n > 1$, so that $P_n < P_{n+1}$ [$\because$ $P_n > 0$ for each n.]

This shows that (1) is true for n whenever (1) is true for $n - 1$. Hence (1) is true for each n by induction.

Ex. 3. *Prove that* (*i*) $\int_{-1}^{1} P_n(x)\, dx = 2$, if $n = 0$ **[Guwahati 2007]**

(*ii*) $\int_{-1}^{1} P_n(x)\, dx = 0$, if $n \geq 1$.

Sol. (*i*) When $n = 0$, $P_n(x) = P_0(x) = 1$. Hence $\int_{-1}^{1} P_n(x)\, dx = \int_{-1}^{1} dx = 2$.

Part (ii) Using Rodrigues' formula, we have

$$\int_{-1}^{1} P_n(x)\, dx = \frac{1}{2^n . n!}\int_{-1}^{1} D^n (x^2 - 1)^n\, dx, \text{ where } D^n \equiv d^n/dx^n$$

$$= \frac{1}{2^n n!}\Big[D^{n-1}(x^2-1)^n\Big]_{-1}^{1} = \frac{1}{2^n n!}\Big[D^{n-1}\{(x-1)^n (x+1)^n\}\Big]_{-1}^{1}$$

$$= \frac{1}{2^n n!}\Big[D^{n-1}(x-1)^n \cdot (x+1)^n + {}^{n-1}C_1\, D^{n-2}(x-1)^n\, D(x+1)^n + ... + (x-1)^n \cdot D^{n-1}(x+1)^n\Big]_{-1}^{1}$$

[$\because$ By Leibnitz Theorem, $D^n(uv) = D^n u \,.\, v + {}^nC_1 D^{n-1} u .\, Dv + ... + u \,.\, D^n v$]

$$= \frac{1}{2^n . n!}\Big[n!\,(x-1)\,(x+1)^n + ... + n!\,(x+1)\,(x-1)^n\Big]_{-1}^{1}$$

$= 0$ $\left[\because D^n(ax+b)^m = a^n \dfrac{m!}{(m-n)!}(ax+b)^{m-n}\right]$

Ex. 4. *If* $m > n - 1$ and n is a positive integer, prove that

$$\int_0^1 x^m P_n(x)\, dx = \frac{m(m-1)(m-2)\ldots(m-n+2)}{(m+n+1)(m+n-1)\ldots(m-n+3)}$$

Sol. Using Rodrigues' formula, we have

$$\int_0^1 x^m P_n(x)\, dx = \frac{1}{2^n n!}\int_0^1 x^m D^n (x^2-1)^n\, dx, \text{ where } D^n \equiv d^n/dx^n$$

$$= \frac{1}{2^n n!}\left[\left\{x^m D^{n-1}(x^2-1)^n\right\}_0^1 - m\int_0^1 x^{m-1} D^{n-1}(x^2-1)^n\, dx\right] \qquad ...(1)$$

$$= \frac{(-1)^1 m}{2^n n!}\int_0^1 x^{m-1} D^{n-1}(x^2-1)^n\, dx$$

[$\because$ the first term in (1) vanishes on using Leibnitz theorem as shown in Ex. 3.]

$$= \frac{(-1)^1 m}{2^n\ n!}\left[\left\{x^{m-1}\ D^{n-2}(x^2-1)^n\right\}_0^1 - (m-1)\int_0^1 x^{m-2}\ D^{n-2}(x^2-1)^n\ dx\right]$$

$$= \frac{(-1)^2 m(m-1)}{2^n\ n!}\int_0^1 x^{m-2}\ D^{n-2}(x^2-1)^n\ dx \qquad ...(2)$$

[$\because$ the first term is again zero as shown in Ex. 3.]

$$= \frac{(-1)^n\ m(m-1)\dots(m-n+1)}{2^n\ n!}\int_0^1 x^{m-n}\ (x^2-1)^n\ dx,$$ on continuing the similar steps $n-2$ times more

$$= \frac{(-1)^n\ m(m-1)\dots(m-n+1)}{2^n\ n!}\int_0^1 x^{m-n}\ (-1)^n(1-x^2)^n\ dx$$

$$= \frac{m(m-1)\dots(m-n+1)}{2^n\ n!}\int_0^1 (t^{1/2})^{m-n}\ (1-t)^n\ \frac{dt}{2t^{1/2}},$$ taking $x^2 = t$ so that $dx = \dfrac{dt}{2x} = \dfrac{dt}{2t^{1/2}}$

$$= \frac{m(m-1)\dots(m-n+1)}{2^{n+1}\ n!}\int_0^1 t^{\frac{1}{2}(m-n-1)}\ (1-t)^n\ dt = \frac{m(m-1)\dots(m-n+1)}{2^{n+1}\ n!}\int_0^1 t^{\frac{1}{2}(m-n+1)-1}\ (1-t)^{(n+1)-1}\ dt$$

$$= \frac{m(m-1)\dots(m-n+1)}{2^{n+1}\ n!}\ \frac{\Gamma\left(\dfrac{m-n+1}{2}\right)\Gamma(n+1)}{\Gamma\left(\dfrac{m-n+1}{2}+n+1\right)} \qquad \left(\because \int_0^1 t^{p-1}(1-t)^{q-1}\ dt = B(p, q) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}\right)$$

$$= \frac{m(m-1)\dots(m-n+1)}{2^{n+1}\ n!}\ \frac{\Gamma\left(\dfrac{m-n+1}{2}\right).\,n!}{\Gamma\left(\dfrac{m+n+3}{2}\right)}$$

$$= \frac{m(m-1)\dots(m-n+1)\,\Gamma\left(\dfrac{m-n+1}{2}\right)}{2^{n+1}\times\dfrac{m+n+1}{2}.\dfrac{m+n-1}{2}\dots\dfrac{m-n+1}{2}\,\Gamma\left(\dfrac{m-n+1}{2}\right)} \qquad [\because\ \Gamma(p+1) = p\Gamma(p)]$$

$$= \frac{m(m-1)\dots(m-n+1)}{(m+n+1)(m+n-1)\dots(m-n+1)}$$

Ex. 5. (*i*) If $m < n$, show that $\int_{-1}^1 x^m\ P_n(x)\ dx = 0$.

[Ranchi 2010, Ravishankar 2010, Kanpur 2011]

Deduce that $\int_{-1}^1 x^4 P_6(x)dx = 0$ **[Meerut 2006]**

(*ii*) Prove that $\int_{-1}^1 x^n\ P_n(x)\ dx = \dfrac{\Gamma(1/2)\,\Gamma(n+1)}{2^n\ \Gamma(n+3/2)} = \dfrac{2^{n+1}(n!)^2}{(2n+1)!}$. **[Kanpur 2006, 10]**

Sol. (*i*) Using Rodrigues' formula, we have

$$\int_{-1}^{1} x^m P_n(x)\, dx = \frac{1}{2^n \cdot n!}\int_{-1}^{1} x^m\ D^n (x^2-1)^n\ dx \text{ , where } D^n \equiv d^n/dx^n$$

$$= \frac{(-1)^2\ m\,(m-1)}{2^n \cdot n!}\int_{-1}^{1} x^{m-2}\ D^{n-2}(x^2-1)^n\ dx, \text{ doing upto equation (2) as in Ex. 4}$$

$$= \frac{(-1)^m\ m\,(m-1)....3\cdot 2\cdot 1}{2^n \cdot n!}\times\int_{-1}^{1} x^{m-m}\ D^{n-m}(x^2-1)^n\ dx \qquad ...(A)$$

(On continuing the similar steps $m-2$ times more and noting that $m < n$)

$$= \frac{(-1)^m\ m!}{2^n \cdot n!}\int_{-1}^{1}\frac{d}{dx}\left\{\frac{d^{n-m-1}}{dx^{n-m-1}}(x^2-1)^n\right\}dx$$

$$= \frac{(-1)^m\, m!}{2^n \cdot n!}\left[\frac{d^{n-m-1}}{dx^{n-m-1}}(x^2-1)^n\right]_{-1}^{1} = \frac{(-1)^m\ m!}{2^n \cdot n!}\left[D^{n-m-1}\{(x-1)^n\cdot(x+1)^n\}\right]_{-1}^{1} = 0$$

[By using Leibnitiz theorem and simplifying as before]

Deduction : Taking $m = 4$ and $n = 6$ in the above result, we get the required result because $4 < 6$ satisfies condition $m < n$.

Part (ii). Here $m = n$. So proceeding as above upto (A), we obtain

$$\int_{-1}^{1} x^n P_n(x)\, dx = \frac{(-1)^n\, n!}{2^n\ n!}\int_{-1}^{1} x^{n-n} D^{n-n}(x^2-1)^n\, dx = \frac{(-1)^n}{2^n}\int_{-1}^{1}(x^2-1)^n\ dx = \frac{(-1)^n}{2^n}\int_{-1}^{1}(-1)^n(1-x^2)^n\ dx$$

$$= \frac{1}{2^n}\times 2\int_{0}^{1}(1-x^2)^n\ dx = \frac{1}{2^{n-1}}\int_{0}^{\pi/2}\cos^{2n+1}\theta\ d\theta, \text{ putting } \quad x=\sin\theta \quad \text{and} \quad dx = \cos\theta\ d\theta$$

$$= \frac{1}{2^{n-1}}\times\frac{\Gamma(n+1)\,\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\dfrac{2n+3}{2}\right)}, \qquad \text{as} \qquad \int_{0}^{\pi/2}\cos^p\theta\,\sin^q\theta\ d\theta = \frac{\Gamma\left(\dfrac{p+1}{2}\right)\Gamma\left(\dfrac{q+1}{2}\right)}{2\Gamma\left(\dfrac{p+q+2}{2}\right)}$$

$$= \frac{1}{2^n}\times\frac{n!\sqrt{\pi}}{\dfrac{2n+1}{2}\cdot\dfrac{2n-1}{2}\cdots\cdots\dfrac{3}{2}\cdot\dfrac{1}{2}\sqrt{\pi}} = \frac{n!}{2^n}\times\frac{2^{n+1}}{(2n+1)\,(2n-1)\ldots 3\cdot 1} \qquad (\because\ \Gamma(n+1) = n!)$$

$$= 2(n!)\times\frac{(2n)\,(2n-2)\ldots 4\cdot 2}{(2n+1)\,(2n)\,(2n-1)\,(2n-2)\ldots 4\cdot 3\cdot 2\cdot 1}$$

$$= 2(n!)\times\frac{(2\cdot n)\cdot[2\cdot(n-1)]\ldots(2\cdot 2)\cdot(2\cdot 1)}{(2n+1)!} = 2(n!)\times\frac{2^n\ n!}{(2n+1)!} = \frac{2^{n+1}(n!)^2}{(2n+1)!}.$$

Ex. 6. *Deduce from Rodrigue's formula* $\displaystyle\int_{-1}^{1} f(x)\,P_n(x)\,dx = \frac{(-1)^n}{2^n\ n!}\int_{-1}^{1}(x^2-1)^n\ f^{(n)}(x)\ dx$.

Sol. Using Rodrigue's formula, we have

$$\int_{-1}^{1} f(x)\,P_n(x)\,dx = \frac{1}{2^n\,n!}\int_{-1}^{1} f(x)\,D^n(x^2-1)^n, \text{ where } D^n \equiv d^n/dx^n$$

$$= \frac{1}{2^n\,n!}\left[\left\{f(x)\,D^{n-1}(x^2-1)^n\right\}_{-1}^{1} - \int_{-1}^{1} f'(x)\,D^{n-1}(x^2-1)^n\,dx\right] \qquad \text{...(1)}$$

(On integration by parts)

$$= \frac{(-1)^1}{2^n\,n!}\int_{-1}^{1} f'(x)\,D^{n-1}(x^2-1)^n\,dx$$

(the first term in (1) vanishes on using Leibnitz theorem as explained in Ex. 3)

$$= \frac{(-1)^1}{2^n\,n!}\left[\left\{f'(x)\,D^{n-2}(x^2-1)^n\right\}_{-1}^{1} - \int_{-1}^{1} f''(x)\,D^{n-2}(x^2-1)^n\,dx\right], \text{ on integration by parts again}$$

$$= \frac{(-1)^2}{2^n\,n!}\int_{-1}^{1} f''(x)\,D^{n-2}(x^2-1)^n\,dx \qquad (\because \text{ the first term vanishes as shown in Ex. 3})$$

$$= \frac{(-1)^n}{2^n\,n!}\int_{-1}^{1} f^{(n)}(x)\,D^{n-n}(x^2-1)^n\,dx, \qquad \text{on continuing the similar steps } n-2 \text{ times more}$$

$$= \frac{(-1)^n}{2^n\,n!}\int_{-1}^{1} (x^2-1)^n\,f^{(n)}(x)\,dx.$$

Ex.7. *Using Rodrigue's formula, show that $P_n(x)$ satisfies*

$$\frac{d}{dx}\left\{(1-x^2)\frac{d}{dx}P_n(x)\right\}+n(n+1)P_n(x)=0 \qquad \textbf{[CDLU 2004]}$$

Sol. Rodrigue's formula is $\qquad P_n(x)=\dfrac{1}{2^n\,n!}\dfrac{d^n}{dx^n}(x^2-1)^n \qquad$ ... (1)

Let $\qquad y = (x^2-1)^n \qquad$ ... (2)

Differentiating (2) w.r.t. 'x', $\;y_1 = 2nx(x^2-1)^{n-1}\;$ so that $\;(x^2-1)y_1 = 2nx\,(x^2-1)^n$

or $\qquad (x^2-1)\,y_1 = 2nxy$, using (2) $\qquad$ ... (3)

Differentiating (3) w.r.t. 'x', $\qquad (x^2-1)y_2 + 2xy_1 = 2n\,(xy_1+y)$

or $\qquad (x^2-1)\,y_2 + 2(1-n)xy_1 - 2ny = 0 \qquad$ ... (4)

Differentiating both sides of (4) w.r.t. 'x' n times, we have

$D^n\{(x^2-1)y_2\} + 2(1-n)\,D^n(xy_1) - 2nD^n\,(y) = 0$, where $D^n \equiv d^n/dx^n \qquad$ ... (5)

Using Leibnitz' theorem, (5) yields

$$y_{n+2}\,(x^2-1) + {}^nC_1\,y_{n+1}(2x) + {}^nC_2\,y_n\cdot 2 + 2(1-n)\,(y_{n+1}x + {}^nC_1\;y_n\cdot 1) - 2ny_n = 0$$

or $\qquad (x^2-1)\,y_{n+2} + 2x\,y_{n+1} + \{n(n-1) + 2n(1-n) - 2n\}\,y_n = 0$

or $\qquad (1-x^2)\,y_{n+2} - 2x\,y_{n+1} + n(n+1)\,y_n = 0$

or $\qquad \dfrac{d}{dx}\left\{(1-x^2)y_{n+1}\right\}+n(n+1)y_n=0 \qquad$ or $\qquad \dfrac{d}{dx}\left\{(1-x^2)\times\left(\dfrac{dy_n}{dx}\right)\right\}+n(n+1)y_n=0$

or $\qquad \dfrac{d}{dx}\left\{(1-x^2)\dfrac{d}{dx}\left(\dfrac{d^n}{dx^n}(x^2-1)^n\right)\right\}+n(n+1)\dfrac{d^n}{dx^n}(x^2-1)^n=0$, using (2)

Dividing by $2^n n!$ and using (1), we get $\qquad \dfrac{d}{dx}\{(1-x^2)\dfrac{d}{dx}P_n(x)\}+n(n+1)P_n(x)=0$

Ex. 8. Using Rodrigue's formula, prove that

(*i*) $(1-t)^n P_n\left(\dfrac{1+t}{1-t}\right) = \sum\limits_{k=0}^{\infty}\left({}^n c_k\right)^2 t^k$ **(Ravishankar 1993)**

(*ii*) $P_n(\cosh u) \geq 1$ (*ii*) $|P_n(x)| \geq 1$, if $|x| \geq 1$. **(Bilaspur 1994)**

Sol. (*i*) Rodrigues formula is given by $$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n \quad \text{... (1)}$$

Let $D \equiv d/dx$. Then (1) can be re-written as

$$P_n(x) = \frac{1}{2^n n!}D^n(x^2-1)^n = \frac{1}{2^n n!}D^n\left\{(x+1)^n(x-1)^n\right\}$$

or $$P_n(x) = \frac{1}{2^n n!}\sum_{k=0}^{n}{}^nC_k D^{n-k}(x+1)^n D^k(x-1)^n, \text{ by Leibnitz's rule} \quad \text{... (2)}$$

From Differential calcules, we know that $$D^n(ax+b)^m = \frac{m!}{(m-n)!}(ax+b)^{m-1}a^n \quad \text{... (3)}$$

Using (3), $$D^{n-k}(x+1)^n = \frac{n!}{[n-(n-k)]!}(x+1)^{n-(n-k)} = \frac{n!}{k!}(x+1)^k \quad \text{... (4)}$$

and $$D^k(x-1)^n = \frac{n!}{(n-k)!}(x-1)^{n-k} \quad \text{... (5)}$$

Using (4) and (5), (3) reduces to

$$P_n(x) = \frac{1}{2^n n!}\sum_{k=0}^{n}{}^nC_k\frac{n!}{k!}(x+1)^k\frac{n!}{(n-k)!}(x-1)^{n-k} \quad \text{or} \quad P_n(x) = \sum_{k=0}^{\infty}({}^nC_k)^2\left(\frac{x-1}{2}\right)^{n-k}\left(\frac{x+1}{2}\right)^k \quad \text{... (6)}$$

Put $x = \dfrac{1+t}{1-t}$ so that $x-1 = \dfrac{2t}{1-t}$ and $x+1 = \dfrac{2}{1-t}$ ... (7)

Using (7), (6) reduces to

$$P_n\left(\frac{1+t}{1-t}\right) = \sum_{k=0}^{\infty}({}^nC_k)^2\left(\frac{t}{1-t}\right)^{n-k}\frac{1}{(1-t)^k} \quad \text{or} \quad (1-t)^n P_n\left(\frac{1+t}{1-t}\right) = \sum_{k=0}^{\infty}({}^nC_k)^2 t^k \quad \text{... (8)}$$

(*ii*) Let $t = \tanh^2\dfrac{u}{2}$, then $t \geq 0$ and $\dfrac{1+t}{1-t} = \cosh u \geq 1$.

Hence (8) reduces to $$P_n(\cosh u) = (1+t)^{-n}\sum_{k=0}^{n}({}^nC_k)^2 t^k \quad \text{... (9)}$$

Since the first term on R.H.S. of (9) is 1 and all the subsequent terms are positive, hence (9) yields $P_n(\cosh u) \geq 1$, as required

(*iii*) Left as an exercise.

Ex. 9. *Show that all the roots of* $P_n(x) = 0$ *are real and lie between* -1 *and* 1. **(Bilaspur 1998)**

Sol. Let $$f(x) = (x^2-1)^n = (x-1)^n(x+1)^n. \quad \text{...(1)}$$

From (1), we notice that $f(x)$ vanishes for $x = 1$ and $x = -1$, hence by *Rolle's theorem, $f'(x)$ must vanish at least once for some value α of x lying between -1 and 1. Now, from (1), we have

$$f'(x) = n(x-1)^{n-1}(x+1)^n + n(x-1)^n(x+1)^{n-1}. \qquad ...(2)$$

(2) shows that $f'(x)$ vanishes at $x = 1$ and $x = -1$. But we have just proved that $f'(x)$ vanishes at $x = \alpha$, where $-1 < \alpha < 1$. Hence applying Rolle's theorem to function f' two times, we conclude that $f''(x)$ must vanish at $x = \beta$ such that $-1 < \beta < \alpha$ and also at $x = \gamma$ such that $\alpha < \gamma < 1$. Proceeding likewise we conclude that $f^{(n)}(x) = 0$ must have n real roots lying between -1 and 1.

Using (1), Rodrigue's formula gives

$$P_n(x) = \frac{1}{2^n \cdot n!}\frac{d^n}{dx^n}(x^2-1)^n = \frac{1}{2^n \cdot n!}\cdot f^{(n)}(n). \qquad ...(3)$$

$$\therefore \qquad f^{(n)}(x) = 0 \qquad \Rightarrow \qquad P_n(x) = 0, \quad \text{by (3)}. \qquad ...(4)$$

Since $f^n(x) = 0$ has n real roots lying between -1 and 1, so (4) shows that $P_n(x) = 0$ has n real roots between -1 and 1.

Remarks. The roots of $P_n(x) = 0$ are also known as *zeros of $P_n(x)$*.

Ex. 10. *Prove that all the roots of $P_n(x)$ are distinct.*

Sol. If possible, let the roots of $P_n(x) = 0$ be not all different. Then at least two roots must be equal. Let α be the repeated root, then from the theory of equations, we have

$$P_n(\alpha) = 0 \qquad \text{and} \qquad P'_n(\alpha) = 0. \qquad ...(1)$$

Since $P_n(x)$ satisfies Legendre's equation, $\quad (1-x^2)P''_n - 2xP'_n + n(n+1)P_n = 0. \qquad ...(2)$

Differentiating r times and using Leibnitz theorem, (2) gives

$$(1-x^2)\frac{d^{r+2}}{dx^{r+2}}P_n(x) + {}^rC_1 \times (-2x) \times \frac{d^{r+1}}{dx^{r+1}}P_n(x) + {}^rC_2 \times (-2) \times \frac{d^r}{dx^r}P_n(x)$$

$$-2\left[x\frac{d^{r+1}}{dx^{r+1}}P_n(x) + {}^rC_1 \times 1 \times \frac{d^r}{dx^r}P_n(x)\right] + n(n+1)\frac{d^r}{dx^r}P_n(x) = 0$$

or $\quad (1-x^2)\dfrac{d^{r+2}}{dx^{r+2}}P_n(x) - 2x({}^rC_1+1)\dfrac{d^{r+1}}{dx^{r+1}}P_n(x) - \{2\times{}^rC_2 + 2\times{}^rC_1 - n(n+1)\}\dfrac{d^r}{dx^r}P_n(x) = 0$

....(3)

Putting $r = 0$ and $x = \alpha$ in (3) and using (1), we get

$$(1-\alpha^2)P''_n(\alpha) - 0 - 0 = 0 \qquad \text{or} \qquad P''_n(\alpha) = 0. \qquad ...(4)$$

Next, putting $r = 1$ and $x = \alpha$ in (3) and using (1) and (4), we get

$$(1-\alpha^2)P_r'''(\alpha) - 0 - 0 = 0 \qquad \text{or} \qquad P'''_n(\alpha) = 0. \quad ...(5)$$

Putting $r = 2, 3, ..., n-3, n-2$ in (3) and doing as above stepwise, we finally arrive at

$$P_n^{(n)}(\alpha) = 0 \qquad \textit{i.e.} \qquad \left[\frac{d^n}{dx^n}P_n(x)\right]_{x=\alpha} = 0. \qquad ...(6)$$

But $\qquad P_n(x) = \dfrac{1\cdot3\cdot5\,...(2n-1)}{n!}\left[x^n - \dfrac{n(n-1)}{2(2n-1)}x^{n-2} + ...\right]$

$$\therefore \qquad \frac{d^n}{dx^n}P_n(x) = \frac{1\cdot3\cdot5\,...(2n-1)}{n!}\times n! \qquad \Rightarrow \qquad \left[\frac{d^n}{dx^n}P_n(x)\right]_{x=\alpha} \neq 0. \qquad ...(7)$$

***Rolle's theorem:** If $f(x)$ vanishes for $x = a$ and $x = b$, then $f'(x)$ vanishes at least once for some value of x between a and b.

Since (6) and (7) are contradictory results, it follows that our assumption about not distinct roots of $P_n(x)$ is absurd. Hence all the roots of $P_n(x) = 0$ must be distinct.

EXERCISE 9 (C)

1. Define Legendre's differential equation and show that $y = \dfrac{d^n}{dx^n}(x^2 - 1)^n$ satisfies it.

2. Using Rodrigue's formula, prove that $P'_{n+1} - P'_{n-1} = (2n + 1)P_n$.

3. Show that $P_n(x) = \dfrac{1}{2^n\, n!}\dfrac{d^n}{dx^n}(x^2 - 1)^n$ is a solution of the Legendre's equation $(1 - x^2)y_2 - 2xy_1 + n(n + 1)y = 0$, where n is +ve integer. Hence or otherwise show that

(*i*) $xP'_n(x) - P'_{n-1}(x) = nP_n(x)$. (*ii*) $(2n + 1)xP_n(x) = (n + 1)P_{n+1}(x) + nP_{n-1}(x)$.

4. Use Rodrigue's formula to derive the orthogonal property for $P_n(x)$ and show that

$$\int_{-1}^{1} P_n^2(x)\, dx = \frac{2}{2n+1}.$$

5. Prove that the function $y = \dfrac{d^n}{dx^n}(x^2 - 1)^n$ satisfies the Legendre's differential equation $(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$. Hence obtain Rodrigue's formula for Legendre–polynomial $P_n(x)$.

Using this formula prove that $\displaystyle\int_{-1}^{1} x^m P_n(x)\, dx = 0$ for $m < n$.

9.16. Legendre series for f(x) when f(x) is a polynomial.

Theorem (a). If $f(x)$ is a polynomial of degree n, then $f(x) = \displaystyle\sum_{r=0}^{n} C_r\, P_r(x)$, ...(*i*)

where
$$c_r = (r + 1/2)\int_{-1}^{1} f(x)\, P_r(x)\, dx. \quad ...(ii)$$

(b) If $f(x)$ is even (or odd), only those C_r with even (or odd) suffixes are non–zero.

Proof (a). Since $f(x)$ is a polynomial of degree n, we write

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_1 x + a_0. \quad ...(i)$$

Again, we know that $P_n(x)$ is a polynomial of degree n of the form

$$P_n(x) = k_n x^n + k_{n-1}x^{n-1} + ... + k_1 x + k_0. \quad ...(1)$$

Consider $f(x) - (a_n/k_n)\, P_n(x)$. Two cases may arise :

Case (i). $f(x) - (a_n/k_n)P_n(x) = 0$ so that $f(x) = (a_n/k_n)\, P_n(x)$, which proves the required result (*i*).

Case (ii). $f(x) - (a_n/k_n)\, P(x) = g_{n-1}(x)$, $g_{n-1}(x)$ being a polynomial of degree $n - 1$. Taking $c_n = a_n/k_n$, we may write

$$f(x) = c_n P_n(x) + g_{n-1}(x). \quad ...(3)$$

Taking $g_{n-1}(x)$ in place of $f(x)$ and proceeding as above, we have

$$g_{n-1}(x) = c_{n-1}P_{n-1}(x) + g_{n-2}(x). \quad ...(4)$$

Making use of (4), (3) may be re–written as $f(x) = c_n P(x) + c_{n-1}P_{n-1}(x) + g_{n-2}(x)$. ...(5)

Making use of similar method for $g_{n-2}(x)$ etc., we finally obtain (noting that $P_0(x) = 1$).

$$f(x) = c_n P_n(x) + c_{n-1}P_{n-1}(x) + ... + c_1 P_1(x) + c_0 P_0(x)$$

or
$$f(x) = \sum_{r=0}^{\infty} c_r\, P_r(x). \quad ...(6)$$

Since $\displaystyle\sum_{r=0}^{\infty} c_r\, P_r(x) = \sum_{s=0}^{\infty} c_s\, P_s(x)$, (6) gives $\displaystyle f(x) = \sum_{s=0}^{\infty} c_s\, P_s(x)$. ...(7)

Multiplying both sides of (7) by $P_r(x)$ and then integrating w.r.t. 'x' from –1 to 1, we have

$$\int_{-1}^{1} f(x)\, P_r(x)\, dx = \sum_{s=0}^{n} \left\{ c_s \int_{-1}^{1} P_s(x)\, P_r(x)\, dx \right\}. \qquad ...(8)$$

But $$\int_{-1}^{1} P_s(x)\, P_r(x)\, dx = \begin{cases} 0, \text{ if } r \neq s \\ 2/(2r+1), \text{ if } r = s \end{cases} \qquad ...(9)$$

Using (9), (8) reduces to $$\int_{-1}^{1} f(x)\, P_r(x)\, dx = c_r \times \frac{2}{2r+1}$$

so that $$c_r = \left(r + \tfrac{1}{2}\right) \int_{-1}^{1} f(x)\, P_r(x)\, dx. \qquad ...(10)$$

Part (b). We now prove that if $f(x)$ is even, only those c_r with even suffixes are non–zero. We know that $P_r(x)$ is even when r is even, and odd when r is odd. Thus $f(x)\, P_r(x)$ is even when r is even and odd when r is odd. But, it is known that $\int_{-1}^{1} F(x)\, dx = 0$, if $F(x)$ is odd. Hence (10) show that $c_r = 0$ if r is odd. Thus if r is even, only those c_r with even suffixes are non–zero.

Similarly, we can prove that if $f(x)$ is odd, only those c_r with odd suffixes are non–zero.

9.17. Solved examples based on Art 9.16

Ex. 1. *If $f(x)$ is a polynomial of degree less than l, prove that* $\int_{-1}^{1} f(x)\, P_l(x)\, dx = 0.$

Sol. Let $f(x)$ be a polynomial of degree n such that $n < l$. Then we have (Do upto equation (6) as explained in Art 9.16) $$f(x) = \sum_{r=0}^{n} c_r\, P_r(x)$$

Multiplying both sides by $P_l(x)$ and then integrating from –1 to 1, we have

$$\int_{-1}^{1} f(x)\, P_l(x)\, dx = \sum_{r=0}^{n} \left\{ c_r \int_{-1}^{1} P_r(x)\, P_l(x)\, dx \right\}. \qquad ...(A)$$

Since $n < l$, for each $r = 0, 1, 2, ..., n$ we find that $r \neq l$. But we know that

$$\int_{-1}^{1} P_r(x)\, P_l(x) dx = 0 \quad \text{if} \quad r \neq l.$$

Hence (A) reduces to $$\int_{-1}^{1} f(x)\, P_l\,(x) dx = 0.$$

Ex. 2. *Expand $f(x) = x^2$ in a series of the form $\Sigma c_r\, P_r(x)$*

Sol. Since x^2 is a polynomial of degree two, from Legendre series, we have

$$x^2 = \sum_{r=0}^{2} c_r\, P_r(x) = c_0 P_0(x) + c_1 P_1(x) + c_2 P_2(x), \qquad ...(1)$$

where $$c_r = \left(r + \tfrac{1}{2}\right) \int_{-1}^{1} x^2\, P_r(x)\, dx. \qquad ...(2)$$

But $P_0(x) = 1$, $P_1(x) = x$ and $P_2(x) = \tfrac{1}{2}(3x^2 - 1)$.
...(3)

Putting $r = 0, 1, 2$ successively in (2) and using (3), we have

$$c_0 = \tfrac{1}{2} \int_{-1}^{1} x^2\, dx = \tfrac{1}{2}\left[\frac{x^3}{3}\right]_{-1}^{1} = \frac{1}{3},\ c_1 = \frac{3}{2}\int_{-1}^{1} x^3\, dx = 0,\ c_2 = \frac{1}{2} \times \frac{5}{2} \int_{-1}^{1} x^2 (3x^2 - 1)\, dx = \tfrac{5}{4}\left[3 \times \frac{x^5}{5} - \frac{x^3}{3}\right]_{-1}^{1} = \frac{2}{3}.$$

With the above values of c_0, c_1 and c_2, (1) gives $x^3 = (1/3) \times P_0(x) + (2/3) \times P_2(x)$.

Ex. 3. *Expand* $x^4 - 3x^2 + x$ *in a series of form* $\Sigma c_r\, P_r(x)$.

Ans. $x^4 - 3x^2 + x = -(4/5) \times P_0(x) + P_1(x) - (10/7) \times P_2(x) + (8/35) \times P_4(x)$.

9.18. Expansion of function f(x) in a series of Legendre Polynomials.

Supposing the expansion of $f(x)$ in a series of Legendre polynomials to be possible, we write

$$f(x) = \sum_{r=0}^{\infty} c_r\, P_r(x). \qquad ...(1)$$

But (1) may also be expressed as

$$f(x) = \sum_{s=0}^{\infty} c_s\, P_s(x) \qquad ...(2)$$

where $c_0, c_1, c_2, ... c_r, ...$ are constants. Multiplying both sides of (2) by $P_r(x)$ and then integrating it w.r.t. 'x' from -1 to 1, we have

$$\int_{-1}^{1} f(x)\, P_r(x)\, dx = \sum_{s=0}^{\infty} \left\{ c_s \int_{-1}^{1} P_r(x)\, P_s(x)\, dx \right\}. \qquad ...(3)$$

But

$$\int_{-1}^{1} P_r(x)\, P_s(x)\, dx = \begin{cases} 0, \text{ if } r \neq s \\ 2/(2r+1), \text{ if } r = s \end{cases} \qquad ...(4)$$

Using (4), (3) reduces to $\displaystyle\int_{-1}^{1} f(x)\, P_r(x)\, dx = c_r \times \frac{2}{2r+1} \quad \Rightarrow \quad c_r = \left(r + \frac{1}{2}\right) \int_{-1}^{1} f(x)\, P_r(x)\, dx$.

Remark 1. Fourier–Legendre expansion of f(x). If $f(x)$ be defined from $x = -1$ to $x = 1$, then

$$f(x) = \sum_{r=0}^{\infty} c_r\, P_r(x), \qquad \text{where} \qquad c_r = \left(r + \tfrac{1}{2}\right) \int_{-1}^{1} f(x)\, P_r(x)\, dx.$$

Remark 2. Suppose that function f is continuous and has continuous derivatives in $[-1, 1]$, then we prove that series (1) converges uniformly in $[-1, 1]$ and the series expansion (1) is unique.

Ex. 1. *Expand* $f(x)$ *in the form* $\displaystyle\sum_{r=0}^{\infty} c_r\, P_r(x)$, *where* $f(x) = \begin{cases} 0, \text{ where } -1 < x < 0 \\ 1, \text{ where } 0 < x < 1. \end{cases}$

Sol. Given that

$$f(x) = \begin{cases} 0, \text{ if } -1 < x < 0 \\ 1, \text{ if } 0 < x < 1 \end{cases} \qquad ...(1)$$

We know that

$$f(x) = \sum_{r=0}^{\infty} c_r\, P_r(x), \text{ where} \qquad ...(2)$$

$$c_r = \left(r + \tfrac{1}{2}\right) \int_{-1}^{1} f(x)\, P_r(x)\, dx = \frac{2r+1}{2} \left[\int_{-1}^{0} f(x)\, P_r(x)\, dx + \int_{0}^{1} f(x)\, P_r(x)\, dx \right]$$

$$\therefore \qquad c_r = \frac{2r+1}{2} \int_{0}^{1} P_r(x)\, dx, \text{ by (1)} \qquad ...(3)$$

Putting $r = 0, 1, 2, ...$ successively in (3), we get

$$c_0 = \frac{1}{2} \int_{0}^{1} P_0(x)\, dx = \frac{1}{2} \int_{0}^{1} (1)\, dx = \frac{1}{2}, \qquad c_1 = \frac{3}{2} \int_{0}^{1} P_1(x)\, dx = \frac{3}{2} \int_{0}^{1} x\, dx = \frac{3}{4},$$

$$c_2 = \frac{5}{2}\int_0^1 P_2(x)\,dx = \int_0^1 \frac{3x^2-1}{2}\,dx = 0, \qquad c_3 = \frac{7}{2}\int_0^1 P_3(x)\,dx = \frac{7}{2}\int_0^1 \frac{5x^2-2x}{2}\,dx = -\frac{7}{16}$$

and so on. Using these values in (2), we get

$f(x) = (1/2) \times P_0(x) + (3/4) \times P_1(x) - (7/16) \times P_3(x) + \ldots\ldots + c_r P_r(x) + \ldots$, where c_r is given by (3)

EXERCISE 9 (D)

1. Expand $f(x)$ in a series of Legendre polynomials, if

(*i*) $f(x) = x,\ 0 < x < 1;\ f(x) = 0,\ -1 < x < 0.$ (*ii*) $f(x) = 0,\ -1 \le x < \alpha;\ f(x) = 1,\ \alpha < x \le 1.$

(*iii*) $f(x) = \frac{1}{2},\ 0<x<1;\ f(x) = -\frac{1}{2},\ -1<x<0.$ (*iv*) $f(x) = 2x+1,\ 0<x,\le 1;\ f(x)=0,\ -1\le x<0.$

(*v*) Given $f(x) = |\,x\,|$ for $-1 \le x \le 1$.

2. If $f(x) = \sum_{r=0}^{\infty} C_r\, P_r(x)$, obtain Parseval's identity $\displaystyle\int_{-1}^{1} [f(x)]^2\,dx = \sum_{r=0}^{\infty} \frac{C_r^2}{2r+1}$

and illustrate it by making use of $f(x) = x^4 - 3x^2 + x$.

3. Obtain the first three terms in the expansion of the following function f in terms of Legendre's polynomial: $f(x) = \begin{cases} 0, & \text{if } -1 < x < 0 \\ x, & \text{if } 0 < x < 1. \end{cases}$

4 (a). Express $P(x) = x^4 + 2x^3 + 5x^2 - x - 2$ in terms of Legendre's polynomials.

(b). Prove that $x^4 + 3x^3 - x^2 + 5x - 2 = -(31/15) \times P_0(x) + (34/5) \times P_1(x) - (2/21) \times P_2(x) - (6/5) \times P_3(x) + (8/35) \times P_4(x)$.

5. Prove that, for all x, (*a*) $x^2 = (1/3) \times P_0(x) + (2/3) \times P_2(x)$.

(*b*) $x^3 = (3/5) \times P_1(x) + (2/5) \times P_3(x)$.

(*c*) $x^4 = (1/5) \times P_0(x) + (4/7) \times P_2(x) + (8/35) \times P_4(x)$.

(*d*) $x^5 = (3/7) \times P_1(x) + (4/9) \times P_3(x) + (8/63) \times P_5(x)$.

6. Express $f(x) = 4x^3 + 6x^2 + 7x + 2$ in terms of Legendre's polynomials. **[Kanpur 2005]**

7. Express $f(x) = x^4 + 2x^3 - 2x^2 - x - 3$ in terms of Legendre's polynomials. **[Kanpur 2011]**

ANSWERS

1. (*i*) $f(x) = (1/4) + (1/2) \times P_1(x) + (5/16) \times P_2(x) + \ldots + C_r P_r(x) + \ldots$ where $C_r = \left(r + \frac{1}{2}\right)\int_0^1 x\, P_r(x)\,dx$.

(*ii*) $f(x) = \frac{1}{2}(1-\alpha) - \sum_{r=0}^{\infty} [P_{r+1}(\alpha) - P_{r-1}(\alpha)]\, P_r(x)$.

(*iii*) $C_r = 0$ if r is even, and $C_r = (-1)^{r-1/2} \dfrac{\left(r+\frac{1}{2}\right)(r-1)!}{2^r\{(r+1)/2\}!\,\{(r-1)/2\}!}$, if r is odd

(*iv*) $f(x) = P_0(x) + (7/4) \times P_1(x) + (5/8) \times P_2(x) - (7/16)P_2(x) + \ldots$

(*v*) $f(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^{n+1}(4n+1)(2n-2)!}{2^{2n}(n+1)!\,(n-1)!} P_{2n}(x)$.

3. $f(x) = -(1/4) \times P_0(x) + (1/6) \times P_1(x) + (1/16) \times P_2(x)$.

4 (a). $P(x) = (8/35) \times P_4(x) + (4/5) \times P_3(x) + (82/21) \times P_2(x) + (1/5) \times P_1(x) - (2/15) \times P_0(x)$.

6. $f(x) = (8/5) \times P_3(x) + (32/5) \times P_2(x) + 7\, P_1(x) + 4P_0(x)$.

9.19. Even and odd fuctions. An important observation.

(*i*) **Even function.** f is called an even function of x if $f(-x) = f(x)$. Suppose an even function f can be sum of a series of functions $f_1, f_2, \ldots, f_n, \ldots$ such that

$$f(x) = f_1(x) + f_2(x) + \ldots + f_n(x) + \ldots\ldots \qquad \ldots(1)$$

Then all functions $f_1, f_2, \ldots, f_n$ must be even functions of x because if any of the above function is not even, then $f(-x) \neq f(x)$ and so f ceases to be an even function.

(*ii*) **Odd function.** f is called an odd functions of x if $f(-x) = -f(x)$. Suppose an odd function f can be sum of a series as (1).

Then all the functions $f_1, f_2, \ldots, f_n$ must be odd functions of x because if any of the above function is not odd, then $f(-x) \neq -f(x)$ and so f ceases to be an odd function.

9.20. Expansion of x^n in Legendre's polynomials. **(Bilaspur 1994, 96)**

Let $\quad x^n = C_n P_n(x) + C_{n-2} P_{n-2}(x) + C_{n-4} P_{n-4}(x) + \ldots + C_r P_r(x) + \ldots, \qquad \ldots(1)$

where $P_n(x)$, $P_{n-2}(x)$, ... are all even or odd functions of x according as x^n is an even or odd function (Refer Art 9.19 for more details). Again $P_{n-1}(x)$, $P_{n-3}(x)$, ... cannot occur in the proposed series (1) because they are odd or even functions of x according as x^n is even or odd function x respectively. Since $P_n(x)$ contains terms of degree n and lower, hence the expansion (1) cannot contain any P with suffix higher than n.

Multiplying both sides of (1) by $P_m(x)$ and integrating between the limits -1 and 1, we have

$$\int_{-1}^{1} x^n P_m(x)\,dx = C_m \int_{-1}^{1} [P_m(x)]^2\,dx, \quad \text{as} \quad \int_{-1}^{1} P_n(x) P_m(x)\,dx = \begin{cases} 0, \text{ if, } m \neq n \\ 2/(2m+1), \text{ if } m = n \end{cases}$$

$$\text{or} \quad \int_{-1}^{1} x^n P_m(x)\,dx = C_m \cdot \frac{2}{2m+1} \qquad \text{or} \qquad C_m = \frac{(2m+1)}{2} \int_{-1}^{1} x^n P_m(x)\,dx \qquad \ldots(2)$$

$$(2) \Rightarrow C_m = \frac{(2m+1)}{2} \int_{-1}^{1} x^n \left[\frac{1}{2^m\, m!} \frac{d^m}{dx^m}(x^2-1)^m\right] dx, \text{ by Rodrigue's formula}$$

$$= \frac{(2m+1)}{2^{m+1}\, m!} \int_{-1}^{1} x^n \left[\frac{d^m}{dx^m}(x^2-1)^m\right] dx = \frac{(2m+1)}{2^{m+1}\, m!} \left[\left\{x^n \frac{d^{m-1}}{dx^{m-1}}(x^2-1)^m\right\}_{-1}^{1} - \int_{-1}^{1} n x^{n-1} \frac{d^{m-1}}{dx^{m-1}}(x^2-1)^m\,dx\right]$$

(Integrating by parts taking x^n as first function)

$$= (-1)\frac{(2m+1)n}{2^{m+1}\, m!} \int_{-1}^{1} x^{n-1} \left\{\frac{d^{m-1}}{dx^{m-1}}(x^2-1)^m\right\} dx = (-1)^2 \frac{(2m+1)}{2^{m+1}\, m!} n(n-1) \int_{-1}^{1} x^{n-1} \left\{\frac{d^{m-1}}{dx^{m-1}}(x^2-1)^m\right\} dx$$

[Again Integrating by parts taking x^{n-1} as first function and simplifying as before]

$$= (-1)^m \frac{(2m+1)}{2^{m+1}\, m!} n(n-1)\ldots(n-m+1) \int_{-1}^{1} x^{n-m} (x^2-1)^m\,dx$$

[Repeating the above process of integration by parts $m-2$ times more]

$$= (-1)^m \frac{(2m+1)}{2^{m+1}\, m!} n(n-1)\ldots(n-m+1)(-1)^m \int_{-1}^{1} x^{n-m} (1-x^2)^m\,dx$$

$$\therefore \qquad C_m = \frac{(2m+1)}{2^{m+1}\, m!} n(n-1)(n-2)\ldots(n-m+1) \int_{-1}^{1} x^{n-m} (1-x^2)^m\,dx. \qquad \ldots(3)$$

In our discussion m can be one of integers n, $n-2$, $n-4$, ... only and hence $(n-m)$ can be one of the integers 0, 2, 4, 6, ... etc only. Hence $x^{n-m}(1-x^2)^m$ is an even function of x.

$$\therefore \qquad \int_{-1}^{1} x^{n-m} (1-x^2)^m\,dx = 2\int_{0}^{1} x^{n-m} (1-x^2)^m\,dx. \qquad \ldots(4)$$

Using (4), (3) reduces to

$$C_m = \frac{(2m+1)}{2^{m+1}\ m!} \cdot \frac{n!}{(n-m)!} . 2\int_0^1 x^{n-m}\ (1-x^2)^m\ dx\ = \frac{(2m+1)}{2^{m+1}\ m!} \cdot \frac{n!}{(n-m)!}\int_0^1 x^{n-m-1}\ (1-x^2)^m \cdot 2xdx$$

$$= \frac{(2m+1)}{2^{m+1}\ m!} \cdot \frac{n!}{(n-m)!} \cdot \int_0^1 t^{(n-m-1)/2}\ (1-t)^m\ dt \qquad \text{[Putting } x^2 = t \text{ and } 2xdx = dt]$$

$$= \frac{(2m+1)}{2^{m+1}\ m!} \cdot \frac{n!}{(n-m)!} \cdot \int_0^1 t^{(n-m+1)/2-1}\ (1-t)^{(m+1)-1} \cdot dt$$

$$= \frac{(2m+1)}{2^{m+1}\ m!} \cdot \frac{n!}{(n-m)!} \cdot \frac{\Gamma[(n-m+1)/2]\,\Gamma(m)}{\Gamma[(n-m+1)/2+m]} \quad \left[\because B(p, q) = \int_0^1 t^{p-1}\ (1-t)^{q-1}\ dt = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}\right]$$

$$= \frac{(2m+1)\, n!}{2^{m+1}\ (n-m)!} \times \frac{\Gamma\{(n-m+1)/2\}}{\frac{n+m+1}{2} \cdot \frac{n+m-1}{2} \cdot \frac{n+m-3}{2} \cdots \frac{n-m+1}{2}\, \Gamma\left(\frac{n-m+1}{2}\right)}$$

$$= \frac{(2m+1)}{(n-m)!} \cdot \frac{n(n-1)\,(n-2) \ldots (n-m+2)\,(n-m+1)\,(n-m)!}{(n+m+1)\,(n+m-1) \ldots (n-m+3)\,(n-m+1)}.$$

Thus, $$C_m = \frac{n(n-1)\,(n-2) \ldots (n-m+2)}{(n+m+1)\,(n+m-1) \ldots (n-m+3)}(2m+1). \qquad \ldots(5)$$

Putting $m = n$, $(n-2)$, $(n-4)$, ... in (5), we obtain

$$C_n = \frac{n(n-1)\,(n-2) \ldots 3 \cdot 2}{(2n+1)\,(2n-1) \ldots 5 \cdot 3}(2n+1) = \frac{n!}{3 \cdot 5 \cdot (2n-1)\,(2n+1)}(2n+1),$$

$$C_{n-2} = \frac{n(n-1) \ldots 5 \cdot 4}{(2n-1)\,(2n-3) \ldots 7 \cdot 5}(2n-3) = \frac{n!}{3 \cdot 5 \cdot 7 \cdot (2n-1)\,(2n+1)}\ \frac{(2n+1)}{2}(2n-3),$$

$$C_{n-4} = \frac{n(n-1) \ldots 7 \cdot 6}{(2n-3)\,(2n-5) \ldots 9 \cdot 7}(2n-7) = \frac{6 \cdot 7 \ldots (n-1)n}{7 \cdot 9 \ldots (2n-5)\,(2n-3)}\ (2n-7)$$

$$= \frac{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \ldots (n+1)n}{3 \cdot 5 \cdot 7 \ldots (2n-1)\,(2n+1)} \cdot \frac{(2n+1)\,(2n-1)}{2.4}(2n-7) = \frac{n!}{3 \cdot 5 \cdot 7 \ldots (2n+1)} \cdot \frac{(2n+1)\,(2n-1)}{2.4}(2n-7)$$

and so on. Putting these values in (1), we have

$$x^n = \frac{n!}{3 \cdot 5 \ldots (2n+1)}\left[(2n+1)\, P_n(x) + (2n-3) \cdot \frac{(2n+1)}{2}\, P_{n-2}(x) + (2n-7)\frac{(2n+1)(2n-1)}{2 \cdot 4}\, P_{n-4}(x)\right.$$

$$\left. + \ldots + \frac{1}{(n+1)} P_0(x)\right], \text{ if } n \text{ is even} \qquad \ldots 6(A)$$

and $$x^n = \frac{n!}{3 \cdot 5 \ldots (2n+1)}\left[(2n+1)\, P_n(x) + (2n-3) \cdot \frac{(2n+1)}{2} P_{n-2}(x)\right.$$

$$\left. + (2n-7)\frac{(2n+1)\,(2n-1)}{2 \cdot 4} P_{n-4}(x) + \ldots + \frac{3P_1(x)}{(n+2)}\right], \text{ if } n \text{ is odd} \qquad \ldots 6\,(B)$$

To compute C_0 and C_1 it is convenient to use (2) as shown below. From (2), we have

$$C_0 = \frac{1}{2}\int_{-1}^{1} x^n\, P_0(x)\, dx = \frac{1}{2}\int_{-1}^{1} x^n\, dx = \frac{1}{2}\cdot 2\int_{0}^{1} x^n\, dx = \left[\frac{n^{n+1}}{n+1}\right]_0^1 = \frac{1}{n+1}.$$

[using the facts that $P_0(x) = 1$ and n is even for C_0]

and
$$C_1 = \frac{3}{2}\int_{-1}^{1} x^n\, P_1(x)\, dx = \frac{2}{3}\int_{-1}^{1} x^n \cdot x\, dx, \text{ as } P_1(x) = x$$

$$= \frac{3}{2}\cdot 2\int_0^1 x^{n+1}\, dx, \text{ as } n \text{ is odd so } (n+1) \text{ is even for } C_1$$

$$= 3\left[\frac{x^{n+2}}{n+2}\right]_0^1 = \frac{3}{n+2}.$$

Remark. The result of this article can also put in compact form

$$x^n = \frac{n!}{2^n}\sum_{k=0}^{[n/2]} \frac{(2n-4k+1)\, P_{n-2k}(x)}{k!\,(3/2)_{n-k}}, \qquad \text{where} \qquad [n/2] = \begin{cases} n/2, & \text{if } n \text{ is even} \\ (n-1)/2, & \text{if } n \text{ is odd.} \end{cases}$$

and the symbol $(\alpha)_n$ is defined as below $\qquad (\alpha)_0 = 1, \quad (\alpha)_n = \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n-1).$

Corollary. Let $f(x) = a_0 + a_1x + a_2x^2 + \ldots. + a_nx^n + \ldots..$, where $a_0, a_1, \ldots, a_n, \ldots$ are constants. Then $f(x)$ can be expanded in Legendre's polynomials in the form

$$f(x) = \sum_{n=0}^{\infty} C_nP_n(x), \quad \text{where}$$

$$C_n = \frac{n!}{3\cdot 5 \ldots (2n+1)}\left[a_n + \frac{(n+1)(n+2)}{2(2n+3)}a_{n+2} + \frac{(n+1)(n+2)(n+3)(n+4)}{2\cdot 4\cdot (2n+3)(2n+5)}a_{n+4} + \ldots\right]$$

Proof. Given
$$f(x) = a_0 + a_1x + a_2x^2 + \ldots + a_nx^n + a_{n+1}x^{n+1} + \ldots \qquad \ldots(7)$$

Assume that
$$f(x) = \sum_{n=0}^{\infty} C_nP_n(x). \qquad \ldots(8)$$

With help of formula (4), we replace every power of x in (7) by its expansion in terms of Legendre's polynomials and then we collect the terms involving $P_n(x)$.

Clearly when $x^n, x^{n+2}, x^{n+4}, \ldots$ are expanded in terms of Legendre's polynomials, each one of them involves a term containing $P_n(x)$. Again $P_n(x)$ will not be involved in any expansion containing power of x less than n. Thus, we see that only a_nx^n, $a_{n+2}x^{n+2}$, $a_{n+4}x^{n+4}$, ... etc in (7) will contain $P_n(x)$. Now, using (6), we have

$$a_nx^n = a_n \frac{n!}{3\cdot 5\ldots(2n+1)}\left[(2n+1)\,P_n(x) + \ldots\ldots.\right]$$

$$a_{n+2}x^{n+2} = a_{n+2}\frac{(n+2)!}{3\cdot 5\ldots(2n+5)}\left[(2n+5)P_{n+2}(x) + (2n+1)\frac{(2n+5)}{2}P_n(x) + \ldots\right]$$

$$a_{n+4}x^{n+4} = a_{n+4}\frac{(n+4)!}{3\cdot 5\ldots(2n+9)}\Big[(2n+9)\,P_{n+4}(x)$$

$$+ (2n+5)\frac{(2n+9)}{2}P_{n+2}(x) + (2n+1)\,.\,\frac{(2n+9)(2n+7)}{2\cdot 4}P_n(x) + \ldots.\Big]$$

Putting the above values in (7) and equating the coefficients of $P_n(x)$ from (7) and (8),

$$C_n = \frac{n!}{3\cdot 5\ldots(2n+1)}(2n+1)a_n + \frac{(n+2)!}{3\cdot 5\ldots(2n+5)}\left[\frac{(2n+1)(2n+5)}{2}\right]a_{n+2}$$

$$+\frac{(n+4)!}{3\cdot 5\ldots(2n+9)}\left[(2n+1)+\frac{(2n+9)(2n+7)}{2.4}\right]a_{n+4}+\ldots$$

or $$C_n = \frac{n!}{3\cdot 5\ldots(2n-1)}\left[a_n + \frac{(n+1)(n+2)}{2(2n+3)}a_{n+2} + \frac{(n+1)(n+2)(n+3)(n+4)}{2\cdot 4\cdot(2n+3)(2n+5)}a_{n+4}+\ldots\right]$$

9.21. Solved examples based on Art 9.20

Ex. 1. *Prove that for all x* (*a*) $x^4 = (8/35)\times P_4(x) + (4/7)\times P_2(x) + (1/5)\times P_0(x)$.
(*b*) $x^5 = (8/63)\times P_5(x) + (4/9)\times P_3(x) + (3/7)\times P_1(x)$.

Sol. We have, $$x^n = \frac{n!}{3\cdot 5\ldots(2n+1)}\left[(2n+1)P_n(x) + (2n-3)\frac{(2n+1)}{2}P_{n-2}(x)\right.$$

$$\left. + (2n-7)\frac{(2n+1)(2n-1)}{2\cdot 4}P_{n-4}(x)+\ldots\right] \quad \ldots(1)$$

(*a*) Putting $n = 4$ in (1), we have

$$x^4 = \frac{4!}{3\cdot 5\cdot 7\cdot 9}\left[9P_4(x) + 5\times\frac{9}{2}P_2(x) + 1\times\frac{9\cdot 7}{2\cdot 4}P_0(x)\right] = \frac{8}{35}P_4(x)+\frac{4}{7}P_2(x)+\frac{1}{5}P_0(x)$$

(*b*) Putting $n = 5$ in (1), we have

$$x^5 = \frac{5!}{3\cdot 5\cdot 7\cdot 9\cdot 11}\left[11P_5 + 7\times\frac{11}{2}P_3(x) + 3\times\frac{11\cdot 9}{2\cdot 4}P_1(x)\right] = \frac{8}{63}P_5(x)+\frac{4}{9}P_3(x)+\frac{3}{7}P_1(x)$$

Ex. 2. *Prove* (*a*) $x^2 = (1/3)\times P_0(x) + (2/3)\times P_1(x)$. (*b*) $x^3 = (3/5)\times P_1(x) + (2/5)\times P_3(x)$.

Ex. 3. *Express* $f(x) = x^4 + 3x^2 - x^2 + 5x - 2$ *in terms of Legendre's polynomials.*

Sol. As in Ex. 1 and 2, prove yourself that

$x^4 = (8/35)P_4(x) + (4/7)P_2(x) + (1/5)P_0(x)$, $x^3 = (3/5)P_1(x) + (2/5)P_3(x)$,

$x^2 = (1/3)P_0(x) + (2/3)P_1(x)$, $x = P_1(x)$ and $1 = P_0(x)$.

$\therefore$ $f(x) = (8/35)P_4(x) + (4/7)P_2(x) + (1/5)P_0(x) + 3[(3/5)P_1(x) + (2/5)P_3(x)] - [(1/3)P_0(x) + (2/3)P_1(x)] + 5P_1(x) - 2P_0(x)$, on putting values of x^4, x^3, x^2, x and 1

$= (8/35)P_4(x) + (6/5)P_3(x) + [(4/7) - (2/3)]P_2(x) + (9/5 + 5)P_1(x) + [(1/5) - (1/3) - 2]P_0(x)$

$= (8/35)P_4(x) + (6/5)P_3(x) - (2/21)P_2(x) + (34/5)P_1(x) - (32/15)P_0(x)$.

EXERCISE

Express the following polynomials in terms of Legendre polynomials :

1. $x^3 + 2x^2 - x - 3$ **(KU Kurukshetra 2004)**

2. $4x^3 - 2x^2 - 3x + 8$ **(KU Kurukshtra 2005)**

3. $x^4 + 3x^3 - x^2 + 5x - 2$

4. $x^4 + 2x^3 - 5x^2 - x - 2$

5. $1 + x - x^2$ **(Gulbarga 2005, Purvanchal 2004)**

6. $5x^3 + x$ **[Agra 2007]**

ANSWERS

1. $(2/5) \times P_3(x) + (4/3) \times P_2(x) - (2/5) \times P_1(x) - (7/3) \times P_0(x)$

2. $(8/5) \times P_3(x) - (4/3) \times P_2(x) - (3/5) \times P_1(x) + (22/3) \times P_0(x)$

3. $(8/35) \times P_4(x) + (6/5) \times P_3(x) - (2/21) \times P_2(x) + (34/5) \times P_1(x) - (32/25) \times P_0(x)$

4. $(8/35) \times P_4(x) + (4/5) \times P_3(x) + (82/21) \times P_2(x) + (1/5) \times P_1(x) - (2/15) \times P_0(x)$

5. $(2/3) \times P_0(x) + (1/3) \times P_0(x)$ **6.** $2P_3(x) + 4P_1(x)$

MISCELLANEOUS EXAMPLES ON CHAPTER 9

1. *Verify that the Legendre polynomise* $P_4(x) = (35x^3 - 30x^2 + 3)/8$ *satisfies the Legendre equation when the parameter n is equal to* 4.

[Sol. The Legendre equation $(1-x^2)y'' - 2xy' + n(n+1)y = 0$ for $n = 4$ becomes

$$(1-x^2)y'' - 2xy' + 20y = 0 \qquad \text{... (1)}$$

Let $$y = P_4(x) = (35x^4 - 30x^2 + 3)/8 \qquad \text{... (2)}$$

From (2), $y' = (35x^3 - 15x)/2$ and $y'' = (105x^2 - 15)/2$

Substituting the above value of y, y' and y'' in (1), we get

$(1 - x^2) \times (1/2) \times (105x^2 - 15) - (2x) \times (1/2) \times (35x^3 - 15x) + 20 \times (1/8) \times (35x^4 - 15x^2 + 3) = 0$

or $(-105/2 - 35 + 175/2)x^4 + (105/2 + 15/2 + 15 - 75)\, x^2 + (-15/2 + 15/2) = 0$

or $0 = 0$, which is true. Hence $P_4(x)$ is a solution of (1).]

2. Prove that $P_n(\cos\theta) = \sum_{r=0}^{p} {}^nC_r \cos(n-2r)\theta$, where $p = n/2$ or $(n-1)/2$ according as n is even or odd. Deduc that $|P_n(\cos\theta)| \le 1$

3. Prove that $\displaystyle\int_{-1}^{1} x^m P_n(x)dx = \begin{cases} 0, \text{ if } m < n \\ \dfrac{m!\,\Gamma(m/2 - n/2 + 1/2)}{2^n (m-n)!\,\Gamma(m/2 + n/2 + 3/2)}, \text{ if } m - n\ (\ge 0) \text{ is even} \\ 0, \text{ if } m - n\ (> 0) \text{ is odd} \end{cases}$

4. If $U_n = \displaystyle\int_{-1}^{1} P_n(x)P_{n-1}(x)\frac{dx}{x}$, show that $(n + 1)\, U_{n+1} + n\, U_n = 2$. Hence evaluate U_n.

5. If R denotes the operator

$\dfrac{d}{dx}\left\{(1-x^2)\dfrac{d}{dx}\right\}$, then show that $\displaystyle\int_{-1}^{1} P_n(x) R\{f(x)\}\, dx = -n(n+1)\int_{-1}^{1} P_n(x) f(x)\, dx$

provided that $f(x)$ and $f'(x)$ are finite at $x = \pm 1$. Deduce that $\displaystyle\int_{-1}^{1} \log(1-x) P_n(x)\, dx = -\frac{2}{n(n+1)}$.

6. If n is a positive integer, show that $\displaystyle\int_{-1}^{1} P_n(x)(1 - 2xz + z^2)^{-1/2} dx = \frac{2z^n}{2n+1}$

and hence making use of Rodrigue's formula, deduce that

$$\int_{-1}^{1} (1-x^2)^n (1 - 2xz + z^2)^{-n-1/2} dx = \frac{2^{2n+1}(n!)^2}{(2n+1)!}$$

From this result, deduce $$\int_{-1}^{1} \frac{P_n(x)dx}{(1-x)^{1/2}} = \frac{2\sqrt{2}}{2n+1}$$

7. Using Rodrigue's formula show that
(i) $P_n(-x) = (-1)^n P_n(x)$ (ii) $P_n(1) = 1$ **(Bangalore 2005)**

OBJECTIVE PROBLEMS IN CHAPTER 9

Ex 1. *The value of* $(1/2^n n!) \times \{d^n(x^2-1)^n/dx^n\}$ is (a) 0 (b) 1 (c) $P_n(x)$ (d) *None of these*.
Sol. Ans. (c) Refer Art. 9.14. **[Agra 2005, 06]**

Ex. 2. *Let* $P_n(x)$ *be the Legendre polynomial of degree* $n \geq 0$. If $1+x^{10} = \sum_{n=0}^{10} C_n P_n(x)$, *then* C_5 *equals* : (a) 0 (b) 2/11 (c) 1 (d) 11/2 **[GATE 2004]**

Sol. Ans. (a) Refer theorem 9.16. Here $f(x) = 1 + x^{10}$ is a polynomial of degree 10, which is even. Hence, only those C_n with even suffuses are non-zero.

Ex. 3. *Let* $y = \psi(x)$ *be a bounded solution for the equation* $(1-x^2)y'' - 2xy' + 30y = 0.$

Then (a) $\int_{-1}^{1} x^3 \psi(x) dx \neq 0$ (b) $\int_{-1}^{1} (1+x^3+x^4)\, \psi\,(x) \neq 0$

(c) $\int_{-1}^{1} x^3 \psi(x)\, dx = 0$ (d) $\int_{-1}^{1} x^{2n} \psi(x) dx = 0$ for all $n \in \boldsymbol{N}$ **[GATE 2003]**

Sol. Ans. (d) Re-writing given equation, $(1-x^2)y'' - 2xy' + 5(5+1)y = 0$ ... (1)

Comparing (1) with $(1-x^2)y'' - 2xy' + n(n+1)y = 0,$ we find that (1) is Legendre equation with $n = 5$. Since $\psi(x)$ is a bounded solution of (1), we have $\psi(x) = P_5(x) = (63x^2 - 70x^3 + 15x)/8,$ by Art. 9.2. Since $x^{2n}\psi(x)$ is an odd function, conclusion (d) is true.

Ex. 4. Let $P_n(x)$ denote the Legendre polynmial of degree n. If

$f(x) = \begin{cases} x, & -1 \leq x \leq 0 \\ 0, & 0 \leq x \leq 1 \end{cases}$, and $f(x) = a_0 P_0(x) + a_1 P_1(x) + a_2 P_2(x) + \ldots$, then

(a) $a_0 = -1/4$, $a_1 = -1/2$ (b) $a_0 = -1/4$, $a_1 = 1/2$
(c) $a_0 = 1/2$, $a_1 = -1/4$ (d) $a_0 = -1/2$, $a_1 = -1/4$ **(GATE 2005)**

Sol. Ans. (b). Here $f(x) = \sum_{r=0}^{\infty} a_r P_r$. Proceed exactly as in Ex. 1 of Art. 9.18 to get a_0, a_1.

PART II : ASSOCIATED LEGENDRE FUNCTIONS.

9.22. Associated Legendre Functions

Theorem. If z is a solution of Legendre equation $(1-x^2)y'' - 2xy' + n(n+1)y = 0$ ...(1)

then to show that $(1-x^2)^{m/2} \dfrac{d^m z}{dx^m}$ is a solution of the equation

$$(1-x^2)y'' - 2xy' + \left\{ n(n+1) - \frac{m^2}{1-x^2} \right\} y = 0 . \qquad \text{...(2)}$$

Proof. Since z is a solution of (1), we get $$(1-x^2)\frac{d^2z}{dx^2} - 2x\frac{dz}{dx} + n(n+1)z = 0. \quad ...(3)$$

Differentiation (3) m times with help of Leibnitz' theorem, we have

$$\frac{d^m}{dx^m}\left[\frac{d^2z}{dx^2}.(1-x^2)\right] + \frac{d^m}{dx^m}\left[\frac{dz}{dx}(-2x)\right] + n(n+1)\frac{d^mz}{dx^m} = 0$$

or $$\frac{d^{m+2}z}{dx^{m+2}}(1-x^2) + {}^mC_1\frac{d^{m+1}z}{dx^{m+1}}(-2x) + {}^mC_2\frac{d^mz}{dx^m}(-2) + \frac{d^{m+1}z}{dx^{m+1}}(-2x) + {}^mC_1\frac{d^mz}{dx^m}(-2) + n(n+1)\frac{d^mz}{dx^m} = 0$$

or $$(1-x^2)\frac{d^{m+2}z}{dx^{m+2}} - 2x(m+1)\frac{d^{m+1}z}{dx^{m+1}} + \{n(n+1) - m(m-1) - 2m\}\frac{d^mz}{dx^m} = 0$$

or $$(1-x^2)\frac{d^{m+2}z}{dx^{m+2}} - 2x(m+1)\frac{d^{m+1}z}{dx^{m+1}} + \{n(n+1) - m(m+1)\}\frac{d^mz}{dx^m} = 0 \quad ...(4)$$

Let $$u = d^mz/dx^m. \quad ...(5)$$

Then (4) reduces to

$$(1-x^2)u'' - 2x(m+1)u'' + [n(n+1) - m(m+1)\}u = 0. \quad ...(6)$$

Let $$v = (1-x^2)^{m/2}u \quad ...(7)$$

so that $$u = (1-x^2)^{-m/2}\, v. \quad ...(8)$$

From (8), $$u' = (1-x^2)^{-m/2}v' + (-m/2) \times (1-x^2)^{-(m/2)-1}(-2x)\, v.$$

or $$u' = (1-x^2)^{-m/2}\, v' + mvx(1-x^2)^{-(m/2)-1}.$$

From (9), $$u'' = (1-x^2)^{-m/2}v'' + (-m/2) \times (1-x^2)^{(m/2)-1} \times (-2x) \times v' + mvx\{-(m/2)-1\}\,(1-x^2)^{-(m/2)-1}(-2x) + m(1-x^2)^{-(m/2)-1}(xv' + v). \quad ...(10)$$

Substituting the values of u, u' and u'' given by (8), (9) and (10) into (6), we have

$$(1-x^2)^{-(m/2)+1}v'' + mx(1-x^2)^{-m/2}v' + mx^2v(m+2)(1-x^2)^{-(m/2)-1} + mx(1-x^2)^{-m/2}v' + mv(1-x^2)^{-m/2}$$
$$-2(m+1)x\,(1-x^2)^{-m/2}v' - 2m(m+1)\,x^2v(1-x^2)^{-m/2-1} + \{n(n+1) - m^2 + m\}\,(1-x^2)^{-m/2}v = 0.$$

Dividing throughout by $(1-x^2)^{-(m/2)}$, the above equation becomes

$$(1-x^2)v'' - 2xv''\left[n(n+1) - m^2 - m - \frac{2m(m+1)}{1-x^2}x^2 + m + \frac{m(m+2)}{1-x^2}x^2\right]v = 0.$$

or $$(1-x^2)v'' - 2xv' + \{n(n+1) - m^2/(1-x^2)\}\,v = 0 \quad ...(11)$$

Thus, using (11), (7) and (5), we get $$v = (1-x^2)^{m/2}\,u = (1-x^2)^{m/2}\frac{d^mz}{dx^m}$$

is a solution of (11) and hence it is solution of (2).

Remark 1. Equation (2) is called the *associated Legendre equation*. Since $P_n(x)$ and $Q_n(x)$ are solutions of (3), we conclude that

$$P_n^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x) \qquad \text{and} \qquad Q_n^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}Q_n(x)$$

are solutions of (2). $P_n^m(x)$ and $Q_n^m(x)$ are called the *associated Legendre's functions of degree n and order m of the first and second kind* respectively. Since these are independent solutions of (2), the general solutions of (2) is $y = AP_n^m(x) + BQ_n^m(x)$, where A and B are arbitrary constants.

Note that if $m > n$, $P_n^m(x) = 0$. The functions $Q_n^m(x)$ are unbounded for $x = \pm 1$.

Remark 2. For $m \geq 0$, we define $$P_n^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x).$$

Using Rodeigue's formula this gives $$P_n^m(x) = (1-x^2)^{m/2} \frac{1}{2^n n!} \frac{d^{m+n}}{dx^{m+n}} (x^2-1)^n,.$$

where R.H.S.is well defined for negative values of m such that $m+n \geq 0$ *i.e.*, $m \geq -n$.

Thus, we define $$P_n^m(x) = \begin{cases} (1-x^2)^{m/2} \dfrac{d^m}{dx^m} P_n(x), \text{ for } m \geq 0 \\ (1-x^2)^{m/2} \dfrac{1}{2^n n!} \dfrac{d^{m+n}}{dx^{m+n}} (x^2-1)^n, \text{for } m \geq n. \end{cases}$$

Corollary. *To find the general solutions of the following equations:*

(*a*) $$\Theta'' + \cot\theta \,.\, \Theta' + \{n(n+1) - m^2/\sin^2\theta\}\Theta. \qquad ...(12)$$

(*b*) $$\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\theta}\right) + n(n+1)\sin\theta\,\Theta = 0 \qquad ...(13A)$$

or $$\Theta'' + \cot\theta.\ \Theta' + n(n+1)\Theta = 0. \qquad ...(13B)$$

Proof. (*a*) Put $\cos\theta = \mu$. Then, $$\Theta' = \frac{d\Theta}{d\theta} = \frac{d\Theta}{d\mu}\frac{d\mu}{d\theta} = -\sin\theta \,.\, \frac{d\Theta}{d\mu} \qquad ...(14)$$

$$\therefore \qquad \frac{d}{d\theta} \equiv -\sin\theta \,.\, \frac{d}{d\mu} \qquad ...(15)$$

Now using (14), $$\Theta'' = \frac{d}{d\theta}\left(\frac{d\Theta}{d\theta}\right) = -\frac{d}{d\theta}\left(\sin\theta \frac{d\Theta}{d\mu}\right) = -\cos\theta \frac{d\Theta}{d\mu} - \sin\theta \frac{d}{d\theta}\left(\frac{d\Theta}{d\mu}\right)$$

$$= -\cos\theta \frac{d\Theta}{d\mu} - \sin\theta \times (-\sin\theta)\ \frac{d}{d\mu}\left(\frac{d\Theta}{d\mu}\right), \text{ by (15)}$$

$$= -\cos\theta \frac{d\Theta}{d\mu} + \sin^2\theta \frac{d^2\Theta}{d\mu^2} = -\mu \frac{d\Theta}{d\theta} + (1-\mu^2)\frac{d^2\Theta}{d\mu^2}, \text{ as } \sin^2\theta = 1 - \cos^2\theta = 1 - \mu^2$$

Using these values of Θ' and Θ'' in (12), we have

$$-\mu \frac{d\Theta}{d\mu} + (1-\mu^2)\frac{d^2\Theta}{d\mu^2} + \frac{\cos\theta}{\sin\theta}\left(-\sin\theta \frac{d\Theta}{d\mu}\right) + \left\{n(n+1) - \frac{m^2}{1-\cos^2\theta}\right\}\Theta = 0$$

or $$(1-\mu^2)\frac{d^2\Theta}{d\mu^2} - 2\mu \frac{d\Theta}{d\mu} + \left\{n(n+1) - \frac{m^2}{1-\mu^2}\right\}\Theta = 0, \text{ as } \mu = \cos\theta \qquad ...(16)$$

which is same as (2). Hence the general solution of (16) is $\Theta = AP_n^m(\mu) + BQ_n^m(\mu)$

or $\Theta = AP_n^m(\cos\theta) + BQ_n^m(\cos\theta)$, as $\mu = \cos\theta$. ...(17)

Part (a). Putting $\cos\theta = \mu$ and doing as before, (13A) or (13B) gives

$$(1-\mu^2)\frac{d^2\Theta}{d\mu^2} - 2\mu \frac{d\Theta}{d\mu} + n(n+1)\Theta = 0, \qquad ...(18)$$

which is Legendre equation and so solution of (13A) or (13B) is

$\Theta = AP_n(\mu) + BQ_n(\mu)$ or $\Theta = AP_n(\cos\theta) + BQ_n(\cos\theta)$.

9.23. Properties of the Associated Legendre Functions.

(*i*) $P_n^0(x) = P_n(x)$ (*ii*) $P_n^m(x) = 0$ if $m > n$.

Proof. We have $$P_n^m(x) = (1-x^2)^{m/2} \frac{d^m}{dx^m} P_n(x). \qquad ...(1)$$

(i) Putting $m = 0$ in (1), $P_n^0(x) = P_n(x)$.

(ii) Since $P_n(x)$ a polynomial of degree n, so when $m > n$, (1) gives $P_n^m(x) = (1-x^2)^{m/2} \times 0 = 0$.

9.24. Orthogonality relation for $\mathbf{P_n^m(x)}$. *To show that*

$$\int_{-1}^{1} P_n^m(x) P_l^m(x)\, dx = \frac{2(n+m)!}{(2n+1)\,(n-m)!}\delta_{nl}, \qquad \text{where} \qquad \delta_{nl} = \begin{cases} 0, \text{ if } n \neq l \\ 1, \text{ if } n = l \end{cases}$$

Proof. Case I. Let $l \neq n$. Since $P_n^m(x)$ satisfy the corresponding associated equation

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \left\{n(n+1) - \frac{m^2}{1-x^2}\right\}y = 0. \quad i.e., \quad \frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\} + \left\{n(n+1) - \frac{m^2}{1-x^2}\right\}y = 0.$$

$$\therefore \qquad \frac{d}{dx}\left\{(1-x^2)\frac{d}{dx}P_n^m(x)\right\} + \left\{n(n+1) - \frac{m^2}{1-x^2}\right\}P_n^m(x) = 0. \qquad \ldots (1)$$

Similarly, for $P_l^m(x)$, $\dfrac{d}{dx}\left\{(1-x^2)\dfrac{d}{dx}P_l^m(x)\right\} + \left\{l(l+1) - \dfrac{m^2}{1-x^2}\right\}P_l^m(x) = 0.$ ... (2)

Multiplying (1) by $P_l^m(x)$ and (2) by $P_n^m(x)$ and subtracting the resulting equations gives

$$P_l^m \frac{d}{dx}\left\{(1-x^2)\frac{d}{dx}P_n^m(x)\right\} - P_n^m(x)\frac{d}{dx}\left\{(1-x^2)\frac{d}{dx}P_l^m(x)\right\} + [n(n+1) - l(l+1)]P_n^m(x)P_l^m(x) = 0.$$

Integrating between limits –1 to 1, we get

$$\int_{-1}^{1} P_l^m(x)\frac{d}{dx}\left\{(1-x^2)\frac{d}{dx}P_n^m(x)\right\}dx - \int_{-1}^{1} P_n^m(x)\frac{d}{dx}\left\{(1-x^2)\frac{d}{dx}P_l^m(x)\right\}dx + [n(n+1) - l(l+1)]\int_{-1}^{1} P_n^m(x)P_l^m(x)dx = 0.$$

Integrating by parts, we now obtain

$$\left[P_l^m(x)(1-x^2)\frac{d}{dx}P_n^m(x)\right]_{-1}^{1} - \int_{-1}^{1}\frac{d}{dx}P_l^m(1-x^2)\frac{d}{dx}P_n^m(x)dx - \left[P_n^m(x)(1-x^2)\frac{d}{dx}P_l^m(x)\right]_{-1}^{1}$$

$$+\int_{-1}^{1}\frac{d}{dx}P_n^m(1-x^2)\frac{d}{dx}P_l^m(x)\,dx + [(n^2 - l^2) + (n-l)]\int_{-1}^{1} P_n^m(x)P_l^m(x)dx = 0$$

or $\qquad (n-1)\,(n+l+1)\displaystyle\int_{-1}^{1} P_n^m(x)P_l^m(x)dx = 0 \qquad$ or $\qquad \displaystyle\int_{-1}^{1} P_n^m(n)P_l^m(x)dx = 0,$ if $n \neq 1$ ... (3)

Case II. Let $l = n$, If $m > 0$, we have $\qquad P_n^m(x) = (1-x^2)^{m/2}\dfrac{d^m}{dx^m}P_n(x).$ ... (4)

$$\therefore \quad \int_{-1}^{1} P_n^m(x)P_l^m(x)dx = \int_{-1}^{1}\{P_n^m(x)\}^2 dx = \int_{-1}^{1}\left[(1-x^2)^m\frac{d^m}{dx^m}P_n(x)\right]\frac{d^m}{dx^m}P_n(x)dx, \text{ using (4)}$$

$$= \left[\left\{(1-x^2)^m\frac{d^m}{dx^m}P_n(x)\right\}\left\{\frac{d^{m-1}}{dx^{m-1}}P_l(x)\right\}\right]_{-1}^{1} - \int_{-1}^{1}\left\{\frac{d^{m-1}}{dx^{m-1}}P_n(x)\right\}\frac{d}{dx}\left\{(1-x^2)^m\frac{d^m}{dx^m}P_n(x)\right\}dx$$

[On integrating by parts]

$$\therefore \qquad \int_{-1}^{1} \{P_n^m(x)\}^2 dx = -\int_{-1}^{1} \left\{ \frac{d^{m-1}}{dx^{m-1}} P_n(x) \right\} \frac{d}{dx} \left\{ (1-x^2)^m \frac{d^m}{dx^m} P_n(x) \right\} dx \qquad \text{... (5)}$$

Since $P_n(x)$ satisfies the Legendre equations $(1-x^2)y'' - 2xy' + n(n+1)y = 0.$

$$\therefore \qquad (1-x^2)\frac{d^2 P_n(x)}{dx^2} - 2x\frac{dP_n(x)}{dx} + n(n+1)P_n(x) = 0 \qquad \text{... (6)}$$

Differentiating both sides of (6) w.r.t. 'x' $(m-1)$ times with help of Leibnitz's theorems gives

$$(1-x^2)\frac{d^{m+1}}{dx^{m+1}}P_n(x) - 2mx\frac{d^m}{dx^m}P_n(x) + [n(n+1) - m(m-1)]\frac{d^{m-1}}{dx^{m-1}}P_n(x) = 0$$

Multiplying by $(1 - x^2)^{m-1}$, we get

$$(1-x^2)^m \frac{d^{m+1}}{dx^{m+1}} P_n(x) \; -2xm(1-x^2)^{m-1}\frac{d^n}{dx^m}P_n(x) \; +[n(n+1)-m(m-1)](1-x^2)^{m-1}\frac{d^{m-1}}{dx^{m-1}}P_n(x) = 0$$

or
$$\frac{d}{dx}\left\{(1-x^2)^m \frac{d^m}{dx^m}P_n(x)\right\} = -(n+m)\,(n-m+1)\,(1-x^2)^{m-1}\frac{d^{m-1}}{dx^{m-1}}P_n(x) \qquad \text{... (7)}$$

Using (7), (5) reduces to

$$\int_{-1}^{1}\{P_n^m(x)\}^2 dx = \int_{-1}^{1}\left\{\frac{d^{m-1}}{dx^{m-1}}P_n(x)\right\}(n+m)\,(n-m+1)\,(1-x^2)^{m-1}\frac{d^{m-1}}{dx^{m-1}}P_n(x)dx$$

$$= (n+m)\,(n-m+1)\int_{-1}^{1}\left\{(1-x^2)^{(m-1)/2}\frac{d^{m-1}}{dx^{m-1}}P_n(x)\right\}^2 dx$$

$$= (n+m)\,(n-m+1)\int_{-1}^{1}\left\{P_n^{m-1}(x)\right\}^2 dx, \text{ using definition (4) for } P_n^m(x)$$

$$= \{(n+m)\,(n-m+1)\}\;\{(n+m-1)\,(n-m+2)\}\;\ldots\ldots\{(n+1)n\}\int_{-1}^{1}\{P_n^0(x)\}^2 dx$$

[on repeating the similar method $(m-1)$ times more]

$$= (n+m)\,(n+m-1)\;\ldots.\;(n+1)\,n\,(n-1)\;\ldots\ldots(n-m+2)\,(n-m+1)\int_{-1}^{1}\{P_n(x)\}^2 dx,$$

$$= \frac{(n+m)!}{(n-m)!}\times\frac{2}{2n+1}, \text{ using Art. 9.8.} \qquad [\because \text{ From Art 9.23 } (i),\; P_n^0(x) = P_n(x)]$$

Finally, let $m < 0$. Then if $k > 0$, we write $m = -k$. Then, we know that

$$P_n^{-k}(x) = (-1)^k \frac{(n-k)!}{(n+k)!}P_n^k(x) \qquad \text{... (9)}$$

$$\therefore \qquad \int_{-1}^{1}\{P_n^m(x)\}^2\,dx = \int_{-1}^{1}\{P_n^{-k}(x)\}^2\,dx = (-1)^{2k}\left[\frac{(n-k)!}{(n+k)!}\right]^2\int_{-1}^{1}\{P_n^k(x)\}^2\,dx, \text{ using (9)}$$

$$= \left[\frac{(n-k)!}{(n+k)!}\right]^2 \times \frac{(n+k)!}{(n-k)!}\times\frac{2}{2n+1}, \quad = \frac{(n-k)!}{(n+k)!}\cdot\frac{2}{2n+1} = \frac{(n+m)!}{(n-m)!}\times\frac{2}{2n+1}, \text{ using (8) and } k = -m$$

From (8) and (10), we find
$$\int_{-1}^{1}\{P_n^m(x)\}^2\,dx = \frac{(n+m)!}{(n-m)!}\times\frac{2}{2n+1} \qquad \text{... (11)}$$

$\therefore$ (3) and (11) $\Rightarrow$
$$\int_{-1}^{1} P_n^m(x)P_l^m(x)\,dx = \frac{2(n+m)!}{(2n+1)!(n-m)!}\delta_{nl},$$

9.24. Recurrence relations (Formulae) for $P_n^m(x)$. Prove that

(*i*) $P_n^{m+1}(x) - \dfrac{2mx}{(1-x^2)^{1/2}}P_n^m(x) + \{n(n+1)-m(m-1)\}P_n^{m-1}(x) = 0.$

(*ii*) $(2n+1)xP_n^m(x) = (n+m)P_{n-1}^m(x) + (n-m-1)P_{n+1}^m(x).$

(*iii*) $(1-x^2)^{1/2}P_n^m(x) = \dfrac{1}{2n+1}\{P_{n+1}^{m+1}(x) - P_{n-1}^{m+1}(x)\}.$

(*iv*) $(1-x^2)^{1/2}P_n^m(x) = \dfrac{1}{2n+1}\{(n+m)(n+m-1)P_{n-1}^{m-1}(x) - (n-m+1)(n-m+2)P_{n+1}^{m-1}(x)\}.$

Proof. (*i*) Since $P_n(x)$ is a solution of Legendre's equation, $(1-x^2)y'' - 2xy' + n(n+1)y = 0,$

hence
$$(1-x^2)\frac{d^2}{dx^2}P_n(x) - 2x\frac{d}{dx}P_n(x) + n(n+1)P_n(x) = 0. \qquad \text{... (1)}$$

Differentiating both sides of (1) w.r.t. 'x' (m – 1) times with help of Leibnitz's theorem, gives

$$\left\{(1-x^2)\frac{d^{m+1}}{dx^{m+1}}P_n(x) - 2x(m-1)\frac{d^m}{dx^m}P_n(x) - 2\frac{(m-1)(m-2)}{2!}\cdot\frac{d^{m-1}}{dx^{m-1}}P_n(x)\right\}$$

$$-2\left\{x\frac{d^m}{dx^m}P_n(x) + (m-1)\cdot 1\cdot\frac{d^{m-1}}{dx^{m-1}}P_n(x)\right\} + n(n+1)\frac{d^{m-1}}{dx^{m-1}}P_n(x) = 0.$$

or
$$(1-x^2)\frac{d^{m+1}}{dx^{m+1}}P_n(x) - 2xm\frac{d^m}{dx^m}P_n(x) + [n(n+1)-m(m-1)]\frac{d^{m-1}}{dx^{m-1}}P_n(x) = 0.$$

Multiplying both sides by $(1-x^2)^{(m-1)/2}$, we get

$$(1-x^2)^{(m+1)/2}\frac{d^{m+1}}{dx^{m+1}}P_n(x) - 2xm(1-x^2)^{-1/2}(1-x^2)^{m/2}\frac{d^m}{dx^m}P_n(x)$$

$$+[n(n+1)-m(m-1)](1-x^2)^{(m-1)/2}\frac{d^{m-1}}{dx^{m-1}}P_n(x) = 0$$

Using the definition $P_n^m(x) = (1-x^2)^{m/2} \dfrac{d^m}{dx^m} P_n(x)$, the above result becomes

$$P_n^{m+1}(x) - \frac{2mx}{(1-x^2)^{1/2}} P_n^m(x) + [n(n+1) - m(m-1)]P_n^{m-1}(x) = 0.$$

(ii) From recurrence relations for Legendre polynomial (Refer recurrence relations I and II of Art. 9.8), we have

$$(2n + 1)\, x\, P_n(x) = (n + 1)\, P_{n+1}(x) + n\, P_{n-1}(x) \qquad \text{... (1)}$$

and

$$(2n+1)P_n(x) = \frac{d}{dx} P_{n+1}(x) - \frac{d}{dx} P_{n-1}(x) \qquad \text{... (2)}$$

Differentiating both sides of (1) w.r.t. 'x' m times with help of Leibnitz theorem, we get

$$(2n+1)\left\{x \frac{d^m}{dx^m} P_n(x) + m \cdot 1 \cdot \frac{d^{m-1}}{dx^{m-1}} P_n(x)\right\} = (n+1)\frac{d^m}{dx^m} P_{n+1}(x) + n \frac{d^m}{dx^m} P_{n-1}(x) \qquad \text{... (3)}$$

Now, differentiating (2) w.r.t 'x' $(m - 1)$ times, we get

$$(2n+1)\frac{d^{m-1}}{dx^{m-1}} P_n(x) = \frac{d^m}{dx^m} P_{n+1}(x) - \frac{d^m}{dx^m} P_{n-1}(x). \qquad \text{... (4)}$$

Putting the value of $(2n+1)\dfrac{d^{m-1}}{dx^{m-1}} P_n(x)$ given by (4) in (3), we get

$$(2n+1)x \frac{d^m}{dx^m} P_n(x) + m\left\{\frac{d^m}{dx^m} P_{n+1}(x) - \frac{d^m}{dx^m} P_{n-1}(x)\right\} = (n+1)\frac{d^m}{dx^m} P_n(x) + n \frac{d^m}{dx^m} P_{n-1}(x)$$

or

$$(2n+1)x \frac{d^m}{dx^m} P_m(x) = (n+m)\frac{d^m}{dx^m} P_{n-1}(x) + (n-m+1)\frac{d^m}{dx^m} P_{n+1}(x)$$

Multiplying both sides by $(1 - x^2)^{m/2}$, we get

$$(2n+1)x(1-x^2)^{m/2} \frac{d^m}{dx^m} P_n(x) = (n+m)(1-x^2)^{m/2} \frac{d^m}{dx^m} P_{n-1}(x) + (n-m+1)(1-x^2)^{m/2} \frac{d^m}{dx^m} P_{n+1}(x).$$

Using the definition $P_n^m(x) = (1-x^2)^{m/2} \dfrac{d^m}{dx^m} P_n(x)$, the above result becomes

$$(2n+1)x\, P_n^m(x) = (n+m)P_{n-1}^m(x) + (n-m+1)P_{n+1}^m(x).$$

***(iii)*.** From recurrence relation for Legendre polynomial (Refer relation III of Art. 9.9), we have

$$(2n+1)P_n(x) = \frac{d}{dx} P_{n+1}(x) - \frac{d}{dx} P_{n-1}(x). \qquad \text{... (1)}$$

Differentiating (1) m times w.r.t. 'x', we get

$$(2n+1)\frac{d^m}{dx^m} P_n(x) = \frac{d^{m+1}}{x^{m+1}} P_{n+1}(x) - \frac{d^{m+1}}{dx^{m+1}} P_{n-1}(x).$$

Multiplying both sides by $(1 - x^2)^{(m+1)/2}$, we get

$$(2n+1)(1-x^2)^{m/2}(1-x^2)^{1/2} \frac{d^m}{dx^m} P_n(x) = (1-x^2)^{(m+1)/2} \frac{d^{m+1}}{dx^{m+1}} P_{n+1}(x) - (1-x^2)^{(m+1)/2} \frac{d^{m+1}}{dx^{m+1}} P_{n-1}(x)$$

Using the definition $P_n^m(x) = (1-x^2)^{m/2} \dfrac{d^m}{dx^m} P_n(x)$, the above result becomes

$$(2n+1)(1-x^2)^{1/2} P_n^m(x) = P_{n+1}^{m+1}(x) - P_{n-1}^{m+1}(x) \quad \text{or} \quad (1-x^2)^{1/2} P_n^m(x) = \frac{1}{2n+1}\left\{P_{n+1}^{m+1}(x) - P_{n-1}^{m+1}(x)\right\}.$$

(iv). From recurrence relation (*ii*), $xP_n^m(x) = \dfrac{1}{2n+1}\{(n+m)P_{n-1}^m(x) + (n-m+1)P_{n+1}^m(x)\}$

Putting this values of $xP_n^m(x)$ in recurrence relation (*i*), we get

$$P_n^{m+1}(x) - \frac{2m}{(1-x^2)^{1/2}} \cdot \frac{1}{2n+1}\{(n+m)P_{n-1}^m(x) + (n-m+1)P_{n+1}^m(x)\} + \{n(n+1) - m(m-1)\}P_n^{m-1}(x) = 0 \quad \ldots (1)$$

Replacing m by $(m-1)$ in recurrence relation (*iii*), we get

$$(1-x^2)^{1/2} P_n^{m-1}(x) = \frac{1}{2n+1}\left(P_{n+1}^m(x) - P_{n-1}^m(x)\right) \text{ or } P_n^{m-1}(x) = \frac{1}{(2n+1)(1-x^2)^{1/2}}\{P_{n+1}^m(x) - P_{n-1}^m(x)\}.$$

Putting this value of $P_n^{m-1}(x)$ in (1), we get

$$P_n^{m+1}(x) - \frac{2m}{(2n+1)(1-x^2)^{1/2}}\{(n+m)P_{n-1}^m(x) + (n-m+1)P_{n+1}^m(x)\} + \frac{1}{(2n+1)(1-x^2)^{1/2}}\{n(n+1)$$

$$-m\,(m-1)\}P_{n+1}^m(x) - P_{n-1}^m(x)\} = 0$$

or $$(1-x^2)^{1/2} P_n^{m+1}(x) = \frac{1}{2n+1}[\{2m(n+m) + n(n-1) - m(m-1)]P_{n-1}^m(x)$$

$$+\{2m\,(n-m+1) - n(n+1) + m(m-1)P_{n+1}^m(x)]$$

or $$(1-x^2)^{1/2} P_n^{m+1} = \frac{1}{2n+1}[(n+m)\,(n+m+1)P_{n-1}^m(x) \; -(n-m)\,(n-m+1)P_{n+1}^m(x)] \quad \ldots (2)$$

Replacing m by $(m-1)$ in (2), we get

$$(1-x^2)^{1/2} P_n^m(n) = \frac{1}{2n+1}[(n+m-1)\,(n+m)P_{n-1}^{m-1}(n) \; -(n-m+1)\,(n-m+2)P_{n+1}^{m-1}(x)].$$

EXERCISE

1. Show that $\sum\limits_{n=0}^{\infty} P_{n+m}^m(x)t^n = \dfrac{(2m)!(1-x^2)^{m+1/2}}{2^m m!(1-2tx+t^2)^{m+1/2}}$.

2. Prove that (*a*) $P_n^m(x) = \dfrac{1}{2^n \cdot n!}(1-x^2)^{m/2} \dfrac{d^{n+m}}{dx^{n+m}}(x^2-1)^n$ (*b*) $P_n^{-m}(x) = (-1)^m \dfrac{(n-m)!}{(n+m)!} P_n^m(x)$.

3. Prove that $P_n^m(x) = \dfrac{(n+m)!(1-x^2)^{m/2}}{(n-m)!2^m m!}\, {}_2F_1\left(m-n,\, m+n+1;\, m+1;\, \dfrac{1-x}{2}\right)$.

4. Define associated Legendre's polynomials and prove their orthoganality condition.

5. Prove that $P_m^m(x) = \dfrac{(n+m)!}{(n-m)!} \times \dfrac{(1-x^2)^{m/2}}{2^m m!}\, {}_2F_1\left(m-n,\, m+n+1; m+1;\, \dfrac{1-x}{2}\right)$.

6. Prove that $P_n^m(-x) = (-1)^{m+n} P_n^m(x)$ **(Utkal 2003)**

10

Legendre Functions of the Second Kind — $Q_n(x)$

10.1. Some useful results: From Art. 9.2 of chapter 9, we have

$$Q_n(x) = \frac{n!}{1\cdot 3\cdot 5\cdot \ldots \cdot (2n+1)}\left[x^{-(n+1)} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-(n+3)} + \ldots\right]$$

Multiplying by $2\cdot 4\cdot 6\cdot \ldots\ldots \cdot 2n$ in the numerator and denominator, we get

$$Q_n(x) = \frac{n!\,(2\cdot 4\cdot 6 \ldots 2n)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot \ldots \cdot 2n\cdot (2n+1)}\left[x^{-(n+1)} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-(n+3)} + \ldots\right]$$

or
$$Q_n(x) = \frac{n!\cdot 2^n\, n!}{(2n+1)!}\left[x^{-(n+1)} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-(n+3)} + \ldots\right]$$

or
$$Q_n(x) = \frac{2^n\,(n!)^2}{(2n+1)!}\left[x^{-(n+1)} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-(n+3)} + \ldots\right], \qquad \ldots(1)$$

which is a solution of Legendre's equation in descending powers of x, all the powers of x being negative. Re–writing (1), we have

$$Q_n(x) = \frac{2^n\,(n!)}{(2n+1)!}\left[(n!)\,x^{-(n+1)} + \frac{(n+2)!}{2(2n+3)}x^{-(n+3)} + \frac{(n+4)!}{2\cdot 4(2n+3)\,(2n+5)}x^{-(n+5)} + \ldots\right]$$

$$= \frac{2^n\,(n!)}{(2n+1)!}\left[(n!)\,x^{-(n+1)} + \frac{(n+2)!}{2(2n+3)}x^{-(n+2+1)} + \frac{(n+4)!}{2\cdot 4(2n+3)\,(2n+5)}x^{-(n+4+1)} + \ldots\right]$$

$$= \frac{2^n\,(n!)}{(2n+1)!}\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2\cdot 4\cdot \ldots\cdot 2r\,(2n+3)\,(2n+5)\ldots(2n+2r+1)}$$

$\therefore$
$$Q_n(x) = \frac{2^n\,(n!)}{(2n+1)!}\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2^r\,(r!)(2n+3)\,(2n+5)\ldots(2n+2r+1)} \qquad \ldots(2)$$

Differentiating (2) w.r.t. 'x' we get

$$Q_n'(x) = -\frac{2^n.\,n!}{(2n+1)!}\sum_{r=0}^{\infty}\frac{(n+2r+1)!\,x^{-(n+2r+2)}}{2^r\,r!(2n+3)\,(2n+5)\ldots(2n+2r+1)}. \qquad \ldots(3)$$

Putting $n-1$ for n in (3), we get

$$Q'_{n-1}(x) = -\frac{2^{n-1}(n-1)!}{(2n-1)!}\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2^r\,r!(2n+1)(2n+3)\dots(2n+2r-1)}$$

$$= -\frac{2n\cdot 2^{n-1}(n-1)!}{(2n-1)!\cdot 2n}\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2^r\cdot r!(2n+1)(2n+3)\dots(2n+2r-1)}$$

$$= -\frac{2^n\cdot n!}{(2n)!}\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2^r\cdot r!(2n+1)(2n+3)\dots(2n+2r-1)}. \qquad \text{...(4)}$$

Again, putting $n+1$ for n in (3), we get

$$Q'_{n+1}(x) = -\frac{2^{n+1}\cdot(n+1)!}{(2n+3)!}\sum_{r=0}^{\infty}\frac{(n+2r+2)!\,x^{-(n+2r+3)}}{2^r\cdot r!(2n+5)\dots(2n+2r+1)(2n+2r+3)}$$

$$= -\frac{2^n\cdot n!\cdot(2n+2)}{(2n+3)(2n+2)(2n+1)(2n)!}\sum_{r=0}^{\infty}\frac{(n+2r+2)!\,x^{-(n+2r+3)}}{2^r\cdot r!(2n+5)\dots(2n+2r+3)}$$

$$= -\frac{2^n\cdot n!}{(2n)!}\sum_{r=0}^{\infty}\frac{(n+2r+2)!\,x^{-(n+2r+3)}}{2^r\cdot r!(2n+1)(2n+3)\dots(2n+2r+3)}. \qquad \text{...(5)}$$

10.2. Recurrence Relations (formulae) for $Q_n(x)$

I. $Q'_{n+1} - Q'_{n-1} = (2n+1)Q_n$. **[Bilaspur 1997, 98]**

Proof. We have, $Q'_{n-1} + (2n+1)Q_n$

$$= -\frac{2^n(n!)}{(2n)!}\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2^r(r!)(2n+1)(2n+3)\dots(2n+2r-1)} + (2n+1)\frac{2^n(n!)}{(2n+1)!}\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2^r(r!)(2n+3)(2n+5)\dots(2n+2r+1)}$$

[using results (4) and (2) Art. of 10.1.]

$$= \frac{2^n(n!)}{(2n)!}\left[\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2^r(r!)(2n+3)(2n+5)\dots(2n+2r+1)} - \sum_{r=0}^{\infty}\frac{(n+2r)\,x^{-(n+2r+1)}}{2^r(r!)(2n+1)(2n+3)\dots(2n+2r+1)}\right]$$

$$= \frac{2^n(n!)}{(2n)!}\left[\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2^r(r!)(2n+1)(2n+3)\dots(2n+2r+1)}\times\{(2n+1)-(2n+2r+1\}\right]$$

$$= \frac{2^n(n!)}{(2n)!}\left[\sum_{r=0}^{\infty}\frac{-2r(n+2r)!\,x^{-(n+2r+1)}}{2^r(r!)(2n+1)(2n+3)\dots(2n+2r+1)}\right] = -\frac{2^n(n!)}{(2n)!}\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2^{r-1}(r-1)!(2n+1)(2n+3)\dots(2n+2r+1)}$$

$$= -\frac{2^n(n!)}{(2n)!}\left[0+\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2^{r-1}(r-1)!(2n+1)(2n+3)\dots(2n+2r+1)}\right], \text{ as } \frac{1}{(-1)!}=0$$

$$= -\frac{2^n(n!)}{(2n)!}\sum_{s=0}^{\infty}\frac{(n-2s+2)!\,x^{-(n+2s+3)}}{2^s(s)!(2n+1)(2n+3)\dots(2n+2s+3)}, \text{ putting } r=s+1 \text{ so that } s=r-1$$

$= Q'_{n+1}$ by result (5) of Art. 10.1.

II. $nQ'_{n+1} + (n+1)Q'_{n-1} = (2n+1)xQ'_n$. **(Bilaspur 1998)**

Proof. We have, $(2n+1)xQ'_n - (n+1)Q'_{n-1}$

$$= -(2n+1)x \cdot \frac{2^n\,(n!)}{(2n+1)!}\sum_{r=0}^{\infty}\frac{(n+2r+1)!\,x^{-(n+2r+2)}}{2^r \cdot r!\,(2n+3)\,(2n+5)\ldots(2n+2r+1)}$$

$$-(n+1)\cdot(-1)\frac{2^n\,(n!)}{(2n)!}\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2^r\,(r!)\,(2n+1)\,(2n+3)\ldots(2n+2r-1)}, \text{putting the values of } Q'_n \text{ and } Q'_{n-1}$$

$$= -\frac{2^n\,(n!)}{(2n)!}\sum_{r=0}^{\infty}\frac{(n+2r+1)!\,x^{-(n+2r+1)}\,(2n+1)}{2^r\,(r!)\,(2n+1)\,(2n+3)\ldots(2n+2r+1)} + \frac{2^n\,(n!)}{(2n)!}\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}\,(n+1)\,(2n+2r+1)}{2^r\,(r!)\,(2n+1)\,(2n+3)\ldots(2n+2r+1)}$$

$$= -\frac{2^n\,(n!)}{(2n)!}\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2^r\,(r!)\,(2n+1)\,(2n+3)\ldots(2n+2r+1)} \times [(n+2r+1)\,(2n+1) - (n+1)\,(2n+2r+1)]$$

$$= -\frac{n\cdot 2^n\,(n!)}{(2n)!}\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2^{r-1}(r-1)!\,(2n+1)\,(2n+3)\ldots(2n+2r+1)}$$

$$= -\frac{n\cdot 2^n\,(n!)}{(2n)!}\left[\sum_{r=1}^{\infty}\left\{\frac{(n+2r)!\,x^{-(n+2r+1)}}{2^{r-1}(r-1)!\,(2n+1)\,(2n+3)\ldots(2n+2r+1)}\right\}+0\right]$$

$$= -\frac{n\cdot 2^n\,(n!)}{(2n)!}\sum_{s=0}^{\infty}\frac{(n+2s+2)!\,x^{-(n+2s+3)}}{2^s\,(s!)\,(2n+1)\,(2n+3)\ldots(2n+2s+3)}, \text{ putting } r = s+1$$

$= nQ'_{n+1}$, by result (5) of Art. 10.1.

III. $(2n+1)\,xQ_n = (n+1)\,Q_{n+1} + nQ_{n-1}$.

Or $\quad xQ_n = \dfrac{n+1}{2n+1}\,Q_{n+1} + \dfrac{n}{2n+1}Q_{n-1}$ $\quad$ *Or* $\quad (n+1)\,Q_{n+1} - (2n+1)\,xQ_n + nQ_{n-1} = 0.$

Proof. We have, $nQ_{n-1} - (2n+1)\,xQ_n$

$$= n\times\frac{2^{n-1}\,(n-1)!}{(2n-1)!}\sum_{r=0}^{\infty}\frac{(n+2r-1)!\,x^{-(n+2r)}}{2^r\,(r!)\,(2n+3)\,(2n+5)\ldots(2n+2r-1)}$$

$$-(2n+1)\times\frac{2^n\,(n!)}{(2n+1)!}\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r+1)}}{2^r\,(r!)\,(2n+3)\,(2n+5)\ldots(2n+2r+1)}, \text{ putting values of } Q_{n-1} \text{ and } Q_n$$

$$= \frac{2^{n-1}\,(n!)\cdot 2n}{(2n+1)!}\sum_{r=0}^{\infty}\frac{(n+2r-1)!\,x^{-(n+2r)}\cdot(2n+2r+1)}{2^r\,(r!)\,(2n+3)\,(2n+5)\ldots(2n+2r-1)\,(2n+2r+1)}$$

$$-\frac{(2n+1)\,2^n\,(n!)}{(2n+1)!}\sum_{r=0}^{\infty}\frac{(n+2r)!\,x^{-(n+2r)}}{2^r\,(r!)\,(2n+3)\,(2n+5)\ldots(2n+2r+1)}$$

$$= \frac{2^n(n!)}{(2n+1)!}\left[\sum_{r=0}^{\infty}\frac{(n+2r-1)!\,x^{-(n+2r)}}{2^r\,(r!)\,(2n+3)\,(2n+5)\ldots(2n+2r+1)}\times\{n(2n+2r+1)-(2n+1)\,(n+2r)\}\right]$$

$$= \frac{2^n (n!)}{(2n+1)!} \sum_{r=0}^{\infty} \frac{(n+2r-1)!\, x^{-(n+2r)}\, (-2r)\, (n+1)}{2^r (r!)\, (2n+3)\, (2n+5) \ldots (2n+2r+1)}$$

$$= -\frac{2^n (n+1)!}{(2n+1)!} \sum_{r=0}^{\infty} \frac{(n+2r-1)!\, x^{-(n+2r)}}{2^{r-1} (r-1)!\, (2n+3)\, (2n+5) \ldots (2n+2r+1)}$$

$$= -(n+1) \frac{2^n (n!)}{(2n+1)!} \left[\sum_{r=1}^{\infty} \left\{ \frac{(n+2r-1)!\, x^{-(n+2r)}}{2^{r-1} (r-1)!\, (2n+3)\, (2n+5) \ldots (2n+2r+1)} \right\} + 0 \right]$$

$$= -(n+1) \times \frac{2^n (n!)}{(2n+1)!} \left[\sum_{s=0}^{\infty} \frac{(n+2s+1)!\, x^{-(n+2s+2)}}{2^s (s!)\, (2n+3)\, (2n+5) ... (2n+2s+3)} \right], \text{ taking } r = s+1$$

$$= -(n+1) \times \frac{2^n (n!)\, (2n+2)}{(2n+2)!} \sum_{s=0}^{\infty} \frac{(n+2s+1)!\, x^{-(n+2s+2)}}{2^s (s!)\, (2n+3)\, (2n+5) ... (2n+2s+3)}$$

$$= -(n+1) \times \frac{2^n (n+1)!}{(2n+3)!} \sum_{s=0}^{\infty} \frac{(n+2s+1)!\, x^{-(n+2s+2)}}{2^s (s!)\, (2n+5) ... (2n+2s+3)}$$

$= -(n+1)Q_{n+1}$, be definition (2).

IV. $(2n + 1)(1 - x^2)Q_n' = n(n + 1)(Q_{n-1} - Q_{n+1})$.

Proof. Since Q_n is a solution of Legendre's equation, namely,

$$\frac{d}{dx}\left[(1-x^2)\frac{dy}{dx}\right] + n(n+1)\, y = 0 \qquad \text{so} \qquad \frac{d}{dx}[(1-x^2)Q'_n] = -n(n+1)Q_n. \qquad ...(1)$$

Integrating both sides of (1) between the limits ∞ to x, we get

$$\left[(1-x^2)Q'_n\right]_{\infty}^{x} = -n(n+1)\int_{\infty}^{x} Q_n\, dx$$

or $\qquad (1-x^2)\, Q'_n(x) = -n(n+1)\int_{\infty}^{x} Q_n\, dx$, as $\{Q_n'\}_{x=\infty} = 0$ and $\{x^2 Q_n'\}_{x=\infty} = 0$.

But by recurrence relation I, we get $\qquad Q'_{n+1} - Q'_{n-1} = (2n+1)Q_n. \qquad ...(3)$

Integrating both sides of (3) between the limits ∞ to x, we get

$$\left[Q_{n+1} - Q_{n-1}\right]_{\infty}^{x} = \int_{\infty}^{x} (2n+1)Q_n\, dx \quad \text{or} \quad Q_{n+1}(x) - Q_{n-1}(x) = \int_{\infty}^{x} (2n+1)Q_n\, dx. \quad ...(4)$$

$$[\because \{Q_{n+1}\}_{x=\infty} = 0 = \{Q_{n-1}\}_{x=\infty}]$$

Now, from (4) and (2), $\qquad (1-x^2)\, Q_n'(x) = -n(n+1)\, \dfrac{Q_{n+1}(x) - Q_{n-1}(x)}{2n+1}$

or $\qquad (2n+1)(1-x^2)\, Q_n'(x) = n(n+1)\, [Q_{n-1}(x) - Q_{n+1}(x)].$

V. $xQ_n' - Q_{n-1}' = nQ_n$.

Proof. From recurrence relation III, $\quad (n+1)\, Q_{n+1} - (2n+1)x\, Q_n + n\, Q_{n-1} = 0. \qquad ...(1)$

Differentiating (1) w.r.t. 'x', we have

$$(n+1)Q'_{n+1} - (2n+1)\, \{x\, Q_n' + 1.Q_n\} + nQ'_{n-1} = 0. \qquad ...(2)$$

Now, by recurrence relation I, $\qquad Q'_{n+1} = Q'_{n-1} + (2n+1)Q_n. \qquad ...(3)$

Putting the value of Q'_{n+1} given by (3) in (2), we get

$$(n+1)\{Q'_{n-1} + (2n+1)Q_n\} - (2n+1)(x\,Q_n' + Q_n) + nQ'_{n-1} = 0$$

or $$(2n+1)\,Q'_{n-1} - (2n+1)\,xQ_n' + (2n+1)\,Q_n\,(n+1-1) = 0$$

or $$Q'_{n-1} - xQ_n' + nQ_n = 0 \qquad \textit{i.e.} \qquad xQ_n' - Q'_{n-1} = nQ_n.$$

VI. $\mathbf{Q'_n - xQ'_{n-1} = nQ_{n-1}}$.

Proof. From recurrence relation I, $$Q'_{n+1} - Q'_{n-1} = (2n+1)\,Q_n. \qquad ...(1)$$

Multiplying both sides of (1) by x, $$x\,Q'_{n+1} - x\,Q'_{n-1} = (2n+1)x\,Q_n. \qquad ...(2)$$

Now, from recurrence relation III, $$(n+1)\,Q_{n+1} - (2n+1)\,x\,Q_n + n\,Q_{n-1} = 0. \qquad ...(3)$$

Putting the value of $(2n+1)xQ_n$ from (2) in (3), $$(n+1)Q_{n+1} - xQ'_{n+1} + xQ'_{n-1} + nQ_{n-1} = 0. \qquad ...(4)$$

From recurrence relation V, $$nQ_n = xQ_n' - Q'_{n-1}. \qquad ...(5)$$

Replacing n by $n+1$ in (5), we get $$(n+1)\,Q_{n+1} = xQ'_{n+1} - Q_n'. \qquad ...(6)$$

Putting the value of $(n+1)Q_{n+1}$ from (6) in (4), we get

$$xQ'_{n+1} - Q'_n - xQ'_{n+1} + xQ'_{n-1} + nQ_{n-1} = 0 \qquad \text{or} \qquad Q_n' - xQ'_{n-1} = nQ_{n-1}.$$

VII. $\mathbf{(x^2-1)Q'_n = nxQ_n - nQ_{n-1}}$.

Proof. Recurrence relation V, we have $$xQ_n' - Q'_{n-1} = nQ_n. \qquad ...(1)$$

Multiplying both sides of (1) by x, we get $$x^2Q_n' - xQ'_{n-1} = nxQ_n. \qquad ...(2)$$

Again, from recurrence relation VI, we get $$Q_n' - xQ'_{n-1} = nQ_{n-1}. \qquad ...(3)$$

Subtracting (3) from (2), we have

$$x^2Q'_n - Q'_n = nxQ_n - nQ_{n-1} \qquad \text{or} \qquad (x^2-1)Q'_n = nxQ_n - nQ_{n-1}.$$

VIII. $\mathbf{(x^2-1)Q_n' = (n+1)\,Q_{n+1} - (n+1)xQ_n}$.

Proof. Recurrence relations V and VI are $$xQ'_n - Q'_{n-1} = nQ_n \qquad ...(1)$$

and $$Q'_n - xQ'_{n-1} = nQ_{n-1}. \qquad ...(2)$$

Replacing n by $n+1$ in (1) and (2), $$xQ'_{n+1} - Q'_n = (n+1)Q_{n+1} \qquad ...(3)$$

and $$Q_n'{}_{+1} - xQ_n' = (n+1)Q_n. \qquad ...(4)$$

Multiplying both sides of (4) by x, $$xQ'_{n+1} - x^2Q'_n = (n+1)x\,Q_n. \qquad ...(5)$$

Subtracting (5) from (3), we have $$-Q'_n + x^2Q'_n = (n+1)Q_{n+1} - (n+1)xQ_n$$

or $$(x^2-1)Q_n' = (n+1)Q_{n+1} - (n+1)xQ_n.$$

10.3. Theorem. *The associated Legendre function of the second kind defined by*

$$Q_n^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}\,Q_n(x)$$

satisfy Legendre's associated equation $$(1-x^2)\frac{d^2y}{dx^2} - 2x\,\frac{dy}{dx} + \left\{n(n+1) - \frac{m^2}{1-x^2}\right\}y = 0.$$

Proof. Proceed as in chapter 9.

10.4. Theorem. *Assuming P_n as a solution of Legendre's equation, show that the complete solution of this equation is given by $aP_n + bQ_n$, where $Q_n = cP_n \int \frac{dx}{(1-x^2)P_n^2}$, c being a constant.*

Proof. The Legendre's equation is given by $$(1-x^2)\frac{d^2y}{dx^2} - 2x\,\frac{dy}{dx} + n(n+1)\,y = 0. \qquad ...(1)$$

Let $y = uP_n$ be the complete solution of (1), where u is a function of x.

Now, $$\frac{dy}{dx} = u\frac{dP_n}{dx} + P_n\frac{du}{dx} \qquad \text{and} \qquad \frac{d^2y}{dx^2} = u\frac{d^2P_n}{dx^2} + 2\frac{dP_n}{dx}\frac{du}{dx} + P_n\frac{d^2u}{dx^2}.$$

Putting these values of (1), we have

$$(1-x^2)\left(u\frac{d^2P_n}{dx^2}+2\frac{dP_n}{dx}\frac{du}{dx}+P_n\frac{d^2u}{dx^2}\right)-2x\left(u\frac{dP_n}{dx}+P_n\frac{du}{dx}\right)+n(n+1)uP_n=0$$

or $$(1-x^2)\left(P_n\frac{d^2u}{dx^2}+2\frac{dP_n}{dx}\frac{du}{dx}\right)-(1-x^2)\,u\frac{d^2P_n}{dx^2}-2xu\frac{dP_n}{dx}-2xP_n\frac{du}{dx}+n(n+1)uP_n=0$$

or $$(1-x^2)\left(P_n\frac{d^2u}{dx^2}+2\frac{du}{dx}\frac{dP_n}{dx}\right)+u\left\{(1-x^2)\frac{d^2P_n}{dx^2}-2x\frac{dP_n}{dx}+n(n+1)P_n\right\}-2xP_n\frac{du}{dx}=0$$

or $$(1-x^2)\left(P_n\frac{d^2u}{dx^2}+2\frac{du}{dx}\frac{dP_n}{dx}\right)-2xP_n\frac{du}{dx}=0$$

[$\because$ P_n is a solution of Legendre's equation $\Rightarrow (1-x^2)\dfrac{d^2P_n}{dx^2}-2x\dfrac{dP_n}{dx}+n(n+1)P_n=0.$]

Now dividing throughout by $(1-x^2)P_n\,(du/dx)$, we have

$$\frac{d^2u/dx^2}{du/dx}+2\frac{dP_n/dx}{P_n}-\frac{2x}{1-x^2}=0. \qquad ...(2)$$

Integrating (2), $\log(du/dx)+2\log P_n+\log(1-x^2)=\log k$

or $\log\{(du/dx)\,.\,P_n^{\,2}(1-x^2)\}=\log k$, k being an arbitrary constant.

or $\dfrac{du}{dx}\,.\,P_n^{\,2}(1-x^2)=k$ or $\dfrac{du}{dx}=\dfrac{k}{(1-x^2)P_n^2}$.

Integrating, $u=k\displaystyle\int\frac{1}{(1-x^2)P_n^2}dx+a$, where a and k are arbitrary constants of integration.

Hence the complete solution of Legendre's equation is

$$y=uP_n=\left[k\int\frac{dx}{(1-x^2)P_n^2}+a\right]P_n=aP_n+kP_n\int\frac{dx}{(1-x^2)P_n^2}=aP_n+\frac{k}{c}\times cP_n\int\frac{dx}{(1-x^2)P_n^2}$$

or $$y=aP_n+bQ_n,\quad\text{where}\quad b=k/c.$$

10.5. Christoffel's second summation formula

From recurrence relations of $P_n(x)$ and $Q_n(x)$, we have

$$(2n+1)\,xP_n(x)=(n+1)\,P_{n+1}(x)+nP_{n-1}(x), \qquad ...(1)$$

and $$(2n+1)\,yQ_n(y)=(n+1)\,Q_{n+1}(y)+nQ_{n-1}(y), \qquad ...(2)$$

Multiplying both sides of (1) by $Q_n(y)$ and (2) by $P_n(x)$ and then subtracting, we have

$$(2n+1)\,(x-y)\,P_n(x)\,Q_n(y)=(n+1)\,\{P_{n+1}(x)\,Q_n(y)-P_n(x)\,Q_{n+1}(y)\}$$
$$+\,n\{P_{n-1}(x)\,Q_n(y)-Q_{n-1}(y)\,P_n(x)\}$$

or $$(2n+1)\,P_n(x)\,Q_n(y)\,(x-y)+n\{P_n(x)\,Q_{n-1}(y)-P_{n-1}(x)\,Q_n(y)\}$$
$$=(n+1)\,\{P_{n+1}(x)\,Q_n(y)-P_n(x)Q_{n+1}(y)\}. \qquad ...(3)$$

Putting $n=1,2,3,\ldots n$ in succession in (3), we have

$$3P_1(x)\,Q_1(y)\,(x-y)+\{P_1(x)\,Q_0(y)-Q_1(y)\,P_0(x)\}=2\{P_2(x)\,Q_1(y)-P_1(x)\,Q_2(y)\}, \qquad ...(A_1)$$
$$5P_2(x)\,Q_2(y)\,(x-y)+2\{P_2(x)\,Q_1(y)-P_1(x)\,Q_2(y)\}=3\{P_3(x)\,Q_2(y)-P_2(x)\,Q_3(y)\}, \qquad ...(A_2)$$
$$7P_3(x)\,Q_3(y)\,(x-y)+3\{P_3(x)\,Q_2(y)-P_2(x)\,Q_3(y)\}=4\{P_4(x)\,Q_3(y)-P_3(x)\,Q_4(y)\}, \qquad ...(A_3)$$

... ...

$$(2n+1)P_n(x)Q_n(y)\,(x-y)+n\{P_n(x)Q_{n-1}(y)-P_{n-1}(x)Q_n(y)\}=(n+1)\{P_{n+1}(x)Q_n(y)-P_n(x)Q_{n+1}(y)\}. \quad ...(A_n)$$

Adding (A_1), (A_2), (A_3), (A_4) etc. upto (A_n), we have

$$(x-y)\sum_{r=1}^{n}(2r+1)\,P_r(x)\,Q_r(y)+\{P_1(x)\,Q_0(y)-Q_1(y)\,P_0(x)\}=(n+1)\,\{P_{n+1}(x)\,Q_n(y)-P_n(x)\,Q_{n+1}(y)\}$$

or $$(y-x)\sum_{r=1}^{n}(2r+1)\,P_r(x)\,Q_r(y)+\{Q_1(y)\,P_0(x)-P_1(x)\,Q_0(y)\}=-(n+1)\,\{P_{n+1}(x)\,Q_n(y)-P_n(x)\,Q_{n+1}(y)\}$$

or $$(y-x)\sum_{r=1}^{n}(2r+1)\,P_r(x)\,Q_r(y)+[\{yQ_0(y)-1\}\times P_0(x)-xQ_0(y)\,P_0(x)]$$
$$=-(n+1)\,\{P_{n+1}(x)\,Q_n(y)-Q_{n+1}(y)\,P_n(x)\} \qquad [\because\ Q_1(y)=yQ_0(y)-1,\ P_1(x)=x,\ P_0(x)=1]$$

or $$(y-x)\sum_{r=1}^{n}(2r+1)\,P_r(x)\,Q_r(y)+(y-x)\,P_0(x)\,Q_0(y)-P_0(x)=-(n+1)\,\{P_{n+1}(x)\,Q_n(y)-Q_{n+1}(y)\,P_n(x)\}$$

or $$\mathbf{(y-x)\sum_{r=1}^{n}(2r+1)\,P_r(x)\,Q_r(y)=1-(n+1)\,[P_{n+1}(x)\,Q_n(y)-Q_{n+1}(y)\,P_n(x)],}$$

which is called *Christoffel's second summation formula.*

10.6. A relation connecting $P_n(x)$ and $Q_n(x)$: $\dfrac{1}{y-x}=\sum_{m=0}^{\infty}(2m+1)\,P_m(x)\,Q_m(y)$.

Proof. Let $$f(x)=1/(y-x).$$

Now, $$f(x)=\frac{1}{y(1-x/y)}=\frac{1}{y}\left(1-\frac{x}{y}\right)^{-1}=y^{-1}\left(1+\frac{x}{y}+\frac{x^2}{y^2}+\ldots+\frac{x^m}{y^m}+\ldots\right)$$
$$=y^{-1}+y^{-2}x+y^{-3}x^2+\ldots+y^{-m-1}x^m+y^{-m-2}x^{m+1}+y^{-m-3}x^{m+2}+y^{-m-4}x^{m+3}+\ldots \quad ...(1)$$

Let $$f(x)=A_0+A_1x+A_2x^2+\ldots, \quad ...(2)$$

where A's are constant. Further suppose that $f(x)$ is also expressed as

$$f(x)=\sum_{m=0}^{\infty}B_mP_m(x), \quad ...(3)$$

then, we know that

$$B_m=\frac{1\cdot2\cdot3...m}{1\cdot3\cdot5...(2m-1)}\left[A_m+\frac{(m+1)\,(m+2)}{2(2m+3)}A_{m+2}+\frac{(m+1)\,(m+2)\,(m+3)\,(m+4)}{2\cdot4\cdot(2m+3)\,(2m+5)}A_{m+4}+\ldots\right].$$

Comparing (1) and (2), $A_0=y^{-1}$, $A_1=y^{-2}$, $\ldots A_m=y^{-(m+1)}$, $A_{m+1}=y^{-(m+2)}$,...

$$\therefore B_m=\frac{m!}{1\cdot3\cdot5...(2m-1)}\left[y^{-(m+1)}+\frac{(m+1)\,(m+2)}{2(2m+3)}y^{-(m+3)}+\ldots\right]=(2m+1)\ Q_n(y),$$ by definition

$$\therefore \qquad f(x)=\frac{1}{y-x}=\sum_{m=0}^{\infty}B_m\,P_m(x)=\sum_{m=0}^{\infty}(2m+1)P_m(x)Q_m(y).$$

10.7. Neumann's Integral for $Q_m(y)$.

To show that $$Q_m(y)=\frac{1}{2}\int_{-1}^{1}\frac{P_m(x)}{y-x}\,dx, \quad (y>1).$$ **(Bilaspur 1996, 98)**

Proof. From Art. 10.6 we have $$\frac{1}{y-x}=\sum_{m=0}^{\infty}(2m+1)P_m(x)Q_m(y).$$

$$\therefore \frac{1}{2}\int_{-1}^{1} P_m(x)\times\frac{1}{y-x}\,dx = \int_{-1}^{1} P_m(x)\times\left\{\sum_{m=0}^{\infty}(2m+1)P_m(x)\,Q_m(y)\right\}dx = Q_m(y)\int_{-1}^{1}\{P_m(x)\}^2\,.(2m+1)dx$$

$$\left(\because \int_{-1}^{1} P_m(x)\,P_n(x)dx = 0,\text{ if } m\neq n\right)$$

$$= (2m+1)\,Q_m(y)\times\frac{2}{2m+1} = 2Q_m(y). \qquad \left[\because \int_{-1}^{1}(P_m)^2\,dx = \frac{2}{2m+1}\right]$$

Hence $$Q_m(y) = \frac{1}{2}\int_{-1}^{1}\frac{P_m(x)}{y-x}\,dx.$$

10.8 SOLVED EXAMPLES ON CHAPTER 8

Ex. 1. *Prove that* (*i*) $(x^2-1)(Q_nP'_n - P_nQ'_n) = c$. **[Agra 2006]** (*ii*) $\dfrac{Q_n}{P_n} = \displaystyle\int_x^{\infty}\frac{dx}{(x^2-1)P_n^2}$.

(*iii*)*From* (*ii*) *deduce that* (*a*) $Q_0(x) = \dfrac{1}{2}\log\dfrac{x+1}{x-1}$. (*b*) $Q_1(x) = \dfrac{x}{2}\log\dfrac{x+1}{x-1} - 1$.

Sol. (*i*) Legendre's equation is $(1-x^2)y'' - 2xy' + n(n+1)y = 0.$ ...(1)

Since P_n and Q_n are both solutions of (1), $(1-x^2)P''_n - 2xP'_n + n(n+1)P_n = 0$ ...(2)

and $(1-x^2)Q''_n - 2xQ'_n + n(n+1)Q_n = 0.$ ...(3)

Multiplying (2) by Q_n, (3) by P_n and then subtracting, we get

$$(1-x^2)(P''_n\,Q_n - Q''_n\,P_n) - 2x(P'_n\,Q_n - Q'_n\,P_n) = 0$$

or $$(1-x^2)\frac{d}{dx}(P'_n\,Q_n - Q'_n\,P_n) - 2x(P'_n\,Q_n - Q'_n\,P_n) = 0$$

or $$\frac{d}{dx}\{(1-x^2)(P'_n\,Q_n - Q'_n\,P_n)\} = 0 \qquad ...(4)$$

Integrating w.r.t. 'x', (4) gives $(1-x^2)(Q_n\,P_n' - P_n\,Q_n') = -c$

or $(x^2-1)(Q_n\,P_n' - P_n\,Q_n') = c$, c being an arbitrary constant. ...(5)

Part (*ii*) From part (*i*) above, we have

$$Q_n\,P_n' - P_n\,Q_n' = \frac{c}{x^2-1} = \frac{c}{x^2}\left(1-\frac{1}{x^2}\right)^{-1} \quad\text{or}\quad Q_n\,P_n' - P_n\,Q_n' = \frac{c}{x^2}\left(1+\frac{1}{x^2}+\frac{1}{x^4}+\dots\right). \qquad ...(6)$$

From Art. 9.2, $$Q_n = \frac{n!}{1\cdot3\cdot5...(2n+1)}\left[x^{-(n+1)} + \frac{(n+1)(n+2)}{2\cdot(2n+3)}x^{-(n+3)} + ...\right] \qquad ...(7)$$

and $$P_n = \frac{1\cdot3\cdot5...(2n-1)}{n!}\left[x^n - \frac{n(n-1)}{(2n-1)\cdot2}x^{n-2} + ...\right]. \qquad ...(8)$$

Using (7) and (8), L.H.S. of (6) $$= \frac{n!}{1\cdot3\cdot5...(2n+1)}\left\{x^{-(n+1)} + \frac{(n+1)(n+2)}{2\cdot(2n+3)}x^{-(n+3)} + ...\right\}$$

$$\times\left[\frac{1\cdot3\cdot5...(2n-1)}{n!}\left\{nx^{n-1} - \frac{n(n-1)(n-2)}{(2n-1)\cdot2}x^{n-3} + ...\right\}\right]$$

$$-\left[\frac{1\cdot 3\cdot 5 ...(2n-1)}{n!}\left\{x^n-\frac{n(n-1)}{(2n-1)\cdot 2}x^{n-2}+...\right\}\right]\times\left[\frac{n!}{1\cdot 3\cdot 5...(2n+1)}\left\{-(n+1)x^{-(n+2)}\right.\right.$$

$$\left.\left.-\frac{(n+1)(n+2)(n+3)x^{-(n+4)}}{2\cdot(2n+3)}+...\right\}\right]$$

$\therefore$ the coefficient of $1/x^2$ in L.H.S. of (6)

$$=\frac{n!}{1\cdot 3\cdot 5...(2n+1)}\times\frac{1\cdot 3\cdot 5...(2n-1)}{n!}n-\frac{1\cdot 3\cdot 5...(2n-1)}{n!}\times\frac{n!}{1\cdot 3\cdot 5...(2n+1)}(-n-1)$$

$$=\frac{n}{2n+1}+\frac{n+1}{2n+1}=\frac{2n+1}{2n+1}=1.$$

Also the coefficient of $1/x^2$ in R.H.S. of (6) = 1

Hence, by equating the coefficients of $1/x^2$ on both sides of (6), we have $c = 1$. With this value of c, (5) becomes

$$(x^2-1)(P_n' Q_n - P_n Q_n') = 1 \qquad \text{or} \qquad -(x^2-1)(Q_n' P_n - P_n' Q_n) = 1 \;...\;(*)$$

or $$\frac{Q_n' P_n - P_n' Q_n}{P_n^2} = -\frac{1}{(x^2-1)P_n^2} \qquad \text{or} \qquad \frac{d}{dx}\left(\frac{Q_n}{P_n}\right) = -\frac{1}{(x^2-1)P_n^2}.$$

Integrating both sides w.r.t. x from ∞ to x, we get

$$\left[\frac{Q_n}{P_n}\right]_\infty^x = -\int_\infty^x \frac{dx}{(x^2-1)P_n^2} = \int_x^\infty \frac{dx}{(x^2-1)P_n^2} \qquad \text{or} \qquad \frac{Q_n(x)}{P_n(x)} - \lim_{x\to\infty}\frac{Q_n(x)}{P_n(x)} = \int_x^\infty \frac{dx}{(x^2-1)P_n^2} \qquad ...(9)$$

Now, $$\lim_{x\to\infty}\frac{Q_n(x)}{P_n(x)} = \lim_{x\to\infty}\frac{\frac{d^n}{dx^n}Q_n(x)}{\frac{d^n}{dx^n}P_n(x)}, \text{ by L'Hospital's rule}$$

$$=\lim_{x\to\infty}\frac{\frac{n!}{1\cdot 3\cdot 5\,(2n+1)}\{(-1)^n(n+1)(n+2)...2n\,x^{-(2n+1)}+...\}}{\frac{1\cdot 3\cdot 5...(2n-1)}{n!}n!}, \text{ using (7) and (8)}$$

$= 0$, on taking limit as $x\to\infty$.

$\therefore$ (9) reduces to $$\frac{Q_n(x)}{P_n(x)} = \int_x^\infty \frac{dx}{(x^2-1)P_n^2}. \qquad ...(10)$$

Part (iii). Deductions from part (ii).

(a)Replacing n by 0 in (10), we get $$\frac{Q_0(x)}{P_0(x)} = \int_x^\infty \frac{dx}{(x^2-1)P_0^2(x)}.$$

$$\therefore \qquad Q_0(x) = \int_x^\infty \frac{dx}{(x^2-1)}, \text{ as } P_0(x) = 1$$

$$=\frac{1}{2}\left[\log\frac{x-1}{x+1}\right]_x^\infty = -\frac{1}{2}\log\frac{x-1}{x+1} = \frac{1}{2}\log\frac{x+1}{x-1}, \text{ as } \lim_{x\to\infty}\log\left(\frac{x-1}{x+1}\right) = \lim_{x\to\infty}\left(\frac{1-1/x}{1+1/x}\right) = 0$$

(*b*) Replacing n by 1 in (10), we get $$\frac{Q_1(x)}{P_1(x)} = \int_x^\infty \frac{dx}{(x^2-1)P_1^2(x)}.$$

$$\therefore \quad Q_1(x) = x\int_x^\infty \frac{dx}{(x^2-1)x^2}, \text{ as } P_1(x) = x$$

$$= x\int_x^\infty \left[\frac{1}{x^2-1} - \frac{1}{x^2}\right] dx = x\left[\frac{1}{2}\log\frac{x-1}{x+1} + \frac{1}{x}\right]_x^\infty = -x\,.\left[\frac{1}{2}\log\frac{x-1}{x+1} + \frac{1}{x}\right], \text{ as before}$$

$$= -\frac{x}{2}\log\frac{x-1}{x+1} - 1 = \frac{x}{2}\log\frac{x+1}{x-1} - 1$$

Ex. 2. *Prove that*

(*i*) $n(Q_nP_{n-1} - P_{n-1}Q_n) = (n-1)\,(Q_{n-1}P_{n-2} - Q_{n-2}P_{n-1})$ *and deduce that*

(*ii*) $n(Q_nP_{n-1} - Q_{n-1}P_n) = -1$ *or* $P_nQ_{n-1} - Q_nP_{n-1} = 1/n$. **(Bilaspur 1997)**

(*iii*) $P_nQ_{n-2} - Q_nP_{n-2} = \{(2n-1)x/n(n-1)\}$.

Sol. (*i*) From recurrence relations of P_n and Q_n, we have

$$(2n+1)xP_n = (n+1)P_{n+1} + nP_{n-1} \qquad ...(1)$$

and $$(2n+1)xQ_n = (n+1)Q_{n+1} + nQ_{n-1}. \qquad ...(2)$$

Replacing n by $n-1$ in (1) and (2), we get

$$(2n-1)xP_{n-1} = nP_n + (n-1)P_{n-2} \qquad ...(3)$$

and $$(2n-1)xQ_{n-1} = nQ_n + (n-1)Q_{n-2}. \qquad ...(4)$$

Multiplying (3) by Q_{n-1} and (4) by P_{n-1} and then subtracting, we have

$$0 = n(P_n\,Q_{n-1} - P_{n-1}\,Q_n) + (n-1)\,(P_{n-2}\,Q_{n-1} - P_{n-1}\,Q_{n-2})$$

or $$n(Q_n\,P_{n-1} - Q_{n-1}\,P_n) = (n-1)\,(Q_{n-1}\,P_{n-2} - Q_{n-2}\,P_{n-1}). \qquad ...(5)$$

Part (ii) Deduction. Let $$U_n = n(Q_n\,P_{n-1} - Q_{n-1}\,P_n). \qquad ...(6)$$

Then (5) may be written as $U_n = U_{n-1}$, which gives $U_{n-1} = U_{n-2} = U_{n-3} = ... = U_3 = U_2 = U_1$.

Thus, we have $U_n = U_1$. or $n(Q_n\,P_{n-1} - Q_{n-1}\,P_n) = Q_1P_0 - Q_0P_1$, by (5)

or $n(Q_n\,P_{n-1} - Q_{n-1}\,P_n) = Q_1 - xQ_0$, as $P_0 = 1$ and $P_1 = x$ ...(7)

But $$Q_1 - xQ_0 = \frac{x}{2}\log\frac{x+1}{x-1} - 1 - x\times\frac{1}{2}\log\frac{x+1}{x-1} = -1.$$ [Do as in part (*iii*) of Ex. 1]

With this value of $Q_1 - xQ_0$, (7) gives the required result.

Part (iii). Multiplying (3) by Q_{n-2} and (4) by P_{n-2} and then subtracting, we get

$$(2n-1)x\,(P_{n-1}\,Q_{n-2} - Q_{n-2}\,P_{n-1}) = n(P_n\,Q_{n-2} - Q_n\,P_{n-2})$$

or $$P_n\,Q_{n-2} - Q_n\,P_{n-2} = \frac{(2n-1)x}{n}\,(P_{n-1}\,Q_{n-2} - Q_{n-1}\,P_{n-2}). \qquad ...(9)$$

Replacing n by $n-1$ in (1), $P_{n-1}\,Q_{n-2} - Q_{n-1}\,P_{n-2} = 1/(n-1)$. ...(10)

Using (10), (9) reduces to $P_n\,Q_{n-2} - Q_n\,P_{n-2} = \{(2n-1)x/n(n-1)\}$.

Ex. 3. *Prove that* : $Q_2(x) = \dfrac{1}{2}\,P_2(x)\log\dfrac{x+1}{x-1}x - \dfrac{3}{2}x$.

Sol. From recurrence relation III for $Q_n(x)$, $(n+1)Q_{n-1} = (2n+1)xQ_n - nQ_{n-1}$. ...(1)

Replacing n by 1 in (1), we get

$$2Q_2 = 3xQ_1 - Q_0 = 3x\left[\frac{x}{2}\log\frac{x+1}{x-1} - 1\right] - \frac{1}{2}\log\frac{x+1}{x-1} \qquad \text{[Do as in part (iii) of Ex. 1]}$$

$$= \frac{3x^2-1}{2}\log\frac{x+1}{x-1} - 3x = P_2(x)\log\frac{x+1}{x-1} - 3x \qquad \left[\because P_2(x) = \frac{3x^2-1}{2}\right]$$

$$\therefore \qquad Q_2(x) = \frac{1}{2}P_2(x)\log\frac{x+1}{x-1} - \frac{3}{2}x.$$

Ex. 4. *Prove that* $\dfrac{d^{n+1}}{dx^{n+1}}[Q_n(x)] = -\dfrac{(-2)^n\, n!}{(x^2-1)^{n+1}}$.

Sol. We know that

$$Q_n(x) = \frac{n!}{1\cdot3\cdot5\ldots(2n+1)}\left[x^{-(n+1)} + \frac{(n+1)(n+2)}{2(2n+3)}x^{-(n+3)} + \frac{(n+1)(n+2)(n+3)(n+4)}{2\cdot4\cdot(2n+3)(2n+5)}x^{-(n+5)} + \ldots\right] \quad \ldots(1)$$

Differentiating both sides of (1) $(n+1)$ times w.r.t. x, we get

$$\frac{d^{n+1}}{dx^{n+1}}[Q_n(x)] = \frac{n!}{1\cdot3\cdot5\ldots(2n+1)}\left[\frac{d^{n+1}}{dx^{n+1}}(x^{-n-1}) + \frac{(n+1)(n+2)}{2(2n+3)}\frac{d^{n+1}}{dx^{n+1}}(x^{-n-3})\right.$$

$$\left. + \frac{(n+1)(n+2)(n+3)(n+4)}{2\cdot4\cdot(2n+3)(2n+5)}\frac{d^{n+1}}{dx^{n+1}}(x^{-n-5}) + \ldots\right]$$

$$= \frac{n!}{1\cdot3\cdot5\ldots(2n+1)}\ [(-n-1))(-n-2)\ldots(-n-1-n)x^{-n-1-n-1} + \frac{(n+1)(n+2)}{2(2n+3)}\ .\ (-n-3)(-n-4)\ldots$$

$$(-n-3-n)x^{-n-3-n-1} + \frac{(n+1)(n+2)(n+3)(n+4)}{2\cdot4\cdot(2n+3)(2n+5}\ (-n-5)(-n-6)\ldots(-n-5-n)x^{-n-5-n-1} + \ldots]$$

$$\left[\because \frac{d^n x^m}{dx^n} = m(m-1)(m-2)\ldots(m-n+1)x^{m-n}\right]$$

$$= \frac{n!}{1\cdot3\cdot5\ldots(2n+1)}\left[(-1)^{n+1}(n+1)(n+2)\ldots(2n+1)x^{-2n-2}\right.$$

$$\left. + (-1)^{n+1}\frac{(n+1)(n+2)(n+3)\ldots(2n+3)}{2\cdot(2n+3)}x^{-2n-4} + (-1)^{n+1}\frac{(n+1)(n+2)\ldots(2n+5)}{2\cdot4\cdot(2n+3)(2n+5)}x^{-2n-6} + \ldots\right]$$

$$= \frac{(-1)^{n+1}\, n!\,(n+1)(n+2)\ldots(2n+1)x^{-2n-2}}{1\cdot3\cdot5\ldots(2n+1)}\left[1 + \frac{(2n+2)(2n+3)}{2\cdot(2n+3)}x^{-2} + \frac{(2n+2)(2n+3)(2n+4)(2n+5)}{2\cdot4\cdot(2n+3)(2n+5)}x^{-6} + \ldots\right]$$

$$= \frac{(-1)^{n+1}(2n+1)!\,x^{-2n-2}}{1\cdot3\cdot5\ldots(2n+1)}\left[1 + (n+1)x^{-2} + \frac{(n+1)(n+2)}{2!}x^{-4} + \ldots\right]$$

$$= (-1)^{n+1}\ 2\cdot4\cdot6\ldots(2n)\,x^{-2n-2}\left(1 - \frac{1}{x^2}\right)^{-n-1} = (-1)^{n+1}2^n\, n!\, x^{-2n-2}\left(\frac{x^2-1}{x^2}\right)^{-n-1} = -\frac{(-2)^n\, n!}{(x^2-1)^{n+1}}.$$

EXERCISE

1. Prove that $Q_n(x) = 2^n\, n! \int_x^{\infty} dx \int_x^{\infty} dx \ldots \int_x^{\infty} (x^2 - 1)^{-n-1} dx$. **(Bilaspur 1994, 96)**

2. Prove that $Q_n(x) = \int_0^{\infty} \dfrac{dt}{\{x + \Gamma(x^2 - 1) \cosh t\}^{n+1}}$.

3. By using the definition of $Q_n(x)$, find value of $Q_0(x)$ and $Q_1(x)$.

4. Show that the Wronskian $W(Q_n, P_x)$ of $Q_n(x)$ and $P_n(x)$ is $1/(x^2 - 1)$, *i.e.*, show that $W(Q_n, P_n) = 1/(x^2 - 1)$. **(Ravishankar 2004)**

[**Hint.** Refer equation (*) of part (*ii*) of Ex. 1 of Art. 10.8. Proceeding as indicated, get

$$(x^2 - 1)\,(Q_n P_n' - Q_n' P_n) = 1 \quad \text{or} \quad Q_n P_n' - Q_n' P_n = 1/(x^2 - 1) \quad \text{or} \quad W(Q_n, P_n) = 1/(x^2 - 1)$$

5. $(x^2 - 1)\,(Q_n P_n' - P_n\, Qn'') =$ (*a*) n^2 (*b*) 0 (*c*) k (*d*) None of these **[Agra 2006]**

[Hint: Ans (c) Refer Ex1, Art 10.8.]

11

Bessel Functions

11.1. Bessel's equation and its solution. **[Garhwal 2004; Kanpur 2009]**

The differential equation of the form $$x^2y'' + xy' + (x^2 - n^2)y = 0 \quad ...(1)$$

or $$y'' + (1/x) \times y' + (1 - n^2/x^2)y = 0 \quad ...(1)'$$

is called *Bessel's equation of order n*, *n* being a non–negative constant. We now solve (1) in series by using the well known method of Frobenius.

Let the series solution (1) be $$y = \sum_{m=0}^{\infty} c_m\, x^{k+m},\ c_0 \neq 0. \quad ...(2)$$

$$\therefore\ y' = \sum_{m=0}^{\infty} c_m\,(k+m)\,x^{k+m-1} \qquad \text{and} \qquad y'' = \sum_{m=0}^{\infty} c_m\,(k+m)(k+m-1)\,x^{k+m-2}.$$

Substitution for y, y', y'' in (1) now gives

$$x^2 \sum_{m=0}^{\infty} c_m\,(k+m)(k+m-1)\,x^{k+m-2} + x\sum_{m=0}^{\infty} c_m\,(k+m)\,x^{k+m-1} + (x^2 - n^2)\sum_{m=0}^{\infty} c_m\, x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} c_m\,\{(k+m)(k+m-1) + (k+m) - n^2\}\,x^{k+m} + \sum_{m=0}^{\infty} c_m\, x^{k+m+2} = 0.$$

But the bracketed expression in the above identity

$$= (k+m)^2 - (k+m) + (k+m) - n^2 = (k+m)^2 - n^2 = (k+m+n)(k+m-n).$$

So the above identity becomes

$$\sum_{m=0}^{\infty} c_m\,(k+m+n)(k+m-n)\,x^{k+m} + \sum_{m=0}^{\infty} c_m\, x^{k+m+2} \equiv 0. \quad ...(3)$$

Equating to zero the smallest power of x, namely x^k, (3) gives the indicial equation

$c_0(k+n)(k-n) = 0$ *i.e.*, $(k+n)(k-n) = 0$ as $c_0 \neq 0$. Its roots are $k = n, -n$.

Next equating to zero the coefficient of x^{k+1} in (3) gives

$$c_1(k+1+n)(k+1-n) = 0, \qquad \text{so that} \qquad c_1 = 0 \qquad \text{for} \qquad k = n \qquad \text{and} \qquad k = -n.$$

Finally equating to zero the coefficient of x^{k+m} in (3) gives

$$c_m(k+m+n)(k+m-n) + c_{m-2} = 0 \qquad \text{or} \qquad c_m = \frac{1}{(k+m+n)(n-k-m)}c_{m-2}. \quad ...(4)$$

Putting $m = 3, 5, 7, ...$ in (4) and using $c_1 = 0$, we find

$$c_1 = c_3 = c_5 = c_7 = ... = 0. \quad ...(5)$$

Putting $m = 2, 4, 6, ...$ in (4) gives $$c_2 = \frac{1}{(k+2+n)(n-k-2)}c_0,$$

$$c_4 = \frac{1}{(k+4+n)(n-k-4)}c_2 = \frac{1}{(k+4+n)(n-k-4)(k+2+n)(n-k-2)}c_0$$

and so on. Putting these values in (2), we get

$$y = c_0 x^k \left[1 + \frac{x^2}{(n+k+2)(n-k-2)} + \frac{x^4}{(n+k+2)(n-k-2)(n+k+4)(n-k-4)} + \dots\right]$$

Replacing k by n and $-n$ and also replacing c_0 by a and b in the above equation gives

$$y = ax^n \left\{1 - \frac{x^2}{4(1+n)} + \frac{x^4}{4 \cdot 8(1+n)(2+n)} - \dots\right\} \qquad \text{...(6)}$$

and

$$y = bx^{-n} \left\{1 - \frac{x^2}{4(1+n)} + \frac{x^4}{4 \cdot 8(1-n)(2-n)} - \dots\right\}. \qquad \text{...(7)}$$

The particular solution of (1) obtained from (6) above by taking the arbitrary constant $a = 1/\{2^n \, \Gamma(n+1)\}$, is called the *Bessel function of the first kind of order n*. It will be denoted by $J_n(x)$. Thus, we have

$$J_n(x) = \frac{x^n}{2^n \, \Gamma(n+1)} \left[1 - \frac{x^2}{4(n+1)} + \frac{x^4}{4 \cdot 8(n+1)(n+2)} - \dots\right] \qquad \text{...(8)}$$

or

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r\,!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{2r+n} \qquad \text{...(9)}$$

Replacing b by $1/\{2^n \, \Gamma(n+1)\}$ in (7) and proceeding as above gives

$$J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r\,!\,\Gamma(-n+r+1)} \left(\frac{x}{2}\right)^{2r-n} \qquad \text{...(10)}$$

Let n be non–integral. We know that $\Gamma(m) = \infty$ if m is zero or a negative integer and $\Gamma(m)$ is finite otherwise. Since n is not an integer and r is always integral, the factor $\Gamma(-n+r+1)$ in (10) is always finite and non–zero. For $2r < n$, (10) shows that $J_{-n}(x)$ contains negative powers of x. On the other hand, (9) shows that $J_n(x)$ is not containing negative powers of x at all. Therefore, we find that at $x = 0$, $J_n(x)$ is finite while $J_{-n}(x)$ is infinite, and so one cannot be expressed as a constant multiple of the other. From these arguments we conclude that $J_n(x)$ and $J_{-n}(x)$ are two independent solutions of (1) when n is not an integer (this condition being stronger than $2n$ non–integral which was assumed earlier). Thus, the general solution of Bessel equation (1) when n is not an integer is

$y = AJ_n(x) + BJ_{-n}(x)$, where A and B are arbitrary constant. ... (11)

11.2. Bessel's functions of the first kind of order n. Definition **[Nagpur 2003]**

Bessel's function of the first kind and of order n is denoted by $J_n(x)$ and is defined as

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r\,!\,\Gamma(n+r+1)} \left(\frac{x}{2}\right)^{2r+n}, \text{ where } n \text{ is any non–negative constant.} \qquad \text{...(1)}$$

Remark 1. When n is an integer, $\Gamma(n+r+1) = (n+r)\,!$ and so (1) may be rewritten as

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r\,!\,(n+r)\,!} \left(\frac{x}{2}\right)^{2r+n}. \qquad \text{...(2)}$$

Replacing n by 0 and 1 in turn in (2), Bessel's functions of orders 0 and 1 are given by

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \dots. \qquad \textbf{Kanpur 2005]}$$

and $$J_1(x) = \frac{x}{2} - \frac{x^3}{2^2 \cdot 3} + \frac{x^5}{2^2 \cdot 4^2 \cdot 6} - \ldots.$$

Remark 2. When there is no confusion regarding the variable, we shall write J_n for $J_n(x)$ and J'_n for $d\, J_n(x)/dx$ etc.

11.3. List of important results of Gamma function Γ(n) and Beta function B(m, n).

For more details, refer Chapter 6.

(*i*) $\Gamma(n) = \int_0^\infty e^{-x}\, x^{n-1}\, dx,\ n > 0$ (*ii*) $\Gamma(n) = 2\int_0^\infty e^{-x^2}\, x^{2n-1}\, dx$

(*iii*) $\Gamma(1) = 1$ (*iv*) $\Gamma(1/2) = \sqrt{\pi}$

(*v*) $\Gamma(n+1) = n\,\Gamma(n),\ n > 0$ (*vi*) $\Gamma(n+1) = n\,!$, if n is +ve integer

(*vii*) $B(m, n) = B(n, m)$ (*viii*) $\Gamma(n)\,\Gamma(1-n) = \pi/\sin n\pi$.

(*ix*) $\Gamma(m) = \infty$, so that $1/\Gamma(m) = 0$ if $m = 0$ or –ve integer

(*x*) $B(m, n) = \int_0^i x^{m-1}(1-x)^{n-1}\, dx = \int_0^1 x^{n-1}(1-x)^{m-1}\, dx$, where $m > 0,\ n > 0$

(*xi*) $B(m, n) = \dfrac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$ (*xii*) $\Gamma(2n) = \dfrac{2^{2n-1}}{\sqrt{\pi}}\Gamma(n)\,\Gamma(n+\tfrac{1}{2})$

11.4. Relation between $J_n(x)$ and $J_{-n}(x)$, n being an integer

Theorem. I Show that when n is

(*i*) positive integer, $J_{-n}(x) = (-1)^n J_n(x)$. **[Agra 2006; Kanpur 2006, 07, 08 MDU Rohtak 2004; Purvanchal 2006; Meerut 2006; Nagpur 1995;]**

(*ii*) any integer, $J_{-n}(x) = (-1)^n J_n(x)$ **[Kanpur 2004, 08; Ranchi 2010; Meerut 1993]**

Proof. Part (*i*). Let *n* be a +ve integer. We know that

$$J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r \frac{1}{r\,!\,\Gamma(-n+r+1)}\left(\frac{x}{2}\right)^{2r-n}. \qquad \ldots(1)$$

Since $n > 0$, so $\Gamma(-n + r + 1)$ is infinite (and so $1/\Gamma(-n + r + 1)$ is zero) for $r = 0, 1, 2, \ldots\ldots, (n-1)$. Keeping this in mind we see that the sum over r in (1) must be taken from n to infinity. Thus,

$$J_{-n}(x) = \sum_{r=n}^{\infty} (-1)^r \frac{1}{r\,!\,\Gamma(-n+r+1)}\left(\frac{x}{2}\right)^{2r-n} \qquad \ldots(2)$$

From (2), $J_{-n}(x) = \sum_{m=0}^{\infty} (-1)^{m+n} \dfrac{1}{(m+n)\,!\,\Gamma(m+1)}\left(\dfrac{x}{2}\right)^{2(m+n)-n}$, (on changing the variable of summation to $m = r - n$ so that $r = m + n$ and so $m = 0$ when $r = n$ and $m = \infty$ when $r = \infty$)

$$\therefore \quad J_{-n}(x) = \sum_{m=0}^{\infty} (-1)^m (-1)^n \frac{1}{\Gamma(m+n+1)\,m!}\left(\frac{x}{2}\right)^{2m+n} = (-1)^n \sum_{r=0}^{\infty} (-1)^r \frac{1}{\Gamma(r+n+1)\,r!}\left(\frac{x}{2}\right)^{2m+n},$$

(on changing the variable of summation from m to r while keeping the limits of summation unchanged.)

Thus, for $n > 0$ $\quad J_{-n}(x) = (-1)^n J_n(x)$, by the definition of $J_n(x)$. ... (3)

Part (*ii*). Let n < 0. Let p be a positive integer such that $n = -p$. Since $p > 0$, from part (*i*) above, we have $J_{-p}(x) = (-1)^p J_p(x)$ so that $J_p(x) = (-1)^{-p} J_{-p}(x)$.

But $p = -n$ hence the above result becomes $J_{-n}(x) = (-1)^n J_n(x)$, ...(4)

which is of the same form as (3). Hence the required result holds for any integer.

Remark. When n is an integer $J_{-n}(x)$ is not independent of $J_n(x)$, because $J_{-n}(x)$ is a constant multiple of $J_n(x)$ as shown above. Hence $y = AJ_n(x) + BJ_{-n}(x)$ is not the general solution of Bessel equation when n is an integer. Of course, when n is not an integer, the most general solution of Bessel equation is given by $y = AJ_n(x) + BJ_{-n}(x)$. When n is an integer, the nature of general solution is indicated by the following theorem.

Theorem II. The two independent solutions of Bessel's equation may be taken to be $J_n(x)$ and

$$Y_n(x) = \frac{\cos n\pi\, J_n(x) - J_{-n}(x)}{\sin n\pi}, \text{ for all values of } n. \qquad ...(5)$$

Proof. Case I. Let n be not an integer. Since n is not an integer, $\sin n\pi \neq 0$. Hence (5) shows that $Y_n(x)$ is a linear combination of $J_n(x)$ and $J_{-n}(x)$. But we know that $J_n(x)$ and $J_{-n}(x)$ are independent solutions if n is not an integer. Hence $J_n(x)$ and a linear combination of $J_n(x)$ and $J_{-n}(x)$ will also be independent solutions. Thus we find that $J_n(x)$ and $Y_n(x)$ are two independent solutions of Bessel's equation.

Case II. Let n be an integer. Then we have $\cos n\pi = (-1)^n$, $\sin n\pi = 0$ and $J_{-n}(x) = (-1)^n J_n(x)$.

Using these values in (5), we find that $Y_n(x)$ has the form 0/0 and so $Y_n(x)$ is undefined. To make $Y_n(x)$ meaningful, we define it as

$$Y_n(x) = \lim_{v \to n} Y_v(x) = \lim_{v \to n} \frac{\cos v\pi\, J_v(x) - J_{-v}(x)}{\sin v\pi} \qquad ...(6)$$

$$= \frac{[(\partial/\partial v)\{(\cos v\pi\, J_v(x) - J_{-v}(x)\}]_{v=n}}{[(\partial/\partial v)\cos v\pi)]_{v=n}}, \text{ by L' Hospital's rule}$$

$$= \frac{\left[-\pi \sin v\pi\, J_v(x) + \cos v\pi \frac{\partial}{\partial v} J_v(x) - \frac{\partial}{\partial v} J_{-v}(x)\right]_{v=n}}{[\pi \cos v\pi]_{v=n}} = \frac{\cos n\pi \left[\frac{\partial}{\partial v} J_v(x)\right]_{v=n} - \left[\frac{\partial}{\partial v} J_{-v}(x)\right]_{v=n}}{\pi \cos n\pi}$$

$$= \frac{(-1)^n \left[\frac{\partial}{\partial v} J_v(x)\right]_{v=n} - (-1)^{2n}\left[\frac{\partial}{\partial v} J_{-v}(x)\right]_{v=n}}{\pi(-1)^n} = \frac{1}{\pi}\left[\frac{\partial}{\partial v} J_v(x) - (-1)^n \frac{\partial}{\partial v} J_{-v}(x)\right]_{v=n} \qquad ...(7)$$

We now establish the following two results about $Y_n(x)$ as given by (6).

(*i*) $Y_n(x)$ is a solution of Bessel's equation. (*ii*) $Y_n(x)$ is a solution independent of $J_n(x)$.

Proof of (i). Since $J_v(x)$ and $J_{-v}(x)$ are solutions of Bessel's equation of order v, we must have

$$x^2 \frac{d^2 J_v}{dx^2} + x\frac{dJ_v}{dx} + (x^2 - v^2)\, J_v = 0 \qquad ...(8)$$

and

$$x^2 \frac{d^2 J_{-v}}{dx^2} + x\frac{dJ_{-v}}{dx} + (x^2 - v^2)\, J_{-v} = 0. \qquad ...(9)$$

Differentiating (8) and (9) w.r.t. 'v', we obtain

$$x^2 \frac{d^2}{dx^2}\left(\frac{\partial J_v}{\partial v}\right) + x\frac{d}{dx}\left(\frac{\partial J_v}{\partial v}\right) + (x^2 - v^2)\frac{\partial J_v}{\partial v} - 2vJ_v = 0 \qquad ...(10)$$

$$x^2 \frac{d^2}{dx^2}\left(\frac{\partial J_{-v}}{\partial v}\right) + x\frac{d}{dx}\left(\frac{\partial J_{-v}}{\partial v}\right) + (x^2 - v^2)\frac{\partial J_{-v}}{\partial v} - 2vJ_{-v} = 0. \qquad ...(11)$$

Multiplying (11) by $(-1)^v$ and subtracting from (10) gives

$$x^2 \frac{d^2}{dx^2}\left\{\frac{\partial J_v}{\partial v} - (-1)^v \frac{\partial}{\partial v} J_{-v}\right\} + x\frac{d}{dx}\left\{\frac{\partial}{\partial v} J_v - (-1)^v \frac{\partial}{\partial v} J_{-v}\right\} + (x^2 - v^2)\left\{\frac{\partial}{\partial v} J_v - (-1)^v \frac{\partial}{\partial v} J_{-v}\right\}$$

$$- 2v\{J_v - (-1)^v J_{-v}\} = 0 \qquad ...(12)$$

Taking $v = n$ in (12) and using (7), we have

$$x^2 \frac{d^2}{dx^2}\{\pi\, Y_n(x)\} + x\frac{d}{dx}\{\pi\, Y_n(x)\} + (x^2 - n^2)\,\pi Y_n(x) - 2n\{J_n(x) - (-1)^n\, J_{-n}(x)\} = 0$$

Since n is an integer, $J_{-n}(x) = (-1)^n J_n(x)$ by theorem I and hence the last term in the above equation vanishes. So the above equation reduces to

$$x^2 Y_n'' + x Y_n' + (x^2 - n^2) Y_n = 0, \qquad \text{...(13)}$$

showing that $Y_n(x)$ is also a solution of Bessel's equation of order n.

Proof of (ii). We know that an explicit expression $Y_n(x)$ for n integral is given by

$$Y_n(x) = \frac{2}{\pi}\left\{\log\frac{x}{2} + \gamma - \frac{1}{2}\sum_{r=1}^{n}\frac{1}{r}\right\} J_n(x) - \frac{1}{\pi}\sum_{m=0}^{\infty}(-1)^m \frac{1}{m!\,(n+m)!}\left(\frac{x}{2}\right)^{n+2m}\sum_{r=1}^{m}\left\{\frac{1}{r} + \frac{1}{r+n}\right\}$$

$$- \frac{1}{\pi}\sum_{m=0}^{n-1}\frac{(n-m-1)!}{m!}\left(\frac{x}{2}\right)^{-n+2m}, \qquad \text{...(14)}$$

where γ is Euler's constant. From (14) we find that $Y_n(x)$ is infinite when $x = 0$, whereas $J_n(x)$ is infinite when $x = 0$. So $Y_n(x)$ as given by (6) and $J_n(x)$ are two independent solutions of Bessel's equation of order n.

Remark 1. General solution of Bessel's equation when n is an integer is

$$y = AJ_n(x) + BY_n(x), \text{ A and B being arbitrary constants.} \qquad \text{...(15)}$$

where $Y_n(x)$ is given by (6). $Y_n(x)$ is known as *Bessel's function of order n of the second kind.* $Y_n(x)$ is also called the *Neumann function of order n* and is denoted by $N_n(x)$.

Remark 2. Equations reducible to Bessel's equation

Consider $$x^2 y'' + xy' + (\lambda^2 x^2 - n^2)y = 0 \qquad \text{...(16)}$$

Let $z = \lambda x$ so that $\dfrac{dy}{dx} = \lambda\dfrac{dy}{dz}$ and $\dfrac{d^2 y}{dx^2} = \lambda^2\dfrac{d^2 y}{dz^2}$. Then (16) becomes

$$z^2(d^2 y/dz^2) + z(dy/dz) + (z^2 - n^2)y = 0, \qquad \text{...(17)}$$

which is Bessel's equation of order n. As explained in remark 1 above, the general solution of (17) is

$$y = AJ_n(z) + BY_n(z) \quad \text{or} \quad y = AJ_n(\lambda x) + BY_n(\lambda x). \qquad \text{...(18)}$$

Thus $J_n(\lambda x)$ and $Y_n(\lambda x)$ are solutions of (16), which is called the *modified Bessel's equation.*

11.5. Bessel's function of the second kind of order n. Definition

This is denoted by $Y_n(x)$ and is defined by

$$Y_n(x) = \frac{J_n(x)\cos n\pi - J_{-n}(x)}{\sin n\pi}, \; n \neq \text{integer}$$

and $$Y_n(x) = \lim_{v\to n}\frac{J_v(x)\cos v\pi - J_{-v}(x)}{\sin v\pi}, \; n \text{ is an integer.}$$

11.6. Integration of Bessel's equation xy″ + y′ + xy = 0 in series for n = 0. Bessel's function of zeroth order, i.e. $J_0(x)$. [Kakitiya 1997]

Bessel's equation for $n = 0$ is given by $$xy'' + y' + xy = 0 \qquad \text{...(1)}$$

Let its series solution be $$y = \sum_{m=0}^{\infty} c_m x^{k+m}, \; c_0 \neq 0. \qquad \text{...(2)}$$

$$\therefore\; y' = \sum_{m=0}^{\infty}(k+m)\,c_m x^{k+m-1} \quad \text{and} \quad y'' = \sum_{m=0}^{\infty}(k+m)(k+m-1)\,c_m x^{k+m-2}. \qquad \text{...(3)}$$

Substituting for y, y', y'' in (1), we obtain

$$x\sum_{m=0}^{\infty}(k+m)(k+m-1)c_m x^{k+m-2} + \sum_{m=0}^{\infty}(k+m)c_m x^{k+m-1} + x\sum_{m=0}^{\infty}c_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty}(k+m)(k+m-1)c_m x^{k+m-1} + \sum_{m=0}^{\infty}(k+m)c_m x^{k+m-1} + \sum_{m=0}^{\infty}c_m x^{k+m+1} = 0$$

or $$\sum_{m=0}^{\infty}\{(k+m)(k+m-1)+(k+m)\}c_m x^{k+m-1} + \sum_{m=0}^{\infty}c_m x^{k+m+1} = 0$$

or $$\sum_{m=0}^{\infty}(k+m)^2 c_m x^{k+m-1} + \sum_{m=0}^{\infty}c_m x^{k+m+1} = 0, \qquad ...(4)$$

which is an identity. Equating to zero the coefficient of the lowest power of x, namely x^{k-1}, we have

$k^2 c_0 = 0$ so that $k = 0, 0$ (as $c_0 \neq 0$)

Now equating to zero the coefficient of next higher power of x, namely x^k, in (4), we have

$c_1(k+1)^2 = 0$ so that $c_1 = 0$ as $(k+1)^2 \neq 0$ for $k = 0$.

Finally, equating to zero the coefficient of x^{k+m-1} in (4), we get

$$(k+m)^2 c_m + c_{m-2} = 0 \quad \text{or} \quad c_m = -\frac{c_{m-2}}{(k+m)^2}, \text{ for all } m \geq 2.$$

When $k = 0$, we have $c_m = -(1/m^2)c_{m-2}$ for all $m \geq 2$ ...(5)

Putting $m = 3, 5, 7, ...$ in (5) and noting that $c_1 = 0$, we obtain

$$c_1 = c_3 = c_5 = c_7 = ... = 0. \qquad ...(6)$$

Next, putting = 2, 4, 6, ... in (5), we have

$$c_2 = -\frac{c_0}{2^2}, \qquad c_4 = -\frac{c_2}{4^2} = \frac{c_0}{2^2 \cdot 4^2}, \qquad c_6 = -\frac{c_4}{6^2} = -\frac{c_0}{2^2 \cdot 4^2 \cdot 6^2}, \text{ ... and so on}$$

Putting the above values in (2) for $k = 0$, $y = \sum_{m=0}^{\infty} c_m x^m = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + ...$

or $$y = c_0\left(1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + ... \text{ad. inf}\right)$$

If $c_0 = 1$, the above solution is denoted by $J_0(x)$ so that

$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + ... \text{ad. inf}, \qquad ...(7)$$

where $J_0(x)$ is known as *Bessel's function of zeroeth order.*

Note 1. Replacing n by 0 in Art. 11.2, we can deduce (7).

Note 2. From (7), we have $J_0(0) = 1$. **[Nagpur 1995]**

11.6.A. Solved examples based on Art. 11.1 to 11.6

Ex. 1. *Prove that*

(*i*) $J_{-1/2}(x) = \sqrt{(2/\pi x)}\cos x$. **[Garhwal 2005; Nagpur 2005; Kanpur 2009, 10; Agara 2010; Bhopal 2010; Ranchi 2010]**

(*ii*) $J_{1/2}(x) = \sqrt{(2/\pi x)}\sin x$. **[Nagpur 2003, 05; Garhwal 2004; Kanpur 2004; 07]**

(*iii*) $[J_{1/2}(x)]^2 + [J_{-1/2}(x)]^2 = 2/(\pi x)$. **[Lucknow 2010; Meerut 1992, 93]**

Sol. By the definition of $J_n(x)$, we have

$$J_n(x) = \frac{x^n}{2^n \Gamma(n+1)}\left[1 - \frac{x^2}{2 \cdot 2(n+1)} + \frac{x^4}{2 \cdot 4 \cdot 2^2 (n+1)(n+2)} - ...\right] \qquad ...(1)$$

Part (i). Replacing n by – (1/2) in (1) and simplifying, we get

$$J_{-1/2}(x) = \frac{x^{-1/2}}{2^{-1/2}\Gamma\left(\frac{1}{2}\right)}\left[1-\frac{x^2}{1\cdot 2}+\frac{x^4}{1\cdot 2\cdot 3\cdot 4}-...\right] = \sqrt{\frac{2}{\pi x}}\ \cos x, \text{ as } \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

Part (ii). Replacing n by 1/2 in (1) and simplifying, we get

$$J_{1/2}(x) = \frac{x^{1/2}}{2^{1/2}\Gamma(3/2)}\cdot\left[1-\frac{x^2}{1\cdot 2\cdot 3}+\frac{x^4}{1\cdot 2\cdot 3\cdot 4\cdot 5}-...\right] = \sqrt{\left(\frac{x}{2}\right)}\frac{1}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}\cdot\frac{1}{x}\left[x-\frac{x^3}{1\cdot 2\cdot 3}+\frac{x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}-...\right]$$

$[\because\ \Gamma(p+1) = p\Gamma(p)]$

$$= \sqrt{\frac{2}{\pi x}}\left[x-\frac{x^3}{3\,!}+\frac{x^5}{5\,!}-...\right] = \sqrt{\frac{2}{\pi x}}\sin x$$

Part (iii). Squaring and adding the results of (i) and (ii), we get

$$[J_{1/2}(x)]^2 + J_{-1/2}(x)]^2 = (2/\pi x)\ (\sin^2 x + \cos^2 x) = 2/\pi x.$$

Ex. 2. *Prove that* $\lim\limits_{z\to 0}\dfrac{J_n(z)}{z^n} = \dfrac{1}{2^n\Gamma(n+1)^n}$, *where* $n > -1$. **(Kanpur 2005, 07)**

Sol. By defination, $\quad J_n(z) = \dfrac{z^n}{2^n\ \Gamma(n+1)}\left[1-\dfrac{z^2}{4(n+1)}+\dfrac{z^4}{4\,.\,8\,.\,(n+1)\,.\,(n+2)}-...\right]$

$$\therefore \quad \lim_{z\to 0}\frac{J_n(z)}{z^n} = \frac{1}{2^n\ \Gamma(n+1)}\left[1-\frac{z^2}{4(n+1)}+\frac{z^4}{4\,.\,8\,.\,(n+1)\,.\,(n+2)}-...\right] = \frac{1}{2^n\Gamma(n+1)}$$

Ex. 3. *Write the general solution of the following equations:*

(i) $x^2(d^2y/dx^2) + x(dy/dx) + (x^2 - 25)y = 0$ **(MDU Rohtak 2005)**

(ii) $x^2(d^2y/dx^2) + x(dy/dx) + (x^2 - 9/16)y = 0$

(iii) $d^2y/dx^2 + (1/x) \times (dy/dx) + (1 - 1/6.25x^2)\ y = 0$

(iv) $x^2\ (d^2z/dx^2) + x(dz/dx) + (x^2 - 64)\ z = 0$

(v) $z(d^2y/dz^2) + (dy/dz) + zy = 0$

Sol. In what follows, we shall use the following solutions of Bessel's equation

$$x^2y'' + xy' + (x^2 - n^2)y = 0$$

Result I: $y = A\ J_n\ (x) + B\ J_{-n}(x)$, where n is not an integer. A, B being arbitrary constants.

Result II: $y = A\ J_n(x) + B\ Y_n(x)$, where n is an integer, A, B being arbitrary constants.

(i) Given $x^2y'' + xy' + (x^2 - 5^2)y = 0$, which is Bessel's equation of order 5, which is an integer. Its general solution is $y = A\ J_5(x) + B\ Y_5(x)$, where A and B are arbitrary constants.

(ii) Given $x^2y'' + xy' + \{x^2 - (3/4)^2\} = 0$, which is Bessel's equation of order 3/4, which is not an integer. Its solution is $y = A\ J_{3/4}(x) + B\ J_{-3/4}\ (x)$, where A and B are arbitrary constants

(iii) Re-writing, given equating becomes $\quad x^2y'' + xy' + \{x^2 - (2/5)^2\}y = 0$

As in part (ii), solution is $\quad y = A\ J_{2/5}(x) + B\ J_{-2/5}(x)$, A, B being arbitrary constants

(iv) Given $\quad x^2(d^2z/dx^2) + x(dz/dx) + (x^2 - 8^2)\ z = 0$

As in part (i), solution is $\quad z = A\ J_8\ (x) + B\ Y_8(x)$, A, B being arbitrary constants

(v) Re-writing the given equation, $\quad z^2(d^2y/dz^2) + z(dy/dz) + z^2y = 0$

which is a Bessel equation of order 0, which is an integer. Its solution is $y = A\ J_0(z) + BY_0(z)$.

Ex. 4. *Solve the following differential equation*:

(i) $x^2(d^2y/dx^2) + x(dy/dx) + (4x^4 - 1/4)y = 0$

(ii) $x(d^2y/dx^2) + dy/dx + (y/4) = 0$ by using the subtitution $z = \sqrt{x}$

Sol. (i) Suppose that $$z = x^2 \qquad \text{... (1)}$$

$$\therefore \qquad \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = 2x\frac{dy}{dz}, \text{ by (1)} \qquad \text{... (2)}$$

and $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(2x\frac{dy}{dz}\right) = 2\frac{dy}{dz} + 2x\frac{d}{dx}\left(\frac{dy}{dz}\right) = 2\frac{dy}{dz} + 2x\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx} = 2\frac{dy}{dz} + (2x)^2\frac{d^2y}{dz^2}, \text{ by (1)}$$

Substituting the above values in the given equation, we get

$$x^2\left(2\frac{dy}{dz} + 4x^2\frac{d^2y}{dz^2}\right) + 2x^2\frac{dy}{dz} + \left(4x^4 - \frac{1}{4}\right)y = 0 \quad \text{or} \quad 4z^2\frac{d^2y}{dz^2} + 4z\frac{dy}{dz} + \left(4z^2 - \frac{1}{4}\right)y = 0$$

or $$z^2(d^2y/dz^2) + z(dy/dz) + \{z^2 - (1/4)^2\}y = 0 \qquad \text{... (3)}$$

(3) is a Bessel's equation of order 1/4. Since 1/4 is a positive non-integral real number, hence solution of (3) is $y = A\,J_{1/4}(z) + B\,J_{-1/4}(z)$ or $y = A\,J_{1/4}(x^2) + B\,J_{-1/4}(x^2)$.

(ii) Given $$z = \sqrt{x} \qquad \text{... (1)}$$

$$\therefore \qquad \frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = \frac{1}{2\sqrt{x}}\frac{dy}{dz}, \text{ using (1)} \qquad \text{... (2)}$$

and $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{1}{2\sqrt{x}}\frac{dy}{dz}\right) = -\frac{x^{-3/2}}{4}\frac{dy}{dz} + \frac{1}{2\sqrt{x}}\frac{d}{dx}\left(\frac{dy}{dz}\right)$$

or $$\frac{d^2y}{dx^2} = -\frac{1}{4x^{3/2}}\frac{dy}{dz} + \frac{1}{2\sqrt{x}}\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx} = -\frac{1}{4x^{3/2}}\frac{dy}{dx} + \frac{1}{4x}\frac{d^2y}{dz^2} \qquad \text{... (3)}$$

Using (1), (2) and (3), the given equation, reduces to

$$x\left(-\frac{1}{4x^{3/2}}\frac{dy}{dz} + \frac{1}{4x}\frac{d^2y}{dz^2}\right) + \frac{1}{2\sqrt{x}}\frac{dy}{dz} + \frac{1}{4}y = 0 \quad \text{or} \quad -\frac{1}{4z}\frac{dy}{dz} + \frac{1}{4}\frac{d^2y}{dz^2} + \frac{1}{2z}\frac{dy}{dz} + \frac{1}{4}y = 0$$

or $$z^2(d^2y/dz^2) + z(dy/dz) + (z^2 - 0^2)y = 0,$$

which is a Bessel's equation of order 0 and so its solution is

$$y = A\,J_0(z) + B\,Y_0(z) \quad \text{or} \quad y = A\,J_0(\sqrt{x}) + B\,Y_0(\sqrt{x}),$$

which is the general solution of the given equation, A, B being arbitrary constants.

Ex. 5 (a) *Solve* $x(d^2y/dx^2) + 2(dy/dx) + (xy)/2 = 0$ *in terms of Bessel's functions.*

(KU Kurukshetra 2005)

(b) Solve $x(d^2z/dx^2) - 2(dz/dx) + xz = 0$ by using the substitution $y = z/x^{3/2}$.

Sol. (a) Given $$x(d^2y/dx^2) + 2(dy/dx) + (xy)/2 = 0 \qquad \text{... (1)}$$

Assume that $z = y\sqrt{x}$ so that $y = z/\sqrt{x}$ ... (2)

$$\therefore \qquad \frac{dy}{dx} = \frac{d}{dx}(x^{-1/2}z) = x^{-1/2}\frac{dz}{dx} - \frac{x^{-3/2}}{2}z$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(x^{-1/2}\frac{dz}{dx}\right) - \frac{1}{2}\frac{d}{dx}(x^{-3/2}z) = x^{-1/2}\frac{d^2z}{dx^2} - \frac{x^{-3/2}}{2}\frac{dz}{dx} - \frac{1}{2}\left(x^{-3/2}\frac{dz}{dx} - \frac{3x^{-5/2}}{2}z\right)$$

or $$\frac{d^2y}{dx^2} = \frac{1}{x^{1/2}}\frac{d^2z}{dx^2} - \frac{1}{x^{3/2}}\frac{dz}{dx} + \frac{3}{4x^{5/2}}z$$

Substituting the above values of y, dy/dx and d^2y/dx^2 in (1), we get

$$x\left(\frac{1}{x^{1/2}}\frac{d^2z}{dx^2} - \frac{1}{x^{3/2}}\frac{dz}{dx} + \frac{3z}{4x^{5/2}}\right) + 2\left(\frac{1}{x^{1/2}}\frac{dz}{dx} - \frac{z}{2x^{3/2}}\right) + \frac{x}{2}\times\frac{z}{x^{1/2}} = 0$$

or $$x^{1/2}\frac{d^2z}{dx^2} + \frac{1}{x^{1/2}}\frac{dz}{dx} - \frac{z}{4x^{3/2}} + \frac{1}{2}x^{1/2}z = 0 \qquad \text{... (3)}$$

Multiplying both sides of (3) by $x^{3/2}$, $\quad x^2(d^2z/dx^2) + x(dz/dx) + (x^2/2 - 1/4)\,z = 0 \qquad$ ... (4)

Let $$u = x/\sqrt{2} \qquad \text{... (5)}$$

$\therefore$ $$\frac{dz}{dx} = \frac{dz}{du}\frac{du}{dx} = \frac{dz}{du}\times\frac{1}{\sqrt{2}}, \text{ by (5)} \qquad \text{... (6)}$$

From (*6*), we have $$\frac{d}{dx} \equiv \frac{1}{\sqrt{2}}\frac{d}{du} \qquad \text{... (7)}$$

$\therefore$ $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{1}{\sqrt{2}}\frac{d}{du}\left(\frac{1}{\sqrt{2}}\frac{dz}{du}\right), \text{ by (6) and (7)}$$

Thus, $$d^2y/dx^2 = (1/2)\times(d^2z/du^2) \qquad \text{... (8)}$$

Substituting the above values in (4), we get

$$2u^2\times\frac{1}{2}\frac{d^2z}{du^2} + u\sqrt{2}\times\frac{1}{\sqrt{2}}\frac{dz}{du} + \left(u^2 - \frac{1}{4}\right)z = 0 \quad \text{or} \quad u^2\frac{d^2z}{du^2} + u\frac{dz}{du} + \left\{u^2 - \left(\frac{1}{2}\right)^2\right\}z = 0 \qquad \text{... (9)}$$

which is a Bessel equation of order 1/2. Since 1/2 a positive non-negative integer, hence the required solution is given by

$$z = A\,J_{1/2}(u) + B\,J_{1/2}(u) \quad \text{or} \quad y\sqrt{x} = A\,J_{1/2}(x/\sqrt{2}) + BJ_{-1/2}(x/\sqrt{2}), \text{ by (2) and (5)}$$

(b) Ans. $z = c_1\,x^{3/2}J_{3/2}(x) + c_2x^{3/2}J_{-3/2}(x)$, c_1, c_2 being arbitrary constants

Ex. 6. *Verify that the Bessel function* $J_{1/2}(x) = (\sin x)\times(2/\pi x)^{1/2}$ *satisfies the Bessel equation of order* 1/2. **(MDU Rohtak 2006)**

Sol. Bessel equation of order 1/2 is given by

$$x^2(d^2y/dx^2)\ \ x(dy/dx) + (x^2 - 1/4)y = 0 \qquad \text{... (1)}$$

Let $$y = J_{1/2}(x) = (2/\pi)^{1/2}\times(x^{-1/2}\sin x) \qquad \text{... (2)}$$

$\therefore$ $$dy/dx = (2/\pi)^{1/2}\times\left\{x^{-1/2}\cos x + (-1/2)\times x^{-3/2}\sin x\right\}$$

$$d^2y/dx^2 = (2/\pi)^{1/2}\times\left\{-(1/2)\times x^{-3/2}\cos x - x^{-1/2}\sin x + (3/4)\times x^{-5/2}\sin x + (-1/2)\times x^{-3/2}\cos x\right\}$$

Substituting the above values of y, dy/dx and d^2y/dx^2 in (1), we get

$$x^2\times(2/\pi)^{1/2}\left\{-x^{-3/2}\cos x - x^{-1/2}\sin x + (3/4)\times x^{-5/2}\sin x\right\} + x\times(2/\pi)^{1/2}\left\{x^{-1/2}\cos x - (1/2)\times x^{-3/2}\sin x\right\} + (x^2 - 1/4)\times(2/\pi)^{1/2}\times(x^{-1/2}\sin x) = 0$$

or $$(2/\pi)^{1/2}\left\{-x^{1/2}\cos x - x^{3/2}\sin x + (3/4)\times x^{-1/2}\sin x + x^{1/2}\cos x - (1/2)\times x^{-1/2}\sin x + x^{3/2}\sin x - (1/4)\times x^{-1/2}\sin x\right\} = 0$$

or $\quad 0 = 0$, which is true. Hence $y = J_{1/2}\,(x)$ satisfies the Bessel equation (1) of order 1/2.

Ex. 7. *Show that* $d\,(x^n J_n(ax)/dx = a\,x^n J_{n-1}(ax)$ and hence deduce that $d(x\,J_1(x))/dx = x\,J_0(x)$.

Sol. We know that
$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{2r+n} \quad \text{... (1)}$$

$$\therefore \quad x^n J_n(ax) = x^n \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\Gamma(n+r+1)}\left(\frac{ax}{2}\right)^{2r+n} \quad \text{or} \quad x^n J_n(ax) = \sum_{r=0}^{\infty} \frac{(-1)^r a^{2r+n} x^{2r+2n}}{2^{2r+n} r!\Gamma(n+r+1)}$$

$$\therefore \quad \frac{d}{dx}\left(x^n J_n(ax)\right) = \sum_{r=0}^{\infty} \frac{(-1)^r a^{2r+n} \times 2(r+n) x^{2r+2n-1}}{2^{2r+n} r!\,\Gamma\,(n+r+1)} \quad \text{... (2)}$$

From (1),
$$J_{n-1}(ax) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma\,(n+r)}\left(\frac{ax}{2}\right)^{2r+n-1}$$

$$\therefore \quad ax^n J_{n-1}(ax) = \sum_{r=0}^{\infty} \frac{(-1)^r a^{2r+n} x^{2r+2n-1} \times 2(r+n)}{2^{2r+n-1} r!\,\Gamma\,(n+r) \times 2(r+n)}$$

or
$$ax^n J_{n-1}(ax) = \sum_{r=0}^{\infty} \frac{(-1)^r a^{2r+n} \times 2(r+n) x^{2r+2n-1}}{2^{2r+n} r!\,\Gamma\,(r+n+1)} \quad \text{... (3)}$$

From (2) and (3),
$$\frac{d}{dx}\left(x^n J_n(ax)\right) = ax^n J_{n-1}(x) \quad \text{... (4)}$$

Putting $n = 1$ and $a = 1$ in (4), we have
$$\frac{d}{dx}(xJ_1(x)) = x\,J_0(x)$$

Ex. 8. *Show that* $\displaystyle\int_0^1 \frac{u\,J_0(xu)}{(1-u^2)^{1/2}}\,du = \frac{\sin x}{x}$

Sol. We have (see Art. 11.2),
$$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2 \cdot 4^2} - \frac{x^6}{2^2 \cdot 4^2 \cdot 6^2} + \ldots.. \quad \text{... (1)}$$

$$\therefore \quad \int_0^1 \frac{uJ_0(xu)}{(1-u^2)^{1/2}}\,du = \int_0^1 \frac{u}{(1-u^2)^{1/2}}\left(1 - \frac{x^2}{4}u^2 + \frac{x^4}{4 \cdot 16}u^4 \ldots\right)du$$

$$= \int_0^{\pi/2} \frac{\sin\theta}{\cos\theta}\left(1 - \frac{x^2}{4}\sin^2\theta + \frac{x^4}{64}\sin^4\theta - \ldots\right)\cos\theta\,d\theta = \int_0^{\pi/2} \sin\theta\,d\theta - \frac{x^2}{4}\int_0^{\pi/2} \sin^3\theta\,d\theta + \frac{x^4}{64}\int_0^{\pi/2} \sin^5\theta\,d\theta \ldots.$$

[On putting $u = \sin\theta$ and $du = \cos\theta\,d\theta$]

$$= [-\cos\theta]_0^{\pi/2} - \frac{x^2}{4} \times \frac{2}{3} + \frac{x^4}{64} \times \frac{4 \cdot 2}{5 \cdot 3} \ldots.$$

[using Walli's formula of Integral calculus]

$$= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \ldots = \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right) = \frac{\sin x}{x}$$

Ex. 9. *Show that* $\displaystyle\int_0^{\pi/2} J_1(z\cos\theta)\,d\theta = \frac{1 - \cos z}{z}$ **[Bilaspur 1997]**

Sol. We have
$$J_n(x) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{2r+n} \quad \text{... (1)}$$

$$(1) \quad \Rightarrow \quad J_1(z\cos\theta) = \sum_{r=0}^{\infty} \frac{(-1)^r}{r\,!\,\Gamma(1+r+1)}\left(\frac{z\cos\theta}{2}\right)^{2r+1}$$

$$\therefore \quad \int_0^{\pi/2} J_1\left(z\cos\theta\right)d\theta = \sum_{r=0}^{\infty} \frac{(-1)^r z^{2r+1}}{r\,!\,(r+1)\,!\,2^{2r+1}}\int_0^{\pi/2}\cos^{2r+1}\theta\,d\theta$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r z^{2r+1}}{r\,!\,(r+1)!2^{2r+1}}\cdot\frac{2r(2r-2)...4\cdot 2}{(2r+1)(2r-1)...5\cdot 3} \qquad \text{[using a standard result of Integral calculus]}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r z^{2r+1}}{r!\,(r+1)!2^{2r+1}}\cdot\frac{[2r(2r-2)...4\cdot 2]^2}{(2r+1)\,2r\,(2r-1)\,(2r-2)...5\cdot 4\cdot 3\cdot 2\cdot 1}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r z^{2r+1}}{r!\,(r+1)!2^{2r+1}}\cdot\frac{2^{2r}(r1)^2}{(2r+1)!} = \sum_{r=0}^{\infty}\frac{(-1)^r z^{2r+1}}{2(r+1)(2r+1)!} \quad = \frac{1}{z} - \frac{1}{z} + \frac{z}{2!} - \frac{z^3}{4!} + \frac{z^5}{6!} - ... = \frac{1-\cos z}{z}$$

Ex. 10. *Prove that* $J_n(x) = \dfrac{x^n}{2^{n-1}\Gamma(n)}\displaystyle\int_0^{\pi/2}\sin\theta\cos^{2n-1}\theta\,J_0(x\sin\theta)d\theta,$ *where* $n > -1/2$.

[Ravishankar 2002]

Sol. We have $$J_n(x) = \sum_{r=0}^{\infty}\frac{(-1)^r}{r\,!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{2r+n} \qquad ... (1)$$

$$(1) \quad \Rightarrow \quad J_0(x\sin\theta) = \sum_{r=0}^{\infty}\frac{(-1)^r}{r\,!\,\Gamma(r+1)}\left(\frac{x\sin\theta}{2}\right)^{2r} \qquad ... (2)$$

Let $$I = \int_0^{\pi/2}\sin\theta\,\cos^{2n-1}\theta\,J_0(x\sin\theta)d\theta \qquad ... (3)$$

$$\therefore \quad I = \int_0^{\pi/2}\sin\theta\,\cos^{2n-1}\theta\left(\sum_{r=0}^{\infty}\frac{(-1)^r}{r\,!\,\Gamma(r+1)}\frac{x^{2r}}{2^{2r}}\sin^{2r}\theta\right)d\theta, \text{ using (2)}$$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r x^{2r}}{(r!)^2\,2^{2r}}\int_0^{\pi/2}\cos^{2n-1}\theta\,\sin^{2r+1}\theta\,d\theta = \sum_{r=0}^{\infty}\frac{(-1)^r x^{2r}}{(r!)^2 2^{2r}}\cdot\frac{\Gamma(n)\,\Gamma(r+1)}{\Gamma(n+r+1)} = \frac{\Gamma(n)}{2}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{2r}$$

or $$\frac{x^n}{2^{n-1}\Gamma(n)}I = \left(\frac{x}{2}\right)^n\sum_{r=0}^{\infty}\frac{(-1)^r}{r\,!\,\Gamma\,(n+r+1)}\left(\frac{x}{2}\right)^{2r} = \sum_{r=0}^{\infty}\frac{(-1)^r}{r!(n+r+1)!}\left(\frac{x}{2}\right)^{2r+n}$$

or $$\frac{x^n}{2^{n-1}\Gamma(n)}\int_0^{\pi/2}\sin\theta\cos^{2n-1}\theta\,J_0(x\sin\theta)\,d\theta = J_n(x), \text{ by (1) and (3)}$$

Ex. 11. *Show that* $y = A\,J_n(x)\displaystyle\int_0^x\frac{dx}{xJ_n^2(x)} + B\,J_n(x)$ *is the complete solution of Bessel's equation.* **[Bilaspur 1994]**

Sol. The Bessel's equation is $$y'' + (1/x)\times y + (1 - n^2/x^2)y = 0 \qquad ... (1)$$

We know that a solution (1) is $u = J_n(x)$ ... (2)

Let the complete solution of (1) be $y = u\,v$ ... (3)

Comparing (1) with $y'' + Py' + Qy = R$, $\quad P = 1/x \quad$ and $\quad Q = 1 - n^2/x^2, \quad R = 0$

*Then, we know that v is given by

$$\frac{d^2v}{dx^2}+\left(P+\frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx}=\frac{R}{u} \qquad \text{or} \qquad \frac{d^2v}{dx^2}+\left(\frac{1}{x}+\frac{2\,J_n'(x)}{J_n(x)}\right)\frac{dv}{dx}=0 \qquad \ldots (4)$$

Let $\quad dv/dx = q \quad$ so that $\quad d^2v/dx^2 = dq/dx \quad \ldots (5)$

Then (4) yields $\quad \dfrac{dq}{dx}+\left(\dfrac{1}{x}+\dfrac{2J_n'}{J_n}\right)q=0 \quad$ or $\quad \dfrac{dq}{q}+\left(\dfrac{1}{x}+\dfrac{2J_n'}{J_n}\right)dx=0.$

Integrating, $\quad \log q + \log x + 2\log J_n = \log A \quad$ or $\quad q\,x\,J_n^2 = A$

or $\quad q = dv/dx = A/(x J_n^2) \quad$ or $\quad dv = \left\{A/(x J_n^2)\right\}dx$

Integrating, $$v = A\int_0^x \frac{dx}{x J_n^2}+B, \qquad \ldots (6)$$

where A and B are arbitrary constants. From (2), (3) and (6) the required complete integral is

$$y = J_n(x)\left(A\int_0^x \frac{dx}{x J_n^2}+B\right) \qquad \text{or} \qquad y = A\,J_n(x)\int_0^x \frac{dx}{x J_n^2(x)}+BJ_n(x)$$

Ex. 12. *Prove that if* $n > m-1$, *then* $J_n(x) = \dfrac{2(x/2)^{n-m}}{\Gamma(n-m)}\displaystyle\int_0^1 (1-t^2)^{n-m-1}\,t^{m+1}\,J_m(xt)\,dt$.

Sol. Let $$I = \int_0^1 (1-t^2)^{n-m-1}\,t^{m+1}\,J_m(xt)\,dt.$$

Then using the definition of $J_m(xt)$, we have

$$I = \int_0^1 (1-t^2)^{n-m-1}\,t^{m+1}\sum_{r=0}^{\infty}(-1)^r\frac{1}{r!\,\Gamma(m+r+1)}\left(\frac{xt}{2}\right)^{2r+m}dt$$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r(x/2)^{2r+m}}{r!\,\Gamma(m+r+1)}\int_0^1(1-t^2)^{n-m-1}\,t^{2m+2r+1}\,dt = \sum_{r=0}^{\infty}\frac{(-1)^r(x/2)^{2r+m}}{r!\,\Gamma(m+r+1)}\int_0^1(1-t^2)^{n-m-1}\,(t^2)^{m+r}\,t\,dt$$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r(x/2)^{2r+m}}{r!\,\Gamma(m+r+1)}\cdot\frac{1}{2}\int_0^1(1-z)^{n-m-1}\,z^{(m+r+1)-1}\,dz, \text{ on putting } t^2 = z \text{ and } t\,dt = \frac{dz}{2}$$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r(x/2)^{2r+m}}{r!\,\Gamma(m+r+1)}\cdot\frac{1}{2}\cdot\frac{\Gamma(n-m)\,\Gamma(m+r+1)}{\Gamma(n+r+1)}, \text{provided } n-m>0 \text{ and } m+r+1>0 \text{ i.e. } n>m>-1$$

$$\left[\because \int_0^1(1-z)^{\alpha-1}z^{\beta-1}dz = B(\alpha,\beta) = \frac{\Gamma(\alpha)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \text{ if } \alpha>0, \beta>0\right]$$

$$= \frac{\Gamma(n-m)}{2}\left(\frac{x}{2}\right)^{m-n}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r} = \frac{\Gamma(n-m)}{2}\left(\frac{x}{2}\right)^{m-n}J_n(x), \text{ by def. of } J_n(x)$$

$$\therefore\; J_n(x) = 2\frac{(x/2)^{m-n}}{\Gamma(n-m)}I \qquad \text{or} \qquad J_n(x) = 2\frac{(x/2)^{m-n}}{\Gamma(n-m)}\int_0^1(1-t^2)^{n-m-1}t^{m+1}J_m(xt)\,dt, \text{ using (1)}$$

*Refer Chapter 10 in part I of "Ordinary and Partial Differential Equations" by Dr. M.D. Raisinghania published by S.Chand & Co., New Delhi

Ex. 13. *Prove that* (*i*) $J_n(x) = \dfrac{(x/2)^n}{\sqrt{\pi}\,\Gamma(n+1)}\displaystyle\int_{-1}^{1}(1-t^2)^{n-1/2}e^{ixt}\,dt$, if $n \geq -\frac{1}{2}$

(*ii*) $J_n(x) = \dfrac{x^n}{2^{n-1}\,\Gamma(\frac{1}{2})\,\Gamma(n-1)}\displaystyle\int_0^1(1-t^2)^{n-1/2}\cos xt\,dt.$

Sol. Part (i). Let $\quad I = \displaystyle\int_{-1}^{1}(1-t^2)^{n-1/2}e^{ixt}\,dt.$

But $\quad e^A = 1 + A + \dfrac{A^2}{2\,!} + \dots + \dfrac{A^r}{r\,!} + \dots = \displaystyle\sum_{r=0}^{\infty}\dfrac{A^r}{r\,!}$

$\therefore \quad I = \displaystyle\int_{-1}^{1}(1-t^2)^{n-1/2}\left\{\sum_{r=0}^{\infty}\frac{(ixt)^r}{r\,!}\right\}dt \quad$ or $\quad I = \displaystyle\sum_{r=0}^{\infty}\frac{(ix)^r}{r\,!}\int_{-1}^{1}(1-t^2)^{n-1/2}\,t^r dt.$ ...(2)

Now, if r is odd (*i.e.* $r = 2m+1$), the integrand in the above integral is an odd function of t and hence it vanishes whereas if r is even (*i.e.* $r = 2m$), the integral is even function of t and so by a property of definite integral $\displaystyle\int_{-1}^{1}(1-t^2)^{n-1/2}\,t^{2m}dt = 2\int_0^1(1-t^2)^{n-1/2}\,t^{2m}dt$. So (2) gives

$$I = \sum_{m=0}^{\infty}\frac{(ix)^m}{(2m)\,!}\cdot 2\int_0^1(1-t^2)^{n-1/2}\,t^{2m}\,dt = \sum_{m=0}^{\infty}\frac{(i^2)^m\,x^{2m}}{(2m)\,!}\int_0^1(1-t^2)^{n-1}\,t^{2m-1}\cdot 2t\,dt, \text{ by (1)}$$

$$= \sum_{m=0}^{\infty}\frac{(-1)^m\,x^{2m}}{(2m)\,!}\int_0^1(1-z)^{n-1/2}\,(z^{1/2})^{2m-1}.dz, \qquad \text{on putting} \quad t^2 = z \quad \text{so that} \quad 2t\,dt = dz$$

$$= \sum_{m=0}^{\infty}\frac{(-1)^m\,x^{2m}}{(2m)\,!}\int_0^1(1-z)^{(n+1/2)-1}\,z^{(m+1/2)-1}\,dz = \sum_{m=0}^{\infty}\frac{(-1)^m\,x^{2m}}{(2m)\,!}\times\frac{\Gamma(n+\frac{1}{2})\,\Gamma(m+\frac{1}{2})}{\Gamma(n+m+1)},$$

provided $n + \frac{1}{2} > 0$, $m + \frac{1}{2} > 0$ *i.e.* $n > -\frac{1}{2}$, $m > -\frac{1}{2}$

$$\left[\because \int_0^1(1-x)^{\alpha-1}\,x^{\beta-1}\,dx = B(\alpha,\beta) = \frac{\Gamma(a)\,\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \textit{ if } \alpha > 0, \beta > 0\right]$$

$$= \Gamma(n+\tfrac{1}{2})\sum_{m=0}^{\infty}\frac{(-1)^m\,x^{2m}}{(2m)\,!\,\Gamma(n+m+1)}\times\frac{(2m)\,!}{2^{2m}}\times\frac{\sqrt{\pi}}{m\,!}$$

$$\left[\because \text{ By duplication formula,when } m \text{ is a+ve integer,} \Gamma(m+\tfrac{1}{2}) = \frac{(2m)!\sqrt{\pi}}{2^{2m}\,m!}\right]$$

$$= \Gamma(n+\tfrac{1}{2})\sqrt{\pi}\left(\frac{x}{2}\right)^{-n}\sum_{m=0}^{\infty}\frac{(-1)^m}{m\,!\,\Gamma(n+m+1)}\left(\frac{x}{2}\right)^{2m+n} = \Gamma(n+\tfrac{1}{2})\sqrt{\pi}\left(\frac{x}{2}\right)^{-n}J_n(x), \text{ by definition of } J_n(x)$$

$$\therefore \qquad J_n(x) = \frac{(x/2)^n}{\Gamma(n+\frac{1}{2})\sqrt{\pi}}\,I \qquad \text{...(3)}$$

or $$J_n(x) = \frac{(x/2)^n}{\sqrt{\pi}\,\Gamma(n+\frac{1}{2})}\int_{-1}^{1}(1-t^2)^{n-1/2}\,e^{ixt}dt, \text{ by (1).}$$

Part (ii). Since $e^{ixt} = \cos xt + i\sin xt$, (1) gives

$$I = \int_{-1}^{1}(1-t^2)^{n-1/2}\,(\cos xt + i\sin xt)\,dt = \int_{-1}^{1}(1-t^2)^{n-1/2}\cos xt + i\int_{-1}^{1}(1-t^2)^{n-1/2}\sin xt\,dt$$

or $$I = 2\int_0^1 (1-t^2)^{n-1/2} \cos xt\, dt + 0 \qquad ...(4)$$

[$\because$ the integrand in the first integral is an even function of t, while the integrand in the second integral is an odd function of t and it is known that

$$\int_{-a}^{a} f(t)\, dt = \begin{cases} 2\int_0^a f(t)\, dt, \text{ if } f(t) \text{ is even function} \\ 0, \text{ if } f(t) \text{ is odd function} \end{cases}$$

Using (4) and noting that $\sqrt{\pi} = \Gamma(1/2)$, (3) gives

$$J_n(x) = \frac{(x/2)^n}{\Gamma(\frac{1}{2})\,\Gamma(n+\frac{1}{2})} \cdot 2\int_0^1 (1-t^2)^{n-1/2} \cos xt\, dt = \frac{x^n}{2^{n-1}\Gamma(\frac{1}{2})\,\Gamma(n+\frac{1}{2})} \int_0^1 (1-t^2)^{n-1/2} \cos xt\, dt.$$

Ex. 14. (*i*) *Prove that* $J_n J'_{-n} - J'_n J_{-n} = -\dfrac{2 \sin n\pi}{x\pi}$. **[Vikram 2004]**

(*ii*) Prove that $\dfrac{d}{dx}\left(\dfrac{J_{-n}}{J_n}\right) = -\dfrac{2 \sin n\pi}{\pi x\, J_n^2}$. **[Bilaspur 1998]**

(*iii*) Prove that $J'_n(x)\, J_{-n}(x) - J_n(x)\, J'_{-n}(x) = c/x$ where c is a constant. By considering the behaviour for the large values of x, show that $c = (2 \sin n\pi)/\pi$.

(*iv*) Show that the Wronskian $W(J_n, J_{-n})$ of J_n and J_{-n} is given by

$W(J_n, J_{-n}) = -(2/\pi x) \times \sin n\pi$. **[Ravishankar 1998, 2000]**

Sol. (i) We know that J_n and J_{-n} are solutions of Bessel's equation

$$y'' + (1/x)y' + (1 - n^2/x^2)y = 0.$$

$$\therefore \qquad J_n'' + (1/x)J_n' + (1 - n^2/x^2)J_n = 0 \qquad ...(1)$$

and $$J_{-n}'' + (1/x)J_{-n}' + (1 - n^2/x^2)J_{-n} = 0 \qquad ...(2)$$

Multiplying (1) by J_{-n} and (2) by J_n and then subtracting, we have

$$J_n'' J_{-n} - J_{-n}'' J_n + (1/x) \times (J_n' J_{-n} - J_{-n}' J_n) = 0 \qquad ...(3)$$

Let $$J_n' J_{-n} - J_{-n}' J_n = v. \qquad ...(4)$$

Differentating w.r.t. 'x' (4) gives

$$J_n'' J_{-n} + J_n' J_{-n}' - (J_{-n}'' J_n + J_{-n}' J_n') = v' \qquad \text{or} \qquad J_n'' J_{-n} - J_{-n}'' J_n = v'. \quad ...(5)$$

Using (4) and (5), (3) becomes

$v' + (1/x)v = 0$ or $(dv/dx) + (1/x)v = 0$ or $(1/v)dv + (1/x)dx = 0.$

Integrating, $\log v + \log x = \log c$ or $v = c/x$

or $$J'_n J_{-n} - J'_{-n} J_n = c/x, \text{ by (4)} \qquad ...(6)$$

Now, $$J_n(x) = \frac{1}{2^n\,\Gamma(n+1)}\left[x^n - \frac{x^{n+2}}{4(n+1)} + \frac{x^{n+4}}{4 \cdot 8 \cdot (n+1)(n+2)} - ...\right]$$

and $$J_{-n}(x) = \frac{1}{2^{-n}\,\Gamma(-n+1)}\left[x^{-n} - \frac{x^{2-n}}{4(1-n)} + \frac{x^{4-n}}{4 \cdot 8 \cdot (1-n)(2-n)} - ...\right]$$

$\therefore$ Using the above values of J_n and J_{-n}, (6) gives

$$\frac{1}{2\Gamma(n+1)}\left[nx^{n-1}-\frac{(n+2)\,x^{n+1}}{4(n+1)}+\frac{(n+4)\,x^{n+3}}{4\cdot 8\cdot(n+1)(n+2)}-\dots\right]$$

$$\times\frac{1}{2^{-n}\,\Gamma(-n+1)}\left[x^{-n}-\frac{x^{2-n}}{4(1-n)}+\frac{x^{4-n}}{4\cdot 8\cdot(1-n)(2-n)}-\dots\right]$$

$$-\frac{1}{2^{-n}\,\Gamma(-n+1)}\left[-n\,x^{-n-1}-\frac{(2-n)\,x^{1-n}}{4(1-n)}+\frac{(4-n)\,x^{3-n}}{4\cdot 8\cdot(1-n)(2-n)}-\dots\right]$$

$$\times\frac{1}{2^{n}\,\Gamma(n+1)}\left[x^{n}-\frac{x^{n+2}}{4(n+1)}+\frac{x^{n+4}}{4\cdot 8\cdot(n+1)(n+2)}-\dots\right]=\frac{c}{x}$$

Now comparing the coefficients of $1/x$ from both sides, we get

$$\frac{n}{2^n\Gamma(n+1)\cdot 2^{-n}\Gamma(-n+1)}+\frac{n}{2^{-n}\Gamma(-n+1)\cdot 2^{n}\Gamma(n+1)}=c \qquad \text{or} \qquad c=\frac{2n}{n\Gamma(n)\,\Gamma(1-n)}$$

or $$c=\frac{2}{(\pi/\sin n\pi)}=\frac{2\sin n\pi}{\pi} \qquad \left[\because\ \Gamma(n)\,\Gamma(1-n)=\frac{\pi}{\sin n\pi}\right]$$

Putting this value of c in (6) and multiplying both sides by (–1), we get

$$J_n\,J'_{-n}-J'_nJ_{-n}=-\frac{2\sin n\pi}{\pi x}. \qquad \text{...(7)}$$

Part (ii). Dividing both sides of (7) by $J_n^{\,2}$, we get

$$\frac{J_nJ'_{-n}-J'_nJ_{-n}}{J_n^2}=-\frac{2\sin n\pi}{\pi x\,J_n^2} \qquad \text{or} \qquad \frac{d}{dx}\left(\frac{J_{-n}}{J_n}\right)=-\frac{2\sin n\pi}{\pi x\,J_n^2}.$$

Part (iii) Refer part (i).

(*iv*) By definition, $W(J_n,J_{-n})=\begin{vmatrix}J_n & J_{-n}\\ J'_n & J'_{-n}\end{vmatrix}=J_n\,J'_{-n}-J_n{}'\,J_{-n}=-(2/\pi x)\times\sin n\pi$, by (7)

Ex. 15(a). *Prove that* $J_n(x)=(-2)^n\,x^n\,\dfrac{d^nJ_0(x)}{d(x^2)^n}$. **[Ranchi 2010]**

(***b***) *Show that if* $J_n(x)=kx^n\left(\dfrac{d}{dx^2}\right)^n J_0(x)$, *then* $k=-(2)^n$. **[Ravishankar 1998]**

Sol. (*a*) We know that $J_0(x)$ is a solution of Bessel's equation of order zero (*i.e.* $n = 0$) namely,

$$d^2y/dx^2+(1/x)\times(dy/dx)+y=0. \qquad \text{...(1)}$$

Let $x^2=X$ so that $x=\sqrt{X}$ and $dX/dx=2x=2\sqrt{X}$. ...(2)

$$\therefore \qquad \frac{dy}{dx}=\frac{dy}{dX}\frac{dX}{dx}=2\sqrt{X}\frac{dy}{dX},\ \text{by (2)} \qquad \text{...(3)}$$

From (3) $$\frac{d}{dx}\equiv 2\sqrt{X}\frac{d}{dX}. \qquad \text{...(3)}'$$

$$\therefore \qquad \frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=2\sqrt{X}\frac{d}{dX}\left(2\sqrt{X}\frac{dy}{dx}\right),\ \text{by (3) and (3)}'$$

$$=4\sqrt{X}\frac{d}{dX}\left(\sqrt{X}\frac{dy}{dX}\right)=4\sqrt{X}\left[\frac{1}{2}(X)^{-1/2}\frac{dy}{dX}+\sqrt{X}\frac{d^2y}{dX^2}\right]=4X\frac{d^2y}{dX^2}+2\frac{dy}{dX} \qquad \text{... (4)}$$

Putting the values of x, dy/dx and d^2y/dx^2 from (2), (3) and (4) in (1), we get

$$4X\frac{d^2y}{dX^2}+2\frac{dy}{dX}+\frac{1}{\sqrt{X}}.2\sqrt{X}\frac{dy}{dX}+y=0 \qquad \text{or} \qquad 4\sqrt{X}\frac{d^2y}{dX^2}+4\frac{dy}{dX}+y=0. \quad ...(5)$$

Differentiation of (5) n times w.r.t. 'X' by Leibnitz theorem gives

$$4\left[X\frac{d^{n+2}y}{dX^{n+2}}+n.1\frac{d^{n+1}y}{dX^{n+1}}\right]+4\frac{d^{n+1}y}{dX^{n+1}}+\frac{d^ny}{dX^n}=0 \ \text{ or } \ 4X\frac{d^{n+2}y}{dX^{n+2}}+4(n+1)\frac{d^{n+1}y}{dX^{n+1}}+\frac{d^ny}{dX^n}=0 \quad ....(6)$$

Since $y=J_0(x)$ is a solution of (1), we now take

$$Y=\frac{d^ny}{dX^n}=\frac{d^nJ_0(x)}{d(x^2)^n} \qquad \text{so that} \qquad \frac{dY}{dX}=\frac{d^{n+1}y}{dX^{n+1}} \quad \text{and} \quad \frac{d^2y}{dX^2}=\frac{d^{n+2}y}{dX^{n+2}} \quad ...(7)$$

$\therefore$ (6) gives

$$4X\frac{d^2y}{dX^2}+4(n+1)\frac{dy}{dX}+X=0. \quad ...(8)$$

But $J_n(x)$ is a solution of Bessel equation $\dfrac{d^2y}{dx^2}+\dfrac{1}{x}\dfrac{dy}{dx}+\left(1-\dfrac{n^2}{x^2}\right)y=0.$ ...(9)

Let $y=x^nz$ so that $J_n(x)=x^nz.$ ...(10)

Differentiating (10), $\dfrac{dy}{dx}=x^n\dfrac{dz}{dx}+nx^{n-1}z$ and $\dfrac{d^2y}{dx^2}=x^n\dfrac{d^2z}{dx^2}+2nx^{n-1}\dfrac{dz}{dx}+n(n-1)z^{n-2}z.$

Using the above values of y, dy/dx and d^2y/dx^2, (9) gives

$$x^n\frac{d^2z}{dx^2}+2nx^{n-1}\frac{dz}{dx}+n(n-1)x^{n-2}z+\frac{1}{x}\left(x^n\frac{dz}{dx}+nx^{n-1}z\right)+\left(1-\frac{n^2}{x^2}\right)x^nz=0$$

or
$$x^n\frac{d^2z}{dx^2}+(2n+1)\frac{x^n}{x}.\frac{dz}{dx}+x^nz=0 \qquad \text{or} \qquad \frac{d^2z}{dx^2}+(2n+1)\frac{1}{x}\frac{dz}{dx}+z=0. \quad ...(11)$$

Using (2) and also (3) and (4) [after replacing y by z here], (11) gives

$$\left(4X\frac{d^2z}{dX^2}+2\frac{dz}{dX}\right)+\frac{(2n+1)}{\sqrt{x}}.2\sqrt{X}\frac{dz}{dX}+z=0$$

or
$$4X(d^2z/dX^2)+4(n+1)(dz/dX)+z=0. \quad ...(12)$$

Comparing (8) and (12), we have $z=kY$, where k is a constant to be determined. Using (7) and (10), we have

$$x^nz=kx^nY \qquad \text{or} \qquad J_n(x)=kx^n\frac{d^nJ_0(x)}{d(x^2)^n}. \quad ...(13)$$

But
$$J_0(x)=\sum_{r=0}^{\infty}\frac{(-1)^r x^{2r}}{(2^r\, r!)^2}=\sum_{r=0}^{\infty}\frac{(-1)^r X^r}{(2^r\, r!)^2} \qquad (\because\ x^2=X)$$

Differentiating both sides w.r.t. x^2 (*i.e.* X) n times, we have

$$\frac{d^nJ_0(x)}{d(x^2)^n}=\sum_{r=0}^{\infty}\frac{(-1)^r}{(2^r\, r!)^2}\frac{d^n}{dX^n}X^r \quad ...(14)$$

$$=\frac{(-1)^n}{(2^n\, n!)^2}n!+\text{terms involving } X\ (i.e.\ x^2)$$

(Note that on differentiation all those terms in R.H.S. of (14) for which $r < n$ will vanish).

Using value of $J_n(x)$ and the above expression, (13) gives

$$\frac{x^n}{2^n \Gamma(n+1)}\left[1-\frac{x^2}{2 \cdot 2(n+1)}+\ldots\right] = kx^n\left[\frac{(-1)^n\, n\,!}{(2^n\, n\,!)^2} + \text{terms contaning } x^2\right]$$

Equaing the coefficients of x^n from both sides, we get

$$\frac{1}{2^n\Gamma(n+1)} = k\frac{(-1)^n\ n\,!}{(2^n n\,!)^2} \qquad \text{or} \qquad k = (-1)^n\, 2^n = (-2)^n \qquad \text{... (15)}$$

[$\because$ $\Gamma(n + 1) = n\,!$, n being +ve integer]

With this value of k, (13) gives $\qquad J_n(x) = (-2)^n x^n \dfrac{d^n J_0(x)}{d(x^2)^n}$.

(*b*) Proceed as in part (*a*). From (15), we have $k = (-2)^n$

EXERCISE 11 (A)

1. Show that $y = x^{-n/2}\, J_n(2\sqrt{x}\,)$ satisfies the equation $xy'' + (n + 1)y' + y = 0$.

2. Show that $\displaystyle\int_0^{\pi/2} \sqrt{(\pi x)}\ J_{1/2}(2x)\, dx = 1$.

3. (*a*) Prove : $\displaystyle\int_0^{\infty} J_n(bx)x^n e^{-ax} dx = \frac{2^n\Gamma(n+\frac{1}{2})}{\sqrt{\pi}} \cdot \frac{b^n}{(a^2+b^2)^{n+1/2}}$

(*b*) Prove that $\displaystyle\int_0^{\infty} J_n(bx)\, x^{n+1}\, e^{-ax} dx = \frac{2^{n+1}\Gamma(n+\frac{1}{2})}{\pi} \cdot \frac{ab^n}{(a^2+b)^{n+3/2}},\ a > 0.$

4. Prove : $\displaystyle\int_0^{\infty} J_n(bx)\, x^{n+1} e^{-ax^2} dx = \frac{b^n}{(2a)^{n+1}} \exp(-b^2/4a)$, where $\exp(p) = e^p$

5. Prove that $\displaystyle\int_0^{\infty} J_n(bx) x^{n+1} e^{-ax^2} dx = \frac{b^n}{2^{n+1}a^{n+2}}\left(n+1-\frac{b^2}{4a}\right)\exp\left(-\frac{b^2}{4a}\right), a>0.$

6. For what value of n the general solution of Bessel's differential equation will be of the form $y = AJ_n(x) + BJ_n(x)$.

7. Write the differential equation satisfied by Bessel's function of order n. Express the following Bessel's functions in terms of trigonometric functions :

(*i*) $J_{1/2}(x)$, (*ii*) $J_{-1/2}(x)$, (*iii*) $J_{3/2}(x)$, (*iv*) $J_{-3/2}(x)$.

11.7. Recurrence Relations (Formulae) for $J_n(x)$. Prove that

I. $\dfrac{d}{dx}\{x^n J_n(x)\} = x^n J_{n-1}(x)$. **[Agra 2005; Guwahati 2007; Kanpur 2004, 09; Nagpur 1996]**

II. $\dfrac{d}{dx}\{x^{-n}J_n(x)\} = -x^{-n}J_{n+1}(x)$. **[Agra 2008; Gulbarga 2005; Kanpur 2009]**

III. $J'_n(x) = J_{n-1}(x) - (n/x)\, J_n$ or $xJ'_n = -nJ_n + xJ_{n-1}$. **[Agra 2010; Bilaspur 2004; KU Kurukshetra 2005; Agra 1997; Kanpur 2005, 09, 11; Meerut 2010; Nagpur 2010]**

IV. $J'_n(x) = (n/x)\, J_n(x) - J_{n+1}(x)$ or $xJ'_n = nJ_n - xJ_{n+1}$. **[Nagpur 2005; Bangalore 93, 94; Meerut 2005, 07, 11; Kanpur 1999; Bilaspur 1998]**

V. $J'_n(x) = \frac{1}{2}\{J_{n-1}(x) - J_{n+1}(x)\}$ or $J_{n-1} - J_{n+1} = 2\, J'_n$. **Agra 2009; Jiwaji 2004, 07; Ravishankar 2004; Nagpur 1995; Kanpur 2007]**

VI. $J_{n-1}(x) + J_{n+1}(x) = (2n/x)J_n(x)$ or $xJ_{n+1}(x) + xJ_{n-1}(x) = 2nJ_n(x)$ or $2J_n = x(J_{n-1} + J_{n+1})$. **[Agra 2008; Bangalore 2005; Meerut 2006; Kanpur 2006, 11; Nagpur 2010]**

Proof 1. Using the definition of $J_n(x)$, we have

$$\frac{d}{dx}\{x^n J_n(x)\} = \frac{d}{dx}\left\{x^n \sum_{r=0}^{\infty}(-1)^r \frac{1}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{2r+n}\right\}$$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma(n+r+1)}\cdot\frac{1}{2^{2r+n}}\cdot\frac{d}{dx}(x^{2r+2n}) = \sum_{r=0}^{\infty}\frac{(-1)^r\,(2r+2n)\,x^{2r+2n-1}}{r!\Gamma(n+r+1)\,2^{2r+n}}$$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r\cdot 2\cdot(r+n)}{r!\,(n+r)\,\Gamma(n+r)}\cdot\frac{x^n\cdot x^{2r+n-1}}{2^{2r+n}} \qquad [\because\ \Gamma(n+1) = n\Gamma(n)]$$

$$= x^n\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma(n-1+r+1)}\left(\frac{x}{2}\right)^{2r+n-1} = x^n J_{n-1}(x).$$ by the definition of $J_{n-1}(x)$.

II. Using the definition of $J_n(x)$, we have

$$\frac{d}{dx}\{x^{-n}J_n(x)\} = \frac{d}{dx}\left\{x^{-n}\sum_{r=0}^{\infty}(-1)^r\frac{1}{r!\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{2r+n}\right\}$$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r}{r!\Gamma(n+r+1)}\cdot\frac{1}{2^{2r+n}}\cdot\frac{d}{dx}(x^{2r}) = \sum_{r=0}^{\infty}\frac{(-1)^r\,2rx^{2r-1}}{r(r-1)!\Gamma(n+r+1)\,2^{2r+n}}$$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r\,x^{2r-1}}{(r-1)!\Gamma(n+r+1)}\cdot\frac{1}{2^{2r+n-1}} = \sum_{r=1}^{\infty}\frac{(-1)^r\,x^{2r-1}}{(r-1)!\Gamma(n+r+1)}\cdot\frac{1}{2^{2r+n-1}}$$

(since $(r-1)! = \infty$ when $r = 0$ so the term corresponding to $r = 0$ vanishes)

$$= \sum_{m=0}^{\infty}\frac{(-1)^{m+1}\,x^{2m+2-1}}{m!\Gamma(n+m+2)}\cdot\frac{x^n\cdot x^{-n}}{2^{2m+2+n-1}},$$ (on changing the variable of summation to $m = r - 1$

so that $r = m + 1$. Then $m = 0$ when $r = 1$ and $m = \infty$ when $r = \infty$)

$$= -x^{-n}\sum_{m=0}^{\infty}\frac{(-1)^m}{m!\Gamma(n+m+2)}\left(\frac{x}{2}\right)^{n+1+2m} = -x^{-n}\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\Gamma(n+1+r+1)}\left(\frac{x}{2}\right)^{n+1+2r}$$

(on changing the variable of summation from m to r)

$= -x^{-n}J_{n+1}(x)$, by the definition of $J_{n+1}(x)$.

III. Recurrence relation I is

$\frac{d}{dx}\{x^n J_n(x)\} = x^n J_{n-1}(x)$ or $n\,x^{n-1}J_n(x) + x^n J'_n(x) = x^n J_{n-1}(x).$

Dividing both sides by x^{n-1}, $n\,J_n(x) + x\,J'_n(x) = x\,J_{n-1}(x)$

or $(n/x)\,J_n(x) + J_n'(x) = J_{n-1}(x)$ or $J'_n(x) = J_{n-1}(x) - (n/x)\,J_n(x)$

IV. Recurrence relation II is

$\frac{d}{dx}\{x^{-n}\,J_n(x)\} = -x^{-n}\,J_{n+1}(x)$ or $-n\,x^{-n-1}J_n(x) + x^{-n}\,J'_n(x) = -x^{-n}J_{n+1}(x)$

Dividing both sides by x^{-n}, $n\,x^{-1}\,J_n(x) + J'_n(x) = -J_{n+1}(x)$

or $(-n/x)\,J_n + J'_n = J_{n+1}$ or $J'_n = (n/x)\,J_n - J_{n+1}$.

V. From recurrence relations III and IV, we have $J'_n(x) = J_{n-1}(x) - (n/x)J_n(x)$ ...(1)

and $J'_n(x) = (n/x)\, J_n(x) - J_{n+1}(x)$. ...(2)

Adding (1) and (2), $2J'_n(x) = J_{n-1}(x) - J_{n+1}(x)$ or $J'_n(x) = \frac{1}{2}\{J_{n-1}(x) - J_{n+1}(x)\}$.

VI. From recurrence relations III and IV, we have $J'_n(x) = J_{n-1}(x) - (n/x)J_n(x)$ ...(1)

and $J'_n(x) = (n/x)\, J_n(x) - J_{n+1}(x)$. ...(2)

Subtracting (2) from (1), we get

$0 = J_{n-1} + J_{n+1} - 2(n/x)\, J_n$ or $J_{n-1} + J_{n+1} = (2n/x)\, J_n$.

11.7. A. Solved examples based on recurrence relations

Ex. 1(*a*). *Show that* $x^n J_n(x)$ is a solution of $x(d^2y/dx^2) + (1-2n) \times (dy/dx) + xy = 0$.

[CDLU 2004]

(*b*) *Show that* $x^{-n} J_n(x)$ is a solution of $x(d^2y/dx^2) + (1+2n) \times (dy/dx) + xy = 0$.

[MDU Rohtak 2005]

Sol. (*a*) Given $x(d^2y/dx^2) + (1-2n) \times (dy/dx) + xy = 0$ ... (1)

Let $y = x^n J_n(x)$ ... (2)

From recurrence relation I, $\dfrac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$

or $dy/dx = x^n J_{n-1}$, using (2) ... (3)

From (3), $d^2y/dx^2 = x^n J'_{n-1} + n\, x^{n-1} J_{n-1}$

Substituting the above values of y, dy/dx and d^2y/dx^2 in (1), we get

$$x(x^n J'_{n-1} + nx^{n-1} J_{n-1}) + (1-2n)x^n J_{n-1} + x^{n+1} J_n = 0 \quad \text{or} \quad x^{n+1} J'_{n-1} - (n-1)x^n J_{n-1} + x^{n+1} J_n = 0$$

or $$x^{n+1}\left[J'_{n-1} - \{(n-1)/x\} J_{n-1}\right] + x^{n+1} J_n = 0 \qquad \text{... (4)}$$

From recurrence relation VI, we have

$$J'_n(x) = \frac{n}{x} J_n - J_{n+1}(x) \qquad \text{so that} \qquad J'_{n-1} - \frac{n-1}{x} J_{n-1} = -J_n(x) \qquad \text{... (5)}$$

Using (5), (4) reduces to $-x^{n+1} J_n + x^{n+1} J_n = 0$, *i.e.*, $0 = 0$, which is true.

Hence $x^n J_n$ is a solution of (1).

(*b*) Given $x(d^2y/dx^2) + (1+2n) \times (dy/dx) + xy = 0$ ... (1)

Let $y = x^{-n} J_n(x)$ ... (2)

From recurrence relation II, $\dfrac{d}{dx}\left\{x^{-n} J_n(x)\right\} = -x^{-n} J_{n+1}(x)$

or $dy/dx = -x^{-n} J_{n+1}$, using (2)

From (3), $d^2y/dx^2 = -x^{-n} J'_{n+1} + n\, x^{-n-1} J_{n+1}$ ... (3)

Substituting the above values of y, dy/dx and d^2y/dx^2 in (1), we get

$$x(-x^{-n} J'_{n+1} + nx^{-n-1} J_{n+1}) + (1+2n) \times (-x^{-n} J_{n+1}) + x^{-n+1} J_n = 0$$

or $-x^{-n+1} J'_{n+1} - (n+1)x^{-n} J_{n+1} + x^{-n+1} J_n = 0$ or $-x^{-n+1}\left[J'_{n+1} + \{(n+1)/x\} J_{n+1}\right] + x^{-n+1} J_n = 0$ ... (4)

From recurrence relation III, we have

$$J'_n(x) = J_{n-1}(x) - \frac{n}{x} J_n \qquad \text{so that} \qquad J'_{n+1} + \frac{n+1}{x} J_{n+1} = J_n(x) \qquad \text{... (5)}$$

Using (5), (4) reduce to $-x^{-n+1} J_n + x^{-n+1} J_n = 0$, *i.e.*, $0 = 0$, which is true. Hence $x^{-n} J_n$ is a solution of (1).

Ex. 2. (Lommel theorems) Prove that

$$(i)\ \left(\frac{1}{x}\frac{d}{dx}\right)^m \left(x^n J_n\right) = x^{n-m} J_{n-m}, \text{ where } m \text{ is positive integer and } m < n.$$

$$(ii)\ \left(\frac{1}{x}\frac{d}{dx}\right)^m (x^{-n} J_n) = (-1)^m x^{-n-m} J_{n+m}$$

$$(iii)\ J_n(x) = (-1)^n x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n J_0(x), \ n \text{ being positive integer.}$$

Sol. (*i*) From recurrence relation I, we have

$$\frac{d}{dx}(x^n J_n) = x^n J_{n-1} \qquad \text{so that} \qquad \left(\frac{1}{x}\frac{d}{dx}\right)(x^n J_n) = x^{n-1} J_{n-1} \qquad \ldots (1)$$

$$\text{Now, } \left(\frac{1}{x}\frac{d}{dx}\right)^m (x^n J_n) = \left(\frac{1}{x}\frac{d}{dx}\right)^{m-1} \left(\frac{1}{x}\frac{d}{dx}\right)(x^n J_n) = \left(\frac{1}{x}\frac{d}{dx}\right)^{m-1} (x^{n-1} J_{n-1}), \text{ using (1)}$$

....

$$= x^{n-m} J_{n-m}, \text{ on preceeding as before } m \text{ times more}$$

(*ii*) From recurrence relation II, we have

$$\frac{d}{dx}(x^{-n} J_n) = -x^{-n} J_{n+1} \qquad \text{so that} \qquad \left(\frac{1}{x}\frac{d}{dx}\right)(x^{-n} J_{n+1}) = (-1)^1 x^{-n-1} J_{n+1} \qquad \ldots (2)$$

$$\text{Now, } \left(\frac{1}{x}\frac{d}{dx}\right)^m (x^{-n} J_n) = \left(\frac{1}{x}\frac{d}{dx}\right)^{m-1} \left(\frac{1}{x}\frac{d}{dx}\right)(x^{-n} J_n) = \left(\frac{1}{x}\frac{d}{dx}\right)^{m-1} (-1)^1 x^{-n-1} J_{n+1}, \text{ using (2)}$$

....

$$= (-1)^m x^{-n-m} J_{n+m}, \text{ on proceeding as before } m \text{ times more} \qquad \ldots (3)$$

(*iii*) Replacing n by 0 and m by n in part (*ii*), we get

$$\left(\frac{1}{x}\frac{d}{dx}\right)^n J_0 = (-1)^n x^{-n} J_n \qquad \text{or} \qquad J_n(x) = (-1)^n x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n J_0(x)$$

Ex. 3. *Prove that* $\int_0^1 t\{J_n(t)\}^2 \, dt = \frac{1}{2} x^2 \{J_n^2(x) - J_{n-1}(x) J_{n+1}(x)\}$.

Sol. We have, $\frac{d}{dt}\left[\frac{t^2}{2}\{J_n^2(t) - J_{n-1}(t) J_{n+1}(t)\}\right]$

$$= t\{J_n^2(t) - J_{n-1}(t) J_{n+1}(t)\} + \tfrac{1}{2} t^2 \{2J_n(t) J'_n(t) - J'_{n-1}(t) J_{n+1}(t) - J_{n-1}(t) J'_{n+1}(t)\}$$

$$= t\{J_n^2(t) - J_{n-1}(t) J_{n+1}(t)\} + \tfrac{1}{2} t^2 \{2J_n(t) \times \tfrac{1}{2}\{J_{n-1}(t) - J_{n+1}(t)\} - J_{n+1}\left\{\frac{n-1}{t} J_{n-1}(t) - J_n(t)\right\}$$

$$- J_{n-1}(t)\left\{J_n(t) - \frac{n+1}{t} J_{n+1}(t)\right\}, \text{ using recurrence relations III, IV and V}$$

$= t\, J_n^2(t)$, on simplification.

$$\therefore \qquad tJ_n^{\,2}(t) = \frac{d}{dt}\left[\frac{t^2}{2}\{J_n^2(t) - J_{n-1}(t)\, J_{n+1}(t)\}\right]. \qquad ...(1)$$

Integrating both sides of (1) w.r.t. 'x' from 0 to x, we get

$$\int_0^x t\, J_n^2(t)\, dt = \left[\frac{t^2}{2}\{J_n^2(t) - J_{n-1}(t)\, J_{n+1}(t)\}\right]_0^x = \frac{x^2}{2}\{J_n^{\,2}(x) - J_{n-1}(x)\, J_{n+1}(x)\}$$

$$[\because\ J_n(0) = J_{n+1}(0) = J_{n-1}(0) = 0]$$

Ex. 4. *Prove that* (*i*) $J_{-3/2}(x) = \sqrt{\left(\frac{2}{\pi x}\right)}\left(\frac{-\cos x}{x} - \sin x\right)$ **[Kanput 2005, 10]**

(*ii*) $J_{3/2}(x) = \sqrt{\left(\frac{2}{\pi x}\right)}\left(\frac{\sin x}{x} - \cos x\right)$ *i.e.* $\sqrt{\left(\frac{\pi x}{2}\right)}\, J_{3/2}(x) = \frac{\sin x}{x} - \cos x$.

[Purvanchal 2005, Agra 2005, Kakitiya 1997; Kanpur 2008, 11; Bangalore 1997; Meerut 2007; KU Kurukshetra 2004]

Sol. Proceed as in Ex. 1 of Art. 11.6A and prove that

$$J_{-1/2}(x) = \sqrt{(2/\pi x)}\ \cos x \qquad ...(1)$$

and

$$J_{1/2}(x) = \sqrt{(2/\pi x)}\ \sin x. \qquad ...(2)$$

Recurrence relation (VI) is $\quad J_{n-1}(x) + J_{n+1}(x) = (2n/x) \times J_n(x). \qquad ...(3)$

Part (i). Replacing n by $-\frac{1}{2}$ in (3), we have $\quad J_{-3/2}(x) + J_{1/2}(x) = -(2/2x) \times J_{-1/2}(x)$

or $\quad J_{-3/2}(x) = -J_{1/2}(x) - (1/x)\times J_{-1/2}(x) = -\sqrt{\left(\frac{2}{\pi x}\right)}\ \sin x - \frac{1}{x}\sqrt{\left(\frac{2}{\pi x}\right)}\ \cos x$, by (1) and (2)

$$= \sqrt{\left(\frac{2}{\pi x}\right)}\left(-\frac{\cos x}{x} - \sin x\right).$$

Part (ii). Replacing n by $\frac{1}{2}$ in (3), we have $\quad J_{-1/2}(x) + J_{3/2}(x) = (2/2x) \times J_{1/2}(x)$

or $\quad J_{3/2}(x) = -J_{-1/2}(x) + (1/x) \times J_{1/2}(x) = -\sqrt{\left(\frac{2}{\pi x}\right)}\ \cos x + \frac{1}{x}\sqrt{\left(\frac{2}{\pi x}\right)}\ \sin x$, by (1) and (2)

$$= \sqrt{\left(\frac{2}{\pi x}\right)}\left(\frac{\sin x}{x} - \cos x\right).$$

Ex. 5. *Prove that* (*i*) $J_{5/2}(x) = (2/\pi x)^{1/2}\left\{\left\{(3-x^2)/x^2\right\}\sin x - (3/x)\times\cos x\right\}$

[Kanpur 2006, 07]

(*ii*) $J_{-5/2}(x) = (2/\pi x)^{1/2}\left\{\left\{(3-x^2)/x^2\right\}\cos x + (3/x)\times\sin x\right\}$ **[KU Kurukshetra 2006]**

Sol. (*i*) From recurrence relation VI, we have

$J_n(x) = (x/2n)\ \{J_{n-1}(x) + J_{n-1}(x)\}$ or $J_{n+1} = (2n/x)J_n(x) - J_{n-1}(x)$ ... (1)

Putting $n = 3/2$ in (1), we have,

$$J_{5/2}(x) = \frac{3}{x}J_{3/2}(x) - J_{1/2}(x) = \frac{3}{x}\times\sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right) - \sqrt{\frac{2}{\pi x}}\sin x$$

[using values of $J_{3/2}(x)$ and $J_{1/2}(x)$ as obtained in Ex. 4 (*ii*), Art. 11.7A and Ex. 1 (*ii*), Art 11.6A]

$$= \sqrt{\frac{2}{\pi x}}\left(\frac{3\sin x}{x^2} - \frac{3\cos x}{x} - \sin x\right) = \sqrt{\frac{2}{\pi x}}\left(\frac{3-x^2}{x^2}\sin x - \frac{3\cos x}{x}\right)$$

(*ii*) Re-writing (1), $J_{n-1}(x) = (2n/x)\, J_n(x) - J_{n+1}(x)$ ... (2)

Putting $n = -3/2$ in (1), we have

$$J_{-5/2}(x) = -\frac{3}{x} J_{-3/2}(x) - J_{-1/2}(x) = -\frac{3}{x} \times \sqrt{\frac{2}{\pi x}} \left(-\frac{\cos x}{x} - \sin x \right) - \sqrt{\frac{2}{\pi x}} \cos x$$

[using values of $J_{-3/2}(x)$ and $J_{-1/2}(x)$ as obtained in Ex. 4 (*i*), Art. 11.7A and Ex. 1 (*i*), Art. 11.6A]

$$= \left(\frac{2}{\pi x} \right)^{1/2} \left(\frac{3\cos x}{x^2} + \frac{3\sin x}{x} - \cos x \right) = \left(\frac{2}{\pi x} \right)^{1/2} \left(\frac{3 - x^2}{x^2} \cos x + \frac{3\sin x}{x} \right)$$

Ex. 6. *Express* $J_4(x)$ in terms of J_0 and J_1.

Sol. Recurrence relation VI is $J_{n+1}(x) = (2n/x)J_n(x) - J_{n-1}(x)$. ...(1)

Replacing n by 3 in (1), we get $J_4(x) = (6/x)J_3(x) - J_2(x)$. ...(2)

Now replacing n by 2 in (1), we get $J_3(x) = (4/x)J_2(x) - J_1(x)$. ...(3)

Using (3), (2) becomes

$$J_4(x) = \frac{6}{x}\left[\frac{4}{x} J_2(x) - J_1(x)\right] - J_2(x) \quad \text{or} \quad J_4(x) = \left(\frac{24}{x^2} - 1\right) J_2(x) - \frac{6}{x} J_1(x). \quad ...(4)$$

Next, replacing n by 1 in (1) gives $J_2(x) = (2/x)J_1(x) - J_0(x)$. ...(5)

Using (5), (4) becomes

$$J_4(x) = \left(\frac{24}{x^2} - 1\right)\left[\frac{2}{x} J_1(x) - J_0(x)\right] - \frac{6}{x} J_1(x) \quad \text{or} \quad J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x}\right) J_1(x) - \left(\frac{24}{x^2} - 1\right) J_0(x).$$

Ex. 7(a). *Prove that* (*i*) $\frac{1}{2} x J_n = (n+1)J_{n+1} - (n+3)J_{n+3} + (n+5)J_{n+5} \ldots$.

(*ii*) $J_{n-1} = (2/x)\,[n\, J_n - (n+2)J_{n+2} + (n+4)J_{n+4} - \ldots]$.

Proof (i). Recurrence relation VI is $2nJ_n = x(J_{n-1} + J_{n+1})$.

Replacing n by $n+1$ in the above relation, we get

$2(n+1)J_{n+1} = x(J_n + J_{n+2})$ or $\frac{1}{2} x J_n = (n+1)\, J_{n+1} - \frac{1}{2} x J_{n+2}$. ...(1)

Replacing n by $n+2$ in (1), we get $\frac{1}{2} x J_{n+2} = (n+3)J_{n+3} - \frac{1}{2} x J_{n+4}$. ...(2)

Putting the value of $\frac{1}{2} x J_{n+2}$ from (2) in (1), $\frac{1}{2} x J_n = (n+1)J_{n+1} - (n+3)J_{n+3} + \frac{1}{2} x J_{n+4}$. ...(3)

Repalcing n by $n+4$ in (1) gives $\frac{1}{2} x J_{n+4} = (n+5)J_{n+5} - \frac{1}{2} x J_{n+6}$. ...(4)

Putting the value of $\frac{1}{2} x J_{n+4}$ from (4) in (3) gives

$$\frac{1}{2} x J_n = (n+1)J_{n+1} - (n+3)J_{n+3} + (n+5)J_{n+5} - \frac{1}{2} x J_{n+6}$$

Proceeding likewise and noting that $J_n \to 0$ as $n \to \infty$, we get

$$\frac{1}{2} x J_n = (n+1)J_{n+1} - (n+3)J_{n+3} + (n+5)J_{n+5} - \ldots \quad ...(5)$$

(*ii*) Replacing n by $n-1$ in (5) and then multiplying both sides by $(2/x)$, we get

$$J_{n-1} = (2/x)\,[nJ_n - (n+2)J_{n+2} + (n+4)J_{n+4} - \ldots]$$

Ex. 7(b). *Prove that* $J_{n-1} = (2/x)[nJ_n - (n+2)J_{n+2} + (n+4)J_{n+4} - \ldots]$ and hence deduce that $(x/2)J_n = (n+1)J_{n+1} - (n+3)J_{n+3} - (n+5)J_{n+5} + \ldots$ **[Meerut 1993]**

Hint : Proceed as in Ex. 7(*a*).

Ex. 8. *Prove that* $J'_n = (2/x)\left[(n/2)J_n - (n+2)J_{n+2} + (n+4)J_{n+4} - \ldots\right]$

Sol. Recurrence relation III is $xJ'_n = -nJ_n + xJ_{n-1}$ or $J'_n = -(n/x)J_n + J_{n-1}$. ...(1)

From Ex. 7(a) part (*ii*), $J_{n-1} = (2/x)[nJ_n - (n+2)J_{n+2} + (n+4)J_{n+4} - \ldots]$ ... (2)

Putting the value of J_{n-1} from (2) in (1), we get

$$J_n' = -(n/x)J_n + (2/x)\,[nJ_n - (n+2)J_{n+2} + (n+4)\,J_{n+4} - \ldots]$$

or
$$J'_n = (2/x)\Big[(n/2)\,J_n - (n+2)J_{n+2} + (n+4)J_{n+4} - \ldots\Big].$$

Ex. 9. *Prove* : $\dfrac{d}{dx}(J_n^2 + J^2_{n+1}) = 2\left(\dfrac{n}{x}J_n^2 - \dfrac{n+1}{x}J_{n+1}^2\right)$ **[Agra 2005; Meerut 1998]**

Sol. From recurrence relation III and IV, we have

$$J'_n = -(n/x)J_n + J_{n-1} \qquad \ldots(1)$$

and
$$J'_n = (n/x)J_n - J_{n+1}. \qquad \ldots(2)$$

Replacing n by $n + 1$ in (1), we get $\quad J'_{n+1} = -\dfrac{n+1}{x}J_{n+1} + J_n. \qquad \ldots(3)$

$$\frac{d}{dx}\,(J^2_n + J^2_{n+1}) = 2J_nJ'_n + 2J_{n+1}J'_{n+1} = 2J_n\left(\frac{n}{x}J_n - J_{n+1}\right) + 2J_{n+1}\left(-\frac{n+1}{x}J_{n+1} + J_n\right), \text{ by (2) and (3)}$$

$$= 2\left(\frac{n}{x}J_n^2 - \frac{n+1}{x}J_{n+1}^2\right), \text{ on simplification.}$$

Ex. 10. *Prove that* (*i*) $J_0^2 + 2(J_1^2 + J^2_2 + J_3^2 + \ldots) = 1.$

[Agra 2009, 10; Meerut 2008; Kanpur 2011]

(*ii*) $|\,J_0(x)\,| \leq 1$ **[Agra 2009; Meerut 1996, 97, 98; Kanpur 2011]**

(*iii*) $|\,J_n(x)\,| \leq 2^{-1/2}$, *when* $n \geq 1$. **[Meerut 1996, 97, 98; Kanpur 2011]**

Sol. (i) From Ex. 9 above, $\quad \dfrac{d}{dx}\,(J^2_n + J^2_{n+1}) = 2\left(\dfrac{n}{x}J_n^2 - \dfrac{n+1}{x}J_{n+1}^2\right). \qquad \ldots(1)$

Replacing n by 0, 1, 2, 3 ... successively in (1), we get

$$\frac{d}{dx}(J_0^2 + J_1^2) = 2\left(0 - \frac{1}{x}J_1^2\right)$$

$$\frac{d}{dx}(J_1^2 + J_2^2) = 2\left(\frac{1}{x}J_1^2 - \frac{2}{x}J_2^2\right)$$

$$\frac{d}{dx}(J_2^2 + J_3^2) = 2\left(\frac{2}{x}J_2^2 - \frac{3}{x}J_3^2\right)$$

$$\ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots$$

Adding these columnwise and noting that $J_n \to 0$ as $n \to \infty$, we get

$$\frac{d}{dx}[J_0^2 + 2(J_1^2 + J_2^2 + \ldots)] = 0$$

Integrating, $\quad J_0^2(x) + 2[J_1^2(x) + J_2^2(x) + \ldots] = C. \qquad \ldots(2)$

Replacing x by 0 in (2) and noting that $J_0(0) = 1$ and $J_n(0) = 0$ for $n \geq 1$, we get

$1 + 2(0 + 0 + \ldots) = C$ or $C = 1$. Hence (2) becomes $\quad J_0^2 + 2(J_1^2 + J_2^2 + \ldots) = 1 \quad \ldots(3)$

Part (ii). From (3), $\quad J_0^2 = 1 - 2(J_1^2 + J_2^2 + \ldots + J^2_{n-1} + J^2_n + J^2_{n+1} + \ldots.) \qquad \ldots(4)$

Since $J_1^2, J_2^2, J_3^2 \ldots$ are all positive or zero, (4) gives $\quad J_0^2 \leq 1 \quad$ so that $\quad |\,J_0(x)\,| \leq 1.$

Part (iii). Solving (4) for J_n^2, we have

$$J_n^2 = (1/2) \times (1 - J_0^2) - (J_1^2 + J_2^2 + \ldots + J^2_{n-1} + J^2_{n+1} + \ldots). \qquad \ldots(5)$$

Since J_0^2, J_1^2, J_2^2 ... are all positive or zero, (5) gives $J^2_n \leq 1/2$ or $|\,J_n(x)\,| \leq 2^{-1/2}$, where $n \geq 1$.

Ex. 11. *Prove that* (*i*) $\dfrac{d}{dx}\,\{xJ_n(x)\,J_{n+1}(x)\} = x\{J_n^2(x) - J^2_{n+1}(x)\}$

[Agra 2009; Sager 2004; Meerut 2005; Kanpur 2007]

(*ii*) $x = 2J_0J_1 + 6J_1J_2 + \ldots + 2(2n+1)J_nJ_{n+1} + \ldots.$ **[Bilaspur 1998]**

Sol. (i) $\dfrac{d}{dx}(xJ_n J_{n+1}) = J_n J_{n+1} + x(J'_n J_{n+1} + J_n J'_{n+1}) = J_n J_{n+1} + J_{n+1}(xJ'_n) + J_n(xJ'_{n+1})$. ...(1)

From recurrence relation III and IV, we have $\qquad xJ'_n = nJ_n - xJ_{n+1}$ ...(2)

and $\qquad xJ'_n = -nJ_n + xJ_{n-1}$ ...(3)

Replacing n by $n + 1$ in (3), $\qquad xJ'_{n+1} = -(n+1)J_{n+1} + xJ_n$. ...(4)

Putting the values of xJ'_n and xJ'_{n+1} from (2) and (4) in (1), we get

$\dfrac{d}{dx}(xJ_n J_{n+1}) = J_n J_{n+1} + J_{n+1}(nJ_n - xJ_{n+1}) + J_n[-(n+1)J_{n+1} + xJ_n] = x(J_n^2 - J^2_{n+1})$, other terms cancel.

Part (ii). From part (*i*) above we have $\qquad \dfrac{d}{dx}(xJ_n J_{n+1}) = x(J_n^2 - J^2_{n-1})$.

Replacing n by 0, 1, 2, ... successively, we get

$$\frac{d}{dx}(xJ_0 J_1) = x(J_0^2 - J_1^2) \qquad ...(1)$$

$$\frac{d}{dx}(xJ_1 J_2) = x(J_1^2 - J_2^2) \qquad ...(2)$$

$$\frac{d}{dx}(xJ_2 J_3) = x(J_2^2 - J^2_3) \qquad ...(3)$$

....

Multiplying (1), (2), (3) ... by 1, 3, 5, ... respectively and adding, we have

$$\frac{d}{dx}[x(J_0 J_1 + 3J_1 J_2 + 5J_2 J_3 + \ldots)] = x[(J_0^2 - J_1^2) + 3(J_1^2 - J_2^2) + 5(J_2^2 - J^2_3) + \ldots]$$

$$= x[J_0^2 + 2(J_1^2 + J_2 + \ldots)] = x \times 1 = x \text{, by Ex. 10. } (i)$$

Integrating, $\qquad x(J_0 J_1 + 3J_1 J_2 + 5J_2 J_3 + \ldots) = \int x\,dx + C = \tfrac{1}{2}x^2 + C.$ ...(4)

Putting $x = 0$ in (4) gives $C = 0$. Putting $C = 0$ in (4) and simplifying, we get

$$2J_0 J_1 + 6J_1 J_2 + 10J_2 J_3 + \ldots = x.$$

Ex. 12. *Prove that* (*i*) $J_0' = -J_1$. **[Agra 2010; Kanpur 2008]** (*ii*) $J_2 - J_0 = 2J''_0$. **[Agra 2008]**

(*iii*) $J_2 = J_0'' - (1/x)J_0'$. **[Bilaspur 1997]** (*iv*) $J_2 + 3J'_0 + 4J_0''' = 0$.

Sol. (*i*) Recurrence relation IV is $\qquad xJ'_n = nJ_n - xJ_{n+1}$. ...(1)

Replacing n by 0 in (1), we have $\qquad xJ'_0 = -xJ_1 \qquad$ or $\qquad J_0' = -J_1$.

Part (ii). Recurrence relation V is $\qquad 2J'_n = J_{n-1} - J_{n+1}$. ...(2)

Differentiating (2), w.r.t. 'x', we get $\qquad 2J_n'' = J'_{n-1} - J'_{n+1}$ ...(3)

Replacing n by $n - 1$ and $n + 1$ successively in (2), we have

$$2J'_{n-1} = J_{n-2} - J_n \qquad ...(4)$$

and $$2J'_{n+1} = J_n - J_{n+2}. \qquad ...(5)$$

Putting the values of J'_{n-1} and J'_{n+1} from (4) and (5) in (3), we have

$2J_n'' = \tfrac{1}{2}(J_{n-2} - J_n) - \tfrac{1}{2}(J_n - J_{n+2}) \qquad$ or $\qquad 4J_n'' = J_{n-2} - 2J_n + J_{n+2}$. ...(6)

Replacing n by 0 in (6), we have

$$4J_0'' = J_{-2} - 2J_0 + J_2 = (-1)^2 J_2 - 2J_0 + J_2 \qquad [\because J_{-n} = (-1)^n J_n]$$

$\therefore \qquad 4J_0'' = 2(J_2 - J_0) \qquad$ or $\qquad 2J_0'' = J_2 - J_0$.

Part (iii). Replacing n by 1 in (1), we get

$xJ_1' = J_1 - xJ_2 \qquad$ or $\qquad J_2 = x^{-1}J_1 - J_1'$. ...(7)

From part (*i*), $\quad J_1 = -J'_0 \qquad$ so that $\qquad J'_1 = -J_0''$.

$\therefore$ (7) gives $\qquad J_2 = x^{-1}(-J'_0) + J''_0 = J_0'' - x^{-1} J_0'$.

Part (iv). Differentating (6), we have $4J_n'' = J'_{n-2} - 2J'_n + J'_{n+2}$. ...(8)

Replacing n by $n - 2$ and $n + 2$ successively in (2), we get

$$2J'_{n-2} = J_{n-3} - J_{n-1} \quad ...(9)$$

and

$$2J'_{n+2} = J_{n+1} - J_{n+3}. \quad ...(10)$$

Putting the values of J'_{n-2}, J'_{n+2} and J'_n from (9), (10) and (2) in (8), we get

$$J_n''' = \tfrac{1}{2}(J_{n-3} - J_{n-1}) - (J_{n-1} - J_{n+1}) + \tfrac{1}{2}(J_{n+1} - J_{n+3})$$

or

$$8J_n''' = J_{n-3} - 3J_{n-1} + 3J_{n+1} - J_{n+3}. \quad ...(11)$$

Replacing n by 0 in (11), we get

$$8J_0''' = J_{-3} - 3J_{-1} + 3J_1 - J_3 = -2J_3 + 6J_1 \quad [\because J_{-n} = (-1)^n J_n]$$

$$= -2J_3 - 6J_0' \quad [\because J_1 = -J'_0]$$

$$\therefore \quad 8J_0''' + 2J_3 + 6J_0' = 0 \quad \text{or} \quad J_3 + 3J_0' + 4J_0''' = 0.$$

Ex. 13. *Show that $J_n(x) = 0$ has no repeated roots except at $x = 0$.*

Sol. If possible suppose $J_n(x) = 0$ has repeated roots; then at least two roots must be equal (say α), that is, α is a double root of $J_n(x) = 0$. Then from the theory of equations, we have

$$J_n(\alpha) = 0 \quad \text{and} \quad J'_n(\alpha) = 0. \quad ...(1)$$

Recurrence relations III and IV, $J_{n-1}(x) = (n/x)J_n(x) + J'_n(x)$. ...(2)

and

$$J_{n+1}(x) = (n/x)\, J_n(x) - J'_n(x). \quad ...(3)$$

Replacing x by α in (2) and (3) and using (1), we get

$$J_{n+1}(\alpha) = 0 \quad \text{and} \quad J_{n-1}(\alpha) = 0 \quad \text{except} \quad \text{when} \quad x = 0.$$

Since two different power series have distinct sum functions, so $J_{n+1}(\alpha) = 0 = J_{n-1}(\alpha)$ must be absurd. Hence $J_n(x) = 0$ has no repeated roots except at $x = 0$.

Ex. 14. *From the recurrence formula $2J_n' = J_{n-1} - J_{n+1}$, deduce the result*

$$2^r J_n^r(x) = J_{n-r} - r\, J_{n-r+2} + \frac{r(r-1)}{2!} J_{n-r+4} + \ldots + (-1)^r J_{n+r}. \quad ...(1)$$

Sol. Given that $2J'_n = J_{n-1} - J_{n+1}$. ...(2)

Clearly (2) shows that (1) is true for $r = 1$. We assume that (1) is true for some particular value of r, say $r = p$. Then we have

$$2^p J^p{}_n(x) = J_{n-p} - p\, J_{n-p+2} + \frac{p\,(p-1)}{2!} J_{n-p+4} + \ldots + (-1)^p J_{n+p}$$

or

$$2^p J_n^p(x) = J_{n-p} - {}^pC_1 J_{n-p+2} + {}^pC_2 J_{n-p+4} + \ldots + (-1)^p J_{n+p}. \quad ...(3)$$

Differentiating (3) w.r.t. x and then multiplying by 2, we get

$$2^{p+1} J_n^{p+1}(x) = 2J'_{n-p} - 2 \times {}^pC_1 J_{n-p+2} + 2 \times {}^pC_2 J'_{n-p+4} + \ldots + 2(-1)^p J'_{n+p}. \quad ...(4)$$

Replacing n by $n - p$, $n - p + 2$, $n - p + 4$ ..., $n + p$ successively in (2),

$$2J'_{n-p} = J_{n-p-1} - J_{n-p+1}$$

$$2J'_{n-p+2} = J_{n-p+1} - J_{n-p+2}$$

$$2J'_{n-p+4} = J_{n-p+3} - J_{n-p+5}$$

$$\ldots\ldots\ldots \qquad \ldots.\ \ldots.\ \ldots.$$

$$2J'_{n+p} = J_{n-p-1} - J_{n+p+1}$$

Substituting these values in (4), we have

$$2^{p+1} J_n^{p+1}(x) = J_{n-p-1} - J_{n-p+1} - {}^pC_1(J_{n-p+1} - J_{n-p-3})$$
$$+ {}^pC_2(J_{n-p+4} - J_{n-p+5}) + \ldots + (-1)^p (J_{n-p-1} - J_{n+p+1})$$

$$= J_{n-p-1} - (1 + {}^pC_1)J_{n-p+1} + ({}^pC_1 + {}^pC_2)J_{n-p+2} + \dots + (-1)^{p+1}J_{n+p+1}. \qquad ...(5)$$

Since ${}^pC_r + {}^pC_{r-1} = {}^{p+1}C_r$, we have $1 + {}^pC_1 = {}^pC_0 + {}^pC_1 = {}^{p+1}C_1$, ${}^pC_1 + {}^pC_2 = {}^{p+1}C_2$ and so on. Then (5) becomes

$$2^{p+1}J_n^{p+1}(x) = J_{n-p-1} - {}^{p+1}C_1\, J_{n-p+1} + {}^{p+1}C_2\, J_{n-p+2} + \dots + (-1)^{p+1}\, J_{n+p+1},$$

showing that (1) is true for $r = p + 1$ if it were true for $r = p$. So (1) is true for all natural numbers by mathematical induction.

Ex. 15. *Prove that* $x^2J''_n(x) = (n^2 - n - x^2)J_n(x) + xJ_{n+1}(x)$, where $n = 0, 1, 2, \dots$.

Sol. Recurrence relation IV is $\quad xJ_n'(x) = nJ_n(x) - xJ_{n+1}(x). \qquad ...(1)$

Differentiating both sides of (1) w.r.t. 'x', we have

$$xJ''_n(x) + J'_n(x) = nJ'_n(x) - [xJ'_{n+1}(x) + J_{n+1}(x)]$$

or
$$x^2J''_n(x) = (n-1)xJ'_n(x) - x[xJ'_{n+1}(x)] - xJ_{n+1}(x). \qquad ...(2)$$

Recurrence relation III is $\quad xJ'_n(x) = -nJ_n(x) + xJ_{n+1}(x)$

Replacing n by $(n + 1)$ in this relation, we obtain

$$xJ'_{n+1}(x) = -(n+1)\, J_{n+1}(x) + x\, J_n(x). \qquad ...(3)$$

Substituting for xJ'_n from (1) and for $xJ'_{n+1}(x)$ from (3) in (2), we get

$$x^2J''_n(x) = (n-1)\,[n\, J_n(x) - xJ_{n+1}(x) - x[-(n+1)J_{n+1}(x) + xJ_n(x)] - x\, J_{n+1}(x)$$
$$= [(n-1)n - x^2]J_n(x) + [-(n-1) + (n+1) - 1]xJ_{n+1}(x)$$

$\therefore$
$$x^2J''_n(x) = (n^2 - n - x^2)\, J_n(x) + x\, J_{n+1}(x).$$

Ex. 16. *Show that** $\dfrac{J_{n+1}}{J_n} = \dfrac{(x/2)}{(n+1)-}\,\dfrac{(x/2)^2}{(n+2)-}\,\dfrac{(x/2)^3}{(n+3)-}\dots$

Sol. Recurrence relation VI is $\quad J_{n-1} + J_{n+1} = (2n/x) \times J_n$

or $\quad J_{n-1} = \dfrac{2n}{x}J_n - J_{n+1} \quad$ or $\quad \dfrac{J_{n-1}}{J_n} = \dfrac{2n}{x} - \dfrac{J_{n+1}}{J_n}. \qquad ...(1)$

Replacing n by $(n + 1)$ in (1), we get $\quad \dfrac{J_n}{J_{n+1}} = \dfrac{2(n+1)}{n} - \dfrac{J_{n+2}}{J_{n+1}} \qquad ...(2)$

$\therefore$
$$\frac{J_{n+1}}{J_n} = \frac{1}{\dfrac{J_n}{J_{n+1}}} = \frac{1}{\dfrac{2(n+1)}{x} - \dfrac{J_{n+2}}{J_{n+1}}}, \text{ using (2)}$$

$$= \frac{1}{\dfrac{2(n+1)}{x} - \dfrac{1}{(J_{n+1}/J_{n+2})}} = \frac{1}{\dfrac{2(n+1)}{x} - \dfrac{1}{\dfrac{2(n+2)}{x} - \dfrac{J_{n+3}}{J_{n+2}}}}$$

[With help of (2) by replacing n by $n + 1$]

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*Student should read a chapter on continued fractions in some book on Algebra to understand the notations, namely,

$$\frac{a_1}{a_2 + \dfrac{a_3}{a_4 + \dfrac{a_5}{a_6 + \dots}}} = \frac{a_1}{a_2 +}\,\frac{a_3}{a_4 +}\,\frac{a_5}{a_6 + \dots}$$

$$= \cfrac{1}{\cfrac{2(n+1)}{x} - \cfrac{1}{\cfrac{2(n+2)}{x} - \cfrac{1}{(J_{n+2}/J_{n+3})}}}$$

$$= \cfrac{1}{\cfrac{2(n+1)}{x} - \cfrac{1}{\cfrac{2(n+2)}{x} - \cfrac{1}{\cfrac{2(n+3)}{x} - \cfrac{J_{n+4}}{J_{n+3}}}}} \qquad \left[\begin{array}{l}\text{With help of (2) by}\\ \text{replacing } n \text{ by } n+2\end{array}\right]$$

$$= \cfrac{x/2}{(n+1) - \cfrac{x/2}{\cfrac{2(n+2)}{x} - \cfrac{1}{\cfrac{2(n+3)}{x} + \ldots}}} \qquad \left[\begin{array}{l}\text{Multiply numerator and}\\ \text{denominator by } x/2\end{array}\right]$$

$$= \cfrac{x/2}{(n+1) - \left[\cfrac{(x/2)^2}{(n+2) - \cfrac{(x/2)}{\cfrac{2(n+3)}{x} + \ldots}}\right]} \qquad \left[\begin{array}{l}\text{Repeating the}\\ \text{similar operations}\end{array}\right]$$

$$= \frac{(x/2)}{(n+1)-}\ \frac{(x/2)^2}{(n+2)-}\ \frac{(x/2)^3}{(n+3)-\cdots}.$$

EXERCISE 11 (B)

1. Show that all roots of $J_n(x)$ are real.

2. Prove that between any two zeros of $J_n(x)$ lie one and only one zero of $J_{n+1}(x)$ as well as $J_{n-1}(x)$.

3.(a) Prove that between any two consecutive positive roots of the equation $J_n(x) = 0$, there is one and only one root of the equation $J_{n+1}(x) = 0$.

(b) Show that between two consecutive positive zeros of $J_n\,(x)$ there is precisely one zero of $J_{n-1}\,(x)$.

4. Prove (*i*) $J_{n+3} + J_{n+5} = (2/x)\,(n+4)J_{n+4}$. **[Kanpur 2009]**

(*ii*) $4J''_n = J_{n-2} - 2J_n + J_{n+2}$.

5. For Bessel's functions $J_n(x)$, find out a and b, where

$d\{J_n\,(x)\}/dx = aJ_{n-1}(x) + bJ_{n+1}(x)$. **Ans.** $a = 1/2$; $b = -(1/2)$

6. Show that $J_2 - J_0 = 2J_0''$ **[Kanpur 2006]**

7. Evaluate $J_3(x)$ in terms of $J_0(x)$ and $J_1(x)$. **Ans.** $J_3(x) = \{(8 - x^2)/x^2\}J_1(x) - (4/x)J_0(x)$

8. Prove that (*a*) $x^2J''_n(x) + J'_n(x) = (n^2/x)J_n(x) - xJ_n(x)$. **[Meerut 1998]**

(*b*) $\int J_{n+1}(x)\,dx = \int J_{n-1}(x)\,dx - 2\,J_n(x)$. **[Kanpur 1998]**

11.7.B. Solved Example involving integration and recurrence relations

Ex. 1. *If* $n > -1$, show that $\displaystyle\int_0^x x^{n+1}J_n(x)\,dx = x^{n+1}J_{n+1}(x)$. **[Bilaspur 1998]**

Sol. From recurrence relations I, $\qquad \dfrac{d}{dx}\{x^nJ_n(x)\} = x^nJ_{n-1}(x)$. ...(1)

Replacing n by $n+1$ in (1), $\qquad \dfrac{d}{dx}\{x^{n+1}J_{n+1}(x)\} = x^{n+1}J_n(x)$. ...(2)

Integrating (2) w.r.t. 'x' between the limits 0 and x, we get

$$[x^{n+1}J_{n+1}(x)]_0^x = \int_0^x x^{n+1}J_n(x)\,dx \qquad \text{or} \qquad \int_0^x x^{n+1}J_n(x)\,dx = x^{n+1}J_{n+1}(x).$$

Ex. 2. *Prove that* $J_{n+1}(x) = x\int_0^1 J_n(xy)\,y^{n+1}dy$.

Sol. Let $xy = t$ so that $x\,dy = dt$

$\therefore$ R.H.S. of (1) $= x\int_0^x J_n(t)\,(t/x)^{n-1}(dt/x) = x^{-n-1}\int_0^x t^{n+1}J_n(t)\,dt$

$$= x^{-n-1}\int_0^x x^{n+1}J_n(x)\,dx = x^{-n-1}\,x^{n+1}J_{n+1}(x), \text{ by Ex. 1}$$

$$= J_{n+1}(x) = \text{L.H.S. of (1).}$$

Ex. 3. *Show that* (*a*) $\int_0^x x^{-n}J_{n+1}(x)\,dx = \dfrac{1}{2^n\,\Gamma(n+1)} - x^{-n}J_n,\ n > 1.$ **[Bilaspur 1997]**

(*b*) $\int_0^\infty x^{-n}J_{n+1}(x)\,dx = \dfrac{1}{2^n\,\Gamma(n+1)},\ n > -\dfrac{1}{2}.$

Sol. (a) From recurrence relation II, $\dfrac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x).$ ...(1)

Integrating (1) w.r.t. 'x' between the limits 0 and x, we get

$$\left[x^{-n}J_n(x)\right]_0^x = -\int_0^x x^{-n}J_{n+1}(x)\,dx \quad \text{or} \quad x^{-n}J_n(x) - \lim_{x\to 0}\frac{J_n(x)}{x^n} = -\int_0^x x^{-n}J_{n+1}(x)\,dx. \qquad \text{...(2)}$$

But $$\lim_{x\to 0}\frac{J_n(x)}{x^n} = \lim_{x\to 0}\frac{1}{x^n}\frac{x^n}{2^n\,\Gamma(n+1)}\left[1 - \frac{x^2}{2\cdot 2(n+1)} + \ldots\right] = \frac{1}{2^n\,\Gamma(n+1)}$$

Hence (2) may be written as $$\int_0^x x^{-n}J_{n+1}(x)\,dx = \frac{1}{2^n\,\Gamma(n+1)} - x^{-n}J_n(x).$$

Part (b). Integrating (1) w.r.t. 'x' from 0 to ∞, we get

$$\left[x^{-n}J_n(x)\right]_0^\infty = -\int_0^\infty x^{-n}J_{n+1}(x)\,dx \quad \text{or} \quad \lim_{x\to\infty}\frac{J_n(x)}{x^n} - \lim_{x\to 0}\frac{J_n(x)}{x^n} = -\int_0^\infty x^{-n}J_{n+1}(x)\,dx. \qquad \text{...(3)}$$

As in part (*a*), $$\lim_{x\to 0}\frac{J_n(x)}{x^n} = \frac{1}{2^n\Gamma(n+1)}. \qquad \text{...(4)}$$

We know that for large values of x the approximate value of $J_n(x)$ is

$$J_n(x) \sim \left(\frac{2}{\pi x}\right)^{1/2}\cos\left\{x - \left(n+\frac{1}{2}\right)\frac{\pi}{2}\right\},\ n > \frac{1}{2} \qquad \text{...(5)}$$

Using (5), $$\lim_{x\to\infty}\frac{J_n(x)}{x^n} = 0. \qquad \text{...(6)}$$

Using (4) and (6), (3) reduces to $$\int_0^\infty x^{-n}J_{n+1}(x)\,dx = \frac{1}{2^n\,\Gamma(n+1)}.$$

Ex. 4. *Prove* (*i*) $\dfrac{d}{dx}\{xJ_1(x)\} = xJ_0(x).$ (*ii*) $\int_0^b xJ_0(ax)\,dx = \dfrac{b}{a}J_1(ab).$

Sol. (*i*) Recurrence relation I is $\dfrac{d}{dx}\{x^nJ_n(x)\} = x^nJ_{n-1}(x).$ ...(1)

Replacing n by 1 in (1), we have $\dfrac{d}{dx}\{xJ_1(x)\} = xJ_0(x).$

Part (ii). Put $ax = t$, so that $adx = dt$. Then, we get

$$\int_0^b x\, J_0(ax)\, dx = \frac{1}{a^2}\int_0^{ab} t\, J_0(t)\, dt = \frac{1}{a^2}\int_0^{ab} \frac{d}{dt}\{t\, J_1(t)\}\, dt, \text{ by part } (i)$$

$$= \frac{1}{a^2}\left[tJ_1(t)\right]_0^{ab} = \frac{1}{a^2}\,[abJ_1(ab) - 0], \text{ as } J_1(0) = 0.$$

$\therefore$ $$\int_0^b x\, J_0(ax)\, dx = \frac{b}{a} J_1(ab).$$

Ex. 5. *Prove that* (i) $\dfrac{d}{dx} J_0(x) = -J_1(x)$.

(ii) $\displaystyle\int_a^b J_0(x)\, J_1(x)\, dx = \tfrac{1}{2}[J_0^2(a) - J_0^2(b)]$. **[Nagpur 2005]**

Sol. (i) From recurrence relation II, $\dfrac{d}{dx}\{x^{-n}J_n(x)\} = -x^n J_{n+1}(x)$. ...(1)

Put $n = 0$ in (1). Then $\dfrac{d}{dx} J_0(x) = -J_1(x)$. ...(2)

Part (ii). Using (2), we have

$$\int_a^b J_0(x)\, J_1(x)\, dx = -\int_a^b J_0(x)\, J_0'(x)\, dx = -\left[\frac{[J_0(x)]^2}{2}\right]_a^b = \tfrac{1}{2}[J_0^2(a) - J_0^2(b)]$$

Ex. 6. *Evaluate* $\int J_3(x)\, dx$ *and express the result in terms of* J_0 *and* J_1.

Sol. From recurrence II, we have $x^{-n}J_{n+1} = \dfrac{d}{dx}\{x^{-n}J_n(x)\}$.

Integrating it, $\int x^{-n} J_{n+1}\, dx = -x^{-n}J_n$. ...(1)

Now, $\int J_3(x)\, dx = \int x^2\,(x^{-2}\, J_3)\, dx = x^2(-x^{-2}J_2) - \int 2x\,(-x^{-2}\, J_2)\, dx$.

[Integrating by parts and using (1) for $n = 2$]

$= -J_2 + 2\int x^{-1}\, J_2\, dx = -J_2 + 2(-x^{-1}J_1) + c$ [using (1) for $n = 1$]

$\therefore$ $\int J_3(x)\, dx = -J_2 - 2x^{-1}J_1 + c$. ...(2)

From recurrence relation VI, $(2n/x)J_n = J_{n-1} + J_{n+1}$ ...(3)

Put $n = 1$ in (3). Then $J_2 = 2J_1/x - J_0$ ...(4)

Using (4), (2) gives $\int J_3(x)\, dx = -(2J_1/x - J_0) - 2x^{-1}\, J_2 + c$

$\therefore$ $\int J_3(x)\, dx = J_0 - 4J_1/x + c$, being an arbitrary constant.

Ex. 7. *Evaluate* $\int x^3 J_3(x)\, dx$. **[Gulbarga 2005]**

Sol. Since $\dfrac{d}{dx}\{x^{-n}J_n\} = -x^n J_{n+1}$ so $\int x^{-n} J_{n+1}\, dx = -x^{-n}J_n$. ...(1)

Now, $\displaystyle\int x^3 J_3(x)\, dx = \int x^5\,(x^{-2}J_3)\, dx = x^5(-x^{-2}J_2) - \int 5x^4\,(-x^{-2}\, J_2)\, dx$

(On integration by parts and using (1) for $n = 2$)

$$= -x^3J_2 + 5\int x^2 J_2\, dx = -x^3 J_2 + 5\int x^3(x^{-1}\, J_2)\, dx = -x^3J_2 + 5\left[x^3\,(-x^{-1}J_1) - \int 3x^2(-x^{-1}J_1)\, dx\right]$$

(on integration by parts and using (1) for $n = 1$)

$= -x^3J_2 - 5x^2J_1 + 15\int x\, J_1\, dx = -x^3J_2 - 5x^2J_1 + 15\int x\,(-J_0')\, dx$ $[\because J_1 = -J_0']$

$= -x^3J_2 - 5x^2J_1 - 15\int x\, J_0'\, dx = -x^3J_2 - 5x^2J_1 - 15\left[x\, J_0 - \int 1 \,.\, J_0\, dx\right]$, integrating by part,

$= -x^3J_2 - 5x^3J_1 - 15xJ_0 + 15x\int J_0\, dx\,.$

Remark. From Ex. 6. and 7 note that, in general, an integral of the form $\int x^m\, J_n(x)\, dx$, $m + n \geq 0$, can be completely integrated if $m + n$ is an odd integer, while if $m + n$ is even, then the integral can be put in terms of $\int J_0(x)\, dx$. Note that $\int J_0(x)\, dx$ cannot be expressed in closed form and so it must be left as such in final answer.

Ex. 8. *Evaluate* $\int x^4 J_1(x)\, dx$.

Sol. Since $\dfrac{d}{dx}\{x^nJ_n\} = x^nJ_{n-1}$ so $\int x^n J_{n-1}\, dx = x^nJ_n.$ ...(1)

Now $\int x^4J_1\, dx = \int x^2\,(x^2J_1)\, dx = x^2(x^2J_2) - \int 2x(x^2J_2)\, dx$

[on integrating by parts and using (1) for $n = 2$]

$= x^4J_2 - 2\int x^3J_2\, dx = x^4J_2 - 2x^3J_3 + c$ [using (1) for $n = 3$]

Ex. 9. Express $\int x^{-3}\, J_4\,(x)\, dx$ in terms of J_0 and J_1.

Sol. Putting $n = 3$ in recurrence relation II $\dfrac{d}{dx}\{x^{-n}J_n\} = -x^{-n}J_{n+1}$ gives $\dfrac{d}{dx}\{x^{-3}J_3\} = -x^{-3}J_4$.

Integrating, $\int x^{-3}\, J_4\,(x)\, dx = -x^{-3}J_3 + c$, c being an arbitrary constant ...(1)

Recurrence relation VI is $J_{n+1} = (2n/x)J_n - J_{n-1}$. ...(2)

Replacing n by 2 and 1 successively in (2) gives $J_3 = (6/x)J_2 - J_1$ ...(3)

and $J_2 = (4/x)J_1 - J_0$. ...(4)

Using (4), (3) becomes $J_2 = (6/x)[(4/x)J_1 - J_0] - J_1 = (24x^{-2} - 1)J_1 - (6/x)J_0$ ...(5)

Using (5), (1) becomes $\int x^{-3}\, J_4\,(x)\, dx = -x^{-3}[(24x^{-2} - 1)J_1 - 6x^{-1}J_0] + c.$

Ex. 10. *Prove* $\int J_{n+1}\, dx = \int J_{n-1}(x)\, dx - 2J_n(x)$.

Sol. From recurrence relations, we have

$2J'_n(x) = J_{n-1}(x) - J_{n+1}(x)$ or $J_{n+1}(x) = J_{n-1}(x) - 2J'_n(x)$

Integrating $\int J_{n+1}(x)\, dx = \int J_{n-1}(x)\, dx - 2J_n(x).$

Ex. 11(a). *Prove that* $\int x^{-1}J_4(x)\, dx = -x^{-1}J_3(x) - 2x^{-2}J_2(x) + c.$

Sol. Recurrence relation II is $\dfrac{d}{dx}\{x^{-n}J_n(x) = -x^{-n}J_{n+1}(x).$

Integrating it, we get $\int x^{-n}\, J_{n+1}(x)\, dx = -\, x^{-n}J_n(x).$ ...(1)

$\therefore\ \int x^{-1}\, J_4(x)\, dx = \int x^2[x^{-3}J_4(x)]\, dx = x^2\left[-x^{-3}\, J_3(x)\right] - \int 2x \times [-x^{-3}J_3(x)]\, dx$

(Integrating by parts taking x^2 as first function and using result (1) for $n = 3$)

$= -x^{-1}J_3(x) + 2\int x^{-2}\, J_3(x)\, dx = -x^{-1}J_3(x) + 2[-x^{-2}J_2(x)] + c$, using result (1) for $n = 2$

$= -x^{-1}J_3(x) - 2x^{-2}J_2(x) + c.$

Ex. 11(b). *Prove that* $\int J_3(x)dx = -J_2(x) - (2/x) \times J_0(x) + C$ **[KU Kurukshetra 2005]**

Sol. From recurrence relation II, $\frac{d}{dx}\left\{x^{-n}J_n(x)\right\} = -x^{-n}J_{n+1}(x)$

Integrating, $\int x^{-n}J_{n+1}(x)dx = -x^{-n}J_n(x)$ ... (1)

Now, $\int J_3(x)dx = \int x^2\left\{x^{-2}J_3(x)\right\}dx = x^2\int x^{-2}J_3(x)dx - \int 2x\left(\int x^{-2}J_3(x)dx\right)dx$

$= x^2\left(-x^{-2}J_2(x)\right) - \int 2x(-x^{-2}J_2(x))dx$, using (1) for $n = 2$

$= -J_2(x) + 2\int x^{-1}J_2(x)dx = -J_2(x) - 2x^{-1}J_1(x) + C$, using (1) again for $n = 1$

Ex. 12. *Show that* (*i*) $\int_0^x x^3\, J_0(x)\, dx = x^3J_1(x) - 2x^2J_2(x)$. **[GATE 2003]**

(*ii*) $\int_0^1 x^3\, J_0(x)\, dx = 2J_0(1) - 3J_1(1)$.

Sol. (i) Since $\frac{d}{dx}\{x^nJ_n(x)] = x^nJ_{n-1}(x)$ so $\int x^n J_{n-1}(x)\, dx = x^nJ_n(x)$ ...(1)

$\therefore\ \int_0^x x^3J_0(x)\, dx = \int_0^x x^2\,[x\, J_0(x)]\, dx = [x^2\{xJ_1(x)\}]_0^x - \int_0^x 2x\,\{x\, J_1(x)\}\, dx$

(Integrating by parts and using (1) for $n = 1$)

$= x^3J_1(x) - 2\int_0^x x^2J_1(x)\, dx = x^3J_1(x) - 2[x^2J_2(x)]^x{}_0$, using result (1) for $n = 2$

$= x^3\, J_1(x) - 2x^2\, J_2(x)$, as $J_2(0) = 0$

(ii) Proceed as in part (*i*) and prove that $\int_0^x x^3J_0(x)\, dx = x^3J_1(x) - 2x^2J_2(x)$. ...(2)

Putting $x = 1$ in (2), $\int_0^1 x^3J_0(x)\, dx = J_1(1) - 2J_2(1)$. ...(3)

Recurrence relation VI is $J_{n-1}(x) + J_{n+1}(x) = (2n/x)J_n(x)$.

Putting $n = 1$ and re–writing it, we have $J_2(1) = 2J_1(1) - J_0(1)$ ...(4)

Substituting the above value of $J_2(1)$ in (3), we have

$$\int_0^1 x^3J_0(x)\, dx = J_1(1) - 2\{2J_1(1) - J_0(1)] = 2J_0(1) - 3J_1(1).$$

EXERCISE 11(C)

1. Prove that $\int x^2J_0(x)\, dx = x^2J_1(x) + xJ_0(x) - \int J_0(x)\, dx$.
2. Prove that $\int_0^x x^2J_0(x)\, J_1(x)\, dx = \frac{1}{2}x^2J_1{}^2(x)$.
3. Prove that $\int x^{-2}J_1(x)\, dx = -\frac{1}{3}x^{-1}J_2(x) - \frac{1}{3}J_1(x) + \frac{1}{2}\int J_0(x)\, dx$.
4. Prove that $\int J_0(x)\sin x\, dx = xJ_0(x)\sin x - xJ_1(x)\cos x + c$.
5. Show that $\int J_3 dx = -3J_0 + 4J_1'$ **[Bilaspur 1997]**

11.8. Generating function for the Bessel's function $J_n(x)$

Prove : $\exp\left\{\frac{1}{2}x\left(z - \frac{1}{z}\right)\right\} = \sum_{n=-\infty}^{\infty} z^nJ_n(x)$ **[Kanpur 2007; Meerut 1996, 97]**

Or *Show that when n is a positive integer,* $J_n(x)$ *is the coefficient of* z^n *in the expansion of*

$\exp\{(x/2)\times(z-1/z)\}$ *i.e.,* $e^{(x/2)(z-1/z)}$ *in ascending and descending power of z. Also show that* J_n *is coefficient of* z^{-n} *multiplied by* $(-1)^n$ *in the expansion of the above expression.*

[Kanpur 2007; Kakitiya 1997; Kanpur 2005, 06]

Note. $\text{Exp}\{(x/2)\times(z-1/z)\}$ is called the generating function for $J_n(x)$. Here $\exp A = e^A$.

Proof. We have $\exp\{(x/2)\times(z-1/z)\} = e^{\frac{xz}{2}-\frac{x}{2z}} = e^{\frac{xz}{2}}\cdot e^{-\frac{x}{2z}}$

$$= \left[1+\left(\frac{x}{2}\right)z+\left(\frac{x}{2}\right)^2\frac{z^2}{2!}+\ldots+\left(\frac{x}{2}\right)^n\frac{z^n}{n!}+\left(\frac{x}{2}\right)^{n+1}\frac{z^{n+1}}{(n+1)}+\ldots\right]\times\left[1-\left(\frac{x}{2}\right)z^{-1}\right.$$

$$\left.+\left(\frac{x}{2}\right)^2\frac{z^{-2}}{2!}+\ldots+\left(\frac{x}{2}\right)^n\frac{(-1)^n z^{-n}}{n!}+\left(\frac{x}{2}\right)^{n+1}\frac{(-1)^{n+1}z^{-(n+1)}}{(n+1)!}+\ldots\right] \quad \ldots(1)$$

The coeff. of z^n in the product (1) is obtained by multiplying the coefficients of z^n, z^{n+1}, z^{n+2}, ... in the first bracket with the coeff. of z^0, z^{-1}, z^{-2}, ... in the second bracket respectively

$$\therefore \text{ coefficient of } z^n \text{ in product (1)} = \left(\frac{x}{2}\right)^n\frac{1}{n!}-\left(\frac{x}{2}\right)^{n+2}\frac{1}{(n+1)!}+\left(\frac{x}{2}\right)^{n+4}\frac{1}{(n+2)!\,2!}-\ldots$$

$$=\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,(n+r)!}\left(\frac{x}{2}\right)^{n+2r}=\sum_{r=0}^{\infty}\frac{(-1)^r}{r!\,\Gamma(n+r+1)!}\left(\frac{x}{2}\right)^{n+2r}=J_n(x)$$

[$\because$ $(n+r)! = \Gamma(n+r+1)$, $n+r$ being positive integer].

The coeff. of z^{-n} in the product (1) is obtained by multiplying the coefficients of z^{-n}, z^{-n-1}, z^{-n-2}... of the second bracket with the coefficients of z^0, z^1, z^2, ... in the first bracket respectively

$$\therefore \text{coeff. of } z^{-n} \text{ in product (1)} = \left(\frac{x}{2}\right)^n\frac{(-1)^n}{n!}+\left(\frac{x}{2}\right)^{n+1}\frac{(-1)^{n+1}}{(n+1)!}\frac{x}{2}+\left(\frac{x}{2}\right)^{n+2}\frac{(-1)^{n+2}}{(n+2)!\,2!}\left(\frac{x}{2}\right)^2+\ldots$$

$$=(-1)^n\left[\left(\frac{x}{2}\right)^n\frac{1}{n!}-\left(\frac{x}{2}\right)^{n+2}\frac{1}{(n+1)!}+\left(\frac{x}{2}\right)^{n+4}\frac{1}{(n+2)\,2!}\ldots\right]=(-1)^nJ_n(x), \text{ as before}$$

Thus the coefficient of $z^{-n} = (-1)^nJ_n(x)$ $\Rightarrow$ $J_n(x) = (-1)^n \times$ the coefficient of z^{-n}.

Finally, in the product (1) the coefficient of z^0 is obtained by multiplying the coefficient of z^0, z^1, z^2, ... in the first bracket with the coefficients of z^0, z^{-1}, z^{-2}, ... in the second bracket and is thus

$$= 1-\left(\frac{x}{2}\right)^2+\left(\frac{x}{2}\right)^4\left(\frac{1}{2!}\right)^2-\left(\frac{x}{2}\right)^6\left(\frac{1}{3!}\right)^2+\ldots=1-\frac{x^2}{2^2}+\frac{x^4}{2^2\cdot 4^2}-\ldots=J_0(x).$$

We observe that the coefficients of z^0, $(z-z^{-1})$, (z^2+z^{-2}) ..., $[z^n+(-1)^nz^{-n}]$ ... are $J_0(x)$, $J_1(x)$, $J_2(x)$ ..., $J_n(x)$ ... respectively. Thus, (1) gives

$\exp\{(x/2)\times(z-1/z)\} = J_0(x)+(z-z^{-1})J_1(x)+(z^2+z^{-2})J_2(x)+\ldots+[z^2+(-1)^nz^{-n}]J_n(x)+\ldots$

$$= \sum_{n=-\infty}^{\infty} z^nJ_n(x), \text{ as } J_{-n}(x)=(-1)^nJ_n(x)$$

11.9. Trigonometric expansions involving Bessel's functions. *Show that*

(*i*) $\cos(x\sin\phi) = J_0 + 2\cos 2\phi\cdot J_2 + 2\cos 4\phi\cdot J_4 + ...$ **[Meerut 1995, KU Kurukshetra 2005]**

(*ii*) $\sin(x\sin\phi) = 2\sin\phi\cdot J_1 + 2\sin 3\phi\cdot J_2 + ...$ **[KU Kurukshreta 2006]**

(*iii*) $\cos(x\cos\phi) = J_0 - 2\cos 2\phi\cdot J_2 + 2\cos 4\phi\cdot J_4 - ...$

(*iv*) $\sin(x\cos\phi) = 2\cos\phi\cdot J_1 - 2\cos 3\phi\cdot J_3 + 2\cos 5\phi\cdot J_5 - ...$

(*v*) $\cos x = J_0 - 2J_4 + 2J_4 .... = J_0(x) + 2\sum_{n=1}^{\infty}(-1)^n J_{2n}(x)$ **[Kanpur 2011]**

(*vi*) $\sin x = 2J_1 - 2J_3 + 2J_5 ... = 2\sum_{n=0}^{\infty}(-1)^n J_{2n+1}(x)$ **[Kanpur 2011]**

Proof. We know that

$$e^{(x/2)(z-1/z)} = J_0 + (z - z^{-1})J_1 + (z^2 + z^{-2})J_2 + (z^3 - z^{-3})J_3 + ... \quad ...(1)$$

Let $z = e^{i\phi}$ so that $z^n = e^{in\phi}$ and $z^{-n} = e^{-in\phi}$. Then (1) gives

$$e^{(x/2)(e^{i\phi}-e^{-i\phi})} = J_0 + (e^{i\phi} - e^{-i\phi})J_1 + (e^{2i\phi} + e^{-2i\phi})J_2 + (e^{3i\phi} - e^{-3i\phi})J_3 + ... \quad ...(2)$$

Since $\cos n\phi = (e^{ni\phi} + e^{-ni\phi})/2$ and $\sin n\phi = (e^{ni\phi} - e^{-ni\phi})/2i$, (2) gives

$$e^{xi\sin\phi} = J_0 + 2i\sin\phi\cdot J_1 + 2\cos 2\phi\cdot J_2 + 2i\sin 3\phi\cdot J_3 + ...$$

or $\cos(x\sin\phi) + i\sin(x\sin\phi) = (J_0 + 2\cos 2\phi\cdot J_2 + ...) + 2i(\sin\phi\cdot J_1 + \sin 3\phi\cdot J_3 + ...)$ ...(3)

Part (i). Equating real parts in (3), we get

$$\cos(x\sin\phi) = J_0 + 2\cos 2\phi\cdot J_2 + 2\cos 4\phi\cdot J_4 + ... \quad ...(4)$$

Part (ii). Equating imaginary parts in (3), we get

$$\sin(x\sin\phi) = 2\sin\phi\cdot J_1 + 2\sin 3\phi\cdot J_3 + 2\sin 5\phi\cdot J_5 + ... \quad ...(5)$$

Part (iii). Replacing ϕ by $\pi/2 - \phi$ in (4) and simplifying, we have

$$\cos(x\cos\phi) = J_0 - 2\cos 2\phi\cdot J_2 + 2\cos 4\phi\cdot J_4 - ... \quad ...(6)$$

Part (iv). Replacing ϕ by $\pi/2 - \phi$ in (5) and simplifying, we get

$$\sin(x\cos\phi) = 2\cos\phi\cdot J_1 - 2\cos 3\phi\cdot J_3 + 2\cos 5\phi\cdot J_5 - ... \quad ...(7)$$

Part (v) & (vi). Replacing ϕ by 0 in (6) and (7), we get

$$\cos x = J_0 - 2J_1 + 2J_4 - ... = J_0(x) + 2\sum_{n=1}^{\infty}(-1)^n J_{2n}(x)$$

and

$$\sin x = 2J_1 - 2J_3 + 2J_5 - ... = 2\sum_{n=0}^{\infty}(-1)^n J_{2n+1}(x).$$

11.9A. Solved examples based on Art 11.8 and 11.9.

Ex. 1. *Show that* (*i*) $x\sin x = 2(2^2J_2 - 4^2J_4 + 6^2J_6 - ...)$. **[Agra 2010]**

(*ii*) $x\cos x = 2(1^2J_1 - 3^2J_3 + 5^2J_5 - ...)$.

Sol. We know that $\cos(x\sin\phi) = J_0 + 2J_2\cos 2\phi + 2J_4\cos 4\phi + ...$ ...(1)

Differentiating (1) w.r.t. 'ϕ', $-\sin(x\sin\phi)\cdot x\cos\phi = 0 - 2\cdot 2J_2\sin 2\phi - 2\cdot 4J_4\sin 4\phi + ...$ ...(2)

Differentiating (2) w.r.t. 'ϕ', $-\cos(x\sin\phi)\cdot(x\cos\phi)^2 + \sin(x\sin\phi)\cdot(x\sin\phi)$

$$= -2\cdot 2^2J_2\cos 2\phi - 2\cdot 4^2J_4\cos 4\phi - 2\cdot 6^2J_6\cos 6\phi + ... \quad ...(3)$$

Replacing ϕ by $\pi/2$ in (3), we get $x\sin x = 2(2^2J_2 - 4^2J_4 + 6^2J_6 - ...)$

Part (ii). Start with $\sin(x \sin \phi) = 2J_1 \sin \phi + 2J_3 \sin 3\phi + 2J_5 \sin \phi + ...$

Differentiate this twice w.r.t. 'ϕ' as in part (*i*) and then replace ϕ by $\pi/2$. This will lead to the desired answer. Complete the solution yourself.

Ex. 2. ***Bessel's Integrals.*** *Show that*

(*i*) $J_n(x) = \dfrac{1}{\pi}\displaystyle\int_0^\pi \cos(n\phi - x \sin \phi)\, d\phi$, where n is a positive integer

[Purvanchal 2004, 07; Punjab 2005]

(*ii*) $J_n(x) = \dfrac{1}{\pi}\displaystyle\int_0^\pi \cos(n\phi - x \sin \phi)\, d\phi$, where n is any integer **[KU Kurukshetra 2004]**

(*iii*) $J_0(x) = \dfrac{1}{\pi}\displaystyle\int_0^\pi \cos(x \sin \phi)\, d\phi = \dfrac{1}{\pi}\displaystyle\int_0^\pi \cos(x \cos \phi)\, d\phi$. **[Agra 2006]**

(*iv*) Deduce that $J_0(x) = 1 - \dfrac{x^2}{2^2} + \dfrac{x^4}{2^2 \cdot 4^2} - ... = \displaystyle\sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{(2^r \cdot r!)^2}$.

Sol. (i). We shall use the following results :

$$\left.\begin{aligned}\int_0^\pi \cos m\phi \cos n\phi\, d\phi = \int_0^\pi \sin m\phi \sin n\phi\, d\phi &= \pi/2 \textit{ when } m = n \\ &= 0 \textit{ when } m \neq n\end{aligned}\right\} \quad ...(1)$$

$$\cos(x \sin \phi) = J_0 + 2J_2 \cos 2\phi + 2J_4 \cos 4\phi + ... \quad ...(2)$$

and
$$\sin(x \sin \phi) = 2J_1 \sin \phi + 2J_3 \sin 3\phi + 2J_5 \sin 5\phi + ... \quad ...(3)$$

Multiplying both sides of (2) by $\cos n\phi$ and then integrating w.r.t. 'ϕ' between limits 0 to π and using (1), we have

$$\int_0^\pi \cos(x \sin \phi) \cos n\phi\, d\phi = 0, \text{ if } n \text{ is odd} \quad ...(4)$$

$$= \pi J_n, \text{ if } n \text{ is even.} \quad ...(5)$$

Again, multiplying both sides of (3) by $\sin n\phi$ and then integrating w.r.t. 'ϕ' between limits 0 to π and using (1), we get

$$\int_0^\pi \sin(x \sin \phi) \sin n\phi\, d\phi = \pi J_n, \text{ if } n \text{ is odd} \quad ...(6)$$

$$= 0, \text{ if } n \text{ is even.} \quad ...(7)$$

Let n be odd. Adding (4) and (6), we get

$$\int_0^\pi [\cos(x \sin \phi) \cos n\phi + \sin(x \sin \phi) \sin n\phi]\, d\phi = \pi J_n.$$

or $\displaystyle\int_0^\pi \cos(n\phi - x \sin \phi)\, d\phi = \pi J_n$ or $J_n(x) = \dfrac{1}{\pi}\displaystyle\int_0^\pi \cos(n\phi - x \sin \phi)\, d\phi$. ...(8)

Next, let n be even. Then adding (5) and (7) as before, we again get (8). Thus (8) holds for each positive integer (even as well as odd).

Part (ii). Let n be any integer. Then as in part (*i*), if n is positive integer, we have

$$J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(n\phi - x \sin \phi)\, d\phi \quad ...(9)$$

Next, let n be a negative integer so that $n = -m$, where m is a positive interger. To prove the required result for a negative integer, we prove that

$$J_{-m}(x) = \frac{1}{\pi}\int_0^\pi \cos(-m\phi - x \sin \phi)\, d\phi. \quad ...(10)$$

Let $\phi = \pi - \theta$ so that $d\phi = -d\theta$. Then, we have

R.H.S. of (10)

$$= \frac{1}{\pi}\int_{\pi}^{0} \cos\{-m(\pi-\theta)-x\sin(\pi-\theta)\}\,(-d\,\theta) = \frac{1}{\pi}\int_{0}^{\pi} \cos[(m\theta - x\sin\theta)-m\pi]\,d\,\theta$$

$$= \frac{1}{\pi}\int_{0}^{\pi} [\cos(m\theta - x\sin\theta)\cos m\pi + \sin(m\theta - x\sin\theta)\sin m\pi]\,d\,\theta$$

$$= \frac{1}{\pi}\int_{0}^{\pi} (-1)^m \cos(m\theta - x\sin\theta)\,d\,\theta \qquad [\because \sin m\pi = 0 \text{ and } \cos m\pi = (-1)^m]$$

$$= \frac{1}{\pi}(-1)^m \int_0^{\pi} \cos(m\phi - x\sin\phi)\,d\phi = (-1)^m J_m(x) \qquad [\text{Using (9) as } m \text{ is + ve integer}]$$

$$= J_{-m}(x) = \text{L.H.S. of (10)} \qquad [\because J_{-m}(x) = (-1)^m J_m(x)]$$

Thus, (10) is true. (9) and (10), show that the required result holds for each integer.

Part (iii). Integrating (2) w.r.t. 'ϕ' between the limits 0 to π and using the result

$\int_0^{\pi} \cos p\phi\,d\phi = 0$, if p is an even interger, we have ...(11)

$$\int_0^{\pi} \cos(x\sin\phi)\,d\phi = J_0(x)\int_0^{\pi} d\phi + 0 + 0 + \dots = J_0(x).\ \pi.$$

$$\therefore \qquad J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\phi)\,d\phi. \qquad \text{...(12)}$$

Replacing ϕ by $\pi/2 - \phi$ in (2) and simplifying, we get

$$\cos(x\cos\phi) = J_0 - 2J_2\cos 2\phi + 2J_4\cos 4\phi - \dots \qquad \text{...(13)}$$

Integrating (13) w.r.t. 'ϕ' and using (11), we get

$$\int_0^{\pi} \cos(x\cos\phi)\,d\phi = J_0(x).\ \pi - 0 - 0\dots \quad \text{or} \quad J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\cos\phi)\,d\phi. \qquad \text{...(14)}$$

(iv) Deduction. From (14), $J_0(x) = \dfrac{1}{\pi}\displaystyle\int_0^{\pi} \left(1 - \frac{x^2\cos^2\phi}{2\,!} + \frac{x^4\cos^4\phi}{4\,!} - \dots\right)d\phi$ ...(15)

But $$\int_0^{\pi} \cos^{2n}\phi\,d\phi = \frac{1\cdot 3\cdot 5\dots(2n-1)}{2\cdot 4\cdot 6\dots(2n)}\pi. \qquad \text{...(16)}$$

Using (16), (14) becomes $$J_0(x) = \frac{1}{\pi}\left[x - \frac{x^2}{2\,!}\cdot\frac{1}{2}\pi + \frac{x^4}{4\,!}\cdot\frac{1\cdot 3}{2\cdot 4}\pi - \frac{x^6}{6\,!}\cdot\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\pi + \dots\right]$$

or $$J_0(x) = 1 - \frac{x^2}{2^2} + \frac{x^4}{2^2\cdot 4^2} - \frac{x^6}{2^2\cdot 4^2\cdot 6^2} + \dots = \sum_{r=0}^{\infty} \frac{(-1)^r x^{2r}}{(2^r\cdot r\,!)^2}.$$

Ex. 3. *Use the generating function to show that* $J_n(-x) = (-1)^n J_n(x)$.

[Agra 2008; Meerut 2006, 11; Kanpur 2008]

Sol. We have $$\sum_{n=-\infty}^{\infty} J_n(x)\,z^n = \exp\left\{\frac{x}{2}\left(z - \frac{1}{z}\right)\right\}. \qquad \text{...(1)}$$

Replacing x by $-x$ in (1), we get

$$\sum_{n=-\infty}^{\infty} J_n(-x)\,z^n = \exp\left\{-\frac{x}{2}\left(z - \frac{1}{z}\right)\right\} = \exp\left\{\frac{x}{2}\left(-z - \frac{1}{-z}\right)\right\} = \sum_{n=-\infty}^{\infty} J_n(x)\,.\,(-z)^n, \text{ by (1)}$$

$$\therefore \qquad \sum_{n=-\infty}^{\infty} J_n(-x)\, z^n = \sum_{n=-\infty}^{\infty} J_n(x)\,(-1)^n\, z^n. \qquad ...(2)$$

Equating the coefficients of z^n from both sides of (2), we have $\qquad J_n(-x) = (-1)^n J_n(x).$

Ex. 4. *Use the generating function to prove that* $\qquad J_n(x+y) = \sum_{n=-\infty}^{\infty} J_r(x)\, J_{n-r}(y).$

Sol. By the generating function, $\quad \exp\left\{\frac{1}{2}(x+y)\left(z - \frac{1}{z}\right)\right\} = \sum_{n=-\infty}^{\infty} J_n(x+y)\, z^n. \qquad ...(1)$

So we see that $J_n(x+y)$ is the coefficient of z^n on R.H.S. of (1). We next obtain the coefficient of z^n on L.H.S. of (1). Now, we have

L.H.S. of (1) $= \exp\left\{\frac{1}{2}x\left(z - \frac{1}{z}\right)\right\}.\exp\left\{\frac{1}{2}y\left(z - \frac{1}{z}\right)\right\}$, as $\exp(A+B) = e^{A+B} = \exp A.\exp B.$

$$= \sum_{r=-\infty}^{\infty} J_r(x)\, z^r \,.\, \sum_{s=-\infty}^{\infty} J_s(y)\, z^s \qquad \text{[by definition of generating function]}$$

$$= \sum_{r=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} J_r(x)\, J_s(y)\, z^{r+s}. \qquad ...(2)$$

For a fixed value of r, we get z^n by taking $r + s = n$ *i.e.* $s = n - r$. Thus keeping r fixed, the coefficient of z^n in (1) is $J_r(x)\, J_{n-r}(y)$. So the total coefficient of z^n will be given by summing all such terms from $r = -\infty$ to $r = \infty$ and is given by $\qquad \sum_{r=-\infty}^{\infty} J_r(x)\, J_{n-r}(y).$

Hence equating the coefficient of z^n from both sides of (1), $\qquad J_n(x) = \sum_{r=-\infty}^{\infty} J_r(x)\, J_{n-r}(y).$

Ex. 5. *If* $a > 0$, *prove that* $\displaystyle\int_0^\infty e^{-ax} J_0(bx)\, dx = \frac{1}{\sqrt{(a^2+b^2)}}.$

[Ravishanker 1999; Purvanchal 2007; Lucknow 2010]

Sol. We know that (Refer Ex. 2. Part (*iii*)) $\qquad J_0(x) = \dfrac{1}{\pi}\displaystyle\int_0^\pi \cos(x \sin\phi)\, d\phi.$

$$\therefore \qquad J_0(bx) = \frac{1}{\pi}\int_0^\pi \cos(bx \sin\phi)\, d\phi. \qquad ...(1)$$

$$\therefore \quad \int_0^\infty e^{-ax} J_0(bx)\, dx = \int_0^\infty e^{-ax}\left\{\frac{1}{\pi}\int_0^\infty \cos(bx\sin\phi)\, d\phi\right\} dx, \text{ using (1)}$$

$$= \frac{1}{\pi}\int_0^\infty \left\{\int_0^\pi e^{-ax}\cos(bx\sin\phi)\, d\phi\right\} dx = \frac{1}{\pi}\int_0^\pi \left\{\int_0^\infty e^{-ax}\cos(bx\sin\phi)\, dx\right\} d\phi$$

(On interchanging the order of integration)

$$= \frac{1}{\pi}\int_0^\pi \left\{\int_0^\infty e^{-ax}\,\frac{e^{i\,bx\sin\phi} + e^{-i\,bx\sin\phi}}{2}\, dx\right\} d\phi = \frac{1}{2\pi}\int_0^\pi \left\{\int_0^\infty [e^{-(a-ib\sin\phi)x} + e^{-(a+ib\sin\phi)x}]\, dx\right\} d\phi$$

$$= \frac{1}{2\pi}\int_0^{\pi}\left[\frac{e^{-(a-ib\sin\phi)x}}{-(a-ib\sin\phi)}+\frac{e^{-(a+ib\sin\phi)x}}{-(a+ib\sin\phi)}\right]_0^{\infty} d\phi = \frac{1}{2\pi}\int_0^{\pi}\left[\frac{1}{a-ib\sin\phi}+\frac{1}{a+ib\sin\phi}\right]d\phi$$

$$= \frac{a}{\pi}\int_0^{\pi}\frac{d\phi}{a^2+b^2\sin^2\phi} = \frac{a}{\pi}\int_0^{\pi}\frac{\operatorname{cosec}^2\phi\, d\phi}{b^2+a^2\operatorname{cosec}^2\phi} = \frac{a}{\pi}\int_0^{\pi}\frac{\operatorname{cosec}^2\phi\, d\phi}{b^2+a^2(1+\cot^2\phi)} = \frac{2a}{\pi}\int_0^{\pi/2}\frac{\operatorname{cosec}^2 d\phi}{(a^2+b^2)+a^2\cot^2\phi}$$

$$= \frac{2a}{\pi}\int_{\infty}^{0}\frac{(-dt)}{(a^2+b^2)+a^2t^2}, \text{ putting } \cos\phi = t \text{ so that } -\operatorname{cosec}^2\phi\, d\phi = dt$$

$$= -\frac{2a}{\pi a^2}\int_{\infty}^{0}\frac{dt}{t^2+(a^2+b^2)/a^2} = \frac{2}{a\pi}\int_0^{\infty}\frac{dt}{t^2+[\sqrt{(a^2+b^2)}/a]^2}$$

$$= \frac{2}{a\pi}\cdot\frac{1}{\sqrt{(a^2+b^2)}/a}\left[\tan^{-1}\frac{t}{\sqrt{(a^2+b^2)}/a}\right]_0^{\infty} = \frac{2}{\pi\sqrt{(a^2+b^2)}}\left(\frac{\pi}{2}-0\right) = \frac{1}{\sqrt{a^2+b^2}}.$$

Ex. 6. *Using the generating function, prove that*

(*i*) $2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$ (*ii*) $2n\, J_n(x) = x\{J_{n+1}(x)+J_{n-1}(x)\}$

Sol. Generating function is given by $$\sum_{n=-\infty}^{\infty} J_n(x)z^n = e^{(x/2)\times(z-1/z)} \qquad \text{... (1)}$$

Diff. both sides of (1) w.r.t. 'x', $$\sum_{n=-\infty}^{\infty} J_n'(x)z^n = e^{(x/2)\times(z-1/z)}\times(1/2)\times(z-1/z)$$

or $$2\sum_{n=-\infty}^{\infty} J_n'(x)z^n = (z-z^{-1})\sum_{n=-\infty}^{\infty} J_n(x)\,z^n = \sum_{n=-\infty}^{\infty} J_n(x)\,(z^{n+1}-z^{n-1}), \text{ using (1)}$$

or $$2\sum_{n=-\infty}^{\infty} J_n'(x)\,z^n = \sum_{n=-\infty}^{\infty} J_n(x)\,z^{n+1} - \sum_{n=-\infty}^{\infty} J_n(x)\,z^{n-1} \qquad \text{... (2)}$$

Equating the coefficients of z^n on both sides of (2) yields $2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$

(*ii*) Differentiating both sides of (1) w.r.t. 'z', we get

$$\sum_{n=-\infty}^{\infty} n\,J_n(x)z^{n-1} = e^{(x/2)\times(z-1/z)}\times(1/2)\times(1+z^{-2})$$

or $$2\sum_{n=-\infty}^{\infty} n\,J_{n-1}(x)z^{n-1} = x\,(1+z^{-2})\sum_{n=-\infty}^{\infty} J_n(x)z^n, \text{ using (1)}$$

or $$2\sum_{n=-\infty}^{\infty} n\,J_n(x)z^{n-1} = x\sum_{n=-\infty}^{\infty} J_n(x)z^n + x\sum_{n=-\infty}^{\infty} J_n(x)z^{n-2} \qquad \text{... (3)}$$

Equating the coefficients of z^{n-1} on both sides of (3), $2n\, J_n(x) = x\{J_{n-1}(x) + J_{n+1}(x)\}$

Ex. 7. *Show that* $\displaystyle\int_0^y \frac{x\sin ax}{(y^2-x^2)^{1/2}}dx = \frac{\pi y}{2}J_1(ay)$

Sol. From result (ii) of Art. 11.9, $\sin(x\sin\phi) = 2\sin\phi\cdot J_1 + 2\sin 3\phi\cdot J_2 + \ldots$ ... (1)

Multiplying both sides of (1) by $\sin\phi$ and then integrating between the limits 0 and π, we have

$$\int_0^{\pi}\sin(x\sin\phi)\sin\phi\, d\phi = J_1\int_0^{\pi}(2\sin^2\phi)d\phi + J_2\int_0^{\pi}(2\sin\phi\sin 3\phi)\,d\phi + \ldots$$

$$= J_1\int_0^\pi (1-\cos 2\phi)\, d\phi + J_2\int_0^\pi (\cos 2\phi - \cos 4\phi) d\phi + ... = J_1\left[\phi - \frac{\sin 2\phi}{2}\right]_0^\pi + J_2\left[\frac{\sin 2\phi}{2} - \frac{\sin 4\phi}{2}\right]_0^\pi + ...$$

$= J_1 \cdot \pi$, since the remaining terms vanish

Thus, $$\pi\, J_1(x) = \int_0^\pi \sin\,(x \sin\phi) \sin\phi\, d\phi \qquad ... (2)$$

Let $F(\phi) = \sin(x\sin\phi)\sin\phi$. Then, clearly $F(\pi-\phi) - F(\phi)$.

Hence, using a property of definite integrals, (2) yields $\pi\, J_1(x) = 2\int_0^{\pi/2} \sin\,(x\sin\phi)\,\sin\phi\, d\phi$

$$\Rightarrow \qquad \pi\, J_1(ay) = 2\int_0^{\pi/2} \sin\,(ay\sin\phi)\,\sin\phi\, d\phi \qquad ... (3)$$

Put $y\sin\phi = x$ so that $y\cos\phi\, d\phi = dx$ and hence

$$d\phi = \frac{dx}{y\cos\phi} = \frac{dx}{y(1-\sin^2\phi)^{1/2}} = \frac{dx}{y(1-x^2/y^2)^{1/2}} = \frac{dx}{(y^2-x^2)^{1/2}}$$

Hence (3) yields $\pi\, J_1(ay) = 2\int_0^y \left(\sin ax \times \frac{x}{y} \times \frac{1}{(y^2-x^2)^{1/2}}\right) dx \quad \Rightarrow \quad \int_0^y \frac{x \sin ax\, dx}{(y^2-x^2)^{1/2}} = \frac{\pi y}{2} J_1(ay)$

Ex. 8. *Prove that* (*i*) $J_n(x) = \dfrac{1}{\sqrt{\pi}\ \Gamma(n+\frac{1}{2})}\left(\dfrac{x}{2}\right)^n \displaystyle\int_0^\pi \cos\,(x\,\sin\,\phi)\cos^{2n}\phi\, d\phi.$

[Garhwal 2005, Ravishanker 2004; Ranchi 2010]

(*ii*) $J_n(x) = \dfrac{1}{\sqrt{\pi}\ \Gamma(n+\frac{1}{2})}\left(\dfrac{x}{2}\right)^n \displaystyle\int_0^\pi \cos\,(x\,\cos\,\phi)\sin^{2n}\phi\, d\phi.$ **[Bilaspur 1998]**

Sol. (i). Let $$I = \int_0^\pi \cos\,(x\,\sin\,\phi)\cos^{2n}\phi\, d\phi. \qquad ...(1)$$

$\therefore\ I = 2\int_0^{\pi/2} \cos\,(x\,\sin\,\phi)\cos^{2n}\phi\, d\phi$, since $\cos(x\sin\phi)\cos^{2n}\phi$ is an even function.

or $$I = 2\int_0^{\pi/2} \cos^{2n}\phi\left[1 - \frac{x^2\sin^2\phi}{2\,!} + \frac{x^4\sin^4\phi}{4\,!} - \ldots\right] d\phi. \qquad ...(2)$$

But $$2\int_0^{\pi/2} \cos^p\phi\,\cos^q\phi\, d\phi = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}. \qquad ...(3)$$

Using (3), (2) may be written as

$$I = 2\left[\frac{\Gamma(n+\frac{1}{2})\,\Gamma(\frac{1}{2})}{2\Gamma(n+1)} - \frac{x^2}{2\,!}\,\frac{\Gamma(n+\frac{1}{2})\,\Gamma(3/2)}{2\Gamma(n+2)} + \frac{x^4}{4\,!}\,\frac{\Gamma(n+\frac{1}{2})\,\Gamma(5/2)}{2\Gamma(n+3)} - \ldots\right]$$

$$= \Gamma(n+\tfrac{1}{2})\left[\frac{\sqrt{\pi}}{\Gamma(n+1)} - \frac{x^2}{2\,!}\times\frac{\frac{1}{2}\sqrt{\pi}}{(n+1)\,\Gamma(n+1)} + \frac{x^4}{4\,!}\times\frac{(3/2)\times\frac{1}{2}\sqrt{\pi}}{(n+2)\,(n+1)\Gamma(n+1)} - ..\right]$$

$$= \frac{\Gamma(n+\frac{1}{2})\sqrt{\pi}}{\Gamma(n+1)}\left[1 - \frac{x^2}{4(n+1)} + \frac{x^4}{4\cdot 8(n+1)\,(n+2)} - ...\right] \qquad ...(4)$$

Also, $$J_n(x) = \frac{x^n}{2^n \Gamma(n+1)}\left[1 - \frac{x^2}{4(n+1)} + \frac{x^4}{4 \cdot 8(n+1)(n+2)} - \ldots\right]. \qquad \ldots(5)$$

Multiplying both sides of (4) by $\dfrac{x^n}{2^n \Gamma(n+\frac{1}{2})\sqrt{\pi}}$, we get

$$\frac{x^n I}{2^n \Gamma(n+\frac{1}{2})\sqrt{\pi}} = \frac{x^n}{2^n \Gamma(n+\frac{1}{2})}\left[1 - \frac{x^2}{4(n+1)} + \frac{x^4}{4 \cdot 8 \cdot (n+1)(n+2)} - \ldots\right] \qquad \ldots(6)$$

Using (1) and (5), (6) becomes $\quad \dfrac{1}{\sqrt{\pi}\,\Gamma(n+1/2)}\left(\dfrac{x}{2}\right)^n \displaystyle\int_0^\pi \cos(x \sin\phi)\cos^{2n}\phi \, d\phi = J_n(x).$

Part (ii). Proceed as in part (*i*),

Ex. 9. *Prove that* $J_n(x) = \dfrac{x^n}{2^{n-1}\,\Gamma(n+\frac{1}{2})\sqrt{\pi}}\displaystyle\int_0^1 (1-t^2)^{n-1/2}\cos xt \, dt.$ **[Bilaspur 1994, 97]**

Sol. From part (*i*) of Ex. 8, $\quad J_n(x) = \dfrac{1}{\sqrt{\pi}\,\Gamma(n+\frac{1}{2})}\left(\dfrac{x}{2}\right)^n \displaystyle\int_0^\pi \cos(x\sin\phi)\cos^{2n}\phi\, d\phi. \quad \ldots(1)$

Let $$I = \int_0^\pi \cos(x\sin\phi)\cos^{2n}\phi\, d\phi = 2\int_0^{\pi/2}\cos(x\sin\phi)\cos^{2n}\phi\, d\phi \qquad \ldots(2)$$

Let $\sin\phi = t$ so that $\cos\phi\, d\phi = dt$. Then (2) gives

$$I = 2\int_0^1 \cos xt \cos^{2n}\phi \frac{dt}{\cos\phi} = 2\int_0^1 \cos xt \cos^{2n-1}\phi\, dt$$

$$= 2\int_0^1 \cos xt\,(\cos^2\phi)^{(2n-1)/2} dt = 2\int_0^1 \cos xt\,(1-\sin^2\phi)^{n-1/2} dt.$$

$$\therefore \qquad I = 2\int_0^1 \cos xt.\,(I - t^2)^{n-1/2}\, dt. \qquad \ldots(3)$$

Using (2), (3) gives $\quad \displaystyle\int_0^\pi \cos(x\sin\phi)\cos^{2n} d\phi = 2\int_0^1 (1-t^2)^{n-1/2}\cos xt\, dt. \qquad \ldots(4)$

Using (4), (1) $\quad \Rightarrow \quad J_n(x) = \dfrac{1}{\sqrt{\pi}\,\Gamma(n+\frac{1}{2})2^n} \times \dfrac{x^n}{2^n} \times 2\displaystyle\int_0^1 (1-t^2)^{n-1/2}\cos xt\, dt$

$$\therefore \qquad J_n(x) = \frac{x^n}{2^{n-1}\,\Gamma(n+\frac{1}{2})\sqrt{\pi}}\int_0^1 (1-t^2)^{n-1/2}\cos xt\, dt.$$

Ex. 10. *Verify directly that representation* $J_0(x) = \dfrac{1}{\pi}\displaystyle\int_0^\pi \cos(x\sin\theta)\, d\theta$ *satisfies Bessel's equation in which* $n = 0$. **[Indore 2004]**

Sol. Given $$J_0(x) = \frac{1}{\pi}\int_0^\pi \cos(x\sin\theta)\, d\theta. \qquad \ldots(1)$$

Bessel's equation of order n is $\quad x^2y'' + xy' + (x^2 - n^2)y = 0. \qquad \ldots(2)$

For $n = 0$, (2) reduces to $\quad x^2y'' + xy' + x^2y = 0. \qquad \ldots(3)$

In order to show that $J_0(x)$ satisfies (3), we must prove that

$$x^2 J_0''(x) + xJ_0'(x) + x^2 J_0(x) = 0. \qquad \ldots(4)$$

Differentiating both sides of (1) w.r.t. 'x', $\quad J_0'(x) = -\frac{1}{\pi}\int_0^\pi \sin(x\sin\theta)\sin\theta\, d\theta.$ ...(5)

Differentiating both sides of (5) w.r.t. 'x', $\quad J_0''(x) = -\frac{1}{\pi}\int_0^\pi \cos(x\sin\theta)\sin^2\theta\, d\theta.$ ...(6)

Integrating R.H.S. of (5) by parts taking $\sin\theta$ as second function, we get

$$J_0'(x) = -\frac{1}{\pi}\left[\{-\sin(x\sin\theta)(\cos\theta)\}_0^\pi + \int_0^\pi \cos(x\sin\theta)\cdot x\cos\theta\cos\theta\, d\theta\right]$$

or
$$J_0'(x) = -\frac{1}{\pi}\left[0 + x\int_0^\pi \cos(x\sin\theta)\cos^2\theta\, d\theta\right] = -\frac{x}{\pi}\int_0^\pi \cos(x\sin\theta)\cos^2\theta\, d\theta. \quad ...(7)$$

Using (1), (6) and (7), L.H.S. of (4)

$$= -\frac{x^2}{\pi}\int_0^\pi \cos(x\sin\theta)\sin^2\theta\, d\theta - \frac{x^2}{\pi}\int_0^\pi \cos(x\sin\theta)\cos^2\theta\, d\theta + \frac{x^2}{\pi}\int_0^\pi \cos(x\sin\theta)\, d\theta$$

$$= -\frac{x^2}{\pi}\int_0^\pi \cos(x\sin\theta)(\sin^2\theta + \cos^2\theta)\, d\theta + \frac{x^2}{\pi}\int_0^\pi \cos(x\sin\theta)\, d\theta = 0 = \text{R.H.S. of (4)}$$

Thus (4) is true. Hence we get the required result.

EXERCISE 11 (D)

1. Prove that (*i*) $J_0(x) = \frac{2}{x}\int_0^1 \frac{\cos xt\, dt}{\sqrt{(1-t^2)}}$ $\qquad$ (*ii*) $\int_0^\infty \frac{J_n(x)}{x}\, dx = \frac{1}{n}.$

2. If n is non–negative, prove that $\int_0^\infty J_n(bx)\, dx = \frac{1}{b}.$

3. Prove that (*i*) $\int_0^\infty \sin ax\, J_0(bx)\, dx = \begin{cases} 0, & b > a \\ 1/(a^2+b^2)^{1/2}, & b < a \end{cases}$

(*ii*) $\int_0^\infty \cos ax\, J_0(bx)\, dx = \begin{cases} 1/(b^2-a^2)^{1/2}, & b > a \\ 0, & b < a \end{cases}$

4. Show that (*i*) $\int_0^{\pi/2} J_0(x\cos\theta)\cos\theta\, d\theta = \frac{\sin x}{x}$ $\qquad$ (*ii*) $\int_0^{\pi/2} J_1(x\cos\theta)\, d\theta = \frac{1-\cos x}{x}.$

5. For Bessel function $J_n(x)$ prove that $J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(n\theta - x\sin\theta)\, d\theta$ and hence show that

$J_0(x) = \frac{2}{\pi}\int_0^1 \frac{\cos xt\, dt}{\sqrt{(1-t^2)}}.$ **[Purvanchal 2007]**

6. Show that the recurrence relation $J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$, follows directly from differentiation of $J_n(x) = \frac{1}{\pi}\int_0^\pi \cos(n\theta - x\sin\theta)\, d\theta.$

7. Show that coefficient of t^n in the expansion of $e^{-(x/2)(t-1/t)}$ equals $\frac{1}{\pi}\int_0^\pi \cos(n\phi - x\sin\phi)\, d\phi.$

11.10. Orthogonality of Bessel functions

If λ_i and λ_j are roots of the equation $J_n(\lambda a) = 0$, then

$$\int_0^a x\, J_n(\lambda_i x)\, J_n(\lambda_j x)\, dx = \begin{cases} 0, \text{ if } i \neq j \text{ (different roots)} \\ \dfrac{a^2}{2} J_{n+1}^2(\lambda_i a), \text{ if } i = j \text{ (equal roots)} \end{cases}$$

i.e. $\displaystyle\int_0^a x\, J_n(\lambda_i x)\, J_n(\lambda_j x)\, dx = \frac{a^2}{2} J_{n+1}^2(\lambda_i a)\, \delta_{ij}.$ where δ_{ij} = Kronecker delta = $\begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$

[Meerut 2010; Nagpur 2005; Purvanchal 2005, 06; Delhi Physics (H) 2002]

Proof. Case I. Let $i \neq j$, *i.e.*, let λ_i and λ_j be unequal roots of $J_n(\lambda a) = 0$.

$\therefore$ $J_n(\lambda_i a) = 0$ and $J_n(\lambda_j a) = 0$ ...(1)

Let $u(x) = J_n(\lambda_i x)$ and $v(x) = J_n(\lambda_j x)$. ...(2)

Then u and v are Bessel functions satisfying the modified Bessel's equation [Refer Art. 11.4]

$$x^2 y'' + xy' + (\lambda^2 x^2 - n^2) y = 0. \quad \text{...(3)}$$

$$\therefore \quad x^2 u'' + xu' + (\lambda_i^2 x^2 - n^2) u = 0. \quad \text{...(4)}$$

and

$$x^2 v'' + xv' + (\lambda_j^2 x^2 - n^2) v = 0. \quad \text{...(5)}$$

Multiplying (4) by v and (5) by u and then subtracting, we get

$$x^2(vu'' - uv'') + x(vu' - uv') + x^2(\lambda_i^2 - \lambda_j^2)uv = 0 \quad \text{or} \quad x(vu'' - uv'') + (vu' - uv') = x(\lambda_j^2 - \lambda_i^2)uv$$

or

$$x\frac{d}{dx}(vu' - uv') + (vu' - uv') = x(\lambda_j^2 - \lambda_i^2)uv$$

or

$$\frac{d}{dx}\{x(vu' - uv')\} = x(\lambda_j^2 - \lambda_i^2)uv. \quad \text{...(6)}$$

Integrating (6) w.r.t. x from 0 to a, $\displaystyle(\lambda_j^2 - \lambda_i^2) \int_0^a xuv\, dx = \big[x(vu' - uv')\big]_0^a.$ ...(7)

Using (2), (7) reduces to

$$(\lambda_j^2 - \lambda_i^2)\int_0^a x\, J_n(\lambda_i x)\, J_n(\lambda_j x)\, dx = \Big[x\{J_n(\lambda_j x) J_n'(\lambda_i x) - J_n(\lambda_i x)\, J_n'(\lambda_j x)\}\Big]_0^a$$

$$= a\{J_n(\lambda_j a) J'_n(\lambda_i a) - J_n(\lambda_i a) J'_n(\lambda_j a)\} = 0, \text{ using (1)}$$

Since $\lambda_i \neq \lambda_j$, the above equation gives $\displaystyle\int_0^a x\, J_n(\lambda_i x)\, J_n(\lambda_j x)\, dx = 0$, when $i \neq j$. ...(8)

Case II. Let $i = j$ (equal roots). Multiplying (4) by $2u'$, we have

$$2x^2 u'' u' + 2xu'^2 + 2(\lambda_i^2 x^2 - n^2) uu' = 0 \quad \text{or} \quad \frac{d}{dx}\{x^2 u'^2 - n^2 u^2 + \lambda_i^2 x^2 u^2\} - 2\lambda_i^2 xu^2 = 0$$

$$\therefore \quad 2\lambda_i^2 xu^2 = \frac{d}{dx}(x^2 u'^2 - n^2 u^2 + \lambda_i^2 x^2 u^2). \quad \text{...(9)}$$

Integrating (9) w.r.t. 'x' from 0 to a, $\displaystyle 2\lambda_i^2 \int_0^a xu^2\, dx = \Big[x^2 u'^2 - n^2 u^2 + \lambda_i^2 x^2 u^2\Big]_0^a$ ...(10)

Using (1) and (2) and noting that $J_n(0) = 0$, we have

$$2\lambda_i^2 \int_0^a x\, J_n^2(\lambda_i x)\, dx = a^2\Big[\{J_n'(\lambda_i x)\}^2\Big]_{x=a} \quad \text{...(11)}$$

From recurrence relation IV, we have $\displaystyle\frac{d}{dx} J_n(x) = \frac{n}{x} J_n(x) - J_{n+1}(x).$ ...(12)

Replacing x by $\lambda_i x$ in (12), we have

$$\frac{d}{d(\lambda_i x)} J_n(\lambda_i x) = \frac{n}{\lambda_i x} J_n(\lambda_i x) - J_{n+1}(\lambda_i x) \quad \text{or} \quad \frac{1}{\lambda_i}\frac{d}{dx} J_n(\lambda_i x) = \frac{n}{\lambda_i x} J_n(\lambda_i x) - J_{n+1}(\lambda_i x)$$

or $$J'_n(\lambda_i x) = \frac{n}{x} J_n(\lambda_i x) - \lambda_i J_{n+1}(\lambda_i x).$$

$$\therefore \quad \left[\{J_n'(\lambda_i x)\}^2\right]_{x=a} = \left[\left\{\frac{n}{x} J_n(\lambda_i x) - \lambda_i J_{n+1}(\lambda_i x)\right\}^2\right]_{x=a} = \{0 - \lambda_i J_{n+1}(\lambda_i a)\}^2, \text{ by (1)}$$
$$= \lambda_i^2 J^2_{n+1}(\lambda_i a).$$

Using this value in (11) and dividing both sides of the resulting equation by $2\lambda_i^2$, we get

$$\int_0^a x J_n^2(\lambda_i x)\, dx = \frac{a^2}{2} J_{n+1}^2(\lambda_i a). \qquad \text{...(13)}$$

Combining (8) and (13), we have $$\int_0^a x J_n(\lambda_i x) J_n(\lambda_j x)\, dx = \frac{a^2}{2} J_{n+1}^2(\lambda_i a)\, \delta_{ij}.. \qquad \text{...(14)}$$

11.11. Bessel–series or Fourier–Bessel expansion for f(x).

If $f(x)$ is defined in the region $0 \le x \le a$ and has an expansion of the form

$$f(x) = \sum_{i=1}^{\infty} c_i J_n(\lambda_i x), \qquad \text{...(1)}$$

where the λ_i are the roots of the equation $$J_n(\lambda a) = 0, \qquad \text{...(2)}$$

then $$c_i = \frac{2\int_0^a x f(x) J_n(\lambda_i x)\, dx}{a^2 J_{n+1}^2(\lambda_i a)}. \qquad \text{...(3)}$$

Proof. Multiplying both sides of (1) by $xJ_n(\lambda_j x)$, $xf(x) J_n(\lambda_j x) = \sum_{i=1}^{\infty} c_i x J_n(\lambda_i x) J_n(\lambda_j x)$ ...(4)

Integrating both sides of (4) w.r.t. 'x' from 0 to a, we get

$$\int_0^a x f(x) J_n(\lambda_j x)\, dx = \sum_{i=1}^{\infty} c_i \int_0^a x J_n(\lambda_i x) J_n(\lambda_j x)\, dx. \qquad \text{...(5)}$$

From the orthogonality property of Bessel functions, we have

$$\int_0^a x J_n(\lambda_i x) J_n(\lambda_j x)\, dx = \begin{cases} 0, \text{ if } i \neq j \\ \dfrac{a^2}{2} J_{n+1}^2(\lambda_j a), \text{ if } i = j \end{cases} \qquad \text{...(6)}$$

Using (6), (5) reduces to $$\int_0^a x f(x) J_n(\lambda_j x)\, dx = c_j \frac{a^2}{2} J^2_{n+1}(\lambda_j a). \qquad \text{...(7)}$$

Replacing j by i in (7), we have

$$c_i \frac{a^2}{2} J^2_{n+1}(\lambda_i a) = \int_0^a x f(x) J_n(\lambda_i x)\, dx \quad \text{or} \quad c_i = \frac{2\int_0^a x f(x) J_n(\lambda_i x)\, dx}{a^2 J_{n+1}^2(\lambda_i a)}.$$

...(8)

11.11A. Solved Exaples based on Art. 11.11

Ex. 1. *Expand the function* $f(x) = 1$, $0 \le x \le a$ in a series of the form $\sum_{i=1}^{\infty} c_i J_0(\lambda_i x)$, where λ_i are the roots of the equation $J_0(\lambda a) = 0$.

Sol. Given $$f(x) = 1 = \sum_{i=1}^{\infty} c_i J_0(\lambda_i x), \qquad \text{...(1)}$$

where $$J_0(\lambda_i a) = 0. \qquad ...(2)$$

Then from Art. 11. 11 (with $n = 0$), we know that

$$c_i = \frac{2\int_0^a x\ f(x)\ J_0(\lambda_i x)\ dx}{a^2 J_1^2(\lambda_i a)} = \frac{2\int_0^a x\ J_0(\lambda_i x)\ dx}{a^2 J_1^2(\lambda_i a)},\ \text{as } f(x) = 1 \qquad ...(3)$$

Let $\lambda_i x = t$ so that $dx = dt/\lambda_i$. Then, we have

$$\int_0^a x\ J_0(\lambda_i x)\ dx = \frac{1}{\lambda_i^2}\int_0^{a\lambda_i} t\ J_0(t)\ dt$$

$$= \frac{1}{\lambda_i^2}\int_0^{a\lambda_i} \frac{d}{dt}\{t\ J_1(t)\}\ dt,\ \text{as}\ \ \frac{d}{dx}[x^n\ J_n(x)] = x^n J_{n-1}(x) \Rightarrow \frac{d}{dx}[x\ J_1(x)] = x\ J_0(x)$$

$$= \frac{1}{\lambda_i^2}\left[t\ J_1(t)\right]_0^{a\lambda_i} = \frac{1}{\lambda_i^2}[a\lambda_i\ J_1(a\lambda_i) - 0],\ \text{as } J_1(0) = 0.$$

$$\therefore \qquad \int_0^a x\ J_0(\lambda_i x)\ dx = (a/\lambda_i) \times J_1(a\lambda_i) \qquad ...(4)$$

Using (4), (3) becomes $$c_i = \frac{2 \times (a/\lambda_i) \times J_1(a\lambda_i)}{a^2\ J_1^2\ (a\lambda_i)} = \frac{2}{a\lambda_i J_1(a\lambda_i)} \qquad ...(5)$$

Using (5), (1) becomes $$1 = \frac{2}{a}\sum_{i=1}^{\infty}\frac{J_0(\lambda_i x)}{\lambda_i\ J_0(\lambda_i a)}.$$

Ex. 2. *Expand x in a series of the form* $\sum_{r=1}^{\infty} C_r\ J_1(\lambda_r x)$ *valid for the region* $0 \le x \le 1$, *where* λ_r *are the roots of the equation* $J_1(\lambda) = 0$.

Sol. Given $$f(x) = x = \sum_{r=1}^{\infty} C_r\ J_1(\lambda_r x), \qquad ...(1)$$

where $$J_1(\lambda_r) = 0 \qquad ...(2)$$

Then from Art. 11.11 (with $n = 1$, $i = r$ and $a = 1$), $$C_r = \frac{2\int_0^1 x^2\ J_1(\lambda_r x)\ dx}{J_2^2(\lambda_r)} \qquad ...(3)$$

Let $\lambda_r x = t$, so that $dx = dt/\lambda_r$. Then we have

$$\int_0^1 x^2\ J_1(\lambda_r x)\ dx = \frac{1}{\lambda_r^3}\int_0^{\lambda_r} t^2\ J_1(t)\ dt$$

$$= \frac{1}{\lambda_r^3}\int_0^{\lambda_r}\frac{d}{dt}\{t^2\ J_2(t)\}\ dt, \quad \text{as} \quad \frac{d}{dx}[x^n\ J_n(x)] = x^n J_{n-1}(x) \Rightarrow \frac{d}{dx}[x^2\ J_2(x)] = x^2\ J_1(x)$$

$$= \frac{1}{\lambda_r^3}\left[t^2\ J_2(t)\right]_0^{\lambda_r} = \frac{1}{\lambda_r^3}[\lambda_r^2\ J_2(\lambda_r) - 0],\ \text{as } J_2(0) = 0$$

$$\therefore \qquad \int_0^1 x^2\ J_1(\lambda_r x)\ dx = \frac{1}{\lambda_r}\ J_2(\lambda_r). \qquad ...(4)$$

Using (4), (3) becomes $$C_r = \frac{2}{\lambda_r\ J_2(\lambda_r)} \qquad ...(5)$$

Using (5), (1) becomes $$x = 2\sum_{r=1}^{\infty}\frac{J_1(\lambda_r x)}{\lambda_r\ J_2(\lambda_r)},\ 0 \le x \le 1.$$

EXERCISE 11 (E)

1. Expand x^2 in a series of the form $\sum_{r=1}^{\infty} C_r\, J_0(\lambda_r x)$ valid for the region $0 \le x \le a$, where λ_r are the roots of the equation $J_0(\lambda a) = 0$. **Ans.** $x^2 = \dfrac{2}{a}\sum_{r=1}^{\infty} \dfrac{\{(\lambda_r a)^2 - 4\}\, J_0(\lambda_r x)}{\lambda_r^3\, J_0(\lambda_r a)}$

2. Prove that $1 = \sum_{n=1}^{\infty} \dfrac{2}{\alpha_n} \dfrac{J_0(\alpha_n x)}{J_1(\alpha_n)}$.

3. If λ_i are the solutions of $J_0(\lambda) = 0$, show that $\sum_{i=1}^{\infty} \dfrac{J_0(\lambda_i x)}{\{\lambda_i J_1(\lambda_n)\}^2} = -\dfrac{1}{2}\log x$, where $0 < x < 1$.

4. If $f(x) = \sum_{i=1}^{\infty} C_i\, J_0(\lambda_i x)$ where $J_0(\lambda_i) = 0$, $i = 1, 2, 3, \ldots$, show that $\int_0^1 x[f(x)]^2\, dx = \sum_{i=1}^{\infty} \lambda_i^2\, J_1^2(\lambda_i)$.

5. If λ_i are the positive roots of $J_0(\lambda) = 0$, show that $\dfrac{1-x^2}{8} = \sum_{i=1}^{\infty} \dfrac{J_0(\lambda_i x)}{\lambda_i^3\, J_1(\lambda_i)}$, where $-1 < x < 1$

6. If λ_i are the positive roots of $J_1(\lambda) = 0$, show that $x^3 = 2\sum_{i=1}^{\infty} \dfrac{J_1(\lambda_i x)}{\lambda_i\, J_2(\lambda_i)}$, where $-1 < x < 1$

7. If λ_i are the positive roots of $J_1(\lambda) = 0$, show that $x^3 = 2\sum_{i=1}^{\infty} \dfrac{(8-\lambda_i^2)\, J_1(\lambda_i x)}{\lambda_i^3\, J_1'(\lambda_i)}$, where $-1 < x < 1$

OBJECTIVE PROBLEMS ON CHAPTER 11

1. *Write* (*a*), (*b*), (*c*) *or* (*d*) *whichever is correct*

(*a*) $x\{J_{n-1}(x) + J_{n+1}(x)\}$ is equal to

(*a*) $2J_n(x)$ (*b*) $2J_n'(x)$ (*c*) $2n\, J_n(x)$ (*d*) None of these **[Agra 2005, 06]**

Sol. Ans. (*c*). Refer recurrence relation VI of Art. 11.7.

2. The Bessel's equation is (*a*) $z^2(d^2w/dz^2) - z\,(dw/dz) + (z^2 - n^2)\, w = 0$
(*b*) $z^2(d^2w/dz^2) + z\,(dw/dz) + (z^2 + n^2)\, w = 0$ (*c*) $z^2(d^2w/dz^2) + z(dw/dz) + (z^2 - n^2)w = 0$
(*d*) None of these. **[Agra 2005, 07]**

Sol. Ans. (c). Refer equation (1) of Art. 11.1

3. $d\,\{x^n J_n(x)\}/dx$ is equal to
(*a*) $x^n J_{n-1}(x)$ (*b*) $x^{n-1} J_n(x)$ (*c*) $x^n J_{n+1}(x)$ (*d*) $x^{n+1} J_n(x)$ **[Bhopal 2010]**

Sol. Ans. (*a*). Refer recurrence relation I of Art. 11.7.

4. *The Bessel's functions* $\{J_0(\alpha_k x)\}_{k=1}^{\infty}$ *with* α_k *denoting the* k^{th} *zero of* $J_0(x)$ *form an orthogonal system on* [0, 1] with respect to weight function (*a*) 1 (*b*) x^2 (*c*) x (*d*) $\sqrt{x}$.

Sol. Ans. (*c*) Refer Art. 11.10. **[GATE 2002]**

5. *If* $J_n(x)$ *and* $Y_n(x)$ *denote Bessel functions of order n of the first and second kind, then the general solution of the differential equation* $x(d^2y/dx^2) + (dy/dx) + xy = 0$ *is given by*

(*a*) $y(x) = \alpha x\, J_1(x) + \beta x Y_1(x)$ (*b*) $y(x) = \alpha\, J_1(x) + \beta Y_1(x)$

(*c*) $y(x) = \alpha J_0(x) + \beta Y_0(x)$ (*d*) $y(x) = \alpha x J_0(x) + \beta x Y_0(x)$. **[GATE 2005]**

Sol. Ans. (*c*) Refer remark of theorem I, Art. 11.4.

MISCELLANEOUS PROBLEMS ON CHAPTER 11

1. If $x > a$, show that $\int_0^{\pi} e^{a\cos\theta} \cos(x\sin\theta)\, d\theta = J_0\{(x^2 - a^2)^{1/2}\}$

2. Prove that $J_0(x) = \dfrac{1}{\pi}\int_0^{\pi} \cos(x \cos\phi)\, d\phi$ and deduce that (*i*) $\int_0^{\infty} e^{-ax} J_0(bx)dx = \dfrac{1}{(a^2 + b^2)^{1/2}}$

(*ii*) $\int_0^{\infty} \cos(ax) J_0(bx)dx = 0$ or $\dfrac{1}{(b^2 - a^2)^{1/2}}$, according $a^2 >$ or $< b^2$

(*iii*) $\int_0^{\infty} \sin(ax) J_0(bx)\, dx = -\dfrac{1}{(a^2 - b^2)^{1/2}}$ or 0, according as $a^2 >$ or $< b^2$.

3. Show that (*i*) $\int_0^{\pi/2} J_{2n}(2x\cos\theta)\, d\theta = \dfrac{\pi}{2}[J_n(x)]^2$ (*ii*) $J_0(x) = \dfrac{2}{\pi}\int_0^{\infty} \dfrac{\sin(x+y)}{x+y} J_0(y)\, dy$

(*iii*) $x^n = 2^n \sum\limits_{r=0}^{\infty} \dfrac{(n+2r)\,(n+r-1)}{r\,!} J_{n+2r}(x)$

4. Show that $J_n(x)$ is even or odd function of x according as n is even or odd, respectively.

5. Show that $\int_0^1 J_0\left(\{x\,(t-x)\}^{1/2}\right) = 2\sin(t/2)$. **[Kanpur 2008]**

Sol. Setting $n = 0$ and $x = \{x\,(t-x)\}^{1/2}$ in result (1) of Art. 11.2, we have

$$j_0(\{x\,(t-x)\}^{1/2}) = \sum_{r=0}^{\infty}(-1)^r \frac{1}{r!\,\Gamma(r+1)}\left[\frac{\{x(t-x\}^{1/2}}{2}\right]^{2r} = \sum_{r=0}^{\infty}\frac{(-1)^r x^r (t-x)^r}{2^{2r} r!\,\Gamma(r+1)}$$

$$\therefore \int_0^1 J_0\left(\{x(t-x)\}^{1/2}\right)dx = \int_0^1 \sum_{r=0}^{\infty}\frac{(-1)^r x^r (t-x)^r}{2^{2r} r!\Gamma(r+1)}dx = \sum_{r=0}^{\infty}\frac{(-1)^r}{2^{2r} r!\Gamma(r+1)}\int_0^1 x^r (t-x)^r\, dx$$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r t^{2r+1}}{2^{2r} r!\Gamma(r+1)}\int_0^1 y^r (1-y)^r\, dy, \text{ putting } x = ty \text{ and } dx = tdy$$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r t^{2r+1}}{2^{2r} r!\Gamma(r+1)} B(r+1, r+1), \text{ by definition 6.2, page 6.1}$$

$$= \sum_{r=0}^{\infty}\frac{(-1)^r t^{2r+1}\, r!}{2^{2r} r!\Gamma(2r+1)}, \text{ since } \Gamma(r+1) = r!, r \text{ being a positive integer.}$$

$$= 2\sum_{r=0}^{\infty}(-1)^r \frac{(t/2)^{2r+1}}{(2r+1)!} = 2\left\{\frac{t}{2} - \frac{(t/2)^3}{3!} + \frac{(t/5)^5}{5!} - \ldots\ldots\right\}$$

$= 2\sin(t/2)$, as $\sin x = x - x^3/3! + x^3/5! - x^7/7! + \ldots$

6. Show that every non-trivial solution of Bessel's equation $x^2 y'' + xy' + (x^2 - n^2) y = 0$ has infinitely many zeros. **[Mumbai 2010]**

7. Find the normal form of Bessel's equation $x^2y'' + xy' + (x^2 - p^2) y = 0$ and use it to show that every non-trival solution has infinitely many positive zeros. **[Himachal 2010]**

8. Show that if x is real, between two consecutive zeros of $x^{-n} J_n (x)$, there lies one and only one zero of $x^{-n} \mathrm{J}_{n+1} (x)$.

12

Hermite Polynomials

12.1. Hermite's equation and its solution. [Meerut 1993, 95, 96]

Hermite's equation is $$(d^2y/dx^2) - 2x(dy/dx) + 2ny = 0, \qquad ...(1)$$
where n is a constant. We now solve (1) in series by using the method of Frobenius.

Let the series solution of (1) be $$y = \sum_{m=0}^{\infty} C_m x^{k+m}, \quad C_0 \neq 0 \qquad ...(2)$$

Differentiating (2) and then putting the value of y, dy/dx and d^2y/dx^2 in (1), we get

$$\sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2} - 2xC_m(k+m) x^{k+m-1} + 2n \sum_{m=0}^{\infty} C_m x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2} - 2\left[\sum_{m=0}^{\infty} C_m (k+m) x^{k+m} - \sum_{m=0}^{\infty} C_m n x^{k+m}\right] = 0$$

or $$\sum_{m=0}^{\infty} C_m (k+m)(k+m-1) x^{k+m-2} - 2\sum_{m=0}^{\infty} C_m (k+m-n) x^{k+m} = 0. \qquad ...(3)$$

(3) is an identity. To get the indicial equation, we equate to zero the coefficient of the smallest power of x, namely x^{k-2}, in (3) and obtain

$$C_0\, k(k-1) = 0 \quad \text{or} \quad k(k-1) = 0, \quad \text{as} \quad C_0 \neq 0. \qquad ...(4)$$

So the roots of indicial equation (4) are $k = 0, 1$. They are distinct and differ by an integer. The next smallest power of x is $k - 1$. So equating to zero the coefficient of x^{k-1} in (3), we get

$$C_1 (k+1) k = 0. \qquad ...(5)$$

When $k = 0$ (one of the roots of the indicial equation), (5) shows that C_1 it indeterminate. Hence C_0 and C_1 may be taken as arbitrary constants. Equating to zero the coefficient of x^{k+m-2}, (3) gives $$C_m (k+m)(k+m-1) - 2\, C_{m-2} (k+m-2-n) = 0$$

or $$C_m = \frac{2(k+m-2-n)}{(k+m)(k+m-1)} C_{m-2}. \qquad ...(6)$$

Putting $k = 0$ in (6) gives $$C_m = \frac{2(m-2-n)}{m(m-1)} C_{m-2}. \qquad ...(7)$$

Putting $m = 2, 4, 6, ..., 2m$, in (7), we have

$$C_2 = -\frac{2n}{2\cdot 1} C_0 = -\frac{2n}{2!} C_0 = -\frac{(-1)^1 \cdot 2^1 \cdot n}{2!} C_0,$$

$$C_4 = \frac{2(2-n)}{4\cdot 3} C_2 = \frac{(-1)^2 \cdot 2(2-n)}{4\cdot 3} \times \frac{2n}{2!} C_0 = \frac{(-1)^2\, 2^2 n(n-2)}{4!} C_0,$$

$$\cdots \qquad \cdots \quad \cdots \quad \cdots \qquad \cdots \quad \cdots$$

and $$C_{2m} = \frac{(-1)^m\, 2^m \cdot n(n-2)...(n-2m+2)}{(2m)!} C_0$$

Next, putting $m = 3, 5, 7, ..., 2m + 1$, in (7), we get

$$C_3 = \frac{2(1-n)}{3\cdot 2} C_1 = -\frac{2(n-1)}{3!} C_1 = \frac{(-1)^1 2^1 (n-1)}{3!} C_1, \; C_5 = \frac{2(3-n)}{5\cdot 4} C_3 = \frac{(-1)^2 2^2 (n-1)(n-3)}{5!} C_1,$$

$$\ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots \quad \ldots$$

and
$$C_{2m+1} = \frac{(-1)^m 2^m (n-1)(n-3)\ldots(n-2m+1)}{(2m+1)!} C_1.$$

Putting the above values in (2) with $k = 0$, we get $y = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + ...$

i.e.
$$y = (C_0 + C_2 x^2 + C_4 x^4 + ...) + (C_1 x + C_3 x^3 + C_5 x^5 + ...),$$

i.e.
$$y = C_0\left[1 - \frac{2n}{2!}x^2 + \frac{2^2 n(n-2)}{4!} x^4 \ldots + \frac{(-2)^m n(n-2)\ldots(n-2m+2)}{(2m)!} x^{2m} + \ldots\right]$$

$$+ C_1\left[x - \frac{2(n-1)}{3!} x^3 + \frac{2^2 (n-1)(n-3)}{5!} x^5 + \ldots + \frac{(-2)^m (n-1)(n-3)\ldots(n-2m+1)}{(2m+1)!} x^{2m+1} + \ldots\right] \quad \ldots(8)$$

or
$$y = C_0 u + C_1 v, \text{ say,} \quad \ldots(9)$$

Since u/v is not merely a constant, u and v form a fundamental set (*i.e.* linearly independent) of solutions of (1). Hence (8) or (9) is the most general solution of (1) with C_0 and C_1 as two arbitrary constants.

Remarks. In practice we require a solution of (1) such that

(*i*) it is finite for all finite values of x and (*ii*) as $x \to \infty$, exp. $(1/2x^2)\, y(x) \to 0$.

The solution (8) in ascending powers of x does not satisfy the condition exp. $(1/2x^2)\, y(x) \to 0$ as $x \to \infty$. However, this requirement is easily seen to be satisfied provided the series terminate. Replacing m by $m + 2$ in (7), we have

$$C_{m+2} = \frac{2(m-n)}{(m+1)(m+2)} C_m. \quad \ldots(10)$$

Let n be a non–negative integer. Then (10) shows that C_{m+2} and all subsequent coefficients in (2) will vanish and so the corresponding series terminate. We shall now obtain the series solution of (1) in descending powers of x by assuming n to be a non–negative integer. Re–writing (2) for $k = 0$, we have as explained above

$$y = C_n x^n + C_{n-2} x^{n-2} + C_{n-4} x^{n-4} + \ldots \quad \ldots(11)$$

From (10),
$$C_m = -\frac{(m+1)(m+2)}{2(n-m)} C_{m+2}. \quad \ldots(12)$$

Putting $m = n - 2, n - 4, ...,$ in (12),
$$C_{n-2} = -\frac{(n-1)n}{2(n-n+2)} C_n = -\frac{n(n-1)}{2\cdot 2} C_n$$

$$C_{n-4} = -\frac{(n-3)(n-2)}{2(n-n+4)} C_{n-2} = -\frac{n(n-1)(n-2)(n-3)}{2^2\cdot 2\cdot 4} C_n$$

and so on. Putting these is (11), we have $y = a_n\{x^n$

$$-\frac{n(n-1)}{2\cdot 2} x^{n-2} + \frac{n(n-1)(n-2)(n-3)}{2^2\cdot 2\cdot 4} x^{n-4} + ... + (-1)^r \frac{n(n-1)(n-2r+1)}{2^r\cdot 2\cdot 4\ldots 2r} x^{n-2r} + \ldots\Big\}$$

or
$$y = a_n \sum_{r=0}^{[n/2]} (-1)^r \frac{n(n-1)(n-2r+1)}{2^r\cdot 2\cdot 4\ldots 2r} x^{n-2r} = a_n \sum_{r=0}^{[n/2]} (-1)^r \frac{n!}{2^{2r} r!(n-2r)!} x^{n-2r},$$

where $$[n/2] = \begin{cases} n/2, & \text{if } n \text{ is even} \\ (n-1)/2, & \text{if } n \text{ is odd} \end{cases}$$

Taking $a_n = 2^n$ and denoting the solution by $H_n(x)$, we obtain the standard solution of (1), known as *Hermite polynomial of order n*.

12.2. Hermite polynomial of order n. **[Meerut 1994, 97, 98]**

Hermite polynomial of order n is denoted and defined by

$$H_n(x) = \sum_{r=0}^{[n/2]} (-1)^r \frac{n!}{r!(n-2r)!}(2x)^{n-2r}, \qquad \text{where,} \qquad [n/2] = \begin{cases} n/2, \text{ if } n \text{ is even} \\ (n-1)/2, \text{ if } n \text{ is odd} \end{cases}$$

12.3. Generating function for Hermite polynomials.

Theorem. *Prove that* $e^{2tx-t^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)$ **[Meerut 1992, 94]**

Proof. Using the well known expansion for exponential function, we get

$$e^{2tx-t^2} = e^{2tx}.e^{-t^2} = \sum_{s=0}^{\infty} \frac{(2tx)^s}{s!} \sum_{r=0}^{\infty} \frac{(-t^2)^r}{r!} = \sum_{s=0}^{\infty}\sum_{r=0}^{\infty} (-1)^r \frac{(2x)^s}{r!\,s!} t^{s+2r}. \quad ...(1)$$

Let $s + 2r = n$ so that $s = n - 2r$. So for a fixed value of r, the coefficient of t^n is given by

$$(-1)^r \frac{(2x)^{n-2r}}{r!(n-2r)!}. \quad ...(2)$$

Now, $s \geq 0 \Rightarrow n - 2r \geq 0 \Rightarrow n \geq 2r \Rightarrow r \leq n/2$, which gives all values of r for which (2) is the coefficient of t^n. If n is even, $r \leq n/2$ shows that r varies from 0 to $n/2$. Again, if n is odd, $r \leq n/2$ shows that r varies from 0 to $(n-1)/2$. Here note that r is an integer. Combining these results we see that r varies from 0 to $[n/2]$, where,

$$[n/2] = \begin{cases} n/2, \text{ if } n \text{ is even} \\ (n-1)/2, \text{ if } n \text{ is odd} \end{cases}$$

Hence the total coefficient of t^n in the expansion of e^{2tx-t^2} is given by

$$\sum_{r=0}^{[n/2]} (-1)^r \frac{1}{r!(n-2r)!}(2x)^{n-2r} \quad i.e. \quad \frac{1}{n!}\sum_{r=0}^{[n/2]} (-1)^r \frac{n!}{r!(n-2r)!}(2x)^{n-2r}, \quad i.e. \quad \frac{1}{n!}H_n(x).$$

Again the coefficient of t^n in $\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)$ is also $\frac{1}{n!}H_n(x)$.

This proves the required result.

12.4. Alternative expressions for the Hermite polynomials.

Theorem 1. *Prove that* $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$.

[Kanpur 2005, 07, 09; Garhwal 1996; Meerut 1997]

Proof. Using the generating function, we have $$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) = e^{2tx-t^2}. \quad ...(1)$$

Expanding the function on the R.H.S. by Taylor's theorem, (1) gives

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x) = \sum_{n=0}^{\infty} \left[\frac{\partial^n}{\partial t^n} e^{2tx-t^2}\right]_{t=0} \frac{t^n}{n!}. \quad ...(2)$$

Equating coefficient of t^n in (2) and cancelling n ! from both sides, we get

$$H_n(x) = \left[\frac{\partial^n}{\partial t^n} e^{2tx - t^2}\right]_{t=0} = \left[\frac{\partial^n}{\partial t^n} e^{x^2 - (x-t)^2}\right]_{t=0} = e^{x^2}\left[\frac{\partial^n}{\partial t^n} e^{-(x-t)^2}\right]_{t=0} = e^{x^2}\left[(-1)^n \frac{\partial^n}{\partial x^n} e^{-(x-t)^2}\right]_{t=0}$$

$$\left\{\because \frac{\partial^n}{\partial t^n} f(x-t) = (-1)^n \frac{\partial^n}{\partial x^n} f(x-t)\right\}$$

$$= e^{x^2}\left[(-1)^n \frac{d^n}{dx^n} e^{-x^2}\right] = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}.$$

Theorem II. *Prove that* $H_n(x) = 2^n \left\{\exp\left(-\frac{1}{4}\frac{d^2}{dx^2}\right)\right\} x^n$. **[Meerut 1995, 96, 97]**

Proof. Since $\exp x = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, we have

$$\exp\left(-\frac{1}{4}\frac{d^2}{dx^2}\right) e^{2tx} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(-\frac{1}{4}\frac{d^2}{dx^2}\right)^n e^{2tx} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\left(\frac{1}{2}\frac{d}{dx}\right)^{2n} e^{2tx} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\frac{1}{2^{2n}}\frac{d^{2n}}{dx^{2n}} e^{2tx}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{1}{2^{2n}} (2t)^{2n} e^{2tx} = e^{2tx} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^{2n} = e^{2tx} \sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!} = e^{2tx} \cdot e^{-t^2}. \quad \text{...(1)}$$

Using the generating function, we have $\quad e^{2tx} \cdot e^{-t^2} = e^{2tx - t^2} = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}. \quad$ ...(2)

Also, $$e^{2tx} = \sum_{n=0}^{\infty} \frac{(2tx)^n}{n!} = \sum_{n=0}^{\infty} \frac{2^n x^n}{n!} t^n. \quad \text{...(3)}$$

Using (2) and (3), (1) becomes $$\sum_{n=0}^{\infty} \exp\left(-\frac{1}{4}\frac{d^2}{dx^2}\right) \frac{2^n x^n}{n!} t^n = \sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!}. \quad \text{...(4)}$$

On equating the coefficients of t^n on both sides of (4), we have

$$\exp\left(-\frac{1}{4}\frac{d^2}{dx^2}\right)\frac{2^n x^n}{n!} = \frac{H_n(x)}{n!} \quad \text{or} \quad \frac{2^n}{n!}\left\{\exp\left(-\frac{1}{4}\frac{d^2}{dx^2}\right)\right\} x^n = \frac{H_n(x)}{n!}$$

or $$H_n(x) = 2^n \left\{\exp\left(-\frac{1}{4}\frac{d^2}{dx^2}\right)\right\} x^n.$$

12.5. Hermite Polynomials for some special values of n.

By definition, $$H_n(x) = \sum_{r=0}^{[n/2]} (-1)^r \frac{n!}{r!(n-2r)!} (2x)^{n-2r} \quad \text{...(1)}$$

where, $$[n/2] = \begin{cases} n/2, \text{ if } n \text{ is even} \\ (n-1)/2, \text{ if } n \text{ is odd} \end{cases} \quad \text{...(2)}$$

Putting $n = 0, 1, 2, 3, \ldots$ in (1), we have

$$H_0(x) = \sum_{r=0}^{0} (-1)^r \frac{0!}{r!(-2r)!} (2x)^{-2r} = (-1)^0 \frac{1}{0!\ 0!} (2x)^0 = 1,$$

$$H_1(x) = \sum_{r=0}^{0} (-1)^r \frac{1!}{r!(1-2r)!}(2x)^{1-2r} = (-1)^0 \frac{1}{0!\,1!}(2x)^1 = 2x,$$

$$H_2(x) = \sum_{r=0}^{1} (-1)^r \frac{2!}{r!(2-2r)!}(2x)^{2-2r} = (-1)^0 \frac{2!}{0!2!}(2x)^2 + (-1)^1 \frac{2!}{1!0!}(2x)^0 = 4x^2 - 2,$$

$$H_3(x) = \sum_{r=0}^{1} (-1)^r \frac{3!}{r!(3-2r)!}(2x)^{3-2r} = (-1)^0 \frac{3!}{0!3!}(2x)^3 + (-1)^1 \frac{3!}{1!1!}(2x)^1 = 8x^3 - 12x,$$

$$H_4(x) = \sum_{r=0}^{2} (-1)^r \frac{4!}{r!(4-2r)!}(2x)^{4-2r} = (-1)^0 \frac{4!}{0!4!}(2x)^4 + (-1)^1 \frac{4!}{1!2!}(2x)^2 + (-1)^2 \frac{4!}{2!0!}(2x)^0$$

or $\qquad H_4(x) = 16x^4 - 48x^2 + 12 \qquad$ and so on.

12.6. Evaluation values of $H_{2n}(0)$ and $H_{2n+1}(0) = 0$.

Theorem. *Prove that* $H_{2n}(0) = (-1)^n \dfrac{(2n)!}{n!}$; $H_{2n+1}(0) = 0$.

[Meerut 1997, Kanpur 2006, 08, 09, 10]

Proof. Using the generating function, we have $\qquad \displaystyle\sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n = e^{2tx - t^2}$. $\qquad$ ...(1)

Replacing x by 0 in (1), we have

$$\sum_{n=0}^{\infty} \frac{H_n(0)}{n!} t^n = e^{-t^2} = \sum_{n=0}^{\infty} \frac{(-t^2)^n}{n!} \quad \text{or} \quad \sum_{n=0}^{\infty} \frac{H_n(0)}{n!} t^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} t^{2n}. \qquad \text{...(2)}$$

Equating coefficients of t^{2n} on both sides of (2), we have

$$\frac{H_{2n}(0)}{(2n)!} = \frac{(-1)^n}{n!} \quad \text{or} \quad H_{2n}(0) = (-1)^n \frac{(2n)!}{n!}.$$

Since the R.H.S. of (2) does not contain odd powers of t. equating coefficients of t^{2n+1} on both sides of (2) gives $\qquad \dfrac{H_{2n+1}(0)}{(2n+1)!} = 0 \qquad$ so that $\qquad H_{2n+1}(0) = 0.$

12.7. Orthogonality properties of the Hermite Polynomials.

Theorem. *Prove that* : $\displaystyle\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx = 2^n\, n!\sqrt{\pi}\,\delta_{nm}$

or $$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx = \begin{cases} 0, & \text{if } m \neq n \\ \sqrt{\pi}\, 2^n\, n!, & \text{if } m = n \end{cases}$$

or Show that Hermite polynomials are orthogonal over $(-\infty, \infty)$ *with respect to the weight function* e^{-x^2}.

Proof. Using the generating functions, we have

$$\sum_{n=0}^{\infty} H_n(x) \frac{t^n}{n!} = e^{2tx - t^2} \quad \text{and} \quad \sum_{m=0}^{\infty} H_m(x) \frac{s^m}{m!} = e^{2sx - s^2}.$$

Multiplying their corresponding sides, gives $\displaystyle\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{H_n(x) H_m(x)}{n!\,m!} t^n s^m = e^{2tx - t^2 + 2sx - s^2}$

Multiplying both sides by e^{-x^2} and then integerating both sides w.r.t. 'x' from $-\infty$ to ∞, we have

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx\right] \frac{t^n s^m}{n!\, m!}$$

$$= \int_{-\infty}^{\infty} e^{-x^2+2x(t+s)-(t^2+s^2)}\, dx = \int_{-\infty}^{\infty} e^{-x^2+2x(t+s)-(t+s)^2} \times e^{(t+s)^2-(t^2+s^2)}\, dx$$

$$= e^{2ts}\int_{-\infty}^{\infty} e^{-[x-(t+s)]^2} dx = e^{2ts}\int_{-\infty}^{\infty} e^{-y^2}\, dy, \text{ putting } x-(t+s)=y \text{ so that } dx = dy$$

$$= e^{2ts}\sqrt{\pi}, \quad \text{as} \int_{-\infty}^{\infty} e^{-y^2}\, dy = \sqrt{\pi}$$

$$= \sqrt{\pi}\sum_{n=0}^{\infty}\frac{(2ts)^n}{n!} = \sum_{n=0}^{\infty}\frac{2^n\sqrt{\pi}}{n!} t^n s^n. \qquad \text{...(1)}$$

We note that powers of t and s are always equal in each term on R.H.S. of (1). Hence when $m \neq n$, equating coefficients of $t^n s^m$ on both sides of (1), we have

$$\frac{1}{n!\, m!}\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx = 0 \Rightarrow \int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx = 0, \text{ when } n \neq m. \qquad \text{...(2)}$$

Again equating coefficients of $t^n s^n$ on both sides (1), we have

$$\frac{1}{n!\, n!}\int_{-\infty}^{\infty} e^{-x^2} [H_n(x)]^2\, dx = \frac{2^n\sqrt{\pi}}{n!} \quad \text{or} \quad \int_{-\infty}^{\infty} e^{-x^2} [H_n(x)]^2\, dx = 2^n\, n!\, \sqrt{\pi}. \qquad \text{...(3)}$$

[Kanpur 2007; Utkal 2003]

Let $$\delta_{nm} = \begin{cases} 0, & \text{if } n \neq m \\ 1, & \text{if } n = m. \end{cases} \qquad \text{...(4)}$$

Combining results (2) and (3) with help of (4), we get

$$\int_{-\infty}^{\infty} e^{-x^2} H_n(x) H_m(x)\, dx = 2^n\, n!\, \sqrt{\pi}\, \delta_{nm.}$$

12.8. Recurrence Relations (or formulae)

Theorem. (*i*) $H'_n(x) = 2n\, H_{n-1}(x)$ $(n \geq 1)$; $H'_0(x) = 0.$ **[Kanpur 2007]**

(*ii*) $H_{n+1}(x) = 2x\, H_n(x) - 2n\, H_{n-1}(x)$ $(n \geq 1)$; $H_1(x) = 2x\, H_0(x).$ **[Kanpur 2007, 11]**

(*iii*) $H_n'(x) = 2x\, H_n(x) - H_{n+1}(x).$

(*iv*) $H_n''(x) - 2x\, H_n'(x) + 2n\, H_n(x) = 0.$

Proof. (*i*) We know that $$\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} = e^{2tx-t^2} \qquad \text{...(1)}$$

Differentiating both sides of (1) w.r.t. 'x' we have

$$\sum_{n=0}^{\infty} H_n'(x)\frac{t^n}{n!} = 2t\, e^{2tx-t^2} = 2t\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!}, \text{ by (1).}$$

Thus, $$\sum_{n=0}^{\infty} H_n'(x)\frac{t^n}{n!} = 2\sum_{n=0}^{\infty} H_n(x)\frac{t^{n+1}}{n!}. \qquad \text{...(2)}$$

Equating coefficients of t^n from both sides for $n = 0$, (2) gives $H_0'(x) = 0.$

Again equating coefficient of t^n from both sides for $n \geq 1$, (2) gives

$$\frac{H_n'(x)}{n\,!} = \frac{H_{n-1}(x)}{(n-1)\,!} \qquad \text{so that} \qquad H_n'(x) = 2n\,H_{n-1}(x) \qquad [\because\; n\,! = n(n-1)\,!]$$

(*ii*) We know that
$$e^{2tx - t^2} = \sum_{n=0}^{\infty} H_n(x)\,\frac{t^n}{n\,!}. \qquad \text{...(3)}$$

Differentiating both sides of (3) w.r.t. 't' gives
$$(2x - 2t)\,e^{2tx - t^2} = \sum_{n=0}^{\infty} \frac{nt^{n-1}}{n\,!}\,H_n(x)$$

or
$$(2x - 2t)\sum_{n=0}^{\infty} H_n(x)\,\frac{t^n}{n\,!} = \frac{0.t^{0-1}}{0!}\,H_0(x) + \sum_{n=1}^{\infty} \frac{nt^{n-1}}{n!}\,H_n(x)$$

or
$$2x\sum_{n=0}^{\infty} H_n(x)\,\frac{t^n}{n\,!} - 2\sum_{n=0}^{\infty} H_n(x)\frac{t^{n+1}}{n\,!} = \sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)\,!}\,H_n(x) \qquad \text{...(4)}$$

$[\because\; 0\,! = 1,\; H_n(x) = 1 \text{ and } n\,! = n(n-1)\,!]$

Equating coefficients of t^n from both sides for $n = 0$, (4) gives $\qquad 2x\,H_0(x) = H_1(x)$.
Again equating coefficient of t^n from both sides for $n \geq 1$, (2) gives

$$2x \times \frac{H_n(x)}{n\,!} - 2\,\frac{H_{n-1}(x)}{(n-1)\,!} = \frac{H_{n+1}(x)}{n\,!}. \qquad \text{...(5)}$$

On multiplying both sides of (5) by $n!$ and noting that $n! = n(n-1)!$, (5) gives

$$2x\,H_n(x) - 2n\,H_{n-1}(x) = H_{n+1}(x).$$

(*iii*) From recurrence relations (1) and (2), we get $\qquad H_n'(x) = 2n\,H_{n-1}(x) \quad$ ...(1)

and $\qquad H_{n+1}(x) = 2x\,H_n(x) - 2n\,H_{n-1}(x). \qquad$ ...(2)

Adding (1) and (2), $\quad H_n'(x) + H_{n+1}(x) = 2x\,H_n(x) \quad$ or $\quad H_n'(x) = 2x\,H_n(x) - H_{n+1}(x)$.

(*iv*) Since $H_n(x)$ is a solution of Hermite's differential equation $\qquad y'' - 2xy' + 2ny = 0$.

$\therefore \qquad H_n''(x) - 2x\,H_n'(x) + 2n\,H_n(x) = 0.$

12.9. SOLVED EXAMPLES

Ex. 1. *Express* $H(x) = x^4 + 2x^3 + 2x^2 - x - 3$ *in terms of Hermite's polynomials.*

[Kanpur 2008, 10]

Sol. We know that $H_0(x) = 1$, $H_1(x) = 2x$, $H_2(x) = 4x^2 - 2$, $H_3(x) = 8x^3 - 12x$ and $H_4(x) = 16x^4 - 48x^2 + 12$. From these, we have

$$x^4 = (1/16) \times H_4(x) + 3x^2 - (3/4), \qquad \text{...(1)}$$
$$x^3 = (1/8) \times H_3(x) + (3x/2) \qquad \text{...(2)}$$
$$x^2 = (1/4) \times H_2(x) + 1/2 \qquad \text{...(3)}$$

and
$$x = (1/2) \times H_1(x), \qquad 1 = H_0(x). \qquad \text{...(4)}$$

$\therefore\;\; H(x) = x^4 + 2x^3 + 2x^2 - x - 3 = (1/16) \times H_4(x) + 3x^2 - (3/4) + 2x^3 + 2x^2 - x - 3$, by (1)

$= (1/16) \times H_4(x) + 2x^3 + 5x^2 - x - 15/4 = (1/16) \times H_4(x) + 2\,[(1/8) \times H_3(x) + 3x/2] + 5x^2 - x - 15/4$, by (2)

$= (1/16) \times H_4(x) + (1/4) \times H_3(x) + 5x^2 + 2x - (15/4)$

$= (1/16) \times H_4(x) + (1/4) \times H_3(x) + 5\,[(1/4) \times H_2(x) + (1/2)] + 2x - (15/4)$, by (3)

$= (1/16) \times H_4(x) + (1/4) \times H_3(x) + (5/4) \times H_2(x) + 2x - (5/4)$,

$= (1/16) \times H_4(x) + (1/4) \times H_3(x) + (5/4) \times H_2(x) + H_1(x) - (5/4) \times H_0(x)$, by (4)

Ex. 2. *Prove that, if* $m < n$, $\dfrac{d^m}{dx^m}\{H_n(x)\} = \dfrac{2^m.n!}{(n-m)!}\,H_{n-m}(x)$. **[Garhwal 2004, 05; Meerut 98]**

Sol. We know that $$\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n\,!} = e^{2tx-t^2}. \qquad ...(1)$$

Differentiating both sides of (1) w.r.t. 'x' m times, we have

$$\sum_{n=0}^{\infty} \frac{t^n}{n\,!}\frac{d^m}{dx^m}\{H_n(x)\} = \frac{d^m}{dx^m} e^{2tx-t^2} = (2t)^m e^{2tx-t^2} = 2^m t^m \sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n\,!}, \text{ by (1)}$$

$$\therefore \qquad \sum_{n=0}^{\infty} \frac{t^n}{n\,!}\frac{d^m}{dx^m}\{H_n(x)\} = 2^m \sum_{n=0}^{\infty} H_n(x)\frac{t^{n+m}}{n\,!}. \qquad ...(2)$$

Equating coefficients of t^n from both sides for $m < n$, (2) gives

$$\frac{1}{n\,!}\frac{d^m}{dx^m}\{H_n(x)\} = 2^m \frac{H_{n-m}(x)}{(n-m)\,!} \qquad \text{or} \qquad \frac{d^m}{dx^m}\{H_n(x)\} = \frac{2^m n\,!}{(n-m)\,!}H_{n-m}(x).$$

Ex. 3. *Prove that* $\displaystyle\int_{-\infty}^{\infty} x^2 e^{-x^2} [H_n(x)]^2\, dx = (\sqrt{\pi})\, 2^n\, n\,!\,(n+1/2)$.

Sol. From the recurrence relations, we know that

$$H_{n+1}(x) = 2x\, H_n(x) - 2n\, H_{n-1}(x) \qquad \text{or} \qquad x\, H_n(x) = n\, H_{n-1}(x) + (1/2) \times H_{n+1}(x) \qquad ...(1)$$

or $$x^2 H_n(x) = nx\, H_{n-1}(x) + (x/2) \times H_{n+1}(x) \qquad ...(2)$$

Replacing n by $n-1$ and $n+1$ successively in (1), we have

$$x\, H_{n-1}(x) = (n-1)\, H_{n-2}(x) + (1/2) \times H_n(x) \qquad ...(3)$$

and $$x\, H_{n+1}(x) = (n+1)\, H_n(x) + (1/2) \times H_{n+2}(x). \qquad ...(4)$$

Using (3) and (4), (2) becomes

$$x^2 H_n(x) = n\,[(n-1)\, H_{n-2}(x) + (1/2) \times H_n(x)] + (1/2) \times [(n+1)\, H_n(x) + (1/2) \times H_{n+2}(x)]$$

or $$x^2 H_n(x) = n\,(n-1)\, H_{n-2}(x) + (1/4) \times H_{n+2}(x) + (n+1/2)\, H_n(x) \qquad ...(5)$$

Multiplying both sides of (5) by $e^{-x^2} H_n(x)$ and then integrating w.r.t. 'x' from $-\infty$ to ∞, gives

$$\int_{-\infty}^{\infty} x^2 e^{-x^2} \{H_n(x)\}^2\, dx = n\,(n-1)\int_{-\infty}^{\infty} e^{-x^2} H_n(x)\, H_{n-2}(x)\, dx$$
$$+\frac{1}{4}\int_{-\infty}^{\infty} e^{-x^2} H_n(x)\, H_{n+2}(x) + \left(n+\frac{1}{2}\right)\int_{-\infty}^{\infty} e^{-x^2} \{H_n(x)\}^2\, dx$$

$$= 0 + 0 + (n+1/2)\left(\sqrt{\pi}\right) 2^n\, n\,!, \qquad \text{as} \qquad \int_{-\infty}^{\infty} e^{-x^2} H_n(x)\, H_m(x)\, dx = \left(\sqrt{\pi}\right) 2^n\, n\,!\,\delta_{nm}$$

$$= (n+1/2)\left(\sqrt{\pi}\right) 2^n\, n\,!.$$

Ex. 4. *Evaluate* $\displaystyle\int_{-\infty}^{\infty} xe^{-x^2} H_n(x)\, H_m(x)\, dx$. **[Meerut 1997]**

Sol. From the recurrence relations, we have $\qquad H_{n+1}(x) = 2x\, H_n(x) - 2n\, H_{n-1}(x)$

or $$xH_n(x) = n\, H_{n-1}(x) + (1/2) \times H_{n+1}(x). \qquad ...(1)$$

Multiplying both sides of (1) by $e^{-x^2} H_m(x)$ and then integrating w.r.t. 'x' from $-\infty$ to ∞, we have

$$\int_{-\infty}^{\infty} xe^{-x^2} H_m(x)\, H_n(x)\, dx = n\int_{-\infty}^{\infty} e^{-x^2} H_m(x)\, H_{n-1}(x)\, dx + \frac{1}{2}\int_{-\infty}^{\infty} e^{-x^2} H_m(x)\, H_{n+1}(x)\, dx$$

$$= n\sqrt{\pi}\, 2^{n-1}\,(n-1)!\,\delta_{n-1,m} + (1/2)\times\sqrt{\pi}\, 2^{n+1}\,(n+1)!\,\delta_{n+1,m}$$

$$\left[\because \int_{-\infty}^{\infty} e^{-x^2}\, H_n(x)\, H_m(x)\, dx = \sqrt{\pi}\, 2^n\, n!\,\delta_{nm}\right]$$

$$= \sqrt{\pi}\, 2^{n-1}\, n!\,\delta_{n-1,m} + \sqrt{\pi}\, 2^n\,(n+1)!\,\delta_{n+1,m}.$$

Ex. 5. *Show that* $P_n(x) = \dfrac{2}{\sqrt{\pi}\, n!}\displaystyle\int_0^{\infty} t^n\, e^{-t^2} H_n(xt)\, dt$.

Sol. By the definition of Hermite polynomial (on replacing x by xt here), we have

$$H_n(xt) = \sum_{r=0}^{[n/2]} (-1)^r \frac{n!}{r!\,(n-2r)!}(2xt)^{n-2r}. \qquad ...(1)$$

Making use of (1), we have

$$\frac{2}{\sqrt{\pi}\, n!}\int_0^{\infty} t^n e^{-t^2} H_n(xt)\, dt$$

$$= \frac{2}{\sqrt{\pi}\, n!}\int_0^{\infty} t^n e^{-t^2}\left\{\sum_{r=0}^{[n/2]}(-1)^r \frac{n!\,2^{n-2r}\, x^{n-2r}}{r!\,(n-2r)!}\, t^{n-2r}\right\} dt = \sum_{r=0}^{[n/2]} \frac{2^{n-2r+1}\,(-1)^r\, x^{n-2r}}{\sqrt{\pi}\, r!\,(n-2r)!}\int_0^{\infty} e^{-t^2}\, t^{2n-2r}\, dt$$

$$= \sum_{r=0}^{[n/2]} \frac{2^{n-2r+1}\,(-1)^r\, x^{n-2r}}{\sqrt{\pi}\, r!\,(n-2r)!}\,\frac{1}{2}\,\Gamma(n-r+1/2)$$

$$\left[\because \int_0^{\infty} e^{-t^2}\, t^{2n-2r}\, dt = \int_0^{\infty} e^{-t^2}\, t^{2(n-r+1/2)-1}\, dt = \tfrac{1}{2}\Gamma\left(n-r+\tfrac{1}{2}\right), \text{ as } \Gamma(x) = 2\int_0^{\infty} e^{-t^2}\, e^{2x-1}\, dt\right]$$

$$= \sum_{r=0}^{[n/2]} \frac{2^{n-2r}\,(-1)^r\, x^{n-2r}}{\sqrt{\pi}\, r!\,(n-2r)!}\times\frac{(2n-2r)!}{2^{2n-2r}\,(n-r)!}\sqrt{\pi} \qquad \left(\begin{array}{l}\because \text{ by using duplication formula, if n is} \\ \text{positive integer then } \Gamma\left(n+\dfrac{1}{2}\right) = \dfrac{(2n)!}{2^{2n} n!}\sqrt{\pi}\end{array}\right)$$

$$= \sum_{r=0}^{[n/2]} (-1)^r \frac{(2n-2r)!}{2^n\, r!\,(n-2r)!\,(n-r)!}\, x^{n-2r} = P_n(x), \text{ by the definition of Legendre polynomial.}$$

Ex. 6. *If* $\psi_n(x) = e^{-x^2/2} H_n(x)$, *where* $H_n(x)$ *in a Hermite's polynomial of degree n, then*

(*a*) $\displaystyle\int_{-\infty}^{\infty} \psi_m(x)\,\psi_n(x)\, dx = 2^n\,!\sqrt{\pi}\,\delta_{m,n}$

(*b*) $\displaystyle\int_{-\infty}^{\infty} \psi_m(x)\,\psi_n'(x)\, dx = \begin{cases} 0, & \text{if } m \neq n\pm 1 \\ 2^{n-1}\, n!\sqrt{\pi}, & \text{if } m = n-1 \\ -2^n\,(n+1)!\sqrt{\pi}, & \text{if } m = n+1 \end{cases}$

Sol. Part (a) Given $\qquad \psi_n(x) = e^{-x^2/2} H_n(x). \qquad ...(1)$

$$\therefore \int_{-\infty}^{\infty} \psi_m(x)\,\psi_n(x)\, dx = \int_{-\infty}^{\infty} e^{-x^2/2} H_m(x).e^{-x^2/2} H_n(x)\, dx, \text{ using (1)}$$

$$= \int_{-\infty}^{\infty} e^{-x^2} H_m(x) H_n(x)\, dx = 2^n\, n!\sqrt{\pi}\,\delta_{nm},$$

[using orthogonal properties of the Hermite polynomials]

Part (b) Here $\psi_n(x) = e^{-x^2/2}\, H_n(x)$ ...(2)

and $\psi_m(x) = e^{-x^2/2}\, H_m(x).$ ...(3)

From (2), $\psi_n'(x) = -xe^{-x^2/2}\, H_n(x) + e^{-x^2/2}\, H_n'(x)$ ...(4)

From recurrence relations, we have $xH_n(x) = n\, H_{n-1}(x) + (1/2)\times H_{n+1}(x)$ ...(5)

and $H_n'(x) = 2n\, H_{n-1}(x).$ ...(6)

Using (5) and (6), (4) reduces to

$$\psi_n'(x) = -e^{-x^2/2}\,[n\, H_{n-1}(x) + (1/2)\times H_{n+1}(x)] + e^{-x^2/2}\times 2n\, H_{n-1}(x)$$

or $$\psi_n'(x) = e^{-x^2/2}\,[n\, H_{n-1}(x) - (1/2)\times H_{n+1}(x)] \quad ...(7)$$

$\therefore \int_{-\infty}^{\infty} \psi_m(x)\,\psi_n'(x)\,dx = \int_{-\infty}^{\infty} e^{-x^2/2}\, H_m(x).e^{-x^2/2}\,[n\, H_{n-1}(x) - \frac{1}{2}\, H_{n+1}(x)]\,dx$, by (3) and (7)

$= n\int_{-\infty}^{\infty} e^{-x^2}\, H_{n-1}\, H_m(x)\,dx - \frac{1}{2}\int_{-\infty}^{\infty} e^{-x^2}\, H_{n+1}(x)\, H_m(x)\,dx$

$= n\,2^{n-1}(n-1)!\,\sqrt{\pi}\,\delta_{n-1.m} - (1/2)\times 2^{n+1}\,(n+1)!\,\sqrt{\pi}\,\delta_{n+1,m}$, using orthogonal properties

$= 2^{n-1}\, n!\,\sqrt{\pi}\,\delta_{n-1,m} - 2^n\,(n+1)!\,\sqrt{\pi}\,\delta_{n+1,m}$

$$= \begin{cases} 0, & \text{if } m \neq n \pm 1 \\ 2^{n-1}\, n!\,\sqrt{\pi}, & \text{if } m = n-1 \\ -2^n\,(n+1)!\,\sqrt{\pi}, & \text{if } m = n+1 \end{cases}$$

Ex. 7. *Using the Rodrigue's formula for $H_n(x)$ and integrating by parts iteratively, show that*

$$\psi = \int_{-\infty}^{\infty} \exp(-x^2)\, H_n(x)\, H_m(x)\,dx = 0, \text{ if } m \neq n$$

$$= 2^n\, n!\,\sqrt{\pi}, \text{ if } m = n$$

Sol. Rodrigue's formula for $H_n(x)$ is given by $H_n(x) = (-1)^n\, e^{x^2}\,\dfrac{d^n}{dx^n}\, e^{-x^2}.$

$\therefore$ $$\exp(-x^2)\, H_n(x) = (-1)^n\,\frac{d^n}{dx^n}\, e^{-x^2}. \quad ...(1)$$

$\therefore \psi = \int_{-\infty}^{\infty} (-1)^n \exp(-x^2)\, H_n(x)\, H_m(x)\,dx = \int_{-\infty}^{\infty} (-1)^n \exp\left\{\dfrac{d^n}{dx^n}\, e^{-x^2}\right\} H_m(x)\,dx$

$= (-1)^n \left[\left\{\left(\dfrac{d^{n-1}}{dx^{n-1}}\, e^{-x^2}\right) H_m(x)\right\}_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \left(\dfrac{d^{n-1}}{dx^{n-1}}\, e^{-x^2}\right)\dfrac{d}{dx}\, H_m(x)\,dx\right]$, integrating by parts

$= 0 - \int_{-\infty}^{\infty} \left(\dfrac{d^{n-1}}{dx^{n-1}}\, e^{-x^2}\right)\dfrac{d}{dx}\, H_m(x)\,dx$, as first term is zero due to presence of e^{-x^2}

$= (-1)^n\,(-1)^n \int_{-\infty}^{\infty} \dfrac{d^n}{dx^n}\, H_m(x)\,.\,e^{-x^2}\,dx$, integrating by parts iteratively

$$\therefore \qquad \psi = \int_{-\infty}^{\infty} \frac{d^n}{dx^n} H_m(x) \,.\, e^{-x^2}\, dx. \qquad ...(2)$$

We know that $H_m(x)$ is a polynomial of degree m. Hence if $n > m$, $\dfrac{d^n}{dx^n} H_m(x) = 0.$

$\therefore$ If $n > m$, then from (2), $\psi = 0.$ ...(3)

Since ψ is symmetrical in m and n, it follows that if $m > n$, then $\psi = 0.$ ...(4)

From (3) and (4), we see that $\psi = 0,$ for $m \neq n$ ...(5)

When, $m = n$, (2) gives

$$\psi = \int_{-\infty}^{\infty} \frac{d^n}{dx^n} H_n(x) \,.\, e^{-x^2}\, dx = \int_{-\infty}^{\infty} 2^n\, n!\, e^{-x^2}\, dx, \qquad \text{as} \qquad \frac{d^n}{dx^n} H_n(x) = 2^n\, n!$$

$$= 2^n\, n!\, \sqrt{\pi}, \qquad \text{as} \qquad \int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$$

$$\therefore \qquad \psi = 2^n\, n!\, \sqrt{\pi} \qquad \text{when} \qquad m = n \qquad ...(6)$$

From (5) and (6), we get the required results.

Ex. 8. *Show that* $\displaystyle\sum_{k=0}^{n} \frac{H_k(x)\, H_k(y)}{2^k\, k!} = \frac{H_{n+1}(y)\, H_n(x) - H_{n+1}(x)\, H_n(y)}{2^{n+1}\, n!\,(y-x)}.$

Sol. From the recurrence relations, we have $H_{n+1}(x) = 2x\, H_n(x) - 2n\, H_{n-1}(x)$

or $\qquad x\, H_n(x) = n\, H_{n-1}(x) + (1/2) \times H_{n+1}(x).$ ...(1)

Replacing x by y in (1), $\quad y\, H_n(y) = n\, H_{n-1}(y) + (1/2) \times H_{n+1}(y).$ ...(2)

Multiplying (2) by $H_n(x)$ and (1) by $H_n(y)$ and then subtracting, we have

$$(y-x)\, H_n(x)\, H_n(y) = (1/2) \times [H_{n+1}(y)\, H_n(x) - H_{n+1}(x)\, H_n(y)]$$
$$- 2\, [H_{n-1}(x)\, H_n(y) - H_{n-1}(y)\, H_n(x)]. \quad ...(3)$$

Putting $n = 0, 1, 2, 3, \ldots, (n-1), n$ successively in (1), we have

$$(y-x)\, H_0(x)\, H_0(y) = (1/2) \times [H_1(y)\, H_0(x) - H_1(x)\, H_0(y)] - 0 \qquad ...(\mathrm{E}_0)$$

$$(y-x)\, H_1(x)\, H_1(y) = (1/2) \times [H_2(y)\, H_1(x) - H_2(x)\, H_1(y)]$$
$$- [H_0(x)\, H_1(y) - H_0(y)\, H_1(x)]. \quad ...(\mathrm{E}_1)$$

$$(y-x)\, H_2(x)\, H_2(y) = (1/2) \times [H_3(y)\, H_2(x) - H_3(x)\, H_2(y)]$$
$$-2\, [H_1(x)\, H_2(y) - H_1(y)\, H_2(x)]. \quad ...(\mathrm{E}_2)$$

... ...

$$(y-x)\, H_{n-1}(x)\, H_{n-1}(y) = (1/2) \times [H_n(y)\, H_{n-1}(x) - H_n(x)\, H_{n-1}(y)]$$
$$-2\, [H_{n-2}(x)\, H_{n-1}(y) - H_{n-2}(y)\, H_{n-1}(x)]. \quad ...(\mathrm{E}_{n-1})$$

$$(y-x)\, H_n(x)\, H_n(y) = (1/2) \times [H_{n+1}(y)\, H_n(x) - H_{n+1}(x)\, H_n(y)]$$
$$-2\, [H_{n-1}(x)\, H_n(y) - H_{n-1}(y)\, H_n(x)]. \quad ...(\mathrm{E}_n)$$

Multiplying $(E_0), (E_1), (E_2), \ldots, (E_{n-1}), (E_n)$ by $1, \dfrac{1}{2 \,.\, 1!}, \dfrac{1}{2^2 \,.\, 2!}, \dfrac{1}{2^3 \,.\, 3!}, \ldots, \dfrac{1}{2^{n-1}\,(n-1)!}, \dfrac{1}{2^n \,.\, n!}$

respectively and adding (note that all terms, except the first term on R.H.S. of (E_n), cancel in pairs), we have

$$(y-x) \sum_{k=0}^{\infty} \frac{H_k(x)\, H_k(y)}{2^k\, k!} = \frac{H_{n+1}(y)\, H_n(x) - H_{n+1}(x)\, H_n(y)}{2^{n+1}\, n!}$$

or

$$\sum_{k=0}^{\infty} \frac{H_k(x)\, H_k(y)}{2^k\, k!} = \frac{H_{n+1}(y)\, H_n(x) - H_{n+1}(x)\, H_n(y)}{2^{n+1}\, n!\,(y-x)}.$$

EXERCISE

1. Show that $H_n(x) = 2^{n+1}\, e^{x^2} \int_x^{\infty} e^{-t^2}\, t^{n+1}\, P_n\left(\frac{x}{t}\right) dt$.

2. If $f(x)$ is a polynomial of degree m, show that $f(x)$ may be expressed in the form

$$f(x) = \sum_{r=0}^{m} C_r\, H_r(x), \qquad \text{where,} \qquad C_r = \frac{1}{2^r\, r!\,\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-x^2} f(x)\, H_r(x)\, dx.$$

Deduce that $\int_{-\infty}^{\infty} e^{-x^2} f(x)\, H_n(x) = 0$, if $f(x)$ is a polynomial of degree less than n.

3. Using the generating function for Hermite polynomials, evaluate the values of
(*i*) $H_0(x)$ (*ii*) $H_1(x)$ (*iii*) $H_2(x)$ (*iv*) $H_3(x)$.

4. Show that $H_n(x)$ defined by $e^{2tx - t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$ satisfies the differential equation $H_n''(x) - 2x\, H_n'(x) + 2x\, H_n(x) = 0$.

5. The Hermite polynomial is defined for integral values of x by the identity $e^{2tx - t^2} = \sum_{n=0}^{\infty} \frac{H_n(x)}{n!} t^n$. Show that $H_n(x)$ satisfies the differential equation $H_n''(x) - 2x\, H_n'(x) + 2x\, H_n(x) = 0$ and $H_n(x)$ is given by $H_n(x) = (-1)^n e^{x^2} \frac{d^n}{dx^n} e^{-x^2}$.

Further show that $$\int_{-\infty}^{\infty} \{H_n(x)\}^2 e^{-x^2}\, dx = \sqrt{\pi}\, 2^n\, n!.$$

6. Show that $\int_{-\infty}^{\infty} e^{-x^2} \{H_n(x)\}^2\, dx = 2^n\, n! \int_{-\infty}^{\infty} e^{-x^2}\, dx = 2^n\, n!\,\sqrt{\pi}$.

7. Show that (*a*) $H_5(x) = 32x^5 - 160x^3 + 120x$. (*b*) $H_6(x) = 64x^6 - 480x^4 + 720x^2 - 120$.

8. Show that for $n = 0, 1, 2 \ldots$.

(*i*) $$H_{2n}(x) = \frac{2^{n+1}.(-1)^n e^{x^2}}{\sqrt{(\pi)}} \int_{\theta}^{\infty} e^{-t^3}\, t^{2n} \cos 2xt\, dt.$$

(*ii*) $$H_{2n}(x) = \frac{2^{2n+2}.(-1)^n e^{x^2}}{\sqrt{(\pi)}} \int_{\theta}^{\infty} e^{-t^2}\, t^{2n+1} \sin 2xt\, dt.$$ **[Kanpur-2004]**

9. Prove that $H_n(-x) = (-1)^n H_n(x)$. **[Kanpur-2004, 09]**

10. Prove that $x^n = \sum_{k=0}^{[n/2]} \frac{n!\, H_{n-2k}(x)}{2^n\, k!\,(n-2k)!}$ and express x^6 in terms of Hermite polynomials. **[Kanpur 2004, 09]**

11. Show that $d^n H_o(x)/dx^n = 2^n n!$ **(Kanpur 2008, 09)**

12. Show that $\int_0^x H_n(y)dy = \frac{1}{2(n+1)}\left[H_{n+1}(x) - H_{n-1}(o)\right]$ **(Kanpur 2010)**

13

Laguerre Polynomials

13.1. Laguerre's equation and its solution. [Meerut 1995, 96]

Laguerre's equation of order n is

$$x(d^2y/dx^2)+(1-x)(dy/dx)+ny=0, \qquad \text{...(1)}$$

where n is a positive integer. We obtain a solution of (1) which is finite for all values of x and which tends to infinity no faster than $e^{x/2}$ as $x \to \infty$. For this we use the well known method of Frobenius. Let the series solution of (1) be

$$y=\sum_{m=0}^{\infty} C_m\, x^{k+m}, \qquad C_0 \neq 0. \qquad \text{...(2)}$$

Differentiating (2) and then putting the values of y, dy/dx and d^2y/dx^2 in (1), we have

$$x\sum_{m=0}^{\infty} C_m(k+m)(k+m-1)x^{k+m-2}+(1-x)\sum_{m=0}^{\infty} C_m(k+m)x^{k+m-1}+n\sum_{m=0}^{\infty} C_m x^{k+m}=0$$

or $$\sum_{m=0}^{\infty} C_m(k+m)(k+m-1)x^{k+m-1}+\sum_{m=0}^{\infty} C_m(k+m)x^{k+m-1}$$
$$-\sum_{m=0}^{\infty} C_m(k+m)x^{k+m}+n\sum_{m=0}^{\infty} C_m x^{k+m-1}=0$$

or $$\sum_{m=0}^{\infty} C_m(k+m)x^{k+m-1}\{(k+m-1)+1\}-\sum_{m=0}^{\infty} C_m x^{k+m}(k+m-n)=0$$

or $$\sum_{m=0}^{\infty} C_m(k+m)^2x^{k+m-1}-\sum_{m=0}^{\infty} C_m(k+m-n)x^{k+m}=0. \qquad \text{...(3)}$$

(3) is an identity. To get the indicial equation, we equate to zero the coefficient of the smallest power of x, namely x^{k-1} in (3) and obtain

$$C_0k^2=0, \qquad \text{so that} \qquad k^2=0 \qquad (\because\ C_0 \neq 0) \qquad \text{...(4)}$$

From (4) we see that the roots of indicial equation are equal. Next equating to zero the coefficient of x^{k+m-1}, we have

$$C_m(k+m)^2-C_{m-1}(k+m-1-n)=0 \qquad \text{or} \qquad C_m=\frac{k+m-1-n}{(k+m)^2}C_{m-1}. \qquad \text{...(5)}$$

The two independent solutions in the present case are $(y)_{k=0}$ and $(\partial y/\partial k)_{k=0}$. But $(\partial y/\partial k)_{k=0}$ involves a term of the form $\log x$, and so is infinite when $x = 0$. Since we wish to obtain a solution finite for all finite values of x, we consider only the former solution, *i.e.*, $(y)_{k=0}$, as follows.

With $k = 0$, (5) and (2) reduce to $$C_m=\frac{m-1-n}{m^2}C_{m-1} \qquad \text{...(6)}$$

and $$y=\sum_{m=0}^{\infty} C_m x^m=C_0+C_1x+C_2x^2+\dots. \qquad \text{...(7)}$$

Putting $m = 1, 2, 3, \ldots$ in (6), we have

$$C_1 = \frac{-n}{1^2} C_0 = \frac{(-1)}{(1!)^2} nC_0, \qquad C_2 = \frac{1-n}{2^2} C_1 = -\frac{(n-1)}{2^2} \times (-1) nC_0 = (-1)^2 \frac{n(n-1)}{(2!)^2} C_0,$$

$$C_3 = \frac{2-n}{3^2} C_2 = -\frac{(n-2)}{3^2} \times (-1)^2 \frac{n(n-1)}{(2!)^2} C_0 = (-1)^3 \frac{n(n-1)(n-2)}{(3!)^2} C_0,$$

....

Thus, $$C_r = (-1)^r \frac{n(n-1)\ldots(n-r+1)}{(r!)^2} C_0, \quad \text{for } r \leq n.$$

Also, $$C_{n+1} = C_{n+2} = C_{n+3} = \ldots = 0.$$

With these values, (7) reduces to

$$y = C_0 \left[1 - \frac{n}{(1!)^2} x + \frac{n(n-1)}{(2!)^2} x^2 + \ldots + (-1)^r \frac{n(n-1)\ldots(n-r+1)}{(r!)^2} x^r + \ldots \right]$$

$$= C_0 \sum_{r=0}^{\infty} (-1)^r \frac{n(n-1)\ldots(n-r+1)}{(r!)^2} x^r = C_0 \sum_{r=0}^{n} (-1)^r \frac{n(n-1)\ldots(n-r+1)(n-r)(n-r-1)\ldots 3\cdot 2\cdot 1}{(n-r)(n-r-1)\ldots 3\cdot 2\cdot 1.(r!)^2} x^r$$

Thus, $$y = C_0 \sum_{r=0}^{n} (-1)^r \frac{n!}{(n-r)!(r!)^2} x^r.$$

Taking $C_0 = 1$, we define the corresponding solution as the Laguerre polynomial of order n, and denote it by $L_n(x)$. Thus, we have $$L_n(x) = \sum_{r=0}^{n} (-1)^r \frac{n!}{(n-r)!\,(r!)^2} x^r.$$

13.2A. Laguerre polynomial of order (or degree) n. Definition. [Meerut 1997]

Laguerre polynomial of order n is denoted and defined by

$$L_n(x) = \sum_{r=0}^{n} (-1)^r \frac{n!}{(n-r)!\,(r!)^2} x^r.$$

13.2B. Alternative definition of Laguerre polynomial of order (or degree) n.

In Art. 13.1, we took $C_0 = 1$ to define $L_n(x)$ and the same is given in Art. 13.2A. However, some authors take $C_0 = n!$ to define $L_n(x)$. Thus, another definition of Laguerre polynomial is

$$L_n(x) = \sum_{r=0}^{n} \frac{(-1)^r (n!)^2 x^r}{(n-r)!\,(r!)^2}.$$

13.3. Generating function for Laguerre polynomials.

Theorem. *Prove that* $$\frac{\exp\{-xt/(1-t)\}}{1-t} = \sum_{n=0}^{\infty} L_n(x)\, t^n.$$

Proof. In order to prove the required result we must show that the coefficient of t^n in the expansion of L.H.S. (in ascending powers of t) is $L_n(x)$. Now,

$$\frac{\exp\{-xt/(1-t)\}}{1-t} = \frac{1}{1-t} \sum_{r=0}^{\infty} \left(\frac{-xt}{1-t} \right)^r . \frac{1}{r!}, \qquad \text{as} \qquad \exp\, x = e^x = \sum_{r=0}^{\infty} \frac{x^r}{r!}$$

$$= \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} x^r t^r (1-t)^{-(r+1)} = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} x^r t^r \sum_{s=0}^{\infty} \frac{(r+s)!}{r!\,s!}, \text{ by the binomial theorem}$$

$$= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} (-1)^r \frac{(r+s)!}{(r!)^2 s!} x^r t^{r+s} \quad .$$

Let r be fixed. Then the coefficient of t^n can be obtained by setting $r + s = n$ *i.e.* $s = n - r$.

Hence, for the chosen fixed value of r, the coefficient of t^n is $(-1)^r \dfrac{n!}{(r!)^2 (n-r)!} x^r$.

Now, $s \geq 0 \Rightarrow n - r \geq 0 \Rightarrow r \leq n$, which gives all allowed values of r for finding coefficient of t^n. Thus, the total coefficient of t^n is given by

$$\sum_{r=0}^{n} (-1)^r \frac{n!}{(r!)^2 (n-r)!} x^r \qquad i.e. \qquad L_n(x), \text{ by definition 13.2A}$$

This proves the desired result.

Remark. If we use definition 13.2B, then $\dfrac{\exp\{-xt/(1-t)\}}{1-t} = \sum_{n=0}^{\infty} \dfrac{L_n(x)\, t^n}{n!}$.

13.4. Alternative expression for the Laguerre polynomials

Prove that $L_n(x) = \dfrac{e^x}{n!} \dfrac{d^n}{dx^n}(x^n e^{-x})$. **[Kanpur 1992; Meerut 1992, 93; Garhwal 2005]**

Proof. By the Leibnitz's theorem, we have

$$D^n(uv) = d^n(uv)/dx^n = D^n u.v + {}^nC_1 D^{n-1} u.Dv + \ldots + {}^nC_r D^{n-r} u.D^r v + \ldots + uD^n v$$

i.e.,

$$D^n(uv) = \sum_{r=0}^{n} {}^nC_r\, D^{n-r} u\, D^r v.$$

$$\therefore \quad \frac{e^x}{n!} \frac{d^n}{dx^n}(x^n e^{-x}) = \frac{e^x}{n!} \sum_{r=0}^{n} {}^nC_r\, D^{n-r} x^n\ D^r e^{-x}, \text{. by (1)}$$

$$= \frac{e^x}{n!} \sum_{r=0}^{n} {}^nC_r \frac{n!}{\{n-(n-r)\}!} x^{n-(n-r)} . (-1)^r e^{-x}, \qquad \text{as} \qquad D^n x^m = \frac{m!}{(m-n)!} x^{m-n} \text{ and } D^n e^{ax} = a^n e^{ax}$$

$$= \sum_{r=0}^{n} \frac{e^x}{n!} \times \frac{n!}{r!(n-r)!} \frac{n!}{r!} x^r \times (-1)^r e^{-x} = \sum_{r=0}^{n} \frac{(-1)^r n!}{(r!)^2 (n-r)!} x^r = L_n(x), \text{ by definition.}$$

Remark. If the use definition 13.2B, we get $L_n(x) = e^x \dfrac{d^n}{dx^n}(x^n e^{-x})$.

13.5. First few Laguerre polynomials.

We know that (Refer Art. 13.4) $L_n(x) = \dfrac{e^x}{n!} \dfrac{d^n}{dx^n}(x^n e^{-x})$. ...(1)

Putting $n = 0, 1, 2, 3, 4, \ldots$ in succession in (1), we obtain

$$L_0(x) = \frac{e^x}{0!}(x^0 e^{-x}) = 1, \qquad L_1(x) = \frac{e^x}{1!} \frac{d}{dx}(xe^{-x}) = e^x(e^{-x} - xe^{-x}) = 1 - x,$$

$$L_2(x) = \frac{e^x}{2!} \frac{d^2}{dx^2}(x^2 e^{-x}) = \frac{e^x}{2!} \frac{d}{dx}\left[\frac{d}{dx}(x^2 e^{-x})\right] = \frac{e^x}{2!} \frac{d}{dx}(2xe^{-x} - x^2 e^{-x})$$

$$= \frac{e^x}{2!}[2e^{-x} + 2x(-e^{-x}) - \{2xe^{-x} + x^2(-e^{-x})\}] = \frac{1}{2!}(2 - 4x + x^2),$$

$$L_3(x) = \frac{e^x}{3!}\frac{d^3}{dx^3}(x^3 e^{-x}) = \frac{e^x}{3!}\frac{d^2}{dx^2}\left[\frac{d}{dx}(x^3 e^{-x})\right] = \frac{e^x}{3!}\frac{d^2}{dx^2}(3x^2 e^{-3} - x^3 e^{-x}) = \frac{e^x}{3!}\frac{d}{dx}\left[\frac{d}{dx}\{(3x^2 - x^3)e^{-x}\}\right]$$

$$= \frac{e^x}{3!}\frac{d}{dx}[(6x - 3x^2)e^{-x} - (3x^2 - x^3)e^{-x}] = \frac{e^x}{3!}\frac{d}{dx}\{(6x - 6x^2 + x^3)\, e^{-x}\}$$

$$= (e^x/3!) \times [(6 - 12x + 3x^2)e^{-x} - (6x - 6x^2 + x^3)e^{-x}] = (6 - 18x + 9x^2 - x^3)/3!,$$

$$L_4(x) = \frac{e^x}{4!}\frac{d^4}{dx^4}(x^4 e^{-x}) = \frac{e^x}{4!}\frac{d^3}{dx^3}\left[\frac{d}{dx}(x^4 e^{-x})\right] = \frac{e^x}{4!}\frac{d^3}{dx^3}[4x^3 e^{-x} - x^4 e^{-x}] = \frac{e^x}{4!}\frac{d^2}{dx^2}\left[\frac{d}{dx}\{(4x^3 - x^4)e^{-x}\}\right]$$

$$= \frac{e^x}{4!}\frac{d^2}{dx^2}[(12x^2 - 4x^3)e^{-x} - (4x^3 - x^4)e^{-x}] = \frac{e^x}{4!}\frac{d}{dx}\left[\frac{d}{dx}(12x^2 - 8x^3 + x^4)e^{-x}\right]$$

$$= \frac{e^x}{4!}\frac{d}{dx}[(24x - 24x^2 + 4x^3)e^{-x} - (12x^2 - 8x^3 + x^4)e^{-x}] = \frac{e^x}{4!}\frac{d}{dx}[(24x - 36x^2 + 12x^3 - x^4)e^{-x}]$$

$$= (e^x/4!)[(24 - 72x + 36x^2 - 4x^3)e^{-x} - (24x - 36x^2 + 12x^3 - x^4)e^{-x}] = (24 - 96x + 72x^2 - 16x^3 + x^4)/4!$$

Note : If we use the result given remark of Art. 13.4, namely, $L_n(x) = e^x \dfrac{d^n}{dx^n}(x^n e^{-x})$

and proceed as before, then we have $L_0(x) = 1$, $L_1(x) = 1 - x$, $L_2(x) = 2 - 4x + x^2$, $L_3(x) = 6 - 18x + 9x^2 - x^3$, $L_4(x) = 24 - 96x + 72x^2 - 16x^3 + x^4$ and so on.

13.6. Orthogonality properties of Laguerre's polynomials.

Theorem. Prove that $\int_0^\infty e^{-x} L_n(x)\, L_m(x)\, dx = 0$, *if* $m \neq n$

and $\int_0^\infty e^{-x}\{L_n(x)\}^2 = 1$ **(Kanpur 2004)**

or *prove that* $\int_0^\infty e^{-x} L_n(x) L_m(x)\, dx = \delta_{mn} = \begin{cases} 0, & m \neq n \\ 1, & m = n \end{cases}$

[Arga 1992, Rohilkhand 1996, Meerut 1996, 97]

Proof. Using the generating function, we get

$$\sum_{n=0}^{\infty} L_n(x)\, t^n = \frac{\exp\left(-\dfrac{xt}{1-t}\right)}{1-t} \quad \text{and} \quad \sum_{m=0}^{\infty} L_m(x)\, s^m = \frac{\exp\left(-\dfrac{xs}{1-s}\right)}{1-s}.$$

Multiplying the corresponding sides, we have

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} L_n(x)\, L_m(x)\, t^n s^m = e^{-x\{t/(1-t) + s/(1-s)\}} \times \frac{1}{(1-t)(1-s)}. \quad \text{...(1)}$$

$[\because \exp x . \exp y = e^x . e^y = e^{x+y}]$

Multiplying both sides of (1) by e^{-x} and then integrating both sides w.r.t. 'x', from 0 to ∞, gives

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\left[\int_0^\infty e^{-x} L_n(x)\, L_m(x)\, dx\right] t^n s^m = \frac{1}{(1-t)(1-s)}\int_0^\infty e^{-x\{1 + t/(1-t) + s/(1-s)\}}\, dx$$

$$= \frac{1}{(1-t)(1-s)}\left[\frac{e^{-x\{1 + t/(1-t) + s/(1-s)\}}}{-\{1 + t/(1-t) + s/(1-s)\}}\right]_0^\infty = \frac{1}{(1-t)(1-s)} \times \frac{1}{1 + \dfrac{t}{1-t} + \dfrac{s}{1-s}}$$

$$= \frac{1}{(1-t)(1-s)+t(1-s)+s(1-t)} = \frac{1}{1-st} = (1-st)^{-1} = \sum_{n=0}^{\infty} s^n t^n \text{, by the binomial theorem}$$

$$\therefore \qquad \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \left[\int_0^{\infty} e^{-x} L_n(x)\, L_m(x)\, dx \right] t^n s^m = \sum_{n=0}^{\infty} s^n t^n . \qquad \text{...(2)}$$

We note that the indices of t and s are always equal in each term on R.H.S. of (2). When $m \neq n$, equating coefficients of $t^n s^m$ on both sides of (2) gives

$$\int_0^{\infty} e^{-x} L_n(x)\, L_m(x)\, dx = 0, \quad \text{if } m \neq n. \qquad \text{...(3)}$$

Again equating coefficients of $t^n s^n$ on both sides of (2) gives $\int_0^{\infty} e^{-x} [L_n(x)]^2\, dx = 1$. ...(4)

Using (5), (3) and (4) can be combined to give

$$\int_0^{\infty} e^{-x} L_m(x)\, L_n(x)\, dx = \delta_{mn} = \begin{cases} 0, \text{ if } m \neq n \\ 1, \text{ if } m = n \end{cases}.$$

Remark. If we use result of remark given in Art. 13.3, namely

$$\sum_{n=0}^{\infty} \frac{L_n(x) t^n}{n!} = \frac{\exp\{-xt/(1-t)\}}{1-t}, \qquad \text{we get} \qquad \int_0^{\infty} e^{-x} L_n(x)\, L_m(x)\, dx = n!\, m!\, \delta_{mn}.$$

13.7. Expansion of a polynomial in a series of Laguerre polynomials

Theorem. *If $f(x)$ is polynomials of degree m, show that $f(x)$ may be expressed in the form*

$$f(x) = \sum_{r=0}^{m} C_r L_r(x), \qquad \text{where} \qquad C_r = \int_0^{\infty} e^{-x} L_r(x) f(x)\, dx .$$

Proof. Since $f(x)$ is polynomal of degree m, we write

$$f(x) = a_m x^m + a_{m-1} x^{m-1} + \ldots + a_1 x + a_0. \qquad \text{...(1)}$$

Again, we know that $L_m(x)$ is a polynomial of degree m of the form

$$L_m(x) = k_m x^m + k_{m-1} x^{m-1} + \ldots + k_1 x + k_0. \qquad \text{...(2)}$$

Consider $f(x) - (a_m/k_m) L_m(x)$. Two cases may arise :

Case (i) Let $f(x) - (a_m/k_m) L_m(x) = 0$ so that $f(x) = (a_m/k_m) L_m(x)$, which proves the required result.

Case (ii) Let $f(x) - (a_m/k_m) L_m(x) = g_{m-1}(x)$, $g_{m-1}(x)$ being a polynomial of degree $m - 1$. Taking $C_m = a_m/k_m$, we may write $f(x) = C_m L_m(x) + g_{m-1}(x)$. ...(3)

Taking $g_{m-1}(x)$ in place of $f(x)$ and proceeding as above, we have

$$g_{m-1}(x) = C_{m-1} L_{m-1}(x) + g_{m-2}(x). \qquad \text{...(4)}$$

Making use of (4), (3) may be re–written as

$$f(x) = C_m L_m(x) + C_{m-1} L_{m-1}(x) + g_{m-2}(x). \qquad \text{...(5)}$$

Noting that $L_0(x) = 1$ and proceeding as above, we finally obtain

$$f(x) = C_m L_m(x) + C_{m-1} L_{m-1}(x) + \ldots + C_1 L_1(x) + C_0 L_0(x) \qquad \text{or} \qquad f(x) = \sum_{r=0}^{m} C_r\, L_r(x). \qquad \text{...(6)}$$

$$\text{Since} \qquad \sum_{r=0}^{m} C_r\, L_r(x) = \sum_{s=0}^{m} C_s\, L_s(x), \text{ (6) gives} \qquad f(x) = \sum_{s=0}^{m} C_s\, L_s(x). \qquad \text{...(7)}$$

Multiplying both sides of (7) by $e^{-x}L_r(x)$ and then integrating w.r.t. 'x' between the limits 0 to ∞, we have
$$\int_0^\infty e^{-x} f(x)\, L_r(x)\, dx = \sum_{s=0}^{m} C_s \left\{\int_0^\infty e^{-x} L_r(x) L_s(x)\, dx\right\} \qquad ...(8)$$

But
$$\int_0^\infty e^{-x} L_r(x) L_s(x)\, dx = \begin{cases} 0, & r = s \\ 1, & r \neq s. \end{cases} \qquad ...(9)$$

Using (9), (8) gives
$$C_r = \int_0^\infty e^{-x} f(x) L_r(x)\, dx \qquad ...(10)$$

(6) and (10) prove the required result.

13.8. Relations between Laguerre polynomials and their derivatives:

Recurrence relations (formulae). *Show that*

I. $(n + 1)L_{n+1}(x) = (2n + 1 - x)L_n(x) - nL_{n-1}(x)$. **[Kanpur 2005, 07–10; Meerut 1994]**

II. $xL_n'(x) = nL_n(x) - nL_{n-1}(x)$. **[Kanpur 2009; Meerut 1992, 95]**

III. $L_n'(x) = -\sum_{r=0}^{n-1} L_r(x)$ **[Meerut 1997]**

Proof I. We know that
$$\sum_{n=0}^{\infty} L_n(x)\, t^n = \frac{\exp\{-xt/(1-t)\}}{1-t}. \qquad ...(1)$$

Differentiating both sides of (1) w.r.t. 't', we get
$$\sum_{n=0}^{\infty} L_n(x).nt^{n-1} = \frac{1}{(1-t)^2}\exp\left\{-\frac{xt}{1-t}\right\} - \frac{1}{1-t}\times\exp\left\{-\frac{xt}{1-t}\right\}\times\frac{x}{(1-t)^2}$$

i.e.,
$$\sum_{n=0}^{\infty} L_n(x).nt^{n-1} = \frac{1}{1-t}\sum_{n=0}^{\infty} L_n(x)\, t^n - \frac{x}{(1-t)^2}\sum_{n=0}^{\infty} L_n(x)\, t^n \text{ , by (1)}$$

Multiplying both sides by $(1 - t)^2$, the above equation becomes
$$(1 - 2t + t^2)\sum_{n=0}^{\infty} L_n(x).nt^{n-1} = (1-t)\sum_{n=0}^{\infty} L_n(x)\, t^n - x\sum_{n=0}^{\infty} L_n(x)\, t^n$$

or
$$\sum_{n=0}^{\infty} nL_n(x)t^{n-1} - 2\sum_{n=0}^{\infty} nL_n(x)t^n + \sum_{n=0}^{\infty} nL_n(x)t^{n+1} = \sum_{n=0}^{\infty} L_n(x)t^n - \sum_{n=0}^{\infty} L_n(x)t^{n+1} - x\sum_{n=0}^{\infty} L_n(x)t^n. \qquad ...(2)$$

Equating the coefficients of t^n from both sides, (2) gives
$$(n + 1)L_{n+1}(x) - 2nL_n(x) + (n - 1)L_{n-1}(x) = L_n(x) - L_{n-1}(x) - xL_n(x)$$
or
$$(n + 1)L_{n+1}(x) = (2n + 1 - x)L_n(x) - nL_{n-1}(x). \qquad ...(3)$$

Remark. If we use
$$\sum_{n=0}^{\infty} \frac{t^n L_n(x)}{n!} = \frac{1}{1-t} e^{-xt/(1-t)}$$
and proceed as above, we have
$$L_{n+1}(x) = (2n + 1 - x)L_n(x) - n^2 L_{n-1}(x).$$

II. We know that
$$\sum_{n=0}^{\infty} L_n(x)\, t^n = \frac{\exp\{-xt/(1-t)\}}{1-t}. \qquad ...(1)$$

Differentiating both sides of (1) w.r.t. 'x', we get

$$\sum_{n=0}^{\infty} L_n'(x)\, t^n = \frac{1}{1-t} \times \exp\left\{-\frac{xt}{1-t}\right\} \times \left(\frac{-t}{1-t}\right) \quad \text{or} \quad \sum_{n=0}^{\infty} L_n'(x)\, t^n = \frac{-t}{1-t} \sum_{n=0}^{\infty} L_n(x)\, t^n \text{, by (1)}$$

or $$(1-t) \sum_{n=0}^{\infty} L_n'(x)\, t^n = -t \sum_{n=0}^{\infty} L_n(x)\, t^n \quad \text{or} \quad \sum_{n=0}^{\infty} L_n'(x)\, t^n - \sum_{n=0}^{\infty} L_n'(x)\, t^{n+1} = -\sum_{n=0}^{\infty} L_n(x)\, t^{n+1} \quad \text{...(2)}$$

Equating the coefficients of t^n from both sides, (2) gives $L'_n(x) - L'_{n-1}(x) = -L_{n-1}(x)$. ...(3)

From recurrence relation (1), $\quad (n+1)L_{n+1}(x) = (2n+1-x)L_n(x) - nL_{n-1}(x)$. ...(4)

Differentiating (4) w.r.t. 'x', we have

$$(n+1)L'_{n+1}(x) = (2n+1-x)L'_n(x) - L_n(x) - nL'_{n-1}(x). \quad \text{...(5)}$$

From (3), $\quad L'_{n-1}(x) = L'_n(x) + L_{n-1}(x)$. ...(6)

Now, replacing n by $n+1$ in (3), we have

$L'_{n+1}(x) - L'_n(x) = -L_n(x) \quad$ or $\quad L'_{n+1}(x) = L'_n(x) - L_n(x)$. ...(7)

Putting the values of $L'_{n-1}(x)$ and $L'_{n+1}(x)$ from (6) and (7) in (5), we get

$$(n+1)[L'_n(x) - L_n(x)] = (2n+1-x)L'_n(x) - L_n(x) - n[L'_n(x) + L_{n-1}(x)]$$

or $$xL'_n(x) = nL_n(x) - nL_{n-1}(x), \text{ on simplification}$$

Remark. If we use $\displaystyle\sum_{n=0}^{\infty} \frac{t^n L_n(x)}{n!} = \frac{1}{1-t} e^{-tx/(1-t)}$ and proceed as above, we have

$$L'_n(x) - xL'_{n-1} + nL_{n-1}(x) = 0.$$

III. We know that $$\sum_{n=0}^{\infty} L_n(x) t^n = \frac{\exp\{-xt/(1-t)\}}{1-t}. \quad \text{...(1)}$$

Differentiating both sides of (1) w.r.t. 'x', we get

$$\sum_{n=0}^{\infty} L_n'(x)\, t^n = \frac{1}{1-t} \exp\left\{-\frac{xt}{1-t}\right\} \times \left(-\frac{t}{1-t}\right)$$

$$= -\frac{t}{1-t} \sum_{r=0}^{\infty} L_r(x)\, t^r \text{, using (1) and writing } r \text{ in place of } n$$

$$= -t(1-t)^{-1} \sum_{r=0}^{\infty} L_r(x)\, t^r = -t \sum_{s=0}^{\infty} t^s \sum_{r=0}^{\infty} L_r(x)\, t^r \text{, by the binomial theorem}$$

$$\therefore \qquad \sum_{n=0}^{\infty} L_n'(x)\, t^n = -\sum_{r=0}^{\infty} \sum_{s=0}^{\infty} L_r(x)\, t^{r+s+1}. \quad \text{...(2)}$$

Clearly the coefficient of t^n on L.H.S. of (2) is $L'_n(x)$. We now obtain the coefficient of t^n on R.H.S. of (2). Let $r + s + 1 = n$ so that $s = n - r - 1$. Hence, for a fixed value of r, the coefficient of t^n on R.H.S. of (2) is $-L_r(x)$.

But $s \geq 0 \Rightarrow n - r - 1 \geq 0 \Rightarrow r \leq n - 1$, which gives all the values of r for which $-L_r(x)$ is the coefficient of t^n. Hence the total coefficient of tn on R.H.S. of (2) is given by $\displaystyle -\sum_{r=0}^{n-1} L_r(x)$

Thus, equating the coefficients of t^n from both sides of (2), we get $\displaystyle L'_n(x) = -\sum_{r=0}^{n-1} L_r(x)$.

13.9. SOLVED EXAMPLES

Ex. 1. *Prove that* (*i*) $L_n(0) = 1$. **[Meerut 1993]** (*ii*) $L_n(0) = n!$. **[Meerut 1996]**

Sol. (*i*) We know that (Refer Art. 13.3.) $$\sum_{n=0}^{\infty} t^n L_n(x) = \frac{1}{1-t} e^{-tx/(1-t)} \qquad ...(1)$$

Putting $x = 0$ in (1), $\sum_{n=0}^{\infty} t^n L_n(0) = \frac{1}{1-t} = (1-t)^{-1} = 1 + t + t^2 + .,$ by the binomial theorem

or $$\sum_{n=0}^{\infty} t^n L_n(0) = \sum_{n=0}^{\infty} t^n, \qquad ...(2)$$

which is an identity. Equating the coefficient of t^n on both sides of (2), we get $L_n(0) = 1$.

(*ii*) We know that (Refer remark of Art. 13.3) $$\sum_{n=0}^{\infty} \frac{t^n L_n(x)}{n!} = \frac{1}{1-t} e^{-tx/(1-t)}. \qquad ...(3)$$

Putting $x = 0$ in (3), $\sum_{n=0}^{\infty} \frac{t^n L_n(0)}{n!} = (1-t)^{-1} = \sum_{n=0}^{\infty} t^n$, by binomial theorem

Equating the coefficient of t^n on both sides, we get

$(1/n!)L_n(0) = 1$ or $L_n(0) = n!$.

Ex. 2. *Prove that* (*i*) $L'_n(0) = -n$. (*ii*) $L''_n(0) = \{n(n-1)\}/2$

Part (i). Since $L_n(x)$ is a solution of the Laguerre's equation

$$xy'' + (1-x)y' + ny = 0, \qquad ...(1)$$

we get $$xL_n''(x) + (1-x)L_n'(x) + nL_n(x) = 0. \qquad ...(2)$$

Putting $x = 0$ and using $L_n(0) = 1$, (3) gives

$0 + (1-0)L_n'(0) + n \times 1 = 0$ or $L_n'(0) = -n$.

Part (ii). We know that $$\frac{1}{1-t} \exp\{-xt/(1-t)\} = \sum_{n=0}^{\infty} t^n L_n(x). \qquad ... (3)$$

Differentiating twice w.r.t. 'x', (3) gives $$\frac{\exp\{-xt/(1-t)\}}{1-t}\left(-\frac{t}{1-t}\right)^2 = \sum_{n=0}^{\infty} L_n''(x)\, t^n. \qquad ...(4)$$

Putting $x = 0$ in (4), we have $$\sum_{n=0}^{\infty} L_n''(0)\, t^n = t^2(1-t)^{-3}. \qquad ...(5)$$

Equating the coefficients of t^n on both sides of (5), we get

$L_n''(0)$ = coefficients of t^n in the expansion of $t^2(1-t)^{-3}$ = coeff. of t^{n-2} in the explansion of $(1-t)^{-3}$

$$= \frac{(-3)(-3-1)\ldots\{-3-(n-2)+1\}}{(n-2)!}(-1)^{n-2} = \frac{(-3)(-4)\ldots(-n)}{(n-2)!}(-1)^{n-2} = \frac{3\cdot 4\cdot 5\ldots n}{(n-2)!}(-1)^{n-2}(-1)^{n-2}$$

$$= \frac{1\cdot 2\cdot 3\cdot 4\cdots n}{1\cdot 2\cdot (n-2)!} = \frac{n!}{2(n-2)!} = \frac{n(n-1)(n-2)!}{2(n-2)!} = \frac{n(n-1)}{2}. \qquad [\because (-1)^{2n-2} = 1]$$

Ex. 3. *Prove that* $xL_n''(x) + (1-x)L_n'(x) + L_n(x) = 0$ *and hence deduce that* $L_n'(0) = -n$.

Sol. Since $L_n(x)$ satisfies Laguerre's equation $x(x^2y/dx^2) + (1-x)(dy/dx) + ny = 0$,

therefore $$xL_n''(x) + (1-x)L_n'(x) + nL_n(x) = 0. \qquad ...(1)$$

Replacing x by 0 in (1), we have

$L_n'(0) + nL_n(0) = 0$ or $L_n'(0) = -n$, as $L_n(0) = 1$.

Ex. 4. *Expand* $x^3 + x^2 - 3x + 2$ *in a series of Laguerre polynomials.* **[Meerut 1994]**

Sol. We know that $L_0(x) = 1, L_1(x) = 1-x, L_2(x) = (2-4x+x^2)/2$ and $L_3(x) = (6-18x+9x^2-x^3)/6$.

$$\therefore \qquad x^3 = 6 - 18x + 9x^2 - 6L_3(x), \qquad ...(1)$$

$$x^2 = 4x - 2 + 2L_2(x), \qquad ...(2)$$

$$x = 1 - L_1(x) \qquad \text{and} \qquad 1 = L_0(x). \qquad ...(3)$$

Now, $x^3 + x^2 - 3x + 2 = 6 - 18x + 9x^2 - 6L_3(x) + x^2 - 3x + 2$, by (1)

$= 8 - 21x + 10x^2 - 6L_3(x) = 8 - 21x + 10[4x - 2 + 2L_2(x)] - 6L_3(x)$, by (2)

$= -12 + 19x + 20L_2(x) - 6L_3(x) = -12 + 19[1 - L_1(x)] + 20L_2(x) - 6L_3(x)$, by (3)

$= 7 - 19L_1(x) + 20L_2(x) - 6L_3(x) = 7L_0(x) - 19L_1(x) + 20L_2(x) - 6L_3(x)$, by (3).

Ex. 5. *Find the values of* (*i*) $\int_0^\infty e^{-x} L_3(x) L_5(x)\, dx$ (*ii*) $\int_0^\infty e^{-x} \{L_4(x)\}^2\, dx$.

Sol. By results of Art 13.6, we have

(*i*) $\int_0^\infty e^{-x} L_3(x) L_5(x)\, dx = 0$ (*ii*) $\int_0^\infty e^{-x} \{L_4(x)\}^2\, dx = 1$.

Ex. 6. *Taking* $L_n(x)$ *to be the coefficient of* t^n *in the expansion of*

$\dfrac{1}{1-t} \exp\left(\dfrac{-xt}{1-t}\right)$, *prove that* $\displaystyle\int_0^{\pi/2} \frac{e^{-\tan\theta}}{\cos^2\theta} L_n(\tan\theta)\, L_m(\tan\theta)\, d\theta = \delta_{mn.}$

Sol. $\displaystyle\int_0^{\pi/2} \frac{e^{-\tan\theta}}{\cos^2\theta} L_n(\tan\theta)\, L_m(\tan\theta)\, d\theta = \int_0^\infty e^{-x} L_n(x)\, L_m(x)\, dx$

[on putting $x = \tan\theta$ so that $dx = \sec^2\theta\, d\theta$]

$= \delta_{m,\, n,}$ by orthogonal property of $L_n(x)$.

Ex. 7. *Taking* $\displaystyle\sum_{n=0}^{\infty} \frac{t^n}{n!} L_n(x) = \frac{1}{1-t} \exp\left\{\frac{-tx}{1-t}\right\}$, *prove that*

(*i*) $L_n'(x) = n[L'_{n-1}(x) - L_{n-1}(x)]$ (*ii*) $xL'_n(x) = nL_n(x) - n^2 L_{n-1}(x)$.

Sol. Part (*i*) Given $\displaystyle\sum_{n=0}^{\infty} \frac{t^n}{n!} L_n(x) = \frac{1}{1-t} e^{-tx/(1-t)}. \qquad ...(1)$

Differentiating both sides of (1) w.r.t. 'x', we get

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} L_n'(x) = \frac{1}{1-t} e^{-tx/(1-t)} \left(-\frac{t}{1-t}\right) \quad \text{or} \quad \sum_{n=0}^{\infty} \frac{t^n}{n!} L_n'(x) = -\frac{t}{1-t} \sum_{n=0}^{\infty} \frac{t^n}{n!} L_n(x), \text{ using (1)}$$

$$(1-t)\sum_{n=0}^{\infty} \frac{t^n}{n!} L_n'(x) = -t \sum_{n=0}^{\infty} \frac{t^n}{n!} L_n(x) \quad \text{or} \quad \sum_{n=0}^{\infty} \frac{t^n}{n!} L_n'(x) = \sum_{n=0}^{\infty} \frac{t^{n+1}}{n!} L_n'(x) - \sum_{n=0}^{\infty} \frac{t^{n+1}}{n!} L_n(x). \quad ...(2)$$

Equating the coefficient of t^n on both sides of (2), we have

$$\frac{1}{n!} L_n'(x) = \frac{1}{(n-1)!} L_{n-1}'(x) - \frac{1}{(n-1)!} L_{n-1}(x) \quad \text{or} \quad L'_n(x) = n[L'_{n-1}(x) - L_{n-1}(x)].$$

Part (ii) : We have (From Remark of recurrence relation 1).

$$L_{n+1}(x) = (2n + 1 - x)L_n(x) - n^2 L_{n-1}(x). \qquad ...(1)$$

Differentiating both sides of (1) w.r.t. 'x', we get

$$L'_{n+1}(x) = (2n + 1 - x)L'_n(x) - L_n(x) - n^2 L'_{n-1}(x). \qquad ...(2)$$

From part (*i*) above, we have $L'_n(x) = n[L'_{n-1}(x) - L_{n-1}(x)]. \qquad ...(3)$

Replacing n by $n + 1$ in (3), we get $L'_{n+1}(x) = (n + 1)[L'_n(x) - L_n(x)]. \qquad ...(4)$

Again, from (3), we get $nL'_{n-1}(x) = L'_n(x) + nL_{n-1}(x). \qquad ...(5)$

Using (4) and (5), (2) reduces to

$$(n + 1)[L'_n(x) - L_n(x)] = (2n + 1 - x)L'_n(x) - L_n(x) - n[L'_n(x) + nL_{n-1}(x)]$$

or $$xL'_n(x) = nL_n(x) - n^2 L_{n-1}(x).$$

Ex. 8. *Prove that* $\int_0^\infty e^{-st} L_n(t)\, dt = (1/s) \times (1 - 1/s)^n$. **[Kanpur 2011]**

Sol. By definition of $L_n(x)$, we have $$L_n(t) = \sum_{r=0}^{n} \frac{(-1)^r n!\, t^r}{(n-r)!\,(r!)^2}. \qquad ...(1)$$

Now, L.H.S. of the problem

$$= \int_0^\infty e^{-st} L_n(t)\, dt = \int_0^\infty e^{-st}\left(\sum_{r=0}^{n} \frac{(-1)^r n!\, t^r}{(n-r)!(r!)^2}\right) dt = \sum_{r=0}^{n} \frac{(-1)^r n!}{(n-r)!(r!)^2} \int_0^\infty e^{-st} t^{(r+1)-1}\, dt$$

$$= \sum_{r=0}^{n} \frac{(-1)^r n!}{(n-r)!(r!)^2} \cdot \frac{\Gamma(r+1)}{s^{r+1}} = \frac{1}{s}\sum_{r=0}^{n} \frac{(-1)^r n!}{(n-r)!(r!)}\left(\frac{1}{s}\right)^r, \qquad \text{as} \qquad \Gamma(r+1) = r!$$

$$= \frac{1}{s}\sum_{r=0}^{n} {}^nC_r \left(-\frac{1}{s}\right)^r = \frac{1}{s}\left(1 - \frac{1}{s}\right)^n, \text{ by the binomial theorem.}$$

Ex. 9. *Prove that* $\int_x^\infty e^{-y} L_n(y)\, dy = e^{-x}[L_n(x) - L_{n-1}(x)]$. **[Kanpur 2007]**

Sol. $\int_x^\infty e^{-y} L_n(y)\, dy = [-e^{-y} L_n(y)]_x^\infty - \int_x^\infty (-e^{-y}) L'_n(y)\, dy$, integrating by parts

$$= e^{-x} L_n(x) + \int_x^\infty e^{-y} L'_n(y)\, dy \quad = e^{-x} L_n(x) + \int_x^\infty e^{-y}\left(-\sum_{r=0}^{n-1} L_r(y)\right) dy, \quad \text{as} \quad L'_n(y) = -\sum_{r=0}^{n-1} L_r(y)$$

$$= e^{-x} L_n(x) - \sum_{r=0}^{n-1} \int_x^\infty e^{-y} L_r(y)\, dy.$$

$$\therefore \qquad \int_x^\infty e^{-y} L_n(y)\, dy + \sum_{r=0}^{n-1} \int_x^\infty e^{-y} L_r(y)\, dy = e^{-x} L_n(x). \qquad ...(1)$$

Re–writing (1), we have $$\sum_{r=0}^{n} \int_x^\infty e^{-y} L_r(y)\, dy = e^{-x} L_n(x). \qquad ...(2)$$

Subtracting (1) from (2), we have

$$\sum_{r=0}^{n} \int_x^\infty e^{-y} L_r(y)\, dy - \int_x^\infty e^{-y} L_n(y)\, dy - \sum_{r=0}^{n-1} \int_x^\infty e^{-y} L_r(y)\, dy = 0$$

or $$\int_x^\infty e^{-y} L_n(y)\, dy = \sum_{r=0}^{n} \int_x^\infty e^{-y} L_r(y)\, dy - \sum_{r=0}^{n-1} \int_x^\infty e^{-y} L_r(y)\, dy$$

$= e^{-x} L_n(x) - e^{-x} L_{n-1}(x)$, using (2) for $n = n$ and $n = n - 1$

$= e^{-x}[L_n(x) - L_{n-1}(x)]$.

Ex. 10. *Prove that* $\int_0^\infty e^{-y} x^k L_n(x)\, dx = \begin{cases} 0, \text{ if } k < n \\ (-1)^n n!, \text{ if } k = n. \end{cases}$ **[Kanpur 2004]**

Sol. From Art. 13.7, we know that if $f(x)$ is a polynomial of degree m, we have

$$f(x) = \sum_{r=0}^{m} C_r L_r(x), \qquad ...(1)$$

where $$C_r = \int_0^\infty e^{-x} f(x) L_r(x)\, dx\,. \qquad \text{...(2)}$$

Let $f(x) = x^k$ so that $f(x)$ is a polynomial of degree k. Then (1) and (2) reduce to

$$x^k = \sum_{r=0}^{k} C_r L_r(x), \qquad \text{...(3)}$$

where $$C_r = \int_0^\infty e^{-x} x^k L_r(x)\, dx. \qquad \text{...(4)}$$

Case (i) Let $k < n$. Then expansion (3) shows that $C_n = 0$. Hence from (4), we have (with $r = n$)

$$\int_0^\infty e^{-x} x^k L_n(x)\, dx = 0. \qquad \text{...(5)}$$

Case (ii). Let $k = n$. Then (3) and (4) give $x^n = \sum_{r=0}^{n} C_r L_r(x) = C_0 L_0(x) + \ldots + C_n L_n(x)$ ...(6)

and $$C_r = \int_0^\infty e^{-x} x^n L_n(x)\, dx\,. \qquad \text{...(7)}$$

We have, by definition of $L_n(x)$ $$L_n(x) = \sum_{r=0}^{n} (-1)^r \frac{n!}{(n-r)!(r!)^2} x^r\,. \qquad \text{...(8)}$$

From (8), we see that, coefficient of x^n in $L_n(x) = (-1)^n \dfrac{n!}{0!(n!)^2} = \dfrac{(-1)^n}{n!}$. ...(9)

Equating the coefficients of x^n from both sides of (6) and using (9), we get
$1 = C_n(-1)^n/n!$ so that $C_n = (-1)^n n!$.

With this value of C_n, (7) (with $r = n$) reduces to $\int_0^\infty e^{-x} x^n L_n(x)\, dx = (-1)^n n!$. ...(10)

(5) and (10) together may be written as $$\int_0^\infty e^{-x} x^k L_n(x)\, dx = \begin{cases} 0, \text{ if } k < n \\ (-1)^n n!, \text{ if } k = n. \end{cases}$$

EXERCISE

1. Express $10 - 23x + 10x^2 - x^3$ in terms of Laguerre polynomials.

Ans. $L_0(x) + L_1(x) + 2L_1(x) - 6L_3(x)$

2. Prove that $L_n(2x) = n! \sum_{n=0}^{\infty} \dfrac{(-1)^m 2^{n-m}}{m!(n-m)!} L_{n-m}(x)$.

3. If $e^{-x} = \sum_{n=0}^{\infty} C_n L_n(x)$, then show that $C_n = 1/(2^{n+1} n!)$.

4. Show that $L_n(x)$ defined by $$\exp\left(-\frac{xt}{1-t}\right) = (1-t)\sum_{n=0}^{\infty} \frac{L_n(x) t^n}{n!}$$

satisfies the differential equation $xL''_n(x) + (1-x)L'_n(x) + nL_n(x) = 0$.

5. Laguerre polynomial $L_q(x)$ is defined by $$e^{-xs/(1-s)} = \sum_{q=0}^{\infty} \frac{L_q(x)}{q!} s^q\,,\ s < 1.$$

Show that $L'_q = qL'_{q-1} - qL_{q-1}$ and $L_{q+1} = (2q + 1 - x)L_q - q^2 L_{q-1}$.

6. State and prove that generating function for Laguerre polynomial. **[Meerut 1993]**

7. Show that (*i*) $L_n^{(n)}(0) = (-1)^n$ (*ii*) $L_n^{(n)}(0) = (-1)^n n!$

where $L_n^{(n)}$ stands for $d^n L_n(x)/dx^n$ **[Kanpur 2009]**

14

Hypergeometric Function

14.1. Pochhammer symbol. Definition.

Let n be a positive integer. Then Pochhammer symbol is denoted and defined by

$$(\alpha)_n = \alpha(\alpha + 1)...(\alpha + n - 1) \qquad ...(1)$$

with $$(\alpha)_0 = 1. \qquad ...(2)$$

Deductions. By definition, we have

I. $(\alpha)_n = \alpha(\alpha+1)...(\alpha+n-1) = \dfrac{1\cdot 2\cdot 3...(\alpha-1)\alpha(\alpha+1)...(\alpha+n-1)}{1\cdot 2\cdot 3...(\alpha-1)} = \dfrac{\Gamma(\alpha+n)}{\Gamma(\alpha)}$ as $\Gamma(p) = (p-1)\Gamma(p-1)$

Thus, $$(\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}. \qquad ...(3)$$

II. $(\alpha)_{n+1} = \alpha(\alpha+1)(\alpha+2)...[\alpha+(n+1)-1] = \alpha[(\alpha+1)(\alpha+2)...(\alpha+1+n-1)] = \alpha(\alpha+1)_n$

Thus, $$(\alpha)_{n+1} = \alpha(\alpha + 1)_n. \qquad ...(4)$$

III. $$(\alpha + n)(\alpha)_n = \alpha(\alpha + 1)...(\alpha + n - 1)(\alpha + n)$$
$$= \alpha(\alpha + 1)...(\alpha + n - 1)(\alpha + n + 1 - 1) = (\alpha)_{n+1}$$

Thus, $$(\alpha + n)(\alpha)_n = (\alpha)_{n+1}. \qquad ...(5)$$

14.2. General hypergeometric function. Definition.

The general hypergeometric function is denoted and defined by

$${}_mF_n(\alpha_1, \alpha_2,...\alpha_m ; \beta_1, \beta_2...\beta_n ; x) = \sum_{r=1}^{\infty} \frac{(\alpha_1)_r(\alpha_2)_r...(\alpha_m)_r}{(\beta_1)_r(\beta_2)_r...(\beta_n)_r}.\frac{x^r}{r!} \qquad ....(1)$$

The general hypergeometric function is also denoted by $\quad {}_mF_n\left[\begin{matrix}\alpha_1, \alpha_2,\ldots\alpha_m ; \\ \beta_1, \beta_2\ldots\beta_n ;\end{matrix} x\right]. \quad ...(2)$

Remark. We shall consider only two special cases of (1) in the present chapter. These are given in the following two articles for $m = n = 1$ and $m = 2$, $n = 1$ respectively.

14.3. Confluent hypergeometric (or Kummer) function. Definition.

Confluent hypergeometric function is denoted by ${}_2F_1(\alpha; \beta; x)$ or $F(\alpha; \beta; x)$ or $M(\alpha, \beta, x)$ and is defined by

$$F(\alpha; \beta; x) = \sum_{r=0}^{\infty} \frac{(\alpha)_r}{(\beta)_r}.\frac{x^r}{r!}. \qquad ...(1)$$

Remark. Sometimes we use the following modified definition of the confluent hypergeometric function

$$F(\alpha; \beta; x) = 1 + \frac{x}{1\cdot\beta}x + \frac{\alpha(\alpha+1)}{1\cdot 2\cdot\beta(\beta+1)}x^2 + ... \qquad ...(2)$$

14.4. Hypergeometric function. Definition. [Ranchi 2010]

Hypergeometric function is denoted by ${}_2F_1(\alpha; \beta; \gamma; x)$ or simply $F(\alpha, \beta; \gamma; x)$ and is defined by

$$F(\alpha, \beta; \gamma; x) = \sum_{r=0}^{\infty} \frac{(\alpha)_r (\beta)_r}{(\gamma)_r} \cdot \frac{x^r}{r!}. \qquad \text{...(1)}$$

Remark 1. The series on the R.H.S. of (1) is $1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1 \cdot 2 \cdot \gamma(\gamma+1)} \cdot x^2 + \dots$ ...(2)

In particular, if $\alpha = 1$, $\beta = \gamma$, then the series (2) takes the form

$$1 + x + x^2 + x^3 + \dots,$$

which is a geometric series. Since (2) reduces to a geometric series as a particular case, (2) is called *hypergeometric series*.

Remark 2. Sometimes we use the following modified definition of the hypergeometric series

$$F(\alpha, \beta; \gamma; x) = 1 + \frac{\alpha \cdot \beta}{\gamma} \frac{x}{1!} + \frac{\alpha(\alpha+1) \cdot \beta(\beta+1)}{\gamma(\gamma+1)} \frac{x^2}{2!} + \frac{\alpha(\alpha+1)(\alpha+2) \cdot \beta(\beta+1)(\beta+2)}{\gamma(\gamma+1)(\gamma+2)} \frac{x^3}{3!} + \dots \qquad \text{...(3)}$$

Remark 3. Hypergeometric function $F(a, b; c; x)$ can be put in the following different forms:

$$F(a, b; c; x) = (1-x)^{c-a-b} F(c-a, c-b; c; x) \quad \textbf{[Lucknow2010]} \qquad \text{...(4)}$$

$$= (1-x)^{-a} F\left(a, c-b; c; \frac{x}{x-1}\right) \quad \textbf{[Kanpur 2009]} \qquad \text{...(5)}$$

$$= (1-x)^{-b} F\left(b, c-a; c; \frac{x}{x-1}\right). \qquad \text{...(6)}$$

14.5. Gauss's hypergeometric equation or Gauss's equation or hypergeometric equation. Definition.

$$x(1-x)(d^2y/dx^2) + \{\gamma - (\alpha + \beta + 1)x\}(dy/dx) - \alpha\beta y = 0$$

is called *hypergeometric equation.*

14.6. Solution of the hypergeometric eqution. [Kanpur 2006]

For solution refer Ex. 7 of Art. 8.9 of chapter 8. Let α, β and γ be constants, then

$$x(1-x)y'' + \{\gamma - (\alpha + \beta + 1)x\} y' - \alpha \beta y = 0 \qquad \text{... (1)}$$

is known the *hypergeometric equation*. Let $\gamma \neq 0, -1, -2, \dots$. Then a solution of (1) is given by

$${}_2F_1(\alpha, \beta; \gamma, x) = 1 + \frac{\alpha \cdot \beta}{1 \cdot \gamma} x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1 \cdot 2\, \gamma(\gamma+1)} x^2 + \dots. \qquad \text{... (2)}$$

${}_2F_1(\alpha; \beta; \gamma; x)$ called the *hypergeometric function*.

If γ is not an integer, then the other solution of (1) and linearly independent of the solution ${}_2F_1(\alpha, \beta; \gamma, x)$ is $x^{1-\gamma} {}_2F_1(\alpha+1-\gamma, \beta+1-\gamma; 2-\gamma; x)$.

Thus, if γ is not an integer, the general solution of (1) is

$y = a\, {}_2F_1(\alpha, \beta; \gamma; x) + b x^{1-\gamma} {}_2F_1(\alpha+1-\gamma, \beta+1-\gamma; 2-\gamma; x)$, where a and b are arbitrary constants. ... (3)

Note 1. The hypergeometric function $F(\alpha, \beta; \gamma; x)$ is defined only if (*i*) α and β are real numbers (*ii*) γ is any real number such that $\gamma \neq 0, -1, -2, \dots$ (*iii*) the variable x satisfies $|x| < 1$.

Note 2. The general solution (3) of (1) exists if γ is not an integer.

Note 3. If either α or β is a negative integer, then $F(\alpha, \beta; \gamma; x)$ reduces to a polynomial, because after finite number of terms, the coefficient of each term will be zero. For examples, consider the following:

$$F(-2, b; \gamma; c) = 1 + \frac{(-2)b}{1 \cdot c} x + \frac{(-2)(-1)b(b+1)}{1 \cdot 2 \cdot c(c+1)} x^2 + 0 = 1 - \frac{2b}{c} x + \frac{b(b+1)}{c(c+1)} x^2 + \ldots$$

which is a polynomial of degree 2

Similarly, $$F(a, -2; \gamma : c) = 1 - (2a/c)x + \{a(a+1)/c(c+1)\}x^2.$$

14.7. Symmetric property of hypergeometric function.

Hypergeometric function does not change if the parameters α and β are interchanged, keeping γ fixed. Thus, $$F(\alpha, \beta; \gamma; x) = F(\beta, \alpha; \gamma; x).$$

Proof. We have, by definition $$F(\alpha, \beta; \gamma; x) = \sum_{r=0}^{\infty} \frac{(\alpha)_r (\beta)_r}{(\gamma)_r} . \frac{x^r}{r!} \qquad \ldots(1)$$

and $$F(\beta, \alpha; \gamma; x) = \sum_{r=0}^{\infty} \frac{(\beta)_r (\alpha)_r}{(\gamma)_r} . \frac{x^r}{r!} . \qquad \ldots(2)$$

From (1) and (2), we have $$F(\alpha, \beta; \gamma; x) = (\beta, \alpha; \gamma; x).$$

14.8. Differentiation of hypergeometric functions.

Show that

$$\frac{d}{dx} F(\alpha, \beta; \gamma; x) = \frac{\alpha\beta}{\gamma} F(\alpha + 1, \beta + 1; \gamma + 1; x)$$

[Garhwal 2005; Kurukshreta 2006; Kanpur 2005, 08, 10]

and deduce that (*i*) $$\frac{d^n}{dx^n} F(\alpha, \beta; \gamma; x) = \frac{(\alpha)_n (\beta)_n}{(\gamma)_n} F(\alpha + n, \beta + n; \gamma + n; x).$$

(*ii*) $$\left[\frac{d^n}{dx^n} F(\alpha, \beta; \gamma; x) \right]_{x=0} = \frac{(\alpha)_n (\beta)_n}{(\gamma)_n}.$$

Proof. By definition, we have $$F(\alpha, \beta; \gamma; x) = \sum_{r=0}^{\infty} \frac{(\alpha)_r (\beta)_r}{(\gamma)_r r!} x^r.$$

Differentiating both sides w.r.t. 'x', we have

$$\frac{d}{dx} F(\alpha, \beta; \gamma; x) = \sum_{r=0}^{\infty} \frac{(\alpha)_r (\beta)_r}{(\gamma)_r r!} . r x^{r-1} = \sum_{r=1}^{\infty} \frac{(\alpha)_r (\beta)_r}{(\gamma)_r (r-1)!} x^{r-1} \qquad (\because \text{ the term with } r = 0 \text{ vanishes})$$

$$= \sum_{m=0}^{\infty} \frac{(\alpha)_{m+1} (\beta)_{m+1}}{(\gamma)_{m+1} m!} x^m$$ (taking m as the new variable of summation such that $r = m + 1$ *i.e.* $m = r - 1$ so that when $r = 1$, $m = 0$, and $r = \infty$, $m = \infty$)

$$= \sum_{m=0}^{\infty} \frac{\alpha\,(\alpha+1)_{m+1} . \beta\,(\beta+1)_m}{\gamma\,(\gamma+1)_m\; m!} x^m$$, by Art. 14.1

$$= \frac{\alpha\beta}{\gamma} \sum_{m=0}^{\infty} \frac{(\alpha+1)_m (\beta+1)_m}{(\gamma+1)_m m!} x^m = \frac{\alpha\beta}{\gamma} F(\alpha + 1, \beta + 1; \gamma + 1; x).$$

$$\therefore \qquad \frac{d}{dx}F(\alpha, \beta; \gamma; x) = \frac{\alpha\beta}{\gamma}F(\alpha+1, \beta+1; \gamma+1; x). \qquad ...(1)$$

Deduction. (*i*) For each positive integer, we must show that

$$\frac{d^n}{dx^n}F(\alpha, \beta; \gamma; x) = \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}F(\alpha+n, \beta+n; \gamma+n; x) \qquad ...(2)$$

Since $\alpha = (\alpha)_1$, $\beta = (\beta)_1$ and $\gamma = (\gamma)_1$, (1) shows that (2) is true for $n = 1$. We now assume that (2) is true for a particular value of n (say $n = m$) so that

$$\frac{d^m}{dx^m}F(\alpha, \beta; \gamma; x) = \frac{(\alpha)_m(\beta)_m}{(\gamma)_m}F(\alpha+m, \beta+m; \gamma+m; x). \qquad ...(3)$$

Differentiating both sides of (3) w.r.t. 'x' we get

$$\frac{d^{m+1}}{dx^{m+1}}F(\alpha, \beta; \gamma; x) = \frac{(\alpha)_m(\beta)_m}{(\gamma)_m}\frac{d}{dx}F(\alpha+m, \beta+m; \gamma+m; x) = \frac{(\alpha)_m(\beta)_m}{(\gamma)_m}\frac{(\alpha+m)(\beta+m)}{(\gamma+m)}F(\alpha+m+1, \beta+m+1; \gamma+m+1; x)$$

[using (1) for $\alpha + m$, $\beta + m$, $\gamma + m$ in place of α, β, γ respectively]

$$\therefore \qquad \frac{d^{m+1}}{dx^{m+1}}F(\alpha, \beta; \gamma; x) = \frac{(\alpha)_{m+1}(\beta)_{m+1}}{(\gamma)_{m+1}}F(\alpha+m+1, \beta+m+1; \gamma+m+1; x), \qquad ...(4)$$

where we have used relation (5) of Art. 14.1. (4) shows that (2) is true for $n = m + 1$. Thus if (2) is true for $n = m$, then (2) is also true $n = m + 1$. Hence by mathematical induction, (2) is true for each positive integer.

Deduction (*ii*). Putting $x = 0$ in (2), we have

$$\left[\frac{d^n}{dx^n}F(\alpha, \beta; \gamma; x)\right]_{x=0} = \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}F(\alpha+n, \beta+n; \gamma+n; 0) = \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}\left[\sum_{r=0}^{\infty}\frac{(\alpha+n)_r\,(\beta+n)_r}{(\gamma+n)_r}\frac{x^r}{r!}\right]_{x=0} = \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}.$$

14.9. Integral representation for the hypergeometric function

$$F(\alpha, \beta; \gamma; x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-xt)^{-\alpha}\,dt$$

[Purvanchal 2007; Garhwal 2004; Kanpur 2004, 07, 10]

or $\quad F(\alpha, \beta; \gamma; x) = \dfrac{1}{B(\beta, \gamma-\beta)}\displaystyle\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-xt)^{-\alpha}\,dt$, if $\gamma > \beta > 0$.

Proof. By definition, we have

$$F(\alpha, \beta; \gamma; x) = \sum_{n=0}^{\infty}\frac{(\alpha)_n\,(\beta)_n}{(\gamma)_n}.\frac{x^n}{n!} = \sum_{n=0}^{\infty}(\alpha)_n\frac{\Gamma(\beta+n)}{\Gamma(\beta)}.\frac{\Gamma(\gamma)}{\Gamma(\gamma+n)}.\frac{x^n}{n!}, \text{ by Art. 14.1}$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\gamma-\beta)\Gamma(\beta)}\sum_{n=1}^{\infty}(\alpha)_n\frac{\Gamma(\beta+n)\Gamma(\gamma-\beta)}{\Gamma(\beta+n+\gamma-\beta)}.\frac{x^n}{n!}, \text{ multiplying and dividing by } \Gamma(\gamma-\beta)$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\sum_{n=0}^{\infty}(\alpha)_n\left\{\int_0^1 t^{\beta+n-1}(1-t)^{\gamma-\beta-1}\,dt\right\}\frac{x^n}{n!}, \text{ where } \gamma-\beta > 0,\ \beta+n > 0 \text{ so that } \gamma > \beta > 0$$

$$\left[\because \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)} = B(p, q) = \int_0^1 t^{p-1}(1-t)^{q-1}\,dt\right]$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1} \left(\sum_{n=0}^{\infty} (\alpha)_n \frac{(xt)^n}{n!} \right) dt$$

Thus, $$F(\alpha, \beta, \gamma, x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-xt)^{-\alpha}\, dt \qquad ...(1)$$

[$\because$ the general term in the expansion of $(1 - xt)^{-\alpha} = \dfrac{(-\alpha)(-\alpha-1)...(-\alpha-n+1)}{n!}(-xt)^n$

$$= (-1)^n \frac{\alpha(\alpha+1)...(\alpha+n-1)}{n!} \times (-1)^n x^n t^n = (\alpha)_n \frac{x^n t^n}{n!}, \text{ by Art. } 14.1]$$

Also, $B(\beta, \gamma-\beta) = \dfrac{\Gamma(\beta)\,\Gamma(\gamma-\beta)}{\Gamma(\beta+\gamma-\beta)} = \dfrac{\Gamma(\beta)\,\Gamma(\gamma-\beta)}{\Gamma(\gamma)} \Rightarrow \dfrac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} = \dfrac{1}{B(\beta, \gamma-\beta)}$. ...(2)

Using (2), (1) may be re–written as

$$F(\alpha, \beta; \gamma; x) = \frac{1}{B(\beta, \gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-xt)^{-\alpha}\, dt. \qquad ...(3)$$

Thus (1) and (3) are the required results.

14.10. Gauss Theorem. $F(\alpha, \beta; \gamma; 1) = \dfrac{\Gamma(\gamma)\,\Gamma(\gamma-\beta-\alpha)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}$ **[Kanpur 2005, 06, 07 Ranchi 2010]**

Proof. From Art. 14.9, for $x = 1$ we have

$$F(\alpha, \beta; \gamma; 1) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-t)^{-\alpha}\, dt$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-\alpha-1}\, dt = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \frac{\Gamma(\beta)\,\Gamma(\gamma-\beta-\alpha)}{\Gamma(\beta+\gamma-\beta-\alpha)}$$

$$\left(\because \int_0^1 t^{p-1}(1-t)^{q-1}\, dt = B(p, q) = \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)} \right)$$

$$= \frac{\Gamma(\gamma)\,\Gamma(\gamma-\beta-\alpha)}{\Gamma(\gamma-\alpha)\,\Gamma(\gamma-\beta)}.$$

14.11. Vandermonde's theorem. $F(-n, \beta; \gamma; 1) = \dfrac{(\gamma-\beta)_n}{(\gamma)_n}$.

Proof. From Art. 14.10, with $\alpha = -n$, we get

$$F(-n, \beta; \gamma; 1) = \frac{\Gamma(\gamma)\,\Gamma(\gamma-\beta+n)}{\Gamma(\gamma+n)\,\Gamma(\gamma-\beta)} = \frac{\Gamma(\gamma)\,\Gamma(\gamma-\beta+n-1)\,(\gamma-\beta+n-2)...(\gamma-\beta)\,\Gamma(\gamma-\beta)}{(\gamma+n-1)(\gamma+n-2)...\gamma\Gamma(\gamma)\Gamma(\gamma-\beta)}$$

$$= \frac{(\gamma-\beta+n-1)\,(\gamma-\beta+n-2)...(\gamma-\beta)}{(\gamma+n-1)\,(\gamma+n-2)...\gamma} = \frac{(\gamma-\beta)_n}{(\gamma)_n}, \text{ by Art. } 14.1$$

14.12. Kummer's theorem.

$$F(\alpha, \beta; \beta-\alpha+1; -1) = \frac{\Gamma(\beta-\alpha+1)\,\Gamma(\beta/2+1)}{\Gamma(\beta+1)\,\Gamma(\beta/2-\alpha+1)}.$$ **[Purvanchal 2005]**

Proof. From Art. 14.9, with $x = -1$ and $\gamma = \beta - \alpha + 1$, we get
$F(\alpha, \beta; \beta-\alpha+1; -1)$

$$= \frac{\Gamma(\beta-\alpha+1)}{\Gamma(\alpha)\,\Gamma(\beta-\alpha+1-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\beta-\alpha+1-\beta-1}(1-t)^{-\alpha}\,dt = \frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)\,\Gamma(1-\alpha)}\int_0^1 t^{\beta-1}(1-t^2)^{-\alpha}\,dt$$

$$= \frac{\Gamma(\beta-\alpha+1)}{\Gamma(\beta)\,\Gamma(1-\alpha)}\int_0^1 (u^{1/2})^{\beta-1}(1-u)^{-\alpha}\,(du/2\sqrt{u})$$ (putting $t^2 = u$ so that $dt = du/2\sqrt{u}$)

$$= \frac{\Gamma(\beta-\alpha+1)}{2\Gamma(\beta)\,\Gamma(1-\alpha)}\int_0^1 u^{(\beta/2)-1}(1-u)^{1-\alpha-1}\,du = \frac{\Gamma(\beta-\alpha+1)}{2\Gamma(\beta)\,\Gamma(1-\alpha)}\cdot\frac{\Gamma(\beta/2)\,\Gamma(1-\alpha)}{\Gamma(\beta/2+1-\alpha)}$$

$$\left(\because \int_0^1 u^{p-1}(1-u)^{q-1} = B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}\right)$$

$$= \frac{\Gamma(\beta-\alpha+1)\cdot(\beta/2)\cdot\Gamma(\beta/2)}{\Gamma(\beta/2+1-\alpha)\beta\,\Gamma(\beta)} = \frac{\Gamma(\beta-\alpha+1)\,\Gamma(\beta/2+1)}{\Gamma(\beta/2+1-\alpha)\,\Gamma(\beta+1)}$$

14.13. More about the confluent hypergeometric function and solution of confluent hypergeometric equation The hypergeometric differential equation is

$$(x^2 - x)y'' + [(1+\alpha+\beta)x - \gamma]y' + \alpha\beta y = 0. \qquad ...(1)$$

Replacing x by x/β in (1), we get $$x\left(1-\frac{x}{\beta}\right)y'' + \left\{\gamma - \left(1+\frac{\alpha+1}{\beta}\right)x\right\}y' - \alpha y = 0 \qquad ...(2)$$

Its solution is represented by the function $F(\alpha, \beta; \gamma; x/\beta)$

When $\beta \to \infty$, the equation (2) reduces to $$xy'' + (\gamma - x)y' - \alpha y = 0 \qquad ...(3)$$

whose solution is given by $$\lim_{\beta\to\infty} F(\alpha, \beta; \gamma; x/\beta). \qquad ...(4)$$

The equation (3) is known as the *confluent hypergeometric differential equation* or *Kummer's equation*.

Now, $$\lim_{\beta\to\infty}\frac{(\beta)_r}{\beta^r} = \lim_{\beta\to 0}\frac{\beta(\beta+1)(\beta+2)...(\beta+r-1)}{\beta\cdot\beta\cdot\beta...r \text{ times}} = \lim_{\beta\to\infty}\left(1+\frac{1}{\beta}\right)\left(1+\frac{2}{\beta}\right)...\left(1+\frac{r-1}{\beta}\right) = 1$$

Hence solution (4) may be written as

$$\lim_{\beta\to\infty} F\left(\alpha, \beta; \gamma; \frac{x}{\beta}\right) = \lim_{\beta\to\infty}\sum_{r=0}^{\infty}\frac{(\alpha)_r\,(\beta)_r}{r!\,(\gamma)_r}\left(\frac{x}{\beta}\right)^r = \lim_{\beta\to\infty}\sum_{r=0}^{\infty}\frac{(\alpha)_r\,(\beta)_r}{r!\,(\gamma)_r\beta^r}x^r = \sum_{r=0}^{\infty}\frac{(\alpha)_r}{r!\,(\gamma)_r}x^r = F(\alpha; \gamma; x). \qquad ...(5)$$

The function $F(a; \gamma; x)$ is called the *confluent hypergeometric function*.

Solution of differential equation (3) may also be obtained directly by the series integration method. Considering the equation (3), we find that $x = 0$ is a removable (non–essential) singularity and so the series representing the solution can be developed about the point $x = 0$.

Solution of the confluent hypergeometric differential equation when x = 0 and γ is not an integer. **[Kanpur 2010]**

Let us consider the solution of the hypergeometric differential equation (3) in the ascending powers of x as

$$y = x^k(a_0 + a_1x + a_2x^2 + ...) = \sum_{r=0}^{\infty} a_0 x^{k+r},\ a_0 \neq 0. \quad ...(6)$$

$\therefore \quad y' = \sum_{r=0}^{\infty} a_r(k+r)x^{k+r-1}$ and $y'' = \sum_{r=0}^{\infty} a_r(k+r)(k+r-1)x^{k+r-2}$.

Now putting the values of y, y' and y'' in (3) we get

$$x\sum_{r=0}^{\infty} a_r(k+r)(k+r-1)\,x^{k+r-2} + (\gamma - x)\sum_{r=0}^{\infty} a_r(k+r)\,x^{k+r-1} - \alpha\sum_{r=0}^{\infty} a_r\,x^{k+r} = 0$$

or
$$\sum_{r=0}^{\infty} a_r[(k+r)(k+r-1) + \gamma(k+r)]\,x^{k+r-1} - \sum_{r=0}^{\infty}[a_r(k+r)+\alpha]\,x^{k+r} = 0. \quad ...(7)$$

which is an identity and so coefficients of various powers of x must be zero.

Equating the coefficients of x^{k-1} (lowest powers of x) to zero, we get

$a_0[k(k-1) + \gamma k] = 0$ or $k(k-1) + \gamma k = 0$ as $a_0 \neq 0$.

Hence $k = 0$ and $k = 1 - \gamma$ are the roots of the indical equation.

Now equating to zero the coefficients of x^{k+i}, in (7) we get

$$[(k+i+1)(k+i) + \gamma(k+i+1)]a_{i+1} - [(k+i+\alpha)]a_i = 0.$$

$$\therefore \quad a_{i+1} = \frac{(k+i)+\alpha}{(k+i+1)\,(k+i+\gamma)}a_i. \quad ...(8)$$

Case I. When k = 0. Then (8) gives $a_{i+1} = \dfrac{(i+\alpha)}{(i+1)(i+\gamma)}a_i$. ...(9)

Putting $i = 0, 1, 2, 3, ...$ in (9), we get $a_1 = \dfrac{\alpha}{\gamma}a_0$, $a_2 = \dfrac{1+\alpha}{2(1+\gamma)}a_1 = \dfrac{\alpha(\alpha+1)}{1.2.\gamma(\gamma+1)}a_0$

....

$$a_m = \frac{\alpha(\alpha+1)(a+2)...(a+m-1)}{1\cdot 2\cdot 3...m\cdot\gamma(\gamma+1)...(\gamma+m-1)}a_0 = \frac{(\alpha)_m}{m!\,(\gamma)_m}a_0.$$

Substituting the values of $a_1.a_2...a_m$ in the series (6), we get

$$y = a_0\left[1 + \frac{\alpha}{\gamma}x + \frac{\alpha(\alpha+1)}{1\cdot 2\cdot\gamma(\gamma+1)}x^2 + ...\right] = a_0\sum_{m=0}^{\infty}\frac{(\alpha)_m}{m!\,(\gamma)_m}x^m.$$

Taking $a_0 = 1$, we have $y = F(\alpha; \gamma; x)$ which is called the *confluent hypergeometric function of the first kind*.

Case II. When $k = 1 - \gamma$. Then (8) gives $a_{i+1} = \dfrac{1-\gamma+i+\alpha}{(i+2-\gamma)(i+1)}a_i = \dfrac{(\alpha-\gamma+1)+i}{(2-\gamma+i)(i+1)}a_i$

$\therefore \quad a_{i+1} = \dfrac{\alpha'+1}{(\gamma'+i)(i+1)}a_i$ where $\alpha - \gamma + 1 = \alpha'$ and $2 - \gamma = \gamma'$.

Putting $i = 0, 1, 2, 3, ...$ in the above relation, we have

$a_1 = \dfrac{\alpha'}{\gamma'}a_0$, $a_2 = \dfrac{\alpha'+1}{(\gamma'+1)\cdot 2}a_1 = \dfrac{\alpha'(\alpha'+1)}{1\cdot 2\cdot\gamma'(\gamma'+1)}a_0$ Similarly, $a_m = \dfrac{(\alpha')_m}{m!\,(\gamma')_m}a_0$.

Substituting the values of a_1, a_2, ... in (6), we have

$$y = a_0 x^{1-\gamma}\left[1 + \frac{\alpha'}{\gamma'}x + \frac{\alpha'(\alpha'+1)}{1\cdot 2\cdot\gamma'(\gamma'+1)}x^2 + \ldots\right] = a_0 x^{1-\gamma}\sum_{m=0}^{\infty}\frac{(\alpha')_m}{m!\,(\gamma')_m}x^m$$

or $$y = a_0 x^{1-\gamma} F(\alpha';\gamma';x) = a_0 x^{1-\gamma} F(\alpha-\gamma+1;\, 2-\gamma;\, x).$$

Putting $a_0 = 1$, we have $y = x^{1-\gamma} F(\alpha - \gamma + 1;\, 2 - \gamma,\, x)$ which is called the *confluent hypergeometric function of the second kind*. Thus the *general solution* of the confluent hypergeometric differential equation is given by

$$y = AF(\alpha;\gamma;x) + Bx^{1-\gamma} F(\alpha-\gamma+1;\, 2-\gamma;\, x), \qquad \text{where } \gamma > 0.$$

14.14. Differentiation of hypergeometric confluent functions.

Show that $\dfrac{d}{dx}F(\alpha;\beta;x) = \dfrac{\alpha}{\beta}F(\alpha+1;\beta+1;x)$ and deduce that

(*i*) $\dfrac{d^n}{dx^n}F(\alpha;\beta;x) = \dfrac{(\alpha)_n}{(\beta)_n}F(\alpha+n;\beta+n;x)$ (*ii*) $\left[\dfrac{d^n}{dx^n}F(\alpha;\beta;x)\right]_{x=0} = \dfrac{(\alpha)_n}{(\beta)_n}.$

Proof. By definition, we have $F(\alpha;\beta;x) = \displaystyle\sum_{r=0}^{\infty}\frac{(\alpha)_r}{(\beta)_r}x^r.$

Now the proof is similar to that of Art. 14.8.

14.15. Integral representation for confluent hypergeometric function.

$$F(\alpha;\beta;x) = \frac{\Gamma(\beta)}{\Gamma(\alpha)\,\Gamma(\beta-\alpha)}\int_0^1 (1-t)^{\beta-\alpha-1}t^{\alpha-1}e^{xt}\,dt$$ **[Purvanchal 2006]**

or $$F(\alpha;\beta;x) = \frac{1}{B(\alpha,\beta-\alpha)}\int_0^1 (1-t)^{\beta-\alpha-1}t^{\alpha-1}e^{xt}\,dt, \text{ where } \beta > \alpha > 0.$$

Proof. By definition, we have

$$F(\alpha;\beta;x) = \sum_{n=0}^{\infty}\frac{(\alpha)_n}{(\beta)_n}\frac{x^n}{n!} = \sum_{n=0}^{\infty}\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}\frac{\Gamma(\beta)}{\Gamma(\beta+n)}\frac{x^n}{n!}, \text{ by Art. 14.1}$$

$$= \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\sum_{n=0}^{\infty}\frac{\Gamma(\beta-\alpha)\,\Gamma(\alpha+n)}{\Gamma(\beta-\alpha+\alpha+n)}\frac{x^n}{n!} = \frac{\Gamma(\beta)}{\Gamma(\alpha)\,\Gamma(\beta-\alpha)}\sum_{n=0}^{\infty}\left\{\int_0^1(1-t)^{\beta-\alpha-1}t^{\alpha+n-1}\,dt\right\}\frac{x^n}{n!}, \text{ if } \beta > \alpha$$

$$\left(\because \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)} = B(p,q) = \int_0^1(1-t)^{p-1}t^{q-1}\,dt, \text{ if } p>0, q>0\right)$$

$$= \frac{\Gamma(\beta)}{\Gamma(\alpha)\,\Gamma(\beta-\alpha)}\int_0^1(1-t)^{\beta-\alpha-1}t^{\alpha-1}\left(\sum_{n=0}^{\infty}\frac{(xt)^n}{n!}\right)dt$$

$$\therefore \qquad F(\alpha;\beta;x) = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)}\int_0^1(1-t)^{\beta-\alpha-1}t^{\alpha-1}e^{xt}\,dt \qquad \ldots(1)$$

or $$F(\alpha;\beta;x) = \frac{1}{B(\alpha,\beta-\alpha)}\int_0^1(1-t)^{\beta-\alpha-1}t^{\alpha-1}e^{xt}\,dt \qquad \ldots(2)$$

$$\left(\because B(\alpha,\beta-\alpha) = \frac{\Gamma(\alpha)\,\Gamma(\beta-\alpha)}{\Gamma(\alpha+\beta-\alpha)} \Rightarrow \frac{1}{B(\alpha,\beta-\alpha)} = \frac{\Gamma(\beta)}{\Gamma(\alpha)\,\Gamma(\beta-\alpha)}\right)$$

Thus (1) and (2) are the required results.

14.16. Theorem (Kummer's Relation). *Show that* $F(\alpha; \beta; x) = e^x F(\beta - \alpha; \beta; -x)$.

Proof. From Art. 14.15, we know that

$$F(\alpha\,;\,\beta\,;\,x) = \frac{\Gamma(\beta)}{\Gamma(\alpha)\,\Gamma(\beta-\alpha)}\int_0^1 (1-t)^{\beta-\alpha-1} t^{\alpha-1}\, e^{xt}\, dt\,. \qquad ...(1)$$

Replacing α by $\beta - \alpha$ and x by $-x$ in (1), we get

$$F(\beta-\alpha; \beta; -x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)\,\Gamma[\beta-(\beta-\alpha)]}\int_0^1 (1-t)^{\beta-(\beta-\alpha)-1} t^{\beta-\alpha-1}\, e^{-xt}\, dt$$

$$= \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)\,\Gamma(\alpha)}\int_0^1 (1-t)^{\alpha-1} t^{\beta-\alpha-1}\, e^{-xt}\, dt = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)\,\Gamma(\alpha)}\int_1^0 u^{\alpha-1}(1-u)^{\beta-\alpha-1}\, e^{-x(1-u)}\,(-du)$$

(putting $1 - t = u$ so that $dt = -\,du$ and $t = 1 - u$)

$$= \frac{\Gamma(\beta)e^{-x}}{\Gamma(\beta-\alpha)\,\Gamma(\alpha)}\int_0^1 (1-u)^{\beta-\alpha-1}\, u^{\alpha-1}\, e^{xu}\, du \;=\; e^{-x}\frac{\Gamma(\beta)}{\Gamma(\alpha)\,\Gamma(\beta-\alpha)}\int_0^1 (1-t)^{\beta-\alpha-1}\, t^{\alpha-1}\, e^{xt}\, dt$$

$= e^{-x}F(\alpha; \beta; x)$, by (1)

$\therefore\; F(\beta - \alpha; \beta; -x) = e^{-x}\, F(\alpha; \beta; x)$ so that $F(\alpha; \beta; x) = e^x(\beta - \alpha; \beta; -x)$.

14.17. Contiguous hypergeometric functions. Definitions.

According to Gauss, the function $F(\alpha', \beta'\,;\,\gamma'\,;\,x)$ is said to be contiguous to $F(\alpha, \beta\,;\,\gamma\,;\,x)$ when it is increased or decreased by one and only one of the parameters α, β, γ by unity.

According to above definition, there exist six hypergeometric functions contiguous to $F(\alpha, \beta\,;\,\gamma\,;\,x)$. These are denoted and defined as given below :

$F_{\alpha+} = F(\alpha + 1, \beta\,;\,\gamma\,;\,x)$, $F_{\beta+} = F(\alpha, \beta + 1\,;\,\gamma\,;\,x)$, $F_{\gamma+} = F(\alpha, \beta\,;\,\gamma + 1\,;\,x)$

$F_{\alpha-} = F(\alpha - 1, \beta\,;\,\gamma\,;\,x)$, $F_{\beta-} = F(\alpha, \beta - 1\,;\,\gamma\,;\,x)$, $F_{\gamma-} = F(\alpha, \beta\,;\,\gamma - 1\,;\,x)$.

14.18. To prove the contiguity relationship

$$(\alpha - \beta)F(\alpha, \beta\,;\,\gamma\,;\,x) = \alpha F(\alpha + 1, \beta\,;\,\gamma\,;\,x) - \beta F(\alpha, \beta + 1\,;\,\gamma\,;\,x)$$

or

$$(\alpha - \beta)F(\alpha, \beta, \gamma, x) = \alpha F_{\alpha+} - \beta F_{\beta+}.$$

Proof. We have, by definition 14.4, $\alpha F(\alpha + 1, \beta\,;\,\gamma\,;\,x) - \beta F(\alpha, \beta + 1\,;\,\gamma\,;\,x)$

$$= \alpha\sum_{r=0}^{\infty}\frac{(\alpha+1)_r(\beta)_r}{(\gamma)_r\; r!}x^r - \beta\sum_{r=0}^{\infty}\frac{(\alpha)_r(\beta+1)_r}{(\gamma)_r\; r!}x^r = \sum_{r=0}^{\infty}\frac{\alpha(\alpha+1)_r(\beta)_r}{(\gamma)_r\; r!}x^r - \sum_{r=0}^{\infty}\frac{(\alpha)_r\beta(\beta+1)_r}{(\gamma)_r\; r!}$$

$$= \sum_{r=0}^{\infty}\frac{(\alpha+r)\,(\alpha)_r(\beta)_r}{(\gamma)_r\; r!}x^r - \sum_{r=0}^{\infty}\frac{(\alpha)_r(\beta)_r(\beta+r)_r}{(\gamma)_r\; r!}x^r$$

[$\because\; \alpha(\alpha + 1)_r = \alpha(\alpha + 1)(\alpha + 2)...(\alpha + 1 - r)(\alpha + 1 + r - 1)$, by Art. 14.1

$= [\alpha(\alpha + 1)(\alpha + 2)...(\alpha + r - 1)](\alpha + r) = (\alpha + r)(\alpha)_r$, by Art. 14.1 again

Similarly, $\beta(\beta + 1)_r = (\beta + r)(\beta)_r$]

$$= \sum_{r=0}^{\infty}[(\alpha+r)-(\beta+r)]\frac{(\alpha)_r(\beta)_r}{(\gamma)_r\; r!}x^r = (\alpha-\beta)\sum_{r=0}^{\infty}\frac{(\alpha)_r(\beta)_r}{(\gamma)_r r!}x^r = (\alpha-\beta)F(\alpha,\beta\,;\gamma\,;x)$$

14.19. Contiguity relationship for confluent hypergeometric functions

$(\alpha - \beta)xF(\alpha\,;\,\beta + 1\,;\,x) + \beta(\alpha + \beta - 1)F(\alpha\,;\,\beta\,;\,x) - \beta(\beta - 1)F(\alpha\,;\,\beta - 1\,;\,x) = 0.$

Proof. Proceed as explained in Art. 14.18.

14.20. A SOLVED EXAMPLES

Ex. 1. *Prove that*

(*i*) $e^x = {}_1F_1(\alpha\,;\,\alpha\,;\,x)$.

(*ii*) $(1-x)^{-\alpha} = {}_2F_1(\alpha,\,\beta\,;\,\beta\,;\,x)$. **[Kanpur 2008, 09, Lucknow 2010]**

(*iii*) $(1-x)^{-1} = F(1,\,1\,;\,1\,;\,x),\ |\,x\,| < 1$. **[Kanpur 2004]**

(*iv*) $(1+x)^n = F(-n,\,1\,;\,1\,;\,-x)$.

(*v*) $ln(1+x) = \log_e(1+x) = x\;{}_2F_1(1,\,1\,;\,2\,;\,-x)$.

(*vi*) $\log(1-x) = -x\;{}_2F_1(1,\,1\,;\,2\,;\,x)$.

(*vii*) $\log\dfrac{1+x}{1-x} = 2x\,F\left(1/2, 1\,;\,3/2\,;\,x^2\right) = \ . = F\begin{bmatrix} 1/2, 1; & \\ 3/2 & ;\ x^2 \end{bmatrix}$ **[Kanpur 2005, 10]**

(*viii*) $\sin^{-1} x = xF(1/2;\ 1/2;\ 3/2;\ x^2)$.

(*ix*) $\tan^{-1} x = x\,F(1/2;\ 1;\ 3/2;\ -x^2)$. **[Kanpur 2006]**

Sol. (*i*) We have, by definition 14.3

$$ {}_1F_1(\alpha\,;\,\beta\,;\,x) = 1 + \frac{\alpha}{\beta}.\frac{x}{1!} + \frac{\alpha(\alpha+1)}{\beta(\beta+1)}.\frac{x^2}{2!} + \ldots \text{ ad. inf.} \qquad \ldots(1)$$

Replacing β by α in (1), we have

$$ {}_1F_1(\alpha\,;\,\alpha\,;\,x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \ldots \qquad \text{or} \qquad {}_1F_1(\alpha\,;\,\alpha\,;\,x) = e^x.$$

(*ii*) We have, by definition 14.4

$$ {}_2F_1(\alpha,\,\beta\,;\,\gamma\,;\,x) = 1 + \frac{\alpha\,\beta}{\beta}.\frac{x}{1!} + \frac{\alpha(\alpha+1)\,\beta(\beta+1)}{\gamma(\gamma+1)}\frac{x^2}{2!} + \ldots \qquad \ldots(1)$$

Replacing γ by β in (1), we have

$$ {}_2F_1(\alpha,\,\beta\,;\,\beta\,;\,x) = 1 + \frac{\alpha\beta}{\beta}.\frac{x}{1!} + \frac{\alpha(\alpha+1)\beta(\beta+1)}{\beta(\beta+1)}\frac{x^2}{2!} + \ldots = 1 + \alpha.\frac{x}{1!} + \alpha(\alpha+1)\frac{x^2}{2!} + \alpha(\alpha+1)(\alpha+2)\frac{x^2}{3!} + \ldots$$

$$ = 1 + (-\alpha)(-x) + \frac{(-\alpha)(-\alpha-1)}{2!}(-x)^2 + \frac{(-\alpha)(-\alpha-1)(-\alpha-2)}{3!}(-x)^3 + \ldots$$

$= (1-x)^{-\alpha}$, by the binomial theorem.

(*iii*) and **(*iv*).** Procced like part (*ii*) above.

(*v*) We have, $\quad {}_2F_1(\alpha,\,\beta\,;\,\gamma\,;\,x) = 1 + \dfrac{\alpha\cdot\beta}{\gamma}\dfrac{x}{1!} + \dfrac{\alpha(\alpha+1)\cdot\beta(\beta+1)}{\gamma(\gamma+1)}.\dfrac{x^2}{2!} + \ldots \qquad \ldots(1)$

Replacing α, β, γ, x by 1, 1, 2 and $-x$ respectively in (1), we get

$$ {}_2F_1(1,\,1\,;\,2\,;\,1-x) = 1 + \frac{1\cdot 1}{2}\cdot\frac{(-x)}{1!} + \frac{1\cdot 2\cdot 1\cdot 2}{2\cdot 3}\cdot\frac{(-x)^2}{2!} + \frac{1\cdot 2\cdot 3\cdot 1\cdot 2\cdot 3}{2\cdot 3\cdot 4}\cdot\frac{(-x)^3}{3!} + \ldots$$

Multiplying both sides of the above equation by x, we get

$$ x\;{}_2F_1(1,\,1\,;\,2\,;\,-x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \text{ ad. inf.} = \log(1+x).$$

(*vi*) and **(*vii*).** Proceed like part (*v*) above

(*viii*) We have, $\quad F(\alpha,\,\beta\,;\,\gamma\,;\,x) = 1 + \dfrac{\alpha\cdot\beta}{\gamma}\dfrac{x}{1!} + \dfrac{\alpha(\alpha+1)\cdot\beta(\beta+1)}{\gamma(\gamma+1)}\dfrac{x^2}{2!} + \ldots \qquad \ldots(1)$

Replacing α, β, γ, x by 1/2, 1/2, 3/2 and x^2 respectively in (1), we get

$$F\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};x^2\right) = 1+\frac{(1/2)\times(1/2)}{(3/2)}\cdot\frac{x^2}{1!}+\frac{(1/2)\times(3/2)\times(1/2)\times(3/2)}{(3/2)\times(5/2)}\cdot\frac{x^4}{2!}+\ldots$$

$$\therefore \qquad xF\left(\frac{1}{2},\frac{1}{2};\frac{3}{2};x\right) = x+1^2\cdot\frac{x^3}{3!}+1^2\cdot3^2\cdot\frac{x^5}{5!}+1^2\cdot3^3\cdot5^2\cdot\frac{x^7}{7!}+\ldots = \sin^{-1}x.$$

(*ix*) We have, $\quad F(\alpha,\beta;\gamma;x) = 1+\dfrac{\alpha\cdot\beta}{\gamma}\dfrac{x}{1!}+\dfrac{\alpha(\alpha+1)\cdot\beta(\beta+1)}{\gamma(\gamma+1)}\dfrac{x^2}{2!}+\ldots \qquad ...(1)$

Replacing α, β, γ and x by 1/2, 1, 3/2 and $-x^2$ respectively in (1), we have

$$F\left(\frac{1}{2},1;\frac{3}{2};-x^2\right) = 1+\frac{(1/2)\times1}{(3/2)}\cdot\frac{(-x^2)}{1!}+\frac{(1/2)\times(3/2)\times1\times2}{(3/2)\times(5/2)}\cdot\frac{(-x^2)^2}{2!}+\ldots \quad \text{so that}$$

$$F\left(\frac{1}{2},1;\frac{3}{2};-x^2\right) = 1-\frac{x^2}{3}+\frac{x^4}{5}-\ldots\infty \Rightarrow xF\left(\frac{1}{2},1;\frac{3}{2};-x^2\right) = x-\frac{x^3}{3}+\frac{x^5}{5}+\ldots\infty = \tan^{-1}x.$$

Ex. 2. *Show that* $\lim\limits_{a\to\infty} {}_2F_1(1, a; 1; x/a) = e^x$.

Sol. By definition, $\quad {}_2F_1(1, a; 1; x/a) = 1+\dfrac{1\cdot a}{1\cdot1}\left(\dfrac{x}{a}\right)+\dfrac{1\cdot2\cdot a\,(a+1)}{1\cdot2\cdot1\cdot2}\left(\dfrac{x}{a}\right)^2+....$

or $\quad {}_2F_1(1, a; 1; x/a) = 1+\dfrac{x}{1!}+\left(1+\dfrac{1}{a}\right)\dfrac{x^2}{2!}+\left(1+\dfrac{1}{a}\right)\left(1+\dfrac{2}{a}\right)\dfrac{x^3}{3!}+\ldots$

$$\therefore \qquad \lim_{a\to\infty} {}_2F_1(1, a; 1; x/a) = 1+x/1!+x^2/2!+x^3/3!+\ldots = e^x$$

Ex. 3. *Show than* ${}_2F_1(a, 1; a; x) = (1-x)^{-1}$.

Sol. $F(a, 1; a; x) = 1+\dfrac{a\cdot1}{1\cdot a}x+\dfrac{a\,(a+1)\cdot1\cdot2}{1\cdot2\,a(a+1)}x^2+\dfrac{a\,(a+1)\,(a+2)\cdot1\cdot2\cdot3}{1\cdot2\cdot3\cdot a\,(a+1)\,(a+2)}x^3+\ldots$

$$= 1+x+x^2+x^3+.... = (1-x)^{-1}$$

Ex. 4. *Show that* $\left[\dfrac{d}{dx}\,{}_2F_1(a, b; c; x)\right]_{x=c} = \dfrac{ab}{c}$.

Sol. By definition, $\quad {}_2F_1(a, b; c; x) = 1+\dfrac{ab}{1\cdot c}x+\dfrac{a\,(a+1)\,b\,(b+1)}{1\cdot2\cdot c\,(c+1)}x^2+\ldots$

$$\therefore \qquad \frac{d}{dx}\,{}_2F_1(a, b; c; x) = 0+\frac{ab}{c}\cdot1+\frac{a\,(a+1)\,b\,(b+1)}{2\cdot c\,(c+1)}\cdot2x+\ldots \Rightarrow \left[\frac{d}{dx}\,{}_2F_1(a, b; c; x)\right]_{x=0} = \frac{ab}{c}$$

Ex. 5. *Show that* $1-n\,x\,{}_2F_1(1-n, 1; 2; x) = (1-x)^n$, where n is any natural number.

Sol. Using formula $F(\alpha, \beta; \gamma; x) = 1+\dfrac{\alpha\beta}{1\cdot\gamma}x+\dfrac{\alpha(\alpha+1)\,\beta(\beta+1)}{1\cdot2\cdot\gamma(y+1)}x^2+\ldots,$ we get

$$1-n\,x\,{}_2F_1\,(1-n, 1; 2; x)$$

$$= 1 - nx\left\{1 + \frac{(1-n).1}{1\cdot 2}x + \frac{(1-n)(2-n)\cdot 1\cdot 2}{1\cdot 2\cdot 2\cdot 3}x^2 + \frac{(1-n)(2-n)...(n-1-n)\cdot 1\cdot 2...(n-1)}{1\cdot 2....(n-1)\cdot 2\cdot 3....n}x^{n-1} + 0\right\}$$

$$= 1 - nx\left\{1 + \frac{n-1}{2!}x + \frac{(n-1)(n-2)}{3!}x^2 + ... + \frac{(-1)^{n-1}(n-1)(n-2)\cdots 2\cdot 1}{n!}x^{n-1}\right\}$$

$$= 1 - nx + \frac{n(n-1)}{2!}x^2 - \frac{n(n-1)(n-2)}{3!}x^3 + ... + \frac{(-1)^n n!}{n!}x^n$$

$$= 1 + {}^nc_1(-x) + {}^nc_2(-x)^2 + {}^nc_3(-x)^3 + ... + {}^nc_n(-x)^n = (1-x)^n$$

Ex. 6. *Show that* $\lim\limits_{a,b\to\infty} {}_2F_1(a, b; 1/2; x^2/4ab) = \cosh x$

Sol. Using formula ${}_2F_1(\alpha, \beta; \gamma; x) + 1 + \frac{\alpha\beta}{1\cdot\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\gamma(\gamma+1)}x^2 + ...,$ we get

$${}_2F_1(a, b; 1/2; x^2/4ab) = 1 + \frac{ab}{1\times(1/2)}\times\frac{x^2}{4ab} + \frac{a(a+1)b(b+1)}{1\times 2\times(1/2)\times(3/2)}\times\left(\frac{x^2}{4ab}\right)^2 + ...$$

$$= 1 + x^2/2 + (1+1/a)(1+1/b)\times(x^4/24) + ...$$

$$\therefore \quad \lim_{a,b\to\infty} {}_2F_1(a, b; 1/2; x^2/4ab) = 1 + x^2/2! + x^4/4! + ... = \cosh x$$

Ex. 7. *Show that* ${}_2F_1(a-1, b-1, c; x) - F(a, b-1); c; x) = (x/c)\times(1-b)\ {}_2F_1(a, b; c+1; x)$

Sol. Using formula ${}_2F_1(\alpha, \beta; \gamma; x) = 1 + \frac{\alpha\beta}{1\cdot\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1\cdot 2\gamma(\gamma+1)}x^2 + ...,$ we get

$$F(a-1, b-1; c; x) = 1 + (a-1)(b-1)\times(x/c)$$
$$+ \frac{(a-1)a(b-1)b}{1\cdot 2\cdot c(c+1)}x^2 + \frac{(a-1)a(a+1)(b-1)b(b+1)}{1\cdot 2\cdot 3\cdot c(c+1)(c+2)}x^3 + ... \quad ...(1)$$

and $\quad F(a, b-1; c; x) = 1 + a(b-1)\times(x/c)$

$$+ \frac{a(a+1)(b-1)b}{1\cdot 2\cdot c(c+1)}x^2 + \frac{a(a+1)(a+2)(b-1)b(b+1)}{1\cdot 2\cdot 3\cdot c(c+1)(c+2)}x^3 + ... \quad ...(2)$$

Subtracting (2) from (1), $F(a-1, b-1; c; x) - F(a, b-1; c; x)$

$$= \frac{(a-1-a)(b-1)}{c}x + \frac{(a-1-a-1)a(b-1)b}{1\cdot 2\cdot c(c+1)}x^2 + \frac{(a-1-a-2)a(a+1)(b-1)b(b+1)}{1\cdot 2\cdot 3c(c+1)(c+2)}x^3 + ...$$

$$= \frac{1-b}{c}x\left\{1 + \frac{2ab}{1\cdot 2\cdot(c+1)}x + \frac{3a(a+1)b(b+1)}{1\cdot 2\cdot 3(c+1)(c+2)}x^2 + ...\right\} = (x/c)\times(1-b)\ {}_2F_1(a, b; c+1; x)$$

Ex. 8. *Find the third derivative of* ${}_2F_1(2, 3; 1; x)$ *w.r.t. 'x'*

(*b*) *Find the fourth derivative of the following hypergeonetre functions w.r.t 'x':*

(*i*) $2F_1(2, 1; 4; x)$ (*ii*) ${}_2F_1(2, -2; 5; x)$.

Sol. (*a*) By Art. 14.8, $$\frac{d^n}{dx^n}\,{}_2F_1(\alpha, \beta; \gamma; x) = \frac{(\alpha)_n(\beta)_n}{(\gamma)_n}F(\alpha+n, \beta+n; \gamma+n; x) \qquad \text{... (1)}$$

$$\therefore \qquad \frac{d^3}{dx^3}\,{}_2F_1(2, 3; 1; x) = \frac{(2)_3(3)_3}{(1)_3}F(2+3, 3+3; 1+3; x) \qquad \text{... (2)}$$

But $\qquad (\alpha)_n = \alpha\,(\alpha+1)\,(\alpha+2)...(\alpha+n-1)$, by Art. 14.1

$\therefore\ (1)_3 = 1\cdot 2\cdot 3,\qquad (2)_3 = 2\times(2+1)\times(2+2) = 24;\qquad (3)_3 = 3\times(3+1)\times(3+2) = 60$

Hence, (2) yields $\quad \dfrac{d^3}{dx^3}\,{}_2F_1(2, 3; 1; x) = \dfrac{24\times 60}{6}\,{}_2F_1(5, 6; 4; x) = 240\ {}_2F_1(5, 6; 4; x)$

(*b*) Proceed as in part (*a*) **Ans.** (*i*) $(24/7)\times F(6, 5; 8; x)$ (*ii*) 0

Ex. 9. *Find the solutions of the following equations :*

(*i*) $x(1-x)\,y''+(3/2-2x)\,y'+2y=0$ about $x = 0$ **(KU Kurukshetra 2004)**

(*ii*) $(x-x^2)y''+(3/2-2x)y'-(y/4)=0$ about $x = 0$ **(KU Kurukshetra 2004)**

(*iii*) $8x(1-x)y''+(4-14x)y'-y=0$ about $x = 0$

(*iv*) $4x(1-x)y''+y'+8y=0$ about $x = 0$

Sol. In what follows, we shall use the following results:

I. Hypergeometric equation is given by $\qquad x(1-x)y''+\{\gamma-(\alpha+\beta+1)x\}\,y'-\alpha\beta y=0$

II. General solution of hypergeometric equation is

$y = a\ {}_2F_1(\alpha, \beta; \gamma; x)+b\,x^{\gamma-1}\ {}_2F_1(1-\gamma+\alpha, 1-\gamma+\beta; 2-\gamma; x)$, where a and b are arbitrary constants

III. $${}_2F_1(\alpha, \beta; \gamma; x) = 1+\frac{\alpha\cdot\beta}{1\cdot\gamma}x+\frac{\alpha(\alpha+1)\,\beta(\beta+1)}{1\cdot 2\,\gamma\,(\gamma+1)}x^2+....$$

Part (*i*) Re-writing the given equation, we have, $\quad x(1-x)y''+(3/2-2x)y'+2y=0 \qquad$ ... (1)

Comparing (1) with $x(1-x)y''+\{\gamma-(\alpha+\beta+1)x\}y'-\alpha\beta y=0$, we have $\gamma=3/2$, $\alpha+\beta+1=2$ and $\alpha\beta=-2$. Solving these, $\alpha=2$, $\beta=-1$, $\gamma=3/2$.

Here γ is not an integer. The general solution of (1) is $y = au+bv$, where

$$u = {}_2F_1(\alpha, \beta; \gamma; x) = F(2, -1; 3/2; x) = 1+\frac{2\times(-1)}{1\times(3/2)}x+\frac{2\times 3\times(-1)\times 0}{1\times 2\times(3/2)\times(5/2)}x^2+... = 1-\frac{4x}{3}$$

and $\quad v = x^{1-\gamma}{}_2F_1(\alpha+1-\gamma, \beta+1-\gamma; 2-\gamma; x)$

$$= x^{1-(3/2)}{}_2F_1(2+1-3/2, -1+1-3/2; 2-3/2; x) = x^{-1/2}{}_2F_1(3/2, -3/2; 1/2; x)$$

Hence the general solution of (1) is given by $\quad y = a(1-4x/3)+b\,x^{-1/2}\ {}_2F_1(3/2, -3/2; 1/2; x)$

(*ii*) Given $$x(1-x)y''+(3/2-2x)\,y'-(1/4)\times y=0 \qquad \text{... (1)}$$

Comparing (1) with $x(1-x)y''+\{\gamma-(\alpha+\beta+1)x\}y'-\alpha\beta y=0$, we have

$\alpha = 1/2, \beta = 1/2, \gamma = 3/2.$ Hence γ is not an integer. The general solution of (1) is $y = au = bv,$

where $\quad u = {}_2F_1(\alpha, \beta; \gamma; x) = {}_2F_1(1/2, 1/2; 3/2; x) \quad$ and

$$v = x^{1-\gamma}{}_2F_1(\alpha+1-\gamma, \beta+1-\gamma; 2-\gamma; x) = x^{1-3/2}{}_2F_1(1/2+1-3/2, 1/2+1-3/2; 2-3/2; x)$$

$$= x^{-1/2}{}_2F_1(0; 0; 1/2; x) = x^{-1/2}\left(1 + \frac{0\times 0}{1\times(1/2)}x + \ldots\right) = \frac{1}{\sqrt{x}}$$

General solution of (1) is $y = a\ {}_2F_1(1/2, 1/2; 3/2; x) + b/\sqrt{x},$ a, b being arbitrary constants

(*iii*) Ans. $y = a\,(1-x)^{-1/4} + bx^{1/2}\ {}_2F_1(1, 3/4; 3/2; x)$

(*iv*) Ans. $y = a\,(1 - 8x + 32x^2/5) + bx^{3/4}\ 2F_1(7/4 - 5/4; 7/4; x)$

Ex. 10. *Solve the Legendre equation* $(1-x^2)\,(d^2y/dx^2) - 2x\,(dy/dx) + n(n+1)y = 0$ by changing it to a hypergeometre equation. **(MDU Rohtak 2004)**

Sol. Given $\quad (1-x^2)\,(d^2y/dx^2) - 2x\,(dy/dx) + n(n+1)y = 0 \quad$ ... (1)

Let $\quad z = x^2 \quad$ ... (2)

Now, $$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx} = 2x\frac{dy}{dz} \qquad \text{... (3)}$$

and $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(2x\frac{dy}{dz}\right) = 2\frac{dy}{dz} + 2x\frac{d}{dx}\left(\frac{dy}{dz}\right) = 2\frac{dy}{dz} + 2x\frac{d}{dz}\left(\frac{dy}{dz}\right)\frac{dz}{dx} = 2\frac{dy}{dz} + (2x)^2\frac{d^2y}{dz^2}$$

Substituing the above values in (1), $\quad (1-x^2)\left(2\frac{dy}{dz} + 4x^2\frac{d^2y}{dz^2}\right) - (2x)^2\frac{dy}{dz} + n(n+1)y = 0$

or $$(1-z)\left(2\frac{dy}{dz} + 4z\frac{d^2z}{dz^2}\right) - 4z\frac{dy}{dz} + n(n+1)y = 0$$

or $$4z(1-z)(d^2y/dz^2) + (2-6z)(dy/dz) + n(n+1)y = 0$$

or $$z(1-z)\frac{d^2y}{dz^2} + \left(\frac{1}{2} - \frac{3}{2}z\right)\frac{dy}{dz} + \frac{n(n+1)}{4}y = 0 \qquad \text{... (4)}$$

Refer Ex. 9 for hypergeometric equation and its solution. Comparing (4) with hypergeometre equation $$z(1-z)\,(d^2y/dz^2) + \{\gamma - (\alpha+\beta+1)z\}\,(dy/dz) - \alpha\beta\, y = 0, \qquad \text{... (5)}$$

we have, $\quad \gamma = 1/2, \quad \alpha+\beta+1 = 3/2 \quad$ and $\quad \alpha\beta = -(1/4)\times n(n+1)$

Solving these, $\quad \alpha = (n+1)/2, \quad \beta = -(n/2) \quad$ and $\quad \gamma = 1/2$

Note that here γ is not an integer. The general solution of (5) is given by

$$y = a\ {}_2F_1(\alpha, \beta; \gamma; z) + b\,z^{\gamma-1}{}_2F_1(1-\gamma+\alpha, 1-\gamma+\beta; 2-\gamma; z)$$

Hence the required solution of (1) is given by

$$y = a\ {}_2F_1\left(\frac{n+1}{2}, -\frac{n}{2}; \frac{1}{2}; z\right) + b\,z^{1/2}{}_2F_1\left(\frac{n+1}{2} + 1 - \frac{1}{2}, -\frac{n}{2} + 1 - \frac{1}{2}; 2 - \frac{1}{2}; z\right)$$

or $y = a\ {}_2F_1\left(\dfrac{n+1}{2}, -\dfrac{n}{2}; \dfrac{1}{2}; x^2\right) + bx\ {}_2F_1\left(\dfrac{n+2}{2}, -\dfrac{n-1}{2}; \dfrac{3}{2}; x^2\right)$, where a and b are arbitrary constants.

Ex. 11. *Show that* (*i*) $\dfrac{2}{\pi}\displaystyle\int_0^{\pi/2}\frac{d\theta}{(1-x^2\sin^2\theta)^{1/2}} = {}_2F_1(1/2, 1/2; 1; x^2), |x|<1$

(*ii*) $\dfrac{2}{\pi}\displaystyle\int_0^{\pi/2}(1-x^2\sin\theta)^{1/2}\,d\theta = {}_2F_1(-1/2, 1/2; 1; x^2), |x|<1$

Sol. By Art. 14.9, $${}_2F_1(\alpha,\beta;\gamma;x)=\frac{\Gamma(\gamma)}{\Gamma(\beta)\,\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-xt)^{-\alpha}\,dt$$

so that $$\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-xt)^{-\alpha}\,dt=\frac{\Gamma(\beta)\,\Gamma(\gamma-\beta)}{\Gamma(\gamma)}\times{}_2F_1(\alpha,\beta;\gamma;x) \qquad \text{... (A)}$$

Part (*i*) Putting $t=\sin^2\theta$ and $dt=2\sin\theta\cos\theta\,d\theta$, we have

$$\frac{2}{\pi}\int_0^{\pi/2}\frac{d\theta}{(1-x^2\sin^2\theta)^{1/2}}=\frac{2}{\pi}\int_0^1\frac{1}{(1-x^2t)^{1/2}}\frac{dt}{2\sin\theta\cos\theta}=\frac{2}{\pi}\int_0^1\frac{1}{(1-x^2t)^{1/2}}\frac{dt}{2\,t^{1/2}(1-t)^{1/2}}$$

$$=\frac{1}{\pi}\int_0^1 t^{-1/2}(1-t)^{-1/2}(1-x^2t)^{-1/2}\,dt=\frac{1}{\pi}\int_0^1 t^{\frac{1}{2}-1}(1-t)^{1-\frac{1}{2}-1}(1-x^2t)^{-\frac{1}{2}}\,dt$$

$$=\frac{1}{\pi}\times\frac{\Gamma(1/2)\Gamma(1-1/2)}{\Gamma(1)}\;{}_2F_1(1/2;1/2;1;x^2),\text{ using result (A) taking } \beta=1/2, \gamma=1, \alpha=1/2, x=x^2$$

$$=\frac{1}{\pi}\times\frac{\sqrt{\pi}\times\sqrt{\pi}}{1}\times{}_2F_1(1/2,1/2,1;x^2)={}_2F_1(1/2,1/2;1;x^2)$$

(*ii*) Putting $t=\sin^2\theta$ and $dt=2\sin\theta\cos\theta\,d\theta$, we have

$$\frac{2}{\pi}\int_0^{\pi/2}(1-x^2\sin^2\theta)^{1/2}\,d\theta=\frac{2}{\pi}\int_0^1(1-x^2t)^{1/2}\frac{dt}{2\sin\theta\cos\theta}=\frac{1}{\pi}\int_0^1(1-x^2t)^{1/2}\frac{dt}{t^{1/2}(1-t)^{1/2}}$$

$$=\frac{1}{\pi}\int_0^1 t^{-1/2}(1-t)^{-1/2}(1-x^2t)^{1/2}\,dt=\frac{1}{\pi}\int_0^1 t^{\frac{1}{2}-1}(1-t)^{1-\frac{1}{2}-1}(1-x^2t)^{1/2}\,dt$$

$$=\frac{1}{\pi}\times\frac{\Gamma(1/2)\Gamma(1-1/2)}{\Gamma(1)}\;{}_2F_1(-1/2,1/2;1;x^2)={}_2F_1(-1/2,1/2;1;x^2)$$

[using result (A), taking $\beta=1/2, \gamma=1, \alpha=-1/2, x=x^2$]

Ex. 12. *Show that if* $|x|<1$ *and* $|x/(1-x)|<1$, *then*

$${}_2F_1(\alpha,\beta;\gamma;x)=(1-x)^{-\alpha}\,{}_2F_1\left(\alpha,\gamma-\beta;\gamma;\frac{x}{x-1}\right)$$

or $$F(\alpha,\beta;\gamma;x)=(1-x)^{-\alpha}\,F\left[\alpha,\gamma-\beta;\gamma;\frac{-x}{1-x}\right].$$

Sol. By integral representation for the hypergeometric function (refer Art. 14.9), we have

$${}_2F_1(\alpha,\beta;\gamma;x)=\frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-xt)^{-\alpha}\,dt. \qquad \text{...(1)}$$

Putting $u = 1 - t$ so that $dt = -du$ and $t = 1 - u$, (1) gives

$${}_2F_1(\alpha, \beta ; \gamma ; x) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_1^0 (1-u)^{\beta-1}u^{\alpha-\beta-1}(1-x+xu)^{-\alpha}\,(-du)$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 u^{\gamma-\beta-1}(1-u)^{\beta-1}(1-x)^{-\alpha}\left\{1+\frac{xu}{1-x}\right\}^{-\alpha} du$$

$$= \frac{(1-x)^{-\alpha}\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 u^{\gamma-\beta-1}(1-u)^{\beta-1}\left\{1-\frac{x}{x-1}u\right\}^{-\alpha} du. \qquad \text{...(2)}$$

Replacing β and x by $\gamma - \beta$ and $x/(x - 1)$ in (1), we get

$${}_2F_1\left(\alpha, \gamma-\beta ; \gamma ; \frac{x}{x-1}\right) = \frac{\Gamma(\gamma)}{\Gamma(\gamma-\beta)\Gamma[\gamma-(\gamma-\beta)]}\int_0^1 t^{\gamma-\beta-1}(1-t)^{\gamma-(\gamma-\beta)-1}\left\{1-\frac{xt}{x-1}\right\}^{-\alpha} dt$$

$$= \frac{\Gamma(\gamma)}{\Gamma(\gamma-\beta)\Gamma(\beta)}\int_0^1 u^{\gamma-\beta-1}(1-u)^{\beta-1}\left\{1-\frac{x}{x-1}u\right\}^{-\alpha} du. \qquad \text{...(3)}$$

Using (3), (2) reduces to $\quad {}_2F_1(\alpha, \beta ; \gamma ; x) = (1-x)^{-\alpha}\ {}_2F_1\left(\alpha, \gamma-\beta ; \gamma ; \dfrac{x}{x-1}\right)$

Ex. 13. *Prove that* $P_n(x) = {}_2F_1\left(-n, n+1 ; 1 ; \dfrac{1-x}{2}\right)$

[Punjab 2005, Purvanchal 2005,06, Kanpur 2004, 06]

Sol. $P_n(x) = \dfrac{1}{n!\,2^n}\dfrac{d^n}{dx^n}(x^2-1)^n$, by Rodrigue's formula

$$= \frac{(-1)^n}{n!\,2^n}\frac{d^n}{dx^n}(1-x^2)^n = \frac{(-1)^n}{n!}\frac{d^n}{dx^n}\left\{(1-x)^n.\frac{(1+x)^n}{2^n}\right\} = \frac{(-1)^n}{n!}\frac{d^n}{dx^n}\left[(1-x)^n.\left\{1-\frac{1-x}{2}\right\}^n\right]$$

$$= \frac{(-1)^n}{n!}\frac{d^n}{dx^n}\left[(1-x)^n.\left\{1-n\left(\frac{1-x}{2}\right)+\frac{n(n-1)}{2!}\left(\frac{1-x}{2}\right)^2-\ldots\right\}\right], \text{ by the binomial theorem}$$

$$= \frac{(-1)^n}{n!}\frac{d^n}{dx^n}\left[(1-x)^n-\frac{n}{2}(1-x)^{n+1}+\frac{n(n-1)}{2!\,2^2}(1-x)^{n+2}+\ldots\right]$$

$$= \frac{(-1)^n}{n!}\left[(-1)^n n! - (-1)^n\frac{n}{2}\frac{(n+1)!}{1!}(1-x)+(-1)^n\times\frac{n(n-1)}{2!\,2^2}\times\frac{(n+2)}{2!}(1-x)^2+\ldots\right]$$

$$\left[\because \frac{d^n}{dx^n}(a-bx)^m = (-1)^n b^n.\frac{m!}{(m-n)!}(a-bx)^{m-n}\right]$$

$$= \frac{(-1)^{2n}}{n!}\left[n!-\frac{n(n+1)}{2}n!(1-x)+\frac{n(n-1)}{2!\,2^2}(n+2)(n+1)n!(1-x)^2+\ldots\right]$$

$$= 1+\frac{(-n)(n+1)}{1\cdot 1!}\left(\frac{1-x}{2}\right)+\frac{(-n)(-n+1)(n+1)(n+2)}{2\cdot 1\cdot 2!}\left(\frac{1-x}{2}\right)^2+\ldots = {}_2F_1\left(-n, n+1 ; 1 ; \frac{1-x}{2}\right), \text{ by definition.}$$

Ex. 14. *Show that* $P_n(\cos\theta) = \cos^n\theta\ {}_2F_1\left(-\frac{n}{2}, -\frac{n-1}{2}; 1; -\tan^2\theta\right)$.

Sol. From Laplace's first integral for $P_n(x)$, (Refer Art. 9.6 in chapter 9), we have

$$P_n(x) = \frac{1}{\pi}\int_0^\pi [x \pm (x^2-1)^{1/2}\cos\phi]^n\, d\phi. \qquad ...(1)$$

Let $x = \cos\theta$. Then, we have $(x^2-1)^{1/2} = (\cos^2\theta - 1)^{1/2} = \{(-1)\times(1-\cos^2\theta)\}^{1/2} = i\sin\theta$. With these values and taking positive sign in (1), we get

$$P_n(\cos\theta) = \frac{1}{\pi}\int_0^\pi (\cos\theta + i\sin\theta\cos\phi)^n\, d\phi = \frac{\cos^n\theta}{\pi}\int_0^1 (1 + i\tan\theta\cos\phi)^n\, d\phi$$

$$= \frac{\cos^n\theta}{\pi}\int_0^\pi \left\{1 + i\tan\theta\cos\phi + \frac{n(n-1)}{2!}i^2\tan^2\theta\cos^2\phi + ...\right\}d\,\phi, \text{ by the binomial theorem}$$

$$= \frac{\cos^n\theta}{\pi}\left[\int_0^\pi d\phi + 0 + \frac{n(n-1)}{2}i^2\tan^2\theta\left(2\int_0^{\pi/2}\cos^2\phi\, d\phi\right) + 0\right.$$

$$\left. + \frac{n(n-1)(n-2)(n-3)}{3\cdot 2\cdot 1}i^4\tan^4\theta\left(2\int_0^{\pi/2}\cos^4\phi\, d\phi\right) + ...\right], \text{ as } \int_0^{2a} f(x)\,dx = \begin{cases} 0, \text{ if } f(2a-x) = -f(x) \\ 2\int_0^a f(x)\,dx, \text{ if } f(2a-x) = f(x) \end{cases}$$

$$= \frac{\cos^n\theta}{\pi}\left[\pi - \frac{n(n-1)}{2}\tan^2\theta \times 2 \times \frac{1}{2}\times\frac{\pi}{2} + \frac{n(n-1)(n-2)(n-3)}{3\cdot 2\cdot 1}\tan^2\theta \times \frac{3}{4}\times\frac{1}{2}\times\frac{\pi}{2} + ...\right]$$

$$= \cos^n\theta\left[1 - \frac{\left(-\frac{n}{2}\right)\left(-\frac{n-1}{2}\right)}{1}\frac{(-\tan^2\theta)}{1!} + \frac{\left(-\frac{n}{2}\right)\left(-\frac{n}{2}+1\right)\left(-\frac{n-1}{2}\right)\left(-\frac{n-1}{2}+1\right)}{1\cdot 2}\frac{(-\tan^2\theta)^2}{2!} + ...\right]$$

$$= \cos^n\theta\ {}_2F_1\left(-\frac{n}{2}, -\frac{n-1}{2}; 1; \tan^2\theta\right), \text{ by definition.}$$

EXERCISE

1. Show that ${}_1F_1(\beta\,;\gamma\,;x) = \lim_{\beta\to\infty} {}_2F_1(\alpha, \beta\,;\gamma\,;x/\beta)$.

2. Show that ${}_2F_1(\alpha, \beta\,;\beta-\alpha+1\,;-1) = \dfrac{\Gamma(1+\beta-\alpha)\Gamma(1+\beta/2)}{\Gamma(1+\beta)\Gamma(1+\beta/2-\alpha)}$ and deduce that

$${}_2F_1(\alpha, 1-\alpha\,;\gamma\,;1/2) = \frac{\Gamma(\gamma/2)\Gamma(\gamma/2+1/2)}{\Gamma(\alpha/2+\gamma/2)\Gamma(1/2-\alpha/2+\gamma/2)}.$$

3. Evaluate the integral $\int_0^\infty e^{-sx}{}_1F_1(\alpha\,;\beta\,;x)\,dx$. [**Ans.** $(1/s)_2 \times F_1(\alpha, 1\,;\beta\,;s)$]

   [**Hint.** Use Art. 14.15]

4. Prove that $F(\alpha, \beta+1\,;\gamma+1\,;x) - F(\alpha, \beta\,;\gamma\,;x) = \dfrac{\alpha(\gamma-\beta)}{\gamma(\gamma+1)}xF(\alpha+1, \beta+1\,;\gamma+2\,;x)$.

5. The complete elliptic integral of the first kind is

$$K = \int_0^{\pi/2} \frac{d\phi}{\sqrt{(1-k^2 \sin^2 \phi)}}. \text{ Show that } K = \frac{\pi}{2} F\left(\frac{1}{2}, \frac{1}{2}; 1; k^2\right).$$ **[Kanpur 2011]**

6. The complete elliptic integral of the second kind is

$$E = \int_0^{\pi/2} \sqrt{(1-k^2 \sin^2 \phi)}\, d\phi. \text{ Show that } E = \frac{\pi}{2} F\left(-\frac{1}{2}, \frac{1}{2}; 1; k^2\right), |k| < 1.$$

7. Prove the following relations :

(*i*) $F(\alpha-1, \beta-1; \gamma; x) - F(\alpha, \beta-1; \gamma; x) = \dfrac{(1-\beta)x}{\gamma} F(\alpha, \beta; \gamma+1; x)$

(*ii*) $\alpha F(\alpha+1, \beta; \gamma; x) - (\gamma-1)F(\alpha, \beta; \gamma-1; x) = (\alpha+1-\gamma)F(\alpha, \beta; \gamma; x)$.

8. Prove that $F(\alpha, \beta; \gamma; 1/2) = 2^{\alpha} F(\alpha, \gamma-\beta; \gamma; -1)$.

9. Show that (*i*) $e^x - 1 = xF(1; 2; x)$.

(*ii*) $(1 + x/\alpha)e^x = F(\alpha+1; \alpha; x)$.

10. The incomplete Gamma function is defined by the equation $\gamma(\alpha, x) = \int_0^{\pi} e^{-t} t^{\alpha - t}\, dt$, $\alpha > 0$.

Prove that $\gamma(\alpha, x) = \alpha^{-1} x^{\alpha} F(\alpha; \alpha+1; -x)$.

11. Prove that following relations : (*i*) $\beta F(\alpha; \beta; x) = \beta F(\alpha-1; \beta; x) + xF(\alpha; \beta+1; x)$.

(*ii*) $\alpha F(\alpha+1; \beta; x) - (\beta-1)F(\alpha; \beta-1; x) = (\alpha-\beta+1)F(\alpha; \beta; x)$.

12. Prove the following relations :

(*i*) $F(\alpha, \beta; \gamma; x) - F(\alpha, \beta; \gamma-1; x) = -\dfrac{\alpha\beta x}{\gamma(\gamma-1)} F(\alpha+1, \beta+1; \gamma+1; x)$

(*ii*) $F(\alpha+1; \gamma; x) - F(\alpha; \gamma; x) = (x/\gamma) \times F(\alpha+1, \gamma+1; x)$.

13. Hypergeometric function ${}_2F_1(\alpha, \beta; \gamma; x) = 1 + \dfrac{\alpha \cdot \beta}{1 \cdot \gamma} x + \dfrac{\alpha(\alpha+1).\beta(\beta+1)}{1 \cdot 2 \cdot \gamma(\gamma+1)} x^2 + \ldots$ is the solution of the differential equation $x(1-x)y'' + [\gamma - (\alpha+\beta+1)x]y - \alpha\beta y = 0$. Show that

(*a*) $\left[\dfrac{d}{dx}\, {}_2F_1(\alpha, \beta; \gamma; x)\right]_{x=0} = \dfrac{\alpha\beta}{\gamma}$. (*b*) ${}_2F_1\left(\alpha, \beta; \frac{1}{2}\alpha + \frac{1}{2}\beta + \frac{1}{2}; \frac{1}{2}\right) = \dfrac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2} + \frac{1}{2}\alpha + \frac{1}{2}\beta\right)}{\Gamma\left(\frac{1}{2} + \frac{1}{2}\alpha\right)\Gamma\left(\frac{1}{2} + \frac{1}{2}\beta\right)}$.

(*c*) ${}_2F_1\left(\alpha, \beta; \gamma; \frac{1}{2}\right) = 2^{\alpha}\, {}_2F_1(\alpha, \gamma-\beta; \gamma; -1)$.

14. Prove that (*a*) $P_n(\cos\theta) = {}_2F_1(-n; n+1; 1; \sin^2 \theta/2)$.

(*b*) $P_n(\cos\theta) = (-1)^n\, {}_2F_1(n+1, -n; 1; \cos^2 \theta/2)$.

(*c*) $P_n(x) = \left(\dfrac{x-1}{2}\right)^n {}_2F_1\left(-n, -n; 1; \dfrac{x+1}{x-1}\right)$. **[Purvanchal 2007]**

15. Show that

(*a*) $H_{2n}(x) = (-1)^n \dfrac{(2n!)}{n!}\, {}_1F_1\left(-n; \dfrac{1}{2}; x^2\right)$. (*b*) $H_{2n+1}(x) = (-1)^n \dfrac{2(2n+1)!}{n!} x\; {}_1F_1\left(-n; \dfrac{3}{2}; x^2\right)$.

16. Show that $L_n(x) = n!\; {}_1F_1(-n; 1; x)$.

17. Prove that, for $|x| < a$, ${}_2F_1(a, b, c, x) = (1-x)^{c-a-b}\, {}_2F_1(c-a, c-b, c_1, x)$. **[Lucknow 2010]**

18. State the confluent hyper geometric equation and explain its solution. **[Lucknow 2010]**

15

Orthogonal Sets of Functions And Strum Liouville Problem

15.1. Orthogonality. Two functions $f(x)$ and $g(x)$ defined on some interval $a \le x \le b$ are said to be *orthogonal* on $a \le x \le b$ if $$\int_a^b f(x)\, g(x)\, dx = 0.$$

The norm $\| f(x) \|$ of $f(x)$ is defined by $$\| f(x) \| = \left\{ \int_a^b f^2(x)\, dx \right\}^{1/2}$$

15.2. Orthogonal set of functions. Let $\{f_n(x)\}$, where $n = 1, 2, 3, \ldots$ be a set of functions defined on some interval $a \le x \le b$. Then the set $\{f_n(x)\}$ is said to be an *orthogonal set of functions on the interval* $a \le x \le b$ *if* $$\int_a^b f_m(x)\, f_n(x)\, dx = 0 \text{, whenever } m \ne n..$$

15.3. Orthonormal set of functions. Let $\{f_n(x)\}$, where $n = 1, 2, 3, \ldots$ be a set of functions defined on some interval $a \le x \le b$. Then the set $\{f_n(x)\}$ is said to be *orthonormal* on $a \le x \le b$ if they are orthogonal on $a \le x \le b$ and all have norm 1. Thus, set $\{f_n(x)\}$ is orthonormal on $a \le x \le b$, if $\int_a^b f_m(x)\, f_n(x)\, dx = \begin{cases} 0, & \text{when } m \ne n \\ 1, & \text{when } m = n \end{cases}$ *i.e.,* $\int_a^b f_m(x)\, f_n(x)\, dx = \delta_{mn}$,

where $$\delta_{mn} = \text{Kronecker delta} = \begin{cases} 0, & \text{when } m \ne n \\ 1, & \text{when } m = n \end{cases}$$

15.4. Orthogonality with respect to a weight function. Let $p(x) > 0$. Then two functions $f(x)$ and $g(x)$ defined on some interval $a \le x \le b$ are said to be *orthogonal* on $a \le x \le b$ with respect to weight function $p(x)$, if $$\int_a^b p(x)\, f(x)\, g(x)\, dx = 0 .$$

Then, the norm $\| f(x) \|$ of $f(x)$ is defined by $$\| f(x) \| = \left\{ \int_a^b p(x)\, f^2(x)\, dx \right\}^{1/2}$$

15.5. Orthogonal set of functions with respect to a weight function. Let $\{f_n(x)\}$, where $n = 1, 2, 3, \ldots$ be a set of functions defined on some interval $a \le x \le b$. Then the set $\{f_n(x)\}$ is said to be orthogonal on $a \le x \le b$ with respect to weight function $p(x) > 0$, if

$$\int_a^b p(x)\, f_m(x)\, f_n(x)\, dx = 0, \text{ whenever } m \ne n.$$

15.6. Orthonormal set of functions with respect to a weight function. Let $\{f_n(x)\}$, where $n = 1, 2, 3, \ldots$ be a set of functions defined on some interval $a \le x \le b$. Then the set $\{f_n(x)\}$ is said to be orthonormal with respect to a weight function $p(x) > 0$ if they are orthogonal with respect to weight function $p(x)$ on $a \le x \le b$ and all have norm 1. Thus, set $\{f_n(x)\}$ is orthonormal with respect to weight function $p(x)$, if

$$\int_a^b p(x)\, f_m(x)\, f_n(x)\, dx = \delta_{mn} = \begin{cases} 0, & \text{when } m \neq n \\ 1, & \text{when } m = n \end{cases}$$

Remark 1. Terms defined in Art. 15.1, 15.2 and 15.3 are particular cases of Art. 15.4, 15.5 and 15.6 respectively for $p(x) = 1$.

15.7. Working rule for getting orthonormal set {f$_n$(*x*)} of functions corresponding to a known orthogonal set {*f$_n$*(*x*)}, where *n* = 1, 2, 3, ..., where none of the functions *f$_n$*(*x*) have zero norm. Divide each function $f_n(x)$ by its norm $\| f_n(x) \|$ and get a new function $\phi_n(x) = f_n(x)/\| f_n(x) \|$. Then, we have

$$\| \phi_n(x) \| = \left\{ \int_a^b \phi_n^2(x)\, dx \right\}^{1/2} = \left[\int_a^b \left\{ \frac{f_n(x)}{\| f_n(x) \|} \right\}^2 dx \right]^{1/2} = \frac{1}{\| f_n(x) \|} \left\{ \int_a^b f_n^2(x)\, dx \right\}^{1/2} = \frac{1}{\| f_n(x) \|} \cdot \| f_n(x) \| = 1,$$

showing that norm of $\phi_n(x) = 1$. Hence the set $\{\phi_n(x)\}$ *i.e.*, $\{f_n(x)/\| f_n(x) \|\}$ is an orthonormal set of functions.

15.8. Gram–Schmidt process of Orthonormalization. **[Kanpur 2010]**

Let $\{f_n(x)\}$, where $n = 1, 2, 3, \ldots$ be a set of a linearly independent functions for each of which norm $\| f_n(x) \|$ exists and is non–zero. Then we wish to obtain an orthonormal set $\{\phi_n(x)\}$, where $n = 1, 2, 3, \ldots$ such that

$$\int_a^b \phi_m(x)\, \phi_n(x)\, dx = \begin{cases} 0, & \text{when } m \neq n \\ 1, & \text{when } m = n \end{cases} \qquad \ldots(1)$$

We select $f_1(x)$ and obtain $\qquad \phi_1(x) = f_1(x)/\| f_1(x) \| \qquad \ldots(2)$

We next choose $f_2(x)$ and let $\qquad F_2(x) = f_2 + c\, \phi_1, \qquad \ldots(3)$

where c is chosen in such a manner so that F_2 may be orthogonal to ϕ_1,

i.e., $\int_a^b F_2\, \phi_1\, dx = 0$ or $\int_a^b (f_2 + c\, \phi_1)\, \phi_1\, dx = 0$, using (3)

or $\int_a^b f_2 \phi_1\, dx + c \int_a^b \phi_1^2 dx = 0$ or $\int_a^b f_2 \phi_1\, dx + c = 0$, by (1)

$\therefore \quad c = -\int_a^b f_2 \phi_1\, dx$. With this value of c, (3) gives $\qquad F_2 = f_2 - \phi_1 \int_a^b f_2 \phi_1\, dx \qquad \ldots(4)$

We now take $\qquad \phi_2(x) = F_2 /\| F_2 \| \qquad \ldots(5)$

Now choose f_3 and let $\qquad F_3(x) = f_3 + c_1 \phi_1 + c_2 \phi_2, \qquad \ldots(6)$

where c_1 and c_2 are chosen in such a manner so that F_2 may be orthogonal to ϕ_1 and ϕ_2, *i.e.*,

$\int_a^b (f_3 + c_1 \phi_1 + c_2 \phi_2) \phi_1\, dx$ and $\int_a^b (f_3 + c_1 \phi_1 + c_2 \phi_2) \phi_2\, dx = 0$

or $\int_a^b f_3 \phi_1\, dx + c_1 = 0$ and $\int_a^b f_3\, \phi_2\, dx + c_2 = 0$, using (1)

$\therefore \quad c_1 = -\int_a^b f_3 \phi_1\, dx$ and $c_2 = -\int_a^b f_3 \phi_2\, dx \qquad \ldots(7)$

Using (7), (6) gives $\qquad F_3 = f_3 - \phi_1 \int_a^b f_3 \phi_1\, dx - \phi_2 \int_a^b f_3 \phi_2\, dx \qquad \ldots(8)$

So we take $\qquad \phi_3(x) = F_3/\| F_3 \| \qquad \ldots(9)$

By continuing the above process, the nth normalized function ϕ_n is given by

$$\phi_n = F_n/\| F_n \|, \qquad \ldots(10)$$

where $\quad F_n = f_n - \phi_1 \int_a^b f_n \phi_1\, dx - \phi_2 \int_a^b f_n \phi_2\, dx - \ldots - \phi_{n-1} \int_a^b f_n\, \phi_{n-1}\, dx \qquad \ldots(11)$

Above process will fail if and only if at some stage $F_r = 0$ for $1 \le r \le n$.

But, $F_r = 0 \Rightarrow f_r - \phi_1 \int_a^b f_r \phi_1 \, dx - \phi_2 \int_a^b f_r \phi_2 \, dx - ... - \phi_{r-1} \int_a^b f_r \phi_{r-1} \, dx = 0$

$\Rightarrow$ f_r is a linear combination of $\phi_1, \phi_2, ..., \phi_{r-1} \Rightarrow$ f_r is a linear combination of $f_1, f_2, ..., f_{r-1}$ (by our construction of new functions $\phi_1, \phi_2, ..., \phi_{r-1}$)

$\Rightarrow$ $f_1, f_2, ..., f_r$ is a linearly dependent set for $1 \le r \le n$.

But this a contradiction because $\{f_n(x)\}$, where $n = 1, 2, 3, ...$, is linearly independent set and its every subset would also be linearly independent. Hence $F_r \ne 0$ for $1 \le r \le n$ and so we would always get a set $\{\phi_n(x)\}$, where $n = 1, 2, 3, ...$ which would be orthonormal.

Remark. Sometimes orthonormalization is required with respect to a weight function $p(x) > 0$. Then (1) takes the form

$$\int_a^b p(x)\, \phi_m(x)\, \phi_n(x)\, dx = \begin{cases} 0, & \text{when } m \ne n \\ 1, & \text{when } m = n \end{cases}$$

Also, $\int_a^b F_2 \phi_1 \, dx = 0$ would take the form $\int_a^b p(x)\, F_2 \phi_2 \, dx = 0$ etc. Rest of the procedure is similar.

15.9. Illustrative Solved Examples

Ex. 1. *Show that the set of functions {sin (nπx/c)}, n = 1, 2, 3, ... is orthogonal on the interval (0, c) and find the corresponding orthonormal set.*

Sol. Here the given functions are $f_n(x) = \sin(n\pi x/c)$, $n = 1, 2, 3, ...$ For $m \ne n$, we have

$$\int_0^c f_m(x)\, f_n(x)\, dx = \int_0^c \sin\frac{m\pi x}{c} \sin\frac{n\pi x}{c}\, dx = \frac{1}{2}\int_0^c \left\{\cos\frac{(m-n)\pi x}{c} - \cos\frac{(m+n)\pi x}{c}\right\} dx$$

$$= \frac{1}{2}\left[\frac{c}{\pi(m-n)}\sin\frac{(m-n)\pi x}{c} - \frac{c}{\pi(m+n)}\sin\frac{(m+n)\pi x}{c}\right]_0^c = 0,$$

showing that the given set of functions is orthogonal.

$$\text{Norm of } f_n(x) = \| f_n(x) \| = \left\{\int_0^c \sin^2(n\pi x/c)\, dx\right\}^{1/2} = \left[\frac{1}{2}\int_0^c \{1 - \cos(2n\pi x/c)\}\right]^{1/2}$$

$$= \left[\{x/2 - (c/4n\pi)\sin(2n\pi x/c)\}^{1/2}\right]_0^c = (c/2)^{1/2}.$$

Let $\phi_n(x) = f_n(x)/\| f_n(x) \| = (2/c)^{1/2} \sin(n\pi x/c)$ Hence the required orthonormal set is given by $\{\phi_n(x)\}$ *i.e.*, $\{(2/c)^{1/2} \sin(n\pi x/c)\}$ where $n = 1, 2, 3, ...$

Ex. 2.(*a*) *Show that the set of functions {cos nx}, n = 0, 1, 2, 3, ... is orthogonal on the interval –π ≤ x ≤ π, and find the corresponding orthonormal set of functions.*

(*b*) *Show that the set of functions {cos nx}, n = 0, 1, 2, 3, ... is an orthogonal set of functions on 0 ≤ x ≤ π, and find the corresponding orthonormal set of functions.*

Sol. (*a*) Here the given functions are $f_n(x) = \cos nx$, $n = 0, 1, 2, ...$ For $m \ne n$, we have

$$\int_{-\pi}^{\pi} f_m(x)\, f_n(x)\, dx = \int_{-\pi}^{\pi} \cos mx \cos nx\, dx = 2\int_0^{\pi} \cos mx \cos nx\, dx$$

$$= \int_0^{\pi} \{\cos(m+n)x + \cos(m-n)x\}\, dx = \left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]_0^{\pi} = 0,$$

showing that the given set of functions is orthogonal.

Norm of $f_n(x)$, *i.e.,* $\| f_n(x) \|$, for $n \neq 0$

$$= \left\{\int_{-\pi}^{\pi} f_n^2(x)\,dx\right\}^{1/2} = \left\{\int_{-\pi}^{\pi} \cos^2 nx\,dx\right\}^{1/2} = \left\{2\int_0^{\pi} \cos^2 nx\,dx\right\}^{1/2} = \left\{\int_0^{\pi} (1+\cos 2nx)\,dx\right\}^{1/2} = \left[\{x+(1/2n)\sin 2nx\}^{1/2}\right]_0^{\pi} = \sqrt{\pi}$$

Again, for $n = 0$, $\qquad \| f_n(x) \| = \| 1 \| = \left[\int_{-\pi}^{\pi} (1)^2\,dx\right]^{1/2} = \sqrt{2\pi}$

Thus, $\qquad \phi_n(x) = \dfrac{f_n(x)}{\| f_n(x) \|} = \begin{cases} 1/\sqrt{2\pi}, & \text{if } n = 0 \\ (\cos nx)/\sqrt{\pi}, & \text{if } n \neq 0 \end{cases}$

Hence, the required orthonormal set is $1/(2\pi)^{1/2}, (\cos x)/\sqrt{\pi}, (\cos 2x)/\sqrt{\pi}, (\cos 3x)/\sqrt{\pi}, \ldots$

(***b***) Proceed as in part (*a*), **Ans.** $1/\sqrt{\pi}, (2/\pi)^{1/2}\cos x, (2/\pi)^{1/2}\cos 2x, (2/\pi)^{1/2}\cos 3x, \ldots$

Ex. 3. (***a***) *Show that the functions sin x, sin 2x, sin 3x, ..., 1, cos x, cos 2x, cos 3x, ... constitute an orthogonal set on the interval (–π, π). Normalize the set.* **[Lucknow 2010]**

Sol. For $m \neq n$, we have the following results :

(*i*) $\displaystyle\int_{-\pi}^{\pi} \sin mx \sin nx\,dx = 2\int_0^{\pi} \sin mx \sin nx\,dx$, as $\sin mx \sin nx$ is an even function

$$= \int_0^{\pi} \{\cos(m-n)x - \cos(m+n)x\}\,dx = \left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right]_0^{\pi} = 0$$

(*ii*) $\displaystyle\int_{-\pi}^{\pi} \sin mx \cos nx\,dx = 0,$ as $\sin mx \cos nx$ is an odd function of x.

(*iii*) $\displaystyle\int_{-\pi}^{\pi} \cos mx \cos nx\,dx = 2\int_0^{\pi} \cos mx \cos nx\,dx,$ as $\cos mx \cos nx$ is an even function of x

$$= \int_0^{\pi} \{\cos(m+n)x + \cos(m-n)x\}\,dx = \left[\frac{\sin(m+n)x}{m+n} + \frac{\sin(m-n)x}{m-n}\right]_0^{\pi} = 0$$

(*iv*) $\displaystyle\int_{-\pi}^{\pi} 1.\sin mx\,dx = 0,$ as $\sin mx$ is an odd function.

(*v*) $\displaystyle\int_{-\pi}^{\pi} 1.\cos nx\,dx = 2\int_0^{\pi} \cos nx\,dx = (2/n) \times \left[\sin nx\right]_0^{\pi} = 0$

Relations (*i*), (*ii*), (*iii*), (*iv*) and (*v*) together show that the given set of functions is orthogonal.

Now, $\| \sin nx \| = \left\{\int_{-\pi}^{\pi} \sin^2 nx\,dx\right\}^{1/2} = \left\{2\int_0^{\pi} \sin^2 nx\,dx\right\}^{1/2} = \left\{\int_0^{\pi} (1-\cos 2nx)\,dx\right\}^{1/2} = \left\{\left[x - \frac{1}{2n}\sin 2nx\right]_0^{\pi}\right\}^{1/2} = \sqrt{\pi}$

$$\| \cos nx \| = \left\{\int_{-\pi}^{\pi} \cos^2 nx\,dx\right\}^{1/2} = \left\{2\int_0^{\pi} \cos^2 nx\,dx\right\}^{1/2} = \left\{\int_0^{\pi} (1+\cos 2nx)\,dx\right\}^{1/2} = \left\{\left[x + \frac{1}{2n}\sin 2nx\right]_0^{\pi}\right\}^{1/2} = \sqrt{\pi}$$

and $\qquad \| 1 \| = \left\{\int_{-\pi}^{\pi} 1^2\,dx\right\}^{1/2} = \left\{[x]_{-\pi}^{\pi}\right\}^{1/2} = \sqrt{2\pi}$

Hence the required orthonormal set is

$$(\sin x)/\sqrt{\pi}, (\sin 2x)/\sqrt{\pi}, \ldots, 1/\sqrt{2\pi}, \quad (\cos x)/\sqrt{\pi}, (\cos 2x)/\sqrt{\pi}, \ldots$$

Ex. 3. (***b***) *Show that the functions 1, cos (2nπx/T), sin (2nπx/T), n = 1, 2, 3, ... are orthogonal on the interval –T/2 ≤ x ≤ T/2, and find the corresponding orthonormal set.*

Sol. Do as in Ex. 3(*a*). **Ans.** $\left\{1/\sqrt{T}, (2/T)^{1/2}\cos(2n\pi x/T), (2/T)^{1/2}\sin(2n\pi x/T)\right\}, n = 1, 2, 3, \ldots$

Ex. 4. (***a***) *Show that the functions $f_1(x) = 1$, $f_2(x) = x$ are orthogonal on the interval $(-1, 1)$ and determine the constants A and B so that the function $f_3(x) = 1 + Ax + Bx^2$ is orthogonal to both f_1 and f_2 on the interval $(-1, 1)$.* **[Meerut 2007; Kanpur 2011]**

(***b***) *Show that the functions $f_1(x) = 4$ and $f_2(x) = x^3$ are orthogonal on the interval $(-2, 2)$ and determine constants A and B so that the function $f_3(x) = 1 + Ax + Bx^2$ is orthogonal to both f_1 and f_2.*

Sol. (***a***) Here $$\int_{-1}^{1} f_1(x)\, f_2(x)\, dx = \int_{-1}^{1} x\, dx = \left[x^2/2\right]_{-1}^{1} = 0,$$

showing that $f_1(x)$ and $f_2(x)$ are orthgonal on the interval $(-1, 1)$.

If $f_3(x)$ is orthogonal to both $f_1(x)$ and $f_2(x)$ on the interval $(-1, 1)$, then by definition,

$$\int_{-1}^{1} f_1(x)\, f_3(x)\, dx = 0 \qquad \text{and} \qquad \int_{-1}^{1} f_2(x)\, f_3(x)\, dx = 0$$

or $$\int_{-1}^{1} (1 + Ax + Bx^2)\, dx = 0 \qquad \text{and} \qquad \int_{-1}^{1} x(1 + Ax + Bx^2)\, dx = 0$$

or $$\left[x + \frac{A}{2}x^2 + \frac{B}{3}x^3\right]_{-1}^{1} = 0 \qquad \text{and} \qquad \left[\frac{1}{2}x^2 + \frac{1}{3}Ax^3 + \frac{1}{4}Bx^4\right]_{-1}^{1} = 0$$

or $2 + (2/3)\times B = 0$ and $(2/3)\times A = 0$ so that $A = 0$ and $B = -3$.

So f_3 is orthogonal to both f_1 and f_2 if $A = 0$ and $B = -3$.

(***b***) Proceed as in part (*a*). **Ans.** $A = 0$ and $B = -3/4$.

Ex. 5. *Given that $f_1(x) = a_0$, $f_2(x) = b_0 + b_1 x$ and $f_3(x) = c_0 + c_1 x + c_2 x^2$. Determine the constants a_0, b_0, c_0, b_1, c_1 and c_2 so that the given functions form an orthonormal set on the interval $-1 \le x \le 1$.*

Sol. Since f_1, f_2 and f_3 form an orthonormal set on $-1 \le x \le 1$, we have

$$(i)\ \int_{-1}^{1} f_1 f_2\, dx = 0 \qquad (ii)\ \int_{-1}^{1} f_2 f_3\, dx = 0 \qquad (iii)\ \int_{-1}^{1} f_3 f_1\, dx = 0,$$

$$(iv)\ \| f_1 \| \ne 0 \qquad (v)\ \| f_2 \| \ne 0 \qquad (vi)\ \| f_3 \| \ne 0.$$

Now, $(i) \Rightarrow \displaystyle\int_{-1}^{1} a_0(b_0 + b_1 x)\, dx = 0 \Rightarrow \left[a_0 b_0 x + \frac{1}{2} a_0 b_1 x^2\right]_{-1}^{1} = 0 \Rightarrow a_0\, b_0 = 1$ ...(1)

$$(ii) \Rightarrow \qquad \int_{-1}^{1} (b_0 + b_1 x)(c_0 + c_1 x + c_2 x^2)\, dx = 0$$

or $$\left[b_0 c_0 x + \frac{1}{2} b_0 c_1 x^2 + \frac{1}{3} b_0 c_2 x^3 + \frac{1}{2} b_1 c_0 x^2 + \frac{1}{3} b_1 c_1 x^3 + \frac{1}{4} b_1 c_2 x^4\right]_{-1}^{1} = 0$$

or $2b_0 c_0 + (2/3)\times b_0 c_2 + (2/3)\times b_1 c_1 = 0$ or $b_0(3c_0 + c_2) + b_1 c_1 = 0$ ...(2)

$(iii) \Rightarrow \displaystyle\int_{-1}^{1} a_0(c_0 + c_1 x + c_2 x^2)\, dx = 0 \Rightarrow \left[a_0 c_0 x + \frac{1}{2} a_0 c_1 x^2 + \frac{1}{3} a_0 c_2 x^3\right]_{-1}^{1} = 0$

or $2a_0 c_0 + (2/3)\times a_0 c_2 = 0$ or $a_0(3c_0 + c_2) = 0$ ...(3)

$(iv) \Rightarrow \displaystyle\left[\int_{-1}^{1} a_0^2\, dx\right]^{1/2} \ne 0 \Rightarrow (2a_0^2)^{1/2} \ne 0 \Rightarrow a_0 \ne 0$ ...(4)

Now, (1) and (4) give $b_0 = 0.$ ...(5)

Again, since $b_0 \neq 0$, so by (2), $b_1 c_1 = 0$ ...(6)

$$\therefore\ (v) \Rightarrow \left[\int_{-1}^{1} (b_0 + b_1 x)^2 dx\right]^{1/2} \neq 0 \Rightarrow \left[\int_{-1}^{1} b_1^2 x^2 dx\right]^{1/2} \neq 0, \text{ using (5)}$$

Thus, $[(2/3)\times b^2{}_1]^{1/2} \neq 0$ so that $b_1 \neq 0$ ...(7)

$\therefore$ from (6), we get $c_1 = 0$, as $b_1 \neq 0$ ...(8)

$$\text{Finally, } (vi) \Rightarrow \left[\int_{-1}^{1} (c_0 + c_1 x + c_1 x^2)^2 dx\right]^{1/2} \neq 0 \Rightarrow \left[\int_{-1}^{1} (c_0 + c_2 x^2)^2 dx\right]^{1/2} \neq 0,$$

$$\therefore \quad \left\{\left[c_0^2 x + (2/3)\times c_0 c_2 x^3 + (1/5)\times c_2^2 x^5\right]_{-1}^{1}\right\}^{1/2} \neq 0$$

or $\{2c_0^2 + (4/3)\times c_0 c_2 + (2/5)\times c_2^2\}^{1/2} \neq 0$ or $\{c_0^2 + (2/3)\times c_0 c_2 + (1/5)\times c_2^2\}^{1/2} \neq 0$ ...(9)

From (3) and (5), we get $3c_0 + c_2 = 0$ or $c_2 = -3c_0$ ...(10)

Using (10), (9) can be re–written as

$\{c_0^2 - 2c_0^2 + (9/5)\times c_0^2\}^{1/2} \neq 0$ so that $c_0 \neq 0$ ...(11)

From (4), (5), (7), (8), (10) and (11), we find that $b_0 = 0$, $c_1 = 0$, $c_2 = -3c_0$ and a_0, c_0 can take arbitrary real values.

Ex. 6. *Show that the functions $1 - x$, $1 - 2x + x^2/2$ and $1 - 3x + 3x^2/2 - x^3/6$ are orthogonal with respect to e^{-x} on $0 \le x < \infty$. Determine the corresponding orthonormal functions.*

Sol. Let $f_1(x) = 1 - x$, $f_2(x) = 1 - 2x + x^2/2$ and $f_3(x) = 1 - 3x + 3x^2/2 - x^3/6$

Here the given weight function $= p(x) = e^{-x}$.

$$\text{Now, } \int_0^\infty p(x) f_1 f_2\, dx = \int_0^\infty e^{-x}(1-x)(1-2x+x^2/2)\, dx = \int_0^\infty e^{-x}(1-3x+5x^2/2-x^3/2)\, dx$$

$$= \int_0^\infty e^{-x} x^0\, dx - 3\int_0^\infty e^{-x} x\, dx + \frac{5}{2}\int_0^\infty e^{-x} x^2\, dx - \frac{1}{2}\int_0^\infty e^{-x} x^3\, dx \quad ...(1)$$

To evaluate integrals involved in (1), we make use of the following result of Gamma function,

$$\int_0^\infty e^{-x} x^n\, dx = n!\,, \text{ if } n \text{ is a non–negative integer} \quad ...(2)$$

$$\text{Using (2), (1)} \Rightarrow \int_0^\infty p(x) f_1 f_2\, dx = 0! - 3\cdot 1! + (5/2)\cdot 2! - (1/2)\cdot 3! = 1 - 3 + 5 - 3 = 0 \quad ...(3)$$

$$\text{Next, } \int_0^\infty p(x) f_2 f_3\, dx = \int_0^\infty e^{-x}(1-2x+x^2/2)(1-3x+3x^2/2-x^3/6)\, dx$$

$$= \int_0^\infty e^{-x}(1-5x+8x^2-14x^3/3+13x^4/12-x^5/12)\, dx = 0! - 5\cdot 1! + 8\cdot 2! - (14/3)\cdot 3! + (13/12)\cdot 4! - (1/12)\cdot 5!, \text{ using (2)}$$

$$\text{Thus, } \int_0^\infty p(x) f_2 f_3\, dx = 1 - 5 + 16 - 28 + 26 - 10 = 0 \quad ...(4)$$

$$\text{Again, } \int_0^\infty p(x) f_3 f_1\, dx = \int_0^\infty e^{-x}(1-3x+3x^2/2-x^3/6)(1-x)\, dx$$

$$= \int_0^\infty e^{-x}(1-4x+9x^2/2-5x^3/3+x^4/6)\,dx = 0!-4\cdot 1!+(9/2)\cdot 2!-(5/3)\cdot 3!+(1/6)\cdot 4!, \text{ using (2)}$$

Thus, $$\int_0^\infty p(x)\, f_1 f_3\, dx = 1-4+9-10+4=0 \qquad \text{...(5)}$$

From (3), (4) and (5) we find that f_1, f_2, f_3 are orthogonal with respect to weight function $p(x)$ on $0 \le x < \infty$. Also, we have

$$\| f_1 \| = \left[\int_0^\infty e^{-x}(1-x)^2 dx\right]^{1/2} = \left[\int_0^\infty e^{-x}(1-2x+x^2)\, dx\right]^{1/2} = [0! - 2\cdot 1! + 2!]^{1/2} = 1$$

$$\| f_2 \| = \left[\int_0^\infty e^{-x}(1-2x+x^2/2)^2\, dx\right]^{1/2} = \left[\int_0^\infty e^{-x}(1-4x+5x^2-2x^3+x^4/4)\, dx\right]^{1/2}$$

$$= [0!-4\cdot 1!+5\cdot 2!-2\cdot 3!+(1/4)\cdot 4!]^{1/2} = (1-4+10-12+6)^{1/2} = 1$$

$$\| f_3 \| = \left[\int_0^\infty e^{-x}(1-3x+3x^2/2-x^3/6)^2 dx\right]^{1/2}$$

$$= \left[\int_0^\infty e^{-x}(1-6x+12x^2-28x^3/3+13x^4/4-x^5/2+x^6/36)\, dx\right]^{1/2}$$

$$=[0!-6\cdot 1!+12\cdot 2!-(28/3)\cdot 3!+(13/4)\cdot 4!-(1/2)\cdot 5!+(1/36)\cdot 6!]^{1/2} = (1-6+24-56+78-60+20)^{1/2} = 1$$

Since norm of each of the functions f_1, f_2 and f_3 is unity, it follows that the given set of functions is an orthonormal set.

Ex. 7. *With help of 1, x, x^2 construct three functions ϕ_0, ϕ_1 and ϕ_2 which are orthogonal over $-1 \le x \le 1$.*

Sol. We take $$\phi_0(x) = 1 \qquad \text{...(1)}$$

Next, choose $$\phi_1 = x + c\ \ \phi_0(x) = x + c \qquad \text{...(2)}$$

Let ϕ_1 be orthogonal to ϕ_0 so that

$$\int_{-1}^1 \phi_0\phi_1\, dx = 0 \quad \text{or} \quad \int_{-1}^1 (x+c)\, dx = 0 \quad \text{or} \quad \left[\frac{1}{2}x^2+cx\right]_{-1}^1 = 0$$

giving $c = 0$. Hence (2) gives $$\phi_1 = x. \qquad \text{...(3)}$$

Next, we take $$\phi_2 = x^2 + c_1\phi_0 + c_2\phi_1 = x^2 + c_1 + c_2 x \qquad \text{...(4)}$$

Let ϕ_2 be orthogonal to both ϕ_0 and ϕ_1 so that

$$\int_{-1}^1 \phi_2\phi_0\, dx = 0 \quad \text{and} \quad \int_{-1}^1 \phi_2\phi_1\, dx = 0$$

i.e., $$\int_{-1}^1 (x^2+c_1+c_2x)\, dx = 0 \quad \text{and} \quad \int_{-1}^1 (x^2+c_1+c_2x)x\, dx = 0$$

i.e., $$\left[(1/3)\times x^3+c_1x+(1/2)\times c_2x^2\right]_{-1}^1 = 0 \text{ and } \left[(1/4)\times x^4+(1/2)\times c_1x^2+(1/3)\times c_2x^3\right]_{-1}^1 = 0$$

i.e., $(2/3) + 2c_1 = 0$ and $c_2 \times (2/3) = 0$ so that $c_1 = -1/3$, $c_2 = 0$

$\therefore$ (4) becomes $$\phi_2(x) = x^2 - (1/3).$$

So the required functions are $\phi_0 = 1$, $\phi_1 = x$, $\phi_2 = x^2 - (1/3)$.

Ex. 8. *With the help of 1, x, x^2 construct three functions ϕ_0, ϕ_1 and ϕ_2 which are orthogonal with respect to e^{-x} over $0 \le x < \infty$.*

Sol. We take $\phi_0(x) = 1$ ...(1)

Next, choose $\phi_1 = x + c\, \phi_0(x) = x + c$ ...(2)

Let ϕ_1 be orthogonal to ϕ_0 with respect to e^{-x}. Then, we have

$$\int_0^\infty e^{-x}(x+c)\,dx = 0 \qquad \text{or} \qquad \int_0^\infty e^{-x} x\,dx + c\int_0^\infty e^{-x}dx = 0 \qquad ...(3)$$

We know that $$\int_0^\infty e^{-x}x^n\,dx = n!, \quad \text{where} \quad n = 0, 1, 2, 3, ... \qquad ...(4)$$

Using (4), (3) gives $1! + c \times 0! = 0$ or $1 + c = 0$ or $c = -1$.

Hence (2) reduces to $\phi_1 = x - 1$ ...(5)

Finally, take $\phi_2 = x^2 + c_1\phi_0 + c_2\phi_1 = x^2 + c_1 + c_2(x - 1)$ ...(6)

Let ϕ_2 be orthogonal to ϕ_0 and ϕ_1 with respect to e^{-x}. Then, we have

$$\int_0^\infty e^{-x}\phi_2\phi_0\,dx = 0 \qquad i.e., \qquad \int_0^\infty e^{-x}(x^2 + c_2x + c_1 - c_2)\,dx = 0 \qquad ...(7)$$

and $$\int_0^\infty e^{-x}\phi_2\phi_1\,dx = 0 \qquad i.e., \qquad \int_0^\infty e^{-x}(x^2 + c_2x + c_1 - c_2)(x-1)\,dx = 0 \qquad ...(8)$$

Using (4), (7) gives $2! + c_2 \times 1! + (c_1 - c_2) \times 0! = 0$

or $2 + c_2 + c_1 - c_2 = 0$ so that $c_1 = -2$ ...(9)

Now, by (8), $$\int_0^\infty e^{-x}\left\{x^3 + (c_2 - 1)x^2 + (c_1 - 2c_2)x + (c_2 - c_1)\right\}dx = 0$$

or $3! + (c_2 - 1) \times 2! + (c_1 - 2c_2) \times 1! + (c_2 - c_1) \times 0! = 0$, using (4)

or $6 + 2(c_2 - 1) + c_1 - 2c_2 + c_2 - c_1 = 0$ so that $c_2 = -4$.

Since $c_1 = -2$, and $c_2 = -4$, so (6) gives $\phi_2 = x^2 - 4x + 2$.

So required functions are $\phi_0(x) = 1$, $\phi_1(x) = x - 1$ and $\phi_2(x) = x^2 - 4x + 2$.

Ex. 9. *Given the set of functions 1, x, x^2, x^3, ...,. Obtain from these a set of functions which are mutually orthonormal in (–1, 1).* **[Kanpur 2009]**

Sol. Let $\{\phi_n(x)\}$ be the required orthonormal set of function so that

$$\int_{-1}^1 \phi_n\phi_m\,dx = \begin{cases} 0, & \text{if } m \ne n \\ 1, & \text{if } m = n \end{cases} \qquad ...(1)$$

Step 1. Choose $f_1(x) = 1$ and take $\phi_1(x) = f_1(x)/\| f_1(x) \|$.

Now, $$\| f_1(x) \| = \left[\int_{-1}^1 f_1^2(x)\,dx\right]^{1/2} = \left[\int_{-1}^1 1^2\,dx\right]^{1/2} = \sqrt{2}. \text{ So } \qquad \phi_1(x) = \frac{1}{\sqrt{2}} \qquad ...(2)$$

Step 2. Choose $f_2(x) = x$ and take a function $g_2(x) = f_2(x) + c\,\phi_1(x) = x + (c/\sqrt{2})$ ...(3)

Let $g_2(x)$ and $\phi_1(x)$ be orthogonal on the interval (–1, 1)

$$\therefore \int_{-1}^1 g_2(x)\,\phi_1(x)\,dx = 0 \qquad \text{or} \qquad \int_{-1}^1 (x + c/\sqrt{2})\cdot(1/\sqrt{2})\,dx = 0$$

or $$\left[x^2/2 + cx/\sqrt{2}\right]_{-1}^1 = 0 \qquad \text{so that} \qquad c = 0$$

Hence, by (3), $g_2(x) = x$

Now, $\| g_2(x) \| = \left[\int_{-1}^{1} g_2^2(x)\,dx\right]^{1/2} = \left[\int_{-1}^{1} x^2\,dx\right]^{1/2} = \left(\frac{2}{3}\right)^{1/2}$

$\therefore$ $\phi_2(x) = g_2(x)/\| g_2(x) \| = (3/2)^{1/2} \cdot x$ ...(4)

Step 3. Choose $f_3(x) = x^2$ and take a function

$$g_3(x) = f_3(x) + c_1\,\phi_1(x) + c_2\,\phi_2(x) = x^2 + c_1\,\phi_1 + c_2\,\phi_2 \qquad ...(5)$$

Let $g_3(x)$ be orthgonal to $\phi_1(x)$ and $\phi_2(x)$ so that

(*i*) $\int_{-1}^{1} g_3(x)\phi_1(x)\,dx = 0$ (*ii*) $\int_{-1}^{1} g_3(x)\phi_2(x)\,dx = 0$

(*i*) $\Rightarrow$ $\int_{-1}^{1} (x^2 + c_1\phi_1 + c_2\phi_2)\,\phi_1\,dx = 0$ or $\int_{-1}^{1} x^2\phi_1\,dx + c_1\int_{-1}^{1} \phi_1^2\,dx + c_2\int_{-1}^{1} \phi_2\phi_1\,dx = 0$

or $\int_{-1}^{1} x^2 \cdot (1/\sqrt{2})\,dx + c_1 \cdot 1 + c_2 \cdot 0 = 0$, using (1) and (2)

or $(2/3) \times (1/\sqrt{2}) + c_1 = 0$ or $c_1 = -\left(\sqrt{2}/3\right)$ ...(6)

Next, from (*ii*), $\int_{-1}^{1} (x^2 + c_1\phi_1 + c_2\phi_2)\,\phi_2\,dx = 0$ or $\int_{-1}^{1} x^2\phi_2\,dx + c_1\int_{-1}^{1} \phi_1\phi_2\,dx + c_2\int_{-1}^{1} \phi_2^2\,dx = 0$

or $\int_{-1}^{1} x^2 \times (3/2)^{1/2} \times x\,dx + 0 + c_2 \times 1 = 0$, using (1) and (4)

or $(3/2)^{1/2}\left[x^4/4\right]_{-1}^{1} + c_2 = 0$ or $c_2 = 0$

$\therefore$ (5) gives $g_3(x) = x^2 - (\sqrt{2}/3) \times (1/\sqrt{2}) = (3x^2 - 1)/3$

Now, $\| g_3(x) \| = \left[\int_{-1}^{1} g_3^2(x)\,dx\right]^{1/2} = \left[\frac{1}{9}\int_{-1}^{1} (3x^2 - 1)^2\,dx\right]^{1/2}$

$$= \left[\frac{1}{9}\int_{-1}^{1} (9x^4 - 6x^2 + 1)\,dx\right]^{1/2} = \frac{1}{3}\left\{\left[\frac{9}{5}x^5 - 2x^3 + x\right]_{-1}^{1}\right\}^{1/2} = \frac{2\sqrt{2}}{3\sqrt{5}}$$

$\therefore$ $\phi_3(x) = g_3(x)/\| g_3(x) \| = (1/2) \times (5/2)^{1/2} \times (3x^2 - 1)$ ...(7)

Proceeding like wise, we obtain

$\phi_4(x) = \left(\frac{7}{2}\right)^{1/2} \cdot \left(\frac{5x^3 - 3x}{2}\right)$, $\phi_5(x) = \left(\frac{9}{2}\right)^{1/2} \cdot \left(\frac{5x^4 - 30x^2 + 3}{8}\right)$ ...(8)

and so on. The required orthonormal set of functions $\{\phi_n(x)\}$ is given by (2), (4), (7), (8) and so on.

EXERCISE 15(A)

1. Show that the functions $\sin x$, $\sin 2x$, $\sin 3x$, ... are orthogonal on the interval $(0, \pi)$.

2. Show that the functions 1, $\cos 2x$, $\cos 4x$, $\cos 6x$, ... are orthogonal on interval $0 \le x \le \pi$, and find the corresponding orthonormal set. **Ans.** $\{1/\sqrt{\pi}, (2/\pi)^{1/2} \cos 2nx\}$, $n = 1, 2, 3, ...$

3. Show that the functions $\sin \pi x$, $\sin 2\pi x$, $\sin 3\pi x$, ... form an orthogonal set on the interval $-1 \le x < 1$ and obtain the corresponding orthonormal set.

Ans. $\sin \pi x$, $\sin 2\pi x$, $\sin 3\pi x$, ... *i.e.,* the given set itself.

4. Show that the functions 1, $\cos \pi x$, $\sin \pi x$, $\cos 2\pi x$, $\sin 2\pi x$, ... form an orthogonal set on $-2 \le x \le 2$ and find the corresponding orthonormal set.

Ans. $1/2$, $(\cos \pi x)/2$, $(\sin \pi x)/2$, $(\cos 2\pi x)/2$, $(\sin 2\pi x)/2$, ...

5. Show that each of the following set is orthogonal on the given interval and find the corresponding orthonormal sets :

(*i*) $1, \cos(2\pi x/c), \cos(4\pi x/c), \cos(6\pi x/c), ..., 0 \le x \le c$ **Ans.** $\{1/\sqrt{c}, (2/c)^{1/2} \cos(2n\pi x/c)\}$, $n = 1, 2, 3, ...$

(*ii*) $1, \cos 2x, \sin 2x, \cos 4x, \sin 4x, ..., -\pi/2 \le x \le \pi/2$ **Ans.** $1/\sqrt{\pi}, (2/\pi)^{1/2} \cos 2x, (2/\pi)^{1/2} \sin 2x, ...$

15.10. Strum–Liouville equation. Strum–Liouville problem. Eigen (or characteristic) functions and eigen (or characteristic) values.

[Meerut 2010; Ravishankar 1998; Himanchal 2010]

Definitions. A differential equation of the form $[r(x)\, y']' + [q(x) + \lambda\, p(x)]y = 0$ ... (1) is known as *Strum–Liouville equation.*

We assume that the functions p, q, r and r' in (1) are continuous in $a \le x \le b$ and $p(x) > 0$. Here λ is a parameter independent of x.

Equation (1) is considered on some interval $a \le x \le b$, satisfying boundary conditions at the two end points a and b,

$$a_1 y(a) + a_2 y'(a) = 0 \quad \text{and} \quad b_1 y(b) + b_2 y'(b) = 0 \quad ...(2)$$

with the real constants a_1, a_2, b_1, b_2. Suppose that a_1, a_2 in (2) are not both zero and so are b_1, b_2. The boundary value problem consisting of (1) and (2) is called a *Strum–Liouville problem.*

Clearly $y = 0$ is always a solution of Strum–Liouville problem for any value of the parameter λ. $y = 0$ known as a *trivial solution* is of no practical use. The non–zero solutions of the Strum–Liouville problem given by (1) and (2) are called the *eigenfunctions* of the problem and the values of λ for which such solutions exist, are called *eigenvalues* of the problem.

Remark. A special case of (1) and (2). Let $p = r = 1$ and $q = 0$ in (1). Also, let $a_1 = b_1 = 1$ and $a_2 = b_2 = 0$ in (2). Then (1) and (2) reduce to $y'' + \lambda y = 0$ with $y(a) = 0$, $y(b) = 0$. This is the simplest form of Strum–Liouville problem.

15.11. Orthogonality of eigenfunctions

Theorem. *Suppose that the functions $p(x)$, $q(x)$, $r(x)$ and $r'(x)$ in the Strum–Liouville equation $[r(x)\, y']' + \{q(x) + \lambda p(x)\}y = 0$ are real valued and continuous and $p(x) > 0$ on the interval $a \le x \le b$. Let $y_m(x)$ and $y_n(x)$ be eigenfunctions of the Strum–Liouville problem (given by the above Strum Liouville equation and boundary conditions $a_1\, y(a) + a_2\, y'(a) = 0$ and $b_1\, y(b) + b_2\, y'(b) = 0$) that correspond to different eigenvalues λ_m and λ_n respectively. Then y_m, y_n are orthogonal on that interval with respect to the weight function $p(x)$.*

Prove that eigenfunctions corresponding to different eigenvalues are orthogonal with respect to some weight function. **[Himanchal 2009]**

Proof. Consider the following Strum–Lioville problem :

$$[r(x)\, y']' + [q(x) + \lambda\, p(x)]y = 0 \quad ...(1)$$

$$a_1\, y(a) + a_2\, y'(a) = 0 \quad ...(2a)$$

$$b_1\, y(b) + b_2\, y'(b) = 0, \quad ...(2b)$$

where p, q, r and r' are real valued and continuous and $p(x) > 0$ on $a \le x \le b$. Let a_1, a_2 in $(2a)$ be given constants, not both zero and so be b_1, b_2 in $(2b)$.

Let y_m and y_n be eigenfunctions of the above Strum–Liouville problem that correspond to different eigenvalues λ_m and λ_n. Then, by definition of eigen functions, y_m and y_n both satisfy (1).

Hence $$(ry'_m)' + (q + \lambda_m p)\, y_m = 0 \qquad ...(3)$$

and $$(ry'_n)' + (q + \lambda_n p)\, y_n = 0 \qquad ...(4)$$

Multiplying (3) by y_n and (4) by y_m and subtracting, we get

$$(ry'_m)'\, y_n - (ry'_n)'\, y_m + (\lambda_m - \lambda_n) p\, y_m y_n = 0 \quad \text{or} \quad (\lambda_m - \lambda_n) p\, y_m y_n = (ry'_n)'\, y_m - (ry'_m)'\, y_n$$

or $$(\lambda_m - \lambda_n) p\, y_m y_n = \frac{d}{dx}\{(ry'_n) y_m - (ry'_m) y_n\}, \qquad ...(5)$$

which can be verified by performing the indicated differentiation of the expression in brackets on R.H.S. of (5). Since $r(x)$ and $r'(x)$ are continuous by assumption and y_m, y_n are solutions of (1), it follows that the expression within brackets on R.H.S. of (5) is continuous on $a \le x \le b$. Integrating both sides of (5) over x from a to b, we thus obtain

$$(\lambda_m - \lambda_n) \int_a^b p\, y_m y_n\, dx = \left[r(y'_n y_m - y'_m y_n) \right]_a^b$$

or $$(\lambda_m - \lambda_n) \int_a^b p\, y_m y_n\, dx = r(b)\, \{y'_n(b)\, y_m(b) - y'_m(b)\, y_n(b)\} - r(a)\, \{y'_n(a)\, y_m(a) - y'_m(a)\, y_n(a)\} \qquad ...(6)$$

We now have to consider several cases depending on whether $r(x)$ vanishes or does not vanish at a or b.

Case I. Let $r(a) = r(b) = 0$. Then (6) reduces to $$(\lambda_m - \lambda_n) \int_a^b p\, y_m y_n\, dx = 0 \qquad ...(7)$$

Cae II. Let $r(b) = 0$ but $r(a) \neq 0$. Then (6) reduces to

$$(\lambda_m - \lambda_n) \int_a^b p\, y_m y_n\, dx = -r(a) \{y'_n(a)\, y_m(a) - y'_m(a)\, y_n(a)\} \qquad ...(8)$$

Since y_m and y_n both satisfy (2a), we have

$$a_1 y_m(a) + a_2 y'_m(a) = 0 \qquad ...(9)$$

and $$a_1 y_n(a) + a_2 y'_n(a) = 0 \qquad ...(10)$$

Let $a_2 \neq 0$. Multiplying (10) by $y_m(a)$ and (9) by $y_n(a)$ and then subtracting, we get

$$a_2\{y'_n(a)\, y_m(a) - y'_m(a)\, y_n(a)\} = 0$$

Since $a_2 \neq 0$, so $$y'_n(a)\, y_m(a) - y'_m(a)\, y_n(a) = 0 \qquad ...(11)$$

Using (11), (8) reduces to (7). If $a_2 = 0$, then let $a_1 \neq 0$. Now, multiplying (9) by $y'_n(a)$ and (10) by $y'_m(a)$ and then subtracting, we get

$$a_1\{y'_n(a)\, y_m(a) - y'_m(a)\, y_n(a)\} = 0$$

Since $a_1 \neq 0$, so $$y'_n(a)\, y_m(a) - y'_m(a)\, y_n(a) = 0$$

Hence as before (8) reduces to (7).

Case III. Let $r(a) = 0$ but $r(b) \neq 0$. Then (6) reduces to

$$(\lambda_m - \lambda_n) \int_a^b p\, y_m y_n\, dx = r(b) \{y'_n(b)\, y_m(b) - y'_m(b)\, y_n(b)\} \qquad ...(12)$$

Since y_m and y_n both satisfy (2b), we have

$$b_1 y_m(b) + b_2 y'_m(b) = 0 \qquad ...(13)$$

and $$b_1 y_n(b) + b_2 y'_n(b) = 0 \qquad ...(14)$$

Let $b_2 \neq 0$. Multiplying (14) by $y_m(b)$ and (13) by $y_n(b)$ and then subtracting, we get

$$b_2\{y'_n(b)\, y_m(b) - y'_m(b)\, y_n(b)\} = 0$$

Since $b_2 \neq 0$, so $$y'_n(b)\, y_m(b) - y'_m(b)\, y_n(b) = 0 \qquad ...(15)$$

Using (15), (12) reduces to (7). If $b_2 = 0$, then let $b_1 \neq 0$. Now, multiplying (13) by $y'_n(b)$ and (14) by $y'_m(b)$ and then subtracting, we get

$$b_1\{y'_n(b)\, y_m(b) - y'_m(b)\, y_n(b)\} = 0$$

Since $b_1 \neq 0$, so $$y'_n(b)\, y_m(b) - y'_m(b)\, y_n(b) = 0$$

Hence, as before, (12) reduces to (7).

Case IV. Let $r(a) \neq 0$ and $r(b) \neq 0$. There is no loss of generality by assuming that $a_2 \neq 0$ and $b_2 \neq 0$. Then, proceeding as in cases II and III, relations (11) and (15) can be proved. Then, using (11) and (15), (6) reduces to (7).

Case V. Let $r(a) = r(b)$. Proceed as in case IV to show that (6) reduces to (7).

From the above discussion, we see that in all situations, we get (7). Since λ_m and λ_n are different, (7) reduces to $$\int_a^b p(x)\, y_m(x)\, y_n(x)\, dx = 0,$$

showing that $y_m(x)$ and $y_n(x)$ are orthogonal with respect to weight function $p(x)$.

15.12. Reality of eigenvalues **[Meerut 2010; Purvanchal 2005; Lucknow 2010]**

Theorem. *To prove that all eigenvalues of Strum Liouville problem are real.*

Proof. Consider the following Strum–Liouville problem :

$$[r(x)\, y']' + [q(x) + \lambda\, p(x)]y = 0 \qquad ...(1)$$

$$a_1 y(a) + a_2 y'(a) = 0 \qquad ...(2a)$$

$$b_1 y(b) + b_2 y'(b) = 0, \qquad ...(2b)$$

where p, q, r and r' are real valued and continuous and $p(x) > 0$ on $a \leq x \leq b$. Let a_1, a_2 in (2*a*) be given constants, not both zero, and so be b_1, b_2 in (2*b*).

Let $y(x)$ be an eigenfunction corresponding to an eigenvalue $\lambda = \alpha + i\beta$, where α, β are real constants. This eigenfunction $y(x)$ satisfies (1), (2*a*) and (2*b*) and may be a complex valued function.

Taking the complex conjugates of all the terms in (1), (2*a*) and (2*b*), we get

$$[r(x)\, \bar{y}'\,]' + [q(x) + \bar{\lambda}\, p(x)]\, \bar{y} = 0 \qquad ...(3)$$

$$a_1 \bar{y}\,(a) + a_2 \bar{y}'\,(a) = 0 \qquad ...(4a)$$

$$b_1 \bar{y}\,(b) + b_2 \bar{y}'\,(b) = 0 \qquad ...(4b)$$

The above equations (3), (4*a*) and (4*b*) show that $\bar{y}(x)$ is the eigenfunction corresponding to the eigenvalue $\bar{\lambda} = \alpha - i\beta$. Multiplying (1) by $\bar{y}$ and (3) by y and subtracting, we get

$$(r\, y')'\, \bar{y} - \left(r\, \bar{y}'\right)'\, y + (\lambda - \bar{\lambda}) p\, y\, \bar{y} = 0 \qquad \text{or} \qquad (\lambda - \bar{\lambda}) p\, y\, \bar{y} = \left(r\, \bar{y}'\right)'\, y - (r\, y')'\, \bar{y}$$

or $$(\lambda - \bar{\lambda}) p\, y\, \bar{y} = \frac{d}{dx}\{(r\, \bar{y}')y - (r\, y')\bar{y}\}, \qquad ...(5)$$

which can be verified by performing the indicated differentiation of the expression in brackets on R.H.S. of (5). Integrating both sides of (5) w.r.t. 'x' from a to b, we thus obtain

$$(\lambda - \bar{\lambda}) \int_a^b p\, y\, \bar{y}\, dx = [r(\bar{y}'\, y - y'\, \bar{y})]_a^b$$

or $$(\lambda - \bar{\lambda}) \int_a^b p\, y\, \bar{y}\, dx = r(b)\{\bar{y}'(b)\, y(b) - y'(b)\, \bar{y}(b)\} - r(a)\{\bar{y}'(a)\, y(a) - y'(a)\, \bar{y}(a)\} \qquad ...(6)$$

We now have to consider several cases depending on whether $r(x)$ vanishes or does not vanish at a or b.

Case I. Let $r(a) = r(b) = 0$. Then (6) reduces to $$(\lambda - \bar{\lambda}) \int_a^b p\, y\, \bar{y}\, dx = 0 \qquad ...(7)$$

Case II. Let $r(b) = 0$ but $r(a) \neq 0$. Then (6) reduces to

$$(\lambda - \bar{\lambda})\int_a^b p\, y\, \bar{y}\, dx = -r(a)\{\bar{y}'(a)\, y(a) - y'(a)\, \bar{y}(a)\} \qquad ...(8)$$

Consider relations (2a) and (4a). Let $a_2 \neq 0$. Multiplying (4a) by $y(a)$ and (2a) by $\bar{y}(a)$ and then subtracting, we get $\quad a_2\{\bar{y}'(a)\, y(a) - y'(a)\, \bar{y}(a)\} = 0$

Since $a_2 \neq 0$, so $\quad \bar{y}'(a)\, y(a) - y'(a)\, \bar{y}(a) = 0$. ...(9)

Using (9), (8) reduces to (7). If $a_2 = 0$, then assume that $a_1 \neq 0$. Now, multiplying (2a) by $\bar{y}'(a)$ and (4a) by $y'(a)$ and then subtracting, we get

$$a_1\{\bar{y}'(a)\, y(a) - y'(a)\, \bar{y}(a)\} = 0$$

Since $a_1 \neq 0$, so $\quad \bar{y}'(a)\, y(a) - y'(a)\, \bar{y}(a) = 0$

Hence as before (8) reduces to (7).

Case III. Let $r(a) = 0$ but $r(b) \neq 0$. Then (6) reduces to

$$(\lambda - \bar{\lambda})\int_a^b p\, y\, \bar{y}\, dx = r(b)\{\bar{y}'(b)\, y(b) - y'(b)\, \bar{y}(b)\} \qquad ...(10)$$

Consider relations (2b) and (4b). Let $b_2 \neq 0$. Multiplying (4b) by $y(b)$ and (2b) by $\bar{y}(b)$ and then subtracting, we get $\quad b_2\{\bar{y}'(b)\, y(b) - y'(b)\, \bar{y}(b)\} = 0$.

Since $\quad b_2 \neq 0, \quad$ so $\quad \bar{y}'(b)\, y(b) - y'(b)\, \bar{y}(b) = 0$ ...(11)

Using (11), (10) reduces to (7). If $b_2 = 0$, then assume that $b_1 \neq 0$. Now, multiplying (2b) by $\bar{y}'(b)$ and (4b) by $y'(b)$ and then subtracting, we get

$$b_1\{\bar{y}'(b)\, y(b) - y'(b)\, \bar{y}(b)\} = 0$$

Since $b_1 \neq 0$, so $\quad \bar{y}'(b)\, y(b) - y'(b)\, \bar{y}(b) = 0$

Hence as before (10) reduces to (7).

Case IV. Let $r(a) \neq 0$ and $r(b) \neq 0$. There is no loss of generality by assuming that $a_2 \neq 0$ and $b_2 \neq 0$. Then proceeding as in cases II and III, relations (9) and (11) can be proved. Then, using (9) and (11), (6) reduces to (7).

Case V. Let $r(a) = r(b)$. Proceed as in case IV to show that (6) reduces to (7).

From the above discussion, we see that in all situations we get (7).

Now, $\lambda = \alpha + i\beta \Rightarrow \bar{\lambda} = \alpha - i\beta$ and hence $\lambda - \bar{\lambda} = (\alpha + i\beta) - (\alpha - i\beta) = 2i\beta$.

Again $y\bar{y} = |y|^2$, where $|y|$ stands for modulus of y. Then, (7) reduces to

$$2i\beta\int_a^b p(x)\,|\,y(x)\,|^2\, dx = 0 \qquad \text{or} \qquad \beta\int_a^b p(x)\,|\,y(x)\,|^2\, dx = 0 \qquad ...(12)$$

Since $\int_a^b p(x)\,|\,y(x)\,|^2\, dx$ has a positive value in the given interval $a \leq x \leq b$, (12) reduces to $\beta = 0$ and hence $\lambda = \alpha + \iota\beta = \alpha$, which is real. Since λ is an arbitrary eigenvalue, it follows that eigenvalues of Strum–Liouville problem are all real.

15.13. SOLVED EXAMPLES

Ex. 1. *Find the eigenvalues and the corresponding eigenfunctions of* $X'' + \lambda X = 0$, $X(0) = 0$ *and* $X'(L) = 0$. **[Himanchal 2009; Jiwaji 2004; Meerut 2006; Ravishakar 2005]**

Sol. Given $X'' + \lambda X = 0$ ...(1)

with boundary conditions $X(0) = 0$ and $X'(L) = 0$ ...(2)

Case I. Let $\lambda = 0$. Then solution of (1) is $X(x) = Ax + B$. ...(3)

From (3), $X'(x) = A$ ...(4)

Replacing x by 0 in (3) and using (2), we get $B = 0$. Again, replacing x by L in (4) and using (2), we get $A = 0$. With $A = 0$, $B = 0$, (3) reduces to $X(x) = 0$. Since $X(x) \neq 0$, so there is no eigen function corresponding to $\lambda = 0$.

Case II. Let $\lambda = -\mu^2$, where $\mu \neq 0$ (*i.e.,* λ is negative). So (1) gives $X'' - \mu^2 X = 0$

whose solution is $X(x) = Ae^{\mu x} + Be^{-\mu x}$ ...(5)

From (5), $X'(x) = A\mu e^{\mu x} - B\mu e^{-\mu x}$ ...(6)

Using (2), (5) and (6) reduces to

$0 = A + B$ and $A\mu e^{\mu L} - B\mu e^{-\mu L} = 0$ ...(7)

Solving (7), we find $A = B = 0$. Hence (5) gives $X(x) = 0$. Since $X(x) \neq 0$, so there is no eigen function corresponding to $\lambda = -\mu^2$.

Case III. Let $\lambda = \mu^2$ where $\mu \neq 0$ (*i.e.,* λ is positive). So (1) gives $X'' + \mu^2 X = 0$

whose solution is $X(x) = A \cos \mu x + B \sin \mu x$ ...(8)

From (8), $X'(x) = -A\mu \sin \mu x + B\mu \cos \mu x$ ...(9)

Using (2), (8) and (9) reduce to $0 = A$ and $0 = B\mu \cos \mu L$

Thus $A = 0$ and $B \cos \mu L = 0$, as $\mu \neq 0$ ...(10)

Now consider $B \cos \mu L = 0$ ...(11)

If $B = 0$, then with $A = 0$, (8) reduces to $X(x) = 0$, which is not an eigen function. So $B \neq 0$ for the existence of eigen functions. Since $B \neq 0$, (11) gives

$\cos \mu L = 0$ so that $\mu L = (2n - 1)\pi/2$, $n = 1, 2, 3, \ldots$

$\therefore$ $\mu = (2n - 1)\pi/2L$ ...(12)

Using $A = 0$ and (12), (8) reduces to $X(x) = B \sin \{(2n - 1)\pi/2L\}$, $n = 1, 2, 3, \ldots$

and then $\lambda = \mu^2 = (2n - 1)^2\pi^2/4L^2$, $n = 1, 2, 3, \ldots$

So required eigenfunctions $X_n(x)$ with corresponding eigenvalues λ_n are

$X_n(x) = B_n \sin \{(2n - 1)\pi/2L\}$, and $\lambda_n = (2n - 1)^2\pi^2/4L^2$, $n = 1, 2, 3, \ldots$

Note. We can take $B_n = 1$ while writing eigenfunctions.

Ex. 2. *Find the eigenvalues and eigenfunctions of the Strum–Liouville problem*

$X'' + \lambda X = 0,\ X'(0) = 0,\ X'(L) = 0.$ **(Kanpur 2009; Meerut 1995)**

Sol. Given $X'' + \lambda X = 0$ ...(1)

with boundary conditions $X'(0) = 0$ and $X'(L) = 0$ ...(2)

Case I. Let $\lambda = 0$. Then solution of (1) is $X(x) = Ax + B$ ...(3)

From (3), $X'(x) = A$ ...(4)

Using (2), (4) gives $0 = A$ and $0 = A$

These gives $A = 0$, while B is arbitrary. Hence (3) reduces to $X(x) = B$ which is non–zero. Hence taking $B = 1$, $X(x) = 1$ is an eigenfunction with $\lambda = 0$ as the corresponding eigenvalue.

Case II. Let $X = -\mu^2$, where $\mu \neq 0$. Then (11) reduces to $X'' - \mu^2 X = 0$ whose solution is

$X(x) = Ae^{\mu x} + Be^{-\mu x}$ ...(5)

From (5), $X'(x) = A\mu e^{\mu x} - B\mu e^{-\mu x}$ ...(6)

Using (2), (6) reduces to $0 = A\mu - B\mu$ and $A\mu e^{\mu L} - B\mu e^{-\mu L} = 0$

i.e., $A - B = 0$ and $Ae^{\mu L} - Be^{-\mu L} = 0$, as $\mu \neq 0$.

Solving these equations, $A = B = 0$. So (5) gives $X(x) = 0$, which is not an eigenfunction.

Case III. Let $\lambda = \mu^2$, where $\mu \neq 0$. Then (1) reduces to $X'' + \mu^2 X = 0$ whose solution is

$$X(x) = A\cos\mu x + B\sin\mu x \qquad ...(7)$$

From (7), $$X'(x) = -A\mu\sin\mu x + B\mu\cos\mu x \qquad ...(8)$$

Using (2), (8) reduces to $\quad 0 = B\mu \quad$ and $\quad 0 = -A\mu\sin\mu L + B\mu\cos\mu L$

i.e., $\quad B = 0 \quad$ and $\quad A\sin\mu L = 0, \quad$ as $\quad \mu \neq 0 \qquad ...(9)$

We now consider $$A\sin\mu L = 0. \qquad ...(10)$$

If $A = 0$, then with $B = 0$, (8) reduces to $X(x) = 0$, which is not an eigenfunction. So we take $A \neq 0$ for the existence of eigenfunctions. Since $A \neq 0$, (10) reduce to

$\sin\mu L = 0 \quad$ so that $\quad \mu L = n\pi, \quad n = 1, 2, 3, ...$

$$\therefore \qquad \mu = (n\pi/L), \qquad n = 1, 2, 3, ... \qquad ...(11)$$

($n = 0$ is omited as $n = 0 \Rightarrow \mu = 0$ which is contrary to our assumption $\mu \neq 0$).

Using $B = 0$ and (11), (7) reduces to $\quad X(x) = A\cos(n\pi x/L)$, $n = 1, 2, 3, ...$

and then $$\lambda = \mu^2 = n^2\pi^2/L^2, \; n = 1, 2, 3, ...$$

Hence the required eigenfunctions $X_n(x)$ with the corresponding eigenvalues λ_n are given by (taking $A = 1$). $\quad X_n(x) = \cos(n\pi x/L), \quad \lambda_n = n^2\pi^2/L^2, \quad n = 0, 1, 2, 3, ...$

Ex. 3. *Find all the eigenvalues and eigenfunctions of the Strum–Liouville problem $y'' + \lambda y = 0$ with $y(0) + y'(0) = 0$ and $y(1) + y'(1) = 0$.*

Sol. Given $$y'' + \lambda y = 0 \qquad ...(1)$$

with boundary conditions $$y(0) + y'(0) = 0 \qquad ...(2)$$

$$\therefore \qquad y(1) + y'(1) = 0 \qquad ...(3)$$

Case I. Let $\lambda = 0$. Then solution of (1) is $$y(x) = Ax + B \qquad ...(4)$$

From (4), $$y'(x) = A \qquad ...(5)$$

From (4) and (5), $y(0) = B$, $y'(0) = A$. With these values, (2) gives

$$B + A = 0 \qquad ...(6)$$

From (4) and (5), $y(1) = A + B$ and $y'(1) = A$. With these values (3) gives

$$2A + B = 0 \qquad ...(7)$$

Solving (6) and (7), $A = B = 0$. Hence (4) reduces to $X(x) = 0$, which is not an eigenfunction and so $\lambda = 0$ is not an eigenvalue.

Case II. Let $\lambda = -\mu^2$, where $\mu \neq 0$. Then (1) reduces to $y'' - \mu^2 y = 0$ whose solution is

$$y(x) = Ae^{\mu x} + Be^{-\mu x} \qquad ...(8)$$

From (8), $$y'(x) = A\mu e^{\mu x} - B\mu e^{-\mu x} \qquad ...(9)$$

From (8) and (9), $\quad y(0) = A + B \quad$ and $\quad y'(0) = \mu(A - B) \qquad ...(10)$

Using (10), (2) reduces to $$A + B + \mu(A - B) = 0$$

i.e., $$A(1 + \mu) + B(1 - \mu) = 0 \qquad ...(11)$$

Again, from (8) and (9), we have

$y(1) = Ae^{\mu} + Be^{-\mu} \quad$ and $\quad y'(1) = \mu(Ae^{\mu} - Be^{-\mu}) \qquad ...(12)$

Using (12), (3) reduces to $$Ae^{\mu} + Be^{-\mu} + \mu(Ae^{\mu} - Be^{-\mu}) = 0$$

i.e., $$Ae^{\mu}(1 + \mu) + Be^{-\mu}(1 - \mu) = 0 \qquad ...(13)$$

We now use the theory of determinants for solving (11) and (13). For non–trivial solution of these equations, we must have

$$\begin{vmatrix} 1+\mu & 1-\mu \\ e^{\mu}(1+\mu) & e^{-\mu}(1-\mu) \end{vmatrix} = 0 \;\Rightarrow\; (1 + \mu)(1 - \mu)(e^{-\mu} - e^{\mu}) = 0, \quad \text{giving} \quad \mu = -1 \text{ and } \mu = 1.$$

When $\mu = -1$, (11) and (13) give $B = 0$, while A will be arbitrary. So (8) reduces to $y(x) = Ae^{-x}$ and the corresponding eigenvalue is given by $\quad \lambda = -\mu^2 = -(-1)^2 = -1$.

Next when $\mu = 1$, (11) and (13) give $A = 0$, while B will be arbitrary. So (8) reduces to $y(x) = Be^{-x}$ and the corresponding eigenvalue is given by $\lambda = -\mu^2 = -1^2 = -1$.

Taking $A = B = 1$, $y(x) = e^{-x}$ is an eigen function and $\lambda = -1$ is the corresponding eigenvalue.

Case III. Let $\lambda = \mu^2$, where $\mu \neq 0$. Then (1) reduces to $y'' + \mu^2 y = 0$ whose solution is

$$y(x) = A \cos \mu x + B \sin \mu x. \qquad ...(14)$$

From (14), $\quad y'(x) = -A\mu \sin \mu x + B\mu \cos \mu x \qquad ...(15)$

From (14) and (15), $\quad y(0) = A \quad$ and $\quad y'(0) = B\mu. \qquad ...(16)$

Using (16), (2) reduces to $\quad A + B\mu = 0 \qquad ...(17)$

Again, from (14) and (15), we have

$y(1) = A \cos \mu + B \sin \mu \quad$ and $\quad y'(1) = -A\mu \sin \mu + B\mu \cos \mu \qquad ...(18)$

Using (18), (3) reduces to $\quad A \cos \mu + B \sin \mu - A\mu \sin \mu + B\mu \cos \mu = 0. \qquad ...(19)$

From (17), $A = -B\mu$. With this value of A, (19) gives

$$-B\mu \cos \mu + B \sin \mu + B\mu^2 \sin \mu + B\mu \cos \mu = 0$$

or $\quad B(1 + \mu^2) \sin \mu = 0 \quad$ or $\quad B \sin \mu = 0, \quad$ as $\quad (\mu^2 + 1) \neq 0. \qquad ...(20)$

If $B = 0$, then (17) gives $A = 0$. Hence (14) reduces to $y(x) = 0$, which is not an eigen function. So take $B \neq 0$. Then (20) gives

$\sin \mu = 0 \quad$ so that $\quad \mu = n\pi,\ n = 1, 2, 3, ... \qquad ...(21)$

(Here we omit $n = 0$, for $n = 0$ gives $\mu = 0$ so that $\lambda = \mu^2 = 0$ which has been considered in case I)

Using $A = -B\mu = -Bn\pi$ and (21), (14) reduces to

$y(x) = B(\sin n\pi x - n\pi \cos n\pi x)$ $n = 1, 2, 3, ...$ and then $\lambda = \mu^2 = n^2\pi^2,\ n = 1, 2, 3, ...$

Hence the required eigenfunctions $y_n(x)$ with the corresponding eigenvalues λ_n are given by (taking $B = 1$) $\quad y_n(x) = \sin n\pi x - n\pi \cos n\pi x,\ n = 1, 2, 3, ...$ and $\lambda_n = n^2\pi^2,\ n = 1, 2, 3, ...$

Ex. 4. *Find all the eigenvalues and eigenfunctions of* $4(e^{-x}y')' + (1 + \lambda)e^{-x} y = 0$, $y(0) = 0$, $y(1) = 0$.

Sol. Re–writing the given equation, we have

$4(e^{-x}y'' - e^{-x}y') + (1 + \lambda)e^{-x}y = 0 \quad$ or $\quad 4y'' - 4y' + (1 + \lambda)y = 0 \qquad ...(1)$

Also, given $\quad y(0) = 0 \quad$ and $\quad y(1) = 0 \qquad ...(2)$

Case I. Let $\lambda = 0$. Then (1) reduces to

$4y'' - 4y' + y = 0 \quad$ *i.e.,* $\quad (4D^2 - 4D + 1)y = 0, \quad$ where $\quad D = d/dx. \qquad ...(3)$

Here auxiliary equation is $4D^2 - 4D + 1 = 0$ *i.e.,* $(2D - 1)^2 = 0$. This gives $D = 1/2, 1/2$.

Hence solution of (3) is $\quad y(x) = (A + Bx)e^{x/2} \qquad ...(4)$

Using (2), (4) reduces to $\quad 0 = A \quad$ and $\quad 0 = (A + B)e^{1/2}$

These give $A = B = 0$. So (4) reduces to $y(x) = 0$, which is not an eigenfunction.

Case II. Let $\lambda = -\mu^2$ where $\mu \neq 0$. Then (1) reduces to

$4y'' - 4y' + (1 - \mu^2)y = 0 \quad$ *i.e.,* $\quad (4D^2 - 4D + 1 - \mu^2)y = 0. \qquad ...(5)$

Here auxiliary equation is $\quad 4D^2 - 4D + 1 - \mu^2 = 0 \quad$ giving

$$D = [4 \pm \{16 - 16(1 - \mu^2)\}^{1/2}] / 8 = 1/2 \pm \mu/2$$

Hence solution of (5) is $\quad y = Ae^{(1/2 + \mu/2)x} + Be^{(1/2 - \mu/2)x} \qquad ...(6)$

Using (2), (6) reduces to $\quad 0 = A + B, \quad 0 = Ae^{1/2 + \mu/2} + Be^{1/2 - \mu/2}$

These give $A = B = 0$. So (6) reduces to $y(x) = 0$, which is not an eigenfunction.

Case III. Let $\lambda = \mu^2$, where $\mu \neq 0$. Then (1) reduces to

$4y'' - 4y' + (1 + \mu^2)y = 0 \quad$ *i.e.,* $\quad (4D^2 - 4D + 1 + \mu^2)y = 0 \qquad ...(7)$

Here auxiliary equation is $\quad 4D^2 - 4D + 1 + \mu^2 = 0$

giving $$D = [4 \pm \{16 - 16(1+\mu^2)\}^{1/2}]/8 = 1/2 \pm i(\mu/2)$$

Hence solution of (7) is given by $\quad y(x) = e^{x/2}\{A \cos(\mu x/2) + B \sin(\mu x/2)\}$ ...(8)

Using (2), (8) reduces to

$0 = A$ and $0 = e^{1/2}\{\cos(\mu/2) + B \sin(\mu/2)\}$ *i.e.,* $A = 0$ and $B \sin(\mu/2) = 0$ ...(9)

Consider $\quad B \sin(\mu/2) = 0$ ...(10)

If $B = 0$, then with $A = 0$, (8) reduces to $y(x) = 0$, which is not an eigen function. So we take $B \neq 0$ for the existence of eigen function. As $B \neq 0$, (10) gives

$\sin(\mu/2) = 0$ so that $\mu/2 = n\pi$ or $\mu = 2n\pi$, $n = 1, 2, 3, \ldots$ ...(11)

($n = 0$ is not being considered here as $n = 0 \Rightarrow \mu = 0$ which is contrary to our assumption.)

Using $A = 0$ and (11), (8) reduces to

$y(x) = Be^{x/2} \sin n\pi x$ and $\lambda = \mu^2 = 4n^2\pi^2$, $n = 1, 2, 3, \ldots$.

Hence the required eigenfunctions $y_n(x)$ with the corresponding eigenvalues λ_n are given by (taking $B = 1$) $\quad y_n(x) = e^{x/2} \sin n\pi x$, $\quad \lambda_n = 4n^2\pi^2$, $\quad n = 1, 2, 3, \ldots$

Ex. 5. *For the eigen–value problem given below, obtain the set of orthogonal eigenfunctions in the interval (0, 2c) :* $X'' + \lambda X = 0$, $X(0) = X(2c)$, $X'(0) = X'(2c)$.

Sol. Given $\quad X'' + \lambda X = 0$ ...(1)

with boundary conditions $\quad X(0) = X(2c)$ ...(2)

and $\quad X'(0) = X'(2c)$ ...(3)

Case I. Let $\lambda = 0$. Then solution of (1) is $\quad X(x) = Ax + B$ ...(4)

From (4), $\quad X'(x) = A$ ...(5)

From (4), $X(0) = B$ and $X(2c) = 2cA + B$. So (2) reduces to

$B = 2cA + B$ and hence $A = 0$.

Next, from (5), $X'(0) = X'(2c) = A$. So (3) gives $\quad A = A$.

Hence corresponding to the eigen value $\lambda = 0$, the eigenfunction is

$X(x) = B$ or $X(x) = 1$, taking $B = 1$.

Case II. Let $\lambda = -\mu^2$, where $\mu \neq 0$. Then (1) becomes $X'' - \mu^2 X = 0$ whose solution is

$$X(x) = Ae^{\mu x} + Be^{-\mu x} \quad \text{...(6)}$$

From (6), $\quad X'(x) = A\mu e^{\mu x} - B\mu e^{-\mu x}$ ...(7)

From (6), $X(0) = A + B$ and $X'(2c) = Ae^{2\mu c} + Be^{-2\mu c}$. So (2) gives

$A + B = Ae^{2\mu c} + Be^{-2\mu c}$ or $A(1 - e^{2\mu c}) + B(1 - e^{-2\mu c}) = 0$ ...(8)

From (7), $X'(0) = \mu(A - B)$ and $X'(2c) = \mu(Ae^{2\mu c} - Be^{-2\mu c})$

$\therefore$ (3) gives $\mu(A - B) = \mu(Ae^{2\mu c} - Be^{-2\mu c})$ or $A(1 - e^{2\mu c}) - B(1 - e^{-2\mu c}) = 0$ ...(9)

Solving (8) and (9), $A = B = 0$. So (6) reduces to $X(x) = 0$, which is not an eigen function. So there is no eigenfunction corresponding to $\lambda = -\mu^2$.

Case III. Let $\lambda = \mu^2$, where $\mu \neq 0$. Then (1) becomes $X'' + \mu^2 X = 0$ whose solution is

$$X(x) = A \cos \mu x + B \sin \mu x \quad \text{...(10)}$$

From (10), $\quad X'(x) = -A\mu \sin \mu x + B\mu \cos \mu x$ ...(11)

From (10), $\quad X(0) = A$ and $X(2c) = A \cos 2\mu c + B \sin 2\mu c$

$\therefore$ (2) gives $A = A \cos 2\mu c + B \sin 2\mu c$ or $A(1 - \cos 2\mu c) - B \sin 2\mu c = 0$ ...(12)

From (11), $\quad X'(0) = B\mu$ and $X'(2c) = \mu(-A \sin 2\mu c + B \cos 2\mu c)$

$\therefore$ (3) gives $B\mu = \mu(-A \sin 2\mu c + B \cos 2\mu c)$ or $A \sin 2c\mu + B(1 - \cos 2c\mu) = 0$ ...(13)

For non–trival solution of (12) and (13), we must have

$$\begin{vmatrix} 1 - \cos 2\mu c & -\sin 2\mu c \\ \sin 2\mu c & 1 - \cos 2\mu c \end{vmatrix} = 0$$

or $(1 - \cos 2\mu c)^2 + \sin^2 2\mu c = 0$ or $\cos 2\mu c = 1 = \cos 0$

$\therefore$ $2\mu c = 2n\pi$ or $\mu = n\pi/c,\ n = 1, 2, 3, \ldots$ ...(14)

($n = 0$ is omitted here as $n = 0 \Rightarrow \mu = 0$, which is contrary to our assumption $\mu \neq 0$.)

With this value of μ, (10) becomes $X(x) = A \cos (n\pi x/c) + B \sin (n\pi x/c)$ ...(15)

Taking $A = 1$ and $B = 0$ in (15), the eigenfunctions are given by $X(x) = \cos (n\pi x/c)$. Again taking $A = 0$ and $B = 1$ in (15), the eigenfunctions are given by $X(x) = \sin (n\pi x/c)$.

Note that $\int_0^{2c} 1 \cdot \cos \frac{n\pi x}{c}\, dx = 0,\ \int_0^{2c} 1 \cdot \sin \frac{n\pi x}{c}\, dx = 0\ ,\ \int_0^{2c} \sin \frac{n\pi x}{c} \cos \frac{m\pi x}{c}\, dx = 0$, for $m \neq n$ ... (16)

In view of (16) the required eigenfunctions which are orthogonal on $(0, 2c)$ are given by $\{1, \cos (n\pi x/c), \sin (n\pi x/c)\}$, $(n = 1, 2, 3, \ldots)$

Ex. 6. *Find the eigenvalues and eigenfunctions of* $[xy'(x)]' + (\lambda/x)\, y(x) = 0,\ y'(1) = y'(e^{2\pi}) = 0$

[Kanpur 2009; Himanchal 2009]

Sol. Re–writing the given equation, $xy'' + y' + (\lambda/x)y = 0$

or $x^2y'' + xy' + \lambda y = 0$ *i.e.,* $(x^2D^2 + xD + \lambda)y = 0$, where $D \equiv d/dx$ ...(1)

This is a homogeneous differential equation. To solve it, we take

$x = e^z$ so that $z = \log x$ ...(2)

We know that, if $D_1 \equiv d/dz \equiv x(d/dx)$, then $xD = D_1$ and $x^2D^2 = D_1(D_1 - 1)$ ...(3)

Using (3), (1) reduces to $\{D_1(D_1 - 1) + D_1 + \lambda\}y = 0$ or $(D_1^{\,2} + \lambda)y = 0$ ...(4)

Also, given that $y'(1) = 0$ and $y'(e^{2\pi}) = 0$, ...(5)

Case I. Let $\lambda = 0$. Then solution of (4) *i.e.,* $D_1^{\,2}y = 0$ is

$y = Az + B$ or $y = A \log x + B$, using (2) ...(6)

From (6), $y'(x) = A/x$. ...(7)

Using (5), (7) reduces to $0 = A$ and $0 = A/e^{2\pi}$

These give $A = 0$, while B may be taken as arbitrary. With these values and taking $B = 1$, (6) gives $y(x) = 1$ as the eigenfunction corresponding to eigenvalue $\lambda = 0$.

Case II. Let $\lambda = -\mu^2$, where $\mu \neq 0$. Then solution of (4) is

$y = Ae^{\mu z} + Be^{-\mu z} = A(e^z)^\mu + B(e^z)^{-\mu}$ or $y(x) = Ax^\mu + Bx^{-\mu}$, as $x = e^z$ ...(8)

From (8), $y'(x) = A\mu x^{\mu - 1} - B\mu x^{-\mu - 1}$ ...(9)

Using (5), (9) reduces to $0 = A\mu - B\mu$ and $0 = A\mu e^{2\pi(\mu - 1)} - B\mu e^{-2\pi(\mu + 1)}$

i.e., $A - B = 0$ and $Ae^{2\pi\mu} - Be^{-2\pi\mu} = 0$, as $\mu \neq 0$,

giving $A = B = 0$. So (8) reduces to $y(x) = 0$, which is not an eigenfunction.

Case III. Let $\lambda = \mu^2$, where $\mu \neq 0$. Then solution of (4) is $y = A \cos \mu z + B \sin \mu z$,

i.e., $y(x) = A \cos (\mu \log x) + B \sin (\mu \log x)$, by (2) ...(10)

From (10), $y'(x) = -(A\mu/x) \sin (\mu \log x) + (B\mu/x) \cos (\mu \log x)$. ...(11)

Using (5), (11) reduces to (noting that $\log 1 = 0$ and $\log e^{2\pi} = 2\pi$)

$0 = B\mu$ and $0 = -(A\mu/e^{2\pi}) \sin 2\pi\mu + (B\mu/e^{2\pi}) \cos 2\pi\mu$,

i.e., $B = 0$ and $A \sin 2\pi\mu = 0$, as $\mu \neq 0$. ...(12)

Consider $A \sin 2\pi\mu = 0$ ...(13)

If $A = 0$, then with $B = 0$, (10) reduces to $y(x) = 0$, which is not an eigenfunction. So we take $A \neq 0$ for the existence of eigenfunctions. Since $A \neq 0$, (13) gives

$\sin 2\pi\mu = 0$ so that $2\pi\mu = n\pi$, $n = 1, 2, 3, \ldots$

Thus, $\mu = n/2$, $n = 1, 2, 3, ..$ ...(14)

$n = 0$ is not being considered because $n = 0 \Rightarrow \mu = 0 \Rightarrow \lambda = 0$, which has already been considered in case I. Using $B = 0$ and (14), (10) reduces to

$y(x) = A \cos \{(n/2) \times \log x\}$, with $\lambda = \mu^2 = n^2/4$, $n = 1, 2, 3, \ldots.$

So the required eigenfunctions $y_n(x)$ (taking A=1) with the corresponding eigenvalues λ_n are $y_n(x) = \cos\{(n/2) \times \log x\}$, $n = 1, 2, 3, \ldots$, $\lambda_n = n^2/4$, $n = 1, 2, 3, \ldots$; and $y(x) = 1$ with $\lambda = 0$.

Ex. 7. *Find all the eigenvalues and eigenfunctions of the Strum–Liouville problem* $(x^3y')' + \lambda xy = 0,\ y(1) = 0,\ y(e) = 0.$

Sol. Re–writing the given equation $\qquad x^3y'' + 3x^2y' + \lambda xy = 0$

or $\qquad x^2y'' + 3xy' + \lambda y = 0 \qquad$ or $\qquad (x^2D^2 + 3xD + \lambda)y = 0, \qquad D \equiv d/dx \quad$ ...(1)

This is a homogeneous differential equation. To solve it, we take

$x = e^z \qquad$ so that $\qquad z = \log x \qquad$ ...(2)

We know that, if $D_1 \equiv d/dz \equiv x(d/dx)$, then $xD = D_1$ and $x^2D^2 = D_1(D_1 - 1) \quad$ ...(3)

Using (3), (1) reduces to $\{D_1(D_1 - 1) + 3D_1 + \lambda\}y = 0$ or $(D_1^2 + 2D_1 + \lambda)y = 0 \quad$ ...(4)

Also, given that $\qquad y(1) = 0 \qquad$ and $\qquad y(e) = 0 \qquad$ ...(5)

Case I. Let $\lambda = 0$. Then (4) reduces to $D_1(D_1 + 2)y = 0$ whose solution is

$y = Ae^{0.z} + Be^{-2z} \qquad$ or $\qquad y(x) = A + B/x^2 \qquad$ ...(6)

Putting $x = 1$ in (6) and using (5), $\qquad A + B = 0 \qquad$ ...(7)

Putting $x = e$ in (6) and using (5), $\qquad A + B/e^2 = 0 \qquad$ ...(8)

Solving (7) and (8), we get $A = B = 0$ and so (6) gives $y(x) = 0$, which is not an eigenfunction.

Case II. Let $\lambda = -\mu^2$, where $\mu \neq 0$. Then (4) reduces to $(D_1^2 + 2D_1 - \mu^2)y = 0 \quad$ ...(9)

whose auxiliary equation is $\qquad D_1^2 + 2D_1 - \mu^2 = 0$

Solving it, $\qquad D_1 = \{-2 \pm (4 + 4\mu^2)^{1/2}\}/2 = -1 \pm (1 + \mu^2)^{1/2}$

or $\qquad D_1 = -1 \pm k, \qquad$ where $\qquad k = (1 + \mu^2)^{1/2} \qquad$ ...(10)

Hence solution of (9) is $\qquad y(x) = Ae^{(-1+k)z} + Be^{(-1-k)z}$

or $\qquad y(x) = Ax^{(-1+k)} + Bx^{(-1-k)} \qquad$ or $\qquad y(x) = x^{-1}(Ax^k + B/x^k) \qquad$ ...(11)

Putting $x = 1$ in (11) and using (5), $\qquad A + B = 0 \qquad$ ...(12)

Putting $x = e$ in (11) and using (5), $\qquad Ae^{k-1} + Be^{-k-1} = 0 \qquad$ ...(13)

Solving (12) and (13), we get $A = B = 0$. Hence (11) gives $y(x) = 0$, which is not an eigenfunction.

Case III. Let $\lambda = \mu^2$, where $\mu \neq 0$. Then (4) reduces to $(D_1^2 + 2D_1 + \mu^2)y = 0 \quad$ ...(14)

whose auxiliary equation is $\qquad D_1^2 + 2D_1 + \mu^2 = 0 \qquad$ ...(15)

Solving it, $\qquad D_1 = \{-2 \pm (4 - 4\mu^2)^{1/2}\}/2 = -1 \pm i(\mu^2 - 1)^{1/2}$

or $\qquad D_1 = -1 \pm ip, \qquad$ where $\qquad p = (\mu^2 - 1)^{1/2} \qquad$ and $\qquad \mu^2 > 1 \qquad$ ...(16)

Hence solution of (14) is $\qquad y(x) = e^{-z}(A\cos pz + B\sin pz)$

i.e., $\qquad y(x) = (1/x) \times \{A\cos(p\log x) + B\sin(p\log x)\} \qquad$ ...(17)

Putting $x = 1$ in (17) and using (5), we get $\qquad A = 0.$

Putting $x = e$ and $A = 0$ in (17) and using (5), we get $(B/e)\sin(p) = 0$. But we must take $B \neq 0$ for non–trivial solution (and hence for getting eigenfunction).

$\therefore \sin p = 0 \qquad$ so that $\qquad p = n\pi \qquad$ or $\qquad (\mu^2 - 1)^{1/2} = n\pi$, by (16)

or $\qquad \mu^2 - 1 = n^2\pi^2 \qquad$ or $\qquad \mu^2 = 1 + n^2\pi^2 \qquad$ or $\qquad \lambda = 1 + n^2\pi^2$

Hence the required eigenfunctions are $y_n(x) = (1/x) \times \sin(n\pi\log x)$ (taking $B = 1$) and the corresponding eigenvalues are $\lambda_n = n^2\pi^2$, where $n = 1, 2, 3, \ldots$

Ex. 8. *Find the eigenvalues and the corresponding eigenfunctions of the boundary value problem :* $y'' + 2y' + (1 + \lambda)y = 0,\ y(0) = 0,\ y'(a) = 0$

Sol. Given $\qquad (D^2 + 2D + 1 + \lambda)y = 0, \qquad D \equiv d/dx \qquad$ ...(1)

with $\qquad y(0) = 0, \qquad y'(a) = 0 \qquad$ ...(2)

Case I. Let $\lambda = 0$. Then solution of (1) is $\qquad y(x) = (A + Bx)e^{-x} \qquad$ ...(3)

From (3), $\qquad y'(x) = Be^{-x} - (A + Bx)e^{-x} \qquad$ ...(4)

Using (2), (3) and (4) reduce to $\qquad 0 = A \qquad$ and $\qquad Be^{-a} - (A + Ba)e^{-a} = 0,$

i.e., $A = 0$ and $Be^{-a}(1 - a) = 0$. These give $A = B = 0$.

So (3) reduces to $y(x) = 0$, which is not an eigenfunction.

Case II. Let $\lambda = -\mu^2$, where $\mu \neq 0$. Then (1) reduces to

$$[D^2 + 2D + 1 - \mu^2]y = 0 \qquad ...(5)$$

whose auxiliary equation is $D^2 + 2D + (1 - \mu^2) = 0$.

This gives $D = [-2 \pm \{4 - 4(1 - \mu^2)\}^{1/2}]/2 = -1 \pm \mu$

Hence solution of (5) is $$y(x) = Ae^{(-1+\mu)x} + Be^{-(1+\mu)x} \qquad ...(6)$$

so that $$y'(x) = A(-1 + \mu)\, e^{(-1+\mu)x} - B(1 + \mu)e^{-(1+\mu)x} \qquad ...(7)$$

Using (2), (6) and (7) reduce to

$0 = A + B$ and $0 = A(-1 + \mu)\, e^{(-1+\mu)a} - B(1 + \mu)e^{-(1+\mu)a}$,

i.e., $A + B = 0$ and $A(\mu - 1)e^{\mu a} - B(\mu + 1)e^{-\mu a} = 0$.

These give $A = B = 0$. So (6) reduces to $y(x) = 0$, which is not an eigenfunction.

Case III. Let $\lambda = \mu^2$, where $\mu \neq 0$. Then (1) reduces to

$$(D^2 + 2D + 1 + \mu^2)y = 0 \qquad ...(8)$$

whose auxiliary equation is $D^2 + 2D + 1 + \mu^2 = 0$.

This gives $D = [-2 \pm \{4 - 4(1 + \mu^2)\}^{1/2}]/2 = -1 \pm i\mu$.

Hence solution of (8) is $$y(x) = e^{-x}\,(A \cos \mu x + B \sin \mu x) \qquad ...(9)$$

From (9), $$y'(x) = -e^{-x}\,(A \cos \mu x + B \sin \mu x) + e^{-x}\,(-A\mu \sin \mu x + B\mu \cos \mu x) \qquad ...(10)$$

Using (2), (9) and (10) reduce to

$0 = A$, $\quad 0 = -e^{-a}\,(A \cos \mu a + B \sin \mu a) + \mu e^{-a}\,(-A \sin \mu a + B \cos \mu a)$

$\therefore\quad A = 0$ and $B(\mu \cos \mu a - \sin \mu a) = 0 \qquad ...(11)$

Consider $$B(\mu \cos \mu a - \sin \mu a) = 0 \qquad ...(12)$$

If $B = 0$, then with $A = 0$, (9) reduces to $y(x) = 0$, which is not an eigenfunction. So we take $B \neq 0$ for the existence of eigenfunctions. Since $B \neq 0$, (12) reduces to

$$\mu \cos \mu a - \sin \mu a = 0 \quad \text{or} \quad \tan \mu a = \mu, \qquad ...(13)$$

which is a trigonometrical equation in μ. Let μ_n ($n = 1, 2, 3, ...$) be positive roots of (13). With $A = 0$, (9) reduces to $y(x) = Be^{-x} \sin \mu x$

Hence the required eigenfunctions $y_n(x)$ with the corresponding eigenvalues λ_n are given by (taking $B = 1$) $\quad y_n(x) = e^{-x} \sin \mu_n x \quad$ and $\quad \lambda_n = \mu_n^2, \quad n = 1, 2, 3, ...,$ where μ_n ($n = 1, 2, 3, ...$) are positive roots of (13).

EXERCISE 15(B)

1. For the Strum–Liouville problem $X'' + \lambda X = 0$, $X(0) = 0$, $X(\pi) = 0$, obtain the eigenfunctions and the corresponding eigenvalues. **[Nagpur 2005; Bilaspur 2004; Bhopal 2004; Meerut 2005, 11; Ravishaker 2004; Vikram 2004; Lucknow]** **Ans.** $X_n(x) = \sin nx$, $\lambda_n = n^2$, $n = 1, 2, 3, ...$

2. Find all eigenvalues and eigenfunctions of the Strum–Liouville problem $X'' + \lambda X = 0$, $X(0) = 0$, $X'(\pi/2) = 0$. **[Jabalpur 2004]** **Ans.** $X_n(x) = \sin (2n - 1)x$, $\lambda_n = (2n - 1)^2$, $n = 1, 2, 3, ...$

3. Find all eigenvalues and eigenfunctions of the problem $X'' + \lambda X = 0$, $X'(-\pi) = 0$, $X'(\pi) = 0$

Ans. $X_n = \cos[(n/2) \times (\pi + x)]$, $n = 0, 1, 2, ...$, $\lambda_n = n^2/4$, $n = 1, 2, 3, ...$

4. Find the eigenvalues and the corresponding eigenfunctions of the boundary value problem $y'' + \lambda y = 0$, $y(0) + \pi\, y'(0) = 0$, $y(\pi) = 0$.

Ans. $y_n(x) = \sin \mu_n x - \pi\mu_n \cos \mu_n x$, $\lambda_n = \mu_n^2$, $n = 1, 2, 3, ...$ where μ_n are +ve roots of $\tan \mu\pi = \mu\pi$; $y(x) = x - \pi$ is eigenfunction corresponding to eigenvalue $\lambda = 0$.

5. Find all the eigenvalues and eigenfunctions of Strum-Liouville problem : $y'' + \lambda^2 y = 0$, $y'(0) = y'(l) = 0$, $0 \leq x \leq l$ **[Nagpur 2005]**

Hint. Refer case III of Ex. 2 Art 15.13. Here take $X = y$, $\mu = \lambda$ and L = l. Then eigenvalues $= \lambda_n = n\pi / l$ and the corresponding eigenfunctions $= y_n = \cos(n\pi x / l), n = 1, 2, 3, ...$

6. Find the eigenvalues and eigenfunctions of the Strum-Liouville problem $y'' + \lambda y = 0, y'(0) = 0, y'(\pi) = 0.$ **[CDVU 2004; M.D.U. Rohtak 2005]**

[Hint : This is a particular case of Ex. 2 of Art 15.13. Here $X = y$ and $L = \pi$.

Ans. Eigenfunctions: $y_n(x) = \cos nx$ and eigenvalues $\lambda_n = n^2$, where n = 0, 1, 2, ...]

7. Find the eigenvalues and the corresponding eigenfunctions of the eigen value problem $X'' + \lambda X = 0$, $X(0) = 0$, $X(1) - X'(1) = 0$.

Ans. $X_n = \sin \mu_n x$, $\lambda_n = \mu_n^2$, $n = 1, 2, 3, ...$ where μ_n are positive roots of $\tan \mu = \mu$. Again $X(x) = x$ is eigen function corresponding to the eigen value $\lambda = 0$.

8. Find the eigenvalues and eigenfunctions of the Strum Liouville problem $y'' + \lambda y = 0$, $y(0) = y(\pi) = 0$. **Ans.** $\lambda_n = n^2$, $y_n = \sin n\pi$, $n = 1, 2, ..., ...$

9. Solve the Strum–Liouville problem $(xX')' + \lambda(1/x)X = 0$, $X'(1) = 0$, $X(b) = 0$ $(b > 1)$ and normalize the eigen functions. **Ans.** $\lambda_n = \dfrac{(2n-1)\pi}{2 \log b}$, $X_n = \cos\left\{\dfrac{(2n-1)\pi \log x}{2 \log b}\right\}$, $n = 1, 2, 3, ...$

10. Find the eigenvalues and eigenfunctions of $y'' - 4\lambda y' + 4\lambda^2 y = 0, y'(0) = 0, y(2) + 2y'(2) = 0.$

[Purvanchal 2006]

15.14. Orthogonality of Legendre polynomials

We know that Legendre's differential equation is $(1 - x^2)y'' - 2xy' + n(n + 1)y = 0$ which can be re–written in the form $[(1 - x^2)y']' + \lambda y = 0$, where $\lambda = n(n + 1)$ ...(1)
and is therefore in the form of Strum–Liouville equation $[r(x)\, y']' + [q(x) + \lambda\, p(x)]y = 0$ ...(2)

Comparing (1) with (2), here $r(x) = 1 - x^2$, $q(x) = 0$ and $p(x) = 1$.

Since $r(-1) = r(1) = 0$, we need no boundary conditions to form a Strum–Liouville problem. Further, we know that $P_n(x)$ for $n = 0, 1, 2, 3, ...$ are solutions of (1) and so they are eigenfunctions. Again they have continuous derivatives and hence it follows that Legendre polynomials $P_n(x)$, $n = 0, 1, 2, ...$ are orthogonal on the interval $-1 \le x \le 1$ with respect to the weight function $p(x) = 1$,

i.e., $$\int_{-1}^{1} P_m(x)\, P_n(x)\, dx = 0, \text{ when } \quad m \ne n$$

Remark. For alternative proof, refer Art. 9.8 of chapter 9.

15.15. Orthogonality of Bessel functions

The Bessel functions $J_n(u)$ with fixed integer $n \ge 0$ satisfies Bessel's equation (refer equation (1) of Art. 11.1 of chapter 11, by taking u as independent variable in place of x)

$$u^2(d^2y/du^2) + u(dy/du) + (u^2 - n^2)y = 0$$

$$\therefore \quad u^2 \frac{d^2 J_n(u)}{du^2} + u \frac{d\, J_n(u)}{du} + (u^2 - n^2) J_n(u) = 0 \quad ...(1)$$

Let $u = \lambda x$, where λ is a constant. ...(2)

Then, $$\frac{d\, J_n(u)}{du} = \frac{d\, J_n(\lambda x)}{dx} \cdot \frac{dx}{du} = \frac{1}{\lambda} J_n'(\lambda x) \text{, using (2)} \quad ...(3)$$

and $$\frac{d^2 J_n(u)}{du^2} = \frac{d}{du}\left(\frac{d\, J_n(u)}{du}\right) = \frac{d}{du}\left(\frac{J_n'(\lambda x)}{\lambda}\right) \text{, by (3)}$$

$$= \frac{d}{dx}\left(\frac{J_n'(\lambda x)}{\lambda}\right)\cdot\frac{dx}{du} = \frac{1}{\lambda^2}J_n''(\lambda x)\text{, by (2)} \qquad ...(4)$$

Using (2), (3) and (4), (1) reduces to

$$x^2\ J_n''\ (\lambda x) + x\ J_n'(\lambda x) + (\lambda^2 x^2 - n^2)\ J_n(\lambda x) = 0$$

which can be re–written in the form $\qquad [x\ J_n'(\lambda x)]' + (-n^2/x + \lambda^2 x)\ J_n(\lambda x) = 0 \qquad ...(5)$

and is therefore in the form of Strum–Liouville equation for each fixed n

$$[r(x)\ y']' + [q(x) + \lambda\ p(x)]y = 0 \qquad ...(6)$$

Comparing (5) with (6), here $r(x) = x$, $q(x) = -n^2/x$ and $p(x) = x$ and the parameter is λ^2 in place of λ.

Since $r(x) = 0$ for $x = 0$, it follows from Art. 15.11 that solutions of (5) on an interval $0 \le x \le a$ satisfying the boundary condition $\qquad J_n(\lambda a) = 0, \quad (n \text{ fixed}) \qquad ...(7)$

form an orthogonal set with respect to the weight function $p(x) = x$.

We know that $J_n(u)$ has infinitely many real zeros, say, $u = \alpha_1 < \alpha_2 < ....$ Hence (7) gives

$\lambda a = \alpha_i \qquad$ and $\qquad$ so $\qquad \lambda = \lambda_i = \alpha_i/a, \quad i = 1, 2, 3, ... \qquad ...(8)$

Further $J_n'(\lambda x)$ is continuous at $x = 0$. Hence for each fixed non–negative integer n the sequence of Bessel functions of the first kind $J_n(\lambda_1 x)$, $J_n(\lambda_2 x)$, $J_n(\lambda_3 x)$, ..., with λ_i as in (8), forms an orthogonal set on the interval $0 \le x \le a$ with respect to the weight function $p(x) = x$, that is

$$\int_0^a x\ J_n(\lambda_i x)\ J_n(\lambda_j x)\,dx = 0\text{, for } i \ne j.$$

Remark. From the above discussion, it follows that we obtain infinitely many orthogonal sets, each corresponding to one of the fixed values of n.

15.16. Orthogonality on an infinite interval

(*i*) Orthogonality of Hermite Polynomial $H_n(x)$. *Hermite polynomials are orthogonal over $]-\infty, \infty[$ with respect to the weight function* $e^{-x^2/2}$, that is,

$$\int_{-\infty}^{\infty} e^{-x^2} H_m(x)\ H_n(x)\,dx = 0, \quad \text{for} \quad m \ne n.$$

For proof, refer Art. 12.7 of chapter 12.

(*ii*) Orthogonality of Laguerre's polynomial $L_n(x)$. *Laguerre polynomials* are orthogonal over $[0, \infty[$ with respect to the weight function e^{-x}, that is,

$$\int_0^{\infty} e^{-x} L_m(x)\ L_n(x)\,dx = 0, \quad \text{for} \quad m \ne n.$$

For proof, refer Art. 13.6 of chapter 13.

15.17. Orthogonal expansion or genralized Fourier series

Let $\{y_n(x)\}$, $n = 0, 1, 2, 3, ...$ be an orthogonal set of functions with respect to weight function $p(x)$ on the interval $a \le x \le b$ and $f(x)$ a function that can be represented by a convergent series

$$f(x) = C_0 y_0(x) + C_1 y_1(x) + C_2 y_2(x) + ... = \sum_{n=0}^{\infty} C_n\ y_n(x) \qquad ...(1)$$

This is called an *orthogonal expansion or generalised Fourier series.* Coefficients C_0, C_1, C_2, C_3, ... are called *Fourier Constants.* Since $\{y_n(x)\}$ is an orthogonal set with respect to weight function $p(x)$, we get

$$\int_a^b p(x)\, y_n(x)\, y_m(x)\, dx = 0, \text{ when } m \neq n. \qquad ...(2)$$

Again, $\quad \| y_n(x) \| = \text{Norm of } y_n(x) = \int_a^b p(x)\, y_n^2(x)\, dx \qquad ...(3)$

Multiply both sides of (1) by $p(x)\, y_n(x)$ (n fixed). Then integrating over $a \le x \le b$ and assuming that term–by–term integration is permissible, we get

$$\int_a^b p(x)\, f(x)\, y_n(x)\, dx = C_0 \int_a^b p(x)\, y_0(x)\, y_n(x)\, dx + C_1 \int_a^b p(x)\, y_1(x)\, y_n(x)\, dx$$

$$+ \dots C_{n-1} \int_a^b p(x)\, y_{n-1}(x)\, y_n(x) + C_n \int_a^b p(x)\, y_n^2(x)\, dx + C_{n+1} \int_a^b p(x)\, y_{n+1}(x) y_n(x)\, dx + \dots$$

or $\quad \int_a^b p(x)\, f(x)\, y_n(x)\, dx = C_n \| y_n(x) \|^2$, using (2) and (3)

or $\quad C_n = \dfrac{1}{\| y_n(x) \|^2} \int_a^b p(x)\, f(x)\, y_n(x)\, dx, \quad n = 0, 1, 2, 3, \dots \qquad ...(4)$

We now discuss two particular cases of the generalised Fourier series.

Particular Case I. Fourier–Legendre Series. We know that the set $\{P_n(x)\}$, $n = 0, 1, 2, 3, \dots$ of Legendre polynomials is an orthogonal set on the interval $-1 \le x \le 1$. Hence, we get

$$f(x) = \sum_{n=0}^{\infty} C_n P_n(x), \qquad -1 \le x \le 1 \qquad ...(i)$$

where $\quad C_n = \dfrac{1}{\| P_n(x) \|^2} \int_{-1}^{1} f(x)\, P_n(x)\, dx, \qquad n = 0, 1, 2, \dots \qquad ...(ii)$

But, $\quad \| P_n(x) \|^2 = \int_{-1}^{1} P_n^2(x)\, dx = \dfrac{2}{2n+1}$, by Art 9.8, chapter 9.

So (ii) becomes $\quad C_n = \left(n + \dfrac{1}{2}\right) \int_{-1}^{1} f(x)\, P_n(x)\, dx \qquad ...(iii)$

Expansion (i) of $f(x)$ in a series of Legendre's polynomials is known as *Fourier–Legendre series*. The constants C_n are given by (iii).

Note : For altenative proof refer, Art. 9.15, of chapter 9.

Particular Case II : Fourier–Bessel series. From Art. 15.15, we know that for a fixed n, the set of Bessel functions of first kind $J_n(\lambda_1 x)$, $J_n(\lambda_2 x)$, $J_n(\lambda_3 x)$, ..., with λ_i given by

$$\lambda_i = \alpha_i / a, \quad \text{where} \quad i = 1, 2, 3, \dots, \quad (n \text{ fixed}) \qquad ...(i)$$

form an orthogonal set on the internal $0 \le x \le a$ with respect to the weight function $p(x) = x$.

$\therefore \quad f(x) = \sum_{i=1}^{\infty} C_i J_n(\lambda_i x) \qquad ...(ii)$

where $\quad C_i = \dfrac{1}{\| J_n(\lambda_i x) \|^2} \int_0^a x\, f(x)\, J_n(\lambda_i x)\, dx, \quad i = 1, 2, 3, \dots \qquad ...(iii)$

But, $\quad \| J_n(\lambda_i x) \|^2 = \int_0^a x\, J_n^2(\lambda_i x)\, dx = \dfrac{a^2}{2} J_{n+1}^2(a\lambda_i)$, by Art. 11.10

$\therefore$ (*iii*) becomes, $$C_i = \frac{2}{a^2 J_{n+1}^2(a\lambda_i)} \int_0^a x\, f(x)\, J_n(\lambda_i x)\, dx\,,\ n = 1, 2, 3, \ldots \qquad \ldots(iv)$$

Expansion (*ii*) of $f(x)$ in a series of Bessel functions is known as *Fourier–Bessel series.* The constants C_i are given by (*iv*).

Note. For alternative proof refer, Art. 11.11, of chapter 11.

Ex. 1. *Obtain the formal expansion of the function* $f(x) = \pi x - x^2,\ 0 \le x \le \pi$, *in the series of orthonormal characteristic functions* $\{\phi_n\}$ *of Strun–Liouville problem* $y'' + \lambda y = 0,\ y(0) = y(\pi) = 0$.

Sol. Proceed as usual to show that $\phi_n(x) = (2/\pi)^{1/2} \sin nx$. Then the orthogonal expansion or generalised Fourier series of $f(x)$ is given by (refer Art. 15.17)

$$f(x) = \sum_{n=1}^{\infty} C_n\, \phi_n(x)\,, \qquad \text{where} \qquad C_n = \int_0^{\pi} f(x)\, \phi_n(x)\, dx\,, \qquad \ldots(1)$$

noting that $\| \phi_n(x) \| = 1,\quad p(x) = 1,\quad a = 0,\quad b = \pi.$

Now, $$C_n = \int_0^{\pi} (\pi x - x^2)(2/\pi)^{1/2} \sin nx\, dx$$

$$= \left(\frac{2}{\pi}\right)^{1/2} \left[(\pi x - x^2)\left(-\frac{1}{n}\cos nx\right) - (\pi - 2x)\left(-\frac{1}{n^2}\sin nx\right) + (-2)\left(\frac{1}{n^3}\cos nx\right)\right]_0^{\pi}$$

(By chain rule of integration by parts)

$$= \left(\frac{2}{\pi}\right)^{1/2} \cdot \left(\frac{2}{n^3}\right)(1 - \cos n\pi) = \begin{cases} (2/\pi)^{1/2} \times (4/n^3), & \text{if } n \text{ is odd} \\ 0, & \text{if } n \text{ is even} \end{cases}$$

Hence the required expansion of $f(x)$ is given by

$$\pi x - x^2 = \sum_{n=1}^{\infty} \left(\frac{2}{\pi}\right)^{1/2} \cdot \frac{4}{(2n-1)^3} \cdot \left(\frac{2}{\pi}\right)^{1/2} \sin(2n-1)x, \qquad i.e., \qquad \pi x - x^2 = \frac{8}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)x}{(2n-1)^3}\,,\ 0 \le x \le \pi$$

Ex. 2. *Obtain the formal expansion of* $f(x) = \log x,\ 1 \le x \le e^{2\pi}$ *in a seies of orthogonal eigen functions of Strum–Liouville problem* $[xy']' + (1/x)y = 0,\ y(0) = y(e^{2\pi}) = 0$.

Sol. Do as in Ex. 1. **Ans.** $\log x = -\dfrac{8}{\pi} \displaystyle\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \cos\left\{\frac{2n-1}{2} \log |x|\right\}$

OBJECTIVE PROBLEMS ON CHAPTER 15

Write (a), (b), (c) or (d) whichever is correct

1. For the Strum-Liouville problem $(1+x^2)y'' + 2xy' + \lambda x^2 y = 0$ with $y'(1) = 0$ and $y'(10) = 0$, the eigenvalues, λ, satisfy

(*a*) $\lambda \ge 0$ (*b*) $\lambda < 0$ (*c*) $\lambda \ne 0$ (*d*) $\lambda \le 0$ **[GATE 2003]**

2. Let n be non-negative integer. The eigenvalues of the Strum-Liouville problem $y'' + \lambda y = 0$ with boundary conditions $y(0) = y(2\pi),\ y'(0) = y'(2\pi)$ are

(*a*) n (*b*) $n^2\pi^2$ (*c*) $n\pi$ (*d*) n^2 **[GATE 2002]**

3.The eigenvalues of the Strum-Liouville problem $y'' + \lambda y = 0,\ \ 0 \le x \le \pi,\ y(0) = 0,\ y'(\pi) = 0$ are (*a*) $n^2/4$ (*b*) $(2n-1)^2 \pi^2 / 4$ (*c*) $(2n-1)^2 / 4$ (*d*) $n^2 \pi^2 / 4$ **[GATE 2001]**

4. The eigenvalues of the boundary value problem $x'' + \lambda x = 0,\ x(0) = 0,\ x(\pi) + x'(\pi) = 0$ satisfy (*a*) $\lambda + \tan \lambda\pi = 0$ (*b*) $\sqrt{\lambda} + \tan \lambda\pi = 0$

(*c*) $\sqrt{\lambda} + \tan \sqrt{\lambda}\pi = 0$ (*d*) $\lambda + \tan \sqrt{\lambda}\pi = 0.$ **[GATE 2000]**

ANSWERS

1. (*a*) 2. (*d*) 3. (*c*) 4. (*b*)

MISCELLANEOUS PROBLEM ON CHAPTER 15

1. Show that the set {1, cos $(2n\pi/T)x$, n = 1, 2, 3, ... in orthogonal set of functions on an interval $0 \le x \le T$. **[Lucknow 2010]**

2. Find eigenfunctions of the system $u'' + \lambda u = 0$, $-\pi \le x \le \pi$ with the boundary condition $u(-\pi) = u(\pi)$, $u'(-\pi) = u'(\pi)$.

Ans. $\{1, \cos nx, \sin nx\}$, n = 1, 2, 3 .... **[Nagpur 2010]**

3. Prove that the set of eigenfunctions of strum-Liouville problem $\dfrac{d}{dx}\left\{p(x)\dfrac{dy}{dx}\right\} + \lambda q\,(x) = 0,$ $y(0) = 0$, $y(1) = 0$, $q(x) > 0$, form a set of orthogonal functions **[Himanchal 2008]**

4. Define norm of a function. **[Meerut 2011]**

Hints. Refer Art. 15.1 and 15.4.

MISCELLANEOUS PROBLEMS BASED ON THIS PART OF THE BOOK

Ex. 1. *The value of* $P_n(-1)$ *is*

(a) 1 *(b)* 0 *(c)* -1 *(d)* $(-1)^n$ **[Agra 2008]**

Sol. Ans. (*d*). Refer part (*ii*) of Ex. 2, page 9.6

Ex. 2. *The value of* $P_0(x)$ *is*

(a) 0 *(b)* ∞ *(c)* 1 *(d) None of these* **[Agra 2007]**

Sol. Ans. (*c*). Refer Art. 9.2

Ex. 3. *Rodrigue's formula for* $P_n(x)$ *is*

(a) $P_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}(x^2-1)^n$ *(b)* $P_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}(x^2-1)^{-n}$ *(c)* $P_n(x) = \dfrac{1}{2^n n!}\dfrac{d^n}{dx^n}(x^2+1)^n$

(d) None of these **[Agra 2007]**

Sol. Ans. (*a*). Refer Art-9.14

Ex. 4. *Statement for the following linked questions 14 (i) and 14 (ii).*

Let $n \geq 3$ *be an integer. Let y be the polynomial solution of*

$$(1-x^2)\,y'' - 2xy' + n(n-1)\,y = 0 \qquad \text{satisfying} \qquad y(1) = 1.$$

4. (*i*): *Then the degree of y is*

(a) n *(b)* $n-1$ *(c) less than* $n-1$ *(d) greater than* $n+1$

4. (*ii*): *If* $I = \int_{-1}^{1} y(x)x^{n-3}dx$ *and* $J = \int_{-1}^{1} y(x)x^n dx,$ *then*

(*a*) $I \neq 0,\ J \neq 0$ (*b*) $I \neq 0, J = 0$ (*c*) $I = 0, J \neq 0$ (*d*) $I = 0, J = 0$ **[GATE 2008]**

Sol: 4. (*i*): Ans. (*a*). Refer Ex. 2 (*i*), page 9.6

4. (*ii*): Ans. (*c*). Refer Ex. 5, page 9.29

5. *Using Rodrigue's formula, find value of* $P_4(x)$ *at* $x = 1$. **[Kanpur 2008]**

[Hint. Do as in Ex. 1, page 9.37 to get $P_4(x) = (35x^4 - 30x^2 + 3)/8$ and so the value of $P_4(x)\, at\, x = 1$ is (35 – 30 + 3)/8, i.e. 1.

6. Show that the condition of integrability of (*i*) $Pdx + Qdy + Rdz = 0$ implies the orthoganality of any pair of intersecting curves of the families (*ii*) $(dx)/P = (dy)/Q = (dz)/R$ and (*iii*) $\dfrac{dx}{\partial Q/\partial z - \partial R/\partial y} = \dfrac{dy}{\partial R/\partial x - \partial P/\partial z} = \dfrac{dz}{\partial P/\partial y - \partial Q/\partial x}$. Hence show that the curves of (*iii*) all lie on the surfaces of (*i*). Verify this conclusion for $P = ny - mz,\ Q = lz - nx,\ R = mx - ly$. **[I.A.S. 2001]**

7. If f_1, f_2, f_3 are homogeneous functions of the same degree in x, y and z and if $xf_1 + yf_2 + zf_3 = 0$, then show that the equation $f_1dx + f_2dy + f_3dz = 0$ is integrable.

[Himanchal 2000; Kerla 2001, 07, Pune 2010]

8. If the equation $x_1\,dx_1 + x_2\,dx_2 + \ldots.. + x_n\,dx_n = 0$ has integrating factor, show that it has infinitely many. If $n = 2$, prove that the equation always has an integrating factor.

[Himanchal 2002; I.A.S. 2002; Calicut 2004, 05; Osmania 2000, 06; U.P.(P.C.S.) 2003]

9. $xdx + ydy + zdz = 0$ *is the first order differential equation of*

(a) sphere *(b) ellipsoid* *(c) right circular cone* *(d) hyperboloid.* **[I.A.S. (Prel) 2001]**

Sol. Ans. (*a*). Integrating the given differential equation, $x^2/2 + y^2/2 + z^2/2 = a^2/2$, where $a^2/2$ is an arbitrary constant of integration. Thus, we get $x^2 + y^2 + z^2 = a^2$, which is a sphere.

10. Show that ${}_1F_1\,(a\,;\,b;\,z) = \dfrac{1}{\Gamma(a)} \displaystyle\int_0^{\infty} e^{-t} t^{a-1}\ {}_0F_1(-;\,b;\,z\,t)\ dt$ **[Kanpur 2009]**

Sol. Using Art. 14.3 for the value of ${}_0F_1$ $(-;\, b;\, zt)$, we get

$$\text{R.H.S.} = \frac{1}{\Gamma(a)}\int_0^{\infty} e^{-t}\ t^{a-1}\left(\sum_{r=0}^{\infty}\frac{(zt)^r}{(b)_r\, r!}\right) dt$$

$$= \frac{1}{\Gamma(a)}\sum_{r=0}^{\infty}\frac{z^r}{(b)_r\, r!}\int_0^{\infty} e^{-t}\ t^{a+r-1}\ dt = \frac{1}{\Gamma(a)}\sum_{r=0}^{\infty}\frac{z^r}{(b)_r\, r!}\,\Gamma\,(a+r)$$

[Using definition of Gamma function 6.2 page 6.1]

$$= \sum_{r=0}^{\infty}\frac{(a)_r}{(b)_r}\frac{z^r}{r!} = F\ \ (a;\, b;\, z), \text{ using art-14.3}$$

11. Apply the method of Frobenius to the equation $xy'' + 2y' + xy = 0$ to derive its general solution $y = c_0$ $\{(\cos x)/x\}$ + c_1 $\{(\sin x)/x\}$ **[Delhi B.Sc. (Hons.) II 2011, Nagpur 1996]**

Sol. Given $$xy'' + 2y' + xy = 0 \qquad \text{... (1)}$$

Let the series solution of (1) be $$y = \sum_{m=0}^{\infty} c_m\, x^{kem}, \text{ where } c_0 \neq 0 \qquad \text{... (2)}$$

From (2), $$y' = \sum_{m=0}^{\infty}(k+m)c_m\ x^{k+m-1} \text{ and } y'' = \sum_{m=0}^{\infty} c_m\ (k+m)(k+m-1)x^{k+m-2} \qquad \text{... (3)}$$

Substituting the values of y, y' and y'' given by (2) and (3) in (1), we get

$$x\sum_{m=0}^{\infty} c_m\ (k+m)(k+m-1)x^{k+m-2} + 2\sum_{m=0}^{\infty} c_m\ (k+m)\ x^{k+m-1} + x\sum_{m=0}^{\infty} c_m\ x^{k+m} = 0$$

or $$\sum_{m=0}^{\infty} c_m\ (k+m)(k+m-1)x^{k+m-1} + 2\sum_{m=0}^{\infty} c_m\ (k+m)\ x^{k+m-1} + \sum_{m=0}^{\infty} c_m\ x^{k+m+1} = 0$$

or $$\sum_{m=0}^{\infty} c_m\ (k+m)\{(k+m-1)+2\}x^{k+m-1} + \sum_{m=0}^{\infty} c_m\ x^{k+m+1} = 0$$

or $$\sum_{m=0}^{\infty} c_m\ (k+m)\ (k+m+1)\ x^{k+m-1} + \sum_{m=0}^{\infty} c_m\ x^{k+m+1} = 0 \qquad \text{... (4)}$$

Equating to zero the hcoefficient of the smallest power of x, namely x^{k-1}, the above identity (4) yields the indicial equation

$c_0 k\ (k + 1) = 0$ or $k\ (k + 1) = 0$, as $c_0 \neq 0$ so that $k = 0$ or $k = -\ 1y$,

which are uequal and differ by an integer. Next, equations the coefficient of x^{k+m-1}, we arrive at the recurrence relation.

$$c_m(k+m)\ (k+m+1) + c_{m-2} = 0 \quad \text{so that } c_m = -\{1/(k+m)(k+m+1)\}c_{m-2} \qquad \text{... (5)}$$

Finally, we equate to zero the coefficient of x^k in the identity (4) and get

$$c_1(k+1)(k+2)=0. \qquad \text{... (6)}$$

If we take $k = -1$, (6) shows that c_1 is ideterminate. With $k = -1$ and using (5), we now proceed to express c_2, c_4, c_6 in terms of c_0 and $c_3, c_5, c_7, \ldots$ in terms of c_1 if c_1 is assume to be finite. Setting $k = -1$ in (5), we have

$$c_m = -\{1/m(m-1)\}\, c_{m-2} \qquad \text{... (7)}$$

Putting $m = 3, 5, 7, \ldots$ in (7) by turn, we obtain

$$c_3 = -\frac{1}{3\times 2}c_1 = -\frac{c_1}{3!},\; c_5 = -\frac{1}{5\times 4}c_3 = -\frac{1}{1\times 2\times 3\times 4\times 5}c_1 = \frac{c_1}{5!} \text{ and so on}$$

Putting m = 2, 4, 6, ... in (7) by turn, we obtain

$$c_2 = -\frac{1}{2\times 1}c_0 = -\frac{c_0}{2!},\; c_4 = \frac{1}{4\times 3}c_2 = \frac{1}{1\times 2\times 3\times 4}c_0 = \frac{c_0}{4!} \text{ and so on}$$

With $k = -1$ and substituting the above values of $c_2, c_3, c_4, c_5, \ldots$ in (1) the required solution is

$$y = x^{-1}(c_0 + c_1x + c_2x^2 + c_3x^3 + \ldots) = x^{-1}(c_0 + c_2x^2 + c_4x^4 + \ldots) + x^{-1}(c_1x + c_3x^3 + c_5x^5 + \ldots)$$

or $\quad y = c_0x^{-1}(1 - x^2/2! + x^4/4! \ldots) + c_1x^{-1}(x - x^3/3! + x^5/5! - \ldots)$

or $\quad y = c_0\{(\cos x)/x\} + c_1\{(\sin x)/x\}$,

$[\because \cos x = 1 - x^2/2! + x^4/4! - \ldots, \sin x = x - x^3/3! + x^5/5 - \ldots]$

Ex. 12. If $y = \sum_{m=0}^{\infty} c_m x^{r+m}$ is assumed to be a solution of the differential equation $x^2y'' - xy' - 3(1+x^2)y = 0$, then the value of r are

(*a*) 1 and 3 (*b*) –1 and 3 (*c*) 1 and – 3 (*d*) –1 and –3 **[GATE 2012]**

Hint. Ans. (*b*). As usual find indicial equation $r^2 - 2r - 3 = 0$ giving $r = -1, 3$

PARTIAL DIFFERENTIAL EQUATIONS

WHERE IS WHAT

CONTENTS

1

Origin of Partial Differential Equations

1.1 INTRODUCTION

Partial differential equations arise in geometry, physics and applied mathematics when the number of independent variables in the problem under consideration is two or more. Under such a situation, any dependent variable will be a function of more than one variable and hence it possesses not ordinary derivatives with respect to a single variable but partial derivatives with respect to several independent variables. In the present part of the book, we propose to study various methods to solve partial differential equations.

1.2 PARTIAL DIFFERENTIAL EQUATION (P.D.E.) [Delhi Maths (H) 2001]

Definition. An equation containing one or more partial derivatives of an unknown function of two or more independent variables is known as a *partial differential equation.*

For examples of partial differential equations we list the following:

$\partial z/\partial x + \partial z/\partial y = z + xy$ ... (1) $\quad (\partial z/\partial x)^2 + \partial^3 z/\partial y^3 = 2x(\partial z/\partial x)$ ... (2)

$z(\partial z/\partial x) + \partial z/\partial y = x$ ... (3) $\quad \partial u/\partial x + \partial u/\partial y + \partial u/\partial z = xyz$ ... (4)

$\partial^2 z/\partial x^2 = (1 + \partial z/\partial y)^{1/2}$ ... (5) $\quad y\left\{(\partial z/\partial x)^2 + (\partial z/\partial y)^2\right\} = z(\partial z/\partial y)$ ... (6)

1.3 ORDER OF A PARTIAL DIFFERENTIAL EQUATION [Delhi Maths (H) 2001]

Definition. The *order* of a partial differential equation is defined as the order of the highest partial derivative occuring in the partial differential equation.

In Art. 1.2, equations (1), (3), (4) and (6) are of the first order, (5) is of the second order and (2) is of the third order.

1.4 DEGREE OF A PARTIAL DIFFERENTIAL EQUATION [Delhi Maths (H) 2001]

The *degree* of a partial differential equation is the degree of the highest order derivative which occurs in it after the equation has been rationalised, *i.e.*, made free from radicals and fractions so far as derivatives are concerned.

In 1.2, equations (1), (2), (3) and (4) are of first degree while equations (5) and (6) are of second degree.

1.5 LINEAR AND NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS

Definitions. A partial differential equation is said to be *linear* if the dependent variable and its partial derivatives occur only in the first degree and are not multiplied. A partial differential equation which is not linear is called a *non-linear* partial differential equation.

In Art. 1.2, equations (1) and (4) are linear while equations (2), (3), (5) and (6) are non-linear.

1.6 NOTATIONS

When we consider the case of two independent variables we usually assume them to be x and y and assume z to be the dependent variable. We adopt the following notations throughout the study of partial differential equations

$p = \partial z/\partial x, \qquad q = \partial z/\partial y, \qquad r = \partial^2 z/\partial x^2, \qquad s = \partial^2 z/\partial x \partial y \qquad$ and $\qquad t = \partial^2 z/\partial y^2$

In case there are n independent variables, we take them to be x_1, x_2....., x_n and z is then regarded as the dependent variable. In this case we use the following notations :

$p_1 = \partial z/\partial x_1$, $\quad p_2 = \partial z/\partial x_2$, $\quad p_3 = \partial z/\partial x_3$, $\quad$ and $\quad p_n = \partial z/\partial x_n$.

Sometimes the partial differntiations are also denoted by making use of suffixes. Thus we write $u_x = \partial u/\partial x$, $\; u_y = \partial u/\partial y$, $\; u_{xx} = \partial^2 u/\partial x^2$, $\; u_{xy} = \partial^2 u/\partial x \partial y$ and so on.

1.7 Classification of first order partial differential equations into linear, semi-linear, quasi-linear and non-linear equations with examples. [Delhi Maths (H) 2001; 2004]

Linear equation. *A first order equation* $f(x, y, z, p, q) = 0$ is known as linear if it is linear in p, q and z, that is, if given equation is of the form $P(x, y)\, p + Q(x, y)\, q = R(x, y)\, z + S(x, y)$.

For examples, $\quad yx^2p + xy^2q = xyz + x^2y^3 \quad$ and $\quad p + q = z + xy$

are both first order linear partial differential equations.

Semi-linear equation. *A first order partial differential equation* $f(x, y, z, p, q) = 0$ *is known as a semi-linear equation, if it is linear in p and q and the coefficients of p and q are functions of x and y only i.e. if the given equation is of the form* $\quad P(x, y)\, p + Q(x, y)\, q = R(x, y, z)$

For examples, $\quad xyp + x^2yq = x^2y^2z^2 \quad$ and $\quad yp + xq = (x^2z^2/y^2)$

are both first order semi-linear partial differential equations.

Quasi-linear equation. *A first order partial differential equation* $f(x, y, z, p, q) = 0$ *is known as quasi-linear equation, if it is linear in p and q, i.e., if the given equation is of the form*

$$P(x, y, z)\, p + Q(x, y, z)\, q = R(x, y, z)$$

For examples, $\quad x^2zp + y^2zp = xy \quad$ and $\quad (x^2 - yz)\, p + (y^2 - zx)\, q = z^2 - xy$

are first order quasi-linear partial differential equations.

Non-linear equation. *A first order partial differential equation* $f(x, y, z, p, q) = 0$ *which does not come under the above three types, in known as a non-liner equation.*

For examples, $\quad p^2 + q^2 = 1, \quad p\,q = z \quad$ and $\quad x^2\,p^2 + y^2\,q^2 = z^2$

are all non-linear partial differential equations.

1.8 Origin of partial differential equations. We shall now examine the interesting question of how partial differential equations arise. We show that such equations can be formed by the elimination of arbitrary constants or arbitrary functions.

1.9 Rule I. Derivation of a partial differential equation by the elimination of arbitrary constants.

Consider an equation $\qquad F(x, y, z, a, b) = 0, \qquad$ ...(1)

where a and b denote arbitrary constants. Let z be regarded as function of two independent variables x and y. Differentiating (1) with respect to x and y partially in turn, we get

$$\partial F/\partial x + p(\partial F/\partial z) = 0 \quad \text{and} \quad \partial F/\partial y + q(\partial F/\partial z) = 0 \qquad ..(2)$$

Eliminating two constants a and b from three equations of (1) and (2), we shall obtain an equation of the form

$$f(x, y, z, p, q) = 0, \qquad ...(3)$$

which is partial differential equation of the first order.

In a similar manner it can be shown that if there are more arbitrary constants than the number of independent variables, the above procedure of elimination will give rise to partial differential equations of higher order than the first.

Working rule for solving problems: For the given relation $F(x, y, z, a, b) = 0$ involving variables x, y, z and arbitrary constants a, b, the relation is differentiated partially with respect to independent variables x and y. Finally arbitrary constants a and b are eliminated from the relations

$$F(x, y, z, a, b) = 0, \qquad \partial F/\partial x = 0 \qquad \text{and} \qquad \partial F/\partial y = 0.$$

The equation free from a and b will be the required partial differential equation.

Three situations may arise :

Situation I. *When the number of arbitrary constants is less than the number of independent variables, then the elimination of arbitrary constants usually gives rise to more than one partial differential equation of order one.*

For example, consider $$z = ax + y, \qquad \text{... (1)}$$
where a is the only arbitrary constant and x, y are two independent variables.

Differentiating (1) partially w.r.t. 'x', we get $$\partial z/\partial x = a \qquad \text{... (2)}$$

Differentiating (1) partially w.r.t. 'y', we get $$\partial z/\partial y = 1 \qquad \text{... (3)}$$

Eliminating a between (1) and (2) yields $$z = x(\partial z/\partial x) + y \qquad \text{... (4)}$$

Since (3) does not contain arbitrary constant, so (3) is also partial differential under consideration. Thus, we get two partial differential equations (3) and (4).

Situation II. *When the number of arbitrary constants is equal to the number of independent variables, then the elimination of arbitrary constants shall give rise to a unique partial differential equation of order one.*

Example: *Eliminate a and b from* $$az + b = a^2 x + y \qquad \text{... (1)}$$

Differentiating (1) partially w.r.t 'x' and 'y', we have

$$a(\partial z/\partial x) = a^2 \quad \text{... (2)} \qquad a(\partial z/\partial y) = 1 \quad \text{... (3)}$$

Eliminating a from (2) and (3), we have $$(\partial z/\partial x)\,(\partial z/\partial y) = 1,$$

which is the unique partial differential equation of order one.

Situation III. *When the number of arbitrary constants is greater than the number of independent variables, then the elimination of arbitrary constants leads to a partial differential equation of order usually greater than one.*

Example: *Eliminate a, b and c from* $$z = ax + by + cxy \qquad \text{... (1)}$$

Differentiating (1) partially w.r.t., 'x' and 'y', we have

$$\partial z/\partial x = a + cy \quad \text{... (2)} \qquad \partial z/\partial y = b + cx \quad \text{... (3)}$$

From (2) and (3), $$\partial^2 z/\partial x^2 = 0, \qquad \partial^2 z/\partial y^2 = 0 \qquad \text{... (4)}$$

and $$\partial^2 z/\partial x\partial y = c \qquad \text{... (5)}$$

Now, (2) and (3) $\Rightarrow$ $$x(\partial z/\partial x) = ax + cxy \qquad \text{and} \qquad y(\partial z/\partial y) = by + cxy$$

$$\therefore \qquad x(\partial z/\partial x) + y(\partial z/\partial y) = ax + by + cxy + cxy$$

or $$x(\partial z/\partial x) + y(\partial z/\partial y) = z + xy(\partial^2 z/\partial x\partial y), \text{ using (1) and (5)} \qquad \text{... (6)}$$

Thus, we get three partial differential equations given by (4) and (6), which are all of order two.

1.10 SOLVED EXAMPLES BASED ON RULE I OF ART 1.9

Ex. 1. *Find a partial differential equation by eliminating a and b from* $z = ax + by + a^2 + b^2$.

Sol. Given $$z = ax + by + a^2 + b^2. \qquad \text{...(1)}$$

Differentiating (1) partially with respect to x and y, we get

$$\partial z/\partial x = a \qquad \text{and} \qquad \partial z/\partial y = b.$$

Substituting these values of a and b in (1) we see that the arbitrary constants a and b are eliminated and we obtain,

$$z = x(\partial z/\partial x) + y(\partial z/\partial y) + (\partial z/\partial x)^2 + (\partial z/\partial y)^2,$$

which is the required partial differential equation.

Ex. 2. *Eliminate arbitrary constants a and b from* $z = (x - a)^2 + (y - b)^2$ *to form the partial differential equation.* **[Jiwaji 1999; Banglore 1995]**

Sol. Given $$z = (x - a)^2 + (y - b)^2. \qquad ...(1)$$

Differentiating (1) partially with respect to a and b, we get

$\partial z/\partial x = 2(x - a)$ and $\partial z/\partial y = 2(y - b)$.

Squatring and adding these equations, we have

$$(\partial z/\partial x)^2 + (\partial z/\partial y)^2 = 4(x - a)^2 + 4(y - b)^2 = 4\ [(x - a)^2 + (y - b)^2]$$

or $$(\partial z/\partial x)^2 + (dz/\partial y)^2 = 4z, \text{ using (1).}$$

Ex. 3. *Form partial differential equations by eliminating arbitrary constants a and b from the following relations :*

(a) $z = a(x + y) + b$. (b) $z = ax + by + ab$. **[Bhopal 2010, Rewa 1996]**

(c) $z = ax + a^2y^2 + b$. **[Agra 2010]** (d) $z = (x + a)\ (y + b)$. **[Madurai Kamraj 2008]**

Sol. (a) Given $$z = a(x + y) + b \qquad ...(1)$$

Differentiating (1) partially with respect to x and y, we get

$\partial z/\partial x = a$ and $\partial z/\partial y = a$.

Eliminating a between these, we get $\partial z/\partial x = \partial z/\partial y$,

which is the required partial differential equation.

(b) Given $$z = ax + by + ab. \qquad ...(1)$$

Differentiating (1) partially with respect to x and y, we get

$\partial z/\partial x = a$ and $\partial z/\partial y = b$ ....(2)

Substituting the values of a and b from (2) in (1), we get

$$z = x(\partial z/\partial x) + y(\partial z/\partial y) + (\partial z/\mathrm{d}x)(\partial z/\partial y),$$

which is the required partial differential equation.

(c) Try yourself. **Ans.** $\partial z/\partial y = 2y(\partial z/\partial x)^2$.

(d) Try yourself. **Ans.** $z = (\partial z/\partial y)\ (\partial z/\partial x)$.

Ex. 4. *Eliminate a and b from* $z = axe^y + (1/2) \times a^2e^{2y} + b$. **[Meerut 2006]**

Sol. Given $$z = axe^y + (1/2) \times a^2e^y + b. \qquad ...(1)$$

Differentiating (1) partially with respect to x and y, we get

$$\partial z/\partial x = ae^y \qquad ...(2)$$

and $$\partial z/\partial y = axe^y + a^2e^{2y} = x(ae^y) + (ae^y)^2. \qquad ...(3)$$

Substituting the value of ae^y from (2) in (3), we get $\partial z/\partial y = x(\partial z/\partial x) + (\partial z/\partial x)^2$.

Ex. 5(*a*). *Form the partial differential equation by eliminating h and k from the equation* $(x - h)^2 + (y - k)^2 + z^2 = \lambda^2$. **[Gulbarga 2005; I.A.S. 1996]**

Sol. Given $$(x - h)^2 + (y - k)^2 + z^2 = \lambda^2. \qquad ...(1)$$

Differentiating (1) partially with respect to x and y, we get

$$2(x - h) + 2z(\partial z/\partial x) = 0 \quad \text{or} \quad (x - h) = -z(\partial z/\partial x) \qquad ...(2)$$

and $$2(y - k) + 2z(\partial z/\partial y) = 0 \quad \text{or} \quad (y - k) = -z(\partial z/\partial y). \qquad ...(3)$$

Substituting the values of $(x - h)$ and $(y - k)$ from (2) and (3) in (1) gives

$$z^2(\partial z/\partial x)^2 + z^2(\partial z/\partial y)^2 + z^2 = \lambda^2 \quad \text{or} \quad z^2[(\partial z/\partial x)^2 + (\partial z/\partial y)^2 + 1] = \lambda^2,$$

which is the required partial differential equation.

Ex. 5(*b*). *Find the differential equation of all spheres of radius λ, having centre in the xy-plane.* **[M.D.U. Rohtak 2005; I.A.S. 1996, K.U. Kurukshetra 2005]**

Sol. From the coordinate geometry of three-dimensions, the equation of any sphere of radius λ, having centre $(h, k, 0)$ in the xy-plane is given by

$$(x - h)^2 + (y - k)^2 + (z - 0)^2 = \lambda^2 \quad \text{or} \quad (x - h)^2 + (y - k)^2 + z^2 = \lambda^2, \qquad \text{...(1)}$$

where h and k are arbitray constants. Now, proceed exactly in the same way as in Ex. 5(a).

Ex. 6. *Form the differential equation by eliminating a and b from* $z = (x^2 + a)(y^2 + b)$.

[Madras 2005; Sagar 1997, I.A.S. 1997]

Sol. Given $$z = (x^2 + a)(y^2 + b). \qquad \text{...(1)}$$

Differentiating (1) partially with respect to x and y, we get

$$\partial z/\partial x = 2x(y^2 + b) \quad \text{or} \quad (y^2 + b) = (1/2x) \times (\partial z/\partial x) \qquad \text{...(2)}$$

and $$\partial z/\partial y = 2y(x^2 + a) \quad \text{or} \quad (x^2 + a) = (1/2y) \times (\partial z/\partial y). \qquad \text{...(3)}$$

Substituting the values of $(y^2 + b)$ and $(x^2 + a)$ from (2) and (3) in (1) gives

$$z = (1/2y) \times (\partial z/\partial y) \times (1/2x) \times (\partial z/\partial x) \quad \text{or} \quad 4xyz = (\partial z/\partial x)(\partial z/\partial y),$$

which the required partial differential equation.

Ex. 7. *Form differential equation by eliminating constants A and p from* $z = A\,e^{pt} \sin px$.

Sol. Given $$z = A\,e^{pt} \sin px. \qquad \text{...(1)}$$

Differentiating (1) partially with respect to x and t, we get

$$\partial z/\partial x = Ap\,e^{pt} \cos px \qquad \text{...(2)} \qquad\qquad \partial z/\partial t = Ap\,e^{pt} \sin px. \qquad \text{...(3)}$$

Differentiating (2) and (3) partially with respect to x and t respectively gives

$$\partial^2 z/\partial x^2 = -Ap^2 e^{pt} \sin px. \qquad \text{...(4)} \qquad\qquad \partial^2 z/\partial t^2 = Ap^2 e^{pt} \sin px. \qquad \text{...(5)}$$

Adding (4) and (5), $$\partial^2 z/\partial x^2 + \partial^2 z/\partial t^2 = 0,$$

which is the required partial differential equation.

Ex. 8. *Find the differential equation of the set of all right circular cones whose axes coincide with z-axix.* **[I.A.S. 1998]**

Sol. The general equation of the set of all right circular cones whose axes coincide with z-axis, having semi-vertical angle α and vertex at (0, 0, c) is given by

$$x^2 + y^2 = (z - c)^2 \tan^2 \alpha, \qquad \text{... (1)}$$

in which both the constants c and α are arbitrary.

Differentiating (1) partially, w.r.t. x and y, we get

$$2x = 2(z - c)\,(\partial z/\partial x) \tan^2 \alpha \quad \text{and} \quad 2y = 2(z - c)\,(\partial z/\partial y) \tan^2 \alpha$$

$$\Rightarrow \quad y(z - c)\,(\partial z/\partial x) \tan^2 \alpha = xy \quad \text{and} \quad x(z - c)\,(\partial z/\partial y) \tan^2 \alpha = xy$$

$$\Rightarrow \quad y(z - c)\,(\partial z/\partial x) \tan^2 \alpha = x(z - c)\,(\partial z/\partial y) \tan^2 \alpha$$

Thus, $y\,(\partial z/\partial x) = x\,(\partial z/\partial y)$, which is the required partial differential equation.

Ex. 9. *Show that the differential equation of all cones which have their vertex at the origin is* $px + qy = z$. *Verify that* $yz + zx + xy = 0$ *is a surface satisfying the above equation.*

[I.A.S. 1979, 2009]

Sol. The equation of any cone with vertex at origin is

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0, \qquad \text{...(1)}$$

where a, b, c, f, g, h are parameters. Differentiating (1) partially w.r.t. 'x' and 'y' by turn, we have (noting that $p = \partial z/\partial x$ and $q = \partial z/\partial y$)

$$2ax + 2czp + 2fyp + 2g(px + z) + 2hy = 0 \quad \text{or} \quad ax + gz + hy + p(cz + gx + fy) = 0 \quad \text{...(2)}$$

and $2by + 2czq + 2f(yq + z) + 2gxq + 2hx = 0$ or $by + fz + hx + q(cz + fy + gx) = 0.$ ...(3)

Multiplying (2) by x and (3) by y and adding, we have

$$(ax^2 + by^2 + gzx + fyz + 2hxy) + (cz + fy + gx)(px + qy) = 0.$$

$$-(cz^2 + fyz + gxz) + (cz + fy + gx)(px + qy) = 0, \text{ using (1)}$$

or $(cz + fy + gx)(px + qy - z) = 0$ or $px + qy - z = 0,$ ...(4)

which is required partial differential equation.

Second Part : Given surface is $yz + zx + xy = 0$ ...(5)

Differentiating (5) partially w.r.t. 'x' and 'y' by turn, we get

$yp + px + z + y = 0$ and $z + qy + xq + x = 0.$ ...(6)

Solving (6) for p and q, $p = -(z + y)/(x + y)$ and $q = -(z + x)/(x + y)$.

$$\therefore \quad px + qy - z = -\frac{x(z+y)}{x+y} - \frac{y(z+x)}{x+y} - z = -\frac{2(xy+yz+zx)}{x+y} = 0, \text{ using (5)}$$

Hence (5) is a surface satisfying (4).

Ex. 10. *Form partial differential equations by eliminating arbitrary constants a and b from the following relations:*

(a) $2z = x^2/a^2 + y^2/b^2$ **[Nagpur 1995; M.D.U. Rohtak 2006]**

(b) $2z = (ax + y)^2 + b$ **[Nagpur 1996; Delhi Maths (G) 2006; Pune 2010]**

Sol. (*a*) Given $2z = x^2/a^2 + y^2/b^2$ ... (1)

Differentiating (1) partially w.r.t. 'x' and 'y', we get

$2(\partial z/\partial x) = 2x/a^2$ ... (2) $2(\partial z/\partial y) = 2y/b^2$ ... (3)

From (2) and (3), $p = x/a^2$, $q = y/b^2$ $\Rightarrow$ $a^2 = x/p$, $b^2 = y/q$

Substituting these values of a^2 and b^2 in (1), we get

$2z = px + qy$, which is the required partial differential equation

(b) Given $2z = (ax + y)^2 + b$ ... (1)

Differentiating (1) partially w.r.t. 'x' and 'y', we get

$2p = 2a(ax + y)$ ... (2) $2q = 2(ax + y)$ ... (3)

where $p = \partial z/\partial x$ and $q = \partial z/\partial y$. Dividing (2) by (3) yields $p/q = a$.

Substituting this value of a in (3), we get $q = (p/q)\,x + y$ or $px + qy = q^2$.

Ex. 11. *Eliminate a, b and c from $z = a(x + y) + b(x - y) + abt + c$* **[I.A.S. 1998]**

Sol. Given $z = a(x + y) + b(x - y) + abt + c$ ... (1)

Differentiating (1) partially w.r.t. 'x', 'y' and 't', we get

$\partial z/\partial x = a + b$ ... (2) $\partial z/\partial y = a - b$ ... (3) $\partial z/\partial t = ab$ ... (4)

We have the identity: $(a + b)^2 - (a - b)^2 = 4ab$

$$\therefore \quad (\partial z/\partial x)^2 - (\partial z/\partial y)^2 = 4(\partial z/\partial t), \text{ using (2), (3) and (4)}$$

Ex. 12. *Form the partial differential equation by eliminating the arbitrary constants a and b from log $(az - 1) = x + ay + b$.* **[I.A.S. 2002]**

Sol. (*a*) Given $\log(az - 1) = x + ay + b$ ... (1)

Differentiating (1) partially w.r.t. 'x', we get $\dfrac{a}{az-1}\dfrac{\partial z}{\partial x} = 1$ ... (2)

Differentiating (1) partially w.r.t. 'y', we get $\dfrac{a}{az-1}\dfrac{\partial z}{\partial y} = a$ ... (3)

From (3), $az - 1 = \dfrac{\partial z}{\partial y}$ so that $a = \dfrac{1 + (\partial z/\partial y)}{z}$ ... (4)

Putting the above values of $az - 1$ and a in (2), we have

$$\frac{1+(\partial z/\partial y)}{z(\partial z/\partial y)}\,\frac{\partial z}{\partial x} = 1 \qquad \text{or} \qquad \left(1 + \frac{\partial z}{\partial y}\right)\frac{\partial z}{\partial x} = z\frac{\partial z}{\partial y}.$$

Ex. 13. *Find a partial differential equation by eliminating a, b, c, from* $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1$.

[Bhopal 2004; Jabalpur 2000, 03, Jiwaji 2000, Vikram 2002, 04; Ravishanker 2010]

Sol. Given $x^2/a^2 + y^2/b^2 + z^2/c^2 = 1.$ ... (1)

Differentiating (1) partially with respect to x and y, we get

$$\frac{2x}{a^2} + \frac{2z}{c^2}\frac{dz}{dx} = 0 \qquad \text{or} \qquad c^2x + a^2z\frac{dz}{dx} = 0 \quad \text{... (2)}$$

and
$$\frac{2y}{b^2} + \frac{2x}{c^2}\frac{\partial z}{\partial y} = 0 \qquad \text{or} \qquad c^2y + b^2z\frac{\partial z}{\partial y} = 0. \quad \text{... (3)}$$

Differentiating (2) with respect to x and (3) with respect to y, we have

$$c^2 + a^2\left(\frac{\partial z}{\partial x}\right)^2 + a^2z\frac{\partial^2 z}{\partial x^2} = 0 \quad \text{... (4)} \qquad c^2 + b^2\left(\frac{\partial z}{\partial x}\right)^2 + b^2z\frac{\partial^2 z}{\partial y^2} = 0. \quad \text{... (5)}$$

From (2), $c^2 = -(a^2z/x) \times (\partial z/\partial x)$ ... (6)

Putting this value of c^2 in (4) and dividing by a^2, we obtain

$$-\frac{z}{x}\frac{\partial z}{\partial x} + \left(\frac{\partial z}{\partial x}\right)^2 + z\frac{\partial^2 z}{\partial x^2} = 0 \qquad \text{or} \qquad zx\frac{\partial^2 z}{\partial x^2} + x\left(\frac{\partial z}{\partial x}\right)^2 - z\frac{\partial z}{\partial x} = 0. \quad \text{... (7)}$$

Similarly, from (3) and (5), $zy\dfrac{\partial^2 z}{\partial y^2} + y\left(\dfrac{\partial z}{\partial y}\right)^2 - z\dfrac{\partial z}{\partial y} = 0.$ ... (8)

Differentiating (2) partially w.r.t. y, $0 + a^2\left\{(\partial z/\partial y)\,(\partial z/\partial x) + z(\partial^2 z/\partial x\partial y)\right\} = 0$

or $(\partial z/\partial x)\,(\partial z/\partial y) + z(\partial^2 z/\partial x\partial y) = 0$ ... (9)

(7), (8) and (9) are three possible forms of the required partial differential equations.

Ex. 14. *Find the partial differential equation of all planes which are at a constant distance 'a' from the origin.*

Sol. Let $lx + my + nz = a$ ... (1)

be the equation of the given plane where l, m, n are direction colines of the normal to the plane so that $l^2 + m^2 + n^2 = 1$, l, m, n being parameters ... (2)

Differentiating (1) partially w.r.t. 'x' and 'y', we have

$l + np = 0$ ...(3) $m + nq = 0$, ... (4)

where $p = \partial z/\partial x$ and $q = \partial z/\partial y$. From (3) and (4), $l = -np$ and $m = -nq$. Substituting these values in (2), we have

$n^2(p^2 + q^2 + 1) = 1$ so that $n = (p^2 + q^2 + 1)^{-1/2}$ ... (5)

$\therefore\ l = -np = -p(p^2 + q^2 + 1)^{-1/2}$ and $m = -nq = -q(p^2 + q^2 + 1)^{-1/2}$ ... (6)

Substituting the values of l, m, n given by (5) and (6) in (1), we get

$$-px(p^2+q^2+1)^{-1/2} - qy(p^2+q^2+1)^{-1/2} + z(p^2+q^2+1)^{-1/2} = a$$

or $\quad z = px + qy + a(p^2+q^2+1)^{1/2}$, which is the required partial differential equation.

Ex. 15. *Show that the partial differential equation obtained by eliminating the arbitrary constants a and c from $z = ax + g(a)y + c$, where $g(a)$ is an arbitrary function of a, is free of the variables x, y, z.*

Sol. Differentiating $z = ax + g(a)y + c$ partially w.r.t. 'x' and 'y' yields $p = a$ and $q = g(a)$. Eliminating a between them leads to $q = g(p)$ or $f(p, q) = 0$, where f is an arbitrary function of p and q. Clearly, the resulting partial differential equation contains p and q but none of the variables x, y, z.

Ex. 16. *Show that the partial differential equation obtained by eliminating the arbitrary constants a and b from $z = ax + by + f(a, b)$ is given by $z = px + qy + f(p, q)$.*

Sol. Differentiating $\qquad z = ax + by + f(a, b) \qquad \ldots (1)$

partially with respect to 'x' and 'y', we get $\qquad p = a \quad$ and $\quad q = b \qquad \ldots (2)$

Eliminating a and b from (1) and (2) yields $\qquad z = px + qy + f(p, q)$

Ex. 17. *Form a partial differential equation by eliminating a, b and c from the relation* $ax^2 + by^2 + cz^2 = 1$. **[Mysore 2004]**

Sol. Given $\qquad ax^2 + by^2 + cz^2 = 1. \qquad \ldots (1)$

Differentiating (1) partially w.r.t. 'x' and 'y', we have

$$2ax + 2cz(\partial z/\partial x) = 0 \quad \ldots (2) \qquad\qquad 2by + 2cz(\partial z/\partial y) = 0 \quad \ldots (3)$$

Differentiating (2) partially w.r.t. 'y', we get

$$0 + 2c\left\{(\partial z/\partial y)(\partial z/\partial x) + z(\partial^2 z/\partial y\partial x\right\} = 0 \quad \text{or} \quad (\partial z/\partial x)(\partial z/\partial y) + z(\partial^2 z/\partial x\partial y) = 0, \quad \ldots (4)$$

since c is an arbitrary constant. (4) is the desired partial differential equation.

Again, differentiating partially (2) w.r.t. x and (3) w.r.t. y, we get

$$2a + 2c\left\{(\partial z/\partial x)^2 + z(\partial^2 z/\partial x^2)\right\} = 0 \quad \ldots (5) \qquad 2b + 2c\left\{(\partial z/\partial y)^2 + z(\partial^2 z/\partial y^2)\right\} = 0 \quad \ldots (6)$$

From (2), $a = -(cz/x) \times (\partial z/\partial x)$. Putting this in (5), we get

$$-(cz/x) \times (\partial z/\partial x) + c\left\{(\partial z/\partial x)^2 + z(\partial^2 z/\partial x^2\right\} = 0 \quad \text{or} \quad zx(\partial^2 z/\partial x^2) + x(\partial z/\partial x)^2 - z(\partial z/\partial x) = 0 \quad \ldots (7)$$

Similarly, from (3) and (6), we get $\qquad zy(\partial^2 z/\partial y^2) + y(\partial z/\partial y)^2 - z(\partial z/\partial y) = 0. \quad \ldots (8)$

(4), (7) and (8) are three possible forms of the required partial differential equations.

EXERCISE 1 (A)

Eliminate the arbitrary constants indicated in brackets from the following equations and form corresponding partial differential equations.

1. $z = Ae^{pt}\sin px$, (p and A). **Ans.** $\partial^2 z/\partial x^2 + \partial^2 z/\partial t^2 = 0$.

2. $z = Ae^{-p^2 t}\cos px$, (p and A) **(Sagar 1999; Ranchi 2010)** **Ans.** $\partial^2 z/\partial x^2 = dz/\partial t$

3. $z = ax^3 + by^3$; (a, b) **Ans.** $x(\partial z/\partial x) + y(\partial z/\partial y) = 3z$

4. $4z = [ax + (y/a) + b]^2$; (a, b). **(Delhi B.A. (Prog) II 2011)** **Ans.** $z = (dz/\partial x)(\partial z/\partial y)$

5. $z = ax^2 + bxy + cy^2$, (a, b,c) **Ans.** $x^2(\partial^2 z/\partial x^2) + 2xy(\partial^2 z/\partial x\partial y) + y^2(\partial^2 z/\partial y^2) = 2z$

6. $z^2 = ax^3 + by^3 + ab$, (a, b)

Ans. $9x^2y^2z = 6x^3y^2(\partial z/\partial x) + 6x^2y^3(\partial z/\partial y) + 4z(\partial z/\partial x)(\partial z/\partial y)$

7. $e^{1/\{z-(x^2/y)\}} = \dfrac{ax^2}{y^2} + \dfrac{b}{y}, (a, b)$ **Ans.** $x\dfrac{\partial z}{\partial x} + 2y\dfrac{\partial z}{\partial y} = 2(z - x^2/y)^2$

8. Find the differential equation of the family of spheres of radius 4 with centres on the *xy*-plane. **Ans.** $(x-y)^2[(\partial z/\partial x)^2 + (\partial z/\partial y)^2 + 1] = 16(\partial z/\partial x - \partial z/\partial y)^2$

9. Find the P.D.E of planes having equal x and y intercepts. **Ans.** $p - q = 0$

10. Find the partial differential equation of the family of spheres of radius 7 with centres on the plane $x - y = 0$. **Ans.** $(p^2 + q^2 + 1)(x - y)^2 = 49(p - q)$

11. Find the partial differential equation of all spheres whose centres lie on z-axis.

Ans. $xq - yp = 0$

1.11 Rule II. Derivation of partial differential equation by the elimination of arbitrary function ϕ from the equation $\phi(u, v) = 0$, where u and v are functions of x, y and z.

[Meerut 1995]

Proof. Given $\phi(u, v) = 0$. ...(1)

We treat z as dependent variable and x and y as independent variables so that

$\partial z/\partial x = p$, $\partial z/\partial y = q$, $\partial y/\partial x = 0$ and $\partial x/\partial y = 0$.

Differentiating (1) partially with respect to x, we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial v}{\partial z}\frac{\partial z}{\partial x}\right) = 0$$

or
$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0$$

or
$$\frac{\partial \phi}{\partial u}\Big/\frac{\partial \phi}{\partial v} = -\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right)\Big/\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right). \qquad ...(3)$$

Similarly, differentiating (1) partially w.r.t. 'y', we get

$$\frac{\partial \phi}{\partial u}\Big/\frac{\partial \phi}{\partial v} = -\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right)\Big/\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) \qquad ...(4)$$

Eliminating ϕ with the help of (3) and (4), we get

$$\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right)\Big/\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) = \left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right)\Big/\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right)$$

or
$$\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = \left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right)$$

or
$$Pp + Qq = R, \qquad ...(5)$$

where $P = \dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial z} - \dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial y}$, $Q = \dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial z}$, $R = \dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial y} - \dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial x}$.

Thus we obtain a linear partial differntial equation of first order and of first degree in p and q.

Note. If the given equation between x, y, z contains two arbitrary functions, then in general, their elimination gives rise to equations of higher order.

1.12 SOLVED EXAMPLES BASED ON RULE II OF ART. 1.11.

Ex. 1. *Form a partial differential equation by eliminating the arbitrary function* ϕ *from* $\phi(x + y + z, x^2 + y^2 - z^2) = 0$. *What is the order of this partial differential equation ?*

[Bilaspur 2003; Indore 2003; Jiwaji 2003; Vikram 2001]

Sol. Given $\phi(x + y + z, x^2 + y^2 - z^2) = 0$. ...(1)

Let $u = x + y + z$ and $v = x^2 + y^2 - z^2$. ...(2)

Then (1) becomes $\qquad \phi(u, v) = 0.$ ...(3)

Differentiating (3) w.r.t., 'x' partially, we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right) = 0. \qquad ...(4)$$

From (2), $\quad \dfrac{\partial u}{\partial x} = 1, \quad \dfrac{\partial u}{\partial z} = 1, \quad \dfrac{\partial v}{\partial x} = 2x, \quad \dfrac{\partial v}{\partial z} = -2z, \quad \dfrac{\partial u}{\partial y} = 1, \quad \dfrac{\partial v}{\partial y} = 2y.$ ..(5)

From (4) and (5), $\qquad (\partial\phi/\partial u)(1 + p) + 2(\partial\phi/\partial v)(x - pz) = 0$

or $\qquad (\partial\phi/\partial u)/(\partial\phi/\partial v) = -2(x - pz)/(1 + p).$ ...(6)

Again, differentiating (3) w.r.t., 'y' partially, we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)+\frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right) = 0$$

or $\qquad (\partial\phi/\partial u)(1 + q) + 2(\partial\phi/\partial v)(y - zq) = 0,$ by (5)

or $\qquad (\partial\phi/\partial u)/(\partial\phi/\partial v) = -2(y - qz)/(1 + q).$ ...(7)

Eliminating ϕ from (6) and (7), we obtain

$(x - pz)/(1 + p) = (y - qz)/(1 + q) \qquad$ or $\qquad (1 + q)(x - pz) = (1 + p)(y - qz)$

or $\quad (y + z)p - (x + z)q = x - y$, which is the desired partial differential equation of first order.

Ex. 2. *Form a partial differential equation by eliminating the arbitrary function f from the equation* $x + y + z = f(x^2 + y^2 + z^2)$. **(Kanpur 2011)**

Sol. Given $\qquad x + y + z = f(x^2 + y^2 + z^2).$ ...(1)

Differentiating partially w.r.t. 'x' and 'y', (1) gives

$$1 + p = f'(x^2 + y^2 + z^2).(2x + 2zp). \qquad ...(2)$$

and $\qquad 1 + q = f'(x^2 + y^2 + z^2).(2y + 2zq).$ ...(3)

Eliminating $f'(x^2 + y^2 + z^2)$ from (2) and (3), we obtain

$(1 + p)/(2x + 2zp) = (1 + q)/(2y + 2zq) \qquad$ or $\qquad (1 + p)(y + zq) = (1 + q)(x + zp)$

or $\quad (y - z)p + (z - x)q = x - y$, which is the required partial differential equations.

Ex. 3. *Eliminate the arbitrary functions f and F from* $y = f(x - at) + F(x + at)$.

(Sagar 1997; Vikram 1995; Jabalpur 2002)

Sol. Given $\qquad y = f(x - at) + F(x + at).$ ...(1)

From (1), $\qquad \partial y/\partial x = f'(x - at) + F'(x + at)$

and hence $\qquad \partial^2 y/\partial x^2 = f''(x - at) + F''(x + at).$ ...(2)

Also, $\qquad \partial y/\partial t = f'(x - at) . (-a) + F'(x + at) . (a)$

and hence $\qquad \partial^2 y/\partial t^2 = f''(x - at) . (-a)^2 + F''(x + at) . (a)^2$

or $\qquad \partial^2 y/\partial t^2 = a^2[f''(x - at) + F''(x + at)].$ ...(3)

Then, $\qquad$ (2) and (3) $\quad \Rightarrow \quad \partial^2 y/\partial t^2 = a^2(\partial^2 y/\partial x^2).$

Ex. 4. *Eliminate arbitrary function f from*

(*i*) $z = f(x^2 - y^2)$. **[Bilaspur 1996; Sagar 1996; Bangalore 1995]**

(*ii*) $z = f(x^2 + y^2)$. **[Meerut 1995; Pune 2010]**

Sol. (*i*) Given $\qquad z = f(x^2 - y^2).$ ...(1)

Differentiating (1) partially with respect to x and y, we get

$\partial z/\partial x = f'(x^2 - y^2) \times 2x \qquad$ so that $\qquad f'(x^2 - y^2) = (1/2x) \times (\partial z/\partial x)$ ...(2)

and $\quad \partial z/\partial y = f'(x^2 - y^2) \times (-2y) \qquad$ so that $\qquad f'(x^2 - y^2) = -(1/2y) \times (\partial z/\partial y).$ ...(3)

Eliminating $f'(x^2 - y^2)$ between (2) and (3), we have

$$\frac{1}{2x}\frac{\partial z}{\partial x} = -\frac{1}{2y}\frac{\partial z}{\partial y} \qquad \text{or} \qquad y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0.$$

(*ii*) Proceed as in part (1). **Ans.** $y(\partial z/\partial x) - x(\partial z/\partial y) = 0$

Ex. 5. *Form a partial differential equation by eliminating the function f from*

(*i*) $z = f(y/x)$. **[Sagar 2000]** (*ii*) $z = x^n f(y/x)$.

Sol. Given $z = f(y/x)$. ...(1)

Differentiating (1) partially with respect to x and y, we get

$\partial z/\partial x = f'(y/x) \times (-y/x^2)$ or $f'(y/x) = -(x^2/y) \times (\partial z/\partial x)$ ...(2)

and $\partial z/\partial y = f'(y/x) \times (1/x)$ or $f'(y/x) = x(\partial z/\partial y)$. ...(3)

Eliminating $f'(y/x)$ between (2) and (3), we have

$$-\frac{x^2}{y}\frac{\partial z}{\partial x} = x\frac{\partial z}{\partial y} \qquad \text{or} \qquad x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0.$$

which is the required partial differential equation.

(*ii*) Given $z = x^n f(y/x)$. ...(1)

Differentiating (1) partially with respect to x and y, we get

$$\partial z/\partial x = n\, x^{n-1} f(y/x) + x^n f'(y/x) \times (-y/x^2) \qquad ...(2)$$

and $$\partial z/\partial y = x^n f'(y/x) \times (1/x). \qquad ...(3)$$

Multiplying both sides of (2) by x, we have $x(\partial z/\partial x) = n\, x^n f(y/x) - yx^{n-1} f'(y/x)$. ...(4)

Multiplying both sides of (3) by y, we have $y(\partial z/\partial y) = y\, x^{n-1} f'(y/x)$. ...(5)

Adding (4) and (5), $x(\partial z/\partial x) + y(\partial z/\partial y) = n\, x^n f(y/x)$

or $x(\partial z/\partial x) + y(\partial z/\partial y) = nz$, by (1)

Ex. 6. *Form a partial differential equation by eliminating the function* ϕ from $lx + my + nz = \phi(x^2 + y^2 + z^2)$. **[Ravishankar 2003; Vikram 2003]**

Sol. Given $lx + my + nz = \phi(x^2 + y^2 + z^2)$. ...(1)

Differentiating (1) partially with respect to x and y, we get

$$l + n(\partial z/\partial x) = \phi'(x^2 + y^2 + z^2) \times \{2x + 2z(\partial z/\partial x)\} \qquad ...(2)$$

and $$m + n(\partial z/\partial y) = \phi'(x^2 + y^2 + z^2) \times \{2y + 2z(\partial z/\partial y)\} \qquad ...(3)$$

Dividing (2) by (3), we get $$\frac{l + n(\partial z/\partial x)}{m + n(\partial z/\partial y)} = \frac{2\{x + z(\partial z/\partial x)\}}{2\{y + z(\partial z/\partial y)\}}$$

or $(ny - mz)(\partial z/\partial x) + (lz - nx)(\partial z/\partial y) = mx - ly$, which is the required partial differential equation.

Ex. 7. *Form partial differential eqn. by eliminating the function f from* $z = e^{ax+by} f(ax - by)$.

Sol. Given $z = e^{ax+by} f(ax - by)$. ...(1)

Differentiating (1) partially with respect to x and y, we get

$$\partial z/\partial x = e^{ax+by}\, a\, f'(ax - by) + a\, e^{ax+by} f(ax - by) \qquad ...(2)$$

and $$\partial z/\partial y = e^{ax+by} \{-b\, f'(ax - by)\} + b\, e^{ax+by} f(ax - by). \qquad ...(3)$$

Multiplying (2) by b and (3) by a and adding, we get

$b(\partial z/\partial x) + a(\partial z/\partial y) = 2ab\, e^{ax+by} f(ax - by)$ or $b(\partial z/\partial x) + a(\partial z/\partial y) = 2abz$, by (1)

Ex. 8. *Form a partial differential equation by eliminating the arbitrary functions f and F from* $z = f(x + iy) + F(x - iy)$, *where* $i^2 = -1$. **[Bilaspur 2004; Jiwaji 1998; Meerut 2010]**

Sol. Given $z = f(x + iy) + F(x - iy)$. ...(1)

Differentiating (1) partially with respect to x and y, we get

$$\partial z/\partial x = f'(x + iy) + F'(x - iy) \qquad ...(2)$$

and $$\partial z/\partial y = i f'(x + iy) - iF'(x - iy). \qquad ...(3)$$

Differentiating (2) and (3) partial w.r.t. x and y respectively, we get

$$\partial^2 z/\partial x^2 = f''(x + iy) + F''(x - iy) \qquad ...(4)$$

and $$\partial^2 z/\partial y^2 = i^2 f''(x + iy) + i^2 F''(x - iy) = -\{f''(x + iy) + F''(x + iy)\}. \qquad ...(5)$$

Adding (4) and (5), $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 = 0$, which is the required equation.

Ex. 9. *Form partial differential equation by eliminating arbitrary functions f and g from* $z = f(x^2 - y) + g(x^2 + y)$. **[Nagpur 1996 ; I.A.S. 1996; Kanpur 2011]**

Sol. Given $$z = f(x^2 - y) + g(x^2 + y). \qquad ...(1)$$

Differentiating (1) partially with respect to x and y, we get

$$\partial z/\partial x = 2xf'(x^2 - y) + 2xg'(x^2 + y) = 2x\{f'(x^2 - y) + g'(x^2 + y)\}. \qquad ...(2)$$

and $$\partial z/\partial y = -f'(x^2 - y) + g'(x^2 + y). \qquad ...(3)$$

Differentiating (2) and (3) w.r.t. x and y respectively, we get

$$\partial^2 z/\partial x^2 = 2\{f'(x^2 - y) + g'(x^2 + y)\} + 4x^2\{f''(x^2 - y) + g''(x^2 + y)\} \qquad ...(4)$$

and $$\partial^2 z/\partial y^2 = f''(x^2 - y) + g''(x^2 + y). \qquad ...(5)$$

Again, (2) $\Rightarrow$ $f'(x^2 - y) + g'(x^2 + y) = (1/2x) \times (\partial z/\partial x)$. ...(6)

Substituting the values of $f''(x^2 - y) + g''(x^2 + y)$ and $f'(x^2 - y) + g'(x^2 + y)$ from (5) and (6) in (4), we have

$$\frac{\partial^2 z}{\partial x^2} = 2 \times \left(\frac{1}{2x}\right)\frac{\partial z}{\partial x} + 4x^2\frac{\partial^2 z}{\partial y^2} \quad \text{or} \quad x\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial x} + 4x^3\frac{\partial^2 z}{\partial y^2},$$

which is the required partial differential equation.

Ex. 10. *Find the differential equation of all surfaces of revolution having z-axis as the axis of rotation.* **[I.A.S. 1997]**

Sol. From coordinate geometry of three dimensions, equation of any surface of revolution having z-axis as the axis of rotation may be taken as

$$z = \phi[(x^2 + y^2)^{1/2}], \text{ where } \phi \text{ is an arbitrary function.} \qquad ...(1)$$

Differentating (1) partially with respect to x and y, we get

$$\partial z/\partial x = \phi'[(x^2 + y^2)^{1/2}] \times (1/2) \times (x^2 + y^2)^{-1/2} \times 2x \qquad ...(2)$$

and $$\partial z/\partial y = \phi'[(x^2 + y^2)^{1/2}] \times (1/2) \times (x^2 + y^2)^{-1/2} \times 2y. \qquad ...(3)$$

Dividing (2) by (3), $$\frac{\partial z/\partial x}{\partial z/\partial y} = \frac{x}{y} \quad \text{or} \quad y\frac{\partial z}{\partial x} = x\frac{\partial z}{\partial y}.$$

Ex. 11. *Form a partial differential equation by eliminating the arbitrary functions f and g from* $z = y\, f(x) + x\, g(y)$. (*Guwahati 2007*)

Sol. Given $$z = y\,f(x) + x\,g(y). \qquad ...(1)$$

Differentiating (1) partially w.r.t. 'x' and 'y', we get

$$\partial z/\partial x = y\,f'(x) + g(y) \quad ...(2) \qquad \partial z/\partial y = f(x) + x\,g'(y). \quad ...(3)$$

Differentiating (3) with respect to x, $$\partial^2 z/\partial x\partial y = f'(x) + g'(y). \qquad ...(4)$$

From (2) and (3), $$f'(x) = \frac{1}{y}\left[\frac{\partial z}{\partial x} - g(y)\right] \quad \text{and} \quad g'(y) = \frac{1}{x}\left[\frac{\partial z}{\partial y} - f(x)\right].$$

Substituting these values in (4), we have

$$\frac{\partial^2 z}{\partial x\partial y} = \frac{1}{y}\left[\frac{\partial z}{\partial x} - g(y)\right] + \frac{1}{x}\left[\frac{\partial z}{\partial y} - f(x)\right]$$

or $$xy\frac{\partial^2 z}{\partial x\partial y} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - \{x\,g(y) + y\,f(x)\} \quad \text{or} \quad xy\frac{\partial^2 z}{\partial x\partial y} = x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} - z, \text{ by (2)}$$

Ex. 12. *Form a partial differential equation by eliminating the arbitrary function* ϕ *from* $\phi(x^2 + y^2 + z^2, z^2 - 2xy) = 0$. **[Nagpur 1996; 2002]**

Sol. Given $\phi(x^2 + y^2 + z^2, z^2 - 2xy) = 0.$ ...(1)

Let $u = x^2 + y^2 + z^2$ and $v = z^2 - 2xy.$ ...(2)

Then, (1) becomes $\phi(u, v) = 0.$ ...(3)

Differentiating (3) partially w.r.t. 'x', we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0, \qquad \text{...(4)}$$

where $p = \partial z/\partial x$ and $q = \partial z/\partial y$. Now, from (2), we have

$\partial u/\partial x = 2x$, $\partial u/\partial y = 2y$, $\partial u/\partial z = 2z$, $\partial v/\partial x = -2y$, $\partial v/\partial y = -2x$, $\partial v/\partial z = 2z$. ...(5)

Using (5), (4) reduces to $(\partial\phi/\partial u)\,(2x + 2pz) + (\partial\phi/\partial v)\,(-2y + 2pz) = 0$

or $(x + pz)\,(\partial\phi/\partial u) = (y - pz)\,(\partial\phi/\partial v).$ ...(6)

Again, differentiating (3) partially w.r.t. 'y', we get

$$\frac{\partial \phi}{\partial u}\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) + \frac{\partial \phi}{\partial v}\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial y}\right) = 0$$

or $(\partial\phi/\partial u)\,(2y + 2qz) + (\partial\phi/\partial v)\,(-2x + 2qz) = 0$, by (5)

or $(y + qz)\,(\partial\phi/\partial u) = (x - qz)\,(\partial\phi/\partial v).$ ...(7)

Dividing (6) by (7), $(x + pz)/(y + qz) = (y - pz)/(x - qz)$

or $pz(y + x) - qz(y + x) = y^2 - x^2$ or $(p - q)z = y - x.$

Ex. 13. *Eliminate the arbitrary function f and obtain the partial differential equation from* $z = e^y f(x + y)$ **[Madras 2005]**

Sol. Given $z = e^y f(x + y)$ ... (1)

Differentiating (1) partially w.r.t. x and y, we get

$\partial z/\partial x = e^y f'(x+y)$ and $\partial z/\partial y = e^y f(x+y) + e^y f'(x+y)$ ... (2)

From (1) and (2), we have $\partial z/\partial y = z + \partial z/\partial x$

Ex. 14. *If* $z = f(x+ay) + \phi(x-ay)$, *prove that* $\partial^2 z/\partial y^2 = a^2(\partial^2 z/\partial x^2)$

Hint. Refer solved Ex. 3. **[Madurai Kamraj 2008; Jabalpur 2002]**

Ex. 15. *Equation of any cone with vertex at P* (a, b, c) *is of the form* $f\left(\dfrac{x-a}{z-c}, \dfrac{y-b}{z-c}\right) = 0.$ *Find the differential equation of the cone.*

Sol. Let $(x - a) / (z - c) = u$ and $(y - b) / (z - c) = v$ ... (1)

Then, the equation of the given cone becomes $f(u, v) = 0$ ... (2)

Differentiating (2) partially with respect to 'x', we have

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = 0 \quad \text{or} \quad \frac{\partial f}{\partial u}\left(\frac{1-0}{z-c} - \frac{x-a}{(z-c)^2}\frac{\partial z}{\partial x}\right) + \frac{\partial f}{\partial v}\left(-\frac{y-b}{(z-c)^2}\frac{\partial z}{\partial x}\right) = 0, \text{ using (1)}$$

or
$$\frac{\partial f}{\partial u}\left(\frac{1}{z-c} - p\frac{x-a}{(z-c)^2}\right) - \frac{\partial f}{\partial v}\left(p\frac{y-b}{(z-c)^2}\right) = 0, \quad \text{where} \quad p = \frac{\partial z}{\partial x} \qquad \text{... (3)}$$

Differentiating (2) partially with respect to 'y', we have

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = 0 \quad \text{or} \quad \frac{\partial f}{\partial u}\left(-\frac{x-a}{(z-c)^2}\frac{\partial z}{\partial y}\right) + \frac{\partial f}{\partial v}\left(\frac{1-0}{z-c} - \frac{y-b}{(z-c)^2}\frac{\partial z}{\partial y}\right) = 0, \text{ using (1)}$$

or $$-\frac{\partial f}{\partial u}\left(q\frac{x-a}{(z-c)^2}\right)+\frac{\partial f}{\partial v}\left(\frac{1}{z-c}-q\frac{y-b}{(z-c)^2}\right)=0,\quad \text{where}\quad q=\frac{\partial z}{\partial y} \qquad \text{... (4)}$$

Eliminating $\partial f/\partial u$ and $\partial f/\partial v$ from (3) and (4), we have

$$\begin{vmatrix} \dfrac{1}{z-c}-p\dfrac{x-a}{(z-c)^2} & -p\dfrac{y-b}{(z-c)^2} \\ -q\dfrac{x-a}{(z-c)^2} & \dfrac{1}{z-c}-q\dfrac{y-b}{(z-c)^2} \end{vmatrix}=0$$

or $$\begin{vmatrix} z-c-p(x-a) & -p(y-b) \\ -q(x-a) & z-c-q(y-b) \end{vmatrix}=0$$

or $$\{z-c-p\,(x-a)\}\ \{z-c-q(y-b)\}-pq(x-a)\,(y-b)=0$$

or $(z-c)^2-p(x-a)\,(z-c)-q\,(y-b)\,(z-c)=0$ or $(x-a)p+(y-b)q=z-c.$

which in the required partial differential equation of the given cone.

EXERCISE 1 (B)

Eliminate the arbitrary functions and hence obtain the partial differential equations:

1. $z=e^{mx}\phi(x+y)$. **Ans.** $p-q=mz$

2. $z=f(x+ay)$ **[Bilaspur 1997; Jabalpur 1999]** **Ans.** $q=ap$

3. $z=xy+f(x^2+y^2)$ **[Delhi B.A./B.Sc. (Maths) (Prog.) 2007]** **Ans.** $py-qx=y^2-x^2$

4. $z=x+y+f(xy)$ **[Delhi B.A. (Prog) II 2010]** **Ans.** $px-qy=x-y$

5. $z=f(xy/z)$ **[Nagpur 1995 KU Kurukshetra 2004]** **Ans.** $px-qy=0$

6. $z=f(x-y)$ **[Delhi B.A. (Prog.) II 2011]** **Ans.** $p+q=0$

7. $z=(x-y)\,\phi(x^2+y^2)$ **Ans.** $(x-y)yp-(x-y)xq=(x+y)z$

8. $z=f(x^2+2y^2)$ **Ans.** $xq-yp=x^2-y^2$

9. $x=f(z)+g(y)$ **Ans.** $ps-qr=0$

10. $z=f(y+ax)+g(y+bx)$, $a\neq b$. **Ans.** $r-(a+b)s+abt=0$

11. $f(x+y+z)=xyz$ **Ans.** $x(y-z)p+y(z-x)q=z\,(x-y)$

12. $z=(x+y)\,f(x^2-y^2)$ **Ans.** $yp+xq=z$

13. $z=f(x)+e^y\,g(x)$ **Ans.** $t-q=0$

14. $f(x+y+z, x^2+y^2-z^2)=0$ **(CDLU 2004)** **Ans.** $p(y+z)-(x+z)q=x-y$

15. $z=f(xy)+g(x/y)$ **Ans.** $x^2(\partial^2 z/\partial x^2)-y^2(\partial^2 z/\partial y^2)+x(\partial z/\partial x)-y(\partial z/\partial y)=0$

16. $z=f(x-z)+g(x+y)$ **Ans.** $\dfrac{\partial z}{\partial y}\dfrac{\partial^2 z}{\partial x^2}+\left(1-\dfrac{\partial z}{\partial x}-\dfrac{\partial z}{\partial y}\right)\dfrac{\partial^2 z}{\partial x\partial y}-\left(1-\dfrac{\partial z}{\partial x}\right)\dfrac{\partial^2 z}{\partial y^2}=0$

17. $z=f(x\cos\alpha+y\sin\alpha-at)+\phi(x\cos\alpha+y\sin\alpha+at)$.

Ans. $\partial^2 z/\partial x^2+\partial^2 z/\partial y^2=(1/a^2)\times(\partial^2 z/\partial t^2)$

18. $y=f(x+at)+xg(x+at)$ **Ans.** $a^2(\partial^2 z/\partial x^2)-2a(\partial^2 z/\partial x\partial t)+(\partial^2 z/\partial t^2)=0$

19. $y=f(x-at)+xg(x-at)+x^2h(x-at)$. **(Jabalpur 1994)**

Ans. $\partial^3 y/\partial t^3+3a(\partial^3 y/\partial x\partial t^2)+3a^2(\partial^3 y/\partial x^2\partial t)+a^3(\partial^3 y/\partial x^3)=0$

20. $z=f(xy)+g(x+y)$ **Ans.** $x\,(y-x)r-(y^2-x^2)s+y\,(y-x)t+(p-q)\,(x+y)=0$

1.13 CAUCHY'S PROBLEM FOR FIRST ORDER EQUATIONS

The aim of an existence theorem is to establish conditions under which we can decide whether or not a given partial differential equation has a solution at all; the next step of proving that the solution, when it exists, is unique requires a uniqueness theorem. The conditions to be satisfied in the case of a first order partial differential equation are easily contained in the classic problem of Cauchy, which for the two independent variables can be stated as follows:

Cauchy's problem for first order partial differential equation

If (a) $x_0(\mu)$, $y_0(\mu)$ and $z_0(\mu)$ *are functions which, together with their first derivatives, are continuous in the interval I defined by* $\mu_1 < \mu < \mu_2$.

(b) And if f (x, y, z, p, q) is a continuous function of x, y, z, p and q in a certain region U of the xyzpq space, then it is required to establish the existence of a function $\Phi(x, y)$ *with the following properties :*

(i) $\Phi(x, y)$ *and its partial derivatives with respect to x and y are continuous functions of x and y in a region R of the xy space.*

(ii) For all values of x and y lying in R, the point $\{x, y, \Phi(x,y), \Phi_x(x,y), \Phi_y(x,y)\}$ *lies in* $\dfrac{U}{x}$ *and* $f[x, y, \Phi(x,y), \Phi_x(x,y), \Phi_y(x,y)] = 0$.

(iii) For all μ *belonging to the interval I, the point* $\{x_0(\mu), y_0(\mu)\}$ *belongs to the region R, and* $\Phi\{x_0(\mu), y_0(\mu)\} = z_0$

Stated geometrically, what we wish to prove is that there exists a surface $z = \Phi(x, y)$ *which passes through the curve C whose parametric equations are given by* $x = x_0(\mu)$, $y = y_0(\mu)$, $z = z_0(\mu)$ *and at every point of which the *direction (p, q, – 1) of the normal is such that* $f(x, y, z, p, q) = 0$

Problem 1. *State the properties of* $\Phi(x, y)$ *if there exists a surface* $z = \Phi(x, y)$ *which passes through the curve C with parametric equations* $x = x_0(\mu)$, $y = y_0(\mu)$, $z = z_0(\mu)$ *and at every point of which the direction* $(p, q, -1)$ *of the normal is such that* $f(x, y, z, p, z) = 0$. **(Delhi B.Sc. (H) 2002)**

Sol. Hint. Refer conditions (*i*), (*ii*) and (*iii*) of the above Art. 1.13

Problem 2. *Solve the Cauchy's problem for* $zp + q = 1$, *when the initial data curve is* $x_0 = \mu$, $y_0 = \mu$, $z_0 = \mu/2$, $0 \le \mu \le 1$. **[Bangalore 2003; I.A.S. 2004]**

Sol. Given $$f(x, y, z, p, q) = zp + q - 1 = 0 \qquad \text{... (1)}$$

Given inital data curve $$x_0 = \mu, \quad y_0 = \mu, \quad z_0 = \mu/2, \quad 0 \le \mu \le 1 \qquad \text{... (2)}$$

From (1), $$\partial f/\partial p = z, \qquad \partial f/\partial q = 1,$$

and $$\frac{\partial f}{\partial q}\frac{dx_0}{d\mu} - \frac{\partial f}{\partial p}\frac{dy_0}{d\mu} = 1 \times 1 - z \times 1 = 1 - \frac{1}{2}\mu \ne 0, \text{ for } 0 \le \mu \le 1.$$

Now, we have the following ordinary differential equations :

---

*Let $$z = \Phi(x, y) \qquad \text{... (1)}$$
be the equation of the given surface

Let $$F(x, y, z) = \Phi(x, y) - z. \qquad \text{... (2)}$$

From (1) and (2), $$\frac{\partial F}{\partial x} = \frac{\partial \phi}{\partial x} = \frac{\partial z}{\partial x} = p, \qquad \frac{\partial F}{\partial y} = \frac{\partial \phi}{\partial y} = \frac{\partial z}{\partial y} = q, \qquad \frac{\partial F}{\partial z} = -1$$

Since ∇F is normal to the surface $F(x, y, z) = 0$, $\partial F/\partial x$, $\partial F/\partial y$, $\partial F/\partial z$ i.e., $p, q - 1$ are direction ratios of the normal to $F(x, y, z) = 0$ or $z = \Phi(x, y)$.

$$\frac{dx}{dt} = \frac{\partial f}{\partial p}, \qquad \frac{dy}{dt} = \frac{\partial f}{\partial q} \qquad \text{and} \qquad \frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

or $\qquad dx/dt = z, \qquad dy/dt = 1 \qquad \text{... (3)}$

and $\qquad dz/dt = p(\partial f/\partial p) + q\,(\partial f/\partial q) = pz + q = 1,\ by\ (1) \qquad \text{... (4)}$

Integrating (3) and (4), $\quad y = t + C_1 \quad$ and $\quad z = t + C_2 \quad$ ... (5)

From (2), at $t = 0$, $\quad x(\mu, 0) = \mu, \quad y(\mu, 0) = \mu \quad$ and $\quad z(\mu, 0) = \mu/2 \quad$ ... (6)

Using (6), (5) reduces to $\quad y = t + \mu \quad$ and $\quad z = t + \mu/2 \quad$ ... (7)

Then, from (3) and (7), $dx/dt = t + \mu/2$ so that $x = (1/2) \times t^2 + (1/2) \times \mu t + C_3$ ... (8)

Using (6), (8) reduces to $\quad x = (1/2) \times t^2 + (1/2) \times \mu t + \mu \quad$ ... (9)

Solving $y = t + \mu$ with (9) for μ and t in terms of x and y, we get

$$t = \frac{y - x}{1 - (y/2)} \qquad \text{and} \qquad \mu = \frac{x - (y^2/2)}{1 - (y/2)}$$

Putting these values in $z = t + \mu/2$, the required solution passing through the initial data curve is $\quad z = \{2(y - x) + x - y^2/2\}/(2 - y)$.

OBJECTIVE PROBLEMS ON CHAPTER 1

Indicate the correct answer by writing (a), (b), (c) *or* (d)

1. Equation $p \tan y + q \tan x = \sec^2 z$ is of order
   (a) 1 (b) 2 (c) 0 (d) none of these **[Agra 2005, 2008]**
2. Equation $\partial^2 z/\partial x^2 - 2(\partial^2 z/\partial x \partial y) + (\partial z/\partial y)^2 = 0$ is of order
   (a) 1 (b) 2 (c) 3 (d) none of these **[Agra 2005, 2006]**
3. The equation $(2x + 3y)p + 4xq - 8pq = x + y$ is
   (a) linear (b) non-linear (c) quasi-linear (d) semi-linear **[Agra 2005, 06]**
4. $(x + y - z)(\partial z/\partial x) + (3x + 2y)(\partial z/\partial y) + 2z = x + y$ is
   (a) linear (b) quasi-linear (c) semi-linear (d) non-linear

Answers **1.** (a) **2.** (b) **3.** (b) **4.** (b)

MISCELLANEOUS EXAMPLES ON CHAPTER 1

Ex.1. *Formulate a partial differential equation by eliminating arbitrary constants a and b from the equation* $(x + a)^2 + (y + b)^2 + z^2 = 1$. *Examine whether the partial differential equation is linear or non-linear. Also,* find its order and degreee. **[Delhi Maths (H) 2008]**

Hint. Proceed as in Ex. 5(a), page 1.6 with $\lambda = 1$. *Thus we get the partial differential equation* $z^2\left\{(\partial z/\partial x)^2 + (\partial z/\partial y)^2 + 1\right\} = 1$, *which is non-linear partial differential equation of order one and degree two.*

Ex. 2. *Eliminate arbitrary constants a and b from the following equations :*

(i) $ax^2 + by^2 + z^2 = 1$ **(Delhi B.A. (Prog.) II 2010)**

(ii) $z = ax + (1 - a)\,y + b$ **(Lucknow 2010)**

Ans. (i) $z\,(z - px - qy) = 1$ (ii) $p + q = 1$, where $p = \partial z/\partial x, q = \partial z/\partial y$

Ex. 3. (i) *Eliminate the arbitrary function* ϕ *from* $p + x - y = \phi\,(q - x + y)$ **(Ranchi 2010)**

(*ii*) *State true or false with justification. Eliminating arbitrary function f from* $z = f(x^2 + y^2)$*, we get first order non-linear partial differential equation.* **(Pune 2010)**

Ans. (*i*) $(1+\partial^2 z/\partial x^2)(1+\partial^2 z/\partial y^2) = (\partial^2 z/\partial x\partial y - 1)^2$ (*ii*) False. see Ex. 4 (ii), page 1.21.

Ex. 4. (*i*) *Obtain the partial differential equation by eliminating arbitrary function of f and g from the equation* $v = \{f(r-at) + g(r+at)\}/r$ **(Nagpur 2010)**

Ans. Given $v = (1/r) \times \{f(r-at) + g(r+at)\}$ ...(1)

(1) $\Rightarrow \partial v/\partial t = (1/r) \times \{-af'(r-at) + ag'(r+at)\} = -(a/r) \times \{f'(r-at) - g'(r+at)\}$ ...(2)

(2) $\Rightarrow \partial^2 v/\partial t^2 = -(a/r) \times \{-af''(r-at) - ag''(r+at)\} = (a^2/r) \times \{f''(r-at) + g''(r+at)\}$ ...(3)

(1) $\Rightarrow \partial v/\partial r = (1/r) \times \{(f'(r-at) + g'(r+at)\} - (1/r^2) \times \{f(r-at) + g(r+at)\}$ ...(4)

(4) $\Rightarrow \partial^2 v/\partial r^2 = (1/r) \times \{f''(r-at) + g''(r+at)\} - (1/r^2) \times \{f'(r-at) + g'(r+at)\}$

$= -(1/r^2) \times \{f'(r-at) + g'(r+at)\} + (2/r^3) \times \{f(r-at) + g(r+at)\}$

$= (1/a^2) \times (\partial^2 v/\partial t^2) - (2/r^2) \times \{f'(r-at)\} + g'(r+at)\} + (2/r^2) \times v$, using (1) and (3)

$= (1/a^2) \times (\partial^2 v/\partial t^2) - (2/r) \times [\partial v/\partial r + (1/r^2) \times \{f(r-at)\} + g(r+at)\} + (2/r^2) \times v$

[Since from (4), $(1/r) \times \{f'(r-at) + g'(r+at)\} = \partial v/\partial r + (1/r^2) \times \{f(r-at) + g(r+at)\}$]

Thus, $\partial^2 v/\partial r^2 = (1/a^2) \times (\partial^2 v/\partial r^2) - (2/r) \times \{\partial v/\partial r + (1/r) \times v\} + (2/r^2) \times v$, using (1)

or $\partial^2 v/\partial r^2 = (1/a^2) \times (\partial^2 v/\partial t^2) - (2/r) \times (\partial^2/\partial r)$, which is the required equation

2

Linear Partial differential equations of order one

2.1. LAGRANGE'S EQUATION

A quasi–linear partial differential equation of order one is of the form $Pp + Qq = R$, where P, Q and R are functions of x, y, z. Such a partial differential equation is known as *Lagrange equation.*

For Example $xyp + yzq = zx$ is a Lagrange equation.

2.2. Lagrange's method of solving $Pp + Qq = R$, when P, Q and R are functions of x, y, z (Delhi Maths (H) 2009; Meerut 2003; Poona 2003, 10; Lucknow 2010)

Theorem. *The general solution of Lagrange equation*

$$Pp + Qq = R, \qquad \text{... (1)}$$

is

$$\phi(u, v) = 0 \qquad \text{... (2)}$$

where ϕ *is an arbitrary function and*

$$u(x, y, z) = c_1 \quad \text{and} \quad v(x, y, z) = c_2 \qquad \text{... (3)}$$

are two independent solutions of

$$(dx)/P = (dy)/Q = (dz)/R \qquad \text{... (4)}$$

Here, c_1 and c_2 are arbitrary constants and at least one of u, v must contain z. Also recall that u and v are said to be independent if u/v is not merely a constant.

Proof. Differentiating (2) partially w.r.t. 'x' and 'y', we get

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right) + \frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0 \qquad \text{... (5)}$$

and

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right) + \frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) = 0 \qquad \text{... (6)}$$

Eliminating $\partial\phi/\partial u$ and $\partial\phi/\partial v$ between (5) and (6), we have

$$\begin{vmatrix} \partial u/\partial x + p(\partial u/\partial z) & \partial v/\partial x + p(\partial v/\partial z) \\ \partial u/\partial y + q(\partial u/\partial z) & \partial v/\partial y + q(\partial v/\partial z) \end{vmatrix} = 0$$

or

$$\left(\frac{\partial u}{\partial x} + p\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial y} + q\frac{\partial v}{\partial z}\right) - \left(\frac{\partial u}{\partial y} + q\frac{\partial u}{\partial z}\right)\left(\frac{\partial v}{\partial x} + p\frac{\partial v}{\partial z}\right) = 0$$

or

$$\left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial z}\right)p + \left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial x}\right)q + \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = 0$$

$$\therefore \quad \left(\frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y}\right)p + \left(\frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z}\right)q = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \qquad \text{... (7)}$$

Hence (2) is a solution of the equation (7)

Taking the differentials of $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$, we get

$$(\partial u/\partial x)dx + (\partial u/\partial y)dy + (\partial u/\partial z)dz = 0 \qquad \text{... (8)}$$

and $$(\partial v/\partial x)dx + (\partial v/\partial y)dy + (\partial v/\partial z)dz = 0 \qquad \text{... (9)}$$

Since u and v are independent functions, solving (8) and (9) for the ratios $dx : dy : dz$, gives

$$\frac{dx}{\dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial z} - \dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial y}} = \frac{dy}{\dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial z}} = \frac{dz}{\dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial y} - \dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial x}} \qquad \text{... (10)}$$

Comparing (4) and (10), we obtain

$$\frac{\dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial z} - \dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial y}}{P} = \frac{\dfrac{\partial u}{\partial z}\dfrac{\partial v}{\partial x} - \dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial z}}{Q} = \frac{\dfrac{\partial u}{\partial x}\dfrac{\partial v}{\partial y} - \dfrac{\partial u}{\partial y}\dfrac{\partial v}{\partial x}}{R} = k, \text{ say}$$

$$\Rightarrow \quad \frac{\partial u}{\partial y}\frac{\partial v}{\partial z} - \frac{\partial u}{\partial z}\frac{\partial v}{\partial y} = kP, \qquad \frac{\partial u}{\partial z}\frac{\partial v}{\partial x} - \frac{\partial u}{\partial x}\frac{\partial v}{\partial z} = kQ \qquad \text{and} \qquad \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = kR$$

Substituting these values in (7), we get $\quad k(Pp + Qq) = kR \quad$ or $\quad Pp + Qq = R$, which is the given equation (1).

Therefore, if $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ are two independent solutions of the system of differential equations $(dx)/P = (dy)/Q = (dz)/R$, then $\phi(u, v) = 0$ is a solution of $Pp + Qq = R$, ϕ being an arbitrary function. This is what we wished to prove.

Note. Equations (4) are called *Lagrange's auxillary* (*or subsidiary*) equations for (1).

2.3. Working Rule for solving *Pp* + *Qq* = *R* by Lagrange's method.

[Delhi Maths Hons. 1998]

Step 1. Put the given linear partial differential equation of the first order in the standard form

$$Pp + Qq = R. \qquad \text{...(1)}$$

Step 2. Write down Lagrange's auxiliary equations for (1) namely,

$$(dx)/P = (dy)/Q = (dz)/R \qquad \text{...(2)}$$

Step 3. Solve (2) by using the well known methods (refer Art. 2.5, 2.7, 2.9 and 2.11). Let $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ be two independent solutions of (2).

Step 4. The general solution (or integral) of (1) is then written in one of the following three equivalent forms :

$\phi(u, v) = 0, \qquad u = \phi(v) \qquad$ or $\qquad v = \phi(u)$, ϕ being an arbitrary function.

2.4. Examples based on working rule 2.3. In what follows we shall discuss four rules for getting two independent solutions of $(dx)/P = (dy)/Q = (dz)/R$. Accordingly, we have four types of problems based on $Pp + Qq = R$.

2.5. Type 1 based on Rule I for solving $(dx)/P = (dy)/Q = (dz)/R$. ...(1)

Suppose that one of the variables is either absent or cancels out from any two fractions of given equations (1). Then an integral can be obtained by the usual methods. The same procedure can be repeated with another set of two fractions of given equations (1).

2.6. SOLVED EXAMPLES BASED ON ART. 2.5

Ex. 1. *Solve* $(y^2z/x)p + xzq = y^2$. **[Indore 2004; Sagar 1994]**

Sol. Given $$(y^2z/x)p + xzq = y^2. \qquad \text{...(1)}$$

The Lagrange's auxiliary equations for (1) are $$\frac{dx}{(y^2z/x)} = \frac{dy}{xz} = \frac{dz}{y^2}. \qquad \text{..(2)}$$

Taking the first two fractions of (2), we have

$$x^2zdx = y^2zdy \qquad \text{or} \qquad 3x^2dx - 3y^2dy = 0, \qquad \text{...(3)}$$

Integrating (3), $x^3 - y^3 = c_1$, c_1 being an arbitrary constant ...(4)

Next, taking the first and the last fractions of (2), we get

$xy^2 dx = y^2 z dz$ or $2x dx - 2z dz = 0.$ ...(5)

Integrating (5), $x^2 - z^2 = c_2$, c_2 being an arbitrary constant ...(6)

From (4) and (6), the required general integral is

$$\phi(x^3 - y^3, x^2 - z^2) = 0, \ \phi \text{ being an arbitrary function.}$$

Ex. 2. *Solve* (*i*) $a(p + q) = z$. **[Bangalore 1997]** (*ii*) $2p + 3q = 1$. **[Bangalore 1995]**

Sol. (*i*) Given $ap + aq = z.$ ...(1)

The Lagrange's auxiliary equation for (1) are $(dx)/a = (dy)/a = (dz)/1.$ ...(2)

Taking the first two members of (1), $dx - dy = 0.$ ...(3)

Integrating (3), $x - y = c_1$, c_1 being an arbitrary constant ...(4)

Taking the last two members of (1), $dy - a dz = 0.$ ...(5)

Integrating (5), $y - az = c_2$, c_2 being an arbitrary constant. ...(6)

From (4) and (6), the required solution is given by

$$\phi(x - y, y - az) = 0, \ \phi \text{ being an arbitrary function.}$$

Ex. 3. *Solve* $p \tan x + q \tan y = \tan z$. **[Madras 2005 ; Kanpur 2007]**

Sol. Given $(\tan x)p + (\tan y)q = \tan z.$ ...(1)

The Lagrange's auxiliary equations for (1) are $\dfrac{dx}{\tan x} = \dfrac{dy}{\tan y} = \dfrac{dz}{\tan z}.$ ...(2)

Taking the first two fractions of (2), $\cot x \, dx - \cot y \, dy = 0.$

Integrating, $\log \sin x - \log \sin y = \log c_1$ or $(\sin x)/(\sin y) = c_1.$ ...(3)

Taking the last two fractions of (2), $\cot y \, dy - \cot z \, dz = 0.$

Integrating, $\log \sin y - \log \sin z = \log c_2$ or $(\sin y)/(\sin z) = c_2.$ ...(4)

From (3) and (4), the required general solution is

$$\sin x/\sin y = \phi(\sin y/\sin z), \ \phi \text{ being an arbitrary function.}$$

Ex. 4. *Solve* $zp = -x$.

Sol. Given $zp + 0.q = -x.$ ...(1)

The Lagrange's subsidiary equations for (1) are $(dx)/z = (dy)/0 = (dz)/(-x)$ ...(2)

Taking the first and the last members of (2), we get

$-x dx = z dz$ or $2x dx + 2z dz = 0.$ ...(3)

Integrating (3), $x^2 + z^2 = c_1$, c_1 being an arbitrary constant. ...(4)

Next, the second fraction of (2) implies that $dy = 0$ giving $y = c_2$ ...(3)

From (4) and (5), the required solution is $x^2 + z^2 = \phi(y)$, ϕ being an arbitrary function.

Ex. 5. *Solve* $y^2 p - xyq = x(z - 2y)$ **[Delhi Maths Hons. 1995, Delhi Meths(G) 2006]**

Sol. Here Lagrange's auxiliary equations are $\dfrac{dx}{y^2} = \dfrac{dy}{-xy} = \dfrac{dz}{x(z-2y)}.$ ...(1)

Taking the first two fractions of (1) and re–writing, we get

$2x dx + 2y dy = 0$ so that $x^2 + y^2 = c_1.$ ...(2)

Now, taking the last two fractions of (1) and re–writing, we get

$\dfrac{dz}{dy} = -\dfrac{z-2y}{y}$ or $\dfrac{dz}{dy} + \dfrac{1}{y} z = 2$ ...(3)

which is linear in z and y. Its I.F. $= e^{\int (1/y) dy} = e^{\log y} = y$. Hence solution of (3) is

$z \,.\, y = \int 2y\,dy + c_2$ or $zy - y^2 = c_2$. ...(4)

Hence $\phi(x^2 + y^2, zy - y^2) = 0$ is the desired solution, where ϕ is an arbitrary function.

Ex. 6. *Solve* $(x^2 + 2y^2)p - xyq = xz$ **[K.U. Kurukshetra 2005]**

Sol. The Lagrange's auxiliary equation for the given equation are

$$\frac{dx}{x^2+2y^2} = \frac{dy}{-xy} = \frac{dz}{xz} \qquad \text{... (1)}$$

Taking the last two fractions of (2) and re–writing, we get

$(1/y)\,dy + (1/z)dz = 0$ so that $\log y + \log z = \log c_1$ or $yz = c_1$ ... (2)

Taking the first two fractions of (1), we have

$$\frac{dx}{dy} = \frac{x^2+2y^2}{-xy} \qquad \text{or} \qquad 2x\frac{dx}{dy} + \left(\frac{2}{y^2}\right)x^2 = -4y \qquad \text{... (3)}$$

Putting $x^2 = v$ and $2x(dx/dy) = dv/dx$, (3) yields

$dv/dx + (2/y)\,v = -\,4y$, which is a linear equation.

Its integrating factor $= e^{\int (2/y)dy} = e^{2\log y} = y^2$ and hence its solution is

$$yv^2 = \int \left\{(-\,4y)xy^2\right\} dy + c_2 \qquad \text{or} \qquad y^2x^2 + y^4 = c_2 \qquad \text{... (4)}$$

From (2) and (4), the required solution is $\phi(yz,\, y^2x^2 + y^4) = 0$, ϕ being an arbitrary function.

EXERCISE 2 (A)

Solve the following partial differential equations

1. $(-a + x)p + (-b + y)q = (-c + z)$. **Ans.** $\phi\{(x-a)/(y-b), (y-b)/(z-c)\} = 0$
2. $xp + yq = z$ **(Kanpur 2011)** **Ans.** $\phi(x/z,\, y/z) = 0$
3. $p + q = 1$ **Ans.** $\phi(x-y,\, x-z) = 0$
4. $x^2p + y^2p = z^2$ **[Bilaspur 2001, Jabalpur 2000, Sagar 2000, Vikram 1999]**
   Ans. $\phi(1/x - 1/y, 1/y - 1/z) = 0$
5. $x^2p + y^2q + z^2 = 0$ **Ans.** $\phi(1/x - 1/y, 1/y + 1/z) = 0$
6. $\partial z/\partial x + \partial z/\partial y = \sin x$ **[Meerut 1995]** **Ans.** $\phi(x-y,\, z+\cos x) = 0$
7. $yzp + 2xq = xy$ **[Nagpur 1996]** **Ans.** $\phi(x^2 - z^2,\, y^2 - 4z) = 0$
8. $xp + yq = z$ **[Bangalore 1995]** **Ans.** $\phi(x/y,\, x/z) = 0$
9. $yzp + zxq = xy$ **[M.S. Univ. T.N. 2007, Lucknow 2010, Revishankar 2004]**
   Ans. $\phi(x^2 - y^2,\, x^2 - z^2) = 0$
10. $zp = x$ **Ans.** $\phi(y,\, x^2 - z^2) = 0$
11. $y^2p^2 + x^2q^2 = x^2y^2z^2$ **Ans.** $\phi(x^3 - y^3,\, y^3 + 3z^{-1}) = 0$

2.7. Type 2 based on Rule II for solving $(dx)/P = (dy)/Q = (dz)/R$. ...(1)

Suppose that one integral of (1) is known by using rule I explained in Art 2.5 and suppose also that another integral cannot be obtained by using rule I of Art. 2.5. Then one integral known to us is used to find another integral as shown in the following solved examples. Note that in the second integral, the constant of integration of first integral should be removed later on.

2.8. SOLVED EXAMPLES BASED ON ART. 2.7

Ex. 1. *Solve* $p + 3q = 5z + \tan(y - 3x)$.

[Agra 2006; Meerut 2003; Indore 2002; Ravishankar 2003]

Sol. Given $\qquad p + 3q = 5z + \tan(y - 3x)$. ...(1)

The Lagrange's subsidiary equations for (1) are $\dfrac{dx}{1} = \dfrac{dy}{3} = \dfrac{dz}{5z+\tan(y-3x)}$. ...(2)

Taking the first two fractions, $\qquad dy - 3dx = 0$. ...(3)

Integrating (3), $y - 3x = c_1$, c_1 being an arbitrary constant. ...(4)

Using (4), from (2) we get $\qquad \dfrac{dx}{1} = \dfrac{dz}{5z + \tan c_1}$. ...(5)

Integrating (5), $x - (1/5) \times \log(5z + \tan c_1) = (1/5) \times c_2$, c_2 being an arbitrary constant.

or $\qquad 5x - \log[5z + \tan(y - 3x)] = c_2$, using (4) ...(6)

From (4) and (6), the required general integral is

$$5x - \log[5z + \tan(y - 3x)] = \phi(y - 3x), \text{ where } \phi \text{ is an arbitrary function.}$$

Ex. 2. *Solve* $z(z^2 + xy)(px - qy) = x^4$.

Sol. Given $\qquad xz(z^2 + xy)p - yz(z^2 + xy)q = x^4$. ...(1)

The Lagrange's subsidiary equations for (1) are $\dfrac{dx}{xz(z^2+xy)} = \dfrac{dy}{-yz(z^2+xy)} = \dfrac{dz}{x^4}$. ...(2)

Cancelling $z(z^2 + xy)$, the first two fractions give

$(1/x)\,dx = -(1/y)\,dy \qquad$ or $\qquad (1/x)\,dx + (1/y)\,dy = 0$. ...(3)

Integrating (3), $\quad \log x + \log y = \log c_1 \qquad$ or $\qquad xy = c_1$. ...(4)

Using (4), from (2) we get $\qquad \dfrac{dx}{xz(z^2+c_1)} = \dfrac{dz}{x^4}$

or $\qquad x^3dx = z(z^2 + c_1)dz \qquad$ or $\qquad x^3dx - (z^3 + c_1z)dz = 0$. ...(5)

Integrating (5), $x^4/4 - z^4/4 - (c_1z^2)/2 = c_2/4 \qquad$ or $\qquad x^4 - z^4 - 2c_1z^2 = c_2$

or $\qquad x^4 - z^4 - 2xy\,z^2 = c_2$, using (4) ...(6)

From (4) and (6), the required general integral is

$$\phi(xy,\ x^4 - z^4 - 2xy\,z^2) = 0, \ \phi \text{ being an arbitrary function.}$$

Ex. 3. *Solve* $xyp + y^2q = zxy - 2x^2$. **[Garhwal 2005]**

Sol. Given $\qquad xyp + y^2q = zxy - 2x^2$. ...(1)

The Lagrange's subsidiary equations for (1) are $\dfrac{dx}{xy} = \dfrac{dy}{y^2} = \dfrac{dz}{zxy - 2x^2}$. ...(2)

Taking the first two fractions of (2), we have

$(dx)/xy = (dy)/y^2 \qquad$ or $\qquad (1/x)dx - (1/y)dy = 0$ ...(3)

Integrating (3), $\quad \log x - \log y = \log c_1 \qquad$ or $\qquad x/y = c_1$. ...(4)

From (4), $x = c_1y$. Hence from second and third fractions of (2), we get

$\dfrac{dy}{y^2} = \dfrac{dz}{c_1zy^2 - 2c_1^2y^2} \qquad$ or $\qquad c_1dy - \dfrac{dz}{z - 2c_1^2} = 0$. ...(5)

Integrating (5), $c_1y - \log(z - 2c_1^2) = c_2 \quad$ or $\quad x - \log[z - 2(x^2/y^2)] = c_2$, using (4). ...(6)

From (4) and (6), the required general solution is

$$x - \log[z - 2(x^2/y^2)] = \phi(x/y), \ \phi \text{ being an arbitrary function.}$$

Ex. 4. *Solve* $xzp + yzq = xy$. **[Bhopal 1996; Jabalpur 1999; Jiwaji 2000; Punjab 2005; Agra 2007; Ravishanker 1996; Vikram 2000]**

Sol. Given $$xzp + yzq = xy. \qquad ...(1)$$

The Lagrange's subsidiary equations for (1) are $$\frac{dx}{xz} = \frac{dy}{yz} = \frac{dz}{xy}. \qquad ...(2)$$

Taking the first two fractions of (2), $(1/x)dx - (1/y)dy = 0$ ...(3)

Integrating (3), $\log x - \log y = \log c_1$ or $x/y = c_1$. ...(4)

From (4), $x = c_1 y$. Hence, from second and third fractions of (2), we get

$(1/yz)dy = (1/c_1 y^2)dz$ or $2c_1 y\, dy - 2z\, dz = 0$. ...(5)

Integrating (5), $c_1 y^2 - z^2 = c_2$ or $xy - z^2 = c_2$, using (4). ...(6)

From (4) and (6), the required solution is $\phi(xy - z^2, x/y) = 0$, ϕ being an arbitrary function.

Ex. 5. *Solve* $py + qx = xyz^2 (x^2 - y^2)$.

Sol. Given $$py + qx = xyz^2 (x^2 - y^2). \qquad ...(1)$$

The Lagrange's auxiliary equations for (1) are $$\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{xyz^2(x^2 - y^2)}. \qquad ...(2)$$

Taking the first two fractions of (2), $2xdx - 2ydy = 0$. ...(3)

Integrating. $x^2 - y^2 = c_1$, c_1 being an arbitrary constant. ...(4)

Using (4), the last two fractions of (2) give

$(dy)/x = (dz)/(xyz^2 c_1)$ or $2c_1 y\, dy - 2z^{-2} dz = 0$. ...(5)

Integrating (5), $c_1 y^2 + (2/z) = c_2$, c_2 being an arbitrary constant.

or $$y^2 (x^2 - y^2) + (2/z) = c_2, \text{ using (4)}. \qquad ...(6)$$

From (4) and (6), the required general solution is

$$y^2 (x^2 - y^2) + (2/z) = \phi(x^2 - y^2), \text{ where } \phi \text{ is an arbitrary function.}$$

Ex. 6. *Solve* $xp - yq = xy$ **[Madras 2005]**

Sol. The Lagrange's auxiliary equations for the given equation are

$$(dx)/x = (dy)/(-y) = (dz)/(xy) \qquad ... (1)$$

Taking the first two fractions of (1), $(1/x)dx + (1/y)dy = 0$

Intergrating, $\log x + \log y = c_1$ so that $xy = c_1$ ... (2)

Using (2), (1) yields $(1/x)dx = (1/c_1)\, dz$ so that $\log x - \log c_2 = z/c_1$

or $\log (x/c_2) = z/c_1$ or $\log (x/c_2) = z/(xy)$, by (2)

Thus, $x/c_2 = e^{z/(xy)}$ or $xe^{-z/(xy)} = c_2$, c_2 being an arbitrary constant. ... (3)

From (2) and (3), the required solution is $x\, e^{-z/(xy)} = \phi(xy)$, ϕ being an arbitrary function

Ex. 7. *Solve* $p + 3q = z + \cot (y - 3x)$. **[M.D.U Rohtak 2006]**

Sol. The Lagrange's auxiliary equation for the given equation are

$$\frac{dx}{1} = \frac{dy}{3} = \frac{dz}{z + \cot(y - 3x)} \qquad ... (1)$$

Taking the first two fractions of (1), $dy - 3\, dx = 0$ so that $y - 3x = c_1$ ... (2)

Taking the first and last fraction of (1), we have

$$dx = \frac{dz}{z + \cot(y - 3x)} \quad \text{or} \quad dx = \frac{dz}{z + \cot c_1}, \text{ using (2)}$$

Intergrating, $x = \log | z + \cot c_1| + c_2$, c_1 and c_2 being an arbitrary constants.

or $\qquad x - \log|z + \cot(y - 3x)| = c_2$, using (2) $\qquad$ ... (3)

From (2) and (3), the required general solution is

$$x - \log|z + \cot(y - 3x)| = \phi(y - 3x), \quad \phi \text{ being an arbitrary function.}$$

Ex. 8. *Solve* $px(z - 2y^2) = (z - qy)(z - y^2 - 2x^3)$ **[Delhi B.Sc. II 2008; Delhi B.A. II 2010]**

Sol. Re–writing the given equation, we have

$$x(z - 2y^2)\,p + y\,(z - y^2 - 2x^3)\,q = z(z - y^2 - 2x^3) \qquad \text{... (1)}$$

The Lagrange's subsidiary equations for (1) are

$$\frac{dx}{x(z-2y^2)} = \frac{dy}{y(z-y^2-2x^3)} = \frac{dz}{z(z-y^2-2x^3)} \qquad \text{... (2)}$$

Taking the last two fraction, we get $\qquad (1/z)dz = (1/y)dy$

Integrating, $\qquad \log z = \log y + \log a \qquad$ or $\qquad z/y = a \qquad$ ... (3)

where a is an arbitrary constant. Using (3), (2) yields

$$\frac{dx}{x(ay-2y^2)} = \frac{dy}{y(ay-y^2-2x^3)} \qquad \text{so that} \quad (ay - y^2 - 2x^3)\,dx + x\,(2y - a)\,dy = 0 \quad \text{... (4)}$$

Comparing (4) with $Mdx + Ndy = 0$, here $M = ay - y^2 - 2x^3$ and $N = x\,(2y - a)$. Then $\partial M/\partial y = a - 2y$ and $\partial N/\partial x = 2y - a$. Now, we have

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = \frac{1}{x(2y-a)} \times 2(a-2y) = -\frac{2}{x}, \text{ which is a function of } x \text{ alone.}$$

Hence, by usual rule, integrating factor of (1) $= e^{\int(-2/x)dx} = e^{-2\log x} = e^{x^{-2}} = x^{-2}$

Multiplying (4) by x^{-2}, we get exact equation $(ayx^{-2} - y^2x^{-2} - 2x)dx + x^{-1}(2y - a)\,dy = 0$

By the usual rule of solving an exact equation, its solution is

$$\int\left\{(ay-y^2)x^{-2} - 2x\right\}dx + \int x^{-1}(2y-a)dy \qquad = b$$

(Treating y as constant) $\qquad$ (Integrating terms free from x)

or $\quad (ay - y^2) \times (-1/x) - x^2 = b \qquad$ or $\qquad (y^2 - ax)/x - x^2 = b$

or $\quad (y^2 - ax - x^3)/x = b$, where b is an arbitrary constant. $\qquad$ ... (5)

From (3) and (5), required solution is $(y^2 - ax - x^3)/x = \phi(z/y)$, ϕ being an arbitrary function

EXERCISE 2 (B)

Solved the following differential equations:

1. $p - 2q = 3x^2 \sin(y + 2x)$. **Ans.** $x^2 \sin(y+2x) - z = \phi(y+2x)$

2. $p - q = z/(x + y)$. **Ans.** $x - (x+y)\log z = \phi(x+y)$

3. $xy^2p - y^3q + axz = 0$. **Ans.** $\log z + (ax/3y^2) = \phi(xy)$

4. $(x^2 - y^2 - z^2)p + 2xyq = 2xz$. **Ans.** $(x^2 + y^2 + z^2)/z = \phi(y/z)$

5. *(a)* $z(p - q) = z^2 + (x + y)^2$. **(Meerut 2011)** **Ans.** $e^{2y}[z^2 + (x+y)^2] = \phi(x+y)$

(b) $z(p+q) = z^2 + (x-y)^2$ **Ans.** $e^{2y}[z^2 + (x-y)^2] = \phi(x-y)$

6. $p - 2q = 3x^2 \sin(y + 2x)$. **Ans.** $x^3 \sin(y+2x) - z = \phi(y+2x)$

7. $p - q = z/(x + y)$. **Ans.** $x - (x+y)\log z = \phi(x+y)$

8. $zp - zq = x + y$. **Ans.** $2x(x+y) - z^2 = \phi(x+y)$

9. $xyp + y^2q + 2x^2 - xyz = 0$. **Ans.** $x - \log|z - (2x/y)| = \phi(x/y)$

2.9. Type 3 based on Rule III for solving $(dx)/P = (dy)/Q = (dz)/R$. ...(1)

Let P_1, Q_1 and R_1 be functions of x, y and z. Then, by a well–known principle of algebra, each fraction in (1) will be equal to $(P_1 dx + Q_1 dy + R_1 dz)/(P_1 P + Q_1 Q + R_1 R)$. ...(2)

If $P_1P + Q_1Q + R_1R = 0$, then we know that the numerator of (2) is also zero. This gives $P_1dx + Q_1dy + R_1dz = 0$ which can be integrated to give $u_1(x, y, z) = c_1$. This method may be repeated to get another integral $u_2(x, y, z) = c_2$. P_1, Q_1, R_1 are called multipliers. As a special case, these can be constants also. Sometimes only one integral is possible by use of multipliers. In such cases second integral should be obtained by using rule I of Art. 2.5 or rule II of Art. 2.7 as the case may be.

2.10. SOLVED EXAMPLES BASED ON ART. 2.9

Ex.1. *Solve* $\{(b - c)/a\}yzp + \{(c - a)/b\}zxq = \{(a - b)/c\}xy$.

Sol. Given $\{(b - c)/a\}yzp + \{(c - a)/b\}zxq = \{(a - b)/c\}xy$. ...(1)

The Lagrange's subsidiary equations of (1) are $\dfrac{a\,dx}{(b-c)yz} = \dfrac{b\,dy}{(c-a)zx} = \dfrac{c\,dz}{(a-b)xy}$. ...(2)

Choosing x, y, z as multipliers, each fraction for (2)

$$= \frac{a\,xdx + by\,dy + cz\,dz}{xyz\,[(b-c)+(c-a)+(a-b)]} = \frac{ax\,dx + by\,dy + cz\,dz}{0}.$$

$\therefore$ $ax\,dx + by\,dy + cz\,dz = 0$ or $2axdx + 2bydy + 2czdz = 0$.

Integrating, $ax^2 + by^2 + cz^2 = c_1$, c_1 being an arbitrary constant. ...(3)

Again, choosing ax, by, cz as multipliers, each fraction of (2)

$$= \frac{a^2xdx + b^2ydy + c^2\,zdz}{xyz\,[a(b-c)+b(c-a)+c(a-b)]} = \frac{a^2xdx + b^2ydy + c^2\,zdz}{0}.$$

$\therefore$ $a^2xdx + b^2ydy + c^2zdz = 0$ or $2a^2xdx + 2b^2ydy + 2c^2zdz = 0$.

Integrating, $a^2x^2 + b^2y^2 + c^2z^2 = c_2$, c_2 being an arbitrary constant. ...(4)

From (3) and (4), the required general solution is given by

$$\phi\,(ax^2 + by^2 + cz^2,\ a^2x^2 + b^2y^2 + c^2z^2) = 0, \text{ where } \phi \text{ is an arbitrary function.}$$

Ex. 2. *Solve* $z(x + y)p + z(x - y)q = x^2 + y^2$.

Sol. Given $z(x + y)p + z(x - y)q = x^2 + y^2$. ...(1)

The Langrange's subsidiary equations for (1) are $\dfrac{dx}{z(x+y)} = \dfrac{dy}{z(x-y)} = \dfrac{dz}{x^2+y^2}$. ...(2)

Choosing x, $-y$, $-z$, as multipliers, each fraction

$$= \frac{x\,dx - y\,dy - z\,dz}{xz(x+y) - yz(x-y) - z(x^2-y^2)} = \frac{x\,dx - y\,dy - z\,dz}{0}.$$

$\therefore$ $x\,dx - y\,dy - z\,dz$ or $2x\,dx - 2y\,dy - 2z\,dz = 0$.

Integrating, $x^2 - y^2 - z^2 = c_1$, c_1 being an arbitrary constant. ...(3)

Again, choosing y, x, $-z$ as multipliers, each fraction

$$= \frac{y\,dx + x\,dy - z\,dz}{yz(x+y) + xz(x-y) - z(x^2+y^2)} = \frac{y\,dx + x\,dy - z\,dz}{0}.$$

$\therefore$ $y\,dx + x\,dy - z\,dz = 0$ or $2d(xy) - 2zdz = 0$.

Integrating, $2xy - z^2 = c_2$, c2 being an arbitrary constant. ...(4)

From (3) and (4), the required general solution is given by

$$\phi\,(x^2 - y^2 - z^2,\ 2xy - z^2) = 0, \quad \phi \text{ being an arbitrary function.}$$

Ex. 3. *Solve* $(mz - ny)p + (nx - lz)q = ly - mx$. **[Patna 2003; Madras 2005; Delhi Maths Hons. 1991; Bhopal 2004; Meerut 2008, 10; Sagar 2002; I.A.S. 1977; Kanpur 2005, 06]**

Sol. The Lagrange's auxiliary equations for the given equation are

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}. \qquad ...(1)$$

Choosing x, y, z as multipliers, each fraction of (1)

$$= \frac{xdx + ydy + zdz}{x(mz - ny) + y(nx - lz) + z(ly - mx)} = \frac{xdx + ydy + zdz}{0}$$

$\therefore\ xdx + ydy + zdz = 0$ or $2xdx + 2ydy + 2zdz = 0$

Integrating, $x^2 + y^2 + z^2 = c_1$, c_1 being an arbitrary constant. ...(2)

Again, choosing l, m, n as multipliers, each fraction of (1)

$$= \frac{ldx + mdy + n\,dz}{l(mx - ny) + m(nx - lz) + n(ly - mx)} = \frac{ldx + mdy + n\,dz}{0}.$$

$\therefore\ ldx + mdy + n\,dz = 0$ so that $l\,x + m\,y + n\,z = c_2$. ...(3)

From (2) and (3), the required general solution is given by

$$\phi\,(x^2 + y^2 + z^2,\ l\,x + m\,y + n\,z) = 0, \ \phi \text{ being an arbitrary function.}$$

Ex. 4. *Solve* $x(y^2 - z^2)q - y(z^2 + x^2)q = z(x^2 + y^2)$.

Sol. The lagrange's auxiliary equations for the given equation are

$$\frac{dx}{x(y^2 - z^2)} = \frac{dy}{-y(z^2 + x^2)} = \frac{dz}{z(x^2 + y^2)}. \qquad ...(1)$$

Choosing x, y, z, as multipliers, each fraction of (1)

$$= \frac{xdx + ydy + zdz}{x^2(y^2 - z^2) - y^2(z^2 + x^2) + z^2(x^2 + y^2)} = \frac{xdx + ydy + zdz}{0}$$

$\Rightarrow\ xdx + ydy + zdz = 0$ so that $x^2 + y^2 + z^2 = c_1$. ...(2)

Choosing $1/x, -1/y, -1/z$ as multipliers, each fraction of (1)

$$= \frac{(1/x)dx - (1/y)dy - (1/z)dz}{y^2 - z^2 + z^2 + x^2 - (x^2 + y^2)} = \frac{(1/x)dx - (1/y)dy - (1/z)dz}{0}$$

$\Rightarrow\ (1/x)dx - (1/y)dy - (1/z)dz = 0$ so that $\log x - \log y - \log z = \log c_2$

$\Rightarrow\ \log\{x/(yz)\} = \log c_2 \ \Rightarrow\ x/yz = c_2$. ...(3)

$\therefore$ The required solution is $\phi\,(x^2 + y^2 + z^2,\ x/yz) = 0$, ϕ being an arbitrary function.

Ex. 5. *Solve* $(y - zx)p + (x + yz)q = x^2 + y^2$.

Sol. The Lagrange's auxiliary equations for the given equation are

$$\frac{dx}{y - zx} = \frac{dy}{x + yz} = \frac{dz}{x^2 + y^2}. \qquad ...(1)$$

Choosing $x, -y, z$ as multipliers, each fraction of (1)

$$= \frac{xdx - ydy + zdz}{x(y - zx) - y(x + yz) + z(x^2 + y^2)} = \frac{xdx - ydy + zdz}{0}$$

$\Rightarrow$ $2xdx - 2ydy + 2zdz = 0$ so that $x^2 - y^2 + z^2 = c_1$. ...(2)

Choosing $y, x, -1$ as multipliers, each fraction of (1)

$$= \frac{ydx + xdy - dz}{y(y - zx) + x(x + yz) - (x^2 + y^2)} = \frac{d(xy) - dz}{0}$$

$\Rightarrow$ $d(xy) - dz = 0$ so that $xy - z = c_2$. ...(3)

$\therefore$ From (2) and (3) solution is $\phi(x^2 - y^2 + z^2, xy - z) = 0$, ϕ being an arbitrary function.

Ex. 6. *Solve* $x(y^2 + z)p - y(x^2 + z)q = z(x^2 - y^2)$. **[I.A.S. 2004; Agra 2005 ; Delhi Maths (H) 2006; M.S. Univ. T.N. 2007; Indore 2003; Meerut 2009; Purvanchal 2007]**

Sol. Here Lagrange's subsidiary equations for given equation are

$$\frac{dx}{x(y^2 + z)} = \frac{dy}{-y(x^2 + z)} = \frac{dz}{z(x^2 - y^2)}. \quad ...(1)$$

Choosing $1/x, 1/y, 1/z$ as multipliers, each fraction of (1)

$$= \frac{(1/x)dx + (1/y)dy + (1/z)dz}{y^2 + z - (x^2 + z) + x^2 - y^2} = \frac{(1/x)dx + (1/y)dy + (1/z)dz}{0}$$

$\Rightarrow (1/x)dx + (1/y)dy + (1/z)dz = 0$ so that $\log x + \log y + \log z = \log c_1$

$\Rightarrow$ $\log(xyz) = \log c_1$ $\Rightarrow$ $xyz = c_1$. ...(2)

Choosing $x, y, -1$ as multipliers, each fraction of (1)

$$= \frac{xdx + ydy - dz}{x^2(y^2 + z) - y^2(x^2 + z) - z(x^2 - y^2)} = \frac{xdx + ydy - dz}{0}$$

$\Rightarrow$ $x\,dx + y\,dy - z\,dz = 0$ so that $x^2 + y^2 - 2z = c_2$. ...(3)

$\therefore$ From (2) and (3), solution is $\phi(x^2 + y^2 - 2z, xyz) = 0$, ϕ is being an arbitrary function.

Ex. 7. *Solve* $(x + 2z)q + (4zx - y)q = 2x^2 + y$. **[Meerut 2005]**

Sol. Here Lagrange's auxiliary equations are $\dfrac{dx}{x + 2z} = \dfrac{dy}{4zx - y} = \dfrac{dz}{2x^2 + y}$. ...(1)

Choosing $y, x, -2z$ as multipliers, each fraction of (1)

$$= \frac{ydx + xdy - 2zdz}{y(x + 2z) + x(4zx - y) - 2z(2x^2 + y)} = \frac{d(xy) - 2zdz}{0}$$

$\Rightarrow$ $d(xy) - 2zdz = 0$ so that $xy - z^2 = c_1$. ...(2)

Choosing $2x, -1, -1$ as multipliers, each fraction of (1)

$$= \frac{2xdx - dy - dz}{2x(x + 2z) - (4zx - y) - (2x^2 + y)} = \frac{2xdx - dy - dz}{0}$$

$\Rightarrow$ $2xdx - dy - dz = 0$ so that $x^2 - y - z = c_2$. ...(3)

$\therefore$ From (2) and (3), solution is $\phi(xy - z^2, x^2 - y - z) = 0$, ϕ being an arbitrary function.

Ex. 8. *Solve* $(z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx$. **[Ranchi 2010; Meerut 1994]**

If the solution of the above equation represents a sphere, what will be the coordinates of its centre.

Sol. Here Lagrange's auxiliary equations for given equation are

$$\frac{dx}{z^2 - 2yz - y^2} = \frac{dy}{x(y + z)} = \frac{dz}{x(y - z)}. \quad ...(1)$$

Taking the last two fractions of (1), we have

$(y - z)dy = (y + z)dz$ or $2ydy - 2zdz - 2(zdy + ydz) = 0.$

Integrating, $y^2 - z^2 - 2yz = c_1$, c_1 being an arbitrary constant. ...(2)

Choosing x, y, z as multipliers, each fractrion of (1)

$$= \frac{xdx + ydy + zdz}{x(z^2 - 2yz - y^2) + xy(y+z) + xz(y-z)} = \frac{xdx + ydy + zdz}{0}$$

$\Rightarrow$ $2xdx + 2ydy + 2zdz = 0$ so that $x^2 + y^2 + z^2 = c_2$. ...(3)

From (2) and (3), solution is $\phi\,(y^2 - z^2 - 2yz, x^2 + y^2 + z^2) = 0$, ϕ being an arbitrary function.

From the solution of the given equation, it follows that if it represents a sphere, then its centre must be at (0,0,0), *i.e.*, origin.

Ex. 9. *Solve* $(y^3x - 2x^4)p + (2y^4 - x^3y)q = 9z(x^2 - y^3)$. **[Jabalpur 2004; M.S. Univ. T.N. 2007]**

Sol. Here Lagrange's auxiliary equations for the given equation are given by

$$\frac{dx}{y^3x - 2x^4} = \frac{dy}{2y^4 - x^3y} = \frac{dz}{9z(x^3 - y^3)}. \quad ...(1)$$

Taking first two fractions of (1), we have $(2y^4 - x^3y)dx = (y^3x - 2x^4)dy$

Dividing both sides by x^3y^3 gives $\left(\frac{2y}{x^3} - \frac{1}{y^2}\right)dx = \left(\frac{1}{x^2} - \frac{2x}{y^3}\right)dy$

or $\left(\frac{1}{x^2}dy - \frac{2y}{x^3}dx\right) + \left(\frac{1}{y^2}dx - \frac{2x}{y^3}dy\right) = 0$ or $d\left(\frac{y}{x^2}\right) + d\left(\frac{x}{y^2}\right) = 0.$

Integrating, $(y/x^2) + (x/y^2) = c_1$, c_1 being an arbitrary constant. ...(2)

Choosing $1/x$, $1/y$, $1/3z$ as multipliers, each fraction of (1)

$$= \frac{(1/x)dx + (1/y)dy + (1/3z)dz}{(y^3 - 2x^3) + (2y^3 - x^3) + 3(x^3 - y^3)} = \frac{(1/x)dx + (1/y)dy + (1/3z)dz}{0}$$

$\Rightarrow$ $(1/x)dx + (1/y)dy + (1/3z)dz = 0$ so that $\log x + \log y + (1/3) \times \log z = \log c_2$

$\Rightarrow$ $\log(xy\,z^{1/3}) = \log c_2$ $\Rightarrow$ $xyz^{1/3} = c_2$. ...(3)

From (2) and (3) solution is $\phi\,(xyz^{1/3}, y/x^2 + x/y^2) = 0$, ϕ being an arbitrary function.

Ex. 10. *Solve* $x^2p + y^2q = nxy$. **[Ravishankar 1998; Bhopal 1998; Jabalpur 2002]**

Sol. Here Lagrange's auxiliary equations are $(dx)/x^2 = (dy)/y^2 = (dz)/nxy$ ... (1)

Taking the first two fractions of (1), we get $x^{-2}dx - y^{-2}dy = 0.$

Integrating, $-1/x + 1/y = -c_1$ so that $(y - x)/xy = c_1$. ...(2)

Choosing $1/x$, $-1/y$, c_1/n as multipliers, each fraction of (2)

$$= \frac{(1/x)dx - (1/y)dy + (c_1/n)dz}{x - y + c_1xy} = \frac{(1/x)dx - (1/y)dy + (c_1/n)dz}{x - y + y - x}, \text{ by (2)}$$

$$= \frac{(1/x)dx + (1/y)dy + (c_1/n)dz}{0} \quad \text{so that} \quad \frac{1}{x}dx - \frac{1}{y}dy + \frac{c_1}{n}dz = 0.$$

Integrating, $\log x - \log y + (c_1/n)\,z = (c_1/n)c_2$, c_2 being an arbitrary constant.

or $z - (n/c_1)(\log y - \log x) = c_2$ or $z - (n/c_1)\log(y/x) = c_2$

or $$z - \frac{nxy}{y-x}\log\frac{y}{x} = c_2, \text{ using (2)}. \quad ...(3)$$

From (2) and (3), the required general solution is

$$\phi\left(\frac{y-x}{xy}, z-\frac{nxy}{y-x}\log\frac{y}{x}\right) = 0, \quad \phi \text{ being an arbitrary function.}$$

Ex. 11. *Solve* $(x-y)p + (x+y)q = 2xz.$

Sol. Here the Lagrange's subsidiary equations are $\dfrac{dx}{x-y} = \dfrac{dy}{x+y} = \dfrac{dz}{2xz}$. ...(1)

Taking the first two fractions of (1), $\dfrac{dy}{dx} = \dfrac{x+y}{x-y} = \dfrac{1+(y/x)}{1-(y/x)}$. ...(2)

Let $y/x = v$ *i.e.*, $y = xv$. ...(3)

From (3), $(dy/dx) = v + x(dv/dx)$. ...(4)

Using (3) and (4), (2) gives $v + x\dfrac{dv}{dx} = \dfrac{1+v}{1-v}$

or $$x\frac{dv}{dx} = \frac{1+v}{1-v} - v = \frac{1+v-v(1-v)}{1-v} = \frac{1+v^2}{1-v}$$

or $\dfrac{1-v}{1+v^2}\, dv = \dfrac{dx}{x}$ or $\left(\dfrac{2}{1+v^2} - \dfrac{2v}{1+v^2}\right)dv = \dfrac{2dx}{x}$

Integrating, $2\tan^{-1} v - \log(1+v^2) = 2\log x - \log c_1$

or $\log x^2 - \log(1+v^2) - \log c_1 = 2\tan^{-1} v$

or $\log\{x^2(1+v^2)/c_1\} = 2\tan^{-1} v$ or $x^2(1+v^2) = c_1 e^{2\tan^{-1} v}$

or $x^2[1+(y^2/x^2)] = c_1 e^{2\tan^{-1}(y/x)}$, as $v = y/x$ by (3)

or $(x^2+y^2)\, e^{-2\tan^{-1}(y/x)} = c_1$, c_1 being an arbitrary constant. ...(5)

Choosing $1, 1, -1/z$ as multipliers, each fraction of (1)

$$= \frac{dx+dy-(1/z)dz}{(x-y)+(x+y)-(1/z)\times(2xz)} = \frac{dx+dy-(1/z)dz}{0}$$

$\Rightarrow dx + dy - (1/z)dz = 0$ so that $x + y - \log z = c_2$. ...(6)

From (5) and (6), the required general solution is

$$\phi\,(x+y-\log z,\ (x^2+y^2)\, e^{-2\tan^{-1}(y/x)}) = 0, \quad \text{where } \phi \text{ is an arbitrary function.}$$

Ex. 12. *Solve* $y^2p + x^2q = x^2y^2z^2.$

Sol. Here Lagrange's auxiliary equations are $(dx)/y^2 = (dy)/x^2 = (dz)/x^2y^2z^2$. ...(1)

Taking the first two fractions of (1), we have

$3x^2dx - 3y^2dy = 0$ so that $x^3 - y^3 = c_1$. ...(2)

Choosing $x^2, y^2, -2/z^2$ as multipliers, each fraction of (1) $= \{x^2dx + y^2dy - (2/z^2)dz\}/0$

so that $3x^2dx + 3y^2dy - (6/z^2)dz = 0.$

Integrating, $x^3 + y^3 + (6/z) = c_2$, c_2 being an arbitrary constant. ...(3)

From (2) and (3), the required general solution is

$$\phi\,[x^3 - y^3,\ x^3 + y^3 + (6/z)] = 0, \quad \phi \text{ being an arbitrary function.}$$

Ex. 13. *Solve* $(3x+y-z)p + (x+y-z)q = 2(z-y).$ **[Bangalore 1992]**

Sol. Here Lagrange's auxiliary equations are $\dfrac{dx}{3x+y-z} = \dfrac{dy}{x+y-z} = \dfrac{dz}{2(z-y)}$ ...(1)

Choosing 1, –3, 1 as multipliers, each ratio of (1) $=\{dx-3dy-dz\}/0$

so that $$dx - 3dy - dz = 0.$$

Integrating, $x - 3y - z = c_1$, c_1 being an arbitrary constant. ...(2)

From (2), $$z = c_1 - x + 3y. \qquad ...(3)$$

Substituting the above value of z, the first two fractions of (2) reduce to

$$\frac{dx}{3x+y-(c_1-x+3y)} = \frac{dy}{x+y-(c_1-x+3y)} \quad \text{or} \quad \frac{dx}{2x+4y+c_1} = \frac{dy}{4y+c_1}. \qquad ...(3)$$

Let $u = 4y + c_1$ so that $dy = (1/4) \times du$. ...(4)

Then, (3) $\Rightarrow$ $\dfrac{dx}{2x+u} = \dfrac{(1/4)du}{u}$ or $\dfrac{dx}{du} = \dfrac{1}{4}\dfrac{2x+u}{u}$ or $\dfrac{dx}{du} - \dfrac{1}{2u}x = \dfrac{1}{4}$, which is linear. ...(5)

Integrating factor of (5) $= e^{-\int(1/2u)du} = e^{-(1/2)\log u} = e^{\log(u)^{-1/2}} = u^{-1/2} = 1/\sqrt{u}$.

Hence solution of (5) is $$x \times \frac{1}{\sqrt{u}} = \int \frac{1}{4}\frac{1}{\sqrt{u}}\,du + c = \frac{1}{2}\sqrt{u} + c_2$$

or $$\frac{2x-u}{\sqrt{u}} = c_2 \quad \text{or} \quad \frac{2x-(4y+c_1)}{\sqrt{4y+c_1}} = c_2, \text{ by (4)}$$

or $$\frac{2x-4y-(x-3y-z)}{\sqrt{4y+x-3y-z}} = c_2, \text{ using (2)} \quad \text{or} \quad \frac{x-y+z}{\sqrt{x+y-z}} = c_2 \qquad ...(6)$$

From (2) and (6), the required general solution is

$$\phi\left(x-3y-z, (x-y+z)/\sqrt{x+y-z}\right) = 0, \ \phi \text{ being an arbitrary function.}$$

Ex. 14. *Solve* $x(x^2 + 3y^2)p - y(3x^2 + y^2)q = 2z(y^2 - x^2)$. **[Delhi Maths Hons 95, 2000]**

Sol. Here the Lagrange's auxiliary equations for the given equation are

$$\frac{dx}{x(x^2+3y^2)} = \frac{dy}{-y(3x^2+y^2)} = \frac{dz}{2z(y^2-x^2)}. \qquad ...(1)$$

Choosing $1/x$, $1/y$, $-1/z$ as multipliers, each fraction of (1)

$$= \frac{(1/x)dx + (1/y)dy - (1/z)dz}{0} \quad \text{so that} \quad \frac{1}{x}dx + \frac{1}{y}dy - \frac{1}{z}dz = 0.$$

Integrating, $\log x + \log y - \log z = \log c_1$ so that $(xy)/z = c_1$. ...(2)

Taking the first two ratios of (1), $$\frac{dy}{dx} = -\frac{y(3x^2+y^2)}{x(x^2+3y^2)} = -\left(\frac{y}{x}\right)\frac{3+(y/x)^2}{1+3(y/x)^2}. \qquad ...(3)$$

Put $y/x = v$ or $y = xv$ so that $(dy/dx) = v + x(dv/dx)$. ...(4)

Using (4), (3) reduces to $$v + x\frac{dv}{dx} = -v\,\frac{3+v^2}{1+3v^2} \quad \text{or} \quad x\frac{dv}{dx} = -v\left[\frac{3+v^2}{1+3v^2}+1\right]$$

or $$x\frac{dv}{dx} = -\frac{4(1+v^2)v}{1+3v^2} \quad \text{or} \quad 4\frac{dx}{x} + \frac{1+3v^2}{v(1+v^2)}dv = 0$$

or $$4\,\frac{dx}{x} + \left(\frac{1}{v} + \frac{2v}{1+v^2}\right)dv, \text{ on resolving into partial fractions}$$

Integrating, $4 \log x + \log v + \log(1 + v^2)$ or $x^4 v(1 + v^2) = c_2'$

$$x^4(y/x)[1+(y/x)^2]=c_2' \quad \text{or} \quad xy(x^2+y^2)=c_2' \quad \text{or} \quad c_1 z(x^2+y^2)=c_2', \text{ by (2)}$$

or $\quad z(x^2+y^2) = c_2'/c_1 \quad$ or $\quad z(x^2+y^2)=c_2$, where $c_2 = c_2'/c_1$. ... (5)

$\therefore$ From (2) and (5) solution is $\phi(z(x^2+y^2), xy/z) = 0$, ϕ being an arbitrary function.

Ex. 15. *Solve* $(y-z)p + (z-x)q = x - y$. **[Agra 2010; Delhi Maths Hons. 1992]**

Sol. Here the Lagrange's auxiliary equations are $\quad \dfrac{dx}{y-z} = \dfrac{dy}{z-x} = \dfrac{dz}{x-y}$. ...(1)

Choosing 1, 1, 1 as multipliers, each fraction of (1)

$$= \frac{dx+dy+dz}{(y-z)+(z-x)+(x-y)} = \frac{dx+dy+dz}{0}.$$

$\therefore$ $dx + dy + dz = 0$ so that $x + y + z = c_1$. ...(2)

Choosing x, y, z as multipliers, each fraction of (1)

$$= \frac{x\,dx + y\,dy + z\,dz}{x(y-z)+y(z-x)+z(x-y)} = \frac{x\,dx+y\,dy+z\,dz}{0}$$

$\therefore$ $2x\,dx + 2y\,dy + 2z\,dz = 0$ so that $x^2 + y^2 + z^2 = c_2$ ...(3)

$\therefore$ From (2) and (3) solution is $\phi(x+y+z, x^2+y^2+z^2) = 0$, ϕ being an arbitrary of function.

Ex. 16. *Solve the general solution of the equation* $(y + zx)p - (x + yz)q + y^2 - x^2 = 0$.
[Delhi B.Sc. (Prog) II 2011; GATE 2001; Delhi Math Hons. 1997, 98]

Sol. Given $\quad (y + zx)p - (x + yz)q = x^2 - y^2$. ...(1)

Here the Lagrange's auxiliary equations are $\quad \dfrac{dx}{y+zx} = \dfrac{dy}{-(x+yz)} = \dfrac{dz}{x^2-y^2}$. ...(2)

Choosing $x, y, -z$ as multipliers, each fraction of (2)

$$= \frac{xdx+ydy-zdz}{x(y+zx)-y(x+yz)-z(x^2-y^2)} = \frac{xdx+ydy-zdz}{0}$$

$\therefore$ $xdx + ydy - zdz = 0$ so that $2xdx + 2ydy - 2zdz = 0$.

Integrating, $x^2 + y^2 - z^2 = c_1$, c_1 being an arbitrary constant. ...(3)

Choosing y, x, 1 as multipliers, each fraction of (2)

$$= \frac{ydx+xdy+dz}{y(y+zx)-x(x+yz)+x^2-y^2} = \frac{ydx+xdy+dz}{0}$$

$\therefore$ $ydx + xdy + dz = 0$ or $d(xy) + dz = 0$.

Integrating, $xy + z = c_2$, c_2 being an arbitrary constant. ...(4)

$\therefore$ The required solution is $\phi(x^2 + y^2 - z^2, xy + z) = 0$, ϕ being an arbitrary function.

Ex. 17. *Solve* $x(y - z)\,p + y\,(z - x)q = z(x - y)$, *i.e.*, $\{(y - z)/(yz)\}p + \{(z - x)/(zx)\}q = (x - y)/(xy)$. **[Delhi B.A (Prog) II 2010; I.A.S. 2005, M.S. Univ. T.N. 2007; Vikram 2003]**

Sol. Given $\quad x(y - z)p + y(z - x)q = z(x - y)$ ... (1)

The Lagrange's auxiliary equations for (1) are $\quad \dfrac{dx}{x(y-z)} = \dfrac{dy}{y(z-x)} = \dfrac{dz}{z(x-y)}$ ... (2)

Choosing $1/x$, $1/y$, $1/z$ as multipliers each fraction of (1)

$$= \frac{(1/x)dx+(1/y)dy+(1/z)dz}{(y-z)+(z-x)+(x-y)} = \frac{(1/x)dx+(1/y)dy+(1/z)dz}{0}$$

$\Rightarrow$ $(1/x)dx + (1/y)dy + (1/z)dz = 0$ so that $\log x + \log y + \log z = \log c_1$

$\therefore$ $\log (xyz) = c_1$ or $xyz = c_1$ ... (3)

Choosing 1, 1, 1 as multipliers, each fraction of (1)

$$= \frac{dx + dy + dz}{(xy - xz) + (yz - yx) + (zx - zy)} = \frac{dx + dy + dz}{0}$$

$\Rightarrow$ $dx + dy + dz = 0$ so that $x + y + z = c_2$ ... (4)

From (3) and (4), solution is $\phi(x + y + z, xyz) = 0, \phi$ being an arbitrary function.

Ex. 18. *Solve* $2y(z - 3)p + (2x - z)q = y(2x - 3)$ **[Delhi Math (H) 1999]**

Sol. The Lagrange's auxiliary equations for given equation are

$$\frac{dx}{2y(z-3)} = \frac{dy}{2x - z} = \frac{dz}{y(2x-3)} \quad \text{... (1)}$$

Taking the first and third fractions, $(2x - 3)dx = 2(z - 3)dz$.

Integrating, $x^2 - 3x = z^2 - 6z + C_1$ or $x^2 - 3x - z^2 + 6z = C_1$ ... (2)

Choosing 1, 2y, –2 as multipliers, each fraction of (1)

$$= \frac{dx + 2ydy - 2dz}{2y(z-3) + 2y(2x - z) - 2y(2x-3)} = \frac{dx + 2ydy - 2dz}{0}$$

$\therefore$ $dx + 2ydy - 2dz = 0$ so that $x + y^2 - 2z = C_2$ ... (3)

From (2) and (3), solution is $\phi(x^2 - 3x - z^2 + 6z, x + y^2 - 2z) = 0, \phi$ being an arbitrary function.

Ex. 19. *Solve* $x^2(\partial z/\partial x) + y^2(\partial z/\partial y) = (x + y)z.$ **[Delhi Maths (H) 2001]**

Sol. Re-writing the given equation $x^2p + y^2q = (x + y)z$ ... (1)

The Lagrange's auxiliary equations for (1) are $\dfrac{dx}{x^2} = \dfrac{dy}{y^2} = \dfrac{dz}{(x+y)z}$ ... (2)

Taking the first two fractions of (2), $(1/x^2)dx - (1/y^2)dy = 0.$

Integrating, $-(1/x) + (1/y) = C_1$ or $1/y - 1/x = C_1$ ... (3)

Choosing $1/x$, $1/y$, $-1/z$ as multipliers, each fraction of (2)

$$= \frac{(1/x)dx + (1/y)dy - (1/z)dz}{x + y - (x+y)} = \frac{(1/x)dx + (1/y)dy - (1/z)dz}{0}$$

$\therefore$ $(1/x)dx + (1/y)dy - (1/z)dz = 0$ so that $xy/z = C_2$ ... (4)

From (3) and (4), solution is $\Phi(1/y - 1/x, xy/z) = 0$, Φ being an arbitrary function.

Ex. 20. *Solve* $z(x + 2y)\, p - z(y + 2x)\, q = y^2 - x^2$ **[Vikram 1999]**

Sol. The Lagrange's subsidiary equations are $\dfrac{dx}{z(x+2y)} = \dfrac{dy}{-z(y+2x)} = \dfrac{dz}{y^2 - x^2}$ ... (1)

Taking the first two fraction of (1), we have

$(y + 2x)\, dx + (x + 2y)\, dy = 0$ or $2xdx + 2y\, dy + d\,(xy) = 0$

Integrating, $x^2 + y^2 + xy = C_1$, C_1 being an arbitrary constant ... (2)

Choosing x, y, z as multipliers, each fraction of (1)

$$= \frac{xdx + ydy + zdz}{(x^2z + 2xyz) - (y^2z + 2xyz) + (zy^2 - zx^2)} = \frac{xdx + ydy + zdz}{0}$$

$\Rightarrow$ $2x\,dx + 2y\,dy + 2z\,dz = 0$ so that $x^2 + y^2 + z^2 = C_2$ ... (3)

From (2) and (3), solution is $\phi(x^2 + y^2 + z^2, x^2 + y^2 + xy) = 0$, ϕ being an arbitrary function

EXERCISE 2(C)

Solve the following partial differential equations:

1. $x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$ **Ans.** $\phi(x^2 + y^2 + z^2, xyz) = 0$

[Mysore 2004, Delhi B.Sc. (Prog). II 2007, M.S. Unit. T.N. 2007]

2. $z(xp - yq) = y^2 - x^2$ **Ans.** $\phi(x^2 + y^2 + z^2, xy) = 0$

3. $(y^2 + z^2)p - xyq + xz = 0$ **[I.A.S. 1990]** **Ans.** $\phi(x^2 + y^2 + z^2, y/z) = 0$

4. $yp - xq = 2x - 3y$ **[M.S. Univ. T.N. 2007]** **Ans.** $\phi(x^2 + y^2, 3x + 2y + z) = 0$

5. $x^2(y - z)p + y^2(z - x)q = z^2(x - y)$ **Ans.** $\phi(xyz, 1/x + 1/y + 1/z) = 0$

[Meerut 2007, Bilaspur 2004, Rewa 2003]

2.11. Type 4 based on Rule IV for solving $(dx)/P = (dy)/Q = (dz)/R$. ...(1)

Let P_1, Q_1 and R_1 be functions of x, y and z. Then, by a well–known principle of algebra, each fraction of (1) will be equal to $(P_1dx + Q_1dy + R_1dz)/(P_1P + Q_1Q + R_1R)$. ...(2)

Suppose the numerator of (2) is exact differential of the denominator of (2). Then (2) can be combined with a suitable fraction in (1) to give an integral. However, in some problems, another set of multipliers P_2, Q_2 and R_2 are so chosen that the fraction

$$(P_2dx + Q_2dy + R_2dz)/(P_2P + Q_2Q + R_2R) \qquad ...(3)$$

is such that its numerrator is exact differential of denominator. Fractions (2) and (3) are then combined to given an integral. This method may be repeated in some problems to get another integral. Sometimes only one integral is possible by using the above rule IV. In such cases second integral should be obtained by using rule 1 of Art. 2.5 or rule 2 of Art. 2.7 or rule 3 of Art. 2.9.

2.12. SOLVED EXAMPLES BASED IN ART. 2.11

Ex. 1. *Solve* $(y + z)p + (z + x)q = x + y$. **[Indore 2000; Jabalpur 2000, Jiwaji 2002, Kanpur 2008; Purvanchal 2007, Ravishankar 2002, 2005; Delhi BA (Prog.) II 2011]**

Sol. Here the Lagrange's auxiliary equations are $\dfrac{dx}{y+z} = \dfrac{dy}{z+x} = \dfrac{dz}{x+y}$. ...(1)

Choosing 1, –1, 0 as multipliers, each fraction of (1) $= \dfrac{dx - dy}{(y+z)-(z+x)} = \dfrac{d(x-y)}{-(x-y)}$. ...(2)

Again, choosing 0, 1, –1 as multipliers, each fraction of (1) $= \dfrac{dy - dz}{(z+x)-(x+y)} = \dfrac{d(y-z)}{-(y-z)}$. ...(3)

Finally, choosing 1, 1, 1 as multipliers, each fraction of (1)

$$= \frac{dx + dy + dz}{(y+z)+(z+x)+(x+y)} = \frac{d(x+y+z)}{2(x+y+z)}. \qquad ...(4)$$

(2), (3) and (4) $\Rightarrow$ $\dfrac{d(x-y)}{-(x-y)} = \dfrac{d(y-z)}{-(y-z)} = \dfrac{d(x+y+z)}{2\,(x+y+z)}$. ...(5)

Taking the first two fractions of (5), $\dfrac{d(x-y)}{x-y} = \dfrac{d(y-z)}{y-z}$.

Integrating, $\log(x - y) = \log(y - z) + \log c_1$, c_1 being an arbitrary constant.

or $\log \{(x-y)/(y-z)\} = \log c_1$ or $(x-y)/(y-z) = c_1$. ...(6)

Taking the first and the third fractions of (5), $2\,\dfrac{d(x-y)}{(x-y)} + \dfrac{d(x+y+z)}{x+y+z} = 0$

Integrating, $2 \log (x-y) + \log (x+y+z) = \log c_2$ or $(x-y)^2 (x+y+z) = c_2$. ...(7)

From (6) and (7), the required general solution is

$$\phi\,[(x-y)^2(x+y+z),\ (x-y)/(y-z)] = 0, \quad \phi \text{ being an arbitrary function.}$$

Ex. 2. *Solve* $y^2(x-y)p + x^2(y-x)q = z(x^2+y^2)$ **[Delhi Maths Hons 1997; Nagpur 2010]**

Sol. Here the Lagrange's auxiliary equations for the given equation are

$$\frac{dx}{y^2\,(x-y)} = \frac{dy}{-x^2\,(x-y)} = \frac{dz}{z\,(x^2+y^2)}. \qquad ...(1)$$

Taking the first two fractions of (1), $x^2dx = -y^2dy$ or $3x^2dx + 3y^2dy = 0$.

Integrating, $x^3 + y^3 = c_1$, c_1 being an arbitrary as constant. ...(2)

Choosing 1, –1, 0 as multipliers, each fraction of (1)

$$= \frac{dx-dy}{y^2\,(x-y)+x^2\,(x-y)} = \frac{dx-dy}{(x-y)(x^2+y^2)}. \qquad ...(3)$$

Combining the third fraction of (1) with fraction (3), we get

$$\frac{dx-dy}{(x-y)(x^2+y^2)} = \frac{dz}{z\,(x^2+y^2)} \quad \text{or} \quad \frac{d(x-y)}{x-y} - \frac{dz}{z} = 0.$$

Integrating, $\log (x-y) - \log z = \log c_2$ or $(x-y)/z = c_2$. ...(4)

From (3) and (4), solution is $\phi\,(x^3+y^3,\ (x-y)/z) = 0$, ϕ being an arbitrary function.

Ex. 3. *Solve* $(x^2-y^2-z^2)p + 2xyq = 2xz$ or $(y^2+z^2-x^2)p - 2xyq = -2xz$.

[Bangalore 1993, I.A.S. 1973; P.C.S. (U.P.) 1991; Bhopal 2010]

Sol. Here the Lagrange's auxiliary equations for the given equation are

$$\frac{dx}{y^2+z^2-x^2} = \frac{dy}{-2xy} = \frac{dz}{-2xz}. \qquad ...(1)$$

Taking the last two fractions of (1), we have

$(1/y)dy = (1/z)dz$ so that $(1/y)dy - (1/z)dz = 0$.

Integrating, $\log y - \log z = \log c_1$ or $y/z = c_1$. ...(2)

Choosing x, y, z as mnultipliers, each fraction of (1)

$$= \frac{x\,dx+y\,dy+z\,dz}{xy^2+xz^2-x^3-2xy^2-2xz^2} = \frac{x\,dx+y\,dy+z\,dz}{-x\,(x^2+y^2+z^2)}. \qquad ...(3)$$

Combining the third fraction of (1) with fraction (3), we have

$$\frac{x\,dx+y\,dy+z\,dz}{-x\,(x^2+y^2+z^2)} = \frac{dz}{-2xz} \quad \text{or} \quad \frac{2x\,dx+2y\,dy+2z\,dz}{x^2+y^2+z^2} - \frac{dz}{z} = 0.$$

Integrating, $\log (x^2+y^2+z^2) - \log z = \log c_2$ or $(x^2+y^2+z^2)/z = c_2$. ... (4)

From (2) and (4) solution is $\phi\,(y/z,\ (x^2+y^2+z^2)/z) = 0$, ϕ being an arbitrary function.

Ex. 4. *Solve* $(1+y)p + (1+x)q = z$. **[M.S. Univ. T.N. 2007; Kanpur 2011]**

Sol. Here the Lagrange's auxiliary equations are $\dfrac{dx}{1+y} = \dfrac{dy}{1+x} = \dfrac{dz}{z}$. ...(1)

Taking the first two fractions of (1), we have

$(1+x)dx = (1+y)dy$ or $2(1+x)dx - 2(1+y)dy = 0$.

Integrating, $(1 + x)^2 - (1 + y)^2 = c_1$, c_1 being an arbitrary constant. ...(2)

Taking 1, 1, 0 as multipliers, each fraction of (1) $= \dfrac{dx+dy}{1+y+1+x} = \dfrac{d(2+x+y)}{2+x+y}$. ...(3)

Combining the last fraction of (1) with fraction (3), we get

$$\frac{d(2+x+y)}{2+x+y} = \frac{dz}{z} \quad \text{or} \quad \frac{d(2+x+y)}{2+x+y} - \frac{dz}{z} = 0.$$

Integrating, $\log (2 + x + y) - \log z = \log c_2$ or $(2 + x + y)/z = c_2$. ...(4)

From (2) and (4), the required general solution is given by

$\phi\,[(1 + x)^2 - (1 + y^2), (2 + x + y)/z] = 0$, ϕ being an arbitrary function.

Ex. 5. *Find the general integral of $xzp + yzq = xy$.*

Sol. Here the Lagrange's auxiliary equations are $(dx)/xz = (dy)/yz = (dz)/xy$ ... (1)

From the first two fractions of (1), $(1/x)dx = (1/y)dy$.

Integrating, $\log x = \log y + \log c_1$ or $x/y = c_1$. ...(2)

Choosing $1/x$, $1/y$, 0 as multipliers, each fraction of (1) $= \dfrac{(1/x)dx+(1/y)dy}{(1/x)xz+(1/y)yz} = \dfrac{ydx+xdy}{2xyz}$ ...(3)

Combining the last fraction of (1) with fraction (3), we have

$$\frac{ydx+xdy}{2xyz} = \frac{dz}{xy} \quad \text{or} \quad ydx + xdy = 2zdz \quad \text{or} \quad d(xy) = 2zdz \quad \text{or} \quad d(xy) - 2zdz = 0$$

Integrating, $xy - z^2 = c_2$, c_2 being an arbitrary constant. ...(4)

From (2) and (4) solution is $\phi\,(x/y, xy - z^2) = 0$, ϕ being an arbitrary function.

Ex. 6. *Solve $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$.* **Delhi Math (H) 2005, 11, M.D.U. Rohtak 2005; Agra 2008, 09; Guwahati 2007; Meerut 2006; Sagar 2000; Ravishankar 2000; Lucknow 2010]**

Sol. Here the Lagrange's auxiliary equations are $\dfrac{dx}{x^2-yz} = \dfrac{dy}{y^2-zx} = \dfrac{dz}{z^2-xy}$. ...(1)

Choosing 1, –1, 0 and 0, 1, –1 as multipliers in turn, each fraction of (1)

$$= \frac{dx-dy}{x^2-y^2+z(x-y)} = \frac{dy-dz}{(y-z)(y+z+x)}$$

so that $$\frac{dx-dy}{(x-y)(x+y+z)} = \frac{dy-dz}{(y-z)(y+z+x)} \quad \text{or} \quad \frac{d(x-y)}{x-y} - \frac{d(y-z)}{y-z} = 0.$$

Integrating, $\log (x - y) - \log (y - z) = \log c_2$ or $(x - y)/(y - z) = c_1$. ...(2)

Choosing x, y, z as multipliers, each fraction of (1)

$$= \frac{xdx+ydy+zdz}{x^3+y^3+z^3-3xyz} = \frac{xdx+ydy+zdz}{(x+y+z)(x^2+y^2+z^2-xy-yz-zx)}. \qquad ...(3)$$

Again, choosing 1, 1, 1 as multipliers, each fraction of (1)

$$= \frac{dx+dy+dz}{x^2+y^2+z^2-xy-yz-zx}. \qquad ...(4)$$

From (3) and (4), $$\frac{xdx+ydy+zdz}{x+y+z} = dx + dy + dz$$

or $$2(x + y + z)\, d(x + y + z) - (2xdx + 2ydy + 2zdz) = 0.$$

Integrating, $$(x + y + z)^2 - (x^2 + y^2 + z^2) = 2c_2$$

or $$(x^2 + y^2 + z^2 + 2xy + 2yz + 2zx) - (x^2 + y^2 + z^2) = 2c_2$$

or $xy + yz + zx = c_2$, c_2 being an arbitrary constant. ...(5)

From (2) and (5), the required general solution is given by

$$\phi\,[xy + yz + zx, (x - y)/(y - z)] = 0, \quad \phi \text{ being an arbitrary function.}$$

Ex. 7. *Solve* $(x^2 - y^2 - yz)p + (x^2 - y^2 - zx)q = z(x - y)$.

Sol. Here Lagrange's auxiliary equations for the given equation are

$$\frac{dx}{x^2 - y^2 - yz} = \frac{dy}{x^2 - y^2 - zx} = \frac{dz}{z(x - y)} \qquad ...(1)$$

Choosing 1, –1, 0 as multipliers, each fraction of (1)

$$= \frac{dx - dy}{(x^2 - y^2 - yz) - (x^2 - y^2 - zx)} = \frac{dx - dy}{z(x - y)}. \qquad ...(2)$$

Choosing x, $-y$, 0 as multipliers each fraction of (1)

$$= \frac{xdx - ydy}{x(x^2 - y^2 - yz) - y(x^2 - y^2 - zx)} = \frac{xdx - ydy}{(x - y)(x^2 - y^2)}. \qquad ...(3)$$

From (1), (2), (3) we have

$$\frac{dz}{z(x - y)} = \frac{dx - dy}{z(x - y)} = \frac{xdx - ydy}{(x - y)(x^2 - y^2)} \quad \text{or} \quad \frac{dz}{z} = \frac{dx - dy}{z} = \frac{2xdx - 2ydy}{2(x^2 - y^2)}. \qquad ...(4)$$

Taking the first two fractions of (4), we have

$dz = dx - dy$ so that $z - x + y = c_1$ ...(5)

Again , taking the first and third fractions of (4), $d(x^2 - y^2)/(x^2 - y^2) - (2/z)\,dz = 0$

Integrating, $\log (x^2 - y^2) - 2 \log z = c_2$ or $(x^2 - y^2)/z^2 = c_2$. ...(6)

From (5) and (6), solution is $\phi\,(z - x + y, (x^2 + y^2)/z^2) = 0$, ϕ being an arbitrary function.

Ex. 8. *Solve* $(x^2 + y^2 + yz)p + (x^2 + y^2 - xz)q = z(x + y)$.

Sol. Here the Lagrange's, auxiliary equations are $\dfrac{dx}{x^2 + y^2 + yz} = \dfrac{dy}{x^2 + y^2 - xz} = \dfrac{dz}{z(x + y)}$. ...(1)

Choosing 1, –1, 0 as multipliers, each fraction of (1)

$$= \frac{dx - dy}{(x^2 + y^2 + yz) - (x^2 + y^2 - xz)} = \frac{dx - dy}{z(x + y)}. \qquad ...(2)$$

Choosing x, y, 0 as multipliers, each fraction of (1)

$$= \frac{xdx + ydy}{x(x^2 + y^2 + yz) + y(x^2 + y^2 - xz)} = \frac{xdx + ydy}{(x + y)(x^2 + y^2)}. \qquad ...(3)$$

From (1), (2) and (3), we have

$$\frac{dz}{z(x + y)} = \frac{dx - dy}{z(x + y)} = \frac{xdx + ydy}{(x + y)(x^2 + y^2)} \quad \text{or} \quad \frac{dz}{z} = \frac{dx - dy}{z} = \frac{xdx + ydy}{x^2 + y^2}. \qquad ...(4)$$

Taking the first two fractions of (4), we have

$dz = dx - dy$ or $dz - dx + dy = 0$.

Integrating, $z - x + y = c_1$, c_1 being an arbitrary constant. ...(5)

Taking the first and third fractions of (4), we have

$$\frac{2xdx + 2ydy}{x^2 + y^2} = 2\frac{dz}{z} \quad \text{or} \quad \frac{d(x^2 + y^2)}{x^2 + y^2} - 2\frac{dz}{z} = 0.$$

Integrating, $\log (x^2 + y^2) - 2\log z = \log c_2$ or $(x^2 + y^2)/z^2 = c_2$. ...(6)

From (5) and (6), solution is $\phi\,(z - x + y, (x^2 + y^2)/z^2) = 0$, ϕ being an arbitrary function.

Ex. 9. *Solve cos* $(x + y)p + \sin (x + y)q = z$. **[Garhwal 2010, Vikram 1998; Meerut 2007; Delhi Maths (H) 2007; Rajasthan 1994; Delhi B.A./B.Sc. (Prog.) Maths 2007]**

Sol. Here the Lagrange's auxiliary equations are $\dfrac{dx}{\cos(x+y)} = \dfrac{dy}{\sin(x+y)} = \dfrac{dz}{z}$. ...(1)

Choosing 1, 1, 0 as multipliers, each fraction of (1)

$$= \frac{dx+dy}{\cos(x+y)+\sin(x+y)} = \frac{d(x+y)}{\cos(x+y)+\sin(x+y)}. \qquad ...(2)$$

Choosing 1, –1, 0 as multipliers, each fraction of (1) $= \dfrac{dx-dy}{\cos(x+y)-\sin(x+y)}$. ...(3)

From (1), (2) and (3), $$\frac{dz}{z} = \frac{d(x+y)}{\cos(x+y)+\sin(x+y)} = \frac{dx-dy}{\cos(x+y)-\sin(x+y)}. \qquad ...(4)$$

Taking the first two fractions of (4), $$\frac{dz}{z} = \frac{d(x+y)}{\cos(x+y)+\sin(x+y)}. \qquad ...(5)$$

Putting $x + y = t$ so that $d(x + y) = dt$, (5) reduces to

$$\frac{dz}{z} = \frac{dt}{\cos t+\sin t} = \frac{dt}{\sqrt{2}\left\{(1/\sqrt{2})\cos t+(1/\sqrt{2})\sin t\right\}} = \frac{dt}{\sqrt{2}\{\sin(\pi/4)\cos t+\cos(\pi/4)\sin t\}} = \frac{dt}{\sqrt{2}\sin(t+\pi/4)}$$

Thus, $$(\sqrt{2}/z)dz = \operatorname{cosec}(t+\pi/4)\ dt.$$

Integrating, $\sqrt{2}\log z = \log\tan\dfrac{1}{2}\left(t+\dfrac{\pi}{4}\right) + \log c_1$, or $z^{\sqrt{2}} = c_1\tan\left(\dfrac{t}{2}+\dfrac{\pi}{8}\right)$

or $$z^{\sqrt{2}}\cot\left(\frac{x+y}{2}+\frac{\pi}{8}\right) = c_1.\ \text{as } t = x + y \qquad ...(6)$$

Taking the last two fraction of (4), $dx - dy = \dfrac{\cos(x+y)-\sin(x+y)}{\cos(x+y)+\sin(x+y)}d(x + y)$. ...(7)

On R.H.S. of (7), putting $x + y = t$, so that $d(x + y) = dt$, (7) reduces to

$dx - dy = \dfrac{\cos t-\sin t}{\cos t+\sin t}dt.$ so that $x - y = \log(\sin t + \cos t) - \log c_2$

or $(\sin t + \cos t)/c_2 = e^{x-y}$ or $e^{-(x-y)}(\sin t + \cos t) = c_2$

or $$e^{y-x}[\sin(x + y) + \cos(x + y)] = c_2, \quad \text{as } t = x + y. \qquad ...(8)$$

From (6) and (8), the required general solution is

$$\phi\left[z^{\sqrt{2}}\cot\left(\frac{x+y}{2}+\frac{\pi}{8}\right), e^{y-x}\{\sin(x+y)+\cos(x+y)\}\right] = 0,$$ where ϕ is an arbitrary function.

Ex. 10. *Solve cos* $(x + y)p + \sin(x + y)q = z + (1/z)$. **[Delhi B.A. (Prog.) 2011]**

Sol. Do like Ex. 9. **Ans.** $\phi\left[(z^2+1)^{1/\sqrt{2}}\tan\left(\dfrac{3\pi}{8}-\dfrac{x+y}{2}\right), e^{y-x}\{\cos(x+y)+\sin(x+y)\}\right] = 0$

Ex. 11. *Solve* $xp + yq = z - a\sqrt{(x^2+y^2+z^2)}$. **[Meerut 1997; Jiwaji 1997; Rawa 1999]**

Sol. Here the lagrange's auxiliary equations are $\dfrac{dx}{x} = \dfrac{dy}{y} = \dfrac{dz}{z-a\sqrt{(x^2+y^2+z^2)}}$. ...(1)

Taking the first two fractions of (1), we have

$(1/x)dx = (1/y)dy$ or $(1/x)dx - (1/y)dy = 0.$

Integrating, $\log x - \log y = \log c_1$ or $x/y = c_1.$ ...(2)

Choosing x, y, z as multipliers, each fraction of (1) $= \dfrac{xdx + ydy + zdz}{x^2 + y^2 + z^2 - az\sqrt{(x^2 + y^2 + z^2)}}$ ...(3)

Combining first and third fractions of (1) with fraction (3), we get

$$\frac{dx}{x} = \frac{dz}{z - a\sqrt{(x^2 + y^2 + z^2)}} = \frac{xdx + ydy + zdz}{x^2 + y^2 + z^2 - az\sqrt{(x^2 + y^2 + z^2)}}. \quad ...(4)$$

Putting $x^2 + y^2 + z^2 = t^2$ so that $xdx + ydy + zdz = tdt$, (4) gives

$$\frac{dx}{x} = \frac{dz}{z - at} = \frac{tdt}{t^2 - azt} \quad \text{or} \quad \frac{dx}{x} = \frac{dz}{z - at} = \frac{dt}{t - az}. \quad ...(5)$$

Choosing 0, 1, 1 as multipliers, each fraction of (5) $= \dfrac{dz + dt}{(z + t) - a(t + z)} = \dfrac{d(z + t)}{(1 - a)(z + t)}$. ...(6)

Combining the first fraction of (5) with fraction (6), we get

$$\frac{dx}{x} = \frac{d(z + t)}{(1 - a)(z + t)} \quad \text{or} \quad (1 - a)\frac{dx}{x} - \frac{d(z + t)}{z + t} = 0.$$

Integrating, $(1 - a) \log x - \log (z + t) = \log c_2$, c_2 being an arbitrary constant.

or $$\frac{x^{a-1}}{z + t} = c_2 \quad \text{or} \quad \frac{x^{a-1}}{z + \sqrt{(x^2 + y^2 + z^2)}} = c_2, \quad \text{as} \quad t = (x^2 + y^2 + z^2)^{1/2} \quad ...(7)$$

From (2) and (7), the required general solution is

$$\phi\, [x^{a-1}/\{z + \sqrt{(x^2 + y^2 + z^2)}\},\, x/y] = 0, \ \phi \text{ being an arbitrary function.}$$

Ex. 12. *Solve* $(x^3 + 3xy^2)p + (y^3 + 3x^2y)q = 2z(x^2 + y^2)$. **[I.A.S. 1993]**

Sol. Here the Lagrange's subsidiary equations are $\dfrac{dx}{x^3 + 3xy^2} = \dfrac{dy}{y^3 + 3x^2y} = \dfrac{dz}{2z(x^2 + y^2)}$. ...(1)

Choosing 1, 1, 0 as multipliers, each fraction of (1) $= \dfrac{dx + dy}{x^3 + 3xy^2 + 3x^2y + y^3} = \dfrac{d(x + y)}{(x + y)^3}$. ...(2)

Choosing 1, –1, 0 as multipliers, each fraction of (1) $= \dfrac{dx - dy}{x^3 + 3xy^2 - y^3 - 3x^2y} = \dfrac{d(x - y)}{(x - y)^3}$. ...(3)

From (2) and (3), $(x + y)^{-3}\, d(x + y) = (x - y)^{-3}\, d(x - y)$

or $u^{-3}du - v^{-3}dv = 0$, on putting $u = x + y$ and $v = x - y$.

Integrating, $u^{-2}/(-2) - v^{-2}/(-2) = c_1/2$ or $v^{-2} - u^{-2} = c_1$

or $(x - y)^{-2} - (x + y)^{-2} = c_1$, as $u = x + y$ and $v = x - y$. ...(4)

Choosing $1/x$, $1/y$, 0 as multipliers, each fraction of (1)

$$= \frac{(1/x)\,dx + (1/y)\,dy}{(1/x) \times (x^3 + 3xy^2) + (1/y) \times (y^3 + 3x^2y)} = \frac{(1/x)dx + (1/y)dy}{4(x^2 + y^2)}. \quad ...(5)$$

Combining the last fraction of (1) with fraction (5), we have

$$\frac{dz}{2z(x^2 + y^2)} = \frac{(1/x)dx + (1/y)dy}{4(x^2 + y^2)} \quad \text{or} \quad \frac{dx}{x} + \frac{dy}{y} - 2\frac{dz}{z} = 0.$$

Integrating, $\log x + \log y - 2 \log z = \log c_2$ or $(xy)/z^2 = c_2$. ...(6)

From (4) and (6), the required general solution is given by

$$\phi\,[(x-y)^{-2}-(x+y)^{-2},\ (xy)/z^2]=0,\quad \phi \text{ being an arbitrary function.}$$

Ex. 13. *Solve* $p+q=x+y+z$. **[Bhopal 2010, Bilaspur 2000, 02; I.A.S. 1975; Gulberge 2005]**

Sol. Here Lagrange's auxiliary equations are $\dfrac{dx}{1}=\dfrac{dy}{1}=\dfrac{dz}{x+y+z}$. ...(1)

Taking the first two fractions of (1), $dx-dy=0$ so that $x-y=c_1$. ...(2)

Choosing 1, 1, 1 as multipliers, each fraction of (1) $=\dfrac{dx+dy+dz}{1+1+(x+y+z)}=\dfrac{d(2+x+y+z)}{2+x+y+z}$ ... (3)

Combining the first fraction of (1) with fraction (3), $d(2+x+y+z)/(2+x+y+z)=dx$.

Integrating, $\log(2+x+y+z)-\log c_2=x$ or $(2+x+y+z)/c_2=e^x$

or $e^{-x}(2+x+y+z)=c_2$, c_2 being arbitrary function ...(4)

From (2) and (4), the required general solution is

$$\phi[x-y,\ e^{-x}(2+x+y+z)]=0,\quad \phi \text{ being an arbitrary function.}$$

Ex. 14. *Solve* $(2x^2+y^2+z^2-2yz-zx-xy)p+(x^2+2y^2+z^2-yz-2zx-xy)q=x^2+y^2+2z^2-yz-zx-2xy$. **[Meerut 1996 ; I.A.S. 1992]**

Sol. Here Lagrange's auxiliary equations are

$$\frac{dx}{2x^2+y^2+z^2-2yz-zx-xy}=\frac{dy}{x^2+2y^2+z^2-yz-2zx-xy}=\frac{dz}{x^2+y^2+2z^2-yz-zx-2xy}. \quad ...(1)$$

Choosing 1, –1, 0 ; 0, 1, –1 and –1, 0, 1 as multipliers in turn, each fraction of (1)

$$=\frac{dx-dy}{x^2-y^2-yz+zx}=\frac{dy-dz}{y^2-z^2-zx+xy}=\frac{dz-dx}{z^2-x^2-xy+yz}$$

$$\therefore \quad \frac{dx-dy}{(x-y)\,(x+y+z)}=\frac{dy-dz}{(y-z)\,(x+y+z)}=\frac{dz-dx}{(z-x)\,(x+y+z)}. \quad ...(2)$$

Taking the first two fractions of (2), we have

$$(dx-dy)/(x-y)-(dy-dz)/(y-z)=0.$$

Integrating, $\log(x-y)-\log(y-z)=\log c_1$ or $(x-y)/(y-z)=c_1$. ...(3)

Taking the last two fractions of (2), $(dy-dz)/(y-z)-(dz-dx)/(z-x)=0$.

Integrating, $\log(y-z)-\log(z-x)=\log c_2$ or $(y-z)/(z-x)=c_2$. ...(4)

From (3) and (4), the required general solution is

$$\phi[(x-y)/(y-z),\ (y-z)/(z-x)]=0,\quad \phi \text{ being an arbitrary function.}$$

Ex. 15. *Find the general solution of the partial differential equation* $px(x+y)-qy(x+y)+(x-y)(2x+2y+z)=0$. **[Delhi B.Sc. II (Prog) 2009; Delhi Maths Hons. 2006, 09, 11]**

Sol. Given $x(x+y)p-y(x+y)q=-(x-y)(2x+2y+z)$. ...(1)

Lagrange's auxiliary equations are $\dfrac{dx}{x\,(x+y)}=\dfrac{dy}{-y\,(x+y)}=\dfrac{dz}{-(x-y)\,(2x+2y+z)}$. ...(2)

Taking the first two fractions, $(1/x)dx=-(1/y)dy$ or $(1/x)dx+(1/y)dy=0$.

Integrating, $\log x+\log y=\log c_1$ or $xy=c_1$. ...(3)

Again, each fraction of (2)

$$=\frac{dx+dy}{x(x+y)-y(x+y)}=\frac{dx+dy+dz}{x(x+y)-y(x+y)-(x-y)(2x+2y+z)}$$

$$=\frac{dx+dy}{(x-y)(x+y)}=\frac{dx+dy+dz}{(x-y)(x+y)-(x-y)(2x+2y+z)}$$

Thus, $$\frac{dx+dy}{(x+y)} = \frac{dx+dy+dz}{x+y-(2x+2y+z)} = -\frac{dx+dy+dz}{x+y+z}$$

Thus, $\dfrac{dx+dy}{x+y} + \dfrac{dx+dy+dz}{x+y+z} = 0$, so that $\log(x+y) + \log(x+y+z) = \log c_2$

or $(x+y)(x+y+z) = c_2$, c_2 being an arbitrary constant. ...(4)

From (3) and (4), solution is $\phi[xy, (x+y)(x+y+z)] = 0$, ϕ being an arbitrary function.

Ex. 16. *Solve* $\{my(x+y)-nz^2\}(\partial z/\partial x) - \{lx(x+y)-nz^2\}(\partial z/\partial y) = (lx-my)z$ **[I.A.S. 2001]**

Sol. Re-writing the given equation, $\{my(x+y)-nz^2\}\,p - \{lx(x+y)-nz^2\}q = (lx-my)z$ ...(1)

Lagrange's auxiliary equations for (1) are $\dfrac{dx}{my(x+y)-nz^2} = \dfrac{dy}{-lx(x+y)+nz^2} = \dfrac{dz}{(lx-my)z}$ ...(2)

Each fraction of (2) = $\dfrac{dx+dy}{(my-lx)(x+y)} = \dfrac{dz}{-(my-lx)z}$ so that $\dfrac{d(x+y)}{x+y} = -\dfrac{dz}{z}$

Integrating, $\log(x+y) = -\log z + \log C_1$ or $(x+y)z = C_1$ ... (3)

Taking lx, my, nz as multipliers, each fraction of (2)

$$= \frac{lxdx+mydy+nzdz}{lx\,my(x+y)-lx\,nz^2-mylx(x+y)+mynz^2+nz^2(lx-my)} = \frac{lx\,dx+my\,dy+nz\,dz}{0}$$

$\therefore\ 2lx\,dx+2my\,dy+2nzdz = 0$ so that $lx^2+my^2+nz^2 = C_2$ ...(4)

From (3) and (4), solution is $\Phi(xz+yz,\, lx^2+my^2+nz^2) = 0$, Φ being an arbitrary function.

Ex. 17. *Solve* $px(z-2y^2) = (z-qy)(z-y^2-2x^2)$. **[Delhi Maths (H) 2002]**

Sol. Re-writing the given equation $x(z-2y^2)p + y(z-y^2-2x^2)q = z(z-y^2-2x^2)$ ... (1)

Lagrange's auxiliary equations for (1) are $\dfrac{dx}{x(z-2y^2)} = \dfrac{dy}{y(z-y^2-2x^2)} = \dfrac{dz}{z(z-y^2-2x^2)}$ ...(2)

Taking the last two fractions, $(1/y)\,dy-(1/z)\,dz = 0$ so that $y/z = C_1$ ...(3)

Taking $0, -2y, 1$ as multipliers, each fraction of (2)

$$= \frac{-2ydy+dz}{-2y^2(z-y^2-2x^2)+z(z-y^2-2x^2)} = \frac{d(z-y^2)}{(z-2y^2)(z-y^2-2x^2)} \quad \text{... (4)}$$

Combining fraction (4) with first fraction of (2), we get

$$\frac{dx}{x(z-2y^2)} = \frac{d(z-y^2)}{(z-2y^2)(z-y^2-2x^2)} \quad \text{or} \quad \frac{d(z-y^2)}{dx} = \frac{z-y^2-2x^2}{x}$$

or $du/dx = (u-2x^2)/x$, taking $z-y^2 = u$ ... (5)

or $(du/dx)-(1/x)u = -2x$ which is an ordinary linear differential equation

whose I.F. $= e^{-\int (1/x)dx} = e^{-\log x} = e^{\log x^{-1}} = x^{-1} = 1/x$ and solution is

$$u.\frac{1}{x} = \int (-2x)\left(\frac{1}{x}\right)dx + C_2 \quad \text{or} \quad \frac{z-y^2}{x} = -2x + C_2, \text{ using (5)}$$

or $\quad (z-y^2)/x+2x=C_2 \quad$ or $\quad (z-y^2+2x^2)/x=C_2 \quad$ ... (6)

From (3) and (6), the required general solution of (1)

$\Phi(y/z,\ (z-y^2-2x^2)/x)=0$, Φ being an arbitrary function.

Ex. 18. *Solve* $px(z-2y^2)=(z-qy)\ (z-y^2-2x^3)$. **[I.A.S. 2006]**

Sol. Do like Ex. 17, **Ans.** $\Phi(y/z,\ (z-y^2+x^3)/x)=0$

For another method of solution, refer solved Ex. 8 of Art. 2.8.

Ex. 19. *Solve* $x(z+2a)p+(xz+2yz+2ay)q=z(z+a)$.

Sol. The Lagrange's auxiliary equations for given equation are

$$\frac{dx}{x(z+2a)}=\frac{dy}{xz+2yz+2ay}=\frac{dz}{z(z+a)} \quad \text{... (1)}$$

Each fraction of (1) = $\dfrac{dx+dy}{2(x+y)(z+a)}=\dfrac{dz}{z(z+a)}$ or $\dfrac{d(x+y)}{x+y}=\dfrac{2}{z}dz$

Integrating, $\log (x+y) = 2 \log z + \log C_1$ or $(x+y)/z^2=C_1$ ...(2)

Taking the first and third ratios of (4), $\dfrac{dx}{x}=\dfrac{z+2a}{z(z+a)}dz$ or $\dfrac{dx}{x}=\left(\dfrac{2}{z}-\dfrac{1}{z+a}\right)dz$

Integrating, $\log x = 2 \log z - \log (z + a) + \log C_2$ or $x(z+a)/z^2=C_2$ ... (3)

From (2) and (3), solution is $\Phi\ \{(x+y)/z^2,\ x(z+a)/z^2)\}=0$. ϕ being an arbitrary function.

Ex. 20. *Solve* $2x\ (y+z^2)p+y(2y+z^2)q=z^3$ **[Delhi Maths (Hans.) 2007]**

Sol. The Lagrange's auxiliary equations for the given equation are

$$\frac{dx}{2x(y+z^2)}=\frac{dy}{y(2y+z^2)}=\frac{dz}{z^3} \quad \text{...(1)}$$

Each fraction of (1) = $\dfrac{dx}{2x(y+z^2)}=\dfrac{z\,dy+y\,dz}{2yz(y+z^2)}=\dfrac{d(yz)}{2yz(y+z^2)}$

$\therefore\ (1/x)\ dx+(1/yz)\ d(yz)=0$ so that $x/(yz)=C_1$ ...(2)

From the last two fractions of (1), $\dfrac{dy}{dz}=\dfrac{y(2y+z^2)}{z^3}=\dfrac{2y^2}{z^3}+\dfrac{y}{z}$ or $y^{-2}\dfrac{dy}{dz}-\dfrac{1}{z}y^{-1}=\dfrac{2}{z^3}$ ...(3)

Putting $-y^{-1}=u$ and $(1/y^2)\times(dy/dz)=du/dz$ in (3), we get

$(du/dz)+(1/z)\ u=2/z^3$, which is an ordinary linear equation.

Its I.F. = $e^{\int (1/z)dz}=e^{\log z}=z$ and solution is $u\,z=\int (2/z^3)\ z\ dz-C_2=-2\ z^{-1}-C_2$

or $\quad -y^{-1}z-2z^{-1}=-C_2 \quad$ or $\quad z/y-2/z=C_2 \quad$ ... (4)

From (3) and (4), solution is $\Phi\ (x/yz,\ z/y-2/z)=0$, ϕ being arbitrary function.

Ex. 21. $xp + zq + y = 0$. **[M.D.U. Rohtak 2004]**

Sol. Given equation is $xp + zq = - y$

Its Lagrange's auxiliary equation are $$\frac{dx}{x} = \frac{dy}{z} = \frac{dz}{-y} \qquad \text{... (1)}$$

Taking the last two fractions of (2), $2ydy + 2zdz = 0$ so that $y^2 + z^2 = C_1$ ... (2)
Choosing 0, z, $-y$ as multipliers, each fraction of (1)

$$= \frac{zdy - ydz}{z^2 + y^2} = \frac{(1/z)dy - (y/z^2)dz}{1 + (y/z)^2} = \frac{d(y/z)}{1 + (y/z)^2} \qquad \text{... (3)}$$

Combining the first faction of (1) with fraction (3), we get

$$\frac{dx}{x} = \frac{d(y/z)}{1 + (y/z)^2} \qquad \text{or} \qquad \frac{dx}{x} - d\left(\tan^{-1}\frac{y}{z}\right) = 0$$

Integrating, $\log|x| - \tan^{-1}(y/z) = C_2$, c_2 being an arbitrary constant. ... (4)
From (2) and (4), the required general solution is

$$\log|x| - \tan^{-1}(y/z) = \phi(y^2 + z^2), \ \phi \text{ being an arbitrary function.}$$

Ex. 22. *Find the general solution of the differential equation* $x^2(\partial z/\partial x) + y^2(\partial z/\partial y) = (x + y)z.$ **[Delhi B.A./B.Sc. (Prog.) Maths 2007)**

Sol. Let $p = \partial z/\partial x$ and $q = \partial z/\partial y$. Then, the given equation takes the form

$$x^2p = y^2q = z\,(x + y) \qquad \text{... (1)}$$

The Lagrange's auxiliary equations for (1) are

$$(dx)/x^2 = (dy)/y^2 = (dz)/z\,(x + y) \qquad \text{... (2)}$$

Taking the first two fractions of (2), $(1/x^2)dx - (1/y^2)dy = 0$
Integrating, $-(1/x) + (1/y) = c_1$ or $(x - y)/xy = c_1$ ... (3)

Chossing 1, –1, 0 as multipliers, each fraction of (2) $= \dfrac{dx - dy}{x^2 - y^2}$ ... (4)

Combining the last fraction of (2) with fraction (4), we have

$$\frac{dx - dy}{(x - y)(x + y)} = \frac{dz}{z(x + y)} \qquad \text{or} \qquad \frac{dx - dy}{x - y} - \frac{dz}{z} = 0$$

Integrating, $\log(x - y) - \log z = \sin c^2$ or $(x - y)/z = c^2$ ... (5)
From (5), $x - y = c_2\, z$ ... (6)
using (6), (3) becomes $(c_2 z)/xy = a$ or $(xy)/z = c_2/c_1 = c_3$ say ... (7)
From (5) and (7), the required solution is $\phi\big((x, y)/z, (x - y)/z\big) = 0.$

EXERCISE 2(D)

Solve the following partial differential equations:

1. $(x^2 + y^2)p + 2xy\,q = z(x + y)$ **Ans.** $(x + y)/z = \phi\big(y/(x^2 - y^2)\big)$

2. $\{y\,(x + y) + az\}\,p + \{x\,(x + y) - az\}q = z(x + y)$ **Ans.** $(x + y)/z = \phi(x^2 - y^2 - 2az)$

3. $(y^2 + yz + z^2)p + (z^2 + zx + x^2)q = x^2 + xy + y^2$ **Ans.** $\phi\left(\dfrac{y - z}{x - y}, \dfrac{x - z}{x - y}\right) = 0$

2.13. Miscellaneous Examples on $Pp + Qq = R$

Ex. 1. *Solve* $(x + y - z)(p - q) + a(px - qy + x - y) = 0$.

Sol. Let $u = x + y$ and $v = x - y$. ...(1)

Then $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ using (1)} \qquad ...(2)$$

and $$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}, \text{ using (1)} \qquad ...(3)$$

From (2) and (3), we get $$p - q = 2(\partial z/\partial v). \qquad ...(4)$$

and $$px - qy = x\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v} - y\frac{\partial z}{\partial u} + y\frac{\partial z}{\partial v}$$

or $$px - qy = (x - y)\frac{\partial z}{\partial u} + (x + y)\frac{\partial z}{\partial v} = v\frac{\partial z}{\partial u} + u\frac{\partial z}{\partial v}, \text{using (1)} \qquad ...(5)$$

Using (1), (4) and (5), the given equation reduces to

$$2(u - z)\frac{\partial z}{\partial v} + a\left(v\frac{\partial z}{\partial u} + u\frac{\partial z}{\partial v} + v\right) = 0$$

or $$av(\partial z/\partial u) + (2u - 2z + au)(\partial z/\partial v) = -av, \qquad ...(6)$$

which is Lagrange's linear equation. Its Lagrange's auxiliary equations are

$$\frac{du}{av} = \frac{dv}{2u - 2z + au} = \frac{dz}{-av}. \qquad ...(7)$$

Taking the first and third fractions of (7), we have

$du + dz = 0$ so that $u + z = c_1$. ...(8)

Considering the first two fractions of (7) and eliminating z with help of (8), we have

$$\frac{du}{av} = \frac{dv}{2u - 2(c_1 - u) + au} \quad \text{or} \quad avdv = (4u - 2c_1 + au)du.$$

Integrating, $(1/2) \times av^2 = 2u^2 - 2c_1u + (1/2) \times au^2 + c_2/2$

$av^2 = 4u^2 - 4u(u + z) + au^2 + c_2$, or $av^2 + 4uz - au^2 = c_2$ (using (8) ....(9)

From (8) and (9), the required general solution is given by

$\phi(u + z, av^2 + 4uz - au^2) = 0$, where ϕ is an arbitrary function and u and v are given by (1).

Ex. 2 (*a*). *Find the surface whose tangent planes cut off an intercept of constant length k from the axis of z.*

(*b*) *Formulate partial differential equation for surfaces whose tangent planes form a tetrahedrom of constant volume with the coordinate planes.* **[I.A.S. 2005]**

Sol. (*a*) We know that the equation of the tangent plane at point (x, y, z) to a surface is given by $$p(X - x) + q(Y - y) = Z - z, \qquad ...(1)$$
where X, Y, Z denote current coodrinates of any point on the plane (1). Since (1) cuts an intercept k on the z–axis, it follows that (1) must pass through the point $(0, 0, k)$. Hence putting $X = 0$, $Y = 0$ and $Z = k$ in (1), we obtain

$$px + qy = z - k, \qquad ...(2)$$

which is well known Lagrange's linear equation. For (2), the Lagrange's auxiliary equations are
$$(dx)/x = (dy)/y = (dz)/(z - x). \qquad ...(3)$$

Taking the first two fractions of (3), $(1/x)dx - (1/y)dy = 0$. so that $x/y = c_1$. ...(4)

Again, taking the first and third fraction of (3), $[1/(z - k)]dz - (1/x)dx = 0$

Integrating, $\log(z - k) - \log x = \log c_2$ or $(z - k)/x = c_2$. ...(5)

From (4) and (5), the required surface (solution) is given by

$$\phi[y/x, (z - k)/x] = 0, \quad \phi \text{ being an arbitrary function.}$$

(*b*) Left as an exercise.

EXERCISE 2 (E)

Solve the following partial differential equations :

1. $p - qy \log y = z \log y$. **Ans.** $\phi(yz, e^x \log y) = 0$
2. $(p + q)(x + y) = 1$. **Ans.** $\phi(y - x, e^{-2z}y + x) = 0$
3. $x^2p + y^2q = x + y$. **Ans.** $\phi[(1/y) - (1/x), e^{-z}(x - y)] = 0$
4. $(x^2 + 2y^2)p - xyq = xz$. **Ans.** $\phi(x^2y^2 + y^4, yz) = 0$
5. $px - qy = (z - xy)^2$. **Ans.** $\phi[xy, xe^{1/(z-xy)}] = 0$
6. $zp + zq = z^2 + (x - y)^2$. **Ans.** $\log [z^2 + (x - y)^2] - 2x = \phi(x - y)$.
7. $x(y^n - z^n)p + y(z^n - x^n)q = z(x^n - y^n)$. **Ans.** $x^n + y^n + z^n = \phi(xyz)$.
8. $(xz + y^2)p + (yz - 2x^2)q + 2xy + z^2 = 0$. **Ans.** $\phi(yz + x^2, 2xz - y^2) = 0$.
9. $xyp + y(2x - y)q = 2xz$. **Ans.** $\phi(xy - x^2, z/xy) = 0$.

2.14. Integral surfaces passing through a given curve. In the last article we obtained general integral of $Pp + Qq = R$. We shall now present two methods of using such a general solution for getting the integral surface which passes through a given curve.

Method I. Let
$$Pp + Qq = R \qquad ...(1)$$
be the given equation. Let its auxiliary equations give the following two independent solutions
$$u(x, y, z) = c_1 \quad \text{and} \quad v(x, y, z) = c_2. \qquad ...(2)$$
Suppose we wish to obtain the integral surface which passes through the curve whose equation in parametric form is given by
$$x = x(t), \qquad y = (t), \qquad z = z(t), \qquad ...(3)$$
where t is a parameter. Then (2) may be expressed as
$$u[x(t), y(t), z(t)] = c_1 \quad \text{and} \quad v[x(t), y(t), z(t)] = c_2. \qquad ...(4)$$
We eliminate single parameter t from the equations of (4) and get a relation involving c_1 and c_2. Finally, we replace c_1 and c_2 with help of (2) and obtain the required integral surface.

2.15. SOLVED EXAMPLES BASED ON ART. 2.14.

Ex. 1. *Find the integral surface of the linear partial differential equation* $x(y^2 + z)p - y(x^2 + z)q = (x^2 - y^2)z$ *which contains the straight line* $x + y = 0$, $z = 1$. **[Delhi 2008; Pune 2010]**

Sol. Given
$$x(y^2 + z)p - y(x^2 + z)q = (x^2 - y^2)z. \qquad ...(1)$$
Lagrange's auxiliary equations of (1) are
$$\frac{dx}{x(y^2+z)} = \frac{dy}{-y(x^2+z)} = \frac{dz}{(x^2-y^2)z}. \qquad ...(2)$$
Proceed as in solved Ex. 6, Art. 2.10 and show that
$$xyz = c_1 \quad \text{and} \quad x^2 + y^2 - 2z = c_2. \qquad ...(3)$$
Taking t as parameter, the given equation of the straight line $x + y = 0$, $z = 1$ can be put in parametric form
$$x = t, \qquad y = -t, \qquad z = 1. \qquad ...(4)$$
Using (4), (3) may be re–written as
$$-t^2 = c_1 \quad \text{and} \quad 2t^2 - 2 = c_2. \qquad ...(5)$$
Eliminating t from the equations of (5), we have
$$2(-c_1) - 2 = c_2 \quad \text{or} \quad 2c_1 + c_2 + 2 = 0. \qquad ...(6)$$
Putting values of c_1 and c_2 from (3) in (6), the desired integral surface is
$$2xyz + x^2 + y^2 - 2z + 2 = 0.$$

Ex. 2. *Find the equation of the integral surface of the differential equation* $2y(z - 3)p + (2x - z)q = y(2x - 3)$, *which pass through the circle* $z = 0$, $x^2 + y^2 = 2x$. **[Meerut 2007]**

Sol. Given equation is
$$2y(z - 3)p + (2x - z)q = y(2x - 3). \qquad ...(1)$$
Given circle is
$$x^2 + y^2 = 2x, \qquad z = 0. \qquad ...(2)$$
Lagrange's auxiliary equations for (1) are
$$\frac{dx}{2y(z-3)} = \frac{dy}{2x-z} = \frac{dz}{y(2x-3)}. \qquad ...(3)$$

Taking the first and third fractions of (3), $(2x-3)dx - 2(z-3)dz = 0$.

Integrating, $x^2 - 3x - z^2 + 6z = c_1$, c_1 being an arbitrary constant. ...(4)

Choosing 1/2, y, –1 as multipliers, each fraction of (3)

$$= \frac{(1/2)dx + ydy - dz}{y(z-3) + y(2x-z) - y(2x-3)} = \frac{(1/2)dx + ydy - dz}{0}$$

Hence $(1/2)dx + ydy - dz = 0$ or $dx + 2ydy - 2dz = 0$.

Integrating, $x + y^2 - 2z = c_2$, c2 being an arbitrary constant. ...(5)

Now, the parametric equations of given circle (2) are $x = t$, $y = (2t - t^2)^{1/2}$, $z = 0$. ...(6)

Substituting these values in (4) and (5), we have

$t^2 - 3t = c_1$ and $3t - t^2 = c_2$. ...(7)

Eliminating t from the above equations (7), we have $c_1 + c_2 = 0$. ...(8)

Substituting the values of c_1 and c_2 from (4) and (5) in (8), the desired integral surface is

$x^2 - 3x - z^2 + 6z + x + y^2 - 2z = 0$ or $x^2 + y^2 - z^2 - 2x + 4z = 0$.

Method II. Let $Pp + Qq = R$ ...(1)

be the given equation. Let is Lagrange's auxiliary equations give the following two independent integrals $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$. ...(2)

Suppose we wish to obtain the integral surface passing though the curve which is determined by the following two equations

$\phi(x, y, z) = 0$ and $\psi(x, y, z) = 0$. ...(3)

We eliminate x, y, z from four equations of (2) and (3) and obtain a relation between c_1 and c_2. Finally, replace c_1 by $u(x, y, z)$ and c_2 by $v(x, y, z)$ in that relation and obtain the desired integral surface.

Ex. 3. *Find the integral surface of the partial differential equation* $(x - y)p + (y - x - z)q = z$ *through the circle* $z = 1$, $x^2 + y^2 = 1$. **(Nagpur 2002)**

Sol. Given $(x - y)p + (y - x - z)q = z$. ...(1)

Lagrange's auxiliary equations for (1) are $\dfrac{dx}{x-y} = \dfrac{dy}{y-x-z} = \dfrac{dz}{z}$. ...(2)

Choosing 1, 1, 1 as multipliers, each fraction on (2) = $(dx + dy + dz)/0$

$\therefore\ dx + dy + dz = 0$ so that $x + y + z = c_1$. ...(3)

Taking the last two fractions of (2) and using (3) we get

$\dfrac{dy}{y-(c_1-y)} = \dfrac{dz}{z}$ or $\dfrac{2dy}{2y-c_1} - \dfrac{2dz}{z} = 0$.

Integrating it, $\log\ (2y - c_1) - 2\log z = \log c_2$ or $(2y - c_1)/z^2 = c_2$

or $(2y - x - y - z)/z^2 = c_2$ or $(y - x - z)/z^2 = c_2$. ....(4)

The given curve is given by $z = 1$ and $x^2 + y^2 = 1$. ...(5)

Putting $z = 1$ in (3) and (4), we get $x + y = c_1 - 1$ and $y - x = c_2 + 1$. ...(6)

But $2(x^2 + y^2) = (x + y)^2 + (y - x)^2$. ...(7)

Using (5) and (6), (7) becomes

$2 = (c_1 - 1)^2 + (c_2 + 1)^2$ or $c_1^{\,2} + c_2^{\,2} - 2c_1 + 2c_2 = 0$. ...(8)

Putting the values of c_1 and c_2 from (3) and (4) in (8), required integral surface is

$$(x + y + z)^2 + (y - x - z)^2/z^4 - 2(x + y + z) + 2(y - x - z)/z^2 = 0$$

or
$$z^4(x + y + z)^2 + (y - x - z)^2 - 2z^4(x + y + z) + 2z^2(y - x - z) = 0.$$

Ex. 4. *Find the equation of the integral surface of the differential equation* $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy$ *which passes through the line* $x = 1$, $y = 0$.

Sol. Given $(x^2 - yz)p + (y^2 - zx)q = z^2 - xy.$ ...(1)

Proceed as in solved Ex. 6, Art. 2.12 and show that

$$(x - y)/(y - z) = c_1 \quad ...(2)$$

and $$xy + yz + zx = c_2. \quad ...(3)$$

The given curve is represented by $x = 1$ and $y = 0.$ ...(4)

Using (4) in (2) and (3), we obtain $-1/z = c_1$ and $z = c_2$

so that $(-1/z) \times z = c_1c_2$ or $c_1c_2 + 1 = 0.$ ...(5)

Putting the values of c_1 and c_2 from (2) and (3) in (5), the required integral surface is

$[(x - y)/(y - z)]\,(xy + yz + zx) + 1 = 0$ or $(x - y)\,(xy + yz + zx) + y - z = 0$

Ex. 5. *Find the equation of surface satisfying* $4yzp + q + 2y = 0$ *and passing through* $y^2 + z^2 = 1$, $x + z = 2$. **[I.A.S. 1997]**

Sol. Given $4yzp + q = -2y.$ ...(1)

Given curve is given by $y^2 + z^2 = 1,$ and $x + z = 2.$ ...(2)

The Lagrange's auxiliary equations for (1) are $\dfrac{dx}{4yz} = \dfrac{dy}{1} = \dfrac{dz}{-2y}.$ ...(3)

Taking the first and third fractions of (3), $dx + 2zdz = 0$ so that $x + z^2 = c_1.$ ...(4)

Taking the last two fractions of (3), $dz + 2ydy = 0$ so that $z + y^2 = c_2.$ ...(5)

Adding (4) and (5), $(y^2 + z^2) + (x + z) = c_1 + c_2$

or $1 + 2 = c_1 + c_2,$ using (2) ...(6)

Putting the values of c_1 and c_2 from (4) and (5) in (6), the equation of the required suface is given by $3 = x + z^2 + z + y^2$ or $y^2 + z^2 + x + z - 3 = 0.$

Ex. 6. *Find the general integral of the partial differential equation* $(2xy - 1)p + (z - 2x^2)q = 2(x - yz)$ *and also the particular integral which passes through the line* $x = 1$, $y = 0$. **[I.A.S. 2008]**

Sol. Given $(2xy - 1)p + (z - 2x^2)q = 2(x - yz).$ ...(1)

Given line is given by $x = 1$ and $y = 0.$ ...(2)

Lagrange's auxiliary equations of (1) are $\dfrac{dx}{2xy-1} = \dfrac{dy}{z-2x^2} = \dfrac{dz}{2x-2yz}.$ ...(3)

Taking z, 1, x as multipliers, each fraction of (3) $= (zdx + dy + x\,dz)/0$

so that $zdx + dy + xdz = 0$ or $d(xz) + dy = 0$

Integrating, $xz + y = c_1.$ ...(4)

Again, taking x, y, 1/2 as multipliers, each fraction of (3) $= \{xdx + ydy + (1/2)dz\}/0$

so that $x\,dx + ydy + (1/2) \times dz = 0$ or $2xdx + 2ydy + dz = 0$

Integrating, $x^2 + y^2 + z = c_2.$ ...(5)

Since the required curve given by (4) and (5) passes through the line (2), so putting $x = 1$ and $y = 0$ in (4) and (5), we get

$z = c_1$ and $1 + z = c_2$ so that $1 + c_1 = c_2.$ ...(6)

Substituting the values of c_1 and c_2 from (4) and (5) in (6), the eqution of the required surface is given by

$1 + xz + y = x^2 + y^2 + z$ or $x^2 + y^2 + z - xz - y = 1.$

Ex. 7. *Find the integral surface of* $x^2p + y^2q + z^2 = 0$, $p = \partial z/\partial x$, $q = \partial z/\partial y$ *which passes through the hyperbola* $xy = x + y$, $z = 1$. **[I.A.S. 1994, 2009]**

Sol. Given $x^2p + y^2q + z^2 = 0$ or $x^2p + y^2q = -z^2.$ ...(1)

Given curve is given by $xy = x + y$ and $z = 1.$ ...(2)

Here Lagrange's auxiliary equations for (1) are $(dx)/x^2 = (dy)/y^2 = (dz)/(-z^2).$ ...(3)

Taking the first and third fractions of (1), $x^{-2}dx + z^{-2}dz = 0$.

Integrating, $-(1/x) - (1/z) = -c_1$ or $1/x + 1/z = c_1$. ...(4)

Taking the second and third fractions of (1), $y^{-2}dy + z^{-2}dz = 0$.

Integrating, $-(1/y) - (1/z) = -c_2$ or $1/y + 1/z = c_2$. ...(5)

Adding (4) and (5), $\frac{1}{x}+\frac{1}{y}+\frac{2}{z} = c_1 + c_2$ or $\frac{x+y}{xy}+\frac{2}{z} = c_1 + c_2$

or $(xy)/(xy) + 2 = c_1 + c_2$, using (2) or $c_1 + c_2 = 3$. ...(6)

Substituting the values of c_1 and c_2 from (4) and (5) in (6), we get

$1/x + 1/z + 1/y + 1/z = 3$ or $yz + 2xy + xz = 3xyz$.

Ex. 6. *Find the integral surface of the linear first order partial differential equation* $yp + xq = z - 1$ *which passes through the curve* $z = x^2 + y^2 + z,\ y = 2x$

Sol. Given equation is $yp + xq = z - 1$ ... (1)

and the given curve is given by $z = x^2 + y^2 + 1$ and $y = 2x$ ... (2)

Lagrange's auxiliary equations for (1) are $\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{z-1}$ ... (3)

Taking the first two fractions, $2ydy - 2xdx = 0$

Integrating, it, $y^2 - x^2 = C_1$, C_1 being an arbitrary constant ... (4)

Taking the first and the last fractions of (3) and using (4), we get

$\frac{dx}{(x^2 + C_1)^{1/2}} = \frac{dz}{z-1}$ so that $\log(z-1) - \log\{x + (x^2 + C_1)^{1/2}\} = \log C_2$

or $\log(z - 1) - \log(x + y) = \log C_2$, by (4) or $(z - 1)/(x + y) = C_2$ ... (5)

The parametric form of the given curve (2) is $x = t,\quad y = 2t,\quad z = 5t^2 + 1$ ... (6)

Substituting these values in (4) and (5), we get $3t^2 = C_1$ and $5t/3 = C_2$ ... (7)

Eliminating t from the above equations (7), we get $5\sqrt{C_1}/3\sqrt{3} = C_2$ ... (8)

Substituting the values of C_1 and C_2 from (4) and (5) in (8), the required surface is given by

$$5(y^2 - x^2)^{1/2}/3\sqrt{3} = (z-1)/(x+y).$$

Ex. 7. *Find the integral surface of the partial differential equation* $(x - y)y^2p + (y - x)x^2q = (x^2 + y^2)z$ *passing through the curve* $xz = a^3,\ y = 0$.

Sol. Given equation is $(x-y)y^2p + (y-x)x^2q = (x^2 + y^2)z$ ... (1)

and the given curve is given by $xz = a^3$ and $y = 0$ ... (2)

Lagrange's auxiliary equations for (1) are $\frac{dx}{(x-y)y^2} = \frac{dy}{(y-x)x^2} = \frac{dz}{(x^2+y^2)z}$ ... (3)

Each fraction of (3) = $\frac{dx - dy}{(x-y)\,(y^2 + x^2)} = \frac{dz}{(x^2+y^2)z}$ so that $\frac{d(x-y)}{x-y} - \frac{dz}{z} = 0$

Integrating it, $(x-y)/z = C_1$, C_1 being an arbitrary contant ... (4)

Taking the first two fractions, $3x^2dx + 3y^2dy = 0$

Integrating it, $x^3 + y^3 = C_2$, C^2 being an arbitrary constant. ... (5)

The parameteric form of the given curve (2) is $z = t,\quad x = a^3/t,\quad y = 0$ ... (6)

Substituting these values in (4) and (5), we get

$$a^3/t^2 = C_1 \quad \text{so that} \quad t^2 = a^3/C_1 \qquad ...(7)$$

and $$(a^3/t)^3 = C_2 \quad \text{so that} \quad t^3 = a^9/C_2 \qquad ...(8)$$

Squaring both sides of (8), $t^6 = a^{18}/C_2^2$ or $(t^2)^3 = a^{18}/C_2^2$

or $$(a^3/C_1)^3 = a^{18}/C_2^2, \quad \text{since} \quad t^2 = a^3/C_1, \text{ by (7)}$$

or $$a^9/C_1^3 = a^{18}/C_2^2, \quad \text{or} \quad C_2^2 = a^9\ C_1^3 \qquad ...(9)$$

Substituting the values of C_1 and C_2 from (4) and (5) in (9), the required integral surface of (1) is given by

$$(x^3 + y^3)^2 = a^9\ (x-y)^3/z^3 \quad \text{or} \quad z^3(x^3+y^3)^2 = a^9\ (x-y)^3.$$

EXERCISE 2(F)

1. *Find particular integrals of the following partial differential equations to represent surfaces passing through the given curves :*

(*i*) $p + q = 1$; $x = 0$, $y^2 = z$. **Ans.** $(y - x)^2 = z - x$.

(*ii*) $xp + yq = z$; $x + y = 1$, $yz = 1$. **Ans.** $yz = (x + y)^2$.

(*iii*) $(y - z)p + (z - x)q = x - y$; $z = 0$, $y = 2x$ **Ans.** $5(x + y + z)^2 = 9(x^2 + y^2 + z^2)$.

(*iv*) $x(y - z)p + y(z - x)q = z(x - y)$; $x = y$; $x = y = z$. **Ans.** $(x + y + z)^3 = 27xyz$.

(*v*) $yp - 2xyq = 2xz$; $x = t$, $y = t^2$, $z = t^3$. **Ans.** $(x^2 + y^2)^5 = 32y^2z^2$.

(*vi*) $(y - z)\ [2xyp + (x^2 - y^2)q] + z(x^2 - y^2) = 0$; $x = t^2$, $y = 0$, $z = t^3$.
Ans. $x^3 - 3xy^2 = z^2 - 2yz$.

2. Find the general solution of the equation $2x(y + z^2)p + y(2y + z^2)q = z^2$ and deduce that $yz(z^2 + yz - 2y) = x^2$ is a solution.

3. Find the general solution of $x(z + 2a)p + (xz + 2yz + 2ay)q = z(z + a)$.

Find also the integral surfaces which pass through the curves :

(*i*) $y = 0$, $z^2 = 4ax$. (*ii*) $y = 0$, $z^3 + x(z + a)^2 = 0$.

4. Solve $xp + yq = z$. Find a solution representing a surface meeting the parabola $y^2 = 4x$, $z = 1$. **Ans.** General solution $\phi\ (x/2, y/2) = 0$; surface $y^2 = 4xz$.

2.16. SURFACES ORTHOGONAL TO A GIVEN SYSTEM OF SURFACES

Let $$f(x, y, z) = C \qquad ...(1)$$

represents a system of surfaces where C is parameter. Suppose we wish to obtain a system of surfaces which cut each of (1) at right angles. Then the direction ratios of the normal at the point (x, y, z) to (1) which passes through that point are $\partial f/\partial x$, $\partial f/\partial y$, $\partial f/\partial z$.

Let the surface $$z = \phi(x, y) \qquad ...(2)$$

cuts each surface of (1) at right angles. Then the normal at (x, y, z) to (2) has direction ratios $\partial z/\partial x$, $\partial z/\partial y$, -1 *i.e.*, p, q, -1. Since normals at (x, y, z) to (1) and (2) are at right angles, we have

$$p(\partial f/\partial x) + q(\partial f/\partial y) - (\partial f/\partial z) = 0 \quad \text{or} \quad p(\partial f/\partial x) + q(\partial f/\partial y) = \partial f/\partial z \qquad ...(3)$$

which is of the form $Pp + Qq = R$.

Conversely, we easily verify that any solution of (3) is orthogonal to every surface of (1).

2.17. SOLVED EXAMPLES BASED ON ART. 2.16.

Ex. 1. *Find the surface which intersects the surfaces of the system* $z(x + y) = c(3z + 1)$ *orthogonally and which passes through the circle* $x^2 + y^2 = 1$, $z = 1$. **[I.A.S. 1999]**

Sol. The given system of surfaces is $f(x, y, z) \equiv \{z(x+y\}/(3z+1) = C.$ ...(1)

$\therefore \quad \dfrac{\partial f}{\partial x} = \dfrac{z}{3z+1}, \qquad \dfrac{\partial f}{\partial y} = \dfrac{z}{3z+1}, \qquad \dfrac{\partial f}{\partial z} = (x+y)\dfrac{(3z+1)-z\times 3}{(3z+1)^2} = \dfrac{x+y}{(3z+1)^2}.$

The required orthogonal surface is solution of

$$p\frac{\partial f}{\partial x}+q\frac{\partial f}{\partial y} = \frac{\partial f}{\partial z} \qquad \text{or} \qquad \frac{z}{3z+1}p+\frac{z}{3z+1}q = \frac{x+y}{(3z+1)^2}$$

or $$z(3z + 1)p + z(3z + 1)q = x + y. \qquad ...(2)$$

Lagrange's auxiliary equations for (2) are $\dfrac{dx}{z(3z+1)} = \dfrac{dy}{z(3z+1)} = \dfrac{dz}{x+y}$. ...(3)

Taking the first two fractions of (3), we get $dx - dy = 0$ so that $x - y = C_1$. ...(4)

Choosing $x, y, -z(3z + 1)$ as multipliers, each fraction of (3) $= [xdx + ydy - z(3z + 1)dz]/0$

$\therefore \quad xdx + ydy - 3z^2dz - zdz = 0 \qquad$ or $\qquad 2xdx + 2ydy - 6z^2dz - 2zdz = 0$

Integrating, $x^2 + y^2 - 2z^3 - z^2 = C_2$, C_2 being an arbitrary constant. ...(5)

Hence any surface which is orthogonal to (I) has equation of the form

$$x^2 + y^2 - 2z^3 - z^2 = \phi(x - y),\ \phi \text{ being an arbitrary function} \qquad ...(6)$$

In order to get the desired surface passing through the circle $x^2 + y^2 = 1$, $z = 1$ we must choose $\phi(x - y) = -2$. Thus, the required particular surface is $x^2 + y^2 - 2z^3 - z^2 = -2$.

Ex. 2. *Write down the system of equations for obtaining the general equation of surfaces orthogonal to the family given by* $x(x^2+y^2+z^2)=C_1y^2$. **[I.A.S. 2001]**

Sol. Given family of surfaces is $\qquad x(x^2+y^2+z^2)/y^2 = C_1$

Let $$f(x,y,z) = x(x^2+y^2+z^2)/y^2 = C_1 \qquad ...(1)$$

Then the surfaces orthogonal to the system (1) are the surfaces generated by the integral curves of the equations

$$\frac{dx}{\partial f/\partial x} = \frac{dy}{\partial f/\partial y} = \frac{dz}{\partial f/\partial z} \qquad \text{or} \qquad \frac{dx}{(3x^2+y^2+z^2)/y^2} = \frac{dy}{-2x(x^2+z^2)/y^3} = \frac{dz}{2x/y^2z}$$

or $$\frac{dx}{y(3x^2+y^2+z^2)} = \frac{dy}{-2x(x^2+z^2)} = \frac{dz}{2xyz} \qquad ...(2)$$

Taking x, y, z as multipliers, each fraction of (2)

$$= \frac{xdx+ydy+zdz}{xy(3x^2+y^2+z^2)-2xy(x^2+z^2)+2xyz} = \frac{xdx+ydy+zdz}{xy(x^2+y^2+z^2)} \qquad ...(3)$$

Combining this fraction (3) with the last fraction of (2), we get

$$\frac{xdx+ydy+zdz}{xy(x^2+y^2+z^2)} = \frac{dz}{2xyz} \qquad \text{or} \qquad \frac{2xdx+2ydy+2zdz}{x^2+y^2+z^2} = \frac{\mathrm{dz}}{\mathrm{z}}$$

Integrating, $\log(x^2+y^2+z^2) = \log z + \log C_2 \qquad$ or $\qquad (x^2+y^2+z^2)/z = C_2$ ...(4)

Taking $4x$, $2y$, 0 as multipliers, each fraction of (2)

$$= \frac{4xdx+2ydy}{4xy(3x^2+y^2+z^2)-4xy(x^2+y^2)} = \frac{4xdx+2ydy}{4xy(2x^2+y^2)} \qquad ...(5)$$

Combining this fraction (5) with the last fraction of (2), we get

$$\frac{4xdx+2ydy}{4xy(2x^2+y^2)} = \frac{dz}{2xyz} \qquad \text{or} \qquad \frac{4xdx+2ydy}{2x^2+y^2} = \frac{2dz}{z}$$

Integrating, $\log(2x^2+y^2) = 2\log z + \log C_3 \qquad$ or $\qquad (2x^2+y^2)/y^2 = C_3$ ... (6)

From (4) and (5), the required general equation of the surfaces which are orthogonal to the given family of surfaces (1) is of the form $(x^2+y^2+z^2)/z=\phi\,\{(2x^2+y^2)/z^2\}$, *i.e.*,

or $\quad x^2+y^2+z^2=z\,\phi\,\{(2x^2+y^2)/z^2\}$, where ϕ is an arbitrary function.

Ex. 3. *Find the surface which is orthogonal to the one parameter system* $z=cxy(x^2+y^2)$ *which passes through the hyperbola* $x^2-y^2=a^2,\ z=0$

Sol. The given system of surfaces is $\quad f(x,y,z)=z/(x^3y+xy^3)=C \quad$ ... (1)

$$\frac{\partial f}{\partial x}=-\frac{z(3x^2y+y^3)}{(x^3y+xy^3)^2},\qquad \frac{\partial f}{\partial y}=-\frac{z(3y^2x+x^3)}{(x^3y+xy^3)^2},\qquad \frac{\partial f}{\partial z}=\frac{1}{x^3y+xy^3}$$

The required orthogonal surface is solution of $\quad p(\partial f/\partial x)+q(\partial f/\partial y)=\partial f/\partial z$

or
$$-\frac{z(3x^2y+y^3)}{(x^3y+xy^3)^2}p-\frac{z(3y^2x+x^3)}{(x^3y+xy^3)^2}q=\frac{1}{x^3y+xy^3}$$

or
$$\{(3x^2+y^2)/x\}p+\{(3y^2+x^2)/y\}q=-(x^2+y^2)/z \qquad \text{... (2)}$$

Lagrange's auxiliary equations for (2) are

$$\frac{dx}{(3x^2+y^2)/x}=\frac{dy}{(3y^2+x^2)/y}=\frac{dz}{-(x^2+y^2)/z} \qquad \text{... (3)}$$

Taking the first two fractions of (3), $\quad 2xdx-2ydy=0 \quad$ so that $\quad x^2-y^2=C_1$

Choosing x, y, $4z$ as multipliers, each fraction of (3) = $(xdx+ydy+4zdz)/0$

$\therefore \quad 2xdx+2ydy+8zdz=0 \quad$ so that $\quad x^2+y^2+4z^2=C_2$

Hence any surface which is orthogonal to (1) is of the form

$$x^2+y^2+4z^2=\Phi\,(x^2-y^2),\ \Phi \text{ being an arbitrary function.} \qquad \text{... (4)}$$

For the particular surface passing through the hyperbola $x^2-y^2=a^2,\ z=0$ we must take $\Phi(x^2-y^2)=a^4(x^2+y^2)/(x^2-y^2)^2$. Hence, the required surface is given by

$$(x^2+y^2+4z^2)^2\,(x^2-y^2)^2=a^4(x^2+y^2)$$

Ex. 4. *Find the equation of the system of surfaces which cut orthogonally the cones of the system* $x^2+y^2+z^2=cxy$.

2.18 (*a*). Geometrical description of the solutions of $Pp+Qq=R$ and of the system of equations $dx/P=dy/Q=dz/R$ and to establish relationship between the two.

[G.N.D.U. Amritsar 1998; Meerut 1997; Kanpur 1996]

Proof. Consider $\quad Pp+Qq=R. \quad$ ....(1)

and $\quad (dx)/P=(dy)/Q=(dz)/R, \quad$ ...(2)

where P, Q and R are functions of x.

Let $\quad z=\phi(x,y) \quad$ ...(3)

represent the solution of (1). Then (3) represents a surface whose normal at any point (x, y, z) has direction ratios $\partial z/\partial x$, $\partial z/\partial y$, -1 *i.e.*, p, q, -1. Also we know that the simultaneous equations (2) represent a family of curves such that the tangent at any point has direction ratios P, Q, R. Rewriting (1), we have

$$Pp+Qq+R(-1)=0, \qquad \text{...(4)}$$

showing that the normal to surface (3) at any point is perpendicular to the member of family of curves (2) through that point. Hence the member must touch the surface at that point. Since this holds for each point on (3), we conclude that the curves (2) lie completely on the surface (3) whose differential equation is (1).

2.18 (b). Another geomertical interpretation of Lagrange's equation Pp + Qq = R.

To show that the surfaces represented by $Pp + Qq = R$ are orthogonal to the surfaces represented by $Pdx + Qdy + Rdz = 0$.

We know that the curves whose equations are solutions of

$$(dx)/P = (dy)/Q = (dz)/R \qquad ...(1)$$

are orthogonal to the system of the surfaces whose equation satisfies

$$Pdx + Qdy + Rdz = 0. \qquad ...(2)$$

Again from Art 2.18 (*a*) the curves of (1) lie completely on the surface represented by

$$Pp + Qq = R. \qquad ...(3)$$

Hence we conclude that surfaces represented by (2) and (3) are orthogonal.

2.19. SOLVED EXAMPLES BASED ON ART 2.18(*a*) AND ART. 2.18 (*b*)

Ex. 1. *Find the family orthogonal to $\phi\,[z(x + y)^2, x^2 - y^2] = 0$.*

Sol. Given $\qquad \phi[z(x + y)^2, x^2 - y^2] = 0. \qquad ...(1)$

Let $\qquad u = z(x + y)^2 \qquad$ and $\qquad v = x^2 - y^2 \qquad ...(2)$

Then (1) becomes $\qquad \phi(u, v) = 0. \qquad ...(3)$

Differentiating (3) w.r.t. x and y partially by turn, we get

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial x}+p\frac{\partial u}{\partial z}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial x}+p\frac{\partial v}{\partial z}\right) = 0 \qquad ...(4)$$

and

$$\frac{\partial\phi}{\partial u}\left(\frac{\partial u}{\partial y}+q\frac{\partial u}{\partial z}\right)+\frac{\partial\phi}{\partial v}\left(\frac{\partial v}{\partial y}+q\frac{\partial v}{\partial z}\right) = 0. \qquad ...(5)$$

From (2), $\quad (\partial u/\partial x) = 2z(x + y), \quad (\partial u/\partial y) = 2z(x + y), \quad (\partial u/\partial z) = (x + y)^2,$

$(\partial v/\partial x) = 2x, \quad (\partial v/\partial y) = -2y, \quad (\partial v/\partial z) = 0.$

Putting these values in (4) and (5), we get

$$(\partial\phi / \partial u)\left[2z(x+y)+p(x+y)^2\right]+(\partial\phi / \partial v)\,(2x+0) = 0 \qquad ...(6)$$

and

$$(\partial\phi / \partial u)\left[2z(x+y)+q(x+y)^2\right]+(\partial\phi / \partial v)\,(-2y+0) = 0 \qquad ...(7)$$

Evaluating the values of $-\dfrac{\partial\phi / \partial u}{\partial\phi / \partial v}$ from (6) and (7) and then equating these, we get

$$-\frac{\partial\phi / \partial u}{\partial\phi / \partial v} = \frac{2x}{2z(x+y)+p(x+y)^2} = \frac{-2y}{2z(x+y)+q(x+y)^2}$$

or $\;x(x+y)[2z+q(x+y)] = -y(x+y)[2z+p(x+y)] \quad$ or $\quad 2xz+qx(x+y)+2yz+py(x+y) = 0$

or $\quad py(x + y) + qx(x + y) = -2z(x + y) \quad$ or $\quad py + qx = -2z \qquad ...(8)$

which is differential equation of the family of surfaces given by (1). So the differential equation of the family of surfaces orthogonal to (8) is given by [use Art. 2.18 (b)]

$$ydx + xdy - 2zdz = 0 \qquad \text{or} \qquad d(xy) - 2zdz = 0. \qquad ...(9)$$

Integrating (9), $\qquad xy - z^2 = C,$

which is the desired family of orthogonal surfaces, C being parameter

Ex. 2. *Find the family of surfaces orthogonal to the family of surfaces given by the differential equation $(y + z)p + (z + x)q = x + y$.*

Sol. Let $\qquad P = y + z, \qquad Q = z + x \qquad$ and $\qquad R = x + y. \qquad ...(1)$

Then, the given differential equation can be written as $\qquad Pp + Qq = R. \qquad ...(2)$

Now, the differential equation of the family of surfaces orthogonal to the given family is

$Pdx + Qdy + Rdz = 0$ or $(y + z)dx + (z + x)dy + (x + y)dz = 0$

or $$(ydx + xdy) + (ydz + zdy) + (zdx + xdz) = 0.$$

Integrating, $$xy + yz + zx = C,$$
which is the required family of surfaces, C being a parameter.

2.20. The linear partial differential equation with n independent variables and its solution. Let $x_1, x_2 ..., x_n$ be the n independent variables and let $p_1 = \partial z/\partial x_1, p_2 = \partial z/\partial x_2, ..., p_n = \partial z/\partial x_n$, where z is the dependent variable. Consider the general linear partial differential equation with n independent variables $$P_1 p_1 + P_2 p_2 + .. + P_n p_n = R, \quad ...(1)$$
where $P_1, P_2,.., P_n$ are functions of $x_1, x_2..., x_n$. Let $u_1 = c_1$, $u_2 = c_2$, ..., $u_n = c_n$ be any n independent integrals of the auxiliary equations
$$(dx_1)/P_1 = (dx_2)/P_2 = ... = (dx_n)/P_n. \quad ...(2)$$
Then the general solution of (1) is given by $\phi(u_1, u_2, ...u_n) = 0.$ ...(3)

Note that the above procedure is generalization of Lagrange's method.

2.21. SOLVED EXAMPLES BASED ON ART. 2.20

Ex. 1. *Solve* $x_2x_3 p_1 + x_3x_1 p_2 + x_1x_2 p_3 + x_1x_2x_3 = 0.$

Sol. Re–writing the given equation in standard form, we have
$$x_2x_3 p_1 + x_3x_1 p_2 + x_1x_2 p_3 = -x_1x_2x_3. \quad ...(2)$$

The auxiliary equations for (2) are $\dfrac{dx_1}{x_2x_3} = \dfrac{dx_2}{x_3x_1} = \dfrac{dx_3}{x_1x_2} = \dfrac{dz}{-x_1x_2x_3}.$ ...(3)

Taking the first and the fourth fractions of (3), $x_1dx_1 + dz = 0$ so that $x_1^2 + 2z = c_1$. ...(4)

Taking 1st and 2nd fractions of (3), $x_1dx_1 = x_2dx_2$ so that $x_1^2 - x_2^2 = c_2$. ...(5)

Finally, 2nd and 3rd fractions of (3) give $x_2dx_2 = x_3dx_3$ so that $x_2^2 - x_3^2 = c_3$. ...(6)

Hence the required general integral is
$$\phi(x_1^2 + 2z, x_1^2 - x_2^2, x_2^2 - x_3^2) = 0, \quad \phi \text{ being an arbitrary function.}$$

Ex. 2. *Solve* $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} + t\dfrac{\partial z}{\partial t} = az + \dfrac{xz}{t}$

Sol. Here auxiliary equations for the given equation are
$$\frac{dx}{x} = \frac{dy}{y} = \frac{dt}{t} = \frac{dz}{az + xy/t}. \quad ...(1)$$

From the first two fractions of (1), $(1/x)dx - (1/y)dy = 0$ so that $x/y = C_1$. ...(2)

From the first and third fractions of (1), $(1/x)dx - (1/t)dt = 0$ so that $x/t = C_2$. ...(3)

Dividing (3) by (2), we have $y/t = C_2/C_1$. ...(4)

Taking the first and third fractions of (1) and using (4), we get
$$\frac{dx}{x} = \frac{dz}{az + (C_2/C_1)x} \quad \text{or} \quad \frac{dz}{dx} = \frac{az + (C_2/C_1)x}{x}$$

or $$\frac{dz}{dx} - \left(\frac{a}{x}\right)z = \left(\frac{C_2}{C_1}\right), \text{ which is linear.} \quad ...(5)$$

I.F. of (5) $= e^{-\int (a/x)\,dx} = e^{-a\log x} = e^{\log x^{-a}} = x^{-a}$ and so solution of (5) is given by
$$zx^{-a} = C_3 + \frac{C_2}{C_1}\int x^{-a}dx = C_3 + \frac{C_2}{C_1}\frac{x^{1-a}}{1-a} = C_3 + \frac{y}{t}\frac{x^{1-a}}{1-a}, \text{ using (4)}$$

$\therefore \quad zx^{-a} - \dfrac{y}{t}\dfrac{x^{1-a}}{1-a} = C_3$, C_3 being an arbitrary constant. ...(6)

From (2), (3) and (6), the required general solution is

$$\phi\left(\frac{x}{y},\frac{x}{t},zx^{-a}-\frac{y}{t}\,\frac{x^{1-a}}{1-a}\right)=0,\ \ \phi \text{ being an arbitrary function.}$$

Ex. 3. *Solve* $x(\partial u/\partial x)+y(\partial u/\partial y)+z(\partial u/\partial z)=xyz$. **[Bhopal 1995, 98; I.A.S. 1999]**

Sol. Here the auxiliary equations for the given equation are

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{z}=\frac{du}{xyz}. \qquad ...(1)$$

Taking the first two fractions of (1), $(1/x)dx-(1/y)dy=0$.

Integrating it, $\log x-\log y=\log C_1$ or $x/y=C_1$. ...(2)

Taking the first and third fractions of (1), $(1/x)dx-(1/z)dz=0$

Integrating it, $\log x-\log z=\log C_2$ or $x/z=C_2$. ...(3)

Choosing yz, zx, xy as multipliers, each fraction of (1) $=\dfrac{yzdx+zxdy+xydz}{xyz+xyz+xyz}=\dfrac{d(xyz)}{3xyz}$. ...(4)

Combining the fourth fraction of (1) with fraction (4), we get

$\dfrac{du}{xyz}=\dfrac{d(xyz)}{3xyz}$ or $d(xyz)-3du=0$ so that $xyz-3u=C_3$. ...(5)

From (2), (3) and (5), the required general solution is

$$\phi(x/y,\ x/z,\ xyz-3u)=0,\ \ \phi \text{ being an arbitrary function.}$$

Ex. 4. *Solve* $(y+z+w)(\partial w/\partial x)+(z+x+w)(\partial w/\partial y)+(x+y+w)(\partial w/\partial z)=(x+y+z)$.
[Ravishanker 2004 ; I.A.S. 1995; Indore 1998, Kanpur 2004]

Sol. Here the auxiliary eqautions of the given equation are

$$\frac{dx}{y+z+w}=\frac{dy}{z+x+w}=\frac{dz}{x+y+w}=\frac{dw}{x+y+z}. \qquad ...(1)$$

Each fraction of (1) $=\dfrac{dw-dx}{-(w-x)}=\dfrac{dw-dy}{-(w-y)}=\dfrac{dw-dz}{-(w-z)}=\dfrac{dw+dx+dy+dz}{3(w+x+y+z)}$. ...(2)

Taking the first and the fourth fractions of (2), $\dfrac{dw+dx+dy+dz}{3(w+x+y+z)}+\dfrac{dw-dz}{w-x}=0$.

Integrating, $(1/3)\times\log(w+x+y+z)+\log(w-x)=\log C_1$

or $(w+x+y+z)^{1/3}(w-x)=C_1$. ...(3)

Similarly, $(w+x+y+z)^{1/3}(w-y)=C_2$. ...(4)

and $(w+x+y+z)^{1/3}(x-z)=C_3$. ...(5)

From (3), (4) and (5), the required general solution is

$\phi[(w+x+y+z)^{1/3}(w-x),\ (w+x+y+z)^{1/3}(w-y),\ (w+x+y+z)^{1/3}(w-z)]=0$,
where ϕ is an arbitrary function.

Ex. 5. *Solve* $p_1+p_2+p_3=4z$. **Ans.** $\phi(ze^{-4x_1},ze^{-4x_2},ze^{-4x_3})=0$

Ex. 6. *Solve* $x_2x_3p_1+x_3x_1p_2+x_1x_2p_3+x_1x_2x_3=0$.

Sol. Putting the given equation in standard form, we have

$$x_2x_3p_1+x_3x_1p_2+x_1x_2p_3=-x_1x_2x_3. \qquad ...(1)$$

Here the auxiliary equations for (1) are $\dfrac{dx_1}{x_2x_3}=\dfrac{dx_2}{x_3x_1}=\dfrac{dx_3}{x_1x_2}=\dfrac{dz}{-x_1x_2x_3}$. ...(2)

Taking the first and second fractions of (2), we have

$2x_1dx_1-2x_2dx_2=0$ so that $x_1^{\,2}-x_2^{\,2}=C_1$. ...(3)

Taking the first and third fractions of (2), we have

$2x_1 dx_1 - 2x_3 dx_3 = 0$ so that $x_1^2 - x_3^2 = C_2$. ...(4)

Taking the first and fourth fractions of (2), we have

$2x_1 dx_1 + 2dz = 0$ so that $x_1^2 + 2z = C_3$. ...(5)

From (3), (4) and (5), the required general solution is

$$\phi(x_1^2 - x_2^2, x_1^2 - x_3^2, x_1^2 + 2z) = 0, \quad \phi \text{ being an arbitrary function.}$$

Ex. 7. *Solve* $p_1 + x_1 p_2 + x_1 x_2 p_3 = x_1 x_2 x_3 \sqrt{z}$. **Ans.** $\phi(x_1^2 - 2x_2, x_2^2 - 2x_3, x_3^2 - 4\sqrt{z}) = 0$

Ex. 8. *Solve* $(x_3 - x_2)p_1 + x_2 p_2 - x_3 p_3 + x_2^2 - (x_2 x_1 + x_2 x_3) = 0$.

Sol. Re–writing the given equation in the standard form, we get

$$(x_3 - x_2)p_1 + x_2 p_2 - x_3 p_3 = x_2 x_1 + x_2 x_3 - x_2^2. \quad ...(1)$$

Here the auxiliary equations for (1) are $\dfrac{dx_1}{x_3 - x_2} = \dfrac{dx_2}{x_2} = \dfrac{dx_3}{-x_3} = \dfrac{dz}{x_2 x_1 + x_2 x_3 - x_2^2}$. ...(2)

Taking the second and the third fractions of (2), we have

$(1/x_2)dx + (1/x_3)dx_2 = 0$ so that $\log x_2 + \log x_3 = \log C_1$ or $x_2 x_3 = C_1$....(3)

Each fraction of (2) $= \dfrac{dx_1 + dx_2 + dx_3}{(x_3 - x_2) + x_2 - x_3} = \dfrac{dx_1 + dx_2 + dx_3}{0}$.

$\therefore\ dx_1 + dx_2 + dx_3 = 0$ so that $x_1 + x_2 + x_3 = C_2$. ...(4)

Each fraction of (2) $= \dfrac{x_2 dx_1 + x_1 dx_2}{x_2(x_3 - x_2) + x_1 x_2} = \dfrac{d(x_1 x_2)}{x_1 x_2 + x_2 x_3 - x_2^2}$. ...(5)

Combining the last fraction of (2) with fraction (5), we have

$\dfrac{dz}{x_1 x_2 + x_2 x_3 - x_2^2} = \dfrac{d(x_1 x_2)}{x_1 x_2 + x_2 x_3 - x_2^2}$ or $dz - d(x_1 x_2) = 0$.

Integrating, $z - x_1 x_2 = C_3$, C_3 being an arbitrary constant. ...(6)

From (3), (4) and (5), the required general solution is

$$\phi(x_2 x_3, x_1 + x_2 + x_3, z - x_1 x_2) = 0, \quad \phi \text{ being an arbitrary function.}$$

Ex. 9. *If u is a function of x, y and z which satisfies* $(y - z)(\partial u/\partial x) + (z - x)(\partial u/\partial y) + (x - y)(\partial u/\partial z) = 0$, *show that u contains x, y, z only in combinations of* $x + y + z$ and $x^2 + y^2 + z^2$. **(Nagpur 2002, 05)**

Sol. Here auxiliary equations for given equation are $\dfrac{dx}{y - z} = \dfrac{dy}{z - x} = \dfrac{dz}{x - y} = \dfrac{du}{0}$. ...(1)

Each fraction of (1) $= \dfrac{dx + dy + dz}{(y - z) + (z - x) + (x - y)} = \dfrac{xdx + ydy + zdz}{x(y - z) + y(z - x) + z(x - y)} = \dfrac{du}{0}$

$$= \frac{dx + dy + dz}{0} = \frac{xdx + ydy + zdz}{0} = \frac{du}{0}.$$

$\therefore\ dx + dy + dz = 0$, $2xdx + 2ydy + 2zdz = 0$ and $du = 0$

Integrating, $x + y + z = C_1$, $x^2 + y^2 + z^2 = C_2$ and $u = C_3$.

Hence the required general solution is

$$u = f(x + y + z, x^2 + y^2 + z^2), \quad f \text{ being an arbitrary function.}$$

Ex. 10. *Prove that if* $x_1^3 + x_2^3 + x_3^3 = 1$ *when z = 0, the solution of the equation* $(s - x_1)p_1 + (s - x_2)p_2 + (s - x_2)p_2 + (s - x_3)p_3 = s - z$ can be given in the form $s^3\{(x_1 - z)^3 + (x_2 - z)^3 + (x_3 - z)^3\}^4 = (x_1 + x_2 + x_3 - 3z)^3$, where $s = x_1 + x_2 + x_3 + z$ and $p_i = \partial z/\partial x_i$, $i = 1, 2, 3$. **[I.A.S. 2000]**

Sol. Given $(s - x_1)p_1 + (s - x_2)p_2 + (s - x_3)p_3 = s - z$ ... (1)

where $s = x_1 + x_2 + x_3 + z$ ... (2)

The auxiliary equations for (2) are $\dfrac{dx_1}{s-x_1} = \dfrac{dx_2}{s-x_2} = \dfrac{dx_3}{s-x_3} = \dfrac{dz}{s-z}$

or $$\frac{dx_1}{x_2+x_3+z} = \frac{dx_2}{x_3+x_1+z} = \frac{dx_3}{x_1+x_2+z} = \frac{dz}{x_1+x_2+x_3}, \text{ using (2)} \qquad \dots (3)$$

Each fraction of (3) = $$\frac{dx_1+dx_2+dx_3-3dz}{2(x_1+x_2+x_3)+3z-3(x_1+x_2+x_3)} = \frac{d(x_1+x_2+x_3-3z)}{-(x_1+x_2+x_3-3z)} \qquad \dots (4)$$

Again, each fraction of (3) = $$\frac{dx_1+dx_2+dx_3+dz}{3(x_1+x_2+x_3+z)} = \frac{d(x_1+x_2+x_3+z)}{3(x_1+x_2+x_3+z)} \qquad \dots (5)$$

Then, (4) and (5) give $$\frac{d(x_1+x_2+x_3-3z)}{-(x_1+x_2+x_3-3z)} = \frac{d(x_1+x_2+x_3+z)}{3(x_1+x_2+x_3+z)} \qquad \dots (5)$$

or $$\frac{d(x_1+x_2+x_3+z)}{x_1+x_2+x_3+z} + 3\frac{d(x_1+x_2+x_3-3z)}{x_1+x_2+x_3-3z} = 0$$

Integrating, $\log(x_1+x_2+x_3+z) + 3\log(x_1+x_2+x_3-3z) = \log a$

or $(x_1+x_2+x_3+z)\,(x_1+x_2+x_3-3z)^3 = a$, where a is an arbitrary constant. ... (6)

Given that $x_1^3+x_2^3+x_3^3 = 1$ when $z = 0$ ... (7)

Hence (6) gives $a = (x_1+x_2+x_3)^4$. Then (6) reduces to

$$(x_1+x_2+x_3+z)\,(x_1+x_2+x_3-3z)^3 = (x_1+x_2+x_3)^4 \qquad \dots (8)$$

Now, each fraction of (3) = $$\frac{dx_1-dz}{-(x_1-z)} = \frac{3(x_1-z)^2\,d(x_1-z)}{-3(x_1-z)^3} = \frac{d(x_1-z)^3}{3(x_1-z)^3} \qquad \dots (9)$$

By symmetry, each fraction of (3) is also = $$\frac{d(x_2-z)^3}{-3(x_2-z)^3} = \frac{d(x_3-z)^3}{-3(x_3-z)^3} \qquad \dots (10)$$

Using (9) and (10), we find that each fraction of (3)

$$= \frac{d(x_1-z)^3}{-3(x_1-z)^3} = \frac{d(x_2-z)^3}{-3(x_2-z)^3} = \frac{d(x_3-z)^3}{-3(x_3-z)^3} = \frac{d[(x_1-z)^3+(x_2-z)^3+(x_3-z)^3]}{-3[(x_1-z)^3+(x_2-z)^3+(x_3-z)^3]} \qquad \dots (11)$$

Then, from (4) and (11), we have

$$\frac{3d(x_1+x_2+x_3-3z)}{(x_1+x_2+x_3-3z)} = \frac{d[(x_1-z)^3+(x_2-z)^3+(x_3-z)^3]}{[(x_1-z)^3+(x_2-z)^3+(x_3-z)^3]}$$

Integrating it, $3\log\,(x_1+x_2+x_3-3z) + \log b = \log\,\{(x_1-z)^3+(x_2-z)^3+(x_3-z)^3\}$

or $(x_1-z)^3+(x_2-z)^3+(x_3-z)^3 = b\,(x_1+x_2+x_3-3z)^3$ where b is an arbitrary constant. ...(12)

Putting $z = 0$, (12) gives $x_1^3+x_2^3+x_3^3 = b\,(x_1+x_2+x_3)^3$

or $1 = b\,(x_1+x_2+x_3)^3$, using (7) so that $b = 1/(x_1+x_2+x_3)^3$

$\therefore$ (12) $\Rightarrow$ $(x_1-z)^3+(x_2-z)^3+(x_3-z)^3 = (x_1+x_2+x_3-3z)^3/(x_1+x_2+x_3)^3$ ... (13)

Raising both sides of (8) to power 3, we have

$$(x_1+x_2+x_3+z)^3\,(x_1+x_2+x_3-3z)^9 = (x_1+x_2+x_3)^{12} \qquad \dots (14)$$

Raising both sides of (13) to power 4, we have

$$\{(x_1-z)^3+(x_2-z)^3+(x_3-z)^3\}^4 = (x_1+x_2+x_3-3z)^{12}/(x_1+x_2+x_3)^{12} \qquad \dots (15)$$

Multiplying the corresponding sides of (14) and (15), we have

$$(x_1+x_2+x_3+z)^3\,\{(x_1-z)^3+(x_2-z)^3+(x_3-z)^3\}^4 = (x_1+x_2+x_3-3z)^3$$

or $$s^3\{(x_1-z)^3+(x_2-z)^3+(x_3-z)^3\}^4 = (x_1+x_2+x_3-3z)^3, \text{ using (2)}$$

EXERCISE 2 (H)

Ex. 1. *Solve* $p_2 + p_3 = 1 + p_1$. **Ans.** $\phi(x_1 + x_2, x_1 + x_3, x_1 + z) = 0$

Ex. 2. *Solve* $zx_2x_3p_1 + zx_3x_1p_2 + zx_1x_2p_3 = x_1x_2x_3$. **Ans.** $\phi(x_1^2 - x_2^2, x_1^2 - x_2^2, x_1^2 - z^2) = 0]$

Ex. 3. *Solve* $x_1p_1 + 2x_2p_2 + 3x_3p_3 + 4x_4p_4 = 0$. **Ans.** $\phi(x_1^2/x_2, x_1^3/x_3, x_1^4/x_4, z) = 0$

Ex. 4. *Solve* $p_1 + p_2 + p_3 \{1 - (z - x_1 - x_2 - x_3)^{1/2}\} = 3$.

Ans. $\phi[z - 3x_1, \mathrm{z} - 3x_2, z + 6(z - x_1 - x_2 - x_3)^{1/2}] = 0$

Ex. 5. $p_1 + p_2 + p_3 \{1 + (z + x_1 + x_2 + x_3)^{1/2}\} + 3 = 0$.

Ans. $\phi[z + 3x_1, z + 3x_2, z + 6(z + x_1 + x_2 + x_3)^{1/2}] = 0$

Ex. 6. $x_1 p_1 + x_2 p_2 + x_3 p_3 = az + (x_1x_2)/x_3$. **[Delhi Maths (H) 1998]**

[**Hint.** This is same as Ex. 2 of Art. 2.21. Here $x = x_1$, $y = x_2$, $t = x_3, \partial z/\partial x = \partial z/\partial x = p_1$, $\partial z/\partial y = \partial z/\partial x_2 = p_2$, $\partial z/\partial t = \partial z/\partial x_3 = p_3$]

OBJECTIVE PROBLEMS ON CHAPTER 2

Select correct answer by writing (a), (b), (c) or (d).

1. The equation $Pp + Qq = R$ is known as (*a*) Charpit's equation
(*b*) Lagrange' equation (*c*) Bernoulli's equation (*d*) Clairaut's equation.

[Agra 2005, 06, 08]

2. The Lagrange's auxiliary equations for the partial differential equation $Pp + Qq = R$ are (*a*) $(dx)/P = (dy)/Q = (dz)/R$ (*b*) $(dx)/P = (dy)/Q$ (*c*) $(dx)/P = (dz)/R$. (*d*) none of these. **[Garhwal 2005]**

3. The general solution of $(y - z)\, p + (z - x)\, q = x - y$ is

(*a*) $\phi(x+y+z, x^2+y^2+z^2) = 0$, (*b*) $\phi(xyz, x+y+z) = 0$

(*c*) $\phi(xyz, x^2+y^2+z^2) = 0$ (*d*) $\phi(x^2-y^2-z^2, x-y-z) = 0$ **[M.S. Univ. T.N. 2007]**

[**Hint :** Refer Ex. 15, Art 2.10]

4. Subsidiary equations for equation $(y^2z/x) + zxy = y^2$ are

(*a*) $(dx)/y^2z = (dy)/(zx) = (dz)/y^2$ (*b*) $(dx)/x^2 = (dy)/y^2 = (dz)/zx$

(*c*) $(dx)/x^2 = (dy)/y^2 = (dz)/zx$ (*d*) $(dx)/(1/x^2) = (dy)/(1/y^2) = (dz)/(1/zx)$

[Kanpur 2004]

5. The general solution of the linear partial differential equation $Pp + Qq = R$ is

(*a*) $\phi(u,v) = 1$ (*b*) $\phi(u,v) = -1$ (*c*) $\phi(u,v) = 0$ (d) None of these **[Agra 2007]**

Answers. **1.** (*b*) **2.** (*a*) **3.** (*a*) **4.** (*d*) **5.** (c)

MISCELLANEOUS EXAMPLES ON CHAPTER 2

Ex. 1. *Tranform the equation $yz_x - xz_y = 0$ into one in polar coordinates and thereby show that the solution of the given equation represents surfaces of revolution.* **(I.A.S. 2007)**

Sol. Let $x = r\cos\theta$ and $y = r\sin\theta$ $\Rightarrow$ $r^2 = x^2 + y^2$ and $\theta = \tan^{-1}(y/x)$ ... (1)

$\Rightarrow 2r\ (\partial r/\partial x) = 2x,\ 2r\ (\partial r/\partial y) = 2y$ $\Rightarrow$ $\partial r/\partial x = \cos\theta,\ \partial r/\partial y = \sin\theta$ ... (2)

$$\frac{\partial\theta}{\partial x} = \frac{1}{1+y^2/x^2}\left(-\frac{y}{x^2}\right) = -\frac{r\sin\theta}{r^2} = -\frac{\sin\theta}{r}, \quad \frac{\partial\theta}{\partial y} = \frac{1}{1+y^2/x^3}\left(\frac{1}{x}\right) = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r} \quad \text{... (3)}$$

Now, $$z_x = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial z}{\partial\theta}\frac{\partial\theta}{\partial x} = \cos\theta\frac{\partial z}{\partial r} - \frac{\sin\theta}{r}\frac{\partial z}{\partial\theta}, \text{ using (2) and (3)} \quad \text{...(4)}$$

and $$z_y = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial y} = \sin\theta \frac{\partial z}{\partial r} + \frac{\cos\theta}{r}\frac{\partial z}{\partial \theta} \quad \text{using (2) and (3)} \quad ...(5)$$

Using (1), (4) and (5), the given equation $yz_x - xz_y = 0$ reduce to

$$r\sin\theta\left(\cos\theta\frac{\partial z}{\partial r} - \frac{\sin\theta}{r}\frac{\partial z}{\partial \theta}\right) - r\cos\theta\left(\sin\theta\frac{\partial z}{\partial r} + \frac{\cos\theta}{r}\frac{\partial z}{\partial \theta}\right) = 0 \quad \text{or} \quad \frac{\partial z}{\partial \theta} = 0 \quad ...(6)$$

Integrating (6) w.r.t. 'θ', $z = f(r)$, where f is an arbitrary function .... (7)

Clearly (7) represents surfaces of revolution, as required.

Ex.2. Solve $(y + z)\, p - (x + z)\, q = x - y$ **(Agra 2010)**

Hint. Do like Ex. 15, page 2.14. **Ans.** $\Phi(x + y + z,\ x^2 + y^2 - z^2) = 0$

Ex. 3. The integral surface satisfying equation $y(\partial z / \partial x) - x(\partial z / \partial y) = x^2 + y^2$ and passing through the curve $x = 1 - t$, $y = 1 + t$, $z = 1 + t^2$ is

(*a*) $z = xy + (x^2 - y^2)/2$ (*b*) $z = xy + (x^2 - y^2)^2/8$

(*c*) $z = xy + (x^2 - y^2)^2/4$ (*d*) $z = xy + (x^2 - y^2)^2/16$ **(GATE 2009]**

Ex. 4. Find the partial differential equation whose surfaces are orthogonal to the surface $z(x + y) = 3z + 1$ **[Pune 2010]** **Ans.** $z(p + q) = x + y - 3$

Ex. 5. if $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ are integral curves of $(dx)/P = (dy)/Q = (dz)/R$, then show that $F(u, v) = 0$ is general solution of $Pp + Qq = R,$ where F is an arbitrary function.

[Pune 2010]

3

Non–linear partial differential equations of order one

3.1. Explanation of terms : complete integral (or complete solution), particular integral, singular integral (or singular solution), and general integral (or general solution) as applied to solutions of first order partial differential equations

[I.A.S. 1995; Meerut 1997; Delhi Maths Hons. 1995]

A *solution* or *integral* of a differential equation is a relation between the variables, by means of which and the derivatives obtained there from the equation is satisfied. Let us now discuss various classes of integrals of a partial differential equation of order one.

Complete Integral (C. I.). or complete solution (C.S.) **[Sagar 1995]**

Let us consider a relation $\phi(x, y, z, a, b) = 0$ ...(1)

in which x, y, z are variables such that z is dependent on x and y. Differentiating (1) partially w.r.t x and y respectively, we obtain

$$\frac{\partial \phi}{\partial x}+\frac{\partial \phi}{\partial z} p = 0 \qquad \text{and} \qquad \frac{\partial \phi}{\partial y}+\frac{\partial \phi}{\partial z} q = 0, \qquad ...(2)$$

Since there are two arbitrary constants (namely a and b) connected by the above three equations, these can be eliminated and there will appear a relation of the form

$$f(x, y, z, p, q) = 0, \qquad ...(3)$$

which is a partial differential equation of order one.

Suppose, now, that (1) has been derived from (3), by using some method; then the integral (1), which has as many arbitrary constants as there are independent variables, is called the *complete integral* of (3).

Particular Integral : A particular integral of (3) is obtained by giving particular values to a and b in (1) which is the complete integral of (3).

Singular Integral (S.I.) or singular solution (S.S.) **[Delhi 2009; Sagar 1995]**

We know that the locus of all the points whose co–ordinates along with the values of p and q satisfy (3), represent the doubly infinite system of surfaces given by (1). The system is doubly infinite, since there are two constants a and b and each of these can take an infinite number of values. Since the envelope of all the surfaces given by (1) is touched at each of its points by some one of these surfaces, the coordinates of any point on the envelope along with the values of p and q belonging to the envelope at that point must also satisfy (3). Hence we conclude that the equation of the envelope is a solution of (3). The envelope of the surfaces given by (2) is obtained by eliminating a and b between the equations

$$\phi(x, y, z, a, b) = 0, \qquad \partial \phi / \partial a = 0 \qquad \text{and} \qquad \partial \phi / \partial b = 0. \quad ...(4)$$

The relation between x, y, and z so obtained is called the *singular integral*. In general, it is distinct from the complete integral. However, in exceptional cases it may be contained in the complete integral, that is, singular integral may be obtained by giving particular values to the constants in the complete integral. Since other relations may appear in the process of getting the singular integral, it is necessary to test that the equation of singular integral satisfies the given differential equation.

General Integral (G.I.) or General Solution (G.S.).

Assume that in (1), one of the constants is a function of the other, say $b = F(a)$, then (1) becomes

$$\phi(x, y, z, a, F(a)) = 0. \qquad ...(5)$$

Now (5) represents one of the families of surfaces given by the system (1). As before, the equation of the envelope of the family of surfaces given by (5) must also satisfy (3). Again the equation so obtained will be distinct from that of the envelope of the surfaces, and it is not a particular integral. It is known as the *general integral* and is obtained by eliminating a between

$$\phi(x, y, z, a\ F(a)) = 0 \qquad \text{and} \qquad \partial\phi/\partial a = 0. \qquad ...(6)$$

Since other relations may appear in the process of getting the singular integral, it is necessary to test that the equation of general integral satisfies the given differential equation.

Important Note. While solving a non–linear equation, we must not only obtain the complete integral but should also find the singular and general integrals. In absence of details of singular and general integrals, merely the complete solution is considered to be incomplete solution of the given partial differential equation. However, for reason of space, we have found complete integral only in some problems. The students are advised to find singular and general integrals also for such problems. Note that there is always a simple routine method for the same.

Also, if you are asked to find complete integral of a given equation, then you need not give singular and general integrals. Again, if examiner wants singular integral/general integral, then you must find them.

3.2. Geometrical interpretation of three types of integrals of f(x, y, z, p, q) = 0.

(*i*) **Complete integral.**

A complete integral, being a relation between x, y and z represents equation of a surface. Since it involves two arbitrary parameters, it belongs to a double infinite system of surfaces or to a single infinite system of family of surfaces.

(*ii*) **General integral.**

Let a complete solution of $f(x, y, z, p, q) = 0$ be

$$\phi(x, y, z, a, b) = 0. \qquad ...(1)$$

A general integral is obtained by eliminating 'a' between (1) and the equations

$$b = \psi(a) \qquad ...(2)$$

$$(\partial\phi/\partial a) + (\partial\phi/\partial b)\ \psi'(a) = 0. \qquad ...(3)$$

where ψ is an arbitrary function.

The operation of elimination is equivalent to selecting from the system of families of surfaces a representative family and finding the envelope. Equations (1), (2) and (3) together represent a curve drawn on the surface of the family whose parameter is 'a' whereas the equation obtained by eliminating 'a' between them is the envelope of the family. It follows that the envelope touches the surface represented by (1) and (2) along the curve represented by (1), (2) and (3). This curve is known as *characteristic of the envelope* and the general integral thus represents the envelope of a family of surfaces considered as composed of its characteristics.

(*iii*) **Singular Integral.**

The singular integral is obtained by eliminating 'a' and 'b' between equation (1)

$$\partial\phi/\partial a = 0 \qquad ...(4)$$

and

$$\partial\phi/\partial b = 0. \qquad ...(5)$$

The operation of elimination is equivalent to finding the envelope of all the surfaces included in the complete integral. (1), (4) and (5) give the point of contact of the particular surfaces represented by (1) with the general envelope. It follows that the singular integral represents the general envelope of all surfaces included in the complete integral.

3.3. Method of getting singular integral directly from the partial differential equation of first order.

Let the given partial differential equation be $f(x, y, z, p, q) = 0,$ ...(1)

whose complete integral is of the form $\phi(x, y, z, a, b) = 0,$ ...(2)

where 'a' and 'b' are arbitrary constants.

The singular integral of (1) is obtained by eliminating 'a' and 'b' between equation (2)

$\partial\phi/\partial a = 0$ ...(3) and $\partial\phi/\partial b = 0.$ ...(4)

The values of z, p, q derived from (2) when substituted in (1) will reduce it into an identity and the substitution of the values of p and q (but not of z) will in general render (1) equivalent to the integral equation. By using this substitution p and q are replaced by functions of x, y, z, a and b in (1). It follows that the singular integral is given by (1) and the equations obtained on differentiating (1) partially w.r.t. 'a' and 'b', namely the equations

$$\frac{\partial f}{\partial p}\frac{\partial p}{\partial a}+\frac{\partial f}{\partial q}\frac{\partial q}{\partial a} = 0 \qquad ...(5) \qquad\qquad \frac{\partial f}{\partial p}\frac{\partial p}{\partial b}+\frac{\partial f}{\partial q}\frac{\partial q}{\partial b} = 0. \qquad ...(6)$$

If $\dfrac{\partial f}{\partial p} \neq 0$ and $\dfrac{\partial f}{\partial q} \neq 0$, (5) and (6) hold if $\quad \dfrac{\partial p}{\partial a}\dfrac{\partial q}{\partial b}-\dfrac{\partial p}{\partial b}\dfrac{\partial q}{\partial a} = 0,$

showing that there exists a functional relation between p and q which does not contain a and b explicitly. Let this functional relation be

$$\psi(p, q) = 0. \qquad ...(7)$$

If both the constants a and b occur in p and q (which does nolt always happen), then (7) shows that one of them is a function of the other and the equations using them give general integral which is not now required.

Equations (5) and (6) are also true if

$\partial f/\partial p = 0$ ...(8) and $\partial f/\partial q = 0.$ ... (9)

Elimination of p and q from (1), (7) and (8) will yield a relation between x, y, z free from 'a' and 'b'. If this relation satisfies the given differential equation (1), it must be the singular integral.

3.4. COMPATIBLE SYSTEM OF FIRST–ORDER EQUATIONS

[Delhi Maths (H) 2007; Pune 2010]

Consider first order partial differential equations

$$f(x, y, z, p, q) = 0 \qquad ...(1)$$

and $$g(x, y, z, p, q) = 0. \qquad ...(2)$$

Equations (1) and (2) are known as compatible when every solution of one is also a solution of the other.

To find condition for (1) and (2) to be compatible. [Delhi 2008; Pune 2011]

Let $\quad J = \text{Jacobian of } f \text{ and } g \equiv \partial(f, g)/\partial(p, q) \neq 0.$ ...(3)

Then (1) and (2) can be solved to obtain the explicit expressions for p and q given by

$p = \phi(x, y, z)$ and $q = \psi(x, y, z).$ ...(4)

The condition that the pair of equations (1) and (2) should be compatible reduces then to the condition that the system of equations (4) should be completely integrable, *i.e.*, that the equation

$dz = pdx + qdy$ or $\phi\, dx + \psi dy - dz = 0,$ using (4) ...(5)

should be integrable. (5) is integrable if*

$$\phi\left(\frac{\partial\psi}{\partial z}-0\right)+\psi\left(0-\frac{\partial\phi}{\partial z}\right)+(-1)\left(\frac{\partial\phi}{\partial y}-\frac{\partial\psi}{\partial x}\right) = 0$$

which is equivalent to
$$\frac{\partial\psi}{\partial x}+\phi\frac{\partial\psi}{\partial z} = \frac{\partial\phi}{\partial y}+\psi\frac{\partial\phi}{\partial z}. \qquad ...(6)$$

Substituting from equations (4) in (1) and differentiating *w.r.t.* 'x' and 'z' respectively, we get

$$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial p}\frac{\partial\phi}{\partial x}+\frac{\partial f}{\partial q}\frac{\partial\psi}{\partial x} = 0 \qquad ...(7)$$

and
$$\frac{\partial f}{\partial z}+\frac{\partial f}{\partial p}\frac{\partial\phi}{\partial z}+\frac{\partial f}{\partial q}\frac{\partial\psi}{\partial z} = 0. \qquad ...(8)$$

From (7) and (8),
$$\frac{\partial f}{\partial x}+\phi\frac{\partial f}{\partial z}+\frac{\partial f}{\partial p}\left(\frac{\partial\phi}{\partial x}+\phi\frac{\partial\phi}{\partial z}\right)+\frac{\partial f}{\partial q}\left(\frac{\partial\psi}{\partial x}+\phi\frac{\partial\psi}{\partial z}\right) = 0. \qquad ...(9)$$

Similarly (2) yields
$$\frac{\partial g}{\partial x}+\phi\frac{\partial g}{\partial z}+\frac{\partial g}{\partial p}\left(\frac{\partial\phi}{\partial x}+\phi\frac{\partial\phi}{\partial z}\right)+\frac{\partial g}{\partial q}\left(\frac{\partial\psi}{\partial x}+\phi\frac{\partial\psi}{\partial z}\right) = 0. \qquad ...(10)$$

Solving (9) and (10),
$$\frac{\partial\psi}{\partial x}+\phi\frac{\partial\psi}{\partial z} = \frac{1}{J}\left\{\frac{\partial(f,g)}{\partial(x,p)}+\phi\frac{\partial(f,g)}{\partial(z,p)}\right\}. \qquad ...(11)$$

Again, substituting from equations (4) in (1) and differentiating w.r.t. 'y' and 'z' and proceeding as before, we obtain
$$\frac{\partial\phi}{\partial y}+\psi\frac{\partial\phi}{\partial z} = -\frac{1}{J}\left\{\frac{\partial(f,g)}{\partial(y,q)}+\psi\frac{\partial(f,g)}{\partial(z,q)}\right\} \qquad ...(12)$$

Substituting from equations (11) and (12) in (1) and replacing ϕ, ψ by p, q respectively, we obtain

$$\frac{1}{J}\left\{\frac{\partial(f,g)}{\partial(x,p)}+p\frac{\partial(f,g)}{\partial(z,p)}\right\} = -\frac{1}{J}\left\{\frac{\partial(f,g)}{\partial(y,q)}+q\frac{\partial(f,g)}{\partial(z,q)}\right\} \quad \text{or} \quad [f, g] = 0, \qquad ...(13)$$

where
$$[f, g] \equiv \frac{\partial(f,g)}{\partial(x,p)}+p\frac{\partial(f,g)}{\partial(z,p)}+\frac{\partial(f,g)}{\partial(y,q)}+q\frac{\partial(f,g)}{\partial(z,q)} \qquad ...(14)$$

3.5. A PARTICULAR CASE OF ART. 3.4.

To show that first order partial differential equations $p = P(x, y)$ and $q = Q(x, y)$ are compatible if and only if $\partial P/\partial y = \partial Q/\partial x$. **[Delhi Maths (H) 2009; Pune 2010]**

Proof. Given $\quad \partial z/\partial x = p = P(x, y) \quad$ and $\quad \partial z/\partial z = q = Q(x, y)$ ... (1)

Since $\quad dz = (\partial z/\partial x)dx + (\partial z/\partial y)dy = pdx + qdy,$ ... (2)

it follows that the given partial differential equations (1) are compatible if and only if the single differential equation

$$dz = Pdx + Qdy \qquad ... (3)$$

is integrable.

Since P and Q are functions of two variables x and y, hence $Pdx + Qdy$ is an exact differential if and only if $\partial P/\partial y = \partial Q/\partial x$. Therefore (3) is integrable if and only if $\partial P/\partial y = \partial Q/\partial x$

Remark 1. If $\partial P/\partial y = \partial Q/\partial x$, then the system of two given partial differential equations (1) is compatible and hence these will possess a common solution.

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*$Pdx + Qdy + Rdz = 0$ is integrable if $P\left(\frac{\partial Q}{\partial z}-\frac{\partial R}{\partial y}\right)+Q\left(\frac{\partial R}{\partial x}-\frac{\partial P}{\partial z}\right)+R\left(\frac{\partial P}{\partial y}-\frac{\partial Q}{\partial x}\right) = 0.$

Remark 2. If $\partial P/\partial y \neq \partial Q/\partial x,$ then the system of two given partial differential equations (1) is not compatible and hence these equations possess no solution.

3.6. SOLVED EXAMPLES BASED ON ART. 3.4 AND ART. 3.5.

Ex. 1. (*a*) *Show that the differential equations* $\partial z/\partial x = 5x - 7y$ *and* $\partial z/\partial y = 6x + 8y$ *are not compatible.*

(*b*) $\partial z/\partial x = 5x - 7y,\ \partial z/\partial y = 6x + 8y$ *possess* (*i*) *common solution* (*ii*) *No common solution* (*iii*) *No solution* (*iv*) *None of these. Point out correct choice.* **[Agra 2005, 06]**

(*c*)*Show that the differential equations* $p = x^2 - ay,\ q = y^2 - ax$ *are compactible and find their common solution.*

(*d*) *Show that the differential equations* $\partial z/\partial x = (x+y)^2,\ \ \partial z/\partial y = x^2 + 2xy - y^2$ *are compatible and solve them*

(*e*)*Show that* $p = x - y/(x^2 + y^2),\ q = y + x/(x^2 + y^2)$ *are compatible and find their solution.*

(*f*) Show that $p = 1 + e^{x/y},\ q = e^{x/y}(1 - x/y)$ *are compatible and find their solution.*

Sol. (a) Given $\quad dz/dx = p = 5x - 7y \quad$ and $\quad dz/dy = q = 6x + 8y \quad$ ...(1)

Comparing (1) with $p = P(x, y)$ and $q = Q(x, y)$ ... (2)

here $\quad p = 5x - 7y \quad$ and $\quad Q = 6x + 8y \quad$ ... (3)

We know that $p = P(x, y)$ and $q = Q(x, y)$ are compatible if $\partial P/\partial y = \partial Q/\partial x.$ Hence the system (1) is compatible if $\partial P/\partial y = \partial Q/\partial x.$

From (3), $\partial P/\partial y = -7$ and $\partial Q/\partial x = 6$ and so $\partial P/\partial y \neq \partial Q/\partial x$

Therefore, the given system (1) is not compatible.

(*b*) **Ans.** (*iii*) As in part (a), the given system is not compatible. Hence the given equations have no solution (refer Art. 3.5).

(*c*) We known that the system of equations $\quad p = P(x, y), \quad q = Q(x, y)$ ... (1)

is compatible if and only if $\quad \partial P/\partial y = \partial Q/\partial x.$

Comparing $\quad p = x^2 - ay, \quad$ and $\quad q = y^2 - ax \quad$ ... (2)

with (1), here $\quad P = x^2 - ay, \quad$ and $\quad Q = y^2 - ax \quad$ ... (3)

From (3), $\partial P/\partial y = -a = \partial Q/dx$ and so equations (2) are compatible

To find the common solution of (2).

Substituting the values of p and q given by (2) in $dz = pdx + qdy$, we get

$$dz = (x^2 - ay)\,dx + (y^2 - ax)\,dy = x^2dx + y^2dy - a\,d(xy)$$

Integrating, $\quad z = (x^3 + y^3)/3 - axy + c$, c being an arbitrary contant ... (4)

(4) is the required common solution of the given equation (2).

(*d*) We know that the system of equations $\quad p = P(x, y), \quad q = Q(x, y)$ ...(1)

is compatible if and only if $\quad \partial P/\partial y = \partial Q/\partial x.$

Comparing $\quad \partial z/\partial x = p = (x+y)^2, \quad$ and $\quad \partial z/\partial y = q = x^2 + 2xy - y^2$ ... (2)

with (1), here $\quad P = (x + y)^2 = x^2 + 2xy + y^2 \quad$ and $\quad Q = x^2 + 2xy - y^2 \quad$ ... (3)

From (3), $\partial P/\partial y = 2x + 2y = \partial Q/\partial y$ and hence equations (2) are compatible.

The find the common solution of (2).

Substituting the values of p and q given by (2) in $dz = pdx + qdy$, we get

$$dz = (x^2 + 2xy + y^2)dx + (x^2 + 2xy - y^2)\,dy \quad ... (4)$$

Integrating (4) and noting that R.H.S. of (4) must an exact differential, we have, by method of solving an exact equation

$$z = \underset{\text{(Treating y as a constant)}}{\int (x^2 + 2xy + y^2)dx} + \underset{\text{(Integrating terms free from } x)}{\int (x^2 + 2xy - y^2)} + c$$

or $\quad z = x^3/3 + x^2 y + y^2 x - y^3/3 + c,$ c being an arbitrary constant

(*e*) We know that the system of equation $\quad p = P(x, y), \quad q = Q(x, y)$ ... (1)

is compatible if and only if $\quad \partial P/\partial y = \partial Q/\partial x.$

Comparing $\quad p = x - y/(x^2 + y^2),\quad$ and $\quad q = y + x/(x^2 + y^2)$ ... (2)
with (1), here $\quad P = x - y/(x^2 + y^2)\quad$ and $\quad Q = y + x/(x^2 + y^2)$ ... (3)

From (3),
$$\frac{\partial P}{\partial y} = 0 - \frac{1\cdot(x^2+y^2) - 2y\cdot y}{(x^2-y^2)^2} = \frac{y^2 - x^2}{(x^2+y^2)^2} \qquad \text{... (4)}$$

and
$$\frac{\partial Q}{\partial x} = 0 + \frac{1\cdot(x^2+y^2) - 2x\cdot x}{(x^2+y^2)^2} = \frac{y^2 - x^2}{(x^2-y^2)^2} \qquad \text{... (5)}$$

(4) and (5) $\Rightarrow \quad \partial P/\partial y = \partial Q/\partial x \quad \Rightarrow \quad$ The system (2) is compatible.

To find the solution of the system (2).

Substituting the values of p and q given by (2) in $dz = pdx + qdy$, we get

$$dz = \{x - y/(x^2 + y^2)\}dx + \{y + x/(x^2 + y^2)\}dy \qquad \text{... (6)}$$

Integrating (6) and noting that R.H.S. of (6) must be an exact differential, we obtain

$$z = \underset{\text{(Treating } y \text{ as a constant)}}{\int \left\{x - y/(x^2+y^2)\right\} dx} + \underset{\text{(Integrating terms free from } x)}{\int \left\{y + x/(x^2+y^2)\right\} dy} + c$$

or $\quad z = x^2/2 - y \times (1/y) \times \tan^{-1}(x/y) + y^2/2 + c = (x^2 + y^2)/2 - \tan^{-1}(x/y) + c,$
which is the required solution, c being an arbitrary constant.

(*f*) We know that the system of equations $\quad p = P(x, y), \quad$ and $\quad q = Q(x, y)$ ... (1)

is compatible if and only if $\quad \partial P/\partial y = \partial Q/\partial x.$

Comparing $\quad p = 1 + e^{x/y} \quad$ and $\quad q = e^{x/y}(1 - x/y)$ ... (2)
with (1), here $\quad P = 1 + e^{x/y} \quad$ and $\quad Q = e^{x/y}(1 - x/y)$ ... (3)

$$(3) \;\Rightarrow\; \partial P/\partial y = 0 + e^{x/y}(-x/y^2) = -(x/y^2)\, e^{x/y} \qquad \text{... (4)}$$

and
$$\partial Q/\partial x = e^{x/y} \times (1/y) \times (1 - x/y) + e^{x/y} \times (-1/y) = -(x/y^2)e^{x/y} \qquad \text{... (5)}$$

(4) and (5) $\Rightarrow \quad \partial P/\partial y = \partial Q/\partial x \quad \Rightarrow \quad$ The system (2) is compatible.

To find the solution of (2). Substituting the values of p and q given by (2) in $dz = pdx + qdy$, we get

$$dz = (1 + e^{x/y})dx + e^{x/y}(1 - x/y)dy \qquad \text{... (6)}$$

Integrating (6) and noting that R.H.S. of (6) must be an exact differential, we obtain

$$z = \underset{\text{(Treating } y \text{ as constant)}}{\int (1 + e^{x/y})dx} + \underset{\text{(Integrating terms free from } x)}{\int e^{x/y}(1 - x/y)\, dy} + c$$

or $\quad z = x + y\, e^{x/y} + c,$ c being an arbitrary constant.

Ex. 2. *Show that the equations $xp = yq$ and $z(xp + yq) = 2xy$ are compatible and solve them.*

[Delhi Maths (Hons) 2005, 07, 11]

Sol. Let $$f(x, y, z, p, q) = xp - yq = 0 \qquad ...(1)$$

and $$g(x, y, z, p, q) = z(xp + yq) - 2xy = 0 \qquad ...(2)$$

$$\therefore \quad \frac{\partial(f,g)}{\partial(x,p)} = \begin{vmatrix} \partial f/\partial x & \partial f/\partial p \\ \partial g/\partial x & \partial g/\partial p \end{vmatrix} = \begin{vmatrix} p & x \\ zp-2y & xz \end{vmatrix} = 2xy,$$

$$\frac{\partial(f,g)}{\partial(z,p)} = \begin{vmatrix} \partial f/\partial z & \partial f/\partial p \\ \partial g/\partial z & \partial g/\partial p \end{vmatrix} = \begin{vmatrix} 0 & x \\ xp+yq & xz \end{vmatrix} = -x^2p - xyq,$$

$$\frac{\partial(f,g)}{\partial(y,q)} = \begin{vmatrix} \partial f/\partial y & \partial f/\partial q \\ \partial g/\partial y & \partial g/\partial q \end{vmatrix} = \begin{vmatrix} -q & -y \\ zq-2x & zy \end{vmatrix} = -2xy$$

and $$\frac{\partial(f,g)}{\partial(z,q)} = \begin{vmatrix} \partial f/\partial z & \partial f/\partial q \\ \partial g/\partial z & \partial g/\partial q \end{vmatrix} = \begin{vmatrix} 0 & -y \\ xp+yq & zy \end{vmatrix} = xyp + y^2q.$$

$$\therefore \quad [f, g] = \frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)} = 2xy - x^2p^2 - xyqp - 2xy + xypq + y^2q^2$$

$$= -xp(xp + yq) + yq(xp + yq) = -(xp - yq)(xp + yq) = 0, \text{ using (1)}$$

Hence (1) and (2) are compatible.

Solving (1) and (2) for p and q, $\quad p = y/z \quad$ and $\quad q = x/z. \quad ...(3)$

Using (3) in $dz = pdx + qdy$, we have $\quad dz = (y/z)dx + (x/z)dy \quad$ or $\quad z\,dz = d(xy)$.

Integrating, $\quad z^2/2 = xy + c/2 \quad$ or $\quad z^2 = 2xy + c$, where c is an arbitrary constant.

Ex. 3. *Show that the equations $xp - yq = x$ and $x^2p + q = xz$ are compatible and find their solution.* **[Delhi B.Sc. II (Prog) 2009; Delhi Maths Hons. 2007]**

Sol. Let $$f(x, y, z, p, q) = xp - yq - x = 0. \qquad ...(1)$$

and $$g(x, y, z, p, q) = x^2p + q - xz = 0. \qquad ...(2)$$

$$\therefore \quad \frac{\partial(f,g)}{\partial(x,p)} = \begin{vmatrix} \partial f/\partial x & \partial f/\partial p \\ \partial g/\partial x & \partial g/\partial p \end{vmatrix} = \begin{vmatrix} p-1 & x \\ 2xp-z & x^2 \end{vmatrix} = (p-1)x^2 - x(2xp - z).$$

Similarly, $\quad \dfrac{\partial(f,g)}{\partial(z,p)} = x^2, \qquad \dfrac{\partial(f,g)}{\partial(y,q)} = -q, \qquad \dfrac{\partial(f,g)}{\partial(z,q)} = -xy.$

$$\therefore \quad [f, g] = \frac{\partial(f,g)}{\partial(x,p)} + p\frac{\partial(f,g)}{\partial(z,p)} + \frac{\partial(f,g)}{\partial(y,q)} + q\frac{\partial(f,g)}{\partial(z,q)} = (p-1)x^2 - x(2xp - z) - px^2 - q - xyq$$

$$= -x^2 + zx - q - xyq = -x^2 + x^2p - qxy, \text{ by (2)}$$

$$= x(-x + xp - yq) = 0, \text{ by (1)}$$

Hence (1) and (2) are compatible.

Solving (1) and (2) for p and q, $\quad p = (1 + yz)/(1 + xy) \quad$ and $\quad q = x(z - x)/(1 + xy). \quad ...(3)$

Using (3) in $dz = pdx + qdy$, $\quad dz = [(1 + yz)/(1 + xy)]\,dx + [x(z - x)/(1 + xy)]dy$

or $\quad (1 + xy)dz = (1 + yz)dx + x(z - x)dy \quad$ or $\quad (1 + xy)dz - z(ydx + xdy) = dx - x^2dy$

or $$\frac{(1+xy)\,dz - z\,d(xy)}{(1+xy)^2} = \frac{dx - x^2dy}{(1+xy)^2} = \frac{(dx/x^2) - dy}{(y+1/x)^2} \quad \text{or} \quad d\left(\frac{z}{1+xy}\right) = \frac{-d\,(y+1/x)}{(y+1/x)^2}.$$

Integrating it, $$\frac{z}{1+xy} = \frac{1}{(y+1/x)} + c \quad \text{or} \quad \frac{z}{1+xy} = \frac{x}{1+xy} + c$$

or $\quad z - x = c(1 + xy)$, $\;c$ being an arbitrary constant.

Ex. 4. *Show that the equation* $z = px + qy$ *is compatible with any equation* $f(x, y, z, p, q) = 0$ *which is homogeneous in* x, y, z. **[Delhi Maths, Hons. 2001, 06, 10]**

Sol. Given that differential equation $$f(x, y, z, p, q) = 0 \qquad ...(1)$$
is homogeneous in x, y, z. Then, clearly $f(x, y, z, p, q)$ will be a homogeneous function in variables x, y, z ; say of degree n. Then, by Euler's theorem on homogeneous function, we have

$$x(\partial f/\partial x)+y(\partial f/\partial y)+z(\partial f/\partial z)=nf \quad \text{so that} \quad x(\partial f/\partial x)+z(\partial f/\partial z)=0, \text{ by (1)} \qquad ... (2)$$

We take $$g(x, y, z, p, q) = px + qy - z = 0 \qquad ... (3)$$

Then, using (3), we have

$$\frac{\partial(f,g)}{\partial(x,p)}=\begin{vmatrix}\partial f/\partial x & \partial f/\partial p\\ \partial g/\partial x & \partial g/\partial p\end{vmatrix}=\begin{vmatrix}\partial f/\partial x & \partial f/\partial p\\ p & x\end{vmatrix}=x\frac{\partial f}{\partial x}-p\frac{\partial f}{\partial p},$$

$$\frac{\partial(f,g)}{\partial(z,p)}=\begin{vmatrix}\partial f/\partial z & \partial f/\partial p\\ \partial g/\partial z & \partial g/\partial p\end{vmatrix}=\begin{vmatrix}\partial f/\partial z & \partial f/\partial p\\ -1 & x\end{vmatrix}=x\frac{\partial f}{\partial z}+\frac{\partial f}{\partial p},$$

$$\frac{\partial(f,g)}{\partial(y,q)}=\begin{vmatrix}\partial f/\partial y & \partial f/\partial q\\ \partial g/\partial y & \partial g/\partial q\end{vmatrix}=\begin{vmatrix}\partial f/\partial y & \partial f/\partial q\\ q & y\end{vmatrix}=y\frac{\partial f}{\partial y}-q\frac{\partial f}{\partial q}$$

and $$\frac{\partial(f,g)}{\partial(z,q)}=\begin{vmatrix}\partial f/\partial z & \partial f/\partial q\\ \partial g/\partial z & \partial g/\partial q\end{vmatrix}=\begin{vmatrix}\partial f/\partial z & \partial f/\partial q\\ -1 & y\end{vmatrix}=y\frac{\partial f}{\partial z}+\frac{\partial f}{\partial q}$$

$$\therefore \qquad [f,g]=\frac{\partial(f,g)}{\partial(x,p)}+p\frac{\partial(f,g)}{\partial(z,p)}+\frac{\partial(f,g)}{\partial(y,q)}+q\frac{\partial(f,q)}{\partial(z,q)}$$

$$=x\frac{\partial f}{\partial x}-p\frac{\partial f}{\partial p}+p\left(x\frac{\partial f}{\partial z}+\frac{\partial f}{\partial p}\right)+y\frac{\partial f}{\partial y}-q\frac{\partial f}{\partial q}+q\left(y\frac{\partial f}{\partial z}+\frac{\partial f}{\partial q}\right)$$

$$=x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+(px+qy)\frac{\partial f}{\partial y}=x\frac{\partial f}{\partial x}+y\frac{\partial f}{\partial y}+z\frac{\partial f}{\partial z}, \text{ using (3)}$$

$$= 0, \text{ using (2)}$$

Hence, the differential equation $z = px + qy$ is compatible with any differential equation $f(x, y, z, p, q)$ that is homogeneous in x, y, z.

Ex. 5. *If* $u_1 = \partial u/\partial x$, $u_2 = \partial u/\partial y$, $u_3 = \partial u/\partial z$, *show that the equations* $f(x, y, z, u_1, u_2, u_3) = 0$ *and* $g(x, y, z, u_1, u_2, u_3) = 0$ *are compatible if* $\dfrac{\partial(f,g)}{\partial(x,u_1)}+\dfrac{\partial(f,g)}{\partial(y,u_2)}+\dfrac{\partial(f,g)}{\partial(z,u_3)}=0$. **(Delhi Maths (H) 2004)**

Sol. Treating z as constant, given equations are compatible if

$$\frac{\partial(f,g)}{\partial(x,u_1)}+u_1\frac{\partial(f,g)}{\partial(u,u_1)}+\frac{\partial(f,g)}{\partial(y,u_2)}+u_2\frac{\partial(f,g)}{\partial(u,u_2)}=0. \qquad ...(1)$$

Since f and g do not contain u, we have $\partial f/\partial u = 0$ and $\partial g/\partial u = 0$. ...(2)

$$\therefore \qquad \frac{\partial(f,g)}{\partial(u,u_1)}=0 \qquad \text{and} \qquad \frac{\partial(f,g)}{\partial(u,u_2)}=0. \qquad ...(3)$$

$\therefore$ (1) reduces to $$\frac{\partial(f,g)}{\partial(x,u_1)}+\frac{\partial(f,g)}{\partial(y,u_2)}=0. \qquad ...(4)$$

Similarly treating x and y constant respectively, given equtions are compatible if

$$\frac{\partial(f,g)}{\partial(y,u_2)}+\frac{\partial(f,g)}{\partial(z,u_3)}=0 \quad ...(5) \qquad \frac{\partial(f,g)}{\partial(x,u_1)}+\frac{\partial(f,g)}{\partial(z,u_3)}=0. \quad ...(6)$$

We know that the given equations are compatible when they remain compatible even when any variable is taken as constant, *i.e.*, (4), (5) and (6) hold simultaneously. Hence adding (4), (5) and (6), the required condition for given equations to be compatible is

$$\frac{\partial(f,g)}{\partial(x,u_1)}+\frac{\partial(f,g)}{\partial(y,u_2)}+\frac{\partial(f,g)}{\partial(z,u_3)}=0.$$

Ex. 6. *Show that the equations* $f(x,y,p,q)=0$, $g(x,y,p,q)=0$ *are compatible if* $\partial(f,g)/\partial(x,p)+\partial(f,g)/\partial(y,q)=0$

Verify that the equations $p=P(x,y)$, $q=Q(x,y)$ *are compatible if* $\partial P/\partial y=\partial Q/\partial x$.

Sol. We know that $f(x,y,z,p,q)=0$ and $g(x,y,z,p,q)=0$ ... (1)

are compatible if
$$\frac{\partial(f,g)}{\partial(x,p)}+p\frac{\partial(f,g)}{\partial(z,p)}+\frac{\partial(f,g)}{\partial(y,q)}+q\frac{\partial(f,g)}{\partial(z,q)}=0 \quad ... (2)$$

First part: Comparing the given equations $f(x,y,p,q)=0$ and $g(x,y,p,q)=0$ with (1), we find that z is absent in given equations and so

$\partial f/\partial z=0$ and $\partial g/\partial z=0$ ... (3)

Now,
$$\frac{\partial(f,g)}{\partial(z,p)}=\begin{vmatrix}\partial f/\partial z & \partial f/\partial p\\ \partial g/\partial z & \partial g/\partial p\end{vmatrix}=\begin{vmatrix}0 & \partial f/\partial p\\ 0 & \partial g/\partial p\end{vmatrix}=0$$

and
$$\frac{\partial(f,g)}{\partial(z,q)}=\begin{vmatrix}\partial f/\partial z & \partial f/\partial q\\ \partial g/\partial z & \partial g/\partial q\end{vmatrix}=\begin{vmatrix}0 & \partial f/\partial q\\ 0 & \partial g/\partial q\end{vmatrix}=0$$

Substituting these values in (2), the required condition is

$$\partial(f,g)/\partial(x,p)+\partial(f,g)/\partial(y,q)=0$$

Second Part. Let $f=P(x,y)-p$ and $g=Q(x,y)-q$ ... (4)

Comparing (4) with (1), we find that z and q are absent in f and z and p are absent in g and so $\partial f/\partial z=0$, $\partial f/\partial q=0$, $\partial g/\partial z=0$ and $\partial g/\partial p=0$ ... (5)

$$\therefore \quad \frac{\partial(f,g)}{\partial(x,p)}=\begin{vmatrix}\partial f/\partial x & \partial f/\partial p\\ \partial g/\partial x & \partial g/\partial p\end{vmatrix}=\begin{vmatrix}\partial P/\partial x & -1\\ \partial Q/\partial x & 0\end{vmatrix}=\frac{\partial Q}{\partial x}$$

$$\frac{\partial(f,g)}{\partial(z,p)}=\begin{vmatrix}\partial f/\partial z & \partial f/\partial p\\ \partial g/\partial z & \partial g/\partial p\end{vmatrix}=\begin{vmatrix}0 & -1\\ 0 & 0\end{vmatrix}=0$$

$$\frac{\partial(f,g)}{\partial(y,q)}=\begin{vmatrix}\partial f/\partial y & \partial f/\partial q\\ \partial g/\partial y & \partial g/\partial q\end{vmatrix}=\begin{vmatrix}\partial P/\partial y & 0\\ \partial Q/\partial y & -1\end{vmatrix}=-\frac{\partial P}{\partial y}$$

$$\frac{\partial(f,g)}{\partial(z,q)}=\begin{vmatrix}\partial f/\partial z & \partial f/\partial q\\ \partial g/\partial z & \partial g/\partial q\end{vmatrix}=\begin{vmatrix}0 & 0\\ 0 & -1\end{vmatrix}=0$$

Substituting these values in (2), the required condition is

$\partial Q/\partial x-\partial P/\partial y=0$ or $\partial P/\partial y=\partial Q/\partial x$.

Ex. 7. *Show that* $p^2+q^2=1$ *and* $(p^2+q^2)x=pz$ *are compatible and solve them.*

Hint. Proceed as in solved Ex. 1 Ans. $z^2=x^2+(y+c)^2$.

Ex. 8. *Solve completey the simultaneous equations:* $z = px + qy$ *and* $2xy(p^2 + q^2) = z(yp + xq)$. **[Delhi Math (H) 2006, 10]**

Sol. Given $$z = px + qy \qquad \text{... (1)}$$

and $$2xy(p^2 + q^2) - z(yp + xq) = 0 \qquad \text{... (2)}$$

Let $f(x, y, z, p, q) = 2xy\ (p^2 + q^2) - z\ (yp + xq)\ =\ z^2\ \{2(x/z)\ (y/z)\ (p^2 + q^2) - (y/z)p - (x/z)q\}$,

showing that $f(x, y, z, p, q)$ is homogeneous in x, y, z.

We know that (refer solved example 4) the equation $z = px + qy$ is compatible with any equation $f(x, y, z, p, q) = 0$ which is homogenous in x, y, z. Hence (1) and (2) are compatible.

From (1), we have $$q = (z - px)/y \qquad \text{... (3)}$$

Using (3), (2) gives $$2x(x^2 + y^2)p^2 - z(3x^2 + y^2)\ p + xz^2 = 0$$

or $$(2xp - z)\ \{(x^2 + y^2)p - xz\} = 0$$

so that $$p = z/2x,\ xz/(x^2 + y^2) \qquad \text{... (4)}$$

Using (4), (3) gives $$q = z/2y,\ yz/(x^2 + y^2) \qquad \text{... (5)}$$

Using the corresponding values $p = z/2x,\ q = z/2y$ in $dz = px + qdy$, we get

$dz = (z/2x)dx + (z/2y)dy$ or $2(1/z)dz = (1/x)dx + (1/y)dy$

Integrating, $2\log z = \log x + \log y + \log C_1$ or $z^2 = C_1 xy$ ...(6)

Similarly, using the corresponding values $p = xz/(x^2 + y^2)$ and $q = yz/(x^2 + y^2)$ in $dz = pdx + qdy$, we get

$$dz = \frac{xzdx}{x^2 + y^2} + \frac{yzdy}{x^2 + y^2} \quad \text{or} \quad \frac{2dz}{z} = \frac{2(xdx + ydy)}{x^2 + y^2}$$

Integrating, $2\log z = \log(x^2 + y^2) + \log C_2$ or $z^2 = C_2(x^2 + y^2)$ ... (7)

(6) and (7) give two two common solutions of (1) and (2)

EXERCISE 3(A)

1. Show that $\partial z/\partial x = 7x + /8y - 1$ and $\partial z/\partial y = 9x + 11y - 2$ are not compatible.

2. Show that the partial differential equations $p = 6x - 4y + 1$ and $q = 4x + 6y + 1$ do not possess any common solution.

3. Show that the following system of partial differential equations are compatible and hence solve them

(*i*) $p = 6x + 3y,\ q = 3x - 4y$ **Ans.** $z = 3x^2 + 3xy - 2y^2 + c$

(*ii*) $p = ax + hy + g,\ q = hx + by + f$ **Ans.** $z = (ax^2 + by^2)/2 + hxy + gx + fy + c$

(*iii*) $\partial z/\partial x = y(2ax + by),\ \partial z/\partial y = x(ax + 2by)$ **Ans.** $z = ax^2y + bxy + c$

(*iv*) $p = x^4 - 2xy^2 + y^4,\ q = 4xy^3 - 2x^2y - \sin y$ **Ans.** $z = x^5/5 - x^2y^2 + xy^4 + \cos y + c$

(*v*) $p = (e^y + 1)\cos x,\ q = e^y \sin x$ **Ans.** $z = (e^y + 1)\sin x + c$

(*vi*) $p = y\ (1 + 1/x) + \cos y,\ q = x + \log x - x\sin y$ **Ans.** $z = y\ (x + \log x) + x\cos y + c$

(*vii*) $p = y^2 e^{xy^2} + 4x^3,\ q = 2xy\, e^{xy^2} - 3y^2$ **Ans.** $z = e^{xy^2} + x^4 - y^3 + c$

(*viii*) $p = \sin x \cos y + e^{3x}$, $q = \cos x \sin y + \tan y$

Ans. $x = (1/3) \times e^{3x} - \cos x \cos y + \log \sec x + c$

3.7. Charpit's method.* (General method of solving partial differential equations of order one but of any degree.

[Agra 2003; Delhi Maths (H) 2000, 05, 06, 08-11; Kanpur 1998; Meeerut 2003, 05; Nagpur 2002, 04, 06, 08; Rohilkhand 2001, 04]

Let the given partial equation differential of first order and non–linear in p and q be

$$f(x, y, z, p, q) = 0. \qquad \text{...(1)}$$

We know that

$$dz = p\, dx + q\, dy. \qquad \text{...(2)}$$

The next step consists in finding another relation $F(x, y, z, p, q) = 0$ ...(3)

such that when the values of p and q obtained by solving (1) and (3), are substituted in (2), it becomes integrable. The integration of (2) will give the complete integral of (1).

In order to obtain (3), differentiate partially (1) and (3) with respect to x and y and get

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} p + \frac{\partial f}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial x} = 0, \qquad \text{... (4)}$$

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} p + \frac{\partial F}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial x} = 0, \qquad \text{... (5)}$$

$$\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z} q + \frac{\partial f}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial y} = 0 \qquad \text{... (6)}$$

and

$$\frac{\partial F}{\partial y} + \frac{\partial F}{\partial z} q + \frac{\partial F}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial F}{\partial q}\frac{\partial q}{\partial y} = 0. \qquad \text{... (7)}$$

Eliminating $\partial p/\partial x$ from (4) and (5), we get

$$\left(\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} p + \frac{\partial f}{\partial q}\frac{\partial q}{\partial x}\right)\frac{\partial F}{\partial p} - \left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z} p + \frac{\partial F}{\partial q}\frac{\partial q}{\partial x}\right)\frac{\partial f}{\partial p} = 0$$

or

$$\left(\frac{\partial f}{\partial x}\frac{\partial F}{\partial p} - \frac{\partial F}{\partial x}\frac{\partial f}{\partial p}\right) + \left(\frac{\partial f}{\partial z}\frac{\partial F}{\partial p} - \frac{\partial F}{\partial z}\frac{\partial f}{\partial p}\right)p + \left(\frac{\partial f}{\partial q}\frac{\partial F}{\partial p} - \frac{\partial F}{\partial q}\frac{\partial f}{\partial p}\right)\frac{\partial q}{\partial x} = 0. \qquad \text{...(8)}$$

Similarly, eliminating $\partial q/\partial y$ from (6) and (7), we get

$$\left(\frac{\partial f}{\partial y}\frac{\partial F}{\partial q} - \frac{\partial F}{\partial y}\frac{\partial f}{\partial q}\right) + \left(\frac{\partial f}{\partial z}\frac{\partial F}{\partial q} - \frac{\partial F}{\partial z}\frac{\partial f}{\partial q}\right)q + \left(\frac{\partial f}{\partial p}\frac{\partial F}{\partial q} - \frac{\partial F}{\partial p}\frac{\partial f}{\partial q}\right)\frac{\partial p}{\partial y} = 0. \qquad \text{...(9)}$$

Since $\partial q/\partial x = \partial^2 z/\partial x\partial y = \partial p/\partial y$, the last term in (8) is the same as that in (9), except for a minus sign and hence they cancel on adding (8) and (9).

Therefore, adding (8) and (9) and rearranging the terms, we obtain

$$\left(\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}\right)\frac{\partial F}{\partial p} + \left(\frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}q\right)\frac{\partial F}{\partial q} + \left(-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}\right)\frac{\partial F}{\partial z} + \left(-\frac{\partial f}{\partial p}\right)\frac{\partial F}{\partial x} + \left(-\frac{\partial f}{\partial q}\right)\frac{\partial F}{\partial y} = 0. \qquad \text{...(10)}$$

This is a linear equation of the first order to obtain the desired function F. As in Art 2.20 of chapter 2, integral of (10) is obtained by solving the auxiliary equations

$$\frac{dp}{(\partial f/\partial x) + p(\partial f/\partial z)} = \frac{dq}{(\partial f/\partial y) + q(\partial f/\partial z)} + \frac{dz}{-p(\partial f/\partial p) - q(\partial f/\partial q)} = \frac{dx}{-\partial f/\partial p} = \frac{dy}{-\partial f/\partial q} = \frac{dF}{0}. \qquad \text{...(11)}$$

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*This is general method for solving equations with two independent variables. Since the solution by this method is generally more complicated, this method is applied to solve equations which cannot be reduced to any of the standard forms which will be discussed later on. Thus, Charpit's method is used in two situations (*i*) When you are asked to solve a problem by Charpit's method (*ii*) when the given equation is not of any four standard forms given in Articles 3.10, 3.12, 3.14 and 3.17.

Since any of the integrals of (11) will satisfy (10), an integral of (11) which involves p or q (or both) will serve along with the given equation to find p and q. In practice, however, we shall select the simplest integral.

Note. In what follows we shall use the following standard notations:

$$\partial f / \partial x = f_x, \qquad \partial f / \partial y = f_y, \qquad \partial f / \partial z = f_z, \qquad \partial f / \partial p = f_p, \qquad \partial f / \partial q = f_q.$$

Therefore, Charpit's auxiliary equations (11) may be re–written as

$$\frac{dp}{f_x + p f_z} = \frac{dq}{f_y + q f_z} = \frac{dz}{-p f_p - q f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} = \frac{dF}{0} \qquad \text{... (11)}'$$

3.8A. WORKING RULE WHILE USING CHARPIT'S METHOD

Step 1. Transfer all terms of the given equation to L.H.S. and denote the entire expression by f.

Step 2. Write down the Charpit's auxiliary equations (11) or (11)′.

Step 3. Using the value of f in step 1 write down the values of $\partial f/\partial x$, $\partial f/\partial y$ ..., *i.e.*, f_x, f_y, ... etc. occuring in step 2 and put these in Charpit's equations (11) or (11)′.

Step 4. After simplifying the step 3, select two proper fractions so that the resulting integral may come out to be the simplest relation involving at least one of p and q.

Step 5. The simplest relation of step 4 is solved along with the given equation to determine p and q. Put these values of p and q in $dz = p\,dx + q\,dy$ which on integration gives the complete integral of the given equation.

The Singular and General integrals may be obtained in the usual manner.

Remark. Sometimes Charpit's equations give rise to $p = a$ and $q = b$, where a and b are constants. In such cases, putting $p = a$ and $q = b$ in the given equation will give the required complete integral.

3.8.B. SOLVED–EXAMPLES BASED ON ART. 3.8A.

Ex. 1. *Find a complete integral of* $z = px + qy + p^2 + q^2$.

[Bilaspur 2000l; Bhopal 1996, I.A.S. 1996; Indore 2000; Jabalpur 2000; K.U. Kurukshetra 2005; Ravishankar 2000; 04; Meerut 2010; Garhwal 2010]

Sol. Let $$f(x, y, z, p, q) \equiv z - px - qy - p^2 - q^2 = 0 \qquad \text{... (1)}$$

Charpit's auxiliary equations are $$\frac{dp}{f_x + p f_z} = \frac{dq}{f_y + q f_z} = \frac{dz}{-p f_p - q f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q} \qquad \text{... (2)}$$

From (1), $f_x = -p$, $f_y = -q$, $f_z = 0$, $f_p = -x - 2p$ and $f_q = -y - 2q$ ... (3)

Using (3), (2) reduces to

$$\frac{dp}{0} = \frac{dq}{0} = \frac{dz}{p(x+2p)+q(y+2q)} = \frac{dx}{x+2p} = \frac{dy}{y+2q} \qquad \text{... (4)}$$

Taking the first fraction of (4), $dp = 0$ so that $p = a$ ... (5)

Taking the second fraction of (4), $dq = 0$ so that $q = b$ ... (6)

Putting $p = a$ and $q = b$ in (1), the required complete integral is

$$z = ax + by + a^2 + b^2, \; a, b \text{ being arbitrary constants.}$$

Ex. 2. *Find a complete integral of* $q = 3p^2$. **[Agra 2006]**

Sol. Here given equation is $$f(x, y, z, p, q) \equiv 3p^2 - q = 0. \qquad \text{...(1)}$$

$\therefore$ Charpit's auxiliary equations are $$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

or $$\frac{dp}{0+p.0}=\frac{dq}{0+q.0}=\frac{dz}{-6p^2+q}=\frac{dx}{-6p}=\frac{dy}{1}, \text{ using (1)} \qquad \text{...(2)}$$

Taking the first fraction of (1), $dp = 0$ so that $p = a$. ...(3)

Substituting this value of p in (1), we get $q = 3a^2$. ...(4)

Putting these values of p and q in $dz = pdx + qdy$, we get

$dz = adx + 3a^2dy$ so that $z = ax + 3a^2y + b$,

which is a complete integral, a and b being arbitrary constants.

Ex. 3. *Find the complete integral of* $zpq = p + q$ **[Nagpur 2010; Meerut 2006]**

Sol. Let $f(x, y, z, p, q) = zpq - p - q = 0$ ... (1)

Here Charpit's auxiliary equations are $$\frac{dp}{f_x+pf_z}=\frac{dq}{f_y+qf_z}=\frac{dz}{-pf_p-qf_q}=\frac{dx}{-f_p}=\frac{dy}{-f_q} \qquad \text{... (2)}$$

From (1), $f_x = 0$, $f_y = 0$, $f_z = pq$, $f_p = zq - 1$ and $f_q = zp - 1$ ... (3)

Using (3), (2) reduces to

$$\frac{dp}{p^2q}=\frac{dq}{pq^2}=\text{......} \quad \text{or} \quad \frac{dp}{p}=\frac{dq}{q} \quad \text{so that} \quad p = aq \qquad \text{... (4)}$$

Solving (1) and (2), $p = (1 + a)/z$ and $q = (1 + a)/az$.

$\therefore$ $dz = pdx + qdy = [(1 + a)/z]dx + [(1 + a)/az]dy$ or $2zdz = 2(1 + a)\,[dx + (1/a)dy]$

Integrating, $z^2 = 2(1 + a)\,[x + (1/a)y] + b$, a, b being arbitrary constants

Ex. 4. *Find a complete integral of* $p^2 - y^2q = y^2 - x^2$. **[M.D.U. Rohtak 2006]**

Sol. Here given equation is $f(x, y, z, p, q) = p^2 - y^2q - y^2 + x^2 = 0$. ...(1)

Charpit's auxiliary equations are $$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}$$

or $$\frac{dp}{2x}=\frac{dq}{-2qy-2y}=\frac{dz}{-p(2p)-q(-q^2)}=\frac{dx}{-2p}=\frac{dy}{y^2}, \text{ using (1)} \qquad \text{...(2)}$$

Taking the first and fourth fractions, $pdp + xdx = 0$ so that $p^2 + x^2 = a^2$ ... (3)

Solving (1) and (3) for p and q, $p = (a^2 - x^2)^{1/2}$, $q = a^2y^{-2} - 1$.

$\therefore$ $dz = pdx + qdy = (a^2 - x^2)^{1/2}dx + (a^2y^{-2} - 1)dy$.

Integrating, $z = (x/2) \times (a^2 - x^2)^{1/2} + (a^2/2) \times \sin^{-1}(x/a) - (a^2/y) - y + b$.

Ex. 5. *Find a complete integral of* $z^2(p^2z^2 + q^2) = 1$. **[I.A.S. 1997; Meerut 2007]**

Sol. Here given equation is $f(x, y, z, p, q) = p^2z^4 + q^2z^2 - 1 = 0$. ...(1)

Charpit's auxiliary equations are $$\frac{dp}{f_x+pf_z}=\frac{dq}{f_y+qf_z}=\frac{dz}{-pf_p-qf_q}=\frac{dx}{-f_p}=\frac{dq}{-f_q}$$

or $$\frac{dp}{p(4p^2z^3+2zq^2)}=\frac{dq}{q(4p^2z^3+2zq^2)}=\frac{dz}{-2p^2z^4-2q^2z^2}=\frac{dx}{-2pz^4}=\frac{dy}{-2qz^2}, \text{ by (1)} \qquad \text{... (2)}$$

Taking the first two fractions, $(1/p)dp = (1/q)dq$ so that $p = aq$.

Solving (1) and (2) for p and q, $$p=\frac{a}{z(a^2z^2+1)^{1/2}}, \qquad q=\frac{1}{z(a^2z^2+1)^{1/2}}.$$

$\therefore$ $dz = pdx + qdy = (a\,dx + dy)/z\,(a^2z^2 + 1)^{1/2}$ or $adx + dy = z(a^2z^2 + 1)^{1/2}dz$.

Integrating, $$ax + y = \int (a^2z^2 + 1)^{1/2} . z dz. \qquad ...(3)$$

Putting $a^2z^2 + 1 = t^2$ so that $2a^2zdz = 2tdt$, (3) becomes

$$ax + y = \int (1/a^2)\, t . t\, dt \quad \text{or} \quad ax + y + b = (1/3a^2)t^3, \text{ where } t = (a^2z^2 + 1)^{1/2}$$

or $ax + y + b = (1/3a^2) \times (a^2z^2 + 1)^{3/2}$ or $9a^4(ax + y + b)^2 = (a^2z^2 + 1)^3$,

which is a complete integral, a and b being arbitrary constants.

Ex. 6. *Find a complete integral of* $px + qy = pq$. **[Kurukshetra 2006 Rajasthan 2000, 01, Gulbarga 2005; Meerut 2002; Kanpur 2004; Jiwaji 2004; Rewa 2001; Vikram 2000, 03, 04; Bhopal 2010]**

Sol. Here given equation is $f(x, y, z, p, q) \equiv px + qy - pq = 0. \qquad ...(1)$

Charpit's auxiliary equations are $$\frac{dp}{f_x + p f_z} = \frac{dq}{f_y + q f_z} = \frac{dz}{-p f_p - q f_q} = \frac{dx}{-f_p} = \frac{dq}{-f_q}$$

or $$\frac{dp}{-(x-q)} = \frac{dy}{-(y-q)} = \frac{dz}{-p(x-q)-q(y-p)} = \frac{dp}{p+p.0} = \frac{dq}{q+q.0}, \text{ by (1)} \qquad ...(2)$$

Taking the last two fractions of (2), $(1/p)dp = (1/q)dq$.

Integrating, $\log p = \log q + \log a$ or $p = aq. \qquad ...(3)$

Substituting this value of p in (1), we have

$aqx + qy - aq^2 = 0$ or $aq = ax + y$, as $q \neq 0 \qquad ...(4)$

$\therefore$ From (3) and (4), $q = (ax + y)/a$ and $p = ax + y. \quad ...(5)$

Putting these values of p and q in $dz = pdx + qdy$, we get

$dz = (ax + y)dx + [(ax + y)/a]\, dy$ or $adz = (ax + y)\,(adx + dy)$

or $$adz = (ax + y)\, d(ax + y) = udu, \text{ where } u = ax + y.$$

Integrating, $$az = u^2/2 + b = (ax + y)^2/2 + b,$$

which is a complete integral, a and b being arbitrary constants.

Ex. 7. *Find the complete integrals of following equations:*

(*i*) $q = (z + px)^2$ **[Indore 2004; Ravishanker 2005]**

(*ii*) $p = (z + qy)^2$ **[Meerut 2008, 09; Agra 2001; Delhi B.Sc. (Prog) 2008; Kurukshetra 2005]**

Sol. (*i*). Here given equations is $f(x, y, z, p, q) = (z + px)^2 - q = 0 \qquad ... (1)$

Charpit's auxiliary equations are $$\frac{dp}{f_x + p f_z} = \frac{dq}{f_y + q f_z} = \frac{dz}{-p f_p - q f_q} = \frac{dx}{-f_p} = \frac{dq}{-f_q}$$

or $$\frac{dp}{2p(z+px)+2p(z+px)} = \frac{dq}{2q(z+px)} = \frac{dz}{-2px(z+px)+q} = \frac{dx}{-2x(z+px)} = \frac{dy}{0}, \text{ by (1)}$$

Taking the second and fourth fractions, $(1/q)dq = -(1/x)dx$.

Integrating, $\log q = \log a - \log x$ so that $q = a/x. \qquad ...(2)$

Substituting the above value of q in (1), we have

$(z + px)^2 = a/x$ or $px = \sqrt{a}/\sqrt{x} - z$ or $p = \sqrt{a}/x\sqrt{x} - z/x. \qquad ...(3)$

$$\therefore \quad dz = pdx + qdy = \left(\frac{\sqrt{a}}{x\sqrt{x}} - \frac{z}{x}\right) dx + \frac{a}{x} dy, \text{ by (2) and (3)}$$

or $xdz = \sqrt{a}\, x^{-1/2}dx - zdx + ady$ or $xdz + zdx = \sqrt{a}\, x^{-1/2}dx + ady$

or $$d(xz) = \sqrt{a}\, x^{-1/2}dx + ady.$$

Integrating, $xz = 2\sqrt{a}\sqrt{x} + ay + b$, a, b being arbitrary constants

(*ii*) **Sol.** Do as in part (1). **Ans.** $yz = ax + \sqrt{ay} + b$.

Ex. 8. *Find a complete integral of* $yzp^2 - q = 0$.

Sol. Here $$f(x, y, z, p, q) = yzp^2 - q = 0.$$

Charpit's uxiliary equations are $$\frac{dp}{f_x + p f_z} = \frac{dq}{f_y + q f_z} = \frac{dz}{-p f_p - q f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

or $$\frac{dp}{0 + p(yp^2)} = \frac{dq}{zp^2 + q(yp^2)} = \frac{dz}{-2yzp^2 + q} = \frac{dx}{-2yzp} = \frac{dy}{1}, \text{ by (1)} \qquad \text{...(2)}$$

Taking the first and fifth fractions, $(1/yp^3)\, dp = dy$

or $p^{-3}dp = ydy$ or $-2p^{-3}dp = -2ydy$.

Integrating, $p^{-2} = a^2 - y^2$ so that $p = 1/(a^2 - y^2)^{1/2}$. ...(3)

Using (3), (1) $\Rightarrow$ $q = yzp^2$ $\Rightarrow$ $q = yz/(a^2 - y^2)$. ...(4)

$$\therefore \qquad dz = pdx + qdy = \frac{dx}{(a^2 - y^2)^{1/2}} + \frac{yzdy}{(a^2 - y^2)}$$

or $$(a^2 - y^2)^{1/2}\, dz - \frac{yzdy}{(a^2 - y^2)^{1/2}} = dx \qquad \text{or} \qquad d\,[z\,(a^2 - y^2)^{1/2}] = dx.$$

Integrating, $z\,(a^2 - y^2)^{1/2} = x + b$ or $z^2(a^2 - y^2) = (x + b)^2$, a, b being arbitrary constants.

Ex. 9. *Find a complete integral of* $16p^2z^2 + 9q^2z^2 + 4z^2 - 4 = 0$. **[I.A.S. 1994]**

Sol. Given equation is $f(x, y, z, p, q) = 16p^2z^2 + 9q^2z^2 + 4z^2 - 4 = 0$. ...(1)

Charpit's auxiliary equations are $$\frac{dp}{f_x + p f_z} = \frac{dq}{f_y + q f_z} = \frac{dz}{-pf_p - q f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

or $$\frac{dp}{-p(32p^2z + 18q^2z + 8z)} = \frac{dq}{q(32p^2z + 18q^2z + 8z)} = \frac{dz}{-p(32pz^2) - q(18qz^2)} = \frac{dx}{-32pz^2} = \frac{dy}{-18qz^2}.$$

Taking the first and second fractions, $(1/p)dp = (1/q)dq$ so that $p = aq$ ... (2)

Solving (1) and (2) for p and q, we have

$$q = \frac{2(1 - z^2)^{1/2}}{z(16a^2 + 9)^{1/2}} \qquad \text{and} \qquad p = \frac{2a(1 - z^2)^{1/2}}{z(16a^2 + 9)^{1/2}}. \qquad \text{...(3)}$$

Hence, $$dz = pdx + qdy = \frac{2(1 - z^2)^{1/2}}{z(16a^2 + 9)^{1/2}}\,(adx + dy), \text{ using (3)}$$

or $$(1/2) \times (16a^2 + 9)^{1/2}\,(1 - z^2)^{-1/2}\,(-2zdz) = -\,2(adx + dy). \qquad \text{...(4)}$$

Putting $1 - z^2 = t$ so that $-2zdz = dt$, (4) becomes

or $$(1/2) \times (16a^2 + 9)^{1/2}\, t^{-1/2}\, dt = -\,2\,(adx + dy).$$

Integrating, $(16a^2 + 9)^{1/2}\, t^{1/2} = -\,2(ax + y) + b$, a, b being arbitrary constants.

or $$(16a^2 + 9)^{1/2}\sqrt{(1 - z^2)} + 2(ax + y) = b, \text{ as } t = 1 - z^2.$$

Ex. 10(a). *Find a complete integral of* $(p^2 + q^2)x = pz$.

[Agra 2003; Rajasthan 2005; Ravishankar 2001; Delhi Maths (Hons) 2004, 05]

(b). *Find the complete integral of the partial differential equation* $(p^2 + q^2)x = pz$ *and deduce the solution which passes through the curve* $x = 0$, $z^2 = 4y$. **[Meerut 2007]**

Sol. Let $f(x, y, q, p, q) = (p^2 + q^2)x - pz = 0$. ...(1)

Charpit's auxiliary equations are $\dfrac{dp}{f_x + p\,f_z} = \dfrac{dq}{f_y + q\,f_z} = \dfrac{dz}{-p\,f_p - q\,f_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

giving $dp/q^2 = dq/(-pq)$, by (1) or $2pdp + 2qdq = 0$.

Integrating, $p^2 + q^2 = a^2$, where a is an arbitrary constant. ...(2)

Solving (1) and (2), $p = a^2x/q$ and $q = (a/z) \times \sqrt{(z^2 - a^2x^2)}$. ...(3)

$\therefore$ $dz = pdx + qdy = \dfrac{a^2xdx}{z} + \dfrac{a\sqrt{(z^2 - a^2x^2)}\,dy}{z}$ or $\dfrac{zdz - a^2xdx}{\sqrt{(z^2 - a^2x^2)}} = ady.$

Putting $z^2 - a^2x^2 = t$ so that $2(zdz - a^2xdx) = dt$, we get

$(1/2\sqrt{t})dt = ady$ or $(1/2) \times t^{-1/2} = ady.$

Integrating, $t^{1/2} = ay + b$ or $\sqrt{(z^2 - a^2x^2)} = ay + b,$ as $t = \sqrt{z^2 - a^2x^2}$

or $z^2 - a^2x^2 = (ay + b)^2$ or $z^2 = a^2x^2 + (ay + b)^2$. ... (4)

(b) Proceeding as in part (a), (4) is the complete integral.

The parametric equations of the given curve $x = 0$, $z^2 = 4y$ are given by

$x = 0,$ $y = t^2,$ $z = 2t$ ... (5)

Therefore the intersections of (1) and (2) are determined by

$4t^2 = (at^2 + b)^2$ or $a^2t^4 + 2(ab - 2)t^2 + b^2 = 0$ ... (6)

Equation (6) has equal roots if its discriminant = 0, *i.e.*, if

$4(ab - 2)^2 - 4a^2b^2 = 0$ or $a^2b^2 = 1$ so that $b = 1/a$

Hence from (4), the appropriate one paremeter sub–system is given by

$z^2 = a^2x^2 + (ay + 1/a)^2$ or $a^4(x^2 + y^2) + a^2(2y - z^2) + 1 = 0,$

which is a quatratic equation in parameter 'a'. Therefore, this has for its envelope surface

$(2y - z^2)^2 - 4(x^2 + y^2) = 0$ or $(2y - z^2)^2 = 4(x^2 + y^2)$ ... (7)

The desired solution is given by the function z defined by equation (7).

Ex. 10(c). *Find a complete, singular and general integrals of* $(p^2 + q^2)y = qz$.
[Guwahati 2007; Agra 2001; Bilaspur 1998; Delhi Maths (H) 2003, 05; Garhwal 2005; Meerut 2010, 11; K.V. Kurukshetra 2004; Kanpur 2005; Rohilkhand 2001; Pune 2010]

Sol. Here the given equation is $f(x, y, z, p, q) = (p^2 + q^2)y - qz = 0.$ ...(1)

Charpit's auxiliary equations are $\dfrac{dp}{f_x + p\,f_z} = \dfrac{dq}{f_y + q\,f_z} = \dfrac{dz}{-pf_p - q\,f_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

or $\dfrac{dp}{-pq} = \dfrac{dq}{p^2} = \dfrac{dz}{-2p^2y + qz - 2q^2y} = \dfrac{dx}{-2py} = \dfrac{dy}{-2qy + z}$, by (1) ...(2)

Taking the first two fractions, we get $2pdp + 2qdq = 0$ so that $p^2 + q^2 = a$ ... (3)

Using (3), (1) gives $a^2y = qz$ or $q = a^2y/z.$

Putting this value of q in (3), we get

$$p = \sqrt{(a^2 - q^2)} = \sqrt{a^2 - (a^4y^2/z^2)} = \frac{a}{z}\sqrt{(z^2 - a^2y^2)}\,.$$

Now putting these values of p and q in $dz = pdx + qdy$, we have

$dz = \dfrac{a}{z}\sqrt{(z^2 - a^2y^2)}\,dx + \dfrac{a^2y\,dy}{z}dy$ or $\dfrac{zdz - a^2y\,dy}{\sqrt{(z^2 - a^2y^2)}} = a\,dx.$

Integrating, $\quad (z^2 - a^2y^2)^{1/2} = ax + b \quad$ or $\quad z^2 - a^2y^2 = (ax + b)^2, \quad$ ...(4)
which is a required complete integral, a, b being arbitrary constants.

Singular Integral. Differentiating (4) partially w.r.t. a and b, we have

$$0 = 2ay^2 + 2\,(ax + b)x \qquad \text{...(5)}$$

and
$$0 = 2(ax + b). \qquad \text{...(6)}$$

Eliminating a and b between (4), (5) and (6), we get $z = 0$ which clearly satisfies (1) and hence it is the singular integral.

General Integral. Replacing b by $\phi(a)$ in (4), we get

$$z^2 - a^2y^2 = [ax + \phi(a)]^2. \qquad \text{...(7)}$$

Differentiating (7) partially w.r.t. a, $\quad -2ay^2 = 2[ax + \phi(a)]\,.\,[x + \phi'(a)]. \quad$ ...(8)

General integral is obtained by eliminating a from (7) and (8).

Ex. 11. *Find a complete integral of* $p\,(1 + q^2) + (b - z)q = 0.$ **[Agra 1996]**

Sol. Here given equation is $\quad f(x, y, z, p, q) \equiv p(1 + q^2) + (b - z)q = 0. \quad$ ...(1)

Charpit's auxiliary equations are $\dfrac{dp}{\dfrac{\partial f}{\partial x}+p\dfrac{\partial f}{\partial z}}=\dfrac{dq}{\dfrac{\partial f}{\partial y}+q\dfrac{\partial f}{\partial z}}=\dfrac{dz}{-p\dfrac{\partial f}{\partial p}-q\dfrac{\partial f}{\partial q}}=\dfrac{dx}{-\dfrac{\partial f}{\partial p}}=\dfrac{dy}{-\dfrac{\partial f}{\partial q}}$

or
$$\frac{dp}{pq}=\frac{dq}{p^2}=\frac{dz}{-p(1+q^2)-(b-z)q}=\frac{dx}{-(q^2+1)}=\frac{dy}{-2pq-(b-z)}, \text{ by (1)}$$

First two fractions give $\quad (1/p)dp = (1/q)dq \quad$ so that $\quad q = pc.$

Putting $q = pc$ in (1), we have $\quad p = \sqrt{[c(z-b)-1]}\big/c.$

$\therefore \quad q = pc \quad$ gives $\quad q = \sqrt{[c(z-b)-1]}.$

Putting these values of p and q in $dz = pdx + qdy$, we get

$$dz = \sqrt{[c(z-b)-1]}\left(\frac{dx}{c}+dy\right) \quad \text{or} \quad \frac{cdz}{\sqrt{[c(z-b)-1]}} = dx + c\,dy.$$

Integrating, $\quad 2\sqrt{[c(z-b)-1]} = x + cy + a \quad$ or $\quad 4\{c(z - b) - 1\} = (x + cy + a)^2$
which is a complete integral, a and c being arbitrary constants.

Ex. 12. *Find a complete and singular integrals of* $2xz - px^2 - 2q\,xy + pq = 0$. **[I.A.S. 1991, 93, 2007, 2008; Delhi Hons. 2001, 01, 05; Kanpur 2001, 03; Meerut 2005; Bhopal 2004, 10; Indore 1999; M.D.U. Rohtak 2004, Ravishanker 2004; Rajasthan 2000, 03, 05, 10]**

Sol. Here given equation is $\quad f(x, y, z, p, q) = 2xz - px - 2qxy + pq = 0. \quad$ ...(1)

Charpit's auxiliary equations are $\dfrac{dp}{\dfrac{\partial f}{\partial x}+p\dfrac{\partial f}{\partial z}}=\dfrac{dq}{\dfrac{\partial f}{\partial y}+q\dfrac{\partial f}{\partial z}}=\dfrac{dz}{-p\dfrac{\partial f}{\partial p}-q\dfrac{\partial f}{\partial q}}=\dfrac{dx}{-\dfrac{\partial f}{\partial p}}=\dfrac{dy}{-\dfrac{\partial f}{\partial q}}$

or
$$\frac{dp}{2z-2qy}=\frac{dq}{0}=\frac{dx}{x^2-q}=\frac{dy}{2xy-p}=\frac{dz}{px^2+2xyq-2pq}, \text{ by (1)}$$

The second fraction gives $\quad dq = 0 \quad$ so that $\quad q = a$

Putting $q = a$ in (1), we get $\quad p = 2x(z - ay)/(x^2 - a)$

Putting values p and q in $dz = p\,dx + q\,dy$, we get

$$dz = \frac{2x(z-ay)}{x^2-a}\,dx + a\,dy \quad \text{or} \quad \frac{dz-ady}{z-ay}=\frac{2xdx}{x^2-a}.$$

Integrating, $\quad \log(z - ay) = \log(x^2 - a) + \log b$

or $\quad z - ay = b(x^2 - a)$ $\quad$ or $\quad z = ay + b(x^2 - a)$, ...(2)

which is the complete integral, a and b being arbitrary constants.

Differentiating (2) partially with respect to a and b, we get

$$0 = y - b \quad \text{and} \quad 0 = x^2 - a. \qquad ...(3)$$

Solving (3) for a and b, $\quad a = x^2 \quad$ and $\quad b = y$. ...(4)

Substituting the values of a and b given by (4) in (2), we get $z = x^2 y$, which is the required singular integral.

Ex. 13. *Find a complete integrals of the following partial differential equations:*

(*i*) $q = px + p^2$. **[Sagar 2003; Meerut 1994]**

(*ii*) $q = -px + p^2$.

Sol. (*i*) Here given equation is $\quad f(x, y, z, p, q) \equiv q - px - p^2 = 0$. ... (1)

Charpit's auxiliary equations are $\quad \dfrac{dp}{f_x + p f_z} = \dfrac{dq}{f_y + q f_z} = \dfrac{dz}{-pf_p - q f_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

or $$\frac{dp}{-p} = \frac{dq}{0} = \frac{dz}{-p(-x-2p)-q} = \frac{dx}{-(-x-2p)} = \frac{dy}{-1}, \text{ by (1)}$$

The 2nd fraction gives $\quad dq = 0 \quad$ so that $\quad q = a$.

Putting $\quad q = a$ in (1) gives $\quad p^2 + px - a = 0 \quad$ so that $\quad p = (1/2) \times \left[-x \pm (x^2 + 4a)^{1/2}\right]$

Putting these values of p and q in $dz = pdx + qdy$, we get

$$dz = -(x/2) \times dx \pm (1/2) \times (x^2 + 4a)^{1/2}\, dx + a\, dy.$$

Integrating, the required complete integral is

$$z = -\frac{x^2}{4} \pm \frac{1}{2}\left[\frac{x}{2}(x^2 + 4a)^{1/2} + 2a \log\{x + (x^2 + 4a)^{1/2}\}\right] + ay + b,$$

Part (*ii*). Proceed like part (*i*) yourself. Complete integral is

$$z = \frac{x^2}{4} \pm \frac{1}{2}\left[\frac{x}{2}(x^2 + 4a)^{1/2} + 2a \log\{x + (x^2 + 4a)^{1/2}\}\right] + ay + b.$$

Ex. 14. *Find a complete integral of* $pxy + pq + qy = yz$. **[Delhi B.A. (Prog) H 2010]**
[Garhwal 2001; Rohilkhand 1999; Meerut 2001, 02; Kanpur 2005]

Sol. Given $\quad f(x, y, z, p, q) \equiv pxy + pq + qy - yz = 0$. ...(1)

Charpit's auxiliary equation are $\quad \dfrac{dp}{f_x + p f_z} = \dfrac{dq}{f_y + q f_z} = \dfrac{dz}{-pf_p - q f_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

or $$\frac{dp}{0} = \frac{dq}{(px+q)+qy} = \frac{dz}{-p(xy+q) - q(p+y)} = \frac{dx}{-(xy+q)} = \frac{dy}{-(p+y)}, \text{ by (1)}$$

The first fraction gives $\quad dp = 0 \quad$ so that $\quad p = a$.

Putting $p = a$ in (1) gives $\quad axy + aq + qy = yz \quad$ so that $\quad q = y(z - ax)/(a + y)$.

Putting these values of p and q in $dz = p\,dx + q\,dy$, we get

$$dz = adx + \frac{y(z-ax)}{a+y}dy \quad \text{or} \quad \frac{dz - adx}{z - ax} = \frac{y\,dy}{a+y} = \left(1 - \frac{a}{a+y}\right)dy.$$

Integrating, $\quad \log(z - ax) = y - a\log(a + y) + \log b$, a, b, being arbitrary constants.

or $\quad \log(z - ax) + \log(a + y)^a - \log b = y \quad$ or $\quad (z - ax)(y + a)^a = be^y$

Ex. 15. *Find a complete integral* $p^2 + q^2 - 2px - 2qy + 1 = 0$.

[Patna 2003; Meerut 99, 2003; Delhi Maths Hons 91; Ravishanker 2010]

Sol. Given $\qquad f(x, y, z, p, q) \equiv p^2 + q^2 - 2px - 2qy + 1 = 0. \qquad$ ...(1)

Charpit's auxiliary equations are $\quad \dfrac{dp}{f_x + p\,f_z} = \dfrac{dq}{f_y + q\,f_z} = \dfrac{dz}{-pf_p - q\,f_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

or $$\frac{dp}{-2p} = \frac{dq}{-2q} = \frac{dz}{-p(2p-2x)-q(2q-2y)} = \frac{dx}{-(2p-2y)} = \frac{dy}{-(2q-2y)}, \text{ by (1)}$$

The first two fractions give $\quad (1/p)dp = (1/q)dq \quad$ so that $\quad p = aq$.

Putting $p = aq$ in (1), $a^2q^2 + q^2 - 2aqx - 2qy + 1 = 0$ or $(a^2 + 1)q^2 - 2(ax - y)q + 1 = 0$.

$$\Rightarrow \; q = \frac{2(ax+y) \pm \sqrt{\{4(ax+y)^2 - 4(a^2+1)\}}}{2(a^2+1)}, \; p = aq \; = a\,\frac{2(ax+y) \pm \sqrt{\{4(ax+y)^2 - 4(a^2+1)\}}}{2(a^2+1)}.$$

Putting these values of p and q in $dz = p\,dx + y\,dy$, we get

$$dz \;=\; \frac{(ax+y) \pm \sqrt{\{(ax+y)^2 - (a^2+1)\}}}{(a^2+1)}(adx + dy). \qquad \text{...(2)}$$

Put $ax + y = v$ so that $a\,dx + dy = dv$. Then (2) gives

$$(a^2 + 1)dz = [v \pm \sqrt{\{v^2 - (a^2+1)\}}\,]dv.$$

Integrating, $$(a^2 + 1)\,z = v^2/2 \pm [\,(v/2) \times \sqrt{\{v^2 - (a^2+1)\}}$$
$$- (1/2) \times (a^2 + 1) \log (v + \sqrt{\{v^2 - (a^2+1)\}}\,)] + b$$

is the complete integral, where $v = ax + b$ and a, b are arbitrary constants.

Ex. 16. *Find a complete integral of* $p^2 + q^2 - 2px - 2qy + 2xy = 0$. **[PCS (U.P.) 2001; Garhwal 1993; Delhi 1997; Kanpur 1996; I.A.S. 1999; Meerut 2003; Rohitkhand 1998]**

Sol. Given equation is $\qquad f(x, y, z, p, q) \equiv p^2 + q^2 - 2px - 2qy + 2xy = 0. \qquad$ ...(1)

Charpit's auxiliary equations are $\quad \dfrac{dp}{\dfrac{\partial f}{\partial x} + p\dfrac{\partial f}{\partial z}} = \dfrac{dq}{\dfrac{\partial f}{\partial y} + q\dfrac{\partial f}{\partial z}} = \dfrac{dz}{-p\dfrac{\partial f}{\partial p} - q\dfrac{\partial f}{\partial q}} = \dfrac{dx}{-\dfrac{\partial f}{\partial p}} = \dfrac{dy}{-\dfrac{\partial f}{\partial q}}$

or $$\frac{dp}{-2p+2y} = \frac{dq}{-2q+2x} = \frac{dx}{2x-2p} = \frac{dy}{2y-2q}, \text{ by (1)}$$

which gives $$\frac{dp+dq}{2(x+y-p-q)} = \frac{dx+dy}{2(x+y-p-q)}$$

or $\quad dp + dq = dx + dy \qquad$ *i.e.*, $\qquad dp - dx + dq - dy = 0$.

Integrating, $\qquad (p - x) + (q - y) = a \qquad$ ...(2)

Re–writing (1), $\qquad (p - x)^2 + (q - y)^2 = (x - y)^2. \qquad$ ...(3)

Putting the value of $(q - y)$ from (2) in (3), we get

$(p - x)^2 + [a - (p - x)]^2 = (x - y)^2 \qquad$ or $\qquad 2(p - x)^2 - 2a(p - x) + \{a^2 - (x - y)^2\} = 0$.

$$\therefore\; p - x = \frac{2a \pm \sqrt{[4a^2 - 4.2.\{a^2 - (x-y)^2\}]}}{4} \qquad \Rightarrow \qquad p = x + \frac{1}{2}[a \pm \sqrt{\{2(x-y)^2 - a^2\}}\,],$$

$$\therefore \text{ (2) gives} \qquad q = a + y - p + x \qquad \text{or} \qquad q = y + (1/2) \times [a \mp \sqrt{\{2(x-y)^2 - a^2\}}\,].$$

Putting these value of p and q in $dz = p\,dx + q\,dy$, we get

$$dz = x\,dx + y\,dy + (a/2) \times (dx + dy) \pm (1/2)\sqrt{\{2(x-y)^2 - a^2\}}(dx - dy)$$

or
$$dz = x\,dx + y\,dy + \frac{a}{2}(dx + dy) \pm \frac{1}{\sqrt{2}}\sqrt{(x-y)^2 - a^2/2}.(dx - dy).$$

Integrating, the desired complete integral is

$$z = \frac{x^2 + y^2}{2} + \frac{a(x+y)}{2} \pm \frac{1}{\sqrt{2}}\left(\frac{x-y}{2}\sqrt{(x-y)^2 - a^2/2} \;-\frac{a^2}{4}\log\left[(x-y) + \sqrt{(x-y)^2 - a^2/2}\right]\right)$$

Ex. 17. *Find a complete integral of* $p^2x + q^2y = z$. **[Gujarat 2005; K.U. Kurukshetra 2001; Meerut 2008; Agra 2004; I.A.S. 2004, 06 ; Delhi Maths Hons. 1997; Punjab 2001]**

Sol. Given equation is $\qquad f(x, y, z, p, q) = p^2x + q^2y - z = 0.$ ...(1)

Charpit's auxiliary equations are $\quad \dfrac{dp}{f_x + p\,f_z} = \dfrac{dq}{f_y + q\,f_z} = \dfrac{dz}{-p\,f_p - q\,f_q} = \dfrac{dx}{-f_p} - \dfrac{dy}{-f_q}$

or
$$\frac{dp}{-p + p^2} = \frac{dq}{-q + q^2} = \frac{dz}{-2(p^2x + q^2y)} = \frac{dx}{-2px} = \frac{dy}{-2qy}, \text{ by (1)} \qquad ...(2)$$

Now, each fraction in (2) $\quad = \dfrac{2px\,dp + p^2dx}{2px(-p + p^2) + p^2(-2px)} = \dfrac{2qy\,dq + q^2dy}{2qy(-q + q^2) + q^2(-2qy)}$

or
$$\frac{d(p^2x)}{-2p^2x} = \frac{d(q^2y)}{-2qy} \qquad i.e., \qquad \frac{d(p^2x)}{p^2x} = \frac{d(q^2y)}{q^2y}.$$

Integrating it, $\quad \log(p^2x) = \log(q^2y) + \log a \quad$ or $\quad p^2x = q^2ya.$ ...(3)

Form (1) and (3), $\quad aq^2y + q^2y = z \quad$ or $\quad q = [z/(1 + a)]^{1/2}.$ ...(4)

Form (3) and (4), $\quad p = q\left(\dfrac{ya}{x}\right)^{1/2} = \left\{\dfrac{za}{(1+a)x}\right\}^{1/2}.$

Putting the above values of p and q in $dz = p\,dx + q\,dy$, we get

$$dz = \left\{\frac{za}{(1+a)x}\right\}^{1/2} dx + \left\{\frac{z}{(1+a)y}\right\}^{1/2} dy \qquad \text{or} \qquad (1 + a)^{1/2}z^{-1/2}\,dz = \sqrt{a}x^{-1/2}dx + y^{-1/2}dy.$$

Integrating, $\quad (1 + a)^{1/2}\sqrt{z} = \sqrt{a}\sqrt{x} + \sqrt{y} + b$, a, b being arbitrary constants.

Ex. 18. *Find a complete integral of* $2z + p^2 + qy + 2y^2 = 0$. **[I.F.S. 2005; Meerut 2000; Rohilkhand 1993; Bilaspur 2004, M.D.U Rohtak 2005; Rawa 1999; Ranchi 2010]**

Sol. Given equation is $\qquad f(x, y, z, p, q) = 2z + p^2 + qy^2 + 2y^2 = 0.$ ... (1)

Charpit's auxiliary equations are $\quad \dfrac{dp}{f_x + p\,f_z} = \dfrac{dq}{f_y + q\,f_z} = \dfrac{dz}{-pf_p - q\,f_q} = \dfrac{dp}{-f_p} = \dfrac{dq}{-f_q}$

or
$$\frac{dp}{0 + 2p} = \frac{dq}{(q + 4y) + 2q} = \frac{dz}{-p \times (2p) - qy} = \frac{dx}{-2p} = \frac{dy}{-y}, \text{ by (1)}$$

Taking the first and fourth fractions, $\qquad dp = -\,dx.$

Integrating, $\quad p = a - x \quad$ or $\quad p = -(x - a).$ ...(2)

Using (2), (1) becomes $\qquad 2z + (a - x)^2 + qy + 2y^2 = 0$

$\therefore \qquad q = -[2z + (x - a)^2 + 2y^2]/y.$ ...(3)

$\therefore \qquad dz = p\,dx + q\,dy = -(x-a)\,dx - [\{2z + (x-a)^2 + 2y^2\}/y]dy$, by (2) and (3)

Multiplying both sides by $2y^2$ and re–writing, we have

$$2y^2dz = -2\,(x-a)y^2\,dx - 4zydy - 2y(x-a)^2dy - 4y^3dy$$

or
$$2(y^2dz + 2zy\,dy) + [2(x-a)^2y^2dx + 2y(x-a)^2dy] + 4y^3dy = 0$$

or
$$2d(y^2z) + d[y^2(x-a)^2] + 4y^3dy = 0.$$

Integrating, $\qquad 2y^2z + y^2(x-a)^2 + y^4 = b$, a, b being arbitrary constants

Ex. 19(a). *Find a complete integral of* $2(z + px + qy) = yp^2$.

[Delhi B.A. (Prog.) II 2007, 10; CDLU 2004; Delhi Maths Hons. 1998, 2008]

Sol. Given equation is $\qquad f(x, y, z, p, q) = 2(z + px + qy) - yp^2 = 0 \qquad$ ...(1)

Charpit's auxiliary equations are $\qquad \dfrac{dp}{f_x + p\,f_z} = \dfrac{dq}{f_y + q\,f_z} = \dfrac{dz}{-pf_p - q\,f_q} = \dfrac{dp}{-f_p} = \dfrac{dq}{-f_q}$

or
$$\frac{dp}{2p+2p} = \frac{dq}{2q-p^2+2q} = \frac{dz}{-p\,(2x-2yp)-q\times 2y} = \frac{dx}{-(2x-2yp)} = \frac{dy}{-2y}, \text{ by (1)}$$

Taking the first and the last fractions, $\qquad \dfrac{dp}{4p} = \dfrac{dy}{-2y} \qquad$ or $\qquad \dfrac{dp}{p} + 2\dfrac{dy}{y} = 0.$

Integrating, $\qquad \log p + 2\log y = \log a \qquad$ or $\qquad py^2 = a. \qquad$ ...(2)

Solving (1) and (2) for p and q, $\qquad p = \dfrac{a}{y^2} \qquad$ and $\qquad q = -\dfrac{z}{y} - \dfrac{ax}{y^3} + \dfrac{a^2}{2y^4}.$

$\therefore$
$$dz = p\,dx + q\,dy = \frac{a}{y^2}dx + \left[-\frac{z}{y} - \frac{ax}{y^3} + \frac{a^2}{2y^4}\right]dy$$

Multiplying both sides by y and re–arranging, we get

$$(ydz + zdy) - a\left(\frac{y\,dx - x\,dy}{y^2}\right) - \frac{a^2}{2y^3}\,dy = 0 \qquad \text{or} \qquad d(yz) - ad\left(\frac{x}{y}\right) - \frac{a^2}{2}y^{-3}\,dy = 0.$$

Integrating, $\qquad yz - a(x/y) + (a^2/4y^2) = b$, a, b being arbitrary constants. $\qquad$ ... (3)

Ex. 19(b). *Find the complete integral, general integral and the singular integral of* $2(z + xp + yq) = yp^2$ **[Delhi B.Sc. (H) 1998, 2008]**

Sol. Proceed as in solved Ex. 19(a) to get the complete integral (3).

General integral. Replacing b by $\phi(a)$ in (3), we get

$$yz - a(x/y) + (a^2/4y^2) = \phi(a) \qquad \text{... (4)}$$

Differentiating (4) partially w.r.t. 'a', $\qquad -(x/y) + (a/2y^2) = \phi'(a) \qquad$ ... (5)

Then the general integral is obtained by eliminating a from (4) and (5).

Singular integral. Differentiating (3) partially w.r.t. 'a' and 'b' by turn, we get

$-(x/y) + (a/2y^2) = 0 \qquad$ ... (6) $\qquad\qquad 0 = 1 \qquad$ ... (7)

Relation (7) is absurd and hence there is no singular solution of the given equation.

Ex. 20. *Find a complete integral of* $z^2 = pqxy$. **[Delhi B.A. (Prog) II 2010]**

[Delhi Maths (H) 2004 Jabalpur 2004; Meerut 2006; Lucknow 2010]

Sol. The given equation is $\qquad f(x, y, z, p, q) = z^2 - pqxy = 0. \qquad$ ...(1)

Charpits's auxiliary equations are $\qquad \dfrac{dp}{f_x + p\,f_z} = \dfrac{dq}{f_y + q\,f_z} = \dfrac{dz}{-p\,f_p - q\,f_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

or $$\frac{dp}{-pqy+2pz}=\frac{dq}{-pqx+2qz}=\frac{dz}{-p(-qxy)-q(-pxy)}=\frac{dx}{qxy}=\frac{dy}{pxy}, \text{ by (1)} \qquad ...(2)$$

Each fraction of (2) $= \dfrac{x\,dp+p\,dx}{x(-pqy+2pz)+pqxy} = \dfrac{y\,dq+q\,dy}{y(-pqx+2qz)+pqxy}$

or $$\frac{xdp+pdx}{2pxz}=\frac{ydq+qdy}{2qyz} \quad \text{or} \quad \frac{d(xp)}{xp}=\frac{d(yq)}{yq}.$$

Integrating, $\log(xp) = \log(yq) + \log a^2$ or $xp = a^2yq$. ...(3)

Solving (1) and (2) for p and q, $p = (az)/x$ and $q = z/(ay)$.

$\therefore\ dz = p\,dx + q\,dy = (az/x)\,dx + (z/ay)\,dy$ or $(1/z)\,dz = (a/x)\,dx + (1/ay)\,dy$.

Integrating, $\log z = a\log x + (1/a)\log y + \log b$ or $z = x^a y^{1/a}\, b$.

Ex. 21. *Using Charpit's method, find three complete integrals of $pq = px + qy$.*

(Kanpur 2004; Meerut 2002; Rajasthan 2001)

Sol. Here given equation is $f(x, y, z, p, q) = pq - px - qy = 0$. ...(1)

Charpit's auxiliary equations are $$\frac{dp}{f_x+p\,f_z}=\frac{dq}{f_y+q\,f_z}=\frac{dz}{-p\,f_p-q\,f_q}=\frac{dx}{-f_p}=\frac{dy}{-f_q}$$

or $$\frac{dp}{-p}=\frac{dq}{-q}=\frac{dz}{-p(q-x)-p(p-y)}=\frac{dx}{-(q-x)}=\frac{dy}{-(p-y)}, \text{ by (1)} \qquad ...(2)$$

To find first complete integral. Taking the first two fractions of (2), we get

$(1/p)dp = (1/q)dq$ so that $\log p = \log q + \log a$ or $p = aq$. ...(3)

Using (3), (1)$\Rightarrow$ $aq^2 = q(ax + y)$ $\Rightarrow$ $q = (ax + y)/a$. ...(4)

Hence, from (3), we have $p = ax + y$. ...(5)

$\therefore\ dz = p\,dx + q\,dy = (ax + y)dx + [(ax + y)/a]dy = (1/a)(ax + y)(a\,dx + y)$.

Putting $ax + y = t$ so that $adx + dy = dt$, we get

$dz = (1/a)\times t\,dt$ so that $z = (1/2a)\times t^2 + b$ or $z = (1/2a)\times(ax + y)^2 + b$, as $t = ax + y$.

To find second complete integral. Taking the second and the fourth ratios in (2), we get

$dx/(q - x) = dq/q$ or $q\,dx + x\,dq = q\,dq$.

Integrating, $qx = q^2/2 + a/2$ or $q^2 - 2xq + a = 0$.

$\therefore\ q = [2x \pm 2(x^2-a)^{1/2}]/2$ so that $q = x + (x^2-a)^{1/2}$. ...(6)

Using (6), (1) $\Rightarrow$ $p[x + (x^2-a)^{1/2}] - px - y[x + (x^2-a)^{1/2}] = 0$

so that $$p = \left\{1+x\big/(x^2-a)^{1/2}\right\}y. \qquad ...(7)$$

$$\therefore\quad dz = p\,dx + q\,dy = \left\{1+x\big/(x^2-a)^{1/2}\right\} y\,dx + [x + (x^2-a)^{1/2}]dy$$

or $$dz = (y\,dx + x\,dy) + \left[\frac{xy\,dy}{(x^2-a)^{1/2}}+(x^2-a)^{1/2}dy\right] \quad \text{or} \quad dz = d(xy) + d[y(x^2-a)^{1/2}].$$

Integraing, $z = xy + y(x^2-a)^{1/2} + b$, a, b being arbitrary constants.

To find third complete integral. Taking the first and the fifth ratios of (2) and proceeding as above third complete integral is $z = xy + x(y^2-a)^{1/2} + b$.

Ex. 22. *Find complete integral of $xp + 3yq = 2(z - x^2q^2)$.* **[Delhi B.Sc. (Prog) II 2009; Delhi B.Sc. (Hons) II 2010;]**

Sol. Given equation is $f(x, y, z, p, q) = xp + 3yq - 2z + 2x^2q^2 = 0$. ...(1)

Charpit's auxiliary equations are $\dfrac{dp}{f_x + p\,f_z} = \dfrac{dq}{f_y + q\,f_z} = \dfrac{dz}{-p\,f_p - q\,f_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

or $$\frac{dp}{-p+4xq^2} = \frac{dq}{q} = \frac{dz}{-p\,x - q(3y+4x^2q)} = \frac{dx}{-x} = \frac{dy}{-3y-4x^2q}, \text{ by (1)} \qquad ...(2)$$

(2) $\Rightarrow$ $\dfrac{dq}{q} = \dfrac{dx}{-x}$ $\Rightarrow$ $\log q = \log a - \log x$ $\Rightarrow$ $qx = a$ $\Rightarrow$ $q = \dfrac{a}{x}$. ...(3)

Using (3), (1) $\Rightarrow$ $xp + 3y(a/x) - 2z + 2x^2(a^2/x^2) = 0$ $\Rightarrow$ $p = \dfrac{2(z-a^2)}{x} - \dfrac{3ay}{x^2}$. ...(4)

$\therefore$ $$dz = p\,dx + q\,dy = \left\{\frac{2(z-a^2)}{x} - \frac{3ay}{x^2}\right\}dx + \frac{a}{x}dy$$

or $x^2dz = 2x(z-a^2)dx - 3ay\,dx + ax\,dy$ or $x^2dz - 2x(z-a^2)\,dx = -3ay\,dx + ax\,dy$

or $\dfrac{x^2dz - 2x(z-a^2)dx}{x^4} = -\dfrac{3ay\,dx}{x^4} + \dfrac{a\,dy}{x^3}$ or $d\left(\dfrac{z-a^2}{x^2}\right) = d\left(\dfrac{ay}{x^3}\right)$

Integrating, $(z-a^2)/x^2 = (ay)/x^3 + b$ or $z = a(a + y/x) + bx^2$.

Ex. 23. *Find complete integrals of the following equations* :

(*i*) $(p^2 + q^2)^n\,(qx - py) = 1$.

(*ii*) $qx + py = (p^2 - q^2)^n$.

Sol. (*i*) Given equation is $f(x, y, z, p, q) = (p^2 + q^2)^n\,(qx - py) - 1 = 0$. ...(1)

Charpit's auxiliary equations are $\dfrac{dp}{f_x + p\,f_z} = \dfrac{dq}{f_y + q\,f_z} = \dfrac{dz}{-p\,f_p - q\,f_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

or $\dfrac{dp}{q(p^2+q^2)^n} = \dfrac{dq}{-p(p^2+q^2)^n} = \ldots$ or $\dfrac{dp}{q} = \dfrac{dq}{-p}$ or $pdp + qdq = 0$.

Integrating, $p^2 + q^2 = \text{constant} = (1/a^2)$, say ...(2)

Using (2), (1) $\Rightarrow qx - py = a^{2n}$ or $qx = py + a^{2n}$. ...(3)

Using (3), (2) $\Rightarrow$ $p^2 + (p^2y^2 + a^{4n} + 2a^{2n}yp)/x^2 = 1/a^2$

or $p^2(x^2 + y^2) + 2a^{2n}yp + \{a^{4n} - (x^2/a^2)\} = 0$ so that

$$p = \frac{-ya^{2n} + \sqrt{\left\{a^{4n}y^2 - (x^2+y^2)\left(a^{4n} - x^2/a^2\right)\right\}}}{x^2+y^2} = \frac{-ya^{2n} + x\sqrt{\left[\left\{(x^2+y^2)/a^2\right\} - a^{4n}\right]}}{x^2+y^2} \qquad ...(4)$$

$\therefore$ (3) $\Rightarrow$ $$q = \frac{xa^{2n} + y\sqrt{\left[\left\{(x^2+y^2)/a^2\right\} - a^{4n}\right]}}{x^2+y^2}. \qquad ...(5)$$

Substituting these values in $dz = p\,dx + q\,dy$, we have

$$dz = a^{2n}\left(\frac{x\,dy - y\,dx}{x^2+y^2}\right) + \frac{x\,dx + y\,dy}{x^2+y^2}\sqrt{\left\{\left(\frac{x^2+y^2}{a^2}\right) - a^{4n}\right\}}.$$

Integrating, $z + b = a^{2n}\tan^{-1}\left(\dfrac{y}{x}\right) + \dfrac{1}{2}\displaystyle\int \frac{1}{u}(ua^{-2} - a^{4n})^{1/2}\,du$, where $u = x^2 + y^2$.

Part (ii). Proceed as in part (i). If $u = x^2 + y^2$, then complete integral is

$$z + b = -\frac{1}{2}a^{2n}\log\frac{x-y}{x+y} - \frac{1}{2}\int\frac{1}{u}\sqrt{(a^{4n}+a^2u)}\,du.$$

Ex. 24. *Find complete integral of* $p^2 + q^2 - 2pq\tanh 2y = \text{sech}^2\, 2y$.

Sol. Given $\quad f(x, y, z, p, q) = p^2 + q^2 - 2pq\tanh 2y - \text{sech}^2\, 2y = 0. \quad$ ...(1)

Charpit's auxiliary equations are $\quad \dfrac{dp}{f_x + p f_z} = \dfrac{dq}{f_y + q f_z} = \dfrac{dz}{-p f_p - q f_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

or $\quad \dfrac{dp}{0} = \dfrac{dq}{-4pq\,\text{sech}^2 2y + 4\,\text{sech}^2 2y\tanh 2y} = \cdots$, by (1)

Then, first fraction $\Rightarrow \quad dp = 0 \quad \Rightarrow \quad p = \text{constant} = a$, say. ...(2)

Using (2), (1) $\Rightarrow \quad q^2 - (2a\tanh 2y)q + a^2 - \text{sech}^2\, 2y = 0$

$\Rightarrow \quad q = [2a\tanh 2y \pm 2\sqrt{(a^2\tanh^2 2y - a^2 + \text{sech}^2\, 2y)}\,]/2$

$\Rightarrow \quad q = a\tanh 2y + \sqrt{(1-a^2)}\ .\ \text{sech}\, 2y.$ ...(3)

[Note that $\text{sech}^2 2y = 1 - \tanh^2 2y$]

Using (2) and (3), $dz = p\,dx + q\,dy$ reduces to

$$dz = a\,dx + \{a\tanh 2y + \sqrt{(1-a^2)}\,\text{sech}\, 2y\}dy$$

Integrating, $\quad z + b = ax + \dfrac{a}{2}\log\cosh 2y + \sqrt{(1-a^2)}\displaystyle\int\frac{2dy}{e^{2y}+e^{-2y}}$

or $\quad z + b = ax + \dfrac{a}{2}\log\cosh 2y + \sqrt{(1-a^2)}\displaystyle\int\frac{2e^{2y}dy}{1+(e^{2y})^2}$

or $\quad z + b = ax + \dfrac{a}{2}\log\cosh 2y + \sqrt{(1-a^2)}\tan^{-1}(e^{2y}),$

$\left[\because \text{ on putting } e^{2y} = t \text{ and } 2e^{2y}\,dy = dt,\ \displaystyle\int\frac{2e^{2y}dy}{1+(e^{2y})^2} = \int\frac{dt}{1+t^2} = \tan^{-1} t = \tan^{-1} e^{2y}\right]$

Ex. 25. *Find complete integral of the equation* $q = \{(1 + p^2)/(1 + y^2)\}x + yp(z - px)^2$.

Sol. Let $\quad f(x, y, z, p, q) = \{(1 + p^2)/(1 + y^2)\}x + yp(z - px^2) - q = 0. \quad$ ...(1)

Charpit's auxiliary equations are $\quad \dfrac{dp}{f_x + p f_z} = \dfrac{dq}{f_y + q f_z} = \dfrac{dz}{-pf_p - q f_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

or $\quad \dfrac{dp}{\{(1+p^2)/(1+y^2)\} - 2yp^2(z-px) + 2yp^2(z-px)} = \dfrac{dy}{1} = \ldots\ldots\ldots$, by (1)

or $\quad \dfrac{dp}{1+p^2} = \dfrac{dy}{1+y^2} \quad$ so that $\quad \tan^{-1} p - \tan^{-1} y = \text{constant} = \tan^{-1} a$

$\Rightarrow \quad (p - y)/(1 + py) = a \quad \Rightarrow \quad p = (y + a)/(1 - ay).$ ...(2)

Using (2), (1) $\Rightarrow \quad q = \dfrac{1+a^2}{(1-ay)^2}x + \dfrac{y(y+a)}{(1-ay)^3}\{z\,(1-ay) - x(y + a)\}^2.$ ...(3)

Using (2) and (3), $dz = p\,dx + q\,dy$ reduces to

$$dz = \frac{y+a}{1-ay}dx + \left[\frac{1+a^2}{(1-ay)^2}x + \frac{y(y+a)}{(1-ay)^3}\{z(1-ay) - x(y+a)\}^2\right]dy$$

or $dz = d\left(\dfrac{y+a}{1-ay}x\right)+\dfrac{y(y+a)}{(1-ay)^3}\{z(1-ay)-x(y+a)\}^2 dy$ or $dz = du + \dfrac{y(y+a)}{1-ay}(z-u)^2 dy$, ...(4)

where $u = x(y+a)/(1-ay)$. ...(5)

(4) $\Rightarrow$ $\dfrac{dz-du}{(z-u)^2} = \dfrac{y(y+a)}{1-ay}$dy. or $\dfrac{d(z-u)}{(z-u)^2}=\left\{-1-\dfrac{1}{a^2}(1+ay)+\dfrac{a^2+1}{a^2}\dfrac{1}{1-ay}\right\}dy.$

Integrating, $b-\dfrac{1}{z-u}=-y-\dfrac{1}{a^2}\left(y+\dfrac{ay^2}{2}\right)-\dfrac{a^2+1}{a^3}\log(1-ay)$, where u is given by (5).

Ex. 26. Find complete integral of $xp - yq = xq\,f(z - px - qy)$.

Sol. Let $F(x, y, z, p, q) = xp - yq - xq\,f(z - px - qy) = 0.$ ...(2)

Charpit's auxiliary equations are

$$\frac{dp}{\partial F/\partial x+p(\partial F/\partial z)}=\frac{dq}{\partial F/\partial y+q(\partial F/\partial z)}=\frac{dz}{-p(\partial F/\partial p)-q(\partial F/\partial q)}=\frac{dx}{-(\partial F/\partial p)}=\frac{dy}{-(\partial F/\partial q)}$$

or $$\frac{dp}{p-qf+xqpf'-pqxf'}=\frac{dq}{-q+xq^2f'-xq^2f'}=\cdots\text{, by (2)} \qquad ...(3)$$

Each ratio of (3) = $\dfrac{x\,dp+y\,dq}{xp-yq-qxf}=\dfrac{x\,dp+y\,dq}{0}$, by (2)

$\Rightarrow$ $x\,dp + y\,dq = 0$ $\Rightarrow$ $x\,dp + y\,dq + p\,dx + q\,dy = p\,dx + q\,dy$

$\Rightarrow$ $dz - d(xp) - d(yq) = 0$, as $dz = pdx + qdy$

Integrating, $z - xp - yq = \text{constant} = a$, say ...(4)

$\therefore$ $xp + yq = z - a.$ ...(5)

Using (4), (1) becomes $x\,p - y\,q = x\,q\,f(a).$ ...(6)

Subtracting (6) from (5), $2yq = z - a - xqf(a)$ $\Rightarrow$ $q = (z-a)/\{2y + xf(a)\}$ ...(7)

Using (7), (5) $\Rightarrow$ $p=\dfrac{(z-a)\{y+xf(a)\}}{x\{2y+xf(a)\}}.$ ...(8)

Using (7) and (8), $dz = p\,dx + q\,dy$ reduces to

$$dz = (z-a)\left[\frac{\{y+xf(a)\}\,dx}{x\{2y+xf(a)\}}+\frac{dy}{2y+xf(a)}\right]$$

or $$\frac{2dz}{z-a}=\frac{2y\,dx+2xf(a)dx+2x\,dy}{x\{2y+xf(a)\}}=\frac{2d(xy)+2xf(a)\,dx}{2xy+x^2f(a)}.$$

Integrating, $2\log(z-a) = \log\{2xy + x^2 f(a)\} + \log b$ or $(z-a)^2 = b\,\{2xy + x^2 f(a)\}$.

Ex. 27. *Find a complete integral of* $px + qy = z(1+pq)^{1/2}$

[Meerut 2001, 02; Kanpur 1995, I.A.S. 1992]

Sol. Given $f(x, y, z, p, q) = px + qy - z(1+pq)^{1/2} = 0.$ ...(1)

Charpit's auxiliary equation are $\dfrac{dp}{f_x+pf_z}=\dfrac{dq}{f_y+qf_z}=\dfrac{dz}{-pf_p-qf_q}=\dfrac{dx}{-f_p}=\dfrac{dy}{-f_q}$

or $\dfrac{dp}{p-p(1+pq)^{1/2}}=\dfrac{dq}{q-q(1+pq)^{1/2}}=\ldots\ldots\ldots$ so that $\dfrac{dp}{p}=\dfrac{dq}{q}$, by (1)

$\Rightarrow$ $\log p = \log a + \log q$ $\Rightarrow$ $p = aq.$ ...(2)

Using (2), (1) $\Rightarrow$ $q(ax+y) = z(1+aq^2)^{1/2}$ or $q^2\,[(ax+y)^2 - az^2] = z^2.$

$\therefore\ q = \dfrac{z}{[(ax+y)^2 - az^2]^{1/2}}$ and $p = aq = \dfrac{az}{[(ax+y)^2 - az^2]^{1/2}}$.

Substituting these values in $dz = p\,dx + q\,dy$, we have

$$dz = \frac{z(a\,dx+dy)}{\sqrt{\{(ax+y)^2 - az^2\}}} \quad \text{or} \quad \frac{dz}{z} = \frac{a\,dx+dy}{\sqrt{\{(ax+y)^2 - az^2\}}}. \qquad \text{... (3)}$$

Let $ax + y = \sqrt{a}\,u$ so that $a\,dx + dy = \sqrt{a}\,du$.

$$\therefore\quad (3) \Rightarrow \frac{dz}{z} = \frac{\sqrt{a}\,du}{\sqrt{(au^2 - az^2)}} \quad \text{or} \quad \frac{du}{dz} = \frac{\sqrt{(u^2 - z^2)}}{z} = \sqrt{\left\{\left(\frac{u}{z}\right)^2 - 1\right\}}, \qquad \text{... (4)}$$

which is linear homogeneus equation. To solve it, we put

$\dfrac{u}{z} = v$ or $u = vz$ so that $\dfrac{du}{dz} = v + z\dfrac{dv}{dz}$.

$\therefore$ (4) yields $\quad v + z\dfrac{dv}{dz} = (v^2 - 1)^{1/2}$. or $\quad \dfrac{dz}{z} = \dfrac{dv}{(v^2-1)^{1/2} - v}$

or $\quad (1/z)dz = -\left[(v^2 - 1)^{1/2} + v\right]dv$, on rationalization.

Integrating, $\log z = -\left[\dfrac{v}{2}(v^2-1)^{1/2} - \dfrac{1}{2}\log\{v + (v^2-1)^{1/2}\}\right] - \dfrac{v^2}{2} + b$, where, $v = \dfrac{u}{z} = \dfrac{ax+y}{z\sqrt{a}}$

Ex. 28. *Find complete integral of* $(x^2 - y^2)\,pq - xy(p^2 - q^2) = 1$.

Sol. Let $\quad f(x, y, z, p, q) = (x^2 - y^2)\,pq - xy(p^2 - q^2) - 1 = 0. \qquad \text{...(1)}$

Charpit's auxiliary equations are $\quad \dfrac{dp}{f_x + p f_z} = \dfrac{dq}{f_y + q f_z} = \dfrac{dz}{-p f_p - q f_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

or $\quad \dfrac{dp}{2pqx - z(p^2 - q^2)} = \dfrac{dq}{-2pqy - x(p^2 - q^2)} = \dfrac{dx}{-(x^2 - y^2)y + 2pxy} = \dfrac{dy}{-(x^2 - y^2)p - 2pxy}$, by (1)

Using x, y, p, q as multipliers, each fraction $= \dfrac{x\,dp + y\,dq + p\,dx + q\,dy}{0} = \dfrac{d(xp) + d(yq)}{0}$

$\Rightarrow \quad d(xp + qy) = 0 \quad \Rightarrow \quad xp + yq = a \quad \Rightarrow \quad p = (a - qy)/x. \qquad \text{...(2)}$

Using (2), $\quad (1) \Rightarrow \quad (x^2 - y^2)\left(\dfrac{a - qx}{x}\right)q - xy\left[\left(\dfrac{a - qy}{x}\right)^2 - q^2\right] - 1 = 0$

or $\quad \dfrac{a - qy}{x}\{(x^2 - y^2)q - (a - qy)y\} + xyq^2 - 1 = 0$ or $\{(a - qy)/x\}\,(x^2q - ay) + xyq^2 - 1 = 0$

or $\quad (a - qy)\,(x^2q - ay) + x^2yq^2 - x = 0$ or $\quad aq(x^2 + y^2) = a^2y + x$

$\therefore \quad q = \dfrac{a^2y + x}{a(x^2 + y^2)}$ and $\quad p = \dfrac{1}{x}\left[a - \dfrac{(a^2y + x)y}{a(x^2 + y^2)}\right] = \dfrac{a^2x - y}{a(x^2 + y^2)}$.

Substituting these values in $dz = p\,dx + q\,dy$, we have

$$dz = \frac{(a^2x - y)dx + (a^2y + x)dy}{a(x^2 + y^2)} = a\frac{x\,dx + y\,dy}{x^2 + y^2} + \frac{x\,dy - y\,dx}{a(x^2 + y^2)}.$$

Integrating, $\quad z = (a/2) \times \log(x^2 + y^2) + (1/a) \times \tan^{-1}(y/x) + b.$

Ex. 29. *Find a complete integral of* $2(pq + yp + qx) + x^2 + y^2 = 0$. **[Kanpur 1993]**

Sol. Given equation is $\quad f(x, y, z, p, q) = 2(pq + yp + qx) + x^2 + y^2 = 0. \quad$ ...(1)

Charpit's auxiliary equations are
$$\frac{dp}{\frac{\partial f}{\partial x}+p\frac{\partial f}{\partial z}}=\frac{dq}{\frac{\partial f}{\partial y}+q\frac{\partial f}{\partial z}}=\frac{dz}{-p\frac{\partial f}{\partial p}-q\frac{\partial f}{\partial q}}=\frac{dx}{-\frac{\partial f}{\partial p}}=\frac{dy}{-\frac{\partial f}{\partial q}}$$

or
$$\frac{dp}{2q+2x}=\frac{dq}{2p+2y}=\frac{dz}{-p(2q+2y)-q(2p+2x)}=\frac{dx}{-(2q+2y)}=\frac{dy}{-(2p+2x)}, \text{ by (1)}$$

Each of these above fractions $= \dfrac{dp+dq+dx+dy}{(2q+2x)+(2p+2y)-(2q+2y)-(2p+2x)}$
$= (dp + dq + dx + dy)/0$

This $\Rightarrow \quad dp + dq + dx + dy = 0 \quad$ so that $\quad (p + x) + (q + y) = a. \quad$ ...(2)

Re–writing (1), $2(p + x)(q + y) + (x - y)^2 = 0 \quad$ or $\quad (p + x)(q + y) = -(x - y)^2/2. \quad$ ...(3)

Now, $$(p + x) - (q + y) = \sqrt{\{(p+x)^2+(q+y)\}^2-4(p+x)(q+y)}$$

$\therefore \quad (p + x) - (q + y) = \sqrt{a^2+2(x-y)^2}$, using (2) and (3) ...(4)

Adding (2) and (4), $\quad 2(p + x) = a + \sqrt{a^2+2(x-y)^2}$.

Substracting (4) from (2), $\quad 2(q + y) = a - \sqrt{a^2+2(x-y)^2}$.

These give $\quad p = -x + \dfrac{a}{2}+\dfrac{1}{2}\sqrt{a^2+2(x-y)^2}, \qquad q = -y + \dfrac{a}{2}-\dfrac{1}{2}\sqrt{a^2+2(x-y)^2}$

Substituting the above values of p and q, $dz = p\,dx + q\,dy$ becomes
$$dz = -(x\,dx + y\,dy) + (a/2) \times (dx + dy) + (1/2)\times\sqrt{a^2+2(x-y)^2}\ (dx - dy)$$

or
$$dz = -\frac{1}{2}d(x^2 + y^2) + \frac{a}{2}d(x + y) + \sqrt{2}\times\frac{1}{2}\sqrt{\frac{a^2}{2}+(x-y)^2}\ d(x - y) \qquad \text{...(5)}$$

Put $\quad x - y = t \quad$ so that $\quad d(x - y) = dt$. Then (5) becomes
$$dz = -(1/2) \times d(x^2 + y^2) + (a/2) \times d(x + y) + (1/\sqrt{2}) \times \sqrt{(a/\sqrt{2})^2+t^2}\ dt.$$

$$\therefore \quad z = -\frac{x^2+y^2}{2}+a\frac{x+y}{2}+\frac{1}{\sqrt{2}}\left[\frac{t}{2}\sqrt{(a/\sqrt{2})^2+t^2}+\frac{(a/\sqrt{2})^2}{2}\log\left\{t+\sqrt{(a/\sqrt{2})^2+t^2}\right\}\right]+b$$

Putting the value of t, the required complete integral is
$$z = -\frac{x^2+y^2}{2}+\frac{a(x+y)}{2}+\frac{1}{2\sqrt{2}}\left[(x-y)\sqrt{\frac{a^2}{2}+(x-y)^2}+\frac{a^2}{2}\log\left\{x-y+\sqrt{\frac{a^2}{2}+(x-y)^2}\right\}\right]+b.$$

Ex. 30. *Solve* $z = (1/2)\times(p^2+q^2)+(p-x)(q-y)$ **[I.A.S. 2002]**

Sol. Given $\quad z = (1/2)\times(p^2+q^2)+(p-x)(q-y)$

Re–writing (1), $\quad f(x,y,z,p,q) = (1/2)\times(p^2+q^2)+pq-xq-yp+xy-z = 0 \quad$ ... (2)

Charpit's auxiliary equations are $\quad \dfrac{dp}{f_x+p\,f_z}=\dfrac{dq}{f_y+q\,f_z}=\dfrac{dz}{-p\,f_p-q\,f_q}=\dfrac{dx}{-f_p}=\dfrac{dy}{-f_q}$

or $$\frac{dp}{-q+z-p}=\frac{dq}{-p+x-q}=\frac{dz}{-p(p+q-y)-q(p+q-x)}=\frac{dx}{-(p+q-y)}=\frac{dy}{-(p+q-x)}$$

Taking the first and the fourth fractions, we have

$dp=dx$ so that $p=x+a$, a being an arbitratry constant. ... (3)

Taking the second and the fifth fractions, we have

$dq = dy$ so that $q=y+b$, b being an arbitrary constant ... (4)

Putting $p=x+a$ and $q=y+b$ in (1), the required solution is

$z=(1/2)\times\{(x+a)^2+(y+b)^2\}+ab$, a and b being arbitrary constants.

Ex. 31 *Find a complete integral of* $z=pq$. **[Sagar 2004, Ravishankar 2003, Rewa 2003]**

Sol. Here given equation is $f(x,y,z,p,q)=z-pq=0$ ... (1)

Charpit's auxiliary equations are $$\frac{dp}{f_x+pf_z}=\frac{dq}{f_y+qf_z}=\frac{dz}{-pf_p-qf_q}=\frac{dx}{-f_p}=\frac{dy}{-f_q}$$

or $$\frac{dp}{-p}=\frac{dq}{-q}=\frac{dz}{-2pq}=\frac{dx}{-q}=\frac{dy}{-p}, \text{ by (1)} \quad \text{... (2)}$$

Taking the first and last fractions of (2), $dp=dy$

Integrating $p=y+a$, a being an arbitrary constant ... (3)

Similarly, taking the second and fourth fractions of (2), we get

$dq = dx$ so that $q=x+b$, b being an arbitrary constant. ... (4)

Putting values of p and q given by (3) and (4) in (1), we get

$z=(x+b)(x+a)$, which is the required complete integral.

Ex. 32. *Use Charpit's method to find the complete integral of* $2x\ \{z^2(\partial z/\partial y)^2+1\}=z(\partial z/\partial x)$.

[I.A.S. 1998]

Sol. Given $2x(z\partial z/\partial y)^2+2x-(z\partial z/\partial x)=0$ ... (1)

Let $z\,dz=dZ$ so that $z^2=2Z$ ... (2)

Then (1) becomes $2x(\partial Z/\partial y)^2+2x-(\partial Z/\partial x)=0$ or $2xQ^2+2x-P=0$

where $P=\partial Z/\partial x$ and $Q=\partial Z/\partial y$ ... (3)

Let $f(x,y,Z,P,Q)=2x\,Q^2+2x-P=0$ ... (4)

Charpit's auxiliary equations are $$\frac{dP}{f_x+Pf_Z}=\frac{dQ}{f_y+Qf_Z}=\frac{dZ}{-Pf_P-Qf_Q}=\frac{dx}{-f_P}=\frac{dy}{-f_Q}$$

giving $\dfrac{dP}{2Q^2+2}=\dfrac{dQ}{O}=\ldots$, by (4) so that $dQ=0.$

Integrating, $Q = a$, a being an arbitrars constant ... (5)

Using $Q = a$, (4) gives $P = 2x\,(a^2 + 1)$, $Q = a$ ... (6)

$\therefore$ $dZ=P\,dx+Q\,dy=2x(a^2+1)dx+a\,dy$, by (5) and (6)

Integrating, $Z=x^2(a^2+1)+ay+b/2$, or $z^2/2=x^2(a^2+1)+ay+b/2$, using (2) or $\qquad z^2 = 2x^2(a^2+1)+2ay+b$, which is complete integral of (1)

Ex. 33. *Solve by Charpit's method the partial differential equation.*

$$p^2x(x-1)+2pqxy+q^2y(y-1)-2pxz-2qyz+z^2=0. \qquad \textbf{[I.A.S. 2000]}$$

Sol. Let $\quad f(x, y, z, p, q) = p^2x(x-1)+2pqxy+q^2y(y-1)-2pxz-2qyz+z^2=0 \qquad \text{... (1)}$

Charpit's auxiliary equations are $\dfrac{dp}{f_x+pf_z}=\dfrac{dq}{f_y+q\,f_z}=\dfrac{dz}{-p\,f_p-q\,f_q}=\dfrac{dx}{-f_p}=\dfrac{dy}{-f_q} \quad \text{... (2)}$

From (1), $\qquad f_x = p^2(2x-1)+2pqy-2pz, \qquad f_y = 2pqx+q^2(2y-1)-2qz,$

$f_z = -2px-2qy+2z, \qquad f_p = 2px(x-1)+2qxy-2xz; \qquad f_q = 2pxy+2qy(y-1)-2yz$

and so $\quad f_x + p\,f_z = -p^2, \qquad f_y + q\,f_z = -q^2$. Then (2) becomes

$$\frac{dp}{-p^2}=\frac{dq}{-q^2}=\frac{dz}{-p\{2px(x-1)+2qxy-2xz\}-q\{2pxy+2qy(y-1)-2yz\}}$$

$$=\frac{dx}{-(2px^2-2px+2qxy-2xz)}=\frac{dy}{-(2pxy+2qy^2-2qy-2yz)} \qquad \text{... (3)}$$

$$\text{Each fraction of (3)} = \frac{(1/p)\,dp}{-p}=\frac{(1/q)\;dq}{-q}=\frac{(1/p)\,dp-(1/q)dq}{-p+q} \qquad \text{... (4)}$$

Also, each fraction of (3) = $\dfrac{(1/x)\;dx-(1/y)\;dy}{-2px+2p-2qy+2z+2px+2qy-2q-2z} \qquad \text{... (5)}$

$\therefore \qquad$ (4) and (5) $\Rightarrow \dfrac{(1/p)\,dp-(1/q)\;dq}{-(p-q)}=\dfrac{(1/x)\;dx-(1/y)\,dy}{2(p-q)}$

or $\qquad (1/2)\times\{(1/x)\;dx-(1/y)dy\}=(1/q)\;dq-(1/p)\;dp$

Integrating, $\qquad (1/2)\times\{\log x-\log y\}=\log q-\log p+\log a \qquad$ or $\qquad (x/y)^{1/2}=aq/p$

or $\qquad p=(ay^{1/2}q)/x^{1/2}$, a being an arbitrary constant. $\qquad \text{... (5)}$

Re–writing (1), $(px+qy-z)^2 = p^2x+q^2y \qquad$ or $\qquad px+qy-z=\pm(p^2x+qy^2)^{1/2} \qquad \text{... (6)}$

Taking + ive sign in (7), $\qquad px+qy-z=(p^2x+q^2y)^{1/2} \qquad \text{... (7)}$

[The case of – ve sign in (7) can be discussed similarly]

Substituting the value of p given by (6) in (8), $\qquad aqy^{1/2}x^{1/2}+qy-z=(a^2q^2y+q^2y)^{1/2}$

or $\quad q\{y+a(xy)^{1/2}-(1+a^2)^{1/2}y^{1/2}\}=z \quad$ so that $\quad q=z/y^{1/2}\{y^{1/2}+a\;x^{1/2}-(1+a^2)^{1/2}\} \qquad \text{... (9)}$

Then (6) gives $\qquad p=az/x^{1/2}\;\{y^{1/2}+ax^{1/2}-(1+a^2)^{1/2}\} \qquad \text{... (10)}$

Putting these values of p and q in $dz = p\,dx+q\,dy$, we get

$$dz=\frac{az\;dx}{x^{1/2}\{y^{1/2}+ax^{1/2}-(1+a^2)^{1/2}\}}+\frac{z\;dy}{y^{1/2}\{y^{1/2}+ax^{1/2}-(1+a^2)^{1/2}\}}$$

or $$\frac{dz}{z} = \frac{ay^{1/2}dx + x^{1/2}dy}{(xy)^{1/2}\ \{y^{1/2} + ax^{1/2} - (1+a^2)^{1/2}\}}$$

Integrating, $$\log z = 2\log\ \{y^{1/2} + ax^{1/2} - (1+a^2)^{1/2}\} + \log b$$

or $z = b\ \{y^{1/2} + ax^{1/2} - (1+a^2)^{1/2}\}^2$, a and b being an arbitrary constants.

Ex. 34. *Find the complete integral of* $(p+q)\ (px+qy) = 1$.

[Meerut 2007; Delhi Maths (H) 2007, Purvanchal 2007]

Sol. Let $$f(x, y, z, p, q) = (p+q)\ (px+qy) - 1 = 0 \qquad \text{... (1)}$$

Charpit's auxiliary equations $$\frac{dp}{f_x + p\ f_z} = \frac{dq}{f_y + q\ f_z} = \frac{dz}{-p\,f_p - q\ f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

give $\dfrac{dp}{p(p+q)} = \dfrac{dq}{q(p+q)} = \ldots$ so that $\dfrac{dp}{p} = \dfrac{dq}{q}$, using (1)

Integrating, $p = aq$, a being an arbitrary constant ... (2)

Putting $p = aq$ in (2) gives $(aq+q)\ (aqx+qy) - 1 = 0$ or $q^2\ (1 + a)\ (ax + y) = 1$ ... (3)

$\therefore$ From (2) and (3), $q = 1/(1+a)^{1/2}\ (ax+y)^{1/2}$, $p = a/(1+a)^{1/2}(ax+y)^{1/2}$

Putting these values of p and q in $dz = p\,dx + q\,dy$, we get

$$dz = \frac{a\ dx}{(1+a)^{1/2}(ax+y)^{1/2}} + \frac{dy}{(1+a)^{1/2}(ax+y)^{1/2}} = \frac{d\,(ax+y)}{(1+a)^{1/2}(ax+y)^{1/2}}$$

Integrating, $z(1+a)^{1/2} = 2(ax+y)^{1/2} + b$, a, b being arbitrary constants.

Ex. 35. *Find the complete integral of the following partial differential equations*

(a) $px^5 - 4q^2x^2 + 6x^2z - 2 = 0$. **[Delhi B.Sc. (H) 2002; Delhi B.A. (Proj) II 2011]**

(b) $px^5 - 4q^3x^2 + 6x^2z - 2 = 0$

Sol. (a) Let $$f(x, y, z, p, q) = px^5 - 4q^2x^2 + 6x^2z - 2 = 0 \qquad \text{... (1)}$$

Charpit's auxiliary equations are $$\frac{dp}{f_x + p\,f_z} = \frac{dq}{f_y + q\,f_y} = \frac{dz}{-p\,f_p - q\,f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

or $$\frac{dp}{5px^4 - 8q^2x + 12xz + 6px^2} = \frac{dq}{6qx^2} = \frac{dz}{-px^5 + 8q^2x^2} = \frac{dx}{-x^5} = \frac{dy}{8qx^2}, \text{ by (1)}$$

Taking the second and the last fractions, $4\,dq = 3dy$

Integrating, $4q = 3y + 3a$ or $q = 3(y+a)/4$ ... (2)

Using (2), (1) gives $$p = \{(9/4) \times (y+a)^2 - 6x^2z + 2\}/x^5 \qquad \text{... (3)}$$

Putting the above values of p and q in $dz = p\,dx + q\,dy$, we get

$$dz = (9/4x^3)\ (y+a)^2\ dx - (6z/x^3)\ dx + (2/x^5)dx + (3/4)\ (y+a)dy$$

or $$(6z/x^3)dx + dz = \{(9/4x^3)\ (y+a)^2 dx + (3/4)\ (y+a)\ dy\} + (2/x^5)dx \qquad \text{... (4)}$$

The total differential equation (4) is always integrable. To solve (4), we first proceed to find the integrating factor of the L.H.S. of (4). Comparing L.H.S. of (4) with $M\,dx + N\,dz$ (here on L.H.S. we have variable x, z in place of usual variables x, y), we have $M = 6z/x^3$ and $N = 1$.

$\dfrac{1}{N}\left(\dfrac{\partial M}{\partial z}-\dfrac{\partial N}{\partial x}\right)=\dfrac{6}{x^3}$, which is function x alone and so I.F. $= e^{\int (6/x^3)dx} = e^{-3/x^2}$.

Multiplying both sides of (4) by I.F. e^{-3/x^2}, we get

$$(6z/x^3)\,e^{-3/x^2}dx + e^{-3/x^2}dz = (3/8)\times\{(6/x^3)\,(y+a)^2\,e^{-3/x^2}dx + 2(y+a)\,e^{-3/x^2}dy\} + (2/x^5)\,e^{-3/x^2}dx$$

or $$d(z\,e^{-3/x^2}) = (3/8)\times d\{(y+a)^2\,e^{-3/x^2}\} + (2/x^5)\times e^{-3/x^2}dx$$

Integrating, $$z\,e^{-3/x^2} = (3/8)\times(y+a)^2\,e^{-3/x^2} + 2\int (1/x^2)\,e^{-3/x^2}\,(1/x^3)dx$$

or $z\,e^{-3/x^2} = (3/8)\times(y+a)^2\,e^{-3/x^2} - (1/9)\times\int u\,e^u du$, putting $(-3/x^2) = u$ so that $(6/x^3)dx = du$

or $$z\,e^{-3/x^2} = (3/8)\times(y+a)^2\,e^{-3/x^2} - (1/9)\times(ue^u - e^u) + b$$

or $$z\,e^{-3/x^2} = (3/8)\times(y+a)^2\,e^{-3/x^2} - (1/9)\times(-3/x^2)e^{-3/x^2} + (1/9)\times e^{-3/x^2} + b$$

or $z = (3/8)\times(y+a)^2 + (1/3x^2) + (1/9) + be^{3/x^2}$, a, b being arbitrary constants.

(*b*) Proceed exacts as in part (*a*) **Ans.** $z = (2/3)\times(y+a)^{3/2} + (1/3x^2) + (1/9) + b\,e^{3/x^2}$

Ex. 36. *Find the complete integral of* $(p+y)^2 + (q+x)^2 = 1$

Sol. Let $$f(x,y,z,p,q) = (p+y)^2 + (q+x)^2 - 1 = 0 \quad \text{... (1)}$$

Charpit's auxiliary equations are $$\frac{dp}{f_x + p\,f_z} = \frac{dq}{f_y + q\,f_z} = \frac{dz}{-pf_p - q\,f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

or $$\frac{dp}{2(q+x)} = \frac{dq}{2(p+y)} = \frac{dz}{-2(p^2+q^2+py+qx)} = \frac{dx}{-2(p+y)} = \frac{dy}{-2(q+x)}, \text{ by (1)}$$

Taking the first and the last fractions, $dp + dy = 0$ so that $p + y = a$ ... (2)

Using (2), (1) gives $a^2 + (q+x)^2 - 1 = 0$ or $q + x = (1-a^2)^{1/2}$ ... (3)

Using (2) and (3) in $dz = p\,dx + q\,dy$, we get

$$dz = (a-y)dx + \{(1-a^2)^{1/2} - x\}dy = a\,dx - (1-a^2)^{1/2}dy - (y\,dx + x\,dy)$$

Integrating, $z = ax - (1-a^2)^{1/2}y - xy + b$, a, b being arbitrary constants.

Ex. 37. *Find the complete integral of* $2\,(y + zq) = q(xp + yq)$ **[Nagpur 2003, 06; Delhi B.Sc. (Prog) II 2011; Delhi B.Sc. (Hons) 2011]**

Sol. Let $$f(x, y, z, p, q) = 2y + 2zq - xpq - yq^2 = 0 \quad \text{... (1)}$$

Charpit's auxiliary equations are $$\frac{dp}{f_x + p\,f_z} = \frac{dq}{f_y + q\,f_z} = \frac{dz}{-p\,f_x - q\,f_y} = \frac{dx}{-f_p} = \frac{dy}{-f_q} \quad \text{... (2)}$$

$$\frac{dp}{-pq+2pq}=\frac{dq}{2-q^2+2q^2}=\frac{dz}{2pqx+2qy-2qz}=\frac{dx}{qx}=\frac{dy}{xp+2yq-2z}, \text{ by (1)}$$

Taking the first and fourth fractions, $(1/pq)dp = (1/qx)dx$ or $(1/p)dp = (1/x)dx$

Integrating, $\log p = \log a + \log x$ or $p = ax$, ... (3)

where a is an arbitrary constant. Substituting the value of p given by (3) in (1), we have

$2y + 2zq - ax^2q - yq^2 = 0$ or $yq^2 + q(ax^2 - 2z) - 2y = 0.$

$$\Rightarrow \qquad q=[-(ax^2-2z)\pm\{(ax^2-2z)^2+8y^2\}^{1/2}]/(2y) \qquad \text{... (4)}$$

Substituting the values of p and q given by (3) and (4) in $dz = p\,dx + q\,dy$, we obtain

$$dz = ax\,dx+(1/2y)\times[2z-ax^2\pm\{(2z-ax^2)^2+8y^2\}^{1/2}]dy$$

or
$$\frac{2dz-2ax\,dx}{(2z-ax^2)\pm\{(2z-ax^2)^2+8y^2\}^{1/2}}=\frac{dy}{y} \qquad \text{... (5)}$$

Putting $2z - ax^2 = u$ and $2dz - 2\,ax\,dx = du$, (5) yields

$$\frac{du}{u\pm(u^2+8y^2)^{1/2}}=\frac{dy}{y} \qquad \text{or} \qquad \frac{du}{dy}=\frac{u}{y}\pm\left\{\left(\frac{u}{y}\right)^2+8\right\}^{1/2}, \qquad \text{... (6)}$$

which is linear homogeneous differential equation. To solve it, we put $u/y = v$, *i.e.*, $u = yv$ so that $du/dy = v + y(dv/dy)$ and so (6) reduces to

$$v+y\frac{dv}{dy}=v\pm(v^2+8)^{1/2} \qquad \text{or} \qquad \frac{dv}{(v^2+8)^{1/2}}=\frac{dy}{y},$$

taking positive sign. Integrating it, we have

$$\log\{v+(v^2+8)^{1/2}\}=\log y+\log b \qquad \text{or} \qquad v+(v^2+8)^{1/2}=by$$

or $\quad u/y+\left\{(u/y)^2+8\right\}^{1/2}=by \qquad$ or $\qquad u+(u^2+8y^2)^{1/2}=by^2$

or $\quad 2z-ax^2+\{(2z-ax^2)^2+8y^2\}^{1/2}=by^2$, as $u = 2z - ax^2$; a, b being arbitrary constants

EXERCISE 3(B)

Using Charpit's method, find a complete integral of the following equations :

1. $z = px + qy + pq$. **[Mysore 2004]** **Ans.** $z = ax + by + ab$
2. $pq = xz$. **Ans.** $z = (a + x^2/2)(b + y)$
3. $p^2 + px + q = z$. **Ans.** $z = ax + a^2 + be^y$
4. $(p + q)(z - px - qy) = 1$. **Ans.** $(a + b)(z - ax - by) = 1$
5. $px + qy + pq = 0$ **Ans.** $az = -(1/2) \times (y + ax)^2 + b$
6. $q = px + q^2$ **Ans.** $z = (a - a^2)\log x + ay + b$
7. $p - 3x^2 = q^2 - y$ **Ans.** $z = x^3 - (1/3) \times (a - x)^3 + ay - xy + b$
8. $x^2p^2 + y^2q^2 = 4$ **Ans.** $z = a\log x + (4 - a^2)^{1/2}\log y + b$
9. $xpq + yq^2 = 1$ **Ans.** $(z + b)^2 = 4(ax + b)$
10. $p + q = 3pq$ **Ans.** $az = b - (1/2) \times (y + ax)^2$
11. $pq + x(2y + 1)p + (y^2 + y)q - (2y + 1)z = 0$ **Ans.** $z = ax + b(a + y + y^2)$
12. $z^2(p^2 + q^2) = x^2 + e^{2y}$.

$$\textbf{Ans.} \quad \frac{z^2}{2}=\frac{x\sqrt{(x^2+a)}}{2}+\frac{a}{2}\sinh^{-1}\frac{x}{\sqrt{a}}+\sqrt{(e^{2y}-a)}-\sqrt{a}\tan^{-1}\left(\frac{e^{2y}-a}{a}\right)+b$$

13. $p^2 - y^2 q = x^2 - y^2$ **[Madurai Kamraj 2008]**

Ans. $z = \dfrac{x\sqrt{(x^2+b)}}{2} + \dfrac{b}{2}\sinh^{-1}\dfrac{x}{\sqrt{b}} - \dfrac{b}{2y^2} + \log y + c$

14. $p^2 q\,(x^2 + y^2) = p^2 + q$ **Ans.** $z = \log[x + \sqrt{(x^2+a)}] + \dfrac{1}{2\sqrt{a}}\log\dfrac{y-\sqrt{a}}{y+\sqrt{a}} + b$

15. $yp = 2xy + \log q$. **[Lucknow 2010]** **[Ans.** $z = (a+2x)^2/4 + (1/a)\times e^{ay} + b$]

3.9. Special methods of solutions applicable to certain standard forms:

We now consider equations in which p and q occur other than in the first degree, that is non–linear equations. We have already discussed the general method (*i.e.*, Charpit's method — see Art. 3.7). We now discuss four standard forms to which many equations can be reduced, and for which a complete integral can be obtained by inspection or by other shorter methods.

3.10. Standard Form I. Only p and q present. **[Nagpur 2002; Bhopal 2010]**

Under this standard form, we consider equations of the form $f(p, q) = 0.$ ...(1)

Charpit's auxiliary equations are $\dfrac{dp}{f_x + p f_z} = \dfrac{dq}{f_y + q f_z} = \dfrac{dz}{-p f_p - q f_q} = \dfrac{dx}{-f_p} = \dfrac{dy}{-f_q}$

giving $\dfrac{dp}{0} = \dfrac{dq}{0}$, by (1)

Taking the first ratio, $dp = 0$ so that p = constant = a, say ...(2)

Substituting in (1), we get $f(a, q) = 0$, giving q = constant = b, say, ... (3)

where b is such that $f(a, b) = 0.$ ...(4)

Then, $dz = p\,dx + q\,dy = a dx + b dy$, using (2) and (3).

Integrating, $z = ax + by + c,$ ...(5)

where c is an arbitrary constant. (5) together with (4) give the required solution.

Now solving (4) for b, suppose we obtain $b = F(a)$, say.

Putting this value of b in (5), the *complete integral* of (1) is

$$z = ax + yF(a) + c, \qquad ...(6)$$

which contains two arbitrary constants a and c which are equal to the number of independent variables, namely x and y.

The *singular integral* of (1) is obtained by eliminating a and c between the complete integral (6) and the equations obtained by differentiating (6) partially w.r.t. a and c ; *i.e.*, between

$z = ax + yF(a) + c,$ $0 = x + yF'(a)$ and $0 = 1.$ ...(7)

Since the last equation in (7) is meaningless, we conclude that the equations of standard form I have no singular solution.

In order to find the general integral of (1), we first take $c = \phi(a)$ in (6), ϕ being an arbitrary function and obtain $z = ax + yF(a) + \phi(a).$ ...(8)

Now, we differentiate (8) partially with respect to a and get

$$0 = x + yF'(a) + \phi'(a). \qquad ....(9)$$

Eliminating a between (8) and (9), we get the general solution of (1).

Remark. Sometimes change of variables can be employed to transform a given equation to standard form I.

3.11. SOLVED EXAMPLES BASED ON ART. 3.10

Ex. 1. (a) *Solve $pq = k$, where k is a constant.* **[M.S. Univ. T.N. 2007; Meerut 1995]**

(b) Solve pq = 1 by standard from I **[Bhopal 2010]**

Sol. Given that $pq = k$. ...(1)

Since (1) is of the form $f(p, q) = 0$, its solution is $z = ax + by + c$, ...(2)

where $ab = k$ or $b = k/a$, on putting a for p and b for q in (1).

$\therefore$ From (2), the complete integral is $z = ax + (k/a)y + c$, ...(3)

which contains two arbitrary constants a and c.

For singular solution, differentiating (3) partially with respect to a and c, we get $0 = x - (k/a^2)y$ and $0 = 1$. But $0 = 1$ is absurd. Hence there is no singular solution of (1).

To find the general solution, put $c = \phi(a)$ in (3). Then, we get

$$z = ax + (k/a)y + \phi(a). \qquad \text{...(4)}$$

Differentiating (4) partially with respect to 'a', we get $0 = x - (k/a^2)y + \phi'(a)$. ...(5)

Eliminating a from (4) and (5), we get the required general solution.

(b) Do like part (a) taking $k = 1$

Ex. 2. *Solve (a) $p^2 + q^2 = m^2$, where m is a constant.* **[Kanpur 1993]**

(b) $p^2 + q^2 = 1$ **[Meerut 2011]**

Sol. (a) Given that $p^2 + q^2 = m^2$. ...(1)

Since (1) is of the form $f(b, q) = 0$, its solution is $z = ax + by + c$, ...(2)

where $a^2 + b^2 = m^2$ or $b = (m^2 - a^2)^{1/2}$, on putting a for p and b for b in (1).

$\therefore$ From (2), the complete integral is $z = ax + y(m^2 - a^2)^{½} + c$, ...(3)

which contains two arbitrary constants a and c.

For singular solution, differentiating (3) partially with respect to a and c, we get $0 = x - ay/(m^2 - a^2)^{1/2}$ and $0 = 1$. But $0 = 1$ is absurd. Hence there is no singular solution of (1).

To find the general solution, put $c = \phi(a)$ in (3). Then, we get

$$z = ax + y(m^2 - a^2)^{1/2} + \phi(a). \qquad \text{...(4)}$$

Differentiating (4) partially wit respect to 'a', we get

$$0 = a - ay/(m^2 - a^2)^{1/2} + \phi'(a). \qquad \text{...(5)}$$

Eliminating a from (4) and (5), we get the required general solution.

(b) **Hint.** Do like part (a) by taking $m = 1$

EQUATIONS REDUCIBLE TO STANDARD FORM I

Ex. 3. *Find the complete integral of $z^2p^2y + 6zpxy + 2zqx^2 + 4x^2y = 0$.*

Sol. The given equation can be rewritten as

$$z^2y\,(\partial z/\partial x)^2 + 6zxy(\partial z/\partial x) + 2zx^2(\partial z/\partial y) + 4x^2y = 0$$

or $$\left(\frac{z}{x}\frac{\partial z}{\partial x}\right)^2 + 6\left(\frac{z}{x}\frac{\partial z}{\partial x}\right) + 2\left(\frac{z}{y}\frac{\partial z}{\partial y}\right) + 4 = 0, \text{ dividing by } x^2y \qquad \text{...(1)}$$

Put $x\,dx = dX$, $y\,dy = dY$ and $z\,dz = dZ$. ...(2)

so that $x^2/2 = X$, $y^2/2 = Y$ and $z^2/2 = Z$. ...(3)

Using (2), (1) becomes $(\partial Z/\partial X)^2 + 6\,(\partial Z/\partial X) + 2(\partial Z/dY) + 4 = 0$

or $P^2 + 6P + 2Q + 4 = 0$, where $P = \partial Z/\partial X$, $Q = \partial Z/\partial Y$. ...(4)

Equation (4) is of the form $f(P, Q) = 0$. Note that now we have P, Q, X, Y, Z in place of p, q, x, y, z in usual equations. Accordingly, solution of (4) is

$$Z = aX + bY + c, \qquad \text{...(5)}$$

where $a^2 + 6a + 2b + 4 = 0$ or $b = -(a^2 + 6a + 4)/2$, on putting a for P and b for Q in (4). So, from (5), the required complete integral is

$$Z = aX - \{(a^2 + 6a + 4)/2\}Y + c, \text{ where } a \text{ and } c \text{ are arbitrary constants.}$$

or $$z^2/2 = a(x^2/2) - (a^2 + 6a + 4) \times (y^2/4) + c, \text{ using (3)}$$

or $$z^2 = ax^2 - (2 + 3a + a^2/2)y^2 + c', \text{ where } c' = 2c.$$

Ex. 4. *Find the complete integral of*

(*i*) $x^2p^2 + y^2q^2 = z$ **[Delhi Maths (H) 2004]**

(*ii*) $p^2x + q^2y = z.$ **[Meerut 1994]**

Sol. (*i*) The given equation can be rewritten as

$$\frac{x^2}{z}\left(\frac{\partial z}{\partial x}\right)^2 + \frac{y^2}{z}\left(\frac{\partial z}{\partial y}\right)^2 = 1 \quad \text{or} \quad \left(\frac{x\,\partial z}{\sqrt{z}\,\partial x}\right)^2 + \left(\frac{y\,\partial z}{\sqrt{z}\,\partial y}\right)^2 = 1. \qquad \text{...(1)}$$

Put $(1/x)dx = dX$, $(1/y)dy = dY$ and $(1/\sqrt{z})dz = dZ$ ...(2)

so that $\log x = X$, $\log y = Y$ and $2\sqrt{z} = Z.$ ...(3)

Using (2), (1) becomes $(\partial Z/\partial X)^2 + (\partial Z/\partial Y)^2 = 1$ or $P^2 + Q^2 = 1,$ ...(4)

where $P = \partial Z/\partial X$ and $Q = \partial Z/\partial Y$. (4) is of the form $f(P,Q) = 0$.

$\therefore$ solution of (4) is $Z = aX + bY + c,$ ...(5)

where $a^2 + b^2 = 1$ or $b = \sqrt{1-a^2}$, on putting a for P and b for Q in (4).

$\therefore$ from (5), the required complete integral is

$$Z = aX + Y\sqrt{1-a^2} + c \quad \text{or} \quad 2\sqrt{z} = a \log x + \log y \,.\, \sqrt{1-a^2} + c, \text{ by (3)}$$

or $$\log x^a + \log y^{\sqrt{1-a^2}} - \log c' = 2\sqrt{z}, \text{ taking } c = -\log c'$$

or $$\log \{x^a y^{\sqrt{1-a^2}}/c'\} = 2\sqrt{z} \quad \text{or} \quad x^a\, y^{\sqrt{1-a^2}} = c' e^{2\sqrt{z}}$$

where a and c' are two arbitrary constants.

(*ii*) The given equation can be re–written as

$$\frac{x}{z}\left(\frac{\partial z}{\partial x}\right)^2 + \frac{y}{z}\left(\frac{\partial z}{\partial y}\right)^2 = 1 \quad \text{or} \quad \left(\frac{\sqrt{x}}{\sqrt{z}}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\sqrt{y}}{\sqrt{z}}\frac{\partial z}{\partial y}\right)^2 = 1. \qquad \text{...(1)}$$

Put $\left(1/\sqrt{x}\right)dx = dX$, $\left(1/\sqrt{y}\right)dy = dY$ and $\left(1/\sqrt{z}\right)dz = dZ$ ...(2)

so that $2\sqrt{x} = X$, $2\sqrt{y} = Y$ and $2\sqrt{z} = Z.$ ...(3)

Using (2), (1) becomes $(\partial Z/\partial X)^2 + (\partial Z/\partial Y)^2 = 1$ or $P^2 + Q^2 = 1,$ ...(4)

where $P = \partial Z/\partial X$ and $Q = \partial Z/\partial Y$. (4) is of the form $f(P, Q) = 0$.

$\therefore$ solution of (4) is $z = aX + bY + c,$ ...(5)

where $a^2 + b^2 = 1$ or $b = \sqrt{1-a^2}$, putting a for P and b for Q in (4).

$\therefore$ from (5), the required complete integral is

$$Z = aX + Y\sqrt{1-a^2} + c \quad \text{or} \quad 2\sqrt{z} = 2a\sqrt{x} + 2\sqrt{y}\sqrt{1-a^2} + c, \text{ by (3)}$$

where a and c are two arbitrary constants.

Ex. 5. *Solve* $x^2p^2 + y^2q^2 = z^2$. **[Jabalpur 2000, 03; Gulbarga 2005; Bilaspur 1997; Meerut 2008, Sagar 2004, Vikram 1999; Ravi Shanker 1994, 96; Rohitkhand 2004]**

Sol. The given equation can be rewritten as

$$\frac{x^2}{z^2}\left(\frac{\partial z}{\partial x}\right)^2 + \frac{y^2}{z^2}\left(\frac{\partial z}{\partial y}\right)^2 = 1 \qquad \text{or} \qquad \left(\frac{x\,\partial z}{z\,\partial x}\right)^2 + \left(\frac{y\,\partial z}{z\,\partial y}\right)^2 = 1. \qquad \text{...(1)}$$

Put $(1/x)dx = dX$, $(1/y)dy = dY$ and $(1/z) = dZ$ ...(2)

so that $\log x = X$, $\log y = Y$ and $\log z = Z$. ...(3)

Using (2), (1) becomes $(dZ/dX)^2 + (dZ/dY)^2 = 1$ or $P^2 + Q^2 = 1$, ... (4)

where $P = dZ/dX$ and $Q = dZ/dy$. (4) is of the form $f(P, Q) = 0$.

$\therefore$ solution of (4) is $Z = aX + bY + c$, ... (5)

where $a^2 + b^2 = 1$ or $b = \sqrt{1-a^2}$, on putting a for P and b for Q in (4).

$\therefore$ from (5), the required complete integral is

$Z = aX + Y\sqrt{1-a^2} + c$ or $Z = X \cos \alpha + Y \sin \alpha + \log c'$, taking $a = \cos \alpha$ and $c = \log c'$

or $\log z = \cos \alpha \log x + \sin \alpha \log y + \log c'$ or $z = c' x^{\cos \alpha} y^{\sin \alpha}$. ...(6)

To determine singular integral. Differentiating (6) partially w.r.t. α and c' successively, we obtain $0 = c' \cos \alpha \,.\, x^{\cos \alpha} y^{\sin\alpha} \log y - c' \sin \alpha \,.\, x^{\cos \alpha} y^{\sin \alpha} \log x$ ...(7)

and $0 = x^{\cos \alpha} y^{\sin \alpha}$. ...(8)

Eliminating α and c' from (6), (7) and (8), the singlar solution is $z = 0$.

To determine general integral. Putting $c' = \phi(\alpha)$, where ϕ is an arbitrary function, (4) gives

$$z = \phi(\alpha) \sin \alpha\, x^{\cos \alpha} y^{\sin \alpha}. \qquad \text{...(9)}$$

Differentiating (9), partially, w.r.t. 'α', we get

$$0 = \phi'(\alpha)\, x^{\cos \alpha} y^{\sin \alpha} + \phi(\alpha)\, \{x^{\cos \alpha} y^{\sin \alpha} \cos \alpha - y^{\sin \alpha} x^{\cos \alpha} \sin \alpha\}. \qquad \text{...(10)}$$

The required general integral is obtained by eliminating α from (9) and (10).

Ex. 6. *Find a complete integral of* (*i*) $pq = x^m y^n z^{2l}$ **[Delhi B.Sc. (Prog) II 2007]**

(*ii*) $pq = x^m y^n z^l$ **[I.A.S. 1989, 94]**

Sol. (*i*) The given equation can be rewritten as

$$\frac{z^{-l}z^{-l}}{x^m y^n}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = 1 \qquad \text{or} \qquad \left(\frac{z^{-l}}{x^m}\frac{\partial z}{\partial x}\right)\left(\frac{z^{-l}}{y^n}\frac{\partial z}{\partial y}\right) = 1. \qquad \text{...(1)}$$

Put $x^m dx = dX$, $y^n dy = dY$ and $z^{-l} dz = dZ$ ...(2)

so that $\dfrac{x^{m+1}}{m+1} = X$, $\dfrac{y^{n+1}}{n+1} = Y$ and $\dfrac{z^{1-l}}{1-l} = Z$. ...(3)

Using (2), (1) becomes $(\partial Z/\partial X)(\partial Z/\partial Y) = 1$ or $PQ = 1$, ...(4)

where $P = \partial Z/\partial X$ and $Q = \partial Z/\partial Y$. (4) is of the form $f(P, Q) = 0$.

$\therefore$ Solution of (4) is $z = aX + bY + c$, ... (5)

where $ab = 1$ or $b = 1/a$, on putting a from P and b for Q in (4).

$\therefore$ from (5), the required complete integral is

$$Z = aX + (1/a)Y + c \qquad \text{or} \qquad \frac{z^{1-l}}{1-l} = a\frac{x^{m+1}}{m+1} + \frac{y^{n+1}}{a(n+1)} + c, \text{ using (3)}$$

where a and c are arbitrary constants.

(ii) The given equation can be rewritten as

$$\frac{z^{-l/2}z^{-l/2}}{x^m y^n}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = 1 \quad \text{or} \quad \left(\frac{z^{-l/2}}{x^m}\frac{\partial z}{\partial x}\right)\left(\frac{z^{-l/2}}{y^n}\frac{\partial z}{\partial y}\right) = 1. \qquad ...(1)$$

Put $x^m dx = dX$, $\quad y^n dy = dY \quad$ and $\quad z^{-l/2}dz = dZ$ ...(2)

so that $$\frac{x^{m+1}}{m+1} = X, \quad \frac{y^{n+1}}{n+1} = Y \quad \text{and} \quad \frac{z^{1-(l/2)}}{1-(l/2)} = Z. \qquad ...(3)$$

Using (2), (1) becomes $\quad (\partial Z/\partial X)(\partial Z/\partial Y) = 1 \quad$ or $\quad PQ = 1$, ...(4)

where $P = \partial Z/\partial X$ and $Q = \partial Z/\partial Y$. (4) is of the form $f(P, Q) = 0$.

$\therefore$ Solution of (4) is $\quad z = aX + bY + c$, ... (5)

where $\quad ab = 1 \quad$ or $\quad b = 1/a$, on putting a for P and b for Q in (4).

$\therefore$ from (5), the required complete integral is

$$Z = aX + (1/a)Y + c \quad \text{or} \quad \frac{z^{1-(l/2)}}{1-(l/2)} = a\frac{x^{m+1}}{m+1} + \frac{y^{n+1}}{a(n+1)} + c, \text{ using (3)}$$

where a and c are arbitrary constants.

Ex. 7. *Find complete integral of* $p^m \sec^{2m} x + z^l q^n \operatorname{cosec}^{2n} y = z^{lm/(m-n)}$

Sol. The given equation can be re–written as

$$\frac{1}{z^{lm/(m-n)}}\left(\frac{1}{\cos^2 x}\frac{\partial z}{dx}\right)^m + \frac{z^l}{z^{lm/(m-n)}}\left(\frac{1}{\sin^2 y}\frac{\partial z}{\partial y}\right)^n = 1 \text{ or } \left(\frac{z^{-l/(m-n)}}{\cos^2 x}\frac{\partial z}{\partial x}\right)^m + \left(\frac{z^{-l/(m-n)}}{\sin^2 y}\frac{\partial z}{\partial y}\right)^n = 1. \quad ...(1)$$

Put $\quad \cos^2 x\, dx = dX$, $\quad \sin^2 y\, dy = dY \quad$ and $\quad z^{-l/(m-n)}dz = dZ$ ...(2)

i.e., $\{(1 + \cos 2x)/2\}dx = dX$, $\quad \{(1 - \cos 2y)/2\}dy = dY \quad$ and $\quad z^{-l/(m-n)}\, dz = dZ$

so that $$\frac{1}{2}(x + \frac{1}{2}\sin 2x) = X, \quad \frac{1}{2}(y - \frac{1}{2}\sin 2y) = Y \quad \text{and} \quad \frac{(m-n)z^{(m-n-l)/(m-n)}}{m-n-l} = Z. \qquad ...(3)$$

Using (2), (1) becomes $\quad (\partial Z/\partial X)^m + (\partial Z/\partial Y)^n = 1 \quad$ or $\quad P^m + Q^n = 1$, ...(4)

where $P = \partial Z/dX$ and $Q = \partial Z/\partial Y$. (4) is of the form $f(P, Q) = 0$.

$\therefore$ Solution of (4) is $\quad Z = aX + bY + c$, ... (5)

where $\quad a^m + b^n = 1 \quad$ or $\quad b = (1 - a^m)^{1/n}$, on putting a for P and b for Q is (4).

$\therefore$ from (5), the required complete integral is

$Z = aX + (1 - a^m)^{1/n} Y + c$, $\;a$ and c being two arbitrary constants.

or $$\frac{m-n}{m-n-l}z^{(m-n-l)/(m-n)} = \frac{a}{4}(2x + \sin 2x) + \frac{(1-a^m)^{1/n}}{4}(2y - \sin 2y) + c, \text{ by (3).}$$

Ex. 8. *Find the complete integral of* $(1 - x^2)\, yp^2 + x^2 q = 0$.

Sol. The given equation can be rewritten as

$$\frac{1-x^2}{x^2}\left(\frac{\partial z}{\partial x}\right)^2 + \frac{1}{y}\frac{\partial z}{\partial y} = 0 \quad \text{or} \quad \left(\frac{(1-x^2)^{1/2}}{x}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{1}{y}\frac{\partial z}{\partial y}\right) = 0. \qquad ...(1)$$

Put $\quad \{x/(1 - x^2)^{1/2}\}dx = dX \quad$ and $\quad y\, dy = dY$ ...(2)

so that $$X = \int \frac{x\,dx}{(1-x^2)^{1/2}} = -\frac{1}{2}\int (1-x^2)^{-1/2}(-2x)dx = -(1-x^2)^{1/2} \quad \text{and} \quad Y = \frac{y^2}{2} \qquad ...(3)$$

Using (2), (1) becomes $\quad (\partial z/\partial X)^2 + (\partial z/\partial Y) = 0 \quad$ or $\quad P^2 + Q = 0$, ...(4)

where $\quad P = \partial z/\partial X$ and $Q = \partial z/\partial Y$. Note carefully that here the old variable z remains unchanged even after transformation (2). Here (4) is of the form $f(P, Q) = 0$.

$\therefore$ Solution of (4) is $$z = aX + bY + c, \qquad \text{... (5)}$$

where $a^2 + b = 0$ or $b = -a^2$, on putting a for P and b for Q in (4),

$\therefore$ from (5), the required complete integral is

$z = aX - a^2Y + c$ or $z = -a(1 - x^2)^{1/2} - (a^2 y^2)/2 + c$, by (3).

Ex. 9. *Find the complete integral of* $(y - x)(qy - px) = (p - q)^2$. **[Delhi Maths (H) 2005; Ravishankar 2010; Meerut 1995, 97; Agra 1999; Kanpur 2001, 04, 07, 08]**

Sol. Let X and Y be two new variables such that

$$X = x + y \quad \text{and} \quad Y = xy. \qquad \text{...(1)}$$

Given equation is $$(y - x)(qy - px) = (p - q)^2. \qquad \text{...(2)}$$

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial x} = \frac{\partial z}{\partial X} + y\frac{\partial z}{\partial Y} \qquad \text{...(3)}$$

[$\because$ from (1), $\partial X/\partial x = 1$ and $\partial Y/\partial x = y$]

and $$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial y} + \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial y} = \frac{\partial z}{\partial X} + x\frac{\partial z}{\partial Y}. \qquad \text{...(4)}$$

[$\because$ from (1), $\partial X/\partial y = 1$ and $\partial Y/\partial y = x$]

Substituting the above values of p and q in (2), we have

$$(y - x)\left[y\left(\frac{\partial z}{\partial X} + x\frac{\partial z}{\partial Y}\right) - x\left(\frac{\partial z}{\partial X} + y\frac{\partial z}{\partial Y}\right)\right] = \left[\left(\frac{\partial z}{\partial X} + y\frac{\partial z}{\partial Y}\right) - \left(\frac{\partial z}{\partial X} + x\frac{\partial z}{\partial Y}\right)\right]^2$$

or $$(y - x)^2\frac{\partial z}{\partial X} = (y - x)^2\left(\frac{\partial z}{\partial Y}\right)^2 \quad \text{or} \quad \frac{\partial z}{\partial X} = \left(\frac{\partial z}{\partial Y}\right)^2 \quad \text{or} \quad P = Q^2, \qquad \text{...(5)}$$

where $P = \partial z/\partial X$ and $Q = \partial z/\partial Y$. (4) is of the form $f(P, Q) = 0$.

$\therefore$ Solution of (4) is $$z = aX + bY + c, \qquad \text{...(6)}$$

where $a = b^2$, on putting a for P and b for Q in (5).

$\therefore$ from (6), the required complete integral is

$z = b^2X + bY + c$ or $z = b^2(x + y) + bxy + c$, by (1).

Ex. 10. *Find the complete integral of* $(x + y)(p + q)^2 + (x - y)(p - q)^2 = 1$. **[I.A.S. 1991; Kanpur 2006; Meerut 1997]**

Sol. Let X and Y be two new variables such that

$$X^2 = x + y \quad \text{and} \quad Y^2 = x - y. \qquad \text{...(1)}$$

Given equation is $$(x + y)(p + q)^2 + (x - y)(p - q)^2 = 1. \qquad \text{...(2)}$$

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial x} + \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial x} = \frac{1}{2X}\frac{\partial z}{\partial X} + \frac{1}{2Y}\frac{\partial z}{\partial Y} \qquad \text{...(3)}$$

[$\because$ from (1), $\partial X/\partial x = 1/2X$ and $\partial Y/\partial x = 1/2Y$]

and $$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial X}\frac{\partial X}{\partial y} + \frac{\partial z}{\partial Y}\frac{\partial Y}{\partial y} = \frac{1}{2X}\frac{\partial z}{\partial X} - \frac{1}{2Y}\frac{\partial z}{\partial Y}. \qquad \text{...(4)}$$

[$\because$ from (1), $\partial X/\partial y = 1/2X$ and $\partial Y/\partial y = -1/2Y$]

(3) and (4) $\Rightarrow$ $$p + q = \frac{1}{X}\frac{\partial z}{\partial X} \quad \text{and} \quad p - q = \frac{1}{Y}\frac{\partial z}{\partial Y}. \qquad \text{...(5)}$$

Using (1) and (5), (2) reduces to

$$X^2 \times \frac{1}{X^2}\left(\frac{\partial z}{\partial X}\right)^2 + Y^2 \times \frac{1}{Y^2}\left(\frac{\partial z}{\partial Y}\right)^2 = 1 \quad \text{or} \quad P^2 + Q^2 = 1, \qquad \text{...(6)}$$

where $P = \partial z/\partial X$ and $Q = \partial z/\partial Y$. (4) is of the form $f(P, Q) = 0$.

$\therefore$ Solution of (4) is $\qquad z = aX + bY + c,$ ...(7)

where $a^2 + b^2 = 1$ or $b = \sqrt{1-a^2}$, on putting a for P and b for Q in (6).

$\therefore$ from (7), the required complete integral is

$$z = aX + Y\sqrt{1-a^2} + c \qquad \text{or} \qquad z = a\sqrt{x+y} + \sqrt{x-y}\sqrt{1-a^2} + c, \text{ by (1).}$$

Ex. 11. *Find a complete integral of* $(x^2 + y^2)(p^2 + q^2) = 1$.

[Agra 2008; Indore 2004; Vikram 2000; Meerut 1995; Rohitkhand 1994]

Sol. Put $\qquad x = r\cos\theta \qquad$ and $\qquad y = r\sin\theta.$ ...(1)

Then, $\qquad r^2 = x^2 + y^2 \qquad$ and $\qquad \theta = \tan^{-1}(y/x).$ ...(2)

Differentiating (2) partially with respect to x and y, we get

$$2r(\partial r/dx) = 2x \qquad \text{and} \qquad 2r(\partial r/\partial y) = 2y$$

$$\Rightarrow \qquad \frac{\partial r}{\partial x} = \frac{r\cos\theta}{r} = \cos\theta \qquad \text{and} \qquad \frac{\partial r}{\partial y} = \frac{r\sin\theta}{r} = \sin\theta. \quad ...(3)$$

and
$$\frac{\partial\theta}{\partial x} = \frac{1}{1+(y/x)^2}\times\left(-\frac{y}{x^2}\right) = -\frac{y}{x^2+y^2} = -\frac{r\sin\theta}{r^2} = -\frac{\sin\theta}{r} \quad ...(4)$$

$$\frac{\partial\theta}{\partial y} = \frac{1}{1+(y/x)^2}\times\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2} = \frac{r\cos\theta}{r^2} = \frac{\cos\theta}{r} \quad ...(5)$$

Given equation is $\qquad (x^2 + y^2)(p^2 + q^2) = 1.$ ....(6)

Now, $\qquad p = \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial r}\dfrac{\partial r}{\partial x} + \dfrac{\partial z}{\partial\theta}\dfrac{\partial\theta}{\partial x} = \cos\theta\dfrac{\partial z}{\partial r} - \dfrac{\sin\theta}{r}\dfrac{\partial z}{\partial\theta}$, by (3) and (4)

and $\qquad q = \dfrac{\partial z}{\partial y} = \dfrac{\partial z}{\partial r}\dfrac{\partial r}{\partial y} + \dfrac{\partial z}{\partial\theta}\dfrac{\partial\theta}{\partial y} = \sin\theta\dfrac{\partial z}{\partial r} + \dfrac{\cos 0}{r}\dfrac{\partial z}{\partial\theta}$, by (3) and (5).

Hence $\qquad p^2 + q^2 = (\partial z/\partial r)^2 + (1/r^2)\times(\partial z/\partial\theta)^2.$ ...(7)

$\therefore$ (6) becomes $\qquad r^2[(\partial z/\partial r)^2 + (1/r^2)\times(\partial z/\partial\theta)^2] = 1,$ using (2) and (7)

or
$$\left(r\frac{\partial z}{\partial r}\right)^2 + \left(\frac{\partial z}{\partial\theta}\right)^2 = 1. \quad ...(8)$$

Let R be a new variable such that $(1/r)dr = dR$ so that $\log r = R.$ ...(9)

Then (8) becomes $\qquad (\partial z/\partial R)^2 + (\partial z/\partial\theta)^2 = 1 \qquad$ or $\qquad P^2 + Q^2 = 1,$ ...(10)

where $P = \partial z/\partial R$ and $Q = \partial z/\partial\theta$. (10) is of the form $f(P, Q) = 0$.

$\therefore$ solution of (4) is $\qquad z = aR + b\theta + c,$ ...(11)

where $a^2 + b^2 = 1$ or $b = \sqrt{1-a^2}$, on putting a for P and b for Q in (10)

$\therefore$ from (11), the required complete integral is

$$z = aR + \theta\sqrt{1-a^2} + c \qquad \text{or} \qquad z = a\log r + \theta\sqrt{1-a^2} + c,$$

or
$$z = a\log(x^2 + y^2)^{1/2} + \tan^{-1}(y/x)\,.\,\sqrt{1-a^2} + c, \text{ by (2)}$$

or $\quad z = (a/2)\times\log(x^2 + y^2) + \sqrt{1-a^2}\tan^{-1}(y/x) + c$, a and c being arbitrary constants.

Ex. 12. *Find the complete integral of* $z^2 = pqxy$ **[Meerut 2007; Punjab 2005]**

Sol. The given equation can be re–written as

$$\frac{xy}{z^2}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = 1 \qquad \text{or} \qquad \left(\frac{x}{z}\frac{\partial z}{\partial x}\right)\left(\frac{y}{z}\frac{\partial z}{\partial y}\right) = 1 \quad ...(1)$$

Put $(1/x)dx = dX$, $(1/y)\, dy = dY$ and $(1/z)dz = dZ$

so that $\log x = X,$ $\log y = Y$ and $\log z = Z$ ... (2)

Then (1) becomes $(\partial Z/\partial X)\,(\partial Z/\partial Y) = 1$ or $PQ = 1$ ... (3)

where $P = \partial Z/\partial X$ and $Q = \partial Z/\partial Y$. Then, solution of (3) is

$$Z = aX + bY + C', \quad \text{where} \quad ab = 1 \quad \text{so that} \quad b = 1/a.$$

$\therefore$ $\log z = a\log x + (1/a)\log y + \log C$, taking $C' = \log C$ and using (2)

or $$z = x^a\ y^{1/a}C, \text{ a and } C \text{ being arbitrary constants.}$$

Ex. 13. *Find the complete integral of* $(x/p)^n + (y/q)^n = z^n$.

Sol. The given can be re–written as $(x/zp)^n + (y/zq)^n = 1$ ... (1)

Let $X = x^2/2,$ $Y = y^2/2,$ $Z = z^2/2,$ $P = \partial Z/\partial X$ and $Q = \partial Z/\partial Y$ ... (2)

Now, $p = \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial Z}\dfrac{\partial Z}{\partial X}\dfrac{\partial X}{\partial x} = \dfrac{x}{z}P$. Similarly, $q = \dfrac{y}{z}Q$, using (2)

Hence $x/zp = 1/P$ and $y/zq = 1/Q$ and so (1) reduces to $P^{-n} + Q^{-n} = 1$, whose solution is

$Z = aX + bY + C'$, where $a^{-n} + b^{-n} = 1$ so that $b = (1 - a^{-n})^{-1/n}$ ... (3)

$\therefore$ (2) and (3) $\Rightarrow$ $z^2/2 = a(x^2/2) + (1 - a^{-n})^{-1/n}\,(y^2/2) + C/2$, taking $C' = C/2$

or $$z^2 = ax^2 + (1 - a^{-n})^{-1/n} y^2 + C, \ a \text{ and } C \text{ being arbitrary constants.}$$

Ex. 14. *Find the complete integral of* $p^3 \sec^6 x + z^2 q^2 \operatorname{cosec}^4 y = z^6$

Sol. The given equation can be re–written as

$$\frac{z^{-6}}{\cos^6 x}\left(\frac{\partial z}{\partial x}\right)^3 + \frac{z^{-4}}{\sin^4 y}\left(\frac{\partial z}{\partial y}\right)^2 = 1 \quad \text{or} \quad \left(\frac{z^{-2}}{\cos^2 x}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{z^{-2}}{\sin^2 y}\frac{\partial z}{\partial y}\right)^2 = 1 \quad \text{... (1)}$$

Let $\cos^2 x\, dx = dX,$ $\sin^2 y\, dy = dY,$ $z^{-2}dz = dZ$ ... (2)

$\Rightarrow$ $X = (1/2)\displaystyle\int (1 + \cos 2x)dx,$ $Y = (1/2)\displaystyle\int (1 - \cos 2y)dy,$ $Z = -(1/z)$

$\Rightarrow$ $X = (1/2)\,\{x + (1/2)\sin 2x\},$ $y = (1/2)\,\{y - (1/2)\sin 2y\},$ $Z = -(1/z)$

$\Rightarrow$ $X = (1/2)\ (x + \sin x \cos x),$ $y = (1/2)\ (y - \sin y \cos y),$ $Z = -(1/z)$ ... (3)

Using (2), (1) becomes $(\partial Z/\partial X)^2 + (\partial Z/\partial Y)^2 = 1$ or $P^2 + Q^2 = 1$ ... (4)

where $P = \partial Z/\partial X$ and $Q = \partial Z/\partial Y$. Now, solution of (4) is

$Z = aX + bY + C/2$, where $a^2 + b^2 = 1$ so that $b = (1 - a^2)^{1/2}$ ... (5)

$\therefore$ $-(1/z) = a(1/2)\ (x + \sin x \cos x) + (1 - a^2)^{1/2}(1/2)(y - \sin y \cos y) + C/2$, by (3) and (5)

or $$(2/z) + a(x + \sin x \cos x) + (1 - a^2)^{1/2}(y - \sin y \cos y) + C = 0.$$

Ex. 15. *Find the complete integral of* $yp + xq = pq$.

Sol. The given equation can be re–written as

$$\frac{1}{x}\frac{\partial z}{\partial x} + \frac{1}{y}\frac{\partial z}{\partial y} = \frac{1}{xy}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} \qquad \text{or} \qquad \frac{\partial z}{x\partial x} + \frac{\partial z}{y\partial y} = \left(\frac{\partial z}{x\partial x}\right)\left(\frac{\partial z}{y\partial y}\right) \qquad \text{... (1)}$$

Put $\quad x\,dx = dX, \quad y\,dy = dY \quad$ so that $\quad x^2/2 = X, \quad y^2/2 = Y \qquad$ ... (2)

Then (1) becomes $\partial z/\partial X + \partial z/\partial Y = (\partial z/\partial X)\,(\partial z/\partial Y) \quad$ or $\quad P + Q = PQ \qquad$ ... (3)

where $P = \partial z/\partial X$ and $Q = \partial z/\partial Y$. Then solution of (3) is

$z = aX + bY + c$, $\quad$ where $\quad a + b = ab \quad$ so that $\quad b = a/(a-1) \qquad$ ... (4)

or $\quad z = a(x^2/2) + a(a-1)^{-1}\,(y^2/2) + c$, a and c being arbitrary constants, by (2) and (4)

Ex. 16. *Find the complete integral of* $p^2x^2 + px = q$

Sol. The given equation can be re–written as

$$x^2\left(\frac{\partial z}{\partial x}\right)^2 + x\frac{\partial z}{\partial x} = q \qquad \text{or} \qquad \left(x\frac{\partial z}{\partial x}\right)^2 + x\frac{\partial z}{\partial x} = q \qquad \text{... (1)}$$

Putting $(1/x)dx = dX$ so that $\log x = X$, (1) gives

$(\partial z/\partial X)^2 + \partial z/\partial X = q \quad$ or $\quad P^2 + P = q$, $\quad$ where $\quad P = \partial z/\partial X$.

Its solution is $\quad z = aX + by + c \quad$ where $\quad a^2 + a = b$

or $\quad z = a\log x + (a^2 + a)y + c$, a and c being arbitrary constants. $\qquad [\because\ X = \log x]$

Ex. 17. *Find the complete integral, general integral and singular integral of* $pq = 4xy$. *Show that the equation is satisfied by* $z = 2xy + C$, *C being an arbitrary constant. What is the character of this integral.* **[Delhi Maths (H) 2007]**

Sol. The given equation can be re–written as

$$\frac{pq}{4xy} = 1 \qquad \text{or} \qquad \frac{1}{4xy}\frac{\partial z}{\partial x}\frac{\partial z}{\partial y} = 1 \qquad \text{or} \qquad \left(\frac{1}{2x}\frac{\partial z}{\partial x}\right)\left(\frac{1}{2y}\frac{\partial z}{\partial y}\right) = 1 \qquad \text{... (1)}$$

Putting $2x\ dx = dX,\ 2y\ dy = dY$ so that $x^2 = X,\ y^2 = Y$, (1) gives

$(\partial z/\partial X)\ (\partial z/\partial Y) = 1 \quad$ or $\quad PQ = 1 \quad$ whose solution is

$z = aX + bY + d$, $\quad$ where $ab = 1 \quad$ so that $\quad b = 1/a$.

$$\therefore \qquad z = ax^2 + (1/a)y^2 + d \qquad \text{... (2)}$$

is complete integral of (1) containing two arbitrary constants a and d.

General integral. Putting $d = \phi(a)$ in (2), we get

$$z = ax^2 + (1/a)\ y^2 + \phi(a) \qquad \text{... (3)}$$

Differenting (3) partially w.r.t. 'a', $\qquad 0 = x^2 - (1/a^2)\ y^2 + \phi'(a) \qquad$ ... (4)

Then general integral is obtained by eliminating a from (3) and (4).

Singular integral. Differenting (2) partially w.r.t. 'a' and 'd' by turn, we get

$$0 = x^2 + (-1/a^2)\ y^2 \quad \text{... (5)} \qquad 0 = 1 \quad \text{... (6)}$$

Since (6) is absurd, so (1) has no singular solution.

Discussion of the character of the given integral

$$z = 2xy + C, \ C \text{ being an arbitrary constant} \quad \text{... (7)}$$

Differenting (7) partially w.r.t. x and y, we get $\partial z/\partial x = p = 2x$ and $\partial z/\partial y = q = 2y$. These values of p and q satisfy (1). Hence (1) is satisfied by (7).

Now, (7) can be derived from (2), if the values of p and q given by (7) and (2) are same, that is if $2ax = 2y$ and $2y/a = 2x$, *i.e.,* if we choose $a = y/x$. Putting $a = y/x$ and taking $d = C$ in (2), we have

$$z = (y/x)\ x^2 + (x/y)\ y^2 + C \qquad \text{or} \qquad z = 2xy + C,$$

showing that (7) is a particular case of the complete integral (2)

We now show that (7) is a particular case of the general integral. To this end, replace $\phi(a)$ by C in (3) and write

$$z = ax^2 + (1/a)\ y^2 + C \quad \text{... (8)}$$

Differenting (8) partially w.r.t. 'a', we get

$$0 = x^2 - (1/a^2)y^2 \qquad \text{or} \qquad a = y/x \quad \text{... (9)}$$

Eliminating a from (8) and (9), we get $\qquad z = 2xy + C$

Ex. 18. *Find the complete integral of* $z = p^2 - q^2$. **[Delhi Maths (G) 2006]**

Sol. Re–writing the given equation, we have

$$\frac{1}{z}\left(\frac{\partial z}{\partial x}\right)^2 - \frac{1}{z}\left(\frac{\partial z}{\partial y}\right)^2 = 1 \qquad \text{or} \qquad \left(z^{-1/2}\frac{\partial z}{\partial x}\right)^2 - \left(z^{-1/2}\frac{\partial z}{\partial y}\right)^{1/2} = 1 \quad \text{... (1)}$$

Let X, Y and Z be new variables such that

$$dX = dx, \qquad dY = dy \qquad \text{and} \qquad dZ = z^{-1/2}dz \qquad \text{so that} \quad X = x, \quad Y = y, \quad Z = 2z^{1/2} \quad \text{... (2)}$$

Let $P = \partial Z/\partial X$ and $Q = \partial Z/\partial y$. Using (2), (1) becomes

$$P^2 - Q^2 = 1, \quad \text{... (3)}$$

which is of the form $f(P, Q) = 0$. Hence a solution of (3) is $\qquad Z = aX + by + c, \quad$ ... (4)

where $a^2 - b^2 = 1$. Then $b = \pm(a^2 - 1)^{1/2}$ and so from (4), we have

$$Z = aX \pm (a^2 - 1)^{1/2}\,Y + c \qquad \text{or} \qquad 2z^{1/2} = ax \pm (a^2 - 1)^{1/2}\,y + c,$$

which is the complete integral, a and c being arbitrary constants and $|a| \ge 1$.

EXERCISE 3(C)

Solve the following partial differential equations (1 – 10)

1. $p^2 - q^2 = 1$ **Ans. C.I.** $z = ax + (a^2 - 1)^{1/2} + c$, a and c are arbitrary constants and $|a| \ge 1$; **S.I.** Does not exist, **G.I.** It is given by $z - ax - (a^2 - 1)^{1/2}\,y - \psi(a) = 0$, $-x - a(a^2 - 1)^{-1/2}\ y - \psi'(a)$, where ψ is an arbitrary function.

2. $p^2 - q^2 = \lambda$ **Ans. C.I.** $z = ax + (a^2 - \lambda)^{1/2} y + c,$ where a and c are arbitrary constants and $\lambda \le a^2$; **S.I.** Does not exist, **G.I.** It is given by $z - ax - (a^2 - \lambda)^{1/2} y - \psi(a) = 0,$

$$-x - a(a^2 - \lambda)^{1/2} y - \psi'(a) = 0$$

3. $p + q = pq$ **[Mysore 2004; Gulberga 2005; Kanpur 2011; Pune 2010]**

Ans. C.I. $z = ax + \{a/(a-1)\}y + c$, where a and c are arbitrary constants and $a \ne 1$; **S.S.** Does not exist; **G.S.** It is given by $z - ax - \{a/(a-1)\}y - \psi(a) = 0$ and $-x - y(a-1)^{-2} - \psi'(a) = 0,$ where ψ is an arbitrary function

4. $p + q + pq = 0$. **Ans. C.I.** $z = ax - \{a/(a+1)\}y + c$, where a and c are arbitrary constants and $a \ne -1$; **S.S.** Does not exist; **G.S.** It is given by $z - ax + \{a/(a+1)\}y - \psi(a) = 0,$

$-x - \{(2a+1)/2(a^2 + a)^{1/2}\}y - \psi'(a) = 0,$ where ψ is an arbitrary function.

5. $p^2 + q^2 = npq$. **(M.S. Univ. T.N. 2007), Ans. C.I.** $z = ax + (a/2) \times \{n + (n^2 - 4)^{1/2}y + c$, where a and c are arbitrary constants and $n^2 \ge 4$; **S.S.** Does not exists; **G.S.** It is given by $z - ax - (a/2) \times \{n + (n^2 - 4)^{1/2}\}y - \psi(a) = 0,\ -x - (1/2) \times \{n + (n^2 - 4)^{1/2\}} y - \psi'(a) = 0,$ where ψ is an arbitrary function.

6. $p = 2q^2 + 1$. **Ans. C.I.** $z = ax + \{(a-1)/2\}^{1/2} y + c,$ where a and c are arbitrary constants and $a \ge 1$; **S.S.** Does not exist; **G.S.** It is given by $z - ax - \{(a-1)/2\}^{1/2} y - \psi(a) = 0,$

$-x - (2\sqrt{2}\sqrt{a-1})^{-1} y - \psi'(a) = 0,$ where ψ is an arbitrary function.

7. $p = e^q$. **Ans. C.I.** $z = ax + y \log a + c$, where a and c are arbitrary constants and $a > 0$; **S.S.** Does not exist; **G.S.** It is given by $z - ax - y \log a - \psi(a) = 0, - x - (y/a) - \psi'(a) = 0,$ where ψ is an arbitrary function.

8. $p^2q^3 = 1$ **Ans. C.I.** $z = ax + a^{-2/3}y + c$, where a and c are arbitrary constants and $a > 0$; **S.S.** Does not exist; **G.S.** It is given by $z - ax - a^{-2/3} y - \psi(a) = 0,\ \ -x + (2/3) \times a^{-5/3} y - \psi(a) = 0,$ where ψ is an arbitrary function.

9. $p^2 + p = q^2$. **Ans. C.I.** $z = ax + (a^2 + a)^{1/2}y + c$, where a and c are arbitrary constants and $a \in \boldsymbol{R} - (-1, 0)$; **S.S.** Does not exist; **G.S.** It is given by $z - ax - (a^2 + a)^{1/2} y - \psi(a) = 0,$

$-x - \{(2a+1)/2(a^2 + a)^{1/2}\}y - \psi'(a) = 0,$ where ψ is an arbitrary function.

10. $p^2 + 6p + 2q + 4 = 0$. **C.I.** $z = ax - (2 + 3a + a^2/2)y + c$, where a and c are arbitrary constants; **S.S.** Does not exist; **G.S.** It is given by $z - ax + (2 + 3a + a^2/2)y - \psi(a) = 0,$

$-x + (a+3)y - \psi'(a) = 0,$ where ψ is an arbitrary function.

Find the complete integral (solution) of the following equations (Ex. 11—18).

11. $zy^2p = x(y^2 + z^2q^2)$. **Ans.** $z^2 = ax^2 \pm y^2(a-1)^{1/2} + c$, where $a \ge 1$

12. $z^2(p^2/x^2 + q^2/y^2) = 1$. **Ans.** $z^2 = ax^2 \pm y^2(1 - a^2)^{1/2} + c$, where $-1 \le a \le 1$

13. $yp + x^2q^2 = 2x^2y$. **Ans.** $(3z - ax^3 - b)^2 = 4(2 - a)y^2$

14. $(1 - y^2)\, xq^2 - y^2p = 0$. **Ans.** $(2z - ax^2 - b)^2 = a(1 - y^2)$

15. $p^2y(1 + x^2) = qx^2$. **Ans.** $z = a(1 + x^2)^{1/2} + (1/2) \times a^2y^2 + c$

16. $x^4p^2 + y^2zq - z^2 = 0$. **Ans.** $xy \log z = ay + (a^2 - 1)x + bxy$

17. $p^2 + q^2 = z$. **[Bangalore 1995]** **Ans.** $2z^{1/2} = ax \pm (1 - a^2)^{1/2} y + c$, where $-1 \le a \le 1$

18. $x^2p^2 + y^2q^2 = 4z^2$. **Ans.** $\log z = a \log x + (4 - a^2)^{1/2} + c$, $-2 \le a \le 2$

3.12. Standard form II. Clairaut equation. [Meerut 2009; Nagpur 2002]

A first order partial differential equation is said to be of Clariaut form if it can be written in the form
$$z = px + qy + f(p, q). \qquad ...(1)$$
Let
$$F(x, y, z, p, q) \equiv px + qy + f(p, q) - z. \qquad ...(2)$$

Charpit's auxiliary equations are $\dfrac{dp}{\dfrac{\partial F}{\partial x} + p\dfrac{\partial F}{\partial z}} = \dfrac{dq}{\dfrac{\partial F}{\partial y} + q\dfrac{\partial F}{\partial z}} = \dfrac{dz}{-p\dfrac{\partial F}{\partial p} - q\dfrac{\partial F}{\partial q}} = \dfrac{dx}{-\dfrac{\partial F}{\partial p}} = \dfrac{dy}{-\dfrac{\partial F}{\partial q}}$

or
$$\frac{dp}{0} = \frac{dq}{0} = \frac{dz}{-px - qy - p(\partial f/\partial p) - q(\partial f/\partial q)} = \frac{dx}{-x - (\partial f/\partial p)} = \frac{dy}{-y - (\partial f/\partial q)}, \text{ by (1)}$$

Then, first and second fractions $\Rightarrow$ $dp = 0$ and $dq = 0 \Rightarrow$ $p = a$ and $q = b$.

Substituting these values in (1), the complete integral is $z = ax + by + f(a, b)$

Remark 1. Observe that the complete integral of (1) is obtained by merely replacing p and q by a and b respectively. Singular and general integrals can be obtained by usual methods.

Remark 2. Sometimes change of variables can be employed to transform a given equation to standard form II.

3.13. SOLVED EXAMPLES BASED ON ART. 3.12

Ex. 1. *Solve* $z = px + qy + pq$. **[Ravishanker 1997; Bangalore 2005; Sagar 1995, 96]**

Sol. The complete integral is $z = ax + by + ab$, a, b being arbitrary constants ...(1)

Singular integral. Differentiating (1) partially w.r.t. a and b, we have
$$a = x + b \quad \text{and} \quad 0 = y + a. \qquad ...(2)$$
Eliminating a and b between (1) and (2), we get $z = -xy - xy + xy$ *i.e.*, $z = -xy$, which is the required singular solution, for it satisfies the given equation.

General Integral. Take $b = \phi(a)$, where ϕ denotes an arbitrary function.

Then (1) becomes $z = a\,x + \phi(a)\,y + a\,\phi\,(a)$. ...(3)

Differentiating (3) partially w.r.t. a, $0 = x + \phi'(a)y + \phi(a) - a\,\phi'(a)$. ...(4)

The general integral is obtained by eliminating a between (3) and (4).

Ex. 2. *Prove that complete integral of the equations* $(px + qy - z)^2 = 1 + p^2 + q^2$ is $ax + by + cz = (a^2 + b^2 + c^2)^{1/2}$. **[I.A.S. 1989]**

Sol. Re–writting the given equation, we have
$$px + qy - z = \pm\sqrt{(1 + p^2 + q^2)} \qquad \text{or} \qquad z = px + qy \pm \sqrt{(1 + p^2 + q^2)}$$
which is of standard form II and so its complete integral is
$$z = Ax + By \pm (1 + A^2 + B^2)^{1/2}. \qquad ...(1)$$
To get the desired form of solution we take +ve sign in (1) and set $A = -a/c$ and $B = -b/c$. Then (1) becomes
$$z = -(ax + by)/c + (c^2 + a^2 + b^2)^{1/2}/c$$
or
$$ax + by + cz = (a^2 + b^2 + c^2)^{1/2}.$$

Ex. 3. *Solve* $z = px + qy + c\sqrt{(1 + p^2 + q^2)}$. **[I.A.S. 1989; Meerut 1998]**

Sol. The complete integral of the given equation is
$$z = ax + by + c\sqrt{(1 + a^2 + b^2)}, \; a, b \text{ being arbitrary constants.} \qquad ...(1)$$

Singular Integral. Differentiating (1) partially w.r.t. a and b, we get

$0 = x + ac/\sqrt{(1+a^2+b^2)}$ ...(2) $\qquad 0 = y + bc/\sqrt{(1+a^2+b^2)}$. ...(3)

$\therefore$ From (2) and (3), $\qquad x^2 + y^2 = (a^2c^2 + b^2c^2)/(1 + a^2 + b^2)$.

$$\therefore \qquad c^2 - x^2 - y^2 = c^2 - \frac{a^2c^2+b^2c^2}{1+a^2+b^2} = \frac{c^2}{1+a^2+b^2}$$

so that $\qquad 1 + a^2 + b^2 = c^2/(c^2 - x^2 - y^2)$. ...(4)

From (2), $\qquad a = -\dfrac{x\sqrt{(1+a^2+b^2)}}{c} = -\dfrac{x}{\sqrt{(c^2-x^2-y^2)}}$, by (4)

Similarly from (3) and (4), we obtain $\qquad b = -y/\sqrt{c^2-x^2-y^2}$.

Putting these values of a and b in (1), the singular solution is

$$z = -\frac{x^2}{\sqrt{(c^2-x^2-y^2)}} - \frac{y^2}{\sqrt{(c^2-x^2-y^2)}} + \frac{c^2}{\sqrt{(c^2-x^2-y^2)}} = \frac{c^2-x^2-y^2}{\sqrt{(c^2-x^2-y^2)}}$$

or $\quad z = (c^2 - x^2 - y^2)^{1/2}$ $\quad$ or $\quad z^2 = c^2 - x^2 - y^2$ $\quad$ or $\quad x^2 + y^2 + z^2 = c^2$. ...(5)

We can easily verify that (1) is satisfied by (5).

General Integral. Take $b = \phi(a)$, where ϕ is an arbitrary function.

Then, (1) yeilds $\qquad z = ax + y\phi(a) + c[1 + a^2 + \{\phi(a)\}^2]^{1/2}$. ...(6)

Differentiating both sides of (6) partially w.r.t. 'a', we get

$$0 = x + y\phi'(a) + (c/2) \times [1 + a^2 + \{\phi(a)\}^2]^{-1/2} \times [2a + 2\phi(a)\,\phi'(a)]. \quad ...(7)$$

Eliminating a from (6) and (7), we get the general integral.

Ex. 4. *Find the complete and singular integrals of the following equations:*

(*i*) $z = px + qy + \log(pq)$ **[Indore 2004; K.U. Kurukshetra 2006]**

(*ii*) $z = px + qy - 2\sqrt{pq}$. **[Bangalore 1993; Lucknow 2010]**

Sol. (*i*) The complete integral is $\qquad z = ax + by + \log(ab)$

or $\qquad z = ax + by + \log a + \log b$, a, b being arbitrary constants ...(1)

Differentiating (1) partially with respect to a and b, we get

$0 = x + (1/a)$ $\quad$ and $\quad$ $0 = y + (1/b)$ $\quad$ so that $\quad a = -1/x$ and $b = -1/y$. ...(2)

Eliminating a and b from (1) and (2), the required singular integral is

$$z = -1 - 1 + \log(1/xy) \qquad \text{or} \qquad z = -2 - \log(xy).$$

(*ii*) The complete integral is $\qquad z = ax + by - 2\sqrt{ab}$. ...(1)

Differentiating (1) partially with respect to a and b, we get

$0 = x - \dfrac{2b}{2\sqrt{ab}}$ $\quad$ and $\quad$ $0 = y - \dfrac{2a}{2\sqrt{ab}}$ $\quad$ so that $\quad x = \sqrt{\dfrac{b}{a}}$ and $y = \sqrt{\dfrac{a}{b}}$. ...(2)

Now, using (1) $\qquad x - z = x - (ax + by - 2\sqrt{ab}) = \sqrt{\dfrac{b}{a}} - a\sqrt{\dfrac{b}{a}} - b\sqrt{\dfrac{a}{b}} + 2\sqrt{ab}$, using (2)

$$\therefore \qquad x - z = \sqrt{(b/a)}. \quad ...(3)$$

Similarly, using (1) $\quad y - z = y - (ax + by - 2\sqrt{ab})$, $= \sqrt{\dfrac{a}{b}} - a\sqrt{\dfrac{b}{a}} - b\sqrt{\dfrac{a}{b}} + 2\sqrt{ab}$

$$\therefore \qquad y - z = \sqrt{(a/b)}. \quad ...(4)$$

From (3) and (4), $\qquad (x - z)(y - z) = 1$,

which is singular integral as it satisfies the given equation.

Ex. 5. *Prove that the complete integral of* $z = px + qy - 2p - 3q$ *represents all possible planes through the point* (2, 3, 0). *Also find the envelope of all planes represented by the complete integral* (*i.e.*, *find the singular integral*). **(M.D.U. Rohtak 2006)**

Sol. Given that $z = px + qy - 2p - 3q,$ ...(1)

which is of the form $z = px + qy + f(p, q)$ and so its complete integral is

$$z = ax + by - 2a - 3b, \quad a, b \text{ being arbitrary constants} \qquad ...(2)$$

Since (2) is a linear equation in x, y, z, it follows that (2) represents planes for various values of a and b. Again putting $x = 2$, $y = 3$, $z = 0$ in (2), we have

$$0 = 2a + 3b - 2a - 3b \qquad i.e., \qquad 0 = 0,$$

showing that coordinates of the point (2, 3, 0) satisfy (2). Hence the complete integral (2) of (1) represents all possible planes passing through the point (2, 3,0).

Differentiating (2) partially with respect to a and b, we get

$0 = x - 2$ and $0 = y - 3$ so that $x = 2$ and $y = 3$.

Substituting these values in (2), we get $z = 0$ as the required envelope (*i.e.*, singular integral).

Ex. 6. *Prove that the complete integral of* $z = px + qy + [pq/(pq - p - q)]$ *represents all planes such that the algebraic sum of the intercepts on three coordinate axes is unity.*

Sol. Since the given equation is of the form $z = px + qy + f(p, q)$, so its complete integral is

$$z = ax + by + [ab/(ab - a - b)], \quad a \text{ and } b \text{ being arbitrary constants.} \qquad ...(1)$$

Since (2) is a linear equation in x, y, z, it follows that (1) represents planes for various values of a and b. We now rewrite (1) in the intercept form of a plane as follows :

$$ax + by - z = ab/(a + b - ab)$$

or
$$\frac{x}{[b/(a+b-ab)]} + \frac{y}{[a/(a+b-ab)]} + \frac{z}{[-ab/(a+b-ab)]} = 1.$$

$\therefore$ The algebric sum of the intercepts on three coordinate axes

$$= \frac{b}{a+b-ab} + \frac{a}{a+b-ab} + \frac{(-ab)}{a+b-ab} = \frac{b+a-ab}{a+b-ab} = 1, \text{ as required.}$$

Ex. 7. *Show that the complete integral of the equation* $z = px + qy + (p^2 + q^2 + 1)^{1/2}$ *represents all planes at unit distance from the origin.*

Sol. Given equation is of the form $z = px + qy + f(p, q)$, so its complete integral is

$$z = ax + by + (a^2 + b^2 + 1)^{1/2}, \; a, b \text{ being an arbitrary constants.}$$

or
$$ax + by - z + (a^2 + b^2 + 1)^{1/2} = 0. \qquad ...(1)$$

Since (2) is a linear equation in x, y, z, it follows that (1) represents planes for various values of a and b.

The perpendicular distance of (1) from origin (0, 0, 0)

$$= \frac{a.0 + b.0 - 0 + \sqrt{a^2+b^2+1}}{\sqrt{\{a^2+b^2+(-1)^2\}}} = \frac{\sqrt{a^2+b^2+1}}{\sqrt{a^2+b^2+1}} = 1, \text{ as required}$$

Ex. 8. *Find the complete integral of the following equations:*

(*i*) $(p+q)(z-px-qy) = 1$ **[Pune 2010]**

(*ii*) $pqz = p^2(xq+p^2) + q^2(yp+q^2)$ **[Delhi B.A. (Prog) II 2008, 10]**

Sol. (*i*) Re–writing the given equation in the standard form $z = px + qy + f(p,q)$, we get

$$z - px - qy = 1/(p+q) \qquad \text{or} \qquad z = px + qy + 1/(p+q)$$

$\therefore$ Its complete integral is $z = ax + by + 1/(a+b)$, where a and b are arbitrary constants.

(*ii*) Dividing both sides of the given equation by pq, $z = px + qy + (p^4 + q^4)/pq$,

Its complete integral is $z = ax + by + (a^4 + b^4)/ab$, a, b being arbitrary constants.

Ex. 9. (*a*) *Find the complete integral the equation* $2(y + zq) = q(xp + yq)$.

[Delhi Maths (H) 1999]

Sol. Re–writing the given equation, we have

$2zq = xpq + yq^2 - 2y$ or $z = (1/2)px + (1/2)qy - (y/q)$

or
$$z = x^2\left(\frac{1}{2x}\frac{\partial z}{\partial x}\right) + y^2\left(\frac{1}{2y}\frac{\partial z}{\partial y}\right) - \frac{1}{2}\left(\frac{1}{2y}\frac{\partial z}{\partial y}\right)^{-1} \quad \text{... (1)}$$

Putting $2x\,dx = dX$ and $2y\,dy = dY$ so that $x^2 = X$ and $y^2 = Y$, (1) gives

$z = X\,(\partial z/\partial X) + Y\,(\partial z/\partial Y) - 1/\{2\,(\partial z/\partial Y)\}$ or $z = PX + QY - (1/2Q)$,

where $P = \partial z/\partial X$ and $Q = \partial z/\partial Y$. The above equation is of the form $z = PX + Qy + f(P, Q)$ and hence its complete integral is

$z = aX + bY - (1/2b)$ or $z = ax^2 + by^2 - (1/2b)$, a and b being arbitrary constants.

Ex. 9. (*b*) *Find the complete integral of* $2q(z - px - qy) = 1 + q^2$.

Sol. Re–writing the given equation in the form $z = px + qy + f(p, q)$, we have

$z - px - qy = (1 + q^2)/2q$ or $z = px + qy + (1 + q^2)/2q$,

Its complete integral is $z = ax + by + (1 + b^2)/2b$, a and b being arbitrary constants.

Ex. 10. *Find the compelte integral of* $p^2x + q^2y = (z - 2px - 2qy)^2$.

Sol. Taking positive root, the given equation reduces to

$z - 2px - 2qy = (p^2x + q^2y)^{1/2}$ or $z = 2px + 2qy + (p^2x + q^2y)^{1/2}$

or
$$z = \sqrt{x}\,\frac{\partial z}{(1/2\sqrt{x})\partial x} + \sqrt{y}\,\frac{\partial z}{(1/2\sqrt{y})\partial y} + \frac{1}{2}\left[\left(\frac{\partial z}{(1/2\sqrt{x})\partial x}\right)^2 + \left(\frac{\partial z}{(1/2\sqrt{y})\partial y}\right)^2\right]^{1/2} \quad \text{... (1)}$$

Put $(1/2\sqrt{x})\,dx = dX$ and $(1/2\sqrt{y})dy = dY$ so that $\sqrt{x} = X$ and $\sqrt{y} = Y$ ... (2)

Using (2), (1) gives $z = (\partial z/\partial X)X + (\partial z/\partial Y)Y + (1/2) \times \{(\partial z/\partial X)^2 + (\partial z/\partial Y)^2\}^{1/2}$

or $z = PX + QY + (1/2) \times (P^2 + Q^2)^{1/2}$, where $P = \partial z/\partial X$ and $Q = \partial z/\partial Y$.

It is of the Clairaut's form $z = Px + Qy + f(P, Q)$ and so its complete integral is given by

$z = aX + bY + (1/2) \times (a^2 + b^2)^{1/2}$ or $z = a\sqrt{x} + b\sqrt{y} + (1/2) \times (a^2 + b^2)^{1/2}$

Ex. 11. *Find a complete and the singular integral of* $4xyz = pq + 2px^2y + 2qxy^2$

Sol. The given equation can be rewritten as

$$z = \left(\frac{1}{2x}\frac{\partial z}{\partial x}\right)\left(\frac{1}{2y}\frac{\partial z}{\partial y}\right) + x^2\left(\frac{1}{2x}\frac{\partial z}{\partial x}\right) + y^2\left(\frac{1}{2y}\frac{\partial z}{\partial y}\right). \quad ... (1)$$

Put $2x\,dx = dX$ and $2y\,dy = dY$ ... (2)

so that $x^2 = X$ and $y^2 = Y.$ ... (3)

Using (2), (1) becomes $z = (\partial z/\partial X)(\partial z/\partial Y) + X(\partial z/\partial X) + Y(\partial z/\partial Y)$

or $z = XP + YQ + PQ,$ ... (4)

where $P = \partial z/\partial X$ and $Q = \partial z/\partial Y$. (4) is of the form $z = XP + YQ + f(P, Q)$.

$\therefore$ Solution of (4) is $z = aX + bY + ab$, a, b being arbitrary constants.

or $z = ax^2 + by^2 + ab$, which is complete integral. ... (5)

Differentiating (5) partially w.r.t a and b, we have

$0 = x^2 + b$ and $0 = y^2 + b$ so that $b = -x^2$ and $a = -y^2$ .... (6)

Eliminating a and b between (5) and (6), the required singular integral is

$z = -x^2y^2 - x^2y^2 + x^2y^2$ or $z = -x^2y^2.$

Ex. 12. *Find the complete and singular solutions of* $z = px + qy + p^2q^2$. **[Jabalpur 2000; Sagar 1995; Rewa 2003; Ravishankar 2004]**

Sol. Given $z = px + qy + p^2q^2$ ... (1)

Since (1) is in Clairaut's form, its complete solution is

$z = ax + by + a^2b^2$, a, b being arbitrary constants ... (2)

To find singular solution of (1). Differentiating (2) partially w.r.t. 'a' and 'b' successively,

$0 = x + 2ab^2$ and $0 = y + 2a^2b$ ... (3)

From (3), $a = -(y^2/2x)^{1/3}$ and $b = -(x^2/2y)^{1/3}$ ... (4)

Substituting the values of a and b given by (4) in (2), we get

$z = -x(y^2/2x)^{1/3} - y(x^2/2y)^{1/3} + (x^2y^2/16)^{1/3}$ or $z = -(3/4) \times 4^{1/3}\,x^{2/3}\,y^{2/3}$,

which is the required singular solution of (1)

EXERCISE 3 (D)

Solve the following partial differential equations : (1 – 9)

1. $z = px + qy - 2p - 3q.$ **[M.D.U. Rohtak 2006]**

Ans. C.I. $z = ax + by - 2a - 3b$; **S.S.** $z = 0$; **G.S.** It is given by $z - ax - \psi(a)y + 2a + 3\psi(a) = 0,$

$x + (y-3)\psi'(a) - 2 = 0$

2. $z = px + qy + 5pq.$ **Ans. S.I.** $z = ax + by + 5ab$; **S.S.** $5z + xy = 0$

G.S. $z - ax - \psi(a)y - 5a\,\psi(a) = 0,\;\; x + 5\psi(a) + (y + 5a)\,\psi'(a) = 0$

3. $z = px + qy + p^2 - q^2$ **[Purvanchal 2007]** **Ans. S.I.** $z = ax + by + a^2 - b^2$;

S.S. $x^2 - y^2 + 4z = 0$; **G.S.** $z - ax - \psi(a)y - a^2 + \{\psi(a)\}^2 = 0;\;\; x + 2a + \{y - 2\psi(a)\}\psi'(a) = 0;$

4. $z = px + qy + (q/p) - p.$ **[Madras 2005]**

Ans. C.I. $z = ax + by + (b/a) - a$; **S.S.** $yz = 1 - x$; **G.S.** It is given by

$z - ax + \psi(a)y + (1/a)\times\psi(a) - a,\;\; -x + \psi'(a)y - (1/a^2)\,\psi(a) + (1/a)\times\psi'(a) = 0$

5. $z = px + qy + p/q$ **Ans. C.I.** $= ax + by + a/b$; **S.S.** $xz + 4 = 0$;

G.S. $z - ax - \psi(a)y - a/\psi(a) = 0;\;\; x + \psi'(a)y + 1/\psi(a) - \{a\psi'(a)\}/\{\psi(a)\}^2 = 0$

6. $z = px + qy + 2\sqrt{pq}$ **[Bangalore 1994]**

Ans. C.I. $z = ax + by + 2\sqrt{ab}$; **S.S.** $(x - z)(y - z) = 1$; **G.S.** $z - ax - \psi(a)y - 2\sqrt{a\psi(a)} = 0$,

$$x + \psi'(a) + \{\psi(a) + a\psi'(a)\}/2\sqrt{a\psi(a)} = 0$$

7. $z = px + qy - 2\sqrt{pq}$. **Ans. C.I.** $z = ax + by - 2\sqrt{ab}$; **S.S.** $(x - z)(y - z) = 1$;

G.S. $z - ax - \psi(a)y + 2\sqrt{a\psi(a)} = 0$, $x + \psi'(a)y - \{\psi(a) + a\psi'(a)\}/\sqrt{a\psi(a)} = 0$

8. $z = px + qy + p^2 + pq + q^2$. **[Ranchi 2010]**

Ans. C.I. $z = ax + by + a^2 + ab + b^2$; **S.S.** $x^2 + y^2 - xy + 3z = 0$, **G.S.** $z - ax - \psi(a)y - a^2 - a\psi(a) - \{\psi(a)\}^2 = 0$, $x + \{y + a + 2\psi(a)\}\psi'(a) + 2a + \psi(a) = 0$

9. $z = px + qy + (\alpha p^2 + \beta q^2 + 1)^{1/2}$. **Ans. C.I.** $z = ax + by + (\alpha a^2 + \beta b^2 + 1)^{1/2}$;

S.S. $x^2/\alpha + y^2/\beta + z^2 = 1$; **G.S.** $z - ax - \psi(a)y - [a\alpha^2 + \beta\{\psi(a)\}^2 + 1]^{1/2} = 0; x + \psi'(a)y$

$$+\{a\alpha + \beta\psi(a)\psi'(a)\}/[a\alpha^2 + \beta(\psi(a)\}^2 + 1]^{1/2} = 0$$

10. Find the complete integral of $z = px + qy - \sin(pq)$ **[GATE 2003]**

Ans. $z = ax + by - \sin(ab)$ a, b being arbitrary constants.

11. Find the complete integral and singular integral of the differential equation $z = px + qy + p^2 - q^2$. Find also a developable surface belonging to the general integral of this differential equation. **[I.A.S 1983]**

Ans. Complete integral is $z = ax + by + a^2 - b^2$; singular integral is $4z = 3(x^2 - y^2)$

3.14. Standard form III. Only p, q and z present. [Nagpur 2003; Delhi Maths (H) 2006]

Under this standard form we consider differential equation of the form

$$f(p, q, z) = 0. \qquad ...(1)$$

Charpit's auxiliary equations are $$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}} = \frac{dz}{-p\frac{\partial f}{\partial p} - q\frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}}$$

or $$\frac{dp}{p(\partial f/\partial z)} = \frac{dq}{q(\partial f/\partial z)} = \frac{dz}{-p(\partial f/\partial p) - q(\partial f/\partial q)} = \frac{dx}{-\partial f/\partial p} = \frac{dy}{-\partial f/\partial q}, \text{ using (1)}$$

Taking the first two ratios, $(1/p)dp = (1/q)dq$

Integrating, $q = ap$, a being an arbitrary constant. ...(2)

Now, $dz = p\,dx + q\,dy = p\,dx + ap\,dy$, using (2)

or $dz = p(dx + ady) = pd(x + ay) = p\,du$, ...(3)

where $u = x + ay$. ...(4)

Now, (3) $\Rightarrow$ $p = dz/du$ and so by (2) $q = ap = a(dz/du)$.

Substituting these values of p and q in (1), we get $$f\left(\frac{dz}{du}, a\frac{dz}{du}, z\right) = 0, \qquad ...(5)$$

which is an ordinary differential equation of first order. Solving (5), we get z as a function of u. Complete integral is then obtained by replacing u by $(x + ay)$.

3.15. Working rule for solving equations of the form $f(p,q, z) = 0$. ...(1)

Step I. Let $u = x + ay$, where a is an arbitrary constant. ...(2)

Step II. Replace p and q by dz/du and $a(dz/du)$ respectively in (1) and solve the resulting ordinary differential equation of first order by usual methods.

Step III. Replace u by $x + ay$ in the solution obtained in step II.

Remark 1. Sometimes change of variables can be employed to reduce a given equation in the standard form III.

Remark 2. Singular and general integrals are obtined by well known methods.

3.16. SOLVED EXAMPLES BASED ON ART 3.15.

Ex. 1. *Find a complete integral of* $9(p^2z + q^2) = 4$.

[Delhi Maths (H) 2006; Banglore 1995; I.A.S. 1988; Meerut 1996; Rohilkhand 1995]

Sol. Given equation is $\quad 9(p^2z + q^2) = 4, \quad$ ...(1)

which is of the form $f(p, q, z) = 0$. Let $u = x + ay$, where a is an arbitrary constant. Now, replacing p and q by dz/du and $a(dz/du)$ respectively in (1), we get

$$9\left[z\left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2\right] = 4 \qquad \text{or} \qquad \left(\frac{dz}{du}\right)^2 = \frac{4}{9(z+a^2)}.$$

or $\quad du = \pm\, (3/2) \times (z + a^2)^{1/2}dz$, separating variables u and z.

Integrating, $\quad u + b = \pm\, (3/2) \times [(z + a^2)^{3/2}/(3/2)] \quad$ or $\quad u + b = \pm\, (z + a^2)^{3/2}$

or $\quad (u + b)^2 = (z + a^2)^3 \quad$ or $\quad (x + ay + b)^2 = (z + a^2)^3$, as $u = x + ay$

which is a complete integral contaning two arbitrary constants a and b.

Ex. 2. *Find a complete integral of* $p^2 = qz$. **[Bilaspur 1996; Sagar 2004]**

Sol. Given equation is $\quad p^2 = qz, \quad$ ...(1)

which is of the form $f(p, q, z) = 0$. Let $u = x + ay$, where a is an arbitrary constant. Now, replacing p and q by dz/du and $a(dz/du)$ respectively in (1), we get

$$\left(\frac{dz}{du}\right)^2 = \left(a\frac{dz}{du}\right)z \qquad \text{or} \qquad \frac{dz}{du} = az \qquad \text{or} \qquad \frac{dz}{z} = a\,du.$$

Integrating, $\quad \log z - \log b = au \quad$ or $\quad z = be^{au} \quad$ or $\quad z = be^{a(x + ay)}$,

which is a complete integral containing two arbitrary constants a and b.

Ex. 3.(a) *Find a complete integral of* $z = pq$. **[Meerut 1994]**

Sol. Given equation is $\quad z = pq, \quad$ ... (1)

which is of the form $f(p, q, z) = 0$. Let $u = x + ay$, where a is an arbitrary constant. Now, replacing p and q by dz/du and $a(dz/du)$ respectively in (1), we get

$$z = a\left(\frac{dz}{du}\right)^2 \qquad \text{or} \qquad \frac{dz}{du} = \pm\frac{\sqrt{z}}{\sqrt{a}} \qquad \text{or} \qquad \pm\sqrt{a}\, z^{-1/2}dz = du.$$

Integrating, $\quad \pm 2\sqrt{az} = u + b \quad$ or $\quad 4(az) = (x + ay + b)^2$, as $u = x + ay$

Ex. 3.(b) *Find a complete integral of* $pq = 4z$.

Sol. Proceed as in Ex. 3.(a). **Ans.** $(x + ay + b)^2 = az$

Ex. 4.(a) *Find a complete integral of* $p(1 + q^2) = q(z - \alpha)$. **[Meerut 1999; Bilaspur 2002; Jiwaji 2003; Ravishanker 2005 Rewa 1998, Vikram 2004]**

(b) *Find a complete integral of* $p(1 + q^2) = q(z - 1)$. **[M.S. Univ. T.N. 2007]**

Sol. (*a*) Given equation is $\quad p(1 + q^2) = q(z - \alpha), \quad$ ...(1)

which is of the form $f(p, q, z) = 0$. Let $u = x + ay$, where a is an arbitrary constant. Now, replacing p and q by dz/du and $a(dz/du)$ respectively in (1), we get

$$\frac{dz}{du}\left\{1+\left(a\frac{dz}{du}\right)^2\right\} = a\frac{dz}{du}\,(z - \alpha) \qquad \text{or} \qquad 1 + a^2\left(\frac{dz}{du}\right)^2 = a(z - \alpha)$$

or $$\frac{dz}{du} = \pm\frac{\sqrt{\{a(z-\alpha)-1\}}}{a} \qquad \text{or} \qquad du = \pm\frac{adz}{\sqrt{\{a(z-\alpha)-1\}}}.$$

Integrating, $u + b = \pm 2\sqrt{\{a(z-\alpha)-1\}}$ or $(u + b)^2 = 4\{a(z - \alpha) - 1\}^2$

or $(x + ay + b)^2 = 4\{a(z - \alpha) - 1\}^2$, a and b being arbitrary constants.

(***b***) Proceed as in part (a) by taking $\alpha = 1$.

Ex. 5.(a) *Find a complete integral of* $pz = 1 + q^2$. **[Meerut 1996]**

Sol. Given equation is $$pz = 1 + q^2, \quad ...(1)$$
which is of the form $f(p, q, z) = 0$. Let $u = x + ay$, where a is an arbitrary constant. Now, replacing p and q by dz/du and $a(dz/du)$ respectively in (1), we get

$$z\frac{dz}{du} = 1 + a^2\left(\frac{dz}{du}\right)^2 \quad \text{or} \quad a^2\left(\frac{dz}{du}\right)^2 - z\frac{dz}{du} + 1 = 0.$$

$$\therefore \frac{dz}{du} = \frac{z \pm (z^2 - 4a^2)^{1/2}}{2a^2} \quad \text{or} \quad \frac{dz}{z \pm (z^2 - 4a^2)^{1/2}} = \frac{du}{2a^2}$$

or $$\frac{[z \mp (z^2 - 4a^2)^{1/2}]dz}{[z \pm (z^2 - 4a^2)^{1/2}][z \mp (z^2 - 4a^2)^{1/2}]} = \frac{du}{2a^2} \quad \text{or} \quad \frac{z \mp (z^2 - 4a^2)^{1/2}}{4a^2} = \frac{du}{2a^2}$$

or $$[z \mp (z^2 - 4a^2)^{1/2}]\, dz = 2du.$$

Integrating, $$\frac{z^2}{2} \mp \left[\frac{z}{2}(z^2 - 4a^2)^{1/2} - \frac{4a^2}{2}\log\left\{z + (z^2 - 4a^2)^{1/2}\right\}\right] = 2u + \frac{b}{2}$$

or $$z^2 \mp \left[z(z^2 - 4a^2)^{1/2} - 4a^2\log\left\{z + (z^2 - 4a^2)^{1/2}\right\}\right] = 4(x + ay) + b.$$

Ex. 5. (b) *Find a complete integral of* $1 + p^2 = qz$.

Sol. Proceed as in Ex. 5.(*a*). The requied complete integral is

$$a^2z^2 \mp \left[az\sqrt{(a^2z^2 - 4)} - 4\log\left\{az + \sqrt{(a^2z^2 - 4)}\right\}\right] = 4(x + ay) + b.$$

Ex. 6. *Find complete integrals of the following partial differential equations*

(*i*) $p(z + p) + q = 0$

(*ii*) $p(1 + q) = qz$. **[Gulbarga 2005]**

Sol. (*i*) The given equation is of the form $f(p, q, z) = 0$. Let $u = x + ay$, a being an arbitrary constant. Replacing p by dz/du and q by $a(dz/du)$ in the given equation, we get

$$\frac{dz}{du}\left(z + \frac{dz}{du}\right) + a\frac{dz}{du} = 0 \quad \text{or} \quad \frac{dz}{du} = -(z + a) \quad \text{or} \quad \frac{dz}{z + a} = -du.$$

Integrating, $\log(z + a) - \log b = -u$ or $z + a = be^{-u}$ or $z + a = be^{-(x + ay)}$.

(*ii*) Proceed as in part (*i*). **Ans.** $az - 1 = be^{x + ay}$.

Ex. 7. *Find a complete integral of* $p^3 + q^3 - 3pqz = 0$. **[I.A.S. 1991]**

Sol. The given equation is of the form $f(p, q, z) = 0$. Let $u = x + ay$, a being an arbitrary constant. Replacing p by dz/du and q by $a(dz/du)$ in the given equation,

$$\left(\frac{dz}{du}\right)^3 + a^3\left(\frac{dz}{du}\right)^3 - 3az\left(\frac{dz}{du}\right)^2 = 0 \quad \text{or} \quad (1 + a^3)\frac{dz}{du} = 3az \quad \text{or} \quad \frac{1 + a^3}{z}dz = 3au.$$

Integrating $(1 + a^3)\log z = 3au + b$ or $(1 + a^3)\log z = 3a(x + ay) + b$.

Ex. 8. *Find a complete integrals of* (*i*) $p + q = z/c$.

(*ii*) $p + q = z$.

Sol. (*i*) The given equation is of the form $f(p, q, z) = 0$. Let $u = x + ay$, a being an arbitrary constant. Replacing p by dz/du and q by $a(dz/du)$ in the given equation, we get

$$\frac{dz}{du}+a\frac{dz}{du}=\frac{z}{c} \qquad \text{or} \qquad (1+a)\frac{dz}{du}=\frac{z}{c} \qquad \text{or} \qquad \frac{c(1+a)}{z}dz = du.$$

Integrating, $c(1 + a) \log z = u + b$ or $c(1 + a) \log z = x + ay + b$.

(*ii*) Proceed as in part (*i*). **Ans.** $(1 + a) \log z = x + ay + b$.

Ex. 9. *Find a complete integral of* $p^2 = z^2(1 - pq)$. **[Jiwaji 1998; Meerut 2001]**

Sol. The given equation is of the form $f(p, q, z) = 0$. Let $u = x + ay$, a being an arbitrary constant. Replacing p by dz/du and q by $a(dz/du)$ in the given equation, we have

$$\left(\frac{dz}{du}\right)^2 = z^2\left\{1-a\left(\frac{dz}{du}\right)^2\right\} \qquad \text{or} \qquad \left(\frac{dz}{du}\right)^2(1+az^2) = z^2$$

or $$\frac{dz}{du} = \pm\frac{z}{\sqrt{(1+az^2)}} \qquad \text{or} \qquad \pm\, du = \frac{\sqrt{(1+az^2)}}{z}dz = \pm\frac{(1+az^2)dz}{z\sqrt{(1+az^2)}}.$$

or $$\pm\int du \pm b = \int\frac{dz}{z\sqrt{(1+az^2)}}+\frac{1}{2}\int\frac{2az\,dz}{\sqrt{(1+az^2)}}. \qquad \text{...(1)}$$

Now, $$\int\frac{dz}{z\sqrt{(1+az^2)}}=\int\frac{(-1/t^2)\,dt}{(1/t)\times\sqrt{\{(1+(a/t^2)\}}}, \text{ putting } z = 1/t \text{ so that } dz = -(1/t^2)dt$$

$$= -\int\frac{dt}{(t^2+a)^{1/2}} = -\sinh^{-1}\frac{t}{\sqrt{a}} = -\sinh^{-1}\frac{1}{z\sqrt{a}}, \text{ as } t=\frac{1}{z} \qquad \text{...(2)}$$

and $$\frac{1}{2}\int\frac{2az\,dz}{(1+az^2)^{1/2}} = \frac{1}{2}\int\frac{2v\,dv}{v}, \text{ putting } 1+az^2=v^2 \text{ and } 2az\,dz = 2v\,dv$$

$$= v = (1+az^2)^{1/2}.$$

Using (2) and (3), (1) reduces to $\pm (u + b) = -\sinh^{-1}(1/z\sqrt{a}) + (1 + az^2)^{1/2}$

or $$\pm (x + ay + b) = -\sinh^{-1}(1/z\sqrt{a}) + (1 + az^2)^{1/2}.$$

Ex. 10. *Find complete and singular integrals of* $4(1 + z^3) = 9z^4pq$.

Sol. The given equation is of the form $f(p, q, z) = 0$. Let $u = x + ay$, a being an arbitrary constant. Replacing p by dz/du and q by $a(dz/du)$ in the given equation, we have

$$4(1+z^3) = 9z^4a\left(\frac{dz}{du}\right)^2 \qquad \text{or} \qquad \pm\frac{3\sqrt{a}z^2}{(1+z^3)^{1/2}}dz = 2\,du$$

or $$\pm\left(\sqrt{a}/t\right)\times 2t\,dt = 2\,du, \text{ putting } 1+z^3=t^2 \text{ so that } 3z^2dz = 2t\,dt$$

Integrating, $\pm\sqrt{a}\,t = u + b$ or $\pm\sqrt{a}\,(1 + z^3)^{1/2} = x + ay + b$

or $$a(1+z^3) = (x+ay+b)^2, \qquad \text{...(1)}$$

which is a complete integral containing two arbitrary constants a and b.

Singular Integral. Differentiating (1) partially w.r.t. a and b by turn, we get

$$1 + z^3 = 2y(x + ay + b) \qquad \text{...(2)}$$

and $$0 = 2(x + ay + b). \qquad \text{...(3)}$$

Eliminating a and b from (1), (2) and (3), the singular integral is $1 + z^3 = 0$. ...(4)

From (4), $p = \partial z/\partial x = 0$ and $q = \partial z/\partial y = 0$. Thus these values of p and q together with $1 + z^3 = 0$ satisfy the given equation. Hence $1 + z^3 = 0$ is the required singular integral.

Ex. 11. *Find complete and singular integrals of* $q^2 = z^2p^2(1 - p^2)$.

[Madurai Kamraj 2008; CDLU 2004]

Sol. The given equation is of the form $f(p, q, z) = 0$. Let $u = x + ay$, a being an arbitrary constant. Replacing p by dz/du and q by $a(dz/du)$ in the given equation, we have

$$\left(a\frac{dz}{du}\right)^2 = z^2\left(\frac{dz}{du}\right)^2\left[1-\left(\frac{dz}{du}\right)^2\right] \qquad \text{or} \qquad a^2 = z^2\left[1-\left(\frac{dz}{du}\right)^2\right]$$

or $$\left(\frac{dz}{du}\right)^2 = \frac{z^2-a^2}{z^2} \qquad \text{or} \qquad du = \pm\frac{zdz}{(z^2-a^2)^{1/2}}.$$

Integrating, $\quad u + b = \pm(z^2 - a^2)^{1/2} \quad$ or $\quad (x + ay + b)^2 = z^2 - a^2, \quad$ ...(1)

which is a complete integral containing two arbitrary constants a and b.

Singular Integral. Differentiating (1) partially w.r.t. 'a' and 'b', we get

$$-2a = 2y(x + ay + b) \qquad \text{...(2)}$$

and $$0 = 2(x + ay + b). \qquad \text{...(3)}$$

From (2) and (3), $x + ay + b = 0$ and $a = 0$. Putting these values in (1), we get $z = 0$, which is free from a and b. Again, from $z = 0$, we get $p = \partial z/\partial x = 0$ and $q = \partial z/\partial y = 0$. These values *i.e.*, $z = 0$, $p = 0$ and $q = 0$ satisfy the given equation and hence the required singular integral is $z = 0$.

Ex. 12. *Find complete, singular and general integral of* $p^3 + q^3 = 27z$. **[Ravishankar 2005]**

Sol. The given equation is of the form $f(p, q, z) = 0$. Let $u = x + ay$, a being an arbitrary constant. Replacing p by dz/du and q by $a(dz/du)$ in the given equation, we have

$$\left(\frac{dz}{du}\right)^3 + \left(a\frac{dz}{du}\right)^3 = 27z \qquad \text{or} \qquad \frac{dz}{du}(1 + a^3)^{1/3} = 3z^{1/3}$$

or $$du = (1/3) \times (1 + a^3)^{1/3}\, z^{-1/3}\, dz.$$

Integrating, $\quad u + b = (1/3) \times (1 + a^3)^{1/3} \times [z^{2/3}/(2/3)] \quad$ or $\quad 2(u + b) = (1 + a^3)^{1/3}z^{2/3}$

or $\quad 8(u + b)^3 = (1 + a^3)z^2 \quad$ or $\quad 8(x + ay + b)^3 = (1 + a^3)z^2, \quad$ ...(1)

which is a complete integral containing two arbitrary contants a and b.

Singular Integral. Differentiating (1) partially w.r.t. 'a' and 'b', we get

$$24y(x + ay + b)^2 = 3a^2z^2 \qquad \text{...(2)}$$

and $$24(x + ay + b) = 0. \qquad \text{...(3)}$$

From (2) and (3), $x + ay + b = 0$ and $a = 0$. Putting these values in (1), we get $z = 0$, which is free from a and b. Again, from $z = 0$, we get $p = \partial z/\partial x = 0$ and $q = \partial z/\partial y = 0$. These values *i.e.*, $z = 0$, $p = 0$ and $q = 0$ satisfy the given equation and hence the required singular integral is $z = 0$.

General integral. Let $b = \phi(a)$, where ϕ is an arbitrary function. Then (1) becomes

$$8[x + ay + \phi(a)]^3 = z^2(1 + a^3). \qquad \text{...(4)}$$

Differentiating (4) partially w.r.t. 'a', $\quad 24[x + ay + \phi(a)]^2\,[y + \phi'(a)] = 3a^2z^2. \quad$ ...(5)

General integral is obtained by eliminating a from (4) and (5).

Ex. 13. *Find complete and singular integrals of* $z^2(p^2z^2 + q^2) = 1$.

[Delhi Maths Hons 2005; Meerut 2003]

Sol. The given equation is of the form $f(p, q, z) = 0$. Let $u = x + ay$, a being an arbitrary constant. Replacing p by dz/du and q by $a(dz/du)$ in the given equation, we have

$$z^2\left[z^2\left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2\right] = 1 \qquad \text{or} \qquad z^2(z^2 + a^2)\left(\frac{dz}{du}\right)^2 = 1$$

or $$du = \pm\, z(z^2 + a^2)^{1/2}\, dz = \pm\,(1/2) \times (z^2 + a^2)^{1/2}\,(2zdz)$$

Integrating, $$u + b = \pm\,(1/2) \times [(z^2 + a^2)^{3/2}/(3/2)]$$

or $\quad 9(u + b)^2 = (z^2 + a^2)^3 \quad$ or $\quad 9(x + ay + b)^2 = (z^2 + a^2)^3, \quad$ ...(1)

which is a complete integral containing two arbitrary constants a and b.

Singular Integral. Differentiating (1) partially, w.r.t. 'a' and 'b', we get

$$18(x + ay + b)y = 3(z^2 + a^2) \times 2a \qquad \text{...(2)}$$

and

$$18(x + ay + b) = 0. \qquad \text{...(3)}$$

From (2) and (3), $x + ay + b = 0$ and $a = 0$. Putting these values in (1), we get $z = 0$, which is free from a and b. Again, from $z = 0$, we get $p = \partial z/\partial x = 0$ and $q = \partial z/\partial y = 0$. These values *i.e.*, $z = 0$, $p = 0$ and $q = 0$ do not satisfy the given equation. Hence $z = 0$ is not a singular solution of the given equation.

Ex. 14. (*i*) *Find a complete integral of* $z^2(p^2 + q^2 + 1) = k^2$.

[Jabalpur 2004; Banglore 1993; I.A.S. 1996; Meerut 1997]

(*ii*) *Find a complete and singular integral of* $z^2(p^2 + q^2 + 1) = 1$. **[I.A.S. 1979]**

Sol. (*i*) The given equation is of the form $f(p, q\ z) = 0$. Let $u = x + ay$ where a is an arbitrary constant. Replacing p by (dz/du) and q by $a(dz/du)$ in the given equation, we get

$$z^2\left[\left(\frac{dz}{du}\right)^2 + a^2\left(\frac{dz}{du}\right)^2 + 1\right] = k^2 \qquad \text{or} \qquad (1 + a^2)\left(\frac{dz}{du}\right)^2 = \frac{k^2 - z^2}{z^2}$$

or $$\pm (1 + a^2)^{1/2}\frac{z}{(k^2 - z^2)^{1/2}}\,dz = du \qquad \text{or} \qquad \pm\frac{1}{2}(1 + a^2)(k^2 - z^2)^{-1/2}(-2z dz) = du.$$

Integrating, $\pm (1 + a^2)^{1/2}\,(k^2 - z^2)^{1/2} = u + b$ or $(1 + a^2)(k^2 - z^2) = (u + b)^2$

or $$(1 + a^2)(k^2 - z^2) = (x + ay + b)^2.$$

(*ii*) Here $k = 1$. Proceed as in part (*i*) and get complete integral

$$(1 + a^2)(1 - z^2) = (x + ay + b)^2. \qquad \text{...(1)}$$

Differentiating (1) partially w.r.t. a and b, we get

$$2a(1 - z^2) = 2(x + ay + b) \times y \qquad \text{...(2)}$$

and

$$0 = 2(x + ay + b). \qquad \text{...(3)}$$

From (2) and (3), we get $x + ay + b = 0$ and $a = 0$. With these values (1) reduces to $z^2 = 1$, which is free from a and b. Again, from $z^2 = 1$, $p = \partial z/\partial x = 0$ and $q = \partial z/\partial y = 0$. Now, $p = 0$, $q = 0$ and $z^2 = 1$, satisfy the given equation and hence singular integral of the given equation is $z^2 = 1$.

Ex. 15. *Find a complete integral of* (*i*) $q^2y^2 = z(z - px)$ **[Meerut 1997]**

(*ii*) $p^2x^2 = z(z - qy)$.

Sol. (*i*) Given equation can be rewritten as $$\left(y\frac{\partial z}{\partial y}\right)^2 = z\left(z - x\frac{\partial z}{\partial x}\right). \qquad \text{...(1)}$$

We choose new variables X and Y such that $(1/x)dx = dX$ and $(1/y)dy = dY$. ...(2)

so that $\log x = X$ and $\log y = Y$. ...(3)

Using (2), (1) becomes $$\left(\frac{\partial z}{\partial Y}\right)^2 = z\left(z - \frac{\partial z}{\partial X}\right) \qquad \text{or} \qquad Q^2 = z\,(z - P), \qquad \text{...(4)}$$

where $P = \partial z/\partial X$ and $Q = \partial z/\partial Y$. (4) is of the form $f(P, Q, z) = 0$. Let $u = X + aY$, where a is an arbitrary constant. Replacing P by dz/du and Q by $a(dz/du)$ in (4), we get

$$a^2\left(\frac{dz}{du}\right)^2 = z\left(z - \frac{dz}{du}\right) \qquad \text{or} \qquad a^2\left(\frac{dz}{du}\right)^2 + z\frac{dz}{du} - z^2 = 0.$$

$$\therefore \qquad \frac{dz}{du} = \frac{-z \pm (z^2 + 4a^2z^2)^{1/2}}{2a^2} = \frac{-1 \pm (1 + 4a^2)^{1/2}}{2a^2}z = kz, \qquad \text{...(5)}$$

where $$k = [-1 \pm (1 + 4a^2)^{1/2}]/2a^2. \qquad ...(6)$$

From (5), $(1/kz)dz = du$ so that $(1/k) \log z = u + \log b$

or $$\log z^{1/k} = X + aY + \log b = \log x + a \log y + \log b = \log (xby^a).$$

$\therefore$ $z^{1/k} = xby^a$ is complete integral containing two arbitrary constants a and b and an absolute constant k given by (6).

(*ii*) Proceed as in part (*i*). **Ans.** $xby^a = z^{1/k}$, where $k = [-1 \pm (a^2 + 4)^{1/2}]/2$

Ex. 16. *Solve* $p^2 + q^2 = z$.

Sol. Given equation is $$p^2 + q^2 = z, \qquad ... (1)$$
which is of the form $f(p, q, z) = 0$. Let $u = x + ay$, where a is an arbitrary constant. Now, replacing p and q by dz/du and $a(dz/du)$ respectively in (1), we have

$$(du/dz)^2 + a^2(du/dz)^2 = z \qquad \text{or} \qquad (dz/du)^2 = z/(1 + a^2)$$

or $$\frac{dz}{du} = \pm \frac{z^{1/2}}{(1+a^2)^{1/2}} \qquad \text{or} \qquad \pm z^{-1/2}(1+a^2)^{1/2} dz = du$$

Integrating, $$\pm 2z^{1/2}(1+a^2)^{1/2} = u + b \qquad \text{or} \qquad \pm 2z^{1/2}(1+a^2)^{1/2} = x + ay + b$$

Thus, $4z(1 + a^2) = (x + ay + b)^2$, a, b being arbitrary constants ... (2)

(2) is the complete integral of the given equation (1).

Differentiating (2) partially w.r.t. 'a' and 'b', we get

$$8az = 2y(x + ay + b) \qquad \text{or} \qquad 4az = y(x + ay + b) \qquad ... (3)$$

$$0 = 2(x + ay + b) \qquad \text{or} \qquad x + ay + b = 0 \qquad ... (4)$$

Substituting the value of $x + ay + b$ from (4) in (3), we have

$4az = 0$ or $z = 0$, which is the singular solution.

In order to get the general solution, put $b = \psi(a)$ in (2) and get

$$4z(1+a^2) - \{x + ay + \psi(a)\}^2 = 0 \qquad ... (5)$$

Differentiating (5) partially w.r.t. 'a', $8az - 2\{x + ay + \psi(a)\} \times \{y + \psi'(a)\} = 0$ ... (6)

The required general solution is given by (5) and (6)

Ex. 17. *Find the complete integral of* $16p^2z^2 + 9q^2z^2 + 4(z^2 - 1) = 0$

Sol. Given equation is of the form $f(p, q, z) = 0$

Let $u = x + ay$, a being an arbitrary constant. ... (1)

Now replacing p and q by dz/du and $a(dz/du)$ respectively in the given equation, we have

$$16z^2\ (dz/du)^2 + 9a^2z^2\ (dz/du)^2 + 4(z^2 - 1) = 0$$

or $$(16 + 9a^2)z^2 \left(\frac{dz}{du}\right)^2 = 4(1 - z^2) \qquad \text{or} \qquad \frac{dz}{du} = \frac{2(1-z^2)^{1/2}}{z(16+9a^2)^{1/2}}$$

or $$(-1/2) \times (16 + 9a^2)^{1/2}\ (1 - z^2)^{-1/2}(-2z)dz = du$$

Integrating, $-(16 + 9a^2)^{1/2}\ (1 - z^2)^{1/2} = u + b = x + ay + b$, by (1)

or $(16 + 9a^2)\ (1 - z^2) = (x + ay + b)^2$ is the complete integral, a, b being arbitrary constants

Ex. 18. *Find the complete integral of* $pq = x^4y^3z^2$

Sol. Re–writing the given equation, we get $$(\partial z / x^4 \partial x)\ (\partial z / y^3 \partial y) = z^2 \qquad ... (1)$$

Putting $x^4 dx = dX$, $y^3 dy = dY$ so that $x^5/5 = X$, $y^4/4 = Y$, (1) gives

$$(\partial z/\partial X)\ (\partial z/\partial Y) = z^2 \qquad \text{or} \qquad PQ = z^2 \qquad \text{... (2)}$$

which is of the form $f(P, Q, z) = 0$. Let $u = X + aY$, a being an arbitrary constant. Replacing P and Q be dz/du and $a(dz/du)$ respectively in (2), we get

$$a\,(dz/du)^2 = z^2 \qquad \text{giving} \qquad (\sqrt{a}/z)dz = du$$

Integrating $\sqrt{a}\log z = u + b = X + aY + b$, as $u = X + aY$

or $\sqrt{a}\ \log z = x^5/5 + (ay^4)/4 + b,\ a, b$ being arbitrars constants

EXERCISE 3(E)

Find the complete integral of the following equations (1–7)

1. $z = p^2 - q^2$. **Ans.** $x + ay + b = 4(1 - a^2)z$

2. $zpq = p + q$. **(Delhi 2007)** **Ans.** $x + ay + b = (4az)/(1 - a)$

3. $z^2p^2 + q^2 = p^2q$. **Ans.** $z = a \tan(x + ay + b)$

4. $p^2z^2 + q^2 = 1$. **[Delhi B.A. (Prog) II 2010, 11; M.S. Univ. T.N. 2007, Nagpur 2001; Meerut 2008]** **Ans.** $x + ay + b = \pm[(z/2)\times(z^2 + a^2)^{1/2} + (a^2/2)\times\sinh^{-1}(z/a)]$

5. $p^3 = qz$. **(Delhi B.Sc. (Prog) II 2007)** **Ans.** $4z = (x + ay + b)^2$

6. $16z^2p^2 + 25z^2a^2 + 9z^2 - 81 = 0$. **Ans.** $(16 + 25a^2)(9 - z^2) = 9(x + ay + b)^2$

7. $z^2 = 1 + p^2 + q^2$. **Ans.** $z = \cosh\{x + ay + b)/(1 + a^2)^{1/2}\}$

8. Using Charpit's method, discuss how to solve equations of the form $f(z, p, q) = 0$. Hence find complete integral of the equation $9(p^2z + q^2) = 4$. **[Delhi Maths (H) 2006]**

Hint : Refer Art. 3.14 and Ex. 1 of Art. 3.15

Solve the following partial differential equations (9 – 14)

9. $z^2(p^2 + q^2 + 2) = 1$ **Ans. C.I.** $(1 + a^2)(1 - 2z^2) = 4(x + ay + b)^2$; **S.S.** $2z^2 - 1 = 0$; **G.S.** It is given by $(1 + a^2)(1 - 2z^2) - \{x + ay + \psi(a)\}^2 = 0$, $a(1 - 2z^2) - 4\{x + ay + \psi(a)|\times\{y + \psi'(a)\} = 0$

10. $z = pq$. **Ans. C.I.** $4az = (x + ay + b)^2$; **S.S.** $z = 0$; **G.S.** It is given by $4az - \{x + ay + \psi(a)\}^2$, $2z - \{x + ay + \psi(a)\}\times\{y + \psi'(a)\} = 0$

11. $p(1 - q^2) = q(1 - z)$. **Ans. C.I.** $4(1 - a + az) = (x + ay + b)^2$; **S.S.** Does not exist; **G.S.** It is given by $4(1 - a + az) - \{x + ay + \psi(a)\}^2 = 0$, $2z - 2 - \{x + ay + \psi(a)\}\{y + \psi'(a)\} = 0$

12. $p^2 + pq = 4z$. **Ans. C.I.** $(1 + a)z = (x + ay + b)^2$; **S.S.** $z = 0$; **G.S.** $(1 + a)z - \{x + ay + \psi(a)\}^2 = 0$, $z - 2\{x + ay + \psi(a)\}\times\{y + \psi'(a)\} = 0$

13. $p^3 + q^3 = 3pqz$, $z > 0$. **Ans. C.S.** $(1 + a^3)\log z = 3a(x + ay) + b$; **S.S.** Does not exist. **G.S.** It is given by $(1 + a^3)\log z - 3a(x + ay) - \psi(a) = 0$, $3a^2\log z - 3x - 6ay - \psi'(a) = 0$

14. $p^2 + q^2 = 4z$. **Ans. C.I.** $4(1 + a^2)z - (x + ay + b)^2 = 0$; **S.S.** $z = 0$; **G.S.** It is given by $(1 + a^2)z - \{x + ay + \psi(a)\}^2 = 0$, $az - \{x + ay + \psi(a)\}\times\{y + \psi'(a)\} = 0$.

3.17. Standard form IV. Equation of the form $f_1(x, p) = f_2(y, q)$. *i.e.*, a form in which z does not appear and the terms containing x and p are on one side and those containing y and q on the other side. **[Bhopal 2010; Ravishankar 1999]**

Let $$F(x, y, z, p, q) = f_1(x, p) - f_2(y, q) = 0. \qquad ...(1)$$

Then Charpit's auxiliary equations are

$$\frac{dp}{\frac{\partial F}{\partial x} + p\frac{\partial F}{\partial z}} = \frac{dq}{\frac{\partial F}{\partial y} + q\frac{\partial F}{\partial z}} = \frac{dz}{-p\frac{\partial F}{\partial p} - q\frac{\partial F}{\partial q}} = \frac{dx}{-\frac{\partial F}{\partial p}} = \frac{dy}{-\frac{\partial F}{\partial q}}$$

or $$\frac{dp}{\partial f_1/\partial x} = \frac{dq}{-\partial f_2/\partial y} = \frac{dz}{-p(\partial f_1/\partial p) + q(\partial f_2/\partial q)} = \frac{dx}{-\partial f_1/\partial p} = \frac{dy}{\partial f_2/\partial q}, \text{ by (1)}$$

Taking the first and the fourth ratios, we have

$(\partial f_1 / \partial p)\, dp + (\partial f_1 / \partial x) dx = 0$ or $df_1 = 0$.

Integrating, $f_1 = a$, a being an arbitrary constant.

$\therefore$ (1) $\Rightarrow$ $f_1(x, p) = f_2(y, q) = a$. ...(2)

Now, (2) $\Rightarrow$ $f_1(x, p) = a$ and $f_2(y, q) = a$. ...(3)

From (3), on solving for p and q respectively, we get

$p = F_1(x, a)$, say and $q = F_2(y, a)$, say ...(4)

Substituting these values in $dz = p\, dx + q\, dy$, we get $dz = F_1(x, a)\, dx + F_2(y, a)\, dy$.

Integrating, $$z = \int F_1(x,a)\, dx + \int F_2(y,a) dy + b,$$

which is a complete integral containing two arbitrary constants a and b.

Remark 1. Sometimes change of variables can be employed to reduce a given equation in the standard form IV.

Remark 2. Singular and general integral are obtained by well known methods.

3.18. SOLVED EXAMPLES BASED ON ART 3.17

Ex. 1. *Find a complete integral of* $x(1 + y)p = y(1 + x)q$. **[Agra 1991]**

Sol. Separating p and x from q and y, the given equation reduces to

$$(xp)/(1 + x) = (yq)/(1 + y)$$

Equating each side to an arbitrary constant a, we have

$$\frac{xp}{1+x} = a \quad \text{and} \quad \frac{yq}{1+y} = a \quad \text{so that} \quad p = a\left(\frac{1+x}{x}\right) \quad \text{and} \quad q = a\left(\frac{1+y}{y}\right).$$

Putting these values of p and q in $dz = p\, dx + q\, dy$, we get

$$dz = \frac{a(1+x)}{x}dx + \frac{a(1+y)}{y}dy \quad \text{or} \quad dz = a\left(\frac{1}{x}+1\right)dx + a\left(\frac{1}{y}+1\right)dy.$$

Integrating, $z = a(\log x + x) + a(\log y + y) + b = a(\log xy + x + y) + b$,

which is a complete integral containing two arbitrary constants a and b.

Ex. 2. *Find a complete integral of* $p - 3x^2 = q^2 - y$. **[Meerut 1996]**

Sol. Equating each side to an arbitrary constant a, we get

$p - 3x^2 = a$ and $q^2 - y = a$ so that $p = a + 3x^2$ and $q = (a + y)^{1/2}$.

Putting these values of p and q in $dz = pdx + qdy$, we get

$dz = (a + 3x^2)dx + (a + y)^{1/2}\, dy$ so that $z = ax + x^3 + (2/3) \times (a + y)^{3/2} + b$.

Ex. 3. *Find a complete integral of* $yp = 2yx + \log q$. **[Ravishankar 2005]**

Sol. Rewritting the given equation, $p = 2x + (1/y) \log q$ or $p - 2x = (1/y) \log q$.

Equating each side to an arbitrary constant a, we get

$p - 2x = a$ and $(1/y) \log q = a$ so that $p = a + 2x$ and $q = e^{ay}$.

Putting these values of p and q in $dz = p\,dx + q\,dy$, we get

$dz = (a + 2x)dx + e^{ay}dy$ so that $z = (ax + x^2) + (1/a) \times e^{ay} + b$.

Ex. 4. *Find a complete integral of* $q = px + p^2$.

[Agra 1995; Meerut 1994; Bilaspur 2004; Jabalpur 1998]

Sol. Equating each side of the given equation to an arbitrary constant a, we have

$q = a$ and $px + p^2 = a$ or $q = a$ and $p^2 + px - a = 0$.

$\therefore$ $q = a$ and $p = [-x \pm (x^2 + 4a)^{1/2}]/2$.

Putting thse values of p and q in $dz = p\,dx + q\,dy$, we get

$$dz = (1/2) \times [-x \pm (x^2 + 4a)^{1/2}]dx + a\,dy.$$

Integrating, $$z = -\frac{x^2}{4} \pm \frac{1}{2}\left[\frac{x}{2}\sqrt{(x^2+4a)} + 2a \log\left\{x + \sqrt{(x^2+4a)}\right\}\right] + ay + b.$$

Ex. 5. *Solve* $py + qx + pq = 0$. **[Kurukshetra 2004; I.A.S 1990]**

Sol. Given $py + q(x + p) = 0$. or $p/(p + x) = -q/y$.

Equating each side to an arbitrary constant a, we get

$p/(p + x) = a$ and $-q/y = a$ $\Rightarrow$ $p = (xa)/(1 - a)$ and $q = -ay$.

Putting thse values of p and q in $dz = p\,dx + q\,dy$, we get

$dz = \{a/(1 - a)\}\, x\,dx - ay\,dy$ so that $z = \{a/(1 - a)\} \times (x^2/2) - a \times (y^2/2) + b/2$

or $2z = \{a/(1 - a)\}x^2 - ay^2 + b$, a, b being arbitrary constants.

Ex. 6. *Find a complete integral of* $z^2(p^2 + q^2) = x^2 + y^2$, *i.e.*, $z^2[(\partial z/\partial x)^2 + (\partial z/\partial y)^2] = x^2 + y^2$.

[Agra 2006; Jabalpur 2004; Rewa 2002 Sagar 1999; Vikram 1996 Delhi Maths Hons 1990; I.A.S. 1989; Kanpur 1994; Meerut 2003]

Sol. Given $$z^2\left(\frac{\partial z}{\partial x}\right)^2 + z^2\left(\frac{\partial z}{\partial y}\right)^2 = x^2 + y^2 \quad \text{or} \quad \left(z\frac{\partial z}{\partial x}\right)^2 + \left(z\frac{\partial z}{\partial y}\right)^2 = x^2 + y^2. \quad ...(1)$$

Let $z\,dz = dz$ so that $z^2/2 = Z$. ...(2)

Using (2), (1) becomes $(\partial Z/\partial x)^2 + (\partial Z/\partial y)^2 = x^2 + y^2$ or $P^2 + Q^2 = x^2 + y^2$,

where $P = \partial Z/\partial x$ and $Q = \partial Z/\partial y$. Separating P and x from Q and y, we get

$$P^2 - x^2 = y^2 - Q^2.$$

Equating each side of the above equation to an arbitrary constant a^2, we get

$P^2 - x^2 = a^2$ and $y^2 - Q^2 = a^2$ so that $P = (a^2 + x^2)^{1/2}$ and $Q = (y^2 - a^2)^{1/2}$.

Putting these values of P and Q in $dZ = P\,dx + Q\,dy$, we have

$$dZ = (a^2 + x^2)^{1/2}\,dx + (y^2 - a^2)^{1/2}\,dy.$$

Integrating, $$Z = (x/2) \times (a^2 + x^2)^{1/2} + (a^2/2) \times \log\{x + (a^2 + x^2)^{1/2}\}$$
$$+ (y/2) \times (y^2 - a^2)^{1/2} - (a^2/2) \times \log\{y + (y^2 - a^2)^{1/2}\} + (b/2)$$

or $$z^2 = x^2(a^2 + x^2)^{1/2} + a^2 \log[x + (a^2 + x^2)^{1/2}\} + y(y^2 - a^2)^{1/2} - a^2\log\{y + (y^2 - a^2)^{1/2}\} + b$$

$[\because$ From (2), $Z = z^2/2]$

Ex. 7. *Find a complete integral of* $z(p^2 - q^2) = x - y$.

[Bilaspur 2003; Indore 2002, 02; Jiwaji 2000; Bangalore 1995; I.A.S 1989]

Sol. Re–writting the given equation, $(\sqrt{z}\partial z/\partial x)^2 - (\sqrt{z}\partial z/\partial y)^2 = x - y.$ ...(1)

Let $\sqrt{z}\, dz = dZ$ so that $(2/3) \times z^{3/2} = Z.$ ...(2)

Using (2), (1) becomes $(\partial Z/\partial x)^2 - (\partial Z/\partial y)^2 = x - y$ or $P^2 - Q^2 = x - y,$ where $P = \partial Z/\partial x$ and $Q = \partial Z/\partial y$. Separating P and x from Q and y, we get

$$P^2 - x = Q^2 - y. \qquad ...(3)$$

Equating each side to an arbitrary constant a, we get

$P^2 - x = a$ and $Q^2 - y = a$ so that $P = (x + a)^{1/2}$ and $Q = (y + a)^{1/2}$

Putting these values of P and Q in $dZ = P\,dx + Q\,dy$, $dZ = (x + a)^{1/2}dx + (y + a)^{1/2}\,dy.$

Integrating, $Z = (2/3) \times (x + a)^{3/2} + (2/3) \times (y + b)^{3/2} + 2b/3$

or $(2/3) \times z^{3/2} = (2/3) \times (x + a)^{3/2} + (2/3) \times (y + b)^{3/2} + 2b/3$, as $Z = (2/3) \times z^{3/2}$

or $z^{3/2} = (x + a)^{3/2} + (y + a)^{3/2} + b$, a, b being arbitrary constants.

Ex. 8. *Find a complete integral of* $z(xp - yq) = y^2 - x^2$.

Sol. Re–writting the given equation, we have

$$xz\frac{\partial z}{\partial x} - yz\frac{\partial z}{\partial y} = y^2 - x^2 \qquad \text{or} \qquad x\left(z\frac{\partial z}{\partial x}\right) - y\left(z\frac{\partial z}{\partial y}\right) = y^2 - x^2. \qquad ...(1)$$

Let $z\,dz = dZ$ so that $z^2/2 = Z.$ ...(2)

Using (2), (1) becomes $x(\partial Z/\partial x) - y(\partial Z/\partial y) = y^2 - x^2$ or $xP - yQ = y^2 - x^2,$ where $P = \partial Z/\partial x$ and $Q = \partial Z/\partial y$. Separating P and x from Q and y, we get

$$xP + x^2 = yQ + y^2.$$

Equating each side to an arbitrary constant a, we have

$xP + x^2 = a$ and $yQ + y^2 = a$ so that $P = a/x - x$ and $Q = a/y - y.$

Putting these values of P and Q in $dZ = P\,dx + Q\,dy$, $dZ = (a/x - x)dx + (a/y - y)dy.$

Integrating, $Z = a \log x - (x^2/2) + a \log y - (y^2/2) + b/2$

or $z^2/2 = a(\log x + \log y) - (x^2 + y^2 - b)/2$ or $z^2 = 2a \log (xy) - x^2 - y^2 + b.$

Ex. 9. *Find a complete integral of* $p^2 + q^2 = z^2(x + y)$. **[Agra 2010; M.S. Univ. T.N. 2007]**

Sol. Given $\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 = z^2(x + y)$ or $\left(\frac{1}{z}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{1}{z}\frac{\partial z}{\partial y}\right)^2 = x + y$ ....(1)

Let $(1/z)dz = dZ$ so that $\log z = Z.$ ...(2)

Using (2), (1) becomes $(\partial Z/\partial x)^2 + (\partial Z/\partial y)^2 = x + y$ or $P^2 + Q^2 = x + y,$ where $P = \partial Z/\partial x$ and $Q = \partial Z/\partial y$. Separating P and x from Q and y, we get

$$P^2 - x = y - Q^2.$$

Equating each side to an arbitrary constant a, we have

$P^2 - x = a$ and $y - Q^2 = a$ so that $P = (a + x)^{1/2}$ and $Q = (y - a)^{1/2}.$

Putting these values of P and Q in $dZ = P\,dx + Q\,dy$, $dZ = (a + x)^{1/2}dx + (y - a)^{1/2}dy.$

Integrating, $Z = (2/3) \times [(a + x)^{3/2} + (y - a)^{3/2}] + (2/3) \times b$

$\therefore$ $\log z = (2/3) \times [(a + x)^{3/2} + (y - a)^{3/2} + b]$ is a complete integral, using $Z = \log z$

Ex. 10. *Find a complete integral of* $p^2 + q^2 = (x^2 + y^2)z$. **[Delhi Maths Hons. 1995]**

Sol. The given equation can be rewritten as

$$\frac{1}{z}\left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right] = x^2 + y^2 \qquad \text{or} \qquad \left(\frac{1}{\sqrt{z}}\frac{\partial z}{\partial x}\right)^2 + \left(\frac{1}{\sqrt{z}}\frac{\partial z}{\partial y}\right)^2 = x^2 + y^2. \qquad ...(1)$$

Let $\left(1/\sqrt{z}\right)dz = dZ$ *i.e.*, $z^{-1/2}dz = dZ$ so that $2\sqrt{z} = Z$. ...(2)

Using (2), (1) becomes $(\partial Z/\partial x)^2 + (\partial Z/\partial y)^2 = x^2 + y^2$ or $P^2 + Q^2 = x^2 + y^2$,

where $P = \partial Z/\partial x$ *and* $Q = \partial Z/\partial y$. Separting P and x from Q and y, we get

$$P^2 - x^2 = y^2 - Q^2.$$

Equating each side to an arbitrary constant a^2, we have

$P^2 - x^2 = a^2$ and $y^2 - Q^2 = a^2$ so that $P = (a^2 + x^2)^{1/2}$ and $Q = (y^2 - a^2)^{1/2}$

Putting these values of P and Q in $dZ = P\,dx + Q\,dy$, $dZ = (a^2 + x^2)^{1/2}dx + (y^2 - a^2)^{1/2}dy$.

Integrating, $$Z = \frac{x}{2}(x^2 + a^2)^{1/2} + \frac{a^2}{2}\sinh^{-1}\frac{x}{a} + \frac{y}{2}(y^2 - a^2)^{1/2} - \frac{a^2}{2}\cosh^{-1}\frac{y}{a} + \frac{b}{2}$$

or $4z^{1/2} = x(x^2 + a^2)^{1/2} + a^2\sinh^{-1}(x/a) + y(y^2 - a^2)^{1/2} - a^2\cosh^{-1}(y/a) + b$, as $Z = 2\sqrt{z}$

Ex. 11. *Find a complete integral of* $(p^2/x) - (q^2/y) = (1/z) \times [(1/x) + (1/y)]$.

[Delhi B.Sc. Hons. 1996]

Sol. The given equation can be re–written as

$$\frac{z}{x}\left(\frac{\partial z}{\partial x}\right)^2 - \frac{z}{y}\left(\frac{\partial z}{\partial y}\right)^2 = \frac{1}{x} + \frac{1}{y} \quad \text{or} \quad \frac{1}{x}\left(\sqrt{z}\frac{\partial z}{\partial x}\right)^2 - \frac{1}{y}\left(\sqrt{z}\frac{\partial z}{\partial y}\right)^2 = \frac{1}{x} + \frac{1}{y} \quad ....(1)$$

Let $\sqrt{z}\,dz = dZ$ so that $(2/3) \times z^{3/2} = Z$. ...(2)

Using (2), (1) becomes

$$\frac{1}{x}\left(\frac{\partial Z}{\partial x}\right)^2 - \frac{1}{y}\left(\frac{\partial Z}{\partial y}\right)^2 = \frac{1}{x} + \frac{1}{y} \quad \text{or} \quad \frac{P^2}{x} - \frac{Q^2}{y} = \frac{1}{x} + \frac{1}{y},$$

where $P = \partial Z/\partial x$ and $Q = \partial Z/\partial y$. Separating P and x from Q and y, we get

$$(P^2 - 1)/x = (Q^2 + 1)/y.$$

Equating each side to an arbitrary constant a, we have

$(P^2 - 1)/x = a$ and $(Q^2 + 1)/y = a$ so that $P = (1 + ax)^{1/2}$ and $Q = (ay - 1)^{1/2}$.

Putting these values of P and Q in $dZ = P\,dx + Q\,dy$, $dZ = (1 + ax)^{1/2}dx + (ay - 1)^{1/2}dy$.

Integrating, $Z = (2/3a) \times (1 + ax)^{3/2} + (2/3a) \times (ay - 1)^{3/2} + (2/3a) \times b$

or $az^{3/2} = (1 + ax)^{3/2} + (ay - 1)^{3/2} + b$, as $Z = (2/3) \times z^{3/2}$.

Ex. 12. *Find a complete integral of* $yzp^2 = q$. **[M.S. Univ. T.N. 2007]**

Sol. Given $$yz^2\left(\frac{\partial z}{\partial x}\right)^2 = z\frac{\partial z}{\partial y} \quad \text{or} \quad y\left(z\frac{\partial z}{\partial x}\right)^2 = \left(z\frac{\partial z}{\partial y}\right). \quad ...(1)$$

Let $z\,dz = dZ$ so that $z^2/2 = Z$. ...(2)

Using (2), (1) becomes $y(\partial Z/\partial x)^2 = \partial Z/\partial y$ or $yP^2 = Q$, ...(3)

where $P = \partial Z/\partial x$ and $Q = \partial Z/\partial y$. Separating P from y and Q, we get

$P^2 = Q/y = a^2$, (say) ; a being an arbitrary constant. Hence $P = a$ and $Q = ya^2$.

Then, $dZ = P\,dx + Q\,dy$ reduces to $dZ = a\,dx + ya^2dy$ so that $Z = ax + (a^2/y^2)/2 + b/2$

or $z^2/2 = ax + (a^2y^2)/2 + b/2$ or $z^2 = 2ax + a^2y + b$.

Ex. 13. *Find a complete integral of* $zpy^2 = x(y^2 + z^2q^2)$.

Sol. Given $$y^2\left(z\frac{\partial z}{\partial x}\right) = x\,y^2 + x\left(z\frac{\partial z}{\partial y}\right)^2 \quad ...(1)$$

Let $z\,dz = dZ$ so that $z^2/2 = Z$. ...(2)

Using (2), (1) becomes $y^2(\partial Z/\partial x) = xy^2 + x\,(\partial Z/\partial y)^2$ or $y^2P = x(y^2 + Q^2)$,

where $P = \partial Z/\partial x$ and $Q = \partial Z/\partial y$. Separating P and x from, Q and y, we get

$$P/x = (y^2 + Q^2)/y^2.$$

Equating each side to an arbitrary constant a, we get

$P/x = a$ and $1 + (Q^2/y^2) = a$ so that $P = ax$ and $Q = \pm (a-1)^{1/2}y.$

$\therefore$ $$dZ = P\,dx + Q\,dy = ax\,dx \pm (a-1)^{1/2}y\,dy$$

Integrating, $Z = (ax^2/2) \pm (a-1)^{1/2}(y^2/2) + b/2$ or $z^2 = ax^2 \pm (a-1)^{1/2}y^2 + b$, as $Z = z^2/2$.

Ex. 14. *Find the complete integral of the partial differential equation*

$$2p^2q^2 + 3x^2y^2 = 8x^2q^2(x^2+y^2)$$ **[I.A.S. 2001]**

Sol. Re–writing the given equation, we have

$2q^2(p^2 - 4x^4) = x^2y^2(8q^2-3)$ or $(p^2-4x^4)/x^2 = y^2\ (8q^2-3)/2q^2 = 4a^2$, say

where a is an arbitrary constant. Then, $p^2 = 4x^2(a^2+x^2)$ and $8q^2(y^2-a^2) = 3y^2$

so that $p = 2x(a^2+x^2)^{1/2}$ and $q = (3/2)^{1/2} \times (y/2) \times (y^2-a^2)^{-1/2}$

Substituting these values in $dz = p\,dx + q\,dy$, we get

$$dz = 2x(a^2+x^2)^{1/2}dx + (3/2)^{1/2} \times (y/2) \times (y^2-a^2)^{-1/2}dy$$

Integrating, $z = 2\int x(a^2+x^2)^{1/2}dx + (3/2)^{1/2} \times (1/2) \times \int y(y^2-a^2)^{-1/2}dy + b$ ... (1)

Put $x^2 + a^2 = u$ and $y^2 - a^2 = v$ so that $2xdx = du$ and $2ydy = dv$ ... (2)

i.e., $xdx = (1/2) \times du$ and $ydy = (1/2) \times dv$. Then (1) reduces to

$$z = \int u^{1/2}du + (3/2)^{1/2} \times (1/4) \times \int v^{-1/2}dv + b$$

or $$z = (2/3) \times u^{3/2} + (3/2)^{1/2} \times (1/4) \times 2v^{1/2} + b$$

or $$z = (2/3) \times (x^2+a^2)^{3/2} + (3/2)^{1/2} \times (1/2) \times (y^2-a^2)^{1/2} + b,$$

which is the required complete integral containing a and b as arbitrary constants.

Ex. 15. *Find the complete integral of the partial differential equation* $p^2q^2 + x^2y^2 = x^2q^2(x^2+y^2)$ **[Delhi Maths (H) 2002; Agra 2005]**

Sol. Re–writing, $p^2/x^2 + y^2/q^2 = x^2 + y^2$ or $(p^2/x^2) - x^2 = y^2 - (y^2/q^2) = a^2$, say

$\Rightarrow$ $p = x(x^2+a^2)^{1/2}$, and $q = y/(y^2 - a^2)^{1/2}$

$\therefore$ $dz = p\,dx + q\,dy$ becomes $dz = x(x^2+a^2)^{1/2}dx + y(y^2-a^2)^{-1/2}dy$

Integrating, $$z = (1/3) \times (x^2+a^2)^{1/2} + (y^2-a^2)^{1/2} + b,$$

which is complete integral with a and b as arbitrary constants.

Ex. 16. *Find the complete integral of* $(1-x^2)yp^2 + x^2q = 0$

Sol. Re–writing, we have $(x^2-1)p^2/x^2 = q/y = a^2$, say

$\therefore$ $p = ax/(x^2-1)^{1/2}$ and $q = a^2y$. Hence $dz = p\,dx + q\,dy$ becomes

$dz = ax(x^2-1)^{-1/2}dx + a^2ydy$ so that $z = a(x^2-1)^{1/2} + (a^2y^2)/2 + b$.

Ex. 17. *Find the the complete integral of* $p + q - 2px - 2qy + 1 = 0$.

Sol. Re–writing, $p - 2px = 2qy - q - 1 = a$, say

$\therefore$ $p = \dfrac{a}{1-2x}$, $q = \dfrac{a+1}{2y-1}$ and so $dz = p\,dx + q\,dy = \dfrac{a\,dx}{1-2x} + \dfrac{(a+1)dy}{2y-1}$

Integrating, $z = -(a/2) \times \log|1-2x| + (1/2) \times (a+1) \log|2y+1| + b$.

Ex. 18. *Find the complete integral of* $2x(z^2q^2+1) = pz$

Sol. Re–writing the given equation, we have $2x\,\{(z\partial z/\partial y)^2+1\} = (z\partial z/\partial x)$ ... (1)

Putting $z\,dz = dZ$ so that $z^2/2 = Z$, (1) reduces to

$2x\{(\partial Z/\partial y)^2+1\} = \partial Z/\partial x$ or $2x(Q^2+1) = P$, ... (2)

where $P = \partial Z/\partial x$ and $Q = \partial Z/\partial y$. Re–writing (2), we have

$(P/2x) - 1 = Q^2 = a^2$, say so that $P = 2x(1+a^2)$, $Q = a$

$\therefore\ dZ = P\,dx + Q\,dy$ becomes $dZ = 2x(1+a^2)dx + a\,dy$

Integrating, $Z = (1+a^2)x^2 + ay + b$ or $z^2/2 = (1+a^2)x + ay + b$.

EXERCISE 3 (F)

Find a complete integral of the following equations (1 – 9)

1(a). $p^2 = q + x$. **Ans.** $z = (2/3) \times (a+x)^{3/2} + ay + b$.

(b). $p^2y\,(1+x^2) = qx^2$. **[Delhi B.A (Prog) II 2011]** **Ans.** $z = a(1+x^2)^{1/2} + (a^2y/2) + b$.

2. $p^2 + q^2 = x + y$. **[Agra 2009; Meerut 2007]** **Ans.** $3z = 2(x+a)^{3/2} + 2(y-a)^{3/2} + b$.

3. $p^2 + q^2 = x^2 + y^2$. **[Jiwaji 1999; Ravishankar 2003]**

Ans. $2z = x(x^2+a^2)^{1/2} + a^2\sinh^{-1}(x/a) + y(y^2-a^2)^{1/2} - a^2\cosh^{-1}(y/a) + b$.

4. $pe^y = qe^x$. **[Jiwaji 1996]** **Ans.** $z = ae^x + ae^y + b$.

5. $p^{1/3} - q^{1/3} = 3x - 3y$. **Ans.** $z = 3x^3 - 3ax^2 + a^2x + 2y^4 - 4ay^3 + 3a^2y^2 - a^3y + b$.

6. $q = 2yp^2$. **Ans.** $z = ax + a^2y^2 + b$.

7. $p^2 - y^3q = x^2 - y^2$. **Ans.** $2z = x(x^2+a^2)^{1/2} + a^2\sinh^{-1}(x/a) - (a^2/2) + \log y^2 + b$.

8. $z^2(p^2+q^2) = x^2 + e^{2y}$. **[Delhi Maths (H) 2005]**

Ans. $z^2 = x(x^2+a)^{1/2} + a\sinh^{-1}(x/\sqrt{a}) + 2(e^{2y}-a)^{1/2} - \sqrt{a}\,\tan^{-1}\{(e^{2y}-a)/a\}^{1/2} + b$

9. $p + q = px + qy$. **[Bangalore 1996]** **Ans.** $z = -a\log(1-x) + a\log(y-1) + b$.

Solve the following partial differential equations: (10 – 17)

10. $pq = xy$ **Ans. C.I.** $2z = ax^2 + y^2/a + b$; **S.S.** Does not exist **G.S.** $2z - ax^2 - y^2/a - \psi(a) = 0$,

$x^2 - y^2/a^2 + \psi'(a) = 0$

11. $\sqrt{p} + \sqrt{q} = 2x$. **Ans. C.I.** $z = (2x-a)^3/6 + a^2y + b$; **S.S.** Does not exit; **G.S.**

$z - (2x-a)^3/6 - a^2y - \psi(a) = 0,\ \ (2x-a)^2/2 - 2ay - \psi'(a) = 0$

12. $q\,(p - \cos x) = \cos y$. **Ans.** $z = ax + \sin x + (1/a) \times \sin y + b$; **S.S.** Does not exist

G.S. $z - ax - (1/a) \times \sin y - \psi(a) = 0,\ \ -x - (1/a^2) \times \sin y + \psi'(a) = 0$

13. $q = xyp^2$ **Ans. C.I.** $2z = 4\sqrt{ax} + ay^2 + b$; **S.S.** Does not exist; **G.S.** $2z - 4\sqrt{ax} - ay^2 - \psi(a) = 0$,

$2\sqrt{(x/a)} + y^2 + \psi'(a) = 0$

14. $x^2p^2 = q^2y$. **Ans. C.I.** $z = \sqrt{a}\log x + 2\sqrt{ay} + b$; **S.S.** Does not exist.

G.S. $z - \sqrt{a}\log x - 2\sqrt{ay} - \psi(a) = 0,\ \ \log x + 2\sqrt{y} + 2\sqrt{a}\,\psi'(a) = 0$

15. $p - q = x^2 + y^2$. **Ans. C.I.** $z = (x^3 - y^3)/3 + a(x+y) + b$; **S.S.** Does not exit;

G.S. $z - (x^3 - y^3)/3 - a(x+y) - \psi(a) = 0,\ \ x + y + \psi'(a) = 0$

16. $p^2 - x = q^2 - y$ **Ans. C.I.** $3z = 2(x+a)^{3/2} + 2(y+a)^{3/2} + b$; **S.S.** Does not exist

G.S. $3z-2(x+a)^{3/2}-2(y+a)^{3/2}-\psi(a)=0, 3(x+a)^{1/2}+3(y+a)^{1/2}+\psi'(a)=0$

17. $px+q=p^2$. **Ans. C.I.** $z=(1/4)\times\left\{x^2+x(x^2+4a)^{1/2}\right\}+a\log\{x+(x^2+4a^2)^{1/2}\}\ ay$ $+b$; **S.S.** Does not exist **G.S.** $z-(1/4)\times\{x^2+x(x^2+4a)^{1/2}\}-a\log\{x+(x^2+4a)^{1/2}\}-ay$ $-\psi(a)=0,\ (x/2)\times(x^2+4a)^{-1/2}+\log\{x+(x^2+4a)^{1/2}\}+(2a)/[\{x+(x^2+4a)^{1/2}\}\times(x^2+4a)]$ $+y+\psi'(a)=0$

3.19. JACOBI'S METHOD [Himachel 2005; Meerut 2005, 06, 08; Pune 2010]

This method is used for solving partial differential equations involving three or more independent variables. The central idea of Jacobi's method is almost the same as that of Charpit's method for two independent variables. We begin with the case of three independent variables. The results arrived at are, however, general and will be used with suitable modification for the case of four independent variables and so on.

Let $p_1=\partial z/\partial x_1$, $p_2=\partial z/\partial x_2$ and $p_3=\partial z/\partial x_3$.

Consider a partial differential equation $f(x_1, x_2, x_3, p_1, p_2, p_3)=0$, ...(1)

where the dependent variable z does not occur except by its partial differential coefficients with respect to the three independent variables x_1, x_2, x_3.

The main idea in Jacobi's method is to get two additional partial differential equations of the first order $F_1(x_1, x_2, x_3, p_1, p_2, p_3)=a_1$ ...(2)

and $F_2(x_1, x_2, x_3, p_1, p_2, p_3)=a_2$, ...(3)

where a_1 and a_2 are two arbitrary constants such that (1), (2) and (3) can be solved for p_1, p_2, p_3 in terms of x_1, x_2, x_3 which when substituted in

$$dz=p_1dx_1+p_2dx_2+p_3dx_3, \qquad ...(4)$$

makes it integrable, for which the conditions are

$$\partial p_2/\partial x_1=\partial p_1/\partial x_2, \qquad \partial p_3/\partial x_2=\partial p_2/\partial x_3, \qquad \text{and} \qquad \partial p_1/\partial x_3=\partial p_3/\partial x_1 \qquad ...(5)$$

Differentiating (1) and (2) partially, w.r.t. x_1, we have

$$\frac{\partial f}{\partial x_1}+\frac{\partial f}{\partial p_1}\frac{\partial p_1}{\partial x_1}+\frac{\partial f}{\partial p_2}\frac{\partial p_2}{\partial x_1}+\frac{\partial f}{\partial p_3}\frac{\partial p_3}{\partial x_1}=0 \qquad ...(6)$$

and

$$\frac{\partial F_1}{\partial x_1}+\frac{\partial F_1}{\partial p_1}\frac{\partial p_1}{\partial x_1}+\frac{\partial F_1}{\partial p_2}\frac{\partial p_2}{\partial x_1}+\frac{\partial F_1}{\partial p_3}\frac{\partial p_3}{\partial x_1}=0. \qquad ...(7)$$

Eliminating $\partial p_1/\partial x_1$ from (6) and (7), we have

$$\left(\frac{\partial f}{\partial x_1}\frac{\partial F_1}{\partial p_1}-\frac{\partial f}{\partial p_1}\frac{\partial F_1}{\partial x_1}\right)+\left(\frac{\partial f}{\partial p_2}\frac{\partial F_1}{\partial p_1}-\frac{\partial f}{\partial p_1}\frac{\partial F_1}{\partial p_2}\right)\frac{\partial p_2}{\partial x_1}+\left(\frac{\partial f}{\partial p_3}\frac{\partial F_1}{\partial p_1}-\frac{\partial f}{\partial p_1}\frac{\partial F_1}{\partial p_3}\right)\frac{\partial p_3}{\partial x_1}=0. \quad ...(8)$$

Similarly, differentiating (1) and (2) partially w.r.t. x_2 and then eliminating $\partial p_2/\partial x_2$ from the resulting equations, we have

$$\left(\frac{\partial f}{\partial x_2}\frac{\partial F_1}{\partial p_2}-\frac{\partial f}{\partial p_2}\frac{\partial F_1}{\partial x_2}\right)+\left(\frac{\partial f}{\partial p_1}\frac{\partial F_1}{\partial p_2}-\frac{\partial f}{\partial p_2}\frac{\partial F_1}{\partial p_1}\right)\frac{\partial p_1}{\partial x_2}+\left(\frac{\partial f}{\partial p_3}\frac{\partial F_1}{\partial p_2}-\frac{\partial f}{\partial p_2}\frac{\partial F_1}{\partial p_3}\right)\frac{\partial p_3}{\partial x_2}=0. \quad ...(9)$$

Again, differentiating (1) and (2) partially w.r.t. x_3 and then eliminating $\partial p_3/\partial x_3$ from the resulting equation, we have

$$\left(\frac{\partial f}{\partial x_3}\frac{\partial F_1}{\partial p_3}-\frac{\partial f}{\partial p_3}\frac{\partial F_1}{\partial x_3}\right)+\left(\frac{\partial f}{\partial p_1}\frac{\partial F_1}{\partial p_3}-\frac{\partial f}{\partial p_3}\frac{\partial F_1}{\partial p_1}\right)\frac{\partial p_1}{\partial x_3}+\left(\frac{\partial f}{\partial p_2}\frac{\partial F_1}{\partial p_3}-\frac{\partial f}{\partial p_3}\frac{\partial F_1}{\partial p_2}\right)\frac{\partial p_2}{\partial x_3}=0. \quad ...(10)$$

Adding (8), (9) and (10) and using the relations (5), we have

$$\left(\frac{\partial f}{\partial x_1}\frac{\partial F_1}{\partial p_1}-\frac{\partial f}{\partial p_1}\frac{\partial F_1}{\partial x_1}\right)+\left(\frac{\partial f}{\partial x_2}\frac{\partial F_1}{\partial p_2}-\frac{\partial f}{\partial p_2}\frac{\partial F_1}{\partial x_2}\right)+\left(\frac{\partial f}{\partial x_3}\frac{\partial F_1}{\partial p_3}-\frac{\partial f}{\partial p_3}\frac{\partial F_1}{\partial x_3}\right)=0. \quad ...(11)$$

The L.H.S. of (11) is generally denoted by (f, F_1). Then, (11) becomes

$$(f, F_1)=\sum_{r=1}^{3}\left(\frac{\partial f}{\partial x_r}\frac{\partial F_1}{\partial p_r}-\frac{\partial f}{\partial p_r}\frac{\partial F_1}{\partial x_r}\right)=0. \quad ...(11)'$$

Starting with (1) and (3) in place of (1) and (2) and proceeding as above, we have a similar relation

$$(f, F_2)=\sum_{r=1}^{3}\left(\frac{\partial f}{\partial x_r}\frac{\partial F_2}{\partial p_r}-\frac{\partial f}{\partial p_r}\frac{\partial F_2}{\partial x_r}\right)=0. \quad ...(12)$$

Again, starting with (2) and (3) in place of (1) and (2) and proceeding as above, we again have a similar relation

$$(F_1, F_2)=\sum_{r=1}^{3}\left(\frac{\partial F_1}{\partial x_r}\frac{\partial F_2}{\partial p_r}-\frac{\partial F_1}{\partial p_r}\frac{\partial F_2}{\partial x_r}\right)=0. \quad ...(13)$$

(11) [or (11)′] and (12) are linear equtions of first order with x_1, x_2, x_3, p_1, p_2, p_3 as independent variables and F_1, F_2 as dependent variables respectively. For both of these equations, Lagrange's auxiliary equations are

$$\frac{dp_1}{\partial f/\partial x_1}=\frac{dx_1}{-\partial f/\partial p_1}=\frac{dp_2}{\partial f/\partial x_2}=\frac{dx_2}{-\partial f/\partial p_2}=\frac{dp_3}{\partial f/\partial x_3}=\frac{dx_3}{-\partial f/\partial p_3}, \quad ...(14)$$

which are known as *Jacobi's auxiliary equations*.

We try to find two independent integrals $F_1(x_1, x_2, x_3, p_1, p_2, p_3) = a_1$ and $F_2(x_1, x_2, x_3, p_1, p_2, p_3) = a_2$ with help of (14). If these relations satisfy (13), these are the required two additional relations (2) and (3).

We now solve (1), (2) and (3) for p_1, p_2, p_3 in terms of x_1, x_2, x_3. Substituting these values in (4) and then integrating the resulting equation, we shall obtain a complete integral of the given equation containing three arbitrary constants of integration.

3.20. Working rules for solving partial differential equations with three or more independent variable. Jacobi's method

Step 1 : Suppose the given equation with three independent variables is

$$f(x_1, x_2, x_3, p_1, p_2, p_3)=0. \quad ...(1)$$

Step II. We write Jacobi's auxiliary equations

$$\frac{dp_1}{\partial f/\partial x_1}=\frac{dx_1}{-\partial f/\partial p_1}=\frac{dp_2}{\partial f/\partial x_2}=\frac{dx_2}{-\partial f/\partial p_2}=\frac{dp_3}{\partial f/\partial x_3}=\frac{dx_3}{-\partial f/\partial p_3}.$$

Solving these equation we obtain two additional equations

$F_1(x_1, x_2, x_3, p_1, p_2, p_3) = a_1$ ...(2) $\qquad F_2(x_1, x_2, x_3, p_1, p_2, p_3) = a_2$. ...(3)

where a_1 and a_2 are arbitrary constants.

While obtaining (2) and (3), try to select simple equations so that later on solutions of (1), (2) and (3) may be as easy as possible.

Step III. Verify that relations (2) and (3) satisfy the condition

$$(F_1, F_2)=\sum_{r=1}^{3}\left(\frac{\partial F_1}{\partial x_r}\frac{\partial F_2}{\partial p_r}-\frac{\partial F_1}{\partial p_r}\frac{\partial F_2}{\partial x_r}\right)=0. \quad ...(4)$$

If (4) is satisfied then solve (1), (2) and (3) for p_1, p_2, p_3 in terms of x_1, x_2, x_3. Their substitution in

$$dz = p_1dx_1 + p_2dx_2 + p_3dx_3$$

and subsequent integration leads to a complete integral of the given equation.

Remark 1. Sometime, change of variables can be employed to reduce the given equation in a form solvable by Jacobian method.

Remark 2. While solving a partial differential equation with four independent variables, we modify the above working rule as follows :

Step I. Suppose the given equation with four independent variables is

$$f(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = 0. \qquad ...(1)$$

Step II. We write Jacobi's auxiliary equations

$$\frac{dp_1}{\partial f/\partial x_1} = \frac{dx_1}{-\partial f/\partial p_1} = \frac{dp_2}{\partial f/\partial x_2} = \frac{dx_2}{-\partial f/\partial p_2} = \frac{dp_3}{\partial f/\partial x_3} = \frac{dx_3}{-\partial f/\partial p_3} = \frac{dp_4}{\partial f/\partial x_4} = \frac{dx_4}{-\partial f/\partial p_4}$$

Solving these equations we obtain three additional equations

$F_1(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = a_1$, ...(2) $\qquad F_2(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = a_2$, ...(3)

and $\qquad F_3(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = a_3$, ...(4)

where a_1, a_2 and a_3 are arbitrary constants.

Step IV. Verify that relations (2), (3) and (4) satisfy following three conditions:

$$(F_1, F_2) = \sum_{r=1}^{4}\left(\frac{\partial F_1}{\partial x_r}\frac{\partial F_2}{\partial p_r} - \frac{\partial F_1}{\partial p_r}\frac{\partial F_2}{\partial x_r}\right) = 0, \ ...(4) \qquad (F_2, F_3) = \sum_{r=1}^{4}\left(\frac{\partial F_2}{\partial x_r}\frac{\partial F_3}{\partial p_r} - \frac{\partial F_2}{\partial p_r}\frac{\partial F_3}{\partial x_r}\right) = 0 \ ...(5)$$

and

$$(F_3, F_1) = \sum_{r=1}^{4}\left(\frac{\partial F_3}{\partial x_r}\frac{\partial F_1}{\partial p_r} - \frac{\partial F_3}{\partial p_r}\frac{\partial F_1}{\partial x_r}\right) = 0. \qquad ...(6)$$

If (4), (5) and (6) are satisfied, then solve (1), (2), (3) and (4) for p_1, p_2, p_3 and p_4 in terms of x_1, x_2, x_3 and x_4. Their substitution in $\qquad dz = p_1dx_1 + p_2dx_2 + p_3dx_3 + p_4dx_4$
and subsequent integration leads to a complete integral of the given equation.

3.21 SOLVED EXAMPLES BASED ON ART 3.20.

Ex. 1. *Find a complete integral of* $p_1^3 + p_2^2 + p_3 = 1$. **[I.A.S. 1997; Meerut 2006]**

Sol. Let the given equation be rewritten as

$$f(x_1, x_2, x_3, p_1, p_2, p_3) = p_1^3 + p_2^2 + p_3 - 1 = 0. \qquad ...(1)$$

$\therefore$ Jacobi's auxiliary equations are

$$\frac{dp_1}{\partial f/\partial x_1} = \frac{dx_1}{-\partial f/\partial p_1} = \frac{dp_2}{\partial f/\partial x_2} = \frac{dx_2}{-\partial f/\partial p_2} = \frac{dp_3}{\partial f/\partial x_3} = \frac{dx_3}{-\partial f/\partial p_3}$$

or

$$\frac{dp_1}{0} = \frac{dx_1}{-3p_1^2} = \frac{dp_2}{0} = \frac{dx_2}{-2p_2} = \frac{dp_3}{0} = \frac{dx_3}{-1}, \text{ using (1)}$$

From first and third fractions, $dp_1 = 0$ and $dp_2 = 0$ so that $p_1 = a_1$ and $p_2 = a_2$.

$\therefore$ Here $\qquad F_1(x_1, x_2, x_3, p_1, p_2, p_3) = p_1 = a_1$. ...(2)

and $\qquad F_2(x_1, x_2, x_3, p_1, p_2, p_3) = p_2 = a_2$. ...(3)

Now,

$$(F_1, F_2) = \sum_{r=1}^{3}\left(\frac{\partial F_1}{\partial x_r}\frac{\partial F_2}{\partial p_r} - \frac{\partial F_1}{\partial p_r}\frac{\partial F_2}{\partial x_r}\right)$$

or

$$(F_1, F_2) = \frac{\partial F_1}{\partial x_1}\frac{\partial F_2}{\partial p_1} - \frac{\partial F_1}{\partial p_1}\frac{\partial F_2}{\partial x_1} + \frac{\partial F_1}{\partial x_2}\frac{\partial F_2}{\partial p_2} - \frac{\partial F_1}{\partial p_2}\frac{\partial F_2}{\partial x_2} + \frac{\partial F_1}{\partial x_3}\frac{\partial F_2}{\partial p_3} - \frac{\partial F_1}{\partial p_3}\frac{\partial F_2}{\partial x_3}$$

or $\qquad (F_1, F_2) = (0)(0) - (1)(0) + (0)(1) - (0)\,(0) + (0)(0) - (0)(0) = 0$, by (3) and (4).

Thus, we have verified that for relations (2) and (3), $(F_1, F_2) = 0$. Hence (2) and (3) may be taken as additional equations.

Solving (1), (2) and (3) for p_1, p_2, p_3, $\quad p_1 = a_1, \quad p_2 = a_2, \quad p_3 = 1 - a_1^3 - a_2^2$.

Putting these values in $dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3$, we have

$$dz = a_1 dx_1 + a_2 dx_2 + (1 - a_1^3 - a_2^2) dx_3.$$

Integrating, $\quad z = a_1 x_1 + a_2 x_2 + (1 - a_1^3 - a_2^2)\, x_3 + a_3$,

which is a complete integral of given equation containing three arbitrary constants a_1, a_2, and a_3.

Ex. 2. *Find a complete integral of* $x_3^2 p_1^2 p_2^2 p_3^2 + p_1^2 p_2^2 - p_3^2 = 0$. **[Delhi Maths (H) 2006]**

Sol. Let $\quad f(x_1, x_2, x_3, p_1, p_2, p_3) = x_3^2 p_1^2 p_2^2 p_3^2 + p_1^2 p_2^2 - p_3^2 = 0. \quad$ ...(1)

$\therefore$ Jacobi's auxiliary equations are

$$\frac{dp_1}{\partial f/\partial x_1} = \frac{dx_1}{-\partial f/\partial p_1} = \frac{dp_2}{\partial f/\partial x_2} = \frac{dx_2}{-\partial f/\partial p_2} = \frac{dp_3}{\partial f/\partial x_3} = \frac{dx_3}{-\partial f/\partial p_3}$$

or $$\frac{dp_1}{0} = \frac{dx_1}{-(2p_1 x_3^2 p_2^2 p_3^2 + 2p_1 p_2^2)} = \frac{dp_2}{0} = \frac{dx_2}{-(2p_2 x_3^2 p_1^2 p_3^2 + 2p_2 p_1^2)} = \text{.....}, \text{ by (1)}$$

From first and third fractions, $\quad dp_1 = 0$ and $dp_2 = 0 \quad$ so that $\quad p_1 = a_1$ and $\quad p_2 = a_2$.

$\therefore$ Here $\quad F_1(x_1, x_2, x_3, p_1, p_2, p_3) = p_1 = a_1, \quad$ ...(2)

and $\quad F_2(x_1, x_2, x_3, p_1, p_2, p_3) = p_2 = a_2. \quad$ ...(3)

As in Ex. 1, verify that for relations (2) and (3), $\quad (F_1, F_2) = 0$.

Hence (2) and (3) may be taken as the additional equations.

Solving (1), (2) and (3) for p_1, p_2, p_3, we have $p_1 = a_1,\ p_2 = a_2,\ \ p_3 = \pm a_1 a_2 \big/ \sqrt{(1 - a_1^2 a_2^2 x_3^2)}$.

Putting these values in $dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3$, we get

$$dz = a_1 dx_1 + a_2 dx_2 \pm \left\{ a_1 a_2 \big/ \sqrt{(1 - a_1^2 a_2^2 x_3^2)} \right\} dx_3, \text{ whose integration gives}$$

$$z = a_1 x_1 + a_2 x_2 \pm \sin^{-1}(a_1 a_2 x_3) + a_3,\ a_1, a_2, a_3 \text{ being arbitrary constants.}$$

Ex. 3. *Find a complete integral of* $p_1 x_1 + p_2 x_2 = p_3^2$. **[Meerut 2007]**

Sol. Let $\quad f(x_1, x_2, x_3, p_1, p_2, p_3) = p_1 x_1 + p_2 x_2 - p_3^2 = 0. \quad$ ...(1)

$\therefore$ Jacobi's auxiliary equations are

$$\frac{dp_1}{\partial f/\partial x_1} = \frac{dx_1}{-\partial f/\partial p_1} = \frac{dp_2}{\partial f/\partial x_2} = \frac{dx_2}{-\partial f/\partial p_2} = \frac{dp_3}{\partial f/\partial x_3} = \frac{dx_3}{-\partial f/\partial p_3}$$

or $$\frac{dp_1}{p_1} = \frac{dx_1}{-x_1} = \frac{dp_2}{p_2} = \frac{dx_2}{-x_2} = \frac{dp_3}{0} = \frac{dx_3}{2p_3}, \text{ using (1)} \qquad \text{...(2)}$$

Taking the first two fractions of (2), $(1/x_1)dx + (1/p_1)dp_1 = 0. \Rightarrow \log x_1 + \log p_1 = \log a_1$.

$\therefore x_1 p_1 = a_1 \quad$ and $\quad$ let $\quad F_1(x_1, x_2, x_3, p_1, p_2, p_3) = x_1 p_1 = a_1$. ...(3)

Taking the third and fourth fractions of (2), $\quad (1/x_2)dx_2 + (1/p_2)dp_2 = 0$.

$\therefore x_2 p_2 = a_2 \quad$ and $\quad$ let $\quad F_2(x_1, x_2, x_3, p_1, p_2, p_3) = x_2 p_2 = a_2$. ...(4)

As in Ex. 1, verify that for relations (3) and (4), $(F_1, F_2) = 0$.

Solving (1), (3) and (4) for p_1, p_2, p_3, $\quad p_1 = a_1/x_1, \quad p_2 = a_2/x_2 \quad$ and $\quad p_3 = (a_1 + a_2)^{1/2}$.

Putting these values in $dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3$, we have

$$dz = (a_1/x_1)dx_1 + (a_2/x_2)dx_2 + (a_1 + a_2)^{1/2}\, dx_3.$$

Integrating, $\quad z = a_1 \log x_1 + a_2 \log x_2 + x_3 (a_1 + a_2)^{1/2} + a_3$.

Ex. 4. *Find complete integral of* $2p_1 x_1 x_3 + 3p_2 x_3^2 + p_2^2 p_3 = 0$.

[I.A.S. 1998, Meerut 1999]

Sol. Let $\quad f(x_1, x_2, x_3, p_1, p_2, p_3) = 2p_1 x_1 x_3 + 3p_2 x_3^2 + p_2^2 p_3 = 0. \quad$ ...(1)

$\therefore$ Jacobi's auxiliary equations are

$$\frac{dp_1}{\partial f/\partial x_1}=\frac{dx_1}{-\partial f/\partial p_1}=\frac{dp_2}{\partial f/\partial x_2}=\frac{dx_2}{-\partial f/\partial p_2}=\frac{dp_3}{\partial f/\partial x_3}=\frac{dx_3}{-\partial f/\partial p_3}$$

or $$\frac{dp_1}{2p_1x_3}=\frac{dx_1}{-2x_1x_3}=\frac{dp_2}{0}=\frac{dx_2}{-3x_3^2-2p_2p_3}=\frac{dp_3}{2p_1x_1+6p_2x_3}=\frac{dx_3}{-p_2^2}\text{, by (1)} \qquad ...(2)$$

Taking the first two fractions of (2), $(1/p_1)dp_1 + (1/x_1)dx_1 = 0.$ so $p_1x_1 = a_1$

Let $F_1(x_1, x_2, x_3, p_1, p_2, p_3) = p_1x_1 = a_1.$ ...(3)

From the third fraction of (2), $dp_2 = 0$ so that $p_2 = a_2.$

Let $F_2(x_1, x_2, x_3, p_1, p_2, p_3) = p_2 = a_2.$ ...(4)

As in Ex. 1, verify that for relations (3) and (4), $(F_1, F_2) = 0.$

Solving (1), (3) and (4) for p_1, p_2, p_3, $p_1 = a_1/x_1$, $p_2 = a_2$, $p_3 = -(2a_1x_3 + 3a_2x_3^2)/a_2^2.$

Putting these values in $dz = p_1dx_1 + p_2dx_2 + p_3dx_2$, we have

$dz = (a_1/x_1)dx_1 + a_2dx_2 - \{(2a_1x_3 + 3a_2x_3^2)/a_2^2\}dx_3$, whose integration gives

$z = a_1 \log x_1 + a_2x_2 - (a_1x_3^2 + a_2x_3^3)/a_2^2 + a_3.$ which is required complete integral

Ex. 5. *Find a complete integral of* $p_3x_3 (p_1 + p_2) + x_1 + x_2 = 0.$

Sol. Given $f(x_1, x_2, x_3, p_1, p_2, p_3) = p_3x_3(p_1 + p_2) + x_1 + x_2 = 0.$ ...(1)

$\therefore$ Jacobi's auxiliary equations are

$$\frac{dp_1}{\partial f/\partial x_1}=\frac{dx_1}{-\partial f/\partial p_1}=\frac{dp_2}{\partial f/\partial x_2}=\frac{dx_2}{-\partial f/\partial p_2}=\frac{dp_3}{\partial f/\partial x_3}=\frac{dx_3}{-\partial f/\partial p_3}$$

or $$\frac{dp_1}{1}=\frac{dx_1}{p_3x_3}=\frac{dp_2}{1}=\frac{dx_2}{-p_3x_3}=\frac{dp_3}{p_3(p_2+p_3)}=\frac{dx_3}{-x_3(p_1+p_2)}\text{, by (1)} \qquad ...(2)$$

Taking the two fractions of (2), $dp_1 - dp_2 = 0$ so $p_1 - p_2 = a_1$

Let $F_1(x_1, x_2, x_3, p_1, p_2, p_3) = p_1 - p_2 = a_1.$ ...(3)

Taking the fifth and sixth fractions of (2), $(1/p_3)dp_3 + (1/x_3)dx_3 = 0$ giving $p_3x_3 = a_3$

Let $F_2(x_1, x_2, x_3, p_1, p_2, p_3) = p_3x_3 = a_2.$ ...(4)

Now, $$(F_1, F_2) = \sum_{r=1}^{3}\left(\frac{\partial F_1}{\partial x_r}\frac{\partial F_2}{\partial p_r}-\frac{\partial F_1}{\partial p_r}\frac{\partial F_2}{\partial x_r}\right)$$

$$=\left(\frac{\partial F_1}{\partial x_1}\frac{\partial F_2}{\partial p_1}-\frac{\partial F_1}{\partial p_1}\frac{\partial F_2}{\partial x_1}\right)+\left(\frac{\partial F_1}{\partial x_2}\frac{\partial F_2}{\partial p_2}-\frac{\partial F_1}{\partial p_2}\frac{\partial F_2}{\partial x_2}\right)+\left(\frac{\partial F_1}{\partial x_3}\frac{\partial F_2}{\partial p_3}-\frac{\partial F_1}{\partial p_3}\frac{\partial F_2}{\partial x_3}\right)$$

$= (0)(0) - (1)(0) + (0)(0) - (-1)(0) + (0)(x_3) - (0)(p_3) = 0$ by (3) and (4)

Thus, we have verified that for the relations (3) and (4), $(F_1, F_2) = 0.$

From (1) and (4), $a_2(p_1 + p_2) + x_1 + x_2 = 0$ or $p_1 + p_2 = -(x_1 + x_2)/a_2.$ ...(5)

Solving (3) and (5), $p_1 = \dfrac{a_1}{2}-\dfrac{x_1+x_2}{2a_2}$ and $p_2 = -\dfrac{a_1}{2}-\dfrac{x_1+x_2}{2a_2}.$ ...(6)

Again, from (4), $p_3 = a_2/x_3.$ ...(7)

Putting the values of p_1, p_2, p_3 given by (6) and (7) in $dz = p_1dx_1 + p_2dx_2 + p_3dx_3$, we have

$$dz = \frac{a_1}{2}(dx_1 - dx_2) - \frac{(x_1+x_2)}{2a_2}(dx_1 + dx_2) + \frac{a_2}{x_3}dx_3.$$

Integrating, $z = (a_1/2) \times (x_1 - x_2) - (1/4a_2) \times (x_1 + x_2)^2 + a_2 \log x_3 + a_3.$

Ex. 6. *Find a complete integral of* $(p_1 + x_1)^2 + (p_2 + x_2)^2 + (p_3 + x_3)^2 = 3(x_1 + x_2 + x_3)$.

Sol. Let the given partial differential equation be re–written as

$$f(x_1, x_3, x_3, p_1, p_2, p_3) = (p_1 + x_1)^2 + (p_2 + x_2)^2 + (p_3 + x_3)^2 - 3(x_1 + x_2 + x_3) = 0. \quad ...(1)$$

$\therefore$ Jacobi's auxiliary equations are

$$\frac{dp_1}{\partial f/\partial x_1} = \frac{dx_1}{-\partial f/\partial p_1} = \frac{dp_2}{\partial f/\partial x_2} = \frac{dx_2}{-\partial f/\partial p_2} = \frac{dp_3}{\partial f/\partial x_3} = \frac{dx_3}{-\partial f/\partial p_3}, \text{ giving}$$

$$\frac{dp_1}{2(p_1+x_1)-3} = \frac{dx_1}{-2(p_1+x_1)} = \frac{dp_2}{2(p_2+x_2)-3} = \frac{dx_2}{-2(p_2+x_2)} = \frac{dp_3}{2(p_3+x_3)-3} = \frac{dx_3}{-2(p_3+x_3)}. \quad ...(2)$$

$$\text{Each fraction of (2)} = \frac{dp_1 + dx_1}{-3} = \frac{dp_2 + dx_2}{-3} = \frac{dp_3 + dx_3}{-3} \quad ...(3)$$

Then (3) $\Rightarrow$ $\quad dp_1 + dx_1 = dp_2 + dx_2 \quad$ and $\quad dp_3 + dx_3 = dp_2 + dx_2$

Integrating, $\quad p_1 + x_1 = p_2 + x_2 + a_1 \quad$ and $\quad p_3 + x_3 = p_2 + x_2 + a_2$,

where a_1 and a_2 are arbitrary constants

Let $\quad F_1(x_1, x_2, x_3, p_1, p_2, p_3) = x_1 + p_1 - x_2 - p_2 = a_1. \quad ...(4)$

and $\quad F_2(x_1, x_2, x_2, p_1, p_2, p_3) = x_3 + p_3 - x_2 - p_2 = a_2. \quad ...(5)$

As in Ex. 1, verify that for relations (4) and (5), the condition $(F_1, F_2) = 0$ is satisfied. Hence (4) and (5) may be taken as two additional equations.

With help of (4) and (5), (1) reduces to

$$(x_2 + p_2 + a_1)^2 + (x_2 + p_2)^2 + (x_2 + p_2 + a_2)^2 = 3(x_1 + x_2 + x_3)$$

or

$$3(p_2 + x_2)^2 + 2(p_2 + x_2)(a_1 + a_2) + a_1{}^2 + a_2{}^2 - 3(x_1 + x_2 + x_3) = 0.$$

$$\therefore \quad p_2 + x_2 = \left[-2(a_1 + a_2) \pm \sqrt{[4(a_1+a_2)^2 - 12\{a_1^2 + a_2^2 - 3(x_1 + x_2 + x_3)\}}\right] \Big/ 6$$

$$\Rightarrow \quad p_2 = -x_2 + \left[-(a_1 + a_2) \pm \sqrt{\{9(x_1+x_2+x_3) - 2a_1^2 - 2a_2^2 + 2a_1a_2\}}\right] \Big/ 3$$

For sake of simplification, we take $a_1 = 3c_1$ and $a_2 = 3c_2$. Then, we get

$$p_2 = -x_2 - (c_1 + c_2) \pm \sqrt{\{(x_1+x_2+x_3) - 2c_1^2 - 2c_2^2 + 2c_1c_2\}}. \quad ...(6)$$

$\therefore$ From (4), $\quad p_1 = x_2 + p_2 + 3c_1 - x_1$

$$\Rightarrow \quad p_1 = -x_1 + 2c_1 - c_2 \pm \sqrt{\{(x_1+x_2+x_3) - 2c_1^2 - 2c_2^2 + 2c_1c_2\}}, \text{by (6)}$$

Again, from (5), $\quad p_3 = x_2 + p_2 + 3c_2 - x_3$

$$\Rightarrow \quad p_3 = -x_3 + 2c_2 - c_1 \pm \sqrt{\{(x_1+x_2+x_3) - 2c_1^2 - 2c_2^2 + 2c_1c_2\}}, \text{ by (6)}$$

Substituting these values in $dz = p_1dx_1 + p_2dx_2 + p_3dx_3$, we get

$$dz = -(x_1dx + x_2dx_2 + x_3dx_3) + [(2c_1 - c_2)dx_1 - (c_1 + c_2)dx_2 + (2c_2 - c_1)dx_3]$$
$$\pm (x_1 + x_2 + x_3 - 2c_1^2 - 2c_2^2 + 2c_1c_2)^{1/2} \; (dx_1 + dx_2 + dx_3).$$

Integrating, $z = -(1/2) \times (x_1^2 + x_2^2 + x_3^2) + (2c_1 - c_2)\,x_1 - (c_1 + c_2)x_2 + (2c_2 - c_1)x_3$
$$\pm (2/3) \times (x_1 + x_2 + x_3 - 2c_1^2 - 2c_2^2 + 2c_1c_2)^{3/2} + c_3,$$

which is a complete integral containing c_1, c_2, c_3 as arbitrary constants.

Ex. 7. *Find a complete integral of* $(x_2 + x_3)(p_2 + p_3)^2 + zp_1 = 0$. **[Delhi B.Sc. (Hons) III 2011]**

Sol. Given $\quad (x_2 + x_3)(p_2 + p_3)^2 + zp_1 = 0. \quad ...(1)$

Since the dependent variable z is involved, the given equation (1) is not in the standard form. We shall first reduce it in the standard form and then proceed as usual. Re–writting (1), we have

$$(x_2 + x_3)\left(\frac{1}{z}\frac{\partial z}{\partial x_2} + \frac{1}{z}\frac{\partial z}{\partial x_3}\right)^2 + \frac{1}{z}\frac{\partial z}{\partial x_1} = 0. \qquad ...(2)$$

Let $(1/z)dz = dZ$ so that $Z = \log z$. ...(3)

Then, (2) $\Rightarrow$ $(x_2 + x_3)\,(\partial Z/\partial x_2 + \partial Z/\partial x_3)^2 + \partial Z/\partial x_1 = 0.$ ...(4)

Let $P_1 = \partial Z/\partial x_1$, $P_2 = \partial Z/\partial x_2$, $P_3 = \partial Z/\partial x_3$. Then (4) becomes

$$(x_2 + x_3)(P_2 + P_3)^2 + P_1 = 0.$$

So here $f(x_1, x_2, x_3, P_1, P_2, P_3) \equiv (x_2 + x_3)(P_2 + P_3)^2 + P_1 = 0.$ ...(5)

Jacobi's auxiliary equations take the form

$$\frac{dP_1}{\partial f/\partial x_1} = \frac{dx_1}{-\partial f/\partial P_1} = \frac{dP_2}{\partial f/\partial x_2} = \frac{dx_2}{-\partial f/\partial P_2} = \frac{dP_3}{\partial f/\partial x_3} = \frac{dx_3}{-\partial f/\partial P_3}$$

or $$\frac{dx_1}{-1} = \frac{dP_1}{0} = \frac{dx_2}{-2(x_2+x_3)(P_2+P_3)} = \frac{dP_2}{(P_2+P_3)^2} = \frac{dx_3}{-2(x_2+x_3)(P_2+P_3)} = \frac{dP_3}{(P_2+P_3)^2}. \qquad ...(6)$$

Taking second ratio of (6), we have $dP_1 = 0$ $\Rightarrow$ $P_1 = -a_1$.

Let $F_1(x_1, x_2, x_3, P_1, P_2, P_3) = P_1 = -a_1.$ ...(7)

Taking the fourth and sixth ratios in (6), we get $dP_2 = dP_3$ $\Rightarrow$ $P_2 - P_3 = a_2$.

Let $F_2(x_1, x_2, x_3, P_1, P_2, P_3) = P_2 - P_3 = a_2.$ ...(8)

Using (7), (5) $\Rightarrow P_2 + P_3 = \pm\,\{a_1/(x_2 + x_3)\}^{1/2}.$ ...(9)

Solving (8) and (9) for P_2 and P_3, we have

$$P_2 = \frac{1}{2}\left[a_2 \pm \left(\frac{a_1}{x_2+x_3}\right)^{1/2}\right] \quad \text{and} \quad P_3 = \frac{1}{2}\left[\pm\left(\frac{a_1}{x_2+x_3}\right)^{1/2} - a_2\right]. \qquad ...(10)$$

Using (7) and (10), $dZ = P_1dx_1 + P_2dx_2 + P_3dx_3$ becomes

$$dZ = -a_1dx_1 + \frac{1}{2}\left[a_2 \pm \frac{\sqrt{a_1}}{(x_2+x_3)^{1/2}}\right]dx_2 + \frac{1}{2}\left[\pm\frac{\sqrt{a_1}}{(x_2+x_3)^{1/2}} - a_2\right]dx_3$$

or $$dZ = -\,a_1dx_1 + (1/2)\times a_2dx_2 - (1/2)\times a_2dx_3 \pm (1/2)\times\sqrt{a_1}\,(x_2 + x_3)^{-1/2}(dx_2 + dx_3).$$

Integrating and noting that $dZ = (1/z)dz$, complete integral is given by

$$\log z = -\,a_1x_1 + (a_2/2)\times(x_2 - x_3) \pm \sqrt{a_1}\,(x_2 + x_3)^{1/2} + a_3.$$

Ex. 8. *Find a complete integral of* $p_1p_2p_3 = z^3x_1x_2x_3$. **[Meerut 1998]**

i.e., $(\partial z/\partial x_1)(\partial z/\partial x_2)\,(\partial z/\partial x_3) = z^3x_1x_2x_3.$ **[Delhi Maths (H) 2000, 10; I.A.S. 1995]**

Sol. Given $p_1p_2p_3 = z^3x_1x_2x_3$ or $(\partial z/\partial x_1)(\partial z/\partial x_2)\,(\partial z/\partial x_3) = z^3x_1x_2x_3.$ ...(1)

Since the dependent variable z is involved, the given equation (1) is not in the standard form. We shall first reduce it in the standard form and then proceed as usual. Re–writting (1) we have

$$\left(\frac{1}{z}\frac{\partial z}{\partial x_1}\right)\left(\frac{1}{z}\frac{\partial z}{\partial x_2}\right)\left(\frac{1}{z}\frac{\partial z}{\partial x_3}\right) = x_1x_2x_3. \qquad ...(2)$$

Let $(1/z)dz = dZ$ so that $\log z = Z$. Then (2) becomes

$(\partial Z/\partial x_1)(\partial Z/\partial x_2)\,(\partial Z/\partial x_3) = x_1x_2x_3$ or $P_1P_2P_3 = x_1x_2x_3$.

$\therefore$ Here $f(x_1, x_2, x_3, P_1, P_2, P_3) \equiv P_1P_2P_3 - x_1x_2x_3 = 0.$ ...(3)

$\therefore$ Jacobi's auxilliary equations are

$$\frac{dP_1}{\partial f/\partial x_1} = \frac{dx_1}{-\partial f/\partial P_1} = \frac{dP_2}{\partial f/\partial x_2} = \frac{dx_2}{-\partial f/\partial P_2} = \frac{dP_3}{\partial f/\partial x_3} = \frac{dx_3}{-\partial f/\partial P_3}$$

or $$\frac{dP_1}{-x_2x_3}=\frac{dx_1}{-P_2P_3} = \frac{dP_2}{-x_1x_3}=\frac{dx_2}{-P_1P_3} = \frac{dP_3}{-x_1x_2}=\frac{dx_3}{-P_1P_2}, \text{ by (3)}$$

Since from (3), $P_2P_3 = (x_1\, x_2\, x_3)/P_1$, hence first and second fractions give

$$\frac{dP_1}{-x_2x_3} = \frac{dx_1}{-(x_1x_2x_3/P_1)} \qquad \text{or} \qquad \frac{dP_1}{P_1}=\frac{dx_1}{x_1}.$$

Integrating, $\log P_1 = \log x_1 + \log a_1$ or $P_1 = a_1x_1$.

Thus, here we have $F_1(x_1, x_2, x_3, P_1, P_2, P_3) \equiv P_1 - a_1x_1 = 0.$ ...(4)

Similarly, $F_2(x_1, x_2, x_3, P_1, P_2, P_3) \equiv P_2 - a_2x_2 = 0.$ ...(5)

As in Ex. 1, verify that for (4) and (5) the condition $(F_1, F_2) = 0$ is satisfied. Hence (4) and (5) can be taken as two additional equations. Solving (3), (4) and (5) for P_1, P_2, P_3, we have

$P_1 = a_1x_1$, $P_2 = a_2x_2$ and $P_3 = x_3/(a_1a_2)$.

Putting these values in $dZ = P_1dx_1 + P_2dx_2 + P_3dx_3$, we have

$$dZ = a_1x_1dx + a_2x_2dx_2 + \{x_3/(a_1a_2)\}dx_3.$$

Integrating, $Z = (1/2) \times a_1x_1^2 + (1/2) \times a_2x_2^2 + \{1/(2a_1a_2)\}x_3^2 + a_3/2$

or $$2 \log z = a_1x_1^2 + a_2x_2^2 + \{1/(a_1a_2)\}x_3^2 + a_3, \text{ as } Z = \log z$$

Ex. 9. *Find a complete integral of* $p_1^2 + p_2p_3 - z(p_2 + p_3) = 0.$ **[Delhi Maths (H) 2009]**

Sol. Given equation is $p_1^2 + p_2p_3 - z(p_2 + p_3) = 0.$ ...(1)

Since the dependent variable z is involved, the given equation (1) is not in the standard form. We shall first reduce it in the standard form and then proceed as usual. Dividing each term by z^2, (1) can be re–written as

$$\left(\frac{1}{z}\frac{\partial z}{\partial x_1}\right)^2 + \left(\frac{1}{z}\frac{\partial z}{\partial x_2}\right)\left(\frac{1}{z}\frac{\partial z}{\partial x_3}\right) - \left(\frac{1}{z}\frac{\partial z}{\partial x_2}\right) - \left(\frac{1}{z}\frac{\partial z}{\partial x_3}\right) = 0. \quad \text{...(2)}$$

Let $(1/z)dz = dZ$ so that $\log z = Z.$ ...(3)

Using (3), (2) becomes $P_1^2 + P_2P_3 - P_2 - P_3 = 0,$ ...(4)

Let us write $f(x_1, x_2, x_3, P_1, P_2, P_3) = P_1^2 + P_2P_3 - P_2 - P_3 = 0.$ ...(5)

$\therefore$ Jacobi's auxiliary equations are

$$\frac{dP_1}{\partial f/\partial x_1}=\frac{dx_1}{-\partial f/\partial P_1} = \frac{dP_2}{\partial f/\partial x_2}=\frac{dx_2}{-\partial f/\partial P_2} = \frac{dP_3}{\partial f/\partial x_3}=\frac{dx_3}{-\partial f/\partial P_3}$$

or $$\frac{dP_1}{0}=\frac{dx_1}{-2P_1} = \frac{dP_2}{0}=\frac{dx_2}{-P_3+1} = \frac{dP_3}{0}=\frac{dx_3}{-P_2+1}, \text{ by (5)}$$

Taking the third and fifth fractions, $dP_2 = 0$ and $dP_3 = 0$ so that $P_2 = a_1$ and $P_3 = a_2$.

Let $F_1(x_1, x_2, x_3, P_1, P_2, P_3) = P_2 = a_1.$ ...(6)

and $F_3(x_1, x_2, x_3, P_1, P_2, P_3) = P_3 = a_2.$ ...(7)

As in Ex. 1, verify that for (6) and (7), the condition $(F_1, F_2) = 0$ is satisfied. Hence (6) and (7) can be taken as two additional equations. Solving (4), (6) and (7) for P_1, P_2, P_3, we have

$P_2 = a_1$, $P_3 = a_2$, $P_1 = (a_1 + a_2 - a_1a_2)^{1/2}$.

Putting these values in $dZ = P_1dx_1 + P_2dx_2 + P_3dx_3$, we have

$$dZ = (a_1 + a_2 - a_1a_2)^{1/2}dx_1 + a_1dx_2 + a_2dx_3.$$

Integrating, $Z = (a_1 + a_2 - a_1a_2)^{1/2}x_1 + a_1x_2 + a_2x_3 + a_3$. Then, the complete integral is

$$\log z = (a_1 + a_2 - a_1a_2)^{1/2}x_1 + a_1x_2 + a_2x_3 + a_3, \text{ using (3).}$$

Ex. 10. *Find a complete integral of* $2x_1x_3zp_1p_3 + x_2p_2 = 0.$

Sol. Given equation is $2x_1x_3zp_1p_3 + x_2p_2 = 0.$ ...(1)

Since the dependent variable z is involved, the given equation (1) is not in the standard form. We shall first reduce it in the standard form and then proceed as usual. Multiplying each term by z, (1) can be re–written as

$$2x_1x_3\left(z\frac{\partial z}{\partial x_1}\right)\left(z\frac{\partial z}{\partial x_3}\right)+x_2\left(z\frac{\partial z}{\partial x_2}\right) = 0. \qquad ...(2)$$

Let $\quad zdz = dZ \quad$ so that $\quad z^2/2 = Z. \qquad ...(3)$

Using (3), (2) becomes $\quad 2x_1x_3P_1P_3 + x_2P_2 = 0, \qquad ...(4)$

where $P_1 = \partial Z/\partial x_1$, $P_2 = \partial Z/\partial x_2$ and $P_3 = \partial Z/\partial x_3$. We re–write (4) as

$$f(x_1, x_2, x_3, P_1, P_2, P_3) = 2x_1x_3P_1P_3 + x_2P_2 = 0. \qquad ...(5)$$

$\therefore$ Jacobi's auxilairy equations are

$$\frac{dP_1}{\partial f/\partial x_1}=\frac{dx_1}{-\partial f/\partial P_1} = \frac{dP_2}{\partial f/\partial x_2}=\frac{dx_2}{-\partial f/\partial P_2} = \frac{dP_3}{\partial f/\partial x_3}=\frac{dx_3}{-\partial f/\partial P_3}$$

or
$$\frac{dP_1}{2x_3P_1P_3}=\frac{dx_1}{-2x_1x_3P_3} = \frac{dP_2}{P_2}=\frac{dx_2}{-x_2} = \frac{dP_3}{2x_1P_1P_3}=\frac{dx_3}{-2x_1x_3P_2}, \text{ by (5)} \qquad ...(6)$$

Taking the first and second fractions of (6) and simplifying, we get

$(1/P_1)dP_1 + (1/x_1)dx_1 = 0 \quad$ so that $\quad \log P_1 + \log x_1 = \log a_1 \quad$ or $\quad P_1x_1 = a_1$

So here $\quad F_1(x_1, x_2, x_3, P_1, P_2, P_3) = P_1x_1 = a_1. \qquad ...(7)$

Taking the fifth and sixth fractions of (6) and simplifying, we get

$(1/P_3)dP_3 + (1/x_3)dx_3 = 0 \quad$ so that $\quad \log P_3 + \log x_3 = \log a_3 \quad$ or $\quad P_3x_3 = a_2$

So here $\quad F_2(x_1, x_2, x_3, P_1, P_2, P_3) = P_3x_3 = a_2. \qquad ...(8)$

As in Ex. 1, verify that for (7) and (8), the condition $(F_1, F_2) = 0$ is satisfied. Hence (7) and (8) can be taken as additional equations. Solving (5), (7) and (8) for P_1, P_2, P_3, we have

$$P_1 = a_1/x, \qquad P_3 = a_2/x_3, \qquad P_2 = -(2a, a_2)/x_2.$$

Putting these values in $dZ = P_1dx + P_2dx_2 + P_3dx_3$, we have

$$dZ = (a_1/x_1)dx_1 - \{(2a_1a_2)/x_2\}dx_2 + (a_2/x_3)dx_3.$$

Integrating, $\quad Z = a_1\log x_1 - 2a_1a_2 \log x_2 + a_2 \log x_3 + a_3$

or $\quad z^2/2 = a_1 \log x_1 - 2a_1a_2 \log x_2 + a_2 \log x_3 + a_3$, by (3).

Ex. 11. *Find a complete integral of* $p_1p_2p_3 + p_4^3x_1x_2x_3x_4^3 = 0$.

Sol. [In the present problem we have four independent variables in places of three. According we shall use modified working as explained in remark 2 of Art 3.20]

The given equation can be written as

$$f(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = p_1p_2p_3 + p_4^3x_1x_2x_3x_4^3 = 0. \qquad ...(1)$$

$\therefore$ Jacobi's auxiliary equations are

$$\frac{dp_1}{\partial f/\partial x_1}=\frac{dx_1}{-\partial f/\partial p_1} = \frac{dp_2}{\partial f/\partial x_2}=\frac{dx_2}{-\partial f/\partial p_2} = \frac{dp_3}{\partial f/\partial x_3}=\frac{dx_3}{-\partial f/\partial p_3} = \frac{dp_4}{\partial f/\partial x_4}=\frac{dx_4}{-\partial f/\partial p_4}, \text{ giving}$$

$$\frac{dp_1}{p_4^3x_2x_3x_4^3}=\frac{dx_1}{-p_2p_3} = \frac{dp_2}{p_4^3x_1x_3x_4^3}=\frac{dx_2}{-p_1p_3} = \frac{dp_3}{p_4^3x_1x_2x_4^3}=\frac{dx_3}{-p_1p_2} = \frac{dp_4}{3p_4^3x_1x_2x_3x_4^2}=\frac{dx_4}{-3p_4^3x_1x_2x_3x_4^3}$$

Since from (1), $p_4^3x_2x_3x_4^3 = -p_1p_2p_3/x_1$, the first two fractions give

$$\frac{dp_1}{-(p_1p_2p_3/x_1)} = \frac{dx_1}{-p_2p_3} \quad \text{or} \quad \frac{dp_1}{p_1}=\frac{dx_1}{x_1}.$$

Integrating, $\log p_1 = \log x_1 + \log a_1$ or $p_1 = a_1 x_1$.

Let $F_1(x_1, x_2, x_3, x_4\, p_1, p_2, p_3, p_4) = p_1 - a_1 x_1 = 0.$ ...(2)

Similarly, $F_2(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = p_2 - a_2 x_2 = 0$ ...(3)

and $F_1(x_1, x_2, x_3, x_4, p_1, p_2, p_3, p_4) = p_3 - a_3 x_3 = 0.$ ...(4)

With these values of F_1, F_2 and F_3, we can verify that

$$(F_1, F_2) = \sum_{r=1}^{4}\left(\frac{\partial F_1}{\partial x_r}\frac{\partial F_2}{\partial p_r} - \frac{\partial F_1}{\partial p_r}\frac{\partial F_2}{\partial x_r}\right) = 0.$$

Similarly, we see that $(F_2, F_3) = 0$ and $(F_3, F_1) = 0$. Hence (2), (3) and (4) can be taken as the three desired additional equations. Now solving (1), (2) (3) and (4) for p_1, p_2, p_3 and p_4, we get

$$p_1 = a_1 x_1, \qquad p_2 = a_2 x_2, \qquad p_3 = a_3 x_3 \qquad \text{and} \qquad p_4 = (a_1 a_2 a_3)^{1/3}/x_4.$$

Putting these in $dz = p_1 dx_1 + p_2 dx_2 + p_3 dx_3 + p_4 dx_4$ and integrating the desired complete integral is $z = (1/2) \times (a_1 x_1^{\,2} + a_2 x_2^{\,2} + a_3 x_3^{\,2}) - (a_1 a_2 a_3)^{1/2} \log x_4 + a_4/2$

or $2z = a_1 x_1^{\,2} + a_2 x_2^{\,2} + a_3 x_3^{\,2} - 2(a_1 a_2 a_3)^{1/2} \log x_4 + a_4,$

Ex. 12. *Find a complete integral by Jacobi's method of the equation* $2x^2 y(\partial u/\partial x)^2 (\partial u/\partial z)$ $= x^2(\partial u/\partial y) + 2y(\partial u/\partial x)^2$. **[Delhi Maths (H) 2001]**

Sol. Let $x = x_1$, $y = x_2$, $z = x_3$, $\partial u/\partial x = p_1$, $\partial u/\partial y = p_2$, and $\partial u/\partial z = p_3$

Then given equation becomes $2x_1^2 x_2 p_1^2 p_3 = x_1^2 p_2 + 2x_2 p_1^2$

Dividing by $x_1^2\ x_2$, $2p_1^2 p_3 = (p_2/x_2) + (2p_1^2/x_1^2)$, which can be written as

$$f(x_1, x_2, x_3, p_1, p_2, p_3) = 2p_1^2(p_3 - 1/x_1^2) - p_2/x_2 = 0 \qquad \text{... (1)}$$

$\therefore$ Jacobi's auxiliary equations are

$$\frac{dp_1}{\partial f/\partial x_1} = \frac{dx_1}{-\partial f/\partial p_1} = \frac{dp_2}{\partial f/\partial x_2} = \frac{dx_2}{-\partial f/\partial p_2} = \frac{dp_3}{\partial f/\partial x_3} = \frac{dx_3}{-\partial f/\partial p_3}$$

or
$$\frac{dp_1}{4p_1^2/x_1^3} = \frac{dx_1}{-4p_1(p_3 - 1/x_1^2)} = \frac{dp_2}{p_2/x_2^2} = \frac{dx_2}{1/x_2} = \frac{dp_3}{0} = \frac{dx_3}{-2p_1^2}, \text{ by (1)}$$

Taking the fifth fraction, $dp_3 = 0$ so that $p_3 = a_1$

Taking the second and fourth fractions, $(1/p_2)\, dp_2 = (1/x_2)\, dx_2$

Integrating, $\log p_2 = \log x_2 + \log(2a_2^2)$ or $p_2/x_2 = 2a_2^2$

$\therefore$ Here $F_1(x_1, x_2, x_3, p_1, p_2, p_3) = p_3 = a_1$ ... (2)

and $F_2(x_1, x_2, x_3, p_1, p_2, p_3) = p_2/x_2 = 2a_2^2$ ... (3)

Now, $$(F_1,\ F_2) = \sum_{r=1}^{3}\left(\frac{\partial F_1}{\partial x_r}\frac{\partial F_2}{\partial p_r} - \frac{\partial F_1}{\partial p_r}\frac{\partial F_2}{\partial x_r}\right)$$

or
$$(F_1, F_2) = \frac{\partial F_1}{\partial x_1}\frac{\partial F_2}{\partial p_1} - \frac{\partial F_1}{\partial p_1}\frac{\partial F_2}{\partial x_1} + \frac{\partial F_1}{\partial x_2}\frac{\partial F_2}{\partial p_2} - \frac{\partial F_1}{\partial p_2}\frac{\partial F_2}{\partial x_2} + \frac{\partial F_1}{\partial x_3}\frac{\partial F_2}{\partial p_3} - \frac{\partial F_1}{\partial p_3}\frac{\partial F_2}{\partial x_3}$$

$$= (0)\,(0) - (0)\,(0) + (0)\ (0) - (0)\ (0) + (0)\ (0) - (1)\,(0) = 0$$

Hence (2) and (3) may be taken as additional equations.

Solving (1), (2) and (3) for $p_1, p_2, p_3,$ $\quad p_1 = a_2 x_1 /(a_1 x_1^2 - 1)^{1/2}, \quad p_2 = 2a_2^2 x_2, \quad p_3 = a_1$

Putting these in $du = p_1 dx_1 + p_2 dx_2 + p_3 dx_3 = a_2 x_1 (a_1 x_1^2 - 1)^{-1/2} dx_1 + 2a_2^2 x_2 dx_2 + a_1 dx_3$.

Integrating, $$u = (a_2 / a_1) \times (a_1 x_1^2 - 1)^{1/2} + a_2^2 x_2^2 + a_1 x_3 + a_3,$$

which is the complete integral with a_1, a_2, a_3 as arbitrary constants.

Ex. 13. *Show that a complete integral of the equation* $f(\partial u/\partial x, \partial u/\partial y, \partial u/\partial z) = 0$ *is* $u = ax + by + \theta(a, b)z + c$, *where a, b and c are arbitrary constants and* $f(a, b, \theta) = 0$

(*b*)*Find a complete integral of the equation* $\partial u/\partial x + \partial u/\partial y + \partial u/\partial z = (\partial u/\partial x)(\partial u/\partial y)(\partial u/\partial z)$. **[Allahabad 2004, 06; Meerut 2004, 06; Purvanchal 2003]**

Sol. (*a*) Let $\quad \partial u/\partial x = p_1, \quad \partial u/\partial y = p_2 \quad$ and $\quad \partial u/\partial z = p_3$.

Then gievn equation becomes $$f(p_1, p_2, p_3) = 0 \qquad \text{... (1)}$$

We shall now proceed as in Ex. 1, Art. 3.21. Here Jacobi's auxiliary equations are given by

$$\frac{dp_1}{\partial f/\partial x} = \frac{dx}{-\partial f/\partial p_1} = \frac{dp_2}{\partial f/\partial y} = \frac{dy}{-\partial f/\partial p_2} = \frac{dp_3}{\partial f/\partial z} = \frac{dz}{-\partial f/\partial p_3}$$

$$\Rightarrow \quad \frac{dp_1}{0} = \frac{dp_2}{0}, \text{ using (1)} \quad \Rightarrow \quad dp_1 = 0 \quad \text{and} \quad dp_2 = 0$$

Integrating, $\quad p_1 = a, \quad p_2 = b, \quad a$ and b being arbitrary constants ... (2)

Putting $p_1 = a$ and $p_2 = b$ in (1), $f(a, b, p_3) = 0$ so that

$$p_3 = \text{a function of } a, b = \theta(a, b), \text{ say} \qquad \text{... (3)}$$

Now, we have $$du = (\partial u/\partial x)\, dx + (\partial u/\partial y) dy + (\partial u/\partial z) dz = p_1 dx + p_2 dy + p_3 dz$$

or $$du = a\, dx + b\, dy + \theta(a, b)\, dz, \text{ by (2) and (3)}$$

Integrating, $$u = ax + by + \theta\,(a,b)\, z + c, \qquad \text{... (4)}$$

where c is an arbitrary constant and a, b, θ are connected by relation

$$f(a, b, \theta\,(a, b)) = 0, \text{ by (1), (2) and (3)} \qquad \text{... (5)}$$

(***b***) Given $$\partial u/\partial x + \partial u/\partial y + \partial u/\partial z - (\partial u/\partial x)(\partial u/\partial y)(\partial u/\partial z) = 0 \qquad \text{... (}i\text{)}$$

Let $\quad p_1 = \partial u/\partial x, \quad p_2 = \partial u/\partial y \quad$ and $\quad p_3 = \partial u/\partial z$. Then, (*i*) gives

$$p_1 + p_2 + p_3 - p_1 p_2 p_3 = 0 \qquad \text{... (}ii\text{)}$$

Comparing (*ii*) with (1) of part (*a*), here

$$f(p_1, p_2, p_3) = p_1 + p_2 + p_3 - p_1 p_2 p_3 \qquad \text{... (}iii\text{)}$$

Hence required complete integral is given by (4) and (5) of part (*a*) *i.e.*,

$$u = a\,x + b\,y + \theta\,(a,b)\, z + c, \qquad \text{... (}iv\text{)}$$

where $$a + b + \theta\,(a,b) - ab\,\theta(a, b) = 0 \qquad \text{... (}v\text{)}$$

From (*v*), $$\theta(a, b) = (a + b)/(ab - 1) \qquad \text{... (}vi\text{)}$$

From (*iv*) and (*vi*), $\quad u = ax + by + \{(a + b)/(ab - 1)\} + c,$

which is the required complete integral of (*i*), *a*, *b*, *c* being arbitrary constants.

EXERCISE 3(G)

Find the complete integral of the following equation: (1 – 5)

1. $f(p_1, p_2, p_3) = 0$ **Ans.** $z = a_1x_1 + a_2x_2 + a_3x_3 + a_4$, where $f(a_1, a_2, a_3) = 0$

2. $p_1 + p_2 + p_3 - p_1p_2p_3 = 0$ **Ans.** $z = a_1x_1 + a_2x_2 + a_3x_3 + a_4$, where $a_1 + a_2 + a_3 - a_1a_2a_3 = 0$

3. $p_1x_1^2 - p_2^2 - ap_3^2 = 0$ **Ans.** $z = -(a_1^2 + a a_2^2)x_1^{-1} + a_1x_2 + a_2a_3x_3 + a_3$

4. $x_3(x_3 + p_3) = p_1^2 + p_2^2$ **Ans.** $z = a_1x_1 + a_2x_2 + (a_1^2 + a_2^2)\log x_3 - x_3^2/2 + a_3$

5. $x_3 + 2p_3 - (p_1 + p_3^2) = 0$ **Ans.** $z = a_1x_1 + a_2x_2 + (a_1 + a_2)^2 \times (x_3/2) - (x_3^2/4) + a_3$

6. $x_1 + p_1^2 + x_2 + p_2^2 - x_3p_3^2 = 0$ **Ans.** $z = 2\,(a_1x_1)^{1/2} + 2\,(a_2x_2)^{1/2} + 2\,\{(a_1 + a_2)\,x_3\}^{1/2} + a_3$

7. *Show how to solve, by Jacobi method, a partial differential equation of the type* $f(x, \partial u/\partial x, \partial u/\partial z) = g(y, \partial u/\partial y, \partial u/\partial z)$ *and illustrate the method by finding a complete integral of equation* $2x^2y(\partial u/\partial x)^2(\partial u/\partial z) = x^2(\partial u/\partial y) + 2y(\partial u/\partial x)^2$. **[Meerut 2005]**

Sol. Try yourself **Ans.** $u = (ax^2 - b)^{1/2} + ay^2 + (z/b) + c$

8. *Prove that an equation of the "Clairaut" form* $x(\partial u/\partial x) + y(\partial u/\partial y) + z(\partial u/\partial z)$ $= f(\partial u/\partial x, \partial u/\partial y, \partial u/\partial z)$ *is always solvable by Jacobi's method. Hence solve*

$$(\partial u/\partial x + \partial u/\partial y + \partial u/\partial z)\,\{x(\partial u/\partial x) + y(\partial u/\partial y) + z(\partial u/\partial z)\} = 1$$

3.22. Jacobi's method for solving a non-linear first order partial differential equation in two independent variables.

[Delhi Maths (H) 1997; Amaravati 2001; Himanchal 2003, 05]

Let $$F(x, y, z, p, q) = 0 \qquad \ldots (1)$$

be the non-linear first order equation in two independent variables x, y.

Then we know that a solution of (1) is of the form $$u(x, y, z) = 0 \qquad \ldots (2)$$

showing that u can be treated as a dependent variable and x, y, z as three independent variables.

Differentiating (2) partially w.r.t. 'x' and 'y', respectively, we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} = 0 \qquad \text{and} \qquad \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y} = 0$$

or $$p_1 + p_3 p = 0 \qquad \text{and} \qquad p_2 + p_3\ q = 0 \qquad \ldots (3)$$

where $p = \partial z/\partial x$, $q = \partial z/\partial y$, $p_1 = \partial u/\partial x = \partial u/\partial x_1$, $p_2 = \partial u/\partial y = \partial u/\partial x_2$, $p_3 = \partial u/\partial z = \partial u/\partial x_3$

by taking $$x = x_1, \qquad y = y_2 \qquad \text{and} \qquad z = x_3 \qquad \ldots (4)$$

From (3), $$p = -(p_1/p_3) \qquad \text{and} \qquad q = -(p_2/p_3) \qquad \ldots (5)$$

Using (4) and (5), (1) reduces to $$f(x_1, x_2, x_3, p_1, p_2, p_3) = 0 \qquad \ldots (6)$$

We now solve (6) by Jacobi's method as usual (refer Art. 3.20) to get the complete integral of (6). Finally, putting $x_1 = x$, $x_2 = y$, $x_3 = z$, we obtain solution of (6) containing original variables x, y, z and new dependent variable u. The solution so obtained will contain three arbitrary constants a_1, a_2, a_3 (say). However, for the given equation in the form (1), we need only two arbitrary constants in the final solution. The required solution $u = 0$ of (1) is obtained by making different choices of our third arbitrary constant.

Ex. 1. *Solve* $p^2 x + q^2 y = z$ *by Jacobi's method.* **[Nagpur 2002; Himanchal 2003, 05]**

Sol. Given $p^2 x + q^2 y = z$.

Let a solution of (1) be of the form $u(x, y, z) = 0$ ... (2)

So treating u as dependent variable and x, y, z as three independent variables, differentiation of (2) partially w.r.t 'x' and 'y' respectively gives

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial x} = 0 \quad \text{and} \quad \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial y} = 0 \quad i.e. \quad p_1 + p_3 p = 0 \quad \text{and} \quad p_2 + p_3\, q = 0$$

so that $p = -p_1 / p_3$ and $q = -p_2 / p_3$ .... (3)

where $p_1 = \partial u / \partial x = \partial u / \partial x_1$, $p_2 = \partial u / \partial y = \partial u / \partial x_2$, $p_3 = \partial u / \partial z = \partial u / \partial x_3$, $p = \partial z / \partial x, q = \partial z / \partial y$

by taking $x = x_1$, $y = x_2$ and $z = x_3$ ... (4)

Using (3) and (4), (1) $\Rightarrow$ $x_1 (p_1 / p_3)^2 + x_2 (p_2 / p_3)^2 = x_3$. $\Rightarrow$ $x_1 p_1^2 + x_2 p_2^2 - x_3 p_3^2 = 0$

Let $f(x_1, x_2, x_3, p_1, p_2, p_3) = x_1 p_1^2 + x_2 p_2^2 - x_3\ p_3^2 = 0$ ... (5)

Now, the Jacobi's auxiliary equations are

$$\frac{dp_1}{\partial f / \partial x_1} = \frac{dx_1}{-\partial f / \partial p_1} = \frac{dp_2}{\partial f / \partial x_2} = \frac{dx_2}{-\partial f / \partial p_2} = \frac{dp_3}{\partial f / \partial x_3} = \frac{dx_3}{-\partial f / \partial p_3}$$

or

$$\frac{dp_1}{p_1^2} = \frac{dx_1}{-2p_1 x_1} = \frac{dp_2}{p_2^2} = \frac{dx_2}{-2p_2 x_2} = \frac{dp_3}{-p_3^2} = \frac{dx_3}{2p_3 x_3}, \text{ by (5)}$$

Taking the first two fractions, $(2/p_1)\, dp_1 + (1/x) dx = 0$.

Integrating, $2 \log p_1 + \log x_1 = \log a_1$ so that $x_1 p_1^2 = a_1$ or $p_1 = (a_1 / x_1)^{1/2}$

Similarly, the third and fourth fractions give $p_2 = (a_2 / x_2)^{1/2}$

Substituting these values of p_1 and p_2 in (5), we get $p_3 = \{(a_1 + a_2)/x_3\}^{1/2}$.

Putting the above values of p_1, p_2 and p_3 in $du = p_1 dx_1 + p_2 dx_2 + p_3 dx_3$, we get

$$du = a_1^{1/2}\ x_1^{-1/2} dx_1 + a_2 x_2^{-1/2} dx_2 + (a_1 + a_2)^{1/2} x_3^{-(1/2)} dx_3.$$

Integrating, $u = 2(a_1 x_1)^{1/2} + (a_2 x_2)^{1/2} + 2(a_1 + a_2)^{1/2} x_3^{1/2} + a_3$ ... (6)

Taking $a_2 = 1$ and using (4), the required solution $u = 0$ is given by

$$2(a_1 x)^{1/2} + 2y^{1/2} + 2(a_1 + 1)^{1/2} z^{1/2} + a_3 = 0,$$

which is the complete integral containing two arbitrary constants a_1 and a_3.

Ex. 2. *Solve* $p^2 + q^2 = k^2$ by Jacobi's method **[Delhi B.A./B.Sc. (Prog) Maths 2007]**

Sol. Given $p^2 + q^2 = k^2$ ... (1)

Let a solution of (1) be of the form $u(x, y, z) = 0$ ... (2)

So treating u as dependent variable and x, y, z as three independent variable, differentiation of (2) partially w.r.t. 'x' and 'y' respectively gives

$$\frac{\partial u}{\partial x}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial x}=0 \quad \text{and} \quad \frac{\partial u}{\partial y}+\frac{\partial u}{\partial z}\frac{\partial z}{\partial y}=0 \quad \textit{i.e.,} \quad p_1+p_3p=0 \quad \text{and} \quad p_2+p_3q=0$$

so that $\quad p=-(p_1/p_3) \quad$ and $\quad q=-(p_2/p_3) \quad$ ... (3)

where $p_1=\partial u/\partial x=\partial u/\partial x_1,\ p_2=\partial u/\partial y=\partial u/\partial x_2,\ p_3=\partial u/\partial z=\partial u/\partial x_3,\ p=\partial z/\partial x,\ q=\partial z/\partial y$

by taking $\quad x=x_1, \quad y=x_2 \quad$ and $\quad z=x_3 \quad$ ... (4)

Using (3) and (4), (1) reduces to $\quad p_1^2/p_3^2+p_2^2/p_3^2=k^2 \quad$ or $\quad p_1^2+p_2^2=k^2p_3^2$

Let $\quad f(x_1,x_2,x_3,p_1,p_2,p_3)=p_1^2+p_2^2-k^2p_3^2=0 \quad$ ... (5)

Now, the Jacobi auxilliary equations are given by

$$\frac{dp_1}{\partial f/\partial x_1}=\frac{dx_1}{-\partial f/\partial p_1}=\frac{dp_2}{\partial f/\partial x_2}=\frac{dx_2}{-\partial f/\partial p_2}=\frac{dp_3}{\partial f/\partial x_3}=\frac{dx_3}{-\partial f/\partial p_3}$$

or
$$\frac{dp_1}{0}=\frac{dx_1}{-2p_1}=\frac{dp_2}{0}=\frac{dx_2}{-2p_2}=\frac{dp_3}{0}=\frac{dx_3}{2k^2p_3}, \text{ using (5)}$$

From the first and third fractions of (5), $\quad dp_1=0 \quad$ and $\quad dp_2=0$

Integrating, $\quad p_1=a_1$ and $p_2=a_2$, a_1 and a_2 being arbitrary constants

With $\quad p_1=a_1 \quad$ and $\quad p_2=a_2, \quad$ (5) gives $\ p_3=(a_1^2+a_2^2)^{1/2}/k$

Putting the above values of p_1, p_2 and p_3 in $du=p_1dx_1+p_2dx_2+p_3dx_3$, we get

$$du=a_1dx_1+a_2dx_2+\{(a_1^2+a_2^2)^{1/2}/k\}dx_3$$

Integrating, $\quad u=a_1x+a_2x_2+\{(a_1^2+a_2^2)^{1/2}/k\}x_3+a_3 \quad$ ... (6)

Taking $a_2=1$ and using (4), the required solution $u=0$ is given by

$$a_1x+x_2+\{(a_1^2+1)^{1/2}/k\}x_3+a_3=0,$$

which is the complete integral of (1) containing two arbitrary constants a_1 and a_3.

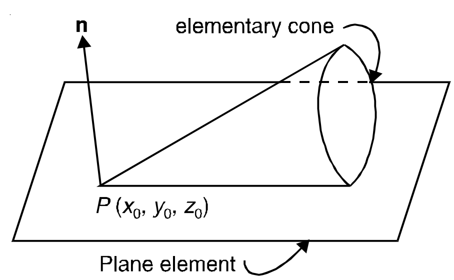
Ex. 3. *Solve the following partial differential equations by Jacobi's method:*

(*i*) $p=(z+qy)^2$ (*ii*) $(p^2+q^2)x=pz$ (*iii*) $xpq+yq^2=1$ **[Nagpur 2005]**

Hint. Proceed as in the above solved Ex. 1

3.23 Cauchy's method of characteristics for solving non-linear partial differential equation $\quad f(x,y,z,\ \partial z/\partial x,\ \partial z/\partial y)=0 \quad$ *i.e.,* $\quad f(x,y,z,p,q)=0 \quad$ ... (1)

We know that the plane passing through the point $P(x_0, y_0 z_0)$ with its normal parallel to the direction **n** whose direction ratios are $p_0, q_0, -1$ is uniquely given by the set of five numbers $D(x_0,y_0,z_0,p_0,q_0)$ and conversely any such set of five numbers defines a plane in three dimensional space. In view of this fact a set of five numbers $D(x,y,z,p,q)$ is known as a *plane element* of a three dimensional space. As a special case a plane element (x_0,y_0,z_0,p_0,q_0) whose components satisfy (1) is known as an *integral element* of (1) at P. Solving (1) for q, suppose we get

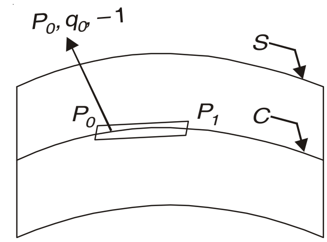


$$q = F(x, y, z, p).$$

which gives a value of q corresponding to known values of x, y, z and p. Then, keeping x_0, y_0 and z_0 fixed and varying p, we shall arrive at a set of plane elements $\{x_0, y_0, z_0, p,\ G(x_0, y_0, z_0, p)\}$ which depend on the single parameter p. As p varies, we get a set of plane elements all of which pass through the point P. Hence the above mentioned set of plane elements envelop a cone with vertex P. The cone thus obtained is known as the *elementary cone of* (1) at the point P.

Consider a surface S with equation $$z = g(x, y) \qquad \text{... (2)}$$

If the function $g(x, y)$ and its first partial derivatives $g_x(x, y)$ and $g_y(x, y)$ are continuous in a certain region R of the xy-plane, then the tangent plane at each point of S determines a plane element of the form $\{x_0, y_0, g(x_0, y_0), g_x(x_0, y_0), g_y(x_0, y_0)\}$ which will be referred as *the tangent element* of the surface S at the point $\{x_0, y_0, g(x_0, y_0)]$.



Consider a curve C with parametric equations

$$x = x(t), \qquad y = y(t), \qquad z = z(t), \qquad \text{t being the parameter.} \qquad \text{... (3)}$$

Then curve C lies on (2) provided $$z(t) = g\{x(t), y(t)\} \qquad \text{... (4)}$$

holds good for all values of t in the appropriate interval I. Let P_0 be a point on curve C corresponding to $t = t_0$. Now, the direction ratios of the tangent line $P_0\ P_1$ are $x'(t_0), y'(t_0), z'(t_0)$ where $x'(t_0), y'(t_0)\ z'(t_0)$ denote the values of $dx/dt,\ dy/dt,\ dz/dt$ respectively at $t = t_0$

This direction will be perpendicular to direction of normal $\boldsymbol{n}$ (with direction ratios $p_0, q_0, -1$)

if $$p_0 x'(t_0) + q_0\ y'(t_0) + (-1)\ z'(t_0) = 0 \qquad \text{or} \qquad z'(t_0) = p_0\ x'(t_0) + q_0\ y'(t_0)$$

It follows that any set $$\{x(t), y(t), z(t), p(t), q(t)\} \qquad \text{...(5)}$$

of five real functions satisfying the condition that $$z'(t) = p(t)\ x'(t) + q(t)\ y'(t) \qquad \text{... (6)}$$

defines a strip at the point (x, y, z) of the curve C. When such a strip is also an integral element of (1), then the strip under consideration is known as an *integral strip* of (1). In other words, the set of functions (5) is known as an integral strip of (1) provided these satisty (6) and the following additional condition

$$f\{x(t), y(t), z(t), p(t), q(t)\} = 0, \quad \text{for all } t \text{ in I.}$$

If at each point of the curve (3) touches a generator of the elementary cone, then the corresponding strip is known as a *characteristic strip.*

Derivation of the equations determining a *characteristic strip*

Clearly, the point $(x + dx,\ y + dy,\ z + dz)$ lies in the tangent plane to the elementary cone at P if

$$dz = p\,dx + q\ dy \qquad \text{... (7)}$$

where p, q satisfy (1). Differentiation (7) w.r.t. 'p', we get

$$0 = dx + (dq/dp)\ dy \qquad \text{... (8)}$$

Again, differentiating (1) partially w.r.t. 'p', we have

$$\partial f/\partial p + (\partial f/\partial q)\ (dq/dp) = 0 \qquad \text{i.e.,} \qquad f_p + f_q (dq/dp) = 0 \qquad \text{... (9)}$$

Here, $\partial f/\partial p = f_p$ and $\partial f/\partial q = f_q$

Solving (7), (8) and (9) for the ratios of dy, dz to dx, we get

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{p f_p + q f_q}. \qquad \text{... (10)}$$

Hence along a characteristic strip $x'(t)$, $y'(t)$, $z'(t)$ will be proportional to f_p, f_q, $p f_p + q f_q$ respectively. If the parameter t be selected satisfying the relations

$$x'(t) = f_p \qquad \text{and} \qquad y'(t) = f_q.$$

then, we have $$z'(t) = p f_p + q f_q$$

Since along a characteristic strip p is a function of t, hence

$p'(t) = (\partial p/\partial x)\ (dx/dt) + (\partial p/\partial y)\ (dy/dt) = (\partial p/\partial x)\ \ (\partial f/\partial p) + (\partial p/\partial y)\ (\partial f/\partial q)$, using (11)

Thus, $$p'(t) = (\partial p/\partial x)\ (\partial f/\partial p) + (\partial q/\partial x)\ (\partial f/\partial q) \qquad \text{... (12)}$$

$$\left[\because\ \frac{\partial p}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial q}{\partial x}\right]$$

Now, differentiating (1) partially w.r.t. 'x', gives $$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial z} p + \frac{\partial f}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial f}{\partial q}\frac{\partial q}{\partial x} = 0$$

or $$f_x + p\ f_z + p'(t) = 0, \text{ using (12)}$$

Hence on a characteristic strip, $$p'(t) = -f_x - p\, f_z \qquad \text{... (13)}$$

Similarly, we have $$q'(t) = -f_y - q\, f_z \qquad \text{... (14)}$$

Here $f_x = \partial f/\partial x$, $\qquad f_y = \partial f/\partial y$, $\qquad f_z = \partial f/\partial z$

From (11), (13) and (14), we get the following system of five ordinary differential equations for the determination of the characteristic strip

$$x'(t) = f_p, \quad y'(t) = f_q, \quad z'(t) = p\ f_p + q f_q, \quad p'(t) = -f_x - p\ f_z \quad \text{and} \quad q'(t) = -f_y - q\, f_z \quad \text{...(15)}$$

The above equations are called the *characteristic equations* of (1). In view of a well known result if the functions which are involved in (15) satisfy a Lipschitz condition, there exists a unique solution of (15) for given set of initial values of the variables. It follows that the characteristic strip is determined uniquely by any initial element $(x_0, y_0, z_0, p_0, q_0)$ and any initial value t_o of t.

Working rule for solving Cauchy's problem. **[Meerut 2005]**

Suppose we wish to find the integral surface of (1) which passes through a given curve with parametric equation $\quad x = f_1(\lambda)$, $y = f_2(\lambda)$, $\quad z = f_3(\lambda)$, $\quad \lambda$ being the parameter ... (16)

then in the solution $$x = x\ (p_0, q_0, x_0, y_0, t_0, t) \text{ etc.} \qquad \text{... (17)}$$

of the characteristic equations (15), we shall assume that

$$x_0 = f_1(\lambda), \qquad y_0 = f_2(\lambda), \qquad z_0 = f_3(\lambda)$$

are the initial values of x, y, z respectively. Then the corresponding initial values of p_0, q_0 can be obtained by the following relations

$$f_3'(\lambda) = p_0\ f_1'(\lambda) + q_0\ f_3'(\lambda\ ) \qquad \text{and} \qquad f\{f_1(\lambda),\ f_2(\lambda),\ f_3(\lambda),\ p_0,\ q_0\} = 0$$

When the above values of x_0, y_0, z_0, p_0, q_0 and the appropriate value of t_0 is substituted in (17), we shall be able to express x, y, z involving the two parameters t and λ of the form

$$x = \phi_1\ (t, \lambda), \qquad y = \phi_2\ (t, \lambda) \qquad \text{and} \qquad z = \phi_3(t, \lambda) \qquad \text{... (18)}$$

which are known as characteristics of (1)

Finally, by eliminating λ and t from (18), we arrive at a relation of the form $G\ (x,y,z) = 0$, which is the required equation of the integral surface of (1) passing through the given curve (16).

3.24 Some Theorems:

Theorems 1. *A necessary and sufficient condition that a surface be an integral surface of a partial differential equation is that at each point its tangent element should touch the elementary cone of the equation.*

Proof. Using geometrical considerations and Art 3.23, complete the proof yourself.

Theorem II. *Along every characteristic strip of the partial differential equation $f\ (x, y, z, p, q) = 0$ the function $f\ (x, y, z, p, q)$ is a constant.*

Proof. Along a characteristic strip, we have

$$\frac{d}{dt} f\,\{x(t),\ y(t),\ z(t),\ p(t), q(t)\} = f_x\ x'(t) + f_y\ y'(t) + f_z\ z'(t) + f_p\ p'(t) + f_q\ q'(t)$$

$$= f_x f_p + f_y f_q + f_z\ (p\ f_p + q\ f_q) - f_p(f_x + p\ f_z) - f_q(f_y + q\ f_z)$$

= 0, using the characteristic equation (15) of Art. 3.23

showing that $f\ (x, y, z, p, q) = K$, a constant along the strip.

Corollary to theorem II. *If a characteristic strip contains at least one integral element of $f(x, y, z, p, q) = 0$ it is an integral strip of the equation $f(x, y, z,\ \partial z/\partial x,\ \partial z/\partial y) = 0$*

Proof. Left as an exercise.

3.25 SOLVED EXAMPLES BASED ON ART. 3.23

Ex. 1. *Find the characteristics of the equation $pq = z$, and determine the integral surface which passes through the parabola $x = 0,\ y^2 = z$.* **[Meerut 2005; I.A.S. 1999]**

Sol. Given equation is $$pq = z \qquad \text{... (1)}$$

We are to find its integral surface which passes through the given parabola given by

$$x = 0, \qquad \text{and} \qquad y^2 = z \qquad \text{... (2)}$$

Re-writing (2) in parametric form, we have

$$x = 0, \qquad y = \lambda, \qquad z = \lambda^2, \qquad \lambda \text{ being a parameter} \qquad \text{... (3)}$$

Let the initial values x_0, y_0, z_0, p_0, q_0 of x, y, z, p, q be taken as

$$x_0 = x_0(\lambda) = 0, \qquad y_0 = y_0(\lambda) = \lambda, \qquad z_0 = z_0(\lambda) = \lambda^2 \qquad \text{... (4A)}$$

Let p_0, q_0 be the initial values of p, q corresponding to the initial values x_0, y_0, z_0. Since initial values (x_0, y_0, z_0, p, q_0) satisfy (1), we have

$$p_0 q_0 = z_0, \qquad \text{or} \qquad p_0\ q_0 = \lambda^2, \text{ by (4A)} \qquad \text{... (5)}$$

Also, we have $$z_0'(\lambda) = p_0\ x_0'(\lambda) + q_0\ y_0'(\lambda)$$

so that $$2\lambda = p_0 \times 0 + q_0 \times 1 \qquad \text{or} \qquad q_0 = 2\lambda, \text{ by (4A)} \qquad \text{... (6)}$$

Solving (5) and (6), $$p_0 = \lambda/2 \qquad \text{and} \qquad q_0 = 2\lambda \qquad \text{... (4B)}$$

Collecting relations (4A) and (4B) together, initial values of x_0, y_0, z_0, p_0, q_0 are given by

$$x_0 = 0, \quad y_0 = \lambda, \quad z_0 = \lambda^2, \quad p_0 = \lambda/2, \quad q_0 = 2\lambda \quad \text{when} \quad t = t_0 = 0 \qquad \text{... (7)}$$

Re-writing (1), let $\quad f(x, y, z, p, q) = pq - z = 0 \qquad$ ... (8)

The usual characteristic equations of (8) are given by

$$dx/dt = \partial f/\partial p = q \qquad \text{... (9)}$$

$$dy/dt = \partial f/\partial q = p \qquad \text{... (10)}$$

$$dz/dt = p(\partial f/\partial p) + q(\partial f/\partial q) = 2pq \qquad \text{... (11)}$$

$$dp/dt = -(\partial f/\partial x) - p(\partial f/\partial z) = p \qquad \text{... (12)}$$

and
$$dq/dt = -(\partial f/\partial y) - q(\partial f/\partial z) = q \qquad \text{... (13)}$$

From (9) and (13), $\quad (dx/dt) - (dq/dt) = 0, \quad$ so that $\quad x - q = C_1, \qquad$ .... (14)

where C_1 is an arbitrary constant. Using initial values (7), (14) gives

$x_0 - q_0 = C_1 \quad$ or $\quad 0 - 2\lambda = C_1 \quad$ or $\quad C_1 = -2\lambda$, Then (14) becomes

$x - q = -2\lambda \quad$ or $\quad x = q - 2\lambda, \qquad$ ... (15)

From (10) and (12), $\quad (dy/dt) - (dp/dt) = 0 \quad$ so that $\quad y - p = C_2, \qquad$ ... (16)

where C_2 is an arbitrary constant. Using initial values (7), (16) gives

$y_0 - p_0 = C_2 \quad$ or $\quad \lambda - (\lambda/2) = C_2 \quad$ or $\quad C_2 = \lambda/2$. Then (16) becomes

$y - p = \lambda/2 \quad$ or $\quad y = p + (\lambda/2) \qquad$ ... (17)

From (12), $\quad (1/p)\, dp = dt \quad$ so that $\quad \log p - \log C_3 = t \quad$ or $\quad p = C_3\, e^t \qquad$ ... (18)

Using initial values (7), (18) gives $\quad p_0 = C_3 e^0 \quad$ or $\quad \lambda/2 = C_3$

Hence (18) reduces to $\quad p = (\lambda/2) \times e^t \qquad$ ... (19)

From (13), $\quad (1/q)\, dq = dt \quad$ so that $\quad \log q - \log C_4 = t \quad$ or $\quad q = C_4\, e^t \quad$ ... (20)

Using initial values (7), (20) gives $\quad q_0 = C_4\, e^0 \quad$ or $2\lambda = C_4$

Hence (20) reduces to $\quad q = 2\lambda\, e^t \qquad$ ... (21)

Using (21), (15) becomes $\quad x = 2\lambda\, e^t - 2\lambda \quad$ or $\quad x = 2\lambda\,(e^t - 1) \qquad$ ... (22)

Using (19), (17) becomes $\quad y = (\lambda/2)\, e^t + \lambda/2 \quad$ or $\quad y = (\lambda/2) \times (e^t + 1) \qquad$ ... (23)

Substituting values of p and q from (19) and (21) in (11), we get

$dz/dt = 2\{(\lambda/2) \times e^t\} \times \{2\lambda\, e^t\} \quad$ or $\quad dz = 2\lambda^2 e^{2t}\, dt$.

Integrating, $\quad z = \lambda^2 e^{2t} + C_5$, C_5 being arbitrary constant $\qquad$ .... (24)

Using initial values (7), (24) gives $\quad z_0 = \lambda^2 e^0 + C_5 \quad$ or $\quad \lambda^2 = \lambda^2 + C_5 \quad$ or $\quad C_5 = 0$

Then, (24) gives $$z = \lambda^2 e^{2t} \quad \text{or} \quad z = \lambda^2 (e^t)^2 \qquad \text{... (25)}$$

The required characteristics of (1)are given by (22), (23) and (25)

To find the required integral surface of (1), we now proceed to eliminate two parameters t and λ from three equations (22), (23)and (25). Solving (22) and (23) for e^t and λ, we have

$$e^t = (x+4y)/(4y-x) \qquad \text{and} \qquad \lambda = (4y-x)/4$$

Substituting these values of e^t and λ in (25), we have

$$z = \{(4y-x)^2/16\} \times \{(x+4y)/(4y-x)\}^2 \qquad \text{or} \qquad 16z = (4y+x)^2,$$

which is the required integral surface of (1) passing through (2).

Ex. 2. *Find the solution of the equation* $z = (p^2+q^2)/2 + (p-x)(q-y)$ *which passes through the x-axis.* **[Himachal 1996; 2004; I.A.S. 2002]**

Sol. Given equation is $$z = (p^2+q^2)/2 + (p-x)(q-y) \qquad \text{... (1)}$$

We are to find its integral surface which passes through x-axis which is given by equations
$$y = 0 \qquad \text{and} \qquad z = 0 \qquad \text{... (2)}$$

Re-writing (2) in parametric form, $x = \lambda, \quad y = 0, \quad z = 0$, λ being the parameter ... (3)

Let the initial values x_0, y_0, z_0, p_0, q_0 of x, y, z, p, q be taken as

$$x_0 = x_0(\lambda) = \lambda, \qquad y_0 = y_0(\lambda) = 0, \qquad z = z_0(\lambda) = 0 \qquad \text{... (4A)}$$

Let p_0, q_0 be the initial values of p, q corresponding to the initial values x_0, y_0, z_0. Since initial values $(x_0, y_0, z_0, p_0, q_0)$ satisfy (1), we have

$$z_0 = (p_0^2+q_0^2)/2 + (p_0 - x_0)(q_0 - x_0) \qquad \text{or} \qquad 0 = (p_0^2+q_0^2)/2 + q_0(p_0 - \lambda), \text{ by (4A)}$$

or $$p_0^2 + q_0^2 + 2q_0 p_0 - 2q_0\lambda = 0 \qquad \text{... (5)}$$

Also, we have $$z_0'(\lambda) = p_0 x_0'(\lambda) + q_0 y_0'(\lambda)$$

so that $$0 = p_0 \times 1 + q_0 \times 0 \qquad \text{or} \qquad p_0 = 0, \text{ by (4A)} \qquad \text{... (6)}$$

Solving (5) and (6), $$p_0 = 0 \qquad \text{and} \qquad q_0 = 2\lambda \qquad \text{... (4B)}$$

Collecting relations (4A) and (4B) together, initial values of x_0, y_0, z_0, p_0, q_0 are given by

$$x_0 = \lambda, \quad y_0 = 0, \quad z_0 = 0, \quad p_0 = 0, \quad q_0 = 2\lambda \quad \text{when} \quad t = t_0 = 0 \qquad \text{... (7)}$$

Let $$f(x,y,z,p,q) = (p^2+q^2)/2 + pq - py - qx + xy - z = 0 \qquad \text{... (8)}$$

The usual characteristic equations of (8) are given by

$$dx/dt = \partial f/\partial p = p + q - y \qquad \text{... (9)}$$

$$dy/dt = \partial f/\partial q = q + p - x \qquad \text{... (10)}$$

$$dz/dt = p\,(\partial f/\partial p) + q(\partial f/\partial q) = p(p+q-y) + q(q+p-x), \qquad \text{... (11)}$$

$$dp/dt = -(\partial f/\partial x) - p(\partial f/\partial z) = p + q - y \qquad \text{... (12)}$$

and $$dq/dt = -(\partial f/\partial y) - q\,(\partial f/\partial z) = p + q - x \qquad \text{... (13)}$$

From (9) and (12), $$(dx/dt) - (dp/dt) = 0 \qquad \text{so that} \qquad x - p = C_1 \qquad \text{... (14)}$$

where C_1 is an arbitrary constant. Using initial conditions (7), (14) gives $\lambda - 0 = C_1$ or $C_1 = \lambda$.

Hence (14) reduces to $\quad x-p=\lambda \quad$ or $\quad x=p+\lambda$ ... (15)

From (10) and (13), $\quad (dy/dt)-(dq/dt)=0 \quad$ so that $\quad y-q=C_2$, ... (16)

where C_2 is an arbitrary constant.

Using initial conditions (7), (16) gives $\quad 0-2\lambda=C_2 \quad$ or $\quad C_2=-2\lambda$.

Hence (16) reduces to $\quad y-q=-2\lambda \quad$ or $\quad y=q-2\lambda$ ... (17)

$$\therefore \quad \frac{d(p+q-x)}{dt}=\frac{dp}{dt}+\frac{dq}{dt}-\frac{dx}{dt}=p+q-y+p+q-x-(p+q-y), \text{ using (9), (12) and (13)}$$

or $\quad \dfrac{d(p+q-x)}{dt}=p+q-x \quad$ or $\quad \dfrac{d(p+q-x)}{p+q-x}=dt\cdot$

Integrating, $\quad \log(p+q-x)-\log C_3=t \quad$ or $\quad p+q-x=C_3\, e^t$, ... (18)

where C_3 is an arbitrary constant. Using initial conditions (7), (18) gives $0+2\lambda-\lambda=C_3$ or $C_3=\lambda$.

Hence (18) reduces to $\quad p+q-x=\lambda e^t$ ... (19)

Now, $\quad \dfrac{d(p+q-y)}{dt}=\dfrac{dp}{dt}+\dfrac{dq}{dt}-\dfrac{dy}{dt}=p+q-y+p+q-x-(q+p-x)$, by (10), (12) and (13)

or $\quad \dfrac{d(p+q-y)}{dt}=p+q-y \quad$ or $\quad \dfrac{d(p+q-y)}{b+q-y}=dt.$

Integrating, $\quad \log(p+q-y)-\log C_4=t \quad$ or $\quad p+q-y-C_4\, e^t$ ... (20)

where C_4 is an arbitrary constant. Using initial conditions (7), (20) gives $0+2\lambda-0=C_4$ or $C_4=2\lambda$.

Hence (20) reduces to $\quad p+q-y=2\lambda\, e^t$ ... (21)

From (9) and (21), $\quad dx/dt=2\lambda\, e^t$ so that $\quad x=2\lambda\, e^t+C_5$ ... (22)

where C_5 is an arbitary constant. Using initial conditions (7), (22) gives $\lambda=2\lambda+C_5$ or $C_5=-\lambda$.

Hence (22) reduces to $\quad x=2\lambda\, e^t-\lambda \quad$ or $\quad x=\lambda\,(2e^t-1)$ ... (23)

From (10) and (19), $\quad dy/dt=\lambda\, e^t$ so that $\quad y=\lambda\, e^t+C_6$ ... (24)

where C_6 is an arbitrary constant. Using initial conditions (7), (24) gives $0=\lambda+C_6$ or $C_6=-\lambda$.

Hence (24) reduces to $\quad y=\lambda e^t-\lambda \quad$ or $\quad y=\lambda\,(e^t-1)$ ... (25)

Substituting value of y from (17) in (12), we get

$dp/dt=p+q-(q-2\lambda) \quad$ or $\quad (dp/dt)-p=2\lambda$, ... (26)

which is a linear equation whose integrating factor = $e^{\int(-1)dt}=e^{-t}$ and solution is

$p\, e^{-t}=\int(2\lambda)\, e^{-t}dt+C_7=-2\lambda e^{-t}+C_3 \quad$ or $\quad p=-2\lambda+C_3\, e^t$ ... (27)

where C_7 is an arbitrary constant. Using initial condition (7), (27) gives $0=-2\lambda+C_7$ or $C_7=2\lambda$.

Hence (27) reduces to $\quad p=-2\lambda+2\lambda e^t \quad$ or $\quad p=2\lambda\,(e^t-1)$ ... (28)

Substituting value of x from (15) in (13), we get

$$dq/dt = p+q-(p+\lambda) \qquad \text{or} \qquad dq/dt - q = -\lambda, \qquad \ldots (29)$$

which is a linear equation whose integrating factor = $e^{\int(-1)dt} = e^{-t}$ and solution is

$$qe^{-t} = \int(-\lambda)\, e^{-t} dt + C_8 = \lambda e^{-t} + C_8 \qquad \text{or} \qquad q = \lambda + C_8\, e^{t} \qquad \ldots (30)$$

where C_8 is an arbitrary constant. Using initial condition (7), (30) gives $2\lambda = \lambda + C_8$ or $C_8 = \lambda$.

Hence (30) reduces to $\quad q = \lambda + \lambda\, e^{t} \quad$ or $\quad q = \lambda\,(1+e^{t})$ ... (31)

Substitutions the values of $p+q-x$ and $p+q-y$ from (13) and (24) respectively in (1) gives

$$dz/dt = p(2\lambda\, e^{t}) + q(\lambda\, e^{t}) = 2\lambda\,(e^{t}-1)\,(2\lambda\, e^{t}) + \lambda(1+e^{t})\,(\lambda\, e^{t})$$

[on putting values of p and q with help of (28) and (31)]

or $\qquad dz/dt = 5\lambda^2 e^{2t} - 3\lambda^2 e^{t} \qquad$ or $\qquad dz = (5\lambda^2 e^{2t} - 3\lambda^2 e^{t})dt$.

Integrating, $\qquad z = (5/2) \times \lambda^2 e^{2t} - 3\lambda^2 e^{t} + C_9$ ... (32)

where C_9 is an arbitrary constant. Using initial conditions (7), namely $z = 0$ where $t = 0$, (32) gives $0 = (5/2) \times \lambda^2 - 3\lambda^2 + C_9$ or $C_9 = 3\lambda^2 - (5/2)\lambda^2$. Hence (32) reduces to

$$z = (5/2) \times \lambda^2 (e^{2t}-1) - 3\lambda^2 (e^{t}-1) \qquad \ldots (33)$$

Solving (23) and (25) for λ and e^t, $\quad \lambda = x-2y \quad$ and $\quad e^{t} = (x-y)/(x-2y)$ ... (34)

Eliminating λ and e^t from (33) and (34), we have

$(p^2+q^2)x = pz$

$$z = \frac{5}{2}(x-2y)^2 \left\{ \left(\frac{x-y}{x-2y} \right)^2 - 1 \right\} - 3(x-2y)^2 \left(\frac{x-y}{x-2y} - 1 \right)$$

or $$z = (5/2) \times \{(x-y)^2 - (x-2y)^2\} - 3\ \{(x-2y)\,(x-y) - (x-2y)^2\}$$

or $$z = (y/2) \times (4x-3y), \text{ on simplification.}$$

Ex. 3. *Determine the characteristics of the equation* $z = p^2 - q^2$ *and find the integral surface which passes through the parabola* $4z + x^2 = 0,\ y = 0$. **[Himachal 2000, 05]**

Sol. Do yourself, the required characteristics are $x = 2\lambda(2-e^{-t})$, $y = 2\sqrt{2}\lambda\,(e^{-t}-1)$, $z = -\lambda^2 e^{-2t}$, λ being parameter. Solution is $4z + (x + y\sqrt{2})^2 = 0$.

Ex. 4. *Determine the characteristics of the equation* $p^2 + q^2 = 4z$ *and find the solution of this equation which reduces to* $z = x^2 + 1$ *when* $y = 0$.

Miscellaneous Problem on Chapter 3

1. Show that the envelope of the family of surfaces touch each member of the family at all points of its characteristics. **[Meerut 2008]**

2. Find a complete integral of the partial differential equation $(p^2+q^2)x = pz$ and deduce the surface solution which passes through the curve $x = 0,\ z^2 = 4y$. **[Meerut 2007]**

3. Solve $p^2 y + p^2 y x^2 = q x^2$ **[Pune 2010]**

Ans. Complete integral is $\quad z = a\,(1+x^2)^{1/2} + (a^2 y^2)/2 + b$.

4. Given thta $(x-a)^2 + (y-b)^2 + z^2 = 1$ is complete integral of $z^2(1+p^2+q^2) = 1$. Find its singular integral. **[Pune 2010]**

Hint. Use definition on page 3.1. **Ans.** $z^2 = 1$

4

Homogenous Linear Partial Differential Equations with Constant Coefficients

4.1. Homogeneous and Non–homogeneous linear equations with constant coefficients. A partial differential equation in which the dependent variable and its derivatives appear only in the first degree and are not multiplied together, their coefficients being constants or functions of x and y, is known as a *linear partial differential equation*. The general form of such an equation is

$$\left(A_0 \frac{\partial^n z}{\partial x^n} + A_1 \frac{\partial^n z}{\partial x^{n-1}\partial y} + \ldots + A_n \frac{\partial^n z}{\partial y^n}\right) + \left(B_0 \frac{\partial^{n-1} z}{\partial x^{n-1}} + B_1 \frac{\partial^{n-1} z}{\partial x^{n-2}\partial y} + \ldots \quad + B_{n-1} \frac{\partial^{n-1} z}{\partial y^{n-1}}\right) + \left(M_0 \frac{\partial z}{\partial x} + M_1 \frac{\partial z}{\partial y}\right)$$

$$+ N_0 z = f(x, y), \quad \ldots(1)$$

where the coefficients $A_0, A_1, \ldots A_n, B_0, B_1, \ldots B_{n-1}, M_0, M_1$ and N_0 are constants or functions of x and y. If $A_0, A_1, \ldots A_n, B_0, B_1, \ldots B_{n-1}, M_0, M_1$ and N_0 are all constants, then (1) is called a *linear partial differential equation with constant coefficients.*

For convenience $\partial/\partial x$ and $\partial/\partial y$ will be denoted by D (or D_x) and D' (or D_y) respectively. Then (1) can be rewritten as

$$[(A_0 D^n + A_1 D^{n-1} D' + \ldots + A_n D'^n) + (B_0 D^{n-1} + B_1 D^{n-2} D' + \ldots + B_{n-1} D'^{n-1})$$
$$+ (M_0 D + M_1 D') + N_0] z = f(x, y), \quad \ldots(2)$$

or, briefly, $$F(D, D')z = f(x, y). \quad \ldots(3)$$

When all the derivatives appearing in (1) are of the same order, then the resulting equation is called a *linear homogeneous partial differential equation with constant coefficients* and it is then of the form

$$(A_0 D^{n-1} + A_1 D^{n-1} D' + \ldots + A_n D'^n) z = f(x, y). \quad \ldots(4)$$

On the other hand, when all the derivatives in (1) are not of the same order, then it is called a *non–homogeneous linear partial differential equation with constant coefficients*.

In this chapter we propose to study the various methods of solving homogeneous linear partial differential equation with constant coefficients, namely, (4)

4.2. Solution of a homogeneous linear partial differential equation with constant coefficients, namely, $$(A_0 D^{n-1} + A_1 D^{n-1} D' + \ldots + A_n D'^n) z = f(x, y), \quad \ldots(1)$$

where $A_0, A_1, \ldots, A_n$ are constants. (1) may rewritten as

$$F(D, D')z = f(x, y), \quad \ldots(2)$$

where $$F(D, D') = A_0 D^{n-1} + A_1 D^{n-1} D' + \ldots + A_n D'^n. \quad \ldots(3)$$

As in the case of linear ordinary differential equation with constant coefficients, we start with the following basic theorems.

Theorem I. *If u is the complementary function and z' a particular integral of a linear partial differential equation $F(D, D')z = f(x, y)$, then $u + z'$ is a general solution of the equation.*

Proof. Given $$F(D, D')z = f(x, y). \quad \ldots(1)$$

The complementary function u of (1) is the most general solution of

$$F(D, D')z = 0. \quad \ldots(2)$$

$$\therefore \qquad F(D, D')u = 0. \qquad ...(3)$$

Note that the complementary function must contain as many arbitrary constants as is the order of equation (2).

Any solution z' of (1) is called a particular integral of (1). Note that particular integral does not contain any arbitrary constant. Thus, by definition, we have

$$F(D, D')z' = f(x, y). \qquad ...(4)$$

Adding (3) and (4), $\qquad F(D, D')(u + z') = f(x, y),$

showing that $u + z'$ is a solution of (1). Since (1) and (2) are of the same order, the general solution $u + z'$ will contain as many arbitrary constants as the general solution of (1) requires.

Theorem II. *If $u_1, u_2, ..., u_n$ are solutions of the homogeneous linear partial differential equation $F(D, D')z = 0$, then $\sum_{r=1}^{n} c_r u_r$ is also a solution, where $C_1, C_2, ..., C_r, ..., C_n$ are arbitrary constants.*

Proof. Given equation is $\qquad F(D, D')z = 0. \qquad ...(1)$

We have $\qquad F(D, D')(c_r u_r) = c_r F(D, D')u_r \qquad ...(2)$

and
$$F(D, D')\sum_{r=1}^{n} v_r = \sum_{r=1}^{n} F(D, D')v_r \qquad ...(3)$$

for any set of functions v_r. Using results (2) and (3), we get

$$F(D, D')\sum_{r=1}^{n}(c_r u_r) = \sum_{r=1}^{n} F(D, D')(c_r u_r) = c_r \sum_{r=1}^{n} F(D, D')u_r \qquad ...(4)$$

Since u_r is solution of (1) for $r = 1, 2, ..., n$, $\quad$ so $\quad F(D, D')u_r = 0 \quad$ for $\; r = 1, 2, 3, ..., n$.

$\therefore \qquad$ (4) gives $F(D, D')\sum_{r=1}^{n}(c_r u_r) = 0$, which proves the required result.

Note. For convenience we shall denote complementary function by C.F. and particular integral by P.I.

4.3. Method of finding the complementary function (C.F.) of the linear homogeneous partial differential equation with constant coefficients, namely, $F(D, D')z = f(x, y)$

i.e.,
$$(A_0 D^n + A_1 D^{n-1} D' + ... + A_n D'^n)z = f(x, y), \qquad ...(1)$$

where $A_0, A_1, ..., A_n$ are all constants.

The complementary function of (1) is the general solution of

$$(A_0 D^n + A_1 D^{n-1} D' + ... + A_n D'^n)z = 0. \qquad ...(2)$$

or
$$[(D - m_1 D')(D - m_2 D') \; ... \; (D - m_n D')]z = 0. \qquad ...(3)$$

where $m_1, m_2, ..., m_n$ are some constants.

Clearly, the solution of any one of the equations

$$(D - m_1 D')z = 0, \qquad (D - m_2 D')z = 0, \; ...., \qquad (D - m_n D')z = 0 \quad ...(4)$$

is also a solution of (3).

We now show that the general solution of $\quad (D - mD')z = 0 \quad$ is $\quad z = \phi(y + mx)$, where ϕ is an arbitrary function.

We have, $\; (D - mD')z = 0 \;$ or $\; (\partial z/\partial x) - m(\partial z/\partial y) = 0 \;$ or $\; p - mq = 0, \quad ...(5)$

which is in Lagrange's form $Pp + Qq = R$. Here Lagrange's auxiliary equations for (5) are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{0}. \qquad ...(6)$$

Taking the first two fractions of (6), $dy + mdx = 0$ so that $y + mx = c_1$ ... (7)

From the third fraction of (6), $dz = 0$ so that $z = c_2$. ...(8)

Hence from (7) and (8), the general solution of (5) is $z = \phi(y + mx)$, where ϕ is an arbitrary function. So, we assume that a solution of (2) is of the form

$$z = \phi(y + mx). \qquad ...(9)$$

From (9),
$$Dz = \partial z/\partial x = m\phi'(y + mx),$$
$$D^2z = \partial^2 z/\partial x^2 = m^2\phi''(y + mx),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

and
$$D^n z = \partial^n z/\partial x^n = m^n\phi^{(n)}(y + mx).$$

Again,
$$D'z = \partial z/\partial y = \phi'(y + mx),$$
$$D'^2 z = \partial^2 z/\partial y^2 = \phi''(y + mx),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

and
$$D'^n z = \partial^n z/\partial y^n = \phi^{(n)}(y + mx).$$

Also, in general,
$$D^r D'^s z = \partial^{r+s} z/\partial x^r \partial y^s = m^r \phi^{(r+s)}(y + mx).$$

Substituting these values in (2) and simplifying, we get

$$(A_0 m^n + A_1 m^{n-1} + A_2 m^{n-2} + \ldots + A_n)\phi^{(n)}(y + mx) = 0,$$

which is true if m is a root of the equation

$$A_0 m^n + A_1 m^{n-1} + A_2 m^{n-2} + \ldots + A_n = 0. \qquad ...(10)$$

The equation (10) is known as the *auxiliary equation* (*A.E.*) and is obtained by putting $D = m$ and $D' = 1$ in $F(D, D') = 0$.

Let $m_1, m_2, \ldots, m_n$ be n roots of A.E. (10). Two cases arise.

Case I. When $\mathbf{m_1, m_2, m_3, \ldots m_n}$ are distinct. Then the part of C.F. corresponding to $m = m_r$ is $z = \phi_r(y + m_r x)$ for $r = 1, 2, 3, \ldots, n$. Since (2) is linear, the sum of the solutions is also a solution.

$$\therefore \quad \text{C.F. of (2)} = \phi_1(y + m_1 x) + \phi_2(y + m_2 x) + \ldots + \phi_n(y + m_n x), \qquad ...(11)$$

where $\phi_1, \phi_2, \ldots, \phi_n$ are arbitrary functions.

Case II. Repeated roots. Let m be repeated root of (10) and so consider

$$(D - mD')(D - mD')z = 0. \qquad ...(12)$$

Let
$$(D - mD')z = v. \qquad ...(13)$$

Then, (12) $\Rightarrow$ $(D - mD')v = 0$ or $(\partial v/\partial x) - m(\partial v/\partial y) = 0$, ...(14)

which is in Lagrange's form. Hence Lagrange's auxiliary equations for (14) are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dv}{0}. \qquad ...(15)$$

As before, two independent integrals of (15) are $y + mx = c_3$ and $v = c_4$.

$$\therefore \quad v = \phi(y + mx) \qquad ...(16)$$

is a solution of (14), ϕ being as arbitrary function.

Using (16), (13) becomes $(\partial z/\partial x) - m(\partial z/\partial y) = \phi(y + mx)$ ...(17)

which is in Lagrange's form. Its Lagrange's auxiliary equations for (7) are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{\phi(y + mx)} \qquad ...(18)$$

Taking the first two fractions of (18), $dy + mdx = 0$ so that $y + mx = c_5$. ... (19)

Taking the first and third fractions of (18) and using (19), we get

$(dx)/1 = (dz)/\phi(c_5)$ so that $dz - \phi(c_5)dx = 0$.

Integrating, $z - x\,\phi(c_5) = c_6$ or $z - x\,\phi(y + mx) = c_6$, using (19). ...(20)

From (19) and (20), the general solution of (12) is

$z - x\phi(y + mx) = \psi(y + mx)$ or $z = \psi(y + mx) + x\phi(y + mx)$, ...(21)

where ϕ and ψ are arbitrary functions. (21) is a part of C.F. corresponding to the two times repeated root m. In general, if a root 'm' is repeated 'r' times, the corresponding part of C.F. is

$$\phi_1(y + mx) + x\phi_2(y + mx) + x^2\phi_3(y + mx) + \dots + x^{r-1}\phi_r(y + mx).$$

4.4.A. Working rule for finding C.F. of linear homogeneous partial differential equation with constant coefficients

Step 1. Put the given equation in standard form $(A_0D^n + A_1D^{n-1}D' + \dots A_nD'^n)z = f(x, y)$....(1)

Step 2. Replacing D by m and D' by 1 in the coefficients of z, we obtain auxiliary equation (A.E.) for (1) as $A_0m^n + A_1m^{n-1} + \dots + A_n = 0$.
...(2)

Step 3. Solve (2) for m. Two cases will arise :

Case (*i*) Let $m = m_1, m_2, \dots, m_n$ (different roots). Then

C.F. $= \phi_1(y + m_1x) + \phi_2(y + m_2x) + \dots + \phi_n(y + m_nx)$, where $\phi_1, \phi_2 \dots, \phi_n$ are arbitrary functions.

If in the above case (*i*), $m = a_1/b_1, a_2/b_2, \dots, a_n/b_n$, then

$$\text{C.F.} = \phi_1(b_1y + a_1x) + \phi_2(b_2y + a_2x) + \dots + \phi_n(b_ny + a_nx)$$

Further if $m = -(a_1/b_1), -(a_2/b_2), \dots, -(a_n/b_n)$, then

$$\text{C.F.} = \phi_1(b_1y - a_1x) + \phi_2(b_2y - a_2x) + \dots + \phi_n(b_ny - a_nx)$$

Case (*ii*) Let $m = m'$ (repeated n times). Then corresponding to these roots

$$\text{C.F.} = \phi_1(y + m'x) + x\,\phi_2(y + m'x) + x^2\phi_2(y + m'x) + \dots + x^{n-1}\phi_n(y + m'x).$$

In the above case (*ii*), if $m = a/b$ (repeated n times), Then corresponding to these n roots,

$$\text{C.F.} = \phi_1(by + ax) + x\,\phi_2(by + ax) + x^2\phi_2(by + ax) + \dots + x^{n-1}\phi_n(by + ax)$$

And, if $m = -(a/b)$, (repeated n times), then

$$\text{C.F.} = \phi_1(by - ax) + x\,\phi_2(by - ax) + x^2\phi_2(by - ax) + \dots + x^n\phi_n(by - ax)$$

Case (*iii*) Corresponding to a non–repeated factor D on L.H.S. of (1),
the part of C.F. is taken as $\phi(y)$.

Case (*iv*) Corresponding to a repeated factor D^m on L.H.S. of (1), the part of C.F. is taken as

$$\phi_1(y) + x\,\phi_2(y) + x^2\phi_3(y) + \dots + x^{m-1}\phi_m(y).$$

Case (*v*) Corresponding to a non–repeated factor D' on L.H.S. of (1),
the part of C.F. is taken as $\phi(x)$.

Case (*vi*) Corresponding to a repeated factor D'^m on L.H.S. of (1), the part of C.F. is taken as

$$\phi_1(x) + y\,\phi_2(x) + y^2\phi_3(x) + \dots + y^{m-1}\phi_m(x).$$

4.4.B. Alternative working rule for finding C.F.

Let the given partial differential equation be $F(D, D')z = f(x, y)$. Factorize $F(D, D')$ into linear factors of the form $(bD - aD')$. Then we use the following results :

(*i*) Corresponding to each non–repeated factor $(bD - aD')$, the part of C.F. is taken as

$$\phi(by + ax).$$

(*ii*) Corresponding to a repeated factor $(bD - aD')^m$, the part of C.F. is taken as

$$\phi_1(by + ax) + x\phi_2(by + ax) + x^2\phi_3(by + ax) + \dots + x^{m-1}\phi_m(by + ax).$$

(*iii*) Corresponding to a non–repeated factor D, part of C.F. is taken as $\phi(y)$.

(*iv*) Corresponding to a repeated factor D^m, the part of C.F. is taken as

$$\phi_1(y) + x\,\phi_2(y) + x^2\phi_3(y) + \dots + x^{m-1}\phi_m(y).$$

(*v*) Corresponding to a non–repeated factor D', part of C.F. is taken as $\phi(x)$.

(*vi*) Corresponding to a repeated factor D'^m, the part of C.F. is taken as

$$\phi_1(x) + y\,\phi_2(x) + y^2\phi_3(x) + \dots + y^{m-1}\phi_m(x).$$

4.5. Solved examples based on articles 4.4A and 4.4B

[Notations $p = \partial z/\partial x$, $q = \partial z/\partial y$, $r = \partial^2 z/\partial x^2$, $s = \partial^2 z/\partial x\partial y$ and $t = \partial^2 z/\partial y^2$ will be used]

Ex. 1. *Solve* (*a*) $r = a^2 t$. **[I.A.S. 1987; Meerut 1991]**

(*b*)$(\partial^2 z/\partial x^2) - (\partial^2 z/\partial y^2) = 0$.

(*c*) $(D^2 - 3aDD' + 2a^2D'^2)z = 0$. **[Kanpur 2007; Meerut 2007]**

Sol. (*a*) Given equation is $\partial^2 z/\partial x^2 = a^2(\partial^2 z/\partial y^2)$ or $(D^2 - a^2D'^2)z = 0$. ...(1)

The auxiliary equation of (1) is $m^2 - a^2 = 0$ so that $m = a, -a$.

$\therefore$ The general solution of (1) is $z = \text{C.F.} = \phi_1(y + ax) + \phi_2(y - ax)$,

where ϕ_1 and ϕ_2 are arbitrary functions.

(*b*) Proceed as in part (*a*). **Ans.** $z = \phi_1(y + x) + \phi_2(y - x)$

(*c*) Proceed as in part (*a*). **Ans.** $z = \phi_1(y + ax) + \phi_2(y + 2ax)$

Ex. 2. *Solve* (*a*) $(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)z = 0$. **[Agra 2005]**

(*b*) $(\partial^3 z/\partial x^3) - 7(\partial^3 z/\partial x\partial y^2) + 6(\partial^3 z/\partial y^3) = 0$. **[Bhopal 2010]**

(*c*) $(D^3 - 3D^2D' + 2DD'^2)z = 0$. **[Meerut 2008; Lucknow 2010]**

Sol. (*a*) The auxiliary equation is $m^3 - 6m^2 + 11m - 6 = 0$

or $(m - 1)(m - 2)(m - 3) = 0$ so that $m = 1, 2, 3$.

$\therefore$ The general solution of the given equation is

$z = \phi_1(y + x) + \phi_2(y + 2x) + \phi_3(y + 3x)$, ϕ_1, ϕ_2, ϕ_3 being arbitrary functions.

(*b*) The given equation can be written as $(D^3 - 7DD'^2 + 6D'^3)z = 0$. ...(1)

Its auxiliary equation is $m^3 - 7m + 6 = 0$ or $(m - 1)(m - 2)(m + 3) = 0$.

Hence $m = 1, 2, -3$ and so the general solution of (1) is

$z = \phi_1(y + x) + \phi_2(y + 2x) + \phi_3(y - 3x)$, ϕ_1, ϕ_2, ϕ_3 being arbitrary functions.

(*c*) Proceed as above. **Ans.** $z = \phi_1(y) + \phi_2(y + x) + \phi_3(y + 2x)$.

Ex. 3. (*a*) *Solve* $2r + 5s + 2t = 0$. **[Meerut 2011]**

(*b*) $2(\partial^2 z/\partial x^2) - 3(\partial^2 z/\partial x\partial y) - 2(\partial^2 z/\partial y^2) = 0$.

Sol. (*a*) Now, $r = \partial^2 z/\partial x^2 = D^2 z$, $s = \partial^2 z/Dx\partial y = DD'z$ and $t = \partial^2 z/\partial y^2 = D'^2 z$. Hence the given equation can be re–written as $(2D^2 + 5\,DD' + 2D'^2)y = 0$. ...(1)

Its auxiliary equation is $2m^2 + 5m + 2 = 0$ or $(2m + 1)(m + 2) = 0$.

So $m = -1/2, -2$ and hence the general solution of (1) is

$z = \phi_1(2y - x) + \phi_2(y - 2x)$, ϕ_1 and ϕ_2 being arbitrary functions.

Alternative method : (1) can be re–written as $(2D + D')(D + 2D') = 0$.

So by using the alternative working rule 4.4B, the general solution of (1) is

$z = \phi_1(2y - x) + \phi_2(y - 2x)$, ϕ_1 ϕ_2 being arbitrary functions.

(*b*) Proceed as in part (*a*). **Ans.** $z = \phi_1(2y - x) + \phi_2(y + 2x)$

Ex. 4. *Solve* (*a*) $r + t + 2s = 0$ **[Kanpur 2009]**

(*b*) $25r - 40s + 16t = 0$ **[Bilaspur 1996; Jabalpur 2002; Sagar 2004];**

(*c*) $(4D^2 + 12DD' + 9D'^2)z = 0$. **[Kanpur 2008; Indore 2004]**

Sol. (*a*) Here $r = \partial^2 z/\partial x^2 = D^2 z$, $t = \partial^2 z/\partial y^2 = D'^2 z$, $s = \partial^2 z/\partial x\partial y = DD'z$.

So the given equation becomes $(D^2 + D'^2 + 2DD')z = 0$. ...(1)

Its auxiliary equation is $m^2 + 1 + 2m = 0$ or $(m + 1)^2 = 0$ or $m = -1, -1$.

So the general solution of (1) is $z = \phi_1(y - x) + x\phi_2(y - x)$, ϕ_1 and ϕ_2 being arbitrary functions.

(*b*) Here $r = \partial^2 z/\partial x^2 = D^2 z$, $s = \partial^2 z/\partial x\partial y = DD'z$, $t = \partial^2 z/\partial y^2 = D'^2 z$. So the given equation becomes $(25D^2 - 40DD' + 16D'^2)z = 0$. ...(1)

Its auxiliary equation is $25m^2 - 40m + 16 = 0$ or $(5m - 4)^2 = 0$ so that $m = 4/5, 4/5$.

Hence the general solution of (1) is $z = \phi_1(5y + 4x) + x\phi_2(5y + 4x)$.

Alternative method (1) may be written as $(5D - 4D')^2 z = 0$.

Using result (*ii*) of Art 4.4B, the solution of (1) is

$$z = \phi_1(5y + 4x) + x\phi_2(5y + 4x),\ \phi_1,\ \phi_2 \text{ being arbitrary functions.}$$

(*c*) Proceed as in part (b). **Ans.** $z = \phi_1(2y - 3x) + x\phi_2(2y - 3x)$

Ex. 5. *Solve* (*a*) $(D^3 - 4D^2D' + 4DD'^2)z = 0$. **[Bhopal 2000, 03]**

(*b*) $(D^4 - 2D^3D' + 2DD'^3 - D'^4)z = 0$. **[Bilaspur 2004]**

(*c*) $(D^4 + D'^2 - 2D^2D'^2)z = 0$.

(*d*) $(D^3 - 3D^2D' + 3DD'^2 - D'^3)z = 0$.

Sol. (*a*) The auxiliary equation of the given equation is

$m^3 - 4m^2 + 4m = 0$ or $m(m - 2)^2 = 0$ so that $m = 0, 2, 2$

Hence the general solution of the given equation is

$$z = \phi_1(y) + \phi_2(y + 2x) + x\phi_3(y + 2x),\ \phi_1,\ \phi_2,\ \phi_3 \text{ being arbitrary functions.}$$

(*b*) The auxiliary equation of the given equation is

$m^4 - 2m^3 + 2m - 1 = 0$ or $(m + 1)(m - 1)^3 = 0$ so that $m = -1, 1, 1, 1$.

Hence the general solution of the given equation is

$z = \phi_1(y - x) + \phi_2(y + x) + x\phi_2(y + x) + x^2\phi_3(y + x)$, where ϕ_1, ϕ_2, ϕ_3 and ϕ_4 are arbitrary functions.

(*c*) Try yourself. **Ans.** $z = \phi_1(y + x) + x\phi_2(y + x) + \phi_3(y - x) + x\phi_4(y - x)$

(*d*) Proceed as in part (*a*). **Ans.** $z = \phi_1(y + x) + x\phi_2(y + x) + x^2\phi_3(y + x)$

Ex. 6. *Solve* (*a*) $(D^3D'^2 + D^2D'^3)z = 0$. (*b*) $(D^3D' - 4D^2D'^2 + 4DD'^3)z = 0$.

Sol. (*a*) The given equation can be re–written as $D^2D'^2(D + D')z = 0$. ...(1)

Hence using the alternative method 4.4B for C.F., the general solution is

$z = \phi_1(y) + x\phi_2(y) + \phi_3(x) + y\phi_4(x) + \phi_5(y - x)$, where ϕ_1, ϕ_2, ϕ_3, ϕ_4 and ϕ_5 are arbitrary functions.

(*b*) The given equation can be re–written as

$DD'(D^2 - 4DD' + 4D'^2) = 0$ *Or* $DD'(D - 2D')^2 = 0$.

Using the working rule 4.4B, the required general solution is

$z = \phi_1(y) + \phi_2(x) + \phi_3(y + 2x) + x\phi_4(y + 2x)$, where ϕ_1, ϕ_2, ϕ_3 and ϕ_4 are arbitrary functions.

Ex. 7. *Solve* (*a*) $(\partial^4 z/\partial x^4) - (\partial^4 z/\partial y^4) = 0$ (*b*) $(D^4 + D'^4)z = 0$.

Sol. (*a*) Rewriting, the given equation is $(D^4 - D'^4)z = 0$. ...(1)

Its auxiliary equation is $m^4 - 1 = 0$ or $(m^2 - 1)(m^2 + 1) = 0$ $\Rightarrow$ $m = 1, -1\ i, -i$.

Hence the general solution of (1) is $z = \phi_1(y - x) + \phi_2(y + x) + \phi_3(y + ix) + \phi_4(y - ix)$, where ϕ_1, ϕ_2, ϕ_3 and ϕ_4 are arbitrary functions.

(*b*) The auxiliary equation of the given equation is $m^4 + 1 = 0$ or $(m^2 + 1)^2 - 2m^2 = 0$

or $(m^2 + 1)^2 - (m\sqrt{2})^2 = 0$ or $(m^2 + 1 + \sqrt{2}m)(m^2 + 1 - \sqrt{2}m) = 0$

so that $m^2 + \sqrt{2}m + 1 = 0$ or $m^2 - \sqrt{2}m + 1 = 0$ $\Rightarrow$ $m = (-1 \pm i)/\sqrt{2}, (1 \pm i)/\sqrt{2}$.

Let $z_1 = (-1 + i)/\sqrt{2}$ and $z_2 = (1 + i)/\sqrt{2}$, then, $m = z_1, \bar{z}_1, z_2, \bar{z}_2$,

where $\bar{z}_1$ and $\bar{z}_2$ denote complex conjugates of z_1 and z_2 respectively.

Hence the general solution of the given equation is

$z = \phi_1(y + z_1x) + \phi_2(y + \bar{z}_1x) + \phi_3(y + z_2x) + \phi_4(y + \bar{z}_2x)$, where ϕ_1, ϕ_2, ϕ_3, ϕ_4 are arbitrary functions.

4.6. Particular integral (P.I.) of homogeneous linear partial differential equation given by

$$F(D, D')y = f(x, y) \qquad \text{... (1)}$$

The inverse operator $1/F(D, D')$ of the operator $F(D, D')$ is defined by the following identity

$$F(D, D')\left(\frac{1}{F(D, D')} f(x, y)\right) = f(x, y)$$

$\therefore$ Particular integral (P.I.) of (1) $= \dfrac{1}{F(D, D')} f(x, y)$

In what follows we shall treat the symbolic functions of D and D' as we do for the symbolic functions of D alone in ordinary differential equations. Thus it will be factorized and resolved into partial fractions or expanded in an infinite series as the case may be. The reader is advised to note carefully the following results :

(*i*) D, D^2, ... will stand for differentiating partially with respect to x once, twice and so on.

For example, $Dx^4y^5 = \dfrac{\partial}{\partial x} x^4 y^5 = 4x^3 y^5$; $D^2x^4y^5 = \dfrac{\partial^2}{\partial x^2} x^4 y^5 = 12x^2 y^5$.

(*ii*) D', D'^2... will stand for differentiating partially with respect to y once, twice and so on.

For example, $D' x^4y^5 = \dfrac{\partial}{\partial y} x^4 y^5 = 5x^4 y^4$; $D'^2x^4y^5 = \dfrac{\partial^2}{\partial y^2} x^4 y^5 = 20x^4 y^3$.

(*iii*) $1/D$, $1/D^2$, ... will stand for integrating partially with respect to x once, twice and so on.

For example, $\dfrac{1}{D} x^4 y^5 = \displaystyle\int x^4 y^5 \, dx = \dfrac{x^5 y^5}{5}$; $\dfrac{1}{D^2} x^4 y^5 = \displaystyle\iint x^4 y^5 \, dx \, dx = \dfrac{x^6 y^5}{30}$

(*iv*) $1/D'$, $1/D'^2$, ... will stand for integrating partially with respect to y once, twice and so on.

For example, $\dfrac{1}{D'} x^4 y^5 = \displaystyle\int x^4 y^5 \, dy = \dfrac{x^4 y^6}{6}$; $\dfrac{1}{D'^2} x^4 y^5 = \displaystyle\iint x^4 y^5 \, dy \, dy = \dfrac{x^4 y^7}{42}$.

4.7. Short methods of finding the P.I. in certain cases.

Before taking up the general method for finding P.I. of $F(D, D')z = f(x, y)$ we begin with cases when $f(x, y)$ is in two special forms. The methods corresponding to these forms are much shorter than the general methods to be discussed in Art. 4.12.

4.8. A Short Method I. When f(x, y) is of the form f(ax + by).

The method under consideration is based on the following theorem.

Theorem I. *If $F(D, D')$ be homogeneous function of D and D' of degree n, then*

$$\frac{1}{F(D, D')} \phi^{(n)}(ax + by) = \frac{1}{F(a, b)} \phi(ax + by),$$

provided $F(a, b) \neq 0$, $\phi^{(n)}$ being the nth derivative of ϕ w.r.t. $ax + by$ as a whole.

Proof. By direct differentiation, we have $D^r\phi(ax + by) = a^r\phi^{(r)}(ax + by)$,

$D'^s\phi(ax + by) = b^s\phi^{(s)}(ax + by)$ and $D^rD'^s\phi(ax + by) = a^rb^s\phi^{(r+s)}(ax + by)$.

Since $F(D, D')$ is homogeneous function of degree n, so we have

$$F(D, D')\phi(ax + by) = F(a, b)\phi^{(n)}(ax + by). \quad ...(1)$$

Operating both sides of (1) by $1/F(D, D')$, we have

$$\phi(ax + by) = F(a, b)\frac{1}{F(D, D')}\phi^{(n)}(ax + by). \quad ...(2)$$

Since $F(a, b) \neq 0$, dividing both sides of (2) by $F(a, b)$, we get

$$\frac{1}{F(D, D')}\phi^{(n)}(ax + by) = \frac{1}{F(a, b)}\phi(ax + by). \quad ...(3)$$

An important deduction from result (3) : Putting $ax + by = v$, (3) gives

$$\frac{1}{F(D, D')}\phi^{(n)}(v) = \frac{1}{F(a, b)}\phi(v). \qquad ...(4)$$

Integrating both sides of (4) n times w.r.t. 'v', we have

$$\frac{1}{F(D, D')}\phi(v) = \frac{1}{F(a, b)}\int\int\ldots\int\phi(v)\,dv\,dv\ldots dv, \qquad \text{where} \qquad v = ax + by.$$

Exceptional case when F(a, b) = 0. When $F(a, b) = 0$, then the above theorem does not hold good. In such a case the new method is based on the following theorem. Note that $F(a, b) = 0$ if and only if $(bD - aD')$ is a factor $F(D, D')$.

Theorem II. $$\frac{1}{(bD - aD')^n}\phi(ax + by) = \frac{x^n}{b^n n!}\phi(ax + by).$$

Proof. Consider the equation $\qquad (bD - aD')z = x^r\phi(ax + by) \qquad ...(1)$

or $\qquad bp - aq = x^r\phi(ax + by). \qquad ...(2)$

Lagrange's subsidiary equations for (2) are $\qquad \dfrac{dx}{b} = \dfrac{dy}{-a} = \dfrac{dz}{x^r\phi(ax + by)}. \qquad ...(3)$

Taking the first two fractions of (3), $\quad a dx + b dy = 0 \quad$ so that $\quad ax + by = c_1 \quad ...(4)$

Taking the first and third members of (4) and using (4), we get

$$\frac{dx}{b} = \frac{dz}{x^r\phi(c_1)} \qquad \text{or} \qquad dz = \frac{x^r\phi(c_1)}{b}dx.$$

Integrating, $$z = \frac{x^{r+1}\phi(c_1)}{b(r+1)} = \frac{x^{r+1}\phi(ax + by)}{b(r+1)}, \text{ by (4)}. \qquad ...(5)$$

(5) is a solution of (1).

Now, from (1), $$z = \frac{1}{(bD - aD')}x^r\phi(ax + by). \qquad ...(6)$$

From (5) and (6), $$\frac{1}{(bD - aD')}x^r\phi(ax + by) = \frac{x^{r+1}}{b(r+1)}\phi(ax + by). \qquad ...(7)$$

Hence, if $\quad z = \dfrac{1}{(bD - aD')^n}\phi(ax + by), \quad$ then we have

$$z = \frac{1}{(bD - aD')^{n-1}}\left[\frac{1}{(bD - aD')}x^0\phi(ax + by)\right], \text{ as } x^0 = 1$$

$$= \frac{1}{(bD - aD')^{n-1}}\frac{x}{b}\phi(ax + by), \text{ using (7) for } r = 0$$

$$= \frac{1}{b}\frac{1}{(bD - aD')^{n-2}}\left[\frac{1}{(bD - aD')}x\phi(ax + by)\right]$$

$$= \frac{1}{b}\frac{1}{(bD - aD')^{n-2}}\frac{x^2}{2b}\phi(ax + by), \text{ using (7) for } r = 1$$

$$= \frac{1}{2!b^2}\frac{1}{(bD - aD')^{n-2}}x^2\phi(ax + by)$$

$$= \frac{1}{n!b^n}\frac{1}{(bD - aD')^{n-n}}x^n\phi(ax + by) = \frac{x^n}{b^n n!}\phi(ax + by)$$

[after repeated use of (7) for $n - 2$ times more]

Working rule for finding particular integral where f(x, y) = ϕ(ax + by).

The following rules will used depending upon the situation in hand.

Formula (i). When $F(a, b) \neq 0$ and $F(D, D')$ is a homogeneous function of degree n, then

$$\text{P.I.} = \frac{1}{F(D, D')}\phi(ax+by) = \frac{1}{F(a, b)}\int\int \dots \int f(v)\, dv\, dv \dots dv, \qquad \text{where} \qquad v = ax + by$$

Note that R.H.S. contains a multiple integral of nth order.

Formula (ii). When $F(a, b) = 0$, we have

$$\frac{1}{(bD - aD')^n}\phi(ax+by) = \frac{x^n}{b^n n!}\phi(ax+by).$$

4.9. Solved Examples based on Short Method I of Art. 4.8

Ex. 1. *Solve* $(D^2 + 3DD' + 2D'^2)z = x + y$. **[I.A.S. 1986, Meerut 2005, 07, 09, 10]**

Sol. The auxiliary equation of the given equation is $m^2 + 3m + 2 = 0$ giving $m = -1, -2$.

$\therefore$ C.F. $= \phi_1(y - x) + \phi_2(y - 2x)$, ϕ_1, ϕ_2 being arbitrary functions.

Now, $\quad$ P.I. $= \dfrac{1}{D^2 + 3DD' + 2D'^2}(x + y)$

$= \dfrac{1}{1^2 + 3 \times 1 \times 1 + 2 \times 1^2}\displaystyle\int\int v\, dv\, dv$, where $v = x + y$, using formula (i) of working rule

$= \int (v^2/2)\, dv = (1/6) \times (v^3/6) = (1/36) \times (x + y)^3$.

Hence the required general solution is $\quad z =$ C.F. + P.I., *i.e.*,

$$z = \phi_1(y - x) + \phi_2(y - 2x) + (1/36) \times (x + y)^3.$$

Ex. 2. *Solve* (*a*) $(2D^2 - 5DD' + 2D'^2)z = 24(y - x)$. **[Vikram 2000]**

(*b*) $(\partial^2 V/\partial x^2) + (\partial^2 V/\partial y^2) = 12(x + y)$ **[Nagpur 2010]**

(*c*) $(D^2 + D'^2)z = 30(2x + y)$. **[Bhopal 1996, Sagar 2004]**

(*d*) $(\partial^2 z/\partial x^2) + 2(\partial^2 z/\partial x\partial y) + (\partial^2 z/\partial y^2) = 2x + 3y$. **[Kumaun 1992]**

(*e*) $(D^2 + 3DD' + 2D'^2)z = 2x + 3y$. **[Kurukshetra 2005]**

(*f*) $r + s - 2t = (2x + y)^{1/2}$. **[Lucknow 2010]**

Sol. (*a*) The auxiliary of the given equation is $2m^2 - 5m + 2 = 0$ giving $m = 1/2$, 2.

$\therefore$ C.F. $= \phi_1(2y + x) + \phi_2(y + 2x)$, ϕ_1, ϕ_2 being arbitrary functions.

Now, P.I. $= \dfrac{1}{2D^2 - 5DD' + 2D'^2}24(y - x) = 24\dfrac{1}{2D^2 - 5DD' + 2D'^2}(y - x)$

$= \dfrac{24}{2 \times (-1)^2 - 5 \times (-1) \times 2 + 2 \times 2^2}\displaystyle\int\int v\, dv\, dv$, where $v = y - x$, using formula (*i*) of working rule

$= (24/20) \times \displaystyle\int (v^2/2)\, dv = (6/5) \times (v^3/6) = (1/5) \times (y - x)^3$.

Hence the required general solution is $\quad z = \phi_1(2y + x) + \phi_2(y + 2x) + (y - x)^3/5$.

(*b*) The given equation can be written as $\quad (D^2 + D'^2)V = 12(x + y)$. ...(1)

Its auxiliary equation is $\quad m^2 + 1 = 0 \quad$ so that $\quad m = \pm\, i$.

$\therefore$ C.F. $= \phi_1(y + ix) + \phi_2(y - ix)$, ϕ_1, ϕ_2 being arbitrary functions.

Now, $\quad$ P.I. $= \dfrac{1}{D^2 + D'^2}12(x + y) = 12\dfrac{1}{D^2 + D'^2}(x + y) = \dfrac{12}{1^2 + 1^2}\displaystyle\int\int v\, dv\, dv$

$$= 6\int (v^2/2)\, dv = v^3 = (x+y)^3 .$$

Hence the required general solution is $V = \phi_1(y + ix) + \phi_2(y - ix) + (x + y)^3$.

(c) Try yourself. **Ans.** $z = \phi_1(y + ix) + \phi_2(y - ix) + (2x + y)^3$

(d) Try yourself. **Ans.** $z = \phi_1(y - x) + x\phi_2(y - x) + (1/150) \times (2x + 3y)^3$

(e) Proceed as in part (d). **Ans.** $y = \phi_1(y - x) + \phi_2(y - 2x) + (1/240) \times (2x + 3y)^3$

(f) Since $r = \partial^2 z/\partial x^2,\ s = \partial^2 z/\partial x\partial y,\ t = \partial^2 z/\partial y^2$, the given equation can be re–written as

$\partial^2 z/\partial x^2 + \partial^2 z/\partial x\,\partial y - 2\,(\partial^2 z/\partial y^2) = (2x + y)^{1/2}$ or $(D^2 + DD' - 2D'^2)z = (2x + y)^{1/2}$.

Its auxiliary equation is $m^2 + m - 2 = 0$ so that $m = 1, -2$.

$\therefore$ C.F. $= \phi_1(y + x) + \phi_2(y - 2x)$, ϕ_1, ϕ_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{D^2 + DD' - 2D'^2}(2x + y)^{1/2} = \frac{1}{2^2 + 2\times 1 - 2\times 1^2}\iint v^{1/2}\, dv\, dv, \text{ where } v = 2x + y$$

$$= \frac{1}{4}\int \frac{2}{3} v^{3/2}\, dv = \frac{1}{4}\times\frac{2}{3}\times\frac{2}{5}v^{5/2} = \frac{1}{15}(2x + y)^{5/2}$$

Hence the required general solution is $z = \phi_1(y + x) + \phi_2(y - 2x) + (1/15)(2x + y)^{5/2}$.

Ex. 3. *Solve* (a) $(D^2 + 2DD' + D'^2)z = e^{2x + 3y}$. **[Bhopal 2010; Indore 1998; Jabalpur 1998; Purvanchal 2007, Sagar 1999; K.V. Kurkshetra 2005]**

(b) $(D^2 - 2DD' + D'^2)z = e^{x + 2y}$. **[Bhopal 1997, 98, Kanpur 2005]**

(c) $(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)z = e^{5x + 6y}$.

Sol. (a) Here auxiliary equation is $m^2 + 2m + 1 = 0$ so that $m = -1, -1$.

$\therefore$ C.F. $= \phi_1(y - x) + x\phi_2(y - x)$, ϕ_1, ϕ_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{D^2 + 2DD' + D'^2}e^{2x + 3y} = \frac{1}{(D + D')^2}e^{2x + 3y} = \frac{1}{(2 + 3)^2}\iint e^v\, dv\, dv, \text{ where } v = 2x + 3y$$

$$= (1/25)\times\int e^v\, dv = (1/25)\times e^v = (1/25)\times e^{2x + 3y}$$

$\therefore$ Solution is $z =$ C.F. + P.I. $= \phi_1(y - x) + x\phi_2(y - x) + (1/25) \times e^{2x + 3y}$.

(b) Proceed as in part (a). **Ans.** $z = \phi_1(y + x) + x\phi_2(y + x) + e^{x + 2y}$

(c) Here auxiliary equation is $m^3 - 6m^2 + 11m - 6 = 0$ giving $m = 1, 2, 3$.

$\therefore$ C.F. $= \phi_1(y + x) + \phi_2(y + 2x) + \phi_3(y + 3x)$, ϕ_1, ϕ_2, ϕ_3 being arbitrary functions.

$$\text{P.I.} = \frac{1}{D^3 - 6D^2D' + 11DD'^2 - 6D'^3}e^{5x + 6y} = \frac{1}{(D - D')(D - 2D')(D - 3D')}e^{5x + 6y}$$

$$= \frac{1}{(5 - 6)(5 - 12)(5 - 18)}\iiint e^v\, dv\, dv\, dv, \text{ where } v = 5x + 6y$$

$$= \frac{1}{-91}\iint e^v\, dv\, dv = -\frac{1}{91}\int e^v\, dv = -\frac{1}{91}e^v = -\frac{1}{91}e^{5x + 6y}$$

Hence the required solution is $z = \phi_1(y + x) + \phi_2(y + 2x) + \phi_3(y + 3x) - (1/91) \times e^{5x + 6y}$.

Ex. 4. *Solve* (a) $r - 2s + t = \sin(2x + 3y)$. **[Meerut 2007; Indore 2002; Vikram 1996;]**

(b) $(D^3 - 4D^2D' + 4DD'^2)z = 2\sin(3x + 2y)$. **[Kanpur 2008; I.A.S. 2006]**

(c) $(D^3 - 4D^2D' + 4DD'^2)z = \cos(2x + 3y)$.

(d) $(D^3 - 3DD'^2 - 2D'^3)z = \cos(x + 2y)$. **[Delhi Maths Hons. 1992]**

Sol. (a) Since $r = \partial^2 z/\partial x^2,\ s = \partial^2 z/\partial x\partial y,\ t = \partial^2 z/\partial t^2$, the given equation becomes

$\partial^2 z/\partial x^2 - 2(\partial^2 z/\partial x\partial y) + \partial^2 z/\partial y^2$ = sin (2*x* + 3*y*) or $(D^2 - 2DD' + D'^2)z = \sin(2x + 2y)$.

Its auxiliary equation is $m^2 - 2m + 1 = 0$ so that $m = 1, 1$.

$\therefore$ C.F. $= \phi_1(y + x) + x\,\phi_2(y + x)$, ϕ_1, ϕ_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{(D - D')^2}\sin(2x + 3y) = \frac{1}{(2-3)^2}\iint \sin v\, dv\, dv, \text{ where } v = 2x + 3y$$

$$= -\int \cos v\, dv = -\sin v = -\sin(2x + 3y).$$

Hence the required general solution is $z = \phi_1(y + x) + x\phi_2(y + x) - \sin(2x + 3y)$.

(*b*) The auxiliary equation of the given equation is $m^3 - 4m^2 + 4m = 0$.

or $m(m^2 - 4m + 4) = 0$ or $m(m - 2)^2 = 0$ so that $m = 0, 2, 2$.

$\therefore$ C.F. $= \phi_1(y) + \phi_2(y + 2x) + x\phi_3(y + 2x)$, ϕ_1, ϕ_2, ϕ_3 being arbitrary functions.

Now, $\text{P.I.} = \dfrac{1}{D^3 - 4D^2D' + 4DD'^2}\, 2\sin(3x + 2y)$

$$= 2 \times \frac{1}{3^3 - 4\times 3^2 \times 2 + 4 \times 3 \times 2^2}\iiint \sin v\, dv\, dv, \text{ where } v = 3x + 2y$$

$$= (2/3) \times \iint (-\cos v)dv\, dv = -(2/3) \times \int \sin v\, dv = (2/3)\times \cos v = (2/3) \times \cos(3x + 2y)$$

The required general solution is $z = \phi_1(y) + \phi_2(y + 2x) + x\phi_3(y + 2x) + (2/3) \times \cos(3x + 2y)$.

(*c*) Proceed as in part (*b*). **Ans.** $z = \phi_1(y) + \phi_2(y + 2x) + x\phi_3(y + 2x) - (1/32) \times \sin(2x + 3y)$

(*d*) Proceed as in part (*b*). **Ans.** $z = \phi_1(y - x) + x\phi_2(y - x) + \phi_3(y + 2x) + (1/27) \times \sin(x + 2y)$

Ex. 5. *Solve* $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 = \cos mx \cos ny$. **[Kanpur 2007; Nagpur 2010]**

Sol. Given equation can be written as $(D^2 + D'^2)z = \cos mx \cos ny$.

Its auxiliary equation is $m^2 + 1 = 0$ so that $m = \pm i$.

$\therefore$ C.F. $= \phi_1(y + ix) + \phi_2(y - ix)$, ϕ_1 and ϕ_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{D^2 + D'^2}\cos mx \cos ny = \frac{1}{D^2 + D'^2}\frac{\cos(mx + ny) + \cos(mx - ny)}{2}$$

$$= \frac{1}{2}\frac{1}{D^2 + D'^2}\cos(mx + ny) + \frac{1}{2}\frac{1}{D^2 + D'^2}\cos(mx - ny)$$

$$= \frac{1}{2}\frac{1}{m^2 + n^2}\iint \cos v\, dv\, dv + \frac{1}{2}\frac{1}{m^2 + (-n)^2}\iint \cos u\, du\, du,$$

where $v = mx + ny$ and $u = mx - ny$

$$= \frac{1}{2}\frac{1}{m^2 + n^2}\int \sin v\, dv + \frac{1}{2}\frac{1}{m^2 + n^2}\int \sin u\, du = \frac{1}{2}\frac{1}{m^2 + n^2}[-\cos v - \cos u]$$

$$= -\frac{1}{2(m^2 + n^2)}[\cos(mx + ny) + \cos(mx - ny)], \text{ as } v = mx + ny,\ u = mx - ny$$

$$= -\frac{1}{2(m^2 + n^2)} \times 2\cos mx \cos ny = -(m^2 + n^2)^{-1}\cos mx \cos ny.$$

Hence the required general solution is $z = \phi_1(y + ix) + \phi_2(y - ix) - (m^2 + n^2)^{-1}\cos mx \cos ny$.

Ex. 5. (*b*) *Solve* $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 = \cos mx \sin ny$. **[Ravishankar 1999, 2001]**

Sol. Do like Ex. 5(*a*) **Ans.** $z = \phi_1(y + ix) + \phi_2(y - ix) + (\sin mx + \sin ny)/(m^2 + n^2)$

Ex. 6. *Solve the following partial differential equations :*

(*a*) $(D^2 - 2DD' + D'^2)z = \tan(y + x)$ or $(D - D')^2 z = \tan(y + x)$ **[Jiwaji 1996]**

(*b*) $(D^2 - 2aDD' + a^2D'^2)z = f(y + ax)$ or $(D - aD')^2 z = f(y + ax)$.

(*c*) $4r - 4s + t = 16 \log(x + 2y)$. **[Agra 2009; Meerut 2009; Ravishankar 2000]**

Sol. (*a*) Here auxiliary equation is $(m - 1)^2 = 0$ so that $m = 1, 1$.

$\therefore$ C.F. $= \phi_1(y + x) + x\phi_2(y + x)$, where ϕ_1 and ϕ_2 are arbitrary functions.

Now, $$\text{P.I.} = \frac{1}{(D - D')^2} \tan(y + x) = \frac{x^2}{1^2 \times 2!} \tan(y + x) = \frac{x^2}{2} \tan(y + x)$$

[Using formula (*ii*) of working rule with $a = 1$, $b = 1$, $m = 2$]

Hence the required general solution is $z = \phi_1(y + x) + x\phi_2(y + x) + (x^2/2) \times \tan(y + x)$.

(*b*) Here auxiliary equation is $(m - a)^2 = 0$ so that $m = a, a$.

$\therefore$ C.F. $= \phi_1(y + ax) + x\phi_2(y + ax)$, ϕ_1, ϕ_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{(D - aD')^2} f(y + ax) + \frac{x^2}{1^2 \times 2!} f(y + ax) = \frac{x^2}{2} f(y + ax).$$

[using formula (*ii*) of working rule with $a = a$, $b = 1$, $m = 2$]

$\therefore$ General solution is $z = \phi_1(y + x) + x\phi_2(y + x) + (x^2/2) \times f(y + ax)$.

(*c*) Since $r = \partial^2 z/\partial x^2$, $s = \partial^2 z/\partial x \partial y$, $t = \partial^2 z/\partial y^2$, the given equation becomes

$4(\partial^2 z/\partial x^2) - 4(\partial^2 z/\partial x\partial y) + (\partial^2 z/\partial y^2) = 16 \log(x + 2y)$ or $(4D^2 - 4DD' + D'^2)z = 16 \log(x + 2y)$

Its auxiliary equation is $4m^2 - 4m + 1 = 0$ so that $m = 1/2.\ 1/2$.

$\therefore$ C.F. $= \phi_1(2y + x) + x\phi_2(2y + x)$, ϕ_1 and ϕ_2 being arbitrary functions.

Now, $$\text{P.I.} = \frac{1}{(2D - D')^2} 16 \log(x + 2y) = 16 \times \frac{x^2}{2^2 \times 2!} \log(x + 2y) = 2x^2 \log(x + 2y)$$

[using formula (*ii*) of working rule with $a = 1$, $b = 2$, $m = 2$]

$\therefore$ The required solution is $z = \phi_1(2y + x) + x\phi_2(2y + x) + 2x^2 \log(x + 2y)$.

Ex. 7. *Solve the following partial differential equations :*

(*a*) $(2D^2 - 5DD' + 2D'^2)z = 5 \sin(2x + y)$. **[M.D.U. Rohtak 2005]**

(*b*) $(D^2 - 5DD' + 4D'^2)z = \sin(4x + y)$. **[Meerut 2006, 08]**

(*c*) $(D^3 - 2D^2D' - DD'^2 + 2D'^3)z = e^{x+y}$. **[Bhopal 2000, 03, Meerut 2007; Jabalpur 2004]**

(*d*) $r - t = x - y$.

(*e*) $2r - s - 3t = 5e^x/e^y$. **[Indore 2003; Jiwaji 2003, Vikram 1998]**

(*f*) $r + 5s + 6t = (y - 2x)^{-1}$ or $(\partial^2 z/\partial x^2) + 5(\partial^2 z/\partial x\partial y) + 6(\partial^2 z/\partial y^2) = 1/(y - 2x)$

[Agra 2009; Indore 2000; I.A.S. 1991; Garhwal 2005]

Sol. (*a*) Here auxiliary equation is $2m^2 - 5m + 2 = 0$ so that $m = 2, 1/2$.

$\therefore$ C.F. $= \phi_1(y + 2x) + \phi_2(2y + x)$, ϕ_1, ϕ_2 being arbitrary functions.

Now, $$\text{P.I.} = \frac{1}{2D^2 - 5DD' + 2D'^2} 5 \sin(2x + y) = 5 \frac{1}{(D - 2D')} \left[\frac{1}{(2D - D')} \sin(2x + y) \right]$$

$$= 5 \frac{1}{D - 2D'} \frac{1}{(2 \times 2) - 1} \int \sin v \, dv$$, where $v = 2x + y$, using formula (*i*) of working rule

$$= \frac{5}{3}\frac{1}{D-2D'}(-\cos v) = -\frac{5}{3}\frac{1}{D-2D'}\cos(2x+y) = -\frac{5}{3}\times\frac{x}{1^1\times 1!}\cos(2x+y)$$

[Using formula (*ii*) with $a = 2$, $b = 1$, $m = 2$]

$\therefore$ The required general solution is $z = \phi_1(y + 2x) + \phi_2(2y + x) - (5x/3) \times \cos(2x + y)$.

(*b*) Do as in part (*a*). **Ans.** $z = \phi_1(y + x) + \phi_2(y + 4x) - (x/3) \times \cos(4x + y)$

(*c*) Here auxiliary equation is $m^3 - 2m^2 - m + 2 = 0$ or $m^2(m - 2) - (m - 2) = 0$

or $(m^2 - 1)(m - 2) = 0$ so that $m = 2, 1, -1$.

$\therefore$ C.F. $= \phi_1(y + 2x) + \phi_2(y + x) + \phi_3(y - x)$, ϕ_1, ϕ_2, ϕ_3 being arbitrary functions.

$$\text{P.I.} = \frac{1}{D^3 - 2D^2D' - DD'^2 + 2D'^3}e^{x+y} = \frac{1}{(D-D')}\left\{\frac{1}{(D^2 - DD' - 2D'^2)}e^{x+y}\right\}$$

$$= \frac{1}{D-D'}\frac{1}{1^2 - (1\times 1) - (2\times 1)^2}\int e^v\, dv, \text{ where } v = x + y, \text{ using formula } (i) \text{ of working rule}$$

$$= \frac{1}{D-D'}\left(-\frac{1}{2}\right)e^v = -\frac{1}{2}\frac{1}{D-D'}e^{x+y} = -\frac{1}{2}\times\frac{x}{1^1\times 1!}e^{x+y} = -\frac{x}{2}e^{x+y}$$

[Using formula (*ii*) with $a = 1$, $b = 1$, $m = 1$]

$\therefore$ Required solution is $z = \phi_1(y + 2x_1) + \phi_2(y + x) + \phi_3(y - x) - (x/2) \times e^{x+y}$.

(*d*) The given equation can be re–written as

$(\partial^2 z/\partial x^2) - (\partial^2 z/\partial y^2) = x - y$ or $(D^2 - D'^2)z = x - y$. ...(1)

Its auxiliary equation is $m^2 - 1 = 0$ so that $m = 1, -1$.

$\therefore$ C.F. $= \phi_1(y + x) + \phi_2(y - x)$, ϕ_1, ϕ_2 being arbitrary functions.

$$\text{Now, P.I.} = \frac{1}{D^2 - D'^2}(x-y) = \frac{1}{D+D'}\cdot\frac{1}{D-D'}(x-y) = \frac{1}{D+D'}\cdot\frac{1}{1-(-1)}\int v\, dv, \text{ where } v = x - y.$$

$$= \frac{1}{2}\frac{1}{D+D'}\frac{v^2}{2} = \frac{1}{4}\frac{1}{D+D'}(x-y)^2 = -\frac{1}{4}\frac{1}{[(-1)\times D - 1\times D']^1}(x-y)^2$$

$$= -\frac{1}{4}\frac{x}{(-1)^1\times 1!}(x-y)^2 = \frac{x}{4}(x-y)^2, \text{ by formula } (ii) \text{ with } a = 1,\ b = -1,\ m = 1.$$

$\therefore$ The required solution is $z = \phi_1(y + x) + \phi_2(y - x) + (x/4) \times (x - y)^2$.

(*e*) The given equation can be re–written as

$2(\partial^2 z/\partial x^2) - (\partial^2 z/\partial x\partial y) - 3(\partial^2 z/\partial y^2) = 5e^{x-y}$ or $(2D^2 - DD' - 3D'^2)z = 5e^{x-y}$

or $(D + D')(2D - 3D')z = 5e^{x-y}$.

C.F. $= \phi_1(y - x) + \phi_2(2y + 3x)$, ϕ_1, ϕ_2 being arbitrary functions.

$$\text{Now, P.I.} = \frac{1}{(D+D')}\times\frac{1}{(2D-3D')}5e^{x-y} = 5\frac{1}{D+D'}\times\frac{1}{(2\times 1) - 3\times(-1)}\int e^v\, dv, \text{ where } v = x - y,$$

$$= \frac{1}{D+D'}e^v = -\frac{1}{[(-1)\times D - 1\times D']}e^{x-y} = -\frac{x}{(-1)^1\times 1!}e^{x-y} = xe^{x-y},$$

[using formula (*ii*) with $a = 1$, $b = -1$, $m = 1$]

$\therefore$ The required solution is $z = \phi_1(y - x) + \phi_2(2y + 3x) + xe^{x-y}$.

(*f*) Given equation can be rewritten as $(D^2 + 5DD' + 6D'^2)z = (y - 2x)^{-1}$. ...(1)

Its auxiliary equation is $m^2 + 5m + 6 = 0$ so that $m = -2, -3$.

$\therefore$ C.F. $= \phi_1(y - 2x) + \phi_2(y - 3x)$, ϕ_1, ϕ_2 being arbitrary functions.

Now, P.I. $= \dfrac{1}{D^2+5DD'+6D'^2}(y-2x)^{-1} = \dfrac{1}{D+2D'}\left[\dfrac{1}{D+3D'}(y-2x)^{-1}\right]$

$= \dfrac{1}{D+2D'} \times \dfrac{1}{-2+(3\times 1)}\displaystyle\int v^{-1}\, dv$, where $v = y - 2x$, by formula (i)

$= \dfrac{1}{D+2D'}\log v = \dfrac{1}{D+2D'}\log(y-2x) = \dfrac{1}{[1\times D-(-2)\times D']^1}\log(y-2x)$

$= \dfrac{x}{1^1\times 1!}\log(y-2x)$, by formula ($ii$) with $a = -2$, $b = 1$, $m = 1$

$\therefore$ The general solution is $z = \phi_1(y-2x) + \phi_2(y-3x) + x\log(y-2x)$.

Ex. 8. *Solve the following partial differential equation :*

(a) $(D^3 - 4D^2D' + 4DD'^2)z = 4\sin(2x+y)$.

[Bilaspur 1995; Indore 2004;Rewa 2000, 01; MDU Rohtak 2004]

(b) $(D^3 - 4D^2D' + 4DD'^2)z = \cos(2x+y)$.

(c) $(D^3 - 4D^2D' + 4DD'^2)z = \sin(y+2x)$.

Sol. (a) Here the auxiliary equation is $m^3 - 4m^2 + 4m = 0$ or $m(m^2 - 4m + 4) = 0$

or $m(m-2)^2 = 0$ so that $m = 0, 2, 2$.

$\therefore$ C.F. $= \phi_1(y) + \phi_2(y+2x) + x\phi_3(y+2x)$, ϕ_1, ϕ_2, ϕ_3 being arbitrary functions.

Now, P.I. $= \dfrac{1}{D^3-4D^2D'+4DD'^2}4\sin(2x+y) = 4\dfrac{1}{(D-2D')^2}\left\{\dfrac{1}{D}\sin(2x+y)\right\}$

$= 4\dfrac{1}{(D-2D')^2}\left\{-\dfrac{1}{2}\cos(2x+y)\right\}$, since $1/D$ stands for integration w.r.t. x treating y as constant.

$= -2\dfrac{1}{(D-2D')^2}\cos(2x+y) = -2\dfrac{x^2}{1^2\times 2!}\cos(2x+y)$,

[Using formula (ii) with $a = 2$, $b = 1$, $m = 2$]

So the required solution is $y = \phi_1(y) + \phi_2(y+2x) + x\phi_3(y+2x) - x^2\cos(2x+y)$.

(b) Try yourself **Ans.** $z = \phi_1(y) + \phi_2(y+2x) + x\phi_3(y+2x) + (x^2/4)\times\sin(2x+y)$

(c) Try yourself **Ans.** $z = \phi_1(y) + \phi_2(y+2x) + x\phi_3(y+2x) - (x^2/4)\times\cos(2x+y)$

Ex. 9. *Solve the following partial differential equations :*

(a) $(D^2 - 3DD' + 2D'^2)z = e^{2x-y} + e^{x+y} + \cos(x+2y)$.

(b) $(D^2 - 3DD' + 2D'^2)z = e^{2x-y} + \cos(x+2y)$ **[Delhi Maths (H) 2006]**

(c) $(D^3 - 4D^2D' + 5DD'^2 - 2D'^3)z = e^{y+2x} + (y+x)^{1/2}$.

(d) $(D_x^3 - 7D_xD_y^2 - 6D_y^3)z = \sin(x+2y) + e^{3x+y}$. **[I.A.S. 1995]**

Sol. (a) Here auxiliary equation is $m^2 - 3m + 2 = 0$ so that $m = 1, 2$.

$\therefore$ C.F. $= \phi_1(y+x) + \phi_2(y+2x)$, ϕ_1, ϕ_2 being arbitrary functions. ...(1)

Now, P.I. corresponds to e^{2x-y}

$= \dfrac{1}{D^2-3DD'+2D'^2}e^{2x-y} = \dfrac{1}{2^2-3\times 2\times(-1)+2\times(-1)^2}\displaystyle\iint e^v\, dv\, dv$, where $v = 2x - y$

$= (1/12)\times\displaystyle\int e^v dv = (1/12)\times e^v = (1/12)\times e^{2x-y}$. ...(2)

P.I. corresponding to e^{x+y} $= \dfrac{1}{D^2 - 3DD' + 2D'^2} e^{x+y} = \dfrac{1}{D-D'}\left\{\dfrac{1}{D-2D'} e^{x+y}\right\}$

$= \dfrac{1}{D-D'}\left\{\dfrac{1}{1-(2\times 1)}\int e^v \, dv\right\}$, where $v = x + y$, using formula (i)

$= -\dfrac{1}{D-D'} e^v = -\dfrac{1}{(D-D')^1} e^{x+y} = -\dfrac{x}{1^1 \times 1!} e^{x+y} = -xe^{x+y}$. ...(3)

[Using formula (ii) with $a = b = 1$, $m = 1$]

Finally, P.I. corresponding to $\cos(x + 2y)$

$= \dfrac{1}{D^2 - 3DD' + 2D'^2} \cos(x+2y) = \dfrac{1}{1^2 - (3\times 1\times 2) + (2\times 2^2)} \iint \cos v \, dv \, dv$, where $v = x + 2y$

$= (1/3) \times \int \sin v dv = -(1/3) \times \cos v = -(1/3) \times \cos(x + 2y)$ ...(4)

From (1), (2), (3) and (4), the required solution is z = C.F. + P.I.

or $z = \phi_1(y + x) + \phi_2(y + 2x) + (1/12) \times e^{2x-y} - xe^{x+y} - (1/3) \times \cos(x + 2y)$.

(b) This problem is same as part (a) except that the term e^{x+y} is missing on R.H.S. So, now you need not compute P.I. corresponding to e^{x+y}. Therefore the solution will take the form

$$y = \phi_1(y+x) + \phi_2(y+2x) + (1/12)\times e^{2x-y} - (1/3)\times\cos(x+2y)$$

(c) Here auxiliary equation is $m^3 - 4m^2 + 5m - 2 = 0$ giving $m = 1, 1, 2$.

$\therefore$ C.F. $= \phi_1(y + x) + x\phi_2(y + x) + \phi_3(y + 2x)$., ϕ_1, ϕ_2, ϕ_3 being arbitrary function ...(1)

Now, P.I. corresponding to e^{y+2x}

$= \dfrac{1}{D^3 - 4D^2D' + 5DD'^2 - 2D'^3} e^{y+2x} = \dfrac{1}{D-2D'}\left\{\dfrac{1}{(D-D')^2} e^{y+2x}\right\}$

$= \dfrac{1}{D-2D'} \dfrac{1}{(2-1)^2} \iint e^v \, dv \, dv$, where $v = y + x$, by formula (i)

$= \dfrac{1}{D-2D'}\int e^v \, dv = \dfrac{1}{D-2D'} e^v = \dfrac{1}{(1\times D - 2\times D')^1} e^{y+2x} = \dfrac{x}{1\times 1!} e^{y+x} = xe^{y+x}$, ...(2)

[using formula (ii) with $a = 2$, $b = 1$, $m = 1$]

Finally, P.I. corresponding to $(y + x)^{1/2}$

$= \dfrac{1}{D^3 - 4D^2D' + 5DD'^2 - 2D'^3} (y+x)^{1/2} = \dfrac{1}{(D-D')^2} \cdot \left\{\dfrac{1}{D-2D'} (y+x)^{1/2}\right\}$

$= \dfrac{1}{(D-D')^2} \times \dfrac{1}{1-(2\times 1)} \int v^{1/2} \, dv$, where $v = y + x$, using formula (i)

$= -\dfrac{1}{D-D'} \times \dfrac{2}{3} v^{3/2} = -\dfrac{2}{3}\dfrac{1}{(D-D')^2}(y+x)^{3/2} = -\dfrac{2}{3}\times\dfrac{x^2}{1^2\times 2!}(y+x)^{3/2}$

$= -(x^2/3) \times (y + x)^{3/2}$, using formula ($ii$) with $a = b = 1$, $m = 2$ ...(3)

From (1), (2) and (3), the required general solution is

$$z = \phi_1(y + x) + x\phi_2(y + x) + \phi_3(y + 2x) + xe^{y+x} - (x^2/3) \times (y + x)^{3/2}.$$

(d) Here note that D_x and D_y stand for D and D' respectively.

$\therefore$ Auxiliary equation is $m^3 - 7m - 6 = 0$ so that $m = -1, -2, 3.$

$\therefore$ C.F. $= \phi_1(y - x) + \phi_2(y - 2x) + \phi_3(y + 3x)$, ϕ_1, ϕ_2, ϕ_3 being arbitrary functions.

Now, P.I. corresponding to $\sin(x + 2y)$

$$= \frac{1}{D_x^3 - 7D_xD_y^2 - 6D_y^3}\sin(x+2y) = \frac{1}{1^3 - (7\times 1\times 2^2) - (6\times 2^3)}\iiint \sin v\, dv\, dv\, dv \text{ ,where } v = x + 2y$$

$$= -(1/75)\times\iint(-\cos v)\, dv\, dv = -(1/75)\times\int(-\sin v)\, dv = -(1/75)\times\cos v = -(1/75)\times\cos(x+2y)$$

and P.I. corresponding to e^{3x+y}

$$= \frac{1}{D_x^3 - 7D_xD_y^2 - 6D_y^3}e^{3x+y} = \frac{1}{D_x - 3D_y}\left[\frac{1}{(D_x + D_y)(D_x + 2D_y)}e^{3x+y}\right]$$

$$= \frac{1}{D_x - 3D_y}\cdot\frac{1}{(3+1)(3+2)}\iint e^v\, dv\, dv, \text{ where } v = 3x + y, \text{ by formula } (i)$$

$$= \frac{1}{20}\frac{1}{D_x - 3D_y}\int e^v\, dv = \frac{1}{20}\frac{1}{D_x - 3D_y}e^v = \frac{1}{20}\frac{1}{(D_x - 3D_y)^1}e^{3x+y}$$

$$= \frac{1}{20}\times\frac{x}{1^1\times 1!}e^{3x+y} = \frac{x}{20}e^{3x+y}, \text{ using formula } (ii) \text{ with } a = 2,\ b = 1,\ m = 1.$$

Hence the required general solution is z = C.F. + P.I.

or $z = \phi_1(y - x) + \phi_2(y - 2x) + \phi_3(y + 3x) - (1/75)\times\cos(x + 2y) + (1/20)\times x\, e^{3x+y}$.

Ex. 10. *Solve (i)* $(D^2 - 6DD' + 9D'^2)z = \tan(y + 3x)$ **[Delhi 2007; Ravishankar 2004]**

(ii) $(D^2 - 6DD' + 9D'^2)z = 6x + 2y$

Sol (i) Here auxiliary equation is $(m - 3)^2 = 0$ so that $m = 3, 3.$

$\therefore$ C.F. $= \phi_1(y+3x) + x\,\phi_2(y+3x)$, ϕ_1, ϕ_2 being arbitrary functions

$$\text{P.I.} = \frac{1}{(D-3D')^2}\tan(y+3x) = \frac{x^2}{1^2\times 2!}\tan(y+3x) = \frac{x^2}{2}\tan(y+3x)$$

$\therefore$ The required solution is $z = \phi_1(y+3x) + x\,\phi_2(y+3x) + (x^2/2)\times\tan(y+3x)$.

(ii) Re-writing the given equation reduces to $(D - 3D')^2 z = 2(3x + y)$

$\therefore$ C.F. $= \phi_1(y + 3x) + x\phi_1(y + 3x)$, ϕ_1, ϕ_2 being arbitrars constants.

Now, $$P.I. = \frac{1}{(D-3D')^2}\,2(3x+y) = 2\,\frac{x^2}{1^2\times 2!}(3x+y) = x^2(3x+y)$$

$\therefore$ The required solution is $z = \phi_1(y+3x) + x\phi_2(y+3x) + 3x^3 + x^2y.$

Ex. 11. *Solve (i)* $(D-3D')^2(D+3D')z = e^{3x+y}$ **[Delhi Maths (Hons) 2000, Agra 2005]**

(ii) $(D-2D')(D+D')^2 z = \cos(2x+y)$

Sol. (i) C.F. $= \phi_1(y+3x) + x\,\phi_2(y+3x) + \phi_3(y-3x)$, where ϕ_1, ϕ_2, ϕ_3 are arbitrary functions

$$\text{P.I.} = \frac{1}{(D-3D')^2}\frac{1}{D+3D'}e^{3x+y} = \frac{1}{(D-3D')^2}\frac{1}{3+(3\times 1)}\int e^v dv, \text{ where } v = 3x + y$$

$$= \frac{1}{6}\frac{1}{(D-3D')^2}e^v = \frac{1}{6}\frac{1}{(D-3D')^2}e^{3x+y} = \frac{1}{6}\frac{x}{1\times 2!}e^{3x+y}$$

The required solution is $z = \phi_1(y+3x) + x\phi_2(y+3x) + \phi_3(y-3x) + (x^2/12)\times e^{3x+y}$

(*ii*) Try your self **Ans.** $z = \phi_1\ (y+2x) + \phi_2(y-x) + x\ \phi_3(y-x) - (x/9)\times\cos(2x+y)$

Ex. 12. Solve (*i*) $r + s - 2t = e^{x+y}$

(*ii*) $(D^3 - 7DD'^2 - 6D'^3)y = \sin(x+2y)$

(*iii*) $(D^3 - 3DD'^2 + 2D'^3)y = (x-2y)^{1/2}$ **[Delhi Maths (H) 2009]**

Sol. (*i*) Re-writing given equation becomes $(\partial^2 z/\partial x^2) + (\partial^2 z/\partial x\,\partial y) - 2\,(\partial^2 z/\partial y^2) = e^{x+y}$

or $(D^2 + DD' - 2D'^2)\ z = e^{x+y}$ or $(D-D')\ (D+2D')z = e^{x+y}$.

Its C.F. $= \phi(y+x) + \phi_2(y-2x)$, ϕ_1, ϕ_2 being arbitrary functions

$$\text{P.I.} = \frac{1}{D-D'}\frac{1}{D+2D'}e^{x+y} = \frac{1}{D-D'}\frac{1}{1+(2\times 1)}\int e^v dv, \text{ where } v = x+y$$

$$= \frac{1}{3}\frac{1}{D-D'}e^v = \frac{1}{3}\frac{1}{D-D'}e^{x+y} = \frac{1}{3}\frac{x}{1!}e^{x+y}$$

$\therefore$ The required solution is $z = \phi_1\ (y+x) + \phi_2(y-2x) + (x/3)\times e^{x+y}$.

(*ii*) Here the auxiliary equation is $m^3 - 7m - 6 = 0$ giving $m = -1, -2, 3.$

$\therefore$ C.F. $= \phi_1\ (y-x) + \phi_2\ (y-2x) + \phi_3(y+3x)$, ϕ_1, ϕ_2, ϕ_3 are arbitrary functions.

$$\text{P.I.} = \frac{1}{D^3 - 7DD'^2 - 6D'^3}\sin\ (x+2y) = \frac{1}{1^3 - (7\times 1\times 2^2) - (16\times 2^3)}\int\iint \sin v(dv)^3, \text{ where } v = x + 2y$$

$$= -\frac{1}{75}\iint(-\cos v)dv\ dv = -\frac{1}{75}\int\ (-\sin v)\ dv = -\frac{1}{75}\cos v = -\frac{1}{75}\cos\ (x+2y)$$

$\therefore$ Required solution $z = \phi_1(y-x) + \phi_2(y-2x) + \phi_3(y+3x) - (1/75)\times\cos(x+2y)$

(*iii*) The auxiliary equation is $m^3 - 3m + 2 = 0$ giving $m = 1, 1, 2.$

$\therefore$ C.F. $= \phi_1(y+x) + x\ \phi_2(y+x) + \phi_3(y+2x)$, ϕ_1, ϕ_2, ϕ_3 are arbitrary functions

$$\text{P.I.} = \frac{1}{D^3 - 3DD'^2 + 2D'^3}(x-2y)^{1/2} = \frac{1}{1^3 - 3\times 1\times(-2)^2 + 2\times(-2)^3}\iiint v^{1/2}(dv)^3, \text{ where } v = x - 2y$$

$$= -\frac{1}{27}\iint\frac{v^{3/2}}{(3/2)}dv\ dv = -\frac{1}{27}\int\frac{v^{5/2}}{(3/2)\times(5/2)}dv = -\frac{1}{27}\frac{v^{7/2}}{(3/2)\times(5/2)\times(7/2)}$$

$$= -(8/2835)\times v^{7/2} = -(8/2835)\times(x-2y)^{7/2}$$

General solution is $z = \phi_1\ (y+x) + x\ \phi_2\ (y+x) + \phi_3(y+2x) - (8/2835)\times(x-2y)^{7/2}$.

Ex. 13. *Solve* $(D^2 - 3DD' + 2D'^2)z = \cos\ (x+2y)$

Sol. The auxiliary equation $m^2 - 3m + 2 = 0$ gives $m = 1, 2.$

$\therefore$ C.F. $= \phi_1(y+x) + \phi_2(y+2x)$, ϕ_1, ϕ_2 being arbitrary functions

$$\text{P.I.} = \frac{1}{D^2 - 3DD' + 2D'^2} \cos(x+2y) = \frac{1}{1^2 - 3\cdot 1\cdot 2 + 2\cdot 2^2} \iint \cos v (dv)^2, \text{ where } v = x + 2y$$

$$= (1/3) \times \int \sin v\, dv = -(1/3) \times \cos v = -(1/3) \times \cos(x+2y)$$

$\therefore$ Solution is $\quad z = \phi_1(y+x) + \phi_2(y+2x) - (1/3) \times \cos(x+2y)$

Ex. 14. *Solve* $(D^2 - DD' - 2D'^2)z = 2x + 3y + e^{3x+4y}$. **[I.A.S. 2000]**

Sol. The auxiliary equation $\quad m^2 - m - 2 = 0 \quad$ giving $\quad m = 2, -1$.

$\therefore$ C.F. $= \phi_1(y+2x) + \phi_2(y-x)$, ϕ_1, ϕ_2 being arbitrary functions

P.I. corresponding to $(2x + 3y)$

$$= \frac{1}{D^2 - DD' - 2D'^2}(2x+3y) = \frac{1}{2^2 - (2\times 3) - (2\times 3^2)} \iint v\,(dv)^2, \text{ where } v = 2x+3y$$

$$= -\frac{1}{20}\int \frac{v^2}{2}\, dv = -\frac{1}{20}\left(\frac{v^3}{2\times 3}\right) = -\frac{1}{60}(2x+3y)^3$$

P.I. corresponding to e^{3x+4y}

$$= \frac{1}{D^2 - DD' - 2D'^2} e^{3x+4y} = \frac{1}{3^2 - (3\times 4) - (2\times 4^2)} \iint e^v (dv)^2, \text{ where } v = 3x+4y$$

$$= -(1/35) \times e^v = -(1/35) \times e^{3x+4y}.$$

$\therefore$ General solution is $\quad z = \phi_1(y+2x) + \phi_2(y-x) - (1/60) \times (2x+3y)^3 - (1/35) \times e^{3x+4y}$.

EXERCISE 4(A)

Solve the following partial differential equations:

1. $(D^2 - DD' - 6D'^2)z = \cos(2x+y)$ **[Agra 2009, 10]**

Ans. $z = \phi_1(y+3x) + \phi_2(y-2x) - (1/4)\times\cos(2x+y)$ ϕ_1, ϕ_2, being arbitrary functions.

2. $r - 4s + 4t = e^{2x+y}$ **[Agra 2010]**

Ans. $z = \phi_1(y+2x) + x\phi_2(y+2x) + (x^2/2)\times e^{2x+y}$ ϕ_1, ϕ_2 being arbitrary functions

3. $(D^3 - 4D^2D'^2 + 4DD'^2)z = 6\sin(3x+2y)$

Ans. $z = \phi_1(y) + \phi_2(y+2x) + x\phi_3(y+2x) + 2\cos(3x+2y)$, ϕ_1, ϕ_2, ϕ_3, being arbitrary functions.

4. $(D-3D')^2(D+3D')z = e^{3x+y}$ **[Agra 2005]**

Ans. $z = \phi_1(y+3x) + x\phi_2(y+3x) + \phi_3(y-3x) + (x/12)\times e^{3x+y}$, ϕ_1, ϕ_2, ϕ_3, being arbitrary functions.

4.10. Short Method II. When f(x, y) is of the form $x^m y^n$ or a rational integral algebraic function of *x* and *y*.

Then the particular integral (P.I.) is evaluated by expanding the symbolic function $1/f(D, D')$ in an infinite series of ascending powers of D or D'. In solved examples 1 and 2 of Art. 4.11, we have shown that P.I. obtained on expanding $1/f(D, D')$ in ascending powers of D is different from that obtained on expanding $1/f(D, D')$ in ascending powers of D'. Since to get the required general solution of given differential equation any P.I. is required, any of the two methods can be used. The difference in the two answers of P.I. is not material as it can be incorporated in the arbitrary functions occuring in C.F. of that given differential equation.

Remark : If $n < m$, $1/f(D, D')$ should be expanded in powers of D'/D whereas if $m < n$, $1/f(D, D')$ should be expanded in powers of D/D'.

4.11 SOLVED EXAMPLES BASED ON SHORT METHOD II

Ex. 1. *Solve* $(D^2 - a^2D'^2)z = x$ or $(\partial^2 z/\partial x^2) - a^2(\partial^2 z/\partial y^2) = x$.

Sol. Here auxiliary equation is $m^2 - a^2 = 0$ so that $m = a, -a$.

$\therefore$ C.F. $= \phi_1(y + ax) + \phi_2(y - ax)$, ϕ_1, ϕ_2 being arbitrary functions. ...(1)

Now, $$\text{P.I.} = \frac{1}{D^2 - a^2D'^2}x = \frac{1}{D^2[1 - a^2(D'^2/D^2)]}x = \frac{1}{D^2}\left(1 - a^2\frac{D'^2}{D^2}\right)^{-1}x$$

$$= \frac{1}{D^2}\left(1 + a^2\frac{D'^2}{D^2} + \dots\right)x = \frac{1}{D^2}x = \frac{x^3}{6}. \quad ...(2)$$

Alternatively, we can compute P.I. on follows :

$$\text{P.I.} = \frac{1}{D^2 - a^2D'^2}x = \frac{1}{-a^2D'^2[1 - (D^2/a^2D'^2)]}x = -\frac{1}{a^2D'^2}\left(1 - \frac{D^2}{a^2D'^2}\right)^{-1}x$$

$$= -\frac{1}{a^2D'^2}\left(1 + \frac{D^2}{a^2D'^2} + \dots\right)x = -\frac{1}{a^2D'^2}x = -\frac{1}{a^2} \times \frac{xy^2}{2}. \quad ...(3)$$

Hence the required general solution is $z =$ C.F. + P.I. that is,

$z = \phi_1(y + ax) + \phi_2(y - ax) + x^3/6$, using (1) and (2).

or $z = \phi_1(y + ax) + \phi_2(y - ax) - (xy^2)/(2a^2)$, using (1) and (3).

Ex. 2. *Solve* $(D^2 + 3DD' + 2D'^2)z = x + y$, *by expanding the particular integral in ascending powers of D as well as in ascending powers of D'.*

[Bhopal 2000, 03; Indore 1999; Jiwaji 1995; Rewa, 2002, 03; I.A.S. 1994]

Sol. Here auxiliary equation is $m^2 + 3m + 2 = 0$ so that $m = -2, -1$.

$\therefore$ C.F. $= \phi_1(y - 2x) + \phi_2(y - x)$, ϕ_1, ϕ_2 being arbitrary functions. ...(1)

Now, by expanding in ascending powers of D, we have

$$\text{P.I.} = \frac{1}{D^2 + 3DD' + 2D'^2}(x + y) = \frac{1}{2D'^2\left[1 + \left(\frac{D^2}{2D'^2} + \frac{3}{2}\frac{D}{D'}\right)\right]}(x + y)$$

$$= \frac{1}{2D'^2}\left[1 + \left(\frac{D^2}{2D'^2} + \frac{3}{2}\frac{D}{D'}\right)\right]^{-1}(x + y) = \frac{1}{2D'^2}\left(1 - \frac{3}{2}\frac{D}{D'} + \dots\right)(x + y)$$

$$= \frac{1}{2D'^2}\left(x + y - \frac{3}{2}y\right) = \frac{1}{2D'^2}\left(x - \frac{y}{2}\right) = \frac{xy^2}{4} - \frac{y^3}{24}. \quad ...(2)$$

Again, by expanding in ascending powers of D, P.I. of given equation is given by

$$\text{P.I.} = \frac{1}{D^2 + 3DD' + 2D'^2}(x + y) = \frac{1}{D^2\left[1 + \left(\frac{3D'}{D} + \frac{2D'^2}{D^2}\right)\right]}(x + y) = \frac{1}{D^2}\left[1 + \left(\frac{3D'}{D} + \frac{2D'^2}{D^2}\right)\right]^{-1}(x + y)$$

$$= \frac{1}{D^2}\left(1 - \frac{3D'}{D} + \dots\right)(x + y) = \frac{1}{D^2}(x + y - 3x) = \frac{yx^2}{2} - \frac{x^3}{3}.$$

Hence the required general solution is given by $z =$ C.F. + P.I., *i.e.*,

$z = \phi_1(y - 2x) + \phi_2(y - x) + (1/4) \times xy^2 - (1/24) \times y^3$, using (1) and (2)

or $z = \phi_1(y - 2x) + \phi_2(y - x) + (1/2) \times yx^2 - (1/3) \times x^3$, using (1) and (3).

Ex. 3. *Solve* $(\partial^3 z/\partial x^3) - (\partial^3 z/\partial y^3) = x^3 y^3$ or $(D^3 - D'^3)z = x^3 y^3$. **[I.A.S. 1997]**

Sol. Here auxiliary equation is $m^3 - 1 = 0$ so that $m = 1, \omega, \omega^2$, where ω and ω^2 are complex cube roots of unity.

$\therefore$ C.F. $= \phi_1(y + x) + \phi_2(y + \omega x) + \phi_3(y + \omega^2 x)$, ϕ_1, ϕ_2, ϕ_3 being arbitrary functions.

$$\text{Now, P.I.} = \frac{1}{D^3 - D'^3} x^3 y^3 = \frac{1}{D^3[1 - (D'^3/D^3)]} x^3 y^3 = \frac{1}{D^3}\left(1 - \frac{D'^3}{D^3}\right)^{-1} x^3 y^3$$

$$= \frac{1}{D^3}\left(1 + \frac{D'^3}{D^3} + \dots\right) x^3 y^3 = \frac{1}{D^3}\left(x^3 y^3 + \frac{1}{D^3} 6x^3\right) = \frac{1}{D^3}\left(x^3 y^3 + 6 \times \frac{x^6}{4 \times 5 \times 6}\right)$$

$$= (1/120) \times x^6 y^3 + (1/10080) \times x^9.$$

Hence the required general solution is $z =$ C.F. + P.I.

or $$z = \phi_1(y + x) + \phi_2(y + \omega x) + \phi_3(y + \omega^2 x) + (1/120) \times x^6 y^3 + (1/10080) \times x^9.$$

Ex. 4. *Solve* $r + (a + b)s + abt = xy$. **[Indore 1998; Vikram 1998, 2000; Rewa 1998]**

Sol. Given equation can be written as $[D^2 + (a + b)DD' + abD'^2] = xy$.

Its auxiliary equation is $m^2 + (a + b)m + ab = 0$ or $(m + a)(m + b) = 0$ so that $m = -a, -b$.

$\therefore$ C.F. $= \phi_1(y - ax) + \phi_2(y - bx)$, ϕ_1, ϕ_2 being arbitrary functions.

$$\text{Now, P.I.} = \frac{1}{D^2 + (a + b)DD' + abD'^2} xy = \frac{1}{D^2\left[1 + (a + b)\dfrac{D'}{D} + ab\dfrac{D'^2}{D^2}\right]} xy$$

$$= \frac{1}{D^2}\left[1 + (a + b)\frac{D'}{D} + ab\frac{D'^2}{D^2}\right]^{-1} xy = \frac{1}{D^2}\left[1 - (a + b)\frac{D'}{D} + \dots\right] xy$$

$$= \frac{1}{D^2}\left\{xy - \frac{a + b}{D} D'(xy)\right\} = \frac{1}{D^2}\left\{xy - \frac{a + b}{D} x\right\} = \frac{1}{D^2}\left\{xy - \frac{(a + b)x^2}{2}\right\}$$

$$= y \times \frac{x^3}{2 \times 3} - \frac{a + b}{2} \times \frac{x^4}{3 \times 4} = \frac{x^3 y}{6} - \frac{(a + b)x^4}{24}.$$

Required general solution $z = \phi_1(y - ax) + \phi_2(y - bx) + (1/6) \times x^3 y - (a + b) \times (x^4/24)$,

Ex. 5. *Solve* (*a*) $(2D^2 - 5DD' + 2D'^2)z = 24(y - x)$.

(*b*) $(D^2 + 3DD' + 2D'^2)z = x + y$. **[Meerut 1996]**

(*c*) $(\partial^2 z/\partial x^2) + 3(\partial^2 z/\partial x \partial y) + 2(\partial^2 z/\partial y^2) = 2x + 3y$.

(*d*) $(\partial^2 z/\partial x^2) + 3(\partial^2 z/\partial x \partial y) + 2(\partial^2 z/\partial y^2) = 6(x + y)$.

(*e*) $(\partial^2 z/\partial x^2) - (\partial^2 z/\partial y^2) = x - y$.

Sol. (*a*) Here auxiliary equation is $2m^2 - 5m + 2 = 0$ so that $m = 2, 1/2$.

$\therefore$ C.F. $= \phi_1(y + 2x) + \phi_2(2y + x)$, ϕ_1, ϕ_2 being arbitrary functions.

$$\text{Now, P.I.} = \frac{1}{2D^2 - 5DD' + 2D'^2}\, 24(y - x) = 24\, \frac{1}{2D^2\left(1 - \dfrac{5D'}{2D} + \dfrac{D'^2}{D^2}\right)}(y - x)$$

$$=\frac{12}{D^2}\left(1-\frac{5D'}{2D}+\frac{D'^2}{D^2}\right)^{-1}(y-x)=\frac{12}{D^2}\left(1+\frac{5D'}{2D}+...\right)(y-x)=\frac{12}{D^2}\left\{(y-x)+\frac{5}{2D}D'(y-x)\right\}$$

$$=\frac{12}{D^2}\left(y-x+\frac{5}{2D}1\right)=\frac{12}{D^2}\left(y-x+\frac{5x}{2}\right)=\frac{12}{D^2}\left(y+\frac{3x}{2}\right)=\left(12y\times\frac{x^2}{2}\right)+18\times\left(\frac{x^3}{2\times 3}\right)$$

Hence the required general solution is $z=\phi_1(y+2x)+\psi_2(2y+x)+6x^2y+3x^3$.

(*b*)Try as in part (*a*). **Ans.** $z=\phi_1(y-2x)+\phi_2(y-x)-(1/3)\times x^3+(1/2)\times x^2y$

(*c*) Try yourself **Ans.** $z=\phi_1(y-x)+\phi_2(y-2x)+(3/2)\times x^2y-(7/6)\times x^3$

(*d*) Try as in part (*c*). [**Ans.** $z=\phi_1(y-x)+\phi_2(y-2x)+3x^2y-2x^3$

(*e*) Try yourself. **Ans.** $z=\phi_1(y+x)+\phi_2(y-x)+(1/6)\times x^3-(1/2)\times x^2y$

Ex. 6. *Solve* $(D^2-6DD'+9D'^2)z=12x^2+36xy$. **[Meerut 1994, Bhuj 1999, Jabalpur 2003]**

Sol. Re-writing the given equation, we get $(D-3D')^2z=12(x^2+3xy)$.

Its auxiliary equation is $(m-3)^2=0$ so that $m=3, 3$.

$\therefore$ C.F. $=\phi_1(y+3x)+x\phi_2(y+3x)$, ϕ_1, ϕ_2 being arbitrary functions.

Now, P.I.$=\dfrac{1}{(D-3D')^2}12(x^2+3xy)=12\dfrac{1}{D^2(1-3D'/D)^2}(x^2+3xy)$

[Take D common as power of y is less than that of x]

$$=\frac{12}{D^2}\left(1-\frac{3D'}{D}\right)^{-2}(x^2+3xy)=\frac{12}{D^2}\left(1+6\frac{D'}{D}+...\right)(x^2+3xy)$$

[Retain upto D' as maximum power of y in (x^2+3xy) is one]

$$=\frac{12}{D^2}\left\{x^2+3xy+\frac{6}{D}D'(x^2+3xy)\right\}=\frac{12}{D^2}\left\{x^2+3xy+\frac{6}{D}(3x)\right\}=\frac{12}{D^2}\left\{x^2+3xy+18\times\frac{x^2}{2}\right\}$$

$$=\frac{12}{D^2}(10x^2+3xy)=120\left(\frac{x^4}{3\times 4}\right)+36y\left(\frac{x^3}{2\times 3}\right)=10x^4+6x^3y.$$

Hence the required general solution is $z=\phi_1(y+3x)+x\phi_2(y+3x)+10x^4+6x^3y$.

Ex. 7. (*a*) Solve $(\partial^2V/\partial x^2)+(\partial^2V/\partial y^2)=-4\pi(x^2+y^2)$.

(*b*) Find a real function V of x and y, satisfying $(\partial^2V/\partial x^2)+(\partial^2V/\partial y^2)=-4\pi(x^2+y^2)$ and reducing to zero, when $y=0$. **[Nagpur 2005; I.A.S. 1998]**

Sol. (*a*) Given equation can be rewritten as $(D^2+D'^2)V=-4\pi(x^2+y^2)$. ...(1)

Its auxiliary equation is $m^2+1=0$ so that $m=i, -i$.

$\therefore$ C.F. $=\phi_1(y+ix)+\phi_2(y-ix)$, where ϕ_1, ϕ_2 are arbitrary functions.

P.I. $=\dfrac{1}{D^2+D'^2}[-4\pi(x^2+y^2)]=-4\pi\dfrac{1}{D^2+D'^2}(x^2+y^2)=-4\pi\dfrac{1}{D^2(1+D'^2/D^2)}(x^2+y^2)$

$$= -4\pi\frac{1}{D^2}\left(1+\frac{D'^2}{D^2}\right)^{-1}(x^2+y^2) = -\frac{4\pi}{D^2}\left(1-\frac{D'^2}{D^2}+\ldots\right)(x^2+y^2)$$

$$= -\frac{4\pi}{D^2}\left\{(x^2+y^2)-\frac{1}{D^2}D'^2(x^2+y^2)\right\} = -\frac{4\pi}{D^2}\left\{(x^2+y^2)-\frac{1}{D^2}.2\right\}$$

$$= -\frac{4\pi}{D^2}\left(x^2+y^2-2\times\frac{x^2}{2}\right) = -\frac{4\pi}{D^2}y^2 = -4\pi y^2\times\frac{x^2}{2} = -2\pi^2x^2y^2.$$

Hence the required general solution is $V = \phi_1(y+ix)+\phi_2(y-ix)-2\pi^2x^2y^2.$ ...(2)

(*b*)Proceed as in part (*a*) upto equation (2).Since we want real function $V(x, y)$ satisfying (1) and reducing to zero when $y = 0$, it follows $\phi_1(y+ix) = \phi_2(y-ix) = 0$ in (2) and hence the required solution is $V = -2\pi^2x^2y^2.$

Ex. 8. *Solve* $(D^2 - 2DD' + D'^2)z = e^{x+2y} + x^3.$ **[Bhopal 1995, 97, 98; Lucknow 2010]**

Sol. Given equation is $(D - D')^2 z = e^{x+2y} + x^3.$ ...(1)

Its auxiliary equation is $(m-1)^2 = 0$ so that $m = 1, 1.$

$\therefore$ C.F. $= \phi_1(y+x) + x\phi_2(y+x)$, ϕ_1, ϕ_2 being arbitrary functions.

Now, P.I. corresponding to e^{x+2y}

$$= \frac{1}{(D-D')^2}e^{x+2y} = \frac{1}{(1-2)^2}\int\int e^v\, dv\, dv, \text{ where } v = x+2y$$

$= \int e^v\, dv = e^v = e^{x+2y}$, as $v = x + 2y$, using formula (*i*) of working rule of Art. 4.8 and P.I. corresponding to x^3

$$= \frac{1}{(D-D')^2}x^3 = \frac{1}{D^2(1-D'/D)^2}x^3 = \frac{1}{D^2}\left(1-\frac{D'}{D}\right)^{-2}x^3 = \frac{1}{D^2}(1+\ldots)x^3 = \frac{x^5}{20}$$

Hence the required general solution is $z = \phi_1(y+x) + x\phi_2(y+x) + e^{x+2y} + x^5/20.$

Ex. 9. *Solve* $\partial^2 z/\partial x^2 - a^2(\partial^2 z/\partial y^2) = x^2.$ **[Ravishanker 2000, 04; Vikram 1995]**

Sol. Let $D \equiv \partial/\partial x$ and $D' = \partial/\partial y$. Then, the given equation becomes $(D^2 - a^2D'^2)y = x^2$

The auxiliary equation is $m^2 - a^2 = 0$ so that $m = a, -a.$

$\therefore$ $C.F. = \phi_1(y+ax) + \phi_2(y-ax)$, ϕ_1, ϕ_2 being arbitrary functions

Now, $$P.I. = \frac{1}{(D^2-a^2D'^2)}x^2 = \frac{1}{D^2\{1-a^2(D'^2/D^2)\}}x^2 = \frac{1}{D^2}\left(1-a^2\frac{D'^2}{D^2}\right)^{-1}x^2$$

$$= \frac{1}{D^2}\left(1+a^2\frac{D'^2}{D^2}+\ldots\right)x^2 = \frac{1}{D^2}x^2 = \frac{1}{D}\frac{x^3}{3} = \frac{x^4}{12}$$

Hence, the required general solution is $z = C.F. + P.I.$ i.e., $z = \phi_1(y+ax) + \phi_2(y-ax) + x^4/12$.

Ex. 10. *Solve* $\partial^3 z/\partial x^2\partial y - 2(\partial^3 z/\partial x\partial y^2) + \partial^3 z/\partial y^3 = 1/x^2$ **(Vikram 1994)**

Sol. Let $D = \partial/\partial x$ and $D' = \partial/\partial y$. Then the given equation becomes

$(D^2D' - 2DD'^2 + D'^3)z = 1/x^2$ or $(D-D')^2D'z = 1/x^2$ ...(1)

Corresponding to repeated factor $(D-D')^2$, the part of C.F. is $\phi_1(y+x) + x\phi_2(y+x)$. Again corresponding to factor D', the part of *C.F.* is $f_1(x)$.

$\therefore \qquad C.F. \text{ of } (1) = \phi_1(y+x)+x, \ \phi_2(y+x)+\phi_3(x)$

$$\text{P.I.} = \frac{1}{(D-D')^2 D'}\frac{1}{x^2} = \frac{1}{(D-D')^2}\int \frac{1}{x^2}\,dy = \frac{1}{(D-D')^2}\frac{y}{x^2} = \frac{1}{D^2}\left(1-\frac{D'}{D}\right)^{-2}\frac{y}{x^2}$$

$$= \frac{1}{D^2}\left(1+\frac{2D'}{D}+\frac{3D'^2}{D^2}+\dots\right)\frac{y}{x^2} = \frac{1}{D^2}\left\{\frac{y}{x^2}+\frac{2}{D}\left(\frac{1}{x^2}\right)\right\} = y\frac{1}{D^2}\frac{1}{x^2}+\frac{2}{D^3}\frac{1}{x^2}$$

$$= y\frac{1}{D}\left(-\frac{1}{x}\right) = -y\log x,$$

where we have omitted a function of x as it can be included in the term $\phi_3(x)$ of C.F.

$\therefore$ Required general solution is $\qquad z = \phi_1(y+x)+x\phi_2(y+x)+\phi_3(x)-y\log x,$

where ϕ_1, ϕ_2 and ϕ_3 are arbitrary functions.

Ex. 11. *Solve* $(D^3-7DD'^2-6D'^3)z = x^2+xy^2+y^3+\cos(x-y)$ **[K.U. Kurukshetra, 2004]**

Sol. Given $\qquad (D^3-7DD'^2-6D'^3)\,z = x^2+xy^2+y^3+\cos(x-y) \qquad ...(1)$

Here auxiliary equation is $\qquad m^3-7m-6=0 \qquad$ so that $\qquad m=-1, -2, 3.$

$\therefore C.F. = \phi_1(y-x)+\phi_2(y-2x)+\phi_3(y+3x)$, ϕ_1, ϕ_2, ϕ_3 being arbitrary functions

P.I. Corresponding to $(x^2+xy^2+y^3)$

$$= \frac{1}{D^3-7DD'^2-6D'^3}(x^2+xy^2+y^3) = \frac{1}{D^3}\left\{1-\left(7\frac{D'^2}{D^2}+6\frac{D'^3}{D^3}\right)\right\}^{-1}(x^2+xy^2+y^3)$$

$$= \frac{1}{D^3}\left\{1+\left(7\frac{D'^2}{D^2}+6\frac{D'^3}{D^3}\right)+\dots\right\}(x^2+xy^2+y^3) = \frac{1}{D^3}(x^2+xy^2+y^3)+\frac{7}{D^5}(2x+6y)+\frac{36}{D^6}1$$

$$= (x^5/60+x^4y^2/24+x^3y^3/6)+7(x^6/360+x^5y/20)+36\times(x^6/720)$$

$$= 5x^6/72+x^5/60+7x^5y/20+x^4y^2/24+x^3y^3/6$$

P.I. Corresponding to $\cos(x-y)$

$$= \frac{1}{D^3-7DD'^2-6D'^3}\cos(x-y) = \frac{1}{(D+D')}\frac{1}{(D^2-DD'-6D'^2)}\cos(x-y)$$

$$= \frac{1}{D+D'}\frac{1}{1^2-4\times 1\times(-1)-6\times(-1)^2}\iint \cos v\,dv\,dv, \text{ where } v = x-y$$

$$= \frac{1}{D+D'}\times\frac{1}{(-4)}\times(-\cos v) = \frac{1}{4}\frac{1}{D+D'}\cos(x-y) - \frac{1}{4}\frac{1}{(-1)\times D-1\times D'}\cos(x-y) = -\frac{1}{4}\frac{x}{(-1)^1\times 1!}\cos(x-y)$$

$$= (x/4)\times\cos(x-y)$$

Hence the required general solution is $\qquad z = \phi_1(y-x)+\phi_2(y-2x)+\phi_3(y+3x)$

$$+(5/72)\times x^6+x^5/60+(7/20)\times x^5y+(1/24)\times x^4y^2+(1/6)\times(x^3y^3)+(x/4)\times\cos(x-y),$$

Ex 12. *Solve* $(D^2 - 2DD' - 15D'^2)z = 12xy$. **(K.U. Kurukshetra 2004; Meerut 2006, 2011)**

Sol. Here auxiliary equation is $m^2 - 2m - 15 = 0$ so that $m = 5, -3$.

$\therefore$ $C.F. = \phi_1(y+5x) + \phi_2(y-3x)$, ϕ_1, ϕ_2 being arbitrary functions

Now, $$\text{P.I.} = \frac{1}{D^2 - 2DD' - 15D'^2}(12xy) = \frac{1}{D^2(1 - 2D'/D - 15D'^2/D^2)}(12xy)$$

$$= \frac{12}{D^2}\left\{1 - \left(\frac{2D'}{D} + \frac{15D'^2}{D^2}\right)\right\}^{-1}(xy) = \frac{12}{D^2}\left(1 + \frac{2D'}{D} + \frac{15D'^2}{D^2} + \dots\right)(xy)$$

$$= \frac{12}{D^2}\left\{xy + \frac{2}{D}D'(xy) + \frac{15}{D^2}D'^2(xy) + \dots\right\} = \frac{12}{D^2}\left(xy + \frac{2}{D}x\right)$$

$$= 12y\frac{1}{D^2}x + \frac{24}{D^3}x = 12y\left(\frac{x^3}{6}\right) + 24\left(\frac{x^4}{24}\right) = 2x^3y + x^4$$

Hence the required general solution is $z = \phi_1(y+5x) + \phi_2(y-3x) + 2x^3y + x^4$

Ex. 13. *Solve* $\partial^3u/\partial x^3 + \partial^3u/\partial y^3 + \partial^3u/\partial z^3 - 3(\partial^3u/\partial x\,\partial y\,\partial z) = x^3 + y^3 + z^3 - 3xyz$.

Sol. Let $D = \partial/\partial x$, $D' = \partial/\partial y$, $D'' = \partial/\partial z$. Then the given equation can be re-written as

$$(D^3 + D'^3 + D''^3 - 3DD'D'')u = x^3 + y^3 + z^3 - 3xyz.$$

or $$(D + D' + D'')(D + \omega D' + \omega^2 D'')(D + \omega^2 D' + \omega D'')u = x^3 + y^3 + z^3 - 3xyz,$$

where ω is a complex cube root of unity. ...(1)

For C.F., let us consider $(D + \omega^2 D' + \omega D'')u = 0.$ ...(2)

Subsidiary equations of (2) are $\dfrac{dx}{1} = \dfrac{dy}{\omega^2} = \dfrac{dz}{\omega} = \dfrac{du}{0}$. ...(3)

Three independent integrals of (3) are

$y - \omega^2 x$ = constant, $z - \omega x$ = constant and u = constant.

Hence, general solution of (2) is $u = \phi_1(y - \omega^2 x, z - \omega x)$.

Similarly, the contributions to complementary function corresponding to other factors in (1) are

$\phi_2(y - \omega x, z - \omega^2 x)$ and $\phi_3(y - x, z - x)$ and hence

C.F. $= \phi_1(y - \omega^2 x, z - \omega x) + \phi_2(y - \omega x, z - \omega^2 x) + \phi_3(y - x, z - x)$,

where ϕ_1, ϕ_2 and ϕ_3 are arbitrary functions.

P.I. corresponding to x^3

$$= \frac{1}{D^3 + D'^3 + D''^3 - 3DD'D''}x^3 = \frac{1}{D^3}\left\{1 + \left(\frac{D'^3}{D^3} + \dots\right)\right\}^{-1}x^3 = \frac{1}{D^3}x^3 = \frac{x^6}{120}.$$

Similarly, P.I. corresponding to $y^3 = y^6/120$ and P.I. corresponding to $z^3 = z^6/120$.

Finally, P.I. corresponding to $(-3xyz)$

$$= -3\frac{1}{D^3 + D'^3 + D''^3 - 3DD'D''}xyz = \frac{1}{DD'D''}\left\{1 + \frac{D^3 + D'^3 + D''^3}{3DD'D''}\right\}^{-1}xyz$$

$$= \frac{1}{DD'D''}\left\{1 - \frac{D^3 + D'^3 + D''^3}{3DD'D''} + \dots\dots\right\}xyz = \frac{1}{DD'D''}xyz = \frac{x^2y^2z^2}{8}.$$

Hence the required general solution of (1) is $z = \phi_1(y - \omega^2 x, z - \omega x) + \phi_2(y - \omega x, z - \omega^2 x)$
$+ \phi_3(y - x, z - x) + (1/120) \times (x^6 + y^6 + z^6) + (1/8) \times x^2y^2z^2.$

Ex.14. *Solve* $\dfrac{\partial^3 u}{\partial x^3} + \dfrac{\partial^3 u}{\partial y^3} + \dfrac{\partial^3 u}{\partial z^3} - 3\dfrac{\partial^3 u}{\partial x \partial y \partial z} = x^3 - 3xyz.$

Sol. Proceed as in Ex. 13. Its general solution is $z = \phi(y - w^2x, z - wx) + \phi_2(y - wx, z - w^2x) + \phi_3(y - x, z - x) + (1/120)x^6 + (1/8)x^2y^2z^2$.

EXERCISE 4(B)

Solve the following partial differential equations:

1. $(D^2 - 2DD' + D'^2)z = 12xy$ **(Jiwaji 1998; Ravishanker 1999; Vikram 1995, 97)**

Ans. $z = \phi_1(y + x) + x\,\phi_2(y + x) + 2x^3y + x^4$, ϕ_1, ϕ_2 being arbitrary functions

2. $(D^2 - DD' - 6D'^2)z = xy$ **(Sagar 2003, Vikram 1995)**

Ans. $z = \phi_1(y - 2x) + \phi_2(y + 3x) + (1/6) \times x^3y + (1/24) \times x^4$

4.12. A general method of finding the particular integral of linear homogeneous equation with constant coefficients.

Let the given equation be $F(D, D')z = f(x, y)$, ...(1)

where $F(D, D')$ is a homogeneous function of D and D' of degree n, (say) so that

$$F(D, D') = (D - m_1D')(D - m_2D') \ldots (D - m_nD').$$

$$\therefore \quad \text{P.I. of (1)} = \frac{1}{F(D, D')} f(x, y) = \frac{1}{(D - m_1D')(D - m_2D') \ldots (D - m_nD')} f(x, y). \qquad \ldots(2)$$

In order to evaluate P.I. given by (2), we consider a solution of the following equation :

$(D - mD')z = f(x, y)$ or $p - mq = f(x, y)$, ...(3)

which is of the form $Pp + Qq = R$. So Lagrange's auxiliary equation for (3) are

$$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{f(x, y)}. \qquad \ldots(4)$$

Taking the first two fractions of (4), $dy + mdx = 0$ so that $y + mx = c$. ...(5)

Next, taking the first and the last fractions of (4), we have

$$dz = f(x, y)dx = f(x, c - mx)dx, \text{ as from (5), } y = c - mx$$

Integrating, $z = \int f(x, c - mx)\, dx$.

Thus, $$z = \frac{1}{D - mD'} f(x, y) = \int f(x, c - mx)\, dx, \qquad \ldots(6)$$

where after integration the constant c must be replaced by $y + mx$ since the P.I. does not contain any arbitrary constant.

Hence the P.I. given by (2) can be obtained by applying the operation (6) by the factors, in succession, starting from the right.

Working rule for finding P.I. (General method) of F(D, D′)z = f(x, y).

$$\text{P.I.} = \frac{1}{(D - m_1D')(D - m_2D') \ldots (D - m_nD')} f(x, y) \qquad \ldots(7)$$

We shall use one of the following formulas :

Formula I : $\dfrac{1}{D - mD'} f(x, y) = \int f(x, c - mx)\, dx$, where $c = y + mx$. ...(8)

Formula II : $\dfrac{1}{D + mD'} f(x, y) = \int f(x, c + mx)\, dx$, where $c = y - mx$. ...(9)

Hence in order to evaluate P.I. (7), we apply (8) or (9) depending on the factor $D - mD'$ and $D + mD'$. Note that result (9) can be obtained from (8) by replacing m by $-m$.

4.13 SOLVED EXAMPLES BASED ON GENERAL METHOD

Ex. 1. *Solve* $(\partial z/\partial x) + (\partial z/\partial y) = \sin x$.

Sol. Rewritting, the given equation is $(D + D')z = \sin x$. ...(1)

Its auxiliary equation is $m + 1 = 0$ so that $m = -1$.

$\therefore$ C.F. $= \phi(y - x)$, where ϕ is an arbitrary function.

and P.I. $= \dfrac{1}{D + D'} \sin x = \int \sin x \, dx = -\cos x$

Hence the required solution is $z =$ C.F. + P.I. $= \phi(y - x) - \cos x$.

Ex. 2. *Solve* (*a*) $(D^2 - DD' - 2D'^2)z = (y - 1)e^x$.

[Delhi Maths (H) 2004, 10; Bhopal 2004; Jiwaji 2000; Rewa 2003; Vikram 2002, 04]

(*b*) $(D - D')(D + 2D')z = (y + 1)e^x$. **[Delhi Maths (H) 1993; I.A.S. 2004]**

Sol. (*a*) Here given $(D^2 - DD' - 2D'^2)z = (y - 1)e^x$ or $(D + D')(D - 2D')z = (y - 1)e^x$.

Its auxiliary equation is $(m + 1)(m - 2) = 0$ so that $m = -1, 2$.

$\therefore$ C.F. $= \phi_1(y - x) + \phi_2(y + 2x)$, ϕ_1, ϕ_2 being arbitrary functions.

P.I. $= \dfrac{1}{(D + D')(D - 2D')}(y - 1)e^x = \dfrac{1}{D + D'}\left\{\dfrac{1}{D - 2D'}(y - 1)e^x\right\}$

$= \dfrac{1}{D + D'} \int (c - 2x - 1)e^x \, dx$, by formula I of working rule of Art. 4.12. and taking $c = y + 2x$

$= \dfrac{1}{D + D'}\left[(c - 2x - 1)e^x - \int (-2)e^x \, dx\right]$, integrating by parts

$= \dfrac{1}{D + D'}[(c - 2x - 1)e^x + 2e^x] = \dfrac{1}{D + D'}(c - 2x + 1)e^x$

$= \dfrac{1}{D + D'}\{(y + 2x) - 2x + 1\}e^x$, replacing c by $y + 2x$ $= \dfrac{1}{D + D'}(y + 1)e^x$

$= \int (c' + x + 1)e^x \, dx$, by formula II of working rule of Art. 4.12 and taking $c' = y - x$

$= (c' + x + 1)\, e^x - \int (1.e^x)\, dx = (c' + x + 1)\, e^x - e^x = ye^x$, since $c' = y - x$

Hence the required general solution is $z =$ C.F. + P.I. or $z = \phi_1(y - x) + \phi_2(y + 2x) + ye^x$.

(*b*) Here auxiliary equation is $(m - 1)(m + 2) = 0$ so that $m = 1, -2$.

$\therefore$ C.F.$= \phi_1(y + x) + \phi_2(y - 2x)$, ϕ_1, ϕ_2 being arbitrary functions.

and P.I. $= \dfrac{1}{(D - D')(D + 2D')}(y + 1)\, e^x = \dfrac{1}{D - D'}\left\{\dfrac{1}{D + 2D'}(y + 1)\, e^x\right\}$

$= \dfrac{1}{D - D'}\int (c + 2x + 1)\, e^x \, dx = \dfrac{1}{(D - D')}\{(c + 2x + 1)\, e^x - 2e^x\}$, where $c = y - 2x$

$= \dfrac{1}{D - D'}(y - 1)\, e^x = \int (c' - x - 1)\, e^x \, dx$, where $c' = y + x$

$= (c' - x - 1)e^x + e^x = ye^x$ as $c' = y + x$.

$\therefore$ General solution is $y = \phi_1(y + x) + \phi_2(y - 2x) + ye^x$.

Ex. 3. $(D^2 - 4D'^2)z = (4x/y^2) - (y/x^2)$. **[Delhi Maths (H) 2004, 08; Meerut 1992, Bhopal 2010]**

Sol. Here auxiliary equation is $m^2 - 4 = 0$ so that $m = 2, -2$.

$\therefore$ C.F. $= \phi_1(y + 2x) + \phi_2(y - 2x)$, ϕ_1, ϕ_2 being arbitrary functions

P.I .$= \dfrac{1}{(D + 2D')(D - 2D')}\left(\dfrac{4x}{y^2} - \dfrac{y}{x^2}\right) = \dfrac{1}{D + 2D'}\int\left\{\dfrac{4x}{(c - 2x)^2} - \dfrac{c - 2x}{x^2}\right\} dx$, where $c = y + 2x$.

$$= \frac{1}{D+2D'}\int\left\{-\frac{2}{c-2x}+\frac{2c}{(c-2x)^2}-\frac{c}{x^2}+\frac{2}{x}\right\}dx = \frac{1}{D+2D'}\left\{\log(c-2x)+\frac{c}{c-2x}+\frac{c}{x}+2\log x\right\}$$

$$= \frac{1}{D+2D'}\left[\log y+\frac{y+2x}{y}+\frac{y+2x}{x}+2\log x\right]$$

$$= \int\left\{\log(c'+2x)+1+2\frac{x}{c'+2x}+\frac{c'+2x}{x}+2+2\log x\right\}dx, \text{ taking } c'=y-2x$$

$= x\log(c'+2x)+5x+c\log x+2x\log x-2x \quad = x\log y+y\log x+3x$, as $c'=y-2x$

$\therefore$ The required solution $\qquad z=\phi_1(y+2x)+\phi_2(y-2x)+x\log y+y\log x+3x$

Ex. 4. *Solve* $(D^2-DD'-2D'^2)z=(2x^2+xy-y^2)\sin xy-\cos xy$.

[I.A.S 2009; Meerut 1994; Delhi Maths (Hons.) 2007]

Sol. Here auxiliary equation is $\quad m^2-m-2=0 \quad$ so that $\quad m=2,-1$.

$\therefore$ C.F. $=\phi_1(y+2x)+\phi_2(y-x)$, ϕ_1, ϕ_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{(D-2D')}\frac{1}{D+D'}\{(2x^2+xy-y)\sin xy-\cos xy\}$$

$$= \frac{1}{D-2D'}\frac{1}{D+D'}\{(2x-y)(x+y)\sin xy-\cos xy\}$$

$$= \frac{1}{D-2D'}\int\{(x-c)(2x+c)\sin x(c+x)-\cos x(c+x)\}\,dx, \text{ taking } c=y-x$$

$$= \frac{1}{D-2D'}\int\{(x-c)(2x+c)\sin(cx+x^2)-\cos(cx+x^2)\}\,dx$$

$$= \frac{1}{D-2D'}\left[-(x-c)\cos(cx+x^2)+\int\cos(cx+x^2)\,dx-\int\cos(cx+x^2)\,dx\right]$$

$$= \frac{1}{D-2D'}(y-2x)\cos xy, \text{ as } c=y-x$$

$= \int(c'-4x)\cos(c'x-2x^2)\,dx$, where $c'=y+2x$

$= \int\cos t\,dt = \sin t$, putting $c'x-2x^2=t$ so that $(c'-4x)dx=dt$

$= \sin(c'x-2x^2)=\sin xy$, as $c'=y+2x$.

$\therefore$ Required solution is $\qquad z=\phi_1(y+2x)+\phi_2(y-x)+\sin xy$.

Ex. 5. *Solve* (*a*) $r+s-6t=y\cos x$. or $\quad (D^2+DD'-6D'^2)z=y\cos x$

[Bilaspur 2002, Indore 2002, Jabalpur 1999 Meerut 2000, 02, Ravishankar 1994, Jiwaji 1999, Garhwal 2005, 10; I.A.S.1992, 2008; Vikram 1999, Delhi Maths (Hons) 2007, Purvanchal 2007; Kanpur 2011]

(*b*) $(D^2+DD'-6D'^2)z=y\sin x$.

Sol. (*a*) Since $r=\partial^2 z/\partial x^2$, $s=\partial^2 z/\partial x\partial y$, $t=\partial^2 z/\partial y^2$, the given equation becomes

$\partial^2 z/\partial x^2+\partial^2 z/\partial x\,\partial y-6(\partial^2 z/\partial y^2)=y\cos x \quad$ or $\quad (D^2+DD'-6D'^2)z=y\cos x$ ....(1)

Its auxiliary equation is $\quad m^2+m-6=0 \quad$ so that $\quad m=2,-3$.

$\therefore$ C.F. $=\phi_1(y+2x)+\phi_2(y-3x)$, ϕ_1, ϕ_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{D^2+DD'-6D'^2}y\cos x = \frac{1}{(D-2D')(D+3D')}y\cos x$$

$$= \frac{1}{D-2D'}\int(3x+c)\cos x\,dx, \text{ where } c=y-3x$$

$= \dfrac{1}{D-2D'}\left[(3x+c)\sin x - \int 3\sin x\, dx\right]$, integrating by parts

$= \dfrac{1}{D-2D'}[y \sin x + 3\cos x]$, as $c = y - 3x$

$= \int [(c' - 2x)\sin x + 3\cos x]\, dx$, where $c' = y + 2x$

$= (c' - 2x)(-\cos x) - \int (-2)(-\cos x)\, dx + 3\sin x$, integrating by parts

$= y(-\cos x) - 2\sin x + 3\sin x$, as $c' = y + 2x$

$= \sin x - y\cos x$.

$\therefore$ General solution is $\qquad z = \phi_1(y+2x) + \phi_2(y-3x) + \sin x - y\cos x$.

(b) Proceed as in part (a). **Ans.** $z = \phi_1(y+2x) + \phi_2(y-3x) - y\sin x - \cos x$

Ex. 6. *Solve* $(D^2 + 2DD' + D'^2)z = 2\cos y - x\sin y$. **[Agra 2009; Meerut 1999; Bilaspur 2002; Indore 2004; Jabalpur 1999; Rewa 2002; Ranchi 2010]**

Sol. Given equation is $\qquad (D + D')^2 z = 2\cos y - x\sin y$. $\qquad$ ...(1)

Its auxiliary equation is $\quad (m+1)^2 = 0 \quad$ so that $\quad m = -1, -1$.

$\therefore$ C.F. $= \phi_1(y - x) = x\phi_2(y-x)$, ϕ_1, ϕ_2 being arbitrary functions.

P.I. $= \dfrac{1}{D+D'}\dfrac{1}{D+D'}(2\cos y - x\sin y) = \dfrac{1}{D+D'}\int [2\cos(x+c) - x\sin(x+c)]\, dx$, where $c = y - x$

$= \dfrac{1}{D+D'}\left[2\int \cos(x+c)\, dx - \int x\sin(x+c)\, dx\right] = \dfrac{1}{D+D'}\left[2\sin(x+c) - \left\{-x\cos(x+c) + \int \cos(x+c)\, dx\right\}\right]$

$= \dfrac{1}{D+D'}[2\sin(x+c) + x\cos(x+c) - \sin(x+c)] = \dfrac{1}{D+D'}(\sin y + x\cos y)$, as $c = y - x$

$= \int [\sin(x+c') + x\cos(x+c')]\, dx = -\cos(x+c') + x\sin(x+c') - \int \{1.\sin(x+c')\}\, dx$, where $c' = y - x$

$= -\cos(x+c') + x\sin(x+c') + \cos(x+c') = x\sin y$, as $c' = y - x$.

So the required solution is $\qquad z = \phi_1(y-x) + x\phi_2(y-x) + x\sin y$.

Ex. 7. *Solve* $r - t = \tan^3 x \tan y - \tan x \tan^3 y$ or $(D^2 - D'^2)z = \tan^3 x \tan y - \tan x \tan^3 y$ **[Agra 2010; Delhi Maths (G) 2006]**

Sol. Given equation is $\qquad (D^2 - D'^2)z = \tan^3 x \tan y - \tan x \tan^3 y$

or $\qquad (D + D')(D - D')z = \tan^3 x \tan y - \tan x \tan^3 y$. $\qquad$ ...(1)

Its auxiliary equation is $\quad (m+1)(m-1) = 0 \quad$ so that $\quad m = -1, 1$.

$\therefore$ C.F. $= \phi_1(y - x) + \phi_2(y + x)$, ϕ_1, ϕ_2 being arbitrary functions.

P.I. $= \dfrac{1}{(D+D')(D-D')}(\tan^3 x \tan y - \tan x \tan^3 y)$

$= \dfrac{1}{D+D'}\int [\tan^3 x \tan(c-x) - \tan x \tan^3 (c-x)]\, dx$, where $c = y + x$

$= \dfrac{1}{D+D'}\int [\tan x \tan(c-x)(\sec^2 x - 1) - \tan x \tan^2(c-x)\{\sec^2 (c-x) - 1\}]\, dx$

$= \dfrac{1}{D+D'}\int [\tan x \sec^2 x \tan(c-x) - \tan(c-x)\sec^2(c-x)\tan x]\, dx$

$$= \frac{1}{D+D'}\left[\frac{\tan^2 x}{2}\tan(c-x) - \int \frac{\tan^2 x}{2}\sec^2(c-x).(-1)\, dx - \left\{\frac{\tan^2(c-x)}{2\times(-1)}\tan x - \int \frac{\tan^2(c-x)}{2\times(-1)}\sec^2 x\, dx\right\}\right]$$

$$= \frac{1}{2(D+D')}[\tan^2 x \tan(c-x) + \tan x \tan^2(c-x)]$$

$$+ \int (\sec^2 x - 1)\sec^2(c-x)\,dx - \int \{\sec^2(c-x)-1\}\sec^2 x\,dx]$$

$$= \frac{1}{2(D+D')}[\tan^2 x \tan(c-x) + \tan x \tan^2(c-x) - \int \sec^2(c-x)\,dx + \int \sec^2 x\,dx]$$

$$= \frac{1}{2(D+D')}[\tan^2 x \tan(c-x) + \tan x \tan^2(c-x) + \tan(c-x) + \tan x]$$

$$= \frac{1}{2(D+D')}(\tan^2 x \tan y + \tan x \tan^2 y + \tan y + \tan x), \text{ as } c = y + x$$

$$= \frac{1}{2(D+D')}[\tan y(\tan^2 x + 1) + \tan x(\tan^2 y + 1)] = \frac{1}{2(D+D')}(\tan y \sec^2 x + \tan x \sec^2 y)$$

$$= \frac{1}{2}\left[\int \tan(c'+x)\sec^2 x\,dx + \int \tan x \sec^2(c'+x)\,dx\right], \text{ where } c' = y - x$$

$$= \frac{1}{2}\left[\tan(c'+x)\tan x - \int \sec^2(c'+x)\tan x\,dx + \int \tan x \sec^2(c'+x)\,dx\right]$$

[On integrating the first integral by parts keeping the second integral unchanged]

$= (1/2) \times \tan(c'+x)\tan x = (1/2) \times \tan y \tan x$, as $c' = y - x$.

$\therefore$ The required solution is $\qquad z = \phi_1(y-x) + \phi_2(y+x) + (1/2) \times \tan y \tan x$.

Ex. 8. *Solve* $(D^2 + DD' - 6D'^2)z = x^2 \sin(x+y)$. **[Meerut 1994]**

Sol. Re-writing the given equation is $\qquad (D+3D')(D-2D')z = x^2 \sin(x+y)$. ...(1)

$\therefore$ C.F. $= \phi_1(y-3x) + \phi_2(y+2x)$, ϕ_1, ϕ_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{(D+3D')(D-2D')}x^2 \sin(x+y) = \frac{1}{D+3D'}\cdot\left\{\frac{1}{D-2D'}, x^2 \sin(x+y)\right\}$$

$$= \frac{1}{D+3D'}\int x^2 \sin(x+c-2x)\,dx = \frac{1}{D+3D'}\int x^2 \sin(c-x)\,dx, \text{ where } c = y + 2x$$

$$= \frac{1}{D+3D'}\left[x^2\cos(c-x) - \int 2x\cos(c-x)\,dx\right], \text{ integrating by parts}$$

$$= \frac{1}{D+3D'}\left[x^2\cos(c-x) - \left\{-2x\sin(c-x) + \int 2\sin(c-x)\,dx\right\}\right], \text{ integrating by parts}$$

$$= \frac{1}{D+3D'}[x^2\cos(c-x) + 2x\sin(c-x) - 2\cos(c-x)]$$

$$= \frac{1}{D+3D'}[(x^2-2)\cos(x+y) + 2x\sin(x+y)], \text{ as } c = y + 2x$$

$$= \int [(x^2-2)\cos(x+c'+3x) + 2x\sin(x+c'+3x)]\,dx, \text{ where } c' = y - 3x$$

$$= \int (x^2-2)\cos(4x+c')\,dx + 2\int x\sin(4x+c')\,dx$$

$$= (x^2-2)\frac{\sin(4x+c')}{4} - \int 2x\frac{\sin(4x+c')}{4}\,dx + 2\int x\sin(4x+c')\,dx$$

[Integrating by part 1st integral and keeping the second integral unchanged]

$$= \frac{1}{4}(x^2-2)\sin(4x+c') + \frac{3}{2}\int x\sin(4x+c')\,dx = \frac{x^2-2}{4}\sin(4x+c') + \frac{3}{2}\left[-\frac{x\cos(4x+c')}{4} + \int \frac{\cos(4x+c')}{4}\,dx\right]$$

$$= \frac{x^2-2}{4}\sin(4x+c') - \frac{3}{8}x\cos(4x+c') + \frac{3}{32}\sin(4x+c')$$

$$= \frac{1}{4}(x^2-2)\sin(4x+y-3x) - \frac{3}{8}x\cos(4x+y-3x) + \frac{3}{32}\sin(4x+y-3x), \text{ as } c' = y - 3x$$

$= \left(x^2/4 - 13/32\right)\sin(x+y) - (3x/8)\times\cos(x+y)$, on simplification

The solution is $z = \phi_1(y-3x) + \phi_2(y+2x) + [(x^2/4) - (13/32)]\sin(x+y) - (3x/8)\times\cos(x+y)$.

Ex. 9. *Solve* $(D^3 + D^2D' - DD'^2 - D'^3)\, z = e^y \cos 2x$

Sol. Here $D^3 + D^2D' - DD'^2 - D'^3 = D^2(D+D') - D'^2(D+D') = (D+D')^2(D-D')$

So the given equation reduces to $(D-D')\,(D+D')^2 z = e^y \cos 2x$

$\therefore$ C.F. $= \phi_1(y+x) + \phi_2(y-x) + x\,\phi_3(y-x)$, ϕ_1, ϕ_2 being arbitrary functions

$$\text{P.I.} = \frac{1}{(D-D')\,(D+D')}\frac{1}{D+D'}\, e^y \cos 2x = \frac{1}{(D-D')\,(D+D')}\int e^{a+x}\cos 2x\, dx, \text{ where } y-x=a$$

$$= \frac{1}{(D-D')(D+D')} e^a \int e^x \cos 2x\, dx = \frac{1}{(D-D')(D+D')} e^{y-x}\frac{1}{1^2+2^2} e^x(\cos 2x + 2\sin 2x)^{*}$$

$$= \frac{1}{5}\frac{1}{(D-D')\,(D+D')} e^y(\cos 2x + 2\sin 2x) = \frac{1}{5}\frac{1}{D-D'}\int e^{x+a}(\cos 2x + 2\sin 2x)\, dx, \text{ where } y-x=a$$

$$= \frac{1}{5}\frac{1}{D-D'}\, e^a\left\{\int e^x \cos 2x\, dx + 2\int e^x \sin 2x\, dx\right\}$$

$$= \frac{1}{5}\frac{1}{D-D'}\, e^{y-x}\left\{\frac{e^x}{1^2+2^2}(\cos 2x + 2\sin 2x) + \frac{2e^x}{1^2+2^2}(\sin 2x - 2\cos 2x)\right\}$$

$$= \frac{1}{25}\frac{1}{D-D'}\, e^y(4\sin 2x - 3\cos 2x) = \frac{1}{25}\int e^{b-x}(4\sin 2x - 3\cos 2x)\, dx, \text{ where } b = y+x$$

$$= \frac{1}{25}\, e^b\left\{4\int e^{-x}\sin 2x\, dx - 3\int e^{-x}\cos 2x\, dx\right\}$$

$$= \frac{1}{25}\, e^{y+x}\left\{\frac{4e^{-x}}{1^2+2^2}(-\sin 2x - 2\cos 2x) - \frac{3e^{-x}}{1^2+2^2}(-\cos 2x + 2\sin 2x)\right\}$$

$$= -(1/25)\times e^y\,(\cos 2x + 2\sin 2x)$$

$\therefore$ Required solution is $z = \phi_1(y+x) + \phi_2(y-x) + x\,\phi_3(y-x) - (e^y/25)\times(\cos 2x + 2\sin 2x)$

Ex. 10. *Find the solution of the equation* $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 = e^{-x}\cos y$ *which* $\to 0$ *as* $x \to \infty$ *and has the value cos y when x = 0.* **[I.A.S. 1999]**

Sol. Given $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 = e^{-x}\cos y$ or $(D^2 + D'^2)z = e^{-x}\cos y$ ... (1)

$$\text{P.I.} = \frac{1}{D^2+D'^2} e^{-x}\cos y = \frac{1}{D^2+D'^2}\, e^{(-1)x+0.y}\cos y = e^{(-1)x+0.y}\frac{1}{(D-1)^2+(D'+0)^2}\cos y$$

$$= e^{-x}\frac{1}{D^2+D'^2-2D-1}\cos y = e^{-x}\frac{1}{0^2+(-1)^2-2D+1}\cos y = e^{-x}\frac{1}{-2D}\cos y = -\frac{1}{2}x e^{-x}\cos y.$$

Now, the general solution of (1) is z = C.F. + P.I.

where C.F. is solution of $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 = 0$ ... (2)

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* We shall use the following results of Integral calculus directly

$$\int e^{ax}\sin bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\sin bx - b\cos bx), \text{ and } \int e^{ax}\cos bx\, dx = \frac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$$

Since $$\lim_{x\to\infty} x e^{-x} = \lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0,$$

we observe that P.I. $\to$ 0 as $x \to \infty$. Also, we have P.I. = 0 when $x = 0$

Here we are to solve (1) satisfying the conditions $z \to 0$ as $x \to \infty$ and $z = \cos y$ when $x = 0$. It follows that C.F. of (1), that is solution of (2) must satisfy the conditions C.F. $\to 0$ as $x \to \infty$ and C.F. = cos y when $x = 0$. In other words, we now * solve (2) subject to conditions :

$$z(x, y) \to 0 \text{ as } x \to \infty \qquad \text{... (3)}$$

and $$z(x, y) = \cos y \text{ when } x = 0 \qquad \text{... (4)}$$

Let a solution of (2) be $$z(x, y) = X(x),\ Y(y) \qquad \text{... (5)}$$

From (5), $\partial^2 z/\partial x^2 = X''(x)\, Y(y)$ and $\partial^2 z/\partial y^2 = X(x)\, Y''(y)$,

where prime denotes the derivative w.r.t. to the relevent variable. Substituting these in (2), we get

$$X''(x)\ Y(y) + X(x)\ Y''(y) = 0 \quad \text{or} \quad X''/X = -Y''/Y \qquad \text{... (6)}$$

Since x and y are independent variables, (6) is true if each side is equal to a constant, say n^2. Since condition (4) involves trigonometric function cos y, we choose n as positive integer.

$\therefore$ (6) $\Rightarrow$ $X''/X = n^2$ and $-Y''/Y = n^2$

or $d^2X/dx^2 - n^2X = 0$ and $d^2Y/dy^2 + n^2Y = 0$

Solving these, $$X(x) = A\, e^{nx} + Be^{-nx} \qquad \text{... (7)}$$

and $$Y(y) = C\cos ny + D\sin ny \qquad \text{... (8)}$$

Take $A = 0$ in (7) for otherwise $X(x) \to \infty$ as $x \to \infty$ and so $z(x, y) \to \infty$ as $x \to \infty$ which contradicts (3) Using (7) and (8), (5) reduces to

$$z(x, y) = e^{-nx}(E\cos ny + F\sin ny) \qquad \text{... (9)}$$

where E (= BC) and F (= BD) are new arbitrary constants.

Now, putting $x = 0$ in (9) and using (4), we get cos y = E cos ny + F sin ny which holds if we choose $n = 1$, $E = 1$ and $F = 0$. Hence from (9), the C.F. of (1) is given by $e^{-x}\cos y$. Keeping in mind this C.F. of (1) and P.I. already obtained, the required solution of (1) is

$$z = e^{-x}\cos y - (x/2) \times e^{-x}\cos y = (1/2) \times (2 - x)\, e^{-x}\cos y.$$

EXERCISE 4(C)

Solve the following partial differential equations:

1. $(D^3 + 2D^2D - DD'^2 - 2D'^3)\ z = (y+2)e^x$ **Ans.** $z = \phi_1(y+x) + \phi_2(y-x) + \phi_3(y-2x) + ye^y$

2. $(D^3 - 3DD'^2 - 2D'^3)z = \cos(x+2y) - e^y(3+2x)$

Ans. $z = \phi_1(y-x) + x\,\phi_2(y-x) + \phi_3(y+2x) + (1/27)\times\sin(y+2x) + xe^x$.

3. $(D^3 - D^2D' - 2DD'^2)z = e^{x+2y}(x^2 + 4y^2)$.

Ans. $z = \phi_1(y) + \phi_2(y+2x) + \phi_3(y-x)\ -(1/81)\times(9x^2 + 36y^2 - 18x - 72y + 76)\ e^{x+2y}$

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* We shall use the method of separation for solving partial differential equation. For details refer part III "Boundary value problems" in author's "Advanced Differential Equations."

4.14. SOLUTIONS UNDER GIVEN GEOMETRICAL CONDITIONS:

We have seen that solutions obtained in above methods involve arbitrary functions of x and y. We shall now determine these under the given geometrical conditions. This will lead to required surface satisfying the given differential equation under the given geometrical conditions.

4.15. SOLVED EXAMPLES BASED ON ART. 4.14.

Ex. 1. *Find a surface passing through the two lines* $z = x = 0$, $z - 1 = x - y = 0$ *satisfying* $r - 4s + 4t = 0$. **[Meerut 97,2000; I.A.S. 1996; Bhopal 2010]**

Sol. The given equation may be written as $\partial^2 z/\partial x^2 - 4(\partial^2 z/\partial x \partial y) + 4(\partial^2 z/\partial y^2) = 0$

or $\qquad (D^2 - 4DD' + 4D'^2)z = 0 \qquad$ or $\qquad (D - 2D')^2 z = 0.$

Its solution is $\qquad z = \phi_1(y + 2x) + x\phi_2(y + 2x)$. ϕ_1, ϕ_2 being arbitrary functions ...(1)

Since (1) passes through $z = x = 0$, we have $0 = \phi_1(y)$ which gives $\qquad \phi_1(y + 2x) = 0.$

$\therefore$ (1) becomes $\qquad z = x\,\phi_2(y + 2x).$ ...(2)

Since (2) passes through $\quad z - 1 = x - y = 0, \quad$ *i.e.* $\quad z = 1 \quad$ and $\quad y = x$, we get

$1 = x\,\phi_2(3x) \qquad$ or $\qquad \phi_2(3x) = 3/(3x) \qquad$ so that $\phi_2(y + 2x) = 3/(y + 2x).$

$\therefore$ from (2), we have $\qquad 3x = z(y + 2x)$, which is the required surface.

Ex. 2. *Find the surface satisfying the equation* $r + t - 2s = 0$ *and the conditions that* $bz = y^2$ *when* $x = 0$ *and* $az = x^2$ *when* $y = 0$.

Sol. Re-writing the given equation, $\qquad \partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 - 2(\partial^2 z/\partial x \partial y) = 0$

or $\qquad (D^2 - 2DD' + D'^2) = 0 \qquad$ or $\qquad (D - D')^2 z = 0.$

Its solution is $\qquad z = \phi_1(y + x) + x\,\phi_2(y + x)$. ϕ_1, ϕ_2 being arbitrary functions. ...(1)

Since $z = y^2/b$ when $x = 0$, (1) gives $\quad y^2/b = \phi_1(y), \quad \Rightarrow \quad \phi_1(y + x) = (y + x)^2/b.$ ...(2)

Again since $z = x^2/a$ when $y = 0$, (1) gives $\qquad x^2 a = x\phi_2(x) + \phi_1(x).$ ...(3)

Since from (2), $\phi_2(x) = x^2/b$, (3) becomes $\quad \dfrac{x^2}{a} = x\phi_2(x) + \dfrac{x^2}{b} \qquad$ *i.e.* $\qquad \phi_2(x) = \dfrac{b - a}{ab} x$

which gives $\qquad \phi_2(y + x) = \dfrac{b - a}{ab}(y + x).$ ...(4)

Using (2) and (4) in (1), the required surface is

$$z = \frac{b - a}{ab} x(y + x) + \frac{(y + x)^2}{b} = \frac{y + x}{b}\left(\frac{b - a}{a} x + y + x\right) \qquad \text{or} \qquad z = (y + x)\left(\frac{x}{a} + \frac{y}{b}\right)$$

Ex. 3. *Find a surface satisfying* $r - 2s + t = 6$ *and touching the hyperbolic paraboloid* $z = xy$ *along its section by the plane* $y = x$. **[Meerut 1997]**

Sol. Re-writing the given equation, $\qquad \partial^2 z/\partial x^2 - 2(\partial^2 z/\partial x \partial y) + \partial^2 z/\partial y^2 = 6$

or $\quad (D^2 - 2DD' + D'^2)z = 6 \qquad$ or $\qquad (D - D')^2 z = 6.$ ... (1)′

Its $\qquad$ C.F. $= \phi_1(y + x) + x\,\phi_2(y + x)$, ϕ_1, ϕ_2 being arbitrary functions

Now, P.I. $= \dfrac{1}{(D - D')^2}.6 = \dfrac{1}{D^2}\left(1 - \dfrac{D'}{D}\right)^{-2} 6 = \dfrac{1}{D^2}\left(1 + \dfrac{2D'}{D} + \ldots\right) 6 = \dfrac{1}{D^2} 6 = 3x^2.$

$\therefore$ General solution of (1)′ is $\qquad z =$ C.F. + P.I. $= \phi_1(y + x) + x\phi_2(y + x) + 3x^2.$ ...(1)

Since the required surface (1) touches the given surface $\qquad z = xy$ ...(2)

along the section $y = x$, the values of p and q for the two surfaces must be equal for any point on the plane $y = x$. ...(3)

Now equating the values of p and q from (1) and (2), we have

$$p = \phi_2(y + x) + x\phi_2'(y + x) + \phi_1'(y + x) + 6x = y \quad ...(4)$$

and $$q = x\phi_2'(y + x) + \phi_1'(y + x) = x. \quad ...(5)$$

Subtracting (5) from (4) and using (3), we get

$$\phi_2(2x) = -6x = -3 \times (2x)$$

which gives $$\phi_2(y + x) = -3(y + x). \quad ...(6)$$

From (6), $\phi_2'(y + x) = -3$. Then (5) becomes

$-3x + \phi_1'(y + x) = x$ so that $\phi_1'(2x) = 2 \times (2x)$, as $y = x$

Now, $$\phi_1'(2x) = 2(2x) \Rightarrow \phi_1'(x) = 2x. \quad ...(7)$$

Integrating (7), $\phi_1(x) = x^2 + c$ which gives $\phi_1(y + x) = (y + x)^2 + c$. ...(8)

Putting the values of $\phi_2(y + x)$ and $\phi_1(y + x)$ given by (6) and (8) in (1), we get

$z = x\{-3(y + x)\} + (y + x)^2 + c + 3x^2$ or $z = x^2 - xy + y^2 + c$.
...(9)

Equating the values of z from (2) and (9), we get

$xy = x^2 - xy + y^2 + c$ or $x^2 = x^2 - x^2 + x^2 + c$, using (3)

giving $c = 0$. Hence the required surface is $z = x^2 - xy + y^2$.

Ex. 4. *A surface is drawn satisfying $r + t = 0$ and touching $x^2 + z^2 = 1$ along its section by $y = 0$. Obtain its equation in the form $x^2(x^2 + z^2 - 1) = y^2(x^2 + z^2)$.* **[Meerut 1998]**

Sol. Given equation $(D^2 + D'^2)z = 0$ *i.e.* $(D + iD)(D - iD')z = 0$.

$\therefore$ Its solution is $z = \phi_1(y + ix) + \phi_2(y - ix)$, ϕ_1, ϕ_2 being arbitrary functions. ...(1)

The given surface is $x^2 + z^2 = 1$ or $z = (1 - x^2)^{1/2}$. ...(2)

Since (1) and (2) touch along their common section by $y = 0$, ...(3)

the values of p and q from (1) and (2) must be the same.

$$\therefore \quad p = i\,\phi_1'(y + ix) - i\,\phi_2'(y - ix) = -\frac{x}{(1-x^2)^{1/2}} \quad \text{and} \quad q = \phi_1'(y + ix) + \phi_2'(y - ix) = 0.$$

Using (3), these reduce to

$$\phi_1'(ix) - \phi_2'(-ix) = \frac{ix}{(1 + x^2 i^2)^{1/2}} \quad \text{and} \quad \phi_1'(ix) + \phi_2'(-ix) = 0, \quad \text{noting that } i^2 = -1$$

Solving these for $\phi_1'(ix)$ and $\phi_2'(-ix)$, $\phi'_1(ix) = \dfrac{ix}{2(1+x^2i^2)^{1/2}}$, $\phi'_2(-ix) = \dfrac{-ix}{2(1+x^2i^2)^{1/2}}$.

Writing $ix = X$ and $-ix = Y$, these give $\phi_1'(X) = \dfrac{X}{2(1+X^2)^{1/2}}$ $\phi_2'(Y) = \dfrac{Y}{2(1+Y^2)^{1/2}}$

Integrating, $\phi_1(X) = (1/2) \times (1 + X^2)^{1/2} + c_1$, $\phi_2(Y) = (1/2) \times (1 + Y^2)^{1/2} + c_2$

These give $\phi_1(y + ix) = (1/2) \times \{1 + (y + ix)^2\}^{1/2} + c_1$, $\phi_2(y - ix) = (1/2) \times \{1 + (y - ix)^2\}^{1/2} + c_2$.

Putting these in (1) and writing $c_1 + c_2 = c$, $z = (1/2) \times [\sqrt{\{1 + (y + ix)^2\}} + \sqrt{\{1 + (y - ix)^2\}}] + c$ ...(4)

Now equating two values of z from (2) and (4) at $y = 0$, we get

$(1/2) \times [\sqrt{(1 - x^2)} + \sqrt{(1 - x^2)}] + c = \sqrt{(1 - x^2)}$ so that $c = 0$.

Then, (4) gives $2z = \sqrt{\{1+(y+ix)^2\}} + \sqrt{\{1+(y-ix)^2\}}$. Squaring its both sides gives

or $$4z^2 = \{1+(y+ix)^2\} + (1+(y-ix)^2\} + 2\sqrt{[\{1+(y+ix)^2\}\ \{1+(y-ix)^2\}]},$$

or $$2z^2 = (1+y^2-x^2) + \sqrt{[\{1+(y+ix)^2\}\{1+(y-ix)^2\}]}. \qquad ...(5)$$

Squaring both sides of (5), we get

$$4z^4 = (1+y^2-x^2)^2 + \{1+(y+ix)^2\}\{1+(y-ix)^2\} + 2(1+y^2-x^2)\sqrt{[\{1+(y+ix)^2\}\{1+(y-ix)^2\}]}$$

or $$4z^4 = (1+y^2-x^2)^2 + \{(1+y^2-x^2)+2ixy\}\{(1+y^2-x^2)-2ixy\}$$
$$+2(1+y^2-x^2)\{2z^2-(1+y^2-x^2)\}, \text{ using (5)}$$

or $$4z^4 = (1+y^2-x^2)^2 + (1+y^2-x^2)^2 + 4x^2y^2 + 4z^2(1+y^2-x^2) - 2(1+y^2-x^2)^2$$

or $$4z^4 = 4x^2y^2 + 4z^2(1+y^2-x^2) \qquad \text{or} \qquad z^2(x^2+z^2-1) = y^2(x^2+z^2).$$

Ex.5. *Find a surface satisfying the equation* $D^2z = 6x+2$ *and touching* $z = x^3+y^3$ *along its section by the plane* $x+y+1=0$. **Ans.** $z = x^3+y^3+(x+y+1)^2$

MISCELLANEOUS PROBLEMS ON CHAPTER 4

1. Solution of differential equation $(D^3 - 6D^2D' + 11DD'^2 - 6D'^3)z = 0$ is

(*a*) $z = \phi_1(y+x)+\phi_2(y+2x)+\phi_3(y-3x)$ (*b*) $z = \phi_1(y-x)+\phi_2(y-2x)+\phi_3(y+3x)$

(*c*) $z = \phi_1(y-x)+\phi_2(y+2x)+\phi_3(y+3x)$ (*d*) $z = \phi_1(y+x)+\phi_2(y+2x)+\phi_3(y+3x)$

Sol. Ans (*a*). Refer solved Ex. 2 (*a*) of Art. 4.5. **[Agra 2005]**

2. P.I. of the equation $r - 2s + t = \cos(2x+3y)$ is

(*a*) $-\cos(2x+3y)$ (*b*) $\cos(2x+3y)$ (*c*) $\sin(2x+3y)$ (*d*) None of these **[Kanpur 2004]**

Sol. Ans (*a*). Proceed like Ex. 4, Art. 4.9.

3. Auxialary equation of $r - 2s + t = \sin(2x+3y)$ is

(*a*) $m^2 - 2m + 1 = \sin(2x+3y)$ (*b*) $m^2 + 2m + 1 = \sin(2x+3y)$

(*c*) $(m-1)^2 = 0$ (*d*) $(m+1)^2 = 0$ **[Bhopal 2010]**

Ans. (*c*)

4. Solve $(D^2 - DD' - 6D'^2)z = \cos(2x+y)$ **[Agra 2009, 10]**

Ans. $z = \phi_1(y+3x) + \phi_2(y-2x) - (1/4)\times\cos(2x+y)$

5

Non–homogeneous Linear Partial Differential Equations with Constant Coefficients

5.1. Non–homogeneous linear partial differential equations with constant coefficients.

Definition. A linear partial differential equation with constant coefficients is known as non–homogeneous linear partial differential equation with coefficients if the orders of all the partial derivatives involved in the equation are not equal.

For example, $(\partial^2 z/\partial x^2)-(\partial^3/\partial y^3)+\partial z/\partial x+z=x+y$ is a non–homogenous partial differential equation with constant coefficients.

5.2. Reducible and irreducible linear differential operators.

A linear differentiable operator $\mathrm{F}(D,\ D')$ is known as *reducible*, if it can be written as the product of linear factors of the form $aD+bD'+c$ with a, b and c as constants.

$F(D,\ D')$ is known as *irreducible*, if it is not reducible.

For example, the operator $D^2-D'^2$ which can be written in the form $(D+D')(D-D')$ is reducible, whereas the operator $D^2-D'^3$ which cannot be decomposed into linear factors is irreducible.

5.3. Reducible and irreducible linear partial differential equations with constant coefficients. [Delhi Maths (H) 2001, 2004, 09]

A linear partial differential equation with constant coefficients $F(D,\ D')z=f(x,y)$ is known as reducible, if $F(D,D')$ is reducible.

$F(D,D')z=f(x,\ y)$ is known as irreducible if $F(D,D')$ is irreducible.

For example, $(D^2-D'^2)z=x^2y^3$ is a reducible partial differential equation, with constant coefficients, since $D^2-D'^2=(D+D')\ (D-D')$ whereas $(D^2-D'^3)\ z=x^2y^3$ is an irreducible partial differential equation with constant coefficients, since $D^2-D'^3$ cannot be decomposed into linear factors.

5.4. Theorem. *If the operator $F(D,\ D')$ is reducible, the order in which the linear factors occur is unimportant.*

Proof. In order to prove the theorem we must show that

$$(a_rD+b_r\ D'+c_r)\ (a_sD+b_sD'_r+c_s)=\ (a_sD+b_sD'+c_s)\ (a_rD+b_rD'+c_r) \qquad \text{... (1)}$$

for any reducible operator can be written in the form

$$F(D,\ D')=\prod_{r=1}^{n}\ (a_rD+b_rD'+c_r) \qquad \text{... (2)}$$

The proof of (1) is immediate, since both sides are equal to

$$a_ra_sD^2+(a_sb_r+a_rb_s)\ DD'+b_rb_sD'^2+(c_sa_r+c_ra_s)D+(c_sb_r+c_rb_s)D'+c_sc_r$$

5.5. Determination of complementary function (C.F.) of a reducible non–homogeneous linear partial differential equation with constant coefficients given by

$$F(D, D')z = 0 \qquad \text{... (1)}$$

Let $F(D, D') = (b_1D - a_1D' - c_1)(b_2D - a_2D' - c_2)\ldots\ldots (b_nD - a_nD' - c_n)$, ...(2)
where a's, b's and c's are constants. Then (1) becomes

$$(b_1D - a_1D' - c_1)(b_2D - a_2D' - c_2)\ldots(b_nD - a_nD' - c_n)\, z = 0. \qquad \text{...(3)}$$

Equation (3) shows that any solution of the equation

$$(b_rD - a_rD' - c_r)z = 0,\ r = 1, 2, \ldots n \qquad \text{...(4)}$$

is a solution of (3) *i.e.* $b_rp - a_rq = c_rz$ which is Lagrange's equation.

Its Lagrange's auxiliary equations are $$\frac{dx}{b_r} = \frac{dy}{-a_r} = \frac{dz}{c_rz}. \qquad \text{...(5)}$$

Proceeding as usual two independent integrals of (5) are $b_ry + a_rx = c_1$

and $z = c_2e^{(c_r/b_r)x}$, if $b_r \neq 0$ or $z = c_2'e^{-(c_r/a_r)y}$, if $a_r \neq 0$

$\therefore$ the general solution of (4) is $z = e^{(c_r/b_r)x}\phi_r(b_ry + a_rx)$, if $b_r \neq 0$ ...(6)

or $z = e^{-(c_r/a_r)y}\psi_r(b_ry + a_rx)$, if $a_r \neq 0$, ...(7)

where ϕ_r and ψ_r are arbitrary functions.

The general solution of (3) is the sum of the solutions of the equations of the form (4) corresponding to each factor in (2).

Case of repeated factors. Let two times repeated factor of (2), be $bD - aD' - c$.

Consider the equation $(bD - aD' - c)\,(bD - aD' - c)z = 0.$ ...(8)

Let $(bD - aD' - c)z = v\,.$ ...(9)

Then (8) reduces to $(bD - aD' - c)\,v = 0.$ ...(10)

As before, the general solution of (10) is $v = e^{(c/b)x}\phi(by + ax)$, if $b \neq 0$. ...(11)

or $v = e^{-(c/a)y}\psi(by + ax)$, $a \neq 0$. ...(12)

where ϕ and ψ are arbitrary functions

Substituting from (11) is (9), we have

$(bD - aD' - c)z = e^{(c/b)x}\,\phi(by + ax)$ or $bp - aq = cz + e^{(c/b)x}\phi(by + ax)$. ...(13)

Lagrange's auxiliary equations of (13) are $$\frac{dx}{b} = \frac{dy}{-a} = \frac{dz}{cz + e^{(c/b)x}\phi(by + ax)}. \qquad \text{...(14)}$$

Taking the first two fractions of (14), $adx + bdy = 0$ so that $by + ax = \lambda$, (say). ...(15)
where λ is an arbitrary constant.

Taking first and third fractions of (14), we get

$$\frac{dz}{dx} - \frac{c}{b}z = \frac{1}{b}e^{(c/b)x}\phi(by + ax) \quad \text{or} \quad \frac{dz}{dx} - \frac{c}{b}z = \frac{1}{b}e^{(c/b)x}\phi(\lambda), \text{ using (15)}$$

This is linear equation. Its I.F. $= e^{-\int (c/b)dx} = e^{-(c/b)x}$ and solution is given by

$$ze^{-(c/b)x} = \int \frac{1}{b}e^{(c/b)x}\phi(\lambda)e^{-(c/b)x}\, dx, \text{ that is,}$$

$ze^{-(c/b)x} - (x/b)\,\phi(\lambda) = \mu$ or $ze^{-(c/b)x} - (x/b)\,\phi(by + ax) = \mu$, by (15) ...(16)

where μ is an arbitrary constant.

From (15) and (16), the general solution of (13) or (8) is

$ze^{-(c/b)x} - (x/b)\phi(by + ax) = \phi_1(by + ax)$ or $z = e^{(c/b)x}[\phi_1(by + ax) + x\,\phi_2(by + ax)]$, if $b \neq 0$ ...(17)

where ϕ_1 and ϕ_2 are arbitrary functions.

Taking (12) and (9), we obtain as before.

$$z = e^{-(c/a)y}[\psi_1(by + ax) + y\,\psi_2(by + ax)], \text{ if } a \neq 0. \qquad ...(18)$$

where ψ_1 and ψ_2 are arbitrary functions.

In general, if $(bD - aD' - c)$ is repeated r times, then

$$z = c^{(c/b)x} \sum_{i=1}^{r} x^{i-1}\phi_i(by + ax), \text{ if } b = 0 \qquad \text{or} \qquad z = c^{-(c/a)y} \sum_{i=1}^{r} y^{i-1}\psi_i(by + ax), \text{ if } a \neq 0.$$

5.6. Working rule for finding C.F. of reducible non–homogeneous linear partial differential equation with constant coefficients. For proofs refer Art. 5.5.

Let the given differential equation be $\qquad F(D, D') = f(x, y)$.

Factorize $F(D, D')$ into linear factors. Then use the following results:

Rule I. Corresponding to each non–repeated factor $(bD - aD' - c)$, the part of C.F. is taken as $e^{(cx/b)}\phi(by + ax)$, if $b \neq 0$

We now have three particular cases of Rule I

Rule IA. Take $c = 0$ in rule I. Hence corresponding to each linear factor $(bD - aD')$, the part of C.F. is $\phi(by + ax),\ b \neq 0$.

Rule IB. Take $a = 0$ in rule I. Hence corresponding to each linear factor $(bD - c)$, the part of C.F. is $e^{(cx/b)}\phi(by),\ b \neq 0$.

Rule IC. Take $a = c = 0$ and $b = 1$ in rule I. Hence corresponding to linear factor D, the part of C.F. is $\phi(y)$.

Rule II. Corresponding to a repeated factor $(bD - aD' - c)^r$, the part of C.F. is taken as

$$e^{(cx/b)}\ [\phi_1(by + ax) + x\phi_2(by + ax) + x^2\phi_3(by + ax) + ... + x^{r-1}\phi_r(by + ax)], \text{ if } b \neq 0$$

We now have three particular cases of Rule II.

Rule II A. Take $c = 0$ in rule II. Hence corresponding to each repeated factor $(bD - aD')^r$, the part of C.F. is $\qquad \phi_1(by + ax) + x\phi_2(by + ax) + x^2\phi_3(by + ax) + ... + x^{r-1}\phi_r(by + ax),\ b \neq 0$.

Rule II B. Take $a = 0$ in rule II. Hence corresponding to a repeated factor $(bD - c)^r$, the part of C.F. is $\qquad e^{(cx/b)}\ [\phi_1(by) + x\phi_2(by) + x^2\phi_3(by) + ... + x^{r-1}\phi_r(by)],\ b \neq 0$

Rule II C. Take $a = c = 0$ and $b = 1$ in rule II. Hence corresponding to repeated factor D^r, the part of C.F. is $\qquad \phi_1(y) + x\,\phi_2(y) + x^2\phi_3(y) + ... + x^{r-1}\phi_r(y)$.

Rule III. Corresponding to each non–repeated linear factor $(bD - aD' - c)$, the part of C.F. is taken as $e^{-(cy/a)}\phi(by + ax)$, if $a \neq 0$.

We now have three particular cases of rule III.

Rule III A. Take $c = 0$ in rule III. Hence corresponding to each linear factor $(bD - aD')$, the part of C.F. is $\phi(by + ax)$, $a \neq 0$.

Rule III B. Take $b = 0$ in rule III. Hence corresponding to each linear factor $(aD' + c)$, the part of C.F. is $e^{-(cy/a)}\phi(ax)$, $a \neq 0$.

Rule III C. Take $b = c = 0$ and $a = 1$ in rule III. Hence corresponding to linear factor D', the part of C.F. is $\phi(x)$

Rule IV. Corresponding to a repeated factor $(bD - aD' - c)^r$, the part of C.F. is taken as

$$e^{-(cy/a)}\,[\phi_1(by+ax) + y\,\phi_2(by+ax) + y^2\phi_3(by+ax) + \dots + y^{r-1}\phi_r(by+ax)], \text{ if } a \neq 0$$

We now have three particular cases of rule IV.

Rule IV A. Take $c = 0$ in rule IV. Hence corresponding to repeated factor $(bD - aD')^r$, the part of C.F. is $\phi_1(by+ax) + y\,\phi_2(by+ax) + y^2\phi_3(by+ax) + \dots + y^{r-1}\phi_r(by+ax)$, $a \neq 0$.

Rule IV B. Take $b = 0$ in rule IV. Hence corresponding to a repeated factor $(aD' + c)^r$, the part of C.F. is $e^{-(cy/a)}[\phi_1(ax) + y\,\phi_2(ax) + y^2\phi_3(ax) + \dots + y^{r-1}\phi_r(ax)]$, $a \neq 0$

Rule IV C. Take $b = c = 0$ and $a = 1$ in rule IV. Hence corresponding to repeated factor D'^r, the part of C.F. is $\phi_1(x) + y\phi_2(x) + y^2\phi_3(x) + \dots + y^{r-1}\phi_r(x)$.

5.7. SOLVED EXAMPLES BASED ON ART. 5.6.

Ex. 1. *Solve* $(D^2 - D'^2 + D - D')z = 0$.

Sol. The given equation can be re-written as $(D - D')(D + D' + 1)z = 0$.

Here R.H.S. = 0 ⇒ P.I. = 0. Hence the required solution is $z =$ C.F.

or $z = \phi_1(y + x) + e^{-x}\phi_2(y - x)$, ϕ_1 and ϕ_2 being arbitrary functions.

Ex. 2. *Solve* $(D^2 - a^2D'^2 + 2abD + 2a^2bD')z = 0$.

Sol. The given equation can be re–written as

$[(D + aD')(D - aD') + 2ab(D + aD')]z = 0$ or $(D + aD')(D - aD' + 2ab)z = 0$.

Hence general solution is $z =$ C.F. $= \phi_1(y - ax) + e^{-2abx}\phi_2(y + ax)$,

where ϕ_1 and ϕ_2 are arbitrary functions.

Ex. 3. *Solve* $r + 2s + t + 2p + 2q + z = 0$.

Sol. The given equation can be re–written as

$$(\partial^2z/\partial x^2) + 2(\partial^2z/\partial x\partial y) + (\partial^2z/\partial y^2) + 2(\partial z/\partial x) + 2(\partial z/\partial y) + z = 0$$

or $$(D^2 + 2DD' + D'^2 + 2D + 2D' + 1)z = 0$$

or $$[(D + D')^2 + 2(D + D') + 1]z = 0 \quad \text{or} \quad (D + D' + 1)^2z = 0.$$

There are repeated linear factors. So the required general solution is

$$z = \text{C.F.} = e^{-x}[\phi_1(y - x) + x\phi_2(y - x)],\ \phi_1, \phi_2 \text{ being arbitrary functions.}$$

Ex. 4. *Solve* $DD'(D - 2D' - 3)z = 0$.

Sol. Using rules I, IC, III C of Art. 5.6, the required solution is $z =$ C.F., i.e.,

$z = \phi_1(y) + \phi_2(x) + e^{3x}\phi_3(y + 2x)$, where ϕ_1, ϕ_2 and ϕ_3 are arbitrary functions.

Ex. 5. *Solve* $(2D - 3)\ (3D - 5D' - 7)^2 z = 0$

Sol. Using rules IB and II of Art. 5.6, the required solution is $z =$ C.F., i.e.,

$z = e^{3x/2}\phi_1(2y) + e^{7x/3}\{\phi_2(3y + 5x) + x\,\phi_3(3y + 5x)\}$, ϕ_1, ϕ_2, ϕ_3 being arbitrary functions.

Ex. 6. *Solve* $(3D - 5)\ (7D' + 2)\ DD'(2D + 3D' + 5)z = 0$

Sol. Using rules IB, IIIB, IC, IIIC and I, the required solution is $z =$ C.F. i.e.,

$$z = e^{5x/3}\phi_1(3y) + e^{-(2y/7)}\phi_2(7x) + \phi_3(y) + \phi_4(x) + e^{-(5x/2)}\phi_5(2y - 3x)$$

Ex. 7. *Solve the partial differential equation $t + s + q = 0$.*

Sol. Re-writing the given equation, $\partial^2 z/\partial x^2 + \partial^2 z/\partial x\partial y + \partial z/\partial y = 0$

or $(D^2 + DD' + D')z = 0$ or $D'(D + D' + 1) = 0$,

whose general solution is $z = \phi_1(x) + e^{-x}\phi_2(y - x)$, ϕ_1, ϕ_2 being arbitrary functions.

EXERCISE 5(A)

Solve the following partial differential equations :

1. $(D + D' - 1)(D + 2D' - 2)z = 0$ **Ans.** $z = e^x\phi_1(y - x) + e^{2x}\phi_2(y - 2x)$

2. $(D - D' + 1)(D + 2D' - 3)z = 0$ **Ans.** $z = e^{-x}\phi_1(y + x) + e^{2x}\phi_2(y - 3x)$

3. $(DD' + aD + bD' + ab)z = 0$ **Ans.** $z = e^{-bx}\phi_1(y) + e^{-ay}\phi_2(x)$

4. $r + 2s + t + 2p + 2q + z = 0$ **Ans.** $z = e^{-x}\{\phi_1(y - x) + x\,\phi_2(y - x)\}$

5. $(D + 1)(D + D' - 1)z = 0$ **Ans.** $z = e^{-x}\phi_1(y) + e^x\phi_2(y - x)$

6. $(D^2 - D'^2 + D - D')z = 0$ **Ans.** $z = \phi_1(y + x) + e^{-x}\phi_2(y - x)$

7. $\partial^2 z/\partial x^2 + \partial^2 z/\partial x\partial y - 6(\partial^2 z/\partial y^2) = 0$ **Ans.** $z = \phi_1(y + 2x) + \phi_2(y - 3x)$

8. $(D^2 - DD' - 2D'^2 + 2D + 2D')z = 0$ **Ans.** $z = \phi_1(y - x) + e^{-2x}\phi_2(y + 2x)$

9. $s + p - q - z = 0$ **Ans.** $z = e^x\phi_1(y) + e^{-y}\phi_2(x)$

10. $(D^2 - DD' + D' - 1)z = 0$ **Ans.** $z = e^x\phi_1(y) + e^{-x}\phi_2(y + x)$

11. $(D^2 + DD' + D' - 1)z = 0$ **Ans.** $z = e^{-x}\phi_1(y) + e^x\phi_2(y - x)$

5.8. Method of finding C.F. of irreducible linear partial differential equation with constant coefficients, namely, $F(D, D')z = f(x, y)$ ... (1)

When the operator $F(D, D')$ in (1) is irreducible, it is not always possible to find a solution with the full number of arbitrary functions, but it is possible to construct solutions which contain as many arbitrary constants as we wish. We now state and prove a theorem which will be used to find C.F. of (1).

Theorem. *To show that* $F(D, D')\,e^{ax+by} = F(a, b)\,e^{ax+by}$

Proof. We know that $F(D, D')$ consists of terms of the form $C_{rs}\,D^r\,D'^s$.

Also $D^r(e^{ax+by}) = a^r e^{ax+by}$ and $D'^s(e^{ax+by}) = b^s\,e^{ax+by}$

so that $(C_{rs}\,D^r D'^s)(e^{ax+by}) = C_{rs}a^r b^s\;e^{ax+by}$

The theorem follows by combining the terms of the operator $F(D, D')$.

We now discuss method of finding C.F. of (1). Consider $F(D, D')\;z = 0$ ... (2)

From the above theorem we see that e^{hx+ky} is a solution of (2) provided $F(h, k) = 0$, so that

$$z = \sum_i A_i\;e^{h_i x + k_i y} \qquad \text{... (3)}$$

in which A_i, h_i, k_i are all constants, is also a solution provided that h_i, k_i are connected by the relation $F(h_i, k_i) = 0$ ... (4)

Thus we can construct solution of (2) containing as many arbitrary constants as we need. The series (3) may not be finite but if it is infinite, it is necessary that it should be uniformly convergent if it has to be a solution of (2).

Remark. We can also present C.F. of irreducible equation (1) in the following manner

$$C.F. = \Sigma A e^{hx+ky}$$

where A, h, k are orbitrary constants such that $F(h, k) = 0$

Working rule for finding C.F. of irreducible non–homogeneous linear partial differential equation with constant coefficients, namely, $F(D, D')z = 0$

Step 1. If necessary, factorize $F(D, D')$ in the form $F_1(D, D')\, F_2(D, D')$, where $F_1(D, D')$ consist of a product of linear factors in D, D' and $F_2(D, D')$ consists of a product of irreducible factors in D, D'.

Step 2. Using Art. 5.6, write down the part of C.F. corresponding to factors of $F_1(D, D')$.

Step 3. Using Art 5.8, write down the part of C.F. corresponding to factors of $F_2(D, D')$.

Step 4. Adding the C.F. corresponding to $F_1(D, D')$ obtained in step 2 and the C.F. corresponding to $F_2(D, D')$ obtained in step 3, we obtain the C.F. of the given equation

$$F(D, D')z = 0, \qquad i.e., \qquad F_1(D, D')F_2(D, D')z = 0.$$

5.9 SOLVED EXAMPLES BASED ON ART. 5.8

Ex. 1. *Solve* $(D - D'^2)z = 0$.

Sol. Here $D - D'^2$ is not a linear factor in D and D'. Let $z = Ae^{hx + ky}$ be a trial solution of the given equation. Then $Dz = Ahe^{hx + ky}$ and $D'^2z = Ak^2e^{hx + ky}$.

Putting these values in the given differential equation, we get

$A(h - k^2)e^{hx + ky} = 0$ so that $h - k^2 = 0$ or $h = k^2$.

Replacing h by k^2, the most general solution of the given equation is

$$z = \Sigma A e^{k^2 x + ky}, \text{ where } A \text{ and } k \text{ are arbitrary constants.}$$

Ex. 2. *Solve* $(D - 2D' - 1)(D - 2D'^2 - 1)z = 0$.

Sol. $(D - 2D' - 1)$ being linear in D and D', the part of C.F. corresponding to it is $e^x\phi(y + 2x)$, where ϕ is an arbitrary function.

To find C.F. corresponding non–linear factor $D - 2D'^2 - 1$, we now proceed as follows :

Let a trial solution of $(D - 2D'^2 - 1)z = 0$ ...(1)

be $z = Ae^{hx + ky}$ ...(2)

$\therefore$ $Dz = Ahe^{hx + ky}$ and $D'^2z = Ak^2e^{hx + ky}$. Hence (1) becomes

$A(h - 2k^2 - 1)e^{hx + ky} = 0$ or $h - 2k^2 - 1 = 0$ or $h = 2k^2 + 1$.

Replacing h by $2k^2 + 1$ in (2), the solution of (1) *i.e.* the part of C.F. corresponding to $(D - 2D'^2 - 1)$ in the given equation is given by

$$\Sigma A e^{(2k^2+1)x+ky}, \; A \text{ and } k \text{ being arbitrary constants.}$$

$\therefore$ The required solution is $z = e^x\phi(y + 2x) + \Sigma A e^{(2k^2+1)x+ky}$.

Ex. 3. *Solve* $(\partial^2 z/\partial x^2) + (\partial^2 z/\partial y^2) = n^2 z$.

Sol. The given equation can be written as $(D^2 + D'^2 - n^2)z = 0$. ...(1)

Let a trial solution of (1) be $z = Ae^{hx + ky}$. ...(2)

$\therefore \quad D^2 z = Ah^2 e^{hx+ky}$ and $D'^2 z = Ak^2 e^{hx+ky}$. Hence (1) gives

$A(h^2 + k^2 - n^2)e^{hx+ky} = 0$ or $h^2 + k^2 = n^2$. ...(3)

Taking α as parameter, we see that (2) is satisfied if $h = n \cos \alpha$ and $k = n \sin \alpha$. Putting these values in (2), the required general solution is

$$z = \Sigma A e^{n(x \cos \alpha + y \sin \alpha)}, \ A \text{ and } \alpha \text{ being arbitrary constants.}$$

Ex. 4. *Solve* $(2D^4 - 3D^2 D' + D'^2)z = 0$

Sol. Re–writing the given equation, we have $(2D^2 - D')(D^2 - D')z = 0$ ... (1)

Let $z = A\, e^{hx+ky}$ be a solution of $(2D^2 - D')z = 0$

$\therefore (2h^2 - k)A e^{hx+ky} = 0$ so that $2h^2 - k = 0$ or $k = 2h^2$

Hence C.F. corresponding to $(2D^2 - D')$ is $\Sigma A e^{hx+2h^2 y}$ ... (2)

Again, let $z = A' e^{h'x+k'y}$ be a solution of $(D^2 - D')z = 0$

$\therefore \ (h'^2 - k')A' e^{h'x+k'y} = 0$ so that $h'^2 - k' = 0$ or $k' = h'^2$.

Hence, C.F. corresponding to $(D^2 - D')$ is $\Sigma A' e^{h'x+h'^2 y}$ ... (3)

From (2) and (3), the general solution of (1) is given by $z =$ Total C.F., i.e.,

$z = \sum_i A_i e^{h_i x + 2h_i^2 y} + \sum_i A_i' e^{h_i' x + h_i'^2 y}$, where A_i, h_i, A_i' and h_i' are arbitrary constants.

Ex. 5. Solve $(D + 2D' - 3)(D^2 + D')z = 0$

Sol. C.F. corresponding to linear factor $(D + 2D' - 3)$ is $e^{3x}\phi(y - 2x)$.

We now find C.F. corresponding to irreducible factor $(D^2 + D')$.

Let $z = Ae^{hx+ky}$ be a solution of $(D^2 + D')z = 0$.

$\therefore \quad (h^2 + k)A e^{hx+ky} = 0$ so that $h^2 + k = 0$ or $k = -h^2$.

Hence C.F. corresponding to $(D^2 + D')$ is $\Sigma A e^{hx - h^2 y}$.

Therefore, the general solution of given equation is $z = e^{3x}\phi(y - 2x) + \sum_i A_i e^{h_i x - h_i^2 y}$,

where ϕ is an arbitrary function and A_i, h_i are arbitrary constants.

Ex. 6. *Solve* $(D^2 - D')\, z = 0$. **[Delhi Maths (H) 2009]**

Sol. Given equation is $F(D, D')z = 0$, where $F(D, D') = D^2 - D'$.

Let $z = e^{hx+ky}$ be a trial solution of the given equation. Then, the required solution is $z = \sum_i A_i \ e^{h_i x + k_i y}$, where $F(h_i, k_i) = h_i^2 - k_i = 0$ so that $k_i = h_i^2$

Hence the required solution is $z = \sum_i A_i \ e^{h_i x + h_i^2 y}$, A_i, h_i being arbitrary constants.

Ex. 7. *Show that* $\partial^2 z/\partial x^2 = (1/k)\times(\partial z/\partial t)$ *possesses solutions of the form*

$$\sum_{n=0}^{\infty} C_n \cos(nx+\in_n)\, e^{-kn^2 t}.$$

Sol. Re–writing giving equation, we get $\quad \{D^2 - (1/k)\times D'\}\, z = 0, \quad \ldots (1)$

where $D = \partial/\partial x$, $D' = \partial/\partial t$. Note that here we have t in place of usual independent variable y. Let $z = e^{ax+bt}$ be a trial solution of (1). Then, $a^2 - (1/k)\, b = 0$ so that $a^2 = b/k$. This relation is satisfied if we take $a = \pm in$ and $b = -kn^2$. Then solution of (1) will be of the form

$z = \sum_{n=0}^{\infty} C_n\, e^{\pm inx - kn^2 t}$, which can be re-written as $\quad z = \sum_{n=0}^{\infty} C_n \cos(nx+\in_n) e^{-kn^2 t}$

Ex. 8. *Write the form of solution possessed by the equation* $\partial^2 y/\partial t^2 + 2k\,(\partial y/\partial t) = c^2\,(\partial^2 y/\partial x^2)$ **[Delhi B.Sc (H) 2002]**

Another form. *Show that the equation* $\partial^2 y/\partial t^2 + 2k\,(\partial y/\partial t) = c^2(\partial^2 y/\partial x^2)$ *possesses solutions of the form* $\quad \sum_{r=0}^{\infty} C_r\, e^{-kt} \cos\,(w_r t + \delta_r) \cos\,(\alpha_r x + \in_r)$,

where C_r, $\alpha_r\, \delta_r$, $\in_r$ are constants and $w_r^2 = \alpha_r^2 c^2 - k^2$.

Sol. Re–writing the given equation, we get $\quad (D'^2 + 2k\ D' - c^2 D^2)\ y = 0 \quad \ldots (1)$

where $D' \equiv \partial/\partial t$, $D \equiv \partial/\partial x$. Here t is independent variable and y is dependent variable.

Let $z = e^{ax + bt}$ be a solution of (1) Then, $\quad b^2 + 2kb - c^2 a^2 = 0$

so that $\quad b = \{-2k \pm (4k^2 + 4c^2 a^2)\}^{1/2}/2 \;= -k \pm (k^2 - c^2\alpha_r^2)^{1/2}$, $\quad$ where $\quad a^2 = -\alpha_r^2$

or $\quad b = -k \pm i w_r$, $\quad$ where $\quad w_r^2 = c^2 \alpha_r^2 - k^2 \quad \ldots (2)$

Hence the solution of (1) takes the form

$$y = \sum_{r=0}^{\infty} C_r\ e^{\pm i\alpha_r x + (-k \pm i w_r)t} = \sum_{r=0}^{\infty} C_r\, e^{(-k \pm i w_r)t}\ e^{\pm i \alpha_r x}$$

which can also be re–written as $\quad y = \sum_{r=0}^{\infty} C_r\ e^{-kt} \cos(w_r t + \delta_r) \cos\,(\alpha_r x + \in_r)$,

where C_r, α_r, δ_r, $\in_r$ are constants. Also, by (2), $\omega_r^2 = \alpha_r^2 c^2 - k^2$.

EXERCISE 5(B)

Solve the following partial differential equations:

1. $(D^2 + D + D')z = 0$ $\quad$ **Ans.** $z = \sum_i A_i e^{h_i - (h_i{}^2 + h_i)y}$, where A_i and h_i are arbitrary constants.

2. $(2D^2 - D'^2 + D)z = 0$

Ans. $z = \sum_i A_i e^{h_i x + k_i y}$, where $2h_i{}^2 - k_i{}^2 + h_i = 0$; A_i, h_i, k_i being arbitrary constants.

3. $(D' + 3D)^2 (D^2 + 5D + D')z = 0$ $\quad$ **Ans.** $z = \phi_1(3y - x) + x\,\phi_2(3y - x)$ $+ \sum_i A_i\, e^{h_i x - (h_i^2 + 5h_i)y}$, where ϕ_1, ϕ_2 are arbitrary functions and A_i, h_i are arbitrary constants.

4. $(2D-3D'+7)^2\,(D^2+3D')z=0$. **Ans.** $z=e^{-(7x/2)}\{\phi_1(2y+3x)+x\,\phi_2(2y+x)\}$

$+\sum_i A_i\, e^{h_i x-(h_i^2 y)/3}$, where ϕ_1, ϕ_2 are arbitrary functions and A_i, h_i are arbitrary constants.

5.10. General solution of non–homogeneous linear partial differential equation with constant coefficients.

Let
$$F(D, D')z = f(x, y) \qquad \text{... (1)}$$
be a non-homogeneous linear partial differential equation with constant coefficients. Let u be the C.F. of (1). Then, by definition u in a solution of $F(D, D')\,z=0$ so that
$$F(D, D')u = 0 \qquad \text{... (2)}$$
Let z' be a particular integral (P.I.) of (1). Hence
$$F(D, D')z' = f(x, y) \qquad \text{... (3)}$$
Now, $\quad F(D, D')\,(u+z') = F(D, D')u + F(D, D')z' = f(x, y)$, by (2) and (3),

Thus $u+z'$ is a solution of (1). Hence, a solution of (1) is $\quad$ z = C.F. + P.I.

5.11. Particular integral of non–homogeneous linear partial differential equation

$$\boldsymbol{F(D, D')y = f(x, y)} \qquad \text{... (1)}$$

The inverse operator $1/F(D, D')$ of the operator $F(D, D')$ is defined by the following indentity:
$$F(D, D')\left(\frac{1}{F(D, D')} f(x, y)\right) = f(x, y)$$
$$\Rightarrow \qquad \text{Particular integral (P.I.)} = \frac{1}{F(D, D')} f(x, y)$$

5.12. Determination particular integral of non–homogeneous linear partial differential equations (reducible or irreducible), namely,

$$\mathbf{F(D, D')z = f(x, y).} \qquad \text{...(1)}$$

The methods of finding particular integrals of non–homogeneous partial differential equations are very similar to those of ordinary linear differential equation with constant coefficients. We now give a list of some cases of finding P.I. of (1).

Case I. When f(x, y) = e^{ax+by} and F(a, b) ≠ 0.

Then,
$$\text{P.I.} = \frac{1}{F(D, D')} e^{ax+by} = \frac{1}{F(a, b)} e^{ax+by}.$$

Thus in this case we replace D by a and D' by b.

Case II. When f(x, y) = sin (ax + by) or cos (ax + by).

Then,
$$\text{P.I.} = \frac{1}{F(D, D')} \sin(ax+by) \qquad \text{or} \qquad \text{P.I.} = \frac{1}{F(D, D')} \cos(ax+by)$$
which is evaluated by putting $D^2=-a^2$, $D'^2=-b^2$, $DD'=-ab$, provided the denominator is non–zero.

Case III. When f(x, y) = $x^m y^n$

Then,
$$\text{P.I.} = \frac{1}{F(D, D')} x^m y^n = [F(D, D')]^{-1}\, x^m y^n,$$
which is evaluated by expanding $[F(D, D')]^{-1}$ in ascending powers of D'/D or D/D' or D or D' as the case may be. In practice, we shall expand in ascending powers of D'/D. However note that if

we expand in ascending powers of D/D', we shall get a P.I. of apparently different form. In this connection remember that both forms of P.I. are correct because the two could be transformed into each other with the help of C.F. of the given equation.

Case IV. When $f(x, y) = Ve^{ax+by}$, when V is a function of x and y.

Then $$\text{P.I.} = \frac{1}{F(D, D')}Ve^{ax+by} = e^{ax+by}\frac{1}{F(D+a, D'+b)}e^{ax+by}$$

Remark. If $F(a, b) = 0$ and $f(x, y) = e^{ax+by}$. Then, we have

$$\text{P.I.} = \frac{1}{F(D, D')}e^{ax+by},$$

in which case I fails. However, by treating e^{ax+by} as product of e^{ax+by} with '1' and applying the result of case IV, we can evaluate P.I. as follows :

$$\text{P.I.} = \frac{1}{F(D, D')}e^{ax+by}\cdot 1 = e^{ax+by}\frac{1}{F(D+a, D'+b)}\cdot 1,$$

which can be evaluated as explained in case III by treating $1 = x^0y^0$.

5.13 SOLVED EXAMPLES BASED ON ARTICLES 5.6, 5.8, AND 5.12.

Type 1 : Examples based on case I of Art. 5.12.

Ex. 1. *Solve* $(DD' + aD + bD' + ab)z = e^{mx+ny}$.

Sol. The given equation can be re–written as $(D + b)(D' + a)z = e^{mx+ny}$

$\therefore$ C.F. $= e^{-bx}\phi_1(y) + e^{-ay}\phi_2(x)$, ϕ_1, ϕ_2 being arbitrary functions.

and $$\text{P.I.} = \frac{1}{(D+b)(D'+a)}e^{mx+xy} = \frac{1}{(m+b)(n+a)}e^{mx+ny}.$$

Hence the required general solution is $z = e^{-bx}\phi_1(y) + e^{-ay}\phi_2(x) + [(m + b)(n + a)]^{-1}e^{mx+ny}$.

Ex. 2. *Solve* $(D^2 - D'^2 + D - D')z = e^{2x+3y}$. **[Ravishankar 2005]**

Sol. The given equation can be re–written as

$[(D - D')(D + D') + (D - D')]z = e^{2x+3y}$ or $(D - D')(D + D' + 1)z = e^{2x+3y}$.

$\therefore$ C.F. $= \phi_1(y + x) + e^{-x}\phi_2(y - x)$, ϕ_1, ϕ_2 being arbitrary functions.

and $$\text{P.I.} = \frac{1}{(D-D')(D+D'+1)}e^{2x+3y} = \frac{1}{(2-3)(2+3+1)}e^{2x+3y} = -\frac{1}{6}e^{2x+3y}$$

Hence the required general solution is z. $z = \phi_1(y + x) + e^{-x}\phi_2(y - x) - (1/6) \times e^{2x+3y}$.

Ex. 3. *Solve* $(D - D' - 1)(D - D' - 2)z = e^{2x-y}$.

Sol. Try yourself. **Ans.** $z = e^x\phi_1(y + x) + e^{2x}\phi_2(y + x) + (1/2) \times e^{2x-y}$

Ex. 4. *Solve* (*a*) $(D^2 - 4DD' + D - 1)z = e^{3x-2y}$.

(*b*) $(D^3 - 3DD' + D + 1)z = e^{2x+3y}$. **[Kanpur 2006]**

Sol. (*a*) Here $(D^2 - 4DD' + D - 1)$ cannot be resolved into linear factors in D and D'. Hence for finding C.F., consider the equation $(D^2 - 4DD' + D - 1)z = 0$. ...(1)

Let a trial solution of (1) be $z = Ae^{hx+ky}$. ...(2)

$\therefore$ $D^2z = Ah^2e^{hx+ky}$, $DD' = Ahke^{hx+ky}$, $Dz = Ahe^{hx+ky}$ and so (1) gives

$A(h^2 - 4hk + h - 1)e^{hx+ky} = 0$ so that $h^2 - 4hk + h - 1 = 0$

giving $k = (h^2 + h - 1)/4h$. ...(3)

$\therefore$ C.F. $= \Sigma Ae^{hx+ky}$, when k is given by (3).

$$\text{Again, P.I.} = \frac{1}{D^2 - 4DD' + D - 1}e^{3x-2y} = \frac{1}{3^2 - 4\times 3\times(-2) + 3 - 1}e^{3x-2y} = \frac{1}{35}e^{3x-2y}$$

Hence the required general solution is $z = \text{C.F.} + \text{P.I.}$ *i.e.* $z = \Sigma\, A e^{hx+ky} + (1/35) \times e^{3x-2y}$, where A, h and k are constants and k and h are related by (3).

(b) Proceed as in part (a). **Ans.** $z = \Sigma\, A e^{hx+ky} - (1/7) \times e^{2x+3y}$, where A, h, k are arbitrary constants and k is given by $k = (h^3 + h + 1)/3h$

Ex. 5. *Solve* $(\partial^2 y/\partial x^2) - (\partial^2 y/\partial z^2) = y + e^{x+z}$.

Sol. Re–writing, $(D^2 - D'^2 - 1)y = e^{x+z}$, where $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial z$.

$$\text{C.F.} = \sum A\, e^{hx+kz}, \qquad \text{where} \qquad h^2 - k^2 - 1 = 0 \qquad \text{... (1)}$$

$$\text{P.I.} = \frac{1}{D^2 - D'^2 - 1} e^{x+z} = \frac{1}{1-1-1} e^{x+z} = -e^{x+z}$$

$\therefore$ The required solution is $z = \Sigma A\, e^{hx+ky} - e^{x+z}$, where A, h and k are arbitrary constants and h and k are connected by relation (1).

EXERCISE 5(C)

Solve the following partial differential equations:

1. $(D^2 - DD' - 2D)z = e^{2x+y}$. **Ans.** $z = \phi_1(y) + e^{2x}\phi_2(y+x) - (1/2) \times e^{2x+y}$

2. $(D^2 - DD' - 2D'^2 + 2D + 2D')z = e^{2x+3y}$.

Ans. $z = \phi_1(y-x) + e^{-2x}\phi_2(y+2x) - (1/10) \times e^{2x+3y}$

3. $(D^2 - D'^2 + D + 3D' - 2)z = e^{x-y}$. **Ans.** $z = e^{-2x}\phi_1(y+x) + e^{x}\phi_2(y-x) - (1/4) \times e^{x-y}$

4. $(D^2 + D' + 4)z = e^{4x-y}$.

Ans. $z = \sum_i A_i\, e^{a_i x - (a_i^2+4)y} + (1/19) \times e^{4x-y}$, where A_i and a_i are arbitrary constants.

5. $(D^2 - D'^2 - 3D')z = e^{x+2y}$ **(Purvanchal 2007)**

Ans. $z = \Sigma\, A e^{(k^2+3k)^{1/2} x + ky} - (1/9) \times e^{x+2y}$

Type 2 : Examples based on case II of Art. 5.12.

Ex. 1. *Solve* $(D^2 + DD' + D' - 1)z = \sin(x + 2y)$.

[Bilaspur 2003; Bhopal 1998; Jiwaji 1997; Ravishankar 2004]

Sol. The given equation can be re–written as $(D + 1)(D + D' - 1)z = \sin(x + 2y)$.

$\therefore$ C.F. $= e^{-x}\phi_1(y) + e^{x}\phi_2(y - x)$, ϕ_1, ϕ_2 being arbitrary functions

and $$\text{P.I.} = \frac{1}{D^2 + DD' + D' - 1}\sin(x+2y) = \frac{1}{-1^2 - (1\cdot 2) + D' - 1}\sin(x+2y)$$

$$= \frac{1}{D' - 4}\sin(x+2y) = (D'+4)\frac{1}{D'^2 - 16}\sin(x+2y) = (D'+4)\frac{1}{-2^2 - 16}\sin(x+2y)$$

$$= -(1/20) \times (D'+4)\sin(x+2y) = -(1/20) \times [D'\sin(x+2y) + 4\sin(x+2y)]$$

$$= -(1/20) \times [2\cos(x+2y) + 4\sin(x+2y)].$$

$\therefore$ Solution is $z = e^{-x}\phi_1(y) + e^{x}\phi_2(y - x) - (1/10) \times [\cos(x + 2y) + 2\sin(x + 2y)]$.

Ex. 2. *Solve* $(\partial^2 z/\partial x^2) - (\partial z/\partial x\partial y) + (\partial z/\partial y) - z = \cos(x + 2y)$.

[Delhi Maths (H) 2001; M.D.U Rohtak 2004]

Sol. The given equation can be re–written as

$(D^2 - DD' + D' - 1)z = \cos(x + 2y)$ or $(D - 1)(D - D' + 1)z = \cos(x + 2y)$.

$\therefore$ C.F. $= e^x\phi_1(y) + e^{-x}\phi_2(y + x)$, ϕ_1, ϕ_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{D^2 - DD' + D' - 1}\cos(x + 2y) = \frac{1}{-1^2 + (1\cdot 2) + D' - 1}\cos(x + 2y) = \frac{1}{D'}\cos(x + 2y)$$

$= (1/2) \times \sin(x + 2y)$, as $1/D'$ stands for integration w.r.t. y keeping x as constant

Hence the required solution is $z = e^x\phi_1(y) + e^{-x}\phi_2(y + x) + (1/2) \times \sin(x + 2y)$.

Ex. 3. *Solve* $2(\partial^2 z/\partial x^2) + (\partial^2 z/\partial y^2) - 3(\partial z/\partial y) = 5\cos(3x - 2y)$.

Ans. $z = \phi_1(x) + e^{3x/2}\phi_2(2y - x) + (1/10) \times [4\cos(3x - 2y) + 3\sin(3x - 2y)]$.

Ex. 4. *Solve* $(D - D' - 1)(D - D' - 2)z = \sin(2x + 3y)$. **[KU Kurukshetra 2005]**

Sol. Here C.F. $= e^x\phi_1(y + x) + e^{2x}\phi_2(y + x)$, ϕ_1, ϕ_2 being arbitrary functions.

and
$$\text{P.I.} = \frac{1}{(D - D' - 1)(D - D' - 2)}\sin(2x + 3y) = \frac{1}{D^2 - 2DD' + D'^2 - 3D + 3D' + 2}\sin(2x + 3y)$$

$$= \frac{1}{-2^2 + 2\times(2\times 3) - 3^2 - 3D + 3D' + 2}\sin(2x + 3y)$$

$$= \frac{1}{-3D + 3D' + 1}\sin(2x + 3y) = D\frac{1}{-3D^2 + 3DD' + D}\sin(2x + 3y)$$

$$= D\frac{1}{-3\times(-2^2) + 3\times(2\times 3) + D}\sin(2x + 3y) = D\frac{1}{D - 6}\sin(2x + 3y)$$

$$= D(D + 6)\frac{1}{D^2 - 36}\sin(2x + 3y) = (D^2 + 6D)\frac{1}{-2^2 - 36}\sin(3x + 2y)$$

$= -(1/40) \times [D^2\sin(2x + 3y) + 6D\sin(2x + 3y)] = -(1/40) \times [-4\sin(2x + 3y) + 12\cos(2x + 3y)]$

$\therefore$ Solution is $z = e^x\phi_1(y + x) + e^{2x}\phi_2(y + x) + (1/10) \times [\sin(2x + 3y) - 3\cos(2x + 3y)]$.

Ex. 5. *Solve* (*a*) $(D - D'^2)z = \cos(x - 3y)$ **[Delhi Maths (Hons.) 1998, 2007, 2009, 2011]**

(*b*) $(D^2 - D')z = \cos(3x - y)$.

Sol. (*a*) Here $(D - D'^2)$ cannot be resolved into linear factors in D and D'. Hence in order to find C.F. of the given equation, consider the equation

$$(D - D'^2)z = 0. \qquad \text{...(1)}$$

Let a trial solution of (1) be $z = Ae^{hx + ky}$. ...(2)

$\therefore$ $Dz = Ahe^{hx + ky}$ and $D'^2 z = Ak^2e^{hx + ky}$. Then (1) gives

$A(h - k^2)e^{hx + ky} = 0$ so that $h - k^2 = 0$ or $h = k^2$.

$\therefore$ C.F. $= \Sigma\, Ae^{k(kx + y)}$, where A, k are arbitrary constants.

Now,
$$\text{P.I.} = \frac{1}{D - D'^2}\cos(x - 3y) = \frac{1}{D - (-3^2)}\cos(x - 3y)$$

$$= (D - 9)\frac{1}{(D + 9)(D - 9)}\cos(x - 3y) = (D - 9)\frac{1}{D^2 - 81}\cos(x - 3y) = \frac{(D - 9)}{-1^2 - 81}\cos(x - 3y)$$

$= -(1/82)\times[D\cos(x - 3y) - 9\cos(x - 3y)] = -(1/82) \times [-\sin(x - 3y) - 9\cos(x - 3y)]$.

$\therefore$ General solution is $z = \Sigma\, Ae^{k(kx + y)} + (1/82) \times [\sin(x - 3y) + 9\cos(x - 3y)]$

(*b*) **Ans.** $z = \Sigma A e^{h(x+hy)} - (1/82) \times [9 \cos(3x-y) - \sin(3x-y)]$, where A and h are arbitrary constants.

Ex. 5. (*c*) *Solve* $(D^2 - D')z = A \cos(lx + my)$, where A, l, m *are constants*.

Sol. Proceed as in Ex. 5(*a*). **Ans.** $z = \Sigma A' e^{hx+h^2 y} - \{A/(m^2 + l^4)\} \times \{m \sin(lx + my)$
$+ \ l^2 \cos(lx + my)\}$, where A' and h are arbitrary constants.

Ex. 6. *Solve* $(D^2 - DD' - 2D)z = \sin(3x + 4y)$. **[Delhi Maths (Hons.) 1997]**

Sol. The given equation can be re–written as $D(D - D' - 2)z = \sin(3x + 4y)$.

$\therefore$ C.F. $= \phi_1(y) + e^{2x}\phi_2(y + x)$, ϕ_1, ϕ_2 being arbitrary functions.

and P.I. $$= \frac{1}{D^2 - DD' - 2D} \sin(3x+4y) = \frac{1}{-3^2 + (3\times 4) - 2D} \sin(3x+4y) = \frac{1}{3-2D} \sin(3x+4y)$$

$$= (3+2D)\frac{1}{9-4D^2} = \frac{3+2D}{9-4D^2} \sin(3x+4y) = \frac{3+2D}{9-4(-3^2)} \sin(3x+4y) = \frac{1}{45}[3\sin(3x+4y) + 2D\sin(3x+4y)]$$

$= (1/45) \times [3 \sin(3x + 4y) + 6 \cos(3x + 4y)] = (1/15) \times [\sin(3x + 4y) + 2 \cos(3x + 4y)]$.

$\therefore$ Solution is $z = \phi_1(y) + e^{2x}\phi_2(y + x) + (1/15) \times [\sin(3x + 4y) + 2 \cos(3x + 4y)]$

Ex. 7. *Solve* $(3DD' - 2D'^2 - D')z = \sin(2x + 3y)$. **[Delhi Maths (H) 2001]**

Sol. Re–writing, given equation becomes $D'(3D - 2D' - 1)\, z = \sin(2x + 3y)$

$\therefore$ C.F. $= \phi_1(x) + e^{x/3}\phi_2(2x + 3y)$, ϕ_1, ϕ_2 are arbitrary functions

P.I. $$= \frac{1}{3DD' - 2D'^2 - D'} \sin(2x+3y) = \frac{1}{-3\times(2\times 3) - 2\times(-3^2) - D'} \sin(2x+3y)$$

$= -(1/D') \sin(2x + 3y) = (1/3) \times \cos(2x + 3y)$

$\therefore$ General solution is $z = \phi_1(x) + e^{x/3}\phi_2(2x + 3y) + (1/3) \times \cos(2x + 3y)$.

Ex. 8. $(D + D')\ (D + D' - 2)z = \sin(x + 2y)$ **[Delhi Maths (H) 2000]**

Sol. Here C.F. $= \phi_1(y - x) + e^{2x}\phi_2(y - x)$, ϕ_1, ϕ_2 being arbitrary function

P.I. $$= \frac{1}{(D+D')\,(D+D'-2)} \sin(x+2y) = \frac{1}{D^2 + 2DD' + D'^2 - 2D - 2D'} \sin(x+2y)$$

$$= \frac{1}{-1^2 - 2\times(1\times 2) - 2^2 - 2D - 2D'} \sin(x+2y)$$

$$= -\frac{1}{9 + 2(D+D')} \sin(x+2y) = -\{9 - 2\,(D+D')\}\cdot \frac{1}{81 - 4\,(D+D')^2} \sin(x+2y)$$

$$= \frac{-(9 - 2D - 2D')}{81 - 4D^2 - 4D'^2 - 8DD'} \sin(x+2y) = -\frac{9 - 2D - 2D'}{81 - 4(-1^2) - 4(-2^2) + 8\times(1\times 2)} \sin(x+2y)$$

$= (1/117)\times\{-9\sin(x+2y) + 2\cos(x+2y) + 4\cos(x+2y)\} = (3/117) \times \{2\cos(x+2y) - 3\sin(x+2y)\}$

$\therefore$ Solution is $z = \phi_1(y - x) + e^{2x}\ \phi_2(y - x) + (3/117) \times \{2\cos(x + 2y) - 3\sin(x + 2y)\}$

Ex. 9. *Solve* $(D^2 - DD' - 2D'^2 + 2D + 2D')z = \sin(2x + y)$

Sol. Re–writing the given equation, we get $(D + D')(D - 2D' + 2)z = \sin(2x + y)$.

$\therefore$ C.F. $= \phi_1(y - x) + e^{-2x}\phi_2(y + 2x)$, ϕ_1 and ϕ_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{D^2 - DD' - 2D'^2 + 2D + 2D'}\sin(2x + y) = \frac{1}{-2^2 + (2\times 1) - 2\times(-1^2) + 2D + 2D'}\sin(2x + y)$$

$$= \frac{1}{2(D + D')}\sin(2x + y) = \frac{D - D'}{2}\frac{1}{D^2 - D'^2}\sin(2x + y) = \frac{D - D'}{2}\frac{1}{-2^2 - (-1^2)}\sin(2x + y)$$

$$= -(1/6)\times(D - D')\sin(2x + y) = -(1/6)\times\{2\cos(2x + y) - \cos(2x + y)\}$$

$\therefore$ General solution is $z = \phi_1(y - x) + e^{-2x}\phi_2(y + 2x) - (1/6)\times\cos(2x + y)$

EXERCISE 5 (D)

Solve the following partial differential equations:

1. $(2DD' + D'^2 - 3D')z = 3\cos(3x - 2y)$. **[MDU Rohtak 2005]**

Ans. $z = \phi_1(x) + e^{3x/2}\phi_2(2y - x) + (3/50)\times\{4\cos(3x - 2y) + 3\sin(3x - 2y)\}$

2. $(D^2 + D')(D - D - D'^2)z = \sin(2x + y)$ **Ans.** $z = \sum_i A_i\, e^{a_i x - a_i^2 y} + \sum_i B_i\, e^{(b_i + b_i^2)x + b_i y}$

$-(1/34)\times\{5\sin(2x + y) - 3\cos(2x + y)\}$, which A_i, a_i, B_i, b_i are arbitrary constants

Type 3. Examples based on case III of Art. 5.12.

Ex. 1. *Solve* $s + p - q = z + xy$. **[Delhi Maths (Hons.) 1995, I.A.S. 1991]**

Sol. The given equation can be rewritten as $(\partial^2 z/\partial x\partial y) + (\partial z/\partial x) - (\partial z/\partial y) - z = xy$

or $(DD' + D - D' - 1)z = xy$ or $(D - 1)(D' + 1)z = xy$. ...(1)

$\therefore$ C.F. $= e^x\phi_1(y) + e^{-y}\phi_2(x)$, ϕ_1 and ϕ_2 being arbitrary functions

$$\text{P.I.} = \frac{1}{(D - 1)(D' + 1)}xy = -\frac{1}{(1 - D)(1 + D')}xy = -(1 - D)^{-1}(1 + D')^{-1}xy$$

$$= -(1 + D + \ldots)(1 - D' + \ldots)xy = -(1 + D - D' - DD' + \ldots)xy = -xy - y + x + 1.$$

$\therefore$ The required solution is $z = e^x\phi_1(y) + e^{-y}\phi_2(x) - xy - y + x + 1$.

Ex. 2. *Solve* (*a*) $r - s + 2q - z = x^2y^2$. **[I.A.S. 1993]**

(*b*) $(D^2 - D' - 1)z = x^2y$.

Sol. (*a*) The given equation can be re–written as

$(\partial^2 z/\partial x^2) - (\partial^2 z/\partial x\partial y) + 2(\partial z/\partial y) - z = x^2y^2$ or $(D^2 - DD' + 2D' - 1)z = x^2y^2$. ...(1)

Since $(D^2 - DD' + 2D' - 1)$ cannot be resolved into linear factors in D and D', hence C.F. of (1) is obtained by considering the equation $(D^2 - DD' + 2D' - 1)z = 0$. ...(2)

Let a trial solution of (2) be $z = Ae^{hx + ky}$ ...(3)

$\therefore$ $D^2z = Ah^2e^{hx + ky}$, $DD'z = Ahke^{hx + ky}$, $D'z = Ake^{hx + ky}$. Then (2) gives

$A(h^2 - hk + 2k - 1)e^{hx + ky} = 0$ or $h^2 - hk + 2k - 1 = 0$

so that $k = (1 - h^2)/(2 - h)$. ...(4)

$\therefore$ C.F. $= \Sigma Ae^{hx + ky}$, where A, h, k are arbitrary constants; h, k being related by (4).

$$\text{P.I.} = \frac{1}{D^2 - DD' + 2D' - 1}x^2y^2 = -\frac{1}{1-(D^2 - DD' + 2D')}x^2y^2 = -[1 - (D^2 - DD' + 2D')]^{-1}x^2y^2$$

$= -[1 + (D^2 - DD' + 2D') + (D^2 - DD' + 2D')^2 + \{D^2 + D'(2 - D)\}^3 + ...]x^2y^2$

$= -[1 + (D^2 - DD' + 2D') + (D^2D'^2 + 4D'^2 + 4D^2D' - 4DD'^2 + ...) + 3D^2D'^2(2 - D)^2 + ...]x^2y^2$

$= -(1 + D^2 - DD' + 2D' + D^2D'^2 + 4D'^2 + 4D^2D' - 4DD'^2 + 12D^2D'^2 + ...)x^2y^2$

$= -x^2y^2 - 2y^2 + 4xy - 4x^2y - 8x^2 - 16x - 16y - 52.$

$\therefore$ Solution is $z = \Sigma Ae^{hx + ky} - x^2y^2 - 2y^2 + 4xy - 4x^2y - 8x^2 - 16x - 16y - 52.$

(*b*) **Ans.** $z = \Sigma Ae^{hx + (h^2 - 1)y} + x^2 - x^2y - 2y + 4$, A and h being arbitrary constants.

Ex. 3. *Solve* (*a*) $(D^2 - D')z = 2y - x^2$. **[Delhi Maths (H.) 2004, 10; Agra 2005]**

(*b*) $(2D^2 - D'^2 + D)z = x^2 - y$.

Sol. (*a*) Here $D^2 - D'$ cannot be resolved into linear factors in D and D'. Hence to find C.F., we consider the equation $\qquad (D^2 - D')z = 0. \qquad$ ...(1)

Let a trial solution of (1) be $\qquad z = Ae^{hx + ky}. \qquad$ ...(2)

So $\quad D^2z = Ah^2e^{hx + ky}$ and $D'z = Ake^{hx + ky}$. Then (1) gives

$A(h^2 - k)e^{hx + ky} = 0 \quad$ or $\quad h^2 - k = 0 \quad$ so that $\quad k = h^2$.

$\therefore \quad$ C.F. $= \Sigma Ae^{hx + ky} = \Sigma Ae^{hx + h^2y}$, A, h being arbitrary constants.

$$\text{Now, P.I.} = \frac{1}{D^2 - D'}(2y - x^2) = \frac{1}{D^2(1 - D'/D^2)}(2y - x^2) = \frac{1}{D^2}\left(1 - \frac{D'}{D^2}\right)^{-1}(2y - x^2)$$

$$= \frac{1}{D^2}\left(1 + \frac{D'}{D^2} + ...\right)(2y - x^2) = \frac{1}{D^2}\left\{(2y - x^2) + \frac{1}{D^2}D'(2y - x^2)\right\} = \frac{1}{D^2}\left(2y - x^2 + \frac{1}{D^2}2\right)$$

$$= \frac{1}{D^2}\left(2y - x^2 + 2 \times \frac{x^2}{2}\right) = \frac{1}{D^2}(2y) = 2y \times \frac{x^2}{2} = x^2y.$$

General solution is $\qquad z = \Sigma Ae^{hx + h^2y} + x^2y$, A and h being arbitrary constants.

(*b*) **Ans.** $z = \Sigma Ae^{hx+ky} - (1/2) \times x^2y^2 + (1/6) \times y^3 - (1/12) \times xy^4 - (1/6) \times y^4 - (1/360) \times y^6$, where h and k are connected by the relation $2h^2 - k^2 + h = 0$

Ex. 4. *Solve* $r - s + p = 1$. **[Meerut 1993; Sagar 2003; Vikram 2004]**

Sol. The given equation can be re–written as $(\partial^2z/\partial x^2) - (\partial^2z/\partial x\partial y) + (\partial z/\partial x) = 1$

or $\quad (D^2 - DD' + D)z = 1 \quad$ or $\quad D(D - D' + 1)z = 1. \qquad$ ...(1)

$\therefore \quad$ C.F. $= \phi_1(y) + e^{-x}\phi_2(y + x)$, ϕ_1, ϕ_2 being arbitrary functions.

and $\quad \text{P.I.} = \dfrac{1}{D(1 + D - D')}1 = \dfrac{1}{D}(1 + D - D')^{-1}\,1 = \dfrac{1}{D}[1 - (D - D') + ...]\,1 = \dfrac{1}{D}1 = x$

So the required general solution is $\qquad z = \phi_1(y) + e^{-x}\phi_2(y + x) + x.$

Ex. 5. *Solve* (*a*) $D(D + D' - 1)(D + 3D' - 2)z = x^2 - 4xy + 2y^2$.

(*b*) $(D + D' - 1)(D + 2D' - 3)z = 2x + 3y$. **[Bhopal 2000, 03, 04]**

Sol. (*a*) Here C.F. $= \phi_1(y) + e^x\phi_2(y - x) + e^{2x}\phi_3(y - 3x)$, ϕ_1, ϕ_2, ϕ_3 being arbitrary functions

$$\text{P.I.} = \frac{1}{D(D + D' - 1)(D + 3D' - 2)}(x^2 - 4xy + 2y^2) = \frac{1}{2D}\{1 - (D + D')\}^{-1}\left\{1 - \frac{D + 3D'}{2}\right\}^{-1}(x^2 - 4xy + 2y^2)$$

$$= \frac{1}{2D}\{1+(D+D')+(D+D')^2+\dots\}\left\{1+\frac{D+3D'}{2}+\left(\frac{D+3D'}{2}\right)^2+\dots\right\}(x^2-4xy+2y^2)$$

$$= \frac{1}{2D}\left\{1+(D+D')+(D+D')^2+\frac{D+3D'}{2}+\left(\frac{D+3D'}{2}\right)^2\right.$$

$$\left.+\frac{(D+D')(D+3D')}{2}+\dots\right\}(x^2-4xy+2y^2)$$

$$= \frac{1}{2D}\left(1+\frac{3D}{2}+\frac{5D'}{2}+\frac{7D^2}{4}+\frac{19D'^2}{4}+\frac{11DD'}{2}+\dots\right)(x^2-4xy+2y^2)$$

$$= \frac{1}{2D}\left\{(x^2-4xy+2y^2)+3(x-2y)+5(2y-2x)+\frac{7}{2}+19-22\right\}$$

$$= \frac{1}{2D}\left(x^2-4xy+2y^2-7x+4y+\frac{1}{2}\right) = \frac{1}{2}\left(\frac{x^3}{3}-2x^2y+2y^2x-\frac{7x^2}{2}+4xy+\frac{x}{2}\right)$$

Hence the required general solution is $z = \text{C.F.} + \text{P.I.}$, *i.e.*,

$z = \phi_1(y) + e^x\phi_2(y-x) + e^{2x}\phi_3(y-3x) - (x^3/6) + x^2y + y^2x - (7x^2/4) + 2xy + x/4.$

(*b*) Try yourself. **Ans.** $z = e^x\phi_1(y-x) + e^{3x}\phi_2(y-2x) + 2x/3 + y + 23/9$

Ex. 6. *Solve* $(D-1)(D-D'+1)z = 1+xy.$ **[Delhi Maths (H) 2001]**

Sol. Here C.F. $= e^x\phi_1(y)+e^{-x}\phi_2(y+x)$, where ϕ_1 and ϕ_2 are arbitrary functions

$$\text{P.I.} = \frac{1}{(D-1)(D-D'+1)}(1+xy) = -(1-D)^{-1}\{1+(D-D')\}^{-1}(1+xy)$$

$$= -(1+D+\dots)\{1-(D-D')+(D-D')^2\dots\}(1+xy) = -(1+D+\dots)(1-D+D'-2DD'+\dots)(1+xy)$$

$$= -(1+D'-DD'+\dots)(1+xy) = -(1+xy+x-1) = -xy-x$$

$\therefore$ The required solution is $z = e^x\phi_1(y)+e^{-x}\phi_2(y+x)-xy-x$

Ex. 7. *Solve* $(D^2-D'^2-3D+3D')z = xy.$

Sol. Re–writing, given equation is $(D-D')(D+D'-3)z = xy.$

Its C.F. $= \phi_1(y+x)+e^{3x}\phi_2(x-y)$, ϕ_1, ϕ_2 being arbitrary functions

$$\text{P.I.} = \frac{1}{(D-D')(D+D'-3)} = -\frac{1}{3D}\left(1-\frac{D'}{D}\right)^{-1}\left(1-\frac{D+D'}{3}\right)xy$$

$$= -\frac{1}{3D}\left(1+\frac{D'}{D}+\dots\right)\left\{1+\frac{D+D'}{3}+\frac{(D+D')^2}{9}+\dots\right\}xy = -\frac{1}{3D}\left(1+\frac{D'}{D}+\dots\right)\left(1+\frac{1}{3}D+\frac{1}{3}D'+\frac{2}{9}DD'+\dots\right)xy$$

$$= -\frac{1}{3D}\left(1+\frac{1}{3}D+\frac{1}{3}D'+\frac{2}{9}DD'+\frac{D'}{D}+\frac{1}{3}D'+\dots\right)xy$$

$$= -\frac{1}{3D}\left(xy+\frac{1}{3}y+\frac{2}{3}x+\frac{2}{9}+\frac{x^2}{2}\right) = -\frac{1}{3}\left(\frac{x^2y}{2}+\frac{xy}{3}+\frac{x^2}{3}+\frac{2x}{9}+\frac{x^3}{6}\right)$$

$\therefore$ solution is $z = \phi_1(y+x)+e^{3x}\phi_2(y-x)-(1/6)\times x^2y-(x^2/9)-(2x/27)-(x^3/18)$

Ex. 8. Solve $(D+D'-1)(D+2D'-3)z=4+3x+6y.$

Sol. Here C.F. $=e^{x}\phi_1(y-x)+e^{3x}\phi_2(y-2x)$, ϕ_1, ϕ_2 being arbitrary functions

$$\text{P.I.} = \frac{1}{(D+D'-1)(D+2D'-1)}(4+3x+6y) = \frac{1}{3}\{1-(D+D')\}^{-1}\left\{1-\frac{D+2D'}{3}\right\}^{-1}(4+3x+6y)$$

$$=\frac{1}{3}(1+D+D'+...)\left(1+\frac{D}{3}+\frac{2D'}{3}+...\right)(4+3x+6y)=\frac{1}{3}\left(1+\frac{4D}{3}+\frac{5D'}{3}+...\right)(4+3x+6y)$$

$$=(1/3)\times(4+3x+6y+4+10)=6+x+2y.$$

$\therefore$ General solution is $z=e^{x}\phi_1(y-x)+e^{3x}\phi_2(y-2x)+6+x+2y.$

EXERCISE 5(E)

Solve the following partial differential equations:

1. $(D-D'-1)(D-D'-2)z=x.$ **Ans.** $z=e^{x}\phi_1(y+x)+e^{2x}\phi_2(y+x)+(2x+3)/4$

2. $(D^2-DD'-2D'^2+2D+2D')z=xy.$ **Ans.** $z=\phi_1(y-x)+e^{-2x}\phi_2(y+2x)+(1/4)\times(x^2y-xy-2x)$ $+(3x^2)/8-(x^3/12)$, ϕ_1, ϕ_2 being arbitrary functions

3. $(D^2-D'^2+D+3D'-2)z=x^2y$ **Ans.** $z=e^{-2x}\phi_1(y+x)+e^{x}\phi_2(y-x)-(4x^2y+4xy+6x^2+6y$ $+12x+21)/8$, ϕ_1, ϕ_2 being arbitrary functions.

Type 4. Examples based on case IV of Art. 5.12.

Ex. 1. *Solve* $(D^2-D')z = xe^{ax+a^2y}$.

Sol. Since (D^2-D') cannot be resolved into linear factors in D and D', C.F. is obtained by considering the equation $(D^2-D')z=0.$ ...(1)

Let a trial solution of (1) be $z=Ae^{hx+ky}.$ ...(2)

$\therefore$ $D^2z=Ah^2e^{hx+ky}$ and $D'z=Ake^{hx+ky}$. Then (1) becomes

$A(h^2-k)e^{hx+ky}=0$ so that $h^2-k=0$ or $k=h^2.$

$\therefore$ From (2), C.F. $=\Sigma A e^{hx+h^2y}$, A, h being arbitrary constants.

$$\text{P.I.} = \frac{1}{D^2-D'}xe^{ax+a^2y}=e^{ax+a^2y}\frac{1}{(D+a)^2-(D'+a^2)}x = e^{ax+a^2y}\frac{1}{D^2+2aD-D'}x$$

$$=e^{ax+a^2y}\frac{1}{2aD}\left(1+\frac{D}{2a}-\frac{D'}{2aD}\right)^{-1}x=e^{ax+a^2y}\frac{1}{2aD}\left\{1-\left(\frac{D}{2a}-\frac{D'}{2aD}\right)+...\right\}x=e^{ax+a^2y}\frac{1}{2aD}\left(x-\frac{1}{2a}\right)$$

$$= e^{ax+a^2y}\left\{(x^2/4a)-(x/4a^2)\right\}.$$

$\therefore$ General solution is $z=\sum Ae^{hx+h^2y}+e^{ax+a^2y}\{(x^2/4a)-(x/4a^2)\}.$

Ex. 2. *Solve* $(D-3D'-2)^2z=2e^{2x}\sin(y+3x)$ **[I.A.S. 2005]**

Sol. Here C.F. $=e^{2x}[\phi_1(y+3x)+x\phi_2(y+3x)]$, ϕ_1, ϕ_2 being arbitrary functions

$$\text{P.I.} = \frac{1}{(D-3D'-2)^2}2e^{2x+0.y}\sin(y+3x)= 2e^{2x+0.y}\frac{1}{\{(D+2)-3(D'+0)-2\}^2}\sin(y+3x)$$

$$= 2e^{2x}\frac{1}{(D'-3D')^2}\sin(y+3x) = 2e^{2x}\frac{x^2}{1^2\,2!}\sin(y+3x), \text{ using formula } (ii) \text{of Art. 4.8}$$

$\therefore$ Required solution is $\qquad z = e^{2x}[\phi_1(y+3x) + x\phi_2(y+3x)] + x^2e^{2x}\sin(y+3x).$

Ex. 3. *Solve* $\dfrac{\partial^2 z}{\partial x^2} - 4\dfrac{\partial^2 z}{\partial x\partial y} + 4\dfrac{\partial^2 z}{\partial y^2} + \dfrac{\partial z}{\partial x} - 2\dfrac{\partial z}{\partial y} = e^{x+y}.$

Sol. Given $(D^2 - 4DD' + 4D'^2 + D - 2D')z = e^{x+y}$ or $(D-2D')(D-2D'+1)z = e^{x+y}$.

$\therefore$ C.F. $= \phi_1(y+2x) + e^{-x}\phi_2(y+2x)$, ϕ_1, ϕ_2 being arbitrary functions

$$\text{P.I.} = \frac{1}{(D-2D'+1)}\left[\frac{1}{D-2D'}e^{x+y}\right] = \frac{1}{D-2D'-1}\,\frac{1}{1-2}e^{x+y}, \text{ by case I of Art. 5.12}$$

$$= -e^{x+y}\frac{1}{(D+1)-2(D'+1)+1}.1, \text{ using result of case IV of Art. 5.12}$$

$$= -e^{x+y}\frac{1}{D-2D'}1 = -e^{x+y}\frac{1}{D}\left(1-\frac{2D'}{D}\right)^{-1}1 = -e^{x+y}\frac{1}{D}1 = -xe^{x+y}$$

$\therefore$ The required solution is $\qquad z = \phi_1(y+2x) + e^{-x}\phi_2(y+2x) - xe^{x+y}.$

Ex. 4. *Solve* $(3D^2 - 2D'^2 + D - 1)z = 4e^{x+y}\cos(x+y)$. **[Delhi Maths (H) 1999, 2008]**

Sol. Since $(3D^2 - 2D'^2 + D - 1)$ cannot be resolved into linear factors in D and D', hence

C.F. $= \sum Ae^{hx+ky}$, where A, h are arbitrary constants connected by $3h^2 - 2k^2 + h - 1 = 0$.

$$\text{P.I.} = \frac{1}{3D^2 - 2D'^2 + D - 1}4e^{x+y}\cos(x+y) = 4e^{x+y}\frac{1}{3(D+1)^2 - 2(D'+1)^2 + (D+1) - 1}\cos(x+y)$$

$$= 4e^{x+y}\frac{1}{3D^2 + 7D - 2D'^2 - 4D' + 1}\cos(x+y) = 4e^{x+y}\frac{1}{3(-1^2) + 7D - 2(-1^2) - 4D' + 1}\cos(x+y)$$

$$= 4e^{x+y}\frac{1}{7D-4D'}\cos(x+y) = 4e^{x+y}(7D+4D')\frac{1}{49D^2 - 16D'^2}\cos(x+y)$$

$$= 4e^{x+y}\frac{7D+4D'}{49(-1^2) - 16(-1^2)}\cos(x+y)$$

$$= -(4/33)\times e^{x+y}(7D+4D')\cos(x+y) = -(4/33)e^{x+y}\times[7D\cos(x+y) + 4D'\cos(x+y)]$$

$$= -(4/33)\times e^{x+y}[-7\sin(x+y) - 4\sin(x+y)] = (4/3)\times e^{x+y}\sin(x+y).$$

Hence general solution is $\qquad z = \Sigma Ae^{hx+ky} + (4/3)\times e^{x+y}\sin(x+y).$

Ex. 5. *Solve* $(D-1)(D-D'+1)z = e^y$. **[Delhi Maths (H) 2001]**

Sol. Here C.F. $= e^x\phi_1(y) + e^{-x}\phi_2(y+x)$, ϕ_1, ϕ_2 being arbitrary functions

$$\text{P.I.} = \frac{1}{(D-1)(D-D'+1)}e^y = \frac{1}{D-D'+1}\,\frac{1}{D-1}e^{0\cdot x+1\cdot y} = \frac{1}{D-D'+1}\,\frac{1}{0-1}e^{0\cdot x+1\cdot y}1$$

$$= -e^{0\cdot x+1\cdot y}\frac{1}{(D+0)-(D'+1)+1}1 = -e^y\frac{1}{D}\left(1-\frac{D'}{D}\right)^{-1}1 = -e^y\frac{1}{D}\left(1+\frac{D'}{D}+\ldots\right) = -xe^y$$

$\therefore$ General solution is $\qquad z = e^x\phi_1(y) + e^{-x}\phi_2(y+x) - x\,e^y$

Ex. 6. *Solve* $(D-3D'-2)^2 z = 2e^{2x}\tan(y+3x)$. **[Delhi Maths (H) 2004]**

Sol. Here C.F. = $e^{2x}\{\phi_1(y+3x)+x\phi_2(y+3x)\}$, ϕ_1 and ϕ_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{(D-3D'-2)^2}\,2e^{2x}\tan(y+3x) = 2\frac{1}{(D-3D'-2)^2}\,e^{2x+0\cdot y}\tan(y+3x)$$

$$= 2e^{2x+0\cdot y}\frac{1}{\{(D+2)-3(D'+0)-2\}^2}\tan(y+3x) = 2e^{2x}\frac{1}{(D-3D')^2}\tan(y+3x)$$

$= 2e^{2x}\times(x^2/2!)\tan(y+3x)$, refer formula (*ii*), of Art. 4.12 of chapter 4

$\therefore$ General solution is $\quad z = e^{2x}\{\phi_1(y+3x)+x\,\phi_2(y+3x)\}+x^2e^{2x}\tan(y+3x)$

Ex. 7. *Solve* $(D^2-D'^2-3D+3D')z = e^{x+2y}$. **[Delhi Maths (H) 2005]**

Sol. Re–writing the given equation, we get $\quad (D-D')(D+D'-3)z = e^{x+2y}$.

Its C.F. $=\phi_1(y+x)+e^{3x}\phi_2(y-x)$, ϕ_1, ϕ_2 being arbitrary functions

$$\text{P.I.} = \frac{1}{D+D'-3}\left\{\frac{1}{D-D'}e^{x+2y}\right\} = \frac{1}{D+D'-3}\,\frac{1}{(1-2)}\,e^{x+2y}\cdot 1$$

$$= -e^{x+2y}\frac{1}{D+1+D'+2-3}1 = -e^{x+2y}\frac{1}{D}\left(1+\frac{D'}{D}\right)^{-1}1$$

$$= -e^{x+2y}(1/D)\,(1-D'/D+.....)\,1 = -e^{x+2y}x$$

$\therefore$ General solution is $\quad z = \phi_1(y+x)+e^{3x}\phi_2(y-x)-x\,e^{x+2y}$

Ex. 8. *Solve* (*i*) $(D^2-D')z = e^{x+y}$

(*ii*) $(D^2-D')z = e^{2x+y}$.

Sol. (*i*) C.F. = $\sum A\,e^{hx+ky}$, where $h^2-k=0$ so that $k=h^2$

$\therefore \quad$ C.F. $= \sum A\,e^{hx+h^2y} = \sum A\,e^{h(x+hy)}$

$$\text{P.I.} = \frac{1}{D^2-D'}e^{x+y}\cdot 1 = e^{x+y}\frac{1}{(D+1)^2-(D'+1)}1 = e^{x+y}\frac{1}{D^2+2D-D'}1$$

$$= -e^{x+y}\frac{1}{D'}\left(1-\frac{D^2+2D}{D'}\right)^{-1}1 = -e^{x+y}\frac{1}{D'}\left(1+\frac{D^2+2D}{D'}+....\right)1 = -e^{x+y}\frac{1}{D'}1 = -e^{x+y}y$$

$\therefore$ The required solution is $z = \sum A e^{h(x+hy)} - y e^{x+y}$, where A and h are arbitrary constants

(*ii*) C.F. is same as in part (*i*). Its P.I. is given by

$$\text{P.I.} = \frac{1}{D^2-D'}\,e^{2x+y} = \frac{1}{2^2-1}\,e^{2x+y} = \frac{1}{3}\,e^{2x+y}$$

$\therefore$ General solution is $z = \sum A\,e^{h(x+hy)} - (1/3)\times e^{2x+y}$, A, h being arbitrary constants.

Ex. 9. *Solve* $(D+D'-1)(D+D'-3)(D+D')z = e^{x+y}\sin(2x+y)$

Sol. C.F. = $e^{x}\phi_1(y-x)+e^{3x}\phi_2(y-x)+\phi_3(y-x)$, ϕ_1,ϕ_2,ϕ_3 being arbitrary functions.

$$\text{P.I.} = \frac{1}{(D+D'-1)(D+D'-3)(D+D')}\, e^{x+y}\sin(2x+y)$$

$$= e^{x+y}\frac{1}{(D+1+D'+1-1)(D+1+D'+1-3)(D+1+D'+1)}\sin(2x+y)$$

$$= e^{x+y}\frac{1}{(D+D'+1)(D+D'-1)(D+D'+2)}\sin(2x+y) = e^{x+y}\frac{1}{(D+D'+2)(D^2+2DD'+D'^2-1)}\sin(2x+y)$$

$$= e^{x+y}\frac{1}{D+D'+2}\,\frac{1}{-2^2-2\times(2\times1)-1^2-1}\sin(2x+y) = -\frac{e^{x+y}}{10}(D+D'-2)\frac{1}{(D+D')^2-4}\sin(2x+y)$$

$$= -\frac{e^{x+y}}{10}(D+D'-2)\frac{1}{D^2+2DD'+D'^2-4}\sin(2x+y) = -\frac{e^{x+y}}{10}(D+D'-2)\frac{1}{-2^2-2\times(2\times1)-1^2-4}\sin(2x+y)$$

$$= (1/130)\times e^{x+y}(D+D'-2)\sin(2x+y) = (1/130)\times e^{x+y}\{2\cos(2x+y)+\cos(2x+y)-2\sin(2x+y)\}$$

$\therefore$ Solution is $z = e^{x}\phi_1(y-x)+e^{3x}\phi_2(y-x)+\phi_3(y-x) + (1/130)\times e^{x+y}\{3\cos(2x+y)-2\sin(2x+y)\}$

Ex. 10. *Solve* $r-3s+2t-p+2q = (2+4x)e^{-y}$

Sol. Re–writing the given equation $\quad (D^2-3DD'+2D'^2-D+2D')z = (2+4x)\,e^{-y}$

or $\quad (D-2D')(D-D'-1)z = (2+4x)\,e^{-y}$

$\therefore$ C.F. = $\phi_1(y+2x)+e^{x}\phi_2(y+x)$, where ϕ_1,ϕ_2 are arbitrary functions.

$$\text{P.I.} = \frac{1}{(D-2D')(D-D'-1)}2\,e^{0\cdot x-y}(1+2x) = 2\,e^{0\cdot x-y}\frac{1}{\{D+0-2(D'-1)\}\,\{D+0-(D'-1)-1\}}(1+2x)$$

$$= 2e^{-y}\frac{1}{(D-2D'+2)(D-D')}(1+2x) = 2e^{-y}\frac{1}{2D}\left(1+\frac{D-2D'}{2}\right)^{-1}\left(1-\frac{D'}{D}\right)^{-1}(1+2x)$$

$$= e^{-y}\frac{1}{D}\left\{1-\frac{1}{2}(D-2D')+\dots\right\}\left(1+\frac{D'}{D}+\dots\right)(1+2x) = e^{-y}\frac{1}{D}\left(1-\frac{D}{2}+\dots\right)(1+2x)$$

$$= e^{-y}(1/D)(1+2x-1) = x^2e^{-y}$$

$\therefore$ The required solution is $\quad z = \phi_1(y+2x)+e^{x}\phi_2(y+x)+x^2e^{-y}$.

Ex. 11. *Solve* $(D^2-D')z = xe^{x+y}$

Sol. As usual C.F. = $\sum A\,e^{hx+ky}$, when $h^2-k=0$ or $k=h^2$. So C.F. $=\Sigma Ae^{hx+h^2y}$

$$\text{P.I.} = \frac{1}{D^2-D'}x\,e^{x+y} = e^{x+y}\frac{1}{(D+1)^2-(D'+1)}x = e^{x+y}\frac{1}{D^2+2D-D'}x$$

$$= e^{x+y}\frac{1}{2D}\left(1+\frac{D^2-D'}{2D}\right)^{-1}x = e^{x+y}\frac{1}{2D}\left(1-\frac{D^2-D'}{2D}+\dots\right)x = e^{x+y}\frac{1}{2D}\left(1-\frac{D}{2}+\frac{D'}{2D}+\dots\right)x$$

$$= e^{x+y}(1/2D)(x-1/2) = e^{x+y}(x^2/4-x/4).$$

$\therefore$ Solution is $z = \sum A\,e^{hx+h^2y} + (x/4)\times(x-1)\,e^{x+y}$, A, h being arbitrary constants.

EXERCISE 5(F)

Solve the following partial differential equations:

1. $D(D-2D')(D+D')z = e^{x+2y}(x^2+4y^2)$. **Ans.** $z = \phi_1(y)+\phi_2(y+2x)+\phi_3(y-x)$

$-(1/81)\times(9x^2+36y^2-18x-72y+76)e^{x+2y}$ ϕ_1, ϕ_2, ϕ_3 being arbitrary functions.

2. $(D^2+DD'+D+D'-1)z = e^{-2x}(x^2+y^2)$.

Ans. $z = \Sigma Ae^{hx+ky}+(1/27)\times e^{-2x}(9x^2+9y^2+18x+6y+14)$, where $h^2 + hk + h + k + 1 = 0$.

3. $(D^2D'+D'^2-2)z = e^{2y}\sin 3x - e^x \cos y$. **Ans.** $z = \Sigma Ae^{hx+ky} - (1/16)\times e^{2y}\sin 3x$

$+(1/20)\times e^x(3\cos 2y - \sin 2y)$, where h and k are related by $h^2k + k^2 - 2 = 0$.

4. $(D^2-DD'+D'-1)z = e^y$. **Ans.** $z = e^x\phi_1(y)+e^{-x}\phi_2(y+x)-xe^y$

5. $(D^2-DD'+D'-1)z = e^x$ **Ans.** $z = e^x\phi_1(y)+e^{-x}\phi_2(y+x)-(1/2)\times xe^x$

MISCELLANEOUS EXAMPLES ON ART. 5.12.

Ex. 1. *Solve* (*a*) $(D^2 - DD' + D' - 1)z = \cos(x + 2y) + e^y$. **[Jabalpur 2004; I.A.S. 1992]**
(*b*) $(D^2 - DD' + D' - 1)z = \cos(x + 2y) + e^x$.

Sol. (*a*) From given equation $(D - 1)(D - D' + 1)z = \cos(x + 2y) + e^y$. ...(1)

$\therefore$ C.F. $= e^x\phi_1(y) + e^{-x}\phi_2(y + x)$, ϕ_1, ϕ_2 being arbitrary functions.

P.I. corresponding to $\cos(x + 2y)$

$$= \frac{1}{D^2 - DD' + D' - 1}\cos(x+2y) = \frac{1}{-1^2 + (1\times 2) + D' - 1}\cos(x+2y)$$

$$= (1/D')\cos(x+2y) = (1/2)\times\sin(x+2y).$$

P.I. corresponding to e^y

$$= \frac{1}{(D-1)(D-D'+1)}e^y = \frac{1}{D-D'+1}\cdot\frac{1}{D-1}e^{0\cdot x+1\cdot y} = \frac{1}{D-D'+1}\cdot\frac{1}{0-1}e^{0\cdot x+1\cdot y}$$

$$= -e^{0\cdot x+1\cdot y}\frac{1}{(D+0)-(D'+1)+1}1 = -e^y\frac{1}{D(1-D'/D)}1 = -e^y\frac{1}{D}\left(1-\frac{D'}{D}+\ldots\right)^{-1}1$$

$$= -e^y(1/D)(1+\ldots).1 = -e^y x.$$

$\therefore$ The general solution is $z = e^x\phi_1(y) + e^{-x}\phi_2(y + x) + (1/2)\times\sin(x + 2y) - xe^y$.

(*b*) As in part (*a*), C.F. $= e^x\,\phi_1(y) + e^{-x}\,\phi_2(y + x)$, ϕ_1, ϕ_2 being arbitrary function.

and P.I. corresponding to $\cos(x + 2y) = (1/2)\times\sin(x + 2y)$.

Now, P.I. corresponding to e^x

$$= \frac{1}{(D-1)(D-D'+1)}e^x = \frac{1}{(D-1)(D-D'+1)}e^{1\cdot x+0\cdot y} = \frac{1}{(D-1)}\frac{1}{(1-0+1)}e^{1\cdot x+0\cdot y}$$

$$= \frac{1}{2}\frac{1}{D-1}e^{1\cdot x+0\cdot y}1 = \frac{1}{2}e^{1\cdot x+0\cdot y}\frac{1}{(D+1)-1}1 = \frac{1}{2}e^x\frac{1}{D}1 = \frac{xe^x}{2}.$$

$\therefore$ The general solution is $z = e^x\phi_1(y) + e^{-x}\phi_2(y + x) + (1/2)\times\sin(x + 2y) + (x/2)\times e^x$.

Ex. 2. *Solve* (*a*) $(D^2 - DD' - 2D'^2 + 2D + 2D')z = e^{2x+3y} + xy + \sin(2x+y)$. **[Delhi 2008]**

(*b*) $(D^2 - DD' - 2D'^2 + 2D + 2D')z = e^{2x+3y} + xy$.

(*c*) $(D^2 - DD' - 2D'^2 + 2D + 2D')z = xy + \sin(2x+y)$.

Sol. (*a*) The given equation can be rewritten as

$$(D + D')(D - 2D' + 2)z = e^{2x+3y} + xy + \sin(2x+y). \qquad \text{...(1)}$$

$\therefore$ C.F. $= \phi_1(y - x) + e^{-2x}\phi_2(y + 2x)$, ϕ_1, ϕ_2 being arbitrary functions.

P.I. corresponding to e^{2x+3y}

$$= \frac{1}{(D+D')(D-2D'+2)}e^{2x+3y} = \frac{1}{(2+3)(2-6+2)}e^{2x+3y} = -\frac{1}{10}e^{2x+3y}.$$

P.I. corresponding to xy

$$= \frac{1}{(D+D')(D-2D'+2)}xy = \frac{1}{D(1+D'/D)\times 2\{1+(D/2-D')\}}xy$$

$$= \frac{1}{2D}\left(1+\frac{D'}{D}\right)^{-1}\left\{1+\left(\frac{D}{2}-D'\right)\right\}^{-1}xy = \frac{1}{2D}\left(1-\frac{D'}{D}+\ldots\right)\left\{1-\left(\frac{D}{2}-D'\right)+\left(\frac{D}{2}-D'\right)^2+\ldots\right\}xy$$

$$= \frac{1}{2D}\left(1-\frac{D'}{D}+\ldots\right)\left(1-\frac{D}{2}+D'-DD'+\ldots\right)xy = \frac{1}{2D}\left(1-\frac{D'}{D}+\ldots\right)\left(xy-\frac{y}{2}+x-1\right)$$

$$= \frac{1}{2D}\left[xy-\frac{y}{2}+x-1-\frac{1}{D}\left(x-\frac{1}{2}\right)\right] = \frac{1}{2D}\left[xy-\frac{y}{2}+x-1-\frac{x^2}{2}+\frac{x}{2}\right]$$

$$= \frac{1}{2}\left[\frac{x^2y}{2}-\frac{xy}{2}+\frac{x^2}{2}-x-\frac{x^3}{6}+\frac{x^2}{4}\right] = \frac{x^2y}{4}+\frac{3x^2}{8}-\frac{xy}{4}-\frac{x}{2}-\frac{x^3}{12}.$$

P.I. corresponding to $\sin(2x+y)$

$$= \frac{1}{D^2-DD'-2D'^2+2D+2D'}\sin(2x+y) = \frac{1}{-2^2+(2\times 1)-2\times(-1^2)+2D+2D'}\sin(2x+y)$$

$$= \frac{1}{2(D+D')}\sin(2x+y) \quad = \frac{1}{2}(D-D')\frac{1}{(D^2-D'^2)}\sin(2x+y)$$

$$= \frac{1}{2}\frac{1}{-2^2-(-1^2)}(D-D')\sin(2x+y)$$

$$= -(1/6)\times(D-D')\sin(2x+y) = -(1/6)\times[D\sin(2x+y) - D'\sin(2x+y)]$$

$$= -(1/6)\times[2\cos(2x+y) - \cos(2x+y)] = -(1/6)\times\cos(2x+y).$$

The required solution is $z = \phi_1(y-x) + e^{-2x}\phi_2(y+2x) - (1/10)\times e^{2x+3y} + (1/4)\times x^2y$
$+ (3/8)\times x^2 - (1/4)\times xy - (x/2) - (x^3/12) - (1/6)\times\cos(2x+y)$

(*b*) As in part (*a*), C.F. $= \phi_1(y-x) + e^{-2x}\phi_2(y+2x)$.

P.I. corresponding to $e^{2x+3y} = -(1/10)\times e^{2x+3y}$

P.I. corresponding to $xy = (1/4)\times x^2y + (3/8)\times x^2 - (1/4)\times xy - (1/2)\times x - (1/12)\times x^3$.

$\therefore$ The required general solution is z = C.F. + P.I., *i.e.*

$$z = \phi_1(y-x) + e^{-2x}\phi_2(y+2x) - (1/10)\times e^{2x+3y} + (1/4)\times x^2y + (3/8)\times x^2 - (1/4)\times xy - (x/2) - (x^3/12).$$

(*c*) As in part (*a*), C.F. $= \phi_1(y-x) + e^{-2x}\phi_2(y+2x)$.

P.I. corresponding to $xy = (1/4) \times x^2 + (3/8) \times x^2 - (1/4) \times xy - (x/2) - (x^3/12)$
and P.I. corresponding to $\sin(2x + y) = -(1/6) \times \cos(2x + y)$.

$\therefore$ The required solution is $z = \text{C.F.} + \text{P.I.}$, *i.e.*, $z = \phi_1(y - x) + e^{-2x}\phi_2(y + 2x)$
$+ (1/4) \times x^2 + (3/8) \times x^2 - (1/4) \times xy - (x/2) - (x^3/12) \times - (1/6) \times \cos(2x + y)$.

Ex. 3. *Find a particular integral of the differential equation* : $(D^2 - D')z = e^{x+y} + 5\cos(x + 2y)$.

Sol. P.I. corresponding to e^{x+y}

$$= \frac{1}{D^2 - D'} e^{x+y} = e^{x+y} \frac{1}{(D+1)^2 - (D'+1)} 1 = e^{x+y} \frac{1}{D^2 + 2D - D'} 1$$

$$= e^{x+y} \frac{1}{2D}\left[1 + \left(\frac{D}{2} - \frac{D'}{2D}\right)\right]^{-1} 1 = e^{x+y} \frac{1}{2D}\{1 + \ldots\}1 = \frac{1}{2} x e^{x+y}$$

P.I. corresponding to $5\cos(x + 2y)$

$$= 5\frac{1}{D^2 - D'} \cos(x + 2y) = 5\frac{1}{-1^2 - D'} \cos(x + 2y) = -\frac{5}{D' + 1} \cos(x + 2y)$$

$$= -5(D' - 1)\frac{1}{D'^2 - 1} \cos(x + 2y) = -5\frac{1}{-2^2 - 1}(D' - 1)\cos(x + 2y)$$

$= (D' - 1)\cos(x + 2y) = D'\cos(x + 2y) - \cos(x + 2y) = -2\sin(x + 2y) - \cos(x + 2y)$

$\therefore$ Required P.I. $= (x/2) \times e^{x+y} - 2\sin(x + 2y) - \cos(x + 2y)$.

Ex. 4. Solve $(D^2 - D'^2 - 3D + 3D')z = xy + e^{x+2y}$. **[Delhi Maths (Prog) 2007; Delhi Maths (H) 2007; Meerut 1998; Bhopal 1995; Indore 1998; KU Kurukshetra 2004]**

Sol. The given equation can be re–written as $(D - D')(D + D' - 3)z = xy + e^{x+2y}$.

$\therefore$ C.F. $= \phi_1(y + x) + e^{3x}\phi_2(y - x)$, ϕ_1, ϕ_2 being arbitrary functions.

P.I. corresponding to $xy = \dfrac{1}{(D - D')(D + D' - 3)} xy = -\dfrac{1}{3D}\left(1 - \dfrac{D'}{D}\right)^{-1}\left(1 - \dfrac{D + D'}{3}\right)^{-1} xy$

$$= -\frac{1}{3D}\left(1 + \frac{D'}{D} + \ldots\right)\left[1 + \frac{D + D'}{3} + \left(\frac{D + D'}{3}\right)^2 + \ldots\right]xy$$

$$= -\frac{1}{3D}\left(1 + \frac{D'}{D} + \ldots\right)\left(1 + \frac{D + D'}{3} + \frac{2DD'}{9} + \ldots\right)xy = -\frac{1}{3D}\left(1 + \frac{D}{3} + \frac{D'}{3} + \frac{D'}{D} + \frac{D'}{3} + \frac{2DD'}{9} + \ldots\right)xy$$

$$= -\frac{1}{3D}\left(xy + \frac{y}{3} + \frac{2x}{3} + \frac{1}{D}x + \frac{2}{9}\right) = -\frac{1}{9}\left(\frac{x^2 y}{2} + \frac{xy}{2} + \frac{x^2}{3} + \frac{x^3}{6} + \frac{2x}{9}\right)$$

P.I. corresponding to e^{x+2y}

$$= \frac{1}{(D + D' - 3)} \frac{1}{D - D'} e^{x+2y} = \frac{1}{D + D' - 3} \frac{1}{(1 - 2)} e^{x+2y}$$

$$= -\frac{1}{D + D' - 3} e^{1\cdot x + 2y}\, 1 = -e^{1\cdot x + 2y} \frac{1}{(D+1) + (D'+2) - 3} 1 = -e^{x+2y} \frac{1}{D + D'} 1$$

$$= -e^{x+2y} \frac{1}{D}\left(1 + \frac{D'}{D}\right)^{-1} 1 = -e^{x+2y} \frac{1}{D}(1 + \ldots)\, 1 = -x\, e^{x+2y}.$$

Hence the required general solution is $z = \text{C.F} + \text{P.I.}$, *i.e.*

$z = \phi_1(y + x) + e^{3x}\phi_2(y - x) - (x^2y/6) - (xy/6) - (x^2/9) - (x^3/18) - (2x/27) - xe^{x+2y}$.

Ex. 5. *Solve* $(D - D' - 1)(D - D' - 2)z = e^{2x-y} + x$. **[Meerut 2008]**

Sol. Here C.F. $= e^x\phi_1(y + x) + e^{2x}\phi_2(y + x)$, ϕ_1, ϕ_2 being arbitrary functions

Now, P.I. corresponding to e^{2x-y}

$$= \frac{1}{(D-D'-1)(D-D'-2)} e^{2x-y} = \frac{1}{\{2-(-1)-1\}\{2-(-1)-2\}} e^{2x-y} = \frac{1}{2} e^{2x-y}$$

and P.I. corresponding to x

$$= \frac{1}{(D-D'-1)(D-D'-2)} x = \frac{1}{2\{1-(D-D')\}\{1-(D-D')/2\}} x$$

$$= \frac{1}{2}[1-(D-D')]^{-1}\left\{1-\frac{D-D'}{2}\right\}^{-1} x = \frac{1}{2}[1+(D-D')+\ldots]\left\{1+\frac{D-D'}{2}+\ldots\right\} x$$

$$= \frac{1}{2}\left\{1+(D-D')+\frac{D-D'}{2}+\ldots\right\} x = \frac{1}{2}\left\{1+\frac{3}{2}D+\ldots\right\} x = \frac{1}{2}\left(x+\frac{3}{2}\right).$$

$\therefore$ General solution is $z = e^x\phi_1(y+x) + e^{2x}\phi_2(y+x) + (1/2)\times e^{2x-y} + x/2 + 3/4.$

Ex. 6. *Solve* (*a*) $(D^2 - DD' - 2D)z = \sin(3x+4y) - e^{2x+y}$. **[Meerut 1995]**

(*b*) $(D^2 - DD' - 2D)z = \sin(3x+4y) + x^2 y$ **[Agra 2009, 10]**

Sol. (*a*) The given equation can be re–written as $D(D-D'-2)z = \sin(3x+4y) - e^{2x+y}$.

$\therefore$ C.F. $= \phi_1(y) + e^{2x}\phi_2(y+x)$, ϕ_1, ϕ_2 being arbitrary functions.

P.I. corresponding to $\sin(3x+4y)$

$$= \frac{1}{D^2 - DD' - 2D}\sin(3x+4y) = \frac{1}{-3^2 + (3\times 4) - 2D}\sin(3x+4y)$$

$$= \frac{1}{3-2D}\sin(3x+4y) = (3+2D)\frac{1}{9-4D^2}\sin(3x+4y) = \frac{3+2D}{9-4(-3^2)}\sin(3x+4y)$$

$= (1/45)\times[3\sin(3x+4y) + 2D\sin(3x+4y)] = (1/45)\times[3\sin(3x+4y) + 6\cos(3x+4y)]$

and P.I. corresponding to $(-e^{2x+y})$

$$= -\frac{1}{D(D-D'-2)} e^{2x+y} = -\frac{1}{2(2-1-2)} e^{2x+y} = \frac{1}{2} e^{2x+y}$$

Hence the required general solution is z = C.F. + P.I., *i.e.*

$z = \phi_1(y) + e^{2x}\phi_2(y+x) + (1/15)\times[\sin(3x+4y) + 2\cos(3x+4y)] + (1/2)\times e^{2x+y}$

(*b*) As in part (*a*), C.F. $= \phi_1(y) + e^{2x}\phi_2(y+x)$, ϕ_1, ϕ_2 being arbitrary functions.

P.I. corresponding to $\sin(3x+4y) = (1/15)\times[\sin(3x+4y) + 2\cos(3x+4y)]$.

$$\text{P.I. corresponding to } x^2y = \frac{1}{D(D-D'-2)} x^2 y = -\frac{1}{2D}\left\{1-\left(\frac{D-D'}{2}\right)\right\}^{-1} x^2 y$$

$$= -\frac{1}{2D}\left\{1+\frac{D-D'}{2}+\left(\frac{D-D'}{2}\right)^2+\left(\frac{D-D'}{2}\right)^3+\ldots\right\}x^2y = -\frac{1}{2D}\left(1+\frac{D}{2}-\frac{D'}{2}+\frac{D^2}{4}-\frac{DD'}{2}-\frac{3D^2D'}{8}+\ldots\right)x^2y$$

$$= -\frac{1}{2D}\left(x^2y+xy-\frac{x^2}{2}+\frac{y}{2}-x-\frac{3}{4}\right) = -\frac{1}{2}\left(\frac{x^3y}{3}+\frac{x^2y}{2}-\frac{x^3}{6}+\frac{xy}{2}-\frac{x^2}{2}-\frac{3x}{4}\right)$$

Hence the solution is z = C.F. + P.I., *i.e.* $z = \phi_1(y) + e^{2x}\phi_2(y+x) + (1/15)\times[\sin(3x+4y)$
$+ 2\cos(3x+4y)] - (1/6)\times x^3y - (1/4)\times x^2y + (1/12)\times x^3 - (1/4)\times xy - (x^2/4) + 3x/8$

Ex. 7. Solve $(\partial^2 z/\partial x^2) - (\partial^2 z/\partial y^2) + (\partial z/\partial x) + 3(\partial z/\partial y) - 2z = e^{x-y} - x^2 y$. **[Rewa 1999]**

Sol. The given equation can be re–written as $(D^2 - D'^2 + D + 3D' - 2)z = e^{x-y} - x^2y$

or $\{(D-D')(D+D') + 2(D+D') - (D-D'+2)\}z = e^{x-y} - x^2y$

or $\{(D+D')(D-D'+2) - (D-D'+2)\}z = e^{x-y} - x^2y$ or $(D-D'+2)(D+D'-1)z = e^{x-y} - x^2y$.

$\therefore$ C.F. $= e^{-2x}\phi_1(y+x) + e^x\phi_2(y-x)$, ϕ_1, ϕ_2 being arbitrary functions.

P.I. corresponding to e^{x-y}

$$= \frac{1}{(D - D' + 2)(D + D' - 1)} e^{x-y} = \frac{1}{\{1-(-1)+2\}(1-1-1)} e^{x-y} = -\frac{1}{4} e^{x-y}$$

and P.I. corresponding to $(-x^2 y)$

$$= \frac{1}{(D - D' + 2)(D + D' - 1)}(-x^2 y) = \frac{1}{2}\left\{1 + \frac{D-D'}{2}\right\}^{-1} \{1 - (D + D')\}^{-1} x^2 y$$

$$= \frac{1}{2}\left\{1 - \frac{D-D'}{2} + \left(\frac{D-D'}{2}\right)^2 - \left(\frac{D-D'}{2}\right)^3 + \dots\right\} \times \{1 + (D + D') + (D + D')^2 + (D + D')^3 + \dots\} x^2 y$$

$$= \frac{1}{2}\left(1 - \frac{D}{2} + \frac{D'}{2} + \frac{D^2}{4} - \frac{DD'}{2} + \frac{3D^2 D'}{8} + \dots\right) \times (1 + D + D' + D^2 + 2DD' + 3D^2 D' + \dots) x^2 y$$

$= (1/2) \times [1 + (1/2) \times D + (3/2) \times D' + (3/4) \times D^2 + (3/2) \times DD' + (21/8) \times D^2 D' + \cdots] x^2 y$

$= (1/2) \times [x^2 y + xy + (3x^2/2) + (3y/2) + 3x + 21/4].$

Hence general solution is $z =$ C.F. + P.I., *i.e.* $z = e^{-2x}\phi_1(y + x) + e^x \phi_2(y - x)$

$- (1/4) \times e^{x-y} + (1/2) \times x^2 y + (1/2) \times xy + (3/4) \times x^2 + (3/4) \times y + (3/2) \times x + 21/8.$

Ex. 8. *Solve* $(D^2 - D')(D - 2D')z = e^{2x+y} + xy.$

Sol. C.F. corresponding to linear factor $(D - 2D')$ is $\phi(y + 2x)$. Now, $(D^2 - D')$ cannot be resolved into linear factor in D and D'. To find C.F. corresponding to it, we consider the equation

$$(D^2 - D')z = 0. \qquad \dots(1)$$

Let a trial solution of (1) be $z = Ae^{hx+ky}$. ...(2)

$\therefore D^2 z = Ah^2 e^{hx+ky}$ and $D'z = Ake^{hx+ky}$. Then (1) becomes

$A(h^2 - k)e^{hx+ky} = 0$ so that $h^2 - k = 0$ or $k = h^2$.

So from (2), C.F. corresponding to $(D^2 - D')$ is $\sum Ae^{hx+h^2 y}$.

Now, P.I. corresponding to e^{2x+y}

$$= \frac{1}{D - 2D'} \cdot \frac{1}{D^2 - D'} e^{2x+y} = \frac{1}{D - 2D'} \frac{1}{2^2 - 1} e^{2x+y} = \frac{1}{3} \frac{1}{D - 2D'} e^{2x+y} \cdot 1$$

$$= \frac{1}{3} e^{2x+y} \frac{1}{(D + 2) - 2(D' + 1)} 1 = \frac{1}{3} e^{2x+y} \frac{1}{D(1 - 2D'/D)} 1 = \frac{1}{3} e^{2x+y} \frac{1}{D}\left(1 - \frac{2D'}{D}\right)^{-1} 1$$

$= (1/3) \times e^{2x+y} \times (1/D)(1 + \dots)\, 1 = (x/3) \times e^{2x+y}$

and P.I. corresponding to xy

$$= \frac{1}{(D - 2D')(D^2 - D')} xy = \frac{1}{(-2D')(1 - D/2D')(-D')(1 - D^2/D')} xy$$

$$= \frac{1}{2D'^2}\left(1 - \frac{D}{2D'}\right)^{-1}\left(1 - \frac{D^2}{D'}\right)^{-1} xy = \frac{1}{2D'^2}\left(1 + \frac{D}{2D'} + \dots\right)(1 + \dots) xy$$

$$= \frac{1}{2D'^2}\left(1 + \frac{D}{2D'} + \dots\right) xy = \frac{1}{2D'^2}\left(xy + \frac{1}{2D'} y\right) = \frac{1}{2D'^2}\left(xy + \frac{y^2}{4}\right) = \frac{1}{2}\left(\frac{xy^3}{6} + \frac{y^4}{3 \times 4 \times 4}\right).$$

$$\therefore \text{ General solution } z = \phi(y + 2x) + \sum Ae^{hx+h^2 y} + (x/3) \times e^{2x+y} + (xy^3)/12 + y^4/96,$$

where ϕ is an arbitrary function and A and h are arbitrary constants.

Ex. 9. *Solve* $(D^2 - DD' + D' - 1)z = \cos(x + 2y) + e^y + xy + 1.$

Sol. The given equation can be re–written as

$$(D - 1)(D - D' + 1)z = \cos(x + 2y) + e^y + xy + 1. \qquad ...(1)$$

Its C.F. $= e^x\phi_1(y) + e^{-x}\phi_2(y + x)$, ϕ_1, ϕ_2 being arbitrary functions.

Now, P.I. corresponding to $\cos(x + 2y)$

$$= \frac{1}{D^2 - DD' + D' - 1}\cos(x + y) = \frac{1}{-1^2 + (1\times 2) + D' - 1}\cos(x + 2y)$$

$$= (1/D')\cos(x + 2y) = (1/2)\times\sin(x + 2y),$$

P.I. corresponding to e^y *i.e.* $e^{0\cdot x + 1\cdot y}$

$$= \frac{1}{D^2 - DD' + D' - 1}e^{0\cdot x + 1\cdot y}\cdot 1 = e^{0\cdot x + 1\cdot y}\frac{1}{(D+0)^2 - (D+0)(D'+1) + (D'+1) - 1}1$$

$$= e^y\frac{1}{D^2 - DD' + D'}1 = e^y\frac{1}{D'}\left\{1 + \left(\frac{D^2}{D'} - D\right)\right\}^{-1}1 = e^y\frac{1}{D'}\{1 + \ldots\}1 = e^y y$$

and P.I. corresponding to $(xy + 1)$

$$= \frac{1}{(D-1)(D-D'+1)}(xy+1) = -(1-D)^{-1}\{1 + (D - D')\}^{-1}(xy + 1)$$

$$= -\{1 + D + \ldots\}\{1 - (D - D') + (D - D')^2 - \ldots\}(xy + 1) = -(1 + D + \ldots)(1 - D + D' - 2DD' + \ldots)(xy + 1)$$

$$= -(1 + D + \ldots)(xy + 1 - y + x - 2) = -(1 + D + \ldots)(xy - y + x - 1) = -(xy - y + x - 1 + y + 1) = -(xy + x)$$

$\therefore$ Solution is $\quad z = e^x\phi_1(y) + e^{-x}\phi_2(y + x) + (1/2)\times\sin(x + 2y) + ye^y - (xy + x).$

EXERCISE 5(G)

Solve the following partial differential equations:

1. $(D^2 - DD' - 2D^2 + 2D + 2D')z = e^{2x+3y} + \sin(2x + y)$ **[KU Kurukshetra 2004]**

Ans. $z = \phi_1(y - x) + e^{-2x}\phi_2(y + 2x) - (1/10)\times e^{2x+3y} - (1/6)\times\cos(2x + y).$

2. $(D^2 - DD' + D' - 1)\,z = e^y + xy.$ **[Delhi Maths (H) 2005]**

Hint : Do like solved Ex. 9, Art. 5.13 **Ans.** $z = e^x\phi_1(y) + e^{-x}\phi_2(y + x) + ye^y - xy - x + 1$

5.14. General method of finding particular integral for only reducible non–homogeneous linear partial differential equation, namely,

$$\boldsymbol{F(D, D')z = f(x, y)}$$

Let $\quad F(D, D') = (a_1D + b_1D' + c_1)(a_2D + b_2D' + c_2)\ldots(a_nD + b_nD' + c_n)$

$\therefore\quad$ P.I. of the given equation $= \dfrac{1}{F(D, D')}f(x, y)$

or $\quad$ P.I. $= \dfrac{1}{(a_1D + b_1D' + c_1)(a_2D + b_2D' + c_2)\ldots(a_nD + b_nD' + c_2)}f(x, y) \qquad \ldots(1)$

In order to evaluate P.I. given by (1), we consider a solution of the following equation (assuming that $a \neq 0$)

$(aD + bD + c)z = f(x, y) \quad$ or $\quad a\,p + b\,q = f(x, y) - c\,z \qquad \ldots(2)$

which is of the form $Pp + Qq = R$. So Lagrange's auxiliary equations are

$$\frac{dx}{a} = \frac{dy}{b} = \frac{dz}{f(x,y)-cz}$$

Taking the first two fraction of (3), $ady - bdx = 0$ ... (3)

Integrating, $ay - bx = d$, d being an arbitrary constant ... (4)

From (4), we have $y = (d+bx)/a$, if $a \neq 0$ ... (5)

Taking the first and last fractions of (3) and using (5), we get

$$\frac{dz}{dx} = \frac{f(x,y)-cz}{a} = -\frac{cz}{a} + \frac{1}{a} f\left(x, \frac{d+bx}{a}\right) \quad \text{or} \quad \frac{dz}{dx} + \frac{c}{a}z = \frac{1}{a} f\left(x, \frac{d+bx}{a}\right) \qquad \text{... (6)}$$

which is linear differential equation whose I.F. $= e^{\int (c/a)dx} = e^{cx/a}$

$\therefore$ Solution of (6) is $z e^{cx/a} = \dfrac{1}{a}\displaystyle\int f\left(x, \frac{d+bx}{a}\right)dx$

so that $z = \dfrac{e^{-(cx/a)}}{a}\displaystyle\int f\left(x, \frac{d+bx}{a}\right) dx, \quad a \neq 0$ and $d = ay - bx.$

$\therefore$ From (2), $$\frac{1}{(aD+bD'+c)} f(x,y) = \frac{e^{-(cx/a)}}{a}\int f\left(x, \frac{d+bx}{a}\right) dx \qquad \text{... (7)}$$

where $ay - bx = d$ and $a \neq 0$ d being arbitrary constant.

Similarly, if $b \neq 0$, we can show that

$$\frac{1}{(aD+bD'+c)} f(x,y) = \frac{e^{-(cy/b)}}{b}\int e^{cy/b} f\left(\frac{d+ay}{b}\right) dy, \qquad \text{... (8)}$$

where $bx - ay = d$ and $b \neq 0$, d being an arbitrary constant.

Results (7) and (8) will be used to evaluate P.I. given by (1).

5.15. Working rule for finding P.I. of any reducible linear partial differential equation (homogeneous or non–homogeneous), namely,

$$\boldsymbol{F(D, D')z = f(x, y)} \qquad \text{... (1)}$$

Rule I. $\dfrac{1}{aD+bD'+c} f(x,y) = \dfrac{e^{-(cx/a)}}{a}\displaystyle\int e^{cx/a} f\left(x, \frac{d+bx}{a}\right) dx,\ a \neq 0$ where $ay - bx = d$

Note that constant d must be replaced by $ay - bx$ after integration is performed.

Rule II. $\dfrac{1}{aD+bD'+c} f(x,y) = \dfrac{e^{-(cy/b)}}{b}\displaystyle\int e^{cy/b} f\left(\frac{d+ay}{b}\right) dy,\ b \neq 0$, where $bx - ay = d$

Note that constant d must be replaced by $bx - ay$ after integration is performed.

We now consider some special cases of the above rules.

Rule III. $\dfrac{1}{aD+c} f(x,y) = \dfrac{e^{-(cx/a)}}{a}\displaystyle\int e^{cx/a} f(x, d/a)\, dx$, where $ay = d$

Rule IV. $\dfrac{1}{D-mD'} f(x,y) = \displaystyle\int f(x, d-mx)\, dx$, where $y + mx = d$

Rule V. $\dfrac{1}{bD'+c} f(x,y) = \dfrac{e^{-(cy/b)}}{b}\displaystyle\int e^{cy/b} f(d/b, y)\, dy$, where $bx = d$

Rule VI. $\dfrac{1}{D'-mD}f(x,y) = \int f(d-my,\ y)\,dy$, where $x+my=d$

Note 1. Results IV and VI have already been obtained in Art. 4.12 of chapter 4.

Note 2. Suppose $F(D, D')$ can be factored as $\prod_{r=1}^{n}(a_rD+b_rD'+c_r)$, then

$$\text{P.I. for (1)} = \frac{1}{F(D,D')}f(x,y) = \frac{1}{(a_1D+b_1D'+c_1)(a_2D+b_2D'+C_2)....(a_nD+b_n\ D'+c_n)}f(x,y),$$

which is evaluated by using the above six rules for each factor, in succession, from right to the left.

5.16 SOLVED EXAMPLES BASED ON ART. 5.15.

Ex. 1. *Solve* $(D+D')\,(D+D'-2)z=\sin(x+2y)$ [**Delhi Maths (H) 2000**]

Sol. Here C.F. = $\phi_1(y-x)+e^{2x}\phi_2(y-x)$, ϕ_1, ϕ_2 being arbitrary function

$$\text{P.I.} = \frac{1}{D+D'-2}\left\{\frac{1}{D+D'}\sin\,(x+2y)\right\} = \frac{1}{D+D'-2}\int \sin\,(3x+2d)\,dx,\ \text{where,}\ y-x=d,$$

[using rule IV of Art. 5.15]

$$= \frac{1}{D+D'-2}\left\{-\frac{\cos(3x+2d)}{3}\right\} = -\frac{1}{3}\,\frac{1}{D+D'-2}\cos\,(2x+y)$$

$$= -\frac{1}{3}\,e^{2x}\int e^{-2x}\cos(3x+2d)dx,\ \text{where}\ y-x=d$$ [using rule I of Art. 5.15]

$$= -\frac{1}{3}e^{2x}\,\frac{1}{(-2)^2+3^2}e^{-2x}\{-2\cos(3x+2d)+3\sin(3x+2d)\} = \frac{2}{39}\cos(x+2y)-\frac{1}{13}\sin(x+2y)$$

$\therefore$ Solution is $\quad z=\phi_1(y-x)+e^{2x}\phi_2(y-x)+(2/39)\times\cos(x+2y)-(1/13)\times\sin(x+2y)$

Ex. 2. *Solve* $(D^3-DD'^2-D^2+DD')z=(x+2)/x^3$

Sol. Re–writing, the given equation $\quad D(D-D')\,(D+D'-1)z=(x+2)/x^3$

Its C.F. = $\phi_1(y)+\phi_2(y+x)+e^x\phi_3(y-x)$, ϕ_1, ϕ_2, ϕ_3 being arbitrary functions

$$\text{P.I.} = \frac{1}{(D-D')(D+D'-1)}\,\frac{1}{D}\left(\frac{1}{x^2}+\frac{2}{x^3}\right) = \frac{1}{(D+D'-1)\,(D-D)}\left(-\frac{1}{x}-\frac{1}{x^2}\right)$$

$$= \frac{1}{D+D'-1}\int\left(-\frac{1}{x}-\frac{1}{x^2}\right)dx = \frac{1}{D+D'-1}\left(-\log x+\frac{1}{x}\right) = e^x\int e^{-x}\left(-\log x+\frac{1}{x}\right)dx$$

$$= -e^x\int e^{-x}\log x\,dx+e^x\int e^{-x}\frac{1}{x}dx = -e^x\left[(-e^{-x})\log x-\int(-e^{-x})\frac{1}{x}dx\right]+e^x\int e^{-x}\frac{1}{x}dx = \log x$$

[on integration by parts first integral only]

$\therefore$ General solution is $\quad z=\phi_1(y)+\phi_2(y+x)+e^x\phi_3(y-x)\ +\log x.$

Ex. 3. *Solve* $(D^2+DD'+D'-1)\,z=4\sinh x$.

Sol. Re–writing, the given equation $\quad (D+1)\,(D+D'-1)\,z=2(e^x-e^{-x})$.

C.F. = $e^{-x}\phi_1(y)+e^x\phi_2(y-x)$, where ϕ_1,ϕ_2 are arbitrary functions

$$\text{P.I.} = \frac{1}{(D+1)(D+D'-1)} 2(e^x - e^{-x}) = \frac{1}{(D+1)} 2e^x \int e^{-x}(e^x - e^{-x})dx$$

$$= \frac{1}{(D+1)} 2e^x \left(x + \frac{1}{2} e^{-2x} \right) = \frac{1}{D+1}(2xe^x + e^{-x}) = e^{-x} \int e^x (2x\ e^x + e^{-x})dx$$

[using rule III of Art. 5.15]

$$= 2e^{-x} \int x\, e^{2x} dx + e^{-x} x = 2e^{-x} \left[x \times (e^{2x}/2) - \int 1 \cdot (e^{2x}/2) dx \right] + xe^{-x}$$

$$= x\, e^x - e^{-x} \int e^{2x} dx + xe^{-x} = xe^x - e^{-x} \times (1/2) \times e^{2x} + xe^{-x} = (x - 1/2)e^x + xe^{-x}$$

$\therefore$ General solution is $\quad z = e^{-x}\phi_1(y) + e^x \phi_2(y - x) + (x - 1/2)e^x + xe^{-x}$

Ex. 4. *Solve* $(D^2 - DD' + D' - 1)z = 1 + xy + e^y + \cos(x + 2y)$ **[Delhi Maths (H) 2001]**

Sol. Re–writing the given equation $\quad (D - 1)\ (D - D' + 1)z = 1 + xy + e^y + \cos(x + 2y)$

C.F. = $e^x \phi_1(y) + e^{-x} \phi_2(y + x)$, ϕ_1, ϕ_2 being arbitrary functions.

$$\text{P.I.} = \frac{1}{(D-1)\ (D - D' + 1)} \{1 + xy + e^y + \cos\ (x + 2y)\}$$

$$= \frac{1}{D-1} e^{-x} \int e^x \{1 + x\ (d - x) + e^{d-x} + \cos(2d - x)\} dx, \quad \text{where } d = y + x$$

$$= \frac{1}{D-1}\ e^{-x} \left\{ \int\ (1 + d\,x - x^2) e^x dx + \int e^a dx + \int e^x \cos(x - 2d) dx \right\}$$

$$= \frac{1}{D-1} e^{-x} [(1 + dx - x^2)e^x - (d - 2x)\, e^x + (-2)e^x + e^d x + \frac{e^x}{1^2 + 1^2} \{\cos\ (x - 2d) + \sin (x - 2d)\}]$$

$$= \frac{1}{D-1} [(1 + d\,x - x^2 - d + 2x - 2) + e^{d-x} x + (1/2) \times \{\cos(x - 2d) + \sin (x - 2d)\}]$$

$$= \frac{1}{D-1} [-1 + x(y + x) - x^2 - (y + x) + 2x + e^y x + (1/2) \times \{\cos(-2y - x) + \sin(-2y - x)\}]$$

$$= \frac{1}{D-1} \{xy - y + x - 1 + x\, e^y + (1/2) \times \cos(2y + x) - (1/2) \times \sin\,(2y + x)\}$$

$$= \frac{1}{D-1} \{(x - 1)\ (y + 1) + xe^y + (1/2) \times \cos(2y + x) - (1/2) \times \sin(2y + x)\}$$

$$= e^x \int e^{-x} \{(x - 1)\ (k + 1) + xe^k + (1/2) \times \cos(2k + x) - (1/2) \times \sin(2k + x)\}, \text{ where } y = k$$

$$= e^x \left[(k + 1) \int e^{-x}(x - 1)dx + e^k \int xe^{-x} dx + \frac{1}{2} \int e^{-x} \cos(2k + x)]\ dx - \frac{1}{2} \int e^{-x} \sin(2k + x) dx \right]$$

$$= e^x (k + 1)\ \{(-e^{-x})\ (x + 1) - (e^{-x})\ (1)\} + e^x e^k \{(-e^{-x})(x) - (e^{-x})(1)\}]$$

$$+ \frac{e^x}{2} \frac{e^{-x}}{(-1)^2 + 1^2} \{-\cos(2k + x) + \sin(2k + x)\} - \frac{e^x}{2} \frac{e^{-x}}{(-1)^2 + 1^2} \{-\sin(2k + x) - \cos(2k + x)]$$

$$= -(k + 1)\, x - e^k\ (x + 1) + (1/2) \times \sin\,(2k + x)] = -(y + 1)x - e^y\,(x + 1) + (1/2)\sin\,(2y + x)$$

$\therefore$ solution is $\quad z = e^x \phi_1(y) + e^{-x} \phi_2(y + x) - x(y + 1) - (x + 1)e^y + (1/2) \times \sin(2y + x))$

Ex 5. *Solve* $(D^2 - DD' - 2D'^2 + 2D + 2D')z = xy + \sin(2x + y)$

Ans. $z = \phi_1(y - x) + e^{-2x}\phi_2(y + 2x) + (x/24) \times (6xy - 6y + 9x - 2x^2 - 12) - (1/6) \times \cos(2x + y).$

5.17. Solutions under given geometrical conditions

We have seen that solutions of non–homogeneous linear partial differential equations involve arbitrary functions of x and y. We shall now determine these functions under the given geometrical conditions. This will lead to the required surface satisfying the given differential equation under the prescribed geometrical conditions.

Example. *Find a surface satisfying* $r + s = 0$, *i.e.*, $(D^2 + DD')z = 0$ *and touching the elliptic paraboloid* $z = 4x^2 + y^2$ *along its section by the plane* $y = 2x + 1$. **[I.A.S. 1994]**

Sol. Given $(D^2 + DD')z = 0.$ or $D(D + D') = 0$ ...(1)

$\therefore$ Solution of (1) is $z = \text{C.F.} = \phi_1(y) + \phi_2(y - x),$ ...(2)

where ϕ_1 and ϕ_2 are arbitrary functions.

Since (2) touches the curve given by $z = 4x^2 + y^2$ ...(3)

and $y = 2x + 1,$ ...(4)

values of $p\ (= \partial z/\partial x)$ and $q\ (= \partial z/\partial y)$ obtained from (2) and (3) must be equal for any point on (4).

$\therefore$ $-\phi_2'(y - x) = 8x$ for $y = 2x + 1$ or $\phi_2'(x + 1) = -8x.$ ...(5)

and $\phi_1'(y) + \phi_2'(y - x) = 2y$ for $y = 2x + 1$ or $\phi_1'(2x + 1) + \phi_2'(x + 1) = 4x + 2$ ...(6)

From (5), $\phi_2'(x) = 8 - 8x$

Integrating it, $\phi_2(x) = 8x - 4x^2 + c_1$, c_1 being an arbitrary constant ...(7)

Subtracting (5) from (6), $\phi_1'(2x + 1) = 12x + 2 = 6(2x + 1) - 4$

so that $\phi_1'(x) = 6x - 4.$

Integrating it, $\phi_1(x) = 3x^2 - 4x + c_2$, c_2 being an arbitrary constant ...(8)

From (8), $\phi_1(y) = 3y^2 - 4y + c_2.$

and from (7), $\phi_2(y - x) = 8(y - x) - 4(y - x)^2 + c_1.$

Putting the above values of $\phi_1(y)$ and $\phi_2(y - x)$ in (2), we get

$$z = 3y^2 - 4y + c_2 + 8(y - x) - 4(y - x)^2 + c_1$$

or $z = -y^2 + 4y - 8x - 4x^2 + 8xy + c_3,$ where $c_3 = c_1 + c_2.$ ...(9)

Equating the values of z from (3) and (9), we get

$4x^2 + y^2 = -y^2 + 4y - 8x - 4x^2 + 8xy + c_3,$ where $y = 2x + 1.$

$\therefore$ $c_3 = 8x^2 + 2y^2 - 4y + 8x - 8xy = 8x^2 + 2(2x + 1)^2 - 4(2x + 1) + 8x - 8x(2x + 1) = -2$

Hence, from (9), the required surface is $4x^2 - 8xy + y^2 - 4y + z + 2 = 0.$

MISCELLANEOUS PROBLEM ON CHAPTER 5

1. Find the solution of the equation $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 = e^{-x}\cos y$, which tends to zero as $x \to \infty$ and has the value *cos y* when $x = 0$ **Ans.** $z = (1 - x/2)e^{-x}\cos y$

6

Partial Differential Equations Reducible to Equations with Constant Coefficients

6.1. INTRODUCTION

In chapters 4 and 5, we have discussed methods of solving linear partial differential equations with constant coefficients. In this chapter, we propose to discuss the method of solving the so-called *Euler-Cauchy type partial differential equations* of the form

$$a_0 x^n \frac{\partial^n z}{\partial x^n} + a_1 x^{n-1} y \frac{\partial^n z}{\partial x^{n-1}\partial y} + a_2 x^{n-2} y^2 \frac{\partial^n z}{\partial x^{n-2}\partial y^2} + \dots + a_n y^n \frac{\partial^n z}{\partial y^n} + \dots = f(x, y), \qquad \dots(1)$$

having variable coefficients in particular form (namely the term $\partial^n z/\partial x^n$ is multiplied by x^n, $\partial^n z/\partial y^n$ is multiplied by y^n, $\partial^n z/\partial x^r \partial y^{n-r}$, $r = 1, 2, \dots, n-1$ is multiplied by $x^r y^{n-r}$ and so on. If $D \equiv \partial/\partial x$ and $D' \equiv \partial/\partial y$, then (1) can be re-written as

$$(a_0 x^n D^n + a_1 x^{n-1} y\, D^{n-1} D' + a_2\, x^{n-2} y^2\, D^{n-2} D'^2 + \dots + a_n y^n D'^n + \dots)z = f(x, y) \qquad \dots(2)$$

Examples of such equations are: $(x^2 D^2 - y^2 D'^2)z = xy; x^2 D^2 - y^2 D'^2 + xD - yD' = \log x$

6.2. METHOD OF REDUCING EULER-CAUCHY TYPE EQUATION TO A LINEAR PARTIAL DIFFERENTIAL EQUATION WITH CONSTANT COEFFICIENTS

Consider Euler-Cauchy type equation

$$a_0 x^n D^n + a_1 x^{n-1} y D^{n-1} D' + a_2 x^{n-2} y^2 D^{n-2} D'^2 + \dots + a_n y^n D'^n + \dots)\, z = f(x, y) \qquad \dots(1)$$

where $D \equiv \partial/\partial x$ and $D' \equiv \partial/\partial y$. Define two new variables u and v by

$$x = e^u \quad \text{and} \quad y = e^v \quad \text{so that} \quad u = \log x \quad \text{and} \quad v = \log y \qquad \dots(2)$$

Let $\quad D_1 \equiv \partial/\partial u \quad$ and $\quad D_1' \equiv \partial/dv \qquad \dots(3)$

Now, $$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} = \frac{1}{x}\frac{\partial z}{\partial u}, \text{ using (2)}$$

$$\therefore \quad x\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \quad \text{so that} \quad x\frac{\partial}{\partial x} = \frac{\partial}{\partial u} \quad \text{or} \quad xD = D_1 \qquad \dots(4)$$

Again, $$x\frac{\partial}{\partial x}\left(x^{n-1}\frac{\partial^{n-1} z}{\partial x^{n-1}}\right) = x^n \frac{\partial^n z}{\partial x^n} + (n-1)x^{n-1}\frac{\partial^{n-1} z}{\partial x^{n-1}}$$

$$\therefore \quad x^n \frac{\partial^n z}{\partial x^n} = \left(x\frac{\partial}{\partial x} - n + 1\right)x^{n-1}\frac{\partial^{n-1} z}{\partial x^{n-1}}$$

so that $$x^n D^n = (D_1 - n + 1)x^{n-1} D^{n-1}. \qquad \dots(5)$$

Putting $n = 2, 3, \dots$ in (5), we have

$$x^2 D^2 = (D_1 - 1)xD \quad \text{or} \quad x^2 D^2 = D_1(D_1 - 1), \text{ using (4)} \qquad \dots(6)$$

$x^3D^3 = (D_1 - 2)x^2D^2$ or $x^3D^3 = D_1(D_1 - 1)(D_1 - 2)$, using (6) ...(7)

and so on. Similarly, we have

$yD' = D_1'$, $y^2D'^2 = D_1'(D_1' - 1)$, $y^3D'^3 = D_1'(D_1' - 1)(D_1' - 2)$ ...(8)

and so on. Also, we have $xyDD' = D_1D_1'$. ...(9)

and $x^my^nD^m D'^n = D_1(D_1 - 1)...(D_1 - m + 1)D_1'(D_1' - 1)...(D_1' - n + 1)$. ...(10)

Using the substitutions (2) and results (4), (5), (6), (7), (8), (9) and (10), the given equation (1) reduces to an equation having constant coefficients and now it can easily be solved by the methods already discussed for homogeneous (refer Chapter 4) and non-homogeneous (refer Chapter 5) linear equations with constant coefficients. Finally, with help of (2), the solution is obtained in terms of old variables x and y.

6.3. WORKING RULE FOR SOLVING EULER-CAUCHY TYPE PARTIAL DIFFERENTIAL EQUATION

$$(a_0x^nD^n + a_1x^{n-1}y\,D^{n-1}D' + a_2x^{n-2}y\,D^{n-2}\,D'^2 + ... + a_ny^nD'^n + ...)z = f(x, y) \quad ...(1)$$

Step 1 Introduce two new variables u and v :

$x = e^u$ and $y = e^v$, *i.e.*, $u = \log x$ and $v = \log y$ ...(2)

Step 2 We have $D \equiv \partial/\partial x$ and $D' \equiv \partial/\partial y$. Also let $D_1 \equiv \partial/\partial u$, $D_1' \equiv \partial/\partial v$

Step 3 Use the following results in (1)

$$\left.\begin{aligned} &xD = D_1, \quad yD' = D_1', \quad x^2D^2 = D_1(D_1 - 1), \quad y^2D'^2 = D_1'(D_1' - 1) \\ &x^3D^3 = D_1(D_1 - 1)(D_1 - 2), \quad y^3D'^3 = D_1'(D_1' - 1)(D_1' - 2) \text{ and so on} \end{aligned}\right\} \quad ...(3)$$

Step 4 Using (2) and (3) in (1), we obtain the following linear partial differential (homogeneous or non-homogeneous) $(b_0\,D_1^n + b_1D_1^{n-1}D_1' + b_2D_1^{n-2}D_1'^2 + ... + b_nD_1'^n + ...)z = g(u, v)$ ...(4)

Step 5 If (4) is homogeneous linear partial differential equation, then it is solved with help of methods of chapter 4. Again, if (4) is non-homogeneous linear partial differential equation, then it is solved with help of methods of chapter 5.

Step 6 Using $u = \log x$ and $v = \log y$ in the solution obtained in step 5, we finally obtain the required solution in terms of the original variables x and y.

6.4. SOLVED EXAMPLES BASED ON ART. 6.3.

Ex. 1. *Solve* $x^2(\partial^2 z/\partial x^2) - y^2(\partial^2 z/\partial y^2) - y(\partial z/\partial y) + x(\partial z/\partial x) = 0$. **(Jabalpur 1996)**

Sol. Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$. ...(1)

Also, let $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$, $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$.

Then the given equation $(x^2D^2 - y^2D'^2 - yD' + xD)z = 0$ becomes

$$[D_1(D_1 - 1) - D_1'(D_1' - 1) - D_1' + D_1]z = 0$$

or $(D_1^2 - D_1'^2)z = 0$ or $(D_1 - D_1')(D_1 + D_1')z = 0$.

Hence the required general solution is $z = \text{C.F.} = \phi_1(v + u) + \phi_2(v - u)$

or $z = \phi_1(\log y + \log x) + \phi_2(\log y - \log x)$, using (1)

or $z = \phi_1 \log(xy) + \phi_2 \log(y/x)$ or $z = f_1(xy) + f_2(y/x)$, where f_1, f_2 are arbitrary functions.

Ex.2. *Solve* $x^2(\partial^2 z/\partial x^2) + 2xy(\partial^2 z/\partial x\partial y) + y^2(\partial^2 z/\partial y^2) = 0$. **[Delhi Maths (H) 1994, CDLU 2004]**

or Solve $x^2r + 2xys + y^2t = 0$ **(Purvanchal 2007)**

Sol. Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$. ...(1)

Also, let $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$, $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$.

Then the given equation can be written as $(x^2D^2 + 2xyDD' + y^2D'^2)z = 0$ which reduces to

$$[D_1(D_1 - 1) + 2DD' + D'(D' - 1)]z = 0$$

$$[(D_1 + D_1')^2 - (D_1 + D_1')]z = 0 \qquad \text{or} \qquad (D_1 + D_1')(D_1 + D_1' - 1)z = 0.$$

Hence the required general solution is $z = \text{C.F.} = \phi_1(v - u) + e^u\phi_2(v - u)$

or $z = \phi_1(\log y - \log x) + x\phi_2(\log y - \log x)$, using (1)

or $z = \phi_1 \log (y/x) + x\phi_2 \log (y/x)$ or $z = f_1(y/x) + xf_2(y/x)$, where f_1 and f_2 are arbitrary functions.

Ex. 3. *Solve* $x^2(\partial^2 z/\partial x^2) - 3xy(\partial^2 z/\partial x\partial y) + 2y^2(\partial^2 z/\partial y^2) + 5y(\partial z/\partial y) - 2z = 0$.

Sol. Let $x = e^u$, $y = e^v$, so that $u = \log x$, $v = \log y$. ...(1)

Also, let $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$, $D_1 \equiv \partial/\partial u$, $D_1' \equiv \partial/\partial v$.

Then the given equation $(x^2D^2 - 3xyDD' + 2y^2D'^2 + 5yD' - 2)z = 0$ becomes

$$[D_1(D_1 - 1) - 3D_1D_1' + 2D_1'(D_1' - 1) + 5D_1' - 2]z = 0$$

or $(D_1^2 - 3D_1D_1' + 2D_1'^2 - D_1 + 3D_1' - 2)z = 0$ or $(D_1 - D_1' - 2)(D_1 - 2D_1' + 1)z = 0.$

Hence general solution is $z = \text{C.F.} = e^{2u}\phi_1(v + u) + e^{-u}\phi_2(v + 2u)$

or $z = (e^u)^2\phi_1(\log y + \log x) + (e^u)^{-1}\phi_2(\log y + 2 \log x) = x^2\phi_1 \log (xy) + x^{-1}\phi_2 \log (yx^2)$, using (1)

or $z = x^2 f_1(xy) + x^{-1}f_2(yx^2)$, f_1, f_2 being arbitrary functions.

Ex. 4. *Solve* $(x^2D^2 + 2xyDD' + y^2D'^2)z = x^my^n$, where $(m + n) \neq 0, 1$.

[Delhi Maths (H) 1994, KU Kurukshetra 2004]

Sol. Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$. ...(1)

Also, let $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/dv$.

Then the given equation reduces to $[D_1(D_1 - 1) + 2D_1D_1' + D_1'(D_1' - 1)]z = e^{mu}.e^{nv}$

or $[(D_1 + D_1')^2 - (D_1 + D_1')]z = e^{mu+nv}$ or $(D_1 + D_1')(D_1 + D_1' - 1) = e^{mu+nv}$

Here C.F.$= \phi_1(v - u) + e^u\phi_2(v - u) = \phi_1(\log y - \log x) + x\phi_2(\log y - \log x)$, using (1)

$\therefore$ C.F. $= \phi_1 \log (y/x) + x\phi_2 \log (y/x) = f_1(y/x) + xf_2(y/x)$, where f_1 and f_2 are arbitrary functions.

$$\text{Now, P.I.} = \frac{1}{(D_1 + D_1')(D_1 + D_1' - 1)}e^{mu+nv} = \frac{1}{(m + n)(m + n - 1)}e^{mu+nv}$$

$$= \frac{1}{(m + n)\,(m + n - 1)}(e^u)^m(e^{v})^n = \frac{1}{(m + n)(m + n - 1)}x^my^n, \text{ using (1).}$$

$\therefore$ Required general solution is $z = f_1(y/x) + xf_2(y/x) + [1/\{(m + n)(m + n - 1)\}]x^my^n$.

Ex. 5. *Solve* $x^2(\partial^2 z/\partial x^2) - 4xy(\partial^2 z/\partial x\partial y) + 4y^2(\partial^2 z/\partial y^2) + 6y(\partial z/\partial y) = x^3y^4$.

[Jabalpur 2004; Vikram 2004; Meerut 1999; Delhi Maths (H) 1995]

Sol. Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$. ...(1)

Also, let $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$, $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$.

Then the given equation $(x^2D^2 - 4xyDD' + 4y^2D'^2 + 6yD')z = x^3y^4$ becomes

$$[D_1(D_1 - 1) - 4D_1D_1' + 4D_1'(D_1' - 1) + 6D_1']z = e^{3u}e^{4v}$$

or $$[(D^2{}_1 - 4D_1D_1' + 4D'^2_1) - (D_1 - 2D_1')]z = e^{3u+4v}$$

or $[(D_1 - 2D_1')^2 - (D_1 - 2D_1')]z = e^{3u+4v}$ or $(D_1 - 2D_1')(D_1 - 2D_1' - 1)z = e^{3u+4v}$

Here C.F. $= \phi_1(v + 2u) + e^u\phi_2(v + 2u) = \phi_1(\log y + 2 \log x) + x\phi_2(\log y + 2 \log x)$, using (1)

$\therefore$ C.F. $= \phi_1 \log (yx^2) + x\phi_2 \log (yx^2) = f_1(yx^2) + xf_2(yx^2)$, where f_1 and f_2 are arbitrary functions.

Now, P.I. $= \dfrac{1}{(D_1 - 2D_1')(D_1 - 2D_1' - 1)} e^{3u+4v} = \dfrac{1}{\{3-(2\times 4)\}\times\{3-(2\times 4)-1\}} e^{3u+4v}$

$= (1/30)\times(e^u)^3(e^v)^4 = (1/30)\times x^3y^4$, using (1)

$\therefore$ The required general solution is $z =$ C.F. + P.I. or $z = f_1(yx^2) + xf_2(yx^2) + (1/30)\times x^3y^4$.

Ex. 6. *Solve* $x^2(\partial^2 z/\partial x^2) + 2xy(\partial^2 z/\partial x\partial y) - x(\partial z/\partial x) = x^3/y^2$.

Sol. Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$. ...(1)

Also, let $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$, $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$.

Then the given equation $(x^2D^2 + 2xyDD' - xD)z = x^3y^{-2}$ becomes

$$[D_1(D_1-1) + 2D_1D_1' - D_1]z = (e^u)^3(e^v)^{-2}$$

or $(D_1^2 + 2D_1D_1' - 2D_1)z = e^{3u-2v}$ or $D_1(D_1 + 2D_1' - 2)z = e^{3u-2v}$

$\therefore$ C.F. $= \phi_1(v) + e^{2u}\phi_2(v-2u) = \phi_1(v) + (e^u)^2\phi_2(v-2u) = \phi_1(\log y) + x^2\,\phi_2(\log y - 2\log x)$, using (1)

$= \phi_1(\log y) + x^2\phi_2(\log(y/x^2)) = f_1(y) + x^2f_2(y/x^2)$, where f_1 and f_2 are arbitrary functions.

Now, P.I. $= \dfrac{1}{D_1(D_1 + 2D_1' - 2)} e^{3u-2v} = \dfrac{1}{3(3-4-2)}(e^u)^3(e^v)^{-2} = -\dfrac{x^3y^{-2}}{9} = -\dfrac{x^3}{9y^2}$

Hence the required general solution is $z =$ C.F. + P.I. or $z = f_1(y) + x^2f_2(y/x^2) - (x^3/9y^2)$.

Ex. 7. *Solve* $x^2r - 3xys + 2y^2t + px + 2qy = x + 2y$.

Sol. The given equation can be re-written as

$$x^2(\partial^2 z/\partial x^2) - 3xy(\partial^2 z/\partial x\partial y) + 2y^2(\partial^2 z/\partial y^2) + x(\partial z/\partial x) + 2y(\partial z/\partial y) = x + 2y$$

or $(x^2D^2 - 3xyDD' + 2y^2D'^2 + xD + 2yD')z = x + 2y$. ...(1)

Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$. ...(2)

Also, let $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$, $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$.

$\therefore$ (1) becomes $[D_1(D_1-1) - 3D_1D_1' + 2D_1'(D_1'-1) + D_1 + 2D_1']z = e^u + 2e^v$

or $(D_1^2 - 3D_1D_1' + 2D_1'^2)z = e^u + 2e^v$ or $(D_1 - D_1')(D_1 - 2D_1')z = e^u + 2e^v$.

$\therefore$ C.F. $= \phi_1(v+u) + \phi_2(v+2u) = \phi_1(\log y + \log x) + \phi_2(\log y + 2\log x)$

or C.F. $= \phi_1\log(xy) + \phi_2\log(x^2y) = f_1(xy) + f_2(x^2y)$, where f_1 and f_2 are arbitrary functions.

Also, P.I. =

$$\frac{1}{(D_1 - D_1')(D_1 - 2D_1')}(e^u + 2e^v) = \frac{1}{(D_1 - D_1')(D_1 - 2D_1')}e^{1\cdot u + 0\cdot v} + 2\frac{1}{(D_1 - D_1')(D_1 - 2D_1')}e^{0\cdot u + 1\cdot v}$$

$$= \frac{1}{(1-0)(1-0)}e^u + 2\frac{1}{(0-1)(0-2)}e^v = x + y, \text{ using (2)}$$

Hence the required general solution is $z =$ C.F. + P.I. or $z = f_1(xy) + f_2(x^2y) + x + y$.

Ex. 8. *Find the general solution of* $x^2(\partial^2 z/\partial x^2) + 2xy(\partial^2 z/\partial x\partial y) + y^2(\partial^2 z/\partial y^2) + nz = n\{x(\partial z/\partial x) + y(\partial z/\partial y)\} + x^2 + y^2 + x^3$.

Sol. Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$. ...(1)

Also, let $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$, $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$.

Then, the given equation reduces to

$$[x^2D^2 + 2xyDD' + y^2D'^2 - n(xD + yD') + n]z = x^2 + y^2 + x^3$$

or $[D_1(D_1-1) + 2D_1D_1' + D_1'(D_1'-1) - n(D_1 + D_1') + n]z = e^{2u} + e^{2v} + e^{3u}$

or $\{(D_1 + D_1')^2 - (D_1 + D_1') - n(D_1 + D_1' - 1)\}z = e^{2u} + e^{2v} + e^{3u}$

or $\{(D_1 + D_1')(D_1 + D_1' - 1) - n(D_1 + D_1' - 1)\}z = e^{2u} + e^{2v} + e^{3u}$

or $(D_1 + D_1' - 1)(D_1 + D_1' - n)z = e^{2u} + e^{2v} + e^{3u}$.

$\therefore$ C.F. $= e^u\phi_1(v-u) + e^{nu}\phi_2(v-u) = e^u\phi_1(v-u) + (e^u)^n\phi_2(v-u)$

$= x\phi_1(\log y - \log x) + x^n\phi_2(\log y - \log x) = x\phi_1\log(y/x) + x^n\phi_2\log(y/x)$, using (1)

$= xf_1(y/x) + x^n f_2(y/x)$, where f_1 and f_2 are arbitrary functions.

Also, $\text{P.I.} = \dfrac{1}{(D_1 + D_1' - 1)(D_1 + D_1' - n)}(e^{2u} + e^{2v} + e^{3u}) = \dfrac{1}{(D_1 + D_1' - 1)(D_1 + D_1' - n)} e^{2u + 0\cdot v}$

$$+ \frac{1}{(D_1 + D_1' - 1)(D_1 + D_1' - n)} e^{0\cdot u + 2v} + \frac{1}{(D_1 + D_1' - 1)(D_1 + D_1' - n)} e^{3u + 0\cdot v}$$

$$= \frac{(e^u)^2}{(2+0-1)(2+0-n)} + \frac{(e^v)^2}{(0+2-1)(0+2-n)} + \frac{(e^u)^3}{(3+0-1)(3+0-n)} = \frac{x^2+y^2}{2-n} + \frac{x^3}{2(3-n)}$$

Hence general solution is $z = xf_1(y/x) + x^n f_2(y/x) + (x^2 + y^2)/(2 - n) + x^3/2(3 - n)$.

Ex. 9. *Solve* $x^2(\partial^2 z/\partial x^2) - y^2(\partial^2 z/\partial y^2) = xy$ or $(x^2D^2 - y^2D'^2)z = xy$.

(Bilaspur 1999, Jabalpur 2003, Jiwaji 2003, 04, Vikram 2004, Ravishankar 2010, I.A.S. 1987, Rohilkhand 1995, Delhi Maths (H) 2004, 06)

Sol. Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$. ...(1)

Also, let $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$, $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$.

Then the given equation $(x^2D^2 - y^2D'^2)z = xy$ becomes

$[D_1(D_1 - 1) - D_1'(D_1' - 1)]z = e^u e^v$ or $(D_1^2 - D_1'^2 - D_1 + D_1')z = e^{u+v}$

or $[(D_1 - D_1')(D_1 + D_1') - (D_1 - D_1')]z = e^{u+v}$ or $(D_1 - D_1')(D_1 + D_1' - 1)z = e^{u+v}$.

$\therefore$ C.F. $= \phi_1(v + u) + e^u\phi_2(v - u) = \phi_1(\log y + \log x) + x\phi_2(\log y - \log x)$, using (1)

or C.F. $= \phi_1 \log(xy) + x\phi_2 \log(y/x) = f_1(xy) + xf_2(y/x)$, where f_1 and f_2 are arbitrary functions.

Also, $\text{P.I.} = \dfrac{1}{(D_1 - D_1')(D_1 + D_1' - 1)} e^{u+v} = \dfrac{1}{D_1 - D_1'} \dfrac{1}{(1+1-1)} e^{u+v} = \dfrac{u}{1!} e^{u+v} = ue^u e^v = xy \log x$

Hence the required general solution is $z = \text{C.F.} + \text{P.I.}$ or $z = f_1(xy) + xf_2(y/x) + xy\log x$.

Ex. 10. *Solve* $yt - q = xy$.

Sol. The given equation can be rewritten as $y(\partial^2 z/\partial y^2) - (\partial z/\partial y) = xy$

or $y^2(\partial^2 z/\partial y^2) - y(\partial z/\partial y) = xy^2$ or $(y^2D'^2 - yD')z = xy^2$. ...(1)

Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$. ... (2)

Also, let $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$, $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$.

Then (1) becomes $[D_1'(D_1' - 1) - D_1']z = e^u e^{2v}$ or $D_1'(D_1' - 2)z = e^{u+2v}$

$\therefore$ C.F. $= \phi_1(u) + e^{2v}\phi_2(u) = \phi_1(\log x) + y^2\phi_2(\log x)$, by (2)

$= f_1(x) + y^2 f_2(x)$, f_1 and f_2 being arbitrary functions.

Also, $\text{P.I.} = \dfrac{1}{(D_1' - 2)D_1'} e^{u+2v} = \dfrac{1}{D_1 - 2} \dfrac{1}{2} e^{u+2v} = \dfrac{1}{2} e^{u+2v} \dfrac{1}{D' + 2 + 2} \cdot 1$

$$= \frac{1}{2} e^{u+2v} \frac{1}{D'} 1 = \frac{1}{2} e^u \times (e^v)^2 \times v = \frac{xy^2}{2} \log y.$$

Hence the required general solution is $z = \text{C.F.} + \text{P.I.}$ or $z = f_1(x) + y^2 f_2(x) + (1/2) \times xy^2 \log y$.

Ex. 11. *Solve* $x^2(\partial^2 z/\partial x^2) + 2xy(\partial^2 z/\partial x\partial y) + y^2(\partial^2 z/\partial y^2) = (x^2 + y^2)^{n/2}$. **[Delhi Maths 1999]**

Sol. Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$. ...(1)

Also, let $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$, $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$.

Then, the given equation can be re-written as $(x^2D^2 + 2xyDD' + y^2D'^2)z = (x^2 + y^2)^{n/2}$

or $[D_1(D_1 - 1) + 2D_1D_1' + D_1'(D_1' - 1)]z = (e^{2u} + e^{2v})^{n/2}$

or $[(D_1 + D_1')^2 - (D_1 + D_1')]z = (e^{2u} + e^{2v})^{n/2}$ or $(D_1 + D_1')(D_1 + D_1' - 1)z = (e^{2u} + e^{2v})^{n/2}$.

$\therefore$ C.F. $= \phi_1(v-u) + e^u\phi_2(v-u) = \phi_1(\log y - \log x) + x\phi_2(\log y - \log x)$, using (1)

or C.F. $= \phi_1 \log(y/x) + x\phi_2 \log(y/x) = f_1(y/x) + xf_2(y/x)$, where f_1 and f_2 are arbitrary functions.

$$\text{P.I.} = \frac{1}{(D_1+D_1')(D_1+D_1'-1)}(e^{2u}+e^{2v})^{n/2} = \frac{1}{(D_1+D_1')(D_1+D_1'-1)}e^{nu}\{1+e^{2(v-u)}\}^{n/2}$$

$$= \frac{1}{(D_1+D_1')(D_1+D_1'-1)}e^{nu}\left\{1+\frac{n}{2}e^{2(v-u)}+\frac{(n/2)\{(n/2)-1\}}{2!}e^{4(v-u)}+\ldots\right\}$$

$$= \frac{1}{(D_1+D_1')(D_1+D_1'-1)}e^{nu+0\cdot v} + \frac{n}{2}\frac{1}{(D_1+D_1')(D_1+D_1'-1)}e^{(n-2)u+2v} + \frac{(n/2)\{(n/2)-1\}}{2!}\frac{1}{(D_1+D_1')(D_1+D_1'-1)}e^{(n-4)u+4v}+\ldots$$

$$= \frac{1}{(n+0)(n+0-1)}e^{nu} + \frac{n}{2}\frac{1}{\{(n-2)+2\}\{(n-2)+2-1\}}e^{(n-2)u+2v} + \frac{(n/2)[(n/2)-1]}{2!}\frac{1}{\{(n-4)+4\}\{(n-4)+4-1\}}e^{(n-4)u+4v}+\ldots$$

$$= \frac{e^{nu}}{n^2-n}\left[1+\frac{n}{2}e^{2(v-u)}+\frac{(n/2)\{(n/2)-1\}}{2!}e^{4(v-u)}+\ldots\right] = \frac{e^{nu}}{n^2-n}\{1+e^{2(v-u)}\}^{n/2}$$

$$= \frac{1}{n^2-n}\{e^{2u}+e^{2u}e^{2(v-u)}\}^{n/2} = \frac{1}{n^2-n}(e^{2u}+e^{2v})^{n/2} = \frac{1}{n^2-n}(x^2+y^2)^{n/2}, \text{ using (1)}$$

Hence the required general solution is $z = f_1(y/x) + xf_2(y/x) + \{1/(n^2-n)\}(x^2+y^2)^{n/2}$.

Ex.12. *Solve* $x^2r - y^2t + px - qy = \log x$. **[KU Kurukshatra 2004; Meerut 2008]**

Or $(x^2D^2 - y^2D'^2 + xD - yD')z = \log x$. **[Delhi Maths (G) 2004; I.A.S. 1997]**

Sol. Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$. ...(1)

Also, let $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$, $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$.

Then the given equation $(x^2D^2 - y^2D'^2 + xD - yD')z = \log x$ becomes

$$[D_1(D_1-1) - D_1'(D_1'-1) + D_1 - D_1']z = u \qquad \text{or} \qquad ({D^2}_1 - {D_1}'^2)z = u$$

or
$$(D_1 + D_1')(D_1 - D_1')z = u.$$

$\therefore$ C.F. $= \phi_1(v-u) + \phi_2(v+u) = \phi_1(\log y + \log x) + \phi_2(\log y - \log x)$

or C.F. $= \phi_1 \log(xy) + \phi_2 \log(y/x) = f_1(xy) + f_2(y/x)$, where f_1 and f_2 are arbitrary functions.

and
$$\text{P.I.} = \frac{1}{D_1^2 - D_1'^2}u = \frac{1}{D_1^2(1-D_1'^2/D_1^2)}u = \frac{1}{D_1^2}\left(1-\frac{D_1'^2}{D_1^2}\right)^{-1}u = \frac{1}{D_1^2}\left(1+\frac{D_1'^2}{D_1^2}+\ldots\right)u$$
$$= u^3/6 = (\log x)^3/6, \text{ using (1)}$$

$\therefore$ Required solution is $z = f_1(xy) + f_2(y/x) + (1/6)\times(\log x)^3$, f_1, f_2 being arbitrary functions.

Ex. 13. *Solve* $(x^2D^2 - xyDD' - 2y^2D'^2 + xD - 2yD')z = \log(y/x) - (1/2)$.

[Delhi B.Sc. (Hons) III 2011]

Sol. Let $x = e^u$, $y = e^v$ so that $u = \log x$, $v = \log y$. ...(1)

Also, let $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$, $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$.

Then the given equation reduces to

$$[D_1(D_1-1) - D_1D_1' - 2D_1'(D_1'-1) + D_1 - 2D_1']z = \log y - \log x - (1/2)$$

or $(D_1^2 - D_1D_1' - 2D_1'^2)z = v - u - (1/2)$ or $(D_1 - 2D_1')(D_1 + D_1') = v - u - (1/2)$.

$\therefore$ C.F. $= \phi_1(v+2u) + \phi_2(v-u) = \phi_1(\log y + 2\log x) + \phi_2(\log y - \log x)$

or C.F. $= \phi_1(\log(yx^2)) + \phi_2(\log(y/x)) = f_1(yx^2) + f_2(y/x)$, where f_1 and f_2 are arbitrary functions.

$$\text{P.I.} = \frac{1}{D_1^2 - D_1D_1' - 2D_1'^2}\left(v - u - \frac{1}{2}\right) = \frac{1}{D_1^2\,(1 - D_1'/D_1 - 2D_1'^2/D_1^2)}\left(v - u - \frac{1}{2}\right)$$

$$= \frac{1}{D_1^2}\left\{1 - \left(\frac{D_1'}{D_1} + \frac{2D_1'^2}{D_1^2}\right)\right\}^{-1}\left(v - u - \frac{1}{2}\right) = \frac{1}{D_1^2}\left(1 + \frac{D_1'}{D_1} + \ldots\right)\left(v - u - \frac{1}{2}\right)$$

$$= \frac{1}{D_1^2}\left\{v - u - \frac{1}{2} + \frac{1}{D_1}D_1'\left(v - u - \frac{1}{2}\right)\right\} = \frac{1}{D_1^2}\left(v - u - \frac{1}{2} + \frac{1}{D_1}\cdot 1\right) = \frac{1}{D_1^2}\left(v - u - \frac{1}{2} + u\right)$$

$$= \frac{1}{D_1^2}\left(v - \frac{1}{2}\right) = \left(v - \frac{1}{2}\right)\frac{u^2}{2} = \frac{1}{2}u^2 v - \frac{1}{4}u^2 = \frac{(\log x)^2 \log y}{2} - \frac{(\log x)^2}{4}, \text{ by (1)}$$

$\therefore$ Required solution is $\quad z = f_1(yx^2) + f_2(y/x) + (1/2) \times (\log x)^2 \log y - (1/4) \times (\log x)^2.$

Ex. 14. *Solve* $(x^2D^2 - 4y^2D'^2 - 4yD' - 1)z = x^2y^2 \log y.$ **[Delhi Maths (H) 2006]**

Sol. Let $\quad x = e^u, \quad y = e^v \quad$ so that $\quad u = \log x, \quad v = \log y.$ ...(1)

Also, let $\quad D_1 \equiv \partial/\partial u \quad$ and $\quad D_1' \equiv \partial/\partial v.$

Then the given equation reduces to $[D_1(D_1 - 1) - 4D_1'(D_1' - 1) - 4D_1' - 1]z = e^{2u}e^{2v}v$

or $$(D^2{}_1 - D_1 - 4D_1'^2 - 1)z = e^{2u + 2v}v. \qquad \text{...(2)}$$

Here $(D^2{}_1 - D_1 - 4D_1'^2 - 1)$ cannot be resolved into linear factors in D_1 and D_1'. To find C.F. corresponding to it, we consider the equation.

$$(D^2{}_1 - D_1 - 4D_1'^2 - 1)z = 0. \qquad \text{...(3)}$$

Let a trial solution of (3) be $\quad z = Ae^{hu + kv}$ ...(4)

$\therefore \quad D_1^2 z = Ah^2e^{hu+kv}, \qquad D_1z = Ahe^{hu+kv}, \qquad$ and $\qquad D_1'^2\, z = Ak^2e^{hu+kv}.$

Then, $\quad (3) \quad \Rightarrow \quad A(h^2 - h - 4k^2 - 1)e^{hu+hv} = 0 \quad \Rightarrow \quad h^2 - h - 4k^2 - 1 = 0.$ ...(5)

$\therefore \qquad$ C.F. of (2) $= \Sigma Ae^{hu+kv} = \Sigma A(e^u)^h\,(e^v)^k = \Sigma Ax^hy^k$

$$\text{P.I. of (2)} = \frac{1}{D_1^2 - D_1 - 4D_1'^2 - 1}e^{2u+2v}v = e^{2u+2v}\frac{1}{(D_1+2)^2 - (D_1+2) - 4(D_1'+2)^2 - 1}v$$

$$= e^{2u+2v}\frac{1}{D_1^2 + 3D_1 - 4D_1'^2 - 16D_1' - 15}v$$

$$= e^{2u+2v}\frac{1}{(-15)\times[1 + (16/15)\times D_1' + (4/15)\times D_1'^2 - (1/5)\times D_1 - (1/15)\times D_1^2]}v$$

$$= e^{2u+2v}\frac{1}{(-15)}\left[1 + \left\{\frac{16}{15}D_1' + \frac{4}{15}D_1'^2 - \frac{1}{5}D_1 - \frac{1}{15}D_1^2\right\}\right]^{-1}v$$

$$= \frac{e^{2u+2v}}{(-15)}\left(1 - \frac{16}{15}D_1' + \ldots\right)v = \frac{e^{2u+2v}}{(-15)}\left(v - \frac{16}{15}\right) = \frac{(e^u)^2 \times (e^v)^2(16 - 15v)}{225}$$

$= (1/225) \times x^2y^2\,(16 - 15 \log y)$, using (1).

The required general solution is $\quad z = \Sigma Ax^hy^k + (1/225) \times x^2y^2(16 - 15 \log y)$ where $h^2 - h - 4k^2 - 1 = 0$, and A, h and k are arbitrary constants.

Ex. 15. *Solve* $(x^2D^2 + 2xy\, DD' + y^2D'^2)\, z = x^2y^2$ **[Delhi Maths (H) 2007]**

Sol. Let $\quad x = e^u, \quad y = e^v \quad$ so that $\quad u = \log x, \quad v = \log y$ ... (1)

Here $D \equiv \partial/\partial x, \quad D' \equiv \partial/\partial y \quad$ let $\quad D_1 \equiv \partial/\partial u \quad$ and $\quad D_1' \equiv \partial/\partial v$

$\therefore$ Given equation reduces to

$\{D_1(D_1-1)+2D_1D_1'+D_1'(D_1'-1)\}z=e^{2u}\ e^{2v}$

or $$(D_1+D_1')\ (D_1+D_1'-1)\ z=e^{2u+2v}$$

Its C.F. $=\ \phi_1(v-u)+e^u\phi_2(v-u)=\phi_1(\log y-\log x)+x\ \phi_2(\log y-\log x)$, by (1)

$=\ \phi_1\{\log(y/x)\}+x\ \phi_2\{\log(y/x)\}=\psi_1(y/x)+x\psi_2(y/x)]$. ψ_1, ψ_2 being arbitrary functions.

P.I. $= \dfrac{1}{(D_1+D_1')\ (D_1+D_1'-1)}e^{2u+2v}=\dfrac{1}{(2+2)(2+2-1)}e^{2u+2v}=\dfrac{(e^u)^2(e^v)^2}{12}=\dfrac{x^2y^2}{12}$

Hence the required solution is given by $z=\psi_1(y/x)+x\ \psi_2\ (y/x)+(1/12)\times x^2y^2$.

Ex. 16. *Solve* $(x^2D^2-4xyDD'+4y^2D'^2+4yD'+xD)z=x^2y$

Sol. Let $x=e^u$, $y=e^v$ so that $u=\log x$, $v=\log y$ ... (1)

Here $D\equiv\partial/\partial x$, $D'\equiv\partial/\partial x$, . Let $D_1\equiv\partial/\partial u$ and $D_1'\equiv\partial/\partial v$

Then given equation reduces to $\{D_1\ (D_1-1)-4\ D_1D_1'+4D_1'(D_1'-1)+4D_1'+D_1)\}z=e^{2u}e^v$

or $(D_1^2-4D_1D_1'+4D_1'^2)z=e^{2u+v}$ or $(D_1-2D_1')^2\,z=e^{2u+v}$.

C.F. $=\ \phi_1\ (v+2u)+u\ \phi_2(v+2u)\ =\phi_1\ (\log y+2\log x)+\log x\ \phi_2(\log y+2\log x)$, by (1)

$=\ \phi_1(\log yx^2)+\log x\ \phi_2(\log yx^2)=\psi_1(yx^2)+\log x\ \psi_2(yx^2)$, ψ_1, ψ_2 being arbitrary functions.

P.I. $= \dfrac{1}{(D_1-2D_1')^2}\ e^{2u+v}=\dfrac{u^2}{2!}\ e^{2u+v}=\dfrac{1}{2}(\log x)^2x^2y$

$\therefore$ General solution is $z=\psi_1\ (yx^2)+(\log x)\ \psi_2(yx^2)+(1/2)\times x^2y\ (\log x)^2$.

Ex. 17. *Solve* $(x^2D-2xy\ DD'+y^2D'^2-xD+3yD')z=8y/x$ **[Delhi Maths (H) 2005]**

Sol. Let $x=e^u$, $y=e^v$ so that $u=\log x$, $v=\log y$ ... (1)

Here $D\equiv\partial/\partial x$, $D'\equiv\partial/\partial y$. Let $D_1\equiv\partial/\partial u$ and $D_1'\equiv\partial/\partial v$.

Then the given equation reduces to $\{D_1(D_1-1)-2D_1D_1'+D_1'(D_1'-1)-D_1+3D_1'\}z=8e^v/e^u$

or $\{(D_1^2-2D_1\ D_1'+D_1'^2)-2\ (D_1-D_1')\}z=8e^{v-u}$ or $(D_1-D_1')\ (D_1-D_1'-2)z=8\ e^{v-u}$

C.F. $=\ \phi_1\ (v+u)+e^{2u}\phi_2\ (v+u)\ =\ \phi_1\ (\log y+\log x)+(e^u)^2\phi_2(\log y+\log x)$, using (1)

$=\ \phi_1(\log xy)+x^2\phi_2(\log xy)=\psi_1(xy)+x^2\psi_2(xy)$, ψ_1 ψ_2 being arbitrary functions

P.I. $= \dfrac{1}{(D_1-D_1')\ (D_1-D_1'-2)}8\ e^{-u+v}=8\dfrac{1}{(-1-1)\ (-1-1-2)}e^{-u+v}=\dfrac{e^v}{e^u}=\dfrac{y}{x}$

$\therefore$ The required solution is $z=\psi_1(xy)+x^2\psi_2(xy)+y/x$.

Ex. 18. $(x^2D^2-2xy\ DD'-3y^2D'^2+xD-3yD')z\ =\ x^2y\cos\ (\log x^2)$

Sol. Let $x=e^u$, $y=e^v$ so that $u=\log x$, $v=\log y$ ... (1)

Here $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$. Let $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$.

Then the given equation reduces to

$$\{D_1(D_1-1)-2D_1\ D_1'-3D_1'\ (D_1'-1)\ +\ D_1-3\ D_1'\}z\ =\ e^{2u}\ e^{v}\cos 2u\,.$$

or $\quad (D_1^2-2D_1D_1'-3D_1'^2)z=e^{2u+v}\cos 2u \quad$ or $\quad (D_1-3D_1')\ (D+D_1')z=e^{2u+v}\cos 2u$

Its C.F. $=\ \phi_1\ (u+3u)+\phi_2(v-u)\ =\ \phi_1(\log y+3\log x)+\phi_2(\log y-\log x)$, using (1)

$=\ \phi_1(\log yx^3)+\phi_2(\log(y/x))=\psi_1(x^3y)+\psi_2(y/x)$, $\psi_1\ \psi_2$ being arbitrary functions.

$$\text{P.I.}\ =\ \frac{1}{(D_1-3D_1')\ (D_1+D_1')}\ e^{2u+v}\cos 2u=\ e^{2u+v}\ \frac{1}{\{(D_1+2)-3(D_1'+1)\}\ (D_1+2+D_1'+1)}\cos 2u$$

$$=\frac{(e^u)^2(e^v)}{(D_1-3D_1'-1)\ (D_1+D_1'+3)}\cos 2u\ =\ \frac{x^2y}{D_1^2-2D_1D_1'-3D_1'^2+2D_1-10D_1'-3}\cos(2u+0\cdot v)$$

$$=\ \frac{x^2y}{-2^2-2(-2\cdot 0)-3\cdot 0^2+2D_1-10D_1'-3}\cos 2u=\ \frac{x^2y}{2D_1-10D_1'-7}\cos 2u=x^2y\frac{2D_1-10D_1'+7}{(2D_1-10D_1')^2-49}\cos 2u$$

$$=\ x^2y\frac{2D_1-10D_1'+7}{4D_1^2-40D_1D_1'+100\,D_1'^2-49}\cos(2u+0.v)=\ x^2\,y\frac{2D_1-10D_1'+7}{4(-2^2)-40(-2\cdot 0)+100(-0^2)-49}\cos(2u+0\cdot v)$$

$= -\ (1/65)\ x^2y\ (-\ 4\ \sin\ 2u\ +\ 7\ \cos\ 2u) = (1/65)\ x^2y\ \{4\ \sin\ (2\ \log\ x) - 7\ \cos\ (2\ \log\ x)\}$

$\therefore$ Required solution is $z=\psi_1(x^3y)+\psi_2(y/x)+(1/65)x^2y\ \ \{4\sin(\log x^2)-7\cos(\log x^2)\}$.

Ex. 19. *Solve* $x^2(\partial^2z/\partial x^2)-y^2(\partial^2z/\partial y^2)+x(\partial z/\partial x)-y(\partial z/\partial y)=x^2y^4$ *by reducing it to the equation with constant coefficients.* **[I.A.S. 2001]**

Sol. Re-writing, the given equation $\quad (x^2D^2-y^2D'^2+xD-yD')z=x^2y^4 \quad$ ... (1)

Let $x=e^u$ and $y=e^v$ so that $u=\log x$ and $v = \log y$ ... (2)

Here $D\equiv\partial/\partial x,\ D'\equiv\partial/\partial y$. Let $D_1\equiv\partial/\partial u$, $D_1'\equiv\partial/\partial v$. Then (1) becomes

$\{D_1\ (D_1-1)-D_1'(D_1'-1)+D_1-D_1'\}z\ =\ e^{2u}e^{4v}$ or $(D_1^2-D_1'^2)z=e^{2u+4v}$

or $\quad (D_1-D_1')\ (D_1+D_1')z=e^{2u+4v}$ ... (3)

C.F. $=\ \phi_1(v+u)+\phi_2\ (v-u)=\phi_1(\log y+\log x)+\phi_2(\log y-\log x)\ =\ \phi_1(\log xy)+\phi_2(\log(y/x)\}$

or $\quad$ C.F. $=\ \psi_1(xy)+\psi_2(y/x)$, ψ_1,ψ_2 being arbitrary functions

$$\text{P.I.}\ =\ \frac{1}{D_1^2-D_1'^2}\ e^{2u+4v}=\frac{1}{(2^2-4^2)}e^{2u+4v}=-\frac{1}{12}\ (e^u)^2\ (e^v)^4\ =-\frac{1}{12}x^2y^4$$

$\therefore\ z=\psi_1\ (xy)+\psi_2(y/x)-(1/12)\times x^2y^4$ is the required solution.

Remark. Sometimes typical substitutions are employed to reduce a given equation into a partial differential equation with constant coefficients as shown in the following Ex. 20.

Ex. 20. *Solve* $\dfrac{1}{x^2}\dfrac{\partial^2 z}{\partial x^2} - \dfrac{1}{x^3}\dfrac{\partial z}{\partial x} = \dfrac{1}{y^2}\dfrac{\partial^2 z}{\partial y^2} - \dfrac{1}{y^3}\dfrac{\partial z}{\partial y}$.

Sol. Let $x^2/2 = u$, $y^2/2 = v$ so that $dx/du = 1/x$, $dy/dv = 1/y$. ...(1)

Now,
$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{dx}{du} = \frac{1}{x}\frac{\partial z}{\partial x}, \text{ using (1)}. \qquad ...(2)$$

and
$$\frac{\partial^2 z}{\partial u^2} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right) = \frac{\partial}{\partial u}\left(\frac{1}{x}\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial z}{\partial x}\right)\frac{dx}{du} = \left(\frac{1}{x}\frac{\partial^2 z}{\partial x^2} - \frac{1}{x^2}\frac{\partial z}{\partial x}\right)\frac{1}{x}, \text{ using (1)}$$

$$\therefore \qquad \frac{\partial^2 z}{\partial u^2} = \frac{1}{x^2}\frac{\partial^2 z}{\partial x^2} - \frac{1}{x^3}\frac{\partial z}{\partial x}. \qquad ...(3)$$

Similarly,
$$\frac{\partial^2 z}{\partial v^2} = \frac{1}{y^2}\frac{\partial^2 z}{\partial y^2} - \frac{1}{y^3}\frac{\partial z}{\partial y}. \qquad ...(4)$$

Using (3) and (4), the given equation reduces to

$\partial^2 z/\partial u^2 = \partial^2 z/\partial v^2$ or $(D_1^2 - D_1'^2)z = 0$ or $(D_1 - D_1')(D_1 + D_1')z = 0$, ...(5)

where $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$. Hence solution of (5) is

$$z = \phi_1(v + u) + \phi_2(v - u) = \phi_1\{(1/2) \times (x^2 + y^2)\} + \phi_2\{(1/2) \times (y^2 - x^2)\}$$

or $\qquad z = f_1(y^2 + x^2) + f_2(y^2 - x^2), f_1, f_2$ being arbitrary functions.

EXERCISE 6

Solve the following partial differential equations:

1. $x^2(\partial^2 y/\partial x^2) + 2xy(\partial^2 z/\partial x\partial y) + y^2(\partial^2 z/\partial y^2) + x(\partial z/\partial x) + y(\partial z/\partial y) - z = 0$

Ans. $z = xf_1(y/x) + x^{-1}f_2(y/x)$, f_1, f_2 being arbitrary functions

2. $(x^2D^2 - y^2D'^2)z = x^2y$. **Ans.** $z = f_1(xy) + x\,f_2(y/x) + (1/2)\times x^2y$

3. $(x^2D^2 - xy\,DD' - 2y^2D'^2 + xD - 2yD')z = \log(y/x)$ **(Delhi Maths (H) 2005)**

Ans. $z = f_1(yx^2) + f_2(y/x) + (1/2)\times(\log x)^2 \log y$, f_1, f_2 being arbitrary functions

4. $(x^2D^2 - 2xy\,DD' - 3y^2D'^2 + xD - 3yD')z = x^2y\sin(\log x^2)$ **[Nagpur 2010]**

Ans. $z = f_1(x^3y) + f_2(y/x) - (1/65)\times\{4\cos(\log x^2) + 7\sin(\log x^2)\}$

6.5. SOLUTIONS UNDER GIVEN GEOMETRICAL CONDITIONS

We have seen that solution of Euler-Cauchy type partial differential equations involve arbitrary functions of x and y. We shall now determine these functions under the given geometrical conditions. This will lead to the required surface satisfying the given differential equation under the prescribed geometrical conditions.

Illustrative example. *Find a surface satisfying equation* $2x^2r - 5xys + 2y^2t + 2(px + qy) = 0$ *and touching the hyperbolic paraboloid* $z = x^2 - y^2$ *along its section by the plane* $y = 1$.

[Meerut 1998]

Sol. Re–writing given equation, $\quad 2x^2\dfrac{\partial^2 z}{\partial x^2} - 5xy\dfrac{\partial^2 z}{\partial x\partial y} + 2y^2\dfrac{\partial^2 z}{\partial y^2} + 2\left(x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y}\right) = 0.$

or $\qquad \{2x^2D^2 - 5xyDD' + 2y^2D'^2 + 2(xD + yD')\}z = 0$ ...(1)

Put $x = e^u$, $y = e^v$ so that $u = \log x$ and $v = \log y$.

If $D_1 \equiv \partial/\partial u$ and $D_1' \equiv \partial/\partial v$, then (1) reduces to

$$[2D_1(D_1 - 1) - 5D_1D_1' + 2D_1'(D_1' - 1) + 2(D_1 + D_1')]z = 0$$

or $\quad (2D_1^2 - 5D_1D_1' + 2D_1'^2) = 0 \quad$ or $\quad (2D_1 - D_1')(D_1 - 2D_1') = 0.$

$\therefore$ solution is $z = \text{C.F.} = \phi_1(2v + u) + \phi_2(u + 2v)$, ϕ_1, ϕ_2 being arbitrary function

or $\quad z = \phi_1(2 \log y + \log x) + \phi_2(\log y + 2 \log x) = \phi_1(\log y^2x) + \phi_2(\log yx^2)$

or $\quad z = f_1(y^2x) + f_2(yx^2)$, f_1 and f_2 being arbitrary functions ...(2)

The given surface is $\quad z = x^2 - y^2.$ ...(3)

Now (2) and (3) are to touch each other along the section by the plane

$$y = 1. \qquad ...(4)$$

Therefore the values of p and q for (2) and (3) must be equal at $y = 1$. Equating values of p and q from (2) and (3), we get

$$y^2f_1'(y^2x) + 2xyf_2'(x^2y) = 2x \qquad ...(5)$$

and $$2xyf_1'(y^2x) + x^2f_2'(x^2y) = -2y. \qquad ...(6)$$

Putting $y = 1$, (5) and (6) reduce to

$f_1'(x) + 2xf_2'(x^2) = 2x \quad$ and $\quad 2xf_1'(x) + x^2f_2'(x^2) = -2.$

Solving these, $\quad f_1'(x) = -(2/3) \times x - (4/3) \times x^{-1}$ ...(7)

and $\quad f_2'(x^2) = (2/3) \times x^{-2} + (4/3)$ ...(8)

Integrating (7), $\quad f_1(x) = -(1/3) \times x^2 - (4/3) \times \log x + c_1$

which gives $\quad f_1(y^2x) = -(1/3) \times y^4x^2 - (4/3) \times \log (y^2x) + c_1.$ ...(9)

Writing X for x^2 in (8), $\quad f_2'(X) = (2/3) \times (1/X) + (4/3)$

Integrating it, $\quad f_2(X) = (2/3) \times \log X + (4/3) \times X + c_2$

which gives $\quad f_2(yx^2) = (2/3) \times \log (yx^2) + (4/3) \times (yx^2) + c_2$ ...(10)

Putting the values of $f_1(y^2x)$ and $f_2(yx^2)$ from (9) and (10) in (2) and writing $c_1 + c_2 = c/3$, the complete solution is

$$z = -(1/3) \times y^4x^2 - (4/3) \times \log (y^2x) + (2/3) \times \log(yx^2) + (4/3) \times (yx^2) + c/3$$

or $$3z = -y^4x^2 - 4(\log x + 2 \log y) + 2(\log y + 2 \log x) + 4yx^2 + c$$

or $$3z = -y^4x^2 - 6 \log y + 4yx^2 + c.$$

Now equating values of z from (3) and (11) and putting $y = 1$, we have

$$x^2 - 1 = (1/3)[-x^2 - 6 \log 1 + 4x^2 + c], \text{ giving } c = -3.$$

So the required surface is $\quad 3z = 4yx^2 - y^4x^2 - 6 \log y - 3.$

MISCELLANEOUS PROBLEMS ON CHAPTER 6

1. Show that a linear partial differential equation of the type

$$\sum C_{qs} x^q y^s \frac{\partial^{q+s} z}{\partial x^q \partial y^s} = f(x, y)$$

may be reduced to one with constant coefficients by the substitutions $\log x = \xi$, $\log y = \eta$.

(Meerut 2008)

2. Find the general solution of $x^2(\partial^2 z/\partial x^2) + y^2(\partial^2 z/\partial y^2) = z$ **[Pune 2010]**

Sol. Let $\quad x = e^u \quad$ and $\quad y = e^v \quad$ so that $\quad u = \log x \quad$ and $\quad v = \log y$ ... (1)

Also, let $\quad D \equiv \partial/\partial x, \quad D' \equiv \partial/\partial y, \quad D_1 \equiv \partial/\partial u \quad$ and $\quad D_1' = \partial/\partial v$ ... (2)

Then, the given equation $(x^2 D^2 + y^2 D'^2 - 1)\, z = 0$ reduce to

$\{D_1(D_1 - 1) + D_1'(D_1' - 1) - 1\}z = 0 \quad$ or $\quad (D_1^2 + D_1' - D_1 - D_1' - 1)z = 0$ ... (3)

Let $z = A\, e^{hu + kv}$ be a trial solution of (3). Then, we have

$D_1 z = Ah\, e^{hu+kv}$, $D_1^2 z = Ah^2 e^{hu+kv}$, $D_1' z = Ak\, e^{hu+kv}$ and $D_1' z = Ak^2\ e^{hu+kv}$

Substituting the above values of $D_1\, z, D_1^2 z, D_1' z$ and $D_1'^2 z$ in (3), we have

$A(h^2 + k^2 - h - k - 1)\, e^{hu+kv} = 0$ so that $h^2 + k^2 - h - k - 1 = 0$, taking $A \neq 0$ ... (4)

Hence the required solution is given by

$z = \sum_i A_i e^{h_i u + k_i v}$ or $z = \sum_i A_i e^{h_i \log x + k_i \log y}$, using (1)

or $z = \sum A_i x^{h_i} y^{k_i}$, A_i, h_i and k_i being arbitrary constants.

7

Partial Differential Equations of order Two With Variable Coefficients

7.1 INTRODUCTION

In the present chapter, we propose to discuss partial differential equations of order two with variable coefficients. An equation is said to be of order two, if it involves at least one of the differential coefficients $r\ (= \partial^2 z/\partial x^2)$, $s\ (= \partial^2 z/\partial x \partial y)$, $t\ (= \partial^2 z/\partial y^2)$, but none of higher order ; the quantities p and q may also enter into the equation. Thus, the general form of a second order partial differential equation is

$$f(x, y, z, p, q, r, s, t) = 0. \qquad ...(1)$$

The most general linear partial differential equation of order two in two independent variables x and y with variable coefficients is of the form

$$Rr + Ss + Tt + Pp + Qq + Zz = F, \qquad ...(2)$$

where R, S, T, P, Q, Z, F are functions of x and y only and not all R, S, T are zero.

In what follows, we shall show how a large class of second order partial differential equations may be solved by using the methods of solving ordinary differential equations.

Note that x and y, being independent variables, are constant with respect to each other in differentiation and integration. To understand this, note the solution of the following equation.

$$s = 2x + 2y. \qquad ...(3)$$

From (3), $$\frac{\partial^2 z}{\partial x \partial y} = 2x + 2y \quad \text{or} \quad \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = 2x + 2y. \qquad ...(4)$$

Integrating (4) w.r.t. 'x', $(\partial z/\partial y) = x^2 + 2xy + f(y)$, where $f(y)$ in an arbitrary functions of y. ...(5)

Integrating (5) w.r.t. 'y', $$z = x^2 y + xy^2 + F(y) + g(x),$$

where F *and* g are arbitrary functions and $F(y)$ is given by $\quad F(y) = \int f(y)\, dy.$

In what follows we shall use the following results.

$$p = \frac{\partial z}{\partial x}, \qquad q = \frac{\partial z}{\partial y}, \qquad r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial p}{\partial x}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial q}{\partial y} \quad \text{and} \quad s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial q}{\partial x} = \frac{\partial p}{\partial y}.$$

We shall now consider some special types of equations based on (2).

7.2. Type I. Under this type, we consider equations of the form

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{F}{R} = f_1(x, y), \qquad t = \frac{\partial^2 z}{\partial y^2} = \frac{F}{T} = f_2(x, y), \qquad s = \frac{\partial^2 z}{\partial x \partial y} = \frac{F}{S} = f_3(x, y).$$

These are homogeneons linear partial differential equations with constant coefficients and can be solved by methods discussed in chapter 4. However a more direct method of solving such equation will be used in practice.

7.3 SOLVED EXAMPLES BASED ON ART 7.2.

Ex 1. *Solve the following partial differential equations:*

(*i*) $r = 6x$. **[Agra 2009; Bhopal 2010]** (*ii*) $ar = xy$ **[Meerut 2001; Vikram 2003]**

(*iii*) $r = x^2 e^y$ **[Indore 2004]** (*iv*) $r = 2y^2$

(*v*) $r = \sin(xy)$

Sol. (*i*) Given equation can be written as $\partial^2 z/\partial x^2 = 6x$. ...(1)

Integrating (1) with respect to 'x', $\partial z/\partial x = 3x^2 + \phi_1(y)$, ...(2)

where $\phi_1(y)$ is an arbitrary function of y.

Integrating (2) with respect to 'x', $z = x^3 + x\phi_1(y) + \phi_2(y)$,

where $\phi_2(y)$ is an arbitrary function of y.

(*ii*) Given equation can be written as $\partial^2 z/\partial x^2 = (1/a) \times xy$. ...(1)

Integrating (1) w.r.t. 'x', $\partial z/\partial x = (y/a) \times (x^2/2) + \phi_1(y)$. ...(2)

Integrating (2) w.r.t. 'x', $z = (y/6a) \times x^3 + x\phi_1(y) + \phi_2(y)$,

which is the required general solution, ϕ_1, ϕ_2 being arbitrary functions.

(*iii*) Try yourself. **Ans.** $z = (e^y/12) \times x^4 + x\phi_1(y) + \phi_2(y)$.

(*iv*) Try yourself. **Ans.** $z = x^2y^2 + x\phi_1(y) + \phi_2(y)$.

(v) Given equation can be written as $\partial^2 z/\partial x^2 = \sin (xy)$. ...(1)

Integrating (1) w.r.t. 'x'. $\partial z/\partial x = -(1/y) \times \cos (xy) + \phi_1(y)$. ...(2)

Integrating (2) w.r.t. 'x', $z = -(1/y^2) \times \sin (xy) + x\phi_1(y) + \phi_2(y)$,

which is the required general solution, ϕ_1, ϕ_2 being arbitrary functions.

Ex. 2. *Solve* (*i*) $t = \sin (xy)$ **(Meerut 2008)** (*ii*) $t = x^2 \cos (xy)$.

Sol.(*i*) Given equation can be written as $\partial^2 z/\partial y^2 = \sin (xy)$. ...(1)

Integrating (1) w.r.t. 'y', $\partial z/\partial y = -(1/x) \times \cos (xy) + \phi_1(x)$. ...(2)

Integrating (2) w.r.t., 'y', $z = -(1/x^2) \times \sin (xy) + y\phi_1(x) + \phi_2(x)$,

which is the required solution, ϕ_1, ϕ_2 being arbitrary functions.

(*ii*) Given equation can be written as $\partial^2 z/\partial y^2 = x^2 \cos (xy)$. ...(1)

Integrating (1) w.r.t. 'y', $\partial z/\partial y = x \sin (xy) + \phi_1(x)$. ...(2)

Integrating (2) w.r.t. 'y', $z = -\cos (xy) + y\,\phi_1(x) + \phi_2(x)$,

which is the required solution, ϕ_1, ϕ_2 being arbitrary functions.

Ex. 3. *Solve the following partial differential equations:*

(*i*) $xys = 1$ **[Agra 2007; Rewa 2004, Vikram 2005]** (*ii*) $xy^2 s = 1 - 2x^2 y$

(*iii*) $\log s = x + y$ (*iv*) $s = x - y$

(*v*) $s = x^2 - y^2$ (*vi*) $x^2 s = \sin y$

(*vii*) $s = (x/y) + a$ (*viii*) s = 0.

Sol. (*i*) Re-written the given equation, $\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{1}{xy}$. ...(1)

Integrating (1) w.r.t. 'x', $\partial z/\partial y = (1/y) \times \log x + \phi_1(y)$.

Integrating (2) w.r.t. 'y', z = $\log x \log y + \int \phi_1(y) dy + \psi_2(x)$

or $z = \log x \log y + \psi_1(y) + \psi_2(x)$, taking $\psi_1(y) = \int \phi_1(y) dy$.

which is the required general solution, ψ_1, ψ_2 being arbitrary functions.

(*ii*) Given equation is $\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{1}{xy^2} - \dfrac{2x}{y}$. ...(1)

Integrating (1) w.r.t. 'x', $\partial z/\partial y = (1/y^2) \times \log x - (x^2/y) + \phi_1(y)$. ...(2)

Integrating (2) w.r.t. 'y', $z = -(1/y) \times \log x - x^2 \log y + \int \phi_1(y)dy + \psi_2(x)$

or $z = -(1/y) \times \log x - x^2 \log y + \psi_1(y) + \psi_2(x)$, taking $\psi_1(y) = \int \phi_1(y)dy$.

which is the required general solution, ψ_1, ψ_2 being arbitrary functions.

(*iii*) The given equation $\log s = x + y$ can be rewritten as

$$s = e^{x+y} \qquad \text{or} \qquad \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = e^x \, . \, e^y. \qquad ...(1)$$

Integrating (1) w.r.t. 'x', $\partial z/\partial y = e^x e^y + \phi_1(y)$. ...(2)

Integrating (2) w.r.t. 'y', $z = e^x e^y + \int \phi_1(y)dy + \psi_2(x)$

or $z = e^{x+y} + \psi_1(y) + \psi_2(x)$, where $\psi_1(y) = \int \phi_1(y)dy$, ψ_1, ψ_2 being arbitrary functions

(*iv*) Try yourself. **Ans.** $z = (1/2) \times (x^2y - xy^2) + \psi_1(y) + \psi_2(x)$.

(*v*) Try yourself. **Ans.** $z = (1/3) \times (x^3y - xy^3) + \psi_1(y) + \psi_2(x)$.

(*vi*)Given equation can be written as $\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{\sin y}{x^2}$. ...(1)

Integrating (1) w.r.t. 'x', $\partial z/\partial y = -(1/x) \times \sin y + \phi_1(y)$. ...(2)

Integrating (2) w.r.t. 'y', $z = (1/x) \cos y + \int \phi_1(y)dy + \psi_2(x)$

or $z = (1/x) \cos y + \psi_1(y) + \psi_2(x)$, where $\psi_1(y) = \int \phi_1(y)dy$, ψ_1, ψ_2 being arbitrary functions

(*vii*) Try yourself. **Ans.** $z = (1/2) \times x^2 \log y + axy + \psi_1(y) + \psi_2(x)$.

(*viii*) Try yourself. **Ans.** $z = \psi_1(y) + \psi_2(x)$.

Ex. 4. *Solve* (*i*) $xr = p$ **[Agra 2007]** (*ii*) $rx = (n-1)p$.

Sol.(*i*) Given equation can be rewritten as

$$x\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial x} \qquad \text{or} \qquad \frac{\partial^2 z/\partial x^2}{\partial z/\partial x} = \frac{1}{x}.$$

Integrating, $\log(\partial z/\partial x) = \log x + \log \phi_1(y)$ or $\partial z/\partial x = x\phi_1(y)$.

Integrating it w.r.t. x, $z = (x^2/2) \times \phi_1(y) + \phi_2(y)$, where $\phi_1(y)$ and $\phi_2(y)$ are arbitrary functions.

(*ii*) Given $x\dfrac{\partial^2 z}{\partial x^2} = (n-1)\dfrac{\partial z}{\partial x}$ or $\dfrac{\partial^2 z/\partial x^2}{\partial z/\partial x} = \dfrac{n-1}{x}$.

Integrating, $\log(\partial z/\partial x) = (n-1)\log x + \log \phi_1(y)$ or $\partial z/\partial x = x^{n-1}\phi_1(y)$.

Integrating it, $z = (x^n/n) \times \phi_1(y) + \phi_2(y)$, ϕ_1, ϕ_2 being arbitrary functions.

Ex. 5. *Solve* (*i*) $xr + 2p = 0$ (*ii*) $2yq + y^2t = 1$.

Sol.(*i*) The given equation can be rewritten as

$$x\frac{\partial p}{\partial x} + 2p = 0 \qquad \text{or} \qquad x^2\frac{\partial p}{\partial x} + 2xp = 0 \qquad \text{or} \qquad \frac{\partial}{\partial x}(x^2 p) = 0. \qquad ...(1)$$

Integrating (1) w.r.t. 'x', $x^2 p = \phi_1(y)$ or $p = \partial z/\partial x = (1/x^2) \times \phi_1(y)$.

Integrating it w.r.t. 'x', $z = -(1/x) \times \phi_1(y) + \phi_2(y)$, ϕ_1, ϕ_2 being arbitrary functions.

(*ii*) The given equation can be rewritten as

$$2yq + y^2\frac{\partial q}{\partial y} = 1 \qquad \text{or} \qquad \frac{\partial}{\partial y}(y^2 q) = 0. \qquad ...(1)$$

Integrating (1) w.r.t. 'y', $y^2 q = \phi_1(x)$ or $q = \partial z/\partial y = (1/y^2) \times \phi_1(x)$.

Integrating it, $z = -(1/y) \times \phi_1(x) + \phi_2(x)$, ϕ_1, ϕ_2 being arbitrary functions.

Ex. 6. *Solve* $xs + q = 4x + 2y + 2$.

Sol. The given equation can be re-written as

$$x\frac{\partial p}{\partial y} + \frac{\partial z}{\partial y} = 4x + 2y + 2 \qquad \text{or} \qquad \frac{\partial}{\partial y}(xp + z) = 4x + 2y + 2.$$

Integrating it w.r.t. 'y', $\quad xp + z = 4xy + y^2 + 2y + \phi_1(x)$

or $\quad x\dfrac{\partial z}{\partial x} + z = 4xy + y^2 + 2y + \phi_1(x) \qquad$ or $\qquad \dfrac{\partial}{\partial x}(xz) = 4xy + y^2 + 2y + \phi_1(x).$

Integrating it w.r.t. 'x', $\quad xz = 2x^2y + xy^2 + 2xy + \int \phi_1(x)dx + \psi_2(y)$

or $\quad \therefore$ Required solution is $\quad xz = 2x^2y + xy^2 + 2xy + \psi_1(x) + \psi_2(y)$, where $\psi_1(x) = \int \phi_1(x)dx$.

Ex. 7. *Solve* $\quad ys + p = \cos(x + y) - y\sin(x + y)$. **[Meerut 1995]**

Sol. The given equation can be rewritten as

$$y\frac{\partial q}{\partial x} + \frac{\partial z}{\partial x} = \cos(x+y) - y\sin(x+y) \qquad \text{or} \qquad \frac{\partial}{\partial x}(yq + z) = \cos(x+y) - y\sin(x+y).$$

Integrating it w.r.t. 'x', $\quad yq + z = \sin(x + y) + y\cos(x + y) + \phi_1(y)$.

or $\quad y\dfrac{\partial z}{\partial y} + z = \sin(x+y) + y\cos(x+y) + \phi_1(y) \qquad$ or $\qquad \dfrac{\partial(yz)}{\partial y} = \sin(x+y) + y\cos(x+y) + \phi_1(y).$

Integrating it w.r.t. 'y', $\quad yz = \int \sin(x+y)dy + \int y\cos(x+y)\,dy + \int \phi_1(y)\,dy + \psi_2(y)$

or $\quad yz = \int \sin(x+y)dy + y\sin(x+y) - \int \sin(x+y)\,dy + \psi_1(y) + \psi_2(y)$

[Integrating by parts and taking $\psi_1(y) = \int \phi_1(y)\,dy$]

Required solution is $\quad yz = y\sin(x + y) + \psi_1(y) + \psi_2(y)$, ψ_1, ψ_2 being arbitrary functions.

7.4. Type II. Under this type, we consider equations of the form:

$Rr + Pp = F$, $\quad$ *i.e.*, $\quad R\dfrac{\partial p}{\partial x} + Pp = F$; $\qquad Ss + Pp = F$, $\quad$ *i.e.*, $\quad S\dfrac{\partial p}{\partial y} + Pp = F$,

$Ss + Qq = F$, $\quad$ *i.e.*, $\quad S\dfrac{\partial q}{\partial x} + Qq = F$; $\qquad Tt + Qq = F$, $\quad$ *i.e.*, $\quad T\dfrac{\partial q}{\partial y} + Qq = F$.

These will be treated as ordinary linear differential equations of order one in which p (or q) is the dependent variable.

7.5 SOLVED EXAMPLES BASED ON ART 7.4

Ex. 1. *Solve* (*i*) $t - xq = x^2$. **[Ravishanker 2010; Nagpur 1996]**

(*ii*) $yt - q = xy$. **[Meerut 1997]**

Sol. (*i*) The given equation can be rewritten as $\quad (\partial q/\partial y) - xq = x^2$, $\quad$ ...(1)

which is linear differential equation in variables q and y, regarding x as constant.

Integrating factor (I.F.) of (1) $= e^{\int(-x)dy} = e^{-xy}$ and solution of (1) is

$q(I.F.) = \int (x^2)(I.F.)dy + \phi_1(x) \qquad$ or $\qquad qe^{-xy} = \int x^2 e^{-xy}\,dy + \phi_1(x)$

or $\quad qe^{-xy} = x^2 \times (-1/x) \times e^{-xy} + \phi_1(x) \qquad$ or $\qquad q = \partial z/\partial y = -x + e^{xy}\phi_1(x)$.

Integrating it w.r.t. 'y', $\quad z = -xy + (1/x) \times \phi_1(x)\,e^{xy} + \psi_2(x)$

or $\quad z = -xy + \psi_1(x)e^{xy} + \psi_2(x)$, where $\psi_1(x) = (1/x) \times \phi_1(x)$.

It is the required solution, ψ_1, ψ_2 being arbitrary functions.

(*ii*) The given equation can be rewritten as $\quad y(\partial q/\partial y) - q = xy \quad$ or $\quad (\partial q/\partial y) - (1/y) \times q = x$,

which is differential equation linear in variables q and y, regarding x as constant.

I.F. of (1) $= e^{\int(-1/y)dy} = e^{-\log y} = 1/y$ and solution of (1) is

$$q \times \frac{1}{y} = \int \left(x \times \frac{1}{y} \right) dy + \phi_1(x) \qquad \text{or} \qquad \frac{q}{y} = x \log y + \phi_1(x)$$

or $\quad q = xy \log y + y\phi_1(x) \qquad$ or $\qquad \partial z/\partial y = xy \log y + y\phi_1(x)$.

Integrating it, $z = x\left[(y^2/2) \times \log y - \int \{(y^2/2) \times (1/y)\}\, dy \right] + (y^2/2) \times \phi_1(x) + \phi_2(x)$

or $\quad z = (1/2) \times xy^2 \log y - (1/4) \times xy^2 + (1/2) \times y^2\, \phi_1(x) + \phi_2(x)$, ϕ_1, ϕ_2 being arbitrary functions

Ex. 2. *Solve* $xs + q = 4x + 2y + 2$.

Sol. Re-writing $\quad x\dfrac{\partial q}{\partial x} + q = 4x + 2y + 2 \quad$ or $\quad \dfrac{\partial q}{\partial x} + \dfrac{1}{x} q = 4 + \dfrac{2y}{x} + \dfrac{2}{x}$.

Its I.F. $= e^{\int (1/x) dx} = e^{\log x} = x$ and hence its solution is

$$qx = \int x\left(4 + \frac{2y}{x} + \frac{2}{x}\right) dx + \phi_1(y) = 2x^2 + 2xy + 2x + \phi_1(y)$$

or $$q = \partial z/\partial y = 2x + 2y + 2 + (1/x) \times \phi_1(y).$$

Integrating, $$z = 2xy + y^2 + 2y + (1/x) \times \int \phi_1(y)\, dy + \psi_2(x)$$

or $$z = 2xy + y^2 + 2y + (1/x) \times \psi_1(y) + \psi_2(x), \text{ where } \psi_1(y) = \int \phi_1(y)\, dy.$$

Ex. 3. *Solve* $xr + p = 9x^2y^3$. **[Ranchi 2010]**

Sol. The given equation can be re-written as $\quad x\dfrac{\partial p}{\partial x} + p = 9x^2y^3 \quad$ or $\quad \dfrac{\partial p}{\partial x} + \dfrac{1}{x} p = 9xy^3$.

Its I.F. $= e^{\int (1/x) dx} = e^{\log x} = x$ and hence solution is

$px = \int \{x \times (9xy^3)\} dx + \phi_1(y) \qquad$ or $\qquad px = 3x^3y^3 + \phi_1(y)$

or $\quad p = 3x^2y^3 + (1/x) \times \phi_1(y) \qquad$ or $\qquad (\partial z/\partial x) = 3x^2y^3 + (1/x) \times \phi_1(y)$.

Integrating, $$z = x^3y^3 + \phi_1(y) \log x + \phi_2(y),$$

which is the required solution, ϕ_1, ϕ_2 being arbitrary functions.

Ex. 4. *Solve* $ys - p = xy^2 \cos (xy)$.

Sol. Re-writing given equation, $\quad y\dfrac{\partial p}{\partial y} - p = xy^2 \cos (xy) \quad$ or $\quad \dfrac{\partial p}{\partial y} - \dfrac{1}{y} p = xy \cos (xy)$.

Its I.F. $= e^{\int (-1/y) dy} = e^{-\log y} = 1/y$ and so its solution is

$$p \times (1/y) = \int (1/y) \times [xy \cos (xy)]\, dx + \phi_1(x) = \sin (xy) + \phi_1(x)$$

or $\quad p = y \sin (xy) + y\, \phi_1(x) \qquad$ or $\qquad \partial z/\partial x = y \sin (xy) + y\phi_1(x)$.

Integrating, $$z = -\cos (xy) + y \int \phi_1(x)\, dx + \psi_2(y)$$

or $$z = -\cos (xy) + y\psi_1(x) + \psi_2(y), \text{ where } \psi_1(x) = \int \phi_1(x)\, dx.$$

Ex. 5. *Solve* $t - xq = -\sin y - x \cos y$.

Sol. Re-writting, $\quad (\partial q/\partial y) - xq = -\sin y - x \cos y$, which is linear differential equation in q and y.

Its I.F. $= e^{\int (-x) dy} = e^{-xy}$ and so its soltion is

$$qe^{-xy} = -\int e^{-xy} (\sin y + x \cos y) dy + \phi_1(x) = -\int e^{-xy} \sin y\, dy - x\int e^{-xy} \cos y\, dy + \phi_1(x)$$

$$= -\int e^{-xy} \sin y\, dy - x\left[-\frac{1}{x} e^{-xy} \cos y - \int \left(\frac{1}{x} e^{-xy} \sin y \right) dy \right] + \phi_1(x)$$

$$= -\int e^{-xy}\sin y\, dy + e^{-xy}\cos y + \int e^{-xy}\sin y\, dy + \phi_1(x)$$

or $(\partial z/\partial y)\, e^{-xy} = e^{-xy}\cos y + \phi_1(x)$ or $(\partial z/\partial y) = \cos y + e^{xy}\,\phi_1(x)$.

Integrating, $z = \sin y + (1/x) \times e^{xy}\,\phi_1(x) + \psi_2(x)$

or $z = \sin y + e^{xy}\,\psi_1(x) + \psi_2(x)$, where $\psi_1(x) = (1/x) \times \phi_1(x)$.

Ex. 6. *Solve* $xys - qy = x^2$. **[Delhi Maths Hons. 1992, 93]**

Sol. Re-writing the given equation, we have

$$xy\frac{\partial q}{\partial x} - qy = x^2 \qquad \text{or} \qquad \frac{\partial q}{\partial x} - \frac{1}{x}q = \frac{x}{y}. \qquad \text{...(1)}$$

which is linear differential equation in variables q and x.

Integrating factor of (1) $= e^{-\int (1/x)\, dx} = e^{-\log x} = (1/x)$. Hence solution of (1) is given by

$$q \times \frac{1}{x} = \int \{(x/y) \times (1/x)\}\, dx = \frac{x}{y} + f(y) \qquad \text{or} \qquad \frac{\partial z}{\partial y} = \frac{x^2}{y} + x\, f(y). \qquad \text{...(2)}$$

Integrating (2), $z = x^2 \log y + x\phi_1(y) + \phi_2(x)$, where $\phi_1(y)$ and $\phi_2(x)$ are arbitrary functions.

Ex. 7. *Solve* $xs + q - xp - z = (1 - y)\,(1 + \log x)$.

Sol. Re-writing the given equations $\dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{1}{x}\dfrac{\partial z}{\partial y} - \left(\dfrac{\partial z}{\partial x} + \dfrac{z}{x}\right) = \dfrac{1-y}{x}\,(1 + \log x)$

or
$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x} + \frac{z}{x}\right) - \left(\frac{\partial z}{\partial x} + \frac{z}{x}\right) = \frac{1-y}{x}\,(1 + \log x). \qquad \text{...(1)}$$

Let $u = (\partial z/\partial x) + (z/x)$. ...(2)

$\therefore$ (1) $\Rightarrow$ $\dfrac{\partial u}{\partial y} - u = \dfrac{1-y}{x}\,(1 + \log x)$, which is linear differential equation ...(3)

Integrating factor of (3) $= e^{-\int dy} = e^{-y}$ and so solution of (3) is

$$ue^{-y} = \int \frac{1-y}{x}\,(1 + \log x)e^{-y}\, dy = \frac{1+\log x}{x}\int (1-y)e^{-y}dy$$

$$= \frac{1+\log x}{x}\left[(1-y)(-e^{-y}) - \int (-1)(e^{-y})dy\right] = \frac{1+\log x}{x}\,[-e^{-y} + y\,e^{-y} + e^{-y}] + \phi(x)$$

or $u = (y/x) \times (1 + \log x) + e^y\,\phi(x)$. ...(4)

Then, using (2), $\dfrac{\partial z}{\partial x} + \dfrac{z}{x} = \dfrac{y}{x}(1 + \log x) + e^y\phi(x)$, which is linear differential equation ...(5)

Integrating factor of (5) $= e^{\int (1/x) dx} = e^{\log x} = x$ and solution of (5) is

$$zx = \int x\left[\frac{y}{x}(1 + \log x) + e^y\phi(x)\right]dx + \phi_2(y) = y\int (1 + \log x)\, dx + e^y \int x\,\phi(x) dx + \phi_2(y)$$

or $zx = y\left[(1 + \log x) \times x - \int \{(1/x) \times x\}\, dx\right] + e^y\phi_1(x) + \phi_2(y)$ or $zx = xy\log x + e^y\phi_1(x) + \phi_2(y)$.

Ex. 8. *Solve* $ys + p = \cos(x + y) - y\sin(x + y)$. **[Meerut 1995]**

Sol. Re-writing given equation, $\dfrac{\partial p}{\partial y} + \dfrac{1}{y}p = \dfrac{1}{y}\cos(x + y) - \sin(x + y)$, ...(1)

which is linear differential equation whose I.F. $= e^{\int (1/y) dy} = y$ and so solution of (1) is

$$py = \int y\left[\frac{1}{y}\cos(x+y) - \sin(x+y)\right]dy = \sin(x + y) - \int y\sin(x+y)\, dy$$

$$= \sin(x + y) - \left[-y\cos(z+y) - \int \{-\cos(x+y)\}\, dy\right]$$

$$= \sin(x + y) + y\cos(x + y) - \sin(x + y) + F(x)$$

$\therefore$ $y(\partial z/\partial x) = y\cos(x + y) + F(x)$ or $(\partial z/\partial x) = \cos(x + y) + (1/y) \times F(x)$.

Integrating, $z = \sin(x+y) + (1/y) \times \phi_1(x) + (1/y) \times \phi_2(y)$, where $\phi_1(x) = \int F(x)\, dx$

or $yz = y \sin(x+y) + \phi_1(x) + \phi_2(y)$, ϕ_1, ϕ_2 being arbitrary functions

Ex. 9. *Solve* $yt + 2q = (9y+6)e^{2x+3y}$.

Sol. Re-writing, $\dfrac{\partial q}{\partial y} + \dfrac{2}{y} q = \left(9 + \dfrac{6}{y}\right) e^{2x+3y}$, which is linear differential equations ...(1)

whose integrating factor $= e^{\int (2/y)dy} = e^{2 \log y} = y^2$ and so solution of (1) is

$$qy^2 = \int y^2 \left(9 + \frac{6}{y}\right) e^{2x+3y} dy = e^{2x} \int (9y^2 + 6y) e^{3y}\, dy$$

or $$qy^2 = e^{2x}\left[(9y^2+6y)\left(\frac{1}{3}e^{3y}\right) - (18y+6)\left(\frac{1}{9}e^{3y}\right) + 18\left(\frac{1}{27}e^{3y}\right)\right] + \phi_1(x)$$

[using chain rule of integrating by parts]

or $qy^2 = 3y^2 e^{2x+3y} + \phi_1(x)$ or $y^2(\partial z/\partial y) = 3y^2 e^{2x+3y} + \phi_1(x)$.

$\therefore$ $(\partial z/\partial y) = 3e^{2x+3y} + (1/y^2) \times \phi_1(x)$.

Integrating, $z = e^{2x+3y} - (1/y) \times \phi_1(x) + \phi_2(x)$, ϕ_1, ϕ_2 being arbitrary functions.

Ex. 10. *Solve* (*i*) $2yq + y^2 t = 1$ (*ii*) $xr + p = 9x^2 y^2$.

Sol. (*i*) Re-writing given equation $2yq + y^2(\partial q/\partial y) = 1$ or $\partial q/\partial y + (2/y)q = 1/y^2$, which is linear differential equation in variables q and y, regarding x as constant.

Its I.F. $= e^{\int (2/y)dy} = e^{2\log y} = y^2$ and solution is $qy^2 = \int (1/y^2)\, dy + \phi_1(x)$

or $qy^2 = y + \phi_1(x)$ or $\partial z/\partial y = (1/y) + (1/y^2) \times \phi_1(x)$.

Integrating it w.r.t. 'y', $z = \log y - (1/y) \times \phi_1(x) + \phi_2(x)$, where ϕ_1 and ϕ_2 are arbitrary functions.

(*ii*) Do as in Ex. 3 of Art 7.5. **Ans.** $z = x^3 y^2 + \log x\ \phi_1(y) + \phi_2(y)$.

7. 6. Type III. Under this type, we consider equations of the form

$Rr + Ss + Pp = F$ or $R(\partial p/\partial x) + S(\partial p/\partial y) = F - Pp$

and $Ss + Tt + Qq = F$ or $S(\partial q/\partial x) + T(\partial q/\partial y) = F - Qq$.

These are linear partial differential equations of order one with p (or q) as dependent variable and x, y, as independent variables. In such situations we shall apply well known Lagrange's method (for more details refer chapter 2).

Recall that $Pp + Qq = R$ is solved by considering its auxiliary equatrions $dx/P = dy/Q = dz/R$. Sometimes the given equation can be reduced to $Pp + Qq = R$ with help of integration of the given equation.

7.7 SOLVED EXAMPLES BASED ON ART 7.6

Ex. 1. *Solve* $t + s + q = 0$. **[Meerut 1994]**

Sol. Re-writing the given equation, $(\partial q/\partial y) + (\partial p/\partial y) + (\partial z/\partial y) = 0$.

Integrating w.r.t. 'y', $q + p + z = f(x)$ or $p + q = f(x) - z$, ...(1)

which is in Lagrange's form $Pp + Qq = R$. Its Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{f(x) - z}. \qquad ...(2)$$

From first and second fractions of (2), $dx - dy = 0$

Integrating, $x - y = c_1$, c_1 being an arbitrary constant. ...(3)

From first and third fraction of (2), $(dz/dx) = f(x) - z$ or $(dz/dx) + z = f(x)$.

Its I.F. $= e^{\int dx} = e^x$ and hence its solution is $\quad ze^x = \int e^x f(x)\, dx + c_2$

or $\quad ze^x - \phi(x) = c_2$, where $\phi(x) = \int e^x (x)\, dx$ and c_2 is an arbitrary constant ...(4)

From (3) and (4), the required general solution is

$ze^x - \phi(x) = \psi(x - y)$ or $ze^x = \phi(x) + \psi(x - y)$, where ϕ and ψ are arbitrary functions.

Ex. 2. *Solve* $p + r + s = 1$. **[Kanpur 2004; Meerut 2005, 10]**

Sol. Re-writing the given equation $\quad (\partial z/\partial x) + (\partial p/\partial x) + (\partial q/\partial x) = 1$

Integrating w.r.t. 'x', $\quad z + p + q = x + f(y) \quad$ or $\quad p + q = x + f(x) - z$, ...(1)

which is in Langrange's form $Pp + Qq = R$. Its Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{1} = \frac{dz}{x + f(y) - z}. \qquad \text{... (2)}$$

From first and second fractions of (2), $\quad dx - dy = 0$.

Integrating, $\quad x - y = c_1$, c_1 being an arbitrary constant. ...(3)

From second and third fractions of (2), $(dz/dy) = x + f(y) - z \quad$ or $\quad (dz/dy) + z = x + f(y)$.

Its I.F. $= e^{\int dy} = e^y$ and hence its solution is

$$ze^y = \int \{x + f(y)\} e^y dy + c_2 = x \int e^y\, dy + \int e^y f(y)\, dy + c_2$$

or $\quad ze^y - xe^y - \phi(y) = c_2$, where $\phi(y) = \int e^y f(y)\, dy$. ...(4)

From (3) and (4), the required general solution is $\quad ze^y - xe^y - \phi(y) = \psi(x - y)$

or $\quad (z - x)e^y = \phi(y) + \psi(x - y)$, where ϕ and ψ are arbitrary functions.

Ex. 3. *Solve* $s - t = x/y^2$. **[Ravishankar 2005; I.A.S. 1988]**

Sol. The given equation can be re-written as $\quad \dfrac{\partial p}{\partial y} - \dfrac{\partial q}{\partial y} = \dfrac{x}{y^2} \quad$ or $\quad \dfrac{\partial}{\partial y}(p - q) = xy^{-2.}$

Integrating it w.r.t. 'y', $\quad p - q = -(x/y) + f(x)$, ...(1)

which is in Lagrange's form $Pp + Qq = R$. Its auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{-(x/y) + f(x)}. \qquad \text{... (2)}$$

Taking first two fractions of (2), $\quad dx + dy = 0 \quad$ so that $\quad x + y = c_1$. ...(3)

Taking first and third fractions of (2), $\quad dz = [-(x/y) + f(x)]\, dx$

or $\quad dz = \left[-\{x/(c_1 - x)\} + f(x)\right] dx$, since from (3), $y = c_1 - x$

or $\quad dz = [1 - \{c_1/(c_1 - x)\} + f(x)]\, dx$.

Integrating, $\quad z = x + c_1 \log(c_1 - x) + \phi(x) + c_2$, where $\phi(x) = \int f(x)\, dx$

or $\quad z - x - (x + y) \log y - \phi(x) = c_2$, using (3). ...(4)

From (3) and (4), the required general solution is $\quad z - x - (x + y) \log y - \phi(x) = \psi(x + y)$

or $\quad z = x + (x + y) \log y + \phi(x) + \psi(x + y)$, where ϕ and ψ are arbitrary functions.

Ex. 4. *Solve* $xyr + x^2 s - yp = x^3 e^y$.

Sol. Re-writing the given equations, $\quad xy(\partial p/\partial x) + x^2(\partial p/\partial y) = yp + x^3 e^y$. ...(1)

Here Lagrange's auxiliary equations for (1) are $\quad \dfrac{dx}{xy} = \dfrac{dy}{x^2} = \dfrac{dp}{yp + x^3 e^y}$. ...(2)

From the first two fractions of (2), $\quad 2x dx - 2y dy = 0 \quad$ so that $\quad x^2 - y^2 = c_1$. ... (3)

From second and third fractions of (2), $\quad dp/dy = (yp + x^3 e^y)/x^2$

or $\quad \dfrac{dp}{dy} - \dfrac{yp}{x^2} = xe^y \quad$ or $\quad \dfrac{dp}{dy} - \dfrac{y}{y^2+c_1} p = (y^2+c_1)^{1/2} e^y,$

[$\because$ from (3), $x^2 = y^2 + c_1$ so that $x = (y^2+c_1)^{1/2}$]

Its I.F. $= e^{-\int \{y/(y^2+c_1)\}dy} = e^{-(1/2)\times\log(y^2+c_1)} = (y^2+c_1)^{-1/2}$ and solution of above equation is

$$p\,(y^2+c_1)^{-1/2} = \int \{(y^2+c_1)^{-1/2} \cdot e^y (y^2+c_1)^{1/2}\}\, dy + c_2 = \int e^y dy + c_2 = e^y + c_2$$

or $\quad px^{-1} = e^y + c_2,\;$ as from (3) $\;y^2 + c_1 = x^2.$ ... (4)

From (3) and (4), the general solution of (1) is $\quad (p/x) - e^y = f(x^2 - y^2)$

or $\quad p = x\,e^y + x\,f(x^2 - y^2) \quad$ or $\quad \partial z/\partial x = x\,e^y + x\,f(x^2-y^2)$, f being an arbitrary function.

Integrating the above equation w.r.t. 'x', $\quad z = e^y \int x\,dx + \int x\,f(x^2-y^2)dx + \phi(y)$

or $\quad z = (1/2) \times x^2 e^y + \psi(x^2 - y^2) + \phi(y)$, where $\psi(x^2-y^2) = \int x\,f(x^2-y^2)dx$

which is the required solution, ϕ and ψ being arbitrary functions.

Ex. 5. (*i*) Solve $xr + ys + p = 10xy^3$ **[Delhi Maths Hons. 1993]**

(*ii*) $xs + yt + q = 10x^3 y$.

Sol. Re-writing the given equation $\quad x(\partial p/\partial x) + y(\partial p/\partial y) = 10xy^3 - p.$ ...(1)

Its Lagrange's auxiliary equations are $\quad \dfrac{dx}{x} = \dfrac{dy}{y} = \dfrac{dp}{10xy^3 - p}.$ ...(2)

Taking the first two ratios (2), $\quad (1/x)dx = (1/y)dy \quad$ so that $\quad x/y = c_1.$ ...(3)

Taking the second and third ratios of (2), we have

$\dfrac{dp}{dy} = \dfrac{10xy^3 - p}{y} \quad$ or $\quad \dfrac{dp}{dy} + \dfrac{1}{y} p = 10xy^2 \quad$ or $\quad \dfrac{dp}{dy} + \dfrac{1}{y} p = 10c_1 y^3$, using (3). ...(4)

I.F. of (4) $= e^{\int (1/y)dy} = e^{\log y} = y$ and so solution is

$py = \int y(10c_1 y^3)\,dy = 2c_1 y^5 + c_2 \quad$ or $\quad py - 2xy^4 = c_2$, using (3) ...(5)

From (3) and (5), the general solution of (1) is

$py - 2xy^4 = \phi(x/y) \quad$ or $\quad (\partial z/\partial x) = 2xy^3 + (1/y) \times \phi(x/y).$

Integrating w.r.t. 'x', $\quad z = x^2 y^3 + \phi_1\left(\dfrac{x}{y}\right) + \phi_2(y), \quad$ where $\quad \dfrac{\partial}{\partial x}\phi_1\left(\dfrac{x}{y}\right) = \dfrac{1}{y}\phi\left(\dfrac{x}{y}\right).$

(*ii*) Do yourself. **Ans.** $zx = x^3 y^2 + \phi_1(y/x) + \phi_2(x)$.

Ex. 6. *Solve* $\quad sy - 2xr - 2p = 6xy$.

Sol. Re-writing the given equation, $\quad 2x(\partial p/\partial x) - y(\partial p/\partial y) = -(6xy + 2p).$ ...(1)

Lagrange's auxiliary equations are $\quad \dfrac{dx}{2x} = \dfrac{dy}{-y} = \dfrac{dp}{-6xy - 2p}.$ ...(2)

Taking the first and second ratios in (2), we get

$(1/x)dx + (2/y)dy = 0 \quad$ so that $\quad xy^2 = c_1.$ ...(3)

Now, $\quad$ each ratio of (2) $= \dfrac{2y^3 dx - (2yp + 2xy^2)\,dy + y^2 dp}{0}$

$\Rightarrow (2yp + 2xy^2)dy - 2y^3 dx - y^2 dp = 0 \Rightarrow 2y\,(p + 2xy)dy - y^2(dp + 2xdy + 2ydx) = 0.$

$\Rightarrow 2y\,(p + 2xy)dy - y^2 d\,(p + 2xy) = 0 \quad \Rightarrow \quad -\dfrac{2dy}{y} + \dfrac{d(p+2xy)}{p+2xy} = 0$

Integrating it, $\quad -2\log y + \log(p + 2xy) = \log c_1$, being an arbitrary constant

or $\quad \log\{(p + 2xy)/y^2\} = \log c_1 \quad$ or $\quad (p + 2xy)/y^2 = c_1.$ ...(4)

From (3) and (4), the general solution of (1) is

$(p + 2xy)/y^2 = \phi(xy^2)$ or $(\partial z/\partial x) = -2xy + y^2\phi(xy^2)$. ... (5)

Integrating (5) w.r.t. 'x', $z = -x^2y + \phi_1(xy^2) + \phi_2(y)$, where $\dfrac{\partial}{\partial x}\phi_1(xy^2) = y^2\phi(xy^2)$.

Ex. 7. *Solve* $r + (y/x)s = 15xy^2$.

Sol. Re-writing the given equation, $(\partial p/\partial x) + (y/x)(\partial p/\partial y) = 15xy^2$. ...(1)

So Lagrange's auxiliary equations are $\dfrac{dx}{1} = \dfrac{dy}{y/x} = \dfrac{dp}{15xy^2}$. ...(2)

From (2), $(1/y)\,dy = (1/x)\,dx \;\Rightarrow\; \log y - \log x = \log c_1 \;\Rightarrow\; y/x = c_1$. ...(3)

Taking the first and third ratios of (2), $dp = 15xy^2dx = 15c_1^2x^3dx$, by (3)

Integrating, $p = (15/4) \times c_1^2x^4 + c_2 \;\Rightarrow\; p - (15/4) \times (y/x)^2 \times x^4 = c_2$, by (3)

or $p - (15/4) \times x^2y^2 = c_2$, c_2 being an arbitrary constant ...(4)

Using (3) and (4), the general solution of (1) is

$$p - \frac{15}{4}x^2y^2 = \phi\left(\frac{y}{x}\right) \quad \text{or} \quad \frac{\partial z}{\partial x} = \frac{15}{4}x^2y^2 + \phi\left(\frac{y}{x}\right). \quad ...(5)$$

Integrating (5) w.r.t. 'x', $z = \dfrac{5}{4}x^3y^2 + y\displaystyle\int \frac{1}{(-y^2/x^2)}\phi\left(\frac{y}{x}\right)d\left(\frac{y}{x}\right) + \phi_2(y)$

or $z = (5/4) \times x^3y^2 + y\phi_1(y/x) + \phi_2(y)$, where ϕ_1 and ϕ_2 are arbitrary functions.

Ex. 8. *Solve the following partial differential equations* :

(*i*) $2xr - ys + 2p = xy^2$. (*ii*) $2yt - xs + 2q = 4yx^2$.

Ans. (*i*) $z = \phi_1(xy^2) + \phi_2(y) + (1/4) \times x^2y^2$. (*ii*) $z = \phi_1(x^2y) + \phi_2(x) + x^2y^2$.

Ex. 9. *Solve* $s + r = x\cos(x + y)$

Sol. Re-writing the given equation, $\partial p/\partial x + \partial p/\partial y = x\cos(x + y)$ ... (1)

Its Lagrange's auxiliary equations are $\dfrac{dx}{1} = \dfrac{dy}{1} = \dfrac{dp}{x\cos(x+y)}$ .... (2)

Taking the first two ratios, $dx - dy = 0$ so that $x - y = c_1$ ... (3)

Taking the first and the last fractions of (2) and using (3), we get

$dp = x\cos(x + x - c_1)$ or $dp = x\cos(2x - c_1)$

Integrating, $p = (1/2) \times x\sin(2x - c_1) - (1/2) \times \displaystyle\int \sin(2x - c_1)dx + c_2$

or $p - (1/2) \times x\sin(2x - c_1) - (1/4) \times \cos(2x - c_1) = c_2$

or $p - (1/2) \times x\sin(x + y) - (1/4) \times \cos(x + y) = c_2$, using (2). ... (4)

From (3) and (4), the general solution of (1) is given by

$$p - (1/2) \times x\sin(x + y) - (1/4) \times \cos(x + y) = f(x - y).$$

or $\partial z/\partial x = (1/2) \times x\sin(x + y) + (1/4) \times \cos(x + y) + f(x - y)$ ... (5)

Integrting (5) w.r.t. 'x', $z = (1/2) \times [-x\cos(x + y) + \displaystyle\int \cos(x + y)\,dx]$

$+ (1/4) \times \sin(x + y) + \phi_1(x - y) + \phi_2(y)$, where $\phi_1(x - y) = \displaystyle\int f(x - y)\,dx$

or $z = -(x/2) \times \cos(x + y) + (3/4) \times \sin(x + y) + \phi_1(x - y) + \phi_2(y)$.

Ex. 10. *Solve* $yt + xs + q = 8yx^2 + 9y^2$.

Sol. Re-writing the given equation, $x(\partial q/\partial x) + y(\partial q/\partial y) = 8yx^2 + 9y^2 - q$ ... (1)

Its Lagrange's auxiliary equations are $$\frac{dx}{x} = \frac{dy}{y} = \frac{dq}{8yx^2 + 9y^2 - q} \qquad \text{... (2)}$$

Taking the first two ratios of (2), $\log y - \log x = \log c_1$ or $y/x = c_1$ ... (3)
Taking the last two ratios of (2), we get

$$\frac{dq}{dy} = 9y + 8x^2 - \frac{q}{y} \qquad \text{or} \qquad \frac{dq}{dy} + \frac{1}{y}q = 9y + 8x^2 = 9y + \frac{8y^2}{c_1^2}, \text{ by (3)}$$

which is linear differential equation. Its I.F. $= e^{\int (1/y)\, dy} = e^{\log y} = y$ and solution is

$$qy = \int y(9y + 8y^2/c_1^2)\, dy + c_2 = 3y^3 + (2y^4/c_1^2) + c_2$$

or $qy - 3y^3 - (2y^4/c_1^2) = c_2$ or $qy - 3y^3 - 2y^2 x^2 = c_2$, by (3) ... (4)
From (3) and (4), the general solution of (1) is

$qy - 3y^3 - 2y^2 x^2 = f(y/x)$ or $\partial z/\partial y = 3y^2 + 2x^2 y + (1/y) \times f(y/x)$
Integrating it w.r.t. 'y' while treating x as constant, we get

$$z = y^3 + x^2y^2 + \int \frac{1}{y} f(y/x)\, dy + \phi_1(x) \qquad \text{or} \qquad z = y^3 + x^2y^2 + \int \frac{f(y/x)}{(y/x)} d\left(\frac{y}{x}\right) + \phi_1(x)$$

or $z = y^3 + x^2y^2 + \phi_2(y/x) + \phi_1(x)$, ϕ_1, ϕ_2 being arbitrary functions

Ex. 11. *Solve* $xyr + x^2 s - yp = x^3 e^y$

Sol. Re-writing the given equation, $xy\,(\partial p/\partial x) + x^2\,(\partial p/\partial y) = yp + x^3 e^y$ ... (1)

Its Lagrange's auxiliary equations are $$\frac{dx}{xy} = \frac{dy}{x^2} = \frac{dp}{yp + x^3 e^y} \qquad \text{... (2)}$$

Taking the first two ratios of (2), $2xdx - 2ydy = 0$ so that $x^2 - y^2 = c_1$ ... (3)
Taking the first and the last ratios of (2), we get

$$\frac{dp}{dx} = \frac{yp + x^3 e^y}{xy} = \frac{p}{x} + \frac{x^2 e^y}{y} \qquad \text{or} \qquad \frac{dp}{dx} - \frac{1}{x}p = \frac{x^2}{(x^2 - c_1)^{1/2}}\, e^{(x^2 - c_1)^{1/2}}, \text{ by (3)}$$

Its I.F. $= e^{\int (-1/x)\, dx} = e^{-\log x} = 1/x$ and solution is

$$p \times \frac{1}{x} = \int \frac{1}{x} \frac{x^2}{(x^2 - c_1)^{1/2}}\, e^{(x^2 - c_1)^{1/2}}\, dx = \int e^t\, dt = e^t + c_2$$

[on putting $(x^2 - c_1)^{1/2} = \text{t}$ and $\{x/(x^2 - c_1^2)^{1/2}\}dx = dt$]

or $(p/x) - e^{(x^2 - c_1)^{1/2}} = c_2$ or $(p/x) - e^y = c_2$, using (3) ... (4)
From (3) and (4), the general solution of (1) is

$(p/x) - e^y = f(x^2 - y^2)$ or $\partial z/\partial x = x\, e^y + xf(x^2 - y^2)$ ... (5)
Integrating (5) w.r.t. 'x' (while treating y as constant), we get

$z = (1/2) \times x^2 e^y + \phi_1(x^2 - y^2) + \phi_2(y)$, where $\phi_1(x^2 - y^2) = \int x\, f(x^2 - y^2)\, dx.$

7.8. Type IV. Under this type, we consider equations of the form

$$Rr + Pp + Zz = F \qquad \text{or} \qquad R\frac{\partial^2 z}{\partial x^2} + P\frac{\partial z}{\partial x} + Zz = F \qquad \text{...(1)}$$

and $\quad Tt + Qq + Zz = F \quad$ or $\quad T\dfrac{\partial^2 z}{\partial y^2} + Q\dfrac{\partial z}{\partial y} + Zz = F, \quad$ ...(2)

which are linear ordinary differential equations of order two with x as independent variable in (1) and y as independent variable in (2).

7.9 SOLVED EXAMPLES BASED ON ART 7.8

Ex. 1. *Solve* $t - 2xq + x^2 z = (x-2)e^{3x+2y}$. **[Delhi Math (G) 1999; Poona 1996]**

Sol. Taking $D' \equiv \partial/\partial y$, the given equation becomes

$$(D'^2 - 2x\,D' + x^2)z = (x-2)e^{3x+2y} \quad \text{or} \quad (D'-x)^2 = (x-2)\,e^{3x+2y} \quad ...(1)$$

$\therefore$ Complementary function of (1) $= e^{xy}\,\{\phi_1(x) + x\phi_2(x)\}$.

and $\quad$ particular integral of (1) $= \dfrac{1}{(D'-x)^2}(x-2)e^{3x+2y} = \dfrac{(x-2)\,e^{3x+2y}}{(2-x)^2} = \dfrac{e^{3x+2y}}{x-2}$.

$\therefore$ Required solution is $z = e^{xy}\,\{\phi_1(x) + x\phi_2(x)\} + \dfrac{e^{3x+2y}}{x-2}$, ϕ_1, ϕ_2 being arbitrary functions.

Ex. 2. *Solve* (*i*) $t - q - (1/x)\,\{(1/x) - 1\}z = xy^2 - x^2y^2 + 2x^3y - 2x^3$. **[Calicut 1999]**

(*ii*) $r - p - (1/y)\,\{(1/y) - 1\}z = x^2y - x^2y^2 + 2xy^3 - 2y^3$.

Sol. (*i*) Let $D' \equiv \partial/\partial y$. Then given equation can be re-written as

$$[D'^2 - D' - (1/x)\,\{(1/x) - 1\}]\,z = xy^2 - x^2y^2 + 2x^3y - 2x^3. \quad ...(1)$$

or

$$\left(D' - \frac{1}{x}\right)\left\{D' + \left(\frac{1}{x} - 1\right)\right\}z = xy^2 - x^2y^2 + 2x^3y - 2x^3. \quad ...(1)'$$

So $\quad$ C.F. $= e^{y/x}\,\phi_1(x) + e^{y-(y/x)}\phi_2(x)$, ϕ_1, ϕ_2 being arbitrary functions

In order to determine a particular integral of (1), we assume that

$$z = F_1 y^2 + F_2 y + F_3, \text{ where } F_1, F_2, F_3 \text{ are functions of } x \text{ or constants.} \quad ...(2)$$

$$(2) \Rightarrow \partial z/\partial y = 2F_1 y + F_2 \quad \Rightarrow \quad \partial^2 z/\partial y^2 = 2F_1.$$

so that $\quad q = 2F_1 y + F_2 \quad$ and $\quad t = 2F_1. \quad$ ...(3)

Using (2) and (3), given equation reduces to

$$2F_1 - (2F_1 y + F_2) - \frac{1}{x}\left(\frac{1}{x} - 1\right)(F_1 y^2 + F_2 y + F_3) = xy^2 - x^2y^2 + 2x^3y - 2x^3.$$

Equation coefficients of various powers of y in the above identity, we obtain

$$-\{(1/x^2) - (1/x)\}F_1 = x\,(1-x), \quad ...(4)$$

$$-2F_1 - \{(1/x^2) - (1/x)\}F_2 = 2x^3 \quad ...(5)$$

and

$$2F_1 - F_2 - \{(1/x^2) - (1/x)\}F_3 = -2x^3. \quad ...(6)$$

From (4), $F_1 = -x^3$. Then, from (5), $F_2 = 0$. So (6) $\Rightarrow$ $F_3 = 0$.

$\therefore$ from (2), P.I. $= -x^3y^2$ and so the required solution is

$$z = e^{y/x}\phi_1(x) + e^{y-(y/x)}\phi_2(x) - x^3y^2.$$

(*ii*) Do your as in part (*i*). **Ans.** $z = e^{(x/y)}\phi_1(y) + e^{x-(x/y)}\phi_2(y) - x^2y^3$.

7.10. SOLUTIONS OF EQUATIONS UNDER GIVEN GEOMETRICAL CONDITIONS.

Working rule. As explained in this chapter, we first find the solution of the given equation containing some arbitrary functions of x and y, which are determined with help of the given geometrical conditions. Substituting the values of arbitrary functions in the general solution, we shall obtain surfaces which satisfy the given geometrical conditions.

7.11 SOLVED EXAMPLES BASED ON ART 7.10

Ex. 1.(a) *Find the surface satisfying* $t = 6x^2y$ *containing two lines* $y = 0 = z$ *and* $y = 2 = z$.

[Kanpur 2001; Sagar 2004]

Sol. Re-writing the given equation, we get $\partial q/\partial y = 6x^2y$.

Integrating it w.r.t. 'y', $q = 3x^2y^2 + f(x)$ or $\partial z/\partial y = 3x^2y^2 + f(x)$.

Integrating it w.r.t. 'y', $z = x^2y^3 + y f(x) + \phi(x)$, ...(1)

which is the general solution, f and ϕ being arbitrary functions.

Since (1) contains the given lines $y = 0 = z$ and $y = 2 = z$, we get $0 = \phi(x)$ ...(2)

and $2 = 8x^2 + 2 f(x) + \phi(x)$. ...(3)

Using (2), (3) becomes $2 = 8x^2 + 2 f(x)$ or $f(x) = 1 - 4x^2$.

Putting $\phi(x) = 0$ and $f(x) = 1 - 4x^2$ in (1), the required surface is $z = x^2y^3 + y(1 - 4x^2)$.

Ex. 1.(b) *Find a surface satisfying* $t = 6x^3y$ *and containing the two lines* $y = 0 = z, y = 1 = z$.

Sol. Re-writing the given equation, we get $(\partial q/\partial y) = 6x^3y$

Integrating w.r.t. y, $q = 3x^3y^2 + f(x)$ or $\partial z/\partial y = 3x^3y^2 + f(x)$

Integrating it w.r.t. y $z = x^3y^3 + y f(x) + \phi(x)$. ...(1)

Since (1) contains the lines $y = 0 = z$ and $y = 1 = z$, we get $0 = \phi(x)$ ...(2)

and $1 = x^3 + f(x) + \phi(x)$. ...(3)

From (2) and (3), $\phi(x) = 0$ and $f(x) = 1 - x^3$.

Putting these values in (1), the required surface is $z = x^3y^3 + y(1 - x^3)$.

Ex. 2. *Find the surface passing through the parabolas* $z = 0$, $y^2 = 4ax$ and $z = 1, y^2 = -4ax$ *and satisfying the equation* $xr + 2p = 0$. **[Kanpur 2000; Agra 1996 ; Meerut 1993 ; I.A.S. 2006]**

Sol. Re-writing the given differential equation, $x(\partial p/\partial x) + 2p = 0$

or $x^2(\partial p/\partial x) + 2px = 0$ or $\partial(x^2p)/\partial x = 0$

Integrating it w.r.t. x, $x^2p = f(y)$ or $p = f(y)/x^2$ or $(\partial z/\partial x) = (1/x^2) \times f(y)$.

Integrating it w.r.t. x, $z = -(1/x) \times f(y) + \phi(y)$. ...(1)

Since (1) passes through $z = 0$, $y^2 = 4ax$, $0 = -(4a/y^2) \times f(y) + \phi(y)$. ...(2)

Again since (1) passes through $z = 1$, $y^2 = -4a x$, $1 = (4a/y^2) \times f(y) + \phi(y)$. ...(3)

Adding (2) and (3), $1 = 2\phi(y)$ so that $\phi(y) = 1/2$. ...(4)

Putting $\phi(y) = 1/2$ in (2), we get $f(y) = y^2/8a$. ...(5)

Putting the values of $\phi(y)$ and $f(y)$ given by (4) and (5) in (1), the desired surface is

$z = -y^2/(8ax) + 1/2$ or $8axy = 4ax - y^2$.

Ex. 3. *Show that a surface satisfying* $r = 6x + 2$ *and touching* $z = x^3 + y^3$ *along its section by the plane* $x + y + 1 = 0$ *is* $z = x^3 + y^3 + (x + y + 1)^2$. **[Agra 1994; KU Kurukshetra 2004]**

Sol. Given $r = 6x + 2$ or $\partial p/\partial x = 6x + 2$. ...(1)

Integrating (1) w.r.t. x, $p = 3x^2 + 2x + f(y)$ or $\partial z/\partial x = 3x^2 + 2x + f(y)$. ...(2)

Integrating (2) w.r.t. 'x', $z = x^3 + x^2 + xf(y) + F(y)$, ...(3)

where $f(y)$ and $F(y)$ are arbitrary functions.

The given surface is $z = x^3 + y^3$ ...(4)

and the given plane is $x + y + 1 = 0$. ...(5)

Since (3) and (4) touch each other along their section by (5), the values of p and q at any point on (5) must be equal. Thus we must have

$$3x^2 + 2x + f(y) = 3x^2 \qquad ...(6)$$

and $$xf'(y) + F'(y) = 3y^2. \qquad ...(7)$$

From (5) and (6), $f(y) = -2x = 2(y + 1)$ ...(8)

From (8), $f'(y) = 2$. Using this value, (7) gives

$2x + F'(y) = 3y^2$ or $F'(y) = 3y^2 - 2x$ or $F'(y) = 3y^2 + 2(y+1)$, using (5)

Integrating it, $$F(y) = y^3 + y^2 + 2y + c, \qquad ...(9)$$

where c is an arbitrary constant. Using (8) and (9), (3) gives

$$z = x^3 + x^2 + 2x(y+1) + y^3 + y^2 + 2y + c \qquad ...(10)$$

Now at the point of contact of (4) and (10) values of z must be the same and hence we have

$x^3 + x^2 + 2x(y+1) + y^3 + y^2 + 2y + c = x^3 + y^3$ or $x^2 + 2x(y+1) + y^2 + 2y + c = 0$

or $x^2 + 2x(-x) + (x+1)^2 - 2(x+1) + c = 0$, as from (5), $y + 1 = -x$ and $y = -(x+1)$

which gives $c = 1$. Putting $c = 1$ in (10), the required surface is

$z = x^3 + x^2 + 2x(y+1) + y^3 + y^2 + 2y + 1$ or $z = x^3 + y^3 + (x+y+1)^2$

Ex. 4(a). *Show that a surface passing through the circle* $z = 0$, $x^2 + y^2 = 1$ *and satisfying the differential equation* $s = 8xy$ *is* $z = (x^2 + y^2)^2 - 1$. **[Agra 1993 ; Meerut 1994]**

Sol. Re-writing the given equation, $$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = 8xy. \qquad ...(1)$$

Integrating (1) w.r.t. 'x', $$\partial z/\partial y = 4x^2 y + f(y). \qquad ...(2)$$

Integrating (2) w.r.t. 'y', $$z = 2x^2y^2 + \int f(y)\,dy + \phi_1(x)$$

or $$z = 2x^2y^2 + \phi_2(y) + \phi_1(x), \qquad ...(3)$$

where $\phi_2(y) = \int f(y)\,dy$ and ϕ_1 ϕ_2 are arbitrary functions.

Given circle is given by $x^2 + y^2 = 1$ and $z = 0$. ...(4)

Putting $z = 0$ in (3), we have $2x^2y^2 + \phi_2(y) + \phi_1(x) = 0$. ...(5)

Now, $x^2 + y^2 = 1 \Rightarrow (x^2 + y^2)^2 = 1^2 \Rightarrow 2x^2y^2 + x^4 + y^4 = 1$. ...(6)

Comparing (5) and (6), $\phi_2(y) + \phi_1(x) = x^4 + y^4 - 1$.

Substituting the above value of $\phi_2(y) + \phi_1(x)$ in (3), we have

$z = 2x^2y^2 + x^4 + y^4 - 1$ or $z = (x^2 + y^2)^2 - 1$.

Ex. 4(b). *Find the surface passing through the cirlce* $x^2 + y^2 = a^2$, $z = 0$ *and satisfying the differential equation* $s = 8xy$.

Sol. Proceed as in Ex. 4(a).

Ex. 5. *Show that a surface of revolution satisfying the differential equation* $r = 12x^2 + 4y^2$ *and touching the plane* $z = 0$ *is* $z = (x^2 + y^2)^2$. **[Kanpur 1999; Agra 2000, 02 ; Meerut 1993, 97]**

Sol. The given equation can be re-written as

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = 12x^2 + y^2 \quad \text{or} \quad \frac{\partial p}{\partial x} = 12x^2 + 4y^2. \qquad ...(1)$$

Integrating (1) w.r.t. 'x', $$p = \partial z/\partial x = 4x^3 + 4xy^2 + f(y). \qquad ...(2)$$

Integrating (2) w.r.t., 'x', $$z = x^4 + 2x^2y^2 + xf(y) + g(y). \qquad ...(3)$$

Given the required surface (3) touches the plane $z = 0$. Now, for $z = 0$, $\partial z/\partial x = 0$ and so (2) reduces to $4x^3 + 4xy^2 + f(y) = 0$ or $-f(y) = 4x^3 + 4xy^2$. ...(4)

Since L.H.S. of (4) is function of y alone and R.H.S. is not a function of y alone, (4) shows that we must take each side of (4) equal to zero. Thus, we take $f(y) = 0$ ...(5)

and $4x^3 + 4xy^2 = 0$ so that $x^2 = -y^2$.

Putting $z = 0$, $x^2 = -y^2$ and $f(y) = 0$ in (3), we have

$0 = y^4 - 2y^4 + 0 + g(y)$ so that $g(y) = y^4$. ...(6)

Putting the values of $f(y)$ and $g(y)$ given by (5) and (6) in (3), desired surface is

$z = x^4 + 2x^2y^2 + y^4$ or $z = (x^2 + y^2)^2$.

8

Classification of P.D.E. Reduction to Canonical or Normal Forms. Riemann Method

8.1. CLASSIFICATION OF PARTIAL DIFFERENTIAL EQUATIONS OF SECOND ORDER.

Consider a general partial differential equation of second order for a function of two independent variables x and y in the form:

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0, \qquad ...(1)$$

where R, S and T are continuous functions of x and y only possessing partial derivatives defined in some domain D on the xy-plane. Then (1) is said to be

(i) *Hyperbolic* at a point (x, y) in domain D if $S^2 - 4RT > 0$

(ii) *Parabolic* at a point (x, y) in domain D if $S^2 - 4RT = 0$

(iii) *Elliptic* at a point (x, y) in domain D if $S^2 - 4RT < 0$.

Observe that the type of (1) is determined solely by its principal part ($Rr + Ss + Tt$, which involves the highest order derivatives of z) and that the type will generally change with position in the xy-plane unless R, S and T are constants

Remark. Some authors use u in place of z. Then, we have

$r = \partial^2 u/\partial x^2$, $s = \partial^2 u/\partial x \partial y$ and $t = \partial^2 u/\partial t^2$. etc.

Examples: (i) Consider the one-dimensional wave equation $\partial^2 z/\partial x^2 = \partial^2 z/\partial y^2$ *i.e.* $r - t = 0$.
Comparing it with (1), here $R = 1$, $S = 0$ and $T = -1$.
Hence $S^2 - 4RT = 0 - \{4 \times 1 \times (-1)\} = 4 > 0$ and so the given equation is hyperbolic.

(ii) Consider the one-dimensional diffusion equation $\partial^2 z/\partial x^2 = \partial z/\partial y$ *i.e.* $r - q = 0$.
Comparing it with (1), here $R = 1$ and $S = T = 0$.
Hence $S^2 - 4RT = 0 - (4 \times 1 \times 0) = 0$ and so the given equation is parabolic.

(iii) Consider two dimensional Laplace's equation $\partial^2 z/\partial x^2 + \partial^2 z/\partial y^2 = 0$ *i.e.* $r + t = 0$.
Comparing it with (1), here $R = 1$, $S = 0$ and $T = 1$.
Hence $S^2 - 4RT = 0 - (4 \times 1 \times 1) = -4 < 0$ and so the given equation is elliptic.

Ex. 2. *Classify the following partial differential equations:*

(i) $2(\partial^2 u/\partial x^2) + 4(\partial^2 u/\partial x \partial y) + 3(\partial^2 u/\partial y^2) = 2$ **[Meerut 2006]**

(ii) $\partial^2 u/\partial x^2 + 4(\partial^2 u/\partial x \partial y) + 4(\partial^2 u/\partial y^2) = 0$ **[I.F.S. 2005]**

(iii) $xyr - (x^2 - y^2)s - xyt + py - qx = 2(x^2 - y^2)$ **[Delhi Maths (G) 2006]**

(iv) $x^2(y-1)r - x(y^2-1)s + y(y-1)t + xyp - q = 0$ **[Delhi Maths (Prog) 2007]**

(v) $x(xy-1)r - (x^2y^2-1)s + y(xy-1)t + xp + yq = 0$ **[Delhi 2008]**

(vi) $(x - y)(xr - xs - ys + yt) = (x + y)(p - q)$ **[Delhi BA (Prog) II 2011]**

Sol. (*i*) Re-writing the given equation, we get $2r+4s+3t-2=0$ ...(1)

Comparing (1) with $Rs+Ss+Tt+f(x,y,u,p,q)=0,$ we get $R=2, S=4$ and $T=3.$ So $S^2-4RT=(4)^2-(4\times 2\times 3)=-8<0,$ showing that the given equation is elliptic at all points .

(*ii*) Re-writing the given equation, we get $r+4s+4t=0$ ...(1)

Comparing (1) with $Rr+Ss+Tt+f(x,y,u,p,q)=0,$ we get $R=1, S=4$ and $T=4.$ So $S^2-4RT=(4)^2-(4\times 1\times 4)=0,$ showing that the given equation is parabolic at all points.

(*iii*) Given $xyr-(x^2-y^2)s-xyt+py-qx-2(x^2-y^2)=0$ ...(1)

Comparing (1) with $Rs+Ss+Tt+f(x, y, z, p, q)=0,$ we get $R=xy, S=-(x^2-y^2)$ and $T=-xy.$ So, here $S^2-4RT=(x^2-y^2)^2+4x^2y^2=(x^2+y^2)^2>0,$ showing that the given equation is hyperbolic at all points.

(*iv*) Hyperbolic (*v*) Hyperbolic

(*vi*) Hyperbolic

8.2. CLASSIFICATION OF A PARTIAL DIFFERENTIAL EQUATION IN THREE INDEPENDENT VARIABLES.

A linear partial differential differential equation of the second order in 3 independent variables x_1, x_2, x_3 is given by

$$\sum_{i=1}^{3}\sum_{j=1}^{3} a_{ij}\frac{\partial^2 u}{\partial x_i \partial x_j}+\sum_{i=1}^{3} b_i \frac{\partial u}{\partial x_i}+cu=0 \qquad \text{... (1)}$$

where $a_{ij}\ (=a_{ji})$, b_i and c are constants or some functions of the independent variables x_1, x_2, x_3 and u is the dependent variable.

Since $a_{ij}=a_{ji}$, $A=[a_{ij}]_{3\times 3}$ is a real symmetric matrix of order 3 × 3. The eigen values of matrix A are roots of the characteristic equation of A, namely, $|A-\lambda I|=0$.

With help of matrix A, (1) is classified as follows:

I. If all the eigenvalues of A are non-zero and have the same sign, except precisely one of them, then (1) is known as *hyperbolic type of equation*.

II. If $|A|=0$, *i.e.,* any one of the eigenvalues of A is zero, then (1) is known as *parabolic type of equation*

III. If all the eigenvalues of A are non-zero and of the same sign, then (1) is known as *elliptic type of equation*.

Note. the matrix A can be remembered as indicated below:

$$A=\begin{bmatrix} \text{Coeff. of } u_{xx} & \text{Coeff. of } u_{xy} & \text{Coeff of } u_{xz} \\ \text{Coeff. of } u_{yx} & \text{Coeff. of } u_{yy} & \text{Coeff of } u_{yz} \\ \text{Coeff. of } u_{zx} & \text{Coeff. of } u_{zy} & \text{Coeff of } u_{zz} \end{bmatrix}$$

8.2.A SOLVED EXAMPLES BASED ON ART. 8.2

Ex. 1. *Classify* $u_{xx}+u_{yy}=u_{zz}$ **[Delhi Maths (H) 2007; Kanpur 2011]**

The matrix A of the given equation is given by $$A=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

The eigenvalues of A are given by $|A-\lambda I|=0$, *i.e.*,

$$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & -1-\lambda \end{vmatrix} = 0 \qquad \text{or} \qquad -(1+\lambda)\,(1-\lambda)^2 = 0.$$

Hence $\lambda = -1$, 1, 1, showing that all the eigenvalues are non-zero and have the same sign except one. Hence the given equation is of hyperbolic type.

Ex. 2. *Classify* $u_{xx} + u_{yy} + u_{zz} + u_{yz} + u_{zy} = 0$.

Sol. The given equation can be re-written as

$$u_{xx} + 0 \cdot u_{xy} + 0 \cdot u_{xz} + 0 \cdot u_{yx} + u_{yy} + u_{yz} + 0 \cdot u_{zx} + u_{zy} + u_{zz} = 0$$

$\therefore$ The matrix A of the given equation is given by $$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

Now, $|A| = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0$, using properties of determinents

Since $|A| = 0$, the given equation is of parabolic type.

Ex. 3. *Classify* $u_{xx} + u_{yy} + u_{zz} = 0$ **[Meerut 2007, 08; Kanpur 2011]**

Sol. The given equation can be re-written as

$$u_{xx} + 0 \cdot u_{xy} + 0 \cdot u_{xz} + 0 \cdot u_{yx} + u_{yy} + 0 \cdot u_{yz} + 0 \cdot u_{zx} + 0 \cdot u_{zy} + u_{zz} = 0$$

$\therefore$ The matrix A of the given equation is given by $$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The eigen values of A are given by $|A - \lambda I| = 0$,

i.e. $$\begin{vmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0 \qquad \text{or} \qquad (1-\lambda)^3 = 0 \quad \text{giving} \quad \lambda = 1,1,1.$$

Since all eigenvalues are non-zero and of the same sign, the given equation is of parabolic type.

Ex. 4. *Classify the following equations:*

(*i*) $u_{xx} + u_{yy} = u_z$ **[Kanpur 2011]** (*ii*) $u_{xx} + 2\,u_{yy} + u_{zz} = 2u_{xy} + 2u_{yz}$. **[Delhi 2008]**

Sol. Try yourself **Ans.** (*i*) parabolic (*ii*) parabolic

8.3. Cauchy's problem for second order partial differential equation. Characteristic equation and characteristic curves (or simply characteristics) of the second order partial differential equations. (Delhi Maths (H) 2001)

Cauchy' problem. Consider the second order partial differential equation

$$Rr + Ss + Tt + f(x, y, z, p, q) = 0 \qquad \ldots (1)$$

in which R, S and T are functions of x and y only. The Cauchy's problem consists of the problem of determining the solution of (1) such that on a given space curve C it takes on prescribed values of z and $\partial z / \partial n$, where n is the distance measured along the normal to the curve.

As an example of Cauchy's problem for the second order partial differential equation, consider the following problem :

To determine solution of $\partial^2 z/\partial x^2 = \partial^2 z/\partial y^2$ *with the following data prescribed on the x-axis:* $z(x, 0) = f(x)$, $z_y(x, 0) = g(x)$. *Observe that y-axis is the normal to the given curve (x-axis here)*

Characteristic equations and characteristic curves.

Corresponding to (1), consider the λ-quadratic

$$R\lambda^2 + S\lambda + T = 0 \qquad \text{... (2)}$$

where $S^2 - 4RT \geq 0$, (2) has real roots. Then, the ordinary differential equations

$$(dy/dx) + \lambda(x, y) = 0 \qquad \text{... (3)}$$

are called the *characteristic equations*.

The solutions of (3) are knwon as *characteristic curves* or simply the *characteristics* of the second order partial differential equation (1).

Now, consider the following three cases:

Case (*i*) If $S^2 - 4RT > 0$ (*i.e.,* if (1) is hyperbolic), then (2) has two distinct real roots λ_1, λ_2 say so that we have two characteristic equations $(dy/dx) + \lambda_1(x, y) = 0$ and $(dy/dx) + \lambda_2(x, y) = 0$.

Solving these we get two distinct families of characteristics.

Case (*ii*). If $S^2 - 4RT = 0$ (*i.e.* (1) is parabolic), then (2) has two equal real roots λ, λ so that we get only one characteristic equation (3). Solving it, we get only one family of characteristics.

Case (*iii*) If $S^2 - 4RT < 0$ (*i.e.* (1) is elliptic), then (2) has complex roots. Hence there are no real characteristics. Thus we get two families of complex characteristics when (1) is elliptic

8.4 ILLUSTRATIVE SOLVED EXAMPLES BASED ON ART. 8.3

Ex. 1. *Find the characteristics of* $y^2 r - x^2 t = 0$ **[I.A.S. 2009]**

Sol. Given $$y^2 r - x^2 t = 0 \qquad \text{... (1)}$$

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = y^2$, $S = 0$ and $T = -x^2$. Then $S^2 - 4RT = 0 - 4 \times y^2 \times (-x^2) = 4x^2y^2 > 0$ and hence (1) is hyperbolaic everywhere except on the coordinate axes $x = 0$ and $y = 0$.

The λ-quadratic is $\quad R\lambda^2 + S\lambda + T = 0 \quad$ or $\quad y^2\lambda^2 - x^2 = 0 \quad$ ... (2)

Solving (2), $\lambda = x/y, -x/y$ (two distinct real roots). Corresponding characteristic equations are

$$(dy/dx) + (x/y) = 0 \quad \text{and} \quad (dy/dx) - (x/y) = 0$$

or $$x\,dx + y\,dx = 0 \quad \text{and} \quad x dx - y\,dy = 0$$

Integrating, $x^2 + y^2 = c_1$ and $x^2 - y^2 = c_2$, which are the required families of characteristics. Here these are families of circles and hyperbolas respectively.

Ex. 2. *Find the characteristics of* $x^2 r + 2xys + y^2 t = 0$.

Sol. Given $$x^2 r + 2xys + y^2 t = 0 \qquad \text{... (1)}$$

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = x^2$, $S = 2xy$ and $T = y^2$. Then, $S^2 - 4RT = 4x^2y^2 - 4x^2y^2 = 0$ and hence (1) is parabolic everywhere.

The λ-quadratic is $\quad R\lambda^2 + S\lambda + T = 0 \quad$ or $\quad x^2\lambda^2 + 2xy\lambda + y^2 = 0 \quad$ ... (2)

Solving (2), $(x\lambda + y)^2 = 0$ so that $\lambda = -y/x, -y/x$ (equal roots). The characteristic equation is

$(dy/dx) - (y/x) = 0$ or $(1/y)\,dy - (1/x)\,dx = 0$ giving $y/x = c_1$ or $y = c_1 x$, which is the required family of characteristics. Here it represents a family of straight lines passing through the origin.

Ex. 3. *Find the characteristics of* $4r + 5s + t + p + q - 2 = 0$.

Sol. Try yourself. **Ans.** $y - x = c_1$ and $y - (x/y) = c_2$.

Ex. 4. *Find the characteristics of* $(\sin^2 x)\, r + (2 \cos x)\, s - t = 0$

Sol. Try yourself **Ans.** $y + \operatorname{cosec} x - \cot x = c_1$, $y + \operatorname{cosec} x + \cot x = c_2$

8.5. Laplace transformation. Reduction to Canonical (or normal) forms.

[Himanchal 2007; Avadh 2001; Delhi Maths (H) 2004, 09]

Consider partial differential equation of the type $\quad Rr + Ss + Tt + f(x, y, z, p, q) = 0, \quad$ ...(1)

where R, S, T are continuous functions of x and y possessing continuous partial derivatives of as high an order as necessary. Laplace transformation on (1) consists of changing the independent variables x, y to new set of continuously differentiable independent variables u, v where

$$u = u(x, y) \qquad \text{and} \qquad v = v(x, y) \qquad ...(2)$$

are to be chosen so that the resulting equation in independent variables u, v is transformed into one of three canonical forms, which are easily integrable. From (2), we have

$$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} \qquad \text{and} \qquad q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} \qquad ...(3)$$

$$(3) \Rightarrow \frac{\partial}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v} \qquad \text{and} \qquad \frac{\partial}{\partial y} = \frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial}{\partial v}. \qquad ...(4)$$

$$\therefore \quad r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}\right)\left(\frac{\partial u}{\partial x}\frac{\partial z}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial z}{\partial v}\right), \text{ by (3) and (4)}$$

$$= \frac{\partial u}{\partial x}\frac{\partial}{\partial u}\left(\frac{\partial u}{\partial x}\frac{\partial z}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial z}{\partial v}\right) + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}\left(\frac{\partial u}{\partial x}\frac{\partial z}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial x}\right)^2 + 2\frac{\partial^2 z}{\partial u \partial v}\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial x}\right)^2 + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial x^2} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial x^2},$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial u}{\partial x}\frac{\partial}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}\right), \text{by (3) and (4)}$$

$$= \frac{\partial^2 z}{\partial u^2}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial^2 z}{\partial u \partial v}\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) + \frac{\partial^2 z}{\partial v^2}\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial y \partial x} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial y \partial x}$$

and

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial u}{\partial y}\frac{\partial}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial}{\partial v}\right)\left(\frac{\partial u}{\partial y}\frac{\partial z}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial z}{\partial v}\right), \text{ by (3) and (4)}$$

$$= \frac{\partial u}{\partial y}\frac{\partial}{\partial u}\left(\frac{\partial u}{\partial y}\frac{\partial z}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial z}{\partial v}\right) + \frac{\partial v}{\partial y}\frac{\partial}{\partial v}\left(\frac{\partial u}{\partial y}\frac{\partial z}{\partial u} + \frac{\partial v}{\partial y}\frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial^2 z}{\partial u^2}\left(\frac{\partial u}{\partial y}\right)^2 + 2\frac{\partial^2 z}{\partial u \partial v}\frac{\partial u}{\partial y}\frac{\partial v}{\partial y} + \frac{\partial^2 z}{\partial v^2}\left(\frac{\partial v}{\partial y}\right)^2 + \frac{\partial z}{\partial u}\frac{\partial^2 u}{\partial y^2} + \frac{\partial z}{\partial v}\frac{\partial^2 v}{\partial y^2}.$$

Putting the above values of p, q, r, s, t, in (1) and simplifying, we get

$$A\frac{\partial^2 z}{\partial u^2} + 2B\frac{\partial^2 z}{\partial u \partial v} + C\frac{\partial^2 z}{\partial v^2} + F(u, v, z, \partial z/\partial u, \partial z/\partial v) = 0, \qquad ...(5)$$

where

$$A = R\left(\frac{\partial u}{\partial x}\right)^2 + S\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + T\left(\frac{\partial u}{\partial y}\right)^2, \qquad ...(6)$$

$$B = R\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{1}{2}S\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right) + T\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}, \qquad ...(7)$$

$$C = R\left(\frac{\partial v}{\partial x}\right)^2 + S\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + T\left(\frac{\partial v}{\partial y}\right)^2 \qquad ...(8)$$

and $F(u, v, z, \partial z/\partial u, \partial z/\partial v)$ is the transformed form of $f(x, y, z, p, q)$.

Now we shall find out u and v so that (5) reduces to simplest possible form. The method of evaluation of desired values of u and v becomes easy when the discriminant $S^2 - 4RT$ of the quadratic equation

$$R\lambda^2 + S\lambda + T = 0 \qquad ...(9)$$

is everywhere either positive, negative or zero, and now we shall present these three cases separately.

Case I. Let $S^2 - 4RT > 0$. When this condition is satisfied, then the roots λ_1, λ_2 of the equation (9) are real and distinct. The coefficients of $\partial^2 z/\partial u^2$ and $\partial^2 z/\partial v^2$ in the equation (5) will vanish if we choose u and v such that

$$\partial u/\partial x = \lambda_1 (\partial u/\partial y) \qquad ...(10)$$

and

$$\partial v/\partial x = \lambda_2 (\partial v/\partial y). \qquad ...(11)$$

Since λ_1 is a root of (9), we have $\qquad R\lambda_1^2 + S\lambda_1 + T = 0.$ ...(12)

Using (10), (6) gives $\qquad A = (R\lambda_1^2 + S\lambda_1 + T)(\partial u/\partial y)^2 = 0$, by (12) ...(13)

Again, since λ_2 is a root of (9), we have $\qquad R\lambda_2^2 + S\lambda_2 + T = 0$ ...(14)

Using (11), (8) gives $\qquad C = (R\lambda_2^2 + S\lambda_2 + T)(\partial v/\partial y)^2 = 0$, by (14) ...(15)

Re-writing (10), we have $\qquad (\partial u/\partial x) - \lambda_1(\partial u/\partial y) = 0.$ ...(16)

Lagrange's auxiliary equation for (16) are $\qquad dx/1 = dy/(-\lambda_1) = du/0$ ...(17)

Taking third fraction of (17), $du = 0$ so that $u = c_1$, c_1 being an arbitrary constant ...(18)

Taking first and second fractions of (17), we get $\qquad (dy/dx) + \lambda_1 = 0$ ...(19)

Let the solution of (19) be $\qquad f_1(x, y) = c_2$, c_2 being an arbitrary constant ...(20)

From (18) and (20), the general solution of (16) [*i.e.* (10)] is $\qquad u = f_1(x, y).$ ...(21)

Similarly, the general solution of (11) can be taken as $\qquad v = f_2(x, y).$ ...(22)

Here f_1 and f_2 are arbitrary function

We can easily verify that $\qquad AC - B^2 = \frac{1}{4}(4RT - S^2)\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)^2$

or $$B^2 = \frac{1}{4}(S^2 - 4RT)\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x}\right)^2, \text{ as } A = C = 0. \qquad ...(23)$$

Let the Jacobian J of u and v be non-zero, i.e., let

$$J = \partial(u, v)/\partial(x, y) = (\partial u/\partial x)(\partial v/\partial y) - (\partial u/\partial y)(\partial v/\partial x) \neq 0$$

Since $S^2 - 4RT > 0$, (23) shows that $B^2 > 0$. Hence we may divide both sides of (5) by B^2. Then noting that $A = C = 0$, (5) transforms to the form $\quad \partial^2 z/\partial u\,\partial v = \phi\left(u, v, z, \partial z/\partial u, \partial z/\partial v\right),$

...(24)

which is the canonical form of (1) in this case.

Case II. Let $S^2 - 4RT = 0$. When this condition is satisfied, the roots λ_1, λ_2 of (9) are real and equal. We now take u exactly as in case I and take v to be any function of x, y which is independent of u. We have, as in case I, $A = 0$. Also, since $S^2 - 4RT = 0$, (23) shows that $B^2 = 0$ so that $B = 0$.

Moreover in this case $C \neq 0$, otherwise v would be a function of u and consequently v would not be independent of u as already assumed.

Putting $A = 0$, $B = 0$ and dividing by C, (5) transforms to the form

$$\partial^2 z/\partial v^2 = \phi(u, v, z, \partial z/\partial u, \partial z/\partial v). \qquad ...(25)$$

which is the canonical form of (1) in this case.

Case III. Let $S^2 - 4RT < 0$. When this condition is satisfied, the roots λ_1, λ_2 of (9) are complex. Hence this case III is formally the same as case I. Therefore, proceeding as in case I, we find that (1) reduces to (24) but that the variables u, v instead of being real are now complex conjugates. To obtain a real canonical form we make further transformation $u = \alpha + i\beta$ and $v = \alpha - i\beta$ so that

$$\alpha = (u + v)/2, \quad \text{and} \quad \beta = i\,(v - u)/2. \qquad ...(26)$$

Now, $$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial u} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial u} = \frac{1}{2}\left(\frac{\partial z}{\partial \alpha} - i\frac{\partial z}{\partial \beta}\right), \text{ by (26)} \qquad ...(27)$$

and $$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial v} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial v} = \frac{1}{2}\left(\frac{\partial z}{\partial \alpha} + i\frac{\partial z}{\partial \beta}\right), \text{ by (26)} \qquad ...(28)$$

$$\therefore \quad \frac{\partial^2 z}{\partial u \partial v} = \frac{\partial}{\partial u}\left(\frac{\partial z}{dv}\right) = \frac{1}{2}\left(\frac{\partial}{\partial \alpha} - i\frac{\partial}{\partial \beta}\right) \times \frac{1}{2}\left(\frac{\partial z}{\partial \alpha} + i\frac{\partial z}{\partial \beta}\right), \text{ by (27) and (28)}$$

$$= \frac{1}{4}\left[\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha} + i\frac{\partial z}{\partial \beta}\right) - i\frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \alpha} + i\frac{\partial z}{\partial \beta}\right)\right] = \frac{1}{4}\left(\frac{\partial^2 z}{\partial \alpha^2} + i\frac{\partial^2 z}{\partial \alpha \partial \beta} - i\frac{\partial^2 z}{\partial \beta \partial \alpha} + i\frac{\partial^2 z}{\partial \beta^2}\right)$$

or $$\frac{\partial^2 z}{\partial u \partial v} = \frac{1}{4}\left(\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2}\right), \qquad \text{as} \qquad \frac{\partial^2 z}{\partial \alpha \partial \beta} = \frac{\partial^2 z}{\partial \beta \partial \alpha} \qquad ...(29)$$

Putting $u = \alpha + i\beta$, $v = \alpha - i\beta$ and using (27), (28) and (29), (24) reduces to

$$(\partial^2 z/\partial \alpha^2) + (\partial^2 z/\partial \beta^2) = \psi(\alpha, \beta, z, \partial z/\partial \alpha, \partial z/\partial \beta), \qquad ...(30)$$

which is the canonical form of (1) in this case.

8.6 Working rule for reducing a hyperbolic equation to its canonical form

Step 1. Let the given equation $\qquad Rr + Ss + Tt + f(x, y, z, p, q) = 0 \qquad ... (1)$

be hyperbolic so that $\qquad S^2 - 4RT > 0.$

Step 2. Write λ-quadratic equation $\qquad R\lambda^2 + S\lambda + T = 0 \qquad ... (2)$

Let λ_1 and λ_2 be its two distinct roots of (2).

Step 3. Then corresponding characteristic equations are

$$(dy/dx) + \lambda_1 = 0 \qquad \text{and} \qquad (dy/dx) + \lambda_2 = 0$$

Solving these, we get $f_1(x, y) = c_1$ and $f_2(x, y) = c_2 \qquad ... (3)$

Step 4. We select u, v such that $u = f_1(x, y)$ and $v = f_2(x, y) \qquad ... (4)$

Step 5. Using relations (4), find p, q, r, s and t in terms of u and v as shown in Art. 8.5.

Step 6. Substituting the values of p, q, r, s, t obtained in step 4 in (1) and simplifying we shall get the following canonical form of (1):

$$\partial^2 z/\partial u \partial v = \phi\,(u, v, z,\ \partial z/\partial u,\ \partial z/\partial u).$$

8.7. SOLVED EXAMPLES BASED ON ART. 8.6

Ex.1. (*a*) *Write canonical form of* $\partial^2 z/\partial x^2 - \partial^2 z/\partial y^2 = 0$. **[Sagar 2004; Delhi Maths (H) 2002]**

(*b*) Reduc $3(\partial^2 z/\partial x^2) + 10\,(\partial^2 z/\partial x \partial y) + 3\,(\partial^2 z/\partial y^2) = 0$ to canonical form and hence solve it **(Himanchal 2008)**

Sol. (*a*) Re-writing the given equation, we get $\qquad r - t = 0 \qquad ... (1)$

Comparing (1) with $Rs + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = 1$, $S = 0$ and $T = -1$ so that $S^2 - 4RT = 4 > 0$, showing that (1) is hyperbolic

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $\qquad \lambda^2 - 1 = 0$

Hence $\lambda = 1, -1$. So $\lambda_1 = 1, \lambda_2 = -1$ (Real and distinct roots).

Then the characteristic equations $dy/dx + \lambda_1 = 0$, $dy/dx + \lambda_2 = 0$ reduces to

$(dy/dx) + 1 = 0$ and $(dy/dx) - 1 = 0.$

Integrating these, $y + x = c_1$ and $y - x = c_2.$

In order to reduce (1) to its canonical form, we choose

$$u = y + x \qquad \text{and} \qquad v = y - x \qquad ... (2)$$

$$\therefore \qquad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}, \text{ using (2)} \qquad ... (3)$$

and $$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ using (2)} \qquad ... (4)$$

From (3) and (4), $$\frac{\partial}{\partial x} = \frac{\partial}{\partial u} - \frac{\partial}{\partial v} \qquad \text{and} \qquad \frac{\partial}{\partial y} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v}. \qquad ... (5)$$

$$\therefore \qquad r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right), \text{ using (3) and (5)}$$

or $$r = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) - \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \qquad ... (6)$$

and $$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right), \text{ by (3) and (5)}$$

or $$t = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \qquad \text{... (7)}$$

Using (6) and (7) in (1), the required canonical form is

$$\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} - \left(\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right) = 0 \qquad \text{or} \qquad \frac{\partial^2 z}{\partial u \partial v} = 0.$$

(*b*) $\partial^2 z / \partial u \partial v = 0;\ z = f(y - 3x) + g(3y - x)$

Ex. 2. *Reduce* $\partial^2 z / \partial x^2 = (1+y)^2 (\partial^2 z / \partial y^2)$ *to canonical form*

Sol. Re-writing the given equation, $\qquad r - (1 + y^2)\, t = 0 \qquad$ ... (1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = 1$, $S = 0$, and $T = -(1 + y)^2$ so that $S^2 - 4RT = (1 + y^2) > 0$ for $y \neq -1$, showing that (1) is hyperbolic. The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $\lambda^2 - (1+y)^2 = 0$ so that $\lambda = 1 + y,\ -(1 + y)$. Hence the corresponding characteristic equations are given by

$$(dy/dx) + (1 + y) = 0 \qquad \text{and} \qquad (dy/dx) - (1 + y) = 0$$

Integrating these, $\quad \log(1 + y) + x = C_1 \quad$ and $\quad \log(1 + y) - x = C_2$.

In order to reduce (1) to its canonical form, we choose

$$u = \log(1 + y) + x \qquad \text{and} \qquad v = \log(1 + y) - x \qquad \text{... (2)}$$

$$\therefore \qquad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}, \text{ using (2)} \qquad \text{... (3)}$$

and $$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{1}{1+y}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) \qquad \text{... (4)}$$

From (3) $$\partial / \partial x \equiv \partial / \partial u - \partial / \partial v \qquad \text{... (5)}$$

$$\therefore \qquad r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial}{\partial u} - \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right), \text{ using (3) and (5)}$$

or $$r = \partial^2 z / \partial u^2 - 2\,(\partial^2 z / \partial u\, \partial v) + \partial^2 z / \partial v^2 \qquad \text{... (6)}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left\{\frac{1}{1+y}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\right\} = -\frac{1}{(1+y)^2}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{1}{1+y}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right), \text{ by (4)}$$

$$= -\frac{1}{(1+y)^2}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{1}{1+y}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$= -\frac{1}{(1+y)^2}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{1}{1+y}\left[\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v}\right)\frac{1}{y+1} + \left(\frac{\partial^2 z}{\partial v \partial u} + \frac{\partial^2 z}{\partial v^2}\right)\frac{1}{1+y}\right], \text{ by (2)}$$

or $$t = \frac{1}{(1+y)^2}\left(\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} - \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) \qquad \text{... (7)}$$

Using (6) and (7) in (1), the required canonical form is

$$\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} - \left(\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} - \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) = 0 \qquad \text{or} \qquad 4\frac{\partial^2 z}{\partial u \partial v} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}..$$

Ex. 3. *Reduce the differential equation* $t - s + p - q(1 + 1/x) + (z/x) = 0$ *to canonical form.*

[Delhi Maths (H) 2004]

Sol. Given $$0\cdot r - s + t + p - q\,(1 + 1/x) + (z/x) = 0 \qquad ...(1)$$

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = 0$, $S = -1$ and $T = 1$. Hence $S^2 - 4RT = 1 > 0$, showing that the given equation is hyperbolic.

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $-\lambda + 1 = 0$ giving $\lambda = 1$. Hence the corresponding characteristic equation $dy/dx + \lambda = 0$ yields $dy/dx + 1 = 0$ or $dx + dy = 0$

Integrating it, $x + y = c$, c being an arbitrary constant

Choose $u = x + y$ and $v = x$, ...(2)

where we have chosen $v = x$ in such a manner that u and v are independent as verified below:

Jacobian of u and $v = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = 1 \neq 0 \Rightarrow$ u and v are independent functions.

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ using (2)} \qquad ...(3)$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u}, \text{ using (2)} \qquad ...(4)$$

From (4), we have $$\partial/\partial y \equiv \partial/\partial u \qquad ...(5)$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right), \text{ using (3) and (5)} \qquad ...(6)$$

and $$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right), \text{ using (5)}$$

or $$t = \partial^2 z/\partial u^2. \qquad ...(6)$$

Using (2), (3), (4), (6) and (7), (1) reduces to

$$-\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v}\right) + \frac{\partial^2 z}{\partial u^2} + \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\left(1 + \frac{1}{v}\right) + \frac{z}{v} = 0$$

or $\partial^2 z/\partial u \partial v - (\partial z/\partial v) + (1/v) \times (\partial z/\partial u) - (z/v) = 0$, which is the required canonical form.

Ex. 4. *Reduce the equation yr + (x + y) s + xt = 0 to canonical form and hence find its general solution.* **(Delhi Maths (Hons) 2007)**

Sol. Given $$yr + (x + y)\,s + xt = 0 \qquad ... (1)$$

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = y$, $S = (x + y)$ and $T = x$ so that $S^2 - 4RT = (x + y)^2 - 4xy = (x - y)^2 > 0$ for $x \neq y$ and so (1) is hyperbolic. Its λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $y\lambda^2 + (x + y)\,\lambda + x = 0$ or $(y\lambda + x)\,(\lambda + 1) = 0$

so that $\lambda = -1$, $-x/y$. Then the corresponding characteristic equations are given by

$(dy/dx) - 1 = 0$ and $(dy/dx) - (x/y) = 0$

Integrating these, $y - x = c_1$ and $y^2/2 - x^2/2 = c_2$

In order to reduce (1) to its canonical form, we choose

$u = y - x$ and $v = y^2/2 - x^2/2$ ... (2)

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = -\left(\frac{\partial z}{\partial u} + x\frac{\partial z}{\partial v}\right), \text{ using (2)} \qquad ... (3)$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + y\frac{\partial z}{\partial v}, \text{ using (2)} \qquad ... (4)$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) - \frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial v}\right), \text{ using (3)}$$

$$= -\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) - \left[x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right) + \frac{\partial z}{\partial v}\right] = -\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) - x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right) - \frac{\partial z}{\partial v}$$

$$= -\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right] - x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right] - \frac{\partial z}{\partial v}$$

$$= -\left(-\frac{\partial^2 z}{\partial u^2} - x\frac{\partial^2 z}{\partial v \partial u}\right) - x\left(-\frac{\partial^2 z}{\partial u \partial v} - x\frac{\partial^2 z}{\partial v^2}\right) - \frac{\partial z}{\partial v}, \text{ using (2)}$$

$$\therefore \qquad r = \frac{\partial^2 z}{\partial u^2} + 2x\frac{\partial^2 z}{\partial u \partial v} + x^2\frac{\partial^2 z}{\partial v^2} - \frac{\partial z}{\partial v} \qquad \text{... (5)}$$

Now, $t = \dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial u} + y\dfrac{\partial z}{\partial v}\right) = \dfrac{\partial}{\partial y}\left(\dfrac{\partial z}{\partial u}\right) + \dfrac{\partial}{\partial y}\left(y\dfrac{\partial z}{\partial v}\right)$, using (4)

$$= \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) + y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right) + \frac{\partial z}{\partial v} = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y} + y\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right\} + \frac{\partial z}{\partial v}$$

$$\therefore\ t = \frac{\partial^2 z}{\partial u^2} + y\frac{\partial^2 z}{\partial u \partial v} + y\left(\frac{\partial^2 z}{\partial u \partial v} + y\frac{\partial^2 z}{\partial v^2}\right) + \frac{\partial z}{\partial v} = \frac{\partial^2 z}{\partial u^2} + 2y\frac{\partial^2 z}{\partial u \partial v} + y^2\frac{\partial^2 z}{\partial v^2} + \frac{\partial z}{\partial v} \qquad \text{...(6)}$$

Also, $s = \dfrac{\partial^2 z}{\partial x \partial y} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial y}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial u} + y\dfrac{\partial z}{\partial v}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial u}\right) + \dfrac{\partial}{\partial x}\left(y\dfrac{\partial z}{\partial v}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial u}\right) + y\dfrac{\partial}{\partial x}\left(\dfrac{\partial z}{\partial v}\right)$

$$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x} + y\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right\} = -\frac{\partial^2 z}{\partial u^2} - x\frac{\partial^2 z}{\partial v \partial u} - y\frac{\partial^2 z}{\partial u \partial v} - xy\frac{\partial^2 z}{\partial v^2}, \text{ using (2)}$$

$$\therefore \qquad s = -\frac{\partial^2 z}{\partial u^2} - (x+y)\frac{\partial^2 z}{\partial u \partial v} - xy\frac{\partial^2 z}{\partial v^2} \qquad \text{... (7)}$$

Using (5) (6) and (7) in (1), we get

$$y\left(\frac{\partial^2 z}{\partial u^2} + 2x\frac{\partial^2 z}{\partial u \partial v} + x^2\frac{\partial^2 z}{\partial v^2} - \frac{\partial z}{\partial v}\right)$$

$$+(x+y)\left\{-\frac{\partial^2 z}{\partial u^2} - (x+y)\frac{\partial^2 z}{\partial u \partial v} - xy\frac{\partial^2 z}{\partial v^2}\right\} + x\left(\frac{\partial^2 z}{\partial u^2} + 2y\frac{\partial^2 z}{\partial u \partial v} + y^2\frac{\partial^2 z}{\partial v^2} + \frac{\partial z}{\partial v}\right) = 0$$

or $\quad \{4xy - (x+y)^2\}\dfrac{\partial^2 z}{\partial u \partial v} - y\dfrac{\partial z}{\partial v} + x\dfrac{\partial z}{\partial v} = 0 \qquad$ or $\qquad (y-x)^2\dfrac{\partial^2 z}{\partial u \partial v} + (y-x)\dfrac{\partial z}{\partial v} = 0$

or $\quad u^2\dfrac{\partial^2 z}{\partial u \partial v} + u\dfrac{\partial z}{\partial v} = 0$, by (2) $\qquad$ or $\qquad u\dfrac{\partial^2 z}{\partial v \partial v} + \dfrac{\partial z}{\partial v} = 0$, as $u \neq 0$ ... (8)

(8) is the required canonical form of (1).

Solution of (8). Multiplying both sides of (8) by v, we get

$uv\,(\partial^2 z/\partial u \partial v) + v(\partial z/\partial v) = 0 \qquad$ or $\qquad (uv\ DD' + vD')z = 0 \qquad$ ... (9)

where $D \equiv \partial/\partial u$ and $D' \equiv \partial/\partial v$. To reduce (9) into linear equation with constant coefficients, we take new variables X and Y as follows. For details refer Art. 6.3.

Let $u = e^X$ and $v = e^Y$ so that $X = \log u$ and $Y = \log v$ ... (10)

Let $D_1 \equiv \partial / \partial X$ and $D_1' \equiv \partial / \partial Y$. Then (9) reduces to

$$(D_1 D_1' + D_1')z = 0 \quad \text{or} \quad D_1'(D_1 + 1)z = 0$$

Its general solution is $z = e^{-X}\phi_1(Y) + \phi_2(X) = u^{-1}\phi_1(\log v) + \phi_2(\log u)$ [See Art. 5.6]

or $z = u^{-1}\psi_1(v) + \psi_2(u) = (y-x)^{-1}\psi_1(y^2 - x^2) + \psi_2(y-x)$, where ψ_1 and ψ_2 are arbitrary functions.

Ex.5. *Reduce the equation* $r - (2\sin x)s - (\cos^2 x)t - (\cos x)q = 0$ *to canonical form and hence solve it.* **(Himanchal 2008)**

Sol. Given $$r - (2\sin x)\, s - (\cos^2 x)t - (\cos x)q = 0 \qquad \text{... (1)}$$

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = 1$, $S = -2\sin x$ and $T = -\cos^2 x$ so that $S^2 - 4RT = 4(\sin^2 x + \cos^2 x) = 4 > 0$, showing that (1) is hyperbolic. The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $\lambda^2 - (2\sin x)\lambda - \cos^2 x = 0$ so that $\lambda = \sin x + 1, \ \sin x - 1$. Hence the corresponding characteristic equations become

$dy/dx + \sin x + 1 = 0$ and $dy/dx + \sin x - 1 = 0$

Integrating these, $y - \cos x + x = c_1$ and $y - \cos x - x = c_2$

Choose $u = y - \cos x + x$ and $v = y - \cos x - x$ ... (2)

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = (1+\sin x)\frac{\partial z}{\partial u} + (\sin x - 1)\frac{\partial z}{\partial v}, \text{ by (2)} \qquad \text{... (3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ using (2)} \qquad \text{... (4)}$$

From (4), we have $$\partial/\partial y = \partial/\partial u + \partial/\partial v \qquad \text{... (5)}$$

$$\therefore \quad t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right), \text{ using (4) and (5)}$$

or $$t = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \qquad \text{... (6)}$$

Now, $$s = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left\{(1+\sin x)\frac{\partial z}{\partial u} + (\sin x - 1)\frac{\partial z}{\partial v}\right\}, \text{ by (3)}$$

$$= (\sin x + 1)\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) + (\sin x - 1)\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)$$

$$= (\sin x + 1)\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}\right\} + (\sin x - 1)\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right\}$$

$$= (\sin x + 1)\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v}\right) + (\sin x - 1)\left(\frac{\partial^2 z}{\partial u \partial x} + \frac{\partial^2 z}{\partial v^2}\right)$$

or $$s = \sin x\left(\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right) + \frac{\partial^2 z}{\partial u^2} - \frac{\partial^2 z}{\partial v^2} \qquad \text{... (7)}$$

$$r = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left\{(\sin x + 1)\frac{\partial z}{\partial u} + (\sin x - 1)\frac{\partial z}{\partial v}\right\} = \cos x\frac{\partial z}{\partial u} + (\sin x + 1)\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \cos x\frac{\partial z}{\partial v} + (\sin x - 1)\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= \cos x\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)+(\sin x+1)\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right\}+(\sin x-1)\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right\}$$

$$= \cos x\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)+(\sin x+1)\left\{(\sin x+1)\frac{\partial^2 z}{\partial u^2}+(\sin x-1)\frac{\partial^2 z}{\partial v^2}\right\}+(\sin x-1)\left\{(\sin x+1)\frac{\partial^2 z}{\partial u\partial v}+(\sin x-1)\frac{\partial^2 z}{\partial v^2}\right\}$$

$$\therefore \qquad r=\cos x\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)+(1+\sin x)^2\frac{\partial^2 z}{\partial u^2}+(\sin x-1)^2\frac{\partial^2 z}{\partial v^2}-2\cos^2 x\frac{\partial^2 z}{\partial u\partial v} \qquad \text{... (8)}$$

Using (4) (6), (7) and (8) in (1), we get

$$\cos x\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)+(1+2\sin x+\sin^2 x)\frac{\partial^2 x}{\partial u^2}+(\sin^2 x+1-2\sin x)\frac{\partial^2 z}{\partial v^2}-2\cos^2 x\frac{\partial^2 z}{\partial u\partial v}$$

$$-2\sin x\left\{\sin x\left(\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}\right)+\frac{\partial^2 z}{\partial u^2}-\frac{\partial^2 z}{\partial v^2}\right\}-\cos^2 x\left(\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}\right)-\cos x\left(\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)=0$$

or $\qquad (1+2\sin x+\sin^2 x-2\sin^2 x-2\sin x-\cos^2 x)\times(\partial^2 z/\partial u^2)+(\sin^2 x+1-2\sin x-2\sin^2 x$

$$+2\sin x-\cos^2 x)\times(\partial^2 z/\partial v^2)-(2\cos^2 x+4\sin^2 x+2\cos^2 x)\times(\partial^2 z/\partial u\,\partial v)=0$$

or $\qquad \partial^2 z/\partial u\,\partial v=0$, on simplification. ... (9)

(9) is the required canonical form of (1).

Solution of (9). Integrating (9) w.r.t. 'u', $\partial z/\partial v=\phi(v),\phi$ being an arbitrary function ... (10)

Integrating (10) w.r.t. 'v', $\qquad z=\int\phi(v)\,dv+F(u)=G(v)+F(u)$,

where $G(v)=\int\phi(v)\,dv$, F and G are arbitrary functions.

$\therefore\quad z=G(y-\cos x-x)+F(y-\cos x+x)$ is the required solution.

Ex. 6. *Reduce $\partial^2 z/\partial x^2=x^2(\partial^2 z/\partial y^2)$ to canonical form.*

[Agra 2005; Himanchal 2005; Delhi B.Sc. (Prog) II 2002, 07; Kurukshetra 2004; Ravishankar 2004; Nagpur 2010, Kanpur 2011]

Sol. Re-writing the given equation becomes $\qquad r-x^2t=0.$ ...(1)

Comparing (1) with $Rr+Ss+Tt+f(x,y,z,p,q)=0$, we have $R=1$, $S=0$, $T=-x^2$.

Now, the λ-quadratic $R\lambda^2+S\lambda+T=0$ gives $\lambda^2-x^2=0$ so that $\lambda=\pm x$.

$\therefore$ Here $\lambda_1=x$ and $\lambda_2=-x$ (Real and disticnt roots)

Hence characterstic equations $dy/dx+\lambda_1=0$ and $dy/dx+\lambda_2=0$

become $dy/dx+x=0$ and $dy/dx-x=0$.

Integrating these, $y+(x^2/2)=c_1$ and $y-(x^2/2)=c_2$.

Hence in order to reduce (1) to canonical form, we change x, y, to u, v by taking

$u=y+(x^2/2)$ and $v=y-(x^2/2)$ ...(2)

Now, $\qquad p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=x\frac{\partial z}{\partial u}-x\frac{\partial z}{\partial v}$, using (2) ...(3)

$\qquad q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}$, using (2) ...(4)

$$\therefore\ r=\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left\{x\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)\right\}=x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)+1.\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right),\text{ using (3)}$$

$$=x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]+\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}=x^2\left(\frac{\partial^2 z}{\partial u^2}-2\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial^2 v^2}\right)+\frac{\partial z}{\partial u}-\frac{\partial z}{\partial v}$$

and $\quad t = \dfrac{\partial^2 z}{\partial y^2} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial}{\partial y}\right) = \left(\dfrac{\partial}{\partial u} + \dfrac{\partial}{\partial v}\right)\left(\dfrac{\partial z}{\partial u} + \dfrac{\partial z}{\partial v}\right) = \dfrac{\partial^2 z}{\partial u^2} + 2\dfrac{\partial^2 z}{\partial u \partial v} + \dfrac{\partial^2 z}{\partial v^2}$, using (4)

Putting the above values of r and t in (1), we get

$$x^2\left(\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right) + \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} - x^2\left(\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right) = 0$$

or $\quad \dfrac{\partial^2 z}{\partial u \partial v} = \dfrac{1}{4x^2}\left(\dfrac{\partial z}{\partial u} - \dfrac{\partial z}{\partial v}\right) \quad$ or $\quad \dfrac{\partial^2 z}{\partial u \partial v} = \dfrac{1}{4(u-v)}\left(\dfrac{\partial z}{\partial u} - \dfrac{\partial z}{\partial v}\right)$, by (2)

which is the required canonical form of the given equation.

Ex. 7. *Reduce the equation* $(n-1)^2(\partial^2 z/\partial x^2) - y^{2n}(\partial^2 z/\partial y^2) = n y^{2n-1}(\partial z/\partial y)$ *to canonical form, and find its general solution*. **[Delhi Maths. (H) 2000, 01, 05; Himanchal 2004; Ravishankar 2004]**

Sol. Given $\quad (n-1)^2 r - y^{2n}\, t - n\, y^{2n-1}\, q = 0. \qquad$ ...(1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, we have $R = (n-1)^2$, $S = 0$, $T = -y^{2n}$.

Now, the λ—quadratic $R\lambda^2 + S\lambda + T = 0$ gives

$(n-1)^2\lambda^2 - y^{2n} = 0 \quad$ so that $\quad \lambda = \pm (n-1)^{-1} y^n$.

$\therefore$ Here $\quad \lambda_1 = (n-1)^{-1} y^n \quad$ and $\quad \lambda_2 = -(n-1)^{-1} y^n$.

Hence, characterstic equations $\quad dy/dx + \lambda_1 = 0 \quad$ and $\quad dy/dx + \lambda_2 = 0$

become $\quad dy/dx + (n-1)^{-1} y^n = 0 \quad$ and $\quad dy/dx - (n-1)^{-1} y^n = 0$.

Integrating these, $\quad x - y^{-n+1} = c_1 \quad$ and $\quad x + y^{-n+1} = c_2$.

Hence in order to reduce (1) to canonical form, we change x, y to u, v by taking

$u = x - y^{-n+1} \quad$ and $\quad v = x + y^{-n+1}. \qquad$ ...(2)

$\therefore\ p = \dfrac{\partial z}{\partial x} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial x} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial x} = \dfrac{\partial z}{\partial u} + \dfrac{\partial z}{\partial v} \quad$ so that $\quad \dfrac{\partial}{\partial x} \equiv \dfrac{\partial}{\partial u} + \dfrac{\partial}{\partial v}$,

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = (n-1)y^{-n}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right),$$

$$r = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}$$

$$t = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left\{(n-1)y^{-n}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\right\} = -n(n-1)y^{-n-1}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) + (n-1)y^{-n}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)$$

$$= -n(n-1)y^{-n-1}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) + (n-1)y^{-n}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$= -n(n-1)y^{-n-1}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right) + (n-1)^2 y^{-2n}\left(\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right)$$

Substituting the above values of r, t, q in (1) and simplifying, we obtain

$$\partial^2 z / \partial u \partial v = 0, \qquad ...(3)$$

which is the required canonical form of the given equation

Integrating (3) w.r.t. 'v', $\quad \partial z/\partial u = F(u)$, where $F(u)$ is an arbitrary function of u, $\qquad$ ...(4)

Integrating (4) w.r.t. 'u', $\quad z = G(u) + H(v)$,

where $G(u) = \int F(u)\, du$ and $G(u)$, $H(v)$ are arbitrary functions

Using (2), the solution of the given equation is $\quad z = G(x - y^{-n+1}) + H(x + y^{-n+1})$.

Ex. 8. *Reduce the equation* $(y-1)\,r-(y^2-1)s+y(y-1)t+p-q=2ye^{2x}(1-y)^3$ to *canonical form and hence solve it.* **[Delhi B.Sc. (Hons) III 2008; Rohilkhand 1992]**

Sol. Given $\quad (y-1)\,r-(y^2-1)s+y(y-1)t+p-q-2ye^{2x}(1-y)^3=0.\quad$ ...(1)

Comparing (1) with $Rr+Ss+Tt+f(x,y,z,p,q)=0$, we get

$R=y-1,\qquad S=-(y^2-1)\qquad$ and $\qquad T=y(y-1).\quad$ ...(2)

$\therefore$ The λ–quardratic $R\lambda^2-S\lambda+T=0$ gives

$(y-1)\lambda^2-(y^2-1)\lambda+y(y-1)=0 \quad\Rightarrow\quad \lambda_1=1$ and $\lambda_2=y$ (real and distinct roots)

Hence characterstic equations $(dy/dx)+\lambda_1=0$ and $(dy/dx)+\lambda_2=0$ become

$(dy/dx)+1=0$ and $(dy/dx)+y=0.$

Integrating these, $\qquad x+y=c_1$ and $y\,e^x=c_2.$

To reduce (1) to canonical form, we change the independent variables x, y, to new independent variables u, v by taking

$$u=x+y \qquad\text{and}\qquad v=y\,e^x. \qquad ...(3)$$

$$\therefore\qquad p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=\frac{\partial z}{\partial u}+y\,e^x\frac{\partial z}{\partial v}=\frac{\partial z}{\partial u}+v\frac{\partial z}{\partial v},\text{ by (3)} \qquad ...(4)$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\frac{\partial z}{\partial u}+e^x\frac{\partial z}{\partial v},\text{ by (3)} \qquad ...(5)$$

$$r=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)=\left(\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u}+v\frac{\partial z}{\partial v}\right)=\frac{\partial^2 z}{\partial u^2}+2v\frac{\partial^2 z}{\partial u\partial v}+v^2\frac{\partial^2 z}{\partial v^2}+v\frac{\partial z}{\partial v},\text{ by (4)}$$

$$s=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}+e^x\frac{\partial z}{\partial v}\right)=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)+e^x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)+e^x\frac{\partial z}{\partial v}$$

$$=\left(\frac{\partial z}{\partial u}+v\frac{\partial z}{\partial v}\right)\left(\frac{\partial z}{\partial u}\right)+e^x\left(\frac{\partial}{\partial u}+v\frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial v}\right)+e^x\frac{\partial z}{\partial v}=\frac{\partial^2 z}{\partial u^2}+(e^x+v)\frac{\partial^2 z}{\partial u\partial v}+ve^x\frac{\partial^2 z}{\partial v^2}+e^x\frac{\partial z}{\partial v}$$

and
$$t=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial y}\left(\frac{\partial}{\partial u}+e^x\frac{\partial z}{\partial v}\right)=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)+e^x\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)$$

$$=\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial z}{\partial y}+e^x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]=\frac{\partial^2 z}{\partial u^2}+2e^x\frac{\partial^2 z}{\partial u\partial v}+e^{2x}\frac{\partial^2 z}{\partial v^2}.$$

Substituting the above values in (1) and simplifying, we have

$$(1-y)^3e^x\frac{\partial^2 z}{\partial u\partial v}=2y\,e^{2x}(1-y)^3 \qquad\text{or}\qquad \frac{\partial^2 z}{\partial u\partial v}=2v, \qquad ...(6)$$

which is the canonical form of (1).

Integrating (6) w.r.t. 'v', $\qquad \partial z/\partial u=v^2+\phi(u)$, $\phi(u)$ being an arbitrary function $\quad$ ...(7)

Integrating (7) w.r.t. 'u', $\qquad z=uv^2+\phi_1(u)+\phi_2(v)$, where $\phi_1(u)=\int\phi(u)\,du$

$\therefore$ Using (3) $\qquad z=(x+y)y^2e^{2x}+\phi_1(x+y)+\phi_2(ye^x)$, where ϕ_1 and ϕ_2 are arbitrary functions.

Ex. 9. *Solve* $x^2(y-1)\,r-x\,(y^2-1)\,s+y(y-1)\,t+xyp-q=0.$

Sol. Given $\qquad x^2(y-1)\,r-x(y^2-1)\,s+y(y-1)t+xyp-q=0.\qquad$ ...(1)

Comparing (1) with $Rr+Ss+Tt+f(x,y,z,p,q)=0$, we get

$R=x^2(y-1),\qquad S=-x\,(y^2-1)\qquad$ and $\qquad T=y\,(y-1).$

$\therefore$ λ–quadratic $R\lambda^2+S\lambda+T=0$ reduces to

$x^2(y-1)\,\lambda^2-x(y^2-1)\,\lambda+y(y-1)=0 \quad\Rightarrow\quad \lambda_1=y/x$ and $\lambda_2=1/x$ (real and distinct)

So characterstic equations $(dy/dx) + \lambda_1 = 0$ and $(dy/dx) + \lambda_2 = 0$ become

$$(dy/dx) + (y/x) = 0 \quad \text{and} \quad (dy/dx) + (1/x) = 0$$

Integrating these, $xy = c_1$ and $xe^y = c_2$ so for canonical form, we take

$$u = xy \quad \text{and} \quad v = xe^y. \qquad ...(2)$$

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = y\frac{\partial z}{\partial u} + e^y\frac{\partial z}{\partial v}, \text{ by (2)} \qquad ...(3)$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = x\frac{\partial z}{\partial u} + xe^y\frac{\partial z}{\partial v}, \text{ by (2)} \qquad ...(4)$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(y\frac{\partial z}{\partial u} + e^y\frac{\partial z}{\partial v}\right) = y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + e^y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right), \text{ by (3)}$$

$$= y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right] + e^y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right] = y^2\frac{\partial^2 z}{\partial u^2} + 2ye^x\frac{\partial^2 z}{\partial u \partial v} + e^{2y}\frac{\partial^2 z}{\partial y^2},$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial u} + xe^y\frac{\partial z}{\partial v}\right) = \frac{\partial z}{\partial u} + x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + e^y\frac{\partial z}{\partial v} + xe^y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= \frac{\partial z}{\partial u} + e^y\frac{\partial z}{\partial v} + x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right] + x\,e^y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]$$

$$= \frac{\partial z}{\partial u} + e^y\frac{\partial z}{\partial v} + xy\frac{\partial^2 z}{\partial u^2} + (yxe^y + e^y x)\frac{\partial^2 z}{\partial u \partial v} + xe^{2y}\frac{\partial^2 z}{\partial v^2}$$

and
$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(x\frac{\partial z}{\partial u} + xe^y\frac{\partial z}{\partial v}\right) = x\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) + xe^y\frac{\partial z}{\partial v} + x\,e^y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)$$

$$= x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}\right] + xe^y\frac{\partial z}{\partial v} + xe^y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$= x^2\frac{\partial^2 z}{\partial u^2} + 2x^2e^y\frac{\partial^2 z}{\partial u \partial v} + x^2e^{2y}\frac{\partial^2 z}{\partial y^2} + xe^y\frac{\partial z}{\partial v}.$$

Substituting the above values in (1) and simplifying, we get $\partial^2 z/\partial u \partial v = 0,$ ... (5)
which is canonical form of (1).

Integrating (5) w.r.t. 'u', $\partial z/\partial v = \phi(v)$, $\phi(v)$ being an arbitrary function.

Integrating it w.r.t. 'v', $z = \phi_1(v) + \phi_2(u)$, where $\phi_1(v) = \int \phi(v)\,dv$.

$\therefore$ $z = \phi_1(xe^y) + \phi_2(xy)$, by (2). This is the required solution, ϕ_1, ϕ_2 being arbitrary functions

Ex. 10. *Solve* (*i*) $xyr - (x^2 - y^2)s - xyt + py - qx = 2(x^2 - y^2)$. **[Delhi Maths (H) 2006]**
(*ii*) $x(y - x)r - (y^2 - x^2)s + y(y - x)t + (y + x)(p - x) = 2x + 2y + 2$.

Sol. (*i*) Given $xyr - (x^2 - y^2)s - xyt + py - qx - 2(x^2 - y^2) = 0$ ...(1)
Comparing (*i*) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, we have

$$R = xy, \quad S = -(x^2 - y^2) \quad \text{and} \quad T = -xy.$$

So λ–quadratic $R\lambda^2 + S\lambda + T = 0$ becomes $xy\lambda^2 - (x^2 - y^2)\lambda - xy = 0$ giving $\lambda = -y/x, x/y$.

$$\therefore \quad \frac{dy}{dx} + \lambda_1 = 0 \quad \text{and} \quad \frac{dy}{dx} + \lambda_2 = 0 \quad \Rightarrow \quad \frac{dy}{dx} - \frac{y}{x} = 0 \quad \text{and} \quad \frac{dy}{dx} + \frac{y}{x} = 0.$$

Integrating, $y/x = c_1$, and $x^2 + y^2 = c_2$. So, we take

$$u = y/x \quad \text{and} \quad v = x^2 + y^2. \qquad ...(2)$$

$\therefore$ Proceeding as usual, we obtain

$$p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=\left(-\frac{y}{x^2}\right)\frac{\partial z}{\partial u}+2x\frac{\partial z}{\partial v},\quad q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=\frac{1}{x}\frac{\partial z}{\partial u}+2y\frac{\partial z}{\partial v},$$

$$r=\left(-\frac{y}{x^2}\right)^2\frac{\partial^2 z}{\partial u^2}+2\times(2x)\left(-\frac{y}{x^2}\right)\frac{\partial^2 z}{\partial v\partial u}+4x^2\frac{\partial^2 z}{\partial v^2}+\frac{2y}{x^3}\frac{\partial z}{\partial u}+2\frac{\partial z}{\partial v}$$

$$s=\left(-\frac{y}{x^2}\right)\left(\frac{1}{x}\right)\frac{\partial^2 z}{\partial u^2}+\left\{2y\left(-\frac{y}{x^2}\right)+2x\times\frac{1}{x}\right\}\frac{\partial^2 z}{\partial u\partial v}+4xy\frac{\partial^2 z}{\partial v^2}-\frac{1}{x^2}\frac{\partial z}{\partial u}$$

and
$$t=\left(\frac{1}{x}\right)^2\frac{\partial^2 z}{\partial u^2}+2\times\frac{1}{x}\times(2y)\frac{\partial^2 z}{\partial u\partial v}+4y^2\frac{\partial^2 z}{\partial v^2}+2\frac{\partial z}{\partial v}.$$

Substituting these in (1) we get

$$(x^2+y^2)^2\frac{\partial^2 z}{\partial u\partial v}=(y^2-x^2)x^2 \qquad \text{or} \qquad \frac{\partial^2 z}{\partial u\partial v}=\frac{(y^2-x^2)x^2}{(x^2+y^2)^2}=\frac{u^2-1}{(u^2+1)^2},\ \text{by (2)} \quad ...(3)$$

Integrating (3) w.r.t. 'u', we have

$$\frac{\partial z}{\partial v}=\int\frac{u^2-1}{(u^2+1)^2}du+\phi(v)=\int\frac{du}{u^2+1}-2\int\frac{du}{(u^2+1)^2}+\phi(v) \qquad ...(4)$$

We have, $$\int 1\cdot\frac{1}{u^2+1}du=u\times\frac{1}{u^2+1}-\int u\times\left(\frac{-2u}{(u^2+1)^2}\right)du,\ \text{integrating by parts}$$

or $$\int\frac{du}{u^2+1}=\frac{u}{u^2+1}+2\int\frac{(u^2+1)-1}{(u^2+1)^2}du=\frac{u}{u^2+1}+2\int\frac{du}{u^2+1}-2\int\frac{du}{(u^2+1)^2}$$

Then, $$\int\frac{du}{u^2+1}-2\int\frac{du}{(u^2+1)^2}=-\frac{u}{u^2+1} \qquad ...(4)$$

Using (5), (4) gives $\quad \partial z/\partial v=-u/(u^2+1)+\phi(v)$, $\phi(v)$ being an arbitrary function ...(6)

Integrating (6) w.r.t. v, $\quad z=-(uv)/(u^2+v^2)+\phi_1(v)+\phi_2(u)$, where $\quad \phi_1(v)=$ $\int\phi(v)\,dv$

$\therefore$ Using (2), $z=-xy+\phi_1(x^2+y^2)+\phi_2(y/x)$, ϕ_1,ϕ_2 being arbitrary functions.

(*ii*) **Hint.** Since $R=x(y-x)$, $S=-(y^2-x^2)$, $T=y(y-x)$, so here $\lambda_1=y/x$, $\lambda_2=1$.

So we get $(dy/dx)+(y/x)=0$ and $(dy/dx)+1=0$ as characteristic equations

These give $\quad xy=c_1 \quad$ and $\quad x+y=c_2$. Hence take

$u=xy \quad$ and $\quad v=x+y.$ ...(1)

As usual, $\quad p=y\dfrac{\partial z}{\partial u}+\dfrac{\partial z}{\partial v} \quad$ and $\quad q=x\dfrac{\partial z}{\partial u}+\dfrac{\partial z}{\partial v},$

$$r=y^2\frac{\partial^2 z}{\partial u^2}+2y\frac{\partial^2 z}{\partial v\partial u}+\frac{\partial^2 z}{\partial v^2},\ t=x^2\frac{\partial^2 z}{\partial u^2}+2x\frac{\partial^2 z}{\partial v\partial u}+\frac{\partial^2 z}{\partial v^2},\ s=xy\frac{\partial^2 z}{\partial u^2}+(x+y)\frac{\partial^2 z}{\partial v\partial u}+\frac{\partial^2 z}{\partial v^2}+\frac{\partial z}{\partial u}.$$

$\therefore$ Given equation becomes $$-(y-x)^3\frac{\partial^2 z}{\partial v\partial u}=2x+2y+2 \qquad ...(2)$$

or $$\frac{\partial^2 z}{\partial v \partial u} = -\frac{2(x+y+1)}{(y-x)^3} = -\frac{2(x+y+1)}{[(y+x)^2-4xy]^{3/2}} = \frac{2(v+1)}{(v^2-4u)^{3/2}}, \text{ by (1)}$$

Integrating (2) w.r.t. 'u', we get $$\frac{\partial z}{\partial v} = \frac{v+1}{\sqrt{(v^2-4u)}} + \phi(v). \qquad \text{... (3)}$$

Integrating, (3) w.r.t. v, $$z = \sqrt{(v^2-4u)} + \log[v + \sqrt{(v^2-4u)}] + \phi_1(v) + \phi_2(u)$$

or $z = x - y + \log(2x) + \phi_1(x+y) + \phi_2(xy)$, ϕ_1, ϕ_2 being arbitrary functions.

Ex. 11. *Solve* (i) $y(x+y)(r-s) - xp - yq - z = 0$ **[Delhi Maths (H) 1998]**

(ii) $xys - x^2r - px - qy + z = -2xy^2y.$

Sol. (i) Given $$y(x+y)r - y(x+y)s - xp - yq - z = 0. \qquad \text{...(1)}$$

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, $R = y(x+y)$, $S = -y(x+y)$, $T = 0$.

So, the λ–quadratic $R\lambda^2 + S\lambda + T = 0$ reduces to

$y(x+y)\lambda^2 - y(x+y)\lambda = 0$, giving $\lambda = 0, 1$. Thus $\lambda_1 = 1$ and $\lambda_2 = 0$ and so

$$\frac{dy}{dx} + \lambda_1 = 0 \quad \text{and} \quad \frac{dy}{dx} + \lambda_2 = 0 \quad \Rightarrow \quad \frac{dy}{dx} + 1 = 0 \quad \text{and} \quad \frac{dy}{dx} = 0$$

Integrating these, $x + y = c_1$, and $y = c_2$.

So we take $u = x + y$ and $v = y$ ...(2)

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u}, \text{ by (2)} \qquad \text{...(3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ by (2)} \qquad \text{....(4)}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right) = \frac{\partial^2 z}{\partial u^2}, \text{ by (3)} \qquad \text{...(5)}$$

$$s = \frac{\partial^2 z}{\partial v \partial u} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v \partial u}, \text{ using (3) and (4)} \qquad \text{...(6)}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial v \partial u} + \frac{\partial^2 z}{\partial v^2}$$

Substituting these values in (1), we have

$$y(x+y)\left(-\frac{\partial^2 z}{\partial v \partial u}\right) - x\frac{\partial z}{\partial u} - y\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) - z = 0 \quad \text{or} \quad uv\frac{\partial^2 z}{\partial v \partial u} + u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v} + z = 0.$$

or $$\frac{\partial^2 z}{\partial v \partial u} + \frac{1}{v}\frac{\partial z}{\partial u} + \frac{1}{u}\frac{\partial z}{\partial v} + \frac{1}{uv}z = 0 \quad \text{or} \quad \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v} + \frac{z}{v}\right) + \frac{1}{u}\left(\frac{\partial z}{\partial v} + \frac{z}{v}\right) = 0. \qquad \text{... (7)}$$

Let $$\partial z/\partial v + (z/v) = w. \qquad \text{...(8)}$$

Then, the above equation (7) becomes $\partial w/\partial u + w/u = 0.$

Integrating, $wu = \phi(v)$ or $w = (1/u) \times \phi(v).$

Substituting this value of w in (8), we have $$\frac{\partial z}{\partial v} + \frac{1}{v}z = \frac{1}{u}\phi(v), \qquad \text{...(9)}$$

I.F. of (9) $= e^{\int (1/v)dv} = v$ and solution of (9) is

$$zv = \frac{1}{u}\int \phi(v)dv + \phi_2(u) \quad \text{or} \quad z = \frac{1}{uv}\phi_1(v) + \frac{1}{v}\phi_2(u), \text{ where } \phi_1(v) = \int \phi(v)\,dv$$

or $z = \dfrac{1}{y(x+y)}\,\phi_1(y) + \dfrac{1}{y}\phi_2(x+y)$, by (2); ϕ_1, ϕ_2 being arbitrary functions

(*ii*) **Hint.** Given $xys - x^2r - px - qy + z = -2x^2y.$ ...(1)

Here, $R = -x^2$, $S = xy$, $T = 0$ and λ–quadratic is $-x^2\lambda^2 + xy\lambda = 0$

so that $\lambda_1 = y/x$ and $\lambda_2 = 0$. Hence, characterstic equations

$$\frac{dy}{dx} + \lambda_1 = 0 \quad \text{and} \quad \frac{dy}{dx} + \lambda_2 = 0 \quad \Rightarrow \quad \frac{dy}{dx} + \frac{y}{x} = 0 \quad \text{and} \quad \frac{dy}{dx} = 0$$

Integrating these, $xy = c_1$, $y = c_2$. So we take $u = xy$ and $v = y$. ...(2)

Then, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u}y = v\frac{\partial z}{\partial u}, \text{ by (2)} \qquad ...(3)$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = x\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} = \frac{u}{v}\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ by (2)} \qquad ...(4)$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = v\frac{\partial}{\partial u}\left(v\frac{\partial z}{\partial u}\right) = v^2\frac{\partial^2 z}{\partial u^2}, \text{ by (3)}$$

$$s = \frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = v\frac{\partial}{\partial u}\left(\frac{u}{v}\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = v\left(\frac{1}{v}\frac{\partial z}{\partial u} + \frac{u}{v}\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u\partial v}\right), \text{ by (3) and (4)}$$

Substituting these values in (1), we have

$$xy\left(\frac{\partial z}{\partial u} + u\frac{\partial^2 z}{\partial u^2} + v\frac{\partial^2 z}{\partial u\partial v}\right) - x^2v^2\frac{\partial^2 z}{\partial u^2} - v\frac{\partial z}{\partial u}x - y\left(\frac{u}{v}\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + z = -2x^2y$$

or $$u\frac{\partial z}{\partial u} + u^2\frac{\partial^2 z}{\partial u^2} + uv\frac{\partial^2 z}{\partial u\partial v} - u^2\frac{\partial^2 z}{\partial u^2} - u\frac{\partial z}{\partial u} - u\frac{\partial z}{\partial u} - v\frac{\partial z}{\partial v} + z = -2(u^2/v^2)v, \text{ by (2)}$$

or $$uv\frac{\partial^2 z}{\partial u\partial v} - u\frac{\partial z}{\partial u} - v\frac{\partial z}{\partial v} + z = -\frac{2u^2}{v} \quad \text{or} \quad \frac{\partial^2 z}{\partial u\partial v} - \frac{1}{v}\frac{\partial z}{\partial u} - \frac{1}{u}\frac{\partial z}{\partial v} + \frac{z}{uv} = -\frac{2u}{v^2}.$$

or $$\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v} - \frac{z}{v}\right) - \frac{1}{u}\left(\frac{\partial z}{\partial v} - \frac{z}{v}\right) = -\frac{2u}{v^2}. \qquad ...(5)$$

Let $$\partial z/\partial v - z/v = w. \qquad ...(6)$$

Then (5) becomes $\dfrac{\partial w}{\partial u} - \dfrac{1}{u}w = -\dfrac{2u}{v^2}$, which is linear differential equation ...(7)

I.F. of (7) $= e^{-\int (1/u)du} = e^{-\log u} = e^{\log u^{-1}} = (1/u)$ and so its solution is

$$\frac{w}{u} = -\int\left(\frac{2u}{v^2} \times \frac{1}{u}\right)du = -\frac{2u}{v^2} + \phi(v) \qquad \text{or} \qquad w = -\frac{2u^2}{v^2} + u\,\phi(v)$$

Substituting this value of w in (6), we get $\dfrac{\partial z}{\partial v} - \dfrac{1}{v}z = -\dfrac{2u^2}{v^2} + u\phi(u).$

Its I.F. $= e^{-\int (1/v)dv} = e^{-\log v} = e^{\log v^{-1}} = (1/v)$ and so its solution is

$$\frac{z}{v} = \int \frac{1}{v}\left[-\frac{2u^2}{v^2} + u\,\phi(v)\right]dv = \frac{u^2}{v^2} + u\psi(v) + \phi_2(u)$$

or $z = (u^2/v) + uv\psi(v) + v\phi_2(u) = (u^2/v) + u\phi_1(v) + v\phi_2(u)$ or $z = x^2y + xy\phi_1(y) + y\phi_2(xy)$, by (2).

Ex. 12. *Solve* $x^2r - y^2t + px - qy = x^2$. **[Kurukshetra 2003; Delhi Maths (H) 1998]**

Sol. Given $x^2r - y^2t + (px - qy - x^2) = 0.$ ...(1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, we get

$R = x^2$, $S = 0$ and $T = -y^2$. ...(2)

Now, the λ-quadratic $R\lambda^2 + S\lambda + T = 0$ and (2) give

$x^2\lambda^2 - y^2 = 0$ so that $\lambda = \pm y/x$. (real and distinct roots)

Take $\lambda_1 = y/x$ and $\lambda_2 = -y/x$.

Hence characterstic equations $(dy/dx) + \lambda_1 = 0$ and $(dy/dx) + \lambda_2 = 0$

become $(dy/dx) + (y/x) = 0$ and $(dy/dx) - (y/x) = 0$

or $(1/x)dx + (1/y)dy = 0$ and $(1/x)dx - (1/y)dy = 0$

Integrating, $\log x + \log y = \log c_1$ and $\log x - \log y = \log c_2$

or $xy = c_1$ and $x/y = c_2$.

To reduce (1) to canonical form, we change the independent variables x, y to new independent variables u, v by taking

$$u = xy \quad \text{and} \quad v = x/y. \qquad \text{...(3)}$$

$$\therefore \quad p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = y\frac{\partial z}{\partial u} + \frac{1}{y}\frac{\partial z}{\partial v}, \text{ using (3).} \qquad \text{...(4)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = x\frac{\partial z}{\partial u} - \frac{x}{y^2}\frac{\partial z}{\partial v}, \text{ using (3).} \qquad \text{...(5)}$$

$$r = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(y\frac{\partial z}{\partial u} + \frac{1}{y}\frac{\partial z}{\partial v}\right) = y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \frac{1}{y}\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right] + \frac{1}{y}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]$$

$$= y\left(\frac{\partial^2 z}{\partial u^2} \times y + \frac{\partial^2 z}{\partial v \partial u} \times \frac{1}{y}\right) + \frac{1}{y}\left(\frac{\partial^2 z}{\partial v \partial u} \times y + \frac{\partial^2 z}{\partial v^2} \times \frac{1}{y}\right), \text{ using (3)}$$

$$\therefore \quad r = y^2\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{1}{y^2}\frac{\partial^2 z}{\partial y^2}. \qquad \text{...(6)}$$

$$t = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(x\frac{\partial z}{\partial u} - \frac{x}{y^2}\frac{\partial z}{\partial v}\right) = x\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) - \left[-\frac{2x}{y^3}\frac{\partial z}{\partial v} + \frac{x}{y^2}\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\right]$$

$$= x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}\right] + \frac{2x}{y^3}\frac{\partial z}{\partial v} - \frac{x}{y^2}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$= x\left[\frac{\partial^2 z}{\partial u^2} \times x + \frac{\partial^2 z}{\partial v \partial u} \times \left(-\frac{x}{y^2}\right)\right] + \frac{2x}{y^3}\frac{\partial z}{\partial v} - \frac{x}{y^2}\left[\frac{\partial^2 z}{\partial u \partial v} \times x + \frac{\partial^2 z}{\partial v^2} \times \left(-\frac{x}{y^2}\right)\right]$$

$$\therefore \quad t = x^2\frac{\partial^2 z}{\partial u^2} - \frac{2x^2}{y^2}\frac{\partial^2 z}{\partial u \partial v} + \frac{2x}{y^3}\frac{\partial z}{\partial v} + \frac{x^2}{y^4}\frac{\partial^2 z}{\partial v^2}. \qquad \text{...(7)}$$

Substituting the values of r, t, p and q given by (6), (7) (3) and (4) in (1), we obtain

$$x^2\left(y^2\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{1}{y^2}\frac{\partial^2 z}{\partial u^2}\right) - y^2\left(x^2\frac{\partial^2 z}{\partial u^2} - \frac{2x^2}{y^2}\frac{\partial^2 z}{\partial u \partial v} + \frac{2x}{y^3}\frac{\partial z}{\partial v} + \frac{x^2}{y^4}\frac{\partial^2 z}{\partial v^2}\right)$$

$$+ x\left(y\frac{\partial z}{\partial u} + \frac{1}{y}\frac{\partial z}{\partial v}\right) - y\left(x\frac{\partial z}{\partial u} - \frac{x}{y^2}\frac{\partial z}{\partial v}\right) - x^2 = 0$$

or $4x^2\dfrac{\partial^2 z}{\partial u \partial v} = x^2$ so that $\dfrac{\partial}{\partial u}\left(\dfrac{\partial z}{\partial v}\right) = \dfrac{1}{4}$, ...(8)

which is the canonical form of (1).

Now, integrating (8) w.r.t. 'u', $\partial z/\partial u = (u/4) + f(v)$. ...(9)

Integrating (9) w.r.t. 'v', $z = (uv)/4 + \int f(v)\,dx + \phi(u)$

or $z = (uv)/4 + \psi(v) + \phi(u)$, where $\psi(v) = \int f(v)\,dv$

or $z = x^2/4 + \psi(x/y) + \phi(xy)$, which is the required solution, ϕ, ψ being arbitrary functions.

Ex. 13. *(a) Reduce* $x^2\,(\partial^2 z/\partial x^2) - y^2\,(\partial^2 z/\partial y^2) = 0$ *to canonical form and hence solve it.*

(b) Reduce $y^2(\partial^2 z/\partial x^2) - x^2(\partial^2 z/\partial y^2) = 0$ *to canonical form.*

Sol. (*a*) Re-writing the given equation, $x^2 r - y^2 t = 0$ ... (1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = x^2$, $S = 0$ and $T = -y^2$ so that $S^2 - 4RT = 4x^2 y^2 > 0$ for $x \neq 0,\ y \neq 0$ and hence (1) is hyperbolic. The λ-quadrate equation $R\lambda^2 + S\lambda + T = 0$ reduces to $\lambda^2 x^2 - y^2 = 0$ so that $\lambda = y/x,\ -y/x$ and hence the corresponding characteristic equations become $(dy/dx) + (y/x) = 0$ and $(dy/dx) - (y/x) = 0$

Integrating these, $xy = c_1$ and $x/y = c_2$

In order to reduce (1) to its canonical form, we choose $u = xy$ and $v = x/y$ ... (2)

Now, doing exactly as in solved Ex. 12, we get

$$r = y^2\,\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{1}{y^2}\frac{\partial^2 z}{\partial v^2} \quad \text{and} \quad t = x^2\,\frac{\partial^2 z}{\partial u^2} - \frac{2x^2}{y^2}\frac{\partial^2 z}{\partial u \partial v} + \frac{2x}{y^3}\frac{\partial z}{\partial v} + \frac{x^2}{y^4}\frac{\partial^2 z}{\partial v^2}$$

Putting these values of r and t in (1), we get

$$x^2\left(y^2\,\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{1}{y^2}\frac{\partial^2 z}{\partial v^2}\right) - y^2\left(x^2\,\frac{\partial^2 z}{\partial u^2} - \frac{2x^2}{y^2}\,\frac{\partial^2 z}{\partial u \partial v} + \frac{2x}{y^3}\frac{\partial z}{\partial v} + \frac{x^2}{y^4}\frac{\partial^2 z}{\partial v^2}\right) = 0$$

or $$4x^2\,\frac{\partial^2 z}{\partial u \partial v} - \frac{2x}{y}\,\frac{\partial z}{\partial v} = 0 \quad \text{or} \quad 2xy\,\frac{\partial^2 z}{\partial u \partial v} - \frac{\partial z}{\partial v} = 0$$

or $2u\,(\partial^2 z/\partial u \partial v) - (\partial z/\partial v) = 0$, using (2). ... (3)

This is the required canonical form of (1).

We now proceed to find solution of (1). Multiplying both sides of (3) by v, we get

$$2uv\,\frac{\partial^2 z}{\partial u\,\partial v} - v\,\frac{\partial z}{\partial v} = 0 \quad \text{or} \quad (2uv\,DD' - vD')z = 0 \qquad \text{... (4)}$$

where $D \equiv \partial/\partial u$ and $D' \equiv \partial/\partial v$. We now reduce (4) to a linear equation with constant coefficients by usual method (refer Art. 6.3 of chapter 6).

Let $u = e^X$ and $v = e^Y$ so that $X = \log u$ and $y = \log v$ ... (5)

Let $D_1 \equiv \partial/\partial X$ and $D_1' \equiv \partial/\partial Y$. Then (4) reduces to

$(2D_1 D_1' - D_1')z = 0$ or $D_1'\,(2D_1 - 1)\,z = 0$

Its general solution is given by (use Art. 5.6 of chapter 5)

$$z = e^{X/2}\,\phi_1(Y) + \phi_2(X) = u^{1/2}\phi_1(\log v) + \phi_2(\log u) = u^{1/2}\psi_1(v) + \psi_2(u), \text{ using (5)}$$

$$= (xy)^{1/2}\psi_1(x/y) + \psi_2(xy) = x(y/x)^{1/2}\,\psi_1(x/y) + \psi_2(xy) = xf(x/y) + \psi_2(xy), \text{ using (2)}$$

where f and ψ_2 are arbitrary functions

(*b*) Try yourself. Choose $u = (y^2 - x^2)/2$, $v = (y^2 + x^2)/2$.

Ans. $$\frac{\partial^2 z}{\partial u \partial v} = \frac{1}{2(u^2 - v^2)}\left(v\frac{\partial z}{\partial u} - u\frac{\partial z}{\partial v}\right).$$

Ex. 14. *Reduce the equation* $x(xy-1)r-(x^2y^2-1)s+y(xy-1)t+(x-1)p+(y-1)q=0$ *to canonical form and hence solve it.*

Sol. Comparing the given equation with $Rr+Ss+Tt+f(x,y,z,p,q)=0$,

here, $R=x(xy-1)$, $S=-(x^2y^2-1)$, $T=y(xy-1)$. ...(1)

Now, the λ-quadratic equation $R\lambda^2+S\lambda+T=0$ and (1) give

$x(xy-1)\lambda^2-(x^2y^2-1)\lambda+y(xy-1)=0$ or $x\lambda^2-(xy+1)\lambda+y=0$

or $(x\lambda-1)(\lambda-y)=0$ so that $\lambda=1/x$, y. Take $\lambda_1=1/x$ and $\lambda_2=y$.

Hence characterstic equations $(dy/dx)+\lambda_1=0$ and $(dy/dx)+\lambda_2=0$

become $(dy/dx)+(1/x)=0$ and $(dy/dx)+y=0$

or $dy+(1/x)dx=0$ and $(1/y)dy+dx=0$. ...(2)

Intergrating (2), $y+\log x=\log c_1$ and $\log y+x=\log c_2$

or $\log e^y+\log x=\log c_1$ and $\log y+\log e^x=\log c_2$

$x\,e^y=c_1$ and $y\,e^x=c_2$.

To reduce the given equation to canonical form, we change the independent variables x, y to new independent variables u, v, by taking

$u=x\,e^y$ and $v=y\,e^x$. ...(3)

$$\therefore\quad p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial x}=e^y\frac{\partial z}{\partial u}+y\,e^x\frac{\partial z}{\partial v},\ \text{using (3)} \qquad ...(4)$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial z}{\partial v}\frac{\partial v}{\partial y}=xe^y\frac{\partial z}{\partial u}+e^x\frac{\partial z}{\partial v},\ \text{using (3).} \qquad ...(5)$$

$$r=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(e^y\frac{\partial z}{\partial u}+y\,e^x\frac{\partial z}{\partial v}\right)=e^y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)+ye^x\frac{\partial z}{\partial v}+y\,e^x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$=e^y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right]+y\,e^x\frac{\partial z}{\partial v}+ye^x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]$$

$$=e^y\left[\frac{\partial^2 z}{\partial u^2}e^y+\frac{\partial^2 z}{\partial v\partial u}ye^x\right]+y\,e^x\frac{\partial z}{\partial v}+y\,e^x\left[\frac{\partial^2 z}{\partial u\partial v}e^y+\frac{\partial^2 z}{\partial v^2}ye^x\right]$$

$$\therefore\quad r=e^{2y}\frac{\partial^2 z}{\partial u^2}+2y\,e^{x+y}\frac{\partial^2 z}{\partial u\partial v}+y^2\,e^{2x}\frac{\partial^2 z}{\partial v^2}+ye^x\frac{\partial z}{\partial v}.$$

$$s=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial x}\left(x\,e^y\frac{\partial z}{\partial u}+e^x\frac{\partial z}{\partial v}\right)=e^y\frac{\partial z}{\partial u}+x\,e^y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)+e^x\frac{\partial z}{\partial v}+e^x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$=e^y\frac{\partial z}{\partial u}+x\,e^y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right]+e^x\frac{\partial z}{\partial v}+e^x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]$$

$$=e^y\frac{\partial z}{\partial u}+xe^y\left[\frac{\partial^2 z}{\partial u^2}e^y+\frac{\partial^2 z}{\partial v\partial u}ye^x\right]+e^x\frac{\partial z}{\partial v}+e^x\left[\frac{\partial^2 z}{\partial u\partial v}e^y+\frac{\partial^2 z}{\partial v^2}ye^x\right]$$

$$=x\,e^{2y}\frac{\partial^2 z}{\partial u^2}+(xy+1)\,e^{x+y}\frac{\partial^2 z}{\partial u\partial v}+y\,e^{2x}\frac{\partial^2 z}{\partial v^2}+e^y\frac{\partial z}{\partial u}+e^x\frac{\partial z}{\partial v}$$

$$t=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial y}\left(x\,e^y\frac{\partial z}{\partial u}+e^x\frac{\partial z}{\partial v}\right)=x\,e^y\frac{\partial z}{\partial u}+x\,e^y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)+e^x\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)$$

$$=x\,e^y\frac{\partial z}{\partial u}+x\,e^y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial u}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}\right]+e^x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y}+\frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right]$$

$$=x\,e^y\frac{\partial z}{\partial u}+x\,e^y\left[\frac{\partial^2 z}{\partial u^2}x\,e^y+\frac{\partial^2 z}{\partial u\partial v}e^x\right]+e^x\left[\frac{\partial^2 z}{\partial u\partial v}x\,e^y+\frac{\partial^2 z}{\partial v^2}e^x\right],$$

$$\therefore \qquad t = x^2 e^{2y}\frac{\partial^2 z}{\partial u^2} + 2x\, e^{x+y}\frac{\partial^2 z}{\partial u \partial v} + x\, e^{y}\frac{\partial z}{\partial u} + e^{2x}\frac{\partial^2 z}{\partial v^2}.$$

Putting the above values of r, s, t, p, q in the given equation and simplifying, we obtain the required canonical form

$$\frac{\partial^2 z}{\partial u \partial v} = 0 \qquad \text{or} \qquad \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right) = 0. \qquad ...(6)$$

Integrating (6) w.r.t. 'v', $\partial z/\partial u = f(u)$, f being an arbitrary function ...(7)

Integrating (7) w.r.t. 'u', $z = \int f(u)\,du + \psi(v)$ or $z = \phi(u) + \psi(v)$, where $\phi(u) = \int f(u)\,du$.

Using (3), the required solution is $z = \phi(xe^y) + \psi(ye^x)$, ϕ and ψ being arbitrary functions.

Ex. 15. (*a*) *Reduce the one-dimensional wave equation* $\partial^2 z/\partial x^2 = (1/c^2)\times(\partial^2 z/\partial t^2)$, *(c > 0) to canonical form and hence find its general solution.*

(b) Find the D'Alembert's solution of the Cauchy's problem: $\partial^2 z/\partial x^2 = (1/c^2)\times(\partial^2 z/\partial t^2)$, *(c > 0) satisfying z (x, 0) = f (x) and* $z_t(x,0) = g(x)$ *where f (x) and g (x) are given functions representing the initial displacement and initial velocity, respectively. Also,* $z_t = \partial z/\partial t$

Sol. (*a*) Given $\qquad \partial^2 z/\partial x^2 - (1/c^2)\times(\partial^2 z/\partial t^2) = 0$, $c > 0$. ... (1)

To re-write (1), put $\qquad y = ct$, ... (2)

Then, (1) reduces to $\qquad \partial^2 z/\partial x^2 - (\partial^2 z/\partial y^2) = 0 \qquad$ or $\qquad r - t = 0$ ... (3)

Proceed now exactly as in solved Ex. 1 to reduce (3) to its canonical form

$$\frac{\partial^2 z}{\partial u \partial v} = 0 \qquad \text{or} \qquad \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right) = 0 \qquad ... (4)$$

where $u = y + x$, $\quad v = y - x \quad$ or $\quad u = ct + x \quad$ and $\quad v = ct - x$. ... (5)

Integrating (4) w.r.t. 'u', $\quad \partial z/\partial v = f(v)$, where f is an arbitrary function ... (6)

Integrating (6) w.r.t. 'v', $\quad z = \int f\ (v)\ dv + \psi(u) = F(v) + \psi(u)$, where $f(v) = \int f(v)\,dv$

or $$z(x,t) = F(ct - x) + \psi\ (ct + x), \text{ using (5)}$$

or $$z(x,t) = \phi(x - ct) + \psi(x + ct), \qquad ... (7)$$

where we take $\phi\ (x - ct) = F\ (ct - x)$ and ϕ, ψ as arbitrary functions.

(7) is the required general solution of (1).

(***b***) We are to solve $\qquad \partial^2 z/\partial x^2 - (1/c^2)\times(\partial^2 z/\partial t^2) = 0$ ... (*i*)

subject to the conditions $\qquad z(x,0) = f(x)$ ... (*ii*)

and $\qquad (\partial z/\partial t)_{t=0} = g(x)$ ... (*iii*)

Proceed exactly as in part (*a*) and get solution of (*i*) as

$$z(x,t) = \phi(x - ct) + \psi(x + ct) \qquad ... (iv)$$

Differentiating (*iv*) partially w.r.t. 't', we get

$$\partial z/\partial t = -c\ \phi'(x - ct) + c\ \psi'(x + ct) \qquad ... (v)$$

where dash denotes the derivative w.r.t. the argument. Putting $t = 0$ in (*iv*) and (*v*) and using (*ii*) and (*iii*) respectively, we get $\qquad \phi(x) + \psi(x) = f(x)$ ... (*vi*)

and $\qquad -c\ \phi'(x) + c\,\psi'(x) = g(x)$ ... (*vii*)

Integrating (*vii*), $\qquad -c\ \phi(x) + c\ \psi\ (x) = \int_a^x g(u)\ du$, ... (*viii*) where a is an arbitrary constant. Solving (vi) and ($viii$) for $\phi(x)$ and $\psi(x)$, we have

$$\phi(x) = \frac{1}{2} f(x) - \frac{1}{2c}\int_a^x g(x)\, du, \qquad \text{and} \qquad \psi(x) = \frac{1}{2} f(x) + \frac{1}{2c}\int_a^x g(u)\, du$$

so that
$$\phi\,(x-ct) = \frac{1}{2} f(x-ct) - \frac{1}{2c}\int_a^{x-ct} g(u)\, du \qquad \dots (ix)$$

and
$$\psi\,(x+ct) = \frac{1}{2} f(x+ct) + \frac{1}{2c}\int_a^{x+ct} g(u)\, du \qquad \dots (x)$$

Using (ix) and (x) in (iv), we get the required so called D'Alembert's solutiion of the Cauchy problem (which represents the vibrations of an infinite string in the present problem)

$$z(x,t) = \frac{1}{2}\left[f(x-ct) + f(x+ct)\right] + \frac{1}{2c}\left[\int_{x-ct}^{a} g(u)\, du + \int_a^{x+ct} g(u)\, dx\right]$$

or
$$z(x,t) = \frac{1}{2}\,\{f\,(x-ct) + f(x+ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(u)\, du \qquad \dots (xi)$$

Particular Case I. If in the above problem, we take $g\,(x) = 0$ so that the initial velocity of the string is zero, then (xi) reduces to

$$z(x,t) = \{f(x-ct) + f(x+ct)\}/2\,,$$

where $f\,(x - ct)$ represents a right travelling wave travelling with the speed c (along OX) and $f\,(x + ct)$ represents a left travelling wave travelling with the speed c.

Particular case II. If $f\,(x) = \sin x$ and $g\,(x) = \cos x$ in the above problem, then the corresponding solution (xi) reduces to

$$z(x,t) = \frac{1}{2}\{\sin\,(x-ct) + \sin(x+ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} \cos u\, du$$

or $z(x,t) = \sin x \cos ct + (1/2c) \times \{\sin(x+ct) - \sin(x-ct)\}$ or $z(x,t) = \sin x \cos ct + (1/c) \times \cos x \sin ct$.

Particular case III. If $f(x) = \sin x$ and $g(x) = x^2$, then (xi) gives

$z(x,\,t) = \sin x \cos ct + x^2 t + (c^3 t^3)/3$, on simplification.

8.8 Working rule for reducing a parabolic equation to its canonical form.

Step 1. Let the given equation $\qquad Rr + Ss + Tt + f(x,y,z,p,q) = 0 \qquad \dots (1)$

be parabolic so that $\qquad S^2 - 4RT = 0$.

Step 2. Write λ-quadratic equation $\qquad R\lambda^2 + S\lambda + T = 0 \qquad \dots (2)$

Let λ_1, λ_1 be two equal roots of (2)

Step 3. Write the characteristic equation corresponding to $\lambda = \lambda_1$, *i.e.*, $(dy/dx) + \lambda_1 = 0$

Solving it, we get $\qquad f_1(x,y) = C_1$, C_1 being an arbitrary constant $\qquad \dots (3)$

Step 4. Choose $\qquad u = f_1(x,y) \qquad$ and $\qquad v = f_2(x,y) \qquad \dots (4)$

where $f_2\,(x, y)$ is an arbitrary function of x and y and is independent of $f_1\,(x, y)$. For this verify that Jacobian J of u and v given by (4) is non-zero,

i.e.
$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} \neq 0 \qquad \dots (5)$$

Step 5. Using relations (4), find p, q, r, s and t in terms of u and v as shown in Art. 8.5.

Step 6. Substituting the values of p, q, r, s and t obtained in step (1) and simplifying we get the following canonical forms of (1)

$$\partial^2 z/\partial u^2 = \phi\,(u, v, z, \partial z/\partial u,\ \partial z/\partial v) \qquad \text{or} \qquad \partial^2 z/\partial v^2 = \phi(u, v, z, \partial z/\partial u,\ \partial z/\partial v)$$

8.9 SOLVED EXAMPLES BASED ON ART. 8.8

Ex. 1. *Reduce the equation* $\partial^2 z/\partial x^2 + 2(\partial^2 z/\partial x\partial y) + \partial^2 z/\partial y^2 = 0$ *to canonical form and hence solve it.* **[Delhi Maths (H) 2000, 06; 08; Jabalpur 2004; Delhi Maths (Prog) II 2008; Delhi B.Sc. (Prog) II 2008, 11; Himanchal 2001; 05 Rajasthan 2003; Lucknow 2010]**

Sol. Re-writing the given equation, we get $\qquad r + 2s + t = 0 \qquad$ ... (1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = 1$, $S = 2$, $T = 1$ so that $S^2 - 4RT = 0$, showing that (1) is parabolic.

The λ-quadrate equation reduces to $\lambda^2 + 2\lambda + 1 = 0$ so that $\lambda = -1, -1$ (equal roots).

The corresponding characteristic equation is $(dy/dx) - 1 = 0$ or $dx - dy = 0$

Integrating, $x - y = c$, c being an arbitrary constant.

Choose $\qquad u = x - y \qquad$ and $\qquad v = x + y, \qquad$ ... (2)

where we have chosen $v = x + y$ in such a manner that u and v are independent functions as verified below.

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = 1\cdot 1 + 1\cdot 1 = 2 \neq 0.$$

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ using (2)} \qquad \text{... (3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = -\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ using (2)} \qquad \text{... (4)}$$

From (3) and (4), $$\frac{\partial}{\partial x} = \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \qquad \text{and} \qquad \frac{\partial}{\partial y} = -\frac{\partial}{\partial u} + \frac{\partial}{\partial v} \qquad \text{... (5)}$$

$\therefore$ $$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right), \text{ by (3) and (5)}$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u\partial v} + \frac{\partial^2 z}{\partial v^2} \qquad \text{... (6)}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left(-\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right), \text{ by (4) and (5)}$$

$$= -\frac{\partial}{\partial u}\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial u}\right) + \frac{\partial}{\partial v}\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 v}{\partial u\partial v} + \frac{\partial^2 z}{\partial v^2} \qquad \text{... (7)}$$

and $$s = \frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right), \text{ by (4) and (5)}$$

$$= \frac{\partial}{\partial u}\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{\partial}{\partial v}\left(-\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = -\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v^2} \qquad \text{... (8)}$$

Using (6), (7) and (8) in (1), the required canonical form is

$$\frac{\partial^2 z}{\partial v^2} = 0 \qquad \text{or} \qquad \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right) = 0 \qquad \dots (9)$$

To find the required solution. Integrating (9) partially w.r.t. 'v', we get

$$\partial z / \partial v = \phi(u), \ \phi \text{ being an arbitrary function.} \qquad \dots (10)$$

Integrating (10) partially w.r.t 'v', $\quad z = \int \phi(u)\, dv + \psi(u) = v\phi(u) + \psi(u)$

or $\quad z = (x+y)\,\phi(x-y) + \psi(x-y)$, which is the desired solution, ϕ, ψ being arbitrary functions.

Ex. 2. *Reduce the equation* $y^2(\partial^2 z/\partial x^2) - 2xy(\partial^2 z/\partial x\,\partial y) + x^2(\partial^2 z/\partial x^2) = (y^2/x)(\partial z/\partial x) + (x^2/y)(\partial z/\partial y)$ *to canonical form and hence solve it.* **[Nagpur 2005; Delhi Maths (H) 2001, 05, 09; Avadh 2001, Himanchal 2009; Delhi B.Sc. (Prog) II 2007; Meerut 2005, 06, 11; G.N.D.U. Amritsar 2005]**

Sol. Re-writing the given equation, $\quad y^2 r - 2xys + x^2 t - (y^2/x)\,p - (x^2/y)q = 0 \qquad \dots (1)$

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = y^2$, $S = -2xy$, $T = x^2$ so that $S^2 - 4RT = 0$, showing that (1) is parabolic.

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to

$$y^2\lambda^2 - 2xy\,\lambda + x^2 = 0 \qquad \text{or} \qquad (y\lambda - x)^2 = 0 \qquad \text{so that} \qquad \lambda = x/y,\ x/y.$$

The corresponding characteristic equation is $\quad dy/dx + x/y = 0$

or $\quad x\,dx + y\,dy = 0 \qquad$ so that $\qquad x^2/2 + y^2/2 = C_1$

Choose $\quad u = x^2/2 + y^2/2 \qquad$ and $\qquad v = x^2/2 - y^2/2, \qquad \dots (2)$

where we have chosen $v = x^2/2 - y^2/2$ in such a manner that u and v are independent functions as verified below

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = -2xy \neq 0.$$

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = x\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right), \text{ using (2)} \qquad \dots (3)$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = y\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right), \text{ using (2)} \qquad \dots (4)$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left\{x\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\right\} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} + x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right), \text{ by (3)}$$

$$= \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} + x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right]$$

$$= \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v} + x^2\left(\frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right), \text{ using (2)} \qquad \dots (5)$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left[y\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\right] = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} + y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right), \text{ by (4)}$$

$$= \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} + y\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}\right\} = \frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} + y^2\left(\frac{\partial^2 z}{\partial u^2} - 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}\right) \quad \ldots (6)$$

and $$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left\{y\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\right\} = y\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}\right\}$$

or $$s = xy\,(\partial^2 z/\partial u^2 - \partial^2 z/\partial v^2) \quad \ldots (7)$$

Using (3), (4), (5), (6) and (7) in (1) and simplifying, we get

$4x^2y^2\ (\partial^2 z/\partial v^2) = 0$ so that $\partial^2 z/\partial v^2 = 0$, ... (8)

which is the required canonical form.

Integrating (8) partially w.r.t. 'v', $\partial z/\partial v = \phi(u)$, ϕ being arbitrary function. ... (9)

Integrating (9) partially w.r.t. 'v', $z = v\ \phi(u) + \psi(u)$, ψ being arbitrary function.

or $$z = [(x^2 - y^2)/2]\ \phi\{(x^2 + y^2)/2\} + \psi\ \{(x^2 + y^2)/2\}, \text{ using (2)}$$

or $$z = (x^2 - y^2)\ F(x^2 + y^2) + G\ (x^2 + y^2), \text{ F, G being arbitrary functions}$$

Ex. 3. *(a) Reduce $r + 2xs + x^2 t = 0$ to canonical form*

(b) Reduce $r - 6s + 9t + 2p + 3q - z = 0$ to canonical form

(c) Reduce $r - 2s + t + p - q = 0$ to canonical form and hence solve it.

Sol. (a) Given $$r + 2xs + x^2 t = 0 \quad \ldots (1)$$

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = 1$, $S = 2x$ and $T = x^2$ so that $S^2 - 4RT = 0$, showing that (1) is parabolic.

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to

$\lambda^2 + 2\lambda x + x^2 = 0$ or $(\lambda + x^2) = 0$ so that $\lambda = -x, -x$.

The corresponding characteristic equation is $(dy/dx) - x = 0$ or $dy - x\,dx = 0$

Integrating, $y - x^2/2 = c_1$, c_1 being an arbitrary constant. ... (2)

Choose $u = y - x^2/2$ and $v = x$ ... (2)

where we have chosen $v = x$ in such a manner that u and v are independent functions as verified below.

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = -1 \neq 0$$

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = -x\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ by (2)} \quad \ldots (3)$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u}, \text{ using (2)} \quad \ldots (4)$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(-x\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = -\frac{\partial z}{\partial u} - x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= -\frac{\partial z}{\partial u} - x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right] + \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}$$

$$= -\frac{\partial z}{\partial u} - x\left(-x\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v \partial u}\right) - x\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} = x^2\frac{\partial^2 z}{\partial u^2} - 2x\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} - \frac{\partial z}{\partial u} \qquad \text{... (5)}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x} = -x\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v \partial u}, \text{ by (4)} \qquad \text{... (6)}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y} = \frac{\partial^2 z}{\partial u^2}, \text{ by (4)} \qquad \text{... (7)}$$

Using (5), (6) and (7) in (1), we finally obtain $\partial^2 z/\partial v^2 = \partial z/\partial u$, which is required canonical form.

3. (*b*) Hint. Here $\lambda = 3$, $u = y + 3x$. Choose $v = y$. The canonical form will be

$$\partial^2 z/\partial v^2 = z/9 - (\partial z/\partial u) + (1/3)\times(\partial z/\partial v).$$

3. (c) Hints. Here $\lambda = 1$, $u = x + y$. Choose $v = y$. The canonical form is $\partial^2 z/\partial v^2 = \partial z/\partial v$.

Solution is $\qquad z = \phi\,(x+y) + e^y \psi\,(x+y)$, ϕ, ψ being arbitrary functions

Ex. 4. *Reduce the following to canonical form and hence solve*

(*a*) $x^2 r + 2xy s + y^2 t = 0$

(*b*) $r - 4s + 4t = 0$

(*c*) $x^2 r + 2xys + y^2 t + xyp + y^2 q = 0$

(*d*) $2r - 4s + 2t + 3z = 0$.

Sol. (a) Given $\qquad x^2 r + 2xys + y^2 t = 0 \qquad$ ... (1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = x^2$, $S = 2xy$ and $T = y^2$ so that $S^2 - 4RT = 0$, showing that (1) is parabolic.

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to

$$x^2\lambda^2 + 2xy\lambda + y^2 = 0 \qquad \text{or} \qquad (x\lambda + y)^2 = 0 \qquad \text{giving} \qquad \lambda = -y/x,\ -y/x.$$

The corresponding characteristic equation is $\qquad dy/dx - y/x = 0$

or $\qquad (1/y)dy - (1/x)dx = 0 \qquad$ so that $\qquad \log y - \log x = c_1 \qquad$ or $\qquad y/x = c_1$

Choose $\qquad u = y/x \qquad$ and $\qquad v = y, \qquad$ ... (2)

where we have chosen $v = y$ in such a manner that u and v are independent functions as verified below.

$$J = \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = -\frac{y}{x^2} \neq 0.$$

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = -\frac{y}{x^2}\frac{\partial z}{\partial u}, \text{ using (2)} \qquad \text{... (3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{1}{x}\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ using (2)} \qquad \text{... (4)}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(-\frac{y}{x^2}\frac{\partial z}{\partial u}\right) = \frac{2y}{x^3}\frac{\partial z}{\partial u} - \frac{y}{x^2}\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right), \text{ by (3)}$$

$$= \frac{2y}{x^3}\frac{\partial z}{\partial u} - \frac{y}{x^2}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right] = \frac{2y}{x^3}\frac{\partial z}{\partial u} + \frac{y^2}{x^4}\frac{\partial^2 z}{\partial u^2} \quad \text{... (5)}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{1}{x}\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = -\frac{1}{x^2}\frac{\partial z}{\partial u} + \frac{1}{x}\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= -\frac{1}{x^2}\frac{\partial z}{\partial u} + \frac{1}{x}\left\{\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right\} + \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}.$$

$$= -\frac{1}{x^2}\frac{\partial z}{\partial u} - \frac{y}{x^3}\frac{\partial^2 z}{\partial u^2} - \frac{y}{x^2}\frac{\partial^2 z}{\partial u \partial v} \quad \text{... (6)}$$

$$t = \frac{\partial^2 y}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{1}{x}\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) \text{ using (4)}$$

$$= \frac{1}{x}\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}\right] + \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial y}$$

$$= \frac{1}{x^2}\frac{\partial^2 z}{\partial u^2} + \frac{2}{x}\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial v^2} \quad \text{... (7)}$$

Using (5), (6) and (7) in (1), we finally get as the canonical form $\partial^2 z/\partial v^2 = 0$ ... (8)

Integrating (8) partially w.r.t. 'v', $\partial z/\partial v = \phi(u)$ ... (9)

Integrating (9) partially w.r.t 'v', $z = v\phi(u) + \psi(u)$

or $z = y\phi(y/x) + \psi(y/x), \phi, \psi$ being arbitrary functions.

(*b*) **Hint.** Here $\lambda = 2$, $u = y + 2x$. Choose $v = y$. The canonical form is $\partial^2 z/\partial v^2 = 0$ and solution is $z = y\phi(y+2x) + \psi(y+2x)$.

(*c*) **Hint.** Here $\lambda = -y/x$, $u = y/x$. Choose $v = y$. The canonical form is $\partial^2 z/\partial v^2 = -(\partial z/\partial v)$ and solution is $z = \phi(y/x) + e^{-y}\,\psi(y/x)$

(*d*) **Hint.** Here $\lambda = 1$, $u = x + y$. Choose $v = y$. The canonical form is $\partial^2 z/\partial v^2 = -(3z/2)$ and solution is $z = e^{(i\sqrt{3}/2)y}\phi(y+x) + e^{-(i\sqrt{3}/2)}\psi(y+x)$,

Ex. 5. *Reduce the following in canonical form and solve them*

(*a*) $r - 2s + t + p - q = e^x(2y-3) - e^y$

(*b*) $r - 2s + t + p - q = e^{x+y}$

Sol. (*a*) Given $r - 2s + t + p - q - e^x(2y - 3) + e^y = 0$ ... (1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = 1$, $S = -2$ and $T = 1$ so that $S^2 - 4RT = 0$, showing that (1) is parabolic.

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to

$\lambda^2 - 2\lambda + 1 = 0$ or $(\lambda - 1)^2 = 0$ so that $\lambda = 1, 1$ (equal roots)

So the corresponding characteristic equation is $dy/dx + 1 = 0$ or $dx + dy = 0$

Integrating it, $x + y = c_1$, c_1 being an arbitrary constant.

Choose $u = x + y$ and $v = y$ ... (2)

where we have chosen $v = y$ in such a manner that u and v are independent functions as verified below

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = 1 \neq 0.$$

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = \frac{\partial z}{\partial u}, \text{ using (2)} \qquad \text{... (3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ using (2)} \qquad \text{... (4)}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right) = \frac{\partial^2 z}{\partial u^2} \text{ by (3)} \qquad \text{... (5)}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \left(\frac{\partial}{\partial u} + \frac{\partial}{\partial v}\right)\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right), \text{ using (4)}$$

$$= \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} \qquad \text{... (6)}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = \frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v}, \text{ using (4)} \qquad \text{... (7)}$$

Using (2) (3), (4), (5), (6) and (7) in (1), we get

$$\frac{\partial^2 z}{\partial u^2} - 2\left(\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial u \partial v}\right) + \frac{\partial^2 z}{\partial u^2} + 2\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} + \frac{\partial z}{\partial u} - \left(\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = e^{u-v}(2v-3) - e^v$$

or $$\partial^2 z/\partial v^2 - \partial z/\partial v = e^{u-v}(2v-3) - e^v \qquad \text{... (8)}$$

which is the required canonical form of (1) Let $D \equiv \partial/\partial x$, $D' \equiv \partial/\partial y$.

Then (8) can be re-written as $$D'(D'-1)\, z = e^{u-v}(2v-3) - e^v, \qquad \text{... (9)}$$

which is non-honmogeneous linear partial differential equation with constant coefficients. To solve it, we shall use results of chapter 5. Accordingly, we have

$$\text{C.F.} = \phi(u) + e^v\ \psi(u) = \phi(x+y) + e^y \psi(x+y), \text{ by (2)}$$

P.I. corresponding to $e^{u-v}(2v-3)$

$$= \frac{1}{D'(D'-1)} e^{u+(-1)v}(2v-3) = e^{u+(-1)v} \frac{1}{(D'-1)\,(D'-1-1)}(2v-3)$$

$$= (1/2) \times e^{u-v}\ (1-D')^{-1}(1-D'/2)^{-1}(2v-3) = (1/2) \times e^{u-v}\ (1+D'+...)(1+D'/2+...)\ (2v-3)$$

$$= (1/2) \times e^{u-v}(1+3D'/2+...)\ (2v-3) = (1/2) \times e^{u-v}(2v-3+3) = v\ e^{u-v} = y\ e^{x+y-y} = ye^x, \text{ using(2)}$$

P.I. Corresponding to $(-\ e^v)$

$$= \frac{1}{D'(D'-1)}(-e^v) = -\frac{1}{D'-1}\frac{1}{D'}e^v = -\frac{1}{D'-1}\ (e^v \times 1) = -e^v \frac{1}{D'+1-1} 1 = -e^y \frac{1}{D'} 1$$

$$= -e^v v = -e^y y, \text{ using (2)}$$

Hence the required general solution is given by $y = \phi\ (x+y) + e^y\ \psi\ (x+y) + y\ e^x - y\ e^y$

or $$y = \phi\ (x+y) + e^y\ \psi\ (x+y) + ye^x - (x+y)\ e^y + x\ e^y$$

or $$y = \phi\ (x+y) + e^y\ \{\phi\ (x+y) + (x+y)\} + y\ e^x + x\ e^y$$

or $$y = \phi\ (x + y) + e^y F (x + y) + y e^x + x e^y,$$

where ϕ and F are arbitrary functions and $F (x + y) = \phi\ (x + y) + x + y$

(*b*) Hint. Here $\lambda = 1$, $u = x + y$, choose $v = y$. The canonical form is $\partial^2 z/\partial v^2 = \partial z/\partial v + e^u$ and solution is $z = \phi\ (x+y) + e^y \psi(x+y) - y\ e^{x+y}$, ϕ, ψ being arbitrary functions.

Ex. 6. *Reduce the equation $x^2 r - 2xys + y^2 t - xp + 3yq = 8y/x$ to canonical form.*

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Sol. Given $$x^2 r - 2xy\ s + y^2 t - xp + 3yq - 8y/x = 0 \qquad \text{... (1)}$$

Comparing (1) with $Rr + Ss + Tt + f(x, y, y, p, q) = 0$, here $R = x^2$, $S = -2xy$, $T = y^2$ so that $S^2 - 4RT = 0$, showing that (1) is parabolic.

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to

$$x^2\lambda^2 - 2xy\lambda + y^2 = 0 \qquad \text{or} \qquad (x\lambda - y)^2 = 0 \qquad \text{so that} \qquad \lambda = y/x,\ y/x.$$

The corresponding characteristic equation is

$$dy/dx + y/x = 0 \qquad \text{or} \qquad (1/y)\, dy + (1/x)\, dx = 0 \qquad \text{so that} \qquad xy = C_1$$

Choose $$u = xy \qquad \text{and} \qquad v = x \qquad \text{... (2)}$$

where we have chosen $v = x$ in such a manner that u and v are independent functions as verified below.

$$J = \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \partial u/\partial x & \partial u/\partial y \\ \partial v/\partial x & \partial v/\partial y \end{vmatrix} = \frac{\partial u}{\partial x}\frac{\partial v}{\partial y} - \frac{\partial u}{\partial y}\frac{\partial v}{\partial x} = -x \neq 0.$$

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x} = y\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}, \text{ by (2)} \qquad \text{... (3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} = x\frac{\partial z}{\partial u}, \text{ by (2)} \qquad \text{... (4)}$$

$$r = \frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(y\frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}\right) = y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right) + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)$$

$$= y\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right] + \frac{\partial}{\partial u}\left(\frac{\partial z}{\partial v}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial v}\right)\frac{\partial v}{\partial x}$$

$$= y\left(y\frac{\partial^2 z}{\partial u^2} + \frac{\partial^2 z}{\partial v \partial u}\right) + y\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2} = y^2\frac{\partial^2 z}{\partial u^2} + 2y\frac{\partial^2 z}{\partial u \partial v} + \frac{\partial^2 z}{\partial v^2}, \qquad \text{.... (5)}$$

$$s = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial u}\right) = \frac{\partial z}{\partial u} + x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)$$

$$= \frac{\partial z}{\partial u} + x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial x} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial x}\right] = \frac{\partial z}{\partial u} + xy\frac{\partial^2 z}{\partial u^2} + x\frac{\partial^2 z}{\partial u \partial v} \qquad \text{... (6)}$$

and $$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(x\frac{\partial z}{\partial u}\right) = x\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right), \text{ by (4)}$$

$$= x\left[\frac{\partial}{\partial u}\left(\frac{\partial z}{\partial u}\right)\frac{\partial u}{\partial y} + \frac{\partial}{\partial v}\left(\frac{\partial z}{\partial u}\right)\frac{\partial v}{\partial y}\right] = x^2\frac{\partial^2 z}{\partial u^2}, \text{ by (2)} \qquad \text{... (7)}$$

Using (2), (3), (4), (5), (6) and (7) in (1), we have

$$x^2\left(y^2\frac{\partial^2 z}{\partial u^2}+2y\frac{\partial^2 z}{\partial u\partial v}+\frac{\partial^2 z}{\partial v^2}\right)-2xy\left(\frac{\partial z}{\partial u}+xy\frac{\partial^2 z}{\partial u^2}+x\frac{\partial^2 z}{\partial u\partial v}\right)+y^2x^2\frac{\partial^2 z}{\partial u^2}-x\left(y\frac{\partial z}{\partial u}+\frac{\partial z}{\partial v}\right)+3y\,x\frac{\partial z}{\partial u}-\frac{8y}{x}=0$$

or $\qquad x^2\dfrac{\partial^2 z}{\partial v^2}--x\dfrac{\partial z}{\partial v}=\dfrac{8y}{x}$ $\qquad$ or $\qquad v^2\dfrac{\partial^2 z}{\partial v^2}-v\dfrac{\partial z}{\partial v}=\dfrac{8u}{v^2}$, by (2)

or $\qquad (v^2D'^2-vD')z=8u/v^2,\qquad$ where $\quad D\equiv\partial/\partial u,\qquad D'\equiv\partial/\partial v$ $\qquad$ ... (8)

As explained in chapter 6, we shall reduce (8) to linear partial differential equation with constant coefficients and then use methods of chapter 5 to solve the resulting equation.

To solve (8), let $\quad u=e^X\quad$ and $\quad v=e^Y\quad$ so that $\quad X=\log u,\qquad Y=\log v$ $\quad$ ... (9)

Then (8) becomes $\quad \{D'\,(D'-1)-D'\}z=8e^{X-2y}\qquad$ or $\qquad D'(D'-2)z=8e^{X-2y}$.

C.F. $=\ \phi(X)+e^{2Y}\psi\,(X)=\phi(\log u)+v^2\psi(\log u)$, using (9)

$\quad = F(u)+v^2G(u)=F(xy)+x^2G(xy)$, using (2)

P.I. $=\dfrac{1}{D'(D'-1)}\,8\,e^{X-2Y}\ =\ 8\,e^{X-2Y}\dfrac{1}{(D'-2)\,(D'-2-2)}\cdot 1$

$$=\frac{8e^X}{(e^Y)^2}\times\frac{1}{8}\left(1-\frac{D'}{2}\right)^{-1}\left(1-\frac{D'}{4}\right)^{-1}\cdot 1=\frac{u}{v^2}\left(1+\frac{D'}{2}+\ldots\right)\left(1+\frac{D'}{4}+\ldots\right)\cdot 1=\frac{u}{v^2}\times 1=\frac{xy}{x^2}=\frac{y}{x},\ \text{by (2)}$$

$\therefore$ Required solution is $\qquad z=F(xy)+x^2G(xy)+y/x$, F, G being arbitrary functions.

8.10 Working rule for reducing an elliptic equation to its canonical form.

Step 1. Let the given equation $\qquad Rr+Ss+Tt+f(x,y,z,p,q)=0$ $\qquad$ ... (1)

be ellipitic so that $\qquad S^2-4RT<0\cdot$ $\qquad$ ... (2)

Step 2. Write λ quadratic equation $\qquad R\lambda^2+S\lambda+T=0$ $\qquad$ ... (2)

Let roots λ_1,λ_2 of (2) be complex conjugates.

Step 3. Then corresponding charactersitic equations are

$(dy/dx)+\lambda_1=0\qquad$ and $\qquad dy/dx+\lambda_2=0$

Solving these, we shall obtain solutions of the form

$f_1(x,y)+i\ f_2(x,y)=c_1\qquad$ and $\qquad f_1(x,y)-i\ f_2(x,y)=c_2$ $\qquad$ ... (3)

Step 4. Choose $\qquad u=f_1(x,y)+i\,f_2(x,y),\qquad v=f_1(x,y)-i\ f_2(x,y)$

Let α and β be two new real independent variables such that $u=\alpha+i\beta$ and $v=\alpha-i\beta$,

so that $\qquad \alpha=f_1(x,y)\qquad$ and $\qquad \beta=f_2(x,y)$ $\qquad$ ... (4)

Step 5. Using relations (4), find *p*, *q*, *r*, *s* and *t* in terms of α and β (in place of *u* and *v* as we did in Art 8.6 and 8.8 corresponding to the cases of hyperbolic and parabolic equations).

Step 6. Substituting the values of *p*, *q*, *r*, *s* and *t* and relations (4) in (1) and simplifying we shall get the following canonical form of (1)

$$\partial^2 z/\partial\alpha^2+\partial^2 z/\partial\beta^2\ =\ \phi\,(\alpha,\beta,z,\partial z/\partial\alpha,\ \partial z/\partial\beta)\,.$$

8.11 SOLVED EXAMPLES ON ART 8.10

Ex. 1. *Reduce the following partial differential equations to canonical forms:*

$(a)\,\partial^2 z/\partial x^2+x^2(\partial^2 z/\partial y^2)=0$ *or* $\qquad r+x^2t=0$

[Delhi B.Sc. (Prog) II 2010; Delhi B.Sc. (Hons) III 2011; Kanpur 2011]

(b) $y^2(\partial^2 z/\partial y^2)+\partial^2 z/\partial x^2 = 0$ **[Delhi Math (Hons.) 1995, 98, 2005]**

Sol. (a) Re-writing the given equations, we get $$r+x^2 t=0 \qquad \text{... (1)}$$

Comparing (1) with $Rr+Ss+Tt+f(x,y,z,p,q)=0$, here $R=1$, $S=0$, $T=x^2$ so that $S^2-4RT=-4x^2<0$, $x\neq 0$, showing that (1) is elliptic.

The λ-quadratic $R\lambda^2+S\lambda+T=0$ reduces to $\lambda^2+x^2=0$ giving $\lambda=ix, -ix$.
The corresponding characteristic equations are given by

$$dy/dx+ix=0 \quad \text{and} \quad dy/dx-ix=0$$

Integrating, $y+i\,(x^2/2)=c_1$ and $y-i\,(x^2/2)=c_2$.

Choose $u=y+i(x^2/2)=\alpha+i\beta$ and $v=y-i(x^2/2)=\alpha-i\beta$,

where $\alpha=y$ and $\beta=x^2/2$ ... (2)

are now two new independent variables.

Now, $$p=\frac{\partial z}{\partial x}=\frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial x}+\frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial x}=x\frac{\partial z}{\partial \beta}, \text{ by (2)} \qquad \text{.... (3)}$$

$$q=\frac{\partial z}{\partial y}=\frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial y}+\frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial y}=\frac{\partial z}{\partial \alpha}, \text{ by (2)} \qquad \text{... (4)}$$

$$r=\frac{\partial^2 z}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)=\frac{\partial}{\partial x}\left(x\frac{\partial z}{\partial \beta}\right)=\frac{\partial z}{\partial \beta}+x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \beta}\right), \text{ by (3)}$$

$$=\frac{\partial z}{\partial \beta}+x\left[\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \alpha}{\partial x}+\frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \beta}{\partial x}\right]=\frac{\partial z}{\partial \beta}+x^2\frac{\partial^2 z}{\partial \beta^2} \qquad \text{... (5)}$$

and $$t=\frac{\partial^2 z}{\partial y^2}=\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)=\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right)=\frac{\partial^2 z}{\partial \alpha^2}, \text{ by (4)} \qquad \text{... (6)}$$

Using (5) and (6) in (1) the required canonical form is

$$\frac{\partial z}{\partial \beta}+x^2\frac{\partial^2 z}{\partial \beta^2}+x^2\frac{\partial^2 z}{\partial \alpha^2}=0 \quad \text{or} \quad \frac{\partial^2 z}{\partial \alpha^2}+\frac{\partial^2 z}{\partial \beta^2}=-\frac{1}{2\beta}\frac{\partial z}{\partial \beta}, \text{ as } \beta=\frac{x^2}{2}.$$

(***b***) Do as in part (a). **Ans.** $\partial^2 z/\partial\alpha^2+\partial^2 z/\partial\beta^2=-(1/2\alpha)\times(\partial z/\partial\alpha)$, where $\alpha=y^2/2$, $\beta=x$.

Ex. 2. *Reduce* $y^2(\partial^2 z/\partial x^2)+x^2(\partial^2 z/\partial y^2)=0$ *to canonical form*

Sol. Re-writing the given equation, we get $$y^2 r+x^2 t=0 \qquad \text{... (1)}$$

Comparing (1) with $Rr+Ss+Tt+f(x,y,z,p,q)=0$, here $R=y^2$, $S=0$, $T=x^2$ so that $S^2-4RT=-4x^2y^2<0$ for $x\neq 0$, $y\neq 0$, showing that (1) is elliptic.

The λ-quadratic equation $R\lambda^2+S\lambda+T=0$ reduces to

$y^2\lambda^2+x^2=0$ or $\lambda^2=-x^2/y^2$ so that $\lambda=ix/y$, $-ix/y$

The corresponding characteristic equations are

$$dy/dx+ix/y=0 \quad \text{and} \quad dy/dx-ix/y=0$$

Integrating, $y^2+ix^2=C_1$ and $y^2-ix^2=C_2$

Choose $u=y^2+ix^2=\alpha+i\beta$ and $v=y^2-ix^2=\alpha-i\beta$, where $\alpha = y^2$ and $\beta = x^2$ ... (2)

are now two new independent variables

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial x} = 2x\frac{\partial z}{\partial \beta}, \text{ by (2)} \quad \text{... (3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial y} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial y} = 2y\frac{\partial z}{\partial \alpha}, \text{ by (2)} \quad \text{... (4)}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(2x\frac{\partial z}{\partial \beta}\right) = 2\frac{\partial z}{\partial \beta} + 2x\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial \beta}\right), \text{ by (3)}$$

$$= 2\frac{\partial z}{\partial \beta} + 2x\left\{\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \beta}{\partial x}\right\} = 2\frac{\partial z}{\partial \beta} + 4x^2\frac{\partial^2 z}{\partial \beta^2} \quad \text{... (5)}$$

and $$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(2y\frac{\partial z}{\partial \alpha}\right) = 2\frac{\partial z}{\partial \alpha} + 2y\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial \alpha}\right)$$

$$= 2\frac{\partial z}{\partial \alpha} + 2y\left\{\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right)\frac{\partial \alpha}{\partial y} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \alpha}\right)\frac{\partial \beta}{\partial y}\right\} = 2\frac{\partial z}{\partial \alpha} + 4y^2\frac{\partial^2 z}{\partial \alpha^2} \quad \text{... (6)}$$

Using (5) and (6) in (1), the required canonical form is

$$2y^2\frac{\partial z}{\partial \beta} + 4x^2y^2\frac{\partial^2 z}{\partial \beta^2} + 2x^2\frac{\partial z}{\partial \alpha} + 4x^2y^2\frac{\partial^2 z}{\partial \alpha^2} = 0 \quad \text{or} \quad 2\alpha\beta\left(\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2}\right) + \alpha\frac{\partial z}{\partial \beta} + \beta\frac{\partial z}{\partial \alpha} = 0$$

or $$\frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} + \frac{1}{2}\left(\frac{1}{\alpha}\frac{\partial z}{\partial \alpha} + \frac{1}{\beta}\frac{\partial z}{\partial \beta}\right) = 0$$

Ex. 3. *Reudce $\partial^2 z/\partial x^2 + y^2(\partial^2 z/\partial y^2) = y$ to canonical form.*

Sol. Re-writing the given equation, we get $r + y^2 t - y = 0$ ... (1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here R = 1, S = 0 and $T = y^2$ so that $S^2 - 4RT = -4y^2 < 0$ for $y \neq 0$, showing that (1) is elliptic.

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $\lambda^2 + y^2 = 0 \Rightarrow \lambda = iy, -iy$.

The corresponding characteristic equations are given by

$dy/dx + iy = 0$ and $dy/dx - iy = 0$

Integrating these, $\log y + ix = c_1$ and $\log y - ix = c_2$

Choose $u = \log y + ix = \alpha + i\beta$ and $v = \log y - ix = \alpha - i\beta$,

where $\alpha = \log y$ and $\beta = x$ ... (2)

are now two new independent variables.

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial x} = \frac{\partial z}{\partial \beta}, \text{ using (2)} \quad \text{... (3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial y} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial y} = \frac{1}{y}\frac{\partial z}{\partial \alpha}, \text{ using (2)} \quad \text{.... (4)}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right) = \frac{\partial^2 z}{\partial \beta^2}, \text{ by (3)} \quad \text{... (5)}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{1}{y}\frac{\partial z}{\partial \alpha}\right) = -\frac{1}{y^2}\frac{\partial z}{\partial \alpha} + \frac{1}{y}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial \alpha}\right)$$

$$= -\frac{1}{y^2}\frac{\partial z}{\partial \alpha} + \frac{1}{y}\left\{\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right)\frac{\partial \alpha}{\partial y} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \alpha}\right)\left(\frac{\partial \beta}{\partial y}\right)\right\}$$

$$= -\frac{1}{y^2}\frac{\partial z}{\partial \alpha} + \frac{1}{y}\left(\frac{\partial^2 z}{\partial \alpha^2}\frac{1}{y}\right) = \frac{1}{y^2}\left(\frac{\partial^2 z}{\partial \alpha^2} - \frac{\partial z}{\partial \alpha}\right) \qquad \text{... (6)}$$

Using (5) and (6) in (1), the required canonical form is

$$\frac{\partial^2 z}{\partial \beta^2} + \frac{\partial^2 z}{\partial \alpha^2} - \frac{\partial z}{\partial \alpha} - y = 0 \qquad \text{or} \qquad \frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = \frac{\partial z}{\partial \alpha} + e^{\alpha}, \quad \text{using (2)}$$

Ex. 4. *Reduce* $x\,(\partial^2 z/\partial x^2) + \partial^2 z/\partial y^2 = x^2$ *(x > 0) to canonical form.* **[Delhi Maths(H) 2007, 11]**

Sol. Re-writing the given equation, we get $\qquad xr + t - x^2 = 0\,,\ (x > 0)$ ... (1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = x$, $S = 0$ and $T = 1$ so that $S^2 - 4RT = -4x < 0$, showing that (1) is elliptic.

The λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to

$$x\lambda^2 + 1 = 0 \qquad \text{or} \qquad \lambda^2 = -(1/x^2) \qquad \text{so that} \qquad \lambda = i/x^{1/2},\ -i/x^{1/2}$$

The corresponding characteristic equations are given by

$$dy/dx + i\ x^{-1/2} = 0 \qquad \text{and} \qquad dy/dx - i\ x^{-1/2} = 0.$$

Integrating these, $\qquad y + 2i\ x^{1/2} = C_1 \qquad$ and $\qquad y - 2i\ x^{1/2} = C_2$

Choose $\qquad u = y + 2i\ x^{1/2} = \alpha + i\beta \qquad$ and $\qquad v = y - 2i\,x^{1/2} = \alpha - i\beta$,

where $\qquad \alpha = y \qquad$ and $\qquad \beta = 2x^{1/2}$ ... (2)

are now two new independent variables.

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial x} = x^{-1/2}\frac{\partial z}{\partial \beta}, \text{ by (2)} \qquad \text{... (3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial y} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial y} = \frac{\partial z}{\partial \alpha}, \text{ by (2)} \qquad \text{... (4)}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(x^{-1/2}\frac{\partial z}{\partial \beta}\right) = -\frac{1}{2}x^{-3/2}\frac{\partial z}{\partial \beta} + x^{-1/2}\left\{\frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \alpha}{\partial x} + \frac{\partial}{\partial \beta}\left(\frac{\partial z}{\partial \beta}\right)\frac{\partial \beta}{\partial x}\right\}$$

or $$r = -\frac{1}{2}x^{-3/2}\frac{\partial z}{\partial \beta} + x^{-1/2}\left(x^{-1/2}\frac{\partial^2 z}{\partial \beta^2}\right) = -\frac{1}{2x^{3/2}}\frac{\partial z}{\partial \beta} + \frac{1}{x}\frac{\partial^2 z}{\partial \beta^2} \qquad \text{... (5)}$$

and $$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial \alpha}\left(\frac{\partial z}{\partial \alpha}\right) = \frac{\partial^2 z}{\partial \alpha^2}, \text{ using (4)} \qquad \text{... (6)}$$

Using (5) and (6) in (1), the required canonical form is

$$x\left(-\frac{1}{2x^{3/2}}\frac{\partial z}{\partial \beta} + \frac{1}{x}\frac{\partial^2 z}{\partial \beta^2}\right) + \frac{\partial^2 z}{\partial \alpha^2} = x^2 \qquad \text{or} \qquad \frac{\partial^2 z}{\partial \alpha^2} + \frac{\partial^2 z}{\partial \beta^2} = x^2 + \frac{1}{2x^{1/2}}\frac{\partial z}{\partial \beta}$$

or $\partial^2 z/\partial\alpha^2 + \partial^2 z/\partial\beta^2 = (\beta^2/4) + (1/\beta)\times(\partial z/\partial\beta)$, as $\beta = 2\,x^{1/2}$.

Ex. 5. *Reduce* $\dfrac{\partial^2 z}{\partial x^2} + 2\dfrac{\partial^2 z}{\partial x\partial y} + 5\dfrac{\partial^2 z}{\partial y^2} + \dfrac{\partial z}{\partial x} - 2\dfrac{\partial z}{\partial y} - 3z = 0$ *to canonical form.*

Sol. Re-writing the given equation, we get $r + 2s + 5t + p - 2q - 3z = 0$ ... (1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = 1$, $S = 2$ and $T = 5$ so that $S^2 - 4RT = -16 < 0$, showing (1) is elliptic.

The λ-quadratic $R\lambda^2 + S\lambda + T = 0$ reduces to

$\lambda^2 + 2\lambda + 5 = 0$ so that $\lambda = \{-2 \pm (4-20)^{1/2}\}/2 = -1 \pm 2i$

The corresponding characterisitc equations are given by

$dy/dx + (-1+2i) = 0$ and $dy/dx + (-1-2i)x = 0.$

Integrating these, $y + (-1+2i)x = C_1$ and $y + (-1-2i)x = C_2$

Let $u = y - x + 2ix = \alpha + i\beta$ and $v = y - x - 2ix = \alpha - i\beta$,

where $\alpha = y - x$ and $\beta = 2x$ ... (2)

are now two new independent variables.

Now, $$p = \frac{\partial z}{\partial x} = \frac{\partial z}{\partial\alpha}\frac{\partial\alpha}{\partial x} + \frac{\partial z}{\partial\beta}\frac{\partial\beta}{\partial x} = -\frac{\partial z}{\partial\alpha} + 2\frac{\partial z}{\partial\beta}, \text{ using (2)} \quad \text{... (3)}$$

$$q = \frac{\partial z}{\partial y} = \frac{\partial z}{\partial\alpha}\frac{\partial\alpha}{\partial y} + \frac{\partial z}{\partial\beta}\frac{\partial\beta}{\partial y} = \frac{\partial z}{\partial\alpha}, \text{ using (2)} \quad \text{... (4)}$$

$$t = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial\alpha}\left(\frac{\partial z}{\partial\alpha}\right) = \frac{\partial^2 z}{\partial\alpha^2}, \text{ using (3)} \quad \text{... (5)}$$

$$r = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}\left(-\frac{\partial z}{\partial\alpha} + 2\frac{\partial z}{\partial\beta}\right) = -\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial\alpha}\right) + 2\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial\beta}\right)$$

$$= -\left\{\frac{\partial}{\partial\alpha}\left(\frac{\partial z}{\partial\alpha}\right)\frac{\partial\alpha}{\partial x} + \frac{\partial}{\partial\beta}\left(\frac{\partial z}{\partial\alpha}\right)\frac{\partial\beta}{\partial x}\right\} + 2\left\{\frac{\partial}{\partial\alpha}\left(\frac{\partial z}{\partial\beta}\right)\frac{\partial\alpha}{\partial x} + \frac{\partial}{\partial\beta}\left(\frac{\partial z}{\partial\beta}\right)\frac{\partial\beta}{\partial x}\right\}$$

$$= -\left(-\frac{\partial^2 z}{\partial\alpha^2} + 2\frac{\partial^2 z}{\partial\beta\partial\alpha}\right) + 2\left(-\frac{\partial^2 z}{\partial\alpha\partial\beta} + 2\frac{\partial^2 a}{\partial\beta^2}\right), \text{ by (2)}$$

$\therefore$ $$r = \partial^2 z/\partial\,\alpha^2 + 4\,(\partial^2 z/\partial\beta^2) - 4\,(\partial^2 z/\partial\alpha\partial\beta) \quad \text{... (6)}$$

$$s = \frac{\partial^2 z}{\partial x\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial\alpha}\right) = \frac{\partial}{\partial\alpha}\left(\frac{\partial z}{\partial\alpha}\right)\frac{\partial\alpha}{\partial x} + \frac{\partial}{\partial\beta}\left(\frac{\partial z}{\partial\alpha}\right)\frac{\partial\beta}{\partial x}$$

or $$s = -(\partial^2 z/\partial\alpha^2) + 2\,(\partial^2 z/\partial\alpha\,\partial\beta), \text{ using (2)} \quad \text{... (7)}$$

Using (3), (4), (5), (6) and (7) in (1), we get

$$\frac{\partial^2 z}{\partial\alpha^2} + 4\frac{\partial^2 z}{\partial\beta^2} - 4\frac{\partial^2 z}{\partial\alpha\partial\beta} + 2\left(-\frac{\partial^2 z}{\partial\alpha^2} + 2\frac{\partial^2 z}{\partial\alpha\partial\beta}\right) + 5\frac{\partial^2 z}{\partial\alpha^2} - \frac{\partial z}{\partial\alpha} + 2\frac{\partial z}{\partial\beta} - 2\frac{\partial z}{\partial\alpha} - 3z = 0$$

or $$4\left(\frac{\partial^2 z}{\partial\alpha^2} + \frac{\partial^2 z}{\partial\beta^2}\right) = 3z + 3\frac{\partial z}{\partial\alpha} - 2\frac{\partial z}{\partial\beta}$$ or $$\frac{\partial^2 z}{\partial\alpha^2} + \frac{\partial^2 z}{\partial\beta^2} = \frac{3z}{4} + \frac{3}{4}\frac{\partial z}{\partial\alpha} - \frac{1}{2}\frac{\partial z}{\partial\beta},$$

which is the required canonical form of given equation (1).

8.12. The solution of linear hyperbolic equations. It what follows we aim at sketching the existence theorems for two types of initial conditions on the linear hyperbolic equation

$$\partial^2 z/\partial x \partial y = f(x, y, z, p, q). \qquad ...(1)$$

For both kinds of initial condition, we assume that the function $f(x, y, z, p, q)$ satisfies the following two conditions :

(*i*) f is continuous at all points of a rectangular region R defined by $\alpha < x < \beta, \gamma < y < \delta$ for all values of x, y, z, p, q concerned.

(*ii*) f satisfies the so called Lipschitz condition, namely,

$$| f(x, y, z_2, p_2, q_2) - f(x, y, z_1, p_1, q_1) | \le M \{| z_2 - z_1 | + | p_2 - p_1 | + | q_2 - q_1 |\}$$

in all bounded subrectangles r of R.

We now state (without proof) two existence theorems.

Theorem 1. Initial conditions of the first kind. *If $F(x)$ and $G(x)$ are defined in the open intervals (α, β), (γ, δ), respectively, and have continuous first derivatives, and if (ξ, η) is a point inside R such that $F(\xi) = G(\eta)$, then (1) has at least one integral $z = \phi(x, y)$ in R such that*

$$\phi(x, y) = \begin{cases} F(x), \text{ when } y = \eta \\ G(y), \text{ when } x = \xi. \end{cases}$$

Theorem II. *Initial conditions of the second kind.* *Let C_1 be a space curve defined by $x = x(\lambda)$, $y = y(\lambda)$, $z = z(\lambda)$ in terms of a single parameter λ and also let C_0 be the projection of C_1 on the xy-plane. If we are given (x, y, z, p, q) along a strip C_1, then (1) has an integral which takes on the given values of z, p, q along the curve C_0. This intergral exists at every point of the region R, which is defined as the smallest rectangle completely enclosing the curve C_0.*

8.13. Riemann method of solution of general linear hypobolic equation of the second order. [Himanchal 2002; Meerut 2005, 07, 08; Delhi Maths (Hons.) 1995, 1999, 2000]

Assume that the given linear hyperbolic equation is reducible to canonical form

$$L(z) = f(x, y), \qquad ...(1)$$

where L denotes the linear operator given by
$$L \equiv \frac{\partial^2}{\partial x \partial y} + a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c, \qquad ...(2)$$

where a, b, c are functions of x and y only.

Let w be another function with continuous derivatives of the first order. Again, let M be another operator defined by the relation

$$Mw = \frac{\partial^2 w}{\partial x \partial y} - \frac{\partial (aw)}{\partial x} - \frac{\partial (bw)}{\partial y} + cw. \qquad ...(3)$$

The operator M defined by (4) is called the *adjoint operator* to the operator L.

$$\therefore \quad w\,Lz - z\,Mw = w\left(\frac{\partial^2 z}{\partial x \partial y} + a\frac{\partial z}{\partial x} + b\frac{\partial z}{\partial y} + c\right) - z\left(\frac{\partial^2 w}{\partial x \partial y} - \frac{\partial (aw)}{\partial x} - \frac{\partial (bw)}{\partial y} + cw\right)$$

$$= \left(w\frac{\partial^2 z}{\partial x \partial y} - z\frac{\partial^2 w}{\partial x \partial y}\right) + \left(wa\frac{\partial z}{\partial x} + z\frac{\partial (aw)}{\partial x}\right) + \left(wb\frac{\partial z}{\partial y} + z\frac{\partial (bw)}{\partial y}\right) = \frac{\partial}{\partial y}\left(w\frac{\partial z}{\partial x}\right) - \frac{\partial}{\partial x}\left(z\frac{\partial w}{\partial y}\right) + \frac{\partial (awz)}{\partial x} + \frac{\partial (bwz)}{\partial y}$$

$$= \frac{\partial}{\partial x}\left(awz - z\frac{\partial w}{\partial y}\right) + \frac{\partial}{\partial y}\left(bwz + w\frac{\partial z}{\partial x}\right) = \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}, \qquad ...(4)$$

where $U = awz - z(\partial w/\partial y)$ and $V = bwz + w(\partial z/\partial x)$. ...(5)

Now if C' is a closed curve enclosing an area S, then

$$\iint_S (w\,Lz - z\,Mw)\,dxdy = \iint_S \left(\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y}\right) dxdy = \int_{C'} (Udy - Vdx), \text{ by Green's theorem} \qquad ...(6)$$

Assume that the values of z and $\partial z/\partial x$ (or $\partial z/\partial y$) are prescribed along a curve C in the xy-plane (refer figure 1) and further assume that we are required to determine the solution of (1) at the point P (ξ, η) agreeing with these boundary conditions. Draw PA, PB parallel to x-axis and y-axis and cutting the curve C in the points A and B respectively. The closed circuit $PABP$ can be taken as the closed curve C'. Then (6) reduces to

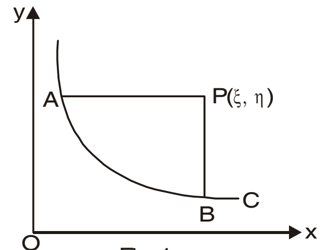


Fig. 1.

$$\iint_S (w\,Lz - z\,Mw)\,dxdy = \int_{AB}(Udy - Vdx) + \int_{BP}(Udy - Vdx) + \int_{PA}(Udy - Vdx)$$
$$= \int_{AB}(Udy - Vdx) + \int_{BP} Udy - \int_{PA} Vdx, \qquad ...(7)$$

where we have used the following facts:

along BP, x = constant so that $dx = 0$ and along PA, y = constant so that $dy = 0$.

Now, $\int_{PA} Vdx = \int_{PA}\left(bwz + w\dfrac{\partial z}{\partial x}\right)dx$, by (5)

$$= \int_{PA} bwzdz + \int_{PA} w\frac{\partial z}{\partial x}\,dx = \int_{PA} bwzdz + [wz]_P^A - \int_{PA} z\frac{\partial w}{\partial x}\,dx, \text{ integrating by parts}$$
$$= [wz]_A - [wz]_P + \int_{PA} z\left(bw - \frac{\partial w}{\partial x}\right)dx. \qquad ...(8)$$

Using (5) and (8), (7) becomes

$$\iint_S (w\,Lz - z\,Mz)\,dxdy = \int_{AB}(Udy - Vdx) + \int_{BP}\left(awz - z\frac{\partial w}{\partial y}\right)dy - [wz]_A + [wz]_P - \int_{PA} z\left(bw - \frac{\partial w}{\partial x}\right)dx.$$

$$\therefore\ [wz]_P = [wz]_A + \int_{PA} z\left(bw - \frac{\partial w}{\partial x}\right)dx - \int_{BP} z\left(aw - \frac{\partial w}{\partial y}\right)dy$$
$$- \int_{AB}(Udy - Vdx) + \iint_S (w\,Lz - z\,Mw)\,dxdy \qquad ...(9)$$

So far we have treated w as an arbitrary function. Now, we choose a function $w\,(x, y, \xi, \eta)$ which has the following four properties, namely,

(*i*) $Mw = 0$, (*ii*) $w = 1$, when $x = \xi$, $y = \eta$ *i.e.*, at $P(\xi, \eta)$,

(*iii*) $\partial w/\partial x = bw$ when $y = \eta$, (*iv*) $\partial w/\partial y = aw$ when $x = \xi$.

Such a function $w(x, y, \xi, \eta)$ is known as *Green's function* for the problem or sometimes a *Riemann-Green function*. Using the above four properties of w, (9) may be re-written as

$$[z]_P = [wz]_A - \int_{AB}(Udy - Vdx) + \iint_S w\,Lz\,dxdy$$
$$= [wz]_A - \int_{AB}\left(awz - z\frac{\partial w}{\partial y}\right)dy + \int_{AB}\left(bwz + w\frac{\partial z}{\partial x}\right)dx + \iint_S (wf)dxdy, \text{ using (1) and (5)}$$
$$= [wz]_A - \int_{AB} wz(ady - bdx) + \int_{AB}\left(z\frac{\partial w}{\partial y}\,dy + w\frac{\partial z}{\partial x}\,dx\right) + \iint_S wf\,dxdy. \qquad .(10)$$

Equation (10) may be used to determine the value of z at the point P when $\partial z/\partial x$ is prescribed along the curce C.

Suppose, in place of the prescribed value of $\partial z/\partial x$, we are now given a prescribed value of $\partial z/\partial y$. Then, we make use of the following relation

$$\int_{AB} d(wz) = \int_{AB}\left(\frac{\partial(wz)}{\partial x}\,dx + \frac{\partial(wz)}{\partial y}\,dy\right)$$

$$\Rightarrow \qquad 0 = [wz]_B - [wz]_A - \int_{AB}\left(\frac{\partial(wz)}{\partial x}\,dx + \frac{\partial(wz)}{\partial y}\,dy\right). \qquad ...(11)$$

Adding the corresponding sides of (10) and (11), we get

$$[z]_P = [wz]_B - \int_{AB} wz\,(ady - bdx) + \int_{AB}\left(z\frac{\partial w}{\partial y}dy + w\frac{\partial z}{\partial x}dx\right) - \int_{AB}\left(\frac{\partial(wz)}{\partial x}dx + \frac{\partial(wz)}{\partial y}dy\right) + \iint_S (wf)\,dxdy$$

$$= [wz]_B - \int_{AB} wz\,(ady - bdx) - \iint_{AB}\left(z\frac{\partial w}{\partial x}\,dx + w\frac{\partial z}{\partial y}\,dy\right) + \iint_S (wf)\,dxdy \qquad ..(12)$$

Equation (12) may be used to determine the z at the point P when $\partial z/\partial y$ is precribed along the curve C.

Finally, by adding (10) and (12), we get the following symmetrical result which can be used to find value of z at the point P when both $\partial z/\partial x$ and $\partial z/\partial y$ are prescribed along the curve C.

$$[z]_P = \frac{1}{2}\{[wz]_A + [wz]_B\} - \int_{AB} wz\,(ady - bdx) + \iint_S (wf)\,dxdy$$

$$-\frac{1}{2}\int_{AB} w\left(\frac{\partial z}{\partial y}dy - \frac{\partial z}{\partial x}dx\right) - \frac{1}{2}\int_{AB} z\left(\frac{\partial w}{\partial x}dx - \frac{\partial w}{\partial y}dy\right). \qquad ...(13)$$

By means of whichever of the formulas (10), (12) and (13) is suitable, we may determine the solution of (1) at any point in terms of the prescribed values of z, $\partial z/\partial x$ or/and $\partial z/\partial y$ along a given curve C.

We now discuss four particular cases:

Particular Case I *Determine the solution of*

$$\partial^2 z/\partial x\,\partial y + a(\partial z/\partial x) + b(\partial z/\partial y) + cz = f(x, y) \qquad ...\ (i)$$

which satisfies the boundary conditions that z and $\partial z/\partial x$ *are prescribed along curve C in the xy-plane.* **[Delhi Maths (H) 1995, 99, 2000, 06, 08; Meerut 2010]**

Hint. Proceed as in Art. 8.13 upto equation (10), *i.e.,*

$$[z]_P = [wz]_A - \int_{AB} wz\,(a\,dy - b\,dx) + \int_{AB}\left(z\,\frac{\partial w}{\partial y}dy + w\frac{\partial z}{\partial x}dx\right) + \iint_S wfdxdy \qquad ...\ (ii)$$

Relation (*ii*) may be used to determine the value of z at the point P when z and $\partial z/\partial x$ are prescribed along a curve C.

Particular case II *To determine the solution of the equation*

$$\partial^2 z/\partial x\,\partial y = f(x, y) \qquad ...\ (iii)$$

which satisfies the boundary conditions that z and $\partial z/\partial x$ *are prescribed along a curve C in the xy-plane.* **[Meerut 2010; Delhi. Maths (H) 1995, 99, 2000, 06, 08, 09, 10]**

Hint. First state and prove that above particular case I. Note that (*ii*) is solution of (*i*). Comparing (*iii*) with (*i*), we have $a = b = c = 0$ and hence for the present equation (*iii*), (*ii*) gives

$$[z]_P = [wz]_A + \int_{AB}\left(z\frac{\partial w}{\partial y}dy + w\frac{\partial z}{\partial x}dx\right) + \iint_S wf\ dx\,dy \qquad ...\ (iv)$$

where the Green's function w satisfies the following four properties (refer Art. 8.13 and note that $a = b = c = 0$ for the present case).

(*a*) $\partial^2 w/\partial x\,\partial y = 0$ at all points of S (*b*) $w = 1$ at $P\ (\xi, \eta)$

(*c*) $\partial w/\partial x = 0$ when $y = \eta$ (*d*) $\partial w/\partial y = 0$ when $x = \xi$

Hence Green's function can be taken as $w = 1$ so as to satisfy the above four conditions. Substituting $w = 1$ in (*iv*), the required solution takes the following form

$$[z]_P = [z]_A + \int_{AB}\frac{\partial z}{\partial x}\,dx + \iint_S f(x, y)\,dx\,dy \qquad ...\ (v)$$

The relation (*v*) may be used to determine the value of z at the point P when z and $\partial z/\partial x$ are prescribed along a curve C.

Particular Case III *Determine the solution of*

$$\partial^2 z/\partial x\partial y + a\,(\partial z/\partial x) + b(\partial z/\partial y) + cz = f(x, y) \qquad \text{... }(vi)$$

which satisfies the boundary conditions that z and $\partial z/\partial y$ are prescribed along a curve C in the xy-plane.

Hint. Proceed as in Art. 8.13 upto equation (12), *i.e.*

$$[z]_P = [wz]_B - \iint_{AB} wz\,(ady - bdx) - \iint_{AB}\left(z\frac{\partial w}{\partial x}dx + w\frac{\partial z}{\partial y}dy\right) + \iint_S wf\ dx\,dy \qquad \text{... }(vii)$$

Relation (*vii*) may be used to determine the value of z at the point P when z and $\partial z/\partial y$ are prescribed along a curve C in the xy-plane.

Particular Case IV To *determine the solution* of

$$\partial^2 z/\partial x\partial y = f(x, y) \qquad \text{... }(viii)$$

which satisfies the boundary conditions that z and $\partial z/\partial y$ are prescribed along a curve C in the xy-plane. **[Delhi Maths (H) 1974, 97, 2001]**

Hint. First state and prove the above particular case III. Note that (*vii*) is solution of (*vi*). Comparing (*viii*) with (*vi*), we have a = b = c = 0 and hence for the present equation (*viii*), (*vii*) reduces to

$$[z]_P = [wz]_B - \iint_{AB}\left(z\frac{\partial w}{\partial x}dx + w\frac{\partial z}{\partial y}dy\right) + \iint_S wf\ dx\,dy \qquad \text{... }(ix)$$

where the Green's function w satisfies the following four properties (refer Art. 8.13 and note that $a = b = c = 0$ for the present case)

(*a*) $\partial^2 w/\partial x\,\partial y = 0$ at all points of S (*b*) $w = 1$ at $P(\xi, \eta)$

(*c*) $\partial w/\partial x = 0$ when $y = \eta$ (*d*) $\partial w/\partial y = 0$ when $x = \xi$.

Hence Green's function can be taken as $w = 1$ so as to satisfy the above four conditions. Substituting $w = 1$ in (*ix*), the required solution takes the following form.

$$[z]_P = [z]_B - \iint_{AB} \frac{\partial z}{\partial y}\,dy + \iint_S f(x, y)\ dx\ dy\,. \qquad \text{... }(x)$$

The relation (*x*) may be used to determine the value of z at the point P when z and $\partial z/\partial y$ are prescribed along a curve.

Note: Relations (10), (12) (13) (*ii*), (*v*), (*vii*) and (*x*) must be remembered and may be used directly in solving problems based on them.

8.14 SOLVED EXAMPLES BASED ON ART 8.13

Ex 1. *Find the solution, valid when x, y>0, xy > 1 of the equation* $\partial^2 z/\partial x\partial y = 1/(x+y)$ *such that z = 0,* $p = (2y)/(x+y)$ *on the hyperbola xy = 1.* **[Meerut 2007 Delhi Maths (H) 1999, 2006 Himanchal 1997, K.U. Kurukshatra 1999]**

Sol. Comparing the given equation with $L(z) = f(x, y)$, we have

$a = b = c = 0$ and $f(x, y) = 1/(x + y)$. Hence the adjoint operator M of the operator L is given by $M \equiv \partial^2/\partial x\partial y$.

So Green's function can be taken as $w = 1$. ...(1)

In the present problem, the values of z and $\partial z/\partial x$ (=p) are given by

$$z = 0, \qquad \partial z/\partial x = (2y)/(x+y), \qquad \text{...(2)}$$

along the curve C, which is hyperbola.

$$xy = 1. \qquad \text{...(3)}$$

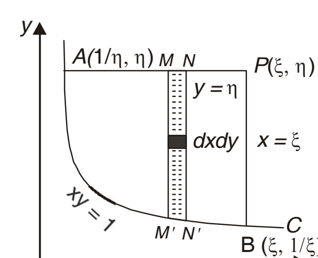


Fig. 2.

Then we wish to find the solution of given equation at the point $P(\xi, \eta)$ agreeing with these boundary conditions. Through P we draw PA parallel to the x-axis and cutting $xy = 1$ in the point A and PB parallel to the y-axis and cutting $xy = 1$ in B. Then region enclosed by $xy = 1$, $x = \xi$, $y = \eta$ is denoted by S. Now, we know that (refer equation (10) of Art 8.13.)

$$[z]_P = [wz]_A - \int_{AB} wz(ady - bdx) + \int_{AB}\left(z\frac{\partial w}{\partial y}dy + w\frac{\partial z}{\partial x}dx\right) + \iint_S wf\ dxdy$$

or
$$[z]_P = \int_{AB}\frac{2y}{x+y}dx + \iint_S \frac{1}{x+y}dxdy, \text{ by (1) and (2).} \qquad ...(4)$$

Now, $$\int_{AB}\frac{2y}{x+y}dx = 2\int_A^B \frac{xy}{x^2+xy}dx = 2\int_{1/\eta}^{\xi}\frac{1}{1+x^2}dx = 2\,\{\tan^{-1}\xi - \tan^{-1}(1/\eta)\} \qquad ...(5)$$

and
$$\iint_S \frac{1}{1+x}dxdy = \int_{x=1/\eta}^{\xi}\left\{\int_{y=1/x}^{\eta}\frac{1}{x+y}dy\right\}dx, \qquad ...(6)$$

since to integrate over area bounded by $PABP$, we first integrate along the strip $MNN'M'$ by fixing x and varying y from $y = 1/x$ at M' to $y = \eta$ at M and then integrate from A to P (keeping y fixed) by varying x from $x = 1/\eta$ to $x = \xi$. Evaluating the double integral on R.H.S. of (6) by the usual rule,

$$\iint_S \frac{1}{x+y}dxdy = \int_{1/\eta}^{\xi}\left[\log(x+y)\right]_{1/x}^{\eta}dx = \int_{1/\eta}^{\xi}[\log(x+\eta) - \log(x+1/x)]\,dx$$

$$= \int_{1/\eta}^{\xi}\{\log(x+\eta) - \log(1+x^2) + \log x\}\,dx$$

$$= \Big[\{\log(x+\eta) - \log(1+x^2) + \log x\}\,x\Big]_{1/\eta}^{\xi} - \int_{1/\eta}^{\xi} x\left(\frac{1}{x+\eta} - \frac{2x}{1+x^2} + \frac{1}{x}\right)dx$$

$$= \xi\,\{\log(\xi+\eta) - \log(1+\xi^2) + \log\xi)\} - \frac{1}{\eta}\left\{\log\left(\frac{1}{\eta}+\eta\right) - \log\left(1+\frac{1}{\eta^2}\right) + \log\frac{1}{\eta}\right\}$$

$$- \int_{1/\eta}^{\xi}\left(\frac{2}{1+x^2} - \frac{\eta}{x+\eta}\right)dx, \text{ on re-arranging*}$$

$$= \xi\log\frac{\xi(\xi+\eta)}{1+\xi^2} - \Big[2\tan^{-1}x - \eta\log(x+\eta)\Big]_{1/\eta}^{\xi} = \xi\log\frac{\xi(\xi+\eta)}{1+\xi^2} - 2\left(\tan^{-1}\xi - \tan^{-1}\frac{1}{\eta}\right) + \eta\log\frac{\eta(\xi+\eta)}{1+\eta^2} \qquad ...(7)$$

Using (5) and (7), (4) reduces to
$$[z]_P = \xi\log\frac{\xi(\xi+\eta)}{1+\xi^2} + \eta\log\frac{\eta(\xi+\eta)}{1+\eta^2}. \qquad ...(8)$$

Replacing ξ and η by x and y respectively in (8), the value of z (*i.e.*, solution of the given equation) at any point (x, y) is given by
$$z = x\log\frac{x(x+y)}{1+x^2} + y\log\frac{y(x+y)}{1+y^2}.$$

Ex. 2. *Prove that, for the equation* $(\partial^2 z/\partial x\partial y) + (z/4) = 0$, *the Green's function is* $w(x, y\,;\,\xi, \eta) = J_0\sqrt{(x-\xi)(y-\eta)}$, *where* $J_0(z)$ *denotes Bessel's function of the first kind of order zero.*

[Himanchal 1998, 2004, Kerala 2001; Kurukshetra 2000; Nagpur 2000, 03, 05 Delhi Maths Hons. 2004, 07, 09]

Sol. Here
$$L(z) = (\partial^2 z/\partial x\partial y) + (z/4) = 0. \qquad ...(1)$$

$$\therefore\quad L \equiv \frac{\partial^2}{\partial x\partial y} + a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c = \frac{\partial^2}{\partial x\partial y} + \frac{z}{4}. \quad\Rightarrow\quad a = 0,\quad b = 0,\quad c = z/4. \qquad ...(2)$$

---

* $$x\left(\frac{1}{x+\eta} - \frac{2x}{1+x^2} + \frac{1}{x}\right) = \frac{x}{x+\eta} - \frac{2x^2}{1+x^2} + 1 = \frac{(x+\eta)-\eta}{x+\eta} - 2\frac{(1+x^2)-1}{1+x^2} + 1 = -\frac{\eta}{x+\eta} + \frac{2}{1+x^2}$$

So the adjoint operator M to the operator L is given by $\quad M \equiv (\partial^2/\partial x\,\partial y)+(1/4)=0.$ ...(3)

Given, $$w = J_0\sqrt{(x-\xi)(y-\eta)}. \qquad ...(4)$$

From (4) $$\frac{\partial w}{\partial x} = \frac{\sqrt{(y-\eta)}}{2\sqrt{(x-\xi)}}J_0' \qquad ...(5)$$

From (5), $$\frac{\partial^2 w}{\partial y \partial x} = \frac{1}{4}\frac{1}{\sqrt{(x-\xi)(y-\eta)}}J_0' + \frac{\sqrt{(y-\eta)}}{2\sqrt{(x-\xi)}} \times \frac{\sqrt{(x-\xi)}}{2\sqrt{(y-\eta)}}J_0''$$

or $$\frac{\partial^2 w}{\partial y \partial x} = \frac{1}{4}\left\{J_0'' + \frac{1}{\sqrt{(x-\xi)(y-\eta)}}J_0'\right\}. \qquad ...(6)$$

So (3) and (6) $\Rightarrow$ $$Mw = \frac{1}{4}\left\{J_0'' + \frac{1}{\sqrt{(x-\xi)(y-\eta)}}J_0' + J_0\right\}. \qquad ...(7)$$

Now, Bessel's eqution of order zero is given by

$x^2y'' + xy' + x^2y = 0$ or $y'' + (1/x) \times y' + y = 0.$ ...(8)

Since $y = J_0\{\sqrt{(x-\xi)(y-\eta)}\}$ is a solution of (8), we get

$$J_0'' + \frac{1}{\sqrt{(x-\xi)(y-\eta)}}J_0' + J_0 = 0 \quad \text{or} \quad Mw = 0, \text{ by (7)} \qquad ...(9)$$

Again, (5) $\Rightarrow$ $(\partial w/\partial x) = 0 = bw$ when $y = \eta$, as $b = 0$ ...(10)

Similarly, $(\partial w/\partial y) = 0 = aw$ when $x = \xi$, as $a = 0$ ...(11)

Finally, when $x = \xi$, $y = \eta$, $w = J_0(0) = 1.$ ...(12)

Since w satisfies four properties (9), (10), (11) and (12) of a Green's function, it follows that w must be a Green's function of the given equation (1).

Ex. 3. *Prove that for the equation* $\dfrac{\partial^2 z}{\partial y \partial x} + \dfrac{2}{x+y}\left(\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y}\right) = 0$, *the Green's function is*

$$w(x, y\,;\,\xi, \eta) = \frac{(x+y)\{2xy + (\xi-\eta)(x+y) + 2\xi\eta\}}{(\xi+\eta)^3}.$$

Hence find the solution of the differential equation which satisfies the conditions $z = 0$, $\partial z/\partial x = 3x^2$ on $y = x$. **[Bangalore 2003, Himanchel 2001; Kurukshetra 2004; Delhi Maths (H) 2001,05,11]**

Sol. Compare the given equation with $L(z) = f(x, y)$ where $L \equiv \dfrac{\partial^2}{\partial y \partial x} + a\dfrac{\partial}{\partial x} + b\dfrac{\partial}{\partial y} + c$, we find

$a = 2/(x+y)$, $\quad b = 2/(x+y)$, $\quad c = 0$, $\quad f(x, y) = 0.$ ...(1)

So the adjoint operator M to the operator L is given by

$$Mw \equiv \frac{\partial^2 w}{\partial y \partial x} - \frac{\partial}{\partial x}\left(\frac{2}{x+y}w\right) - \frac{\partial}{\partial y}\left(\frac{2}{x+y}w\right). \qquad ...(2)$$

Given $$w(x, y\,;\,\xi, \eta) = \frac{(x+y)\{2xy + (\xi-\eta)(x-y) + 2\xi\eta\}}{(\xi+\eta)^3}. \qquad ...(3)$$

$$(3) \Rightarrow \quad \frac{\partial w}{\partial x} = \frac{2xy + (\xi-\eta)(x-y) + 2\xi\eta + (x+y)(2y+\xi-\eta)}{(\xi+\eta)^3}. \qquad ...(4)$$

$$(3) \Rightarrow \quad \frac{\partial w}{\partial y} = \frac{2xy + (\xi-\eta)(x-y) + 2\xi\eta + (x+y)(2y-\xi+\eta)}{(\xi+\eta)^3}. \qquad ...(5)$$

$$(5) \Rightarrow \quad \frac{\partial^2 w}{\partial x \partial y} = \frac{2y+\xi-\eta + 2x-\xi+\eta + 2(x+y)}{(\xi+\eta)^3} = \frac{4(x+y)}{(\xi+\eta)^3}. \qquad ..(6)$$

Using (6) and (3), (2) reduces to

$$Mw = \frac{4(x+y)}{(\xi+\eta)^3} - 2\frac{\partial}{\partial x}\left\{\frac{2xy+(\xi-\eta)(x-y)+2\xi\eta}{(\xi+\eta)^3}\right\} - 2\frac{\partial}{\partial y}\left\{\frac{2xy+(\xi-\eta)(x-y)+2\xi\eta}{(\xi+\eta)^3}\right\}$$

$$= \frac{4(x+y)}{(\xi+\eta)^3} - \frac{2(2y+\xi-\eta)+2(2x-\xi+\eta)}{(\xi+\eta)^3} = 0. \qquad ...(7)$$

At $y = \eta$, $$\frac{\partial w}{\partial x} = \frac{2x\eta+(\xi-\eta)(x-\eta)+2\xi\eta+(x+\eta)(2\eta+\xi-\eta)}{(\xi+\eta)^3}, \text{ by (4)}$$

$$= \frac{2\{x(\xi+\eta)+\eta^2+\xi\eta\}}{(\xi+\eta)^3}. \qquad ...(8)$$

From (1) and (3), $$bw = \frac{2\{2xy+(\xi-\eta)(x-y)+2\xi\eta\}}{(\xi+\eta)^3}. \qquad ...(9)$$

So at $y = \eta$, (9) reduces to

$$bw = \frac{2\{2x\eta+(\xi-\eta)(x-y)+2\xi\eta\}}{(\xi+\eta)^3} = \frac{2\{x(\xi+\eta)+\eta^2+\xi\eta\}}{(\xi+\eta)^3}. \qquad ...(10)$$

From (8) and (10), $\partial w/\partial x = bw$ when $y = \eta$. ...(11)

Similarly, $\partial w/\partial y = aw$ when $x = \xi$. ...(12)

From (3), when $x = \xi$, $y = \eta$, we get $$w = \frac{(\xi+\eta)\{2\xi\eta+(\xi-\eta)^2+2\eta\}}{(\xi+\eta)^3} = 1. \qquad ...(13)$$

Since w satisfies four properties (7), (11), (12) and (13) of a Green's function, it follows that w must be a Green's function of the given equation.

To find the solution of the equation. In the present problem, the values of z and $\partial z/\partial x$ $(= p)$ are given by

$z = 0$ and $\partial z/\partial x = 3x^2$. ...(14)

along the line AB, $y = x$. ...(15)

Then we wish to find the solution of given equation at the point $P(\xi, \eta)$ agreeing with these boundary conditions. Through P we draw PA parallel to the x-axis and cutting $y = x$ in the point A and PB parallel to the y-axis and cutting $y = x$ in B. Then triangular region enclosed by straight lines $y = x$, $y = \eta$ and $x = \xi$ is denoted by S. Then we know that (refer equation (10) of Art 8.13).

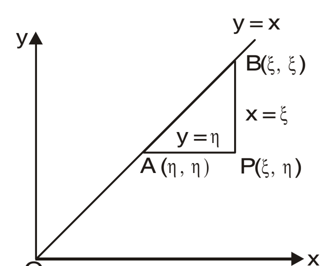


Figure-3

$$[z]_P = [wz]_A - \int_{AB} wz(ady - bdx) + \int_{AB}\left(z\frac{\partial w}{\partial y}dy + w\frac{\partial z}{\partial x}dx\right) + \iint_S wf\, dxdy. \qquad ..(16)$$

Now on line AB, from (3), $$w = \frac{4x(x^2+\xi\eta)}{(\xi+\eta)^2}, \text{ as } y = x. \qquad ...(17)$$

Using (1), (14) and (17), (16) reduces to

$$[z]_P = \int_A^B \frac{4x(x^2+\xi\eta)}{(\xi+\eta)^3}\, 3x^2 dx = \frac{12}{(\xi+\eta)^3}\int_\eta^\xi (x^5+\xi\eta x^3)dx$$

$$= \frac{12}{(\xi+\eta)^3}\left[\frac{x^6}{6}+\xi\eta\frac{x^4}{4}\right]_\eta^\xi = \frac{12}{(\xi+\eta)^3}\left[\frac{\xi^6-\eta^6}{6}+\frac{\xi\eta}{4}(\xi^4-\eta^4)\right]$$

$$= (\xi+\eta)^{-3}\left\{2(\xi^3+\eta^3)(\xi^3-\eta^3)+3\xi\eta(\xi^2-\eta^2)(\xi^2+\eta^2)\right\}$$

$= (\xi+\eta)^{-3}\left\{2(\xi^3+\eta^3)(\xi-\eta)(\xi^2+\eta^2+\xi\eta)+3\xi\eta(\xi-\eta)(\xi+\eta)(\xi^2+\eta^2)\right\}$

$= (\xi+\eta)^{-3}(\xi-\eta)\left\{2(\xi^3+\eta^3)(\xi^2+\eta^2)+2\xi\eta(\xi^3+\eta^3)-3\xi\eta(\xi+\eta)(\xi^2+\eta^2)\right\}$

$= (\xi+\eta)^{-3}(\xi-\eta)\left\{2(\xi^3+\eta^3)(\xi^2+\eta^2)+6\xi\eta(\xi+\eta)(\xi^2+\eta^2)\ -3\xi\eta(\xi+\eta)(\xi^2+\eta^2)+2\xi\eta(\xi^3+\eta^3)\right\}$

$= (\xi+\eta)^{-3}(\xi-\eta)\left[2(\xi^2+\eta^2)\left\{\xi^3+\eta^3+3\xi\eta(\xi+\eta)\right\}\ -\xi\eta\left\{3(\xi+\eta)(\xi^2+\eta^2)-2(\xi^3+\eta^3)\right\}\right]$

$= (\xi+\eta)^{-3}(\xi-\eta)\left\{2(\xi^2+\eta^2)(\xi+\eta)^3-\xi\eta(\xi+\eta)^3\right\} = (\xi-\eta)(2\xi^2+2\eta^2-\xi\eta)$

or $$[z]_P = 2\xi^3+3\xi\eta^2-3\xi^2\eta-2\eta^3. \qquad ....(18)$$

Replacing ξ and η by x and y respectively in (18), the value of z (*i.e.*, solution of the given equation) at any point (x, y) is given by

$$z = 2x^3 + 3xy^2 - 3x^2y - 2y^3.$$

Ex.4. *Obtain the solution of* $\partial^2 z/\partial x\partial y = 1/(x+y)$ *such that* $z = 0$, $p = 2y/(x+y)$ *on* $y = x$.

[Delhi Maths (H) 1998]

Sol. Here we are solve $$\partial^2 z/\partial x\partial y = 1/(x+y), \qquad ... (1)$$

where $z = 0$ and $p = \partial z/\partial x = 2y/(x+y)$ on $y = x$ ... (2)

Here the given curve C is straight line $y = x$. Then we wish to find the solution of (1) at $P(\xi, \eta)$ agreeing with boundary conditions (2). Through P we draw PA parallel to the x-axis and cutting $y = x$ at the point A and PB parallel to the y-axis and cutting $y = x$ in B. The triangular-region enclosed by stright lines $y = x$, $y = \eta$, $x = \xi$ is denoted by S (draw figure as shown in figure 3 of solved Ex. 3). Then we know that (refer particular case II of Art. 8.13).

$$[z]_P = [z]_A + \int_{AB}\frac{\partial z}{\partial x}dx + \iint_S f(x,y)\ dx\ dy \qquad ... (3)$$

Comparing (1) with $\partial^2 z/\partial x\ \partial y = f(x,y)$, here $f(x,y) = 1/(x+y)$.

Since A lies on given curve AB and it is given that $z = 0$ on AB, hence $[z]_A = 0$.

From (2), $\partial z/\partial x = 2y/(x+y)$ on AB. *i.e.* $y = x$

so that $\partial z/\partial x = 2x/(x+x) = 1$ on $y = x$.

Using the above facts, (3) reduces to

$$[z]_P = \int_\eta^\xi dx + \int_{x=\eta}^{\xi}\left\{\int_{y=\eta}^{x}\frac{1}{x+y}dy\right\}dx = \xi-\eta+\int_\eta^\xi\left[\log(x+y)\right]_{y=\eta}^{x}dx = \xi-\eta+\int_\eta^\xi[\log(2x)-\log(x+\eta)]\,dx$$

$$= \xi-\eta+\left[x\log\frac{2x}{x+\eta}\right]_\eta^\xi - \int_\eta^\xi\left(\frac{1}{x}-\frac{1}{x+\eta}\right)x\ dx = \xi-\eta+\xi\log\frac{2\xi}{\xi+\eta}-\eta\log 1-\int_\eta^\xi\frac{\eta}{x+\eta}dx$$

$$= \xi-\eta+\xi\log\frac{2\xi}{\xi+\eta}-\eta\left[\log\,(x+\eta)\right]_\eta^\xi = \xi-\eta+\xi\log\frac{2\xi}{\xi+\eta}+\eta\log\frac{2\eta}{\xi+\eta}$$

Replacing ξ and η by x and y respectively in the above equation, the value of z (*i.e.,* solution of (1) at any point (x, y) is given by

$$z = x-y+x\log\ \{2x/(x+y)\}+y\log\{2y/(x+y)\}.$$

8.15. Riemann-Volterra method for solving the Cauchy problem for the one-dimensional wave equation

The entire procedure of solution will become clear from the following solved examples.

Ex. 1. *Using Riemann-Volterra method, solve* $\partial^2 z/\partial x^2 = \partial^2 z/\partial y^2$, *when* z, $\partial z/\partial x$, $\partial z/\partial y$ *are prescribed along a curve C in the xy-plane*

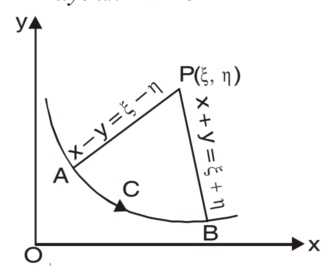
Sol. Given $\qquad \partial^2 z/\partial x^2 - \partial^2 z/\partial y^2 = 0 \qquad$ or $\qquad r - t = 0 \qquad$ ... (1)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, p) = 0$, here R = 1, S = 0 and T = – 1. Hence the λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $\lambda^2 - 1 = 0$ so that $\lambda = 1, -1$. The corresponding characteristic equations of (1) are given by

$$dy/dx + 1 = 0 \qquad \text{and} \qquad dy/dx - 1 = 0$$

Integrating these,

$$x + y = C_1 \qquad \text{and} \qquad x - y = C_2, \qquad \text{... (2)}$$



which are characteristics of (1) and these are two families of straight lines. Let $P(\xi, \eta)$ be any point in xy-plane. We now obtain characteristics of (1) passing through P. So putting $x = \xi$ and $y = \eta$ in (2), we have $C_1 = \xi + \eta$ and $C_2 = \xi - \eta$. Hence the characteristics of (1) passing through P are given by

$$x + y = \xi + \eta \qquad \text{and} \qquad x - y = \xi - \eta, \qquad \text{... (3)}$$

which have been shown by straight lines *PB* and *PA* respectively in the figure. Let the characteristics *PA* and *PB* cut the given curve *C* in *A* and *B* respectively. Let C' denote the closed curve *PABP* (which is made up of straight line *PA*, curve *C* (*i.e. AB*) and straight line *BP*). Let *S* be the region enclosed by C'.

Integrating both sides of (1) over S, we have

$$\iint_S \left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2}\right) dx\,dy = 0 \qquad \text{or} \qquad \iint_S \left\{\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\right\} dx\,dy = 0$$

or $$\oint_{C'} \left(\frac{\partial z}{\partial y} dx + \frac{\partial z}{\partial x} dy\right) = 0, \text{ by Green's theorem.*}$$

or $$\int_C \left(\frac{\partial z}{\partial y} dx + \frac{\partial z}{\partial x} dy\right) + \int_{BP} \left(\frac{\partial z}{\partial y} dx + \frac{\partial z}{\partial x} dy\right) + \int_{PA} \left(\frac{\partial z}{\partial y} dx + \frac{\partial z}{\partial x} dy\right) = 0$$

Equation of *BP* is $x + y = \xi + \eta$ and hence $dx = -\,dy$ on *BP*. Similarly, equation of *PA* is $x - y = \xi - \eta$ and hence $dx = dy$ on *PA*. Using these facts in the above equation, we get

$$\int_C \left(\frac{\partial z}{\partial y}\, dx + \frac{\partial z}{\partial x}\, dy\right) + \int_B^P \left(-\frac{\partial z}{\partial y} dy - \frac{\partial z}{\partial x} dx\right) + \int_P^A \left(\frac{\partial z}{\partial y} dy + \frac{\partial z}{\partial x} dx\right) = 0$$

or $$\int_C \left(\frac{\partial z}{\partial y} dx + \frac{\partial z}{\partial x} dy\right) - \int_B^P dz + \int_P^A dz = 0, \text{ as } dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

or $$\int_C \left(\frac{\partial z}{\partial y} dx + \frac{\partial z}{\partial x} dy\right) - (z_P - z_B) + (z_A - z_P) = 0$$

---

* **Green's theorem.** Let C' be a closed curve bounding the region S on *xy*-plane and $u(x, y)$, $v(x, y)$ be differential functions in *S* and continusou on C', then $\oint_{C'} (u\,dx + v\,dy) = \iint_S \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$

$$\therefore \qquad z_P = \frac{1}{2}(z_A + z_B) + \frac{1}{2}\int_C \left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right), \qquad \ldots (4)$$

which is the required solution of (1) at any point P.

Ex. 2. *Solve one dimensional wave equation by Riemann Volterra method.*

[Kurukshetra 2001; Delhi Maths (H) 1996]

or *Solve homogeneous one-dimensional wave equation* $\partial^2 z/\partial x^2 = (1/c^2)\times(\partial^2 z/\partial t^2)$, *when* $z, \partial z/\partial x, \partial z/\partial t$ *are prescribed along a curve C.*

Sol. Given $\qquad \partial^2 z/\partial x^2 = (1/c^2)\times(\partial^2 z/\partial t^2) \qquad \ldots (i)$

Let y be a new variable such that $\qquad y = ct \qquad \ldots (ii)$

Then (*i*) becomes $\qquad \partial^2 z/\partial x^2 - \partial^2 z/\partial y^2 = 0 \qquad$ or $\qquad r - t = 0 \qquad \ldots (iii)$

for which z, $\partial z/\partial x$ and $\partial z/\partial y$ are now prescribed along *C*.

Proceed with (*iii*) as we did in solved Ex. 1 upto equation (4).

Ex. 3. *Find z (x, y) such that* $\partial^2 z/\partial x^2 = \partial^2 z/\partial y^2$ *and* $z = f(x)$ *and* $\partial z/\partial y = g(x)$ *on y = 0.*

[Kanpur 2003; Delhi Maths (H) 2000, 08]

Sol. Given $\qquad \partial^2 z/\partial x^2 - \partial^2 z/\partial y^2 = 0 \qquad$ or $\qquad r - t = 0, \qquad \ldots (1)$

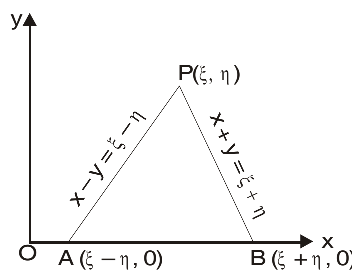
where $\quad z(x, 0) = f(x), \quad$ *i.e.,* $\quad z = f(x) \quad$ on $\quad y = 0 \quad$ *i.e.* x – axis ... (2)

and $\quad (\partial z/\partial y)_{y=0} = g(x) \quad$ *i.e.,* $\quad \partial z/\partial y = g(x) \quad$ on $\quad y = 0 \quad$ *i.e.* x – axis ...(3)

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = 1$, $S = 0$ and $T = -1$. Hence the λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $\lambda^2 - 1 = 0$ so that $\lambda = 1, -1$. The corresponding characteristic equations of (1) are given by $\qquad dy/dx + 1 = 0 \qquad$ and $\qquad dy/dx - 1 = 0$

Integrating these $\quad x + y = C_1 \quad$ and $\quad x - y = C_2 \quad \ldots (4)$

which are the characteristics of (1) and these are two families of straight lines. Let $P(\xi, \eta)$ be any point in *xy*-plane. We now obtain characteristics of (1) passing through *P*. So putting $x = \xi$ and $y = \eta$ in (4), we get $C_1 = \xi + \eta$ and $C_2 = \xi - \eta$. Hence the characteristics of (1) passing through *P* are given by



$$x + y = \xi + \eta \qquad \text{and} \qquad x - y = \xi - \eta \qquad \ldots (5)$$

which have been shown by straight lines *PB* and *PA* respectively in the figure. Let the characteristics *PA* and *PB* cut given curve (here $y = 0$ *i.e.* x-axis) in $A(\xi - \eta, 0)$ and $B(\xi + \eta, 0)$ respectively. Let C' denote the closed curve *PA BP* (which is made up of straight lines *PA*, *AB* and *BP*). Let *S* be the region enclosed by C'.

Integrating both sides of (1) over *S*, we have

$$\iint_S \left(\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2}\right) dxdy = 0 \qquad \text{or} \qquad \iint_S \left\{\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\right\} dx\,dy = 0$$

or $$\oint_{C'} \left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) = 0, \text{ by Green's theorem}$$

or $$\int_{AB}\left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) + \int_{BP}\left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) + \int_{PA}\left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) = 0$$

On *AB* (*i.e.,* *x*-axis), $y = 0$ so that $dy = 0$. Also, from (3), $\partial z/\partial y = g(x)$ on $y = 0$. On *BP* (*i.e.,* $x + y = \xi + \eta$), $dx = -dy$. Similarly, on *PA* (*i.e.,* $x - y = \xi - \eta$), $dx = dy$. Using these facts, the above equation reduces to

$$\int_A^B g(x)dx + \int_B^P \left(-\frac{\partial z}{\partial y}dy - \frac{\partial z}{\partial x}dx\right) + \int_P^A \left(\frac{\partial z}{\partial y}dy + \frac{\partial z}{\partial x}dx\right) = 0$$

or $$\int_A^B g(x)\, dx - \int_B^P dz + \int_P^A dz = 0, \quad \text{as} \quad \frac{\partial z}{\partial y}dy + \frac{\partial z}{\partial x}dx = dz$$

or $$\int_A^B g(x) - (z_P - z_B) + z_A - z_P = 0 \qquad \text{or} \qquad z_P = \frac{1}{2}(z_A + z_B) + \frac{1}{2}\int_A^B g(x)\,dx \qquad \text{... (6)}$$

From (2), $z = f(x)$ on $y = 0$ (*i.e.,* *x*-axis). Since *x*-coordinates of *A* and *B* are $\xi - \eta$ and $\xi + \eta$ respectively, it follows that $z_A = f(\xi - \eta)$ and $z_B = f(\xi + \eta)$. Hence (6) reduces to

$$z_P = \frac{1}{2}\{f(\xi - \eta) + f(\xi + \eta)\} + \frac{1}{2}\int_{\xi-\eta}^{\xi+\eta} g(x)\,dx \qquad \text{... (7)}$$

Replacing ξ and η by *x* and *y* respectively in (7), the value of *z* (*i.e.,* solution of (1) at any point *P* (*x*, *y*)) is given by

$$z(x, y) = \frac{1}{2}\{f(x - y) + f(x + y)\} + \frac{1}{2}\int_{x-y}^{x+y} g(u)\,du$$

Ex. 4. *Find the solution of one-dimensional non-homogeneous wave equation* $\partial^2 z/\partial x^2 - \partial^2 z/\partial y^2 + f(x, y) = 0$ *by Riemann-Vatterra method.*

[A.M.I.E. 2005; Delhi Maths (H) 1998, 2002; Kanpur 1998]

Sol. Given $\quad \partial^2 z/\partial x^2 - \partial^2 z/\partial y^2 + f(x, y) = 0 \quad$ or $\quad r - t + f(x, y) = 0$. ... (1)

Suppose that *z*, $\partial z/\partial x$ and $\partial z/\partial y$ are prescribed along a given curve *C*. Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, here $R = 1$, $S = 0$, $T = -1$ and so λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $\lambda^2 - 1 = 0$ giving $\lambda = 1, -1$. The corresponding characteristic equations $dy/dx + 1 = 0$ and $dy/dx - 1 = 0$ give, on integration,

$$x + y = C_1 \qquad \text{and} \qquad x - y = C_2 \qquad \text{... (2)}$$

which are characteristics of (1). Draw figure as in solved Ex. 1. Let $P(\xi, \eta)$ be any point in *xy*-plane. Then characteristics of (1) passing through $P(\xi, \eta)$ are given by

$$x + y = \xi + \eta \qquad \text{and} \qquad x - y = \xi - \eta \qquad \text{... (3)}$$

which have been shown by straight lines *PB* and *PA* respectively. Let the characteristics *PA* and *PB* cut the given curve *C* in *A* and *B* respectively. Let C' denote the closed curve *PABP* and let *S* denote the region enclosed by C'.

Integrating both sides of (1) over *S*, we have

$$\iint_S \left\{\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\right\} dx\,dy + \iint_S f(x, y)\,dx\,dy = 0$$

or $$\oint_{C'}\left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) + \iint_S f(x, y)\,dx\,dy = 0, \text{ by Green's theorem}$$

or $$\int_C \left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) + \int_{BP}\left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) + \int_{PA}\left(\frac{\partial z}{\partial y}dx + \frac{\partial z}{\partial x}dy\right) + \iint_S f(x, y)\,dxdy = 0$$

Equation of BP is $x+y=\xi+\eta$ and so $dx = -dy$ on BP. Similarly, equation of PA is $x-y = \xi-\eta$ and so $dx = dy$ on PA. Using these facts, the above equation reduces to

$$\int_C \left(\frac{\partial z}{\partial y}dx+\frac{\partial z}{\partial x}dy\right)+\int_B^P\left(-\frac{\partial z}{\partial y}dy-\frac{\partial z}{\partial x}dx\right)+\int_P^A\left(\frac{\partial z}{\partial y}dy+\frac{\partial z}{\partial x}dx\right)+\iint_S f(x,y)\,dx\,dy=0$$

or $$\int_C \left(\frac{\partial z}{\partial y}dx+\frac{\partial z}{\partial x}dy\right)-\int_B^P dz+\int_P^A dz+\iint_S f(x,y)\,dx\,dy=0$$

or $$\int_C \left(\frac{\partial z}{\partial y}dx+\frac{\partial z}{\partial x}dy\right)-(z_P-z_B)+z_A-z_P+\iint_S f(x,y)\,dx\,dy=0$$

or $$z_P=\frac{1}{2}(z_A+z_B)+\frac{1}{2}\int_C\left(\frac{\partial z}{\partial y}dx+\frac{\partial z}{\partial x}dy\right)+\frac{1}{2}\iint_S f(x,y)\,dx\,dy=0,$$

which is the required solution of (1) at any point P.

Ex. 5. *Solve* $\partial^2 z/\partial x^2-\partial^2 z/\partial y^2=1$, *when* $z(x, 0) = \sin x$, $z_y(x, 0) = x$.

Sol. Given $\partial^2 z/\partial x^2-\partial^2 z/\partial y^2-1=0$ or $r-t-1=0$ ... (1)

where $z(x,0)=\sin x$, *i.e.*, $z=\sin x$ on $y=0$, *i.e.*, x-axis ... (2)

and $z_y(x, 0) = x$, *i.e.*, $\partial z/\partial y=x$ on $y=0$, *i.e.*, x-axis ... (3)

Comparing (1) with $Rr+Ss+Tt+f(x,y,z,p,q)=0$, here $R = 1$, $S = 0$ and $T = -1$. Hence the λ-quadratic equation $R\lambda^2+S\lambda+T=0$ reduces to $\lambda^2-1=0$ so that $\lambda=1,-1$. The corresponding characteristic equations of (1) are

$dy/dx + 1 = 0$ and $dy/dx - 1 = 0$

Integrating these, $x + y = C_1$ and $x - y = C_2$ ... (4)

which are the characteristics of (1) and these are two families of straight lines Draw a figure as in solved Ex. 3. Let $P(\xi,\eta)$ be any point in xy-plane. Putting $x = \xi$, $y = \eta$ in (4), we get $C_1=\xi+\eta$, $C_2=\xi-\eta$. Hence the characteristics of (1) passing through P are given by

$x+y=\xi+\eta$ and $x-y=\xi-\eta$ ... (5)

which have been shown by straight lines PB and PA respectively in the figure. Let the characteristics PA and PB cut the given line $y = 0$ *i.e.*, x-axis in $A(\xi-\eta, 0)$ and $B(\xi+\eta, 0)$ respectively. Let C' denote the closed curve $PABP$ and let S denote the region enclosed by C'.

Integrating both sides of (1) over S, we have

$$\iint_S\left\{\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)-\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)\right\}-\iint_S dx\,dy=0$$

or $$\oint_{C'}\left(\frac{\partial z}{\partial y}dx+\frac{\partial z}{\partial x}dy\right)-\iint_S dx\,dy=0, \text{ by Green's theorem} \quad \text{... (5)}$$

Now, $\iint_S dx\,dy$ = area of the triangle SAB = (1/2) × AB × perpendicular distance of P from AB

= (1/2) × $\{\xi+\eta-(\xi-\eta)\}\times\eta=\eta^2$

Hence (5) reduces to

$$\int_{AB}\left(\frac{\partial z}{\partial y}dx+\frac{\partial z}{\partial x}dy\right)+\int_{BP}\left(\frac{\partial z}{\partial y}dx+\frac{\partial z}{\partial x}dy\right)+\int_{PA}\left(\frac{\partial z}{\partial y}dx+\frac{\partial z}{\partial x}dy\right)-\eta^2=0 \quad \text{... (6)}$$

On *AB* (*i.e., x*-axis), $y = 0$ so that $dy = 0$. Also from (3), $\partial z/\partial y = x$ on $y = 0$. On *BP* (*i.e.,* $x + y = \xi+\eta$), $dx = -dy$. Similarly on *PA* (*i.e.,* $x - y = \xi-\eta$), $dx = dy$. Using these facts, (6) reduces to

$$\int_A^B x\,dx + \int_B^P\left(-\frac{\partial z}{\partial y}dy - \frac{\partial z}{\partial x}dx\right) + \int_P^A\left(\frac{\partial z}{\partial y}dy + \frac{\partial z}{\partial x}dx\right) - \eta^2 = 0$$

or $$\int_A^B x\,dx - \int_B^P dz + \int_P^A dz - \eta^2 = 0, \text{ as } dz = \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

or $$\int_A^B x\,dx - (z_P - z_B) + z_A - z_P - \eta^2 = 0 \quad \text{or} \quad z_P = \frac{1}{2}(z_Z + z_B) + \frac{1}{2}\int_A^B x\,dx - \frac{1}{2}\eta^2 \qquad \text{... (7)}$$

From (2), $z = \sin x$ on $y = 0$ (*i.e., x*-axis). Since *x*-coordinates of *A* and *B* are $\xi-\eta$ and $\xi+\eta$ respectively, it follows that $z_A = \sin(\xi-\eta)$ and $z_B = \sin(\xi+\eta)$. Hence (7) reduces to

$$z_P = \frac{1}{2}\{\sin(\xi-\eta) + \sin(\xi+\eta)\} + \frac{1}{2}\int_{\xi-\eta}^{\xi+\eta} x\,dx - \frac{\eta^2}{2} \quad \text{or} \quad z_P = \sin\xi\cos\eta - \frac{1}{4}\left[x^2\right]_{\xi-\eta}^{\xi+\eta} - \frac{\eta^2}{2}$$

or $$z_P = \sin\xi\cos\eta - (1/4)\times\{(\xi+\eta)^2 - (\xi-\eta)^2\} - (1/2)\times\eta^2$$

or $$z_P = \sin\xi\cos\eta - \xi\eta - (\eta^2/2) \qquad \text{... (8)}$$

Replacing ξ and η by *x* and *y* respectively in (8), the value of *z* (*i.e.,* solution of (1)) at any point (x, y) is given by $z(x, y) = \sin x \cos y - xy - (y^2/2)$.

Ex. 6. *A function z (x, y) satisfies the non-homogeneous equation* $\partial^2 z/\partial x^2 - \partial^2 z/\partial y^2 + f(x,y) = 0$ *and the initial conditions* $z = \partial z/\partial y = 0$ *when* $y = 0$. *Show that (using Riemann - Volterra method)*

$$z(x, y) = \frac{1}{2}\iint_\Gamma f(u,v)\,du\,dv,$$

where Γ is the triangle cut off from the upper half of uv-plane by two characteristics through the point (x, y). **[Delhi Maths (Hons) 2002, 07, 11; A.M.I.E. 2005; Amaravati 2003; Kanpur 1999; Rohilkhand 2004]**

Sol. Given $\partial^2 z/\partial x^2 - \partial^2 z/\partial y^2 + f(x,y) = 0$ or $r - t + f(x,y) = 0$, ... (1)

where $z(x, 0) = 0$, *i.e.,* $z = 0$ on $y = 0$ (*x*-axis) ... (2)

and $(\partial z/\partial y)_{y=0} = 0$, *i.e.,* $(\partial z/\partial y) = 0$ on $y = 0$ (*x*-axis) ... (3)

Comparing (1) with $Rr + Ss + Tt + f(x,y,z,p,q) = 0$, here $R = 1$, $S = 0$ and $T = -1$ and so λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $\lambda^2 - 1 = 0$ giving $\lambda = 1, -1$. The corresponding characteristic equations are given by

$$dy/dx + 1 = 0 \qquad \text{and} \qquad dy/dx - 1 = 0$$

Integrating these, $x + y = c_1$ and $x - y = c_2$, ... (4)

which are characteristics of (1). Draw figure as in solved Ex. 3. Let $P(\xi, \eta)$ be any point in *xy*-plane. We now obtain characteristics of (1) passing through *P*. So putting $x = \xi$ and $y = \eta$ in (4), we get $C_1 = \xi+\eta$ and $C_2 = \xi-\eta$. Hence the characteristics of (1) passing through *P* are given by

$$x + y = \xi+\eta \qquad \text{and} \qquad x - y = \xi-\eta \qquad \text{... (5)}$$

which have been shown by straight lines *PB* and *PA* respectively in the figure. Let the characteristics *PA* and *PB* cut $y = 0$ (*i.e., x*-axis) at *A* and *B* respectively. Let C' denote the closed curve *PA BP* and let Γ denote the triangular region enclosed by C'.

Integrating both sides of (1) over Γ, we have

$$\iint_\Gamma \left(\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) \right) dx\, dy + \iint_\Gamma f(x,y)\, dx\, dy = 0$$

or $$\oint_{C'} \left(\frac{\partial z}{\partial y} dx + \frac{\partial z}{\partial x} dy \right) + \iint_\Gamma f(x,y)\, dx\, dy = 0 \text{, using Green's theorem}$$

or $$\int_{AB} \left(\frac{\partial z}{\partial y} dx + \frac{\partial z}{\partial x} dy \right) + \int_{BP} \left(\frac{\partial z}{\partial y} dx + \frac{\partial z}{\partial x} dy \right) + \int_{PA} \left(\frac{\partial z}{\partial y} dx + \frac{\partial z}{\partial x} dy \right) + \iint_\Gamma (x,y)\, dx\, dy = 0 \quad \text{... (6)}$$

On AB (*i.e.*, x-axis), $y = 0$ so that $dy = 0$. Also, from (3), $\partial z / \partial y = 0$ on $y = 0$. On BP (*i.e.*, $x + y = \xi + \eta$), $dx = -dy$. Similarly, on PA (*i.e.*, $x - y = \xi - \eta$), $dx = dy$. Using these facts, (6) reduces to

$$\int_B^P \left(-\frac{\partial z}{\partial y} dy - \frac{\partial z}{\partial x} dx \right) + \int_P^A \left(\frac{\partial z}{\partial y} dy + \frac{\partial z}{\partial x} dx \right) + \iint_\Gamma (x,y)\, dxdy = 0$$

or $$-\int_B^P dz + \int_P^A dz + \iint_\Gamma f(x,y) dxdy = 0, \text{ as } dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

or $$-(z_P - z_B) + z_A - z_P + \iint_\Gamma f(x,y)\, dx\, dy = 0 \quad \text{... (7)}$$

From (2), $z = 0$ on $y = 0$ (*i.e.*, x-axis) and so $z_A = z_B = 0$, because A and B both lie on $y = 0$. Hence (7) becomes

$$2z_P = \iint_\Gamma f(x,y)\, dxdy \qquad \text{or} \qquad z(x, y) = \frac{1}{2} \iint_\Gamma f(u,v)\, du\, dv,$$

which gives the value of z (*i.e.*, solution of (1)) at any point (x, y)

Ex.7. *Find the solution of the non-homogeneous wave equation* $\partial^2 z/\partial x^2 - (1/c^2)(\partial^2 z/\partial t^2) + f(x,t) = 0$ *with initial conditions* $z(x, 0) = f(x)$, $z_t(x, t) = g(x)$.

Sol. Let y be a new variable such that $y = c\,t$ ... (1)

Then the given problem may be re-written as

$$\partial^2 z / \partial x^2 - \partial^2 z / \partial y^2 + F(x,y) = 0 \qquad \text{or} \qquad r - t + F(x,y) = 0, \quad \text{... (2)}$$

with the modified initial conditions given below

$z(x, 0) = f(x)$, *i.e.*, $z = f(x)$ on $y = 0$, *i.e.*, x-axis ... (3)

$(\partial z / \partial y)_{y=0} = G(x)$, *i.e.*, $\partial z / \partial y = G(x)$ on $y = 0$, *i.e.*, x-axis ... (4)

Here $F(x, y) = f(x, t)$ and $G(x) = (1/c) \times g(x)$ ... (5)

Comparing (2) with $Rr + Ss + Tt + f(x,y,z,p,q) = 0$, here $R = 1$, $S = 0$ and $T = -1$. Hence the λ-quadratic equation $R\lambda^2 + S\lambda + T = 0$ reduces to $\lambda^2 - 1 = 0$ so that $\lambda = 1, -1$. The corresponding characteristic equations of (1) are given by

$$dy/dx + 1 = 0 \qquad \text{and} \qquad dy/dx - 1 = 0 \quad \text{... (6)}$$

Integrating these, $x + y = C_1$ and $x - y = C_2$,

which are characteristics of (1). Draw a figure same as in Ex. 3.

Let $P(\xi, \eta)$ be any point in xy-plane. Then characteristics (2) passing through $P(\xi, \eta)$ are

$$x + y = \xi + \eta \qquad \text{and} \qquad x - y = \xi - \eta \quad \text{... (7)}$$

which have been shown by straight lines PB and PA respectively in the figure. Here lines given by (7) cut x-axis (*i.e.*, $y = 0$) in $A(\xi - \eta, 0)$ and $B(\xi + \eta, 0)$ respectively. Let C' denote the closed curve $PABP$ and let S be the region enclosed by C'.

Integrating both sides of (2) over S, we have

$$\iint_S \left\{ \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) \right\} \, dx\,dy + \iint_S F(x, y)\, dx\,dy = 0$$

or $$\oint_{C'} \left(\frac{\partial z}{\partial y} dx + \frac{\partial z}{\partial x} dy \right) + \iint_S F(x, y)\, dx\,dy = 0, \text{ using Green's theorem}$$

or $$\int_{AB} \left(\frac{\partial z}{\partial y} dx + \frac{\partial z}{\partial x} dy \right) + \int_{BP} \left(\frac{\partial z}{\partial y} dx + \frac{\partial z}{\partial x} dy \right) + \int_{PA} \left(\frac{\partial z}{\partial y} dx + \frac{\partial z}{\partial x} dy \right) + \iint_S F(x, y)\, dx\,dy = 0 \quad \ldots (8)$$

On AB (*i.e.,* x-axis), $y = 0$ so that $dy = 0$. Also, from (4), $\partial z / \partial y = G(x)$ on $y = 0$. On BP (*i.e.,* $x + y = \xi + \eta$), $dx = -$ dy. Similarly, on PA (*i.e.,* $x - y = \xi - \eta$), $dx = dy$. Using these facts, (8) reduces to

$$\int_A^B G(x)\, dx + \int_B^P \left(-\frac{\partial z}{\partial y} dy - \frac{\partial z}{\partial x} dx \right) + \int_P^A \left(\frac{\partial z}{\partial y} dy + \frac{\partial z}{\partial x} dx \right) + \iint_S F(x, y)\, dx\,dy = 0$$

or $$\int_A^B G(x) dx - \int_B^P dz + \int_P^A dz + \iint_S F(x, y)\, dx\,dy = 0$$

or $$\int_A^B G(x)\, dx - (z_P - z_B) + z_A - z_P + \iint_S F(x, y)\, dx\,dy = 0$$

or $$z_P = \frac{1}{2}(z_A + z_B) + \frac{1}{2}\int_A^B G(x) \quad dx + \frac{1}{2} \iint_S F(x, y)\, dx\,dy \quad \ldots (9)$$

From (3), $z = f(x)$ on $y = 0$ (*i.e.,* x-axis). Since x-coordinates of A and B are $\xi - \eta$ and $\xi + \eta$ respectively, it follows that $z_A = f(\xi - \eta)$ and $z_B = f(\xi + \eta)$. Hence (9) reduces to

$$z_p = \frac{1}{2}\{f(\xi - \eta) + f(\xi + \eta)\} + \frac{1}{2}\int_{\xi - \eta}^{\xi + \eta} G(x)\, dx + \frac{1}{2}\iint_s F(x, y)\, dx\,dy \quad \ldots (10)$$

Replacing ξ and η by x and y ($= ct$) respectively and using (5), (10) reduces to

$$z(x, y) = \frac{1}{2}\{f(x - y) + f(x + y)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(u) du + \frac{1}{2}\iint_s f(x, t) dx\, dt$$

Miscellaneous Problems on chapter 8

Ex. 1. $\partial^2 u / \partial t^2 = c^2 (\partial^2 u / \partial x^2)$ is hyperbolic or parabolic. Classify it. **[Agra 2008]**

Hint. See Art 8.1 **Ans.** Hyperbolic

Ex. 2. The equation $\partial^2 u / \partial t^2 = \partial^2 u / \partial x^2$ is

(a) parabolic (b) hyperbolic (c) elliptic (d) Nonw of these **[Agra 2007]**

Sol. Ans. (b.) See Art 8.1.

Ex. 3. Classify and solve the following equation $\partial^2 z / \partial x^2 = x^2 (\partial^2 z / \partial y^2)$. **[Bhopal 2010]**

9

Monge's Methods

9.1 INTRODUCTION

The most general form of partial differential equation of order two is

$$f(x, y, z, p, q, r, s, t) = 0. \qquad ...(1)$$

It is only in special cases that (1) can be integrated. Some well known methods of solutions were given by Monge. His methods are applicable to a wide class (but not all) of equations of the form (1). Monge's methods consists in finding one or two first integrals of the form

$$u = \phi(v), \qquad ...(2)$$

where u and v are known functions of x, y, z, p and q and ϕ is an arbitrary function. In other words, Monge's methods consists in obtaining relations of the form (2) such that equation (1) can be derived from (2) by eliminating the arbitrary function. A relation of the form (2) is known as an *intermediate integral* of (1). Every equation of the form (1) need not possess an intermediate integral. However, it has been shown that most general partial differential equations having (2) as an intermediate integral are of the following forms

$$Rr + Ss + Tt = V \quad \text{and} \quad Rr + Ss + Tt + U(rt - s^2) = V, \qquad ...(3)$$

where R, S, T, U and V are functions of x, y, z, p and q. Even equations (3) need not always possess an intermediate integral. In what follows we shall assume that an intermediate integral of (3) exists.

9.2. MONGE'S METHOD OF INTERGRATING Rr + Ss + Tt = V. [Agra 2005; Delhi Maths (Hons) 2000, 02, 08, 09, 11; Garhwal 1994; Patna 2003; Kanpur 1997; Meerut 2000]

Given $$Rr + Ss + Tt = V, \qquad ...(1)$$

where R, S, T and V are functions of x, y, z, p and q.

We know that $\quad p = \partial z/\partial x, \quad q = \partial z/\partial y,$

$$\left.\begin{aligned} r &= \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial p}{\partial x}, & t &= \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial q}{\partial y}, \\ s &= \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial q}{\partial x} \quad \text{and} & s &= \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial p}{\partial y} \end{aligned}\right\} \quad ...(2)$$

Now, $$dp = (\partial p/\partial x)\, dx + (\partial p/\partial y) dy = rdx + sdy, \text{ using (2)} \qquad ...(3)$$

and $$dq = (\partial q/\partial x) dx + (\partial q/\partial y) dy = sdx + tdy, \text{ using (2)} \qquad ...(4)$$

From (3) and (4), $\quad r = (dp - s\, dy)/dx \quad$ and $\quad t = (dq - s\, dx)/dy \qquad ...(5)$

Substituting the values of r and s given by (5) in (1), we get

$$R\left(\frac{dp - sdy}{dx}\right) + Ss + T\left(\frac{dq - sdx}{dy}\right) = V \quad \text{or} \quad R(dp - sdy)dy + Ss\, dxdy + T(dq - sdx)dx = V\, dxdy$$

or $$(Rdpdy + Tdqdx - Vdxdy) - s\{R(dy)^2 - Sdxdy + T(dx)^2\} = 0. \qquad ...(6)$$

Clearly any relation between x, y, z, p and q which satisfies (6) must also satisfy the following two simultaneous equations

$$Rdpdy + Tdq\,dx - Vdxdy = 0. \qquad ...(7)$$

and
$$(dy)^2 - Sdxdy + T(dx)^2 = 0. \qquad ...(8)$$

The equations (7) and (8) are called *Monge's subsidiary equations* and the relations which satisfy these equations are called *intermediate integrals*.

Equation (8) being a quadratic, in general, it can be resolved into two equations, say

$$dy - m_1\,dx = 0 \qquad ...(9)$$

and
$$dy - m_2\,dx = 0. \qquad ...(10)$$

Now the following two cases arise :

Case I. When m_1 and m_2 are distinct in (9) and (10).

In this case (7) and (9), if necessary by using well known result $dz = pdx + qdy$, will give two integrals $u_1 = a$ and $v_1 = b$, where a and b are arbitrary constants. These give

$$u_1 = f_1(v_1), \qquad ...(11).$$

where f_1 is an arbitrary function. It is called an *intermediate integral* of (1).

Next, taking (7) and (10) as before, we get another intermediate integral of (1), say

$u_2 = f_2(v_2)$, where f_2 is an arbitrary function. ...(12)

Thus we have in this case two distinct intermediate integrals (11) and (12). Solving (11) and (12), we obtain values of p and q in terms of x, y and z. Now substituting these values of p and q in well known relation $dz = pdx + qdy$ ...(13)

and then integrating (13), we get the required complete integral of (1).

Case II . When $m_1 = m_2$ i.e., (8) is a perfect square.

As before, in this we get only one intermediate integral which is in Lagrange's form

$$Pp + Qq = R. \qquad ...(14)$$

Solving (14) with help of Lagrange's method (refer Art. 2.3, chapter 2), we get the required complete integral of (1).

Remark 1. Usually while dealing with case I, we obtain second intermediate integral directly by using symmetry. However sometimes in absence of any symmetry, we find the complete integral with help of only one indetermediate integral. This is done with help of using Lagrange's method.

Remark 2. While obtaining an intermediate integral, remember to use the relation $dx = pdx + qdy$ as explained below :

(*i*) $pdx + qdy + 2xdx = 0$ can be re–written as $dz + 2xdx = 0$ so that $z + x^2 = c$.

(*ii*) $xdp + ydq = dx$ can be re–written as $xdp + ydq + pdx + qdy = dx + pdx + qdy$

or $d(xp) + d(yq) = dx + dz$ so that $xp + yq = x + z + c$, on integration

Remark 3. While integrating, we shall use the following types of calculations. In what follows, f and g are arbitrary functions and k and a are a constants.

(*i*) $\int k\,f(t)\,dt = g(t)$ (*ii*) $\int k\frac{1}{t}f(t)\,dt = g(\mathrm{t})$. (*iii*) $\int k\frac{1}{t^2}f(t^2)\,d(t^2) = g(t^2)$

(*iv*) $\int k\,f(x+y)\,d(x+y) = g(x+y)$. (*v*) $\int k\,t^2 f\left(\frac{1}{t}\right)d\left(\frac{1}{t}\right) = \int \frac{k}{(1/t)^2} f\left(\frac{1}{t}\right)d\left(\frac{1}{t}\right) = g\left(\frac{1}{t}\right)$

(*vi*) $\int \frac{k}{t^2} f(at^2)\,d(t^2) = \int \frac{k}{(at^2)} f(at^2)\,d(at^2) = g(at^2)$

Proof of (vi). Putting $at^2 = u$, and $d(at^2) = du$ we have

$$\int \frac{k}{t^2} f(at^2)\,d(t^2) = \int \frac{k}{u} f(u)\,d(u) = g(u) = g(at^2), \text{ as } u = at^2.$$

Similarly, other results can be proved. In examination we shall not use substitution as explained above. With good practice, the students will be able to write direct results of integration very easily.

Important Note. For sake of convenience, we have divided all questions based on $Rr + Ss + Tt = V$ in four types. We shall now discuss them one by one.

9.3. Type 1. When the given equation Rr + Ss + Tt = V leads to two distict intermediate intergrals and both of them are used to get the desired solution.

Working rule for solving problems of type 1.

Step 1. Write the given equation in the standard form $Rr + Ss + Tt = V.$

Step 2. Substitute the values of R, S, T and V in the Monge's subsidiary equations:

$Rdpdy + Tdqdx - Vdxdy = 0$ ...(1) $\qquad R(dy)^2 - Sdxdy + T(dx)^2 = 0$ ... (2)

Step 3. Factorise (1) into two distinct factors.

Step 4. Using one of the factors obtained in (1), (2) will lead to an intermediate integral. In general, the second intermediate integral can be obtained from the first one by inspection, taking advantage of symmetry. In absence of any symmetry, the second factor obtained in step 3 is used in (2) to arrive at second intermediate integral. You should use remark 2 of Art. 9.2 while finding intermediate integrals.

Step 5. Solve the two intermediate integrals obtained in step 4 and get the values of p and q.

Step 6. Substitute the values of p and q in $dz = pdx + q\,dy$ and integrate to arrive at the required general solution. You should use remark 3 of Art. 9.2 while integrating $dz = pdx + qdy$.

9.4. SOLVED EXAMPLES BASED ON ART. 9.3.

Ex. 1. (a) *Solve* $r = a^2 t$. **[Agra 2008; Lucknow 2010; Patna 2003; Meerut 2008]**

(b) $r = t$. **[Agra 2006]**

(c) Solve one-dimensions wave equation by Monge's method: $\partial^2 y/dx^2 = a^2(\partial^2 y/\partial t^2)$. **[Meerut 2003]**

Sol. (*a*) Given equation is $\qquad r - a^2 t = 0.$

Comparing it with $Rr + Ss + Tt = V$, we have $\quad R = 1, \quad S = 0, \quad T = -a^2, \quad V = 0.$

Hence Monge's subsidiary equations

$Rdpdy + Tdq\,dx - Vdxdy = 0 \quad$ and $\quad R(dy)^2 - S\,dxdy + T\,(dx)^2 = 0$

become $\qquad dpdy - a^2 dqdx = 0 \qquad$ ...(1)

and $\qquad (dy)^2 - a^2(dx)^2 = 0. \qquad$ ...(2)

Equation (2) may be factorised as $\qquad (dy - adx)\,(dy + adx) = 0$

Hence two systems of equations to be considered are

$dpdy - a^2\,dqdx = 0, \qquad dy - adx = 0.$ ...(3)

and $\qquad dpdy - a^2\,dqdx = 0, \qquad dy + adx = 0.$ ...(4)

Integrating the second equation of (3), we get $\qquad y - ax = c_1.$ ...(5)

Eliminating dy/dx between the equations of (3), we get

$dp - adq = 0 \qquad$ so that $\qquad p - aq = c_2.$ ...(6)

Hence the intermediate integral corresponding to (3) is $\qquad p - aq = \phi_1(y - ax).$ ...(7)

Similarly another intgermediate integral corresponding to (4) is $\quad p + aq = \phi_2(y + ax).$...(8)

Here ϕ_1 and ϕ_2 are arbitrary functions.

Solving (7) and (8) for p and q, we have

$p = (1/2) \times \{\phi_2(y + ax) + \phi_1(y - ax)\} \quad$ and $\quad q = (1/2a) \times \{\phi_2(y + ax) - \phi_1(y - ax)\}.$

Substituting these values of p and q in $dz = pdx + qdy$, we get

$$dz = (1/2) \times \{\phi_2(y + ax) + \phi_1(y - ax)\}dx + (1/2a) \times \{\phi_2(y + ax) - \phi_1(y - ax)\}dy$$
$$= (1/2a) \times \phi_2(y + ax)\,(dy + adx) - (1/2a) \times \phi_1(y - ax)(dy - adx)$$

Integrating, $\qquad z = \psi_2(y + ax) + \psi_1(y - ax)$, ψ_1, ψ_2 being arbitrary functions.

(b) This is a particular case of part (a). Here $a = 1$. **Ans.** $z = \psi_2(y + x) + \psi_1(y - x)$.

(c) Refer part (a). Note that $\partial^2 y/\partial x^2 = r$ and $\partial^2 y/\partial t^2 = t$

Ex. 2. *Solve* $r + (a + b)s + abt = xy$. **[Vikram 2003]**

Sol. Comparing the given equation with $Rr + Ss + Tt = V$, we have

$R = 1$, $S = a + b$, $T = ab$, $V = xy$. The usual Monge's subsidiary equations

$Rdpdy + Tqdx - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$.

become $$dp\, dy + a\, b\, dq\, dx - xy\, dx\, dy = 0 \qquad \text{...(1)}$$

and $$(dy)^2 - (a + b)\, dxdy + ab\, (dx)^2 = 0. \qquad \text{...(2)}$$

Factorizing, (2) gives $(dy - bdx)\,(dy - adx) = 0$.

Hence two systems to be considered are

$$dp\, dy + ab\, dq\, dx - xy\, dx\, dy = 0, \qquad dy - b\, dx = 0. \qquad \text{...(3)}$$

and $$dp\, dy + ab\, dq\, dx - xy\, dx\, dy = 0, \qquad dy - a\, dx = 0. \qquad \text{...(4)}$$

Integrating the second equation of (3), $y - bx = c_1$. ...(5)

Eliminating dy/dx between the equations of (3), we get

$dp + a\, dq - xy\, dx = 0$ or $dp + a\, dq - x(c_1 + bx)\, dx = 0$, by (5) ...(6)

Integrating (6), $p + aq - (c_1/2)x^2 - (b/3)x^3 = c_2$ or $p + aq - (x^2/2)\,(y - bx) - (b/3)x^3 = c_2$, using (5)

or $$p + aq - (1/2) \times yx^2 + (1/6) \times bx^3 = c_2. \qquad \text{...(7)}$$

Using (5) and (7), the first intermediate integral corresponding to (3) is

$p + aq - (1/2) \times yx^2 + (1/6) \times bx^3 = \phi_1(y - bx)$, ϕ_1 being an arbitrary function ...(8)

Similarly, another intermediate integral corresponding to (4) is

$p + bq - (1/2) \times yx^2 + (1/6) \times ax^3 = \phi_2(y - ax)$, ϕ_2 being an arbitrary functionJ ...(9)

Solving (8) and (9) for p and q, we have

$$p = (1/2) \times x^2y - (1/6) \times (a + b)x^3 + (a - b)^{-1}\,[a\phi_2(y - ax) - b\phi_1(y - ax)]$$

and $$q = (1/6) \times x^3 + (a - b)^{-1}\,[\phi_1(y - bx) - \phi_2(y - ax)].$$

Substituting these values in $dz = pdx + qdy$, we get

$$dz = (1/2) \times x^2ydx - (1/6) \times (a + b)x^3dx + (a - b)^{-1}\,[\phi_2(y - bx)dx - \phi_1(y - ax)dx]$$
$$+ (1/6) \times x^3dy + (a - b)^{-1}\,[\phi_1(y - bx)dy - \phi_2(y - ax)dy]$$

or $$dz = (1/6) \times (3x^2ydx + x^3dy) - (1/6) \times (a + b)\, x^3dx - (b - a)^{-1}\,[\phi_2(y - bx)dx$$
$$- \phi_1(y - ax)dx] - (b - a)^{-1}\,[(\phi_1(y - bx)dy - \phi_2(y - ax)dy]$$

or $$dz = (1/6) \times d(x^3y) - (1/6) \times (a + b)\, x^3dx + (b - a)^{-1}\phi_2(y - ax)\,(dy - adx)$$
$$- (b - a)^{-1}\phi_1(y - bx)\,(dy - bdx)$$

or $$dz = (1/6) \times d(x^3y) - (1/6) \times (a + b)\, x^3dx + (b - a)^{-1}\,\phi_2(y - ax)\, d(y - ax)$$
$$- (b - a)^{-1}\,\phi_1(y - bx)\, d(y - bx)$$

Integrating, $$z = (1/6) \times x^3y - (1/24) \times (a + b)x^4 + \psi_2(y - ax) + \psi_1(y - bx),$$

where ψ_1 and ψ_2 are arbitrary functions.

Ex. 3. *Solve* $r - t\cos^2 x + p\tan x = 0$. **[K.U. Kurukshetra 2005; Meerut 1993]**

Sol. Given $$r - t\cos^2 x = -\, p\tan x \qquad \text{...(1)}$$

Comparing (1) with $Rr + Ss + Tt = V$, we find

$R = 1$, $S = 0$, $T = -\cos^2 x$ and $V = -\,p\tan x$. ...(2)

Monge's subsidiary equations are $$Rdp\, dy + Tdq\, dx - V\, dx\, dy = 0 \qquad \text{...(3)}$$

and $$R\,(dy)^2 - S\, dx\, dy + T\,(dx)^2 = 0 \qquad \text{...(4)}$$

Putting the values of R, S, T and V, (3) and (4) become

$$dp\, dy - \cos^2 x\, dq\, dx + p\tan x\, dx\, dy = 0 \qquad \text{...(5)}$$

and $$(dy)^2 - \cos^2 x\,(dx)^2 = 0 \qquad \text{...(6)}$$

Equation (6) may be factoriesed as $(dy - \cos x\, dx)\,(dy + \cos x\, dx) = 0$

$\therefore$ $$dy - \cos x\, dx = 0 \qquad \text{...(7)}$$

or $$dy + \cos x\, dx = 0 \qquad \text{...(8)}$$

Putting the value of dy from (7) in (5), we get

$dp \cos x\, dx - \cos^2 x\, dq\, dx + p \tan x\, dx \cos x\, dx = 0$ or $dp - \cos x\, dq + p \tan x\, dx = 0$

or $\sec x\, dp + p \sec x \tan x\, dx - dq = 0$ or $d\,(p \sec x) - dq = 0.$

Integrating it, $p \sec - q = c_1$, c_1 being an arbitrary constant ....(9)

Integrating (7), $y - \sin x = c_2$, c_2 being an arbitrary constant ...(10)

From (9) and (10), one integral of (1) is $p \sec x - q = f(y - \sin x)$. ...(11)

In a similar manner, (8) and (5) give another integral of (1)

$$p \sec x + q = g(y + \sin x). \qquad \text{...(12)}$$

Solving (11) and (12) for p and q, we find

$p = (f + g)/2 \sec x = (1/2) \times (f + g) \cos x$ and $q = (g - f)/2$ ...(13)

Now, $dz = p\, dx + q\, dy$ or $dz = (1/2) \times (f + q) \cos x\, dx + (1/2) \times (g - f)\, dy$, by (13)

or $dz = -(1/2) \times f(y - \sin x)\,(dy - \cos x\, dx) + (1/2) \times g(y + \sin x)\,(dy + \cos x\, dx)$

Integrating, $z = F(y - \sin x) + G(y + \sin x)$, F and G being arbitrary functions.

Ex. 4. *Solve* $t - r \sec^4 y = 2q \tan y$. **[Delhi Maths Hons 1995; Kanpur 1995; Meerut 1995]**

Sol. Given $$t - r \sec^4 y = 2q \tan y. \qquad \text{...(1)}$$

Comparing (1) with $Rr + Ss + Tt = V$, $R = -\sec^4 y$, $S = 0$, $T = 1$, $V = 2q \tan y$. ...(2)

Monge's subsidiary equations are $$Rdp\, dy + T\, dq\, dx - V\, dx\, dy = 0 \qquad \text{...(3)}$$

and $$R(dy)^2 - S\, dxdy + T\,(dx)^2 = 0 \qquad \text{...(4)}$$

Putting the values of R, S, T and V, (3) and (4) become

$$-\sec^4 y\, dp\, dy + dq\, dx - 2q \tan y\, dx\, dy = 0 \qquad \text{...(5)}$$

and $$-\sec^4 y\,(dy)^2 + (dx)^2 = 0. \qquad \text{...(6)}$$

Equation (6) may be factorised as $(dx - \sec^2 y\, dy)\,(dx + \sec^2 y\, dy) = 0$ so that

$$dx - \sec^2 y\, dy = 0 \qquad \text{...(7)}$$

or $$dx + \sec^2 y\, dy = 0. \qquad \text{...(8)}$$

Putting the value of dx from (7) in (5), we get

$-\sec^4 y\, dp\, dy + dq \sec^2 y\, dy - 2q \tan y\, dy \times \sec^2 y\, dy = 0$ or $-dp + \cos^2 y\, dq - 2q \sin y \cos y\, dy = 0$

or $dp - (\cos^2 x\, dq - q \times 2 \sin y \cos y\, dy) = 0$ or $dp - d(q \cos^2 y) = 0.$

Integrating it, $p - q \cos^2 y = c_1$, c_1 being an arbitrary constant ...(9)

Integrating (7), $x - \tan y = c_2$, being an arbitrary constant ...(10)

From (9) and (10), one integral of (1) is $p - q \cos^2 y = f(x - \tan y)$. ...(11)

Similarly, from (8) and (5) the other integral of (1) is $p + q \cos^2 y = g(x + \tan y)$. ...(12)

Solving (11) and (12) for p and q, we find

$p = (f + g)/2$ and $q = (g - f)/(2 \cos^2 y) = (1/2) \times (g - f) \times \sec^2 y$ ...(13)

Now, we have $$dz = pdx + qdy$$

or $$dz = (1/2) \times (f + g)dx + (1/2) \times (g - f) \times \sec^2 y\, dy, \text{ using (13)}$$

or $$dz = (1/2) \times f(x - \tan y)\,(dx - \sec^2 y\, dy) + (1/2) \times g(x + \tan y)\,(dx + \sec^2 y\, dy)$$

or $$dz = (1/2) \times f(x - \tan y)\, d(x - \tan y) + (1/2) \times g(x + \tan y)\, d(x + \tan y).$$

Integrating, $z = F(x - \tan y) + G(x + \tan y)$, F, G being arbitrary functions.

Ex. 5. *Solve* $q(yq + z)r - p\,(2yq + z)s + yp^2 t + p^2 q = 0.$ **[Delhi 2008]**

Sol. As usual, here Monge's subsidiary equations are

$$q(yq + z)dp\, dy + yp^2 dqdx + p^2 qdxdy = 0 \qquad \text{... (1)}$$

and $$q(yq + z)(dy)^2 + p(2yq + z)dxdy + yp^2(dx)^2 = 0. \qquad \text{... (2)}$$

On factorization, (2) gives $(qdy + pdx)\ \{(yq + z)dy + ypdx\} = 0.$

Hence two systems to be considered are

$$q(yq + z)dpdy + yp^2dqdx + p^2qdxdy = 0, \qquad qdy + pdx = 0 \qquad \text{... (3)}$$

and $$q(yq + z)dpdy + yp^2dqdx + p^2q\, dxdy = 0, \qquad (yq + z)dy + ypdx = 0 \qquad \text{... (4)}$$

Using $dz = pdx + qdy$, the second equation of (3) reduces to

$dz = 0$ so that $z = c_1$. ... (5)

From second equation of (3), $qdy = -pdx$. Hence first equation of (3) reduces to

$(yq + z)dp - ypdq - pqdy = 0$ or $(yq + z)dp - p\, d(yq) = 0$

or $(yq + z)dp - pd(yq + z) = 0,$ as $dz = 0$, by (5)

or $$\frac{d(yq+z)}{yq+z} - \frac{dp}{p} = 0 \qquad \text{so that} \qquad \log(yq + z) - \log p = \log c_1$$

or $(yq + z)/p = c_2$, c_2 being an arbitrary constant ... (6)

From (5) and (6), the intermediate integral corresponding to (3) is

$(yq + z)/p = \phi_1(z)$ or $yq + z = p\phi_1(z),$ ...(7)

where ϕ_1 is an arbitrary function.

Using $dz = pdx + qdy$, the second equation of (4) becomes

$y(qdy + pdx) + zdy = 0$ or $ydz + zdy = 0$ or $d(yz) = 0.$

Integrating it, $yz = c_3$, c_3 being an arbitrary constant ...(8)

From second equation of (4), $(yq + z)dy = -ypdx.$

Using this fact, first equation of (4) reduces to

$qdp - pdq - (pq/y)dy = 0$ or $-(1/p)dp + (1/q)dq + (1/y)dy = 0.$

Integrating, $-\log p + \log q + \log y = \log c_1$ or $(yq)/p = c_2$ ...(9)

From (8) and (9), another intermediate integral corresponding to (4) is

$$(qy)/p = \phi_2(yz), \text{ where } \phi_2 \text{ is an arbitrary function.} \qquad \text{...(10)}$$

Solving (7) and (10) for p and q, we have $p = \dfrac{z}{\phi_1(z) - \phi_2(yz)}$, $\quad q = \dfrac{z\phi_2(yz)}{y\{\phi_1(z) - \phi_2(yz)\}}$.

Substituting these in $dz = pdx + qdy$, $\quad dz = \dfrac{z}{\phi_1(z) - \phi_2(yz)}\{dx + (1/y) \times \phi_2(yz)\, dy\}$

or $\phi_1(z)dz = zdx + \phi_2(yz)\dfrac{zdy + ydz}{y}$ or $\dfrac{\phi_1(z)\, dz}{z} = dx + \dfrac{\phi_1(yz)\, d(yz)}{yz}$.

Integrating, $\psi_1(z) = x + \psi_2(yz)$, where ψ_1 and ψ_2 are arbitrary functions.

Ex. 6. *Solve* $(r - t)xy - s(x^2 - y^2) = qx - py$. **[Delhi Maths 2005, Kurukshetra 2005 (H)]**

Sol. Usual Monge's auxiliary equations are

$$xydpdy - xydqdx - (qx - py)dxdy = 0 \qquad \text{...(1)}$$

and $$xy(dy)^2 + (x^2 - y^2)\, dxdy - xy\, (dx)^2 = 0. \qquad \text{...(2)}$$

On factorizing, (2)gives $(xdy - ydx)\ (ydx + xdy) = 0.$

Hence, two systems to be considered are

$$xydpdy - xydqdx - (qx - py)\, dxdy = 0, \qquad xdy - ydx = 0 \qquad \text{...(3)}$$

and $$xydpdy - xydqdx - (qx - py)\, dxdy = 0, \qquad ydx + xdy = 0. \qquad \text{...(4)}$$

Second equation of (3) gives $y/z = c_1$, c_1 being an arbitrary constant ...(5)

Using second equation, first equation of (3) reduces to

$ydp - xdq - qdx + pdy = 0$ or $d(yp - xq) = 0$

Integrating, $yp - xq = c_2$, c_2 being an arbitrary constant ...(6)

From (5) and (6), intermediate integral corresponding to (3) is

$yp - xq = \phi_1(y/x)$, where ϕ_1 is an arbitrary function. ...(7)

Second equation of (4) gives $x^2 + y^2 = c_3$, c_3 being arbitrary constant ...(8)

Using second equation, first equation of (4) reduces to

$xdp + ydq + qdy + pdx = 0$ or $d(xp) + d(yq) = 0$

Integrating, $xp + yq = c_4$, c_4 being an arbitrary constant ...(9)

From (8) and (9), another intermediate integral corresponding to (4) is

$xp + yq = \phi_2(x^2 + y^2)$, where ϕ_2 is an arbitrary function. ...(10)

Solving (7) and (10) for p and q, we have

$$p = \frac{1}{x^2+y^2}\left\{y\phi_1\left(\frac{y}{x}\right) + x\,\phi_2(x^2+y^2)\right\} \quad \text{and} \quad q = \frac{1}{x^2+y^2}\left\{y\,\phi_2(x^2+y^2) - x\,\phi_1\left(\frac{y}{x}\right)\right\}.$$

Substituting these values in $dz = pdx + qdy$, we get

$$dz = \frac{1}{x^2+y^2}\left[\left\{y\,\phi_1\left(\frac{y}{x}\right) + x\,\phi_2(x^2+y^2)\right\}dx + \left\{y\,\phi_2(x^2+y^2) - x\,\phi_1\left(\frac{y}{x}\right)dy\right\}\right]$$

or $$dz = \frac{ydx - xdy}{x^2+y^2}\phi_1\left(\frac{y}{x}\right) + \frac{xdx+ydy}{x^2+y^2}\phi_2(x^2+y^2) \quad \text{or} \quad dz = -\frac{\phi_1(y/x)}{1+(y/x)^2}d\left(\frac{y}{x}\right) + \frac{1}{2}\frac{\phi_2(x^2+y^2)}{x^2+y^2}d(x^2+y^2).$$

Integrating, $z = \psi_1(y/x) + \psi_2(x^2 + y^2)$, ψ_1, ψ_2 being arbitrary functions.

Ex. 7. *Solve* $(r - s)\,x = (t - s)\,y$. **(M.D.U Rohtak 2005)**

Sol. Usual Monge's subsidiary equations are $xdpdy - ydqdx = 0$ ...(1)

and $x(dy)^2 + (x - y)\,dxdy - y(dx)^2 = 0.$ ...(2)

Factorising, (2) $\Rightarrow$ $(xdy - ydx)\,(dy + dx) = 0.$

Hence two systems to be considered are

$xdpdy - ydqdx = 0,$ $\quad xdy - ydx = 0$ ...(3)

and $xdpdy - ydqdx = 0,$ $\quad dy + dx = 0.$ ...(4)

Integrating second equation of (3), $y/x = c_1$, c_1 being an arbitrary constant ...(5)

Eliminating dy/dx between equations of (3), we get

$dp - dq = 0$ so that $p - q = c_2$, c_2 being an arbitrary constant ...(6)

Hence the intermediate integral corresponding to (3) is $p - q = \phi_1(y/x)$. ...(7)

Integrating second equation of (4), $x + y = c_3$, c_3 being an arbitrary constant ...(8)

Eliminating dy/dx between equations of (4), we get

$xdp + ydq = 0$ or $xdp + ydq + pdx + qdy = pdx + qdy$

or $d(xp) + d(yq) - dz = 0,$ as $dz = pdx + qdy$.

Integrating, $xp + yq - z = c_4$, c_4 being an arbitrary constant ...(9)

Hence the intermediate integral corresponding to (4) is

$xp + yq - z = \phi_2(x + y)$ or $xp + yq = z + \phi_2(x + y),$ ...(10)

Solving (7) and (10) for p and q, we have

$$p = \frac{1}{x+y}\left\{z + \phi_2(x+y) + y\phi_1\left(\frac{y}{x}\right)\right\} \quad \text{and} \quad q = \frac{1}{x+y}\left\{z + \phi_2(x+y) - x\,\phi_1\left(\frac{y}{x}\right)\right\}.$$

Substituting these values in $dz = pdx + qdy$, we have

$$dz = \frac{1}{x+y}\left[\left\{z + \phi_2(x+y) + y\,\phi_1\left(\frac{y}{x}\right)\right\}dx + \left\{z + \phi_2(x+y) - x\,\phi_1\left(\frac{y}{x}\right)\right\}dy\right]$$

$$\Rightarrow \qquad \frac{(x+y)\,dx - zdx}{(x+y)^2} = \frac{\phi_2(x+y)\,d\,(x+y)}{(x+y)^2} + \frac{(ydx - xdy)\,\phi_1\,(y/x)}{(x+y)^2}$$

$$\Rightarrow \qquad d\left(\frac{z}{x+y}\right) = \frac{\phi_2(x+y)}{(x+y)^2}\, d(x+y) - \frac{\phi_1(y/x)}{1+(y/x)^2}\, d\left(\frac{y}{x}\right).$$

Integrating, $z/(x+y) = \psi_2(x+y) + \psi_1(y/x)$, ψ_1, ψ_2 being arbitrary functions.

Ex. 8. *Solve* $r + ka^2t - 2as = 0$.

Sol. Given $$r - 2as + ka^2t = 0. \qquad ...(1)$$

Comparing (1) with $Rr + Ss + Tt = V$, we have $R = 1$, $S = -2a$, $T = ka^2$, $V = 0$.

Hence the Monge's subsidiary equations

$Rdp\, dy + Tdq\, dx - Vdx\, dy = 0$ and $R(dy)^2 - S\, dx\, dy + T(dx)^2 = 0$

become $$dp\, dy + ka^2\, dq\, dx = 0 \qquad ...(2)$$

and $$(dy)^2 + 2a\, dx\, dy + ka^2\, (dx)^2 = 0. \qquad ...(3)$$

From (3), $dy = [-2a\, dx \pm \{4a^2(dx)^2 - 4ka^2\, (dx)^2\}^{1/2}]/2 = -a\, dx \pm a\sqrt{(1-k)}\, dx$

or $dy + a\, \{1 \pm \sqrt{(1-k)}\}dx = 0$ or $dy + a\,(1 \pm l)\, dx = 0$, where $l = \sqrt{(1-k)}$.

Hence (3) reduces to the following two equations :

$$dy + a(1 + l)dx = 0 \qquad ...(4)$$

and $$dy + a(1 - l)dx = 0. \qquad ...(5)$$

From (2) and (4) , eliminating dy, we have

$dp\{-a\,(1 + l)\, dx\} + ka^2\, dqdx = 0$ or $(1 + l)dp - ka\, dq = 0$.

Integrating it, $(1 + l)p - kaq = c_1$, c_1 being an arbitrary constant ...(6)

Again, integrating (4), $y + a(1 + l)x = c_2$, c_2 being an arbitrary constant ...(7)

From (6) and (7), first intermediate integral is

$(1 + l)p - kaq = f_1\{y + a(1 + l)x\}$, where f_1 is an arbitrary function. ...(8)

Similary, from (2) and (5), second intermediate intgegral is given by (replacing l by $-l$ in (8) since (5) differs from (4) in having $-l$ in place of l)

$(1 - l)p - kaq = f_2\{y + a(1 - l)x\}$, where f_2 is an arbitrary function ...(9)

Solving (8) and (9) for p and q, $p = (1/2l) \times [f_1\{y + a(1 + l)x\} - f_2\{y + a(1 - l)x\}]$

and $q = (1/2akl) \times [(1 - l)f_1\{y + a(1 + l)x\} - (1 + l)\, f_2\{y + a(1 - l)x\}]$.

Substituting these values of p and q in $dz = pdx + qdy$, we get

$$dz = (1/2l) \times [f_1\{y + a(1 + l)x\} - f_2\{y + a(1 - l)x\}]dx$$
$$+ (1/2akl) \times [(1 - l)\, f_1\{y + a(1 + l)x\} - (1 + l)\, f_2\{y + a(1 - l)x\}]dy$$

or $$dz = (1/2l) \times [f_1\{y + a(1 + l)x\} - f_2\{y + a(1 - l)x\}]dx$$

$+ \dfrac{1}{2al(1-l^2)}\, [\,(1 - l)\, f_1\{y + a(1 + l)x\} - (1 + l)\, f_2\{y + a(1 - l)x\}]dy$, as $l = (1 - k)^{1/2} \Rightarrow k = 1 - l^2$

or $dz = (1/2l)\,[dx f_1\,\{y + a(1 + l)x\} - dx f_2\{y + a(1 - l)x\}] + \dfrac{1}{2al}\left[\dfrac{dy}{1+l} f_1\{y + a(1+l)x\} - \dfrac{dy}{1-l} f_2\{y + (1-l)x\}\right]$

$$= \frac{1}{2al(l+1)} f_1\{y + a(1 + l)x\}\{dy + a(1 + l)dx\} - \frac{1}{2al(1-l)}\, f_2\{y + a(1 - l)x\}\ \{dy + a(1 - l)dx\}$$

or $$dz = \frac{1}{2al(l+1)}\, f_1\{y + a(1 + l)x\}\ d\{y + a(1 + l)x\} - \frac{1}{2al(1-l)} f_2\{y + a(1 - l)x\}\ d\{y + a\,(1 - l)x\}.$$

Integrating, $z = F_1\{y + a(1 + l)x\} + F_2\{y + a\,(1 - l)x\}$, where F_1 and F_2 are arbitrary functions.

Ex. 9. *Solve* $x^{-2}r - y^{-2}t = x^{-3}p - y^{-3}q$.

Sol. Comparing the given equation with $Rr + Ss + Tt = V$, we get

$R = x^{-2}$, $S = 0$, $T = y^{-2}$, $V = x^{-3}p - y^{-3}q$. Then Monge's subsidiary equations

$Rdpdy + Tdqdx + Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $$x^{-2}dpdy + y^{-2}dqdx - (x^{-3}p - y^{-3}q)\, dxdy = 0. \qquad ...(1)$$

and $$x^{-2}(dy)^2 - y^{-2}(dx)^2 = 0. \qquad ...(2)$$

Multiplying both sides of (1) by x^3y^3, we get

$$xy^3dpdy - x^3ydqdx - py^3dxdy + qx^3dxdy = 0. \qquad ...(3)$$

Again, (2) $\Rightarrow$ $x^2y^2(y^2dy^2 - x^2dx^2) = 0$ or $x^2y^2(ydy + xdx)\,(ydy - xdx) = 0$

Hence (2) is equivalent to the equations

$$ydy + xdx = 0 \quad i.e., \quad ydy = -xdx \qquad ...(4)$$

and $$ydy - xdx = 0. \qquad ...(5)$$

Integrating (4), $y^2/2 + x^2/2 = c_1/2$ or $x^2 + y^2 = c_1$. ...(6)

From (3), $xy^2dp(ydy) - x^2ydq(xdx) - py^2dx(ydy) + qx^2dy(xdx) = 0$

or $xy^2dp(-xdx) - x^2ydq(xdx) - py^2dx\,(-xdx) + qx^2dy(xdx) = 0$, using (4)

or $-xy^2dp - x^2ydq + py^2dx + qx^2dy = 0$ or $y^2(xdp - pdx) + x^2(ydq - qdy) = 0$

or $$\frac{xdp - pdx}{x^2} + \frac{ydq - qdy}{y^2} = 0 \quad \text{or} \quad d\left(\frac{p}{x}\right) + d\left(\frac{q}{y}\right) = 0.$$

Integrating, $(p/x) + (q/y) = c_2$, c_2 being an arbitrary constant ...(7)

From (6) and (7), an intermediate integral is

$(1/x)p + (1/y)q = f(x^2 + y^2)$, where f is an arbitrary function. ...(8)

Similarly, from (3) and (5), another intermediate integral is

$(1/x)p - (1/y)q = g(x^2 - y^2)$, where g is an arbitrary function ...(9)

Solving (8) and (9) for p and q, we obtain

$p = (x/2) \times \{f(x^2 + y^2) + g(x^2 - y^2)\}$ and $q = (y/2) \times \{f(x^2 + y^2) - g(x^2 - y^2)\}$.

Substituting these values of p and q in $dz = pdx + qdy$, we get

$$dz = (x/2) \times \{f(x^2 + y^2) + g(x^2 - y^2)\}dx + (y/2) \times \{f(x^2 + y^2) - g(x^2 - y^2)\}dy$$

or $$dz = (1/4) \times f(x^2 + y^2)\,(2xdx + 2ydy) + (1/4) \times g(x^2 - y^2)\,(2xdx - 2ydy) \quad ... (10)$$

Putting $x^2 + y^2 = u$, $x^2 - y^2 = v$ so that $2xdx + 2ydy = du$ and $2xdx - 2ydy = dv$, (10) gives

$$dz = (1/4) \times f(u)\, du + (1/4) \times g(v)dv, \qquad ... (11)$$

Integrating (11), $z = F(u) + G(v) = F(x^2 + y^2) + G(x^2 - y^2)$,

where F and G are arbitrary functions.

Ex. 10. *Solve* $rx^2 - 3s\,xy + 2t\,y^2 + px + 2qy = x + 2y$.

Sol. Comparing the given equation with $Rr + Ss + Tt = V$, we get

$R = x^2$, $S = -3xy$, $T = 2y^2$, $V = x + 2y - px - 2qy$.

Hence Monge's subsidiary equations are

$Rdpdy + Tdqdy - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $$x^2\, dpdy + 2y^2dqdx - (x + 2y - px - 2qy)\, dxdy = 0 \qquad ...(1)$$

and $$x^2(dy)^2 + 3xy\, dxdy + 2y^2(dx)^2 = 0. \qquad ...(2)$$

Here (2) $\Rightarrow$ $(xdy + 2ydx)\,(xdy + ydx) = 0$.

Hence (2) resolves into the following two equations

$$xdy + 2ydx = 0 \quad i.e., \quad 2ydx = -\,xdy \qquad ...(3)$$

and $$xdy + ydx = 0. \qquad ...(4)$$

Re–writing (3), $(1/y)dy + 2(1/x)dx = 0$

Integrating, $\log y + 2 \log x = \log c_1$ or $yx^2 = c_1$. ...(5)

Re–writing (1), $(xdp)(xdy) + ydq(2ydx) - dx(xdy) - dy(2ydx) + pdx(xdy) + qdy(2ydx) = 0$

or $(xdp)(xdy) + ydq(-xdy) - dx(xdy) - dy(-xdy) + pdx(xdy) + qdy(-xdy) = 0$, using (3)

or $xdp - ydq - dx + dy + pdx - qdy = 0$

or $(xdp + pdx) - (ydq + qdy) - dx + dy = 0$ or $d(xp) - d(yq) - dx + dy = 0$.

Integrating, $xp - yq - x + y = c_2$, c_2 being an arbitrary constant ...(6)

From (5) and (6), an intermediate integral is

$xp - yq - x + y = f(x^2y)$, where f is an arbitrary function. ...(7)

Similarly from (1) and (4), another intermediate integral is

$xp - 2yq - x + 2y = g(xy)$, where g is an arbitrary function. ...(8)

Solving (7) and (8) for p and q, we have

$p = (1/x) \times \{x + 2f(x^2y) - g(xy)\}$, and $q = (1/y) \times \{y + f(x^2y) - g(xy)\}$.

Substituting these values of p and q in $dz = pdx + qdy$, we get

$$dz = (1/x) \times \{x + 2f(x^2y) - g(xy)\}dx + (1/y) \times \{y + f(x^2y) - g(xy)\}]dy$$

or
$$dz = dx + dy + f(x^2y)\left(\frac{2}{x}dx + \frac{1}{y}dy\right) - g(xy)\left(\frac{dx}{x} + \frac{dy}{y}\right)$$

or
$$dz = dx + dy + f(x^2y)\, d[\log(x^2y)] - g(xy)\, d[\log(xy)].$$

Integrating, $z = x + y + F(x^2y) + G(xy)$, G, and F being arbitrary functions.

Ex. 11. *Find the general solution of the equation $r + 4t = 8xy$, by Monge's method. Find also the particular solution for which $z = y^2$ and $p = 0$, when $x = 0$* **[Delhi Maths (Hons) 2006, 09]**

Sol. Given $r + 4t = 8xy$ ... (1)

Comparing (1) with $Rr + Ss + Tt = V$, here $R = 1$, $S = 0$, $T = 4$ and $V = 8xy$. Hence Monge's subsidians equations $Rdp\,dy + Tdq\,dx - Vdxdy = 0$ and $R(dy)^2 - Sdx\,dy + T(dx)^2 = 0$ become

$$dpdy + 4\,dqdx - 8xydxdy = 0 \quad \text{... (2)}$$

and
$$(dy)^2 + 4\,(dx)^2 = 0 \quad \text{... (3)}$$

Re–writing (3), $dy^2 - 4i^2dx^2 = 0$ or $(dy - 2idx)(dy + 2idx) = 0$

so that $dy - 2idx = 0$ or $dy = 2idx$ ... (4)

and $dy + 2idx = 0$ or $dy = -2idx$ ... (5)

We first consider (4) and (2). Integrating (4), $y - 2ix = C_1$ ... (6)

Using (4) and (6), (2) gives $dp\,(2i\,dx) + 4\,dq\,dx - 8x(C_1 + 2ix)\,(2i\,dx)\,dx = 0$

or $i\,dp + 2dq - 8xi\,(C_1 + 2ix) = 0$, by (6) or $idp + 2dq - 8C_1ix\,dx + 16x^2dx = 0$

Integrating, $ip + 2q - 4C_1ix^2 + (16/3)x^3 = C_2$, C_2 being an arbitrary constant

or $ip + 2q - 4ix^2(y - 2ix) + (16/3) \times x^3 = C_2$, by (6) ... (7)

From (6) and (7) first intermediate integral of (1) is $ip + 2q - 4ix^2(y - 2ix) + (16/3)x^3 = f(y - 2ix)$

or $ip + 2q = (8/3) \times x^3 + 4ix^2y + f(y - 2ix)$, f being an abritrary function ... (8)

Similarly considering the pair (5) and (2), the second intermediate integral of (1) is

$$ip - 2q = -(8/3) \times x^3 + 4ix^2 y + g\ (y + 2ix), \text{ } g \text{ being an arbitrary function} \quad \dots (9)$$

Solving (8) and (9) for p and q, $\quad p = \{8ix^2 y + f(y - 2ix) + g(y + 2ix)\}/2i$

and $\quad q = \{(16/3) \times x^3 + f(y - 2ix) - g(y + 2ix)\}/4$

Putting the above values of p and q in $dz = pdx + qdy$, we get

$$dz = (1/2i) \times \{8ix^2 y + f(y - 2ix) + g(y + 2ix)\}dx + (1/4) \times \{(16/3) \times x^3 + f(y - 2ix) - g(y - 2ix)\}dy$$

$$= (4/3) \times (3x^2 ydx + x^3 dy) + (1/4) \times f(y - 2ix)d(y - 2ix) - (1/4) \times g(y + 2ix)d(y + 2ix)$$

$$\therefore \quad dz = (4/3) \times d(x^3 y) + (1/4) \times f(y - 2ix)\ d(y - 2ix) - (1/4) \times g(y + 2ix)d(y + 2ix)$$

Integrating, $\quad z = (4/3) \times x^3 y + F(y - 2ix) + G\ (y + 2ix), \quad \dots (10)$

which is the general solution of (1) containing F and G as arbitrary functions

To find particular solution of (1) Given conditions are

$z = y^2$ and $p = \partial z / \partial x = 0$ when $x = 0 \quad \dots (11)$

From (11), $\quad \partial z / \partial y = 2y$ when $x = 0 \quad \dots (12)$

Differentiating (10) partially w.r.t. 'x' and 'y', we get

$$\partial z / \partial x = 4x^2 y - 2i\ F'(y - 2ix) + 2i\ G'(y + 2ix) \quad \dots (13)$$

and
$$\partial z / \partial y = (4/3) \times x^3 + F'(y - 2ix) + G'(y + 2ix) \quad \dots (14)$$

Using (11) and (12), (10), (13) and (14) reduce to

$$F(y) + G(y) = y^2 \quad \dots (15)$$

$$F'(y) - G'(y) = 0 \quad \dots (16)$$

and
$$F'(y) + G'(y) = 2y \quad \dots (17)$$

From (16) and (17), $\quad F'(y) = y \quad$ and $\quad G'(y) = y$

Integrating these, $\quad F(y) = y^2/2 \quad$ and $\quad G(y) = y^2/2 \quad \dots (18)$

which also satisfy (15).

From (18), $\quad F(y - 2ix) = (y - 2ix)^2/2 \quad$ and $\quad G(y + 2ix) = (y + 2ix)^2/2$

Putting these values in (10), the required particular solution is

$$z = (4/3) \times x^3 y + (y - 2ix)^2/2 + (y + 2ix)^2/2 \quad \text{or} \quad z = (4/3) \times x^3 y + y^2 - 4x^2.$$

9.5. Type 2. When the given equation $Rr + Ss + Tt = V$ leads to two distinct intermediate integrals and only one is employed to get the desired solution.

Working rule for solving problems of type 2.

Step 1. Write the given equation in the standard form $\quad Rr + Ss + Tt = V$.

Step 2. Substitute the values of R, S, T and V in the Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0 \quad \dots (1) \qquad R(dy)^2 - Sdxdy + T(dx)^2 = 0 \quad \dots (2)$

Step 3. Factorise (1) into two distinct factors.

Step 4. Take one of the factors of step 3 and use (2) to get an intermediate integral. Don't find second intermediate integral as we did in type 1. If required use remark 1 of Art. 9.2.

Step 5. Re–write the intermediate integral of the step 4 in the form of Lagrange equation, namely, $Pp + Qq = R$ (refer chapter 2). Using the well known Lagrange's method we arrive at the desired general solution of the given equation.

9.6 SOLVED EXAMPLES BASED ON ART. 9.5.

Ex. 1. *Solve* $(r - s)y + (s - t)x + q - p = 0$.

Sol. The given can be written as $\quad yr + s(x - y) - tx = p - q. \quad$ ...(1)

Comparing (1) with $Rr + Ss + Tt = V$, $R = y$, $S = x - y$, $T = -x$ and $V = p - q$.

Hence Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $\quad ydpdy - xdqdx + (q - p)dxdy = 0 \quad$ ...(1)

and $\quad y(dy)^2 - (x - y)\, dxdy - x\,(dx)^2 = 0. \quad$ ...(2)

Re–writing (2), $\quad (dy + dx)\,(ydy - xdx) = 0.$

so that $\quad dy + dx = 0 \quad$ or $\quad dy = -\,dx \quad$ ...(3)

and $\quad ydy - xdx = 0. \quad$ ...(4)

Using (3), (1) becomes $\quad -\,ydpdx - xdqdx + q\,dx(-\,dx) - p\,dxdy = 0$

or $\quad ydp + xdq + qdx + pdy = 0 \quad$ or $\quad (ydp + pdy) + (xdq + qdx) = 0$

or $\quad d(yp) + d(xq) = 0 \quad$ so that $\quad yp + xq = c_1. \quad$ ...(5)

Integrating (3), $\quad x + y = c_2$, c_2 being an arbitrary constant $\quad$ ...(6)

From (5) and (6), one intermediate integral is $\quad yp + xq = f(x + y), \quad$ ...(7)

which is of the Lagrange's form and so its subsidiary equations are

$$\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{f(x+y)}. \qquad \text{...(8)}$$

From first and second fractions of (8), $\quad 2xdx - 2ydy = 0.$

Integrating, $\quad x^2 - y^2 = a$, a being an arbitrary constant $\quad$ ...(9)

Taking first and third fractions of (8), we get

$$\frac{dx}{y} = \frac{dz}{f(x+y)} \quad \text{or} \quad \frac{dx}{(x^2-a)^{1/2}} = \frac{dz}{f\left[x+(x^2-a)^{1/2}\right]}, \text{ as (9)} \quad \Rightarrow y = (x^2 - a)^{1/2}$$

or $\quad dz = f[x + (x^2 - a^2)^{1/2}]\,(x^2 - a^2)^{-1/2}dx \quad$ ...(10)

Put $\quad x + (x^2 - a)^{1/2} = v \quad$ so that $\quad \left[1 + x/(x^2 - a)^{1/2}\right]dx = dv \quad$ ...(11)

or $\quad \dfrac{x+(x^2-a)^{1/2}}{(x^2-a)^{1/2}}\, dx = dv \quad$ or $\quad \dfrac{dx}{(x^2-a)^{1/2}} = \dfrac{dv}{v}$, using (11)

Then, (10) reduces to $\quad dz - (1/v)\, f(v)dv = 0.$

Integrating, $\quad z - F(v) = b \quad$ or $\quad z - F[x + (x^2 - a)^{1/2}] = b$, by (11)

or $\quad z - F(x + y) = b$, $\quad$ as $\quad y = (x^2 - a)^{1/2}$, by (9) $\quad$ ...(12)

From (9) and (12), the required general solution is $\quad z - F(x + y) = G(x^2 - y^2)$

or $\quad z = F(x + y) + G(x^2 - y^2)$, where F and G are arbitrary functions.

Ex. 2. *Solve* : $q(1 + q)r - (p + q + 2pq)s + p(1 + p)t = 0$. **[Meerut 1994; I.A.S. 1974]**

Sol. Comparing the given equation with $Rr + Ss + Tt = V$, we find

$R = q(1 + q)$, $\quad S = -\,(p + q + 2pq)$, $\quad T = p(1 + p)$, $\quad V = 0 \quad$ ...(1)

Monge's subsidiary equations are $\quad Rdpdy + Tdq\,dx - Vdxdy = 0 \quad$ ...(2)

and $\quad R(dy)^2 - Sdxdy + T(dx)^2 = 0 \quad$ ...(3)

Using (1), (2) and (3) become $\quad (q + q^2)dpdy + (p + p^2)dqdx = 0 \quad$ ...(4)

and $\quad (q + q^2)\,(dy)^2 + (p + q + 2pq)dxdy + (p + p^2)\,(dx)^2 = 0. \quad$ ...(5)

In order to factorise (5), we re–write it as

$$q(1+q)(dy)^2 + (p+pq)dxdy + (q+pq)dxdy + p(1+p)(dx)^2 = 0$$

or $$q(1+q)(dy)^2 + p(1+q)dxdy + q(1+p)dxdy + p(1+p)(dx)^2 = 0$$

or $$(1+q)dy(qdy+pdx) + (1+p)dx(qdy+pdx) = 0$$

or $$(qdy+pdx)\,[(1+q)dy + (1+p)dx] = 0. \qquad \text{... (6)}$$

Then, from (6), we get $qdy + pdx = 0$ *i.e.,* $qdy = -pdx$ ...(7)

and $$(1+q)dy + (1+p)dx = 0. \qquad \text{...(8)}$$

Keeping (7) in view, (4) may be re–written as $(1+q)dp\,(qdy) - (1+p)dq\,(-pdx) = 0$

From (7), qdy and $(-pdx)$ are equivalent. Hence dividing each term of the above equation by qdy, or its equivalent $(-pdx)$, we get

$(1+q)dp - (1+p)dq = 0$ or $dp/(1+p) - dq/(1+q) = 0.$

Integrating it, $\log(1+p) - \log(1+q) = \log c_1$ or $(1+p)/(1+q) = c_1$. ...(9)

Using $dz = pdx + qdy$, (7) becomes $dz = 0$ so that $z = c_2$. ...(10)

From (9) and (10), one intermediate integral of (1) is given by

$(1+p)/(1+q) = f(z)$ or $p - f(z)q = f(z) - 1$, ...(11)

which is of the form $Pp + Qq = R$. Here Lagrange's auxiliary equations for (11) are

$$\frac{dx}{1} = \frac{dy}{-f(z)} = \frac{dz}{f(z)-1}. \qquad \text{...(12)}$$

Choosing 1, 1, 1 as multipliers, each fraction in (12) = $\dfrac{dx+dy+dz}{1-f(z)+f(z)-1} = \dfrac{dx+dy+dz}{0}$

$\therefore$ $dx + dy + dz = 0$ so that $x + y + z = c_2$. ...(13)

From first and third fractions in (12), we get $dx - [f(z)-1]^{-1}\,dz = 0.$

Integrating it, $x + F(z) = c_4$, c_4 being an arbitrary constant ...(14)

From (13) and (14), the required general solution is

$x + F(z) = G(x+y+z)$, F, G being arbitrary functions.

Ex. 3. *Solve* $(x-y)\,(xr - xs - ys + yt) = (x+y)(p-q)$.

[Delhi Maths (H) 97, 2000; Meerut 1999; Garhwal 1996]

Sol. Given $(x-y)xr - (x^2-y^2)s + (x-y)yt = (x+y)(p-q)$ ...(1)

Comparing (1) with $Rr + Ss + Tt = V$, we find

$R = x(x-y)$, $S = -(x^2-y^2)$, $T = y(x-y)$, $V = (x+y)(p-q)$. ...(2)

Monge's subsidiary equations are $Rdpdy + Tdqdx - Vdxdy = 0$ ...(3)

and $$R(dy)^2 - Sdxdy + T(dx)^2 = 0. \qquad \text{...(4)}$$

Using (2), (3) and (4) become

$$x(x-y)dpdy + y(x-y)dqdx - (x+y)(p-q)dxdy = 0 \qquad \text{...(5)}$$

and $$x(x-y)(dy)^2 + (x^2-y^2)dxdy + y(x-y)(dx)^2 = 0. \qquad \text{...(6)}$$

Since $x^2 - y^2 = (x-y)(x+y)$, dividing (6) by $(x-y)$ gives

$xdy^2 + (x+y)dxdy + ydx^2 = 0$ or $(xdy + ydx)\,(dx+dy) = 0$

Thus we get $xdy + ydx = 0$ or $xdy = -\,ydx$ ...(7)

and $$dx + dy = 0. \qquad \text{...(8)}$$

Keeping (7) in view, (5) may be rewritten as

$$(x-y)dp\,(xdy) - (x-y)\,dq\,(-ydx) - (p-q)\,dx\,(xdy) + (p-q)dy(-\,ydx) = 0.$$

From (7), $x\,dy$ and $(-y\,dx)$ are equal. So dividing each term of the above equation by $x\,dy$, or its equivalent $(-y\,dx)$, we get

$(x-y)\,dp - (x-y)dq - (p-q)\,dx + (p-q)dy = 0$ or $(x-y)\,(dp-dq) - (p-q)\,(dx-dy) = 0$

or $\quad \dfrac{dp-dq}{p-q}-\dfrac{dx-dy}{x-y}=0$ so that $\quad \dfrac{p-q}{x-y}=c_1$ ...(9)

Integrating (7), $\quad xy = c_2$, c_2 being an arbitrary constant ...(10)

From (9) and (10), one intermediate integral of (10) is

$(p-q)/(x-y) = f(xy)$ or $\quad p-q=(x-y)f(xy)$ ...(11)

which is of the form $Pp + Qq = R$. Its Lagrange's auxiliary equations are

$$\frac{dx}{1}=\frac{dy}{-1}=\frac{dz}{(x-y)f(xy)}. \qquad ...(12)$$

Taking the first two fractions of (12), we get

$dx + dy = 0$ so that $x + y = c_3$, c_3 being an arbitrary constant ...(13)

Taking $y\,f(xy)$, $x\,f(xy)$, 1 as multipliers, each fraction of (12) = $\dfrac{y\,f(xy)\,dx+x\,f(xy)\,dy+dz}{0}$

so that $\quad f(xy)\times(ydx + x\,dy) + dz = 0 \quad$ or $\quad f(xy)\times d(xy) + dz = 0.$

Integrating it, $\quad F(xy) + z = c_4$, c_4 being an arbitrary constant ...(14)

From (13) and (14), the required general solution is

$F(xy) + z = G(x + y)$, where F and G are arbitrary functions.

Ex. 4. $xy\,(t-r) + (x^2-y^2)\,(s-2) = py - qx.$ **[Delhi Maths (H) 2001]**

Sol. Given $\quad -xyr + (x^2-y^2)\,s + xyt = py - qx + 2(x^2-y^2).$ ...(1)

Comparing (1) with $Rr + Ss + Tt = V$, we find

$R = -xy$, $\quad S = x^2-y^2$, $\quad T = xy$, $\quad V = py - qx + 2(x^2-y^2)$. ...(2)

Monge's subsidiary equations are $\quad Rdp\,dy + Tdq\,dx - V\,dx\,dy = 0$ ...(3)

and $\quad R(dy)^2 - Sdxdy + T(dx)^2 = 0.$ ...(4)

Using (2), (3) and (4) become

$$-xy\;dp\;dy + xydqdx - [py - qx + 2(x^2-y^2)]dxdy = 0 \qquad ...(5)$$

and $\quad -xy\,(dy)^2 - (x^2-y^2)dxdy + xy(dx)^2 = 0.$ ...(6)

From (6), $\quad xy(dy)^2 + x^2dxdy - y^2dxdy - xy(dx)^2 = 0$

or $\quad xdy(ydy + xdx) - ydx\,(ydy + xdx) = 0 \quad$ or $\quad (xdy - ydx)\,(ydy + xdx) = 0.$

So, we get $\quad xdx + ydy = 0$, $\quad$ *i.e.*, $\quad xdx = -ydy$ ...(7)

and $\quad xdy - ydx = 0.$ ...(8)

Keeping (7) in view, (5) may be re–written as

$$xdp(-ydy) + ydq\,(xdx) + pdx\,(-ydy) + qdy(xdx) - 2xdy(xdx) - 2ydx(-ydy) = 0.$$

From (7), xdx and $(-ydy)$ are equivalent. So dividing each term of the above equation by xdx, or its equivalent $(-ydy)$, we get

$xdp + ydq + pdx + qdy - 2xdy - 2ydx = 0 \quad$ or $\quad (xdp + pdx) + (ydq + qdy) - 2(xdy + ydx) = 0.$

Integrating it, $\quad xp + yq - 2xy = c_1$, being an arbitrary constant ...(9)

Integrating (7), $\quad x^2/2 + y^2/2 = c_2/2 \quad$ or $\quad x^2 + y^2 = c_2.$ ...(10)

From (9) and (10), one intgegral of (1) is

$xp + qy - 2xy = f(x^2 + y^2) \quad$ or $\quad xp + yq = 2xy + f(x^2 + y^2)$, ...(11)

which is of the form $Pp + Qq = R$. So Lagrange's auxiliary equations for (11)are

$$\frac{dx}{x}=\frac{dy}{y}=\frac{dz}{2xy+f(x^2+y^2)}. \qquad ..(12)$$

Taking the first two fractions in (12), we get

$\log y - \log x = \log c_3 \quad$ or $\quad y/x = c_3 \quad$ or $\quad y = xc_3$ ...(13)

Taking the first and the last fractions in (12) and using $y = xc_3$ in it, we get

$dz = (1/x) \times [2c_3x^2 + f(x^2 + x^2c_3^2)]dx$ or $dx = 2c_3xdx + (1/x) \times f\{(1 + c_3^2)x^2\}dx$

or $$dz = 2c_3xdx + (1/2x^2) \times f\{(1 + c_3^2)x^2\}d\,(x^2).$$

Integrating $z - 2c_3(x^2/2) + F\{(1 + c_3^2)x^2\} = c_4$ or $z - (y/x)x^2 + F\{(1 + y^2/x^2)x^2\} = c_4$, by (13)

or $$z - xy + F(x^2 + y^2) = c_4,\ c_4 \text{ being an arbitrary constant} \qquad ...(14)$$

From (13) and (14), the required general solution is

$$z - xy + F(x^2 + y^2) = G(y/x), \text{ where } F \text{ and } G \text{ are arbitrary functions.}$$

Ex. 5. *Solve* $x^2r - y^2t - 2xp + 2z = 0$.

Sol. Given $$x^2r - y^2t = 2xp - 2z. \qquad ...(1)$$

Comparing (1) with $Rr + Ss + Tt = V$, $R = x^2$, $S = 0$, $T = -y^2$, $V = 2xp - 2z$.

Hence the usual Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $$x^2dpdy - y^2dqdx - (2xp - 2z)dxdy = 0 \qquad ...(2)$$

and $$x^2(dy)^2 - y^2(dx)^2 = 0. \qquad ...(3)$$

On factorizing, $(3) \Rightarrow (xdy - ydx)\,(xdy + ydx) = 0$

Thus, we have $xdy - ydx = 0$ *i.e.*, $xdy = ydx$. ...(4)

and $$xdy + ydx = 0. \qquad ...(5)$$

Re–writing (2), $xdp\,(xdy) - ydq(ydx) - 2(xp - z)\,(xdy)(1/x)dx = 0$

or $xdp(xdy) - ydq(xdy) - 2(xp - z)\,(xdy)\,(1/x)dx = 0$, using (4)

or $$xdp - ydq - 2(xp - z)\,(1/x)dx = 0$$

or $xdp - dz + pdx + qdy - ydq - 2(xp - z)\,(1/x)dx = 0$ as $dz = pdx + qdy \Rightarrow -dz + pdx + qdy = 0$

or $$d(xp - z) - d(yq) + 2qdy - 2(xp - z)\,(1/x)dx = 0$$

or $d(xp - yq - z) + 2qy(1/x)dx - 2(xp - z)\,(1/x)dx = 0$, as from (4), $dy = (y/x)dx$

or $d(xp - yq - z) - 2\,(xp - yq - z)\,(1/x)\,dx = 0$ or $\dfrac{d(xp-yq-z)}{xp-yq-z} - \dfrac{2dx}{x} = 0.$

Integrating, $\log\,(xp - yq - z) - 2\log x = \log c_1$ or $(xp - yq - z)/x^2 = c_1$. ...(6)

From (4), $(1/y)dy - (1/x)dx = 0$ so that $\log y - \log x = \log c_2$

or $$y/x = c_2,\ c_2 \text{ being an arbitrary constant} \qquad ...(7)$$

From (6) and (7), an intermediate integral is

$(xp - yq - z)\,/x^2 = \phi_1(y/x)$ or $xp - yq = z + x^2\phi_1(y/x)$. ...(8)

Lagrange's auxiliary equations for (8) are $$\frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{z + x^2\phi_1(y/x)}. \qquad ...(9)$$

From the first two ratios of (9), we get

$(1/x)\,dx + (1/y)\,dy = 0$ so that $xy = c_3$. ...(10)

Taking the second and third ratios of (9), we get

$$\frac{dz}{dy} + \frac{z}{y} = -\frac{x^2}{y}\phi_1\left(\frac{y}{x}\right) = -\frac{c_3^2}{y^3}\phi_1\left(\frac{y^2}{c_3}\right), \text{ by (10)}$$

Its I.F $= e^{\int(1/y)dy} = y$ and so solution is $$zy = -\int \frac{c_3^2}{y^2}\phi_1\left(\frac{y^2}{c_3}\right)dy + c_4$$

or $zy + \dfrac{c_3^{3/2}}{2}\displaystyle\int\left(\frac{c_3}{y^2}\right)\phi_1\left(\frac{y^2}{c_3}\right)\left(\frac{\sqrt{c_3}}{y}\right)d\left(\frac{y^2}{c_3}\right) = c_4$ or $zy + c_3^{3/2}\,\psi_1\left(\dfrac{y^2}{c_3}\right) = c_4$

or $$zy + (xy)^{3/2}\ \psi_1(y/x) = c_4, \text{ using (10).} \qquad ...(11)$$

From (10) and (11), the required general solution is

$zy + (xy)^{3/2}\ \psi_1(y/x) = \psi_2(xy)$, where ψ_1 and ψ_2 are arbitrary functions.

Ex. 6. *Solve* $(r - t)xy - s(x^2 - y^2) = qx - py$.

Sol. Given $$xyr - (x^2 - y^2)s - xyt = qx - py. \qquad ...(1)$$

Hence the usual Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $$xy\ dpdy - xy\ dqdx - (qx - py)\ dxdy = 0 \qquad ...(2)$$

and $$xy\ (dy)^2 + (x^2 - y^2)\ dxdy - xy\ (dx)^2 = 0. \qquad ...(3)$$

Now, (3) $\Rightarrow$ $(xdx + ydy)\ (xdy - ydx) = 0$

Hence, $xdx + ydy = 0$ *i.e.*, $xdx = -\,ydy$ ...(4)

and $$xdy - ydx = 0 \qquad ...(5)$$

Re–writing (2), $(xdp)\ (ydy) - ydq(xdx) - qdy(xdx) + pdx\ (ydy) = 0$

or $(xdp)\ (ydy) - ydq(-ydy) - qdy(-ydy) + pdx(ydy) = 0$, using (4)

or $xdp + ydq + qdy + pdx = 0$ or $d(xp) + d(yq) = 0$.

Integrating, $xp + yq = c_1$, c_1 being an arbitrary constant ...(6)

Integrating (4) $x^2/2 + y^2/2 = c_2/2$ or $x^2 + y^2 = c_2$. ...(7)

From (6) and (7), are intremediate integral is

$$xp + yq = f(x^2 + y^2),\ f \text{ being an arbitrary function.} \qquad ...(8)$$

Lagrange's subsidiary equations for (8) are $$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{f(x^2+y^2)}. \qquad ...(9)$$

Taking the first and second fractions of (9), $(1/y)dy - (1/x)dx = 0$.

Integrating, $\log y - \log x = \log a$ or $y/x = a$, ...(10)

where a is an arbitrary constant.

Taking the first and third fraction of (9), we get

$$\frac{dx}{x} = \frac{dz}{f(x^2+y^2)} \quad \text{or} \quad \frac{dx}{x} = \frac{dz}{f(x^2+a^2x^2)}, \text{ using (10)}$$

or $$dz = (1/x) \times f[x^2\ (1 + a^2)]\ dx = (1/x^2) \times f[x^2\ (1 + a^2)]\ xdx. \qquad ...(11)$$

Putting $x^2(1 + a^2) = v$ and $2x(1+ a^2)dx = dv$, (11) gives

$$dz = \frac{1+a^2}{v}\ f(v) \times \frac{1}{2(1+a^2)}\,dv = \left(\frac{1}{2v}\right) f(v)dv.$$

Integrating, $z = F(v) + b$ or $z - F[x^2\ (1 + a^2)] = b$

or $z - F(x^2 + x^2a^2) = b$ or $z - F(x^2 + y^2) = b$, using (10). ...(12)

Here b is an arbitrary constant. From (10) and (12), general solution of (1) is

$z - F(x^2 + y^2) = G(y/x)$ or $z = F(x^2 + y^2) + G(y/x)$,

where F and G are arbitrary functions.

Ex. 7. *Solve* $2xr - (x + 2y)s + yt = [(x + 2y)\ (2p - q)]/(x - 2y)$

Sol. Comparing the given equation with $Rr + Ss + Tr = V$, we have

$R = 2x$, $S = -(x + 2y)$, $T = y$, $V = [(x - 2y)\ (2p - q)]/(x - 2y)$.

Hence the usual Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $$2xdpdy + ydqdx - \frac{x+2y}{x-2y}\ (2p - q)\ dxdy = 0 \qquad ...(1)$$

and $$2x(dy)^2 + (x + 2y)dxdy + y(dx)^2 = 0. \quad ...(2)$$

The equation (2) can be resolved into the following two equations

$$xdy + ydx = 0 \qquad i.e., \qquad xdy = -\,ydx \quad ...(3)$$

and $$dx + 2ydy = 0. \quad ...(4)$$

Re–writing (1), $$2dp(xdy) + dq(ydx) - \frac{2p-q}{x-2y}[(xdy)dx + 2(ydx)dy]$$

or $$2dp(-ydx) + dq(ydx) - \frac{2p-q}{x-2y}\,\{(-ydx)dx + 2(ydx)dy\} = 0 \text{ using (3)}$$

or $$-2dp + dq - \frac{2p-q}{x-2y}\,(-dx + 2dy) = 0 \qquad \text{or} \qquad \frac{2dp-dq}{2p-q} - \frac{dx-2dy}{x-2y} = 0.$$

Intergrating, $\log(2p - q) - \log(x - 2y) = \log c_1$ or $(2p - q)/(x - 2y) = c_1$. ...(5)

Re–writing (3), $(1/y)dy + (1/x)dx = 0$ so that $\log x + \log y = \log c_2$

$\therefore$ $xy = c_2$, c_2 being an arbitrary constant ...(6)

From (5) and (6), an intermediate integral is

$$(2p - q)/(x - 2y) = f(xy) \qquad \text{or} \qquad 2p - q = (x - 2y)\,f(xy), \quad ...(7)$$

where f is an arbitrary function. The equation (7) is of Lagrange's form $Pp + Qq = R$. So Lagrange's, subsidiary equation for (7) are

$$\frac{dx}{2} = \frac{dy}{-1} = \frac{dz}{(x-2y)\,f(xy)}. \quad ...(8)$$

Taking the first and second fractions of (8), $dx + 2dy = 0$.

Integrating, $x + 2y = a$, a being an arbitrary constant ...(9)

Taking $y\,f(xy)$, $x\,f(xy)$, 1 as multipliers, each fraction of (8)

$$= \frac{y\,f(xy)\,dx + x\,f(xy)\,dy + dz}{2y\,f(xy) - x\,f(xy) + (x-2y)\,f(xy)} = \frac{f(xy)(ydx + x\,dy) + dz}{0}$$

This $\Rightarrow$ $f(xy)\,d(xy) + dz$, as $ydx + xdy = d(xy)$

Integrating, $F(xy) + z = b$, b being an arbitrary constant. ...(10)

From (9) and (10), the required complete integral is

$F(xy) + z = G(x + y)$, F and G being arbitrary functions.

Ex. 8. *Solve $xr + (x + y)s + yt + p + q = 0$ by Monge's method.*

Sol. Given $$xr + (x + y)s + yt = -(p + q) \quad ...(1)$$

Comparing (1) with $Rr + Sr + Tt = V$, here $R = x$, $S = x + y$, $T = y$ and $V = -(p + q)$.

Hence Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$ become

$$xdpdy + ydqdx + (p + q)dxdy = 0 \quad ...(2)$$

and $$x(dy)^2 - (x + y)dxdy + y(dx)^2 = 0 \quad ...(3)$$

Re–writing (3), $$(xdy - ydx)\,(dy - dx) = 0$$

so that $$xdy - ydx = 0 \quad ...(4)$$

and $$dy - dx = 0 \qquad i.e., \qquad dy = dx \quad ...(5)$$

For the required solution, we consider relation (5) only.

Integrating (5), $x - y = c_1$, being an arbitrary constant ... (6)

Using (5), (2) becomes $$xdpdx + ydqdx + (p + q)(dx)^2 = 0$$

or $xdp + ydq + pdx + qdx = 0$, on dividing by dx (as $dx \neq 0$)

or $(xdp + pdx) + (ydq + qdx) = 0$ or $(xdp + pdx) + (ydq + qdy) = 0$ by(5)

or $d(xp) + d(yq) = 0$ so that $xp + yq = c_2$. ...(7)

From (6) and (7), one intermediate integral of (1) is

$$xp + yq = f(x-y), \; f \text{ being an arbitrary function} \qquad ...(8)$$

which is of Lagrange's form. Its Lagrange's auxiliary equations are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{f(x-y)} \qquad ...(9)$$

Taking the first two fractions of (9), $(1/x)dx - (1/y)dy = 0$

Integrating, $\log x - \log y = \log c_3$ or $x/y = c_3$ ...(10)

Now, each fraction of (9) $= \dfrac{dx-dy}{x-y} = \dfrac{d(x-y)}{x-y}$ ...(11)

Combining this fraction with last fraction of (9), we get

$$\frac{dz}{f(x-y)} = \frac{d(x-y)}{x-y} \quad \text{or} \quad dz = \frac{f(x-y)}{x-y}d(x-y) = \frac{f(u)du}{u}, \text{ if } \quad u = x-y$$

Integrating, $z = F(u) + c_4 = F(x-y) + c_4$, where $F(u) = \int \frac{1}{u} f(u)\, du$

or $z - F(x-y) = c_4$, c_4 being an arbitrary constant ...(12)

From (10) and (12), the required solution is

$$z - F(x-y) = G(x/y) \quad \text{or} \quad z = G(x/y) + F(x-y),$$

where F and G are arbitrary functions.

Ex. 9. *Solve $rq^2 - 2pqs + p^2t = pt - qs$ by Monge's method.* **[Delhi Maths (Hons) 2002]**

Sol. Given $q^2r - q(2p-1)s + p(p-1)t = 0$ ...(1)

Comparing (1) with $Rr + Ss + Tt = V$, here $R = q^2$, $S = -q(2p-1)$, $T = p(p-1)$, $V = 0$.

Hence Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - S\,dxdy + T(dx)^2 = 0$ become

$$q^2 dpdy + p(p-1)dqdx = 0 \qquad ...(2)$$

and $$q^2(dy)^2 + q(2p-1)dxdy + p(p-1)(dx)^2 = 0 \qquad ...(3)$$

Re–writing (3), $(qdy + pdx)\{qdy + (p-1)dx\} = 0$

so that $qdy + pdx = 0$ *i.e.*, $qdy = -pdx$ ...(4)

and $qdy + (p-1)dx = 0$ ...(5)

For the required solution, we consider relation (4) only.

Since $dz = pdx + qdy$, (4) reduces to $dz = 0$ and so $z = c_1$ ...(6)

Re–writing (2), $(qdp)(qdy) + (p-1)dq(pdx) = 0$

or $(qdp)(-pdx) + (p-1)dq(pdx) = 0$, since from (4), $qdy = -pdx$

or $-qdp + (p-1)dq = 0$ or $\{1/(p-1)\}dp - \{1/q\}dq = 0$

Integrating, $\log(p-1) - \log q = \log c_2$ or $(p-1)/q = c_2$ ...(7)

From (6) and (7), one intermediate integral of (1) is

$(p-1)/q = f(z)$ or $p - qf(z) = 1$, ...(8)

which is of Lagrange's form. Its Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-f(z)} = \frac{dz}{1} \qquad ...(9)$$

From the first and the last fractions of (9), $dx - dz = 0$ so that $x - z = c_3$ ...(10)

From the last two fractions of (9), $dy - f(z)dz = 0$

Integrating, $y - F(z) = c_4,$ where $F(z) = \int f(z)dz$ ...(11)

From (10) and (11), the required solution is $y - F(z) = G(x - z)$

or $y = F(z) + G(x - z)$, where F, G are arbitrary functions.

Ex. 10. *Solve $e^{2y}(r - p) = e^{2x}(t - q)$ by Monge's method.*

Sol. Given $e^{2y}r - e^{2x}t = pe^{2y} - qe^{2x}$ ...(1)

Comparing (1) with $Rr + Ss + Tt = V$, here $R = e^{2y}$, $S = 0$, $T = -e^{2x}$ and $V = pe^{2y} - qe^{2x}$.

Hence Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $e^{2y}dxdy - e^{2x}dqdx - (pe^{2y} - qe^{2x})dxdy = 0$ ...(2)

and $e^{2y}(dy)^2 - e^{2x}(dx)^2 = 0$ ...(3)

From (3), $(e^y dy - e^x dx)\,(e^y dy + e^x dx) = 0$

so that $e^y dy - e^x dx = 0$, that is, $e^x dx = e^y dy$ ...(4)

and $e^y dy + e^x dx = 0$ ...(5)

For the required solution, we consider relation (4) only.

Integrating (4), $e^x - e^y = c_1$, c_1 being arbitrary constant ...(6)

Rewriting (2), $(e^y dp)(e^y dy) - (e^x dq)(e^x dx) - p(e^y dy)(e^y dx) + q(e^x dx)(e^x dy) = 0$

or $(e^y dp)(e^x dx) - (e^x dq)(e^x dx) - p(e^x dx)(e^y dx) + q(x^x dx)(e^y dy) = 0$, by (4)

or $e^y dp - e^x dq - pe^y dx + qe^x dy = 0$ or $\{d(e^y p) - pe^y dy\} - \{d(e^x q) - qe^x dx\} = pe^y dx - qe^x dy$

or $d(e^y p) - d(e^x q) = pe^y(dx + dy) - qe^x(dx + dy)$ or $d(e^y p - e^x q) = (e^y p - e^x q)\,(dx + dy)$

or $$\frac{d(e^y p - e^x q)}{e^y p - e^x q} = d(x + y)$$

Integrating, $\log(e^y p - e^x q) - \log c_2 = x + y$ or $(e^y p - e^x q)/c_2 = e^{x+y}$

or $(e^y p - e^x q)/e^{x+y} = c_2$, c_2 being an arbitrary constant ...(7)

From (6) and (7), one intermediate integral of (1) is

$(e^y p - e^x q)/e^{x+y} = f(e^x - e^y)$ or $e^y p - e^x q = e^{x+y} f(e^x - e^y)$

which is of Lagrange's form. Its Lagrange's auxiliary equations are

$$\frac{dx}{e^y} = \frac{dy}{-e^x} = \frac{dz}{e^{x+y} f(e^x - e^y)} \qquad ...(8)$$

From the first two fractions of (8), $e^x dx + e^y dy = 0$ so that $e^x + e^y = c_3$ ...(9)

Taking the first and third fraction of (8) and noting that $e^y = c_3 - e^x$ from (9), we get

$$\frac{dx}{e^y} = \frac{dz}{e^x e^y f(e^x - c_3 + e^x)} \quad \text{or} \quad dz = e^x f(2e^x - c_3)dx$$

or $dz - (1/2) \times f(2e^x - c_3)d(2e^x - c_3) = 0$ or $dz - (1/2) \times f(u)du = 0$, taking $u = 2e^x - c_3$

Integrating, $z - F(u) = c_3,$ where $F(u) = \int (1/2) \times f(u)du$

or $z - F(2e^x - c_3) = c_4$ or $z - F(e^x - e^y) = c_4$, by (9) ...(10)

From (9) and (10), the required solution is $z - F(e^x - e^y) = G(e^x + e^y)$

or $z = F(e^x - e^y) + G(e^x + e^y)$, where F, G are arbitrary functions.

Ex. 11. *Solve $x^2r - y^2t = xp - yq$ by Monge's method.*

Sol. Given $$x^2r - y^2t = xp - yq \qquad ...(1)$$

Comparing (1) with $Rr + Ss + Tt = V$, here $R = x^2$, $S = 0$, $T = -y^2$ and $V = xp - yq$.

Hence Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$ become

$$x^2dpdy - y^2dqdx - (xp - yq)dxdy = 0 \qquad ...(2)$$

and $$x^2(dy)^2 - y^2(dx)^2 = 0 \qquad ...(3)$$

Re–writing (3), $$(xdy - ydx)(xdy + ydx) = 0$$

so that $xdy - ydx = 0$ that is, $xdy = ydx$ ...(4)

and $$xdy + ydx = 0 \qquad ...(5)$$

From (4), $(1/y)dy - (1/x)dx = 0$ so that $y/x = c_1$ ...(6)

For the required solution, we consider relation (4) only.

Re–writing (2), $$(xdp)(xdy) - (ydq)(ydx) - (pdx)(xdy) + (qdy)(ydx) = 0$$

or $$(xdp)(ydx) - (ydq)(ydx) - (pdx)(ydx) + (qdy)(ydx) = 0, \text{ by (4)}$$

or $xdp - ydq - pdx + qdy = 0$ or $\{d(xp) - pdx\} - \{d(yq) - qdy\} - pdx + qdy = 0$

or $d(xp - yq) - 2pdx + 2qdy = 0$ or $d(xp - yq) - 2p\,dx + 2(y/x)dx = 0$, by (4)

or $d(xp - yq) - (2/x)(xp - yq)dx = 0$ or $\dfrac{d(xp - yq)}{xp - yq} - \dfrac{2dx}{x} = 0$

Integrating, $\log(xp - yq) - 2\log x = c_2$ or $(xp - yq)/x^2 = c_2$ ...(7)

From (6) and (7), one intermediate integral of (1) is

$(xp - yq)/x^2 = f(y/x)$ or $xp - yq = x^2 f(x/y)$ ...(8)

which is of Lagrange's form. Its Lagrange's auxiliary equations are

$$\frac{dx}{x} = \frac{dy}{-y} = \frac{dz}{x^2 f(y/x)} \qquad ...(9)$$

Taking the first two ratios of (9), $(1/x)\,dx + (1/y)\,dy = 0$ so that $\log x + \log y = c_3$

or $$xy = c_3,\ c_3 \text{ being an arbitrary constant} \qquad ...(10)$$

Taking the first and last fractions of (9), we get

$dz = x\,f(y/x)dx$ or $dz = x\,f(c_3/x^2)$, since by (10), $y = c_3/x$

$$\therefore \quad z = \int\left(-\frac{x^4}{2c_3}\right) f\left(\frac{c_3}{x^2}\right)\left(-\frac{2c_3}{x^3}\right)dx = \int\left(-\frac{c_3^2}{2c_3t^2}\right)f(t)dt, \text{ putting } \frac{c_3}{x^2} = t \text{ and } -\frac{2c_3}{x^3}dx = dt$$

or $z = -\dfrac{c_3}{2}\displaystyle\int\frac{f(t)}{t^2}dt + c_4 = c_3F(t) + c_4,$ where $F(t) = -\dfrac{1}{2}\displaystyle\int\frac{f(t)}{t^2}dt$

or $z - c_3F(c_3/x^2) = c_4$ or $z - xyF(y/x) = c_4$, by (10) ... (11)

From (10) and (11), the required solution is

$z - xy\ F(y/x) = G(xy)$ or $z = x^2(y/x)\ F\ (y/x) + G(xy)$

or $z = x^2H(y/x) + G(xy)$ where $H(y/x) = (y/x)\ F(y/x)$ and H, G are arbitrary functions.

Ex. 12. *Solve* $2x^2r - 5xys + 2y^2t + 2(px + qy) = 0$.
and hence find the surface satisfying the above equation and touching the hyperbolic paraboloid $z = x^2 - y^2$ *along its section by the plane* $y = 1$. **[Meerut 2001, I.A.S. 1978, Ranchi 2010]**

Sol. Given $$2x^2r - 5xys + 2y^2t = -2(px + qy). \qquad ...(1)$$

Comparing (1) with $Rr + Ss + Tt = V$, $R = 2x^2$, $S = -5xy$, $T = 2y^2$, $V = -2(px + qy)$

Hence Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - S\,dxdy + T(dx)^2 = 0$.

become $$2x^2dpdy + 2y^2dqdx + 2(px + qy)dxdy = 0. \qquad ...(2)$$

and $$2x^2(dy)^2 + 5xydxdy + 2y^2(dx)^2 = 0. \qquad ...(3)$$

Re–writing (3), $$(xdy + 2ydy)(2xdy + ydx) = 0.$$

so that $xdy + 2ydx = 0$, *i.e.*, $xdy = -2ydx$ ...(4)

and $$2xdy + ydx = 0. \qquad ...(5)$$

Keeping (4) in view, (2) may be re–written as

$$2xdp(xdy) - ydq(-2ydx) + 2pdx(xdy) - qdy(-2ydx) = 0.$$

or $2xdp(xdy) - ydq(xdy) + 2pdx(xdy) - qdy(xdy) = 0$, using (4)

or $2xdp - ydq + 2pdx - qdy = 0$ or $2(xdp + pdx) - (ydq + qdy) = 0$

or $2d(xp) - d(yq) = 0$ so that $2xp - yq = c_1$. ...(6)

From (4), $(1/y)dy + 2(1/x)dx = 0$ so that $\log y + 2\log x = \log c_2$

or $\log y + \log x^2 = \log c_2$ or $x^2y = c_2$. ...(7)

From (6) and (7), one intermediate integral is

$2xp - yq = f(x^2y)$, f being an arbitrary function. ...(8)

which is of Lagrange's form. Hence Lagrange's subsidiary equations are

$$\frac{dx}{2x} = \frac{dy}{-y} = \frac{dz}{f(x^2y)}. \qquad ...(9)$$

Taking the first two fractions of (9), $2(1/y)dy + (1/x)dx = 0$.

Integrating, $2\log y + \log x = \log a$ or $y^2x = a$ or $x = a/y^2$. ...(10)

Taking the second and third fractions of (9) and using (10), we get

$$\frac{dy}{-y} = \frac{dz}{f(a^2/y^3)} \quad \text{or} \quad dz + \frac{1}{y} f\left(\frac{a^2}{y^3}\right) dy = 0. \qquad ...(11)$$

Putting $(a^2/y^3) = v$ so that $-(3a^2/y^4)\,dy = dv$, (11) gives

$$dz + \frac{1}{y} f(v) \times \left(-\frac{y^4}{3a^2}\right) dv = 0 \quad \text{or} \quad dz - \frac{f(v)}{3(a^2/y^3)} dv = 0$$

or $dz - (1/3v) \times f(v)\,dv = 0$, as $v = a^2/y^3$.

Integrating, $z - F(v) = b$ or $z - F(a^2/y^3) = b$, b being an arbitrary constant.

or $z - F(x^2y) = b$, as $y^2x = a$. ...(12)

From (10) and (12), the required complete solution is

$z - F(x^2y) = G(xy^2)$, F and G being arbitrary fucntions.

or $$z = F(x^2y) + G(xy^2). \qquad ...(13)$$

Second Part. The given surface is $z = x^2 - y^2$. ...(14)

(13) $\Rightarrow$ $p = \partial z/\partial x = 2xy\,F'(x^2y) + y^2G'(xy^2)$ and $q = \partial z/\partial y = x^2F'(x^2y) + 2xyG'(xy^2)$. ...(15)

From (14), $p = \partial z/\partial x = 2x$ and $q = \partial x/\partial y = -2y$. ...(16)

Since (13) and (14) touch each other along their section by the plane $y = 1$, the values of p and q given by (15) and (16) at any point on $y = 1$ must be equal

Thus, $$2xyF'(x^2y) + y^2G'(xy^2) = 2x, \text{ where } y = 1 \qquad ...(17)$$

and $$x^2F'(x^2y) + 2xy\, G'(xy^2) = -2y, \text{ where } y = 1. \qquad ...(18)$$

From (17), $$2xF'(x^2) + G'(x) = 2x. \qquad ...(19)$$

From (18), $$x^2F'(x^2) + 2xG'(x) = -2. \qquad ...(20)$$

Solving (19) and (20) for $F'(x^2)$ and $G'(x)$, we have

$$F'(x^2) = (4/3) + (2/3) \times (1/x^2). \qquad ...(21)$$

and $$G'(x) = -(2/3)\times x - (4/3) \times (1/x). \qquad ...(22)$$

(21) $\Rightarrow$ $$F'(u) = (4/3) + (2/3) \times (1/u), \text{ on putting } x^2 = u$$

Integrating, $F(u) = (4/3) \times u + (2/3) \times \log u + c_1$, c_1 being an arbitrary constant

This $\Rightarrow$ $$F(x^2y) = (4/3) \times x^2y + (2/3) \times \log (x^2y) + c_1. \qquad ...(23)$$

Integrating (22), $G(x) = -(2/3)(x^2/2) - (4/3) \log x + c_2$, being an arbitrary constant

This $\Rightarrow$ $$G(xy^2) = -(1/3) \times x^2y^4 - (4/3) \times \log (xy^2) + c_2. \qquad ...(24)$$

Putting values of $F(x^2y)$ and $G(xy^2)$ given by (23) and (24) in (13), we get

$$z = (4/3) \times x^2y + (2/3) \times \log (x^2y) + c_1 - (1/3) \times x^2y^4 - (4/3) \times \log (xy^2) + c_2$$

or $$z = (4/3) \times x^2y - (1/3)\times x^2y^4 + (2/3) \times [\log (x^2y) - 2 \log (xy^2)] + c, \text{ taking } c_1 + c_2 = c$$

or $$z = (4/3)\times x^2y - (1/3) \times x^2y^4 + (2/3) \times [\log (x^2y) - \log (xy^2)^2]$$

or $$z = (4/3) \times x^2y - (1/3) \times x^2y^4 + (2/3) \times [\log \{(x^2y)/(x^2y^4)\} + c$$

or $$z = (4/3) \times x^2y - (1/3) \times x^2y^4 + (2/3) \times \log y^{-3} + c$$

or $$z = (4/3) \times x^2y - (1/3) \times x^2y^4 - 2 \log y + c. \qquad ...(25)$$

Now at the point of contact of (14) and (25), the values of z must be the same and hence

$$x^2 - y^2 = (4/3) \times x^2y - (1/3) \times x^2y^4 - 2 \log y + c, \text{ where } y = 1$$

$$\Rightarrow \quad x^2 - 1 = (4/3) \times x^2 - (1/3) \times x^2 + c, \text{ putting } y = 1$$

$$\Rightarrow \quad x^2 - 1 = x^2 + c \quad \Rightarrow \quad c = -1.$$

Putting $c = -1$ in (25), the required surface is

$$z = (4/3) \times x^2y - (1/3) \times x^2y^4 - 2 \log y - 1 \quad \text{or} \quad 3z = 4x^2y - x^2y^4 - 6 \log y - 3.$$

9.7. Type 3. When the given equation Rr + Ss + Tt = V leads to two identical intermediate intergrals.

Working rule for solving problems of type 3

Step 1. Write the given equation in the standard form $Rr + Ss + Tt = V$.

Step 2. Substitute the values of R, S, T and V in the Monge's subsidiary equations

$Rpdy + Tdqdx - Vdx\, dy = 0$ ... (1) $\qquad R(dy)^2 - S\, dxdy + T(dx)^2 = 0$ ... (2)

Step 3. R.H.S. of (2) reduces to a perfect square and hence it gives only one distinct factor in place of two as in type 1 and type 2.

Step 4. Start with the only one factor of step 3 and use (2) to get an intermediate integral.

Step 5. Re–write the intermediate integral of the step 4 in the form of $Pp + Qq = R$ and use Lagrange's method to obtain the required general solution of the given equation.

9.8. Solved examples based on Art 9.7

Ex. 1. *Solve* : $(1 + q)^2r - 2(1 + p + q + pq)s + (1 + p)^2t = 0$

[Meerut 2002, Delhi Maths (H) 1999 2007, 10; Rohailkhand 1997; Kanpur 1994]

Sol. Comparing the given equation with $Rr + Ss + Tt = V$, ...(1)

$$R = (1 + q)^2, \qquad S = -2(1 + p + q + pq), \qquad T = (1 + p)^2, \qquad V = 0. \quad ...(2)$$

Monge's subsidiary equations are $Rdpdy + Tdqdx - Vdxdy = 0$ ...(3)

and $R(dy)^2 - Sdxdy + T(dx)^2 = 0.$ ...(4)

Using (2), (3) and (4) become

$$(1 + q)^2 dpdy + (1 + p)^2 dqdx = 0 \quad ...(5)$$

and $(1 + q)^2(dy)^2 + 2(1 + p + q + pq)dxdy + (1 + p)^2\ (dx)^2 = 0.$ ...(6)

Since $1 + p + q + pq = (1 + p)\ (1 + q)$, (6) becomes $[(1 + q)dy + (1 + p)dx]^2 = 0$

so that $(1 + q)dy + (1 + p)dx = 0$ or $(1 + q)dy = -(1 + p)dx.$ ...(7)

Keeping (7) in view, (5) may be re–written as

$$(1 + q)dp\ \{(1 + q)dy\} - (1 + p)dq\ \{-(1 + p)dx\} = 0. \quad ...(8)$$

Dividing each term of (8) by $(1 + q)dy$, or its equivalent $-(1 + p)dx$, we get

$(1 + q)dp - (1 + p)dq = 0$ or $dp/(1 + p) - dq/(1+ q) = 0.$

Integrating it, $(1+ p)/(1 + q) = c_1$, c_1 being an arbitrary constant ...(9)

From (7), $dx + dy + pdx + qdy = 0$ or $dx + dy + dz = 0$, as $dz = pdx + qdy$

Integrating it, $x + y + z = c_2$, c_2 being an arbitrary constant ...(10)

From (9) and (10), one intermediate integral of (1) is

$(1 + p)/(1 + q) = F(x + y + z)$ or $1 + p = (1 + q)\ F(x + y + z)$

or $p - q\ F(x + y + z) = F\ (x + y + z) - 1,$ ...(11)

which is of the form $Pp + Qq = R$. So Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-F(x+y+z)} = \frac{dz}{F(x+y+z)-1} \quad ...(12)$$

Choosing 1, 1, 1 as multipliers, each fraction of (12) = $(dx + dy + dz)/0$

so that $dx + dy + dz = 0$ giving $x + y + z = c_2$ ... (13)

Using (13) and taking the first two fractions of (12), we have

$dx = -\ dy/F(c_2)$ or $dy + F(c_2)dx = 0.$

Integrating it, $y + xF(c_2) = c_3$ or $y + x\ F(x + y + z) = c_3$ ...(14)

From (13) and (14), the required general solution is

$y + x\ F(x + y + z) = G(x + y + z)$, F, G being arbitrary functions.

Ex. 2. *Solve* $y^2r + 2xys + x^2t + px + qy = 0.$ **[Bilaspur 2004]**

Sol. Given $y^2r + 2xys + x^2t = -\ (px + qy).$ ...(1)

Comparing (1) with $Rr + Ss + Tt = V$, here $R = y^2$, $S = 2xy$, $T = x^2$, $V = -\ (px + qy).$ ...(2)

Monge's subsidiary equations are $Rdpdy + Tdqdx + Vdxdy = 0$ ...(3)

and $R(dy)^2 - Sdxdy + T(dx)^2 = 0.$ ...(4)

Using (2), (3) and (4) become

$$y^2dpdy + x^2dqdx + (px + qy)\ dxdy = 0 \quad ...(5)$$

and $y^2(dy)^2 - 2xydxdy + x^2(dx)^2 = 0.$ ...(6)

From (6), $(xdx - ydy)^2 = 0$ so that $xdx - ydy = 0$ or $xdx = ydy.$ ...(7)

Keeping (7) in view, (5) may be re–written as

$$ydp\ (ydy) + xdq\ (xdx) + pdy\ (xdx) + qdx\ (ydy) = 0. \quad ...(8)$$

Dividing each term of (8) by xdx, or its equivalent ydy, we get

$ydp + xdq + pdy + qdx = 0$ or $(ydp + pdy) + (xdq + qdx) = 0$

Integrating it, $yp + xq = c_1$, being an arbitrary constant ...(9)

Integrating (7), $x^2/2 - y^2/2 = c_2/2$ or $x^2 - y^2 = c_2.$ ...(10)

From (9) and (10), one intermediate integral of (1) is $\quad yp + xq = F(x^2 - y^2)$, ...(11)
which is of the form $Pp + Qq = R$. Its Lagrange's auxiliary equations are

$$\frac{dx}{y} = \frac{dy}{x} = \frac{dz}{F(x^2 - y^2)}. \qquad \text{...(12)}$$

From the first two fractions of (2), $\quad xdx - ydy = 0 \quad$ so that $\quad x^2 - y^2 = c_2$. ...(13)

Taking the last two fractions and using (13), we get

$$\frac{dy}{(y^2 + c_2)^{1/2}} = \frac{dz}{F(c_2)} \qquad \text{or} \qquad dz - F(c_2)\frac{dy}{(y^2 + c_2)^{1/2}} = 0.$$

Integrating, $\quad z - F(c_2) \log [y + (y^2 + c_2)^{1/2}] = c_3$

or $\quad z - F(x^2 - y^2) \log [y + \sqrt{(y^2 + x^2 - y^2)}\,] = c_3$, using (13)

or $\quad z - F(x^2 - y^2) \log (x + y) = c_3$, c_3 being an arbitrary constant ...(14)

From (13) and (14), the required general solution is

$z - F(x^2 - y^2) \log (x + y) = G(x^2 - y^2)$, F, G being arbitrary functions.

Ex. 3(a). *Obtain the integral of* $q^2 r - 2pqs + p^2 t = 0$ *in the form* $y + xf(z) = F(z)$.

[Delhi Maths Hons. 1999, 2007; Meerut 1994, 95; Nagpur 2005]

(b) *Show also that this solution represents a surface generated by straight lines that are parallel to a fixed plane.*

Sol. (*a*) Given $\quad q^2 r - 2pqs + p^2 t = 0.$ ...(1)

As ususal Monge's subsidiary equations are $\quad q^2 dpdy + p^2 dp\, dx = 0$ ...(2)

and $\quad q^2 (dy)^2 + 2pqdxdy + p^2 (dx)^2 = 0 \quad$ or $\quad (qdy + pdx)^2 = 0.$ ...(3)

From (3), we have $\quad qdy + pdx = 0 \quad$ or $\quad qdy = -pdx.$ ...(4)

In view of (4), (2) may be re–written as $\quad qdp\,(qdy) - pdq\,(-pdx) = 0.$ ...(5)

Dividing each term of (5) by qdy, or its equivalent $(-pdx)$, we find

$qdp - pdq = 0 \quad$ or $\quad (1/p)dp - (1/q)\, dp = 0.$

Integrating it, $\quad p/q = c_1$, c_1 being an arbitrary constant ...(6)

From (4), $\quad dz = 0$, $\quad$ (as $dz = pdx + qdy$) $\quad$ so that $\quad z = c_2$. ...(7)

From (6) and (7), one integral of (1) is $\quad p/q = f(z) \quad$ or $\quad p - f(z)q = 0$, ...(8)
which is of the form $Pp + Qq = R$. Here f is an arbitrary function. Its Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-f(z)} = \frac{dz}{0}. \qquad \text{...(9)}$$

The last fraction in (9) gives $\quad dz = 0 \quad$ so that $\quad z = c_2$ ...(10)

From the first two fractions in (9) and (10), we find

$$\frac{dx}{1} = \frac{dy}{-f(c_2)} \qquad \text{or} \qquad dy + f(c_2)dx = 0.$$

Integrating, $\quad y + xf(c_2) = c_3 \quad$ or $\quad y + xf(z) = c_3$, by (10). ...(11)

From (10) and (11), the required integral is $\quad y + xf(z) = \mathrm{F}(z).$ ...(12)

Part (*b*). Let $z = k$, k being an arbitrary constant. Then (12) is the locus of the straight lines given by the intersection of the planes

$$z = k \qquad \text{and} \qquad y + xf(k) - F(k) = 0. \qquad \text{...(13)}$$

Clearly the lines are parallel to the plane $z = 0$ (which is a fixed plane) because these lie on the plane $z = k$ for different values of k.

Ex. 4. *Solve* $y^2 r - 2ys + t = p + 6y$. **[Agra 1993; Bhopal 2004; Vikram 2004; Meerut 2009; Delhi Maths Hons 1994, 98, 2006, 09, 10]**

Sol. As usual Monge's subsidiary equations are

$$y^2 dpdy + dqdx - (p + 6y)dxdy = 0 \qquad ...(1)$$

and $y^2(dy)^2 + 2ydydx + (dx)^2 = 0$ or $(ydy + dx)^2 = 0$. ...(2)

From (2), $ydy + dx = 0$ or $dx = -ydy$. ...(3)

Putting the value of dx from (3) in (1), we find

$$y^2 dpdy + dq(-ydy) - (p + 6y)\, dy\, (-ydy) = 0$$

or $ydp - dq + (p + 6y)\, dy = 0$ or $(ydp + pdy) - dq + 6ydy = 0$.

Integrating it, $yp - q + 3y^2 = c_1$, c_1 being an arbitrary constant ...(4)

Integrating (4), $y^2/2 + x = c_2/2$ or $y^2 + 2x = c_2$. ...(6)

From (5) and (6), one integral of (1) is

$yp - q + 3y^2 = F(y^2 + 2x)$ or $yp - q = F(y^2 + 2x) - 3y^2$, ...(7)

which is of the form $Pp + Qq = R$. Its Lagrange's auxiliary equations are

$$\frac{dx}{y} = \frac{dy}{-1} = \frac{dz}{F(y^2 + 2x) - 3y^2}. \qquad ...(8)$$

From the first two fractions of (8), $2ydy + 2dx = 0$ so that $y^2 + 2x = c_2$. ...(9)

Taking the last two fractions of (8) and using (9), $dz + [F(c_2) - 3y^2]dy = 0$.

Integrating, $z + yF(c_2) - y^3 = c_2$ or $z + yF(y^2 + 2x) - y^3 = c_3$. ...(10)

From (9) and (10), the required general solution is

$$z + yF(y^2 + 2x) - y^3 = G(y^2 + 2x), \ F, G \text{ being arbitrary functions.}$$

Ex. 5. *Solve* $(b + cq)^2 r - 2(b + cq)\,(a + cp)s + (a + cp)^2 t = 0$

Sol. Usual Monge's subsidiary equations are $(b + cq)^2\, dpdy + (a + cp)^2\, dqdx = 0$. ...(1)

and $(b + cq)^2\, (dy)^2 + 2(b + cq)\,(a + cp)\, dxdy + (a + cp)^2\, (dx)^2 = 0$. ...(2)

(2) $\Rightarrow$ $\{(b + cq)dy + (a + cp)dx\}^2 = 0$ ...(3)

or $(b + cq)dy + (a + cp)dx = 0$ or $adx + bdy + c(pdx + qdy) = 0$

or $adx + bdy + cdz = 0$, as $dz = pdx + qdy$.

Integrating, $ax + by + cz = c_1$, c_3 being an arbitrary constant ...(4)

From (3), $(b + cq)dy = -(a + cp)dx$. So (1) reduces to $(b + cq)dp - (a + cp)dq = 0$

or $\dfrac{dp}{a + cp} - \dfrac{dq}{b + cq} = 0$ so that $\dfrac{a + cp}{b + cq} = c_2$ ...(5)

So the intermediate integral of the given equation is $(a + cp)/(b + cq) = \phi_1(ax + by + cz)$

or $cp - c\phi_1(ax + by + cz)q = -a + b\,\phi_1(ax + by + cz)$. ...(6)

Lagrange's auxiliary equations are

$$\frac{dx}{c} = \frac{dy}{-c\,\phi_1(ax + by + cz)} = \frac{dz}{-a + b\,\phi_1(ax + by + cz)}. \qquad ...(7)$$

Using a, b, c as multipliers, each fraction of (7) = $(adx + bdy + cdz)/0$

$\therefore$ $adx + bdy + cdz = 0$ so that $ax + by + cz = c_3$. ...(8)

Using (8) and taking the first two ratios of (7), we get

$dx = -\,dy/\phi_1(c_3)$ or $dy + \phi_1(c_3)dx = 0$.

Integrating, $y + x\phi_1(c_3) = c_4$ or $y + x\phi_1(ax + by + cz) = c_4$. ...(9)

From (8) and (9), the required solution is

$$y + x\phi_1(ax + by + cz) = \phi_2(ax + by + cz), \ \phi_1, \phi_2 \text{ being arbitrary functions.}$$

Ex. 6. *Solve* $x^2 r - 2xs + t + q = 0$. **[K.U. Kurukshetra 2004; Ravishankar 2005]**

Sol. Usual Monge's subsidiary equations are $x^2 dpdy + dqdx + qdxdy = 0$ ...(1)

and $x^2(dy)^2 + 2xdxdy + (dx)^2 = 0$. ...(2)

Now, $(2) \Rightarrow \quad (xdy + dx)^2 = 0 \quad \Rightarrow \quad xdy + dx = 0$ ...(3)

$(3) \Rightarrow \quad (dx)/x + dy = 0 \quad \Rightarrow \quad y + \log x = c_1.$ ...(4)

Using (3), (1) reduces to $\quad x^2 dpdy + dq\,(-x\,dy) + q(-x\,dy)dy = 0$

or $\quad dp - \left(\dfrac{dq}{x} - \dfrac{qdx}{x^2}\right) = 0 \quad$ or $\quad d\left(p - \dfrac{q}{x}\right) = 0.$

Integrating, $\quad p - (q/x) = c_2$, c_2 being an arbitrary constant ...(5)

From (4) and (5), the intermediate integral of the given equation is

$p - (q/x) - \phi_1(y + \log x) \quad$ or $\quad xp - q = x\phi_1(y + \log x).$ ...(6)

Lagrange's auxiliary equations for (6) are $\quad \dfrac{dx}{x} = \dfrac{dy}{-1} = \dfrac{dz}{x\,\phi_1(y + \log x)}.$ ...(7)

Taking the first two fractions of (7), $\quad (1/x)dx + dy = 0 \Rightarrow y + \log x = c_3.$ ...(8)

Using (8), first and third fractions of (7) give $\quad \dfrac{dx}{x} = \dfrac{dz}{x\,\phi_1(c_3)} = \Rightarrow z - x\phi_1(c_3) = c_4$

or $\quad z - x\phi_1(y + \log x) = c_4$, c_4 being an arbitrary constant ...(9)

From (8) and (9) the required solution is

$$z - x\phi_1(y + \log x) = \phi_2(y + \log x),\ \phi_1,\ \phi_2 \text{ being arbitrary functions.}$$

Ex. 7. *Solve* $(y - x)\,(q^2 r - 2pqs + p^2 t) = (p + q)^2\,(p - q).$

Sol. The usual Monge's subsidiary equations are

$$(y - x)\,(q^2 dpdy + p^2 dqdx) - (p + q)^2\,(p - q)dxdy = 0 \quad ...(1)$$

and

$$q^2(dy)^2 + 2pqdxdy + p^2(dx)^2 = 0. \quad ...(2)$$

$(2) \Rightarrow \quad (qdy + pdx)^2 = 0 \quad$ or $\quad qdy + pdx = 0.$ ...(3)

$dz = pdx + qdy$ and (3) $\quad \Rightarrow \quad dz = 0 \quad \Rightarrow \quad z = c_1.$ ...(4)

Using (3), (1) reduces to $\quad (y - x)\,(qdp - pdq) - (p^2 - q^2)\,(dx - dy) = 0$

or $\quad q^2 d\left(\dfrac{p}{q}\right) - (p^2 - q^2)\,\dfrac{d(x-y)}{y-x} = 0 \quad$ or $\quad \dfrac{d(x-y)}{x-y} + \dfrac{d(p/q)}{(p/q)^2 - 1} = 0$

Integrating, $\quad \log(x - y) + \dfrac{1}{2}\log\dfrac{(p/q)-1}{(p/q)+1} = \dfrac{1}{2}\log c_2 \quad$ or $\quad (x - y)^2\dfrac{p-q}{p+q} = c_2.$ ...(5)

From (4) and (5), the intermediate integral of the given equation is

$(x - y)^2\dfrac{p-q}{p+q} = \phi_1(z) \quad$ or $\quad (x - y)^2(p - q) = (p + q)\phi_1(z)$

or $\quad p\{(x - y)^2 - \phi_1(z)\} - q\{(x - y)^2 + \phi_1(z)\} = 0.$ ...(6)

Here Lagrange's subsidiary equation for (6) are

$$\frac{dx}{(x-y)^2 - \phi_1(z)} = \frac{dy}{-\{(x-y)^2 + \phi_1(z)\}} = \frac{dz}{0}. \quad ...(7)$$

Now, the third fraction of (7) $\Rightarrow \quad dz = 0 \quad$ so that $\quad z = a,$ ...(8)

where 'a' is an arbitrary constant.

Now, each fraction of (7) $= \dfrac{dx + dy}{-2\phi_1(z)} = \dfrac{dx - dy}{2(x-y)^2} \quad \Rightarrow \quad d(x + y) = -\,\phi_1(a)\dfrac{d(x-y)}{(x-y)^2}$, by (8).

Integrating it, $\ x + y - \phi_1(a)\,(x - y)^{-1} = b \quad$ or $\quad x + y - \phi_1(z)\,(x - y)^{-1} = b$, using (8). ...(9)

From (8) and (9), the required general solution is

$$x + y - (x - y)^{-1}\,\phi_1(z) = \phi_2(z),\ \phi_1,\ \phi_2 \text{ being arbitrary functions.}$$

Ex. 8. *Solve* $x^2r + 2xys + y^2t = 0$. **[Meerut 2003, Garhwal 1993; Delhi Maths (H) 2001]**

Sol. Comparing the given equation with $Rr + Ss + Tt = V$, we get

$R = x^2$, $S = 2xy$, $T = y^2$. Hence the usual Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - S\,dxdy + T(dx)^2 = 0$

become $$x^2dpdy + y^2dqdx = 0 \qquad \text{...(1)}$$

and $$x^2(dy)^2 - 2xydxdy + y^2(dx)^2 = 0. \qquad \text{...(2)}$$

Now, (2) gives $(xdy - ydx)^2 = 0$ so that $xdy - ydx = 0$. ...(3)

Re–writing (1), $(xdp)\,(xdy) + (ydx)\,(ydq) = 0$

or $(xdp)(xdy) + (xdy)\,(ydq) = 0$ [$\because$ from (3), $ydx = xdy$]

or $xdp + ydq = 0$ or $xdp + ydq + pdx + qdy = pdx + qdy$

or $d(xp) + d(yq) - dz = 0$, as $dz = pdx + qdy$.

Integrating (1) $xp + yq - z = c_1$, c_1 being an arbitrary constant ...(4)

Now (3) gives $(1/y)dy - (1/x)dx = 0$.

Integrating, $\log y - \log x = \log c_2$ or $y/x = c_2$. ...(5)

From (4) and (5), the intermediate integral of the given equation is

$xp + yq - z = f(y/x)$ or $xp + yq = z + f(y/x)$, ...(6)

where f is an arbitrary function. Lagrange's subsidiary equation for (6) are

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z + f(y/x)}. \qquad \text{...(7)}$$

Taking the first two fractions of (7), $(1/y)dy - (1/x)dx = 0$.

Integrating, $\log y - \log x = \log a$ so that $y/x = a$. ...(8)

Taking the last two fractions of (7) and using (8), we get $\dfrac{dz}{z + f(a)} - \dfrac{dy}{y} = 0$.

Integrating it, $\log\,[z + f(a)] - \log y = \log b$, b being an arbitrary constant

so that $[z + f(a)]/y = b$ or $[z + f(y/x)]/y = b$, using (8) ...(9)

From (8) and (9), the required solution is

$[z + f(y/x)]/y = g(y/x)$ or $z = yg(y/x) - f(y/x)$, where f and g are arbitrary functions.

Ex. 9. *Solve* $r - 2s + t = \sin\,(2x + 3y)$.

Sol. Comparing the given equation with $Rr + Ss + Tt = V$, we have

$R = 1$, $S = -2$, $T = 1$, $V = \sin\,(2x + 3y)$. So Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $$dpdy + dqdx - \sin\,(2x + 3y)dxdy = 0. \qquad \text{...(1)}$$

and $$(dy)^2 + 2\,dxdy + (dx)^2 = 0. \qquad \text{...(2)}$$

Now, (2) gives $(dy + dx)^2 = 0$ so that $dy + dx = 0$. ...(3)

From (3), $dy = -\,dx$. Then, (1) becomes $-\,dpdx + dqdx + \sin\,(2x + 3y)dxdy = 0$

or $$dp - dq + \sin\,(2x + 3y)dy = 0, \text{ as } dx \neq 0. \qquad \text{...(4)}$$

Now, integrating (3), $x + y = c_1$, c_1 being an arbitrary constant ...(5)

From (4), $dp - dq + \sin\,[2(x + y) + y]dy = 0$ or $dp - dq + \sin\,(2c_1 + y)dy = 0$, using (5).

Integrating, $p - q - \cos\,(2c_1 + y) = c_2$

or $$p - q - \cos(2x + 3y) = c_2, \text{ as } c_1 = x + y \qquad \text{...(6)}$$

From (5) and (6), an intermediate integral is

$p - q - \cos\,(2x + 3y) = f(x + y)$ or $p - q = \cos\,(2x + 3y) + f(x + y)$, ...(7)

where f is an arbitrary function. Its Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-1} = \frac{dz}{\cos(2x+3y)+f(x+y)}. \qquad ...(8)$$

Taking the first two fractions of (8), $dx + dy = 0$ so that $x + y = a$ ... (9)

Taking the last two fractions of (8) and using (9), we get

$$\frac{dy}{-1} = \frac{dz}{\cos(2a+y)+f(a)} \qquad \text{or} \qquad dz + [\cos(2a+y) + f(a)]dy = 0.$$

Integrating it, $z + \sin(2a + y) + y f(a) = b$, b being an arbitrary constant

or $z + \sin(2x + 3y) + y f(x + y) = b$, using (9). ...(10)

From (9) and (10) the required complete integral is

$z + \sin(2x + 3y) + y f(x + y) = g(x + y)$, f and g being an arbitrary functions.

Ex. 10. *Solve* $q^2 r - 2pqs + p^2 t = pq^2$. **[I.A.S. 1986]**

Sol. Comparing the given equation with $Rr + Ss + Tt = V$, we have

$R = q^2$, $S = -2pq$, $T = p^2$, $V = pq^2$. The Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $q^2 dpdy + p^2 dqdx - pq^2 dxdy = 0$ ...(1)

and $q^2(dy)^2 + 2pqdxdy + p^2(dx)^2 = 0$. ...(2)

Re–writing (2), $(qdy + pdx)^2 = 0$ so that $pdx + qdy = 0$. ...(3)

Since $dz = pdx + qdy$, (3) $\Rightarrow$ $dz = 0$ so that $z = c_1$. ...(4)

Re–writing (1), $(qdy)(qdp) + (pdx)(pdq) - (qdy)(pqdx) = 0$

or $(qdy)(qdp) - (qdy)(pdq) - (qdy)(pqdx) = 0$, as from (3), $pdx = -qdy$

or $qdp - pdq - pqdx = 0$ or $(1/p)dp - (1/q)dq = dx$.

Integrating, $\log p - \log q - \log c_2 = x$ or $p/(c_2 q) = e^x$

or $(p/q)e^{-x} = c_2$, c_2 being an arbitrary constant ...(5)

From (4) and (5), the intermediate integral of the given equation is

$(p/q)e^{-x} = f(z)$ or $px^{-x} - f(z)q = 0$. ...(6)

Lagrange's auxiliary equations for (6) are $\dfrac{dx}{e^{-x}} = \dfrac{dy}{-f(z)} = \dfrac{dz}{0}$. ...(7)

The last fraction of (7) $\Rightarrow$ $dz = 0$ so that $z = a$. ...(8)

Taking the first fractions of (7) and using (8), we get

$$\frac{dx}{e^{-x}} = \frac{dy}{-f(a)} \qquad \text{or} \qquad e^x f(a)dx + dy = 0.$$

Integrating, $e^x f(a) + y = b$ or $e^x f(z) + y = b$, as from (8), $a = z$ ...(9)

From (8) and (9), the required complete integral is

$e^x f(z) + y = g(z)$, where f and g are arbitrary functions.

Ex. 11. *Solve* $q^2 r - 2q(1+p)s + (1+p)^2 t = 0$ *by Monge's method.*

Sol. Given $q^2 r - 2q(1+p)s + (1+p)^2 t = 0$ ... (1)

Comparing (1) with $Rr + Ss + Tt = V$, here $R = q^2$, $S = -2q(1+p)$ and $T = (1+p)^2$.

Hence Monge's subsidiary equations

$Rdpdy + Tdqdx - Vdx\,dy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$ become

$$q^2 dpdy + (1+p)^2 \ dqdx = 0 \quad \text{... (2)}$$

and $$q^2 (dy)^2 + 2q(1+p)dxdy + (1+p)^2 \ (dx)^2 = 0 \quad \text{... (3)}$$

Rewriting (3), $\{qdy + (1+p)dx\}^2 = 0$ or $qdy + (1+p)\,dx = 0$ ... (4)

From (4), $dx + (pdx + qdy) = 0$ or $dx + dz = 0$, as $dz = pdx + qdy$

Integrating, $x + z = C_1$, C_1 being an arbitrary constant ... (5)

Re–writing (2), $$(qdy)\ (qdp) + [(1+p)\ dx\} \times \{(1+p)dq\} = 0$$

or $$(qdy)\ (qdp) + (-qdy)\ [(1+p)\ dq] = 0, \text{ using (4)}$$

or $qdp - (1+p)\,dq = 0$ or $\{1/(1+p)\}dp - (1/q)dq = 0$

Integrating, $\log(1+p) - \log q = \log C_2$ or $(1+p)/q = C_2$ ... (6)

From (5) and (6), the intermediate integral of (1) is

$(1+p)/q = f(x+z)$ or $p - q\,f(x+z) = -1$ ... (7)

which is of Lagrange's form. Its Lagrange's auxiliary equations are

$$\frac{dx}{1} = \frac{dy}{-f(x+z)} = \frac{dz}{-1} \quad \text{... (8)}$$

Taking the first and last ratios, $dx + dz = 0$ $\Rightarrow$ $x + z = C_3$ ... (9)

Using (9) and taking the first two ratios of (8), we get

$dy + f(C_3)dx = 0$ so that $y + xF(C_3) = C_4$

or $$y + x\ f(x+z) = C_4, \text{ using (9)} \quad \text{... (10)}$$

From (9) and (10), the required general solution is

$$y + xf(x+z) = g(x+z), \ f, g \text{ are arbitrary functions}$$

Ex. 12. *Solve* $(x-y)\ (x^2 - 2xys + y^2 t) = 2xy\ (p-q)$. **[Delhi B.Sc. (Hons) 2011]**

Sol. Given $$x^2(x-y)r - 2xy(x-y)s + y^2(x-y)t = 2xy(p-q) \quad \text{... (1)}$$

Comparing (1) with $Rr + Ss + Tt = V$, here $R = x^2(x-y)$, $S = -2xy\ (x-y)$, $T = y^2\ (x-y)$ and $V = 2xy\ (p-q)$. Hence Monge's subsidiars equations

$Rdpdy + Tdqdx - Vdxdy = 0$ and $R(dy)^2 - S\,dyxy + T(dx)^2 = 0$ become

$$x^2(x-y)\ dpdy - 2xy\ (p-q)\ dxdy + y^2(x-y)dqdx = 0 \quad \text{... (2)}$$

and $$(x-y)\{x^2(dy)^2 + 2xy\ dx\ dy + y^2(dy)^2\} = 0 \quad \text{... (3)}$$

Since $x \neq y$, (3) gives $(xdy + ydx)^2 = 0$ so that $ydx = -xdy$ ... (4)

From (4), $(1/x)dx + (1/y)dy = 0$ so that $xy = C_1$ ... (5)

Re–writing (2), $x(x-y)\ dp\ (xdy) - 2(p-q)(xdy)\ (ydx) + y(x-y)\,dq(ydx) = 0$

or $$x(x-y)\,dp(xdy) - 2(p-q)\ (xdy)\ (ydx) + y(x-y)dq(-xdy), by(4)$$

or $$x(x-y)dp - 2\ (p-q)\ (ydx) - y(x-y)\ dq = 0$$

or $\quad (x-y)\ (xdp-ydq)=2y\ (p-q)dx \quad$ or $\quad xdp-ydq=\{2y(p-q)dx\}/(x-y)$

or $$(xdp+pdx)-(ydq+qdy)=\{2y(p-q)\ dx\}/(x-y)+pdx-qdy$$

or $$d(xp)-d(yq)=\{2(p-q)\ ydx+(x-y)p\,dx-(x-y)\ qdy\}/(x-y)$$

or $\quad (x-y)d(xp-yq)=2pydx-2qydx+xpdx-ypdx-xqdy+yqdy$

$$=\ pydx-2qydx+xpdx+qydx+yqdy\ =\ -pxdy-qydx+xpdx+yqdy,\ by(4)$$

$\therefore \quad (x-y)\ d(xp-yq)=xp(dx-dy)-yq(dx-dy)=(xp-yq)(dx-dy)$

or $\quad \dfrac{d(xp-yq)}{xp-yq}=\dfrac{dx-dy}{x-y} \quad$ or $\quad \dfrac{d(xp-yq)}{xp-yq}-\dfrac{d(x-y)}{x-y}=0.$

Integrating, $\quad \log(xp-yq)-\log(x-y)=\log C_2 \quad$ or $\quad (xp-yq)/(x-y)=C_2 \quad$ ... (6)

From (5) and (6), the intermediate integral of the given equation is

$(xp-yq)/(x-y)=f(xy) \quad$ or $\quad xp-yq=(x-y)f(xy), \quad$ ... (7)

which is of Lagrange's form. Its auxiliary equations are $\quad \dfrac{dx}{x}=\dfrac{dy}{-y}=\dfrac{dz}{(x-y)f(xy)} \quad$ ... (8)

Taking the first two fractions, $\quad (1/x)dx+(1/y)dy=0 \quad$ so that $\quad xy=C_3 \quad$ ... (9)

Now, $\quad$ each fraction of (8) = $\dfrac{dx+dy}{x-y}=\dfrac{dz}{(x-y)f(xy)}$

or $\quad dz=f(xy)\ d(x+y) \quad$ or $\quad dz=f(C_3)\ d(x+y)$, by (9)

Integrating, $\quad z-(x+y)\ f(C_3)=C_4 \quad$ or $\quad z-(x+y)\ f(xy)=C_4 \quad$ ... (10)

From (9) and (10), the required solution is $\quad z-(x+y)f(xy)=g(xy)$

or $\quad z=(x+y)f(xy)+g(xy)$, f and g being arbitrary functions.

9.9 Type 4. When the given equation Rr + Ss + Tt = V fails to yield an intermediate integral as in cases 1, 2 and 3.

Working rule for solving problems of type 4.

Suppose the R.H.S. of $R(dy)^2 - Sdxdy + T(dx)^2 = 0$ neither gives two factors nor a perfect square (as in Types 1, 2 and 3 above). In such cases factors dx, dy, p, $1+p$ etc. are cancelled as the case may be and an integral of given equation is obtained as usual. This integral is then integrated by methods explained in chapter 7.

9.10 SOLVED EXAMPLES BASED ON ART 9.9

Ex. 1. *Solve* $(q+1)s=(p+1)t$. **[Agra 2009]**

Sol. Given $\quad (q+1)s-(p+1)t=0. \quad$ ...(1)

Comparing (1) with $Rr+Ss+Tt=V$, we find $R=0$, $S=(q+1)$. $T=-(p+1)$, $V=0$....(2)

Monge's subsidiary equations are $\quad Rdpdy+Tdqdx-Vdxdy=0. \quad$ ...(3)

and $\quad R(dy)^2-Sdxdy+T(dx)^2=0. \quad$ ...(4)

Using (2), (3) and (4) become $\quad -(p+1)dqdx=0 \quad$ ...(5)

and $\quad -(q+1)dxdy-(p+1)(dx)^2=0. \quad$ ...(6)

Dividing (5) by $-(p+1)dx$, we obtain $\quad dq=0. \quad$ ...(7)

and dividing (6) by $-dx$ we get $\quad (q+1)+(p+1)dx=0. \quad$ ...(8)

From (8), $\quad dx+dy+pdx+qdy=0 \quad$ or $\quad dx+dy+dz=0, \quad$ as $\quad dz=pdx+qdy$

Integrating it, $x + y + z = c_1$, being an arbitrary constant ...(9)

Integrating (7), $q = c_2$, c_2 being an arbitrary constant ...(10)

From (9) and (10), an integral of (1) is

$q = f(x + y + z)$ or $\partial z/\partial y = f(x + y + z)$ ...(11)

Integrating (11) partially w.r.t. y (treating x as constant), we find

$$z = F(x + y + z) + G(x),\ F,\ G \text{ being arbitrary functions.}$$

Ex. 2. *Solve* $pq = x(ps - qr)$. **[Delhi. Maths (H) 2002, 08]**

Sol. Given $xqr - xps + 0.t = -pq$. ...(1)

Comparing (1) with $Rr + Ss + Tt = V$, $R = xq$, $S = xp$, $T = o$ and $V = -pq$

Monge's subsidiary equations $Rdp\,dy + T\,dq\,dx - V\,dx\,dy = 0$ and $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $xqdpdy + pqdxdy = 0.$ ...(2)

and $xq(dy)^2 + xpdxdy = 0.$ ...(3)

Dividing (2) by qdy we get $xdp + pdx = 0$ ...(4)

and dividing (3) by xdy, we get $qdy + pdx = 0.$ ...(5)

Using $dz = pdx + qdy$, (5) gives $dz = 0$ so that $z = c_1$ ...(6)

Integrating (4), $xp = c_2$, c_2 being an arbitrary constant ...(7)

From (6) and (7), one integral of (1) is

$$xp = f(z) \qquad \text{or} \qquad x\frac{\partial z}{\partial x} = f(z) \qquad \text{or} \qquad \frac{1}{f(z)}\frac{\partial z}{\partial x} = \frac{1}{x}.$$

Integrating it partially w.r.t. x, $F(z) = \log x + G(y)$, F, G being arbitrary functions.

Ex. 3. *Solve* $pt - sqs = q^3$ **[MDU Rohtak 2004; Ravishankar 2004; Delhi Maths (H) 2005; Meerut 2005; 06 ; Rohilkhand 1994]**

Sol. Given $pt - qs = q^3$ ... (1)

Comparing (1) with $Rr + Ss + Tt = V$, here $R = 0$, $S = -q$, $T = p$, $V = q^3$.

$\therefore$ Monge's subsidiary equations $Rdpdy + Tdqdx - Vdxdy = 0$, $R(dy)^2 - Sdxdy + T(dx)^2 = 0$

become $pdqdx - q^3dxdy = 0$ ...(2)

and $qdxdy + p(dx)^2 = 0.$ ...(3)

Dividing (2) by dx, we get $pdq - q^3dy = 0$ ...(4)

and dividing (3) by dx, we get $pdx + qdy = 0.$ ...(5)

From (5), $dy = -(pdx)/q$. Putting this value of dy into (4) gives

$pdq - q^3(pdx/q) = 0$ or $(1/q^2)dq + dx = 0.$

Integrating it, $-1/q + x = C_1$, C_1 being an arbitrary constant ...(6)

Using $dz = pdx + qdy$, (5) gives $dz = 0$ so that $z = C_2$. ...(7)

From (6) and (7), one integral of (1) is

$$-\frac{1}{q} + x = f(z) \qquad \text{or} \qquad \frac{\partial y}{\partial z} = x - f(z), \text{ as } q = \frac{\partial z}{\partial y},$$

Integrating with respect to z partially (treat x as constant), we obtain

$$y = xz - F(z) + G(x),\ F,\ G \text{ being arbitrary functions, where } F(z) = \int f(z)\,dz.$$

Ex. 4. *Solve* $z(qs - pt) = pq^2$. **[Delhi Maths (H) 1998; 2004, 11]**

Sol. Given $zqs - zpt = pq^2$. ...(1)

The usual Monge's subsidiary equations are $-zpdqdx - pq^2dxdy = 0$ ...(2)

and $-zqdxdy - zp(dx)^2 = 0.$ ...(3)

Dividing (2) by $-pdx$, we get $zdq + q^2dy = 0$ ...(4)

and dividing (3) by $-z\,dx$ we get $$q\,dy + p\,dx = 0. \quad ...(5)$$

Using $dz = p\,dx + q\,dy$, (5) gives $dz = 0$ so that $z = C_1$. ...(6)

Using (6) in (4), $C_1 dq + q^2 dy = 0$ or $(1/q^2)dq + (1/C_1)dy = 0.$

Integrating it, $-1/q + y/C_1 = C_2$ or $-1/q + y/z = C_2$, by (6) ...(7)

From (6) and (7), one integral of (1) is

$$-\frac{1}{q} + \frac{y}{z} = f(z) \quad \text{or} \quad \frac{\partial y}{\partial z} - \frac{1}{z}y = -f(z), \quad \text{as } q = \frac{\partial y}{\partial z}$$

which is linear in variables y and z (treating x as constant).

Its integrating factor (I.F.) $= e^{-\int(1/z)dz} = e^{-\log z} = z^{-1}$ and so its solution is

$$yz^{-1} = -\int z^{-1} f(z)\,dz + G(x) \quad \text{or} \quad yz^{-1} = F(z) + G(x), \quad \text{where } F(z) = \int f(z)\,dz$$

or $y = zF(z) + zG(x)$ or $y = H(z) + zG(x)$,

where $H(z)[= zF(z)]$ and $G(x)$ are arbitrary functions.

Ex. 5. *Solve* $2yq + y^2 t = 1$.

Sol. Given equation is $$0.r + 0.s + y^2.t = 1 - 2yq. \quad ...(1)$$

Comparing (1) with $Rr + Ss + Tt = V$, here $R = 0$, $S = 0$, $T = y^2$, $V = 1 - 2yq$.

Hence the usual subsidiary equations

$$R\,dp\,dy + T\,dq\,dx - V\,dx\,dy = 0 \quad \text{and} \quad R(dy)^2 - S\,dx\,dy + T(dx)^2 = 0$$

become $$y^2 dq\,dx - (1 - 2yq)dx\,dy = 0 \quad ...(2)$$

and $$y^2(dx)^2 = 0. \quad ...(3)$$

From (3), $dx = 0$ so that $x = c_1$. ...(4)

From (2), $y^2 dq + 2yq\,dy - dy = 0$ or $d(y^2 q) - dy = 0.$

Integrating it, $y^2 q - y = c_2$, c_2 being an arbitrary constant ...(5)

From (4) and (5), an intermediate integral is

$$y^2 q - y = f(x) \quad \text{or} \quad y^2(\partial z/\partial y) - y = f(x)$$

or $$\partial z/\partial y = 1/y + (1/y^2) \times f(x) \quad ...(6)$$

Integrating (6) w.r.t. y, treating x as constant, we get

$$z = \log y - (1/y)f(x) + g(x) \quad \text{or} \quad yz = y \log y - f(x) + y\,g(x),$$

where f and g being arbitrary functions.

Ex. 6. *Solve* $(e^x - 1)(qr - ps) = pqe^x$.

Sol. Given $$q(e^x - 1)r - p(e^x - 1)s = pqe^x. \quad ...(1)$$

Comparing (1) with $Rr + Ss + Tt = V$, $R = q(e^x - 1)$, $S = -p(e^x - 1)$, $T = 0$, $V = pqe^x$.

Then the usual Monge's subsidiary equations

$$R\,dp\,dy + T\,dq\,dx - V\,dx\,dy = 0 \quad \text{and} \quad R(dy)^2 - S\,dx\,dy + T(dx)^2 = 0$$

become $$q(e^x - 1)dp\,dy - pqe^x dx\,dy = 0 \quad ...(2)$$

and $$q(e^x - 1)(dy)^2 + p(e^x - 1)dx\,dy = 0. \quad ...(3)$$

Now, (3) $\Rightarrow$ $q\,dy + p\,dx = 0$ $\Rightarrow$ $dz = 0$, as $dz = p\,dx + q\,dy$.

Integrating, $z = c_1$, c_1 being an arbitrary constant ...(4)

Again, from (2), $(e^x - 1)dp - pe^x dx = 0$ or $\dfrac{dp}{p} - \dfrac{e^x}{e^x - 1}dx = 0$

Integrating, $\log p - \log(e^x - 1) = \log c_2$ or $p/(e^x - 1) = c_2$. ...(5)

From (4) and (5), an intermediate integral is $p/(e^x - 1) = f(z)$, f being an arbitrary function or $\quad \dfrac{\partial z}{\partial x} = (e^x - 1)f(z), \quad$ or $\quad \dfrac{1}{f(z)}\dfrac{\partial z}{\partial x} = e^x - 1.$

Integrating w.r.t. 'x', treating y as constant, we get

$F(z) = e^x - x + G(y) \quad$ or $\quad x = e^x + G(y) - F(z),$

F and G being arbitrary functions, where $\int (1/f(z))dz = F(z)$.

Miscellaneous problems based on types 1, 2, 3 and 4

Solve the following partial differential equations by using Monge's method:

1. $x^2r - y^2t = xy$. **Ans.** $z = xy \log x + x F(y/x) + G(xy)$

2. $(1 + pq + q^2)r + s(q^2 - p^2) - (1 + pq + p^2)t = 0$ **Ans.** $z\{2 + (x + y)\}^{1/2} = F(x + y) + G(x - y)$

3. $q(1 + q)r - (1 + 2q)(1 + p)s + (1 + p)^2t = 0$ **Ans.** $x = F(x + y + z) + G(x + z)$

4. $x^2r - y^2t - xp + yq = xy$. **Ans.** $z = (xy/4) \times \{(\log x)^2 - (\log y)^2\} + xyF(x/y) + G(xy)$

9.11. Monge's Method of integrating the equation Rr + Ss + Tt + U(rt – s²) = V, where r, s, t have their usual meaning and R, S, T, U, V are functions of x, y, z.

Given $\quad Rr + Ss + Tt + U(rt - s^2) = V. \quad$ ...(1)

We have $\quad dp = (\partial p/\partial x)\,dx + (\partial p/\partial y)\,dy = rdx + sdy$

and $\quad dq = (\partial q/\partial x)\,dx + (\partial q/\partial y)\,dy = sdx + tdy$

which give $\quad r = (dp - sdy)/dx \quad$ and $\quad t = (dq - sdx)/dy.$

Putting these values in (1) and simplifying, we get

$(Rdpdy + Tdqdx - Udpdq - Vdxdy) - s\{R(dy)^2 - Sdxdy + T(dx)^2 + Udpdx + Udqdy\} = 0.$

Hence the usual Monge's subsidiary equations are

$$L \equiv Rdpdy + Tdqdx + Udpdq - Vdxdy = 0 \quad \text{...(2)}$$

and
$$M \equiv R(dy)^2 - Sdxdy + T(dx)^2 + Udpdx + Udqdy = 0. \quad \text{...(3)}$$

We cannot factorise M as we did before (see Art 9.1), on account of the presence of the additional terms, $Udpdx + Udqdy$. Hence let us factorise $M + \lambda L$, where λ is some multiplier to be determined later. Now, we have

$M + \lambda L \equiv R(dy)^2 + T(dx)^2 - (S + \lambda V)dxdy + Udpdx + Udqdy + \lambda Rdpdy + \lambda Tdqdx + \lambda Udpdq = 0.$ ...(4)

Factorising L.H.S. of (4), let k and m be constants such that

$$M + \lambda L \equiv (Rdy + mTdx + kUdp)\left(dy + \frac{1}{m}dx + \frac{\lambda}{k}dq\right) = 0. \quad \text{...(5)}$$

Comparing coefficients in (4) and (5), we get $\quad R/m + mT = -(S + \lambda V), \quad$ ...(6)

$k = m \quad$ and $\quad R\lambda/k = U. \quad$ ...(7)

Now, the two relations of (7) give $\quad m = R\lambda u$

Putting this valus of m in (6) and simplifying, we get $\quad \lambda^2(UV + RT) + \lambda US + U^2 = 0,$ ...(8)

which is quadratic in λ. Let λ_1 and λ_2 be its roots.

When $\lambda = \lambda_1$, (7) $\Rightarrow \quad R\lambda_1/k = U \quad \Rightarrow \quad k = R\lambda_1/U \quad \Rightarrow \quad m = R\lambda_1/U$

Hence (5) gives
$$\left(Rdy + \frac{R\lambda_1}{U}Tdx + R\lambda_1 dp\right)\left(dy + \frac{U}{R\lambda_1}dx + \frac{U}{R}dq\right) = 0$$

or
$$(Udy + \lambda_1 Tdx + \lambda_1 Udp)\,(Udx + \lambda_1 Rdy + \lambda_1 Udq) = 0. \quad \text{...(9)}$$

Similarly for $\lambda = \lambda_2$, (5) gives

$$(Udy + \lambda_2 Tdx + \lambda_2\, Udp)\,(Udx + \lambda_2 Rdy + \lambda_2 Udq) = 0. \quad \text{...(10)}$$

Now one factor of (9) is combined with one factor of (10) to give an intermediate integral. Exactly similarly, the other pair will give rise to another intermediate integral. In this connection remember that we must combine first factor of (9) with the second factor of (10) and similarly the second factor of (9) with the first factor of (10). Thus for the desired solution the proper method is to combine the factors in the following manner :

$$Udy + \lambda_1 Tdx + \lambda_1 Udp = 0, \qquad Udx + \lambda_2 Rdy + \lambda_2 Udq = 0 \qquad \text{...(11)}$$

$$Udy + \lambda_2 Tdx + \lambda_2 Udp = 0, \qquad Udx + \lambda_1 Rdy + \lambda_1 Udq = 0 \qquad \text{...(12)}$$

Let equations (11) give two integrals $u_1 = c$ and $v_1 = d_1$ so that one intermediate integral is

$$u_1 = f_1(v_1), f_1 \text{ being an arbitrary function} \qquad \text{...(13)}$$

Similarly, (12) gives second intermediate integral $\quad u_2 = f_2(v_2), \qquad$ ...(14)

where f_2 is an arbitrary function

We now solve (13) and (14) for p and q and substitute in $dz = pdx + qdy$, which after integration gives the desired general solution.

Remark 1. There are in all four ways of combining factors of (9) and (10). By combining the first factors in these equations, we would get $u\, dy = 0$ on substraction (after dividing equations by λ_1 and λ_2 respectively) and this would not produce any solution. Similarly, combining the second factors in these equations would give $u\, dx = 0$ and hence would produce no solution. Hence for getting integrals of the given equation we must proceed as explained in (11) and (12).

Remark 2. In what follows we shall use the following two results of equation $a\lambda^2 + b\lambda + c = 0$

(*i*) $a = b = 0$, *i.e.*, the coefficients of λ^2 and λ both equal to zero imply that both roots of the equatin are equal to ∞

(*ii*) $a = 0$ but $b \neq 0$, *i.e.*, the coefficient of λ^2 is zero but that of λ is non–zero imply that one root of the equation is ∞ and the other is $-c/b$.

Remark 3. When the two values of λ are equal, we shall have only one intermediate integral $u_1 = f(v_1)$ and proceed as explained in solved examples of type 1 based on $Rr + Ss + Tt + U(rt - s^2) = V$ given below.

An integral of *a* more general form can be obtained by taking the arbitrary function occuring in the intermediate integral to be linear.

Let $u_1 = mv_1 + n$, where m and n are some constants. Then integrating it by Lagrange's method we find the solution of the given equation.

9.12. Type 1: When the roots of λ –quadratic (8) of Art 9.11 are identical.

Solved examples of type 1 based on Rr + Ss + Tt + U(rt – s²) = V

Ex. 1. *Solve* $5r + 6s + 3t + 2(rt - s^2) + 3 = 0.$ **[I.A.S. 1973 ; Meerut 1998]**

Sol. Given equation $\quad 5r + 6s + 3t + 2(rt - s^2) = -3. \qquad$ ...(1)

Comparing the given equation with $Rr + Ss + Tt + U(rt - s^2) = V$, we have $R = 5$, $S = 6$, $T = 3$, $U = 2$ and $V = -3$. Hence the λ–quadratic $\quad \lambda^2(UV + RT) + \lambda SU + U^2 = 0$

becomes $\quad 9\lambda^2 + 12\lambda + 4 = 0 \quad$ or $\quad (3\lambda + 2)^2 = 0 \quad$ so that $\quad \lambda_1 = \lambda_2 = -2/3.$

There is only one intermediate integral given by the equations

$$Udy + \lambda_1 Tdx + \lambda_1 Udp = 0 \qquad \text{and} \qquad Udx + \lambda_2 Rdy + \lambda_2 Udq = 0$$

or $\quad 2dy + (-2/3) \times 3dx + (-2/3) \times 2dp = 0 \quad$ and $\quad 2dx + (-2/3) \times 5dy + (-2/3) \times 2dq = 0$

or $\quad 3dy - 3dx - 2dp = 0 \quad$ and $\quad 3dx - 5dy - 2dq = 0.$

Integrating, $\quad 3y - 3x - 2p = c_1 \quad$ and $\quad 3x - 5y - 2q = c_2. \qquad$ ...(2)

Hence here the only intermediate integral is

$$3y - 3x - 2p = f(3x - 5y - 2q), \text{ where } f \text{ is an arbitrary function.} \qquad \text{...(3)}$$

Solving the two equations of (2) for p and q, we have

$$p = (1/2) \times (3y - 3x - c_1) \qquad \text{and} \qquad q = (1/2) \times (3x - 5y - c_2).$$

Putting these values of p and q in $dz = pdx + qdy$, we have

$$dz = (1/2) \times (3y - 3x - c_1)dx + (1/2) \times (3x - 5y - c_2)dy$$

or $$2dz = 3(ydx + xdy) - 3xdx - 5ydy - c_1dx - c_2dy.$$

Integrating, $$2z = 3xy - (3x^2/2) - (5y^2/2) - c_1x - c_2y + c_3,$$

which is the required complete integral, c_1, c_2 and c_3 being arbitrary constants.

Alternative solution. An integral of a more general form can be obtained by supposing the arbitrary function f occuring in the intermediate integral (3) to be linear, giving

$$3y - 3x - 2p = m(3x - 5y - 2q) + n, \text{ where } m \text{ and } n \text{ are arbitrary constants.} \quad ...(4)$$

Re–writing (4), $$2p - 2mq = 3y - 3x + 5my - 3mx - n. \quad ...(5)$$

Lagrange's auxiliary equations for (5) are $$\frac{dx}{2} = \frac{dy}{-2m} = \frac{dz}{3y-3x+5my-3mx-n}. \quad ...(6)$$

Taking the first two fractions of (6), we have

$$dy + mdx = 0 \qquad \text{so that} \qquad y + mx = a. \quad ...(7)$$

Now, $$\text{each fraction of (6)} = \frac{3xdx+5ydy+2dz}{6x-10my+6y-6x+10my-6mx-2n} \quad ...(8)$$

Hence taking first fraction of (6) and fraction (8), we have

$$\frac{dx}{2} = \frac{3xdx+5ydy+2dz}{6y-6mx-2n} \qquad \text{or} \qquad dx = \frac{3xdx+5ydy+2dz}{3y-3mx-n}$$

or $$3xdx + 5ydy + 2dz = (3y - 3mx - n)dx$$

or $$2dz + 3xdx + 5ydy = \{3(a - mx) - 3mx - n\}dx, \text{ using (7)}$$

or $$2dz + 3xdx + 5ydy = (3a - 6mx - n)dx .$$

Integrating, $$2z + (3x^2/2) + (5y^2/2) = 3ax - 3mx^2 - nx + b/2$$

or $$4z + 3x^2 + 5y^2 = 6x(y + mx) - 6mx^2 - 2xn + b, \text{ using (7)}$$

or $$4z - 6xy + 3x^2 + 5y^2 + 2nx = b. \quad ...(9)$$

From (7) and (9), the required general solution is $4z - 6xy + 3x^2 + 2nx = \phi(y + mx)$, where ϕ is an arbitrary function and m and n are arbitrary constants.

Ex. 2. *Solve* $3r + 4s + t + (rt - s^2) = 1$.

Sol. Comparing the given equation with $Rr + Ss + Tt + U(rt - s^2) = V$, we get $R = 3$, $S = 4$, $T = 1$, $U = 1$, $V = 1$. Then, λ–quadratic $\lambda^2(UV + RT) + \lambda SU + U^2 = 0$ becomes $4\lambda^2 + 4\lambda + 1 = 0$ or $(2\lambda + 1)^2 = 0$ so that $\lambda_1 = \lambda_2 = -1/2$.

There is only one intermediate integral given by the equations

$$Udy + \lambda_1 Tdx + \lambda_1 Udp = 0 \qquad \text{and} \qquad Udx + \lambda_2 Rdy + \lambda_2 Udq = 0$$

or $$dy + (-1/2) \times dx + (-1/2) \times dp = 0 \qquad \text{and} \qquad dx + (-1/2) \times 3dy + (-1/2) \times dq = 0$$

or $$-2dy + dx + dp = 0 \qquad \text{and} \qquad 3dy - 2dx + dq = 0. \quad ...(1)$$

Integrating, $$-2y + x + p = c_1 \qquad \text{and} \qquad 3y - 2x + q = c_2. \quad ...(2)$$

Hence the only intermediate integral is

$$-2y + x + p = f(3y - 2x + q), \text{ where } f \text{ is an arbitrary function.} \quad ...(3)$$

Solving (2) for p and q, $$p = 2y - x + c_1 \qquad \text{and} \qquad q = -3y + 2x + c_2.$$

Putting these values of p and q in $dz = pdx + qdy$, we get

$$dz = (2y - x + c_1)dx + (-3y + 2x + c_2)dy$$

or $$dz = 2(ydx + xdy) - xdx - 3ydy + c_1dx + c_2dy.$$

Integrating, $$z = 2xy - (x^2/2) - (3y^2/2) + c_1x + c_2y + c_3,$$

which is the required complete integral, c_1, c_2, c_3 being arbitrary constants.

Alternative solution. In order to get the more general solution, we assume the arbitrary function ϕ in (3) to be linear. Thus, we take

$$-2y + x + p = m(3y - 2x + q) + n, \ m, n \text{ being arbitrary constants}$$

or $$p - mq = 2y - x + 3my - 2mx + n. \qquad ...(4)$$

Lagrange's auxiliary equations for (4) are $\dfrac{dx}{1} = \dfrac{dy}{-m} = \dfrac{dz}{2y-x+3my-2mx+n}$. ...(5)

Taking the first two fractions of (5), $dy + mdx = 0$ so that $y + mx = a$. ...(6)

Now, each fraction of (5) $= \dfrac{xdx+3ydy+dz}{x-3my+2y-x+3my-2mx+n}$ ...(7)

Taking the first fraction of (5) and the fraction (7), we have $\dfrac{dx}{1} = \dfrac{xdx+3ydy+dz}{2y-2mx+n}$

or $$xdx + 3ydy + dz = (2y - 2mx + n)dx$$

or $$xdx + 3ydy + dz = 2(a - mx)dx - 2mxdx + ndx, \text{ using (6)}$$

Integrating, $$(x^2/2) + (3y^2/2) + z = 2ax - mx^2 - mx^2 + nx + b/2$$

or $$x^2 + 3y^2 + 2z - 2x(y + mx) + 2mx^2 - nx = b, \text{ using (6)} \qquad ...(8)$$

From (6) and (8), the required general solution is $x^2 + 3y^2 + 2z - 2xy - nx = \phi(y + mx)$, where ϕ is an arbitrary function and m and n are arbitrary constants.

Ex. 3. *Solve* $(q^2 - 1)zr - 2pqzs + (p^2 - 1)zt + z^2(rt - s^2) = p^2 + q^2 - 1$.

Sol. Comparing the given equation with $Rr + Ss + Tt + U(rt - s^2) = V$, we have $R = z(q^2 - 1)$, $S = -2pqz$, $T = z(p^2 - 1)$, $U = z^2$ and $V = p^2 + q^2 - 1$.

Hence the λ–quadratic $\lambda^2(UV + RT) + \lambda US + U^2 = 0$ becomes

$p^2q^2\lambda^2 - 2pqz + z^2 = 0$ or $(pq\lambda - z)^2 = 0$ so that $\lambda_1 = \lambda_2 = z/pq$.

There is only one intermediate integral given by equations

$Udy + \lambda_1 Tdx + \lambda_1 Udp = 0$ and $Udx + \lambda_2 Rdy + \lambda_2 Udq = 0$

or $z^2dy + \dfrac{z^2(p^2-1)}{pq}dx + \dfrac{z^3}{pq}dp = 0$ and $z^2dx + \dfrac{z^2(q^2-1)}{pq}dy + \dfrac{z^3}{pq}dq = 0$

or $pqdy + (p^2 - 1)dx + zdp = 0$ and $pqdx + (q^2 - 1)dy + zdq = 0$

or $p(qdy + pdx) - dx + zdp = 0$ and $q(pdx + qdy) - dy + zdq = 0$

or $pdz + zdp - dx = 0$ and $qdz + zdq - dy = 0$, as $dz = pdx + qdy$

or $d(pz) - dx = 0$ and $d(qz) - dy = 0$.

Integrating, $pz - x = c_1$ and $qz - y = c_2$. ...(1)

Hence the only intermediate integral is $pz - x = f(qz - y)$, f being an arbitrary function. ...(2)

Solving (1) for p and q, $p = (c_1 + x)/z$ and $q = (c_2 + y)/z$.

Putting these values of p and q in $dz = pdx + qdy$, we get

$dz = (1/z) \times (c_1 + x)dx + (1/z) \times (c_2 + y)dy$ or $zdz = (c_1 + x)dx + (c_2 + y)dy$.

Integrating, $$(1/2) \times z^2 = (1/2) \times (c_1 + x)^2 + (1/2) \times (c_2 + y)^2 + (1/2) \times c_3'.$$

or $$z^2 = x^2 + y^2 + 2c_1x + zc_2y + c_3, \text{ where } c_3 = c_1^2 + c_2^2 + c_3'$$

which is the complete integral, c_1, c_2, c_3 being arbitrary constants.

Alternative solution. To find the more general solution, we take the arbitrary function f in (2) to be linear. So, let $pz - x = m(qz - y) + n$, m, n being arbitrary constants.

or $$pz - mqz = x - my + n. \qquad ...(3)$$

Lagrange's auxiliary equation for (3) are $\dfrac{dx}{z} = \dfrac{dy}{-mz} = \dfrac{dz}{x-my+n}$. ...(4)

Taking the first two fractions of (4), $dy + mdx = 0$ so that $y + mx = a$. ...(5)

Now, each fraction of (4) = $\dfrac{(-x/z)dx-(y/z)dy+dz}{z\times(-x/z)-mz\times(-y/z)+x-my+n}$. ...(6)

Taking the first fraction of (4) and fraction (6), $\dfrac{dx}{z} = \dfrac{-(x/z)dx-(y/z)dy+dz}{n}$

or $-xdx - ydy + zdz = ndx$ or $-2zdz + 2xdx + 2ydy + 2ndx = 0.$

Integrating, $-z^2 + x^2 + y^2 + 2nx = b$, b being an arbitrary constant ...(7)

From (5) and (7), the required general solution is $-z^2 + x^2 + y^2 + 2nx = \phi(y + mx)$, where ϕ is an arbitrary function and m, n are arbitrary constants.

Ex. 4. *Solve* $2s + (rt - s^2) = 1$. **[Garwhal 1995; Meerut 2000]**

Sol. Comparing the given equation with the equation $Rr + Ss + Tt + U(rt - s^2) = V$, we get $R = 0$, $S = 2$, $T = 0$, $U = 1$, $V = 1$, so λ–quardratic $\lambda^2 (UV + RT) + \lambda SU + U^2 = 0$ becomes $\lambda^2 + 2\lambda + 1 = 0$ so that $\lambda_1 = \lambda_2 = -1.$

Since we have equal values of l, there would be only one intermediate integral given by

$Udy + \lambda_1 Tdx + \lambda_1 Udp = 0$ and $Udx + \lambda_2 Rdy + \lambda_2 Udq = 0$

or $dy - dp = 0$ and $dx - dq = 0$, using (1)

which give $y - p = c_1$, and $x - q = c_2$.

Solving these for p and q, $p = y - c_1$ and $q = x - c_2$.

$\therefore$ $dz = pdx + qdy = (y - c_1)dx + (x - c_2)dy = (ydx + xdy) - c_1dx - c_2dy,$

or $$dz = d(xy) - c_1dx - c_2dy.$$

Integrating, $z = xy - c_1x - c_2y + c_3$, which is solution, c_1, c_2, c_3 being arbitrary constants.

Ex. 5. $z(1 + q^2)r - 2pqzs + z(1 + p^2)t + z^2(s^2 - rt) + 1 + p^2 + q^2 = 0.$

Sol. Comparing the give equation with $Rr + Ss + Tt + U(rt - s^2) = V$, we get

$R = z(1 + q^2)$, $S = -2pqz$, $T = z(1 + p^2)$, $U = z^2$ and $V = -(1 + p^2 + q^2)$. ... (1)

Hence λ–quadratic *i.e.* $\lambda^2(RT + UV) + \lambda US + U^2 = 0$ gives

$\lambda^2(p^2q^2) - 2\lambda zpq + z^2 = 0$ or $(\lambda pq - z)^2 = 0.$

Thus here we obtain $\lambda_1 = \lambda_2 = z/pq$. Hence there would be only one intermediate integral which is given by

$$Udy + \lambda_1 Tdx + \lambda_1 Udp = 0. \quad ...(2)$$

and $$Udx + \lambda_2 Rdy + \lambda_2 Udq = 0 \quad ...(3)$$

Using (1), (2) becomes $pq\,dy + (1 + p^2)dx + zdp = 0$ ...(4)

Using (1), (3) becomes $pqdx + (1 + q^2)dy + zdq = 0$ ...(5)

Now from (4), $p(pdx + qdy) + dx + zdp = 0$ or $pdz + dx + zdp = 0$, as $dz = pdx + qdy$

or $d(zp) + dx = 0$ so that $zp + x = c_1$. ...(6)

Similarly (5) gives $zq + y = c_2$, c_2 being an arbitrary constant ...(7)

Solving (6) and (7), we get $p = (c_1 - x)/z$ and $q = (c_2 - y)/z.$

$\therefore$ $dz = pdx + qdy = \{(c_1 - x)/z\}dx + \{(c_2 - y)/z\}dy$ or $zdz = c_1dx + c_2dy - (xdx + ydy).$

Integrating, $(1/2) \times z^2 = c_1x + c_2y - (x_2 + y^2)/2 + c_3/2$ or $z^2 = 2c_1x + 2c_2y - x^2 - y^2 + c_3$, which is complete integral, c_1, c_2, c_3 being arbitrary constants.

Ex. 6. *Solve* $2r + te^x - (rt - s^2) = 2e^x$.

Sol. Comparing the given equation with $Rr + Ss + Tt + U(rt - s^2) = V$, we get

$R = 2$, $S = 0$, $T = e^x$, $U = -1$ and $V = 2e^x$. ... (1)

Hence the λ–quadratic $\lambda^2(UV + RT) + \lambda SU + U^2 = 0$ gives $\lambda^2(2e^x - 2e^x) + (\lambda \times 0) + 1 = 0$.

Since the coefficient of λ^2 and λ in the above quadratic vanish, it follows from the theory of equations that its both the roots must be infinite. Thus $\lambda_1 = \lambda_2 = \infty$. Since the two roots are equal there would be only one intermediate integral which is given by

$Udy + \lambda_1 Tdx + \lambda_1 Udp = 0$ and $Udx + \lambda_2 Rdy + \lambda_2 Udq = 0$,

i.e., by $(U/\lambda_1)dy + Tdx + Udq = 0$ and $(U/\lambda_2)dx + Rdy + Udq = 0$,

i.e., by $e^x dx - dp = 0$ using (1) and $2dy - dq = 0$, using (1)

Integrating these $e^x - p = c_1$ and $2y - q = c_2$.

Solving these, $p = e^x - c_1$ and $q = 2y - c_2$.

Now, $dz = pdx + qdy = (e^x - c_1)dx + (2y - c_2)dy$.

Integrating, $z = e^x - c_1 x + y^2 - c_2 y + c_3$,

which is complete integral, c_1, c_2, c_3 being arbitrary constants.

Ex. 7. *Solve* $r + t - (rt - s^2) = 1$.

Sol. Comparing the given equation with $Rr + Ss + Tt + U(rt - s^2) = V$,

$R = 1$, $S = 0$, $T = 1$, $U = -1$, $V = 1$. ...(1)

So λ–quadratic $\lambda^2(UV + RT) + \lambda US + U^2 = 0$ becomes $(0 \times \lambda^2) + (0 \times \lambda) + 1 = 0$. Since the coeffieicnts of both λ^2 and λ are zero, so both roots of this quadratic are equal to ∞. So $\lambda_1 = \lambda_2 = \infty$

Now, the only one intermediate integral is given by equations

$Udy + \lambda_1 Tdx + \lambda_1 Udp = 0$ and $\lambda_1 Rdy + Udx + \lambda_1 Udq = 0$

On dividing each term by λ_1 as λ_1 is infinite, the above equations become

or $(1/\lambda_1) \times Udy + Tdx + Udp = 0$ and $Rdy + (1/\lambda_1) \times Udx + Udq = 0$

or $Tdx + Udp = 0$, as $\lambda_1 = \infty$ and $Rdy + Udq = 0$, as $\lambda_1 = \infty$

or $dx - dp = 0$ and $dy - dq = 0$, using (1)

Integrating, $p - x = c_1$ and $q - y = c_2$. ...(2)

Solving (2) for p and q, $p = x + c_1$ and $q = y + c_2$.

Putting these values of p and q in $dz = pdx + qdy$, we get $dz = (x + c_1)dx + (y + c_2)dy$

Integrating, $z = x^2/2 + c_1 x + y^2/2 + c_2 y + c_3$,

which is the required integral, c_1, c_2, c_3 being arbitrary constants.

Ex. 8. *Solve* $2pr + 2qt - 4pq\,(rt - s^2) = 1$.

Sol. Comparing the given equation with $Rr + Ss + Tt + U(rt - s^2) = V$, we have

$R = 2p$, $S = 0$, $T = 2q$, $U = -4pq$, $V = 1$. ...(1)

Then the λ–quadratic $\lambda^2(UV + RT) + \lambda SU + U^2 = 0$ becomes $(0 \times \lambda^2) + (0 \times \lambda) + 4p^2q^2 = 0$. Since the coefficients of both λ^2 and λ are zero, so both roots of the λ– quadratic are equal to ∞. So $\lambda_1 = \lambda_2 = \infty$.

Now the only intermediate integral is given by the equation

$Udy + \lambda_1 Tdx + \lambda_1 Udp = 0$ and $\lambda_1 Rdy + Udx + \lambda_1 Udq = 0$

On dividing each term by λ_1 as λ_1 is infinite, the above equations become

$(1/\lambda_1) \times Udy + Tdx + Udp = 0$ and $Rdy + (1/\lambda_1) \times Udx + Udq = 0$

or $2qdx - 4pqdp = 0$ and $2pdy - 4pqdq = 0$, using (1)

or $2pdp - dx = 0$ and $2qdq - dy = 0$.

Integrating, $p^2 - x = c_1$ and $q^2 - y = c_2$.

Hence $p = \pm (c_1 + x)^{1/2}$ and $q = \pm (c_2 + y)^{1/2}$

Putting values of p and q in $dz = pdx + qdy$ gives $dz = \pm (c_1 + x)^{1/2}dx \pm (c_2 + y)^{1/2}dy$.

Integrating, $z = \pm (2/3) \times (c_1 + x)^{3/2} \pm (2/3) \times (c_2 + y)^{3/2} + c_3/2$

or $3z = \pm 2(c_1 + x)^{3/2} \pm 2(c_2 + y)^{3/2} + c_3$,

which is the complete integral, c_1, c_2, c_3 being arbitrary constants.

Ex. 9. *Solve* $(1 + q^2)r - 2pqs + (1 + p^2)t + (1 + p^2 + q^2)^{-1/2} (rt - s^2) = -(1 + p^2 + q^2)^{3/2}$.

Sol. Comparing the given equation with $Rr + Ss + Tt + U(rt - s^2) = V$, we get

$R = 1 + q^2, \quad S = -2pq, \quad T = 1 + p^2, \quad U = (1 + p^2 + q^2)^{-1/2}, \quad V = -(1 + p^2 + q^2)^{3/2}$ ...(1)

Now, the λ–quadratic $\lambda^2(UV + RT) + \lambda SU + U^2 = 0$ becomes

$\lambda^2 \{-(1 + p^2 + q^2) + (1 + q^2)(1 + p^2)\} - 2pq(1 + p^2 + q^2)^{-1/2}\lambda + (1 + p^2 + q^2)^{-1} = 0$

or $p^2q^2(1 + p^2 + q^2)\lambda^2 - 2pq(1 + p^2 + q^2)^{1/2}\lambda + 1 = 0$

or $\{pq(1 + p^2 + q^2)^{1/2}\lambda - 1\}^2 = 0$ so that $\lambda_1 = \lambda_2 = 1/pq(1 + p^2 + q^2)^{1/2}$.

Here there is only intermediate integral given by equations

$Udy + \lambda_1 Tdx + \lambda_1 Udp = 0$ and $Udx + \lambda_2 Rdy + \lambda_2 Udq = 0$

or $$\frac{1}{(1+p^2+q^2)^{1/2}}dy + \frac{1+p^2}{pq(1+p^2+q^2)^{1/2}}dx + \frac{dp}{pq(1+p^2+q^2)} = 0, \text{ by (1)}$$

and $$\frac{1}{(1+p^2+q^2)^{1/2}}dx + \frac{1+q^2}{pq(1+p^2+q^2)^{1/2}}dy + \frac{dq}{pq(1+p^2+q^2)} = 0, \text{ by (1)}$$

or $pqdy + (1 + p^2)dx + [1/(1 + p^2 + q^2)^{1/2}]dp = 0$ ...(2)

and $pqdx + (1 + q^2)dy + \{1/(1 + p^2 + q^2)^{1/2}\}dq = 0$. ...(3)

Eliminating dy between (2) and (3), $\{(1 + p^2)(1 + q^2) - p^2q^2\}dx + \dfrac{(1+q^2)dp - pqdq}{(1+p^2+q^2)^{1/2}} = 0$

or $$(1 + p^2 + q^2)dx + \frac{(1 + p^2 + q^2)dp - (p^2dp + pqdq)}{(1 + p^2 + q^2)^{1/2}} = 0$$

or $$dx + \frac{dp}{(1+p^2+q^2)^{1/2}} - \frac{p}{2}\frac{2pdp+2qdq}{(1+p^2+q^2)^{3/2}} = 0 \quad \text{or} \quad dx + d\left\{\frac{p}{(1+p^2+q^2)^{1/2}}\right\} = 0$$

Integrating, $x + p(1 + p^2 + q^2)^{-1/2} = a$, where a is an arbitrary constant. ...(4)

Similarly, eliminating dx between (2) and (3), we have

$y + q(1 + p^2 + q^2)^{-1/2} = b$, where b in an arbitrary constant. ...(5)

From (4) and (5), $x - a = -p(1 + p^2 + q^2)^{-1/2}$, $y - b = -q(1 + p^2 + q^2)^{-1/2}$.

$\therefore \quad \dfrac{x-a}{y-b} = \dfrac{p}{q}$ so that $p = \dfrac{x-a}{y-b}q$. ...(6)

Putting the above value of p in (4), we have

$$x + q\frac{x-a}{y-b}\left\{1+q^2\frac{(x-a)^2}{(y-b)^2}+q^2\right\}^{-1/2} = a \quad \text{or} \quad (x-a) + \frac{x-a}{y-b}q\left[1+\frac{(x-a)^2+(y-b)^2}{(y-b)^2}q^2\right]^{-1/2} = 0$$

or $$1 + \frac{(x-a)^2+(y-b)^2}{(y-b)^2}q^2 = \frac{q^2}{(y-b)^2} \quad \text{or} \quad (y-b)^2 = q^2[1 - \{(x-a)^2 + (y-b)^2\}].$$

Thus, $$q = (y-b)/[1-\{(x-a)^2+(y-b)^2\}]^{1/2}. \qquad \text{... (7)}$$

Now, (6) and (7) $\Rightarrow$ $$p = \frac{x-a}{y-b}q = \frac{x-a}{[1-\{(x-a)^2+(y-b)^2\}]^{1/2}}. \qquad \text{... (8)}$$

$$\therefore\ dz = pdx + qdy = \frac{(x-a)\,dx+(y-b)\,dy}{[1-\{(x-a)^2+(y-b)^2\}]^{1/2}}, \text{ by (7) and (8)}$$

Integrating, $z = [1 - \{(x-a)^2 + (y-b)^2\}]^{1/2} + c$ or $(z-c)^2 = 1 - \{(x-a)^2 + (y-b)^2\}$

$\therefore$ $(x-a)^2 + (y-b)^2 + (z-c)^2 = 1$ is the complete integral, a, b, c being arbitrary constants.

9.13 Type 2. When the roots of λ –quadratic (8) of Art 9.11 are distinct.

Solved Examples of Type –2 based on Rr + Ss + Tt + U(rt – s^2) = V

Ex. 1. *Solve* $3s + rt - s^2 = 2$.

Sol. Given $$3s + (rt - s^2) = 2. \qquad \text{...(1)}$$

Comparing (1) with $Rr + Ss + Tt + U(rt - s^2) = V$, $R = 0, S = 3, V = 0, U = 1, V = 2$. ...(2)

λ–quadratic is $$\lambda^2(UV + RT) + \lambda US + U^2 = 0 \qquad \text{...(3)}$$

Using (2), (3) reduces to $2\lambda^2 + 3\lambda + 1 = 0$ so $\lambda_1 = -1, \lambda_2 = -(1/2)$. ... (4)

Two integrals of (1) are given by the following sets

$$\left.\begin{aligned} Udy + \lambda_1 Tdx + \lambda_1 Udp = 0 \\ Udx + \lambda_2 Rdy + \lambda_2 Udq = 0. \end{aligned}\right\} \qquad \text{... (5)}$$

and

$$\left.\begin{aligned} Udy + \lambda_2 Tdx + \lambda_2 Udp = 0 \\ Udx + \lambda_1 Rdy + \lambda_1 Udq = 0. \end{aligned}\right\} \qquad \text{... (6)}$$

Using (2) and (4), (5) and (6) respectively gives

$$\left.\begin{aligned} dy - dp = 0 \quad &\text{or} \quad dp - dy = 0 \\ dx - (1/2)dq = 0 \quad &\text{or} \quad dq - 2dx = 0 \end{aligned}\right\} \qquad \text{... (5A)}$$

and

$$\left.\begin{aligned} dy - (1/2)dp = 0 \quad &\text{or} \quad dp - 2dy = 0 \\ dx - dq = 0 \quad &\text{or} \quad dq - dx = 0. \end{aligned}\right\} \qquad \text{... (6A)}$$

Integration of (5A) and (6A) respectively gives

$$p - y = c_1, \qquad q - 2x = c_2 \qquad \text{...(5B)}$$

and $$p - 2y = c_3, \qquad q - x = c_4, \qquad \text{...(6B)}$$

where c_1, c_2, c_3 and c_4 are arbitrary constants.

From (5B) and (6B), two intermediate integrals of (1) are given by

$$p - y = f(q - 2x) \quad \text{and} \quad p - 2y = F(q - x), \qquad \text{...(7)}$$

where f and F are arbitrary functions.

Let $$q - 2x = \alpha, \qquad \text{...(8)}$$

and $$q - x = \beta. \qquad \text{...(9)}$$

Then from (7) $$p - y = f(\alpha), \qquad \text{...(10)}$$

and $$p - 2y = F(\beta). \qquad \text{...(11)}$$

[If we treat α and β as constants, then solution of four simultaneous equation (8), (9), (10) and (11) would show that x, y, p and q are all constants which is absurd. Hence α and β will be regarded as variables (parameters) and we will get the general solution in parametric form involving α and β as parameters].

Solving (8) and (9) for x and (10) and (11) for y, we have

$$x = \beta - \alpha \qquad \text{...(12)}$$

and $$y = f(\alpha) - F(\beta). \qquad \text{...(13)}$$

From (10) $$p = y + f(\alpha). \qquad \text{...(14)}$$

From (9) $\quad q = x + \beta.$ ...(15)

From (12) and (13), $\quad dx = d\beta - d\alpha, \quad$ and $\quad dy = f'(\alpha)d\alpha - F'(\beta)d\beta.$ ...(16)

$\therefore \quad dz = pdx + qdy = [y + f(\alpha)]dx + (x + \beta)dy$, using (14) and (15)

or $\quad dz = ydx + xdy + f(\alpha)dx + \beta dy = d(xy) + f(\alpha)(d\beta - d\alpha) + \beta[f'(\alpha)d\alpha - F'(\beta)d\beta]$, by (16)

Thus, $\quad dz = d(xy) + [f(\alpha)d\beta + \beta f'(\alpha)d\alpha] - f(\alpha)d\alpha - \beta F'(\beta)d\beta$

or $$dz = d(xy) + d[\beta f(\alpha)] - f(\alpha)d\alpha - \beta F'(\beta)d\beta.$$

Integrating and using integration by parts in the last term on R.H.S. of the above equation, we get
$$z = xy + \beta f(\alpha) - \int f(\alpha)d\alpha - [\beta F(\beta) - \int 1 \cdot F(\beta)d\beta]$$

or $$z = xy + \beta[f(\alpha) - F(\beta)] - \int f(\alpha)d\alpha + \int F(\beta)d\beta. \qquad ...(17)$$

Let $\quad \int f(\alpha)d\alpha = \phi(\alpha) \quad$ and $\quad \int F(\beta)d\beta = \psi(\beta)$ ...(18)

so that $\quad f(\alpha) = \phi'(\alpha) \quad$ and $\quad F(\beta) = \psi'(\beta)$ ...(19)

Using (18) and (19), (12), (13) and (17) give

$x = \beta - \alpha, \qquad y = \phi'(\alpha) - \psi'(\beta) \qquad z = xy + \beta[\phi'(\alpha) - \psi'(\beta)] - \phi(\alpha) + \psi(\beta)$

which is the required solution in parametric form, ϕ and ψ being arbitrary functions and α and β being parameters.

Ex. 2. *Solve* $r + 4s + t + rt - s^2 = 2.$ **[I.A.S. 1979]**

Sol. Given $\quad r + 4s + t + (rt - s^2) = 2.$ ...(1)

Comparing (1) with $Rr + Ss + Tt + U(rt - s^2) = V$, $R = 1$, $S = 4$, $T = 1$, $U = 1$, $V = 2$. ...(2)

λ–quadratic is $\quad \lambda^2(UV + RT) + \lambda US + U^2 = 0.$ ...(3)

Using (2), (3) reduces to $\quad 3\lambda^2 + 4\lambda + 1 = 0 \quad$ so $\quad \lambda_1 = -1, \quad \lambda_2 = -(1/3).$

Two integrals of (1) are given by the following sets

$$\left.\begin{aligned} Udy + \lambda_1 Tdx + \lambda_2 Udp &= 0 \\ Udx + \lambda_2 Rdy + \lambda_2 Udq &= 0 \end{aligned}\right\} \qquad ...(5)$$

$$\left.\begin{aligned} Udy + \lambda_2 Tdx + \lambda_2 Udp &= 0 \\ Udx + \lambda_1 Rdy + \lambda_1 Udq &= 0 \end{aligned}\right\} \qquad ...(6)$$

Using (2) and (4), (5) and (6) respectively gives

$$\left.\begin{aligned} dy - dx - dp = 0 \quad &\text{or} \quad dp + dx - dy = 0 \\ dx - (1/3) \times dy - (1/3) \times dq = 0 \quad &\text{or} \quad dq + dy - 3dx = 0 \end{aligned}\right\} \qquad ...(5A)$$

$$\left.\begin{aligned} dy - (1/3) \times dx - (1/3) \times dp = 0 \quad &\text{or} \quad dp + dx - 3dy = 0 \\ dx - dy - dq = 0 \quad &\text{or} \quad dq + dy - dx = 0 \end{aligned}\right\} \qquad ...(6A)$$

Integration of (5A) and (6A) respectively gives

$p + x - y = c_1, \qquad q + y - 3x = c_2$ ...(5*B*)

and $\quad p + x - 3y = c_3, \qquad q + y - x = c_4,$ ...(6*B*)

where c_1, c_2, c_3 and c_4 are arbitrary constants.

From (5B) and (6B), two intermediate integrals of (1) are given by

$p + x - y = f(q + y - 3x) \quad$ and $\quad p + x - 3y = F(q + y - x).$ ...(7)

Let $\quad q + y - 3x = \alpha,$ ...(8)

and $\quad q + y - x = \beta.$ ...(9)

Then from (7), $\quad p + x - y = f(\alpha),$ ...(10)

and $\quad p + x - 3y = F(\beta).$ ...(11)

Here α and β are treated as parameters. Solving (8) and (9) for x and (10) and (11) for y gives

$$x = (\beta - \alpha)/2 \qquad ...(12)$$

and $$y = [f(\alpha) - F(\beta)]/2 \qquad ...(13)$$

From (10), $$p = y - x + f(\alpha) \qquad ...(14)$$

From (9), $$q = x - y + \beta \qquad ...(15)$$

From (12) and (13), $dx = (1/2) \times (d\beta - d\alpha)$, $dy = (1/2) \times [f'(\alpha)d\alpha - F'(\beta)d\beta]$. ...(16)

$\therefore$ $dz = pdx + qdy = [y - x + f(\alpha)]dx + (x - y + \beta)dy$, by (14) and (15)

$= ydx + xdy - xdx - ydy + f(\alpha)dx + \beta dy$

$= d(xy) - xdx - ydy + f(\alpha) \times (1/2)\times(d\beta - d\alpha) + \beta \times (1/2)\times[f'(\alpha)d\alpha - F'(\beta)d\beta]$, by (16)

$= d(xy) - xdx - ydy + (1/2) \times [f(\alpha)d\beta + \beta f'(\alpha)d\alpha] - (1/2) \times f(\alpha)d\alpha - (1/2) \times \beta F'(\beta)d\beta$

or $$2dz = 2d(xy) - 2xdx - 2ydy + d[\beta f(\alpha)] - f(\alpha)d\alpha - \beta F'(\beta)d\beta.$$

Integrating and using integration by parts in the last term on R.H.S. of the above equation,

we get $$2z = 2xy - x^2 - y^2 + \beta f(\alpha) - \int f(\alpha)d\alpha - [\beta F(\beta) - \int 1 \cdot F(\beta)\, d\beta]$$

or $$2z = 2xy - x^2 - y^2 + \beta[f(\alpha) - F(\beta)] - \int f(\alpha)d\alpha + \int F(\beta)\, d\beta. \qquad ...(17)$$

Let $\int f(\alpha)d\alpha = \phi(\alpha)$ and $\int F(\beta)\, d\beta = \psi(\beta)$ ...(18)

so that $f(\alpha) = \phi'(\alpha)$ and $F(\beta) = \psi'(\beta)$. ...(19)

Using (18) and (19), (12), (13) and (17) give

$2x = \beta - \alpha$, $2y = \phi'(\alpha) - \psi'(\beta)$, $2z = 2xy - x^2 - y^2 + \beta[\phi'(a) - \psi'(\beta)] - \phi(\alpha) + \psi(\beta)$

which is the required solution in parametric form, α and β being parameters and ϕ and ψ being arbitrary functions.

Ex. 3. *Solve* $rt - s^2 + 1 = 0$

Sol. Given that $$0 \cdot r + 0 \cdot s + 0 \cdot t + (rt - s^2) = -1. \qquad ...(1)$$

Comparing (1) with $Rr + Ss + Tt + U(rt - s^2) = V$, $R = 0$, $S = 0$, $T = 0$, $U = 1$ and $V = -1$. ...(2)

Here λ–quadratic $$\lambda^2(UV + RT) + \lambda US + U^2 = 0 \qquad ...(3)$$

becomes $\lambda^2 - 1 = 0$ so that $\lambda_1 = -1$ and $\lambda_2 = 1$. ...(4)

Since the two values of λ are distinct, we shall get two intermediate integrals which are given by the following sets of equations

$$\left.\begin{aligned} Udy + \lambda_1 Tdx + \lambda_1 Udp &= 0 \\ Udx + \lambda_2 Rdy + \lambda_2 Udq &= 0 \end{aligned}\right\} \qquad ... (5A)$$

$$\left.\begin{aligned} Udy + \lambda_2 Tdx + \lambda_2 Udp &= 0 \\ Udx + \lambda_1 Rdy + \lambda_1 Udq &= 0 \end{aligned}\right\} \qquad ... (5B)$$

Using (2) and (4), equations (5) and (6) reduces to

$$\begin{aligned} dy - dp &= 0 \quad i.e., & \left.\begin{aligned} dp - dy &= 0 \\ dq + dx &= 0 \end{aligned}\right\} \quad ... (5A) \\ dx + dq &= 0 \quad i.e., & \\ dy + dp &= 0 \quad i.e., & \left.\begin{aligned} dp + dy &= 0 \\ dq - dx &= 0 \end{aligned}\right\} \quad ... (6A) \\ dx - dq &= 0 \quad i.e., & \end{aligned}$$

Integrating of (5A) and (6A) respectively gives

$$p - y = c_1, \qquad q + x = c_2. \qquad ...(5B)$$

and $$p + y = c_3, \qquad q - x = c_4, \qquad ...(6B)$$

where c_1, c_2, c_3 are c_4 are arbitrary constants

From (5B) and (6B), two intermediate integrals are given by

$$p - y = f(q + x) \quad \text{and} \quad p + y = F(q - x), \qquad ...(7)$$

where f and F are arbitrary functions.

Let $$q + x = \alpha \qquad ...(8)$$

and $$q - x = \beta. \qquad ...(9)$$

Then, from (7), $$p - y = f(\alpha) \qquad ...(10)$$

and $$p + y = F(\beta). \qquad ...(11)$$

In what follows α and β will be regarded as parameters. Solving (8) and (9) for x and (10) and (11) for y, we have

$$x = (\alpha - \beta)/2 \qquad ...(12)$$

and $$y = [F(\beta) - f(\alpha)]/2 \qquad ...(13)$$

From (10), $$p = y + f(\alpha) \qquad ...(14)$$

From (9), $$q = x + \beta. \qquad ...(15)$$

From (12) and (13), $dx = (1/2) \times (d\alpha - d\beta)$, $dy = (1/2) \times [F'(\beta)d\beta - f'(\alpha)d\alpha]$. ...(16)

$\therefore$ $$\begin{aligned} dz &= pdx + qdy = [y + f(\alpha)]dx + (x + \beta)dy, \text{ using (14) and (15)} \\ &= (ydx + xdy) + f(\alpha)dx + \beta dy \\ &= d(xy) + f(\alpha) \times (1/2) \times (d\alpha - d\beta) + \beta \times (1/2) \times [F'(\beta)d\beta - f'(\alpha)d\alpha], \text{ by (16)} \\ &= d(xy) + (1/2) \times f(\alpha)d\alpha - (1/2) \times [f(\alpha)d\beta + \beta f'(\alpha)d\alpha] + (1/2) \times \beta F'(\beta)d\beta \end{aligned}$$

or $$2dz = 2d(xy) + f(\alpha)d\alpha - d[\beta f(\alpha)] + \beta F'(\beta)d\beta.$$

Integrating both sides and using integration by parts in the last term on the R.H.S., we obtain

$$2z = 2xy + \int f(\alpha)d\alpha + \beta f(\alpha) + \beta F(\beta) - \int F(\beta)\, d\beta. \qquad ...(17)$$

Let $\int f(\alpha)\, d\alpha = \phi(\alpha)$ and $\int F(\beta)\, d\beta = \psi(\beta)$ ...(18)

so that $f(\alpha) = \phi'(\alpha)$ and $F(\beta) = \psi'(\beta)$. ...(19)

Using (18) and (19), (12), (13) and (17) may be re–written as

$$2x = (\alpha - \beta), \qquad 2y = \psi'(\beta) - \phi'(\alpha), \qquad 2z = 2xy - \phi(\alpha) + \beta\{\phi'(\alpha) + \psi'(\beta)\} - \psi(\beta)$$

which is the required solution in parametric form, α and β being parameters and ϕ and ψ being arbitrary functions.

Ex. 4. *Solve* $r + 3s + t + (rt - s^2) = 1$. **[Rohilkhand 1995]**

Sol. Given $$r + 3s + t + (rt + s^2) = 1 \qquad ... (1)$$

Comparing (1) with $Rr + Ss + Tt + U(rt - s^2) = V$, $R = 1, S = 3, T = 1, U = 1, V = 1$. ...(2)

Now, λ–quadratic is $$\lambda^2(UV + RT) + \lambda US + U^2 = 0 \qquad ...(3)$$

or $2\lambda^2 + 3\lambda + 1 = 0$ so that $\lambda = -1, \; -1/2$. Here $\lambda_1 = -1$, $\lambda_2 = -1/2$. ...(4)

Two intermediate integrals of (1) are giving by the following sets

$$\left.\begin{aligned} Udy + \lambda_1 Tdx + \lambda_1 Udp = 0 \\ Udx + \lambda_2 Rdy + \lambda_2 Udq = 0 \end{aligned}\right\} \qquad ... (5)$$

$$\left.\begin{aligned} Udy + \lambda_2 Tdx + \lambda_2 Udp = 0 \\ Udx + \lambda_1 Rdy + \lambda_1 Udq = 0 \end{aligned}\right\} \qquad ... (6)$$

Using (2) and (4), equations (5) and (6) reduces to

$$\begin{aligned} &dy - dx - dp = 0 && \textit{i.e.,} && \left.\begin{aligned} dp + dx - dy = 0 \\ dq - 2dx + dy = 0 \end{aligned}\right\} \; ... 5(A) \\ &dx - (1/2) \times dy - (1/2) \times dq = 0 && \textit{i.e.,} && \end{aligned}$$

and $$\begin{aligned} &dy - (1/2) \times dx - (1/2) \times dp = 0 && \textit{i.e.,} && \left.\begin{aligned} dp + dx - 2dy = 0 \\ dq - dx + dy = 0 \end{aligned}\right\} \; ...(6A) \\ &dx - dy - dq = 0 && \textit{i.e.,} && \end{aligned}$$

Integrating of (5*A*) and (6*A*) respectively gives

$$p + x - y = c_1, \qquad q - 2x + y = c_2 \qquad \text{...(5B)}$$

and $$p + x - 2y = c_3, \qquad q - x + y = c_4, \qquad \text{...(6B)}$$

where c_1, c_2, c_3 and c_4 are arbitrary constants

From (5*B*) and (6*B*), two intermediate integrals are given by

$$p + x - y = f(q - 2x + y) \quad \text{and} \quad p + x - 2y = F(q - x + y), \qquad \text{...(7)}$$

where f and F are arbitrary functions

Let $$q - 2x + y = \alpha \qquad \text{...(8)}$$

and $$q - x + y = \beta. \qquad \text{...(9)}$$

Then, from (7) $$p + x - y = f(\alpha) \qquad \text{...(10)}$$

and $$p + x - 2y = F(\beta). \qquad \text{...(11)}$$

In what follows, α and β will be regarded as parameters. Solving (8) and (9) for x and (10) and (11) for y, we have

$$x = \beta - \alpha \qquad \text{...(12)}$$

and $$y = f(\alpha) - F(\beta). \qquad \text{...(13)}$$

From (10), $$p = y - x + f(\alpha) \qquad \text{...(14)}$$

From (9), $$q = x - y + \beta. \qquad \text{...(15)}$$

From (12) and (13), $$dx = d\beta - d\alpha, \qquad dy = f'(\alpha)d\alpha - F'(\beta)d\beta. \qquad \text{...(16)}$$

$\therefore$ $$dz = pdx + qdy = [y - x + f(\alpha)]dx + [x - y + \beta]dy, \text{ using (14) and (15)}$$

$$= -(x - y)(dx - dy) + f(\alpha)dx + \beta dy$$

$$= -(x - y)\, d(x - y) + f(\alpha)(d\beta - d\alpha) + \beta[f'(\alpha)dx - F'(\beta)d\beta], \text{ by (16)}$$

$$= -(x - y)\, d(x - y) - f(\alpha)\, d\alpha + \{f(\alpha)d\beta + \beta f'(\alpha)d\alpha\} - \beta F'(\beta)d\beta$$

or $$dz = -(x - y)\, d(x - y) - f(\alpha)\, d\alpha + d[\beta f(\alpha)] - \beta F'(\beta)d\beta.$$

Integrating both sides and using integration by parts in the last term on the R.H.S., we obtain

$$z = -(1/2) \times (x - y)^2 - \int f(\alpha)\, d\alpha + \beta f(\alpha) - \left[\beta F(\beta) - \int F(\beta)\, d\beta\right]. \qquad \text{...(17)}$$

Let $$\int f(\alpha)\, d\alpha = \phi(\alpha) \quad \text{and} \quad \int F(\beta)\, d\beta = \psi(\beta) \qquad \text{...(18)}$$

so that $$f(\alpha) = \phi'(\alpha) \quad \text{and} \quad F(\beta) = \psi'(\beta). \qquad \text{...(19)}$$

Using (18) and (19), (12), (13) and (17) may be written as

$$x = \beta - \alpha, \qquad y = \phi'(\alpha) - \psi'(\beta), \qquad z = -(1/2)\times (x - y)^2 - \phi(\alpha) + \psi(\beta) + \beta[\phi'(\alpha) - \psi'(\beta)]$$

which is the required solution in parametric form, α and β being parameters, and ϕ and ψ being arbitrary functions.

Ex. 5. *Solve* $rt - s^2 + a^2 = 0$. **[Rohilkhand 1993]**

Sol. Given that $$0 \cdot r + 0 \cdot s + 0 \cdot t + (rt - s^2) = -a^2. \qquad \text{...(1)}$$

Comparing (1) with $Rr + Ss + Tt + U(rt - s^2) = V$, $R = 0$, $S = 0$, $T = 0$, $U = 1$, $V = -a^2$. ...(2)

Then, the λ–quadratic $$\lambda^2(UV + RT) + \lambda SU + U^2 = 0 \qquad \text{...(3)}$$

becomes $-\lambda^2 a^2 + 1 = 0$ or $\lambda = \pm 1/a$. So $\lambda_1 = 1/a$, $\lambda_2 = -1/a$. ...(4)

Two intermediate integrals of (1) are given by the following two sets

$$\left.\begin{aligned} Udy + \lambda_1 Tdx + \lambda_1 Udp &= 0 \\ Udx + \lambda_2 Rdy + \lambda_2 Udq &= 0 \end{aligned}\right\} \qquad \text{... (5)}$$

and $$\left.\begin{aligned} Udy + \lambda_2 Tdx + \lambda_2 Udp &= 0 \\ Udx + \lambda_1 Rdy + \lambda_1 Udq &= 0 \end{aligned}\right\} \qquad \text{... (6)}$$

Using (2) and (4), equations (5) and (6) reduce to

$$\left.\begin{aligned} dy + (1/a) \times dp &= 0 \quad i.e., \quad dp + ady = 0 \\ dx - (1/a) \times dq &= 0 \quad i.e., \quad dq - adx = 0 \end{aligned}\right\} \quad \dots (5A)$$

and

$$\left.\begin{aligned} dy - (1/a) \times dp &= 0 \quad i.e., \quad dp - ady = 0 \\ dx + (1/a) \times dq &= 0 \quad i.e., \quad dq + adx = 0 \end{aligned}\right\} \quad \dots (6A)$$

Integration of (5A) and (6A) respectively gives

$$p + ay = c_1, \qquad q - ax = c_2 \qquad \dots(5B)$$

and

$$p - ay = c_3, \qquad q + ax = c_4. \qquad \dots(6B)$$

where c_1, c_2, c_3 and c_4 are arbitrary constants

From (5*B*) and (6*B*), two intermediate integrals are given by

$$p + ay = f(q - ax) \qquad \text{and} \qquad p - ay = F(q + ax). \qquad \dots(7)$$

where f and F are arbitrary functions

Let $$q - ax = \alpha \qquad \dots(8)$$

and $$q + ax = \beta. \qquad \dots(9)$$

Then, from (7) $$p + ay = f(\alpha) \qquad \dots(10)$$

and $$p - ay = F(\beta). \qquad \dots(11)$$

In what follows, α and β will be regarded as parameters. Solving (8) and (9) for x and (10) and (11) for y, we have

$$x = (1/2a) \times (\beta - \alpha) \qquad \dots(12)$$

and $$y = (1/2a) \times [f(\alpha) - F(\beta)]. \qquad \dots(13)$$

From (10), $$p = f(\alpha) - ay. \qquad \dots(14)$$

From (9), $$q = \beta - ax. \qquad \dots(15)$$

From (12) and (13), $dx = (1/2a) \times (d\beta - d\alpha)$, $dy = (1/2a) \times [f'(\alpha)d\alpha - F'(\beta)d\beta]$ ...(16)

$\therefore$ $dz = pdx + qdy = [f(\alpha) - ay]dx + (\beta - ay)dy$, using (14) and (15)

$$= f(\alpha)dx + \beta dy - a(ydx + xdy)$$

$$= f(\alpha) \times (1/2a) \times (d\beta - d\alpha) + \beta \times (1/2a) \times [f'(\alpha)d\alpha - F'(\beta)d\beta] - ad(xy), \text{ by (16)}$$

or $$2adz = \{f(\alpha)d\beta + \beta f'(\alpha)d\alpha\} - f(\alpha)d\alpha - 2a^2 d(xy) - \beta F'(\beta)d\beta.$$

Integrating both sides and using the formula for integration by parts in the last term on R.H.S., we have

$$2az = \beta f(\alpha) - \int f(\alpha)d\alpha - 2a^2xy - \left[\beta F(\beta) - \int F(\beta)\,d\beta\right]. \qquad \dots(17)$$

Let $$\int f(\alpha)d\alpha = \phi(\alpha) \qquad \text{and} \qquad \int F(\beta)\,d\beta = \psi(\beta) \quad \dots(18)$$

so that $$f(\alpha) = \phi'(\alpha) \qquad \text{and} \qquad F(\beta) = \psi'(\beta). \quad \dots(19)$$

Using (18) and (19), (12), (13) and (17) reduces to

$$2ax = \beta - \alpha, \qquad 2ay = \phi'(\alpha) - \psi'(\beta), \qquad 2az = \beta[\phi'(\alpha) - \psi'(\beta)] - \phi(\alpha) - 2a^2xy + \psi(\beta).$$

which is the required solution in parametric form, α, β, being parameters and $\phi(\alpha)$ and $\psi(\beta)$ being arbitrary functions.

Ex. 6. *Solve* $7r - 8s - 3t + (rt - s^2) = 36$.

Sol. Given that $$7r - 8s - 3t + (rt - s^2) = 36. \qquad \dots(1)$$

Comparing (1) with $Rr + Ss + Tt + U(rt - s^2) = V$, $R = 7$, $S = -8$, $T = -3$, $U = 1$, $V = 36$. ...(2)

The λ–quadratic $$\lambda^2(UV + RT) + \lambda US + U^2 = 0 \qquad \dots(3)$$

becomes $15\lambda^2 - 18\lambda + 1 = 0$ or $(5\lambda - 1)(3\lambda - 1) = 0$. So $\lambda_1 = 1/5$, $\lambda_2 = 1/3$. ...(4)

Two intermediate integrals of (1) are given by the following sets

$$\left.\begin{aligned} Udy + \lambda_1 Tdx + \lambda_1 Udp = 0 \\ Udx + \lambda_2 Rdy + \lambda_2 Udq = 0 \end{aligned}\right\} \qquad \text{... (5)}$$

$$\left.\begin{aligned} Udy + \lambda_2 Tdx + \lambda_2 Udp = 0 \\ Udx + \lambda_1 Rdy + \lambda_1 Udq = 0 \end{aligned}\right\} \qquad \text{... (6)}$$

Using (2) and (4), equations (5) and (6) reduce to

$$\left.\begin{aligned} dy + (1/5) \times (-3)dx + (1/5) \times dp = 0 \quad & i.e., \quad dp - 3dx + 5dy = 0 \\ dx + (1/3) \times 7\, dy + (1/3) \times dq = 0 \quad & i.e., \quad dq + 7dy + 3dx = 0 \end{aligned}\right\} \qquad \text{... (5A)}$$

$$\left.\begin{aligned} dy + (1/3) \times (-3)dx + (1/3) \times dp = 0 \quad & i.e., \quad dp - 3dx + 3dy = 0 \\ dx + (1/5) \times 7dy + (1/5) \times dq = 0 \quad & i.e., \quad dq + 7dy + 5dx = 0 \end{aligned}\right\} \qquad \text{... (6A)}$$

Integrating of (5*A*) and (6*A*) respectively, gives

$$p - 3x + 5y = c_1, \qquad q + 7y + 3x = c_2 \qquad \text{...(5}B\text{)}$$

and $$p - 3x + 3y = c_3, \qquad q + 7y + 5x = c_4, \qquad \text{...(6}B\text{)}$$

where c_1, c_2 c_3 and c_4 are arbitrary constants

From (5*B*) and (6*B*), two intermediatre integrals are given by

$$p - 3x + 5y = f(q + 7y + 3x) \quad \text{and} \quad p - 3x + 3y = F(q + 7y + 5x) \qquad \text{...(7)}$$

where f and F are arbitrary functions

Let $$q + 7y + 3x = \alpha \qquad \text{...(8)}$$

and $$q + 7y + 5x = \beta. \qquad \text{...(9)}$$

Then, from (7) $$p - 3x + 5y = f(\alpha) \qquad \text{...(10)}$$

and $$p - 3x + 3y = F(\beta). \qquad \text{...(11)}$$

In what follows, α and β will be regarded as parameters. Solving (8) and (9) for x and (10) and (11) for y, we have

$$x = (\beta - \alpha)/2 \qquad \text{...(12)}$$

and $$y = [f(\alpha) - F(\beta)]/2 \qquad \text{...(13)}$$

From (10), $$p = f(\alpha) + 3x - 5y. \qquad \text{...(14)}$$

From (9), $$q = \beta - 7y - 5x. \qquad \text{...(15)}$$

From (12) and (13), $$dx = (1/2) \times (d\beta - d\alpha),\ dy = (1/2) \times \{f'(\alpha)d\alpha - F'(\beta)\, d\beta\}. \qquad \text{...(16)}$$

$\therefore$ $dz = pdx + qdy = \{f(\alpha) + 3x - 5y\}dx + \{\beta - 7y - 5x)dy$, using (14) and (15)

$$= 3xdx - 7ydy - 5(ydx + xdy) + f(\alpha)dx + \beta dy$$

$$= 3xdx - 7ydy - 5d(xy) + f(\alpha) \times (1/2) \times (d\beta - d\alpha) + \beta \times (1/2) \times \{f'(\alpha)d\alpha - F'(\beta)d\beta\}$$

or $$2dz = 6xdx - 14ydy - 10d(xy) + \{f(\alpha)d\beta + \beta f'(\alpha)d\alpha\} - f(\alpha)d\alpha - \beta F'(\beta)d\beta$$

or $$2dz = 6xdx - 14ydy - 10d(xy) + d\{\beta f(\alpha)\} - f(\alpha)d\alpha - \beta F'(\beta)d\beta.$$

Integrating both sides and using the formula for integrating by parts in the last term on R.H.S., we have

$$2z = 3x^2 - 7y^2 - 10xy + \beta f(\alpha) - \int f(\alpha)\, d\alpha - [\beta F(\beta) - \int F(\beta)\, d\beta]$$

or $$2z = 3x^2 - 7y^2 - 10xy + \beta[f(\alpha) - F(\beta)] - \int f(\alpha)\, d\alpha + \int F(\beta)\, d\beta. \qquad \text{...(17)}$$

Let $$\int f(\alpha)\, d\alpha = \phi(\alpha) \quad \text{and} \quad \int F(\beta)\, d\beta = \psi(\beta) \qquad \text{...(18)}$$

so that $$f(\alpha) = \phi'(\alpha) \quad \text{and} \quad F(\beta) = \psi'(\beta) \qquad \text{...(19)}$$

Using (18) and (19), relation (12), (13) and (17) become

$x = (1/2) \times (\beta - \alpha)$, $y = (1/2) \times [\phi'(\alpha) - \psi'(\beta)]$, $2z = 3x^2 - 7y^2 - 10xy + \beta[\phi'(\alpha) - \psi'(\beta)] - \phi(\alpha) + \psi(\beta)$.

which is required solution in parametric form, α and β being parameters and $\phi(\alpha)$ and $\psi(\beta)$ being arbitrary functions.

9.14 Miscellaneous examples on Rr + Ss + Tt + U(rt – s²) = V.

In some problems only one intermediate integral is possible. Sometimes even after getting two intermediate integrals, it may not be possible to get p and q from those intermediate integrals. In such problems, final solution is obtained by integrating only one intermediate integral by the methods of solution of first order equation, for example, Charpit's method. Again, we can avoid Charpit's method by taking $u_1 = \phi_1(v_1)$ and u_2 = constant = λ (say) to obtain final solution. [Here we have assumed that $u_1 = \phi_1(v_1)$ and $u_2 = \phi_2(v_2)$ are two intermediate integrals]. Since an arbitrary constant can be regarded as a particular case of an arbitrary function, the values of p and q derived from $u_1 = \phi_1(v_1)$ and $u_2 = \lambda$ will make $dz = pdx + qdy$ integrable. The complete integral so obtained will involve one arbitrary funtion ϕ_1 and two arbitrary constants, namely, λ and the constant of integration. To obtain the general integral, express one of the arbitrary constant as an arbitrary function of the other and eliminate this remaining constant between the equation so obtained and that deduced from it by differentiation with respect to that constant.

Ex. 1. *Obtain the intermediate integral of* $2yr + (px + qy)s + xt - xy(rt - s^2) = 2 - pq$. **[Rohilkhand 1992]**

Sol. Given $\quad 2yr + (px + qy)s + xt - xy(rt - s^2) = 2 - pq. \qquad$ ...(1)

Comparing (1) with $Rr + Ss + Tt + U(rt - s^2) = V$, we have

$R = 2y, \quad S = px + qy, \quad T = x, \quad U = -xy, \quad V = 2 - pq. \qquad$ ...(2)

Now, λ–quadratic $\quad \lambda^2(UV + RT) + \lambda US + U^2 = 0 \qquad$ ...(3)

reduces to $\quad \lambda^2[2yx - xy(2 - pq)] + \lambda[-xy(px + qy)\} + x^2y^2 = 0$

or $\quad \lambda^2 pq - \lambda(px + qy) + xy = 0 \quad$ or $\quad (\lambda p - y)(\lambda q - x) = 0$

or $\quad \lambda = y/p, x/p \quad$ so that $\quad \lambda_1 = y/p \quad$ and $\quad \lambda_2 = x/q. \qquad$ ...(4)

Two intermediate integrals are given by the following sets

$$\left.\begin{aligned} Udy + \lambda_1 Tdx + \lambda_1 Udp = 0 \\ Udx + \lambda_2 Rdy + \lambda_2 Udq = 0 \end{aligned}\right\} \qquad \text{... (5)}$$

$$\left.\begin{aligned} Udy + \lambda_2 Tdx + \lambda_2 Udp = 0 \\ Udx + \lambda_1 Rdy + \lambda_1 Udq = 0 \end{aligned}\right\} \qquad \text{... (6)}$$

Using (2) and (4), equations (5) and (6) reduce to

$-xydy + (y/p)xdx + (y/p)(-xy)dp = 0 \quad$ *i.e.*, $\quad (pdy + ydp) - dx = 0$

and $\quad -xydx + (x/q)(2y)dy + (x/q)(-xy)dq = 0 \quad$ *i.e.*, $\quad (qdx + xdq) - 2dy = 0. \qquad$ ... (5A)

$-xydy + (x/q)xdx + (x/q)(-xy)dp = 0 \quad$ *i.e.*, $\quad -qydy + xdx - xydp = 0$

and $\quad -xydx + (y/p)(2y)dy + (y/p)(-xy)dq = 0 \quad$ *i.e.*, $\quad -pxdx + 2ydy - xydq = 0 \qquad$ ...(6A)

Integrating (5A), $\quad py - x = c_1 \quad$ and $\quad qx - 2y = c_2$

Hence one intermediate integral is $\quad py - x = \phi(qx - 2y). \qquad$ ...(7)

Note, the equation (6A) cannot be integrated. Hence in this problem we can obtain only one intermediate integral, *i.e.*, (7). Here ϕ is an arbitrary function.

Ex. 2. *Solve* $qr + (p + x)s + yt + y(rt - s^2) + q = 0$. **[Rohilkhand 1992]**

Sol. Given $\quad qr + (p + x)s + yt + y(rt - s^2) = -q. \qquad$ ...(1)

Comparing (1) with $Rr + Ss + Tt + U(rt - s^2) = V$, we have

$R = q, \quad S = p + x, \quad T = y, \quad U = y \quad$ and $\quad T = -q. \qquad$ ...(2)

Now, the λ–quadratic $\quad \lambda^2(UV + RT) + \lambda SU + U^2 = 0 \qquad$ ...(3)

reduces to $\quad (0 \times \lambda^2) + \lambda y(p + x) + y^2 = 0$. Since coefficient of λ^2 is zero, it follows that its one root is ∞. The other root is $-y/(p + x)$.

Let $\quad \lambda_1 = -y/(p + x) \quad$ and $\quad \lambda_2 = \infty. \qquad$ ...(4)

One intermediate integral is given by the following sets

$$\left.\begin{aligned} &Udy + \lambda_1 Tdx + \lambda_1 Udp = 0 \quad ,i.e., \quad Udy + \lambda_1 Tdx + \lambda_1 Udp = 0 \\ &Udx + \lambda_2 Rdy + \lambda_2 Udq = 0 \quad\quad (1/\lambda_2)\times Udx + Rdy + Udq = 0. \end{aligned}\right\} \quad ...(5)$$

Using (2) and (4), equations of (5) reduce to

$$\left.\begin{aligned} &ydy - \frac{y^2}{p+x}dx - \frac{y^2}{p+x}dp = 0 \quad i.e. \quad \frac{dp+dx}{p+x} - \frac{dy}{y} = 0 \\ \text{and} \quad &qdy + ydq = 0 \quad i.e., \quad d(yq) = 0 \end{aligned}\right\} \quad ...(5A)$$

Integrating (5A), $\log(p + x) - \log y = \log c_1$ or $(p + x)/y = c_1$ ...(6)

and $yq = c_2$, c_2 being an arbitrary constant ...(7)

From (6) and (7), an intermediate integral is given by $qy = f\left(\frac{p+x}{y}\right)$. ...(8)

Charpit's auxiliary equations for (8) are $$\frac{dx}{-\frac{1}{y}f'\left(\frac{p+x}{y}\right)} = \frac{dy}{y} = \frac{dp}{\frac{1}{y}f'\left(\frac{p+x}{y}\right)}. \quad ...(9)$$

Taking the first and third fractions of (9), we have

$dx + dp = 0$ so that $x + p = c$, where c is an arbitrary constant. ...(10)

Solving (8) and (10) for p and q, $p = c - x$ and $q = (1/y) \times f(c/y)$.

Putting these values of p and q in $dz = pdx + qdy$, $dz = (c - x)dx + (1/y) \times f(c/y)dy$.

Integrating, $z = cx - x^2/2 + F(c/y) + G(\lambda)$,

which is the complete integral, F and G being arbitrary functions.

Ex. 3. *Solve* $qxr + (x + y)s + pyt + xy\,(xt - s^2) = 1 - pq$.

Sol. Given $qxr + (x + y)s + pyt + xy(rt - s^2) = 1 - pq$. ...(1)

Comparing (1) with $Rs + Ss + Tt + U(rt - s^2) = V$, we have

$R = qx$, $S = x+y$, $T = py$, $U = xy$ and $V = 1 - pq$ ...(2)

Now, the λ–qaudratic $\lambda^2(UV+RT) + \lambda US + U^2 = 0$ ...(3)

reduces to $\lambda^2\,[qxpy+xy\,(1 - pq)] + \lambda xy\,(x + y) + x^2y^2 = 0$

or $\lambda^2 + (x + y)\lambda + xy = 0$ or $(\lambda + x)(\lambda + y) = 0$ so that $\lambda = -x, -y$.

Let $\lambda_1 = -x$ and $\lambda_2 = -y$. ...(4)

Two intermediate integrals are given by the following two sets :

$$\left.\begin{aligned} &Udy + \lambda_1 T\,dx + \lambda_1 Udp = 0 \\ &Udx + \lambda_2\, Rdy + \lambda_2 Udq = 0 \end{aligned}\right\} \quad ...(5)$$

and

$$\left.\begin{aligned} &Udy + \lambda_2 Tdx + \lambda_2 Udp = 0 \\ &Udx + \lambda_1\, Rdy + \lambda_1 Udq = 0. \end{aligned}\right\} \quad ...(6)$$

Using (2) and (4), equation (5) and (6) reduce to

$$\left.\begin{aligned} &xydy - xpydx - x^2\,ydp = 0 \quad i.e., \quad (xdp + pdx) - dy = 0 \\ \text{and} \quad &xydx - yqxdy - yxydq = 0 \quad i.e., \quad (ydq + qdy) - dx = 0 \end{aligned}\right\} \quad ...(5A)$$

$$\left.\begin{aligned} &xydy - ypydx - y\,xy\,dp = 0 \quad i.e., \quad xdy - pydx - xydp = 0 \\ \text{and} \quad &xydx - xyqdy - x^2\,ydq = 0 \quad i.e., \quad ydx - qxdy - xydq = 0 \end{aligned}\right\} \quad ...(6A)$$

We observe that (5*A*) can be integrated whereas (6*A*) cannot the integrated. So we shall obtain only one intermediate integral with help of (5*A*) :

Integrating (5*A*), $px - y = c_1$ and $qy - x = c_2$

Hence the only intermediate integral of (1) is given by

$px - y = f(qy - x)$, where f is an arbitrary function. ...(7)

A general solution of (1) can be obtained by supposing the arbitrary function f occuring in the intermediate integral (7) to be linear, giving

$px - y = m(qy - x) + n,$ where m and n are arbitrary constants

or $xp - myq = y - mx + n$, which is in Lagrange's form

Hence here Lagrange axuiliary equations are $$\frac{dx}{x} = \frac{dy}{-my} = \frac{dz}{y - mx + n} \quad ...(8)$$

From first and second fractions of (8), $m(1/x)dx + (1/y)dy = 0$

Integrating, $m \log x + \log y = \log a$ or $x^m y = a.$ ...(9)

Chossing $m,\ 1/m,\ 1$ as multipliers, each fraction of (8)

$$= \frac{mdx + (1/m)dy + dz}{mx + (1/m)(-my) + y - mx + n} = \frac{mdx + (1/m)dy + dz}{n} \quad ...(10)$$

Taking first fraction of (8) and (10), we have

$$\frac{dx}{x} = \frac{mdx + (1/m)dy + dz}{n} \quad \text{or} \quad dz + \frac{1}{m}dy + mdx - \frac{n}{x}dx = 0.$$

Integrating, $z + (1/m)y + mx - n \log x = b$, b being an arbitrary constant ...(11)

From (9) and (11) the required general solution in

$z + (1/m)y + mx - \log x^n = \psi(x^m y)$, ψ being an arbitrary function

Ex. 4. *Solve* $(rt - s^2) - s(\sin x + \sin y) = \sin x \sin y.$ **[Meerut 1999]**

Sol. Given $0{\cdot}r - s(\sin x + \sin y) + 0{\cdot}t + (rt - s^2) = \sin x \sin y.$ ...(1)

Comparing (1) with $Rr + Ss + Tt + U(rt - s^2) = V$, we have

$R = 0, \quad S = -(\sin x + \sin y), \quad T = 0\ , \quad U = 1, \quad V = \sin x \sin y.$ ...(2)

Now, the λ–quadratic $\lambda^2(UV + RT) + \lambda US + U^2 = 0.$ ...(3)

reduces to $\sin x \sin y\, \lambda^2 - (\sin x + \sin y)\, \lambda + 1 = 0$

or $(\lambda \sin x - 1)(\lambda \sin y - 1) = 0$ so that $\lambda = \operatorname{cosec} x$ or $\operatorname{cosec} y.$

Let $\lambda_1 = \operatorname{cosec} x$ and $\lambda_2 = \operatorname{cosec} y.$ ...(4)

Two intermediate integral are given by the following two sets :

$$\left.\begin{aligned} Udy + \lambda_1 Tdx + \lambda_1 Udp = 0 \\ Udx + \lambda_2 Rdy + \lambda_2 Udq = 0 \end{aligned}\right\} \quad ...(5)$$

and

$$\left.\begin{aligned} Udy + \lambda_2 \text{Tdx} + \lambda_2 Udp = 0 \\ Udx + \lambda_1 Rdy + \lambda_1 Udq = 0 \end{aligned}\right\} \quad ...(6)$$

Using (2) and (4), equations (5) and (6) reduce to

$$\left.\begin{aligned} dy + \operatorname{cosec} x\, dp = 0 \\ dx + \operatorname{cosec} y\, dq = 0 \end{aligned}\right\} \quad ...(5A)$$

$dy + \operatorname{cosec} y\, dp = 0$ *i.e.*, $dp + \sin y\, dy = 0$

and $dx + \operatorname{cosec} x\, dq = 0$ *i.e.*, $dq + \sin x\, dx = 0$ ...(6A)

We observe that (5*A*) cannot be integrated where as (6*A*) can be integrated. So we shall obtain only one intermediate integral with help of (6A).

Integrating (6*A*) , $p - \cos y = c_1$ and $q - \cos x = c_2.$

Hence the only intermediate integral of (1) is given by

$p - \cos y = f(q - \cos x)$, f being an arbitrary function. ...(7)

A general solution of (1) can be obtained by supposing the arbitrary function f occuring in the the intermediate integral (7) to be linear, giving

$p - \cos y = m[q - \cos x] + n$, m, n being arbitrary constants

or $p - mq = \cos y - m \cos x + n$, which is in Lagrange's from.

Its Lagrange's auxiliary equations are $$\frac{dx}{1} = \frac{dy}{-m} = \frac{dz}{\cos y - m\cos x + n}. \quad ...(8)$$

From the first two fractions of (8), $dy + mdx = 0$ so that $y + mx = a$. ...(9)

Again, taking the first and third fractions of (8), we have

$dz = (\cos y - m \cos x + n)dx = [\cos (a - mx) - m \cos x + n]dx$, as from (9), $y = a - mx$

Intergrating, $z = -(1/m) \times \sin (a - mx) - m \sin x + nx + (1/m) \times b$

or $mz + \sin y + m^2 \sin x - mnx = b$, b being an arbitrary constants ...(10)

From (9) and (10), the required general solution of (1) is

$$mz + \sin y + m^2 \sin x - mnx = \phi(y + mx), \phi \text{ being an arbitrary function.}$$

Ex. 5. *Solve* $xqr + (p + q)s + ypt + (xy - 1)(rt - s^2) + pq = 0.$

Ans. $z - \log (x - m)^n = \phi\{(x - m)^n (1 - my)$

Ex. 6. *Solve* $2yr + (px + qy)s + xt - xy(rt - s^2) = 2 - pq.$

Ans. $z + (1/m) \times (a^2 - mx^2)^{1/2} + (x / \sqrt{m}) \times \sin^{-1}(x\sqrt{m} / a) + 2mx = \phi(mx^2 + y^2)$

Ex. 7. *Solve* $ar + bs + ct + e(rt - s^2) = h$, *where* a, b, c, e *and* h *are constants.*

Sol. Comparing with $Rr + Ss + Tt + U(rt - s^2) = V$, here $R = a$, $S = b$, $T = c$, $U = e$, $V = h$.

The λ–quadratic $\lambda^2 (UV + RT) + \lambda SU + U^2 = 0$ gives $(ac + eh)\lambda^2 + \lambda be + e^2 = 0.$ ...(1)

Let $\lambda = -e/m.$ ...(2)

$\therefore$ (1) reduces to $m^2 - bm + (ac + eh) = 0.$ ...(3)

Let m_1 and m_2 be the roots of (3). The first intermediate integral is given by

$Udy + \lambda_1 Tdx + \lambda_1 Udp = 0$, where $\lambda_1 = -e/m_1$

and $Udx + \lambda_2 Rdy + \lambda_2 Udq = 0$, where $\lambda_2 = -e/m_2$

i.e., $e\,dy - (e/m_1) \times c\,dx - (e/m_1) \times e\,dp$ and $ex - (e/m_2) \times a\,dy - (e/m_2) \times e\,dq = 0$

i.e., $c\,dx + edp - m_1\,dy = 0$ and $ady + edq - m_2 dx = 0.$

Integrating, $cx + ep - m_1 y = c_1$ and $ay + eq - m_2 x = c_2.$

So the first intermediate integral is $cx + ep - m_1 y = \phi_1(ay + eq - m_2 x).$ ...(4)

Proceding as before, the second intermediate integral is

$$cx + cp - m_2 y = \phi_2(ay + eq - m_1 x). \quad ...(5)$$

Notice that p and q cannot be determined from (4) and (5). Hence we proced as follows :

We also have, $cx + ep - m_2 y = c_3$ ...(6)

From (4) and (6), $(m_2 - m_1)y = \phi_1(ay + eq - m_2 x) - c_3$

$\therefore$ $ay + eq - m_2 x = \psi_1\{(m_2 - m_1)y + c_3\}]$ ...(7)

where ψ_1 is the inverse function of ϕ_1.

From (7). $q = (1/e) \times [-ay + m_2 x + \phi_1\{(m_2 - m_1)y + c_3\}]$

and from (6) $p = (-cx + m_2 y + c_3)/e.$

Putting these values in $dz = pdx + qdy$, we get

$$edz = -(xdx - aydy + m_2(xdy + ydx) + c_3 dx + \phi_1\{(m_2 - m_1)y + c_3\}dy$$

Integrating, $ez = -(1/2) \times cx^2 - (1/2) \times ay^2 + m_2 xy + c_3 x + F\{(m_2 - m_1)y + c_3\} + k.$

EXERCISE

Solve the following partial differential equation:

1. $3r + s + t + (rt - s^2) = -9$ **[K.U. Kurukshetra 2004]**

2. $3s - 2(rt - s^2) = 2$ **3.** $2r - 6s + 2t + (rt - s^2) = 4$

Objective problems

1. The equations $R\,dpdy + T\,dqdx - V\,dxdy = 0$ and $Rdy^2 - S\,dxdy + T\,dx^2 = 0$ are called

Sol. Ans. Monge's subsidiary equations. Refer Art. 9.2. **[Meerut 2003].**

2. Monge's method is used to solve a partial differential equation of

(*a*) nth order (*b*) first order (*c*) second order (*d*) none of these **[Agra 2007]**

Sol. Ans. (**c**). Refer Art 9.2

10

Transport Equation

10.1 INTRODUCTION

The hyperbolic character of a system of first order differential equations exhibits in the fact that it is possible to have solutions whose derivatives are discontinuous and these discontinuities propagate along the characteristic curves. In this chapter we propose to use the above fact and derive a system of linear homogeneous ordinary differential equations known as *transport equation.*

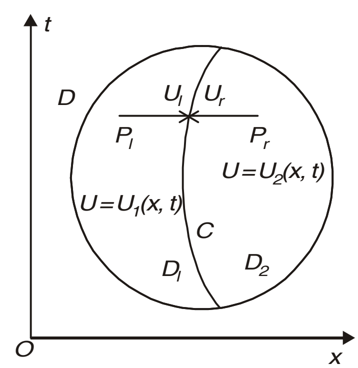
10.2 An IMPORTANT THEOREM

If the first order partial derivatives of a continuous function U (x, t), satisfying a system of quasi-linear equations of first order on both sides of a curve C in xt-plane, are discontinuous across a curve C, then the curve C must be a characteristic curve of the system of equations.

Proof : Suppose that the given first-order system of n quasi-linear partial differential differential equations be given by

$$\left.\begin{aligned} A_{ij}\,(\partial u_j/\partial t) + B_{ij}\,(\partial u_j/\partial x) + C_i = 0, \quad i = 1, 2, ..., n; \quad j = 1, 2, ..., n \\ \text{or} \qquad A(\partial U/\partial t) + B\,(\partial U/\partial t) + C = 0, \end{aligned}\right\} \qquad ...(1)$$

where the n components $u_1, u_2, ..., u_n$ of the column vector U are dependent variables, A and B are $n \times n$ matrices and C is a $n \times 1$ column vector.



Let D be a domain in the xt-plane and let D_1 and D_2 be two portions of D separated by a curve C such that D_1 is on the left and D_2 on the right of C as shown in the adjoining figure. Let U_1 be the genuine solution of (1) in the domain D_1 and U_2 that in the domain D_2. Suppose that the limiting value U_l *of* U_1 as we approach a point P on C from the domain D_1 and the limiting value U_r as we approach P from the domain D_2 exist and are such that $U_l = U_r$ at every point of the curve C. Let a function U be defined in the domain D such that

$$U = \begin{cases} U_1 & \text{in } D_1 \\ U_2 & \text{in } D_2 \end{cases} \qquad ...(2)$$

The function U given by (2) is a genuine solution of (1) in D_1 and D_2 respectively. The function U is continuous in D but its derivatives may be discontinuous across the curve C. Suppose that the limiting values of the derivatives of U as we approach P on the curve C from the two domains D_1 and D_2 exist. Also assume that these derivative, if discontinuous across the curve C, have only a finite jump across the curve C.

Let the equation of the curve C be $\phi(x, t) = 0$ and let $\eta(x,t)$ be any other function independent of ϕ such that ϕ and η are sufficiently smooth and the Jacobian $\partial(\phi\,,\eta)/\partial(x,t) \neq 0$ in the domain D. Therefore, if we can introduce a new set of independent variables (ϕ, η) in place of (x, t), then U_ϕ represents an exterior derivative and U_η is a tangential derivative along the curve C.

We have
$$\left.\begin{aligned} U_x &= \frac{\partial U}{\partial \phi}\frac{\partial \phi}{\partial x} + \frac{\partial U}{\partial \eta}\frac{\partial \eta}{\partial x} = U_\phi\, \phi_x + U_\eta\, \eta_x \\ \text{and} \qquad U_t &= \frac{\partial U}{\partial \phi}\frac{\partial \phi}{\partial t} + \frac{\partial U}{\partial \eta}\frac{\partial \eta}{\partial t} = U_\phi \phi_t + U_\eta \eta_t \end{aligned}\right\} \qquad \text{...(3)}$$

Let us now assume that the first order partial derivatives U_x and U_t are discontinuous across the curve C. Since the function U is continuous across C, its tangential derivative U_η is also continuous across C. Hence from the above two relations (3), it follows that the exterior derivative U_ϕ must be discontinuous across C. Let $(U_\phi)_r$ and $(U_\phi)_l$ be the limiting values of U_ϕ across C, as we approach a point P on C from the domains D_1 and D_2 respectively. Then, the jump $[U_\phi]$ in U_ϕ across C is given by

$$[U_\phi] = (U_\phi)_r - (U_\phi)_l \qquad \text{....(4)}$$

Since U_η is continuous across the curve C, hence from (3), the jumps in the first order derivatives U_x and U_t are related to $[U_\phi]$ by the relations

$$[U_x] = [U_\phi]\phi_x \qquad \text{and} \qquad [U_t] = [U_\phi]\phi_t \qquad \text{...(5)}$$

The quasi-linear equation (1) is valid everywhere in D except at the points on the curve C. Since all the terms appearing in it other than the first order derivatives are continuous across the curve C, hence taking the limit of (1) as we move from the region D_1 to P and again as we more from the region D_2 to P and then subtracting the equations so obtained, we obtain

$$(A\phi_t + B\phi_x)_{at\,P}\, [U_\phi] = 0 \qquad \text{...(6)}$$

Since $[U_\phi]$ is not a zero vector, the matrix $A\phi_t + B\phi_x$ must be singular on the curve C, i.e., at every point of the curve C, we have

$$\left.\begin{aligned} &\det\,(A\phi_t + B\phi_x) = 0 \\ \text{or} \qquad &\det(-\lambda A + B) = 0, \text{ where } \lambda = -\phi_t/\phi_x \end{aligned}\right\} \qquad \text{...(6)}$$

Hence, it follows that C is a characteristic curve.

Note 1. In (6), det X stands for determinant of the matrix X.

Note 2. If the derivatives of a solution U of system (1) upto order $(r \geq 1)$ are continuous across a curve C and the $(r + 1)$ th derivatives are discontinuous across C, then differentiating (1) r times and then proceeding as discussed above, we can show that C is necessarily a characteristic curve.

10.3 GENERALISED OR WEAK SOLUTION

[Allahabad 2003; G.N.D.U. Amritsar 2004; Kanpur 2003, 05)

Consider a general first order quasi-linear hyperbolic system of first order equations

$$A(x,t)U_t + B(x,t)\,U_x + H(x,t)\,J\,(x,t) = 0, \qquad \text{...(1)}$$

where the elements of matrices A, B and H (each of order $n \times n$) and column vector J (of order $n \times 1$) are functions of x and t only. Also note that $U_t = \partial U/\partial t$ and $U_x = \partial U/\partial x$. Let D be a domain in the xt-plane and let D_1 and D_2 be two portions of D separated by a curve C.

Suppose we have a solution U which satisfies (1) in D_1 and D_2 separately but is itself discontinuous across C. It has been established that discontinuities in the solution cannot be discussed for every function U satisfying (1) in D_1 and D_2. However, such discontinuities can be discussed for a "*generalised*" or '*weak*" solution.

To end, we first reduce the system (1) to the characteristic canonical form and note that $(\partial/\partial t)+\lambda_M(x,t)(\partial/\partial x)$ represents the directional derivatives along the characteristics of the Mth field. Integrating it from a point P_M *to a point* (ξ, τ), *both lying on a* characteristic of the Mth field, we obtain.

$$W_M(\xi,\tau)=-\int_{P_M}^{P}[H_{M_i}\{x_M(t,\xi,\tau),t\}\,W_i\{x_M(t,\xi,\tau),t\}]dt$$

$$+(W_M)_{at\,P_M}\int_{P_M}^{P}J_M\{x_M(t,\xi,\tau),t\}dt,\ \text{for } M=1,2,\ldots n \qquad \ldots(2)$$

where $$H_{M_i}=[l^{(M)}A\{\partial r^{(i)}/\partial t+\lambda_M(\partial r^{(i)}/\partial x)\}+(l^{(M)}\,H\,r^{(i)})]/(l^{(M)}A\,r^{(M)}) \qquad \ldots(3)$$

and $$J_M=(l^{(M)}J)/(l^{(M)}\ A\,r^{(M)}) \qquad \ldots(4)$$

with no sum over M in these expressions and $x = x_M\,(t,\ \xi,\tau)$ is the characteristic of Mth field through the point P.

We can rewrite in compact form the expression for H_{M_i} in terms of the operator $\tau\equiv A(\partial/\partial t)+B(\partial/\partial x)+H$. Thus, (3) takes the form

$$H_{M_i}=[l^{(M)}\tau\, r^{(i)}]/[l^{(M)}\ A\ r^{(M)}] \qquad \ldots(5)$$

We now define a generalised or weak solution of (1) to be a function $U(x,\ t)$ obtained from $W_1, W_2, \ldots, W_n$ which satisfy the system of equations (2).

10.4 TRANSPORT EQUATION FOR A LINEAR-HYPERBOLIC SYSTEM

[Calicut 2003 G.N.D.U. (Amritsar 2005; Kanpur 2004; Meerut 2005, 06, 10, 11]

Consider a general first order quasi-linear hyperbolic system of first order equations

$$A(x,t)\,U_t+B(x,t)\,U_x+H(x,t)\,J\,(x,t)=0, \qquad \ldots(1)$$

where the elements of matrices A, B and H(each of order $n \times n$) and column vector J (of order $n \times 1$) are functions of x and t only. Also note that $U_t=\partial u/\partial t$ and $U_x=\partial U/\partial x$. Let D be a domain in the xt-plane and let D_1 and D_2 be two portions of D separated by a curve C. Suppose $U(x,t)$ is a weak solution of (1) which is continuous in the domain D except on the curve C and is a genuine solution of (1) in the domains D_1 and D_2. We also suppose that the function U has a jump discontinuity across C.

We now reduce the system (1) to the characteristic canonical form and note that $(\partial/dt)+\lambda_M(x,t)(\partial/\partial x)$ represents the directional derivatives along the characteristics of the Mth field. Integrating it from a point P_M to a point (ξ,τ), both lying on a characteristic of the Mth field, we have

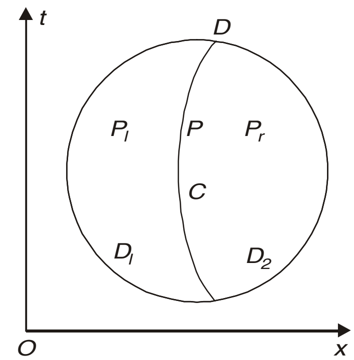
$$W_M(\xi,\tau)=-\int_{P_M}^{P}[H_{M_i}\{x_M(t,\xi,\tau),t\}\,W_i\{x_M(t,\xi,\tau),t]\,dt$$

$$+(W_M)_{at\,P_M}-\int_{P_M}^{P}[J_M\{x_M(t,\xi,\tau),t\}\,dt,\ \text{for } M=1,2,\ldots n \qquad \ldots(2)$$

where $$H_{M_i} = [l^{(M)} A\{\partial r^{(i)}/\partial t + \lambda_M (\partial r^{(i)}/\partial x)\} + l^{(M)}\ Hr^{(i)}]/[l^{(M)} A\, r^{(M)}] \qquad ...(3)$$

and $$J_M = (l^{(M)} J)/[l^{(M)} A\ r^{(M)}] \qquad ...(4)$$

with no sum over M in these expressions and $x = x_M(t, \xi, \tau)$ is the characteristic of Mth field through the point P.



In the case when the function U has a jump discontinuity across the curve C, the integrands on the right hand side of (2) are continuous functions of t except for a finite jump across C. On performing the integration in (2) along a characteristic of the Mth family, we find that the characteristic variable W_M is given by a continuous function of this curve. If a curve C is not tangential to a characteristic of the Mth family, W_M must be continuous across the curve C. However, according to our assumption at least one of $W_1, W_2, \ldots, W_n$ must be discontinuous across the curve C. Therefore, it follows that C, the curve of discontinuity, must be a characteristic curve of jth family (say), and the jump in all characteristic variables W_i, $i \neq j$, must be zero across the carve C.

Now, suppose that the curve of discontinuity C is a characteristic curve of the jth family then, the jump $[W_i]$ in W_i satisfies

$$[W_i] = 0, \text{ for } \quad i = j \qquad \text{and} \qquad [W_j] = 0 \qquad ...(5)$$

Again, we have $$[U] = r^{(j)}[W_j], \text{ on sum over } j \qquad ...(6)$$

Re-writing the equation, we have

$$(\partial W_M/\partial t) + \lambda_M(\partial W_M/\partial x) + C_1\ M_i W_i + J_M = 0, \qquad M = 1, 2, \ldots, n \qquad ...(7)$$

Consider two points P_l and P_r on the two sides of C in the regions D_1 and D_2 respectively as shown in the figure. Taking limit as both these points tend to P on the curve C and substracting the results so obtained we obtain

$$\frac{d}{dt}\left[W_j\right] = -H_{jj}[W_j], \text{ no sum over } j \qquad ...(8)$$

where $$d/dt \equiv (\partial/\partial t) + (\lambda_j)(\partial/\partial x) \qquad ...(9)$$

The above equation (8) is known as the *transport equation*.

Along a given characteristic curve $x = x_j(t)$ of the j the family, the function $H_{jj}(x, t) = H_{jj}(x_j(t), t)$ is a function of t only.

Thus, the transport equation (8) is a linear homogeneous ordinary differential equation of first order and determines the vibration of $[W_j]$, jump in W_j along a characteristic curve of the jth family. From the properties of solutions of linear homogeneous ordinary differential equations, it follows that if there is a discontinuity in U at some point of a characteristic curve C, the discontinuity in U remains non-zero at every point on the curve.

Note. In order to obtain the transport equation for the discontinuities in the derivatives of U of order $n(n \geq 1)$, differentiate (7) $n(n \geq 1)$ times and proceed in exactly same manner as above.

EXERCISE

1. Write short note on the transport equation for a linear hyperbolic system of first order equations. **(Calicut 2003; Meerut 2005; 06)**

2. When all the characteristic velocities λ_i are different from zero, prove that the first order quasi-linear hyperbolic system

$$A(x, t, U)\ (\partial U / \partial t) + B(x, t, U)\ (\partial U / \partial x) + C(x, t, U) = 0$$

can be reduced to a diagonal canonical system of $2n$ equations $(\partial U / \partial t) - RW = 0$ and $(\partial U / \partial t) + A(\partial W / \partial x) + F = 0,$ where the coefficients A, R and F are functions of $x,\ t,\ U$ and W

3. Consider the hyperbolic system $\quad u_t + (x+t)v_x = 0, \quad (x+t)v_t + u_x = 0$

Show that the variation in jump $[v]$ along the characteristic curve $(x-t) = C$ (C = constant) is given by

$$[v] = A/(2t+c)^{1/2},\ \text{A being a constant.}$$

Derive also the transport equation of discontinuities in the first order partial derivatives of u and v.

MISCELLANEOUS PROBLEMS BASED ON THIS PART OF THE BOOK

Ex. 1. *The solution of* $xu_x + yu_y = 0$ *is of the form*

(*a*) $f(y/x)$ (*b*) $f(y + x)$ (*c*) $f(x - y)$ (*d*) $f(xy)$ **[GATE 2008]**

Sol. Ans. (*a*). Given $$xu_x + yu_y = 0 \qquad \ldots(1)$$
which is in the form of Lagrange equation $Pp + Qq = R$, with u in place z. Hence, the Lagrange's auxiliary equations for (1) are given by
$$(dx)/x = (dy)/y = du/0 \qquad \ldots(2)$$
Taking the first two fractions of (2), $(1/y)\, dy - (1/x)\, dx = 0$

Integrating, $\log y - \log x = \log c_1$ or $y/x = c_1$ ...(3)

Again, the last fraction of (2) yields $du = 0$ so that $u = c_2$ ...(4)

From (3) and (4), the required solution is $u = f(y/x)$.

Ex. 2. *Solve* $x(y^2 + z)p + y(z + x^2)q = z(x^2 - y^2)$ **[Madurai Kamraj 2008]**

Sol. Do like Ex. 6, page 2.10. Here Lagrange's auxiliary equations are

or
$$\frac{dx}{x(y^2+z)} = \frac{dy}{y(z+x^2)} = \frac{dz}{z(x^2-y^2)} \qquad \ldots(1)$$
Choosing $1/x$, $-(1/y)$, $1/z$ as multipliers, each fraction of (1)
$$= \frac{(1/x)dx-(1/y)dy+(1/z)dz}{y^2+z-(z+x^2)+x^2-y^2} = \frac{(1/x)dx-(1/y)dy+(1/z)dz}{0}$$
$\Rightarrow$ $(1/x)\, dx - (1/y)\, dy + (1/z)\, dz = 0$ so that $\log x - \log y + \log z = \log c_1$

or $\log (xz/y) = \log c_1$ or $(xz)/y = c_1$ ...(2)

Choosing x, $-y$, -1 as multipliers each fraction of (1)
$$= \frac{xdx-ydy-dz}{x^2(y^2+z)-y^2(z+x^2)-z(x^2-y^2)} = \frac{xdx-ydy-dz}{0}$$
$\Rightarrow$ $xdx - ydy - dz = 0$ or $2xdx - 2ydy - 2dz = 0$

Integrating, $x^2 - y^2 - 2z = c_2$, c_2 being an arbitrary constant ...(3)

From (2) and (3), the required solution is given by

$(xz)/y = \phi\,(x^2 - y^2 - 2z)$, ϕ being an arbitrary function

Ex. 3. *If the partial differential* $(x - 1)^2 u_{xx} - (y - 2)^2 u_{yy} + 2xu_x + 2yu_y + 2xyu = 0$ *is parabolic in* $S \subseteq R^2$ *but not in* $R^2 \setminus S$, *then S is*

(*a*) $\{(x, y) \in R^2 : x = 0 \text{ or } y = 2\}$ (*b*) $\{(x, y) \in R^2 : x = 1 \text{ and } y = 2\}$

(*c*) $\{(x, y) \in R^2 : x = 1\}$ (*d*) $\{(x, y) \in R^2 : y = 2\}$ **[GATE 2008]**

Sol. Ans. (*a*). Refer Art. 8.1. Here R is the set of all real numbers

Ex. 4. *Find the complete integral of* $xp + 3yq = 2(z - x^2q^2)$ **[Delhi Maths (H) 2008]**

Sol. Here given equation is $$f(x, y, z, p, q) = xp + 3yq - 2z + 2x^2q^2 = 0 \qquad \ldots(1)$$

Charpit's auxiliary equations are
$$\frac{dp}{f_x + p f_z} = \frac{dq}{f_y + q f_z} = \frac{dz}{-p f_p - q f_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

i.e., $$\frac{dp}{p+4xq^2-2p}=\frac{dq}{3q-2q}=\frac{dz}{-px-q(3y+4x^2q)}=\frac{dx}{-x}=\frac{dy}{-(3y+4x^2q)}, \text{ using (1)} \qquad \ldots(2)$$

Taking the second and fourth fractions of (2), we have

$(1/q)\,dq + (1/x)\,dx = 0$ so that $\log q + \log x = \log a$, giving

$qx = a$ so that $q = a/x$, a being an arbitrary constant ...(3)

Substituting the value of q given by (3) in (1), we have

$xp + (3ya)/x - 2z + 2a^2 = 0$ or $xp = 2(z - a^2) - (3ya)/x$

Thus, $p = 2(z - a^2)/x - (3ya)/x^2$. ...(4)

Substituting values of q and p given by (3) and (4) in $dz = pdx + qdy$, we get

$dz = \{2(z - a^2)/x - (3ya)/x^2\}dx + (a/x)\,dy$ or $x^2dz = 2x\,(z - a^2)\,dx - 3yadx + axdy$

or $x^2dz - 2x(z - a^2)\,dx = axdy - 3yadx$ or $[x^2dz - 2x\,(z - a^2)dx]/x^4 = ax^{-3}dy - 3ayx^{-4}\,dx$

or $d\{(z - a^2)/x^2\} = d\,(ayx^{-3})$

Integrating, $(z - a^2)/x^2 = (ay)/x^3 + b$, b being an arbitrary constant

or $z = a(a + y/x) + bx^2$, which is the required solution.

Ex.5. *Find the general integral of the partial differential equation $px(z - 2y^2) = (z - qy)\,(z - y^2 - 2x^3)$. Also, find the particular integral which passes through the straight line $x = -1$, $z = 1$.* **(Delhi B.A. (Prog.) 2009)**

Sol. Re-writing the given partial differential equation, we have

$$px\,(z - 2y^2) + qy(z - y^2 - 2x^3) = z\,(z - y^2 - 2x^3) \qquad \ldots(1)$$

Hence the usual Lagrange's, subsidiary equations are given by

$$\frac{dx}{x(z-2y^2)}=\frac{dy}{y(z-y^2-2x^3)}=\frac{dz}{z(z-y^2-2x^3)} \qquad \ldots(2)$$

Taking the last two fractions of (1), we get $(1/y)dy = (1/z)\mathrm{d}z$

Integrating, $\log y = \log z + \log c_1$, c_1 being an arbitrary constant

Thus, $y = c_1 z$. ...(3)

Next, taking the first and third fractions of (2) and using (3), we obtain

$$\frac{dx}{x(z-2c_1^2z^2)}=\frac{dz}{z(z-c_1^2z^2-2x^3)} \quad \text{or} \quad \frac{dz}{dx}=\frac{z-c_1^2z^2-2x^3}{x(1-2c_1^2z)}$$

Re-writing it, $(1-2c_1^2z)\,(dz/dx) - (z-c_1^2z^2)\times(1/x) = -2x^2$ ...(4)

Putting $z-c_1^2z^2 = v$ so that $(1-2c_1^2z)(dz/dx) = dv/dx$, (4) reduces to

$$(dv/dx)-(1/x)v=-2x^2 \qquad \ldots(5)$$

whose integrating factor is $e^{\int(-1/x)dx} = e^{-\log x} = x^{-1}$ and hence solution of (5) is

$v \times x^{-1} = \int\{(-2x^2)\times x^{-1}\}\,dx + c_2 = -x^2 + c_2$, c_2 being an arbitrary constant

or $(z-c_1^2z^2)/x + x^2 = c_2$ or $(z - y^2)/x + x^2 = c_2$, using (3) ...(6)

The required general integral is given by (3) and (6).

We now find the required particular integral. To this end, replacing x by -1 and z by 1 in (3) and (6), we obtain

$y = c_1$ and $-(1 - y^2) + 1 = c_2$ ...(7)

Eliminating y between two relations of (7), we have $$c_1^2 = c_2 \qquad \ldots(8)$$

Substituting the values of c_1 and c_2 given by (3) and (6) in (8), the required particular integral is given by

$$y^2/z^2 = (z - y^2) / x + x^2 \qquad \text{or} \qquad y^2 x = z^2(z - y^2 + x^3)$$

Ex.6. *Solve the partial differential equation $z = px + qy + 3p - 2q$ by Lagrange's method as well as Charpit's method. Hence or otherwise give two different solutions of the above partial differential equation passing through (–3, 2, 0).* **(Delhi B.A. (Prog). II 2009)**

Sol. Solution of the given equation by Lagrange's method:

Re-writing the given equation, $$(x + 3)p + (y - 2)q = z \qquad \ldots(1)$$

Here the usual Lagrange's subsidiary equations are given by

$$(dx)/(x + 3) = (dy)/(y - 2) = (dz) / z \qquad \ldots(2)$$

Taking the first two fractions of (2), $(dx)/(x + 3) = (dy)/(y - 2)$

Integrating, $\log(x + 3) = \log(y - 2) + \log a$ or $(x + 3) = a\,(y - 2)$ ...(3)

Next, taking the last two fractions of (2), $(dy)/(y - 2) = (dz)/z$

Integrating, $\log\,(y - 2) + \log b = \log z$ or $b\,(y - 2) = z$ ...(4)

From (3) and (4), $(x + 3)/a = (y - 2)/1 = (z - 0)/b$, a and b being arbitrary constants ...(5)
which is the required solution of the given equation passing through (–3, 2, 0).

Solution of the given equation by Charpit's method:

Let $$f(x, y, z, p, q) = (x + 3)p + (y - 2)q - z = 0 \qquad \ldots(6)$$

Here Charpit's auxiliary equations $$\frac{dp}{f_x + pf_z} = \frac{dq}{f_y + qf_z} = \frac{dz}{-pf_p - qf_q} = \frac{dx}{-f_p} = \frac{dy}{-f_q}$$

yield $$\frac{dp}{0} = \frac{dq}{0} = \frac{dz}{-p(x+3) - q(y-2)} = \frac{dx}{-(x+3)} = \frac{dy}{-(y-2)}$$

Hence, $dp = 0$ so that $p = c_1$, c_1 being an arbitrary constant ...(7)

From (6) and (7), $(x + 3)c_1 + (y - 2)q - z = 0 \Rightarrow q = \{z - (x + 3)c_1\}/(y - 2)$ ...(8)

Substituting the values of p and q given by (7) and (8), we have

$$dz = pdx + qdy = c_1 dx + [\{z - (x + 3)c_1\}/(y - 2)]dy$$

or $$dz - \frac{z}{y-2}dy = c_1 dx - \frac{(x+3)c_1 dy}{y-2} \qquad \text{or} \qquad \frac{(y-2)\,dz - z\,dy}{(y-2)^2} = \frac{c_1(y-2)\,dx + c_1(x+3)\,dy}{(y-2)^2}$$

or $$d\left(\frac{z}{y-2}\right) = d\left(\frac{c_1(x+3)}{y-2}\right) \qquad \text{giving} \qquad \frac{z}{y-2} = \frac{c_1(x+3)}{y-2} + c_2$$

Thus, $$z = c_1(x + 3) + c_2\,(y - 2),$$
which is the required second solution of the given equation passing through (–3, 2, 0).

Ex. 7. *Define the singular integral of first order partial differential equation. Is it true that singular integral always exists? Justify your answer.* **[Delhi Math (Hons.) 2009]**

Hint: Refer Art. 3.1.

Ex. 8. *Write the form of the solution of the equation $F\,(D, D') = 0$, where $F\,(D, D')$ is not reducible.* **[Delhi Math (Hons). 2009]**

Sol. Let $z = e^{hx + ky}$ be a trial solution of the given equation. Then, the required solution is

$$z = \sum_i A_i\, e^{h_i x + k_i y}, \text{ where } A_i,\ h_i \text{ and } k_i \text{ are arbitrary constants,}$$

and h_i, k_i are connected by the relation $F(h_i, k_i) = 0$.

Ex. 9. *Solve* $z = px + qy + p^2 + q^2$. **[Kanpur 2009]**

Sol. Refer Art 3.12. The complete integral is $z = ax + by + a^2 + b^2$ … (1)

Singular integral. Differentiating (1) partially w.r.t. 'a' and 'b', we get

$$0 = x + 2a \qquad \text{and} \qquad 0 = y + 2b \qquad \text{…(2)}$$

From (2), $a = -(x/2)$ and $b = -(y/2)$. Substituting these values of a and b in (1), we get

$$z = -(x^2/2) - (y^2/2) + x^2/4 + y^2/4 \qquad \text{or} \qquad 4z + x^2 + y^2 = 0.$$

General integral Take $b = \phi(a)$, where ϕ is an arbitrary function

Then, (1) yields $z = ax + y\,\phi(a) + a^2 + [\phi(a)]^2$ …(3)

Differentiating (3) partially w.r.t. 'a', $0 = x + y\phi'(a) + 2a + 2\phi(a)\phi'(a)$ …(4)

The general integral is obtained by eliminating a between (3) and (4).

Ex. 10. *Classify the following partial differential equation into elliptic, parabolic or hyperbolic and find its degree and order* $x(y-x)r - (y^2 - x^2)s + y(y-x)t + (y+x)(p-q) = 0$.

Hint. Use Art 1.3, 1.4 and 8.1. **[Delhi BA (Prog.) II 2009]**

Ans. The given equation is hyperbolic, its degree is one and its order is two.

Ex. 11. *Find the characteristic strips of the equation* $xp + yq - pq = 0$ *and then find the equation of the integral surface through the curve* $z = x/2$, $y = 0$ **[Meerut 2011]**

Sol. Given equation is $xp + yq - pq = 0$ ...(1)

We are to find its integral surface passing through the given curve, namely

$$z = x/2, \qquad y = 0 \qquad \text{…(2)}$$

Re-writing (2) in parametric form, we have

$$x = \lambda, \qquad y = 0, \qquad z = \lambda/2; \qquad \lambda \text{ being a parameter} \qquad \text{…(3)}$$

Let the initial values of x_0, y_0, z_0, p_0 and q_0 of x, y, z, p and q be taken as

$$x_0 = x_0(\lambda) = \lambda \qquad y_0 = y_0(\lambda) = 0, \qquad z_0 = z_0(\lambda) = \lambda/2 \qquad \text{…(4 A)}$$

Let p_0 and q_0 be the initial values of p and q corresponding to the initial values of x_0, y_0, z_0. Since the initial values x_0, y_0, z_0, p_0 and q_0 satisfy (1), we have

$$x_0p_0 + y_0q_0 - p_0q_0 = 0 \quad \text{or} \quad \lambda p_0 - p_0q_0 = 0 \quad \text{or} \quad q_0 = \lambda, \text{ using (4A)} \qquad \text{…(5)}$$

Also, we have $z_0'(\lambda) = p_0x_0'(\lambda) + q_0y_0'(\lambda)$

so that $1/2 = p_0 + 0$ giving $p_0 = 1/2$, using (4A) …(6)

Thus, from (5) and (6), $p_0 = 1/2$ and $q_0 = \lambda$ …(4 B)

Collecting relations (4 A) and (4 B) together, initial values are given by

$$x_0 = \lambda, \quad y_0 = 0, \quad z_0 = \lambda/2, \quad p_0 = 1/2 \quad \text{and} \quad q_0 = \lambda, \text{ when } t = t_0 = 0 \qquad \text{…(7)}$$

Re-writing (1), let $f(x, y, z, p, q) = xp + yq - pq = 0$ …(8)

The usual characteristic equations of (8) are given by

$$dx/dt = \partial f/\partial p = x - p \qquad \text{…(9)}$$

$$dy/dt = \partial f/\partial q = y - p \qquad \text{…(10)}$$

$$dz/dt = p(\partial f/\partial p) + q(\partial f/\partial q) = p(x-q) + q(y-p) = -pq, \text{ using (1)} \qquad \text{…(11)}$$

$$dp/dt = -(\partial f/\partial x) - p(\partial f/\partial z) = -p - (p \times 0) = -p \qquad \text{...(12)}$$

$$dq/dt = -(\partial f/\partial y) - q(\partial f/\partial z) = -q - (q \times 0) = -q \qquad \text{...(13)}$$

From (12), $(1/p)dp = -dt$ so that $\log p - \log c_1 = -t$

Thus, $p = c_1 e^{-t}$, c_1 being an arbitrary constant ...(14)

Similarly, (13) yields $q = c_2 e^{-t}$, c_2 being an a arbitrary constant ...(15)

Using initial values (7), (14) yields $p_0 = c_1 e^{-t_0}$ giving $c_1 = 1/2$

Hence, (14) reduces to $p = (1/2) \times e^{-t}$ ...(16)

Using initial values (7), (15) yields $q_0 = c_2 e^{-t_0}$ giving $c_2 = \lambda$

Hence, (15) reduces to $q = \lambda e^{-t}$ ...(17)

From (9) and (17), $dx/dt = x - \lambda e^{-t}$ or $dx/dt - x = -\lambda e^{-t}$,

whose integrating factor $= e^{\int(-1)dt} = e^{-t}$ and hence its solution is given by

$$xe^{-t} = c_3 + \int\{(-\lambda e^{-t}) \times e^{-t}\}dt, \; c_3 \text{ being an arbitrary constant}$$

or $$xe^{-t} = c_3 + (\lambda/2) \times e^{-2t} \qquad \text{...(18)}$$

Using initial values (7), (18) yields $x_0 e^{-t_0} = c_3 + (\lambda/2) \times e^{-2t_0}$

or $\lambda = c_3 + \lambda/2$ so that $c_3 = \lambda/2$.

Hence, (18) yields $xe^{-t} = (\lambda/2) \times (1 + e^{-2t})$

or $$x = (\lambda/2) \times e^{t}(1 + e^{-2t}) \qquad \text{...(19)}$$

Now, from (10) and (16), $dy/dt = y - (e^{-t})/2$ or $dy/dt - y = -(e^{-t})/2$

whose integrating factor $= e^{\int(-1)dt} = e^{-t}$ and hence its solution is given by

$$ye^{-t} = c_4 + \int\{(-e^{-t}/2) \times e^{-t}\}dt, \; c_4 \text{ being an arbitarary constant}$$

or $$ye^{-t} = c_4 + (1/4) \times e^{-2t} \qquad \text{... (20)}$$

Using initial values (7), (20) yields $y_0 e^{-t_0} = c_4 + (1/4) \times e^{-2t_0}$

or $0 = c_4 + 1/4$ so that $c_4 = -(1/4)$

Hence, (20) reduces to $ye^{-t} = (1/4) \times (e^{-2t} - 1)$

Thus, $y = (1/4) \times e^{t}(e^{-2t} - 1)$ ...(21)

From (11), (16) and (17), $dz/dt = -(e^{-t}/2) \times (\lambda e^{-t}) = -(\lambda/2) \times e^{-2t}$

Thus, $(1/z)dz = -(\lambda/2) \times e^{-2t}\, dt$

Integrating, $z = (\lambda/4) \times e^{-2t} + c_5$, c_5 being an arbitrary constant ...(22)

Using initial values (7), (22) yields $z_0 = (\lambda/4) \times e^{-2t_0} + c_5$

or $\lambda/2 = (\lambda/4) \times e^{0} + c_5$ so that $c_5 = \lambda/4$.

Hence, (22) reduces to $z = (\lambda/4) \times (e^{-2t} + 1)$ ...(23)

The required characteristics of (1) are given by (19), (21) are (23)

In order to obtain the desired integral surface of (1), we now proceed to eliminate two parameters t and λ from (19), (21) and (23).

From (19) and (23), we have $x/z = 2e^t$ giving $e^t = x/2z$. ...(24)

From (21), $y = (1/4) \times (1/e^t - e^t) = (1/4) \times (2z/x - x/2z)$, using (24)

or $8xyz = 4z^2 - x^2$, which is the required integral surface of (1).

Ex.12. *Prove that for the equation $z + px + qy - 1 - pq\, x^2y^2 = 0$ the characteristic strips are given by $x = (B + Ce^{-t})^{-1}$, $y = (A + De^{-t})^{-1}$, $z = E - (AC + BD)e^{-t}$, $p = A(B + Ce^{-t})^2$, $q = B(A + De^{-t})^2$, where A, B, C, D and E are arbitrary constants. Hence, find the integral surface which passes through the line $z = 0$, $x = y$.* **[I.A.S 2001]**

Sol. The given equation is $z + px + qy - 1 - pq\, x^2y^2 = 0$ ...(1)

Let $f(x, y, z, p, q) = z + px + qy - 1 - pq\, x^2y^2$ ...(2)

Then, the characteristic equations of (1) are given by

$$dx/dt = \partial f/\partial p = x - qx^2y^2 \quad \text{...(3)}$$

$$dy/dt = \partial f/\partial q = y - px^2y^2 \quad \text{...(4)}$$

$$dz/dt = p(\partial f/\partial p) + q(\partial f/\partial q) = p(x - qx^2y^2) + q(y - px^2y^2) = px + qy - 2pqx^2y^2 \quad \text{...(5)}$$

$$dp/dt = -(\partial f/\partial x) - p(\partial f/\partial z) = -(p - 2\,pqxy^2) - p = -2p(1 - qxy^2) \quad \text{...(6)}$$

$$dq/dt = -(\partial f/\partial y) - q(\partial f/\partial z) = -(q - 2pqx^2y) - q = -2q\,(1 - px^2y) \quad \text{...(7)}$$

From (3) and (6), $(1/x)\,(dx/dt) = -(1/2p)\,(dp/dt)$ or $(2/x)\,dx + (1/p)\,dp = 0$

Integrating, $2 \log x + \log p = \log A$ or $x^2p = A$, A being an arbitrary constant ...(8)

From (4) and (7), $(1/y)\,(dy/dt) = (-1/2q)\,(dq/dt)$ or $(2/y)dy + (1/q)dq = 0$

Integrating as before, $y^2q = B$, B being an arbitrary constant ...(9)

From (3) and (9), $dx/dt = x - Bx^2$ or $x^{-2}(dx/dt) - x^{-1} = -B$ ...(10)

Putting $x^{-1} = v$ and $-x^{-2}(dx/dt) = dv/dt$, (10) reduces to

$$-(dv/dt) - v = -B \qquad \text{or} \qquad dv/dt + v = B,$$

whose integrating factor is $e^{\int dt}$, i.e., e^t and hence its solution is given by

$$ve^t = C + \int Be^t\,dt, \text{ C being an arbitrary constant}$$

$e^t/x = c + Be^t$ or $1/x = Ce^{-t} + B$ or $x = (B + Ce^{-t})^{-1}$ ...(11)

Similarly, (4) and (8) yield $dy/dt = y - Ay^2$ or $y^{-2}(dy/dt) - y^{-1} = -A$ ...(12)

Putting $y^{-1} = u$ and $-y^{-2}(dy/dt) = du/dt$, (12) yields

$$-(du/dt) - u = -A \qquad \text{or} \qquad du/dt + u = A$$

whose integrating factor is $e^{\int dt}$, i.e., e^t and hence its solution is given by

$$ue^t = D + \int Ae^t\,dt, \text{ D being an arbitrary constant}$$

or $e^t/y = D + Ae^t$ or $1/y = De^{-t} + A$ or $y = (A + De^{-t})^{-1}$ ...(13)

Using (8) and (9), (5) yields $dz/dt = A/x + B/y - 2AB$

or $dz/dt = A(B + Ce^{-t}) + B(A + De^{-t}) - 2AB$, using (11) and (13)

or $dz/dt = (AC + BD)e^{-t}$ or $dz = (AC + BD)\,e^{-t}\,dt$

Integrating, $z = E - (AC + BD)e^{-t}$, E being an arbitrary constant ...(14)

From (8) and (11), $p = Ax^{-2} = A(B + C\,e^{-t})^2$ ...(15)

From (9) and (13), $q = By^{-2} = B(A + De^{-t})^2$ ...(16)

The required characteristics are given by (11), (13), (14), (15) and (16). We now proceed to find the required integral surface passing through the line given by

$$z = 0 \qquad \text{and} \qquad x = y \qquad ...(17)$$

Re-writing (17), $x = \lambda, \quad y = \lambda, \quad z = 0, \quad \lambda$ being a parameter ...(18)

Let the initial values x_0, y_0, z_0, p_0, q_0 of x, y, z, p, q be taken as

$$x_0 = x_0(\lambda) = \lambda, \qquad y_0 = y_0(\lambda) = \lambda, \qquad z_0 = z_0(\lambda) = 0 \qquad ...(19)$$

Let p_0, q_0 be the initial values of p, q corresponding to the initial values x_0, y_0, z_0. Since the initial values statisfy (1), we have

$$z_0 + p_0x_0 + q_0y_0 - 1 - p_0q_0\,x_0^2 y_0^2 = 0 \qquad \text{or} \qquad p_0\lambda + q_0\lambda - 1 - p_0q_0\lambda^4 = 0, \text{ using (19)}$$

Thus, $$\lambda(p_0 + q_0) = p_0q_0\lambda^4 + 1 \qquad ...(20)$$

Also, we have $$z_0'(\lambda) = p_0x_0'(\lambda) + q_0y_0'(\lambda)$$

so that, $0 = p_0 + q_0$ giving $q_0 = -p_0$, using (19) ...(21)

Using (21), (20) yields $p_0q_0\lambda^4 + 1 = 0$ giving $-p_0^2\lambda^4 + 1 = 0$, using (21)

Thus, $p_0 = 1/\lambda^2$ so that $q_0 = -(1/\lambda^2)$, using (21) ...(22)

Using initial values $x = x_0 = \lambda$, $t = t_0 = 0$, (11) reduces to

$$\lambda = (B + C)^{-1} \qquad \text{so that} \qquad B + C = 1/\lambda \qquad ...(23)$$

Using initial values $y = y_0 = \lambda$, $t = t_0 = 0$, (13) reduce to

$$\lambda = (A + D)^{-1} \qquad \text{so that} \qquad A + D = 1/\lambda \qquad ...(24)$$

Using initial values $p = p_0 = 1/\lambda^2$, $t = t_0 = 0$, (15) reduces to

$$p_0 = A(B + C)^2 \quad \text{or} \quad 1/\lambda^2 = A \times (1/\lambda)^2, \text{ by (23)} \quad \text{so that} \quad A = 1 \qquad ...(25)$$

Using initial values $q = q_0 = -(1/\lambda^2)$, $t = t_0 = 0$, (16) reduces to

$$q_0 = B\,(A + D)^2 \quad \text{or} \quad -(1/\lambda^2) = B \times (1/\lambda^2), \text{ by (24)} \quad \text{so that} \quad B = -1 \qquad ...(26)$$

From (23) and (26), $-1 + C = 1/\lambda$ so that $C = 1 + 1/\lambda$ ...(27)

From (24) and (25), $1 + D = 1/\lambda$ so that $D = (1/\lambda) - 1$ ...(28)

Using the initial values $z = z_0 = 0$, $t = t_0 = 0$, (14) reduces to

$$0 = E - (AC + BD) \quad \text{or} \quad E = 1 + 1/\lambda - (1/\lambda - 1) \quad \text{or} \quad E = 2 \qquad ...(29)$$

Substituting the values of A, B, C, D and E given by (25), (26), (27), (28) and (29) in (11), (13) and (14), we obtain

$$x = \{-1 + (1/\lambda + 1)e^{-t}\}^{-1} \qquad ...(30)$$

$$y = \{1 + (1/\lambda - 1)e^{-t}\}^{-1} \qquad ...(31)$$

$$z = 2 - \{1 + 1/\lambda - (1/\lambda - 1)\}e^{-t} = 2(1 - e^{-t}) \qquad ...(32)$$

In order to obtain the required surface, we now eliminate λ from (30), (31) and (32).

From (30), $x^{-1} = -1+(1/\lambda+1)e^{-t}$ so that $1/x + 1 = (1/\lambda+1)e^{-t}$ ...(33)

From (31), $y^{-1} = 1 + (1/\lambda-1)e^{-t}$ so that $1/y - 1 = (1/\lambda-1)e^{-t}$ ...(34)

Subtracting (34) from (33), $1/x - 1/y + 2 = 2e^{-t}$ or $1/x - 1/y = -2(1 - e^{-t})$ ...(35)

From (32) and (35), we get $1/x - 1/y = -z$ which is the required integral surface.

Ex. 13. The general solution of the partial differential equation $\partial^2 z/\partial x\partial y = x+y$ is of the form (*a*) $(1/2)\times xy(x+y)+F(x)+G(y)$ (*b*) $(1/2)\times xy(x-y)+F(x)+G(y)$

(*c*) $(1/2)\times xy(x-y)+F(x)\ G(y)$ (*d*) $(1/2)\times xy(x+y)+F(x)\ G(y)$ **(GATE 2010)**

Sol. Ans. (*a*). Integrating the give equation w.r.t '*x*', we get

$$\partial z/\partial y = x^2/2 + xy + g(y),\ g(y) \text{ being an arbitrary function of } y.$$

Integrating the above equation w.r.t. '*y*', we get

$$z = (x^2 y)/2 + (xy^2)/2 + G(y) + F(x), \quad \text{where} \quad G(y) = \int g(y)dy$$

Ex. 14. Find whether the following is hyperbolic, parabolic or elliptic :

(*i*) $x^2 r - y^2 t - px - qy = x^2$ **[Delhi B.A. (Prog) II 2010, 11]**

(*ii*) $x^2 r + (5/2)\times xys + y^2 t + xp + yq = 0$ **[Delhi B.A. (Prog) II 2010]**

(*iii*) $\partial^2 u/\partial t^2 + \partial^2 u/\partial x\partial t + \partial^2 u/\partial x^2 = 0$ **[Meerut 2010]**

(*iv*) $u_{xx} + u_{yy} + u_{zz} = (1/c^2)\times(\partial u/\partial t)$ **[Meerut 2007, 10]**

(*v*) $(1-x^2)r - 2xys + (1-y^2)t + xp + 3x^2 yq = 0$ **[Ravishankar 2010]**

(*vi*) $\partial^2 z/\partial x^2 = x^2(\partial^2 z/\partial y^2)$ **[Bhopal 2010]**

(*vii*) $\partial^2 u/\partial t^2 + (\partial u/\partial x)(\partial u/\partial t) + \partial^2 u/\partial t^2 = 0$ **[Meerut 2011]**

Hint. Use Articles 8.1, 8.2 and 8.2A.

Ans. (*i*) Hyperbolic (*ii*) Hyperbolic (*iii*) Elliptic (*iv*) Parabolic (*v*) Hyperbolic if $x^2 + y^2 > 1$, parabolic if $x^2 + y^2 = 1$, elliptic if $x^2 + y^2 < 1$ (*vi*) Hyperbolic (*vii*) Elliptic

Ex. 15. *The partial differential equation* $x^2(\partial^2 z/\partial x^2) - (y^2-1)x(\partial^2 z/\partial x\,\partial y)$ $+y(y-1)^2\ (\partial^2 z/\partial y^2) + x(\partial z + \partial x) + y(\partial z/\partial y) = 0$ *is hyperbolic in a region in xy–plane if (a)* $x \neq 0$ *and y = 1 (b) x = 0 and* $y \neq 1$ *(c)* $x \neq 0,$ *and* $y \neq 1$ *(d) x = 0 and y = 1.* **[GATE 2011]**

Sol. Ans (*c*) The given can be re-written as

$$x^2 r - x(y^2-1)s + y(y-1)^2 t + xp + yq = 0 \quad \text{... (1)}$$

Comparing (1) with $Rr + Ss + Tt + f(x, y, z, p, q) = 0$, we have $R = x^2$, $S = -x(y^2-1)$ and $T = y(y-1)^2$. Now, in order that (1) may be hyperbolic we must have $S^2 - rRT > 0$, i.e.,

$$x^2(y^2-1)^2 - 4x^2 y(y-1)^2 > 0 \quad \text{or} \quad x^2(y-1)^2(y+1)^2 - 4x^2 y\,(y-1)^2 > 0$$

or $\quad x^2(y-1)^2\{(y+1)^2-4y\}>0 \quad$ or $\quad x^2(y-1)^2(y-1)^2>0, \quad$ ... (2)

Which is true when $x \neq 0$ and $y \neq 1$.

Ex. 16. *The integral surface for the Cauchy problem* $\partial z/\partial x+\partial z/\partial y=1$ *which passes through the circle* $z = 0,\ x^2 + y^2 = 1$ *is*

(*a*) $x^2+y^2+2z^2+2zx-2yx-2yz-1=0$ (*b*) $x^2+y^2+2z^2+2zx-2yz-1=0$

(*c*) $x^2+y^2+2z^2-2zx-2yz-1=0$ (*d*) $x^2+y^2+2z^2+2zx+2yz-1=0$ **[GATE 2011]**

Sol. Ans. (*c*) In usual symbols, the given equation is $\quad p+q=1 \quad$ ... (1)

Lagrange's auxiliary equation of (1) are $\quad (dx)/1=(dy)/1=(dz)/1 \quad$ ... (2)

Taking the first two fractions of (2), we get $\quad dx-dy=0 \quad$ ... (3)

Integrating (3), $\quad x-y=c_{-1},\ c_1$ being an arbitrary constant ... (4)

Next, taking the first and third fractions of (2), we get $\quad dx-dz=0 \quad$ ... (5)

Integrating (5), $\quad x-z=c_2,\ c_2$ being an arbitrary constant ... (6)

The given curve is defined by $\quad x^2+y^2=1,\ z=0 \quad$ ... (7)

Putting z = 0 in (6), we have $\quad x=c_2, \quad$ ... (8)

Now, from (4) and (8), we have $\quad c_2-y=c_1 \quad$ so that $\quad y=c_1-c_2 \quad$ ... (9)

Substituting the values of x and y given by (8) and (9) in (7), we obtain

$c_2^2+(c_2-c_1)^2=1 \quad$ or $\quad (x-z)^2+(y-z)^2=1, \quad$ using (4) and (6)

or
$$x^2+y^2+2z^2-2zx-2yz-1=0.$$

Ex. 17. *The integral surfaces satisfying the partial differential equation* $(\partial z/\partial y)+z^2(\partial z/\partial y)=0$ *and spassing through the straight line* $x = 1,\ y = z$ *is*

(*a*) $(x-1)z+z^2=y^2$ (*b*) $x^2+y^2-z^2=1$

(c) $(y-z)x+x^2=1$ (*d*) $(x-1)z^2+z=y$ **[GATE 2012]**

Sol. Ans. (d). Given $p+z^2q=0$, where $\quad p=\partial z/\partial x,\ q=\partial z/\partial y \quad$ ... (1)

Lagrange's auxiliary equation (1) are $\quad (dx)/1=(dy)/z^2=(\partial z)/0 \quad$ ... (2)

From third fraction of (1), $\quad dz=0 \quad$ so that $\quad z=c^1 \quad$ ... (3)

Using (3), from first and second fractions of (2), $\quad (dx)/1=(dy)/c_1^2 \quad$ or $\quad c_1^2dx-dy=0$

Integrating, $\quad c_1^2x-y=c_2 \quad$ or $\quad z^2x-y=c_2,\ as\ c_1=z \quad$ ... (4)

In order to get the required integral surfaces, we shall use method II given on page 2.28.

The given straight line is represented by $\quad x=1,\quad y=z \quad$ ... (5)

Using (5) in (3) and (4), we get $\quad y=c_1 \quad$ and $\quad y^2-y=c_2 \quad$ ... (6)

Eliminating *y* from the two equation of (6), we get $\quad c_1^2-c_1=c_2 \quad$ ... (7)

Substituting the values of c_1 and c_2 given by (3) and (4) in (7), we get

$z^2-z=z^2x-y \quad$ or $\quad (x-1)z^2+z^2=y$, which is the required integral surface

Ex.18. *The expression* $\dfrac{1}{D_x^2 - D_y^2}\sin(x-y)$ *is equal to*

(*a*) $-(x/2)\times\cos(x-y)$ (*b*) $-(x/2)\times\sin(x-y)+\cos(x-y)$

(*c*) $-(x/2)\times\cos(x-y)+\sin(x-y)$ (*d*) $(3x/2)\times\sin(x-y)$ **[GATE 2012]**

Sol. Ans. (a). Here note that D_x and D_y stand for D and D' respectively. For solution, processd as in Ex. 7(d). Here, we wish to find only P.I. Thus,

$$\frac{1}{D_x^2 - D_y^2}\sin(x-y) = \frac{1}{D^2 - D'^2}\sin(x-y) = \frac{1}{D+D'}\frac{1}{D-D'}\sin(x-y)$$

$$= \frac{1}{D-D'}\frac{1}{1-(-1))}\int \sin v\,dv, \text{ where } v = x-y$$ [Using formula (i) page 4.9]

$$= \frac{1}{2}\ \frac{1}{D+D'}\ [-\cos(x-y)] = \frac{1}{2}\ \frac{1}{(-1)D-(1)\times D}\ \cos(x-y) = \frac{1}{2}\times\frac{x}{(-1)^1\times 1!}\cos\ (x-y)$$

[Using formula (ii), page 4.9]

$$= -(x/2)\times\cos(x-y)$$