(*b*) A vessel is in the shape of a hollow hemisphere surmounted by a cone held with the axis vertical and vertex uppermost. If it is filled with a liquid so as to submerge half the axis of the cone in the liquid and height of the cone be double the radius (r) of its base, find the resultant downward thrust of the liquid on the vessel in terms of the radius of the hemisphere and density (ρ) of the liquid. **(15)**

(*c*) Derive the Frenet-Serret formulae. Verify the same for the space curve $x = 3 \cos t$, $y = 3 \sin t$, $z = 4t$. **(10)**

8. (*a*) Find the general solution of the differential equation

$$(x-2)y'' - (4x-7)y' + (4x-6)y = 0. \quad \textbf{(10)}$$

(*b*) A shot projected with a velocity u can just reach a certain point on the horizontal plane through the point of projection. So in order to hit a mark h metres above the ground at the same point, if the shot is projected at the same elevation, find increase in the velocity of projection. **(15)**

(*c*) Derive $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2} + \dfrac{\partial^2}{\partial z^2}$

in spherical coordinates and compute

$$\nabla^2\left(\frac{x}{\left(x^2+y^2+z^2\right)^{\frac{3}{2}}}\right)$$

in spherical coordinates. **(15)**

PAPER-II

INSTRUCTIONS: *There are* ***eight*** *questions in all, out of which* ***five*** *are to be attempted. Question Nos. 1 and 5 are compulsory. Out of the remaining six questions,* ***three*** *are to be attempted selecting at least* ***one*** *question from each of the two Sections A and* ***B****. Answers must be written in English only.*

SECTION–A

1. (*a*) Let R be an integral domain. Then prove that ch R (characteristic of R) is 0 or a prime. **(8)**

(*b*) Show that the function $f(x) = \sin\left(\dfrac{1}{x}\right)$ is continuous and bounded in $(0, 2\pi)$, but it is not uniformly continuous in $(0, 2\pi)$. **(8)**

(*c*) Test the Riemann integrability of the function f defined by:

$$f(x) = \begin{cases} 0 & \text{when } x \text{ is rational} \\ 1 & \text{when } x \text{ is irrational} \end{cases}$$

on the interval $[0, 1]$. **(8)**

(*d*) Using Cauchy's Integral formula, evaluate the integral $\oint_c \dfrac{dz}{(z^2+4)^2}$ where $c : |z - i| = 2$. **(8)**

(*e*) A firm manufactures two products A and B on which the profits earned per unit are ₹ 3 and ₹ 4 respectively. Each product is processed on two machines M1 and M2. Product A requires one minute of processing time on M1 and two minutes on M2, while B requires one minute on M1 and one minute on M2. Machine M1 is available for not more than 7 hours 30 minutes, while machine M2 is available for 10 hours during any working day. Find the number of units of products A and B to be manufactures to get maximum profit, using graphical method. **(8)**

2. (*a*) Let I and J be ideals in a ring R. Then prove that the quotient ring $(I + J)/J$ is isomorphic to the quotient ring $I/(I \cap J)$. **(10)**

(*b*) Show that the integral $\int_0^{\pi/2} \log \sin x \, dx$ is convergent and hence evaluate it. **(15)**

(*c*) If $f(z)$ is analytic in a domain D and $|f(z)|$ is a non-zero constant in D, then show that $f(z)$ is constant in D. **(15)**

3. (*a*) If in the group G, $a^5 = e$, $aba^{-1} = b^2$ for some $a, b \in G$, find the order of b. **(10)**

(*b*) Show that the sequence $\{\tan^{-1} nx\}$, $x \geq 0$ is uniformly convergent on any interval $[a, b]$, $a > 0$ but is only pointwise convergent on $[0, b]$. **(15)**

(*c*) Use simplex method to solve the following problem:

Maximize $z = 2x_1 + 5x_2$

subject to $x_1 + 4x_2 \leq 24$

$3x_1 + x_2 \leq 21$

$x_1 + x_2 \leq 9$

$x_1, x_2 \geq 0$ **(15)**

4. (*a*) Show that the smallest subgroup V of A_4 containing (1, 2) (3, 4), (1, 3) (2, 4) and (1, 4), (2, 3) is isomorphic to the Klein 4-group. **(10)**

(*b*) Classify the singular point $z = 0$ of the function $f(z) = \dfrac{e^z}{z + \sin z}$ and obtain the principal part of the Laurent series expansion of $f(z)$. **(15)**

(*c*) A salesman wants to visit cities C1, C2, C3 and C4. He does not want to visit any city twice before completing the tour of all the cities and wishes to return to his home city, the starting station. Cost of going from one city to another in rupees is given below in the table. Find the least cost route. **(15)**

From city \ To City	C1	C2	C3	C4
C1	0	30	80	50
C2	40	0	140	30
C3	40	50	0	20
C4	70	80	130	0

SECTION–B

5. (*a*) Find the solution of the equation:

$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = x - y.$$

(8)

(*b*) The following table gives the values of $y = f(x)$ for certain equidistant values of x. Find the value of $f(x)$ when $x = 0.612$ using Newton's forward difference interpolation formula. **(8)**

x	$y = f(x)$
0.61	1.840431
0.62	1.858928
0.63	1.877610
0.64	1.896481
0.65	1.915541

(*c*) Following values of x_i and the corresponding values of y_i are given. Find $\int_0^3 y \, dx$ using Simpson's one-third rule. **(8)**

x_i	y_i
0.0	0.0
0.5	0.75
1.0	1.0
1.5	0.75
2.0	0.0
2.5	–1.25
3.0	–3.0

(*d*) Consider the flow field given by $\psi = a(x^2 - y^2)$, 'a' being a constant. Show that the flow is irrotational. Determine the velocity potential for this flow and show that the streamlines and equivelocity potential curves are orthogonal. **(8)**

(*e*) Find a complete integral of the equation by Charpit's method $p^2x + q^2y = z$.

Here, $p = \dfrac{\partial z}{\partial x}$, $q = \dfrac{\partial z}{\partial y}$. **(8)**

6. (*a*) Test the integrability of the equation

$$z(z+y^2)\,dx + z(z+x^2)dy - xy(x+y)dz = 0$$

If integrable, then find its solution. **(15)**

(*b*) Solve the following system of equations by Gauss-Jordan elimination method: **(10)**

$$x_1 + x_2 + x_3 = 3$$
$$2x_1 + 3x_2 + x_3 = 6$$
$$x_1 - x_2 - x_3 = -3$$

(*c*) For a dynamical system

$$T = \frac{1}{2}\left\{(1+2k)\dot{\theta}^2 + 2\dot{\theta}\dot{\varphi} + \dot{\varphi}^2\right\},$$

$$V = \frac{n^2}{2}\left\{(1+k)\theta^2 + \varphi^2\right\},$$

where θ, φ are coordinates and n, k are positive constants, write down the Lagrange's equations of motion and deduce that

$$(\ddot{\theta} - \ddot{\varphi}) + n^2\left(\frac{1+k}{k}\right)(\theta - \varphi) = 0$$

Further show that if $\theta = \varphi$, $\dot{\theta} - \dot{\varphi}$ at $t = 0$, then $\theta = \varphi$ for all t. **(15)**

7. (*a*) Given $\frac{dy}{dx} = x^2 + y^2$, $y(0) = 1$. Find $y(0.1)$ and $y(0.2)$ by fourth order Runge-Kutta method. **(15)**

(*b*) Consider a mass-spring system consisting of a mass m and a linear spring of stiffness k hanging from a fixed point. Find the equation of motion using the Hamiltonian method, assuming that the displacement x is measured from the unstretched position of the string. **(10)**

(*c*) Find the equations of the system of curves on the cylinder $2y = x^2$ orthogonal to its intersections with the hyperboloids of the one-parameter system $xy = z + c$. **(15)**

8. (*a*) Consider that the region $0 \le z \le h$ between the planes $z = 0$ and $z = h$ is filled with viscous incompressible fluid. The plane $z = 0$ is held at rest and the plane $z = h$ moves with constant velocity $V\hat{j}$. When conditions are steady, assuming there is no slip between the fluid and either boundary, and neglecting body forces, show that the velocity profile between the plates is parabolic.

Find the tangential stress at any point $P(x, y, z)$ of the fluid and determine the drag per unit area on both the planes. **(15)**

(*b*) State the Newton-Raphson iteration formula to compute a root of an equation $f(x) = 0$ and hence write a program in BASIC to compute a root of the equation.

$$\cos x - xe^x = 0$$

lying between 0 and 1. Use DEF function to define $f(x)$ and $f'(x)$. **(10)**

(*c*) Use Gauss quadrature formula of point six to evaluate $\int_0^1 \frac{dx}{1+x^2}$ given

$x_1 = -0.23861919$, $w_1 = 0.46791393$

$x_2 = -0.66120939$, $w_2 = 0.36076157$

$x_3 = -0.93246951$, $w_3 = 0.17132449$

$x_4 = -x_1$, $x_5 = -x_2$, $x_6 = -x_3$,

$w_4 = w_1$, $w_5 = w_2$

and $w_6 = w_3$. **(15)**

Then, $\frac{dr}{dt} = -3 \sin t\, i + 3 \cos i\, j + 4\, k$

$$\frac{ds}{dt} = \left|\frac{dr}{dt}\right| = \sqrt{\frac{dr}{dt}\cdot\frac{dr}{dt}}$$

$$= \sqrt{(-3\sin t)^2 + (3\cos t)^2 + 4^2} = 5$$

Thus, $T = \frac{dr}{ds} = \frac{dr/dt}{ds/dt}$

$$= -\frac{3}{5}\sin t\, i + \frac{3}{5}\cos t\, j + \frac{4}{5}k$$

(*b*) $\frac{dT}{dt} = \frac{d}{dt}\left(-\frac{3}{5}\sin t\, i + \frac{3}{5}\cos t\, j + \frac{4}{5}k\right)$.

$$= -\frac{3}{5}\cos t\, i - \frac{3}{5}\sin t\, j$$

$$\frac{dT}{ds} = \frac{dT/dt}{ds/dt}$$

$$= -\frac{3}{25}\cos t\, i - \frac{3}{25}\sin t\, j$$

Since $\frac{dT}{ds} = \kappa N$

$$\left|\frac{dT}{ds}\right| = |\kappa||N| = \kappa \text{ as } \kappa \geq 0$$

Then,

$$\kappa = \left|\frac{dT}{ds}\right| = \sqrt{\left(-\frac{3}{25}\cos t\right)^2 + \left(-\frac{3}{25}\sin t\right)^2}$$

$= \frac{3}{25}$ and $\rho = \frac{1}{\kappa}$ From $\frac{dT}{ds} = \kappa N$,

We obtain, $N = \frac{1}{\kappa}\frac{dT}{ds} = -\cos t\, i - \sin t\, j$.

(*c*) $B = T \times N$

$$= \begin{vmatrix} -\frac{3}{5}\sin t & \frac{3}{5}\cos t & \frac{4}{5} \\ -\cos t & -\sin t & 0 \end{vmatrix}$$

$$= \frac{4}{5}\sin t\, i - \frac{4}{5}\cos t\, j + \frac{3}{5}k$$

$$\frac{dB}{dt} = \frac{4}{5}\cos t\, i + \frac{4}{5}\sin t\, j$$

$$\frac{dB}{ds} = \frac{dB/dt}{ds/dt} = \frac{4}{25}\cos t\, i + \frac{4}{25}\sin t\, j$$

$$-\tau N = -\tau(-\cos t\, i - \sin t\, j)$$

$$= \frac{4}{25}\cos t\, i + \frac{4}{25}\sin t\, j$$

or $\tau = \frac{4}{25}$ and $\sigma = \frac{1}{\tau} = \frac{25}{4}$.

Paper-II

1.(*a*) Suppose $ch\text{R} = n \neq 0$ and suppose n is not a prime. Then $n = m_1 \cdot m_2$ where m_1 and m_2 are proper divisors of n. For any $a \in \text{R}$, $a \neq 0$, we have,

$$0 = na^2 = (m_1 m_2)a^2$$
$$= (m_1 a)(m_2 a)$$

Since R is an integral domain,

$$m_1 a = 0 \text{ or } m_2 a = 0$$

Suppose $m_1 a = 0$.

Then we show that $m_1 x = 0$ for any $x \in \text{R}$.

Now $m_1(xa) = (m_1 x)a = x(m_1 a) = x_0 = 0$

Since, $a \neq 0$, and R is an integral domain $m_1 x = 0$.

Thus $m_1 x = 0$ for all x, $m_1 < n$.

This contradicts the assumption that $ch\text{R} = n$.

Hence n is a prime.

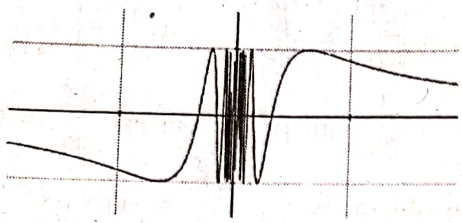
Corollary. The *ch* of a field is either 0 or a prime.

1.(*b*) It suffices to show that for some $\varepsilon > 0$ there is no $\delta > 0$ such that $|x - y| < \delta$ implies $|\sin(1/x) - \sin(1/y)| < \varepsilon$.

We shall do this for $\varepsilon = 1$. Given $\delta > 0$, we choose $x, y \in (0, \delta)$ so that $\sin(1/x) = 1$ and $\sin(1/y) = -1$. (For example, we can let x (resp. y) be the reciprocal of $(n + 1/2)\pi$ (resp. of $(n + 3/2)\pi$) for some integer $n > 1/\delta$).

Then $|x - y| < \delta$ but $|\sin(1/x) - \sin(1/y)| = 2 > \varepsilon$.

FYI, the graph is here,



Noticed that the function starts to oscillate more and more rapidly as x approaches 0. It bounced between -1 and 1.

Hence, the graph is continuous but not uniformily continuous and so not differentiable also.

1.(*c*) Let P be a partition of $[0, 1]$

$P = \{0 = x_0, x_1, x_2, \ldots, x_n = 1\}$

Let its sub-intervals be

$$I_r = [x_{r-1}, x_r] \text{ for } r = 1, 2, \ldots, n$$

If δ_r be the length of this interval I_r, then

$$\delta_r = x_r - x_{r-1}$$

Let M_r and m_r be respectively the l.u.b. and g.l.b. of the function f in I_r.

Then $f(x)$ will have the values 1 as well as -1 in every interval I_r, however small it may be, so we have,

$$M_r = 1 \text{ and } m_r = 1 \text{ as } M_r \geq m_r$$

Now we have,

$$U(P.f) = \sum_{r=1}^{n} M_r\delta_r = \sum_{r=1}^{n} 1(x_r - x_{r-1})$$

$$= (x_1 - x_0) + (x_2 - x_1) + \ldots + (x_n - x_{n-1})$$

$$= x_n - x_0 = 1 - 0 = 1$$

and $$L(P.f) = \sum_{r=1}^{n} m_r\delta_r = \sum_{r=1}^{n} (-1)(x_r - x_{r-1})$$

$$= \sum_{r=1}^{n} (x_{r-1} - x_r)$$

$$= (x_0 - x_1) + (x_1 - x_2) + \ldots + (x_{n-1} - x_n)$$

$$= x_0 - x_n = 0 - 1 = -1$$

$$\therefore \int_0^1 f(x)dx = \text{g.l.b. } \{U(P. f)\} = 1 \quad \ldots(1)$$

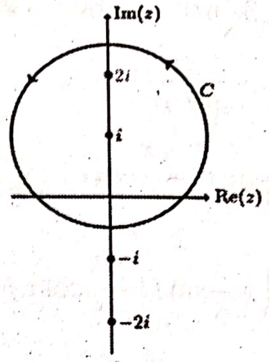
$$\text{and } \int_0^1 f(x)dx = \text{l.u.b. } \{L(P, f)\} = -1 \quad \ldots(2)$$

$\therefore$ From (1) and (2), we find that

$$\int_0^1 f(x)dx \neq \int_0^1 f(x)dx$$

$\therefore$ f is not Reimann integrable over $[0, 1]$.

1.(*d*)



We factor the denominator as

$$\frac{1}{(z^2+4)^2} = \frac{1}{(z-2i)^2(z+2i)^2}$$

Let $$f(z) = \frac{1}{(z+2i)^2}$$

Clearly $f(z)$ is analytic inside C. So, by Cauchy's formula for derivatives:

$$\int_C \frac{1}{(z^2+4)^2}dz = \int_C \frac{f(z)}{(z-2i)^2} = 2\pi i f'(2i)$$

$$= 2\pi i\left[\frac{-2}{(z+2i)^3}\right]_{z=2i} = \frac{4\pi i}{64i} = \frac{\pi}{16}.$$

1.(*e*) Let the firm decide to manufacture x_1 units of product A and x_2 units of product B. To produce these units of products A and B, it requires $x_1 + x_2$ hours of processing time on M_1, $2x_1 + x_2$ hours of processing time on M_2. But the availability of these two machines M_1 and M_2 are 450 minutes and 600 minutes; respectively, therefore the constraints are

$$x_1 + x_2 \leq 450$$

$$2x_1 + x_2 \leq 600$$

$$x_1, x_2 \geq 0$$

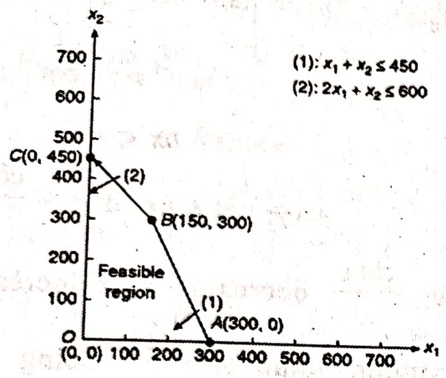
Since the profit from product A is ₹ 3 per unit and from product B is ₹ 4 per unit the total profit is ₹ $3x_1 + 4x_2$. The objective is to maximise the profit $3x_1 + 4x_2$. Hence, the LPP is

$$\text{Maximise } z = 3x_1 + 4x_2$$

subject to $\quad x_1 + x_2 \leq 450$

$$2x_1 + x_2 \le 600$$
$$x_1, x_2 \ge 0$$

Graphical solution: The graph of the LPP is given in Fig.



The feasible region is OABC. The vertices of solution space are O(0, 0), A(300, 0), B(150, 300) and C(0, 450). The values of the objective function at the solution points is given by the Table.

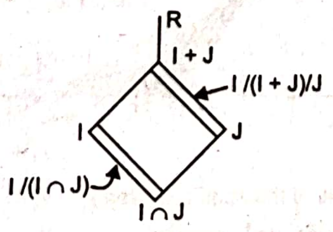
Table

Vertex	Value of z
O(0, 0)	0
A(300, 0)	900
B(150, 300)	1,650
C(0, 450)	1,800

Since the problem is of maximisation and the maximum value of z is attained at a single vertex, this problem has a unique optimal solution. The optimal solution is

$x_1 = 0$, $x_2 = 450$ and maximum of $z = 1,800$.

2.(a) Here is a picture showing the inclusions. The double lines represent the two factor rings.



To prove the result,

Define $\theta : I \to (I + J)/J$ by $i \to i + J$.

Since every coset in $(I + J)/J$ has a representative of this form θ is onto.

If $i \in \ker(\theta)$ then $i + J = J$ and so $i \in J$ and hence is in $I \subseteq J$.

2.(b) 0 is the only point of infinite discontinuity.

Let $f(x) = \log(\sin x)$

$\therefore \quad -f(x) = -\log(\sin x) \ge 0$

for all $x \in \left(0, \frac{\pi}{2}\right]$

Let $g(x) = \frac{1}{\sqrt{x}} > 0$ for all $x \in \left(0, \frac{\pi}{2}\right]$

$$\lim_{x\to 0^+} \frac{-f(x)}{g(x)} = \lim_{x\to 0^+} \frac{-\log\sin x}{x^{-1/2}}$$

$$= \lim_{x\to 0^+} \frac{-\cot x}{-\frac{1}{2}x^{-3/2}}$$

(L' Hospital Rule)

$$= \lim_{x\to 0^+} \frac{2x^{3/2}}{\tan x} = \lim_{x\to 0^+} \frac{3\sqrt{x}}{\sec^2 x}$$

(L' Hospital Rule)

and $\int_0^{\pi/2} g(x)dx = \int_0^{\pi/2} \frac{dx}{\sqrt{x}}$ is convergent as $p = \frac{1}{1} < 1$

Hence, by Comparison test,

$\int_0^{\pi/2} -f(x)dx$ is convergent.

$\therefore \int_0^{\pi/2} f(x)dx = \int_0^{\pi/2} \log(\sin x)\,dx$ is convergent.

Take, $g(x) = \frac{1}{x^m}$, $0 < m < 1$

$$\therefore \lim_{x\to 0^+} \frac{-f(x)}{g(x)} = \lim_{x\to 0^+} \frac{-\log\sin x}{\frac{1}{x^m}} \quad \left(\frac{\infty}{\infty}\right)$$

$$= -\lim_{x\to 0^+} \frac{\log\sin x}{x^{-m}} = -\lim_{x\to 0^+} \frac{\frac{1}{\sin x}\cos x}{-mx^{-m-1}}$$

$$= \lim_{x\to 0^+} \left(\frac{\cos x}{1} \cdot \frac{x}{\sin x} \cdot x^m\right) = 0.$$

2.(c) If $|f(z)| = |u + iv| = C$, then $u^2 + v^2 = C^2$.

Differentiating, we have,

$$uu_x + vv_x = 0,$$
$$uu_y + vv_y = 0, \quad \text{...(i)}$$

An application of Cauchy-Riemann equations to (*i*) yields

$$uu_x - vu_y = 0$$
$$uu_y + vu_x = 0 \quad ...(ii)$$

Eliminating u_y from (*ii*), we get

$$(u^2 + v^2)u_x = 0$$

so that $u_x = 0$. In a similar manner, we can show that $u_y = v_x = v_y = 0$

Thus, we observe that

$$0 = f'(z) = u_x + iu_y$$

which gives that u and v are constants.

3.(*a*) We have,

$$(ab\,a^{-1})^2 = ab\,a^{-1}\,ab\,a^{-1} = ab^2\,a^{-1}$$
$$= aa\,ba^{-1}\,a^{-1} \quad [\because ab\,a^{-1} = b^2]$$
$$= a^2ba^{-2}$$
$$\therefore (ab\,a^{-1})^4 = \{(aba^{-1})^2\}^2 = (a^2ba^{-2})^2$$
$$= a^2ba^{-2}a^2ba^{-2} = a^2b^2a^{-2}$$
$$= a^2aba^{-1}a^{-2} = a^3ba^{-3}$$
$$\therefore (a\,ba^{-1})^8 = \{(aba^{-1})^4\}^2 = (a^3ba^{-3})^2$$
$$= a^3ba^{-3}a^3ba^{-3} = a^3b^2a^{-3}$$
$$= a^3ab\,a^{-1}\,a^{-3} = a^4\,ba^{-4}$$
$$\therefore (a\,ba^{-1})^{16} = \{(aba^{-1})^8\}^2 = (a^4ba^{-4})^2$$
$$= a^4ba^{-4}a^4ba^{-4} = a^4b^2a^{-4}$$
$$= a^4a\,ba^{-1}\,a^{-4} = a^5\,ba^{-5} = ebe$$
$$[\because a^5 = e \text{ and so } a^{-5} = e]$$
$$= b.$$

Thus, $(aba^{-1})^{16} = b$.

$$\therefore \quad (b^2)^{16} = b \quad [\because aba^{-1} = b^2]$$
$$\Rightarrow \quad b^{32} = b \Rightarrow b^{31} = e$$

Since, $bm = e \Rightarrow 0(b) \mid m$,

therefore $0(b) \mid 31$ But 31 is a prime integer.

Therefore $0(b) = 1$ or 31

So, if $b = e$, then $0(b) = 1$ and if $b \neq e$, then $0(b) = 31$.

3.(*b*) Here, $f_n(x) = \tan^{-1}nx,\ x \geq 0$

$$\therefore \quad f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \tan^{-1} nx$$

$$= \begin{cases} \dfrac{\pi}{2}, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \end{cases}$$

Let $\varepsilon > 0$ be given.

For $x > 0$,

$$|f_n(x) - f(x)| = \left|\tan^{-1} nx - \frac{\pi}{2}\right| = |\cot^{-1} nx|$$
$$\left[\because \tan^{-1} nx + \cot^{-1} nx = \frac{\pi}{2}\right]$$
$$= \cot^{-1} nx < \varepsilon$$

If $\quad nx > \cot \varepsilon$ i.e, if $n > \dfrac{\cot \varepsilon}{x}$

Now $\dfrac{\cot \varepsilon}{x}$ decreases as x increases, the maximum value of $\dfrac{\cot \varepsilon}{x}$ being $\dfrac{\cot \varepsilon}{a}$ in $[a, b]$, $a > 0$.

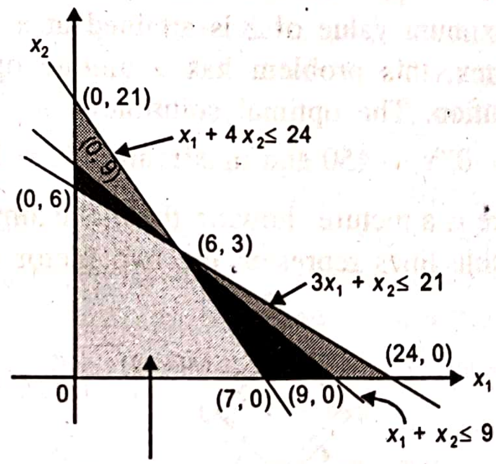
If we choose a positive integer m just $\geq \dfrac{\cot \varepsilon}{a}$, then

$|f_n(x) - f(x)| < \varepsilon \ \forall\ n \geq m$ and $\forall\ x \in [a, b]$, $a > 0$

But as $x \to 0$, $\dfrac{\cot \varepsilon}{x} \to \infty$ so that it is not possible to choose a positive integer m such that $|f_n(x) - f(x)| < \varepsilon \ \forall\ n \geq m$ and $\forall\ x \geq 0$.

Hence, $<f_n>$ is not uniformly convergent on $[0, b]$ but is only pointwise convergent on $[0, b]$.

3.(*c*)



Now, critical point

Given equation:

$$x_1 + 4x_2 \leq 24$$
$$3x_1 + x_2 \leq 21$$
$$x_1 + x_2 \leq 9$$
$$x_1, x_2 \geq 0$$

On solving all, we get the following critical point:

$C_1 = (0, 6)$, $C_2 = (6, 3)$, $C_3 = (7, 0)$

$$z = 2x_1 + 5x_2$$

$$z(0, 6) = 2 \times 0 + 5 \times 6$$
$$= 30 \quad \text{(Maximum)}$$
$$z(6, 3) = 2 \times 6 + 5 \times 3$$
$$= 12 + 15 = 27$$
$$z(7, 0) = 2 \times 7 + 5 \times 0$$
$$= 11 \quad \text{(Minimum)}$$

$\therefore$ z is maximum at $(0, 6) = 30$.

5.(a)
$$\frac{\partial^2 z}{\partial x^2} - \frac{\partial^2 z}{\partial y^2} = x - y$$

Let
$$\frac{\partial^2 z}{\partial x^2} - x = \frac{\partial^2 z}{\partial y^2} - y = a$$

Now,
$$\frac{\partial^2 z}{\partial x^2} - x = a$$

$$\Rightarrow \quad \frac{\partial^2 z}{\partial x^2} = (a + x)$$

$$\Rightarrow \quad \frac{\partial z}{\partial x} = ax + \frac{x^2}{2}$$

$$\therefore \quad f_1(x, a) = ax + \frac{x^2}{2}$$

and
$$\frac{\partial^2 z}{\partial y^2} - y = a$$

$$\Rightarrow \quad \frac{\partial^2 z}{\partial y^2} = (a + y)$$

$$\Rightarrow \quad \frac{\partial z}{\partial y} = ay + \frac{y^2}{2}$$

$$f_2(y, a) = ay + \frac{y^2}{2}$$

$\therefore$ The complete integral is

$$z = \int f_1(x,a).dx + \int f_2(y,a).dy + b$$

$$= \int\left(ax + \frac{x^2}{2}\right).dx + \int\left(ay + \frac{y^2}{2}\right).dy + b$$

$$z = \frac{ax^2}{2} + \frac{x^3}{6} + \frac{ay^2}{2} + \frac{y^3}{6} + b$$

$$z = x^3 + y^3 + 3a(x^2 + y^2) + b$$

5.(b) The value of table for x and y

x	y
0.61	1.840431
0.62	1.858928
0.63	1.87761
0.64	1.896481
0.65	1.915541

Newton's forward difference interpolation method to find solution

Newton's forward differnce table is

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
0.61	1.840431				
		0.018497			
0.62	1.858928		0.000185		
		0.018682		0.000004	
0.63	1.87761		0.000189		-0.000004
		0.018871		0	
0.64	1.896481		0.000189		
		0.01906			
0.65	1.915541				

The value of x at you want to find the $f(x)$:

$$x = 0.612$$
$$h = x_1 - x_0 = 0.62 - 0.61 = 0.01$$
$$p = \frac{x - x_0}{h} = \frac{0.612 - 0.61}{0.01} = 0.2$$

Newton's forward difference interpolation formula is

$$y(x) = y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}.\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}.\Delta^4 y_0$$

$$y(0.612) = 1.840431 + 0.2 \times 0.018497 + \frac{0.2(0.2-1)}{2} \times 0.000185 + \frac{0.2(0.2-1)(0.2-2)}{6} \times 0.000004 + \frac{0.2(0.2-1)(0.2-2)(-0.2-3)}{24} \times -0.000004$$

$$y(0.612) = 1.840431 + 0.003699 - 0.000015 + 0 + 0$$

$$y(0.612) = 1.844116$$

Solution of newton's forward interpolation method $y(0.612) = 1.844116$.

5.(c) Here $f(x) = y,\ n = 6$

$$h = \frac{b-a}{n} = \frac{3-0}{6} = 0.5$$

The value of x and $f(x)$ are given below

x_i	y_i
0	$y_0 = 0$
0.5	$y_1 = 0.75$
1.0	$y_2 = 1$
1.5	$y_3 = 0.75$
2.0	$y_4 = 0$
2.5	$y_5 = -1.25$
3.0	$y_6 = -3.0$

From Simpson's one-third rule, we get,

$$I_S = \int_0^3 y.dx$$

$$= \frac{h}{3}[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)]$$

$$= \frac{0.5}{3}[(0-3) + 4(0.75 + 0.75 - 1.25) + 2(1+0)]$$

$$= \frac{0.5}{3}[-3+1+2] = 0.$$

5.(d) Velocity components are given by

$$u = \frac{\partial\psi}{\partial y} = \frac{\partial\left(a(x^2-y^2)\right)}{\partial y} = -2ay$$

$$v = \frac{-\partial\psi}{\partial x} = \frac{-\partial\left(a(x^2-y^2)\right)}{\partial x} = -2ax$$

Substituting this in the irrotationality condition, we have,

$$\frac{\partial v}{\partial x} - \frac{\partial v}{\partial y} = \frac{\partial(-2ax)}{\partial x} - \frac{\partial(-2ay)}{\partial y}$$

$$-\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2} = -2a + 2a = 0$$

which lead to the condition that

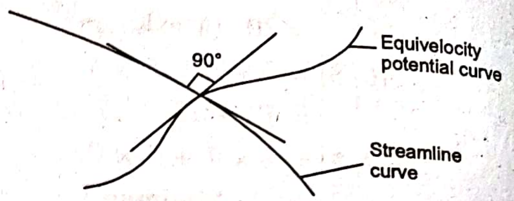
$\Delta^2\psi = 0$ for irrotational flow

Streamlines and equipotential lines are orthogonal to each other.

We have seen that velocity component of the flow are given in terms of velocity potential and stream function by the equation,

$$u = \frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y} = -2ay$$

and $$v = \frac{\partial\phi}{\partial y} = \frac{-\partial\psi}{\partial x} = -2ax$$



We see that equivelocity potential curve and streamline flow curve are orthogonal.

5.(e) Let $f(x, y, z, p, q) \equiv p^2x + q^2y - z = 0$

$$p^2x + q^2y = z \qquad ...(1)$$

Now, $$\frac{\partial f}{\partial x} = p^2;\ \frac{\partial f}{\partial p} = 2px$$

$$\frac{\partial f}{\partial y} = q^2;\ \frac{\partial f}{\partial q} = 2qy$$

$$\frac{\partial f}{\partial z} = -1$$

Consider Charpit's auxiliary equations

$$\frac{dp}{\frac{\partial f}{\partial x} + p\frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q\frac{\partial f}{\partial z}}$$

$$= \frac{dz}{-\left(p\frac{\partial f}{\partial p} + q\frac{\partial f}{\partial q}\right)} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dF}{0}$$

$$\Rightarrow \frac{dp}{p^2 + (-p)} = \frac{dq}{q^2 - q} = \frac{dz}{-(2p^2x + 2q^2y)}$$

$$= \frac{dx}{-2px} = \frac{dy}{-2qy} = \frac{dF}{0}$$

Consider,

$$\frac{p^2dx + 2pxdp}{-2p^3x + 2p^3x - 2p^2x} = \frac{q^2dy + 2qydy}{-2q^3y + 2q^3y - 2q^2y}$$

$$\Rightarrow \frac{p^2dx + 2pxdp}{p^2x} = \frac{q^2dy + 2qydy}{q^2y}$$

Integrating,

$$\int \frac{p^2dx + 2bxdp}{p^2x} = \int \frac{q^2dy + 2qydy}{q^2y}$$

$\Rightarrow \quad \log(p^2x) = \log(q^2y) + \log c = \log q^2yc$

$$\left| \int \frac{f'(x)}{f(x)} dx = \log f(x) \right.$$

$\Rightarrow \quad p^2x = cq^2y \quad \text{...(2)}$

Now from (1), $p^2x + q^2y = z$

$\Rightarrow \quad cq^2y + q^2y = z$

$\Rightarrow \quad (c+1)q^2y = z$

$$\Rightarrow q^2 = \frac{z}{y(c+1)} \Rightarrow q = \sqrt{\frac{z}{y(c+1)}}$$

Hence, from (2), $p^2x = cy.\dfrac{z}{y(c+1)} = \dfrac{c}{c+1}z$

$$\Rightarrow \quad p = \sqrt{\frac{c}{c+1}.\frac{z}{x}}$$

Now consider,

$$dz = pdx + qdy$$

$$\Rightarrow \quad dz = \sqrt{\frac{c}{c+1}.\frac{z}{x}}dx + \sqrt{\frac{1}{c+1}.\frac{z}{y}}dy$$

$$\Rightarrow \quad z^{-1/2}dz = \sqrt{\frac{c}{c+1}}.x^{-1/2}dx + \sqrt{\frac{c}{c+1}}.y^{-1/2}dy$$

Integrating,

$$2\sqrt{z} = \sqrt{\frac{c}{c+1}}2\sqrt{x} + \frac{1}{\sqrt{c+1}}.2y^{1/2} + b$$

$$\Rightarrow \sqrt{c+1}\sqrt{z} = \sqrt{cx} + \sqrt{y} + \frac{b}{2\sqrt{c+1}} \text{ (say)}$$

$$\left| \text{ Take } a = \frac{b}{2\sqrt{c+1}} \right.$$

$$\Rightarrow \sqrt{c+1}\sqrt{z} = \sqrt{cx} + \sqrt{y} + a,$$

is the required complete solution.

6.(a) For this equation,

$X = \{z(z + y^2), z(z + x^2), -xy(x + y)\}$

curl $X = 2(-x^2 - xy - z, y^2 + xy + z, zx - zy)$

and it is soon verified that X . curl $X = 0$, showing that the equation is integrable. An inspection of the equation suggests that it is probably simplest to take $dy = 0$ in Natani's method.

The equation then becomes

$$\left\{\frac{1}{x} - \frac{1}{x+y}\right\}dx + \left\{\frac{1}{z+y^2} - \frac{1}{z}\right\}dz = 0$$

Showing that it has the solution

$$\frac{x(y^2+z)}{z(x+y)} = f(y) \quad \text{...}(i)$$

If we now let $z = 1$ in the original equation, we see that it reduces to the simple form

$$\frac{dx}{1+x^2} + \frac{dy}{1+y^2} = 0 \quad \text{...}(ii)$$

with solution

$\tan^{-1} x + \tan^{-1} y = \text{const.}$

Writing $\tan^{-1}(1/c)$ for the constant and making use of the addition formula

$$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\frac{x+y}{1-xy}$$

we see that the solution of equation (*ii*) is

$$\frac{1-xy}{x+y} = c \quad \text{...}(iii)$$

This solution must be the form assumed by (*i*) in the case $z = 1$; in other words, (*iii*) must be equivalent to the relation

$$\frac{x(y^2+1)}{x+y} = f(y) \quad \text{...}(iv)$$

Eliminating x between equations (*iii*) and (*iv*), we find that

$$f(y) = 1 - cy$$

Substituting this expression in equation (*i*), we find that the solution of the equation is

$$x(y^2 + z) = z(x + y)(1 - cy).$$

6.(b) $A = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 3 & 1 & 6 \\ 1 & -1 & -1 & -3 \end{bmatrix}$

Make zeros in column 1 except the entry at row 1, column 1 (pivot entry).

Subtract row 1 multiplied by 2 from row 2 ($R_2 = R_2 - (2)\, R_1$):

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & -1 & 0 \\ 1 & -1 & -1 & -3 \end{bmatrix}$$

Subtract row 1 from row 3 ($R_3 = R_3 - R_1$):

$$\begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & -1 & 0 \\ 1 & -2 & -2 & -6 \end{bmatrix}$$

Make zeros in column 2 excet the entry at row 2, column 2 (pivot entry).

Subtract row 2 rom row 1 ($R_1 = R_1 - R_2$):

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & 1 & -1 & 0 \\ 1 & -2 & -2 & -6 \end{bmatrix}$$

Add row 2 multiplied by 2 to row 3 ($R_3 = R_3 + (2)\,R_2$):

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & 0 \\ 1 & 0 & -4 & -6 \end{bmatrix}$$

Make zeros in column 3 except the entry at row 3, column 3 (pivot entry).

Divide row 3 by $-4\left(R_3 = \dfrac{R_3}{-4}\right)$

$$\begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & \frac{3}{2} \end{bmatrix}$$

Subtract row 3 multiplied by 2 from row 1 ($R_1 = R_1 - (2)\,R_3$):

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & \frac{3}{2} \end{bmatrix}$$

Add row 3 to row 2 ($R_2 = R_2 + R_3$):

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & \frac{3}{2} \\ 0 & 0 & 1 & \frac{3}{2} \end{bmatrix}$$

Hence, X1 = 0, $X_2 = 3/2$ and $X_3 = 3/2$.

6.(*c*) **Define: Lagrangian Function**

- L = T − V (Kinetic − Potential energies)

Lagrange's Equation

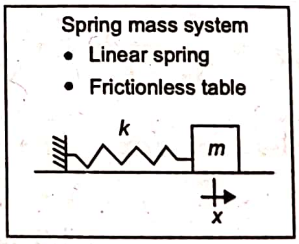
- For conservative systems

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

- Results in the differential equations that describe the equations of motion of the system.

Key point:

- Newton approach requires that you find accelerations in all 3 directions, equate F = *ma*, solve for the constraint forces, and then eliminate these to reduce the problem to "characteristic size".
- Lagrangian approach enables us to immediately reduce the problem to this "characteristic size" ⇒ we only have to solve for that many equations in the first place. The ease of handling constraints really differentiates the two approaches.
- Spring – mass system



- Lagrangian L = T − V

$$L = T - V = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

- Lagrange's Equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

- Do the derivatives

$$\frac{\partial L}{\partial \dot{q}_i} = m\dot{x}, \quad \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) = m\ddot{x}, \quad \frac{\partial L}{\partial q_i} = -kx$$

- Put it all together

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = m\ddot{x} + kx = 0$$