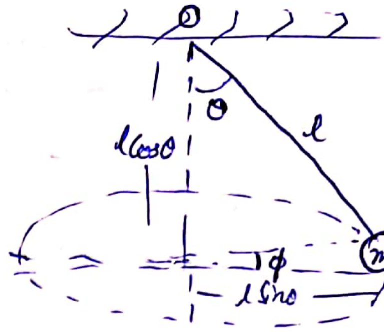
32 (5d)

(d) Obtain the equations governing the motion of a spherical pendulum. 12

2012 → 5(d)

Same IFOS → 6(b)

For mass m,



$$\bar{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta} + r\sin\theta\,\dot{\phi}\hat{\phi}$$

$$\Rightarrow T = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\phi}^2\right)$$

~~Total~~ Kinetic Energy

Since $r = l$ and $\dot{r} = 0$

$$\Rightarrow T = \frac{1}{2}m\left(l^2\dot{\theta}^2 + l^2\sin^2\theta\,\dot{\phi}^2\right)$$

Potential Energy (V) $= -mgl\cos\theta$

$$\therefore L = T - V = \frac{1}{2}m\left[l^2\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right) + 2gl\cos\theta\right]$$

Now Lagrange's equations are given by —

① $$\frac{d}{dt}\left(\frac{dL}{d\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0$$

$$\Rightarrow ml^2\ddot{\theta} - \frac{1}{2}ml^2\sin 2\theta\,\dot{\phi}^2 + mgl\sin\theta = 0 \quad \text{—①}$$

② $$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\phi}}\right) - \frac{\partial L}{\partial\phi} = 0$$

$$\Rightarrow \quad \frac{d}{dt}\left(ml^2 \sin^2\theta\, \dot{\phi}\right) = 0 \quad - \quad (2)$$

$$\cancel{\Rightarrow ml^2\left(\sin 2\theta\, \dot{\theta}\dot{\phi} + \sin^2\theta\, \ddot{\phi}\right) = 0 \quad - \quad (2)}$$

Eqns (1) and (2) govern motion of spherical pendulum

33 (5e)

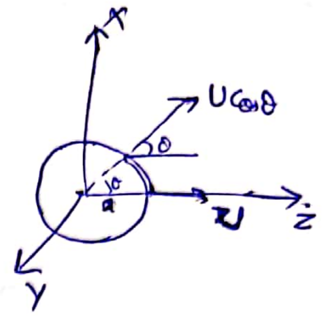
(e) A rigid sphere of radius a is placed in a stream of fluid whose velocity in the undisturbed state is V. Determine the velocity of the fluid at any point of the disturbed stream. 12

2012 5(e) Let ϕ be velocity potential of sphere

Let U be velocity of fluid along Z-axis

Then ϕ satisfies Laplace's eqn

$$\Rightarrow \frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} + \frac{\cot\theta}{r^2}\frac{\partial\phi}{\partial\theta} = 0$$



$\Rightarrow \phi$ is of the form $f(r)\cos\theta$

Let $\phi = \left(Ar + \frac{B}{r^2}\right)\cos\theta$

At Boundary of sphere i.e. $r = a$, Normal velocity of sphere = Velocity of fluid at that point

$$\Rightarrow -\frac{\partial\phi}{\partial r} = 0 \quad \Rightarrow -\left(A - \frac{2B}{a^3}\right)\cos\theta = 0$$

$$\Rightarrow A = \frac{2B}{a^3} \quad — ①$$

$$\Rightarrow \quad \phi = \left[r\left(\frac{2B}{a^3}\right) + \frac{B}{r^2}\right]\cos\theta \quad [\text{Using } ①]$$

$$\Rightarrow \frac{\partial\phi}{\partial r} = \left[\frac{2B}{a^3} - \frac{2B}{r^3}\right]\cos\theta$$

as $r \to \infty$, velocity $= U\cos\theta \left(= -\frac{\partial\phi}{\partial r}\right)$

$$\Rightarrow U\cos\theta = -\left[\frac{2B}{a^3} - \frac{2B}{r^3}\right]\cos\theta$$

As $r \to \infty$

$$\Rightarrow B = -\frac{a^3U}{2} \Rightarrow A = -U \quad [\text{Using } ①]$$

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$$\Rightarrow \phi = (-U)\left[r + \frac{a^3}{2r^2} \right] \cos\theta$$

Now velocity components are given by

$$\boxed{q_r = -\frac{\partial \phi}{\partial r} = U\left(1 - \frac{a^3}{r^3}\right)\cos\theta}$$

and $$\boxed{q_\theta = -\frac{1}{r}\frac{\partial \phi}{\partial \theta} = (-U)\left(1 + \frac{a^3}{2r^3}\right)}$$

34 (8a)

8. (a) A pendulum consists of a rod of length $2a$ and mass m; to one end of which a spherical bob of radius $a/3$ and mass 15 m is attached. Find the moment of inertia of the pendulum :

(i) about an axis through the other end of the rod and at right angles to the rod. 15

(ii) about a parallel axis through the centre of mass of the pendulum.

[Given : The centre of mass of the pendulum is $a/12$ above the centre of the sphere.] 15

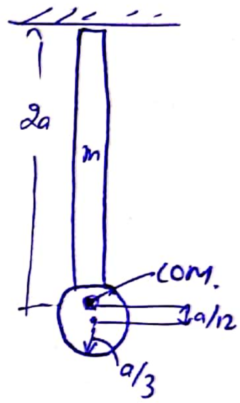
2012
8(a) (i)

Moment of Inertia about axis at A $= \frac{4ma^2}{3} + \frac{2}{5}(15m)\left(\frac{a}{3}\right)^2$

$+ (15m)\left(2a + \frac{a}{3}\right)^2$

$= 2ma^2 + \frac{245ma^2}{3} = \frac{251ma^2}{3}$

(ii) Moment of inertia about Centre of mass $= \frac{4ma^2}{3} + m\left(\frac{a}{3} - \frac{a}{12}\right)^2 +$



$+ \frac{2}{5}(15m)\left(\frac{a}{3}\right)^2 + (15m)\left(\frac{a}{12}\right)^2$

$= \frac{4ma^2}{3} + \frac{ma^2}{16} + \frac{2ma^2}{3} + \frac{5ma^2}{48}$

$= \frac{7ma^2}{3}$

35 (8b)

(b) Show that $\phi = x\, f(r)$ is a possible form for the velocity potential for an incompressible fluid motion. If the fluid velocity $\vec{q} \to 0$ as $r \to \infty$, find the surfaces of constant speed. 30

2012 8(b) $\phi = x f(r)$

For motion of fluid to be possible, $\nabla^2\phi = 0$

$$\Rightarrow \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = 0$$

$$\Rightarrow \left[\frac{x}{r}f'(r) + \frac{x^3}{r^2}f''(r) + \frac{2x}{r}f'(r) - \frac{x^3}{r^3}f'(r)\right] + \left[\frac{x}{r}f'(r) - \frac{xy^2}{r^3}f'(r) + \frac{xy^2}{r^2}f''(r)\right]$$

$$+ \left[\frac{x}{r}f'(r) - \frac{xz^2}{r^3}f'(r) + \frac{xz^2}{r^2}f''(r)\right] = 0$$

$$\Rightarrow xf''(r) + \frac{4x}{r}f'(r) = 0 \Rightarrow \frac{f''(r)}{f'(r)} = -\frac{4}{r}$$

$$\Rightarrow \log f'(r) = -4\log r + c_1 \quad \Rightarrow f'(r) = \frac{c_1}{r^4} \quad -(1)$$

$$\Rightarrow f = \frac{-c_1}{3r^3} + c_2 \quad -(2)$$

Now, $\bar{q} = \nabla\phi = f(r)\,\hat{i} + \frac{x}{r}f'(r)\,\hat{r}$

$$\bar{q} = \left(\frac{-c_1}{3r^3} + c_2\right)\hat{i} + \left(\frac{xc_1}{r^5}\right)\hat{r} \quad [\text{Using } (1) \,\&\, (2)]$$

Since $\bar{q} \to 0$ as $r \to \infty$ $\rightarrow c_2 = 0$

$$\Rightarrow \bar{q} = \frac{c_1}{3r^3}\left(\hat{i} - \frac{3x}{r^2}\hat{r}\right)$$

Surface of constant speed are by $q^2 = \text{Constant}$

$$\Rightarrow \bar{q}\cdot\bar{q} = \text{Constant} = \left(\frac{c_1}{3r^3}\right)^2\left[\left(\hat{i} - \frac{3x}{r^2}\hat{r}\right)\cdot\left(\hat{i} - \frac{3x}{r^2}\hat{r}\right)\right]$$

$$= \frac{c_1^2}{9r^6}\left(1 + \frac{3x^2}{r^2}\right) = \frac{c_1^2}{9r^8}(r^2 + 3x^2)$$

$\therefore$ Surface of constant speed are $\frac{(r^2+3x^2)}{r^8} = \text{Constant}$.