Revision is the most crucial part of mathematics preparation in Civil Services Examination. For that purpose, I had prepared short notes on each topic of the syllabus which I am sharing here. I would suggest aspirants to use this as just a reference for quick revision and not as your primary source. This optional requires good understanding of the topic which cannot be derived from short notes.

It is the process of preparing such notes that gives a candidate confidence about his or her preparation and brings conceptual clarity. Hence I would encourage aspirants to do this exercise themselves even if it is time consuming.

All the best!

Regards,
Yogesh Kumbhejkar
AIR 8 - CSE 2015

Please note that the notes on this particular topic are incomplete. Aspirants are advised to ensure that they cover the remaining part from other sources.

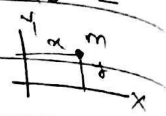
Mechanics

Moment of Inertia (paper 2)

(1) Moment of Inertia $I = \int r^2\, dm$

Radius of gyration is k s.t. $I = Mk^2$

Product of inertia → calculated with respect to 2 perpendicular axes $= mxy$



(2) 3 ⊥ axes → cartesian.

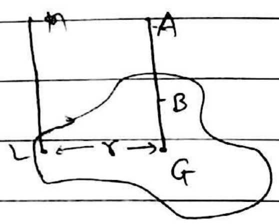
A, B, C, D, E and F defined as

$A = \Sigma m(y^2+z^2)$ $B = \Sigma m(x^2+z^2)$ $C = \Sigma m(x^2+y^2)$

$D = \Sigma myz$ $E = \Sigma mxz$ $F = \Sigma mxy$

(3) Sum of moment of inertia about 3 ⊥ axes meeting at a point is constant & twice the MoI about the point.

(4) Imp. Result → Parallel axis thm.



Finding MOI about axis LM.

G is centre of gravity.

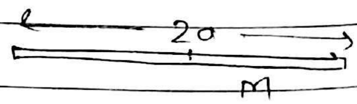
AB is ∥ to LM and passing through G.

Then,

MoI about LM = MoI about AB + Mr^2.

(5) Using this, remember following important result

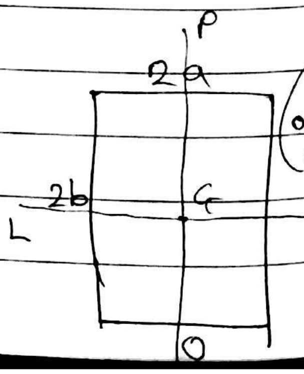
(a) Rod of length 2a & mass M.



Axis through midpoint → $\frac{1}{3}Ma^2$

Axis through endpoint → $\frac{4}{3}Ma^2$

(6) Rectangular plate of sides 2a, 2b & mass M



(obviously same as above)

Axis ~~through~~ PO → $\frac{1}{3}Ma^2$

Axis LM → $\frac{1}{3}Mb^2$

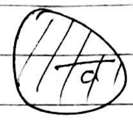
⊥ axis through centre → $\frac{1}{3}M(a^2+b^2)$

(c) Ring of radius a.

$\perp$ axis through centre $= Ma^2$

diameter as axis $= \frac{1}{2}Ma^2$ (obv. from cartesian coordinates dist from z axis $= \sqrt{x^2+y^2}$)

Plate of radius a

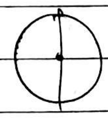


$\perp$ axis through centre $= \frac{1}{2}Ma^2$

diameter as axis $= \frac{1}{4}Ma^2$

So circular plate MoI are half of ring for similar axis.

(d) Shell of radius a



diameter as axis $\rightarrow \frac{2}{3}Ma^2$

Solid sphere of radius a

diameter as axis $\rightarrow \frac{2}{5}Ma^2$

(natural since solid sphere elements are closer to axis & hence its MoI would be less)

(6) Theorem $\rightarrow$ MoI of Tyre like objects:

A closed curve revolves around axis OX and forms a solid of revolution. OX doesn't intersect this curve.

Now let $a =$ distance of centre of curve from OX

$k =$ radius of gyration for an axis $\parallel$ to OX & passing through centre of curve.

then MoI about $OX = M(a^2 + 3k^2)$

M is mass of newly formed mass.

Naturally $\rightarrow$ a solid rubber tyre with radius b & radius of cross section a has M.I $\rightarrow M\left(b^2 + \frac{3a^2}{4}\right)$

if same tyre is hollow $\rightarrow M\left(b^2 + \frac{3a^2}{2}\right)$

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(7) Principle Axes

For every body, $\exists$ a set of 3 $\perp$ axes at every point with property

→ Product of inertia vanish if we take any 2 axes.

These are called principle axes.

The MoI are least & most with these principle axes for axes passing through that point.

(8) ~~Finding~~ Checking Principle axes. (if given axis in principle axis)

We make given axis Z axis & also choose X & Y axis.

Now we calculate A, B, F as earlier.

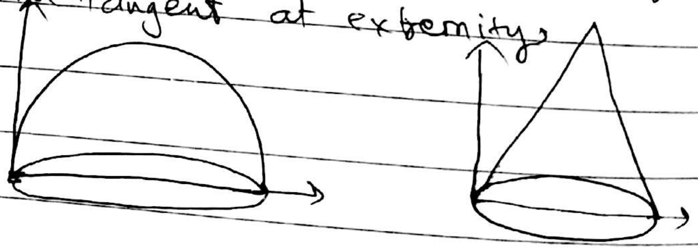
$A = \sum m(y^2+z^2)$ $\quad B = \sum m(x^2+z^2)$ $\quad F = \sum m xy$

Then actual principle axes through origin of this system makes an angle with Z-axis θ given by

$$\tan 2\theta = \left(\frac{2F}{B-A}\right)$$

(If you can't remember this formula, just remember $\tan^{-1}$ & manipulate as per given information in question)

(9) For a cone and hemisphere, an axis through diameter of base and tangent at extremity, they form principle axes.



D'Alembert's Principle (paper 2)

(1) D'Alembert gave a method which enables us to obtain necessary equations of motion without considering unknown inner forces.

This is because by Newton's 3^{rd} law ~~gives~~ all internal actions & reactions are in equilibrium.

(2) Impressed forces = external forces acting on a body.

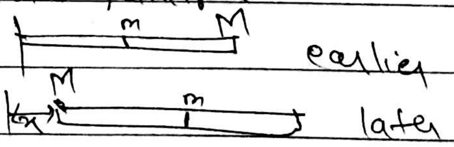
Effective force on a particle is product of its mass and acceleration i.e. $m\ddot{x}$, $m\ddot{y}$ & $m\ddot{z}$

(3) D'Alembert's Principle
The reversed effective forces at each point of a body and ~~external~~ forces are in equilibrium. ∵ virt. work = 0
i.e. $\sum_{i=1}^{N} (F_i - m\ddot{r}_i) \cdot \delta r_i = 0$

(4) Motion of centre of Inertia.
Centre of inertia $(\bar{x}, \bar{y}, \bar{z})$ is given by
$\Sigma m x = M\bar{x}$, $\Sigma m y = M\bar{y}$ & $\Sigma m z = M\bar{z}$
It's motion is as if all mass was concentrated at $(\bar{x}, \bar{y}, \bar{z})$ & all forces acting at ~~that~~ point.

(5) e.g. Rough board of length 2a & mass m rest on smooth horizontal plane. A man of mass M walks from one end to other end. Find movement of board.

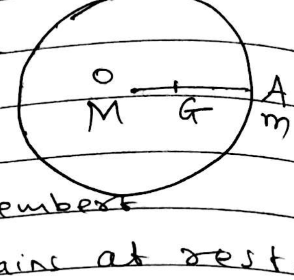
→ No force on Centre of inertia as per D'Alembert's principle.
∴ it shouldn't move



$$\therefore \frac{ma + 2aM}{m+M} = \frac{m(a+x) + Mx}{m+M}$$

$$\therefore x = \frac{2aM}{m+M}$$

(6) Circular board on smooth plane. A boy runs on edge with constant vel. What would be motion of board?



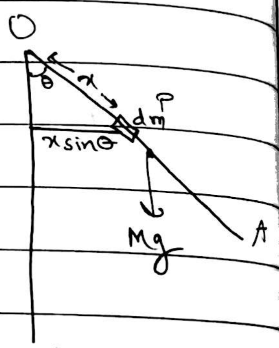
→ No horizontal force. ∴ by D'Alembert principle, centre of inertia remains at rest.

Now $G = \frac{mr}{m+M}$.

∴ G doesn't move ⇒ centre O of disc traces a circle with radius OG & centre G.

(7) Good example needing understanding of D'Alembert.

Q. OA, length 2a, rotates about O with uniform angular velocity ω. Show that inclination would be 0 or $\cos^{-1}\left(\frac{3g}{4a\omega^2}\right)$



→ In D'Alembert, we first find effective force on each particle. Then equate it to external force

Here, $dm = \frac{M}{2a} dx$ & force effective is only corresponding to uniform angular velocity

∴ Reversed effective force on dm = $dm \cdot x\sin\theta \cdot \omega^2 = \frac{Mx\sin\theta\,\omega^2 dx}{2a}$

External forces are Mg & reaction at O.

We take moment about O to avoid reaction.

$$\therefore \int_0^{2a} \frac{M}{2a}\omega^2 x^2 \sin\alpha\cos\alpha\, dx - Mg.a\sin\alpha = 0$$

⇒ on solving $\cos\alpha = \frac{3g}{4a\omega^2}$ or $\sin\alpha = 0$, ∴ solved.

⑧ So, using D'Alembert ⇒ (i) First find effective force for each particle.
(ii) Integrate it & equate to external forces.

Now, angular motion needs calculating moment about a point whose reaction we want to avoid.
If it is normal motion, simple force calculation needed.

So D'Alembert is pretty intuitive.

Ellipsoid of inertia or Momental ellipsoid

→ For a given point, we can draw an ellipsoid indicating MOI along different axes passing through that point.

Now, for an axis l, m, n; the MoI is given by

$I = Al^2 + Bm^2 + Cn^2 + 2D\,mn + 2E\,ln + 2F\,lm$

now, we choose a point on (l, m, n) axis which is inverse square of inertia (I) → say point P.

$$\therefore \; Al^2 + Bm^2 + Cn^2 + 2Dmn + 2Eln + 2Flm = \frac{1}{r^2}$$

$$\therefore \; Al^2r^2 + Bm^2r^2 + Cn^2r^2 + 2Dmnr^2 + 2Elnr^2 + 2Flmr^2 = 1$$

$$\therefore \; Ax^2 + By^2 + Cz^2 + 2Dyz + 2Exz + 2Fxy = 1$$

This becomes our ellipsoid of inertia.

So with principle axes, ellipsoid takes nice shape

$$Ax^2 + By^2 + Cz^2 = 1$$

Lagrange's Equations (paper 2)

① Generalised coordinates
→ They are equal to no. of degrees of freedom. These coordinates are independent of each other and determine position of a dynamic system.

Generally n particles moving freely in space have $3n$ degrees of freedom ((x_i, y_i, z_i) for each particle)

② Holonomic system
Cartesian coordinates are expressed as functions of generalised coordinates.

$x = f_1(t, \theta, \phi, \psi \dots)$

$y = f_2(t, \theta, \phi, \psi \dots)$

$z = f_3(t, \theta, \phi, \psi \dots)$

If these functions do not contain velocities i.e. $\dot{\theta}$ etc. then it is holonomic system.
Otherwise it is non-holonomic system.

③ T = Kinetic energy $= \frac{1}{2}\sum m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$

$= \frac{1}{2}\sum m(\dot{f}_1^2 + \dot{f}_2^2 + \dot{f}_3^2)$ [remember this since use it in all questions

In a conservative field; we have potential function V

Then Lagrange equation for each coordinate becomes

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) - \frac{\partial T}{\partial \theta} = -\frac{\partial V}{\partial \theta}$$

Similarly for generalised coordinate ϕ;

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\phi}}\right) - \frac{\partial T}{\partial \phi} = -\frac{\partial V}{\partial \phi}$$

(easier to remember form $\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{\theta}}\right) = \frac{\partial (T-V)}{\partial \theta}$ ~~function on RHS~~ Lagrangian function

just know this, mug up formula on next page.

(4) Lagrangian Function or (Kinetic Potential)

$L = T - V$

then lagrange θ equation becomes ($\because V$ doesn't contain $\dot{\theta}$ $\therefore \frac{\partial V}{\partial \dot{\theta}} = 0$)

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0}$$

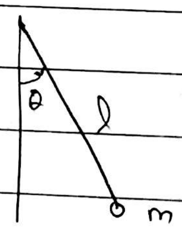
Just mug this up.

(5) So, in all problems (i) first express T & V in terms of min. number of generalised coordinates
(ii) Then get $L = T - V$.
(iii) Then put L in lagrange Equation to get motion of this body $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0.$

(Remember that θ & $\dot{\theta}$ should be considered ind. $\therefore \frac{d\dot{\theta}}{\partial \theta} = 0$)

(6) e.g. Find lagrange function & equation for simple pendulum.

→ The only parameter is θ.



$T = \frac{1}{2}mv^2 = \frac{1}{2}ml^2\dot{\theta}^2$

$V = mgl(1 - \cos\theta)$

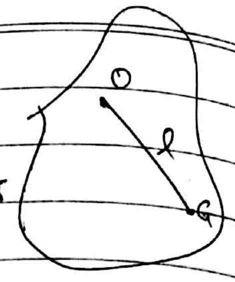
$\therefore L = T - V = \frac{1}{2}ml^2\dot{\theta}^2 - mgl(1 - \cos\theta)$

& Lagrange equation is $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$

$\therefore \frac{d}{dt}\left(ml^2\dot{\theta}\right) - \left(-mgl\sin\theta\right) = 0$

$\therefore l\ddot{\theta} = -g\sin\theta \qquad \therefore \ddot{\theta} = -\frac{g}{l}\sin\theta$

$\therefore \ddot{\theta} = -\frac{g}{l}\theta$ for small θ.



(7) Same question for Compound pendulum

→ Similar to earlier. Just remember that the kinetic energy function changes here:

$$T = \frac{1}{2} M k^2 \dot{\theta}^2$$ where k is radius of gyration

$$V = -mgl\cos\theta$$ (considered from O).

Then $L = T - V = \frac{1}{2} mk^2\dot{\theta}^2 + mgl\cos\theta$

∴ Lagrange eq. $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$

$$\therefore \frac{d}{dt}\left(mk^2\dot{\theta}\right) + mgl\sin\theta = 0$$

$$\therefore \quad k^2\ddot{\theta} = -lg\sin\theta$$

∴ eq. of motion $\ddot{\theta} = \left(\frac{-lg}{k^2}\right)\sin\theta$

(8) Lagrangian function & eq. in cylindrical coordinates.

→ A particle of mass m moves in a conservative field. Find (i) Lagrangian function (ii) eq. of motion

→ In such questions, main thing is finding Kinetic energy T as function of (ρ, ϕ, z) (generalised coordinate). For potential energy, we simply write $V(\rho, \phi, z)$.

Now, we first mention x, y, z in terms of generalised coordinates

$x = \rho\cos\theta \qquad y = \rho\sin\theta \qquad z = z$

$\therefore \dot{x} = \dot{\rho}\cos\theta - \rho\sin\theta\,\dot{\theta} \qquad \dot{y} = \dot{\rho}\sin\theta + \rho\cos\theta\,\dot{\theta} \qquad \dot{z} = \dot{z}$

$\therefore T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}m\left((\dot{\rho}\cos\theta - \rho\sin\theta\,\dot{\theta})^2 + (\dot{\rho}\sin\theta + \rho\cos\theta\,\dot{\theta})^2 + \dot{z}^2\right)$

$= \frac{1}{2}m\left(\dot{\rho}^2 + \rho^2\dot{\theta}^2 + \dot{z}^2\right)$

& $V = V(\rho, \theta, z)$

$\therefore L = \frac{1}{2} m(\dot{\rho}^2 + \rho^2 \dot{\theta}^2 + \dot{z}^2) - V(\rho, \theta, z)$

& now we differentiat for 3 variables as

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0$$

(remember $\frac{\partial V(\rho, \theta, z)}{\partial \dot{\rho}} = 0$ $\because$ we consider ρ & $\dot{\rho}$ independe

9) Cyclic coordinates.

If Lagrangian doesn't contain generalised coordinate q_j. then $\frac{\partial L}{\partial q_j} = 0$ Such q_j is called cyclic or ignorable

As we will see with hamiltonian; $\dot{P}_j = \frac{\partial L}{\partial q_j} = 0$

$\therefore$ Generalised coordinate of momentum becomes 0.
Hence cyclic coordinate implies conservation of generalised momentum.

10) Inertia Tensor

Made by MoI & PoI $= \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$

I_{xx} = MoI about x axis

I_{xy} = -ve of PoI about x & y axis i.e. $\int -xy\, dm$

Now, Torque $\begin{bmatrix} T_x \\ T_y \\ T_z \end{bmatrix} = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zy} & I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} \dot{w}_x \\ \dot{w}_y \\ \dot{w}_z \end{bmatrix}$

Hamilton's Equation (paper 2)

1. Just like lagrange, hamilton uses generalised coordinates but gives 2n equations with n coordinates.

2. Let generalised coordinates be $q_1, q_2 \dots q_n$.
Let generalised coordinates of momentum' Then we define 'generalised coordinates of momentum' as $p_i = \frac{\partial L}{\partial \dot{q}_i}$

Quite intuitive since $K.E = \frac{1}{2}mv^2$ & momentum $= \frac{\partial (K.E.)}{\partial v}$
& if V is P.E. then $\frac{\partial V}{\partial \dot{q}_i} = 0$ ∴ Above result

So if forgotten; just find momentum mv as derivative of K.E. & you get above relation.

3. Then using Lagrange equation $\left(\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = \frac{\partial L}{\partial q_i}\right)$

$$\therefore \dot{p}_i = \frac{\partial L}{\partial q_i}$$

nice parity betn these coordinates; partial derivative of L w.r.t. one guy gives time derivative of other guy.

i.e. $\frac{\partial L}{\partial \dot{q}_r} = p_r$ & $\frac{\partial L}{\partial q_r} = \dot{p}_r$ (~~The 1st~~ This is one of the hamiltons eq.)

4. Hamiltonian Function

$$H = -L + \sum_{r=1}^{n} p_r \dot{q}_r$$

When L is not having t explicitely; H becomes total energy of system ∵ $\sum p_r \dot{q}_r = mv^2 = 2T$ ∴ $H = -(T-V) + 2T = T+V$

Very useful in problems.

5. Hamilton's Equations

Using hamitonian function & applying Lagrange eq; we get

$$\dot{p}_r = -\frac{\partial H}{\partial q_r} \quad \& \quad \dot{q}_r = \frac{\partial H}{\partial P_r}$$

(dimensional analysis helps; $\dot{P}_r$ would have $\frac{mv}{t}$ & hence $\frac{H}{q} \to \frac{mv^2}{S} = \frac{mv}{t}$ & similarly $\dot{q}_r \to V$ $\therefore \frac{\partial H}{P} \to \frac{mv^2}{mv} = v$)

6. In solving problems using hamilton's equations; its easy when you have one variable x like SHM or pend.

Put $K.E = \frac{1}{2}m\dot{x}^2$ & V in terms of x. Don't oversimplify & put K.E. in terms of x.

This gives problems.

Keep L simple.

Now, mostly L will not have explicitely t as part of function. This makes our job easier.

Now $H = T + V$ & then apply $\dot{P}_r = -\frac{\partial H}{\partial q_r}$ $\dot{q}_r = +\frac{\partial H}{\partial P_r}$ & so on

i.e. $\dot{P}_x = \frac{-\partial H}{\partial x}$ & $\dot{x} = +\frac{\partial H}{\partial P_x}$

7. Simple steps in solving problems using hamilton's eq.

→ ① Express T & V in terms of generalised coordinates.

② Get $L = T - V$, generally this would be ind. of t.

Now, find all generalised comp. of momentum $P_\theta = \frac{\partial L}{\partial \dot{\theta}}$

③ Express T in terms of P_θ where possible. Then $H = T + V$ (as L ind. of t). & then find $\dot{P}_\theta = \frac{-\partial H}{\partial \theta}$, $\dot{\theta} = \frac{\partial H}{\partial P_\theta}$

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(7) Then take derivative of $\dot{\theta} = \frac{\partial H}{\partial P_\theta}$ which brings in $\dot{P_\theta}$ in Hamilton eq. $\ddot{\theta} = f(\dot{P_\theta})$ & this becomes $\ddot{\theta} = f(\theta)$ from earlier eq.

So Hamilton eq. problems are simple & straightforward.

Just need proper calculation.

(8) Finding T in these problems.

Always start with $T = \frac{1}{2} m \sum \dot{x}^2 + \dot{y}^2 + \dot{z}^2$

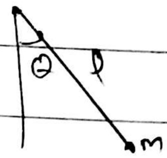
Then if we are 2-D polar $\Rightarrow$ $x = r\cos\theta$ & $y = r\sin\theta$
find $\dot{x}, \dot{y}$ & so on get $T(r, \theta)$.

Similarly for polar, first find $\dot{x}, \dot{y}, \dot{z}$ in terms of new coordinates.

(9) If any info is given about force eg. force depends on distance from origin.
It only means potential V is just a function of r.
So find T & Use $L = T - V(r)$

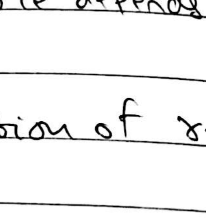
(10) Some Basics of Pendulum

(a) Simple Pendulum



$T = \frac{1}{2} m l^2 \dot{\theta}^2$ $\quad V = -mgl\cos\theta$

(b) Compound Pendulum

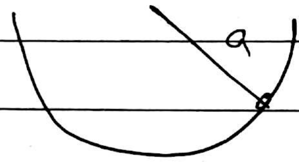


$T = \frac{1}{2} m k^2 \dot{\theta}^2$ $\quad V = -mgl\cos\theta$

k is radius of gyration.

(c) Spherical pendulum

The object moves on a surface of sphere of radius a.



$\therefore$ We consider θ & ϕ as generalised coordinates.

$\therefore \quad x = a\sin\theta\cos\phi \qquad y = a\sin\theta\sin\phi \qquad z = a\cos\theta$

$\therefore T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}ma^2(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta)$

$V = -mga\cos\theta$. (so V doesn't change. T changes with each pendulum)

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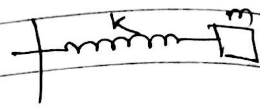
Linear Harmonic Oscillator & Lagrangian / Hamiltonian



1 $T = \frac{1}{2} m \dot{x}^2$

$V = \frac{1}{2} k x^2$

$L = T - V = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$

Lagrange eq. is $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0$

$\therefore \frac{d}{dt}(m\dot{x}) + kx = 0$

$\therefore \boxed{\ddot{x} = -\frac{k}{m} x}$

Using ~~Hami~~ Hamilton's equation

$L = \frac{1}{2} m \dot{x}^2 - \frac{1}{2} k x^2$ This doesn't contain time explicitly.

$\therefore H = T + V = \frac{1}{2} m \dot{x}^2 + \frac{1}{2} k x^2$

now, $P_x = \frac{\partial L}{\partial \dot{x}} = m\dot{x}$

$\therefore H = \frac{P_x^2}{2m} + \frac{1}{2} k x^2$

Hamilton's eq $\Rightarrow$ $\dot{P}_x = -\frac{\partial H}{\partial x} = -kx$

$\dot{x} = \frac{-\partial H}{\partial P_x} = \frac{P_x}{m}$

$\therefore \boxed{\ddot{x} = \frac{\dot{P}_x}{m} = \frac{-kx}{m}}$

So elegant & simple.

FLUID DYNAMICS

(1) Streamlines & Pathlines are 2 different ways of representing fluid flow. Streamline describes flow of all particles at a particular instant of time whereas pathlines indicates what path a particle will follow over a period of time.
When these 2 lines match, we have 'steady flow'.

Streamlines → Tangent at any point indicates velocity at that point at that time.
Pathlines → Curve which a particular particle describes during its motion.
Major formulae depend on velocity vector → $u\hat{i}+v\hat{j}+w\hat{k}$

(2) Streamlines are obtained by solving following equation

$$\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$$ where $u\hat{i}+v\hat{j}+w\hat{k}$ is velocity vector i.e. $\frac{d\vec{r}}{dt} = u\hat{i}+v\hat{j}+w\hat{k}$

Pathlines are obtained by solving

$$\frac{dx}{dt} = u \qquad \frac{dy}{dt} = v \qquad \frac{dz}{dt} = w$$

(3) Velocity Potential

$\phi(x,y,z)$ is called velocity potential where

$$\frac{\partial\phi}{\partial x} = -u \qquad \frac{\partial\phi}{\partial y} = -v \qquad \frac{\partial\phi}{\partial z} = -w$$

(4) Vorticity Vector (As name suggests, a measure of curl)

$$\text{Vorticity} \rightarrow m\hat{i}+n\hat{j}+l\hat{k} = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u & v & w \end{vmatrix}$$

& if vorticity = 0 we say fluid motion is irrotational.

Vortex lines given by $\frac{dx}{m} = \frac{dy}{n} = \frac{dz}{l}$ Quite intuitive

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(5) Condition for given surface to be a boundary surface $F(x, y, t) = 0$

Let eq. of surface be

Then we need normal component of fluid velocity & surface velocity to be same. This gives

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}\right) F = 0$$

(6) Equation of continuity

When fluid is given to be incompressible then this eq. is $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

So when they ask you to check if given motion is possible, check if above eq. is satisfied.

e.g. Don't hesitate to use componendo-dividendo for solving streamline eq. $\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{\omega}$

(7) Streamline eq. can sometimes be helpful in polar form so if Φ is velocity potential then $\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{\omega}$ becomes

$$\frac{dr}{-\frac{\partial \Phi}{\partial r}} = \frac{r\,d\theta}{-\frac{1}{r}\frac{\partial \Phi}{\partial \theta}} = \frac{r\sin\theta\,d\phi}{-\frac{1}{r\sin\theta}\frac{\partial \Phi}{\partial \phi}}$$

The way to remember is, we need differential distance in numerator, so r, $r\,d\theta$ & $r\sin\theta\,d\phi$ are obvious. Also denominator need to maintain dimensional consistancy, so dividing by r & $r\sin\theta$ respectively is natural for $\frac{\partial \Phi}{\partial \theta}$ & $\frac{\partial \Phi}{\partial \phi}$.

8) Orthogonal surfaces are solⁿ of $U\,dx + V\,dy + w\,dz = C$

9) For a given velocity flow, to check if velocity potential is possible, we see if it is irrotational.

Sources, Sinks & Doublets

(1) Motion in 2 dimension means $w=0$ for velocity & only u & v component exist.
Then equation of continuity becomes $\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0$
Also streamline eq. becomes $\frac{dx}{u}=\frac{dy}{v}$ i.e. $v\,dx-u\,dy=0$ (exact)

Hence, we can define stream function (ψ) in this case.

So, in 2 dimension flow, if flow is continuous & incompressible, we can define stream function ψ
s.t. $u=-\frac{d\psi}{dy}$ $\quad v=\frac{\partial\psi}{\partial x}$

(2) Stream function exists whether motion is irrotational or not.
When motion is irrotational; both velocity potential (ϕ) & stream function (ψ) exist & they satisfy laplace equation i.e. $\frac{\partial^2\psi}{\partial^2 x^2}+\frac{\partial^2\psi}{\partial y^2}=0=\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}$

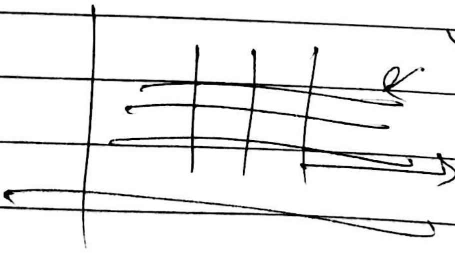
Remember stream function = constant gives stream lines.

e.g. Find stream function ψ for given velocity potential $\phi=cx$. Also draw streamlines & equipotential lines.

$\rightarrow$ $u=-\frac{\partial\phi}{\partial x}=-c$ $\quad v=-\frac{\partial\phi}{\partial y}=0$

Also, $\frac{\partial\psi}{\partial y}=-u=c \Rightarrow \psi=cy+f(x)$

$\frac{\partial\psi}{\partial x}=v=0 \Rightarrow \psi=cy$



$\psi=c \rightarrow$ streamlines

$\phi=c \rightarrow$ equipotential.

(3) Complex Potential

Let ϕ & ψ be velocity potential & stream function.
Then
complex potential is defined as $w = \phi + i\psi$
Obviously $w(z)$ is an analytic function.
now, $\frac{dw}{dz} = -u + iv$ $\therefore \left|\frac{dw}{dz}\right|$ represents magnitude of velocity.

Stagnation points have $\left|\frac{dw}{dz}\right| = 0$

(4) Sources & Sinks

Source – A point from which liquid is emitted radially & symmetrically in all directions.
Sink $\to$ A point to which liquid is flowing symmetrically.

Source has strength m & sink $-m$. It means total volume of flow across any small circle surrounding source is $2\pi m$.

(5) Complex potential due to sources of strength $m_1, m_2 \ldots$ kept at points $a_1, a_2, a_3 \ldots$

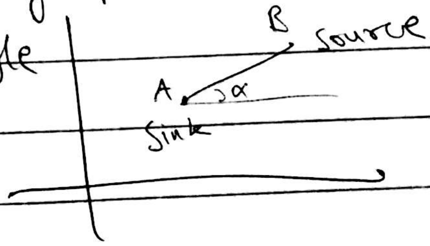
$$W = -m_1 \log(z - a_1) - m_2 \log(z - a_2) \ldots\ldots$$

(6) Doublet

$-m \xrightarrow{\delta s} m$
Sink Source

Strength of doublet
$= \mu = m\delta s$

Position of doublet is identified by position of sink.
Let doublet of strength μ make angle α with horizontal then



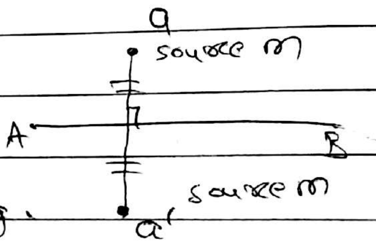
Complex potential $w = \dfrac{\mu e^{i\alpha}}{z - a}$

Image

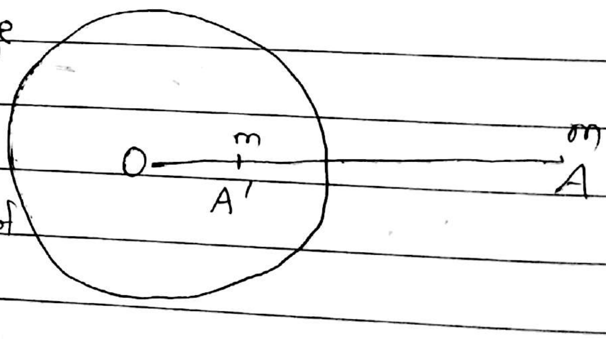
(7) If there is a curve or boundary across which there is no flow; we consider sinks & sources on one side of curve as images of sinks & sources on other side.
These curves can be considered as streamlines as streamlines have property that velocity $\perp$ to curve is 0.

So, if we are given a system with rigid boundaries, we consider equivalent system given by images that ensure zero velocity at these boundaries.

(8) Image of source, sink, doublet in a line
→ Simply find image of that point & put source/sink/doublet of same strength →
In figure, line AB has 0 ~~velocity~~ flow across it.



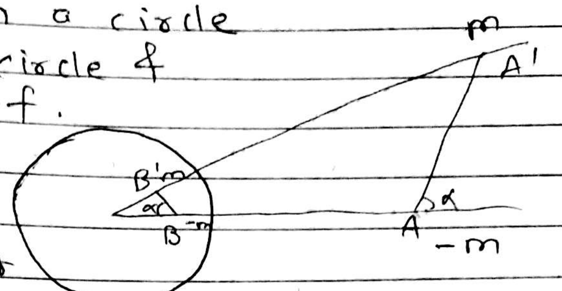
(9) Image of source/sink in a circle
If source/sink is at A, image will be at A' to ensure $\exists$ no flow across circle.
Here, $OA' \cdot OA = r^2$ (r = radius of circle)



i.e. A' is inverse of A.

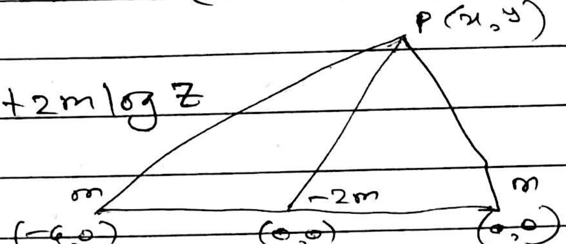
(10) Image of doublet in a circle

Let r be radius of circle &
Distance OA of sink $= f$.
Let μ be strength of
doublet then
strength of image doublet
is $\frac{\mu r^2}{f^2}$ (logical since higher $f \Rightarrow$ lower strength needed)



Again, this is at inverse point B of A but angle with horizontal is $\pi - \alpha$.

e.g. Two sources (m) placed at $(a,0)$ & $(-a,0)$ and sink of strength $-2m$ placed at origin. Show that stream lines are curves $(x^2+y^2)^2 = a^2[x^2-y^2+\lambda xy]$ (proam.)



$\rightarrow$ $w = -m\log(z-a) - m\log(z+a) + 2m\log z$

$\therefore \phi + i\psi = -m\log(x^2-a^2-y^2+2ixy) + m\log(x^2-y^2+2ixy)$

Obv. equalling imaginary parts to get stream function since we want streamlines

$$\psi = -m\tan^{-1}\left(\frac{2xy}{x^2-a^2-y^2}\right) + m\tan^{-1}\left(\frac{2xy}{x^2-y^2}\right)$$

$$= -m\tan^{-1}\frac{2a^2xy}{(x^2-y^2)(x^2-a^2-y^2)+4x^2y^2}$$

Now, streamlines means $\psi = \text{constant}$

$$\therefore \text{let } \frac{2a^2xy}{(x^2-y^2)(x^2-a^2-y^2)+4x^2y^2} = \frac{2}{\lambda}$$

$$\Rightarrow (x^2+y^2)^2 = a^2(x^2-y^2+\lambda xy)$$

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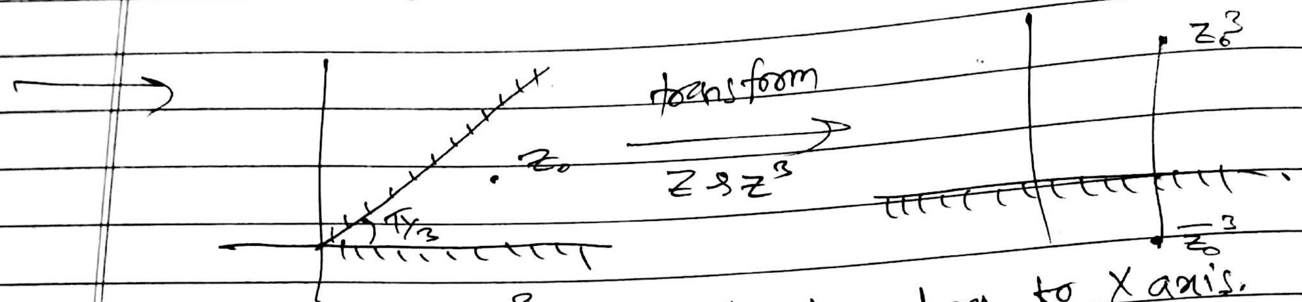
e.g. Using method of images, prove that if there is a source m at z_0 in a fluid bounded by $\theta = 0$ & $\theta = \pi/3$, sol^n is

$$\phi + i\psi = -m \log\left[(z^3 - z_0^3)(z^3 - z_0'^3)\right]$$

where $z_0 = x_0 + iy_0$, $z_0' = x_0 - iy_0$



Mapping $z \to z^3$ converts boundary to X axis.
Let new coordinates be z^*.

$\therefore z_0^* = z_0^3$

$\therefore$ In new system, equation becomes

$$w = -m \log(z^* - z_0^*) - m \log(z^* - \bar{z}_0^*)$$

$$= -m \log\left[(z^3 - z_0^3)(\bar{z}^3 - z_0^3)\right]$$

Vortex

① Complex potential by vortex of strength k

$$= \frac{ik}{2\pi} \log(z - z_0)$$

② By doublet of strength $\mu = \frac{ik}{2\pi}\delta s$

$$= -i\frac{\mu e^{i\alpha}}{z}$$ (keeping centre at origin)