Mains Test Series – 2018

Test – 9 (Paper-I)

Answer Key

1(a). If A, B are square matrices each of order n and I is the corresponding unit matrix, show that the equation $AB - BA = I$ can never hold.

Sol'n: Let us suppose that $AB - BA = I$

Then $AB - BA = I$

$\Rightarrow \text{trace}(AB - BA) = \text{trace } I$

$\Rightarrow \text{trace}(AB) - \text{trace}(BA) = n$ $\quad$ ($\because$ trace I = sum of the elements of I lying along its principal diagonal $= n$]

$\Rightarrow 0 = n$ $\quad$ $[\text{trace}(AB) = \text{trace}(BA)]$

which is not possible because n is a positive integer.

Hence our assumption that $AB - BA = I$ is wrong and so the equation $AB - BA = I$ can never hold.

1(b). If the product of two non-zero square matrices is a zero matrix, show that both of them must be singular matrices.

Sol'n: Let A and B be two non-zero matrices.

of the type $n \times n$.

It is given that AB = a null matrix.

i.e $AB = 0$

we have to show that both of matrices A and B are singular.

Let us suppose that B is non-singular

i.e. $|B| \neq 0$

$\Rightarrow B^{-1}$ exists.

Post multiplying bothsides of $AB=0$ by B^{-1}

we get $(AB)B^{-1} = 0$

$\Rightarrow A(BB^{-1}) = 0$

$\Rightarrow AI_n = 0$

$\Rightarrow A = 0$

But A is not a zero matrix.

Hence $|B| = 0$

i.e. B is singular matrix.

Now suppose $|A| \neq 0$ i.e A is non-singular

$\Rightarrow A^{-1}$ exists

So premultiplying bothsides of

$AB = 0$ by A^{-1}, we get

$A^{-1}(AB) = 0$

$\Rightarrow (A^{-1}A)B = 0$

$\Rightarrow I_n B = 0$

$\Rightarrow B = 0$

But B is not a zero matrix.

Hence $|A| = 0$.

i.e. A is non-singular matrix.

1(c) If $f(x)$ be real value and differentiable on R and $f(x+y) = \dfrac{f(x)+f(y)}{1-f(x)f(y)}$ then $f(x) = \tan(xf'(0))$.

Sol'n: Here $x=y=0$, gives $f(0)=0$, and $y=-x$ gives $f(-x) = -f(x)$ and

$$\frac{f(y-x)}{y-x} = \frac{f(y)-f(x)}{y-x} \cdot \frac{1}{1+f(x)f(y)}, \text{ as } y \to x$$

$$\Rightarrow f'(0) = f'(x) / \{1+f'(x)\}$$

$$\Rightarrow \frac{df}{1+f^2} = f'(0)\,dx$$

$$\Rightarrow \tan^{-1} f(x) = xf'(0) + a, \text{ and for } x=0 \text{ it gives } a=0$$

Hence, $\tan^{-1} f(x) = xf'(0)$

i.e. $f(x) = \tan(xf'(0))$

1(d) If $z = (x+y)\phi(y/x)$, where ϕ is any arbitrary function. Prove that $x\dfrac{\partial z}{\partial x} + y\dfrac{\partial z}{\partial y} = z$.

Sol'n: $\dfrac{\partial z}{\partial x} = \phi(y/x) + (x+y)\phi'(y/x) - (-y/x^2)$

$$\Rightarrow x\frac{\partial z}{\partial x} = x\phi(y/x) - y/x\,(x+y)\,\phi'(y/x) \quad \text{--- (1)}$$

Also $\dfrac{\partial z}{\partial y} = \phi(y/x) + (x+y)\phi'(y/x)(1/x)$

$$\Rightarrow y\frac{\partial z}{\partial y} = y\phi(y/x) + y/x\,(x+y)\,\phi'(y/x) \quad \text{--- (2)}$$

Adding (1) & (2), we get

$$x\frac{\partial z}{\partial y} + y\frac{\partial z}{\partial y} = (x+y)\phi(y/x) = z$$

1(e). Find the equation of the sphere circumscribing the tetrahedron whose faces are $\frac{y}{b}+\frac{z}{c}=0$, $\frac{z}{c}+\frac{x}{a}=0$, $\frac{x}{a}+\frac{y}{b}=0$, $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

Sol'n: the given faces are $\frac{y}{b}+\frac{z}{c}=0$ — ①, $\frac{z}{c}+\frac{x}{a}=0$ — ②

$\frac{x}{a}+\frac{y}{b}=0$ — ③, $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ — ④

Solving ①, ②, ③ we get $(0,0,0)$ as one vertex of the tetrahedron.

To solve (1), (2) and (4): Subtracting (1) from (4), we get $\frac{x}{a}=1 \Rightarrow x=a$.

subtracting ② from (4), we get $\frac{y}{b}=1$
$\Rightarrow y=b$

$\therefore$ From ①, $\frac{z}{c}=\frac{-y}{b}=\frac{-b}{b}=-1 \Rightarrow z=-c$.

Hence $(a,b,-c)$ is another vertex.

Similarly solving (1), (3), ④ and (2), (3), (4), we get $(a,-b,c)$ and $(-a,b,c)$ as the other two vertices.

Hence the vertices of the tetrahedron are

$(0,0,0)$, $(a,b,-c)$, $(a,-b,c)$ and $(-a,b,c)$

Let the equation of the circumscribing sphere be

$x^2+y^2+z^2+2ux+2vy+2wz+d=0$ — ⑤

Since this passes through the origin

$\therefore d=0$ — ⑥

⑤ passes through $(a,b,-c)$.

$\therefore a^2+b^2+c^2+2ua+2bv-2cw+d=0$

(or) $a^2+b^2+c^2+2au+2bv-2cw=0$ — ⑦ $(\because d=0)$

Similarly ⑤ passes through $(a,-b,c)$ and $(-a,b,c)$.

$\therefore a^2+b^2+c^2+2ua-2bv+2cw+d=0$

& $a^2+b^2+c^2-2ua+2bv+2cw+d=0$

(or)

$a^2+b^2+c^2+2ua-2bv+2cw=0$ — ⑧

& $a^2+b^2+c^2-2ua+2bv+2cw=0$ — ⑨ (by using $d=0$)

Adding ⑦ and ⑧, we get

$$2(a^2+b^2+c^2)+4ua=0$$

$$\Rightarrow 2u = -\left(\frac{a^2+b^2+c^2}{a}\right)$$

Adding ⑧ and ⑨, we get

$$2(a^2+b^2+c^2)+4cw=0$$

$$\Rightarrow 2w = -\left(\frac{a^2+b^2+c^2}{c}\right)$$

Adding ⑦ and ⑨, we get

$$2(a^2+b^2+c^2)+4bv=0$$

$$\Rightarrow 2v = -\left(\frac{a^2+b^2+c^2}{b}\right).$$

putting these values of u, v, w in ⑤, we get

$$x^2+y^2+z^2-\frac{a^2+b^2+c^2}{a}x-\frac{a^2+b^2+c^2}{b}y-\frac{a^2+b^2+c^2}{c}z=0$$

$$\Rightarrow \frac{x^2+y^2+z^2}{a^2+b^2+c^2}-\frac{x}{a}-\frac{y}{b}-\frac{z}{c}=0$$

which is the required equation of the sphere.

2(a) Consider the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^2$ defined by $T(x,y,z) = (x-y, x+z)$. Find the matrix of T w.r.t to the bases (u_1, u_2, u_3) and (u_1', u_2') of of $\mathbb{R}^3$ and $\mathbb{R}^2$, where $u_1 = (1,-1,0)$, $u_2 = (2,0,1)$, $u_3 = (1,2,1)$ and $u_1' = (-1,0)$, $u_2' = (0,1)$. Use this matrix to find the image of the vector $u = (3,-4,0)$.

Sol^n: Let $T: \mathbb{R}^3 \to \mathbb{R}^2$ be the given linear transformation defined by

$$T(x,y,z) = (x-y, x+z) \quad \text{---} (1)$$

Let $S_1 = \{u_1, u_2, u_3\}$ and $S_2 = \{u_1', u_2'\}$ be the bases sets of $\mathbb{R}^3$ and $\mathbb{R}^2$ where

$u_1 = (1,-1,0)$, $u_2 = (2,0,1)$, $u_3 = (1,2,1)$

$u_1' = (-1,0)$, $u_2' = (0,1)$.

Let $\alpha = (a,b) \in \mathbb{R}^2$ then

$$(a,b) = x(-1,0) + y(0,1) \quad \text{---} (2)$$

$\Rightarrow x = -a$ and $y = b$.

$$\therefore (2) \equiv (a,b) = -a(-1,0) + b(0,1) \quad \text{---} (3)$$

Now we have

$$T(1,-1,0) = (2,1) = -2(-1,0) + 1(0,1)$$
$$T(2,0,1) = (2,3) = -2(-1,0) + 3(0,1)$$
$$T(1,2,1) = (-1,2) = 1(-1,0) + 2(0,1)$$

$\therefore$ The matrix of L.T (1) w.r.t. given bases S_1 and S_2 is

$$[T : S_1\ S_2] = \begin{bmatrix} -2 & -2 & 1 \\ 1 & 3 & 2 \end{bmatrix}.$$

Let us find the image of the vector $u = (3,-4,0)$ by using the above matrix.

Now we have

$$(3,-4,0) = p(1,-1,0) + q(2,0,1) + r(1,2,1). \quad \text{(4)}$$

$$\Rightarrow p + 2q + r = 3$$
$$p + 2r = -4$$
$$q + r = 0$$
$$\therefore p = 2/3,\ q = 7/3,\ r = -7/3$$

$\therefore$ The image $(3,-4,0)$ by using the above matrix is

$$\begin{bmatrix} -2 & -2 & 1 \\ 1 & 3 & 2 \end{bmatrix} \begin{bmatrix} 2/3 \\ 7/3 \\ -7/3 \end{bmatrix} = \begin{bmatrix} -25/3 \\ 3 \end{bmatrix}$$

we have

$$T(u) = -\frac{25}{3}u_1' + 3u_2'$$
$$= -\frac{25}{3}(-1,0) + 3(0,1)$$
$$= \left(\frac{25}{3}, 3\right)$$

$p = 2,\ q = 1,\ r = -1$

$\begin{pmatrix} -2 & -2 & 1 \\ 1 & 3 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} -7 \\ 3 \end{pmatrix}$

image of $(3,-4,0)$ $= -7u_1' + 3u_2'$ $= (7, 3)$

2(b) → Prove that the following

(i) $\frac{x}{1+x^2} < \tan^{-1}x < x, \ \forall x>0$

(ii) $|\tan^{-1}x - \tan^{-1}y| < |x-y|$, $\forall$ unequal $x, y \in \mathbb{R}$.

Sol^n: Let $f(x) = \tan^{-1}x - \frac{x}{1+x^2}$

$\therefore f(0) = 0$

we have $f'(x) = \frac{1}{1+x^2} - 1\cdot\frac{1}{1+x^2} - x\frac{(-1)}{(1+x^2)^2}\cdot 2x$

$= \frac{2x^2}{(1+x^2)^2}$

Note that $f'(x) > 0$ for all $x > 0$.

Hence $f(x)$ is monotonically increasing in the interval $[0, \infty)$.

$\therefore f(x) > f(0)$ for all $x > 0$.

$\Rightarrow \tan^{-1}x - \frac{x}{1+x^2} > 0 \quad \forall x>0$

$\Rightarrow \tan^{-1}x > \frac{x}{1+x^2} \quad \forall x>0$ — ①

Again, let $\phi(x) = x - \tan^{-1}x$

$\therefore \phi(0) = 0$

we have $\phi'(x) = 1 - \frac{1}{1+x^2}$

$= \frac{x^2}{1+x^2} > 0$ for all $x>0$

$\therefore \phi(x)$ is monotonically increasing in the interval $[0, \infty)$.

$\therefore \phi(x) > \phi(0) \quad \forall x>0$

i.e. $x - \tan^{-1}x > 0 \quad \forall x>0$

$\Rightarrow x > \tan^{-1}x \quad \forall x>0$ — ②

from ① & ②, we get

$\frac{x}{1+x^2} < \tan^{-1}x < x \quad \forall x>0$

(ii) Sol'n: Let $f(x) = \tan^{-1} t$ in $[x, y]$, where $x < y$

By Lagrange's mean value theorem, there exist some $c \in (x, y)$ such that

$$\frac{f(y) - f(x)}{y - x} = f'(c)$$

$$\Rightarrow f(y) - f(x) = (y - x)\frac{1}{1+c^2} \quad \left(\because f'(c) = \frac{1}{1+c^2}\right)$$

$$\Rightarrow f(y) - f(x) \le (y - x) \quad \left(\because 1+c^2 \ge 1 \Rightarrow \frac{1}{1+c^2} \le 1\right)$$

Similarly,

$$f(x) - f(y) \le (x - y), \text{ when } y < x$$

$$\therefore |f(x) - f(y)| \le |x - y|$$

Hence $|\tan^{-1}x - \tan^{-1}y| \le |x - y| \quad \forall x, y \in \mathbb{R}$.

2(c) Find the locus of the points from which three mutually ⊥lar tangents can be drawn to the paraboloid $\left(\frac{x^2}{a^2}\right) - \left(\frac{y^2}{b^2}\right) = 2z$.

Sol'n: [Note:

Here we are to apply the condition that the enveloping cone, of the given paraboloid, with vertex at (α, β, γ) may have three mutually ⊥lar generators and we know that the condition for the same is that the sum of the coefficients of x^2, y^2 and z^2 in the equation of the cone is zero]

we are to apply the condition that the enveloping cone, of the given paraboloid, with vertex at (α,β,γ) may have three mutually perpendicular generators.

Now the equation of the enveloping cone of the given paraboloid with vertex at the point (α,β,γ) is $SS_1 = T^2$ ——— ①

where $S = \frac{x^2}{a^2} - \frac{y^2}{b^2} - 2z$

$S_1 = \frac{\alpha^2}{a^2} - \frac{\beta^2}{b^2} - 2\gamma$

and $T = \frac{\alpha x}{a^2} - \frac{\beta y}{b^2} - (z+\gamma)$

∴ From ①, the equation of the enveloping cone of the given paraboloid with vertex at (α,β,γ) is

$$\left(\frac{x^2}{a^2} - \frac{y^2}{b^2} - 2z\right)\left(\frac{\alpha^2}{a^2} - \frac{\beta^2}{b^2} - 2\gamma\right) = \left(\frac{\alpha x}{a^2} - \frac{\beta y}{b^2} - z - \gamma\right)^2$$

Also we know that if this cone has three mutually perpendicular generators then sum of the coefficients of x^2, y^2, z^2 in it must be zero.

$$\left[\frac{1}{a^2}\left(\frac{\alpha^2}{a^2} - \frac{\beta^2}{b^2} - 2\gamma\right) - \frac{\alpha^2}{a^4}\right] + \left[\frac{-1}{b^2}\left(\frac{\alpha^2}{a^2} - \frac{\beta^2}{b^2} - 2\gamma\right) - \frac{\beta^2}{b^4}\right] + (-1) = 0$$

$$\Rightarrow -\frac{1}{a^2}\left(\frac{\beta^2}{b^2} + 2\gamma\right) - \frac{1}{b^2}\left(\frac{\alpha^2}{a^2} - 2\gamma\right) - 1 = 0$$

$$\Rightarrow \alpha^2 + \beta^2 - 2\gamma(a^2 - b^2) + a^2b^2 = 0$$

∴ Required locus of the point (α,β,γ) is

$$x^2 + y^2 - 2(a^2 - b^2)z + a^2b^2 = 0$$

3(a) Let $M = \begin{bmatrix} 1+i & 2i & i+3 \\ 0 & 1-i & 3i \\ 0 & 0 & i \end{bmatrix}$. Determine the eigen values of the matrix $B = M^2 - 2M + I$

Sol^n.

$$M^2 = \begin{bmatrix} 1+i & 2i & i+3 \\ 0 & 1-i & 3i \\ 0 & 0 & i \end{bmatrix}\begin{bmatrix} 1+i & 2i & i+3 \\ 0 & 1-i & 3i \\ 0 & 0 & i \end{bmatrix}$$

$$= \begin{bmatrix} (1+i)^2 & [(1+i)+(1-i)]2i & (i+3)(1+2i)+6i^2 \\ 0 & (1-i)^2 & 3i(1-i+i) \\ 0 & 0 & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2i & 4i & 7i-5 \\ 0 & -2i & 3i \\ 0 & 0 & -1 \end{bmatrix}$$

Now

$B = M^2 - 2M + I$

$$= \begin{bmatrix} 2i & 4i & 7i-5 \\ 0 & -2i & 3i \\ 0 & 0 & -1 \end{bmatrix} - \begin{bmatrix} 2+2i & 4i & 2i+6 \\ 0 & 2-2i & 6i \\ 0 & 0 & 2i \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 5i-11 \\ 0 & -1 & -3i \\ 0 & 0 & -2i \end{bmatrix}$$

Let λ be the eigen value of B. Then, its augment matrix $|B-\lambda I| = 0$

$$\Rightarrow \begin{vmatrix} -1-\lambda & 0 & 5i-11 \\ 0 & -1-\lambda & -3i \\ 0 & 0 & -2i-\lambda \end{vmatrix} = 0$$

$$(1+\lambda)\left[(1+\lambda)(2i+\lambda)\right] = 0$$

$$\lambda = -1, \ \lambda = -1, \ \lambda = -2i$$

3(b) → Find the area in Positive quadrant enclosed b/w the four curves $a^2y=x^3$, $b^2y=x^3$, $p^2x=y^3$, $q^2x=y^3$.

Solⁿ: Let $x^3/y = X$, $y^3/x = Y$

$$\Rightarrow x^2y^2 = XY \text{ and } x^4/y^4 = X/Y$$

$$\Rightarrow x = X^{3/8}Y^{1/8},\quad y = X^{1/8}Y^{3/8}$$

$$\frac{\partial(x,y)}{\partial(X,Y)} = \begin{vmatrix} \left(\frac{3}{8}X^{-5/8}Y^{1/8}\right) & \left(\frac{1}{8}X^{3/8}Y^{-7/8}\right) \\ \left(\frac{1}{8}X^{-7/8}Y^{3/8}\right) & \left(\frac{3}{8}X^{3/8}Y^{-5/8}\right) \end{vmatrix}$$

$$= \frac{1}{8}(XY)^{-1/2}$$

Now the area $= \iint dx\,dy = \iint \frac{1}{8}(XY)^{-1/2}\,dX\,dY$.

Clearly the limits of X are from a^2 to b^2 and limits of Y are from p^2 to q^2.

$$\therefore \text{ the area} = \frac{1}{8}\int_{a^2}^{b^2}\int_{p^2}^{q^2} X^{-1/2}Y^{-1/2}\,dX\,dY$$

$$= \frac{1}{8}\int_{a^2}^{b^2} X^{-1/2}\cdot 2\left[Y^{1/2}\right]_{p^2}^{q^2} dX$$

$$= \frac{1}{4}\int_{a^2}^{b^2}(q-p)X^{-1/2}\,dX$$

$$= \frac{1}{2}(b-a)(q-p)$$

3(c). If $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ represents one of a set of three mutually $\perp^{lar}$ generators of the cone $5yz-8zx-3xy=0$ find the equation of other two.

Sol^n: The given cone is $5yz-8zx-3xy=0$ —— (1)
and one of its three $\perp^r$ generators is $\frac{x}{1}=\frac{y}{2}=\frac{z}{3}$ —— (2)
The other two $\perp^r$ generators are the lines which lie plane through the vertex $(0,0,0)$ and $\perp^{lar}$ to (2).
i.e. the plane $x+2y+3z=0$ —— (3)
cuts the cone (1)

Let a line of section of (1) and (3) be $\frac{x}{l}=\frac{y}{m}=\frac{z}{n}$ —— (4)

Since (4) lies on cone (1),
$\therefore$ its d.c.s satisfy the equation of the cone
and thus $5mn-8nl-3lm=0$ —— (5)
Since (4) lies on plane (3), it is $\perp^{lar}$ to the normal to the plane
$\therefore l+2m+3n=0$ —— (6)
$\Rightarrow l=-2m-3n$.
putting the value of l in (5),
we get $5mn+8n(2m+3n)+3m(2m+3n)=0$
$\Rightarrow 6m^2+30mn+24n^2=0$
$\Rightarrow (m+n)(m+4n)=0$
$\Rightarrow m+n=0$ or $m+4n=0$
i.e. $0l+m+n=0$ || $0l+m+4n=0$
from (6) $l+2m+3n=0$ || $l+2m+3n=0$
solving we get
$\frac{l}{1}=\frac{m}{1}=\frac{n}{-1}$ or $\frac{l}{5}=\frac{m}{-4}=\frac{n}{1}$
putting these values of l,m,n in (4), the lines of section (3) and (1) are
$\frac{x}{1}=\frac{y}{1}=\frac{z}{-1}$ & $\frac{x}{5}=\frac{y}{-4}=\frac{z}{1}$ which are the other two generators.

4(b). Transform the integral $I = \iiint (x+y+z)^n xyz\, dx\,dy\,dz$ taking over the volume bounded by $x=0, y=0, z=0, x+y+z=1$. Substituting $u = x+y+z$, $x+y = uv$, $y = uvw$, and hence evaluate its value.

Sol'n: we have

$$dx\,dy\,dz = u^2 v\, du\,dv\,dw$$

Also $x = uv - y = uv - uvw = uv(1-w)$

$y = uvw, z = u-(x+y)$

i.e. $z = u - uv = u(1-v)$

$$\therefore (x+y+z)^n xyz = u^n uv(1-w)\, uvw . u(1-v)$$
$$= u^{n+3} . v^2 w (1-v)(1-w)$$

For limits. when $x=0, y=0, z=0$ then $u=0$ $[\because x+y+z=u]$ and when $x+y+z=1$, $u=1$

$\therefore$ the limits of u are from 0 to 1.

Again $w = \frac{y}{uv}$, i.e. $w = \frac{y}{x+y}$, if $y=0$, $w=0$

and if $y = 1-x$; $w = 1-x$ i.e. $w=1$ at $x=0$ and $w=0$ at $x=1$. Thus the limits of w are from 0 to 1.

Now $v = \frac{x+y}{u} = \frac{x+y}{x+y+z}$

$\therefore v = 1$ if $z=0$

Also $x+y+z=1$

$\therefore v = x+y = 1$ at $z=1$

thus limits of v are from 0 to 1.

$\therefore$ The given integral I transforms to

$$= \int_0^1\int_0^1\int_0^1 u^{n+3} v^2 w(1-v)(1-w) . u^2 v\, du\,dv\,dw$$

$$= \int_0^1\int_0^1\int_0^1 u^{n+5}\,(v^3 - v^4)\,(w - w^2)\,du\,dv\,dw$$

$$= \frac{1}{n+6}\left(\frac{1}{4}-\frac{1}{5}\right)\left(\frac{1}{2}-\frac{1}{3}\right) = \frac{1}{120(n+6)}$$

4(c) Show that the enveloping cylinder of the ellipsoid $ax^2+by^2+cz^2=1$ with generators $\perp^r$ to z-axis meet the plane $z=0$ in parabolas.

Sol^n: The d.c's of the z-axis are 0, 0, 1.
$\therefore$ The d.r's of the line perpendicular to z-axis are $l, m, 0$.

Let $P(\alpha,\beta,\gamma)$ be a point on the enveloping cylinder. Then the equations of the generator through $P(\alpha,\beta,\gamma)$ are $\frac{x-\alpha}{l}=\frac{y-\beta}{m}=\frac{z-\gamma}{0}=r$ (say)

$\therefore$ Any point on it is $(\alpha+lr, \beta+mr, \gamma)$

If this point lies on the given conicoid, we get $a(\alpha+lr)^2+b(\beta+mr)^2+c\gamma^2=1$

$r^2(al^2+bm^2)+2r(a\alpha l+b\beta m)+(a\alpha^2+b\beta^2+c\gamma^2-1)=0$ — ①

Since this generator is tangent to the given conicoid so the ~~two~~ values of r obtained from ① must be equal and the condition for same is

$(a\alpha l+b\beta m)^2=(al^2+bm^2)(a\alpha^2+b\beta^2+c\gamma^2-1)$

$\therefore$ The equation of the enveloping cylinder of the given conicoid i.e, the locus of $P(\alpha,\beta,\gamma)$ is $(alx+bmy)^2=(al^2+bm^2)(ax^2+by^2+cz^2-1)$

Its section by the plane $z=0$ is

$(alx+bmy)^2=(al^2+bm^2)(ax^2+by^2-1);\ z=0$

$\Rightarrow a^2l^2x^2+b^2m^2y^2+2ablmxy = a^2l^2x^2+abl^2y^2-al^2+abm^2x^2+b^2m^2y^2-bm^2;\ z=0$

$\Rightarrow ab(m^2x^2+l^2y^2-2lmxy)=al^2+bm^2;\ z=0$

$\Rightarrow ab(mx-ly)^2=al^2+bm^2;\ z=0$

which represents a parabola as the second degree ~~terms~~ form a perfect square.

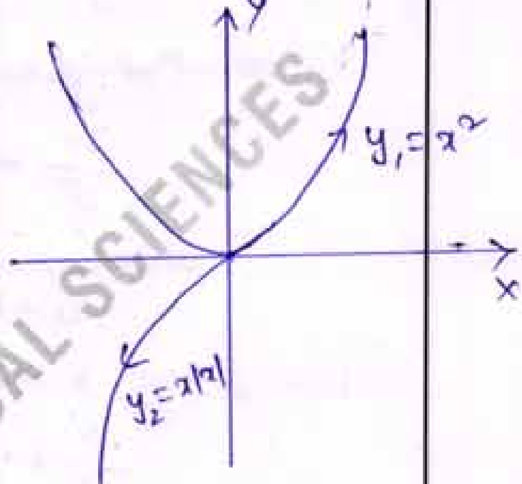
5(a) → Show graphically that $y_1(x) = x^2$ and $y_2(x) = x|x|$ are linearly independent on $-\infty < x < \infty$, however wronskian vanishes for every real value of x.

Sol^n: Given that $y_1(x) = x^2$ and $y_2(x) = x|x|$, $-\infty < x < \infty$

Here $y_2(x) = x|x|$

$$= \begin{cases} x(-x) ; & x < 0 \\ 0 ; & x = 0 \\ x(x) & x > 0 \end{cases}$$

$$= \begin{cases} -x^2 : & x < 0 \\ 0 ; & x = 0 \\ x^2 : & x > 0 \end{cases}$$



According to graph

clearly for $x<0$:

$y_1(x)$ & $y_2(x)$ are linearly independent but wronskian vanishes for every real value of x

i.e. for $x<0$: $W(y_1, y_2) = \begin{vmatrix} x^2 & -x^2 \\ 2x & -2x \end{vmatrix} = 0$

for $x>0$: $W(y_1, y_2) = \begin{vmatrix} x^2 & x^2 \\ 2x & 2x \end{vmatrix} = 0$

5(b) Find the orthogonal trajectories of the family of curves $\frac{x^2}{a^2}+\frac{y^2}{a^2+\lambda}=1$, λ being the parameter.

Sol^n^: The given family of curves is

$$\frac{x^2}{a^2}+\frac{y^2}{a^2+\lambda}=1, \text{ with } \lambda \text{ as parameter} \quad \text{——— (1)}$$

Differentiating (1) w.r.t x, we get

$$\frac{2x}{a^2}+\frac{2y}{a^2+\lambda}\frac{dy}{dx}=0$$

$$\Rightarrow \frac{1}{a^2+\lambda}=\frac{-x}{a^2y}\frac{dx}{dy} \quad \text{—— (2)}$$

Eliminating λ from (1) & (2), we get

$$\frac{x^2}{a^2}+y^2\left(\frac{-x}{a^2y}\cdot\frac{dx}{dy}\right)=1 \Rightarrow \frac{x^2}{a^2}-\frac{xy}{a^2}\frac{1}{dy/dx}=1 \quad \text{— (3)}$$

which is the differential equation of the given family of curves (1).

Replacing dy/dx by $-dx/dy$ in (3), the differential equation of the required orthogonal trajectories is

$$\frac{x^2}{a^2}-\frac{xy}{a^2}\left(-\frac{1}{dx/dy}\right)=1$$

$$\Rightarrow \frac{x^2}{a^2}+\frac{xy}{a^2}\frac{dy}{dx}=1$$

$$\Rightarrow \frac{xy}{a^2}\frac{dy}{dx}=\frac{a^2-x^2}{a^2}$$

$$\Rightarrow y\,dy=\left(\frac{a^2}{x}-x\right)dx$$

Integrating, we get

$$y^2/2=a^2\log x-\tfrac{1}{2}x^2+\tfrac{1}{2}C$$

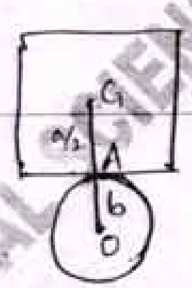
$$x^2+y^2=2a^2\log x+C$$

$$\Rightarrow x^2+y^2-2a^2\log x=C$$

which is the required equation of the orthogonal trajectories.

5(c). A uniform cubical box of edge a is placed on the top of a fixed sphere, the centre of the face of the cube being in contact with the highest point of the sphere. What is the least radius of the sphere for which the equilibrium will be stable?

Sol^n: A uniform cubical box of edge a is placed on the ~~top~~ of a fixed sphere of centre O. The point of contact is A. If G is the centre of Gravity of the box, then for equilibrium the line OAG must be vertical. Let the radius of the sphere be b.



The figure shows the vertical section of the bodies through the point of contact A.

Here ρ_1 = the radius of curvature of the upper body at the point of contact $= \infty$, and ρ_2 = the radius of curvature of the lower body at the point of contact $= b$.

Also h = the height of the C.G. of the box above the point of contact A =

half the edge of the box $= \frac{1}{2}a$

The equilibrium will be stable,

if $\frac{1}{h} > \frac{1}{\rho_1} + \frac{1}{\rho_2}$

i.e, $\frac{1}{\frac{1}{2}a} > \frac{1}{\infty} + \frac{1}{b}$

$\Rightarrow \frac{2}{a} > \frac{1}{b}$

$\Rightarrow b > \frac{1}{2}a.$

Hence the least value of b for the equilibrium to be stable is $\frac{1}{2}a$.

5(d) Prove that Frenet - Serret Formula

(i) $\frac{dT}{ds} = KN$. (ii) $\frac{dB}{ds} = -\tau N$ (iii) $\frac{dN}{ds} = \tau B - KT$

Sol'n: (i) $\frac{dT}{ds} = KN$

Let $\vec{r}(t)$ be the position vector of the point P on the curve then the unit vector T at P is given by $\frac{d\vec{r}}{ds} = T$

Since $|T| = 1$

i.e. T is of constant magnitude.

we have $T \cdot \frac{dT}{ds} = 0$

$\therefore \frac{dT}{ds}$ is $\perp$lar to T.

But we know that $\frac{dT}{ds}$ lies in the osculating plane.

$\therefore \frac{dT}{ds}$ is parallel to $N \Rightarrow \frac{dT}{ds} = \pm KN$ for some scalar K.

By convention, we take +ve sign.

$$\Rightarrow \boxed{\frac{dT}{ds} = KN}$$

(ii) $\frac{dB}{ds} = -\tau N$

Since $|B| = 1$, i.e. B is constant magnitude.

$\therefore B \cdot \frac{dB}{ds} = 0$

$\therefore \frac{dB}{ds}$ is $\perp$lar to B.

we know that $\frac{dB}{ds}$ lies in the osculating plane.

Now we have $B.T = 0$

$$\Rightarrow B.\frac{dT}{ds} + \frac{dB}{ds}.T = 0$$

$$\Rightarrow B.(KN) + \frac{dB}{ds}.T = 0 \quad \text{from (i)}$$

$$\Rightarrow \frac{dB}{ds}.T + (B.N)K = 0$$

$$\Rightarrow \frac{dB}{ds}.T = 0 \qquad (\because B.N = 0)$$

$$\Rightarrow \frac{dB}{ds} \text{ is } \perp\text{lar to } T$$

since $\frac{dB}{ds}$ lies in the osculating plane. so it must be parallel to N. $\therefore \frac{dB}{ds} = \pm \tau N$.

By convention $\frac{dB}{ds} = -\tau N$.

(iii) $\frac{dN}{ds} = \tau B - KT$

Now we have $B \times T = N$

$$\Rightarrow B \times \frac{dT}{ds} + \frac{dB}{ds} \times T = \frac{dN}{ds}$$

$$\Rightarrow B \times (KN) + (-\tau N) \times T = \frac{dN}{ds}$$

$$\Rightarrow \frac{dN}{ds} = K(B \times N) - \tau(N \times T)$$

$$= K(-T) - \tau(-B)$$

$$\frac{dN}{ds} = \tau B - KT$$

Hence the result.

5(e) Show that the vector field defined by $F = (2xy - z^3)\hat{i} + (x^2 + z)\hat{j} + (y - 3xz^2)\hat{k}$ is conservative, and find the scalar potential of F.

Sol^n: we have curl F

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy - z^3 & x^2 + z & y - 3xz^2 \end{vmatrix}$$

$$= \hat{i}\left[\frac{\partial}{\partial y}(y - 3xz^2) - \frac{\partial}{\partial z}(x^2 + z)\right] + \hat{j}\left[\frac{\partial}{\partial z}(2xy - z^3) - \frac{\partial}{\partial x}(y - 3xz^2)\right] + \hat{k}\left[\frac{\partial}{\partial x}(x^2 + z) - \frac{\partial}{\partial y}(2xy - z^3)\right]$$

$$= (1-1)\hat{i} + (-3z^2 + 3z^2)\hat{j} + (2x - 2x)\hat{k}$$

$$= 0$$

$\therefore$ the vector field F is conservative.

Let $F = \nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$. Then

$\frac{\partial\phi}{\partial x} = 2xy - z^3$ when $\phi = x^2y - z^3x + f_1(y,z)$ — (1)

$\frac{\partial\phi}{\partial y} = x^2 + z$ when $\phi = x^2y + zy + f_2(z,x)$ — (2)

$\frac{\partial\phi}{\partial z} = y - 3xz^2$ when $\phi = yz - xz^3 + f_3(x,y)$ — (3)

(1), (2), (3) each represents ϕ. These agree if we choose

$f_1(y,z) = zy$, $f_2(z,x) = -z^3x$, $f_3(x,y) = x^2y$

$\therefore \phi = x^2y - z^3x + zy$ to which may be added any constant.

Hence $\phi = x^2y - z^3x + zy + C$.

6(a) Solve $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + y = \frac{\log x \sin\log x + 1}{x}$

Sol: Given that $x^2\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} + y = \frac{\log x \sin\log x + 1}{x}$ —— (1)

Equation (1) can be written as

$$(x^2D^2 - 3xD + 1)y = x^{-1}\left[1 + \log x \sin\log x\right] \quad \text{—— (2)}$$

Let $x = e^z$ so that $z = \log x$ and let $D_1 = \frac{d}{dz}$ —— (3)

Then $xD = D_1$ and $x^2D^2 = D_1(D_1-1)$ —— (4)

using (3) & (4), (2) reduces to

$$[D_1(D_1-1) - 3D_1 + 1]y = e^{-z}[1 + z\sin z]$$

$$\Rightarrow (D_1^2 - 4D_1 + 1)y = e^{-z} + e^{-z}z\sin z \quad \text{—— (5)}$$

A.E of (5) is $D_1^2 - 4D_1 + 1 = 0$

$$\Rightarrow D_1 = 2 \pm \sqrt{3}$$

$$\therefore C.F. = e^{2z}\left[c_1\cosh(\sqrt{3}z) + c_2\sinh\sqrt{3}z\right]$$

$$y_c = x^2\left[c_1\cosh(\sqrt{3}\log x) + c_2\sinh(\sqrt{3}\log x)\right]$$

P.I. corresponding to e^{-z}

$$= \frac{1}{D_1^2 - 4D_1 + 1}e^{-z} = \frac{1}{1+4+1}e^{-z} = \frac{1}{6}x^{-1} = \frac{1}{6x}$$

and P.I corresponding to $e^{-z}z\sin z$

$$= \frac{1}{D_1^2 - 4D_1 + 1}e^{-z}(z\sin z)$$

$$= e^{-z}\frac{1}{(D_1-1)^2 - 4(D_1-1) + 1}z\sin z$$

$$= e^{-z}\frac{1}{D_1^2 - 6D_1 + 6}z\sin z$$

$$= e^{-z}\left[z\frac{1}{D_1^2 - 6D_1 + 6}\sin z - \frac{2D_1 - 6}{(D_1^2 - 6D_1 + 6)^2}\sin z\right]$$

$$= e^{-z}\left[z\frac{1}{-1-6D_1+6}\sin z-(2D_1-6)\frac{1}{(-1-6D_1+6)^2}\sin z\right]$$

$$= e^{-z}\left[z\frac{1}{5-6D_1}\sin z-\frac{2D_1-6}{(5-6D_1)^2}\sin z\right]$$

$$= e^{-z}\left[z\frac{5+6D_1}{25-36D_1^2}\sin z-(2D_1-6)\frac{1}{25-60D_1+36D_1^2}\sin z\right]$$

$$= e^{-z}\left[z(5+6D_1)\frac{1}{25+36}\sin z-(2D_1-6)\frac{1}{25-60D_1-36}\sin z\right]$$

$$= e^{-z}\left[\frac{z(5+6D_1)}{61}\sin z+(2D_1-6)\frac{1}{11+60D_1}\sin z\right]$$

$$= e^{-z}\left[\frac{z}{61}(5\sin z+6\cos z)+\frac{(2D_1-6)(60D_1-11)}{3600D_1^2-121}\sin z\right]$$

$$= e^{-z}\left[\frac{z}{61}(5\sin z+6\cos z)+\frac{120D_1^2-382D_1+66}{-3600-121}\sin z\right]$$

$$= e^{-z}\left[\frac{z}{61}(5\sin z+6\cos z)+\frac{120(-\sin z)-382\cos z+66\sin z}{-3721}\right]$$

$$= e^{-z}\left[\frac{z}{61}(5\sin z+6\cos z)+\frac{54\sin z+382\cos z}{3721}\right]$$

$$= \frac{1}{x}\left[\frac{\log x}{61}(5\sin\log x+6\cos\log x)+\frac{54\sin(\log x)+382\cos(\log x)}{3721}\right]$$

$$\therefore y = x^2\left[c_1\cosh(\sqrt{3}\log x)+c_2\sinh(\sqrt{3}\log x)\right]+\frac{1}{6x}$$

$$+\frac{1}{x}\left[\frac{\log x}{61}\{5\sin(\log x)+6\cos(\log x)\}+\frac{54\sin\log x+382\cos\log x}{3721}\right]$$

which is the required solution.

6(b) Investigate $(p^2+1)(x-y)^2 = (x+yp)^2$ for singular solution and extraneous loci.

Sol^n: Hint: use $x = r\cos\theta$, $y = r\sin\theta$

G.S. is $(x-c)^2 + (y-c)^2 = c^2$

singular solution $xy = 0$

extraneous loci $y = x$.

6(c) Prove that $L\left\{\dfrac{\cos at - \cos bt}{t}\right\} = \frac{1}{2}\log\dfrac{s^2+b^2}{s^2+a^2}$

Sol^n: Here $L\{\cos at - \cos bt\} = L\{\cos at\} - L\{\cos bt\}$

$$\therefore L\{\cos at - \cos bt\} = \frac{s}{s^2+a^2} - \frac{s}{s^2+b^2} = f(s), \text{ say}$$

$$\therefore L\left\{\frac{\cos at - \cos bt}{t}\right\} = \int_s^\infty f(s)\,ds$$

$$= \int_s^\infty \left\{\frac{s}{s^2+a^2} - \frac{s}{s^2+b^2}\right\} ds$$

$$= \left[\frac{1}{2}\log(s^2+a^2) - \frac{1}{2}\log(s^2+b^2)\right]_s^\infty$$

$$= \frac{1}{2}\left[\log\frac{s^2+a^2}{s^2+b^2}\right]_s^\infty$$

$$= \frac{1}{2}\lim_{s\to\infty}\log\frac{s^2+a^2}{s^2+b^2} - \frac{1}{2}\log\frac{s^2+a^2}{s^2+b^2}$$

$$= \frac{1}{2}\lim_{s\to\infty}\log\frac{1+a^2/s^2}{1+b^2/s^2} + \frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}$$

$$= 0 + \frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}$$

$$= \frac{1}{2}\log\frac{s^2+b^2}{s^2+a^2}$$

6(d) By using Laplace transform, solve $(D^3-D^2-D+1)y = 8te^{-t}$ if $y = Dy = D^2y = 0$ when $t=0$

Sol: Taking the Laplace transform on both sides of the given equation we have

$$L\{y'''\} - L\{y''\} - L\{y'\} + L\{y\} = 8L\{te^{-t}\}$$

$$\Rightarrow [p^3L\{y\} - p^2\{y(0)\} - p\,y(0) - y''(0)] - [p^2L\{y\} - p\,y(0) - y'(0)] - [pL\{y\} - y(0)] + L\{y\} = -8\frac{d}{dp}[L\{e^{-t}\}]$$

$$\Rightarrow [p^3 - p^2 - p + 1]L\{y\} = -8\frac{d}{dp}\left(\frac{1}{p+1}\right)$$

$$\Rightarrow (p^3 - p^2 - p + 1)L\{y\} = \frac{8}{(p+1)^2}$$

$$\Rightarrow L\{y\} = \frac{8}{(p+1)^3(p-1)^2}$$

$$\Rightarrow L\{y\} = -\frac{3}{2(p-1)} + \frac{1}{(p-1)^2} + \frac{3}{2(p+1)} + \frac{2}{(p+1)^2} + \frac{2}{(p+1)^3}$$

$$\Rightarrow y = -\frac{3}{2}L^{-1}\left\{\frac{1}{p-1}\right\} + L^{-1}\left\{\frac{1}{(p-1)^2}\right\} + \frac{3}{2}L^{-1}\left\{\frac{1}{p+1}\right\} + 2L^{-1}\left\{\frac{1}{(p+1)^2}\right\} + 2L^{-1}\left\{\frac{1}{(p+1)^3}\right\}$$

$$= -\frac{3}{2}e^t + e^tL^{-1}\left(\frac{1}{p^2}\right) + \frac{3}{2}e^{-t} + 2e^{-t}L^{-1}\left\{\frac{1}{p^2}\right\} + 2e^{-t}L^{-1}\left\{\frac{1}{p^3}\right\}$$

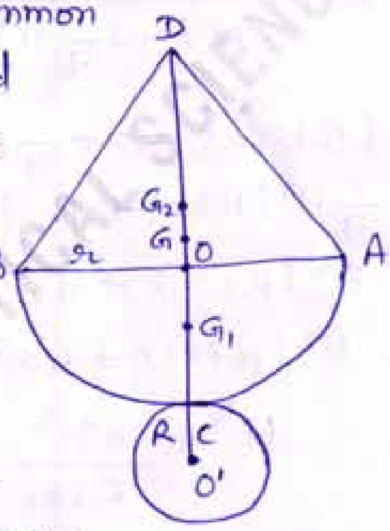
$$= -\frac{3}{2}e^t + e^tt + \frac{3}{2}e^{-t} + 2e^{-t}t + 2e^{-t}\frac{t^2}{2!}$$

$$= -\frac{3}{2}e^t + e^tt + \frac{3}{2}e^{-t} + 2e^{-t}t + t^2e^{-t}$$

$$= \left(t^2 + 2t + \frac{3}{2}\right)e^{-t} + \left(t - \frac{3}{2}\right)e^t$$

7(a) A solid homogeneous hemisphere of radius r has a solid right circular cone of the same substance constructed on the base; the hemisphere rests on the convex side of the fixed sphere of radius R. Show that the length of the axis of the cone consistent with stability for a small rolling displacement is $\frac{r}{R+r}\left[\sqrt{\{(3R+r)(R-r)\}}-2r\right]$.

Sol^n:- Let O be the centre of the common base AB of the hemisphere and the cone. The hemisphere rests on a fixed sphere of Radius R and centre O', their point of contact being C. for equilibrium the line O'COD must be vertical. Let H be the length of the axis OD of the cone.



It is given that OB=OC= r. the radius of the hemisphere.

If G_1 and G_2 are the centres of gravity of the hemisphere and the cone resp., then

$$OG_1 = 3r/8 \text{ and } OG_2 = H/4$$

Let G be the centre of gravity of the combined body composed of the hemisphere and the cone. If h be the height of G above the point of contact C, then

$$h = \frac{\frac{2}{3}\pi r^3 \cdot \frac{5}{8}r + \frac{1}{3}\pi r^2 H\left(r+\frac{1}{4}H\right)}{\frac{2}{3}\pi r^3 + \frac{1}{3}\pi r^2 H} = \frac{H\left(r+\frac{1}{4}H\right)+\frac{5}{4}r^2}{H+2r}$$

Here ρ_1 = the radius of curvature at the point of contact C of the upper body $= r$

and ρ_2 = the radius of curvature at C of the lower body $= R$.

The equilibrium will be stable if,

$$\frac{1}{h} > \frac{1}{\rho_1} + \frac{1}{\rho_2} \quad \text{i.e.,} \quad \frac{1}{h} > \frac{1}{r} + \frac{1}{R}$$

i.e., $$\frac{H+2r}{H\left(r+\frac{1}{4}H\right)+\frac{5}{4}r^2} > \frac{R+r}{rR}$$

i.e. $(R+r)\left\{Hr+\frac{1}{4}H^2+\frac{5}{4}r^2\right\} - rR(H+2r) < 0$

i.e., $\frac{1}{4}H^2(R+r) + H\{(R+r)r - rR\} + \frac{5}{4}r^2(R+r) - 2r^2R < 0$

i.e., $H^2(R+r) + 4r^2H + 5r^3 - 3r^2R < 0$

i.e., $H^2(R+r) + 4r^2H - r^2(3R-5r) < 0$

i.e., $$H^2 + \frac{4r^2}{R+r}H - \frac{r^2(3R-5r)}{R+r} < 0$$

i.e. $$\left(H+\frac{2r^2}{R+r}\right)^2 - \frac{4r^4}{(R+r)^2} - \frac{r^2(3R-5r)}{R+r} < 0$$

i.e. $$\left(H+\frac{2r^2}{R+r}\right)^2 - \frac{4r^4 + r^2(3R-5r)(R+r)}{(R+r)^2} < 0$$

i.e. $$\left(H+\frac{2r^2}{R+r}\right)^2 - \frac{r^2\left[4r^2+3R^2-2rR-5r^2\right]}{(R+r)^2} < 0$$

i.e., $$\left(H+\frac{2r^2}{R+r}\right)^2 - \frac{r^2(3R^2-2rR-r^2)}{(R+r)^2} < 0$$

i.e., $$\left(H+\frac{2r^2}{R+r}\right)^2 < \frac{r^2(3R+r)(R-r)}{(R+r)^2}$$

ie, $H + \frac{2r^2}{R+r} < \frac{r}{R+r}\sqrt{\{(3R+r)(R-r)\}}$

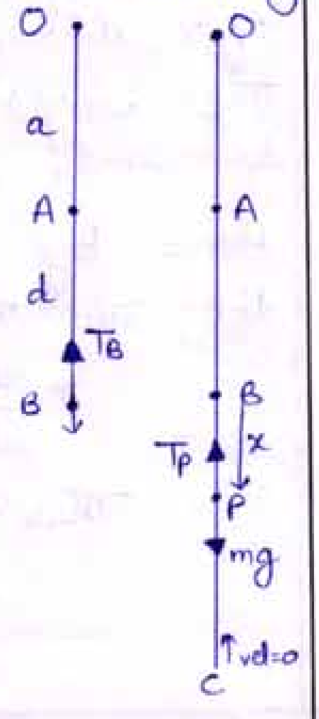
i.e., $H < \frac{r}{R+r}\sqrt{\{(3R+r)(R-r)\}} - \frac{2r^2}{R+r}$

ie $H < \frac{r}{R+r}\left[\sqrt{\{(3R+r)(R-r)\}} - 2r\right]$

Therefore the greatest value of H consistent with the stability of equilibrium is

$$\frac{r}{R+r}\left[\sqrt{\{(3R+r)(R-r)\}} - 2r\right].$$

7(b) → A heavy particle is attached to one end of an elastic string, the other end of which is fixed. The modulus of elasticity of the string is equal to the weight of the particle. The string is drawn vertically down till it is four times its natural length 'a' and then let go. find the time taken by the particle to return to the starting point.



Sol^n :- Let OA = a be the natural length of an elastic string whose one end is fixed at O. Let B be the position of equilibrium of a particle of mass m attached to the other end of the string and AB = d. If T_B is the tension in the string OB, then by Hooke's law,

$$T_B = \lambda\,\frac{OB-OA}{OA} = \lambda\frac{d}{a},$$

Where λ is the modulus of elasticity of the string. Considering the equilibrium of the particle at B, We have,

$$mg = T_B = \lambda \frac{d}{a} = mg\frac{d}{a}, \left[\because \lambda = mg, \text{as given}\right]$$

$$\therefore d = a$$

Now the particle is pulled down to a point C such that OC = 4a and then let go. It starts moving towards B with velocity zero at C. Let P be the position of the particle at time t, where BP = x.

[Note:- that we have taken the position of equilibrium B as origin. The direction BP is that of x increasing and the direction PB is that of x decreasing.]

When the particle is at P, there are two forces acting upon it.

(i) The tension $T_P = \lambda \frac{a+x}{a} = \frac{mg}{a}(a+x)$ in the string OP acting in the direction PO. ie., in the direction of x decreasing.

(ii) The weight mg of the particle acting vertically downwards i.e., in the direction of x increasing.

Hence by Newton's second law of motion (P = mf) the equation of motion of the particle at P is

$$m\frac{d^2x}{dt^2} = mg - \frac{mg}{a}(a+x) = -\frac{mgx}{a}$$

Thus $$\frac{d^2x}{dt^2} = -\frac{g}{a}x \quad \text{—— (1)}$$

Which is the equation of a S.H.M. with centre at the origin B and the amplitude BC = 2a which is greater than AB = a.

Multiplying both sides of (1) by $2(dx/dt)$ and integrating w.r.t. t, we get

$$\left(\frac{dx}{dt}\right)^2 = \frac{-g}{a}x^2 + k,$$ where k is a constant.

At the point C, $x = BC = 2a$, and the velocity $\frac{dx}{dt} = 0$,

$$\therefore\ k = \frac{g}{a}\cdot 4a^2$$

$$\therefore\ \left(\frac{dx}{dt}\right)^2 = \frac{g}{a}(4a^2 - x^2). \quad \text{—— (2)}$$

Taking square root of (2), we get,

$$\frac{dx}{dt} = -\sqrt{\left(\frac{g}{a}\right)}\sqrt{(4a^2 - x^2)},$$

The −ve sign has been taken because the particle is moving in the direction of x de- -creasing.

Separating the variables, we have

$$dt = -\sqrt{\left(\frac{g}{a}\right)}\frac{dx}{\sqrt{(4a^2 - x^2)}} \quad \text{—— (3)}$$

If t_1 be the time from C to A, then integra- -ting (3) from C to A we get

$$\int_0^{t_1} dt = -\sqrt{(g/a)}\int_{2a}^{-a}\frac{dx}{\sqrt{(4a^2 - x^2)}}$$

or $$t_1 = \sqrt{(g/a)}\left[\cos^{-1}\frac{x}{2a}\right]_{2a}^{-a}$$

$$= \sqrt{(a/g)}\left[\cos^{-1}(-1/2) - \cos^{-1}(1)\right] = \sqrt{(a/g)}\cdot\frac{2\pi}{3}.$$

Let v_1 be the velocity of the particle at A.
Then at A :-

$$x = -a \text{ and } (dx/dt)^2 = v_1^2$$

So from (2), we have $v_1^2 = (g/a)(4a^2 - a^2)$
or $v_1 = \sqrt{(3ag)}$; the direction of v_1 being vertically upwards.

Thus the velocity at A is $\sqrt{(3ag)}$ and is in the upwards direction so that the particle rises above A. Since the tension of the string vanishes at A, therefore at A the simple harmonic motion ceases and the particle when string rising above A moves freely under gravity. Thus the particle rising from A with velocity $\sqrt{(3ag)}$ moves upwards till this velocity is destroyed. The time t_2 for this motion is given by

$$0 = \sqrt{(3ag)} - gt_2 \quad \cdot \text{ so that } t_2 = \sqrt{\left(\frac{3a}{g}\right)}$$

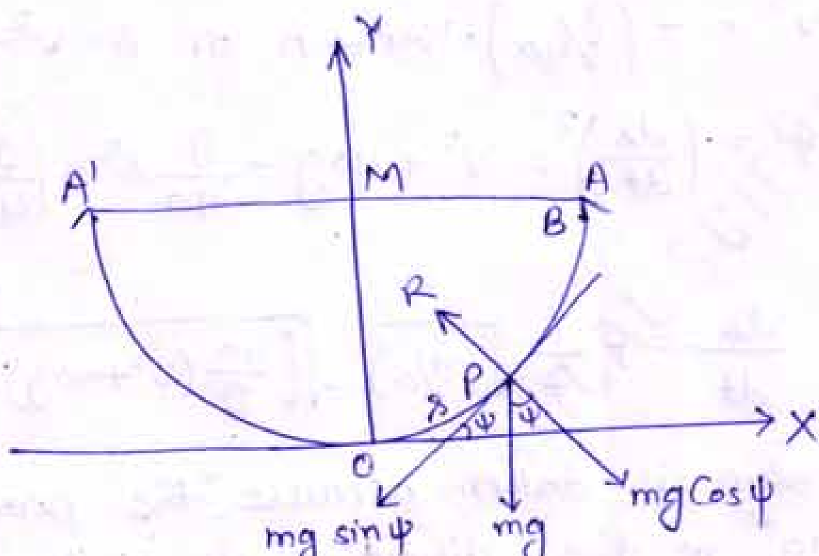
Conditions being the same, the equal time t_2 is taken by the particle in falling freely back to A. from A to C the particle will take the same time t_1 as it takes from C to A. Thus the whole time taken by the particle to return to $C = 2(t_1 + t_2)$

$$= 2\left[\sqrt{(a/g)}\cdot\frac{2\pi}{3} + \sqrt{\left(\frac{3a}{g}\right)}\right]$$

$$= \sqrt{\left(a/g\right)}\left[\frac{4\pi}{3} + 2\sqrt{3}\right].$$

7(c) A particle is projected with velocity V from the cusp of a smooth inverted cycloid down the arc, show that the time of reaching the vertex is $2\sqrt{(a/g)}\tan^{-1}\left[\sqrt{(4ag/V)}\right]$.

Soln:-



Let a particle be projected with velocity V from the cusp A of a smooth inverted cycloid down the arc. If P is the position of the particle at time t such that the tangent at P is inclined at an angle ψ to the horizontal and arc $OP = s$, then the equations of motion of the particle are

$$m\frac{d^2s}{dt^2} = -mg\sin\psi \quad \text{—— (1)}$$

and $$m\frac{v^2}{\rho} = R - mg\cos\psi \quad \text{—— (2)}$$

for the cycloid, $s = 4a\sin\psi$ —— (3)

from (1) and (3), we have $\frac{d^2s}{dt^2} = -\frac{g}{4a}s$.

multiplying both sides by $2(ds/dt)$ and integrating, we have -

$$v^2 = \left(\frac{ds}{dt}\right)^2 = -\frac{g}{4a}s^2 + A$$

But initially at the cusp A, $s = 4a$ and $\left(\frac{ds}{dt}\right)^2 = V^2$.

$$\therefore \quad V^2 = -(g/4a)\cdot 16a^2 + A \quad \text{or} \quad A = V^2 + 4ag$$

$$\therefore \quad v^2 = \left(\frac{ds}{dt}\right)^2 = V^2 + 4ag - \frac{g}{4a}s^2 = \left(\frac{g}{4a}\right)\left[\frac{4a}{g}(V^2+4ag) - s^2\right]$$

or $$\frac{ds}{dt} = -\frac{1}{2}\sqrt{(g/a)}\,\sqrt{\left[\frac{4a}{g}(V^2+4ag) - s^2\right]}$$

(-ve sign is taken because the particle is moving in the direction of s decreasing.)

or $$dt = -2\sqrt{(a/g)}\,\frac{ds}{\sqrt{[(4a/g)(V^2+4ag) - s^2]}}$$

Integrating, the time t_1 from the cusp A to the vertex O is given by

$$t_1 = -2\sqrt{(a/g)}\int_{s=4a}^{0}\frac{ds}{\sqrt{[(4a/g)(V^2+4ag) - s^2]}}$$

$$= 2\sqrt{(a/g)}\int_0^{4a}\frac{ds}{\sqrt{[(4a/g)(V^2+4ag) - s^2]}}$$

$$= 2\sqrt{(a/g)}\left[\sin^{-1}\frac{s}{2\sqrt{(a/g)}\sqrt{(V^2+4ag)}}\right]_0^{4a}$$

$$= 2\sqrt{(a/g)}\sin^{-1}\left\{\frac{2\sqrt{(ag)}}{\sqrt{(V^2+4ag)}}\right\}$$

$$= 2\sqrt{(a/g)}\cdot\theta \qquad \text{—— (4)}$$

where $\theta = \sin^{-1}\left\{\frac{2\sqrt{(ag)}}{\sqrt{(v^2+4ag)}}\right\}$

we have $\sin\theta = \frac{2\sqrt{(ag)}}{\sqrt{(v^2+4ag)}}$

$\therefore \cos\theta = \sqrt{(1-\sin^2\theta)} = \sqrt{\left[1-\frac{4ag}{v^2+4ag}\right]}$

$\therefore \tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{2\sqrt{(ag)}}{v} = \frac{\sqrt{(4ag)}}{v}$

or $\theta = \tan^{-1}\left[\sqrt{(4ag/v)}\right.$

$\therefore$ from (4), the time of reaching the vertex is

$$= 2\sqrt{(a/g)}\tan^{-1}\left[\sqrt{(4ag)}/v\right]$$

8(a) Find the values of constants a, b, c so that the directional derivative of $\phi = axy^2 + byz + cz^2x^3$ at $(1,2,-1)$ has a maximum magnitude 64 in a direction parallel to z-axis.

Sol'n. we have $\text{grad}\,\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{k}$

$$= (ay^2 + 3cz^2x^2)\hat{i} + (2axy + bz)\hat{j} + (by + 2czx^3)\hat{k}$$

$$= (4a+3c)\hat{i} + (4a-b)\hat{j} + (2b-2c)\hat{k} \text{ at the point } (1,2,-1)$$

Now the directional derivative of ϕ at the point $(1,2,-1)$ is maximum in the direction of the vector gradϕ at this point. According to the question this directional derivative is maximum in the direction parallel to z-axis. i.e. in the direction parallel to vector $\hat{k}$.

So if the direction of the vector $(4a+3c)\hat{i}+(4a-b)\hat{j}+(2b-2c)\hat{k}$ is parallel to the vector $\hat{k}$, we must have

$$4a+3c=0 \quad —(1) \quad \text{and} \quad 4a-b=0 \Rightarrow b=4a \quad —(2)$$

then grad ϕ at $(1,2,-1)=(2b-2c)\hat{k}$

Also the maximum value of directional derivative

$$= |\text{grad}\,\phi|$$

$$\therefore |(2b-2c)k| = 64$$

$$\Rightarrow 2b-2c=64$$

$$\Rightarrow b-c=32$$

$$\Rightarrow b=32+c \quad —(3)$$

from (2) & (3) $32+c=4a$

$$\Rightarrow 4a-c=32 \quad —(4)$$

from (1) & (4) $\boxed{c=-8}$

from (3) $\boxed{b=24}$

from (1) $\boxed{a=6}$

$\therefore a=6,\ b=24,\ c=-8$.

8(b)(i) Show that $E = \frac{\vec{r}}{r^2}$ is irrotational find ϕ such that $E = -\nabla\phi$ and such that $\phi(a) = 0$ where $a > 0$.

Sol^n:- E is irrotional if $\nabla \times E = 0$.

Now $\nabla \times E = \nabla \times \left(\frac{\vec{r}}{r^2}\right)$

$= \nabla\left(\frac{1}{r^2}\right) \times \vec{r} + \frac{1}{r^2}(\nabla \times \vec{r})$

$[\because \nabla \times (\phi A) = (\nabla\phi \times A) + \phi(\nabla \times A)]$

$= -\frac{2}{r^4}(\vec{r} \times \vec{r}) + 0$ $\quad (\because \nabla \times \vec{r} = 0)$

$= -\frac{2}{r^4}(0)$ $\quad (\because \vec{r} \times \vec{r} = 0)$

$= 0$

$\therefore \nabla \times E = 0$

$\Rightarrow$ E is irrotational.

Given $E = -\nabla\phi = \frac{\vec{r}}{r^2}$.

$\Rightarrow \nabla\phi = -\frac{\vec{r}}{r^2}$

$\Rightarrow i\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z} = -\frac{\vec{r}}{r^2}$ — ①

$\frac{\partial\phi(r)}{\partial x} = \phi'(r)\frac{\partial r}{\partial x}$ $\quad \left(\because r^2 = x^2+y^2+z^2,\ 2r\frac{\partial r}{\partial x} = 2x,\ \frac{\partial r}{\partial x} = \frac{x}{r}\right)$

$\Rightarrow \frac{\partial\phi}{\partial x}(r) = \phi'(r)\frac{x}{r}$

Similarly $\frac{\partial\phi}{\partial y} = \phi'(r)\frac{y}{r}$ & $\frac{\partial\phi}{\partial z} = \phi'(r)\left(\frac{z}{r}\right)$

$\therefore$ From ①, we have

$\phi'(r)\left(\frac{x}{r}\hat{i}+\frac{y}{r}\hat{j}+\frac{z}{r}\hat{k}\right)=-\frac{\vec{r}}{r^2}$

$\Rightarrow \phi'(r)\cdot\frac{\vec{r}}{r}=-\frac{\vec{r}}{r^2}$

$\Rightarrow \phi'(r)=-\frac{1}{r}$

- Integrating, we get

$\phi(r)=-\log r+\log c$ — (2)

$\Rightarrow \phi(r)=\log\left(\frac{c}{r}\right)$

Given $\phi(a)=0$

$\therefore$ from (2), $\phi(a)=-\log(r)+\log c$

$\Rightarrow 0=-\log a+\log c$

$\Rightarrow \log c=\log a$

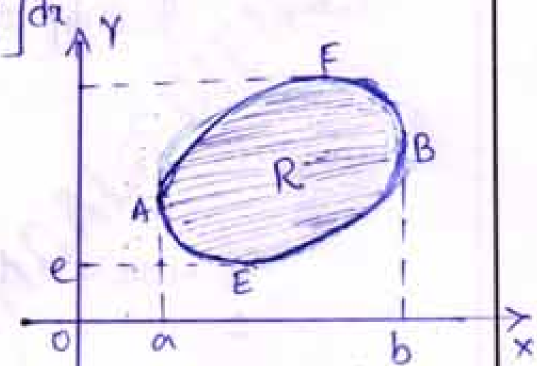
$\Rightarrow \boxed{c=a}$

$\therefore$ from (2) $\boxed{\phi(r)=\log\frac{a}{r}}$

8(C) Prove Green's theorem in the plane if C is a closed curve which has the property that any straight line parallel to the coordinate axes cuts C in atmost two points.

Sol'n: Let the equations of the curves AEB and AFB be $y = Y_1(x)$ and $y = Y_2(x)$ respectively

$$\iint_R \frac{\partial M}{\partial y}\,dx\,dy = \int_{x=a}^{b}\left[\int_{y=Y_1(x)}^{Y_2(x)} \frac{\partial M}{\partial y}\,dy\right]dx$$



$$= \int_{x=a}^{b} M(x,y)\Big|_{Y_1(x)}^{Y_2(x)}\,dx$$

$$= \int_a^b \left[M(x,Y_2) - M(x,Y_1)\right]dx$$

$$= -\int_a^b M(x,Y_1)\,dx - \int_b^a M(x,Y_2)\,dx$$

$$= -\oint_C M\,dx$$

Then $\oint_C M\,dx = -\iint_R \frac{\partial M}{\partial y}\,dx\,dy$ ——— (1)

Similarly let the equations curves EAF and EBF be $x = X_1(y)$ and $x = X_2(y)$ respectively. Then

$$\iint_R \frac{\partial N}{\partial x}\,dx\,dy = \int_{y=e}^{f}\left[\int_{x=X_1(y)}^{X_2(y)} \frac{\partial N}{\partial x}\,dx\right]dy$$

$$= \int_e^f \left[N(x_2, y) - N(x_1, y) \right] dy$$

$$= \int_f^e N(x_1, y)\,dy + \int_e^f N(x_2, y)\,dy$$

$$= \oint_C N\,dy$$

Then $\oint_C N\,dy = \iint_R \frac{\partial N}{\partial x}\,dx\,dy$ —— ②

Adding ① & ②

$$\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dx\,dy.$$

8(d) verify Stokes theorem for the vector $A = 3xi - xzj + yz^2k$, where S is the surface of the paraboloid $2z = x^2+y^2$ bounded by $z=2$ and C is its boundary.

Sol: The boundary C of the surface S is the circle in the plane $z=2$ whose equations are $x^2+y^2=4$, $z=2$. The radius of this circle is 2 and centre $(0,0,2)$

Suppose $x = 2\cos t$, $y = 2\sin t$, $z=2$, $0 \le t < 2\pi$ are parametric equations of C.

By Stokes theorem

$$\oint_C A\cdot dr = \iint_S (\text{curl} A)\cdot n\, ds,$$

where n is a unit vector along outward drawn normal to the surface

we have

$$\oint_C A\cdot dr = \int_C (3xi - xzj + yz^2k)\cdot(dxi + dyj + dzk)$$

$$= \int_C 3x\,dx - xz\,dy + yz^2\,dz$$

$$= \int_C 3x\,dx - 2x\,dy \quad , \text{ Since on C, } z=2 \text{ and } dz=0$$

$$= \int_{2\pi}^{0} \left(3x\frac{dx}{dt} - 2x\frac{dy}{dt}\right) dt$$

$$= \int_{2\pi}^{0} x\left(3\frac{dx}{dt} - 2\frac{dy}{dt}\right) dt$$

Note that here the surface S lies below the curve C and so direction of C is positive if it C is traversed in clockwise sense.

$$= -\int_{0}^{2\pi} 2\cos t\left(3\cdot(-2\sin t) - 2(2\cos t)\right]$$

$$= -\int_{0}^{2\pi} \left(-12\sin t\cos t - 8\cos^2 t\right) dt$$

$$= \int_{0}^{2\pi} (6\sin 2t + 8\cos^2 t)dt = \int_{0}^{2\pi} [6\sin 2t + 4(1-\cos 2t)]dt$$

$$= -\frac{6\cos 2t}{2} + 4t - 2\sin 2t\Big]_{0}^{2\pi}$$

$= 8\pi$ ← (I)

Let S_1 be the plane region bounded by the circle C. If S' is the surface consisting of the surfaces S and S_1, then S' is a closed surface.

Let V be the volume bounded by S'.

By Gauss divergence theorem, we have

$$\iint_{S'} (\text{curl} A)\cdot \vec{n}\, ds = \iiint_V \text{div.}\,\text{curl} A\, dv$$
$$= 0 \quad \text{Since div curl} A = 0$$

$\therefore \iint_S (\text{curl}A\cdot\hat{n})\,ds + \iint_{S_1} (\text{curl}A)\cdot\hat{n}\,ds = 0$

($\because S'$ consists of S & S_1

$\Rightarrow \iint_S (\text{curl}A)\cdot\hat{n}\,ds = -\iint_{S_1} \text{curl}A\cdot\hat{n}\,ds$

$= -\iint_{S_1} \text{curl}A\cdot\hat{k}\,ds$ ($\because$ on S_1, $\hat{n}=\hat{k}$)

Now

$$\text{curl}A = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 3x & -xz & yz^2 \end{vmatrix}$$

$= \hat{i}(z^2+x) - \hat{j}(0-0) + \hat{k}(-z)$

$= (z^2+x)\hat{i} - z\hat{k}$

$\therefore \iint_S \text{curl}A\cdot n\,ds = -\iint_{S_1} \left[(z^2+x)\hat{i} - z\hat{k}\right]\cdot\hat{k}\,ds$

$= \iint_{S_1} z\,ds$

$= \iint_{S_1} 2\,ds$ [$\because$ on S_1, $z=2$]

$= 2S_1$, where S_1 is the area of a circle of radius 2.

$= 2\cdot\pi(2)^2$

$= 8\pi$ ⟶ (2)

$\therefore$ From (1) & (2), we see that $\oint A\cdot dr = \iint \text{curl}A\cdot\hat{n}\,ds$

This verifies Stoke's theorem.